Random matrix theory for underwater sound propagation

K. C. Hegewisch1 and S. Tomsovic1,2(a)

1 Department of Physics and Astronomy, Washington State University - Pullman, WA, 99164-2814, USA
2 Department of Physics, Indian Institute of Technology Madras - Chennai, 600036, India

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Abstract – Ocean acoustic propagation can be formulated as a wave guide with a weakly random medium generating multiple scattering. Twenty years ago, this was recognized as a quantum chaos problem, and yet random matrix theory, one pillar of quantum wave chaos studies, has never been introduced into the subject. The modes of the wave guide provide a representation for the propagation, which in the parabolic approximation is unitary. Scattering induced by the ocean’s internal waves leads to a power-law random banded unitary matrix ensemble for long-range deep-ocean acoustic propagation. The ensemble has similarities, but differs, from those introduced for studying the Anderson metal-insulator transition. The resulting long-range propagation ensemble statistics agree well with those of full wave propagation using the parabolic equation.

Introduction. – Underwater sound provides a means of remote sensing the ocean interior and monitoring global climate change [1,2] amongst other motivations. Beginning in the latter part of the 1980’s decade it was realized that the ray dynamics underlying ocean acoustic propagation in many contexts was chaotic [3–6] and therefore the field serves as a domain with several unique features for studies of wave/quantum chaos [7–9]. To date, one of the theoretical foundations of quantum chaos, random matrix theory (RMT) [10–12], has not been applied previously in the ocean acoustic propagation context. This is in stark contrast to other linear acoustics fields, where random matrix theory has been of growing usefulness since its introduction over twenty years ago [13,14].

The classical chaos observed in ray models of long-range ocean acoustics is thought to be due primarily to the range-dependent effects of the ocean’s internal waves [15], which serve to multiply scatter the acoustic wave as it propagates. The internal waves are in constant motion (as are other parts of the ocean environment) and perpetually changing on the time scale of half an hour. Repeated measurements and statistical treatments are critical elements. Due to practicalities, such as computational demands, typically only a few realizations of internal wave effects have ever been utilized for the analysis of the statistical properties of experimental data [16].

It seems natural therefore to seek an efficient statistical method to describe the acoustic propagation in the ocean. If the acoustic propagation is viewed in terms of scattering between acoustic modes, the propagation can be described solely as the evolution of a matrix of coefficients for mode mixing. A statistical model would then characterize the matrix element distribution.

The earliest RMT application to elastodynamics [13] had a direct experimental connection to the classical, structureless Gaussian/circular ensembles of Wigner and Dyson [12]. Ocean acoustic propagation cannot be represented in this way. Although it had been expected that chaos-induced multiple scattering would greatly degrade information about the ocean, stable properties have been seen experimentally in the acoustic arrivals [17,18] suggesting that there is some mechanism which preserves some core information in the propagation. RMT can be adapted quite well to such situations in that it is equivalent to a kind of maximum entropy approach subject to the inclusion of surviving information; i.e. one can create “deformed” ensembles carrying surviving information and structure. It has the potential to improve the understanding of many of the observed statistical behaviors and connect them to parameters of the ocean environment, and give a new more efficient method of making simulations. Conversely, unique features of ocean acoustics have the potential to influence the future development of RMT through the introduction of new

(a)E-mail: tomsovic@wsu.edu
models. The purpose of this letter therefore is to step through the construction of a structured random matrix ensemble appropriate for ocean acoustic propagation in as straightforward a physical context as possible. The main physical characteristics of the ocean relevant to acoustic propagation are taken fully into account.

Long-range ocean acoustic propagation modeling. – For simplicity, long-range propagation at a fixed, low angular frequency \( \omega \) is considered. There is a so-called sound channel which creates a vertical wave guide [19] in which the ocean’s internal waves [20] generate multiple scatterings (introducing chaos). Over the last few decades there have been several environmental measurements [17], which have led to a great deal of understanding about the nature of the ocean’s wave guide and internal waves. The general shape of the wave guide and how it depends on latitude or the ocean under consideration is well known, as is an analytical method for generating an ensemble of internal waves that possesses the correct power spectrum, dispersion relations, and matches their statistical properties [20,21].

In this physical context, horizontal out-of-plane scattering can be neglected, as can absorption, surface, and bottom interactions. Furthermore, the dominant effect of internal waves is small angle, forward scattering. Incorporation of these approximations into the appropriate scalar wave equation leads to a paraxial optical (parabolic) equation [1]. It has a direct analogy to the Schrödinger wave equation, making the wave and quantum chaos connection even stronger [1]. IthasadirectanalogytotheSchrödingerequation.

\[ \frac{i}{k_0} \frac{\partial \Psi(z,r)}{\partial r} = \left( \frac{1}{2k_0^2} \frac{\partial^2}{\partial z^2} + V(z,r) \right) \Psi(z,r), \quad (1) \]

where the real part of \( \Psi(z,r) \) multiplied by a traveling phase \( r^{-1/2} \) is the sound pressure amplitude. The “potential” is

\[ V(z,r) = \frac{1}{2} \left[ 1 - \frac{c_0^2}{c^2(z,r)} \right] = V_0(z) + \epsilon V_I(z,r), \quad (2) \]

c_0 is a reference sound speed \( \approx 1.5 \text{ km/s} \), and \( k_0^{-1} = c_0/\omega \). Although, the potential itself is unit free, the relevant scales are set by the measured properties of \( c(z,r) \) and the particular \( k_0 \) considered; experiments are generally not done for fixed \( k_0 \), but experimental simulations are straightforwardly constructed as a Fourier integral over solutions of eq. (1) using a range of \( k_0 \) matched to the acoustic source being used. \( V_0(z) \) typically has a minimum at approximately 1 km depth and is modeled with a Munk profile [19]. It possesses an exponential form near the surface due to temperature decrease and a linear dependence deep underwater due to pressure increase. Multiple scattering is induced by the internal wave fluctuations in the range and depth dependences \( V_I(z,r) \) and the form used here is that of ref. [21]; their construction of an internal wave ensemble is rather complicated and not repeated here, but they gave an analytical summation with random phases, which turns out to be quite useful in derivations. The overall strength of the internal waves, \( \epsilon \), depends on location in the ocean. The value used in our calculations is the one most relevant to the northeast Pacific Ocean where the North Pacific Acoustic Laboratory has made a number of measurements (roughly involving the ocean between Hawaii and San Francisco) [22]. More details and further references can be found in the review paper [7]. Note that our calculations have no cutoffs in mode number for the ocean surface or bottom for simplicity.

Wave field propagation for a parabolic equation is unitary and the modes provide a physically relevant basis with which to study the effects of scattering [23,24]. Extensions to adiabatically defined modes could also be incorporated, but here the modes are independent of range. The unitary propagation matrix \( U \) expressed in the mode basis is the natural vehicle for which to construct the random matrix ensemble. The problem is to identify and incorporate into the ensemble’s structure all information that survives long-range propagation up to a maximum of the ocean basin scale (up to several thousand kilometers) and no more.

Without scattering from internal waves, or indeed any plausible scattering mechanism, the unitary propagation matrix would be a diagonal matrix, \( U^{diag} = \Lambda \), and accumulate a phase proportional to the range propagated,

\[ \Lambda_{mn}(r) = \delta_{mn} \exp(-ik_0 E_m r), \quad (3) \]

where \( E_m \) is the unitless energy of the \( m \)-th mode of the vertical wave guide. To propagate any initial fixed-frequency wave field corresponding to an acoustic source just requires its initial modal decomposition and these phases.

Random matrix ensemble construction and derivation. – Perturbation theory is used simultaneously to motivate the structure of the appropriate random matrix ensemble and derive analytically what statistical properties the ensemble must possess. To first order, internal waves generate a propagator

\[ U \approx \Lambda \left( I - i\epsilon k_0 \int_0^r dr' V_I \right), \quad (4) \]

where \( V_I \) is the operator corresponding to \( V_I(z,r) \) in the interaction picture. To construct the ensemble, it is necessary to restore unitarity. One technique applies the Cayley transform of a Hermitian operator \( A \)

\[ U = \Lambda (I + i\epsilon A)^{-1} (I - i\epsilon A), \quad A = \frac{k_0}{2} \int_0^r dr' V_I, \quad (5) \]

which is quite useful because the internal waves generate a weak scattering locally and evaluating the operator
inverse is straightforward. As a consequence, it is natural to construct a fixed-range building block unitary propagator $U_b = U(r = r_b)$ for ranges long enough for the internal waves to generate sufficiently random behaviors, but short enough that perturbation theory is still a viable approximation. The method of constructing building blocks has been introduced previously in the theory of quantum transport for which transfer matrices are used [25]. Long-range propagation follows by multiplying the requisite number of independently drawn members of a $U_b$ ensemble to arrive at an ensemble of $U$ for the full range (i.e. $U(r = r_b) = \prod U_{b,j}$).

Hence, ideally there are three statistical properties of interest to determine a practical value for the range $r_b$: i) correlations between a matrix element of $U_b$ for a given block and its adjacent block should be small, ii) the phases of each $U_{b,mn}$ should be largely randomized, and iii) ideally, dynamical correlations between neighboring matrix elements should be minimal. For extremely short-range propagation, there is little scattering, $U$ is nearly diagonal, and the matrix elements are highly correlated with little randomness. As the propagation range increases, there comes a point at which the phases of the $U_b$ matrix elements become more or less randomized. Figure 1 shows the randomization is well underway by 50 km at 75 Hz (used throughout). At this range the correlations between $U_{b,mn}$ for adjacent blocks have fallen to roughly 10% or less. Conveniently, this is roughly the range at which wave energy has cycled once from the upper to lower turning point and back again.

As the strongest perturbations are close to the surface, choosing this range is consistent with accounting for one strong perturbation cycle. Other than effects due to the unitarity constraints, the off-diagonal matrix elements are beginning to behave like zero-centered, complex, Gaussian random variables. In fact, using eqs. (4), (5) and the ensemble model for $v_1(z,r)$ from ref. [21] gives an analytic construction for the elements $U_{mn}$. This expression is a random phasor sum which generates Gaussian random variables over the ensemble with zero mean and a variance $\sigma^2_{U_{mn}}$. This theoretical variance for off-diagonal elements is shown ahead in fig. 3 to match the variance calculated from direct propagation of the Schrödinger equation. Using eqs. (4), (5), the $A_{mn}$ are taken as independent, complex Gaussian random variables with $\sigma^2_{A_{mn}} = \sigma^2_{U_{mn}}/4$ with the analytic form derived from perturbation theory; the diagonal elements are real. The dynamical correlations amongst neighboring matrix elements are not so small, especially along diagonals of $U_b$; “dynamical” means correlations in addition to those induced by unitarity. Nevertheless, as a starting point, these correlations are ignored. If later, it is found that the ensemble is in some way deficient, one could revisit their incorporation.

**Band width of the ensemble.** – One might anticipate that the mode-mixing from scattering cannot involve modes separated too greatly in mode number rendering the matrix banded, at least initially in the propagation. Figure 2 illustrates the bandedness of $U$ due to the internal wave field. Thus, the variance of the $A_{mn}$ decreases rapidly away from the diagonal. The matrix possesses a rough translational invariance along the diagonal and the matrix element variance is mostly a function of $|n - m|$. Figure 3 displays the behavior of the band, which has a power-law decrease of exponent $\approx 1.3$ for the standard deviation up to a shoulder near $|n - m| \approx 50$, where it suddenly drops off much more sharply. Further investigation of the shoulder is needed to ensure that it is not an artifact of the internal waves construction method. In our calculations, the internal wave summations were cutoff at the nineteenth mode and extending the upper limit may fill in the shoulder. For the purposes of studying the ocean and its environment, this cutoff is well beyond the necessary range and the shoulder is beyond the realm of relevance to the ocean.

The ensemble of eq. (5) with the variances of fig. 3 has some features distinguishing it from the power-law random banded matrices introduced for investigations of the Anderson metal-insulator transition [26]. To begin,
Fig. 2: $U$ for propagation through a single internal wave field is illustrated as a color plot of the magnitudes of the matrix elements $|U_{mn}|$. The top panel is for 50 km propagation, and the bottom panel for 1000 km.

Fig. 3: The variance of the off-diagonal matrix elements of $U$ as a function of distance from the matrix diagonal. The solid line is the result of a perturbation theory not explicitly discussed in the text. The dashed line indicates the power-law decay exponent.

the ensemble is unitary; see ref. [27] for introduction and discussion of a unitary ensemble. In addition the diagonal elements retain a deterministic function and importance depending on the value of $\epsilon$. The bandwidth exponent is roughly independent of wave vector and just a bit greater than unity, which makes this ensemble most similar to those possessing localization and super-diffusive wave packet spreading at short times [26]. Thus, the ensemble is consistent with localization in mode number, which allows for the well-known possibility of isolating and measuring early arrival structures [17,18]. Were the internal waves or other scattering mechanisms in the ocean of a different character with a slower decay away from the diagonal, that would not have been the case. The bandwidth decay exponent is little changed by multiplication of $U_b$, at sufficiently long propagation range the width will grow sufficiently to make wave energy hit the ocean’s surface and bottom, which will then strip out energy.

It is a somewhat brash assumption that dynamical correlations of neighboring matrix elements $U_{b,m,n}$ can be completely ignored. Nevertheless, it is seen in fig. 4 that a typical unitary banded random ensemble matrix $U$ obtained as a product over building blocks and a $U$ constructed by propagation through a single realization of an internal wave field using the parabolic equation give
the same general appearance, including band width, when propagated out to 1000 km. From this figure at least, there is no indication that this correlation information survives intact in propagation to long ranges. Further details are to be published [28].

Summary. – For the first time a random matrix ensemble has been introduced for ocean acoustic propagation. The ensemble is constructed for a building block unitary propagator to a short range, and further propagation is by matrix multiplication of independently chosen $U_b$. The matrix is banded in mode number with an approximate power law whose exponent firmly places the ensemble in the class of those that exhibit localization and super-diffusion. Unlike the ensembles introduced to study the Anderson metal-insulator transition [26], the ensemble here is unitary, and has deterministic structure for the diagonal elements. The relatively slow decay of a power-law cutoff has ramifications for convergence of calculations in determining where one can make numerical approximations such as matrix truncations. The ensemble construction is an ideal tool for addressing a central question in long-range ocean acoustics, “what information survives in acoustic propagation to long ranges and how can it be extracted from experimental data in order to learn more about the state of the ocean?” This is still a poorly understood question after many years of research. Additionally, large ensembles can be generated much faster than full wave propagation and for statistical purposes can be used to simulate the dynamics without having to explicitly construct the potential due to internal waves. There have been a number of investigations of power-law banded random matrices in multiple contexts, and there exists some analytic analysis [29] that may inform ocean acoustics applications. It would be interesting to consider ensemble extensions to shallower water, higher frequencies and surface/bottom scattering. Non-unitary structures would have to be incorporated to derive ensembles with absorption.

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