Marginal operators in quantum field theory with extra dimensions

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Abstract

The classification of relevant, marginal and irrelevant operators is studied in the Randall-Sundrum spacetime. We find that there exist marginal and interacting operators in the Randall-Sundrum spacetime unlike a higher-dimensional effective theory near the free-field fixed point. This gives a direction to treat quantum corrections in the field-theoretical framework with extra dimensions by constructing models out of relevant and marginal operators.
1 Introduction

Physics of extra dimensions is an interesting possibility of particle physics beyond the standard model [1]-[12]. In the standard model, the framework is based on a four-dimensional field theory which is renormalizable and well-defined including quantum effects. Once extra dimensions are introduced, coupling constants can have negative mass dimensions. This gives rise to subsequent divergence for an infinite number of counterterms. At first sight, higher-dimensional field theory seems unsuitable as a framework to include quantum effects. Without corrections treated itself, it seems mere an effective expression where all the parameters are derived from more fundamental theory.

One remarkable feature in the four-dimensional field theory is that super-renormalizable, renormalizable and non-renormalizable operators are identified as relevant, marginal and irrelevant operators. Renormalizability is not necessarily a principle to constrain the theory. The reason why only renormalizable terms are left is because non-renormalizable operators are negligible at low energy. The point to identify the low-energy behavior is to treat renormalization group flows by integrating out the shell of high-momentum degrees of freedom. Indeed, this way leads to the appearance of non-renormalizable terms even if we start with only renormalizable terms. The classification of relevant, marginal and irrelevant operators can be obtained from a distance-rescaling, whose explicit equations will be given at the beginning of Section 2. Although various operators such as higher-dimension operators inevitably arise as quantum effects in theory with extra dimensions, they may be irrelevant operators that are negligible. It is straightforward to analyze operators for uncompactified extra dimensions as a simple extension of this method. It is known that the higher-dimensional effective theory such as the higher-dimensional $\phi^4$ theory at low energy compared to the cutoff yields a free-field theory unlike the four-dimensional case with interacting operators. The idea that relevant and marginal operators only have to be in the starting action would simplify the action. However, if any higher-dimensional theory is free, it would lead to a trivial low-energy theory without other additional interactions such as brane terms.

It has been suggested that a warped spacetime tends to differ in running of couplings from the flat spacetime [13]-[17]. Because couplings in extra-dimensional theories have negative mass dimensions, higher-dimension operators necessarily occur irrespective of the geometry. It has been shown that higher-dimension operators generated by loop effects change the values of physical quantities [18]. Unless higher-dimension operators are taken into account, the difference between the flat and warped spacetimes cannot be concluded. In addition, the discussion must be generally made not only for a finite number of loop calculations. Following the idea of the four-dimensional renormalization group flow, all we have to do might be to claim that the coefficients of higher-dimension operators are negligible at low energies. Can we have a nontrivial low-energy theory with extra dimensions? It needs to be examined whether the description of extra dimensions, interactions and quantum properties can make sense without requiring an additional ultraviolet completion.

In this Letter, we study the issue of the classification of relevant, marginal and irrelevant operators in the Randall-Sundrum spacetime. It is shown that the rescaling of distances is different between the Randall-Sundrum spacetime and the flat spacetime. As a result, we find marginal operators quite similar to the four-dimensional case. These marginal operators are interaction operators. The viewpoint of the classified operators
gives a direction to select Lagrangian terms for model building with extra dimensions.

## 2 Scaling in warped space

Following Ref. [19], a convenient way to treat renormalization group flows is to rescale distances

\[ p' = p/b, \quad x' = xb, \]  

(2.1)

where \( x \) and \( p \) denote the four-dimensional coordinates and the corresponding momenta, respectively and \( b \) is a parameter specifying the scale. The cutoff \( |p| = b\Lambda \) with \( b < 1 \) is read in terms of the rescaled momenta as \( |p'| = \Lambda \). The resolution of a two-dimensional object of the size \( 5^2 \) by unit square lattice is drawn in Figure 1. The object is resolved by \( 5^2 \) squares. For the rescaling \( x' = 5x \), the object is resolved by 1 square shown in Figure 2. The rescaling \( x' = 5x \) decrease the resolution by the factor \( 1/5^2 \). Under the rescaling (2.1), the effective action in \( \phi^4 \) theory,

\[ \int d^d x \mathcal{L}_{\text{eff}} = \int d^d x \left[ \frac{1}{2} Z (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 + C (\partial_\mu \phi)^4 + D \phi^6 \right], \]  

(2.2)

becomes

\[ \int d^d x \mathcal{L}_{\text{eff}} = \int d^d x' \left[ \frac{1}{2} (\partial'_\mu \phi')^2 + \frac{1}{2} m'^2 \phi'^2 + \frac{1}{4!} \lambda' \phi'^4 + C' (\partial'_\mu \phi')^4 + D' \phi'^6 \right]. \]  

(2.3)

Here \( \phi' = [b^{2-d}Z]^{1/2} \phi \), \( m'^2 = m^2 Z^{-1} b^{-2} \), \( \lambda' = \lambda Z^{-2} b^{-d-4} \), \( C' = C Z^{-2} b^d \) and \( D' = D Z^{-3} b^{2d-6} \). Operators whose coefficients are multiplied by negative powers of \( b \) are relevant operators, while operators whose coefficients are multiplied by positive powers of
$b$ are irrelevant operators. For the coefficients of operators multiplied by $b^0$, the operators are marginal operators. The expressions above have been given for the dimension of spacetime $d$. The assumption is that the extra-dimensional coordinates obey the same rescaling as in Eq. (2.1). The action integral (2.3) shows that for $d > 4$, only the relevant operator is the mass operator. In warped space, the scaling law is different from Eq. (2.1). This leads to change of the dependence of operators on $b$.

Let us consider the observation of an object in the Randall-Sundrum spacetime whose metric is given by [10, 11]

$$ds^2 = \frac{1}{z^2} \left( \eta_{\mu\nu} dx^\mu dx^\nu - \frac{1}{k^2} dz^2 \right),$$  

where $k$ is the curvature of the five-dimensional anti-de Sitter space. Here we examine the aspect of the resolution of a two-dimensional object with respect to the extra-dimensional direction. In the Randall-Sundrum spacetime, a small $z$ corresponds to a large cutoff.

Figure 3: The resolution of an object by a unit square lattice at $z = 5$.

Figure 4: The resolution for $z' = 5z$.

The resolution of an object by 1 square at $z = 5$ is shown in Figure 3. The rescaling
We consider the effective action integral in the $\phi$ theory. The rescaling is given by

$$p' = p/b, \quad x' = bx, \quad z' = z/b.$$  \hfill (2.5)

The behavior of the rescaling (2.5) is also seen from that the metric (2.4) with $z \to z/b$. In the next section, the rescaling (2.5) is applied to the $\phi^4$ theory.

### 3 The operators in $\phi^4$ theory

We consider the effective action integral in the $\phi^4$ theory given by

$$\int d^4x dz \sqrt{\det g_{MN}} \left[ \frac{1}{2} Z(\partial_{\mu} \phi)(\partial_{\nu} \phi)g^{\mu\nu} + \frac{1}{2} Z_5(\partial_{z} \phi)^2 g^{zz} \right. $$

$$+ \left. \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 + C((\partial_{\mu} \phi)(\partial_{\nu} \phi)g^{\mu\nu})^2 + C_5((\partial_{z} \phi)^2 g^{zz})^2 + D\phi^6 \right], \hfill \text{ (3.1)}$$

where the capital letters $M, N$ are the five-dimensional indices. Substituting the metric (2.4) into the action integral yields

$$\int d^4x dz \left[ \frac{1}{2z^2} Z(\partial_{\mu} \phi)^2 - \frac{k^2}{2z^2} Z_5(\partial_{z} \phi)^2 \right. $$

$$+ \left. \frac{1}{2z^4} m^2 \phi^2 + \frac{1}{4!z^4} \lambda \phi^4 + C((\partial_{\mu} \phi)^4 + C_5 k^4(\partial_{z} \phi)^4 + \frac{1}{z^4} D\phi^6 \right]. \hfill \text{ (3.2)}$$

Here the abbreviated contraction stands for a contraction with $\eta^{\mu\nu}$ such as $(\partial_{\mu} \phi)^2 = (\partial_{\mu} \phi)(\partial_{\nu} \phi)\eta^{\mu\nu}$. A contraction with $g^{MN}$ is described in the explicit form with $g^{MN}$ as $(\partial_{\mu} \phi)(\partial_{\nu} \phi)g^{\mu\nu}$. This rule will be used throughout this Letter. For the rescaling (2.5), the action integral becomes

$$\int d^4x dz' \left[ \frac{1}{2z'^2} (\partial'_{\mu} \phi')^2 - \frac{k^2}{2z'^2} Z_5'(\partial'_{z} \phi')^2 \right. $$

$$+ \left. \frac{1}{2z'^4} m'^2 \phi'^2 + \frac{1}{4!z'^4} \lambda' \phi'^4 + C'(\partial'_{\mu} \phi')^4 + C_5' k^4(\partial'_{z} \phi')^4 + \frac{1}{z'^4} D'\phi'^6 \right]. \hfill \text{ (3.3)}$$

Here quantities with a prime are given by

$$\phi' = b^{-2} Z^{1/2} \phi, \quad Z_5' = b^{-4} Z^{-1} Z_5, \quad m'^2 = b^{-4} Z^{-1} m^2, \quad \lambda' = b^0 Z^{-2} \lambda, \quad C' = b^{8} Z^{-2} C, \quad C_5' = b^{0} Z^{-2} C_5, \quad D' = b^{4} Z^{-3} D.$$  \hfill (3.4)

Therefore for Eq. (3.3), we find the relevant operators

$$\phi'^4, \quad (\partial'_{\mu} \phi')^2, \quad (\partial'_{z} \phi')^4.$$  \hfill (3.5)

the marginal operators

$$\phi'^4, \quad (\partial'_{\mu} \phi')^2, \quad (\partial'_{z} \phi')^4.$$  \hfill (3.6)
and the irrelevant operators
\[(\partial'_\mu \phi')^4, \quad (\phi')^6.\]  

(3.7)

The equation (3.6) shows that there exist marginal and interacting operators. Therefore we can choose the starting action integral composed of Eqs. (3.5) and (3.6) without the irrelevant operators (3.7). In general, it is necessary to take into account more operators. When the number of constituent fields is large, the operators tend to be irrelevant operators. Since \[-g_{zz} \partial'^2 = k^2 z^2 \partial_z^2 = b^0 k^2 z^2 \partial_z^2,\] operators with a large number of \(\partial_z\) can be marginal operators such as the last term in Eq. (3.6). A formally favorable point is that the derivative \(\partial_z\) corresponds to picking up the mass. Treating these terms may reduce to analysis for algebraic equations. The problem of multiplicative \(\partial_z\) is found also for the pure gauge theory in the next section.

4 The operators in pure gauge theory

In this section, we examine the rescaling behavior of operators in pure gauge theory in the flat spacetime and the Randall-Sundrum spacetime.

Flat spacetime

In the flat spacetime, the action integral is given by
\[
\int d^d x \left[ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} \right]
= \int d^d x \left[ -\frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + ig (\partial_\mu A_\nu - \partial_\nu A_\mu) [A^\mu, A^\nu] + \frac{1}{2} g^2 ([A_\mu, A_\nu])^2 \right].
\]

(4.1)

where \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu].\) For the rescaling (2.4), the action integral becomes
\[
\int d^d x \left[ -\frac{1}{2} (\partial'_\mu A'_\nu - \partial'_\nu A'_\mu)^2 + ig b^{(d-4)/2} (\partial'_\mu A'_\nu - \partial'_\nu A'_\mu) [A'^\mu, A'^\nu] + \frac{1}{2} g^2 b^{d-4} ([A'_\mu, A'_\nu])^2 \right],
\]

(4.2)

where \(A'_\mu = b^{-(d-2)/2} A_\mu.\) From Eq. (4.2), it is seen that the three-point and four-point vertices are irrelevant operators for \(d > 4.\) Thus the bulk action in the flat spacetime could be taken as a free theory.

Randall-Sundrum spacetime

In the Randall-Sundrum spacetime, the most simple gauge-field action integral is given by
\[
\int d^d x dz \sqrt{\text{det} g_{MN}} \left[ -\frac{1}{2} F_{MN} F_{NQ} g^{MP} g^{NQ} \right]
= \int d^d x \left[ -\frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + ig (\partial_\mu A_\nu - \partial_\nu A_\mu) [A^\mu, A^\nu] + \frac{1}{2} g^2 ([A_\mu, A_\nu])^2 \\
+ k^2 (\partial_\mu A_z - \partial_z A_\mu)^2 - 2 ig k^2 (\partial_\mu A_z - \partial_z A_\mu) [A^\mu, A_z] - g^2 k^2 ([A_\mu, A_z])^2 \right].
\]

(4.3)
For the rescaling (2.5), the action integral (4.3) becomes

\[
\int d^4x \frac{dz}{kz} \left[ \frac{1}{2} (\partial_\mu A'_\mu - \partial_\nu A'_\nu)^2 + b^0 g (\partial_\mu A'_\nu - \partial_\nu A'_\mu) [A'^\mu, A'^\nu] \right. \\
\left. + \frac{1}{2} b^0 g^2 (|A'_\mu, A'_\nu|)^2 + k^2 (\partial_\mu A'_\nu - \frac{1}{b^2} \partial_\nu A'_\mu)^2 \\
- 2 b^0 g k^2 (\partial_\mu A'_\nu - \frac{1}{b^2} \partial_\nu A'_\mu) [A'^\mu, A'_\nu] - b^0 g^2 k^2 (|A'_\mu, A'_\nu|)^2 \right]. \tag{4.4}
\]

where \( A'_\mu = A_\mu/b \) and \( A'_\nu = A_\nu/b \). From Eq. (4.4), it is found that the operators with \( \partial_\nu A_\mu \) are relevant operators and that the other terms including the three-point and four-point vertices are marginal operators. Particularly the field strength with four-dimensional indices obeys the rescaling \( F'_{\mu\nu} = F_{\mu\nu}/b^2 \). From the analysis of relevant and marginal operators up to here, all the terms in Eq. (4.3) need to be included in the starting action integral.

To find the rescaling property of higher-dimension operators, we write down several explicit examples. The first example has the form with only four-dimensional indices as

\[
\int d^4x \frac{dz}{kz} \sqrt{\det g_{MN}} \text{tr} \left[ D_\mu F_{\nu\rho} \cdot D_\sigma F_{\tau\lambda} \cdot g^{\mu\nu} g^{\sigma\tau} g^{\rho\lambda} \right] \\
= \int d^4x \frac{dz}{kz} \sqrt{|z|^2} \left[ D_\mu F_{\nu\rho} \cdot D_\sigma F_{\tau\lambda} \cdot \eta^{\mu\nu} \eta^{\sigma\tau} \eta^{\rho\lambda} \right], \tag{4.5}
\]

where \( D_\mu F_{\nu\rho} = \partial_\mu F_{\nu\rho} - ig [A_\mu, F_{\nu\rho}] \). For the rescaling (2.5), Eq. (4.5) becomes

\[
\int d^4x \frac{dz}{kz} \sqrt{|z|^2} \left[ D'_\mu F'_{\nu\rho} \cdot D'_\sigma F'_{\tau\lambda} \cdot \eta^{\mu\nu} \eta^{\sigma\tau} \eta^{\rho\lambda} \right], \tag{4.6}
\]

where \( D'_\mu F'_{\nu\rho} = b^4 D_\mu F_{\nu\rho} \). This shows that the operator in Eq. (4.5) is an irrelevant operator. The next example has the form with the extra-dimensional components,

\[
\int d^4x \frac{dz}{kz} \sqrt{\det g_{MN}} \text{tr} \left[ D_z F_{z\mu} \cdot D_z F_{z\nu} \cdot g^{zz} g^{\mu\nu} \right] \\
= \int d^4x \frac{dz}{kz} \sqrt{|z|^2} \left[ \left( \partial_z F_{z\mu} \right)^2 - 2 b^4 g \partial_z F_{z\mu} \cdot [A_z, F_{z\mu}] - g^2 (|A_z, F_{z\mu}|)^2 \right]. \tag{4.7}
\]

Substituting the rescaling (2.5) into this equation yields

\[
\int d^4x' \frac{dz'}{kz'} \sqrt{|z'|^2} \left[ \left( \partial_z \overline{F}_{z\mu} \right)^2 - 2 b^2 g \partial_z \overline{F}_{z\mu} \cdot [A'_z, \overline{F}_{z\mu}] - g^2 (|A'_z, \overline{F}_{z\mu}|)^2 \right], \tag{4.8}
\]

where \( F_{z\mu} = b^2 \overline{F}_{z\mu} \) and \( \overline{F}_{z\mu} = b^{-2} \partial' z A'_\mu - D'_\mu A'_z \). In Eq. (4.8), it is found that terms with \( \partial'_z \) are relevant or marginal operators and that the other terms without \( \partial'_z \) are irrelevant operators. Even if gauge invariance is required, a large number of terms with \( \partial'_z \) are expected as in the \( \phi^4 \) theory in the previous section. Therefore from the viewpoint of the classification of operators, quadratic field strengths (4.3) are necessary for the starting action integral in the quantum field theory and terms with \( \partial'_z \) must be treated carefully.

## 5 The operators in fermionic theory

In this section, we examine operators for fermions in the flat spacetime and in the Randall-Sundrum spacetime.
Flat spacetime

In the flat spacetime, the fermionic action integral is written as
\[ \int d^d x \bar{\Psi} i \gamma^\mu (\partial_\mu - igA_\mu) \Psi. \] (5.1)

For the rescaling (2.1), this action becomes
\[ \int d^d x' \bar{\Psi}' i \gamma^\mu (\partial'_\mu - igb^{(d-4)/2} A'_\mu) \Psi'. \] (5.2)

Here \( \Psi' = b^{-(d-1)/2} \Psi \). For \( d > 4 \), the gauge interaction is an irrelevant operator.

Randall-Sundrum spacetime

In the Randall-Sundrum spacetime, the action integral is written as
\[ \int d^4 x dz \sqrt{\text{det} g_{MN}} \bar{\Psi} i \Gamma^A e^M_A (\partial_M + \frac{1}{8} \omega_{MBC}[\Gamma^B, \Gamma^C] - igA_M) \Psi, \] (5.3)

where \( e^M_A \) and \( \omega_{MBC} \) denote the five-dimensional vielbein and spin connection, respectively. This action integral includes the interaction terms as
\[ \int d^4 x \frac{dz}{k z^3} \left[ \bar{\Psi} i \Gamma^\mu (\partial_\mu - igA_\mu) \Psi + gk \bar{\Psi} \Gamma^5 A_z \Psi \right]. \] (5.4)

Substituting the rescaling (2.5) into this equation yields
\[ \int d^4 x' \frac{dz'}{k' z'^3} \left[ \bar{\Psi}' i \Gamma^\mu (\partial'_\mu - ib^0 gA'_\mu) \Psi' + b^0 gk \bar{\Psi}' \Gamma^5 A'_z \Psi' \right]. \] (5.5)

Here \( \Psi' = b^{-3} \Psi \). It is found that the gauge and effective Yukawa interactions of fermions are marginal operators.

6 Conclusion

We have found relevant and interacting operators in the Randall-Sundrum spacetime. In the gauge theory, quadratic terms composed of the field strength are relevant and marginal operators. In addition, it has been shown that the gauge and effective Yukawa interactions of fermions are marginal operators. This is different from the flat case which has only irrelevant operators except for the free part. Hence, interacting theory with extra dimensions can be treated in the field-theoretical context. Our analysis here is general and can be applied to various field theories. We have also found the problem including a large number of \( \partial_z \). The rescaling behavior of operators do not seem to constrain the form of terms with \( \partial_z \). It may be useful to examine the effect of multiple \( \partial_z \) in explicit models.

As a complementary aspect of the analysis given here, the effect of loop corrections of higher-dimension operators with respect to the four-derivative have been examined in the flat spacetime. From the result of a diagram calculation, the predictability is connected
to a large cutoff compared to the compactification scale [18], whereas a large cutoff could
deteriorate the validity of perturbation. This subtle situation and the free-field behavior
might be a general defect of the flat spacetime.

In the Randall-Sundrum spacetime, the four-dimensional coordinates are quite differ-
ent from the extra-dimensional coordinate. We have shown that this changes the quantum
aspect in field theory with extra dimensions. To describe particle physics, it would be im-
portant to formulate quantum field theory at the scale much lower than the Planck scale.
In defining the warped space, the largest cutoff could be the intermediate scale instead
of the Planck scale. In this way, corrections have an analogous logarithmic behavior to
the four-dimensional counterpart [20]. It needs to be examined from various viewpoints
whether a quantum field theory with extra dimensions can consistently overcome the
problems of the standard model.

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