Existence of global solutions to the modified Camassa-Holm equation on the line

Yiling YANG\textsuperscript{1*}, Gaozhan LI\textsuperscript{1†} and Engui FAN\textsuperscript{1‡}

Abstract

We address the existence of global solutions to the Cauchy problem of the modified Camassa-Holm (mCH) equation on the line

\[
m_t + (m(u^2 - u_x^2))_x = 0, \quad m = u - u_{xx},
\]

\[
u(x, t = 0) = u_0(x),
\]

where initial data \((u_0(x), m_0(x) - 1) \in H^4(\mathbb{R}) \times H^2(\mathbb{R})\). Our main technical tool is the inverse scattering transform method based on the representation of a Riemann-Hilbert (RH) problem associated with the above Cauchy problem. The existence and uniqueness of the RH problem is shown via a generalization of Zhou Vanishing Lemma. Based on Cauchy integral projection of reflection coefficients and solutions of the time evolution RH problem, the reconstruction formula is directly used to obtain a unique global solution \((u(x, t), m(x, t) - 1) \in C([0, +\infty); W^{2,\infty}(\mathbb{R}) \times W^{2,\infty}(\mathbb{R}))\) to the Cauchy problem for the mCH equation. Further by establishing a new RH problem and its more careful estimates, we obtain regularity of the solution and a unique global solution \((u(x, t), m(x, t) - 1) \in C([0, +\infty); H^4(\mathbb{R}) \times H^{2,1}(\mathbb{R}))\) to the Cauchy problem for the mCH equation.

\textsuperscript{1} School of Mathematical Sciences and Key Laboratory for Nonlinear Science, Fudan University, Shanghai 200433, P.R. China.

\textsuperscript{*} Email address: 19110180006@fudan.edu.cn

\textsuperscript{†} Email address: 20110180004@fudan.edu.cn

\textsuperscript{‡} Corresponding author and email address: faneg@fudan.edu.cn
Keywords: Modified Camassa-Holm equation; Cauchy problem; Riemann-Hilbert problem; Weighted Sobolev space; Lipschitz continuous; Plemelj projection operator; global solution.

MSC 2020: 35P25; 37K15; 35A01; 35Q51; 35Q55; 35Q15; 35G25.

Contents

1 Introduction ................................................. 3

2 Direct scattering transform ................................. 6
   2.1 Spectral analysis on the Lax pair ....................... 7
   2.2 Reflection coefficient and its Lipschitz continuity .... 11

3 Inverse scattering transform .............................. 18
   3.1 A basic RH problem .................................. 18
   3.2 Solvability of the RH problem ......................... 20
   3.3 Estimates on solutions of the RH problem .......... 23

4 Global existence of solutions for mCH equation .......... 27
   4.1 Reconstruction of potential ......................... 27
   4.2 Time evolution and global solution .................. 28

5 Regularity of the solution for mCH equation .......... 35
   5.1 Transitions of spectral problem ..................... 35
   5.2 A new RH problem .................................. 37
   5.3 $L^2$-estimate on the solution ....................... 46
1 Introduction

In this paper, we study the existence of global solutions to the Cauchy problem of the modified Camassa-Holm (mCH) equation on the line

$$m_t + \left( m \left( u^2 - u_x^2 \right) \right)_x = 0, \quad m = u - u_{xx}, \quad (1.1)$$

$$u(x, t = 0) = u_0(x), \quad (1.2)$$

where initial data \((u_0(x), m_0(x) - 1) \in H^4(\mathbb{R}) \times H^2(\mathbb{R})\).

The mCH equation (1.1) is an integrable modification to the well-known Camassa-Holm (CH) equation

$$m_t + (um)_x + um = 0, \quad m = u - u_{xx},$$

which was first introduced by Camassa and Holm as a model for shallow water waves [1], but it already appeared earlier in a list by Fuchssteiner and Fokas [2]. The CH equation has attracted considerable interest and been studied extensively due to their rich mathematical structure and remarkable properties, such as peakon solutions, bi-Hamiltonian, algebro-geometric solutions, local and global well-posedness of its Cauchy problem [3–14].

It is observed that all the nonlinear terms in the CH equation are quadratic. Over the last few years, various modifications and generalizations of the CH equation have been introduced. Novikov applied the perturbation symmetry approach in order to classify integrable equations of the form

$$(1 - \partial_x^2)u_t = F(u, u_x, u_{xx}, \cdots),$$

among which two integrable CH-type equations with cubic nonlinearity which admit peakons have been discovered [15]. One is the mCH equation (1.1) and another one is the so-called Novikov equation

$$m_t + (m_x u + 3mu_x) u = 0, \quad m = u - u_{xx}. \quad (1.3)$$

In an equivalent form, the mCH equation was given by Fokas [16], Fuchssteiner [17], Olver and Rosenau [18] and Qiao [19], in which the equation was derived from
the two-dimensional Euler system and the M/W-shape solitons and peakon/cuspon solutions were presented. So the mCH equation (1.1) is also referred to as the Fokas-Olver-Rosenau-Qiao equation, but is mostly known as the mCH equation. The algebro-geometric quasiperiodic solutions were constructed by using algebro-geometric method [20]. The wave-breaking and peakons for the mCH equation were investigated by Gui, Liu, Olver and Qu [23]. With the aid of reciprocal transformation, the Bäcklund transformation and nonlinear superposition formula for the mCH equation were given [24]. Applying the scaling transformation and taking parameter limit, the mCH equation (1.1) can reduce a short pulse equation [25–27]

\[ u_{xt} = u + \frac{1}{6}(u^3)_{xx}. \]

Note that the soliton-type solutions of the mCH equation (1.1) vanishing at infinity are weak solutions in the form of peaked waves. On the other hand, adding a linear dispersion term \( \kappa u_x \) to the original mCH equation (1.1) will lead to a form of the mCH equation [28–30]

\[ m_t + (m(u^2 - u_x^2))_x + \kappa u_x = 0, \quad m = u - u_{xx}, \]

where \( \kappa > 0 \) characterizes the effect of the linear dispersion. It can be shown that the mCH equation (1.1) on a nonzero background or the mCH equation (1.4) with decaying initial data

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R} \]

may support smooth soliton solutions [30–32]. The mCH equation (1.4) admits a Lax pair and its smooth dark soliton solutions were obtained by the method of inverse scattering transform method [28, 31]. By using a reciprocal transformation and the Hirota bilinear method, Matsuno obtained the smooth bright multisoliton solutions for the mCH equation (1.4) [30, 32]. Boutet de Monvel, Karpenko and Shepelsky first developed a Riemann-Hilbert (RH) approach to deal with the mCH equation (1.4) with nonzero boundary conditions [33]. In recent years, long time asymptotic behavior of the mCH equation has been obtained by using the Deift-Zhou steepest decadent method and \( \bar{\partial} \)-steepest decadent analysis respectively [34–36].
Compared with the classical CH equation, there are only a few results about stability and well-posedness of mCH equation (1.1). Yin, Tian and Fan studied stability problem of negative solitary waves in [37]. In Besov space, local well-posedness for the equation (1.1) is researched in [38]. The stability and orbital stability of peakons for the mCH equation were further considered by Qu and Liu, they show that peakons and periodic peakons for the modified CH equation (1.1) are orbitally stable in the energy space $H^1(\mathbb{R})$ and $H^s(\mathbb{R})$ with $s > 5/2$, respectively [39, 40]. The local well-posedness for classical solutions and global weak solutions to the mCH equation (1.1) were considered in Lagrangian coordinates [41]. However, to the best of our knowledge, the global existence of the mCH equation (1.1) on the line seem still unknown.

In our paper, we try to establish the global existence of the mCH equation (1.1) on the line by using inverse scattering theory. We find that extension of this approach to the mCH equation will confront some substantial difficulties, which are much different from the NLS, AKNS, derivative NLS equation [42, 43, 45]. For example,

1) In the RH problem associated with the mCH equation, there are two singularities $z = \pm 1$, which give rise to difficulty in directly using Zhou vanishing lemma. However, to remove the singularity at $z = \pm 1$ by change of variable, the function $\eta = 1 - 1/m$ will appear in the condition of the RH problem. The boundness and space estimate of $\eta$ must be done to show that $m - 1 \in H^{2,1}(\mathbb{R})$.

2) In the estimate of the variable limit integral, there is an oscillating term $e^{i(z-1/z)x}$, which is difficult to directly apply the Fourier transform in real axis $z \in \mathbb{R}$. We overcome this technical difficulty by splitting the original spectral problem into two new spectral problems by two transformations.

3) In the jump matrix, the reflection coefficient $|r(z)| = 1$, $z = \pm 1$, which prevent to show the uniform boundness of the operator $(1-C_G)^{-1}$. So we cannot obtain Lipschitz continuous and the prior estimate on potentials. Fortunately we can take advantage of the nice properties of the reflection coefficient $r(z;0)$ and $r(z;t)$ to directly obtain global solution form evolution RH problem without
prior estimate on the potentials.

For convenience, we make some deformation to the original mCH equation (1.1). We introducing new functions by

\[ u(x, t) = 1 = v(x - t, t), \quad w = v^2 - v_x^2 + 2v, \quad (1.6) \]

then the initial value problem (1.1)-(1.2) is changed to

\[ m_t + (wm)_x = 0, \quad m = v - v_{xx} + 1, \quad (1.7) \]
\[ v_0(x) = v(x, 0) \to 0, \quad \text{as} \ x \to \pm \infty. \quad (1.8) \]

In what follows, we will study the existence of global solutions to the initial value problem (1.7)-(1.8) with initial data \((u_0(x), m_0(x) - 1) \in H^4(\mathbb{R}) \times H^{2,1}(\mathbb{R})\).

Our paper is arranged as follows. In Section 2, after quickly recall some basic result on the direct scattering transform [33], we then further establish the Lipschitz continuity from initial valve to reflection coefficient \(r(z)\). In Section 3, we construct a RH problem associated with the initial value problem which is further transformed into a regular RH problem. The solvability of the regular RH problem is shown in subsection 3.2. And in Subsection 3.3, we obtain estimates on the solution of the regular RH problem with reflection coefficients and initial data. In Section 4, the solution of the Cauchy problem (1.7)-(1.8) is recovered from the RH problem based on the reconstruction formula. We obtain a unique global solution \((u(x, t), m(x, t) - 1) \in C([0, +\infty); W^{2,\infty}(\mathbb{R}) \times W^{2,\infty}(\mathbb{R}))\) to the Cauchy problem for the mCH equation. In Section 5, to arrive at the \(L^2\)-integrability, we construct a new RH problem and take change of variable. After some careful estimates, we then obtain the existence of global solutions in the space \((u(x, t), m(x, t) - 1) \in C([0, +\infty); H^4(\mathbb{R}) \times H^{2,1}(\mathbb{R}))\) to the Cauchy problem of the mCH equation (1.7)-(1.8). The main result is summarized in the Theorem 1.

2 Direct scattering transform

We fix some notations used this paper. If \(I\) is an interval on the real line \(\mathbb{R}\) and \(X\) is a Banach space, then \(C(I, X)\) denotes the space of continuous functions on \(I\) taking
values in $X$. It is equipped with the norm

$$\|f\|_{C(I,X)} = \sup_{x \in I} \|f(x)\|_X.$$  

We introduce the normed spaces:

- A weighted $L^p(\mathbb{R})$ space is specified by
  $$L^{p,s}(\mathbb{R}) = \{ f(x) \in L^p(\mathbb{R}) : |x|^s f(x) \in L^p(\mathbb{R}) \};$$

- A Sobolev space is defined by
  $$W^{k,p}(\mathbb{R}) = \{ f(x) \in L^p(\mathbb{R}) : \partial^j_x f(x) \in L^p(\mathbb{R}) \text{ for } j = 1, 2, ..., k \};$$

- A weighted Sobolev space is defined by
  $$H^{k,s}(\mathbb{R}) = \{ f(x) \in L^p(\mathbb{R}) : x^s \partial^j_x f(x) \in L^p(\mathbb{R}) \text{ for } j = 1, 2, ..., k \}.$$  

For the simplicity, the norm of $f(x) \in L^p(\mathbb{R})$ and $g(x) \in L^{p,s}(\mathbb{R})$ are abbreviated to $\|f\|_p$, $\|g\|_{p,s}$ respectively. In our paper, we request the initial value $(u_0(x), m_0(x) - 1) \in H^4(\mathbb{R}) \times H^{2,1}(\mathbb{R})$.

### 2.1 Spectral analysis on the Lax pair

We quickly recall main results on the spectral analysis of the Lax pair which will be used to construct a RH problem associated with the Cauchy problem of the mCH equation (1.7). The detailed derivation may refer to [33].

The mCH equation (1.7) admits the Lax pair

$$\Phi_x = X \Phi, \quad \Phi_t = T \Phi, \quad \text{(2.1)}$$

where

$$X = \frac{1}{2} \begin{pmatrix} -1 & \lambda m \\ -\lambda m & 1 \end{pmatrix}, \quad \lambda = \frac{1}{2}(z + z^{-1}),$$

$$T = \begin{pmatrix} \lambda^{-2} + \frac{1}{2}w & -\lambda^{-1}(v - v_x + 1) - \frac{1}{2}\lambda mw \\ \lambda^{-1}(v + v_x + 1) + \frac{1}{2}\lambda mw & -\lambda^{-2} - \frac{1}{2}w \end{pmatrix}.$$
In direct scattering transform, we first consider the partial spectral problem in the Lax pair (2.1) with \( t \) being a parameter, so we omit the variable \( t \) as usual, for example \( \Phi(\lambda; x, t) \) is just written as \( \Phi(\lambda; x) \). We define a transformation

\[
\Psi(\lambda; x) = D(\lambda; x)\Phi(\lambda; x),
\]

(2.2)

where

\[
D(\lambda; x) = \left( \frac{1}{2} \frac{1}{1-\frac{2\lambda}{2+ik}} \right), \quad k = z - z^{-1},
\]

(2.3)

then \( \Psi(\lambda; x) \) satisfies the new spectral problems

\[
\Psi_x = -\frac{1}{4}ikm\sigma_3 \Psi + P\Psi,
\]

(2.4)

where

\[
P = \frac{m-1}{ik} \begin{pmatrix} 1 & \lambda \\ -\lambda & -1 \end{pmatrix},
\]

\[
L = -\frac{2}{ik} \left( v + \frac{m-1}{2} w \right) \sigma_3 + \frac{v_x}{\lambda} \sigma_1
\]

\[
+ \frac{1}{ik} \left( \lambda(m-1)w + \frac{2v}{\lambda} \right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

We further define

\[
\mu(z; x) = \Psi(z; x)e^{p(z; x)\sigma_3},
\]

(2.5)

where

\[
p(z; x) = \frac{1}{4}ik \left( x + \int_{+\infty}^{x} (m(s) - 1)ds \right).
\]

(2.6)

Then \( \mu(z; x) \) satisfies

\[
\mu_x = -\frac{1}{4}ikm[\sigma_3, \mu] + P\mu,
\]

(2.7)

which leads to two Volterra type integrals

\[
\mu^\pm(z; x) = I + \int_{\pm\infty}^{x} e^{(p(z; s) - p(z; x))\sigma_3} \left[ P(z; s) \mu^\pm(z; s) \right] ds.
\]

(2.8)
We denote the matrix in column
\[ \mu^\pm(z; x) = (\mu_1^\pm(z; x), \mu_2^\pm(z; x)) , \]
where the subscript 1 and 2 denote the first and second columns of \( \mu^\pm(z; x) \), respectively, then the following results can be shown

**Proposition 1.** Assume that initial value satisfies \( m_0(x) - 1 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \), then there exist a unique Jost function \( \mu^\pm(z; x) \), which admits the following properties

- **Analyticity:** \( \mu^-_1(z; x) \) and \( \mu^-_2(z; x) \) are analytic in \( \mathbb{C}^+ \) while \( \mu^+_1(z; x) \) and \( \mu^+_2(z; x) \) are analytic in \( \mathbb{C}^- \). And they are continuous up to real line except at \( z = \pm 1 \);

- **Symmetry:** \( \mu^\pm(z; x) \) admit three kinds of reduction conditions

\[
\sigma_1 \mu^\pm(z) \sigma_1 = \sigma_2 \mu^\pm(-z) \sigma_2 = \sigma_1 \mu^\pm(z^{-1}) \sigma_1; \quad (2.9)
\]

- **Asymptotics:** \( \det[\mu^\pm(z; x)] = 1 \) and \( \mu^\pm(z; x) \) admit asymptotics

\[
\mu^\pm(z; x) \sim I, \quad z \to 0, \\
\mu^\pm(z; x) = I + \mu^\pm(x)z^{-1} + \mathcal{O}(z^{-2}), \quad z \to \infty,
\]

with

\[
\mu^\pm(x) = \begin{pmatrix} -i \int_{x}^{\infty} (m - 1)ds & \frac{m-1}{m} \\ \frac{m-1}{m} & i \int_{x}^{\infty} (m - 1)ds \end{pmatrix};
\]

- **Singularity:**

As \( z \to 1, \mu^\pm(z; x) = \frac{i\alpha\pm(x)}{2(x-1)} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \mathcal{O}(1); \)

As \( z \to -1, \mu^\pm(z; x) = \frac{i\alpha\pm(x)}{2(x+1)} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + \mathcal{O}(1). \)

Two Jost function \( \mu^\pm(z; x) \) satisfy scattering relation on real axis \( \mathbb{R} \)

\[
\mu^+(z; x) = \mu^-(z; x)e^{p(z;x)\bar{\sigma}_3}S(z), \quad (2.10)
\]
where $S(z)$ is a scattering matrix

$$S(z) = \begin{pmatrix} a(z) & b(z) \\ b(z) & a(z) \end{pmatrix}, \quad \det[S(z)] = 1.$$ 

Here, $a(z)$ can be analytically continued into $\mathbb{C}^-$, being continuous up to the real line, except at $z = \pm 1$. Specially, at $z = \pm 1$, in generic case, $a(z)$ and $b(z)$ have same order singularity. While in non-generic case, $a(z)$ and $b(z)$ is continuous at $z = \pm 1$, which has better property.

Define a reflection coefficient by

$$r(z) = \frac{b(z)}{a(z)}, \quad z \in \mathbb{R}, \quad (2.11)$$

then we have the following proposition [33]

**Proposition 2.** The reflection coefficient $r(z)$ satisfies the following property

- $r(z)$ is continuous on $\mathbb{R}$. Specially, it is continuous at $z = \pm 1$. In generic case, $r(1) = -r(-1) = -1$;

- $r(z)$ admits symmetries

$$r(z) = r(z^{-1}) = -r(-z); \quad (2.12)$$

- As $z \to \infty$, $r(z) \to 0$, and $r(0) = 0$.

The $\det[S(z)] = 1$ leads to

$$|a(z)|^{-2} + |r(z)|^2 = 1, \quad z \in \mathbb{R}.$$ 

So $|a(z)|^{-1}$ is bounded on $\mathbb{R}$ since $a(z)$ is continuous on $\mathbb{R}$. Furthermore, $|a(z)|^{-2} = 1 - |r(z)|^2 \neq 0$ for $z \neq \pm 1$. And for any positive constant $\delta$, there exist a positive constant $C(r, \delta)$ such that

$$|a(z)|^{-2} = 1 - |r|^2 > C(r, \delta) > 0. \quad (2.13)$$
Lemma 1. If \( m(x) - 1 \in H^{2,1}(\mathbb{R}) \), then there exists \( v(x) \in H^4(\mathbb{R}) \), such that
\[
v - v_{xx} = m - 1.
\]

Proof. Define \( X = H^{1,1}(\mathbb{R}) \), \( T = 1 - \partial_x^2 \), \( D(T) = H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R}) \), and bilinear functional
\[
A(v, u) = (Tv, w) = \langle v - v_{xx}, w \rangle : D(T) \times X \to \mathbb{R},
\]
then for \( v(x) \in D(T) \),
\[
|A(v, v)| = \|v\|^2 + \|v_x\|^2 = \|v\|^2_{H^1},
\]
which that \( A(v, w) \) is coercive on \( X \). By using Lax-Milgram theorem, for \( m(x) - 1 \in X \), there exists \( v(x) \in D(T) \), such that
\[
A(v, w) = \langle m - 1, w \rangle, \quad \forall w \in X.
\]
Further by regularity, if \( m(x) - 1 \in H^{2,1}(\mathbb{R}) \subset X \), we then have \( v(x) \in H^4(\mathbb{R}) \). □

We denote \( \mathcal{G} \) as a set of initial value \( v_0(x) \) satisfying above Lemma 1
\[
\mathcal{G} = \left\{ (v_0(x), m_0(x) - 1) \in H^4(\mathbb{R}) \times H^{2,1}(\mathbb{R}), \quad a(z; v_0) \neq 0 \text{ in } \mathbb{C}^- \right\}.
\]
(2.14)
To avoid the difficulty that the spectral singularities give rise to, in general the initial data is chosen such that scattering data \( a(z) \) admits no eigenvalues or resonances in analysis of a RH problem by using inverse scattering transform and RH method [34].

2.2 Reflection coefficient and its Lipschitz continuity

In this section, we analyze the boundedness and integrability of the scattering coefficient \( r(z) \).

Proposition 3. If initial data \( v(x) \in \mathcal{G} \), then scattering coefficient \( r(z) \in H^{1,1}(\mathbb{R}_z) \), and the mapping
\[
m(x) - 1 \in H^{2,1}(\mathbb{R}_z) \to r(z) \in H^{1,2}(\mathbb{R}_z)
\]
is Lipschitz continuous.
Proposition 2 gives the continuity and symmetry of \( r(z) \), which means we only need to consider the property of \( r(z) \) out of any bounded interval. In fact,

\[
\int_{\mathbb{R}} |r(s)|^2 ds = \int_{\mathbb{R}^{-}} |r(s)|^2 ds + \int_{\mathbb{R}^{+}} |r(s)|^2 ds = 2 \int_{\mathbb{R}^{+}} |r(s)|^2 ds = 2 \int_{0}^{1} |r(s)|^2 ds + 2 \int_{1}^{+\infty} |r(s)|^2 ds = 2 \int_{1}^{+\infty} |r(s)|^2 \left(1 + \frac{1}{s^2}\right) ds
\]

Specially, either in generic case (\( a(z) \) and \( b(z) \) have no singularity at \( z = \pm 1 \)) or in non-generic case (\( a(z) \), \( b(z) \) is continuous at \( z = \pm 1 \)), \( r(z) \) is both continuous at \( z = \pm 1 \) and has well property. So we just analyze the property of \( r(z) \) near \( z = \infty \). Namely, in this section, we only prove \( r(z) \in H_{1,2}^{1,2}(2, \infty) \). Here, we take 2 for convenience. It can be replaced by any constant greater than 1.

To prove this conclusion, we first analyze the property of Jost function. In the following demonstration, we focus on two function spaces \( C(\mathbb{R}^{\pm}, L^2(2, \infty)_{z}) \) and \( L^\infty((2, \infty)_{z} \times \mathbb{R}^{\pm}) \). For briefly, we denote the norm on this two spaces as \( \| \cdot \|_C \) and \( \| \cdot \|_{L^\infty_{xz}} \) respectively.

**Lemma 2.** If initial data \( v(x) \in G \) with \( |m(x)| - 1| < 1 \), the Jost function \( \mu^\pm(z; x) \) exist unique and \( \mu^\pm(z; x) - I \in C(\mathbb{R}^{\pm}, L^2(2, \infty)_{z}) \cap L^\infty((2, \infty)_{z} \times \mathbb{R}^{\pm}) \) with

\[
\| \mu^\pm(z; x) - I \|_{L^\infty_{xz} \cap C} \leq c e^\|m(x)-1\|_{L^1} \| m(x) - 1 \|_{L^1(\mathbb{R})},
\]

where \( c \) is a positive constant independent of initial value \( v(x) \).

**Proof.** The proof is similar as [36]. Recall that Jost functions \( \mu^\pm(z; x) \) can be given by the Volterra integral equation

\[
\mu^\pm(z; x) = I + \int_{\pm\infty}^{x} e^{i \frac{1}{z} (s-1/z)} \int_{\mathbb{R}} m(l) dl \mathcal{P}(s; z) \mu^\pm(s; z) ds.
\]

We prove this lemma by taking the second column \( \mu^\pm_2(z; x) \) as an example.

If we denote vector function

\[
\mu^\pm_2(z; x) = e_2 \triangleq h(z; x) = (h^{(1)}(z; x), h^{(2)}(z; x))^T,
\]

12
then the second column of (2.16) can be written as an operator equation

\[ h(z; x) = h_0(z; x) + Th(z; x), \]

(2.17)

where \( T \) is a linear integral operator defined by

\[ Tf(z; x) = \int_{+\infty}^{x} K(x, s; z) f(s; z) ds, \]

(2.18)

with \( K(x, s; z) = \text{diag}(e^{\frac{1}{2}((z-1)/z)} f_s^{m(l) dl}, 0)P(s, z) \), while

\[ h_0(z; x) = Te_2 = \int_{+\infty}^{x} \frac{m-1}{z^2-1} \left( \frac{z^2+1}{2} e^{\frac{1}{2}((z-1)/z)} f_s^{m(l) dl} \right) ds. \]

(2.19)

Considering the first component of \( n_0(z; x) \) and integrating by part yields

\[ h^{(1)}_0(z; x) = \left( \frac{z^2+1}{z^2-1} \right) \int_{+\infty}^{x} \frac{m-1}{z^2-1} \left( \frac{z^2+1}{2} e^{\frac{1}{2}((z-1)/z)} f_s^{m(l) dl} \right) ds, \]

(2.20)

which implies that

\[ h^{(1)}_0(z; x) = O(z^{-1}), \quad z \to \infty, \]

and so that \( h^{(1)}_0(z; x) \in L^2(2, \infty)_z \) for all \( x \in \mathbb{R}^+ \). Obviously, \( h^{(2)}_0(x, z) \in L^\infty((2, \infty)_z \times \mathbb{R}_{x}^\pm) \). Thus,

\[ h_0(z; x) \in C(\mathbb{R}_{x}^+, L^2(2, \infty)) \cap L^\infty((2, \infty)_z \times \mathbb{R}_{x}^\pm). \]

Next we prove that \( T \) is a bounded linear operator on \( C(\mathbb{R}_{x}^+, L^2(2, \infty)) \cap L^\infty((2, \infty)_z \times \mathbb{R}_{x}^\pm) \) to itself. In fact, for any \( f = (f_1, f_2)^T \in L^\infty((2, \infty)_z \times \mathbb{R}_{x}^\pm) \), we have

\[ |Tf(z; x)| = \left| \int_{+\infty}^{x} \frac{m-1}{z^2-1} \left( e^{\frac{1}{2}((z-1)/z)} f_s^{m(l) dl} \left(-zf_1(s, z) + \frac{z^2+1}{2} f_2(s, z) \right) \right) ds \right| \]

\[ \lesssim \int_{x}^{+\infty} |m-1| ds \| f \|_{L^\infty} \leq \| m-1 \|_{L^1} \| f \|_{L^\infty}. \]

(2.21)
For $f$ in $C \left( \mathbb{R}^+, L^2(2, \infty)_z \right)$,

$$\left[ \int_{\mathbb{R}} |Tf(z; x)|^2 dz \right]^{1/2} \leq \int_{\mathbb{R}} \left| \int_{+\infty}^{x} \frac{m-1}{z^2-1} \left( e^{\frac{z}{z-1}} f_2(z; s, z) \right) ds \right|^2 dz \right]^{1/2}
$$

Finally the operator equation (2.18) implies that

$$h(z; x) \in C \left( \mathbb{R}^+, L^2(2, \infty)_z \right) \cap L^\infty \left( (2, \infty)_z \times \mathbb{R}^+_z \right),$$

and moreover,

$$\| h(z; x) \|_{L^\infty} \lesssim e^{\| m-1 \|_{L^1}} \| m - 1 \|_{L^1}. \quad (2.24)$$
This lemma also gives the boundedness of \( a(z) \) from (2.10). To arrive at the Lipschitz continuity from initial value \( v(x) \) to Jost function \( \mu^\pm(x, z) \), we have the following corollary.

**Corollary 1.** Let two real functions \( v(x), \tilde{v}(x) \in \mathcal{G} \) satisfying \( m - 1 = v - v_{xx}, \tilde{m} - 1 = \tilde{v} - \tilde{v}_{xx} \) respectively; and corresponding Jost functions are \( \mu^\pm(z; x) \) and \( \tilde{\mu}_\pm(z; x) \). If \( |m - 1| < 1 \) and \( |\tilde{m} - 1| < 1 \), then we have

\[
\| \mu^\pm(z; x) - \tilde{\mu}^\pm(z; x) \|_{L^\infty \cap C} \leq c(u, \tilde{u}) \| m - \tilde{m} \|_{H^{1,2}}.
\]  

(2.25)

**Proof.** Similarly as in above Lemma, we give the proof of the second column of \( \mu^+ (z; x) \), we denote \( h(z; x) = \mu_2^+(z; x) - e_2 \) respectively with

\[
h(z; x) = h_0(z; x) + Th(z; x),
\]

where the definitions of \( T \) and \( \tilde{T} \) are same with (2.18) corresponding to \( m \) and \( \tilde{m} \) respectively. Then we have

\[
h(z; x) - \tilde{h}(z; x) = (I - T)^{-1} \left[ h_0(z; x) - \tilde{h}_0(z; x) \right] + (I - T)^{-1}(T - \tilde{T})(T - \tilde{T})^{-1}\tilde{h}_0(z; x),
\]

(2.27)

from which and following the estimation of Lemma 2 as well as \( K \) is a Lipschitz continuous operator, we obtain (2.25).

Next, we consider the estimate on Jost function \( \mu^\pm(z; x) \) with initial data \( m(x) - 1 \).

**Lemma 3.** Let two real functions \( v(x) \in \mathcal{G} \) with \( |m(x) - 1| < 1 \), then

\[
\mu^\pm(z; x) \in C \left( \mathbb{R}_z^+, L^2(2, \infty)_z \right) \cap L^\infty \left( (2, \infty)_z \times \mathbb{R}_x^+ \right),
\]

which admits the estimate

\[
\| \partial_z \mu^\pm(z; x) \|_{L^\infty \cap C} \leq c \| m(x) - 1 \|_{L^1},
\]

(2.28)

where \( c \) is a positive constant dependent of initial value \( v(x) \). Furthermore, Let two real functions \( v(x), \tilde{v}(x) \in \mathcal{G} \) satisfying \( m - 1 = v - v_{xx}, \tilde{m} - 1 = \tilde{v} - \tilde{v}_{xx} \) respectively; and corresponding Jost functions are \( \mu^\pm \) and \( \tilde{\mu}_\pm \), then we have

\[
\| \partial_z \mu^\pm(z; x) - \partial_z \tilde{\mu}^\pm(z; x) \|_{L^\infty \cap C} \leq c(v, \tilde{v}) \| m - \tilde{m} \|_{H^{2,1}}.
\]

(2.29)
Proof. By taking derivative of (2.17) with respect to \(z\), we obtain

\[
\partial_z h(z; x) = h_1(z; x) + T \partial_z h(z; x),
\]

where

\[
h_1(z; x) = \partial_z h_0(z; x) + (\partial_z T) h(z; x).
\]

Similarly as the proof of Lemma 2, we arrive at the consequence.

From (2.8) and (2.10), scattering data \(a(z)\) can be expressed by

\[
a(z) = 1 + \int_{\mathbb{R}} e^{-\frac{i}{2}(z-1/z)[\int_{\mathbb{R}}^{+\infty} (m(t)-1)dt - s]} i(m - 1) \left[ z\mu_{11}^+ + \frac{z^2 + 1}{2} \mu_{21}^+ \right] ds.
\]

Combining with the property in above subsection, we arrive at

**Lemma 4.** Let real function \(v(x) \in \mathcal{G}\) with \(|m(x) - 1| < 1\), then \(a(z)\) is analytically continued into \(\mathbb{C}^-\), and is continuous up to the real line, except at \(z = \pm 1\). And \(z = \pm 1\) are the only poles of \(a(\pm 1)\) on \(\mathbb{R}\). Moreover, \(a \in W^{1,\infty}(2, +\infty)_z\), and the mapping

\[
m - 1 \in H^{2,1}(\mathbb{R}_x) \rightarrow a(z) \in W^{1,\infty}(2, +\infty)_z,
\]

is Lipschitz continuous.

Similarly, from (2.8) and (2.10), scattering data \(b(z)\) is expressed with

\[
b(z) = \int_{\mathbb{R}} e^{-\frac{i}{2}(z-1/z)[\int_{\mathbb{R}}^{+\infty} (m(t)-1)dt - s]} i(m - 1) \left[ z\mu_{12}^+ + \frac{z^2 + 1}{2} \mu_{22}^+ \right] ds. \quad (2.31)
\]

We has the following Lemma:

**Lemma 5.** Let real function \(v(x) \in \mathcal{G}\) with \(|m(x) - 1| < 1\), then \(b(z)\) is \(H^{1,2}(2, \infty)_z\) and

\[
\|b(z)\|_{H^{1,2}(2, \infty)_z} \leq c \|m - 1\|_{H^{2,1}(\mathbb{R}_x)}.
\]

Moreover, the mapping

\[
m(x) - 1 \in H^{2,1}(\mathbb{R}_x) \rightarrow b(z) \in H^{1,2}(2, \infty)_z,
\]

is Lipschitz continuous.
Proof. We write (2.31) in the form
\[
b(z) = \int_{\mathbb{R}} e^{-\frac{i}{2}(z-1/z)[f_s^{+\infty}(m(l)-1)dl-s]} \frac{i(m-1)}{z^2-1} \left[ z\mu_{12} - \frac{z^2+1}{2}(\mu_{22} - 1) \right] ds
\]
\[- \int_{\mathbb{R}} e^{-\frac{i}{2}(z-1/z)[f_s^{+\infty}(m(l)-1)dl-s]} \frac{z^2+1}{2} i(m-1) ds. \quad (2.32)
\]
Lemma 2 gives the \(L^2\) property of the first integral. As for the second integral, we just give estimation of \(\int_{\mathbb{R}} e^{-\frac{i}{2}(z-1/z)[f_s^{+\infty}(m(l)-1)dl-s]} (m - 1) ds\). Recall the change of variable \(k = \frac{1}{4}(z - \frac{1}{z})\). When \(z \in (2, +\infty)\), \(k \in (3/8, +\infty)\). Note that \(1/4|dz| \leq |dk| \leq 5/16|dz|\), so for any function \(f(z)\) well defined in
\[
f(z(k)), f(z) \in L^2(2, +\infty)_z \iff f(z(k)) \in L^2(3/8, +\infty)_k.
\]
Thus we only prove
\[
\int_{\mathbb{R}} e^{-2ik[f_s^{+\infty}(m(l)-1)dl-s]} (m - 1) ds \in L^2(3/8, +\infty)_k.
\]
For any \(f(k) \in L^2(3/8, +\infty)_k\),
\[
\left| \int_{3/8}^{+\infty} f(k) \int_{\mathbb{R}} e^{-2ik[f_s^{+\infty}(m(l)-1)dl-s]} (m - 1) ds dk \right|
\]
\[
= \left| \int_{\mathbb{R}} (m - 1) \hat{f} \left( \int_{3/8}^{+\infty} (m(l) - 1) dl - s \right) ds \right|
\]
\[
\leq \| m - 1 \|_{L^2} \cdot \| \hat{f} \left( \int_{3/8}^{+\infty} (m(l) - 1) dl - s \right) \|_{L^2}.
\]
Here, \(\hat{f}\) denotes the Fourier transform of \(f\). And
\[
\left\| \hat{f} \left( \int_{3/8}^{+\infty} (m(l) - 1) dl - s \right) \right\|^2_{L^2} = \int_{\mathbb{R}} \left| \hat{f} \left( \int_{3/8}^{+\infty} (m(l) - 1) dl - s \right) \right|^2 ds
\]
\[
\lesssim \int_{\mathbb{R}} |\hat{f}(\tau)|^2 d\tau = \| \hat{f} \|^2_{L^2}.
\]
So \(b(z) \in L^2(2, +\infty)\). And by integration by parts,
\[
zb(z) = -\frac{z^3}{(z^2-1)^2} \int_{\mathbb{R}} e^{-\frac{i}{2}(z-1/z)[f_s^{+\infty}(m(l)-1)dl-s]} \frac{d}{ds} \left[ \frac{2(m-1)}{m} \mu_{12}^+ \right] ds
\]
\[
+ \frac{(z^2+1)z^2}{(z^2-1)^2} \int_{\mathbb{R}} e^{-\frac{i}{2}(z-1/z)[f_s^{+\infty}(m(l)-1)dl-s]} \frac{d}{ds} \left[ \frac{m-1}{m} \mu_{22}^+ - \frac{1}{2} \right] ds
\]
\[
+ \frac{(z^2+1)z^2}{(z^2-1)^2} \int_{\mathbb{R}} e^{-\frac{i}{2}(z-1/z)[f_s^{+\infty}(m(l)-1)dl-s]} \frac{d}{ds} \left[ \frac{m-1}{m} \right] ds.
\]
Substitute (2.7) into it, we consider that

\[
\frac{d}{ds} \left[ \frac{2(m-1)}{m} [\mu^+]_{12} \right] = \frac{d}{ds} \left[ \frac{2(m-1)}{m} \right] [\mu^+]_{12} + \frac{i(m-1)}{2} (z-\frac{1}{z}) [\mu^+]_{12} \\
+ \frac{2i(m-1)^2}{m(z^2-1)} \left[ z[\mu^+]_{12} - \frac{z^2+1}{2} [\mu^+]_{22} \right],
\]

\[
\frac{d}{ds} \left[ \frac{m-1}{m} ([\mu^+]_{22}) \right] = \frac{d}{ds} \left[ \frac{m-1}{m} \right] ([\mu^+]_{22}) + \frac{m-1}{m} [P\mu^+]_{22},
\]

(2.33)

which are both in \( L^2 \). Specially, although the Lax pair bring a \( z \) in \( \frac{d}{ds} [\mu^+]_{12} \), above equations give that the \( \frac{m-1}{m} [\mu^+]_{12} \) item in the integral representation (2.33) of \( b(z) \) has

\[
-2 \frac{(z^2-1)^2}{z^2} \int_{\mathbb{R}} e^{-\frac{z}{2}(z-1/z)} [\int_{s}^{+\infty} (m(l)-1)dl] [\mu^+]_{12} (z-\frac{1}{z}) \frac{i(m-1)}{2} \frac{z[\mu^+]_{12}}{z^2-1} dz
\]

\[
= \int_{\mathbb{R}} e^{-\frac{z}{2}(z-1/z)} [\int_{s}^{+\infty} (m(l)-1)dl] \frac{d}{ds} \left[ \frac{2(m-1)}{m} \right] [\mu^+]_{12} ds \\
= \int_{\mathbb{R}} e^{-\frac{z}{2}(z-1/z)} [\int_{s}^{+\infty} (m(l)-1)dl] [\mu^+]_{12} ds \\
+ \frac{2i(m-1)^2}{m(z^2-1)} \left[ z[\mu^+]_{12} - \frac{z^2+1}{2} [\mu^+]_{22} \right] ds.
\]

(2.34)

So although in (2.31), it can only obtain \( b(z) \in L^2(\mathbb{R}) \), through integration by parts, in (2.33), we can obtain \( zb(z) \in L^2(\mathbb{R}) \). Similarly, continue use integration by parts, the weight transform to \( x \)-derivative of \( m \) again, which lead to \( z^2b(z) \in L^2(\mathbb{R}) \). For the \( z \)-derivative of \( b(z) \), we have the same estimation. Therefore we obtain the result.

Combing above lemmas, we finally arrive at Proposition 2.2.

3 Inverse scattering transform

3.1 A basic RH problem

In order to have the data for the RH problem to depend explicitly on the parameters, we use the space variable

\[
y(x) = -x + \int_{+\infty}^{x} (m(s)-1) ds.
\]

(3.1)
Under this new scale, we define a sectionally analytical matrix

\[ M(z) \triangleq M(z; y) = \begin{cases} \frac{\mu_1(z; x)}{a(z)}, \mu_2^+(z; x), & \text{as } z \in \mathbb{C}^+, \\ \mu_1^+(z; x), \frac{\mu_2^-(z; x)}{a(z)}, & \text{as } z \in \mathbb{C}^-, \end{cases} \]  

(3.2)

then \( M(z) \) solves the following RH problem.

**RHP 1.** Find a matrix-valued function \( M(z) = M(z; y) \) which satisfies

\[ 
\begin{align*}
\triangleright & \text{ Analyticity: } M(z) \text{ is analytical in } \mathbb{C} \setminus \mathbb{R}; \\
\triangleright & \text{ Symmetry: } M(z) = \sigma_1 M(\bar{z}) \sigma_1 = \sigma_2 M(-z) \sigma_2 = \sigma_1 M(z^{-1}) \sigma_1; \\
\triangleright & \text{ Jump condition: } M(z) \text{ has continuous boundary values } M_{\pm}(z) \text{ on } \mathbb{R} \text{ and}
\end{align*} \]

(3.3)

where

\[ V(z) = \begin{pmatrix} 1 - |r|^2 & -r e^{-2\theta(z)} \\ r e^{2\theta(z)} & 1 \end{pmatrix}, \]

(3.4)

with

\[ \theta(z) = \vartheta(z; x(y)) = -\frac{i}{4} \left( z - \frac{1}{z} \right) y \]

(3.5)

\[ \triangleright & \text{ Asymptotic behaviors: } \\
\triangleright & \text{ Singularity: } M(z) \text{ has singularity at } z = \pm 1 \text{ with}
\]

(3.6)

(3.7)

(3.8)

Here \( c = 0 \) in generic case, while \( c \neq 0 \) in non-generic case.

To move out the singularity conditions, we reduce the original RH problem \( M(z; y) \) to a regular one \( M^{(1)}(z; y) \) following the idea [34]

\[ \left( I - \frac{\sigma_1}{z} \right) M(z; y) = \left( I - \frac{\sigma_1 M^{(1)}(0; y)}{z} \right) M^{(1)}(z; y), \]

(3.9)

then \( M^{(1)}(z; y) \) satisfies the following RH problem.
RHP 2. Find a matrix-valued function $M^{(1)}(z) = M^{(1)}(z; y)$ which satisfies

- **Analyticity:** $M^{(1)}(z)$ is analytical in $C \setminus R$;
- **Symmetry:** $M^{(1)}(z) = \sigma_1 \overline{M^{(1)}(\bar{z})} \sigma_1 = \sigma_2 M^{(1)}(-z) \sigma_2 = \sigma_1 M^{(1)}(0)^{-1} M^{(1)}(z^{-1}) \sigma_1$;
- **Jump condition:** $M^{(1)}$ has continuous boundary values $M^{(1)}_{\pm}(z)$ on $R$ and
  \[
  M^{(1)}_{+}(z) = M^{(1)}_{-}(z)V(z), \quad z \in R,
  \]
  where
  \[
  V(z) = \begin{pmatrix} 1 - |r|^2 & -re^{-2\theta} \\ \bar{r}e^{2\theta} & 1 \end{pmatrix},
  \]
- **Asymptotic behaviors:**
  \[
  M^{(1)}(z) = I + O(z^{-1}), \quad z \to \infty. \tag{3.11}
  \]

Moreover, as $z \to \infty$,

\[
M^{(1)}(z) = I + \frac{1}{z} \left( \lim_{z \to \infty} [zM] - \sigma_1 M^{(1)}(0) \right) + O(z^{-2}), \quad z \to \infty. \tag{3.12}
\]

3.2 Solvability of the RH problem

In this subsection, we prove the solvability of the RHP 2 for the given scattering data $r(z) \in H^{1,2}(R_z)$. The symmetry require that not all function $r(z) \in H^{1,2}(R_z)$ can be the scattering coefficient of RHP 2. Denote

\[
\mathcal{R} = \left\{ r(z) \in H^{1,2}(R_z); \ r(z) = \overline{r(z^{-1})} = -r(-z), \ r(1) = -r(-1) = -1 \right\}. \tag{3.13}
\]

For a given function $f(z) \in L^p(R_z)$, $1 \leq p < \infty$, the Cauchy operator $\mathcal{C}$ is defined by

\[
\mathcal{C}[f](z) = \frac{1}{2\pi i} \int_{R_z} \frac{f(s)}{s - z} ds, \quad z \in C \setminus R. \tag{3.14}
\]

The function $\mathcal{C}[f]$ is analytic off the real line such that $\mathcal{C}[f](\cdot + i\gamma)$ is in $L^p(R_z)$ for each $\gamma \neq 0 \in R$. When $z$ approaches to a point on the real line transversely from the upper and lower half planes, that is, if $\gamma \to \pm 0$, the Cauchy operator $\mathcal{C}$ becomes the Plemelj projection $\mathcal{C}_\pm$ with

\[
\mathcal{C}_\pm[f](z) = \lim_{\gamma \to 0} \frac{1}{2\pi i} \int_{R} \frac{f(s)}{s - (z \pm \gamma i)} ds, \quad z \in R. \tag{3.15}
\]

We give a proposition summarizes the basic properties of the Cauchy and projection operators [42].
**Proposition 4.** For any \( h \in L^p(\mathbb{R}_z) \), \( 1 \leq p < \infty \), the Cauchy integral \( C[h] \) is analytical off the real line, decays to zero as \( |z| \to \infty \), and approaches to \( C_\pm[h] \) almost everywhere, when a point \( z \in \mathbb{C}_\pm \) approaches to a point on the real axis by any non-tangential contour from \( \mathbb{C}_\pm \). If \( 1 < p < \infty \), then there exists a positive constant \( c_p \) such that

\[
\| C_\pm[h] \|_{L^p} \leq c_p \| h \|_{L^p}.
\]  

(3.16)

When \( h \in L^1(\mathbb{R}_z) \), as \( z \to \infty \), \( C[h] = \mathcal{O}(z^{-1}) \).

Denote \( G(z) = V(z) - I \). Then we rewrite the jump condition of \( M_\pm^{(1)}(z) \) as

\[
M_\pm^{(1)}(z) - M_\mp^{(1)}(z) = M_\pm^{(1)}(z)G(z).
\]  

(3.17)

If \( M_\pm^{(1)}(z) \) is a solution of the Fredholm integral equation

\[
M_\pm^{(1)}(z) = I + C_-[M_\pm^{(1)}G](z), \quad z \in \mathbb{R},
\]  

(3.18)

then RHP 2 has a unique solution given by

\[
M^{(1)}(z) = I + C[M_\pm^{(1)}G](z), \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]  

(3.19)

We denote a new matrix function

\[
N(z) = M^{(1)}(z) - I,
\]  

(3.20)

then it admits

\[
N_-(z) = C_-[G] + C_-[N_-G],
\]  

(3.21)

and \( z \to \infty \), \( N(z) \to 0 \).

**Lemma 6.** For any \( r(z) \in \mathcal{R} \) with \( \mathcal{R} \) defined in (3.13), \( I - C_-^G \) is a bounded linear operator on: \( L^2(\mathbb{R}_z) \to L^2(\mathbb{R}_z) \) with \( C_-^G[h] = C_-[Gh] \). Moreover, \( (I - C_-^G)^{-1} \) exists and is also a bounded linear operator on: \( L^2(\mathbb{R}_z) \to L^2(\mathbb{R}_z) \).
Proof. From the references [49–51], the operator $I - C^G_-$ is known to be a Fredholm operator of the index zero on $L^2(\mathbb{R}_+ \to L^2(\mathbb{R}_+))$. So there only need to prove that the operator $I - C^G_- : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is injective. Consider the homogeneous equation: $(I - C^G_-) f = 0$ with $f$ is on $L^2(\mathbb{R}_+)$. Denote

$$f_1 = C[fG](z), \quad f_2 = \overline{C[fG](\bar{z})}^T.$$ 

Obviously, this two is analytic on $\mathbb{C}^+$. Along the semi-circle $C_R$ of radius $R$ centered at zero in $\mathbb{C}^+$, we multiply the two functions and integral. Then Cauchy-Goursat theorem gives that

$$\int_{-R}^{R} f_1(s)f_2(s)ds = \int_{C_R} f_1(s)f_2(s)ds. \quad (3.22)$$

Taking $R \to +\infty$, we obtain that

$$\int_{\mathbb{R}} f_1(s)f_2(s)ds = 0. \quad (3.23)$$

On the other hand, through $(I - C^G_-) f = 0$, we have

$$0 = \int_{\mathbb{R}} C_+[fg](s)\overline{C_-[fG](s)}^T ds 
= \int_{\mathbb{R}} (C_-[fG] + fG)(s)\overline{C_-[fG](s)}^T ds 
= \int_{\mathbb{R}} f(s)(I + G)\overline{f(s)}^T ds. \quad (3.24)$$

Then we only prove for any $f \in L^2(\mathbb{R}_+)$, $\int_{\mathbb{R}} f(s)(I + G)\overline{f(s)}^T ds = 0$ if and only if $\|f\|_{L^2} = 0$. For convenience we consider the row vector $f = (f^{(1)}, f^{(2)})$ case, then

$$0 = \text{Re} \left[ f(s)(I + G)\overline{f(s)}^T \right] 
= \text{Re}^2(f^{(1)})(1 - |r|^2) + \text{Re}^2(f^{(2)}) + \text{Im}^2(f^{(1)})(1 - |r|^2) + \text{Im}^2(f^{(2)}).$$

However, $1 - |r(s)|^2 \geq 0$, and $1 - |r(s)|^2 = 0$ if and only if $s = \pm 1$. So there have $f^{(2)} \equiv 0$ and $f^{(1)}(s) = 0$ for $s \neq \pm 1$. Thus we obtain $\|f\|_{L^2} = 0$. \hfill \Box

As a consequence of this Lemma, we obtain the solvability of RHP 2.
Corollary 2. For any $r(z) \in \mathcal{R}$ with $\mathcal{R}$ defined in (3.13), RHP 2 exists unique solution with

$$M^{(1)}(z; r) = I + C[M_-^{(1)}G](z), \ z \in \mathbb{C}. \quad (3.25)$$

where

$$M_-^{(1)}(z; r) = (I - C G)^{-1} [C - G], \ z \in \mathbb{R}. \quad (3.26)$$

Moreover, (3.21) also gives that

$$\| M_-^{(1)}(z; r) - I \|_{L^2} \leq \| (I - C G)^{-1} G \|_{L^2}. \quad (3.27)$$

Meanwhile, there exist a positive constant $c(r)$ relies on $r$ with

$$\| M_-^{(1)}(z; r) - I \|_{L^2} \leq c(r) \| r \|_{H^{1,2}}. \quad (3.28)$$

Remark 1. For other equations, like NLS equation, mKdV equation, the jump matrix in their corresponding RHP admit that $V - I$ or $\frac{1}{2}(V + V^H)$ is positive definite matrix. So the norm of their corresponding $(I - C G)^{-1}$ can be controlled by $\| r \|_{H^{1,2}}$. However, in the mCH equation, its corresponding scattering coefficient $r(z)$ must satisfy $|r(\pm 1)| = 1$. Thus, although $(I - C G)^{-1}$ is bound and its norm must rely on $r(z)$, it is hard to prove that its norm can be controlled by $\| r(z) \|_{H^{1,2}}$. So there will lose the Lipschitz continuous of the solution $M^{(1)}$ with respect to scattering coefficient $r(z)$.

3.3 Estimates on solutions of the RH problem

In this subsection, we give a estimation of $N(z)$. For any scattering date $r(z)$ in $\mathcal{R}$ with $\mathcal{R}$ defined in (3.13), (3.21) implies that

$$N(z) = C[N_- G] + C[G], \ z \in \mathbb{C} \setminus \mathbb{R}. \quad (3.29)$$

As $z \to i$, $N(z)$ has expansion:

$$N(z) = N(i) + N^{(1)}(z - i) + O(z - i)^2, \quad (3.30)$$
with
\[
N(i) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{N(s)G(s)}{s-i} ds + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{G(s)}{s-i} ds, \tag{3.31}
\]
\[
N^{(1)} = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{N(s)G(s)}{(s-i)^2} ds + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{G(s)}{(s-i)^2} ds. \tag{3.32}
\]

Lemma 7. When \( r(z) \in \mathcal{R} \) with \( \mathcal{R} \) defined in (3.13), we denote \( N(i), N^{(1)}, N(0) \) and \( N(i)_n, N^{(1)}_n, N_n(0) \) as above respectively. There exists a positive constant \( c(r) \) rely on \( r \) such that \( N(i), N^{(1)} \) and \( N(0) \) have estimation:
\[
|N(i)|, |N^{(1)}|, |N(0)| \leq c(r) \| r \|_{H^{1,2}}. \tag{3.33}
\]

Proof. We only give the proof of estimation under scattering coefficient \( r \). For \( z = 0 \),
\[
|N(0)| = \left| C_- [G] + C_- [N_- G](0) \right|
\leq \frac{1}{2\pi} \lim_{\gamma \to 0} \left| \int_{\mathbb{R}} \frac{N_- (s)G(s)}{s+\gamma i} ds + \int_{\mathbb{R}} \frac{G(s)}{s+\gamma i} ds \right|. \tag{3.34}
\]
Note that
\[
\int_{\mathbb{R}} \frac{N_- (s)G(s)}{s} ds \leq \| N_- \|_{L^2(\mathbb{R}_z)} \| G(s) \|_{L^2(\mathbb{R}_z)},
\]
So we consider two integration \( \int_{\mathbb{R}} \vert s^{-1} G(s) \vert ds \) and \( \int_{\mathbb{R}} \vert s^{-1} G(s) \vert^2 ds \). Considering the definition of \( G \), by \( |r| < 1 \) and the symmetry of \( r \), we just estimate \( \int_{0}^{+\infty} \vert s^{-1} r(s) \vert ds \) and \( \int_{0}^{+\infty} \vert s^{-1} r(s) \vert^2 ds \). These difficult to estimate these two integration are the integration near \( s = 0 \) because \( r(z) \in H^{1,2}(\mathbb{R}_z) \). So we consider \( \int_{0}^{1} \vert s^{-1} r(s) \vert ds \) and \( \int_{0}^{1} \vert s^{-1} r(s) \vert^2 ds \). From the symmetry of \( r \), we give a change of variable and obtain
\[
\int_{0}^{1} \vert s^{-1} r(s) \vert ds = \int_{1}^{+\infty} s^{-1} \vert r(s) \vert ds \leq \| r \|_{L^1} \| r \|_{L^{2,1}}, \tag{3.35}
\]
\[
\int_{0}^{1} \vert s^{-1} r(s) \vert^2 ds = \int_{1}^{+\infty} \vert r(s) \vert^2 ds \leq \| r \|_{L^{2,1}}. \tag{3.36}
\]
Thus the sign of limit and sign of integration can exchange and we obtain the result
of \( N(0) \). And for \( N(i) \) and \( N^{(1)} \), we have

\[
|N(i)| \leq |C[G](i)| + |C[N \cdot G](i)| = \frac{1}{2\pi} \left[ \left| \int_{\mathbb{R}} \frac{N_-(s)G(s)}{s-i} \, ds \right| + \left| \int_{\mathbb{R}} \frac{G(s)}{s-i} \, ds \right| \right]
\]

\[
\leq \frac{1}{2\pi} \left\| N_- \right\|_{L^2(\mathbb{R})} \left\| G \right\|_{L^2(\mathbb{R})} + \frac{1}{2\pi} \int_{\mathbb{R}} |G(s)| \, ds \lesssim |r| \left\| r \right\|_{H^{1,2}, (3.37)}
\]

\[
|N^{(1)}| \leq \frac{1}{2\pi} \left[ \left| \int_{\mathbb{R}} \frac{N_-(s)G(s)}{(s-i)^2} \, ds \right| + \left| \int_{\mathbb{R}} \frac{G(s)}{(s-i)^2} \, ds \right| \right]
\]

\[
\leq \frac{1}{2\pi} \left\| N_- \right\|_{L^2(\mathbb{R})} \left\| G \right\|_{L^2(\mathbb{R})} + \frac{1}{2\pi} \int_{\mathbb{R}} |G(s)| \, ds \lesssim |r| \left\| r \right\|_{H^{1,2}, (3.38)}
\]

from which we obtain the result.

We take derivative of the inhomogeneous equation (3.21) in \( y \) and obtain

\[
\partial_y N_-(z) = C_-= [\partial_y G](z) + C_-[N_- \partial_y G](z) + C_- [\partial_y (N_-)G](z),
\]

\[
\partial_y^2 N_-(z) = C_-[\partial_y^2 G](z) + 2C_- [\partial_y (N_-)\partial_y G](z)
\]

\[
+ C_- [N_-\partial_y^2 G](z) + C_- [\partial_y^2 (N_-)G](z). \tag{3.40}
\]

From the definition of \( G \) we have that

\[
\partial_y G(z) = \left( -\frac{1}{2}(z - \frac{i}{2})^2 e^{2i\theta} - \frac{i}{2}(z - \frac{i}{2}) e^{-2i\theta} \right), \tag{3.41}
\]

\[
\partial_y^2 G(z) = \left( -\frac{1}{4}(z - \frac{i}{2})^3 e^{2i\theta} + \frac{i}{4}(z - \frac{i}{2})^3 e^{-2i\theta} \right), \tag{3.42}
\]

which are in \( L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) from \( r(z) \in H^{1,2}(\mathbb{R}) \). We first analyze the first-order derivative \( \partial_y N_- \) from (3.39). Note that \( N_- \partial_y G \) is also in \( L^2(\mathbb{R}) \), so \( \partial_y N_- \) exists in \( L^2(\mathbb{R}) \) with

\[
\partial_y N_-(z) = (I - C_-^2)^{-1} [C_- (\partial_y G) + C_- [N_- \partial_y G]]. \tag{3.43}
\]

Therefore, for \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
\partial_y N(z) = C(\partial_y G) + C [N_- \partial_y G] + C [\partial_y (N_-)G].
\]

In addition, as \( z \to i \), (3.39) has expansion:

\[
\partial_y N(z) = \partial_y N(i) + \partial_y N^{(1)}(z - i) + \mathcal{O}(z - i)^2, \tag{3.44}
\]

25
with

\[
\partial_y N(i) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{N(s) \cdot \partial_y G(s)}{s - i} ds + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial_y G(s)}{s - i} ds + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial_y N(s) \cdot G(s)}{s - i} ds,
\]

\[
\partial_y N^{(1)} = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{N(s) \cdot \partial_y G(s)}{(s - i)^2} ds + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial_y G(s)}{(s - i)^2} ds + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial_y N(s) \cdot G(s)}{(s - i)^2} ds.
\]

Similarly as in above Lemma, we have:

**Lemma 8.** When \( r(z) \in \mathcal{R} \) with \( \mathcal{R} \) defined in (3.13), there exist a positive constant \( c(r) \) rely on \( r \) such that \( \partial_y N(i) \), \( \partial_y N^{(1)} \) and \( \partial_y N(0) \) have estimation:

\[
|\partial_y N(i)|, |\partial_y N^{(1)}|, |\partial_y N(0)| \leq c(r) \| r \|_{H^{1.2}}. \tag{3.45}
\]

Similarly, \( \partial_y N_0 \partial_y G \) and \( N_0 \partial_y G \) are also in \( L^2(\mathbb{R}_z) \), so \( \partial_y^2 N_0 \) exists in \( L^2(\mathbb{R}_z) \) with

\[
\partial_y^2 N_0(z) = (I - C_0 G)^{-1} [C_0 [\partial_y^2 G] + 2C_0 [\partial_y (N_0) \partial_y G] + C_0 [N_0 \partial_y^2 G]]. \tag{3.46}
\]

Then for \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
\partial_y^2 N(z) = C [\partial_y^2 G](z) + 2C [\partial_y (N_0) \partial_y G](z) + C [N_0 \partial_y^2 G](z) + C [\partial_y^2 (N_0) G].
\]

In addition, as \( z \to i \), (3.39) has expansion:

\[
\partial_y^2 N(z) = \partial_y^2 N(i) + \partial_y^2 N^{(1)}(z - i) + \mathcal{O}(z - i)^2. \tag{3.47}
\]

Thus, there also has:

**Lemma 9.** When \( r(z) \in \mathcal{R} \) with \( \mathcal{R} \) defined in (3.13), there exist a positive constant \( c(r) \) rely on \( r \) such that \( \partial_y^2 N(i) \), \( \partial_y^2 N^{(1)} \) and \( \partial_y^2 N(0) \) have estimation:

\[
|\partial_y^2 N(i)|, |\partial_y^2 N^{(1)}|, |\partial_y^2 N(0)| \leq c(r) \| r \|_{H^{1.2}}. \tag{3.48}
\]
On the other hand, the equation (3.29) gives that for $z \to \infty$,

$$
\lim_{z \to \infty} |zN(z)|_{12} = \int_{\mathbb{R}} r(s) e^{-2\theta(s)} ds + \int_{\mathbb{R}} r(s) e^{-2\theta(s)} [N_-]_{11} (s) ds, \quad (3.49)
$$

$$
\partial_y \lim_{z \to \infty} |zN(z)|_{12} = - \int_{\mathbb{R}} \frac{i}{2} (s - \frac{1}{s}) r(s) e^{-2\theta(s)} ds - \int_{\mathbb{R}} \frac{1}{4} (s - \frac{1}{s})^2 r(s) e^{-2\theta(s)} [N_-]_{11} (s) ds + \int_{\mathbb{R}} r(s) e^{-2\theta(s)} \partial_y [N_-]_{11} (s) ds, \quad (3.50)
$$

$$
\partial_y \lim_{z \to \infty} |zN(z)|_{12} = - \int_{\mathbb{R}} \frac{1}{4} (s - \frac{1}{s})^2 r(s) e^{-2\theta(s)} ds - \int_{\mathbb{R}} \frac{1}{4} (s - \frac{1}{s})^2 r(s) e^{-2\theta(s)} [N_-]_{11} (s) ds + \int_{\mathbb{R}} r(s) e^{-2\theta(s)} \partial_y [N_-]_{11} (s) ds + \int_{\mathbb{R}} r(s) e^{-2\theta(s)} \partial^2_y [N_-]_{11} (s) ds. \quad (3.51)
$$

So we immediately have the estimate for $\lim_{z \to \infty} |zN(z)|_{12}$, $\partial_y \lim_{z \to \infty} |zN(z)|_{12}$ and $\partial^2_y \lim_{z \to \infty} |zN(z)|_{12}$:

**Lemma 10.** When $r(z) \in \mathcal{R}$ with $\mathcal{R}$ defined in (3.13), there exist a positive constant $c(r)$ rely on $\|r\|_{H^1_2}$ such that $\lim_{z \to \infty} |zN(z)|_{12}$, $\partial_y \lim_{z \to \infty} |zN(z)|_{12}$ and $\partial^2_y \lim_{z \to \infty} |zN(z)|_{12}$ are bounded with

$$
|\lim_{z \to \infty} |zN(z)|_{12}|, |\partial_y \lim_{z \to \infty} |zN(z)|_{12}|, |\partial^2_y \lim_{z \to \infty} |zN(z)|_{12}| \leq c(r) \|r\|_{H^1_2}. \quad (3.52)
$$

Here, $c(r)$ is a positive constant relying on $r(z)$.

### 4 Global existence of solutions for mCH equation

#### 4.1 Reconstruction of potential

The solution of the Cauchy problem (1.7)-(1.8) can be recovered from the solution of the RHP1 that is associated with the initial data $v_0(x)$.

**Proposition 5.** If $M(z; y)$ is the solution of RHP1 whose data is associated with the scattering data, as $z \to i$, $M(z; y)$ has expansion:

$$
M(z) = \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} + \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix} (z - i) + \mathcal{O}(z - i)^2. \quad (4.1)
$$
Here, $a_j$ is real valued function independent of $z$. Then $v$ can be recovered from the solution of the RH problem with:

$$v(y) = -[M]_{11}(i) \lim_{z \to i} \left( \frac{[M]_{12}(z) - [M]_{12}(i)}{z - i} \right)$$

$$- [M]_{22}(i) \lim_{z \to i} \left( \frac{[M]_{21}(z) - [M]_{21}(i)}{z - i} \right),$$

(4.2)

$$x = y + 2 \ln ([M]_{11}(i)).$$

(4.3)

Furthermore, we have

$$m = \frac{1}{1 - \lim_{z \to \infty} (z[M]_{12}(z))},$$

(4.4)

### 4.2 Time evolution and global solution

The crucial result of inverse scattering theory is how to recover solution $v(x, t)$ for the initial-value problem (1.7)-(1.8) from scattering data. With the time spectral problem in the Lax pair (2.1) and scattering relation (2.10), we find that time evolution scattering data $a(z; t)$ and $b(z; t)$ satisfy the equations

$$\partial_t a(z; t) = 0, \quad \partial_t b(z; t) = -\frac{2iz(z^2 - 1)}{z^2 + 1},$$

(4.5)

which yields

$$a(z; t) = a(z; 0), \quad b(z; t) = e^{-\frac{2iz(z^2 - 1)t}{z^2 + 1}} b(z; 0).$$

(4.6)

Therefore we can define the time-dependent scattering data

$$r(z; t) = e^{-\frac{2iz(z^2 - 1)t}{(z^2 + 1)^2}} r(z; 0),$$

(4.7)

which implies that $|r(z; t)| = |r(z; 0)|$ where $r(z; 0)$ denote the reflection coefficient associated with initial data $v_0(x)$.

**Proposition 6.** If $r(z; 0) \in \mathcal{R}$ with $\mathcal{R}$ defined in (3.13), then for every $t \in \mathbb{R}^+$, there also has $r(z; t) \in \mathcal{R}$. 

28
Proof. Since \( e^{-\frac{2iz(z^2-1)}{(z^2+1)^2}} \) = 1, so we have
\[
\|zr(z; t)\|_{L^2} = \|ze^{-\frac{2iz(z^2-1)}{(z^2+1)^2}}r(z; 0)\|_{L^2} = \|r(z; 0)\|_{L^2,1}.
\]
\[
\|z\partial_z r(z; t)\|_{L^2} = \left\|\frac{2iz^3t}{(z^2+1)^2}e^{-\frac{2iz(z^2-1)}{(z^2+1)^2}}r(z; 0) + e^{-\frac{2iz(z^2-1)}{(z^2+1)^2}}\partial_z r(z; 0)\right\|_{L^2} \\
\leq \sup_{z \in \mathbb{R}} \left|\frac{3z}{(z^2+1)^2}\right|\|r(z; 0)\|_{L^2} + \|\partial_z r(z; 0)\|_{L^2}.
\]
We conclude that \( r(z; t) \in H^{1,2}(\mathbb{R}_z) \). And it also admits the symmetry. \( \square \)

Then from the result in Section 3.2, we conclude that the time evolution of the RHP 2 has unique solution:

**RHP 3.** Find a matrix-valued function \( M^{(1)}(z; y, t) \) which satisfies:

- **Analyticity:** \( M^{(1)}(z; y, t) \) is analytical in \( \mathbb{C} \setminus \mathbb{R} \);
- **Symmetry:**
  \[
  M^{(1)}(z; y, t) = \sigma_1 M^{(1)}(\bar{z}; y, t)\sigma_1 = \sigma_2 M^{(1)}(-z; y, t)\sigma_2 = \sigma_1 M^{(1)}(0)^{-1} M^{(1)}(z^{-1}; y, t)\sigma_1;
  \]
- **Jump condition:** \( M^{(1)}(z; y, t) \) has continuous boundary values \( M_{\pm}^{(1)}(z; y, t) \) on \( \mathbb{R} \) and
  \[
  M_{+}^{(1)}(z; y, t) = M_{-}^{(1)}(z; y, t) V(z; y, t), \quad z \in \mathbb{R},
  \]
  where
  \[
  V(z; y, t) = \begin{pmatrix} 1 - |r(z; t)|^2 & -r(z; t)e^{-2\theta} \\ r(z; t)e^{2\theta} & 1 \end{pmatrix},
  \]
- **Asymptotic behaviors:**
  \[
  M^{(1)}(z; y, t) = I + O(z^{-1}), \quad z \to \infty.
  \]

We recall the following main result in [33].

**Proposition 7.** Suppose that \( M^{(1)}(z; y, t) \) is the unique solution of above time depended RHP 3, then the matrix function \( M(z; y, t) \) defined by
\[
\left( I - \frac{\sigma_1}{z} \right) M(z; y, t) = \left( I - \frac{\sigma_1 M^{(1)}(y, t; 0)^{-1}}{z} \right) M^{(1)}(z; y, t).
\]
Define
\begin{equation}
\tilde{M}(z; y, t) = M(z; y, t)e^{-\tilde{p}(z; y, t)\sigma_3},
\end{equation}
where \( \tilde{p}(z; y, t) = -\frac{i(z^2-1)}{4z}(-y + \frac{8y^2}{(z^2+1)^2}t) \). Then \( \tilde{M}(z; y, t) \) satisfies the system of linear differential equations
\begin{equation}
\begin{align*}
\tilde{M}_y(z; y, t) &= \tilde{U}\tilde{M}, \\
\tilde{M}_t(z; y, t) &= \tilde{V}\tilde{M},
\end{align*}
\end{equation}
where \( \tilde{U} = -\frac{i(z^2-1)}{4z}\sigma_3 + v, \tilde{V} = \frac{2i(z^2-1)x}{(z^2+1)^2}\sigma_3 + \tilde{V} \) with
\begin{align*}
v &= \frac{if(y, t)}{z-1} \left( \begin{array}{cc}
1 & -1 \\
1 & -1
\end{array} \right) + \frac{if(y, t)}{z+1} \left( \begin{array}{cc}
1 & 1 \\
-1 & -1
\end{array} \right) + if(y, t) \left( \begin{array}{cc}0 & -1 \\
1 & 0
\end{array} \right), \\
\tilde{V} &= \frac{iq(y, t)}{z-1} \left( \begin{array}{cc}1 & -1 \\
1 & -1
\end{array} \right) + \frac{iq(y, t)}{z+1} \left( \begin{array}{cc}1 & 1 \\
-1 & -1
\end{array} \right) \\
&\quad + \frac{1}{z-1} \left( \begin{array}{cc}0 & g_1(y, t) \\
g_2(y, t) & 0
\end{array} \right) + \frac{1}{z+1} \left( \begin{array}{cc}0 & g_2(y, t) \\
g_1(y, t) & 0
\end{array} \right),
\end{align*}
and
\begin{align}
\eta(y, t) &= -\frac{1}{2} \lim_{z \to \infty} [zM(z; y, t)]_{12} = 1 - 1/m, \\
q(y, t) &= -i \lim_{z \to 1} (z-1)[z\tilde{M}_t(z; y, t)\tilde{M}(z; y, t)]_{11}, \\
g_1(y, t) &= 2[M(y, t; i)]_{11} \lim_{z \to i} \frac{[M(z; y, t)]_{12}}{z-i}, \\
g_1(y, t) &= -2[M(y, t; i)]_{22} \lim_{z \to i} \frac{[M(z; y, t)]_{21}}{z-i}.
\end{align}
The compatibility condition between two equations in the system (4.13)
\begin{equation}
\tilde{U}_t - \tilde{V}_x + [\tilde{U}, \tilde{V}] = 0
\end{equation}
yields the mCH equation in the \((y, t)\) variables.

Moreover, we consider to use \( M^{(1)}(z; y, t) \) to recovering the solution \( v(x, t) \) through Lemma 5 and (4.11).
Proposition 8. Suppose that $M^{(1)}(z; y, t)$ is the unique solution of above time depended RHP 3, then

$$M^{(1)}(y, t; 0) = \begin{pmatrix} \alpha(y, t) & i\beta(y, t) \\ -i\beta(y, t) & \alpha(y, t) \end{pmatrix}, \quad \alpha^2 - \beta^2 = 1, \tag{4.15}$$

where $\alpha(y, t)$, $\beta(y, t)$ are real functions. And when $\beta \neq 0$, as $a \to i$, $m(z; y, t)$ has expansion

$$M^{(1)}(z; y, t) = \begin{pmatrix} f_1(y, t) & i\beta f_1(y, t) \\ -i\beta f_1(y, t) & f_2(y, t) \end{pmatrix} + \begin{pmatrix} i\beta g_1(y, t) \\ -i\beta^2 g_1(y, t) \end{pmatrix} (z - i) + \mathcal{O}(z - i)^2, \tag{4.16}$$

where $g_1(y, t), g_2(y, t), f_1(y, t), f_2(y, t)$ are real functions. Then the following formula give the solution $v(x, t)$ of the initial-value problem (1.1) and (1.5):

$$v(y, t) = -\alpha_2(y, t)\alpha_1(y, t) - \alpha_3(y, t)\alpha_1(y, t)^{-1}, \tag{4.17}$$
$$x(y, t) = y + 2\ln(\alpha_1(y, t)). \tag{4.18}$$

where

$$\alpha_1(y, t) = \left(1 - \frac{\beta}{\alpha + 1}\right) f_1, \quad \alpha_2(y, t) = \frac{\beta}{\alpha + 1} f_2 + \left(1 - \frac{\beta}{\alpha + 1}\right) g_2, \tag{4.19}$$
$$\alpha_3(y, t) = \frac{-\beta}{\alpha + 1} f_1 + \left(1 - \frac{-\beta}{\alpha + 1}\right) g_1. \tag{4.20}$$

Remark 2. In the case $\beta = 0$, then $\alpha = \pm 1$. When $\alpha = 1$, it only need to take $\frac{\beta}{\alpha + 1} = 0$ in above formula. But when $\alpha = -1$, by the symmetry of $M^{(1)}(y, t)$ we obtain that as $a \to i$, $M^{(1)}(z; y, t)$ has expansion

$$M^{(1)}(z; y, t) = \begin{pmatrix} 0 & f_1(y, t)i \\ f_1(y, t)^{-1}i & 0 \end{pmatrix} + \begin{pmatrix} g_1(y, t)i \\ 0 \end{pmatrix} (z - i) + \mathcal{O}(z - i)^2. \tag{4.16}$$

Under this case, the function $\alpha_j$ in reconstruction formula become

$$\alpha_1(y, t) = f_1^{-1},$$
$$\alpha_2(y, t) = f_1 + g_2,$$
$$\alpha_3(y, t) = f_1^{-1} + g_1.$$
On the other hand, we declare that $\alpha_1(y, t) \neq 0$, which is from $\det[M^{(1)}(z; y, t)] = 1$.

We now study properties of the potential $u$ recovered by equations (4.17)-(4.18) from properties of the matrix $M^{(1)}(z; y, t)$. Lemma 7, 8, 9 and 10 give

**Lemma 11.** When $r(z; 0) \in \mathcal{R}$, $\eta, \alpha_j$ defined in (4.14), (4.19)-(4.20) are bounded respectively. In addition, their $L^\infty$-norm can by controlled by $\|r(z; 0)\|_{H^{1,2}}$ with

\[
\|\alpha_1 - 1\|_{W^{2,\infty}}, \|\alpha_j\|_{W^{2,\infty}} \leq c(r) \|r(z; 0)\|_{H^{1,2}}, \quad j = 2, 3, \quad \tag{4.21}
\]

\[
\|\eta(y, t)\|_{W^{2,\infty}} \leq c(r) \|r(z; 0)\|_{H^{1,2}} \quad \tag{4.22}
\]

with a positive constant $c(r)$ depending on $r(z; 0)$. Therefore,

\[
|v(y, t)| \leq c(r) \|r(z; 0)\|_{H^{1,2}}. \quad \tag{4.23}
\]

The above Lemma and Proposition 6 give the existence of $v(y, t)$ and $\eta(y, t)$ as a $W^{2,\infty}(\mathbb{R})$ function of $y$ for every $t \in \mathbb{R}^+$. However, for the boundedness of $m$, the reconstruction formula of $m$ in (4.4) is

\[
m = (1 - \eta(y, t))^{-1}, \quad \tag{4.24}
\]

where $\eta(y, t)$ defined in (4.14). So it is hard to obtain the boundedness from reconstruction formula. So we consider the definition of $m$ in (1.7) with $m = v - v_{xx} + 1$. Note that $\partial_x = (1 + 2\partial_y[\alpha_1]/\alpha_1)\partial_y$, so

\[
\partial_x v = (1 + 2\partial_y[\alpha_1]/\alpha_1) \left[-\partial_y[\alpha_2]\alpha_1 - \partial_y[\alpha_1]\alpha_2 - \partial_y[\alpha_3]\alpha_1^{-1} + \partial_y[\alpha_1]\alpha_3\alpha_1^{-2}\right],
\]

\[
\partial_x^2 v_{xx} = 2\frac{\alpha_1\partial_y^2[\alpha_1]}{\alpha_1^2} v_x + \left(1 + \frac{2\partial_y[\alpha_1]}{\alpha_1}\right)^2 \left[-\frac{\partial_y^2[\alpha_2]}{\alpha_1} - 2\partial_y[\alpha_1]\partial_y[\alpha_2] - \partial_y^2[\alpha_1]\alpha_2\right]
\]

\[
+ \left(1 + \frac{2\partial_y[\alpha_1]}{\alpha_1}\right)^2 \left[-\frac{\partial_y^2[\alpha_3]}{\alpha_1} + 2\frac{\partial_y[\alpha_1]\partial_y[\alpha_3]}{\alpha_1^2} + \frac{\partial_y^2[\alpha_1]\alpha_3}{\alpha_1^2} - 2\partial_y[\alpha_1]^{-2}\alpha_3\right].
\]

Similarly as above analysis, Lemma 8 and 9 gives the boundedness of $v_{xx}$.

**Lemma 12.** When $r(z; 0) \in \mathcal{R}$, $v_x(y, t), v_{xx}(y, t)$ defined above are bounded respectively. In addition, their $L^\infty$-norm can by controlled by $\|r\|_{H^{1,2}}$ with

\[
|v_x(y, t)|, \quad |v_{xx}(y, t)| \leq c(r) \|r(z; 0)\|_{H^{1,2}}, \quad \tag{4.25}
\]

with a positive constant $c(r)$ depending on $r(z; 0)$. 

32
Therefore, we have that for every \( t \in \mathbb{R}^+ \), \( m(y, t) \) exists as a bounded function of \( y \).

**Proposition 9.** When \( r(z; 0) \in \mathcal{R} \), \( m(y, t) \) defined above is bounded respectively. In addition, their \( L^\infty \)-norm can by controlled by \( \| r(z; 0) \|_{H^{1,2}} \) with

\[
|m(y, t) - 1| \leq c(r) \| r(z; 0) \|_{H^{1,2}},
\]
with a positive constant \( c(r) \) depending on \( r(z; 0) \).

To arrive at the property of high order derivative of \( v(x, t) \) and prepare for the proof in next section, we analyze the property of \( \eta(y, t) \). Next, we prove the continuity about \( t \) of \( \eta(y, t) \). We first give the following lemma:

**Lemma 13.** For any given \( t_0 > 0 \), operator \( \mathcal{C}_G^G(t) \) which corresponding to \( r(z; t) \) is continuous with respect to \( t \) at \( t_0 \). So operator \( (I - \mathcal{C}_G^G(t))^{-1} \) is also continuous with respect to \( t \) at \( t_0 \).

**Proof.** For any \( t_0 \in \mathbb{R}^+ \), any \( f = (F_1, F_2)^T \in L^2(\mathbb{R}_z) \),

\[
\| \mathcal{C}_G^G(t)_- [f] - \mathcal{C}_G^G(t_0)_- [f] \|_{L^2_z} \\
\leq \| \mathcal{C}_- \left[ e^{-2\theta} f_2 r(z; 0) e^{\frac{2isa(z^2 - 1)(t - t_0)}{(z^2 + 1)^2}} \left( 1 - e^{-\frac{2isa(z^2 - 1)(t - t_0)}{(z^2 + 1)^2}} \right) \right] \|_{L^2_z} \\
+ \| \mathcal{C}_- \left[ e^{2\theta} f_1 \bar{r}(z; 0) e^{\frac{2isa(z^2 - 1)(t - t_0)}{(z^2 + 1)^2}} \left( e^{\frac{2isa(z^2 - 1)(t - t_0)}{(z^2 + 1)^2}} - 1 \right) \right] \|_{L^2_z} \\
\leq \| f \|_{L^2_z} \| e^{\frac{2isa(z^2 - 1)(t - t_0)}{(z^2 + 1)^2}} - 1 \|_{L^\infty_z} \leq c \| f \|_{L^2_z} |t - t_0|,
\]
where \( c \) is a positive constant. So \( \mathcal{C}_G^G(t) \) is continuous about \( t \). Then from the map: \( \mathcal{A} \rightarrow \mathcal{A}^{-1} \) is a homeomorphism in Banach algebra \( \mathcal{B}(L^2(\mathbb{R})) \), so \( (I - \mathcal{C}_G^G(t))^{-1} \) is also continuous with respect to \( t \). \( \square \)

**Lemma 14.** Suppose that \( M^{(1)}(z; y, t) \) is the solution of RHP 2 with scattering coefficient \( r(z; t) = e^{-\frac{2isa(z^2 - 1)t}{(z^2 + 1)^2}} r(z; 0) \), \( r(z; 0) \in \mathcal{R} \). Then for given \( t_0 \in \mathbb{R}^+ \), \( M^{(1)}(z; y, t) \) continuous with respect to \( t \) at \( t_0 \).
Proof. Note that \((I - C(t))^G\) and \(C[G(t)]\) are both continuous with respect to \(t\) at any \(t_0 \in \mathbb{R}^+\). Then continuity from
\[
M^{(1)}(z; y, t) - I = \left( I - C(t) \right)^G - 1 \left[ C[G(t)] \right], \quad z \in \mathbb{R},
\]
and
\[
M^{(1)}(z; y, t) = I + C[M^{(1)}G(t)], \quad z \in \mathbb{C},
\]
immediately.

Then from reconstruction formula we arrive at

**Corollary 3.** Suppose that \(\eta(y, t)\) is reconstituted by the \(M^{(1)}(z; y, t)\) with scattering coefficient \(r(z; t) = e^{-2iz^2(x^2-1)t} r(z; 0), \quad r(z; 0) \in \mathcal{R}\). Then there has \(\eta(y, t) \in C([0, +\infty); W^{2,\infty}(\mathbb{R}_x))\).

From the continuous with respect to \(t\) of \(\eta \in W^{2,\infty}(\mathbb{R}_x)\) and the boundedness of \(m\), we finally have

**Proposition 10.** Supposed that the initial-value \(v_0(x) \in \mathcal{G}\) with \(\mathcal{G}\) defined in (2.14) with small norm, then there exists a unique global solution \(v(x, t)\) for every \(t \in \mathbb{R}^+\) to the Cauchy problem (1.7)-(1.8) for the mCH equation in \(C([0, +\infty); W^{2,\infty}(\mathbb{R}_x))\). Moreover, for \(m = v - v_{xx} + 1\), there also has \(m \in C([0, +\infty); W^{2,\infty}(\mathbb{R}_x))\).

On the other hand, (4.18) also gives
\[
dx = \left( 1 + 2 \frac{\partial}{\partial y} \alpha_1 \right) dy. \tag{4.27}
\]
Then above propositions also leads to the equivalency between the integral norm in \(\mathbb{R}_x\) and \(\mathbb{R}_y\), namely for any function \(h, 1 \leq p \leq \infty\),
\[
h(x) \in L^p(\mathbb{R}_x) \iff h(x(y)) \in L^p(\mathbb{R}_y). \tag{4.28}
\]
5 Regularity of the solution for mCH equation

In this section, we further discuss the regularity of the solution \( m(y, t) \) and show that \((u(x, t), m(x, t) - 1) \in C([0, +\infty); H^4(\mathbb{R}) \times H^{2,1}(\mathbb{R}))\) to the Cauchy problem for the mCH equation.

5.1 Transitions of spectral problem

First we establish a new RH problem by using the Lax pair 2.1. Consider two column function vector

\[
\Phi_1 = \begin{pmatrix} 1 & 0 \\ -\frac{m-1}{m} & 2\lambda \end{pmatrix} \Psi_1,
\]

\[
\Phi_2 = \begin{pmatrix} 2\lambda & -\frac{m-1}{m} \\ 0 & 1 \end{pmatrix} \Psi_2,
\]

which satisfy two new spectral problems

\[
\Phi_{j,x} = -\frac{1}{4} i k m \sigma_3 \Phi_j + P_j \Phi_j, \quad j = 1, 2,
\]

with

\[
P_1 = \frac{i(m-1)}{2k} \begin{pmatrix} \eta - 2 & 1 \\ -\eta^2 - \frac{4}{m} & -\eta + 2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \eta_x & 0 \end{pmatrix},
\]

\[
P_2 = \frac{i(m-1)}{2k} \begin{pmatrix} \eta - 2 & \eta^2 + \frac{4}{m} \\ -1 & -\eta + 2 \end{pmatrix} - \begin{pmatrix} 0 & \eta_x \\ 0 & 0 \end{pmatrix},
\]

where \( \eta(y) = \frac{2m-1}{m} \). In above section, we have already proved \( \eta(y) \in W^{1,\infty} \).

Furthermore, make transformations

\[
\nu_j^\pm(x; k) = e^{(-1)^{j-1} p(x; k)} \Phi_j(x; k),
\]

which satisfy the Volterra integral equations

\[
\nu_1^\pm(x; k) = e_1 + \int_{\pm\infty}^x \begin{pmatrix} 1 & 0 \\ 0 & e^{2(p(x; k) - p(s; k))} \end{pmatrix} P_1(s, k) \nu_1^\pm(s, k) ds,
\]

\[
\nu_2^\pm(x, k) = e_2 + \int_{\pm\infty}^x \begin{pmatrix} e^{-2(p(x; k) - p(s; k))} & 0 \\ 0 & 1 \end{pmatrix} P_2(s, k) \nu_2^\pm(s, k) ds,
\]
where \( e_1 = (1, 0)^T, \ e_2 = (0, 1)^T. \)

**Lemma 15.** If initial data \( v(x) \in \mathcal{G}, \) then the Jost function \( \nu_j^\pm(x; k) \) admit the following asymptotics

\[
\nu_1^\pm(x; k) = e_1 + \mathcal{O}(k^{-1}), \quad k \to \infty, \tag{5.7}
\]
\[
\nu_2^\pm(x; k) = e_2 + \left( \frac{2i\frac{m_\pm}{m}}{b_\pm(x)} \right) k^{-1} + \mathcal{O}(k^{-2}), \quad k \to \infty, \tag{5.8}
\]

with

\[
b_\pm(x) = \frac{i}{2} \int_{\pm\infty}^x \frac{m(s)^2 - 1}{m(s)} ds.
\]

And

\[
\nu_1^\pm(x, k) = -\frac{i\alpha_\pm}{k} \left( \begin{array}{c} 1 \\ 2 - \eta \end{array} \right) + \mathcal{O}(1), \quad k \to 0, \tag{5.9}
\]
\[
\nu_2^\pm(x, k) = \frac{i\alpha_\pm}{k} \left( \begin{array}{c} 2 - \eta \\ 1 \end{array} \right) + \mathcal{O}(1), \quad k \to 0. \tag{5.10}
\]

Similarly, we consider two column function vector

\[
\varphi_1^\pm = \left( \begin{array}{cc} 1 & 0 \\ 0 & 2\lambda \end{array} \right) \mu_1^\pm, \quad \varphi_2^\pm = \left( \begin{array}{cc} (2\lambda)^{-1} & 0 \\ 0 & 1 \end{array} \right) \mu_2^\pm. \tag{5.11}
\]

**Lemma 16.** If initial data \( v(x) \in \mathcal{G}, \) then new Jost function \( \varphi_j^\pm(k) \) admits the following limits

\[
\varphi_1^\pm = \left( \begin{array}{c} 0 \\ m_{-1} \end{array} \right) + \mathcal{O}(k^{-1}), \quad k \to \infty, \tag{5.12}
\]
\[
\varphi_2^\pm = e_2 + k^{-1} \left( \begin{array}{c} 0 \\ b_\pm \end{array} \right) + \mathcal{O}(k^{-2}) \quad k \to \infty. \tag{5.13}
\]

And

\[
\varphi_1^\pm = \frac{i\alpha_\pm}{k} \left( \begin{array}{c} -1 \\ -2 \end{array} \right) + \mathcal{O}(1), \quad k \to 0, \tag{5.14}
\]
\[
\varphi_2^\pm = \frac{i\alpha_\pm}{k} \left( \begin{array}{c} \frac{1}{2} \\ 1 \end{array} \right) + \mathcal{O}(1), \quad k \to 0. \tag{5.15}
\]
5.2 A new RH problem

To arrive at the $L^2$-property of the solution of mCH equation, we construct another RH problem. We use another Jost function in subsection 5.1 to define

$$\frac{\nu^-}{a(\bar{z})} - \nu^+_1 = r_2 p_+ e^{-iky/2}, \quad \frac{p^-}{a} - p_+ = r_1 \nu^+_2 e^{iky/2}. \quad (5.16)$$

Obviously, $p_{\pm}(k; y)$ are both well defined under variable $k = z - 1/z$. By using this two column vectors, we construct another RH problem

$$M^{(2)}(k; y) = \begin{cases} \left( \frac{\nu^-_{-1}(k)}{a(k)}, p_+(k) \right), & \text{as } k \in \mathbb{C}^+, \\ \left( \nu_{+1}(k), \frac{p_-(k)}{a(k)} \right), & \text{as } k \in \mathbb{C}^-. \end{cases} \quad (5.17)$$

**RHP 4.** Find a matrix-valued function $M^{(2)}(k) = M^{(2)}(k; y)$ which satisfies

- **Analyticity:** $M^{(2)}(k)$ is analytical in $\mathbb{C} \setminus \mathbb{R}$;
- **Jump condition:** $M^{(2)}(k)$ has continuous boundary values $M_{\pm}^{(2)}(k)$ on $\mathbb{R}$ and

$$M_{+}^{(2)}(k) = M_{-}^{(2)}(k) \tilde{V}(k), \quad k \in \mathbb{R}. \quad (5.18)$$

- **Asymptotic behaviors:**

$$M^{(2)}(k) = I + O(k^{-1}), \quad k \to \infty \text{ in } \mathbb{C}; \quad (5.19)$$

- **Singularity:** $M^{(2)}(k)$ has singularity at $k = 0$ with

$$M^{(2)}(k) = \frac{i\alpha_+(y)}{k} \begin{pmatrix} -c & \frac{1}{2} \\ -(2-\eta)c & \frac{2-\eta}{2} \end{pmatrix} + O(1), \quad k \to 0 \text{ in } \mathbb{C}^+, \quad (5.20)$$

where $c = 0$ in generic case, while $c \neq 0$ in non-generic case.

In order to remove the singularity, we make a transformation

$$M^{(3)}(k; y) = \begin{pmatrix} \eta - 2, & 1 \end{pmatrix} M^{(2)}(k; y), \quad (5.21)$$

and reduce above RHP 4 to a regular one.
RHP 5. Find a vector-valued function $M^{(3)}(k) = M^{(3)}(y; k)$ which satisfies

- Analyticity: $M^{(3)}(k)$ is analytical in $\mathbb{C} \setminus \mathbb{R}$;
- Jump condition: $M^{(3)}$ has continuous boundary values $M^{(3)}_\pm(k)$ on $\mathbb{R}$ and
  \begin{equation}
  M^{(3)}_+(k) = M^{(3)}_-(k) \tilde{V}(k), \quad k \in \mathbb{R},
  \end{equation}

  where
  \begin{equation}
  \tilde{V}(k) = \begin{pmatrix}
  1 - |r|^2 & -r_1 e^{-2\theta} \\
  r_2 e^{2\theta} & 1
  \end{pmatrix} := I + \tilde{G},
  \end{equation}

  with
  \begin{equation}
  r_1(k) = \frac{z}{z^2 + 1} r(z), \quad r_2(k) = \frac{z^2 + 1}{z} r(z), \quad r_1(k) r_2(k) = |r(z)|^2;
  \end{equation}

- Asymptotic behaviors:
  \begin{equation}
  M^{(3)}(k) = (\eta - 2, 1) + O(k^{-1}), \quad k \to \infty \text{ in } \mathbb{C};
  \end{equation}

Although in the new RHP 5, as $z \to \infty$, $M^{(3)}(k)$ does not asymptotically to identity matrix but has $\eta = \lim_{z \to \infty} [zM]_{21}$ in $\lim_{z \to \infty} M^{(1)}(z)$. We already have proved the existence and boundedness of $\lim_{z \to \infty} [zM]_{21}$ in Section 4. And a simple calculation gives

Lemma 17. When $r(z) \in H^{1,2}(\mathbb{R}_z)$, then $r_1(k) \in H^{1,2}(\mathbb{R}_k)$, $r_2(k) \in W^{1,2}(\mathbb{R}_k)$, and there exists a positive constant $c$ such that

\begin{equation}
\| r_1 \|_{H^{1,2}_k} + \| r_2 \|_{W^{1,2}_k} \leq c \| r \|_{H^{1,2}_z}.
\end{equation}

However, unlike the operator $I - C^G_-$ defined in Lemma 6, here new operator $I - C^{\tilde{G}}_-$ will loss symmetry, which result to difficulty to obtain the existence of its inverse operator $(I - C^{\tilde{G}}_-)^{-1}$. To further use the operator $I - C^G_-$ on the matrix $M^{(3)}(k)$, we introduce two transformations

\begin{equation}
G^{(j)}(z; y) = M^{(3)}(k; y) \tau_j(z) - (\eta - 2, 1) \tau_j(z),
\end{equation}

where

\begin{equation}
\tau_1 = \begin{pmatrix}
  \left( z + \frac{1}{z} \right)^{-1} & 0 \\
  0 & 1
  \end{pmatrix}, \quad \tau_2 = \begin{pmatrix}
  z + \frac{1}{z} \\
  0
  \end{pmatrix} \tau_1.
\end{equation}
Denote $[M^{(3)}]^{(h)}$ as the $h$ item of vector $M^{(3)}$, thus

$$G^{(1)} = \left( ([M^{(3)}]^{(1)} - \eta + 2) \left( z + \frac{1}{z} \right)^{-1}, [M^{(3)}]^{(2)} - 1 \right).$$

$$G^{(2)} = \left( [M^{(3)}]^{(1)} - \eta + 2, ([M^{(3)}]^{(2)} - 1) \left( z + \frac{1}{z} \right) \right).$$

Obviously, $G^{(j)}(z)$ can not be considered as a function under the variate $k$, so we need to analyze its property on $z$-plane. $G^{(j)}(z)$ admits the following RH problem on $z$-plane:

**RHP 6.** Find a vector-valued function $G^{(j)}(z) = G^{(j)}(y; z)$ on $z$-plane which satisfies

- Analyticity: $G^{(j)}(z)$ is analytical in $\mathbb{C} \setminus \mathbb{R}$;
- Jump condition: $G^{(j)}$ has continuous boundary values $G^{(j)}_{\pm}(z)$ on $\mathbb{R}$ and

$$G^{(j)}_{+}(z) = G^{(j)}_{-}(z)V(z) + F^{(j)}(z), \quad z \in \mathbb{R},$$

where $V(z)$ is same in (3.4) and

$$F_{j}(z) = \eta G(z) \tau_{j} = \eta \tau_{j} \bar{G}(z);$$

- Asymptotic behaviors:

$$G^{(j)}(z) = O(z^{-1}), \quad z \to \infty.$$  

In lemma 6, we have proved that $I - C^{-}_{G}$ and $(I - C^{-}_{G})^{-1}$ are two bounded linear operators on: $L^{2}(\mathbb{R}_{z}) \to L^{2}(\mathbb{R}_{z})$. However, we need the estimation of integral property on $k$-plane. To distinguish two Cauchy projection operators on $k$-plane and $z$-plane, in the following lemmas, we use superscript on the two Cauchy projection operators with $C^{k}_{\pm}, C^{k}$ and $C^{z}_{\pm}, C^{z}$. Similarly, there has $C^{k_{G}}_{\pm}$ and $C^{z_{G}}_{\pm}$.

**Lemma 18.** If $f(z) \in L^{1}(\mathbb{R}_{z}) \cap L^{\infty}(\mathbb{R}_{z})$ and $f(z) = f(-\frac{1}{z})$, then

$$C^{z_{G}}_{\pm}(f)(z) = C^{k}_{\pm} [f](k),$$

where

$$C^{z_{G}}_{\pm}(f)(z) = \frac{1}{2\pi i} \left( \int_{-\infty}^{-1} + \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{\infty} \right) \frac{f(s)ds}{s - (z \pm 6i)}. $$

39
If \( g(z) \in L^1(\mathbb{R}_z) \cap L^\infty(\mathbb{R}_z) \) and \( g(z) = -g(-\frac{1}{z}) \), then

\[
C_{\pm}^{z,(o)}[g](z) = C_k^{\pm}[f](k),
\]

where

\[
C_{\pm}^{z,(o)}[g](z) = \frac{1}{2\pi i} \left( \int_{-1}^{\infty} + \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{\infty} \right) \frac{g(s)ds}{s - (z \pm 0i)}. \tag{5.36}
\]

In addition,

\[
\left( z + \frac{1}{z} \right) C_{\pm}(e)[f](z) = C_{\pm}(e) \left[ \left( z + \frac{1}{z} \right) f \right](z), \tag{5.37}
\]

\[
\left( z + \frac{1}{z} \right) C_{\pm}(o)[g](z) = C_{\pm}(o) \left[ \left( z + \frac{1}{z} \right) g \right](z). \tag{5.38}
\]

This lemma is a directly consequence from

\[
\frac{1}{s - z} + \frac{1}{s^2} \cdot \frac{1}{s^2 - z} = \frac{1 + \frac{1}{z^2}}{s - \frac{1}{z} - z + \frac{1}{z}}.
\]

For convenience, we call the function \( f(z) \) satisfying symmetry \( f(z) = f(-\frac{1}{z}) \) as a "even" function, while the function \( g(z) \) satisfying symmetry \( g(z) = -g(-\frac{1}{z}) \) as a "odd" function. Like lemma 6, we can give the following lemma about \( C_{\pm}^{k,G} \) with \( C_{\pm}^{k,G}[f](k) = C_k^{\pm}[fG](k) \) for any \( f \in L^2(\mathbb{R}_k) \).

**Lemma 19.** The operator \( I - C_{\pm}^{k,G} \) is invertible on \( L^2(\mathbb{R}_k) \rightarrow L^2(\mathbb{R}_k) \) and \( (I - C_{\pm}^{k,G})^{-1} \) is also a bounded operator.

Then we obtain solvability of RHP6.

**Corollary 4.** For any \( r(z) \in H^{1,2}(\mathbb{R}_z) \) with small norm, RH problems 6 exists unique solution with

\[
G^{(j)}(z) = C_k^{\pm} \left[ G^{(j)}_\pm + F^{(j)} \right](k) \tag{5.39}
\]

where \( G^{(j)}_\pm \) is a solution of the Fredholm integral equation:

\[
G^{(j)}_\pm(z) = C_k^{\pm} \left[ G^{(j)}_\pm + F^{(j)} \right](k). \tag{5.40}
\]
Moreover,
\[
\| G_{-}^{(j)}(z) \|_{L^2} \leq \left\| (I - C \tilde{G})^{-1} \| C_- \left[ F^{(j)} \right] \|_{L^2} .
\]
(5.41)
Namely, there exists a positive constant \( C(r) \) depending on \( r \) such that
\[
\| G_{-}^{(j)}(z) \|_{L^2} \leq C(r) \| r \|_{H^{1,2}} .
\]
(5.42)
Here, the element of \( G^{(j)}(z) \) is either "even" or "odd". So in the equation (5.40), the operator \( C_k^- \) follows the corresponding integral contour in lemma 18. For example, the first element of \( G^{(1)}(z) \) is
\[
\left( [M^{(3)}]^{(1)} - \eta + 2 \right) \left( z + \frac{1}{z} \right)^{-1} + c_{2\theta} \left( [M^{(3)}]^{(2)} - 1 \right),
\]
which is an "odd" function, so we take
\[
C_k^- \left[ \left( G^{(1)}(z) \right)^{(1)} \right] (k) = C_{z}^{\text{(o)}} \left[ \left( G^{(1)}(z) \right)^{(1)} \right] (z).
\]
The existence of \( G^{(j)}(z) \) also implies the existence of \( M^{(3)}(k) \).

**Lemma 20.** For any \( r_1(k) \in H^{1,1}(\mathbb{R}_k) \), we have
\[
\sup_{y \in \mathbb{R}} \left| \langle y \rangle^2 C_{\pm} [-r_1 e^{iky/2}] \right| \leq c \| r_1 \|_{H^{1,1}_k},
\]
(5.43)
thus,
\[
\sup_{y \in \mathbb{R}} \| C_{\pm} [-r_1 e^{iky/2}] \|_{L^2_k} \leq c \| r_1(k) \|_{H^{1,1}_k},
\]
(5.44)
\[
\sup_{y \in \mathbb{R}} \| \langle y \rangle C_{\pm} [-r_1 e^{iky/2}] \|_{L^2_k} \leq c \| r_1(k) \|_{H^{1,1}_k},
\]
(5.45)
\[
\sup_{y \in \mathbb{R}} \langle y \rangle^2 \int_{\mathbb{R}} r_2(s) e^{-isy/2} C_{\pm} [-r_1 e^{iky/2}] ds \leq c \| r(z) \|^2_{H^{1,1}_k},
\]
(5.46)
where \( \langle y \rangle = (1 + y^2)^{1/2} \) and \( c \) is a positive constant.

**Proof.** Take \( C_- \) as an example. The first two results comes from the property of Cauchy projection operator. Note that for \( \epsilon > 0 \),
\[
\frac{i}{s - (k - \epsilon i)} = - \int_0^{-\infty} e^{-is\lambda} e^{i(k-\epsilon)i\lambda} d\lambda = \int_0^{+\infty} e^{-ik\lambda} e^{i(s+\epsilon)i\lambda} d\lambda.
\]

Then for \( y \geq 0 \), we have

\[
C_-[-r_1e^{iky/2}] (k) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{-r_1(s)e^{isy/2}}{s - (k - \epsilon i)} ds
\]

\[
= \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} r_1(s)e^{isy/2} \int_{0}^{+\infty} e^{-ik\lambda} e^{i(s+\epsilon)\lambda} d\lambda ds
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} r_1(s)e^{isy/2} \int_{0}^{+\infty} e^{-ik\lambda} e^{is\lambda} d\lambda ds
\]

\[
= \frac{1}{4\pi} \int_{y}^{+\infty} \hat{r}_1(\frac{s}{2})e^{-i(s-y)k/2} ds.
\]

Here, \( \hat{r}_1 \) means the Fourier transformation of \( r_1 \) and the last step from the basic property of Fourier transformation. Then for any \( f(k) \) in \( L^2(\mathbb{R}_k) \),

\[
\left| \langle y \rangle \int_{\mathbb{R}} \frac{1}{4\pi} \int_{y}^{+\infty} \hat{r}_1(\frac{s}{2})e^{-i(s-y)k/2} ds f(k) dk \right| = \frac{1}{4\pi} \int_{y}^{+\infty} \langle y \rangle |\hat{r}_1(\frac{s}{2})| ds f(k) dk
\]

\[
\leq c \| r_1 \|_{H^{1.1}} \| f \|_{L^2}.
\]

In the last step we using that \( \langle y \rangle \leq \langle s \rangle \) for \( y \geq 0 \). And for \( y \leq 0 \),

\[
C_-[-r_1e^{iky/2}] (k) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{-r_1(s)e^{isy/2}}{s - (k - \epsilon i)} ds
\]

\[
= \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} r_1(s)e^{isy/2} \int_{0}^{0} e^{-iks\lambda} e^{i(k-\epsilon)i\lambda} d\lambda ds
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} r_1(s)e^{isy/2} \int_{0}^{0} e^{-iks\lambda} e^{ik\lambda} d\lambda ds
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{0} \hat{r}_1(k - y/2)e^{ik\lambda} d\lambda.
\]

Thus, \( \| \langle y \rangle C_-[-r_1e^{iky/2}] \|_{L^2_k} = \| \langle y \rangle \frac{1}{2\pi} X_- \hat{r}_1(y/2 + k) \|_{L^2_k} \), with \( X_- \) representing the characteristic function on \( \mathbb{R}^- \). So

\[
\| \langle y \rangle C_-[-r_1e^{iky/2}] \|_{L^2_k}^2 = \frac{1}{4\pi^2} \int_{-\infty}^{0} \langle y \rangle |\hat{r}_1(k+y/2)|^2 dk
\]

\[
\frac{1}{8\pi^2} \int_{-\infty}^{y} \langle y \rangle |\hat{r}_1(\frac{k}{2})|^2 dk \leq c \| r_1 \|_{H^{1.1}}^2.
\]
And for the last inequality, same as above step, we take \( y \leq 0 \) as an example,

\[
\langle y \rangle^2 \int_{\mathbb{R}} r_2(s)e^{-isy/2}C_\uparrow \left[-r_1e^{iky/2}\right] ds = \langle y \rangle^2 \frac{1}{2\pi} \int_{-\infty}^{\frac{\pi}{2}} \tilde{r}_1(s) \tilde{r}_2(s) ds \\
\leq \| \langle y \rangle \tilde{r}_1 \|_{L^2_k} \| \langle y \rangle \tilde{r}_2 \|_{L^2_k} \leq c \| r \|_{H^{1,1}}^2.
\]

Then we arrive at the consequence.

We define a new vector function

\[
N^{(2)} = \left( [M^{(3)}]^{(1)}_- - \eta + 2, [M^{(3)}]^{(2)}_+ - 1 \right) = \left( [G^{(2)}_-]^{(1)}, [G^{(1)}_+]^{(2)} \right),
\]

and

\[
H_1 = \begin{pmatrix} 0 & 0 \\ r_2e^{-iky/2} & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & -r_1e^{iky/2} \\ 0 & 0 \end{pmatrix},
\]

\[
H_0 = \left( C_- [r_2e^{-iky/2}], C_+ [(2 - \eta)r_1e^{iky/2}] \right).
\]

Thus from the jump of \( N^{(2)} \) we obtain

\[
N^{(2)} (I - H_2) = C_- \left[ N^{(2)} (I - H_2) \tilde{G} \right] + H_0.
\]

Multiply both sides of this equation by \( \tau_j \):

\[
N^{(2)} (I - H_2) \tau_j = C_- \left[ N^{(2)} (I - H_2) \tau_j G \right] + H_0 \tau_j \\
= (I - C\bar{G})^{-1} [H_0 \tau_j].
\]

So there exists a positive constant \( c(r) \) such that

\[
\|r_1e^{iky/2}([M^{(3)}]^{(2)}_+ - 1) + [M^{(3)}]^{(1)}_- - \eta + 2 \|_{L^2(\mathbb{R}_k)} \\
\leq c(r) \| C_+ [(2 - \eta)r_1e^{iky/2}] \|_{L^2(\mathbb{R}_k)},
\]

\[
\| [M^{(3)}]^{(1)}_- - \eta + 2 \|_{L^2(\mathbb{R}_k)} \leq c(r) \| C_- [r_2e^{-iky/2}] \|_{L^2(\mathbb{R}_k)}.
\]

which finally leads to

\[
\| [M^{(3)}]^{(1)}_- - \eta + 2 \|_{L^2(\mathbb{R}_k)}, \quad \sup_{y \in \mathbb{R}} \| [M^{(3)}]^{(1)}_- - \eta + 2 \|_{L^2(\mathbb{R}_k)},
\]

\[
\| [M^{(3)}]^{(2)}_+ - 1 \|_{L^2(\mathbb{R}_k)}, \quad \sup_{y \in \mathbb{R}} \| [M^{(3)}]^{(2)}_+ - 1 \|_{L^2(\mathbb{R}_k)} \leq c(r) \| r \|_{H^{1,2}(\mathbb{R}_k)}.
\]
Consider the $y$-derivative of $N^{(2)}$, we have that

$$\partial_y \left[ N^{(2)}(I - H_2)\tau_j \right] = \partial_y \left[ N^{(2)} \right] (I - H_2)\tau_j + \frac{ik}{2} N^{(2)} H_2\tau_j$$

$$= C_- \left[ \partial_y \left[ N^{(2)} \right] (I - H_2)\tau_j G \right] + C_- \left[ \frac{ik}{2} N^{(2)} h_2\tau_j G \right] + C_- \left[ N^{(2)} (I - H_2)\tau_j \partial_y [G] \right] + \partial_y [H_0] \tau_j. $$

So

$$\partial_y \left[ N^{(2)} \right] (I - H_2)\tau_j = (I - C G)^{-1} \left[ C_- \left[ \frac{ik}{2} N^{(2)} h_2\tau_j G \right] + C_- \left[ N^{(2)} (I - H_2)\tau_j \partial_y [G] \right] \right]$$

$$+ (I - C G)^{-1} \left[ \partial_y [H_0] \tau_j - \frac{ik}{2} N^{(2)} H_2\tau_j \right].$$

Therefore, same as above estimation, we can also obtain that

$$\| \partial_y [N^{(2)}]^{-1} \|_{L^2(\mathbb{R}_+)} \sup_{y \in \mathbb{R}} \| \partial_y [N^{(2)}]^{-1} \|_{L^2(\mathbb{R}_+)} \leq c(r) \| r \|_{H^{1.2}(\mathbb{R}_+)}.$$  

(5.54)

On the other hand, consider

$$M^{(4)}(z) = \begin{pmatrix} [M]_{11} & \frac{z^2 + 1}{z^2 + 1}[M]_{12} \\ \frac{z^2 + 1}{z^2 + 1}[M]_{21} & [M]_{22} \end{pmatrix}. $$

(5.55)

Note that $[M]_{12}(i) = [M]_{21}(0) = 0$, so $z = 0$ and $z = i$ are not the pole of $M^{(4)}$.

Remark: In fact, from the definition of $M(z)$, we have that $M^{(4)}(z)$ is constructed by $\varphi_{\pm, j}$ (defined in subsection 5.1) with same structure with $M(z)$. However, in the analysis of inverse scattering, we only consider the condition of RH problem. Since

$$\theta(k) = \theta(z(k)) = \frac{i}{4} ky,$$

then by the symmetry of $M(z)$, $M^{(4)}(k) = M^{(4)}(z(k))$ is well defined, and it admits the following RH problem
RHP 7. Find a matrix-valued function $M^{(4)}(k) = M^{(4)}(k; y)$ which satisfies

- **Analyticity:** $M^{(4)}(k)$ is analytical in $\mathbb{C} \setminus \mathbb{R}$;

- **Jump condition:** $M^{(4)}$ has continuous boundary values $M^{(4)}_\pm(k)$ on $\mathbb{R}$ and
  \[
  M^{(4)}_+(k) = M^{(4)}_- (k) \tilde{V}(k), \quad k \in \mathbb{R};
  \] (5.56)

- **Asymptotic behaviors:**
  \[
  M^{(4)}(k) = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} + \mathcal{O}(k^{-1}), \quad k \to \infty \text{ in } \mathbb{C}^\pm,
  \] (5.57)
  where $\eta = \lim_{z \to \infty} [z M]_{21} = \frac{m-1}{m}$.

- **Singularity:** $M^{(4)}(z)$ has singularity at $k = 0$ with
  \[
  M^{(4)}(k) = \frac{i\alpha_+(y)}{k} \begin{pmatrix} -c & \frac{1}{2} \\ -2c & 1 \end{pmatrix} + \mathcal{O}(1), \quad k \to 0 \text{ in } \mathbb{C}^+.
  \] (5.58)

Here, in generic case, $c = 0$, while in non-generic case, $c \neq 0$.

To remove the singularity, we reduce this RH problem to a regular vector one:

\[
M^{(5)}(k; y) = \begin{pmatrix} -2 & 1 \end{pmatrix} M^{(4)}(k; y). \] (5.59)

We can easily find that admits same RHP 5 as $M^{(3)}$. Then by the existence and uniqueness of the solution of RHP 5, we arrive at that $M^{(5)}(k; y) = M^{(3)}(k; y)$. In fact, it also can be obtained by the construction of $M^{(5)}(k; y)$ and $M^{(3)}(k; y)$ through Jost function.

Denote $b_1 = \lim_{k \to \infty} ([M^{(5)}]^{(2)} + 1)$ where $[M^{(5)}]^{(2)}$ is the second item of $M^{(5)}(k; y)$. Thus,

\[
b_1 = \lim_{k \to \infty} \left( C \left[ -(2 - \eta) r_1 e^{iky/2} \right] (k) + C \left[ - \left( [M^{(5)}]^{(1)}_1 - 2 + \eta \right) r_1 e^{iky/2} \right] \right)
= \frac{1}{2\pi i} \left( (2 - \eta) \int_{\mathbb{R}} r_1 e^{iky/2} \, dk + \int_{\mathbb{R}} \left( [M^{(5)}]^{(1)}_1 - 2 + \eta \right) r_1 e^{iky/2} \, dk \right). \] (5.60)

Combing above estimations we have

\[|b_1| \leq c(r) \parallel r \parallel_{H^1_{\pm}}.\] (5.61)

It implies $b_1$ is uniform bounded with respect to $y$. Moreover, recall the definition of $G^{(2)}$ in (5.29), we has

\[
b_1 = \lim_{k \to \infty} \left[ k G^{(2)} \right]^{(2)}.
\] (5.62)
5.3 \(L^2\)-estimate on the solution

We shall now recover \(m(x)\) from RH problem in above subsection. Recall the asymptotic expression of Jost function in subsection 5.1:

\[
\frac{2i\text{Im}_x}{m^3} = \lim_{k \to \infty} \left[ k[N(2)]^{(1)} \right],
\]

\[
b_+ = \frac{i}{2} \int_{+\infty}^{x} \frac{m(s)^2 - 1}{m(s)} ds = \lim_{k \to \infty} \left[ k[N(2)]^{(2)} \right].
\]

For the first reconstruction formula, combing the jump condition and the integral equation of \(M(3)(k)\) we arrive at

\[
\lim_{k \to \infty} [k[N(2)]^{(1)}] = - \int_{\mathbb{R}} r_2(s)e^{-isy/2} ds - \int_{\mathbb{R}} r_2(s)[N(2)]^{(2)} e^{-isy/2} ds.
\]

So

\[
\| m_x \|_{L^2_y, 1} \leq \int_{\mathbb{R}} r_2(s)e^{-isy/2} ds \|_{L^2_y, 1} + \| \int_{\mathbb{R}} r_2(s)[N(2)]^{(2)} e^{-isy/2} ds \|_{L^2_y, 1}.
\]

For the second integral, (5.50) gives

\[
\int_{\mathbb{R}} r_2(s)[N(2)]^{(2)} e^{-isy/2} ds = (2 - \eta) \int_{\mathbb{R}} r_2(s)e^{-isy/2} C_+ \left[ r_1 e^{iky/2} \right] ds
\]

\[
+ \int_{\mathbb{R}} r_2(s)e^{-isy/2} C_+ \left[ [N(2)]^{(1)} r_1 e^{iky/2} \right] ds.
\]

For convenience we denote

\[
I_1(y) = (2 - \eta) \int_{\mathbb{R}} r_2(s)e^{-isy/2} C_+ \left[ r_1 e^{iky/2} \right] ds,
\]

\[
I_2(y) = \int_{\mathbb{R}} r_2(s)e^{-isy/2} C_+ \left[ [N(2)]^{(1)} r_1 e^{iky/2} \right] ds,
\]

then Lemma 20 implies that \(I_1(y) \in L^{2,1}(\mathbb{R}_y)\). And further with Lemma 20 and (5.53), we obtain that

\[
\left\| \langle y \rangle^2 I_2(y) \right\| \leq \left\| \langle y \rangle[N(2)]^{(1)} r_1 \|_{L^2_k} \right\| \left\langle \langle y \rangle C_+ [r_2 e^{-iky/2}] \right\|_{L^2_k} \leq c(r) \| r \|_{H^{1,2}_k}.
\]

46
And the first integral can be controlled by \( \| r \|_{H_k^{1,2}} \) from calculation through Fourier transformation. Consider the \( y \)-derivative of the first reconstruction formula, we obtain that

\[
\partial_y \left[ \lim_{k \to \infty} [k[N^{(2)}]^{(1)}] \right] = \int_{\mathbb{R}} \frac{is}{2} r_2(s) e^{-isy/2} ds + \int_{\mathbb{R}} \frac{is}{2} r_2(s) [N^{(2)}]^{(2)} e^{-isy/2} ds - \int_{\mathbb{R}} r_2(s) e^{-isy/2} \partial_y [N^{(2)}]^{(2)} ds.
\]

Similarly as above calculation, and note that \( kr \in H_k^{1,1} \), there also has

\[
\left| \langle y \rangle \partial_y \left[ \lim_{k \to \infty} [k[N^{(2)}]^{(1)}] \right] \right| \leq c(r) \| r \|_{H_k^{1,2}}. \tag{5.67}
\]

And for the second reconstruction formula, as shown in above subsection, \( b_+ \) is uniform bounded with respect to \( y \) with a positive constant \( c(r) \) relying on \( r \):

\[
|b_+| \leq c(r) \| r \|_{H_k^{1,2}}. \tag{5.68}
\]

Similarly, combing the jump condition and the integral equation of \( M^{(3)} \), there has

\[
\| b_+ \|_{L_y^{2,1}} \leq c(r) \| r \|_{H_k^{1,2}}. \tag{5.69}
\]

Moreover, as we proved in previous section 4.2, above estimation of integral norm also holds in the variable \( x \). To obtain the integral property of \( m - 1 \) under the variable \( x \), we consider

\[
\partial_x b_+ = \frac{m^2 - 1}{m}, \quad \partial^2_x b_+ = \frac{m^2 + 1}{m^2} m_x.
\]

Then by Sobolev-Gagliardo-Nirenberg inequality, we have that \( \partial_x b_+ \) is in \( L_x^{3/2} \). And to arrive at the weighted integrable property, we consider

\[
\tilde{b}(x) = x b_+,
\]

with

\[
\partial_x \tilde{b} = x \frac{m^2 - 1}{m} + b_+, \quad \partial^2_x \tilde{b} = 2 \frac{m^2 - 1}{m} + x \frac{m^2 + 1}{m^2} m_x. \tag{5.70}
\]

Similarly by Sobolev-Gagliardo-Nirenberg inequality, we has \( x \frac{m^2 - 1}{m} \) is in \( L_x^{2} \). Finally, combing all above estimations and lemma, we have the following Proposition.
Proposition 11. For any $r(z; 0) \in H^{1,2}(\mathbb{R}_x)$, $m - 1$, $m_x$ and $m_{xx}$ belong to $L^2(\mathbb{R}_y)$, moreover

$$\|m_x\|_{L^2_2}, \|m_{xx}\|_{L^2_2}, \|m - 1\|_{L^2_2} \leq c(r) \|r(z; 0)\|_{H^{1,2}}. \tag{5.71}$$

As we discussed in the above Section, for every $t > 0$, time evolution of $r$ in (4.7) implies that $r(z; t)$ remains in $H^{1,2}(\mathbb{R}_x)$. And for $m(t, x)$ recovered from the scattering data $r(z; t)$ with the inverse scattering transform, for every $t$, we obtain that $m(t, x) - 1 \in H^{2,1}(\mathbb{R}_x)$. Combing with the Proposition 10 and Lemma 13 in Subsection 4.2, we also have the continuity of $m$ respect to $t$.

Proposition 12. Supposed that the initial-value $m(t, 0) - 1 \in H^{2,1}(\mathbb{R})$ with small norm, then there exists a unique global solution $m(t, x) - 1 \in C([0, +\infty); H^{2,1}(\mathbb{R}))$ to the Cauchy problem (1.7)-(1.8) of the mCH equation.

With Lemma 1 and the definition of $m = v - v_{xx} + 1$ with $u(x, t) = v(x - t, t) + 1$, we finally obtain that

Theorem 1. Supposed that the initial-value $(u_0(x), m_0(x) - 1) \in H^4(\mathbb{R}) \times H^{2,1}(\mathbb{R})$ defined in (2.14) with small norm, then there exists a unique global solution $(u(t, x), m(t, x) - 1) \in C([0, +\infty); H^4(\mathbb{R}) \times H^{2,1}(\mathbb{R}))$ to the Cauchy problem (1.1) and (1.5) of the mCH equation.

Acknowledgements
This work is supported by the National Science Foundation of China (Grant No. 11671095, 51879045).

Data Availability Statements
The data that supports the findings of this study are available within the article.

Conflict of Interest
The authors have no conflicts to disclose.
References

[1] R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett., 71 (1993), 1661-1664.

[2] B. Fuchssteiner and A. Fokas, Symplectic structures, their Backlund transforms and hereditary symmetries, Phys. D, 4 (1981), 47-66.

[3] A. Constantin, On the scattering problem for the Camassa-Holm equation, Proc. Royal Soc. A. 457(2001), 953-970.

[4] A. Constantin and A. Strauss, Stability of peakons, Commun. Pure Appl. Math., 153(2000), 0603-0610.

[5] F. Gesztesy and H. Holden, Algebro-geometric solutions of the Camassa-Holm hierarchy, Rev. Mat. Iberoamericana 19(2003), 73-142.

[6] F. Gesztesy and H. Holden, Soliton equations and their algebro-geometric solutions, Volume I: (1+1)-dimensional continuous models, Cambridge University Press, (2003).

[7] Z. J. Qiao, The Camassa-Holm hierarchy, N-dimensional integrable systems, and algebro-geometric solution on a symplectic submanifold, Commun. Math. Phys. 239(2003), 309-341

[8] A. Boutet de Monvel, A. Its and D. Shepelsky, Painleve-type asymptotics for Camassa-Holm equation, SIAM J. Math. Anal. 42(2010), 1854-1873.

[9] A. Minakov, Riemann-Hilbert problem for Camassa-Holm equation with step-like initial data, J. Math. Anal. Appl. 429 (2015), 81-104.

[10] J. Eckhardt and G. Teschl, On the isospectral problem of the dispersionless Camassa-Holm equation, Adv. Math. 235(2013), 469-495.

[11] J. Eckhardt, The inverse spectral transform for the conservative Camassa-Holm flow with decaying initial data, Arch. Rational Mech. Anal. 224(2017), 21-52.
[12] J. Eckhardt and A. Kostenko, The inverse spectral problem for periodic conservative multi-peakon solutions of the Camassa-Holm equation, Int. Meth. Res. Not. 16(2020), 5126-5151.

[13] A. Boutet de Monvel and D. Shepelsky, Riemann-Hilbert approach for the Camassa-Holm equation on the line, Comptes Rendus Mathematique, 343(2006), 627-632.

[14] A. Boutet de Monvel and D. Shepelsky, Riemann-Hilbert problem in the inverse scattering for the Camassa-Holm equation on the line, Math. Sci. Res. Inst. Publ., 55(2007), 53-75.

[15] V. Novikov, Generalizations of the Camassa-Holm type equation, J. Phys. A 42(2009), 342002,14pp.

[16] A. S. Fokas, On a class of physically important integrable equations, Phys. D, 87(1995), 145-150.

[17] B. Fuchssteiner, Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa-Holm equation, Physica D, 95(1996), 229-243.

[18] P. J. Olver and P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, Phys. Rev. E., 53 (1996), 1900-1906.

[19] Z. J. Qiao, A new integrable equation with cuspons and W/M-shape-peaks solitons, J. Math. Phys., 47(2006), 112701.

[20] Y. Hou, E. G. Fan and Z. J. Qiao, The algebro-geometric solutions for the Fokas-Olver-Rosenau-Qiao (FORQ) hierarchy, J. Geom. Phys., 117(2017), 105-133.

[21] X. K. Chang, X. B. Hu and J. Szmigielski, Multipeakons of a two-component modified Camassa-Holm equation and the relation with the finite Kac-van Moerbeke lattice, Adv. Math., 299(2016), 1-35.
[22] X. K. Chang, J. Szmigielski and J. Lax integrability and the peakon problem for the modified Camassa-Holm equation, Commun. Math. Phys. 358(2018), 295-341.

[23] G. L. Gui, Y. Liu, P. J. Olver and C. Z. Qu, Wave-breaking and peakons for a modified Camassa-Holm equation, Comm. Math. Phys., 319(2013), 731-759.

[24] G. H. Wang, Q. P. Liu and H. Mao, The modified Camassa-Holm equation: Backlund transformation and nonlinear superposition formula, J. Phys. A, 53 (2020) 294003 (15pp).

[25] T. Schäfer and C. E. Wayne, Propagation of ultra-short optical pulses in cubic nonlinear media, Phys. D., 196 (2004), 90-105.

[26] A. Boutet de Monvel, D. Shepelsky and L. Zielinski, The short pulse equation by a Riemann-Hilbert approach, Lett. Math. Phys., 107(2017), 1-29.

[27] D. Pelinovsky and A. Sakovich, Global well-posedness of the short-pulse and sine-Gordon equations in energy space, Comm. Partial Differential Equations, 35(2010), 613-629.

[28] Z. Qiao and X. Q. Li, An integrable equation with nonsmooth solitons, Theor. Math. Phys., 267(2011), 584-589.

[29] R. M. Chen, Y Liu, C. Z. Qu and S. H. Zhang, Oscillation-induced blow-up to the modified Camassa-Holm equation with linear dispersion, Adv. Math., 272(2015), 225-251.

[30] Y. Matsuno, Smooth and singular multisoliton solutions of a modified Camassa-Holm equation with cubic nonlinearity and linear dispersion, J. Phys. A: Math. Theor. 47 (2014) 125203.

[31] R. I. Ivanov and T. Lyons, Dark solitons of the Qiao’s hierarchy, J. Math. Phys. 53(2012), 123701.
[32] Y. Matsuno, Backlund transformation and smooth multisoliton solutions for a modified Camassa-Holm equation with cubic nonlinearity, J. Math. Phys. 54(2013), 051504.

[33] A. Boutet de Monvel, I. Karpenko and D. Shepelsky, A Riemann-Hilbert approach to the modified Camassa-Holm equation with nonzero boundary conditions, J. Math. Phys., 61(2020), 031504.1-25.

[34] A. Boutet de Monvel, I. Karpenko and D. Shepelsky, The modified Camassa-Holm equation on a nonzero background: Large-time asymptotics for the Cauchy problem, arXiv, 2011.13235v1, 2020.

[35] J. Xu and E. G. Fan, Long-time asymptotics behavior for the integrable modified Camassa-Holm equation with cubic nonlinearity, arXiv 1911.12554.

[36] Y. L. Yang and E. G. Fan, On the long-time asymptotics of the modified Camassa-Holm equation in space-time solitonic regions, Adv. Math., 402(2022), 108340.

[37] J. L. Yin, L. X. Tian and X. H. Fan, Stability of negative solitary waves for an integrable modified Camassa-Holm equation. J. Math. Phys., 51(2010), 053515.

[38] X. L. Wu, The Cauchy problem of the modified CH and DP equations. J. Appl. Math, 80(2015), 906-930.

[39] C. Z. Qu, X. C. Liu and Y. Liu, Stability of peakons for an integrable modified Camassa-Holm equation with cubic nonlinearity, Commun. Math. Phys. 322(2013), 967-997.

[40] X. C. Liu, Y Liu and C. Z. Qu, Orbital stability of the train of peakons for an integrable modified Camassa-Holm equation, Adv. Math., 255(2014), 1-37.

[41] Y. Gao and J. G. Liu, The modified Camassa-Holm equation in Lagrangian coordinates, Discrete Contin. Dyn. Syst. Ser. B, 23(2018), 2545-2592.
[42] D. E. Pelinovsky, Y. Shimabukuro, Existence of global solutions to the derivative NLS equation with the inverse scattering transform method, International Mathematics Research Notices, 18(2018), 5663-5728.

[43] X. Zhou, $L^2$-Sobolev space bijectivity of the scattering and inverse scattering transforms, Commun. Pure Appl. Math., 20(1989), 697-731.

[44] P. Deift and X. Zhou, Long-time behavior of the non-focusing nonlinear Schrödinger equation-a case study, Lectures in Mathematical Sciences, Graduate School of Mathematical Sciences, University of Tokyo, 1994.

[45] P. Deift and X. Zhou, Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space, Comm. Pure Appl. Math., 56(2003), 1029-1077.

[46] V. Kotlyaroy and D. Shepelsky, Planar unimodular Baker-Akhiezer function for the nonlinear Schrödinger equation, Ann. Math. Sci. Appl., 2(2017), 343-384.

[47] H. Krüger and G. Teschl, Long-time asymptotics of the Toda lattice for decaying initial data revisited, Rev. Math. Phys., 21(2009), 61-109.

[48] P. Deift, X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Ann. Math., 137(1993), 295-368.

[49] X. Zhou, The Riemann-Hilbert problem and inverse scattering. SIAM Journal of Mathematical Analysis, 20(1989), 966-86.

[50] R. Beals and R. R. Coifman, Scattering and inverse scattering for first order systems. Comm. Pure and Appl. Math., 37(1984), 39-90.

[51] R. Beals and R. R. Coifman, Inverse scattering and evolution equations. Communications Comm. Pure and Appl. Math., 38(1985), 29-42.