UNIFORMLY DISTRIBUTED EIGENFUNCTIONS
ON FLAT TORI WITH RANDOM IMPURITIES

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ABSTRACT. We study a random Schrödinger operator, the Laplacian with \( N \) independently uniformly distributed random delta potentials on flat tori \( T^d_L = \mathbb{R}^d/L\mathbb{Z}^d \), \( d = 2, 3 \), where \( L > 0 \) is large. We determine a condition in terms of the size of the torus \( L \), the density of the potentials \( \rho = NL^{-d} \) and the energy of the eigenfunction \( E \) such that with positive probability any such eigenfunctions will be equidistributed on the entire torus. We remark that the equidistribution we prove here is still consistent with a localized regime, where the localization length is much larger than the size of the torus. In fact our result implies a certain polynomial lower bound on the localization length, so the localization length becomes infinitely large as \( E \to \infty \).

1. INTRODUCTION

As was first observed by Anderson [2] the long-term dynamics of a wave packet in a random lattice of impurities can be spatially confined in the presence of sufficiently strong disorder. This phenomenon, known as “Anderson localization”, is generally expected to occur when the wavelength is of size comparable to the elastic mean free path length.

The physical interpretation is that in this localized regime the quantum particle “feels” the effect of scattering from the impurities which leads to an exponential decay in the low energy eigenfunctions of the system at distances significantly larger than the mean free path length. On the other hand if the wavelength is much smaller than the mean free path length (e. g. consider high energy eigenfunctions, or a low density of impurities) then the question is whether there exists a “delocalized regime”.

The scaling theory of Abrahams, Anderson, Licciardello and Ramakrishnan [1] predicts that the localization properties of the eigenfunctions of a disordered quantum system, as described above, ought to depend on the dimension of the system.

Whereas in dimension \( d = 1 \) exponential localization is always expected, independently of the strength of disorder, one expects a phase transition from localization at strong disorder / low energy to delocalization at weak disorder / high energy in dimension \( d = 3 \). The 2-dimensional case is critical, although, generally, exponential localization is always expected to occur as in dimension \( d = 1 \).

The present paper studies flat tori \( T^d \), \( d = 2, 3 \), with independently uniformly distributed random impurities, modeled by Dirac delta potentials. This means given a torus \( T^d_L = \mathbb{R}^d/L\mathbb{Z}^d \), where \( L > 0 \) is a large parameter, we sample \( N \) points \( x_1, \ldots, x_N \), where \( N \) is large, independently from a uniform distribution on \( T^d_L \), i.e. the \( x_j \) are i.i.d. uniform random variables on \( T^d_L \).
Let us consider the formal random Schrödinger operator

\begin{equation}
H_{x_L} = -\Delta + \sum_{j=1}^{N} \alpha_j \delta(x - x_j), \quad x_L = (x_1, \cdots, x_N), \quad \forall j : \alpha_j \in \mathbb{R}
\end{equation}

which models a disordered quantum system (say an electron in a box with \(N\) randomly distributed nuclei). The disorder parameter is given by the density of the impurities \(\rho_L = N/L^d\), where the number of impurities may depend on the size of the torus, \(N = N(L)\).

For sufficiently large disorder \(\rho_L > \rho_0\) the eigenfunctions of the random operator \(H_{x_L}\) are expected to be exponentially localized in configuration space at the bottom of the spectrum (for localization results regarding delta potentials cf. for instance [3] and [5] or smooth Poisson potentials [4]).

However, if we reduce the disorder in the system and at the same time hold the energy fixed, or, alternatively, hold the disorder fixed and increase the energy, a question of great interest is whether we see a transition from localization to delocalization in the spatial geometry of the eigenfunctions. This means there should be a critical value \(\rho_c\) (or in the case of fixed disorder a critical value for the energy \(E_c\)) such that for \(\rho_L < \rho_c\) (\(E > E_c\)) there exist eigenfunctions which are extended across the entire torus. If we are in the localized regime, we should be able to observe the localization on a large torus \(T^d_L\) if we fix the energy \(E\) and make \(L\) sufficiently large (the localization length may be large and depend on \(E\)).

In the delocalized regime however (i.e. \(E > E_c\)), no exponential localization will be observed on any torus \(T^d_L\) for fixed \(E\), no matter how large \(L\) is.

By a scaling argument, this problem can easily be seen to be equivalent (see subsection 2.2) to the delocalization at high energy of the eigenfunctions on a fixed size torus \(T^d = \mathbb{R}^d/\mathbb{Z}^d\) with random impurities. The disorder parameter is given by the density of impurities, \(\rho = N\), and the localization properties of the eigenfunctions depend on the density \(\rho\) and the eigenvalue \(\lambda\).

Let \(x = (x_1, \cdots, x_N)\) be i.i.d. uniform random variables on \(\mathbb{T}^d\). We consider the formal random Schrödinger operator

\begin{equation}
H_{x_L} = -\Delta + \sum_{j=1}^{N} \alpha_j \delta(x - x_j)
\end{equation}

which may be formulated rigorously via applying the theory of self-adjoint extensions (see subsection 2.1) to the restricted Laplacian \(-\Delta|_{C^\infty(\mathbb{T}^d-x)}\).

We denote the family of self-adjoint extensions associated with the formal operator \(H_{x_L}\) by \(\{-\Delta_{x_L} U\}_{U \in U(N)}\). The number of self-adjoint extensions exceeds the number of physical coupling constants. We remark that in particular the subgroup of diagonal unitary matrices \(D(N) \subset U(N)\) corresponds to the case where a non-local interaction between the individual impurities is forbidden.

For given \(U \in U(N)\) the operator \(-\Delta_{x_L} U\) has three types of eigenfunctions:

1. “Old eigenfunctions” of the Laplacian which vanish at all the points \(x_j\), \(j = 1, \cdots, N\), and therefore do not “feel” the effect of any of the impurities.
2a. “Non-generic new eigenfunctions” which vanish at some, but not all, of the points $x_j$. They arise in subspaces of eigenspaces of lower rank perturbations of the Laplacian. There occurrence constitutes a probability 0 event.

2b. “Generic new eigenfunctions” which do not vanish at any of the points $x_j$, rather diverge logarithmically near each of the locations of the impurities. These eigenfunctions feel the effect of all impurities and are the objects of study in this paper.

Since the operator $H_x$ is a rank $N$ perturbation of the Laplacian, it has at most $N$ new eigenfunctions corresponding to new eigenvalues which are “torn off” each old eigenspace of the Laplacian, provided the dimension of the eigenspace is large enough.

The eigenvalues of the Laplacian on the torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$ are given by the set $S = \{ n \mid n = 4\pi^2(x_1^2 + \cdots + x_d^2), \quad x_1, \cdots, x_d \in \mathbb{Z} \} = \{ 0 = n_0 < n_1 < n_2 < \cdots \}$. The multiplicity of a Laplacian eigenvalue $n$ is given by the number of ways the integer $n/4\pi^2$ can be written as a sum of $d$ squares.

If $d = 2$ the multiplicity of $n$ grows on average like $\sqrt{\log n}$, which is a consequence of Landau’s Theorem \cite{83}:

$$\# \{ n \in S \mid n \leq x \} \sim \frac{Bx}{\sqrt{\log x}}.$$  

If $d = 3$ the multiplicity grows on average like $n^{1/2}$.

We have the following theorem which proves that with positive probability there exist uniformly distributed eigenfunctions of the random operator $-\Delta_{\mathbb{Z},U}$ for sufficiently high energy.

**Theorem 1.1.** Fix $U \in U(N)$. Let $a \in C^\infty(T^d)$. Denote by $\{ g_N^{\lambda, x} \}$ the $L^2$-normalized generic new eigenfunctions of the random operator $-\Delta_{\mathbb{Z},U}$. There exists a subsequence $S' \subset S$ of density 1 and a constant $\gamma_d > 0$ such that the following holds: If the points $x_j$, $j = 1, \cdots, N$ are i.i.d. uniform random variables on $T^d$, we have with probability $\geq \frac{1}{12N}$ that for each $n_k \in S'$ and for each $\lambda \in (n_k, n_{k+1})$

$$\int_{T^d} a(y) | g_N^{\lambda, x}(y) |^2 \, dy = \int_{T^d} a(y) \, dy + O(\|a\|_1; N^{1/2} \lambda^{-\gamma_d}).$$

**Remark.** To be precise, $\gamma_2 = \frac{17}{512} - \epsilon$, however we have made no effort to be optimal here. This could probably be improved substantially. Furthermore, $\gamma_3 = \frac{1}{12} - \epsilon$ for the cubic lattice $\mathbb{Z}^3$ (in fact this exponent equals the optimal exponent in section 3 of the paper \cite{11}, where one has to choose $\delta = \frac{1}{6}$ to get this optimal exponent).

By a straightforward scaling argument (cf. section 2.2) we obtain the main result of this paper.

**Theorem 1.2.** Let $T^d_L = \mathbb{R}^d/L\mathbb{Z}^d$. Let $\mathbb{Z}_L$ be an $N$-point uniform random process on $T^d_L$. Fix $U \in U(N)$ and denote by $-\Delta_{\mathbb{Z}_L,U}$ the corresponding self-adjoint extension of $-\Delta|_{C^\infty(\mathbb{Z}_L^d)}$. Denote an $L^2$-normalized generic new eigenfunction of the random operator $-\Delta_{\mathbb{Z}_L,U}$ with eigenvalue $E$ by $\psi_E$.

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\footnote{I.e. the $x_j$, $j = 1, \cdots, N$, are i.i.d. uniform random variables on $T^d_L$.}
Let \( a \in C^\infty(T_L^d) \). If we sample the points \( x_1, \ldots, x_N \in \mathbb{L} \), we have with probability \( \geq \frac{1}{12N} \) for each \( E \) such that \( EL^2 \in (n_k, n_{k+1}), n_k \in S' \),

\[
\int_{T_L^d} a(y)|\psi_E(y)|^2 dy = \frac{1}{L^d} \int_{T_L^d} a(y)dy + O(||a||_1N^{1/2}E^{-\gamma_d}L^{-2\gamma_d})
\]

and if we introduce the density of the impurities \( \rho = NL^{-d} \) we obtain as a condition for the existence of uniformly distributed eigenfunctions in terms of the three parameters the condition

\[
\rho \ll E^{2\gamma_d}L^{-3d+4\gamma_d}, \quad \gamma_2 = \frac{17}{832} - \epsilon,
\]

\[
(1.5)
\]

Remark. A shortcoming of our result is the bad dependence on \( N \) in the error term of \((1.4)\), which prevents us from studying the limit \( L \to \infty \) for a positive density of potentials in order to study the important problem of delocalization for random Schrödinger operators. However, for a different stochastic process, so-called “random displacement models”, we are able to overcome this obstacle, which is the subject of the forthcoming paper [10].

Remark. Strong coupling renormalization.
We remark here that the theorem above in fact holds for a general superposition of Green’s functions, where the spectral parameter \( \lambda \) lies in the interval \((n_k, n_{k+1})\) for \( n_k \in S' \). The exact position of \( \lambda \) inside the interval is not important. In particular our results apply to the strong coupling regime, sometimes studied in the physics literature, which requires a renormalization of the parameters of the self-adjoint extension (cf. for instance [8] and [9], section 3, p. 5).

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2. Background

2.1. Self-adjoint extension theory. Let \( x_1, \ldots, x_N \) be distinct points on \( T^d = \mathbb{R}^d/\mathbb{Z}^d, d = 2, 3 \). Denote \( \mathcal{L} = \{x_1, \ldots, x_N\} \). This section will be concerned with the rigorous mathematical realization of the formal operator

\[
(2.1) \quad -\Delta + \sum_{j=1}^N \alpha_j \delta(x-x_j), \quad \alpha_1, \ldots, \alpha_N \in \mathbb{R}.
\]

Define \( D_{\mathcal{L}} := C^\infty(T^d - \mathcal{L}) \) and consider the restricted Laplacian \( H = -\Delta|_{D_{\mathcal{L}}} \).

Denote the Green’s function of the Laplacian on \( T^d \) by

\[
G_{\lambda}(x,y) = (\Delta + \lambda)^{-1}\delta(x-y).
\]

The operator \( H \) has deficiency indices \((N,N)\) and the deficiency spaces are spanned by the bases of deficiency elements \( \{G_{\pm i}(x,x_1), \ldots, G_{\pm i}(x,x_N)\} \) respectively. There exists a family of self-adjoint extensions of \( H \) which is parameterized
by the group $U(N)$. We denote the self-adjoint extension of $H$ associated with a
matrix $U \in U(N)$ by $-\Delta_{\mathcal{L}} U$.

2.1.1. Spectrum and eigenfunctions. As explained above there are three types of
eigenfunctions of the operator $-\Delta_{\mathcal{L}} U$. Old eigenfunctions, as well as generic (type
2b) and non-generic (type 2a) eigenfunctions.

Our results hold for both types of new eigenfunctions. Since type 2a eigenfunc-
tions only occur with probability 0 and do not feel the presence of all impurities,
we will ignore them for the rest of the paper, and focus on the generic new eigen-
functions of type 2b.

To find the new eigenfunctions of the operator $-\Delta_{\mathcal{L}} U$ we want to solve
\begin{equation}
(\Delta_{\mathcal{L}} U + \lambda)g_\lambda = 0.\tag{2.2}
\end{equation}

We may write $g_\lambda$ in the decomposition
\begin{equation}
g_\lambda = f_\lambda + \langle v, G_i \rangle + \langle Uv, G_{-i} \rangle\tag{2.3}
\end{equation}
where $G_\lambda(x) = (G_\lambda(x, x_1), \cdots, G_\lambda(x, x_N)), v \in \mathbb{C}^N$ and $g_\lambda \in C_c^\infty(\mathbb{T}^d - \mathcal{L})$.

So we have
\begin{equation}
(\Delta + \lambda)f_\lambda + (-i + \lambda )\langle v, G_i \rangle + (i + \lambda )\langle Uv, G_{-i} \rangle = 0.\tag{2.4}
\end{equation}
We apply the resolvent $(\Delta + \lambda)^{-1}$, for $\lambda \notin \sigma(-\Delta)$, and obtain
\begin{equation}
f_\lambda + \frac{-i + \lambda}{\Delta + \lambda} \langle v, G_i \rangle + \frac{i + \lambda}{\Delta + \lambda} \langle Uv, G_{-i} \rangle = 0\tag{2.5}
\end{equation}
By the repeated resolvent identity
\[
\frac{-i + \lambda}{(\Delta + \lambda)(\Delta \pm i)} = -\frac{1}{\Delta + \lambda} + \frac{1}{\Delta \pm i}
\]
we can rewrite this equation as
\begin{equation}
f_\lambda - \langle v, G_\lambda - G_i \rangle - \langle Uv, G_{-i} \rangle = 0\tag{2.6}
\end{equation}
Furthermore, note that we can write more compactly
\[
\langle v, G_\lambda - G_i \rangle + \langle v, U^{-1}(G_\lambda - G_{-i}) \rangle = \langle v, A_\lambda \rangle
\]
where $A_\lambda(x) = (G_\lambda - G_i)(x) + U^{-1}(G_\lambda - G_{-i})(x)$.

Now, since $f_\lambda = \langle v, A_\lambda \rangle \in C_c^\infty(\mathbb{T}^d - \mathcal{L})$, we obtain the equations (set $x = x_k$ for
$k = 1, \cdots, N$)
\begin{equation}
\langle v, A_\lambda(x_k) \rangle = 0, \quad k = 1, \cdots, N,
\end{equation}
which we can rewrite as the matrix equation
\begin{equation}
M_\lambda v = 0\tag{2.8}
\end{equation}
where $F_\mathcal{L}(\lambda) = M_\lambda = (A_\lambda(x_1), \cdots, A_\lambda(x_N))$.

So in order to find nontrivial solutions we need to solve the spectral equation
\begin{equation}
\det M_\lambda = 0.\tag{2.9}
\end{equation}
We note that the determinant $F_\mathcal{L}(\lambda)$ is a meromorphic function of $\lambda$ with poles
at the Laplacian eigenvalues, which we recall are given by the set $S = \{ n \mid n = 4\pi^2(x^2 + y^2), \quad x, y \in \mathbb{Z} \} = \{ 0 = n_0 < n_1 < n_2 < \cdots \}$.

Consider an interval $(n_k, n_{k+1})$. For a generic choice of points $x_1, \cdots, x_N$ (prob-
ability 1 event), the spectral function $F_\mathcal{L}(\lambda)$ will have poles at $n_k$ and $n_{k+1}$ (for
certain non-generic configurations of \(x_1, \ldots, x_N\) the coefficient in front of the poles may vanish).

**Notation 2.1.** Given an eigenvalue \(\lambda\), which is a solution to the spectral equation \(F_\lambda(\lambda) = 0\), we denote the nearest neighboring Laplacian eigenvalues by \(n_+(\lambda)\) and \(n_-(\lambda)\). Where \(n_-(\lambda)\) is the largest Laplacian eigenvalue smaller than \(\lambda\) and \(n_+(\lambda)\) is the smallest Laplacian eigenvalue larger than \(\lambda\). Furthermore, we denote by \(n_-(\lambda)\) the second largest Laplacian eigenvalue smaller than \(\lambda\).

**Remark.** We point out, that almost surely the \(n_\pm(\lambda)\) do not depend on \(x\). This is, because, as we emphasized above, almost surely for \(x \in \mathbb{T}^{2N}\), the function \(F_\lambda(\lambda)\) has poles at the Laplacian eigenvalues \(\{n_k\}_{k=0}^\infty\). More precisely, for given \(j \in \mathbb{N}\) note that as \(\lambda \to n_j\) we have \(F_\lambda(\lambda) \sim (n_j - \lambda)^{-N} \det(\psi_j(x_k - x_l))_{1 \leq k,l \leq N}\) and \(\psi_j(x) = \sum_{|\xi|^2 = n_j} e^{2\pi i \langle \xi, x \rangle}\). But the constraint \(\det(\psi_j(x_k - x_l))_{1 \leq k,l \leq N} = 0\) defines a submanifold of \(\mathbb{T}^{2N}\) of lower dimension. Since the \(n_k\) form a countable set, the function \(F_\lambda(\lambda)\) has singularities at the \(\{n_k\}\) with probability 1 for \(x \in \mathbb{T}^{2N}\).

Given a solution \(\lambda \in \sigma(-\Delta_{x,U})\) the corresponding eigenfunction will be given by

\[
G^N_{\lambda, x}(x) = \langle (\text{Id} + U)v, G_\lambda(x) \rangle = \sum_{j=1}^N d_{\lambda,j}(x)G_\lambda(x, x_j)
\]

which can be seen by substituting identity (2.6) in (2.3). Note that with probability 1 we have that \(d_{\lambda,j}(x) \neq 0\), \(j = 1, \ldots, N\).

2.2. **Scaling.** It can easily be seen that the formal definition of the operator \(-\Delta_{U, x}\) via the theory of self-adjoint extensions corresponds to the standard Laplacian \(-\Delta\) acting on functions \(f \in C^\infty(\mathbb{T}^d - x)\) where \(\Delta f + c_1 \delta_{x_1} + \cdots + c_N \delta_{x_N} \in L^2(\mathbb{T}^d)\), where \(c_j \in \mathbb{C}\), \(j = 1, \ldots, N\), and \(f\) satisfies certain logarithmic boundary conditions at each of the points \(x_j\) which only depend on the choice of the matrix \(U\).

Let \(f \in L^2(\mathbb{T}_L^d)\) and define by \(g(y) = f(Ly)\) a function \(g \in L^2(\mathbb{T}^d)\). Let \(x_L = (Lx_1, \ldots, Lx_N)\). It can easily be seen that the eigenvalue problem

\[
(\Delta_{U, x_L} + E)f = 0
\]

on the large torus \(\mathbb{T}_L^d\) corresponds to the eigenvalue problem

\[
(L^{-2}\Delta_{U, x} + E)g = 0
\]

on the standard torus \(\mathbb{T}^d\). If, in the first problem we study eigenfunctions with bounded eigenvalue \(E \leq E_0\) and the limit of large tori \(L \to \infty\), then in the second problem this corresponds to studying the high energy limit \(\lambda = EL^2 \to \infty\).

3. **Proof of Theorem 1.1**

We give the detailed proof here only in the critical case of two dimensions. The proof works exactly the same in three dimensions, however, instead of Lemma 4.3 we have to use the subsequence constructed in [11] as well as the different exponent \(\gamma_3\).

Let \(\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2\). Let \(x = (x_1, \ldots, x_N)\).

\[
G^N_{\lambda, x}(x) = \sum_{j=1}^N d_{\lambda,j}(x)G_\lambda(x, x_j)
\]
where we assume, without loss of generality, that \(|d_{\lambda,j}(x)| \leq 1\) and \(\exists j : |d_{\lambda,j}(x)| = 1\) (ie. given a superposition vector \(c = (c_{\lambda,1}, \ldots, c_{\lambda,N})\) set \(d_{\lambda,j} = c_{\lambda,j}/\max_j |c_{\lambda,j}|\).

Note that we will be interested in the spatial distribution of the \(L^2\)-normalized eigenfunctions \(g_{\lambda,N}^{\mathcal{L}} := G_{\lambda,N}^{\mathcal{L}}/\|G_{\lambda,N}^{\mathcal{L}}\|_2\) and its dependence on the random variable \(\mathcal{L}\), which is independent of the choice of normalization of the superposition vector \(c = c(\mathcal{L})\).

Let \(e_\xi(x) := e^{2\pi i \langle \xi, x \rangle}\), for \(\xi \in \mathbb{Z}^2\). We have

\[
G_\lambda(x, x_j) = \sum_{\xi \in \mathbb{Z}^2} c_\lambda(\xi) e_\xi(x - x_j)
\]

where \(c_\lambda(\xi) = (4\pi^2|\xi|^2 - \lambda)^{-1}\), so

\[
G_{\lambda,N}^{\mathcal{L}}(x) = \sum_{\xi \in \mathbb{Z}^2} D_{\lambda,N}(\xi) e_\xi(x)
\]

where

\[
D_{\lambda,N}(\xi) = c_\lambda(\xi) d_\lambda(\mathcal{L}, \xi).
\]

and

\[
d_\lambda(\mathcal{L}, \xi) := \sum_{j=1}^N d_{\lambda,j}(\mathcal{L}) e_\xi(-x_j)
\]

### 3.1. Approximation of Green’s functions on thin annuli.

Let \(a \in C^\infty(T^2)\).

Let \(\lambda, L_0 > 0\). Define the annulus

\[
A(\lambda, L_0) = \{\xi \in \mathbb{Z}^2 | ||\xi|^2 - \lambda| \leq L_0\}.
\]

We introduce the truncated Green’s function

\[
G_{\lambda,L_0,N}^{\mathcal{L}}(x, y) = \sum_{\xi \in A(\lambda, L_0)} D_{\lambda,N}(\xi) e_\xi(x).
\]

In section 4.2 we will prove the following Proposition.

**Proposition 3.1.** Denote \(g_{\lambda,N}^{\mathcal{L}} = G_{\lambda,N}^{\mathcal{L}}/\|G_{\lambda,N}^{\mathcal{L}}\|_2\) and \(g_{\lambda,L_0,N}^{\mathcal{L}} = G_{\lambda,L_0,N}^{\mathcal{L}}/\|G_{\lambda,L_0,N}^{\mathcal{L}}\|_2\)

There exists a certain \(\delta \in (0, \frac{1}{4})\) and a subsequence of Laplacian eigenvalues \(S' \subset \mathcal{S}\), of density one, such that we have for the uniform random process \(\mathcal{L} \in T^{2N}\) that there exists an event \(\Omega_1 \subset T^{2N}\) with \(\text{Prob}(\Omega_1) \geq \frac{1}{6N}\) and that for each \(\lambda \in (n_k, n_{k+1}), n_k \in S'\)

\[
\langle ag_{\lambda,N}^{\mathcal{L}}, g_{\lambda,N}^{\mathcal{L}} \rangle = \langle ag_{\lambda,L_0,N}^{\mathcal{L}}, g_{\lambda,L_0,N}^{\mathcal{L}} \rangle + O(\|a\|_\infty N^{1/2}\lambda^{-\alpha})
\]

for some \(\alpha > 0\) and \(L_0 = \lambda^6\).

### 3.2. Uniformly distributed eigenfunctions.

We have the following result which proves the existence of uniformly distributed eigenfunctions at high energy with positive probability.

**Theorem 3.2.** Fix \(U \in U(N)\). Let \(a \in C^\infty(T^2)\). There exists an infinite sequence \(\Lambda \subset \mathbb{R}_+\), such that the \(L^2\)-normalized eigenfunctions \(g_{\lambda,N}^{\mathcal{L}}\), \(\lambda \in \Lambda\), of the random operator \(-\Delta_{\mathcal{L},U}\) and a constant \(\gamma > 0\) such that with probability \(\geq \frac{1}{12N}\) we have

\[
\int_{T^2} a(y) \left| g_{\lambda,N}^{\mathcal{L}}(y) \right|^2 dy = \int_{T^2} a(y) dy + O(a(\lambda^{1/2}\lambda^{-\gamma})).
\]
Proof. We expand the test function \( a \in C^\infty(\mathbb{T}^2) \) into a Fourier series

\[
a = \sum_{\zeta \in \mathbb{Z}^2} \hat{a}(\zeta) e^\zeta.
\]

We compute, for \( \zeta \neq 0 \),

\[
(3.5) \quad \langle e^\zeta G_{\lambda,x}^N, G_{\lambda,x}^N \rangle = \sum_{\xi \in \mathbb{Z}^2} D_{\lambda,x}(\xi) \overline{D_{\lambda,x}(\xi + \zeta)}.
\]

Hence, we have, by Cauchy-Schwarz,

\[
(3.6) \quad \left| \frac{\langle e^\zeta G_{\lambda,x}^N, G_{\lambda,x}^N \rangle}{\|G_{\lambda,x}^N\|^2_2} \right|^2 \leq \sum_{\xi \in A(\lambda,L_0) \cap \mathbb{Z}^2} |D_{\lambda,x}(\xi + \zeta)|^2 \leq \sum_{\xi \in A(\lambda,L_0) \cap \mathbb{Z}^2} \left| \frac{\langle e^\zeta G_{\lambda,x}^N, G_{\lambda,x}^N \rangle}{\|G_{\lambda,x}^N\|^2_2} \right|^2
\]

Before we continue with the estimation, let us define the following functions of the random variable \( x \)

\[
A_\zeta(x) = \sum_{\xi \in A(\lambda,L) \cap \mathbb{Z}^2} c_{\zeta}(\xi + \zeta)^2 |d_\lambda(x,\xi)|^2
\]

(3.7) + \sum_{\xi \in A(\lambda,L) \cap \mathbb{Z}^2} c_{\zeta}(\xi + \zeta)^2 |d_\lambda(x,\xi)|^2

and

\[
B(x) = \frac{|d_\lambda(x,0)|^2}{(n_+(\lambda) - n_-(-\lambda))^2}
\]

where \( \xi_0 \in \mathbb{Z}^2 \) is such that \( |\xi_0|^2 = n_-(\lambda) \).

Now, note that for \( \lambda \in \Lambda \)

\[
\sum_{\xi \in A(\lambda,L_0) \cap \mathbb{Z}^2} |D_{\lambda,x}(\xi + \zeta)|^2 \leq A_\zeta(x)
\]

because \( |\xi + \zeta|^2 < n_-(\lambda) \) implies \( \lambda - |\xi + \zeta|^2 > n_-(\lambda) - |\xi + \zeta|^2 \), and this in turn implies

\[
c_\lambda(\xi + \zeta)^2 = (\lambda - |\xi + \zeta|^2)^{-2} < (n_-(\lambda) - |\xi + \zeta|^2)^{-2} = c_{n_-(\lambda)}(\xi + \zeta)^2
\]

and the analogous argument applies to the sum over \( |\xi + \zeta| > n_+(\lambda) \). Also note that since \( \lambda \in \Lambda \) we have that \( \xi \in A(\lambda,L_0) \cap \mathbb{Z}^2 \) implies \( \xi + \zeta \notin A(\lambda,L_0) \) (cf. the bound (ii) of Lemma 4.3) and therefore \( |\xi + \zeta|^2 \neq n_+(\lambda), n_-(\lambda) \).

We also have the lower bound

\[
\sum_{\xi \in A(\lambda,L_0) \cap \mathbb{Z}^2} |D_{\lambda,x}(\xi)|^2 \geq B(x)
\]

because \( \lambda - n_-(-\lambda) \leq n_+(\lambda) - n_-(-\lambda) \).
Again, by Cauchy-Schwarz,

\[
\left| \hat{a}(0) - \left\langle a g_{\lambda,L_0,x}^N, g_{\lambda,L_0,x}^N \right\rangle \right|^2 \\
\leq \left| \sum_{\zeta \in \mathbb{Z}^2 \setminus \{0\}} \hat{a}(\zeta) \left\langle e^{\zeta \cdot g_{\lambda,L_0,x}^N}, g_{\lambda,L_0,x}^N \right\rangle \right|^2
\]

(3.9)

\[
\leq \sum_{\zeta \in \mathbb{Z}^2 \setminus \{0\}} |\hat{a}(\zeta)| \left( \sum_{\zeta \in \mathbb{Z}^2 \setminus \{0\}} |\hat{a}(\zeta)| \left| \left\langle e^{\zeta \cdot g_{\lambda,L_0,x}^N}, g_{\lambda,L_0,x}^N \right\rangle \right|^2 \right)
\]

\[
= C_a \frac{1}{B(x)} \sum_{\zeta \in \mathbb{Z}^2 \setminus \{0\}} |\hat{a}(\zeta)| A(\zeta) = C_a \frac{A(\zeta)}{B(x)}
\]

where

\[
C_a = \sum_{\zeta \in \mathbb{Z}^2 \setminus \{0\}} |\hat{a}(\zeta)|.
\]

We have the following proposition which we prove in section 4.

**Proposition 3.3.** Let \( x = (x_1, \cdots, x_{N_0}) \) be points from a stochastic process on \( \mathbb{T}^2 \), and denote its joint probability distribution by \( P_{N_0} \). Let \( A, B \subset \mathbb{T}^{2N_0} \) be probability events and \( C_0 > 0 \). We have that

(3.10) \[ A(x) \leq C_0 E(A), \text{ with probability } \geq 1 - \frac{1}{C_0} \]

and

(3.11) \[ B(x) > \frac{1}{3} E(B), \text{ with probability } \geq \frac{2}{3N_0}. \]

We also require the following proposition which is an immediate consequence of the identities (4.9) and (4.10) proven in section 4.

**Proposition 3.4.** We have the following asymptotics, as \( \lambda \to \infty \),

\[
\frac{E(A_\lambda)}{E(B)} \sim (n_-(\lambda) - n_+ (\lambda))^2
\]

(3.12)

\[ \times \left\{ \sum_{\xi \in A(\lambda,L_0) \cap \mathbb{Z}^2 \atop |\xi|^2 < n_-(\lambda)} c_{n_-(\lambda)} (\xi + \zeta)^2 + \sum_{\xi \in A(\lambda,L_0) \cap \mathbb{Z}^2 \atop |\xi|^2 > n_+(\lambda)} c_{n_+(\lambda)} (\xi + \zeta)^2 \right\} \]

The event \( \Omega_1 \) of Proposition 3.1 implies the lower bound (3.10), since a lower bound on the \( L^2 \)-norm is required in the proof (cf. section 4). Proposition 3.3 implies that there exists an event \( \Omega_2 \) with \( \text{Prob}(\Omega_2) \geq 1 - \frac{12N}{N_0} \) such that

\[ A_0(x) \leq 12N E(A_0). \]
We hence have for $x \in \Omega_2$, for any large $\lambda \in (n_k, n_{k+1})$, 
\[
\frac{A_\alpha(x)}{B(x)} < 36N \frac{\mathbb{E}(A_\alpha)}{\mathbb{E}(B)} \ll 36NC_\alpha(n_+ - n_-(\lambda))^2
\]
\[
\ll \alpha, \epsilon N\lambda^{-2\beta + \epsilon}
\]
for some absolute constant $\beta > 0$. Note that, crucially, for $\lambda \in \Lambda$ we have $|\xi + \zeta|^2 - \lambda| \geq L_0 = \lambda^6$ and $n_+(\lambda) - n_-(\lambda) \ll \lambda^4$ and therefore $|\xi + \zeta|^2 - n_+ - n_-(\lambda) \gg \lambda^6$ 

Finally note that $\Pr(\Omega_1 \cap \Omega_2) \geq \frac{N}{12N} - \frac{N}{24N} \geq \frac{N}{12N}$, and hence for $x \in \Omega_1 \cap \Omega_2$ both the approximation (4.14) and the bound (3.13) hold.

Therefore, we have with probability $\geq \frac{N}{12N}$ 
\[
\langle a g^N_{\lambda, x}, g^N_{\lambda, x} \rangle = \langle a g^N_{\lambda, L_0 x}, g^N_{\lambda, L_0 x} \rangle + \mathcal{O}(N^{1/2}\lambda^{-\alpha + \epsilon}) = \hat{a}(0) + \mathcal{O}(\|a\|_2 N^{1/2}\lambda^{-\min\{\alpha, \beta\}+\epsilon})
\]
which proves the claim. 

\[
\square
\]

4. PROOFS OF THE AUXILIARY RESULTS

4.1. SOME EXPECTATION VALUES. We compute the expectation value of $|d_\lambda(x, \xi)|^2$:
\[
\mathbb{E}(\langle d_\lambda(x, \xi) \rangle^2) = \int_{T^2} \cdots \int_{T^2} |d_\lambda(x, \xi)|^2 dx 
\]
\[
= \int_{T^2} \cdots \int_{T^2} \sum_{j=1}^N d_{\lambda, j}(x) e_{\xi}(-x_j)^2 dx
\]
\[
= \sum_{j,k=1}^N \int_{T^2} \cdots \int_{T^2} d_{\lambda, j}(x) d_{\lambda, k}(x) e_{\xi}(x_k - x_j) dx_1 \cdots dx_N
\]
\[
= \sum_{j=1}^N \mathbb{E}(\langle d_{\lambda, j} \rangle^2) + \sum_{1 \leq j, k \leq N \atop j \neq k} (d_{\lambda, j} d_{\lambda, k})(\Xi_{j,k}).
\]
where $\Xi_{j,k} = (0, 0, \cdots, \xi_1, \xi_2, \cdots, -\xi_1, -\xi_2, \cdots, 0, 0)$. 

For convenience, denote $F_{\xi}(d_{\lambda, j} d_{\lambda, k}) = (d_{\lambda, j} d_{\lambda, k})(\Xi_{j,k})$. 

\[
\]
Recall the definition of the functions

\(A_\xi(x) = \left( \sum_{\xi \in A(\lambda, L)^2 \cap 2} c_{n_-} (\xi + \xi)^2 |d_\lambda(x, \xi)|^2 + \sum_{\xi \in A(\lambda, L)^2 \cap 2 \setminus \{\xi + \xi\}^2 > n_+} c_{n_+} (\xi + \xi)^2 |d_\lambda(x, \xi)|^2 \right)\)

and

\(B(x) = \sum_{|\xi|^2 = n_-} c_{n_+} (\xi)^2 |d_\lambda(x, \xi)|^2\)

and in addition define

\(C(x) = \left( \sum_{\xi \in A(\lambda, L)^2 \cap 2 \setminus |\xi|^2 < n_-} c_{n_-} (\xi + \xi)^2 |d_\lambda(x, \xi)|^2 + \sum_{\xi \in A(\lambda, L)^2 \cap 2 \setminus |\xi|^2 > n_+} c_{n_+} (\xi + \xi)^2 |d_\lambda(x, \xi)|^2 \right)\).

Now, denote \(E_N + F_{N, \xi} = \sum_{j=1}^N \mathbb{E}(|d_{\lambda,j}|^2) + \sum_{j \neq k} \mathbb{E}(\bar{d}_{\lambda,j}d_{\lambda,k})\).

We have, almost surely (since \(n_\pm(\lambda)\) almost surely do not depend on \(x\)),

\(E(A_\xi) = \sum_{\xi \in A(\lambda, L)^2 \cap 2 \setminus |\xi|^2 < n_-} c_{n_-} (\xi + \xi)^2 (E_N + F_{N, \xi}) + \sum_{\xi \in A(\lambda, L)^2 \cap 2 \setminus |\xi|^2 > n_+} c_{n_+} (\xi + \xi)^2 (E_N + F_{N, \xi})\)

and

\(E(B) = \sum_{|\xi|^2 = n_-} c_{n_+} (\xi)^2 (E_N + F_{N, \xi})\)

and

\(E(C) = \sum_{\xi \in A(\lambda, L)^2 \setminus |\xi|^2 < n_-} c_{n_-} (\xi)^2 (E_N + F_{N, \xi}) + \sum_{\xi \in A(\lambda, L)^2 \setminus |\xi|^2 > n_+} c_{n_+} (\xi)^2 (E_N + F_{N, \xi})\).

We have the following Lemma.

**Lemma 4.1.** The diagonal terms in the expectation value dominate the off-diagonal terms in the limit \(|\xi| \to \infty\):

\(\sum_{j \neq k} \mathbb{F}_\xi(d_{\lambda,j}\bar{d}_{\lambda,k}) = o \left( \sum_{j=1}^N \mathbb{E}(|d_{\lambda,j}|^2) \right)\)

**Proof.** First of all note that

\(\sum_{j=1}^N \mathbb{E}(|d_{\lambda,j}|^2) = \mathbb{E} \left( \sum_{j=1}^N |d_{\lambda,j}|^2 \right) \geq 1\)

because \(\sum_{j=1}^N |d_{\lambda,j}|^2 \geq 1\), since due to our normalization of the coefficients for each \(x \in T^{2N}\), there exists \(j\) s.t. \(|d_{\lambda,j}(x)| = 1\).
A. the probability of the event \( P \) and denote its joint probability distribution by 
\[
\text{(4.11)}
\]
Proof.

\[
\text{(4.12) } \sum_{1 \leq j, k \leq N, \ j \neq k} d_{\lambda; j} d_{\lambda; k} (\Xi_{j,k}) = o_{\xi \to \infty}(1)
\]

because \( |d_{\lambda; j} d_{\lambda; k} (\Xi_{j,k})| = o(\|d_{\lambda; j} d_{\lambda; k}\|_2) \), as \( |\xi| \to \infty \), in view of 
\[
\sum_{\xi \in \mathbb{Z}^2} |d_{\lambda; j} d_{\lambda; k} (\Xi_{j,k})|^2 \leq \sum_{\xi \in \mathbb{Z}^2} |d_{\lambda; j} d_{\lambda; k} (\xi)|^2 = \|d_{\lambda; j} d_{\lambda; k}\|_2^2 \leq 1
\]
where we recall our normalization of coefficients which ensures \( \forall j : |d_{\lambda; j}| \leq 1 \) □

As a consequence of the Lemma \ref{reduction}, we have the asymptotics (recall that \( c_\lambda (\xi) = (|\xi|^2 - \lambda)^{-1} \) is weighted near \( \lambda \))
\[
\text{(4.9) } \mathbb{E}(A_\xi) \sim_{\lambda \to \infty} \mathbb{E}_N \times \left( \sum_{\xi \in A_\lambda(L) \cap \mathbb{Z}^2 \atop |\xi + \zeta| < n_- (\lambda)} c_{n_- (\lambda)} (\xi + \zeta)^2 + \sum_{\xi \in A_\lambda(L) \cap \mathbb{Z}^2 \atop |\xi + \zeta| > n_+ (\lambda)} c_{n_+ (\lambda)} (\xi + \zeta)^2 \right)
\]

and, where \( \xi_0 \in \mathbb{Z}^2 \) is such that \( |\xi_0|^2 = n_- (\lambda) \),
\[
\text{(4.10) } \mathbb{E}(B) \sim_{\lambda \to \infty} \mathbb{E}_N \times c_{n_+ (\lambda)} (\xi_0)^2 = \frac{\mathbb{E}_N}{(n_+ (\lambda) - n_- (\lambda))^2}
\]

and
\[
\text{(4.11) } \mathbb{E}(C) \sim_{\lambda \to \infty} \mathbb{E}_N \times \left( \sum_{\xi \in A_\lambda(L) \cap \mathbb{Z}^2 \atop |\xi|^2 < n_- (\lambda)} c_{n_- (\lambda)} (\xi)^2 + \sum_{\xi \in A_\lambda(L) \cap \mathbb{Z}^2 \atop |\xi|^2 > n_+ (\lambda)} c_{n_+ (\lambda)} (\xi)^2 \right)
\]

Given \( a \in C^\infty (\mathbb{T}^2) \), we define the function 
\[
A_a (\xi) = \sum_{\zeta \in \mathbb{Z}^2 \setminus \{0\}} |\hat{a} (\zeta)| A_\zeta (\xi).
\]

Recall the following proposition.

**Proposition 4.2.** Let \( \mathbf{x} = (x_1, \ldots, x_{N_0}) \) be points from a stochastic process on \( \mathbb{T}^2 \),
and denote its joint probability distribution by \( \mathbb{P}_{N_0} \). Let \( C_0 > 0 \).
\[
\text{(4.12) } A_a (\mathbf{x}) \leq C_0 \mathbb{E}(A_a), \text{ with probability } \geq \frac{1}{C_0}
\]

and
\[
\text{(4.13) } \mathbb{B}(\mathbf{x}) > \frac{1}{3} \mathbb{E}(\mathbb{B}), \text{ with probability } \geq 1 - \frac{1}{N_0}.
\]

**Proof.** (i): Define the subset \( A_1 := \{ x \in \mathbb{T}^{2N_0} \mid A_a (x) > C_0 \mathbb{E}(A_a) \} \). We denote
the probability of the event \( A_1 \) occurring by
\[
\mathbb{P}(A_1) = \int_{A_1} \mathbb{P}_{N_0} (x) dx.
\]
Now we have (recall $\mathcal{A}_n \geq 0$ by definition):
\[
E(\mathcal{A}_n) \geq \int_{A_1} \mathbb{P}_{N_0}(x) \mathcal{A}_n(x) dx > C_0 E(\mathcal{A}_n) \mathbb{P}(A_1)
\]
and therefore $\mathbb{P}(A_1) < 1/C_0$. This implies
\[
\text{Prob}\{x \in \mathbb{T}^{2N_0} \mid \mathcal{A}_n(x) \leq C_0 E(\mathcal{A}_n)\} > 1 - \frac{1}{C_0}.
\]

(ii): Define the subset $A_2 := \{x \in \mathbb{T}^{2N_0} \mid \mathcal{B}(x) \leq \frac{4}{3} E(\mathcal{B})\}$.

Also note that
\[
\forall x \in \mathbb{T}^{2N_0} : |D_{\lambda, z}(\xi)|^2 \leq |c_\lambda(\xi)|^2 \left( \sum_{j=1}^{N_0} |d_{j, \lambda}(z)|^2 \right) \leq |c_\lambda(\xi)|^2 N_0 \sum_{j=1}^{N_0} |d_{j, \lambda}(z)|^2
\]
and therefore $\forall x \in \mathbb{T}^{2N_0} : \mathcal{B}(x) \leq N_0 E(\mathcal{B})$.

We have
\[
(1 - \mathbb{P}(A_2)) N_0 E(\mathcal{B}) \geq \int_{\mathbb{T}^{2N_0} \setminus A_2} \mathbb{P}_{N_0}(x) \mathcal{B}(x) dx = E(\mathcal{B}) - \int_{A_2} \mathbb{P}_{N_0}(x) \mathcal{B}(x) dx \geq (1 - \frac{1}{3} \mathbb{P}(A_2)) E(\mathcal{B})
\]
so we get
\[
\text{Prob}\{\mathcal{B}(x) > \frac{1}{3} E(\mathcal{B})\} = 1 - \mathbb{P}(A_2) \geq \frac{1 - \frac{2}{3}}{N_0}.
\]

4.2. Proof of Proposition 3.1 Let $a \in C^\infty(\mathbb{T}^2)$. We will show that there exists an infinite sequence of eigenvalues $\Lambda$ such that if $\lambda \in \Lambda$ we have, with probability $\geq \frac{1 - 1/3 - \frac{1}{2N}}{N - 1}$, for $L_0 = \lambda^\prime$ and $\delta \in (0, \frac{1}{4})$

\[
\left\langle a \mathcal{G}^{\mathbb{N}}_{\lambda, z} \mathcal{G}^{\mathbb{N}}_{\lambda, z} \right\rangle = \left| \mathcal{G}^{\mathbb{N}}_{\lambda, z} \right|_2^2 + O_a(N^{1/2} \lambda^{-\alpha})
\]

for some $\alpha > 0$.

The construction of the sequence $\Lambda$ is almost identical to the one given in [7], sections 5, 6 and 7. We have the following Lemma.

Lemma 4.3. Denote by $S = \{n \mid n = 4\pi^2(x^2 + y^2), x, y \in \mathbb{Z}\} = \{0 = n_0 < n_1 < n_2 < \cdots \}$ the set of Laplacian eigenvalues on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, where we ignore multiplicities.

There exists a subsequence $S' \subset S$ of density 1 such that for all $n_k \in S'$:

(i) $n_{k+1} - n_{k-1} \ll n_k$

(ii) $\forall \lambda \in (n_k, n_{k+1}) \forall \zeta \neq 0 \forall \xi \in A(\lambda, L_0) \cap \mathbb{Z}^2 : |c(\xi + \zeta)| \ll \frac{1}{L_0}$

The sequence $\Lambda$ is defined as the following subset of the spectrum of $-\Delta_{\mathbb{T}}$:

\[
\Lambda := \{\lambda \in \sigma(-\Delta_{\mathbb{T}}) \mid \lambda \in (n_k, n_{k+1}), n_k, n_k \in S'\}
\]

Proof. (i): To see this note that the elements of $S$, integers representable as sums of two squares, have mean spacing of order $\sqrt{\log n_k}$. Therefore the subsequences of $n_k$ s. t. $n_{k+1} - n_k \ll n_k$ and those $n_k$ s. t. $n_k - n_{k-1} \ll n_k$ are of density 1 respectively. Consequently, their intersection is a subsequence of density 1.

(ii): This proof is exactly identical to the construction in sections 6 and 7 of [7]. Note the additional factor $4\pi^2$ which is due to the fact that we consider the standard torus $\mathbb{R}^2/\mathbb{Z}^2$ rather than the scaled torus $\mathbb{R}^2/2\pi\mathbb{Z}^2$ considered in [7].
We proceed with the proof of (4.14).

**Proof.** We split $g_{\lambda}^N = g_{\lambda, L_0}^N + g_{\lambda, R}^N$, where $g_{\lambda, L_0}^N = G_{\lambda, L_0}^N/\|G_{\lambda}^N\|_2$.

We then have the bound

$$
\left| \left\langle ag_{\lambda}^N, g_{\lambda}^N \right\rangle - \left\langle ag_{\lambda, L_0}^N, g_{\lambda, L_0}^N \right\rangle \right|
\leq \left| \left\langle ag_{\lambda, R}^N, g_{\lambda, R}^N \right\rangle \right| + \left| \left\langle ag_{\lambda, L_0}^N, g_{\lambda, L_0}^N \right\rangle \right|,
$$

(4.16)

$$
\leq ||a||_\infty (\|g_{\lambda, R}^N\|_2^2 + 2\|g_{\lambda, L_0}^N\|_2^2)
$$

where $g_{\lambda, R}^N = (G_{\lambda}^N - G_{\lambda, L_0}^N)/\|G_{\lambda}^N\|_2$.

We estimate

$$
\|g_{\lambda, R}^N\|_2^2 = \frac{\|G_{\lambda}^N - G_{\lambda, L_0}^N\|_2^2}{\|G_{\lambda, L_0}^N\|_2^2}
$$

(4.17)

We want to bound $\|g_{\lambda, R}^N\|_2$ in terms of an expression which does not depend on $\lambda = \lambda(\varphi)$. In fact, with probability 1 we have that the $n_\pm(\lambda)$ are independent of $\varphi$.

We thus have, almost surely,

$$
\sum_{\xi \in A(\lambda, L_0) \cap \mathbb{Z}^2} |D_{\lambda, \varphi}(\xi)|^2
\leq \sum_{\xi \in A(\lambda, L_0) \cap \mathbb{Z}^2} c_{n_-(\lambda)}(\xi)^2 |d_{\lambda}(\varphi, \xi)|^2 + \sum_{|\xi|^2 < n_-(\lambda)} c_{n_+}(\lambda)(\xi)^2 |d_{\lambda}(\varphi, \xi)|^2
$$

(4.18)

$$
= C(\varphi)
$$

and, because $|n_-(\lambda) - \lambda| \leq |n_-(\lambda) - n_+(\lambda)|$,

$$
\sum_{\xi \in A(\lambda, L_0) \cap \mathbb{Z}^2} |D_{\lambda, \varphi}(\xi)|^2 \geq \sum_{|\xi|^2 = n_-(\lambda)} c_{n_+}(\lambda)(\xi)^2 |d_{\lambda}(\varphi, \xi)|^2 \geq B(\varphi).
$$

(4.19)

With probability $\frac{1}{N} \leq \frac{1}{N} - \frac{1}{2N} \geq \frac{1}{BN}$ we have $C(\varphi) < 2NE(C)$ and $B(\varphi) > \frac{1}{N}E(B)$. And thus, for $\lambda \in \Lambda$, in view of the identities (4.10) and (4.11),

$$
\frac{\|G_{\lambda}^N - G_{\lambda, L_0}^N\|_2^2}{\|G_{\lambda, L_0}^N\|_2^2} \leq \frac{C(\varphi)}{B(\varphi)} < 6N \frac{E(C)}{E(B)}
$$

(4.20)

$$
\ll N(n_+(\lambda) - n_-(\lambda))^2 \left\{ \sum_{\xi \in A(\lambda, L_0) \cap \mathbb{Z}^2} c_{n_-(\lambda)}(\xi)^2 + \sum_{|\xi|^2 < n_-(\lambda)} c_{n_+}(\lambda)(\xi)^2 \right\} \ll \epsilon N\lambda^{-2\alpha + \epsilon}
$$

for some absolute constant $\alpha > 0$ (cf. the proof of Lemma 5.1 in [7]).
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