On the boundary values of Sobolev $W_{p}^{1}$-functions

By Pavel Shvartsman

Department of Mathematics, Technion - Israel Institute of Technology, 32000 Haifa, Israel

e-mail: pshv@tx.technion.ac.il

Abstract

For each $p > n$ we use local oscillations to give intrinsic characterizations of the trace of the Sobolev space $W_{p}^{1}(\Omega)$ to the boundary of an arbitrary domain $\Omega \subset \mathbb{R}^{n}$.

1. Introduction

1.1. The trace problem for the Sobolev space $W_{p}^{1}(\Omega)$.

Let $\Omega$ be a domain in $\mathbb{R}^{n}$. We recall that, for each $p \in [1, \infty]$, the Sobolev space $W_{p}^{1}(\Omega)$ consists of all (equivalence classes of) real valued functions $f \in L_{p}(\Omega)$ whose first order distributional partial derivatives on $\Omega$ belong to $L_{p}(\Omega)$. $W_{p}^{1}(\Omega)$ is normed by

$$\|f\|_{W_{p}^{1}(\Omega)} := \|f\|_{L_{p}(\Omega)} + \|\nabla f\|_{L_{p}(\Omega)}.$$ 

By $L_{p}^{1}(\Omega)$ we denote the corresponding homogeneous Sobolev space, defined by the finiteness of the seminorm

$$\|f\|_{L_{p}^{1}(\Omega)} := \|\nabla f\|_{L_{p}(\Omega)}.$$ 

In this paper we study the following trace problem.

**Problem 1.1** Given $p \in [1, \infty]$ and an arbitrary function $f : \partial \Omega \to \mathbb{R}$, find a necessary and sufficient condition for $f$ to be the trace to $\partial \Omega$ of a function $F \in W_{p}^{1}(\Omega)$.

This problem is of great interest, mainly due to its important applications to boundary-value problems in partial differential equations where it is essential to be able to characterize the functions defined on $\partial \Omega$, which appear as traces to $\partial \Omega$ of Sobolev functions.

There is an extensive literature devoted to the theory of boundary traces in Sobolev spaces. Among the multitude of results we mention the monographs of Grisvard [14], Lions and Magenes [21], Maz’ya and Poborchi [22, 26], and the papers by Gagliardo [12], Nikol’skii [28], Besov [3], Aronszajn and Szepycki [2], Yakovlev [34, 35], Jonsson and Wallin [17, 18, 19], Maz’ya and Poborchi [23, 24, 25, 29], Maz’ya, Poborchi and Netrusov [27], Vasil’chik [33]; we refer the reader to these works and references therein for numerous results and techniques concerning this topic.

In these monographs and papers Problem 1.1 is investigated and solved for various families of...
smooth, Lipschitz and non-Lipschitz domains in $\mathbb{R}^n$ with different types of singularities on the boundary.

In this paper we characterize the traces to the boundary of Sobolev $W^1_p(\Omega)$-functions whenever $p > n$ and $\Omega$ is an arbitrary domain in $\mathbb{R}^n$.

The first challenge that we face in the study of Problem 1.1 for an arbitrary domain $\Omega \subset \mathbb{R}^n$ is the need to find a “natural” definition of the trace of a Sobolev $W^1_p(\Omega)$-function to the boundary of the domain which is compatible with the structure of Sobolev functions on the domain $\Omega$. More specifically, we want to choose this definition in such a way that every $f \in W^1_p(\Omega)$ possesses a well-defined “trace” to $\partial \Omega$ which in a certain sense characterizes the behavior of the function $f$ near the boundary.

We recall that, when $p > n$, it follows from the Sobolev embedding theorem that every function $f \in L^1_p(\Omega)$ coincides almost everywhere with a function satisfying the local Hölder condition of order $\alpha := 1 - \frac{n}{p}$. I.e., after possibly modifying $f$ on a set of Lebesgue measure zero, we have, for every cube $Q \subset \Omega$, that

$$|f(x) - f(y)| \leq C(n, p)\|f\|_{L^1_p(\Omega)} \|x - y\|^{1 - \frac{n}{p}} \quad \text{for every } x, y \in Q. \quad (1.1)$$

This fact enables us to identify each element of $L^1_p(\Omega)$ with its unique continuous representative. Thus we are able to restrict our attention to the case of continuous Sobolev functions defined on $\Omega$.

Since we deal only with continuous Sobolev functions defined on $\Omega$, given $f \in L^1_p(\Omega)$ it would at first seem quite natural to try to define the “boundary values” of $f$ on $\partial \Omega$ to be the continuous extension of $f$ from $\Omega$ to $\partial \Omega$. In other words, we could try to extend the domain of definition of $f$ to the closure of $\Omega$, the set $\overline{\Omega}$, by letting

$$\tilde{f}(y) := \lim_{x \to y, x \in \Omega} f(x), \quad \text{for each } y \in \overline{\Omega}. \quad (1.2)$$

This indeed is the natural definition to use for certain classes of domains in $\mathbb{R}^n$ such as Lipschitz domains or $(\varepsilon, \delta)$-domains, see Jones [16]. But in general it does not work. For an obvious example showing this, consider the planar domain which is a “slit square” $\Omega = (-1, 1)^2 \setminus J$, where $J$ is the line segment $[(−1/2, 0), (1/2, 0)]$. The reader can easily construct a $C^\infty$-function $f \in W^1_p(\Omega)$ which equals zero on the upper “semi-square” $\{x = (x_1, x_2) \in [-1/4, 1/4]^2 : x_2 > 0\}$ and takes the value 1 on the lower “semi-square” $\{x = (x_1, x_2) \in [-1/4, 1/4]^2 : x_2 < 0\}$. Clearly, (1.2) cannot provide a well-defined function $\tilde{f}$ on the segment $[(−1/4, 0), (1/4, 0)]$. The obvious reason for the existence of such kinds of counterexamples is the fact that the continuity of a $W^1_p(\Omega)$-function does not imply its uniform continuity on $\Omega$.

In order to define a notion of “trace to the boundary” which will work for all domains $\Omega$ we have to adopt a somewhat different approach. Its point of departure is an important property of Sobolev functions which will be recalled in more detail below (see definition (1.6) and inequality (1.7)), namely that every $f \in W^1_p(\Omega)$, $p > n$, is uniformly continuous with respect to a certain intrinsic metric $\rho_{\alpha, \Omega}$ defined on $\overline{\Omega}$.

This property motivates us to define the completion of $\Omega$ with respect to this intrinsic metric. We can then define the “trace to the boundary” of each function $f \in W^1_p(\Omega)$ by first extending $f$ by continuity (with respect to $\rho_{\alpha, \Omega}$) to a continuous function $\tilde{f}$ defined on this completion, and then taking the restriction of $\tilde{f}$ to the appropriately defined boundary of this completion.
As is of course to be expected, in all cases where the definition (1.2) is applicable, these new notions of boundary and trace coincide with the “classical” ones.

We begin our formal development of this approach in the next subsection.

1.2. Subhyperbolic metrics in \( \Omega \) and their Cauchy completions. Following the terminology of Buckley and Stanoyevitch [8], given \( \alpha \in [0, 1] \) and a rectifiable curve \( \gamma \subset \Omega \), we define the subhyperbolic length of \( \gamma \) by the line integral

\[
\text{len}_{\alpha, \Omega}(\gamma) := \int_{\gamma} \text{dist}(z, \partial \Omega)^{\alpha - 1} \, ds(z).
\]

(Here \( \text{dist}(z, \partial \Omega) \) denotes the usual Euclidean distance from the point \( z \) to the boundary of \( \Omega \), and \( ds \) denotes usual arc length.)

Then we let \( d_{\alpha, \Omega} \) denote the corresponding subhyperbolic metric on \( \Omega \) given, for each \( x, y \in \Omega \), by

\[
d_{\alpha, \Omega}(x, y) := \inf_{\gamma} \text{len}_{\alpha, \Omega}(\gamma)
\]

where the infimum is taken over all rectifiable curves \( \gamma \subset \Omega \) joining \( x \) to \( y \).

The metric \( d_{\alpha, \Omega} \) was introduced and studied by Gehring and Martio in [13]. See also [1, 20, 6] for various further results using this metric. Note also that \( \text{len}_{0, \Omega} \) and \( d_{0, \Omega} \) are the well-known quasihyperbolic length and quasihyperbolic distance, and \( d_{1, \Omega} \) is the inner (or geodesic) metric on \( \Omega \).

The subhyperbolic metric \( d_{\alpha, \Omega} \) with \( \alpha = (p - n)/(p - 1) \) arises naturally in the study of Sobolev \( W^1_p(\Omega) \)-functions for \( p > n \). In particular, Buckley and Stanoyevitch [7] proved that the local Hölder condition (1.1) is equivalent to the following Hölder-type condition: for every \( x, y \in \Omega \)

\[
|f(x) - f(y)| \leq C(n, p) \|f\|_{L^1_p(\Omega)} \left\{ d_{\alpha, \Omega}(x, y)^{1 - \frac{1}{p}} + \|x - y\|^{1 - \frac{1}{p}} \right\}
\]

(1.5)

provided \( f \in L^1_p(\Omega) \) and \( \alpha = (p - n)/(p - 1) \).

In view of this result it is convenient for us to introduce a new metric \( \rho_{\alpha, \Omega} \) on \( \Omega \) for each \( \alpha \in (0, 1] \), by simply putting

\[
\rho_{\alpha, \Omega}(x, y) := d_{\alpha, \Omega}(x, y) + \|x - y\|^{\alpha},
\]

(1.6)

for each \( x, y \in \Omega \). Then (1.5) can be rewritten in the following form:

\[
|f(x) - f(y)| \leq C(n, p) \|f\|_{L^1_p(\Omega)} \rho_{\alpha, \Omega}(x, y)^{1 - \frac{1}{p}}, \quad x, y \in \Omega,
\]

(1.7)

where \( \alpha = (p - n)/(p - 1) \). Thus every function \( f \in L^1_p(\Omega) \) is uniformly continuous with respect to the metric \( \rho_{\alpha, \Omega} \).

This observation immediately implies the following important fact for each \( p > n \) and \( \alpha = (p - n)/(p - 1) \):

Every Sobolev function \( f \in L^1_p(\Omega) \) admits a unique continuous extension from the metric space \((\Omega, \rho_{\alpha, \Omega})\) to its Cauchy completion.

Let us now recall several standard facts concerning Cauchy completions and fix the notation that we will use here for the particular case of the Cauchy completion of \((\Omega, \rho_{\alpha, \Omega})\).
Throughout this paper we will use the notation \((x_i)\) for a sequence \(\{x_i \in \Omega : i = 1, 2, \ldots\}\) of points in \(\Omega\). Let \(C[\Omega, \rho_{\alpha, \Omega}]\) be the family of all Cauchy sequences in \(\Omega\) with respect to the metric \(\rho_{\alpha, \Omega}\):

\[
C[\Omega, \rho_{\alpha, \Omega}] := \{(x_i) : x_i \in \Omega, \lim_{i, j \to \infty} \rho_{\alpha, \Omega}(x_i, x_j) = 0\}. \tag{1.8}
\]

Observe that, by definition \((1.6)\), the set \(C[\Omega, \rho_{\alpha, \Omega}]\) consists of sequences \((x_i) \subset \Omega\) which converge in \(\Omega\) (in the Euclidean norm) and are fundamental with respect to the metric \(d_{\alpha, \Omega}\).

By "\(\sim\)" we denote the standard equivalence relation on \(C[\Omega, \rho_{\alpha, \Omega}]\),

\[
(x_i) \sim (y_i) \iff \lim_{i \to \infty} \rho_{\alpha, \Omega}(x_i, y_i) = 0. \tag{1.9}
\]

For each sequence \((x_i) \in C[\Omega, \rho_{\alpha, \Omega}]\), we use the notation

\[
[(x_i)]_\alpha \mbox{ is the equivalence class of } (x_i) \mbox{ with respect to } \sim. \tag{1.10}
\]

Let \(\Omega^*, \alpha := \{[(x_i)]_\alpha : (x_i) \in C[\Omega, \rho_{\alpha, \Omega}]\}\) be the set of all equivalence classes with respect to \(\sim\). We let \(\rho_{\alpha, \Omega^*}\) denote the standard metric on \(\Omega^*, \alpha\) defined by the formula

\[
\rho_{\alpha, \Omega^*}([(x_i)]_\alpha, [(y_i)]_\alpha) := \lim_{i \to \infty} \rho_{\alpha, \Omega}(x_i, y_i), \quad [(x_i)]_\alpha, [(y_i)]_\alpha \in \Omega^*, \alpha. \tag{1.11}
\]

As usual we identify every point \(x \in \Omega\) with the equivalence class \(\hat{x} = [(x, x, \ldots)]_\alpha\) of the constant sequence. This identification enables us to consider the domain \(\Omega\) as a subset of \(\Omega^*, \alpha\).

Observe that

\[
\rho_{\alpha, \Omega^*}|_{\Omega \times \Omega} = \rho_{\alpha, \Omega},
\]

i.e., the mapping \(\Omega \ni x \mapsto \hat{x} \in \Omega^*, \alpha\) is an isometry.

**Remark 1.2** Since

\[
\|x - y\|_\alpha \leq \rho_{\alpha, \Omega}(x, y) = d_{\alpha, \Omega}(x, y) + \|x - y\|_\alpha, \quad x, y \in \Omega,
\]

every Cauchy sequence \((x_i) \in C[\Omega, \rho_{\alpha, \Omega}]\) is also a Cauchy sequence with respect to the Euclidean distance. Consequently it converges to a point in \(\Omega^*\). Moreover, all sequences from any given equivalence class \(\omega = [(x_i)]_\alpha \in \Omega^*, \alpha\) converge (in \(\| \cdot \|\)) to the same point. We denote the common (Euclidean) limit point of all these sequences by \(\ell(\omega)\); thus

\[
y_i \xrightarrow{\| \cdot \|} \ell(\omega) \quad \text{as} \quad i \to \infty \quad \text{for every sequence} \quad (y_i) \in \omega.
\]

Now \((1.6)\) and \((1.12)\) imply the following formula: for every \(\omega_1 = [(x_i)]_\alpha, \omega_2 = [(y_i)]_\alpha \in \Omega^*, \alpha\) we have

\[
\rho_{\alpha, \Omega^*}(\omega_1, \omega_2) = \lim_{i \to \infty} d_{\alpha, \Omega}(x_i, y_i) + \|\ell(\omega_1) - \ell(\omega_2)\|_\alpha. \tag{1.13}
\]
Remark 1.3 As we shall see below, $\rho_{\alpha,\Omega}(u, v) \sim \|u - v\|^\alpha$ provided $u, v$ belong to a sufficiently small neighborhood of a point $x \in \Omega$. This shows that the metric $\rho_{\alpha,\Omega}$ and the Euclidean metric determine the same local topology on $\Omega$. In particular, this implies that $\Omega$ is an open subset of $\Omega^{*,\alpha}$ (in the $\rho_{\alpha,\Omega}$-topology).

We are now ready to define a kind of “boundary” of $\Omega$, which is the appropriate replacement of the usual boundary for our purposes here, and will be one of the main objects to be studied in this paper.

Definition 1.4 Let $\alpha \in (0, 1]$ and let $\Omega$ be a domain in $\mathbb{R}^n$. We let $(\partial \Omega)_{\alpha}$ denote the boundary of $\Omega$ (as an open subset of $\Omega^{*,\alpha}$) in the topology of the metric space $(\Omega^{*,\alpha}, \rho_{\alpha,\Omega})$. We call $(\partial \Omega)_{\alpha}$ the $\alpha$-boundary of the domain $\Omega$.

We observe that, by Remark 1.3,

$$(\partial \Omega)_{\alpha} = \Omega^{*,\alpha} \setminus \Omega.$$  \hspace{1cm} (1.14)

Thus $(\partial \Omega)_{\alpha}$ consists of the new elements which appear as a result of taking the completion of $\Omega$ with respect to the metric $\rho_{\alpha,\Omega}$.

By Remark 1.2,

$$\ell(\omega) \in \partial \Omega \quad \text{for each} \quad \omega \in (\partial \Omega)_{\alpha}.$$  

This means that every element of the $\alpha$-boundary can be identified with a point $x \in \partial \Omega$ and an equivalence class $[(x_i)]_{\alpha}$ of Cauchy sequences (with respect to the metric $\rho_{\alpha,\Omega}$) which converge to $x$ in the Euclidean norm.

However, as of course is to be expected from the preceding discussion, in general the set $(\partial \Omega)_{\alpha}$ will not be in one to one correspondence with $\partial \Omega$ because there may be points $x \in \partial \Omega$ which “split” into a family of elements $\omega \in (\partial \Omega)_{\alpha}$ all of which satisfy $\ell(\omega) = x$. Such families may even be infinite. Every $\omega$ in such a family can be thought of as a certain “approach” to the point $x$ by elements of $\Omega$ whose $\rho_{\alpha,\Omega}$-distance to $x$ tends to 0. For example, in Fig. 1 below the point $z_1$ splits into 6 different elements of $(\partial \Omega)_{\alpha}$, while the points $z_2, z_3$ and $z_4$ split, respectively, into 4, 3 and 2 such elements.

Figure 1: A domain $\Omega$ with agglutinated parts of the boundary.
We will use the terminology *agglutinated point* for points (like \( z_i, i = 1, 2, 3 \), in Fig. 1) which “split” into multiple elements of the \( \alpha \)-boundary. Formally, a point \( z \in \partial \Omega \) will be called an \( \alpha \)-agglutinated point of \( \partial \Omega \) if there exist at least two different equivalence classes \( \omega_1, \omega_2 \in \Omega^{*; \alpha} \), \( \omega_1 \neq \omega_2 \), such that \( z = \ell(\omega_1) = \ell(\omega_2) \). We refer to the set of all \( \alpha \)-agglutinated points as the \( \alpha \)-agglutinated part of the boundary.

The reader may care to think of the \( \alpha \)-boundary of \( \Omega \) as a kind of “bundle” of the regular boundary \( \partial \Omega \) under cutting of \( \partial \Omega \) along its “agglutinated” parts, see Fig. 2.

Figure 2: Cutting the domain \( \Omega \) along the agglutinated parts of the boundary

**Remark 1.5** In general, when we associate elements of \((\partial \Omega)_\alpha\) to elements of \(\partial \Omega\) in the way described above, we can also lose a part of \(\partial \Omega\). I.e., there may exist points in \(\partial \Omega\) which do not arise as \(\ell(\omega)\) for any \(\omega \in (\partial \Omega)_\alpha\). We refer to the set of all such points as the \(\alpha\)-inaccessible part of \(\partial \Omega\). More formally, we set

\[
I_\alpha(\Omega) = \{ x \in \partial \Omega : \nexists \, \omega \in \Omega^{*; \alpha} \text{ such that } x = \ell(\omega) \}.
\]

Thus \(I_\alpha(\Omega)\) is the set of all points \(x \in \partial \Omega\) such that every sequence \((x_i)\) in \(\Omega\) which converges to \(x\) in \(\| \cdot \|\)-norm, is not a Cauchy sequence with respect to the metric \(\rho_{\alpha, \Omega}\). Roughly speaking \(I_\alpha(\Omega)\) consists of all points \(x \in \partial \Omega\) for which \(\rho_{\alpha, \Omega}(x, \Omega) = +\infty\).

We call the set \(\partial \Omega \setminus I_\alpha(\Omega)\) the \(\alpha\)-accessible part of \(\partial \Omega\).

It may be helpful to give an explicit example of a domain \(\Omega\) which has a non-empty inaccessible part. Figure 3 shows such a domain, which contains an infinite sequence of rectangular portions with slits. We shall choose \(\alpha = 1\) so that we are dealing with the geodesic metric. In this picture the line segment \([A, B]\) is a part of the boundary of \(\Omega\). Clearly, for every \(x \in [A, B], y \in \Omega\), and every sequence \((x_i)\) in \(\Omega\) such that \(\|x_i - x\| \to 0\), the intrinsic distance \(d_{1, \Omega}(x_i, y) \to \infty\) as \(i \to \infty\). (Of course, every such a sequence \((x_i)\) in \(\Omega\) is not a Cauchy sequence with respect to the geodesic distance in \(\Omega\).) Thus \([A, B]\) is the 1-inaccessible part of \(\partial \Omega\).

The need to deal (or not deal) appropriately with parts of the boundary of some domain \(\Omega\) which are not accessible with respect to an intrinsic metric on \(\Omega\) arises naturally in the study of boundary values of Sobolev functions.

For instance, consider the space \(L^{1}_\infty(\Omega)\) which coincides with the space Lip(\(\Omega, d_{1, \Omega}\)) of functions on \(\Omega\) satisfying a Lipschitz condition with respect to the geodesic metric \(d_{1, \Omega}\). This metric can be extended “by continuity” to the boundary of \(\Omega\). We denote this extension of \(d_{1, \Omega}\) by \(d_{1, \Omega}^\ast\). In this case the inaccessible part of \(\partial \Omega\), the set \(I_1(\Omega)\), consists of points \(x \in \partial \Omega\) such that the geodesic
Figure 3: A domain $\Omega$ with non-empty set of 1-inaccessible points of $\partial \Omega$.

distance from $x$ to $\Omega$, i.e., $d_{1,\overline{\Omega}}(x, \Omega)$, equals $+\infty$. Since $d_{1,\overline{\Omega}}(I_1(\Omega), \Omega) = +\infty$, there are no points in $\Omega$ which are close to $I_1(\Omega)$. Consequently, the notion of the trace of a Sobolev $L^1_{1,\infty}(\Omega)$-function to $I_1(\Omega)$ is meaningless. Conversely, for the same reason, for every function $f : \partial \Omega \to \mathbb{R}$ its values on $I_1(\Omega)$ do not have any influence on whether $f$ can be extended to a Sobolev $L^1_{1,\infty}(\Omega)$-function on all of the domain $\Omega$. (Note that the trace of the space $L^1_{1,\infty}(\Omega)$ to the accessible part of the boundary coincides with the space $\text{Lip}(\partial \Omega \setminus I_1(\Omega), d_{1,\overline{\Omega}})$.)

These observations motivate our approach to the notion of the boundary values of Sobolev functions on $\Omega$ as the restriction to the accessible part of $\partial \Omega$ rather than the restriction to all of the boundary of $\Omega$.

1.3. The trace to the Sobolev boundary of a domain. We are now in a position to define the trace of $L^1_{p}(\Omega)$ to the $\alpha$-boundary of $\Omega$ whenever $\alpha = (p - n)/(p - 1)$.

As already mentioned above, by inequality (1.7), every function $f \in L^1_{p}(\Omega)$ is uniformly continuous with respect to $\rho_{\alpha,\Omega}$. Since $\Omega$ is a dense subset of $\Omega^{*,\alpha}$ (in $\rho_{\alpha,\Omega}$-metric), there exists a (unique) continuous extension $\bar{f}_\alpha$ of $f$ from $\Omega$ to $\Omega^{*,\alpha}$.

From here onwards we will find it convenient to refer to the set $(\partial \Omega)_\alpha$ introduced in Definition 1.4 when $\alpha = (p - n)/(p - 1)$, as the Sobolev $W^1_p$-boundary of $\Omega$ in $\mathbb{R}^n$.

We let $\text{tr}_{(\partial \Omega)_\alpha} f$ denote the restriction of $\bar{f}_\alpha$ to $(\partial \Omega)_\alpha$, i.e.,

$$\text{tr}_{(\partial \Omega)_\alpha} f := \bar{f}_\alpha|_{(\partial \Omega)_\alpha}. \quad (1.15)$$

We refer to the function $\text{tr}_{(\partial \Omega)_\alpha} f$ as the trace of $f$ to the Sobolev $W^1_p$-boundary of $\Omega$.

More specifically, $\text{tr}_{(\partial \Omega)_\alpha} f$ is a function on $(\partial \Omega)_\alpha$ defined as follows: Let $\omega \in \Omega^{*,\alpha}$ be an equivalence class and let $(y_i) \in \omega$ be an arbitrary sequence. Then

$$\text{tr}_{(\partial \Omega)_\alpha} f(\omega) := \lim_{i \to \infty} f(y_i). \quad (1.16)$$

Since $f \in L^1_{p}(\Omega)$ is uniformly continuous with respect to $\rho_{\alpha,\Omega}$, the trace $\text{tr}_{(\partial \Omega)_\alpha} f$ is well defined and does not depend on the choice of the sequence $(y_i) \in \omega$ in (1.16).

An equivalent definition of the trace $\text{tr}_{(\partial \Omega)_\alpha} f$ is given by the formula:

$$\text{tr}_{(\partial \Omega)_\alpha} f(\omega) := \lim \{ f(x) : \rho_{\alpha,\overline{\Omega}}(x, \omega) \to 0, \ x \in \Omega \}. \quad (1.17)$$
For every domain $\Omega$ and for each $p > n$, we can define the Banach space $\text{tr}(\partial \Omega, \alpha)(W^1_p(\Omega))$ of all traces to the Sobolev boundary of $\Omega$:

$$\text{tr}(\partial \Omega, \alpha)(W^1_p(\Omega)) = \{ f : \exists \text{ a continuous } F \in W^1_p(\Omega) \text{ such that } \text{tr}(\partial \Omega, \alpha) F = f \}.$$ 

We equip the space $\text{tr}(\partial \Omega, \alpha)(W^1_p(\Omega))$ with the norm

$$\|f\|_{\text{tr}(\partial \Omega, \alpha)(W^1_p(\Omega))} = \inf \{ \|F\|_{W^1_p(\Omega)} : F \in W^1_p(\Omega) \text{ and continuous, } \text{tr}(\partial \Omega, \alpha) F = f \}.$$ 

We define the space $\text{tr}(\partial \Omega, \alpha)(L^1_p(\Omega))$ in an analogous way.

We now turn to a formulation of the main results of the paper. At this stage, for simplicity, we will present these results for a domain $\Omega$ in $\mathbb{R}^n$ whose boundary does not contain “agglutinated” parts. In other words, we assume for the moment that $\partial \Omega$ does not “split” under Cauchy completion of $\Omega$ with respect to $\rho_{\alpha, \Omega}$. This simplification will allow us to interpret the trace to the boundary as a function defined on (the accessible part of) $\partial \Omega$ rather than a function defined on a certain set of equivalence classes.

**Definition 1.6** We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies the condition $(A_{\alpha})$ if for every “accessible” point $x \in \partial \Omega$, i.e., for every $x \in \partial \Omega \setminus I_\alpha(\Omega)$, (see Remark 1.5), there exists a unique equivalence class $\omega \in \Omega^*_{\alpha}$ such that $\ell(\omega) = x$.

Thus for a domain $\Omega$ satisfying the condition $(A_{\alpha})$ we may identify the set $(\partial \Omega, \alpha)$ with $\partial \Omega \setminus I_\alpha(\Omega)$. Also we may simplify several main definitions and notions introduced above. In particular, we can introduce a metric on the set $\Omega \setminus I_\alpha(\Omega)$ of all accessible points by letting

$$d_{\alpha, \Omega}(x, y) := \lim \inf \{ \text{len}_{\alpha, \Omega}(\tilde{x}, \tilde{y}) : \tilde{x} \xrightarrow{\|\cdot\|} x, \tilde{y} \xrightarrow{\|\cdot\|} y, \tilde{x}, \tilde{y} \in \Omega \}, \quad (1.18)$$

see (1.3). Here $x, y \in \Omega \setminus I_\alpha(\Omega)$. Clearly, $d_{\alpha, \Omega} = d_{\alpha, \Omega}$ on $\Omega$. Moreover, it can be easily seen that $d_{\alpha, \Omega}$ coincides with the metric of the Cauchy completion of the metric space $(\Omega, d_{\alpha, \Omega})$, and that, for every $x, y \in \Omega \setminus I_\alpha(\Omega)$,

$$\rho_{\alpha, \Omega}(x, y) := d_{\alpha, \Omega}(x, y) + \|x - y\|^\alpha,$$

cf. (1.6).

In this setting the set $I_\alpha(\Omega)$ of inaccessible points, see Remark 1.5, coincides with the set

$$I_\alpha(\Omega) = \{ x \in \partial \Omega : d_{\alpha, \Omega}(x, \Omega) = \infty \}.$$

In turn, the trace to $(\partial \Omega, \alpha)$, cf. (1.15), can be defined as

$$\text{tr}(\partial \Omega, \alpha) f(x) = \lim \{ f(y) : d_{\alpha, \Omega}(y, x) \to 0, y \in \Omega \}, \quad x \in (\partial \Omega, \alpha),$$

provided $f \in L^1_p(\Omega)$, $p > n$, and $\alpha = (p - n)/(p - 1)$.

Recall that every function $f \in L^1_p(\Omega)$ satisfies inequality (1.5). In Section 5 we show that this inequality implies the following property of the trace $\tilde{f} = \text{tr}(\partial \Omega, \alpha) f$: for every $x, y \in (\partial \Omega, \alpha)$

$$|\tilde{f}(x) - \tilde{f}(y)| \leq C(n, p) \|f\|_{L^1_p(\Omega)} d_{\alpha, \Omega}(x, y)^{1 - \frac{1}{p}}.$$
In particular, for every function $f \in L^1_p(\Omega)$, $p > n$, its trace $\text{tr}_{(\partial \Omega)_\alpha} f$ to the Sobolev $W^1_p$-boundary $(\partial \Omega)_\alpha$ is a continuous (with respect to the metric $d_{\alpha, R}$) function on $(\partial \Omega)_\alpha$.

1.4. Main results: a variational criterion of the trace and a characterization via sharp maximal functions.

In [30] we studied the problem of characterizing the trace spaces $L^1_p(\mathbb{R}^n)|_S$ and $W^1_p(\mathbb{R}^n)|_S$ to an arbitrary closed set $S \subset \mathbb{R}^n$ whenever $p > n$. We gave various intrinsic characterizations of these trace spaces in terms of local oscillations and doubling measures supported on $S$. The approach introduced and used in [30] was based on an important property of the classical Whitney extension operator, namely that this operator provides an almost optimal extension of each function on $S$ which is the restriction to $S$ of a function in $W^1_p(\mathbb{R}^n)$.

Our approach here to Problem 1.1 is an adaptation of the main ideas and methods of [30]. In particular, we show that the Whitney extension operator has a similar property to the one just mentioned: it provides an almost optimal extension of every function defined on $\partial \Omega$ to a function from $W^1_p(\Omega)$ whenever $p > n$.

This enables us to characterize the boundary values of Sobolev $L^1_p$-functions and then of $W^1_p$-functions in ways similar to those presented in [30]. In particular, our first main result, Theorem 1.8, is an analog of a trace criterion for the space $L^1_p(\mathbb{R}^n)$ given in [30].

Before we recall that result, we will need to specify some more notation: Throughout this paper, the terminology “cube” will mean a closed cube in $\mathbb{R}^n$ whose sides are parallel to the coordinate axes. We let $Q(x, r)$ denote the cube in $\mathbb{R}^n$ centered at $x$ with side length $2r$. Given $\lambda > 0$ and a cube $Q$ we let $\lambda Q$ denote the dilation of $Q$ with respect to its center by a factor of $\lambda$. (Thus $\lambda Q(x, r) = Q(x, \lambda r)$.) The Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$ will be denoted by $|A|$.

We proved in [30] that $f \in L^1_p(\mathbb{R}^n)|_S$ for $n < p < \infty$, if and only if there exists a constant $\lambda > 0$ such that for every finite family $\{Q_i : i = 1, ..., m\}$ of pairwise disjoint cubes in $\mathbb{R}^n$ and every choice of points $x_i, y_i \in (\eta Q_i) \cap S$ the inequality

$$\sum_{i=1}^m \frac{|f(x_i) - f(y_i)|^p}{(\text{diam } Q_i)^{p-n}} \leq \lambda$$

(1.19)

holds. Here $\eta$ is an absolute constant satisfying $\eta \leq 11$. The special case of this result where $S = \mathbb{R}^n$ and $\eta = 1$ was treated earlier by Yu. Brudnyi [4].

We call the criterion expressed by (1.19) the variational or discrete characterization of the trace space $L^1_p(\mathbb{R}^n)|_S$.

Theorem 1.8 which we will formulate in a moment, provides a “variational” intrinsic characterization of the trace space $\text{tr}_{(\partial \Omega)_\alpha}(L^1_p(\Omega))$. This result can be thought of as a slight modification of the variational criterion of (1.19), where the requirement $x_i, y_i \in (\eta Q_i) \cap S$ for the points $x_i$ and $y_i$ is augmented by a certain additional geometrical requirement, which we shall call “$Q$-visibility”.

**Definition 1.7** Given a point $x \in \overline{\Omega}$ and a cube $Q \subset \Omega$, we say that $x$ is $Q$-visible in $\Omega$ if for each $y \in Q$ the semi-open line segment $(x, y]$ lies in $\Omega$.

In other words, a point $x \in \overline{\Omega}$ is $Q$-visible in $\Omega$ if

$$\text{Conv}(Q \cup \{x\}) \setminus \{x\} \subset \Omega.$$  

Here Conv denotes the convex hull of a set. See Fig. 4.
Theorem 1.8 Let $\Omega$ be a domain in $\mathbb{R}^n$ satisfying condition $(A_\alpha)$. Let $p \in (n, \infty)$ and let $\alpha = (p - n)/(p - 1)$. Let $\eta$ be a constant satisfying $\eta \geq 41$.

A function $f : (\partial \Omega)_\alpha \to \mathbb{R}$ is the trace to $(\partial \Omega)_\alpha$ of a (continuous) function $F \in L^1_p(\Omega)$ if and only if $f$ is continuous (with respect to $d_{\alpha, \Gamma}$) and there exists a constant $\lambda > 0$ such that for every finite family $\{Q_i : i = 1, \ldots, m\}$ of pairwise disjoint cubes in $\Omega$ and every choice of $Q_i$-visible points $x_i, y_i \in (\eta Q_i) \cap (\partial \Omega)_\alpha$,

the inequality

$$\sum_{i=1}^m \frac{|f(x_i) - f(y_i)|^p}{(\text{diam } Q_i)^p - n} \leq \lambda$$

holds. Moreover,

$$\|f\|_{\text{tr}(\partial \Omega)_\alpha(L^1_p(\Omega))} \sim \inf \lambda^\frac{1}{p}$$

with constants of equivalence depending only on $n$, $p$ and $\eta$.
We prove Theorem 1.8 in Section 5 as a corollary of Theorem 5.1 which provides a variational trace criterion for an arbitrary domain $\Omega \subset \mathbb{R}^n$. As we shall see, the more general criterion which appears in Theorem 5.1 is a natural modification of the criterion in Theorem 1.8 where the notion of $Q$-visibility is adapted to the case of a domain whose boundary admits “agglutinated” points.

We now turn to the second main result of the paper, Theorem 1.9, which describes the traces of $W^1_p(\Omega)$-functions to the boundary of $\Omega$. Here again, we first need some more terminology. Given $\varepsilon > 0$ we let

$$O_{\varepsilon}(\partial \Omega) := \{ x \in \Omega : \text{dist}(x, \partial \Omega) < \varepsilon \}$$

(1.21)

denote the $\varepsilon$-neighborhood of $\partial \Omega$ in $\Omega$.

Let $\theta > 1$ and let $Q = \{ Q \}$ be a covering of $\Omega$ by non-overlapping cubes satisfying the following condition:

$$\frac{1}{\theta} \text{diam}(Q) \leq \text{dist}(Q, \partial \Omega) \leq \theta \text{diam}(Q).$$

(1.22)

Let $Q$ be a cube in $\Omega$ and let $a_Q$ be a point in $\partial \Omega$ which is nearest to $Q$ (in the Euclidean metric). Let $T_Q : \Omega \to \partial \Omega$ be a mapping defined by the formula

$$T_Q|_Q := a_Q, \quad Q \in \mathcal{Q}.$$  

(1.23)

Since $\mathcal{Q}$ is a family of non-overlapping cubes, the mapping $T_Q$ is well defined a.e. on $\Omega$.

**Theorem 1.9** Let $\Omega$ be a domain in $\mathbb{R}^n$ satisfying condition $(A_n)$ and let $p \in (n, \infty)$. Fix constants $\varepsilon > 0$, $\theta > 1$, $\eta \geq 22\theta^2$, and an arbitrary covering $\mathcal{Q}$ of $\Omega$ consisting of non-overlapping cubes $Q \subset \Omega$ each satisfying inequality (1.22).

A function $f : (\partial \Omega)_a \to \mathbb{R}$ is an element of $\text{tr}_{(\partial \Omega)}(W^1_p(\Omega))$ if and only if $f$ is continuous (with respect to $d_{\alpha, \Omega}$), $f \circ T_Q \in L_p(O_{\varepsilon}(\partial \Omega))$ and there exists a constant $\lambda > 0$ such that the inequality (1.20) holds for every finite family $\{ Q_i : i = 1, ..., m \}$ of pairwise disjoint cubes contained in $O_{\varepsilon}(\partial \Omega)$, and every choice of $Q_i$-visible points $x_i, y_i \in (\eta Q_i) \cap (\partial \Omega)_a$.

Moreover,

$$\| f \|_{\text{tr}_{(\partial \Omega)}(W^1_p(\Omega))} \sim \| f \circ T_Q \|_{L_p(O_{\varepsilon}(\partial \Omega))} + \inf \lambda^\frac{1}{2}$$

with constants of equivalence depending only on $n$, $p$, $\varepsilon$, $\theta$ and $\eta$.

A general version of this result, Theorem 5.3 which characterizes the trace space $\text{tr}_{(\partial \Omega)}(W^1_p(\Omega))$ for an arbitrary domain $\Omega \subset \mathbb{R}^n$, is proven in Section 5.

**Remark 1.10** Let $\mathcal{W}_\Omega$ be a fixed Whitney decomposition of $\Omega$, i.e., a covering of $\Omega$ by non-overlapping cubes such that $\text{diam} Q \sim \text{dist}(Q, \partial \Omega)$, $Q \in \mathcal{W}_\Omega$. Then the main statements of Theorems 1.8 and 1.9 remain true if we only consider cubes $\{ Q_i \}$ which belong to $\mathcal{W}_\Omega$. In fact, the only modification is that in this case the corresponding constants in the formulations of these theorems will also depend on parameters of the Whitney decomposition $\mathcal{W}_\Omega$.

Our next result provides a different kind of description of trace spaces. It is expressed in terms of a certain kind of maximal function with respect to the metric $d_{\alpha, \Omega}$. This maximal function is a variant of $f^p_\alpha$, the familiar fractional sharp maximal function in $L_p$:

$$f^\omega_p(x) := \sup_{r > 0} \frac{1}{r} \left( \frac{1}{|Q(x, r)|} \int_{Q(x, r)} |f(y) - f_{Q(x, r)}|^p dy \right)^{\frac{1}{p}}.$$
Here $f_Q := |Q|^{-1} \int_Q f \, dx$ denotes the average of $f$ on the cube $Q$. For $p = \infty$ the corresponding definition is

$$f_{\infty}^f(x) := \text{ess sup}\{|f(y) - f(z)|/r : r > 0, y, z \in Q(x, r)\}.$$  

In [9] Calderón proved that, for $1 < p \leq \infty$, a function $f$ is in $W_p^1(\mathbb{R}^n)$ if and only if $f$ and $f_{\infty}^f$ are both in $L_p(\mathbb{R}^n)$. See also [10]. We observe that inequality (1.1) allows us to replace $f_{\infty}^f$ in this statement with the (bigger) fractional sharp maximal function $f_{\infty}^f$ so that for $p > n$ and for every $f \in W_p^1(\mathbb{R}^n)$, we have

$$\|f\|_{W_p^1(\mathbb{R}^n)} \sim \|f\|_{L_p(\mathbb{R}^n)} + \|f_{\infty}^f\|_{L_p(\mathbb{R}^n)}.$$  

In [30] we introduced a variant of the fractional sharp maximal function $f_{\infty}^f$ defined with respect to an arbitrary closed set $S \subset \mathbb{R}^n$. For a function $f : S \to \mathbb{R}$ and $x \in \mathbb{R}^n$ it is given by

$$f_{\infty,S}^f(x) := \text{ess sup}\{|f(y) - f(z)|/r : r > 0, y, z \in Q(x, r) \cap S\}.$$  

We proved that $\|f\|_{L_p^1(\mathbb{R}^n)_{|S}} \sim \|f_{\infty,S}^f\|_{L_p(\mathbb{R}^n)}$ for every $f$ defined on $S$, provided that $p > n$.

Theorem 1.11 formulated below presents an analog of this result for the space $tr_{(\partial \Omega)_{\alpha}}(L_p^1(\Omega))$ whenever $\alpha = (p - n)/(p - 1)$ and $p > n$.

Let us introduce yet another variant of the fractional sharp maximal function, this time defined for functions on a domain $\Omega \subset \mathbb{R}^n$. Fix $q, n < q < p$, and put $\beta := (q - n)/(q - 1)$. For simplicity, we will assume that $\Omega$ satisfies the conditions $(A_n)$ and $(A_\beta)$, see Definition 1.6. Thus we can identify the $\alpha$-boundary and the $\beta$-boundary of $\Omega$ with (possibly different) subsets of $\partial \Omega$. In Section 2 we show that $\rho_{\alpha,\Omega} \leq C\rho_{\beta,\Omega}$ for some constant $C = C(\beta, n)$, see Corollary 2.9 so that $(\partial \Omega)_{\beta} \subset (\partial \Omega)_{\alpha} \subset \partial \Omega$.

We recall that the metric $d_{\beta,\overline{\Omega}}$ on $\overline{\Omega}$ is defined by formula (1.18). We introduce a quasi-metric $\delta_{\beta,\overline{\Omega}}$ on $\overline{\Omega}$ by setting

$$\delta_{\beta,\overline{\Omega}}(x, y) := d_{\beta,\overline{\Omega}}^1(x, y), \quad x, y \in \overline{\Omega}.$$  

Given $x \in \overline{\Omega}$ and $r > 0$, we let $B(x, r : \delta_{\beta,\overline{\Omega}})$ denote the closed ball in the quasi-metric space $(\overline{\Omega}, \delta_{\beta,\overline{\Omega}})$ with center $x$ and radius $r$:

$$B(x, r : \delta_{\beta,\overline{\Omega}}) := \{y \in \overline{\Omega} : \delta_{\beta,\overline{\Omega}}(x, y) \leq r\}.$$  

Our new fractional sharp maximal function $f_{\infty,\beta,\Omega}^f$ is defined for each $f : (\partial \Omega)_{\alpha} \to \mathbb{R}$ by

$$f_{\infty,\beta,\Omega}^f(x) := \text{ess sup}\left\{\frac{|f(y) - f(z)|}{r} : r > 0, y, z \in B(x, r : \delta_{\beta,\overline{\Omega}}) \cap (\partial \Omega)_{\beta}\right\}, \quad x \in \Omega.$$  

**Theorem 1.11** Let $n < q < p$ and let $\beta := (q - n)/(q - 1)$, $\alpha = (p - n)/(p - 1)$. Let $\Omega$ be a domain in $\mathbb{R}^n$ satisfying the conditions $(A_n)$ and $(A_\beta)$.

(i). A function $f \in tr_{(\partial \Omega)_{\alpha}}(L_p^1(\Omega))$ if and only if $f$ is continuous on $(\partial \Omega)_{\alpha}$ (with respect to $d_{\alpha,\overline{\Omega}}$) and also $f_{\infty,\beta,\Omega}^f \in L_p(\Omega)$. Moreover,

$$\|f\|_{tr_{(\partial \Omega)_{\alpha}}(L_p^1(\Omega))} \sim \|f_{\infty,\beta,\Omega}^f\|_{L_p(\Omega)}.$$  

(1.24)
(ii). Fix \( \varepsilon > 0, \theta > 1 \) and a covering \( Q \) of \( \Omega \) consisting of non-overlapping cubes \( Q \subset \Omega \) satisfying inequality (1.23). Let \( T_Q \) be the mapping defined by (1.23).

A function \( f : (\partial \Omega)_\alpha \to \mathbb{R} \) is an element of \( \text{tr}_{(\partial \Omega)_\alpha}(W^1_p(\Omega)) \) if and only if \( f \) is continuous on \( (\partial \Omega)_\alpha \) (with respect to \( d_{\alpha,\overline{\Omega}} \)), and \( f \circ T_Q \) and \( f^{\sharp}_\varepsilon \) are both in \( L_p(\mathcal{O}_\varepsilon(\partial \Omega)) \). Furthermore,

\[
\|f\|_{\text{tr}_{(\partial \Omega)_\alpha}(W^1_p(\Omega))} \sim \|f \circ T_Q\|_{L_p(\mathcal{O}_\varepsilon(\partial \Omega))} + \|f^{\sharp}_\varepsilon\|_{L_p(\mathcal{O}_\varepsilon(\partial \Omega))}.
\] (1.25)

The constants of equivalence in (1.24) depend only on \( n, p \) and \( q \), and in (1.25) they only depend on \( n, p, q, \varepsilon \) and \( \theta \).

This theorem is a particular case of a corresponding result for an arbitrary domain, Theorem \ref{thm:1.12} which we prove in Section 6.

As already mentioned above, the classical Whitney extension operator provides an almost optimal extension of a function \( f \) defined on \( (\partial \Omega)_\alpha \) to a function \( F \) in \( W^1_p(\Omega) \), whenever \( p > n \) and \( \alpha = (p - n)/(p - 1) \). Since this extension operator is linear, we obtain the following

**Theorem 1.12** Let \( \alpha = (p - n)/(p - 1) \). For every domain \( \Omega \subset \mathbb{R}^n \) and every \( p > n \) there exists a continuous linear operator \( E : \text{tr}_{(\partial \Omega)_\alpha}(W^1_p(\Omega)) \to W^1_p(\Omega) \) such that \( \text{tr}_{(\partial \Omega)_\alpha}(Ef) = f \) for each \( f \in \text{tr}_{(\partial \Omega)_\alpha}(W^1_p(\Omega)) \). Its operator norm is bounded by a constant depending only on \( n \) and \( p \).

A similar result holds for the space \( L^1_p(\Omega) \).

2. Subhyperbolic metrics on a domain and chains of cubes

Throughout the paper \( C, C_1, C_2, \ldots \) will be generic positive constants which depend only on parameters determining sets (say, \( n, \alpha, \theta \), etc.) or function spaces \( (p, q, \text{etc}) \). These constants can change even in a single string of estimates. The dependence of a constant on certain parameters is expressed, for example, by the notation \( C = C(n, p) \). We write \( X \sim Y \) if there is a constant \( C \geq 1 \) such that \( X/C \leq Y \leq CX \).

Recall that by \( |A| \) we denote the Lebesgue measure of a measurable set \( A \subset \mathbb{R}^n \).

It will be convenient for us to measure distances in \( \mathbb{R}^n \) in the uniform norm

\[
\|x\| := \max\{|x_i| : i = 1, \ldots, n\}, \quad x = (x_1, \ldots, x_i) \in \mathbb{R}^n.
\]

Thus every cube

\[
Q = Q(x, r) := \{y \in \mathbb{R}^n : \|y - x\| \leq r\}
\]

is a “ball” in \( \|\cdot\| \)-norm of “radius” \( r \) centered at \( x \). Given subsets \( A, B \subset \mathbb{R}^n \), we put

\[
\text{diam} A := \sup\{\|a - a'\| : a, a' \in A\}
\]

and

\[
\text{dist}(A, B) := \inf\{\|a - b\| : a \in A, b \in B\}.
\]

For \( x \in \mathbb{R}^n \) we also set \( \text{dist}(x, A) := \text{dist}(\{x\}, A) \). By \( \overline{A} \) we denote the closure of \( A \), and by \( A^\circ \) its interior.

Let \( Q = \{Q\} \) be a family of cubes in \( \mathbb{R}^n \). By \( M(Q) \) we denote its covering multiplicity, i.e., the minimal positive integer \( M \) such that every point \( x \in \mathbb{R}^n \) is covered by at most \( M \) cubes.
from $Q$. Finally, given a cube $Q \subset \Omega$, by $a_Q$ we denote a point in $\partial \Omega$ which is nearest to $Q$ in the Euclidean metric.

In this section we present a series of results related to the proof of the necessity part of the main theorems. We begin with geometrical characterizations of the intrinsic metrics $d_{\alpha, \Omega}$ and $\rho_{\alpha, \Omega}$ introduced in subsection 1.2, see (1.3), (1.4) and (1.6).

In the next two lemmas we estimate the subhyperbolic length of a line segment in a domain.

**Lemma 2.1** Let $\Omega$ be a domain in $\mathbb{R}^n$ and let $x, y \in \Omega$. Assume that

$$\|x - y\| \leq \max\{\text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega)\}. \quad (2.1)$$

Then for every $\alpha \in (0, 1]$ we have

$$\int_{[x, y]} \text{dist}(z, \partial \Omega)^{\alpha - 1} \, ds(z) \leq \frac{1}{\alpha} \|x - y\|^\alpha. \quad (2.2)$$

**Proof.** Suppose that $\|x - y\| \leq \text{dist}(x, \partial \Omega)$ so that $Q(x, \|x - y\|) \subset \Omega$.

Let $z \in [x, y]$. Then $\|x - y\| = \|x - z\| + \|z - y\|$ so that for every $u \in \Omega$ such that $\|u - z\| \leq \|y - z\|$ we have

$$\|x - u\| \leq \|x - z\| + \|z - u\| \leq \|x - z\| + \|z - y\| = \|x - y\|.$$  

We obtain $u \in Q(x, \|x - y\|) \subset \Omega$ so that $Q(z, \|y - z\|) \subset \Omega$. This proves that $\text{dist}(z, \partial \Omega) \geq \|y - z\|$. Hence

$$\int_{[x, y]} \text{dist}(z, \partial \Omega)^{\alpha - 1} \, ds(z) \leq \int_{[x, y]} \|y - z\|^{\alpha - 1} \, ds(z) = \frac{1}{\alpha} \|x - y\|^\alpha. \quad \Box$$

**Lemma 2.2** Let $\Omega$ be a domain in $\mathbb{R}^n$ and let $Q = Q(a, r)$ be a cube in $\Omega$. Let $x \in \overline{\Omega}$ be a $Q$-visible point, see Definition 1.7, and let $\beta \in (0, 1]$. Then:

(i). For every $b_1, b_2 \in (x, a]$ we have

$$\int_{[b_1, b_2]} \text{dist}(z, \partial \Omega)^{\beta - 1} \, ds(z) \leq C(\beta) \left( \frac{\|a - x\|}{\text{diam} Q} \right)^{1 - \beta} \|b_1 - b_2\|^\beta; \quad (2.3)$$

(ii). Every sequence $(x_i) \subset (x, a]$ which tends to $x$ (in $\|\cdot\|$-norm) is fundamental with respect to the metric $\rho_{\beta, \Omega}$. Moreover, every two sequences $(x_i), (y_i) \in (x, a]$ tending to $x$ in the Euclidean norm are equivalent with respect to $\rho_{\beta, \Omega}$, i.e.,

$$\lim_{i \to \infty} \rho_{\beta, \Omega}(x_i, y_i) = 0.$$  

**Proof.** Prove (i). Since $x$ is $Q$-visible, $\text{Conv}(x, Q) \setminus \{x\} \subset \Omega$ so that $tx + (1 - t)Q \subset \Omega$ for every $t \in [0, 1)$. Therefore for every $z \in [a, x)$ we have

$$Q(z, r_z) \subset \Omega \quad \text{where} \quad r_z := \frac{\|z - x\|}{\|a - x\|} r$$

so that

$$\text{dist}(z, \partial \Omega) \geq r_z = \frac{\|z - x\|}{\|a - x\|} r, \quad z \in [a, x).$$
Suppose that $\|b_1 - x\| \leq \|b_2 - x\|$. We have
\[
I = \int_{[b_1, b_2]} \text{dist}(z, \partial \Omega)^{\beta-1} \, ds(z) \leq \int_{[b_1, b_2]} \left( \frac{\|z - x\|}{\|a - x\|} \right)^{\beta-1} \, ds(z) = \left( \frac{\|a - x\|}{r} \right)^{1-\beta} \int_{[b_1, b_2]} \|z - x\|^{\beta-1} \, ds(z) = \left( \frac{\|a - x\|}{r} \right)^{1-\beta} \int_{[b_1-x]} s^{\beta-1} \, ds.
\]
Hence
\[
I \leq \frac{1}{\beta} \left( \frac{\|a - x\|}{r} \right)^{1-\beta} (\|b_2 - x\| - \|b_1 - x\|) \leq \frac{2^{1-\beta}}{\beta} \left( \frac{\|a - x\|}{\text{diam } \Omega} \right)^{1-\beta} \|b_1 - b_2\|^\beta
\]
proving (2.3).

Prove (ii). By (2.3), for every $u, v \subset (x, a]$ we have
\[
d_{\beta, \Omega}(u, v) \leq \int_{[u, v]} \text{dist}(z, \partial \Omega)^{\beta-1} \, ds(z) \leq C(\beta) \left( \frac{\|a - x\|}{\text{diam } \Omega} \right)^{1-\beta} \|u - v\|^{\beta}
\]
so that
\[
\rho_{\beta, \Omega}(u, v) = d_{\beta, \Omega}(u, v) + \|u - v\|^{\beta} \leq A \|u - v\|^{\beta}
\]
where
\[
A = 1 + C(\beta) \left( \frac{\|a - x\|}{\text{diam } \Omega} \right)^{1-\beta}.
\]

Now let $(x_i) \subset (x, a]$ and let $x_i \xrightarrow{\|\|} x$. Then, by (2.4),
\[
\rho_{\beta, \Omega}(x_i, x_j) \leq A \|x_i - x_j\|^{\beta} \to 0 \quad \text{as } i, j \to \infty,
\]
proving that $(x_i)$ is fundamental with respect to $\rho_{\beta, \Omega}$.

Let $(x_i), (y_i) \subset (x, a]$ and $(x_i), (y_i)$ tend to $x$ (in $\|\cdot\|$-norm). Applying (2.4) with $u = x_i$ and $v = y_i$ we obtain
\[
\rho_{\beta, \Omega}(x_i, y_i) \leq A \|x_i - y_i\|^{\beta} \to 0 \quad \text{as } i \to \infty.
\]

The lemma is proved. \qed

Lemma 2.3 Let $Q = Q(x, r), Q' = Q(x', r')$ be cubes in a domain $\Omega$ such that $Q \cap Q' \neq \emptyset$. Assume that either $\text{diam } Q' \leq \text{dist}(Q, \partial \Omega)$ or $Q' \subset Q$.

Let $a_Q$ be a point on $\partial \Omega$ nearest to $Q$ (in the Euclidean norm). Then $a_Q$ is $Q'$-visible in $\Omega$ and for each $y \in (a_Q, x)$ there exists a point $y' \in (a_Q, x']$ such that
\[
\|y - y'\| \leq C \|y - a_Q\|, \quad (2.5)
\]
and
\[
d_{\alpha, \Omega}(y, y') \leq C \|y - a_Q\|^\alpha, \quad \alpha \in (0, 1],
\]
see Fig. 6. Here $C$ is a constant depending only on $x, x', r, r', a_Q$, and $\alpha$. 

15
Figure 6: Lemma 2.3: the case $Q \cap Q' \neq \emptyset$ and diam $Q' \leq \text{dist}(Q, \partial \Omega)$.

Proof. Put $\lambda = \frac{\|a_Q - y\|}{\|a_Q - x\|}$ and $y' = a_Q + \lambda(x' - a_Q)$. Since $y \in (a_Q, x]$, we have $0 < \lambda \leq 1$ so that $y' \in (a_Q, x']$. In addition, $y = a_Q + \lambda(x - a_Q)$. Hence

$$
\|y - y'\| = \lambda\|x - x'\| = \|a_Q - y\| \|x - x'\|/\|a_Q - x\|,
$$
proving (2.5).

Let $z \in [y, y']$ and let $\tilde{Q} := Q(x, \text{dist}(x, \partial \Omega))$. Then $a_Q \in \partial \tilde{Q}$ and $\tilde{Q}^o \subset \Omega$.

If diam $Q' \leq \text{dist}(Q, \partial \Omega)$, then $Q' \subset \tilde{Q}^o$. Clearly, the same is true whenever $Q' \subset Q$. The open cube $\tilde{Q}^o$ is a convex set and the point $a_Q$ lies on its boundary so that $a_Q$ is $Q'$-visible in $\Omega$, see Definition 1.7.

Moreover, by dilation with respect to $a_Q$, we obtain

$$
Q_1 := Q(y, \lambda r), Q_2 := Q(y', \lambda r') \subset \tilde{Q}^o \subset \Omega.
$$

Since $\tilde{Q}^o$ is a convex set, the convex hull $\text{Conv}(Q_1, Q_2) \subset \tilde{Q}^o \subset \Omega$. In particular, if $\theta \in [0, 1]$ and $z = \theta y + (1 - \theta)y'$, then

$$
\theta Q_1 + (1 - \theta)Q_2 \subset \Omega.
$$

Let $r'' = \min\{r, r'\}$. Then

$$
Q(z, \lambda r'') \subset \theta Q_1 + (1 - \theta)Q_2 \subset \Omega,
$$
so that dist$(z, \partial \Omega) \geq \lambda r''$. Hence

$$
d_{a,\Omega}(y, y') \leq \int_{[y, y']} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) \leq (\lambda r'')^{\alpha-1} \int_{[y, y']} 1 ds(z)
$$
so that

$$
d_{a,\Omega}(y, y') \leq (\lambda r'')^{\alpha-1}\|y - y'\| = (\lambda r'')^{\alpha-1}(\lambda\|x - x'\|)
$$

$$
= \lambda^{\alpha} (r'')^{\alpha-1}\|x - x'\| = (\|a_Q - y\|/\|a_Q - x\|)^\alpha (r'')^{\alpha-1}\|x - x'\|.
$$

The lemma is proved. \square
Definition 2.4 Let $\Omega$ be a domain in $\mathbb{R}^n$ and let $x, y \in \Omega$. A finite family of cubes $\{Q_i \subset \Omega : i = 1, \ldots, m\}$ is said to be a chain of cubes joining $x$ to $y$ in $\Omega$ if $x \in Q_1, y \in Q_m$ and $Q_i \cap Q_{i+1} \neq \emptyset$ for every $i = 1, \ldots, m - 1$.

Lemma 2.5 Let $x, y \in \Omega$ and let $\alpha \in (0, 1]$. Then for every chain of cubes $\{Q_i : i = 1, \ldots, m\}$ joining $x$ to $y$ in $\Omega$ the following inequality

$$\rho_{\alpha, \Omega}(x, y) \leq (1 + 2/\alpha) \sum_{i=1}^{m} (\text{diam } Q_i)^{\alpha}$$

holds.

Proof. Let $Q_i = Q(x_i, r_i), i = 1, \ldots, m$, and let $z_i \in Q_i \cap Q_{i+1}, i = 1, \ldots, m - 1$. Put $z_0 := x$ and $z_m := y$. Then

$$\|x - y\|^\alpha = \|z_0 - z_m\|^\alpha \leq \left( \sum_{i=0}^{m-1} \|z_i - z_{i+1}\| \right)^\alpha \leq \sum_{i=0}^{m-1} \|z_i - z_{i+1}\|^\alpha \leq \sum_{i=1}^{m} (\text{diam } Q_i)^{\alpha}. $$

Let $\gamma$ be a broken line with nodes $\{z_0, x_1, z_1, x_2, z_2, \ldots, x_{m-1}, z_{m-1}, x_m, z_m\}$. Then

$$d_{\alpha, \Omega}(x, y) \leq \int_{\gamma} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z)$$

$$= \sum_{i=0}^{m-1} \left\{ \int_{[z_i, x_{i+1}]} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) + \int_{[x_{i+1}, z_{i+1}]} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) \right\}. $$

Since $z_i \in Q_{i+1} = Q(x_{i+1}, r_{i+1}) \subset \Omega$, $\|z_i - x_{i+1}\| \leq r_{i+1} \leq \text{dist}(x_{i+1}, \partial \Omega)$ so that, by Lemma 2.1,

$$\int_{[z_i, x_{i+1}]} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) \leq \frac{1}{\alpha} \|z_i - x_{i+1}\|^\alpha \leq \frac{1}{\alpha} (\text{diam } Q_{i+1})^{\alpha}. $$

In a similar way we prove that

$$\int_{[x_{i+1}, z_{i+1}]} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) \leq \frac{1}{\alpha} (\text{diam } Q_{i+1})^{\alpha}. $$

Hence

$$d_{\alpha, \Omega}(x, y) \leq (2/\alpha) \sum_{i=0}^{m-1} (\text{diam } Q_{i+1})^{\alpha}. $$

Finally,

$$\rho_{\alpha, \Omega}(x, y) = \|x - y\|^\alpha + d_{\alpha, \Omega}(x, y) \leq (1 + 2/\alpha) \sum_{i=1}^{m} (\text{diam } Q_i)^{\alpha}$$

proving the lemma. □
**Lemma 2.6** ([37]) Let \( x, y \in \Omega \) and let \( \gamma \subset \Omega \) be a continuous curve joining \( x \) to \( y \). There exists a chain of cubes \( \mathcal{Q} = \{Q_i = Q(z_i, r_i) : i = 1, \ldots, m\} \) joining \( x \) to \( y \) in \( \Omega \) such that:

(i). \( z_i \in \gamma \) and \( r_i = \frac{1}{8} \text{dist}(z_i, \partial \Omega) \) for every \( i = 1, \ldots, m \);

(ii). \( 2Q_i \subset \Omega, i = 1, \ldots, m \);

(iii). The covering multiplicity \( M(\mathcal{Q}) \) of the family of cubes \( \mathcal{Q} \) is bounded by a constant \( C = C(n) \).

**Lemma 2.7** For every \( x, y \in \Omega \) and every \( \alpha \in (0, 1] \) there exists a chain of cubes

\[
Ch_{\alpha, \Omega}(x, y) = \{Q_i \subset \Omega : i = 1, \ldots, m, \}
\]

joining \( x \) to \( y \) in \( \Omega \) with covering multiplicity \( M(C_{\alpha, \Omega}(x, y)) \leq C(n) \) such that the following inequality

\[
\sum_{i=1}^{m} (\text{diam } Q_i)^{\alpha} \leq C(\alpha, n)\rho_{\alpha, \Omega}(x, y)
\]

holds.

**Proof.** By ([13] and [14]), there exists a rectifiable curve \( \gamma \subset \Omega \) joining \( x \) to \( y \) such that

\[
\int_{\gamma} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) \leq 2d_{\alpha, \Omega}(x, y). \tag{2.6}
\]

Let \( \mathcal{Q} = \{Q_i = Q(z_i, r_i) : i = 1, \ldots, m, \} \) be a chain of cubes joining \( x \) to \( y \) in \( \Omega \) and satisfying conditions (i)-(iii) of Lemma 2.6.

Let us consider two cases. First suppose that \( x, y \in Q_k = Q(z_k, r_k) \subset \Omega \) for some \( k \in \{1, \ldots, m\} \). Put \( a := (x + y)/2 \) and \( \tilde{Q} := Q(a, \|x - y\|/2) \). Clearly, \( x, y \in \tilde{Q} \). Since \( \|x - y\| \leq 2r_k \) and \( a \in Q_k \), we have \( \tilde{Q} \subset Q(a, r_k) \subset 2Q_k \). But, by property (ii) of Lemma 2.6, \( 2Q_k \subset \Omega \) so that \( \tilde{Q} \subset \Omega \).

Moreover,

\[
(diam \tilde{Q})^\alpha = \|x - y\|^\alpha \leq \rho_{\alpha, \Omega}(x, y) = \|x - y\|^\alpha + d_{\alpha, \Omega}(x, y),
\]

so that in this case one can put \( Ch_{\alpha, \Omega}(x, y) := \{\tilde{Q}\} \).

Now suppose that that for every \( Q = Q(z, r) \in \mathcal{Q} \) either \( x \notin Q \) or \( y \notin Q \). This implies the existence of a point \( u \in \partial Q \cap \gamma \) such that \( \gamma_{zu} \subset Q \). Here \( \gamma_{zu} \) denotes the arc of \( \gamma \) joining \( z \) to \( u \).

On the other hand, by property (i) of Lemma of 2.6, \( 8r = \text{dist}(z, \partial \Omega) \) so that for every \( v \in Q \) we have

\[
\text{dist}(v, \partial \Omega) \leq \text{dist}(z, \partial \Omega) + \|v - z\| \leq 8r + r = 9r.
\]

Hence,

\[
\int_{\gamma \cap Q} \text{dist}(v, \partial \Omega)^{\alpha-1} ds(v) \geq \int_{\gamma_{zu}} \text{dist}(v, \partial \Omega)^{\alpha-1} ds(v)
\]

\[
\geq (9r)^{\alpha-1} \int_{\gamma_{zu}} 1 ds(v) = (9r)^{\alpha-1} \text{length}(\gamma_{zu}).
\]

Since \( u \in \partial Q \), we obtain

\[
\text{length}(\gamma_{zu}) \geq \|z - u\| = r
\]
so that
\[ \int_{\gamma \cap Q} \text{dist}(v, \partial \Omega)^{\alpha-1} \, ds(v) \geq (9r)^{\alpha-1} r = 9^{\alpha-1} r^\alpha = 9^{\alpha-1} \left( \text{diam } Q / 2 \right)^\alpha. \]

Thus for every cube \( Q_i \in \mathcal{Q}, i = 1, \ldots, m \), we have
\[
(diam Q_i)^\alpha \leq C(\alpha) \int_{\gamma \cap Q_i} \text{dist}(v, \partial \Omega)^{\alpha-1} \, ds(v)
\]
so that
\[
\sum_{i=1}^{m} (diam Q_i)^\alpha \leq C(\alpha) \sum_{i=1}^{m} \int_{\gamma \cap Q_i} \text{dist}(v, \partial \Omega)^{\alpha-1} \, ds(v) \leq C(\alpha) M(\mathcal{Q}) \int_{\gamma} \text{dist}(v, \partial \Omega)^{\alpha-1} \, ds(v).
\]

Recall that \( M(\mathcal{Q}) \) denotes the covering multiplicity of the family of cubes \( \mathcal{Q} \).

By property (iii) of Lemma 2.6, \( M(\mathcal{Q}) \leq C(n) \) so that
\[
\sum_{i=1}^{m} (diam Q_i)^\alpha \leq C(\alpha, n) \int_{\gamma} \text{dist}(v, \partial \Omega)^{\alpha-1} \, ds(v).
\]

Combining this inequality with (2.6), we obtain
\[
\sum_{i=1}^{m} (diam Q_i)^\alpha \leq C(\alpha, n) d_{a,\Omega}(x, y)
\]
proving the lemma. \( \square \)

Lemma 2.5 and Lemma 2.7 imply the following

**Proposition 2.8** For every \( x, y \in \Omega \) and every \( \alpha \in (0, 1] \)
\[
\rho_{\alpha,\Omega}(x, y) \sim \inf_{Ch} \sum_{i=1}^{m} (diam Q_i)^\alpha
\]
where the infimum is taken over all chains of cubes \( Ch = \{Q_i : i = 1, \ldots, m\} \) joining \( x \) to \( y \) in \( \Omega \).

Moreover, the same equivalence holds whenever \( Ch \) runs over all chains of cubes joining \( x \) to \( y \) in \( \Omega \) with the covering multiplicity \( M(Ch) \leq C \) where \( C = C(n) \) is a constant depending only on \( n \).

In both cases the constants of equivalence depend only on \( n \) and \( \alpha \).

This proposition implies the following

**Corollary 2.9** For every \( 0 < \beta < \alpha \leq 1 \) and every \( x, y \in \Omega \) the following inequality
\[
\rho_{\alpha,\Omega}(x, y) \leq C(\beta, n) \rho_{\beta,\Omega}(x, y)
\]
holds.
Lemma 2.10 For every $x, y \in \Omega$ and every $\alpha \in (0, 1]$

$$\rho_{\alpha, \Omega}(x, y) \sim \begin{cases} 
  d_{\alpha, \Omega}(x, y), & ||x - y|| \geq \min\{\text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega)\}, \\
  ||x - y||^\alpha, & ||x - y|| < \min\{\text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega)\},
\end{cases}$$

with constants of equivalence depend only on $\alpha$.

Proof. Assume that $\text{dist}(x, \partial \Omega) \geq \text{dist}(y, \partial \Omega)$. Let us consider the case

$$||x - y|| \geq \min\{\text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega)\} = \text{dist}(y, \partial \Omega).$$

(2.7)

Prove that in this case

$$||x - y||^\alpha \leq 2^{1-\alpha}d_{\alpha, \Omega}(x, y).$$

(2.8)

In fact, for every $z \in [x, y]$ we have

$$\text{dist}(z, \partial \Omega) \leq \text{dist}(y, \partial \Omega) + ||z - y|| \leq \text{dist}(y, \partial \Omega) + ||x - y||.$$ 

Hence, for every rectifiable curve $\gamma \subset \Omega$ joining $x$ to $y$ we obtain

$$\int_{\gamma} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) \geq \int_{\gamma} (\text{dist}(y, \partial \Omega) + ||x - y||)^{\alpha-1} ds(z)$$

$$= (\text{dist}(y, \partial \Omega) + ||x - y||)^{\alpha-1}\text{length}(\gamma)$$

$$\geq (2||x - y||^{\alpha-1}||x - y|| = 2^{\alpha-1}||x - y||^\alpha.$$ 

Taking in this inequality the infimum over all such $\gamma$ we obtain inequality (2.8).

Thus

$$d_{\alpha, \Omega}(x, y) \leq \rho_{\alpha, \Omega}(x, y) = d_{\alpha, \Omega}(x, y) + ||x - y||^\alpha \leq (1 + 2^{\alpha-1})d_{\alpha, \Omega}(x, y)$$

provided inequality (2.7) is satisfied. Let us consider the case

$$||x - y|| \leq \min\{\text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega)\}.$$ 

In this case inequality (2.1) is satisfied so that, by Lemma 2.1,

$$d_{\alpha, \Omega}(x, y) \leq \int_{[x,y]} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) \leq \frac{1}{\alpha}||x - y||^\alpha.$$ 

Hence,

$$||x - y||^\alpha \leq \rho_{\alpha, \Omega}(x, y) = d_{\alpha, \Omega}(x, y) + ||x - y||^\alpha \leq (1 + \frac{1}{\alpha})||x - y||^\alpha$$

proving the lemma. □
3. Sobolev-Poincaré type inequalities on a domain.

The following proposition presents the classical Sobolev imbedding inequality for the case \( p > n \), see, e.g. [22], p. 61, or [26], p. 55. This inequality is also known in the literature as Sobolev-Poincaré inequality (for \( p > n \)).

**Proposition 3.1** Let \( F \in L^1_p(\Omega) \) be a continuous function defined on a domain \( \Omega \subset \mathbb{R}^n \) and let \( n < q \leq p < \infty \). Then for every cube \( Q \subset \Omega \) and every \( x, y \in Q \) the following inequality

\[
|F(x) - F(y)| \leq C(n, q) \left( \frac{1}{|Q|} \int_Q \|\nabla F(z)\|^q \, dz \right)^{\frac{1}{q}}
\]

holds.

Clearly, inequality (3.1) (for \( p = q \)) implies the local Hölder inequality (1.1).

In this section we present several global versions of inequality (3.1) related to the case of arbitrary points \( x, y \in \Omega \). These variants of the Sobolev-Poincaré inequality on a domain are a slight generalization of the global Hölder-type inequality (1.5) proved by Buckley and Stanoyevitch [8].

Fix \( q \in (n, p] \) and put

\[
\beta = \frac{q - n}{q - 1}.
\]

**Lemma 3.2** Let \( F \in L^1_p(\Omega) \) be a continuous function and let \( x, y \in \Omega \). Let \( Ch = \{Q_1, ..., Q_m\} \) be a chain of cubes joining \( x \) to \( y \) in \( \Omega \) with the covering multiplicity \( M = M(Ch) < \infty \). Then

\[
|F(x) - F(y)| \leq C(n, q) M^\frac{1}{q} \left( \sum_{i=1}^m (\text{diam } Q_i)^\beta \right)^{1 - \frac{1}{q}} \left( \int_U \|\nabla F(z)\|^q \, dz \right)^{\frac{1}{q}}.
\]

where \( U := \bigcup_{i=1}^m Q_i \).

**Proof.** Let \( z_i \in Q_i \cap Q_{i+1}, i = 1, ..., m - 1 \). Put \( z_0 := x, z_m = y \). Then for every \( i = 0, ..., m-1 \), we have \( z_i, z_{i+1} \in Q_{i+1} \subset \Omega \) so that by the Sobolev-Poincaré inequality (3.1)

\[
|F(z_i) - F(z_{i+1})| \leq C(\text{diam } Q_{i+1}) \left( \frac{1}{|Q_{i+1}|} \int_{Q_{i+1}} \|\nabla F(z)\|^q \, dz \right)^{\frac{1}{q}}.
\]

Hence

\[
|F(z_i) - F(z_{i+1})| \leq C(\text{diam } Q_{i+1})^{1 - \frac{1}{q}} \left( \int_{Q_{i+1}} \|\nabla F(z)\|^q \, dz \right)^{\frac{1}{q}}.
\]

(3.2)
Now, by the Hölder inequality,

\[
|F(x) - F(y)| \leq \sum_{i=0}^{m-1} \frac{|F(z_i) - F(z_{i+1})|}{(\text{diam } Q_{i+1})^{1 - \frac{q}{q'}}} (\text{diam } Q_{i+1})^{1 - \frac{q}{q'}}
\]

\[
\leq \left( \sum_{i=0}^{m-1} \frac{|F(z_i) - F(z_{i+1})|}{(\text{diam } Q_{i+1})^{1 - \frac{q}{q'}}} \right)^{\frac{q}{q'}} \left( \sum_{i=0}^{m-1} (\text{diam } Q_{i+1})^{1 - \frac{q}{q'}} \right)^{1 - \frac{q}{q'}}
\]

\[
= \left( \sum_{i=0}^{m-1} \frac{|F(z_i) - F(z_{i+1})|^q}{(\text{diam } Q_{i+1})^{q-n}} \right)^{\frac{q}{q'}} \left( \sum_{i=0}^{m-1} (\text{diam } Q_{i+1})^{q-n} \right)^{1 - \frac{q}{q'}}.
\]

Recall that \( \beta = (q-n)/(q-1) \). Combining the latter inequality with (3.2), we obtain

\[
|F(x) - F(y)| \leq C \left( \sum_{i=1}^{m} (\text{diam } Q_{i})^\beta \right)^{1 - \frac{1}{q'}} \left( \sum_{i=0}^{m-1} \int_{Q_{i+1}} \|\nabla F(z)\|^q dz \right)^{\frac{1}{q'}}
\]

\[
\leq C \left( \sum_{i=1}^{m} (\text{diam } Q_{i})^\beta \right)^{1 - \frac{1}{q'}} \left( M \int_{U} \|\nabla F(z)\|^q dz \right)^{\frac{1}{q'}}.
\]

The lemma is proved.

**Proposition 3.3** Let \( F \in L^q_p(\Omega) \) be a continuous function and let \( q \in (n, p] \), \( \beta = (q-n)/(q-1) \). There exist constants \( \lambda = \lambda(n, q) \) and \( C = C(n, q) \) such that for every \( x, y \in \Omega \)

\[
|F(x) - F(y)| \leq C \rho_{\beta, \Omega}(x, y)^{1 - \frac{1}{q'}} \left( \int_{B} \|\nabla F(z)\|^q dz \right)^{\frac{1}{q'}}
\]

where

\[
B = \{ z \in \Omega : \rho_{\beta, \Omega}(x, z) \leq \lambda \rho_{\beta, \Omega}(x, y) \}.
\]

**Proof.** By Lemma 2.7 there exists a chain of cubes \( Ch_{\alpha, \Omega}(x, y) = \{ Q_1, ..., Q_m \} \) joining \( x \) to \( y \) in \( \Omega \) with covering multiplicity \( M(Ch_{\beta, \Omega}(x, y)) \leq C(n) \) such that

\[
\sum_{i=1}^{m} (\text{diam } Q_{i})^\beta \leq C(\beta, n) \rho_{\beta, \Omega}(x, y).
\]

(3.3)

Then by Lemma 3.2

\[
|F(x) - F(y)| \leq C M^\frac{1}{q'} \left( \sum_{i=1}^{m} (\text{diam } Q_{i})^\beta \right)^{1 - \frac{1}{q'}} \left( \int_{U} \|\nabla F(z)\|^q dz \right)^{\frac{1}{q'}}
\]

where \( U = \bigcup_{i=1}^{m} Q_{i} \). Hence

\[
|F(x) - F(y)| \leq C \rho_{\beta, \Omega}(x, y)^{1 - \frac{1}{q'}} \left( \int_{U} \|\nabla F(z)\|^q dz \right)^{\frac{1}{q'}}.
\]

22
Let \( z \in U \) so that \( z \in Q_k \) for some \( k \in \{1, \ldots, m\} \). Then \( \{Q_1, \ldots, Q_k\} \) is a chain of cubes joining \( x \) to \( z \) in \( \Omega \). By Lemma 2.5

\[
\rho_{\beta,\Omega}(x, z) \leq (1 + 2/\beta) \sum_{i=1}^{k} (\text{diam } Q_i)^\beta.
\]

Hence, by (3.3),

\[
\rho_{\beta,\Omega}(x, z) \leq (1 + 2/\beta) \sum_{i=1}^{m} (\text{diam } Q_i)^\beta \leq \lambda \rho_{\beta,\Omega}(x, y)
\]

with \( \lambda = \lambda(n, q) \). Finally we obtain

\[
U = \bigcup_{i=1}^{m} Q_i \subset B = \{z \in \Omega : \rho_{\beta,\Omega}(x, z) \leq \lambda \rho_{\beta,\Omega}(x, y)\}
\]

proving the proposition. \( \square \)

**Proposition 3.4** Let \( F \in L^1_p(\Omega) \) be a continuous function and let \( x, y \in \Omega \). Let \( q \in (n, p] \) and let \( \beta = (q - n)/(q - 1) \). There exist constants \( \lambda_1 = \lambda_1(n, q) \geq 1 \) and \( C = C(n, q) \) such that for every \( R \geq \lambda_1 \rho_{\beta,\Omega}(x, y)^{1/\beta} \) the following inequality

\[
|F(x) - F(y)| \leq C R \left( \frac{1}{|Q(x, R)|} \int_{Q(x, R) \cap \Omega} \|\nabla F(z)\|^q \, dz \right)^{\frac{1}{q}}
\]

holds.

**Proof.** Let \( \lambda = \lambda(n, q) \) be the constant from Proposition 3.3 and let

\[
B = \{z \in \Omega : \rho_{\beta,\Omega}(x, z) \leq \lambda \rho_{\beta,\Omega}(x, y)\}.
\]

Then for every \( z \in B \) we have

\[
\|x - z\|^\beta \leq \rho_{\beta,\Omega}(x, z) = \|x - z\|^\beta + d_{\beta,\Omega}(x, z) \leq \lambda \rho_{\beta,\Omega}(x, y)
\]

so that \( B \subset Q(x, r) \) with \( r = \lambda^{\frac{1}{\beta}} \rho_{\beta,\Omega}(x, y)^{\frac{1}{\beta}} \).

Hence, by Proposition 3.3

\[
|F(x) - F(y)| \leq C \rho_{\beta,\Omega}(x, y)^{1-\frac{1}{q}} \left( \int_{B} \|\nabla F(z)\|^q \, dz \right)^{\frac{1}{q}}
\]

\[
\leq C (r^\beta)^{1-\frac{1}{q}} \left( \int_{Q(x, r) \cap \Omega} \|\nabla F(z)\|^q \, dz \right)^{\frac{1}{q}} = C r^{1-\frac{1}{q}} \left( \int_{Q(x, r) \cap \Omega} \|\nabla F(z)\|^q \, dz \right)^{\frac{1}{q}}.
\]
Now, for every $R \geq r = \lambda_1 \rho_{\beta, \Omega}(x, y)^{\frac{1}{\beta}}$ we have

$$|F(x) - F(y)| \leq CR^{1-\frac{n}{q}} \left( \int_{Q(x, R) \cap \Omega} \|\nabla F(z)\|^q \, dz \right)^{\frac{1}{q}}$$

$$\leq CR \left( \frac{1}{|Q(x, R)|} \int_{Q(x, R) \cap \Omega} \|\nabla F(z)\|^q \, dz \right)^{\frac{1}{q}}$$

proving the proposition. \qed

**Lemma 3.5** Let $Q = Q(a, r)$ be a cube in $\Omega$ and let $t \geq 1$. Assume that $x \in (tQ) \cap \Omega$ and $x$ is a $Q$-visible point of $\Omega$, see Definition 1.7. Let $F \in L^1_p(\Omega)$ be a continuous function. Then for every $q \in (n, p]$ we have

$$\left( \frac{|F(x) - F(a)|}{\text{diam } Q} \right)^q \leq C \frac{1}{|Q|} \int_{Q \cap \Omega} \|\nabla F(z)\|^q \, dz$$

where $\gamma = \gamma(n, q, t)$ and $C = C(n, q, t)$.

**Proof.** Let $\beta = (q - n)/(q - 1)$. By Lemma 2.2 (with $b_1 = x \in \Omega, b_2 = a$),

$$d_{\beta, \Omega}(x, a) \leq \int_{[x, a]} \text{dist}(z, \partial \Omega)^{\beta - 1} \, ds(z) \leq C(\beta) \left( \frac{\|a - x\|}{\text{diam } Q} \right)^{1-\beta} \|x - a\|^\beta.$$

Since $x \in tQ$, we have $\|a - x\| \leq tr = \frac{1}{2} t \text{ diam } Q$ so that

$$d_{\beta, \Omega}(x, a) \leq C(\beta) t^{1-\beta} \|x - a\|^\beta.$$

Hence

$$\rho_{\beta, \Omega}(x, a) = \|x - a\|^\beta + d_{\beta, \Omega}(x, a) \leq C(\beta) t^{1-\beta} \|x - a\|^\beta \leq C(\beta) t^{1-\beta} (tr)^\beta = C(\beta) t^\gamma r.$$

Let $\lambda_1 = \lambda_1(n, q)$ be the constant from Proposition 3.4. Then

$$\lambda_1 \rho_{\beta, \Omega}(x, a)^{\frac{1}{\beta}} \leq \lambda_1(C(\beta)t)^{\frac{1}{\beta}} r = \gamma r$$

with $\gamma := \lambda_1(C(\beta)t)^{\frac{1}{\beta}}$.

Put $R := \gamma r = \frac{1}{2} \gamma \text{ diam } Q$. Since $R \geq \lambda_1 \rho_{\beta, \Omega}(x, a)^{\frac{1}{\beta}}$, by Proposition 3.4

$$|F(x) - F(a)| \leq CR \left( \frac{1}{|Q(a, R)|} \int_{Q(a, R) \cap \Omega} \|\nabla F(z)\|^q \, dz \right)^{\frac{1}{q}}$$

so that

$$\left( \frac{|F(x) - F(a)|}{\text{diam } Q} \right)^q \leq C 2^{-q\gamma^{q-n}} \frac{1}{|Q|} \int_{Q(a, R) \cap \Omega} \|\nabla F(z)\|^q \, dz.$$
The lemma is proved. □

As usual, given a function $G \in L_{1,\text{loc}}(\mathbb{R}^n)$ by $\mathcal{M}[G]$ we denote the Hardy-Littlewood maximal function

$$
\mathcal{M}[G](x) := \sup_{t > 0} \frac{1}{|Q(x, t)|} \int_{Q(x, t)} |G(y)| dy, \quad x \in \mathbb{R}^n.
$$

(3.4)

The last auxiliary result of the section is the following

**Lemma 3.6** Let $G \in L_{1,\text{loc}}(\mathbb{R}^n)$ and let $\gamma \geq 1, \theta > 0$. Then for every cube $Q \subset \mathbb{R}^n$ we have

$$
\left( \frac{1}{|Q|} \int_{\gamma Q} |G(x)| dx \right)^\theta \leq C(n, \gamma) \frac{1}{|Q|} \int_{Q} \mathcal{M}[G]^\theta(x) dx.
$$

Proof. Let $z \in Q$ and let $K := Q(z, \text{diam}(\gamma Q))$. Then $K \supset \gamma Q$ and $|K| \sim |Q|$ so that

$$
\frac{1}{|Q|} \int_{\gamma Q} |G(x)| dx \leq \frac{1}{|Q|} \int_{K} |G(x)| dx \leq C \frac{1}{|K|} \int_{K} |G(x)| dx \leq C \mathcal{M}[G](z).
$$

where $C = C(n, \gamma)$. Hence

$$
\left( \frac{1}{|Q|} \int_{\gamma Q} |G(x)| dx \right)^\theta \leq C \mathcal{M}[G]^\theta(z), \quad z \in Q.
$$

Integrating this inequality over cube $Q$ we obtain the required inequality

$$
|Q| \left( \frac{1}{|Q|} \int_{\gamma Q} |G(x)| dx \right)^\theta \leq C \int_{Q} \mathcal{M}[G]^\theta(x) dx. \quad \square
$$

4. **Local oscillation properties of the Whitney extension operator.**

In this section we study local oscillation properties of the classical Whitney operator which extends every function defined on the Sobolev boundary of a domain to a function defined on all of the domain. We present several auxiliary results which we will use later in the proofs of the sufficiency part of the main theorems, see Sections 5 and 6.

Since $\Omega$ is an open subset of $\mathbb{R}^n$, it admits a Whitney covering $\mathbb{W}(\Omega)$. In the next lemma we recall the main properties of this covering, see, e.g. [32], or [15].

**Theorem 4.1** $\mathbb{W}(\Omega) = \{Q_k\}$ is a countable family of cubes such that

(i). $\Omega = \bigcup \{Q : Q \in \mathbb{W}(\Omega)\}$;

(ii). For every cube $Q \in \mathbb{W}(\Omega)$

$$
\text{diam } Q \leq \text{dist}(Q, \partial \Omega) \leq 4 \text{diam } Q;
$$

(4.1)

(iii). The covering multiplicity $M(\mathbb{W}(\Omega))$ of the family $\mathbb{W}(\Omega)$ is bounded by a constant $N = N(n)$. Thus every point of $\Omega$ is covered by at most $N$ cubes from $\mathbb{W}(\Omega)$.  

25
We are also needed certain additional properties of Whitney’s cubes which we present in the next lemma. Let \( Q \) be a cube in \( \Omega \) and let \( Q^* := \frac{2}{8}Q \).

**Lemma 4.2** (1). If \( Q, K \in W(\Omega) \) and \( Q^* \cap K^* \neq \emptyset \), then
\[
\frac{1}{4} \text{diam } Q \leq \text{diam } K \leq 4 \text{diam } Q. \tag{4.2}
\]

(2). For every cube \( K \in W(\Omega) \) there are at most \( N = N(n) \) cubes from the family \( W^*(\Omega) := \{ Q^* : Q \in W(\Omega) \} \) which intersect \( K^* \).

(3). Let \( Q, K \in W(\Omega) \). Then \( Q^* \cap K^* \neq \emptyset \) if and only if \( Q \cap K \neq \emptyset \).

We turn to the construction of the Whitney extension operator. Fix \( \alpha \in (0, 1] \). As usual, given a cube \( Q \subset \Omega \), by \( a_Q \) we denote a point of \( \partial \Omega \) nearest to \( Q \) in the Euclidean norm.

We recall that the standard Whitney’s extension algorithm assigns every function \( f : \partial \Omega \rightarrow \mathbb{R} \) a piecewise constant function \( F \) which on every cube \( Q \in W(\Omega) \) takes the value \( f(a_Q) \). Then we smooth \( F \) using a certain smooth partition of unity subordinated to the Whitney decomposition \( W(\Omega) \).

Let now \( f : (\partial \Omega)_\alpha \rightarrow \mathbb{R} \) be a function defined on the \( \alpha \)-boundary of \( \Omega \), see Definition 1.4. Observe that the same extension procedure works well whenever the \( \alpha \)-boundary of \( \Omega \) can be identified with a subset of \( \partial \Omega \). A domain satisfying the \((A_\alpha)\)-condition, see Definition 1.6, provides an example of such a domain; in fact, in this case the boundary \( \partial \Omega \) does not contain “agglutinated” parts and does not split under the Cauchy completion with respect to the metric \( \rho_{\alpha, \Omega} \).

However, in general, the point \( a_Q \) may split into a finite or infinite number of elements of the \( \alpha \)-boundary. In this case we have to assign the pair \( (a_Q, Q) \) an appropriate equivalence class \( \omega_{Q, \alpha} \in (\partial \Omega)_\alpha \) and then to proceed the Whitney algorithm using the value \( f(\omega_{Q, \alpha}) \) rather than \( f(a_Q) \).

We will do this as follows. Clearly, if a cube \( Q = Q(x_Q, r_Q) \subset \Omega \) then
\[
Q^*(x_Q, \text{dist}(x_Q, \partial \Omega)) \subset \Omega
\]
so that the point \( a_Q \) is \( Q \)-visible in \( \Omega \), see Definition 1.7. In particular, \( [x_Q, a_Q) \subset \Omega \). We define a sequence of points \( x_i \in [x_Q, a_Q) \) by letting
\[
x_i := a_Q + \frac{1}{i}(x_Q - a_Q), \quad i = 1, 2, ...
\tag{4.3}
\]
Clearly, \( x_i \xrightarrow{\|\cdot\|} a_Q \) as \( i \rightarrow \infty \) so that, by Lemma 2.2, \( (x_i) \) is fundamental with respect to the metric \( \rho_{\alpha, \Omega} \).

Thus \( (x_i) \in C[\Omega, \rho_{\alpha, \Omega}] \), see (1.8), so that the equivalence class of \( (x_i) \), the set
\[
\omega_{Q, \alpha} = [(x_i)]_\alpha, \tag{4.4}
\]
is an element of \( \Omega^{*, \alpha} \), see (1.11).
Moreover, since \( x_i \xrightarrow{\| \cdot \|} a_Q \), the point \( a_Q \) is the common limit (in \( \| \cdot \|\)-norm) of all sequences \( (y_i) \in \omega_{Q,\alpha} \), i.e.,

\[
\ell(\omega_{Q,\alpha}) = a_Q, \tag{4.5}
\]

see Remark 1.2. On the other hand, since \( a_Q \in \partial \Omega \), by Definition 1.4, the class \( \omega_{Q,\alpha} \in (\partial \Omega)_\alpha \). In other words, \( \omega_{Q,\alpha} \) is an element of the \( \alpha \)-boundary of the domain \( \Omega \):

\[
\omega_{Q,\alpha} \in (\partial \Omega)_\alpha = \Omega^{*,\alpha} \setminus \Omega,
\]

see (1.14).

We will be needed the following property of the element \( \omega_{Q,\alpha} \).

**Lemma 4.3** For every cube \( Q \in \mathcal{W}(\Omega) \) and every \( y \in Q \) the following inequality

\[
\rho_{\alpha,\Omega}(y, \omega_{Q,\alpha}) \leq C(\alpha) \text{dist}(y, \partial \Omega)^\alpha \tag{4.6}
\]

holds.

**Proof.** By (1.13), (4.3) and (4.5),

\[
\rho_{\alpha,\Omega}(x_Q, \omega_{Q,\alpha}) = \lim_{i \to \infty} d_{\alpha,\Omega}(x_Q, x_i) + \| x_Q - a_Q \|.
\]

Since \( a_Q \) is a nearest point to \( Q \) on \( \partial \Omega \),

\[
\| x_Q - a_Q \| \leq r_Q + \text{dist}(Q, a_Q) = r_Q + \text{dist}(Q, \partial \Omega),
\]

so that, by (4.1),

\[
\| x_Q - a_Q \| \leq r_Q + 4 \text{ diam } Q \leq 5 \text{ diam } Q. \tag{4.7}
\]

Let us estimate \( d_{\alpha,\Omega}(x_Q, x_i) \). Since \( a_Q \) is \( Q \)-visible and \( x_i \in [x_Q, a_Q] \), by Lemma 2.2 (i),

\[
d_{\alpha,\Omega}(x_Q, x_i) \leq \int_{[x_i, x_Q]} \text{dist}(z, \partial \Omega)^{\alpha-1} ds(z) \leq C \left( \frac{\| a_Q - x_Q \|}{\text{diam } Q} \right)^{1-\alpha} \| x_i - x_Q \|^\alpha,
\]

so that

\[
\lim_{i \to \infty} d_{\alpha,\Omega}(x_Q, x_i) \leq C \left( \frac{\| a_Q - x_Q \|}{\text{diam } Q} \right)^{1-\alpha} \| a_Q - x_Q \|^\alpha.
\]

Combining this inequality with (4.7), we obtain

\[
\lim_{i \to \infty} d_{\alpha,\Omega}(x_Q, x_i) \leq C(\text{diam } Q)^\alpha.
\]

From this and (4.7) it follows

\[
\rho_{\alpha,\Omega}(x_Q, \omega_{Q,\alpha}) \leq C(\text{diam } Q)^\alpha.
\]
Let us estimate \( \rho_{\alpha, \Omega}(x_Q, y) \). We have \( \|x_Q - y\| \leq r_Q \) so that
\[
\rho_{\alpha, \Omega}(x_Q, y) = d_{\alpha, \Omega}(x_Q, y) + \|x_Q - y\|^\alpha \leq d_{\alpha, \Omega}(x_Q, y) + (\text{diam } Q)^\alpha. \tag{4.8}
\]

But, by (4.1),
\[
\text{dist}(y, \partial \Omega) \geq \text{dist}(Q, \partial \Omega) \geq \text{diam } Q \tag{4.9}
\]
proving that
\[
\|x_Q - y\| \leq \text{diam } Q \leq \max\{\text{dist}(x_Q, \partial \Omega), \text{dist}(y, \partial \Omega)\}.
\]
Hence, by (2.2),
\[
d_{\alpha, \Omega}(x_Q, y) \leq \int_{[x_Q, y]} \text{dist}(z, \partial \Omega)^{\alpha - 1} ds(z) \leq \frac{1}{\alpha} \|x_Q - y\|^\alpha \leq C(\text{diam } Q)^\alpha,
\]
so that, by (4.8), \( \rho_{\alpha, \Omega}(x_Q, y) \leq C(\text{diam } Q)^\alpha \). We obtain
\[
\rho_{\alpha, \Omega}(y, \omega_{Q, \alpha}) \leq \rho_{\alpha, \Omega}(x_Q, \omega_{Q, \alpha}) + \rho_{\alpha, \Omega}(x_Q, y) \leq C(\text{diam } Q)^\alpha.
\]
This inequality and (4.9) imply the required inequality (4.6). \(\square\)

Let \( \sigma > 0, \bar{c} \in \mathbb{R} \), and let \( f : (\partial \Omega)_\alpha \to \mathbb{R} \) be a function defined on the \( \alpha \)-boundary of \( \Omega \). We put
\[
c_Q := \begin{cases} f(\omega_{Q, \alpha}), & \text{diam } Q \leq \sigma, \\ \bar{c}, & \text{diam } Q > \sigma. \end{cases} \tag{4.10}
\]
We define an extension operator \( \tilde{f} = \text{Ext}[f; \sigma, \bar{c}, \alpha, \Omega] \) by letting \( \tilde{f}(\omega) := f(\omega) \), \( \omega \in (\partial \Omega)_\alpha \), and
\[
\tilde{f}(x) := \sum_{Q \in \mathbb{W}(\Omega)} c_Q \varphi_Q(x), \quad x \in \Omega. \tag{4.11}
\]

Here \( \{\varphi_Q : Q \in \mathbb{W}(\Omega)\} \) is a smooth partition of unity subordinated to the Whitney decomposition \( \mathbb{W}(\Omega) \), see, e.g. [32]. Recall the main properties of this partition.

**Lemma 4.4** The partition of unity \( \{\varphi_Q : Q \in \mathbb{W}(\Omega)\} \) has the following properties:
(a). \( \varphi_Q \in C^\infty(\mathbb{R}^n) \) and \( 0 \leq \varphi_Q \leq 1 \) for every \( Q \in \mathbb{W}(\Omega) \);
(b). \( \text{supp } \varphi_Q \subset Q^* := \frac{3}{2}Q \), \( Q \in \mathbb{W}(\Omega) \);
(c). \( \sum_{Q \in \mathbb{W}(\Omega)} \varphi_Q(x) = 1 \) for every \( x \in \Omega \);
(d). \( \|\nabla \varphi_Q(x)\| \leq C(n)/\text{diam } Q \) for every \( Q \in \mathbb{W}(\Omega) \) and every \( x \in \Omega \).

**Lemma 4.5** For every cube \( K \in \mathbb{W}(\Omega) \) with \( \text{diam } K \leq \sigma/4 \) and every \( x \in K \) the following inequality
\[
|\tilde{f}(x) - f(\omega_{K, \alpha})| \leq C(n) \max\{|f(\omega_{Q, \alpha}) - f(\omega_{K, \alpha})| : Q \in \mathbb{W}(\Omega), Q \cap K \neq \emptyset\}
\]
holds.
Proof. By (4.11) and Lemma 4.4 (c), we have
\[ |\tilde{f}(x) - f(\omega_{K,a})| = \left| \sum_{Q \in W(\Omega)} c_Q \varphi_Q(x) - f(\omega_{K,a}) \right| = \left| \sum_{Q \in W(\Omega)} (c_Q - f(\omega_{K,a}))\varphi_Q(x) \right| \]
\[ \leq \sum_{Q \in W(\Omega)} |c_Q - f(\omega_{K,a})| \varphi_Q(x). \]
Hence, by Lemma 4.4 (b), we obtain
\[ |\tilde{f}(x) - f(\omega_{K,a})| \leq \sum \{|c_Q - f(\omega_{K,a})| : Q \in W(\Omega), Q^\ast \ni x \} \]
\[ \leq \sum \{|c_Q - f(\omega_{K,a})| : Q \in W(\Omega), Q \cap K = \emptyset \}. \]
But \(0 \leq \varphi_Q \leq 1\), and, by Lemma 4.2 (3), \(Q^\ast \cap K^\ast = \emptyset\) iff \(Q \cap K = \emptyset\), so that
\[ |\tilde{f}(x) - f(\omega_{K,a})| \leq \sum \{|c_Q - f(\omega_{K,a})| : Q \in W(\Omega), Q \cap K = \emptyset \}. \]
By Lemma 4.2 (2), there are at most \(N(n)\) cubes \(Q \in W(\Omega)\) such that \(Q \cap K = \emptyset\), so that
\[ |\tilde{f}(x) - f(\omega_{K,a})| \leq C \max \{|c_Q - f(\omega_{K,a})| : Q \in W(\Omega), Q \cap K = \emptyset \}. \]
Moreover, for every \(Q \in W(\Omega), Q \cap K \neq \emptyset\), by Lemma 4.2 (1),
\[ \text{diam } Q \leq 4 \text{ diam } K \leq 4(\sigma/4) = \sigma, \]
so that, by (4.10), \(c_Q = f(\omega_{Q,a})\). The lemma is proved. \(\square\)

Observe that \(\tilde{f}|_\Omega \in C^\infty(\Omega)\). The next lemma provides an estimate of the norm of gradient of \(\tilde{f}\) on every Whitney cube \(K \in W(\Omega)\).

Lemma 4.6 Let \(K \in W(\Omega)\) be a Whitney cube. Then
\[ \sup_K \|\nabla \tilde{f}\| \leq C(n) (\text{diam } K)^{-1} \sum \{|c_Q - c_K| : Q \in W(\Omega), Q \cap K \neq \emptyset \}. \]

Proof. For every \(x \in K\) we have
\[ \|\nabla \tilde{f}(x)\| = \|\nabla \left( \sum_{Q \in W(\Omega)} (c_Q - c_K)\varphi_Q(x) \right) \| = \left\| \sum_{Q \in W(\Omega)} (c_Q - c_K)\nabla \varphi_Q(x) \right\|. \]
Since \(\text{supp } \varphi_Q \subset Q^\ast, Q \in W(\Omega)\), and \(x \in K\), in the latter sum one can consider only those cubes \(Q \in W(\Omega)\) for which \(Q^\ast \cap K = \emptyset\). Hence
\[ \|\nabla \tilde{f}(x)\| \leq \sum \{|c_Q - c_K|\|\nabla \varphi_Q(x)\| : Q \in W(\Omega), Q^\ast \cap K \neq \emptyset \} \]
so that, by Lemma 4.4 (d),
\[ \|\nabla \tilde{f}(x)\| \leq C(n) \sum \{|c_Q - c_K|(\text{diam } Q)^{-1} : Q \in W(\Omega), Q^\ast \cap K \neq \emptyset \}. \]
By Lemma 4.2 (3), \(Q \cap K \neq \emptyset\) provided \(Q^\ast \cap K \neq \emptyset\). Moreover, by (4.2), \(\text{diam } Q \sim \text{diam } K\) for every cube \(Q \in W(\Omega)\) such that \(Q^\ast \cap K \neq \emptyset\). Hence
\[ \|\nabla \tilde{f}(x)\| \leq C(n) (\text{diam } K)^{-1} \sum \{|c_Q - c_K| : Q \in W(\Omega), Q \cap K \neq \emptyset \}, \quad x \in K, \]
proving the lemma. \(\square\)

Let us estimate the \(L_p\)-norm of \(\nabla \tilde{f}\).
Lemma 4.7  
\[ \| \nabla \tilde{f} \|^p_{L^p(\Omega)} \leq C(n) \sum \left\{ \frac{|c_Q - c_{Q'}|^p}{(\text{diam } Q + \text{diam } Q')^{p-n}} : Q, Q' \in \mathcal{W}(\Omega), Q \cap Q' \neq \emptyset \right\} . \]

Proof. We have 
\[ \int_{\Omega} \| \nabla \tilde{f} \|^p \, dx \leq \sum_{K \in \mathcal{W}(\Omega)} \int_K \| \nabla \tilde{f} \|^p \, dx \leq \sum_{K \in \mathcal{W}(\Omega)} |K| \sup_K \| \nabla \tilde{f} \|^p \]
so that, by Lemma 4.6,
\[ \int_{\Omega} \| \nabla \tilde{f} \|^p \, dx \leq C \sum_{K \in \mathcal{W}(\Omega)} |K|(\text{diam } K)^{-p} \sum \{|c_Q - c_K|^p : Q \in \mathcal{W}(\Omega), Q \cap K \neq \emptyset\} \]
By (4.2), \( \text{diam } K \sim \text{diam } Q \) for every \( K, Q \in \mathcal{W}(\Omega), Q \cap K \neq \emptyset \), so that 
\[ \int_{\Omega} \| \nabla \tilde{f} \|^p \, dx \leq C \sum_{K \in \mathcal{W}(\Omega)} \sum \left\{ \frac{|c_Q - c_K|^p}{(\text{diam } K)^{p-n}} : Q \in \mathcal{W}(\Omega), Q \cap K \neq \emptyset \right\} . \]

The lemma is proved. \( \square \)

Lemma 4.8 Let \( \bar{c} := 0 \), see formula (4.10). Then the following inequality 
\[ \| \tilde{f} \|^p_{W^{1,p}(\Omega)} \]
\[ \leq C \left( \sum \left\{ \frac{|f(\omega_{Q,a}) - f(\omega_{Q',a})|^p}{(\text{diam } Q + \text{diam } Q')^{p-n}} : Q, Q' \in \mathcal{W}(\Omega), Q \cap Q' \neq \emptyset, Q, Q' \subset O_{5\sigma}(\partial \Omega) \right\} + \sum \{ |f(\omega_{Q,a})|^p |Q| : Q \in \mathcal{W}(\Omega), Q \subset O_{10\sigma}(\partial \Omega) \} \right) \]
holds. Here \( C = C(n, \sigma) \) is a constant depending only on \( n \) and \( \sigma \).

Proof. Recall that the set \( O_\varepsilon(\partial \Omega) \) is defined by (1.21). By Lemma 4.7 
\[ \| \nabla \tilde{f} \|^p_{L^p(\Omega)} \leq C(I_1 + I_2) \]
where 
\[ I_1 := \sum \left\{ |c_Q - c_{Q'}|^p (\text{diam } Q + \text{diam } Q')^{n-p} : Q, Q' \in \mathcal{W}(\Omega), Q \cap Q' \neq \emptyset, \max\{\text{diam } Q, \text{diam } Q'\} \leq \sigma \right\} \]
and 
\[ I_2 := \sum \left\{ |c_Q - c_{Q'}|^p (\text{diam } Q + \text{diam } Q')^{n-p} : Q, Q' \in \mathcal{W}(\Omega), Q \cap Q' \neq \emptyset, \max\{\text{diam } Q, \text{diam } Q'\} > \sigma \right\} . \]
Let us estimate $I_1$. Let $Q \in \mathcal{W}(\Omega)$ and let $\operatorname{diam} Q \leq \sigma$. Then, by (4.1), for every $x \in Q$ we have
\[
\operatorname{dist}(x, \partial \Omega) \leq \operatorname{diam} Q + \operatorname{dist}(Q, \partial \Omega) \leq \operatorname{diam} Q + 4 \operatorname{diam} Q \leq 5\sigma
\]
so that $Q \subset \mathcal{O}_{5\sigma}(\partial \Omega)$. By (4.10),
\[
|c_Q - c_{Q'}| = |f(\omega_{Q,a}) - f(\omega_{Q',a})|
\]
provided $\operatorname{diam} Q \leq \sigma$ and $\operatorname{diam} Q' \leq \sigma$ so that
\[
I_1 \leq \sum \left\{ \frac{|f(\omega_{Q,a}) - f(\omega_{Q',a})|^p}{(\operatorname{diam} Q + \operatorname{diam} Q')^{p-n}} : Q, Q' \in \mathcal{W}(\Omega), Q \cap Q' \neq \emptyset, Q', Q' \subset \mathcal{O}_{5\sigma}(\partial \Omega) \right\}.
\]

Let us estimate $I_2$. Let $Q \in \mathcal{W}(\Omega)$ and let $Q \notin \mathcal{O}_{10\sigma}(\partial \Omega)$ so that there exists $x \in Q$ such that $\operatorname{dist}(x, \partial \Omega) > 10\sigma$. Then for every $Q' \in \mathcal{W}(\Omega), Q' \cap Q \neq \emptyset$, we have
\[
10\sigma < \operatorname{dist}(x, \partial \Omega) \leq \operatorname{diam} Q + \operatorname{diam} Q' + \operatorname{dist}(Q', \partial \Omega)
\]
so that, by (4.1) and (4.2),
\[
10\sigma < 4 \operatorname{diam} Q' + \operatorname{diam} Q' + 4 \operatorname{diam} Q' = 9 \operatorname{diam} Q'.
\]
Thus $\sigma < \operatorname{diam} Q'$ for every $Q' \in \mathcal{W}(\Omega), Q' \cap Q \neq \emptyset$, so that, by (4.10),
\[
c_{Q'} = 0, \quad Q' \in \mathcal{W}(\Omega), Q' \cap Q \neq \emptyset,
\]
provided $Q \notin \mathcal{O}_{10\sigma}(\partial \Omega)$. Hence
\[
I_2 := \sum \left\{ |c_Q - c_{Q'}|^p (\operatorname{diam} Q + \operatorname{diam} Q')^{n-p} : Q, Q' \in \mathcal{W}(\Omega), Q \cap Q' \neq \emptyset, \max\{\operatorname{diam} Q, \operatorname{diam} Q'\} > \sigma, Q, Q' \subset \mathcal{O}_{10\sigma}(\partial \Omega) \right\}.
\]

Observe that for each $Q \subset \mathcal{O}_{10\sigma}(\partial \Omega)$, by (4.1), we have
\[
\operatorname{diam} Q \leq \operatorname{dist}(Q, \partial \Omega) \leq 10\sigma.
\]
Let us consider cubes $Q, Q' \in \mathcal{W}(\Omega)$ satisfying the following conditions: $Q, Q' \subset \mathcal{O}_{10\sigma}(\partial \Omega), Q \cap Q' \neq \emptyset$ and $\max\{\operatorname{diam} Q, \operatorname{diam} Q'\} > \sigma$. Assume that $\operatorname{diam} Q > \sigma$. Then, by (4.2),
\[
\operatorname{diam} Q' \geq \frac{1}{4} \operatorname{diam} Q \geq \sigma/4
\]
so that
\[
\sigma/4 < \operatorname{diam} Q' \leq \operatorname{diam} Q \leq 10\sigma.
\]
Since in (4.10) we put $\bar{c} := 0$,
\[
|c_Q - c_{Q'}| \leq |c_Q| + |c_{Q'}| \leq |f(\omega_{Q,a})| + |f(\omega_{Q',a})|, \quad \text{so that}
\]
\[
I_2 \leq C \sum \left\{ (|f(\omega_{Q,a})|^p + |f(\omega_{Q',a})|^p) (\operatorname{diam} Q + \operatorname{diam} Q')^{n-p} : Q, Q' \in \mathcal{W}(\Omega), Q \cap Q' \neq \emptyset, Q, Q' \subset \mathcal{O}_{10\sigma}(\partial \Omega), \sigma/4 < \operatorname{diam} Q', \operatorname{diam} Q \leq 10\sigma \right\}.
\]
Hence
\[ I_2 \leq C(n, \sigma) \sum \{ |f(\omega_{Q,\alpha})|^p |Q| : Q \in W(\Omega), Q \subset O_{10\sigma}(\partial \Omega) \}. \]

We obtain
\[
\| \nabla \tilde{f} \|^p_{L^p(\Omega)} \leq C(I_1 + I_2)
\]
\[
\leq C \left( \sum \left\{ \frac{|f(\omega_{Q,\alpha}) - f(\omega_{Q',\alpha})|^p}{(\text{diam } Q + \text{diam } Q')^{p-n}} : Q, Q' \in W(\Omega), Q \cap Q' \neq \emptyset, Q, Q' \subset O_{5\sigma}(\partial \Omega) \right\} 
+ \sum \{ |f(\omega_{Q,\alpha})|^p |Q| : Q \in W(\Omega), Q \subset O_{10\sigma}(\partial \Omega) \} \right)
\]
with \( C = C(n, \sigma) \).

Let us estimate \( \| \tilde{f} \|^p_{L^p(\Omega)} \). By (4.12), \( c_Q = 0 \) provided \( Q \not\subset O_{10\sigma}(\partial \Omega) \). Also, since \( \bar{c} = 0 \), by (1.10), \( |c_Q| \leq |f(\omega_{Q,\alpha})| \) for every \( Q \in W(\Omega) \). Hence
\[
\| \tilde{f} \|^p_{L^p(\Omega)} \leq \sum_{Q \in W(\Omega)} |c_Q|^p |Q| \leq \sum \{ |c_Q|^p |Q| : Q \in W(\Omega), Q \subset O_{10\sigma}(\partial \Omega) \}
\]
\[
\leq \sum \{ |f(\omega_{Q,\alpha})|^p |Q| : Q \in W(\Omega), Q \subset O_{10\sigma}(\partial \Omega) \}.
\]

The lemma is completely proved. \( \square \)

Let us extend the notion of \( Q \)-visibility (Definition 1.7) to the case of an arbitrary domain \( \Omega \subset \mathbb{R}^n \).

**Definition 4.9** Let \( \omega \in (\partial \Omega)_\alpha \), \( \alpha \in (0, 1] \), and let \( Q \subset \Omega \) be a cube. We say that \( \omega \) is \((\alpha, Q)\)-visible in \( \Omega \) if the following conditions are satisfied:

(i). The point \( \ell(\omega) \in \partial \Omega \) is \( Q \)-visible in \( \Omega \) (see Remark (1.2) and Definition (1.7));

(ii). There exists a sequence \( y_i \in (\ell(\omega), x_Q) \), \( i = 1, 2, \ldots \), such that the equivalence class of \((y_i)\) with respect to “ \( \sim \)”, see (1.9) and (1.11), coincides with \( \omega \):
\[
[(y_i)]_\alpha = \omega.
\]

Observe that part (ii) of this definition can be replaced with one the following equivalent statement:

(a). There exists a sequence \((y_i) \in \omega \) such that \( y_i \in (\ell(\omega), x_Q) \).

(b). Every sequence \( y_i \in (\ell(\omega), x_Q) \) such that \( y_i \xrightarrow{\ell(\omega)} \ell(\omega) \) as \( i \to \infty \), belongs to \( \omega \), see Lemma 2.12.

Also note that for each cube \( Q \subset \Omega \) the element \( \omega_{Q,\alpha} \in (\partial \Omega)_\alpha \) defined by (4.4) possesses the following property:
\[
\omega_{Q,\alpha} \text{ is } (\alpha, Q)\text{-visible in } \Omega. \tag{4.13}
\]

**Lemma 4.10** Let \( Q_1, Q_2 \) be cubes in \( \Omega \). Suppose that \( Q_2 \in W(\Omega) \), \( Q_1 \cap Q_2 \neq \emptyset \), and for some \( \tau \geq 1 \)
\[
\text{diam } Q_1 \leq \text{diam } Q_2 \leq \tau \text{ diam } Q_1.
\]
Then \( \omega_{Q_2,\alpha} \) is \((\alpha, Q_1)\)-visible in \( \Omega \). In addition, \( \ell(\omega_{Q_2,\alpha}) \in (10\tau + 1)Q_1 \).
Proof. Prove that $\omega_{Q_2,\alpha}$ is $(\alpha, Q_1)$-visible in $\Omega$. Observe that $\omega_{Q_2,\alpha}$ is $(\alpha, Q_2)$-visible in $\Omega$ and $a_{Q_2} = \ell(\omega_{Q_2,\alpha})$, see (4.13). (Recall that $a_{Q_2}$ denotes a point on $\partial \Omega$ nearest to the cube $Q_2$.) Thus $\omega_{Q_2,\alpha} = [(y_i)]_{\alpha}$ where $(y_i)$ is an arbitrary sequence of points such that $y_i \in (a_{Q_2}, x_{Q_2}]$, $i = 1, 2, \ldots$, and

$$\|y_i - a_{Q_2}\| \to 0 \quad \text{as} \quad i \to \infty.$$  \hfill (4.14)

Let us fix such a sequence $(y_i)$ and construct a sequence $(z_i)$ such that $z_i \in (a_{Q_2}, x_{Q_1}]$ for every $i = 1, 2, \ldots$, and $(z_i) \overset{\alpha}{\sim} (y_i)$. Since $Q_2 \in W(\Omega)$, $Q_1 \cap Q_2 \neq \emptyset$ and

$$\text{diam } Q_1 \leq \text{diam } Q_2 \leq \text{dist}(Q_2, \partial \Omega),$$

by Lemma 2.3, $a_{Q_2}$ is $Q_1$-visible in $\Omega$ and for every $i = 1, 2, \ldots$, there exists a point $z_i \in (a_{Q_2}, x_{Q_1}]$ such that $\|y_i - z_i\| \leq C\|y_i - a_{Q_2}\|$, and

$$d_{\alpha, \Omega}(y_i, z_i) \leq C\|y_i - a_{Q_2}\|^\alpha$$

where $C$ is a constant independent of $i$. Hence,

$$\rho_{\alpha, \Omega}(y_i, z_i) = \|y_i - z_i\|^\alpha + d_{\alpha, \Omega}(y_i, z_i) \leq C\|y_i - a_{Q_2}\|^\alpha$$

so that, by (4.14), $\rho_{\alpha, \Omega}(y_i, z_i) \to 0$ as $i \to \infty$.

This proves that $(y_i) \overset{\alpha}{\sim} (z_i)$, see (1.9). Since $\omega_{Q_2,\alpha}$ is the equivalence class of $(y_i)$, and $(y_i) \overset{\alpha}{\sim} (z_i)$, we conclude that $\omega_{Q_2,\alpha} = [z_i]_{\alpha}$ as well.

Thus the point $\ell(\omega_{Q_2,\alpha}) = a_{Q_2}$ is $Q_1$-visible, $(z_i) \in \omega_{Q_2,\alpha}$, and $z_i \in (\ell(\omega), x_{Q_1}]$, so that, by Definition (1.9) $\omega_{Q_2,\alpha}$ is $(\alpha, Q_1)$-visible.

Prove that $a_{Q_2} \in (10\tau + 1)Q_1$. Let $Q_1 = Q(x_{Q_1}, r_1)$. Since $Q_1 \cap Q_2 \neq \emptyset$,

$$\|a_{Q_2} - x_1\| \leq \text{dist}(a_{Q_2}, Q_2) + \text{diam } Q_2 + r_1 = \text{dist}(Q_2, \partial \Omega) + \text{diam } Q_2 + r_1$$

so that, by (4.1),

$$\|a_{Q_2} - x_1\| \leq 4 \text{ diam } Q_2 + \text{ diam } Q_2 + r_1.$$ 

But $\text{diam } Q_2 \leq r \text{ diam } Q_1 = 2\tau r_1$ so that $\|a_{Q_2} - x_1\| \leq (10\tau + 1)r_1$ proving the required property $\ell(\omega_{Q_2,\alpha}) = a_{Q_2} \in (10\tau + 1)Q_1$. \hfill $\square$

This lemma and the property (4.2) of Whitney’s cubes imply the following

**Lemma 4.11** Let $Q_1, Q_2 \in W(\Omega)$, $Q_1 \cap Q_2 \neq \emptyset$, and let $\text{diam } Q_2 \geq \text{diam } Q_1$. Then $\omega_{Q_1,\alpha}$ and $\omega_{Q_2,\alpha}$ are $(\alpha, Q_1)$-visible. In addition, $\ell(\omega_{Q_1,\alpha}), \ell(\omega_{Q_2,\alpha}) \in 41Q_1$.

The last auxiliary result of the section is the following

**Lemma 4.12** Let cubes $Q, Q' \subset \Omega$ and let $Q' \subset Q$. Then $\omega_{Q,\alpha}$ is an $(\alpha, Q')$-visible in $\Omega$. 

The proof of this result relies on Lemma 2.3 and is very similar to the proof of Lemma 4.10. We leave the details to the interested reader.
5. Boundary values of Sobolev functions: restrictions and extensions.

In subsection 1.4 we have formulated Theorem 1.8 and Theorem 1.9 which provide constructive descriptions of the trace spaces $L^1_{p}(\Omega)|_{\partial \Omega}$ and $W^1_{p}(\Omega)|_{\partial \Omega}$ whenever $\Omega$ is a domain in $\mathbb{R}^n$ satisfying the condition $A_\alpha$ with $\alpha = (p-n)/(p-1)$, see Definition 1.6. In this section we present generalizations of these theorems to the case of an arbitrary domain $\Omega \subset \mathbb{R}^n$.

We will be needed several addition definitions and notation. First of them is a definition of the metric $d_{\alpha,\Omega}$ introduced earlier only for the domains satisfying the condition $A_\alpha$, see (1.18). Given $\alpha \in (0, 1]$ and $\omega_1, \omega_2 \in \Omega^{*, \alpha}$ we put

$$d_{\alpha,\Omega}(\omega_1, \omega_2) := \lim_{i \to \infty} d_{\alpha,\Omega}(x_i, y_i)$$

(5.1)

where $(x_i) \in \omega_1$, $(y_i) \in \omega_2$ are arbitrary sequences. Recall that $\Omega^{*, \alpha}$ is the family of all equivalence classes of Cauchy sequences with respect to the metric $\rho_{\alpha,\Omega}$, see (1.11). Since $d_{\alpha,\Omega} \leq \rho_{\alpha,\Omega}$, see (1.6), $d_{\alpha,\Omega}$ is well defined on $\Omega^{*, \alpha}$.

Clearly, $d_{\alpha,\Omega} = d_{\alpha,\Omega}$ on $\Omega$. Moreover, the reader can easily see that $d_{\alpha,\Omega}$ coincides with the metric of the Cauchy completion of the metric space $(\Omega, d_{\alpha,\Omega})$. Now equality (1.13) can be written in the following form: for every $\omega_1, \omega_2 \in \Omega^{*, \alpha}$

$$\rho_{\alpha,\Omega}(\omega_1, \omega_2) = d_{\alpha,\Omega}(\omega_1, \omega_2) + \|\ell(\omega_1) - \ell(\omega_2)\|^\alpha.$$  

(5.2)

In turn, this equality and (1.3) yield: let $f \in tr(\partial \Omega)_\alpha(L^1_{p}(\Omega))$ and let $\omega_1, \omega_2 \in (\partial \Omega)_\alpha$, see Definition 1.4 and (1.14). Then

$$|f(\omega_1) - f(\omega_2)| \leq C\|f\|_{tr(\partial \Omega)_\alpha(L^1_{p}(\Omega))} \left\{d_{\alpha,\Omega}(\omega_1, \omega_2)^{1-\frac{1}{p}} + \|\ell(\omega_1) - \ell(\omega_2)\|^\frac{1}{p}\right\}$$

(5.3)

where $\alpha = (p-n)/(p-1)$ and $C = C(n, p)$. Observe that this inequality is equivalent to the following one:

$$|f(\omega_1) - f(\omega_2)| \leq C\|f\|_{tr(\partial \Omega)_\alpha(L^1_{p}(\Omega))} \rho_{\alpha,\Omega}(\omega_1, \omega_2)^{1-\frac{1}{p}}, \quad \omega_1, \omega_2 \in \Omega^{*, \alpha}.$$  

(5.4)

By Lemma 2.10 $\rho_{\alpha,\Omega}(x, y) \sim d_{\alpha,\Omega}(x, y)$ provided $x, y \in \Omega$ and

$$\|x - y\| \geq \min\{\text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega)\}.$$  

This implies the following inequality:

$$\rho_{\alpha,\Omega}(\omega_1, \omega_2) \sim d_{\alpha,\Omega}(\omega_1, \omega_2), \quad \omega_1 \in (\partial \Omega)_\alpha, \omega_2 \in \Omega^{*, \alpha}.$$  

(5.5)

In particular,

$$\rho_{\alpha,\Omega}(\omega_1, \omega_2) \sim d_{\alpha,\Omega}(\omega_1, \omega_2) \quad \text{for every} \quad \omega_1, \omega_2 \in (\partial \Omega)_\alpha,$$

(5.6)

and

$$\rho_{\alpha,\Omega}(\omega, x) \sim d_{\alpha,\Omega}(\omega, x) \quad \text{for every} \quad \omega \in (\partial \Omega)_\alpha, \ x \in \Omega.$$  

Combining equivalence (5.6) with inequality (5.4), we obtain

$$|f(\omega_1) - f(\omega_2)| \leq C\|f\|_{tr(\partial \Omega)_\alpha(L^1_{p}(\Omega))} d_{\alpha,\Omega}(\omega_1, \omega_2)^{1-\frac{1}{p}}, \quad \omega_1, \omega_2 \in (\partial \Omega)_\alpha.$$  

(5.7)
Thus every function \( f \in \text{tr}_{(\partial \Omega), \alpha}(L^1_p(\Omega)) \) is continuous with respect to the metric \( d_{\alpha, \Omega} \).

Observe also that (5.5) and (1.17) provide the following definition of the trace to \((\partial \Omega), \alpha\): Let \( F \in W^{1}_p(\Omega) \) and let \( f = \text{tr}_{(\partial \Omega), \alpha} F \). Then for every \( \omega \in (\partial \Omega), \alpha \) we have

\[
f(\omega) = \lim\{F(x) : d_{\alpha, \Omega}(x, \omega) \to 0, x \in \Omega\}. \tag{5.7}
\]

**Theorem 5.1** Let \( \Omega \) be a domain in \( \mathbb{R}^n \). Let \( p \in (n, \infty) \) and let \( \alpha = (p - n)/(p - 1) \). Let \( \eta \) be a constant satisfying \( \eta \geq 41 \).

A function \( f : (\partial \Omega), \alpha \to \mathbb{R} \) is the trace to \((\partial \Omega), \alpha\) of a (continuous) function \( F \in L^1_p(\Omega) \) if and only if \( f \) is continuous with respect to \( d_{\alpha, \Omega} \) and there exists a constant \( \lambda > 0 \) such that for every finite family \( \{Q_i : i = 1, ..., m\} \) of pairwise disjoint cubes in \( \Omega \) and every choice of \((\alpha, Q_i)\)-visible elements \( \omega^{(1)}_i, \omega^{(2)}_i \in (\partial \Omega), \alpha \) such that

\[
\ell(\omega^{(1)}_i), \ell(\omega^{(2)}_i) \in (\eta Q_i) \cap \partial \Omega, \tag{5.8}
\]

the following inequality

\[
\sum_{i=1}^{m} \frac{|f(\omega^{(1)}_i) - f(\omega^{(2)}_i)|^p}{\left(\text{diam } Q_i\right)^{p-n}} \leq \lambda \tag{5.9}
\]

holds. Moreover,

\[
\|f\|_{\text{tr}_{(\partial \Omega), \alpha}(L^1_p(\Omega))} \sim \inf \lambda^\frac{1}{p}
\]

with constants of equivalence depending only on \( n, p \) and \( \eta \).

**Proof. (Necessity).** Let \( F \in L^1_p(\Omega) \). Prove that the function \( f := \text{tr}_{(\partial \Omega), \alpha} F \) satisfies the theorem’s conditions. As we have seen above, \( f \) is a continuous function on \((\partial \Omega), \alpha\) with respect to the metric \( d_{\alpha, \Omega} \). Prove that \( f \) satisfies inequality (5.9).

We will be needed an auxiliary lemma. Given a function \( g \) defined on \( \Omega \) we let \( g^\Diamond \) denote its extension by zero to all of \( \mathbb{R}^n \). Thus \( g^\Diamond(x) := g(x), x \in \Omega, \) and \( g^\Diamond(x) := 0, x \notin \Omega \). Also, given \( q > 0 \) we put

\[
G(x) := (\|\nabla F\|^q)^\Diamond(x), \quad x \in \mathbb{R}^n. \tag{5.10}
\]

**Lemma 5.2** Let \( q \in (n, p] \) and let \( \eta > 1 \). Let \( Q = Q(x_Q, r_Q) \) be a cube in \( \Omega \) and let \( \omega \in (\partial \Omega), \alpha \). Suppose that \( \ell(\omega) \in (\eta Q) \cap \partial \Omega \) and \( \omega \) is \((\alpha, Q)\)-visible. Then

\[
\frac{|f(\omega) - F(x_Q)|^p}{\left(\text{diam } Q\right)^{p-n}} \leq C \int_Q \mathcal{M}[G]^\frac{\xi}{p}(z) \, dz \tag{5.11}
\]

where \( C = C(n, q, \eta) \).
Proof. Since $\omega$ is $(\alpha, Q)$-visible, 
\[ \text{Conv}\{\ell(\omega), Q\} \setminus \{\ell(\omega)\} \subset \Omega, \]
see Definition 1.7 and Definition 4.9. Let $y \in (\ell(\omega), x_Q]$. Clearly, $y$ is also $Q$-visible point of $\Omega$ and $y \in (\eta Q) \cap \partial \Omega$. Then, by Lemma 3.5 with $t = \eta$,
\[ \left( \frac{|F(y) - F(x_Q)|}{\text{diam } Q} \right)^q \leq C \frac{1}{|Q|} \int_{\gamma_{Q} \cap \Omega} \|\nabla F(z)\|^q dz \]
where $C = C(n, q, \eta)$ and $\gamma = \gamma(n, q, \eta)$.

Applying Lemma 3.6 with $\theta = p/q$, we obtain
\[ \left( \frac{1}{|Q|} \int_{\gamma_{Q} \cap \Omega} \|\nabla F(z)\|^q dz \right)^{\frac{p}{q}} \leq C \frac{1}{|Q|} \int_{Q} \mathcal{M}[G]^\frac{p}{q}(z) dz. \]
Hence
\[ \left( \frac{|F(y) - F(x_Q)|}{\text{diam } Q} \right)^p \leq C \frac{1}{|Q|} \int_{Q} \mathcal{M}[G]^\frac{p}{q}(z) dz. \]
so that
\[ \frac{|F(y) - F(x_Q)|^p}{(\text{diam } Q)^{p-n}} \leq C \int_{Q} \mathcal{M}[G]^\frac{p}{q}(z) dz. \]  \hspace{1cm} (5.12)

Now let $y_i \in (\ell(\omega), x_Q]$ and let $(y_i)$ tends to $\ell(\omega)$ in the Euclidean norm. Then, by Lemma 2.2 (ii), the sequence $(y_i) \in \omega$. Therefore, by (1.16),
\[ f(\omega) = (\text{tr}_{\partial \Omega})_{a} F(\omega) = \lim_{i \to \infty} F(y_i). \]
But, by (5.12),
\[ \frac{|F(y_i) - F(x_Q)|^p}{(\text{diam } Q)^{p-n}} \leq C \int_{Q} \mathcal{M}[G]^\frac{p}{q}(z) dz, \]
so that, letting $i$ tend to $\infty$, we obtain (5.11). □

Using this lemma, we prove the necessity as follows. Put $q := (n + p)/2$. Now, let $\omega^{(1)}, \omega^{(2)} \in (\partial \Omega)$ be $(\alpha, Q)$-visible elements such that
\[ \ell(\omega^{(1)}), \ell(\omega^{(2)}) \in (\eta Q) \cap \partial \Omega. \]
Then, by (5.11),
\[ \frac{|f(\omega^{(1)}) - f(\omega^{(2)})|^p}{(\text{diam } Q)^{p-n}} \leq 2^p \left( \frac{|f(\omega^{(1)}) - F(x_Q)|^p}{(\text{diam } Q)^{p-n}} + \frac{|f(\omega^{(2)}) - F(x_Q)|^p}{(\text{diam } Q)^{p-n}} \right) \leq C \int_{Q} \mathcal{M}[G]^\frac{p}{q}(z) dz. \]
Finally, let \( \omega_i^{(1)}, \omega_i^{(2)} \in (\partial \Omega)_\alpha, i = 1, \ldots, m \), and let \( \{Q_i : i = 1, \ldots, m\} \) be a family of pairwise disjoint cubes in \( \Omega \) such that \( \omega_i^{(1)}, \omega_i^{(2)} \) are \((\alpha, Q_i)\)-visible and

\[
\ell(\omega_i^{(1)}), \ell(\omega_i^{(2)}) \in (\eta Q_i) \cap \partial \Omega
\]

for every \( i = 1, \ldots, m \). Then

\[
I := \sum_{i=1}^{m} \frac{|f(\omega_i^{(1)}) - f(\omega_i^{(2)})|^p}{(\text{diam } Q_i)^{p-n}} \leq C \sum_{i=1}^{m} \int_{Q_i} \mathcal{M}[G]^{\frac{p}{q}}(z) \, dz = C \int_{U} \mathcal{M}[G]^{\frac{p}{q}}(z) \, dz
\]

where \( U := \cup_{i=1}^{m} Q_i \). Hence

\[
I \leq C \int_{\mathbb{R}^n} \mathcal{M}[G]^{\frac{p}{q}}(z) \, dz. \tag{5.13}
\]

Since \( p/q > 1 \), by the Hardy-Littlewood maximal theorem,

\[
\int_{\mathbb{R}^n} \mathcal{M}[G]^{\frac{p}{q}}(z) \, dz \leq C \int_{\mathbb{R}^n} |G|^\frac{p}{q}(z) \, dz = C \int_{\mathbb{R}^n} (\|\nabla F\|^q)^\frac{p}{q}(z) \, dz = C \int_{\Omega} (\|\nabla F\|^q)^\frac{p}{q}(z) \, dz
\]

so that

\[
\int_{\mathbb{R}^n} \mathcal{M}[G]^{\frac{p}{q}}(z) \, dz \leq C \|\nabla F\|_{L^p(\Omega)}^p. \tag{5.14}
\]

Hence \( I \leq C \|\nabla F\|_{L^p(\Omega)}^p \). Taking in this inequality the infimum over all functions \( F \in L^1_p(\Omega) \) such that \( f = \operatorname{tr}_{(\partial \Omega)_\alpha} F \) we obtain the required inequality \( I^{\frac{p}{q}} \leq C \| f \|_{\operatorname{tr}_{(\partial \Omega)_\alpha}(L^1_p(\Omega))} \).

The proof of the necessity is finished.

(Sufficiency.) Let \( f : (\partial \Omega)_\alpha \rightarrow \mathbb{R} \) be a continuous function with respect to the metric \( d_{\alpha,\Pi} \), see (5.1). Let \( \lambda \) be a positive constant such that for every finite family \( \{Q_i : i = 1, \ldots, m\} \) of pairwise disjoint cubes in \( \Omega \) and every \((\alpha, Q_i)\)-visible elements \( \omega_i^{(1)}, \omega_i^{(2)} \in (\partial \Omega)_\alpha \) satisfying (5.8) the inequality (5.9) holds.

We put \( \sigma = +\infty \) in formula (4.10); thus \( c_Q = f(\omega_{Q,\alpha}) \) for every \( Q \in \mathcal{W}(\Omega) \). Then we define a function \( \tilde{f} : \Omega \rightarrow \mathbb{R} \) by formula (4.11). Thus

\[
\tilde{f}(x) := \sum_{Q \in \mathcal{W}(\Omega)} f(\omega_{Q,\alpha}) \varphi_Q(x), \quad x \in \Omega. \tag{5.15}
\]

Prove that \( \tilde{f} \in L^1_p(\Omega) \) and \( \|\tilde{f}\|_{L^1_p(\Omega)} \leq C \lambda^\frac{1}{p} \). Since \( c_Q = f(\omega_{Q,\alpha}) \) for every \( Q \in \mathcal{W}(\Omega) \), by Lemma 4.7

\[
\|\nabla \tilde{f}\|_{L^p_p(\Omega)} \leq C V_p(f; \Omega)
\]

where

\[
V_p(f; \Omega) := \sum \left\{ \frac{|f(\omega_{Q,\alpha}) - f(\omega_{Q',\alpha})|^p}{(\text{diam } Q + \text{diam } Q')^{p-n}} : Q, Q' \in \mathcal{W}(\Omega), Q \cap Q' \neq \emptyset \right\}. \tag{5.16}
\]

37
Hence

\[ V_p(f; \Omega) \leq \sum_{Q \in \mathcal{W}(\Omega)} \sum_{Q' \in A_Q} \frac{|f(\omega_{Q,a}) - f(\omega_{Q',a})|^p}{(\text{diam } Q)^{p-n}}. \]

where

\[ A_Q := \{ Q' \in \mathcal{W}(\Omega) : Q' \cap Q \neq \emptyset, \text{diam } Q' \geq \text{diam } Q \}. \]

Let \( K_Q \in A_Q \) be a cube such that

\[ \max_{Q' \in A_Q} |f(\omega_{Q,a}) - f(\omega_{Q',a})| = |f(\omega_{Q,a}) - f(\omega_{K_Q,a})|. \]

Since the family \( A_Q \) consists of at most \( N(n) \) cubes, see Lemma 4.2 (2), we have

\[ V_p(f; \Omega) \leq C \sum_{Q \in \mathcal{W}(\Omega)} \frac{|f(\omega_{Q,a}) - f(\omega_{K_Q,a})|^p}{(\text{diam } Q)^{p-n}}. \]

By Theorem 4.1 (iii), the family \( \mathcal{W}(\Omega) \) has the covering multiplicity \( M(\mathcal{W}(\Omega)) \leq N(n) \). Therefore this family can be partitioned into at most \( N_1 = N_1(n) \) families \( \{ \pi_j : j = 1, \ldots, N_1 \} \) of pairwise disjoint cubes. See \[5, 11\].

Observe that for each \( Q \in \mathcal{W}(\Omega) \) the cube \( K_Q \in \mathcal{W}(\Omega) \), \( K_Q \cap Q \neq \emptyset \) and \( \text{diam } Q \leq \text{diam } K_Q \), so that, by Lemma 4.11, \( \omega_{Q,a} \) and \( \omega_{K_Q,a} \) are \((a, Q)\)-visible and

\[ \ell(\omega_{Q,a}), \ell(\omega_{K_Q,a}) \in 41Q. \]

Hence, by (5.9), for every \( j = 1, \ldots, N_1 \), we have

\[ \sum_{Q \in \pi_j} \frac{|f(\omega_{Q,a}) - f(\omega_{K_Q,a})|^p}{(\text{diam } Q)^{p-n}} \leq \lambda, \]

so that

\[ V_p(f; \Omega) \leq C \sum_{j=1}^{N_1} \sum_{Q \in \pi_j} \frac{|f(\omega_{Q,a}) - f(\omega_{K_Q,a})|^p}{(\text{diam } Q)^{p-n}} \leq C \lambda. \]  

(5.17)

Hence \( \|\nabla \tilde{f}\|_{L^p(\Omega)} \leq C \lambda \) proving that \( \tilde{f} \in L^1_p(\Omega) \). This also proves that the trace \( \text{tr}_{(\partial \Omega)_a} \tilde{f} \) is well-defined.

Prove that \( \text{tr}_{(\partial \Omega)_a} \tilde{f} = f \). Let \( \omega \in (\partial \Omega)_a \) and let a sequence \( (y_i) \in \omega \). Thus

\[ \lim_{i \to \infty} \rho_{a,\overline{\Omega}}(y_i, \omega) = 0. \]

By (5.2),

\[ \rho_{a,\overline{\Omega}}(y_i, \omega) = d_{a,\overline{\Omega}}(y_i, \omega) + \|y_i - \ell(\omega)\|^a, \]

so that \( d_{a,\overline{\Omega}}(y_i, \omega) \to 0 \) and \( \|y_i - \ell(\omega)\| \to 0 \) as \( i \to \infty \). Recall that, by (1.16),

\[ \text{tr}_{(\partial \Omega)_a} \tilde{f}(\omega) := \lim_{i \to \infty} \tilde{f}(y_i). \]  

(5.18)
We let $K_i \in \mathcal{W}(\Omega)$ denote a Whitney’s cube such that $K_i \ni y_i, i = 1, 2, \ldots$. By Lemma 4.3,

$$\rho_{\alpha, \Omega}(y_i, \omega_{K_i, \alpha}) \leq C \operatorname{dist}(y_i, \partial\Omega)^\alpha.$$ 

Recall that the element $\omega_{K_i, \alpha} \in (\partial\Omega)_\alpha$ is defined as an equivalence class $\omega_{K_i, \alpha} = [(x_i)]_\alpha$ where

$$x_i := a_{K_i} + \frac{1}{i}(x_{K_i} - a_{K_i}), \quad i = 1, 2, \ldots,$$

see (4.3) and (4.4). Since $\ell(\omega) \in \partial\Omega$, we obtain

$$\operatorname{dist}(y_i, \partial\Omega) \leq \|y_i - \ell(\omega)\| \to 0 \quad \text{as} \quad i \to \infty,$$

so that

$$d_{\alpha, \Omega}(y_i, \omega_{K_i, \alpha}) \leq \rho_{\alpha, \Omega}(y_i, \omega_{K_i, \alpha}) \leq C \operatorname{dist}(y_i, \partial\Omega)^\alpha \to 0 \quad \text{as} \quad i \to \infty.$$ 

Hence

$$d_{\alpha, \Omega}(\omega_{K_i, \alpha}, \omega) \leq d_{\alpha, \Omega}(\omega_{K_i, \alpha}, y_i) + d_{\alpha, \Omega}(y_i, \omega) \to 0 \quad \text{as} \quad i \to \infty. \quad (5.20)$$

Let us prove that

$$\lim_{i \to \infty} |f(\omega_{K_i, \alpha}) - \tilde{f}(y_i)| = 0. \quad (5.21)$$

Put

$$I(K_i) := \{Q \in \mathcal{W}(\Omega) : Q \cap K_i \neq \emptyset\}.$$ 

By Lemma 4.3,

$$|\tilde{f}(y_i) - f(\omega_{K_i, \alpha})| \leq C \max_{Q \in I(K_i)} |f(\omega_{Q, \alpha}) - f(\omega_{K_i, \alpha})|. \quad (5.22)$$

On the other hand, by Lemma 4.11 for every cube $Q \in I(K_i)$ with diam $Q \geq \operatorname{diam} K_i$ the elements $\omega_{Q, \alpha}$ and $\omega_{K_i, \alpha}$ are $(\alpha, K_i)$-visible. In addition, $\ell(\omega_{Q, \alpha}), \ell(\omega_{K_i, \alpha}) \in 41K_i$.

Put $\omega_1^{(1)} := \omega_{Q, \alpha}, \omega_1^{(2)} := \omega_{K_i, \alpha}$ and $Q_1 := K_i$. Then the triple $\omega_1^{(1)}, \omega_1^{(2)}, \{Q_1\}$ satisfies the conditions of Theorem 5.1 (with $m = 1$) so that, by the assumption, the inequality (5.9) holds for this triple. By this inequality,

$$\frac{|f(\omega_1^{(1)}) - f(\omega_1^{(2)})|^p}{(\operatorname{diam} Q_1)^{p-n}} \leq \lambda,$$

so that

$$|f(\omega_{Q, \alpha}) - f(\omega_{K_i, \alpha})| \leq \lambda^{\frac{1}{p}} (\operatorname{diam} K_i)^{1-\frac{n}{p}}$$

provided $Q \in I(K_i)$ and diam $Q \geq \operatorname{diam} K_i$.

If $Q \in I(K_i)$ and diam $Q < \operatorname{diam} K_i$, in the same way we prove that

$$|f(\omega_{Q, \alpha}) - f(\omega_{K_i, \alpha})| \leq \lambda^{\frac{1}{p}} (\operatorname{diam} Q)^{1-\frac{n}{p}}.$$ 

But diam $Q \leq 4 \operatorname{diam} K_i$ for every $Q \in I(K_i)$, see (4.2), so that

$$|f(\omega_{Q, \alpha}) - f(\omega_{K_i, \alpha})| \leq C \lambda^{\frac{1}{p}} (\operatorname{diam} K_i)^{1-\frac{n}{p}} \quad \text{for every} \quad Q \in I(K_i).$$
Hence, by (5.22),
\[ |\tilde{f}(y_i) - f(\omega_{K_i,\alpha})| \leq C \lambda^p \left( \text{diam } K_i \right)^{1-p}. \]
Since \( K_i \in W(\Omega) \) and \( y_i \in K_i \), by (4.1),
\[ \text{diam } K_i \leq \text{dist}(K_i, \partial \Omega) \leq \text{dist}(y_i, \partial \Omega) \]
so that
\[ |\tilde{f}(y_i) - f(\omega_{K_i,\alpha})| \leq C \lambda^p \text{dist}(y_i, \partial \Omega)^{1-p}. \]
But, by (5.19), \( \text{dist}(y_i, \partial \Omega) \to 0 \) as \( i \to \infty \), proving (5.21).

It remains to note that the function \( f : (\partial \Omega)_\alpha \to \mathbb{R} \) is continuous with respect to the metric \( d_{\alpha,\Omega} \) so that, by (5.20),
\[ \lim_{i \to \infty} f(\omega_{K_i,\alpha}) = f(\omega). \]
Combining this equality with (5.21), we conclude that
\[ \lim_{i \to \infty} \tilde{f}(y_i) = f(\omega) \]
so that, by (5.18), \( \text{tr}_{(\partial \Omega)_\alpha} \tilde{f}(\omega) = f(\omega) \).

Theorem 5.1 is completely proved. \( \square \)

Our next result, Theorem 5.3, extends the trace criterion for the Sobolev space given in Theorem 1.9 to the case of an arbitrary domain in \( \mathbb{R}^n \).

Let \( \theta > 1 \) and let \( Q = \{Q\} \) be a covering of \( \Omega \) by non-overlapping cubes such that
\[ \frac{1}{\theta} \text{diam } Q \leq \text{dist}(Q, \partial \Omega) \leq \theta \text{diam } Q. \] (5.23)

By \( T_Q : \Omega \to (\partial \Omega)_\alpha \) we denote a mapping defined by the following formula:
\[ T_Q|_Q := \omega_{Q,\alpha}, \quad Q \in Q. \] (5.24)
Recall that the element \( \omega_{Q,\alpha} \) is defined by equalities (4.3) and (4.4).

**Theorem 5.3** Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and let \( p \in (n, \infty) \). Fix constants \( \varepsilon > 0, \theta > 1, \eta \leq 22\theta^2 \), and an arbitrary covering \( Q \) of \( \Omega \) consisting of non-overlapping cubes \( Q \subset \Omega \) each satisfying inequality (5.23).

A function \( f : (\partial \Omega)_\alpha \to \mathbb{R} \) is an element of \( \text{tr}_{(\partial \Omega)_\alpha}(W^1_p(\Omega)) \) if and only if \( f \) is continuous with respect to the metric \( d_{\alpha,\Omega} \), the function \( f \circ T_Q \in L_p(\mathcal{O}_{\varepsilon}(\partial \Omega)) \), and there exists a constant \( \lambda > 0 \) such that for every finite family \( \{Q_i : i = 1, \ldots, m\} \) of pairwise disjoint cubes contained in \( \mathcal{O}_{\varepsilon}(\partial \Omega) \) and every choice of \( (\alpha, Q_i) \)-visible elements \( \omega^{(1)}_i, \omega^{(2)}_i \in (\partial \Omega)_\alpha \) such that
\[ \ell(\omega^{(1)}_i), \ell(\omega^{(2)}_i) \in (\eta Q_i) \cap \partial \Omega, \] (5.25)
the following inequality
\[ \sum_{i=1}^m \frac{|f(\omega^{(1)}_i) - f(\omega^{(2)}_i)|^p}{(\text{diam } Q_i)^{p-n}} \leq \lambda \] (5.26)
holds. Moreover,
\[ \|f\|_{\text{tr}_{(\partial \Omega)_\alpha}(W^1_p(\Omega))} \sim \|f \circ T_Q\|_{L_p(\mathcal{O}_{\varepsilon}(\partial \Omega))} + \inf \lambda^\frac{1}{p} \] (5.27)
with constants of equivalence depending only on \( n, p, \varepsilon, \theta \) and \( \eta \).
Proof. (Necessity). Let \( F \in W^1_p(\Omega) \) and let \( f = \text{tr}_{(\partial \Omega)_a} F \). Since \( F \in L^1_p(\Omega) \), the function \( f \) is continuous with respect to the metric \( d_{\alpha,\Omega} \). In turn, we obtain inequality (5.26) by repeating the proof of the necessity part of Theorem 5.1.

Thus it remains to show that \( f \circ T_Q \in L_p(\mathcal{O}_\varepsilon(\partial \Omega)) \) and

\[
\| f \circ T_Q \|_{L_p(\mathcal{O}_\varepsilon(\partial \Omega))} \leq C \| F \|_{W^1_p(\Omega)}.
\]

We put

\[
\tilde{Q} := \{ Q \in \mathcal{Q} : \text{diam } Q \leq \theta \varepsilon \}.
\]

Let \( Q \in \mathcal{Q} \) be a cube such that \( Q \cap \mathcal{O}_\varepsilon(\partial \Omega) \neq \emptyset \). Then \( \text{dist}(Q, \partial \Omega) \leq \varepsilon \), so that

\[
\text{diam } Q \leq \theta \text{dist}(Q, \partial \Omega) \leq \theta \varepsilon
\]

proving that

\[
\mathcal{O}_\varepsilon(\partial \Omega) \subset \cup \{ Q : Q \in \tilde{Q} \}.
\]

Hence

\[
\| f \circ T_Q \|_{L_p(\mathcal{O}_\varepsilon(\partial \Omega))} \leq \sum_{Q \in \tilde{Q}} \int_Q (f \circ T_Q)^p(x) \, dx = \sum_{Q \in \tilde{Q}} |Q| |f(\omega_{Q,\alpha})|^p.
\]

We let \( F_Q := |Q|^{-1} \int_Q F \, dx \) denote the average of \( F \) over cube \( Q \). Then

\[
\| f \circ T_Q \|_{L_p(\mathcal{O}_\varepsilon(\partial \Omega))} \leq C \left( \sum_{Q \in \tilde{Q}} |Q| |f(\omega_{Q,\alpha}) - F(x_Q)|^p + \sum_{Q \in \tilde{Q}} |Q| |F_Q|^p \right) = C (I + J + K).
\]

Let us consider the element \( \omega_{Q,\alpha} \in (\partial \Omega)_a \) defined by formulas (4.3) and (4.4). We recall that \( \omega_{Q,\alpha} \) is an \((\alpha, Q)\)-visible element, see (4.13), and \( \ell(\omega_{Q,\alpha}) = a_Q \), i.e., \( \ell(\omega_{Q,\alpha}) \) is a point nearest to \( Q \) on \( \partial \Omega \) in the Euclidean norm. Hence,

\[
|\ell(\omega_{Q,\alpha}) - x_Q| = |a_Q - x_Q| \leq \text{dist}(a_Q, Q) + r_Q = \text{dist}(Q, \partial \Omega) + r_Q,
\]

so that, by (5.23),

\[
|\ell(\omega_{Q,\alpha}) - x_Q| \leq \theta \text{diam } Q + r_Q = (2\theta + 1)r_Q \leq \eta r_Q.
\]

(Recall that \( \eta \geq 22\theta^2 \) and \( \theta \geq 1 \).) Thus \( \ell(\omega_{Q,\alpha}) = a_Q \in (\eta Q) \cap \partial \Omega \).

We put \( q := (n + p)/2 \) and apply Lemma 5.2 to the cube \( Q \) and the element \( \omega_{Q,\alpha} \). We obtain

\[
\frac{|f(\omega_{Q,\alpha}) - F(x_Q)|^p}{(\text{diam } Q)^p - n} \leq C \int_Q \mathcal{M}[G]^{\frac{p}{q}}(z) \, dz, \quad Q \in \tilde{Q}.
\]
Recall that $G$ is a function defined by (5.10). Since $\text{diam } Q \leq \theta \epsilon$ for every $Q \in \tilde{Q}$, we have

$$I := \sum_{Q \in \tilde{Q}} |Q| |f(\omega_{Q,a}) - F(x_Q)|^p \leq C \sum_{Q \in \tilde{Q}} (\text{diam } Q)^p \int \mathcal{M}[G]^p(z) \, dz$$

$$\leq C (\theta \epsilon)^p \sum_{Q \in \tilde{Q}} \int \mathcal{M}[G]^p(z) \, dz \leq C \int \mathcal{M}[G]^p(z) \, dz = C \int \mathcal{M}[G]^p(z) \, dz.$$

Thus we have obtained the same estimate of $I$ as in inequality (5.13). Hence

$$I \leq C \|\nabla F\|_{L^p(\Omega)}^p,$$

see (5.14). Let us estimate the quantity

$$J := \sum_{Q \in \tilde{Q}} |Q| |F(x_Q) - F_Q|^p.$$

By the Sobolev-Poincaré inequality (3.1),

$$|F(x_Q) - F_Q|^p |Q| \leq |Q| \sup_{x,y \in Q} |F(x) - F(y)|^p \leq C (\text{diam } Q)^p \int \|\nabla F(z)\|^p \, dz \leq C \int \|\nabla F(z)\|^p \, dz,$$

so that

$$J \leq C \sum_{Q \in \tilde{Q}} \int \|\nabla F(z)\|^p \, dz.$$

Since every two cubes of the family $Q$ are pairwise disjoint and $\tilde{Q} \subset Q$, we obtain

$$J \leq C \int_{\Omega} \|\nabla F(z)\|^p \, dz = C \|\nabla F\|_{L^p(\Omega)}^p.$$

It remains to estimate the quantity $K := \sum\{|F_Q|^p |Q| : Q \in \tilde{Q}\}$. We have

$$K \leq \sum_{Q \in \tilde{Q}} |Q| \left( \frac{1}{|Q|} \int_{Q} |F| \, dz \right)^p \leq \sum_{Q \in \tilde{Q}} \int_{Q} |F|^p \, dz$$

so that

$$K \leq \int_{\Omega} |F|^p \, dz = \|F\|_{L^p(\Omega)}^p.$$

Summarizing the estimates for $I, J$ and $K$, we finally obtain

$$\|f \circ T_Q\|_{L^p(\partial \Omega)}^p \leq C (I + J + K) \leq C (\|\nabla F\|_{L^p(\Omega)}^p + \|F\|_{L^p(\Omega)}^p) \leq C \|F\|_{W^1_p(\Omega)}^p.$$
The necessity part of the theorem is proved.

( Sufficiency.) Let \( f : (\partial \Omega)_\sigma \rightarrow \mathbb{R} \) be a continuous function with respect to the metric \( d_{\alpha,\Omega} \) and let \( f \circ T_{\Omega} \in L_p(\Omega_\varepsilon(\partial \Omega)) \). Also assume that there exists \( \lambda > 0 \) such that for every finite family \( \{ Q_i : i = 1, \ldots, m \} \) of pairwise disjoint cubes contained in \( \Omega_\varepsilon(\partial \Omega) \) and every \( (\alpha, Q_i) \)-visible elements \( \omega_i^{(1)}, \omega_i^{(2)} \in (\partial \Omega)_\alpha \) satisfying (5.25) the inequality (5.26) holds.

We put \( \sigma := \sigma(\varepsilon, \theta) = \varepsilon/(80\theta) \),

and \( \bar{c} := 0 \) in formula (4.10); thus \( c_Q := f(\omega_{Q,\alpha}) \) if \( Q \in \mathcal{W}(\Omega) \) and \( \text{diam} \, Q \leq \sigma \) and \( c_Q := 0 \) if \( \text{diam} \, Q > \sigma \). Then we define a function \( \tilde{F} : \Omega \rightarrow \mathbb{R} \) by formula (4.11):

\[
\tilde{F}(x) := \sum_{Q \in \mathcal{W}(\Omega)} c_Q \varphi_Q(x) = \sum_{Q \in \mathcal{W}(\Omega), \text{diam} \, Q \leq \sigma} f(\omega_{Q,\alpha}) \varphi_Q(x), \quad x \in \Omega.
\]

It can be easily seen that the extension \( \tilde{f} \) defined by formula (5.15) and the extension \( \tilde{F} \) coincide in a \( \sigma/2 \)-neighborhood of \( \partial \Omega \). In fact, assume that \( \text{dist}(x, \partial \Omega) < \sigma/2 \). By Lemma 4.4 (b), if \( \varphi_Q(x) \neq 0 \) then \( Q^* = (9/8)Q \ni x \) so that

\[
\text{dist}(Q, \partial \Omega) \leq \text{dist}(x, Q) + \text{dist}(x, \partial \Omega) \leq \text{diam} \, Q/8 + \sigma/2.
\]

But, by (4.1), \( \text{diam} \, Q \leq \text{dist}(Q, \partial \Omega) \) so that

\[
\text{diam} \, Q \leq \text{diam} \, Q/8 + \sigma/2,
\]

proving that \( \text{diam} \, Q < \sigma \).

Recall that in the sufficiency part of the proof of Theorem 5.1 we have shown that \( f = \text{tr}(\Omega_\varepsilon) \tilde{f} \). The proof of this equality relies only on inequality (5.9) which we apply to cubes \( \{ Q_i \} \) contained in a small neighborhood of \( \partial \Omega \). For such cubes corresponding inequality of Theorem 5.3 i.e., inequality (5.26), holds as well. Since \( \tilde{F} \) coincides with \( \tilde{f} \) in a neighborhood of \( \partial \Omega \) and \( f = \text{tr}(\Omega_\varepsilon) \tilde{f} \), we conclude that \( f = \text{tr}(\Omega_\varepsilon) \tilde{F} \).

Let us estimate the \( W^1_p(\Omega) \)-norm of \( \tilde{F} \). By Lemma 4.8

\[
\| \tilde{F} \|_{W^1_p(\Omega)} \leq C \langle I_1 + I_2 \rangle
\]

where

\[
I_1 := \sum \left\{ \frac{|f(\omega_{Q,\alpha}) - f(\omega_{Q',\alpha})|^p}{(\text{diam} \, Q + \text{diam} \, Q')^{p-n}} : Q, Q' \in \mathcal{W}(\Omega), Q \cap Q' \neq \emptyset, Q, Q' \subset \mathcal{O}_{5\sigma}(\partial \Omega) \right\},
\]

and

\[
I_2 := \sum \left\{ |f(\omega_{Q,\alpha})|^p |Q| : Q \in \mathcal{W}(\Omega), Q \subset \mathcal{O}_{10\sigma}(\partial \Omega) \right\}.
\]

Since \( 5\sigma < \varepsilon \), we obtain \( I_1 \leq V_p(f; \mathcal{O}_\varepsilon(\partial \Omega)) \) where

\[
V_p(f; \mathcal{O}_\varepsilon(\partial \Omega)) := \sum \left\{ |f(\omega_{Q,\alpha}) - f(\omega_{Q',\alpha})|^p (\text{diam} \, Q + \text{diam} \, Q')^{n-p} : Q, Q' \in \mathcal{W}(\Omega), Q \cap Q' \neq \emptyset, Q, Q' \subset \mathcal{O}_\varepsilon(\partial \Omega) \right\}.
\]
Observe that the definition of $V_p(f; \mathcal{O}_\varepsilon(\partial \Omega))$ is similar to that of the quantity $V_p(f; \Omega)$ where the cubes $Q, Q'$ run over all cubes from $W(\Omega)$ such that $Q \cap Q' \neq \emptyset$, see (5.16). In turn, the definition of $V_p(f; \mathcal{O}_\varepsilon(\partial \Omega))$ involves the same family of cubes with the additional requirement $Q, Q' \in \mathcal{O}_\varepsilon(\partial \Omega)$.

This allows us to repeat the proof presented in the sufficiency part of Theorem 5.1 and to show that an analog of inequality (5.17) holds for the quantity $V_p(f; \mathcal{O}_\varepsilon(\partial \Omega))$ as well. In other words,

$$V_p(f; \mathcal{O}_\varepsilon(\partial \Omega)) \leq C \lambda$$

proving that

$I_1 \leq C \lambda$

where $C = C(n, p)$. It remains to estimate the quantity $I_2$ defined by (5.29). Let $Q \in W(\Omega)$ and let $Q \subset \mathcal{O}_{10\sigma}(\partial \Omega)$. Since $Q$ is a covering of $\Omega$, there exists a cube $K_Q \in Q$ such that $K_Q \cap Q \neq \emptyset$.

We let $S_Q$ denote a cube of diameter

$$\text{diam } S_Q := \min(\text{diam } Q, K_Q)$$

such that

$$S_Q \subset K_Q \quad \text{and} \quad S_Q \cap Q \neq \emptyset. \quad (5.31)$$

Let us compare diameters of $K_Q$ and $Q$. Let $y \in K_Q \cap Q$. Then

$$\text{diam } Q \leq \text{dist}(Q, \partial \Omega) \leq \text{dist}(y, \partial \Omega) \leq \text{diam } K_Q + \text{dist}(K_Q, \partial \Omega)$$

so that, by (5.23),

$$\text{diam } Q \leq (1 + \theta) \text{ diam } K_Q. \quad (5.32)$$

On the other hand,

$$\text{dist}(K_Q, \partial \Omega) \leq \text{dist}(y, \partial \Omega) \leq \text{diam } Q + \text{dist}(Q, \partial \Omega)$$

so that, by (4.1),

$$\text{dist}(K_Q, \partial \Omega) \leq 5 \text{ diam } Q. \quad (5.34)$$

Note that (5.33) and (4.1) also imply the following:

$$\text{dist}(K_Q, \partial \Omega) \leq \text{ diam } Q + \text{dist}(Q, \partial \Omega) \leq 2 \text{ dist}(Q, \partial \Omega). \quad (5.35)$$

Now, by (5.34) and (5.23),

$$\text{diam } K_Q \leq \theta \text{ dist}(K_Q, \partial \Omega) \leq 5\theta \text{ diam } Q$$

proving that

$$\frac{1}{1 + \theta} \text{ diam } Q \leq \text{ diam } K_Q \leq 5\theta \text{ diam } Q. \quad (5.36)$$

Recall that $\text{diam } S_Q = \text{ diam } Q$ provided $\text{ diam } Q \leq \text{ diam } K_Q$ so that in this case

$$\text{diam } K_Q \leq 5\theta \text{ diam } Q = 5\theta \text{ diam } S_Q. \quad (5.37)$$
If diam $K_Q \leq \text{diam } Q$, then \(\text{diam } S_Q = \text{diam } K_Q\), so that (5.37) holds as well.

Let us estimate the distance from $S_Q$ to the points $a_Q$ and $a_{K_Q}$. By (5.23) and (5.37),

\[
\|x_{S_Q} - a_{K_Q}\| \leq \text{diam } K_Q + \text{dist}(K_Q, \partial \Omega) \leq \text{diam } K_Q + \theta \text{ diam } K_Q
\]

\[
\leq (1 + \theta)(5\theta \text{ diam } S_Q) = 10\theta(1 + \theta)r_{S_Q}
\]

proving that $a_{K_Q} \in \gamma_1 S_Q$ where $\gamma_1 := 10\theta(1 + \theta)$.

Recall that $Q \cap S_Q \neq \emptyset$ and diam $S_Q = \min\{\text{diam } Q, \text{ diam } K_Q\}$. Hence, by (5.32),

\[
\text{diam } S_Q \leq \text{diam } Q \leq (1 + \theta) \text{ diam } S_Q.
\]

By Lemma 4.10 $\omega_{Q,\alpha}$ is $(\alpha, S_Q)$-visible in $\Omega$. Moreover,

\[
a_Q = \ell(\omega_{Q,\alpha}) \in (10\tau + 1)S_Q \quad \text{where } \tau = (1 + \theta).
\]

Observe that the element $\omega_{K_Q,\alpha}$ is $(\alpha, K_Q)$-visible, see (4.13). Since $S_Q \subset K_Q$, by Lemma 4.12 $\omega_{K_Q,\alpha}$ is $(\alpha, S_Q)$-visible.

Summarizing the properties of the elements $\omega_{Q,\alpha}$ and $\omega_{K_Q,\alpha}$, we conclude that $\omega_{Q,\alpha}$ and $\omega_{K_Q,\alpha}$ are $(\alpha, S_Q)$-visible and $\ell(\omega_{Q,\alpha}), \ell(\omega_{K_Q,\alpha}) \in \eta S_Q$. (Recall that $\eta \geq 22\theta^2$.)

We also note that $Q \subset \mathcal{O}_{10\sigma}(\partial \Omega)$. Since \(\text{diam } S_Q \leq \text{diam } Q\), we have $S_Q \subset \mathcal{O}_{20\sigma}(\partial \Omega)$. Since $20\sigma < \varepsilon$, see (5.28), we obtain $S_Q \subset \mathcal{O}_\varepsilon(\partial \Omega), Q \in \mathcal{W}(\Omega)$.

Now can we estimate the quantity $I_2$, see (5.29), as follows. Let $Q \in \mathcal{W}(\Omega)$ and let $Q \subset \mathcal{O}_{10\sigma}(\partial \Omega)$. We have

\[
|f(\omega_{Q,\alpha})|_p|Q| \leq C(|f(\omega_{Q,\alpha}) - f(\omega_{K_Q,\alpha})|_p|Q| + |f(\omega_{K_Q,\alpha})|_p|Q|).
\]

Since \(\text{diam } Q \leq \text{dist}(Q, \partial \Omega) \leq 10\sigma\), by (5.38),

\[
\text{diam } S_Q \leq \text{diam } Q \leq 10\sigma.
\]

Hence

\[
|f(\omega_{Q,\alpha}) - f(\omega_{K_Q,\alpha})|_p|Q| = \frac{|f(\omega_{Q,\alpha}) - f(\omega_{K_Q,\alpha})|^p}{(\text{diam } S_Q)^{p-n}}(\text{diam } S_Q)^{p-n}|Q|
\]

\[
\leq C(\sigma) \frac{|f(\omega_{Q,\alpha}) - f(\omega_{K_Q,\alpha})|^p}{(\text{diam } S_Q)^{p-n}}.
\]

On the other hand, by (5.36),

\[
|f(\omega_{K_Q,\alpha})|_p|Q| \leq (1 + \theta)^n |f(\omega_{K_Q,\alpha})|_p |K_Q|.
\]

Hence,

\[
I_2 := \sum \{|f(\omega_{Q,\alpha})|_p|Q| : Q \in \mathcal{W}(\Omega), Q \subset \mathcal{O}_{10\sigma}(\partial \Omega)\} \leq C(J_1 + J_2)
\]

where

\[
J_1 := \sum \left\{ \frac{|f(\omega_{Q,\alpha}) - f(\omega_{K_Q,\alpha})|^p}{(\text{diam } S_Q)^{p-n}} : Q \in \mathcal{W}(\Omega), Q \subset \mathcal{O}_{10\sigma}(\partial \Omega) \right\},
\]

and

\[
J_2 := \sum \left\{ |f(\omega_{K_Q,\alpha})|^p |K_Q| : Q \in \mathcal{W}(\Omega), Q \subset \mathcal{O}_{10\sigma}(\partial \Omega) \right\}.
\]

(5.39)
Recall that $Q \to S_Q$ is a mapping defined on the family of cubes
\[ W(\Omega)_{10\sigma} := \{ Q \in W(\Omega) : Q \subset O_{10\sigma}(\partial \Omega) \} \]
and satisfying conditions (5.30) and (5.31). Without loss of generality we may assume that this mapping $Q \to S_Q$ is one-to-one so that the converse mapping $S \to Q_S$ is well defined on the family of cubes
\[ S := \{ S : \exists Q \in W(\Omega)_{10\sigma} \text{ such that } S = S_Q \}. \]

Put $\omega_S^{(1)} := \omega_{Q_S, \alpha}$ and $\omega_S^{(2)} := \omega_{K_{Q_S}, \alpha}$. Then
\[ J_1 = \sum_{S \in S} \frac{|f(\omega_S^{(1)}) - f(\omega_S^{(2)})|^p}{(\text{diam } S)^{p-n}}. \tag{5.40} \]

Since the family $Q$ is Whitney’s type decomposition of $\Omega$, it has finite covering multiplicity $M(Q) \leq N(n, \theta)$. But $S_Q \subset K_Q \in Q$ so that
\[ M(S) \leq M(Q) \leq N(n, \theta). \]
Consequently, the family $S$ can be partitioned into at most $N_1(n, \theta)$ families of pairwise disjoint cubes, see [5, 11]. This allows us to assume that the family $S$ itself consists of pairwise disjoint cubes.

Since the cubes $S \in S$ and the elements $\omega_S^{(1)}, \omega_S^{(2)}$ from (5.40) satisfy the conditions of Theorem 5.3, we can apply inequality (5.26) of this theorem to the quantity $J_1$. We obtain $J_1 \leq C \lambda$.

It remains to estimate the quantity $J_2$ defined by (5.39). We let $K$ denote a family of cubes
\[ K := \{ K_Q \in Q : Q \in W(\Omega), Q \subset O_{10\sigma}(\partial \Omega) \}. \]

Given $K \in K$ we put
\[ G_K := \{ Q \in W(\Omega) : Q \subset O_{10\sigma}(\partial \Omega), K_Q = K \}. \]
Since diam $K_Q \sim$ diam $Q$, see (5.36), the family $G_K$ consists of at most $N(n, \theta)$ cubes. Hence
\[ J_2 = \sum_{K \in K} \sum_{Q \in G_K} |f(\omega_{K, \alpha})|^p |K| \leq C N(n, \theta) \sum_{K \in K} |f(\omega_{K, \alpha})|^p |K|. \]
On the other hand, by definition (5.24) of the mapping $T_Q : \Omega \to (\partial \Omega)_\alpha$,
\[ |f(\omega_{K, \alpha})|^p |K| = \int_K |f \circ T_Q|^p(x) \, dx \]
so that
\[ J_2 \leq C \sum_{K \in K} \int_K |f \circ T_Q|^p(x) \, dx = C \int_U |f \circ T_Q|^p(x) \, dx \]
where $U := \cup\{ K : K \in K \}$. 46
Prove that $U \subset O_\varepsilon(\partial \Omega)$. Recall that for every $K \in \mathcal{K}$ there exists a cube $Q \in \mathbb{W}(\Omega)$ such that $K = K_Q$. Thus $Q \subset O_{10\sigma}(\partial \Omega)$, $K \cap Q \neq \emptyset$,

$$\text{dist}(K, \partial \Omega) \leq 2 \text{dist}(Q, \partial \Omega) \quad \text{and} \quad \text{diam} K \leq 5\theta \text{diam} Q,$$

see (5.35) and (5.36). Hence, for each $y \in K$ we have

$$\text{dist}(y, \partial \Omega) \leq \text{dist}(K, \partial \Omega) + \text{diam} K \leq 2 \text{dist}(Q, \partial \Omega) + 5\theta \text{diam} Q = (5\theta + 2) \text{dist}(Q, \partial \Omega).$$

But $Q \subset O_{10\sigma}(\partial \Omega)$ so that $\text{dist}(y, \partial \Omega) \leq 10\sigma$. Hence, $\text{dist}(y, \partial \Omega) \leq (5\theta + 2)10\sigma$ proving that $K \subset O_\xi(\partial \Omega)$ with $\xi := 10(5\theta + 2)\sigma$.

Since $\theta \geq 1$, we have $\xi \leq 70\theta\sigma$ so that $\xi < \varepsilon$, see (5.28). Consequently, $K \subset O_\varepsilon(\partial \Omega)$ for every $K \in \mathcal{K}$ proving that $U := \bigcup \{K : K \in \mathcal{K}\} \subset O_\varepsilon(\partial \Omega)$.

Hence, $J_2 \leq C \int_{O_\varepsilon(\partial \Omega)} |f \circ T_Q|^p(x) \, dx = C \|f \circ T_Q\|^p_{L^p(O_\varepsilon(\partial \Omega))}$.

Finally, summarizing the estimates for the quantities $I_1$, $J_1$ and $J_2$, we obtain

$$\|\tilde{F}\|^p_{W^{1,p}_0(\Omega)} \leq C (I_1 + I_2) \leq C (I_1 + J_1 + J_2) \leq C \left(\lambda + \lambda + \|f \circ T_Q\|^p_{L^p(O_\varepsilon(\partial \Omega))}\right).$$

Theorem 5.3 is completely proved. \qed

6. Sharp maximal functions on the Sobolev boundary of a domain.

We turn to the last result of the paper, Theorem 6.1 which is a generalization of Theorem 1.11 formulated in Section 1.

Fix $q \in (n, p)$ and put $\beta := (q - n)/(q - 1)$ and $\alpha := (p - n)/(p - 1)$. By $\delta_{\beta,\Omega}$ we denote a quasi-metric on $\Omega^{*\beta} = \Omega \cup (\partial \Omega)_\beta$, see (1.14), defined by the formula

$$\delta_{\beta,\Omega}(\omega_1, \omega_2) := d_{\beta,\Omega}^\beta(\omega_1, \omega_2), \quad \omega_1, \omega_2 \in \Omega^{*\beta}.$$

Given $x \in \Omega$ and $r > 0$ by $B(x, r : \delta_{\beta,\Omega})$ we denote the closed ball in the quasi-metric space $(\Omega^{*\beta}, \delta_{\beta,\Omega})$ with center $x$ and radius $r$:

$$B(x, r : \delta_{\beta,\Omega}) := \{\omega \in \Omega^{*\beta} : \delta_{\beta,\Omega}(x, \omega) \leq r\}. \quad (6.1)$$

Let $\omega = [(x_i)]_{\beta} \in \Omega^{*\beta}$. Since $0 < \beta < \alpha$, by Corollary 2.9

$$\rho_{\alpha,\Omega}(x, y) \leq C \rho_{\beta,\Omega}(x, y), \quad x, y, \in \Omega.$$

Consequently, every Cauchy sequence $(x_i)$ with respect to $\rho_{\beta,\Omega}$ is a Cauchy sequence with respect to $\rho_{\alpha,\Omega}$. Moreover, by definition (1.9),

$$(x_i) \overset{\beta}{\sim} (y_i) \Rightarrow (x_i) \overset{\alpha}{\sim} (y_i).$$
Thus for every \((x_i, y_i) \in \omega\) we have \([x_i]_\alpha = [y_i]_\alpha\) so that all sequences of the equivalence class \(\omega\) (with respect to \(\sim\)) belong to the same equivalence class with respect to \(\ow\). We denote this equivalence class by \(\omega^{[\alpha]}\); thus

\[
\omega^{[\alpha]} := [(x_i)_\alpha]_\infty, \quad (x_i) \in \omega.
\] (6.2)

Moreover, if \(\omega \in (\partial\Omega)_\beta\), then \(\omega^{[\alpha]} \in (\partial\Omega)_\alpha\) which shows that it is defined a mapping

\[
(\partial\Omega)_\beta \ni \omega \mapsto \omega^{[\alpha]} \in (\partial\Omega)_\alpha.
\]

(Observe that in general \(\omega \nsubseteq \omega^{[\alpha]}\) (as families of sequences) while \(\ell(\omega) = \ell(\omega^{[\alpha]}).\))

This enables us given a function \(f : (\partial\Omega)_\alpha \to \mathbb{R}\) to define its fractional sharp maximal function \(f^{\sharp}_{\infty, \beta, \Omega}\) on \(\Omega\) as follows: for every \(x \in \Omega\) we put

\[
f^{\sharp}_{\infty, \beta, \Omega}(x) := \sup \left\{ \frac{|f(\omega^{[\alpha]}_1) - f(\omega^{[\alpha]}_2)|}{r} : r > 0, \omega_1, \omega_2 \in B(x, r : \delta_{\beta, \Omega}) \cap (\partial\Omega)_\beta \right\}.
\] (6.3)

**Theorem 6.1** Let \(\Omega\) be a domain in \(\mathbb{R}^n\) and let \(n < q < p\), \(\beta = \frac{(q - n)}{(q - 1)}\) and \(\alpha = \frac{(p - n)}{(p - 1)}\).

(i). A function \(f \in \text{tr}_{(\partial\Omega)_\alpha}(L^1_p(\Omega))\) if and only if \(f\) is continuous on \((\partial\Omega)_\alpha\) (with respect to \(d_{\alpha, \Omega}\)) and \(f^{\star}_{\infty, \beta, \Omega} \in L^p_\Omega\). Moreover,

\[
\|f\|_{\text{tr}_{(\partial\Omega)_\alpha}(L^1_p(\Omega))} \sim \|f^{\star}_{\infty, \beta, \Omega}\|_{L^p_\Omega}.
\] (6.4)

(ii). Fix \(\varepsilon > 0, \theta > 1\) and a covering \(\mathcal{Q}\) of \(\Omega\) consisting of non-overlapping cubes \(Q \subset \Omega\) satisfying inequality (5.22). Let \(T_{\mathcal{Q}}\) be the mapping defined by (5.24).

A function \(f : (\partial\Omega)_\alpha \to \mathbb{R}\) is an element of \(\text{tr}_{(\partial\Omega)_\alpha}(W^1_p(\Omega))\) if and only if \(f\) is continuous on \((\partial\Omega)_\alpha\) (with respect to \(d_{\alpha, \Omega}\)), and

\[
f \circ T_{\mathcal{Q}}\quad \text{and}\quad f^{\star}_{\infty, \beta, \Omega}\quad \text{are both in}\quad L^p_\Omega(O_{(\partial\Omega)}).
\]

Furthermore,

\[
\|f\|_{\text{tr}_{(\partial\Omega)_\alpha}(W^1_p(\Omega))} \sim \|f \circ T_{\mathcal{Q}}\|_{L^p_\Omega(O_{(\partial\Omega)})} + \|f^{\star}_{\infty, \beta, \Omega}\|_{L^p_\Omega(O_{(\partial\Omega)})}.
\] (6.5)

The constants of equivalence in (6.4) depend only on \(n, p, q, \varepsilon\) and in (6.5) they only depend on \(n, p, q, \varepsilon\) and \(\theta\).

**Proof.** (Necessity). (i). Let \(F \in L^1_p(\Omega)\) and let \(f = \text{tr}_{(\partial\Omega)_\alpha}F\).

Recall that, by (5.7), for every \(\omega \in (\partial\Omega)_\alpha\)

\[
f(\omega) = \lim\{F(x) : d_{\alpha, \Omega}(x, \omega) \to 0, x \in \Omega\}.
\] (6.6)

Let

\[
G(x) := (\|\nabla F\|^q)^\lambda(x), \quad x \in \mathbb{R}^n.
\]
Prove that
\[ f_{1,0}^\beta(x) \leq C(n, q) (\mathcal{M}[G])^{\frac{1}{n}}(x), \quad x \in \Omega. \tag{6.7} \]
(Recall that \( \mathcal{M} \) stands for the Hardy-Littlewood maximal operator, see (3.4).)

Let \( \omega_1, \omega_2 \in (\partial \Omega)_\beta \), \( \omega_1 \neq \omega_2 \), and let \( \omega_1, \omega_2 \in B(x, r : \delta_{\beta, \Omega}) \). Hence
\[ d_{\beta, \Omega}(x, \omega_1), d_{\beta, \Omega}(x, \omega_2) \leq r^\beta. \tag{6.8} \]

Since \( \omega_1, \omega_2 \in (\partial \Omega)_\beta \), there exist sequences \( (x_i) \in \omega_1 \), \( (y_i) \in \omega_2 \) such that
\[ \rho_{\beta, \Omega}(x_i, \omega_1) \to 0 \quad \text{and} \quad \rho_{\beta, \Omega}(y_i, \omega_2) \to 0 \quad \text{as} \quad i \to \infty. \tag{6.9} \]
Recall that \( \omega_1^{[\alpha]} = [(x_i)_\alpha \text{ and } \omega_2^{[\alpha]} = [(y_i)_\alpha \text{ so that } \rho_{\alpha, \Omega}(x_i, \omega_1^{[\alpha]}) \text{ and } \rho_{\alpha, \Omega}(y_i, \omega_2^{[\alpha]}) \text{ tend to 0 as } i \to \infty. \]
Consequently, by (6.6),
\[ f(\omega_1^{[\alpha]}) = \lim_{i \to \infty} F(x_i) \quad \text{and} \quad f(\omega_2^{[\alpha]}) = \lim_{i \to \infty} F(y_i). \tag{6.10} \]

By (6.9),
\[ \rho_{\beta, \Omega}(x_i, y_i) = \rho_{\beta, \Omega}(x_i, y_i) \to \rho_{\beta, \Omega}(\omega_1, \omega_2) \quad \text{as} \quad i \to \infty, \]
so that there exists \( N_0 \in \mathbb{N} \) such that
\[ \rho_{\beta, \Omega}(x_i, y_i) \leq 2 \rho_{\beta, \Omega}(\omega_1, \omega_2), \quad i \geq N_0. \]
Combining this inequality with (5.6), we obtain
\[ \rho_{\beta, \Omega}(x_i, y_i) \leq C d_{\beta, \Omega}(\omega_1, \omega_2), \quad i \geq N_0, \]
so that, by (6.8),
\[ \rho_{\beta, \Omega}(x_i, y_i) \leq C (d_{\beta, \Omega}(\omega_1, x) + d_{\beta, \Omega}(x, \omega_2)) \leq C (r^\beta + r^\beta) \leq C r^\beta, \quad i \geq N_0. \]

Let \( \lambda_1 = \lambda_1(n, q) \) be the constant from Proposition 3.4. Then
\[ \lambda_1 \rho_{\beta, \Omega}(x_i, y_i)^{\frac{1}{n \beta}} \leq \lambda_1 C^{\frac{1}{n}} r, \quad i \geq N_0. \]
By this proposition,
\[ |F(x_i) - F(y_i)| \leq C R \left( \frac{1}{|Q(x_i, R)|} \int_{Q(x_i, R)} G(z) \, dz \right)^{\frac{1}{n}}, \quad i \geq N_0, \tag{6.11} \]
provided \( R := \lambda_1 C^{\frac{1}{n}} r \). On the other hand,
\[ \|x_i - x\|^\beta \leq \rho_{\beta, \Omega}(x_i, x) \leq \rho_{\beta, \Omega}(x_i, \omega_1) + \rho_{\beta, \Omega}(x, \omega_1) \]
so that, by (6.9), there exists \( N_1 \in \mathbb{N} \) such that
\[ \|x_i - x\|^\beta \leq 2 \rho_{\beta, \Omega}(x, \omega_1), \quad i \geq N_1. \]

49
Combining this inequality with (5.5) and (6.8), we obtain
\[ \| x_i - x \|_\beta \leq C d_{\beta,\Omega}(x, \omega_1) \leq Cr^\beta, \quad i \geq N_1, \]
proving that
\[ \| x_i - x \| \leq Cr \leq \gamma R, \quad i \geq N_1, \]
with \( \gamma = \gamma(n, q, \beta) \). By this inequality,
\[ Q(x_i, R) \subset \tilde{Q} := Q(x, \gamma + 1)R \]
promised \( i \geq N_1 \). In addition, \( |Q(x_i, R)| \sim |\tilde{Q}| \), so that, by (6.11),
\[ |F(x_i) - F(y_i)| \leq C R \left( \frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} G(z) \, dz \right)^{\frac{1}{q}} \]
so that
\[ |f(\omega_1^{[\alpha]}) - f(\omega_2^{[\alpha]})| \leq C R \left( \frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} G(z) \, dz \right)^{\frac{1}{q}} \]
and
\[ |f(\omega_1^{[\alpha]}) - f(\omega_2^{[\alpha]})|/r \leq C (\mathcal{M}[G](x))^{\frac{1}{q}}. \]
Taking the supremum in the left-hand side of this inequality over all \( \omega_1, \omega_2 \in (\partial \Omega)_\beta \) satisfying (6.8), we obtain inequality (6.7).

By this inequality,
\[ \| f_\circ T_{\Omega} \|_{L_p(\Omega)} \leq C \| f \|_{L_p(\Omega)} \]
so that
\[ \| f_\circ T_{\Omega} \|_{L_p(\Omega)} \leq C \| \nabla F \|_{L_p(\Omega)}. \]
Taking the supremum in the left-hand side of this inequality over all \( \omega_1, \omega_2 \in (\partial \Omega)_\beta \) satisfying (6.8), we obtain inequality (6.7).

By this inequality,
\[ \| f_\circ T_{\Omega} \|_{L_p(\Omega)} \leq C \| f \|_{L_p(\Omega)} \]
so that
\[ \| f_\circ T_{\Omega} \|_{L_p(\Omega)} \leq C \| \nabla F \|_{L_p(\Omega)}. \]
Taking the supremum in the left-hand side of this inequality over all \( \omega_1, \omega_2 \in (\partial \Omega)_\beta \) satisfying (6.8), we obtain inequality (6.7).

By this inequality,
\[ \| f_\circ T_{\Omega} \|_{L_p(\Omega)} \leq C \| f \|_{L_p(\Omega)} \]
so that
\[ \| f_\circ T_{\Omega} \|_{L_p(\Omega)} \leq C \| \nabla F \|_{L_p(\Omega)}. \]
Taking the supremum in the left-hand side of this inequality over all \( \omega_1, \omega_2 \in (\partial \Omega)_\beta \) satisfying (6.8), we obtain inequality (6.7).

(ii). The latter inequality yields
\[ \| f_\circ T_{\Omega} \|_{L_p(\Omega)} \leq C \| f \|_{W^{1}_p(\Omega)}. \]
In turn, by Theorem 5.3,
\[ \| f \circ T_{\Omega} \|_{L_p(\Omega)} \leq C \| f \|_{W^{1}_p(\Omega)}. \]

50
These two inequalities show that the right-hand side of equivalence (6.5) is bounded by $$C \| f \|_{W^{1,\infty}(\Omega)}$$.

The necessity part of the statements (i) and (ii) is proved.

(Sufficiency).

(i). Fix a constant $$\tau \geq 1$$. Let $$Q \subset \Omega$$ be a cube and let $$\omega \in (\partial \Omega)_\alpha$$ be an $$(\alpha, Q)$$-visible element such that $$\ell(\omega) \in \tau Q \cap \partial \Omega$$.

Let $$(y_i)$$ be a sequence of points in $$\Omega$$ such that $$y_i \in (\ell(\omega), x_Q]$$, $$i = 1, 2, \ldots$$, and $$\omega = [(y_i)]_\alpha$$, see Definition 4.9. Recall that $$\ell(\omega)$$ is a $$Q$$-visible point, the line segment $$(\ell(\omega), x_Q] \subset \Omega$$, see Definition 1.7, and $$\lim_{i \to \infty} y_i = \ell(\omega)$$.

By Lemma 2.2, $$(y_i)$$ is a Cauchy sequence with respect to the metric $$\rho_{\beta, \Omega}$$. We put $$\tilde{\omega} = [(y_i)]_\beta$$.

Observe that, by Lemma 2.2, $$\tilde{\omega}$$ is well defined and does not depend on the choice of the sequence $$(y_i) \in \omega$$ such that $$y_i \in (\ell(\omega), x_Q]$$, $$i = 1, 2, \ldots$$. Also note that $$\tilde{\omega} \in (\partial \Omega)_\beta$$. Since $$\omega = [(y_i)]_\alpha$$, by (6.2), $$\omega = \tilde{\omega} [\alpha]$$.

Let us estimate the distance $$d_{\beta, \Omega}(\tilde{\omega}, x_Q)$$. By Lemma 2.2 (i),

$$d_{\beta, \Omega}(y_i, x_Q) \leq \int_{[y_i, x_Q]} \text{dist}(z, \partial \Omega)^{\beta-1} ds(z) \leq C(\beta) \left( \frac{\|\ell(\omega) - x_Q\|}{\text{diam } Q} \right)^{1-\beta} \|y_i - x_Q\|^\beta.$$  

Recall that $$\ell(\omega) \in \tau Q$$. Since $$y_i \in (\ell(\omega), x_Q]$$, the point $$y_i \in \tau Q$$ as well so that

$$\|\ell(\omega) - x_Q\| \leq \tau r_Q = \tau \text{diam } Q/2,$$

and

$$\|y_i - x_Q\| \leq \tau r_Q = \tau \text{diam } Q/2, \quad i = 1, 2, \ldots.$$  

Hence

$$d_{\beta, \Omega}(y_i, x_Q) \leq C(\text{diam } Q)^\beta,$$

where $$C = C(\beta, \tau)$$. Since $$\tilde{\omega} = [(y_i)]_\beta$$, we have

$$d_{\beta, \Omega}(\tilde{\omega}, x_Q) = \lim_{i \to \infty} d_{\beta, \Omega}(y_i, x_Q) \leq C(\text{diam } Q)^\beta.$$  

Now, let $$y$$ be an arbitrary point in $$Q$$. Then

$$\|x_Q - y\| \leq r_Q \leq \text{dist}(x_Q, \partial \Omega) \leq \max \{ \text{dist}(x_Q, \partial \Omega), \text{dist}(y, \partial \Omega) \},$$

so that, by Lemma 2.1,

$$d_{\beta, \Omega}(x_Q, y) \leq \int_{[x_Q, y]} \text{dist}(z, \partial \Omega)^{\beta-1} ds(z) \leq \frac{1}{\beta} \|x_Q - y\|^\beta \leq \frac{1}{\beta} (\text{diam } Q/2)^\beta.$$  

Hence

$$d_{\beta, \Omega}(\tilde{\omega}, y) \leq d_{\beta, \Omega}(\tilde{\omega}, x_Q) + d_{\beta, \Omega}(x_Q, y) \leq C(\text{diam } Q)^\beta.$$
proving that
\[ \delta_{\bar{\beta}\Pi}(\bar{\omega}, y) := d_{\bar{\beta}\Pi} \bar{\mathbf{p}}(\bar{\omega}, y) \leq R := C^\frac{1}{\beta} \text{diam } Q. \]
Thus
\[ \bar{\omega} \in B(y, R : \delta_{\bar{\beta}\Pi}) \quad \text{where} \quad R = C^\frac{1}{\beta} \text{diam } Q, \tag{6.12} \]
see (6.1).

Let \( \omega_1, \omega_2 \in (\partial\Omega)_\alpha \) be two \((\alpha, Q)\)-visible elements such that
\[ \ell(\omega_1), \ell(\omega_2) \in (\tau Q) \cap (\partial\Omega). \]
Prove that
\[ I := \frac{|f(\omega_1) - f(\omega_2)|^p}{(\text{diam } Q)^{p-n}} \leq C |Q| (f^\sharp_{\infty, \beta, \Omega})^p(y). \tag{6.13} \]
In fact, \( \omega_1 = \bar{\omega}_1^{[\alpha]}, \omega_2 = \bar{\omega}_2^{[\alpha]} \), and, by (6.12),
\[ \bar{\omega}_1, \bar{\omega}_2 \in B(y, R : \delta_{\bar{\beta}\Pi}) \quad \text{with} \quad R = C^\frac{1}{\beta} \text{diam } Q. \tag{6.14} \]
Hence
\[
I := \frac{|f(\omega_1) - f(\omega_2)|^p}{(\text{diam } Q)^{p-n}} = \frac{|f(\bar{\omega}_1^{[\alpha]} - f(\bar{\omega}_2^{[\alpha]})|^p}{(\text{diam } Q)^{p-n}} \\
\leq C |Q| \left( \frac{|f(\bar{\omega}_1^{[\alpha]} - f(\bar{\omega}_2^{[\alpha]})|}{\text{diam } Q} \right)^p \leq C |Q| \left( \frac{|f(\bar{\omega}_1^{[\alpha]} - f(\bar{\omega}_2^{[\alpha]})|}{R} \right)^p.
\]
By (6.14) and (6.3),
\[ |f(\bar{\omega}_1^{[\alpha]} - f(\bar{\omega}_2^{[\alpha]})| / R \leq f^\sharp_{\infty, \beta, \Omega}(y), \]
and inequality (6.13) follows.

Integrating this inequality over cube \( Q \), we obtain
\[ I := \frac{\int |f(\omega_1) - f(\omega_2)|^p}{(\text{diam } Q)^{p-n}} \leq C \int_Q (f^\sharp_{\infty, \beta, \Omega})^p(y) \, dy \tag{6.15} \]
where \( C = C(n, p, \beta, \tau) \).

Let \( A = \{Q_i : i = 1, \ldots, m\} \) be a finite family of pairwise disjoint cubes in \( \Omega \). Let \( \omega_1^{(1)}, \omega_1^{(2)} \in (\partial\Omega)_\alpha \) be \((\alpha, Q)\)-visible elements such that
\[ \ell(\omega_1^{(1)}), \ell(\omega_1^{(2)}) \in (\tau Q) \cap \partial\Omega, \quad i = 1, \ldots, m, \]
where \( \tau = 41 \). Then, by (6.15),
\[ J(f; A) := \sum_{i=1}^m \frac{|f(\omega_1^{(1)}(Q_i)) - f(\omega_1^{(2)}(Q_i))|^p}{(\text{diam } Q_i)^{p-n}} \leq C \sum_{i=1}^m \int_{Q_i} (f^\sharp_{\infty, \beta, \Omega})^p(y) \, dy. \]
Since the cubes of the family $\mathcal{A}$ are pairwise disjoint, we have

$$J(f; \mathcal{A}) \leq C \| f^\sharp_{\infty, \beta, \Omega} \|^p_{L^p(U)} \quad (6.16)$$

where $U := \bigcup \{ Q_i : i = 1, \ldots, m \}$ and $C = C(n, p, \beta)$. Hence $J(f; \mathcal{A}) \leq \lambda$ provided $\lambda := C \| f^\sharp_{\infty, \beta, \Omega} \|^p_{L^p(O_\epsilon)}$.

Now, let $f : (\partial \Omega)_\alpha \rightarrow \mathbb{R}$ be a continuous function (with respect to the metric $d_{\alpha, \Omega}$) and let $f^\sharp_{\infty, \beta, \Omega} \in L^p(\Omega)$. Then, by Theorem 5.1, $f \in tr(\partial \Omega)_\alpha(L^1_p(\Omega))$ and the following inequality

$$\| f \|^p_{tr(\partial \Omega)_\alpha(L^1_p(\Omega))} \leq C \lambda^\frac{2}{p} \leq C \| f^\sharp_{\infty, \beta, \Omega} \|^p_{L^p(\Omega)}$$

holds.

The sufficiency part of the statement (i) of Theorem 6.1 is proved.

(ii). By inequality (6.16) with $\tau = 22\theta^2$, we have

$$J(f; \mathcal{A}) \leq C \| f^\sharp_{\infty, \beta, \Omega} \|^p_{L^p(O_\epsilon(\partial \Omega))}$$

provided $\mathcal{A} = \{ Q_i : i = 1, \ldots, m \}$ is a finite family of pairwise disjoint cubes contained in $O_\epsilon(\partial \Omega)$. Hence $J(f; \mathcal{A}) \leq \lambda$ where $\lambda := C \| f^\sharp_{\infty, \beta, \Omega} \|^p_{L^p(O_\epsilon(\partial \Omega))}$.

Consequently, if $f : (\partial \Omega)_\alpha \rightarrow \mathbb{R}$ is a continuous function (with respect to the metric $d_{\alpha, \Omega}$), and the functions $f \circ T_Q$ and $f^\sharp_{\infty, \beta, \Omega}$ are both in $L^p(O_\epsilon(\partial \Omega))$, then, by Theorem 5.3, the function $f \in tr(\partial \Omega)_\alpha(W^1_p(\Omega))$. Moreover,

$$\| f \|^p_{tr(\partial \Omega)_\alpha(W^1_p(\Omega))} \leq C(\| f \circ T_Q \|^p_{L^p(O_\epsilon(\partial \Omega))} + \lambda^\frac{2}{p}) \leq C(\| f \circ T_Q \|^p_{L^p(O_\epsilon(\partial \Omega))} + \| f^\sharp_{\infty, \beta, \Omega} \|^p_{L^p(O_\epsilon(\partial \Omega))}).$$

Theorem 6.1 is completely proved. □

Acknowledgement. I am very thankful to M. Cwikel and V. Mazya for useful suggestions and remarks. I am also very grateful to C. Fefferman, N. Zobin and all participants of Whitney Problems Workshop, College of William and Mary, August 2009, for stimulating discussions and valuable advice.

References

[1] K. Astala, K. Hag, P. Hag, V. Lappalainen, Lipschitz Classes and the Hardy-Littlewood Property, Mh. Math 115 (1993) 267–279.

[2] N. Aronszajn, P. Szeptycki, Theory of Bessel potentials. IV, Ann. Inst. Fourier, 25 (1975) 27–69.

[3] O. V. Besov, The behavior of differentiable functions on a non-smooth surface. Studies in the theory of differentiable functions of several variables and its applications, IV. Trudy Mat. Inst. Steklov. 117 (1972) 3–10. (Russian)

[4] Yu. A. Brudnyi, Spaces that are definable by means of local approximations, Trudy Moscov. Math. Obshch., 24 (1971) 69–132; English transl. in Trans. Moscow Math. Soc. 24 (1974) 73–139.

53
Yu. A. Brudnyi, B. D. Kotliar, A certain problem of combinatorial geometry, Sibirsk. Mat. Z. 11 (1970) 1171-1173; English transl. in Siberian Math. J. 11 (1970) 870-871.

S. Buckley, P. Koskela, Criteria for imbeddings of Sobolev-Poincaré type, Internat. Math. Res. Notices 18 (1996) 881–902.

S. Buckley, A. Stanoyevitch, Weak slice conditions and Hölder imbeddings, J. London Math. Soc. 66 (2001) 690–706.

S. Buckley, A. Stanoyevitch, Weak slice conditions, product domains, and quasiconformal mappings, Rev. Math. Iberoam. 17 (2001) 1–37.

A. P. Calderón, Estimates for singular integral operators in terms of maximal functions, Studia Math. 44 (1972) 563–582.

A. P. Calderón, R. Scott, Sobolev type inequalities for $p > 0$, Studia Math. 62 (1978) 75–92.

V. L. Dolnikov, The partitioning of families of convex bodies, Sibirsk. Mat. Z. 12 (1971) 664-667 (Russian); English transl. in Siberian Math. J. 12 (1971) 473-475.

E. Gagliardo, Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in $n$ variabili, Rend. Sem. Mat. Univ. Padova 27 (1957) 284–305.(Italian)

F.W. Gehring, O. Martio, Lipschitz classes and quasiconformal mappings, Ann. Acad. Sci. Fenn. Ser. AI Math. 10 (1985) 203-219.

P. Grisvard, Elliptic problems in non-smooth domains. Monographs and Studies in Mathematics, 24. Pitman, Boston, 1985. xiv+410 pp.

M. de Guzmán, Differentiation of integrals in $\mathbb{R}^n$, Lect. Notes in Math. 481, Springer-Verlag, 1975.

P. W. Jones, Quasiconformal mappings and extendability of functions in Sobolev spaces, Acta Math. 147 (1981) 71–78.

A. Jonsson, The trace of potentials on general sets, Ark. Mat. 17 (1979) 1–18.

A. Jonsson, Besov spaces on surfaces with singularities, Manuscripta Math. 71 (1991) 121–152.

A. Jonsson, H. Wallin, Function Spaces on Subsets of $\mathbb{R}^n$, Harwood Acad. Publ., London, 1984, Mathematical Reports, Volume 2, Part 1.

V. Lappalainen, $Lip_p$-extension domains, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertations 56 (1985) 1-52.

J-L. Lions, E. Magenes, Non-homogeneous boundary value problems and applications, Berlin, New York, Springer-Verlag, 1972.

V.G. Maz’ja, Sobolev spaces, Springer-Verlag, Berlin, 1985, xix+486 pp.
V. Maz’ya, S. Poborchi, Traces of functions from S. L. Sobolev spaces on small and large components of the boundary, Mat. Zametki 45 (1989), no. 4, 69–77, 126; Engl. transl. in Math. Notes 45 (1989), no. 3-4, 312–317.

V. Maz’ya, S. Poborchi, Traces of functions with a summable gradient in a domain with a cusp at the boundary, Mat. Zametki 45 (1989), no. 1, 57–65, 140; Engl. transl. in Math. Notes 45 (1989), no. 1-2, 39–44.

V. Maz’ya, S. Poborchi, Boundary traces of functions from Sobolev spaces on a domain with a cusp, Trudy Inst. Mat. (Novosibirsk) 14 (1989) 182–208; Engl. transl. in Siberian Adv. Math. 1 (1991), no. 3, 75–107.

V. Maz’ya, S. Poborchi, Differentiable Functions on Bad Domains, Word Scientific, River Edge, NJ, 1997.

V. Maz’ya, S. Poborchi, Yu. Netrusov, Boundary values of functions from Sobolev spaces in some non-Lipschitzian domains. (Russian) Algebra i Analiz 11 (1999), no. 1, 141–170; Eng. transl. in St. Petersburg Math. J. 11 (2000), no. 1, 107–128.

S. M. Nikol’skii, Boundary properties of functions defined on a region with angular points. I–III. Mat. Sb. 40 (1956) 303–318, 43 (1957) 127–144, 45 (1958) 181–194. (Russian); Eng. transl.in Amer. Math. Soc. Transl., Series 2, 83 (1958) 101–158.

S. Poborchi, Continuity of the boundary trace operator $W^1_p(\Omega) \to L_Q(\partial\Omega)$ for a domain with outward peak. Vestnik St. Petersburg Univ. Math. 38 (2005), no. 3, 37–44 (2006).

P. Shvartsman, Sobolev $W^1_p$-spaces on closed subsets of $\mathbb{R}^n$, Advances in Math. 220 (2009) 1842-1922.

P. Shvartsman, On Sobolev extension domains in $\mathbb{R}^n$, J. Funct. Anal. 258 (2010) 2205-2245.

E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, New Jersey, 1970.

M. Yu. Vasil’chik, The boundary behavior of functions of Sobolev spaces defined on a planar domain with a peak vertex on the boundary [Translation of Mat. Tr. 6 (2003), no. 1, 3–27; MR1985623]. Siberian Adv. Math. 14 (2004), no. 2, 92–115.

G. N. Yakovlev, Boundary properties of functions of the class $W_p^{(l)}$ in regions with corners. (Russian) Dokl. Akad. Nauk SSSR 140 (1961) 73–76.; Eng. transl. in Soviet Math. 2 (1961) 1177–1180.

G. N. Yakovlev, Dirichlet problem for a region with a non-Lipschitz boundary. Differencial’nye Uravnenija 1 (1965) 1085–1098 (Russian); Eng. transl. in Differential Equations 1 (1965), 847-858.