LIOUVILLE THEOREM FOR MHD SYSTEM AND ITS APPLICATIONS

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Abstract. In this paper, we construct Liouville theorem for the MHD system and apply it to study the potential singularities of its weak solution. And we mainly study weak axi-symmetric solutions of MHD system in $\mathbb{R}^3 \times (0, T)$.

1. Introduction.

1.1. Model and related works. Let $\Omega \subset \mathbb{R}^n$ be smooth bounded domain. The $n$-dimensional incompressible magnetohydrodynamics (MHD) system are the following coupled equations

\[
\begin{cases}
  u_t - \mu_1 \Delta u + u \cdot \nabla u + \nabla (p + \frac{1}{2}|b|^2) = b \cdot \nabla b, \\
  b_t - \mu_2 \Delta b + u \cdot \nabla b = b \cdot \nabla u, \\
  \text{div } u = 0, \text{div } b = 0,
\end{cases}
\]

where $u : \Omega \times (0, T) \mapsto \mathbb{R}^n$ is fluid velocity, $p : \Omega \times (0, T) \mapsto \mathbb{R}$ is the press and $b : \Omega \times (0, T) \mapsto \mathbb{R}^n$ is the magnetic field. $\mu_1, \mu_2$ are two positive constants. For simplicity, we denote $\Pi = p + \frac{1}{2}|b|^2$. And, along with (1.1), the initial and boundary values are:

\[
\begin{align*}
  u(x, 0) &= u_0(x), \quad b(x, 0) = b_0, \quad \text{for all } x \in \Omega \\
  u &= 0, \quad b \cdot \mathbf{n} = 0, \quad \text{on } \partial \Omega
\end{align*}
\]

for a given datum $(u_0, b_0) : \Omega \mapsto \mathbb{R}^n \times \mathbb{R}^n$, with $\text{div } u_0 = 0$ and $\text{div } b_0 = 0$ and $\mathbf{n}$ is the outward normal on $\partial \Omega$.

There are lots of works on the solution of the MHD equations (1.1). In particular, Duvaut and Lions [4] constructed a class of global weak solutions and the local strong solutions to the initial boundary value problem, and Sermange and Temam in [17] discussed some properties of such solutions. For the 2-dimensional case, the smoothness and uniqueness of solutions have been shown. But for $n$-dimensional ($n \geq 3$), the problem is still open in general case like Navier-Stokes equations

\[
\begin{cases}
  v_t - \Delta v + v \cdot \nabla v + \nabla q = 0, \\
  \text{div } v = 0,
\end{cases}
\]

where $v$ is fluid velocity and $q$ is the press. For Navier-Stokes equations, many regularity criteria have been established (e.g. [18, 19, 1, 24, 9]), and some of these criteria can be extended to the 3-D MHD equations by assumptions only on $u$, see

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He and Xin [8]. We note that 3-D MHD system need assumptions both on $u$ and $b$ for Ladyzhenskaya-Prodi-Serrin class $L_{3,\infty}$, see Mahalov, Nicolaenko and Shilkin [14]. But there is still a gap between the case of existence and the case of regularity. Scheffer [16] began to study the partial regularity theory of Navier-Stokes equations. Scheffer’s works were improved by Caffarelli-Kohn-Nirenberg [2], Tian-Xin [21]. We mention that, according to [2, 21], the 1-D Hausdorff measure of the blow-up set must be zero. For self-similar singularities in the Navier-Stokes equations, the work of Nečas, Ružička and Šverák [15] and Tsai, Tai-Peng [22] showed the trivial solutions with some integration conditions. And then Koch, Nadirashvili, Seregin and Šverák [11] directly studied the potential singularities of weak axi-symmetric solutions in $\mathbb{R}^3 \times (0, T)$, by Liouville theorem for the Navier-Stokes equations. For MHD system, He and Xin [7] extend the result of [21] to it. And [14] extend the result of [15] and [22] to the MHD system. Moreover, Lei [13] constructed one kind of smooth axially symmetric solutions $(u_\theta = b_r = b_z = 0)$ of MHD in three dimensions. In this paper, we construct Liouville theorem for the MHD system and apply it to study the potential singularities of its weak solution. Because the term $b \cdot \nabla b$ in the equations velocity $u$ and equations about magnetic field $b$, the problem becomes more complicated.

1.2. Main result and outline. Under some conditions about $u, b$, we prove the Liouville Theorems for MHD system. And, we mainly study weak axi-symmetric solutions of the MHD system in $\mathbb{R}^3 \times (0, T)$. For 2-D case, with the condition $b_{2,1} - b_{1,2} = 0$ or $b_1 = 0$ or $b_2 = 0$ or $b_1 u_2 = b_2 u_1$, we have the Liouville Theorems for the bounded weak solutions of MHD system. Meanwhile, with the integral condition $|u| + |b| \in L^{s,r}_{x,t}(\mathbb{R}^2 \times (-\infty, 0))(2/s + 2/r \geq 1, s \geq 3, 3 \leq r \leq < \infty)$, we prove the Liouville Theorem for the solutions in $C^{2,1}$. For 3-D case, with the conditions $u_\theta = 0$ (no swirl), $b_r = b_z = 0$ or $|u(x, t)| \leq \frac{C}{\sqrt{x_1^2 + x_2^2}}$, $b_r = 0$ or $|u(x, t)| \leq \frac{C}{\sqrt{x_1^2 + x_2^2}}$, $u_r b_z = u_z b_r$, we have the Liouville Theorems for the weak bounded axi-symmetric solutions of the MHD system. And, when $\mu_1 = \mu_2$, with the conditions $u_{z,z} = b_{z,z} = 0$, we get the Liouville Theorem for the weak bounded axi-symmetric solutions. Moreover, when $u = 0$, we obtain that $b$ must be a constant vector, this means $b$ is smoother than $u$, see section 6. As in [11], by scaling transformations and the Liouville Theorems which we have proved, we proved the regularity for axi-symmetric solutions of MHD systems with the conditions $|u(x, t)| \leq \frac{C}{\sqrt{x_1^2 + x_2^2}}$, ”$b_z$ is bounded in $\mathbb{R}^3 \times (0, T)$” or some other conditions.

The paper is organized as follows. In section 2, we introduce the strong maximum principle which is very essential in this article. In section 3, we introduce the mild solution and bounded weak solution and their properties, for Stokes equations and heat equations. In section 4, we study the regularity of mild solution and bounded weak solution of MHD system, and the limit properties of bounded mild solution of MHD system. In section 5, we construct the Liouville theorem for MHD system, we mention that we need conditions of $b$ to prove the Liouville theorem in $\mathbb{R}^2 \times (-\infty, 0)$. And for 3-dimensional case, we study the axi-symmetric solutions of MHD system. In section 6, we apply the Liouville theorem constructed in section 5, to study the potential singularities of the finite time weak solution of MHD system.

2. maximum principle. The strong maximum principle plays essential role in this article, and we mainly use the form in [11]. For reader’s convenient, we write
it as Lemma 2.1. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and \( T > 0 \). We consider the parabolic equation

\[
    u_t + a(x,t) \nabla u - \Delta u = 0
\]

in \( Q = \Omega \times (0,T) \). Besides \( a \in L^\infty_{x,t}(\Omega \times (0,T)) \). And we mention that \( a \) is a scalar valued function.

**Lemma 2.1.** [11, Lemma 2.1] Assume that \( u \) is a bounded solution of the equation (2.1). Let \( K \subset \Omega \subset \Omega' \subset \Omega \), and \( \tau > 0 \). Let \( M = \sup_{\Omega \times (0,T)} |u| \). Then, for each \( \varepsilon > 0 \), there exists \( \delta = \delta(\Omega, \Omega', K, \tau, \varepsilon) > 0 \) such that \( \sup_{x \in K} |u| \geq M(1-\delta) \), then \( u(x,t) \geq M(1-\varepsilon) \) in \( \Omega' \times (\tau,T) \).

3. Mild solution and bounded weak solution for linear case. Let \( u = (u_1, \cdots, u_n), b = (b_1, \cdots, b_n) : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}^n \). We first consider the following linear equations about cauchy problem:

\[
    \begin{cases}
        u_t - \Delta u + \nabla \Pi = \frac{\partial}{\partial x_k} f_k , \\
        b_t - \Delta b = \frac{\partial}{\partial x_k} g_k , \\
        \text{div } u = 0, \text{div } b = 0 ,
    \end{cases}
\]

in \( \mathbb{R}^n \times (0, \infty) \), and

\[
    u(\cdot, 0) = u_0, \quad b(\cdot, 0) = b_0 \quad \text{in } \mathbb{R}^n
\]

Here \( f_k = (f_{1k}, \cdots, f_{nk}), g_k = (g_{1k}, \cdots, g_{nk}) \) for \( k = 1, \cdots, n \). Let \( P \) denote the Helmholtz projection of vector fields on divergence free fields and let \( S \) be solution operator of the heat equation. Then we get the formula

\[
    u(t) = S(t)u_0 + \int_0^t S(t-s)P\frac{\partial}{\partial x_k} f_k(s)ds , \tag{3.3}
\]

\[
    b(t) = S(t)b_0 + \int_0^t S(t-s)\frac{\partial}{\partial x_k} g_k(s)ds , \tag{3.4}
\]

where \( u(t), b(t) \) denote the two functions \( u(\cdot,t), b(\cdot,t) \).

These can be written as more clearly in terms of some kernels. We first deal with the formula about \( u \). This is similar to the [11], and for reader’s convenient, we write it as follows. Let

\[
    K_{ij}(x,t) = \left( -\delta_{ij} \Delta + \frac{\partial^2}{\partial x_i \partial x_j} \right) \Phi(x,t) , \tag{3.5}
\]

where \( \Phi \) is defined in terms of the fundamental solution of Laplace operator \( G \) and the heat kernel \( \Gamma \):

\[
    \Phi(x,t) = \int_{\mathbb{R}^n} G(y) \Gamma(x-y,t)dy ,
\]

then let

\[
    K_{ijk} = \frac{\partial}{\partial x_k} K_{ij} ,
\]

then we can rewrite the equality (3.3) as

\[
    u_i(x,t) = \int_{\mathbb{R}^n} \Gamma(x-y,t)u_{0i}(y)dy + \int_0^t \int_{\mathbb{R}^n} K_{ijk}(x-y,t-s)f_{jk}(y,s)dyds \tag{3.6}
\]
Note that $K_{ij}$ has the following estimates:

$$|K_{ij}(x, t)| \leq \frac{C}{(|x|^2 + t)^n/2} \quad (3.7)$$

$$|\nabla_x K_{ij}(x, t)| \leq \frac{C}{(|x|^2 + t)(\alpha + 1)/2} \quad (3.8)$$

Now we deal with (3.4). From the theory of heat equation, one can easily rewrite it as

$$b(x, t) = \int_{\mathbb{R}^n} \Gamma(x - y, t)b_0(y)dy - \int_0^t \int_{\mathbb{R}^n} \Gamma_k(x - y, t - s)g_k(y, s)dyds \quad (3.9)$$

where

$$\Gamma_k = \frac{\partial}{\partial x_k} \Gamma.$$ 

Now we give the definition of mild solution and bounded weak solution of the linear system (3.1) and (3.2).

**Definition 3.1.** If $(u(t), b(t))$ is continuous and satisfies the formulas (3.6) and (3.9), then we call it a mild solution of the Cauchy problem (3.1).

Let $V_T = \{ \phi : \mathbb{R}^n \times (0, T) \to \mathbb{R}^n; \phi \text{ is smooth and } \text{div } \phi = 0 \} \quad (3.10)$

**Definition 3.2.** Let $u, b : \mathbb{R}^n \times (0, T) \to \mathbb{R}^n$ be two vector fields in the space $L_{x,t}^\infty$. If $u, b$ satisfy:

(i) $\text{div } u = 0, \text{div } b = 0$ in $\mathbb{R}^n \times (0, T)$ (in the sense of distribution)

(ii) for each $\phi \in V_T$ (see (3.10))

$$\int_0^T \int_{\mathbb{R}^n} u(\phi_t + \Delta \phi)dxdt = \int_0^T \int_{\mathbb{R}^n} f_k \frac{\partial}{\partial x_k} \phi dxdt.$$

(iii) for each $\psi \in C_0^\infty(\mathbb{R}^n \times (0, T))$

$$\int_0^T \int_{\mathbb{R}^n} b(\psi_t + \Delta \psi)dxdt = \int_0^T \int_{\mathbb{R}^n} g_k \frac{\partial}{\partial x_k} \psi dxdt.$$

Then we call $(u, b)$ is a bounded weak solution of the Cauchy problem (3.1).

Let $f, g \in L_{x,t}^\infty$, we have some standard estimates for $u, b$. In particular, if $u_0 = 0, b_0 = 0$, then for any $\alpha \in (0, 1)$ and $p \in (0, \infty)$

$$||u||_{C^\alpha_{par}(Q(z_0, R))} \leq C(\alpha, R)||f||_{L_{x,t}^\infty(\mathbb{R}^n \times (0, T))}, \quad (3.11)$$

$$||\nabla_x u||_{L_{x,t}^p(Q(z_0, R))} \leq C(p, R)||f||_{L_{x,t}^\infty(\mathbb{R}^n \times (0, T))}, \quad (3.12)$$

$$||b||_{C^\alpha_{par}(Q(z_0, R))} \leq C(\alpha, R)||g||_{L_{x,t}^\infty(\mathbb{R}^n \times (0, T))}, \quad (3.13)$$

$$||\nabla_x b||_{L_{x,t}^p(Q(z_0, R))} \leq C(p, R)||g||_{L_{x,t}^\infty(\mathbb{R}^n \times (0, T))}. \quad (3.14)$$

where $Q(z_0, R) = Q((x_0, t_0), R) = B(x_0, R) \times (t_0 - R^2, t_0)$ is any parabolic ball contained in $\mathbb{R}^n \times (0, \infty)$. And the space $C^\alpha_{par}$ is defined by means of the parabolic distance $\sqrt{|x - x'|^2 + |t - t'|}$.

Taking difference on both sides of the equations, we obtain:

$$||\nabla_x u||_{C^\alpha_{par}(Q(z_0, R))} \leq C(\alpha, R)||\nabla_x f||_{L_{x,t}^\infty(\mathbb{R}^n \times (0, T))}, \quad (3.15)$$

$$||\nabla_x b||_{L_{x,t}^p(Q(z_0, R))} \leq C(p, R)||\nabla_x f||_{L_{x,t}^\infty(\mathbb{R}^n \times (0, T))}, \quad (3.16)$$

$$||\nabla_x b||_{C^\alpha_{par}(Q(z_0, R))} \leq C(\alpha, R)||\nabla_x g||_{L_{x,t}^\infty(\mathbb{R}^n \times (0, T))}. \quad (3.17)$$
4. Bounded solutions of MHD.\[4\]

The system (3.1).

Lemma 3.4. For fixed \( f, g \in L^\infty_{x,t}(\mathbb{R}^n \times (0, T)) \), let \( u, b \in L^\infty_{x,t}(\mathbb{R}^n \times (0, T)) \) be any weak solution of (3.1) in \( \mathbb{R}^n \times (0, T) \). And let \( v, e \) be the mild solution of the Cauchy problem (3.1) and (3.2) with \( u_0 = 0, b_0 = 0 \). Then

\[
\begin{align*}
\|\nabla_x b_t\|_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))} & \leq C(p, R)\|\nabla_x g\|_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))}, \\
\|\nabla_x u_t\|_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))} & \leq C(T, k)\|\nabla^{k+2}_x f\|_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))}, \\
\|\nabla_x u\|_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))} & \leq C(T, k)\|\nabla^{k+2}_x g\|_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))},
\end{align*}
\]

for \( u_0 = 0, b_0 = 0 \).

Remark 3.3. The estimates (3.3), (3.5), (3.6), (3.7), (3.8), (3.11), (3.12), (3.15), (3.16), (3.19) are from the theory of linear Stokes equations, see [11, 20, 6]. Estimates (3.4), (3.9), (3.13), (3.14), (3.17), (3.18), (3.20) are from the theory of heat kernel, see [12, 6, 18].

Now, we consider the relation between mild solution and bounded weak solution of the system (3.1).

Lemma 3.4. For fixed \( f, g \in L^\infty_{x,t}(\mathbb{R}^n \times (0, T)) \), let \( u, b \in L^\infty_{x,t}(\mathbb{R}^n \times (0, T)) \) be any weak solution of (3.1) in \( \mathbb{R}^n \times (0, T) \). And let \( v, e \) be the mild solution of the Cauchy problem (3.1) and (3.2) with \( u_0 = 0, b_0 = 0 \). Then

\[
\begin{align*}
\nabla_x u &= 0, \\
\nabla_x b &= 0,
\end{align*}
\]

where \( u_1, w_2 \) satisfy the heat equations \( \nabla_x u - \Delta w = 0 \) in \( \mathbb{R}^n \times (0, T) \) and \( d_1 \) is bounded measurable \( \mathbb{R}^n \)-valued functions on \( (0, T) \). Moreover,

\[
\begin{align*}
\|u_1\|_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))} & \leq C(T)\|u\|_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))}, \\
\|d_1\|_{L^\infty(0, T)} & \leq C(T)\|u\|_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))}, \\
\|w_2\|_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))} & \leq C(T)\|b\|_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))},
\end{align*}
\]

Proof. The proof is similar to the way of G.Koch, N.Nadirashvili, G.A.Seregin, V.Šverák(see [11], Lemma 3.1), so we omit it.

4. Bounded solutions of MHD. Now we consider the Cauchy problem for the MHD system:

\[
\begin{align*}
\begin{cases}
\begin{align*}
\nabla_x u - \Delta u + u \cdot \nabla u + \nabla \Pi &= b \cdot \nabla b, \\
\nabla_x b - \Delta b + u \cdot \nabla b &= b \cdot \nabla u, \\
\text{div } u &= 0, \\
\text{div } b &= 0,
\end{align*}
\end{cases} \\
u(\cdot, 0) = u_0, \quad b(\cdot, 0) = b_0, \quad \text{in } \mathbb{R}^n
\end{align*}
\]

(4.1) (4.2)

The considerations of the section 3 can be repeated with \( f_k = -u_k u + b_k b, \quad g_k = -u_k b + b_k u \). Moreover, if the solutions of (4.1) and (4.2) \( u, b \) are in \( L^\infty_{x,t}(\mathbb{R}^n \times (0, T)) \), then its definitions of mild solution and bounded weak solution follow as the definition 3.1 and the definition 3.2.

We define two bilinear form as follows:

\[
\begin{align*}
B_1 : L^\infty_{x,t}(\mathbb{R}^n \times (0, T)) \times L^\infty_{x,t}(\mathbb{R}^n \times (0, T)) & \rightarrow L^\infty_{x,t}(\mathbb{R}^n \times (0, T)), \\
B_2 : L^\infty_{x,t}(\mathbb{R}^n \times (0, T)) \times L^\infty_{x,t}(\mathbb{R}^n \times (0, T)) & \rightarrow L^\infty_{x,t}(\mathbb{R}^n \times (0, T)),
\end{align*}
\]

where

\[
\begin{align*}
B_1(u, v)(x, t) &= \int_0^t \int_{\mathbb{R}^n} K_{ij}(x - y, t - s)u_j(y, s)v_i(y, s)dyds, \\
B_2(u, v)(x, t) &= \int_0^t \int_{\mathbb{R}^n} \Gamma_k(x - y, t - s)u_k(y, s)v(y, s)dyds.
\end{align*}
\]

(4.3) (4.4)
And let $U_1, U_2$ be the heat extension of the initial datum $u_0, b_0$. Then the solutions $u, b$ become
\[ u = U_1 + B_1(u, u) - B_1(b, b), \]
\[ b = U_2 + B_2(u, b) - B_2(b, u). \]

Since estimate (3.8), one can easily obtain
\[ \|B_1(u, v)\|_{L^\infty_t\left(L^r_x(\mathbb{R}^n \times (0, T))\right)} \leq C\sqrt{T}\|u\|_{L^\infty_t\left(L^r_x(\mathbb{R}^n \times (0, T))\right)}\|v\|_{L^\infty_t\left(L^r_x(\mathbb{R}^n \times (0, T))\right)}, \tag{4.5} \]

and by heat kernel theory, we also get
\[ \|B_2(u, v)\|_{L^\infty_t\left(L^r_x(\mathbb{R}^n \times (0, T))\right)} \leq C\sqrt{T}\|u\|_{L^\infty_t\left(L^r_x(\mathbb{R}^n \times (0, T))\right)}\|v\|_{L^\infty_t\left(L^r_x(\mathbb{R}^n \times (0, T))\right)}. \tag{4.6} \]

Now we give some regularity properties of mild solutions in $L^\infty_{x,t}(\mathbb{R}^n \times (0, T))$.

**Lemma 4.1.** Let $(u, b) \in L^\infty_{x,t}(\mathbb{R}^n \times (0, T))$ be a mild solution of (4.1) and (4.2) with $u_0, b_0 \in L^\infty$. Then for $k, l = 0, 1, \cdots$ the functions $t^{k/2+1}\nabla_x^k\partial_t^lu, t^{k/2+1}\nabla_x^k\partial_t^lb$ are bounded for $T' = \varepsilon(k, l)(\|u_0\|_{L^\infty(\mathbb{R}^n)} + \|b_0\|_{L^\infty(\mathbb{R}^n)})^2$ (where $\varepsilon(k, l) > 0$ is a small constant), and we have
\[ \|t^{k/2+1}\nabla_x^k\partial_t^lu\|_{L^\infty_t\left(L^r_x(\mathbb{R}^n \times (0, T'))\right)} \leq C(k, l)(\|u_0\|_{L^\infty(\mathbb{R}^n)} + \|b_0\|_{L^\infty(\mathbb{R}^n)}), \tag{4.7} \]
\[ \|t^{k/2+1}\nabla_x^k\partial_t^lb\|_{L^\infty_t\left(L^r_x(\mathbb{R}^n \times (0, T'))\right)} \leq C(k, l)(\|u_0\|_{L^\infty(\mathbb{R}^n)} + \|b_0\|_{L^\infty(\mathbb{R}^n)}). \tag{4.8} \]

**Proof.** We use the method of local existence and uniqueness of the solution to Navier-Stokes equations (see [6, 10]). For simplicity, we just give the estimate of the $u, b, \nabla u, \nabla b, \partial_t u, \partial_t b$. We define the approximation $u_j, b_j (j = 1, 2, \cdots)$ by
\[ u_1(t) = S(t)u_0, \]
\[ b_1(t) = S(t)b_0, \]
\[ u_{j+1}(t) = S(t)u_0 + B_1(u_j, u_j)(\cdot, t) - B_1(b_j, b_j)(\cdot, t), \]
\[ b_{j+1}(t) = S(t)b_0 + B_2(u_j, b_j)(\cdot, t) - B_2(b_j, u_j)(\cdot, t), \]
where $u_0, b_0$ are the initial data of the MHD system and the definition of operator $B_1$ and $B_2$ follows (4.3) and (4.4). Let
\[ K_j = K_j(T) = \sup_{0 \leq t \leq T} (\|u_j\|_{L^\infty(\mathbb{R}^n)} + \|b_j\|_{L^\infty(\mathbb{R}^n)}), \]
\[ K'_j = K'_j(T) = \sup_{0 \leq t \leq T} (t^{1/2}\|\nabla u_j\|_{L^\infty(\mathbb{R}^n)} + t^{1/2}\|\nabla b_j\|_{L^\infty(\mathbb{R}^n)}), \]
and
\[ K''_j = K''_j(T) = \sup_{0 \leq t \leq T} (t\|\partial_t u_j\|_{L^\infty(\mathbb{R}^n)} + t\|\partial_t b_j\|_{L^\infty(\mathbb{R}^n)}). \]

Note that $K_0 = \|u_0\|_{L^\infty(\mathbb{R}^n)} + \|b_0\|_{L^\infty(\mathbb{R}^n)}$. By heat kernel theory and the estimates (4.5) and (4.6), we have
\[ K_{j+1}(T) \leq K_0 + C_1T^{1/2}K_j(T)^2, \]
\[ K'_{j+1}(T) \leq CK_0 + C_2T^{1/2}K'_j(T)K_j(T), \]
and
\[ K''_{j+1}(T) \leq CK_0 + C_3T^{1/2}K''_j(T)K_j(T) + C_4TK_j^2(T). \]

We take $T_1$ small so that max, $4C_1T_1^{1/2}K_0 < 1$ and $T^{1/2} < 1/2$, then it is easy to prove that
\[ \sup_{j} K_j(T) \leq 2K_0, \quad \sup_{j} K'_j(T) \leq 2CK_0 \quad \text{and} \quad \sup_{j} K''_j(T) \leq 2CK_0 \tag{4.9} \]
for any $T \leq T_1(C \geq 1)$. Let

$$L_j(T) = \sup_{0 \leq t \leq T} (|u_j(t) - u_j(t)|_{L^\infty(\mathbb{R}^n)} + |b_j(t) - b_j(t)|_{L^\infty(\mathbb{R}^n)}),$$

$$L_j'(T) = \sup_{0 \leq t \leq T} (t^{1/2}||\nabla u_j(t) - \nabla u_j(t)||_{L^\infty(\mathbb{R}^n)} + t^{1/2}||\nabla b_j(t) - \nabla b_j(t)||_{L^\infty(\mathbb{R}^n)}),$$

and

$$L_j''(T) = \sup_{0 \leq t \leq T} (t||\partial_t u_j(t) - \partial_t u_j(t)||_{L^\infty(\mathbb{R}^n)} + t||\partial_t b_j(t) - \partial_t b_j(t)||_{L^\infty(\mathbb{R}^n)}),$$

By direct calculation, we have

$$L_{j+1}(T) \leq C_5 T^{1/2} K_0 L_j(T),$$

$$L_{j+1}'(T) \leq C_6 T^{1/2} K_0 (L_j(T) + L_j'(T)),$$

$$L_{j+1}''(T) \leq C_7 T^{1/2} K_0 (L_j(T) + L_j'(T)) + C_8 T K_0 L_j(T)$$

for $T \leq T_1$ with $C_5, C_6, C_7, C_8$ independent of $K_0, T$. Take $T_2 \leq T_1$ small such that $(C_5 + C_6 + C_7 + C_8 T^{1/2}) T_2^{1/2} K_0 < 1/2$ (that means $T_2 < CK_0^{-2}$), then $L_{j+1}(T)/L_j(T) < 1/2$, $(L_{j+1}'(T) + L_{j+1}(T))/L_j(T) < 1/2$, $(L_{j+1}'(T) + L_{j+1}(T))/L_j(T) < 1/2$ for any $T \leq T_2$ and $j \geq 1$. Therefore, $\{L_j(T)\}_{j \geq 1}$ and $\{L_j'(T)\}_{j \geq 1}$ converge to 0 as $j \to \infty$, then we conclude that the approximation $\{u_j(t)\}_{j \geq 1}$, $\{b_j(t)\}_{j \geq 1}$, $\{\nabla u_j(t)\}_{j \geq 1}$, $\{\nabla b_j(t)\}_{j \geq 1}$, $\{t \partial_t u_j(t)\}_{j \geq 1}$, $\{t \partial_t b_j(t)\}_{j \geq 1}$ respectively has a unique limit function $u(t), b(t), v_1(t), v_2(t), d_1(t), d_2(t)$. The uniqueness of solution can be proved by estimating the difference $\sup_{0 \leq t \leq T_2} (|u_j(t) - u_j(t)|_{L^\infty(\mathbb{R}^n)} + |b_j(t) - b_j(t)|_{L^\infty(\mathbb{R}^n)})$ of two solutions $(u_1, b_1)$ and $(u_2, b_2)$, where $T_2$ is very small (use the equation (3.6) and (3.9)). Finally, by the estimate (4.9), we obtain the result.

**Lemma 4.2.** Let $u^{(k)}, b^{(k)} \in L^\infty_{x,t}(\mathbb{R}^n \times (0, T))$ be a sequence of mild solution of (4.1) and (4.2) with initial conditions $u_0^{(k)}, b_0^{(k)}$. Assume that $||u^{(k)}||_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))} + ||b^{(k)}||_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))} \leq C$, where $C$ is independent of $k$. Then a subsequence of the sequence $u^{(k)}$ converges locally uniformly in $\mathbb{R}^n \times (0, T)$ to a mild solution $u \in L^\infty_{x,t}(\mathbb{R}^n \times (0, T))$ with initial datum $u_0$, where $u_0$ is the weak* limit of a suitable subsequence of the sequence $u_0^{(k)}$.

**Proof.** It is easy to get this result by lemma 4.1 and the decay estimates (3.8) and heat kernel theory.

We now consider the regularity of bounded weak solutions of (4.1) and (4.2). Let $u, b \in L^\infty_{x,t}(\mathbb{R}^n \times (0, T))$ be the weak solutions of (4.1) in $\mathbb{R}^n \times (0, T)$, and let $M = \|u\|_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))} + \|b\|_{L^\infty_{x,t}(\mathbb{R}^n \times (0, T))}$. Then, like [11], we have the following estimates:

$$||\nabla u||_{L^\infty_{x,t}(Q(z_0, R))} + ||\nabla b||_{L^\infty_{x,t}(Q(z_0, R))} \leq C(k, \delta, R, M),$$

(4.10)

for each $Q(z_0, R) \subset \mathbb{R}^n \times (\delta, T)$ and $k = 0, 1, 2, \cdots$.

And for each $k = 0, 1, 2, \cdots$, we also have

$$||\nabla u||_{L^\infty_{x,t}(\mathbb{R}^n \times (\delta, T))} + ||\nabla b||_{L^\infty_{x,t}(\mathbb{R}^n \times (\delta, T))} \leq C(k, \delta, T, M)$$

(4.11)

The we obtain

$$||\nabla \partial_t (u - d_1)||_{L^\infty_{x,t}(\mathbb{R}^n \times (\delta, T))} + ||\nabla \partial_t b||_{L^\infty_{x,t}(\mathbb{R}^n \times (\delta, T))} \leq C(k, \delta, T, M)$$

(4.12)

for each $k = 0, 1, 2, \cdots$.\[\]
5. Liouville theorem in MHD. First, we consider the MHD system in two dimensions space.

**Theorem 5.1.** Let \((u, b)\) be a bounded weak solution of the MHD system in \(\mathbb{R}^2 \times (-\infty, 0)\). If one of the following conditions is satisfied:

1. \(b_{2,1} - b_{1,2} = 0\) in \(\mathbb{R}^2 \times (-\infty, 0)\);
2. \(b_1 = 0\) in \(\mathbb{R}^2 \times (-\infty, 0)\);
3. \(b_2 = 0\) in \(\mathbb{R}^2 \times (-\infty, 0)\);
4. \(b_1u_2 = b_2u_1\) in \(\mathbb{R}^2 \times (-\infty, 0)\).

then \(u(x, t) = d_1(t), b(x, t) = d_0\), where \(d_1\) is bounded measurable functions from \((-\infty, 0)\) to \(\mathbb{R}^2\) and \(d_0\) is a constant vector.

**Proof.** In two dimensions space, the vorticity is a scalar, which is defined by

\[\omega = u_{2,1} - u_{1,2}\]  
(5.1)

where \(u_{k,j} = \frac{\partial u_k}{\partial x_j}\), that is say, the indices after comma mean derivatives. For magnetic field, we also use the definition of vorticity

\[\omega' = b_{2,1} - b_{1,2}\]  
(5.2)

We first consider the first equation of system (4.1). Then the vorticity \(\omega\) satisfies

\[\omega_t + u \cdot \nabla \omega - \Delta \omega = b \cdot \nabla \omega'\]  
(5.3)

Through conditions (1)-(4), we aim to prove the term \(b \nabla \omega'\) is bounded in \(\mathbb{R}^2 \times (-\infty, 0)\). According to section 4, we know that \(\nabla_b^k u, \nabla_b^k b\) are bounded.

1. If condition (1) holds, we have \(b \nabla \omega' = 0\) in \(\mathbb{R}^2 \times (-\infty, 0)\), then equation (5.3) can be written as

\[\omega_t + u \cdot \nabla \omega - \Delta \omega = 0\]  
(5.4)

This is similar to the Navier-Stokes equations, we know that \(u(x, t) = d_1(t)\), where \(d_1\) is bounded measurable functions from \((-\infty, 0)\) to \(\mathbb{R}^2\) (see [11], Theorem 5.1).

With \(\omega' = b_{2,1} - b_{1,2} = 0\) and \(\text{div} \, b = 0\), we find that \(b_1, b_2\) are harmonic functions, then \(b\) is constant in \(x\) for each \(t\) by the classical Liouville theorem. Take it into the second equation of the system (4.1), we find that \(b\) is constant in \(x\) and \(t\).

2. If condition (2) holds, then \(b_{2,2} \equiv 0\), thus \(b \cdot \nabla b \equiv 0\). Therefore, the equation (5.3) can be written as (5.4), so \(u(x, t) = d_1(t)\), where \(d_1\) is bounded measurable functions from \((-\infty, 0)\) to \(\mathbb{R}^2\). Then, the second equation can be written as

\[b_t - \Delta b + u \cdot \nabla b = 0\]  
(5.5)

Then \(b_{1,1}\) satisfies

\[(b_{1,1})_t - \Delta b_{1,1} + u \cdot \nabla b_{1,1} = 0\]  
(5.6)

Let

\[M_1 = \sup_{\mathbb{R}^2 \times (-\infty, 0)} b_{1,1} \quad \text{and} \quad M_2 = \inf_{\mathbb{R}^2 \times (-\infty, 0)} b_{1,1},\]

and assume that \(M_1 > 0\). Applying Lemma 2.1 to \(b_{1,1} - \frac{1}{2}(M_1 + M_2)\), we get that there exist arbitrarily large parabolic balls \(Q((\bar{x}, \bar{t}), R) = B(\bar{x}, R) \times (\bar{t} - R^2, \bar{t}) \subset \mathbb{R}^2 \times (-\infty, 0)\) such that \(b_{1,1} \geq \frac{1}{2}M_1\) in \(Q((\bar{x}, \bar{t}), R)\). For such parabolic balls, we have

\[\int_{Q_R} b_{1,1} \, dx \, dt \geq \frac{1}{2} \pi M_1 R^4.\]  
(5.7)

But, on the other hand, we can obtain

\[\int_{Q_R} b_{1,1} \, dx \, dt = \int_{Q_R} b_1 n_1 \, dx \, dt \leq CR^3\]  
(5.8)
where $n$ is the normal to the boundary of $B(\bar{x}, R)$. When $R$ is big enough, we find that (5.6) contradicts to (5.7), unless $M_1 \leq 0$. By the same way, we conclude that $M_2 \geq 0$. Therefore, $b_{1,1} = 0$ in $\mathbb{R}^2 \times (-\infty, 0)$. In the same way, we conclude that $b_{1,2} = 0, b_{2,1} = 0, b_{2,2} = 0$ in $\mathbb{R}^2 \times (-\infty, 0)$. Therefore, $b$ is constant in $x$ for each $t$. Take it into the second equation of the system (4.1), we find that $b$ is constant in $x$ and $t$.

3. If the condition (3) holds, then the proof is similar to (2), we omit it.

4. If the condition (4) holds, then the second equation of (4.1) can be written as

$$b_t - \Delta b = 0$$

Then $b_{1,1}$ satisfies

$$(b_{1,1})_t - \Delta b_{1,1} = 0$$

Let

$$M_1 = \sup_{\mathbb{R}^2 \times (-\infty, 0)} b_{1,1} \quad \text{and} \quad M_2 = \inf_{\mathbb{R}^2 \times (-\infty, 0)} b_{1,1},$$

and assume that $M_1 > 0$. Applying Lemma 2.1 to $b_{1,1} - \frac{1}{2}(M_1 + M_2)$, we get that there exist arbitrarily large parabolic balls $Q((\bar{x}, \bar{t}), R) = B(\bar{x}, R) \times (\bar{t} - R^2, \bar{t}) \subset \mathbb{R}^2 \times (-\infty, 0)$ such that $b_{1,1} \geq \frac{1}{2} M_1$ in $Q((\bar{x}, \bar{t}), R)$. For such parabolic balls, we have $u(x, t) = d_1(t)$, where $d_1$ is bounded measurable functions from $(-\infty, 0)$ to $\mathbb{R}^2$.

$$\int_{Q_R} b_{1,1} dxdt \geq \frac{1}{2} \pi M_1 R^4. \quad (5.8)$$

But, on the other hand, we can obtain

$$\int_{Q_R} \omega^1 dxdt = \int_{Q_R} b_1 n_1 dxdt \leq CR^3 \quad (5.9)$$

where $n$ is the normal to the boundary of $B(\bar{x}, R)$. When $R$ is big enough, we find that (5.8) contradicts to (5.9), unless $M_1 \leq 0$. By the same way, we conclude that $M_2 \geq 0$. Therefore, $b_{1,1} = 0$ in $\mathbb{R}^2 \times (-\infty, 0)$. In the same way, we conclude that $b_{1,2} = 0, b_{2,1} = 0, b_{2,2} = 0$ in $\mathbb{R}^2 \times (-\infty, 0)$. Therefore, $b$ is constant in $x$ for each $t$. Take it into the second equation of the system (4.1), we find that $b$ is constant in $x$ and $t$. Therefore, $b \cdot \nabla b \equiv 0$, then we have $u(x, t) = d_1(t)$, where $d_1$ is bounded measurable functions from $(-\infty, 0)$ to $\mathbb{R}^2$. \qed

Then to next, we prove a Liouville theorem under integration condition. We get the idea from the way of dealing with Steady-state problems, see [5, 23, 3].

**Theorem 5.2.** Let $(u, b)$ be a weak solution of the MHD system in $\mathbb{R}^2 \times (-\infty, 0)$. Assume that $u, b \in C_{x,t}^{2,1}(\mathbb{R}^2 \times (-\infty, 0) : \mathbb{R}^2)$ and satisfies

$$|u| + |b| \in L^{2r}_{x,t}(\mathbb{R}^2 \times (-\infty, 0)), \quad (5.10)$$

where $2/s + 2/r \geq 1, s \geq 3, 3 \leq r < \infty$. Then, $u(x, t) = d_1(t), b(x, t) = 0$ where $d_1$ is bounded measurable functions from $(-\infty, 0)$ to $\mathbb{R}^2$.

**Proof.** We consider a standard cut-off function $\psi \in C_{c}^{\infty}(\mathbb{R})$ such that

$$\psi(y) = \begin{cases} 1, & \text{if } |y| < 1, \\ 0, & \text{if } |y| > 2, \end{cases}$$

and $0 \leq \psi(y) \leq 1$ for $1 < |y| < 2$. For each $R$, define $\phi_R(x, t) = \psi(|x|/R)\psi(t/R^2)$, $(x, t) \in \mathbb{R}^2 \times (-\infty, 0)$. 
Taking the inner product of (4.1) with $u\phi_R$, (4.1)2 with $b\phi_R$ in $L^2(\mathbb{R}^2 \times (-\infty, 0))$. Adding the two resulting integrations together, and integrating by parts, then we have
\[
\frac{1}{2} \int_{\mathbb{R}^2} (|u|^2 + |b|^2)(x, 0)\phi_R(x, 0)\, dx + \int_{\mathbb{R}^2 \times (-\infty, 0)} (|\nabla u|^2 + |\nabla b|^2)\phi_R \, dxdt
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2 \times (-\infty, 0)} (|u|^2 + |b|^2)\partial_t \phi_R \, dxdt + \frac{1}{2} \int_{\mathbb{R}^2 \times (-\infty, 0)} (|u|^2 + |b|^2)\Delta \phi_R \, dxdt
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}^2 \times (-\infty, 0)} |u|^2 (u \cdot \nabla) \phi_R \, dxdt + \frac{1}{2} \int_{\mathbb{R}^2 \times (-\infty, 0)} |b|^2 (u \cdot \nabla) \phi_R \, dxdt
\]
\[
+ \int_{\mathbb{R}^2 \times (-\infty, 0)} \Pi(u \cdot \nabla) \phi_R \, dxdt - \int_{\mathbb{R}^2 \times (-\infty, 0)} (u \cdot b)(b \cdot \nabla) \phi_R \, dxdt = 6 \sum_{i=1}^I I_i
\]
We estimate $I_i$ for $i = 1, 2, \cdots, 6$ one by one. For $I_1$, Hölder inequality implies
\[
2|I_1| \leq \frac{1}{R^2} \|\partial_t \phi_1\|_{L^\infty} \int_{-2R}^R \int_{\mathbb{R}^2} (|u|^2 + |b|^2) \, dxdt
\]
\[
\leq C \|\partial_t \phi_1\|_{L^\infty} R^{2(1-2/s-2/r)} (J_1 + J_2),
\]
Where
\[
J_1 = \left( \int_{-2R}^R \left( \int_{\mathbb{R}^2} |u|^s \, dx \right)^{r/s} \, dt \right)^{2/r},
J_2 = \left( \int_{-2R}^R \left( \int_{\mathbb{R}^2} |b|^s \, dx \right)^{r/s} \, dt \right)^{2/r}.
\]
It is easy to see that $R^{2(1-2/s-2/r)} J_1 \to 0$ and $R^{2(1-2/s-2/r)} J_2 \to 0$ as $R \to \infty$ by the condition (5.10). In the same way, we can estimate $I_2$. For $I_3$, we have
\[
2|I_3| \leq \frac{1}{R} \|\nabla \phi_1\|_{L^\infty} \int_{-\infty}^0 \int_{B_{2R} \setminus B_R} |u|^3 \, dxdt
\]
\[
\leq C \|\nabla \phi_1\|_{L^\infty} R^{3(1-2/s-2/r)} \left( \int_{-\infty}^0 \left( \int_{B_{2R} \setminus B_R} |u|^s \, dx \right)^{r/s} \, dt \right)^{3/r} \to 0 \text{ as } R \to \infty.
\]
To estimate $I_4$, we first estimate the following term
\[
\frac{1}{R} \|\nabla \phi_1\|_{L^\infty} \int_{-\infty}^0 \int_{B_{2R} \setminus B_R} |b|^3 \, dxdt
\]
\[
\leq C \|\nabla \phi_1\|_{L^\infty} R^{3(1-2/s-2/r)} \left( \int_{-\infty}^0 \left( \int_{B_{2R} \setminus B_R} |b|^s \, dx \right)^{r/s} \, dt \right)^{3/r} \to 0 \text{ as } R \to \infty.
\]
Then
\[
2|I_4| \leq \frac{1}{R} \|\nabla \phi_1\|_{L^\infty} \int_{-\infty}^0 \int_{B_{2R} \setminus B_R} |b|^2 |u| \, dxdt
\]
\[
\leq \frac{1}{R} \|\nabla \phi_1\|_{L^\infty} \left( \int_{-2R}^R \int_{B_{2R} \setminus B_R} |b|^3 \, dxdt \right)^{2/3} \left( \int_{-2R}^R \int_{B_{2R} \setminus B_R} |u|^3 \, dxdt \right)^{1/3}
\]
\[
\to 0 \text{ as } R \to \infty.
\]
By the Calderón-Zygmund theorem, we have $\Pi \in L^{s/2,r/2}_{x,t}(\mathbb{R}^2 \times (-\infty, 0))$, then

$$|I_5| \leq \frac{1}{R} \|\nabla \phi_1\|_{L^\infty} \int_{-\infty}^{0} \int_{B_{2R} \setminus B_R} |\Pi| |u| dx dt$$

$$\leq \frac{1}{R} \|\nabla \phi_1\|_{L^\infty} \left( \int_{-2R}^{0} \int_{B_{2R} \setminus B_R} |\Pi|^{3/2} dx dt \right)^{2/3} \left( \int_{-2R}^{0} \int_{B_{2R} \setminus B_R} |b|^3 dx dt \right)^{1/3}$$

$$\to 0 \text{ as } R \to \infty$$

Finally, the way of estimating $I_6$ is similar to $I_4$. From the above estimations, we conclude that $|\nabla u| + |\nabla b| = 0$ almost everywhere in $\mathbb{R}^2 \times (-\infty, 0)$. With Theorem (5.1), we have the result. □

Remark 5.3. For 2-D, $\int_0^R (\int_{B_R} (|u|^s + |b|^s) dx)^{r/s} dr$, where $2/s + 2/r = 1$, $s \geq 3, 3 < r < \infty$, is scale-invariant for natural scaling of MHD equations. Therefore, we can use the Liouville Theorem 5.2 to study its regularity. Unfortunately, for $n$-D case ($3 \leq n \leq 4$), the integration condition of the Liouville Theorem is not scale-invariant by the method of Theorem 5.2, we just obtain that Liouville Theorem holds when the condition (5.10) is replaced by

$$|u| + |b| \in L^{s,r}_{x,t}(\mathbb{R}^n \times (-\infty, 0)),$$

(5.11)

where $n/s + 2/r \geq n/2, s \geq 3, 3 \leq r < \infty$. That means more decay, like, if $|u| + |b| \leq C(|x| + \sqrt{t})^{-2}$ in $\mathbb{R}^3 \times (-\infty, 0)$, then $|u| + |b| \in L^3(\mathbb{R}^3 \times (-\infty, 0))$ for the bounded weak solutions, then the Liouville theorem holds.

Since the vorticity is no longer a scalar function in three dimensions space, the problem becomes very different. But one can obtain the similar result under the additional assumption when the solutions are axi-symmetric. A vector field $u : \mathbb{R}^3 \to \mathbb{R}^3$ is called axi-symmetric if it is invariant under rotations about a suitable axis, and here we choose $x_3$-coordinate as the "suitable axis". That is to say, the velocity field $u$ is axi-symmetric if $u(Rx) = Ru(x)$ for every rotation $R$ of the form

$$R = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let

$$\vec{e}_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right)^T, \quad \vec{e}_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right)^T, \quad \vec{e}_z = (0, 0, 1)^T,$$

then, in cylindrical coordinates $(r, \theta, z)$, the axi-symmetric fields are given by

$$u = u_r \vec{e}_r + u_\theta \vec{e}_\theta + u_z \vec{e}_z$$

where the coordinate functions $u_r, u_\theta$ and $u_z$ depend only on $r, z$ and time $t$. The axi-symmetric magnetic fields also have the similar representation

$$b = b_r \vec{e}_r + b_\theta \vec{e}_\theta + b_z \vec{e}_z$$
where the coordinate functions $b_r, b_\theta$ and $b_z$ depend only on $r, z$ and time $t$. Therefore, in these coordinates, the MHD system (4.1) becomes

\begin{align}
(u_r)_t + u_r u_{r,r} + u_z u_{r,z} - \frac{u_r^2}{r} - b_r b_{r,r} - b_z b_{r,z} + \frac{b_r}{r} + \Pi, r &= \Delta u_r - \frac{u_r}{r^2}, \\
(u_\theta)_t + u_r u_{\theta,r} + u_z u_{\theta,z} + \frac{u_\theta u_r}{r} - b_r b_{\theta,r} - b_z b_{\theta,z} - \frac{b_\theta}{r} &= \Delta u_\theta - \frac{u_\theta}{r^2}, \\
(u_z)_t + u_r u_{z,r} + u_z u_{z,z} - b_r b_{z,r} - b_z b_{z,z} + \Pi, z &= \Delta u_z,
\end{align}

Therefore, in these coordinates, the MHD system (4.1) becomes

\begin{align}
\frac{(ru_r)_r}{r} + u_z &= 0, \\
(b_r)_t + u_r b_{r,r} + u_z b_{r,z} - b_r u_{r,r} - b_z u_{r,z} &= \Delta b_r - \frac{b_r}{r^2}, \\
(b_\theta)_t + u_r b_{\theta,r} + u_z b_{\theta,z} + \frac{u_\theta u_r}{r} - b_r u_{\theta,r} - b_z u_{\theta,z} &= \Delta b_\theta - \frac{b_\theta}{r^2}, \\
(b_z)_t + u_r b_{z,r} + u_z b_{z,z} - b_r u_{z,r} - b_z u_{z,z} &= \Delta b_z,
\end{align}

where $\Delta = \partial_{rr} + \partial_{rr} + \partial_{zz}$ is the scalar Laplacian (expressed in the coordinates $(r, \theta, z)$), and the indices after comma mean derivatives, i.e. $u_{r,z} = \partial u_r/\partial z$.

Then we consider the case of axi-symmetric flows without swirl in the velocity fields $(u_\theta = 0)$. As usual, $\omega = \text{curl} u$, and in cylindrical coordinates we write

$$\omega = \omega_r \hat{e}_r + \omega_\theta \hat{e}_\theta + \omega_z \hat{e}_z,$$

(5.20)

For axi-symmetric flows $u$ without swirl we have $\omega_r = \omega_z = 0$ by direct calculation. Thus we can write

$$\omega = \omega_\theta \hat{e}_\theta.$$

(5.21)

And in the magnetic fields, we assume that $b_r = b_z = 0$. where $\Delta = \partial_{rr} + \partial_{rr} + \partial_{zz}$ is the scalar Laplacian (expressed in the coordinates $(r, \theta, z)$), and the indices after comma mean derivatives, i.e. $u_{r,z} = \partial u_r/\partial z$.

**Theorem 5.4.** Let $(u, b)$ be a bounded weak solution of the MHD system in $\mathbb{R}^3 \times (0, \infty)$. Assume that $u, b$ are axi-symmetric, and $u_\theta = 0$ (no swirl), $b_r = b_z = 0$ in $\mathbb{R}^3 \times (-\infty, 0)$. Then, $u = (0,0,d_3(t))^T$ and $b = (0,0,0)^T$, where $d_3 : (-\infty, 0) \rightarrow \mathbb{R}$ is a bounded measurable function.

**Proof.** With $b_r = b_z = 0$, we can write (5.17) as

$$(b_\theta)_t + u_r b_{\theta,r} + u_z b_{\theta,z} = \frac{\partial}{\partial r} u_r = \Delta b_\theta - \frac{b_\theta}{r^2}$$

Then $b_\theta$ satisfies

$$\left( \frac{b_\theta}{r} \right)_t + u_r \left( \frac{b_\theta}{r} \right)_r + u_z \left( \frac{b_\theta}{r} \right)_z = \Delta \left( \frac{b_\theta}{r} \right) + \frac{2}{r} \left( \frac{b_\theta}{r} \right)_r.$$

(5.22)

The term on the right side of equation (5.22) can be treated as the 5-D Laplacian acting on SO(4)-invariant function in $\mathbb{R}^5$. We write $r = \sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2}$ and $y_5 = z$, and let $f(y_1, \cdots, y_5) = f(r, z)$, then we have

$$\Delta_y f(y_1, \cdots, y_5) = \left( \frac{\partial^2 f}{\partial y^2} + 3\frac{\partial f}{r \partial r} + \frac{\partial^2 f}{\partial z^2} \right)(r, z)$$
Therefore, with a slight abuse of notation, we can write the equation (5.22) as
\[
\left( \frac{b_\theta}{r} \right)_t + u_r \left( \frac{b_\theta}{r} \right)_r + u_z \left( \frac{b_\theta}{r} \right)_z = \Delta_\theta \left( \frac{b_\theta}{r} \right).
\]
(5.23)

From section 4, we know that $\nabla^2 b$ are bounded in $\mathbb{R}^3 \times (-\infty, 0)$, and by the condition, $b$ is axi-symmetric, thus $b_\theta/r$ is bounded $\mathbb{R}^3 \times (-\infty, 0)$. Let
\[
M_1 = \sup_{\mathbb{R}^3 \times (-\infty, 0)} \frac{b_\theta}{r} \quad \text{and} \quad M_2 = \inf_{\mathbb{R}^3 \times (-\infty, 0)} \frac{b_\theta}{r},
\]
and assume that $M_1 > 0$. Applying Lemma 2.1 to the solution $b_\theta/r - \frac{1}{2}(M_1 + M_2)$ of equation (5.23), considered as an equation in $\mathbb{R}^5 \times (-\infty, 0)$. With suitable centers, we see that $b_\theta/r \geq \frac{1}{2}M_1$ in arbitrarily large parabolic balls, this means $b_\theta$ is unbounded, a contradiction. Therefore, $M_1 \leq 0$. In the same way, we find that $M_2 \geq 0$. Thus $b_\theta \equiv 0$. Therefore, we can write (5.12), (5.14), and (5.15) as
\[
\begin{align*}
(u_r)_t + u_r u_{r,r} + u_z u_{r,z} + \Pi_r &= \Delta u_r - \frac{u_r}{r^2}, \\
(u_z)_t + u_r u_{z,r} + u_z u_{z,z} + \Pi_z &= \Delta u_z, \\
\left( \frac{ru_r}{r} \right)_r + u_{z,z} &= 0,
\end{align*}
\]
This is similar to the Navier-Stokes equations, and $\omega_\theta$ satisfies
\[
\left( \frac{\omega_\theta}{r} \right)_t + u_r \left( \frac{\omega_\theta}{r} \right)_r + u_z \left( \frac{\omega_\theta}{r} \right)_z = \Delta \left( \frac{\omega_\theta}{r} \right) + \frac{2}{r} \left( \frac{\omega_\theta}{r} \right)_r.
\]
Therefore, $u(x, t) = (0, 0, d_3(t))^T$ (see [11], Theorem 5.2).

Next, we will prove other conditions that makes Liouville theorem hold.

**Theorem 5.5.** Let $(u, b)$ be a bounded weak solution of the MHD system in $\mathbb{R}^3 \times (0, \infty)$. Assume that $u, b$ are axi-symmetric. $u$ satisfies
\[
|u(x, t)| \leq C \sqrt{x_1^2 + x_2^2} \quad \text{in} \quad \mathbb{R}^3 \times (-\infty, 0),
\]
and $b_z = 0$ in $\mathbb{R}^3 \times (-\infty, 0)$.

Then, $u = b = 0$ in $\mathbb{R}^3 \times (-\infty, 0)$.

**Proof.** With the condition $b_z = 0$, the equation (5.19) can be written as
\[
\frac{b_r}{r} + br_r = 0
\]
(5.25)
For fix $z, t$, (5.25) is an ordinary differential equation of $b_r$ with respect to $r$, and $b_r \equiv C r^{-1}$ or $b_r = 0$. Since $b_r$ is bounded in $\mathbb{R}^3 \times (-\infty, 0)$, thus $b_r = 0$. With $b_r = b_z = 0$ in $\mathbb{R}^3 \times (-\infty, 0)$, we conclude that $b_\theta = 0$ in $\mathbb{R}^3 \times (-\infty, 0)$ (see the proof of Theorem 5.4). Therefore, equation (5.13) can be rewrite as
\[
(u_\theta)_t + u_r u_{\theta,r} + u_z u_{\theta,z} + \frac{u_\theta u_r}{r} = \Delta u_\theta - \frac{u_\theta}{r^2},
\]
(5.26)
then we use the equations expressed in the cylindrical coordinates $(r, \theta, z)$ for the equation (5.26), and we set $f = ru_\theta$, we have
\[
f_t + u_r f_r + u_z f_z = \Delta f - \frac{2}{r} f_r,
\]
and $f \leq C$. Then we have $u_\theta = 0$ (this is the result of Theorem 5.3 in [11], we omit the proof).
Theorem 5.6. Let \((u, b)\) be a bounded weak solution of the MHD system in \(\mathbb{R}^3 \times (-\infty, 0)\). Assume that \(u, b\) are axi-symmetric. \(u\) satisfies
\[
|u(x, t)| \leq \frac{C}{\sqrt{x_1^2 + x_2^2}} \quad \text{in} \quad \mathbb{R}^3 \times (-\infty, 0),
\]
and \(u_r b_z = u_z b_r\) in \(\mathbb{R}^3 \times (-\infty, 0)\). In addition, let \(\mu_1 = \mu_2\). Then, \(u = (0, 0, 0)^T\) and \(b = (0, 0, 0)^T\).

Proof. With the condition \(u_r b_z = u_z b_r\), equations (5.16) and (5.18) can be written as
\[
(b_r)_t = \Delta b_r - \frac{b_r}{r^2}, \tag{5.28}
\]
\[
(b_z)_t = \Delta b_z. \tag{5.29}
\]
Then we rewrite (5.28) as
\[
\left(\frac{b_r}{r}\right)_t = \Delta \left(\frac{b_r}{r}\right) + \frac{2}{r}\left(\frac{b_r}{r}\right)\frac{b_r}{r}. \tag{5.30}
\]
Like (5.23), the right term of (5.30) also can be treated as Laplacian acting on \(\text{SO}(4)\)-invariant functions in \(\mathbb{R}^3\):
\[
\left(\frac{b_r}{r}\right)_t = \Delta_b \left(\frac{b_r}{r}\right). \tag{5.31}
\]
By the condition, \(b\) is axi-symmetric, and from section 4, we know that \(\nabla_z b\) are bounded in \(\mathbb{R}^3 \times (-\infty, 0)\), thus \(b_r/r\) is bounded \(\mathbb{R}^3 \times (-\infty, 0)\). Let
\[
M_1 = \sup_{\mathbb{R}^3 \times (-\infty, 0)} \frac{b_r}{r} \quad \text{and} \quad M_2 = \inf_{\mathbb{R}^3 \times (-\infty, 0)} \frac{b_r}{r},
\]
and assume that \(M_1 > 0\). Applying Lemma 2.1 to the solution \(b_r/r - \frac{1}{2}(M_1 + M_2)\) of equation (5.23), considered as an equation in \(\mathbb{R}^3 \times (-\infty, 0)\). With suitable centers, we see that \(b_r/r \geq \frac{1}{2}M_1\) in arbitrarily large parabolic balls, this means \(b_r\) is unbounded, a contradiction. Therefore, \(M_1 \leq 0\). In the same way, we find that \(M_2 \geq 0\). Thus \(b_r \equiv 0\) and \(b_z u_r \equiv 0\). Then, by the equation (5.19), \(b_{z,z} = 0\). Therefore, \(b_z\) is a function in \(\mathbb{R}^2 \times (-\infty, 0)\).

From equation (5.29), we know that \(b_{z,x_1}\) and \(b_{z,x_2}\) satisfy
\[
(b_{z,x_1})_t = \Delta b_{z,x_1}, \tag{5.32}
\]
\[
(b_{z,x_2})_t = \Delta b_{z,x_2}. \tag{5.33}
\]
From section 4, we know that \(\nabla_z b\) are bounded in \(\mathbb{R}^2 \times (-\infty, 0)\). Let
\[
M_1 = \sup_{\mathbb{R}^2 \times (-\infty, 0)} b_{z,x_1} \quad \text{and} \quad M_2 = \inf_{\mathbb{R}^2 \times (-\infty, 0)} b_{z,x_1},
\]
and assume that \(M_1 > 0\). Applying Lemma 2.1 to \(b_{z,x_1} - \frac{1}{2}(M_1 + M_2)\), we get that there exist arbitrarily large parabolic balls \(Q((\bar{x}, \bar{t}), R) = B(\bar{x}, R) \times (\bar{t} - R^2, \bar{t}) \subset \mathbb{R}^2 \times (-\infty, 0)\) such that \(b_{z,x_1} \geq \frac{1}{2}M_1\) in \(Q((\bar{x}, \bar{t}), R)\). For such parabolic balls, we have
\[
\int_{Q_R} b_{z,x_1} \, dx \, dt \geq \frac{1}{2} \pi M_1 R^4. \tag{5.34}
\]
But, on the other hand, we can obtain
\[
\int_{Q_R} b_{z,x_1} \, dx \, dt = \int_{Q_R} b_z u_1 \, dx \, dt \leq CR^3 \tag{5.35}
\]
where $n$ is the normal to the boundary of $B(\bar{x}, R)$. When $R$ is big enough, we find that (5.35) contradicts to (5.34), unless $M_1 \leq 0$. By the same way, we conclude that $M_2 \geq 0$. Therefore, $b_{z, x_1} \equiv 0$. Use the same method, we have $b_{z, x_2} \equiv 0$. Therefore, $b_{z}$ is a constant. Then $b_{z} \equiv 0$ or $u_{r} \equiv 0$.

If $b_{z} \equiv 0$, then from Theorem 5.5, we know that $u = 0, b = 0$ in $\mathbb{R}^3 \times (-\infty, 0)$. If $u_{r} \equiv 0$, by the equation (5.15), we conclude that $u_{z, z} = 0$ in $\mathbb{R}^3 \times (-\infty, 0)$. Therefore, $u = 0, b = 0$ in $\mathbb{R}^3 \times (-\infty, 0)$ by the Theorem 5.7.

**Theorem 5.7.** Let $(u, b)$ be a bounded weak solution of the MHD system in $\mathbb{R}^3 \times (\infty, 0)$. Assume that $u, b$ are axi-symmetric and $u_{z, z} = b_{z, z} = 0$ in $\mathbb{R}^3 \times (-\infty, 0)$. In addition, let $\mu_1 = \mu_2$. Then, $u = (0, 0, d_4(t))^T$ and $b = (0, 0, d_0)^T$, where $d_4 : (-\infty, 0) \to \mathbb{R}$ is a bounded measurable function and $d_0$ is a constant.

**Proof.** With the condition $u_{z, z} = 0$, the equation (5.15) can be written as

$$u_{r} + u_{r, r} = 0 \quad (5.36)$$

For fix $z, t$, (5.36) is an ordinary differential equation of $u_{r}$ with respect to $r$, and $u_{r} = Cr^{-1}$ or $u_{r} = 0$. Since $u_{r}$ is bounded in $\mathbb{R}^3 \times (-\infty, 0)$, thus $u_{r} = 0$. In the same way, we conclude that $b_{z} = 0$. Therefore, equations (5.13) and (5.17) can be written as

$$(u_{\theta})_{t} + u_{z}u_{\theta, z} - b_{z}b_{\theta, z} = \Delta u_{\theta} - \frac{u_{\theta}}{r^2}, \quad (5.37)$$

$$(b_{\theta})_{t} + u_{z}b_{\theta, z} - b_{z}u_{\theta, z} = \Delta b_{\theta} - \frac{b_{\theta}}{r^2}. \quad (5.38)$$

Then we rewrite (5.37) and (5.38) as

$$\left(\frac{u_{\theta}}{r}\right)_{t} + u_{z}\left(\frac{u_{\theta}}{r}\right)_{z} - b_{z}\left(\frac{b_{\theta}}{r}\right)_{z} = \Delta \left(\frac{u_{\theta}}{r}\right) + \frac{2}{r}\left(\frac{u_{\theta}}{r}\right)_{, r}, \quad (5.39)$$

$$\left(\frac{b_{\theta}}{r}\right)_{t} + u_{z}\left(\frac{b_{\theta}}{r}\right)_{z} - b_{z}\left(\frac{u_{\theta}}{r}\right)_{z} = \Delta \left(\frac{b_{\theta}}{r}\right) + \frac{2}{r}\left(\frac{b_{\theta}}{r}\right)_{, r}. \quad (5.40)$$

Let $h_{\theta}^{+} = u_{\theta} + b_{\theta}$, $h_{\theta}^{-} = u_{\theta} - b_{\theta}$, then by (5.39) and (5.40), $h_{\theta}^{+}$ and $h_{\theta}^{-}$ satisfy

$$\left(\frac{h_{\theta}^{+}}{r}\right)_{t} + (u_{z} - b_{z})\left(\frac{h_{\theta}^{+}}{r}\right)_{z} = \Delta \left(\frac{h_{\theta}^{+}}{r}\right) + \frac{2}{r}\left(\frac{h_{\theta}^{+}}{r}\right)_{, r}, \quad (5.41)$$

$$\left(\frac{h_{\theta}^{-}}{r}\right)_{t} + (u_{z} + b_{z})\left(\frac{h_{\theta}^{-}}{r}\right)_{z} = \Delta \left(\frac{h_{\theta}^{-}}{r}\right) + \frac{2}{r}\left(\frac{h_{\theta}^{-}}{r}\right)_{, r}. \quad (5.42)$$

Like (5.23), the right term of (5.41) and (5.42) can be treated as Laplacian acting on $\text{SO}(4)$-invariant functions in $\mathbb{R}^3$:

$$\left(\frac{h_{\theta}^{+}}{r}\right)_{t} + (u_{z} - b_{z})\left(\frac{h_{\theta}^{+}}{r}\right)_{z} = \Delta_{5} \left(\frac{h_{\theta}^{+}}{r}\right), \quad (5.43)$$

$$\left(\frac{h_{\theta}^{-}}{r}\right)_{t} + (u_{z} + b_{z})\left(\frac{h_{\theta}^{-}}{r}\right)_{z} = \Delta_{5} \left(\frac{h_{\theta}^{-}}{r}\right). \quad (5.44)$$

By the condition, $b, u$ are axi-symmetric, and from section 4, we know that $\nabla_x b, \nabla_x u$ are bounded in $\mathbb{R}^3 \times (-\infty, 0)$, thus $h_{\theta}^{+}/r, h_{\theta}^{-}/r$ are bounded $\mathbb{R}^3 \times (-\infty, 0)$. Let

$$M_1 = \sup_{\mathbb{R}^3 \times (-\infty, 0)} \frac{h_{\theta}^{+}}{r} \quad \text{and} \quad M_2 = \inf_{\mathbb{R}^3 \times (-\infty, 0)} \frac{h_{\theta}^{-}}{r},$$

**THE END**
and assume that \( M_1 > 0 \). Applying Lemma 2.1 to the solution \( h^+ \) of equation (5.45), considered as an equation in \( \mathbb{R}^5 \times (-\infty, 0) \). With suitable centers, we see that \( h^+ \geq \frac{1}{2} M_1 \) in arbitrarily large parabolic balls, this means \( h^+ \) is unbounded, a contradiction. Therefore, \( M_1 \leq 0 \). In the same way, we find that \( M_2 \geq 0 \). Thus \( h^- = 0 \). By the same proof, we conclude \( h^- = 0 \). Therefore, \( u_0 = b_0 = 0 \) by the definition of \( h^- \) and \( h^+ \). Then,

\[
\Pi_t = 0, \quad \text{by the equation (5.12),}
\]

\[
(u_z)_t + \Pi_z = \Delta u_z, \quad \text{by the equation (5.14),}
\]

\[
(b_z)_t = \nabla b_z, \quad \text{by the equation (5.18)}.
\]

Then, \( u_z, b_z \) satisfy

\[
(u_z)_t = \Delta u_z - \frac{u_z}{r^2},
\]

\[
(b_z)_t = \Delta b_z - \frac{b_z}{r^2}.
\]

Therefore, applying Lemma 2.1 to \( u_z/r \) and \( b_z/r \), we conclude that \( u_z = b_z = 0 \). Thus \( u_z \) and \( b_z \) are constant functions in \( x \) for each \( t \). Take \( b_z \) into the equation (5.18), we have the result.

Now we introduce a special case for MHD system in \( \mathbb{R}^n \times (-\infty, 0) \).

**Theorem 5.8.** Let \((u, b)\) be a bounded weak solution of the MHD system in \( \mathbb{R}^n \times (-\infty, 0) \). If \( u = 0 \) in \( \mathbb{R}^n \times (-\infty, 0) \), then \( b(x, t) = \bar{d}_0 \), where \( \bar{d}_0 \) is a constant vector.

**Proof.** By section 4, with the condition, we know that \( \nabla b \) is bounded in \( \mathbb{R}^n \times (-\infty, 0) \). Because \( u = 0 \) in \( \mathbb{R}^n \times (-\infty, 0) \), the second equation of (4.1) can be written as

\[
b_t - \Delta b = 0.
\]

Then \( b_{1,1} \) satisfies

\[
(b_{1,1})_t - \Delta b_{1,1} = 0.
\]

Let

\[
M_1 = \sup_{\mathbb{R}^n \times (-\infty, 0)} b_{1,1} \quad \text{and} \quad M_2 = \inf_{\mathbb{R}^n \times (-\infty, 0)} b_{1,1},
\]

and assume that \( M_1 > 0 \). Applying Lemma 2.1 to \( b_{1,1} - \frac{1}{2} (M_1 + M_2) \), we get that there exist arbitrarily large parabolic balls \( Q((\bar{x}, t), R) = B(\bar{x}, R) \times (t - R^2, t) \subset \mathbb{R}^n \times (-\infty, 0) \) such that \( b_{1,1} \geq \frac{1}{2} M_1 \) in \( Q((\bar{x}, t), R) \). For such parabolic balls, we have

\[
\int_{Q_R} b_{1,1} dx dt \geq \frac{1}{2} \pi M_1 R^{n+2}.
\]

But, on the other hand, we can obtain

\[
\int_{Q_R} b_{1,1} dx dt = \int_{Q_R} b_{1,1} n_1 dx dt \leq CR^{n+1},
\]

where \( n \) is the normal to the boundary of \( B(\bar{x}, R) \). When \( R \) is big enough, we find that (5.45) contradicts to (5.46), unless \( M_1 \leq 0 \). By the same way, we conclude that \( M_2 \geq 0 \). Therefore, \( b_{1,1} = 0 \) in \( \mathbb{R}^n \times (-\infty, 0) \). In the same way, we conclude that \( b_{i,j} = 0(i, j = 1, 2, \ldots, n) \) in \( \mathbb{R}^n \times (-\infty, 0) \). Therefore, \( b \) is constant in \( x \) for each \( t \). Take it into the second equation of the system (4.1), we obtain the result.
6. Singularities. In this section we will consider the potential singularity in the solutions of the Cauchy problem for the MHD system (4.1) and (4.2). We aim to show that singularities generate bounded ancient solution, which are the solutions defined in \( \mathbb{R}^n \times (-\infty, 0) \). More precisely, an ancient weak solution of the MHD system is a weak solution defined in \( \mathbb{R}^n \times (-\infty, 0) \), and \((u, b)\) is an ancient mild solution if there is a sequence \( T_l \to -\infty \) such that \((u(\cdot, T_l), b(\cdot, T_l))\) is well defined and \((u, b)\) is a mild solution of the Cauchy problem in \( \mathbb{R}^n \times (T_l, 0) \) with initial data \((u(\cdot, T_l), b(\cdot, T_l))\).

**Lemma 6.1.** Let \((u_l, b_l)\) be a sequence of bounded mild solution of the MHD system defined in \( \mathbb{R}^n \times (-\infty, 0) \) (for some initial data) with a uniform bound \(|u_l| + |b_l| \leq C\), and \( T_l \to -\infty \). Then, there exist a subsequence along which \((u_l, b_l)\) converges locally uniformly in \( \mathbb{R}^n \times (-\infty) \) to an ancient mild solution \((u, b)\) satisfying \(|u| \leq C\) in \( \mathbb{R}^n \times (-\infty)\).

**Proof.** It is easy to prove with the regularity results in section 4. \( \square \)

Now assume that the mild solution develops a singularity in finite time, and that \((0, T)\) is its maximal time interval of the existence. There are two situations:

1. there exists a positive number \( C_0 \), such that
   \[
   \lim_{t \to T^-} \sup_{x \in \mathbb{R}^n \times (0, t)} |b(x, \tau)| \leq C_0,
   \]  
   \[ (6.1) \]

2. there exists a positive sequence \( \{t_k\}_{k \geq 1} \) such that \( t_k \to T^- \) as \( k \to \infty \) and
   \[
   \lim_{k \to \infty} \sup_{x \in \mathbb{R}^n \times (0, t_k)} |u(x, \tau)| = 0.
   \]  
   \[ (6.2) \]

First, we assume that case (1) holds((6.1) holds). Let \( h(t) = \sup_{x \in \mathbb{R}^n} |u(x, t)| \) and \( H(t) = \sup_{0 \leq s \leq t} h(s) \). It is easy to see that there exist a sequence \( t_k \uparrow T \) such that \( h(t_k) = H(t_k) \). Now we choose a sequence of numbers \( \gamma_k \downarrow 1 \). For all \( k \), let \( N_k = H(t_k) \) and choose \( x_k \in \mathbb{R}^n \) such that \( M_k = |u(x_k, t_k)| \geq N_k / \gamma_k \). Then we set

\[
v^{(k)}(y, s) = \frac{1}{M_k} u(x_k + \frac{y}{M_k}, t_k + \frac{s}{M_k^2}),
\]

\[ v^{(k)} \]  

\[
e^{(k)}(y, s) = \frac{1}{M_k} b(x_k + \frac{y}{M_k}, t_k + \frac{s}{M_k^2}).
\]

\[ e^{(k)} \]  

The functions \( v^{(k)} \) and \( e^{(k)} \) are defined in \( \mathbb{R}^n \times (A_k, B_k) \), where

\[
A_k = -M_k^2 t_k, \quad \text{and} \quad B_k = M_k^2 (T - t_k) > 0
\]

\[ (6.5) \]

and satisfy

\[
|v^{(k)}| \leq \gamma_k, \quad |e^{(k)}| \leq \gamma_k C_0 \quad \text{in} \quad \mathbb{R}^n \times (A_k, 0) \quad \text{and} \quad |v^{(k)}(0, 0)| = 1.
\]

\[ (6.6) \]

Also, \( v^{(k)}, e^{(k)} \) are mild solution of the MHD system in \( \mathbb{R}^n \times (A_k, 0) \) with initial data \( u_0^{(k)}(y) = (1/M_k) u_0(x_k + y/M_k), e_0^{(k)}(y) = (1/M_k) b_0(x_k + y/M_k) \). By Lemma 6.1, there is a subsequence of \( v^{(k)}, e^{(k)} \) converging to an ancient mild solution \((v, e)\) of the MHD system. Note that, by the construction, we have \(|v| \leq 1, |e| \leq C_0 \) in \( \mathbb{R}^n \times (-\infty, 0) \) and \( v(0, 0) = 1 \).

Then we consider the situation (2)((6.2) holds). Now we let \( h(t) = \sup_{x \in \mathbb{R}^n} |b(x, t)| \) and \( H(t) = \sup_{0 \leq s \leq t} h(s) \), and do the same scaling like (6.3) and (6.4). Therefore, by the construction, we have \(|e| \leq 1\) in \( \mathbb{R}^n \times (\gamma, 0) \) and \( e(0, 0) = 1 \), particularly, \( v = 0 \) in \( \mathbb{R}^n \times (-\infty, 0) \). Note that \( \lim_{t \to -\infty} \sup_{x \in \mathbb{R}^n} |e(x, t)| = 0 \) with the bounded
initial data of \((u, b)\), then by Theorem 5.8, we have \(b = 0 \in \mathbb{R}^n \times (-\infty, 0)\), a contradiction. Therefore, we just need consider the case (1) in the following content.

**Theorem 6.2.** Let \((u, v)\) be a weak axi-symmetric solution in \(\mathbb{R}^3 \times (0, T)\) which belongs to \(L_{x,t}^\infty(\mathbb{R}^3 \times (0, T'))\) for each \(T' < T\). Assume that \(u\) satisfies

\[
|u(x, t)| \leq \frac{C}{\sqrt{x_1^2 + x_2^2}} \quad \text{in} \quad \mathbb{R}^3 \times (0, T),
\]

and \(b\) is bounded in \(\mathbb{R}^3 \times (0, T)\).

Then, \(|u| + |b| \leq M = M(C)\) in \(\mathbb{R}^3 \times (0, T)\). Moreover, \((u, b)\) is a mild solution of the MHD system (for a suitable initial data).

**Proof.** We first prove the statement assuming that \(u\) is a mild solution (for a suitable initial data). Arguing by contradiction, assume that \((u, b)\) is a mild solution which is bounded in \(\mathbb{R}^3 \times (0, T')\) for each \(T' < T\) and develops a singularity at time \(T\) and case (1) is true, that means (6.1) holds. Let \(v^{(k)}\) and \(b^{(k)}\) be as in the construction (6.3) and (6.4). We write \(x_k = (x'_k, x_{3k})\), where \(x'_k = (x_{1k}, x_{2k})\). With the assumption (6.7), we find that \(|x'_k| \leq C/M_k\). This implies that the functions \(v^{(k)}, e^{(k)}\) are axi-symmetric with respect to an axis parallel to the \(y_3\)-axis and at distance at most \(C\) from it. Therefore we can assume (by passing to suitable subsequence) that the limit function \(v\) is axi-symmetric with respect to a suitable axis. Moreover, since assumption (6.7) is scale-invariant, it will again be satisfied (in suitable coordinates) by \(v\), and in addition \(e_z = 0\) by the assumption of ”\(b_z\) is bounded in \(\mathbb{R}^3 \times (0, T)\)”. Then applying Theorem 5.5, we see that \(v = 0\). On the other hand, \(|v(0, 0)| = 1\), this is a contradiction.

Recall that Lemma 3.4, weak solution \(u\) can be decomposed as \(u = v + \omega_1 + d_1\). Thus applying the condition (6.7), we can obtain that \(d_1 = 0\). Therefore, \(u, b\) are mild solutions of the MHD system. \(\square\)

**Remark 6.3.** If the condition ”\(b_z\) is bounded in \(\mathbb{R}^3 \times (0, T)\)” in Theorem 6.2 is replaced by ”\(b_z u_r = u_r b_z\) in \(\mathbb{R}^3 \times (0, T)\)” , the result also holds in the same way (applying Theorem 5.6).

If the condition in Theorem 6.2 is replaced by ”\(u, b\) are axi-symmetric and \(u_{zz} = b_z = 0\) in \(\mathbb{R}^3 \times (0, T)\), and \(\mu_1 = \mu_2\)”, then, the result still holds in the same way (applying Theorem 5.7).

**Theorem 6.4.** Let \((u, v)\) be a weak axi-symmetric solution in \(\mathbb{R}^3 \times (0, T)\) which belongs to \(L_{x,t}^\infty(\mathbb{R}^3 \times (0, T'))\) for each \(T' < T\). Assume that \(u\) satisfies

\[
|u| \leq \frac{C}{\sqrt{T - t}} \quad \text{in} \quad \mathbb{R}^3 \times (0, T),
\]

and there exists some \(R_0 > 0\) such that

\[
|b(x, t)| + |u(x, t)| \leq \frac{C}{\sqrt{x_1^2 + x_2^2}} \quad \text{for} \quad x_1^2 + x_2^2 \geq R_0 \quad \text{and} \quad 0 \leq t \leq T.
\]

In addition, assume that \(b_z\) is bounded in \(\mathbb{R}^3 \times (0, T)\). Then, \(|u| + |b| \leq M = M(C)\) in \(\mathbb{R}^3 \times (0, T)\). Moreover, \((u, b)\) is a mild solution of the MHD system (for a suitable initial data).

**Proof.** Condition (6.9) implies that \((u, b)\) is a mild solution of the MHD system for a suitable initial data. Therefore, it is smooth in open subsets of \(\mathbb{R}^3 \times (0, T)\). By
(6.1) and the condition (6.8), we have

$$|b| \leq \frac{CC_1}{\sqrt{T - t}} \quad \text{in} \quad \mathbb{R}^3 \times (0, T),$$

(6.10)

for some $C_1 \geq C_0$. Let

$$f(x, t) = |x'| |u(x, t)| = \sqrt{x_1^2 + x_2^2 |u(x, t)|},$$

(6.11)

and

$$g(x, t) = |x'| |u(x, t)| = \sqrt{x_1^2 + x_2^2 |b(x, t)|},$$

(6.12)

where $x' = (x_1, x_2)$. According to Theorem 6.2, it is enough to prove that $f$ is bounded in $\mathbb{R}^3 \times (0, T)$. We prove it by contradiction. There are two situations:

1. there exists a positive number $C_2$, such that

$$\lim_{t \to T^-} \sup_{(x, \tau) \in \mathbb{R}^3 \times (0, t)} \frac{|g(x, \tau)|}{|f(x, \tau)|} \leq C_2,$$

(6.13)

2. there exists a positive sequence $\{t_k\}_{k \geq 1}$ such that $t_k \to T^-$ as $k \to \infty$ and

$$\lim_{k \to \infty} \sup_{(x, \tau) \in \mathbb{R}^3 \times (0, t_k)} \frac{|f(x, \tau)|}{|g(x, \tau)|} = 0.$$

(6.14)

Let $F(t) = \sup_{\mathbb{R}^3 \times (0, t)} f(x, t)$ and $G(t) = \sup_{\mathbb{R}^3 \times (0, t)} g(x, t)$. Assume that $f$ is not bounded and (6.13) holds. Choose $t_k \uparrow T$ and $x_k \in \mathbb{R}^3$ such that $M_k = f(x_k, t_k) = F(t_k) \uparrow \infty$. Let $\lambda_k = |x'_k|$, then for $y \in \mathbb{R}^3$ and $s \in (-T\lambda_k^{-2}, 0)$, define

$$v^{(k)}(y, s) = v^{(k)}(y', y_3, s) = \lambda_k u(\lambda_k y', \lambda_k y_3 + x_{3k}, T + \lambda_k^2 s),$$

$$e^{(k)}(y, s) = e^{(k)}(y', y_3, s) = \lambda_k b(\lambda_k y', \lambda_k y_3 + x_{3k}, T + \lambda_k^2 s),$$

We mention that the sequence $\lambda_k$ is bounded because of (6.9). According to the definition, $v^{(k)}, e^{(k)}$ satisfy

$$|v^{(k)}| \leq \frac{C}{\sqrt{-s}} \quad \text{in} \quad \mathbb{R}^3 \times (-T\lambda_k, 0)$$

(6.15)

and

$$|e^{(k)}| \leq \frac{CC_1}{\sqrt{-s}} \quad \text{in} \quad \mathbb{R}^3 \times (-T\lambda_k, 0).$$

(6.16)

Let $s_k = -(T - t_k)\lambda_k^{-2}$. From the construction, we have

$$|v^{(k)}(y, s)| \leq \frac{M_k}{|y'|} \quad \text{in} \quad \mathbb{R}^3 \times (-T\lambda_k, s_k)$$

(6.17)

and

$$|e^{(k)}(y, s)| \leq \frac{C_2 M_k}{|y'|} \quad \text{in} \quad \mathbb{R}^3 \times (-T\lambda_k, s_k).$$

(6.18)

By the inequality $\min\{1/a, 1/b\} \leq 2/(a + b)(a, b > 0)$, the above estimates can lead to

$$|v^{(k)}(y, s)| \leq \frac{2CM_k}{M_k \sqrt{-s} + C|y'|} \quad \text{in} \quad \mathbb{R}^3 \times (-T\lambda_k, s_k),$$

(6.19)

and

$$|e^{(k)}(y, s)| \leq \frac{2CC_1 C_2 M_k}{M_k \sqrt{-s} + C|y'|} \quad \text{in} \quad \mathbb{R}^3 \times (-T\lambda_k, s_k).$$

(6.20)
Let $\gamma = \{y \in \mathbb{R}^3 : |(y_1, y_2)| = 1, y_3 = 0\}$, then, by the construction, $|w^{(k)}(x, s_k)| = M_k$. Now let $e_1 = (1, 0, 0)$. For $x \in \mathbb{R}^3$ and $\tau \in (A_k, 0]$, where $A_k = M_k^2(-T\lambda_k^2 - s_k)$, we define

$$w^{(k)}(x, \tau) = \frac{1}{M_k} v^{(k)}(e_1 + \frac{x}{M_k}, s_k + \frac{\tau}{M_k^2})$$

and

$$h^{(k)}(x, \tau) = \frac{1}{M_k} e^{(k)}(e_1 + \frac{x}{M_k}, s_k + \frac{\tau}{M_k^2}).$$

Let $A_k = \{x \in \mathbb{R}^3 : \sqrt{(x_1 + M_k)^2 + x_2^2} \leq M_k/2\}$, then we have

$$|w^{(k)}(0, 0)| = 1 \quad \text{and} \quad |w^{(k)}(x, \tau)| \leq 2 \text{ in } (\mathbb{R}^3 \setminus A_k) \times (A_k, 0), \quad (6.21)$$

and

$$|h^{(k)}(x, \tau)| \leq 2C_2 \text{ in } (\mathbb{R}^3 \setminus A_k) \times (A_k, 0). \quad (6.22)$$

Note that $(6.19)$, $(6.20)$, $(6.15)$, $(6.16)$ imply

$$|w^{(k)}(x, \tau)| \leq \frac{2CM_k}{M_k\sqrt{-\tau} + C\sqrt{(x_1 + M_k)^2 + x_2^2}} \text{ in } A_k \times (A_k, 0), \quad (6.23)$$

$$|h^{(k)}(x, \tau)| \leq \frac{2CC_1C_2M_k}{M_k\sqrt{-\tau} + C\sqrt{(x_1 + M_k)^2 + x_2^2}} \text{ in } A_k \times (A_k, 0), \quad (6.24)$$

$$|w^{(k)}(x, \tau)| \leq \frac{C}{\sqrt{-\tau}} \text{ in } \mathbb{R}^3 \times (-T\lambda_k, 0) \quad (6.25)$$

and

$$|h^{(k)}(x, \tau)| \leq \frac{CC_1}{\sqrt{-\tau}} \text{ in } \mathbb{R}^3 \times (-T\lambda_k, 0). \quad (6.26)$$

Note that $(w^{(k)}, h^{(k)})$ are the mild solution of the MHD system. Therefore, from the estimate $(6.25)$ and $(6.26)$, we can choose a subsequence of the sequence $(w^{(k)}, h^{(k)})$, which we denote by $(w^{(k)}_1, h^{(k)}_1)$, such that the sequence $(w^{(k)}_1, h^{(k)}_1)$ converge uniformly on compact subsets of $\mathbb{R}^3 \times (-\infty, 0)$ to an ancient mild solution $(w, h)$. Since the solutions $(w^{(k)}, h^{(k)})$ are axi-symmetric and $M_k \uparrow +\infty$, it is easy to prove that $w$ is independent of the $x_2$-variable. In addition, $\lambda_k$ is bounded, thus $w$ is also independent of the $x_2$-variable. Moreover, because $b_z$ is bounded in $\mathbb{R}^3 \times (0, T)$, we have $\lambda_z = 0$ in $\mathbb{R}^3 \times (-\infty, 0)$. Applying Theorem 5.1 to $(\tilde{w}, \tilde{h})$, where $\tilde{w} = (w_1, w_3)^T$, $\tilde{h} = (h_1, h_3)^T$. We conclude that $(\tilde{w}, \tilde{h})$ must vanish identically, that means $w = h = 0$ in $\mathbb{R}^3 \times (-\infty, 0)$. This would give a contradiction with $|w^{(k)}(0, 0)| = 1$, if we could prove that $w^{(k)}(0, 0) \to w(0, 0)$, which is not obvious since $\sup_x |w^{(k)}(x, t)|$ may not be uniformly bounded as $\tau \to 0$. To this aim, we use the representation formula $(3.6)$ and $(3.9)$ with the initial data $w^{(k)}_1(x, -1), h^{(k)}_1(x, -1)$. Let $f_{jl} = -w^{(k)}_{1l}w^{(k)}_{1j} + h^{(k)}_{1l}h^{(k)}_{1j}$, then

$$w^{(k)}_{1l}(x, t) = \int_{\mathbb{R}^n} \Gamma(x - y, t)w^{(k)}_{1l}(y, -1)dy + \int_0^t \int_{\mathbb{R}^n} K_{ijl}(x - y, t - s)f_{jl}(y, s)dyds
$$

$$= I_1 + \int_{-1}^t \int_{\mathbb{R}^n \setminus \Lambda_k} K_{ijl}(x - y, t - s)f_{jl}(y, s)dyds$$

$$+ \int_{-1}^t \int_{\Lambda_k} K_{ijl}(x - y, t - s)f_{jl}(y, s)dyds$$

$$= I_1^{(k)} + I_2^{(k)} + I_3^{(k)},$$
where \((x,t) \in \bar{B}(0,1) \times [-1,0]\). By (6.21), (6.22), the estimate (3.11) and heat kernel theory, it is easy to see the sequence \(I_1^{(k)}, I_2^{(k)}\) have subsequence converge uniformly. By (6.23), (6.24) and the kernel decay (3.8), we have

\[
|I_3^{(k)}| \leq C \int_{-1}^{0} \int_{-\infty}^{+\infty} \int_{|x'| \leq M_k/2} \frac{1}{(\sqrt{-\tau} + |x'|/M_k)^2} \frac{1}{(M_k^2/4 + |x|^2)^2} dx' dx_3 d\tau
\]

By direct calculation, \(I_3^{(k)} \to 0\) as \(k \to \infty\). Therefore, we can choose a subsequence of the sequence \(w_1^{(k)}\), which we again denote by \(w^{(k)}\), such that the sequence \(w^{(k)}\) converge uniformly in \(\bar{B}(0,1) \times [-1,0]\).

For the case (2) ((6.14) holds), we take \(M_k = g(x_k, t_k) = G(t_k)\), in the same way, we get the result (When we estimate \(I_3^{(k)}\) for \(I_1^{(k)}\), we use (3.13) and heat kernel theory).

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