Nonperturbative renormalization of scalar QED in $d=3$

J. Dimock *
Dept. of Mathematics
SUNY at Buffalo
Buffalo, NY 14260
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Abstract

For scalar QED on a three-dimensional toroidal lattice with a fine lattice spacing we consider
the renormalization problem of choosing counter terms depending on the lattice spacing, so that the
theory stays finite as the spacing goes to zero. We employ a renormalization group method which
analyzes the flow of the mass and the vacuum energy as a problem in discrete dynamical systems.
The main result is that counter terms can be chosen so that at the end of the iteration these
quantities take preassigned values. No use is made of perturbation theory. The renormalization
group transformations are defined with bounded fields, an approximation which can be justified
in Balaban’s approach to the renormalization group.

1 Introduction

1.1 overview

We study scalar quantum electrodynamics (QED) as a Euclidean field theory on a toroidal lattice in
dimension $d = 3$. Our concern is the ultraviolet problem of controlling the limit as the lattice spacing
goes to zero. We have nothing to say about the infinite volume limit, and for convenience we take unit
volume.

The renormalization problem is to choose counter terms so that the model remains well-behaved
as the lattice spacing goes to zero. We carry this out in the framework of the renormalization group
(RG) defined with block averaging. Working in a bounded field approximation we study the flow of the
renormalization group transformations as a problem in discrete dynamical systems. In this framework
we show that counter terms can be chosen so that superficially divergent quantities (for this model
the vacuum energy and the scalar mass) flow to preassigned values, and the other parameters in the
model stay bounded, so the model is well-defined. This is nonperturbative renormalization: there are
no expansions in the coupling constant and no Feynman diagrams.

Our bounded field approximation fits nicely with the formulation of the RG developed by Balaban
[1] - [4] who also studies scalar QED in $d = 3$. In this approach at each stage of the iteration the field
space is split into large and small (=bounded) fields. The renormalization problem is confined to the
small field region which we consider here. This is supplemented by a treatment of the large field region
which gives tiny corrections to the analysis. This is carried out by Balaban and leads to an ultraviolet
stability estimate on the partition function.

However in the papers [1] - [4] renormalization is accomplished by picking specific counter terms
suggested by perturbation theory and then exhibiting the cancellations. The final result is non-
perturbative, but in intermediate steps one is obliged to consider Feynman diagrams of rather high
order. In this respect the present paper is an improvement.

*dimock@buffalo.edu
Another feature of [4] - [6] inviting improvement is that the gauge field is taken to be massive. Here the analysis is carried out with the more physical massless gauge field.

A third feature of [4] - [6] that wants improvement is that the full flow of the RG is not developed. Instead estimates above and below are taken after each transformation. This makes it awkward to extend results beyond the partition function, for example to control the correlation functions. Balaban fixes this in subsequent papers on other models, but it remains undeveloped for scalar QED.

Closest to the present paper is the series of paper [24], [25], [26] by the author. These are on the $\phi^4$ model in $d = 3$, essentially the present model without the gauge field. The first paper in the series introduced the non-perturbative renormalization technique employed here. The remaining papers completed the analysis of the large field region, developed the full flow of the RG, and obtained a stability bound. This could be a model for the completion of the program for scalar QED, but it is not undertaken here.

Another important source of ideas for the present work are the papers [17], [18] by Balaban, Imbrie, and Jaffe. They study the abelian Higgs model which is scalar QED with a special scalar potential. We also mention earlier work by Brydges, Fröhlich, and Seiler [20], [21], [22] who treat scalar QED in $d = 2$.

In this paper our nonperturbative renormalization method is applied to a model that is super-renormalizable with no coupling constant renormalization. However there is no obstacle in principle to applying it to renormalizable models, or possibly even some nonrenormalizable models.

1.2 the model

The model is defined as follows. Let $L$ be a (large) positive odd integer. We work on three dimensional lattices

$$T_M^N = (L^{-N}\mathbb{Z}/L^M\mathbb{Z})^3$$

with lattice spacing $\epsilon = L^{-N}$ and linear size $L^M$. At first we take a fine lattice with unit volume $T_0^{-N}$. On this lattice we consider scalar fields $\phi : T_0^{-N} \to \mathbb{R}^2$. The field $\phi = (\phi_1, \phi_2)$ is often regarded as a complex valued field $\phi = \phi_1 + i\phi_2$, but not here. The gauge group is $SO(2)$ with Lie algebra the real numbers $\mathbb{R}$. Elements of the group can be written $e^{i\theta}$ where $\theta \in \mathbb{R}$ and

$$q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

There is also an abelian gauge field (electromagnetic potential, connection) $A : \{\text{bonds in } T_0^{-N}\} \to \mathbb{R}$. A bond from $x$ to a nearestneighbor $x'$ is the ordered pair $b = [x, x']$. We require that $A(b) = A(x, x') = -A(x', x)$. A covariant derivative on scalar fields is defined on bonds by

$$(\partial A)(b) = (\partial_A \phi)(x, x') = \left(e^{q\epsilon A(x, x')} \phi(x') - \phi(x)\right)\epsilon^{-1}$$

where $\epsilon$ is the scalar charge. If $\{\epsilon_\mu\}$ is the standard basis then oriented bonds have the form $[x, x'] = [x, x + \epsilon_\mu]$, we write $A_\mu(x) = A(x, x + \epsilon_\mu)$ and define

$$(\partial_{A_\mu} \phi)(x) = (\partial_A \phi)(x, x + \epsilon_\mu) = \left(e^{q\epsilon A_\mu(x)} \phi(x + \epsilon_\mu) - \phi(x)\right)\epsilon^{-1}$$

The ordinary derivative $\partial_\mu \phi$ has $A = 0$. The gauge field $A$ has field strength $dA$ defined on plaquettes (squares) by

$$dA(p) = \prod_{b \in \partial p} A(b)\epsilon^{-1} \quad \text{or} \quad (dA)_{\mu\nu}(x) = dA\left(x, x + \epsilon_\mu, x + \epsilon_\mu + \epsilon_\nu, x + \epsilon_\nu\right)$$

\footnote{Later we allow complex valued fields, but then each component will be separately complex. With this formulation the action will be analytic in the fields}
The action is

$$S(A,\phi) = \frac{1}{2}\|dA\|^2 + \frac{1}{2}\|\partial A\phi\|^2 + V(\phi)$$  \hspace{1cm} (6)$$

with potential

$$V(\phi) = \varepsilon^N \text{Vol}(\mathbb{T}_0^{-N}) + \frac{1}{2}\mu^N\|\phi\|^2 + \frac{1}{4}\lambda \int |\phi(x)|^4dx$$  \hspace{1cm} (7)$$

Here the norms are $L^2$ norms and integrals are weighted sums, for example

$$\|\phi\|^2 = \int |\phi(x)|^2dx = \sum_x c^3|\phi(x)|^2 = \sum_x c^3(\phi_1(x)^2 + \phi_2(x)^2)$$  \hspace{1cm} (8)$$

The norms involving derivatives or gauge potentials are sums over oriented bonds and oriented plaquettes:

$$\|\partial A\phi\|^2 = \sum_{\mu} \int |\partial A\phi(b)|^2db \equiv \sum_{\mu} \sum_x c^3|\partial A_{\mu}\phi(x)|^2$$

$$\|dA\|^2 = \int |\partial A(p)|^2dp \equiv \sum_{\mu<\nu} \sum_x c^3|(dA)_{\mu\nu}(x)|^2$$  \hspace{1cm} (9)$$

In the potential $\lambda > 0$ is the scalar coupling constant. The vacuum energy $\varepsilon^N$ and the scalar mass $\mu^N$ will be chosen to depend on $N$. The $N \to \infty$ limit formally gives the standard continuum theory. We are interested in bounds uniform in $N$ on things like the partition function

$$\int \exp(-S(A,\phi))\,DA\,D\phi$$  \hspace{1cm} (10)$$

where

$$DA = \prod_b d(A(b)) \quad D\phi = \prod_x d(\phi(x))$$  \hspace{1cm} (11)$$

However the integral will need gauge fixing to enable convergence.

The action is gauge invariant. For $\lambda : \mathbb{T}^{-N}_0 \to \mathbb{R}$ a gauge transformation is defined by

$$\phi^\lambda(x) = e^{\varepsilon(x)\lambda} \phi(x) \quad A^\lambda(x, x') = A(x, x') - \partial \lambda(x, x')$$  \hspace{1cm} (12)$$

Then $\partial A^\lambda \phi^\lambda = (\partial A\phi)^\lambda$ and $|\phi^\lambda| = |\phi|$ and so $S(A^\lambda, \phi^\lambda) = S(A, \phi)$.

Another symmetry is charge conjugation invariance. We defined by

$$C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$  \hspace{1cm} (13)$$

Then $Cq = -qC$ and so $\partial_{-A}C\phi = C\partial A\phi$. Since also $|C\phi| = |\phi|$ we have $S(-A, C\phi) = S(A, \phi)$.

### 1.3 the scaled model

We scale up to the large unit lattice $\mathbb{T}^0_N$, so the ultraviolet problem is recast as in infrared problem, the natural home of the renormalization group. Let $\Phi : \mathbb{T}^0_N \to \mathbb{R}^2$ and $A : \{\text{bonds in } \mathbb{T}^0_N\} \to \mathbb{R}$ be fields on this lattice. These scale down to fields on the original lattice $\mathbb{T}^{-N}_0$ by

$$A_{L^{-N}}(b) = L^{N/2}A(L^N b) \quad \Phi_{L^{-N}}(x) = L^{N/2}\Phi(L^N x)$$  \hspace{1cm} (14)$$

The action on the new lattice $\mathbb{T}^0_N$ is $S_0(A, \Phi) = S(A_{L^{-N}}, \Phi_{L^{-N}})$ which is

$$S_0(A, \Phi) = \frac{1}{2}\|dA\|^2 + \frac{1}{2}\|\partial A\Phi\|^2 + V_0(\Phi)$$  \hspace{1cm} (15)$$
where
\[ V_0(\Phi) = \varepsilon_0^N \text{Vol}(\mathbb{T}_N^0) + \frac{1}{2} \mu_0^N \|\Phi\|^2 + \frac{1}{4} \lambda_0^N \sum_x |\Phi(x)|^4 \] (16)

Now norms are defined with unweighted sums, and derivatives are unit lattice derivatives such as
\[ (\partial_A \Phi)(x, x') = e^{q_{\infty} A(x, x')} \Phi(x') - \Phi(x) \] (17)

The scaled coupling constants are now tiny and given by
\[ \varepsilon_0^N = L^{-\frac{1}{2} N} \varepsilon \quad \lambda_0^N = L^{-N} \lambda \] (18)

The scaled counter terms are
\[ \varepsilon_0^N = L^{-3 N} \varepsilon \quad \mu_0^N = L^{-2 N} \mu \] (19)

In the following we omit the superscript \( N \) writing \( \varepsilon_0, \lambda_0 \) and \( \varepsilon_0, \mu_0 \).

As we proceed with the RG analysis the volume will shrink back down. After \( k \) steps the torus will be \( \mathbb{T}_{N-k} \). The coupling constants scale up to
\[ e_k = L^{\frac{1}{2} k} \varepsilon_0 = L^{-\frac{1}{2} (N-k)} e \quad \lambda_k = L^k \lambda_0 = L^{-(N-k)} \lambda \] (20)

The other coupling constants \( e_k, \mu_k \) will evolve in a more complicated manner.

**Convention:** Throughout the paper the convention is that \( O(1) \) is a constant independent of all parameters. Also \( C, \gamma \) are constants \( (C \geq 1, \gamma \leq 1) \) which may depend on \( L \) and which may change from line to line.

## 2 RG transformation for scalars

We explain how the RG transformation is defined for scalars, but with a gauge field background. The discussion follows [1], [2], [11], [24], [27].

### 2.1 Block averages

We start with fields \( A, \Phi \) on the unit lattice \( \mathbb{T}_N^0 \). We want to define a covariant block averaging operator \( Q(A) \) taking \( \Phi \) to a function \( Q(A)\Phi \) on the \( L \)-lattice \( \mathbb{T}_N^L \). In any 3-dimensional lattice let \( B(y) \) be a cube of \( L^3 \) sites (\( L \) on a side) centered on a point \( y \). Here the lattice is \( \mathbb{T}_N^0 \), and for \( y \in \mathbb{T}_N^1 \) we have
\[ B(y) = \{ x \in \mathbb{T}_N^0 : |x - y| < L/2 \} \quad |x - y| = \sup_\mu |x_\mu - y_\mu| \] (21)

The \( B(y) \) partition the lattice. For \( x \in B(y) \) let \( \pi \) be a permutation of \( (1, 2, 3) \) and let \( \Gamma^\pi(y, x) \) be that path from \( y \) to \( x \) obtained by varying each coordinate to its final value in the order \( \pi \). There are \( 3! \) of these. For any path \( \Gamma \) let \( A(\Gamma) = \sum_{b \in \Gamma} A(b) \) and define an average over the various paths from \( y \) to \( x \) by
\[ (\tau A)(y, x)) = \frac{1}{3!} \sum_\pi A(\Gamma^\pi(y, x)) \] (22)

Then we define the averaging operator
\[ (Q(A)\Phi)(y) = L^{-3} \sum_{x \in B(y)} e^{q_{\infty}\lambda}(\tau A)(y, x)\Phi(x) \quad y \in \mathbb{T}_N^1 \] (23)

This is constructed to be gauge covariant. We have \( (\tau A\lambda)(y, x) = (\tau A)(y, x) - (\lambda(x) - \lambda(y)) \) for \( \lambda : \mathbb{T}_N^0 \to \mathbb{R} \) and so
\[ Q(A\lambda)\Phi = (Q(A)\Phi)^{\lambda(x)} \] (24)
where $\lambda^{(1)}$ is $\lambda$ restricted to the lattice $\mathbb{T}_N^1$.

Because we average over paths, rather than taking a fixed path, our definition is covariant under symmetries of the lattice $\mathbb{T}_N^1$. In particular if $r$ is a rotation by a multiple of $\pi/2$ or a reflection and $\Phi_r(y) = \Phi(r^{-1}y)$ and $A_r(b) = A(r^{-1}b)$ then

$$Q(A_r)\Phi = (Q(A)\Phi)_r$$

(25)

The adjoint operator (transpose operator) maps functions $f$ on $\mathbb{T}_N^1$ to functions on $\mathbb{T}_N^0$. It is computed with sums on $\mathbb{T}_N^1$ weighted by $L^d$ and is given by

$$(Q^T(A)f)(x) = e^{-q_0(\tau A)(y,x)}f(y) \quad x \in B(y)$$

(26)

Then we have

$$Q(A)Q^T(A) = I$$

(27)

and $Q^T(A)Q(A)$ is an orthogonal projection.

### 2.2 the transformation

Suppose we start with a density $\rho(A, \phi)$ with scalar field $\phi$ and background gauge field $A$ on $\mathbb{T}_0^{-N}$. It scales up to a density

$$\rho_0(A, \Phi_0) \equiv \rho_L^{-N}(A, \Phi_0) \equiv \rho(A_{L^{-N}} \Phi_0, L^{-N})$$

(28)

where $A, \Phi_0$ are defined on $\mathbb{T}_N^0$. Starting with $\rho_0(A, \Phi_0)$ we define a sequence of densities $\rho_k(A, \Phi_k)$ defined for $A$ on $\mathbb{T}_{N-k}^{-k}$ and $\Phi_k$ on $\mathbb{T}_{N-k}^{N-k}$. They are defined recursively first by

$$\tilde{\rho}_{k+1}(A, \Phi_{k+1}) = \int \delta_G \left( \Phi_{k+1} - Q(A)\Phi_k \right) \rho_k(A, \Phi_k) D\Phi_k$$

(29)

where $\Phi_{k+1}$ is defined on the coarser lattice $\mathbb{T}_{N-k}^{-k}$. The $\delta_G$ is a Gaussian approximation to the delta function.

$$\delta_G \left( \Phi_{k+1} - Q(A)\Phi_k \right) = \left( \frac{aL}{2\pi} \right)^{s_N-k-1} \exp \left( -\frac{aL}{2} |\Phi_{k+1} - Q(A)\Phi_k|^2 \right)$$

(30)

Here $|\Phi_{k+1} - Q(A)\Phi_k|$ is the $L^2$ norm with a simple sum over points in $\mathbb{T}_N^1$, whereas in $\|\Phi_{k+1} - Q(A)\Phi_k\|$ it is the $L^2$ norm with the sum over points weighted by the factor $L^3$ natural for this lattice. The averaging operator $Q(A)$ is taken to be a modification of (23):

$$(Q(A)\Phi_k)(y) = L^{-d} \sum_{x \in B(y)} e^{q_0 L^{-k}(\tau A)(y,x)} \Phi_k(x)$$

(31)

Here $(\tau A)(y,x)$ is still defined by (22), but now in $A(\Gamma)$ the sum is over bonds of length $L^{-k}$ hence the weighting factor $L^{-k}$ in the exponent. In general $s_N \equiv L^{3N}$ is the number of sites in a three dimensional tidal lattice with $L^N$ sites on a side. The normalization factor $(aL/2\pi)^{s_N-k-1}$ in (30) is chosen so that $\int d\Phi_{k+1} \delta_G \left( \Phi_{k+1} - Q(A)\Phi_k \right) = 1$. (Recall that there are two components per site.)

Therefore

$$\int \tilde{\rho}_{k+1}(A, \Phi_{k+1}) D\Phi_{k+1} = \int \rho_k(A, \Phi_k) D\Phi_k$$

(32)

Next one scales back to the unit lattice. If $A$ is a field on $\mathbb{T}_{N-k}^{-k-1}$ and $\Phi_{k+1}$ is a field on $\mathbb{T}_{N-k-1}^{N-k-1}$ then

$$A_L(b) = L^{-1/2}A(L^{-1}b) \quad \Phi_{k+1,L}(x) = L^{-1/2}\Phi_{k+1,L}(L^{-1}x)$$

(33)
are fields on $T_{N-k}^{-k}$ and $T_{N-k}^1$ respectively, and we define

$$\rho_{k+1}(A, \Phi_{k+1}) = \tilde{\rho}_{k+1}(A, L, \Phi_{k+1}, L) L^{sN-sN-k-1}$$

(34)

If $\Phi'_{k+1} = \Phi_{k+1}$, then $D\Phi'_{k+1} = L^{-sN-k-1} D\Phi_{k+1}$ and we have by (32)

$$\int \rho_{k+1}(A, \Phi_{k+1}) D\Phi_{k+1} = L^{sN} \int \tilde{\rho}_{k+1}(A, L, \Phi_{k+1}, L) D\Phi_{k+1}$$

$$= L^{sN} \int \tilde{\rho}_{k+1}(A, L, \Phi'_{k+1}) D\Phi'_{k+1} = L^{sN} \int \rho_k(A, \Phi_k) D\Phi_k$$

(35)

**Lemma 1.** For $A$ on $T_{N-k}^{-k}$ and $\Phi_k$ on $T_{N-k}^0$

$$\int \rho_k(A, \Phi_k) D\Phi_k = \int \rho_0(A_{L^k}, \phi_{L^k}) D\phi$$

(36)

where the integral is over $\phi$ on $T_{N-k}^{-k}$.

**Remark.** In particular since $\rho_0(A_{L^N}, \phi_{L^N}) = \rho(A, \phi)$

$$\int \rho_N(A, \Phi_N) D\Phi_N = \int \rho_0(A_{L^N}, \phi_{L^N}) D\phi = \int \rho(A, \phi) D\phi$$

(37)

and we are back to the integral of our original density. The right side is the integral over a many dimension space, but can be computed as the left side which is the integral over one dimensional space. This is the point of the renormalization group approach.

**Proof.** It is true for $k = 0$; suppose it is true for $k$. If $\phi = \phi'_L$ then $D\phi = L^{-sN} D\phi'$ and so by (34)

$$\int \rho_{k+1}(A, \Phi_{k+1}) D\Phi_{k+1} = L^{sN} \int \rho_{k+1}(A, L, \Phi_{k+1}, L) D\Phi_{k+1}$$

$$= L^{sN} \int \rho_0(A_{L^k+1}, \phi_{L^k}) D\phi = \int \rho_0(A_{L^k+1}, \phi'_{L^k+1}) D\phi'$$

(38)

Hence it is true for $k+1$.

### 2.3 compositions of averaging operators

To investigate the sequence $\rho_k(A, \Phi_k)$ we first study how averaging operators compose. Suppose we have already defined a $k$-fold averaging operator $Q_k(A)$ depending on $A$ on $T_{N-k}^{-k}$ and sending functions on $T_{N-k}^{-k}$ to functions on $T_{N-k}^0$. We define the same for $k+1$ as follows. First define for the same $A$ an operator

$$Q_{k+1}(A) = Q(A)Q_k(A)$$

(39)

which maps functions on $T_{N-k}^{-k}$ to functions on $T_{N-k}^1$. Then for $A$, $f$ on $T_{N-k-1}^{-k-1}$ define

$$Q_{k+1}(A)f = (Q_{k+1}(A)L)f_{L^{-1}}$$

(40)

which maps functions on $T_{N-k-1}^{-k-1}$ to functions on $T_{N-k}^0$.

We need an explicit representation for $Q_k(A)$. For any lattice let $B_k(y)$ be a block with $L^{dk}$ sites ($L^k$ on a side) centered on $y$. Suppose $x \in T_{N-k}^{-k}$ and $y \in T_{N-k}^0$ satisfy $x \in B_k(y)$, which is the same as $|x-y| < \frac{1}{L}$. There is an associated sequence $x = y_0, y_1, y_2, \ldots y_k = y$ such that $y_j \in T_{N-k}^{-k-j}$ and $x \in B_j(y_j)$. Define

$$(\tau_k A)(y, x) = \sum_{j=0}^{k-1} (\tau A)(y_{j+1}, y_j)$$

(41)
Lemma 2. For $A, f$ on $\mathbb{T}_{N-k}^{-k}$

\[
(Q_k(A)f)(y) = \int_{|x-y|<\frac{1}{2}} e^{q_k L^{-k}(\tau_k A)(y,x)} f(x) \, dx \\
(Q_k^T(A)f)(x) = e^{-q_k L^{-k}(\tau_k A)(y,x)} f(y) \quad x \in B_k(y)
\]

Proof. The proof is by induction on $k$. Assuming it is true for $k$ we have with $\eta = L^{-k}$

\[
(Q_{k+1}(A)f)(y') = (Q(A)Q_k(A)f)(y')
\]

\[
= L^{-3} \sum_{y \in B(y')} e^{q_k \eta(\tau A)(y',y)} \int_{|x-y'|<\frac{1}{2}} e^{q_k \eta(\tau_k A)(y',x)} f(x) \, dx
\]

\[
= L^{-3} \int_{|x-y'|<L/2} e^{q_k \eta(\tau_k A)(y',x)} f(x) \, dx
\]

Here we have defined

\[
(\tilde{\tau}kA)(y', x) = (\tau A)(y', y) + (\tau_k A)(y, x) = \sum_{j=0}^{k} (\tau A)(y_{j+1}, y_j)
\]

where $y_{k+1} = y'$, $y_k = y$, $y_0 = x$. Now we scale by $L^3$ and get for $y' \in \mathbb{T}_{N-k-1}^0$ and $x' \in \mathbb{T}_{N-k-1}^{-k-1}$

\[
(Q_{k+1}(A)f)(y') = L^{-3} \int_{|x-y'|<L/2} e^{q_k \eta(\tilde{\tau}_{k+1} A_L)(Ly', x)} f(x/L) \, dx
\]

\[
= \int_{|x'-y'|<\frac{1}{2}} L^{-3} e^{q_k \eta(\tilde{\tau}_{k+1} A_L)(Ly', Lx')} f(x') \, dx'
\]

Taking into account that $A_L(\Gamma) = L^{-\frac{k}{2}} A(L^{-1} \Gamma)$ we have

\[
e^{q_k \eta(\tilde{\tau}_{k+1} A_L)(Ly', Lx')} = e^{q_k \eta \sum_{j=0}^{k} (\tau A_L)(y_{j+1}, y_j)} \bigg|_{y_{k+1} = Ly', y_0 = Lx'}
\]

\[
= e^{q_k \eta L^{-\frac{k}{2}} \sum_{j=0}^{k} (\tau A)(L^{-1} y_{j+1}, L^{-1} y_j)} \bigg|_{y_{k+1} = Ly', y_0 = Lx'}
\]

\[
= e^{q_k L^{-k-1}(\tau_{k+1} A)(y', x')}
\]

This gives the first result. The expression for the adjoint is a short calculation.

Lemma 3. For $A$ on $\mathbb{T}_{N-k}^{-k}$ and $\Phi_k$ on $\mathbb{T}_{N-k}^{0}$ the density $\rho_{k,A}(\Phi_k)$ can be written

\[
\rho_{k,A}(\Phi_k) = \left(\frac{a_k}{2\pi}\right)^{s_{N-k}} \int \exp \left( -\frac{a_k}{2} \|\Phi_k - Q_k(A)\phi\|^2 \right) \rho_{0,A_L}(\phi_{L^k}) \, D\phi
\]

where $\phi, A$ are on $\mathbb{T}_{N-k}^{-k}$ and

\[
a_k = a \frac{1 - L^{-2k}}{1 - L^{-2k}}
\]
We compute the value at the minimum using Equation (34) we have for $A$ the constant $\Phi$ must be $a^2.4$ free flow over $\rho$. Insert this in (47) and use for $A$:

$$\text{compute } \min_{\Phi} \text{expression inside the exponential.}$$

This has the solution $\Phi_k = \Phi_k^{\text{min}}(A) = \Phi_k^{\text{min}}(A; k, \Phi_k)$ where

$$\Phi_k^{\text{min}}(A) = Q_k(A) - \frac{aL^2}{a_k + aL^2}Q^T(A)Q_{k+1}(A)\phi + \frac{aL^2}{a_k + aL^2}Q^T(A)\Phi_{k+1}$$

We compute the value at the minimum using $Q(A)Q^T(A) = 1$ and $a_{k+1} = a_k/(a_k + aL^2)$ and find (see [24] for details)

$$\frac{a}{2L^2}\|\Phi_k - Q(A)\Phi_k^{\text{min}}(A)\|^2 + \frac{1}{2}a_k\|\Phi_k^{\text{min}}(A) - Q_k(A)\phi\|^2 = \frac{a_{k+1}}{2L^2}\|\Phi_{k+1} - Q_{k+1}(A)\phi\|^2$$

Now in the integral (49) expand around the minimizer. We write $\Phi_k = \Phi_k^{\text{min}}(A) + Z$ and integrate over $Z$. The term with no $Z$’s is (52). The linear terms in $Z$ vanish and the terms quadratic in $Z$ when integrated over $Z$ yield a constant. Thus we have

$$\hat{\rho}_{k+1, A}(\Phi_{k+1}) = \text{const} \int \exp \left( -\frac{1}{2L^2}\|\Phi_{k+1} - Q_{k+1}(A)\phi\|^2 \right) \rho_{0, A_L_k}(\phi_{L_k}) \, D\phi$$

Scaling by (34) we have for $A, \phi'$ on $T_{N-k-1}^*$ and $\Phi_{k+1}$ on $T_{N-k-1}^*$

$$\rho_{k+1, A}(\Phi_{k+1}) = \text{const} \int \exp \left( -\frac{1}{2L^2}\|\Phi_{k+1} - Q_{k+1}(A)\phi'\|^2 \right) \rho_{0, A_L_{k+1}}(\phi_{L_{k+1}}') \, D\phi'$$

The constant must be $(a_{k+1}/2\pi)^{N-k}$ to preserve the identity (30). This completes the proof.

Hereafter we abbreviate the normalization factors in (30) and (47) by

$$N_k = \left(\frac{aL}{2\pi}\right)^{N-k} \quad N_k = \left(\frac{a_k}{2\pi}\right)^{N-k}$$

### 2.4 free flow

Now consider an initial density which is a perturbation of the free action:

$$\rho_0(A, \Phi_0) = F_0(\Phi_0) \exp \left( -\frac{1}{2}\|\partial_A \Phi_0\|^2 \right)$$

Insert this in (47) and use for $A, \phi$ on $T_{N-k}^*$

$$\frac{1}{2}\|\partial_A \phi_{L_k}\|^2 = \frac{1}{2}\|\partial_A \phi\|^2 = \frac{1}{2} < \phi, (-\Delta_A) \phi >$$

where $-\Delta_A \equiv \partial_A^2 \partial_A$ is defined with covariant derivatives containing the coupling constant $e_k$. Then with $F_{0, L-k}(\phi) = F_0(\phi_{L_k})$ we have from (47)

$$\rho_k(A, \Phi_k) = N_k \int F_{0, L-k}(\phi) \exp \left( -\frac{ak}{2}\|\Phi_k - Q_k(A)\phi\|^2 - \frac{1}{2} < \phi, (-\Delta_A) \phi > \right) \, D\phi$$
The minimizer in $\phi$ of the expression in the exponential is

$$\mathcal{H}_k(\mathcal{A})\Phi_k \equiv a_k G_k(\mathcal{A}) Q_k^T(\mathcal{A}) \Phi_k$$  \hspace{1cm} (59)

where $G_k(\mathcal{A})$ is the Green’s function

$$G_k(\mathcal{A}) = \left( -\Delta_\mathcal{A} + a_k Q_k^T(\mathcal{A}) Q_k(\mathcal{A}) \right)^{-1}$$  \hspace{1cm} (60)

The inverse exists since this is a strictly positive operator.

Expanding the exponential around the minimizer with $\Phi_k$ we find

$$\rho_{k,\mathcal{A}}(\Phi_k) = N_k Z_k(\mathcal{A}) F_k \left( \mathcal{H}_k(\mathcal{A})\Phi_k \right) \exp \left( -\frac{1}{2} < \Phi_k, \Delta_k(\mathcal{A})\Phi_k > \right)$$  \hspace{1cm} (61)

where

$$< \Phi_k, \Delta_k(\mathcal{A})\Phi_k > = a_k \frac{1}{2} \| \Phi_k - Q_k(\mathcal{A}) \mathcal{H}_k(\mathcal{A}) \Phi_k \|^2 + \frac{1}{2} < \mathcal{H}_k(\mathcal{A})\Phi_k, (-\Delta_\mathcal{A}) \mathcal{H}_k(\mathcal{A})\Phi_k >$$

$$\Phi_k, \left( a_k - a_k^2 Q_k^T(\mathcal{A}) G_k(\mathcal{A}) Q_k(\mathcal{A}) \right) \Phi_k \right)$$

$$F_k \left( \mathcal{H}_k(\mathcal{A})\Phi_k \right) = Z_k(\mathcal{A})^{-1} \int F_{0,L-k} \left( \mathcal{H}_k(\mathcal{A})\Phi_k + Z \right)$$

$$\exp \left( -\frac{1}{2} < Z, \left( -\Delta_\mathcal{A} + a_k Q_k^T(\mathcal{A}) Q_k(\mathcal{A}) \right) Z > \right)$$

$$Z_k(\mathcal{A}) = \int \exp \left( -\frac{1}{2} < Z, \left( -\Delta_\mathcal{A} + a_k Q_k^T(\mathcal{A}) Q_k(\mathcal{A}) \right) Z > \right) DZ$$

### 2.5 the next step

If we start with the expression (61) for $\rho_{k,\mathcal{A}}$ and apply another renormalization transformation we again get $\rho_{k+1,\mathcal{A}}$. Working out the details will give us some useful identities. We have first

$$\hat{\rho}_{k+1}(\mathcal{A}, \Phi_{k+1}) = N_k N_k Z_k(\mathcal{A})$$

$$\int F_k \left( \mathcal{H}_k(\mathcal{A})\Phi_{k+1} \right) \exp \left( -\frac{1}{2L^2} \| \Phi_{k+1} - Q(\mathcal{A}) \Phi_k \|^2 - \frac{1}{2} < \Phi_k, \Delta_k(\mathcal{A})\Phi_k > \right) D\Phi_k D\phi$$  \hspace{1cm} (63)

Here $\Phi_{k+1}, \Phi_k$ are fields on $T^{1}_{N-k}, T^{0}_{N-k}$ respectively. The minimizer of the expression in the exponential in $\Phi_k$ is

$$H_k(\mathcal{A})\Phi_{k+1} \equiv \frac{a}{L^2} C_k(\mathcal{A}) Q_k^T(\mathcal{A})\Phi_{k+1}$$  \hspace{1cm} (64)

where

$$C_k(\mathcal{A}) = \left( \Delta_k(\mathcal{A}) + \frac{a}{L^2} Q_k^T(\mathcal{A}) Q(\mathcal{A}) \right)^{-1}$$  \hspace{1cm} (65)

Expanding around the minimizer with $\Phi_k = H_k(\mathcal{A})\Phi_{k+1} + Z$ we obtain

$$\hat{\rho}_{k+1}(\mathcal{A}, \Phi_k) = N_k N_k Z_k(\mathcal{A}) Z^T_k(\mathcal{A})$$

$$F_k \left( \mathcal{H}_k(\mathcal{A}) H_k(\mathcal{A}) \Phi_{k+1} \right) \exp \left( -\frac{1}{2} < H_k(\mathcal{A})\Phi_{k+1}, \Delta_k(\mathcal{A}) H_k(\mathcal{A})\Phi_{k+1} > \right)$$  \hspace{1cm} (66)

Here

$$Z^T_k(\mathcal{A}) = \int \exp \left( -\frac{1}{2} < Z, \left( \Delta_k + \frac{a}{L^2} Q^T(A)(A) \right) Z > \right) DZ$$  \hspace{1cm} (67)
and

\[ F^*_k\left(\mathcal{H}_k(A)\mathcal{H}_k(A)\Phi_{k+1}\right) = Z_k^2(A)^{-1} \int F^*_k\left(\mathcal{H}_k(A)\mathcal{H}_k(A)\Phi_{k+1} + \mathcal{H}_k(A)Z\right) \exp\left(-\frac{1}{2} < Z, \left(\Delta_k + \frac{a}{L^2}Q^T(A)Q(A)\right)Z >\right) \]

(68)

\[ = \int F^*_k\left(\mathcal{H}_k(A)\mathcal{H}_k(A)\Phi_{k+1} + \mathcal{H}_k(A)Z\right) d\mu_{C_k(A)}(Z) \]

where \(\mu_{C_k(A)}\) is the Gaussian measure with covariance \(C_k(A)\).

Next we scale by (34) and get

\[ \rho_{k+1}(A, \Phi_{k+1}) = N_k N_k Z_k(A_L)Z_k^T(A_L) L^{sN-sN-k-1} F^*_k\left(\mathcal{H}_k(A_L)\mathcal{H}_k(A_L)\Phi_{k+1,L}\right) \exp\left(-\frac{1}{2} < \Phi_{k+1,L}, \Delta_{k+1}(A)\Phi_{k+1,L} >\right) \]

(69)

Taking \(F_0 = 1\) we have \(F_k = 1\) and \(F_k^* = 1\). Then taking \(\Phi_{k+1} = 0\) and comparing this expression with (61) for \(k + 1\) we find

\[ N_{k+1} Z_{k+1}(A) = N_k N_k Z_k(A_L)Z_k^T(A_L) L^{sN-sN-k-1} \]

(70)

Furthermore the exponential must be \(\exp\left(-\frac{1}{2} < \Phi_{k+1,L}, \Delta_{k+1}(A)\Phi_{k+1,L} >\right)\). Thus in general

\[ \rho_{k+1,A}(\Phi_{k+1}) = N_{k+1} Z_{k+1}(A) F^*_k\left(\mathcal{H}_k(A_L)\mathcal{H}_k(A_L)\Phi_{k+1,L}\right) \exp\left(-\frac{1}{2} < \Phi_{k+1,L}, \Delta_{k+1}(A)\Phi_{k+1,L} >\right) \]

(71)

Comparing this with (61) for \(k + 1\) we find

\[ F^*_k\left(\mathcal{H}_k(A_L)\mathcal{H}_k(A_L)\Phi_{k+1,L}\right) = F_{k+1}\left(\mathcal{H}_k(A)\Phi_{k+1}\right) \]

(72)

Next take \(F_0(\Phi_0) = < \Phi_0, f >\). Then \(F_k(\phi) = < \phi_L, f >\) for all \(k\) and \(F_k^*(\phi) = < \phi_L, f >\) for all \(k\), and (72) says

\[ < \left(\mathcal{H}_k(A_L)\mathcal{H}_k(A_L)\Phi_{k+1,L}\right)_{L^k}, f > = < \left(\mathcal{H}_k(A)\Phi_{k+1}\right)_{L^{k+1}}, f > \]

(73)

and so

\[ \mathcal{H}_k(A_L)\mathcal{H}_k(A_L)\Phi_{k+1,L} = (\mathcal{H}_k(A)\Phi_{k+1})_{L} \]

(74)

Now (68) evaluated at \(\Phi_{k+1,L}\) can be written

\[ F_{k+1}(\mathcal{H}_k(A)\Phi_{k+1}) = \int F_k\left(\mathcal{H}_{k+1}(A)\Phi_{k+1} + \mathcal{H}_k(A)Z\right) d\mu_{C_k(A)}(Z) \]

(75)

More generally for any \(\phi\) on \(T_{N-k-1}\) one can define the fluctuation integral

\[ F_{k+1}(\phi) = \int F_k\left(\phi_L + \mathcal{H}_k(A)Z\right) d\mu_{C_k(A)}(Z) \]

(76)

The identities (70), (74), (76) are what we were after.

### 3 Greens functions

We study the Green’s function \(G_k(A) = \left(-\Delta_A + a_k(Q^T_k Q_k)(A)\right)^{-1}\), an operator on functions on \(T_{N-k}\) defined for a background field \(A\) on \(T_{N-k}\). These results are mostly due to Balaban [1], but there are some minor differences.
3.1 basic properties

We collect some general facts. As before the Laplacian is \(-\Delta_A = \partial_A^T \partial_A\) where with \(\eta = L^{-k}\)

\[
(\partial_A, \mu, f)(x) = (e^{\eta e_k A, \mu} (x) f(x + \eta e_\mu) - f(x)) \eta^{-1} \\
(\partial_A^T, \mu, f)(x) = (e^{-\eta e_k A, \mu} (x - \eta e_\mu) f(x - \eta e_\mu) - f(x)) \eta^{-1}
\]  
(77)

Note that these differ by a phase factor for we have

\[
(\partial_A^T) f(x) = - e^{-\eta e_k A, \mu} (x - \eta e_\mu) (\partial_A, \mu, f)(x - \eta e_\mu)
\]  
(78)

Explicitly

\[
(-\Delta_A) f(x) = \sum_\mu \left( - e^{\eta e_k A, \mu} (x) f(x + \eta e_\mu) + 2 f(x) - e^{-\eta e_k A, \mu} (x - \eta e_\mu) f(x - \eta e_\mu) \right) / \eta^2
\]  
(79)

We note for later reference the product rules:

\[
\partial_A (h f) = h(\cdot + \eta e_\mu) \partial_A f + (\partial h) f \\
\partial_A^T (h f) = h(\cdot - \eta e_\mu) \partial_A^T f + (\partial h) f
\]  
(80)

We also record the symmetries of the Green’s functions. The Laplacian \(\Delta_A\) is covariant under \(T_{N-k}^{-1}\) lattice symmetries and \((Q_k^T Q_k) (A)\) is covariant under \(T_{N-k}^0\) lattice symmetries. Hence \(G_k (A)\) are covariant under \(T_{N-k}^0\) lattice symmetries which means

\[
G_k (A) f_r = (G_k (A) f)_r
\]  
(81)

With gauge transformation \(\lambda\) on \(T_{N-k}^{-1}\) defined as in (12) we have

\[
\Delta_A \lambda = e^{\eta e_k A} \Delta_A e^{-\eta e_k A} \\
Q_k (A^\lambda) = e^{\eta e_k \lambda^{(0)}} Q_k (A) e^{-\eta e_k A}
\]  
(82)

where \(\lambda^{(0)}\) is the restriction to the unit lattice \(T_{N-k}^0\). It follows that

\[
G_k (A^\lambda) = e^{\eta e_k \lambda} G_k (A) e^{-\eta e_k A} \\
H_k (A^\lambda) = e^{\eta e_k \lambda} H_k (A) e^{-\eta e_k \lambda^{(0)}}
\]  
(83)

Similarly we have the charge conjugation invariance

\[
G_k (-A) = C G_k (A) C
\]  
(84)

We also consider the Green’s function for a region \(\Omega \subset T_{N-k}^{-1}\). This has the form

\[
G_k (\Omega, A) = \left( -\Delta_A^N + a_k (Q_k^T Q_k) (A) \right)^{-1}_\Omega
\]  
(85)

The notation \((-\Delta_A^N)_\Omega\) denotes the Laplacian with Neumann boundary conditions, i.e. as a quadratic form, \(< f, (-\Delta_A^N)_\Omega f > = ||\partial_A f||^2_\Omega\), only bonds contained in \(\Omega\) contribute. Thus \((\partial_A, \mu, f)(x)\) is given by (77) if \(x, x + \eta e_\mu \in \Omega\) and is zero otherwise. We still have \((-\Delta_A^N)_\Omega = \partial_A^T \partial_A\) but now \((\partial_A^T, \mu, f)(x)\) is given by (77) if \(x - \eta e_\mu, x \in \Omega\) and is zero otherwise. The expression (79) for the Laplacian must be modified near the boundary.

The operator \(G_k (\Omega, A)\) has the same symmetry properties as \(G_k (\Omega)\), provided \(\Omega\) is transformed as well for lattice symmetries.
3.2 changes in background field

On $T_{N-k}^{-k}$ we consider changing from a background field $A$ to a background field $A + A'$ by studying

$$U_k(A, A') \equiv \left( - \Delta_A + A_k(Q_k^T Q_k)(A + A') \right) - \left( - \Delta_A + A_k(Q_k^T Q_k)(A) \right)$$

Define

$$(F_\mu(A))(x) = \left( e^{q_{\epsilon_\mu} \eta A_\mu(x)} - 1 \right) \eta^{-1}$$

Then we have

$$(\partial_A A_\mu f)(x) = e^{q_{\epsilon_\mu} \eta A_\mu(x)} \left((\partial_A f)(x) - (F_\mu(-A'))(x)f(x)\right)$$

We also define

$$(F^q(A, A')f)(y) = \int_{|x-y| < \frac{1}{4}} \left( e^{q_{\epsilon_\mu} \eta (\tau_k A')(y,x)} - 1 \right) e^{q_{\epsilon_\mu} \eta (\tau_k A')(y,x)} f(x) \, dx$$

and then

$$Q_k(A + A') = Q_k(A) + F^q(A, A')$$

Expanding $\partial_A A_\mu$ and $\partial^T_A A_\mu$ by (88) and $Q_k(A + A')$ by (90) we find

$$U_k(A, A') = - F^T(-A') \cdot \partial_A - \partial^T_A F(-A' \cdot F(-A') + a_k F_{\epsilon_\mu}(A, A') Q_k(A) + a_k Q_k^T(A, A') F^q(A, A') + a_k F_{\epsilon_\mu}(A, A' \cdot F^q(A, A')$$

On a function $f$ the second term is by (90)

$$\sum_\mu \left( \partial^T_A A_\mu f \right)(x) = \sum_\mu (F_\mu(-A'))(x - \eta_{\epsilon_\mu})(\partial^T_A f)(x) + \sum_\mu \left( \partial^T_A F_\mu(-A') \right)(x)f(x)$$

The pair (91), (92) gives our final representation of $U_k(A, A')f$.

For the next results let $\Delta = \Delta_y \subset T_{N-k}^{-k}$ be a unit cube centered on a unit lattice point $y$ and let $\Delta$ be the enlargement to a cube with three unit cubes on a side.

**Lemma 4.** Let $A, A'$, $f$ be complex valued fields on $T_{N-k}^{-k}$ satisfying $e_k|\text{Im} A|, e_k|\text{Im} A'| \leq 1$. Then for $x \in \Delta = \Delta_y$

$$\|U_k(A, A')f\|_{\Delta, \infty} \leq O(1)e_k \left( \|A'\|_{\Delta, \infty} + \|A'\|_{\Delta, \infty} \right) \left( \|f\|_{\Delta, \infty} + \|\partial A f\|_{\Delta, \infty} \right)$$

$$\|U_k(A, A')f\|_{\Delta, 2} \leq O(1)e_k \left( \|A'\|_{\Delta, \infty} + \|A'\|_{\Delta, \infty} \right) \left( \|f\|_{\Delta, 2} + \|\partial A f\|_{\Delta, 2} \right)$$

where $\|\partial A\|_{\Delta, \infty} = \sup_{\mu, \nu} \sup_{x \in \Delta} |\partial A_{\mu}(x)|$, etc.

**Proof.** We give the proof for the $L^\infty$ norm, the proof for the $L^2$ norm is very similar. Consider the various terms in (91). We write for $x \in \Delta$

$$(F_\mu(-A'))(x) = \left( e^{q_{\epsilon_\mu} \eta A_\mu'(x)} - 1 \right) \eta^{-1} = - \int_0^1 dt \ e^{-t q_{\epsilon_\mu} \eta A_\mu'(x)} q_{\epsilon_\mu} A_\mu'(x)$$

For $v \in \mathbb{C}^2$

$$|e^{-t q_{\epsilon_\mu} \eta A_\mu'(x)}| = |e^{-t q_{\epsilon_\mu} \eta \text{Im} A_\mu'(x)}| \leq e^{\epsilon_\mu \eta |\text{Im} A_\mu'(x)|} \leq O(1)|v|$$

$$|e^{-t q_{\epsilon_\mu} \eta A_\mu'(x)}| \leq O(1)|v|$$

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and this gives the estimate

\[ |(F_\mu(-A\prime))(x)v| \leq O(1)e_k |A\prime(x)||v| \leq O(1)e_k |A\prime|_{\Delta,\infty}|v| \]  

(96)

The same holds for the transpose \( F_1^T(-A\prime) \).

Next we note that since \((\tau A)(y_j+1, y_j)\) is an average over paths of length less than \(1/2L^{-(k-j-1)}\) we have

\[ e_k \eta|\text{Im} \ (\tau_k A)(y, x)| \leq \sum_{j=0}^{k-1} e_k \eta|\text{Im} \ (\tau A)(y_j+1, y_j)| \leq \frac{1}{2} \sum_{j=0}^{k-1} L^{-(k-j-1)} \leq 1 \]  

(97)

It follows that

\[ |(Q_k(A)f)(y)| \leq O(1)|f|_{\Delta,\infty} \]  

(98)

The adjoint \( Q^T(A) \) satisfies the same bound. We also have \( e_k \eta|\text{Im} (\tau_k A')(y, x)| \leq e_k |A'|_{\Delta,\infty} \) and this gives

\[ |(F'(A,A')f)(y)| \leq O(1)e_k |A'|_{\Delta,\infty}|f|_{\Delta,\infty} \]  

(99)

and similarly for the adjoint.

Now consider the terms in (92). These terms involve points just outside \(\Delta\) which we accommodate by enlarging to \(\tilde{\Delta}\). In particular for the first term in (92) we have by (78)

\[ |(F_\mu(-A')(x - \eta e_\mu)v| \leq O(1)e_k |A'(x - \eta e_\mu)v| \leq O(1)e_k |A'|_{\Delta,\infty}|v| \] \[ |(\partial_{A,\mu}^T f)(x)| \leq O(1)|\partial_{A,\mu} f|(x - \eta e_\mu) | \leq O(1)|\partial_{A,\mu} f|_{\Delta,\infty} \]  

(100)

Finally for the second term in (92)

\[ |(\partial_{A}^T F_\mu(-A')(x)v) = |(F_\mu(-A', x - \eta e_\mu) - F_\mu(-A', x))v| \eta^{-1} \] \[ = |(e^{-q_\mu x n A'}(x - \eta e_\mu) - e^{-q_\mu x n A'}(x))v| \eta^{-2} \leq O(1)e_k |(A'(x - \eta e_\mu) - A'(x))v| \eta^{-1} \] \[ \leq O(1)e_k |\partial A'|_{\Delta,\infty}|v| \]  

(101)

Now all the terms in \((U_k(\Omega, A, A')f)(x)\) can be estimated and we have the result.

**Remark.** Let \(\Omega \subset \mathbb{T}^{-k}_{N-k}\) which is a union of unit cubes. Consider the difference with Neuman boundary conditions on \(\Omega\).

\[ U_k(\Omega, A, A') = \left[ -\Delta_{A+A'} + a_k(Q_k^T Q_k)(A+A') \right] \Omega - \left[ -\Delta_A + a_k(Q_k^T Q_k)(A) \right] \Omega \]  

(102)

The representation \([91], [92]\) still holds but now everything is restricted to \(\Omega\). The estimate \([93]\) still holds, but the enlargement \(\tilde{\Delta}\) only adds cubes in \(\Omega\).

### 3.3 Local estimates

Partition the lattice \(\mathbb{T}^{-k}_{N-k}\) into large cubes \(\square\) of linear size \(M = L^m\) centered on points in \(\mathbb{T}^m_{N-k}\) for some integer \(m > 1\). Let \(\square\) be cube of linear size \(3M\) centered on the same points.

We quote some estimates on the local Green’s functions \(G_k(\square, A)\). We want to bound \(G_k(\square, A), \partial A G_k(\square, A),\) and a certain Holder derivative \(\delta_{A,A} \partial A G_k(\square, A)\). The Holder derivative for \(0 < \alpha < 1\) is defined by

\[ (\delta_{A,A} f)(x,y) = \frac{e^{q_\mu y n A(x,y)} f(y) - f(x)}{|x-y|^\alpha} \]  

(103)

where again \(\Gamma_{xy}\) is one of the standard paths from \(x\) to \(y\). There is an associated norm

\[ \|\delta_{A,A} f\|_{\infty} = \sup_{|x-y| \leq 1} |(\delta_{A,A} f)(x,y)| \]  

(104)
Lemma 5. Let $e_k$ be sufficiently small depending on $L, M$. Let $A$ on $\square$ be real-valued and gauge equivalent to a field $A'$ ($A \sim A'$) satisfying $|A'|, |\partial A'| < e_k^{-1+\epsilon}$ for some small positive constant $\epsilon$.

1. With Holder derivative $\delta_{\alpha,A}$ of order $\alpha < 1$

$$\|G_k(\square, A)f\|, \|\partial_A G_k(\square, A)f\|, \|\delta_{\alpha,A} \partial_A G_k(\square, A)f\|_\infty \leq C\|f\|_\infty \quad (105)$$

2. Let $\Delta_y, \Delta_{y'}$ be unit squares centered on unit lattice points $y, y' \in \square$ and let $\zeta_y$ be a smooth partition on unity with supp $\zeta_y \subset \Delta_{y'}$. Then for a constants $C, \gamma$

$$|1_{\Delta_y} G_k(\square, A) 1_{\Delta_{y'}}, f|, \quad |1_{\Delta_y} \partial_A G_k(\square, A) 1_{\Delta_{y'}}, f|, \quad \|\delta_{\alpha,A} \zeta_y \partial_A G_k(\square, A) 1_{\Delta_{y'}}, f\|_\infty \leq C e^{-\gamma d(y, y')}\|f\|_\infty \quad (106)$$

3. The same bounds hold with the $L^2$ norm replacing the $L^\infty$ norm.

Proof. The result holds for $A = 0$ see [4], [24]. The $L^2$ result for $A = 0$ is actually an input for the $L^\infty$ result. The $L^2$ result can be found for example as a special case of lemma 34 in [24].

For the general case if $A' = A - \partial \lambda$ then

$$G_k(\square, A) = e^{-qe_k \lambda} G_k(\square, A') e^{qe_k \lambda}$$

$$\partial_A G_k(\square, A) = e^{-qe_k \lambda} \partial A G_k(\square, A') e^{qe_k \lambda}$$

$$\delta_{\alpha,A} \partial_A G_k(\square, A) = e^{-qe_k \lambda} \delta_{\alpha} \partial A G_k(\square, A') e^{qe_k \lambda} \quad (107)$$

Thus it suffices to prove the result with $A'$.

The Green’s function $G_k(\square, A')$ satisfies

$$G_k(\square, A') = G_k(\square, 0) - G_k(\square, 0) U_k(0, A') G_k(\square, A') \quad (108)$$

and so is given by

$$G_k(\square, A) = G_k(\square, 0) \sum_{n=0}^\infty \left(- U_k(0, A') G_k(\square, 0)\right)^n \quad (109)$$

provided the the series converges, which we now establish. It follows from [23] and our hypotheses on $A'$ that

$$|U_k(\square, 0, A') f| \leq O(1) e_k \left(\|f\|_\infty + \|\partial f\|_\infty\right) \quad (110)$$

Then by the result for $G_k(\square, 0)$

$$|U_k(\square, 0, A') G_k(\square, 0) f| \leq O(1) e_k \left(\|G_k(\square, 0) f\|_\infty + \|\partial G_k(\square, 0) f\|_\infty\right) \leq C e_k \|f\|_\infty \quad (111)$$

Since \(\sum_n (C e_k)^n\) converges for $e_k$ small, this is sufficient to establish the convergence of (109) and give (105).

Next using the estimate on $U_k(0, A')$ and the local estimate on $G_k(\square, 0)$ we can establish a local version of (111)

$$|1_{\Delta_y} U_k(\square, 0, A') G_k(\square, 0) 1_{\Delta_{y'}}, f|$$

$$\leq O(1) e_k \left(\|1_{\Delta_y} G_k(\square, 0) 1_{\Delta_{y'}}, f\|_\infty + \|1_{\Delta_{y'}} \partial G_k(\square, 0) 1_{\Delta_{y'}}, f\|_\infty\right)$$

$$\leq C e_k \sum_{|y'' - y| \leq 1} e^{-\gamma d(y'', y')} \|f\|_\infty$$

$$\leq C e_k e^{-\gamma d(y, y')} \|f\|_\infty \quad (112)$$

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Now we have
\[
G_k(\square, A) = \sum_{n=0}^{\infty} \prod_{y_1, \ldots, y_{n-1}} G_k(\square, 0) 1_{A_1} \left( - U_k(\square, 0, A') G_k(\square, 0) \right)
\]  \hspace{1cm} (113)

Then with \( y_n = y' \) the estimate (112) gives
\[
|1_{A_1} G_k(\square, A) 1_{A_1} f| \leq \sum_{n=0}^{\infty} \prod_{j=1}^{n} Ce^{-\gamma d(y, y_j)} \prod_{j=1}^{n} Ce^{-\gamma d(y_{n-1}, y_n)} \|f\|_\infty
\]
\[
\leq Ce^{-\frac{1}{2} \gamma d(y,y')} \sum_{n=0}^{\infty} (Ce_k')^n \|f\|_\infty \leq Ce^{-\frac{1}{2} \gamma d(y,y')} \|f\|_\infty
\]  \hspace{1cm} (114)

Thus the bound holds with a new \( \gamma \). The estimate on derivatives is similar. This completes the proof.

Next we extend the previous result to complex fields \( A \) on \( T_{\gamma-k} \) of the form
\[
A = A_0 + A_1
\]
\[
A_0 \text{ is real and on each } \square \text{ admits } A_0' \sim A_0 \text{ satisfying } |A_0'|, |\partial A_0'| < e_k^{-1+\varepsilon},
\]
\[
A_1 \text{ is complex and satisfies } |A_1|, |\partial A_1| < e_k^{-1+\varepsilon}
\]  \hspace{1cm} (115)

This is an open set in some \( \mathbb{C}^n \) and we can consider functions analytic in this domain.

**Lemma 6.** Under the same hypotheses \( G_k(\square, A) \) has an analytic extension to the region (112), and for such fields \( G_k(\square, A) \) again satisfies bounds of the form (105), (106).

**Proof.** We again have
\[
G_k(\square, A) = G_k(\square, A_0) \sum_{n=0}^{\infty} \left( - U_k(\square, A_0, A_1) G_k(\square, A_0) \right)^n
\]  \hspace{1cm} (116)

and by lemma 4 and lemma 5
\[
|U_k(\square, A_0, A_1) G_k(\square, A_0) f| \leq Ce_k' \|f\|_\infty
\]  \hspace{1cm} (117)

which gives (105). The bound (112) also holds, and the local version (106) follows as before.

### 3.4 Random Walk Expansion

We study the global Green’s functions \( G_k(A) \) by a random walk expansions.

Again partition the lattice \( T_{\gamma-k} \) into cubes \( \square \) of linear size \( M = L^m \). We write \( T_{\gamma-k} = \bigcup_{z} \square_z \) where \( z \) is a point on the \( M \) lattice \( T_{\gamma-k}^m \) and \( \square_z \) is the \( M \) cube centered on \( z \). Let \( \square_z \) be the 3\( M \) cube centered on \( z \). The random walk expansion is based on the operators \( G_k(\square_z, A) \), discussed previously. We assume that \( A \) is in the domain (115) so that these have good estimates by lemma 6.

Let \( h_z^2 \) be a partition of unity with \( \sum_z h_z^2 = 1 \) and supp \( h_z \) well inside \( \square_z \). We define a parametrix
\[
G^*_k(A) = \sum_z h_z G_k(\square_z, A) h_z
\]  \hspace{1cm} (118)
On supp $h_z$ there is no distinction between $-\Delta_A$ and $[-\Delta_A^N]|_{\mathbb{Z}^d}$ and so we can compute

$$(-\Delta_A + a_k P_k(A))G_k^*(A) = I - \sum z K_z(A)G_k^*(\mathbb{Z}_z, A)h_z = I - K$$

where

$$K_z(A) = -\left[-\Delta_A + a_k (Q_k^T Q_k)(A), h_z\right]$$

Then

$$G_k(A) = G_k^*(A)(I - K)^{-1} = G_k^*(A) \sum_{n=0}^{\infty} K^n$$

if it converges. This can be written as the random walk expansion

$$G_k(A) = \sum_{\omega} G_{k,\omega}(A)$$

where a path $\omega$ is a sequence of points $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ in $\mathbb{T}_{N-k}^d$ such that $\omega_i, \omega_{i+1}$ are nearest neighbors (in a sup metric), and

$$G_{k,\omega}(A) = h_{\omega_0} G_k(\mathbb{Z}_{\omega_0}, A)h_{\omega_0} K_{w_1}(A)G_k(\mathbb{Z}_{\omega_1}, A)h_{\omega_1} \cdots K_{\omega_n}(A)G_k(\mathbb{Z}_{\omega_n}, A)h_{\omega_n}$$

Note that $G_{k,\omega}(A)$ only depends on $A$ in the set $\bigcup_{i=0}^{n} \mathbb{Z}_{\omega_i}$

**Lemma 7.** Let $M$ be sufficiently large (depending on $L$), and $e_k$ sufficiently small (depending on $L, M$), and let $A$ be in the domain $\{1\}$. Then

1. The random walk expansion (122) for $G_k(A)$ converges to a function analytic in $A$ which satisfies $|G_k(A)f|$, $|\partial A G_k(A)f|, \|\delta_{\alpha,A} \partial A G_k(A)f\|_\infty \leq C\|f\|_\infty$ (124)

2. Let $\Delta_y, \Delta_{y'}$ be unit squares centered on unit lattice points $y, y' \in \mathbb{T}_{N-k}^d$ and let $\zeta_y$ be a smooth partition on unity with supp $\zeta_y \subset \Delta_{y'}$. Then there are constants $C, \gamma$ so

$$|\Delta_y G_k(A)\Delta_{y'} f|, |\Delta_y \partial A G_k(A)\Delta_{y'} f|, \|\delta_{\alpha,A} \zeta_y \partial A G_k(A)\Delta_{y'} f\|_\infty \leq C e^{-\gamma d(y,y')}\|f\|_\infty$$

(125)

3. The same bounds hold with the $L_2$ norm replacing the $L_\infty$ norm.

**Remark.** The same bounds hold for $\mathcal{H}_k(A)$, for example

$$|\mathcal{H}_k(A)f|, |\partial A \mathcal{H}_k(A)f|, \|\delta_{\alpha,A} \partial A \mathcal{H}_k(A)f\|_\infty \leq C\|f\|_\infty$$

(126)

**Proof.** [4]. We give the proof for the $L_\infty$ norm. We compute using (80)

$$\left([-\Delta_A, h_z] f\right)(x) = (\partial^T h)(x + \eta e_\mu) \cdot \partial A f(x) + (\partial h_z)(x - \eta e_\mu) \cdot \partial A f(x) + (-\Delta h_z)(x) f(x)$$

(127)

and with $x \in \Delta_y$

$$\left([Q_k^T Q_k], h_z] f\right)(x) = \int_{|x' - y| \leq \frac{1}{2}} e^{-\eta e_\mu \eta(\tau_A)(y,x')}e^{\eta e_\mu \eta(\tau_A)(y,x')}\left(h(x'') - h(x)\right) f(x')dx'$$

(128)
The functions \( \{ h_z \} \) can be chosen so that \( |\partial h_z| \leq O(1)M^{-1} \) and \( |\partial h| \leq O(1)M^{-2} \). Then the representations (127), (128) lead to the bound
\[
|K_z(A)f| \leq O(1)M^{-1}(\|f\|_\infty + \|\partial_A f\|_\infty) \tag{129}
\]
and therefore by (105)
\[
|K_z(A)G_k(\bar{\Box}_z, A)f| \leq CM^{-1}\|f\|_\infty \tag{130}
\]
These imply that if \( |\omega| = n \) then
\[
|G_{k,\omega}(A)f| \leq C(M^{-1})^n\|f\|_\infty \tag{131}
\]
This is sufficient to establish the convergence of the expansion for \( M \) large, since the number of paths with a fixed length \( n \) is bounded by \( (3d)^n = 9^n \). The bounds on derivatives follow as well.

For the local estimates use the locality of \( K_z(A) \) and (106) to obtain
\[
|\Delta y K_z(A)G_k(A, \bar{\Box}_z)\Delta y f| \leq CM^{-1}e^{-\gamma d(y, y')}\|f\|_\infty \tag{132}
\]
Proceeding as in lemma 5 we have the result with a new \( \gamma \).

**Remark.** We introduce weakening parameters \( \{ s_{\Box} \} \) with \( 0 \leq s_{\Box} \leq 1 \) and define
\[
s_{\omega} = \prod_{\Box \in X_\omega} s_{\Box} \quad X_\omega = \bigcup_{i=1}^n \bar{\Box}_{\omega_i} \tag{133}
\]
If \( \omega = \omega_0 \) is a single point then \( |\omega| = 0 \) and \( X_\omega = \emptyset \). In this case we define \( s_{\omega} = 1 \).

Weakened propagators are defined by
\[
G_k(s, A) = \sum_{\omega} s_{\omega} G_{k,\omega}(A) \tag{134}
\]
If \( s_{\Box} \) is small then the coupling through \( \Box \) is reduced. The \( G_k(s, A) \) interpolate between \( G_k(A) = G_k(1, A) \) and a strictly local operator \( G_k(0, A) \).

The results of lemma 7 hold for the weakened Green’s functions \( G_k(s, A) \). In fact we can allow complex \( s_{\Box} \) satisfying \( |s_{\Box}| \leq M^\frac{1}{2} \) and still get estimates of the same form. Also \( G_k(s, A) \) has the analyticity and symmetries of \( G_k(A) \).

### 3.5 more random walk expansions

We also need a random walk expansion for \( C_k(A) = \left( \Delta_k(A) + aL^{-2}(Q^T Q)(A) \right)^{-1} \) or even better for \( C_k^{3/2}(A) \). These are treated for \( A = 0 \) in Balaban and [24] and the treatment is similar here. The operator \( C_k(A) \) has a simple expression in terms of \( G_{k+1}(A) \) (see (133)), and this gives the expansion. The analysis for \( C_k^{3/2}(A) \) is based on the representation
\[
C_k^{1/2}(A) = \frac{1}{\pi} \int_0^\infty \frac{dx}{\sqrt{x}} \left( \Delta_k(A) + aL^{-2}(Q^T Q)(A) + x \right)^{-1} \tag{135}
\]
As in appendix C in [24] one can show that
\[
\left( \Delta_k(A) + aL^{-2}(Q^T Q)(A) + x \right)^{-1} = A_k,x(A) + a_k^2 \left( A_k,x Q_k G_{k,x} Q_k^T A_k,x \right)(A) \tag{136}
\]
where
\[
A_{k,x}(A) = \frac{1}{a_k + x}(I - (Q^T Q)(A)) + \frac{1}{a_k + x L^{-2} + x}(Q^T Q)(A)
\]
\[
G_{k,x}(A) = \left[ -\Delta_A + \frac{a_k x}{a_k + x} (Q^T Q_k)(A) + \frac{a_k^2 a L^{-2}}{(a_k + x)(a_k + x L^{-2} + x)} (Q^T_{k+1} Q_{k+1})(A) \right]^{-1}
\]  

(137)

Since all the other operators are local it suffices to establish a random walk expansion for \(G_{k,x}(A)\), and it turns out that an \(L^2\) expansion suffices. The expansion follows from good local estimates on the local operator \(G_{k,x}(..., A)\) defined just as \(G_{k,x}(A)\) but restricting the operator to \(\Box\) before taking the inverse. We claim that if real \(A\) is gauge equivalent to \(A'\) satisfying \(|A'|, |\partial A'| < \epsilon_k^{1+\epsilon}\)

\[
\|1_{\Delta_g} G_{k,x}(A, \Box) 1_{\Delta_g} f \|_2 \leq C e^{-\gamma d(y, y')} \|f\|_2
\]

\[
\|1_{\Delta_g} \partial_A G_{k,x}(A, \Box) 1_{\Delta_g} f \|_2 \leq C e^{-\gamma d(y, y')} \|f\|_2
\]  

(138)

As before it suffices to prove the result for \(A'\). This is known for \(A = 0\), see Appendix E in [24]. For the general case we expand

\[
G_{k,x}(\Box, A') = G_{k,x}(\Box, A') \sum_{n=0}^\infty (-U_{k,x}(\Box, 0, A')G_{k,x}(\Box, 0))^n
\]  

(139)

where now \(U_{k,x}(\Box, 0, A') = G_{k,x}(\Box, A')^{-1} - G_{k,x}(\Box, 0)^{-1} \). As in [122] one establishes

\[
\|U_{k,x}(\Box, 0, A') f \|_2 \leq \mathcal{O}(1)e_k \left( |A'| \|_\Delta, \infty + \|\partial A'| \|_{\Delta, \infty} \right) \left( \|f\|_{\Delta, 2} + \|\partial A f\|_{\Delta, 2} \right)
\]

\[
\leq \mathcal{O}(1)e_k^{-1} \left( \|f\|_{\Delta, 2} + \|\partial A f\|_{\Delta, 2} \right)
\]  

(140)

This gives the convergence of the series and the estimate [138]. We can also extend the result to \(A\) in the complex domain [115].

As in [122] the control over \(G_{k,x}(\Box, A)\) leads to a random walk expansion

\[
G_{k,x}(A) = \sum_\omega G_{k,x,\omega}(A)
\]  

(141)

and \(L^2\) bounds like [125] for \(G_{k,x}(A)\) follow. By [136] we get a random walk expansion for \(G_{k,x}(A)\). This also gives the bound.

\[
|G_{k,x}^+(A) f | \leq C \|f\|_{\infty}
\]  

(142)

4 RG transformations for gauge fields

4.1 axial gauge

For gauge fields we more or less follow the treatment of Balaban [5, 6, 8] and Balaban, Imbrie, and Jaffe [17, 18, 19, 28]. This differs from the treatment of the scalar field in that we need to employ gauge fixing and we use a different definition of the averaging operators. Even so the gauge fixing here is not exactly the axial gauge employed in the above references, but a covariant axial gauge introduced by Balaban in [11, 12] and further developed in [27] to which we refer for more details.

Start with an integral over fields on \(\mathbb{T}_0^N\) of the form

\[
\int f(A) \exp \left( -\frac{1}{2} \|dA\|^2 \right) DA \quad DA = \prod_b d(A(b))
\]  

(143)
This generally does not converge since \(dA\) has a large null space; we proceed formally. We scale up to the lattice \(T^0_N\). Let \(\rho_0\) be the function \(f(A) \exp\left(-\frac{1}{2}\|dA\|^2\right)\) scaled up. For \(A_0\) on \(T^0_N\) it is

\[
\rho_0(A_0) = F_0(A_0) \exp\left(-\frac{1}{2}\|dA_0\|^2\right)
\]

where \(F_0(A_0) = f_{L^N}(A_0) = f(A_{0,L^{-N}})\).

On this lattice we define an averaged field on oriented bonds in \(T^1_N\) by (for reverse oriented bonds take minus this)

\[
(QA)(y, y + Le_\mu) = \sum_{x \in B(y)} L^{-4}A(\Gamma_{x,x+Le_\mu})
\]

where \(\Gamma_{x,x+Le_\mu}\) is the straight line between the indicated points. Note however that \(Q^TQ\) is not a projection operator. The means that an exponential RG transformation cannot be treated as they were in scalar case. Instead we use a delta function RG transformation which has other advantages and difficulties.

We would like to define a sequence of densities \(\rho_0, \rho_1, \ldots, \rho_N\) with \(\rho_k(A_k)\) defined for \(A_k\) on \(T^0_{N-k}\). First consider

\[
\hat{\rho}_{k+1}(A_{k+1}) = \int \delta(A_{k+1} - QA_k)\rho_k(A_k) DA_k
\]

For convergence we introduce an axial gauge fixing function (justified by a Fadeev-Popov argument)

\[
\delta(\tau A_k) = \prod_{y \in T^1_{N-k}} \prod_{x \in B(y), x \neq y} \delta((\tau A_k)(y, x))
\]

where \((\tau A_k)(y, x)\) is defined in \(\text{(22)}\). Instead of \(\text{(146)}\) we define \(\hat{\rho}_{k+1}(A_{k+1})\) for \(A_{k+1}\) on \(T^1_{N-k}\) by

\[
\hat{\rho}_{k+1}(A_{k+1}) = \int \delta(A_{k+1} - QA_k)\delta(\tau A_k)\rho_k(A_k) DA_k
\]

and then \(\rho_{k+1}(A_{k+1})\) for \(A_{k+1}\) on \(T^0_{N-k-1}\) by

\[
\rho_{k+1}(A_{k+1}) = \hat{\rho}_{k+1}(A_{k+1+L})L^{-\frac{1}{2}(bN - b_{N-k-1})}L^{-\frac{1}{2}(sN - s_{N-k-1})}
\]

Here \(b_n = 3L^3N\) is the number of bonds in a three dimensional toroidal lattice with \(L^N\) sites on a side, and \(s_N = L^3N\) is the number of sites.

The delta function averaging operators compose nicely and we have

\[
\rho_k(A_k) = \int \delta(A_k - Q_k A)\delta_k^r(A)\rho_{0,L^{-k}}(A) DA
\]

where now \(A\) is defined on bonds in \(T^0_{N-k}\) and the \(k\)-fold averaging operator is defined by \(Q_k = Q \circ \cdots \circ Q\). Then \(Q_k A\) is given on oriented bonds in \(T^0_{N-k}\) by

\[
(Q_k A)(y, y + e_\mu) = \int_{|x-y| < \frac{1}{2}} L^{-k}A(\Gamma_{x,x+e_\mu}) \, dx
\]

and the gauge fixing function is now

\[
\delta_k^r(A) = \prod_{j=0}^{k-1} \delta(\tau Q_j A)
\]
One can show that $ρ_k(Δ_k)$ is well-defined \[16, 27\].

The integral of final density $ρ_N(Δ_N)$ gives back the original integral \[143\] but now with a hierarchical axial gauge fixing function which enables convergence. See \[27\] for details.

For future reference we note the identity

$$(dQ_k) A(y, y + e_µ, y + e_ν, y + e_ν) = \int_{|x - y| < \frac{1}{2}} L^{-2k} dA(Σ_{x,x+e_µ,x+e_ν,e_ν+e_µ}) \, dx$$  \(153\)

Here $Σ_{x,x+e_µ,...}$ is the square with the indicated corners, and in general $A(Σ) = \sum_{p∈Σ} dA(p)$.

### 4.2 free flow

Scaling the $ρ_0$ we can also write \(150\) as

$$ρ_k(Δ_k) = ∫ δ(Δ_k - Q_k) δ(Δ_k) F_{0,L-k}(A) \exp \left( -\frac{1}{2} ||dA||^2 \right) D A$$  \(154\)

which we analyze further.

Let $H^k_k A_k$ be the minimizer of $||dA||^2$ subject to the constraints of the delta functions in \(150\). We give an explicit representation later on in \(194\). It has the property that it preserves gauge equivalence: if $A_k ≈ A'_k$ then $H^k_k A_k ≈ H^k_k A'_k$.

Expanding around the minimizer by $A = H^k_k A_k + Z$ we find

$$ρ_k(Δ_k) = Z_k F_k(H^k_k A_k) \exp \left( -\frac{1}{2} < A_k, Δ_k A_k > \right)$$  \(155\)

where for $H^k_k A_k$ and $Z$ defined on $T_{N-k}$

$$< A_k, Δ_k A_k > = ||dH^k_k A_k||^2$$

$$F_k(H^k_k A_k) = Z_k^{-1} \int δ(Q_k Z) δ(Δ_k) F_{0,L-k}(H^k_k A_k + Z) \exp \left( -\frac{1}{2} ||dZ||^2 \right) D Z$$  \(156\)

$$Z_k = \int δ(Q_k Z) δ(Δ_k) \exp \left( -\frac{1}{2} ||dZ||^2 \right) D Z$$

### 4.3 the next step

Suppose we are starting with the expression \(155\) for $ρ_k(Δ_k)$. In the next step generated by \(148\) we have

$$ρ_{k+1}(Δ_{k+1}) = Z_k ∫ δ(Δ_{k+1} - Q_k A_k) δ(τ A_k) F_k(H^k_k A_k) \exp \left( -\frac{1}{2} < A_k, Δ_k A_k > \right) D A_k$$  \(157\)

Let $H^k_{k+1} A_{k+1}$ be the minimizer for $\frac{1}{2} < A_k, Δ_k A_k >$ subject to the constraints. Expanding around the minimizer with $A_k = H^k_k A_k + Z$ we again get the representation

$$ρ_{k+1}(Δ_k) = Z_{k+1} F_{k+1}(H^k_{k+1} A_{k+1}) \exp \left( -\frac{1}{2} < A_{k+1}, Δ_{k+1} A_{k+1} > \right)$$  \(158\)

But now with the identifications

$$Z_{k+1} = Z_k Z_k^L L_{N - b N - k - 1,1} L_{s N - s N - k - 1,1}$$

$$(H^k_{k+1} A_{k+1}) = H^k_k A_{k+1}$$

$$(H^k_{k+1} A_{k+1}) = (Z_k^L)^{-1} \int δ(Q Z) δ(τ Z) F_k(H^k_{k+1} A_{k+1}) + H^k_k Z \exp \left( -\frac{1}{2} < Z, Δ_k Z > \right) D Z$$

$$Z_k^L = \int δ(Q Z) δ(τ Z) \exp \left( -\frac{1}{2} < Z, Δ_k Z > \right) D Z$$  \(159\)
See [27] for details. More generally we define for any \( A \) on \( T_{N-k-1} \)
\[
F_{k+1}(A) = (Z_k^l)^{-1} \int \delta(QZ)\delta(\tau Z) \exp \left( -\frac{1}{2} < Z, \Delta_k Z > \right) DZ \tag{160}
\]
Note that if \( F_0 \) is gauge invariant then \( F_k \) is gauge invariant for any \( k \).

4.4 other gauges

Restrict now to the case \( F_0 = 1 \) so \( \rho_0 = \exp(-\frac{1}{2} ||dA||^2) \). Instead of (150) the density \( \rho_k(A_k) \) can be expressed in the modified Feynman gauge for any \( \alpha > 0 \) by [5], [27]
\[
\rho_k(A_k) = \text{const} \int \delta(A_k - Q_k A) \exp \left( -\frac{1}{2} ||dA||^2 - \frac{1}{2\alpha} < \delta A, R_k \delta A > \right) DA \tag{161}
\]
where \( \delta = d^T \) on 1-forms (functions on bonds) is the adjoint of \( d = \partial \) on scalars, and \( R_k \) is the projection onto the subspace \( \Delta(\ker Q_k) \). It is explicitly given by
\[
R_k = I - G_k Q_k^T (Q_k G_k Q_k^T)^{-1} Q_k G_k \tag{162}
\]
where \( G_k = (-\Delta + a Q_k^T Q_k)^{-1} \) for any \( a \geq 0 \) (essentially the same as \( G_k(0) \) in (60)). This includes the Landau gauge at \( \alpha = 0 \) in which case
\[
\rho_k(A_k) = \text{const} \int \delta(A_k - Q_k A) \exp \left( -\frac{1}{2} ||dA||^2 \right) DA \tag{163}
\]

Let \( H_k A_k \) be the minimizer of \( ||dA||^2 + \alpha^{-1} < \delta A, R_k \delta A > \) subject to the constraint \( Q_k A = A_k \) imposed in (161). An explicit expression for the minimizer can be given using Green’s function for this gauge defined by
\[
G_k = \left( \delta d + \frac{1}{2\alpha} d R_k \delta + a Q_k^T Q_k \right)^{-1} \tag{164}
\]
Then one can show [5] that \( Q_k G_k Q_k^T \) is invertible and for any \( a > 0 \)
\[
H_k = G_k Q_k^T (Q_k G_k Q_k^T)^{-1} \tag{165}
\]
It turns out that \( H_k \) is independent of \( \alpha \) and is also the minimizer for Landau gauge. Furthermore
\[
H_k^* = H_k + \partial D_k \tag{166}
\]
for some operator \( D_k \). This means that in gauge invariant expression we can can replace \( H_k^* \) by \( H_k \).

In particular we can make this replacement in the fluctuation integral (160). This is useful because \( H_k \) is more regular than the axial \( H_k^* \).

The relation (166) also shows that \( \Delta_k \) can be expressed in the Landau gauge as
\[
< Z, \Delta_k Z > = ||dH_k Z||^2 \tag{167}
\]
Using this one can show [6], [27] that there are constants \( C_{\pm} \) depending only on \( L \) such that on the subspace \( QZ = 0, \tau Z = 0 \)
\[
C_- ||Z||^2 \leq < Z, \Delta_k Z > \leq C_+ ||Z||^2 \tag{168}
\]
4.5 parametrization of the fluctuation integral

We parametrize the fluctuation integral (160) as in [19, 27]. Let \( Z = (Z_1, Z_2) \) where \( Z_1 \) is defined on bonds that lie in some \( B(y) \) and \( Z_2 \) is defined on bonds joining neighboring cubes \( B(y), B(y') \). The delta function \( \delta(\tau Z) = \delta(\tau Z_1) \) is fulfilled by taking \( Z_1 = \tilde{Z}_1 \in \ker \tau \). Let \( Z_2 \) be defined on bonds joining \( B(y), B(y') \), but not the central bond denoted \( b(y, y') \). The delta function \( \delta(\mathcal{Z}Z) \) selects \( b(y, y') = S(\tilde{Z}_1, \tilde{Z}_2) \) for some local linear operator \( S \). See \ref{footnote} in the appendix for the explicit formula. Then the integral is parametrized by \( Z = (\tilde{Z}_1, \tilde{Z}_2, S(\tilde{Z}_1, \tilde{Z}_2)) \). Or if we let \( \tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2) \) then it is parametrized by

\[
Z = C\tilde{Z} \equiv (\tilde{Z}, S\tilde{Z})
\]  

(169)

The fluctuation integral (160) can now be written

\[
F_{k+1}(A) = \int F_k\left(A_L + \mathcal{H}_k C\tilde{Z}\right) \exp\left(-\frac{1}{2} < C\tilde{Z}, \Delta_k C\tilde{Z}> \right) D\tilde{Z} / \{F_k = 1\}
\]  

(170)

If we define

\[
C_k = (C^T \Delta_k C)^{-1}
\]

(171)

then the integral can be expressed with the Gaussian measure \( \mu_{C_k} \) with covariance \( C_k \) as

\[
F_{k+1}(A) = \int F_k\left(A_L + \mathcal{H}_k C\tilde{Z}\right) d\mu_{C_k}(\tilde{Z}) = \int F_k\left(A_L + \mathcal{H}_k C\tilde{Z} \right) d\mu_f(\tilde{W})
\]

(172)

By (168) \( C^T \Delta_k C \) is uniformly bounded above and below. Hence the same is true of the inverse \( C_k \) and

\[
\|C_k^{1/2}Z\| \leq C\|Z\|
\]

(173)

These are basic facts for controlling the integrals (172), but we will still need more.

We note also that the integral can be written

\[
F_{k+1}(A) = \int F_k\left(A_L + \mathcal{H}_k Z\right) d\mu_{C_k}(Z) = C_k' \equiv CC_k C^T
\]

(174)

where \( C_k' \) is now defined on functions on all of \( T^N_{N-k} \).

4.6 representation for \( C_k', C_k^T \)

We will need a representation of \( C_k \) which admits a random walk expansion. It is easier to treat \( C_k' \) and we consider that first. The following from [27] is a simpler version of an analysis by Balaban [8].

For \( \lambda, A \) on \( T^0_{N-k} \) let \( \lambda = MA \) be the solution of the equations

\[
(\tau(A + d\lambda))(y, x) = 0 \quad x \neq y \quad Q\lambda(y) = 0 \quad x \in B(y)
\]

(175)

This is

\[
\lambda(x) = MA(x) = -(\tau A)(y, x) + L^{-3} \sum_{x' \neq y} (\tau A)(y, x')
\]

(176)

Also define

\[
\tilde{G}_{k+1} = G_{k+1}^0 - G_{k+1}^0 Q_{k+1} \left(Q_{k+1} Q_{k+1}^T \right)^{-1} Q_{k+1} G_{k+1}^0
\]

(177)

where for any \( a > 0 \)

\[
G_{k+1}^0 = \left(\delta d + dR_{k+1} + aQ_{k+1}^T Q_{k+1}\right)^{-1}
\]

(178)
The operator \( G_{k+1}^0 \) is defined on functions on \( \mathbb{T}_{N-k} \). Then the representation is

\[
C'_k = \left( I + \partial\mathcal{M} \right) Q_k \tilde{G}_{k+1}^0 Q_k^T \left( I + \partial\mathcal{M} \right)^T
\]

(179)

We also need a better representation of \( C^\perp_k \) or \((C^\perp_k)' = CC^\perp_k C^T\). We have

\[
C^\perp_k = \frac{1}{\pi} \int_0^\infty \frac{dx}{\sqrt{x}} C_{k,x} \quad C_{k,x} = \left( C^T \Delta_k C + x \right)^{-1}
\]

(180)

So it is sufficient to find a representation for \( C_{k,x} \) or \( C'_{k,x} = C C_{k,x} C^T \). Define

\[
\tilde{G}_{k+1,x} = G_{k+1,x}^0 - G_{k+1,x}^0 \left( Q_{k+1}^0 G_{k+1,x} Q_{k+1}^T \right)^{-1} Q_{k+1} G_{k+1,x}^0
\]

(181)

where for any \( a > 0 \)

\[
G_{k+1,x}^0 = \left( \delta d + d R_{k+1} + a Q_{k+1}^T + x Q_{k+1}^T (I + \partial\mathcal{M}) \chi^T (I + \partial\mathcal{M}) Q_k \right)^{-1}
\]

(182)

and \( \chi^T \) suppresses the contribution of central bonds \( b(y,y') \) joining \( L \)-cubes. Then the representation is

\[
C'_{k,x} = \left( I + \partial\mathcal{M} \right) Q_k \tilde{G}_{k+1,x} Q_k^T \left( I + \partial\mathcal{M} \right)^T
\]

(183)

### 4.7 random walk expansions

We quote some results about various random walk expansions, almost all due to Balaban.

**Lemma 8.** The Green’s function \( \mathcal{G}_k \) has a random walk expansion based on blocks of size \( M \), convergent for \( M \) sufficiently large. These yield the bounds for \( \Delta_y, \Delta_y' \) unit squares centered on unit lattice points \( y, y' \in \mathbb{T}_{N-k}^0 \) and \( \zeta_y \) a smooth partition on unity with \( \text{supp} \, \zeta_y \subset \Delta_y' \):

\[
|1_{\Delta_y} \mathcal{G}_k 1_{\Delta_y'} f|, \ |1_{\Delta_y} \partial_x^\alpha \mathcal{G}_k 1_{\Delta_y'} f|, \ |\delta_{x,y} \partial_{x,y}^\alpha \mathcal{G}_k 1_{\Delta_y'} f| \|_{\infty} \leq C e^{-\gamma d(y,y') \|f\|_\infty}
\]

(184)

Here \((1_{\Delta_y} f)(x, x+\eta e_\mu) = 1_{\Delta_y}(x) f(x, x+\eta e_\mu)\) and \((\partial_x f)(x, x+\eta e_\mu) = (\partial_x f \mu)(x)\). The constant for the Holder derivative \( \delta_\alpha \) depends on \( \alpha \). The statement that \( \mathcal{G}_k \) has a random walk expansion means that \( \mathcal{G}_k = \sum_\omega \mathcal{G}_{k,\omega} \), and just as in [131]

\[
|\mathcal{G}_{k,\omega} f|, \ |\partial_x \mathcal{G}_{k,\omega} f|, \ |\delta_{x,y} \partial_{x,y} \mathcal{G}_{k,\omega} f| \|_{\infty} \leq C \|f\|_{\infty}
\]

(185)

and similarly for the derivatives. It also means that bounds of the same form hold for \( \mathcal{G}_k(s) \) defined with weakening parameters \( s \) as in [134]. The estimates [184] also have a global version:

\[
|\mathcal{G}_k f|, \ |\partial_x \mathcal{G}_k f|, \ |\delta_{x,y} \partial_{x,y} \mathcal{G}_k f| \|_{\infty} \leq C \|f\|_{\infty}
\]

(186)

Similar remarks can be added after each of the following lemmas.

**Lemma 9.** The operators \((Q_k \Delta^{-2} Q_k^T)^{-1}\) and \((Q_k G_k Q_k^T)^{-1}\) have random walk expansions based on blocks of size \( M \), convergent for \( M \) sufficiently large. These yield the bounds

\[
|Q_k G_k Q_k^T|^{-1}(x,x')| \leq C e^{-\gamma d(x,x')}
\]

(187)
These operators are not inverses of local operators so the expansions are more complicated.

**Lemma 10.** [3] The operators $R_k$ and $H_k$ have random walk expansions based on blocks of size $M$, convergent for $M$ sufficiently large. This yields the bounds

$$
|1_{\Delta_y}R_k 1_{\Delta_y'} f|, |1_{\Delta_y}\partial R_k 1_{\Delta_y'} f|, |\delta_0 \zeta_y \partial R_k 1_{\Delta_y'} f| \leq Ce^{-\gamma d(y,y')} \|f\|_{\infty}
$$

(188)

The expansion for $R_k$ follows from the expansion for $G_k = G_k(0)$ in section 5.3.4 and the expansion for $(Q_kG_k^2Q_k^T)^{-1}$ and the representation [162]. The expansion for $H_k$ follows from the expansion for $G_k$ and the expansion for $(Q_kG_k^2Q_k^T)^{-1}$ and the representation [165].

For future reference we record the global estimate on $H_k$:

$$
|H_k f|, |\partial H_k f|, |\delta_0 \partial H_k f| \leq C \|f\|_{\infty}
$$

(189)

Next we consider operators like $C_k$ which act on functions of the type $\tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2)$ defined in section 4.5. For such functions define $1_{B(y)}\tilde{Z} = (1_{B(y)}\tilde{Z}_1, 1_{B(y)}\tilde{Z}_2)$ where $(1_{B(y)}\tilde{Z}_2)(x, x + e_\mu) = 1_{B(y)}(x)\tilde{Z}_2(x, x + e_\mu)$. This is again a variable of the same type and $Z = \sum_y 1_{B(y)}\tilde{Z}$.

**Lemma 11.** [3], [112] The operators $C_k, C_{k,x}, C_k^{\perp}$ have random walk expansions based on blocks of size $M$, convergent for $M$ sufficiently large. These yield the bounds for $y, y'$ on $T^0_{N-k}$.

$$
|1_{B(y)}C_k 1_{B(y')} f|, |1_{B(y)}C_{k,x} 1_{B(y')} f|, |1_{B(y)}C_k^{\perp} 1_{B(y')} f| \leq Ce^{-\gamma (b,b')} \|f\|_{\infty}
$$

(190)

We sketch the proof. For $x \geq 0$ the Green’s function $G^0_{k+1,x}$ has a random walk expansion just as for $G_k$, and $(Q_{k+1}G^0_{k+1,x}Q_{k+1}^T)^{-1}$ has a random walk expansion just as for $(Q_kG_kQ_k^T)^{-1}$. The other operators in [181] are local so we have an expansion for $\tilde{G}_{k+1,x}$. Then the other operators in [183] are local so this yields an expansion for $C_{k,x} = CC_{k,x}C^T$. Next we write

$$
C_{k,x} = C^{-1} C'_{k,x} (C^T)^{-1}
$$

(191)

Since $C^{-1}, (C^T)^{-1}$ are not local this does not immediately give a random walk expansion for $C_{k,x}$. However $C^{-1}, (C^T)^{-1}$ themselves have random walk expansions which we develop in appendix A. Together with the expansion for $C'_{k,x}$ we get an expansion of $C_{k,x}$ and $C_k$ is the special case $x = 0$.

We cannot use the expansion for $C_{k,x}$ directly in [180] unless we can establish that $C_{k,x} = O(x^{-1})$ to ensure the convergence of the integral over $x$. This bound which is not readily available. Instead we use the modified representation. Break the integral over $x$ at some $\gamma_1$. Then for $x > \gamma_1$ write

$$
C_{k,x} = \left(C^T \Delta_k C + x\right)^{-1} = \sum_{n=0}^{\infty} x^{-(n-1)}(-1)^n (C^T \Delta_k C)^n
$$

(192)

This coverages for $\gamma_1$ sufficiently large. Doing the integral over $x$ in the sum yields

$$
C_k^{\perp} = \frac{1}{\pi} \int_{0}^{\gamma_1} \frac{dx}{\sqrt{x}} C_{k,x} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n + \frac{3}{2} \gamma_1} x^{-n - \frac{3}{2}} (C\Delta_k C)^n
$$

(193)

Then for $C_{k,x}$ we use the random walk expansion above, which is uniformly bounded in $x$. For $C^T \Delta_k C$ and powers we can use the representation $\Delta_k = H_k^T dH_k$ from [167] and the expansion for $H_k$.

The bound (190) also has the global version:

$$
|C_k^{\perp} f| \leq C \|f\|_{\infty}
$$

(194)
5 Polymer functions

5.1 a preliminary lemma

Before defining polymer functions we first show that every gauge potential $A$ is locally equivalent to a field depending only on the field strength $dA$. We only need this on a unit lattice.

Lemma 12. Let $A$ be a gauge field on a unit lattice lattice. For any reference point $y$ on any any cube centered on $y$ we have $A = A' + \partial \lambda$ where $A'$ depends only on $dA$ and satisfies

$$|A'(b)| \leq d(b, y) \|dA\|_{\infty}$$

(195)

Proof. We go to an axial gauge. Let $\Gamma(y, x)$ be the path from $y$ to $x$ in which coordinates are increased in the standard order, and let $\lambda(x) = A(\Gamma(y, x))$. If $b = [x, x']$ is one of the paths $\Gamma(y, x)$ then

$$A(x, x') = A(\Gamma(y, x')) - A(\Gamma(y, x)) = \partial \lambda(x, x')$$

(196)

and so $A' = A - \partial \lambda$ vanishes on such bonds and hence on the paths $\Gamma(y, x)$.

Now for any bond $b = [x, x']$ we have that $\Gamma(y, x) + [x, x'] - \Gamma(y, x')$ is a closed path which bounds a surface $\Sigma_{y,x,x'}$ made up of at most $d(b, y)$ unit plaquettes. Therefore by the lattice Stoke's theorem

$$A'(x, x') = A'(\Gamma(y, x) + [x, x'] - \Gamma(y, x')) = dA'(\Sigma_{y,x,x'}) = dA(\Sigma_{y,x,x'})$$

(197)

and the result follows.

5.2 a regularity result

The Landau gauge minimizer $A_k = \mathcal{H}_k A_k$ will play an important role in the following. In particular we want to use it as a background field in the boson Green’s function $G_k(A_k)$. Hence it must satisfy the conditions (115). However as we explain later we only want to assume bounds on $dA_k$ not $A_k$ or general derivatives $\partial A_k$. To obtain the result will require some gymnastics, roughly following [17], [18].

Recall that $\mathcal{H}_k$ is gauge equivalent to the axial gauge $\mathcal{H}_k^x$. The explicit expression for the latter is

$$\mathcal{H}_k^x = Q_k^T - G_k^x \delta Q_k^x d$$

(198)

The operator $Q_k^x$ averages over the faces unit cubes and $\mathcal{Q}_k^x$ averages over plaquettes on the corners of unit cubes. (See [17] or [28] for the exact definition.) Here the operator $\delta = d^T$ on two-forms (functions on plaquettes) is the adjoint $d$ on one-forms (functions on bonds). The operator $G_k^x$ on $T_{N-k}$ is the axial Green’s function defined by

$$\exp \left( \frac{1}{2} \langle f, G_k^x f \rangle \right) = Z_k^{-1} \int \delta(Q_k A) \delta_k^x(A) \exp \left( - \frac{1}{2} \|dA\|^2 + \langle f, A \rangle \right) DA$$

(199)

After a calculation using the identity (160) one finds that the kernel satisfies

$$G_k^x(b, b') = LG_k^x(Lb, Lb') + L(H_k^x C_j^x H^x)(Lb, Lb')$$

(200)

Iterating this we see that the Green’s function admits the decomposition

$$G_k^x(b, b') = \sum_{j=0}^{k-1} L^{k-j} (H_k^x C_j^x H^x)(L^{k-j} b, L^{k-j} b')$$

(201)

\[^2\text{It is not identical with the Green’s function of [17] since the axial gauge fixing is a little different.}\]
In the combination $G_k^c \delta$ we have on the right $H_j^c \delta = (dH_j^c)^T = (dH_j)^T = H_j^T \delta$, and on the left we use $H_j^c \sim H_j$. Thus $G_k^c \delta$ is gauge equivalent to $D_k \delta$ where $D_k$ is defined by the kernel

$$D_k(b, b') = \sum_{j=0}^{k-1} L^{k-j}(\partial_{\gamma} C_j^c \partial_{\gamma} T^c_j)(L^{k-j} b, L^{k-j} b')$$

(202)

and $H_k^c$ (and hence $H_k$) is gauge equivalent to

$$H_k^c = Q_k^{c,T} - D_k \delta Q_k^{c,T} d$$

(203)

The only discontinuous part of $H_k^c$ is $Q_k^{c,T}$. The operator $D_k$ has good regularity and decay bounds as we now show.

We claim that

$$|D_k f(b)|, |\partial D_k f(b)|, |\delta \partial D_k f(b)| \leq C e^{-\gamma d(b, \text{supp} f)} \|f\|_{\infty}$$

(204)

To see this temporarily drop the scaling factors and let $f_L(b) = f(b/L)$. Then with $\tilde{C}_k = H_k C_j^c H_j^T$ we have

$$D_k f(b) = \sum_{j=0}^{k-1} L^{-2(k-j)}(\tilde{C}_j f_{L^{k-j}})(L^{k-j} b)$$

(205)

But it can be deduced from (188) (190) that

$$|\tilde{C}_k f(x)|, |\partial \tilde{C}_k f(b)|, |\delta \partial \tilde{C}_k f(b)| \leq C e^{-\gamma d(b, \text{supp} f)} \|f\|_{\infty}$$

(206)

Therefore

$$|D_k f(b)| \leq C \sum_{j=0}^{k-1} L^{-2(k-j)} e^{-\gamma L^{k-j} d(b, \text{supp} f)} \|f\|_{\infty}$$

(207)

this yields the first bound in (204). The derivatives reduce the $L^{-2(k-j)}$ to $L^{-(k-j)}$ or $L^{-(1-\alpha)(k-j)}$, and we still have convergence. Thus (204) is established. Note that we cannot allow two derivatives unless $d(b, \text{supp} f) \geq O(1) > 0$.

We also need a local version of $D$. Again let $\zeta_{\Delta_y}$ be a smooth partition of unity with $\text{supp} \ z_{\Delta_y} \subset \tilde{\Delta}_y$ and define

$$D_k^{\text{loc}} = \sum_{y, y': d(y, y') < 4} \zeta_{\Delta_y} D \zeta_{\Delta_{y'}}$$

(208)

$$H_k^{\text{loc}} = Q_k^{c,T} - D_k^{\text{loc}} \delta Q_k^{c,T} d$$

Then $D_k^{\text{loc}}$ again satisfies the bounds (204), since if a derivative falls on a $\zeta_{\Delta_y}$ nothing important is changed.

The difference $D_k - D_k^{\text{loc}}$ has no short distance singularity and we can allow more derivatives, also on the right. We have instead of (205)

$$\left( (D_k - D_k^{\text{loc}}) f \right)(b) = \sum_{y, y': d(y, y') \geq 4} \zeta_{\Delta_y}(b) \sum_{j=0}^{k-1} L^{-2(k-j)}(\tilde{C}_j(\zeta_{\Delta_{y'}} f)_{L^{k-j}})(L^{k-j} b)$$

(209)

and since $d(y, y') \geq 4$ implies $d(\tilde{\Delta}_y, \tilde{\Delta}_{y'}) \geq 1$ we have instead of (204)

$$\left| (D_k - D_k^{\text{loc}}) f \right|(b) \leq C \sum_{j=0}^{k-1} L^{-2(k-j)} \sum_{y, y': d(y, y') \geq 4} \zeta_{\Delta_y}(b) e^{-\gamma L^{k-j} d(\tilde{\Delta}_y, \tilde{\Delta}_{y'})} \|f\|_{\infty}$$

$$\leq C \sum_{j=0}^{k-1} L^{-2(k-j)} e^{-\gamma L^{k-j}} \|f\|_{\infty}$$

(210)
Now we can allow any number of extra derivatives, each derivative adds a factor $L^{k-j}$ to the last estimate but the factor $e^{-\frac{1}{2}r L^{k-j}}$ still gives convergence. In particular we have

$$|\langle D_k - D_k^{\text{loc}} \rangle \delta F|, |\partial \langle D_k - D_k^{\text{loc}} \rangle \delta F|, |\delta_a \partial \langle D_k - D_k^{\text{loc}} \rangle \delta F| \leq C \| F \|_{\infty}$$  \hspace{1cm} (211)

With these preliminaries out of the way we can now state the regularity result:

**Lemma 13.** $A_k = H_k A_k$ has the property that in each $\tilde{Q}$ it is gauge equivalent to some $A'$ satisfying

$$|A'|, |\partial A'|, |\delta_a \partial A'| \leq CM \| dA_k \|_{\infty}$$  \hspace{1cm} (212)

**Proof.** We write

$$A_k = (H_k - H_k^D) A_k + (H_k^D - H_k^{\text{loc}}) A_k + H_k^{\text{loc}} A_k$$  \hspace{1cm} (213)

and argue that each term has the stated property. The first is globally pure gauge and hence pure gauge on any $\tilde{Q}$.

The second is the same as $\langle D_k - D_k^{\text{loc}} \rangle \delta Q_k^{T} dA_k$ and by (211) we have globally

$$|\langle D_k - D_k^{\text{loc}} \rangle \delta Q_k^{T} dA_k|, \leq C \| dA_k \|_{\infty}$$  \hspace{1cm} (214)

and the same for the derivatives.

For the third term note that by lemma 12 we have $A_k = A'_k + \partial \lambda$ on a suitable neighborhood of $\tilde{Q}$ and

$$|A'_k| \leq CM \| dA_k \|_{\infty} \leq CM \| dA_k \|_{\infty}$$  \hspace{1cm} (215)

The last inequality follows since $A_k = Q_k A_k$ hence by (153) $|dA_k| \leq \| \partial A_k \|_{\infty}$. Next $Q_k^{T} d\lambda = dQ_k^{T} \lambda$ and so in $\tilde{Q}$

$$H_k^{\text{loc}} A_k = H_k^{\text{loc}} A'_k + dQ_k^{T} \lambda$$  \hspace{1cm} (216)

Thus it suffices to show that $H_k^{\text{loc}} A'_k$ is a sum of terms with the stated properties. The function has a good bound, but we have to work harder for the derivative.

Extend the definition of $A'_k$ to the whole lattice by defining it to be zero off the neighborhood of $\tilde{Q}$. The extension is still bounded by $CM \| dA_k \|_{\infty}$, as are derivatives since we are on a unit lattice. Now write on $\tilde{Q}$

$$H_k^{\text{loc}} A'_k = (H_k^{\text{loc}} - H_k^D) A'_k + (H_k^D - H_k) A'_k + H_k A'_k$$  \hspace{1cm} (217)

The first term $\langle D_k - D_k^{\text{loc}} \rangle \delta Q_k^{T} dA'_k$ and its derivatives are again bounded by (211), the second term is again pure gauge, and the third term has good bounds by (189). This completes the proof.

### 5.3 bounded fields

We define some bounded field conditions. To motivate the definitions consider the minimizers

$$A_k = H_k A_k \quad \phi_k(A) = H_k(A) \Phi_k$$  \hspace{1cm} (218)

As suggested by our discussion to this point, and as we show in detail, the action after $k$ steps will have the leading terms

$$\frac{1}{2} \| A_k \|^2 + \frac{1}{2} \| \Phi_k - Q_k A_k \phi_k(A_k) \|^2 + \frac{1}{2} \| \partial A_k \phi_k(A_k) \|^2 + \lambda_k \int \phi_k(A_k)^4 + \ldots$$  \hspace{1cm} (219)

We choose the small field conditions so that if they are violated somewhere, then some piece of this action is large and the contribution to the density is suppressed. To specify the conditions let

$$p_k = (- \log \lambda_k)^p$$  \hspace{1cm} (220)

for some positive integer $p$. We assume $\lambda_k$ is small so that $p_k$ is large. Further we assume that $e^{k-2} \leq \lambda_k$ so that $p_k \leq \lambda_k e^{k-2} \leq e_k^{-2}$.  


Definition 1. The small field domain $S_k$ is all real-valued fields $A_k, \Phi_k$ on $T_{N-k}^0$ such that

$$|dA_k| \leq p_k \tag{221}$$

and

$$|\Phi_k - Q_k(A_k)\phi_k(A_k)| \leq p_k \quad |\partial_{A_k} \phi_k(A_k)| \leq p_k \quad |\phi_k(A_k)| \leq \lambda_k^{-\frac{2}{3}} p_k \tag{222}$$

The bounds on $S_k$ imply the bounds on the fundamental fields

$$|dA_k| \leq p_k \quad |\partial \Phi_k| \leq 3p_k \quad |\Phi_k| \leq 2\lambda_k^{-\frac{2}{3}} p_k \tag{223}$$

The first follows from $A_k = Q_k(A_k)$ and the identity \((153)\). The other two follow in a straightforward manner (see for example \([24]\)).

We also want a larger complex domain for the polymer functions we are about to introduce.

Definition 2. Let $\epsilon > 0$ be a fixed small number and consider the bounds

$$|A| < e_k^{-1+\epsilon} \quad |\partial A| < e_k^{-1+2\epsilon} \quad |\partial_\alpha \partial A| < e_k^{-1+3\epsilon} \tag{224}$$

and

$$|\phi| < \lambda_k^{-\frac{2}{3}-\epsilon} \quad |\partial A \phi| < \lambda_k^{-\frac{1}{3}-2\epsilon} \quad |\partial_\alpha \partial A \phi| < \lambda_k^{-\frac{2}{3}-\epsilon} \tag{225}$$

The small field domain $R_k$ is all complex-valued fields $A, \phi$ on $T_{N-k}^0$ such that

1. $A = A_0 + A_1$ where $A_0$ is real and each $\Box$ is gauge equivalent to some $A'_0$ satisfying \((224)\) with a factor $\frac{1}{3}$ on the right and $A_1$ is complex and satisfies \((227)\) with a factor $\frac{1}{3}$ on the right.
2. $\phi$ satisfies the bounds \((226)\).

We also say $A \in R_k$ if $A$ satisfies condition 1. Then $A$ is locally gauge equivalent to a field $A'$ satisfying \((224)\), and if $\phi$ satisfies \((227)\) the pair $(A, \phi)$ is locally gauge equivalent to a pair $(A', \phi')$ satisfying \((224), (225)\). (The latter since $|\phi'| = |\phi|, |\partial A' \phi'| = |\partial A \phi|$, etc.) We also note that if $A \in R_k$ then

$$|dA| \leq O(1) e_k^{-1+2\epsilon} \quad |\text{Im } A| \leq O(1) e_k^{-1+\epsilon} \tag{226}$$

These bounds are somewhat arbitrary. They must be large enough so that $S_k \subset R_k$, a fact we establish next. The conditions on $A$ are more restrictive that the domain \((113)\) and hence $G_k(A), H_k(A)$ and derivatives of order less than two have good bounds. The conditions also ensure that the polymer functions do not become too large and hence erode the convergence of our expansions. Also it is convenient to have slightly sharper bounds for higher derivatives.

Lemma 14. If $\alpha < 2/3$ then $A_k, \Phi_k$ in $S_k$ implies $A_k, \phi_k(A_k)$ in $\Box R_k$.

Proof. By Lemma 13, $A_k$ is gauge equivalent in each $\Box$ to some $A'$ satisfying

$$|A'|, |\partial A'|, |\partial_\alpha \partial A'| < CM p_k \leq e_k^{-\epsilon} \tag{227}$$

Hence the bounds \((224)\) are easily satisfied. The bounds on $\phi_k(A_k), \partial_{A_k} \phi_k(A_k)$ are also immediate. For the last we write for $d(x, y) \leq 1$ and $A = A_k$

$$\left| \left( \partial_\alpha \partial A \phi_k(A) \right)(x, y) \right| = \left| \frac{e^{Q_k A(x, y)} \left( \partial A \phi_k(A) \right)(y) - \partial A \phi_k(A)(x) }{|x - y|^{\alpha}} \right|$$

$$= \left| \frac{e^{Q_k A(x, y)} \left( \partial A \phi_k(A) \right)(y) - \partial A \phi_k(A)(x) }{|x - y|^{\frac{2}{3}}} \right| \left| e^{Q_k A(x, y)} \left( \partial A \phi_k(A) \right)(y) - \partial A \phi_k(A)(x) \right|^{1/3} \tag{228}$$

$$\leq (C \lambda_k^{-\frac{2}{3}} p_k)^{2/3} (2p_k)^{1/3} \leq \frac{1}{2} \lambda_k^{-\frac{2}{3}-\epsilon}$$
Here we used \[ \text{(126)} \] and \[ \text{(223)} \] for the first factor and \[ \text{(222)} \] for the second factor. This completes the proof.

For future reference we also note the following result

**Lemma 15.** If \( A, \phi \in \mathcal{R}_{k+1} \), then in any \( \square \) the pair \((A_L, \phi_L)\) is gauge equivalent to \((A', \phi')\) satisfying

\[
|A'| < L^{-1+\varepsilon}[e_k^{1+\varepsilon}] \quad |\partial A'| < L^{-2+\varepsilon}[e_k^{1+2\varepsilon}] \quad |\delta_0 \partial A'| < L^{-2-\alpha+2\varepsilon}[e_k^{1+3\varepsilon}] \quad (229)
\]

and

\[
|\phi'| < L^{-\frac{3}{4}-\varepsilon}[\lambda_k^{\frac{1}{4}-\varepsilon}] \quad |\partial A' \phi| < L^{-\frac{3}{4}-\varepsilon}[\lambda_k^{\frac{1}{4}-2\varepsilon}] \quad |\delta_\alpha \partial A' \phi'| < L^{-\frac{3}{4}-\alpha-\varepsilon}[\lambda_k^{\frac{1}{4}-\varepsilon}] \quad (230)
\]

In particular \((A', \phi') \in L^{-\frac{3}{4}-\varepsilon}\mathcal{R}_k\).

**Proof.** Choose an \( M \)-cube \( \tilde{\square}' \) in \( T_{N-k-1}^{-1} \) so that \( \tilde{\square} \subset L\tilde{\square}' \). We have \( A = A_0 + A_1 \) with \( A_0 \sim A_0' \) in \( \tilde{\square}' \) and \( A_0', A_1 \) satisfy the bounds for \( k + 1 \). Hence \( A_L \sim A_0'_{0,L} + A_{1,L} \equiv A' \) in \( L\tilde{\square}' \) and hence in \( \tilde{\square} \).

Since \( \varepsilon_{k+1} = L^{1/2} \varepsilon_k \) we have in \( \tilde{\square} \)

\[
|A_{0,L}'| \leq L^{-\frac{1}{2}} \|A_0' \|_\infty \leq \frac{1}{2} L^{-\frac{1}{2}} e_k^{1+\varepsilon} \leq \frac{1}{2} L^{-1+\varepsilon}[e_k^{1+\varepsilon}] \quad (231)
\]

The bound for \( \partial A_{0,L}' = L^{-1}(\partial A_0')_L \) is similar as is the bound for \( \delta_\alpha \partial A_{0,L}' \). The same hold for \( A_{1,L} \). Therefore \( A' \) satisfies \[ \text{(229)} \]. Similarly for \( \phi' \sim \phi_L \) since \( \lambda_{k+1} = L \lambda_k \)

\[
|\phi'| = |\phi_L| \leq L^{-\frac{1}{2}} \|\phi\|_\infty \leq L^{-\frac{1}{2}} \lambda_k^{-\frac{1}{4}+\varepsilon} \leq L^{-\frac{3}{4}-\alpha-\varepsilon}[\lambda_k^{\frac{1}{4}-\varepsilon}] \quad (232)
\]

Since \( \partial A \phi' \sim \partial A \phi_L = L^{-1}(\partial A \phi)_L \), etc. the derivatives add extra powers of \( L^{-1} \) as indicated.

### 5.4 definition of polymer functions

A polymer \( X \) in \( T_{N-k}^{-k} \) is a connected union of \( M \) cubes, with the convention that two cubes are connected if they have an entire face in common. The set of all polymer functions is denoted \( \mathcal{D}_k \). Our interaction terms will be expressed in terms of polymer functions \( E(X,A,\phi) \) which depend on the fields \( A, \phi \) only in \( X \).

We require that \( E(X,A,\phi) \) is bounded and analytic on the domain \( \mathcal{R}_k \) so the norm

\[
\|E(X)\|_k = \sup_{A,\phi \in \mathcal{R}_k} \|E(X,A,\phi)\| \quad (233)
\]

is finite.

We also require that \( E(X,A,\phi) \) be exponentially decaying in the size of \( X \). Size is measured on the \( M \)-scale. Define \( d_M(X) \) by

\[
M d_M(X) = \text{length of the shortest continuum tree joining the } M \text{-cubes in } X. \quad (234)
\]

The requirement is that \( E(X,A,\phi) \) be bounded by a constant times \( e^{-\kappa d_M(X)} \) for some \( \kappa = \mathcal{O}(1) \). To put it another way the norm

\[
\|E\|_{k,\kappa} = \sup_{X \in \mathcal{D}_k} \|E(X)\|_k e^{\kappa d_M(X)} \quad (235)
\]

must be finite. The space of all polymer functions with finite norm is is a Banach space called \( \mathcal{K}_k \).

We also note that if \( |X|_M \) is the number of \( M \) cubes in \( X \), then

\[
d_M(X) \leq |X|_M \leq \mathcal{O}(1)(d_M(X) + 1) \quad (236)
\]
Also there are constants $\kappa_0, K_0 = \mathcal{O}(1)$ such that for any $M$-cube $\square$
\[
\sum_{X \in \mathcal{D}_{k}, X \supset \square} e^{-\kappa_0 d_M(X)} \leq K_0 \tag{237}
\]

We assume $\kappa \geq \kappa_0$.

To scale polymer functions we first introduce a blocking operation. If $Y$ is a polymer in $T_{N-k}^{-k}$ which is a connected union of $LM$-cubes we define
\[
(BE)(Y) = \sum_{X: X = Y} E(X) \tag{238}
\]

where $\tilde{X}$ is the union of all $LM$-polymers intersecting $X$. Then
\[
|(BE)(Y, A, \phi)| \leq \mathcal{O}(1)L^3 e^{-L(k-\kappa_0-1)d_{LM}(Y)} \|E\|_{k, \kappa} \tag{239}
\]

This can be scaled down to a polymer function $(BE)_{L^{-1}}$ on $T_{N-k-1}^{-k-1}$ by
\[
(BE)_{L^{-1}}(X, A, \phi) = (BE)(LX, A_L, \phi_L) \tag{240}
\]

and then
\[
|(BE)_{L^{-1}}\|_{k+1, L(k-\kappa_0-1)} \leq \mathcal{O}(1)L^3 \|E\|_{k, \kappa} \tag{241}
\]

Note that if $\kappa$ is large enough then $L(k-\kappa_0-1) > \kappa$ and we can take $\kappa$ on the left. But the $L^3$ means that the size can grow.

### 5.5 symmetries

We consider polymer functions $E(X, \phi, A) \in \mathcal{K}_k$ which are invariant under the following symmetries

1. (lattice symmetries) If $r$ is a $T_{N-k}^0$ unit lattice symmetry and $A_r, \phi_r$ are the transformed fields then $E(rX, A_r, \phi_r) = E(X, A, \phi)$.

2. (gauge invariance) $E(X, A^\lambda, \phi^\lambda) = E(X, A, \phi)$.

3. (charge conjugation invariance) $E(X, -A, C\phi) = E(X, A, \phi)$.

Here are some consequences. The $n^{th}$ derivative of $E(X, \phi, A)$ in $A$ at $\phi = 0, A = 0$ is is the multilinear functional
\[
\frac{\delta^n E}{\delta A^n}(X, 0; f_1, \ldots, f_n) = \frac{\partial^n(t_1 f_1 + \cdots + t_n f_n, 0)}{t_n} \bigg|_{t_n = 0} \tag{242}
\]

If one of the functions $f_i = \partial \lambda$ then by gauge invariance there is no dependence on $t_i$ and the derivative vanishes. Thus we have the Ward identity
\[
\frac{\delta^n E}{\delta A^n}(X, 0; f_1, \ldots, \partial \lambda, \ldots, f_n) = 0 \tag{243}
\]

A special case of gauge invariance is rotation in charge space. If $e_k \lambda = \theta = \text{constant}$ then
\[
E(X, A, e^{\theta} \phi) = E(X, A, \phi) \tag{244}
\]

A rotation by $\theta = \pi$ in charge space gives $E(X, A, -\phi) = E(X, A, \phi)$. Hence any odd number of $\phi$ derivatives at $\phi = 0$ gives zero. Therefore
\[
\begin{align*}
\frac{\delta E}{\delta \phi}(X, 0) &= 0 \\
\frac{\delta^2 E}{\delta \phi \delta A}(X, 0) &= 0 \\
\frac{\delta^3 E}{\delta \phi \delta A^2}(X, 0) &= 0 \\
\frac{\delta^3 E}{\delta \phi^2}(X, 0) &= 0 \tag{245}
\end{align*}
\]

Charge conjugation invariance gives $E(X, -A, 0) = E(X, A, 0)$ and this implies
\[
\begin{align*}
\frac{\delta E}{\delta A}(X, 0) &= 0 \\
\frac{\delta^3 E}{\delta A^3}(X, 0) &= 0 \tag{246}
\end{align*}
\]

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5.6 normalization

As we iterate the RG transformations the scaling operation can increase the size of the polymer functions by as much as $O(L^3)$ as is evident from (241). We have to watch this carefully and start with a discussion of what criteria we need to avoid this growth. The following generalizes the analysis in [19], [24].

**Definition 3.** A polymer function $E(X, \phi, A)$ with the stated symmetries is said to be normalized if in addition to the vanishing derivatives (243), (245), (246) we have for $1 \leq i, j \leq 2$ and some $x_0 \in X$

$$E(X, 0) = 0 \quad \frac{\delta^2 E}{\delta \phi^2} (X, 0; e_i, e_j) = 0 \quad \frac{\delta^2 E}{\delta \phi^2} (X, 0; e_i, (\cdot - x_0)_\mu e_j) = 0$$

Define a polymer $X$ to be small if $d_M(X) < L$ and large if $d_M(X) \geq L$. The set of all small polymers in denoted $\mathcal{S}$. Next we show that a polymer function normalized for small polymers under scaling.

**Lemma 16.** Let $E(X, A, \phi)$ be normalized for small polymers. Then for $L$ sufficiently large and $e_k, \lambda_k$ sufficiently small (depending on $L, M$) and $\frac{\partial}{\partial e_k} \leq \alpha < \frac{\delta}{\lambda}$

$$\| (BE)_{L^{-1}} \|_{k+1, \alpha} \leq O(1)L^{-\alpha}\|E\|_{k, \alpha}$$

**Proof.** This follows a similar proof in [24], where one can find more details. For large sets $d_M(X) \geq L$, We can borrow a factor $e^{-L}$ from the decay factor $e^{-\kappa d_M(X)}$. This beats the $L^3$ and and gives an estimate $O(1)L^{-n}$ for any $n$.

For small sets $X$ we will show that for $A, \phi \in \mathcal{R}_{k+1}$

$$|E(X, A, \phi, L)| \leq O(1)L^{-3}\|E(X)\|_k$$

This improves on the general bound $|E(X, A, L, \phi, \phi)| \leq O(1)|E(X)|_k$ which was the input to (241). The extra factor $L^{-3}$ beats the $L^3$ and yields the result.

Every small set $X$ intersects some $M$-cube $\Box$ and hence is contained in some $\Box$. By lemma 15 $(A, \phi)$ is gauge equivalent in $\Box$ to $(A', \phi')$ satisfying the bounds (229), (230). Since $E(X)$ is gauge invariant it suffices show that $E(X, A', \phi')$ satisfies (241) for fields satisfying (229), (230).

We make a further gauge transformation. Pick a point $x_0 \in X$. Since the constant $A'_\mu(x_0) = \partial_\mu(A'(x_0) \cdot (x - x_0)) \equiv \partial \lambda$ is pure (complex) gauge in $\Box$ we can define

$$\tilde{A}(x) = \lambda'(x) - \partial \lambda = A'(x) - A'(x_0)$$

$$\tilde{\phi}(x) = e^{i\delta}e^{\lambda(x)}e^{\partial}(x)$$

We claim that the new fields satisfy the bounds

$$|\tilde{A}| \leq L^{-2+\epsilon}[e_k^{-1+\epsilon}] \quad |\partial \tilde{A}| \leq L^{-2+\epsilon}[e_k^{-1+2\epsilon}] \quad |\delta_\alpha \partial \tilde{A}| \leq L^{-2+2\epsilon}[e_k^{-1+3\epsilon}]$$

and

$$|\tilde{\phi}| \leq 3L^{-\frac{\epsilon}{2}}[e_k^{-\frac{1}{2}-\epsilon}] \quad |\partial_\alpha \tilde{\phi}| \leq 3L^{-\frac{\epsilon}{2}-\epsilon}[e_k^{-\frac{1}{2}-2\epsilon}] \quad |\delta_\alpha \partial_\alpha \tilde{\phi}| \leq 3L^{-\frac{\epsilon}{2}-\alpha-\epsilon}[e_k^{-\frac{1}{2}-\epsilon}]$$

Indeed since $X$ is small it has diameter less than $M|X|_M \leq O(1)M(d_M(X) + 1) \leq O(1)ML$. We assume $e_k$ is small enough so $O(1)MLe_k \leq 1$. Then we have the improved bound

$$|\tilde{A}| \leq O(1)ML \|\partial A'\|_\infty \leq (O(1)ML)\left[ e_k^{-1+2\epsilon} \right] \leq (O(1)ML)e_k^{-1+2\epsilon} \leq L^{-2+\epsilon}[e_k^{-1+\epsilon}]$$

(253)
The bounds on derivatives stay the same. The gauge function satisfies $|\lambda| \leq O(1)ML|A'(x_0)| \leq CM^{-1+\epsilon} \leq e^{-1}$ and so the bounds on the scalar field are only altered by the inconsiderable $|e^{\epsilon k|\lambda(x)|}| \leq e^{\epsilon k|\lambda(x)|} \leq e < 3$.

We now have $(\tilde{A}, \tilde{\phi}) \in 3L^{-\frac{3}{4}+\epsilon}R_k$ and since $E(A', \phi') = E(\tilde{A}, \tilde{\phi})$ it suffices to prove the bound\(^{249}\) for fields satisfying \(^{251}, \ 252\).

We make a Taylor expansion of $t \to E(X, t\tilde{A}, t\tilde{\phi})$ around $t = 0$ and evaluate at $t = 1$. For complex $t$ with $|t| \leq \frac{1}{4}L^{\frac{3}{4}+\epsilon}$ we have $(t\tilde{A}, t\tilde{\phi}) \in \frac{1}{4}R_k$. Taking account the vanishing derivatives and choosing $r = \frac{1}{4}L^{\frac{3}{4}+\epsilon}$ the expansion is then

$$E(X, \tilde{A}, \tilde{\phi}) = \frac{1}{2} \frac{\delta^2 E}{\delta \tilde{A}^2} (0; \tilde{A}, \tilde{\phi}) + \frac{1}{2} \frac{\delta^2 E}{\delta \tilde{\phi}^2} (0; \tilde{A}, \tilde{\phi}) + \frac{1}{2} \frac{\delta^2 E}{\delta \tilde{A} \delta \tilde{\phi}} (0; \tilde{A}, \tilde{\phi})$$

(254)

Since $|E(X, t\tilde{A}, t\tilde{\phi})| \leq \|E(X)\|_k$ the last term in (254) is bounded by $O(1)L^{-3-4\epsilon}\|E(X)\|_k$ which suffices.

With $\phi = 0$ the first term can be expressed in a larger analyticity domain as

$$1 \frac{\delta^2 E}{\delta \tilde{A}^2} (0; \tilde{A}, \tilde{\phi}) = \frac{1}{2\pi i} \int_{|t| = L^{3-\epsilon}} \frac{dt}{t^3} E(X, t\tilde{A}, 0)$$

(255)

Then this term is bounded by $O(1)L^{-4+2\epsilon}\|E(X)\|_k$ which suffices.

Next consider the term $(\delta^3 E/\delta A \delta \phi^2)(0; \tilde{A}, \tilde{\phi}, \tilde{\phi})$ in (254). Now $(t, s) \to E_k(X, tA, s\phi)$ is analytic in $|t| \leq L^{1+\epsilon}$ and $|s| \leq L^{\frac{3}{4}+\epsilon}$ and so

$$1 \frac{\delta^3 E}{\delta \tilde{A} \delta \tilde{\phi}^2} (0, \tilde{A}, \tilde{\phi}, \tilde{\phi}) = \frac{1}{(2\pi i)^2} \int_{|t| = L^{1+\epsilon}} \frac{dt}{t^2} \int_{|s| = L^{\frac{3}{4}+\epsilon}} \frac{ds}{s^3} E_k(X, t\tilde{A}, s\tilde{\phi})$$

(256)

Then this term is bounded by $O(1)L^{-2+\epsilon}L^{-\frac{3}{4}+2\epsilon}\|E(X)\|_k = O(1)L^{-\frac{3}{4}+3\epsilon}\|E(X)\|_k$ which suffices.

For the analysis of the term $(\delta^2 E/\delta \phi^2)(0; \tilde{\phi}, \tilde{\phi})$ in (254) we write

$$\tilde{\phi}(x) = \phi(x_0) + (x - x_0) \cdot \partial \tilde{\phi}(x_0) + \Delta(x, x_0)$$

(257)

and expand taking account the vanishing derivatives

$$\frac{\delta^2 E}{\delta \tilde{\phi}^2} (X, 0; \tilde{\phi}, \tilde{\phi}) = \frac{\delta^2 E}{\delta \phi^2} (X, 0; (-x_0) \cdot \partial \tilde{\phi}(x_0), (-x_0) \cdot \partial \tilde{\phi}(x_0))$$

$$+ \frac{\delta^2 E}{\delta \phi^2} (X, 0; (-x_0) \cdot \partial \tilde{\phi}(x_0), \Delta)$$

$$+ \frac{\delta^2 E}{\delta \phi^2} (X, 0; \Delta, \Delta) + \frac{\delta^2 E}{\delta \phi^2} (X, 0; \tilde{\phi}(x_0), \Delta)$$

(258)

All these terms can be estimated by Cauchy bounds and the information that

$$\tilde{\phi}(x_0) \in L^{-\frac{5}{4}+\epsilon}R_k \quad (x - x_0) \cdot \partial \tilde{\phi}(x_0) \in L^{-\frac{5}{4}-2\epsilon}R_k \quad \Delta \in L^{-\frac{5}{4}+\epsilon}R_k$$

(259)

See \(^{241}\) for estimates of this form (where the exponents are a little different). The first term is then $O(1)L^{-10/3-4\epsilon}\|E(X)\|_k$ which suffices. The second and third terms are even smaller. The last term is less than $O(1)L^{-29/12-\alpha-2\epsilon}\|E(X)\|_k$, which suffices since we are assuming $\alpha \geq \frac{7}{12}$.

Thus (249) is established and the lemma is proved.

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5.7 arranging normalization

The next result shows that if we remove certain relevant terms from the polymer function, the remainder is normalized.

Given $E(X,A,\phi)$ on $\mathbb{T}^{k}_{N_{-k}}$ satisfying lattice, gauge, and charge conjugation symmetries we define $(RE)(X,A,\phi)$ as follows. If $X$ is large then $(RE)(X,A,\phi) = E(X,A,\phi)$. If $X$ is small ($X \in S$) then $(RE)(X)$ is defined by

$$E(X,A,\phi) = \alpha_0(E,X)Vol(X) + \alpha_2(E,X) \int_X |\phi|^2 + \sum_{\mu} \alpha_{2,\mu}(E,X) \int_X \phi \cdot \nabla_{A,\mu} \phi + (RE)(X,A,\phi)$$

(260)

where

$$\nabla_{A,\mu} = \frac{1}{2} (\partial_{A,\mu} - \partial_{A,\mu}^T) \quad \text{Vol}_\mu(X) = \sum_{x \in X,x = \eta e_\mu \in X} \eta^3$$

(261)

and

$$\alpha_0(E,X) = \frac{1}{\text{Vol}(X)} E(X,0) \quad \alpha_2(E,X) = \frac{1}{2} \frac{\delta^2 E}{\delta \phi^2}(X,0;e_i,e_j)$$

$$\alpha_{2,\mu}(E,X) = \frac{1}{\text{Vol}_\mu(X)} \left( \frac{\delta^2 E}{\delta \phi^2}(X,0;e_i,\cdot - x^0)e_j) - \frac{1}{\text{Vol}(X)} \frac{\delta^2 E}{\delta \phi^2}(X,0;e_i,e_j) \int_X (x_\mu - x^0_\mu)dx \right)$$

(262)

The expression for $\alpha_{2,\mu}(E,X)$ is independent of the base point $x^0$, which we take to be in $X$. To see that $\delta^2 E/\delta \phi^2(X,0;e_i,e_j)$ is proportional to $\delta_{ij}$ first note that charge conjugation says $\delta^2 E/\delta \phi^2(X,0;e_1,e_2) = \delta^2 E/\delta \phi^2(X,0;C e_1,C e_2)$. But $C e_1 = e_1$ and $C e_2 = - e_2$ so this is zero. The identity $\delta^2 E/\delta \phi^2(X,0;e_1,e_1) = \delta^2 E/\delta \phi^2(X,0;e_2,e_2)$ follows by rotation invariance. The same argument works for $\delta^2 E/\delta \phi^2(X,0;e_1,(x - x^0)\mu e_j)$.

The term $\int_X \phi \cdot \nabla_{A,\mu} \phi$ requires some additional comment. The derivative $\nabla_{A,\mu}$ is the average of a forward and a backward derivative, and we use it because transforms like a vector field under lattice symmetries - see appendix B. This would not be the case with just the forward derivative $\partial_{A,\mu}$. In the expression $\int_X \phi \cdot \nabla_{A,\mu} \phi$ we only include bonds in $X$. To accomplish this write it as $\int_X \phi_X \cdot \nabla_{A,\mu} \phi_X$ where

$$\phi_X(x) = \begin{cases} \phi(x) & x \in X \\ \phi(x \pm \eta e_\mu) & x \notin X, x \pm \eta e_\mu \in X \end{cases}$$

(263)

Then if $r$ is a lattice symmetry $(\phi_{r})_{X} = (\phi_{X})_{r}$ and so $\int_X \phi_X \cdot \nabla_{A,\mu} \phi_X$ is covariant. We need this property to guarantee that $RE$ is covariant under lattice symmetries.

**Lemma 17.** $RE$ is invariant under lattice, gauge, and charge symmetries. $RE$ is normalized for small polymers and satisfies for $c_k, \lambda_k$ sufficiently small

$$\|RE\|_{k,\kappa} \leq O(1) \|E\|_{k,\kappa}$$

(264)

**Proof.** The invariance follows since everything else in (260) is invariant. The derivatives in question match on the left and right except for the term $RE$, hence its derivatives vanish. The bound holds since everything else in (260) satisfies the bound. See [24] for more details.

For global quantities we only have to remove energy and mass terms.

---

3 In an equation like (267) we are allowed to use a forward derivative since we are estimating something we already know to be invariant. The substitution $\partial_{\mu} \rightarrow \nabla_{\mu}$ should also be made in equation (157) in [24].
Corollary 1. 
\[
\sum_X E(X) = -\varepsilon(E) \text{Vol}(T_{n-k}) - \frac{1}{2} \mu(E) \|\phi\|^2 + \sum_X R E(X) 
\]  
(265) 

where 
\[
\varepsilon(E) = -\sum_{X \supset \square, X \in S} \alpha_0(E, X) \\
\frac{1}{2} \mu(E) = -\sum_{X \supset \square, X \in S} \alpha_2(E, X) 
\]  
(266) 

Furthermore 
\[
|\varepsilon(E)| \leq O(1) \|E\|_{k,\kappa} \quad \mu(E) \leq O(1) \lambda_1^{2+2\epsilon} \|E\|_{k,\kappa} 
\]  
(267) 

Proof. Sum (260) over \(X\) and rearrange. The \(\phi \cdot \nabla_{\mathcal{A},\mu} \phi\) term vanishes since 
\[
\sum_{X \supset \square, X \in S} \alpha_{2,\mu}(E, X) = 0 
\]  
(268) 

This follows since if \(r\) is a reflection in the \(\mu\) direction \(\alpha_{2,\mu}(E, rX) = -\alpha_{2,\mu}(E, X)\). Take a reflection through the center of \(\square\). 

The bound on \(\varepsilon(E)\) follows directly, and the bound on \(\mu(E)\) uses a Cauchy bound. See [24] for details.

5.8 localized Green’s functions

We can also localize the scalar Green’s functions with polymers using the random walk expansion (122). For a walk \(\omega = (\omega_0, \omega_1, \ldots, \omega_n)\) define \(X'_\omega = \bigcup_{i=0}^n \square \omega_i\). Then write 
\[
G_k(\mathcal{A}) = \sum_{X \in D_k} G_k(X, \mathcal{A}) 
\]  
(269) 

where 
\[
G_k(X, \mathcal{A}) = \sum_{\omega: X'_\omega = X} G_{k,\omega}(\mathcal{A}) = \sum_{n=0}^{\infty} \sum_{\omega: |\omega| = n, X'_\omega = X} G_{k,\omega}(\mathcal{A}) 
\]  
(270) 

Then \(G_k(X, \mathcal{A})\) only depends on \(\mathcal{A}\) in \(X\), and the kernel \(G_k(X, \mathcal{A}, x, y)\) vanishes unless \(x, y \in X\). 

Recall that if \(|\omega| = n\) 
\[
|G_{k,\omega}(\mathcal{A})f| \leq C(CM^{-1})^n \|f\|_{\infty} 
\]  
(271) 

But \(d_M(X) \leq |X|_M = |X'_\omega|_M \leq 27(n + 1)\) so we can make the estimate 
\[
(CM^{-1})^{n/2} \leq O(1)(CM^{-1})^{d_M(X)/54} \leq O(1)e^{-\kappa d_M(X)} 
\]  
(272) 

for \(M\) sufficiently large. The remaining factor \((CM^{-1})^{n/2}\) still gives the overall convergence of the series. Thus we have the bound 
\[
|G_k(X, \mathcal{A})f| \leq Ce^{-\kappa d_M(X)} \|f\|_{\infty} 
\]  
(273) 

as well as bounds on the derivatives and \(L^2\) bounds.
6 The main theorem

6.1 The theorem

The starting density on $\mathbb{T}^0_N$ from (15) is

$$\rho_0(A_0, \Phi_0) = \exp \left( -\frac{1}{2} \|dA_0\|^2 - \frac{1}{2} \|d\Phi_0\|^2 - V_0(\Phi_0) \right)$$

(274)

For the full analysis of the model we define a sequence of densities $\rho_k(A_k, \Phi_k)$ for fields on $\mathbb{T}^0_{N-k}$ by successive RG transformations. First for fields on $\mathbb{T}^1_{N-k}$ we define as in (29) and (115)

$$\hat{\rho}_{k+1}(A_{k+1}, \Phi_{k+1}) = \int \delta \left( A_{k+1} - QA_k \right) \delta(\tau A_k) \delta_G \left( \Phi_{k+1} - Q(\tilde{A}_{k+1}) \Phi_k \right) \rho_k(A_k, \Phi_k) DA_k D\Phi_k$$

(275)

We have chosen a background field $\tilde{A}_{k+1}$ which is a smeared out version of $A_{k+1}$ and defined precisely later on. Then we scale to fields on $\mathbb{T}^0_{N-k-1}$ by

$$\rho_{k+1}(A_{k+1}, \Phi_{k+1}) = \hat{\rho}_{k+1}(A_{k+1}, \Phi_{k+1})$$

(276)

and scaling is the same. New are the characteristic functions $\chi_k \chi^w_k$ enforcing bounds on the fields. Here $\chi_k = \chi((A_k, \Phi_k) \in \mathcal{S}_k)$ is the characteristic function of the small field region $\mathcal{S}_k$ as defined in section 5.3. The other characteristic function $\chi^w_k$ restricts the fluctuation field and is defined by

$$\chi^w_k = \chi_k \left( C_k(\tilde{A}_k) - \frac{1}{2} (\Phi_k - H_k(\tilde{A}_k) \Phi_k) \right) \chi^w_k \left( C_k^{-1} (A_k - H_k^2 \tilde{A}_k) \right)$$

(278)

where $\chi^w_k(W)$ is the characteristic function of $|W| \leq p_{0,k}$ and $p_{0,k} = (-\log \lambda_k)^p$ for some $p_0 < p$. These restrictions are natural in Balaban’s formulation of the renormalization group. Our goal is to study the flow of these modified transformations. As noted earlier this is the location of the renormalization problem.

We are going to claim that after $k$ steps we have a density $\rho_k$ defined on the domain $\mathcal{S}_k$ essentially of the form

$$\rho_k(A_k, \Phi_k) = N_k Z_k Z_k(A_k) \exp \left( -\frac{1}{2} \|\partial A_k\|^2 - S_{k, A_k}(\Phi_k, \phi_k(A_k)) - V_k(\phi_k(A_k)) + E_k(A_k, \phi_k(A_k)) \right)$$

(279)

where

$$A_k = A_k(A_k) = H_k A_k \quad \phi_k(A) = \phi_k(A, \Phi_k) = H_k(A) \Phi_k$$

(280)

and where

$$S_{k, A}(\Phi_k, \phi) = \frac{a_k}{2} \|\Phi_k - Q_k(A)\|^2 + \frac{1}{2} \|\partial A \phi\|^2$$

$$V_k(\phi) = \epsilon_k \text{Vol}(\mathbb{T}_{N-k}) + \frac{1}{2} \mu_k \|\phi\|^2 + \frac{1}{4} \lambda_k \int |\phi(x)|^4 dx$$

(281)

$$E_k(A, \phi) = \sum_x E_k(X, A, \phi)$$
Note that this is true for $k = 0$ with $Z_0 = Z_0(A) = 1$, $E_0 = 0$, and the convention that $A_0 = A_0$ and $\phi_0(A_0) = \Phi_0$ and $Q_0(A_0) = I$ so that $\Phi_0 = Q_0(A_0)\phi_0(A_0) = 0$.

We assume that $L$ is sufficiently large, $M$ is sufficiently large (depending on $L$), and that $e, \lambda$ are sufficiently small (depending on $L, M$). For definiteness we take $e \leq \lambda^{1/2}$ and then $e_k \leq \lambda_k^{1/2}$ for all $k$.

**Theorem 1.** Under these assumptions suppose $\rho_k(A_k, \Phi_k)$ has the representation (279) for $A_k, \Phi_k \in S_k$. Suppose the polymer function $E_k(X, A, \phi)$ is defined on $R_k$, has all the symmetries of section 5, and is normalized for small polymers. Finally suppose

$$|\mu_k| \leq \frac{1}{2} \lambda_k^{1/2} \quad ||E_k||_{k, \kappa} \leq 1 \quad (282)$$

Then up to a phase shift $\rho_{k+1}(A_{k+1}, \Phi_{k+1})$ has a representation of the same form for $A_{k+1}, \Phi_{k+1} \in S_{k+1}$, now with $e_{k+1} = L^{1/2}e_k$ and $\lambda_{k+1} = L\lambda_k$. The bounds are not the same but we do have

$$\begin{align*}
\varepsilon_{k+1} &= L^3\varepsilon_k + L_1E_k + \varepsilon_k^*(\mu_k, E_k) \\
\mu_{k+1} &= L^2\mu_k + L_2E_k + \mu_k^*(\mu_k, E_k) \\
E_{k+1} &= L_3E_k + E_k^*(\mu_k, E_k)
\end{align*} \quad (283)$$

The $L_i$ are linear operators which satisfy

$$|L_1E_k| \leq O(1)L^{-\gamma}||E_k||_{k, \kappa}$$

$$|L_2E_k| \leq O(1)L^{-\gamma}\lambda_k^{1/2+2\gamma}||E_k||_{k, \kappa} \quad (284)$$

and we have the bounds

$$\begin{align*}
|\varepsilon_k^*| &\leq O(1)\lambda_k^{-11\epsilon} \\
|\mu_k^*| &\leq O(1)\lambda_k^{-11\epsilon} \\
||E_k^*||_{k+1, \kappa} &\leq O(1)\lambda_k^{-11\epsilon}
\end{align*} \quad (285)$$

**Remarks.**

1. The phrase ”up to a phase shift” means we actually show that $\rho_{k+1}(A_{k+1}, e^q\Phi_{k+1})$ has the form (279) for some real function $\theta_{k+1} = \theta_{k+1}(A_{k+1})$. Changing it back to $\rho_{k+1}(A_{k+1}, \Phi_{k+1})$ changes the definition of the RG transformation, but does not change the basic property that the integral over $\Phi_{k+1}$ is the same for each $k$.

2. By lemma 14 we have that $A_k, \Phi_k \in S_k$ implies $A_k, \phi_k(A_k) \in R_k$ so that $E_k(X, A, \phi_k(A_k))$ is well-defined.

3. The polymer functions $E_k$ contain all parts of the interaction not in $S_{k,A_k}$ or $V_k$. These are growing at a controlled rate because we have extracted corrections $\varepsilon_k^*(\mu_k, E_k)$ to the energy density and $\mu_k^*(\mu_k, E_k)$ to the mass squared.

The terms $L_i(E_k)$ are the result of normalizing terms which newly qualify as small polymers. (They are not the full linearization of the mapping.) The starred terms are the result of the fluctuation integral and include contributions from both $E_k$ and $V_k$.

4. We have the weak bound $||E_k||_{k, \kappa} \leq 1$ or $|E_k(X, A, \phi)| \leq e^{-\kappa d_M(X)}$ because we are allowing the fields to be somewhat large. But $E_k$ is actually small. For example if $A, \phi$ and derivatives are $O(1)$, then for $|t| \leq \lambda_k^{-1/4} \leq \varepsilon_k^{-1/2}$ we have $tA, t\phi \in R_k$. Hence since $E_k$ is normalized

$$E_k(X, A, \phi) = E_k(X, A, \phi) - E_k(X, 0) = \frac{1}{2\pi i} \int_{|t|=\lambda_k^{-1/4}} \frac{dt}{t(t-1)} E_k(X, tA, t\phi) \quad (286)$$

This gives the bound $|E_k(X, A, \phi)| \leq O(1)\lambda_k^{1/4} e^{-\kappa d_M(X)}$. 

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6.2 proof of the theorem

The proof follows the broad outlines of Balaban, Imbrie, and Jaffe [18], but differs in many details.

6.2.1 preliminaries

We define operators \( \mathcal{H}_{k+1}^0, \mathcal{H}_{k+1}^0(A) \) on \( T_{N-k}^0 \) and fields \( A_{k+1}^0, \phi_{k+1}^0(A) \) on \( T_{N-k}^0 \) which are scalings of \( A_{k+1}, \phi_{k+1}(A) \). For \( A_{k+1}, \Phi_{k+1} \) on \( T_{N-k}^1 \) we define

\[
A_{k+1}^0(A_{k+1}) = \mathcal{H}_{k+1}^0 A_{k+1} = (\mathcal{H}_{k+1} A_{k+1,L^{-1}}) L = (A_{k+1}(A_{k+1,L^{-1}})) L \\
\phi_{k+1}^0(A, \Phi_{k+1}) = \mathcal{H}_{k+1}^0(A) \Phi_{k+1} = (\mathcal{H}_{k+1}(A_{L^{-1}}) \Phi_{k+1,L^{-1}}) L = (\phi_{k+1}(A_{L^{-1}}, \Phi_{k+1,L^{-1}})) L
\]

(287)

These scale to \( A_{k+1}, \phi_{k+1}(A) \), for examples if \( A_{k+1} \) on \( T_{N-k-1}^{-1} \) then \( A_{k+1}^0(A_{k+1}) = (A_{k+1}(A_{k+1})) L \)

We can also write

\[
\phi_{k+1}^0(A, \Phi_{k+1}) = \mathcal{H}_{k}(A) H_k(A) \Phi_{k+1}
\]

(288)

by the identity [4]. But the analogous formula for the gauge field would only hold if we were using the axial gauge at this point.

We study \( \tilde{\rho}_{k+1}(A_{k+1}, \Phi_{k+1}) \) for fields \( A_{k+1}, \Phi_{k+1} \) in \( \mathcal{S}_{k+1}^0 \), the scaled version of \( \mathcal{S}_{k+1} \). The space \( \mathcal{S}_{k+1}^0 \) is defined as all \( A_{k+1}, \Phi_{k+1} \) on \( T_{N-k}^1 \) satisfying

\[
|dA_{k+1}^0| \leq L^{-\frac{k}{2}} p_{k+1}
\]

and

\[
|\Phi_{k+1} - Q_{k+1}(A_{k+1})| \leq L^{-\frac{k}{4}} p_{k+1} \\
|\partial \phi_{k+1}^0(A_{k+1})| \leq L^{-\frac{k}{4}} p_{k+1} \\
|\phi_{k+1}^0(A_{k+1})| \leq L^{-\frac{k}{4}} p_{k+1} \lambda_{k+1}^{-\frac{1}{4}}
\]

(290)

Then \( A_{k+1,L^{-1}}, \phi_{k+1,L^{-1}} \) on \( T_{N-k-1}^0 \) satisfy the conditions for \( \mathcal{S}_{k+1} \) and we conclude by lemma [14] that \( A_{k+1}(A_{k+1,L^{-1}}), \phi_{k+1}(A_{k+1,L^{-1}}) \) satisfy the bounds for \( \frac{1}{2} R_{k+1} \). This is the same as saying \( (A_{k+1}(A_{k+1})), \phi_{k+1}(A_{k+1}(A_{k+1})), \Phi_{k+1}(A_{k+1}) \) satisfy the bounds for \( \frac{1}{2} R_{k+1} \). Then lemma [13] says that \( A_{k+1}(A_{k+1}), \phi_{k+1}(A_{k+1}(A_{k+1})), \Phi_{k+1}(A_{k+1}) \) satisfies the bounds (229), (230) and in particular

\[
(A_{k+1}^0, \phi_{k+1}^0(A_{k+1})) \in L^{-\frac{k}{2}-\epsilon} R_k
\]

(291)

6.2.2 gauge field translation

Now for \( A_{k+1}, \Phi_{k+1} \in \mathcal{S}_{k+1}^0 \)

\[
\tilde{\rho}_{k+1}(A_{k+1}, \Phi_{k+1}) = \int D\Phi_k D\Phi_k \chi_{k}^w \chi_{k}^w \delta(A_{k+1} - QA_k) \delta(\tau A_k - Q(\tilde{A}_{k+1})) \Phi_{k+1} \exp \left( -\frac{1}{2} \|\partial A_k\|^2 - S_{k, A_k}(\Phi_k, \phi_k(A_k)) - V_k(\phi_k(A_k)) + E_k(A_k, \phi_k(A_k)) \right)
\]

(292)

We translate to the minimum of \( S_k(A_k) \) on the surface \( QA_k = A_{k+1}, \tau A_k = 0 \) as before. Write \( A_k = H_k A_k + Z \) and integrate over \( Z \) instead of \( A_k \). Then \( A_k = \mathcal{H}_k A_k \) becomes \( A_{k+1} + Z_k \) where

\[
\tilde{A}_{k+1} = \mathcal{H}_k H_k A_{k+1} \\
Z_k = \mathcal{H}_k Z
\]

(293)
This is the $\tilde{A}_{k+1}$ that appears in (274) and in (278). Next we use $H_k^x = H_k + \partial D_k$ and the scaled version $H_k^0 = H_k^0 + \partial D_k^0$ to change from the axial gauge to the Landau gauge. Using also (159) we obtain

$$\tilde{A}_{k+1} = H_k^0 A_{k+1}$$

$$= H_k^0 A_{k+1} - \partial D_k H_k^0 A_{k+1}$$

$$= H_k^0 A_{k+1} - \partial D_k H_k^0 A_{k+1}$$

$$= H_k^0 A_{k+1} - \partial D_k H_k^0 A_{k+1}$$

$$= H_k^0 A_{k+1} - \partial (D_k H_k^0 - D_k^0) A_{k+1}$$

$$= A_{k+1} - \partial \omega_{k+1}$$

where the last line defines $\omega_{k+1} = \omega_k(A_{k+1})$. As in section 4.1 we find that $\frac{1}{2}||dA_k||^2$ become $\frac{1}{2}||dA_k^0||^2 + \frac{1}{2}Z, \Delta_k Z$ and since $Z_k(A)$ is gauge invariant we have

$$\tilde{A}_{k+1} = N_k Z_k \exp \left( -\frac{1}{2} ||A_k^0||^2 \right) \int D\Phi_k^k Z_k \exp \left( -\frac{1}{2} \langle Z, \Delta_k Z \rangle \right) \delta(QZ) \delta(\tau Z)$$

$$\chi_k \chi_k^w \delta_c \left( \Phi_k^k - Q(A_k^0 \partial \omega_k) \Phi_k \right) Z_k + A_{k+1}^0 + Z_k$$

$$\exp \left( -S_{k,A}^k A_{k+1}^0 + Z_k - \partial \omega_k \right) \left( \Phi_k, \phi_k(A_k^0 + Z_k - \partial \omega_k) \right)$$

$$\exp \left( -V_k \left( \phi_k(A_k^0 + Z_k - \partial \omega_k) \right) + E_k \left( A_{k+1}^0 + Z_k - \partial \omega_k, \phi_k(A_k^0 + Z_k - \partial \omega_k) \right) \right)$$

As in section 4.2 we replace $Z$ by $CZ$ and identify $(Z_k^1)^{-1} \delta(QZ) \delta(\tau Z)$ as the Gaussian measure $d\mu_{C_k}(Z)$. We now understand $Z_k$ as $Z_k = H_k C Z$.

If $\omega_k^{(0)}$ the restriction of $\omega_k$ to the unit lattice $T_{N-k}^N$ then by (33)

$$\phi_k(A - \partial \omega_k) = H_k(A - \partial \omega_k) \Phi_k = e^{q\epsilon k \omega_k} H_k(A) e^{-q\epsilon k \omega_k^{(0)}} \Phi_k$$

We also change variables by $\Phi_k \rightarrow e^{q\epsilon k \omega_k^{(0)}} \Phi_k$. This is a rotation so the Jacobian is one. Then $\phi_k(A - \partial \omega_k)$ becomes $e^{q\epsilon k \omega_k} \phi_k(A)$ and

$$S_{k,A} - \partial \omega_k \left( e^{q\epsilon k \omega_k} \Phi_k, e^{q\epsilon k \omega_k} \phi_k(A) \right) = S_{k,A} \left( \Phi_{k+1}, \phi_k(A) \right)$$

The $\omega_k$ also disappears from the gauge invariant $V_k, E_k$.

We also note that by (241)

$$\delta_c \left( \Phi_k^k - Q(A_k^0 - \partial \omega_k) e^{q\epsilon k \omega_k^{(0)}} \Phi_k \right) = \delta_c \left( \Phi_k^k - e^{q\epsilon k \omega_k^{(1)}} Q(A_k^0) \Phi_k \right)$$

where $\omega_k^{(1)}$ is the restriction of $\omega_k$ to $T_{N-k}^N$. We replace $\Phi_{k+1}$ by $e^{q\epsilon k \omega_k^{(1)}} \Phi_{k+1}$ so the phase factor here disappears as well.

Similar considerations show that the bounds enforced by the characteristic function $\chi_k$ are now

$$|d(A_{k+1}^0 + Z_k)| \leq p_k \quad |\Phi_k - Q_k(A_k^0 + Z_k) \phi_k(A_k^0 + Z_k)| \leq p_k$$

$$|\partial A_k^0 + Z_k \phi_k(A_k^0 + Z_k)| \leq p_k \quad |\phi_k(A_k^0 + Z_k)| \leq x^{-\frac{4}{14}} p_k$$

From the representation (155) we have

$$C_k(A_{k+1}) \frac{n}{2} = C_k(A_{k+1}^0 - \partial \omega_k) = e^{q\epsilon k \omega_k^{(0)}} C_k(A_{k+1}^0)^{q\epsilon k \omega_k^{(0)}}$$
The same holds for \( C_k(\tilde{A}_{k+1})^{-\frac{1}{2}} \) and with the phase shifts on \( \Phi_k, \Phi_{k+1} \) we now have

\[
\chi_k^w = \chi_k^w \left( C_k(A_{k+1}^0)^{-\frac{1}{2}}(\Phi_k - H_k(A_{k+1}^0)\Phi_{k+1}) \right) \chi_k^w \left( C_k^{-\frac{1}{2}}\tilde{Z} \right)
\]  

(301)

With these changes:

\[
\tilde{\rho}_{k+1}(A_{k+1}, e^{q_{k+1}^w} \Phi_{k+1}) = N_k Z_k Z_k^T \exp \left( -\frac{1}{2} \| \partial A_{k+1}^0 \|^2 \right) \int d\mu_{C_k}(\tilde{Z}) D\Phi_k \\
\chi_k \chi_k^w \delta_G \left( \Phi_{k+1} - Q(A_{k+1}^0)\Phi_k \right) Z_k \left( A_{k+1}^0 + \tilde{Z}_k \right) \exp \left( -S_{k,A_{k+1}^0 + \tilde{Z}_k} \left( \Phi_k, \phi_k(A_{k+1}^0 + \tilde{Z}_k) \right) \right)
\]  

(302)

Next we separate out leading terms in an expansion in the fluctuation field \( Z_k \). First for general \( \tilde{Z} \) on \( T_{-k}^N \) define:

\[
\delta \phi_k(A, Z, \Phi_k) \equiv \phi_k(A + Z, \Phi_k) - \phi_k(A, \Phi_k)
\]  

(303)

Then in \( \tilde{Z} \) we can make the replacement \( \phi_k(A_{k+1}^0 + \tilde{Z}_k) = \phi_k(A_{k+1}^0) + \delta \phi_k(A_{k+1}^0, \tilde{Z}_k) \). Next define \( E^{(2)}, E^{(3)}, E^{(4)} \) by:

\[
V_k(\phi + \delta \phi_k(A, Z)) = V_k(\phi) + E_k^{(2)}(A, Z, \phi, \Phi_k) \\
E_k(A + Z, \phi + \delta \phi_k(A, Z)) = E_k(A, \phi) + E_k^{(3)}(A, Z, \phi, \Phi_k) \\
Z_{k+1}(A + Z) = Z_{k+1}(A) \exp(E_k^{(4)}(A, Z))
\]  

(304)

We want to do the same thing with \( S_{k,A_{k+1}^0 + \tilde{Z}_k} \left( \Phi_k, \phi_k(A_{k+1}^0) + \delta \phi_k(A_{k+1}^0, \tilde{Z}_k) \right) \). But first we express \( \Phi_k \) in terms of \( \phi_k(A_{k+1}^0) \). One has the identity:

\[
\Phi_k = T_k(A)\phi_k(A)
\]  

(305)

where

\[
T_k(A) = a_k^{-1}Q_k(A) \left( -\Delta_k + akQ_k^2(A)Q_k(A) \right) = a_k^{-1}Q_k(A)(-\Delta_k) + Q_k(A)
\]  

(306)

Use this in place of \( \Phi_k \) and then:

\[
S_{k,A + Z}(\Phi_k, \phi_k(A + Z)) = S_{k,A + Z}^{\prime} (\phi_k(A + Z))
\]  

(307)

where

\[
S_{k,A}^{\prime}(\phi) = \frac{a_k^{-1}}{2} \| (T_k(A) - Q_k(A))\phi \|^2 + \frac{1}{2} \| \partial A_k \|^2
\]  

(308)

Now define \( E^{(1)} \) by:

\[
S_{k,A + Z}^{\prime}(\phi + \delta \phi_k(A, Z)) = S_{k,A}^{\prime}(\phi) + E_k^{(1)}(A, Z, \phi, \Phi_k)
\]  

(309)

Now we have with \( \tilde{E}(A, Z, \phi, \Phi_k) = \sum_{i=1}^4 E^{(i)}(A, Z, \phi, \Phi_k) \):

\[
\tilde{\rho}_{k+1}(A_{k+1}, e^{q_{k+1}A_{k+1}^0} \Phi_{k+1}) = N_k Z_k Z_k^T Z_k \exp \left( -\frac{1}{2} \| \partial A_{k+1}^0 \|^2 \right) \int d\mu_{C_k}(\tilde{Z}) D\Phi_k \\
n_k \chi_k \chi_k^w \delta_G \left( \Phi_{k+1} - Q(A_{k+1}^0)\Phi_k \right) \exp \left( -S_{k,A_{k+1}^0 + \Phi_k}^{\prime} \left( \phi_k(A_{k+1}^0) \right) - V_k(\phi_k(A_{k+1}^0)) \right) \\
\exp \left( E_k \left( A_{k+1}^0, \phi_k(A_{k+1}^0), Z_k, \phi_k(A_{k+1}^0), \Phi_k \right) \right)
\]  

(310)

\[\text{One could also use } \Phi_k = a_k^{-1} \left( Q_k(A)G_k(A)Q_k^2(A) \right)^{-1} Q_k(A)\phi_k(A) \text{ but this adds another layer of non-locality}\]
6.2.3 first localization

We want to localize the terms contributing to \( \hat{E}_k(A, Z, \phi, \Phi_k) \). These will be treated in the region

\[
A, \phi \in \frac{1}{2} R_k \quad |Z|, |\partial Z|, |\delta_s \partial Z| \leq \lambda_k^{-\epsilon} \quad |\Phi_k| \leq \lambda_k^{-\frac{1}{2}-2\epsilon} \tag{311}
\]

Since \( \epsilon_k \leq \lambda_k^{\frac{1}{2}} \) the bounds on \( Z \) imply

\[
|Z|, |\partial Z|, |\delta_s \partial Z| | \leq \epsilon_k^{-2\epsilon} \tag{312}
\]

Note that the characteristic function \( \chi_k(C_k^{\frac{1}{2}}Z) \) enforces that \( |C_k^{-\frac{1}{2}}Z| \leq p_{0,k} \). Since \( |C_k^{\frac{1}{2}}f| \leq C\|f\|_\infty \) by [194] it follows that \( |\hat{Z}| \leq C\lambda_{0,k} \). Then by the bounds [189] on \( H_k \) the fluctuation field \( Z_k = H_k C\hat{Z} \) satisfies \( |Z_k| \leq C\lambda_{0,k} \leq \lambda_k^{-\epsilon} \) and similarly for the derivative. Thus \( Z_k \) qualifies for the domain \( 311 \).

In lemma [20] below we show that on the domain \( 311 \) \((A + Z, \phi + \delta \phi_k) \in R_k \). Therefore the \( E_k^{(i)}(A, Z, \phi, \Phi_k) \) as given by \( 301 \), \( 309 \) are well-defined on this domain.

First some preliminary results:

**Lemma 18.** In the region \( 311 \)

\[
|\delta \phi_k|, |\partial_s \delta \phi_k|, |\delta_s A \partial_s \delta \phi_k| \leq \lambda_k^{\frac{1}{2}-5\epsilon} \tag{313}
\]

**Proof.** If \( A \in R_k \) then by the bounds \( 126 \) on \( H_k(A) \)

\[
|\phi_k(A, \Phi_k)| \leq C\|\Phi_k\|_\infty \leq C\lambda_k^{-\frac{1}{2}-2\epsilon} \tag{314}
\]

We write for \( r > 1 \)

\[
\delta \phi_k(A, Z, \Phi_k) = \phi_k(A + Z, \Phi_k) - \phi_k(A, \Phi_k) = \frac{1}{2\pi i} \int_{|t|=r} \frac{dt}{t(t-1)} \phi_k(A + tZ, \Phi_k) \tag{315}
\]

If we take \( |t| = \epsilon_k^{-1+5\epsilon} \) then \( |tZ| \leq C\epsilon_k^{-1+5\epsilon} e_k^{-2\epsilon} \leq \frac{1}{2} \epsilon_k^{-1+3\epsilon} \) with the same bound for the derivatives.

Hence \( A + tZ \in R_k \) and we can use \( 314 \) to get the bound

\[
|\delta \phi_k(A, Z, \Phi_k)| \leq \epsilon_k^{1-5\epsilon} (C\lambda_k^{-\frac{1}{2}-2\epsilon}) \leq \lambda_k^{\frac{1}{2}-5\epsilon} \tag{316}
\]

The derivatives have the same bound.

**Remark.** We will also need a version in which the coupling is weakened. In \( \phi_k(A + Z), \phi_k(A) \) replace \( G_k(A + Z), G_k(A) \) by weakened versions \( G_k(s, A + Z), G_k(s, A) \). This gives weakened fields \( \phi_k(s, A + Z), \phi_k(s, A) \) depending on \( s = \{s_\square\} \), and hence a weakened \( \delta \phi_k(s) = \delta \phi_k(s, A, Z, \Phi_k) \).

All the above analysis holds and we still have the same bounds on \( \delta \phi_k(s) \) even for \( s_\square \) complex and satisfying \( |s_\square| \leq M^+ \).

**Lemma 19.** For \( |Im A|, |Im Z| \leq \epsilon_k^{-1} \)

\[
|\partial(A+z-\partial A)f| \leq \epsilon_k\|Z\|_\infty\|f\|_\infty \\
|\partial(\alpha, A+z-\delta_s A)f| \leq \epsilon_k\|Z\|_\infty\|f\|_\infty \\
|\delta_s A f\|_\infty \leq \|\partial A f\|_\infty \tag{317}
\]
**Proof.** The first follows from

$$ (\partial A + z, \mu, f)(x) - (\partial A, \mu) f(x) = e^{q_k \eta A}(x) F_{\mu}(Z_k) f(x + \eta e_\mu) $$  \hspace{1cm} (318)

and the bound

$$ |F_{\mu}(Z_k) f| = \left| \left( e^{q_k \eta Z_k, \mu}(x) - \frac{1}{\eta} \right) f \right| \leq e_k \| Z \| \| f \| $$  \hspace{1cm} (319)

(This is essentially (96) again.) The second follows from

$$ (\delta_{\alpha, A + z} f)(x, y) - (\delta_{\alpha, A} f)(x, y) = e^{q_k A}(\Gamma_{xy}) \left( e^{q_k Z(\Gamma_{xy})} \frac{-1}{d(x, y)^\alpha} \right) f(y) $$  \hspace{1cm} (320)

and the bound for $d(x, y) \leq 1$

$$ |(e^{q_k Z(\Gamma_{xy})} - 1) f| \leq e_k d(x, y) \| Z \| \| f \| \leq d(x, y)^\alpha e_k \| Z \| \| f \| $$  \hspace{1cm} (321)

The last follows from the representation

$$ e^{q_k A}(\Gamma_{xy}) f(y) - f(x) = \int_{\Gamma(x,y)} e^{q_k A}(\Gamma_{xy}) (\partial A f)(z) \cdot dz $$  \hspace{1cm} (322)

which yields the bound for $d(x, y) \leq 1$

$$ |e^{q_k A}(\Gamma_{xy}) f(y) - f(x)| \leq d(x, y) \| \partial A f \| \leq d(x, y)^\alpha \| \partial A f \| $$  \hspace{1cm} (323)

**Lemma 20.** In the region (311) and for $|t| \leq \lambda_k^{-\frac{1}{4} - 5 \epsilon}$ we have

$$ (A + tZ, \phi + t \delta \phi_k) \in \mathcal{R}_k $$  \hspace{1cm} (324)

**Proof.** Let $\phi_t = \phi + t \delta \phi_k$. By lemma 18

$$ |t \delta \phi_k|, |t \partial A \delta \phi_k|, |t \delta_{\alpha, A, \partial A} \delta \phi_k| \leq \lambda_k^{-\frac{1}{4} - 5 \epsilon} \lambda_k^{\frac{1}{4} - 5 \epsilon} = \lambda_k^{-\frac{1}{2} \epsilon} < \frac{1}{4} \lambda_k^{-\frac{1}{2} \epsilon} $$  \hspace{1cm} (325)

Hence $(A, t \delta \phi_k) \in \frac{1}{2} \mathcal{R}_k$ and it follows that $(A, \phi_t) \in \frac{3}{8} \mathcal{R}_k$.

The lemma claim that $(A + tZ, \phi_t) \in \mathcal{R}_k$. For the $A$ conditions it suffices to show that $tZ \in \frac{1}{2} \mathcal{R}_k$.

Since $|t| \leq e_k e^{\frac{3}{4} + 10 \epsilon}$, this follows from

$$ |t Z|, |t dZ|, |t \delta A dZ| \leq e_k e^{\frac{3}{4} + 10 \epsilon} e_k^{-2 \epsilon} < e_k^{\frac{3}{2} \epsilon} $$  \hspace{1cm} (326)

For the $\phi$ conditions we already have $|\phi_t| < \frac{3}{4} \lambda_k^{\frac{1}{2} - \epsilon}$. For the derivatives use (317) and $|t Z| \leq \lambda_k^{\frac{3}{4} + 4 \epsilon}$ to estimate

$$ |\partial A + t Z \phi_t| \leq |\partial A \phi_t| + |(\partial A + t Z - \partial A) \phi_t| $$

$$ \leq \frac{3}{4} \lambda_k^{\frac{1}{2} - 2 \epsilon} + e_k \lambda_k^{\frac{3}{4} + 4 \epsilon} \lambda_k^{\frac{1}{2} - \epsilon} $$

$$ \leq \frac{3}{4} \lambda_k^{\frac{1}{2} - 2 \epsilon} + \lambda_k^{\frac{3}{4} + 3 \epsilon} \leq \lambda_k^{\frac{1}{2} - 2 \epsilon} $$  \hspace{1cm} (327)
Finally we estimate the Holder derivative
\[
\delta_{\alpha, A+1^Z} \partial_{A+1^Z} \phi_t = (\delta_{\alpha, A+1^Z} - \delta_{\alpha, A}) \partial_{A+1^Z} \phi_t \\
+ \delta_{\alpha, A} (\partial_{A+1^Z} - \partial_A) \phi_t + \delta_{\alpha, A} \partial_A \phi_t
\]  
(328)

We know the last term is bounded by \(\frac{3}{8} \lambda^\frac{1}{5} e^{-\epsilon}\). For the first term we use the bounds \((317)\) and \((327)\) to obtain
\[
| (\delta_{\alpha, A+1^Z} - \delta_{\alpha, A}) \partial_{A+1^Z} \phi_t | \leq e_k \lambda^\frac{\alpha^2 + 44}{\lambda^\frac{1}{5} - 4 + \alpha^2 - 2 \alpha + \lambda^\frac{1}{5} + 44 + \nu^\frac{1}{5} - \epsilon}
\]  
(329)

For the second term in \((328)\) we use the bound from \((317)\)
\[
\| \delta_{\alpha, A} (\partial_{A+1^Z} - \partial_A) \phi_t \|_\infty \leq \| \partial_A (\partial_{A+1^Z} - \partial_A) \phi_t \|_\infty
\]  
(330)

We write \(e^{\eta t} \eta A_\nu(x) \phi_t (x + \eta \nu) = \eta \partial_{A, \nu} \phi_t (x) + \phi_t (x)\) and then \((318)\) says
\[
(\partial_{A+1^Z, \nu} - \partial_{A, \nu}) \phi_t = F, (tZ) \left( \eta \partial_{A, \nu} \phi_t + \phi_t \right)
\]  
(331)

Then by \((30)\)
\[
\left( \partial_{A, \mu} (\partial_{A+1^Z, \nu} - \partial_{A, \nu}) \phi_t \right) (x) = \left( F, (tZ) \right) (x + \eta \nu) \left( \eta \partial_{A, \mu} \partial_{A, \nu} \phi_t (x) + \partial_{\nu} \phi_t (x) \right)
\]  
(332)

Note that \(\eta | \partial_{A, \mu} f | \leq O(1) \| f \|_\infty\). Using this and \((A, \phi_t) \in R_k\) and bounds like \((30)\) and \((101)\) on \(F, (tZ)\) we have
\[
\| \partial_A (\partial_{A+1^Z} - \partial_A) \phi_t \|_\infty \leq O(1) e_k \left( \| tZ \|_\infty \| \partial_A \phi_t \|_\infty + \| tZ \|_\infty \| \phi_t \|_\infty \right)
\]  
\[
\leq O(1) \lambda^\frac{\alpha^2 + 44}{\lambda^\frac{1}{5} - 4 + \alpha^2 - 2 \alpha + \lambda^\frac{1}{5} + 44 + \nu^\frac{1}{5} - \epsilon}
\]  
(333)

This is the bound on the second term in \((328)\). Combined with the bounds on the other two terms it gives the required \(| \delta_{\alpha, A+1^Z} \partial_{A+1^Z} \phi_t | < \lambda^\frac{1}{5} e^{-\epsilon}\).

**Lemma 21.** \(E^{(1)}_k\) has a local expansion \(E^{(1)}_k = \sum X E^{(1)}_k (X)\) where \(E^{(1)}_k (X, A, Z, \phi, \Phi_k)\) depends on these fields only in \(X\), is analytic in \((311)\) and satisfies there
\[
\left| E^{(1)}_k \left( X, A, Z, \phi, \Phi_k \right) \right| \leq O(1) \lambda^\frac{\alpha^2 - 100 \alpha}{\lambda^\frac{1}{5} - (\kappa - \kappa_0 - 1) d_M (X)}
\]  
(334)

**Proof.** First split up \(S^{(1)}_{k, A}\) into \(M\)-cubes \(\square\) by
\[
S^{(1)}_{k, A} (\phi) = \sum_\square S^{(1)}_{k, A} (\square, \phi) \quad S^{(1)}_{k, A} (\square, \phi) = \frac{\alpha_k}{2} \| (T_k (A) - Q_k (A)) \phi \|_\infty^2 + \frac{1}{2} \| \partial_A \phi \|_\infty^2
\]  
(335)

In \(\| \partial_A \phi \|_\infty^2\), the star indicates that terms \(| (\partial_A \phi) (x, x') |\) for bonds \((x, x')\) which cross \(M\)-cubes \(\square\) have been divided between the two cubes. Then \(S^{(1)}_{k, A} (\square, \phi)\) depends on \(\phi\) at sites which neighbor \(\square\) but are not in \(\square\). Hence we regard \(S^{(1)}_{k, A} (\square, \phi)\) as localized in the \(3M\)-cube \(\square\) centered on \(\square\). We define
\[
S^{(1)}_{k, A} (\square, \phi) = S^\#_{k, A} (\square, \phi)
\]  
(336)
Then the field is strictly localized in \( \tilde{\Box} \) and we have

\[
S_{k,A}(\phi) = \sum_{\Box} S_{k,A}^\#(\tilde{\Box}, \phi)
\]  

(337)

There is a corresponding split \( E_k^{(1)} = \sum_{\Box} E_k^{(1)}(\tilde{\Box}) \). where

\[
E_k^{(1)}(\tilde{\Box}, A, Z, \phi, \Phi_k) = S_{k,A+\epsilon}^\#(\tilde{\Box}, \phi, \delta \phi_k(A, Z, \Phi_k)) - S_{k,A}^\#(\tilde{\Box}, \Phi_k, \phi)
\]  

(338)

In appendix [C we establish that

\[
|\langle T_k(A) - Q_k(A) \rangle \phi| = a_{k-1}^1|Q_k(A)\Delta_A \phi| \leq C\|\partial_A \phi\|_\infty
\]  

(339)

For \( A, \phi \in R_k \) we have \( |\partial_A \phi| \leq \lambda_k^{-\frac{1}{2}-2\epsilon} \) and so

\[
|S_{k,A}^\#(\tilde{\Box}, \phi)| \leq C M^3 \lambda_k^{-\frac{1}{2} - 4\epsilon} \leq \lambda_k^{-\frac{1}{2} - 5\epsilon}
\]  

(340)

According to lemma [20] in \( S_{k,A+\epsilon Z}^\#(\tilde{\Box}, \phi + t \delta \phi_k(A, Z, \Phi_k)) \) we can take \( |t| \leq \lambda_k^{-5/12 + 5\epsilon} \) and stay in the analyticity region \( R_k \). Hence for \( r = \lambda_k^{-5/12 + 5\epsilon} \) we have the representation

\[
E_k^{(1)}(\tilde{\Box}, A, Z, \phi, \Phi_k) = \frac{1}{2\pi i} \int_{|t|=r} \frac{dt}{t(t-1)} S_{k,A+\epsilon Z}^\#(\tilde{\Box}, \phi + t \delta \phi_k(A, Z, \Phi_k))
\]  

(341)

and the bound (340) yields

\[
|E_k^{(1)}(\tilde{\Box}, A, Z, \phi, \Phi_k)| \leq O(1) \lambda_k^{5/12 - 5\epsilon} \lambda_k^{-\frac{1}{2} - 5\epsilon} \leq O(1) \lambda_k^{1/12 - 10\epsilon}
\]  

(342)

Since \( d_M(\tilde{\Box}) = O(1) \) we can insert a factor \( e^{-(\kappa - \kappa_0 - 1)d_M(\tilde{\Box})} \). Hence the result with \( \tilde{E}_k^{(1)}(X) = E_k^{(1)}(\tilde{\Box}) \) if \( X = \tilde{\Box} \) and zero otherwise.

**Lemma 22.** \( E_k^{(2)}, E_k^{(3)} \) have local expansions \( E_k^{(i)} = \sum_X \hat{E}_k^{(i)}(X) \) where \( \hat{E}_k^{(i)}(X, A, Z, \phi, \Phi_k) \) depends on these fields only in \( X \), is analytic in (333) and satisfies there

\[
\left| \hat{E}_k^{(i)}(X, A, Z, \phi, \Phi_k) \right| \leq O(1) \lambda_k^{T/2 - 10\epsilon} e^{-(\kappa - \kappa_0 - 1)d_M(X)}
\]  

(343)

**Proof.** The potential has the local decomposition \( V_k(\phi) = \sum_{\Box} V_k(\Box, \phi) \) over \( M \)-cubes \( \Box \). Then \( E^{(2)}(\Box) = V_k(\Box, \phi + \delta \phi_k) - V_k(\Box, \phi) \) can be written

\[
E^{(2)}(\Box, A, Z, \phi, \Phi_k) = \frac{1}{2\pi i} \int_{|t|=\lambda_k^{-5/12+5\epsilon}} \frac{dt}{t(t-1)} V_k(\Box, \phi + t \delta \phi_k(A, Z, \Phi_k))
\]  

(344)

Here the circle \( |t| = \lambda_k^{-5/12 + 5\epsilon} \) is chosen so inside the circle \( \phi + t \delta \phi_k \in R_k \) by lemma [20]. On \( R_k \) we have the bound (\( \epsilon \) is irrelevant here)

\[
|V_k(\Box, \phi)| \leq M^3 (\mu k (\lambda_k^{-\frac{1}{2} - \epsilon})^2 + \lambda_k (\lambda_k^{-\frac{1}{2} - \epsilon})^4) \leq M^3 \lambda_k^{-4\epsilon} \leq \lambda_k^{-5\epsilon}
\]  

(345)

and this implies

\[
|E_k^{(2)}(\Box)| \leq O(1) \lambda_k^{5/12 - 5\epsilon} \lambda_k^{-5\epsilon} \leq \lambda_k^{5/12 - 10\epsilon}
\]  

(346)
The term $E^{(3)}_k$ inherits an expansion in $X$ from $E_k$ and we have

$$E^{(3)}_k(X, \Phi_k, A, Z, \phi, \Phi_k) = \frac{1}{2\pi i} \int_{|t| = \lambda_k^{\frac{5}{12} + 5\varepsilon}} \frac{dt}{t(t - 1)} E_k(X, A + tZ, \phi + t\delta\phi_k(A, Z, \Phi_k)) (347)$$

where again $(A + tZ, \phi + t\delta\phi_k) \in \mathcal{R}_k$ by lemma 20. Then the bound $|E_k(X, A, \phi)| \leq e^{-\kappa d_M(X)}$ on $\mathcal{R}_k$ now implies that

$$E^{(3)}_k(X) = O(1) \lambda_k^{5/12 - 5\varepsilon} e^{-\kappa d_M(X)} (348)$$

We are not finished because $E^{(2)}(X), E^{(3)}(X)$ depends on fields outside of $X$ through $\delta\phi_k$. Consider $E^{(3)}(X)$. We replace $\delta\phi_k$ by $\delta\phi_k(s)$ in the above formula and define $E^{(3)}(s, X)$ (see remark after lemma 13). This still satisfies the bound (348). Now in each variable $s = 0$ by

$$f(s = 1) = f(s = 0) + \int_0^1 ds \frac{\partial f}{\partial s} (349)$$

This yields

$$E^{(3)}_k(X) = \sum_{Y \supset X} E_k(X, Y) (350)$$

$E_k(X, Y; A, Z, \phi, \Phi_k) = \int ds_{Y - X} \frac{\partial}{\partial s_{Y - X}} [E_k(X, A, Z, \phi, \delta\phi_k(s, A, Z, \Phi_k))]_{s_{Y - X} = 0, s_{X - Y} = 1}$

The latter only only depends on $A, Z, \phi, \Phi_k$ in $Y$ since there is no coupling through $Y^c$. Now we write

$$E^{(3)} = \sum_X E^{(3)}_k(X) = \sum_X \sum_{Y \supset X} E_k(X, Y) = \sum_Y \hat{E}^{(3)}_k(Y) (351)$$

where the sum is over connected polymers $Y$ and

$$\hat{E}^{(3)}_k(Y) = \sum_{X \subset Y} E_k(X, Y) (352)$$

is strictly local in the fields.

To estimate the new function $\hat{E}^{(3)}_k(Y)$ we argue as follows, see [24] for more details. Since $\delta\phi_k(s, A, Z, \Phi_k)$ is analytic in $|s| \leq M^2$ we can use a Cauchy bound to estimate the derivatives in (351). Each derivative contributes a factor $M^{-\frac{1}{2}}$ and $M^{-\frac{1}{2}} \leq e^{-\kappa}$ for $M$ sufficiently large. Hence in an estimate on $E_k(X, Y)$ we gain a factor $e^{-\kappa |Y - X| M}$. Using also (348) yields

$$|\hat{E}^{(3)}_k(Y)| \leq O(1) \lambda_k^{5/12 - 5\varepsilon} \sum_{X \subset Y} e^{-\kappa |Y - X| M - \kappa d_M(X)} (353)$$

But one can show that

$$|Y - X| M + d_M(X) \geq d_M(Y) (354)$$

Hence one can extract a factor $e^{-(\kappa - \kappa_0)d_M(X)}$ leaving a factor $e^{-\kappa_0 d_M(X)}$ for the convergence of the sum over $X$. The sum is bounded by $O(1) |Y| M \leq O(1) (d_M(Y) + 1)$ and so we have

$$|\hat{E}^{(3)}_k(Y)| \leq O(1) \lambda_k^{5/12 - 5\varepsilon} e^{-(\kappa - \kappa_0 - 1)d_M(Y)} (355)$$

which is more than enough. The construction of $\hat{E}^{(2)}_k(Y)$ follows the same steps.

**Lemma 23.** In the region (311) we have the local expansion $E^{(4)}_k = \sum_X \hat{E}^{(4)}_k(X)$

$$\left| \hat{E}^{(4)}_k(X, A, Z) \right| \leq O(1) \epsilon_k^{1 - 6\varepsilon} e^{-\kappa d_M(X)} (356)$$
Hence on the domain \((311)\):

\[
\|E_k(X,A,Z,\phi,\Phi_k)\| \leq O(1)\lambda_k^{\frac{1}{12} - 10\epsilon} e^{-(\kappa - \kappa_0 - 1)d_M(X)}
\]

\(6.2.4\) restoration of dressed fields

We have some direct dependence on \(\Phi_k\) on the unit lattice \(T_{N-k}^0\). We would like to express this in terms of the dressed field \(\phi_k(A_k^{0+1})\) on the fine lattice \(T_{N-k}^{-k}\). We again use the identity \(\Phi_k = T_k(A)\phi_k(A)\) where \(T_k(A)\) is defined in \((306)\). Our new definition is

\[
\hat{E}_k(X,A,Z,\phi) \equiv \hat{E}_k(X,A,Z,\phi,T_k(A))\phi
\]

(same symbol, different variables). Then in \((310)\) we can make the replacement

\[
\hat{E}_k(X,A_k^{0+1},Z_k,\phi_k(A_k^{0+1}),\Phi_k) = \hat{E}_k(X,A_k^{0+1},Z_k,\phi_k(A_k^{0+1}))
\]

Using the estimate \(|T_k(A) - Q_k(A)|\phi| \leq C\|\partial_A \phi\|\) from appendix \([C]\) and the estimate \(|Q_k(A)|\phi| \leq \|\phi\|\) we have

\[
|T_k(A)|\phi| \leq C\left(\|\phi\| + \|\partial_A \phi\|\right)
\]

Hence on the domain \((311)\):

\[
|T_k(A)|\phi| \leq C\lambda_k^{-\frac{1}{12} - \epsilon} \leq \lambda_k^{-\frac{1}{12} - 2\epsilon}
\]

Thus we are still in the analyticity domain for \(\hat{E}_k(X)\), and the bound \((311)\) still holds.

We are not finished because \(\hat{E}_k(X,A,Z,\phi)\) depends on \(\phi\) in \(\hat{X}\) through \(Q_k(A)\Delta A \phi\). \((\hat{X} = \) union of \(M\) blocks touching \(X\)). We define

\[
\hat{E}_k(Y) = \sum_{X: \hat{X} = Y} \hat{E}_k(X)
\]

Then \(\hat{E}_k = \sum_Y \hat{E}_k(Y)\), and \(\hat{E}_k(Y)\) is strictly local, and

\[
|\hat{E}_k(Y)| \leq O(1)\lambda_k^{\frac{1}{12} - 10\epsilon} \sum_{X: X = Y} e^{-(\kappa - \kappa_0 - 1)d_M(X)}
\]

But \(d_M(\hat{X}) \leq d_M(X) + O(1)|X|_M\) and \(|X|_M \leq O(1)(d_M(X) + 1)\) so there is a constant \(c = O(1)\) such that

\[
\lambda_k \leq d_M(X)(d_M(X) + 1)
\]

We use this to extract a factor \(O(1)e^{-c(\kappa - 2\kappa_0 - 1)d_M(Y)}\). This leaves \(e^{-\kappa_0 d_M(X)}\) for convergence of the sum which is bounded by \(O(1)|Y|_M \leq O(1)(d_M(Y) + 1)\). Hence we end with the bound on the domain \((311)\) (without the condition on \(\Phi_k\))

\[
|\hat{E}_k(Y,A,Z,\phi)| \leq O(1)\lambda_k^{\frac{1}{12} - 10\epsilon} e^{-(\kappa - \kappa_0 - 2)d_M(Y)}
\]
6.2.5 scalar field translation

Now in (310), with $A = A^0_{k+1}$, we translate to the minimum of

$$\frac{1}{2} < \Phi_k, \Delta_k(A) \Phi_k >= \frac{a}{2L^2} ||\Phi_{k+1} - Q(A)\Phi_k||^2 + S_{k,A}(\Phi_k, \phi_k(A))$$

(366)

As in section 2.5 this is $\Phi^1_k$ and we write

$$\Phi_k = H_k(A) \Phi_{k+1} + Z' \quad \phi_k(A) = \phi^0_{k+1}(A) + Z_k(A) \quad Z_k(A) \equiv H_k(A) Z'$$

(367)

At the minimum we have

$$\frac{1}{2} < H_k(A) \Phi_{k+1}, \Delta_k(A) H_k(A) \Phi_{k+1} > + \frac{1}{2} \left< Z', (\Delta_k(A) + \frac{a}{L^2} (Q^T Q)(A) ) Z' \right>$$

(368)

We know the first term here scales to $\frac{1}{2} < \Phi_{k+1}, \Delta_{k+1}(A) \Phi_{k+1} >$ so it must be

$$S^0_{k+1,A}(\Phi_{k+1}, \phi^0_{k+1}(A)) = \frac{a_{k+1}}{2L^2} ||\Phi_{k+1} - Q_{k+1}(A)\phi^0_{k+1}||^2 + \frac{1}{2} ||\partial_\phi \phi^0_{k+1}||^2$$

(369)

Hence (310) becomes

$$\tilde{\rho}_{k+1}(A_{k+1}, e^{q_0^{\omega_k} + \Phi_{k+1}})$$

$$= N_k N_{k+1} Z_k Z^\dagger_{k+1} Z_k(A^0_{k+1}) \exp \left( - \frac{1}{2} ||\partial A^0_{k+1}||^2 - S^0_{k+1,A^0_{k+1}}(\Phi_{k+1}, \phi^0_{k+1}(A^0_{k+1})) \right)$$

$$\int d\mu_{C_k}(\tilde{Z}) DZ' \chi_k \chi_k \exp \left( - \frac{1}{2} \left< Z', (\Delta_k(A^0_{k+1}) + \frac{a}{L^2} (Q^T Q)(A^0_{k+1}) ) Z' \right> \right)$$

$$\exp \left( - V_k(\phi^0_{k+1}(A^0_{k+1}) + Z_k(A^0_{k+1})) + E_k(A^0_{k+1}, \phi^0_{k+1}(A^0_{k+1}) + Z_k(A^0_{k+1})) \right)$$

(370)

Now identify the Gaussian measure $d\mu_{C_k(A^0_{k+1})}(Z')$ by

$$d\mu_{C_k(A^0_{k+1})}(Z') = Z_k(A)^{-1} \exp \left( - \frac{1}{2} \left< Z', (\Delta_k(A) + \frac{a}{L^2} (Q^T Q)(A) ) Z' \right> \right) DZ'$$

(371)

We also define

$$V_k(\phi + Z_k(A)) = V_k(\phi) + E^{(5)}_k(\phi, Z_k(A))$$

$$E_k(A, \phi + Z_k(A)) = E_k(A, \phi) + E^{(6)}_k(A, \phi, Z_k(A))$$

(372)

The $E^{(5)}_k, E^{(6)}_k$ inherit local expansions. Now we have

$$\tilde{\rho}_{k+1}(A_{k+1}, e^{q_0^{\omega_k} + \Phi_{k+1}})$$

$$= N_k N_{k+1} Z_k Z^\dagger_{k+1} Z_k(A^0_{k+1}) \exp \left( - \frac{1}{2} ||\partial A^0_{k+1}||^2 - S^0_{k+1,A^0_{k+1}}(\Phi_{k+1}, \phi^0_{k+1}(A^0_{k+1})) \right)$$

$$\exp \left( - V_k(\phi^0_{k+1}(A^0_{k+1}) + Z_k(A^0_{k+1})) + E_k(A^0_{k+1}, \phi^0_{k+1}(A^0_{k+1})) \right) Z_k(A^0_{k+1})$$

(373)

Here we have isolated a fluctuation integral

$$Z_k(A^0_{k+1}, \phi^0_{k+1}(A^0_{k+1})$$

$$= \int d\mu_{C_k}(\tilde{Z}) d\mu_{C_k(A^0_{k+1})}(Z') \chi_k \chi_k \exp \left( E^{(5)}_k(A^0_{k+1}, Z_k, \phi^0_{k+1}(A^0_{k+1}), Z_k(A^0_{k+1})) \right)$$

(374)
where
\[
E_k^1(A, Z_k, \phi, Z_k(A)) = \mathcal{E}_k(A, Z_k, \phi + Z_k(A)) + E_k^{(5)}(\phi, Z_k(A)) + E_k^{(6)}(A, \phi, Z_k(A))
\] (375)

We make another change of variables writing \(\tilde{Z} = C_k^\frac{1}{2} \tilde{W}\) and \(Z' = C_k(A)^\frac{1}{2} \tilde{W}\). Then \(Z_k, Z_k(A)\) become \(W_k, W_k(A)\) where
\[
W_k = \mathcal{H}_k C_k^\frac{1}{2} \tilde{W} \quad W_k(A) = \mathcal{H}_k(A) C_k^\frac{1}{2} (A) \tilde{W}
\] (376)
The fluctuation integral is then
\[
\Xi_k \left( A_{k+1}^0, \phi_{k+1}^0 (A_{k+1}^0) \right) = \int d\mu_I(\tilde{W}) d\mu_I(W) \chi_k^w \chi_k \exp \left( E_k^1(A_{k+1}^0, W_k, \phi_{k+1}^0 (A_{k+1}^0), W_k(A_{k+1}^0) \right)
\] (377)
The characteristic function \(\chi_k^w\) has simplified (as it was designed to do) so that now
\[
\chi_k^w = \chi_k^w(\tilde{W}) \chi_k^w(W)
\] (378)
These enforce that \(|\tilde{W}|, |W| \leq p_{0,k}\). The bounds \(299\) enforced by characteristic function \(\chi_k\) are now with \(A = A_{k+1}^0\)
\[
|d(A + W_k)| \leq p_k
\]
\[
\left| \left( H_k(A) \phi_{k+1}^0 (A) W_k \right) - Q_k(A + W_k) \left( \phi_{k+1}^0 (A + W_k) + W_k(A + W_k) \right) \right| \leq p_k
\]
\[
\left| \partial_{A + W_k} \left( \phi_{k+1}^0 (A + W_k) + W_k(A + W_k) \right) \right| \leq p_k
\]
\[
\left| \phi_{k+1}^0 (A + W_k) + W_k(A + W_k) \right| \leq \lambda_k^{-\frac{1}{2}} p_k
\] (379)

6.2.6 estimates

We first note that for \(A \in \mathcal{R}_k\)
\[
|C_k^\frac{1}{2} \tilde{W}|, |W_k|, |\partial W_k|, |\delta W_k| \leq C \|\tilde{W}\| \infty
\]
\[
|C_k^\frac{1}{2}(A) W_k|, |W_k(A)|, |\partial_A W_k(A)|, |\delta_{\alpha,A} \partial_A W_k(A)| \leq C \|W\| \infty
\] (380)
The bounds on \(C_k^\frac{1}{2}, C_k^\frac{1}{2}(A)\) were already established in (142), (194). The others follows by the bounds (126), (139) on \(\mathcal{H}_k, \mathcal{H}_k(A)\).

Lemma 24. Let \(A, \phi \in \frac{1}{4} \mathcal{R}_k\) and and \(|\tilde{W}|, |W_k| \leq p_{0,k}\). Then \(E_k^1 = \sum X E_k^1(X)\) where
\[
|E_k^1(X, A, W_k, \phi, W_k(A))| \leq O(1) \lambda_k^{1/12 - 10\epsilon} e^{-c(\kappa - 2\kappa_0 - 2) d_{3r}(X)}
\] (381)

Proof. We have \(E_k^1(X) = \mathcal{E}_k^1(X) + E_k^{(5)}(X) + E_k^{(6)}(X)\). The bound on \(\mathcal{E}_k^1(X, A, W_k, \phi + W_k(A)\) follows from (365). For this we need the fact that our assumptions and the bounds (380) imply that \((A, W_k, \phi + W_k(A))\) is in the domain (311).

The bounds on \(E_k^{(5)}(X), E_k^{(6)}(X)\) are very similar to the bounds on \(E_k^{(2)}(X), E_k^{(3)}(X)\) given in lemma 22. For example
\[
E_k^{(6)}(X, A, \phi, W_k(A)) = \frac{1}{2\pi i} \int_{|t| = \lambda_k^{-\frac{1}{2}}} \frac{dt}{\Gamma(t - 1)} E_k \left( X, A, \phi + t W_k(A) \right)
\] (382)
By (380) we have for such \( t \)
\[
|\mathcal{W}_k(A)|, \quad |t\partial A \mathcal{W}_k(A)|, \quad |t\delta_{\alpha A} \partial_A \mathcal{W}_k(A)| \leq C p_{0,k} \lambda_k^{-\frac{1}{2}} \leq \frac{1}{2} \lambda_k^{-\frac{1}{2} - \epsilon} \tag{383}
\]
and so \((A, t\mathcal{W}_k(A)) \in \mathcal{R}_k\) and we are in the analyticity region for \( E_k(X) \). Together with \( |E_k(X)| \leq e^{-\kappa d M(X)} \) this gives the bound
\[
|E_k^{(6)}(X, A, \phi, \mathcal{W}_k(A))| \leq \mathcal{O}(1) \lambda_k^{\frac{1}{2}} e^{-\kappa d M(X)} \tag{384}
\]
which is sufficient. The bound on \( E_k^{(5)}(X) \) is a little weaker, but still sufficient.

### 6.2.7 adjustments

We make two adjustments. The first is to reblock from polymers \( X \) which are unions of \( M \) blocks to polymers \( Y \) which are unions of \( LM \) blocks. We have as in section 5.4
\[
E_k^1 = \sum_X E_k^1(X) = \sum_Y BE_k^1(Y) \equiv BE_k^1 \tag{385}
\]
Then for \( A, \phi \in \frac{1}{2} \mathcal{R}_k \) and \(|\tilde{W}|, |W| \leq p_{0,k} \)
\[
|BE_k^1(Y, A, \phi, \mathcal{W}_k, \mathcal{W}_k(A))| \leq \mathcal{O}(1) L^3 \lambda_k^{\frac{3}{2} - 10\epsilon} e^{-L(c_k - 3\kappa_0 - 3)d_{LM}(Y)} \tag{386}
\]
We do the same to the leading term \( E_k \), introducing \( BE_k \).

The second adjustment involves the characteristic function \( \chi_k \) which enforces the conditions \( (279) \). The next lemma shows that if we assume \((A_{k+1}, \Phi_{k+1}) \in S_{k+1}^0_0\) as defined in \( (299) \) and if \(|\tilde{W}|, |W| \leq p_{0,k} \) as enforced by \( (380) \), then we can drop this characteristic function entirely, a key simplification.

**Lemma 25.** If \((A_{k+1}, \Phi_{k+1}) \in S_{k+1}^0_0\) and \(|\tilde{W}|, |W| \leq p_{0,k}\) then the bounds \( (379) \) are satisfied and hence \( \chi_k = 1 \).

**Proof.** For the gauge field it suffices to show separately that \( |dA_{k+1}^0| \leq \frac{1}{2} p_k \) and \( |dW_k| \leq \frac{1}{2} p_k \). The first follows by \( (289) \) and \( p_{k+1} \leq p_k \). For the second we have by \( (380) \) \( |dW_k| \leq C p_{0,k} \). But for \( \lambda_k \) sufficiently small \( p_{0,k}/p_k = \frac{1}{(\log \lambda_k)^{p_0-p}} \) is as small as we like since \( p_0 < p \). Hence the result.

It remains to show that the scalar bounds in \( (379) \) are satisfied. The bounds with all the \( W \)’s gone and with a factor of \( \frac{1}{2} \) follow more or less directly from from the assumption \((A_{k+1}, \Phi_{k+1}) \in S_{k+1}^0_0\) just as for the gauge field. Thus it suffices to show that the difference between the expression with and without the \( W \)’s satisfy the indicated bounds with a factor \( \frac{1}{2} \). We have for example
\[
\partial_{A_{k+1}^0, tW_k} \phi_{k+1}^0(A_{k+1}^0 + tW_k) - \partial_{A_{k+1}^0, tW_k} \phi_{k+1}^0(A_{k+1}^0 + tW_k) = \frac{1}{2\pi i} \int_{|t| = e_k^{1+4\epsilon}} \frac{dt}{t(t-1)} \partial_{A_{k+1}^0, tW_k} \phi_{k+1}^0(A_{k+1}^0 + tW_k) \tag{387}
\]
To justify this representation we need control over \( \partial_A \mathcal{H}^0_{k+1}(A) \) for \( A = A_{k+1}^0 + tW_k \) and \(|t| \leq e_k^{-1+4\epsilon} \).
Since \( \partial_A \mathcal{H}^0_{k+1}(A)f = L^{-1} \left( \partial_{A_{L-1}^0, \mathcal{H}_{k+1}^0(A_{L-1}^0)} f_{L-1}^0 \right) \) we need \( A_{L-1}^0 \in \mathcal{R}_{k+1} \) and it suffices that \( A_{k+1, L-1}^0 \in \frac{1}{2} \mathcal{R}_{k+1} \) and \( tW_{k-1} \in \frac{1}{2} \mathcal{R}_{k+1} \). We already know the former. The latter follows by
\[
|tW_{k-1}, |t|W_{k-1}, |\delta_{\alpha} W_{k-1}, |\delta_{\alpha} \partial W_{k-1}, \leq e_k^{-1+4\epsilon} C p_{0,k} \leq \frac{1}{2} e_k^{1+3\epsilon} \tag{388}
\]
Thus we are in the region of analyticity for $\partial A \mathcal{H}^0_{k+1}(A)$ and so $\partial A \phi^0_{k+1}(A)$. Then $|\partial A \mathcal{H}^0_{k+1}(A)f| \leq C\|f\|_\infty$ and

$$|\partial A \phi^0_{k+1}(A)| \leq C\|\Phi_{k+1}\|_\infty \leq C p_{k+1} \lambda^{-\frac{1}{4}}_{k+1}$$

(389)

and then (357) gives

$$\left| \partial A^0_{k+1} \phi^0_{k+1}(A^0_{k+1} + W_k) - \partial A^0_{k+1} \phi^0_{k+1}(A^0_{k+1}) \right| \leq e^{1-4\varepsilon}(C p_{k+1} \lambda^{-\frac{1}{4}}_{k+1}) \leq C p_{k+1} \lambda^{-\varepsilon}_{k+1} \leq \frac{1}{2} p_k$$

(390)

Similarly $A^0_{k+1} + W_k \in \mathcal{R}_k$ and so

$$|\partial A^0_{k+1} + W_k(A^0_{k+1} + W_k)| \leq C p_{0,k} \leq \frac{1}{2} p_k$$

(391)

This completes the bound for the derivative term in (379).

The bounds on the other terms in (379) are similar. Note in particular that $A^0_{k+1} \in \mathcal{R}_k$ and so $|C(A^0_{k+1})| \leq C p_{0,k} < \frac{1}{2} p_k$ by (380). This completes the proof.

6.2.8 second localization

With the characteristic function gone the fluctuation integral is $\Xi_k(A^0_{k+1}, \phi^0_{k+1}(A^0_{k+1}))$ where now for any $A, \phi \in \frac{1}{2} \mathcal{R}_k$

$$\Xi_k(A, \phi) = \int d\mu_k(W_k) \chi_k^\varepsilon \exp \left( B E_k^I(A, W_k, \phi, W_k(A)) \right)$$

(392)

As explained in section 5.4, the Green’s functions $G_k, G_k(A)$ have random walk expansions based on $M$-cubes for $M$ sufficiently large. We use these expansions but now based on $LM$ cubes. With them we define weakened Green’s functions $G_k(s, G_k(s, A)$ and so minimizers $\mathcal{H}_k(s), \mathcal{H}_k(s, A)$. Similarly we weaken $C_k^{1/2}, C_k^{1/2}(A)$ to $C_k^{1/2}(s), C_k^{1/2}(s, A)$, now based on the random walk expansions of (141) and lemma 11 with $LM$-cubes. Then define instead of (376)

$$W_k(s) = \mathcal{H}_k(s) C_k^{1/2}(s) \tilde{W} \quad W_k(s, A) = \mathcal{H}_k(s, A) C_k^{1/2}(s, A) W$$

(393)

The term $B E_k^I(Y, A, \phi, W_k, W_k(A))$ is local in $(A, W_k, \phi, W_k(A))$, but not in $\tilde{W}, W$. To remedy this we write

$$B E_k^I(Y) = \sum_{Z \supset Y} B E_k^I(Y, Z)$$

$$B E_k^I(Y, Z; A, \tilde{W}, \phi, W) = \int ds_{Z - Y} \frac{\partial}{\partial s_{Z - Y}} \left[ B E_k^I(Y, A, \phi, W_k(s), W_k(s, A)) \right]_{s_{Z - Y} = 0, s_{Y - 1}}$$

(394)

Now we write

$$B E_k^I = \sum_Y B E_k^I(Y) = \sum_{Z \supset Y} B E_k^I(Y, Z) = \sum_{Z} E_k^I(Z) \equiv E_k^I$$

(395)

where the sum is over $LM$-polymers $Z$ and

$$E_k^I(Z) = \sum_{Y \subset Z} B E_k^I(Y, Z)$$

(396)

is strictly local $(A, \tilde{W}, \phi, W)$.

Now $B E_k^I(Y, A, \phi, W_k(s), W_k(s, A))$ has a bound of the form (380) even for $|s_{\square}| \leq M^{2+}$, and one can use Cauchy bounds in $s_{\square}$ to prove the following (see for example lemma 19 in [24] for details).

Lemma 26. For $A, \phi \in \frac{1}{2} \mathcal{R}_k$ and $|\tilde{W}, W| \leq p_{0,k}$

$$|E_k^I(X, A, \tilde{W}, \phi, W)| \leq O(1) L^3 \lambda_k^{1/2 - 10\varepsilon} e^{-L(c_\varepsilon - 4\kappa_0 - 4)d_{LM}(X)}$$

(397)
Lemma 27.

We normalized the measure introducing
\[
\Xi_k(A, \phi) = \int \exp \left( \sum_Y E_k^\text{loc}(Y, A, \phi, W) \right) \chi_k(W) d\mu_k(W) d\mu_L(W) \tag{398}
\]

We normalized the measure introducing
\[
d\mu^*_k(W) = \frac{\chi_k(W) d\mu_L(W)}{\int \chi_k(W) d\mu_L(W)} \quad d\mu^*_k(W) = \frac{\chi_k(W) d\mu_L(W)}{\int \chi_k(W) d\mu_L(W)} \tag{399}
\]

The fluctuation integral is now
\[
\Xi_k(A, \phi) = \exp \left( -\varepsilon_k^0 \text{Vol}(\mathbb{T}^0_{N-k}) \right) \Xi_k(A, \phi) \tag{400}
\]

The cluster expansion gives this a local structure. As in \cite{24} using the bound \cite{3897} we have

**Lemma 27. (cluster expansion)** For \( A, \phi \in \frac{1}{2} \mathcal{R}_k \)
\[
\Xi_k(A, \phi) = \exp \left( \sum_Y E_k^\#(Y, A, \phi) \right) \tag{401}
\]

where the sum is over LM polymers \( Y \) and
\[
|E_k^\#(Y, A, \phi)| \leq O(1) L^3 k^{1/2-10c} e^{-L(c\kappa_0 - 7)d_{LM}(Y)} \tag{402}
\]

It is straightforward to check that the construction of \( E_k^\#(Y, A, \phi) \) preserves all the symmetries. Now \cite{3773} becomes
\[
\hat{\rho}_{k+1}(A_{k+1}^+ e^{q\epsilon k^0\phi_{k+1}} \Phi_{k+1}) = \mathcal{N}_k \mathcal{N}_k \mathcal{Z}_k \mathcal{Z}_k^f \mathcal{Z}_k^f (A_{k+1}^0) Z_k(A_{k+1}^0)
\]

Define a scaled phase shift \( \theta_{k+1} = \theta_{k+1}(A_{k+1}) \) on \( \mathbb{T}^0_{N-k} \) by
\[
\theta_{k+1}(A_{k+1}) = \left( \omega_{k+1}^{(i)}(A_{k+1}, L) \right)_{L^-1} \tag{404}
\]

Then
\[
\rho_{k+1}(A_{k+1}^+ e^{q\epsilon k^0\phi_{k+1}} \Phi_{k+1}) = \hat{\rho}_k(A_{k+1}, L, e^{q\epsilon k^0\phi_{k+1}} \Phi_{k+1}, L) L^{\frac{1}{2}(b_{N-b_{N-k}}) + \frac{1}{2}(s_{N-s_{N-k})}} \tag{405}
\]

and so we make the substitutions \( A_{k+1} \to A_{k+1}, L \) and \( \Phi_{k+1} \to \Phi_{k+1}, L \) in \cite{103}. With this substitution \( A_{k+1}^0 \) becomes \( A_{k+1}, L \) and we identify by \cite{76}, \cite{155}
\[
\left( Z_k \mathcal{Z}_k^f L^{\frac{1}{2}(b_{N-b_{N-k}}) + \frac{1}{2}(s_{N-s_{N-k})}} \right) \mathcal{N}_k \mathcal{N}_k \mathcal{Z}_k (A_{k+1}, L) \mathcal{Z}_k^f (A_{k+1}, L) L^{s_{N-s_{N-k}}} \tag{406}
\]

\[
= \mathcal{N}_{k+1} Z_k \mathcal{Z}_k^f (A_{k+1})
\]
We also have that \( \phi^0_{k+1}(\mathbb{A}_{k+1}) \) becomes \((\phi_{k+1}(\mathbb{A}_{k+1})))L\), and \(\|\partial A^0_{k+1}\|^2\) becomes \(\|\partial A_{k+1}\|^2\), and \(\mathcal{S}^0_{k+1,\mathbb{A}_{k+1}}(\Phi_{k+1}, \phi^0_{k+1}(\mathbb{A}_{k+1}))\) becomes \(\mathcal{S}_{k+1,\mathbb{A}_{k+1}}(\Phi_{k+1}, \phi_{k+1}(\mathbb{A}_{k+1}))\). We have also \(\varepsilon_k^0\Vol(T_{N-k-1}) = L^3\varepsilon_k^0\Vol(T_{N-k-1})\) becomes

\[
L^3\varepsilon_k^0\Vol(T_{N-k-1}) + \frac{1}{2}L^2\mu_k\|\phi_{k+1}(A_{k+1})\|^2 + \frac{1}{4}L\lambda_k \int \left(\phi_{k+1}(A_{k+1})\right)^4 \tag{407}
\]

The function \(BE_k\) becomes \(BE_k(\mathbb{A}_{k+1}A_{k+1}, (\phi_{k+1}(A_{k+1})))L\) \(\equiv (BE_k)_{L^{-1}}(A_{k+1}, \phi_{k+1}(A_{k+1}))\). Then we have \(BE_{k,L^{-1}} = \sum_X BE_{k,L^{-1}}(X)\) where \(BE_{k,L^{-1}}(X, \mathcal{A}, \phi) = BE_k(LX, \mathcal{A}_L, \phi_L)\). Since \(E_k\) is normalized for small polymers we have by lemma [19]

\[
\|BE_{k,L^{-1}}\|_{k+1,\kappa} \leq O(1)L^{-t} E_k \tag{408}
\]

Similarly \(E^\#_{k,L^{-1}}\) becomes \(E^\#_{k,L^{-1}}(\mathbb{A}_{k+1}A_{k+1}, (\phi_{k+1}(A_{k+1})))L\) \(\equiv E^\#_{k,L^{-1}}(A_{k+1}, \phi_{k+1}(A_{k+1}))\). Then we have that \(E^\#_{k,L^{-1}} = \sum_X E^\#_{k,L^{-1}}(X)\) where \(E^\#_{k,L^{-1}}(X, \mathcal{A}, \phi) = E^\#_k(LX, \mathcal{A}_L, \phi_L)\). If \(\mathcal{A}, \phi \in \mathcal{R}_{k+1}\) then \(\mathcal{A}_L, \phi_L \in \frac{1}{2}R_k\) and so we can use the bound (402). Since \(d_{L,M}(LX) = d_M(X)\) this gives

\[
|E^\#_{k,L^{-1}}(X, \mathcal{A}, \phi)| \leq O(1)L^3\lambda_k^{1/12-10e} e^{-L(\kappa 7\kappa_0 - 7)d_M(X)} \tag{409}
\]

But for \(L\) sufficiently large \(L(\kappa 7\kappa_0 - 7) \geq \kappa\), so the decay factor can be taken as \(e^{-kd_M(X)}\). Then the bound is

\[
\|E^\#_{k,L^{-1}}\|_{k+1,\kappa} \leq O(1)L^3\lambda_k^{1/12-10e} \tag{410}
\]

Altogether then

\[
\rho_{k+1}(\mathbb{A}_{k+1}, e^{\kappa e + \theta_{k+1} \Phi_{k+1}}) = N_{k+1}Z_{k+1}Z_{k+1}(A_{k+1}) \exp \left( -\frac{1}{2}||\partial A_{k+1}\|^2 - S_{k+1,A_{k+1}}(\Phi_{k+1}, \phi_{k+1}(A_{k+1})) \right) \exp \left( -L^3(\varepsilon_k + \varepsilon_k^0)\Vol(T_{N-k-1}) - \frac{1}{2}L^2\mu_k\|\phi_{k+1}(A_{k+1})\|^2 - \frac{1}{4}L\lambda_k \int \left(\phi_{k+1}(A_{k+1})\right)^4 \right) \tag{411}
\]

6.2.11 completion of the proof

Neither \((BE_k)_{L^{-1}}\) nor \(E^\#_{k,L^{-1}}\) are normalized for small polymers, and we need this feature to complete the proof. We remove energy and mass terms to normalize them.

We have by (255)

\[
(BE_k)_{L^{-1}}(\mathcal{A}, \phi) = -(\mathcal{L}_1 E_k)\Vol(T_{N-k-1}) - \frac{1}{2}(\mathcal{L}_2 E_k)\|\phi\|^2 + (\mathcal{L}_3 E_k)(\mathcal{A}, \phi) \tag{412}
\]

where

\[
\mathcal{L}_1 E_k = \varepsilon \left((BE_k)_{L^{-1}}\right) \\
\mathcal{L}_2 E_k = \mu \left((BE_k)_{L^{-1}}\right) \\
\mathcal{L}_3 E_k = \mathcal{R} \left((BE_k)_{L^{-1}}\right) \tag{413}
\]

By (267) and (408) \(\|\mathcal{L}_1 E_k\| \leq O(1)L^{-t} E_k\|_{k,'}\) and \(\|\mathcal{L}_2 E_k\| \leq O(1)\lambda_k^{1/4+2e} L^{-t} E_k\|_{k,'}\). By (264) and (408) \(\|\mathcal{L}_3 E_k\|_{k+1,\kappa} \leq O(1)L^{-t} E_k\|_{k,\kappa}\). These are the required bounds.
We also apply (265) to \( E_{k,L-1}^\# \) but now tack on the extra term \( \varepsilon_k^0 \). We have

\[
E_{k,L-1}^\#(A, \phi) - L^3 \varepsilon_k^0 \text{Vol}(\mathbb{T}_{N-k-1}) = -\varepsilon_k^0 \text{Vol}(\mathbb{T}_{N-k-1}) - \frac{1}{2} \mu_k^* \|\phi^2\| + E_k^\#(A, \phi)
\]

(414)

where

\[
\varepsilon_k^* = L^3 \varepsilon_k^0 + \varepsilon(E_{k,L-1}^\#)
\]

\[
\mu_k^* = \mu(E_{k,L-1}^\#)
\]

\[
E_k^* = \mathcal{R}(E_{k,L-1}^\#)
\]

By (267) and (410) \( \|\varepsilon_k^*\| \leq \mathcal{O}(1) L^3 \lambda_k^{-10}\epsilon \leq \lambda_k^{1/12-11\epsilon} \) and \( |\mu_k^*| \leq \mathcal{O}(1) L^3 \lambda_k^{-8\epsilon} \leq \lambda_k^{1/12-11\epsilon} \). By (264) and (410) \( \|E_k^*\|_{k+1,\kappa} \leq \mathcal{O}(1) L^{3-\epsilon} \lambda_k^{-10\epsilon} \leq \lambda_k^{1/12-11\epsilon} \). These are the required bounds.

Insert these expansions into (411) and define as in (283)

\[
\varepsilon_{k+1} = L^3 \varepsilon_k + L_1 E_k + \varepsilon_k^*(\mu_k, E_k)
\]

\[
\mu_{k+1} = L^2 \mu_k + L_2 E_k + \mu_k^*(\mu_k, E_k)
\]

\[
E_{k+1} = L_3 E_k + E_k^*(\mu_k, E_k)
\]

(416)

This gives the final form

\[
\rho_{k+1}(A_{k+1}, e^{q_{k+1} + \theta_{k+1}}, \Phi_{k+1}) = Z_{k+1} Z_{k+1}(A_{k+1}) \exp \left( -\frac{1}{2} \|\partial A_{k+1}\|^2 - S_{k+1, A_{k+1}}(\Phi_{k+1}, \phi_{k+1}(A_{k+1})) \right)
\]

\[
\exp \left( -V_{k+1}(\phi_{k+1}(A_{k+1})) + E_{k+1}(A_{k+1}, \phi_{k+1}(A_{k+1})) \right)
\]

(417)

where

\[
V_{k+1}(\phi) = \varepsilon_{k+1} \text{Vol}(\mathbb{T}_{N-k-1}) + \frac{1}{2} \mu_{k+1} \|\phi\|^2 + \frac{1}{4} \lambda_k \int |\phi|^4
\]

(418)

This completes the proof of theorem 1 except for lemma 23.

7 Normalization factor

In this section we prove the missing lemma 23. We need to understand how the normalization factor \( Z_k(A) \) changes under a change in \( A \). This is somewhat involved since \( Z_k(A) \) is nonlocal and we need to express the answer in a local form. In particular we want to write

\[
\frac{Z_k(A + Z)}{Z_k(A)} = \exp(E^z(A, Z))
\]

(419)

with \( E^z(A, Z) \equiv E^{(4)}(A, Z) \) given as a sum of local pieces.

There are two ways to approach this. On the one hand from (92) we have

\[
\frac{Z_k(A + Z)}{Z_k(A)} = \left[ \frac{\det G_k(A + Z)}{\det G_k(A)} \right]^{\frac{1}{2}}
\]

(420)

On the other hand we have from the recursion relation (70)

\[
\frac{Z_k(A + Z)}{Z_k(A)} = \prod_{j=0}^{k-1} \left[ \frac{\det C_j(A_{L-k-j} + Z_{L-k-j})}{\det C_j(A_{L-k-j})} \right]^{\frac{1}{2}}
\]

(421)
In the first representation we are working only on the fine lattice $T_{N-k}^k$ and have to deal with explicit ultraviolet divergences. In the second case we have a product over unit lattice operators on $T_{N-j}^k$ with gauge fields on $T_{N-j}^{-j}$ followed by scalings down to $T_{N-k}^{-k}$. In this case we have no explicit ultraviolet divergences but have to carefully track the scaling behavior. Either approach should work in principle. We prefer to take the second approach which is more in tune with the rest of the paper. However at one point we have to revert to the first approach to make the argument.

### 7.1 single scale

We need estimates on $C_k(A) = \left( \Delta_k(A) + aL^{-2}(Q^T Q)(A) \right)^{-1}$ and on

$$
\Upsilon_k(A, Z) \equiv C_k(A + Z)^{-1} - C_k(A)^{-1} = \left( \Delta_k(A + Z) - \Delta_k(A) \right) + aL^{-2}\left((Q^T Q)(A + Z) - (Q^T Q)(A)\right)
$$

which is defined to satisfy

$$
C_k(A + Z) - C_k(A) = C_k(A)\Upsilon_k(A, Z)C_k(A + Z)
$$

These are all unit lattice operators defined on functions on the lattice $T_{N-k}^k$.

We study these operators for $A \in \frac{1}{2}R_k$ and $Z \in \frac{1}{2}R_k'$ where $R_k'$ is all fields complex valued vector fields $Z$ on $T_{N-k}^k$, satisfying

$$
|Z| < e_k^{-1+3\epsilon} \quad |\partial Z| < e_k^{-1+4\epsilon} \quad |\delta\partial Z| < e_k^{-1+5\epsilon}
$$

We have $R_k' \subset e_k^{2\epsilon}R_k$.

**Lemma 28.** For $A \in \frac{1}{2}R_k$, $Z \in \frac{1}{2}R_k'$ the matrix elements satisfy

$$
|[C_k(A)]_{yy'}| \leq Ce^{-\gamma d(y,y')}
$$

$|$\Upsilon_k(A, Z) |_{yy'}| \leq C e_k 2e^{-\gamma d(y,y')}

**Proof.** First consider $\Upsilon_k$. Define

$$
D_k(A) = Q_k(A)G_k(A)Q_k^T(A)
$$

Then since $\Delta_k(A) = a_k - a_k^2D_k(A)$ we have

$$
\Upsilon_k(A, Z) = a_k^2\left(D_k(A + Z) - D_k(A)\right) + aL^{-2}\left((Q^T Q)(A + Z) - (Q^T Q)(A)\right)
$$

For matrix elements we have

$$
[D_k(A)]_{yy'} = \delta_{y, y'} D_k(A) \delta_{y'} = Q_k^T(A)\delta_{y', y'}
$$

Since $\text{supp}(Q_k^T(A)\delta_y) \subset \Delta_y$ we have by (125) with $L^2$ bounds

$$
|[D_k(A)]_{yy'}| \leq Ce^{-\gamma d(y, y')} ||Q^T(A)\delta_y||_2||Q^T(A)\delta_{y'}||_2 \leq Ce^{-\gamma d(y, y')}
$$

Also consider $|(Q^T Q)(A)]_{yy'} = Q^T(A)\delta_y, Q^T(A)\delta_{y'}$. This vanishes unless $y, y'$ are in the same $L$-cube and satisfies $|(Q^T Q)(A)]_{yy'} \leq O(1)$. Next we use the analyticity in the fields to write for $r \geq 1$

$$
[D_k(A + Z)]_{yy'} = \frac{1}{2\pi i} \int_{|t| = r} \frac{dt}{t(t - 1)} [D_k(A + tZ)]_{yy'}
$$
Here we can take \( r = e_k^{-2\epsilon} \) since then \( t\mathcal{I} \in \frac{1}{2}\mathcal{R}_k \) and we are in the domain of analyticity. Then \([229]\) yields
\[
\left| [D_k(A + \mathcal{I})]_{yy'} - [D_k(A)]_{yy'} \right| \leq Ce_k^{2\epsilon}e^{-\gamma d(y,y')}
\]
(431)

Similarly one shows that
\[
\left| ((Q^TQ)(A + \mathcal{I}))_{yy'} - ((Q^TQ)(A))_{yy'} \right| \leq Ce_k^{2\epsilon}
\]
(432)

This is a local operator so the decay factor is optional here. The bound on \([\Upsilon_k(A, \mathcal{I})]_{yy'}\) follows.

Now consider \( C_k(A) \). We have the identity (this is \([156]\) at \( x = 0 \) )
\[
C_k(A) = A_k(A) + a_k^2A_k(A)Q_k(A)G_{k+1}^0(A)Q_k^T(A)A_k(A)
\]
\[
A_k(A) = \frac{1}{a_k}(I - (Q^TQ)(A)) + \frac{1}{a_k + aL^2}((Q^TQ)(A))^{-1}
\]
(433)

\[ G_{k+1}^0(A) = \left( -\Delta_A + \frac{a_{k+1}}{L^2}(Q^T_{k+1}Q_{k+1})(A) \right)^{-1} \]

Note that \( G_{k+1}^0(A) \) scales to \( G_{k+1}(A) \). Just as for \( D_k(A) \) we have
\[
||Q_k(A)G_{k+1}^0(A)Q_k^T(A)||_{yy'} \leq Ce^{-\gamma d(y,y')} \]
(434)

Every other operator in \([155]\) is local so we have the result. This completes the proof.

Here is a variation of these results. As noted in section \([5.8]\) we can introduce a local version of the Green’s function \( G_k(X,A) \) so that \( G_k(A) = \sum_X G_k(X,A) \) and the same is true for \( G_{k+1}^0(A) \). Using these local Green’s function we define local operators
\[
D_k(X,A) = Q_k(A)G_k(X,A)Q_k^T(A)
\]
\[
\Upsilon_k(X,A,\mathcal{I}) = a_k^2(D_k(X,A+\mathcal{I}) - D_k(X,A)) + aL^{-2}\left( (Q^TQ)(A+\mathcal{I}) - (Q^TQ)(A) \right)\mathbb{I}_X
\]
(435)

\[
C_k(X,A) = A_k(A)\mathbb{I}_X + a_k^2A_k(A)Q_k(A)G_{k+1}^0(X,A)Q_k^T(A)A_k(A)
\]

Here \( \mathbb{I}_X(x) = 1 \) if \( |X|_M = 1 \) and \( x \in X \), and is zero otherwise. Summing over \( X \) we recover \( D_k(A), \Upsilon_k(A,\mathcal{I}), C_k(A) \). Repeat the above proof using \( \|G_k(X,A)f\|_2 \leq Ce^{-\gamma d_M(X)}\|f\|_2 \) and the same for \( G_{k+1}(X,A) \). This yields for the matrix elements
\[
||D_k(X,A)||_{yy'} \leq Ce^{-\gamma d_M(X)}
\]
\[
||\Upsilon_k(X,A,\mathcal{I})||_{yy'} \leq Ce_k^{2\epsilon}e^{-\gamma d_M(X)}
\]
\[
||C_k(X,A)||_{yy'} \leq Ce^{-\gamma d_M(X)}
\]
(436)

These quantities vanish unless \( y,y' \in X \) and only depend on \( A \) in \( X \).

**Lemma 29.** Let \( \epsilon_k \) be sufficiently small depending on \( L,M \). For \( A \in \frac{1}{2}\mathcal{R}_k \), \( \mathcal{I} \in \frac{1}{2}\mathcal{R}_k' \) we have
\[
\left[ \frac{\det C_k(A + \mathcal{I})}{\det C_k(A)} \right]^{\frac{1}{2}} = \exp \left( \sum_{X \in \mathcal{D}_k} E_k^c(X,A,\mathcal{I}) \right) \equiv \exp \left( E_k^c(A,\mathcal{I}) \right)
\]
(437)

where \( E_k^c(X,A,\mathcal{I}) \) is analytic in \( A,\mathcal{I} \), depends on the fields only in \( X \), satisfies \( E_k^c(X,A,0) = 0 \) and
\[
|E_k^c(X,A,\mathcal{I})| \leq \mathcal{O}(1)e_k^{-\epsilon(\kappa - \kappa_0 - 3)d_M(X)}
\]
(438)

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Proof. Since \( C_k(A + Z) = C_k(A) + C_k(A)Y_k(A, Z)C_k(A + Z) \)
\[
\frac{\det C_k(A + Z)}{\det C_k(A)} = \det \left( C_k(A)C_k(A + Z)^{-1} \right)^{-1}
\]
\[
= \det \left( I - C_k(A)Y_k(A, Z) \right)^{-1}
\]
\[
= \exp \left( - \text{Tr} \log \left( I - C_k(A)Y_k(A, Z) \right) \right)
\]
\[
= \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \left( C_k(A)Y_k(A, Z) \right)^n \right)
\]

Now in the sum insert \( Y_k(A, Z) = \sum X Y_k(X, A, Z) \) and \( C_k(A) = \sum Y C_k(Y, A) \). The \( n \)th term is then expressed as a sum over sequences of polymers \( (X_1, Y_1, \ldots, X_n, Y_n) \). The polymer \( X_1 \) must overlap \( Y_1 \) and \( Y_{i-1} \) and so the union is connected. We group together terms with the same union and get the representation \([37] \) with

\[
E_k^c(X, A, Z) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{X_1, \ldots, X_n, Y_1, \ldots, Y_n \rightarrow X} \text{Tr} \left( Y_k(X_1, A, Z)C_k(Y_1, A) \cdots Y_k(X_n, A, Z)C_k(Y_n, A) \right)
\]

Here \( X_1, \ldots, X_n, Y_1, \ldots, Y_n \rightarrow X \) means the overlap conditions are satisfied and \( \cup_{i=1}^{n} (X_i \cup Y_i) = X \). The trace is evaluated as

\[
\sum_{x_1, \ldots, x_n, y_1, \ldots, y_n} [Y_k(x_1, A, Z)]_{y_1} [C_k(Y_1, A)]_{y_1} \cdots [Y_k(x_n, A, Z)]_{y_n} [C_k(Y_n, A)]_{y_n} \]  (441)

Bound the \([Y_k(x_1, A, Z)]_{y_1} \) and \([C_k(Y_1, A)]_{y_1} \) by \([36] \) and bound the sums by estimates like

\[
\sum_{x \in X} 1 \leq \text{Vol}(X) \leq M^3 |X|_M \leq O(1)M^3 e^{d_M(X)}
\]

Thus the trace has an overall factor \((O(1)e^2C^2M^6)^n \leq \epsilon_k^n\) and dropping the \( 1/n \) we have

\[
|E_k^c(X, A, Z)| \leq O(1) \sum_{n=1}^{\infty} \sum_{X_1, \ldots, X_n, Y_1, \ldots, Y_n \rightarrow X} \epsilon_k^n \prod_{i=1}^{n} e^{-(\kappa-1)d_M(X_i)} e^{-(\kappa-1)d_M(Y_i)}
\]

Now we use

\[
\sum_{i=1}^{n} (d_M(X_i) + d_M(Y_i)) \geq d_M(X)
\]

\[
|E_k^c(X, A, Z)| \leq O(1)e^{-(\kappa-\kappa_0-2)d_M(X)} \sum_{n=1}^{\infty} \sum_{X_1, \ldots, X_n, Y_1, \ldots, Y_n \rightarrow X} \epsilon_k^n \prod_{i=1}^{n} e^{-(\kappa_0+1)d_M(X_i)} e^{-(\kappa_0+1)d_M(Y_i)}
\]

We drop the condition that the union is \( X \), retaining only the condition \( X_1 \subset X \), and estimate

\[
\sum_{Y_n \cap X_n \neq \emptyset} e^{-(\kappa_0+1)d_M(Y_n)} \leq O(1)|Y_n|_M
\]

\[
\sum_{X_n \cap Y_{n-1} \neq \emptyset} |X_n|_M e^{-(\kappa_0+1)d_M(X_n)} \leq O(1) \sum_{X_n \cap Y_{n-1} \neq \emptyset} e^{-\kappa_0d_M(X_n)} \leq O(1)|Y_{n-1}|_M
\]

\[
\cdots
\]

\[
\sum_{X_1 \subset X} |X_1|_M e^{-(\kappa_0+1)d_M(X_1)} \leq O(1) \sum_{X_1 \subset X} e^{-\kappa_0d_M(X_1)} \leq O(1)|X|_M \leq O(1)e^{d_M(X)}
\]
The estimate is now
\[ |E_k^\epsilon(X, A, Z)| \leq O(1) e^{-(\kappa - \kappa_0 - 3)d_M(X)} \sum_{n=1}^{\infty} (O(1) e_k^n)^n \leq O(1) e_k^n e^{-(\kappa - \kappa_0 - 3)d_M(X)} \]  
(447)
to finish the proof.

**Remark.** Note that \( E_k^\epsilon(X, A, Z) \) has the symmetries
\[ E_k^\epsilon(X, -A, -Z) = E_j^\epsilon(X, A, Z) \]
\[ E_k^\epsilon(X, A + \partial \lambda, Z) = E_j^\epsilon(X, A, Z) \]  
(448)
These can be deduced from the gauge covariance and charge conjugation covariance of \( \Upsilon_k(X, A, Z) \) and \( C_k(Y, A) \), which in turn follows from the same properties for \( G_k(X, A) \) and \( Q_k(A) \). It is not the case that \( E_k^\epsilon(X, A, Z) \) is gauge invariant in \( Z \). But for the global version we do have
\[ E_k^\epsilon(A, Z + \partial \lambda) = E_j^\epsilon(A, Z) \]  
(449)
Indeed the gauge invariance of \( \det C_k(A + Z) \) implies that the exponentials are equal, hence the identity holds for real fields, and hence for all fields.

### 7.2 Improved single scale

We want to improve the last bound to show it is small when the fields are small. Let \( R = L' \) be a (variable) multiple of \( L \) and let \( \Box_R \) be a partition of \( T_{N-k}^- \) into \( MR \) cubes. We define a new domain based on the inequalities
\[ |A| < R^{-1+\epsilon} e_k^{-1+\epsilon} \]
\[ |
\]  
(450)
|\( \partial A \)\rangle < R^{-2+\epsilon} e_k^{-1+2\epsilon} \]
\[ |\delta_a \partial A\rangle | < R^{-2-\alpha+2\epsilon} e_k^{-1+3\epsilon} \]
We define \( \mathcal{R}_k(R) \) to be all complex-valued \( A \) on \( T_{N-k}^- \) such that \( A = A_0 + A_1 \) where \( A_0 \) is real and in each \( 3MR \) cube \( \Box_R \) is gauge equivalent to some \( A_0' \) satisfying the bounds (450) with a factor \( \frac{1}{2} \) and \( A_1 \) is complex and satisfies the bounds (450) with a factor \( \frac{1}{2} \). We also define \( \mathcal{R}'_k(R) \) by
\[ |Z| < R^{-1+2\epsilon} e_k^{-1+3\epsilon} \]
\[ |\partial Z \rangle | < R^{-2+2\epsilon} e_k^{-1+4\epsilon} \]
\[ |\delta_a \partial Z \rangle | < R^{-2-\alpha+3\epsilon} e_k^{-1+5\epsilon} \]  
(451)
If \( R = 1 \) these are the domains \( \mathcal{R}_k, \mathcal{R}'_k \) we have been discussing. Eventually large \( R \) will be supplied by scaling.

Now define
\[ \hat{E}_k^\epsilon(A, Z) = E_k^\epsilon(A, Z) - \frac{1}{2} \frac{\delta^2 \hat{E}_k^\epsilon}{\delta Z^2}(0; Z, Z) \]  
(452)
This inherits a local expansion \( \hat{E}_k^\epsilon(A, Z) = \sum X \hat{E}_k^\epsilon(X, A, Z) \) from the expansion for \( E_k^\epsilon(X, A, Z) \). We study \( \hat{E}_k^\epsilon(A, Z) \) postponing the treatment of the second derivative term.

**Lemma 30.** For \( A \in \frac{1}{2} \mathcal{R}_k(R) \) and \( Z \in \frac{1}{2} \mathcal{R}'_k(R) \) there is a new localization
\[ \hat{E}_k^\epsilon(A, Z) = \sum_{X \in D_k} \hat{E}_k^\epsilon(X, A, Z) \]  
(453)
where \( \hat{E}_k^\epsilon(X, A, Z) \) is analytic in \( A, Z \), depends on the fields only in \( X \), satisfies \( \hat{E}_k^\epsilon(X, A, 0) = 0 \) and for a constant \( c \leq 1 \) independent of all parameters
\[ |\hat{E}_k^\epsilon(X, A, Z)| \leq O(1) R^{-10/3} e_k^{c(\kappa - \kappa_0 - 4)d_M(X)} \]  
(454)
**Remark.** The key point is that the negative exponent 10/3 is greater than $d = 3$; the specific value is not important.

**Proof.** Let $\delta$ be a fixed small positive number, say $\delta = \frac{1}{8}$. If $d_M(X) \geq LR^d$ then

$$|E_k^\epsilon(X, A, Z)| \leq O(1) e^{\epsilon d_M(X)} = O(1) e^{\epsilon d_M(X)} e^{-(\epsilon-1) d_M(X)}$$

$$\leq O(1) e^{\epsilon L R^d} e^{-\epsilon d_M(X)} \leq O(1) R^{-10/3} e^{\epsilon -\epsilon d_M(X)}$$

(455)

If $|t| \leq R^{1-2\epsilon}$ the $tZ \in \frac{1}{2} R'_k$ and so

$$\frac{1}{2} \delta^2 E_k^\epsilon(X, 0; Z, Z) = \frac{1}{2\pi i} \int_{|t| = R^{1-\epsilon}} \frac{dt}{t^3} E_k^\epsilon(X, 0, tZ)$$

(456)

satisfies a stronger bound than (455). Hence $E_k^\epsilon(X, A, Z)$ satisfies the bound (455) and it qualifies as a contribution to $E_k^\epsilon(X, A, Z)$. Thus it suffices to consider $d_M(X) < LR^d$ which we write as $X \in \mathcal{S}(R)$

The first step is to regroup into terms with greater symmetry. Again let $\square_z$ be the $M$-cubes centered on points $z$ in the $M$-lattice and write

$$\sum_{X \in \mathcal{S}(R)} \tilde{E}_k^\epsilon(X, A, Z) = \sum_z \sum_{X \in \mathcal{S}(R), X \supset \square_z} \frac{1}{|X|^M} \tilde{E}_k^\epsilon(X, A, Z)$$

(457)

Let $\mathcal{O}_z$ be the group of all lattice symmetries that leave $z$ fixed. Each $X \supset \square_z$ determines another polymer $X_z^{sym}$ which is symmetric around $z$ by taking

$$X_z^{sym} = \bigcup_{r \in \mathcal{O}_z} rX$$

(458)

This has $d_M(X_z^{sym}) \leq O(1) d_M(X) \leq O(1) LR^d$. We group together polymers with the same symmetrization and write

$$\sum_{X \in \mathcal{S}(R)} \tilde{E}_k^\epsilon(X, A, Z) = \sum_z \sum_Y \sum_{X \in \mathcal{S}(R), X \supset \square_z, X_z^{sym} = Y} \frac{1}{|X|^M} \tilde{E}_k^\epsilon(X, A, Z)$$

(459)

Change the order of the outside sums and we get $\sum_Y \tilde{E}_k(Y)$ where

$$\tilde{E}_k(Y, A, Z) = \sum_{z: rY = Y \text{ for } r \in \mathcal{O}_z} \left( \sum_{X \in \mathcal{S}(R), X \supset \square_z, X_z^{sym} = Y} \frac{1}{|X|^M} \tilde{E}_k^\epsilon(X, A, 0, Z) \right)$$

(460)

This is zero unless $Y$ is symmetric under some $\mathcal{O}_z$. If $rY = Y$ for $r \in \mathcal{O}_z$ for some $z$, then $z$ is unique and we have $z = z(Y)$. To see this we claim that $|Y|z = \sum_{z' \in Y} z'$. Indeed on the one hand $\sum_{z' \in Y} z' - |Y|z$ is invariant under $\mathcal{O}_z$. On the other hand since it can be written as $\sum_{z' \in Y} (z' - z)$ it changes sign under the reflection $r(z' - z) = -(z' - z)$. Thus it must be zero. Thus outside sum in (460) selects $z = z(Y)$ and we have

$$\tilde{E}_k(Y, A, Z) = \sum_{X \in \Omega(Y)} \frac{1}{|X|^M} \tilde{E}_k^\epsilon(X, A, Z)$$

(461)

where we abbreviate

$$\Omega(Y) = \{ X \in \mathcal{D}_k : X \in \mathcal{S}(R), X \supset \square_{z(Y)}, X_z^{sym} = Y \}$$

(462)

For any unit lattice symmetry $\Omega(rY) = r\Omega(Y)$ and so $\tilde{E}_k(Y, A, Z)$ is still invariant.
Pick a fixed symmetric $Y$. This intersects some cube $\bar{\Box}$, and since $d_M(Y) \leq O(1) LR^3 \leq MR$ it is contained in some enlargement $\Box_R$. Hence in (461) we can replace $A$ by $A'$ satisfying the conditions (450). In each term $E_k^c(X, A', Z)$ contributing to this sum expand around $A' = 0, Z = 0$ taking account the the function is even and that $E_k^c(X, A', 0) = 0$ and that the second derivative in $Z$ is zero. We find for $r \geq 1$

$$E_k^c(X, A', Z) = \frac{\delta^2 E_k^c}{\delta A \delta Z} \left( X, 0; A', Z \right) + \frac{1}{2\pi i} \int_{|t| = r} \frac{dt}{t^4 (t - 1)} E_k^c(X, tA', tZ)$$  \hspace{1cm} (463)

In the last term we can take $r = R$ and then for $|t| = R$ we have that $tA', tZ$ satisfies the $R_k, R'_k$ bounds. Hence we are in the domain of analyticity for $E_k^c(X, A, Z)$ and the the formula holds. From the bound (453) on $E_k^c(X, A, Z)$ we get that the last term in (463) is bounded by $R^{-4} e^{x} e^{-(x - \kappa_0 - 3)d_M(X)}$. For the first term in (463) we have the following:

**Lemma 31.** Under the assumptions of lemma (31) $\left( \frac{\delta^2 E_k^c}{\delta A \delta Z} \right)(X, 0; A', Z)$ for $X \in \Omega(Y)$ can be written as a finite sum of terms which either do not contribute to the sum over $X$ in (461) or are bounded on the domain (450), (451) by $O(1) R^{-10/3} e^{x} e^{-(x - \kappa_0 - 3)d_M(X)}$.

Assuming the lemma, $|E_k^c(X, A', Z)| \leq O(1) R^{-10/3} e^{x} e^{-x d_M(X)}$ and so

$$|E_k^c(Y)| \leq O(1) R^{-10/3} e^{x} \sum_{X \in \Omega(Y)} e^{-x d_M(X)}$$  \hspace{1cm} (464)

But one can show that $d_M(X) \leq \left| d_M(X) \right|$ where $|d_M| = O(1)$ is the number of elements in $O_X$. Then with $c = |d_M|^{-1}$ we have $d_M(X) \geq c \delta d_M(X)$. Hence we can extract from the sum $e^{-x d_M(X)}$ and leave

$$|E_k^c(Y)| \leq O(1) R^{-10/3} e^{x} e^{-c \delta d_M(Y)} \sum_{X \subset Y} e^{-x d_M(X)}$$  \hspace{1cm} (465)

The sum over $X$ is bounded by $O(1)|Y|_M \leq O(1) e^{c \delta d_M(Y)}$ and so

$$|E_k^c(Y)| \leq O(1) R^{-10/3} e^{x} e^{-c \delta d_M(Y)}$$  \hspace{1cm} (466)

This completes the proof of lemma (31) except for lemma (31).

**Proof.** (lemma 31) Expand $A$ around $z = z(Y)$:

$$A'_\nu(x) = A'_\nu(z) + \sum_{\sigma} (x - z)_{\sigma} (\partial_{\sigma} A'_\nu)(z) + \Delta_{\nu}(x, z)$$  \hspace{1cm} (467)

As before the constant vector field $A'_\nu(z)$ is pure gauge in $X$ and disappears. Thus we have

$$\frac{\delta^2 E_k^c}{\delta A \delta Z} \left( X, 0; A', Z \right) = \frac{\delta^2 E_k^c}{\delta A \delta Z} \left( X, 0; (\cdot - z) \cdot \partial A'(z), Z(z) \right)$$
\begin{align*}
&\quad + \frac{\delta^2 E_k^c}{\delta A \delta Z} \left( X, 0; (\cdot - z) \cdot \partial A'(z), Z - Z(z) \right) \\
&\quad + \frac{\delta^2 E_k^c}{\delta A \delta Z} \left( X, 0; \Delta_{\nu}(\cdot, z), Z \right)
\end{align*}

(468)

We claim that the first term in (468) gives zero when summed over $X$ in (461). Writing $Z(z) = \sum_{\nu} Z_{\mu}(z) e_{\mu}$ and $(x - z) \cdot \partial A(z) = \sum_{\nu} (x - z)^{\nu} \partial_{\nu} A(z) e_{\nu}$. It suffices to show that for any $\mu, \nu, \sigma$ the following sum vanishes:

$$\sum_{X \in \Omega(Y)} \frac{1}{|X|_M} \frac{\delta^2 E_k^c}{\delta A \delta Z} \left( X, 0; (\cdot - z)^{\nu} e_{\nu}, e_{\mu} \right)$$  \hspace{1cm} (469)
Let \( r \) the reflection through the point \( z \), so \( r(x - z) = -(x - z) \). Reflection through a unit lattice point is a symmetry of the theory so

\[
\frac{\delta^2 E_k^c}{\delta A \delta Z}(X, 0; f, g) = \frac{\delta^2 E_k^c}{\delta A \delta Z}(rX, 0; f_r, g_r)
\]

(470)

where (taking account \( r^{-1} = r \))

\[
(f_r)_\mu (x) = f_r([x, x + \eta e_\mu]) = f([r(x, x + \eta e_\mu)]) = f([r(x, x - \eta e_\mu)])
\]

\[-= f([r(x - \eta e_\mu, rx]) = -f_\mu(rx - \eta e_\nu)\]

Here under reflection \( e_\mu \) goes to \( -e_\mu \) and \( (x - z)_\sigma e_\nu \) goes to \((x - z)_\sigma + \eta \delta_{\sigma\nu})e_\nu \). Since also \( |rX|_M = |X|_M \), (469) can be written

\[-\sum_{X \in \Omega(Y)} \frac{1}{|rX|_M} \frac{\delta^2 E_k^c}{\delta A \delta Z}(rX, 0; (z - z)_\sigma e_\nu, e_\mu) - \sum_{X \in \Omega(Y)} \frac{1}{|rX|_M} \frac{\delta^2 E_k^c}{\delta A \delta Z}(rX, 0; e_\nu, e_\mu) \delta_{\sigma\nu}
\]

(472)

However the second term vanishes since we have gauge invariance in the first slot (the \( A \) derivative) and the constant vector field \( \eta e_\mu \) is pure gauge. In the first term since \( r \in \mathcal{O}_x \) we have \( r\Omega(Y) = \Omega(Y) \) and summing over \( rX \) here is the same as the sum over \( X \). Hence the first term is exactly minus (469) and therefore zero.

For the second term in (469) note that since \( X \in S(R) \) it has a diameter smaller than \( M|X|_M \leq O(1)M(d_{\mathcal{M}(X)} + 1) \leq O(1)MLR^3 \). Therefore for \( x \in X \)

\[
|Z(x) - Z(z)| \leq O(1)MLR^3|\partial Z|_\infty \leq O(1)MLR^3(R^{-2+\delta+2\epsilon}e_k^{-1+3\epsilon}) \leq \frac{1}{2}R^{-2+\delta+2\epsilon}e_k^{-1+3\epsilon}
\]

(473)

Together with similar bounds on the derivatives this gives \( Z - \tilde{Z}(z) \in \frac{1}{2}R^{-2+\delta+2\epsilon}r_k \). Also

\[
|(x - z) \cdot \partial A'(z)| \leq O(1)MLR^3|\partial A'|_\infty \leq O(1)MLR^3(R^{-2+\epsilon}e_k^{-1+2\epsilon}) \leq \frac{1}{2}R^{-2+\delta+2\epsilon}e_k^{-1+\epsilon}
\]

(474)

Together with similar bounds on derivatives this implies that \( (\cdot - z) \cdot \partial A \in \frac{1}{2}R^{-2+\delta+2\epsilon}r_k \). Then by the bound (435) on \( E_k \) and a Cauchy bound

\[
\left| \frac{\delta^2 E_k^c}{\delta A \delta Z}(X, 0; (\cdot - z) \cdot \partial A(z), Z - \tilde{Z}(z)) \right| \leq O(1)R^{-4+2\delta+3\epsilon}e_k^\epsilon e^{-(\kappa - \kappa_0 - 3)d_{\mathcal{M}(X)}}
\]

(475)

which is more than enough.

For the third term in (468) we write

\[
\Delta_\nu(x, z) = \int_{\Gamma} (\partial A_\nu(y) - \partial A_\nu(z)) \cdot dy
\]

(476)

where \( \Gamma \in G(z, x) \) is any of the standard paths from \( z \) to \( x \). Then

\[
|\Delta_\nu(x, z)| \leq O(1)(MLR^3)^{1+\alpha}|\partial_\nu A|_\infty \leq O(1)(MLR^3)^{1+\alpha}(R^{-2-\alpha+2\epsilon}e_k^{-1+3\epsilon}) \leq \frac{1}{2}R^{-2-\alpha+2\delta+2\epsilon}e_k^{-1+\epsilon}
\]

(477)

Together with similar bounds on the derivatives this implies that \( \Delta \in \frac{1}{2}R^{-2-\alpha+2\delta+2\epsilon}r_k \). Then by a Cauchy bound

\[
\left| \frac{\delta E_k^c}{\delta A \delta Z}(X, 0; \Delta(\cdot, z), Z) \right| \leq O(1)R^{-3-\alpha+2\delta+4\epsilon}e_k^\epsilon e^{-(\kappa - \kappa_0 - 3)d_{\mathcal{M}(X)}}
\]

(478)

This is sufficient since with \( \alpha > \frac{7}{12} \) and \( \delta = \frac{1}{8} \) and \( \epsilon \) sufficiently small we have \(-3 - \alpha + 2\delta + 4\epsilon \geq -\frac{10}{9} \).
7.3 resummation

Combining (421) and (437) we have

\[ \frac{Z_k(A + Z)}{Z_k(A)} = \exp \left( E_k(A, Z) \right) \]

\[ E_k(A, Z) = \sum_{j=0}^{k-1} E_j(A_{L^{k-j}}, Z_{L^{k-j}}) \]  

(479)

**Lemma 32.** $E_k$ has the partial local expansion

\[ E_k(A, Z) = \frac{1}{2} \delta^2 E_k^\varepsilon (0; Z, Z) + \sum_{X \in D_k} \frac{\partial}{\partial t \partial s} \left( 0, tZ_{L^{k-j}} + sZ_{L^{k-j}} \right) \] 

where $E_k^\varepsilon (X, A, Z)$ depends on the fields only in $X$, is analytic in $A \in \frac{1}{2} R_k$ and $Z \in \frac{1}{2} R'_k$ and satisfies there

\[ \left| E_k^\varepsilon (X, A, Z) \right| \leq O(1) e^\varepsilon e^{-\kappa d_M(X)} \]  

(480)

**Remark.** The term $\frac{1}{2} (\delta^2 E_k^\varepsilon / \delta Z^2) (0; Z, Z)$ is localized in the next section.

**Proof.** In (479) we insert the representation of $E_j^\varepsilon (A, Z)$ from lemma 30. Since

\[ \sum_{j=0}^{k-1} \delta^2 E_j^\varepsilon (0; Z_{L^{k-j}}, Z_{L^{k-j}}) = \sum_{j=0}^{k-1} \frac{\partial}{\partial t} \left( 0, tZ_{L^{k-j}} + sZ_{L^{k-j}} \right) \] 

\[ = \frac{\partial}{\partial t} \left( 0, tZ + sZ \right) \] 

(482)

this gives

\[ E_k(A, Z) = \frac{1}{2} \delta^2 E_k^\varepsilon (0; Z, Z) + \sum_{j=0}^{k-1} \sum_{X \in D_j} \tilde{E}_j^\varepsilon (X, A_{L^{k-j}}, Z_{L^{k-j}}) \]  

(483)

As in lemma 15 our assumption $A \in \frac{1}{2} R_k$ implies that in each $L^{k-j}$ we have $A_{L^{k-j}} \sim A_{0, L^{k-j}} + A_{1, L^{k-j}}$ where $A_{0, L^{k-j}}$ is real and satisfies

\[ |A_{0, L^{k-j}}| < \frac{1}{4} (L^{k-j})^{-1+\varepsilon} e^{-1+2\varepsilon} \]

\[ |\partial A_{0, L^{k-j}}| < \frac{1}{4} (L^{k-j})^{-2+2\varepsilon} e^{-1+3\varepsilon} \] 

\[ |\delta ^2 A_{0, L^{k-j}}| < \frac{1}{4} (L^{k-j})^{-2+2\varepsilon} e^{-1+3\varepsilon} \] 

(484)

and $A_{1, L^{k-j}}$ is complex and satisfies the same bounds. Therefore $A_{L^{k-j}} \in \frac{1}{2} R_j (R)$ with $R = L^{k-j}$. Similarly our assumption that $Z_k \in \frac{1}{2} R'_k$ implies that $Z_{L^{k-j}} \in \frac{1}{2} R'_j (R)$ with $R = L^{k-j}$. Thus we can apply lemma 30 and and obtain (using also $\varepsilon' < \varepsilon_k$)

\[ \left| \tilde{E}_j^\varepsilon (X, A_{L^{k-j}}, Z_{L^{k-j}}) \right| \leq L^{-\frac{3}{2} (k-j)} e_k^\varepsilon e^{-c (\kappa - 2 \kappa_0 - 4) d_M(X)} \]  

(485)

Now we reblock. The sum in (483) is now written in the required form

\[ \sum_{Y \in D_k} \tilde{E}_k^\varepsilon (Y) \] 

where for $Y \in D_k$

\[ \tilde{E}_k(A, Z) = \sum_{j=0}^{k-1} \mathcal{B}^{(k-j)} (Y, A, Z) \] 

\[ \mathcal{B}^{(k-j)} (Y, A, Z) = \sum_{X \in Y, L^{k-j}, Z_{L^{k-j}}} \tilde{E}_j^\varepsilon (X, A_{L^{k-j}}, Z_{L^{k-j}}) \] 

(486)
Here $\tilde{X}^{(k-j)}$ is the union of all $L^k-j$ blocks intersecting $X$. A minima spanning tree on the $M$ blocks in $X$ is also a spanning tree on the $L^k-j$ blocks in $\tilde{X}^{(k-j)}$. Therefore $Md_M(X) \geq L^k-jMd_L(L^k-jY)$ or just $d_M(X) \geq L^k-jd_M(Y)$. Then the decay factor in (485) satisfies

$$e^{-c(\kappa-2\kappa_0-4)d_M(X)} \leq e^{-L^k-j(c(\kappa-2\kappa_0-4)-\kappa_0)d_M(Y)} e^{-\kappa_0d_M(X)} \leq e^{-L^k-jd_M(Y)} e^{-\kappa_0d_M(X)}$$

the last since $k-j \geq 1$ and for $L$ sufficiently large $L(c(\kappa-2\kappa_0-4)-\kappa_0) \geq \kappa + 1$. The sum over $X$ in (486) is estimated by

$$\sum_{X \subset L^k-jY} e^{-\kappa_0d_M(X)} \leq O(1)L^{3(k-j)}|L^k-jY|_M = O(1)L^{3(k-j)}e^{d_M(Y)}$$

Hence we have

$$|(B^{(k-j)}\tilde{E}_j^r)(Y,A,Z)| \leq O(1)L^{-\frac{1}{2}(k-j)}e^{\frac{1}{2}e^{-\kappa_0d_M(Y)}} \leq O(1)e^{\frac{1}{2}e^{-\kappa_0d_M(Y)}}$$

The factor $L^{-\frac{1}{2}(k-j)}$ ensures the convergence of the sum over $j$ and we have the required estimate $|\tilde{E}_j^r(Y,A,Z)| \leq O(1)e^{\frac{1}{2}e^{-\kappa_0d_M(Y)}}$

### 7.4 photon self-energy

We treat the term $\frac{1}{2}(\delta^2E_k^x/\delta Z^2)(0,0;Z,Z)$ omitted until now. The background field is now zero so we shorten the notation to $G_k = G_k(0)$ and $U_k(Z) = U_k(0,Z)$ and $E_k^x(Z) = E_k^x(0,Z)$. Since $G_k(Z) = G_k + G_kU_k(Z)G_k(Z)$ we have

$$\frac{\det G_k(Z)}{\det G_k} = \det \left( I - G_kU_k(Z) \right)^{-1} = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \left( G_kU_k(Z) \right)^n \right)$$

The function $\exp(E_k^x(Z))$ is the square root of the last expression so

$$E_k^x(Z) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \left( (G_kU_k(Z))^n \right)$$

The derivative $\frac{1}{2}(\delta^2E_k^x/\delta Z^2)(0;Z,Z)$ is a symmetric quadratic form in $Z$. It is called the photon self-energy and denoted $\Pi_k$. Thus

$$<Z,\Pi_kZ> = \frac{1}{2} \frac{\delta^2 E_k^x}{\delta Z^2}(0;Z,Z)$$

Taking account that $U_k(0) = 0$ we compute it from (491) as

$$<Z,\Pi_kZ> = \frac{1}{4} \text{Tr} \left( \frac{\delta^2 U_k}{\delta Z^2}(0;Z,Z)G_k \right) + \frac{1}{4} \text{Tr} \left( \frac{\delta U_k}{\delta Z}(0;Z)G_k \frac{\delta U_k}{\delta Z}(0;Z)G_k \right)$$

Now $\det G_k(Z)$ is gauge invariant, and it follows that $E_k^x(Z)$ is gauge invariant. So $<f_1,\Pi_k f_2> = \frac{1}{2}(\delta^2 E_k^x/\delta Z^2)(0,f_1,f_2)$ is gauge invariant in either variable. This implies the Ward identity

$$<\partial \lambda,\Pi_k f> = \frac{1}{2} <f,\Pi_k \partial \lambda> = 0 \quad \text{or} \quad \partial^T \Pi_k = \Pi_k \partial = 0$$

Our goal is to prove the following local decomposition (which is not gauge invariant).
Lemma 33.

\[ <Z, \Pi_k Z> = \sum_X E^\gamma_k(X, Z) \]

(495)

where \( E^\gamma_k(X, Z) \) only depends on \( Z \) in \( X \), is invariant under unit lattice symmetries, and

\[ |E^\gamma_k(X, Z)| \leq e^{\gamma_k} \left( \|Z\|_\infty + \|\partial Z\|_\infty + \|\delta_\alpha \partial Z\|_\infty \right)^2 e^{-\alpha d_M(X)} \]

(496)

and so for \( Z \in 1/2 \mathcal{R}'_k \)

\[ |E^\gamma_k(X, Z)| \leq C_k e^{-\alpha d_M(X)} \]

(497)

7.4.1 estimates

We collect some estimates we will need. It is now more convenient to use pointwise estimates than the local \( L^\infty \) estimates employed earlier. We define on \( \mathbb{T}^{N-k}_{-k} \)

\[ d'(x, y) = \begin{cases} 
  d(x, y) & x \neq y \\
  L^k & x = y 
\end{cases} \]

(498)

This is not a true metric since \( d'(x, x) \neq 0 \), but it does satisfy the triangle inequality.

Lemma 34.

\[ |G_k(x, y)| \leq C d'(x, y)^{-\gamma d(x, y)} \]

(499)

\[ |\partial_\mu G_k(x, y)| \leq C d'(x, y)^{-\gamma d(x, y)} \]

\[ |(\partial_\mu G_k \partial_\mu^T)(x, y)| \leq C d'(x, y)^{-\gamma d(x, y)} \]

Proof. We start with the representation on \( \mathbb{T}^{N-k}_{-k} \) (see [1, 24])

\[ G_k(x, y) = \sum_{j=0}^{k-1} L^{k-j} \tilde{C}_j(L^{k-j} x, L^{k-j} y) \]

(500)

where on \( \mathbb{T}^{N-k}_{-j} \)

\[ \tilde{C}_j(x, y) = (H_j C_j H_j^T)(x, y) \]

(501)

and \( \tilde{C}_0 = C_0 = (-\Delta + aL^{-2}Q^T Q)^{-1} \). Now \( C_k, H_j, \) and \( \partial H_j \) all have exponential decay and no short distance singularity. They satisfy (see Appendix D in [24]: \( L^2 \) estimates suffice for \( \tilde{C}_0 = C_0 \))

\[ |\tilde{C}_k(x, y)|, |\partial_\mu \tilde{C}_k(x, y)|, |(\partial_\mu \tilde{C}_k \partial_\mu^T)(x, y)| \leq C e^{-\gamma d(x, y)} \]

(502)

Thus we have

\[ |G_k(x, y)| \leq C \sum_{j=0}^{k-1} L^{k-j} \gamma^{-\gamma d(x, y)} = C \sum_{\ell=1}^k L^\ell e^{-\gamma L^\ell d(x, y)} \]

(503)

Now we split into three cases. For \( x = y \) we have \( |G_k(x, x)| \leq C L^k = C d(x, x)^{-1} \). For \( 0 < d(x, y) \leq 1 \) we need a bound \( C d(x, y)^{-1} \). We choose \( 0 \leq \ell^* \leq k-1 \) so that \( L^{\ell^*} \leq d(x, y)^{-1} \leq L^{\ell^*+1} \) and break the sum into a sum from 1 to \( \ell^* \) (empty if \( \ell^* = 0 \)) and a sum from \( \ell^* + 1 \) to \( k \). The first sum is dominated by

\[ C \sum_{\ell=1}^{\ell^*} L^\ell \leq C L^{\ell^*} \leq C d(x, y)^{-1} \]

(504)
Lemma 35.

The second sum is dominated by

\[ C \sum_{\ell=\ell^*+1}^{\infty} L^\ell e^{-\gamma L^\ell d(x,y)} = C \sum_{j=1}^{\infty} L^j e^{-\gamma L^j d(x,y)} \]

\[ \leq C L^{\ell^*} \sum_{j=1}^{\infty} L^j e^{-\gamma L^j - 1} \leq C L^{\ell^*} \leq C d(x,y)^{-1} \quad (505) \]

For \( d(x,y) \geq 1 \) we have

\[ |d(x,y) G_k(x,y)| \leq C \sum_{\ell=1}^{k} L^\ell d(x,y) e^{-\gamma L^\ell d(x,y)} \]

\[ \leq C \sum_{\ell=1}^{k} e^{-\frac{\ell}{\ell^*} L^\ell d(x,y)} \leq C e^{-\frac{\ell}{\ell^*} Ld(x,y)} \sum_{\ell=1}^{k} e^{-\frac{\ell}{\ell^*} L^\ell} \leq C e^{-\gamma d(x,y)} \}

(506)

For the derivatives we argue similarly starting with expressions like

\[ \partial_\mu G_k(x,y) = \sum_{j=0}^{k-1} L^{2(k-j)} (\partial_\mu \tilde{C}_j) (L^{k-j} x, L^{k-j} y) \]

(507)

This completes the proof.

Next consider the operator \( U_k(\mathcal{Z}) \) which we divide as \( U_k(\mathcal{Z}) = U_k^\delta(\mathcal{Z}) + U_k^q(\mathcal{Z}) \) where

\[ U_k^\delta(\mathcal{Z}) = -\Delta \mathcal{Z} + \Delta_0 \]

\[ U_k^q(\mathcal{Z}) = a_k \left( (Q_k^T Q_k)(\mathcal{Z}) - (Q_k^T Q_k)(0) \right) \]

(508)

Here \( U_k^\delta \) are the standard pieces and \( U_k^q \) are the pieces involving averaging operators \( Q_k \).

To analyze the contribution of \( U_k^q \) we will need the following

\[ \left| \left( \frac{\delta U_k^q}{\delta \mathcal{Z}} (0; \mathcal{Z}) G_k \right)(x,y) \right| \leq C e_k \| \mathcal{Z} \|_\infty e^{-\gamma d(x,y)} \]

\[ \left| \left( \frac{\delta^2 U_k^q}{\delta \mathcal{Z}^2} (0; \mathcal{Z}) G_k \right)(x,y) \right| \leq C e_k^2 \| \mathcal{Z} \|_\infty e^{-\gamma d(x,y)} \]

(509)

**Proof.** Recall that \( \Delta_\mathcal{Z} \) is the unit cube centered on \( z \in \mathbb{T}_{N-k}^\mathcal{Z} \). The operator \( (Q_k^T Q_k)(\mathcal{Z}) \) is local and has the kernel

\[ (Q_k^T Q_k)(\mathcal{Z}; x, y) = \begin{cases} \exp \left( -q e_k \eta(z, \mathcal{Z})(z, x) + q e_k \eta(z, \mathcal{Z})(z, y) \right) & \text{if } x, y \in \Delta_\mathcal{Z} \\ 0 & \text{otherwise} \end{cases} \]

(510)

It is analytic and bounded by \( O(1) \) for \( \| \mathcal{Z} \|_\infty \leq e_k^{-1} \). Then the kernel \( (Q_k^T Q_k)(\mathcal{Z}) G_k \) is analytic for \( \| \mathcal{Z} \|_\infty \leq e_k^{-1} \) and if \( x \in \Delta_\mathcal{Z} \) then by \( 499 \)

\[ |(Q_k^T Q_k)(\mathcal{Z}) G_k(x,y)| \leq C \int_{\Delta_\mathcal{Z}} d'(x', y)^{-1} e^{-\gamma d(x', y)} \ dx' \leq C e^{-\gamma d(x,y)} \]

(511)

63
See appendix [D] for the integrability of $d'(x', y)^{-1}$. Then for $\|Z\|_\infty \leq 1$ we have

$$
\left( \frac{\delta U^q}{\delta Z} (0; Z) G_k \right) (x, y) = a_k \left( \frac{\delta (Q^T_k Q_k)}{\delta Z} (0; Z) G_k \right) (x, y) = a_k \frac{dt}{t^2} \int_{|t| = \epsilon_k^{-1}} \left( (Q^T_k Q_k)(tZ) G_k \right) (x, y)
$$

(512)

This leads to the bound for $\|Z\|_\infty \leq 1$

$$
\left| \left( \frac{\delta U^q}{\delta Z} (0; Z) G_k \right) (x, y) \right| \leq C e_k e^{-\gamma d(x, y)}
$$

(513)

This is sufficient since $(\delta U^q_k / \delta Z)(0; Z) G_k)(x, y)$ is linear in $Z$. The proof for the second derivative is similar.

**7.4.2 removal of averaging operators from interaction**

In the expression for $< Z, \Pi_k Z >$ we insert the decomposition $U_k = U^s_k + U^q_k$. Let $\Pi^q_k$ be the part with only $U^q_k$. We estimate it first. It is written

$$
< Z, \Pi^q_k Z > = \frac{1}{4} \text{Tr} \left( \frac{\delta^2 U^q_k}{\delta Z^2} (0; Z, Z) G_k \right) + \frac{1}{4} \text{Tr} \left( \frac{\delta U^q_k}{\delta Z} (0; Z) G_k \frac{\delta U^q_k}{\delta Z} (0; Z) G_k \right)
$$

(514)

Taking account that the trace over charge indices gives a factor of 2 this can be written

$$
< Z, \Pi^q_k Z > = \frac{1}{2} \int dx \left( \frac{\delta^2 U^q_k}{\delta Z^2} (0; Z, Z) G_k \right) (x, x)
$$

$$
+ \frac{1}{2} \int dx dy \left( \frac{\delta U^q_k}{\delta Z} (0; Z) G_k \right) (x, y) \left( \frac{\delta U^q_k}{\delta Z} (0; Z) G_k \right) (y, x)
$$

(515)

**Lemma 36.**

$$
< Z, \Pi^q_k Z > = \sum_X E^q_k (X, Z)
$$

(516)

where

$$
|E^q_k (X, Z)| \leq e_k^2 \|Z\|_\infty^2 e^{-\kappa d_M (X)}
$$

(517)

**Proof.** The estimates of lemma [33] show that there is no short distance singularity, and that

$$
|< Z, \Pi^q_k Z | \leq \left( C e_k^2 \int dx + C e_k^2 \int e^{-2\gamma d(x, y)} dx dy \right) \|Z\|_\infty^2
$$

(518)

This bound is proportional to the volume.

We need to write $< Z, \Pi^q_k Z >$ as a sum of local pieces, and do it a way that preserves invariance under lattice symmetries. This is best accomplished by regarding $Z$ as a function on bonds. We have

$$
< Z, \Pi^q_k Z > = \int Z(b) \Pi^q (b, b') Z(b') db db'
$$

(519)

where the integral is over oriented bonds and $\int f(b) db = \sum_\mu \int f ([x, x + \eta] \mu) dx$. Alternatively we take an extended definition of $Z(b), \Pi^q (b, b')$ to all bonds with $Z(x, x') = -Z(x', x)$, etc. Then a representation like [33] stills holds, but ranging over all bonds and with an extra factor of $1/2$ for each integral. In this representation the invariance $< Z_r, \Pi^q_k Z_r >= < Z, \Pi^q_k Z >$ under $T^0_{N-k}$ lattice symmetries $r$ implies that

$$
\Pi^q_k (b, b') = \Pi^q_k (rb, rb')
$$

(520)
Again let $\Delta_z$ be the unit cube centered on the unit lattice point $z \in T_{N-k}$. Define a modified characteristic function $\chi_z$ on all bonds by

$$
\chi_z(b) = \begin{cases} 
1 & \text{if } b \subset \Delta_z \\
\frac{1}{2} & \text{if } b \cap \Delta_z \neq \emptyset, b \cap \Delta_z^c \neq \emptyset \\
0 & \text{if } b \cap \Delta_z = \emptyset
\end{cases} \quad (521)
$$

Then $\sum_z \chi_z = 1$ and $(\chi_z)_r(b) = \chi_z(r^{-1}b) = \chi_{rz}(b)$. We make the decomposition

$$< \mathcal{Z}, \Pi_k^q \mathcal{Z} > = \sum_{z,w} < (\chi_z \mathcal{Z}), \Pi_k^q (\chi_w \mathcal{Z}) > \quad (522)
$$

The characteristic functions are insensitive to orientation, so we can evaluated this with either oriented or unoriented bonds. We have $< (\chi_{rz} \mathcal{Z}), \Pi_k^q (\chi_{rw} \mathcal{Z}) > = < (\chi_z \mathcal{Z}), \Pi_k^q (\chi_w \mathcal{Z}) >$ and

$$< (\chi_z \mathcal{Z}), \Pi_k^q (\chi_w \mathcal{Z}) > = \frac{1}{2} \int dx \left( \frac{\delta U_k^q}{\delta Z^2} (0; \chi_z \mathcal{Z}, \chi_w \mathcal{Z}) G_k \right) (x, x) + \frac{1}{2} \int dxdy \left( \frac{\delta U_k^q}{\delta Z} (0; \chi_z \mathcal{Z}) G_k \right) (x, y) \left( \frac{\delta U_k^q}{\partial Z} (0; \chi_w \mathcal{Z}) G_k \right) (y, x) \quad (523)
$$

Again we estimate using the bounds of lemma 35. Because $U_k(Z)$ and its derivatives are local operators the integrals over $x, y$ are restricted to the immediate neighborhood of $\Delta_z, \Delta_w$, denoted $\Delta_z^*, \Delta_w^*$. Thus we have

$$| < (\chi_z \mathcal{Z}), \Pi_k^q (\chi_w \mathcal{Z}) > | \leq \left(C c_k^2 \int_{\Delta_z^* \cap \Delta_w^*} dx + C c_k^2 \int_{\Delta_z^* \times \Delta_w^*} e^{-2\gamma d(x,y)} dxdy \right) \|Z\|_2^2 \quad (524)
$$

The first term only contributes when $\Delta_z, \Delta_w$ touch and in the second term we use Now we use $d(x, y) \geq d(z, w) - O(1)$. Hence

$$| < (\chi_z \mathcal{Z}), \Pi_k^q (\chi_w \mathcal{Z}) > | \leq C c_k^2 \|Z\|_2^2 e^{-2\gamma d(z, w)} \quad (525)
$$

The expansion (522) localizes the expression, but not yet in polymers since $\Delta_z \cup \Delta_w$ is generally not connected.

For any unit lattice points $z, w$ let

$$X_{zw} = \text{the smallest polymer containing } \Delta_z^* \text{ for all } x$$

in any of the paths $T^n(z, w)$ from $z$ to $w$ \quad (526)

It is roughly the thickened edges of a cube with $zw$ on opposite corners. Then we have the required

$$< \mathcal{Z}, \Pi_k^q \mathcal{Z} > = \sum_X E_k^q(X, \mathcal{Z}) \quad (527)
$$

This satisfies $E_k^q(rX, \mathcal{Z}_r) = E_k^q(X, \mathcal{Z})$ and from (524)

$$|E_k^q(X, \mathcal{Z})| \leq C e_k^2 \|Z\|_2^2 \sum_{z,w:X_{zw}=X} e^{-2\gamma d(z,w)} \quad (528)
$$

But $Md_M(X) \leq c d(z, w)$ for some $c = O(1)$. Hence $2\gamma d(z, w) \geq 2\gamma \epsilon^{-1} Md_M(X) \geq \kappa d_M(X)$ for $M$ sufficiently large. Also the number of points $z, w$ with $X_{zw} = X$ is bounded by $O(1) M^6$. Hence for $e_k$ sufficiently small

$$|E_k^q(X, \mathcal{Z})| \leq C M^6 e_k^2 \|Z\|_2^2 e^{-\kappa d_M(X)} \leq e_k^2 \|Z\|_2^2 e^{-\kappa d_M(X)} \quad (529)
$$
This completes the proof of the lemma.

There is also a term one \( U^q_k \) and one of \( U^s_k \). It has the form

\[
< Z, \Pi^q_k Z > = \frac{1}{4} \text{Tr} \left( \frac{\delta U^q_k}{\delta Z}(0; Z)G_k \frac{\delta U^s_k}{\delta Z}(0; Z)G_k \right)
\]

This has integrable short distanced singularities and can be treated using the estimate just established on \( (\delta U^q_k / \delta Z)(0; Z)G_k \) and estimates on \( (\delta U^s_k / \delta Z)(0; Z)G_k \) from the next section. We omit the details.

### 7.4.3 an explicit representation

Now we are reduced to an expression with standard potential but still non-standard propagators. It is partially standard. It is

\[
< Z, \Pi^p_k Z > = \frac{1}{4} \text{Tr} \left( \frac{\delta^2 U^p_k}{\delta Z^2}(0; Z,G_k)G_k \right) + \frac{1}{4} \text{Tr} \left( \frac{\delta U^p_k}{\delta Z}(0; Z,G_k)\frac{\delta U^p_k}{\delta Z}(0; Z,G_k) \right)
\]

Lemma 37.

\[
< Z, \Pi^p_k Z > = \sum_{\mu \nu} \int Z_{\mu}(x)\Pi^{p,0}_{k,\mu}(x,y)Z_{\nu}(y) \, dxdy
\]

where

\[
\Pi^{p,0}_{k,\mu}(x,y) = \delta_{\mu \nu} \delta(x - y)\Pi^{p,0}_{k,\mu}(x) + \Pi^{p,1}_{k,\mu}(x,y)
\]

and

\[
\Pi^{p,0}_{k,\mu}(x) = e^2_k G_k(x + \eta e_\mu, x + \eta e_\mu) - e^2_k \eta \partial_\mu G_k(x + \eta e_\mu, x)
\]

\[
\Pi^{p,1}_{k,\mu
\nu}(x,y) = - e^2_k (\partial_\mu G_k)(x, y + \eta e_\nu)G_k(y, x + \eta e_\mu)
\]

+ \[ e^2_k (\partial_\mu G_k \partial^T_\nu)(x, y)G_k(y + \eta e_\nu, x + \eta e_\mu) \]

**Remark.** Note that \( \Pi^{p,0}_{k,\mu}(x) = \mathcal{O}(\eta^{-1}) \) and

\[
|\Pi^{p,1}_{k,\mu
\nu}(x,y)| \leq C e^2_d d'(x,y)^{-4} e^{-2\gamma d(x,y)}
\]

There is a linear ultraviolet divergence which must be canceled.

**Proof.** Define an operator \( Z_{\mu}^{(1)} \) by

\[
(Z_{\mu}^{(1)} f)(x) = \frac{d}{dt} \left[ (\partial_{t}Z_{\mu}f)(x) \right]_{t=0} = q e_k \partial_{\mu} f(x + \eta e_\mu)
\]

Then

\[
< f, \frac{\delta U^p_k}{\delta Z}(0; Z)f > = \frac{d}{dt} \left[ < f, U^p_k(tZ)f > \right]_{t=0}
\]

\[
= \frac{d}{dt} \left[ < \partial_t Z f, \partial_t Z f > \right]_{t=0} = \sum_{\mu} < Z_{\mu}^{(1)} f, \partial_\mu f > + < \partial_\mu f, Z_{\mu}^{(1)} f >
\]

which is equivalent to the operator identity

\[
\frac{\delta U^p_k}{\delta Z}(0; Z) = \sum_{\mu} Z_{\mu}^{(1)} \partial_\mu + \partial_\mu Z_{\mu}^{(1)}
\]
Also define
\[ (Z^{(2)}_\mu(x)) = \frac{d^2}{dt^2} \left[ (\partial_t Z_\mu f)(x) \right]_{t=0} = -e_k^2 \eta(Z_\mu(x))^2 f(x + \eta \epsilon_\mu) \] (539)

Then
\[ < \ell, \frac{\delta^2 U_k^\ell(0; Z, Z)f >}{\delta Z^2} > = \frac{d^2}{dt^2} \left[ < \ell, U_k^\ell(tZ)f > \right]_{t=0} \]
\[ = \frac{d^2}{dt^2} \left[ < \partial t Z f, \partial t Z f > \right]_{t=0} = 2 \sum_\mu < Z_\mu^{(1)} f, Z_\mu^{(1)} f > + < \partial_\mu f, Z_\mu^{(2)} f > \] (540)
or
\[ \frac{\delta^2 U_k^\ell}{\delta Z^2}(0; Z, Z) = 2 \sum_\mu Z_\mu^{(1), T} Z_\mu^{(1)} + \partial_\mu^T Z_\mu^{(2)} \] (541)

Inserting these in (531) we find
\[ < Z, \Pi_k^\ell Z > = \frac{1}{2} \sum_\mu \Tr \left( Z_\mu^{(1), T} Z_\mu^{(1)} G_k \right) + \eta \sum_\mu \Tr \left( \partial_\mu^T Z_\mu^{(2)} G_k \right) \]
\[ + \frac{1}{2} \sum_{\mu \nu} \Tr \left( Z_\mu^{(1), T} (\partial_\mu G_k) Z_{\nu}^{(1), T} (\partial_\nu G_k) \right) + \sum_{\mu \nu} \Tr \left( Z_\mu^{(1), T} (\partial_\mu G_k \partial_\nu^T) Z_{\nu}^{(1)} G_k \right) \] (542)

Evaluating this with \( (Z_\mu^{(1), T} f)(x) = -q e_k Z_\mu(x - \eta \epsilon_\mu) f(x - \eta \epsilon_\mu) \) and gaining an extra factor of two from the trace over charge indices gives the result.

### 7.4.4 removal of averaging operators from propagators

Next we change to more standard propagators (which have more symmetry) replacing the propagator \( G_k = (-\Delta + a_k Q_k^g Q_k)^{-1} \) by \( G_k^* = (\Delta + I)^{-1} \). This satisfies
\[ |G_k^*(x, y)| \leq O(1)d'(x, y)^{-1}e^{-\gamma d(x, y)} \]
\[ |(\partial G_k^*)^\ell(x, y)| \leq O(1)d'(x, y)^{-2}e^{-\gamma d(x, y)} \]
\[ |(\partial G_k^* \partial_\nu^T)(x, y)| \leq O(1)d'(x, y)^{-3}e^{-\gamma d(x, y)} \] (543)

This is probably well-known; nevertheless we include a proof in appendix E.

Let \( \Pi_k^* \) be the photon self energy with this replacement. It is given by
\[ < Z, \Pi_k^* Z > = \frac{1}{4} \Tr \left( \frac{\delta^2 U_k^*(0; Z, Z) G_k^*}{\delta Z^2} \right) + \frac{1}{4} \Tr \left( \frac{\delta^2 U_k^*(0; Z) G_k^* \delta^2 U_k^*(0; Z) G_k^*}{\delta Z^2} \right) \] (544)
or by an expression like (532) with kernel where
\[ \Pi_k^{(0)}(x, y) = \delta_{\mu \nu} \delta(x - y) \Pi_k^{(0)}(x) + \Pi_k^{(1)}(x, y) \] (545)
and
\[ \Pi_k^{(0)}(x) = e_k^2 G_k^*(x + \eta \epsilon_\mu, x + \eta \epsilon_\mu) - e_k^2 \eta(\partial_\mu G_k^*)(x + \eta \epsilon_\mu, x) \]
\[ \Pi_k^{(1)}(x, y) = -e_k^2(\partial_\mu G_k^*(x, y + \eta \epsilon_\nu)(\partial_\nu G_k^*)(y, x + \eta \epsilon_\mu) \]
\[ + e_k^2(\partial_\mu G_k^* \partial_\nu^T)(x, y)(G_k^*)(y + \eta \epsilon_\nu, x + \eta \epsilon_\mu) \] (546)

Again we have
\[ |\Pi_k^{(1)}(x, y)| \leq C e_k^2 d'(x, y)^{-4}e^{-2\gamma d(x, y)} \] (547)
and still there is an apparent linear ultraviolet divergence.

Note also that $\Pi^*$ can be obtained directly from $G^*_k(Z) \equiv (-\Delta_Z + I)^{-1}$ just as $\Pi$ was obtained from $G_k(Z)$, namely

$$< Z, \Pi_k Z > = \frac{1}{2} \log \left[ \frac{\det G_k(Z)}{\det G^*_k(Z)} \right]$$

(548)

Hence just like $\Pi_k$ we have that $\Pi^*_{k, \mu \nu}$ satisfies the Ward identity

$$\partial T \Pi^*_k = 0 \quad \Pi^*_k \partial = 0$$

(549)

For the difference we have the following two results:

Lemma 38.

$$|G_k(x, y) - G^*_k(x, y)| \leq C e^{-\frac{1}{2} \gamma d(x, y)}$$

(550)

$$|\partial G_k(x, y) - \partial G^*_k(x, y)| \leq C d(x, y)^\epsilon e^{-\frac{1}{2} \gamma d(x, y)}$$

Proof. For the difference we have

$$G_k - G^*_k = G_k(I - a_k Q^T_k Q_k) G^*_k$$

(551)

We focus on the term $G_k G^*_k$; the other term $a_k G_k Q^T_k Q_k G^*_k$ is less singular. We have $(G_k G^*_k)(x, y) = \int G_k(x, z) G_k(z, y) dz$ and so by (499) and (543)

$$|G_k G^*_k(x, y)| \leq C \int d'(x, z)^{-1} e^{-\gamma (d'(x, z) + d'(z, y))} dz$$

(552)

We can extract a factor $e^{-\frac{1}{2} \gamma d'(x, y)}$ here and still have enough decay left for convergence in $z$ at large distances. For short distances we have an integrable singularity, for example by a Schwarz inequality. Hence the first bound. For the second bound we focus on $(\partial G_k G^*_k)(x, y) = \int \partial G_k(x, z) G_k(z, y) dz$ which has the bound

$$|\partial G_k G^*_k(x, y)| \leq C \int d'(x, z)^{-2} d'(z, y)^{-1} e^{-\gamma (d'(x, z) + d'(z, y))} dz$$

(553)

Again we extract a factor $e^{-\frac{1}{2} \gamma d'(x, y)}$ and have no long distance problem. If either $d'(x, z)$ or $d'(z, y)$ is greater than one we have an integrable singularity and get the result. If both $d'(x, z) \leq 1$ and $d'(z, y) \leq 1$ then we use from appendix [3]

$$\int d'(x, z)^{-2} d'(z, y)^{-1} \leq O(1) d'(x, y)^{-\epsilon}$$

(554)

and hence the result.

Lemma 39.

$$< Z, \Pi_k^0 Z > = \sum X E_k^n(X, Z)$$

(555)

where

$$|E_k^n(X, Z)| \leq C_k \epsilon^2 \|Z\|_\infty e^{-\kappa d_M(X)}$$

(556)

Proof. For the first term in $\Pi_k^{(1)}(x, y) - \Pi_k^{*,(1)}(x, y)$ we use lemma [38] to get estimates like

$$|e_k^2 (\partial_\mu G_k - \partial_\mu G^*_k)(x, y + \eta \epsilon \nu)(\partial_\nu G^*_k)(y, x + \eta \epsilon \mu)| \leq C e_k^2 d'(x, y)^{-2} e^{-\gamma d(x, y)}$$

(557)
which has an integrable singularity. The second term is essentially the same. In $\Pi_k^{\rho(0)}(x, y) - \Pi_k^{\tau(0)}(x, y)$ the second term is estimated by

$$|e_k^2\eta(\partial G_k(x + \eta e_\mu, x) - \partial G_k^r(x + \eta e_\mu, x)| \leq O(1)e_k^2\eta(x + \eta e_\mu, x)^{-\epsilon} \leq C e_k^2\eta^{1-\epsilon} \leq C e_k^2$$ \hspace{1cm} (558)

The first term is equally easy.

To localize we proceed as in the proof of lemma 36 writing

$$< Z, [\Pi_k^p - \Pi_k^p](\chi_w Z) = \sum_{z,w} < (\chi_z Z), [\Pi_k^p - \Pi_k^p](\chi_w Z) >$$ \hspace{1cm} (559)

Then $< (\chi_z Z), [\Pi_k^p - \Pi_k^p](\chi_w Z) >$ is finite and estimated by

$$| < (\chi_z Z), [\Pi_k^p - \Pi_k^p](\chi_w Z) > | \leq C e_k^2\|Z\|^2_\infty e^{-\gamma d(z,w)}$$ \hspace{1cm} (560)

Rewriting the expression as a sum over polymers and using $C e_k^2 \leq e_k^{2-\epsilon}$ gives the result.

### 7.4.5 proof of lemma 33

Our standard photon self-energy has the advantage of being invariant under the full $T_{-k}$ lattice symmetries, not just $T_{N-k}$ symmetries. It can be written in two ways

$$< Z, \Pi_k^p Z > = \int dbdb' Z(b)\Pi^p(x, b')Z(b') = \sum_{\mu\nu} \int dx dy Z(x)\Pi_k^p(x, y)Z(y)$$ \hspace{1cm} (561)

where the integral is over oriented bonds and the kernels are related by

$$\Pi_{k,\mu\nu}(x, y) = \Pi_k^p([x, x + \eta e_\mu], [y, y + \eta e_\nu])$$ \hspace{1cm} (562)

If the first form is extended to all bonds as before then $\Pi_k^p(b, b') = \Pi_k^p(rb, rb')$. This symmetry is more complicated in the other notation. For example is r is the complete inversion $rx = -x$ it says

$$\Pi_{k,\mu\nu}(x, y) = \Pi_k^p([x, x + \eta e_\mu], [y, y + \eta e_\nu]) = \Pi_k^p([-x, -x - \eta e_\mu], [-y, -y - \eta e_\nu])$$

$$= \Pi_k^p([-x - \eta e_\mu, -x], [-y - \eta e_\nu, -y]) = \Pi_{k,\mu\nu}(x, y)$$ \hspace{1cm} (563)

We break up $< Z, \Pi_k^p Z >$ into pieces. Each piece should be covariant under lattice symmetries so at first we work with the representation $\Pi^p(b, b')$. First let $\theta$ be a smooth function on $\mathbb{R}$ such that $0 \leq \theta \leq 1$ and $\theta = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and supp$\theta \subset [-\frac{2}{3}, \frac{2}{3}]$. Let $d_\theta(b, b')$ be the Euclidean distance between $b, b'$. Then $\theta(d_\theta(b, b'))$ does not depend on orientation and we can make the split

$$< Z, \Pi_k^p Z > = \int dbdb' Z(b)\left(1 - \theta(d_\theta(b, b'))\right)\Pi^p(b, b')Z(b')$$

$$+ \int dbdb' Z(b)\theta(d_\theta(b, b'))\Pi^p(b, b')Z(b')$$ \hspace{1cm} (564)

without spoiling covariance.

In the first term note that $(1 - \theta(d_\theta(b, b')))$ vanishes unless $d_\theta(b, b') \geq \frac{1}{\epsilon}$. Thus there is no ultraviolet divergence here. We localize it as

$$\sum_{x, w} \int dbdb' \chi_z(b)Z(b)\left(1 - \theta(d_\theta(b, b'))\right)\Pi^p(b, b')\chi_w(b')Z(b')$$ \hspace{1cm} (565)
By \( (547) \) the summand is bounded by \( C e_k^2 \| Z \|_\infty^2 e^{-\gamma d(z,w)} \) and the sum is bounded by \( C e_k^2 \| Z \|_\infty^2 \). As in lemma \( \text{[546]} \) we write the expression as a sum over polymers and get a contributions to \( E_k^2(X) \) bounded by \( C e_k^2 \| Z \|_\infty^2 e^{-\gamma d_M(x)} \) which suffices.

For the second term in \( \text{[501]} \) we localize in the \( b \) variable only write it as

\[
\sum_z \int dbdb' \chi_z(b)Z(b)\theta(d_2(b,b'))\Pi_k^s(b,b')Z(b')
\]  

(566)

Since \( \theta(d_2(b,b')) \) vanishes for \( d_2(b,b') \geq \frac{1}{4} \), for each \( z \) the term is localized in the threefold enlargement \( \Delta_z \). Let \( \Delta^M_z \) be the smallest polymer containing \( \Delta_z \). Then we get a contribution to \( E_k^2(X) \) of the form

\[
\sum_{z: \Delta^M_z = X} \int dbdb' \chi_z(b)Z(b)\theta(d_2(b,b'))\Pi_k^s(b,b')Z(b')
\]  

(567)

The term can also be written

\[
\sum_{z: \Delta^M_z = X} \left[ \sum_{\mu \nu} \int dxdy \chi_z([x,x + \eta e_\mu])Z_\mu(x)\theta \left( d_2([x,x + \eta e_\mu],[y,y + \eta e_\nu]) \right) \Pi_{k,\mu\nu}^s(x,y)Z_\nu(y) \right]
\]  

(568)

We will show that the bracketed expression is bounded by \( C e_k^2 (\| Z \|_\infty + \| \partial Z \|_\infty + \| \delta_\alpha \partial Z \|)^2 \). Since \( X \) contains at most \( O(1) M \)-cubes, the sum over \( z \) contributes a factor \( O(1) M^3 \). Then using \( CM^2 e_k^2 \leq e_k^{2-\epsilon} \) the expression \( \text{[568]} \) is bounded by \( e_k^{2-\epsilon} (\| Z \|_\infty + \| \partial Z \|_\infty + \| \delta_\alpha \partial Z \|^2)^2 \) as required. The term is invariant under lattice symmetries, but in estimating it we break it up into pieces that are not invariant.

The first step is to replace \( \theta \left( d_2([x,x + \eta e_\mu],[y,y + \eta e_\nu]) \right) \) by \( \theta(d_2(x,y)) \). So for the difference we must consider

\[
\sum_{\mu \nu} \int dxdy \chi_z([x,x + \eta e_\mu])Z_\mu(x) \left[ \theta \left( d_2([x,x + \eta e_\mu],[y,y + \eta e_\nu]) \right) - \theta(d_2(x,y)) \right] \Pi_{k,\mu\nu}^s(x,y)Z_\nu(y)
\]  

(569)

If \( d_2([x,x + \eta e_\mu],[y,y + \eta e_\nu]) \leq \frac{1}{4} \) and \( d_2(x,y) \leq \frac{1}{4} \) then \( \theta \left( d_2([x,x + \eta e_\mu],[y,y + \eta e_\nu]) \right) - \theta(d_2(x,y)) = 1 - 1 = 0 \) and the term vanishes. Thus there is no ultraviolet divergence, only \( \Pi_{k,\mu\nu}^{s,(1)} \) contributes, and the term can be estimated by \( C e_k^2 \| Z \|_\infty^2 \) as before.

So now we consider

\[
\sum_{\mu \nu} \int dxdy \chi_z([x,x + \eta e_\mu])Z_\mu(x)\theta(d_2(x,y))\Pi_{k,\mu\nu}^s(x,y)Z_\nu(y)
\]  

(570)

We generate three terms in this expression by making the expansion

\[
Z_\nu(y) = Z_\nu(x) + \sum_\sigma (y - x)_\sigma \partial_\sigma Z_\nu(x) + \Delta_\nu(y,x)
\]  

(571)

The first term is

\[
\sum_{\mu \nu} \int dxdy \chi_z([x,x + \eta e_\mu])Z_\mu(x)\theta(d_2(x,y))\Pi_{k,\mu\nu}^s(x,y)Z_\nu(x)
\]  

(572)

We write

\[
Z_\nu(x) = \frac{\partial}{\partial y_\nu} \left( \sum_\sigma (y - x)_\sigma Z_\sigma(x) \right)
\]  

(573)
It is pure gauge and the expression would vanish by the Ward identity were it not for the factor \( \theta(d_2(x,y)) \). With this factor we get two terms when we integrate by parts in \( y \) (see (80)). They are

\[
\sum_{\mu
u}
\int dxdy \chi_z([x, x + \eta e_\mu])Z_\mu(x)\theta(d_2(x, y - \eta e_\mu)) \left( \frac{\partial}{\partial y_\nu} \Pi_{k,\mu
u}(x, y) \right) \left( \sum_\sigma (y-x)_\sigma Z_\sigma(x) \right)
\]

\[
+ \sum_{\mu
u}
\int dxdy \chi_z([x, x + \eta e_\mu])Z_\mu(x) \left( \frac{\partial}{\partial y_\nu} \theta(d_2(x, y)) \right) \Pi_{k,\mu
u}(x, y) \left( \sum_\sigma (y-x)_\sigma Z_\sigma(x) \right)
\]

(574)

Since \( \Pi_{k,\mu
u}(x, y) = \Pi_{k,\nu\mu}(y, x) \) the first term does indeed vanish by the Ward identity. For the second term we have

\[
\left( \frac{\partial}{\partial y_\nu} \theta(d_2(x, y)) \right) = \theta'(d_2(x, y)) \left( \frac{\partial}{\partial y_\nu} d_2(x, y) \right)
\]

(575)

and this is bounded by \( O(1) \). Furthermore \( \theta'(d_2(x, y)) \) keeps \( x, y \) separate. There is no ultraviolet divergence, only \( \Pi^{(1)}_{k,\mu\nu} \) contributes, and the term is again estimated by \( C\epsilon^2_k\|Z\|_\infty^2 \).

The second term arising from the expansion (574) is

\[
\sum_{\mu\nu\sigma}
\int dxdy \chi_z([x, x + \eta e_\mu])Z_\mu(x)\theta(d_2(x, y))\Pi_{k,\mu\nu}(x, y)(y-x)_\sigma \partial_\sigma Z_\nu(x)
\]

(576)

This almost changes sign under reflection through \( x \), i.e. under the change of variables \( (y-x) \rightarrow -(y-x) \) or \( y \rightarrow -y + 2x \), in which case it would be zero. The problem is that in spite of all our work to get to this point \( \Pi_{k,\mu\nu}^*(x, y) \) is not quite invariant. We have

\[
\Pi_{k,\mu\nu}^*(x, y) = \Pi_{k,\nu\mu}(0, y - x)
\]

\[
\rightarrow \Pi_{k,\mu\nu}(0, -(y-x)) = \Pi_{k,\mu\nu}(-x, -y) = \Pi_{k,\nu\mu}(x - \eta e_\mu, y - \eta e_\nu)
\]

(577)

the last by (576). Making the change of variables the expression (576) is the same as

\[
- \sum_{\mu\nu\sigma}
\int dxdy \chi_z([x, x + \eta e_\mu])Z_\mu(x)\theta(d_2(x, y))\Pi_{k,\mu\nu}(x - \eta e_\mu, y - \eta e_\nu)(y-x)_\sigma \partial_\sigma Z_\nu(x)
\]

(578)

and hence it is also the average of the two which is

\[
\frac{1}{2} \sum_{\mu\nu\sigma}
\int dxdy \chi_z([x, x + \eta e_\mu])Z_\mu(x)\theta(d_2(x, y)) \left( \Pi_{k,\mu\nu}(x, y) - \Pi_{k,\mu\nu}(x - \eta e_\mu, y - \eta e_\nu) \right)(y-x)_\sigma \partial_\sigma Z_\nu(x)
\]

(579)

In the second term make the replacement \( \Pi_{k,\mu\nu}(x - \eta e_\mu, y - \eta e_\nu) = \Pi_{k,\mu\nu}(x, y + \eta e_\mu - \eta e_\nu) \), followed by the change of variables \( y \rightarrow y - \eta e_\mu + \eta e_\nu \), which yields

\[
\frac{1}{2} \sum_{\mu\nu\sigma}
\int dxdy \chi_z([x, x + \eta e_\mu])Z_\mu(x)\theta(d_2(x, y))\Pi_{k,\mu\nu}(x, y)(y-x)_\sigma \partial_\sigma Z_\nu(x)
\]

\[
- \frac{1}{2} \sum_{\mu\nu\sigma}
\int dxdy \chi_z([x, x + \eta e_\mu])Z_\mu(x)\theta(d_2(x, y - \eta e_\mu + \eta e_\nu))\Pi_{k,\mu\nu}(x, y)(y-x - \eta e_\mu + \eta e_\nu)_\sigma \partial_\sigma Z_\nu(x)
\]

(580)

In the second term replace \( \theta(d_2(x, y - \eta e_\mu + \eta e_\nu)) \) by \( \theta(d_2(x, y)) \). The difference is only non-zero if \( x, y \) are well separated. There is no ultraviolet divergence, only \( \Pi^{(1)}_{k,\mu\nu} \) contributes, and this term is
bounded by $Ce_k^2 \|Z\|_\infty \|\partial Z\|_\infty$. Since $(y - x)_\sigma - (y - x - \eta e_\mu + \eta e_\nu)_\sigma = \eta \delta_{\mu\sigma} - \eta \delta_{\nu\sigma}$ we are left with

$$
\frac{1}{2} \sum_{\mu\nu} \int dx\ dy\ \chi_z([x, x + \eta e_\mu]) Z_\mu(x)\theta(d_2(x, y)) \Pi^{(e)}_{k,\mu\nu}(x, y) \eta \left(\partial_\mu Z_\nu(x) - \partial_\nu Z_\nu(x)\right)
$$

(581)

Now write

$$
\left(\partial_\mu Z_\nu(x) - \partial_\nu Z_\nu(x)\right) = \frac{\partial}{\partial y_\nu} \left( \sum_{\sigma} (y - x)_\sigma (\partial_\mu Z_\sigma(x) - \partial_\sigma Z_\sigma(x)) \right)
$$

(582)

As before we integrate by parts to transfer the $y$-derivative to the other factors. On the $\Pi^{(e)}_{k,\mu\nu}(x, y)$ we again get zero by the Ward identity. On the factor $\theta(d_2(x, y))$ it again forces $x, y$ to be separate and so destroys the UV divergence, only $\Pi^{(1)}_{k,\mu\nu}$ contributes and this term can be estimated by $Ce_k^2 \|Z\|_\infty \|\partial Z\|_\infty$.

The final term arising from the expansion (577) is

$$
\sum_{\mu\nu} \int dx\ dy\ \chi_z([x, x + \eta e_\mu]) Z_\mu(x)\theta(d_2(x, y)) \Pi^{(s)}_{k,\mu\nu}(x, y) \Delta_\nu(y, x)
$$

(583)

We have the representation

$$
\Delta_\nu(y, x) = \sum_{\sigma} \int_\Gamma (\partial_\sigma Z_\nu(z) - \partial_\sigma Z_\nu(x)) dz_\sigma
$$

(584)

where $\Gamma$ is any one of the standard paths from $x$ to $y$. For $d(y, x) \leq 1$ we use

$$
|\partial_\sigma Z_\nu(z) - \partial_\sigma Z_\nu(x)| \leq d(z, x) \|\partial_\sigma Z\|_\infty
$$

(585)

which yields

$$
|\Delta_\nu(y, x)| \leq O(1) d(y, x)^{1+\alpha} \|\partial Z\|_\infty
$$

(586)

If $d(y, x) \geq 1$ we have $|\Delta_\nu(y, x)| \leq O(1) d(y, x)^\alpha \|\partial Z\|_\infty$. In either case there is no UV divergence, only $\Pi^{(1)}_{k,\mu\nu}$ contributes since $\Delta_\nu(x, x) = 0$, and the term is bounded by $Ce_k^2 \|Z\|_\infty (\|\partial Z\|_\infty + \|\partial Z\|_\infty)$. This completes the analysis of $<Z, \Pi_k Z>$.

We have $\Pi_k = \Pi^{(e)}_k + \Pi^{(s)}_k + (\Pi^{(e)}_k - \Pi^{(s)}_k) + \Pi^{(1)}_k$. Combining the polymer decompositions and estimates on each of these completes the proof of lemma 33.

**Proof.** (lemma 23). From lemma 32 and lemma 33 we have

$$
E^{(4)}_k(X, A, Z) \equiv E^e_k(X, A, Z) + E^s_k(X, A, Z)
$$

(587)

and this is bounded by $O(1)e_k^e e^{-\kappa d_M(X)}$ on the domain $A \in \frac{1}{2}R_k$, $Z \in \frac{1}{2}R'_k$. We want to show this is bounded by $O(1)e_k^{1-6\nu} e^{-\kappa d_M(X)}$ on the smaller domain $A \in \frac{1}{2}R_k$ and $|Z|, |\partial Z|, \|\delta_{\nu}\partial Z\|_\infty \leq e_k^{2\nu}$. To get the better bound we use a Cauchy inequality. Since $E^e_k(X, A, Z)$ vanishes at $Z = 0$ we have

$$
E^e_k(X, A, Z) = \frac{1}{2\pi i} \int_{|t| = r} \frac{dt}{t(t - 1)} E^e_k(X, A, tZ)
$$

(588)

If we take $r = \frac{1}{2} e^{-1+7\gamma}$ then $|tZ| \leq e_k^{-1+5\gamma}$ with the same bound for the derivatives and so $tZ \in \frac{1}{2}R'_k$. Then we can use the above bound to obtain

$$
|E^e_k(X, A, Z)| \leq O(1)e_k^{1-7\gamma} e_k^{-\kappa d_M(X)} = O(1)e_k^{1-6\nu} e^{-\kappa d_M(X)}
$$

(589)

This completes the proof of lemma 23 and theorem 1.
8 The flow

We seek well-behaved solutions of the RG equations (283). Thus we study

\[ \begin{align*}
\varepsilon_{k+1} &= L^3 \varepsilon_k + L_1 E_k + \varepsilon_k^k(\mu_k, E_k) \\
\mu_{k+1} &= L^2 \mu_k + L_2 E_k + \mu_k^k(\mu_k, E_k) \\
E_{k+1} &= L_3 E_k + E_k^k(\mu_k, E_k)
\end{align*} \tag{590} \]

Our goal is to show that for any \( N \) we can choose the initial point so that the solution exists for \( k = 0, 1, \ldots, N \) and finishes at a values \( (\varepsilon_N, \mu_N) = (\varepsilon_N^N, \mu_N^N) \) independent of \( N \). (Note that at \( k = N \) we are on the lattice \( T_0^N \) with the dressed fields back on the original lattice \( T_{-N}^0 \)). This procedure is nonperturbative renormalization - the initial values for \( (\varepsilon_0, \mu_0) = (\varepsilon_0^N, \mu_0^N) \) will depend \( N \) and in fact be divergent in \( N \). The problem is now formally exactly the same as the pure scalar problem in [24].

The functions \( \varepsilon_k^*, \mu_k^*, E_k^* \) are different, they now contain all radiative corrections, but the analysis is essentially the same as we explain.

Arbitrarily fixing the final values at zero, and starting with \( E_0 = 0 \) as dictated by the model, we look for solutions \( \varepsilon_k, \mu_k, E_k \) for \( k = 0, 1, 2, \ldots, N \) satisfying

\[ \begin{align*}
\varepsilon_N &= 0 \\
\mu_N &= 0 \\
E_0 &= 0
\end{align*} \tag{591} \]

At this point we temporarily drop the equation for the energy density \( \varepsilon_k \) and just study

\[ \begin{align*}
\mu_{k+1} &= L^2 \mu_k + L_2 E_k + \mu_k^k(\mu_k, E_k) \\
E_{k+1} &= L_3 E_k + E_k^k(\mu_k, E_k)
\end{align*} \tag{592} \]

Let \( \xi_k = (\mu_k, E_k) \) be an element of the real Banach space \( \mathbb{R} \times \text{Re} \mathcal{K}_k \) where \( \text{Re} \mathcal{K}_k \) is the real subspace of \( \mathcal{K}_k \) defined in section 5.4. Consider sequences

\[ \xi = (\xi_0, \ldots, \xi_N) \tag{593} \]

Pick a fixed \( \beta \) satisfying \( 0 < \beta < \frac{1}{12} - 11\epsilon \) and let \( B \) be the real Banach space of all such sequences with norm

\[ \|\xi\| = \sup_{0 \leq k \leq N} \{ \lambda_k^{\frac{1}{2} - \beta} |\mu_k|, \lambda_k^{-\beta} \|E_k\|_{k,k} \} \tag{594} \]

Let \( B_0 \) be the subset of all sequences satisfying the boundary conditions. Thus

\[ B_0 = \{ \xi \in B : \mu_N = 0, E_0 = 0 \} \tag{595} \]

This is a complete metric space with distance \( \|\xi - \xi\| \). Finally let

\[ B_1 = B_0 \cap \{ \xi \in B : \|\xi\| < 1 \} \tag{596} \]

Next define an operator \( \xi' = T\xi \) by

\[ \begin{align*}
\mu'_k &= L^{-2}(\mu_{k+1} - L_2 E_k - \mu_k^k) \\
E'_k &= L_3 E_{k-1} + E_k^{k-1}
\end{align*} \tag{597} \]

Then \( \xi \) is a solution of (590) iff it is a fixed point for \( T \) on \( B_0 \). We look for such fixed points in \( B_1 \).

Lemma 40. Let \( \lambda \) be sufficiently small. Then for all \( N \)

1. The transformation \( T \) maps the set \( B_1 \) to itself.

2. There is a unique fixed point \( T\xi = \xi \) in this set.
**Proof.** (1.) We use the bounds of theorem [1] for $L_2, L_3$ (replacing $O(1)L^{-r}$ by $1$) and for $\mu_k^*, E_k^*$. To show the the map sends $B_1$ to itself we estimate for $L$ sufficiently large and $\lambda_k$ sufficiently small
\[
\lambda_k^{-\frac{1}{2} - \beta} |\mu_k'| \leq \lambda_k^{-\frac{1}{2} - \beta} L^{-2} \left( |\mu_{k+1}| + \lambda_{k+1}^{1/2 + 2\epsilon} \|E_k\|_{k,\kappa} + O(1) \lambda_k^{\frac{1}{2} - 11\epsilon} \right)
\]
\[
\leq L^{-\frac{1}{2} - \beta} \left[ \lambda_k^{-\frac{1}{2} - \beta} |\mu_{k+1}| \right] + L^{-2} \lambda_k^{2\epsilon} \left[ \lambda_k^{-\beta} \|E_k\|_{k,\kappa} \right] + O(1) \lambda_k^{\frac{1}{2} - \beta - 11\epsilon} \tag{598}
\]
\[
\leq \frac{1}{2} (\|\xi\| + 1) < 1
\]
We also have
\[
\lambda_k^{-\beta} \|E_k\|_{k,\kappa} \leq \lambda_k^{-\beta} \left( \|E_{k-1}\|_{k-1,\kappa} + O(1) \lambda_k^{\frac{1}{2} - 11\epsilon} \right)
\]
\[
\leq L^{-\beta} \left[ \lambda_k^{-\beta} \|E_{k-1}\|_{k-1,\kappa} \right] + O(1) L^{-\beta} \lambda_k^{\frac{1}{2} - \beta - 11\epsilon} \tag{599}
\]
\[
\leq \frac{1}{2} (\|\xi\| + 1) < 1
\]
Hence $\|T(\xi)\| < 1$ as required.

(2.) It suffices to show that the mapping is a contraction. We show that under our assumptions

\[
\|T(\xi_1) - T(\xi_2)\| \leq \frac{1}{2} \|\xi_1 - \xi_2\| \tag{600}
\]

First consider the $\mu$ terms. We have as above
\[
\lambda_k^{-\frac{1}{2} - \beta} |\mu'_{1,k} - \mu_{2,k}| \leq L^{-\frac{1}{2} - \beta} \left[ \lambda_k^{-\frac{1}{2} - \beta} |\mu_{1,k+1} - \mu_{2,k+1}| \right]
\]
\[
+ L^{-2} \lambda_k^{2\epsilon} \left[ \lambda_k^{-\beta} \|E_{1,k} - E_{2,k}\|_{k,\kappa} \right] + L^{-2} \lambda_k^{2\epsilon} \left[ \mu_k^*(\mu_{1,k}, E_{1,k}) - \mu_k^*(\mu_{2,k}, E_{2,k}) \right] \tag{601}
\]

The first two terms are bounded by a small constant times $\|\xi_1 - \xi_2\|$. For the last term we use the fact that $\mu_k^*(\mu_k, E_k)$ is actually an analytic function of $\mu_k, E_k$ on its domain $|\mu_k| \leq \lambda_k^{\frac{1}{2}}$ and $\|E_k\|_{k,\kappa} \leq 1$. We write
\[
\mu_k^*(\mu_{1,k}, E_{1,k}) - \mu_k^*(\mu_{2,k}, E_{2,k})
\]
\[
= \left( \mu_k^*(\mu_{1,k}, E_{1,k}) - \mu_k^*(\mu_{2,k}, E_{1,k}) \right) + \left( \mu_k^*(\mu_{2,k}, E_{1,k}) - \mu_k^*(\mu_{2,k}, E_{2,k}) \right) \tag{602}
\]

For the first term we write for $r > 1$
\[
\mu_k^*(\mu_{1,k}, E_{1,k}) - \mu_k^*(\mu_{2,k}, E_{1,k}) = \frac{1}{2\pi i} \int_{|t|=r} \frac{dt}{t(t-1)} \mu_k^* \left( \mu_{2,k} + t(\mu_{1,k} - \mu_{2,k}), E_{1,k} \right) \tag{603}
\]

We use the bound $|\mu'| \leq O(1) \lambda_k^{\frac{1}{2} - 11\epsilon}$ on its domain of analyticity. We take $r = 4 \lambda_k^{\frac{1}{2} + \beta} |\mu_{1,k} - \mu_{2,k}|^{-1}$. This keeps us in the domain of analyticity and is greater than one since $|\mu_{1,k} - \mu_{2,k}| \leq \lambda_k^{\frac{1}{2} + \beta} \|\xi_1 - \xi_2\| \leq 2 \lambda_k^{\frac{1}{2} + \beta}$. Hence this term is bounded by
\[
O(1) \left( \lambda_k^{-\frac{1}{2} - \beta} |\mu_{1,k} - \mu_{2,k}| \right) \lambda_k^{\frac{1}{2} - 11\epsilon} \leq O(1) \lambda_k^{\frac{1}{2} - \beta - 11\epsilon} |\mu_{1,k} - \mu_{2,k}| \tag{604}
\]

For the second term in (602) we write for $r > 1$
\[
\left( \mu_k^*\right)_{\mu_{2,k}, E_{1,k}} - \left( \mu_k^*\right)_{\mu_{2,k}, E_{2,k}} = \frac{1}{2\pi i} \int_{|t|=r} \frac{dt}{t(t-1)} \mu_k^* \left( \mu_{2,k}, E_{2,k} + t(E_{1,k} - E_{2,k}) \right) \tag{605}
\]

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Now we take \( r = 4\lambda_k^\beta \|E_{1,k} - E_{2,k}\|_{E_{k-1}}^{-1} \). This keeps us in the domain of analyticity and is greater than one since \( \|E_{1,k} - E_{2,k}\|_{E_{k-1}} \leq \lambda_k^\beta \|\xi_1 - \xi_2\| \leq 2\lambda_k^\beta \). Hence this term is bounded by

\[
O(1) \left( \lambda_k^\beta \|E_{1,k} - E_{2,k}\|_{E_{k-1}} \right) \lambda_k^{-11r} \leq O(1) \lambda_k^{-\beta-11r} \|E_{1,k} - E_{2,k}\|_{E_{k-1}} \quad (606)
\]

Now for the last term in (601) we have

\[
L^{-2}\lambda_k^{\frac{1}{2}-\beta} \left| \mu_k^* (\mu_{1,k}, E_{1,k}) - \mu_k^* (\mu_{2,k}, E_{2,k}) \right|
\leq O(1) \lambda_k^{-\frac{1}{2}-11r} \left[ \lambda_k^{-\frac{3}{2}-\beta} |\mu_{1,k} - \mu_{2,k}| \right] + O(1) \lambda_k^{-\frac{1}{2}-11r} \left[ \lambda_k^{-\beta} \|E_{1,k} - E_{2,k}\|_{E_{k-1}} \right]
\leq O(1) \lambda_k^{-\frac{1}{2}-11r} \|\xi_1 - \xi_2\|
\quad (607)
\]

Altogether then

\[
\lambda_k^{-\frac{1}{2}-\beta} |\mu_{1,k} - \mu_{2,k}| \leq \frac{1}{2} \|\xi_1 - \xi_2\| \quad (608)
\]

Now consider the \( E \) terms. We have

\[
E_{1,k} - E_{2,k} = E_3 (E_{1,k-1} - E_{2,k-1})
+ E_{1,k-1}^* (\mu_{1,k-1}, E_{1,k-1}) - E_{2,k-1}^* (\mu_{2,k-1}, E_{2,k-1})
\quad (609)
\]

Then

\[
\lambda_k^{-\beta} \|E_{1,k}^* - E_{2,k}^*\|_{E_{k-1}} \leq L^{-\beta} \lambda_k^{-\beta} \left( \|E_{1,k-1} - E_{2,k-1}\|_{E_{k-1}} + \|E_{1,k-1}^* - E_{2,k-1}^*\|_{E_{k-1}} \right)
\quad (610)
\]

The first term is bounded by \( L^{-\beta} \|\xi_1 - \xi_2\| \). For the second term we write

\[
\lambda_k^{-\beta} \|E_{1,k}^*|E_{2,k}^*\|_{E_{k-1}} \leq \lambda_k^{-\beta} \|E_{1,k-1}^* - E_{2,k-1}^*\|_{E_{k-1}} \quad (611)
\]

For the first term we write for \( r > 1 \)

\[
E_{k-1}^* (\mu_{1,k-1}, E_{1,k-1}) - E_{k-1}^* (\mu_{2,k-1}, E_{1,k-1})
= \frac{1}{2\pi i} \int_{|t|=r} \frac{dt}{t(t-1)} E_{k-1}^* \left( \mu_{2,k-1} + t(\mu_{1,k-1} - \mu_{2,k-1}), E_{1,k-1} \right)
\quad (612)
\]

We use the bound \( \|E_{k-1}^*\|_{E_{k-1}} \leq \lambda_k^{-11r} \) on its domain, and take \( r = 4\lambda_k^{\frac{3}{2}+\beta} \|\mu_{1,k-1} - \mu_{2,k-1}\|^{-1} \). Hence this term is bounded by

\[
O(1) \left( \lambda_k^{\frac{3}{2}+\beta} |\mu_{1,k-1} - \mu_{2,k-1}| \right) \lambda_k^{-11r} \leq O(1) \lambda_k^{\frac{3}{2}+\beta-11r} |\mu_{1,k-1} - \mu_{2,k-1}| \quad (613)
\]

For the second term we write for \( r > 1 \)

\[
E_{k-1}^* (\mu_{2,k-1}, E_{1,k-1}) - E_{k-1}^* (\mu_{2,k}, E_{2,k-1})
= \frac{1}{2\pi i} \int_{|t|=r} \frac{dt}{t(t-1)} E_{k-1}^* \left( \mu_{2,k} + t(\mu_{1,k-1} - \mu_{2,k-1}), E_{1,k-1} \right)
\quad (614)
\]

Again we use \( \|E_{k-1}^*\|_{E_{k-1}} \leq \lambda_k^{-11r} \) and take \( r = 4\lambda_k^\beta \|E_{1,k-1} - E_{2,k-1}\|_{E_{k-1}}^{-1} \). Hence this term is bounded by

\[
O(1) \left( \lambda_k^\beta \|E_{1,k-1} - E_{2,k-1}\|_{E_{k-1}} \right) \lambda_k^{-11r} \leq O(1) \lambda_k^{\frac{1}{2}+\beta-11r} \|E_{1,k-1} - E_{2,k-1}\|_{E_{k-1}} \quad (615)
\]
Combining these bounds gives

\[ L^{-\beta} \lambda_{k-1}^{-\beta} \parallel E_{\ast}^{k-1} - E_{\ast}^{k-1} \parallel_{k-1,\kappa} \leq O(1) \lambda_{k-1}^{\frac{1}{2} - \beta} \left[ \lambda_{k-1}^{-\beta} \parallel \mu_{1,k-1} - \mu_{2,k-1} \parallel \right] + O(1) \lambda_{k-1}^{\frac{1}{2} - \beta} \left[ \lambda_{k-1}^{-\beta} \parallel E_{1,k-1} - E_{2,k-1} \parallel_{k-1,\kappa} \right] \]

(616)

Altogether then for \( L \) sufficiently large and \( \lambda \) sufficiently small

\[ \lambda_{k}^{-\beta} \parallel E_{1,k}' - E_{2,k}' \parallel_{k,\kappa} \leq \frac{1}{2} \parallel \xi_{1} - \xi_{2} \parallel \]

(617)

Finally combining (608) and (617) yields the result

\[ \parallel \xi_{1}' - \xi_{2}' \parallel \leq \frac{1}{2} \parallel \xi_{1} - \xi_{2} \parallel \]

Now we can state:

**Theorem 2.** Let \( \lambda \) be sufficiently small. Then for each \( N \) there is a unique sequence \( \epsilon_{k}, \mu_{k}, E_{k} \) for \( k = 0, 1, 2, \ldots, N \) satisfying of the dynamical equation (590), the boundary conditions (591), and

\[ |\mu_{k}| \leq \lambda_{k}^{\frac{1}{2} + \beta} \quad ||E_{k}||_{k,\kappa} \leq \lambda_{k}^{\beta} \]

(618)

Furthermore

\[ |\epsilon_{k}| \leq O(1) \lambda_{k}^{\beta} \]

(619)

**Proof.** This solution is the fixed point from the previous lemma and the bounds (618) are a consequence. The bound on the energy density follows from the others, see [24].

**Remarks.** Much remains to be done on this model. The large field region needs to be analyzed along the lines of [25], [26]. Then one could prove an ultraviolet stability bound (proved in [3] for a massive gauge field). Next one would want prove bounds on the correlation functions uniform in the lattice spacing, and then show they have a limit as the lattice spacing goes to zero.

There is also the question of the infinite volume limit. In this connection we remark that although our final mass parameter \( \mu_{N} \) was tuned to zero we could equally well have tuned it to any sufficiently small value. If this analysis could be extended to allow \( \mu_{N} = -1 \) we would have the abelian Higgs model. Then one could proceed along the lines suggested in [17], [18] demonstrating mass generation for the gauge field, exponentially decaying correlations, and a robust infinite volume limit.

**A random walk expansion for \( C^{-1} \)**

The unit lattice operator \( C \) defined in section 4.3 has the form \( C \hat{Z} = (\hat{Z}, S \hat{Z}) \) and so

\[ \parallel C \hat{Z} \parallel^{2} = \parallel \hat{Z} \parallel^{2} + \parallel S \hat{Z} \parallel^{2} \]

(620)

and hence

\[ C^{T} C = I + S^{T} S \]

(621)

which implies

\[ C^{-1} = (I + S^{T} S)^{-1} C^{T} \]

(622)

We will show that \( (I + S^{T} S)^{-1} \) has a random walk expansion. Since \( C^{T} \) is local this gives an expansion for \( C^{-1} \).
Lemma 41. \((I + S^T S)^{-1}\) has a random walk expansion based on blocks of size \(M\), convergent for \(M\) sufficiently large. For \(y, y'\) in the \(L\)-lattice

\[
|1_{B(y)}(I + S^T S)^{-1}1_{B(y')}f| \leq Ce^{-\gamma d(y,y')}\|f\|_\infty
\]

(623)

**Proof.** We follow the proof of lemma 4. Let \(A = I + S^T S\) and let \(A_{\square_z}\) be the restriction to the \(3M\)-cubes \(\square_z\) centered on \(z\) in the \(L\)-lattice. The quadratic form \(A_{\square_z}\) is bounded above and below and has an exponentially decaying kernel (actually a finite range kernel). By a lemma of Balaban the same is true of \(\tilde{G}_{\square_{\tilde{z}}} \equiv A_{\square_{\tilde{z}}}^{-1}\). Since \(A\) is naturally localized in terms of the \(L\)-cubes \(B(y)\) we state it as

\[
|1_{B(y)} \tilde{G}_{\square_{\tilde{z}}} 1_{B(y')}f| \leq Ce^{-\gamma d(y,y')}\|f\|_\infty
\]

(624)

To create the expansion take the partition of unity \(1 = \sum_z h_z^2\) as before (but now defined on bonds) and define

\[
\mathcal{G}^* = \sum_z h_z \tilde{G}_{\square_{\tilde{z}}} h_z
\]

(625)

Then

\[
AG^* = \sum_z h_z A_{\square_z} h_z + \sum_z [A, h_z]\tilde{G}_{\square_{\tilde{z}}} h_z
\]

(626)

But on the support of \(h_z\) we have \(A\tilde{G}_{\square} = A_{\square} \tilde{G}_{\square} = 1\) and so

\[
AG^* = I + \sum_z [A, h_z]\tilde{G}_{\square} h_z = I - \sum_z K_z = I - K
\]

(627)

Hence

\[(I + S^T S)^{-1} = G^*(I - K)^{-1}
\]

(628)

Expanding \((I - K)^{-1}\) yields the random walk expansion.

For convergence we must show \([A, h_z] = [S^T S, h_z] = \mathcal{O}(M^{-1})\) and since \([S^T S, h_z] = [S^T, h_z]S + S^T[S, h_z]\) it suffices to show \([S, h_z] = \mathcal{O}(M^{-1})\). This follows since \(S\) is a strictly local operator. For the details we need an explicit representation of \(S\). We write for \(f\) on the unit lattice and \(y' = y + L e_\mu\)

\[
(Qf)(y, y') = \sum_{x \in B(y)} L^{-4} \sum_{b \in \Gamma(x, x + L e_\mu)} f(b) = \sum_b f(b) L^{-4} n_\mu(b)
\]

(629)

Here \(n_\mu(b)\) is the number of elements in the set \(\{ x \in B(y) : \Gamma(x, x + L e_\mu) \ni b \}\). For example if \(b \in B(y, y')\) then \(n_\mu(b) = L\) and if \(b\) is not in the direction \(e_\mu\) then \(n_\mu(b) = 0\). In general \(0 \leq n_\mu(b) \leq L\). Breaking the sum up by the different categories of bonds we have

\[
(Qf)(y, y') = L^{-3} f(b(y, y')) + L^{-3} \sum_{b \in B(y, y') \setminus B(y)} f(b) n_\mu(b)
\]

(630)

\[
+ \sum_{b \in B(y)} f(b) \frac{n_\mu(b)}{L^4} + \sum_{b \in B(y')} f(b) \frac{n_\mu(b)}{L^4}
\]

Thus the equation \(Qf = 0\) is solved by \(f(b(y, y')) = (Sf)(b(y, y'))\) where

\[
(Sf)(b(y, y')) = -\sum_{b \in B(y, y') \setminus B(y)} f(b) - \sum_{b \in B(y')} f(b) \frac{n_\mu(b)}{L} - \sum_{b \in B(y')} f(b) \frac{n_\mu(b)}{L}
\]

(631)

\[\text{The operator is defined on functions } \tilde{f} = (\tilde{Z}_1, \tilde{Z}_2) \text{ where } \tilde{Z}_1 \text{ is defined on points (actually bonds) and } \tilde{Z}_2 \text{ is defined on } \ker \tau. \text{ Once we pick a basis for } \ker \tau \text{ (localized in each } B(y) \text{) we can regard } \tilde{Z}_1 \text{ as defined on points as well, namely the coefficients in the expansion in the basis. Hence the notion of a kernel for the operator makes sense.} \]
Lemma 43. Let’s look at the contribution of the second term here to \( [S, h_z] f \)(\( b(y, y') \)). It is

\[
\sum_{b \in B(y)} \left( h_z(b(y, y')) - h_z(b) \right) \frac{n_x(b)}{L} f(b)
\]

But

\[
\left| h_z(b(y, y')) - h_z(b) \right| \leq L \| \partial h_z \|_\infty \leq CM^{-1}
\]

so the term is bounded by \( CM^{-1} \| f \|_\infty \). The other terms have the same bound and this gives the convergence of the expansion. The decay factor is extracted using the local estimate \((624)\) as before.

B \  \textbf{a covariant derivative}

We show that the forward/backward covariant derivative \( \nabla_A f = \frac{1}{2}(\partial_A f - \partial^A f) \) transforms like a vector field under lattice symmetries. For notational convenience we work on a unit lattice and absorb the coupling constant into the gauge potential.

Lemma 42. Let \( r \) be a unit lattice symmetry fixing the origin with matrix elements \( r_{\mu \nu} \), and let \( f_r(x) = f(r^{-1}x) \) and \( A_r(x, x') = A(r^{-1}x, r^{-1}x') \). Then

\[
(\nabla_A f_r)_\mu(x) = \sum_\nu r_{\mu \nu} (\nabla_A f)_\nu(r^{-1}x)
\]

Proof. Start with

\[
(\nabla_A f)_\mu(x) = e^{qA(x, x+e_\mu)} f(x + e_\mu) - e^{qA(x, x-\epsilon_\mu)} f(x - e_\mu)
\]

Given \( \mu \) suppose \( r^{-1}e_\mu = e_\rho \) for some \( \rho \). Then \( r_{\mu \nu} = (r^{-1})_{\nu \mu} = \delta_{\rho \nu} \) for all \( \nu \) and

\[
(\nabla_A f_r)_\mu(x) = e^{qA(r^{-1}x, r^{-1}x+e_\rho)} f(r^{-1}x + e_\rho) - e^{qA(r^{-1}x, r^{-1}x-\epsilon_\rho)} f(r^{-1}x - e_\rho)
\]

\[
= (\nabla_A f)_\rho(r^{-1}x)
\]

\[
= \sum_\mu r_{\mu \nu} (\nabla_A f)_\nu(r^{-1}x)
\]

The other possibility is that \( r^{-1}e_\mu = -e_\rho \). Then \( r_{\mu \nu} = -\delta_{\rho \nu} \)

\[
(\nabla_A f_r)_\mu(x) = e^{qA(r^{-1}x, r^{-1}x-\epsilon_\rho)} f(r^{-1}x - e_\rho) - e^{qA(r^{-1}x, r^{-1}x+e_\rho)} f(r^{-1}x + e_\rho)
\]

\[
= - (\nabla_A f)_\rho(r^{-1}x)
\]

\[
= \sum_\mu r_{\mu \nu} (\nabla_A f)_\nu(r^{-1}x)
\]

C \  \textbf{an estimate on} \( Q_k(A) \Delta_A \)

First prove a special case of the divergence theorem on the lattice \( \mathbb{T}_{N-k}^{-k} \) with spacing \( \eta = L^{-k} \). Let \( \Delta_y \) be the unit cube centered on the unit lattice point \( y \). For a vector field \( f_\mu \) let \( \int_{\partial \Delta_y} n \cdot f \) be the inward surface integral

\[
\int_{\Delta_y} \partial^T f = \int_{\partial \Delta_y} n \cdot f
\]
Proof. Take \( y = 0 \) for simplicity. We compute

\[
\sum_{\mu} \int_{\Delta_0} (\partial^T \mu \mathbf{f}_\mu)(x)dx = \sum_{\mu} \sum_{|x|<\frac{\mathbf{8}}{\mathbf{3}}} \eta^{-1}(f_\mu(x) - \eta \epsilon_\mu - f_\mu(x))
= \sum_{\mu} \sum_{|x|<\frac{\mathbf{8}}{\mathbf{3}}, \text{\&} \neq \mu} \eta^{-1}(\epsilon_\mu(x)\mid_{x=\frac{\mathbf{8}}{\mathbf{3}} - \frac{\mathbf{8}}{\mathbf{3}} \eta} - \mid f_\mu(x)\mid_{x=\frac{\mathbf{8}}{\mathbf{3}} - \frac{\mathbf{8}}{\mathbf{3}} \eta})
\]  

(639)

The last expression is identified as \( \int_{\partial \Delta_0} n \cdot f \).

Lemma 44. Let \( |\text{Im} \ A|, \|dA\| < \epsilon_k^{-1} \).

1. For a vector field \( f_\mu \) on \( T_{N-k}^{-1} \)

\[
\|Q_k(A)(\partial^T \mu \cdot f)\|_\infty \leq C\|f\|_\infty
\]  

(640)

2. For a scalar \( \phi \) on \( T_{N-k}^{-1} \)

\[
\|Q_k(A)\Delta A \phi\|_\infty \leq C\|\partial A \phi\|_\infty
\]  

(641)

Proof. The second follows from the first with \( f_\mu \equiv \partial_{A, \mu} \phi. \) For the first let \( U(A, y, x) = e^{\eta \epsilon_\mu \eta(\tau_k A)(y, x)} \) with \( (\tau_k A)(y, x) \) defined in (431). Then we have for \( y \in T_{N-k}^{-1} \)

\[
(Q_k(A)(\partial^T \mu \cdot f)(y) = \sum_{\mu} \int_{\Delta_y} dx \ U(A, y, x) \left( e^{-\eta \epsilon_\mu \eta A, \mu(x) - \eta \epsilon_\mu} - f_\mu(x) \right) \eta^{-1}
= \sum_{\mu} \int_{\Delta_y} dx \left( U(A, y, x) e^{-\eta \epsilon_\mu \eta A, \mu(x) - \eta \epsilon_\mu} - U(A, y, x - \eta \epsilon_\mu) \right) f_\mu(x - \eta \epsilon_\mu) \eta^{-1}
+ \sum_{\mu} \int_{\Delta_y} dx \left( U(A, y, x - \eta \epsilon_\mu) f_\mu(x - \eta \epsilon_\mu) - U(A, y, x) f_\mu(x) \right) \eta^{-1}
\]  

(642)

For the second term here we use the divergence theorem of lemma 33 to write it as

\[
\sum_{\mu} \int_{\Delta_y} dx \ (\partial / \partial x_\mu)^T (U(A, y, x) f_\mu(x)) = \sum_{\mu} \int_{\Delta_y} dx \ U(A, y, x) (\eta \mathbf{f}_\mu)(x)
\]  

(643)

Bounding \( U(A, y, x) \) using (377), this term is bounded by \( C(1)\|f\|_\infty. \) For the first term in (642) it suffices to show

\[
\int_{\Delta_y} dx \ |(\tau_k A)(y, x) - A(x - \eta \epsilon_\mu, x) - (\tau_k A)(y, x - \eta \epsilon_\mu)| \leq C\|dA\|_\infty
\]  

(644)

Then the term is bounded by \( C\|dA\|_\infty\|f\|_\infty \leq C\|f\|_\infty \) as required.

To prove (644) recall that \( (\tau_k A)(y, x) = \sum_{j=0}^{k-1} (\tau A)(y_{j+1}, y_j) \) is defined by the unique sequence \( x = y_0, y_1, \ldots, y_k = y \) with \( y_j \in T_{N-k}^{-1+j} \) and \( x \in B_j(y_j) \). Also \( (\tau_k A)(y, x - \eta \epsilon_\mu) \) is defined by a similar sequence \( x - \eta \epsilon_\mu = y_0', y_1', \ldots, y_k' = y \). Suppose \( x, x - \eta \epsilon_\mu \) are in the same \( L \eta \) cube \( B(y_1) = B(y_1') \). Then \( y_i = y_i' \) for \( i = 1, 2, \ldots, k \). Hence

\[
\eta \epsilon_\mu \eta(\tau_k A)(y, x) - \eta \epsilon_\mu \eta(\tau_k A)(y, x - \eta \epsilon_\mu)
= \eta \epsilon_\mu \eta(\tau A)(y, x) - \eta \epsilon_\mu \eta(\tau A)(y, x - \eta \epsilon_\mu)
= \frac{1}{d} \sum_{\pi} \eta \epsilon_\mu \eta(\Gamma^\pi(y_1, x)) - \eta \epsilon_\mu \eta(\Gamma^\pi(y_1, x - \eta \epsilon_\mu))
\]  

(645)

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But for each $\pi$ the indicated path is a closed and hence is the boundary of a surface $\Sigma^\pi$. By the lattice Stokes theorem with unsigned sums we have

$$\eta A \left( \Gamma^\pi(y_1, x) + [x, x - \eta e_\mu] - \Gamma^\pi(y_1, x - \eta e_\mu) \right) = \eta^2 dA(\Sigma^\pi) \tag{646}$$

The surface $\Sigma^\pi$ is made up of $\eta$-plaquettes in an $L\eta$-cube. Hence the number of plaquettes in $\Sigma^\pi$ is bounded by a constant and so $|\eta^2 dA(\Sigma^\pi)| \leq C \eta^2 \|dA\|_\infty$. Thus the integrand in (644) for points with $x, x - \eta e_\mu$ in the same $L\eta$ cube is bounded by $C \eta^2 \|dA\|_\infty$ which is better than we need.

This would take care of most points in (644), but not all, and not the most important. More generally let $X_j \subset \Delta_y$ be the set of points $x$ with the property that $x, x - \eta e_\mu$ are in the same $L^{i+1}\eta$ cube but not in any smaller cube. Equivalently $x, x - \eta e_\mu$ crosses a $L^i\eta$ face but no larger face. Then $\cup_{j=0}^{k-1} X_j = \Delta_y$ and in (644) we write

$$\int_{\Delta_y} dx \cdots = \sum_{x \in \Delta_y} \eta^3 \cdots = \sum_{j=0}^{k-1} \sum_{x \in X_j} \eta^3 \cdots \tag{647}$$

Note that the number of points in $X_j$ is bounded by the number of points $x$ such that $x, x - \eta e_\mu$ crosses a $L^i\eta$ face and so

$$|X_j| \leq L^{2k-j} L^{k-j} = L^{3k-j} \tag{648}$$

Suppose $x \in X_j$. Then $y_i = y_i'$ for $i = j+1, \ldots, k$ and so

$$\eta (\tau_k A)(y) - \eta A(x - \eta e_\mu, x) - \eta(\tau_k A)(y, x - \eta e_\mu)$$

$$= \sum_{i=0}^{j} \eta \partial \left( \sum_{i=0}^{j} \Gamma^\pi(i, y_i) + [x, x - \eta e_\mu] - \sum_{i=0}^{j} \Gamma^\pi(y_i', y_i') \right) \tag{649}$$

The last step follows since the added lines cancel out, except for $i = 0$ when $[y_0, y_0'] = [x, x - \eta e_\mu]$ and $i = j$ when $[y_j', y_j+1] = \emptyset$. For each $\pi, i$ the indicated path is closed and so the boundary of a surface $\Sigma^\pi$. Hence the last expression is the same as

$$\frac{1}{d} \sum_{\pi} \sum_{i=0}^{j} \eta^2 dA(\Sigma^\pi) \tag{650}$$

For each $i$ the path is made up of $L^i\eta$-segments in an $L^{i+1}\eta$-cube, so the number of $L^i\eta$-squares is bounded by a constant $C$ and so the number of $\eta$-plaquettes is bounded by $CL^{2i}$. Therefore $|\eta^2 dA(\Sigma^\pi)| \leq C \eta^2 L^{2i} \leq CL^{k+2i}$. Hence the expression is bounded by

$$\frac{1}{d} \sum_{\pi} \sum_{i=0}^{j} |\eta^2 dA(\Sigma^\pi)| \leq \sum_{i=0}^{j} CL^{k+2i} \eta \|dA\|_\infty \leq CL^{k+2j} \eta \|dA\|_\infty \tag{651}$$

Therefore, referring to (644), (648), the integral (644) is estimated by

$$C \sum_{j=0}^{k-1} \sum_{x \in X_j} \eta^3 L^{k+2j} \eta \|dA\|_\infty \leq C \sum_{j=0}^{k-1} L^{k+j} \eta \|dA\|_\infty \leq C \eta \|dA\|_\infty \tag{652}$$

which is the result we want.
D integrals

In $\mathbb{T}_{N-k}$ or $L^{-k}\mathbb{Z}^3$ we consider integrals of the form

$$\int f(x)dx = \sum_x \eta^3 f(x) \quad \eta = L^{-k} \quad (653)$$

Recall that $d'(x, y) = d(x, y) = \sup_{\mu} |x_{\mu} - y_{\mu}|$ for $x \neq y$ and $d'(x, x) = L^k = \eta^{-1}$.

**Lemma 45.** For $\alpha < 3$

$$\int_{d'(x, y) \leq 1} d'(x, y)^{-\alpha}dx \leq O(1) \quad (654)$$

**Proof.** It suffices to consider $y = 0$. Isolate the $x = 0$ term. Then with $r \in L^{-k}\mathbb{Z}$

$$\int_{d'(x, 0) \leq 1} d'(x, 0)^{-\alpha}dx \leq \eta^{3-\alpha} + \sum_{|x| \leq \frac{1}{r}, x \neq 0} \eta^3 |x|^{-\alpha}$$

$$\leq 1 + \sum_{0 < r \leq \frac{1}{\eta}} \eta r^{-\alpha} \sum_{x: |x| = r} \eta^2$$

$$\leq 1 + O(1) \sum_{0 < r \leq \frac{1}{\eta}} \eta^{2-\alpha} \leq O(1) \quad (655)$$

**Lemma 46.**

$$\int_{d'(x, z) \leq 1, d'(y, z) \leq 1} d'(x, z)^{-\alpha}d'(y, z)^{-\beta}dz \leq O(1) \quad \alpha + \beta < 3 \quad (656)$$

$$\int_{d'(x, z) \leq 1, d'(y, z) \leq 1} d'(x, z)^{-1}d'(y, z)^{-2}dz \leq O(1)d'(x, y)^{-2}\quad (657)$$

$$\int_{d'(x, z) \leq 1, d'(y, z) \leq 1} d'(x, z)^{-2}d'(y, z)^{-2}dy \leq O(1)d'(x, y)^{-1-\epsilon}\quad (658)$$

**Proof.** For the first inequality consider separately the regions $d'(x, z) \leq d'(y, z)$ and $d'(y, z) \leq d'(x, y)$ and use the previous result. For the second inequality we need

$$\int_{d'(x, z) \leq 1, d'(y, z) \leq 1} d'(x, z)^{-1}d'(y, z)^{-2}d'(x, y)^{-\epsilon}dz \leq O(1) \quad (659)$$

We take $d'(x, y) \leq d'(x, z) + d'(z, y)$. In the region $d'(x, z) \leq d'(y, z)$ we have $d'(x, y) \leq 2d'(y, z)$ and so the integral is dominated by

$$O(1) \int_{d'(x, z) \leq 1, d'(y, z) \leq 1} d'(x, z)^{-1}d'(y, z)^{-2+\epsilon}dz \quad (660)$$

which is $O(1)$ by the previous result. Similarly for the region $d'(y, z) \leq d'(x, z)$. For the third inequality we need

$$\int_{d'(x, z) \leq 1, d'(y, z) \leq 1} d'(x, z)^{-2}d'(y, z)^{-2}d'(x, y)^{1+\epsilon}dz \leq O(1) \quad (661)$$

In the region $d'(x, z) \leq d'(y, z)$ we again have $d'(x, y) \leq 2d'(y, z)$ and so the integral is dominated by

$$O(1) \int_{d'(x, z) \leq 1, d'(y, z) \leq 1} d'(x, z)^{-2}d'(y, z)^{-1+\epsilon}dz \quad (662)$$

which is $O(1)$ by the first inequality. Similarly for the region $d'(y, z) \leq d'(x, z)$.
E Green’s functions on a lattice

We study the standard Greens function $G_k^s = (-\Delta + I)^{-1}$ defined on $\mathbb{T}^k_{N-k}$ or $L^{-k}\mathbb{Z}^3$. We are interested in both short and long distance behavior.

**Lemma 47.** There is a constant $\gamma = \mathcal{O}(1)$ such that

$$
|G_k^s(x, y)| \leq \mathcal{O}(1) d(x, y)^{-1} e^{-\gamma d(x, y)}
$$

$$
|\partial_\mu G_k^s(x, y)| \leq \mathcal{O}(1) d(x, y)^{-2} e^{-\gamma d(x, y)}
$$

$$
|\partial_\mu \partial_\nu G_k^s(x, y)| \leq \mathcal{O}(1) d(x, y)^{-3} e^{-\gamma d(x, y)}
$$

*(Proof)* It suffices to consider the infinite lattice $\eta\mathbb{Z}^3 = L^{-k}\mathbb{Z}^3$ since toroidal case can be obtained by periodizing. Also if $x \neq y$ so $d'(x, y) = d(x, y) = \sup_\mu |x_\mu - y_\mu|$ it suffices to consider the sector where $d'(x, y) = |x_0 - y_0|$.

On the infinite lattice we have the representation

$$
G_k^s(x, y) = \frac{1}{(2\pi)^3} \int_{|p_\mu| < \eta^{-1}\pi} \frac{e^{ip(x-y)}}{1 + \Delta(p)}
$$

where

$$
\Delta(p) = \sum_\mu 2\eta^{-2}(1 - \cos q_\mu) = \sum_\mu \sin^2\left(\frac{1}{2}\eta p_\mu\right) \left(\frac{\pi}{\eta}\right)^2
$$

For $p = (p_0, p_1, p_2)$ let $p = (0, p_1, p_2)$. The denominator in (662) vanishes when

$$
1 + \Delta(p) = 1 + 2\eta^{-2}(1 - \cos \eta p_0) + \Delta(p) = 0
$$

or

$$
\cos(\eta p_0) = 1 - \frac{1}{2}\eta^2 \left(1 + \Delta(p)\right)
$$

So we find poles at $p_0 = \pm i\omega(p)$ where

$$
\cosh(\eta \omega(p)) = 1 + \frac{1}{2}\eta^2 \left(1 + \Delta(p)\right)
$$

Note that if $\eta$ is small $\Delta(p) \approx |p|^2$ and comparing power series gives $\omega(p) \approx (1 + \Delta(p))^{\frac{1}{2}} \approx (1 + |p|^2)^{\frac{1}{2}}$ as expected.

Now deform the $p_0$ contour to a rectangle with large imaginary part. The sides of the rectangle cancel by periodicity and the far piece goes to zero. We only pick up the pole at $p_0 = \pm i\omega(p)$ depending on the sign of $x_0 - y_0$. Compute the residue at the pole using

$$
\frac{\partial}{\partial p_0} \left[2\eta^{-2}(1 - \cos \eta p_0)\right]_{p_0 = \pm i\omega(p)} = [2\eta^{-1} \sin \eta q_0]_{p_0 = \pm i\omega(p)} = \pm 2\eta^{-1} \sinh \eta \omega(p)
$$

and find

$$
G_k^s(x, y) = \frac{1}{(2\pi)^3} \int_{|p_\mu| < \eta^{-1}\pi} \frac{e^{-\omega(p)|x_\mu - y_\mu|} e^{ip(x-y)}}{2\eta^{-1} \sinh \eta \omega(p)} d^3p
$$

To estimate this start with

$$
\frac{1}{2} \leq \left|\frac{\sin x}{x}\right| \leq 1 \quad |x| \leq \frac{\pi}{2}
$$

It follows that $\frac{1}{2}|x|^2 \leq \Delta(p) \leq |x|^2$ and so

$$
1 + \frac{1}{2}\eta^2 \left(1 + \frac{1}{2}|p|^2\right) \leq \cosh(\eta \omega(p)) \leq 1 + \frac{1}{2}\eta^2 \left(1 + |p|^2\right)
$$

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But \( \sinh^2 x = \cosh^2 x - 1 \geq \cosh x - 1 \) so
\[
\frac{1}{2} \eta^2 \left( 1 + \frac{1}{2} |p|^2 \right) \leq \sinh^2(\eta \omega(p)) \quad \text{hence} \quad \frac{1}{2} \eta \sqrt{1 + |p|^2} \leq \sinh(\eta \omega(p)) \quad (671)
\]

Next we claim that there is a constant \( c = \mathcal{O}(1) \) such that
\[
c \sqrt{1 + |p|^2} \leq \omega(p) \leq \sqrt{1 + |p|^2} \quad (672)
\]

For the upper bound note that (670) implies that \( \cosh(\eta \omega(p)) \leq \cosh(\eta \sqrt{1 + |p|^2}) \) and hence \( \omega(p) \leq \sqrt{1 + |p|^2} \). For the lower bound we first note that the upper bound implies
\[
\eta \omega(p) \leq \sqrt{1 + \eta^2 |p|^2} \leq \sqrt{1 + 2\pi^2} \leq 5 \quad (673)
\]

For \( 0 \leq x \leq 5 \) we have \( \sinh x \leq \int_0^x \cosh t \, dt \leq x \cosh 5 \). Hence by (671)
\[
\frac{1}{2} \eta \sqrt{1 + |p|^2} \leq \sinh(\eta \omega(p)) \leq \eta \omega(p) \cosh 5 \quad (674)
\]

which gives the lower bound with \( c = (2 \cosh 5)^{-1} \).

Using (671) and (672) we have for \( x \neq y \) and \( |x_0 - y_0| = d(x, y) \neq 0 \)
\[
|G_k^* (x, y)| \leq \mathcal{O}(1) \int e^{-c \sqrt{1 + |p|^2} |x_0 - y_0|} \frac{dp}{\sqrt{1 + |p|^2}} \quad (675)
\]

Now with \( \gamma = \frac{1}{2} c \) we can extract a factor \( \exp(\gamma |x_0 - y_0|) \) and obtain
\[
|G_k^* (x, y)| \leq \mathcal{O}(1) e^{-\gamma |x_0 - y_0|} \int e^{-\frac{c}{2} \sqrt{1 + |p|^2} |x_0 - y_0|} \frac{dp}{\sqrt{1 + |p|^2}} \quad (676)
\]

Change variables to \( q = |x_0 - y_0| p \) and find that the integral here is
\[
|x_0 - y_0|^{-1} \int e^{-c \sqrt{|x_0 - y_0|^2 + |q|^2}} \frac{dq}{\sqrt{|x_0 - y_0|^2 + |q|^2}} \frac{dq}{\sqrt{|x_0 - y_0|^2 + |q|^2}} \leq \mathcal{O}(1) |x_0 - y_0|^{-1} \int e^{-c |q|} \left| \frac{1}{|q|} \right| dq = \mathcal{O}(1) |x_0 - y_0|^{-1} \quad (677)
\]

Thus we get
\[
|G_k^* (x, y)| \leq \mathcal{O}(1) |x_0 - y_0|^{-1} e^{-\gamma |x_0 - y_0|} \quad (678)
\]

If \( x = y \) the estimate comes down to
\[
|G_k^* (0, 0)| \leq \mathcal{O}(1) \int_{|p_k| \leq \eta^{-1} \pi} \frac{dp}{\sqrt{1 + |p|^2}} \quad (679)
\]

Enlarge the integration region to \( |p| \leq 2\eta^{-1} \pi \) and go to polar coordinates to get
\[
|G_k^* (0, 0)| \leq \mathcal{O}(1) \int_0^{2\eta^{-1} \pi} r \, dr = \mathcal{O}(1) \eta^{-1} = \mathcal{O}(1) L_k = \mathcal{O}(1) d'(0, 0)^{-1} \quad (680)
\]

For derivatives we note that for \( x_0 > y_0 \)
\[
(x_0 - y_0) \frac{\partial}{\partial x_0} G_k^* (x, y) = - \frac{1}{(2\pi)^2} \int_{|p_k| < \eta^{-1} \pi} \omega(p) (x_0 - y_0) e^{-\omega(p) (x_0 - y_0)} e^{i p \cdot (x - y)} \frac{dp}{2\eta^{-1} \sinh \eta \omega(p)} \quad (681)
\]
Now use \( |\omega(p)(x_0 - y_0)| e^{-\frac{1}{2} \omega(p)(x_0 - y_0)}| \leq O(1) \) and proceed as before to estimate the quantity by \( O(1) \). The same works for \( y_0 > x_0 \) so we have

\[
\left| \frac{\partial}{\partial x_0} G_k^s(x, y) \right| \leq O(1)|x_0 - y_0|^{-2} e^{-\gamma|x_0 - y_0|}
\]

(682)

For the other derivatives we have for \( k = 1, 2 \)

\[
(x_0 - y_0) \frac{\partial}{\partial x_k} G_k^s(x, y) = \frac{1}{(2\pi)^2} \int_{|p_k| < \eta^{-1} \pi} \int \frac{ip_k(x_0 - y_0) e^{-\omega(p)(x_0 - y_0)}}{2\eta^{-1} \sinh \eta \omega(p)} \ dp
\]

(683)

and this again yields the bound (682). Higher derivatives are similar.

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