STATISTICAL INFERENCE FOR A PARTIALLY OBSERVED INTERACTING SYSTEM OF HAWKES PROCESSES

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ABSTRACT. We observe the actions of a $K$ sub-sample of $N$ individuals up to time $t$ for some large $K \leq N$. We model the relationships of individuals by i.i.d. Bernoulli($p$)-random variables, where $p \in (0, 1]$ is an unknown parameter. The rate of action of each individual depends on some unknown parameter $\mu > 0$ and on the sum of some function $\phi$ of the ages of the actions of the individuals which influence him. The function $\phi$ is unknown but we assume it rapidly decays. The aim of this paper is to estimate the parameter $p$ asymptotically as $N \to \infty$, $K \to \infty$, and $t \to \infty$. Let $m_t$ be the average number of actions per individual up to time $t$. In the subcritical case, where $m_t$ increases linearly, we build an estimator of $p$ with the rate of convergence $\frac{1}{\sqrt{K}} + \frac{N}{m_t \sqrt{K}} + \frac{N}{K \sqrt{m_t}}$. In the supercritical case, where $m_t$ increases exponentially fast, we build an estimator of $p$ with the rate of convergence $\frac{1}{\sqrt{K}} + \frac{N}{m_t \sqrt{K}}$.

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1. Introduction

1.1. Motivation. Hawkes processes have been used to model interactions between multiple entities evolving through time. For example in neuroscience, Reynaud-Bouret et al. [12] use multivariate Hawkes processes to model the spikes of different neurons. In finance, Bauwens and Hautsch in [3] give an order book model. Social networks interactions are considered in Blundell et al. [4], Simma-Jordan [15], Zhou et al. [16]. There are even some application in criminology, see e.g. Mohler, Short, Brantingham, Schoenberg and Tita in [9].

Concerning the statistical inference for Hawkes processes, mainly the case of fixed finite dimension $N$ has been studied in the asymptotic $t \to \infty$. In parametric models, Ogata has studied the maximum likelihood estimator in [10]. Non-parametric models were considered by Bacry and Muzzy [2], Hansen et al. [7], Reynaud-Bouret et al. [13, 14, 12] and Rasmussen [11] with a Bayesian approach, see remarks 1.1 and 1.2 for details.

However, in the real world, we often need to consider the case when the number of individuals is large. For example, in the neuroscience, the number of the neurons are usually enormously large. So it is natural to consider the double asymptotic $t \to \infty$ and $N \to \infty$. The studies about this case are rare. As far as we are aware, the only paper which consider this case is [6].

1.2. Model. We have $N$ individuals. Each individual $j \in \{1, \ldots, N\}$ is connected to the set of individuals $S_j = \{i \in \{1, \ldots, N\} : \theta_{ij} = 1\}$. The only possible action of the individual $i$ is to send a message to all the individuals of $S_i$. Here $Z_{i,N}^{t}$ stands for the number of messages sent by $i$ during $[0, t]$. The counting process $(Z_{i,N}^{s})_{i=1\ldots N, 0 \leq s \leq t}$ is determined by its intensity process $(\lambda_{i,N}^{s})_{i=1\ldots N, 0 \leq s \leq t}$. It is informally defined by

$$P\left( Z_{i,N}^{t} \text{ has a jump in } [t, t + dt] \bigg| \mathcal{F}_t \right) = \lambda_{i,N}^{t} dt, \ i = 1, \ldots, N$$

where $\mathcal{F}_t$ denotes the sigma-field generated by $(Z_{i,N}^{s})_{i=1\ldots N, 0 \leq s \leq t}$ and $(\theta_{ij})_{i,j=1\ldots N}$.

The rate $\lambda_{i,N}^{t}$ at which $i$ sends messages can be decomposed as the sum of two effects:

- he sends new messages at rate $\mu$;
- he forwards the messages he received, after some delay (possibly infinite) depending on the age of the message, which induces a sending rate of the form $\frac{1}{N} \sum_{j=1}^{N} \theta_{ij} \int_{0}^{t} \phi(t - s) dZ_{j,N}^{s}$. 


If for example $\phi = 1_{[0,K]}$, then \( N^{-1} \sum_{j=1}^{N} \theta_{ij} \int_{0}^{t} \phi(t - s) dZ_{s}^{i,N} \) is precisely the number of messages that the \( i \)-th individual received between time \( t - K \) and time \( t \), divided by \( N \).

**Remark 1.1.** In Bacry and Muzzy [2], Hansen et al [7], Reynaud-Bouret et al [13, 14, 12] consider the non-parametric estimation of the following system: for fixed \( N \geq 1 \), and \( i, j = 1, \ldots, N \), the counting process \( (Z_{s}^{i,N})_{i=1\ldots,N,0 \leq s \leq t} \) is determined by its intensity process \( (\lambda_{s}^{i,N})_{i=1\ldots,N,0 \leq s \leq t} \), which is a family of i.i.d. Bernoulli(\( \theta \)) random variables which is independent of \( Z_{s}^{i,N} \). In Bacry and Muzzy [2], Hansen et al [7], Reynaud-Bouret et al [13, 14, 12] consider the case when one observes the whole sample (super-critical case), it increases exponentially. Thus the estimation procedure will be different in the two cases.

**Remark 1.2.** In [11], Rasmussen consider the Bayesian inference of the following one dimensional system: the counting process \( (Z_{s})_{0 \leq s \leq t} \) is determined by its intensity process \( (\lambda_{s})_{0 \leq s \leq t} \), which is measurable and locally integrable. They provided estimators of \( \mu_{s} \) and the functions \( \phi_{ij} \). For fixed \( N \), our model can be seen as a special case of (1) for \( \mu_{s} = \mu \) and \( \phi_{ij}(s) = \frac{\theta_{ij}}{N} \phi(s) \).

**Remark 1.3.** Since the family of \( (Z_{s}^{i,N})_{i=1\ldots,N} \) is exchangeable, considering that the observation is given by the first \( K \) processes is not restriction.

Let \( \Lambda = \int_{0}^{\infty} \phi(t) dt \in (0, \infty] \). In [9], we see that growth of \( Z_{t}^{i,N} \) depends on the value of \( \Lambda p \). When \( \Lambda p < 1 \) (subcritical case), \( Z_{t}^{i,N} \) increases (in average) linearly with time, while when \( \Lambda p > 1 \) (supercritical case), it increases exponentially. Thus the estimation procedure will be different in the two cases.

**Remark 1.4.** We can find simulations for \( K < N \) already in [6]. The simulations are about the case \( K = \frac{N}{t} \).

2. Main results

2.1. Setting. We consider some unknown parameters \( p \in (0, 1], \mu > 0 \) and \( \phi : [0, \infty) \rightarrow [0, \infty) \). We always assume that the function \( \phi \) is measurable and locally integrable. For \( N \geq 1 \), we consider an i.i.d. family \( (\Pi^{i} dt, dz)_{i=1\ldots,N} \) of Poisson measures on \([0, \infty) \times [0, \infty)\) with intensity \( dt dz \), together with \( (\theta_{ij})_{i,j=1\ldots,N} \) is a family of i.i.d. Bernoulli(\( p \)) random variables which is independent
of the family \((\Pi^i(dt, dz))_{i=1,\ldots,N}\). We consider the following system: for all \(i \in \{1, \ldots, N\}\), all \(t \geq 0\),

\[
(3) \quad Z^{i,N}_t := \int_0^t \int_0^\infty 1_{\{z \leq \lambda^{i,N}_s\}} \Pi^i(ds,dz), \text{ where } \lambda^{i,N}_t := \mu + \frac{1}{N} \sum_{j=1}^N \theta_{ij} \int_0^{t-} \phi(t-s)dZ^{j,N}_s.
\]

In this paper, \(\int_0^t\) means \(\int_{[0,t]}\), and \(\int_0^{t-}\) means \(\int_{[0,t)}\). The solution \(((Z^{i,N}_t)_{t \geq 0})_{i=1,\ldots,N}\) is a family of counting processes. By [6, Proposition 1], the system (1) has a unique \((\mathcal{F}_t)_{t \geq 0}\)-measurable càdlàg solution, where

\[
\mathcal{F}_t = \sigma(\Pi^i(A) : A \in \mathcal{B}([0,t] \times [0,\infty)), i = 1, \ldots, N) \cup \sigma(\theta_{ij}, i, j = 1, \ldots, N),
\]
as soon as \(\phi\) is locally integrable.

2.2. Assumptions. Recall that \(\Lambda = \int_0^\infty \phi(t)dt \in (0, \infty]\). We will work under one of the two following conditions: either for some \(q \geq 1\),

\[
(H(q)) \quad \mu \in (0, \infty), \quad \Lambda p \in (0, 1) \quad \text{and} \quad \int_0^\infty s^q \phi(s)ds < \infty
\]
or

\[
(A) \quad \mu \in (0, \infty), \quad \Lambda p \in (1, \infty] \quad \text{and} \quad \int_0^t |d\phi(s)| \text{ increases at most polynomially.}
\]

Remark 2.1. In many applications, \(\phi\) is smooth and decays fast. Hence what we have in mind is that in the subcritical case, \((H(q))\) is satisfied for all \(q \geq 1\). In the supercritical case, \((A)\) seems very reasonable: all the nonegative polynomial functions satifies the condition.

2.3. The result in the subcritical case. For \(N \geq 1\) and for \(((Z^{i,N}_t)_{t \geq 0})_{i=1,\ldots,N}\) the solution of (3), we set \(\bar{Z}^{N} := N^{-1} \sum_{i=1}^N Z^{i,N}_t\) and \(\bar{Z}^{N,K} := K^{-1} \sum_{i=1}^K Z^{i,N}_t\). Next, we introduce

\[
\varepsilon^{N,K}_t := t^{-1}\left(\bar{Z}^{N,K}_t - \bar{Z}^{N,K}_0\right), \quad \gamma^{N,K}_t := \frac{N}{K} \sum_{i=1}^K \left[\frac{Z^{i,N}_t - Z^{i,N}_0}{K} - \varepsilon^{N,K}_t\right]^2 - \frac{N}{t} \varepsilon^{N,K}_t.
\]

For \(\Delta > 0\) such that \(t/(2\Delta) \in \mathbb{N}^*\), we set

\[
(4) \quad \mathcal{W}^{N,K}_{\Delta,t} := 2\bar{Z}^{N,K}_{2\Delta} - \bar{Z}^{N,K}_{\Delta}, \quad \lambda^{N,K}_{\Delta,t} := \mathcal{W}^{N,K}_{\Delta,t} - \frac{N-K}{K} \varepsilon^{N,K}_t
\]

\[
(5) \quad \text{where } \bar{Z}^{N,K}_{\Delta,t} := \frac{N}{t} \sum_{a=t/\Delta}^{2t/\Delta} (\bar{Z}^{N,K}_{a\Delta} - \bar{Z}^{N,K}_{(a-\Delta)} - \Delta \varepsilon^{N,K}_{t,\Delta})^2.
\]

Remark 2.2. The estimators we defined above were already appearing in [6]. The aim of this paper is to prove the convergence of these estimators.

Theorem 2.3. We assume \((H(q))\) for some \(q > 3\). There exists constants \(C < \infty, C' > 0\) depending only on \(q, p, \mu, \phi\) such that for all \(\varepsilon \in (0,1),\ all \ 1 \leq K \leq N, \ all \ t \geq 1,\ setting \ \Delta_t = \mu = t/[2(1-q/(q+1))]\),

\[
P\left(\left|\mathcal{W}^{N,K}_{\Delta,t} - (\mu, \Lambda, p)\right| \geq \varepsilon\right) \leq C \frac{1}{\varepsilon} + \frac{N}{Kt} + \frac{N}{t\sqrt{K}} + CNe^{-C'K}
\]
with $\Psi := 1_D \Phi : \mathbb{R}^3 \to \mathbb{R}^3$, the function $\Phi := (\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)})$ being defined on $D := \{(u, v, w) \in \mathbb{R}^3 : w > 0 \text{ and } u, v \geq 0\}$ by

$$\Phi^{(1)}(u, v, w) := u \sqrt{\frac{u}{w}}, \quad \Phi^{(2)}(u, v, w) := \frac{v + [u - \Phi^{(1)}(u, v, w)]^2}{u[u - \Phi^{(1)}(u, v, w)]},$$

$$\Phi^{(3)}(u, v, w) := \frac{1 - u^{-1}\Phi^{(1)}(u, v, w)}{\Phi^{(2)}(u, v, w)}.$$

We quote [6, Remark 2], which says that the mean number of actions per individual per unit of time increases linearly.

**Remark 2.4.** Assume $H(1)$. Then for all $\varepsilon > 0$,

$$\lim_{(N, t) \to (\infty, \infty)} P\left(\left|\frac{Z_{t,K}^{N}}{t} - \frac{\mu}{1 - \Lambda p}\right| \geq \varepsilon\right) = 0.$$ 

So roughly, if observing $((Z_{t}^{N}u, v, w))_{u=0}^{\infty}$, we observe approximately $K t$ actions.

**Remark 2.5.** If the function $\phi$ decays fast, for example $\phi(s) = a e^{-bs}$ or $c1_D$ where $D$ is some compact set. In these situations, the function $\phi$ can satisfy the assumptions for arbitrary $q > 0$. Hence, we can almost replace $\frac{N}{K \sqrt{t}}$ by $\frac{N}{K \sqrt{t^{1-\varepsilon}q}}$.

**Remark 2.6.** We are going to consider two special cases:

- When $K \sim N$, we have

$$\left(\frac{1}{\sqrt{K}} + \frac{N}{K^{\varepsilon}} + \frac{N}{t} + \frac{N}{t^{1-\varepsilon}}\right) + C Ne^{-C'K} \sim \left(\frac{1}{\sqrt{N}} + \frac{1}{\sqrt{t^{-\varepsilon}}} + \frac{1}{t^{1-\varepsilon}}\right) + C Ne^{-C'N}.$$ 

Hence, in order to ensure the convergence, we just need $\sqrt{\frac{N}{K}} \to 0$.

- Assume $K \sim \gamma \log N$ and $\gamma C' > 1$, where $C'$ is as in theorem 2.3, we have

$$\left(\frac{1}{\sqrt{K}} + \frac{N}{K^{\varepsilon}} + \frac{N}{t} + \frac{N}{t^{1-\varepsilon}}\right) + C Ne^{-C'K} \sim \left(\frac{1}{\log N} + \frac{N}{\log N^{t^{1-\varepsilon}}} + \frac{N}{t^{1-\varepsilon}}\right) + C N^{1-\gamma C'}.$$ 

Hence, in order to ensure the convergence, we just need $\frac{N}{\log N^{t^{1-\varepsilon}}} + \frac{N}{t^{1-\varepsilon}} \to 0$, which equivalent to $\frac{N}{\log N^{t^{1-\varepsilon}}} \to 0$.

2.4. The result in the supercritical case. Here we define $Z_{t,K}^{N,K}$ as previously and we set

$$U_t^{N,K} := \left[\frac{N}{K} \sum_{i=1}^{K} \left(\frac{Z_{t,i}^{N} - Z_{t,K}^{N,K}}{Z_{t,K}^{N,K}}\right)^2 - \frac{N}{Z_{t,K}^{N,K}}\right] 1_{\{Z_{t,K}^{N,K} > 0\}},$$

and $P_t^{N,K} := \frac{1}{U_t^{N,K} + 1} 1_{\{U_t^{N,K} > 0\}}.$

**Theorem 2.7.** We assume $(A)$ and define $\alpha_0$ by $p \int_0^\infty e^{-\alpha_0 t} \phi(t) dt = 1$ (recall that by $(A)$, $\Lambda p = p \int_0^\infty \phi(t) dt > 1$). For all $\eta > 0$, there is a constant $C_\eta > 0$ (depending on $p, \mu, \phi, \eta$), such that for all $N \geq K \geq 1$, all $\varepsilon \in (0, 1)$,

$$P(|P_t^{N,K} - p| \geq \varepsilon) \leq \frac{C_\eta e^{A\eta t}}{\varepsilon} \left(\frac{N}{\sqrt{K} e^{\alpha_0 t}} + \frac{1}{\sqrt{K}}\right).$$

Next, we quote [6] Remark 5].
Remark 2.8. Assume \((A)\) and consider \(\alpha_0 > 0\) such that \(p \int_0^\infty e^{-\alpha_0 t} \phi(t) dt = 1\). Then for all \(\eta > 0\),
\[
\lim_{t \to \infty} \lim_{(N,K) \to (\infty,\infty)} P(\tilde{Z}^{N,K}_t \in [\epsilon^{(\alpha_0-\eta)t}, \epsilon^{(\alpha_0+\eta)t}]) = 1.
\]
So roughly, if observing \((Z^{i,N}_s)_{s \in [0,t]}\) for \(i = 1, \ldots, K\), we observe around \(K\) \(e^{\alpha_0 t}\) actions.

3. ON THE CHOICE OF THE ESTIMATORS

In the whole paper, we denote by \(E_\theta\) the conditional expectation knowing \((\theta_{ij})_{i,j=1,\ldots,N}\). Here we explain informally why the estimators should converge.

3.1. The subcritical case. We define \(A_N(i,j) := N^{-1} \theta_{ij}\) and the matrix \((A_N(i,j))_{i,j \in \{1,\ldots,N\}}\), as well as \(Q_N := (I - \Lambda A_N)^{-1}\) on the event on which \(I - \Lambda A_N\) is invertible.

Define \(\tilde{\varepsilon}^{N,K}_t := \int_0^t Z^{N,K}_s ds\), \(K \leq N\). We expect that, for \(t\) large enough, \(Z^{i,N}_t \simeq E_\theta[Z^{i,N}_t]\). And, by definition of \(Z^{i,N}_t\), see (3), it is not hard to get
\[
E_\theta[Z^{i,N}_t] = \mu t + N^{-1} \sum_{i,j=1}^N \theta_{ij} \int_0^t \phi(t-s) E_\theta[Z^{i,N}_s] ds.
\]
Hence, assuming that \(\gamma_N(i) = \lim_{t \to \infty} t^{-1} E_\theta[Z^{i,N}_t]\) exists for each \(i = 1, \ldots, N\) and observing that \(\int_0^t \phi(t-s) ds \simeq \Lambda t\), we find that the vector \(\gamma_N = (\gamma_N(i))_{i=1,\ldots,N}\) should satisfy \(\gamma_N = \mu 1_N + \Lambda A_N \gamma_N\), where \(1_N\) is the vector defined by \(1_N(i) = 1\) for all \(i = 1, \ldots, N\). Thus we deduce that \(\gamma_N = \mu(I - \Lambda A_N)^{-1} 1_N = \mu \ell_N\), where we have set
\[
\ell_N := Q_N 1_N, \quad \ell_N(i) := N \sum_{j=1}^N Q_N(i,j), \quad \tilde{\ell}_N := \frac{1}{N} \sum_{i=1}^N \ell_N(i), \quad \tilde{\ell}_N^K := \frac{1}{K} \sum_{i=1}^K \ell_N(i)
\]
So we expect that \(Z^{i,N}_t \simeq E_\theta[Z^{i,N}_t] \simeq \mu \ell_N(i) t\), whence \(\tilde{Z}^{N,K}_t = t^{-1}\tilde{\varepsilon}^{N,K}_t \simeq \mu \tilde{\ell}_N^K t\).

We informally show that \(\ell_N(i) \simeq 1 + \Lambda (1 - \Lambda p)^{-1} L_N(i)\), where \(L_N(i) := \sum_{j=1}^N A_N(i,j)\): when \(N\) is large, \(\sum_{j=1}^N A_N(i,j) = N^{-2} \sum_{j=1}^N \sum_{k=1}^N \theta_{ik} \theta_{kj} \simeq p N^{-1} \sum_{k=1}^N \theta_{ik} = p L_N(i)\). And one gets convinced similarly that for any \(n \in \mathbb{N}_+\), roughly, \(\sum_{j=1}^N A_N^N(i,j) \simeq p^{-1} L_N(i)\). So
\[
\ell_N(i) = \sum_{j=1}^N A_N^N(i,j) \simeq 1 + p N^{-1} L_N(i) = 1 + \frac{\Lambda}{1 - \Lambda p} L_N(i).
\]
But \((N L_N(i))_{i=1,\ldots,N}\) are i.i.d. Bernoulli(\(N,p\)) random variables, so that \(\tilde{\ell}_N^K \simeq 1 + \Lambda p (1 - \Lambda p)^{-1} = (1 - \Lambda p)^{-1}\). Finally, we have explained why \(\tilde{\varepsilon}^{N,K}_t\) should resemble \(\mu (1 - \Lambda p)^{-1}\).

Knowing \((\theta_{ij})_{i,j=1,\ldots,N}\), the process \(Z^{1,N}_t\) resembles a Poisson process, so that \(\text{Var}_\theta(Z^{1,N}_t) \simeq E_\theta[Z^{1,N}_t]\), whence
\[
\text{Var}(Z^{1,N}_t) = \text{Var}(E_\theta[Z^{1,N}_t]) + E(\text{Var}_\theta(Z^{1,N}_t)) \simeq \text{Var}(E_\theta[Z^{1,N}_t]) + E[Z^{1,N}_t].
\]
Writing an empirical version of this equality, we find
\[
\frac{1}{K} \sum_{i=1}^K (Z^{i,N}_t - \bar{Z}^{N,K}_t)^2 \simeq \frac{1}{K} \sum_{i=1}^K \left( E_\theta[Z^{i,N}_t] - E_\theta[\bar{Z}^{N,K}_t] \right)^2 + \bar{Z}^{N,K}_t.
\]
And since $Z_{t}^{i,N} \approx \mu \ell_{N}(i) t \approx \mu [1 + (1 - \Lambda p)^{-1} \Lambda L_{N}(i)] t$ as already seen a few lines above, we find

$$\frac{1}{K} \sum_{i=1}^{K} (Z_{t}^{i,N} - \bar{Z}_{t}^{N,K})^2 \approx \frac{\mu^2 t^2 \Lambda^2}{(1 - \Lambda p)^2} \sum_{i=1}^{K} (L_{N}(i) - \bar{L}_{N}^{K})^2 + Z_{t}^{N,K}.$$ 

But $(NL_{N}(i))_{i=1,\ldots,N}$ are i.i.d. Bernoulli$(N, p)$ random variables, so that

$$\bar{V}_{t}^{N,K} := \frac{N}{K} \sum_{i=1}^{K} \left[ \frac{Z_{t}^{i,N}}{t} - \frac{Z_{t}^{N,K}}{t} \right]^2 - \frac{N}{t} \frac{Z_{t}^{N,K}}{t}$$

$$\approx \frac{N \mu^2 \Lambda^2}{K (1 - \Lambda p)^2} \sum_{i=1}^{K} (L_{N}(i) - \bar{L}_{N}^{K})^2 \approx \frac{\mu^2 \Lambda^2}{(1 - \Lambda p)^2}.$$ 

We finally build a third estimator. The temporal empirical variance

$$\frac{\Delta}{t} \sum_{k=1}^{t/\Delta} \left[ \tilde{Z}_{k\Delta}^{N,K} - \frac{1}{t} \tilde{Z}_{t}^{N,K} \right]^2$$

should resemble $\text{Var}_{\theta}[\tilde{Z}_{\Delta}^{N,K}]$ if $1 \ll \Delta \ll t$. So we expect that:

$$\hat{W}_{\Delta, t}^{N,K} := \frac{N}{t} \sum_{k=1}^{t/\Delta} \left[ \tilde{Z}_{k\Delta}^{N,K} - \frac{1}{t} \tilde{Z}_{t}^{N,K} \right]^2 \approx \frac{N}{\Delta} \text{Var}_{\theta}[\tilde{Z}_{\Delta}^{N,K}].$$

To understand what $\text{Var}_{\theta}[\tilde{Z}_{\Delta}^{N,K}]$ looks like, we introduce the centered process $U_{t}^{i,N} := Z_{t}^{i,N} - \mathbb{E}_{\theta}[Z_{t}^{i,N}]$ and the martingale $M_{t}^{i,N} := Z_{t}^{i,N} - C_{t}^{i,N}$ where $C_{t}^{i,N}$ is the compensator of $Z_{t}^{i,N}$. An easy computation, see [6] Lemma 11, shows that, denoting by $U_{t}^{N}$ and $M_{t}^{N}$ the vectors $(U_{t}^{i,N})_{i=1,\ldots,N}$ and $(M_{t}^{i,N})_{i=1,\ldots,N}$,

$$U_{t}^{N} = M_{t}^{N} + A_{N} \int_{0}^{t} \phi(t - s) U_{s}^{N} \, ds.$$ 

So for large times, we conclude that $U_{t}^{N} \approx M_{t}^{N} + A_{N} U_{t}^{N}$, whence finally $U_{t}^{N} \approx Q M_{t}^{N}$ and thus

$$\frac{1}{K} \sum_{i=1}^{K} U_{t}^{i,N} \approx \frac{1}{K} \sum_{i=1}^{K} \sum_{j=1}^{N} Q(i, j) M_{t}^{j,N} = \frac{1}{K} \sum_{j=1}^{N} c_{N}^{K}(j) M_{t}^{j,N},$$

where we have set $c_{N}^{K}(j) = \sum_{i=1}^{K} Q_{N}(i, j)$. But we obviously have $[M_{t}^{j,N}, M_{t}^{j,N}]_{t} = 1_{\{i=j\}} Z_{t}^{j,N}$ (see [6] Remark 10), so that

$$\text{Var}_{\theta}[Z_{\Delta}^{N,K}] = \text{Var}_{\theta}[U_{\Delta}^{N,K}] \approx \frac{1}{K^2} \sum_{j=1}^{N} \left( c_{N}^{K}(j) \right)^2 Z_{t}^{j,N}.$$ 

Recalling that $Z_{t}^{i,N} \approx \mu \ell_N(j) t$, we conclude that $\text{Var}_{\theta}[Z_{\Delta}^{N,K}] \approx K^{-2} \mu t \sum_{j=1}^{N} \left( c_{N}^{K}(j) \right)^2 \ell_{N}(j),$ 

whence

$$\hat{W}_{\Delta, t}^{N,K} \approx \frac{N}{\Delta} \text{Var}_{\theta}[\tilde{Z}_{\Delta}^{N,K}] \approx \mu \frac{N}{K^2} \sum_{j=1}^{N} \left( c_{N}^{K}(j) \right)^2 \ell_{N}(j).$$
To compute this last quantity, we start from \( c_N^K(j) = \sum_{n \geq 0} \sum_{i=1}^K \Lambda^n A_N(i, j) \). But we have \( \sum_{i=1}^K A_N^2(i, j) = N^{-2} \sum_{i=1}^K \sum_{k=1}^N \theta_{ik} k_j \simeq p K N^{-2} \sum_{k=1}^N \theta_{kj} = p K N^{-1} C_N(j) \). And one gets convinced similarly that for any \( n \in \mathbb{N}_+ \), roughly, \( \sum_{i=1}^K A_N^n(i, j) \simeq K N^{-1} p^{-n} C_N(j) \). So we conclude that \( c_N^K(j) \simeq A_N^0(i, j) + \frac{K^2 \Lambda p}{N(1-\Lambda p)} C_N(j) \). Consequently, \( c_N^K(j) \simeq 1 + \frac{K^2 \Lambda p}{N(1-\Lambda p)} \) for \( j \in \{1, ..., K\} \) and \( c_N^K(j) \simeq \frac{K^2 \Lambda p}{N(1-\Lambda p)} \) for \( j \in \{K+1, ..., N\} \). We finally get, recalling that \( \ell_N(j) \simeq (1-\Lambda p)^{-1} \),

\[
\sim \frac{1}{\mu (1-\Lambda p)}.
\]

All in all, we should have \( \tilde{X}^{N,K}_{\Delta,t} \simeq \frac{1}{\mu (1-\Lambda p)} \).

It readily follows that \( \Psi(\tilde{E}_t^{N,K}, \tilde{V}_t^{N,K}, \tilde{X}^{N,K}_{\Delta,t}) \) should resemble \((\mu, \Lambda, p)\).

The three estimators \( \tilde{E}_t^{N,K}, \tilde{V}_t^{N,K}, \tilde{X}^{N,K}_{\Delta,t} \) are very similar to \( \tilde{E}_t^{N,K}, \tilde{V}_t^{N,K}, \tilde{X}^{N,K}_{\Delta,t} \) and should converge to the same limits. Let us explain why we have introduced \( \tilde{E}_t^{N,K}, \tilde{V}_t^{N,K}, \tilde{X}^{N,K}_{\Delta,t} \) of which the expressions are more complicated. The main idea is that, see [6] Lemma 16 (ii), \( \mathbb{E}[Z_t^{i,N}] = \mu t N(i) + \chi^N i + t^{1-\eta} \) (under \((H(q))\)), for some finite random variable \( \chi^N i \). As a consequence, \( t^{-1}E[Z_t^{i,N} - Z_t^{i,N}] \) converges to \( \mu t N(i) \) considerably much faster, if \( q \) is large, than \( t^{-1}E[\theta | Z_t^{i,N}] \) (for which the error is of order \( t^{-1} \)).

### 3.2. The supercritical case.

We now turn to the supercritical case where \( \Lambda p > 1 \). We introduce the \( N \times N \) matrix \( A_N(i, j) = N^{-1} \theta_{ij} \).

We expect that \( Z_t^{i,N} \simeq H_N \mathbb{E}[Z_t^{i,N}] \), when \( t \) is large, for some random \( H_N > 0 \) not depending on \( i \). Since \( \Lambda p > 1 \), the process should increase like an exponential function, i.e. there should be \( \alpha_N > 0 \) such that for all \( i = 1, ..., N \), \( \mathbb{E}[Z_t^{i,N}] \simeq \gamma_N(i) e^{\alpha_N t} \) for \( t \) very large, where \( \gamma_N(i) \) is some positive random constant. We recall that \( \mathbb{E}[Z_t^{i,N}] = \mu t + N^{-1} \sum_{j=1}^N \theta_{ij} \int_0^t \phi(t-s) ds \). We insert \( \theta_{ij} | Z_t^{i,N} | \simeq \gamma_N(i) e^{\alpha_N t} \) in this equation and let \( t \) go to infinite: we informally get \( \gamma_N = A_N \gamma_N \int_0^\infty e^{-\alpha_N s} \phi(s) ds \). In other words, \( \gamma_N = (\gamma_N(i))_{i=1, ..., N} \) is an eigenvector of \( A_N \) for the eigenvalue \( \rho_N := (\int_0^\infty e^{-\alpha_N s} \phi(s) ds)^{-1} \).

But \( A_N \) has nonnegative entries. Hence by the Perron-Frobenius theorem, it has a unique (up to normalization) eigenvector \( V_N \) with nonnegative entries (say, such that \( \|V_N\|_2 = \sqrt{N} \)), and this vector corresponds to the maximum eigenvalue \( \rho_N \) of \( A_N \). So there is (a random) constant \( \kappa_N \) such that \( \gamma_N \simeq \kappa_N V_N \). All in all, we find that \( Z_t^{i,N} \simeq \kappa_N N \rho_N e^{\alpha_N t} V_N(i) \). We define \( V_N^K = I_K V_N \), where \( I_K \) is the \( N \times N \)-matrix defined by \( I_K(i, j) = 1 \) for \( i \leq K \).

As in the subcritical case, the variance \( K^{-1} \sum_{i=1}^K (Z_t^{i,N} - \bar{Z}_t^{N,K})^2 \) should look like

\[
\frac{1}{K} \sum_{i=1}^K \left( \mathbb{E}[Z_t^{i,N}] - \mathbb{E}[\bar{Z}_t^{N,K}] \right)^2 + \bar{Z}_t^{N,K} \simeq \frac{\kappa_N H_N^2 e^{2\alpha_N t}}{K} \sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2 + \bar{Z}_t^{N,K},
\]
where as usual \( \bar{V}_N^{K_i} := K^{-1} \sum_{i=1}^K V_N(i) \). We also get \( \bar{Z}_{i,N}^{K_i} \simeq \kappa_N H_N \bar{V}_N^{K_i} e^{\alpha_N t} \). Finally,

\[
U_t^{N,K} = \frac{N}{K(\bar{Z}_{i,N}^{K_i})^2} \left[ \sum_{i=1}^K (Z_{i,t}^{N,K} - \bar{Z}_{i,N}^{K_i})^2 - K \bar{Z}_{i,N}^{K_i} \right] 1_{\{\bar{Z}_{i,N}^{K_i} > 0\}} \simeq \frac{N}{K(\bar{V}_N^K)^2} \left[ \sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2 \right].
\]

Next, we consider the term \( (\bar{V}_N^K)^{-2} \sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2 \). By a rough estimation, \( A_{i,N}^2(i,j) \simeq \frac{e^N}{N} \).

Because \( I_K A_N^2 V_N = \rho_N^2 V_N^K \), we have \( \rho_N^2 V_N^K \simeq p^2 \bar{V}_N^K \), where \( 1_K \) is the \( N \) dimensional vector of which the first \( K \) elements are 1 and others are 0. By the same reason, we have \( \rho_N^2 V_N \simeq p^2 \bar{V}_N 1_N \).

So \( V_N^K = I_K A_N V_N / \rho_N \simeq k_N I_K A_N 1_N \), where \( k_N = (p^2 / \rho_N^2) \bar{V}_N \). In other words, the vector \((k_N)^{-1} V_N^K\) is almost like the vector \( L_N = I_K A_N 1_N \). Finally, we expect that

\[
U_t^{N,K} \simeq \frac{N}{K(\bar{V}_N^K)^2} \left[ \sum_{i=1}^K (V_N(i) - \bar{V}_N^K)^2 \right] \simeq \frac{N}{K(\bar{L}_N)^2} \left[ \sum_{i=1}^K (L_N(i) - \bar{L}_N)^2 \right] \simeq p^{-2} p(1 - p) = \frac{1}{p} - 1,
\]

whence \( P_t^{N,K} \simeq p \).

4. Optimal rates in some toy models

The goal of this section is to verify, using some toy models, that the rates of convergence of our estimators, see Theorems 2.3 and 2.7, are not far from being optimal.

4.1. The first example. Consider \( \alpha_0 \geq 0 \) and two unknown parameters \( \Gamma > 0 \) and \( p \in (0,1] \). Consider an i.i.d. family \( \{\theta_{ij}\}_{i,j=1,...,N} \) of Bernoulli \((p)\)-distributed random variables, where \( N \geq 1 \).

We set \( \lambda_i^{1,N} = N^{-1} \Gamma e^{\alpha_0 t} \sum_{j=1}^N \theta_{ij} \) and we introduce the processes \( (Z_i^{1,N})_{i \geq 0}, \ldots, (Z_i^{N,N})_{i \geq 0} \) which are, conditionally on \( \{\theta_{ij}\} \), independent inhomogeneous Poisson process with intensities \( \{\lambda_i^{1,N}\}_{i \geq 0}, \ldots, \{\lambda_i^{N,N}\}_{i \geq 0} \). We only observe \( (Z_i^{s,N})_{s \in [0,t_i], i=1,...,K} \), where \( K \leq N \) and we want to estimate the parameter \( p \) in the asymptotic \((K,N,t) \to (\infty,\infty,\infty)\). This model is a simplified version of the one studied in our paper. And roughly speaking, the mean number of jumps per individuals until time \( t \) resembles \( m_t = \int_0^t e^{\alpha_0 s} ds \). When \( \alpha_0 = 0 \), this mimics the subcritical case, while when \( \alpha_0 > 0 \), this mimics the supercritical case. Remark that \( (Z_i^{s,N})_{i=1,...,K} \) is a sufficient statistic, since \( \alpha_0 \) is known.

We use the central limit theorem in order to perform a Gaussian approximation of \( Z_t^{i,N} \). It is easy to show that:

\[
\lambda_i^{1,N} = \Gamma e^{\alpha_0 t} \left[ \frac{1}{\sqrt{N}} \sqrt{p(1 - p)} \sum_{j=1}^N (\theta_{ij} - p) + \frac{1}{\sqrt{N p(1 - p)}} \sum_{j=1}^N (\theta_{ij} - p) + p \right]
\]

and \( \frac{1}{\sqrt{N p(1 - p)}} \sum_{j=1}^N (\theta_{ij} - p) \) converges in law to a Gaussian random variable \( G_i \sim \mathcal{N}(0,1) \), where \( G_i \) is an i.i.d Gaussian family, as \( N \to \infty \), for each \( i \). Thus

\[
\lambda_i^{1,N} \simeq \Gamma e^{\alpha_0 t} \sqrt{\sqrt{1 - p} G_i + p}.
\]

Moreover, conditionally on \( \{\theta_{ij}\}_{i,j=1,...,N} \), \( Z_t^{i,N} \) is a Poisson random variable with mean \( \int_0^t \lambda_i^{1,N} ds \).

Thus, as \( t \) is large, we have \( Z_t^{i,N} \simeq \int_0^t \lambda_i^{1,N} ds + \int_0^t \lambda_i^{1,N} ds H_i \) where \( \{H_i\}_{i=1,...,N} \) is a family of \( \mathcal{N}(0,1) \)-distributed random variables, independent of \( \{G_i\}_{i=1,...,N} \). Since \( (m_t)^{-1} N^{-1/2} \ll (m_t)^{-1} \), we obtain \( (m_t)^{-1} Z_t^{i,N} \simeq \Gamma p + \Gamma \sqrt{1 - p} G_i + \sqrt{(m_t)^{-1} p} H_i \), of which the law is nothing but \( \mathcal{N}(\Gamma p, N^{-1} \Gamma^2 p(1-p) + (m_t)^{-1} \Gamma p) \).
By the above discussion, we construct the following toy model: one observes \((X_t^{i,N})_{i=1,...,T}\), where \((X_t^{i,N})_{i=1,...,N}\) are i.i.d and \(N(\Gamma p, N^{-1}\Gamma^2 p(1-p) + (m_t)^{-1}\Gamma p)\)-distributed. Moreover we assume that \(\Gamma p\) is known. So we can use the well-known statistic result: the empirical variance \(\hat{\sigma}_t^{N,K} = K^{-1}\sum_{i=1}^K (X_t^{i,N} - \bar{\Gamma}_p)^2\) is the best estimator of \(N^{-1}\Gamma^2 p(1-p) + (m_t)^{-1}\Gamma p\) (in any reasonable sense). So \(T_t^{N,K} = N(\Gamma p)^{-2}(S_t^{N,K} - (\Gamma p)/m_t)\) is the best estimator of \((\frac{1}{p} - 1)\). As

\[
\text{Var}(S_t^{N,K}) = \frac{1}{K}\text{Var}[(X_t^{1,N} - \Gamma_p)^2] = \frac{2}{K}\left(\frac{\Gamma^2 p(1-p)}{N} + \frac{\Gamma p}{m_t}\right)^2,
\]

we have

\[
\text{Var}(T_t^{N,K}) = \frac{2}{(\Gamma p)^4}\left(\frac{\Gamma^2 p(1-p)}{\sqrt{K}} + \frac{N\Gamma p}{m_t\sqrt{K}}\right)^2.
\]

In other words, we cannot estimate \((\frac{1}{p} - 1)\) with a precision better than \(\left(\frac{1}{\sqrt{K}} + \frac{N}{m_t\sqrt{K}}\right)^2\), which implies that we cannot estimate \(p\) with a precision better than \(\left(\frac{1}{\sqrt{K}} + \frac{N}{m_t\sqrt{K}}\right)^2\).

### 4.2. The second example

In the second part of this section, we are going to explain why there is a term \(\frac{N}{m_t\sqrt{K}}\) in the subcritical case.

We consider discrete times \(t = 1,...,T\) and two unknown parameters \(\mu > 0\) and \(p \in (0,1]\). Consider an i.i.d. family \((\theta_{ij})_{i,j=1,...,N}\) of Bernoulli\((p)\)-distributed random variables, where \(N \geq 1\). We set \(Z_t^{i,N} = 0\) for all \(i = 1,...,N\) and assume that, conditionally on \((\theta_{ij})_{i,j=1,...,N}\) and \((Z_t^{i,N})_{s=0,...,t,j=1,...,N}\), the random variables \((Z_t^{i,N} - Z_t^{j,N})\) (for \(i = 1,...,N\)) are independent and \(\mathcal{P}(\lambda_t^{i,N})\)-distributed, where \(\lambda_t^{i,N} = \mu + \frac{1}{N}\sum_{j=1}^N \theta_{ij}(Z_t^{i,N} - Z_t^{j,N})\). This process \((Z_t^{i,N})_{i=1,...,N,t=0,...,T}\) resembles the system of Hawkes processes studied in the present paper.

By [1] theorem 2, we have when time \(t\) is large, the process \(Z_t^{i,N}\) is similar to a \(d\)-dimensional diffusion process \((I - A_N)^{-1}\Sigma \tilde{\beta} B_t + \mathbb{E}_\theta[Z_t^{i,N}]\), where \(B_t\) is a \(N\)-dimensional Brownian Motion and \(\Sigma\) is the diagonal matrix such that \(\Sigma_{ii} = ((I - A_N)^{-1}\mu\)\). Hence \((Z_t^{i,N} - Z_t^{j,N}) - \mathbb{E}_\theta[Z_t^{i,N} - Z_t^{j,N}]\) (for \(i = 1,...,N\) and \(t = 1,...,T\)) are independent. Since \(\mathbb{E}_\theta[Z_t^{i,N}]\) is similar to \(\frac{\mu}{1-p}\) when both \(N\) and \(t\) are large. Hence \(\lambda_t^{i,N} \simeq \mathbb{E}_\theta[\lambda_t^{i,N}] \simeq \frac{\mu}{1-p}\). Then by Gaussian approximation, we can roughly replace \((Z_t^{i,N} - Z_t^{j,N})_{j=1,...,N}\) in the expression of \((\lambda_t^{i,N})_{i=1,...,N}\) by \((\frac{\mu}{1-p} + Y_t^{j,N})_{j=1,...,N}\), for an i.i.d. array \((Y_t^{j,N})_{j=1,...,N,t=1,...,T}\) of \(\mathcal{N}(0,\frac{\mu}{1-p})\)-distributed random variables. Also, we replace the \(\mathcal{P}(\lambda_t^{i,N})\) law by its Gaussian approximation.

We thus introduce the following model, with unknown parameters \(\mu > 0\) and \(p \in (0,1]\). We start with three independent families of i.i.d. random variables, namely \((\theta_{ij})_{i,j=1,...,N}\) with law Bernoulli\((p)\), and \((Y_t^{j,N})_{j=1,...,N,t=1,...,T}\) with law \(\mathcal{N}(0,\frac{\mu}{1-p})\) and \((A_t^{j,N})_{j=1,...,N,t=1,...,T}\) with law \(\mathcal{N}(0,1)\). We then set, for each \(t = 1,...,T\) and each \(i = 1,...,N\),

\[
a_t^{i,N} = \mu + \frac{1}{N} \sum_{j=1}^N \theta_{ij} \left(\frac{\mu}{1-p} + Y_t^{j,N}\right)\quad \text{and}\quad X_t^{i,N} = a_t^{i,N} + \sqrt{a_t^{i,N} A_t^{i,N}}.
\]
We compute the covariances. First, for all \( i = 1, \ldots, N \) and all \( t = 1, \ldots, T \),

\[
\text{Var}(X_i^{i,N}) = \mathbb{E}[(a_i^{i,N} + \sqrt{a_i^{i,N} A_t^{i,N}} - \frac{\mu}{1-p})^2] \\
= \mathbb{E} \left[ \frac{\mu^2}{N(1-p)} \sum_{k=1}^{N} (\theta_{ik} - p) + \frac{1}{N} \sum_{k=1}^{N} \theta_{ik} Y_t^{k,N} + \sqrt{a_i^{i,N} A_t^{i,N}} \right]^2 \\
= \frac{p\mu^2}{N(1-p)} + \frac{\mu^2}{N(1-p)^2} + \frac{\mu}{1-p}.
\]

Next, for \( i \neq j \) and all \( t = 1, \ldots, T \),

\[
\text{Cov}(X_i^{i,N}, X_j^{j,N}) = \mathbb{E} \left[ (a_i^{i,N} + \sqrt{a_i^{i,N} A_t^{i,N}} - \frac{\mu}{1-p}) (a_j^{j,N} + \sqrt{a_j^{j,N} A_t^{j,N}} - \frac{\mu}{1-p}) \right] \\
= \mathbb{E} \left[ \frac{1}{N^2} \sum_{k=1}^{N} \theta_{jk} \theta_{ik} (Y_t^{k,N})^2 \right] = \frac{p^2\mu^2}{N(1-p)^2}.
\]

For \( s \neq t \) and \( i = 1, \ldots, N \),

\[
\text{Cov}(X_i^{i,N}, X_s^{j,N}) = \mathbb{E} \left[ (a_i^{i,N} + \sqrt{a_i^{i,N} A_t^{i,N}} - \frac{\mu}{1-p}) (a_s^{j,N} + \sqrt{a_s^{j,N} A_t^{j,N}} - \frac{\mu}{1-p}) \right] \\
= \left( \frac{\mu}{1-p} \right)^2 \text{Var} \left( \frac{1}{N} \sum_{j=1}^{N} \theta_{ij} \right) = \frac{p\mu^2}{N(1-p)}.
\]

Finally, for \( s \neq t \) and \( i \neq j \),

\[
\text{Cov}(X_i^{i,N}, X_s^{j,N}) = \mathbb{E} \left[ (a_i^{i,N} + \sqrt{a_i^{i,N} A_t^{i,N}} - \mu - p) (a_s^{j,N} + \sqrt{a_s^{j,N} A_t^{j,N}} - \mu - p) \right] = 0.
\]

Over all we have \( \text{Cov}(X_i^{i,N}, X_s^{j,N}) = C_{\mu,p,N}((i,t),(j,s)) \), where

\[
C_{\mu,p,N}((i,t),(j,s)) = \begin{cases} 
\frac{p\mu^2}{N(1-p)} + \frac{p\mu^2}{(1-p)^2} + \frac{\mu}{(1-p)} & \text{if } i = j, t = s, \\
\frac{p^2\mu^2}{N(1-p)^2} & \text{if } i \neq j, t = s, \\
\frac{p\mu^2}{(1-p)^2} & \text{if } i = j, t \neq s, \\
0 & \text{if } i \neq j, t \neq s.
\end{cases}
\]

Form the covariance function above, we can ignore the covariance when \( t \neq s \). So, we construct a new covariance function:

\[
\tilde{C}_{\mu,p,N}((i,t),(j,s)) = \begin{cases} 
\frac{p\mu^2}{N(1-p)} + \frac{p\mu^2}{(1-p)^2} + \frac{\mu}{(1-p)} & \text{if } i = j, t = s, \\
\frac{p^2\mu^2}{N(1-p)^2} & \text{if } i \neq j, t = s, \\
\frac{p\mu^2}{(1-p)^2} & \text{if } i = j, t \neq s, \\
0 & \text{if } i \neq j, t \neq s.
\end{cases}
\]

We thus consider the following toy model: for two unknown parameters \( \mu > 0 \) and \( p \in (0,1) \), we observe \((U_1^{s,s}), \ldots, U_{K,s}, \ldots, T\)

for some Gaussian array \((U_1^{s,s}), \ldots, U_{N,s}, \ldots, T\)

with covariance matrix \( \tilde{C}_{\mu,p,N} \) defined above and we want to estimate \( p \). If assuming that \( \frac{1}{1-p} \) is known, it is well-known that the temporal empirical variance \( \tilde{S}_T^{N,K} = \frac{1}{T} \sum_{t=1}^{T} (\tilde{U}_t^{N,K} - \frac{\mu}{1-p})^2 \), where \( \tilde{U}_t^{N,K} = \)
\[ \frac{1}{K} \sum_{i=1}^{K} U_{i,N}^2 \] is the best estimator of \( \frac{(2r-p)^2 \mu^2}{N(K-1)} + \frac{\mu^2}{(1-p)} \) (in all the usual senses). Consequently, \( C_{K}^{N,K} = \frac{N}{K-1} \left( \frac{\mu^2}{1-p} + \frac{\mu^2}{p} \right) - \frac{2(K-1)r}{(1-p)} \) is the best estimator of \( p^2 \). And
\[ \text{Var}(C_{K}^{N,K}) = \frac{N^2}{T} \left( \frac{K^2}{2} - \frac{2(K-1)\alpha}{N} \right)^2 \approx \frac{N^2}{TK^2} \]
where \( \alpha = \frac{(2r-p)^2 \mu^2}{N(1-p)^2} + \frac{\mu^2}{(1-p)} \) and \( \alpha = \frac{r^2 \mu^2}{(1-p)^2} \) Hence for this Gaussian toy model, it is not possible to estimate \( p^2 \) (and thus \( p \)) with a precision better than \( \frac{N}{K} \).

4.3. Conclusion. Using the first example, it seems that it should not be possible to estimate \( p \) faster than \( N/(\sqrt{K} e^{-\alpha t}) + 1/\sqrt{K} \) in the supercritical case. Using the two examples, it seems that it should not be possible to estimate \( p \) faster than \( N/(t\sqrt{K}) + 1/\sqrt{K} + N/(K\sqrt{t}) \) in the subcritical case.

5. Analysis of a Random Matrix in the Subcritical Case

5.1. Some notations. For \( r \in [1, \infty) \) and \( x \in \mathbb{R}^N \), we set \( ||x||_r = (\sum_{i=1}^{N} |x_i|^r)^{\frac{1}{r}} \), and \( ||x||_{\infty} = \max_{i=1,...,N} |x_i| \). For a \( N \times N \) matrix, we denote by \( ||M||_r \) the operator norm associated to \( || \cdot ||_r \), that is \( ||M||_r = \sup_{x \in \mathbb{R}^N} ||Mx||_r/||x||_r \). We have the special cases
\[ ||M||_1 = \sup_{j=1,...,N} \sum_{i=1}^{N} |M_{ij}|, \quad ||M||_{\infty} = \sup_{i=1,...,N} \sum_{j=1}^{N} |M_{ij}|. \]
We also have the inequality
\[ ||M||_r \leq ||M||_1^{\frac{1}{r}} ||M||_{\infty}^{\frac{1}{\infty}} \quad \text{for any} \quad r \in [1, \infty). \]

We define \( A_N(i,j) := N^{-1} \theta_{ij} \) and the matrix \((A_N(i,j))_{i,j \in \{1,...,N\}}\), as well as \( Q_N := (I - \Lambda A_N)^{-1} \) on the event on which \( I - \Lambda A_N \) is invertible.

Next, we are going to give the event \( \Omega_{N,K} \) which we mainly work on it this paper.

For \( 1 \leq K \leq N \), we introduce the \( N \)-dimensional vector \( 1_K \) defined by \( 1_K(i) = 1_{\{1 \leq i \leq K\}} \) for \( i = 1, \ldots, N \), and the \( N \times N \)-matrix \( K \) defined by \( K(i,j) = 1_{\{i=j \leq K\}} \).

We assume here that \( \Lambda p \in (0,1) \) and we set \( a = \frac{1+\Lambda p}{2} \in (0,1) \). Next, we introduce the events
\[ \Omega_K := \left\{ A_{||A_N||_r} \leq a, \text{ for all } r \in [1, \infty] \right\}, \]
\[ F_{K}^{1,1} := \left\{ A_{||IK_{1,N}A_N||_r} \leq \left( \frac{a}{K} \right)^{\frac{1}{r}} a, \text{ for all } r \in [1, \infty] \right\}, \]
\[ F_{K}^{2,2} := \left\{ A_{||A_N A_1||_r} \leq \left( \frac{a}{K} \right)^{\frac{1}{r}} a, \text{ for all } r \in [1, \infty] \right\}, \]
\[ \Omega_{K}^{1} := \Omega_{N} \cap F_{K}^{1,1}, \quad \Omega_{N,K}^{1} := \Omega_{1} \cap F_{K}^{2,2}, \quad \Omega_{N,K} = \Omega_{N,K}^{1} \cap \Omega_{N,K}^{2}. \]

We set \( \ell_N := Q_N 1_N, \quad \ell_N(i) := \sum_{j=1}^{N} Q_N(i,j), \quad \ell_N := \frac{1}{K} \sum_{i=1}^{N} \ell_N(i), \quad \ell_N := \frac{1}{K} \sum_{i=1}^{K} \ell_N(i). \)
We also set \( c_K^{N}(j) := \sum_{i=1}^{K} Q_N(i,j), \quad c_K^{N} := \frac{1}{K} \sum_{j=1}^{N} c_K^{N}(j). \)
We let \( L_N := A_N 1_N, \quad L_N(i) := \sum_{j=1}^{N} A_N(i,j), \quad L_N := \frac{1}{K} \sum_{i=1}^{N} L_N(i), \quad L_N := \frac{1}{K} \sum_{i=1}^{K} L_N(i) \) and \( C_N := A_N 1_N, \quad C_N(i) := \sum_{j=1}^{N} A_N(i,j), \quad C_N := \frac{1}{K} \sum_{j=1}^{N} C_N(j), \quad C_N := \frac{1}{K} \sum_{j=1}^{K} C_N(i) \) and consider the event \( A_N := \{ ||L_N - p 1_N||_2 + ||C_N - p 1_N||_2 \leq N^{\frac{1}{2}} \}. \)
where $L_N$ is the vectors $(L_N(i))_{i=1,...,N}$. We also set $x_N(i) = \ell_N(i) - \bar{\ell}_N$, $x_N = (x_N(i))_{i=1,...,N}$, $X_N(i) = L_N(i) - L_N$ and $X_N = (X_N(i))_{i=1,...,N}$. We finally put $X^k_N(i) = (L_N(i) - L^k_N)1_{\{i \leq k\}}$ and $X^k_N = (X^k_N(i))_{i=1,...,N} = \bar{\ell}^k_N - L^k_N 1_K$, as well as $x^k_N(i) = (\ell_N(i) - \bar{\ell}^k_N)1_{\{i \leq k\}}$ and $x^k_N = (x^k_N(i))_{i=1,...,N} = \bar{\ell}^k_N - L^k_N 1_K$.

### 5.2. Review of some lemmas found in [6].

In this subsection we recall results from [6] showing that $A_N$ and $\Omega_N^1$ are big, and upper-bounds concerning $x_n$ and $X_N$.

**Lemma 5.1.** We assume that $\Lambda p < 1$. Then $\Omega_N, K \subset \Omega_N \subset \{|||Q_N|||_r \leq C, \text{ for all } r \in [1, \infty]\} \subset \{\sup_{i=1,...,N} \ell_N(i) \leq C\}$, where $C = (1 - a)^{-1}$. For any $a > 0$, there exists a constant $C_\alpha$ such that

$$P(A_N) \geq 1 - C_\alpha N^{-\alpha}.$$

**Proof.** See [6] Notation 12 and Proposition 14, Step 1.

**Lemma 5.2.** Assume that $\Lambda p < 1$. Then,

$$P(\Omega_N^1) \geq 1 - C \exp(-cN)$$

for some constants $C > 0$ and $c > 0$.

**Proof.** See [6] Lemma 13.

**Lemma 5.3.** Assume that $\Lambda p < 1$. Then

$$\mathbb{E}\left[1_{\Omega_N^1} \left| \bar{\ell}_N - \frac{1}{1 - \Lambda p} \right|^2 \right] \leq \frac{C}{N^2}.$$

**Proof.** See [6] Proposition 14.

**Lemma 5.4.** Assume that $\Lambda p < 1$, set $b = \frac{2 + \Lambda p}{3}$ and consider $N_0$ the smallest integer such that $a + \Lambda N_0^{\frac{1}{3}} \leq b$. For all $N \geq N_0$,

(i) $1_{\Omega_N^1 \cap A_N} ||x_N||_2 \leq C||X_N||_2$,  
(ii) $\mathbb{E}[||X_N||_2^2] \leq C$,  
(iii) $\mathbb{E}[||A_N X_N||_2^2] \leq CN^{-1}$.

**Proof.** See [6] Proof of Proposition 14, Steps 2 and 4.

**Remark 5.5.** In Lemma 5.4 the condition $\Lambda p < 1$ is not necessary for (ii) and (iii).

**Lemma 5.6.** Assume that $\Lambda p < 1$ and set $k = \Lambda^{-1} \int_0^\infty s \phi(s)ds$, then for $n \geq 0$, $t \geq 0$,

$$\int_0^t s \phi^n(t - s)ds = \Lambda^n t - n\Lambda^n k + \varepsilon_n(t),$$

where $0 \leq \varepsilon_n(t) \leq C \min\{n^2\Lambda^n t^{1-q}, n^2\Lambda^n k\}$ and where $\phi^n(s)$ is the $n$-times convolution of $\phi$. We adopt the convention that $\phi^n(0) = \delta_0$, whence in particular $\int_0^t s \phi^n(t - s)ds = t$.

**Proof.** See [6] Lemma 15.
5.3. **Probabilistic lower bound.** In this subsection, we are going to prove that the set \( \Omega_{N,K} \) has high probability, which will allow to work on the set \( \Omega_{N,K} \) for all our study.

**Lemma 5.7.** Assume that \( \Delta p < 1 \). It holds that

\[
P(\Omega_{N,K}) \geq 1 - CNe^{-cK}
\]

for some constants \( C > 0 \) and \( c > 0 \).

**Proof.** On \( \Omega_{N,K} \), we have

\[
N||I_{K}A_{N}||_{1} = \sup_{j=1,...,N} \sum_{i=1}^{K} \theta_{ij} = \max\{X_{1}^{N,K},...,X_{N}^{N,K}\},
\]

where \( X_{i}^{N,K} = \sum_{j=1}^{K} \theta_{ij} \) for \( i = 1,...,N \) are i.i.d and Binomial(\( K, p \))-distributed. So,

\[
P\left( \frac{N}{K}||I_{K}A_{N}||_{1} \geq a \right) = P\left( \max\{X_{1}^{N,K},...,X_{N}^{N,K}\} \geq \frac{Ka}{N} \right) \leq NP\left(X_{1}^{N,K} \geq \frac{Ka}{N} \right)
\]

\[
\leq NP\left(|X_{1}^{N,K} - Kp| \geq K\left(\frac{a}{N} - p\right)\right) \leq 2Ne^{-2K(\frac{a}{N} - p)^{2}}.
\]

The last equality follows from Hoeffding inequality. On the event \( \Omega_{N,K}^{1} \cap \{\Lambda_{N}^{K}||I_{K}A_{N}||_{1} \leq a\} \), we have

\[
||I_{K}A_{N}||_{r} \leq ||I_{K}A_{N}||_{1} \left(\frac{1}{r}\right) ||I_{K}A_{N}||_{1} \left(1 - \frac{1}{r}\right) \leq ||I_{K}A_{N}||_{1} \leq \left(\frac{a}{N}\right)^{1/r} \left(\frac{a}{N}\right)^{1-1/r} = \frac{a}{(K)^{1/r}}.
\]

We conclude that \( \Omega_{N,K}^{1} = \Omega_{N}^{1} \cap \{\Lambda_{N}^{K}||I_{K}A_{N}||_{1} \leq a\} \). And by Lemma 5.2, we deduce that \( P(\Omega_{N,K}^{1}) \geq 1 - CNe^{-cK} \). By the same way, we prove that \( P(\Omega_{N,K}^{2}) \geq 1 - CNe^{-cK} \). Finally by the definition of \( \Omega_{N,K} \), we have \( P(\Omega_{N,K}) \geq 1 - CNe^{-cK} \).

\[\square\]

5.4. **Matrix analysis for the first estimator.** The aim of this subsection is to prove that \( \bar{\ell}_{N} \simeq 1/(1 - \Delta p) \) and to study the rate of convergence.

**Lemma 5.8.** Assume \( \Delta p < 1 \). Then

\[
E\left[1_{\Omega_{N,K}}||\bar{\ell}_{N} - \ell_{N}||^{2}\right] \leq \frac{C}{NK}.
\]

**Proof.** Recall that \( \ell_{N} = Q_{N}1_{N} \), whence \( Q_{N}^{-1}\ell_{N} = 1_{N} \). And since, \( Q_{N} = (I - \Lambda A_{N})^{-1} \), we have \( Q_{N}^{T}\ell_{N} = (I - \Lambda A_{N})\ell_{N} = 1_{N} \) and thus \( \ell_{N} = 1_{N} + \Lambda A_{N}\ell_{N} \). We conclude that

\[
\bar{\ell}_{N} = \frac{1}{K}(\ell_{N}, 1_{K}) = 1 + \frac{\Lambda}{K} \sum_{i=1}^{K} \sum_{j=1}^{N} A(i,j)\ell_{N}(j) = 1 + \frac{\Lambda}{K} \sum_{j=1}^{N} C_{N}^{K}(j)\ell_{N}(j),
\]

where \( C_{N}^{K}(j) := \sum_{i=1}^{K} A(i,j) = \frac{1}{K} \sum_{i=1}^{K} \theta_{ij} \). By some easy computing, we have

\[
E\left[\left(\sum_{j=1}^{N} \left[C_{N}^{K}(j) - \frac{Kp}{N}\right]^{2}\right)\right] \leq \frac{C}{NK^{2}}.
\]
whence

\[
\begin{align*}
\mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left| \bar{\ell}_N^K - 1 - \Lambda p\bar{\ell}_N \right|^2 \right] = & \mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left| \sum_{j=1}^{N} \left(C_N^K(j) - \frac{Kp}{N}\right)\ell_N(j) \right|^2 \right] \\
= & \mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left| \sum_{j=1}^{N} \left(C_N^K(j) - \frac{Kp}{N}\right)(\ell_N(j) - \bar{\ell}_N) + \bar{\ell}_N \sum_{j=1}^{N} \left(C_N^K(j) - \frac{Kp}{N}\right) \right|^2 \right] \\
\leq & 2\mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left| \sum_{j=1}^{N} \left(C_N^K(j) - \frac{Kp}{N}\right)\ell_N(j) - \bar{\ell}_N \right|^2 \right] \\
+ & 2\mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left| \sum_{j=1}^{N} \left(C_N^K(j) - \frac{Kp}{N}\right) \right|^2 \right] .
\end{align*}
\]

Consequently,

\[
\begin{align*}
\mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left| \bar{\ell}_N^K - 1 - \Lambda p\bar{\ell}_N \right|^2 \right] \\
\leq & 2\left(\frac{A}{R}\right)^2 \mathbb{E}\left[1_{\Omega_{N,K} \cap A_N} \left\| x_N \right\|^2 \right] \mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left( \sum_{j=1}^{N} \left(C_N^K(j) - \frac{Kp}{N}\right)^2 \right)^{\frac{1}{2}} \right] \\
+ & 2\frac{A^2}{K^2} \mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left| \bar{\ell}_N \right|^2 \right] \mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left( \sum_{j=1}^{N} \left(C_N^K(j) - \frac{Kp}{N}\right)^2 \right) \right] \\
\leq & \frac{C}{NK}\mathbb{E}\left[1_{\Omega_{N,K} \cap A_N} \left\| x_N \right\|^2 \right] + 2\frac{A^2}{K^2} \mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left| \bar{\ell}_N \right|^2 \right] \mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left( \sum_{j=1}^{N} \left(C_N^K(j) - \frac{Kp}{N}\right)^2 \right) \right] .
\end{align*}
\]

By Lemma 5.1, we know that \(\mathbb{E}[1_{\Omega_{N,K} \cap A_N} \left\| x_N \right\|^2] \leq C\). By Lemma 5.1, \(\bar{\ell}_N\) and \(\bar{\ell}_N^K\) are bounded on the set \(\Omega_{N,K}\), whence, recalling (S), and since \(\{C_N^K(j) - \frac{Kp}{N}\}_{j=1}^{N}\) are independent, we conclude that

\[
\begin{align*}
\mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left| \bar{\ell}_N \right|^2 \right] \sum_{j=1}^{N} \left(C_N^K(j) - \frac{Kp}{N}\right)^2 \\
\leq C \mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left( \sum_{j=1}^{N} \left(C_N^K(j) - \frac{Kp}{N}\right)^2 \right) \right] \\
\leq C \mathbb{E}\left[ \sum_{j=1}^{N} \left(C_N^K(j) - \frac{Kp}{N}\right)^2 \right] \\
\leq \frac{CK}{N}.
\end{align*}
\]

Hence

\[
\begin{align*}
\mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left| \bar{\ell}_N^K - 1 - \Lambda p\bar{\ell}_N \right|^2 \right] \leq \frac{C}{NK}.
\end{align*}
\]

We finally apply Lemma 5.1 with \(\alpha = 2\) and get

\[
\begin{align*}
\mathbb{E}\left[1_{\Omega_{N,K}} \left| \bar{\ell}_N^K - 1 - \Lambda p\bar{\ell}_N \right|^2 \right] = & \mathbb{E}\left[1_{\Omega_{N,K} \cap A_N}\left| \bar{\ell}_N^K - 1 - \Lambda p\bar{\ell}_N \right|^2 \right] + \mathbb{E}\left[1_{\Omega_{N,K} \cap A_N^{c}}\left| \bar{\ell}_N^K - 1 - \Lambda p\bar{\ell}_N \right|^2 \right] \\
\leq & \frac{C}{NK} + \frac{C}{N^2} \leq \frac{C}{NK} .
\end{align*}
\]
The next lemma is the main result of the subsection.

Lemma 5.9. If $\Lambda p < 1$, we have
\[
E\left[\mathbf{1}_{\Omega_{N,K}} \left(\hat{\ell}^K_N - \frac{1}{1 - \Lambda p}\right)^2\right] \leq \frac{C}{NK}.
\]

Proof. Observing that $1/(1 - \Lambda p) = 1 + \Lambda p/(1 - \Lambda p)$, we write
\[
E\left[\mathbf{1}_{\Omega_{N,K}} \left(\hat{\ell}^K_N - \frac{1}{1 - \Lambda p}\right)^2\right] \leq 2E\left[\mathbf{1}_{\Omega_{N,K}} \left(\hat{\ell}^K_N - 1 - \Lambda p\hat{\ell}^K_N\right)^2\right] + 2\Lambda p^2E\left[\mathbf{1}_{\Omega_{N,K}} \left(\hat{\ell}^K_N - \frac{1}{1 - \Lambda p}\right)^2\right].
\]

We complete the proof applying Lemmas 5.3 and 5.8.

5.5. Matrix analysis for the second estimator. The aim of this subsection is to prove that $\frac{N}{\Omega} \|x^K_N\|_2^2 \simeq \Lambda^2 p(1-p)/(1 - \Lambda p)^2$ and to study the rate of convergence.

Lemma 5.10. Assume that $p \in (0,1]$. It holds that
\[
E[\|I_K A_N X_N\|_2^2] \leq CKN^{-2}.
\]

Proof. By Lemma 5.4, we already know that $E[\|A_N X_N\|_2^2] \leq \frac{C}{N}$, whence
\[
E\left[\|I_K A_N X_N\|_2^2\right] = \sum_{i=1}^{K} E\left[\left(\sum_{j=1}^{N} \frac{\theta_{ij}}{N} (L_N(j) - \bar{L}_N)\right)^2\right] = \frac{K}{N^2} E\left[\left(\sum_{j=1}^{N} \theta_{ij}(L_N(j) - \bar{L}_N)\right)^2\right],
\]

which equals $\frac{K}{N}E[\|A_N X_N\|_2^2]$ and thus is bounded by $CKN^{-2}$.

Lemma 5.11. Assume that $\Lambda p < 1$. It holds that
\[
E\left[\mathbf{1}_{\Omega_{N,K} \cap A_N}\|x^K_N - \bar{\ell}_N A \bar{X}^K_N\|_2^2\right] \leq CN^{-1}.
\]

Proof. By definition, $\ell^K_N = I_K \ell^K_N = 1_K + \Lambda A_N \ell_N$, so that
\[
\hat{\ell}^K_N = \frac{1}{K}(1_K, \ell^K_N) = \frac{1}{K}(1_K, I_K \ell_N) = \frac{1}{K}(1_K, 1_K + \Lambda A_N \ell_N) = 1 + \frac{\Lambda}{K}(I_K A_N \ell_N, 1_K).
\]

And, recalling that $x^K_N = \ell^K_N - \bar{\ell}_N 1_K$, we find
\[
x^K_N = 1_K + \Lambda A_N \ell_N - [1 + \frac{\Lambda}{K}(I_K A_N \ell_N, 1_K)] 1_K
\]
\[
= \Lambda A_N \ell_N - \frac{\Lambda}{K}(I_K A_N \ell_N, 1_K) 1_K
\]
\[
= \Lambda A_N \ell_N - \frac{\Lambda}{K}(I_K A_N (\ell_N - \bar{\ell}_N 1_N), 1_K) 1_K
\]
\[
+ \bar{\ell}_N[\Lambda A_N 1_N - \frac{\Lambda}{K}(I_K A_N 1_N, 1_K)] 1_K
\]
\[
= \Lambda A_N x_N - \frac{\Lambda}{K}(I_K A_N x_N, 1_K) 1_K + \bar{\ell}_N[\Lambda A_N 1_N - \frac{\Lambda}{K}(I_K A_N 1_N, 1_K)] 1_K
\]
\[
= \Lambda A_N x_N - \frac{\Lambda}{K}(I_K A_N x_N, 1_K) 1_K + \Lambda \bar{\ell}_N X^K_N.
\]

□
We deduce that
\[ x^K_N - \bar{\ell}_N X^K_N = \Lambda I_K A_N x_N - \frac{\Lambda}{K}(I_K A_N x_N, 1_K)1_K \]
\[ = \Lambda I_K A_N (x_N - \bar{\ell}_N X_N) - \frac{\Lambda}{K}(I_K A_N x_N, 1_K)1_K + \bar{\ell}_N \Lambda^2 I_K A_N X_N \]
\[ = \Lambda I_K A_N (x_N - \bar{\ell}_N X_N) + \bar{\ell}_N \Lambda^2 I_K A_N X_N - \frac{\Lambda}{K} \sum_{i=1}^{K} \sum_{j=1}^{N} A_N(i,j)x_N(j)1_K \]
\[ = \Lambda I_K A_N (x_N - \bar{\ell}_N X_N) + \bar{\ell}_N \Lambda^2 I_K A_N X_N - \frac{\Lambda}{K} \sum_{j=1}^{N} \left[ C^K_N(j) - \frac{K}{N} p \right] x_N(j)1_K. \]

In the last step, we used that \( \sum_{i=1}^{N} x(i) = 0 \). As a conclusion,
\[ \|x^K_N - \bar{\ell}_N \Lambda X^K_N\|^2 \leq 3(\Lambda \|I_K A_N (x_N - \bar{\ell}_N X_N)\|^2 + 3(\Lambda^2 \bar{\ell}_N \|I_K A_N X_N\|^2) \]
\[ + 3\Lambda^2 K^{-1} \left( \sum_{j=1}^{N} \left[ C^K_N(j) - \frac{K}{N} p \right] x_N(j) \right)^2. \]

By the Cauchy-Schwarz inequality, (8) and Lemma 5.4, we have
\[ \mathbb{E}\left[ \sum_{j=1}^{N} \left[ C^K_N(j) - \frac{K}{N} p \right] x_N(j) \right] \leq C K^{-1}. \]

We also know from [6, Proposition 14, step 7, line 12] that \( \mathbb{E}\left[ \|x_N - \bar{\ell}_N X_N\|^2 \right] \leq \frac{C}{N} \).

And also, by the definition, \( \|A_N\|_2 \) is bounded on \( A_N \). So
\[ \mathbb{E}\left[ \|I_K A_N\|_2^2 \|x_N - \bar{\ell}_N X_N\|^2 \right] \leq \frac{C}{N}. \]

Recalling Lemma 5.10 and that \( \bar{\ell}_N \) is bounded on \( \Omega_{N,K} \), the conclusion follows. \( \square \)

**Lemma 5.12.** Assume that \( \Delta p < 1 \). It holds that
\[ \mathbb{E}\left[ \|x^K_N\|^2 - (\Lambda \bar{\ell}_N)^2 \|X^K_N\|^2 \right] \leq \frac{C \sqrt{K}}{N}. \]

**Proof.** We start from
\[ \|x^K_N\|^2 - (\Lambda \bar{\ell}_N)^2 \|X^K_N\|^2 \leq \|x^K_N - (\Lambda \bar{\ell}_N)X^K_N\|^2 + (\Lambda \bar{\ell}_N)\|X^K_N\|^2, \]
whence
\[ \mathbb{E}\left[ \|x^K_N\|^2 - (\Lambda \bar{\ell}_N)^2 \|X^K_N\|^2 \right] \leq \mathbb{E}\left[ \|x^K_N - (\Lambda \bar{\ell}_N)X^K_N\|^2 \right] + \mathbb{E}\left[ (\Lambda \bar{\ell}_N)\|X^K_N\|^2 \right]. \]

By the Cauchy-Schwarz inequality,
\[ \mathbb{E}\left[ \|x^K_N - (\Lambda \bar{\ell}_N)X^K_N\|^2 \right] \leq \mathbb{E}\left[ (\|x^K_N\|^2 + (\Lambda \bar{\ell}_N)\|X^K_N\|^2) \right]^2. \]
Lemma 5.11 directly tells us that \( E[\mathbf{1}_{\Omega_{N,K} \cap A_N} \| \mathbf{x}_N^K - \bar{\ell}_N \Lambda \mathbf{X}_N^K \|_2^2] \leq C/N. \)

Next, it is easy to prove, using that \( \| \mathbf{X}_N^K \|_2^2 = \sum_{i=1}^{K} (L_N(i) - \bar{L}_N(i)) \), that \( NL_N(1), \ldots, NL_N(K) \) are i.i.d. and Binomial(\( N, p \)), that

\[
E \left[ \left( \frac{N}{K} \| \mathbf{x}_N^K \|_2^2 - p(1-p) \right)^2 \right] \leq CK^{-1},
\]

whence, recalling that \( \bar{\ell}_N \) is bounded on \( \Omega_{N,K} \),

\[
E \left[ \mathbf{1}_{\Omega_{N,K} \cap A_N} \left( \frac{N}{K} \| \mathbf{x}_N^K \|_2^2 - p(1-p) \right)^2 \right] \leq C \left( \frac{N}{K} \right) \| \mathbf{X}_N^K \|_2^2 - p(1-p) \right)^2 + C \leq C.
\]

Then, by Lemma 5.11 again,

\[
E \left[ \mathbf{1}_{\Omega_{N,K} \cap A_N} \frac{N}{K} \| \mathbf{x}_N^K \|_2^2 \right] \leq 2E \left[ \mathbf{1}_{\Omega_{N,K} \cap A_N} \| \mathbf{x}_N^K - \bar{\ell}_N \Lambda \mathbf{X}_N^K \|_2^2 \right] + 2E \left[ \mathbf{1}_{\Omega_{N,K} \cap A_N} \| \bar{\ell}_N \Lambda \mathbf{X}_N^K \|_2^2 \right] \leq C \frac{K}{N}.
\]

The conclusion follows. \( \square \)

**Lemma 5.13.** Assume that \( \Lambda p < 1 \). It holds that

\[
E \left[ \mathbf{1}_{\Omega_{N,K} \cap A_N} \left( \frac{N}{K} \right)^2 \| \mathbf{X}_N^K \|_2^2 - \frac{p(1-p)}{(1-\Lambda p)^2} \right] \leq \frac{C}{\sqrt{K}}.
\]

**Proof.** We define

\[
d^K_N = E \left[ \mathbf{1}_{\Omega_{N,K} \cap A_N} \left( \frac{N}{K} \right)^2 \| \mathbf{X}_N^K \|_2^2 - \frac{p(1-p)}{(1-\Lambda p)^2} \right].
\]

Then \( d^K_N \leq a^K_N + b^K_N \), where

\[
a^K_N = \frac{N}{K} E \left[ \mathbf{1}_{\Omega_{N,K} \cap A_N} \left( \frac{N}{K} \right)^2 \| \mathbf{X}_N^K \|_2^2 \right],
\]

\[
b^K_N = \left( 1 - \Lambda p \right)^{-2} E \left[ \mathbf{1}_{\Omega_{N,K} \cap A_N} \left( \frac{N}{K} \right)^2 \| \mathbf{X}_N^K \|_2^2 \right].
\]

First, (9) directly implies that \( b^K_N \leq C/\sqrt{K} \). Next, (9) also implies that \( E[\left( \frac{N}{K} \right)^2 \| \mathbf{X}_N^K \|_2^2] \leq C \), whence \( a^K_N \leq C/\sqrt{K} \) by Lemma 5.3. This completes the proof. \( \square \)

Here is the main lemma of this subsection.

**Lemma 5.14.** Assume that \( \Lambda p < 1 \). It holds that

\[
E \left[ \mathbf{1}_{\Omega_{N,K} \cap A_N} \left( \frac{N}{K} \right)^2 \| \mathbf{x}_N^K \|_2^2 - \frac{\Lambda^2 p(1-p)}{(1-\Lambda p)^2} \right] \leq \frac{C}{\sqrt{K}}.
\]

**Proof.** It directly follows from Lemmas 5.12 and 5.13 that

\[
E \left[ \mathbf{1}_{\Omega_{N,K} \cap A_N} \left( \frac{N}{K} \right)^2 \| \mathbf{x}_N^K \|_2^2 - \frac{\Lambda^2 p(1-p)}{(1-\Lambda p)^2} \right] \leq \Lambda^2 E \left[ \mathbf{1}_{\Omega_{N,K} \cap A_N} \left( \frac{N}{K} \right)^2 \| \mathbf{X}_N^K \|_2^2 - \frac{p(1-p)}{(1-\Lambda p)^2} \right] \]

\[
+ \frac{N}{K} E \left[ \mathbf{1}_{\Omega_{N,K} \cap A_N} \| \mathbf{x}_N^K \|_2^2 - \Lambda^2 \| \mathbf{X}_N^K \|_2^2 \right] \]

\[
\leq C \left( \frac{1}{\sqrt{K}} + \frac{N \sqrt{K}}{N} \right),
\]

from which the conclusion. \( \square \)
5.6. Matrix analysis for the third estimator. We define \( W_{\infty,\infty}^{N,K} := \frac{1}{N} \sum_{j=1}^{N} (c_N^K(j))^2 \ell_N(j) - \frac{N-K}{N} \), \( X_{\infty,\infty}^{N,K} := W_{\infty,\infty}^{N,K} - \mu \). The aim of this subsection is to prove that \( X_{\infty,\infty}^{N,K} \) is correct for \( \mu \) and to study the rate of convergence.

**Lemma 5.15.** Assume that \( \lambda p < 1 \). It holds that
\[
\mathbb{E}[\| F_N^K \|^2] \leq \frac{C K}{N},
\]
where \( F_N^K := 1_N^T A_N - \frac{1}{N} (1_N^T A_N, 1_N^T) 1_N^K \) is a row vector.

**Proof.** Since the inequality \( \sum_{i=1}^n (x_i - \bar{x})^2 \leq \sum_{i=1}^n (x_i - m)^2 \), where \( \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \), is correct for any real sequence \( \{ x_i \}_{i=1}^n \) and real number \( p \). By definition,
\[
\mathbb{E}[\| F_N^K \|^2] = \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^K \theta_{ij} - \frac{1}{N^2} \sum_{i=1}^K \sum_{j=1}^N \theta_{ij} \right)^2 \right] 
\leq \frac{1}{N} \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^K \theta_{ij} \right)^2 \right] \leq \frac{1}{N} \mathbb{E} \left[ \sum_{i=1}^K (\theta_{i1} - p)^2 \right] \leq \frac{C K}{N}.
\]
\( \Box \)

**Lemma 5.16.** Assume that \( \lambda p < 1 \). It holds that
\[
\mathbb{E} \left[ 1_{\Omega_{N,K}} \| t_N^K \|^2 \right] \leq \frac{C K^2}{N^2},
\]
where \( c_N^K = 1_N^T Q_N \), \( \tilde{c}_N^K := \frac{1}{N} \sum_{j=1}^N c_N^K(j) \), and
\[
t_N^K := c_N^K - \tilde{c}_N^K 1_N^T - 1_N^K + \frac{K}{N} 1_N^T.
\]

**Proof.** By definition, \( c_N^K := 1_N^T Q_N \), \( \tilde{c}_N^K := \frac{1}{N} (c_N^K, 1_N) \), \( Q_N = (I - \Lambda A_N)^{-1} \), so that
\[
c_N^K = 1_N^T A_N, \quad \tilde{c}_N^K = \frac{1}{N} (c_N^K, 1_N) = \frac{K}{N} + \frac{\Lambda c_N^K A_N, 1_N^T}{N^2}.
\]
We deduce that
\[
t_N^K = c_N^K - \tilde{c}_N^K 1_N^T - 1_N^K + \frac{K}{N} 1_N^T
= \Lambda c_N^K A_N - \frac{\Lambda}{N} (c_N^K A_N, 1_N^T) 1_N^K
= \Lambda t_N^K A_N - \frac{\Lambda}{N} (t_N^K A_N, 1_N^T) 1_N^K + \Lambda c_N^K 1_N^T A_N - \frac{\Lambda}{N} \tilde{c}_N^K (1_N A_N, 1_N^T) 1_N^K
+ \Lambda 1_N^T A_N - \frac{\Lambda}{N} (1_N A_N, 1_N^T) 1_N^K - \frac{\Lambda}{N} \tilde{c}_N^K 1_N^T A_N + \frac{\Lambda}{N^2} (1_N A_N, 1_N^T) 1_N^K
\]
where \( X_N^K = 1_N^T A_N - \frac{1}{N} (1_N A_N, 1_N^T) 1_N^K \). And it is clear that \( \frac{N}{K} \tilde{c}_N^K = \tilde{\ell}_N^K \).

By Lemma 5.11, \( \ell_N^K \) and \( \tilde{\ell}_N^K \) are bounded on the set \( \Omega_{N,K} \), whence, using Lemma 5.14,
\[
\mathbb{E} \left[ 1_{\Omega_{N,K}} \left( \| \Lambda c_N^K X_N^K \|^2 + \| \Lambda \frac{K}{N} X_N^K \|^2 \right) \right] \leq C \frac{K^2}{N^2}.
\]
Next, Lemma 5.15 tells us that
\[
E \left[ 1_{\Omega, N} \parallel F^K_N \parallel_2^2 \right] \leq E \left[ \parallel F^K_N \parallel_2^2 \right] \leq C K^2 / N^2.
\]
Observing that \( \sum_{i=1}^N t^K_N(i) = 0 \), we see that
\[
\left\| \frac{\Lambda}{N} (t^K_N A_N, 1^T_N) 1^T_N \right\|_2^2 = \frac{\Lambda^2}{N^3} \left( \sum_{i=1}^N \sum_{j=1}^N (\theta_{ij} - p)t^K_N(i) \right)^2
\]
\[
= \frac{\Lambda^2}{N} \left( \sum_{i=1}^N (C_N(i) - p)t^K_N(i) \right)^2,
\]
so that
\[
E \left[ 1_{\Omega, N} \cap A_N \parallel t^K_N \parallel_2^2 \right] \leq \Lambda^2 \frac{E \left[ 1_{\Omega, N} \cap A_N \parallel t^K_N \parallel_2^2 \parallel C_N - p1_N \parallel_2^2 \right]}{N^1/2} \leq \Lambda^2 \frac{E \left[ 1_{\Omega, N} \cap A_N \parallel t^K_N \parallel_2^2 \right]}{N^1/2}
\]
by definition of \( A_N \). Since finally \( \parallel \Lambda t^K_N A_N \parallel_2 \leq \parallel \Lambda A_N \parallel_2 \parallel t^K_N \parallel_2 \leq a \parallel \Lambda t^K_N \parallel_2 \) on \( \Omega, N, K \) with \( a = (1 + \Lambda p)/2 \), we conclude that
\[
E \left[ 1_{\Omega, N} \cap A_N \parallel t^K_N \parallel_2^2 \right] \leq C K^2 / N^2 + (a + \Lambda^2 N^{-1/2})E \left[ 1_{\Omega, N} \cap A_N \parallel t^K_N \parallel_2^2 \right].
\]
Since \( (a + \Lambda^2 N^{-1/2}) < (a + 1)/2 < 1 \) for all \( N \) large enough, we conclude that, for some constant \( C > 0 \), for all \( N \geq 1 \),
\[
E \left[ 1_{\Omega, N} \cap A_N \parallel t^K_N \parallel_2^2 \right] \leq C K^2 / N^2.
\]
Finally, observing that \( \parallel t^K_N \parallel_2^2 \) is obviously bounded by \( CN \) on \( \Omega, N, K \) and recalling that \( \mathbb{P}(A_N) \geq 1 - C/N^3 \) by Lemma 5.1, we easily conclude that
\[
E \left[ 1_{\Omega, N} \parallel t^K_N \parallel_2^2 \right] \leq C K^2 + 1 / N^2 \leq C K^2 / N^2
\]
as desired. \( \square \)

**Lemma 5.17.** Assume that \( \Lambda p < 1 \). It holds that
\[
E \left[ 1_{\Omega, N} \parallel f^K_N \parallel_2^2 \right] \leq C K^2 / N^2, \quad E \left[ 1_{\Omega, N} \parallel (f^K_N, 1^T_N) \parallel_2 \right] \leq C K / N
\]
where \( f^K_N := t^K_N 1_K \).
Proof. The first inequality is obvious from Lemma 5.16 because \( \| f_N^K \| \leq \| t_N^K \| \). For the second inequality, by (10), we have

\[
\begin{align*}
\mathbb{E}\left[ 1_{\Omega_{N,K}} \left( f_N^K, 1_N^K \right) \right] &= \mathbb{E}\left[ 1_{\Omega_{N,K}} \left( t_N^K, 1_N^K \right) \right] \\
&= \mathbb{E}\left[ 1_{\Omega_{N,K}} \left( \Lambda t_N^K A_N I_K - \frac{\Lambda}{N} (t_N^K A_N, 1_N^K) t_N^K + \Lambda e_N^K X_N I_K - \Lambda F_N^K I_K - \Lambda \frac{K}{N} X_N I_K, 1_N^K \right) \right] \\
&\leq \frac{CK}{N} \mathbb{E}\left[ 1_{\Omega_{N,K}} \left( X_N I_K, 1_N^K \right) \right] + C \mathbb{E}\left[ 1_{\Omega_{N,K}} \left( F_N^K I_K, 1_N^K \right) \right] \\
&\quad + \frac{CK}{N} \mathbb{E}\left[ 1_{\Omega_{N,K}} \left( t_N^K A_N, 1_N^K \right) \right] + \Lambda \mathbb{E}\left[ 1_{\Omega_{N,K}} \left( t_N^K A_N I_K, 1_N^K \right) \right].
\end{align*}
\]

We used that \( (N/K) \hat{e}_N^K = \hat{e}_N^K \) is bounded on \( \Omega_{N,K} \). First,

\[
\mathbb{E}\left[ 1_{\Omega_{N,K}} \left( X_N I_K, 1_N^K \right) \right] = \mathbb{E}\left[ \sum_{i=1}^K X_N(i) \right] = \mathbb{E}\left[ \sum_{i=1}^K \left( L_N(i) - p \right) + K(p - \bar{L}_N) \right]^2 \\
\leq 2 \mathbb{E}\left[ \left( \sum_{i=1}^K \left( L_N(i) - p \right) \right)^2 \right] + 2K^2 \mathbb{E}\left[ (p - L_N)^2 \right] \leq \frac{CK}{N} \leq C,
\]

using only that \( NL_N(1), \ldots, NL_N(N) \) are i.i.d. and Binomial\((N,p)\)-distributed. Next,

\[
(F_N^K I_K, 1_K) = \frac{1}{N} \left( \sum_{j=1}^K \sum_{i=1}^K \theta_{ij} - \frac{K}{N} \sum_{i=1}^K \sum_{j=1}^K \theta_{ij} \right) \\
= \frac{1}{N} \left[ \frac{N - K}{N} \sum_{j=1}^K \sum_{i=1}^K \theta_{ij} - \frac{K}{N} \sum_{i=1}^K \sum_{j=k+1}^K \theta_{ij} - p \right],
\]

so that

\[
\mathbb{E}\left\| (F_N^K I_K, 1_N^K) \right\| \leq \mathbb{E}\left[ \left( F_N^K I_K, 1_N^K \right)^2 \right] \leq \frac{C}{N} \left( \frac{N - K}{N} + \frac{K}{N} \sqrt{K(N - K)} \right) \leq \frac{CK}{N}.
\]

Next, since \( \sum_{i=1}^N t_N^K(i) = 0 \),

\[
\mathbb{E}\left[ 1_{\Omega_{N,K}} \left( t_N^K A_N, 1_N^K \right) \right] = \mathbb{E}\left[ 1_{\Omega_{N,K}} \left( \frac{1}{N} \sum_{i,j=1}^N \theta_{ij} t_N^K(i) \right) \right] \\
= \mathbb{E}\left[ 1_{\Omega_{N,K}} \left( \frac{1}{N} \sum_{i,j=1}^N (\theta_{ij} - p) t_N^K(i) \right) \right] \\
\leq \frac{C}{N} \mathbb{E}\left[ 1_{\Omega_{N,K}} \left( \sum_{i=1}^N \left( t_N^K(i) \right)^2 \right)^{\frac{1}{2}} \right] \mathbb{E}\left[ \sum_{i=1}^N \left( \sum_{j=1}^N (\theta_{ij} - p) \right)^2 \right] \leq C.
\]
by Lemma \[5.16\] Finally,

\[
\mathbb{E}\left[ 1_{\Omega_{N,K}} \left( t_N^K A_N I_K, 1_K^K \right) \right] = \frac{1}{N} \mathbb{E}\left[ 1_{\Omega_{N,K}} \sum_{i,j=1}^{K} t_N^K(i) \theta_{ij} \right]
\]

\[
\leq \frac{1}{N} \mathbb{E}\left[ 1_{\Omega_{N,K}} \sum_{i,j=1}^{K} t_N^K(i)(\theta_{ij} - p) \right] + \frac{Kp}{N} \mathbb{E}\left[ 1_{\Omega_{N,K}} \sum_{i=1}^{K} t_N^K(i) \right] 
\]

\[
\leq \frac{1}{N} \mathbb{E}\left[ 1_{\Omega_{N,K}} \sum_{i=1}^{K} \left( t_N^K(i) \right)^2 \right] + \frac{Kp}{N} \mathbb{E}\left[ 1_{\Omega_{N,K}} \left( f_N^K, 1_K^K \right) \right]
\]

\[
\leq C \frac{K}{N} + \frac{Kp}{N} \mathbb{E}\left[ 1_{\Omega_{N,K}} \left( f_N^K, 1_K^K \right) \right]
\]
by Lemma \[5.16\] All this proves that

\[
\mathbb{E}\left[ 1_{\Omega_{N,K}} \left( f_N^K, 1_K^K \right) \right] \leq C \frac{K}{N} + \frac{Kp}{N} \mathbb{E}\left[ 1_{\Omega_{N,K}} \left( f_N^K, 1_K^K \right) \right],
\]
whence the conclusion since \( Kp/N \leq p < 1 \).

\[\square\]

**Lemma 5.18.** Assume that \( \Lambda p < 1 \). It holds that

\[
\mathbb{E}\left[ 1_{\Omega_{N,K}} \left( t_N^K A_N I_K, 1_K^K \right) \right] \leq \frac{C}{NK} 
\]

**Proof.** Recall that \( \frac{N}{K} \tilde{c}_N^K = \tilde{c}_N^K \), whence

\[
\left( \tilde{c}_N^K + \frac{N-K}{N} \right)^2 \sum_{j=1}^{K} \ell_N(j) + \left( \tilde{c}_N^K - \frac{K}{N} \right)^2 \sum_{j=K+1}^{N} \ell_N(j)
\]

\[
= \left( \frac{K}{N} \right)^2 (\tilde{c}_N^K - 1)^2 N \tilde{c}_N^K + 2 \frac{K}{N} (\tilde{c}_N^K - 1) K \tilde{c}_N^K 
\]

\[
= \frac{K^2}{N} (\tilde{c}_N^K - 1)^2 \tilde{c}_N^K + 2 \frac{K^2}{N} (\tilde{c}_N^K - 1) \tilde{c}_N^K + K \tilde{c}_N^K 
\]

\[
= 2 \left( \tilde{c}_N^K \tilde{c}_N^K - (\tilde{c}_N^K)^2 \right) + \frac{K^2}{N} (\tilde{c}_N^K - \tilde{c}_N^K)^2 \tilde{c}_N^K + \frac{K^2}{N} (\tilde{c}_N^K)^2 \tilde{c}_N^K - \frac{K^2}{N} \tilde{c}_N^K + K \tilde{c}_N^K.
\]

Consequently,

\[
\frac{N}{K^2} \left( \tilde{c}_N^K + \frac{N-K}{N} \right)^2 \sum_{j=1}^{K} \ell_N(j) + \frac{N}{K^2} \left( \tilde{c}_N^K - \frac{K}{N} \right)^2 \sum_{j=K+1}^{N} \ell_N(j)
\]

\[
= \left( \tilde{c}_N^K \tilde{c}_N^K - (\tilde{c}_N^K)^2 \right) + \left( \tilde{c}_N^K - \tilde{c}_N^K \right)^2 \tilde{c}_N^K + \frac{N-K}{K} \tilde{c}_N^K.
\]
On the event $\Omega_{N,K}$, we have $\tilde{\ell}_N^K$, $\bar{\ell}_N$ are bounded. Hence,

$$E\left[1_{\Omega_{N,K}} \left| -\tilde{\ell}_N^K - \bar{\ell}_N + \left(\tilde{\ell}_N^K\right)^2 \right|^2 \right] = E\left[1_{\Omega_{N,K}} \left| \frac{1}{1 - \Lambda p^3} \right| - \bar{\ell}_N \right|^2 \right]$$

$$\leq C E\left[1_{\Omega_{N,K}} \left| \frac{1}{1 - \Lambda p^3} - \bar{\ell}_N \right|^2 \right]$$

$$\leq C E\left[1_{\Omega_{N,K}} \left| \tilde{\ell}_N^K - \bar{\ell}_N \right|^2 \right] + C E\left[1_{\Omega_{N,K}} \left| \frac{1}{1 - \Lambda p^3} - \bar{\ell}_N \right|^2 \right] \leq \frac{C}{NK}$$

by Lemmas 5.3 and 5.9. Similarly,

$$E\left[1_{\Omega_{N,K}} \left[ \left(\tilde{\ell}_N^K\right)^2 - \frac{1}{(1 - \Lambda p^3)N} \right]^2 \right]$$

$$\leq C E\left[1_{\Omega_{N,K}} \left| \tilde{\ell}_N^K - \bar{\ell}_N \right|^2 \right] + C E\left[1_{\Omega_{N,K}} \left| \frac{1}{1 - \Lambda p^3} - \bar{\ell}_N \right|^2 \right] \leq \frac{C}{NK}.$$ 

The conclusion follows.

Here is the main result of this subsection.

**Lemma 5.19.** Assume that $\Lambda p < 1$. We have that

$$E\left[1_{\Omega_{N,K}} \left| \chi_{\infty}^{N,K} - \frac{\mu}{(1 - \Lambda p^3)^3} \right| \right] \leq \frac{C}{K}.$$ 

**Proof.** By definition,

$$\chi_{\infty}^{N,K} = \frac{\mu}{(1 - \Lambda p^3)^3} - \sum_{j=1}^{N} \left( \frac{c_N^K(j)}{N - K} \right)^2 \bar{\ell}_N(j) - \frac{\mu(N - K)}{K} \bar{\ell}_N^K - \frac{\mu}{(1 - \Lambda p^3)^3} = \mu \sum_{\alpha=1}^{3} I_{N,K}^\alpha,$$

where

$$I_{N,K}^1 = \frac{N}{K^2} \sum_{j=1}^{K} \left[ \frac{c_N^K(j) - c_N^K}{N - K} \right]^2 \ell_N(j) + \frac{N^2}{K^2} \sum_{j=K+1}^{N} \left[ \frac{c_N^K(j) - c_N^K}{N - K} \right]^2 \ell_N(j),$$

$$I_{N,K}^2 = 2 \frac{N}{K^2} \left[ \frac{c_N^K - \frac{K}{N}}{N} \right] \sum_{j=1}^{K} \ell_N(j) \left[ \frac{c_N^K(j) - c_N^K}{N - K} \right]$$

$$+ 2 \frac{N}{K^2} \left[ \frac{c_N^K - \frac{K}{N}}{N} \right] \sum_{j=K+1}^{N} \ell_N(j) \left[ \frac{c_N^K(j) - c_N^K + \frac{K}{N}}{N} \right],$$

$$I_{N,K}^3 = \frac{N}{K^2} \left[ \frac{c_N^K - \frac{K}{N}}{N} \right] \sum_{j=1}^{K} \ell_N(j) + \frac{N}{K^2} \left[ \frac{c_N^K - \frac{K}{N}}{N} \right] \sum_{j=K+1}^{N} \ell_N(j) - \frac{\mu(N - K)}{K} \bar{\ell}_N - \frac{1}{(1 - \Lambda p^3)^3}.$$ 

By Lemma 5.16, $\bar{\ell}_N$ and $\ell_N(j)$ are bounded on the set $\Omega_{N,K}$ for any $j = 1, ..., N$, whence

$$E\left[1_{\Omega_{N,K}} \left| I_{N,K}^1 \right| \right] \leq C \frac{N}{K^2} E\left[1_{\Omega_{N,K}} \left| c_N^K - c_N^K + \frac{K}{N} \left\| I_N^K \right\|_2^2 \right| \right] = C \frac{N}{K^2} E\left[1_{\Omega_{N,K}} \left\| I_N^K \right\|_2^2 \right] \leq \frac{C}{N}.$$ 

Recall the result from Lemma 5.16 we have

$$E\left[1_{\Omega_{N,K}} \left| I_{N,K}^3 \right| \right] \leq \frac{C}{\sqrt{NK}} \leq \frac{C}{K}.$$
Next, we have $I^2_{N,K} = 2I^2_{N,K} + 2I^2_{N,K}$, where

\[ I^2_{N,K} = \frac{N}{K^2} \sum_{j=1}^{\bar{\ell}_N} \ell_N(j) \left[ c^K_N(j) - \bar{c}^K_N - \frac{N - K}{N} \right], \]

\[ I^2_{N,K} = \frac{N}{K^2} \left[ c^K_N - \frac{N - K}{N} \right] \left\{ \sum_{j=1}^{\bar{\ell}_N} x_N(j) \left[ c^K_N(j) - \bar{c}^K_N - \frac{N - K}{N} \right] + \sum_{j=K+1}^{N} \ell_N(j) \left[ c^K_N(j) - \bar{c}^K_N + \frac{K}{N} \right] \right\}. \]

Since

\[ \sum_{j=1}^{K} \left[ c^K_N(j) - \bar{c}^K_N - \frac{N - K}{N} \right] + \sum_{j=K+1}^{N} \left[ c^K_N(j) - \bar{c}^K_N + \frac{K}{N} \right] = 0, \]

we may write

\[ I^2_{N,K} = \frac{N}{K^2} \left[ c^K_N - \frac{N - K}{N} \right] \left\{ \sum_{j=1}^{\bar{\ell}_N} x_N(j) \left[ c^K_N(j) - \bar{c}^K_N - \frac{N - K}{N} \right] + \sum_{j=K+1}^{N} x_N(j) \left[ c^K_N(j) - \bar{c}^K_N + \frac{K}{N} \right] \right\} = \frac{N}{K^2} \left[ c^K_N - \frac{N - K}{N} \right] (x_N, t_N). \]

Recalling that $c^K_N = K \tilde{\ell}^K_N / N$ and that $\tilde{\ell}^K_N$ is bounded on $\Omega_{N,K}$, we conclude that $\mathbb{1}_{\Omega_{N,K}} |I^2_{N,K}| \leq C ||x_N||_2 ||t_N||_2 / K$. Using Lemmas 5.4 and 5.16, we readily conclude that $\mathbb{E}[\mathbb{1}_{\Omega_{N,K}} |I^2_{N,K}|] \leq C / N$. Finally,

\[ \mathbb{E}[\mathbb{1}_{\Omega_{N,K}} |I^2_{N,K}|] \leq \frac{N}{K^2} \mathbb{E} \left[ \mathbb{1}_{\Omega_{N,K}} \left| \sum_{j=1}^{\bar{\ell}_N} \left( \ell_N(j) - \bar{\ell}^K_N \right) \left[ c^K_N(j) - \bar{c}^K_N - \frac{N - K}{N} \right] \right| \right] + \frac{N}{K^2} \mathbb{E} \left[ \mathbb{1}_{\Omega_{N,K}} \left| \sum_{j=1}^{\bar{\ell}_N} \tilde{\ell}^K_N \left[ c^K_N(j) - \bar{c}^K_N - \frac{N - K}{N} \right] \right| \right] \leq \frac{N}{K^2} \mathbb{E} \left[ \mathbb{1}_{\Omega_{N,K}} ||x_N||_2 \right]^2 \mathbb{E} \left[ \mathbb{1}_{\Omega_{N,K}} ||f^K_N||_2 \right]^2 \frac{1}{2} + \frac{N}{K^2} \mathbb{E} \left[ \mathbb{1}_{\Omega_{N,K}} \left| (f^K_N, 1_K^T) \right| \right] \leq \frac{C}{K} \]

by Lemma 5.17. The proof is complete. \hfill \Box

6. Some auxiliary processes

We first introduce a family of martingales: for $i = 1, \ldots, N$, recalling (3),

\[ M_{t,N}^i = \int_0^t \int_0^\infty 1_{\{z \leq \lambda^i_{t,N}\}} \bar{\pi}^i(ds, dz). \]

where $\bar{\pi}^i(ds, dz) = \pi^i(ds, dz) - dsdz$. We also introduce the family of centered processes $U_{t,N}^i = Z_{t,N}^i - \mathbb{E}[Z_{t,N}^i]$. 


We denote by $Z^N_t$ (resp. $U^N_t$, $M^N_t$) the $N$ dimensional vector with coordinates $Z^i_t$ (resp. $U^i_t$, $M^i_t$) and set

$$Z^{N,K}_t = I_K Z^N_t, \quad U^{N,K}_t = I_K U^N_t,$$

as well as $Z^{N,K}_t = K^{-1} \sum_{i=1}^K Z^i_t$, and $U^{N,K}_t = K^{-1} \sum_{i=1}^K U^i_t$. By (9) Remark 10 and Lemma 11, we have the following equalities

$$E_{\theta} [Z^{N,K}_t] = \mu \sum_{n \geq 0} \left[ \int_0^t s \phi^{*n}(t-s)ds \right] I_K A^K_{N} 1_N,$$

$$U^{N,K}_t = \sum_{n \geq 0} \int_0^t \phi^{*n}(t-s)I_K A^K_N M^N_s ds,$$

$$[M^{i,N}, M^{j,N}]_t = 1_{i=j} Z^{i,N}_t.$$

We recall that $\phi^{*0} = \delta_0$, whence in particular $\int_0^t \phi^{*0}(t-s)ds = t$.

**Lemma 6.1.** Assume $H(q)$ for some $q \geq 1$. There exists a constant $C$ such that

(i) for all $r$ in $[1, \infty]$, all $t \geq 0$, a.s.,

$$1_{\Omega_{N,K}} \|E_{\theta} [Z^{N,K}_t]\|_r \leq Ct^{1/r},$$

(ii) for all $r$ in $[1, \infty]$, all $t \geq s \geq 0$, a.s.,

$$1_{\Omega_{N,K}} \|E_{\theta} [Z^{N,K}_t - Z^{N,K}_s - \mu(t-s)\ell^K_N]\|_r \leq C(\min\{1, s^{1-q}\}) K^\frac{1}{r}.$$

**Proof.** (i) We start from (11). Recall that $\Lambda = \int_0^\infty \phi(s)ds$, whence

$$\int_0^s \phi^{*n}(s)ds \leq \Lambda^n, \quad \int_0^t \phi^{*n}(s)ds \leq t \int_0^\infty \phi^{*n}(s)ds \leq t\Lambda^n.$$

So on the event $\Omega_{N,K}$, on which we have $\Lambda||I_K A_N||_r \leq (K/N)^{1/r}$ and $\Lambda||A_N||_r \leq a < 1$, we have (observe that $\|1_N\|_r = K^{1/r}$)

$$\|E_{\theta} [Z^{N,K}_t]\|_r \leq \mu K^{\frac{1}{r}} + \mu t \sum_{n \geq 1} \Lambda^n \|I_K A^K_N\|_r \|1_N\|_r \leq \mu K^{\frac{1}{r}} + \mu t \sum_{n \geq 1} \Lambda^n \|I_K A_N\|_r \|A_N\|^{n-1}_{r'} \|1_N\|_r \leq Ct^{1/r}.$$

(ii) By (11) and Lemma 5.6 we have

$$E_{\theta} [Z^{N,K}_t] - E_{\theta} [Z^{N,K}_s] = \mu(t-s) \sum_{n \geq 0} \Lambda^n I_K A^K_N 1_N + \mu \left( \sum_{n \geq 0} [\varepsilon_n(t) - \varepsilon_n(s)] I_K A^K_N 1_N \right)$$

with $0 \leq \varepsilon_n(t) \leq C \min\{n^q, \Lambda^{n-1}, n\Lambda^n\}$. Since $\sum_{n \geq 0} \Lambda^n I_K A^K_N 1_N = I_K \ell^K_N$ on the event $\Omega_{N,K}$,

$$\|E_{\theta} [Z^{N,K}_t] - E_{\theta} [Z^{N,K}_s] - \mu(t-s)\ell^K_N\|_r \leq C\min\{1, s^{1-q}\} \|1_N\|_r \sum_{n \geq 0} n^q \Lambda^n \|I_K A^K_N\|_r \|A_N\|^{n-1}_{r'} \leq C\min\{1, s^{1-q}\} K^{\frac{1}{r}}.$$

We used the very same arguments as in point (i).
7. The first estimator in the subcritical case

Here we prove that \( \varepsilon_{i}^{N,K} = t^{-1}(Z_{2i}^{N,K} - \bar{Z}_{i}^{N,K}) \approx \frac{\mu}{1-\lambda p} \) and to study the rate of convergence.

**Theorem 7.1.** Assume \((H(q))\) for some \( q \geq 1 \). There are some positive constants \( C, C' \) depending only on \( p, \mu, \phi \) and \( q \) such that for all \( \varepsilon \in (0,1) \), all \( N \geq K \geq 1 \), all \( t \geq 1 \),

\[
P\left( \left| \varepsilon_{i}^{N,K} - \frac{\mu}{1-\lambda p} \right| \geq \varepsilon \right) \leq C N e^{-C'K} + C \left( \frac{1}{\sqrt{NK}} + \frac{1}{\sqrt{Kt}} + \frac{1}{t^q} \right).
\]

**Lemma 7.2.** Assume \((H(q))\) for some \( q \geq 1 \). There is a constant \( C > 0 \) such that a.s.,

\[
(i) \quad 1_{\Omega_{N,K}} \left| \mathbb{E}_{\theta}[\varepsilon_{i}^{N,K}] - \mu \bar{\varepsilon}_{N}^{K} \right| \leq \frac{C}{t^q}, \quad (ii) \quad 1_{\Omega_{N,K}} \mathbb{E}_{\theta}[|\bar{U}_{i}^{N,K}|^2] \leq \frac{Ct}{K}.
\]

**Proof.** By Lemma 6.1 (ii),

\[
|\mathbb{E}_{\theta}[\varepsilon_{i}^{N,K}] - \mu \bar{\varepsilon}_{N}^{K}| \leq \frac{1}{K} \left| \mathbb{E}_{\theta}\left[ \frac{Z_{2i}^{N,K} - Z_{i}^{N,K}}{t} \right] - \mu \bar{\varepsilon}_{N}^{K} \right| \leq \frac{C}{t^q},
\]

which proves (i). Using (12),

\[
\bar{U}_{i}^{N,K} = \frac{1}{K} \sum_{n=0}^{N} \int_{0}^{t} \phi^\ast(t-s) \sum_{i=1}^{K} \sum_{n=1}^{N} A_{N}^{n}(i,j) M_{s}^{i,n} ds.
\]

Recalling (13), it is obvious that for \( n \geq 1 \),

\[
\mathbb{E}_{\theta}\left[ \left( \sum_{i=1}^{K} \sum_{j=1}^{N} A_{N}^{n}(i,j) M_{s}^{i,n} \right)^2 \right] = \sum_{j=1}^{n} \left( \sum_{i=1}^{K} A_{N}^{n}(i,j) \right)^2 \mathbb{E}_{\theta}[Z_{s}^{i,n}] \leq |||I_{K} A_{N}||| \sum_{j=1}^{N} \mathbb{E}_{\theta}[Z_{s}^{i,n}].
\]

By Lemma 6.1 (i) with \( r = 1 \), we have \( 1_{\Omega_{N,K}} \mathbb{E}_{\theta}[\sum_{i=1}^{N} Z_{i}^{i,n}] \leq CtN \) and \( 1_{\Omega_{N,K}} \mathbb{E}_{\theta}[\sum_{i=1}^{K} Z_{i}^{i,n}] \leq CtK \). Thus on \( \Omega_{N,K} \),

\[
\mathbb{E}_{\theta}[|\bar{U}_{i}^{N,K}|^2] \leq \frac{1}{K} \mathbb{E}_{\theta}\left[ \left( \sum_{i=1}^{K} M_{s}^{i,n} \right)^2 \right] + \frac{1}{K} \sum_{n=1}^{N} \int_{0}^{t} \phi^\ast(t-s) \mathbb{E}_{\theta}[\left( \sum_{i=1}^{K} \sum_{j=1}^{N} A_{N}^{n}(i,j) M_{s}^{i,n} \right)^2]^{1/2} ds
\]

\[
\leq \frac{1}{K} \mathbb{E}_{\theta}\left[ \sum_{i=1}^{K} Z_{s}^{i,n} \right] + \frac{C}{K} \sum_{n=1}^{N} |||I_{K} A_{N}||| |||Z_{s}^{i,n}||| \int_{0}^{t} \mathbb{E}_{\theta}\left[ \sum_{i=1}^{K} Z_{s}^{i,n} \right]^{1/2} \phi^\ast(t-s) ds
\]

\[
\leq C \sqrt{t} + C \frac{tN}{K} \sqrt{A} \sum_{n=1}^{N} \Lambda^{n} |||I_{K} A_{N}||| |||Z_{s}^{i,n}|||^{n-1}.
\]

We used that \( \int_{0}^{t} \sqrt{\phi^\ast(t-s)} ds \leq \sqrt{\int_{0}^{t} \phi^\ast(t-s) ds} \leq \sqrt{t} \Lambda^{n} \). As a conclusion, still on \( \Omega_{N,K} \), since \( |||I_{K} A_{N}|||_{1} \leq CK/N \) and \( \Lambda |||Z_{s}^{i,n}|||_{1} \leq a < 1 \),

\[
\mathbb{E}_{\theta}[|\bar{U}_{i}^{N,K}|^2] \leq C \sqrt{t} \left( \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{N}} \right) \leq \frac{C \sqrt{t}}{\sqrt{K}}
\]

as desired. \( \square \)

**Lemma 7.3.** Assume \((H(q))\) for some \( q \geq 1 \). There is \( C > 0 \) such that a.s.,

\[
1_{\Omega_{N,K}} \mathbb{E}_{\theta}\left[ |\varepsilon_{i}^{N,K} - \mu \bar{\varepsilon}_{N}^{K}|^2 \right] \leq C \left( \frac{1}{t^{2q}} + \frac{1}{tK} \right).
\]
Proof. It suffices to write
\[ \mathbb{E}_\theta \left[ |\varepsilon_t^{N,K} - \mu \bar{\varepsilon}_N^K|^2 \right] \leq 2 \mathbb{E}_\theta \left[ |\varepsilon_t^{N,K} - \varepsilon_t^{N,K} - \mu \bar{\varepsilon}_N^K|^2 \right] + 2 \mathbb{E}_\theta \left[ |\varepsilon_t^{N,K} - \mu \bar{\varepsilon}_N^K|^2 \right] \]
\[ \leq \frac{4}{t^2} \left( \mathbb{E}_\theta [\hat{U}_{2t}^{N,K}^2] + \mathbb{E}_\theta [\hat{U}_{t}^{N,K}^2] \right) + 2 \mathbb{E}_\theta \left[ |\varepsilon_t^{N,K} - \mu \bar{\varepsilon}_N^K|^2 \right] \]
and to use Lemma 7.2

Finally, we can give the proof of Theorem 7.1.

Proof. By Lemmas 5.9 and 7.3, we have
\[ \mathbb{E} \left[ 1_{\Omega_{N,K}(t)} |\varepsilon_t^{N,K} - \frac{\mu}{1-\Lambda p} | \right] \leq \mathbb{E} \left[ 1_{\Omega_{N,K}} |\varepsilon_t^{N,K} - \mu \bar{\varepsilon}_N^K|^2 \right] ^{\frac{1}{2}} + \mu \mathbb{E} \left[ 1_{\Omega_{N,K}} \bar{\varepsilon}_N^K - \frac{1}{1-\Lambda p} |^2 \right] ^{\frac{1}{2}} \]
\[ \leq C \left( \frac{1}{\sqrt{K t}} + \frac{1}{t^q} + \frac{1}{\sqrt{NK}} \right) \]
By Chebyshev's Inequality, we deduce
\[ P \left( |\varepsilon_t^{N,K} - \frac{\mu}{1-\Lambda p} | \geq \varepsilon \right) \leq P(\Omega_{N,K}) + P \left( \left| \varepsilon_t^{N,K} - \frac{\mu}{1-\Lambda p} \right| \geq \varepsilon \right) \cap \Omega_{N,K} \]
\[ \leq P(\Omega_{N,K}) + \frac{1}{\varepsilon} \mathbb{E} \left[ 1_{\Omega_{N,K}} |\varepsilon_t^{N,K} - \mu \bar{\varepsilon}_N^K|^2 \right] \]
\[ \leq C N e^{-C' K} + \frac{C}{\varepsilon} \left( \frac{1}{\sqrt{NK}} + \frac{1}{t^q} + \frac{1}{\sqrt{K t}} \right) \]
by Lemma 5.7.

8. The second estimator in the subcritical case

We now prove that \( \varepsilon_t^{N,K} := \frac{N}{K} \sum_{i=1}^K \frac{Z_i^{t,N} - Z_i^{t,N}}{t} - \varepsilon_t^{N,K} \), \( \varepsilon_t^{N,K} \sim \frac{\mu^2 N^2 p (1-p)}{(1-\Lambda p)^2} \).

Theorem 8.1. Assume \( H(q) \) for some \( q \geq 1 \). There is \( C > 0 \) such that for all \( t \geq 1 \), a.s.,
\[ 1_{\Omega_{N,K}} \mathbb{E}_\theta \left[ \left| \varepsilon_t^{N,K} - \varepsilon_t^{N,K} \right| \right] \leq \frac{N^2}{K} \frac{K}{\sqrt{t}} \frac{\| x_{N,K} \|_2}{\| x_N \|_2}, \]
where \( \varepsilon_t^{N,K} := \frac{\mu^2 N^2 p (1-p)}{(1-\Lambda p)^2} \).

We write \( \varepsilon_t^{N,K} - \varepsilon_t^{N,K} = \Delta_t^{N,K,1} + \Delta_t^{N,K,2} + \Delta_t^{N,K,3} \), where
\[ \Delta_t^{N,K,1} = \frac{N}{K} \sum_{i=1}^K \left( \frac{Z_i^{t,N} - Z_i^{t,N}}{t} - \varepsilon_t^{N,K} \right)^2 - \frac{N}{K^2} \frac{K}{\sqrt{t}} \left( \frac{Z_i^{t,N} - Z_i^{t,N}}{t} - \mu \bar{\varepsilon}_N^K \right)^2, \]
\[ \Delta_t^{N,K,2} = \frac{N}{K} \sum_{i=1}^K \left( \frac{Z_i^{t,N} - Z_i^{t,N}}{t} - \mu \bar{\varepsilon}_N^K \right)^2 - \left( \frac{K}{\sqrt{t}} \varepsilon_t^{N,K} \right)^2, \]
\[ \Delta_t^{N,K,3} = \frac{N}{K} \sum_{i=1}^K \left( \frac{Z_i^{t,N} - Z_i^{t,N}}{t} - \mu \bar{\varepsilon}_N^K \right) \left( \mu \varepsilon_t^{N,K} - \mu \bar{\varepsilon}_N^K \right). \]
We also write $\Delta^N_{t,K,2} \leq \Delta^N_{t,K,21} + \Delta^N_{t,K,22} + \Delta^N_{t,K,23}$, where

$$
\Delta^N_{t,K,21} = \frac{N}{K} \sum_{i=1}^{K} \left[ \frac{(Z_{2t}^i - Z_t^i)^2}{t} - \mathbb{E}_\theta \left[ \frac{Z_{2t}^i - Z_t^i}{t} \right] \right],
$$

$$
\Delta^N_{t,K,22} = \frac{N}{K} \sum_{i=1}^{K} \left[ \mathbb{E}_\theta \left[ \frac{(Z_{2t}^i - Z_t^i)^2}{t} \right] - \frac{1}{2} \mathbb{E}_\theta \left[ \frac{(Z_{2t}^i - Z_t^i)^2}{t} \right] - \frac{1}{2} \mathbb{E}_\theta \left[ \frac{(Z_{2t}^i - Z_t^i)^2}{t} \right] \right],
$$

$$
\Delta^N_{t,K,23} = \frac{N}{K} \sum_{i=1}^{K} \left[ \frac{(Z_{2t}^i - Z_t^i)^2}{t} - \mathbb{E}_\theta \left[ \frac{(Z_{2t}^i - Z_t^i)^2}{t} \right] - \mu \bar{\ell}_N(t) \right],
$$

We next write $\Delta^N_{t,K,21} \leq \Delta^N_{t,K,211} + \Delta^N_{t,K,212} + \Delta^N_{t,K,213}$, where

$$
\Delta^N_{t,K,211} = \frac{N}{K} \sum_{i=1}^{K} \left[ \frac{(U_{2t}^i - U_t^i)^2}{t} - \mathbb{E}_\theta \left[ \frac{(U_{2t}^i - U_t^i)^2}{t} \right] \right],
$$

$$
\Delta^N_{t,K,212} = \frac{N}{K} \sum_{i=1}^{K} \left[ \mathbb{E}_\theta \left[ \frac{(U_{2t}^i - U_t^i)^2}{t} \right] - \mathbb{E}_\theta \left[ \frac{(U_{2t}^i - U_t^i)^2}{t} \right] - \mathbb{E}_\theta \left[ \frac{(U_{2t}^i - U_t^i)^2}{t} \right] \right],
$$

$$
\Delta^N_{t,K,213} = \frac{N}{K} \left[ \frac{(U_{2t}^i - U_t^i)^2}{t} - \mathbb{E}_\theta \left[ \frac{(U_{2t}^i - U_t^i)^2}{t} \right] \right].
$$

At last, we write $\Delta^N_{t,K,31} \leq \Delta^N_{t,K,31} + \Delta^N_{t,K,32}$, where

$$
\Delta^N_{t,K,31} = \frac{N}{K} \sum_{i=1}^{K} \left[ \frac{(Z_{2t}^i - Z_t^i)^2}{t} - \mathbb{E}_\theta \left[ \frac{(Z_{2t}^i - Z_t^i)^2}{t} \right] - \mu \bar{\ell}_N(t) \right],
$$

$$
\Delta^N_{t,K,32} = \frac{N}{K} \sum_{i=1}^{K} \left[ \mathbb{E}_\theta \left[ \frac{(Z_{2t}^i - Z_t^i)^2}{t} \right] - \mu \bar{\ell}_N(t) \right].
$$

Lemma 8.2. Assume $H(q)$ for some $q \geq 1$. Then, on the set $\Omega_{N,K}$, for $t \geq 1$, a.s.,

(i) $\mathbb{E}_\theta [\Delta^N_{t,K,1}] \leq C(Nt^{-2q} + NK^{-1}t^{-1})$,

(ii) $\mathbb{E}_\theta [\Delta^N_{t,K,22}] \leq CN/t^{2q}$,

(iii) $\mathbb{E}_\theta [\Delta^N_{t,K,23}] \leq CN/t^q$,

(iv) $\mathbb{E}_\theta [\Delta^N_{t,K,213}] \leq CNK^{-q}t^{-1}$,

(v) $\mathbb{E}_\theta [\Delta^N_{t,K,32}] \leq CN/t^q$.

Proof. (i) Recalling the definition $\varepsilon^N_{t,K} = t^{-1}(\bar{Z}_{2t}^N - Z_t^N)$,

$$
\Delta^N_{t,K,1} = \frac{N}{K} \sum_{i=1}^{K} \left[ \mu \bar{\ell}_N - \varepsilon^N_{t,K} \right] \left[ 2(\bar{Z}_{2t}^i - Z_t^i)/t - \mu \bar{\ell}_N - \varepsilon^N_{t,K} \right] = N \left( \varepsilon^N_{t,K} - \mu \bar{\ell}_N \right)^2,
$$

whence by Lemma 7.3:

$$
\mathbb{E}_\theta [\Delta^N_{t,K,1}] = N \mathbb{E}_\theta \left[ \left( \varepsilon^N_{t,K} - \mu \bar{\ell}_N \right)^2 \right] \leq C \left( Nt^{-2q} + NK^{-1}t^{-1} \right).
$$

(ii) We use Lemma 6.1(ii) with $r = 2$:

$$
\mathbb{E}_\theta [\Delta^N_{t,K,22}] = \frac{N}{K} \sum_{i=1}^{K} \left[ \mathbb{E}_\theta \left[ \frac{(Z_{2t}^i - Z_t^i)^2}{t} \right] - \mu \bar{\ell}_N(t) \right]^2 \leq CN/t^{2q}.
$$
and thanks to Lemma 7.2-(ii), we deduce that
\[ r_\Omega \]
where, recalling (12), we have
\[ \text{Lemma 8.3.} \]
Assume \( \Box \)
The proof is complete.

(iv) Since
\[ \text{(iv)} \]
we then
\[ \text{(iv)} \]
and thanks to Lemma 7.3(ii), we deduce that
\[ \mathbb{E}_\theta [\Delta_{t,2}^{N,K}] \leq CNK^{-\frac{t}{2}} t^{-3}. \]

(v) Since \( \max_{j=1,\ldots,N} \{\ell_N(j)\} \) is bounded on the set \( \Omega_{N,K} \), by Lemma 6.1(ii) with \( r = 1 \),
\[ \mathbb{E}_\theta [\Delta_{t,2}^{N,K}] \leq C t^{-2}. \]
The proof is complete. \( \Box \)

**Lemma 8.3.** Assume \( H(q) \) for some \( q \geq 1 \). We have, for all \( t \geq 1 \), on the set \( \Omega_{N,K} \), a.s.
\[ \mathbb{E}_\theta [\Delta_{t,2}^{N,K}] \leq C t. \]

**Proof.** We write \( \mathbb{E}_\theta [\Delta_{t,2}^{N,K}] \leq t^{-2} N \sum_{i=1}^{\infty} a_i \), where \( a_i = |\mathbb{E}_\theta [(U_{2t,N}^{i,N} - U_t^{i,N})^2 - (Z_{2t,N}^{i,N} - Z_t^{i,N})]| \), and then
\[ a_i = b_i + d_i \quad \text{where} \quad a_i = \mathbb{E}_\theta [(R_{t,N}^{i,N})^2] \quad \text{and} \quad b_i = 2 \mathbb{E}_\theta [(M_{2t,N}^{i,N} - M_t^{i,N})R_{t,N}^{i,N}], \]
where, recalling (12), we have \( U_{2t,N}^{i,N} - U_t^{i,N} = M_{2t,N}^{i,N} - M_t^{i,N} + R_{t,N}^{i,N} \), with
\[ R_{t,N}^{i,N} = \sum_{n=1}^{\infty} \int_0^{2t} \beta_n(t,2t,s) \sum_{j=1}^N A_{N}^{i,j}(i,j) M_{2t,N}^{j,j} ds \quad \text{with} \quad \beta_n(t,2t,s) = \phi^*(2t-s) - \phi^*(t-s). \]

This uses that \( \mathbb{E}_\theta [(M_{2t,N}^{i,N} - M_t^{i,N})^2] = \mathbb{E}_\theta [Z_{2t,N}^{i,N} - Z_t^{i,N}] \) by (13). By the proof of [6] Lemma 21, lines 10 and 15, we have \( b_i \leq CtN^{-1} \) and \( d_i \leq CtN^{-1} \), whence the conclusion. \( \Box \)

Before considering the term \( \Delta_{t,2}^{N,K} \), we review [6] Lemma 22] (observing that \( \Omega_{N,K} \subset \Omega_{1,1} \).

**Lemma 8.4.** Assume \( H(q) \) for some \( q \geq 1 \). Then for all \( t \geq 1 \) and \( k,l,a,b \in \{1,\ldots,N\} \), all \( r,s,u,v \in [0,t] \), on the set \( \Omega_{N,K} \) a.s.

(i) \(|\text{Cov}_\theta (Z_r^{k,N},Z_s^{l,N})| = |\text{Cov}_\theta (U_r^{k,N},U_s^{l,N})| \leq Ct(N^{-1} + 1_{\{k=l\}}), \)

(ii) \(|\text{Cov}_\theta (Z_r^{k,N},M_s^{l,N})| = |\text{Cov}_\theta (U_r^{k,N},M_s^{l,N})| \leq Ct(N^{-1} + 1_{\{k=l\}}), \)

(iii) \(|\text{Cov}_\theta (Z_r^{k,N},\int_0^s M_{r-}dM_s^{l,N})| = |\text{Cov}_\theta (U_r^{k,N},\int_0^s M_{r-}dM_s^{l,N})| \leq Ct^2(N^{-1} + 1_{\{k=l\}}), \)

(iv) \(|\text{E}_\theta [M_r^{k,N}M_s^{l,N}M_u^{a,N}M_v^{b,N}]| \leq C t, \quad \text{if} \quad \#\{k,l\} = 2, \)

(v) \(|\text{Cov}_\theta (M_r^{k,N}M_s^{l,N},M_a^{a,N}M_b^{b,N})| = 0, \quad \text{if} \quad \#\{k,l,a,b\} = 4, \)

(vi) \(|\text{Cov}_\theta (M_r^{k,N}M_s^{l,N},M_a^{a,N}M_b^{b,N})| \leq Ct/N^2, \quad \text{if} \quad \#\{k,a,b\} = 3, \)

(vii) \(|\text{Cov}_\theta (M_r^{k,N}M_s^{l,N},M_a^{a,N}M_b^{b,N})| \leq C N^{-1} t^2, \quad \text{if} \quad \#\{k,a\} = 2, \)

(viii) \(|\text{Cov}_\theta (M_r^{k,N}M_s^{l,N},M_a^{a,N}M_b^{b,N})| \leq Ct^2. \)
Lemma 8.5. Assume $H(q)$ for some $q \geq 1$. Then for $t \geq 1$ on $\Omega_{N,K}$ a.s.,

$$\mathbb{E}_\theta[(\Delta_{t}^{N,K,31})^2] \leq \frac{CN^2}{tK^2} \sum_{i=1}^{K} \left( \ell_{N}(i) - \bar{\ell}_{N}^{K} \right)^2.$$ 

Proof. By definition of $\Delta_{t}^{N,K,31}$,

$$\mathbb{E}_\theta[(\Delta_{t}^{N,K,31})^2] = \frac{4\mu^2N^2}{t^2K^2} \sum_{i,j=1}^{K} (\ell_{N}(i) - \bar{\ell}_{N}^{K})(\ell_{N}(j) - \bar{\ell}_{N}^{K}) \text{Cov}_\theta(U_{2t}^{i,N} - U_{t}^{i,N}, U_{2t}^{j,N} - U_{t}^{j,N}).$$

By Lemma [8.4 (i)], we have $\text{Cov}_\theta[U_{2t}^{i,N} - U_{t}^{i,N}, U_{2t}^{j,N} - U_{t}^{j,N}] \leq Ct(\mathbf{1}_{\{i=j\}} + \frac{1}{N})$. We deduce that

$$\mathbb{E}_\theta[(\Delta_{t}^{N,K,31})^2] \leq \frac{C\mu^2N^2}{t^2K^2} \sum_{i,j=1}^{K} \left( \mathbf{1}_{\{i=j\}} + \frac{1}{N} \right) \left\{ (\ell_{N}(i) - \bar{\ell}_{N}^{K})^2 + (\ell_{N}(j) - \bar{\ell}_{N}^{K})^2 \right\}$$

$$\leq C \frac{N^2}{tK^2} \sum_{i=1}^{K} \left( \ell_{N}(i) - \bar{\ell}_{N}^{K} \right)^2.$$ 

We finally used that $K/N \leq 1$. □

Next, we deal with the term $\Delta_{t}^{N,K,211}$.

Lemma 8.6. Assume $H(q)$ for some $q \geq 1$. Then for all $t \geq 1$, a.s. on the set $\Omega_{N,K}$, we have

$$\mathbb{E}_\theta[(\Delta_{t}^{N,K,211})^2] \leq \frac{CN^2}{tK^2}.$$ 

Proof. First, $\mathbb{E}_\theta[(\Delta_{t}^{N,K,211})^2] = \frac{N^2}{t^2K^2} \sum_{i,j=1}^{K} a_{ij}$, where $a_{ij} = \text{Cov}_\theta[(U_{2t}^{i,N} - U_{t}^{i,N})^2, (U_{2t}^{j,N} - U_{t}^{j,N})^2]$.

Let $\Gamma_{k,t,a,b}(t) = \sup_{r,s,a,v \in [0,2t]} |\text{Cov}_\theta(M_{t}^{k,N}M_{a}^{N}, M_{v}^{k,N}M_{b}^{N})|$. By the proof of [8] Lemma 24 lines 9 to 12, we have

$$a_{ij} \leq C \sum_{k,l,a,b=1}^{N} \left\{ (\mathbf{1}_{\{i=k\}} + N^{-1})(\mathbf{1}_{\{i=l\}} + N^{-1})(\mathbf{1}_{\{j=a\}} + N^{-1})(\mathbf{1}_{\{j=b\}} + N^{-1}) \Gamma_{k,l,a,b}(t) \right\}$$

Hence,

$$\sum_{i,j=1}^{K} a_{ij} \leq C[R_{1}^{K} + R_{2}^{K} + R_{3}^{K} + R_{4}^{K} + R_{5}^{K} + R_{6}^{K}],$$
By Lemma 8.4-(v)-(viii), we see that \( \Gamma \)

Lemmas 8.2, 8.3, 8.5 and 8.6 allow us to conclude that

Recalling that

as desired. \( \square \)

All in all, we deduce that

\[
R_1^K = \frac{1}{N^2} \sum_{i,j=1}^{K} \sum_{k,l,a,b=1}^{N} \Gamma_{k,l,a,b}(t) = \frac{K^2}{N^2} \sum_{k,l,a,b=1}^{N} \Gamma_{k,l,a,b}(t),
\]

\[
R_2^K = \frac{1}{N^3} \sum_{i,j=1}^{K} \sum_{k,l,a,b=1}^{N} 1_{\{i=k\}} \Gamma_{k,l,a,b}(t) = \frac{K}{N^2} \sum_{i=1}^{K} \sum_{l,a,b=1}^{N} \Gamma_{i,l,a,b}(t),
\]

\[
R_3^K = \frac{1}{N^2} \sum_{i,j=1}^{K} \sum_{k,l,a,b=1}^{N} 1_{\{i=k\}} 1_{\{j=a\}} \Gamma_{k,l,a,b}(t) = \frac{K}{N^2} \sum_{k,a=1}^{K} \sum_{b, l=1}^{N} \Gamma_{k,k,a,b}(t),
\]

\[
R_4^K = \frac{1}{N} \sum_{i,j=1}^{K} \sum_{k,l,a,b=1}^{N} 1_{\{i=k\}} 1_{\{i=l\}} \Gamma_{k,l,a,b}(t) = \frac{K}{N} \sum_{k,a=1}^{K} \sum_{b, l=1}^{N} \Gamma_{k,k,a,b}(t),
\]

\[
R_5^K = \frac{1}{N} \sum_{i,j=1}^{K} \sum_{k,l,a,b=1}^{N} 1_{\{i=k\}} 1_{\{i=l\}} 1_{\{j=a\}} \Gamma_{k,l,a,b}(t) = \frac{K}{N} \sum_{k,a=1}^{K} \sum_{b, l=1}^{N} \Gamma_{k,k,a,b}(t),
\]

\[
R_6^K = \sum_{i,j=1}^{K} \sum_{k,l,a,b=1}^{N} 1_{\{i=k\}} 1_{\{i=l\}} 1_{\{j=a\}} \Gamma_{k,l,a,b}(t) = \sum_{k,a=1}^{K} \Gamma_{k,k,a,a}(t).
\]

By Lemma 8.4-(v)-(viii), we see that \( \Gamma_{k,l,a,b}(t) \leq C t^2 1_{\{\#(k,l,a,b)<4\}} \), so that

\[
R_1^K \leq C t^2 \frac{K^2}{N}, \quad R_2^K \leq C t^2 \frac{K^2}{N}, \quad \text{and} \quad R_3^K \leq C t^2 K.
\]

Also, from Lemma 8.4-(vi)-(viii), we have \( \Gamma_{k,k,a,b}(t) \leq C (N^{-2} + 1_{\{\#(k,a,b)<3\}} t^2) \), whence

\[
R_4^K \leq C \left( \frac{K^2}{N^2} t + \frac{K^2}{N} t^2 \right) \leq C \frac{K^2}{N} t^2 \quad \text{and} \quad R_5^K \leq C \left( K t^2 + \frac{K^2}{N^2} t \right) \leq C K t^2.
\]

Finally, notice that from Lemma 8.4-(vii)-(viii), \( \Gamma_{k,k,a,a}(t) \leq C (N^{-1} t^2 + 1_{\{\#(k,a)=1\}} t^2) \), so that

\[
R_6^K \leq C \left( \frac{K^2}{N} t^2 + K t^2 \right) \leq C K t^2.
\]

All in all, we deduce that \( \sum_{i,j} a_{ij} \leq C K t^2 \). \( \square \)

Then we can give prove of Theorem 8.1

Proof. Recalling that

\[
\| \nu_t^{N,K} - \nu_\infty^{N,K} \| = \Delta_t^{N,K,1} + \Delta_t^{N,K,211} + \Delta_t^{N,K,212} + \Delta_t^{N,K,22} + \Delta_t^{N,K,23} + \Delta_t^{N,K,31} + \Delta_t^{N,K,32},
\]

Lemmas 8.2, 8.3, 8.5 and 8.6 allow us to conclude that

\[
1_{\Omega_{N,K}} \mathbb{E}_\theta \| \nu_t^{N,K} - \nu_\infty^{N,K} \| \leq C \left( \frac{N}{t \sqrt{K}} + \frac{N}{K} + \frac{N}{t^2} + \frac{N}{t K} + \frac{N}{K \sqrt{t}} \left[ \sum_{i=1}^{K} (\ell_N(i) - \ell_N^K) \right]^2 \right)
\]

\[
\leq C \left( \frac{N}{t \sqrt{K}} + \frac{N}{t^2} + \frac{N}{K \sqrt{t}} \|a_N^K\|_2 \right)
\]

as desired. \( \square \)
Corollary 8.7. Assume $H(q)$ for some $q > 3$. There exists some constants $C > 0$ and $C' > 0$ depending only on $p$, $\mu$, $\phi$, $q$ such that for all $\varepsilon \in (0, 1)$, such that, for $t \geq 1$,
\[
P\left(\left| V_{t}^{N,K} - \frac{\mu^2 \Lambda^2 (1-p)}{(1-\Lambda p)^2} \right| \geq \varepsilon \right) \leq CN e^{-CK} + \frac{C}{\varepsilon} \left( \frac{1}{\varepsilon} + \frac{N}{t\sqrt{K}} \right).
\]

Proof. By Theorem 8.1 and Lemma 5.13 (since $V_{\infty}^{N,K} = \frac{\mu^2 N}{K} \|x_N^K\|^2 = \mu^2 K \sum_{i=1}^{K} (\ell_N(i) - \bar{\ell}_N^K)^2$), we have
\[
E \left[ 1_{\Omega_{N,K}} \left| V_{t}^{N,K} - \frac{\mu^2 \Lambda^2 (1-p)}{(1-\Lambda p)^2} \right| \right] 
\leq E \left[ 1_{\Omega_{N,K}} \left| V_{t}^{N,K} - V_{\infty}^{N,K} \right| \right] + \mu^2 E \left[ 1_{\Omega_{N,K}} \left| \frac{N}{K} \sum_{i=1}^{K} (\ell_N(i) - \bar{\ell}_N^K)^2 - \frac{\Lambda^2 p (1-p)}{(1-\Lambda p)^2} \right| \right]
\leq C E \left[ 1_{\Omega_{N,K}} \left( \frac{N}{\sqrt{K}} + \frac{N}{K} \right) + \frac{N}{K} \right] + \frac{C}{\sqrt{K}}
\leq C \left( \frac{1}{\sqrt{K}} + \frac{N}{t\sqrt{K}} \right).
\]
By the classical inequality $\frac{N}{\sqrt{K}} + \frac{1}{\sqrt{K}} \geq 2\frac{N}{t\sqrt{K}}$, we have with
\[
E \left[ 1_{\Omega_{N,K}} \left| V_{t}^{N,K} - \frac{\mu^2 \Lambda^2 (1-p)}{(1-\Lambda p)^2} \right| \right] \leq C \left( \frac{1}{\sqrt{K}} + \frac{N}{t\sqrt{K}} \right).
\]
Using Lemma 5.14 and Chebyshev’s inequality, we conclude that
\[
P\left(\left| V_{t}^{N,K} - \frac{\mu^2 \Lambda^2 (1-p)}{(1-\Lambda p)^2} \right| \geq \varepsilon \right) \leq CN e^{-CK} + \frac{C}{\varepsilon} \left( \frac{1}{\sqrt{K}} + \frac{N}{t\sqrt{K}} \right).
\]
Next, we get rid of the term $\frac{N}{t\sqrt{K}}$. We assume without loss of generality that $C \geq 1$. When $t \leq \sqrt{K}$, then $\frac{N}{t\sqrt{K}} \geq 1$, so that
\[
P\left(\left| V_{t}^{N,K} - \frac{\mu^2 \Lambda^2 (1-p)}{(1-\Lambda p)^2} \right| \geq \varepsilon \right) \leq 1 \leq CN e^{-CK} + \frac{C}{\varepsilon} \left( \frac{1}{\sqrt{K}} + \frac{N}{t\sqrt{K}} \right).
\]
When now $t \geq \sqrt{K}$, then $\frac{N}{t\sqrt{K}} \geq \frac{N}{t^2} \geq \frac{N}{t^2}$. So
\[
P\left(\left| V_{t}^{N,K} - \frac{\mu^2 \Lambda^2 (1-p)}{(1-\Lambda p)^2} \right| \geq \varepsilon \right) \leq CN e^{-CK} + \frac{C}{\varepsilon} \left( \frac{1}{\sqrt{K}} + \frac{N}{t\sqrt{K}} \right).
\]
This completes the proof. \(\square\)

9. THE THIRD ESTIMATOR IN THE SUBCRITICAL CASE

Recall that by definition,
\[
\mathcal{W}_{\Delta t}^{N,K} = 2-Z_{\Delta t}^{N,K} - Z_{\Delta t}^{N,K}, \quad \mathcal{Z}_{\Delta t}^{N,K} = \frac{N}{t} \sum_{i=\Delta t+1}^{\Delta t+1} \left( Z_{\Delta t}^{N,K} - \bar{Z}_{(i-1)\Delta}^{N,K} - \Delta \varepsilon_{i}^{N,K} \right)^2,
\]
\[
\mathcal{X}_{\Delta t}^{N,K} = \mathcal{W}_{\Delta t}^{N,K} - \frac{N-K}{K} \varepsilon_{i}^{N,K}.
\]
The goal of this section is to check that $\mathcal{X}_{\Delta t}^{N,K} \simeq \frac{\mu^2}{(1-\Lambda p)^2}$, and more precisely to prove the following estimate.
Theorem 9.1. Assume \( H(q) \) for some \( q \geq 3 \). Then a.s., for all \( t \geq 4 \) and all \( \Delta \in [1, t/4] \) such that \( t/(2\Delta) \) is a positive integer,

\[
\mathbb{E}\left[ 1_{\Omega_{N,K}} \left| \mathcal{X}^{N,K}_{\Delta,t} - \frac{\mu}{(1 - \Delta p)^3} \right| \right] \leq C \left( \frac{N}{K} \sqrt{\frac{\Delta}{t}} + \frac{N^2}{K \Delta^{\frac{1}{2}(q+1)}} + \frac{N t}{K \Delta^{\frac{3}{2}+1}} + \frac{N}{K \sqrt{K t}} \right).
\]

In the whole section, we assume that \( t \geq 4 \) and that \( \Delta \in [1, t/4] \) is such that \( t/(2\Delta) \) is a positive integer. First, we recall that \( \mathcal{W}^{N,K}_{\infty,\Delta} := \frac{N}{K^2} \sum_{j=1}^{N} (c_{N}^j(j))^2 t_{N}(j) \) and write

\[
|\mathcal{X}^{N,K}_{\Delta,t} - \mathcal{X}^{N,K}_{\infty,\Delta}| \leq |\mathcal{W}^{N,K}_{\Delta,t} - \mathcal{W}^{N,K}_{\infty,\Delta}| + \frac{N - K}{K} \left| \bar{\xi}^{N,K}_{\Delta,t} - \bar{\xi}^{K}_{N} \right|
\]

\[
\leq D_{\Delta,t}^{N,K,1} + 2D_{\Delta,t}^{N,K,2} + 2D_{\Delta,t}^{N,K,3} + D_{\Delta,t}^{N,K,4} + \frac{N}{K} \left| \bar{\xi}^{N,K}_{\Delta,t} - \bar{\xi}^{K}_{N} \right|
\]

where

\[
D_{\Delta,t}^{N,K,1} = \frac{N}{t} \left| \sum_{a = \frac{1}{t} + 1}^{\frac{2}{t}} \left( \bar{Z}_{a}^{N,K} - \bar{Z}_{(a-1)\Delta} - \Delta \bar{\xi}^{N,K}_{t} \right) \right|^2 - \sum_{a = \frac{1}{t} + 1}^{\frac{2}{t}} \left( \bar{Z}_{a}^{N,K} - \bar{Z}_{(a-1)\Delta} - \Delta \bar{\xi}^{N,K}_{t} \right)^2,
\]

\[
D_{\Delta,t}^{N,K,2} = \frac{N}{t} \left| \sum_{a = \frac{1}{t} + 1}^{\frac{2}{t}} \left( \bar{Z}_{a}^{N,K} - \bar{Z}_{(a-1)\Delta} - \Delta \bar{\mu}^{N,K}_{t} \right)^2 \right|
\]

\[
- \sum_{a = \frac{1}{t} + 1}^{\frac{2}{t}} \left( \bar{Z}_{a}^{N,K} - \bar{Z}_{(a-1)\Delta} - \Delta \bar{\mu}^{N,K}_{t} \right)^2,
\]

\[
D_{\Delta,t}^{N,K,3} = \frac{N}{t} \left| \sum_{a = \frac{1}{t} + 1}^{\frac{2}{t}} \left( \bar{Z}_{a}^{N,K} - \bar{Z}_{(a-1)\Delta} - \Delta \bar{\mu}^{N,K}_{t} \right)^2 \right|
\]

\[
- \mathbb{E}_{\|} \left[ \sum_{a = \frac{1}{t} + 1}^{\frac{2}{t}} \left( \bar{Z}_{a}^{N,K} - \bar{Z}_{(a-1)\Delta} - \Delta \bar{\mu}^{N,K}_{t} \right)^2 \right],
\]

and finally

\[
D_{\Delta,t}^{N,K,4} = \frac{2N}{t} \mathbb{E}_{\|} \left[ \sum_{a = \frac{1}{t} + 1}^{\frac{2}{t}} \left( \bar{Z}_{a}^{N,K} - \bar{Z}_{2(a-1)\Delta} - \Delta \bar{\mu}^{N,K}_{t} \right)^2 \right]
\]

\[
- \frac{N}{t} \mathbb{E}_{\|} \left[ \sum_{a = \frac{1}{t} + 1}^{\frac{2}{t}} \left( \bar{Z}_{a}^{N,K} - \bar{Z}_{(a-1)\Delta} - \Delta \bar{\mu}^{N,K}_{t} \right)^2 \right] - \mathcal{W}^{N,K}_{\infty,\Delta}.
\]

For the first term \( D_{\Delta,t}^{N,K,1} \), we have the following lemma.

Lemma 9.2. Assume \( H(q) \) for some \( q \geq 1 \). Then a.s. on the set \( \Omega_{N,K} \)

\[
\mathbb{E}_{\|}[D_{\Delta,t}^{N,K,1}] \leq C \Delta \left( \frac{N}{t^{2q}} + \frac{N}{K t} \right).
\]
Proof. Recalling that 
\[ \varepsilon^{N,K}_t := t^{-1}(\tilde{Z}^{N,K}_{2t} - \tilde{Z}^{N,K}_t), \]
we have
\[
D_{\Delta,t}^{N,K,1} = \frac{N}{t} \sum_{a = \frac{a}{\Delta} + 1}^{\frac{t}{\Delta}} \left[ \tilde{Z}^{N,K}_{a\Delta} - \tilde{Z}^{N,K}_{(a-1)\Delta} - \Delta \varepsilon^{N,K}_t \right]^2 - \sum_{a = \frac{a}{\Delta} + 1}^{\frac{t}{\Delta}} \left[ \tilde{Z}^{N,K}_{a\Delta} - \tilde{Z}^{N,K}_{(a-1)\Delta} - \Delta \mu K_N \right]^2
\]
\[= N \Delta (\mu K_N - \varepsilon^{N,K}_t)^2, \]
Lemma 9.3 completes the proof. \( \square \)

Next, we consider the term \( D_{\Delta,t}^{N,K,2} \).

**Lemma 9.3.** Assume \( H(q) \) for some \( q \geq 1 \). Then a.s. on the set \( \Omega_{N,K} \),
\[
\mathbb{E}_\theta[D_{\Delta,t}^{N,K,2}] \leq CN t^{1-q}
\]
Proof. First, we have
\[
D_{\Delta,t}^{N,K,2} = \frac{2N}{t} \sum_{a = \frac{a}{\Delta} + 1}^{\frac{t}{\Delta}} \left( \Delta \mu K_N - \mathbb{E}_\theta[\tilde{Z}^{N,K}_{a\Delta} - \tilde{Z}^{N,K}_{(a-1)\Delta}] \right)
\]
\[
\left( 2(\tilde{Z}^{N,K}_{a\Delta} - \tilde{Z}^{N,K}_{(a-1)\Delta}) - \mathbb{E}_\theta[\tilde{Z}^{N,K}_{a\Delta} - \tilde{Z}^{N,K}_{(a-1)\Delta}] - \Delta \mu K_N \right),
\]
whence
\[
\mathbb{E}_\theta[D_{\Delta,t}^{N,K,2}] \leq \frac{2N}{t} \sum_{a = \frac{a}{\Delta} + 1}^{\frac{t}{\Delta}} \left| \Delta \mu K_N - \mathbb{E}_\theta[\tilde{Z}^{N,K}_{a\Delta} - \tilde{Z}^{N,K}_{(a-1)\Delta}] \right| \left( \mathbb{E}_\theta[\tilde{Z}^{N,K}_{a\Delta} - \tilde{Z}^{N,K}_{(a-1)\Delta}] + \Delta \mu K_N \right).
\]
By Lemma 6.1(i)-(ii) with \( r = 1 \), since \( (a-1)\Delta \geq t \), we conclude that on \( \Omega_{N,K} \), a.s.,
\[
\left| \Delta \mu K_N - \mathbb{E}_\theta[\tilde{Z}^{N,K}_{a\Delta} - \tilde{Z}^{N,K}_{(a-1)\Delta}] \right| \leq Ct^{1-q} \text{ and } \mathbb{E}_\theta[\tilde{Z}^{N,K}_{a\Delta} - \tilde{Z}^{N,K}_{(a-1)\Delta}] \leq C \Delta \mu K_N + C \leq C \Delta
\]
since \( \mu K_N \) is bounded on \( \Omega_{N,K} \). The conclusion follows. \( \square \)

Next, we consider the term \( D_{\Delta,t}^{N,K,4} \).

**Lemma 9.4.** Assume \( H(q) \) for some \( q \geq 1 \). On \( \Omega_{N,K} \), there is a \( \sigma((\theta_{ij}), i,j=1,...N) \)-measurable finite random variable \( Y^{N,K} \) such that for all \( 1 \leq \Delta \leq \frac{t}{2} \), a.s. on \( \Omega_{N,K} \),
\[
\text{Var}_\theta(\bar{U}_{x+\Delta}^{N,K} - \bar{U}_{x}^{N,K}) = \frac{1}{N} \mathcal{W}_{x+\Delta}^{N,K} - Y^{N,K} + r_{N,K}(x,\Delta),
\]
where, for some constant \( C \), \( |r_{N,K}(x,\Delta)| \leq Cx \Delta^{-q} K^{-1} \).
Proof. Recalling \( 12 \), we write
\[
\bar{U}_{x+\Delta}^{N,K} - \bar{U}_{x}^{N,K} = \sum_{n \geq 0} \int_0^{x+\Delta} \beta_n(x, x + \Delta, s) \frac{1}{K} \sum_{i=1}^{K} \sum_{j=1}^{N} A_{N,i}^n(i, j) M_s^{i,N} ds,
\]
where \( \beta_n(x, x + \Delta, s) = \phi^*(x + \Delta - s) - \phi^*(x - s) \). Set \( V_{x+N,K} = \text{Var}_\theta(\bar{U}_{x+\Delta}^{N,K} - \bar{U}_{x}^{N,K}) \). Recall that \( \mathbb{E}[M_t^{i,N} M_t^{j,N}] = 1_{(i,j)} \mathbb{E}_\theta[Z_s^{i,N}] \), see \( 13 \). We thus have
\[
V_{x+\Delta}^{N,K} = \sum_{m,n \geq 0} \int_0^{x+\Delta} \int_0^{x+\Delta} \beta_m(x, x + \Delta, r) \beta_n(x, x + \Delta, s) \frac{1}{K^2} \sum_{i,k=1}^{K} \sum_{j=1}^{N} A_{N,i}^n(i, j) A_{N,k}^m(k, j) \mathbb{E}_\theta[Z_s^{j,N}] dr ds.
\]
In view of [9] Lemma 28, Step 2, we have 
\[ E_N = \mu \kappa \sum_{n \geq 0} n A_n^N(j, l) \] and 
\[ R_N^N = \mu \sum_{n \geq 0} \varepsilon_n(s) \sum_{l=1}^N A_n^N(j, l). \]

Recall that \( \kappa \) and \( \varepsilon_n(s) \) were defined in Lemma [9, Lemma 15 (ii)]. Also, there is a constant \( C \) such that, for all \( j = 1, \ldots, N \), we have \( 0 \leq X_j^N \leq C \) and \( |R_j^N(s)| \leq C(s^{1-q} \wedge 1) \). Then we can write that
\[ \mathcal{V}_{X, x, \Delta} = I - M + Q, \]
where
\[ I = \sum_{n,m \geq 0} \int_{0}^{\infty} \int_{0}^{\infty} \beta_n(x, x + \Delta, s) \beta_m(x, x + \Delta, r) \frac{1}{K^2} \sum_{i,k=1}^{N} A_N^m(i, j) A_N^n(k, j) \mu N(j)(r \wedge s) dr ds, \]
\[ M = \sum_{n,m \geq 0} \int_{0}^{\infty} \int_{0}^{\infty} \beta_n(x, x + \Delta, s) \beta_m(x, x + \Delta, r) \frac{1}{K^2} \sum_{i,k=1}^{N} A_N^m(i, j) A_N^n(k, j) X_j^N dr ds, \]
\[ Q = \sum_{n,m \geq 0} \int_{0}^{\infty} \int_{0}^{\infty} \beta_n(x, x + \Delta, s) \beta_m(x, x + \Delta, r) \frac{1}{K^2} \sum_{i,k=1}^{N} A_N^m(i, j) A_N^n(k, j) R_j^N(r \wedge s) dr ds. \]

First, we consider \( M \). Using that \( |\int_{0}^{\infty} \beta_n(x, x + \Delta, r) dr| \leq C n^q \Lambda^n x^{-q} \), see [9, Lemma 15 (ii)] and that \( X_j^N \) is bounded by some constant not depending on \( t \), we conclude that on \( \Omega_{N,K} \),
\[ |M| \leq C \sum_{m,n \geq 0} m^q n^q \Lambda^{m+n} x^{-2q} K^{-2} \sum_{i,k=1}^{N} A_N^m(i, j) A_N^n(k, j), \]
\[ \leq C x^{-2q} N K^{-2} \sum_{m,n \geq 1} m^q n^q \Lambda^{m+n} \| I_N A_N^m \|_1 \| I_N A_N^n \|_1 \]
\[ \leq C x^{-2q} N K^{-2} \sum_{m,n \geq 1} m^q n^q \Lambda^{m+n} \| I_N A_N \|_1^2 \| A_N \|_1^{m+n-2}, \]
\[ \leq C x^{-2q} N^{-1} \leq C x \Delta^{-q} K^{-1}. \]

Next, we consider \( Q \). We write
\[ |Q| \leq C \sum_{m,n \geq 1} \int_{0}^{\infty} \int_{0}^{\infty} |\beta_m(x, x + \Delta, r)| |\beta_n(x, x + \Delta, s)| \]
\[ \leq \frac{N}{K^2} \| I_N A_N \|_1^2 \| A_N \|_1^{m+n-2} (r \wedge s)^{1-q} \int_{0}^{\infty} dr ds \]
\[ + 2C \sum_{m \geq 0} \int_{0}^{\infty} \int_{0}^{\infty} |\beta_0(x, x + \Delta, s)| |\beta_m(x, x + \Delta, r)| \frac{1}{K^2} \| I_N A_N^m \|_1 (r \wedge s)^{1-q} \int_{0}^{\infty} dr ds \]
\[ \leq Q_1 + 2Q_2 + 2Q_3 + 2Q_4. \]
where, using that \( x - \Delta \geq \frac{\delta}{2} \) and that \((r \wedge s)^{1-q} \leq x^{1-q}\) if \(r \wedge s \geq x - \Delta\),

\[
Q_1 = \frac{C}{x^q} \sum_{m,n \geq 1} \int_{x - \Delta}^{x + \Delta} \int_{x - \Delta}^{x + \Delta} \left| \beta_m(x, x + \Delta, r) || \beta_n(x, x + \Delta, s) \right| \frac{N}{K^2} ||| I_K A_N |||_1 ||| A_N |||_1^{m+n-2} dr ds,
\]

\[
Q_2 = C \sum_{m,n \geq 1} \int_{x - \Delta}^{x + \Delta} \int_{x - \Delta}^{x + \Delta} \left| \beta_m(x, x + \Delta, r) || \beta_n(x, x + \Delta, s) \right| \frac{N}{K^2} ||| I_K A_N |||_1 ||| A_N |||_1^{m+n-2} dr ds,
\]

\[
Q_3 = \frac{C}{x^q} \sum_{m \geq 0} \int_{x - \Delta}^{x + \Delta} \int_{x - \Delta}^{x + \Delta} \left| \beta_0(x, x + \Delta, s) || \beta_m(x, x + \Delta, r) \right| \frac{1}{K} ||| I_K A_N^m |||_1 dr ds,
\]

\[
Q_4 = C \sum_{m \geq 0} \int_{x - \Delta}^{x + \Delta} \int_{x - \Delta}^{x + \Delta} \left| \beta_0(x, x + \Delta, s) || \beta_m(x, x + \Delta, r) \right| \frac{1}{K} ||| I_K A_N^m |||_1 dr ds.
\]

In view of [E] Lemma 15-(ii)], we have the inequalities \( \int_{0}^{x+\Delta} |\beta_n(x, x + \Delta, s)| ds \leq 2A^n \) and \( \int_{0}^{x-\Delta} |\beta_m(x, x + \Delta, r)| dr \leq Cm^qA^m\Delta^{-q} \). Hence, on \( \Omega_{N,K} \),

\[
Q_1 \leq C x^{1-q} \sum_{m,n \geq 1} A^m A^{m+n} K^{-2} ||| I_K A_N |||_1 ||| A_N |||_1^{m+n-2} \leq CN^{-1} x^{1-q} \leq C x^{1-q} K^{-1},
\]

\[
Q_2 \leq C \Delta^{-q} \sum_{m,n \geq 1} m^q A^m A^{m+n} K^{-2} ||| I_K A_N |||_1 ||| A_N |||_1^{m+n-2} \leq C \Delta^{-q} N^{-1} \leq C x^{1-q} K^{-1}.
\]

Since furthermore \( |\beta_0(x, x + \Delta, s)| = |\delta_{s=x+\Delta} - \delta_{s=x} + \delta_{s=x+\Delta} \), we have

\[
Q_3 \leq C x^{1-q} \sum_{m \geq 0} A^m K^{-1} ||| I_K A_N^m |||_1 \leq C x^{1-q} K^{-1} \leq C x^{1-q} K^{-1},
\]

\[
Q_4 \leq C \Delta^{-q} \sum_{m \geq 0} m^q A^m K^{-1} ||| I_K A_N^m |||_1 \leq C \Delta^{-q} K^{-1} \leq C \Delta^{-q} K^{-1}.
\]

All in all, on \( \Omega_{N,K} \), we have \( Q \leq C x^{1-q} K^{-1} \).

Finally we consider \( I \). We recall from [E] Lemma 15 (iii)] that there are \( 0 \leq \kappa_{m,n} \leq (m+n)\kappa \) and a function \( \varepsilon_{m,n} : (0, \infty)^2 \rightarrow \mathbb{R} \) satisfying \( |\varepsilon_{m,n}(t, t + \Delta)| \leq C(m+n)^q A^{m+n} t^{1-q} \) such that

\[
\gamma_{m,n}(x, x + \Delta) = \int_{0}^{x+\Delta} \int_{0}^{x+\Delta} (s \wedge u) \beta_m(x, x + \Delta, s) \beta_n(x, x + \Delta, u) duds = \Delta^{m+n} - \kappa_{m,n} A^{m+n} + \varepsilon_{m,n}(x, x + \Delta).
\]

Then we can write \( I \) as:

\[
I = \mu \sum_{m,n \geq 0} \gamma_{m,n}(x, x + \Delta) \frac{1}{K^2} \sum_{i,k=1}^{K} \sum_{j=1}^{N} A_N^m(i,j) A_N^n(k,j) \varepsilon_N(j) = I_1 - I_2 + I_3,
\]
where

\[ I_1 = \mu \Delta \sum_{m,n \geq 0} \Lambda^{m+n} \frac{1}{K^2} \sum_{i,k=1}^{N} \sum_{j=1}^{N} A_N^m(i,j)A_N^n(k,j)\ell_N(j), \]

\[ I_2 = \mu \sum_{m,n \geq 0} \kappa_{m,n} \Lambda^{m+n} \frac{1}{K^2} \sum_{i,k=1}^{N} \sum_{j=1}^{N} A_N^m(i,j)A_N^n(k,j)\ell_N(j), \]

\[ I_3 = \mu \sum_{m,n \geq 0} \varepsilon_{m,n}(x,x + \Delta) \frac{1}{K^2} \sum_{i,k=1}^{N} \sum_{j=1}^{N} A_N^m(i,j)A_N^n(k,j)\ell_N(j). \]

Recalling that \( W_{\infty,\infty}^{N,K} := \frac{\mu N}{K^2} \sum_{j=1}^{N} (c_N(j))^2 \ell_N(j) \) by definition and that \( \sum_{m \geq 0} \Lambda^m(i,j) = Q_N(i,j) \),

\[ I_1 = \mu \Delta \sum_{m,n \geq 0} \Lambda^{m+n} \frac{1}{K^2} \sum_{i,k=1}^{N} \sum_{j=1}^{N} A_N^m(i,j)A_N^n(k,j)\ell_N(j) = \mu \Delta \frac{1}{K^2} \sum_{j=1}^{N} (c_N(j))^2 \ell_N(j) = \Delta \frac{1}{N} W_{\infty,\infty}^{N,K}. \]

Next, we consider the term \( I_3 \). It is obvious that \( Y_{\infty,\infty}^{N,K} \) is a \((\theta_t)_{t=1,...,N}\) measurable function and well-defined on \( \Omega_{N,K} \). Finally, using that \( \varepsilon_{m,n}(x,x + \Delta) \leq C(m + n)^q \Lambda^{m+n}x^\Delta - q \) and that \( \ell_N \) is bounded on \( \Omega_{N,K} \) (we have to treat separately the case \( n = 0 \) or \( m = 0 \)),

\[ I_3 \leq C \frac{x}{K^2 \Delta^q} \sum_{m \geq 0} m^q \Lambda^m \sum_{i,k=1}^{N} A_N^m(k,i) + C \frac{xN}{\Delta^q K^2} \sum_{m,n \geq 1} (n + m)^q \Lambda^{m+n}||I_K A_N^m||_1 ||I_K A_N^n||_1 \]

\[ \leq C \frac{x}{K^2 \Delta^q} \sum_{m \geq 0} m^q \Lambda^m ||I_K A_N^m||_1 + C \frac{xN}{\Delta^q K^2} \sum_{m,n \geq 1} (n + m)^q \Lambda^{m+n}||I_K A_N^m||_1 ||I_K A_N^n||_1 \]

\[ \leq C \frac{x}{N \Delta^q}, \]

still on \( \Omega_{N,K} \). All in all, we have verified that \( V_{\infty,\infty}^{N,K} = I - M + Q \), with

\[ |M| + |Q| + |I - \Delta N^{-1} W_{\infty,\infty}^{N,K} + Y_{\infty,\infty}^{N,K}| \leq C x \Delta^{-q} K^{-1}, \]

which completes the proof. \( \square \)

Next, we consider the term \( D_{\Delta,t}^{N,K,4} \).

**Lemma 9.5.** Assume \( H(q) \) for some \( q \geq 1 \). Then a.s. on \( \Omega_{N,K} \), for \( 1 \leq \Delta \leq \frac{t}{4} \), we have:

\[ \mathbb{E}_\theta[D_{\Delta,t}^{N,K,4}] \leq C \frac{N t}{K \Delta^{1+q}}. \]

**Proof.** Recalling that \( U_{t}^{i,N} = Z_{t}^{i,N} - \mathbb{E}_\theta[Z_{t}^{i,N}] \), we see that

\[ D_{\Delta,t}^{N,K,4} = \left| \frac{2N}{t} \sum_{a=t/(2\Delta)+1}^{t/\Delta} \operatorname{Var}(\bar{U}_{2a\Delta}^{N,K} - \bar{U}_{(a-1)\Delta}^{N,K} - N^{-1} \sum_{a=t/(2\Delta)+1}^{2t/\Delta} \operatorname{Var}(\bar{U}_{a\Delta}^{N,K} - \bar{U}_{(a-1)\Delta}^{N,K} - W_{\infty,\infty}^{N,K}) \right|. \]

By Lemma 9.4, we have

\[ \operatorname{Var}_\theta(\bar{U}_{x+\Delta}^{N,K} - \bar{U}_x^{N,K}) = \frac{\Delta}{N} W_{\infty,\infty}^{N,K} - Y_{\infty,\infty}^{N,K} + r_{N,K}(x,\Delta). \]
Since \( a \in \{t/(2\Delta) + 1, \ldots, t/\Delta\} \), \( x = 2(a-1)\Delta \geq t \) satisfies \( 2\Delta \leq \frac{x}{2} \) and for \( a \in \{t/\Delta + 1, \ldots, 2t/\Delta\} \), \( x = (a-1)\Delta \geq t \) satisfies \( \Delta \leq x/2 \). Then we conclude that

\[
D_{\Delta,t}^{N,K,4} = \frac{2N}{t} \sum_{a = t/(2\Delta)+1}^{t/\Delta} \left[ \frac{2\Delta}{N} \mathcal{W}_{\infty,\infty}^{N,K} - \mathcal{Y}^{N,K} + \mathcal{R}_{N,K}(2(a-1)\Delta, 2\Delta) \right] - \frac{N}{t} \sum_{a = t/(2\Delta)+1}^{2t/\Delta} \mathcal{R}_{N,K}(2(a-1)\Delta, 2\Delta) - N \sum_{a = t/\Delta + 1}^{t/\Delta} \mathcal{R}_{N,K}((a-1)\Delta, \Delta)
\]

But \( |\mathcal{R}_{N,K}(x, \Delta)| \leq Cx\Delta^{-q}K^{-1} \), whence finally

\[
D_{\Delta,t}^{N,K,4} \leq C \frac{N}{t^q} \left( \frac{t}{\Delta} \right) = \frac{CNt}{K^{1+q}}
\]

as desired. \( \square \)

To treat the last term \( D_{\Delta,t}^{N,K,3} \), we need this following Lemma.

**Lemma 9.6.** Assume \( H(q) \) for some \( q \geq 1 \). On the set \( \Omega_{N,K} \), for all \( t, x, \Delta \geq 1 \), we have

\[
\text{Var} \left[ (\bar{U}_{x+\Delta} - \bar{U}_x)^2 \right] \leq C \frac{\Delta^2}{K^2} + \frac{t^2}{K^2 \Delta^{2q}} \quad \text{if} \quad \frac{t}{\Delta} \leq x - \Delta \leq x + \Delta \leq 2t
\]

and

\[
\text{Cov} \left( (\bar{U}_{x+\Delta} - \bar{U}_x)^2, (\bar{U}_{y+\Delta} - \bar{U}_y)^2 \right) \leq C \left( \frac{\sqrt{t}}{K\Delta^{q-1}} + \frac{t^2}{K^2 \Delta^{2q}} + \frac{\sqrt{t}}{K^2 \Delta^{q-1}} \right)
\]

if \( \frac{t}{\Delta} \leq y - \Delta \leq y + \Delta \leq x - 2\Delta \leq x + \Delta \leq 2t \).

**Proof.** Step 1: recalling (12), for \( z \in [x, x + \Delta] \), we write

\[
U_{z}^{i,N} = \sum_{n \geq 0} \int_{0}^{z} \beta_n(x, z, r) \sum_{j=1}^{N} A_N^N(i,j)M_{t}^{j,N} dr = \Gamma_{x,z}^{i,N} + X_{x,z}^{i,N},
\]

where \( \beta_n(x, z, r) = \phi^*(z - r) - \phi^*(x - r) \) and where

\[
\Gamma_{x,z}^{i,N} = \sum_{n \geq 0} \int_{x-\Delta}^{z} \beta_n(x, z, r) \sum_{j=1}^{N} A_N^N(i,j)(M_{t}^{j,N} - M_{x-\Delta}^{j,N}) dr,
\]

\[
X_{x,z}^{i,N} = \sum_{n \geq 0} \left( \int_{x-\Delta}^{z} \beta_n(x, z, r) dr \right) \sum_{j=1}^{N} A_N^N(i,j)M_{x-\Delta}^{j,N} + \sum_{n \geq 0} \int_{x-\Delta}^{z} \beta_n(x, z, r) \sum_{j=1}^{N} A_N^N(i,j)M_{x}^{j,N} dr.
\]

We set \( \Gamma_{x,z}^{N,K} = K^{-1} \sum_{i=1}^{K} \Gamma_{x,z}^{i,N} \) and \( X_{x,z}^{N,K} = K^{-1} \sum_{i=1}^{K} X_{x,z}^{i,N} \). We write

\[
\bar{X}_{x,z}^{N,K} = \sum_{n \geq 0} \left( \int_{x-\Delta}^{z} \beta_n(x, z, r) dr \right) O_{x-\Delta}^{N,K,n} + \sum_{n \geq 0} \int_{x-\Delta}^{z} \beta_n(x, z, r) O_{x}^{N,K,n} dr.
\]

where

\[
O_{x}^{N,K,n} = \frac{1}{K} \sum_{i=1}^{K} \sum_{j=1}^{N} A_N^N(i,j)M_{x}^{j,N}.
\]
By (13), we have \([M^{i,N}, M^{j,N}]_t = 1_{(i=j)} Z_{t}^{i,N}.\) Hence, for \(n \geq 1,\)
\[
[O^{N,K,n}, O^{N,K,n}]_t = \frac{1}{K^2} \sum_{j=1}^{N} \left( \sum_{i=1}^{K} A_N^0(i,j) \right)^2 Z_t^{j,N} \leq \frac{N}{K^2} \|I_K A_N\|_4^2 \|A_N\|_{2n-2} Z_t^{N,K}.
\]
And when \(n = 0,\) we have
\[
[O^{N,K,0}, O^{N,K,0}]_t = \frac{1}{K^2} \sum_{j=1}^{N} \left( \sum_{i=1}^{K} A_N^0(i,j) \right)^2 Z_t^{j,N} = \frac{1}{K} Z_t^{N,K}.
\]
By Lemma 7.2, we have, on \(\Omega_{N,K},\)
\[
\mathbb{E}_\theta[\bar{Z}_t^{N,K}]^2 \leq 2\mathbb{E}_\theta[\bar{Z}_t^{N,K}]^2 + 2\mathbb{E}_\theta[(\bar{U}_t^{N,K})^2] \leq C t^2.
\]
Hence, by the Doob’s inequality, when \(n \geq 1:\)
\[
(16) \quad \mathbb{E}_\theta\left[ \sup_{[0,t]} \left( O_{r}^{N,K,n} - O_{r-x}^{N,K,n} \right)^4 \right] \leq \frac{C N^2}{t} \|I_K A_N\|_4^2 \|A_N\|_{2n-4} \mathbb{E}_\theta\left[ (\bar{Z}_t^{N,K})^2 \right] \leq \frac{C}{N^2} \|A_N\|_4^2 t^2.
\]
By the same way,
\[
(17) \quad \mathbb{E}_\theta\left[ \sup_{[0,t]} \left( O_{r}^{N,K,0} - O_{r-x}^{N,K,0} \right)^4 \right] \leq \frac{C K^{-2}}{t^2} \|A_N\|_4^2 t^2.
\]
and in the case \(n = 0,\) by Doob’s inequality,
\[
(18) \quad \mathbb{E}_\theta\left[ \sup_{[0,t]} \left( O_{r}^{N,K,0} - O_{r-x}^{N,K,0} \right)^4 \right] \leq C K^{-2} t^2.
\]
Step 2: We recall the result of [6] Lemma 15:
\[
\left| \int_{x} Z_t^{N,K} \right| \leq C n^2 \Lambda_n \Delta^{-q}.
\]
So we conclude that
\[
|\bar{X}^{N}_{x,z}| \leq C \sum_{n \geq 1} n^2 \Lambda_n \Delta^{-q} \sup_{[0,t]} |O_{r}^{N,K,n}| = C \sum_{n \geq 1} n^2 \Lambda_n \Delta^{-q} \sup_{[0,t]} |O_{r}^{N,K,n}|.
\]
Recalling (10), on the set \(\Omega_{N,K},\) by using the Minkowski inequality we conclude that
\[
\mathbb{E}[\bar{X}^{N}_{x,z}]^4 \leq C \sum_{n \geq 1} n^2 \Lambda_n \|A_N\|_4^2 |\Delta^{-q} N^{1/2} \sqrt{t} \leq C \Delta^{-q} N^{1/2} \sqrt{t}.
\]
Step 3: We rewrite
\[
\bar{\Gamma}^{N,K}_{x,z} = \sum_{n \geq 1} \int_{x} \beta_n(x,z) |O_{r}^{N,K,n} - O_{r-x}^{N,K,n}| dr.
\]
Since \(\int_{x} \beta_n(x,z) dr \leq 2 \Lambda_n \) by [6] Lemma 15, using (17)-(18) and the Minkowski inequality,
\[
\mathbb{E}[(\bar{\Gamma}^{N,K}_{x,z})^4] \leq C \left\{ \left| \Delta^{\frac{1}{2}} K^{-\frac{1}{2}} \right|^4 + \sum_{n \geq 1} \Lambda_n \frac{1}{\sqrt{N}} \|A_N\|_4^2 |\Delta^{-q} N^{1/2} \sqrt{t} \right\} \leq C \Delta^{\frac{1}{2}} (K^{-\frac{1}{2}} N^{1/2}) \leq C \Delta^{\frac{1}{2}} K^{-\frac{1}{2}}.
\]
Step 4: Since, see Step 1,
\[
(\bar{U}^{N,K}_{x,x} - \bar{\Gamma}^{N,K}_{x,x})^4 = (\bar{U}^{N,K}_{x,x} + \bar{\Gamma}^{N,K}_{x,x})^4 \leq \mathbb{E}[(\bar{\Gamma}^{N,K}_{x,x})^4] \leq 8 \left[ (\bar{\Gamma}^{N,K}_{x,x})^4 + (\bar{X}^{N,K}_{x,x})^4 \right],
\]
we deduce from Steps 2 and 3 that (13) holds true.
Step 5: The aim of this step is to show that, for \(x, y, \Delta\) as in the statement, it holds true that

\[
\text{Cov}_\theta \left( (\tilde{U}_{x,x+\Delta}^N - \tilde{U}_x^N)^2, (\tilde{U}_{y,y+\Delta}^N - \tilde{U}_y^N)^2 \right) \leq \text{Cov}_\theta \left[ (\tilde{\Gamma}_{x,x+\Delta}^N, (\tilde{\Gamma}_{y,y+\Delta}^N)^2 \right] + \frac{C}{K^2} \left( \frac{t^2}{\Delta^4 \nu} + \frac{\sqrt{t}}{\Delta^{q-\frac{1}{2}}} \right).
\]

We write

\[
(\tilde{U}_{x,x+\Delta} - \tilde{U}_x^N)(\tilde{U}_{y,y+\Delta} - \tilde{U}_y^N) = (\tilde{\Gamma}_{x,x+\Delta}^N)^2 + (\tilde{\Gamma}_{y,y+\Delta}^N)^2 + 2\tilde{\Gamma}_{x,x+\Delta}^N \tilde{\Gamma}_{y,y+\Delta}^N \tilde{\Gamma}_{x,x\Delta}^N \tilde{\Gamma}_{y,y\Delta}^N + \text{and the same formula for } y.
\]

Then we use the bilinearity of the covariance. We have the term \(\text{Cov}_\theta [(\tilde{\Gamma}_{x,x+\Delta}^N)^2, (\tilde{\Gamma}_{y,y+\Delta}^N)^2]\), and it remains to verify that

\[
R := \mathbb{E}_\theta \left( (\tilde{\Gamma}_{x,x+\Delta}^N)^2 (\tilde{\Gamma}_{y,y+\Delta}^N)^2 + 2(\tilde{\Gamma}_{x,x+\Delta}^N)^2 |\tilde{\Gamma}_{y,y+\Delta}^N| + (\tilde{\Gamma}_{x,x+\Delta}^N)^2 (\tilde{\Gamma}_{y,y+\Delta}^N)^2 + (\tilde{\Gamma}_{x,x+\Delta}^N)^2 (\tilde{\Gamma}_{y,y+\Delta}^N)^2 + 2 (\tilde{\Gamma}_{y,y+\Delta}^N)^2 |\tilde{\Gamma}_{x,x+\Delta}^N| + 2 |\tilde{\Gamma}_{x,x+\Delta}^N| (\tilde{\Gamma}_{y,y+\Delta}^N)^2 + 4 |\tilde{\Gamma}_{y,y+\Delta}^N| (\tilde{\Gamma}_{x,x+\Delta}^N)^2 \right)
\]

is bounded by \(\frac{C^4}{K^2} \left( \frac{t^2}{\Delta^4 \nu} + \frac{\sqrt{t}}{\Delta^{q-\frac{1}{2}}} \right)\).

By Step 2 and 3, we know that \(\mathbb{E}[|\tilde{\Gamma}_{x,x+\Delta}^N|^4] \leq C\Delta^2 K^{-2}\) and \(\mathbb{E}[|\tilde{\Gamma}_{y,y+\Delta}^N|^4] \leq C\Delta^2 \Delta^{-4q} N^{-2}\), and the same inequalities hold true with \(y\) instead of \(x\). Using furthermore the Hölder inequality, one may verify that, setting \(a = C\Delta^2 K^{-2}\) and \(b = C\Delta^2 \Delta^{-4q} N^{-2}\), we have

\[
R \leq \sqrt{ab} + 2a^{3/4} b^{1/4} + \sqrt{ab} + b + 2a^{1/4} b^{3/4} + 2a^{3/4} b^{1/4} + 2a^{1/4} b^{3/4} + 4\sqrt{ab},
\]

which is easily bounded by \(C(b + b^{1/4} a^{3/4})\), from which the conclusion follows.

Step 6: Here we want to verify that

\[
\mathcal{I} := \left| \text{Cov}_\theta [(\tilde{\Gamma}_{x,x+\Delta}^N)^2, (\tilde{\Gamma}_{y,y+\Delta}^N)^2] \right| \leq \frac{C \sqrt{t}}{K \Delta^{\nu-1}}.
\]

We recall from [6] Lemma 30, Step 6) that for any \(r, s\) in \([x-\Delta, x+\Delta]\), any \(u, v\) in \([y-\Delta, y+\Delta]\), any \(j, l, \delta, \varepsilon\) in \(\{1, \ldots, N\}\),

\[
|\text{Cov}_\theta \left( (M_{r,l}^N - M_{x-\Delta}^N)(M_{s,l}^N - M_{x-\Delta}^N), (M_{u,\delta}^N - M_{y-\Delta}^N)(M_{v,\delta}^N - M_{y-\Delta}^N) \right) | \leq C \mathbf{1}_{(j=l)} \sqrt{\Delta}^{1-q}.
\]

We start from

\[
\tilde{\Gamma}_{x,x+\Delta}^N = \sum_{n \geq 0} \int_{x-\Delta}^{x+\Delta} \beta_n(x, x + \Delta, r) \frac{1}{K} \sum_{i=1}^{K} \sum_{j=1}^{N} A_N(i, j)(M_{r,l}^N - M_{x-\Delta}^N) dr.
\]

So

\[
\mathcal{I} = \sum_{m,n,a,b \geq 0} \int_{x-\Delta}^{x+\Delta} \int_{x-\Delta}^{x+\Delta} \int_{y-\Delta}^{y+\Delta} \int_{y-\Delta}^{y+\Delta} \beta_m(x, x + \Delta, r) \beta_n(x, x + \Delta, s)
\]

\[
\beta_a(y, y + \Delta, u) \beta_b(y, y + \Delta, v) \frac{1}{K^4} \sum_{i,k,a,\gamma=1}^{K} \sum_{j,l,\delta,\varepsilon=1}^{N} A_{N}^m(i, j) A_{N}^n(k, l) A_{N}^a(\alpha, \delta) A_{N}^b(\gamma, \varepsilon)
\]

\[
|\text{Cov}_\theta \left( (M_{r,l}^N - M_{x-\Delta}^N)(M_{s,l}^N - M_{x-\Delta}^N), (M_{u,\delta}^N - M_{y-\Delta}^N)(M_{v,\delta}^N - M_{y-\Delta}^N) \right) | dvdu dsdr
\]

\[
\leq C \sqrt{\Delta}^{1-q} \sum_{m,n,a,b \geq 0} \Lambda^{m+n+a+b} \frac{1}{K^4} \sum_{i,k,a,\gamma=1}^{K} \sum_{j,l,\delta,\varepsilon=1}^{N} A_{N}^m(i, j) A_{N}^n(k, l) A_{N}^a(\alpha, \delta) A_{N}^b(\gamma, \varepsilon).
\]
We used again the result \[ \int_{x-\Delta}^{x+\Delta} |\beta_m(x, x + \Delta, r)|dr \leq 2\Lambda^m \]. And we observe one more time that \( A_{m}^N(i, j) = 1 \) if \( m, n, a, b \) and, when \( m \geq 1 \), \( \sum_{i=1}^{K} A_{m}^N(i, j) \leq ||I_{K}A_{N}||_1 ||A_{N}||_1^{m-1} \). We now treat separately the cases where \( m, n, a, b \) vanish and find, on \( \Omega_{N,K} \),

\[
I \leq \frac{C\sqrt{t}}{K\Delta^{q-1}} \sum_{m,n,a,b \geq 1}^{N} \sum_{j,\delta,\epsilon=1}^{K} A^{m+n+a+b} \|I_{K}A_{N}\|_1^{2} ||A_{N}||_1^{m+n+a+b-4} \\
+ \frac{4C\sqrt{t}}{K\Delta^{q-1}} \sum_{n,a,b \geq 1}^{N} \sum_{j,\delta,\epsilon=1}^{K} 1(i=j) A^{n+a+b} \|I_{K}A_{N}\|_1^{2} ||A_{N}||_1^{n+a+b-3} \\
+ \frac{2C\sqrt{t}}{K\Delta^{q-1}} \sum_{a,b \geq 1}^{N} \sum_{j,\delta,\epsilon=1}^{K} \left\{ \sum_{i,k=1}^{K} 1(i=k) \right\} A^{a} \|I_{K}A_{N}\|_1^{2} ||A_{N}||_1^{a+b-2} \\
+ \frac{2C\sqrt{t}}{K\Delta^{q-1}} \sum_{a,b \geq 1}^{N} \sum_{j,\delta,\epsilon=1}^{K} \left\{ 2 \sum_{i,k=1}^{K} 1(i=k) \right\} A^{a} \|I_{K}A_{N}\|_1^{2} ||A_{N}||_1^{a+b-2} \\
+ \frac{2C\sqrt{t}}{K\Delta^{q-1}} \sum_{i,a,\gamma=1}^{N} 1(i=a) \gamma \right\} \leq \frac{2C\sqrt{t}}{K\Delta^{q-1}} \left( \frac{K^{4}}{N^{2}} \frac{N^{3}}{N^{2}} K^{2} N^{2} K \right) \leq \frac{C\sqrt{t}}{K\Delta^{q-1}}.
\]

Step 7: We conclude from Steps 5 and 6 that on the set \( \Omega_{N,K} \),

\[
\text{Cov}_{\theta} \left[ (\hat{U}_{x+\Delta} - \hat{U}_{x})^{2}, (\hat{U}_{y+\Delta} - \hat{U}_{y})^{2} \right] \leq C \left[ \frac{\sqrt{t}}{K\Delta^{q-1}} + \frac{t^{2}}{K^{2}\Delta^{q+1}} + \frac{\sqrt{t}}{K^{2}\Delta^{q-1}} \right],
\]

which proves \([15]\). \( \square \)

We can now study \( D^{N,K,3}_{\Delta,t} \).

**Lemma 9.7.** Assume \( H(q) \) for some \( q \geq 1 \). On the set \( \Omega_{N,K} \), for all \( 1 \leq \Delta \leq \frac{t}{2} \),

\[
\text{E}_{\theta}(D^{N,K,3}_{\Delta,t}^{2}) \leq C \left( \frac{N^{2}}{K^{2}} \frac{\Delta}{t} + \frac{N^{2}}{K^{2}} \frac{\Delta}{\Delta^{q+1}} + \frac{N^{2}}{K^{2}} \frac{\sqrt{t}}{\Delta^{q+1}} + \frac{N^{2}}{K^{2}} \frac{t^{2}}{\Delta^{q+2}} + \frac{N^{2}}{K^{2}} \frac{\sqrt{t}}{\Delta^{q+2}} \right).
\]

**Proof.** Recall that by definition

\[
D^{N,K,3}_{\Delta,t} = \frac{N}{t} \left| \sum_{a=t/\Delta+1}^{2t/\Delta} \left( \hat{Z}_{a}\Delta - \hat{Z}_{(a-1)}\Delta - \text{E}_{\theta}[\hat{Z}_{a}\Delta - \hat{Z}_{(a-1)}\Delta] \right)^{2} \\
- \text{E}_{\theta} \left[ \sum_{a=t/\Delta+1}^{2t/\Delta} \left( \hat{Z}_{a}\Delta - \hat{Z}_{(a-1)}\Delta - \text{E}_{\theta}[\hat{Z}_{a}\Delta - \hat{Z}_{(a-1)}\Delta] \right)^{2} \right] \right|.
\]

Since now \( \hat{U}_{r}^{N,K} = \hat{Z}_{r}^{N,K} - \text{E}_{\theta}[\hat{Z}_{r}^{N,K}] \),

\[
\text{E}_{\theta}(D^{N,K,3}_{\Delta,t}^{2}) = \frac{N^{2}}{t^{2}} \text{Var}_{\theta} \left( \sum_{a=t/\Delta+1}^{2t/\Delta} \left( \hat{U}_{a}\Delta - \hat{U}_{(a-1)}\Delta \right)^{2} \right) = \sum_{a,b=t/\Delta+1}^{2t/\Delta} \sum_{a,b=t/\Delta+1}^{2t/\Delta} K_{a,b},
\]
where \( K_{a,b} = \text{Cov}_\theta[(\bar{U}_{a\Delta}^{N,K} - \bar{U}_{(a-1)\Delta}^{N,K})^2, (\bar{U}_{b\Delta}^{N,K} - \bar{U}_{(b-1)\Delta}^{N,K})^2] \). By Lemma 9.6 for \(|a-b| \leq 2,

\[ |K_{a,b}| \leq \left\{ \text{Var}_\theta[(\bar{U}_{a\Delta}^{N,K} - \bar{U}_{(a-1)\Delta}^{N,K})^2] \text{Var}_\theta[(\bar{U}_{b\Delta}^{N,K} - \bar{U}_{(b-1)\Delta}^{N,K})^2] \right\} \leq C \left( \frac{\Delta^2}{K^2} + \frac{t^2}{K^2\Delta^{2q}} \right) \]

If now \(|a-b| \geq 3\), we set \( x = (a-1)\Delta, y = (b-1)\Delta \) in (15) and get

\[ |K_{a,b}| \leq C \left( \frac{\sqrt{t}}{K\Delta^{q+1}} + \frac{t^2}{K^2\Delta^{2q+1}} \right). \]

Finally we conclude that

\[ \mathbb{E}_\theta[(D_{\Delta,t}^{N,K,3})^2] \leq C \left( \frac{N^2}{t^2} \Delta \left( \frac{\Delta^2}{K^2} + \frac{t^2}{K^2\Delta^{2q}} \right) + C_{N,K} \frac{N^2}{t^2} \Delta^2 \left( \frac{\sqrt{t}}{K\Delta^{q-1}} + \frac{t^2}{K^2\Delta^{2q+1}} + \frac{\sqrt{t}}{K^2\Delta^{q-\frac{1}{2}}}, \right) \right) \leq C \left( \frac{N^2}{K^2 t} + \frac{\sqrt{t}}{K^2\Delta^{q+1}} + \frac{\sqrt{t}}{K^2\Delta^{q+\frac{1}{2}}}, \right). \]

which completes the proof. \( \square \)

**Lemma 9.8.** Under the assumption \( H(q) \) for some \( q \geq 3 \) and the the set \( \Omega_{N,K} \), we have:

\[ \mathbb{E}_\theta \left[ \left| W_{\Delta,t}^{N,K} - W_{\infty,\infty}^{N,K} \right| \right] \leq C \left( \frac{\overline{\Delta}}{t} + \frac{N^2}{K\Delta^{\frac{q}{2}+1}} + \frac{Nt}{K\Delta^{q+1}} \right). \]

**Proof.** We summarize all the above Lemmas and conclude that, on \( \Omega_{N,K}, \)

\[ \mathbb{E}_\theta \left[ \left| W_{\Delta,t}^{N,K} - W_{\infty,\infty}^{N,K} \right| \right] \leq \mathbb{E}_\theta \left[ D_{\Delta,t}^{N,K,1} + 2D_{2\Delta,t}^{N,K,1} + 2D_{\Delta,t}^{N,K,2} + 2D_{2\Delta,t}^{N,K,2} + 2D_{\Delta,t}^{N,K,3} + 2D_{2\Delta,t}^{N,K,3} + 2D_{\Delta,t}^{N,K,4} \right] \]

\[ \leq C \left( \frac{N\Delta}{K} + \frac{\sqrt{t}}{K^{\frac{q}{2}}} + \frac{Nt}{K\Delta^{q+1}} + \frac{Nt}{K\Delta^{q+\frac{1}{2}}}, \right). \]

Since \( 1 \leq \Delta \leq t \) and \( q \geq 3 \), we conclude, after some tedious but direct computations, that

\[ \mathbb{E}_\theta \left[ \left| W_{\Delta,t}^{N,K} - W_{\infty,\infty}^{N,K} \right| \right] \leq C \left( \frac{\overline{\Delta}}{t} + \frac{N^2}{K\Delta^{\frac{q}{2}+1}} + \frac{Nt}{K\Delta^{q+1}} \right). \]

The most difficult terms are

\[ \sqrt{\frac{N^2}{K\Delta^{q+1}}} = \sqrt{\frac{N^2}{K\Delta^{(q+1)/2}}} \leq \frac{N^2}{K\Delta^{(q+1)/2}} + \frac{t^{1/2}}{\Delta^{(q+1)/2}} \leq \frac{N^2}{K\Delta^{(q+1)/2}} + \frac{Nt}{K\Delta^{q/2+1}} \]

and

\[ \sqrt{\frac{N^2}{K^2\Delta^{q+\frac{1}{2}}}} \leq \frac{N}{K} \left( \sqrt{\frac{\overline{\Delta}}{t}} + \frac{t}{\Delta^{q+1}} \right) \leq \frac{N}{K} \left( \sqrt{\frac{\overline{\Delta}}{t}} + \frac{t}{\Delta^{q/2+1}} \right). \]

The proof is complete. \( \square \)

Next we prove the main result of this section.
Proof of Theorem 9.1. We start from
\[
E\left[1_{\Omega_{N,K}} \left| \frac{\chi_{N,K}^{\Delta,t}}{(1 - \Lambda p)^3} \right| \right]
\leq E\left[1_{\Omega_{N,K}} \left| \psi_{N,K}^{\Delta,t} - \psi_{N,K}^{\infty} \right| \right] + E\left[1_{\Omega_{N,K}} \left| \psi_{N,K}^{\infty} - \frac{\mu}{(1 - \Lambda p)^3} \right| \right]
\leq C\left(\frac{N}{K} \sqrt{\frac{\Delta}{t}} + \frac{N^2}{K \Delta^{3/2}} + \frac{N t}{K \Delta^{3/2}} + \frac{1}{\sqrt{K}}\right) + C\frac{N}{K \sqrt{K t}} + C\frac{N}{K \sqrt{K t}}
\]
by Lemmas 9.3, 7.3, and 5.19. Since \( t \geq \Delta \geq 1 \), we have \( \frac{N}{K \Delta^{3/2}} \leq \frac{N t}{K \Delta^{3/2}} \) and we conclude that
\[
E\left[1_{\Omega_{N,K}} \left| \frac{\chi_{N,K}^{\Delta,t}}{(1 - \Lambda p)^3} \right| \right] \leq C\left(\frac{N}{K} \sqrt{\frac{\Delta}{t}} + \frac{N^2}{K \Delta^{3/2}} + \frac{N t}{K \Delta^{3/2}} + \frac{1}{\sqrt{K}} + \frac{N}{K \sqrt{K t}}\right),
\]
which was our goal.

Next, we write down the probability estimate.

Corollary 9.9. Assume \( H(q) \) for some \( q \geq 1 \). We have
\[
P\left(\left| \chi_{N,K}^{\Delta,t} - \frac{\mu}{(1 - \Lambda p)^3} \right| \geq \varepsilon \right)
\leq \frac{C}{\varepsilon} \left(\frac{N}{K} \sqrt{\frac{\Delta}{t}} + \frac{N^2}{K \Delta^{3/2}} + \frac{N t}{K \Delta^{3/2}} + \frac{1}{\sqrt{K}} + \frac{N}{K \sqrt{K t}}\right) + C N e^{-C'K}.
\]
Under \( H(q) \) for some \( q \geq 3 \) and with the choice \( \Delta_t \sim \frac{t^{1/3}}{t^{1/2}} \), this gives
\[
P\left(\left| \chi_{N,K}^{\Delta,t} - \frac{\mu}{(1 - \Lambda p)^3} \right| \geq \varepsilon \right) \leq \frac{C}{\varepsilon} \left(\frac{1}{K} + \frac{N}{K t^{1/4}} + \frac{N^2}{K t^{3/4}}\right) + C N e^{-C'K}.
\]
Proof. The first assertion immediately follows from Theorem 9.1 and Lemma 5.7. The second assertion is not difficult.

10. The final result in the subcritical case.

We summarize the rates we obtained for the three estimators: by Theorem 7.1 and Corollaries 8.7 and 9.9, we have, under \( H(q) \) for some \( q \geq 3 \), for all \( \varepsilon \in (0, 1) \), all \( t \geq 1 \), all \( N \geq K \geq 1 \),
\[
P\left(\left| \chi_{N,K}^{\Delta,t} - \frac{\mu}{(1 - \Lambda p)^3} \right| \geq \varepsilon \right) \leq C N e^{-C'K} + \frac{C}{\varepsilon} \left(\frac{1}{\sqrt{N K}} + \frac{1}{\sqrt{N t}} + \frac{1}{t^3}\right),
\]
\[
P\left(\left| \psi_{N,K}^{\Delta,t} - \frac{\mu(1 - p)}{(1 - \Lambda p)^2} \right| \geq \varepsilon \right) \leq C N e^{-C'K} + \frac{C}{\varepsilon} \left(\frac{1}{\sqrt{N K}} + \frac{N}{t^{1/4}}\right),
\]
\[
P\left(\left| \chi_{N,K}^{\Delta,t} - \frac{\mu}{(1 - \Lambda p)^3} \right| \geq \varepsilon \right) \leq \frac{C}{\varepsilon} \left(\frac{1}{K} + \frac{N}{K t^{1/4}} + \frac{N^2}{K t^{3/4}}\right) + C N e^{-C'K}.
\]
Proof of Theorem 2.3. One easily verifies that \( \Psi \) is \( C^\infty \) in the domain \( D \), that
\[
(u, v, w) = \left(\frac{\mu}{1 - \Lambda p}, \frac{\mu^2(1 - p)}{(1 - \Lambda p)^2}, \frac{\mu^3}{(1 - \Lambda p)^3}\right) \ni D
\]
and that \( \Psi(u, v, w) = (\mu, \Lambda, p) \). Hence there is a constant \( c \) such that for any \( N \geq 1, t \geq 1, \) any \( \varepsilon \in (0, 1/c) \),

\[
P \left( \left| \Psi(\varepsilon_t^{N,K}, \nu_t^{N,K}, \lambda_{\Delta_t,t}^{N,K}) - (\mu, \Lambda, p) \right| \geq \varepsilon \right)
\leq P \left( \left| \varepsilon_t^{N,K} - u \right| + \left| \nu_t^{N,K} - v \right| + \left| \lambda_{\Delta_t,t}^{N,K} - w \right| \geq c\varepsilon \right)
\leq \frac{C}{\varepsilon} \left( \frac{1}{\sqrt{K}} + \frac{N}{K \sqrt{t}} + \frac{N}{t \sqrt{K}} \right) + C N e^{-C'K},
\]

which completes the proof. \( \square \)

11. Analysis of a Random Matrix for the Supercritical Case

We define the matrix \( A_N \) by \( A_N(i, j) := N^{-1} \theta_{ij}, i, j \in \{1, ..., N\} \). We assume here that \( p \in (0, 1] \) and we introduce the events:

\[
\Omega_N^2 := \left\{ \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} A_N(i, j) > \frac{p}{2} \right\}
\quad \text{and} \quad
\left| NA_N^2(i, j) - p^2 \right| < \frac{p^2}{2N^{3/8}} \quad \text{for all} \quad i, j = 1, ..., N.
\]

\[
\Omega_N^{K,2} := \left\{ \frac{1}{K} \sum_{i=1}^{K} \sum_{j=1}^{N} A_N(i, j) > \frac{p}{2} \right\} \cap \Omega_N^2.
\]

**Lemma 11.1.** One has

\[
P(\Omega_N^{K,2}) \geq 1 - C e^{-cN^{1/4}}.
\]

**Proof.** By [6] lemma 33], we already have \( P(\Omega_N^2) \geq 1 - C e^{-cN^{1/4}} \). We recall the Hoeffding inequality for the Binomial(\(n, q\)) random variables. For all \( x \geq 0 \) and \( X \) is a Binomial(\(n, q\)) distributed, we have:

\[
P \left( \left| X - nq \right| \geq x \right) \leq 2 \exp(-2x^2/n).
\]

Since \( N \sum_{i=1}^{K} \sum_{j=1}^{N} A_N(i, j) = \sum_{i=1}^{K} \sum_{j=1}^{N} \theta_{ij} \) is Binomial(\(NK, p\)) distributed,

\[
P \left( K^{-1} \sum_{i=1}^{K} \sum_{j=1}^{N} A_N(i, j) \leq \frac{p}{2} \right) \leq P \left( \left| N \sum_{i=1}^{K} \sum_{j=1}^{N} A_N(i, j) - NKp \right| \geq \frac{NKp^2}{2} \right) \leq 2 \exp \left( - \frac{NKp^2}{2} \right).
\]

So we have

\[
P(\Omega_N^{K,2}) \geq 1 - 2 \exp \left( - \frac{NKp^2}{2} \right) - Ce^{-cN^{1/4}} \geq 1 - Ce^{-cN^{1/4}}.
\]

\( \square \)

Next we apply the Perron-Frobenius theorem and recall some lemma in [1].

**Lemma 11.2.** On the event \( \Omega_N^{K,2} \), the spectral radius \( \rho_N \) of \( A_N \) is a simple eigenvalue of \( A_N \) and \( \rho_N \in \left[p \left(1 - \frac{1}{2N^{3/8}}\right), p \left(1 + \frac{1}{2N^{3/8}}\right)\right] \). There is a row eigenvector \( V_N \in \mathbb{R}_+^N \) of \( A_N \) for the eigenvalue \( \rho_N \) such that \( \|V_N\|_2 = \sqrt{N} \). We also have \( V_N(i) > 0 \) for all \( i = 1, ..., N \).

**Proof.** See [6] lemma 34].

We set \( V_N^K := I_K V_N \) and let \( (e_1, \ldots, e_N) \) the canonical basis of \( \mathbb{R}^N \). Recall that \( 1_N = \sum_{i=1}^{N} e_i \).
Lemma 11.3. There exists $N_0 \geq 1$ (depending only on $p$) such that for all $N \geq N_0$, on the set $\Omega_0^{K^2}$, these properties hold true for all $i, j, k, l = 1, \ldots, N$:

(i) for all $n \geq 2$, $A_N^n(i, j) \leq \left(\frac{2}{3}\right) A_N^n(k, l)$,

(ii) $V_N(i) \in [\frac{1}{2}, 2]$,

(iii) for all $n \geq 0$, $\|A_N^n 1_N\|_2 \in [\sqrt{\frac{N}{2}}, 2\sqrt{N}\rho_N^n]$,

(iv) for all $n \geq 2$, $A_N^n(i, j) \in \left[p(n)/(3N), 3\rho_N^n/N\right]$,

(v) for all $n \geq 0$, all $r \in [1, \infty]$, $\|A_N^n e_j\|_r - V_N/\|V_N\|_r \leq 12(2N^{-\frac{3}{4}})^{\frac{1}{2} + 1}$,

(vi) for all $n \geq 1$, $\|A_N^n e_j\|_r \leq 3\rho_N^n/(p\sqrt{N})$ and for all $n \geq 0$, $\|A_N^n 1_N\|_\infty \leq 3\rho_N^n/p$.

Proof. The proof of (i)-(vi) see [6] Lemma 35]. For the point (vii), we set for $x, y \in (0, \infty)^N$

$$d_K(x, y) = \log \left[\frac{\max_{i=1, \ldots, K} \left(\frac{x_i}{y_i}\right)}{\min_{i=1, \ldots, K} \left(\frac{x_i}{y_i}\right)}\right].$$

Clearly one has $d_K(I_K A_N^n 1_N, I_K V_N) \leq d_N(A_N^n 1_N, V_N)$. Moreover from [6] Step 3 of the proof of Lemma 35 one has $d_N(A_N^n 1_N, V_N) \leq (2N^{-3/8})(n/2)^{1+1}$. Therefore we can apply [6] Lemma 39] and we obtain that

$$\|I_K A_N^n 1_N - V_N K/\|V_N\|_r \leq 3d_K(I_K A_N^n 1_N, I_K V_N) \leq 3(2N^{-\frac{3}{4}})^{\frac{1}{2} + 1}.$$

Let us prove (viii). The case $n \in \{0, 1\}$ is straightforward. In [6] Lemma 35 step 4], we already have for all $n \geq 2$, $d_N(A_N^n e_j, V_N) \leq 4(2N^{-3/8})(n/2)^{1}$. Therefore

$$\|I_K A_N^n e_j - I_K A_N^n e_j - V_N K/\|V_N\|_r \leq 3d_K(I_K A_N^n e_j, I_K V_N) \leq 4(2N^{-\frac{3}{4}})^{\frac{1}{2}}.$$

which finishes the proof of (viii).

We now verify (ix). We write $A_N^n 1_N = \|A_N^n 1_N\|_2 (\|V_N\|_2^{-1} V_N + Z_{N,n})$, where $Z_{N,n} = \|A_N^n 1_N\|_2^{-1} A_N^n 1_N - \|V_N\|_2^{-1} V_N$. By (vii), we already have $\|Z_{N,n}\|_2 \leq 3(2N^{-3/8})(n/2)^{1+1}$. Multiplying each side by $I_K$, we obtain that $I_K A_N^n 1_N = \|A_N^n 1_N\|_2 (\|V_N\|_2^{-1} V_N + I_K Z_{N,n})$

Thus

$$\frac{\|I_K A_N^n 1_N\|_2 - \|A_N^n 1_N\|_2}{\|A_N^n 1_N\|_2} \leq \|I_K Z_{N,n}\|_2 \leq \|Z_{N,n}\|_2 \leq 3(2N^{-\frac{3}{4}})^{\frac{1}{2} + 1}.$$

So for all $n \geq 0$, we have

$$\|I_K A_N^n 1_N\|_2 \leq \left(\left(\frac{\|V_N\|_2}{\|V_N\|_2} - CN^{-\frac{3}{4}}\right)\|A_N^n 1_N\|_2, \left(\frac{\|V_N\|_2}{\|V_N\|_2} + CN^{-\frac{3}{4}}\right)\|A_N^n 1_N\|_2\right).$$

Finally, recalling (ii) and (iii), we deduce (ix).

Lemma 11.4. We have

$$\mathbb{E}\left[\|\mathcal{L}_N^K - (\bar{L}_N^K)^2\mathcal{L}_N^K\|_2^2\right] \leq \frac{C}{N}$$

where $\mathcal{L}_N := A_N^6 1_N$, $\mathcal{L}_N^K = I_K \mathcal{L}_N$ and $\mathcal{L}_N(i) = \sum_{j=1}^N A_N^6(i, j)$, $\bar{L}_N^K = \frac{1}{K} \sum_{i=1}^K L_N(i)$. 

□
Proof. We write
\[ \|L_N^K - (L_N^K)^5L_N^K\|_2 = \|I_K A_N^k 1_N - (L_N^K)^5 I_K A_N 1_N\|_2 \]
(19)
\[ \leq \sum_{k=1}^{5} \| (L_N^K)^{5-k} I_K A_N^{k+1} 1_N - (L_N^K)^{6-k} I_K A_N^k 1_N\|_2 \]
(20)
\[ \leq \sum_{k=1}^{5} \| I_K A_N^{k+1} 1_N - (L_N^K) I_K A_N^k 1_N\|_2 \]

First we study the term corresponding to \( k = 1 \). We have
\[ E[|I_K A_N^1 1_N - L_N^K I_K A_N 1_N|^2] \leq 2E[|I_K A_N L_N - L_N I_K A_N 1_N|^2 + \|L_N - L_N^K\| I_K A_N 1_N|^2] \]
By Lemma 5.10 we have \( E[|I_K A_N (L_N - L_N 1_N)|^2] \leq \frac{C}{N} \). Besides we have
\[ E\left[\| (L_N - L_N^K) I_K A_N 1_N\|^2 \right] \leq 2E\left[\| (L_N - p) I_K A_N 1_N\|^2 \right] + 2E\left[\| (p - L_N^K) I_K A_N 1_N\|^2 \right] \]
\[ \leq 2E[(L_N - p)^4]^\frac{1}{2} E\left[\| I_K A_N 1_N\|^4 \right]^\frac{1}{2} + 2E[(p - L_N^K)^4]^\frac{1}{2} E\left[\| I_K A_N 1_N\|^4 \right]^\frac{1}{2} \]
using the Cauchy-Schwarz inequality
\[ \leq C\left(\frac{1}{N^2} K + \frac{1}{NK} K \right) \leq \frac{C}{N} \]
since \( \|I_K A_N 1_N\|_2 \leq \sqrt{K} \), \( E[(L_N - p)^4] \leq \frac{C}{N} \) and \( E[(p - L_N^K)^4] \leq \frac{C}{NK} \) (\( N L_N(1, \ldots, N L_N(K) \) are i.i.d. and Binomial\( (N, p) \)). So
\[ E\left[\| I_K A_N^2 1_N - L_N^K I_K A_N 1_N\|^2 \right] \leq \frac{C}{N} \]

Next, we consider the other terms, for any \( k \geq 2 \). We have
\[ E\left[\| I_K A_N^{k+1} 1_N - (L_N^K) I_K A_N^k 1_N\|^2 \right] \leq E\left[\|I_K A_N\|^2\|A_N\|^2\| I_K A_N^{k-4} (L_N^K) A_N 1_N\|^2 \right] \]
\[ \leq \left(\frac{K}{N} \right)^2 E\left[\| A_N^2 1_N - (L_N^K) A_N 1_N\|^2 \right] \]
since \( \|I_K A_N\|_2 \leq K/N \)
\[ \leq 2\left(\frac{K}{N} \right)^2 \left\{ E\left[\| A_N^2 1_N - L_N A_N 1_N\|^2 \right] + E\left[\| L_N - L_N^K\|^2\| A_N 1_N\|^2 \right] \right\} \]
\[ \leq 2\left(\frac{K}{N} \right)^2 \left\{ E\left[\| A_N X_N\|^2 \right] \right. \]
\[ + 2E\left[\| L_N - p\|^2\| A_N 1_N\|^2 \right] + 2E\left[\| p - L_N^K\|^2\| A_N 1_N\|^2 \right] \right\} \]
\[ \leq C\left(\frac{K}{N} \right)^2 \left\{ \frac{1}{N} + \frac{1}{N^2} N + \frac{1}{NK} N \right\} \leq \frac{C}{N} \]
Recalling (19), we conclude that
\[ E\left[\| L_N^K - (L_N^K)^5 L_N^K\|^2 \right] \leq \frac{C}{N} \]
which completes the proof. □

**Lemma 11.5.** We have

\[
\mathbb{E}\left[1_{\Omega_N^{K,2}} \left| H_N^K - \left(\frac{1}{p} - 1\right) \right| \right] \leq \frac{C}{\sqrt{K}} \quad \text{where} \quad H_N^K := \frac{N}{K} \sum_{i=1}^{K} \left( \frac{L_N(i) - \bar{L}_N^K}{\bar{L}_N^K} \right)^2.
\]

**Proof.** Since \( \bar{L}_N^K \geq p/2 \) on \( \Omega_N^{K,2} \), we have

\[
\left| H_N^K - \left(\frac{1}{p} - 1\right) \right| \leq \left| \frac{N}{K} \frac{\|L_N^K - \bar{L}_N^K\|_2}{(\bar{L}_N^K)^2} - p(1 - p) \right| + p(1 - p) \frac{1}{(\bar{L}_N^K)^2} - \frac{1}{p^2}
\]

\[
\leq C \left| \frac{N}{K} \frac{\|L_N^K\|_2^2 - p(1 - p)}{\left(\bar{L}_N^K\right)^2} \right| + C|\bar{L}_N^K - p|.
\]

Using (38) and the fact that \( \mathbb{E}[(\bar{L}_N^K - p)^2] \leq \frac{C}{\sqrt{K}} \), we obtain

\[
\mathbb{E}\left[1_{\Omega_N^{K,2}} \left| H_N^K - \left(\frac{1}{p} - 1\right) \right| \right] \leq C \mathbb{E}\left[ \left| \frac{N}{K} \frac{\|X_N^K\|_2^2 - p(1 - p)}{\left(\bar{L}_N^K\right)^2} + |\bar{L}_N^K - p| \right| \right] \leq \frac{C}{\sqrt{K}}.
\]

□

**Proposition 11.6.** We set \( \bar{V}_N^K = \frac{1}{K} \sum_{i=1}^{K} V_N(i) \) and

\[
\mathcal{U}_N^{K,i} := \frac{N}{K} (\bar{V}_N^K)^2 \sum_{i=1}^{K} (V_N(i) - \bar{V}_N^K)^2 \quad \text{on} \quad \Omega_N^{K,2}.
\]

There exists \( N_0 \geq 1 \) and \( C > 0 \) (depending only on \( p \)) such that for all \( N \geq N_0 \),

\[
\frac{N}{K} \mathbb{E}\left[1_{\Omega_N^{K,2}} \left| \|V_N^K - \bar{V}_N^K 1_K\|_2^2 \right| \right] \leq C, \quad \mathbb{E}\left[1_{\Omega_N^{K,2}} \left| \mathcal{U}_N^{K,i} - \left(\frac{1}{p} - 1\right) \right| \right] \leq \frac{C}{\sqrt{K}}.
\]

**Proof.** We start from

\[
\left| \mathcal{U}_N^{K,i} - \left(\frac{1}{p} - 1\right) \right| \leq \left| \mathcal{U}_N^{K,i} - \mathcal{H}_N^K \right| + \left| \mathcal{H}_N^K - H_N^K \right| + \left| H_N^K - \left(\frac{1}{p} - 1\right) \right|
\]

where \( \mathcal{H}_N^K = \frac{N}{K} \sum_{i=1}^{K} \left( \frac{L_N(i) - \bar{L}_N^K}{\bar{L}_N^K} \right)^2 \) and \( \bar{L}_N^K = \frac{1}{K} \sum_{i=1}^{K} L_N(i) \).

Step 1: First we check that \( \mathbb{E}\left[1_{\Omega_N^{K,2}} \left| H_N^K - \mathcal{H}_N^K \right| \right] \leq C/\sqrt{K} \). We notice that

\[
H_N^K = \frac{N}{K} \| (\bar{L}_N^K)^5 L_N^K (\bar{L}_N^K)^6 1_K \|_2^2 / (\bar{L}_N^K)^{12}.
\]

Thus

\[
\left| H_N^K - \mathcal{H}_N^K \right| \leq \frac{N}{K} \left( \| (\bar{L}_N^K)^5 (L_N^K - \bar{L}_N^K 1_K) \|_2^2 \left( 1 / (\bar{L}_N^K)^{12} - 1 / (\bar{L}_N^K)^2 \right) + (1 / (\bar{L}_N^K)^2) \left( \| (\bar{L}_N^K)^5 (L_N^K - \bar{L}_N^K 1_K) \|_2^2 - \| L_N^K - \bar{L}_N^K 1_K \|_2^2 \right) \right).
\]
On the set $\Omega_N^{K^2}$, by Lemma 11.3 (iv), we have that $(\bar{L}_N^K)^6 \leq \frac{6^6}{K^6}$ and $\ell_N^K \geq \frac{6^6}{K^2} \geq \frac{6^6}{2^{192}}$, and the function $\frac{1}{x^2}$ is globally Lipschitz and bounded on the interval $\left[\frac{6^6}{2^{192}}, \infty\right)$. So

$$1_{\Omega_N^{K^2}} |H_N^K - H_N^K| \leq \frac{N}{K} \left( \frac{1}{(\bar{L}_N^K)^2} - \frac{1}{(L_N^K)^2} \right) \left( \frac{1}{(\bar{L}_N^K)^6} - \frac{1}{(L_N^K)^6} \right)$$

$$+ \left( \frac{1}{L_N^K} \right)^2 \left( \frac{1}{(\bar{L}_N^K)^6} - \frac{1}{(L_N^K)^6} \right) \left( \frac{1}{(\bar{L}_N^K)^6} - \frac{1}{(L_N^K)^6} \right)$$

$$\leq C \frac{N}{\bar{L}_N^K} \left( \frac{1}{(\bar{L}_N^K)^6} \right) \left( \frac{1}{(L_N^K)^6} \right) \left( \frac{1}{(\bar{L}_N^K)^6} - \frac{1}{(L_N^K)^6} \right)$$

$$+ \left( \frac{1}{L_N^K} \right)^2 \left( \frac{1}{(\bar{L}_N^K)^6} - \frac{1}{(L_N^K)^6} \right) \left( \frac{1}{(\bar{L}_N^K)^6} - \frac{1}{(L_N^K)^6} \right) + \frac{1}{(J_N^K)^2}.$$ 

Next, we use the inequality $|a^2 - b^2| \leq (a - b)^2 + 2|a - b|$ for $a, b \geq 0$. So

$$1_{\Omega_N^{K^2}} |H_N^K - H_N^K| \leq C \frac{N}{K} \left( \frac{1}{(\bar{L}_N^K)^6} \left( \frac{1}{(L_N^K)^6} - \frac{1}{(\bar{L}_N^K)^6} \right) \right)$$

$$+ \left( \frac{1}{L_N^K} \right)^2 \left( \frac{1}{(\bar{L}_N^K)^6} - \frac{1}{(L_N^K)^6} \right) \left( \frac{1}{(\bar{L}_N^K)^6} - \frac{1}{(L_N^K)^6} \right) + \frac{1}{(J_N^K)^2}.$$ 

where

$$J_N^K = \| \mathbf{L}_N^K - (L_N^K)^6 \|_2 = \sqrt{R} \left( \mathbf{L}_N^K - (L_N^K)^6 \right), \quad I_N^K = \| \mathbf{L}_N^K - (L_N^K)^5 (L_N^K - \bar{L}_N^K) \|_2.$$ 

Because

$$\left( (\mathbf{L}_N^K - (L_N^K)^6 \|_2 - (L_N^K)^6 (L_N^K - \bar{L}_N^K), 1_K \right) = 0.$$

it implies

$$(J_N^K)^2 + (I_N^K)^2 = \| \mathbf{L}_N^K - (L_N^K)^5 \|_2.$$ 

And by Lemma 11.4 we conclude

$$\frac{N}{K} E \left[ \left( (I_N^K)^2 + (J_N^K)^2 \right) \right] = \frac{N}{K} E \left[ \| \mathbf{L}_N^K - (L_N^K)^5 \|_2 \right] \leq \frac{C}{K}.$$ 

By (9), we conclude that $E \left[ \| \mathbf{X}_N^K \|_2 \right] \leq C$. Finally,

$$E \left[ 1_{\Omega_N^{K^2}} |H_N^K - H_N^K| \right] \leq C \frac{N}{K} E \left[ \left( \frac{1}{\sqrt{K}} J_N^K + (I_N^K)^2 + \| \mathbf{X}_N^K \|_2 \right) \right]$$

$$\leq C \frac{N}{K} E \left[ \left( \frac{1}{\sqrt{K}} J_N^K + (I_N^K)^2 + \| \mathbf{X}_N^K \|_2 \right) \right]$$

$$\leq C \frac{N}{K} E \left[ \frac{(J_N^K)^2 + (I_N^K)^2}{\| \mathbf{X}_N^K \|_2} \right] + C E \left[ \frac{(J_N^K)^2 + (I_N^K)^2}{\| \mathbf{X}_N^K \|_2} \right] \frac{1}{2} E \left[ \left( (I_N^K)^2 + (J_N^K)^2 \right) \right]$$

$$\leq \frac{C}{\sqrt{K}}.$$
Step 2: By (ii) and (iv) in Lemma [11.3] we have the following inequality under the set $\Omega_{N}^{K,2}$:
\[
\|U_{\infty}^{N,K} - H_{N,K}\| = \frac{N}{K} \left| \sum_{i=1}^{K} \left( \frac{V_{N}(i)/V_{N}^{K}}{\mathcal{L}_{N}(i)/\mathcal{L}_{N}^{K}} \right)^{2} \right| \\
\leq C \frac{N}{K} \sum_{i=1}^{K} \left| \frac{V_{N}(i)/V_{N}^{K}}{\mathcal{L}_{N}(i)/\mathcal{L}_{N}^{K}} \right|
\]
Then we use the lemma [11.3] (v): on the set $\Omega_{N}^{K,2}$ we have
\[
N \sum_{i=1}^{K} \left| \frac{V_{N}(i)/V_{N}^{K}}{\mathcal{L}_{N}(i)/\mathcal{L}_{N}^{K}} \right| = N \left\| I_{K} A_{N}^{0} 1_{N} \right\|_{1}^{-1} I_{K} A_{N}^{0} 1_{N} - \left\| V_{N}^{K} \right\|_{1}^{-1} V_{N}^{K} \right\|_{1} \\
\leq C N (N^{-\frac{1}{2}})^{3+1} \leq \frac{C}{\sqrt{N}}
\]
So we have the following inequality:
\[
\mathbb{E} \left[ I_{\Omega_{N}^{K,2}} \left| U_{\infty}^{N,K} - H_{N,K} \right| \right] \leq \frac{C}{\sqrt{N}}.
\]
Step 3: From the two previous steps and lemma 11.5 it follows that
\[
\mathbb{E} \left[ I_{\Omega_{N}^{K,2}} \left| U_{\infty}^{N,K} - \left( \frac{1}{p} - 1 \right) \right| \right] \leq \frac{C}{\sqrt{K}}.
\]
Moreover, by lemma 11.3 (ii), $V_{N}^{K}$ is bounded by 2 on the set $\Omega_{N}^{K,2}$, thus
\[
\frac{N}{K} \mathbb{E} \left[ I_{\Omega_{N}^{K,2}} \left| V_{N}^{K} - V_{N}^{K} 1_{K} \right|^{2} \right] = \mathbb{E} \left[ I_{\Omega_{N}^{K,2}} \left( V_{N}^{K} \right)^{2} \left| U_{\infty}^{N,K} \right| \right] \leq C.
\]
\]
\]
Lemma 12.1. Assume (A). For all $\eta > 0$, there exists $N_\eta \geq 1$ and $C_\eta < \infty$ such that for all $N \geq N_\eta$, $t \geq 0$, on the set $\Omega_N^{K,2}$, we have

$$\|I_t^{N,K}\|_2 \leq C_\eta t\sqrt{K} N^{-\frac{3}{2}}.$$ 

Proof. In view of (26), Lemma 11.3 (vii) yields

$$\|I_t^{N,K}\|_2 \leq \mu \sum_{n \geq 0} \left[ \int_0^t s\phi^n(t-s)ds \right] \left[ I_K A_N^n 1_N - \frac{\|I_K A_N^n 1_N\|_2}{\|V_N^K\|_2} V_N^K \right]$$

$$\leq C_\eta t\sqrt{K} \sum_{n \geq 0} \left[ \int_0^t \phi^n(t-s)ds \right] (N^{-\frac{3}{2}})^{\frac{1}{2}} \leq C_\eta t\sqrt{K} N^{-\frac{3}{2}} \sum_{n \geq 0} \Delta^n (N^{-\frac{3}{2}})^{\frac{1}{2}} \leq C_\eta t\sqrt{K} N^{-\frac{3}{2}}.$$

Lemma 12.2. Assume (A). For all $\eta > 0$, there exists $N_\eta \geq 1$ and $C_\eta < \infty$ such that for all $N \geq N_\eta$, $t \geq 0$, on the set $\Omega_N^{K,2}$, we have

$$\mathbb{E}_\theta \left[ \left\| J_t^{N,K} - J_t^{N,K} 1_K \right\|_2 \right]^{\frac{1}{2}} \leq C_\eta \sqrt{K} \frac{e^{\beta_0 t}}{N} + \frac{\|V_N^K - \bar{V}_N^K 1_K\|_2 e^{(\alpha_0 + \eta)t}}{\|V_N^K\|_2}$$

where $J_t^{N,K} = \frac{1}{K} (J_t^{N,K}, 1_K)$.

Proof. In view of (26), by Minkowski inequality we have

$$\mathbb{E}_\theta \left[ \left\| J_t^{N,K} - J_t^{N,K} 1_K \right\|_2 \right]^{\frac{1}{2}} \leq \sum_{n \geq 1} \int_0^t \phi^n(t-s) \mathbb{E}_\theta \left[ \left\| I_K A_N^n M_s^{N} - I_K A_N^n M_s^{N} 1_K \right\|_2 \right]^{\frac{1}{2}}.$$

where $I_K A_N^n M_s^{N} := \frac{1}{K} \sum_{j=1}^N \sum_{i=1}^K A_N^n(i, j)M_s^{i,N}$.

In [6] Lemma 44 (i), it is shown that $\max_{i=1, \ldots, N} \mathbb{E}_\theta [(Z_t^{i,N})^2] \leq C_\eta e^{2(\alpha_0 + \eta)t}$ on $\Omega_N^3$. Using (13), we conclude that on $\Omega_N^3$:

$$\mathbb{E}_\theta \left[ \left\| I_K A_N^n M_s^{N} - I_K A_N^n M_s^{N} 1_K \right\|_2 \right]^{\frac{1}{2}} = \sum_{j=1}^N \left( A_N^n(i, j) - \frac{1}{K} \sum_{k=1}^N A_N^n(k, j) \right)^2 \mathbb{E}_\theta [Z_t^{i,N}] \leq C_\eta e^{(\alpha_0 + \eta)t} \sum_{j=1}^N \left\| I_K A_N^n e_j - I_K A_N^n e_j 1_K \right\|_2.$$

Proof. In view of (26), Lemma 11.3 (vii) yields

$$\|I_t^{N,K}\|_2 \leq \mu \sum_{n \geq 0} \left[ \int_0^t s\phi^n(t-s)ds \right] \left[ I_K A_N^n 1_N - \frac{\|I_K A_N^n 1_N\|_2}{\|V_N^K\|_2} V_N^K \right]$$

$$\leq C_\eta t\sqrt{K} \sum_{n \geq 0} \left[ \int_0^t \phi^n(t-s)ds \right] (N^{-\frac{3}{2}})^{\frac{1}{2}} \leq C_\eta t\sqrt{K} N^{-\frac{3}{2}} \sum_{n \geq 0} \Delta^n (N^{-\frac{3}{2}})^{\frac{1}{2}} \leq C_\eta t\sqrt{K} N^{-\frac{3}{2}}.$$
Using (viii) in Lemma 11.3 and the inequality \( \|x - \bar{x}1_N\|_2 - \|y - \bar{y}1_N\|_2 \leq \|x - y\|_2 \) for all \( x, y \in \mathbb{R}^N \), we deduce that on \( \Omega_{N,K}^{K,2} \):

\[
\|I_K A_n^N e_j - \overline{I_K A_n^N} e_j 1_K\|_2 \\
\leq \|I_K A_n^N e_j - \frac{1}{\|V_N^K\|_2} I_K A_n^N e_j\|_2 V_N^K\|_2 + \frac{\|I_K A_n^N e_j\|_2}{\|V_N^K\|_2} \|V_N^K - \overline{V_N^K} 1_K\|_2 \\
= \|I_K A_n^N e_j\|_2 \left( \|I_K A_n^N e_j\|_2 - \frac{\|V_N^K - \overline{V_N^K} 1_K\|_2}{\|V_N^K\|_2} \right) \\
\leq C\|I_K A_n^N e_j\|_2 \left( N^{-\frac{3}{4}} + \frac{\|V_N^K - \overline{V_N^K} 1_K\|_2}{\|V_N^K\|_2} \right).
\]

From Lemma 11.3 (iv) it follows that on the event \( \Omega_{N,K}^{K,2} \) for all \( n \geq 2 \), \( \|I_K A_n^N e_j\|_2 \leq \frac{3\sqrt{K}}{N} \rho_N^2 \). So on the event \( \Omega_{N,K}^{K,2} \),

\[
E_{0}\|I_K J_{N,K} - j_{N,K} 1_K\|_2^2 \leq C_{\eta} \sqrt{\frac{K}{N}} \left( 2N^{-\frac{3}{4}} 1 + \frac{\|V_N^K - \overline{V_N^K} 1_K\|_2}{\|V_N^K\|_2} \right) \int_0^t \phi^{-n}(t - s) e\left( (\alpha_0 + \eta)s \right) ds.
\]

Using \( \square \) lemma 43 (iii) and (iv), we deduce that on the event \( \Omega_{N,K}^{K,2} \)

\[
E_{0}\|I_K J_{N,K} - j_{N,K} 1_K\|_2^2 \leq C_{\eta} \sqrt{\frac{K}{N}} \left( e^{2(\alpha_0 + \eta)t} + \frac{\|V_N^K - \overline{V_N^K} 1_K\|_2}{\|V_N^K\|_2} e(\alpha_0 + \eta)t \right).
\]

**Lemma 12.3.** There exists \( N_0 \geq 1 \) such that for all \( N \geq N_0 \), for all \( t \geq 0 \), on the event \( \Omega_{N,K}^{K,2} \cap \{ \bar{Z}_{t}^{N,K} \geq \frac{1}{4} \nu_{t}^{N,K} > 0 \} \), we have the following inequality:

\[
D_t^{N,K} \leq 16D_t^{N,K,1} + 128 \frac{N}{K} \|V_N^K - \overline{V_N^K} 1_K\|_2^2 D_t^{N,K,2} + \left| U_t^{N,K} - \left( \frac{1}{p} - 1 \right) \right|
\]

where

\[
(27) \quad D_t^{N,K} = \left| U_t^{N,K} - \left( \frac{1}{p} - 1 \right) \right|,
\]

\[
(28) \quad D_t^{N,K,1} = \frac{1}{(v_t^{N,K})^2} \left( \frac{N}{K} \|Z_t^{N,K} - \bar{Z}_t^{N,K} 1_K\|_2^2 - N \bar{Z}_t^{N,K} - \frac{N}{K} (v_t^{N,K})^2 \|V_N^K - \overline{V_N^K} 1_K\|_2^2 \right),
\]

\[
(29) \quad D_t^{N,K,2} = \left| \frac{Z_t^{N,K}}{v_t^{N,K}} - \bar{V}_K^N \right|.
\]

**Proof.** Recall definitions \( 49 \) and \( 24 \). On the event \( \Omega_{N,K}^{K,2} \cap \{ \bar{Z}_{t}^{N,K} \geq \frac{1}{4} \nu_{t}^{N,K} > 0 \} \), we have

\[
\|U_t^{N,K} - U_{\infty}^{N,K}\|_2 \leq \frac{1}{(Z_t^{N,K})^2} \left( \frac{N}{K} \|Z_t^{N,K} - \bar{Z}_t^{N,K} 1_K\|_2^2 - N \bar{Z}_t^{N,K} - (v_t^{N,K})^2 \frac{N}{K} \|V_N^K - \overline{V_N^K} 1_K\|_2^2 \right) + \frac{N}{K} \|V_N^K - \overline{V_N^K} 1_K\|_2^2 \left( v_t^{N,K} \right)^2 - \frac{1}{(V_N^K)^2}.
\]
Lemma 12.5. Assume (A). For all \( N \) already have \( v_t^{N,K} > 0 \), we have
\[
\left( \frac{v_t^{N,K}}{Z_t^{N,K}} \right)^2 - \frac{1}{(V_N^k)^2} \leq 128D_t^{N,K,2}.
\]
Finally on the event \( \Omega_{N,2}^{K} \cap \{ Z_t^{N,K} \geq \frac{1}{4}v_t^{N,K} > 0 \} \), we obtain
\[
D_t^{N,K} \leq \left| U_t^{N,K} - U_{\infty}^{N,K} \right| + \left| U_{\infty}^{N,K} - \left( \frac{1}{p} - 1 \right) \right|
\leq 16D_t^{N,K,1} + 128N^\frac{N}{K}||V_N^K - V_N^K 1_K||_2^2D_t^{N,K,2} + \left| U_{\infty}^{N,K} - \left( \frac{1}{p} - 1 \right) \right|.
\]
\[\square\]

Before the analysis of the term \( D_t^{N,K,2} \), we still need the following fact:

Lemma 12.4. Assume (A). For any \( \eta > 0 \), we can find \( N_\eta \geq 1, t_\eta > 0 \) and \( 0 < c_\eta < C_\eta < \infty \), such that for all \( N \geq N_\eta, t \geq t_\eta \) on the set \( \Omega_{N,2}^{K} \),
\[c_\eta e^{(\alpha_0-\eta)t} \leq v_t^{N,K} \leq C_\eta e^{(\alpha_0+\eta)}\]
where \( v_t^{N,K} \) is defined in (24).

Proof. We work on the set \( \Omega_{N,2}^{K} \). Recall Lemma 12.3 (ii) and (ix). We can conclude that \( \frac{1}{2}\sqrt{\kappa} \leq ||V_N^K||_2 \leq 2\sqrt{\kappa} \) and \( ||I_KA_N^t1_N||_2 \in [\sqrt{\kappa}\rho_N^k/8, 8\sqrt{\kappa}\rho_N^k] \). So there exists \( 0 < c < C < \infty \) such that
\[c||A_N^t1_N||_2 \leq \frac{||I_KA_N^t1_N||_2}{||V_N^K||_2} \leq C||A_N^t1_N||_2.
\]
Therefore we have \( cN_i^{N,K} \leq v_t^{N,K} \leq Cv_t^{N,K} \). Moreover, in view of [6] (i) and (ii) Lemma 43, we already have \( c_\eta e^{(\alpha_0-\eta)t} \leq v_t^{N,K} \leq C_\eta e^{(\alpha_0+\eta)} \). The proof is finished. \[\square\]

Lemma 12.5. Assume (A). For all \( \eta > 0 \), there exists \( N_\eta \geq 1, t_\eta \geq 0 \) and \( 0 < C_\eta < \infty \) such that for all \( N \geq N_\eta, t \geq t_\eta \), on the event \( \Omega_{N,2}^{K} \),
\[
(i) \quad \mathbb{E}_\theta[D_t^{N,K,2}] \leq C_\eta e^{2\eta t} \left( \frac{1}{\sqrt{\kappa}} + e^{-\alpha_0t} \right).
\]
\[
(ii) \quad P_\theta \left( \bar{Z}_t^{N,K} \leq \frac{1}{4}v_t^{N,K} \right) \leq C_\eta e^{2\eta t} \left( \frac{1}{\sqrt{\kappa}} + e^{-\alpha_0t} \right).
\]

Proof. Recalling (22) and (23), we can write
\[
D_t^{N,K,2} = \left( \frac{Z_t^{N,K}}{v_t^{N,K}} - \bar{V}_N^K \right) \leq \left( v_t^{N,K} \right)^{-1} \left( |\bar{I}_t^{N,K}| + |\bar{U}_t^{N,K}| \right).
\]
We fix \( \eta > 0 \) and work with \( N \) large enough and on \( \Omega_{N,2}^{K} \). By the lemma 12.1 we have: \( |\bar{I}_t^{N,K}| \leq \frac{1}{2\sqrt{\kappa}} ||I_t^{N,K}||_2 \leq C_\eta tN^{-\frac{1}{2}} \).
From [6] proof of Lemma 44, step 3], we have \( \mathbb{E}_\theta[(J_i^{N,K})^2] \leq C_\eta N^{-1}e^{2(\alpha_0+\eta)t} \). Thus
\[
\mathbb{E}_\theta[(J_i^{N,K})^2] \leq K^{-1} \sum_{i=1}^K \mathbb{E}_\theta[(J_i^{N,K})^2] \leq C_\eta \frac{1}{N} e^{2(\alpha_0+\eta)t}.
\]
In view of [6] Lemma 44 (i), we already have \( \max_{i=1,...,N} \mathbb{E}_\theta[(Z_t^{i,N})^2] \leq C_\eta e^{2(\alpha_0+\eta)t} \). Then by (13) we deduce that
\[
\mathbb{E}[(\bar{M}_t^{i,N,K})^2] = \frac{1}{K^2} \sum_{i=1}^K \mathbb{E}_\theta[Z_t^{i,N}] \leq C_\eta \frac{1}{K} e^{(\alpha_0+\eta)t}.
\]

Over all, we deduce that \( \mathbb{E}[[\bar{U}_t^{i,N,K}]] \leq K C_\eta e^{(\alpha_0+\eta)t} \). According to Lemma [12.4] there exists \( t_\eta \geq 0 \) such that for all \( t \geq t_\eta \), \( \bar{v}_t^{i,N,K} \geq c_\eta e^{(\alpha_0-\eta)t} \) and we finally obtain (i):
\[
\mathbb{E}_\theta[D_t^{i,N,K,2}] = \mathbb{E}_\theta\left[ (\bar{v}_t^{i,N,K})^{-1} (|\bar{I}_t^{i,N,K}| + |\bar{U}_t^{i,N,K}|) \right] \leq C_\eta e^{2\eta t} \left( \frac{1}{\sqrt{K}} + e^{-\alpha_0 t} \right).
\]

Now we prove (ii). Because of \( \bar{V}_t^K \geq \frac{1}{2} \) we have \( \{ \bar{Z}_t^{N,K} \leq \frac{1}{4} \} \subset \{ D_t^{i,N,K,2} = \frac{2}{\bar{v}_t^{i,N,K}} - \bar{V}_t^K \geq \frac{1}{4} \} \). Hence
\[
P_\theta(\bar{Z}_t^{N,K} \leq \frac{1}{4} \bar{v}_t^{i,N,K}) \leq 4\mathbb{E}_\theta[D_t^{i,N,K,2}] \leq C_\eta e^{2\eta t} \left( \frac{1}{\sqrt{K}} + e^{-\alpha_0 t} \right).
\]

**Lemma 12.6.** Assume (A). For all \( \eta > 0 \), there exists \( N_\eta \geq 1 \) and \( C_\eta < \infty \) such that for all \( N \geq N_\eta \), all \( t \geq 0 \), on \( \Omega_N^{K,2} \):

(i) \( \mathbb{E}_\theta[(M_t^{i,N,K} - \bar{M}_t^{i,N,K}1_K, V_t^K - \bar{V}_t^K 1_K)^2] \leq C_\eta \| V_t^K - \bar{V}_t^K 1_K \|^2_2 e^{(\alpha_0+\eta)t} \).

(ii) \( \mathbb{E}_\theta[|X_t^{i,N,K}|] \leq C_\eta N e^{(\alpha_0+\eta)t} \), where \( X_t^{i,N,K} := \frac{N}{K} (\| M_t^{i,N,K} - \bar{M}_t^{i,N,K}1_K \|^2_2 - K \bar{Z}_t^{i,N,K}) \).

(iii) \( \mathbb{E}_\theta[\| M_t^{i,N,K} - \bar{M}_t^{i,N,K}1_K \|^2] \leq C N e^{(\alpha_0+\eta)t} \).

**Proof.** We fix \( \eta > 0 \) and work with \( N \) large enough and on \( \Omega_N^{K,2} \). We already from [6] Lemma 44 (i) that \( \max_{i=1,...,N} \mathbb{E}_\theta[(Z_t^{i,N})^2] \leq C_\eta e^{2(\alpha_0+\eta)t} \). Thus
\[
\mathbb{E}_\theta\left[ (M_t^{i,N,K} - \bar{M}_t^{i,N,K}1_K, V_t^K - \bar{V}_t^K 1_K)^2 \right] = \sum_{i=1}^K (V_N(i) - \bar{V}_t^K)^2 \mathbb{E}_\theta[Z_t^{i,N}] \leq C_\eta \| V_t^K - \bar{V}_t^K \|^2_2 e^{(\alpha_0+\eta)t}
\]

which completes the proof of (i).

By Itô's formula, we have
\[
\| M_t^{i,N,K} \|^2 = \sum_{i=1}^K (M_t^{i,N})^2 = 2 \sum_{i=1}^K \int_0^t M_t^{i,N}_s dM_t^{i,N}_s + \sum_{i=1}^K Z_t^{i,N},
\]
hence
\[
X_t^{i,N,K} = \frac{N}{K} \left( \| M_t^{i,N,K} \|^2 - K (\bar{M}_t^{i,N,K})^2 - K \bar{Z}_t^{i,N,K} \right)
= \frac{N}{K} \left( 2 \sum_{i=1}^K \int_0^t M_t^{i,N}_s dM_t^{i,N}_s - K (\bar{M}_t^{i,N,K})^2 \right).
\]
It follows that
\[
\mathbb{E}_\theta[|X_t^{i,N,K}|] \leq \frac{N}{K} \left( 2\mathbb{E}_\theta\left[ \left| \sum_{i=1}^K \int_0^t M_t^{i,N}_s dM_t^{i,N}_s \right| \right] + \mathbb{E}_\theta[\bar{Z}_t^{i,N,K}] \right).
\]
Besides, using Cauchy-Schwartz inequality
\[
E_\theta \left[ \left( \sum_{i=1}^{K} \int_0^t M^{i,N}_s dM^{i,N}_s \right)^2 \right] = \sum_{i=1}^{K} E_\theta \left[ \int_0^t (M^{i,N}_s)^2 dZ^{i,N}_s \right] 
\leq \sum_{i=1}^{K} \left[ \sup_{[0,t]} (M^{i,N}_s)^4 \right] \frac{1}{2} E_\theta \left[ (Z^{i,N}_t)^2 \right] 
\leq C \sum_{i=1}^{K} \left[ (Z^{i,N}_t)^2 \right]
\]
since \( E_\theta [\sup_{[0,t]} (M^{i,N}_s)^4] \leq C E_\theta [(Z^{i,N}_t)^2] \) by Doob’s inequality. So
\[
E_\theta [\|X^{N,K}_t\|] \leq \frac{N}{K} \left( 2 E_\theta \left[ \left( \sum_{i=1}^{K} \int_0^t M^{i,N}_s dM^{i,N}_s \right)^2 \right] + E_\theta [\tilde{Z}^{N,K}_t] \right) \leq C_{\eta} \frac{N}{\sqrt{K}} e^{(\alpha_0 + \eta) t}
\]
This completes the proof of (ii). Finally, we have
\[
\frac{N}{K} E_\theta \left[ \|M^{N,K}_t - \tilde{M}^{N,K}_t 1_K\|^2 \right] \leq E_\theta [\|X^{N,K}_t\|] + N E_\theta [\tilde{Z}^{N,K}_t] \leq C_{\eta} N e^{(\alpha_0 + \eta) t}.
\]
This completes the proof of (iii).

Next we consider the term \( D^{N,K,1}_t \).

**Lemma 12.7.** Assume (A). For all \( \eta > 0 \), there are \( N_\eta \geq 1 \), \( t_\eta \geq 0 \) and \( C_{\eta} < \infty \) such that for all \( N \geq N_\eta \), all \( t \geq t_\eta \), we have:

\[
E[1_{\Omega_N^{K,2}} D^{N,K,1}_t] \leq C_{\eta} e^{4\int t} \left( \frac{1}{\sqrt{K}} + \left( \frac{\sqrt{N}}{\theta e^{\alpha_0}} \right)^{1/2} + \frac{N}{\sqrt{K}} e^{-\alpha_0 t} \right).
\]

**Proof.** Recalling (22) and (23), we start from \( Z^{N,K}_t = M^{N,K}_t + J^{N,K}_t + v^{N,K}_t V^{K}_N + I^{N,K}_t \). In view of (23), we have:

\[
D^{N,K,1}_t = \frac{1}{\left( v^{N,K}_t \right)^2} \left[ \frac{N}{K} \|I^{N,K}_t - \tilde{I}^{N,K}_t 1_K\|^2 + \frac{N}{K} \|M^{N,K}_t - \tilde{M}^{N,K}_t 1_K\|^2 \right] - N Z^{N,K}_t + 2 \frac{N}{K} \left( I^{N,K}_t - \tilde{I}^{N,K}_t 1_K + J^{N,K}_t - \tilde{J}^{N,K}_t 1_K, v^{N,K}_t (V^{K}_N - \tilde{V}^{K}_N) 1_K \right) + M^{N,K}_t - \tilde{M}^{N,K}_t 1_K \right) + 2 \frac{N}{K} v^{N,K}_t \left( V^{K}_N - \tilde{V}^{K}_N 1_K, M^{N,K}_t - \tilde{M}^{N,K}_t 1_K \right) \right] \leq \frac{1}{\left( v^{N,K}_t \right)^2} \left[ \frac{N}{K} \|I^{N,K}_t - \tilde{I}^{N,K}_t 1_K\|^2 + 2 \frac{N}{K} \|J^{N,K}_t - \tilde{J}^{N,K}_t 1_K\|^2 + \|X^{N,K}_t\| \right] + 2 \frac{N}{K} \left( I^{N,K}_t - \tilde{I}^{N,K}_t 1_K + J^{N,K}_t - \tilde{J}^{N,K}_t 1_K, v^{N,K}_t (V^{K}_N - \tilde{V}^{K}_N) 1_K \right) \right] + \|M^{N,K}_t - \tilde{M}^{N,K}_t 1_K\|^2 + 2 \frac{N}{K} \left( v^{N,K}_t \left( V^{K}_N - \tilde{V}^{K}_N 1_K, M^{N,K}_t - \tilde{M}^{N,K}_t 1_K \right) \right) \right].
\]

We fix \( \eta > 0 \) and work with \( N \) and \( t \) large enough and on \( \Omega^{K,2}_N \). Using Lemmas 12.1, 12.2, 12.3 together with the fact that \( c \sqrt{K} \leq \|V^{K}_N\| \leq C \sqrt{K} \) on \( \Omega^{K,2}_N \) (by Lemma 11.3 (ii)), we deduce
the following bound on the set $\Omega_{N}^{K,2}$:

$$
\mathbb{E}[D_t^{N,K,1}] \leq C_N e^{-2(\alpha_0-\eta)t} \left\{ N \frac{2}{\sqrt{K}} e^{(\alpha_0+\eta)t} + e^{2(\alpha_0+\eta)t} \frac{1}{\|V_N^K - V_N^1\|_2^2} \right\}^2
$$

By Lemma 12.8, we finally obtain:

$$
\mathbb{E}[1_{\Omega_{N}^{K,2}}] \leq C_N e^{-2(\alpha_0-\eta)t} \left\{ N \frac{2}{\sqrt{K}} e^{(\alpha_0+\eta)t} + e^{2(\alpha_0+\eta)t} \frac{1}{\|V_N^K - V_N^1\|_2^2} \right\}^2
$$

Since $\frac{N}{\sqrt{K}} e^{-\alpha_0t} + \frac{1}{\sqrt{N}} \geq e^{-\frac{\alpha_0}{2}t}$, one gets

$$
\mathbb{E}[1_{\Omega_{N}^{K,2}}] \leq C_N e^{-2(\alpha_0-\eta)t} \left( \frac{1}{\sqrt{N}} + \left( \frac{\sqrt{N}}{e^{\alpha_0t}} \right)^2 + \frac{N}{\sqrt{K}} e^{-\alpha_0t} \right).
$$

13. Proof of the main theorem in the supercritical case.

In this section we prove Theorem 2.1 and Remark 2.8.

13.1. Proof of Theorem 2.1 By Lemma 12.8 on the event $\Omega_{N}^{K,2} \cap \{ Z_t^{N,K} \geq \frac{1}{4} \gamma_t^{N,K} > 0 \}$, we already have the following inequality:

$$
D_t^{N,K} \leq 16 D_t^{N,K,1} + 128 \frac{N}{K} \|V_N^K - V_N^1\|_2^2 D_t^{N,K,2} + \|U_t^{N,K} - \left( \frac{1}{p} - 1 \right) \|
$$

Thus

$$
1_{\Omega_{N}^{K,2}} \mathbb{E}_0 \left[ 1_{\{Z_t^{N,K} \geq \gamma_t^{N,K} / 4 > 0 \}} \right] \leq 1_{\Omega_{N}^{K,2}} \left[ U_t^{N,K} - \left( \frac{1}{p} - 1 \right) \right] + C_N e^{\alpha_0t} \left( \frac{1}{\sqrt{K}} \right)^2 + \frac{N}{\sqrt{K}} e^{-\alpha_0t}.
$$

From Proposition 11.6 and Lemmas 12.8, 12.9, it follows that

$$
\mathbb{E}[1_{\Omega_{N}^{K,2}}] \leq C_N e^{2\alpha_0t} \left( \frac{1}{\sqrt{K}} \right)^2 + \frac{N}{\sqrt{K}} e^{-\alpha_0t}.
$$

Moreover, by Lemmas 11.1 and 12.3 we have:

$$
P(\Omega_{N}^{K,2}) \geq 1 - C e^{-\alpha_0t}, \quad P(\tilde{Z}_t^{N,K} \leq \frac{1}{4} \gamma_t^{N,K}) \leq C_N e^{2\alpha_0t} \left( \frac{1}{\sqrt{K}} + e^{-\alpha_0t} \right).
$$
Hence, by the Chebyshev’s inequality, we obtain:

\[ P(\|P_t^{N,K} - p\| \geq \varepsilon) \leq (C_\eta/\varepsilon)e^{4\eta t} \left( \frac{1}{\sqrt{K}} + \frac{N}{\sqrt{K}}e^{-\alpha t} \right) + Ce^{-3N^2} + C \eta e^{2\eta t} \left( \frac{1}{\sqrt{K}} + e^{-\alpha t} \right) \]

\[ \leq (C_\eta/\varepsilon)e^{4\eta t} \left( \frac{1}{\sqrt{K}} + \frac{N}{\sqrt{K}}e^{-\alpha t} \right). \]

Finally, using that \( \left( \sum_{i \in N,K} \varepsilon_i \right)^2 \leq \frac{N}{\sqrt{K}}e^{-\alpha t} \), we get:

\[ P(\|P_t^{N,K} - p\| \geq \varepsilon) \leq \frac{C_\eta e^{4\eta t}}{\varepsilon} \left( \frac{N}{\sqrt{K}}e^{-\alpha t} + \frac{1}{\sqrt{K}} \right). \]

The proof is complete.

13.2. **Proof of Remark 2.8.** By the Lemma [12.5] for \( N \geq N_\eta \), we have that:

\[ 1_{\Omega_{N,K}^2}E[\left| \tilde{Z}_t^{N,K} - \tilde{V}_t^{N,K} \right|] = 1_{\Omega_{N,K}^2}E[|D_t^{N,K,2}|] \leq C_\eta e^{2\eta t} \left( \frac{1}{\sqrt{K}} + e^{-\alpha t} \right). \]

From Lemma [11.3] (ii), we have for all \( V_N(i) \in \left[ \frac{1}{K}, 2 \right] \). So \( \tilde{V}_N^K = (1-K) \sum_{i=1}^K V_N(i) \in \left[ \frac{1}{K}, 2 \right] \) on the set \( \Omega_{N,K}^2 \). From Lemma [12.3] for \( t \geq t_\eta \) we get \( v_t^{N,K} \in [a_\eta e^{(\alpha_0 + \eta)t}, b_\eta e^{(\alpha_0 - \eta)t}] \) for some \( a_\eta < b_\eta \). So we deduce that for \( N \geq N_\eta, t \geq t_\eta \),

\[ P\left( \tilde{Z}_t^{N,K} \in [a_\eta e^{(\alpha_0 + \eta)t}, 2b_\eta e^{(\alpha_0 - \eta)t}] \right) \geq 1 - Ce^{-3N^2} - C_\eta e^{2\eta t} \left( \frac{1}{\sqrt{K}} + e^{-\alpha t} \right). \]

This implies that for any \( \eta > 0 \),

\[ \lim_{t \to \infty} \lim_{(N,K) \to (\infty,\infty)} P(\tilde{Z}_t^{N,K} \in [e^{(\alpha_0 - \eta)t}, e^{(\alpha_0 + \eta)t}]) = 1. \]

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