Double shell stars as source of the Kerr metric in the CMMR approximation

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Abstract. The Cabezas, Martín, Molina and Ruiz (CMMR) method allows us to build global analytic solutions of Einstein’s equations for stationary isolated and rigidly rotating perfect fluid solutions. We start from a double approximation, postminkowskian and slow rotation, and end up getting a matched global solution from the inner and outer metrics. The metrics this way obtained have some uses. In particular, we will show the application of the scheme to the equation of state $\mu + (1 - n)p = \mu_0$ and how it can be applied to build a source with two concentric comoving shells of fluid with different $\mu_0$. We will also analyse the conditions under which this configuration can be a source of the Kerr metric.

1. Introduction. The CMMR approximation and previous applications
In Cabezas \textit{et al.}[1], some of us described a new analytic approximation method to deal with the problem of obtaining interior solutions of stationary and axisymmetric perfect fluid configurations and their matched asymptotically flat vacuum exterior. To solve the Einstein’s equations iteratively we use the postminkowskian approximation. Their general solution to each order gives rise to an infinite expansion in tensorial spherical harmonics that can be cut considering slow rotation. Using this, we have built and studied the global metrics of a compact ball of constant density, a polytropic fluid\textsuperscript{2} and also one with equation of state (EOS) $\mu + (1 - n)p = \mu_0$. This last case has also been used to establish a relation between the $r_0$ constant of Wahlquist’s metric and the physical parameters of the source, as well as to confirm that there is no asymptotically flat exterior for it\textsuperscript{3,4}. We have also faced the open problem of the existence of a suitable source for the Kerr metric. There have been studies of sources with singularities, and some clues\textsuperscript{5} from the local behaviour of the interior near the surface indicating possible perfect fluid candidates, as well some others indicating the opposite\textsuperscript{6}. Nevertheless, the common stream tends to believe that no reasonable physical interior exists. We have checked all our previously built metrics and none of them can generate a Kerr exterior.

With this idea in mind, we start building the first order solution for a rotating ball of perfect fluid with constant energy density $\mu_1$ (core) surrounded by a corotating thick shell with energy density $\mu_2$ (crust) and again perfect fluid. Then, we will match it to an asymptotically flat vacuum spacetime and compare the resulting values of the multipole moments with the Kerr ones. After that, we will check how much would the results change if we introduced a singular term in the core metric.
2. The spacetime considered

We want to find the metric for three different zones. All of them are stationary and axisymmetric. Being the material content vacuum or perfect fluid, we can adapt coordinates to the symmetries so that the metrics have Papapetrou’s structure[7].

The fluid is rigidly rotating (rotational speed as seen from the infinite \( \omega \) is constant) everywhere. The first spacetime, the core of the perfect fluid, has no convective motion so its 4-velocity is \( u^\alpha = \psi_1 (\xi^\alpha + \omega \zeta^\alpha) \), with \( \xi \) and \( \zeta \) the timelike and the closed orbits-spacelike Killings, respectively. Its EOS is \( \mu_1 + (1 - n_1) p = \epsilon_1 \) and its surface can be written as \( r_I = r_i [1 + S_i \Omega^2 P_2 (\cos \theta)] \) using the slow rotation to cut its expansion in Legendre polynomials. Here \( r_i \) is the value of \( r \) when \( \omega = 0 \) and \( S_i \) a constant to be determined. The crust of the fluid is also non-convective, its EOS is \( \mu_2 + (1 - n_2) p = \epsilon_2 \) and its surface \( r_\Sigma = r_s [1 + S_s \Omega^2 P_2 (\cos \theta)] \).

Solving Euler’s equations, the pressure and energy densities are

\[
p_i = \frac{\epsilon_i}{n_i} \left[ -1 + k_i \left( \frac{\psi_i}{\psi_{fi}} \right)^{n_i} \right] \quad i = 1, 2;
\]

and \( k_i \) constants to adjust imposing \( p_1|_{r_I} = p_2|_{r_I} \) and \( p_2|_{r_\Sigma} = 0 \).

The outer vacuum region shares the symmetries and we impose asymptotical flatness upon it.

3. Building the global solution

We use harmonic Cartesian-like coordinates \( \{x^a\} \), although some results are expressed in the orthonormal spherical associated cobasis \( \{\omega^a\} \). We choose the parameter controlling the postminkowskian approximation as \( \lambda = \frac{2}{3} \pi r_0^2 \) and the slow rotation one as \( \Omega = \lambda^{-1/2} \omega r_0 \), with \( r_0 \) an arbitrary length. We can now start to solve the field equations. Decomposing the exact metric \( g \) as \( g(\lambda, \Omega) = \eta + h(\lambda, \Omega) \), with \( \eta \) the flat metric, the system to solve is, to first order in \( \lambda \),

\[
\begin{cases}
\Delta h_{\alpha\beta} = -16\pi t_{\alpha\beta}, \\
\partial^k \left[ h_{k\alpha} - \frac{1}{2} h \eta_{\alpha\beta} \right] = 0,
\end{cases}
\]

where \( t_{\alpha\beta} \) comes from the energy-momentum tensor.

The general vacuum solution is

\[
h = h_{\text{part}} + 2 \sum_{n=0}^{\infty} \frac{M_n}{r^{n+1}} (T_n + D_n) + 2 \sum_{n=1}^{\infty} \frac{J_n}{r^{n+1}} Z_n + \sum_{n=0}^{\infty} \frac{A_n}{r^{n+3}} E_{n+2} + \frac{B_2}{r^3} F_2.
\]

Here \( M_n \) and \( J_n \) are multipole moments; \( A_n \) and \( B_n \) are pure gauge constants; \( D_0 \equiv \delta_{ij} dx^i \otimes dx^j \), \( T_2 \equiv P_2 (\cos \theta) \omega^j \otimes \omega^j \), \( D_2 \equiv P_2 (\cos \theta) \delta_{ij} dx^i \otimes dx^j \), \( T_0 \equiv \omega^j \otimes \omega^j \) and \( Z_1 \equiv P_1 (\cos \theta) (\omega^i \otimes \omega^i + \omega^i \otimes \omega^i) \) are tensorial spherical harmonics, as well as \( E_n, F_n \) through the last ones have a more cumbersome definition that can be essentially found in [1].

The crust solution is

\[
h = h_{\text{part}} + \sum_{n=0}^{\infty} \tilde{m}_n r^n (T_n + D_n) + \sum_{n=1}^{\infty} \tilde{j}_n r^n Z_n + \sum_{n=0}^{\infty} \tilde{a}_n r^n E^* + \sum_{n=0}^{\infty} \tilde{b}_n r^n F^*
\]

\[
+ 2 \sum_{n=0}^{\infty} \frac{M_n}{r^{n+1}} (T_n + D_n) + 2 \sum_{n=1}^{\infty} \frac{\tilde{J}_n}{r^{n+1}} Z_n + \sum_{n=0}^{\infty} \frac{\tilde{A}_n}{r^{n+3}} E_{n+2} + \frac{\tilde{B}_2}{r^3} F_2.
\]
and the one in the core

\[ h = h_{\text{part}} + \sum_{n=0}^{\infty} m_n r^n (T_n + D_n) + \sum_{n=1}^{\infty} j_n r^n Z_n + \sum_{n=0}^{\infty} a_n r^n E^*_n + \sum_{n=0}^{\infty} b_n r^n F^*_n . \]  

(5)

We need to parametrize the multipole moments and gauge constants. As explained in [1], based on symmetry and newtonian results arguments, their assumed expressions are

\[
\begin{align*}
M_n &\to \lambda^2 \Omega^n \frac{\bar{M}_n}{r^2_0}, \\
J_n &\to \lambda^2 \Omega^n \frac{\bar{J}_n}{r^2_0}, \\
A_n &\to \lambda^2 \Omega^n \frac{\bar{A}_n}{r^2_0+3}, \\
B_2 &\to \lambda \Omega^2 \frac{\bar{B}_2}{r^2_0}.
\end{align*}
\]

(6)

where \( \bar{X} = X^{(1)} + \lambda X^{(2)} + \ldots \)

and \( X^{(n)} \) can still contain \( \Omega \) terms. The infinite series can now be cut using the slow rotation approximation. We will retain only terms up to \( O(\Omega^3) \). The particular solution \( h_{\text{part}} \) in each case is easy to find, and the last step towards a global metric is the matching of spacetimes.

Considering that we are interested in the multipole moments and they are coordinate independent, imposing Lichnerowicz matching conditions \( [g_{\alpha \beta}]^\Sigma = [\partial_\alpha g_{\beta \gamma}]^\Sigma = 0 \) is not restrictive. We get

\[
M_0^{(1)} = \frac{r_\ell^3 (\epsilon_1 - \epsilon_2) + r_s^3 \epsilon_2}{r_0^3},
\]

(7)

\[
J_1^{(1)} = \frac{2 (r_\ell^5 (\epsilon_1 - \epsilon_2) + r_s^5 \epsilon_2)}{5r_0^5} + \Omega^2 \times
\]

\[ \times \quad 3r_0^5 (9r_\ell^8 (\epsilon_1 - \epsilon_2) \epsilon_2 - 2r_s^8 \epsilon_2 (2\epsilon_1 + 3\epsilon_2) - 5r_\ell^4 r_s^2 (2\epsilon_1^2 + \epsilon_1 \epsilon_2 - 3\epsilon_2^2)) \]

(8)

\[
\bar{M}_2^{(1)} = \frac{r_\ell^5 (\epsilon_1 - \epsilon_2) + r_s^5 \epsilon_2}{r_0^5} - 2r_\ell^5 \epsilon_2 (2\epsilon_1 + 3\epsilon_2) - 5r_\ell^3 r_s^2 (2\epsilon_1^2 + \epsilon_1 \epsilon_2 - 3\epsilon_2^2)) \times
\]

\[ \times \quad 12r_\ell^5 (9r_\ell^8 (\epsilon_1 - \epsilon_2) \epsilon_2 - 2r_s^8 \epsilon_2 (2\epsilon_1 + 3\epsilon_2) - 5r_\ell^4 r_s^2 (2\epsilon_1^2 + \epsilon_1 \epsilon_2 - 3\epsilon_2^2)) \]

\[ \times \quad 2r_s^2 (5r_\ell^0 (\epsilon_1 - \epsilon_2) + r_s^0 (\epsilon_1 - \epsilon_2) \epsilon_2 + 2r_s^0 \epsilon_2 (2\epsilon_1 + 3\epsilon_2)) \]

(9)

\[
J_3^{(1)} = \frac{5r_\ell^6 (\epsilon_1 - \epsilon_2) \epsilon_2 - 2r_s^6 \epsilon_2 (2\epsilon_1 + 3\epsilon_2) - 5r_\ell^4 r_s^2 (2\epsilon_1^2 + \epsilon_1 \epsilon_2 - 3\epsilon_2^2)) \times
\]

\[ \times \quad 3r_\ell^5 r_s^5 (\epsilon_1 - \epsilon_2) \epsilon_2 + r_s^0 \epsilon_2 (2\epsilon_1 + 3\epsilon_2)) \quad .
\]

(10)

4. Matching, Kerr and singular solutions

Our vacuum metric can be used to describe Kerr metric near infinity choosing suitable values of the multipole moments[8], but we have to check if it is still possible with the values imposed by the matching. Kerr multipole moments are given by \( M_n^{\text{Kerr}} = m(na)^n \) where \( m \) and \( \alpha \) are the standard parameters of Kerr in Boyer-Lindquist coordinates. If we relate them with ours we have

\[
M_0^{\text{Kerr}} = m 
\]

(11)

\[
M_1^{\text{Kerr}} \equiv J_1^{\text{Kerr}} = ma
\]

(12)

\[
M_2^{\text{Kerr}} = -ma^2 
\]

(13)

\[
M_3^{\text{Kerr}} \equiv J_3^{\text{Kerr}} = ma^3
\]

(14)

Using the relations (11) and (12), we can build \(-ma^2 = -\lambda^2 \Omega^2 r_0^3 J_2^1 / M_0\), a quantity of order \( O(\lambda^2) \). Also, relation (13) gives \( M_2^{\text{Kerr}} = -ma^2 = \lambda^2 \Omega^2 r_0^3 (M_2^{(1)} + \lambda \bar{M}_2^{(2)} + \ldots) \). Comparing this

\[
\frac{r_0^3 J_3^{(1)}}{r_0^3 J_3^{(2)}} = \lambda^3
\]

(15)

\[
\frac{\lambda^2 J_1^{(1)}}{r_0^3 J_3^{(1)}} = \lambda^4
\]

(16)

\[
\frac{\lambda^5 J_4^{(1)}}{r_0^3 J_3^{(1)}} = \lambda^5
\]

(17)
two results we can see that \( \tilde{M}_2^{(1)} \) must be zero. A similar analysis shows that the same holds for \( \tilde{J}_3^{(1)} \). This two conditions are only compatible with \( \epsilon_1 = \epsilon_2 = 0 \), no fluid at all.

We wonder now what would happen if we manually introduced a singular term \( A r \) in the core solution. If we do it in the way

\[
h = h_{\text{part}} + \sum_{n=0}^{\infty} \left( m_n r^n + \frac{A}{r} \right) (T_n + D_n) + \sum_{n=1}^{\infty} j_n r^n Z_n + \sum_{n=0}^{\infty} a_n r^n E^*_n + \sum_{n=0}^{\infty} b_n r^n F^*_n ,
\]

(15)

then, repeating all the process, the multipole moments after the matching are

\[
\tilde{M}_0^{(1)} = \frac{r_s 5 \epsilon_2}{r_0 6} , \quad \tilde{M}_2^{(1)} = \frac{r_s 5 \left( r_0 5 + r_1 5 (\epsilon_1 - \epsilon_2) \right)}{3r_0 10} , \quad \tilde{M}_2^{(1)} = \frac{r_s 5 \left( r_0 5 + r_1 5 (\epsilon_1 - \epsilon_2) \right)}{3r_0 10} ,
\]

\[
\tilde{J}_1^{(1)} = \frac{2r_s 5 \epsilon_2}{5r_0 6} + \Omega^2 - 20r_0 8 r_s 2 \epsilon_2 + 2r_1 5 r_s 5 (- \epsilon_1 + \epsilon_2) , \quad \tilde{J}_3^{(1)} = \frac{2r_s 7 (\epsilon_1 - \epsilon_2) + 20r_0 8 r_s 2 \epsilon_2}{35r_0 12} .
\]

(16)

Once again, \( \tilde{M}_2^{(1)} \) and \( \tilde{J}_3^{(1)} \) must be zero, but in this case this leads us to

\[
\epsilon_1 = - \frac{10r_0 8 - r_1 3 r_s 5}{10r_0 4 r_1 5} , \quad \epsilon_2 = \frac{-r_s 5}{10r_0 4 r_1 5} ,
\]

(17)

thus, the double shell + singularity configuration is compatible with Kerr to this order.

5. Remarks

We have built within the CMMR approximation the global solution for an asymptotically flat vacuum containing a source composed of two thick shells of perfect fluid up to order \( O(\lambda, \Omega^3) \). We have analysed its multipole moments discarding it as a possible source of the Kerr metric and compared it with the case in which a singular term is allowed in the core zone. This last case has been shown compatible with Kerr up to this order.

6. Acknowledgments

This work was supported by projects FIS2009-07238 and FIS2007-63034 from the Spanish Ministerio de Educació y Ciencia. JEC thanks Junta de Castilla y León for predoctoral fellowship EDU/1165/2007.

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