Synchronization analysis of coupled planar oscillators by averaging

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March 15, 2010

Abstract
Sufficient conditions for synchronization of coupled Lienard-type oscillators are investigated via averaging technique. Coupling considered here is pairwise, unidirectional, and described by a nonlinear function (whose graph resides in the first and third quadrants) of some projection of the relative distance (between the states of the pair being coupled) vector. Under the assumption that the interconnection topology defines a connected graph, it is shown that the solutions of oscillators can be made converge arbitrarily close to each other, while let initially be arbitrarily far apart, provided that the frequency of oscillations is large enough and the initial phases of oscillators all lie in an open semicircle. It is also shown that (almost) synchronized oscillations always take place at some fixed magnitude independent of the initial conditions. Similar results are generated for nonlinerly-coupled harmonic oscillators.

1 Introduction
Synchronization in coupled dynamical systems has been a common ground of investigation for researchers from different disciplines. Most of the work in the area studies the case where the coupling between individual systems is linear; see, for instance, [21, 12, 14, 20, 4, 10]. Nonlinear coupling is also of interest since certain phenomena cannot be properly modelled by linear coupling. A particular system exemplifying nonlinear coupling that attracted much attention is Kuramoto model and its like [5, 13]. Among more general results allowing nonlinear coupling are [2, 17] where passivity theory is employed to obtain sufficient conditions for synchronization under certain symmetry or balancedness assumptions on the coupling graph.

In this paper we study the synchronization behavior of an array of nonlinear planar oscillators. We let the individual oscillators share identical dynamics and the coupling between them be nonlinear. We consider Lienard-type oscillators...
that have been much studied due to their close relation to real-life systems and applications [10]. A particular example is van der Pol oscillator [7].

What we investigate here is the relation between frequency of oscillations and synchronization of the oscillators forming the array. The array is formed such that some of the oscillators are coupled to some others via a nonlinear function. Coupling considered is of partial-state nature. That is, if an oscillator affects the dynamics of another, the associated coupling term is not a function of the state vector of the oscillator that is affecting, but only of some projection of that vector. We make no symmetry nor balancedness assumption on the coupling graph.

Our finding in the paper is roughly that (almost) synchronization occurs among the oscillators (at some magnitude independent of the initial conditions, the coupling, and the frequency of oscillations $\omega$) if the following conditions hold: (a) the frequency of oscillations is high, (b) there is at least one oscillator that directly or indirectly affects all others, and (c) the initial phases of oscillators, when each is represented by a point on the unit circle, all lie in an open semicircle. More formally, what we show is that if the coupling graph is connected and the initial phases of oscillators lie in an open semicircle, then the solutions of oscillators can be made converge arbitrarily close to each other, while initially being arbitrarily far from one another, by choosing large enough $\omega$. Incidentally, as sort of a byproduct of our analysis for nonlinear oscillators, we also generate a similar result for an array of nonlinearly-coupled harmonic oscillators. We show that harmonic oscillators (almost) synchronize provided that the coupling graph is connected and the frequency of oscillations is high. Different from nonlinear oscillators, the initial phases of harmonic oscillators do not have any effect on synchronization, at least when $\omega$ is sufficiently high. Also, again unlike nonlinear oscillators, synchronized oscillations can take place at any magnitude, depending on the initial conditions.

In establishing our results we use tools from averaging theory [3]. Our three-step approach is as follows. We first apply a time-varying change of coordinates to the array, which keeps the relative distances between the states of oscillators intact (Section 4). This change of coordinates renders the array periodically time-varying. Then we take the time-average of this new array and show, for the average array, that the oscillators synchronize if their initial phases lie in an open semicircle and the coupling graph is connected (Section 5). Finally we work out what that result means for the original array. Namely, we show that the higher the frequency of oscillations the more the original array behaves like its average (Section 6).

2 Preliminaries

Let $\mathbb{R}_{\geq 0}$ denote the set of nonnegative real numbers. Let $|\cdot|$ denote Euclidean norm. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. It is said to belong to class-$\mathcal{K}_\infty$ if it is also unbounded. Given a closed set $\mathcal{S} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_{\mathcal{S}}$
Either \( \gamma > \angle \). Given a general model for a planar oscillator is given by system (1) to admit a unique, stable limit cycle encircling the origin of the phase plane; see, for instance, [1, Thm. 1]. In this paper we adopt those general conditions on function \( f \) which are stated later in the section. Regarding function \( g \), we assume linearity. Namely, we study the case \( g(q) = \omega^2 q \), where \( \omega \) is some constant. Although this assumption is restrictive compared to what is assumed in [1, Thm. 1], it nevertheless still allows us to cover some physically possible cases.

3 Problem statement

A general model for a planar oscillator is given by Lienard’s equation

\[
\ddot{q} + f(q)\dot{q} + g(q) = 0.
\]  

Sufficient conditions have been established on functions \( f \) and \( g \) in order for system (1) to admit a unique, stable limit cycle encircling the origin of the phase plane; see, for instance, [1, Thm. 1]. In this paper we adopt those general conditions on function \( f \), which are stated later in the section. Regarding function \( g \), we assume linearity. Namely, we study the case \( g(q) = \omega^2 q \), where \( \omega \) is some constant. Although this assumption is restrictive compared to what is assumed in [1, Thm. 1], it nevertheless still allows us to cover some physically possible cases.
important cases. Most famous example that falls into the class of systems being studied here would be van der Pol oscillator \[8\]. In this paper we search for sufficient conditions that yield synchronization of a number of coupled Lienard oscillators. Below we give the precise description of the problem.

Consider the following array of coupled Lienard oscillators

\[
\begin{align*}
\dot{q}_i &= \omega p_i \\
\dot{p}_i &= -\omega q_i - f(q_i)p_i + \sum_{j \neq i} \gamma_{ij}(p_j - p_i), \quad i = 1, \ldots, m
\end{align*}
\]  

(2a)  

(2b)

where \( \omega > 0 \) is the frequency of oscillations and \( \{\gamma_{ij}\} \) is a connected interconnection. Let \( \xi_i \in \mathbb{R}^2 \) denote the state of \( i \)th oscillator, i.e., \( \xi_i = [q_i \ p_i]^T \). Sometimes we choose to handle this array (2) of \( m \) planar oscillators as a single system in \( \mathbb{R}^{2m} \). If we let \( \xi := [\xi_1^T \ldots \xi_m^T]^T \) and

\[
\ell(\xi, \omega) := \begin{bmatrix}
\omega p_1 \\
-\omega q_1 - f(q_1)p_1 + \sum \gamma_{1j}(p_j - p_1) \\
\vdots \\
\omega p_m \\
-\omega q_m - f(q_m)p_m + \sum \gamma_{mj}(p_j - p_m)
\end{bmatrix}
\]

Figure 1: Some examples of coupling functions.
then array (2) defines the below system
\[
\dot{\xi} = \ell(\xi, \omega).
\] (3)

We assume throughout the paper that $\gamma_{ij}$ and $f : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz. Letting $F(s) := \int_0^s f(\sigma)d\sigma$, we further assume the following.

(L1) $f$ is an even function.

(L2) $F(s_0) = 0$ for some $s_0 > 0$; $F$ is negative on $(0, s_0)$; $F$ is positive, nondecreasing, and unbounded on $(s_0, \infty)$.

In this paper the question we ask ourselves is the following.

What is the effect of $\omega$ on the synchronization behavior of array (2)?

Our approach to the problem is as follows. We first apply a norm-preserving, time-varying change of coordinates to array (2) and obtain a new array whose
righthand side is periodic in time. Since this coordinate change does not alter the relative distances between the states of oscillators, it is safe to look at the new array to understand the synchronization behavior of the original one. Hence we focus on the new array. Although the righthand side of the new array does not look any pleasanter than the original one, it nevertheless has an average since it is periodic in time. Once this average is computed we realize that it is very simple to understand from synchronization point of view. From then on we start going backwards. Using averaging theory, we establish what our findings on the average array imply, first for the time-varying array and eventually for the original array.

4 Change of coordinates

We define $S(\omega) \in \mathbb{R}^{2 \times 2}$ and $H, V \in \mathbb{R}^{1 \times 2}$ as

$$S(\omega) := \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \quad H := [1 \ 0], \quad V := [0 \ 1].$$

Then we rewrite (2) as

$$\dot{\xi}_i = S(\omega)\xi_i - f(H\xi)_i V^T V \xi_i + V^T \sum_{j \neq i} \gamma_{ij} (V(\xi_j - \xi_i)).$$  (4)
Let $x_i(t) := e^{-S(\omega)t}\xi_i(t)$ and $x := [x_1^T \ldots x_m^T]^T$. We can by (4) write
\[
\dot{x}_i = -f(He^{S(\omega)t}x_i)He^{S(\omega)t} + \sum_{j \neq i} e^{S(\omega)t}V^T\gamma_{ij}(Ve^{S(\omega)t}(x_j - x_i))
\]
\[
= -f([\cos \omega t \sin \omega t]x_i)\begin{bmatrix} -\sin \omega t \\ \cos \omega t \end{bmatrix} -\sin \omega t \cos \omega t)x_i
\]
\[
+ \sum_{j \neq i} \begin{bmatrix} -\sin \omega t \\ \cos \omega t \end{bmatrix} \gamma_{ij}([-\sin \omega t \cos \omega t](x_j - x_i))
\]
\[
= \ell_i^{BC}(x, \omega t).
\]

Then system in $\mathbb{R}^{2m}$ corresponding to array (5) reads
\[
\dot{x} = \ell^{BC}(x, \omega t).
\]

Remark 1. Since the change of coordinates is realized via rotation matrix $e^{-S(\omega)t}$, the relative distances are preserved, that is $|x_i(t) - x_j(t)| = |\xi_i(t) - \xi_j(t)|$ for all $t$ and all $i, j$. This means from synchronization point of view that the behavior of array (2) will be inherited by array (5).

Exact analysis of (5) seems far from yielding. Therefore we attempt to understand this system via its approximation.

5 Average array

Observe that the righthand side of (5) is periodic in time. Time-average functions $\bar{f} : \mathbb{R}^2 \to \mathbb{R}^2$ and $\bar{\gamma}_{ij} : \mathbb{R}^2 \to \mathbb{R}$ are given by
\[
\bar{f}(v) := \frac{1}{2\pi}\int_0^{2\pi} f([\cos \varphi \sin \varphi]v)\begin{bmatrix} -\sin \varphi \\ \cos \varphi \end{bmatrix} [-\sin \varphi \cos \varphi]vd\varphi
\]
and
\[
\bar{\gamma}_{ij}(v) := \frac{1}{2\pi}\int_0^{2\pi} \begin{bmatrix} -\sin \varphi \\ \cos \varphi \end{bmatrix} \gamma_{ij}([-\sin \varphi \cos \varphi]v)d\varphi.
\]

Then the average array dynamics read
\[
\dot{\eta}_i = -\bar{f}(\eta_i) + \sum_{j \neq i} \bar{\gamma}_{ij}(\eta_j - \eta_i).
\]
Let $\eta := [\eta_1^T \ldots \eta_m^T]^T$ and

$$\bar{\ell}^{\text{RC}}(\eta) := \begin{bmatrix} -\bar{f}(\eta_1) + \sum \gamma_{1j}(\eta_j - \eta_1) \\ \vdots \\ -\bar{f}(\eta_m) + \sum \bar{\gamma}_{mj}(\eta_j - \eta_m) \end{bmatrix}$$

Then system in $\mathbb{R}^{2m}$ corresponding to array (8) reads

$$\dot{\eta} = \bar{\ell}^{\text{RC}}(\eta). \quad (9)$$

**Remark 2** Note that instead of array (4) if we start with the below array

$$\dot{\xi}_i = S(\omega)\xi_i - f(H\xi_i)V^T V \xi_i + C^T \sum_{j \neq i} \gamma_{ij}(C(\xi_j - \xi_i)),$$

where $C \in \mathbb{R}^{1 \times 2}$ satisfies $CC^T = 1$, we still reach the same average array (8). Therefore for the analysis to follow the coupling need not be through velocities as is the case in (2). Any projection of the state $C\xi_i$ is as good as any other for coupling. For instance, the results in this paper hold true for the below array

$$\dot{q}_i = \omega p_i + \sum_{j \neq i} \gamma_{ij}(q_j - q_i)$$
$$\dot{p}_i = -\omega q_i - f(q_i)p_i.$$

Theory of perturbations [3, Ch. 4 §17] tells us that, starting from close initial conditions, the solution of a system with a periodic righthand side and the solution of the time-average approximate system stay close for a long time provided that the period is small enough. Therefore (8) should tell us a great deal about the behavior of (5) when $\omega \gg 1$. Understanding (8) requires understanding average functions $\bar{f}$ and $\bar{\gamma}_{ij}$. Let us begin with the former.

**Lemma 1** We have $f(v) = \kappa(|v|) \frac{v}{|v|}$ where $\kappa : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is

$$\kappa(s) := \frac{2}{\pi} \int_0^s f(\sigma) \sqrt{1 - \frac{\sigma^2}{s^2}} d\sigma. \quad (10)$$
Proof. Given \( v \in \mathbb{R}^2 \), let \( r = |v| \) and \( \theta \in [0, 2\pi) \) be such that \( r[\cos \theta \sin \theta]^T = v \). Then, by using standard trigonometric identities,

\[
\bar{f}(v) = \frac{1}{2\pi} \int_0^{2\pi} f(r(\cos \varphi \cos \theta + \sin \varphi \sin \theta)) \times r(-\sin \varphi \cos \theta + \cos \varphi \sin \theta) \left[ \begin{array}{c} -\sin \varphi \\ \cos \varphi \end{array} \right] d\varphi
\]

\[
= -\frac{r}{2\pi} \int_0^{2\pi} f(r(\varphi - \theta)) \sin(\varphi - \theta) \left[ \begin{array}{c} -\sin(\varphi + \theta) \\ \cos(\varphi + \theta) \end{array} \right] d\varphi
\]

\[
= -\frac{r}{2\pi} \int_0^{2\pi} f(r \cos \varphi) \sin \varphi \left[ -\sin \varphi \cos \theta - \cos \varphi \sin \theta \right] d\varphi
\]

\[
= \left( \frac{r}{2\pi} \int_0^{2\pi} f(r \cos \varphi) \sin^2 \varphi d\varphi \right) \left[ \cos \theta \right] - \left( \frac{r}{2\pi} \int_0^{2\pi} f(r \cos \varphi) \sin \varphi \cos \varphi d\varphi \right) \left[ \frac{\sin \theta}{\sin \theta} \right]
\]

\[
= \left( \frac{r}{2\pi} \int_0^{2\pi} f(r \cos \varphi) \sin^2 \varphi d\varphi \right) \left[ \cos \theta \right] \sin \theta
\]

\[
= \left( \frac{r}{2\pi} \int_0^{2\pi} f(r \cos \varphi) \sin^2 \varphi d\varphi \right) \left[ \frac{\sin \theta}{\sin \theta} \right]
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(r \cos \varphi) \sin \varphi \cos \varphi d\varphi
\]

where the second term is zero since \( f(r \cos \varphi) \sin 2\varphi \) is an odd function and we can write

\[
\int_0^{2\pi} f(r \cos \varphi) \sin \varphi \cos \varphi d\varphi = \frac{1}{2} \int_{-\pi}^{\pi} f(r \cos \varphi) \sin 2\varphi d\varphi.
\]

Therefore

\[
\bar{f}(v) = \left( \frac{r}{2\pi} \int_0^{2\pi} f(r \cos \varphi) \sin^2 \varphi d\varphi \right) \left[ \cos \theta \right] - \left( \frac{r}{2\pi} \int_0^{2\pi} f(r \cos \varphi) \sin \varphi \cos \varphi d\varphi \right) \left[ \frac{\sin \theta}{\sin \theta} \right]
\]

\[
= \left( \frac{r}{2\pi} \int_0^{2\pi} f(r \cos \varphi) \sin^2 \varphi d\varphi \right) \left[ \frac{\sin \theta}{\sin \theta} \right]
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(r \cos \varphi) \sin \varphi \cos \varphi d\varphi
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(r \cos \varphi) \sin^2 \varphi d\varphi.
\]

Since \( f(r \cos \varphi) \sin^2 \varphi \) and \( f(r \sin \varphi) r \cos^2 \varphi \) are even functions we can write

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(r \cos \varphi) r \sin^2 \varphi d\varphi = \frac{1}{\pi} \int_0^{\pi} f(r \cos \varphi) r \sin^2 \varphi d\varphi
\]

\[
= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(r \sin \varphi) r \cos^2 \varphi d\varphi
\]

\[
= \frac{2}{\pi} \int_0^{\pi/2} f(r \sin \varphi) r \cos^2 \varphi d\varphi.
\]

Then by change of variables \( \sigma := r \sin \varphi \) we obtain

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(r \cos \varphi) r \sin^2 \varphi d\varphi = \frac{2}{\pi} \int_0^{r} f(\sigma) \sqrt{1 - \frac{\sigma^2}{r^2}} d\sigma.
\]

Combining (11) and (12) gives the result. \( \blacksquare \)
Claim 1 Defined in (10), function \( \kappa \) satisfies (L2).

Proof. Recall that \( F(s) = \int_0^s f \) satisfies (L2). Therefore there exists \( s_0 > 0 \) such that \( F(s_0) = 0 \), \( F(s) < 0 \) for \( s \in (0, s_0) \), \( F \) is positive, nondecreasing on \((s_0, \infty)\), and \( F(s) \to \infty \) as \( s \to \infty \). Given \( s > 0 \), let \( \delta_s : [0, s] \to [0, 1] \)
be \( \delta_s(\sigma) := \sqrt{1 - (\sigma/s)^2} \). Then we can write \( \kappa(s) = \frac{2}{\pi} \int_0^s f(\sigma)\delta_s(\sigma)d\sigma \). We observe the following properties.

(D1) Given \( s > 0 \), function \( \delta_s(\cdot) \) is strictly decreasing.

(D2) Given \( s > t > s \), map \( \sigma \mapsto \delta_s(\sigma) - \delta_t(\sigma) \) is strictly increasing on \([0, t] \).

(D3) For each \( t > 0 \) there exists \( s > t \) such that \( \delta_s(\sigma) > \frac{1}{2} \) for \( \sigma \in (0, t) \).

It follows from integration by parts that

\[
\int_0^s f(\sigma)\delta_s(\sigma)d\sigma < 0 \quad \forall s \in (0, s_0).
\]

(13)

Let us convince ourselves that (13) is true. Let \( \delta_s'(\sigma) := d\delta_s(\sigma)/d\sigma \). Given \( s \in (0, s_0) \) we can write

\[
\int_0^s f(\sigma)\delta_s(\sigma)d\sigma = F(\sigma)\delta_s(\sigma)|_0^s - \int_0^s F(\sigma)\delta_s'(\sigma)d\sigma
\]

\[
= - \int_0^s F(\sigma)\delta_s'(\sigma)d\sigma
\]

\[
< 0
\]

since for \( \sigma \in (0, s) \) we have \( \delta_s'(\sigma) < 0 \) by (D1) and \( F(\sigma) < 0 \) by (L2).

Next we show

\[
\int_0^{s_0} f(\sigma)\delta_s(\sigma)d\sigma > \int_0^{s_0} f(\sigma)\delta_t(\sigma)d\sigma \quad \text{for} \quad s > t > s_0.
\]

(14)

Given \( s > t > s_0 \), let \( \Delta(\sigma) := \delta_s(\sigma) - \delta_t(\sigma) \) and \( \Delta'(\sigma) := d\Delta(\sigma)/d\sigma \). We can write

\[
\int_0^{s_0} f(\sigma)\delta_s(\sigma)d\sigma = \int_0^{s_0} f(\sigma)\Delta(\sigma)d\sigma + \int_0^{s_0} f(\sigma)\delta_t(\sigma)d\sigma
\]

\[
= F(\sigma)\Delta(\sigma)|_0^{s_0} - \int_0^{s_0} F(\sigma)\Delta'(\sigma)d\sigma + \int_0^{s_0} f(\sigma)\delta_t(\sigma)d\sigma
\]

\[
= - \int_0^{s_0} F(\sigma)\Delta'(\sigma)d\sigma + \int_0^{s_0} f(\sigma)\delta_t(\sigma)d\sigma
\]

\[
> \int_0^{s_0} f(\sigma)\delta_t(\sigma)d\sigma
\]

since for \( \sigma \in (0, s_0) \) we have \( \Delta'(\sigma) > 0 \) by (D2) and \( F(\sigma) < 0 \) by (L2).
Since $F$ is positive, nondecreasing on $(s_0, \infty)$, we have $f(z) > 0$ for $z > s_0$. Therefore (D2) implies

$$\int_{s_0}^{s} f(\sigma) \delta_s(\sigma) d\sigma > \int_{s_0}^{t} f(\sigma) \delta_t(\sigma) d\sigma \quad \text{for} \quad s > t > s_0. \quad (15)$$

Combining (13) and (15) we obtain

$$\int_{s}^{s_0} f(\sigma) \delta_s(\sigma) d\sigma > \int_{t}^{s_0} f(\sigma) \delta_t(\sigma) d\sigma \quad \text{for} \quad s > t > s_0. \quad (16)$$

Moreover, since $F(s) \to \infty$ as $s \to \infty$, (D3) readily yields

$$\lim_{s \to \infty} \int_{s}^{s_0} f(\sigma) \delta_s(\sigma) d\sigma = \infty. \quad (17)$$

Combining (13), (16), and (17) gives the result. □

Lemma 2 We have $\bar{\gamma}_{ij}(v) = \rho_{ij}(|v|) \frac{v}{|v|}$ where $\rho_{ij} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is

$$\rho_{ij}(s) := \frac{1}{2\pi} \int_{0}^{2\pi} \gamma_{ij}(s \sin \varphi) \sin \varphi d\varphi. \quad (18)$$

Proof. Given $v \in \mathbb{R}^2$, let $r = |v|$ and $\theta \in [0, 2\pi)$ be such that $r[\cos \theta \sin \theta]^T = v$. Then, by using standard trigonometric identities,

$$\bar{\gamma}_{ij}(v) = \frac{1}{2\pi} \int_{0}^{2\pi} \left[ -\frac{\sin \varphi}{\cos \varphi} \right] \gamma_{ij}(r(\sin \varphi \cos \theta + \cos \varphi \sin \theta)) d\varphi$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \left[ -\frac{\sin \varphi}{\cos \varphi} \right] \gamma_{ij}(r \sin(\varphi + \theta)) d\varphi$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \left[ -\frac{\sin(\varphi - \theta)}{\cos(\varphi - \theta)} \right] \gamma_{ij}(r \sin \varphi) d\varphi$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \left[ -\frac{\sin \varphi \cos \theta + \cos \varphi \sin \theta}{\cos \varphi \cos \theta + \sin \varphi \sin \theta} \right] \gamma_{ij}(r \sin \varphi) d\varphi$$

$$= \left( \frac{1}{2\pi} \int_{0}^{2\pi} \gamma_{ij}(r \sin \varphi) \sin \varphi d\varphi \right) \left[ -\frac{\cos \theta}{\sin \theta} \right]$$

$$+ \left( \frac{1}{2\pi} \int_{0}^{2\pi} \gamma_{ij}(r \sin \varphi) \cos \varphi d\varphi \right) \left[ \frac{\sin \theta}{\cos \theta} \right]. \quad (19)$$

We focus on the second term in (19). Observe that

$$\int_{0}^{\pi} \gamma_{ij}(r \sin \varphi) \cos \varphi d\varphi = \int_{-\pi/2}^{\pi/2} \gamma_{ij} \left( r \sin \left( \varphi + \frac{\pi}{2} \right) \right) \cos \left( \varphi + \frac{\pi}{2} \right) d\varphi$$

$$= 0$$
since the integrand is an odd function on the interval of integration. Likewise,
\[
\int_0^{2\pi} \gamma_{ij}(r \sin \varphi) \cos \varphi d\varphi = \int_{-\pi/2}^{\pi/2} \gamma_{ij} \left( r \sin \left( \varphi + \frac{3\pi}{2} \right) \right) \cos \left( \varphi + \frac{3\pi}{2} \right) d\varphi = 0.
\]

Therefore
\[
\int_0^{2\pi} \gamma_{ij}(r \sin \varphi) \cos \varphi d\varphi = \int_0^{\pi/2} \gamma_{ij}(r \sin \varphi) \cos \varphi d\varphi + \int_{\pi/2}^{2\pi} \gamma_{ij}(r \sin \varphi) \cos \varphi d\varphi = 0.
\]

Combining (19) and (20) we obtain
\[
\bar{\gamma}_{ij}(v) = \left( \frac{1}{2\pi} \int_0^{2\pi} \gamma_{ij}(r \sin \varphi) \sin \varphi d\varphi \right) \left[ -\cos \theta \sin \theta \right] = \rho_{ij}(|v|) \frac{v}{|v|}.
\]

Hence the result.

\textbf{Claim 2} Defined in (18), function \( \rho_{ij}(s) \equiv 0 \) if \( \gamma_{ij} = 0 \). Otherwise, there exists \( \alpha \in \mathcal{K} \) such that \( \rho_{ij}(s) \geq \alpha(s) \) for all \( s \).

\textbf{Proof.} That \( \rho_{ij}(s) \equiv 0 \) if \( \gamma_{ij} = 0 \) is evident from the definition. Suppose \( \gamma_{ij} \neq 0 \). Then by (G2) there exists \( \alpha_1 \in \mathcal{K} \) such that \( |\gamma_{ij}(s)| \geq \alpha_1(|s|) \). Then we can write
\[
\rho_{ij}(s) = \frac{1}{2\pi} \int_0^{2\pi} \gamma_{ij}(s \sin \varphi) \sin \varphi d\varphi = \frac{1}{2\pi} \int_0^{2\pi} |\gamma_{ij}(s \sin \varphi)| \sin \varphi d\varphi \geq \frac{1}{2\pi} \int_0^{2\pi} \alpha_1(s \sin \varphi)|\sin \varphi| d\varphi =: \alpha_2(s).
\]

Note that \( \alpha_2 \) is a class-\( \mathcal{K} \) function. Hence the result.

In the remainder of the section we generate two results on average array (8). In the first of those results (Theorem 1) we establish that each of the oscillators (8) eventually oscillates with some magnitude no greater than some constant \( \rho \) which only depends on function \( f \) and not on initial conditions. In the second result (Theorem 2) we assert that if the initial conditions are right then

\[\text{We admit that the word oscillator may have been an unfortunate choice here since there is no oscillation (in the standard meaning of the word) taking place in the average array. Averaging gets rid of oscillations.}\]
the oscillators eventually synchronize both in phase and magnitude, where the magnitude equals $\rho$. Initial conditions' being right roughly corresponds to the following condition. If we depict each oscillator’s initial phase with a point on the unit circle then those points should all lie in an open semicircle. Constant $\rho$ we mentioned above is indeed defined as the unique positive number satisfying

$$\kappa(\rho) = 0.$$ 

Existence and uniqueness of $\rho$ is guaranteed by Claim 1. Note then that the first result is about the asymptotic behavior of the solutions of system (9) with respect to the following set

$$B := \{ \eta \in \mathbb{R}^{2m} : |\eta| \leq \rho \}.$$ 

Likewise, the second result has to do with the asymptotic behavior of the solutions of system (9) with respect to

$$R := \{ \eta \in \mathbb{R}^{2m} : \eta_i = \eta_j, |\eta| = \rho \text{ for all } i, j \}.$$ 

**Theorem 1** Consider system (9). Set $B$ is globally asymptotically stable.

**Proof.** We will establish the result by constructing a Lyapunov function. Claim 1 implies that there exists $\alpha_1 \in K$ such that $\alpha_1(s - \rho) \leq \kappa(s)$ for $s \geq \rho$. Then we can find $\alpha_2 \in K$ such that

$$\alpha_2(|\eta|_B) \leq \alpha_1 \left( \max \{0, \max_i \{|\eta_i| - \rho\}\} \right) \max_i |\eta_i|.$$

Let our candidate Lyapunov function $V : \mathbb{R}^{2m} \to \mathbb{R}_{\geq 0}$ be

$$V(\eta) := \frac{1}{2} \max \{0, \max_i \{ |\eta_i|^2 - \rho^2 \} \}.$$ 

Then there exist $\alpha_3, \alpha_4 \in K_\infty$ such that

$$\alpha_3(|\eta|_B) \leq V(\eta) \leq \alpha_4(|\eta|_B).$$

(21)

Now, given some $\eta$ with $|\eta|_B > 0$, let (nonempty) set of indices $I$ be such that $\frac{1}{2}(|\eta_i|^2 - \rho^2) = V(\eta)$ iff $i \in I$. Observe that for $i \in I$, point $\eta_i$ lies on the boundary of the smallest disk (centered at the origin) that contains all points $\eta_j$. Therefore $i \in I$ implies $\eta_i^T (\eta_j - \eta_i) \leq 0$ for all $j$. Then by Lemma 1 and

---

\footnote{That is, endpoints are not included.}
Lemma 2 we can write (almost everywhere)

\[
\langle \nabla V(\eta), \bar{g}(\eta) \rangle = \max_{i \in I} \eta_i^T \left( -\bar{f}(\eta_i) + \sum_{j \neq i} \bar{\gamma}_{ij} (\eta_j - \eta_i) \right)
\]

\[
= \max_{i \in I} \left( -\eta_i^T \bar{f}(\eta_i) + \sum_{j \neq i} \eta_i^T \bar{\gamma}_{ij} (\eta_j - \eta_i) \right)
\]

\[
= \max_{i \in I} \left( -\kappa(\max_i |\eta_i|) |\eta_i| + \sum_{j \neq i} \rho_{ij} (|\eta_j - \eta_i|) \eta_i^T (\eta_j - \eta_i) \right)
\]

\[
\leq -\kappa \left( \max_i |\eta_i| \right) \max_i |\eta_i|
\]

\[
\leq -\alpha_1 \left( \max_i |\eta_i| - r \right) \max_i |\eta_i|
\]

\[
\leq -\alpha_2 (|\eta| B).
\]

(22)

Result follows from (21) and (22).

\[\Box\]

Remark 3 Theorem 1 does not require interconnection \{\gamma_{ij}\} to be connected. This is clear with (or even without) the proof.

Theorem 2 Consider system (9). Let \( R \) be locally asymptotically stable. In particular, \( \{\eta_1(0), \ldots, \eta_m(0)\} \cap \{0\} = \emptyset \) implies \( \eta(t) \to \eta^* \) as \( t \to \infty \) for some \( \eta^* \in \mathcal{R} \).

To prove the theorem we use an invariance principle (similar to that of LaSalle’s) for which we need to tailor an invariant set for our system. To this end we introduce some notation. Given \( \eta \in \mathbb{R}^{2m} \) with \( \text{co}\{\eta_1, \ldots, \eta_m\} \cap \{0\} = \emptyset \) we let \( C(\eta) \subset \mathbb{R}^2 \) and \( \mathcal{W}_\rho(\eta) \subset \mathbb{R}^2 \) respectively be the smallest cone and smallest \( \rho \)-wedge containing set \{\eta_1, \ldots, \eta_m\}. (Recall that \( \rho > 0 \) satisfies \( \kappa(\rho) = 0 \).) Fig. 5 depicts \( \mathcal{W}_\rho(\eta) \). Note that

\[
\mathcal{W}_\rho(\eta) \subset C(\eta).
\]

(23)

For \( \zeta \in \mathbb{R}^{2m} \), when we write \( \zeta \in \mathcal{W}_\rho(\eta)^m \) we mean that \( \zeta_i \in \mathcal{W}_\rho(\eta) \) for all \( i = 1, \ldots, m \). Note that

\[
\zeta \in \mathcal{W}_\rho(\eta)^m \implies \mathcal{W}_\rho(\zeta) \subset \mathcal{W}_\rho(\eta).
\]

(24)

Proof of Theorem 2. We first show local stability of \( \mathcal{R} \). Let \( \eta \in \mathbb{R}^{2m} \) be such that \( \text{co}\{\eta_1, \ldots, \eta_m\} \cap \{0\} = \emptyset \). Consider the first term of the righthand side in (8). Lemma 1 and Claim 1 tell us that vector \( -\bar{f}(\eta_i) \) (tail placed at point \( \eta_i \)) points toward the origin if \( |\eta_i| > \rho \) and away from the origin if \( |\eta_i| < \rho \). Therefore, for \( \eta_i \) on the boundary of \( \mathcal{W}_\rho(\eta) \), vector \( -\bar{f}(\eta_i) \) never points outside \( \mathcal{W}_\rho(\eta) \). Now consider the sum term in (8). Lemma 2 and Claim 2 tell us that each summand \( \bar{\gamma}_{ij}(\eta_j - \eta_i) \), if nonzero, is a vector pointing from \( \eta_i \) to \( \eta_j \). Hence
θ > 0, we have for i

Let H

we can find a node n

tem (9) with ζ

ρ

that

W

lim

bounded from below by definition. Therefore there exists θ

ζ

Now, without loss of generality, assume that

i

there exist

the set of indices at time

t

l /

zero at zero. Moreover, if there is no edge of

G

ρ

by assumption. By Lemma 2 and Claim 2 we know that

forward invariance of

W

Next we show attractivity.

Figure 5: An example with η ∈ R^{10}. Dots represent the positions of η_ℓ on plane. The shaded region is W_ρ(η).

sum \sum_{j \neq i} \bar{\gamma}_{ij} (\eta_j - \eta_i) cannot be pointing outside convex hull co{\eta_1, \ldots, \eta_m} for η_ℓ on the boundary of the convex hull. Since co{\eta_1, \ldots, \eta_m} ⊂ W_ρ(η) we deduce therefore that the right-hand side of (9), that is, vector - \bar{f}(\eta_ℓ) + \sum_{j \neq i} \bar{\gamma}_{ij} (\eta_j - \eta_i), never points outside W_ρ(η) for η_ℓ on the boundary of W_ρ(η). Hence compact set W_ρ(η)^m is forward invariant with respect to system (9). Observe also that for η^* ∈ R we have W_ρ(η)^m → {η^*} as η → η^*. Hence set R is locally stable. Next we show attractivity.

Let G denote the graph of interconnection \{γ_{ij}\}. Note that G is connected by assumption. By Lemma 2 and Claim 2 we know that ρ_{ij} is continuous and zero at zero. Moreover, if there is no edge of G from node i to node j then ρ_{ij}(s) ≡ 0. Otherwise there exists α ∈ K such that ρ_{ij}(s) ≥ α(s) for all s.

Consider system (9). Let co{\eta_1(0), \ldots, \eta_m(0)} ∩ \{0\} = \emptyset. Due to (24) and forward invariance of W_ρ(η(t))^{m}, for t_2 ≥ t_1 ≥ 0 we can write W_ρ(η(t_2)) ⊂ W_ρ(η(t_1)). By (23) map t ↦ ∆C(η(t)) is hence nonincreasing. It is also bounded from below by definition. Therefore there exists θ ∈ [0, π) such that lim_{t→∞} ∆C(η(t)) = θ. We now claim that θ = 0.

Suppose not. Then, by continuity, there must exist a solution ζ(⋅) to system (9) with ζ(0) ∈ W_ρ(η(0))^{m} such that ∆C(ζ(t)) = θ for all t ≥ 0. Recall that W_ρ(ζ(0)) is forward invariant. By (23) therefore

\[ \mathcal{C}(\zeta(t)) = \mathcal{C}(\zeta(0)) \quad \forall t ≥ 0. \]  

Let \mathcal{H}_1 and \mathcal{H}_2 be two half lines convex hull of which equals \mathcal{C}(\zeta(0)). Since θ > 0, we have \mathcal{H}_1 ≠ \mathcal{H}_2. Then we observe that \{ζ_1(0), \ldots, ζ_m(0)\} ∩ \mathcal{H}_i ≠ \emptyset for i = 1, 2. Let n_1, \ldots, n_m denote the nodes of graph G. Since G is connected we can find a node n_l to which there exists a path from every other node. Now, without loss of generality, assume that ζ_i(0) ∈ \mathcal{H}_1. Let I_i(t) denote the set of indices at time t ≥ 0 such that ζ_i(t) ∈ \mathcal{H}_1 iff i ∈ I_i(t). Clearly, l \notin I_l(0) and 1 ≤ #I_l(0) ≤ m - 1. Hence connectedness of G implies that there exist i_0 ∈ I_l(0) and j_0 ∈ \{1, \ldots, m\} \setminus I_l(0) such that (n_{i_0}, n_{j_0}) is an
edge of $G$. Existence of edge $(n_{i_0}, n_{j_0})$ implies that $\gamma_{i_0,j_0} \neq 0$. Now, observe that $\zeta_{i_0}(0) - \zeta_{i_0}(0) \neq 0$ since $\zeta_{i_0}(0) \in \mathcal{H}_1$ and $\zeta_{j_0}(0) \notin \mathcal{H}_1$. By Lemma 2, $\overline{\gamma}_{i_0,j_0}(\zeta_{j_0}(0) - \zeta_{i_0}(0))$ is a nonzero vector that is not parallel to $\mathcal{H}_1$. By (8) we can write

$$\dot{\zeta}_{i_0}(0) = v + \overline{\gamma}_{i_0,j_0}(\zeta_{j_0}(0) - \zeta_{i_0}(0))$$

where

$$v = -f(\zeta_{i_0}(0)) + \sum_{j \neq i_0,j_0} \overline{\gamma}_{i_0,j}(\zeta_{j}(0) - \zeta_{i_0}(0)).$$

By earlier arguments (see the first paragraph of the proof) we know that vector $v$ (tail placed at point $G(0)$ cannot point outside $C(\zeta(0))$. Therefore vector vector $\dot{\zeta}_{i_0}(0)$ is nonzero and not parallel to $\mathcal{H}_1$. This lets us be able to find $t_1 > 0$ such that $\zeta_{i_0}(t_1) \notin \mathcal{H}_1$ and $\#I_1(t_1) \leq \#I_1(0) - 1$. We can continue the procedure and find a sequence of instants $t_k > 0$ such that $\#I_1(t_k) \leq \#I_1(0) - k$. Since $\#I_1(0)$ is finite, there should exist $t^* > 0$ such that $\#I_1(t^*) = 0$, that is, $I_1(t^*) = \emptyset$. This observation translates to $\mathcal{H}_1 \cap \{\zeta_1(t^*), \ldots, \zeta_m(t^*)\} = \emptyset$, which yields $\mathcal{C}(\zeta(t^*)) \neq \mathcal{C}(\zeta(0))$ which contradicts with (20). Hence our claim is valid and $\lim_{t \to \infty} \angle C(\eta(t)) = 0$.

That $\angle C(\eta(t)) \to 0$ means that $\mathcal{C}(\eta(t)) \to \mathcal{H}$ for some half line $\mathcal{H}$. We also know that $\eta(t) \in \mathcal{W}_\rho(\eta(0))^m$ for all $t \geq 0$ and, by Theorem 1, $\eta(t) \to \mathcal{B}$. Therefore solutions $\eta_{t}(i)$, for $i = 1, \ldots, m$, converge to the following set

$$\mathcal{S} := \mathcal{H} \cap \mathcal{W}_\rho(\eta(0)) \cap \{v \in \mathbb{R}^2 : |v| \leq \rho\}.$$ 

Note that $\mathcal{S}$ is a line segment that satisfies $\mathcal{S} = \{\lambda u : \delta \leq \lambda \leq \rho\}$ for some unit vector $u \in \mathbb{R}^2$ and $\delta > 0$. Solution $\eta(t)$ will converge to the largest invariant set in $\mathcal{S}^m$. (Note that $\mathcal{S}^m$ itself is forward invariant.) Take any solution $\zeta(t)$ to system (4) with $\zeta(0) \in \mathcal{S}^m$. At any given time $t \geq 0$ let $i$ be such that point $\zeta_i(t)$ is farthest from the point $\rho u$. Then by Lemma 1, Claim 1, Lemma 2 and Claim 2 we have

$$\dot{\zeta}_i(t) = \begin{cases} 0 & \text{if } \zeta_i(t) = \rho u \\ \lambda u & \text{if } \zeta_i(t) \neq \rho u \end{cases}$$

for some $\lambda \geq -\kappa(\angle \zeta(t)) > 0$. Therefore $\max_i |\zeta_i(t) - \rho u| \to 0$ as $t \to \infty$. We then deduce that $\{\rho u\}^m$ is the largest invariant set in $\mathcal{S}^m$. Hence the result. ■

**Remark 4** In Theorem 2 we require condition $\co\{\eta_1(0), \ldots, \eta_m(0)\} \cap \{0\} = \emptyset$ for synchronization. If we express each $\eta_i$ in polar coordinates $(r_i, \theta_i)$, where $r_i \geq 0$ and $\theta_i \in [0, 2\pi)$, then the required condition is equivalent to that $r_i(0) \neq 0$ and $\theta_i(0) \in (\theta^*, \theta^* + \pi)$ for all $i$ and some $\theta^*$. In other words, if we denote each phase $\theta_i(0)$ by a point on the unit circle, then the condition is equivalent to that they all lie in an open semicircle. Is this condition really essential for synchronization? Theoretically speaking, yes. That is, one can easily find initial conditions that make an equilibrium but are not on the synchronization manifold.
However those equilibria may be practically disregardable if they are unstable, since slightest disturbance would rescue the system from being stuck at those points. Now we ask the second question. Does there exist a stable equilibrium outside synchronization manifold? This we find difficult to answer. We would nevertheless like to report that our attempts to hunt one via (limited) simulations on coupled van der Pol oscillators have failed.

Let us recapitulate what has hitherto been done. We have begun by an array of coupled Lienard oscillators \( \dot{\xi} = \ell(\xi, \omega) \) with frequency of oscillations \( \omega \). By looking at it from a rotating (at frequency \( \omega \)) reference frame we have obtained new array \( \dot{x} = \ell^{\text{BC}}(x, \omega t) \) whose righthand side is periodic in time. Exploiting this periodicity, we have then computed average dynamics \( \dot{\eta} = \overline{\ell}^{\text{BC}}(\eta) \). Finally we have shown for the average array that if oscillators’ initial phases all reside in an open semicircle then they synchronize both in phase and magnitude, where the magnitude should equal \( \rho \).

In the next section, based on our findings on the synchronization behavior of average array, we show that oscillators of array \( \dot{x} = \ell^{\text{BC}}(x, \omega t) \) and therefore of array \( \dot{\xi} = \ell(\xi, \omega) \) get arbitrarily close to synchronization for \( \omega \) large enough. We use tools from averaging theory to establish the results.

6 Synchronization of Lienard oscillators

The following definition is borrowed with slight modification from [18].

**Definition 1** Consider system \( \dot{x} = g(x, \omega, t) \). Closed set \( S \) is said to be semiglobally practically asymptotically stable (with respect to \( \omega \)) if for each pair \( (\Delta, \delta) \) of positive numbers, there exists \( \omega^* > 0 \) such that for all \( \omega \geq \omega^* \) the following hold.

(a) For each \( r > \delta \) there exists \( \varepsilon > 0 \) such that

\[ |x(t_0)|_S \leq \varepsilon \implies |x(t)|_S \leq r \quad \forall t \geq t_0. \]

(b) For each \( \varepsilon < \Delta \) there exists \( r > 0 \) such that

\[ |x(t_0)|_S \leq \varepsilon \implies |x(t)|_S \leq r \quad \forall t \geq t_0. \]

(c) For each \( r < \Delta \) and \( \varepsilon > \delta \) there exists \( T > 0 \) such that

\[ |x(t_0)|_S \leq r \implies |x(t)|_S \leq \varepsilon \quad \forall t \geq t_0 + T. \]

Result [18, Thm. 2] tells that the origin of a time-varying system is semiglobally practically asymptotically stable provided that the origin of the average system is globally asymptotically stable. The analysis therein, with almost no extra effort, can be extended to cover the case where the attractor is not a singleton but only compact. Note that set \( B \) is compact. Theorem [1] therefore yields the following result.
**Theorem 3** Consider system (6). Set $B$ is semiglobally practically asymptotically stable.

The following result is an adaptation of a general theorem on averaging [15, Thm. 2.8.1].

**Lemma 3** Let map $g : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ be locally Lipschitz and satisfy $g(x, \varphi + 2\pi) = g(x, \varphi)$ for all $x \in \mathbb{R}^n$ and $\varphi \in \mathbb{R}$. Define average function $\bar{g} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as

$$\bar{g}(x) := \frac{1}{2\pi} \int_0^{2\pi} g(x, \varphi) d\varphi.$$

Let $x(\cdot)$ and $\eta(\cdot)$ denote, respectively, the solutions of systems $\dot{x} = g(x, \omega t)$ and $\dot{\eta} = \bar{g}(\eta)$, where $\omega > 0$. Then for each compact set $D \subset \mathbb{R}^n$ and pair of positive real numbers $(T, \varepsilon)$ there exists $\omega^* > 0$ such that if

- $\omega \geq \omega^*$,
- $\eta(0) = x(0)$, and
- $x(t) \in D$ for all $t \in [0, T]

hold then $|\eta(t) - x(t)| \leq \varepsilon$ for all $t \in [0, T]$.

Lemma 3 lets us establish the following result.

**Theorem 4** Consider system (6). For each pair $(D, \delta)$, where $D \subset \mathbb{R}^2$ is a compact convex set that does not include the origin and $\delta > 0$, there exists $\omega^* > 0$ such that for all $\omega \geq \omega^*$ the following hold.

(a) There exists $\varepsilon > 0$ such that

$$|x(0)|_R \leq \varepsilon \implies |x(t)|_R \leq \delta \quad \forall t \geq 0.$$

(b) There exists $r > 0$ such that

$$x(0) \in D^m \implies |x(t)|_R \leq r \quad \forall t \geq 0.$$

(c) There exists $T > 0$ such that

$$x(0) \in D^m \implies |x(t)|_R \leq \delta \quad \forall t \geq T.$$

**Proof.** Let us be given $(D, \delta)$. Let $x(\cdot)$ and $\eta(\cdot)$ respectively denote the solutions of system (6) and system (9). We first work on part (a). Choose some $\delta_1 > 0$ such that $|x|_R \leq \delta_1$ implies $\text{co}\{x_1, \ldots, x_m\} \cap \{0\} = \emptyset$. Then set $\delta_2 := \frac{1}{2} \min\{\delta_1, \delta\}$. Choose $\varepsilon \in (0, \delta_2)$ such that

$$|\eta(0)|_R \leq \varepsilon \implies |\eta(t)|_R \leq \delta_2 \quad \forall t \geq 0.$$
Such $\varepsilon$ exists since $R$ is locally asymptotically stable for system $\mathcal{H}$ by Theorem $2$. Moreover, $|\eta(0)|_R \leq \varepsilon$ implies $\{\eta_1(0), \ldots, \eta_m(0)\} \cap \{0\} = \emptyset$. Therefore we can find $T_1 > 0$ such that

$$|\eta(0)|_R \leq \varepsilon \implies |\eta(t)|_R \leq \frac{\varepsilon}{2} \quad \forall t \geq T_1.$$ 

Let $\varepsilon_1 := \min\{\frac{\varepsilon}{2}, \delta_2\}$. By Lemma $3$ there exists $\omega_1 > 0$ such that $|x(t) - \eta(t)| \leq \varepsilon_1$ for all $t \in [0, T_1]$ provided that $\omega \geq \omega_1$, $|x(t)|_R \leq \delta$ for all $t \in [0, T_1]$, and $x(0) = \eta(0)$. Let now $\omega \geq \omega_1$ and $|x(0)|_R \leq \varepsilon$. Set $\eta(0) = x(0)$. Then for all $t \in [0, T_1]$ we can write

$$|x(t)|_R \leq |\eta(t)|_R + |x(t) - \eta(t)| \leq \delta_2 + \varepsilon_1 \leq \delta.$$ 

Also

$$|x(T_1)|_R \leq |\eta(T_1)|_R + |x(T_1) - \eta(T_1)| \leq \frac{\varepsilon}{2} + \varepsilon_1 \leq \varepsilon.$$ 

We can repeat this procedure for the following intervals $[kT_1, (k + 1)T_1]$, $k = 1, 2, \ldots$. Therefore $|x(t)|_R \leq \varepsilon$ implies $|x(t)|_R \leq \delta$ as long as $\omega \geq \omega_1$.

Part (b). By Theorem $3$ we can find $\omega_2 > 0$ and $r_1 > 0$ such that $\omega \geq \omega_2$ and $x(0) \in \mathcal{D}^m$ imply $|x(t)|_R \leq r_1$ for all $t \geq 0$. Then we can find $r > 0$ such that $|x(t)|_R \leq r_1$ implies $|x|_R \leq r$. Therefore $\omega \geq \omega_2$ and $x(0) \in \mathcal{D}^m$ imply $|x(t)|_R \leq r$ for all $t \geq 0$.

Part (c). First recall, from the proof of Theorem $4$, that any $\rho$-wedge is forward invariant with respect to system $\mathcal{H}$. Choose a $\rho$-wedge $\mathcal{W}_\rho$ such that $\mathcal{D} \subset \mathcal{W}_\rho$. Note that $\mathcal{W}_\rho$ is compact, convex, and does not include the origin.

By Theorem $4$ there exists $T > 0$ such that

$$\eta(0) \in \mathcal{W}_\rho^m \implies |\eta(t)|_R \leq \varepsilon_1$$

where $\varepsilon_1$ is as found in part (a). Then define $\mathcal{S} \subset \mathbb{R}^{2m}$ as

$$\mathcal{S} := \{x + \varepsilon_1 v : x \in \mathcal{W}_\rho^m, |v| \leq 1\}.$$ 

By Lemma $5$ there exists $\omega_3 > 0$ such that $|x(t) - \eta(t)| \leq \varepsilon_1$ for all $t \in [0, T]$ provided that $\omega \geq \omega_3$, $x(t) \in \mathcal{S}$ for all $t \in [0, T]$, and $x(0) = \eta(0)$. Let now $\omega \geq \omega_3$ and $x(0) \in \mathcal{D}^m$. Set $\eta(0) = x(0)$. Then for all $t \in [0, T]$ we have $|x(t) - \eta(t)| \leq \varepsilon_1$. Therefore

$$|x(T)|_R \leq |\eta(T)|_R + |x(T) - \eta(T)| \leq \varepsilon_1 + \varepsilon_1 \leq \varepsilon.$$ 

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From part (a) we know that $|x(T)|_\mathcal{R} \leq \varepsilon$ implies $|x(t)|_\mathcal{R} \leq \delta$ for all $t \geq T$, which was to be shown.

We complete the proof by setting $\omega^* := \max\{\omega_1, \omega_2, \omega_3\}$.

We next present the below result for the array of coupled Lienard oscillators. This result directly follow from the observation mentioned in Remark 1 and the fact that sets $\mathcal{B}$ and $\mathcal{R}$ are invariant under rotations in $\mathbb{R}^2$.

**Theorem 5** Theorem 3 and Theorem 4 hold true for system (3).

Theorem 3 roughly says that solutions $x_i(\cdot)$ of oscillators of array (5) can be made converge to an arbitrarily small neighborhood of the disk $\{v \in \mathbb{R}^2 : |v| \leq \rho\}$ starting from arbitrarily large initial conditions by choosing oscillation frequency $\omega$ arbitrarily large. Theorem 4 says that by choosing $\omega$ arbitrarily large, solutions $x_i(\cdot)$ can be made eventually become arbitrarily close to each other and to the ring $\{v \in \mathbb{R}^2 : |v| = \rho\}$, starting from within an arbitrary convex compact set that does not contain the origin. Finally Theorem 5 says that these two results should hold also for array of Lienard oscillators (2).

Theorem 5 marks the end of our answer to the question that we asked for coupled Lienard oscillators in Section 3. It is hard not to realize that the basic idea forming the skeleton of this answer serves as a solution approach also for the problem of understanding the synchronization behavior of coupled harmonic oscillators. Therefore in the next section we analyze the relation between the frequency of oscillations and the synchronization in an array of harmonic oscillators. The results of next section will be similar to the results of previous sections, however there will be two main differences. The first difference is that, unlike Lienard oscillators, where the magnitude of oscillations at synchronization is fixed, i.e., independent of initial conditions, with harmonic oscillators synchronization can occur at any magnitude depending on the initial conditions. The second difference is that the initial phases of harmonic oscillators do not play any role in determining whether synchronization will take place or not provided that the frequency of oscillations is large enough. Recall that this was not the case with Lienard oscillators.

# 7 Synchronization of harmonic oscillators

Consider the following array of coupled harmonic oscillators

\[
\begin{align*}
\dot{q}_i &= \omega p_i \\
\dot{p}_i &= -\omega q_i + \sum_{j \neq i} \gamma_{ij}(p_j - p_i), \quad i = 1, \ldots, m
\end{align*}
\]

where, as before, $\omega > 0$ is the frequency of oscillations and $\{\gamma_{ij}\}$ is a connected interconnection. Functions $\gamma_{ij}$ are assumed to be locally Lipschitz. We let $\xi_i = [q_i, p_i]^T$ denote the state of $i$th oscillator. Array (26) defines the below
system in $\mathbb{R}^{2m}$

$$\dot{\xi} = h(\xi, \omega)$$  \hspace{1cm} (27)

where $\xi = [\xi_1^T \ldots \xi_m^T]^T$ and what $h$ is should be clear. Following the same procedure adopted for Lienard array, applying change of coordinates $x_i(t) = e^{-S(\omega)t}x_i(t)$ yields

$$\dot{x}_i = \sum_{j \neq i} \left[ -\sin \omega t \cos \omega t \right] \gamma_{ij} \left( [-\sin \omega t \cos \omega t](x_j - x_i) \right)$$  \hspace{1cm} (28)

which defines the below system in $\mathbb{R}^{2m}$

$$\dot{x} = h^{\circ \circ}(x, \omega t)$$  \hspace{1cm} (29)

for $x = [x_1^T \ldots x_m^T]^T$. Then, due to the periodicity of righthand side of (28) we can talk about the average array

$$\dot{\eta}_i = \sum_{j \neq i} \bar{\gamma}_{ij} (\eta_j - \eta_i)$$  \hspace{1cm} (30)

where $\bar{\gamma}_{ij}$ is as defined in (7). Array (30) defines the below system in $\mathbb{R}^{2m}$

$$\dot{\eta} = \bar{h}^{\circ \circ}(\eta)$$  \hspace{1cm} (31)

for $\eta = [\eta_1^T \ldots \eta_m^T]^T$.

In the remainder of the section we establish via sequence of three theorems the relation between frequency of oscillations and synchronization of coupled harmonic oscillators. In the first of those results (Theorem 6) we show that the oscillators of average array (30) globally synchronize. Then from our first result we deduce (in Theorem 7) that solutions $x_i(\cdot)$ of array (28) should eventually become arbitrarily close to each other, while initially being arbitrarily far from each other, provided that frequency of oscillations is arbitrarily large. Finally we claim (in Theorem 8) that what is true for array (28) is also true for array (26) due to the norm-preserving nature of the transformation being used to transition between two arrays.

Let us now define synchronization manifold $\mathcal{A} \subset \mathbb{R}^{2m}$ as

$$\mathcal{A} := \{ \eta \in \mathbb{R}^{2m} : \eta_i = \eta_j \text{ for all } i, j \}$$

to be used in the theorems to follow.

**Theorem 6** Consider system (31). Synchronization manifold $\mathcal{A}$ is globally asymptotically stable.

**Proof.** By Lemma 2 we can write

$$\dot{\eta}_i = \sum_{j \neq i} \rho_{ij}(|\eta_j - \eta_i|) \frac{\eta_j - \eta_i}{|\eta_j - \eta_i|} =: \bar{h}_{i}^{\circ \circ}(\eta)$$

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for \( i = 1, \ldots, m \). Let \( G \) denote the graph of interconnection \( \{ \gamma_{ij} \} \). Graph \( G \) is connected by assumption. Note that \( \rho_{ij} \) is continuous and zero at zero. Also, by Claim 2 if there is no edge of \( G \) from node \( i \) to node \( j \) then \( \rho_{ij}(s) \equiv 0 \). If there is an edge from node \( i \) to node \( j \) then \( \rho_{ij}(s) > 0 \) for \( s > 0 \).

Therefore \( \bar{h}_{ii} \) is continuous; and vector \( \bar{h}_{ii}(\eta) \) always points to the (relative) interior of the convex hull of the set \( \{ \eta_i \} \cup \{ \eta_j : \text{there is an edge of } G \text{ from node } i \text{ to node } j \} \). These two conditions together with connectedness of \( G \) yield by [9, Cor. 3.9] that system (9) has the globally asymptotic state agreement property, see [9, Def. 3.4]. Another property of the system is invariance with respect to translations. That is, \( \bar{h}_{ii}(\eta + \zeta) = \bar{h}_{ii}(\eta) \) for \( \zeta \in \mathcal{A} \). These properties let us write the following.

(a) There exists \( \alpha \in \mathcal{K} \) such that \( |\eta(t)|_{\mathcal{A}} \leq \alpha(|\eta(0)|_{\mathcal{A}}) \) for all \( t \geq 0 \).

(b) For each \( r > 0 \) and \( \varepsilon > 0 \), there exists \( T > 0 \) such that \( |\eta(0)|_{\mathcal{A}} \leq r \) implies \( |\eta(t)|_{\mathcal{A}} \leq \varepsilon \) for all \( t \geq T \).

Finally, (a) and (b) give us the result by [19, Prop. 1].

**Theorem 7** Consider system (29). Synchronization manifold \( \mathcal{A} \) is semiglobally practically asymptotically stable.

**Proof.** Consider array (28). Observe that the righthand side depends only on the relative distances \( x_j - x_i \). This allows us to reduce the order of the system. Let us define

\[
y := \begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ \vdots \\ x_m - x_1 \end{bmatrix}
\]

Note that \( \dot{y} = h^r(y, \omega t) \) for some \( h^r: \mathbb{R}_+^{2m-2} \times \mathbb{R}_{>0} \to \mathbb{R}_+^{2m-2} \). Since functions \( \gamma_{ij} \) are assumed to be locally Lipschitz, \( h^r \) is locally Lipschitz in \( y \) uniformly in \( t \). Also, \( h^r \) is periodic in time by (28). Now consider array (30). Again the righthand side depends only on the relative distances \( \eta_j - \eta_i \). Define

\[
z := \begin{bmatrix} \eta_2 - \eta_1 \\ \eta_3 - \eta_1 \\ \vdots \\ \eta_m - \eta_1 \end{bmatrix}
\]

Then \( \dot{z} = \bar{h}^r(z) \) where \( \bar{h}^r \) is the time average of \( h^r \) and locally Lipschitz both due to that \( \bar{\gamma}_{ij} \) is the time average of \( \gamma_{ij} \).

Theorem 6 implies that the origin of \( \dot{z} = \bar{h}^r(z) \) is globally asymptotically stable. Then [18, Thm. 2] tells us that the origin of \( \dot{y} = h^r(y, \omega t) \) is semiglobally practically asymptotically stable. All there is left to complete the proof is to realize that semiglobal practical asymptotic stability of the origin of \( \dot{y} = h^r(y, \omega t) \) is
equivalent to semiglobal practical asymptotic stability of synchronization manifold $\mathcal{A}$ of system (29).

Recall that system (29) is obtained from system (27) by a time-varying change of coordinates that is a rotation in $\mathbb{R}^2$. Since rotation is a norm-preserving operation and set $\mathcal{A}$ is invariant under rotations, Theorem 7 yields the below result.

**Theorem 8** Consider harmonic oscillators (27). Synchronization manifold $\mathcal{A}$ is semiglobally practically asymptotically stable.

8 Conclusion

We have shown that nonlinearly-coupled Lienard-type oscillators (almost) synchronize provided that their phases initially lie in a semicircle and the frequency of oscillations is high enough. We have generated the same result for nonlinearly-coupled harmonic oscillators without any requirement on their initial phases. We have employed averaging techniques to establish our main theorems.

References

[1] E. Abd-Elrady, T. Soderstrom, and T. Wigren. Periodic signal modeling based on Lienard’s equation. *IEEE Transactions on Automatic Control*, 49:1773–1778, 2004.

[2] M. Arcak. Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, 52:1380–1390, 2007.

[3] V.I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer, second edition, 1988.

[4] I. Belykh, V. Belykh, and M. Hasler. Generalized connection graph method for synchronization in asymmetrical networks. *Physica D*, 224:42–51, 2006.

[5] L.L. Bonilla, C.J. Vicente, and R. Spigler. Time-periodic phases in populations of nonlinearly coupled oscillators with bimodal frequency distributions. *Physica D*, 113:79–97, 1998.

[6] A.M. dos Santos, S.R. Lopes, and R.L. Viana. Synchronization regimes for two coupled Lienard-type driven oscillators. *Chaos Solitons & Fractals*, 36:901–910, 2008.

[7] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer, 1997.

[8] H.K. Khalil. *Nonlinear Systems*. Prentice Hall, 1996.
[9] Z. Lin, B. Francis, and M. Maggiore. State agreement for continuous-time coupled nonlinear systems. *SIAM Journal on Control and Optimization*, 46:288–307, 2007.

[10] X. Liu and T. Chen. Boundedness and synchronization of y-coupled Lorenz systems with or without controllers. *Physica D*, 237:630–639, 2008.

[11] K. Murali and M. Lakshmanan. Transmission of signals by synchronization in a chaotic Van der Pol-Duffing oscillator. *Physical Review E*, 48:R1624–R1626, 1993.

[12] L.M. Pecora and T.L. Carroll. Master stability functions for synchronized coupled systems. *Physical Review Letters*, 80:2109–2112, 1998.

[13] A. Pikovsky and M. Rosenblum. Self-organized partially synchronous dynamics in populations of nonlinearly coupled oscillators. *Physica D*, 238:27–37, 2009.

[14] A. Pogromsky, G. Santoboni, and H. Nijmeijer. Partial synchronization: from symmetry towards stability. *Physica D*, 172:65–87, 2002.

[15] J.A. Sanders, F. Verhulst, and J. Murdock. *Averaging Methods in Nonlinear Dynamical Systems*. Springer, second edition, 2007.

[16] T.J. Slight, B. Romeira, L. Wang, J.M.L. Figueiredo, E. Wasige, and C.N. Ironside. A Lienard oscillator resonant tunnelling diode-laser diode hybrid integrated circuit: Model and experiment. *IEEE Journal of Quantum Electronics*, 44:1158–1163, 2008.

[17] G.-B. Stan and R. Sepulchre. Analysis of interconnected oscillators by dissipativity theory. *IEEE Transactions on Automatic Control*, 52:256–270, 2007.

[18] A.R. Teel, J. Peuteman, and D. Aeyels. Semi-global practical asymptotic stability and averaging. *Systems & Control Letters*, 37:329–334, 1999.

[19] A.R. Teel and L. Praly. A smooth Lyapunov function from a class-$KL$ estimate involving two positive semidefinite functions. *ESAIM: Control, Optimisation and Calculus of Variations*, 5:313–367, 2000.

[20] C.W. Wu. Synchronization in networks of nonlinear dynamical systems coupled via a directed graph. *Nonlinearity*, 18:1057–1064, 2005.

[21] C.W. Wu and L.O. Chua. Synchronization in an array of linearly coupled dynamical systems. *IEEE Transactions on Circuits and Systems-I*, 42:430–447, 1995.