Higher order Dehn functions for horospheres in products of Hadamard spaces

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Abstract

Let $X$ be a product of $r$ locally compact Hadamard spaces. In this note we prove that the horospheres in $X$ centered at regular boundary points of $X$ are Lipschitz-$(r-2)$-connected. Using the filling construction by R. Young in [You14] this gives sharp bounds on higher order Dehn functions for such horospheres. Moreover, if $\Gamma \subset \text{Is}(X)$ is a lattice acting cocompactly on $X$ minus a union of disjoint horoballs, we get a sharp bound on higher order Dehn functions for $\Gamma$. We therefore deduce that apart from the Hilbert modular groups already considered by R. Young every irreducible $\mathbb{Q}$-rank one lattice acting on a product of $r$ symmetric spaces of the noncompact type is undistorted up to dimension $r-1$ and has $k$-th order Dehn function asymptotic to $V^{(k+1)/k}$ for all $k \leq r-2$.

1 Introduction

In this note we apply the filling construction introduced by R. Young in [You14] to horospheres in products of arbitrary locally compact Hadamard spaces. The only new result we need for the construction is our Proposition 3.1 which implies that the so-called "slices" in every horosphere $\mathcal{H}$ centered at a regular boundary point are bilipschitz-equivalent to a Hadamard space. Using these slices one can then construct in an identical way as in [You14, Lemma 3.3] a map from the $(r-1)$-simplex $\Delta^{(r-1)} \to \mathcal{H}$ with properties that assure Lipschitz-$(r-2)$-connectivity of $\mathcal{H}$. Theorem 1.3 of [You14] then gives the following

**Theorem A** Any horosphere centered at a regular boundary point of a product of $r$ locally compact Hadamard spaces is undistorted up to dimension $r-1$. Moreover, for any $k \leq r-2$ the $k$-th order Dehn function of such a horosphere is asymptotic to $V^{(k+1)/k}$.

Moreover, if $\Gamma \subset \text{Is}(X)$ is a lattice acting cocompactly on $X$ minus a union of disjoint horoballs centered at regular boundary points, we get a sharp bound on higher order Dehn functions for $\Gamma$. According to Section 2.4 in [Dru04] every irreducible $\mathbb{Q}$-rank one lattice $\Gamma$ acting on a product $X$ of symmetric spaces of the noncompact type satisfies this assumption. We therefore get the following
Theorem B  Let $\Gamma$ be an irreducible $\mathbb{Q}$-rank one lattice acting on a product $X$ of $r$ symmetric spaces of the noncompact type. Then $\Gamma$ is undistorted up to dimension $r - 1$. Moreover, for any $k \leq r - 2$ the $k$-th order Dehn function of $\Gamma$ is asymptotic to $V^{(k+1)/k}$.

We remark that as soon as the symmetric space $X$ possesses at least one rank one factor $X_i$, then by Theorem 13.19 in [Rag72] every irreducible lattice $\Gamma \subset \text{Is}(X)$ has $\mathbb{Q}$-rank one.

2 Preliminaries

The purpose of this section is to introduce terminology and notation and to summarize basic results about (products of) Hadamard spaces and their horospheres. The main references here are [BH99] and [Bal95] (see also [BGS85]).

Let $(X, d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map $\sigma$ from a closed interval $[0, l] \subset \mathbb{R}$ to $X$ such that $\sigma(0) = x$, $\sigma(l) = y$ and $d(\sigma(t), \sigma(t')) = |t - t'|$ for all $t, t' \in [0, l]$. We will denote such a geodesic path $\sigma_{x,y}$.

$X$ is called geodesic if any two points in $X$ can be connected by a geodesic path; if this path is unique we say that $X$ is uniquely geodesic. A geodesic ray in $X$ is a map $\sigma : [0, \infty) \to X$ such that for all $t' > t > 0$ $\sigma_{[t, t']}$ is a geodesic path; a geodesic (line) in $X$ is a map $\sigma : \mathbb{R} \to X$ such that the above holds for all $t' > t$. Two geodesic rays are called asymptotic if they are at bounded Hausdorff distance from each other.

A metric space $(X, d)$ is called a Hadamard space if it is complete, geodesic and if all triangles satisfy the CAT(0)-inequality. This implies in particular that $X$ is simply connected and uniquely geodesic. From here on we assume that $X$ is a locally compact Hadamard space. The geometric boundary $\partial X$ of $X$ is defined as the set of equivalence classes of geodesic rays, where two geodesic rays are equivalent if they are asymptotic.

Let $x, y \in X$, $\xi \in \partial X$ and $\sigma$ a geodesic ray in the class of $\xi$. We set

$$\beta_\xi(x, y) := \lim_{s \to \infty} \left( d(x, \sigma(s)) - d(y, \sigma(s)) \right).$$

This number is independent of the chosen ray $\sigma$, and the function

$$\beta_\xi(\cdot, y) : \quad X \to \mathbb{R}, \quad x \mapsto \beta_\xi(x, y)$$

is called the Busemann function centered at $\xi$ based at $y$ (see also [Bal95], chapter II). From the definition one immediately gets the following properties of the Busemann function which we will need in the sequel: For all $x, y, z \in X$ and $\xi \in \partial X$ the cocycle identity

$$\beta_\xi(x, z) = \beta_\xi(x, y) + \beta_\xi(y, z)$$

(2)
holds, and we have
\[ |\beta_\xi(x,y)| \leq d(x,y) \]  
and \( \beta_\xi(x,y) = d(x,y) \) if and only if \( y \) is a point on the geodesic ray \( \sigma_{x,\xi} \).

For \( \xi \in \partial X \) and \( x \in X \), the horoball centered at \( \xi \) based at \( x \) is defined as the set
\[ \{ y \in X \mid \beta_\xi(x,y) < 0 \}; \]
its boundary
\[ \mathcal{H}_\xi(x) := \{ y \in X \mid \beta_\xi(x,y) = 0 \} \]
in \( X \) is called the horosphere centered at \( \xi \) based at \( x \). If \( \sigma \) is a geodesic line, then the projection along horospheres \( p_\sigma \) is defined by
\[ p_\sigma : X \to \sigma(\mathbb{R}), \quad x \mapsto p_\sigma(x) = \sigma(\beta_{\sigma(\infty)}(\sigma(0),x)). \]
Notice that \( p_\sigma(x) \) is the unique intersection point of \( \sigma(\mathbb{R}) \) with the horosphere \( \mathcal{H}_{\sigma(\infty)}(x) \) centered at \( \sigma(\infty) \) based at \( x \).

In this note we consider the Cartesian product \( X = X_1 \times X_2 \times \cdots \times X_r \) of \( r \) locally compact Hadamard spaces \( (X_1,d_1), (X_2,d_2), \ldots, (X_r,d_r) \) endowed with the distance \( d = \sqrt{d_1^2 + d_2^2 + \cdots + d_r^2} \). Notice that \( (X,d) \) is again a locally compact Hadamard space.

We denote \( p_i : X \to X_i, \ i \in \{1,2,\ldots,r\} \), the natural projections. Every geodesic path \( \sigma : [0,l] \to X \) can be written as a product \( \sigma(t) = (\sigma_1(t\cdot\theta_1), \sigma_2(t\cdot\theta_2), \ldots, \sigma_r(t\cdot\theta_r)) \), where \( \sigma_i \) are geodesic paths in \( X_i, i = 1,2,\ldots,r \), and the \( \theta_i \geq 0 \) satisfy
\[ \sum_{i=1}^r \theta_i^2 = 1. \]

The unit vector
\[ \text{sl}(\sigma) := \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_r \end{pmatrix} \in E := \{ \theta \in \mathbb{R}^r : \|\theta\| = 1, \ \theta_i \geq 0 \ \text{for all} \ i \in \{1,2,\ldots,r\} \} \]
is called the slope of \( \sigma \); a geodesic path \( \sigma \) is said to be regular if its slope does not possess a coordinate zero, i.e. if
\[ \text{sl}(\sigma) \in E^+ := \{ \theta \in \mathbb{R}^r : \|\theta\| = 1, \ \theta_i > 0 \ \text{for all} \ i \in \{1,2,\ldots,r\} \}, \]
and singular otherwise. In other words, \( \sigma \) is regular if none of the projections \( p_i(\sigma([0,l])) \), \( i \in \{1,2,\ldots,r\} \), is a point.

It is an easy exercise to verify that two asymptotic geodesic rays \( \sigma \) and \( \sigma' \) necessarily have the same slope. So the slope \( \text{sl}(\xi) \) of a point \( \xi \in \partial X \) can be defined as the slope of an arbitrary geodesic ray representing \( \xi \). The regular geometric boundary \( \partial X^{\text{reg}} \) and the singular geometric boundary \( \partial X^{\text{sing}} \) of \( X \) are defined by
\[ \partial X^{\text{reg}} := \{ \xi \in \partial X : \text{sl}(\xi) \in E^+ \}, \ \partial X^{\text{sing}} := \partial X \setminus \partial X^{\text{sing}}; \]
notice that the singular boundary $\partial X^{sing}$ consists of equivalence classes of geodesic rays in $X$ which project to a point in at least one of the factors $X_i$. Moreover, given $\theta = (\theta_1, \theta_2, \ldots, \theta_r) \in E$ we can define the subset

$$\partial X_\theta := \{ \xi \in \partial X : sl(\tilde{\xi}) = \theta \}$$

(4)

of the geometric boundary which is homeomorphic to the Cartesian product of the geometric boundaries $\partial X_i$ with $i \in I^+(\theta) := \{ i \in \{1, 2, \ldots, r \} : \theta_i > 0 \}$. So if $\xi \in \partial X_\theta$, then for all $i \in I^+(\theta)$ the projection $\xi_i$ of $\xi$ to $\partial X_i$ is well-defined. Notice that in the particular case $\theta \in E^+$ we have $\partial X_\theta \subseteq \partial X^{reg}$, $I^+(\theta) = \{1, 2, \ldots, r \}$ and hence $\partial X_\theta$ is homeomorphic to $\partial X_1 \times \partial X_2 \times \cdots \times \partial X_r$.

The following easy lemma relates the Busemann function (1) of the product to the Busemann functions on the factors. For a proof we refer the reader to Lemma 3.3 in [Lin15].

**Lemma 2.1** Let $\theta = (\theta_1, \theta_2, \ldots, \theta_r) \in E$, $x = (x_1, x_2, \ldots, x_r)$, $y = (y_1, y_2, \ldots, y_r) \in X$ and $\xi \in \partial X_\theta$. If $\xi_i$ denotes the projection of $\xi$ to $\partial X_i$ then

$$\beta_\xi(x, y) = \sum_{i \in I^+(\theta)} \theta_i \cdot \beta_{\xi_i}(x_i, y_i).$$

### 3 Horospheres and slices

For the remainder of this note we fix a base point $o = (o_1, o_2, \ldots, o_r) \in X$, a point $\tilde{\xi} \in \partial X$ of slope $\theta \in E$ and abbreviate $\mathcal{H} := \mathcal{H}_{\tilde{\xi}}(o)$.

We recall the definition of

$$I^+(\theta) = \{ i \in \{1, 2, \ldots, r \} : \theta_i > 0 \}$$

and remark that for all $i \in I^+(\theta)$ the projection $\xi_i$ of $\tilde{\xi}$ to $\partial X_i$ is well-defined. Moreover, according to Lemma [2.1] we have

$$\mathcal{H} = \{ y = (y_1, y_2, \ldots, y_r) \in X : \beta_{\tilde{\xi}}(o, y) = \sum_{i \in I^+(\theta)} \theta_i \cdot \beta_{\xi_i}(o_i, y_i) = 0 \}.$$ 

Let $q \in \{1, \ldots, r \}$ denote the cardinality of the set $I^+(\theta)$. Notice that if $\tilde{\xi} \in \partial X^{sing}$ then $q \leq r - 1$.

As in [You14] we consider the following important subsets of the horosphere $\mathcal{H}$: For $i \in I^+(\theta)$ we let $s_i \subseteq X_i$ be either all of $X_i$ or the image of a geodesic line $\sigma_i$ in $X_i$ with $\sigma_i(\infty) = \xi_i$. For $i \notin I^+(\theta)$ we set $s_i = \{ o_i \}$. If precisely $k$ of the sets $s_i$ are all of $X_i$, then the set

$$(s_1 \times s_2 \times \cdots \times s_r) \cap \mathcal{H}$$

is called a $k$-slice; notice that necessarily $k \in \{0, 1, \ldots, q - 1\}$. For a $k$-slice $S = (s_1 \times s_2 \times \cdots \times s_r) \cap \mathcal{H}$ we denote

$$I(S) := \{ i \in \{1, 2, \ldots, r \} : s_i = X_i \}$$

(5)
the index set of the "full factors" which by definition has cardinality $k$.

We now state the key proposition for horospheres $H$ centered at a boundary point $\tilde{\xi} \in \partial X$; recall that $q = \#I^+(\theta)$:

**Proposition 3.1** If $k \in \{0, 1, \ldots, q - 1\}$ and $S = (s_1 \times s_2 \times \cdots \times s_r) \cap H \subset H$ is a $k$-slice, then $S$ together with the length metric $d_S$ induced from the metric $d$ on $X$ is bilipschitz-equivalent to the Cartesian product of the factors $X_i$ with $i \in I(S)$ times $\mathbb{R}^{q-k-1}$ (endowed with the product metric).

**Proof.** For simplicity of notation we first assume that $I^+(\theta) = \{1, 2, \ldots, q\}$ and hence

$$H = \{y = (y_1, y_2, \ldots, y_r) \in X : \beta_{\xi}(o, y) = \sum_{i=1}^{q} \theta_i \cdot \beta_{\xi_i}(o_i, y_i) = 0\}.$$ 

So for a $k$-slice the last $r - q$ of the $s_i$ are always equal to $\{o_i\}$, and we moreover assume that the first $k$ of the $s_i$ are equal to $X_i$. For $l \in \{k+1, \ldots, q\}$ we let $\tilde{\sigma}_l \subset X_l$ be a geodesic line with $\tilde{\sigma}_l(\mathbb{R}) = s_l$. Reparametrizing $\tilde{\sigma}_l$ if necessary we may further require that $\beta_{\xi_l}(o_l, \tilde{\sigma}_l(0)) = 0$. We prove that the map

$$pr : S \to X_1 \times \cdots \times X_k \times \mathbb{R}^{q-k-1},$$

$$(x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_q, o_{q+1}, \ldots, o_r)$$

$$\mapsto (x_1, \ldots, x_k, \beta_{\xi_{k+1}}(o_{k+1}, x_{k+1}), \ldots, \beta_{\xi_{q-1}}(o_{q-1}, x_{q-1}))$$

is the desired bilipschitz-equivalence.

We first show that $pr$ is bijective: For that we let $(x_1, \ldots, x_k, t_{k+1}, \ldots, t_{q-1}) \in X_1 \times \cdots \times X_k \times \mathbb{R}^{q-k-1}$ be arbitrary. In order to construct the preimage by $pr$ we first set $x_l = \tilde{\sigma}_l(t_l)$ for $l \in \{k+1, \ldots, q-1\}$; in particular we have

$$\beta_{\xi_l}(o_l, x_l) = \beta_{\xi_l}(o_l, \tilde{\sigma}_l(t_l)) = \beta_{\xi_l}(o_l, \tilde{\sigma}_l(0)) + \beta_{\xi_l}(\tilde{\sigma}_l(0), \tilde{\sigma}_l(t_l)) = t_l$$

since $\tilde{\sigma}_l(\infty) = \xi_l$. Finally we set

$$t_q := -\frac{1}{\theta_q} \left( \sum_{i=1}^{q-1} \theta_i \beta_{\xi_i}(o_i, x_i) \right)$$

and $x_q := \tilde{\sigma}_l(t_q)$.

We next prove that $pr$ is bilipschitz. For that we will denote the product metric on $X_1 \times \cdots \times X_k \times \mathbb{R}^{q-k-1}$ by $d$. Let $x = (x_1, x_2, \ldots, x_r)$, $y = (y_1, y_2, \ldots, y_r) \in S$ be arbitrary. Since $x_i = y_i = o_i$ for $i \in \{q+1, \ldots, r\}$ we clearly have

$$d_S(x, y)^2 \geq d_X(x, y)^2 = \sum_{i=1}^{r} d_i(x_i, y_i)^2 = \sum_{i=1}^{q} d_i(x_i, y_i)^2$$

$$\geq \sum_{i=1}^{q-1} d_i(x_i, y_i)^2 = d(pr(x), pr(y))^2,$$
because for \( k + 1 \leq i \leq q - 1 \) the points \( x_i \) and \( y_i \) lie on the geodesic line \( \hat{\sigma}_i \).

In order to prove the converse estimate, we will construct a path \( c \in S \) joining \( x \) and \( y \) with length bounded by a constant times \( \overline{d}(\text{pr}(x), \text{pr}(y)) \). We abbreviate

\[
\overline{d} = \overline{d}(\text{pr}(x), \text{pr}(y)) \quad \text{and} \quad d_i = d_i(x_i, y_i) \quad \text{for} \quad 1 \leq i \leq q.
\]

For \( 1 \leq i \leq q - 1 \) we consider the geodesic segment \( \sigma_i : [0, d_i] \rightarrow X_i \) joining \( x_i \) and \( y_i \).

Then there exists a unique curve \( \gamma : [0, \overline{d}] \rightarrow \mathbb{R} \) such that the path

\[
c : [0, \overline{d}] \rightarrow X, \quad t \mapsto (\sigma_1(t\theta_1), \sigma_2(t\theta_2), \ldots, \sigma_q-1(t\theta_{q-1}), \hat{\sigma}_q(\gamma(t)), o_{q+1}, \ldots, o_r)
\]

is contained in the \( k \)-slice \( S \). Indeed, the curve \( \gamma \) is determined by the condition

\[
\theta_q \beta_{\xi_q}(o_q, \hat{\sigma}_q(\gamma(t)))) = -\sum_{i=1}^{q-1} \theta_i \beta_{\xi_i}(o_i, \sigma_i(t\theta_i)) \quad \text{for all} \quad t \in [0, \overline{d}]. \tag{6}
\]

Using the properties of the Busemann functions as well as the Cauchy-Schwarz inequality, we estimate for \( t, t' \in [0, \overline{d}] \)

\[
|\gamma(t') - \gamma(t)| = |\beta_{\xi_q}(\hat{\sigma}_q(\gamma(t)), \hat{\sigma}_q(\gamma(t')))\ |
\]

\[
\leq \frac{1}{\theta_q} \sum_{i=1}^{q-1} \theta_i |\beta_{\xi_i}(\sigma_i(t\theta_i), o_i) + \beta_{\xi_i}(o_i, \sigma_i(t'\theta_i))|
\]

\[
\leq \frac{1}{\theta_q} \sum_{i=1}^{q-1} \theta_i |\beta_{\xi_i}(\sigma_i(t\theta_i), \sigma_i(t'\theta_i))| \leq \frac{1}{\theta_q} \sum_{i=1}^{q-1} \theta_i d_i(\sigma_i(t\theta_i), \sigma_i(t'\theta_i))
\]

\[
\leq \frac{1}{\theta_q} \sqrt{\sum_{i=1}^{q-1} \theta_i^2 \sum_{i=1}^{q-1} d_i(\sigma_i(t\theta_i), \sigma_i(t'\theta_i))^2} = \sqrt{\frac{1 - \theta_q^2}{\theta_q} \sum_{i=1}^{q-1} |t - t'|^2 \theta_i^2}
\]

\[
= |t - t'| \frac{1 - \theta_q^2}{\theta_q} \leq \frac{1}{\theta_q} |t - t'|.
\]

We conclude that the length of the curve \( \hat{\sigma}_q \circ \gamma \) in \( X_q \) (which is equal to the length of the curve \( \gamma \) in \( \mathbb{R} \)) is smaller than or equal to \( \overline{d}/\theta_q \). So for the length \( L(c) \) of the curve \( c \) in \( X \) we get

\[
L(c) \leq \sqrt{\sum_{i=1}^{q-1} d_i^2 + \left(\frac{\overline{d}}{\theta_q}\right)^2} = \overline{d} \sqrt{1 + \frac{1}{\theta_q^2}},
\]

and hence

\[
d_S(x, y) \leq \sqrt{1 + \frac{1}{\theta_q^2}} \overline{d}(\text{pr}(x), \text{pr}(y)).
\]

Finally, for \( I^+(\theta) \) arbitrary of cardinality \( q \) and any \( k \)-slice \( S \subset \mathcal{H} \) with index set \( I(S) \subset \{1, 2, \ldots, r\} \) of the “full factors” in \( S \) we denote by \( \text{pr} \) the bijection from \( S \) to
the Cartesian product of the factors \(X_i\) with \(i \in I(S)\) times \(\mathbb{R}^{q-k-1}\), and \(\overline{d}\) its product metric. Then for all \(x, y \in S\) we have

\[
\overline{d}(\text{pr}(x), \text{pr}(y)) \leq d_S(x, y) \leq \sqrt{1 + \max \left\{ \frac{1}{\theta^2} : i \in I^+(\theta) \right\}} \overline{d}(\text{pr}(x), \text{pr}(y)),
\]

which proves the claim. \(\square\)

### 4 Statement of results

In order to deduce Theorems A and B from the introduction we use Lemma 3.2 in [You14], which is due to Gromov and provides an equivalent formulation of Lipschitz connectivity. More precisely, if \(H\) is a horosphere centered at a boundary point \(\tilde{\xi} \in \partial X_\theta\) with \(q = \#I^+(\theta)\), we construct a map from the \((q-1)\)-simplex \(\Delta^{q-1}\) to the horosphere \(H\) with the properties (1) and (2) of Lemma 3.2 in [You14]. This can be done word by word as in Section 3 of the article by R. Young, since our Proposition 3.1 ensures that for \(k \in \{0, 1, \ldots, q-2\}\) every \(k\)-slice of the horosphere \(H\) is bilipschitz-equivalent to a Hadamard space. This proves that \(H\) is Lipschitz-(\(q-2\))-connected, hence applying Theorem 1.3 and Corollary 1.4 of [You14] we get the following stronger version of Theorem A:

**Theorem 4.1** Let \(X\) be a product of \(r\) locally compact Hadamard spaces and \(\tilde{\xi} \in \partial X\). If \(\tilde{\xi}\) is represented by a geodesic ray which does not project to a point in precisely \(q\) factors, then every horosphere \(H\) centered at \(\tilde{\xi}\) is undistorted up to dimension \(q - 1\). Moreover, for any \(k \leq q - 2\) the \(k\)-th order Dehn function of such a horosphere is asymptotic to \(V^{(k+1)/k}\).

We want to remark that for horospheres centered at a singular boundary point \(\tilde{\xi} \in \partial X_\theta\), the \(k\)-slices could be optionally defined by setting \(s_i = X_i\) (instead of \(s_i = \{o_i\}\) for \(i \notin I^+(\theta)\)). Mimicking our proof of Proposition 3.1 one can show that every such \(k\)-slice \(S\) is bilipschitz-equivalent to the Cartesian product of the factors \(X_i\) with \(i \in I(S)\) (as defined by (5)) times \(\mathbb{R}^{q-k-1}\) via the map \(\text{pr}\) which is the identity on the product of the "full factors" \(X_i\) with \(i \in I(S)\) as in the proof of Proposition 3.1. However, now we always have \(\#I(S) \geq \#I^+(\theta) = r - q\), so the minimal dimension of a \(k\)-slice would be \(r - q \geq 1\); but since in Young’s construction one needs to start with 0-slices, this would not yield a statement about Lipschitz-connectivity.

We finally turn to the proof of Theorem B. Since undistortedness and (higher order) Dehn functions are quasi-isometry invariants, Theorem 4.1 implies

**Theorem 4.2** Let \(X\) be a product of \(r\) locally compact Hadamard spaces and \(\Gamma \subset Is(X)\) a lattice acting cocompactly on \(X\) minus a union of disjoint horoballs. If all the centers of these disjoint horoballs are represented by a geodesic ray which does not project to a point in at least \(q\) factors, then \(\Gamma\) is undistorted up to dimension \(q - 1\). Moreover, for any \(k \leq q - 2\) the \(k\)-th order Dehn function of \(\Gamma\) is asymptotic to \(V^{(k+1)/k}\).
Obviously, if all of the disjoint horoballs are centered at regular boundary points then the conclusion of the above theorem holds with $q = r$. Moreover, by Prasad [Pra73] and Raghunathan [Rag72] every irreducible lattice of $\mathbb{Q}$-rank one in a semi-simple Lie group acts cocompactly on the associated symmetric space $X$ minus a finite disjoint union of horoballs; Proposition 2.4.3 in [Dru04] further states that the centers of these horoballs belong to the regular boundary. So Theorem B follows.

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