Iterated integrals over higher dimensional loops

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Abstract: We give a definition of higher dimensional iterated integrals based on integration over membranes. We prove basic properties of this definition and formulate a conjecture which extends Chen’s de Rham Theorem for iterated integrals to the membrane case.

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Introduction

The classical, or one-dimensional iterated integral is defined for a smooth manifold $X$ as well as 1-forms $\omega_1, \ldots, \omega_m$ and a path $\gamma : I \to X$ by

$$\int_\gamma \omega_1 \cdots \omega_m = \int_{t_1 < t_2 < \cdots < t_m} \gamma^* \omega_1 \wedge \cdots \wedge \gamma^* \omega_m.$$ 

See [Che77] for more on this. One of the most important results of Chen’s is the so called de Rham Theorem of Chen, which we state now. Fix a base point $x_0 \in X$ and let $B_s(X)_{x_0}$ denote the real vector space of all maps from the set of loops at $x_0$ to $\mathbb{R}$ which are linear combinations of iterated integrals of length at most $s$. Let $B_s(X)_{x_0}^{hom}$ denote the subspace of homotopy-invariant maps. Chen’s de Rham Theorem says that the evaluation map

$$B_s(X)_{x_0}^{hom} \to \text{Hom}_\mathbb{Z}(\mathbb{Z}[\pi_1(X, x_0)]/J^{s+1}, \mathbb{R})$$

is an isomorphism of real vector spaces, where $\mathbb{Z}[\pi_1(X, x_0)]$ is the group ring and $J$ the augmentation ideal.

In this paper we propose a higher dimensional version of iterated integrals. We investigate their basic properties and state the conjecture that the resulting map

$$B^n_s(X)_{x_0}^{hom} \to \text{Hom}_\mathbb{Z}(\mathbb{Z}[\pi_n/\pi_n^{<n}]/J^{s+1}, \mathbb{R})$$

is an isomorphism. Here $\pi_n$ is the higher homotopy group and $\pi_n^{<n}$ is the subgroup generated by maps from spheres of lower dimension. We prove this conjecture in the case $s = 1$.

Similar, but less general iterated integrals over higher dimensional membranes were first defined in [Hora], where they are applied to Hilbert modular forms. They also are used in a construction of Multiple Dedekind zeta functions [Horb]. The definition of such iterated integrals was done in a search of a higher dimensional analogue of Manin’s non-commutative modular symbol [Man06].

1 Iterated integrals on a membrane

Let $n \in \mathbb{N}$. Instead of paths we consider membranes. A $n$-dimensional membrane on $X$ is a continuous map $g : I^n \to X$, which is continuously
differentiable outside a closed set \( S \subset I^n \) of measure zero. On the set \( I^n \) we introduce the partial order

\[ t < s \iff t_1 < s_1, \ldots, t_n < s_n. \]

The direct generalization of the one-dimensional iterated integral would be this: For \( n \)-forms \( \omega_1, \ldots, \omega_r \) we define the *n-dimensional iterated integral* as

\[
\int g \omega_1 \ldots \omega_r = \int_{t_1 < \cdots < t_r} g^* \omega_1 \wedge \cdots \wedge g^* \omega_r.
\]

In this paper we present a more general approach. The following definition of an \( n \)-dimensional iterated integral is motivated by physics. Consider a number of

- \( s \) events, watched by
- \( n \) observers

in special relativity. Let \( t_\nu \) be the time coordinate for observer \( \nu \), where \( \nu = 1, \ldots, n \). Now consider \( s \) events, \( e_1, \ldots, e_s \). The simplest case is, when each observer sees the events in the same order. However, time is relative, so it is possible that each observer sees the events in a different order. Let \( \rho_\nu \) be a permutation of \( s \) elements, describing the order in which observer \( \nu \) sees the events. For each \( \nu = 1, \ldots, n \), we cut the admissible interval for the time variable \( t_\nu \) into subintervals, where the different events occur. For fixed \( \nu \), let

\[ 0 < t^1_\nu < t^2_\nu < \cdots < t^m_\nu < 1 \]

values of \( t_\nu \) in each subinterval.

Consider variables by \( t^\sigma_\nu \) for \( \nu = 1, \ldots, n \) and \( \sigma = 1, \ldots, s \). The subscript \( \nu \) corresponds to \( \nu \)-th time direction for the observer \( \nu \).

Let \( D \) be a domain in terms of the variables \( t^\sigma_\nu \), defined by

\[
D = \{(t^\sigma_\nu)|0 < t^1_\nu < t^2_\nu < \cdots < t^s_\nu < 1, \ \nu = 1, \ldots, n\} \subset I^{sn},
\]

where \( I = [0,1] \) is the unit interval.

We associate differential \( n \)-forms \( \omega_\sigma \) on a manifold \( X \) to each event \( \sigma \), for \( \sigma = 1, 2, \ldots, s \). Let

\[ g : [0,1]^n \to X \]
be a membrane. The observer $\nu$ sees the event $\sigma$, realized as a differential form $g^*\omega_{\sigma}$ at time $t^\nu_{\rho_\nu}(\sigma)$. Therefore $g^*\omega_{\sigma}$ depends on $(t^1_{\rho_1}(\sigma), \ldots, t^n_{\rho_n}(\sigma))$, which are the time coordinates for each observer, in which they see the event $\sigma$.

To each $\sigma = 1, \ldots, s$ and the permutations $\rho_1, \ldots, \rho_n$ we associate a map

$$\phi_{\sigma, \rho}: I^s \to I^n$$

$$(t^\sigma_{\nu})_{\sigma, \nu} \mapsto (t^\rho_1(\sigma), \ldots, t^\rho_n(\sigma)).$$

**Definition 1.1** An $n$-dimensional iterated integral in terms of $n$-forms $\omega_{\sigma}$, $\sigma = 1, \ldots, s$, permutations $\rho_\nu$, for $\nu = 1, \ldots, n$ and a smooth map $g$, is defined as

$$\int_g^\rho \omega_1 \cdots \omega_s \overset{\text{def}}{=} \int_D \phi^*_{\rho, 1}\phi^*_{\rho, s} \omega_1 \wedge \cdots \wedge \omega_s.$$

In the case when $\rho_\nu = \text{Id}$ for all $\nu$, we leave out the superscript $\rho$.

**Examples 1.2**

- In the classical case $n = 1$ of one observer there is only one permutation $\rho \in \text{Per}(s)$ and one gets

$$\int_g^\rho \omega_1 \cdots \omega_s = \det(\rho) \int_g \omega_{\rho(1)} \cdots \omega_{\rho(s)}.$$

- In the special case of 2 observers one may use the following type of diagram for visualization. We have the observers $A$ and $B$, and we draw a time axis for each of them. Say there are three events $e_1, e_2, e_3$, then the diagram

```
  B
  \[
  \begin{array}{ccc}
  e_3 & & \\
  e_2 & & \\
  e_1 & & \\
  \end{array}
  \]
  \]
  A
```

\[
  \begin{array}{ccc}
  e_1 & e_2 & e_3 \\
  \end{array}
  \]

\[
  \begin{array}{ccc}
  e_1 & e_2 & e_3 \\
  \end{array}
  \]

A
means that observer A sees the events in the order \(e_3, e_1, e_2\) and B sees them as \(e_2, e_3, e_1\). The shaded region indicates, where the integration takes place. Note that these diagrams only cover the case \(\rho_1\rho_2 = \text{Id}\).

There is also a different way of presenting the iterated integral as follows. Define \(\phi_\rho : I^{sn} \to I^{sn}\) by

\[
\phi_\rho(t)_\nu^\sigma = t_\nu^\rho(\sigma).
\]

Let \(p_\sigma : X^s \to X\) be the \(\sigma\)th projection and consider the \(sn\)-form on \(X^s\),

\[
\omega = p_1^s\omega_1 \wedge \cdots \wedge p_s^s\omega_s.
\]

We denote the map \((I^n)^s \to X^s\) with coordinates \(g\) also by \(g\).

**Lemma 1.3** We have

\[
\int_g^\rho \omega_1 \cdots \omega_s = \int_D(g \circ \phi_\rho)^s\omega = \int_{D_\rho} g^s\omega = \int_{g(D_\rho)} \omega.
\]

here \(D_\rho\) is the set of all \(t \in I^{sn}\) with \(0 < t_\nu^\rho(\sigma) < t_\nu^\rho(\sigma+1) < 1\) for all \(\sigma, \nu\).

**Proof:** Clear. \(\square\)

## 2 Reparametrization

Let \(\varphi : I^n \to I^n\) be a piecewise diffeomorphism, that is \(\varphi\) is a homeomorphism, such that \(\varphi\) and \(\varphi^{-1}\) are continuously differentiable outside a closed set of measure zero. We say that \(\varphi\) is *monotonic*, if

\[x \leq y \implies \varphi(x) \leq \varphi(y),\]

where we say \(x \leq y\) if \(x_j \leq y_j\) for every \(1 \leq j \leq n\). We define \(\varphi^s : (I^n)^s \to (I^n)^s\) by

\[
\varphi^s(t_1, \ldots, t_s) = (\varphi(t_1), \ldots, \varphi(t_s)).
\]

**Lemma 2.1** If \(\varphi\) is a monotonic homeomorphism, then

\[
\varphi^s(D_\rho) = D_\rho.
\]
Proof: In our notation, \( D \) is the set of all \( x \in (I^n)^s \) with \( x_1 < x_2 < \cdots < x_s \).
For \( x \in D \) we have \( \varphi(x_1) \leq \varphi(x_2) \leq \cdots \leq \varphi(x_n) \), hence \( \varphi^s(x) \in \overline{D} \), the closure of \( D \), i.e., \( \varphi(D) \subset \overline{D} \). But as \( \varphi \) is a homeomorphism, the image \( \varphi(D) \) is open. Every open subset of \( \overline{D} \) lies in \( D \), so \( \varphi(D) \subset D \). The inverse map \( \varphi^{-1} \) is monotonic as well, hence \( \varphi(D) = D \). We have \( \varphi^s \circ \phi = \phi \circ \varphi^s \), so that finally
\[
\varphi^s(\phi(D)) = \phi(\varphi^s(D)) = \phi(D).
\]
□

Proposition 2.2
(a) We have
\[
\int_g^\rho \omega_1 \cdots \omega_s = \int_{g \circ \varphi}^\rho \omega_1 \cdots \omega_s
\]
for every monotonic piecewise diffeomorphism \( \varphi : I^n \to I^n \).

(b) We have
\[
\int_{F \circ g}^\rho \omega_1 \cdots \omega_s = \int_g^\rho (F^* \omega_1) \cdots (F^* \omega_s)
\]
for every smooth map \( F : X \to Y \).

Proof: (a) By \( g_\rho = g^s \circ \phi \rho \) we get
\[
\int_{g \circ \varphi}^\rho \omega_1 \cdots \omega_s = \int_D (g^s \circ \varphi \circ \phi \rho)^* \omega = \int_D (\varphi \circ \phi \rho)^*(g^s)^* \omega
\]
\[
= \int_{\varphi \circ \phi \rho(D)} (g^s)^* \omega = \int_{\phi \rho(D)} (g^s)^* \omega
\]
\[
= \int_D \phi \rho(g^s)^* \omega = \int_D g_\rho^* \omega = \int_g^\rho \omega_1 \cdots \omega_s.
\]
For (b) note \( (F \circ g)_\rho = F^s \circ g_\rho \), so that
\[
\int_{F \circ g}^\rho \omega_1 \cdots \omega_s = \int_D (F^s \circ g_\rho)^* \omega = \int_D g_\rho^*(F^s)^* \omega
\]
\[
= \int_g^\rho (F^* \omega_1) \cdots (F^* \omega_s).
\]
□
3 Homotopy invariance

In this section we are going to show that for holomorphic differential forms, the iterated integral is invariant under homotopic deformations of $g$, which preserve certain foliations, see below.

For each $\nu = 1, \ldots, n$ the fibers of the projection $p_\nu : I^n \to I; (x_1, \ldots, x_n) \mapsto x_\nu$ form the leaves of a foliation $\mathcal{P}_\nu$.

Let $X$ be a complex manifold of complex dimension $n$ and let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be non-trivial foliations with complex submanifolds as leaves. We say that a membrane $g : T^n \to X$ is $\mathcal{F}$-admissible, if $g$ maps $\mathcal{P}_\nu$-leaves to $\mathcal{F}_\nu$-leaves.

A homotopy $h : I^n \times I \to X$ is called $\mathcal{F}$-admissible, if each intermediate $g_u = h(\cdot, u)$ is $\mathcal{F}$-admissible.

**Theorem 3.1** Let $X$ be a complex manifold of complex dimension $n$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be complex foliations on $X$. Let $\omega_1, \ldots, \omega_s$ be holomorphic $n$-forms on $X$.

Assume that two $\mathcal{F}$-admissible membranes $g_0, g_1$ are homotopic with a $\mathcal{F}$-admissible homotopy. Then we have

$$\int_{g_0}^{\rho} \omega_1 \cdots \omega_s = \int_{g_1}^{\rho} \omega_1 \cdots \omega_s.$$

**Proof:** Write $F_i = \int_{g_i}^{\rho} \omega_1 \cdots \omega_s$. Let $I$ denote the unit interval $[0, 1]$. Let $\omega = p^*_1 \omega_1 \wedge \cdots \wedge p^*_s \omega_s$. Then $\omega$ is a holomorphic $sn$-form on $X^s$, so $d\omega = 0$.

Define $h_\rho : D \times I \to X^s$ by

$$h_\rho(t, u) = (h(\phi_{\rho,1}(t), u), \ldots, h(\phi_{\rho,s}(t), u)).$$

Note that $h_\rho(t, i) = g_{i,\rho}$ for $i = 0, 1$. By Stokes’s Theorem we get

$$0 = \int_{D \times I} d(h_\rho^* \omega) = \int_{\partial(D \times I)} h_\rho^* \omega = \int_{D \times \{1\}} h_\rho^* \omega - \int_{D \times \{0\}} h_\rho^* \omega + \sum_{\nu=1}^n \sum_{\sigma=0}^s \varepsilon_{\sigma, \nu} \int_{D_{\sigma, \nu} \times I} h_\rho^* \omega.$$
where $\varepsilon_{\sigma, \nu} \in \{\pm 1\}$ is a sign and $D_{\sigma, \nu}$ is defined as follows. Put $t^0_\nu = 0$ and $t^{s+1}_\nu = 1$ and define

$$D_{\sigma, \nu} = \left\{ (t^1_\alpha)_{1 \leq \alpha \leq n} : 0 < t^1_\alpha < \cdots < t^s_\alpha < 1 \text{ for } \alpha \neq \nu, \right. \left. t^0_\nu < \cdots < t^s_\nu = t^{s+1}_\nu < \cdots < t^{s+1}_\nu \right\} \subset I^{sn}. $$

We will show that $\int_{D_{\sigma, \nu} \times I} h^*\rho \omega = 0$ for all $\sigma, \nu$. Note that $\int_{D_{\sigma, \rho} \times I} h^*\rho \omega = \int_{h(D_{\sigma, \rho} \times I)} \omega$. Since each $g_u$ is $F$-admissible, $g_{u, \rho}$ maps $D_{\sigma, \nu}$ to the set $X_{\sigma, \nu}$ of all $x_1, \ldots, x_s \in X^s$ for which $x_{\rho(\sigma)}$ and $x_{\rho(\sigma+1)}$ lie in the same $F_{\nu}$-leaf. The latter is a complex submanifold of $X^s$ of dimension $sn - 1$. Since this is true for all $u$, it follows that

$$h(D_{\sigma, \nu} \times I) \subset X_{\sigma, \nu},$$

so that we integrate a holomorphic $sn$-form over a submanifold of dimension $\leq sn - 1$, where all holomorphic $sn$-forms vanish, hence the integral is zero.

\[\square\]

### 4 Shuffle relations

Let $s, s' \in \mathbb{N}$. An $(s, s')$-shuffle is a permutation $\sigma \in \text{Per}(s + s')$ such that

$$\sigma(1) < \cdots < \sigma(s) \quad \text{and} \quad \sigma(s + 1) < \cdots < \sigma(s + s').$$

For two permutations $\rho \in \text{Per}(s)$ and $\rho' \in \text{Per}(s')$, a $(\rho, \rho')$-shuffle is a permutation $\sigma \in \text{Per}(s + s')$ of the form

$$\sigma = \tau \circ (\rho, \rho'),$$

where $\tau$ is an $(s, s')$-shuffle. Let $\text{Sh}(\rho, \rho')$ denote the set of all $(\rho, \rho')$-shuffles. A given permutation $\sigma \in \text{Per}(s + s')$ lies in $\text{Sh}(\rho, \rho')$ if and only if

$$\sigma(\rho(1)) < \cdots < \sigma(\rho(s)) \quad \text{and} \quad \sigma(n + \rho'(1)) < \cdots < \sigma(s + \rho'(s')).$$

Let $\rho = (\rho_1, \ldots, \rho_n)$ and $\rho' = (\rho'_1, \ldots, \rho'_n)$ be tuples of permutations in $\text{Per}(s)$ and $\text{Per}(s')$ respectively. Then define

$$\text{Sh}(\rho, \rho') = \prod_{\nu=1}^n \text{Sh}(\rho_\nu, \rho'_\nu).$$
Proposition 4.1 (Shuffle relations) We have

\[
\left( \int_g \omega_1 \ldots \omega_s \right) \left( \int_g' \omega_{s+1} \ldots \omega_{s+s'} \right) = \sum_{\tau \in \text{Sh}(\rho, \rho')} \int_g \omega_1 \ldots \omega_{s+s'}.
\]

Proof: We write the product on the left as

\[
\left( \int_{t_1^{\rho_1(1)}} \ldots < \int_{t_1^{\rho_1(s)}} \omega_1 \wedge \cdots \wedge \omega_s \right) \times \left( \int_{s_1^{\rho_1'}(1)} \ldots < \int_{s_1^{\rho_1'(s)}} \omega_1 \wedge \cdots \wedge \omega_s \right).
\]

Consider points \( t, x \) in the respective domains of integration, which are chosen generically in the sense that no coordinates of \( t \) match any coordinates of \( x \). Then the order of the points \( t_1^{\rho_1(1)}, \ldots, t_1^{\rho_1(s)}, x_1^{\rho_1(1)}, \ldots, x_1^{\rho_1'(s)} \) determines an \((\rho, \rho')\)-shuffle \( \tau_1 \). Repeat this in the other rows to get a \((\rho, \rho')\)-shuffle \( \tau \). All such shuffles appear and the domain of integration is, up to a set of measure zero, the disjoint union over all \( \tau \). This gives the claim. \( \square \)

5 Composition

We say that a membrane \( g \) is closed with base point \( x_0 \in X \) if \( g \) maps the entire boundary of \( I^n \) to the singleton \( \{x_0\} \).

Let the membranes \( g_1, g_2 \) be closed with the same base point \( x_0 \). We define a membrane or \( g_1g_2 \) by

\[
g_1g_2(t) = \begin{cases} 
g_1(2t) & 0 \leq t_\nu \leq \frac{1}{2} \forall \nu \\
g_2(2t-1) & \frac{1}{2} < t_\nu \leq 1 \forall \nu \\
x_0 & \text{otherwise.}
\end{cases}
\]
Proposition 5.1 Let \( g_1, g_2 \) be closed membranes with the same base point. Then
\[
\int_{g_1 g_2} \omega_1 \ldots \omega_s = \sum_{j=0}^{s} \left( \int_{g_1} \omega_1 \ldots \omega_j \right) \left( \int_{g_2} \omega_{j+1} \ldots \omega_s \right).
\]

Proof: The left hand side equals
\[
\int_{t_1 \prec \ldots \prec t_s} (g_1 g_2)^* \omega_1 \wedge \ldots \wedge (g_1 g_2)^* \omega_s.
\]
Let \( F \) be the set of all \( t \in I^n \) such that either all coordinates are \( \leq \frac{1}{2} \) or all coordinates are \( \geq \frac{1}{2} \). If any of the \( t^1, \ldots, t^m \) is outside \( F \), then locally the integral is zero as one of the forms \((g_1 g_2)^* \omega_\sigma\) is zero. Therefore, the integral can be restricted to \( F^s \) and the order conditions force in that if all coordinates of \( t_\sigma \) are \( \geq \frac{1}{2} \), then the same holds for \( t_{\sigma+1} \). This gives the claim. \( \square \)

Let \( P^n X \) denote the set of all membranes \( g : I^n \to X \) and write \( B^n_s(X) \) for the set of all maps \( P^n X \to \mathbb{R} \) which are linear combinations of iterated integrals of length \( \leq s \).

We write \( L^n_{x_0} X \) or just \( L^n X \) for the space of loops at \( x_0 \), i.e., the set of \( g \in P^n X \) with \( g(\partial(I^n)) = \{x_0\} \). The set of restrictions of elements of \( B^n_s(X) \) to \( L^n X \) is denoted by \( B^n_s(X)_{x_0} \).

We define the space \( L^n X_{\text{par}} \) to be the quotient of \( L^n X \) modulo the equivalence relation given by monotonic reparametrization. The composition law \((g_1, g_2) \mapsto g_1 g_2\) is easily seen to be associative on \( L^n X_{\text{par}} \). This being the case, the composition turns the free abelian group \( \mathbb{Z}[L^n X_{\text{par}}] \) into a ring.

Proposition 5.2 Every \( \omega \in B^n_s(X)_{x_0} \) factors over \( L^n_{x_0} X_{\text{par}} \). We extend it linearly to a map on the ring \( \mathbb{Z}[L^n X_{\text{par}}] \). For \( \alpha_1, \ldots, \alpha_r \in L^n X_{\text{par}} \) we set \( \eta = (\alpha_1 - 1) \cdots (\alpha_r - 1) \). For \( n \)-forms \( \omega_1, \ldots, \omega_s \) we then have
\[
\int_{\eta} \omega_1 \ldots \omega_s = \begin{cases} 
\prod_{j=1}^{r} \int_{\alpha_j} \omega_j & \text{if } s = r \\
0 & \text{if } s < r.
\end{cases}
\]

Proof: The proof is an iterated application of the composition formula
\[
\int_{g_1 g_2} \omega_1 \ldots \omega_s = \sum_{j=0}^{s} \left( \int_{g_1} \omega_1 \ldots \omega_j \right) \left( \int_{g_2} \omega_{j+1} \ldots \omega_s \right).
\]
Let \( a_k = (\alpha_1 - 1) \cdots (\alpha_k - 1)\alpha_{k+1} \cdots \alpha_r \). By induction one shows
\[
\int_{a_k} \omega_1 \cdots \omega_s = \sum_{0 < i_1 < \cdots < i_k \leq i_{k+1} \leq \cdots \leq i_s \leq s} \int_{\alpha_1} \omega_1 \cdots \omega_{i_1} \cdots \int_{\alpha_{i_s}} \omega_{i_s+1} \cdots \omega_s.
\]
If \( k = r \), then the sum becomes empty for \( s < r \) and has one term only for \( r = s \).

6 The conjectural de Rham Theorem

Let \( B^n_s(X)_{x_0} \) be the set of all elements of \( B^n_s(X)_{x_0} \) which are invariant under homotopies which leave the boundary of \( I^n \) fixed. Proposition 5.2 implies that each \( \omega \in B^n_s(X)_{x_0} \) induces a map \( \mathbb{Z}[\pi_n(X, x_0)] / J \rightarrow \mathbb{R} \), where \( J \) is the augmentation ideal in the group ring \( \mathbb{Z}[\pi_n(X, x_0)] \). So we get an injection
\[
B^n_s(X)_{x_0}^{\text{hom}} \hookrightarrow \text{Hom}_\mathbb{Z}(\mathbb{Z}[\pi_n(X, x_0)], \mathbb{R}).
\]
In the case \( n = 1 \) Chen’s de Rham Theorem says that this map is also onto. For higher \( n \), this can’t be the case, because if \( n \) is bigger than the dimension of \( X \), then the space \( B^n_s(X) \) is zero, as every \( n \)-form vanishes.

Question: What is the image of \( B^n_s(X)_{x_0}^{\text{hom}} \) inside the space \( \text{Hom}_\mathbb{Z}(\mathbb{Z}[\pi_n(X, x_0)], \mathbb{R}) \)?

We formulate a conjecture: Let \( \pi_n^{<n}(X, x_0) \) denote the subgroup of \( \pi_n(X, x_0) \) generated by the images of all maps
\[
\alpha_s : \pi_n(S^k) \rightarrow \pi_n(X),
\]
where \( \alpha \) ranges over \( \pi_k(X, x_0) \) and \( 1 \leq k < n \).

Lemma 6.1 For \( n = 1 \) one has \( \pi_n^{<n}(X, x_0) = \pi_n(X, x_0) \). For \( n \geq 2 \) the group \( \pi_n^{<n}(X, x_0) \) is the kernel of the Hurewicz map \( h_n : \pi_n(X) \rightarrow H_n(X) \). One has
\[
\pi_n^{<n}(X) \otimes \mathbb{Q} = \bigcup_{x_0 \in Y \subset X, \dim Y < n} \text{Im}(\pi_n(Y, x_0) \rightarrow \pi_n(X, x_0)) \otimes \mathbb{Q},
\]
where the union is extended over all connected subsets \( Y \), which are finite unions of submanifolds of dimension \( < n \).
HIGHER LOOP SPACES

Proof: For each non-trivial \( \alpha \in \pi_k(X, x_0) \) with \( k < n \) take \( \alpha \) to be the glueing prescription for glueing a new \((k+1)\)-cell onto \( X \) and let \( W \) denote the resulting CW-complex. Then \( W \) satisfies the conditions for Hurewicz’s theorem, so the map \( \pi_n(W) \to H_n(W) \) is an isomorphism. By the long exact homology sequence for the pair \((W, X)\) one gets that the natural map \( H_n(X) \to H_n(W) \) is an isomorphism. Let \( \phi : \pi_n(X) \to \pi_n(W) \) be the induced map. The commutative square

\[
\begin{array}{ccc}
\pi_n(X) & \xrightarrow{\phi} & \pi_n(W) \\
\downarrow h_n & & \downarrow \cong \\
H_n(X) & \xrightarrow{\cong} & H_n(W)
\end{array}
\]

shows that \( \ker(h_n) = \ker(\phi) \). Let \( H \) be the subgroup generated by all images of \( \alpha_s \) as in the lemma. Then \( H \subset \ker(\phi) \). For the converse, let \( \alpha \in \ker(\phi) \). Then there is a contracting homotopy \( h \) in \( W \). As the image of \( h \) is compact, it hits only finitely many new cells. By modifying \( h \), where it leaves \( X \), one gets a homotopy that stays in \( X \) and moves \( \alpha \) not to zero, but to an element of \( H \). This proves the first assertion.

For the second, note that

\[
\bigcup_{x_0 \in Y \subset X \atop \dim Y < n} \text{Im}(\pi_n(Y, x_0) \to \pi_n(X, x_0))
\]

is a subgroup of \( \pi_n \) which contains \( H = \pi_n^{< n} \) and which maps to zero under the Hurewicz map followed by the de Rham map

\[
\pi_n(X) \otimes \mathbb{Q} \to H_n(X) \otimes \mathbb{Q} \hookrightarrow H^n(X, \mathbb{R})^*.
\]

Conjecture 6.2 (Higher dimensional de Rham Theorem) We conjecture that iterated integration gives an isomorphism

\[
B^n_s(X)_{x_0}^{\text{hom}} \cong \hom_{\mathbb{Z}}(\mathbb{Z}[\pi_n/\pi_n^{< n}]/J^{s+1}, \mathbb{R}),
\]

where \( J \) is the augmentation ideal of the group ring.

For \( n = 1 \), this is Chen’s Theorem. For \( s = 1 \) we prove it below.

Proposition 6.3 Let \( n \geq 2 \) and let \( X \) be a smooth and compact manifold. Then the conjecture holds for \( s = 1 \). That is, we have

\[
B^n_1(X)_{x_0}^{\text{hom}} \cong \hom_{\mathbb{Z}}(\mathbb{Z}[\pi_n/\pi_n^{< n}]/J^2, \mathbb{R}).
\]
**Proof:** Observe first, that a single $n$-form $\omega$ is homotopy invariant (either as element of $B_1^n(X)$ or of $B_1^n(X)_{x_0}$), if and only if the form $\omega$ is closed. Since $\Gamma = \pi_n/\pi_n^{<n}$ is abelian, the map $\Gamma \to J/J^2; \gamma \to [\gamma - 1]$ is an isomorphism. Therefore,

$$\text{Hom}_\mathbb{Z}(\mathbb{Z}[\Gamma]/J^2, \mathbb{R}) = \text{Hom}_\mathbb{Z}(\mathbb{Z} \oplus J/J^2, \mathbb{R})$$

$$= \mathbb{R} \oplus \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{R})$$

$$= \mathbb{R} \oplus \text{Hom}_\mathbb{Z}(\text{Im}(h_n), \mathbb{R}).$$

The pairing between homology and de Rham cohomology is given by integration and it identifies $H_d^{dR}(\mathbb{X}, \mathbb{R})$ with $\text{Hom}_\mathbb{Z}(H_n(\mathbb{X}), \mathbb{R})$. Recall that the space $B_1^n(\mathbb{X})_{x_0}^{hom}$ is the direct sum of $\mathbb{R}$ and the set of restrictions of elements of $\text{Hom}_\mathbb{Z}(H_n(\mathbb{X}), \mathbb{R})$ to $\text{Im}(h_n)$. The proposition follows. $\square$

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