A remark on the rational cohomology of $\bar{S}_{1,n}$

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Abstract

We focus on the rational cohomology of Cornalba’s moduli space of spin curves of genus 1 with $n$ marked points. In particular, we show that both its first and its third cohomology group vanish and the second cohomology group is generated by boundary classes.

1 Introduction

The moduli space of spin curves $\bar{S}_g$ was constructed by Cornalba in [6] in order to compactify the moduli space of pairs \{ smooth genus $g$ complex curve $C$, theta-characteristic on $C$ \}. Cornalba’s compactification turns out to be a normal projective variety equipped with a finite morphism:

$$\chi : \bar{S}_g \rightarrow \mathcal{M}_g$$

onto the Deligne-Mumford moduli space of stable curves of genus $g$ (see [6], Proposition (5.2)). The geometry of $\bar{S}_g$ (in particular, its Picard group) was investigated by Cornalba himself in [6] and in [7]; here instead we begin the study of the rational cohomology of $\bar{S}_g$.

As shown by Arbarello and Cornalba in [3], the rational cohomology of $\mathcal{M}_g$ vanishes in low odd degree, so it seems reasonable to expect that the same holds also for $\bar{S}_g$; however, a priori it is not clear at all that the morphism $\chi$ does not increase cohomology. The inductive method of [3] provides indeed an effective tool to check our guess, but the set up of the induction requires to work with moduli of pointed spin curves. Namely, for all integers $g, n$ such that $2g - 2 + n > 0$, we consider the moduli spaces

$$\bar{S}_{g,n} := \{(C, p_1, \ldots, p_n; \zeta; \alpha) : (C, p_1, \ldots, p_n) \text{ is a genus } g \text{ quasi-stable projective curve with } n \text{ marked points; } \zeta \text{ is a line bundle of degree } g - 1 \text{ on } C \text{ having degree 1 on every exceptional component of } C, \text{ and } \alpha : \zeta \otimes 2 \rightarrow \omega_C \text{ is a homomorphism which is not zero at a general point of every non-exceptional component of } C\}$$
In order to put an analytic structure on $\mathcal{S}_{g,n}$, we can easily adapt Cornalba’s construction in [6]: from the universal deformation of the stable model of $(\mathcal{C}, p_1, \ldots, p_n)$ we obtain exactly as in [6], § 4, a universal deformation $U_X \to B_X$ of $X = (\mathcal{C}, p_1, \ldots, p_n; \zeta; \alpha)$; next, we transplant on $\mathcal{S}_{g,n}$ the structure of $B_X/\text{Aut}(X)$ following [6], § 5. Alternatively, we can regard $\mathcal{S}_{g,n}$ as the coarse moduli space associated in the easiest case $r = 2$ to the stack of $r$-spin curves constructed by Jarvis in [10] and revisited by Abramovich and Jarvis in [1].

We recall that $\mathcal{S}_{g,n}$ is the union of two connected components, $\mathcal{S}_{g,n}^+$ and $\mathcal{S}_{g,n}^-$, which correspond to even and odd theta-characteristics, respectively. The main result of the present paper, which completes the research project started in [4] and continued in [5], is the following:

**Theorem 1.** For every $n$,

$$H^1(\mathcal{S}_{1,n}^+, \mathbb{Q}) = H^3(\mathcal{S}_{1,n}^+, \mathbb{Q}) = 0,$$

and $H^2(\mathcal{S}_{1,n}^+, \mathbb{Q})$ is generated by boundary classes.

We note that a similar statement holds true for the moduli space of odd theta-characteristics (see [8]) since $\mathcal{S}_{1,n}^- \cong \overline{\mathcal{M}}_{1,n}$.

In what follows, we work over the field $\mathbb{C}$ of complex numbers; all cohomology groups are implicitly assumed to have rational coefficients.

### 2 The inductive approach

As pointed out in the Introduction, we are going to apply the inductive strategy developed by Arbarello and Cornalba in [3] for the moduli space of curves. Namely, we consider the long exact sequence of cohomology with compact supports:

$$\cdots \to H_c^k(S_1,n) \to H^k(\mathcal{S}_{1,n}) \to H^k(\partial S_{1,n}) \to \cdots$$

(1)

Hence, whenever $H_c^k(S_1,n) = 0$, there is an injection $H^k(\mathcal{S}_{1,n}) \hookrightarrow H^k(\partial S_{1,n})$. Moreover, from [6], § 3, it follows that each irreducible component of the boundary of $\mathcal{S}_{1,n}$ is the image of a morphism:

$$\mu_i : X_i \to \mathcal{S}_{1,n},$$

where either

$$X_i = \overline{\mathcal{M}}_{0,s+1} \times \mathcal{S}_{1,t+1}$$

where $s + t = n$; or

$$X_i = \overline{\mathcal{M}}_{0,n+2}.$$

Finally, exactly as in [3], Lemma 2.6, a bit of Hodge theory implies that the map $H^k(\mathcal{S}_{1,n}) \to \bigoplus_i H^k(X_i)$ is injective whenever $H^k(\mathcal{S}_{1,n}) \to H^k(\partial S_{1,n})$.
Thus, we obtain the first claim of Theorem 1 by induction, provided we show that \( H^1_c(S_1,n) = H^3_c(S_1,n) = 0 \) for almost all values of \( n \), and we check that \( H^1(S_1,n) = H^3(S_1,n) = 0 \) for all remaining values of \( n \). The first task is accomplished by the following

**Lemma 1.** We have \( H_k(S_1,n) = 0 \) for \( k > n \).

Indeed, \( M_{1,1} \cong \mathbb{A}^1 \) is affine. Moreover, it is well-known that the forgetful morphism \( M_{1,n} \to M_{1,1} \) is affine. Finally, the morphism \( S_{1,n} \to M_{1,n} \) is finite since it is the restriction of the finite morphism \( \overline{S}_{1,n} \to \overline{M}_{1,n} \), hence the claim holds.

Now, we give a closer inspection to \( \overline{S}_{1,n} \). Of course, it is the disjoint union of \( \overline{S}^+_{1,n} \) and \( \overline{S}^-_{1,n} \), corresponding to even and odd spin structures respectively. However, since the unique odd theta characteristic on a smooth elliptic curve \( E \) is \( \mathcal{O}_E \), there is a natural isomorphism \( \overline{S}^-_{1,n} \cong \overline{M}_{1,n} \), so we may restrict our attention to \( \overline{S}^+_{1,n} \). First of all, the following holds:

**Proposition 1.** \( H^1(\overline{S}^+_{1,n}) = 0 \).

**Proof.** By the above argument, it is enough to check that \( H^1(\overline{S}^+_{1,n}) \) vanishes for \( n = 1 \). In order to do so, we claim that there is a surjective morphism

\[
\overline{M}_{0,4} \to \overline{S}^+_{1,1}.
\]

Indeed, let \( (C;p_1,p_2,p_3,p_4) \) be a 4-pointed stable genus zero curve. The morphism \( f \) associates to it the admissible covering \( E \) of \( C \) branched at the \( p_i \)'s, pointed at \( q_1 \) and equipped with the line bundle \( \mathcal{O}_E(q_1 - q_2) \), where \( q_i \) denotes the point of \( E \) lying above \( p_i \). It follows that

\[
H^1(\overline{S}^+_{1,1}) \subseteq H^1(\overline{M}_{0,4}) = H^1(\mathbb{P}^1) = 0
\]

and Proposition 1 is completely proved.

Recall that the boundary components of \( \overline{M}_{1,n} \) are \( \Delta_{\text{irr}} \), whose general member is an irreducible \( n \)-pointed curve \( C \) of geometric genus zero with exactly one node, and \( \Delta_{1,I} \), whose general member is the union of two smooth curves meeting at one node, \( C_1 \) of genus 1 with marked points labelled by \( I \subseteq \{1, \ldots, n\} \), and \( C_2 \) of genus 0 with marked points labelled by \( \{1, \ldots, n\} \setminus I \) (of course \( |I| \leq n - 2 \)). The corresponding boundary components of \( \overline{S}^+_{1,n} \) are:

- \( A^+_{\text{irr}} \), with an even spin structure on \( C \);
- \( B^+_{\text{irr}} \), with an even spin structure on \( C \) blown up at the node;
- \( A^+_{1,I} \), with even theta-characteristics on \( C_1 \) and \( C_2 \).
Notice that in this case $B^+_{1,I}$, whose general member should carry odd theta-characteristics on both $C_1$ and $C_2$, is empty since a smooth rational curve has no odd theta-characteristic.

Hence on $S^+_{1,n}$ we have the boundary classes $\alpha^+_{\text{irr}}, \beta^+_{\text{irr}},$ and $\alpha^+_{1,I}$; there are also the classes

\[\delta_{\text{irr}} = p^*(\delta_{\text{irr}})\]
\[\delta_{1,I} = p^*(\delta_{1,I})\]

where

\[p : S^+_{1,n} \rightarrow \mathcal{M}_{1,n}\]

is the natural projection. Exactly as in [6], § 7, there are relations

\[\delta_{\text{irr}} = \alpha^+_{\text{irr}} + 2\beta^+_{\text{irr}} \tag{2}\]
\[\delta_{1,I} = 2\alpha^+_{1,I}. \tag{3}\]

**Lemma 2.** The vector space $H^2(S^+_{1,2})$ is generated by boundary classes.

**Proof.** We are going to deduce this from an Euler characteristic computation. Indeed, we are going to show that

\[\chi(S^+_{1,2}) = 4. \tag{4}\]

Since

\[\chi(S^+_{1,2}) = 2h^0(S^+_{1,2}) - 2h^1(S^+_{1,2}) + h^2(S^+_{1,2}) = 2 + h^2(S^+_{1,2})\]

from (4) we may deduce that $h^2(S^+_{1,2}) = 2$. On the other hand, since the natural projection $S^+_{1,2} \rightarrow \mathcal{M}_{1,2}$ is surjective, $H^2(\mathcal{M}_{1,2})$ injects into $H^2(S^+_{1,2})$. It follows that $H^2(S^+_{1,2})$ is generated by $\delta_{\text{irr}}$ and $\delta_{1,\emptyset}$, which are linear combinations of $\alpha^+_{\text{irr}}, \beta^+_{\text{irr}},$ and $\alpha^+_{1,\emptyset}$ by (2) and (3).

First of all, we compute $\chi(S^+_{1,1})$. It is clear that

\[\chi(S^+_{1,1}) = 2h^0(S^+_{1,1}) - h^1(S^+_{1,1}) = 2.\]

On the other hand, $\partial S^+_{1,1}$ consists of exactly two points, corresponding to a 3-pointed rational curve with two marked points either identified or joined by an exceptional component. Hence

\[\chi(S^+_{1,1}) = \chi(S^+_{1,1}) - \chi(\partial S^+_{1,1}) = 0. \tag{5}\]

Next, we compute $\chi(S^+_{1,2})$. The natural projection $S^+_{1,2} \rightarrow \mathcal{M}_{1,2}$ is generically three-to-one, but there are a few special fibers with less than three points. Indeed, let $(E:p_1,p_2)$ be a smooth 2-pointed elliptic curve.
The linear series \(|2p_1|\) provides a realization of \(E\) as a two-sheeted covering of \(\mathbb{P}^1\) ramified over \(\infty, 0, 1\) and \(\lambda\). Denote by \(q_0, q_1,\) and \(q_\lambda\) the points of \(E\) lying above \(0, 1,\) and \(\lambda,\) so that the three even theta-characteristics of \(E\) are given by \(O_E(p_1 - q_0), O_E(p_1 - q_1),\) and \(O_E(p_1 - q_\lambda).\) If \(\lambda = \frac{1}{2}\) and \(p_2 = q_\lambda,\) then the projectivity of \(\mathbb{P}^1\) defined by \(z \mapsto 1 - \frac{1}{z}\) induces an automorphism of \((E; p_1, p_2)\) exchanging \(O_E(p_1 - q_0)\) and \(O_E(p_1 - q_1).\) If \(\lambda = -\omega (\text{with } \omega^3 = 1)\) and \(p_2\) is one point lying above \(\omega - 1,\) then the projectivity of \(\mathbb{P}^1\) defined by \(z \mapsto z + \omega\) induces an automorphism of \((E; p_1, p_2)\) that exchanges cyclically its three even theta-characteristics. Since it is clear (for instance, from [9], IV, proof of Corollary 4.7) that the above ones are the only exceptional cases, we have:

\[
\chi(S_{1,2}^+) = 3\chi(M_{1,2} \setminus \{2 \text{ points}\}) + 2\chi(\text{point}) + \chi(\text{point}) = 0. \tag{6}
\]

In fact, \(\chi(M_{1,2}) = 1,\) as observed in [3], (5.4). Finally we turn to the Euler characteristic of \(S_{1,2}^+.\) From [6], Examples (3.2) and (3.3), and [3], Figure 1, we may deduce that

\[
\chi(S_{1,2}^+) = \chi(S_{1,2}^+) + 2\chi(M_{0,4}') + \chi(S_{1,1}^+) + 4,
\]

where \(M_{0,n}'\) denotes the quotient of \(M_{0,n}\) modulo the operation of interchanging the labelling of two of the marked points.

Since \(\chi(M_{0,4}') = 0\) (see [3], (5.4)), relation (4) follows from (5) and (6).

Let \(P\) a finite set with \(|P| = n\) and let \(x\) and \(y\) be distinct and not belonging to \(P;\) define

\[
\xi: \overline{M}_{0,P\cup\{x,y\}} \longrightarrow B^+_{\text{irr}} \hookrightarrow S^+_{1,n}
\]

by joining the points labelled \(x\) and \(y\) with an exceptional component and taking the unique even theta characteristic on the resulting curve. Then the analogue of Lemma 4.5 in [3] holds:

**Lemma 3.** The kernel of

\[
\xi^*: H^2(S^+_{1,n}) \longrightarrow H^2(\overline{M}_{0,P\cup\{x,y\}})
\]

is one-dimensional and generated by \(\delta_{\text{irr}}.\)

**Proof.** By [3], Lemma 3.16, it is clear that \(\xi^*(\delta_{\text{irr}}) = 0.\) Moreover, from Lemma 2 it follows that \(H^2(S^+_{1,2})\) is generated by \(\delta_{\text{irr}}\) and \(\delta_{1,0};\) since \(\delta_{1,0}\) pulls back to \(\delta_{0,\{x,y\}},\) which is not zero, the claim holds for \(n = 2.\) Hence we can apply the inductive argument of [3], pp. 113–114. It follows that if \(\xi^*(\alpha) = 0\) for \(\alpha \in H^2(S^+_{1,n})\) then there exists a constant \(a\) such that \(\alpha - a\delta_{\text{irr}}\) restricts to zero on all boundary components of \(S^+_{1,n}\) different from \(A^+_{\text{irr}}.\) However, we claim that \(A^+_{\text{irr}}\) is linearly equivalent to \(2B^+_{\text{irr}}.\) Indeed,
this is clear in $\overline{S}_{1,1}^+ \cong \mathbb{P}^1$. If $\pi : \overline{S}_{1,n}^+ \to \overline{S}_{1,1}^+$ is the natural forgetful map, then $A_{\text{irr}}^{+} = \pi^*(A_{\text{irr}}^+)$ and $B_{\text{irr}}^{+} = \pi^*(B_{\text{irr}}^+)$. Hence $\alpha - a_\delta^{\text{irr}}$ restricts to zero on all boundary components of $\overline{S}_{1,n}^+$ and the claim follows exactly as in [3], Lemma 4.5, from Proposition [4] and the analogue of [2], Proposition 2.8.

\newline

**Proposition 2.** The vector space $H^2(\overline{S}_{1,n}^+)$ is generated by boundary classes.

**Proof.** Let $V$ be the subspace of $H^2(\overline{S}_{1,n}^+)$ generated by the elements $\alpha_{i,l}^+$. In view of Lemma [3] and (2), it will be sufficient to show that the morphism $\xi^*$ vanishes modulo $V$. The proof is by induction on $n$: for the inductive step we refer to [3], pp. 114–118, while the basis of the induction is provided by Lemma [2].

Finally, we are also able to prove the last part of Theorem [1].

**Lemma 4.** We have $H^3(\overline{S}_{1,n}^+) = 0$.

**Proof.** Once again, by the exact sequence (1) and Lemma 1, it is enough to check that $H^3(\overline{S}_{1,n}^+) = 0$ for $n \leq 3$. First, we deal with the case $n = 3$. By Proposition [2] $H^2(\overline{S}_{1,3}^+)$ is generated by the six boundary classes $\alpha_{\text{irr}}^+, \beta_{\text{irr}}^+, \alpha_{i,0}^+, \alpha_{i,(1)}^+, \alpha_{i,(2)}^+$, and $\alpha_{i,(3)}^+$. Notice further that $\alpha_{\text{irr}}^+$ and $\beta_{\text{irr}}^+$ are linearly dependent. Indeed, if $\pi : \overline{S}_{1,3}^+ \to \overline{S}_{1,1}^+$ is the natural forgetful map, from [3], Lemma 3.1 (iii) it follows that $\alpha_{\text{irr}}^+ = \pi^*(\alpha_{\text{irr}}^+)$ and $\beta_{\text{irr}}^+ = \pi^*(\beta_{\text{irr}}^+)$, while Poincaré duality yields $H^2(\overline{S}_{1,1}^+) \cong H^0(\overline{S}_{1,1}^+) \cong \mathbb{Q}$. Hence we deduce $h^2(\overline{S}_{1,3}^+) \leq 5$; next, we claim that

$$\chi(\overline{S}_{1,3}^+) = 12. \quad (7)$$

The statement is a direct consequence of the claim, since

$$\chi(\overline{S}_{1,3}^+) = 2h^0(\overline{S}_{1,3}^+) + 2h^2(\overline{S}_{1,3}^+) - 2h^1(\overline{S}_{1,3}^+) - h^3(\overline{S}_{1,3}^+) \leq 12 - h^3(\overline{S}_{1,3}^+).$$

First of all, we compute $\chi(S_{1,3}^+)$. The natural projection $S_{1,3}^+ \to M_{1,3}$ is generically three-to-one, but there is a special fiber with only one point. Indeed, if $(E ; p_1, p_2, p_3)$ is a smooth 3-pointed elliptic curve realized by the linear series $|2p_1|$ as a two-sheeted covering of $\mathbb{P}^1$ ramified over $\infty$, 0, 1 and $-\omega$ (with $\omega^3 = 1$) and $p_2$, $p_3$ are the two points lying above $\frac{\omega}{\omega - 1}$, then the projectivity of $\mathbb{P}^1$ defined by $z \mapsto \frac{z + \omega}{\omega}$ induces automorphisms of $(E ; p_1, p_2, p_3)$ exchanging cyclically its three even theta-characteristics. Therefore we have:

$$\chi(S_{1,3}^+) = 3\chi(M_{1,3} \setminus \{ \text{point} \}) + \chi(\text{point}) = -2. \quad (8)$$
Recall that $\chi(\mathcal{M}_{1,3}) = 0$, as observed in [3], (5.4). Finally we turn to the Euler characteristic of $\mathcal{S}_{1,3}^+$. From [6], Examples (3.2) and (3.3), and [3], Figure 2, it is clear that

$$
\chi(\mathcal{S}_{1,3}^+) = \chi(S_{1,3}^+) + 2\chi(M_{0,5}^0) + \chi(S_{1,1}^+)\chi(M_{0,4}) + 3\chi(S_{1,2}^+) + 2\chi(M_{0,4}) + 12\chi(M_{0,4}) + 3\chi(S_{1,1}^{(0),+}) + 14.
$$

Since $\chi(M_{0,4}) = -1$, $\chi(M_{0,4}^0) = 0$ and $\chi(M_{0,5}^0) = 1$ (see [3], (5.4)), now (7) follows from [3], [6] and [5]. Finally, by Hodge theory of complex projective orbifolds, the surjective morphism $\mathcal{S}_{1,3} \to \mathcal{S}_{1,n}$ for $n \leq 3$ induces an injective morphism $H^3(\mathcal{S}_{1,n}) \to H^3(\mathcal{S}_{1,3})$ for $n \leq 3$. Hence the claim follows.

\[\Box\]

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References

[1] D. Abramovich and T. J. Jarvis: Moduli of twisted spin curves. Proc. Amer. Math. Soc. 131 (2003), 685–699.
[2] E. Arbarello and M. Cornalba: The Picard groups of the moduli spaces of curves. Topology 26 (1987), 153–171.
[3] E. Arbarello and M. Cornalba: Calculating cohomology groups of moduli spaces of curves via algebraic geometry. Inst. Hautes Études Sci. Publ. Math. 88 (1998), 97–127.
[4] G. Bini, C. Fontanari: Moduli of curves and spin structures via algebraic geometry, Trans. Amer. Math. Soc. 358 (2006), 3207-3217.
[5] G. Bini, C. Fontanari: On the geometry of $\mathcal{S}_2$, Internat. J. Math. (to appear).
[6] M. Cornalba: Moduli of curves and theta-characteristics. Lectures on Riemann surfaces (Trieste, 1987), World Sci. Publishing (1989), 560–589.
[7] M. Cornalba: A remark on the Picard group of spin moduli space. Rend. Mat. Acc. Lincei s. 9, v. 2 (1991), 211–217.
[8] E. Getzler: The semi-classical approximation for modular operads. Comm. Math. Phys. 194 (1998), 481–492.
[9] R. Hartshorne: Algebraic Geometry. Graduate Texts in Math. 52, Springer (1977).

[10] T. J. Jarvis: Geometry of the moduli of higher spin curves. Internat. J. Math. 11 (2000), 637–663.

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