ON STRONGLY SEPARATELY CONTINUOUS FUNCTIONS
ON SEQUENCE SPACES

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Abstract. We study strongly separately continuous real-valued function defined on the Banach spaces \( \ell_p \).

1. Introduction

Let \((X_t : t \in T)\) be a family of sets \(X_t\) with \(|X_t| > 1\) for all \(t \in T\). For \(S \subseteq T\) we put \(X_S = \prod_{t \in S} X_t\). If \(S \subseteq S_1 \subseteq T\), \(x = (x_t)_{t \in T} \in X_T\), \(a = (a_t)_{t \in S_1} \in X_{S_1}\), then we denote by \(x_S^a\) the point \((y_t)_{t \in T} \in X_T\) defined by

\[
y_t = \begin{cases} a_t, & t \in S, \\ x_t, & t \in T \setminus S. \end{cases}
\]

In the case of \(S = \{s\}\) we write \(x_s^a\) instead of \(x_{s_1}^{a_1}\).

If \(S \subseteq T\), then we put \(\pi_S(x) = (x_t)_{t \in S}\).

For \(n \in \mathbb{N}\) let \(\sigma_n(x) = \{y = (y_t)_{t \in T} \in X_T : |\{t \in T : y_t \neq x_t\}| \leq n\}\) and \(\sigma(x) = \bigcap_{n=1}^{\infty} \sigma_n(x)\). Each of the sets of the form \(\sigma(x)\) we call a \(\sigma\)-product of \(X_T\).

A set \(X \subseteq X_T\) is called \(S\)-open [1] if \(\sigma_1(x) \subseteq X\) for all \(x \in X\). Notice that the definition of an \(S\)-open set develops the definition of a set of the type \((P_1)\) introduced in [1]. Observe that \(\sigma\)-products of two distinct points of \(X_T\) either coincide, or do not intersect. Thus, the family of all \(\sigma\)-products of an arbitrary \(S\)-open set \(X \subseteq X_T\) generates a partition of \(X\) on mutually disjoint \(S\)-open sets, which we will call \(S\)-components of \(X\).

Let \(X \subseteq X_T\) be an \(S\)-open set, \(\tau\) be a topology on \(X\) and let \((Y,d)\) be a metric space. A mapping \(f : (X, \tau) \to Y\) is called strongly separately continuous at a point \(x \in X\) with respect to the \(t\)-th variable if

\[
\lim_{x \to a, t} d(f(x), f(x_t^a)) = 0.
\]

A mapping \(f : X \to Y\) is strongly separately continuous at a point \(a \in X\) if \(f\) is strongly separately continuous at \(a\) with respect to every variable \(t \in T\); and \(f\) is strongly separately continuous on the set \(X\). A mapping \(f : X \to Y\) is strongly separately continuous at every point \(a \in X\) with respect to every variable \(t \in T\). Strongly separately continuous functions we will also call ssc-functions for short.

The notion of real-valued ssc-function defined on \(\mathbb{R}^n\) was introduced by Omar Dzagnidze [2], who proved that a function \(f : \mathbb{R}^n \to \mathbb{R}\) is strongly separately continuous at \(x^0\) if and only if \(f\) is continuous at \(x^0\).

In [1] the authors extended the notion of the strong separate continuity to functions defined on the Hilbert space \(\ell_2\) equipped with the norm topology. They proved that there exists a real-valued ssc-function on \(\ell_2\) which is everywhere discontinuous. Determining sets (see Definition [2]) for for the class of all ssc-functions were also studied in [1].

The second named author continued to study properties of strongly separately continuous functions on \(\ell_2\) in [3] and constructed a strongly separately continuous function \(f : \ell_2 \to \mathbb{R}\) which belongs to the third Baire class and is not quasi-continuous at every point. Moreover, he gave a sufficient condition for a strongly separately continuous function to be continuous on \(\ell_2\) and a sufficient condition for a subset of \(\ell_2\) to be determining in the class of real-valued ssc-functions.

The first named author extended the concept of an ssc-function on any \(S\)-open subset of a product of topological spaces [4]. A characterization of the set of all points of discontinuity of strongly separately continuous functions...
continuous functions defined on a σ-product of a sequence of finite-dimensional normed spaces was given in [3]. Further, the Baire classification of ssc-functions defined on the space $\mathbb{R}^\omega$ equipped with the topology of pointwise convergence was investigated in [3]. Moreover, it was shown in [5] that if $X$ is a product of normed spaces and $a \in X$ then for any open set $G \subseteq \sigma(a)$ there is a strongly separately continuous function $f : \sigma(a) \to \mathbb{R}$ such that the discontinuity point set of $f$ is equal to $G$. Strongly separately continuous functions defined on a box-product of topological spaces were considered in [3].

Here we study ssc-functions defined on the spaces $\ell_p$ with $1 \leq p < +\infty$ of all sequences $(x_n)_{n=1}^\infty$ of real numbers for which the series $\sum_{n=1}^\infty |x_n|^p$ is convergent. In Section 2 we find a necessary and sufficient condition on a subset of $\ell_p$ to be determining in the class of all real-valued ssc-functions on $\ell_p$. In the third section we show that for any ordinal $\alpha \in [1, \omega_1)$ there exists an $\mathcal{S}$-open set $E \subseteq \ell_p$ which is of the $\alpha$th multiplicative Borel class and does not belong to the $\alpha$th additive Borel class. Using this fact we construct a real-valued ssc-function on $\ell_p$ which belongs to the $(\alpha + 1)$th Baire class and does not belong to the $\alpha$-th Baire class. In Section 3 we prove that for any open nonempty set $G \subseteq \ell_p$ and $1 \leq p < \infty$ there exists a strongly separately continuous function $f : \ell_p \to \mathbb{R}$ which is discontinuous exactly on $G$.

2. Determining sets for strongly separately continuous functions

Let $(X, Y)$ be a pair of topological spaces and $\mathcal{F}(X, Y)$ be a class of mappings between $X$ and $Y$.

**Definition 2.1.** A set $E \subseteq X$ is called determining for the class $\mathcal{F}(X, Y)$ if for any mappings $f, g \in \mathcal{F}(X, Y)$ the equality $f|_E = g|_E$ implies that $f = g$ on $X$.

It is well-known that any everywhere dense subset $E$ of a topological space $X$ is determining in the class $C(X, \mathbb{R})$ of all continuous real-valued functions on $X$. The theorem of Sierpiński [7] tells us that any everywhere dense subset $E \subseteq \mathbb{R}^2$ is determining in the class $CC(\mathbb{R}^2, \mathbb{R})$ of all separately continuous real-valued functions on $\mathbb{R}^2$.

In this section we give necessary and sufficient conditions on a subset of an $\mathcal{S}$-open set $X \subseteq \prod_{n=1}^N X_n$ to be determining for the class $\text{SSC}(X)$ of all strongly separately continuous real-valued functions on $X$.

The following two notions were introduced in [4].

**Definition 2.2.** A set $A \subseteq X_T$ is called projectively symmetric with respect to a point $a \in A$ if $x_T^a \in A$ for all $t \in T$ and $x \in A$.

**Definition 2.3.** Let $X \subseteq X_T$ and $\tau$ be a topology on $X$. Then $(X, \tau)$ is said to be locally projectively symmetric if every $x \in X$ has a base of projectively symmetric neighborhoods with respect to $x$.

**Definition 2.4.** Let $(X_t : t \in T)$ be a family of topological spaces and $X \subseteq X_T$ be an $\mathcal{S}$-open set. We say that a topology $\tau$ on $X$ is finitely generated if for every $a \in X$ and every finite set $S \subseteq T$ the space $(X_S \times_{t \in T \setminus S} \{a_t\}, \tau)$ is homeomorphic to the space $X_S$ with the topology of pointwise convergence.

Notice that an arbitrary $\mathcal{S}$-open subset of a product $X_T$ of topological spaces $X_t$ equipped with the topology of pointwise convergence is a locally projectively symmetric space with a finitely generated topology. All classical spaces of sequences as the space $\mathcal{C}$ of all convergence sequences or the spaces $\ell_p$ with $0 < p \leq \infty$ are locally projectively symmetric with a finitely generated topology.

Throughout the paper we consider only finitely generated topologies. We need the following result [4] Theorem 4.4.

**Theorem 2.1.** Let $X \subseteq X_T$ be an $\mathcal{S}$-open set equipped with a locally projectively symmetric topology $\tau$ and $x_0 \in X$.

(i) If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are ssc-functions at $x_0$, then $f(x) \pm g(x)$, $f(x) \cdot g(x)$, $|f(x)|$, $\min\{f(x), g(x)\}$ and $\max\{f(x), g(x)\}$ are ssc-functions at $x_0$.

(ii) If $(f_n)_{n=1}^\infty$ is a sequence of ssc-functions at the point $x_0$ and the series $\sum_{n=1}^\infty f_n(x)$ is convergent uniformly on $X$, then the sum $f(x)$ is an ssc-function at $x_0$.

(iii) If $(f_i)_{i \in I}$ is a locally finite family of ssc-functions $f_i : X \to \mathbb{R}$ at $x_0$, then $f(x) = \sum_{i \in I} f_i(x)$ is an ssc-function at $x_0$.

It is worth noting that the quotient of two real-valued ssc-functions need not be an ssc-function [3].
Definition 2.5. Let \((X_t : t \in T)\) be a family of topological spaces \(X_t\) and \(a \in X_T\). A set \(W \subseteq \sigma(a)\) is called nearly open in \(\sigma(a)\) if for any finite set \(T_0 \subseteq T\) the set
\[
W_{T_0} = \{ x \in X_{T_0} : a_{T_0}^x \in W \}
\]
is open in the space \(X_{T_0}\) equipped with the topology of pointwise convergence.

Definition 2.6. Let \((X_t : t \in T)\) be a family of topological spaces \(X_t\), \(X \subseteq X_T\) be an \(S\)-open set and \((\sigma_i : i \in I)\) be a partition of \(X\) on \(S\)-components. A set \(W \subseteq X\) is called nearly open in \(X\) if \(W \cap \sigma_i\) is nearly open in \(\sigma_i\) for every \(i \in I\).

Definition 2.7. Let \(X \subseteq X_T\) be an \(S\)-open set. A set \(H \subseteq X\) is nearly open in \(X\) if the complement \(X \setminus H\) is nearly open in \(X\).

For a set \(A \subseteq \sigma(a)\) we put
\[
\overline{A} = \{ x \in \sigma(a) : W \cap A \neq \emptyset \ \forall W - \text{ nearly open}, x \in W \}.
\]

Definition 2.8. A set \(A \subseteq \sigma(a)\) is said to be super dense (\(p\)-dense) in \(\sigma(a)\) if \(\overline{A} = \sigma(a)\) (respectively, \(\overline{A}^0 = \sigma(a)\), where \(\overline{A}^0\) stands for the closure of \(A\) in the topology of pointwise convergence on \(\sigma(a)\)).

Definition 2.9. A subset \(A\) of an \(S\)-open set \(X = \bigcup_{i \in I} \sigma_i \subseteq X_T\), where \((\sigma_i : i \in I)\) is a family of all \(S\)-components of \(X\), is called super dense in \(X\) if for every \(i \in I\) the set \(A_i = A \cap \sigma_i\) is super dense in \(\sigma_i\) whenever \(A_i \neq \emptyset\).

Observe that \(A\) is nearly closed if and only if \(\overline{A}^0 = A\). Notice also that every super dense set is \(p\)-dense.

Theorem 2.2. [6 Theorem 1] Let \((X_t)_{t \in T}\) be a family of topological spaces, \(a \in X_T\), \(\tau\) be a topology on \(\sigma(a)\) and \(f : \sigma(a) \to \mathbb{R}\) be an ssc-function. Then \(f^{-1}(V)\) is nearly open in \(\sigma(a)\) for any open set \(V \subseteq \mathbb{R}\).

Proposition 2.3. Let \((X_t : t \in T)\) be a family of topological spaces and \(E\) be a super dense set in an \(S\)-open space \(X \subseteq X_T\) equipped with a locally projectively symmetric topology \(\tau\). Then \(E\) is determining set for the class \(SSC(X)\).

Proof. Consider ssc-functions \(f, g : X \to \mathbb{R}\) such that \(f|_E = g|_E\) and let \((\sigma_i : i \in I)\) be a partition of \(X\) on mutually disjoint \(S\)-components. Denote
\[
H = \{ x \in X : f(x) - g(x) = 0 \}
\]
and for every \(i \in I\) we put \(E_i = E \cap \sigma_i\) and \(H_i = H \cap \sigma_i\). Since the function \(h(x) = f(x) - g(x)\) is strongly separately continuous on \(X\) by Theorem 2.1 every set \(H_i = (h^{-1}(0)) \cap \sigma_i\) is nearly closed in \(\sigma_i\) according to Theorem 2.2. Then the inclusion \(E_i \subseteq H_i\) implies that \(\sigma_i = E_i^\circ \subseteq H_i^\circ = H_i\). Consequently, \(H_i = \sigma_i\) for every \(i \in I\). Therefore, \(H = \bigcup_{i \in I} H_i = \bigcup_{i \in I} \sigma_i = X\).

Lemma 2.4. Let \((X_n)_{n=1}^\infty\) be a sequence of locally compact Hausdorff spaces, \(a \in \prod_{n=1}^\infty X_n\) and \(W \subseteq \sigma(a)\) be a nearly open set. Then for every \(x \in W\) there exists a sequence \((U_n)_{n=1}^\infty\) of functionally open sets \(U_n \subseteq X_n\) such that \(x \in \left( \prod_{n=1}^N U_n \right) \cap \sigma(a) \subseteq W\).

Proof. For every \(n \in \mathbb{N}\) we denote \(Y_n = \prod_{k=1}^n X_k\) and \(W_n = W_{[1,\ldots,n]} = \{ x \in Y_n : a^x_{[1,\ldots,n]} \in W \}\). Take a number \(N\) such that \(x_n = a_n\) for all \(n > N\). Notice that \(W_n\) is an open subset of the locally compact Hausdorff (and, consequently, completely regular) space \(Y_n\). Therefore, for every \(n = 1, \ldots, N\) there exists a functionally open neighborhood \(U_n\) of \(x_n\) with compact closure such that \(K_1 = \prod_{n=1}^N U_n \subseteq W_N\). For every \(x \in K_1\) we take a functionally open neighborhood \(V_x \times G_x\) of \((x_1, \ldots, x_N, a_{N+1})\) with compact closure in \(Y_N \times X_{N+1}\) such that \((x_1, \ldots, x_N, a_{N+1}) \in V_x \times G_x \subseteq W_{N+1}\). Since the set \(K_1 \times \{ a_{N+1} \}\) is compact, there exists a finite set \(I \subseteq K_1\) such that \(K_1 \times \{ a_{N+1} \} \subseteq \bigcup_{x \in I} (V_x \times G_x)\). Put \(U_{N+1} = \bigcap_{x \in I} G_x\). Then \(U_{N+1}\) is a functionally open neighborhood of \(a_{N+1}\) in \(X_{N+1}\) and \(K_2 = K_1 \times U_{N+1} \subseteq W_{N+1}\). Proceeding inductively in this way we obtain a sequence \((U_n)_{n=1}^\infty\) of functionally open sets \(U_n \subseteq X_n\) with \(x \in \left( \prod_{n=1}^N U_n \right) \cap \sigma(a) \subseteq W\).

The following result follows from [6 Lemma 2].
Proposition 2.5. Let \((X_n)_{n=1}^{\infty}\) be a sequence of topological spaces, \(a \in \prod_{n=1}^{\infty} X_n\), \((U_n)_{n=1}^{\infty}\) be a sequence of functionally open sets \(U_n \subseteq X_n\), \(W = \left( \bigcap_{n=1}^{\infty} U_n \right) \cap \sigma(a)\) and let \(\tau\) be the topology of pointwise convergence on \(\sigma(a)\). Then there exists an ssc-function \(f : (\sigma(a), \tau) \to [0,1]\) such that \(W = f^{-1}((0,1))\).

Proposition 2.6. Let \(X \subseteq \prod_{n=1}^{\infty} X_n\) be an \(S\)-open subset of the product of a sequence of locally compact Hausdorff spaces \(X_n\), \(T\) is a topology on \(X\) which is finer than the topology \(\tau\) of pointwise convergence and \(E \subseteq X\) be a determining set for the class \(SSC(X)\). Then \(E\) is super dense in \(X\).

Proof. Consider a partition \((\sigma_i : i \in I)\) of \(X\) on mutually disjoint \(S\)-open components. Assume that \(E\) is not super dense in \(X\). Then there exists \(i \in I\) such that \(0 \notin \bigcup_{\sigma_i} E \cap \sigma_i \neq \sigma_i\). Since \(W = \sigma_i \bigcap E \cap \sigma_i\) is a nonempty nearly open set in \(\sigma_i\), by Lemma 2.4 there exists a sequence \((U_n)_{n=1}^{\infty}\) of functionally open sets \(U_n \subseteq X_n\) such that \(G = \left( \bigcap_{n=1}^{\infty} U_n \right) \cap \sigma_i \subseteq W\). According to Proposition 2.5 there exists an ssc-function \(f : (\sigma_i, \tau) \to [0,1]\) such that \(G = f^{-1}((0,1))\). Since \(\tau \subseteq T\), \(f\) is strongly separately continuous on \((\sigma_i, T)\). Notice that \(f|_{E \cap \sigma_i} = 0\) and \(f(x) > 0\) for every \(x \in G\), which implies a contradiction, since \(E \cap \sigma_i\) is determining in \(\sigma_i\).

Since every space \(\ell_p\) is an \(S\)-open subset of a countable product \(\mathbb{R}^\omega\) and the standard topology on \(\ell_p\) is finer than the topology of pointwise convergence, Propositions 2.3 and 2.6 immediately imply the following result.

Theorem 2.7. For \(p \in [1, +\infty)\) a set \(E \subseteq \ell_p\) is determining for the class \(SSC(\ell_p)\) if and only if \(E\) is super dense in \(\ell_p\).

3. Baire classification of ssc-functions on \(\ell_p\)

For \(p \in [1, +\infty)\) and \(x = (x_n)_{n=1}^{\infty}\), \(y \in \ell_p\) we denote

\[
\|x\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \quad \text{and} \quad d_p(x, y) = \|x - y\|_p.
\]

Let \(\pi_n(x) = x_n\) for every \(x = (x_n)_{n=1}^{\infty} \in \ell_p\) and \(n \in \mathbb{N}\).

Lemma 3.1. For any \(\alpha \in [1, \omega_1)\) and \(p \in [1, +\infty)\) there exists an \(S\)-open set \(E\) in \(\ell_p\) such that \(E\) belongs to the \(\alpha\)'th additive class and does not belong to the \(\alpha\)'th multiplicative class.

Proof. Fix \(p \in [1, +\infty)\).

We define inductively sequences \((\tilde{A}_n)_{1 \leq \alpha < \omega_1}\) and \((\tilde{B}_\alpha)_{1 \leq \alpha < \omega_1}\) of subsets of \(\mathbb{R}^\omega\) in the following way. Put

\[
\tilde{A}_1 = \{ x = (x_n)_{n=1}^{\infty} \in \mathbb{R}^\omega : \exists m \forall n \geq m \ x_n = 0 \} \quad \text{and} \quad \tilde{B}_1 = \mathbb{R}^\omega \setminus \tilde{A}_1.
\]

Let \(\mathbb{N} = \bigcup_{n=1}^{\infty} T_n\) be a union of a sequence of mutually disjoint infinite sets \(T_n = \{ t_{n1}, t_{n2}, \ldots \}\), where \((t_{nm})_{m=1}^{\infty}\) is a strictly increasing sequence of numbers \(t_{nm} \in \mathbb{N}\). For every \(n \in \mathbb{N}\) we denote by \(\tilde{A}_1 \cap \tilde{B}_1 / f\) the copy of the set \(\tilde{A}_1 / \tilde{B}_1 / f\), which is contained in the space \(\mathbb{R}^{T_n}\). Assume that for some \(\alpha \geq 1\) we have already defined sequences \((\tilde{A}_\beta)_{1 \leq \beta < \alpha}\) and \((\tilde{B}_\beta)_{1 \leq \beta < \alpha}\) (and their copies \((\tilde{A}^n_\beta)_{1 \leq \beta < \alpha}\) and \((\tilde{B}^n_\beta)_{1 \leq \beta < \alpha}\) in \(\mathbb{R}^{T_n}\)) of subsets of \(\mathbb{R}^\omega\). Now we put

\[\tilde{A}_\alpha = \left\{ \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \pi_n^{-1}(\tilde{B}^n_\beta) \middle| \beta = \alpha + 1 \right\} \quad \text{and} \quad \tilde{B}_\alpha = \mathbb{R}^\omega \setminus \tilde{A}_\alpha.
\]

Let for every \(\alpha \in [1, \omega_1)\)

\[
A_\alpha = \tilde{A}_\alpha \cap \ell_p, \quad B_\alpha = \tilde{B}_\alpha \cap \ell_p,
\]

\[
A^n_\alpha = \tilde{A}^n_\alpha \cap \ell_p(\mathbb{R}^{T_n}), \quad B^n_\alpha = \tilde{B}^n_\alpha \cap \ell_p(\mathbb{R}^{T_n}).
\]

Claim 1. For every \(\alpha \in [1, \omega_1)\) the sets \(A_\alpha\) and \(B_\alpha\) are \(S\)-open in \(\ell_p\).
Proof of Claim 1. Evidently, $A_1$ and $B_1$ are $S$-open. Assume that for some $\alpha < \omega_1$ the claim is valid for all $\beta < \alpha$. Let $\alpha = \beta + 1$ be an isolated ordinal. Take any $x \in A_\alpha$ and $y \in \sigma_1(x)$. Then there exists $m \in \mathbb{N}$ such that $\pi_{T_\alpha}(x) \in B_\beta^n$ for all $n \geq m$. Since $\pi_{T_\alpha}(y) \in \sigma_1(\pi_{T_\alpha}(x))$ and $B_\beta^n$ is $S$-open, $\pi_{T_\alpha}(y) \in B_\beta^n$. Therefore, $y \in A_\alpha$. We argue similarly in the case where $\alpha$ is a limit ordinal.

Consider the equivalent metric

$$d(x, y) = \min\{d_p(x, y), 1\}$$

on the space $\ell_p$.

**Claim 2.** For every $\alpha \in [1, \omega_1)$ the following condition holds:

(1) for every set $C \subseteq (\ell_p, d)$ of the additive/multiplicative class $\alpha$ there exists a contracting mapping $f : (\ell_p, d) \to (\ell_p, d)$ with the Lipschitz constant $q = \frac{1}{2}$ such that

$$C = f^{-1}(A_n) / C = f^{-1}(B_n) /$$

(3.1)

$$|\pi_n(f(x))| \leq 1 \quad \forall x \in \ell_p \quad \forall n \in \mathbb{N}. \quad (3.2)$$

**Proof of Claim 2.** We will argue by the induction on $\alpha$. Let $\alpha = 1$ and $C$ be an arbitrary $F_\sigma$-subset of $(\ell_p, d)$. Then $C = \bigcup_{n=1}^\infty C_n$ is a union of an increasing sequence of closed sets $C_n \subseteq (\ell_p, d)$. For every $x \in \ell_p$ we put

$$f(x) = \left(\frac{1}{3}d(x, C_1), \ldots, \frac{1}{3^m}d(x, C_n), \ldots\right).$$

Since every $d(x, C_n) \leq 1$, $f(x) \in \ell_p$ for every $x \in \ell_p$. Show that $C = f^{-1}(A_1)$. Take $x \in C$ and choose $m \in \mathbb{N}$ such that $x \in C_n$ for all $n \geq m$. Then $d(x, C_n) = 0$ and $\pi_n(f(x)) = 0$ for all $n \geq m$. Hence, $x \in f^{-1}(A_1)$. The inverse inclusion follows from the closedness of $C_n$. Since

$$d(f(x), f(y)) \leq d_p(f(x), f(y)) = \left(\sum_{n=1}^{\infty} \frac{1}{3^m}d(x, C_n) - d(y, C_n)\right)^{\frac{1}{m}} \leq d(x, y)\left(\sum_{n=1}^{\infty} \frac{1}{3^m}\right)^{\frac{1}{m}} \leq \frac{1}{2}d(x, y)$$

for all $x, y \in \ell_p$, the mapping $f : (\ell_p, d) \to (\ell_p, d)$ is contracting with the Lipschitz constant $q = \frac{1}{2}$. Moreover, $|\pi_n(f(x))| = \frac{1}{3^m}d(x, C_n) \leq 1$ for every $n \in \mathbb{N}$.

Assume that for some $\alpha < \omega_1$ the condition (1) is valid for all $\beta < \alpha$. Let $C \subseteq (\ell_p, d)$ be any set of the $\alpha'$th additive class. Take an increasing sequence of sets $C_n$ such that $C = \bigcup_{n=1}^\infty C_n$, where every $C_n$ belongs to the multiplicative class $\beta$ if $\alpha = \beta + 1$, and in the case $\alpha = \sup \beta_n$ we can assume that $C_n$ belongs to the additive class $\beta_n$ for every $n \in \mathbb{N}$. By the inductive assumption for every $n \in \mathbb{N}$ there exists a contracting mapping $f_n : (\ell_p, d) \to (\ell_p, d)$ with the Lipschitz constant $q = \frac{1}{2}$ such that

$$C_n = \left\{\begin{array}{ll}
 f_n^{-1}(B_\beta), & \alpha = \beta + 1, \\
 f_n^{-1}(A_\beta_n), & \alpha = \sup \beta_n,
\end{array}\right.$$

(3.3)

$$|\pi_m(f_n(x))| \leq 1 \quad \forall x \in \ell_p \quad \forall n, m \in \mathbb{N}. \quad (3.4)$$

For every $k \in \mathbb{N}$ we choose a unique pair $(n(k), m(k)) \in \mathbb{N}^2$ such that

$$k = t_{n(k)m(k)} \in T_{n(k)}.$$ 

For every $x \in \ell_p$ we put

$$f(x) = \left(\frac{1}{3}f_{n(k)m(k)}(1)(x), \ldots, \frac{1}{3^{k+1}}f_{n(k)m(k)}(k)(x), \ldots\right).$$

It is easy to see that $f(x) \in \ell_p$ for every $x \in \ell_p$.

Since

$$\frac{1}{3}|f_{nm}(x) - f_{nm}(y)| \leq d_p(f_n(x), f_n(y)) \quad \text{and} \quad \frac{1}{3}|f_{nm}(x) - f_{nm}(y)| \leq \frac{2}{3} \leq 1,$$

we have

$$\frac{1}{3}|f_{nm}(x) - f_{nm}(y)| \leq d(f_n(x), f_n(y)) \leq \frac{1}{2}d(x, y)$$
for all \( x, y \in \ell_p \) and \( n, m \in \mathbb{N} \). Consequently,
\[
d(f(x), f(y)) \leq d_p(f(x), f(y)) = \left( \sum_{k=1}^{\infty} \frac{1}{3^p} \left( \frac{1}{3}|f_n(k)m(k)| - f_n(k)m(k)(y) \right) \right)^{\frac{1}{p}} \leq \left( \sum_{k=1}^{\infty} \frac{1}{3^p} \right)^{\frac{1}{p}} \leq \frac{1}{2} d(x, y)
\]
for all \( x, y \in \ell_p \). Therefore, \( f \) has the Lipschitz constant \( q = \frac{1}{2} \).

Finally, it is easy to verify that \( C = f^{-1}(A_\alpha) \).

**Claim 3.** For every \( \alpha \in [1, \omega_1) \) the set \( A_\alpha \) belongs to the additive class \( \alpha \) and does not belong to the multiplicative class \( \alpha \) in \( \ell_p \).

**Proof of Claim 3.** We first prove that the set \( \tilde{A}_\alpha = f(\sigma_\alpha) \) is of the \( \alpha \)'th additive /multiplicative/ class in \( \mathbb{R}^\omega \).

If \( \alpha = 1 \), then \( \tilde{A}_1 = \bigcup_{n=1}^{\infty} \sigma_n(0) \) is an \( F_\sigma \)-subset of \( \mathbb{R}^\omega \), since every \( \sigma_n(0) \) is closed in \( \mathbb{R}^\omega \). Consequently, \( \tilde{B}_1 \) is \( G_\delta \) in \( \mathbb{R}^\omega \). Suppose that for some \( \alpha \geq 2 \) the set \( \tilde{A}_\beta \) belongs to the additive /multiplicative/ class \( \beta \) in \( \mathbb{R}^\omega \) for every \( \beta < \alpha \). Since every projection \( \pi_{\mathbb{R}^n} : \mathbb{R}^\omega \rightarrow \mathbb{R}^n \) is continuous, the set \( \tilde{A}_\alpha \) belongs to the additive class \( \alpha \) in \( \mathbb{R}^\omega \) and the set \( \tilde{B}_\beta \) belongs to the multiplicative class \( \alpha \) in \( \mathbb{R}^\omega \).

Since the topology of pointwise convergence on \( \ell_p \) is weaker than the topology generated by the norm \( \| \cdot \| \) for every \( \alpha \) the set \( A_\alpha / B_\alpha / \) is of the \( \alpha \)'th additive /multiplicative/ class in \( \ell_p \).

Fix \( \alpha \in [1, \omega_1) \). In order to show that \( A_\alpha \) does not belong to the \( \alpha \)'th multiplicative class we assume the contrary. Then there exists a contracting mapping \( f : (\ell_p, d) \rightarrow (\ell_p, d) \) such that \( A_\alpha = f^{-1}(B_\alpha) \). By the Contraction Mapping Principle, there exists a fixed point for the mapping \( f \), which implies a contradiction.

It remains to put \( E = A_\alpha \).

**Theorem 3.2.** Let \( \alpha \in [1, \omega_1) \) and \( p \in [1, +\infty) \). Then there exists an ssc-function \( f : \ell_p \rightarrow \mathbb{R} \) which belongs to the \( (\alpha + 1) \)'th Baire class and does not belong to the \( \alpha \)'th Baire class.

**Proof.** By Lemma 3.1 there exists an \( S \)-open set \( E \) in \( \ell_p \) such that \( E \) belongs to the \( \alpha \)'th additive class and does not belong to the \( \alpha \)'th multiplicative class. Then the function \( f : \ell_p \rightarrow \mathbb{R} \),
\[
f(x) = \chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E, \end{cases}
\]
satisfies the required properties.

The existence of an ssc-function \( f : \mathbb{R}^\omega \rightarrow \mathbb{R} \) which is not Baire measurable was proved in [5, Proposition 3.2].

4. Discontinuities of ssc-functions on \( \ell_p \)

We denote by \( \ell_p^\omega \) the set \( \ell_p \subseteq \mathbb{R}^\omega \) endowed with the topology of pointwise convergence induced from \( \mathbb{R}^\omega \). Evidently, \( \text{SSC}(\ell_p^\omega) \subseteq \text{SSC}(\ell_p) \). The converse is not true as the following example shows.

**Example 1.** There exists a continuous function \( f : \ell_2 \rightarrow \mathbb{R} \) which is not strongly separately continuous on \( \ell_2^\omega \).

**Proof.** For every \( n \in \mathbb{N} \) we set
\[
x_n = \left( \frac{1}{n}, 0, \ldots, 0, n, 0, \ldots \right) \quad \text{and} \quad y_n = (0, \ldots, 0, n, 0, \ldots).
\]
Since the sets \( F_1 = \{ x_n : n \in \mathbb{N} \} \) and \( F_2 = \{ y_n : n \in \mathbb{N} \} \) are disjoint and closed in \( \ell_2 \), the function \( f : \ell_2 \rightarrow \mathbb{R} \) defined by the formula
\[
f(x) = \frac{d_2(x, F_1)}{d_2(x, F_1) + d_2(x, F_2)}
\]
is continuous and \( F_1 = f^{-1}(0) \), \( F_2 = f^{-1}(1) \). Notice that \( x_n \rightarrow 0 \) in \( \ell_2^\omega \) and \( y_n = (x_n)^0 \). But \( f(x_n) - f(y_n) = 1 \) for every \( n \), which implies that \( f \) is not strongly separately continuous on \( \ell_2^\omega \) at the point \( x = 0 \) with respect to the first variable.

By \( C(f) \) (\( D(f) \)) we denote the set of all points of continuity (discontinuity) of a mapping \( f \).
Theorem 4.1. For any open nonempty set \( G \subseteq \ell_p \) with \( 1 \leq p < \infty \) there exists a strongly separately continuous function \( f : \ell_p \to \mathbb{R} \) such that \( D(f) = G \).

Proof. Fix \( p \in [1, +\infty) \) and let \( \sigma = \sigma(0) = \{(x_n)_{n=1}^\infty \in \ell_p : (\exists k \in \mathbb{N}) (\forall n \geq k) (x_n = 0)\}. \) Denote \( F = \ell_p \setminus G \). For every \( x = (x_n)_{n \in \mathbb{N}} \in \ell_p \) we put
\[
\varphi(x) = \begin{cases} 
\min\{d_p(x,F),1\}, & F \neq \emptyset, \\
1, & F = \emptyset,
\end{cases}
\]
and let
\[
g(x) = \begin{cases} 
\exp\left(-\sum_{n=1}^\infty |x_n|\right), & x = (x_n)_{n=1}^\infty \in \sigma, \\
1, & x \in \ell_p \setminus \sigma,
\end{cases}
\]
\[
f(x) = \varphi(x) \cdot g(x).
\]

**Claim 1.** \( F \subseteq C(f) \).

Proof. Fix \( x^0 \in F \) and take any convergent sequence \((x^m)_{m=1}^\infty \) to \( x^0 \) in \( \ell_p \). Notice that
\[
\lim_{m \to \infty} \varphi(x^m)g(x^m) = 0,
\]
because \( \varphi(x) \) is continuous at \( x^0 \) and \( g(x) \) is bounded. Then
\[
\lim_{m \to \infty} f(x^m) = 0 = f(x^0).
\]
Hence, \( x^0 \in C(f) \).

**Claim 2.** \( G \subseteq D(f) \).

Proof. Fix \( x^0 \in G \). Then \( f(x^0) > 0 \). We put \( \varepsilon = \frac{1}{2} f(x^0) \) and take an arbitrary \( \delta > 0 \). Since the set \( D = \ell_p \setminus \sigma \) is dense in \( \ell_p \), there exists \( x = (x_n)_{n \in \mathbb{N}} \in \ell_p \) such that
\[
\|x - x^0\|_p < \frac{\delta}{2} \quad \text{and} \quad x \notin \sigma.
\]
Take a number \( N \) such that
\[
\sum_{n=1}^N |x_n| > \ln\left(\frac{1}{f(x^0) - \varepsilon}\right) \quad \text{and} \quad \sum_{n=N+1}^\infty |x_n|^p < \left(\frac{\delta}{2}\right)^p.
\]
We put
\[
y = (x_1,\ldots,x_N,0,0,\ldots).
\]
Then \( y \in \sigma \) and
\[
\|y - x^0\|_p \leq \|y - x\|_p + \|x - x^0\|_p = \left(\sum_{n=N+1}^\infty |x_n|^p\right)^{\frac{1}{p}} + \|x - x^0\|_p < \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\]
But
\[
f(x^0) - f(y) = f(x^0) - \varphi(y) \cdot \exp\left(-\sum_{n=1}^N |x_n|\right) = f(x^0) - \exp(-\sum_{n=1}^N |x_n|) > f(x^0) + \varepsilon - f(x^0) = \varepsilon,
\]
which implies that \( f \) is discontinuous at \( x^0 \).

**Claim 3.** \( f : \ell_p \to \mathbb{R} \) is strongly separately continuous.

Proof. Let \( x^0 \in \ell_p \). Evidently, \( f \) is strongly separately continuous at \( x^0 \) if \( x^0 \in F \). Therefore, we assume that \( x^0 \in G \). Fix \( k \in \mathbb{N} \) and \( \varepsilon > 0 \). Take \( \delta_1 > 0 \) with \( B(x^0,\delta_1) \subseteq G \). Since \( \varphi(x) \) is continuous at \( x^0 \), there exists \( \delta_2 > 0 \) such that
\[
|\varphi(x) - \varphi(x^0)| < \frac{\varepsilon}{4}
\]
for all \( x \in B(x^0,\delta_2) \). Put
\[
\delta = \min\{\delta_1,\delta_2,\ln(1 + \frac{\varepsilon}{2})\}.
\]
Now let \( x \in B(x^0,\delta) \) and \( y = x^0_k \in B(x^0,\delta) \). If \( x \notin \sigma \), then \( y \notin \sigma \). In this case
\[
|f(x) - f(y)| = |\varphi(x) - \varphi(y)| \leq |\varphi(x) - \varphi(x^0)| + |\varphi(x^0) - \varphi(y)| < \varepsilon.
\]
Assume that $x \in \sigma$. Then $y \in \sigma$ and

$$|f(x) - f(y)| \leq |g(x)||\varphi(x) - \varphi(y)| + |\varphi(y)||g(x) - g(y)| <$$

$$< \frac{\varepsilon}{2} + \left| \exp(-\sum_{n=1}^{\infty} |x_n|) - \exp(-\sum_{n=1}^{\infty} |y_n|) \right| < \frac{\varepsilon}{2} + |\exp(\sum_{n=1}^{\infty} |y_n|) - \sum_{n=1}^{\infty} |x_n|) - 1|.$$ 

Taking into account that

$$\exp(-\sum_{n=1}^{\infty} |x_n - y_n|) \leq \exp(\sum_{n=1}^{\infty} |x_n| - \sum_{n=1}^{\infty} |y_n|) \leq \exp(\sum_{n=1}^{\infty} |x_n - y_n|),$$

we obtain that

$$|f(x) - f(y)| < \frac{\varepsilon}{2} + \exp(|x_k - x_0^{|\delta|}) - 1 < \frac{\varepsilon}{2} + \exp(\delta) - 1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ 

Hence, $f$ is strongly separately continuous at $x^0$ with respect to the $k$'th variable. $\square$

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