DETECTING ELEMENTS AND LUSTERNIK–SCHNIRELMANN CATEGORY OF 3-MANIFOLDS

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ABSTRACT. In this paper, we give a new simplified calculation of the Lusternik–
Schnirelmann category of closed 3-manifolds. We also describe when 3-manifolds
have detecting elements and prove that 3-manifolds satisfy the equality of the
Ganea conjecture.

1. INTRODUCTION

The Lusternik–Schnirelmann category of a space $X$, denoted $\text{cat}(X)$, is defined
to be the minimal integer $k$ such that there exists an open covering $\{A_0, \ldots, A_k\}$
of $X$ with each $A_i$ contractible to a point in $X$. Category, while easy to define, is
notoriously difficult to compute in general. In particular, except for $K(\pi,1)$’s, it
cannot be expected that the category of a space is determined by its fundamental
group. In GoGe, however, the following interesting result was proved.

1.1. Theorem. Let $M^3$ be a closed 3-dimensional manifold. Then

$$
cat(M) = \begin{cases} 
1 & \text{if } \pi_1(M) = \{1\} \\
2 & \text{if } \pi_1(M) \text{ is free} \\
3 & \text{otherwise}
\end{cases}
$$

In this paper, we will give a somewhat simplified proof of this theorem using the
relatively new approximating invariant for category, category weight. Throughout,
we use only basic results about 3-manifolds found, for instance, in H. But we shall
also do more. We will prove that most 3-manifolds possess a detecting element;
that is, an element whose category weight is equal to the category of $M$ (see R3). It
is known that a detectable space (i.e., a space possessing detecting elements)
has some special properties which allow solutions of certain well-known problems
(R3). For example, from the existence of detecting elements, we prove that closed
3-manifolds satisfy the Ganea conjecture.

1.2. Corollary. For every closed 3-manifold $M$,

$$
cat(M \times S^n) = cat(M) + 1.
$$

This result is not obtainable from knowing the category alone, so the detecting
element approach is a significant embellishment of Theorem 1.1. Another well-
known problem is the relationship between degree 1 maps of manifolds and LS-
category. For closed, 3-manifolds, we have

1.3. Corollary. Let $f : M \rightarrow N$ be a degree 1 map of oriented 3-manifolds. Then
cat$f \geq cat M = cat N$.

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2. Preliminaries on 3-Manifolds

2.1. Definition. A 3-manifold $M$ is irreducible if every embedded two-sphere $S^2 \hookrightarrow M$ bounds an embedded disk $D^3 \hookrightarrow M$.

A 3-manifold $M$ is prime if $M = P \# Q$ implies that either $P = S^3$ or $Q = S^3$. Here, “=” denotes diffeomorphism and # is the connected sum.

The following two results clarify the relation between prime and irreducible manifolds.

2.2. Lemma. If $M^3$ is irreducible, then it is prime.

Proof. Suppose $M$ is irreducible. In order to split $M$ as $M = P \# Q$, there must be an embedded $S^2$ which separates $M$ into two components (i.e. $P - D^3$ and $Q - D^3$). But any such $S^2$ bounds an embedded disk $D^3$ by irreducibility, so $M$ can only split as $M = M' \# S^3$ (since $S^3 - D^3$ is a disk $D^3$). This says that $M$ is prime. \qed

2.3. Lemma. If $M$ is a prime 3-manifold and $M$ is not irreducible, then $M$ is the total space of a 2-sphere bundle over $S^1$.

Proof. See [H, Lemma 3.13] \qed

The fundamental structural result about 3-manifolds is the following

2.4. Theorem (Prime Decomposition). A 3-manifold $M$ may be written as

$$M = M_1 \# M_2 \# \ldots \# M_k,$$

where each $M_j$ is prime. Furthermore, such a prime decomposition is unique up to re-arrangement of summands.

Proof. See [H, Theorems 3.15 and 3.21] \qed

The Sphere theorem says that, for an orientable 3-manifold $M$, $\pi_2(M) \neq 0$ implies that some element of $\pi_2(M)$ is represented by an embedding $S^2 \hookrightarrow M$. We will require the following generalization.

2.5. Theorem (The Projective Plane Theorem). Let $M$ be a 3-manifold with $\pi_2(M) \neq 0$. Then there exists a map $g: S^2 \to M$ with the following properties.

1. The map $g$ is not null-homotopic.
2. The map $g: S^2 \to g(S^2)$ is a covering map.
3. $g(S^2)$ is a 2-sided submanifold (2-sphere or projective plane) in $M$.

Proof. See [H, Theorem 4.12]. \qed

With these preliminaries, we can prove the following important characterization.

2.6. Proposition. Let $M$ be a closed 3-manifold. Then,

1. If $\pi = \pi_1(M)$ is infinite and $\pi_2(M) = 0$, then $M = K(\pi, 1)$.
2. If $\pi_1(M)$ is finite, then the universal covering of $M$ is a homotopy 3-sphere and $M$ is orientable.
**Proof.** For (1), assume that $\pi_1(M)$ is infinite. Let $p: \tilde{M} \to M$ be the universal covering of $M$. Since $\pi_2(M) = 0$, we conclude that $H_2(\tilde{M}) = 0$. Since $\pi_1(M)$ is infinite, we conclude that $\tilde{M}$ is not compact, and therefore $H_3(\tilde{M}) = 0$. Hence, $\tilde{M}$ is acyclic. Moreover, $\tilde{M}$ is simply-connected, and, by the Whitehead theorem, it is therefore contractible. Hence, $\tilde{M} = K(\pi_1(M), 1)$.

For (2), assume that $\pi_1(M)$ is finite. Then the universal cover $\tilde{M}$ of $M$ is a closed simply connected manifold. So, by Poincaré duality, $H_3(\tilde{M}) = 0$, and hence, by the Hurewicz theorem, $\pi_2(\tilde{M}) = 0$. Thus, $\pi_2(M) = 0$. Furthermore,

$$H_3(\tilde{M}) = \mathbb{Z} = \pi_3(\tilde{M}),$$

again by the Hurewicz theorem. Therefore, the generator of $\pi_3(\tilde{M}) = \mathbb{Z}$ provides a degree 1 map $S^3 \to \tilde{M}$ (i.e. an isomorphism on $H_3$). Since $\tilde{M}$ and $S^3$ are simply connected, the Whitehead theorem implies that $\tilde{M} \simeq S^3$.

To see that $M$ is orientable, we simply note that each $g \in \pi_1(M)$, thought of as a covering transformation on the orientable manifold $\tilde{M}$, acts to preserve orientation. This is seen by supposing the opposite; namely, that $g$ reverses orientation. Now, because $\tilde{M} \simeq S^3$, homotopy classes of maps $\tilde{M} \to M$ are classified by degree. Since $g$ is a homeomorphism which reverses orientation, its degree is $-1$. But then the Lefschetz number of $g$ is $L(g) = 2$, implying the existence of a fixed point and contradicting the fact that $g$ is a covering transformation. Hence, all covering transformations preserve orientation, so $M = \tilde{M}/\pi_1(M)$ is orientable. \hfill $\Box$

These are the only ingredients from 3-manifold theory that we shall need. In the next section, we introduce the main technical tool, the approximating invariant category weight.

### 3. Category Weight and Detecting Elements

#### 3.1. Definition ([BG, Fe, F]).

Let $f: X \to Y$ be a map of finite CW-spaces. The **Lusternik-Schnirelmann category** of $f$, denoted $\text{cat}(f)$, is defined to be the minimal integer $k$ such that there exists an open covering $\{A_0, \ldots, A_k\}$ of $X$ with the property that each of the restrictions $f|A_i: A_i \to Y$, $i = 0, 1, \ldots, k$ is null-homotopic.

Clearly, $\text{cat}(X) = \text{cat}(1_X)$ and. Also, it is easy to see that $\text{cat}(f) \leq \text{cat}(X)$ since $f$ is null-homotopic on any subset which is contractible in $X$.

#### 3.2. Definition.

The **category weight** of a non-zero cohomology class $u \in H^*(X; R)$ (for some, possibly local, coefficient ring $R$) is defined by

$$\text{wgt}(u) \geq k \text{ if and only if } \phi^*(u) = 0 \text{ for any } \phi: A \to X \text{ with } \text{cat}(\phi) < k.$$

#### 3.3. Remarks.

1. The idea of category weight was suggested by Fadell and Husseini (see [FH]). In fact, they considered an invariant similar to our wgt (denoted in [FH] by cwgt), but where the defining maps $\phi: A \to X$ were required to be inclusions instead of general maps. Because of this, cwgt was not a homotopy invariant, and this made it a delicate quantity in homotopy calculations. Rudyak in [R2, R3] and Strom in [S] suggested the homotopy invariant version of category weight as defined in Definition 3.2. Rudyak called it **strict category weight** (using the notation $\text{swgt}(u)$) and Strom called it **essential category weight** (using the notation...
to a fibration \( p \). The composition only if \( k \) only if \( k \) can also be shown that, for a cohomology class \( u \in \tilde{X} \) having fibre \( \Omega \) is defined inductively starting with the path fibration \( X \). Then it is easy to see that \( \Omega X \) is equivalent definition of wgt given in Remark 3.3 (3), we see that wgt\((u)\) is detectable. If \( \tilde{X} \) we will only prove (4) since the other results are proven in the references cited. If \( X = K(\pi, 1) \), then \( \Omega X \) has the homotopy type of a discrete set of points and, consequently, \( F_1 = \Omega X \ast \Omega X \) is, up to homotopy, a wedge of circles. Also, \( G_0(X) = PX \simeq \ast \), so the cofibre of \( \Omega X \to G_0(X) \) has the type of a wedge of circles. Then \( G_1(X) \) has the homotopy type of a 1-dimensional space. Similarly, it is easy to see that \( G_k(X) \) has the homotopy type of a \( k \)-dimensional space. If \( u \in H^*(K(\pi, 1); R) \), then \( p_{k-1}^*(u) = 0 \) since \( G_{k-1}(X) \) is \( s \)-dimensional. By the equivalent definition of wgt given in Remark 3.3 (3), we see that wgt\((u)\) \( \geq s \).

3.4. Proposition \((R3, S)\). Category weight has the following properties.

1. \( 1 \leq \text{wgt}(u) \leq \text{cat}(X) \), for all \( u \in \tilde{H}^*(X; R) \), \( u \neq 0 \).
2. For every \( f : Y \to X \) and \( u \in \tilde{H}^*(X; R) \) with \( f^*(u) \neq 0 \) we have \( \text{cat}(f) \geq \text{wgt}(u) \) and \( \text{wgt}(f^*(u)) \geq \text{wgt}(u) \).
3. \( \text{wgt}(u \cup v) \geq \text{wgt}(u) + \text{wgt}(v) \).
4. For every \( u \in \tilde{H}^*(K(\pi, 1); R) \), \( u \neq 0 \), we have \( \text{wgt}(u) \geq s \).

Proof. We will only prove (4) since the other results are proven in the references cited. If \( X = K(\pi, 1) \), then \( \Omega X \) has the homotopy type of a discrete set of points and, consequently, \( F_1 = \Omega X \ast \Omega X \) is, up to homotopy, a wedge of circles. Also, \( G_0(X) = PX \simeq \ast \), so the cofibre of \( \Omega X \to G_0(X) \) has the type of a wedge of circles. Then \( G_1(X) \) has the homotopy type of a 1-dimensional space. Similarly, it is easy to see that \( G_k(X) \) has the homotopy type of a \( k \)-dimensional space. If \( u \in \tilde{H}^*(K(\pi, 1); R) \), then \( p_{k-1}^*(u) = 0 \) since \( G_{k-1}(X) \) is \( s \)-dimensional. By the equivalent definition of wgt given in Remark 3.3 (3), we see that wgt\((u)\) \( \geq s \).

3.5. Definition. We say that \( u \in \tilde{H}^*(X; R) \) is a detecting element for \( X \) if \( \text{wgt}(u) = \text{cat}(X) \). We say that a space \( X \) is detectable if it possesses a detecting element.

Recall that the cup-length of a space \( X \) with respect to a ring \( R \) is defined as

\[ \text{cl}_R(X) = \max\{k \mid u_1 \cup \cdots \cup u_k \neq 0 \text{ for some } u_i \in \tilde{H}^*(X; R) \}. \]

3.6. Lemma. If \( \text{cat}(X) = \text{cl}_R(X) \) for some ring \( R \) then the space \( X \) is detectable.

Proof. It is well known that \( \text{cat}(X) \geq \text{cl}_R(X) \) for every \( R \). Now, let \( \text{cat}(X) = k \) and suppose that there are \( u_1, \ldots, u_k \in \tilde{H}^*(X; R) \) with \( u_1 \cup \cdots \cup u_k \neq 0 \). Then, using the first and third properties of Proposition 3.4, we conclude that \( \text{wgt}(u_1 \cup \cdots \cup u_k) = k \). Thus, \( u_1 \cup \cdots \cup u_k \) is a detecting element for \( X \).
4. Basic Special Cases

First, recall that $\text{cat}(X) \leq \dim(X)$ for every connected CW-space $X$. In particular, $\text{cat}(M) \leq 3$ for every (connected) 3-manifold $M$. We also notice that, by Lemma 3.6, a space $X$ is detectable whenever $\text{cat}(X) = \text{cl}_R(X)$ for some $R$. Here is a first step in understanding the category of 3-manifolds.

**4.1. Proposition.** If $M$ is a 3-manifold with finite fundamental group of order $d > 1$, then $\text{cat}(M) = 3$, and every non-zero element of $H^3(M; \mathbb{Z}/d)$ is a detecting element for $M$. Moreover, if $d$ is even, then every non-zero element of $H^3(M; \mathbb{Z}/2)$ is a detecting element for $M$ as well.

**Proof.** Since $\pi_1(M)$ is finite, $\pi_2(M) = 0$ because, by Proposition 2.6, the universal cover is a homotopy sphere. Hence, there is the Hopf exact sequence

$$\pi_3(M) \xrightarrow{h} H_3(M) \xrightarrow{q} H_3(\pi) \to 0$$

where $h$ is the Hurewicz homomorphism (e.g. see [Br] Theorem II.5.2). Since, by Proposition 2.6, the $d$-fold universal covering $\tilde{M} \to M$ is a $d$-sheeted covering, $M$ is orientable and $\tilde{M}$ is a homotopy sphere, we conclude that $h$ has the form

$$\pi_3(M) = \mathbb{Z} \to \mathbb{Z} = H_3(M), \quad a \mapsto d \cdot a.$$ 

Hence, $H_3(\pi) = \mathbb{Z}/d$. Also consider the induced homomorphism $\text{Hom}(H_3(\pi); \mathbb{Z}/d) \to \text{Hom}(H_3(M); \mathbb{Z}/d)$. It is certainly injective since $H_3(M) \to H_3(\pi)$ is surjective. However, it is also true that, for any $\phi \in \text{Hom}(H_3(M); \mathbb{Z}/d)$, $\text{Im}(h) = d\mathbb{Z} \subseteq \text{Ker}(\phi)$, so there exists $\tilde{\phi} \in \text{Hom}(H_3(\pi); \mathbb{Z}/d)$ with $\tilde{\phi} \mapsto \phi$. Thus, we have an isomorphism $\text{Hom}(H_3(\pi); \mathbb{Z}/d) \xrightarrow{\sim} \text{Hom}(H_3(M); \mathbb{Z}/d)$.

Now consider the diagram

$$\begin{array}{ccc}
H^3(\pi; \mathbb{Z}/d) & \xrightarrow{q^*} & H^3(M; \mathbb{Z}/d) \\
\downarrow & & \downarrow \\
\text{Hom}(H_3(\pi); \mathbb{Z}/d) & \xrightarrow{q^*} & \text{Hom}(H_3(M); \mathbb{Z}/d).
\end{array}$$

By Proposition 3.4 (4), a non-zero element of $H^3(\pi; \mathbb{Z}/d)$ has category weight at least 3. The right arrow is an isomorphism because $H_2(M)$ is free abelian since $M$ is orientable. The bottom arrow is an isomorphism by the argument above. Finally, the left arrow is a surjection by the Universal Coefficient Formula. Therefore, the top arrow is a surjection as well. In particular, by Proposition 3.4 (2), every non-zero element of $H^3(M; \mathbb{Z}/d)$ has category weight at least 3. But $\text{cat}(M) \leq \dim(M) = 3$, so $\text{cat}(M) = 3$, and every non-zero element of $H^3(M; \mathbb{Z}/d)$ is a detecting element for $M$. \[\square\]

**4.2. Remark.** Using the approach as in Proposition 4.1, it is also possible to prove the following result originally due to Krasnoselski [Kra] and, in fact, re-proved in [GoGo]:

For a free action of the finite group $G$ on a homotopy sphere $S \simeq S^{2n+1}$,

$$\text{cat}(S/G) = 2n + 1 = \dim(S/G).$$

Here is another basic result which follows from the characterization of prime non-irreducible 3-manifolds.
4.3. Proposition. Let $M$ be a prime 3-manifold which is not irreducible. Then $\operatorname{cat}(M) = 2 = \operatorname{cl}_{\mathbb{Z}/2}(M)$, and $M$ is detectable.

Proof. In view of Lemma 2.3, $M$ is the total space of a 2-sphere bundle over $S^1$. So, $M$ is either $S^1 \times S^2$ or the mapping torus of the map

$$r: S^2 \to S^2, \quad r(x) = -x$$

where $S^2$ is regarded as the set of unit vectors in $\mathbb{R}^3$. It is easy to see that, in both of the cases, $M = (S^1 \vee S^2) \cup e^3$ where $e^3$ is a 3-cell attached to the wedge $S^1 \vee S^2$. Thus, because a wedge of spheres has category one and a mapping cone can increase category by at most one, we obtain $\operatorname{cat}(M) \leq 2$.

Furthermore, because $\pi_1(M) = \mathbb{Z}$, we conclude that $H_1(M; \mathbb{Z}/2) = \mathbb{Z}/2$. So, because of Poincaré duality (with $\mathbb{Z}/2$-coefficients), we have $\operatorname{cl}_{\mathbb{Z}/2}(M) \geq 2$. Thus, $\operatorname{cl}_{\mathbb{Z}/2}(M) = 2 = \operatorname{cat}(M)$, and $M$ is detectable.

The next two results treat the case of infinite fundamental group, excluding the $S^2$-bundles over $S^1$.

4.4. Proposition. If $M$ is a 3-manifold with $\pi_1(M)$ infinite and $\pi_2(M) = 0$, then $\operatorname{cat}(M) = 3$ and $M$ is detectable.

Proof. By Proposition 2.6, $M = K(\pi_1(M), 1)$, so, by Proposition 3.4, every non-zero element of $H^3(M; \mathbb{R})$ has category weight 3. (Notice that, for example, $H^3(M; \mathbb{Z}/2) \neq 0$). Thus, because $\operatorname{cat}(M) \leq \dim(M) = 3$, each of these elements is a detecting element.

4.5. Proposition. If $M$ is an irreducible 3-manifold with $\pi_1(M)$ infinite and $\pi_2(M) \neq 0$, then $\operatorname{cat}(M) = 3 = \operatorname{cl}_{\mathbb{Z}/2}(M)$. In particular, $M$ is detectable. Furthermore, $M$ is non-orientable.

Proof. Consider a map $g: S^2 \to M$ as in Theorem 2.5. Since $M$ is irreducible, we conclude that $g(S^2)$ is a 2-sided projective plane in $M$. Let $i: \mathbb{R}P^2 \to M$ be the corresponding embedding, and let $[\mathbb{R}P^2] \in H_2(\mathbb{R}P^2; \mathbb{Z}/2)$ denote the fundamental class modulo 2 of $\mathbb{R}P^2$.

Let $w_k$ and $\overline{w}_k$ denote the $k$-th Stiefel–Whitney class of $M$ and $\mathbb{R}P^2$, respectively. Since the 1-dimensional normal bundle of $i$ is trivial, we conclude that $i^*w_k = \overline{w}_k$.

We can now compute the Kronecker products

$$\langle w_2, i_*[\mathbb{R}P^2] \rangle = (i^*w_2, [\mathbb{R}P^2]) = \langle \overline{w}_2, [\mathbb{R}P^2] \rangle = 1,$$

and so $i_*[\mathbb{R}P^2] \neq 0 \in H_2(M; \mathbb{Z}/2)$. Now, since $\langle \overline{w}_1^2, [\mathbb{R}P^2] \rangle = 1$, we conclude that $i^*w_2^2 = \overline{w}_1^2 \neq 0$, and so $w_2^2 \neq 0$. So, by Poincaré duality, there exists $x \in H^1(M; \mathbb{Z}/2)$ with $xw_2^2 \neq 0$. Thus, $\operatorname{cl}_{\mathbb{Z}/2}(M) = 3$.

We also need the following fact which, in a sense, is a converse of Lemma 2.3.

4.6. Corollary. If $M$ is a closed 3-manifold with non-trivial free fundamental group, then $M$ is not irreducible.

Proof. Notice that $\pi_2(M) \neq 0$. Indeed, if $\pi_2(M) = 0$ then, by Proposition 2.6 and the hypothesis that $\pi_1(M)$ is free,

$$M = K(\pi_1(M), 1) = \vee S^1.$$ 

But this is wrong since a wedge of circles has vanishing homology above degree 1 for any coefficients.
Now, if $M$ is irreducible then, by Proposition 4.3, $\text{cl}_{\mathbb{Z}/2} (M) = 3$. But this is impossible. Indeed, let $f: M \to K(\pi_1 (M), 1) = \vee S^1$ be a map which induces an isomorphism of fundamental groups. Then

$$f^*: H^1 (K(\pi_1 (M), 1); \mathbb{Z}/2) \to H^1 (M; \mathbb{Z}/2)$$

is an isomorphism. Thus, $x \cup y = 0$ for all $x, y \in H^1 (M; \mathbb{Z}/2)$, and so $\text{cl}_{\mathbb{Z}/2} (M) < 3$. This is a contradiction. \hfill \Box

4.7. Remark. If $\pi_1 (M) = \mathbb{Z}$ then $M = P \# \Sigma$ where $\Sigma$ is a homotopy sphere and $P$ is prime. So, $\pi_1 (P) = \mathbb{Z}$. But $P$ is not irreducible by Corollary 4.4, so, because of Lemma 2.3, $\pi_2 (P) = \mathbb{Z}$. In other words, $\pi_2 (M) = \mathbb{Z}$ whenever $\pi_1 (M) = \mathbb{Z}$. Actually, the following general fact holds: for every closed 3-manifold $M$, the group $\pi_1 (M)$ completely determines $\pi_2 (M)$, see e.g. [R1].

5. Detectability of 3-Manifolds

5.1. Proposition. Let $M^3$ be a closed 3-manifold with $\pi_1 (M)$ free and non-trivial. Then $\text{cat} (M) = 2$, and $M$ is detectable.

Proof. Write $M = M_1 \# \ldots \# M_k$ with each $M_j$ prime. Because $\pi_1 (M) = \pi_1 (M_1) \ast \ldots \ast \pi_1 (M_k)$ is free, each $\pi^j = \pi_1 (M_j)$ must be free (where we agree that the trivial group is free). If $M_j$ is irreducible with $\pi^j \neq \{1\}$, then this contradicts Corollary 4.6. Therefore, all such $M_j$ are non-irreducible primes; that is, the $M_j$ are the manifolds considered in Proposition 4.3. Because of Lemma 2.3, these are the total spaces of $S^2$-bundles over $S^1$. There are only two such manifolds: one orientable and one non-orientable, and we denote both of them by $S^1 \ltimes S^2$. Of course, the $M_j$ with $\pi^j = \{1\}$ are homotopy spheres $\Sigma_j$. The key point now is that, for $M = P \# Q$ with $P = \# k (S^1 \ltimes S^2)$ and $Q = \# j \Sigma_j$, $M - D^3$ deformation retracts onto the 2-skeleton $\bigvee_k (S^1 \vee S^2)$. Because of Proposition 4.3, $\text{cat} (S^1 \ltimes S^2) = 2$. This handles the “trivial” case where the connected sum degenerates to a single summand. Now suppose $M = \# j M_j = P \# Q$, where $M_j$ is either a homotopy sphere or $S^1 \ltimes S^2$ and $P = \# j M_j$, $Q = \# j M_j$, arbitrarily split $M$. If we remove a disk from a 3-manifold $N$, then the inclusion $S^2 \hookrightarrow N - D^3$ is the inclusion of a subcomplex; so therefore a cofibration. Thus, the pushout diagram

$$\begin{array}{ccc}
S^2 & \longrightarrow & P - D^3 \\
\downarrow & & \downarrow \\
Q - D^3 & \longrightarrow & P \# Q = M
\end{array}$$

is a homotopy pushout as well. But then we may apply the standard estimate for the category of a double mapping cylinder (see [Hat]) to obtain

$$\text{cat} (M) \leq \text{cat} (S^2) + \max \{\text{cat} (P - D^3), \text{cat} (Q - D^3)\}$$

$$= 1 + \max \{\text{cat} (\vee_j (S^1 \vee S^2)), \text{cat} (\vee_j (S^1 \vee S^2))\}$$

$$= 1 + 1$$

$$= 2.$$

Of course, cup-length then shows that $\text{cat} (M) = 2$ and this completes the proof. \hfill \Box
5.2. Theorem. Let $M$ be a 3-manifold whose fundamental group is non-trivial and not a free group. Then $\text{cat}(M) = 3$. Further, $M$ is detectable unless it is non-orientable of the form $P \# Q$, where $P$ is non-orientable and $Q$ is prime with odd torsion. Also, in the last case, the orientable double cover of $M$ has category 3.

Proof: The case of finite $\pi_1$ is considered in Proposition 4.1. So, we assume that $\pi_1(M)$ is infinite. We represent $M$ as a connected sum $M = N \# P$, where $P$ is prime and $\pi_1(P) \neq \{1\}$. Furthermore, we can always assume that $\pi_1(P) \neq \mathbb{Z}$, and therefore $P$ is irreducible in view of Corollary 4.6. Now, because of the results of §4, $P$ possesses a detecting element $u \in H^3(P; R)$ for suitable $R$.

Now suppose that $M$ is orientable. Then there is a map $f : M \to P$ of degree 1. (In greater detail, $M = (N \setminus D) \cup (P \setminus D)$ where $D$ is a 3-disk, and $f : M \to P$ maps $N \setminus D$ to the disk $D$ in $P$ and is the identity on $P \setminus D$.) Then $f^* : H^3(P; R) \to H^3(M : R)$ is an isomorphism for every coefficient ring (group) $R$. Now, for the detecting element $u$ above, $f^*(u) \neq 0$, and therefore, $\text{wgt}(f^*(u)) = 3$. Thus, $f^*(u)$ is a detecting element for $M$.

Now, if $M$ is not orientable, then let $\overline{M} \to M$ be its orientable double cover (which also is a closed 3-manifold). If $\pi_1(M)$ has odd torsion, then so does $\pi_1(\overline{M})$. Because $\overline{M}$ is orientable, the argument above says that $\text{cat}(\overline{M}) = 3$. But because $\overline{M}$ covers $M$, we know that $\text{cat}(\overline{M}) \leq \text{cat}(M)$. Therefore, $\text{cat}(M) = 3$. If, on the other hand, there is a prime component of $M$ with non-free fundamental group having no odd torsion, then this component has a detecting element in 3-dimensional $\mathbb{Z}/2$-cohomology. Therefore, $M$ has a detecting element in $\mathbb{Z}/2$-cohomology as well and $\text{cat}(M) = 3$.

Now, if $\pi_1(M)$ has odd torsion, then this occurs in individual prime components. So, $M$ may not have a detecting element only if we can write $M = P \# Q$, where $P$ is non-orientable and $Q$ is a prime manifold having odd torsion.

For completeness, note that $\text{cat}(\Sigma) = 1$ for every simply connected 3-manifold (= homotopy sphere) $\Sigma$, and, therefore, every non-zero element $u \in H^3(\Sigma)$ is a detecting element. Therefore, we now have proved Theorem 6.1 and augmented it by showing that most closed 3-manifolds possess detecting elements. The significance of this will be apparent in §5.3.

5.3. Remark. In fact, if we allow local coefficients, then all 3-manifolds with non-trivial and non-free fundamental groups have detecting elements. More specifically, by [Ber], $\text{cat}(X) = n = \dim(X)$ if and only if a certain element $u \in H^1(X; I(\pi))$ has $u^n \neq 0$ in $H^n(X; I(\pi) \otimes \ldots \otimes I(\pi))$. Here, $\pi = \pi_1(X)$ and $I(\pi)$ is the augmentation ideal in the group ring $\mathbb{Z}\pi$. Since $u^n$ is a cup product (with local coefficients), it is a detecting element.

6. Two Applications

A prime motivating problem in the study of Lusternik-Schnirelmann category has been the the Ganea conjecture: $\text{cat}(X \times S^n) = \text{cat}(X) + 1$. We now know that the conjecture is not true in general, so it is even more interesting to understand when it is valid. For 3-manifolds, we have the following.
6.1. Corollary. For every closed 3-manifold $M$, 
\[ \text{cat}(M \times S^n) = \text{cat}(M) + 1. \]
That is, the Ganea conjecture holds for $M$.

Proof. First, suppose that $M$ is detectable. Then the equality follows from the general result [R3, Corollary 2.3], but the argument in this case is easy. Let $u \in H^3(M; R)$ have $\text{wgt}(u) = \text{cat}(M)$ and let $v \in H^n(S^n; R)$ be non-trivial, where, by the results above, we can always take $R = \mathbb{Z}$ or $R = \mathbb{Z}/d$. Let $\tilde{u} = p_M^*(u)$ and $\tilde{v} = p_{S^n}^*(v)$, where $p_M : M \times S^n \to M$ and $p_{S^n} : M \times S^n \to S^n$ are the respective projections. Clearly, $\tilde{u} \neq 0$ and $\tilde{v} \neq 0$ since the compositions $\text{cat}(M) + 1 \geq \text{wgt}(\tilde{u} \cup \tilde{v}) \geq \text{wgt}(\tilde{u}) + \text{wgt}(\tilde{v}) \geq \text{cat}(M) + 1$. Hence, $\text{cat}(M \times S^n) = \text{cat}(M) + 1$.

Now, suppose that $M$ is not detectable. Then, by Theorem 5.2, the oriented double cover $\tilde{M}$ of $M$ is detectable, and $\text{cat}(\tilde{M}) = 3$. Therefore, in view of what we said above, $\text{cat}(\tilde{M} \times S^n) = 4$. But $\tilde{M} \times S^n$ covers $M \times S^n$, and so $\text{cat}(M \times S^n) \geq 4$. On the other hand, $\text{cat}(M \times S^n) \leq \text{cat}(M) + 1 = 4$ for general reasons. Thus, $\text{cat}(M \times S^n) = 4$.

6.2. Corollary. Let $f : M \to N$ be a degree 1 map of oriented 3-manifolds. Then $\text{cat}M \geq \text{cat}f = \text{cat}N$.

Proof. Let $u \in H^3(N; A)$ be a detecting element for $N$. (Recall that orientable 3-manifolds always have detecting elements.) Since $\deg(f) = 1$, we conclude that $f^*(u) \neq 0$. So, $\text{cat}(f) \geq \text{wgt}(u)$ by Proposition 5.4 (2). Thus $\text{cat}(M) \geq \text{cat}(f) \geq \text{wgt}(u) = \text{cat}(N)$.

Of course, $\text{cat}(f) = \text{cat}(N)$ holds since $\text{cat}(f) \leq \text{cat}(N)$ for general reasons.

6.3. Corollary. Let $f : M \to N$ be a degree 1 map of oriented 3-manifolds. If $\pi_1(M)$ is free, then $\pi_1(N)$ is.

Proof. By Corollary 6.2, $\text{cat}(N) \leq 2$, and so $\pi_1(N)$ is free by Theorem 5.2.
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