Roundness of the ample cone and existence of double Lagrangian fibrations on hyperkähler manifolds

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Abstract. Let $M$ be a hyperkähler manifold of maximal holonomy (that is, an IHS manifold), and let $K$ be its Kähler cone, which is an open, convex subset in the space $H^{1,1}(M, \mathbb{R})$ of real $(1,1)$-forms. This space is equipped with a canonical bilinear symmetric form of signature $(1, n)$ obtained as a restriction of the Bogomolov-Beauville-Fujiki form. The set of vectors of positive square in the space of signature $(1, n)$ is a disconnected union of two convex cones. The “positive cone” is the component which contains the Kähler cone. We say that the Kähler cone is “round” if it is equal to the positive cone. The manifolds with round Kähler cones have a unique bimeromorphic model and correspond to Hausdorff points in the corresponding Teichmüller space. We prove that any maximal holonomy hyperkähler manifold with $b_2 > 5$ has a deformation with round Kähler cone and the Picard lattice of signature $(1,1)$, admitting two non-collinear integer isotropic classes. This is used to show that all known examples of hyperkähler manifolds admit a deformation with two transversal Lagrangian fibrations, and their Kobayashi metric vanishes, unless the Picard rank is maximal.

1 Introduction

This paper gives a simple solution for a construction problem of hyperkähler geometry. We construct a hyperkähler manifold with rank 2 Picard lattice of signature $(1,1)$ containing isotropic vectors and maximal possible Kähler cone. For our purposes, “a hyperkähler manifold” is a compact, holomorphic symplectic compact manifold $M$ of Kähler type, which satisfies “the maximal holonomy condition”, that is, $\pi_1(M) = 0$, $\dim H^{2,0}(M) = 1$. This condition is also known as IHS (“irreducible holomorphic symplectic”).

The shape of the Kähler cone of a hyperkähler manifold is more or less understood by now (see [AV3]). However, finding examples of manifolds with prescribed shape of their Kähler cone is a complicated task. Such constructions are either very explicit or based on convoluted arguments from number theory. The automorphism group of a hyperkähler manifold and its set of Lagrangian fibrations can be described explicitly in terms of its periods and the shape of the Kähler cone ([AV4]). Therefore, finding manifolds with prescribed Kähler cones has many practical applications.

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Recall that the second cohomology of a maximal holonomy hyperkähler manifold is equipped with a bilinear symmetric form of signature $(3, b_2 - 3)$, which is essentially of topological origin ([Bea, Bo, F]). This form, denoted by $q$ further on, is called the Bogomolov-Beauville-Fujiki form: it is positive on the Kähler cone, and has signature $(1, b_2 - 3)$ on $H^{1,1}(M, \mathbb{R})$.

This implies that the set of “positive vectors” (the vectors with positive square) in $H^{1,1}(M, \mathbb{R})$ has two connected components, both of them convex cones. However, only one of these two components may contain Kähler forms. We call this component “the positive cone” of a hyperkähler manifold.

Every hyperkähler manifold is equipped with a collection $\mathcal{S}$ of primitive integer cohomology classes in $H^2(M, \mathbb{Z})$ with negative squares, called the MBM classes ([AV1]). This set is invariant on deformations of $M$ and under the action of the monodromy group $\Gamma$ generated by the monodromy of the Gauss-Manin connection for all deformations of $M$. The group $\Gamma$ (originally defined by E. Markman, [Mar1, Mar2], who called it the monodromy group) is mapped to the orthogonal lattice $O(H^2(M, \mathbb{Z}))$ with finite kernel, and its image is a finite index sublattice in $O(H^2(M, \mathbb{Z}))$ ([V1]).

In [AV2], it was shown that the monodromy group $\Gamma$ acts on the set of MBM classes with a finite number of orbits, which were computed for some deformational classes of hyperkähler manifolds in [BM], [HT1], [HT2], [AV5].

An MBM bound for a hyperkähler manifold is the number

$$C := \max \{- q(x, x) \mid x \in \mathcal{S}\},$$

where $\mathcal{S}$ denotes the set of MBM classes. Since $\Gamma \subset O(H^2(M, \mathbb{Z}))$ acts on $\mathcal{S}$ with finitely many orbits (by [AV2]), this number is finite.

As shown in [AV3] the positive cone $\text{Pos}(M, I)$ of a hyperkähler manifold is cut into pieces by hyperplanes orthogonal to the MBM classes which lie in $H^{1,1}(M, I)$, and each of the connected components of this complement can be realized as a Kähler cone of a certain hyperkähler birational model of $(M, I)$. In other words, the Kähler cone is a connected component of the set

$$\text{Pos}^\circ(M, I) := \text{Pos}(M, I) \setminus \bigcup_{\alpha \in \mathcal{S} \cap H^{1,1}(M, I)} \alpha^\perp,$$

where $\mathcal{S}$ is the set of all MBM classes in $H^2(M, \mathbb{Z})$, and all connected components are realized as Kähler cones for birational models of $(M, I)$.

The automorphism group of a hyperkähler manifold $(M, I)$ is expressed in terms of its Kähler cone and the monodromy group as follows. Let $\Gamma_I \subset \Gamma$ be the subgroup of the monodromy group preserving the Hodge decomposition on $H^2(M)$. Then $\text{Aut}(M, I)$ is a subgroup of all elements in $\Gamma_I$ which preserve the Kähler cone (the result is essentially due to E. Markman, [Mar2]).

We say that a manifold $M$ has round Kähler cone if $\text{Kah}(M) = \text{Pos}(M)$, or, equivalently, when the set of MBM classes in $H^{1,1}(M, I)$ is empty.

Our main results are the following two theorems.
Theorem 4.1: Let $M$ be a compact maximal holonomy hyperkähler manifold with $b_2(M) \geq 4$, satisfying the SYZ conjecture. Assume that $H^2(M, \mathbb{Q})$ has non-zero isotropic vectors. Then $M$ admits a deformation with two distinct Lagrangian fibrations. If, in addition, $M$ satisfies one of the two assumptions
(a) $(H^{2,0}(M) \oplus H^{0,2}(M)) \cap H^2(M, \mathbb{Q}) = 0$,
(b) $b_2(M) \geq 6$, and $M$ has Picard lattice of non-maximal rank, then the Kobayashi pseudometric on $M$ vanishes.

Corollary 4.2: All known compact hyperkähler examples have vanishing Kobayashi pseudometric.

2 Integral quadratic lattices of rank 2

Recall that a sublattice $R \subset \mathbb{Z}^n$ is called primitive if $(R \otimes \mathbb{Z} \mathbb{Q}) \cap \mathbb{Z}^n = R$. The saturation of a sublattice $R \subset \mathbb{Z}^n$ is the sublattice $R_1 := (R \otimes \mathbb{Z} \mathbb{Q}) \cap \mathbb{Z}^n$. Clearly, $R_1 \supset R$ is a lattice of the same rank.

We say that an integral quadratic lattice $(\Lambda, q)$ represents $n \in \mathbb{Z}$ if $q(x, x) = n$ for some $x \in \Lambda$. In this paper, we prove the following technical result, used to obtain deformations of a hyperkähler manifold without MBM classes.

Proposition 2.1: Let $\Lambda$ be a quadratic integral lattice of signature $(3, k)$, $k \geq 1$, representing 0. Then for any $N > 0$ the lattice $\Lambda$ contains a primitive integral sublattice of signature $(1, 1)$ representing 0 and not representing integers in the interval $[-N, -1]$.

Proof. Step 1: To find a primitive integral sublattice of signature $(1, 1)$ not representing 0 and not representing integers in the interval $[-N, -1]$, we could apply [AV4, Theorem 3.6]. Now we need to prove the same result for lattices representing 0.

Choose an intermediate lattice $B \subset \Lambda$, not necessarily primitive, and let $B^*$ be its saturation. Let $d$ be the index of $B$ in $B^*$. For any sublattice $L \subset B$ primitive in $B$, denote by $L^*$ the saturation of $L$ in $\Lambda$. Consider the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & L & \longrightarrow & L^* & \longrightarrow & L^*/L & \longrightarrow & 0 \\
\big\downarrow \nu & & \big\downarrow \nu_s & & \big\downarrow \nu_c & \\
0 & \longrightarrow & B & \longrightarrow & B^* & \longrightarrow & B^*/B & \longrightarrow & 0
\end{array}
\]

The first two vertical arrows $\nu$ and $\nu_s$ are injective by construction. The Snake Lemma implies the long exact sequence

\[0 \longrightarrow \ker \nu_c \xrightarrow{\delta} \coker \nu \longrightarrow \coker \nu_s \longrightarrow \ldots\]
Since \( L \) is primitive in \( B \), then \( \text{coker} \, \nu_c \) is torsion-free. However, \( L^*/L \) is a torsion group by construction. Therefore, the map \( \delta \) in this exact sequence vanishes, which implies the vanishing of \( \text{ker} \, \nu_c \). We obtain that \( L^*/L \) is embedded into \( B^*/B \), and \( |L^*/L| \leq R := |B^*/B| \).

Assume that the quadratic form \( q_{\mid L} \) does not take any values on the interval \([-N, -1] \). Since all elements of \( L^* \) are proportional to elements in \( L \), with the coefficient of proportionality bounded by \( |L^*/L| \), this implies that \( q_{\mid L^*} \) does not take any of the values in \([-\frac{N}{|L^*/L|}, -1] \). Therefore, \( |L^*/L| \leq R \) implies that \( q_{\mid L^*} \) does not take values in \([-\frac{N}{R}, -1] \).

Any non-degenerate lattice over \( \mathbb{Q} \) has an orthogonal basis. If it represents 0 and its signature is \((3, k), k > 0 \), it also has a basis \((x, y, z_1, ..., z_k) \) such that \( x \) and \( y \) are isotropic, \( q(x, y) = 1 \), and \( z_i \) are pairwise orthogonal and orthogonal to \( x \) and \( y \). The corresponding Gram matrix has the form

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \alpha_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \alpha_{k+1}
\end{pmatrix}
\]

we will call a quadratic form with such a matrix hyperbolic-orthogonal.

Let \( B \subset \Lambda \) be generated by integral vectors proportional to such a basis in \( \Lambda \otimes \mathbb{Q} \). To prove Proposition 2.1, it remains to show that any hyperbolic-orthogonal integral lattice \( B \) of signature \((3, k), k > 1 \) contains a primitive signature \((1, 1) \) sublattice representing 0 and not representing any integers in the interval \([-R^2N, -1] \).

Let \( v = \sum_{i=1}^{k+1} \lambda_i z_i \) be an integral vector in \( B \); it is primitive if and only if the greatest common divisor of \( \lambda_i \) is 1. A vector \( x + t(y + v) \) is isotropic if and only if

\[
0 = 2tq(x, v + y) + t^2q(v + y, v + y) = 2t + t^2q(v, v)
\]

or, equivalently, if \( t = -\frac{2}{q(v, v)} \). This gives a isotropic integral vector \( q(v, v)x - 2(y + v) \), which is primitive whenever \( q(v, v) \) is odd. When \( q(v, v) \) is even, the vector \( \frac{q(v, v)}{2}x - v - y \) is primitive and isotropic. Denote this isotropic vector by \( x' \). Then the rank 2 lattice \( \langle y, x' \rangle \) is primitive in the lattice \( B \) defined above. Indeed, \( \alpha y + \beta x' = \alpha y + \beta (q(v, v)x - 2v) \) is primitive when \( \alpha \) and \( \beta \) are coprime, because \( x \) and \( y \) are parts of the base of \( B \).

It remains to show that \( \langle y, x' \rangle \), for an appropriate choice of \( v \), does not represent integers in the interval \([-R^2N, -1] \).

Clearly, \( q(y, x') \) is either \( q(v, v) \) or \( \frac{2q(v, v)}{\alpha \beta} \), depending on parity of \( q(v, v) \). A general vector in the integral lattice \( L := \langle y, x' \rangle \) has the form \( \beta(q(v, v)x - 2v) + \alpha y \) or \( \beta (\frac{q(v, v)}{2}x - v) + \alpha y \) depending on parity of \( q(v, v) \), where \( \alpha \) and \( \beta \) are arbitrary integers. The square of this vector is \( 4\beta^2q(v, v) + 2q(v, v)\alpha \beta \) in the first case, and \( \beta^2q(v, v)^2 + \frac{4q(v, v)}{\alpha \beta} \) in the second case. In both cases, \( L \) does not represent any integer \( t \in (-\frac{2q(v, v)}{\alpha \beta}, \frac{q(v, v)}{2}) \setminus 0 \). Choosing a vector \( v \) with sufficiently big square,
we can make sure that the bound \( \frac{g(v,v)}{2} > R^2 N \) is reached, and the lattice \( L^s \) does not represent any integer in the interval \( [-N, -1] \).

## 3 Kobayashi metric on hyperkähler manifolds

We apply the results of the current paper to the vanishing of the Kobayashi pseudometric on hyperkähler manifolds. In this section we summarize some of our results in \([KLV]\) joint with S. Lu. The aim of the current paper is to improve some of the bounds imposed on the Betti numbers, and also to show vanishing of the Kobayashi pseudometric for all of the known compact hyperkähler examples. This generalizes our result on Kobayashi non-hyperbolicity of all known examples, \([KV]\).

**Definition 3.1:** An **ergodic complex structure** is a complex structure \( I \) on \( M \) such that for any complex structure \( I' \) in the same deformation class there exists a sequence of diffeomorphisms \( \nu_i \in \text{Diff}(M) \) such that \( \lim_i \nu_i(I) = I' \), where the limit of \( \nu_i(I) \in \text{End}(TM) \) is taken with respect to the \( C^\infty \)-topology on the space of tensors. We denote the space of all integrable complex structures with this topology by \( \text{Comp} \).

**Theorem 3.2:** Any complex structure of hyperkähler type on a hyperkähler manifold with \( b_2 \geq 5 \) with \( (H^{2,0}(M) \oplus H^{0,2}(M)) \cap H^2(M, \mathbb{Q}) = 0 \) is ergodic.

**Proof:** \([V2, V2bis] \).

We will need another diffeomorphism orbit, which is smaller than the maximal one, but has many of the same properties.

**Theorem 3.3:** Let \((M, I)\) be a hyperkähler manifold such that \((H^{2,0}(M) \oplus H^{0,2}(M)) \cap H^2(M, \mathbb{Q})\) is a rank 1 space generated by a class \( \alpha \in H^2(M, \mathbb{Q}) \), and \( \text{Teich}_\alpha \) the Teichmüller space of all complex structure with \( \alpha \in (H^{2,0}(M) \oplus H^{0,2}(M)) \cap H^2(M, \mathbb{Q}) \) and deformationally equivalent to \( I \). Then \( \text{Diff}(M) \cdot I \) is dense in \( \text{Teich}_\alpha \).

**Proof:** \([V2bis, Theorem 2.5, Theorem 3.1] \).

**Theorem 3.4:** \([KLV]\) Let \((M, I)\) be a complex manifold with vanishing Kobayashi pseudometric. Then the Kobayashi pseudometric vanishes for all ergodic complex structures in the same deformation class. Moreover, for each complex structure \( I_1 \) such that the closure of \( \text{Diff}(M)I_1 \) in \( \text{Comp} \) contains \( I \), the pseudometric on \((M, I_1)\) also vanishes.

**Proof:** The proof follows easily from semicontinuity of the diameter of the Kobayashi pseudometric, considered as a function on \( \text{Comp} \) \(([KLV])\).

**Theorem 3.5:** \([KLV]\) Let \( M \) be a hyperkähler manifold admitting two Lagrangian fibrations associated with two non-proportional parabolic classes. Then
the Kobayashi pseudometric on $M$ vanishes.

To prove that a given hyperkähler manifold admits a deformation with two distinct Lagrangian fibrations, in [KLV] we used an argument based on [AV4].

**Theorem 3.6:** [KLV] Let $M$ be a maximal holonomy hyperkähler manifold with $b_2(M) > 13$. Then $M$ admits a projective deformation with Picard lattice of signature $(1,2)$, with round Kähler cone (that is, with the Kähler cone equal to the positive cone), and its automorphism group has finite index in the arithmetic group $SO(\text{Pic}(M))$ of orthogonal automorphisms of its Picard lattice.

**Proof:** From [AV4, Theorem 3.11] it follows that there exists a projective deformation with Picard rank 3, isotropic classes in $H^{1,1}(M) \cap H^2(M, \mathbb{Q})$ and without MBM classes of type $(1,1)$. From [AV4, Theorem 2.10] it follows that for such a manifold the Kähler cone is equal to the positive cone, and from [AV4, Theorem 2.6, Theorem 2.7, Corollary 2.12] it follows that its automorphism group has finite index in the arithmetic group $SO(\text{Pic}(M))$. $\blacksquare$

**Theorem 3.7:** [KLV] Let $M$ be a projective, maximal holonomy hyperkähler manifold with two non-collinear isotropic rational classes in $H^{1,1}(M)$ and with round Kähler cone. Assume that $M$ satisfies the SYZ conjecture, that is, any nef bundle on $M$ is semiample. Then $M$ admits at least two transversal holomorphic Lagrangian fibrations. In particular, the Kobayashi pseudometric on $M$ vanishes.

**Proof:** Since the Kähler cone of $M$ is round, there exist rational vectors on the boundary of the Kähler cone of $M$. These points correspond to rational points in the real quadric $\{ l \in \mathbb{P}H^2(M, \mathbb{Q}) \mid q(l, l) = 0 \}$.

Each of such points corresponds to a Lagrangian fibration, because we assume that the SYZ conjecture holds. $\blacksquare$

**Theorem 3.8:** [KLV] Let $(M, I)$ be a maximal holonomy, compact hyperkähler manifold with non-maximal Picard rank. Suppose that it has a deformation which has two transversal Lagrangian fibrations. Then the Kobayashi pseudometric on $(M, I)$ vanishes.

**Proof:** The vanishing of the Kobayashi pseudometric then immediately follows from **Theorem 3.7** and **Theorem 3.4**. Indeed, there exists a complex structure $I'$ with vanishing Kobayashi pseudometric, and a sequence of diffeomorphisms such that $\lim_i \nu_i(I) = I'$. Then the Kobayashi pseudometric of $(M, I)$ vanishes by semicontinuity properties of the diameter of the Kobayashi pseudometric. $\blacksquare$

**Theorem 3.9:** Let $M$ be a compact, maximal holonomy hyperkähler manifold with $b_2(M) \geqslant 6$. Suppose that all its deformations satisfy the SYZ conjecture. Then the Kobayashi pseudometric on $M$ vanishes.
Proof: See Consider a rank 2 primitive sublattice \( L \subset H^2(M, \mathbb{Z}) \) not representing any integers in \([-N, -1]\), for \( N \) sufficiently big. By the global Torelli theorem ([V1]), for any K3-type Hodge structure on \( H^2(M, \mathbb{R}) \) there exists a complex structure \( I \) of hyperkähler type on \( M \) with the same Hodge decomposition. Let \( I_L \) be a complex structure such that \( H^1_{I_L}(M, \mathbb{Z}) = L \) and \( \langle H_{I_L}^{2,0}(M) + H^{0,2}_{I_L}(M) \rangle \cap H^2(M, \mathbb{Z}) = 0 \). Then Theorem 3.2 implies that \( I_L \) is ergodic. Using Theorem 3.7, Theorem 3.4 and Theorem 3.5, we obtain that the Kobayashi pseudometric vanishes on \((M, I)\) for any complex structure \( I \) of hyperkähler type in the same deformation class as \( I_L \).

Remark 3.10: All known examples of hyperkähler manifolds have \( b_2(M) \geq 7 \) and satisfy the SYZ conjecture. By the results above, the Kobayashi pseudometric of all known manifolds vanishes, unless their Picard rank is maximal.

Remark 3.11: The SYZ conjecture is true for all known hyperkähler examples. Using the Fourier-Mukai transform and the deformations to moduli spaces, it was proven, in projective case, for the deformations of Hilbert schemes of points on K3 surfaces (Bayer-Macrì [BM, Theorem 1.5]; Markman [Mar3, Theorems 1.3 and 6.3]), for the deformations of the generalized Kummer varieties (Yoshioka [Y, Proposition 3.38]), for the O’Grady’s sixfolds (Mongardi-Rapagnetta [MR, Corollary 1.3 and 7.3]), and the for O’Grady’s tenfolds (Mongardi-Onorati, [MO, Theorem 2.2]). In fact, in all these references except [Y], the result is stated for all Kähler deformations. However, the passage from SYZ conjecture for all projective deformations to SYZ for all Kähler is known in general, as shown in [M, Theorem 1.2] (see also [KV, Theorem 3.4]).

4 Main results

The main result of this note is the following.

Theorem 4.1: Let \( M \) be a compact maximal holonomy hyperkähler manifold with \( b_2(M) \geq 4 \), satisfying the SYZ conjecture. Assume that \( H^2(M, \mathbb{Q}) \) has non-zero isotropic vectors. Then \( M \) admits a deformation with two distinct Lagrangian fibrations. If, in addition, \( M \) satisfies one of the two assumptions

(a) \( \langle H^{2,0}(M) \oplus H^{0,2}(M) \rangle \cap H^2(M, \mathbb{Q}) = 0 \),

(b) \( b_2(M) \geq 6 \), and \( M \) has Picard lattice of non-maximal rank,

then the Kobayashi pseudometric on \( M \) vanishes.

Proof: Consider a primitive lattice \( \Lambda \subset H^2(M, \mathbb{Z}) \) of signature \((p, q)\), \( p \leq 1, q \leq b_2 - 3 \). Using the global Torelli theorem [V1], we can find a deformation \((M, I_1)\) of \( M \) with Picard lattice \( \Lambda \).
Since the rank of the indefinite lattice $H^2(M, \mathbb{Z})$ is at least 5, by Meyer’s theorem [Me] there exists an isotropic vector $x \in H^2(M, \mathbb{Z})$. Applying Proposition 2.1 we can find a primitive integral sublattice $L^s \subset H^2(M, \mathbb{Z})$ of signature $(1,1)$ representing 0 and not representing all integers in $[-C, -1]$, where $C$ is the MBM bound.

Choose the complex structure $I$ such that $L^s$ is the Picard lattice $H^{1,1}_I(M, \mathbb{Z})$ of $(M, I)$. Then $(M, I)$ has round Kähler cone, hence both integer isotropic generators of $U_N$ are nef.

These classes correspond to Lagrangian fibrations since $(M, I)$ satisfies SYZ, hence the Kobayashi metric of $(M, I)$ vanishes. This takes care of the first statement of Theorem 4.1.

Applying Theorem 3.4, we obtain that the Kobayashi metric vanishes for all ergodic complex structures, that is, for all complex structures $I_1$ such that $(H^{2,0}(M) \oplus H^{0,2}(M)) \cap H^2(M, \mathbb{Q}) = \emptyset$. This proves the case (a) of Theorem 4.1.

It remains to prove Theorem 4.1 (b). Suppose that

$$\left( H^{2,0}(M, I) \oplus H^{0,2}(M, I) \right) \cap H^2(M, \mathbb{Q})$$

has rank one and is generated by $\alpha$. Since $\alpha^\perp$ has rank $\geq 5$, it contains isotropic vectors. Applying Proposition 2.1 again, we find a deformation $(M, I')$ of $M$ which satisfies $H^{1,1}_I(M, \mathbb{Z}) = L^s$ and $(H^{2,0}(M, I') \oplus H^{0,2}(M, I')) \cap H^2(M, \mathbb{Q}) = \langle \alpha \rangle$. For an appropriate choice of diffeomorphisms $\pi_i \in \text{Diff}(M)$, the sequence $\nu_i(I')$ converges to $I$ (by Theorem 3.3), hence the Kobayashi metric on $(M, I)$ also vanishes. $\blacksquare$

**Corollary 4.2:** All known compact hyperkähler examples have vanishing Kobayashi pseudometric.

**Proof:** See Remark 3.11 and Theorem 4.1. $\blacksquare$

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