Improvement of sphericity test for large-dimensional covariance matrix

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Abstract. This paper discusses the problem of testing the sphericity of population covariance matrix when the sample size \( n \) and the dimensionality \( p \) both tend to infinity with \( \frac{p}{n} \to c \in (0,1) \). A new test statistic is presented by utilizing an inequality, and asymptotic properties of the proposed statistic are derived for generally distributed population under the null hypothesis. Numerical simulations demonstrate that the proposed statistic has a significant improvement on test powers under the alternative hypothesis by comparing with some congeneric statistics.

1. Introduction
Modern applications are often characterized by large-dimensional data sets. Typical examples include financial portfolios involving large numbers of assets[1], biomedical data set involving large nucleotide or protein arrays[2-3], or modern wireless communication signals with large numbers of antennas[4-5]. An important problem in these applications is the hypothesis test for covariance matrix when data dimensionality and the sample size both tend to infinity. If it happens, then some of the conventional multivariate procedures may not work properly. These procedures are justified under the classical regime with large sample size and fixed dimensionality.

Suppose that \( X_1, \ldots, X_n \) are independent and identically distributed (i.i.d.) \( p \)-dimensional vectors with mean zero and covariance matrix \( \Sigma_p \). The sample covariance matrix is \( S_n = \frac{1}{n} \sum_{i=1}^{n} X_i X_i' \). We focus on testing the structure for population covariance matrix:

\[
H_0 : \Sigma_p = \sigma^2 I_p \quad \text{vs.} \quad H_1 : \Sigma_p \neq \sigma^2 I_p
\]

where \( \sigma \) is an unknown but finite positive constant, and \( I_p \) is the \( p \times p \) identity matrix. Traditional test for covariance matrix based on the likelihood ratio, in [6], show poor performance when the dimensionality diverges at the same ratio as the sample size, since the sample covariance matrix can not converge to its population counterpart under large-dimensional regime.

In the need of theoretical research and practical application, the study of the above hypothesis tests has attracted statisticians’ attentions in large-dimensional regime. Ledoit and Wolf (LW)[7] restudied certain tests originally setting up by[8] and[9] in a fixed \( p \) context and generalized the locally best invariant (LBI) tests to accommodate the situation that both \( p \) and \( n \) go to infinity with \( \frac{p}{n} \to c \in (0,\infty) \). These results were later extended to the cases \( c = 0 \) and \( c = \infty \) in [10]. Meanwhile,
Srivastava [11] refined the LBI tests by applying the unbiased estimators of \( \text{tr}\Sigma_p^k / p, k=1,2 \). Subsequently, some test statistics were investigated based on these unbiased estimators, such as in [12-13]. Moreover, Fisher [14] studied homogeneous statistics constructed from unbiased estimators of \( \text{tr}\Sigma_p^k / p, k=1,2,3,4 \). Although these tests overcame the constraint of the classical regime, they relied heavily on the normality assumption. For non-normal case, Chen et al. [15] developed a method where the statistics are constituted by some well-selected U-statistics. Bai et al. [16] modified the likelihood ratio tests to accommodate the situations where \( p \) and \( n \) both go to infinity with \( p/n \to c \in (0,1) \). In [17], the likelihood ratio test and LW test proposed in [7] for sphericity were modified in the general population using random matrix theory (RMT). Most recently, Hu et al. [18] considered testing the sphericity of elliptical populations by using the linear spectral statistics of high-dimensional sample covariance matrix.

Motivated from [16], we think that it is necessary to relax the normality assumption in testing procedure, since many actual data do not always satisfy the normality. Meanwhile, we attempt to construct new statistic in order to pursue the higher test power. In this paper, we discuss the testing problem (1) from the viewpoint of RMT. Our study is based on the sample covariance matrix \( S_n \) for general population in an asymptotic framework where the sample size and the dimension both go to infinity with \( p/n \to c \in (0,1) \). Under this framework, the asymptotic behaviour of the proposed statistic is deduced under the null hypothesis using RMT. Moreover, some numerical simulations are provided to show the performance of the proposed statistic.

2. Preliminaries
Suppose \( H_p \) and \( F_n \) are spectral distributions of the population covariance matrix \( \Sigma_p \) and the sample covariance matrix \( S_n \) respectively. We define the integer-order moments of and \( H_p \) and \( F_n \):

\[
\alpha_k := \int t^k dH_p(t) = \frac{1}{p} \text{tr}(\Sigma_p^k),
\]

\[
\hat{\beta}_k := \int t^k dF_n(t) = \frac{1}{p} \text{tr}(S_n^k),
\]

\[
\alpha_0 := \int \log(t) dH_p(t) = \frac{1}{p} \log|\Sigma_p|,
\]

\( k=1,2,\cdots \). Assuming that the observations are normally distributed, the estimators \( \hat{\alpha}_i \) of \( \alpha_i \), \( i=1,2 \), were proved to be consistent, unbiased and asymptotically normal as \( n,p \to \infty \) and adopted in [11]. Meanwhile, these estimators can be expressed as follows:

\[
\hat{\alpha}_i = \hat{\beta}_i, \quad \hat{\alpha}_2 = \frac{n^2}{(n-1)(n+2)} \left( \hat{\beta}_2 - \frac{p}{n} \hat{\beta}_2^2 \right).
\]

If the underlying distribution is non-normal, we note that the unbiasedness does not hold any more for \( \hat{\alpha}_2 \), but the consistency and asymptotic normality can be retained under some suitable assumptions from [19].

**Assumption 1**: The sample size \( n \) and the dimensionality \( p \) of covariance matrix \( \Sigma_p \) both tend to infinity with \( c_n = p/n \to c \in (0,1) \).

**Assumption 2**: There is a doubly infinite array composed of i.i.d. random variables \( w_{ij} \) satisfying

\[
E(w_{ij}) = 0, \quad E(w_{ij}^2) = 1, \quad E(w_{ij}^4) < \infty,
\]

\( i, j \geq 1 \). For each pair of \( (p,n) \), the observation vectors can be represented as \( X_j = \Sigma_p^{1/2} w_j \), where \( w_j = (w_{ij})_{1 \leq i \leq p} \) is the \( j \)-th column of the matrix \( W_n = (w_{ij})_{1 \leq i \leq n, 1 \leq j \leq p} \).
Assumption 3: The spectral norm of $\Sigma_p$ is bounded by a positive constant and the population spectral distribution $H_p$ converges weakly to a non-random distribution $H$ as $p \to \infty$.

Assumptions 1—3 are typical conditions of the CLT for linear spectral statistics of sample covariance matrix, see [20]. In Assumption 2, we are accustomed to assume $E(w_i^d) = 3 + \Delta$ where $\Delta$ is a finite constant which is 0 if $w_y$ is normal random variable. Under these assumptions, we know that the estimator $\hat{\alpha}_k$ is strongly consistent from [19], i.e.,

$$\hat{\alpha}_k - \alpha_k \xrightarrow{a.s.} 0, \quad k = 1, 2.$$  (2)

Lemma [17] Under Assumptions 1–3, if $\Sigma_p = I_p$, then

$$p \begin{pmatrix} \hat{\beta}_1 - 1 \\ \hat{\beta}_2 - (1 + c_n) \\ \hat{\alpha}_0 - (c_n - 1) \log(1 - c_n) / c_n + 1 \end{pmatrix} \xrightarrow{d} N_3(\mu, \Gamma),$$  (3)

Where

$$\mu = (0, c(1 + \Delta), \log(1 - c) / 2 - c\Delta / 2)^\tau,$$

and

$$\Gamma = \begin{pmatrix} 2c & 4c(c + 1) & 2c \\ 4c(c + 1) & 4c(2c^2 + 5c + 2) & 2c(c + 2) \\ 2c & 2c(c + 2) & -2 \log(1 - c) \end{pmatrix} + \begin{pmatrix} c & 2c(c + 1) & c \\ 2c(c + 1) & 4c(c + 1)^2 & 2c(c + 1) \end{pmatrix} \Delta.$$  

3. Test procedure

The testing problem (1) remains invariant under the scalar transformation $x \mapsto ax (a \neq 0)$ and the orthogonal transformation $x \mapsto Gx$, where $G$ is an orthogonal matrix. Thus, we can assume without loss of generality that $\Sigma_p = \text{diag}(\lambda_1, \ldots, \lambda_p)$ with $\lambda_i > 0$ for $i = 1, \ldots, p$.

From Jensen inequality, we know that

$$\frac{1}{p} \sum_{i=1}^p \log \lambda_i' \leq \log \left( \frac{1}{p} \sum_{i=1}^p \lambda_i' \right)$$  (4)

with equality holding if and only if $\lambda_i = \lambda$ for some constant $\lambda$. It is noticed that the statistic LRT from [17] is the case of $r = 1$. The case of $r = 2$ is explored here.

When $r = 2$, the inequality in (4) becomes

$$\frac{2}{p} \log |\Sigma_p| \leq \log \left( \frac{1}{p} \text{tr}(\Sigma_p^2) \right).$$  (5)

The equality in (5) holds if and only if $\Sigma_p = \sigma^2 I_p$. Denote

$$\gamma := \log \alpha_2 - 2 \alpha_0 \geq 0,$$  (6)

then $\gamma = 0$ if and only if the hypothesis of sphericity holds. Hence, for the sphericity test, we can consider testing the equivalent hypothesis

$$H_0 : \gamma = 0 \quad vs. \quad H_1 : \gamma > 0$$

and propose the following new test statistic

$$\hat{\gamma} = \log \hat{\alpha}_2 - 2 \hat{\alpha}_0.$$  (7)

Theorem Suppose that Assumptions 1–3 hold, under the null hypothesis, we have

$$n \left( \hat{\gamma} + 2(c_n - 1) \log(1 - c_n) / c_n \right) \xrightarrow{d} N(\bar{\mu}, \bar{\Gamma}),$$
Where $\tilde{\mu} = 1 - \log(1-c)/c + 2\Delta$, $\Gamma = 12 - 8/c - 8\log(1-c)/c^2$.

Further, if $E(\hat{w}_i) = 3$, then we have

$$n\left(\hat{\beta}_1 - 1, \hat{\beta}_2 - (1+c_n), \hat{\alpha}_n - (c_n - 1)\log(1-c_n) + 1\right) \overset{d}{\to} N(\mu, \Gamma).$$

**Proof:** Using Lemma, we know

$$n\left(\hat{\beta}_1 - 1, \hat{\beta}_2 - (1+c_n), \hat{\alpha}_n - (c_n - 1)\log(1-c_n) + 1\right) \overset{d}{\to} N(\mu, \Gamma).$$

Let $t = (x, y, z)'$ and define a function $f_n(t)$,

$$f_n(t) = \log\left(\frac{n^2}{(n-1)(n+2)} (y - c_n x^2)\right) - 2z.$$

It is clear that $f_n(t)$ has a continuous partial derivative at $t = (1, 1+c_n, (c_n - 1)\log(1-c_n)/c_n)'$ and the Jacobian $J_n(t) = \frac{\partial f(t)}{\partial t}$ converges to a limit $J(t_0)$, as $(n, p) \to \infty$,

$$J(t_0) = (-2c, 1, -2)'\cdot$$

Using the Delta method and Slutsky’s theorem, we get

$$n\left(\hat{\beta}_1 - 1, \hat{\beta}_2 - (1+c_n), \hat{\alpha}_n - (c_n - 1)\log(1-c_n) + 1\right) \overset{d}{\to} N(\mu, \Gamma).$$

On simply calculating, we know $f_n(t_0) = \frac{2(c_n - 1)\log(1-c_n)}{c_n} - 2$, $J(t_0)\mu = -\frac{\log(1-c)}{c} + 1 + 2\Delta$ and

$$\frac{J(t_0)\Gamma J(t_0)}{c^2} = 12 - 8/c - 8\log(1-c)/c^2.$$

Denote $\tilde{\mu} = J(t_0)\mu / c, \tilde{\Gamma} = J(t_0)\Gamma J(t_0) / c^2$, Thus

$$n\left(\hat{\beta}_1 - 1, \hat{\beta}_2 - (1+c_n), \hat{\alpha}_n - (c_n - 1)\log(1-c_n) + 1\right) \overset{d}{\to} N(\tilde{\mu}, \tilde{\Gamma}).$$

If $E(\hat{w}_i) = 3$, then $\Delta = 0$. Therefore,

$$n\left(\hat{\beta}_1 - 1, \hat{\beta}_2 - (1+c_n), \hat{\alpha}_n - (c_n - 1)\log(1-c_n) + 1\right) \overset{d}{\to} N(0, 1).$$

It is worth pointing out that $E(\hat{w}_i) = 3$ includes the case of normal population.

**4. Simulation study**

In this section, we conduct simulations to demonstrate the asymptotic normality of the proposed statistic, and to perform a comparative study on several congenic statistics in testing the hypothesis that the covariance matrix is the identity.

**4.1 Normality and the attained significance levels**

Firstly, consider the following two cases of population distributions:

Case 1: $w_j$ is a $p$-dimensional normal random vector with mean zero and covariance matrix

$$\Sigma_p = I_p.$$

...
Case 2: \( w_j \) consists of the i.i.d. random variables \( w_{ij}, i=1,\ldots,p \), following the standardized Gamma(4, 0.5) so that \( E(w_{ij})=0 \), \( E(w_{ij}^2)=1 \), and \( \Sigma_p=I_p \). It is easy to see \( E(w_{ij}^4)=4.5 \) and \( \Delta=1.5 \).

For the sphericity test, we compare the proposed statistic with two test statistics given respectively in [16] (denoted as CLRT) and [17] (denoted as CJ). Note that the test statistic in [17] is the modified version of [7] for general distribution. The significant level is set as \( \alpha = 0.05 \), and we run 10000 independent trials.

For illustrating the effectiveness of the proposed test statistic, we simulate its attained significant level (ASL). The ASL is used to assess how to close \( \alpha \). Table 1 shows the ASLs of CLRT, CJ and the proposed statistic for Cases 1 and 2. It is seen that the ASLs of these tests are close to \( \alpha \) as \( p \) and \( n \) both increase. In particular, the ASLs of the proposed statistic perform better in most situations when the population is gamma distribution.

| \( n \) | \( p \) | Case 1 | | Case 2 |
|-----|-----|------||------|
| 32 | 2 | 0.0480 | 0.0376 | 0.0409 | 0.0539 | 0.0357 | 0.0413 |
| 6 | 0.0602 | 0.0463 | 0.0499 | 0.0588 | 0.0695 | 0.0535 |
| 12 | 0.0552 | 0.0542 | 0.0465 | 0.0577 | 0.0731 | 0.0531 |
| 18 | 0.0557 | 0.0528 | 0.0493 | 0.0531 | 0.0724 | 0.0501 |
| 24 | 0.0536 | 0.0515 | 0.0469 | 0.0551 | 0.0690 | 0.0493 |
| 32 | 0.0610 | 0.0529 | 0.0507 | 0.0561 | 0.0735 | 0.0491 |
| 64 | 4 | 0.0542 | 0.0519 | 0.0495 | 0.0656 | 0.0706 | 0.0605 |
| 12 | 0.0566 | 0.0505 | 0.0506 | 0.0633 | 0.0740 | 0.0622 |
| 24 | 0.0537 | 0.0531 | 0.0485 | 0.0536 | 0.0661 | 0.0518 |
| 36 | 0.0525 | 0.0531 | 0.0486 | 0.0572 | 0.0671 | 0.0537 |
| 48 | 0.0486 | 0.0579 | 0.0506 | 0.0508 | 0.0648 | 0.0483 |
| 60 | 0.0552 | 0.0535 | 0.0516 | 0.0522 | 0.0623 | 0.0486 |
| 128 | 8 | 0.0597 | 0.0541 | 0.0532 | 0.0704 | 0.0746 | 0.0678 |
| 24 | 0.0522 | 0.0495 | 0.0504 | 0.0525 | 0.0672 | 0.0520 |
| 48 | 0.0528 | 0.0540 | 0.0487 | 0.0497 | 0.0574 | 0.0496 |
| 72 | 0.0491 | 0.0524 | 0.0475 | 0.0530 | 0.0602 | 0.0514 |
| 96 | 0.0516 | 0.0514 | 0.0498 | 0.0497 | 0.0601 | 0.0495 |
| 120 | 0.0564 | 0.0535 | 0.0509 | 0.0527 | 0.0578 | 0.0512 |

We also check the QQ plot of the proposed statistic under the null hypothesis for sphericity test. Figure 1 shows the QQ plots of the 10000 observations of the test statistic under the null hypothesis with \( n = 100 \) and \( p = 90 \) for Cases 1 and 2 respectively. In both case, the normality appears to be satisfied in the QQ plots for large \( n, p \).

4.2 Power studies

To compare the powers of the three test statistics, we study two models under the alternative hypotheses:

Model 1: \( \Sigma_p = I_p + \text{diag} \{2, \ldots, 2, 0, \ldots, 0\} \); Model 2: \( \Sigma_p = I_p + \text{diag} \{1, \ldots, 1, 0, \ldots, 0\} \).
In the following simulations, the sample size $n$ is set as 32, 64, 128 for normal and gamma distributions. Table 2 shows the powers of CLRT, CJ and the proposed statistic under the normality assumption. Table 3 shows the powers of CLRT, CJ and the proposed statistic under the gamma assumption.

Whether the population is normal or not, Tables 2 and 3 show that the consistency of these test statistics since their empirical powers can fast approach to 1 when $n$ is sufficiently large. At the same time, we also note that our proposed test is powerful than CLRT and CJ in most situations. When the dimensionality $p$ is close to the sample size $n$, the power of the proposed statistic is higher than CLRT, but less than CJ.

Table 2. Empirical powers of CLRT, CJ and proposed test for Models 1 and 2 when the population distribution is normal.

| $n$ | $p$ | CLRT | CJ | Proposed | CLRT | CJ | Proposed |
|-----|-----|------|-----|----------|------|-----|----------|
| 32  | 2   | 0.8675 | 0.8405 | 0.8704  | 0.4009 | 0.3951 | 0.4184  |
|     | 6   | 0.8586 | 0.9234 | 0.9108  | 0.4140 | 0.4278 | 0.4432  |
|     | 12  | 0.7445 | 0.9123 | 0.8676  | 0.4271 | 0.4517 | 0.4888  |
|     | 18  | 0.5308 | 0.8658 | 0.7594  | 0.3881 | 0.4920 | 0.4913  |
|     | 24  | 0.3379 | 0.8146 | 0.6425  | 0.3216 | 0.4909 | 0.4333  |
|     | 32  | 0.1748 | 0.7402 | 0.4557  | 0.1839 | 0.4843 | 0.3296  |
| 64  | 4   | 0.9998 | 0.9999 | 0.9999  | 0.8549 | 0.8366 | 0.8603  |
|     | 12  | 1     | 1     | 1       | 0.9023 | 0.9081 | 0.9155  |
|     | 24  | 0.9975 | 0.9999 | 0.9997  | 0.8950 | 0.9232 | 0.9251  |
|     | 36  | 0.9605 | 0.9996 | 0.9976  | 0.8602 | 0.9308 | 0.9275  |
|     | 48  | 0.8047 | 0.9978 | 0.9847  | 0.7730 | 0.9488 | 0.9116  |
Table 3. Empirical powers of CLRT, CJ and proposed test for Models 1 and 2 when the population distribution is normal.

| n   | p   | CLRT | CJ | Proposed | CLRT | CJ | Proposed |
|-----|-----|------|----|----------|------|----|----------|
| 32  | 2   | 0.7401 | 0.6672 | 0.7223 | 0.3279 | 0.2850 | 0.3158   |
| 6   | 0.7496 | 0.8092 | 0.7996 | 0.3584 | 0.3260 | 0.3610   |
| 12  | 0.6712 | 0.7884 | 0.7568 | 0.3610 | 0.3652 | 0.4107   |
| 18  | 0.4757 | 0.7412 | 0.6574 | 0.3169 | 0.3840 | 0.4071   |
| 24  | 0.3208 | 0.6848 | 0.5528 | 0.3216 | 0.4352 | 0.4333   |
| 32  | 0.1823 | 0.6419 | 0.4292 | 0.2097 | 0.4324 | 0.3345   |
| 64  | 4   | 0.9958 | 0.9886 | 0.9947 | 0.7474 | 0.7059 | 0.7370   |
| 12  | 0.9978 | 0.9986 | 0.9987 | 0.8253 | 0.7974 | 0.8282   |
| 24  | 0.9857 | 0.9982 | 0.9972 | 0.8469 | 0.8396 | 0.8682   |
| 36  | 0.9385 | 0.9978 | 0.9913 | 0.8132 | 0.8760 | 0.8942   |
| 48  | 0.7726 | 0.9934 | 0.9666 | 0.7427 | 0.9045 | 0.8829   |
| 60  | 0.4544 | 0.9854 | 0.8779 | 0.5291 | 0.9177 | 0.8063   |
| 128 | 8   | 0.9967 | 0.9965 | 0.9988 | 1     | 0.9998 | 1        |
| 24  | 1   | 1     | 1     | 1      | 0.9998 | 1      | 1        |
| 48  | 1   | 1     | 1     | 0.9998 | 1      | 1      | 1        |
| 72  | 1   | 1     | 1     | 0.9998 | 1      | 1      | 1        |
| 96  | 0.9999 | 1     | 1     | 0.9978 | 1      | 0.9999  |
| 120 | 0.9251 | 1     | 0.9997 | 0.9650 | 1      | 0.9995  |

5. Conclusion

The problem of testing the sphericity of covariance matrix is addressed in large-dimensional case. Like those in [11, 14], our test statistic is induced by inequality. Whereas, from the technical point of view, our approach mainly relies on RMT. We deduce the asymptotic properties under the null hypothesis when the sample size n and the dimensionality p both tend to infinity. Simulations indicate whether the underlying distribution is normal or not, the proposed test statistic appear to perform better in most situations.

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