THE HESSIANS OF THE COMPLETE AND COMPLETE BIPARTITE GRAPHS AND ITS APPLICATION TO THE STRONG LEFSCHETZ PROPERTY

AKIKO YAZAWA

Abstract. We consider the Hessian matrix of the weighted generating function for spanning trees. We call it the Hessian matrix of a graph. In this paper, we show that the Hessians of the complete and the complete bipartite graphs do not vanish by calculating the eigenvalues of the Hessian matrix of the graphs. As an application, we show the strong Lefschetz property for the Artinian Gorenstein algebra associated to the graphic matroids of the complete and complete bipartite graphs with at most five vertices.

1. Introduction

In [5], Maeno and Numata introduced algebras \( Q/J_M \) and \( A_M \) for a matroid \( M \) to give an algebraic proof of the Sperner property for the lattice \( \mathcal{L}(M) \) consisting of flats of \( M \). The algebra \( Q/J_M \) is isomorphic to the vector space with basis as vector spaces. The algebra \( A_M \) is defined to be the quotient algebra of the ring of the differential polynomials by the annihilator of the base generating function of \( M \). By definition, the algebra \( A_M \) is an Artinian Gorenstein algebra. They show that \( Q/J_M \) has the strong Lefshetz property in the narrow sense (see Definition 4.1 for the definition, see also [2]) if and only if the lattice \( \mathcal{L}(M) \) is modular, and that \( Q/J_M = A_M \) if and only if \( \mathcal{L}(M) \) is modular. In [3], for a simple matroid \( M \) with rank \( r \) and the ground set \( E \), Huh and Wang introduced a graded algebra \( B_*(M) = \bigoplus_{p=0}^r B^p(M) \), which is isomorphic to \( Q/J_M \). They show that the multiplication map \( \times L^{r-2p} : B^p(M) \to B^{r-p}(M) \)

is injective for \( p \leq \frac{r}{2} \) and \( L = \sum_{i \in E} y_i \).

Maeno and Numata conjectured that the algebra \( A_M \) has the strong Lefschetz property for an arbitrary matroid \( M \) in an extended abstract [4] of the paper [5]. In this paper, we discuss the strong Lefschetz

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property of the Artinian Gorenstein algebra $A_M$ for graphic matroids of the complete and the complete bipartite graphs.

Let us recall the definition of $A_M$ in the case of a graphic matroids. Let $\Gamma$ be a connected graph with $N$ edges. Define $B_\Gamma$ to be the set of spanning trees in $\Gamma$. Then the graphic matroid $M(\Gamma)$ of $\Gamma$ is a matroid whose ground set is the set $E(\Gamma)$ of edges $\Gamma$ and whose basis set is the set $B_\Gamma$ of spanning trees in $\Gamma$. We assign the variable $x_i$ to each edge $i$ of $\Gamma$. For the graph $\Gamma$, we define the weighted generating function $F_\Gamma$ for spanning trees in $\Gamma$ by

$$F_\Gamma = \sum_{T \in B_\Gamma} \prod_{e \in E(T)} x_e.$$ 

Then $A_{M(\Gamma)}$ is $\mathbb{K}[x_1, \ldots, x_N]/\text{Ann}(F_\Gamma)$ (see Section 4 for the definition of $\text{Ann}(F_\Gamma)$).

There is a criterion for the strong Lefschetz property of a grade $d$ Artinian Gorenstein algebra (see Theorem 4.3 for the detail). Roughly speaking, a graded Artinian Gorenstein algebra $A = \bigoplus_{k=0}^d A_k$ has the strong Lefshetz property if and only if the determinants of the $k$th Hessian matrices of the algebra do not vanish for all $k$, where the $k$th Hessian matrix is obtained from a linear basis for the homogeneous space $A_k$ of degree $k$.

Now we define the matrix $H_\Gamma$ by

$$H_\Gamma = \left( \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} F_\Gamma \right)_{i,j \in E(\Gamma)}$$

for a connected graph $\Gamma$. We call $H_\Gamma$ the Hessian matrix of the graph $\Gamma$ and $\det H_\Gamma$ the Hessian of the graph $\Gamma$. If the determinant of $H_\Gamma$ does not vanish, then the set of all variables is a linear basis for the homogeneous space of degree one. Hence the Hessian matrix of the graph coincidences with the first Hessian matrix of the algebra $A_{M(\Gamma)}$.

In this paper, we will show that the determinants of the Hessian matrices of the complete and complete bipartite graphs do not vanish by calculating the eigenvalues. The main results imply the strong Lefschetz property of the Artinian Gorenstein algebra corresponding to the graphic matroid of the complete graph and the complete bipartite graph with at most five vertices.

This article is organized as follows: In Section 2, we will calculate the eigenvectors and eigenvalues of some block matrices. Then we will compute the Hessians of the complete graph and the complete bipartite graph in Section 3. In Section 4, we will discuss the strong Lefschetz property of the algebra $A_M$ corresponding to a graphic matroid.
2. THE EIGENVECTORS AND EIGENVALUES OF BLOCK MATRICES

In this section, we give the eigenvectors and eigenvalues of some block matrices. We consider three kinds of block matrices $C, D$ and $M(A, \lambda, d)$ (Theorems 2.5, 2.9 and 2.11).

Let $l \in \mathbb{Z}$, $d = (d_1, d_2, \ldots, d_l)$, and $\delta = d_1 + d_2 + \cdots + d_l$. Let $A^{ij}$ be a $d_i \times d_j$ matrix. We consider the $\delta \times \delta$ matrix $A$ defined by

$$A = (A^{ij})_{1 \leq i,j \leq l}$$

with the $d_i \times d_j$ matrix $A^{ij}$.

**Lemma 2.1.** Let $A^{ij}v_j = \bar{a}_{ij}v_i$ and $\bar{A} = (\bar{a}_{ij})_{1 \leq i,j \leq l}$. If $(w_i)_{1 \leq i \leq l} \in \mathbb{C}^l$ is an eigenvector of $\bar{A}$ belonging to the eigenvalue $\lambda$, then $x = (w_i v_i)_{1 \leq i \leq l} \in \mathbb{C}^\delta$ satisfies $Ax = \lambda x$.

**Proof.** Since $(w_i)_{1 \leq i \leq l}$ is the eigenvector of $\bar{A}$ belonging to $\lambda$,

$$\bar{A} (w_i)_{1 \leq i \leq l} = \lambda (w_i)_{1 \leq i \leq l}.$$  

The $i$th row of the equation implies that

$$
\begin{pmatrix}
\bar{a}_{i1} & \bar{a}_{i2} & \cdots & \bar{a}_{il} \\
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_l \\
\end{pmatrix}
= \lambda w_i
$$

for all $i$. Since $A^{ij}v_j = \bar{a}_{ij}v_i$,

$$
w_1 A^{i1}v_1 + w_2 A^{i2}v_2 + \cdots + w_l A^{il}v_l = \bar{a}_{i1}w_1v_1 + \bar{a}_{i2}w_2v_1 + \cdots + \bar{a}_{il}w_lv_i \\
= (\bar{a}_{i1}w_1 + \bar{a}_{i2}w_2 + \cdots + \bar{a}_{il}w_l)v_i \\
= \lambda w_i v_i
$$

for all $i$. Hence

$$Ax = 
\begin{pmatrix}
A^{11} & A^{12} & \cdots & A^{1l} \\
A^{21} & A^{22} & \cdots & A^{2l} \\
\vdots & \vdots & \ddots & \vdots \\
A^{l1} & A^{l2} & \cdots & A^{ll}
\end{pmatrix}
\begin{pmatrix}
w_1 v_1 \\
w_2 v_2 \\
\vdots \\
w_l v_l
\end{pmatrix}
= 
\begin{pmatrix}
w_1 A^{11} v_1 + w_2 A^{12} v_2 + \cdots + w_l A^{1l} v_l \\
w_1 A^{21} v_1 + w_2 A^{22} v_2 + \cdots + w_l A^{2l} v_l \\
\vdots \\
w_1 A^{l1} v_1 + w_2 A^{l2} v_2 + \cdots + w_l A^{ll} v_l
\end{pmatrix}
= \lambda x.$$
\[
\begin{pmatrix}
w_1v_1 \\
w_2v_2 \\
\vdots \\
w_lv_l
\end{pmatrix}
= \lambda
\begin{pmatrix}
x \\
x \\
\vdots \\
x
\end{pmatrix}
= \lambda x.
\]

\textbf{Remark 2.2.} The vector \( x \) may be equal to the zero vector \( 0_\delta \) of size \( \delta \). Hence the vector \( x \) may not be an eigenvector of \( A \).

Let \( C_n \) be the square matrix
\[
\begin{pmatrix}
0 & 1 \\
0 & 1 \\
\vdots & \vdots \\
1 & 0
\end{pmatrix}
\]
of size \( n \). Let \( \zeta_n \) be the \( n \)th primitive root of unity. Then the eigenvalues of \( C_n \) are \( 1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1} \). Let \( z_{n,k} \) be an eigenvector of \( C_n \) belonging to the eigenvalue \( \zeta_n^k \). Note that \( z_{n,k} \) is also an eigenvector of \((C_n)^t\) belonging to the eigenvalue \( \zeta_n^{-k} \).

Let \( I_n \) be the identity matrix of size \( n \), and \( J_{mn} \) the all-one matrix of size \( m \times n \). For an \( n \times n \) matrix \( X \), \( \chi_X(t) \) denotes the characteristic polynomial \( \det(tI_n - X) \) of \( X \) in the variable \( t \), and \( X^{(1)} \) denotes the first row of \( X \). Note that the product \( X^{(1)} z_{n,k} \) is a complex number.

\textbf{Lemma 2.3.} Let \( A \) be an \( n \times n \) cyclic matrix. The vector \( z_{n,k} \) is an eigenvector of \( A \) belonging to the eigenvalue \( A^{(1)} z_{n,k} \). Hence
\[
\chi_A(t) = \prod_{k=0}^{n-1} (t - A^{(1)} z_{n,k}),
\]
\[
\det A = \prod_{k=0}^{n-1} A^{(1)} z_{n,k}.
\]

We obtain the following from Lemma 2.3

\textbf{Lemma 2.4.} Let \( \alpha, \lambda \in \mathbb{C} \) and \( A = \alpha J_{nn} + \lambda I_n \). The vector \( z_{n,0} \) is an eigenvector of \( A \) belonging to the eigenvalue \( \lambda + n\alpha \). For \( 1 \leq k \leq n-1 \), the vector \( z_{n,k} \) is an eigenvector of \( A \) belonging to the eigenvalue \( \lambda \). Hence
\[
\chi_A(t) = (t - \lambda)^{n-1}(t - (\lambda + n\alpha)),
\]
\[
\det A = \lambda^{n-1}(\lambda + n\alpha).
\]

Now we consider the block matrices \( C, D \) and \( M(A, \lambda, \mathbf{d}) \).

First we consider the case where \( \mathbf{d} = (n, \ldots, n) \) and each block is cyclic. Let us consider the block matrix \( C \) whose blocks are \( n \times n \) cyclic
matrices. Let $C$ be $(C^{ij})_{1\leq i,j\leq l}$, $C^{ij}$ an $n \times n$ matrix for each $i, j$, and $c_{ij}^{(k)}$ an eigenvalue of $C^{ij}$ associate with an eigenvector $z_{n,k}$. For $C$ and $0 \leq k \leq n-1$, we define the $l \times l$ matrix $\bar{C}^{(k)}$ by

$$
\bar{C}^{(k)} = \begin{pmatrix} c_{ij}^{(k)} \end{pmatrix}_{1\leq i,j\leq l}.
$$

**Theorem 2.5.** Let $(w_i)_{1\leq i\leq l} \in \mathbb{C}^l$ be an eigenvector of $\bar{C}^{(k)}$ belonging to the eigenvalue $\lambda$. Then $(w_i z_{n,k})_{1\leq i\leq l} \in \mathbb{C}^{nl}$ is an eigenvector of $C$ belonging to $\lambda$. Hence

$$
\chi_C(t) = \prod_{k=0}^{n-1} \chi_{\bar{C}^{(k)}}(t),
$$

$$
\det C = \prod_{k=0}^{n-1} \det \bar{C}^{(k)}.
$$

**Proof.** Since $z_{n,k}$ is a nonzero vector for any $k$, the vector $(w_i z_{n,k})_{1\leq i\leq l}$ is also a nonzero vector. Since $C^{ij}$ is cyclic, it follows from Lemma 2.3 that $\bar{C}^{(k)}$ satisfies the assumption in Lemma 2.1. $\square$

Next we consider the case where $d = (2n, 2n, \ldots, 2n, n)$. Let $D$ be the block matrix $D = (D^{ij})_{1\leq i,j\leq l}$. We assume that $D^{ij}$ is a $2n \times 2n$ cyclic matrix if $1 \leq i, j \leq l-1$ and that $D^{il}$ is an $n \times n$ cyclic matrix. Moreover we assume that

$$(1) \quad D^{il} = \begin{pmatrix} X_i \\ X_i^* \end{pmatrix}, \quad D^{ij} = \begin{pmatrix} Y_j & Y_j^* \end{pmatrix}$$

for $1 \leq i, j \leq l - 1$, where $X_i$ and $Y_j$ are $n \times n$ cyclic matrices.

For $0 \leq k \leq 2n - 1$, we define the $l \times l$ matrix $\bar{D}^{(k)} = (d_{ij}^{(k)})_{1\leq i,j\leq l}$ as follows: If $k$ is even, then we define

$$
d_{ij}^{(k)} = \begin{cases} (D^{ij})^{(1)} z_{2n,k} & \text{if } 1 \leq j \leq l - 1, \\
(D^{ij})^{(1)} z_{n,\frac{k}{2}} & \text{if } j = l.
\end{cases}
$$

If $k$ is odd, then

$$
d_{ij}^{(k)} = \begin{cases} (D^{ij})^{(1)} z_{2n,k} & \text{if } 1 \leq j \leq l - 1, \\
0 & \text{if } j = l.
\end{cases}
$$

**Lemma 2.6.** Let $0 \leq k \leq 2n - 1$ and $k$ be even. Let $(w_i)_{1\leq i\leq l} \in \mathbb{C}^l$ be an eigenvector of $\bar{D}^{(k)}$ belonging to the eigenvalue $\lambda$. Then

$$
\begin{pmatrix} w_1 z_{2n,k} \\
\vdots \\
w_{l-1} z_{2n,k} \\
w_l z_{n,\frac{k}{2}} \end{pmatrix} \in \mathbb{C}^{2(l-1)n}
$$

is an eigenvector of $D$ belonging to $\lambda$. 
Proof. Now the matrix $D$ satisfies the equation (1), and we have
\[
 z_{2n,k} = \begin{pmatrix} z_{n,k}^1 \\ z_{n,k}^2 \end{pmatrix}.
\]
Hence Lemmas 2.1 and 2.3 imply Lemma 2.6. □

Lemma 2.7. Let $0 \leq k \leq 2n - 1$ and $k$ be odd. Let $w = (w_i)_{1 \leq i \leq l}$ be an eigenvector of $\bar{D}^{(k)}$ belonging to the eigenvector $\lambda$. If the vector $w$ is linearly independent of the vector
\[
 \begin{pmatrix} 0_{l-1} \\ 1 \end{pmatrix},
\]
then
\[
 \begin{pmatrix} w_1 z_{2n,k} \\ \vdots \\ w_{l-1} z_{2n,k} \\ 0_n \end{pmatrix} \in \mathbb{C}^{2(l-1)n}
\]
is an eigenvector of $D$ belonging to $\lambda$.

Proof. It can be also shown by Lemmas 2.1 and 2.3. □

Remark 2.8. The vector
\[
 \begin{pmatrix} 0_{l-1} \\ 1 \end{pmatrix}
\]
is an eigenvector of $\bar{D}^{(k)}$ belonging to the eigenvalue zero. The eigenvector, however, does not induce an eigenvector of $D$. See also Remark 2.2.

Theorem 2.9 follows from Lemmas 2.6 and 2.7.

Theorem 2.9. The characteristic polynomial of $D$ is
\[
 \chi_D(t) = \left( \prod_{k: \text{even}} \chi_{D^{(k)}}(t) \right) \left( \prod_{k: \text{odd}} \frac{1}{t} \chi_{D^{(k)}}(t) \right)
\]
\[
 = \frac{1}{t^n} \prod_{k=0}^{2n-1} \chi_{D^{(k)}}(t).
\]

Finally we consider the block matrix $M(A, \lambda, d)$. For a square matrix $A$ of size $l$, $d = (d_1, d_2, \ldots, d_l)$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, we define
\[
T(A, d) = (a_{ij} I_{d_i d_j})_{1 \leq i,j \leq l},
\]
\[
D(\lambda, d) = \begin{pmatrix} \lambda_1 I_{d_1} & 0 \\ \lambda_2 I_{d_2} & \ddots \\ 0 & \ddots & \lambda_l I_{d_l} \end{pmatrix}.
\]
We define the square matrix $M(A, \lambda, d)$ of size $d_1 + \cdots + d_l$ by

$$M(A, \lambda, d) = T(A, d) + D(\lambda, d).$$

We define also the square matrix $\bar{M}(A, \lambda, d)$ of size $l$ by

$$\bar{M}(A, \lambda, d) = \text{diag}(d_1, \ldots, d_l)A + \text{diag}(\lambda_1, \ldots, \lambda_l),$$

where $\text{diag}(x_1, \ldots, x_l)$ is the diagonal matrix with entries $x_1, \ldots, x_l$.

**Lemma 2.10.** Let $\lambda = (\lambda_1, \ldots, \lambda_l)$, $d = (d_1, d_2, \ldots, d_l)$, and $d = d_1 + d_2 + \cdots + d_l$.

1. Let $i \in \{1, 2, \ldots, l\}$. Let $1 \leq k \leq d_i - 1$. We suppose that $w_i = 1$ and that for $j \in \{1, 2, \ldots, l\} \setminus \{i\}$, $w_j = 0$. The vector $(w_jz_{d_j, k})_{1 \leq j \leq l} \in \mathbb{C}^\delta$ is an eigenvector of $M(A, \lambda, d)$ belonging to the eigenvalue $\lambda_i$.

2. If $(w_i)_{1 \leq i \leq l} \in \mathbb{C}^l$ is an eigenvector of $\bar{M}(A, \lambda, d)$ belonging to the eigenvalue $\mu$, then $(w_jz_{d_j, 0})_{1 \leq i \leq l} \in \mathbb{C}^\delta$ is an eigenvector of $M(A, \lambda, d)$ belonging to the eigenvalue $\mu$.

**Proof.** Let $A = (a_{ij})$. Since $1 \leq k \leq d_i - 1$, it holds that $\zeta^k_{d_i} \neq 1$. Hence

$$a_{ij}J_{d_i, j}z_{d_i, 0} = 0_{d_i}.$$

Therefore the claim (1) follows from Lemmas 2.1 and 2.4. It follows that

$$a_{ij}J_{d_i, j}z_{d_i, 0} = d_ja_{ij}z_{d_i, 0}$$

for $i \neq j$, and that

$$(a_{ii}J_{d_i, i} + \lambda_iI_{d_i})z_{d_i, 0} = (d_ia_{ii} + \lambda_i)z_{d_i, 0}$$

for all $i$. Then the claim (2) follows from Lemma 2.1. \qed

**Theorem 2.11.** For a matrix $A$ of size $l$, $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $d = (d_1, d_2, \ldots, d_l)$, we have

$$\chi_{M(A, \lambda, d)}(t) = \chi_{\bar{M}(A, \lambda, d)}(t) \prod_{i=1}^l(t - \lambda_i)^{d_i-1},$$

$$\det M(A, \lambda, d) = \det \bar{M}(A, \lambda, d) \prod_{i=1}^l\lambda_i^{d_i-1}.$$
Terms of graphs in this section follows mainly [1].

3.1. **The Hessian of the complete graph.** Here we compute the Hessians of the complete graphs (Theorem 3.3).

Let \( n \geq 3 \). The \((i, j)\)-entry in \( \tilde{H}_{K_n} \) is the number of trees including edges \( i, j \) in \( K_n \) with \( n \) vertices. Moon gave the following formula for the number of trees containing some edges.

**Theorem 3.1** (Moon [7]). Let \( F \) be the forest with \( k \) connected components. The number of the trees with \( n \) vertices containing \( F \) is

\[
n^{k-2} \prod_{i=1}^{k} j_i,
\]

where \( j_i \) is the number of vertices of each component of \( F \).

Let \( \tilde{H}_{K_n} = (h_{ij}) \). It follows from Theorem 3.1 that

\[
h_{ij} = \begin{cases} 
0 & (i = j), \\
3n^{n-4} & (\#i \cap j = 1), \\
4n^{n-4} & (\#i \cap j = 0).
\end{cases}
\]

We can prove the following.

**Proposition 3.2.** The eigenvalues of \( \frac{1}{n^{n-4}} \tilde{H}_{K_n} \) are \(-2\), \(-n\) and \(2n(n-2)\). The dimensions of the eigenspaces associate with \(-2\), \(-n\) and \(2n(n-2)\) are \( \binom{n}{2} - n \), \( n - 1 \) and \( 1 \), respectively.

Proposition 3.2 implies Theorem 3.3.

**Theorem 3.3.** Let \( n \geq 3 \). Then the determinant of \( \tilde{H}_{K_n} \) is

\[
(-1)^{\binom{n}{2}-1} 2^{\binom{n}{2}} n^{n+\binom{n}{2}(n-4)}(n-2).
\]

Hence the Hessian \( \det H_{K_n} \) does not vanish for \( n \geq 3 \).

Now we prove Proposition 3.2. Let \( n \geq 3 \). We define \( a(e, e') \) by

\[
a(e, e') = \begin{cases} 
0 & (e = e'), \\
3 & (\#(e \cap e') = 1), \\
4 & \text{otherwise},
\end{cases}
\]

for \( e, e' \in E(K_n) \).

Let \( V(K_n) = \{0, 1, \ldots, n - 1\} \). We consider a group action on \( V(K_n) \) as follows: Let \( G \) be the cyclic group generated by \( \sigma \) of the order \( n \). For a vertex of \( K_n \), define

\[
\sigma(i) = i + 1 \pmod{n}.
\]

The action of \( G \) on \( V(K_n) \) induces an action on \( E(K_n) \) by

\[
\sigma \{ i, j \} = \{ \sigma(i), \sigma(j) \}
\]

for \( \{ i, j \} \in E(K_n) \).
Consider the case where $n$ is odd. Let $n = 2l + 1$. For $1 \leq i, j \leq l$, we define

$$C^{ij} = \left(a(\sigma^k e_i, \sigma^{k'} e_j)\right)_{0 \leq k, k' \leq n-1},$$

$$C = \left(C^{ij}\right)_{1 \leq i, j \leq l},$$

where $e_i = \{0, i\} \in E(K_n)$. Note that $C^{ij}$ is an $n \times n$ cyclic matrix by the definition of the action. Since

$$E(K_n) = \bigcup_{i=1}^{l} \{\sigma^k e_i \mid 0 \leq k \leq n-1\},$$

the matrix $n^{n-4}C$ is $\tilde{H}_{K_n}$. Let us calculate eigenvalues of $C$ by Theorem 2.5. For $0 \leq k \leq n-1$, let

$$\bar{C}^{(k)} = \left((C^{ij})^{(1)} z_{n,k}\right)_{1 \leq i, j \leq l};$$

First we consider the case where $1 \leq k \leq n-1$.

**Lemma 3.4.** Let $1 \leq k \leq n-1$. Then

$$\bar{C}^{(k)} - (-2)I_l = (-\xi_i \xi_j')_{1 \leq i, j \leq l},$$

where

$$\xi_i = \sum_{v \in e_i} \zeta_{vn}^k,$$

$$\xi_j' = \sum_{v \in e_j} \zeta_{vn}^{-k}$$

for all $i, j$. Moreover the rank of $\bar{C}^{(k)} - (-2)I_l$ is one.

**Proof.** We fix $k$ and compute $(C^{ij})^{(1)} z_{n,k}$. First we consider the case where $e_i \neq e_j$. The edges $e_i$ and $\sigma^l e_j$ share their vertices if and only if $l = 0$, $l = i$, $j + l = 0$, and $j + l = i$. Since $e_i \neq e_j$, there does not exist $l$ such that $e_i = \sigma^l e_j$. Hence if $l \in \{0, i, -j, i - j\}$, then

$$a(e_i, \sigma^l e_j) = 3,$$

and if $l \notin \{0, i, -j, i - j\}$, then

$$a(e_i, \sigma^l e_j) = 4.$$

Therefore

$$(C^{ij})^{(1)} z_{n,k} = 3(\sum_{l \in \{0, i, -j, i - j\}} \zeta_{n}^{kl}) + 4(\sum_{l \notin \{0, i, -j, i - j\}} \zeta_{n}^{kl})$$

$$= -1 - \zeta_{n}^{ki} - \zeta_{n}^{-kj} - \zeta_{n}^{k(i-j)}$$

$$= -\xi_i \xi_j'.$$
Next we consider the case where \( e_i = e_j \). The edges \( e_i \) and \( \sigma^l e_i \) share their vertices if and only if \( l = 0, l = i \) and \( l + i = 0 \). If \( l = 0 \), then
\[
a(e_i, e_i) = 0,
\]
if \( l = i \) or \( l = -i \), then
\[
a(e_i, \sigma^l e_i) = 3.
\]
Hence if \( l \not\in \{ 0, i, -i \} \), then
\[
a(e_i, \sigma^l e_i) = 4.
\]
Therefore
\[
(C^{ij})^{(1)} z_{n,k} = 0 \zeta_n^0 + 3\left( \sum_{l \not\in \{ i, -i \}} \zeta_{+l} \right) + 4\left( \sum_{l \not\in \{ 0, i, -i \}} \zeta_{+l} \right)
\]
\[
= -4 - \zeta_{-ki} - \zeta_{-ki}^n
\]
\[
= -\xi_i \xi_i - 2.
\]
We have
\[
\tilde{C}^{(k)} - (-2)I_l = (-\xi_i \xi_j)_{1 \leq i, j \leq l}
\]
\[
= - \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_l \end{pmatrix} (\xi_1 \cdots \xi_l).
\]
Hence the rank of \( \tilde{C}^{(k)} - (-2)I_l \) is one. \( \square \)

Next we obtain all eigenvalues of \( \tilde{C}^{(k)} \) for \( 1 \leq k \leq n-1 \) by computing the trace of \( \tilde{C}^{(k)} \).

**Lemma 3.5.** For \( 1 \leq k \leq n-1 \), the eigenvalues of \( \tilde{C}^{(k)} \) are \(-2\) and \(-n\). The dimensions of the eigenspaces of \( \tilde{C}^{(k)} \) associate with \(-2\) and \(-n\) are \( l-1 \) and 1, respectively.

**Proof.** The trace of \( \tilde{C}^{(k)} \) is
\[
\sum_{i=1}^{l} -\xi_i \xi'_i = \sum_{i=1}^{l} (-4 - \zeta_{ki}^n - \zeta_{-ki}^n)
\]
\[
= -4l + 1
\]
\[
= -4 \frac{n-1}{2} + 1
\]
\[
= -2n + 3.
\]
Therefore it follows from Lemmas 2.5 and 3.4 that the eigenvalues of \( \tilde{C}^{(k)} \) are \(-2\) and \(-n\), and the dimensions of the eigenspaces of \( \tilde{C}^{(k)} \) associate with \(-2\) and \(-n\) are \( l-1 \) and 1, respectively. \( \square \)

Similarly, we obtain the consequence in the case where \( k = 0 \).
Lemma 3.6. The eigenvalues of $\tilde{C}^{(0)}$ are $-2$ and $2n(n-2)$. The dimensions of the eigenspaces of $\tilde{C}^{(0)}$ associate with $-2$ and $2n(n-2)$ are $l-1$ and 1, respectively.

We obtain the following result by Theorem 2.5, and Lemmas 3.5 and 3.6.

Lemma 3.7. The eigenvalues of $C$ are $-2$, $-n$ and $2n(n-2)$. The dimensions of the eigenspaces of $C$ associate with $-2$, $-n$ and $2n(n-2)$ are $\binom{n}{2}$, $n$, $n-1$ and 1, respectively.

Next we consider the case where $n$ is even. Let $n = 2l$. We define the matrix $D^{ij}$ by

$$
D^{ij} = \begin{cases}
(a(\sigma^k e_i, \sigma^k e_j))_{0 \leq k, k' \leq n-1} & \text{for } 1 \leq i, j \leq l-1, \\
(a(\sigma^k e_i, \sigma^k e_j))_{0 \leq k \leq n-1, 0 \leq k' \leq l} & \text{for } 1 \leq i \leq l-1, j = l, \\
(a(\sigma^k e_i, \sigma^k e_j))_{0 \leq k \leq l, 0 \leq k' \leq n-1} & \text{for } i = l, 1 \leq j \leq l-1, \\
(a(\sigma^k e_i, \sigma^k e_j))_{0 \leq k, k' \leq l} & \text{for } i = j = l,
\end{cases}
$$

and the block matrix $D$ by $D = (D^{ij})_{1 \leq i, j \leq l}$. Note that the matrix $n^{n-4}D$ is $\tilde{H}_{K_n}$ since

$$
E(K_n) = \left( \bigsqcup_{i=1}^{l-1} \{ \sigma^k e_i \mid 0 \leq k \leq n-1 \} \right) \cup \{ \sigma^k e_l \mid 0 \leq k \leq l \}.
$$

For $0 \leq j \leq l-1$, the $(1, k)$ entry and $(1, k+l)$ entry in the $(l, j)$ block satisfy

$$
a(e_i, \sigma^k e_j) = a(\sigma^l e_i, \sigma^k e_j) = a(\sigma^l \sigma^l e_i, \sigma^l \sigma^k e_j) = a(e_i, \sigma^{k+l} e_j).
$$

For $0 \leq i \leq l-1$, the $(i, l)$ block also satisfies the assumption (1) since $D$ is a symmetric matrix. Hence the matrix $D$ in this section satisfies the assumption of the matrix $D$ in Section 2.

Let us calculate eigenvalues of $D$ by Lemmas 2.6 and 2.7. For $0 \leq k \leq 2n-1$, we define the $l \times l$ matrix $D^{(k)} = (d^{(k)}_{ij})_{1 \leq i, j \leq l}$ as follows: If $k$ is even, then we define

$$
d^{(k)}_{ij} = \begin{cases}
(D^{ij})^{(1)} z_{n,k} & \text{if } 1 \leq j \leq l-1, \\
(D^{ij})^{(1)} z_{l,l} & \text{if } j = l.
\end{cases}
$$

If $k$ is odd, then we define

$$
d^{(k)}_{ij} = \begin{cases}
(D^{ij})^{(1)} z_{n,k} & \text{if } 1 \leq j \leq l-1, \\
0 & \text{if } j = l.
\end{cases}
$$

First we consider the case where $1 \leq k \leq 2n-1$. 

Lemma 3.8. Let $1 \leq k \leq 2n - 1$. Then
\[
\bar{D}^{(k)} - (-2) I_l = (-\xi_i \xi'_j)_{1 \leq i, j \leq l},
\]
where
\[
\xi_i = \sum_{v \in e_i} \zeta_n^{-vk}
\]
for all $i$, and
\[
\xi'_j = \begin{cases} 
\sum_{v \in e_j} \zeta_n^{-vk} & \text{if } 1 \leq j \leq l - 1, \\
\frac{1}{2} \sum_{v \in e_j} \zeta_n^{-vk} & \text{if } j = l.
\end{cases}
\]
Moreover the rank of $\bar{D}^{(k)} - (-2) I_l$ is one.

Proof. Let $k$ be even. We compute the entries $(D^{ij})^{(1)} z_{n,k}$, $(D^{il})^{(1)} z_{l,k}^2$ and $(D^{il})^{(1)} z_{l,k}^2$ in $\bar{D}^{(k)}$.

First we consider the case where $1 \leq j \leq l - 1$. In this case we can show the Lemma in the same way as Lemma 3.4.

Next let us consider the case where $1 \leq i \leq l - 1$ and $j = l$. We compute $(D^{il})^{(1)} z_{l,k}^2$ for $1 \leq i \leq l - 1$. In this case $j = -j$. The edges $e_i$ and $\sigma^* e_l$ share their vertices if and only if $s = 0$, $s = i$. Since $e_i \neq e_l$, there does not exist $s$ such that $e_i = \sigma^* e_l$. Hence if $s \in \{0, i\}$, then
\[
a(e_i, \sigma^* e_l) = 3,
\]
and if $s \notin \{0, i\}$, then
\[
a(e_i, \sigma^* e_l) = 4.
\]
Therefore
\[
(D^{il})^{(1)} z_{l,k}^2 = 3\left( \sum_{s \in \{0, i\}} \zeta_i^{ks} \right) + 4\left( \sum_{s \notin \{0, i\}} \zeta_i^{ks} \right)
= -1 - \zeta_i^{ki}
= -\xi_i \xi'_l.
\]

Finally we consider the case where $i = j = l$. If $s = 0$, then
\[
a(e_l, \sigma^* e_l) = 0.
\]
If $s \neq 0$, then $e_l$ and $\sigma^* e_l$ do not share their vertices. Hence if $s \neq 0$, then
\[
a(e_s, \sigma^* e_l) = 4.
\]
Therefore
\[
(D^{il})^{(1)} z_{l,k} = 0 \cdot \zeta_i^0 + 4\left( \sum_{s=1}^{l-1} \zeta_i^{sk} \right)
= -4
= -\xi_i \xi'_l - 2.
\]
Let \( k \) be odd. The \( l \text{th} \) column of \( \bar{D}^{(k)} \) is the zero vector by definition. On the other hand, since \( k \) is odd, \( \zeta_n^{lk} = -1 \). Hence \( \xi_l' = 0 \).

Similarly to Lemma 3.4, we can show that the rank of \( \bar{D}^{(k)} - (-2)I_l \) is one.

**Lemma 3.9.** Let \( 1 \leq k \leq 2n - 1 \) and \( k \) be even. The eigenvalues of \( \bar{D}^{(k)} \) are \(-2\) and \(-n\). The dimensions of the eigenspaces of \( \bar{D}^{(k)} \) associate with \(-2\) and \(-n\) are \( l - 1 \) and \( 1 \), respectively.

**Proof.** The trace of \( \bar{D}^{(k)} \) is
\[
\sum_{i=1}^{l-1} -\xi_i \xi_i' - 4 = \sum_{i=1}^{l-1} (-4 - \zeta_n^{ki} - \zeta_n^{-ki}) - 4
\]
\[
= -4(l - 1) + 2 - 4
\]
\[
= -4l + 2
\]
\[
= -2n + 2.
\]
Therefore it follows from Lemmas 2.6 and 3.8 that the eigenvalues of \( \bar{D}^{(k)} \) are \(-2\) and \(-n\), and the dimensions of the eigenspaces of \( \bar{D}^{(k)} \) associate with \(-2\) and \(-n\) are \( l - 1 \) and \( 1 \), respectively. \( \square \)

**Lemma 3.10.** Let \( 1 \leq k \leq 2n - 1 \) and \( k \) be odd. The eigenvalues of \( \bar{D}^{(k)} \) are \(-2\), \(-n\) and \(0\). The dimensions of the eigenspaces of \( \bar{D}^{(k)} \) associate with \(-2\), \(-n\) and \(0\) are \( l - 2 \), \( 1 \) and \( 1 \), respectively.

**Proof.** Similarly to Lemma 3.9, we compute the trace of \( \bar{D}^{(k)} \) and apply Lemma 2.7 and Remark 2.8. \( \square \)

Similarly, we obtain the consequence in the case where \( k = 0 \).

**Lemma 3.11.** The eigenvalues of \( \bar{D}^{(0)} \) are \(-2\) and \(2n(n - 2)\). The dimensions of the eigenspaces of \( \bar{D}^{(0)} \) associate with \(-2\) and \(2n(n - 2)\) are \( l - 1 \) and \( 1 \), respectively.

We obtain the following result by Theorem 2.9, and Lemmas 3.9, 3.10 and 3.11.

**Lemma 3.12.** The eigenvalues of \( D \) are \(-2\), \(-n\) and \(2n(n - 2)\). The dimensions of the eigenspaces of \( D \) associate with \(-2\), \(-n\) and \(2n(n - 2)\) are \( \binom{n}{2} - n \), \( n - 1 \) and \( 1 \), respectively.

On combining Lemma 3.7 with Lemma 3.12, we obtain Proposition 3.2.

**3.2. The Hessian of the complete bipartite graph.** Here we compute the Hessians of the complete bipartite graphs (Theorem 3.16).

For a graph \( \Gamma \), we define the **degree matrix** \( D_\Gamma \) to be a diagonal matrix indexed by vertices of \( \Gamma \) whose entries are degrees of vertices. We also define the **adjacency matrix** \( A_\Gamma \) to be a matrix indexed by vertices of \( \Gamma \) whose \((i, j)\)-entry is the number of edges with the ends
$v_i$ and $v_j$. Note that the entries in $A_{\Gamma}$ are one or zero for a simple graph $\Gamma$. We define the Laplacian matrix $L_{\Gamma}$ by $L_{\Gamma} = D_{\Gamma} - A_{\Gamma}$. For an arbitrary graph, we have the theorem which is the number of spanning trees in the graph. See [1, Theorem 6.3] for the detail.

**Theorem 3.13** (The matrix-tree theorem). *Every cofactor of $L_{\Gamma}$ is equal to the number of spanning trees in $\Gamma$.*

For a graph $\Gamma$ and an edge $e$ with ends $v, v'$ of $\Gamma$, we define the *contraction* $\Gamma/e$ to be the graph obtained by removing the edge $e$ from $\Gamma$ and by putting $v$ in $v'$. Let $e, e'$ be edges of $\Gamma$. Then $(\Gamma/e)/e' = (\Gamma/e')/e$. We write $\Gamma/e, e'$ to denote $(\Gamma/e)/e'$.

Let $e$ be an edge of a graph $\Gamma$. If we apply the matrix-tree theorem to the graph $\Gamma/e$, then we obtain the number of spanning trees including the edge $e$.

Let $X = \{ 0', 1', \ldots, (m - 1)' \}$ and $Y = \{ 0, 1, \ldots, n - 1 \}$. We consider the complete bipartite graph $K_{m,n} = K_{XY}$. Theorems 2.11 and 3.13 implies the following.

**Lemma 3.14.** Let $h_{ij}$ be the number of spanning trees containing the edges $i, j, i \neq j$ of $K_{m,n}$. Then

$$
 h_{ij} = \begin{cases} 
 n^{m-3}m^{n-3}m(2m + n - 2) & \text{if } i \cap j \in X, \\
 n^{m-3}m^{n-3}n(2n + m - 2) & \text{if } i \cap j \in Y, \\
 n^{m-3}m^{n-3}(m + n)(m + n - 2) & \text{otherwise.}
\end{cases}
$$

*Proof.* The Laplacian matrix $L_{K_{m,n}}$ is

$$
 M \left( \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, (n, m), (m, n) \right).
$$

The number of spanning trees including edges $i, j$ is every cofactor of the graph obtained by contraction edges $i, j$ of $K_{m,n}$. If $i \cap j \in X$, then the Laplacian matrix $L_{K_{m,n}/i,j}$ is

$$
 M \left( A, (0, n, m), (1, m - 1, n - 2) \right),
$$

where

$$
 A = \begin{pmatrix} 2m + n - 4 & -2 & -1 \\
 -2 & 0 & -1 \\
 -1 & -1 & 0 \\
\end{pmatrix}.
$$

It follows from Proposition 2.11 that the $(1, 1)$ cofactor of $L_{K_{m,n}/i,j}$ is

$$
 -m^{n-3}n^{m-3}m(2m + n - 2).
$$

Similarly if $i \cap j \in Y$, then the $(1, 1)$ cofactor of $L_{K_{m,n}/i,j}$ is

$$
 -n^{m-3}m^{n-3}n(2n + m - 2).
$$
Finally we consider the case of $i \cap j = \emptyset$. Then the Laplacian matrix $L_{K_{m,n}/i,j}$ is
\[
M (A', (0, 0, n, m), (1, 1, m - 2, n - 2)),
\]
where
\[
A' = \begin{pmatrix}
m + n - 2 & -2 & -1 & -1 \\
-2 & m + n - 2 & -1 & -1 \\
-1 & -1 & 0 & -1 \\
-1 & -1 & -1 & 0 \\
\end{pmatrix}.
\]
By Proposition 2.11, the $(1, 1)$ cofactor of $L_{K_{m,n}/i,j}$ is
\[
m^{n-3}mn - 3(m + n)(m + n - 2).
\]
□

We can prove the following.

**Proposition 3.15.** Let $m, n \geq 1$ and $m + n \geq 3$. Then eigenvalues of $\frac{1}{m^{n-3}mn - 3(m + n)(m + n - 2)}H_{K_{m,n}}$ are $-2mn, -mn^2, -m^2n$ and $mn(m + n - 1)(m + n - 2)$. The dimensions of eigenspaces of $-2mn, -mn^2, -m^2n$ and $mn(m + n - 1)(m + n - 2)$ are $(m - 1)(n - 1), n - 1, m - 1$ and 1, respectively.

Proposition 3.15 implies Theorem 3.16.

**Theorem 3.16.** Let $m, n \geq 1$ and $m + n \geq 3$. Then the determinant of $H_{K_{m,n}}$ is
\[
(-1)^{mn-1}2^{(m-1)(n-1)}m^{(mn-m-1)(n-1)}n^{(mn-n-1)(m-1)}(m + n - 1)(m + n - 2).
\]
Hence the Hessian $\det H_{K_{m,n}}$ does not vanish.

Let us prove Proposition 3.15. We compute the eigenvalues of the Hessian matrices of $K_{X,Y} = K_{m,n}$. For $e, e' \in E(K_{X,Y})$, we define $a(e, e')$ by
\[
a(e, e') = \begin{cases}
0 & (e = e'), \\
m(2m + n - 2) & (e \cap e' \in X), \\
n(2n + m - 2) & (e \cap e' \in Y), \\
(m + n)(m + n - 2) & \text{otherwise}.
\end{cases}
\]

We consider a group action to $V(K_{m,n})$ as follows: Let $G$ be the cyclic group generated by $\sigma$ of order $n$. Let
\[
\begin{align*}
\sigma(i') &= i' & \text{for } i' \in X = \{ 0', 1', \ldots, (m - 1)' \}, \\
\sigma(i) &= i + 1 \pmod n & \text{for } i \in Y = \{ 0, 1, \ldots, n - 1 \}.
\end{align*}
\]
The action of $G$ on $V(K_{m,n})$ induces an action on $E(K_{m,n})$ by
\[
\sigma \{ i', j \} = \{ \sigma(i'), \sigma(j) \} = \{ i', \sigma(j) \}
\]
for \( i' \in X \) and \( j \in Y \). We define
\[
C_{ij} = \left( a(\sigma^k e_{i}, \sigma^{k'} e_{j}) \right)_{0 \leq k, k' \leq n-1},
\]
where \( e_i = \{ i', 0 \} \in E(K_{m,n}) \). Note that \( C_{ij} \) is an \( n \times n \) cyclic matrix by the definition of the action. We have
\[
E(K_{m,n}) = \bigcup_{i=0}^{m-1} \{ \sigma^k e_i \mid 0 \leq k \leq n - 1 \}.
\]
Hence \( m^{n-3}n^{m-3}C \) is \( \tilde{H}_{K_{m,n}} \) by Lemma 3.14.

Let us calculate eigenvalues of \( C \) by Theorem 2.5. Let \( a = m(2m + n - 2), b = n(2n + m - 2), c = (m + n)(m + n - 2) \), \( A = aJ_m + (-a)I_n \), and \( B = cJ_m + (b - c)I_n \). Then we have \( C_{ij} = A \) if \( i = j \), otherwise \( C_{ij} = B \). By Lemma 2.4, we obtain the eigenvalues of \( A \) and \( B \). The eigenvalues of \( A \) are \((n - 1)a\) and \(-a\). The dimensions of eigenspaces of \((n - 1)a\) and \(-a\) are 1 and \(n - 1\), respectively. The eigenvalues of \( B \) are \(b + (n - 1)c\) and \(b - c\). The dimensions of eigenspaces of \(b + (n - 1)c\) and \(b - c\) are 1 and \(n - 1\), respectively. For \( k = 0 \), define
\[
c_{ij}^{(k)} = \begin{cases} (n - 1)a & (i = j), \\ b + (n - 1)c & (i \neq j). \end{cases}
\]
For \( 1 \leq k \leq m - 1 \), define
\[
c_{ij}^{(k)} = \begin{cases} -a & (i = j), \\ b - c & (i \neq j). \end{cases}
\]
For \( 0 \leq k \leq m - 1 \), we define
\[
\tilde{C}^{(k)} = \left( c_{ij}^{(k)} \right)_{1 \leq i, j \leq m}
\]
Then we have
\[
\tilde{C}^{(0)} = (b + (n - 1)c)J_{mm} + ((n - 1)a - (b + (n - 1)c))I_m,
\]
\[
\tilde{C}^{(k)} = (b - c)J_{mm} + (-a - (b - c))I_m
\]
for \( 1 \leq k \leq m - 1 \). By Lemma 2.4, we obtain the eigenvalues of \( \tilde{C}^{(0)} \) and \( \tilde{C}^{(k)} \). The eigenvalues of \( \tilde{C}^{(0)} \) are
\[
(n - 1)a - (b + (n - 1)c) + m(b + (n - 1)c)
= mn(m + n - 1)(m + n - 2),
\]
and
\[
(n - 1)a - (b + (n - 1)c) = -m^{n-1}n^{m-2}.
\]
The dimensions of the eigenspaces are 1 and \( m - 1 \), respectively. For \( 1 \leq k \leq m - 1 \), the eigenvalues of \( \overline{C}^k \) are

\[-a - (b - c) + m(b + (n - 1)c) = -m^{n-2}n^{m-1}\]

and

\[-a - (b - c) = -2mn.\]

The dimensions of the eigenspaces are 1 and \( m - 1 \), respectively. Theorem 2.5 implies Proposition 3.15.

4. The Lefschetz property for an algebra associated to a graphic matroid

In this section, we will show the Lefschetz property of the algebra associated to the graphic matroid of the complete graph and the complete bipartite graph with at most five vertices (Theorems 4.5 and 4.6).

**Definition 4.1.** Let \( A = \bigoplus_{k=0}^s A_k, \ A_s \neq 0, \) be a graded Artinian algebra. We say that \( A \) has the *strong Lefschetz property* if there exists an element \( L \in A_1 \) such that the multiplication map \( \times L^{s-2k} : A_k \rightarrow A_{s-k} \) is bijective for all \( k \leq \frac{s}{2} \). We call \( L \in A_1 \) with this property a *strong Lefschetz element*.

Let \( K \) be a field of characteristic zero. For a homogeneous polynomial \( F \in K[x_1, x_2, \ldots, x_N] \), we define \( \text{Ann}(F) \) by

\[\text{Ann}(F) = \left\{ P \in K[x_1, \ldots, x_N] \mid P \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N} \right) F = 0 \right\} .\]

Then \( \text{Ann}(F) \) is a homogeneous ideal of \( K[x_1, \ldots, x_N] \). We consider \( A = K[x_1, \ldots, x_N]/\text{Ann}(F) \). Since \( \text{Ann}(F) \) is homogeneous, the algebra \( A \) is graded. Furthermore \( A \) is an Artinian Gorenstein algebra. Conversely, a graded Artinian Gorenstein algebra \( A \) has the presentation

\[A = K[x_1, \ldots, x_N]/\text{Ann}(F)\]

for some homogeneous polynomial \( F \in K[x_1, x_2, \ldots, x_N] \). We decompose \( A \) into the homogeneous spaces \( A_k \). Then \( A_k \) is a vector space over \( K \) for all \( k \). Let \( \Lambda_k \) be the basis for \( A_k \). We define the matrix \( H_F^{(k)} \) by

\[H_F^{(k)} = \left( e_i \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N} \right) e_j \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N} \right) F \right)_{e_i, e_j \in \Lambda_k} .\]

The determinant of \( H_F^{(k)} \) is called the \( k \)th *Hessian* of \( F \) with respect to the basis \( \Lambda_k \).

**Remark 4.2.** Since \( A_0 \cong K \) in this case, we can take the basis \( \{ 1 \} \) for \( A_1 \). Hence the 0th Hessian of \( F \) with respect to the basis \( \{ 1 \} \) is \( F \).
There is a criterion for the strong Lefschetz property for a graded Artinian Gorenstein algebra.

**Theorem 4.3** (Watanabe [8], Maeno–Watanabe [6]). Consider the graded Artinian Gorenstein algebra $A$ with the following presentation and decomposition:

$$A = \mathbb{K}[x_1, x_2, \ldots, x_N] / \text{Ann}(F) = \bigoplus_{k=0}^{s} A_k.$$ 

Let $L = a_1x_1 + a_2x_2 + \cdots + a_Nx_N$. The multiplication map $\times L^{s-2k} : A_k \to A_{s-k}$ is bijective if and only if

$$\det H^{(k)}_{F_k}(a_1, a_2, \ldots, a_N) \neq 0.$$ 

**Definition 4.4.** For a graph $\Gamma$ with $N$ edges, we define the graded Artinian Gorenstein algebra $A_{M(\Gamma)}$ by

$$A_{M(\Gamma)} = \mathbb{K}[x_1, x_2, \ldots, x_N] / \text{Ann}(F_{\Gamma}).$$

If a graph $\Gamma$ has $n + 1$ vertices, then the top degree of $A_{M(\Gamma)}$ is $n$.

**Theorem 4.5.** The algebra $A_{M(K_n)}$ has the strong Lefschetz property for $n \leq 5$. The element $x_1 + \cdots + x_N$ is a strong Lefschetz element.

**Proof.** Let $n \leq 5$. Let us compute the 0th and first Hessians of $F_{K_n}$. It follows from Remark 4.2 that the determinant of the 0th Hessian of $F_{K_n}$ with basis $\{1\}$ is $F_{K_n}$. Since $F_{K_n}(1,1,\ldots,1)$ is the number of spanning trees in $K_n$, we have $\det H^{(0)}_{F_{K_n}}(1,\ldots,1) > 0$. It follows from Theorem 3.3 that $H^{(1)}_{F_{K_n}} = H_{K_n}$, and $\det H^{(1)}_{F_{K_n}}(1,\ldots,1) = \det \tilde{H}_{K_n} \neq 0$. Hence $A_{K_n}$ has the strong Lefschetz property, and the element $x_1 + \cdots + x_N$ is a strong Lefschetz element.

Similarly Theorem 4.6 follows from Remark 4.2 and Theorem 3.16.

**Theorem 4.6.** The algebra $A_{M(K_{m,n})}$ has the strong Lefschetz property for $m + n \leq 5$ and $m, n \geq 1$. The element $x_1 + \cdots + x_N$ is a strong Lefschetz element.

**References**

[1] Norman Biggs, *Algebraic graph theory*, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1993. MR 1271140

[2] Tadahito Harima, Toshiaki Maeno, Hideaki Morita, Yasuhide Numata, Akihito Wachi, and Junzo Watanabe, *The Lefschetz properties*, Lecture Notes in Mathematics, vol. 2080, Springer, Heidelberg, 2013, URL https://doi.org/10.1007/978-3-642-38206-2. MR 3112920

[3] June Huh and Botong Wang, *Enumeration of points, lines, planes, etc.*, Acta Math. 218 (2017), no. 2, 297–317, URL https://doi.org/10.4310/ACTA.2017.v218.n2.a2. MR 3733101

[4] Toshiaki Maeno and Yasuhide Numata, *Sperner property, matroids and finite-dimensional Gorenstein algebras*, Tropical geometry and integrable systems, Contemp. Math., vol. 580, Amer. Math. Soc., Providence, RI, 2012, pp. 73–84, URL https://doi.org/10.1090/conm/580/11496. MR 2985388

[5] ______, *Sperner property and finite-dimensional Gorenstein algebras associated to matroids*, J. Commut. Algebra 8 (2016), no. 4, 549–570, URL https://doi.org/10.1216/JCA-2016-8-4-549. MR 3566530
[6] Toshiaki Maeno and Junzo Watanabe, *Lefschetz elements of Artinian Gorenstein algebras and Hessians of homogeneous polynomials*, Illinois J. Math. 53 (2009), no. 2, 591–603, URL http://projecteuclid.org/euclid.ijm/1266934795. MR 2594646

[7] J. W. Moon, *Enumerating labelled trees*, Graph Theory and Theoretical Physics, Academic Press, London, 1967, pp. 261–272. MR 0231755

[8] Junzo Watanabe, *A remark on the Hessian of homogeneous polynomials*, The curves seminar at Queen’s, vol. XIII, Queen’s Papers in Pure and Appl. Math., vol. 119, Queen’s Univ., Kingston, ON, 2000, pp. 171–178.

(Akiko Yazawa) Department of Mathematics, Graduate School of Science and Technology, Shinshu University, Matsumoto, Nagano 390-8621, Japan

E-mail address: yazawa@math.shinshu-u.ac.jp