RAUZY INDUCTION OF POLYGON PARTITIONS
AND TORAL $\mathbb{Z}^2$-ROTATIONS

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Abstract. We extend the notion of Rauzy induction of interval exchange transformations to the case of toral $\mathbb{Z}^2$-rotation, i.e., $\mathbb{Z}^2$-action defined by rotations on a 2-torus. If $\mathcal{X}_{P,R}$ denotes the symbolic dynamical system corresponding to a partition $P$ and $\mathbb{Z}^2$-action $R$ such that $R$ is Cartesian on a sub-domain $W$, we express the 2-dimensional configurations in $\mathcal{X}_{P,R}$ as the image under a 2-dimensional morphism (up to a shift) of a configuration in $\mathcal{X}_{\tilde{P}|_W,\tilde{R}|_W}$, where $\tilde{P}|_W$ is the induced partition and $\tilde{R}|_W$ is the induced $\mathbb{Z}^2$-action on $W$.

We focus on one example $\mathcal{X}_{P_0,R_0}$ for which we obtain an eventually periodic sequence of 2-dimensional morphisms. We prove that it is the same as the substitutive structure of the minimal subshift $X_0$ of the Jeandel-Rao Wang shift computed in an earlier work by the author. As a consequence, $P_0$ is a Markov partition for the associated toral $\mathbb{Z}^2$-rotation $R_0$. It also implies that the subshift $X_0$ is uniquely ergodic and is isomorphic to the toral $\mathbb{Z}^2$-rotation $R_0$ which can be seen as a generalization for 2-dimensional subshifts of the relation between Sturmian sequences and irrational rotations on a circle. Batteries included: the algorithms and code to reproduce the proofs are provided.

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Date: December 2, 2021.

2020 Mathematics Subject Classification. Primary 37A05; Secondary 37B51, 52C23.

Key words and phrases. Rauzy induction and Markov partition and self-induced and self-similar and SFT and polyhedron exchange transformation and aperiodic tiling and Sturmian.

The author acknowledges financial support from the Laboratoire International Franco-Québécois de Recherche en Combinatoire (LIRCO), the Agence Nationale de la Recherche through the project CODYS (ANR-18-CE40-0007) and the Horizon 2020 European Research Infrastructure project OpenDreamKit (676541).
1. Introduction

A person is walking on a sidewalk made of alternating dark bricks of size 1 and light bricks of size \(\alpha > 0\) where step zero is made at position \(p \in \mathbb{R}\). They walk from left to right with steps of length 1 and uses their right foot (\(R\)) on dark bricks and their left foot (\(L\)) on light bricks, thus constructing a bi-infinite binary sequence \(w \in \{L, R\}^\mathbb{Z}\) in \(\{L, R\}^\mathbb{Z}\) (see Figure 1). For a fixed \(\alpha\), let \(X_\alpha \subset \{L, R\}^\mathbb{Z}\) be the set of sequences obtained when \(p \in \mathbb{R}\). When the brick length \(\alpha\) is irrational, the partition of the circle \(\mathbb{R}/(1 + \alpha)\mathbb{Z}\) into the two colored bricks yields a symbolic representation of the map \(x \mapsto x + 1\) having a nice combinatorial interpretation known as Sturmian sequences [Lot02, Chap. 2]:

1. Every sequence \(w \in X_\alpha\) is obtained from a unique starting point in \(\mathbb{R}/(1 + \alpha)\mathbb{Z}\).
2. \(X_\alpha\) is the set of sequences in \(\{L, R\}^\mathbb{Z}\) whose language has \(n + 1\) patterns of length \(n\) for all \(n \geq 0\) and whose letter frequencies exist and satisfies \(\lim_{n \to \infty} \frac{\#\{-n \leq i < n \mid w_i = L\}}{\#\{-n \leq i < n \mid w_i = R\}} = \alpha\).

For example, the patterns of length from 1 to 4 that we see in \(w\) in the figure are \(\{L, R\} \{LL, LR, RL\}, \{LLL, LLR, LRL, RLL\}, \{LLLL, LLLL, LLRL, LRLR, RLLL\}\) and one can show in general that \(w\) has \(n + 1\) patterns of length \(n\). The converse is the difficult part of the proof: any bi-infinite sequence whose letter frequencies exist and having \(n + 1\) distinct patterns of size \(n\) belongs to \(X_\alpha\) for some \(\alpha \in \mathbb{R}_{>0} \setminus \mathbb{Q}\) and for some starting point \(p \in \mathbb{R}/(1 + \alpha)\mathbb{Z}\).

This result is a fundamental result in symbolic dynamics known as Morse-Hedlund’s theorem [MH40], although the connection with the number of fixed-length patterns was done in [CH73]. Its proof uses important tools from dynamical systems and number theory and is explained nowadays in terms of \(S\)-adic developments, first return maps (Rauzy induction), continued fraction expansion of real numbers, and Ostrowski numeration system [Arn02].

A generalization of the result of Morse and Hedlund was provided by Rauzy for a single example [Rau82]. Based on the right-infinite sequence often called the Tribonacci word

\[
T = 12131211213121121312112131211213121...
\]

which is fixed by \(1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1\), Rauzy proved that the system \((X_T, \sigma)\), where \(\sigma\) is the shift action, is measurably conjugate to the toral rotation \((\mathbb{T}^2, x \mapsto x + (\beta^{-1}, \beta^{-2}))\) where \(\beta\) is the real root of \(x^3 - x^2 - x - 1\), the characteristic polynomial of the incidence matrix of the substitution. The coding of the toral translation is made through the partition into three parts of a fundamental domain of \(\mathbb{T}^2\) known as the Rauzy fractal [Fog02, §7.4]. Proving that this holds
for all Pisot substitution is known as the Pisot Conjecture [ABB+15], an important and still open question.

Finding further generalizations was coined the term of \textit{Rauzy program} in [BFZ05], a survey divided into three parts: the \textit{good} coding of \textit{k}-interval exchange transformations (IETs); the \textit{bad} coding of a rotation on $\mathbb{T}^k$; and the \textit{ugly} coding of two rotations on $\mathbb{T}^k$ for $k = 1$. The IETs are the good part since they behave well with induced transformations and admit continued fraction algorithms [Rau79, Vee82, Yoc06, AF07]. The bad part was much improved since then with various recent results using multidimensional continued fraction algorithms including Brun’s algorithm [Bru58] which provides measurable-theoretic conjugacy with symbolic systems for almost every toral rotations on $\mathbb{T}^2$ [BST19, Thu20]. As the authors wrote in [BFZ05], the term ugly “refers to some esthetic difficulties in building two-dimensional sequences by iteration of patterns”. Indeed, digital planes [ABEI01, ABI02, BFZ05, Ber09, BBJS13] are typical objects that are described by the coding of two rotations on $\mathbb{T}^1$ and they are not built by rectangular shaped substitutions. In this article, we want to add some shine on the ugly part by providing a method based on induced transformations on $\mathbb{T}^2$ to prove that a toral partition is a Markov partition for a toral $\mathbb{Z}^2$-rotation.

\textbf{Markov partitions for automorphisms of the torus.} While Morse-Hedlund’s theorem deals with the coding of irrational rotations, other kinds of dynamical systems admit a symbolic representation. Hyperbolic automorphisms of the torus are one example [LM95, Kit98]. Suppose that one starts at some position $v \in \mathbb{R}^2$ and moves according to the successive images under the application of the map $v \mapsto Mv$ with $M = (1, 1)$ as shown in Figure 2. The map $v \mapsto Mv$ is an automorphism of $\mathbb{R}^2/\mathbb{Z}^2$ which is \textit{hyperbolic} since $M$ has no eigenvalue of modulus 1. It allows one to code the orbit $(M^kv)_{k \in \mathbb{Z}}$ as a sequence in $\{A, B, C\}^\mathbb{Z}$ according to a well-chosen partition $\mathcal{P}$ of a fundamental domain of $\mathbb{R}^2/\mathbb{Z}^2$ into three rectangles indexed by letters in the set $\{A, B, C\}$. In Figure 2, the positive orbit $(M^kv)_{k \geq 0}$ of the starting point $v = \left(\frac{-7}{10}, \frac{14}{10}\right)^T$ is coded by the sequence $\text{CBABABCB}\ldots$ which avoids the patterns in $\mathcal{F} = \{AA, BB, CC, CA\}$. We denote the set of obtained sequences as $\mathcal{X}_{\mathcal{P}, M}$. The partition of $\mathbb{R}^2/\mathbb{Z}^2$ is a Markov partition for the automorphism because it has two important properties [LM95, §6.5]:

\begin{itemize}
\item[(C1)] every sequence in $\mathcal{X}_{\mathcal{P}, M}$ is obtained from a unique starting point in $\mathbb{R}^2/\mathbb{Z}^2$,
\item[(C2')] the set $\mathcal{X}_{\mathcal{P}, M}$ is a shift of finite type (SFT), i.e., there exists a finite set $\mathcal{F}$ of patterns such that $\mathcal{X}_{\mathcal{P}, M}$ is the set of sequences in $\{A, B, C\}^\mathbb{Z}$ which avoids the patterns in $\mathcal{F}$.
\end{itemize}

Such Markov Partitions exist for all hyperbolic automorphisms of the torus [AW70, Sin68, Bow75] and various kinds of diffeomorphisms [Bow70], see also [Kit95, Adl98, KV98]. Surprisingly, it turns out that Markov partitions also exist for toral $\mathbb{Z}^2$-rotations $\mathcal{R}$ and 2-dimensional subshifts $\mathcal{X}_{\mathcal{P}, \mathcal{R}}$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The automorphism of $\mathbb{R}^2/\mathbb{Z}^2$ defined as $v \mapsto Mv$ admits a Markov partition.}
\end{figure}
**Results.** In this article, we propose a method for proving that a 2-dimensional toral partition is a Markov partition for a given toral $\mathbb{Z}^2$-rotation. The method is inspired from the proof of Morse-Hedlund’s theorem and its link with continued fraction expansion and induced transformations. We extend the notion of Rauzy induction of IETs to the case of $\mathbb{Z}^2$-actions and we introduce the notion of induced partitions. We apply the method on one example related to the golden mean and Jeandel-Rao aperiodic Wang shift.

![Diagram of a polygon partition](image)

**Figure 3.** For every starting point $p \in \mathbb{R}^2$, the coding of the shifted lattice $p + \mathbb{Z}^2$ under the polygon partition $\mathcal{P}_0$ yields a configuration which is a symbolic representation of $p$. We show that the set of such configurations is a shift of finite type (SFT) and hence that $\mathcal{P}_0$ is a Markov partition for the toral $\mathbb{Z}^2$-rotation $R_0$.

Let
\[
\Gamma_0 = ((\varphi, 0), (1, \varphi + 3))_\mathbb{Z}
\]
be a lattice in $\mathbb{R}^2$ with $\varphi = \frac{1 + \sqrt{5}}{2}$. We consider the dynamical system defined by the following toral $\mathbb{Z}^2$-rotation, i.e., a $\mathbb{Z}^2$-action on the 2-dimensional torus $\mathbb{R}^2/\Gamma_0$:
\[
R_0 : \mathbb{Z}^2 \times \mathbb{R}^2/\Gamma_0 \to \mathbb{R}^2/\Gamma_0 \\
(n, x) \mapsto x + n.
\]

A polygon partition $\mathcal{P}_0$ of $\mathbb{R}^2/\Gamma_0$ indexed by integers from the set $\{0, 1, 2, \ldots, 10\}$ was introduced in [Lab21a], see Figure 3. The set $\mathcal{X}_{\mathcal{P}_0, R_0} \subset [0, 10]^{\mathbb{Z}^2}$ is the set of 2-dimensional configurations obtained by coding the orbits of points in $\mathbb{R}^2/\Gamma_0$ under the $\mathbb{Z}^2$-action $R_0$ by the atoms of the partition $\mathcal{P}_0$. It is a subshift and, in particular, it is closed under the shift action $\sigma : \mathbb{Z}^2 \times \mathcal{X}_{\mathcal{P}_0, R_0} \to \mathcal{X}_{\mathcal{P}_0, R_0}$ which is defined as $(\sigma^n(w))_k = w_{k+n}$ for every $n, k \in \mathbb{Z}^2$ and $w \in \mathcal{X}_{\mathcal{P}_0, R_0}$. It was shown that $\mathcal{P}_0$ gives a symbolic representation of $R_0$, thus the 2-dimensional subshift $\mathcal{X}_{\mathcal{P}_0, R_0}$ forms a symbolic dynamical system $(\mathcal{X}_{\mathcal{P}_0, R_0}, \mathbb{Z}^2, \sigma)$ that satisfies condition (C1). It was also shown that $\mathcal{X}_{\mathcal{P}_0, R_0}$ is a strict subset of the Jeandel-Rao Wang shift $\Omega_0$ [JR21], but proving that $\mathcal{X}_{\mathcal{P}_0, R_0}$ is itself a SFT satisfying condition (C2)’ was left open.

**Theorem 1.1.** $\mathcal{P}_0$ is a Markov partition for the dynamical system $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$.

It may sounds counter-intuitive for the reader since Markov partitions are usually associated with hyperbolic systems and not with rotations. In this article, we use a more inclusive interpretation of the conditions (C1) and (C2)’ by considering higher dimensional subshifts of finite type as it was done already in [ES97]. See the definition and a discussion in Section 5.3.

The fact that $\mathcal{X}_{\mathcal{P}_0, R_0} \subset \Omega_0$ corresponds to the easy direction in the proof of Morse-Hedlund’s theorem, namely that codings of irrational rotations have pattern complexity $n + 1$. Proving the
Theorem 1.2. Let that for every Finally and as in the one-dimensional case, we say that
is given in terms of More precisely,
lattices \( \Gamma \) arbitrarily large in width and in height as \( m \) goes to infinity. A shift \( X \subseteq \mathcal{A}^\mathbb{Z}^2 \) is self-similar if there exists an expansive 2-dimensional morphism \( \omega: \mathcal{A} \to \mathcal{A}^\mathbb{Z}^2 \) such that \( X = \omega(X) \). Finally and as in the one-dimensional case, we say that \( \omega \) is primitive if there exists \( m \in \mathbb{N} \) such that for every \( a, b \in \mathcal{A} \) the letter \( b \) occurs in \( \omega^m(a) \).

Theorem 1.2. Let \( \mathcal{X}_{\mathcal{P}_0, R_0} \) be the symbolic dynamical system associated to \( \mathcal{P}_0, R_0 \). There exist lattices \( \Gamma_i \subseteq \mathbb{R}^2 \), alphabets \( \mathcal{A}_i \), \( \mathbb{Z}^2 \)-actions \( R_i : \mathbb{Z}^2 \times \mathbb{R}^2/\Gamma_i \to \mathbb{R}^2/\Gamma_i \), and topological partitions \( \mathcal{P}_i \) of \( \mathbb{R}^2/\Gamma_i \) indexed by letters from the alphabet \( \mathcal{A}_i \) that provide the substitutive structure of \( \mathcal{X}_{\mathcal{P}_0, R_0} \).

More precisely,

(i) There exists a 2-dimensional morphism \( \beta_0 : \mathcal{A}_1 \to \mathcal{A}_0^\mathbb{Z}^2 \)

\[
\mathcal{X}_{\mathcal{P}_0, R_0} \xleftarrow{\beta_0} \mathcal{X}_{\mathcal{P}_1, R_1}
\]

that is onto up to a shift, i.e., \( \mathcal{X}_{\mathcal{P}_0, R_0} = \beta_0(\mathcal{X}_{\mathcal{P}_1, R_1}) \).

(ii) There exists a shear conjugacy

\[
\mathcal{X}_{\mathcal{P}_1, R_1} \xleftarrow{\beta_1} \mathcal{X}_{\mathcal{P}_2, R_2}
\]

shearing configurations by the action of the matrix \( M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), i.e., satisfying \( \sigma^M \circ \beta_1 = \beta_1 \circ \sigma^k \) for every \( k \in \mathbb{Z}^2 \).

(iii) There exist 2-dimensional morphisms \( \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \) and \( \beta_7 \):

\[
\mathcal{X}_{\mathcal{P}_2, R_2} \xleftarrow{\beta_2} \mathcal{X}_{\mathcal{P}_3, R_3} \xleftrightarrow{\beta_3} \mathcal{X}_{\mathcal{P}_4, R_4} \xleftrightarrow{\beta_4} \mathcal{X}_{\mathcal{P}_5, R_5} \xleftrightarrow{\beta_5} \mathcal{X}_{\mathcal{P}_6, R_6} \xleftrightarrow{\beta_6} \mathcal{X}_{\mathcal{P}_7, R_7} \xleftrightarrow{\beta_7} \mathcal{X}_{\mathcal{P}_8, R_8}
\]

that are onto up to a shift, i.e., \( \mathcal{X}_{\mathcal{P}_i, R_i} = \beta_i(\mathcal{X}_{\mathcal{P}_{i+1}, R_{i+1}}) \) for each \( i \in \{2, 3, 4, 5, 6, 7\} \).

(iv) The subshift \( \mathcal{X}_{\mathcal{P}_8, R_8} \) is self-similar satisfying \( \mathcal{X}_{\mathcal{P}_8, R_8} = \beta_8 \beta_9 \tau(\mathcal{X}_{\mathcal{P}_8, R_8}) \). More precisely, there exist two 2-dimensional morphisms \( \beta_8, \beta_9 \) and a bijection \( \tau : \mathcal{A}_8 \to \mathcal{A}_{10} \)

\[
\mathcal{X}_{\mathcal{P}_8, R_8} \xleftarrow{\beta_8} \mathcal{X}_{\mathcal{P}_9, R_9} \xleftrightarrow{\beta_9} \mathcal{X}_{\mathcal{P}_{10}, R_{10}} \xleftrightarrow{\tau} \mathcal{X}_{\mathcal{P}_8, R_8}
\]

that are onto up to a shift, i.e., \( \mathcal{X}_{\mathcal{P}_8, R_8} = \beta_8(\mathcal{X}_{\mathcal{P}_9, R_9}) \), \( \mathcal{X}_{\mathcal{P}_9, R_9} = \beta_9(\mathcal{X}_{\mathcal{P}_{10}, R_{10}}) \), and \( \mathcal{X}_{\mathcal{P}_{10}, R_{10}} = \tau(\mathcal{X}_{\mathcal{P}_8, R_8}) \) and the product \( \beta_8 \beta_9 \tau \) is an expansive and primitive self-similarity.

(v) The subshift \( \mathcal{X}_{\mathcal{P}_8, R_8} \) is topologically conjugate to the subshift \( \mathcal{X}_{\mathcal{P}_{10}, R_{10}} \) introduced in \([\text{Lab21a}]\) as there exists a bijection \( \zeta : \mathcal{U} \to \mathcal{A}_8 \) such that \( \zeta(\mathcal{X}_{\mathcal{P}_{10}, R_{10}}) = \mathcal{X}_{\mathcal{P}_8, R_8} \).
Theorem 1.2 must be compared with the main result of [Lab21b], recalled herein as Theorem 13.1 giving the substitutive structure of a minimal subshift $X_0$ of the Jeandel-Rao Wang shift $\Omega_0$. That result proved the existence of sets of Wang tiles $\{T_i\}_{1 \leq i \leq 12}$ together with their associated Wang shifts $\{\Omega_i\}_{1 \leq i \leq 12}$ and 2-dimensional morphisms $\omega_i : \Omega_{i+1} \to \Omega_i$ that provide the substitutive structure of Jeandel-Rao Wang shift, see Figure 4. In fact, the consequence of the two theorems is that the subshifts $X_0$ and $X_{P_0,R_0}$ have the exact same substitutive structure given as the inverse limit of the same eventually periodic sequence of 2-dimensional morphisms.

\[
\begin{align*}
&\Omega_0 \xleftarrow{\omega_0} \Omega_1 \xleftarrow{\omega_1} \Omega_2 \xleftarrow{\omega_2} \Omega_3 \xleftarrow{\omega_3} \Omega_4 \\
&\cup \xleftarrow{X_0 \omega_0} \cup \xleftarrow{X_1 \omega_1} \cup \xleftarrow{X_2 \omega_2} \cup \xleftarrow{X_3 \omega_3} \cup \xleftarrow{X_4 \omega_4} \Omega_5 \xleftarrow{\eta} \Omega_6 \xleftarrow{\omega_6} \Omega_7 \xleftarrow{\omega_7 \omega_8 \omega_9 \omega_{10} \omega_{11}} \Omega_{12} \\
&\uparrow \beta_0 \quad \uparrow \beta_1 \quad \uparrow \beta_2 \quad \uparrow \beta_3 \beta_4 \beta_5 \beta_6 \beta_7 \quad \uparrow \beta_8 \beta_9 \tau \quad \zeta^{-1} \beta_8 \beta_9 \zeta \\
&X_{P_0,R_0} \quad X_{P_1,R_1} \quad X_{P_2,R_2} \quad X_{P_3,R_3} \quad X_{P_4,R_4} \quad X_{P_5,R_5} \quad X_{P_6,R_6} \quad X_{P_7,R_7} \quad X_{P_8,R_8} \quad X_{P_9,R_9} \quad X_{P_{10},R_{10}} \quad X_{P_{11},R_{11}}.
\end{align*}
\]

**Figure 4.** We prove that the subshifts $X_0 \subset \Omega_0$ and $X_{P_0,R_0}$ are equal since they have a common substitutive structure. The substitutive structure of $X_0$ computed in [Lab21b] and the substitutive structure of $X_{P_0,R_0}$ satisfy $\beta_0 = \omega_0 \omega_1 \omega_2 \omega_3$, $\beta_1 \beta_2 = j \eta \omega_6$, $\beta_3 = \omega_7$, $\beta_4 = \omega_8$, $\beta_5 = \omega_9$, $\beta_6 = \omega_{10}$, $\beta_7 = \omega_{11}$, $\zeta = \rho$ and $\beta_8 \beta_9 \tau = \rho \omega_4 \rho^{-1}$. We deduce that $X_{P_8,R_8} = \Omega_{12}$, $X_{P_9,R_9} = \Omega_7$, $X_{P_{11},R_{11}} = X_4$ and finally $X_{P_0,R_0} = X_0$.

**Theorem 1.3.** The symbolic dynamical system $X_{P_0,R_0}$ and the minimal subshift $X_0 \subset \Omega_0$ of the Jeandel-Rao Wang shift have the same substitutive structure in the sense that the following equalities hold:

\[
\begin{align*}
\beta_0 &= \omega_0 \omega_1 \omega_2 \omega_3, \\
\beta_1 &= j \eta \omega_6, \\
\beta_3 &= \omega_7, \\
\beta_4 &= \omega_8, \\
\end{align*}
\]

where $\omega_0, \omega_1, \omega_2, \omega_3, j, \eta, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}, \omega_{11}, \rho$ were computed in [Lab21b] and $\omega_4$ was first defined in [Lab19].

**Remark 1.4.** To obtain equalities between substitutions computed from totally different objects, we use a common convention for the definition of the $\beta_i$ from the induction of toral partitions, and in [Lab21b] for the construction of the $\omega_i$ from sets of Wang tiles. When constructing the substitutions, short images of letter come before longer ones, and words of the same length are sorted lexicographically. See Definition 7.1.

The description of the symbolic dynamical system $X_{P_0,R_0}$ and the minimal subshift $X_0$ of Jeandel-Rao aperiodic subshift $\Omega_0$ by their substitutive structure allows to prove their topological conjugacy.

**Corollary 1.5.** The subshifts $X_{P_0,R_0}$ and $\Omega_0$ are topologically conjugate and are equal to the minimal aperiodic substitutive subshift $X_{\omega_4}$. The subshifts $X_{P_8,R_8}$ and $\Omega_{12}$ are topologically conjugate. The subshifts $X_{P_0,R_0}$ and $X_0$ are topologically conjugate, and the same holds for intermediate subshifts:

\[
X_{P_1,R_1} = j(\Omega_5), \quad X_{P_3,R_3} = \Omega_7, \quad X_{P_4,R_4} = \Omega_8, \quad X_{P_5,R_5} = \Omega_9, \quad X_{P_6,R_6} = \Omega_{10} \quad \text{and} \quad X_{P_7,R_7} = \Omega_{11}.
\]

The fact that $X_0$ and $X_{P_0,R_0}$ are equal implies Theorem 1.1 since it was proved in [Lab21b] that $X_0$ is a shift of finite type. It also implies the following corollary which can be seen as a generalization of what happens for Sturmian sequences.
Corollary 1.6. The dynamical system \((X_0, \mathbb{Z}^2, \sigma)\) is strictly ergodic and the measure-preserving dynamical system \((X_0, \mathbb{Z}^2, \sigma, \nu)\) is isomorphic to the toral \(\mathbb{Z}^2\)-rotation \((\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0, \lambda)\) where \(\nu\) is the unique shift-invariant probability measure on \(X_0\) and \(\lambda\) is the Haar measure on \(\mathbb{R}^2/\Gamma_0\).

Although Jeandel-Rao Wang shift \(\Omega_0\) is not minimal as it contains the proper minimal subshift \(X_{P_0,R_0} = X_0\), we believe that it is uniquely ergodic.

Conjecture 1.7. The Jeandel-Rao subshift \(\Omega_0\) is uniquely ergodic.

That conjecture is equivalent to prove that \(\Omega_0 \setminus X_0\) has measure 0 for any shift-invariant probability measure on \(\Omega_0\) which was stated as a conjecture in [Lab21b] and where some progress was done. That would also imply the existence of an isomorphism of measure-preserving dynamical systems between Jeandel-Rao Wang shift \((\Omega_0, \mathbb{Z}^2, \sigma, \nu)\) and \((\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0, \lambda)\) where \(\nu\) would be the unique shift-invariant probability measure on \(\Omega_0\) and \(\lambda\) is the Haar measure on \(\mathbb{R}^2/\Gamma_0\).

This calls for a general theory of \(d\)-dimensional subshifts of finite type coded by Markov partitions of the \(d\)-dimensional torus and admitting induced subsystems.

Structure of the article. The article is divided into three parts. Part 1 gathers the preliminary notions including two-dimensional words and morphisms, toral \(\mathbb{Z}^2\)-rotations and polyhedron exchange transformations (PETs). Part 2 defines the induction of toral \(\mathbb{Z}^2\)-rotations, induced toral partitions and renormalization schemes. In Part 3, we induce the partition \(P_0\) introduced in [Lab21a] to get a sequence of substitutions until we reach a self-induced partition. We prove that the substitutive structure obtained from the induction procedure is the same as the substitutive structure computed directly from the Jeandel-Rao Wang tiles in [Lab21b]. Some remarks about further research can be found in the conclusion.

Batteries included. Two algorithms are given in Section 9 at the end of Part 2 to compute induced partitions and induced PETs. They are implemented in the optional package slabbe [Lab20] of SageMath [Sag20]. The code to reproduce the proof of Theorem 1.2 is included directly in the proofs of intermediate results stated in Part 3. The code present in this article is also contained in the file demos/arXiv_1906_01104.rst of the slabbe package. It allows to make sure it remains reproducible in the future with new versions of SageMath with the command sage -t demos/arXiv_1906_01104.rst which should return something like [87 tests, 8.19 s] and All tests passed!.

Acknowledgments. I am thankful to Vincent Delecroix for many helpful discussions at LaBRI in Bordeaux during the preparation of this article which allowed me to improve my knowledge on dynamical systems and Rauzy induction of interval exchange transformations. I also want to thank the referee for her/his valuable comments which led to many improvements in the article.

Part 1. Preliminaries

This part is divided into four sections on dynamical systems, subshifts and Wang shift; two-dimensional words and morphisms; toral \(\mathbb{Z}^2\)-rotations and polygonal exchange transformations; symbolic representation of toral \(\mathbb{Z}^2\)-rotations.

2. Dynamical systems and subshifts

In this section, we introduce dynamical systems, subshifts and shifts of finite type. We let \(\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}\) denote the integers and \(\mathbb{N} = \{0, 1, 2, \ldots\}\) be the nonnegative integers.
2.1. Topological dynamical systems. Most of the notions introduced here can be found in [Wal82]. A dynamical system is a triple \((X, G, T)\), where \(X\) is a topological group and \(T\) is a continuous function \(G \times X \to X\) defining a left action of \(G\) on \(X\): if \(x \in X\), \(e\) is the identity element of \(G\) and \(g, h \in G\), then using additive notation for the operation in \(G\) we have \(T(e, x) = x\) and \(T(g + h, x) = T(g, T(h, x))\). In other words, if one denotes the transformation \(x \mapsto T(g, x)\) by \(T^g\), then \(T^{g+h} = T^g T^h\). In this work, we consider the Abelian group \(G = \mathbb{Z} \times \mathbb{Z}\).

If \(Y \subset X\), let \(\overline{Y}\) denote the topological closure of \(Y\) and let \(T(Y) := \bigcup_{g \in G} T^g(Y)\) denote the \(T\)-closure of \(Y\). Alternatively, we also use the notation \(\overline{Y}^T = T(Y)\) to denote the \(T\)-closure of \(Y\).

A subset \(Y \subset X\) is \(T\)-invariant if \(T(Y) = Y\). A dynamical system \((X, G, T)\) is called minimal if \(X\) does not contain any nonempty, proper, closed \(T\)-invariant subset. The left action of \(G\) on \(X\) is free if \(g = e\) whenever there exists \(x \in X\) such that \(T^g(x) = x\).

Let \((X, G, T)\) and \((Y, G, S)\) be two dynamical systems with the same topological group \(G\). A homomorphism \(\theta : (X, G, T) \to (Y, G, S)\) is a continuous function \(\theta : X \to Y\) satisfying the commuting property that \(S^g \circ \theta = \theta \circ T^g\) for every \(g \in G\). A homomorphism \(\theta : (X, G, T) \to (Y, G, S)\) is called an embedding if it is one-to-one, a factor map if it is onto, and a topological conjugacy if it is both one-to-one and onto and its inverse map is continuous. If \(\theta : (X, G, T) \to (Y, G, S)\) is a factor map, then \((Y, G, S)\) is called a factor of \((X, G, T)\) and \((X, G, T)\) is called an extension of \((Y, G, S)\). Two subshifts are topologically conjugate if there is a topological conjugacy between them.

Let \((X, G, T)\) and \((Y, G, S)\) be two dynamical systems with the same topological group \(G = \mathbb{Z}^d\). We say that a \(GL_d(\mathbb{Z})\)-homomorphism \(\theta : (X, \mathbb{Z}^d, T) \to (Y, \mathbb{Z}^d, S)\) is a continuous function \(\theta : X \to Y\) equipped with some matrix \(M \in GL_d(\mathbb{Z})\) satisfying the commuting property that \(S^M_k \circ \theta = \theta \circ T^k\) for every \(k \in \mathbb{Z}^d\). A \(GL_d(\mathbb{Z})\)-homomorphism is called a \(GL_d(\mathbb{Z})\)-conjugacy if it is both one-to-one and onto and its inverse map is continuous. A \(GL_d(\mathbb{Z})\)-conjugacy is called a shear conjugacy if the matrix \(M \in GL_d(\mathbb{Z})\) is a shear matrix. The notion of \(GL_d(\mathbb{Z})\)-conjugacy corresponds to flip-conjugacy when \(d = 1\) [BT98] and to extended symmetry when \(X = Y\) [BRY18].

A measure-preserving dynamical system is defined as a system \((X, G, T, \mu, \mathcal{B})\), where \(\mu\) is a probability measure defined on the Borel \(\sigma\)-algebra \(\mathcal{B}\) of subsets of \(X\), and \(T^g : X \to X\) is a measurable map which preserves the measure \(\mu\) for all \(g \in G\), that is, \(\mu(T^g(B)) = \mu(B)\) for all \(B \in \mathcal{B}\). The measure \(\mu\) is said to be \(T\)-invariant. In what follows, when it is clear from the context, we omit the Borel \(\sigma\)-algebra \(\mathcal{B}\) of subsets of \(X\) and write \((X, G, T, \mu)\) to denote a measure-preserving dynamical system.

The set of all \(T\)-invariant probability measures of a dynamical system \((X, G, T)\) is denoted by \(\mathcal{M}^T(X)\). A \(T\)-invariant probability measure on \(X\) is called ergodic if for every set \(B \in \mathcal{B}\) such that \(T^g(B) = B\) for all \(g \in G\), we have that \(B\) has either zero or full measure. A dynamical system \((X, G, T)\) is uniquely ergodic if it has only one invariant probability measure, i.e., \(|\mathcal{M}^T(X)| = 1\). A dynamical system \((X, G, T)\) is said strictly ergodic if it is uniquely ergodic and minimal.

Let \((X, G, T, \mu, \mathcal{B})\) and \((X', G, T', \mu', \mathcal{B}')\) be two measure-preserving dynamical systems. We say that the two systems are isomorphic if there exist measurable sets \(X_0 \subset X\) and \(X_0' \subset X'\) of full measure (i.e., \(\mu(X \setminus X_0) = 0\) and \(\mu'(X' \setminus X_0') = 0\)) with \(T^g(X_0) \subset X_0\), \(T'^g(X_0') \subset X'_0\) for all \(g \in G\) and there exists a map \(\phi : X_0 \to X_0'\), called an isomorphism, that is one-to-one and onto and such that for all \(A \in \mathcal{B}'(X_0')\),

- \(\phi^{-1}(A) \in \mathcal{B}(X_0)\),
- \(\mu(\phi^{-1}(A)) = \mu'(A)\), and
- \(\phi \circ T^g(x) = T'^g \circ \phi(x)\) for all \(x \in X_0\) and \(g \in G\).
The role of the set $X_0$ is to make precise the fact that the properties of the isomorphism need to hold only on a set of full measure.

2.2. Subshifts and shifts of finite type. In this section, we introduce multidimensional subshifts, a particular type of dynamical systems [LM95, §13.10], [Sch01, Lin04, Hoc16]. Let $A$ be a finite set, $d \geq 1$, and let $A^{Z^d}$ be the set of all maps $x : Z^d \to A$, equipped with the compact product topology. An element $x \in A^{Z^d}$ is called configuration and we write it as $x = (x_m) = (x_m : m \in Z^d)$, where $x_m \in A$ denotes the value of $x$ at $m$. The topology on $A^{Z^d}$ is compatible with the metric defined for all configurations $x, x' \in A^{Z^d}$ by $d(x, x') = 2^{-\min\{\|n\| : x_n \neq x'_n\}}$ where $\|n\| = |n_1| + \cdots + |n_d|$. The shift action $\sigma : n \mapsto \sigma^n$ of $A^{Z^d}$ on $A^{Z^d}$ is defined by

\begin{equation}
(\sigma^n(x))_m = x_{m+n}
\end{equation}

for every $x = (x_m) \in A^{Z^d}$ and $n \in Z^d$. If $X \subset A^{Z^d}$, let $X$ denote the topological closure of $X$ and let $X^\sigma := \{\sigma^n(x) \mid x \in X, n \in Z^d\}$ denote the shift-closure of $X$. A subset $X \subset A^{Z^d}$ is shift-invariant if $X^\sigma = X$. A closed, shift-invariant subset $X \subset A^{Z^d}$ is a subshift. If $X \subset A^{Z^d}$ is a subshift we write $\sigma = \sigma^X$ for the restriction of the shift action (1) to $X$. When $X$ is a subshift, the triple $(X, Z^d, \sigma)$ is a dynamical system and the notions presented in the previous section hold.

A configuration $x \in X$ is periodic if there is a nonzero vector $n \in Z^d \setminus \{0\}$ such that $x = \sigma^n(x)$ and otherwise it is said nonperiodic. We say that a nonempty subshift $X$ is aperiodic if the shift action $\sigma$ on $X$ is free.

For any subset $S \subset Z^d$ let $\pi_S : A^{Z^d} \to A^S$ denote the projection map which restricts every $x \in A^{Z^d}$ to $S$. A pattern is a function $p \in A^S$ for some finite subset $S \subset Z^d$. To every pattern $p \in A^S$ corresponds a subset $\pi_S^{-1}(p) \subset A^{Z^d}$ called cylinder. A nonempty set $X \subset A^{Z^d}$ is a subshift if and only if there exists a set $F$ of forbidden patterns such that

\begin{equation}
X = \{x \in A^{Z^d} \mid \pi_S \circ \sigma^n(x) \notin F \text{ for every } n \in Z^d \text{ and } S \subset Z^d\},
\end{equation}

see [Hoc16, Prop. 9.2.4]. A subshift $X \subset A^{Z^d}$ is a shift of finite type (SFT) if there exists a finite set $F$ such that (2) holds. In this article, we consider shifts of finite type on $Z \times Z$, that is, the case $d = 2$.

3. Two-dimensional words and morphisms

In this section, we reuse the notations from [Lab19] and [Lab21b] on $d$-dimensional words, morphisms and self-similar subshifts.

3.1. $d$-dimensional word. In this section, we recall the definition of $d$-dimensional word that appeared in [CKR10] and we keep the notation $u \odot^j v$ they proposed for the concatenation.

We denote by $\{e_k \mid 1 \leq k \leq d\}$ the canonical basis of $Z^d$ where $d \geq 1$ is an integer. If $i \leq j$ are integers, then $[i, j]$ denotes the interval of integers $\{i, i+1, \ldots, j\}$. Let $n = (n_1, \ldots, n_d) \in N^d$ and $A$ be an alphabet. We denote by $A^n$ the set of functions $u : [0, n_1 - 1] \times \cdots \times [0, n_d - 1] \to A$. An element $u \in A^n$ is called a $d$-dimensional word $u$ of shape $n = (n_1, \ldots, n_d) \in N^d$ on the alphabet $A$. We use the notation $\text{shape}(u) = n$ when necessary. The set of all finite $d$-dimensional words is $A^{d^n} = \{A^n \mid n \in N^d\}$. A $d$-dimensional word of shape $e_k + \sum_{i=1}^d e_i$ is called a domino in the direction $e_k$. When the context is clear, we write $A$ instead of $A^{(1, \ldots, 1)}$. When $d = 2$, we represent a $d$-dimensional word $u$ of shape $(n_1, n_2)$ as a matrix with Cartesian coordinates:

$$u = \begin{pmatrix} u_{0, n_2-1} & \cdots & u_{n_1-1, n_2-1} \\ \vdots & \ddots & \vdots \\ u_{0, 0} & \cdots & u_{n_1-1, 0} \end{pmatrix}. $$
Let \( n, m \in \mathbb{N}^d \) and \( u \in A^n \) and \( v \in A^m \). If there exists an index \( i \) such that the shapes \( n \) and \( m \) are equal except maybe at index \( i \), then the **concatenation of \( u \) and \( v \) in the direction \( e_i \)** is **defined**: it is the \( d \)-dimensional word \( u \otimes^i v \) of shape \((n_1, \ldots, n_{i-1}, n_i + m_i, n_{i+1}, \ldots, n_d) \in \mathbb{N}^d \) given as

\[
(u \otimes^i v)(a) = \begin{cases} u(a) & \text{if } 0 \leq a_i < n_i, \\ v(a - n_i e_i) & \text{if } n_i \leq a_i < n_i + m_i. \end{cases}
\]

If the shapes \( n \) and \( m \) are not equal except at index \( i \), we say that the concatenation of \( u \in A^n \) and \( v \in A^m \) in the direction \( e_i \) is **not defined**. The following equation illustrates the concatenation of words in the direction \( e_2 \) when \( d = 2 \):

\[
\begin{pmatrix} 4 & 5 \\ 10 & 5 \end{pmatrix} \otimes^2 \begin{pmatrix} 3 & 10 \\ 9 & 9 \\ 0 & 0 \\ 10 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 10 \\ 9 & 9 \\ 0 & 0 \\ 4 & 5 \\ 10 & 5 \end{pmatrix}.
\]

Let \( n, m \in \mathbb{N}^d \) and \( u \in A^n \) and \( v \in A^m \). We say that \( u \) **occurs in \( v \) at position \( p \in \mathbb{N}^d \)** if \( v \) is large enough, i.e., \( m - p - n \in \mathbb{N}^d \) and

\[
v(a + p) = u(a)
\]

for all \( a = (a_1, \ldots, a_d) \in \mathbb{N}^d \) such that \( 0 \leq a_i < n_i \) with \( 1 \leq i \leq d \). If \( u \) occurs in \( v \) at some position, then we say that \( u \) is a \( d \)-dimensional **subword** or **factor** of \( v \).

### 3.2. \( d \)-dimensional language

A subset \( L \subseteq A^d \) is called a \( d \)-dimensional **language**. The **factorial closure** of a language \( L \) is

\[
L^{\text{Fact}} = \{ u \in A^d \mid u \text{ is a } d\text{-dimensional subword of some } v \in L \}.
\]

A language \( L \) is **factorial** if \( L^{\text{Fact}} = L \). All languages considered in this contribution are factorial. Given a configuration \( x \in A^\mathbb{Z}^d \), the language \( \mathcal{L}(x) \) defined by \( x \) is

\[
\mathcal{L}(x) = \{ u \in A^d \mid u \text{ is a } d\text{-dimensional subword of } x \}.
\]

The language of a subshift \( X \subseteq A^{\mathbb{Z}^d} \) is \( \mathcal{L}_X = \bigcup_{x \in X} \mathcal{L}(x) \). Conversely, given a factorial language \( L \subseteq A^{d^2} \) we define the subshift

\[
\mathcal{X}_L = \{ x \in A^{\mathbb{Z}^d} \mid \mathcal{L}(x) \subseteq L \}.
\]

A \( d \)-dimensional subword \( u \in A^{d^2} \) is **allowed** in a subshift \( X \subseteq A^{\mathbb{Z}^d} \) if \( u \in \mathcal{L}_X \) and it is **forbidden** in \( X \) if \( u \notin \mathcal{L}_X \). A language \( L \subseteq A^{d^2} \) is forbidden in a subshift \( X \subseteq A^{\mathbb{Z}^d} \) if \( L \cap \mathcal{L}_X = \emptyset \).
3.3. $d$-dimensional morphisms. In this section, we generalize the definition of $d$-dimensional morphisms [CKR10] to the case where the domain and codomain are different as for $S$-adic systems [BD14].

Let $\mathcal{A}$ and $\mathcal{B}$ be two alphabets. Let $L \subseteq \mathcal{A}^d$ be a factorial language. A function $\omega : L \to \mathcal{B}^d$ is a $d$-dimensional morphism if for every $i$ with $1 \leq i \leq d$, and every $u, v \in L$ such that $u \circ^i v$ is defined and is in $L$ we have that the concatenation $\omega(u) \circ^i \omega(v)$ in direction $e_i$ is defined and

$$\omega(u \circ^i v) = \omega(u) \circ^i \omega(v).$$

Note that the left-hand side of the equation is defined since $u \circ^i v$ belongs to the domain of $\omega$. A $d$-dimensional morphism $L \to \mathcal{B}^d$ is thus completely defined from the image of the letters in $\mathcal{A}$, so we sometimes denote a $d$-dimensional morphism as a rule $A \to B$ when the language $L$ is unspecified.

Given a language $L \subseteq \mathcal{A}^d$ of $d$-dimensional words and a $d$-dimensional morphism $\omega : L \to \mathcal{B}^d$, we define the image of the language $L$ under $\omega$ as the language

$$\omega(L)^{\text{Fact}} = \{ u \in \mathcal{B}^d \mid u \text{ is a } d\text{-dimensional subword of } \omega(v) \text{ with } v \in L \} \subseteq \mathcal{B}^d.$$

Let $L \subseteq \mathcal{A}^d$ be a factorial language and $\mathcal{X}_L \subseteq \mathcal{A}^{zd}$ be the subshift generated by $L$. A $d$-dimensional morphism $\omega : L \to \mathcal{B}^d$ can be extended to a continuous map $\omega : \mathcal{X}_L \to \mathcal{B}^{zd}$ in such a way that the origin of $\omega(x)$ is at zero position in the word $\omega(x_0)$ for all $x \in \mathcal{X}_L$. More precisely, the image under $\omega$ of the configuration $x \in \mathcal{X}_L$ is

$$\omega(x) = \lim_{n \to \infty} \sigma^{f(n)} \omega \left( \sigma^{-n_1}(x|_{[-n_1,n_1]}) \right) \in \mathcal{B}^{zd}$$

where $1 = (1, \ldots, 1) \in \mathbb{Z}^d$, $f(n) = \text{SHAPE} \left( \omega(\sigma^{-n_1}(x|_{[-n_1,n_1]})) \right)$ for all $n \in \mathbb{N}$ and $\lfloor m, n \rfloor = \lfloor m_1, n_1 \rfloor \times \cdots \times \lfloor m_d, n_d \rfloor$.

In general, the closure under the shift of the image of a subshift $X \subseteq \mathcal{A}^{zd}$ under $\omega$ is the subshift

$$\omega(X)^\ominus = \{ \sigma^k \omega(x) \in \mathcal{B}^{zd} \mid k \in \mathbb{Z}^d, x \in X \} \subseteq \mathcal{B}^{zd}.$$

The next lemma states that $d$-dimensional morphisms preserve minimality of subshifts.

**Lemma 3.1.** [Lab21b] Let $\omega : X \to \mathcal{B}^{zd}$ be a $d$-dimensional morphism for some $X \subseteq \mathcal{A}^{zd}$. If $X$ is a minimal subshift, then $\omega(X)^\ominus$ is a minimal subshift.

3.4. Self-similar subshifts. In this section, we consider languages and subshifts defined from morphisms leading to self-similar structures. In this situation, the domain and codomain of morphisms are defined over the same alphabet. Formally, we consider the case of $d$-dimensional morphisms $A \to B^d$ where $A = B$.

The definition of self-similarity depends on the notion of expansiveness. It avoids the presence of lower-dimensional self-similar structure by having expansion in all directions. We say that a $d$-dimensional morphism $\omega : A \to A^d$ is expansive if for every $a \in A$ and $K \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\min(\text{SHAPE}(\omega^m(a))) > K$.

**Definition 3.2.** A subshift $X \subseteq \mathcal{A}^{zd}$ is self-similar if there exists an expansive $d$-dimensional morphism $\omega : A \to A^d$ such that $X = \omega(X)^\ominus$. The map $\omega$ is called the self-similarity of $X$.

Respectively, a language $L \subseteq \mathcal{A}^d$ is self-similar if there exists an expansive $d$-dimensional morphism $\omega : A \to A^d$ such that $L = \omega(L)^{\text{Fact}}$. Self-similar languages and subshifts can be constructed by iterative application of a morphism $\omega$ starting with the letters. The language $L_\omega$ defined by an expansive $d$-dimensional morphism $\omega : A \to A^d$ is

$$L_\omega = \{ u \in A^d \mid u \text{ is a } d\text{-dimensional subword of } \omega^n(a) \text{ for some } a \in A \text{ and } n \in \mathbb{N} \}.$$
It satisfies $L_\omega = (L_\omega)^{fact}$ and thus is self-similar. The **substitute subshift** $X_\omega = X_{L_\omega}$ defined from the language of $\omega$ is a self-similar subshift since $X_\omega = \omega(L_\omega)^p$ holds.

Substitute shift obtained from expansive and primitive morphisms are interesting for their properties. As in the one-dimensional case, we say that $\omega$ is **primitive** if there exists $m \in \mathbb{N}$ such that for every $a, b \in A$ the letter $b$ occurs in $\omega^m(a)$.

**Lemma 3.3.** Let $\omega : A \to A^d$ be an expansive and primitive $d$-dimensional morphism. Then the self-similar substitute subshift $X_\omega$ is minimal.

**Proof.** The substitute shift of $\omega$ is well-defined since $\omega$ is expansive and it is minimal since $\omega$ is primitive using standard arguments [Que10 §5.2].

4. **Toral $\mathbb{Z}^2$-rotations and polygon exchange transformations (PETs)**

Let $\Gamma$ be a **lattice** in $\mathbb{R}^2$, i.e., a discrete subgroup of the additive group $\mathbb{R}^2$ with 2 linearly independent generators. This defines a 2-dimensional torus $T = \mathbb{R}^2/\Gamma$. By analogy with the rotation $x \mapsto x + \alpha$ on the circle $\mathbb{R}/\mathbb{Z}$ for some $\alpha \in \mathbb{R}/\mathbb{Z}$, we use the terminology of rotation to denote the following $\mathbb{Z}^2$-action defined on a 2-dimensional torus.

**Definition 4.1.** For some $\alpha, \beta \in T$, we consider the dynamical system $(T, \mathbb{Z}^2, R)$ where $R : \mathbb{Z}^2 \times T \to T$ is the continuous $\mathbb{Z}^2$-action on $T$ defined by

$$R^n(x) := R(n, x) = x + n_1\alpha + n_2\beta$$

for every $n = (n_1, n_2) \in \mathbb{Z}^2$. We say that the $\mathbb{Z}^2$-action $R$ is a **toral $\mathbb{Z}^2$-rotation** or a **$\mathbb{Z}^2$-rotation on $T$** which defines a dynamical system $(T, \mathbb{Z}^2, R)$.

When the $\mathbb{Z}^2$-action $R$ is a $\mathbb{Z}^2$-rotation on the torus $T$, the maps $R^{e_1}$ and $R^{e_2}$ can be seen as polygon exchange transformations [Hoo13, Sch14] on a fundamental domain of $T$.

**Definition 4.2.** [AKY19] Let $X$ be a polygon together with two topological partitions of $X$ into polygons

$$X = \bigcup_{k=0}^N P_k = \bigcup_{k=0}^N Q_k$$

such that for each $k$, $P_k$ and $Q_k$ are translation equivalent, i.e., there exists $v_k \in \mathbb{R}^2$ such that $P_k = Q_k + v_k$. A **polygon exchange transformation (PET)** is the piecewise translation on $X$ defined for $x \in P_k$ by $T(x) = x + v_k$. The map is not defined for points $x \in \bigcup_{k=0}^N \partial P_k$.

The fact that a rotation on a circle can be seen as an exchange of two intervals is well-known as noticed for example in [Rau77]. It generalizes in higher dimension where a generic translation on a $d$-dimensional torus is a polyhedron exchange transformation defined by the exchange of at most $2^d$ pieces on a fundamental domain having for shape a $d$-dimensional parallelootope. We state a 2-dimensional version of this lemma restricted to the case of rectangular fundamental domain because we use this connection several times in the following sections to prove that induced $\mathbb{Z}^2$-actions are again $\mathbb{Z}^2$-rotations on a torus.

**Lemma 4.3.** Let $\Gamma = \ell_1 \mathbb{Z} \times \ell_2 \mathbb{Z}$ be a lattice in $\mathbb{R}^2$ and its rectangular fundamental domain $D = [0, \ell_1) \times [0, \ell_2)$. For every $\alpha = (\alpha_1, \alpha_2) \in D$, the dynamical system $(\mathbb{R}^2/\Gamma, \mathbb{Z}, x \mapsto x + \alpha)$ is measurably conjugate to the dynamical system $(D, \mathbb{Z}, T)$ where $T : D \to D$ is the polygon exchange transformation shown in Figure 5.

**Proof.** It follows from the fact that toral rotations and such polygon exchange transformations are the Cartesian product of circle rotations and exchange of two intervals.  \[ \square \]
The closures

Definition 5.1.

Configuration is the set of points whose orbit under the

Definition 5.2.

in $T$

Symbolic representation of a $(\alpha_1, \alpha_2)$ on the torus

$\mathbb{R}^2/ (\ell_1 \mathbb{Z} \times \ell_2 \mathbb{Z})$.

5. Symbolic dynamical systems for toral $\mathbb{Z}^2$-rotations

5.1. Symbolic dynamical systems. Let $\Gamma$ be a lattice in $\mathbb{R}^2$ and $T = \mathbb{R}^2/\Gamma$ be a 2-dimensional torus. Let $(T, \mathbb{Z}^2, R)$ be the dynamical system given by a $\mathbb{Z}^2$-rotation $R$ on $T$. For some finite set $A$, a topological partition of $T$ is a finite collection $\{P_a\}_{a \in A}$ of disjoint open sets $P_a \subset T$ such that $T = \bigcup_{a \in A} P_a$. If $S \subset \mathbb{Z}^2$ is a finite set, we say that a pattern $\sigma \in A^S$ is allowed for $\mathcal{P}, R$ if

$$\bigcap_{k \in S} R^{-k}(P_{\sigma_k}) \neq \emptyset.$$ 

Let $\mathcal{L}_{\mathcal{P}, R}$ be the collection of all allowed patterns for $\mathcal{P}, R$. The set $\mathcal{L}_{\mathcal{P}, R}$ is the language of a subshift $\mathcal{X}_{\mathcal{P}, R} \subseteq \mathcal{A}^{\mathbb{Z}^2}$ defined as follows, see [Hoc16 Prop. 9.2.4],

$$\mathcal{X}_{\mathcal{P}, R} = \{ x \in \mathcal{A}^{\mathbb{Z}^2} | \pi_S \circ \sigma^n(x) \in \mathcal{L}_{\mathcal{P}, R} \text{ for every } n \in \mathbb{Z}^2 \}.$$ 

Definition 5.1. We call $\mathcal{X}_{\mathcal{P}, R}$ the symbolic dynamical system corresponding to $\mathcal{P}, R$.

For each $w \in \mathcal{X}_{\mathcal{P}, R} \subset \mathcal{A}^{\mathbb{Z}^2}$ and $n \geq 0$ there is a corresponding nonempty open set

$$D_n(w) = \bigcap_{\|k\| \leq n} R^{-k}(P_{\sigma_k}) \subset T.$$ 

The closures $\overline{D}_n(w)$ of these sets are compact and decrease with $n$, so that $\overline{D}_0(w) \supseteq \overline{D}_1(w) \supseteq \overline{D}_2(w) \supseteq \ldots$. It follows that $\bigcap_{n=0}^\infty \overline{D}_n(w) \neq \emptyset$. In order for points in $\mathcal{X}_{\mathcal{P}, R}$ to correspond to points in $T$, this intersection should contain only one point. This leads to the following definition.

Definition 5.2. A topological partition $\mathcal{P}$ of $T$ gives a symbolic representation of $(T, \mathbb{Z}^2, R)$ if for every $w \in \mathcal{X}_{\mathcal{P}, R}$ the intersection $\bigcap_{n=0}^\infty \overline{D}_n(w)$ consists of exactly one point $x \in T$. We call $w$ a symbolic representation of $x$.

The set

$$\Delta_{\mathcal{P}, R} := \bigcup_{n \in \mathbb{Z}^2} R^n \left( \bigcup_{a \in A} \partial P_a \right) \subset T$$

is the set of points whose orbit under the $\mathbb{Z}^2$-action of $R$ intersect the boundary of the topological partition $\mathcal{P} = \{P_a\}_{a \in A}$. From Baire Category Theorem [LM95 Theorem 6.1.24], the set $T \setminus \Delta_{\mathcal{P}, R}$ is dense in $T$.

A topological partition $\mathcal{P} = \{P_a\}_{a \in A}$ of $T = \mathbb{R}^2/\Gamma$ is associated to a coding map

$\text{Code : } T \setminus (\bigcup_{a \in A} \partial P_a) \rightarrow A$

$$x \mapsto a \text{ if and only if } x \in P_a.$$ 

For every starting point $x \in T \setminus \Delta_{\mathcal{P}, R}$, the coding of its orbit under the $\mathbb{Z}^2$-action of $R$ is a configuration $\text{Config}^{\mathcal{P}, R}_x \in \mathcal{A}^{\mathbb{Z}^2}$ defined by

$$\text{Config}^{\mathcal{P}, R}_x(n) = \text{Code}(R^n(x)).$$
for every $n \in \mathbb{Z} \times \mathbb{Z}$.

**Lemma 5.3.** The symbolic dynamical system $\mathcal{X}_{\mathcal{P}, R}$ corresponding to $\mathcal{P}$, $R$ is the topological closure of the set of configurations:

$$\mathcal{X}_{\mathcal{P}, R} = \{ \text{CONFIG}_{x}^{\mathcal{P}, R} | x \in \mathbb{T} \setminus \Delta_{\mathcal{P}, R} \}.$$ 

**Proof.** $(\supset)$ Let $x \in \mathbb{T} \setminus \Delta_{\mathcal{P}, R}$. The patterns appearing in the configuration $\text{CONFIG}_{x}^{\mathcal{P}, R}$ are in $L_{\mathcal{P}, R}$. Thus $\text{CONFIG}_{x}^{\mathcal{P}, R} \in \mathcal{X}_{\mathcal{P}, R}$. The topological closure of such configurations is in $\mathcal{X}_{\mathcal{P}, R}$ since $\mathcal{X}_{\mathcal{P}, R}$ is topologically closed.

$(\subset)$ Let $w \in \mathcal{A}^{S}$ be a pattern with finite support $S \subset \mathbb{Z}^{2}$ appearing in $\mathcal{X}_{\mathcal{P}, R}$. Then $w \in L_{\mathcal{P}, R}$ and from Equation (5) there exists $x \in \mathbb{T} \setminus \Delta_{\mathcal{P}, R}$ such that $x \in \bigcap_{k \in S} R^{-k}(P_{w_{k}})$. The pattern $w$ appears in the configuration $\text{CONFIG}_{x}^{\mathcal{P}, R}$. Any configuration in $\mathcal{X}_{\mathcal{P}, R}$ is the limit of a sequence $(w_{n})_{n \in \mathbb{N}}$ of patterns covering a ball of radius $n$ around the origin, thus equal to some limit $\lim_{n \to \infty} \text{CONFIG}_{x_{n}}^{\mathcal{P}, R}$ with $x_{n} \in \mathbb{T} \setminus \Delta_{\mathcal{P}, R}$ for every $n \in \mathbb{N}$.

$\square$

### 5.2. Factor map.

An important consequence of the fact that a partition $\mathcal{P}$ gives a symbolic representation of the dynamical system $(\mathbb{T}, \mathbb{Z}^{2}, R)$ is the existence of a factor map $f : \mathcal{X}_{\mathcal{P}, R} \to \mathbb{T}$ which commutes the $\mathbb{Z}^{2}$-actions. In the spirit of [LM95, Prop. 6.5.8] for $\mathbb{Z}$-actions, we have the following proposition whose proof can be found in [Lab21a].

**Proposition 5.4.** [Lab21a, Prop. 5.1] Let $\mathcal{P}$ give a symbolic representation of the dynamical system $(\mathbb{T}, \mathbb{Z}^{2}, R)$. Let $f : \mathcal{X}_{\mathcal{P}, R} \to \mathbb{T}$ be defined such that $f(w)$ is the unique point in the intersection $\bigcap_{n=0}^{\infty} D_{n}(w)$. The map $f$ is a factor map from $(\mathcal{X}_{\mathcal{P}, R}, \mathbb{Z}^{2}, \sigma)$ to $(\mathbb{T}, \mathbb{Z}^{2}, R)$ such that $R^{k} \circ f = f \circ \sigma^{k}$ for every $k \in \mathbb{Z}^{2}$. The map $f$ is one-to-one on $f^{-1}(\mathbb{T} \setminus \Delta_{\mathcal{P}, R})$.

Using the factor map, one can prove the following lemma that we use herein in the proof of Corollary 1.5.

**Lemma 5.5.** [Lab21a, Lemma 5.2] Let $\mathcal{P}$ give a symbolic representation of the dynamical system $(\mathbb{T}, \mathbb{Z}^{2}, R)$. Then

(i) if $(\mathbb{T}, \mathbb{Z}^{2}, R)$ is minimal, then $(\mathcal{X}_{\mathcal{P}, R}, \mathbb{Z}^{2}, \sigma)$ is minimal,

(ii) if $R$ is a free $\mathbb{Z}^{2}$-action on $\mathbb{T}$, then $\mathcal{X}_{\mathcal{P}, R}$ aperiodic.

Of course, Lemma 5.5 whose proof can also be found in [Lab21a], does not hold if $\mathcal{P}$ does not give a symbolic representation of $(\mathbb{T}, \mathbb{Z}^{2}, R)$. For example, consider the $\mathbb{Z}^{2}$-rotation on $\mathbb{T}^{2}$ defined by $R(\mathbf{n}, \mathbf{x}) = \mathbf{x} + (n_{1}\alpha, n_{2}\beta)$ for every $\mathbf{n} = (n_{1}, n_{2}) \in \mathbb{Z}^{2}$ for some fixed irrational $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$. If $\mathcal{P}$ is a partition of the fundamental domain $[0, 1)^{2}$ consisting of horizontal rectangles cutting all the way across the domain, all configurations of the shift space $\mathcal{X}_{\mathcal{P}, R}$ will be fixed by the horizontal shift. Even though $R$ is a free $\mathbb{Z}^{2}$-action, $\mathcal{X}_{\mathcal{P}, R}$ contains periodic configuration. This is not a contradiction since the partition into horizontal rectangles does not give a symbolic representation of $(\mathbb{T}^{2}, \mathbb{Z}^{2}, R)$. Indeed, for every $\mathbf{w} \in \mathcal{X}_{\mathcal{P}, R}$, the intersection $\bigcap_{n=0}^{\infty} D_{n}(\mathbf{w})$ is an horizontal segment in the fundamental domain. In general, the existence of an atom of the partition of the torus $\mathbb{T}$ which is invariant only under the trivial translation is a sufficient condition for the partition to give a symbolic representation of a minimal $\mathbb{Z}^{2}$-rotation on $\mathbb{T}$, see [Lab21a, Lemma 3.4].

### 5.3. Markov partitions for toral $\mathbb{Z}^{2}$-rotations.

Markov partitions were originally defined for one-dimensional dynamical systems $(\mathbb{T}, \mathbb{Z}, R)$ and were extended to $\mathbb{Z}^{2}$-actions by automorphisms of compact Abelian group in [ES97]. Following [Lab21a], we allow ourselves to use the same terminology and extend the definition proposed in [LM95, §6.5] for dynamical systems defined by higher-dimensional actions by rotations.

**Definition 5.6.** A topological partition $\mathcal{P}$ of $\mathbb{T}$ is a Markov partition for $(\mathbb{T}, \mathbb{Z}^{2}, R)$ if
\begin{itemize}
\item \( \mathcal{P} \) gives a symbolic representation of \((T, \mathbb{Z}^2, R)\) and
\item \( \mathcal{X}_{\mathcal{P}, R} \) is a shift of finite type (SFT).
\end{itemize}

In this article, we consider Markov partitions associated with aperiodic subshifts of finite type over \( \mathbb{Z}^2 \) coded by toral rotations (thus of zero entropy). This may seem counter-intuitive for the reader since Markov partitions are usually associated with hyperbolic systems (thus with positive entropy). Moreover, the coding of an irrational rotation on the circle leads to aperiodic Sturmian sequences which are not SFT. Our opinion is that positive entropy and all associated intuitions follow from the restriction of Definition 5.6 to the case of \( \mathbb{Z} \)-actions, but not from the notion of Markov partition itself. In this article, we made a choice by using the terminology of Markov partitions in the unusual setup of \( \mathbb{Z}^2 \)-rotations.

Of course, SFTs over \( \mathbb{Z}^2 \) are much different than SFTs over \( \mathbb{Z} \). The emptiness of \( \mathbb{Z}^2 \)-SFTs is undecidable [Ber66] and the possible entropies achievable by a \( \mathbb{Z}^2 \)-SFT are exactly the non-negative numbers obtainable as the limit of computable decreasing sequences of rationals [HM10], as opposed to be given by an algebraic characterization in the case of \( \mathbb{Z} \)-SFT, see [LM95, §4]. In particular, there exist aperiodic \( \mathbb{Z}^2 \)-SFTs of zero entropy which is not possible in the one-dimensional case, since infinite \( \mathbb{Z} \)-SFTs have positive entropy and contain a periodic configuration.

**Part 2. Induction of \( \mathbb{Z}^2 \)-actions and induced partitions of the 2-torus**

This part is divided into four sections on induced \( \mathbb{Z}^2 \)-actions; toral partitions induced by \( \mathbb{Z}^2 \)-rotations, renormalization schemes and algorithms.

**6. Induced \( \mathbb{Z}^2 \)-actions**

Renormalization schemes also known as *Rauzy induction* were originally defined for dynamical system including interval exchange transformations (IETs) [Rau79]. A natural way to generalize it to higher dimension is to consider polygon exchange transformations [Hoo13, AKY19] or even polytope exchange transformations [Sch14, Sch11] where only one map is considered. But more dimensions also allows to induce two or more (commuting) maps at the same time.

In this section, we define the induction of \( \mathbb{Z}^2 \)-actions on a sub-domain. We consider the torus \( T = \mathbb{R}^2 / \Gamma \) where \( \Gamma \) is a lattice in \( \mathbb{R}^2 \). Let \((T, \mathbb{Z}^2, R)\) be a minimal dynamical system given by a \( \mathbb{Z}^2 \)-action \( R \) on \( T \). For every \( n \in \mathbb{Z}^2 \), the toral translation \( R^n \) can be seen as a polygon exchange transformation on a fundamental domain of \( T \).

The **set of return times** of \( x \in T \) to \( W \subset T \) under the \( \mathbb{Z}^2 \)-action \( R \) is the subset of \( \mathbb{Z} \times \mathbb{Z} \) defined as:

\[ \delta_W(x) = \{ n \in \mathbb{Z} \times \mathbb{Z} \mid R^n(x) \in W \}. \]

**Definition 6.1.** Let \( W \subset T \). We say that the \( \mathbb{Z}^2 \)-action \( R \) is **Cartesian on** \( W \) if the set of return times \( \delta_W(x) \) can be expressed as a Cartesian product, that is, for all \( x \in T \) there exist two strictly increasing sequences \( r^{(1)}_x, r^{(2)}_x : \mathbb{Z} \rightarrow \mathbb{Z} \) such that

\[ \delta_W(x) = r^{(1)}_x(\mathbb{Z}) \times r^{(2)}_x(\mathbb{Z}). \]

We always assume that the sequences are shifted in such a way that

\[ r^{(i)}_x(n) \geq 0 \iff n \geq 0 \quad \text{for } i \in \{1, 2\}. \]

In particular, if \( x \in W \), then \((0, 0) \in \delta_W(x)\), so that \( r^{(1)}_x(0) = r^{(2)}_x(0) = 0 \).

When the \( \mathbb{Z}^2 \)-action \( R \) is Cartesian on \( W \subset T \), we say that the tuple

\[ (r^{(1)}_x(1), r^{(2)}_x(1)) \]
is the first return time of a starting point $x \in T$ to $W \subset T$ under the action $R$. When the $\mathbb{Z}^2$-action $R$ is Cartesian on $W \subset T$, we may consider its return map on $W$ and we prove in the next lemma that this induces a $\mathbb{Z}^2$-action on $W$.

**Lemma 6.2.** If the $\mathbb{Z}^2$-action $R$ is Cartesian on $W \subset T$, then the map $\hat{R}|_W : \mathbb{Z}^2 \times W \to W$ defined by

$$(\hat{R}|_W)^n(x) := \hat{R}|_W(n, x) = R^{(r_x^{(1)}(n_1), r_x^{(2)}(n_2))}(x)$$

for every $n = (n_1, n_2) \in \mathbb{Z}^2$ is a well-defined $\mathbb{Z}^2$-action.

We say that $\hat{R}|_W$ is the induced $\mathbb{Z}^2$-action of the $\mathbb{Z}^2$-action $R$ on $W$.

**Proof.** Let $x \in W$. We have that

$$\hat{R}|_W(0, x) = R^{(r_x^{(1)}(0), r_x^{(2)}(0))}(x) = R^{(0,0)}(x) = x.$$

Firstly, using

$$r_x^{(i)}(k + n) = r_{R^{(i)}(x)}^{(i)}(k) + r_x^{(i)}(n),$$

and skipping few details, we get

$$\hat{R}|_W(k e_i + n e_i, x) = \hat{R}|_W(k e_i, (\hat{R}|_W(n e_i, x))).$$

Secondly, using the fact that

$$r_x^{(i)}(k_1) = r_y^{(i)}(k_1)$$

whenever $y = R^{(0, r_x^{(2)}(k_2))}x = \hat{R}|_W(k_2 e_2, x)$, we obtain

$$\hat{R}|_W(k, x) = R^{(r_x^{(1)}(k_1), r_x^{(2)}(k_2))}(x) = R^{(r_x^{(1)}(k_1), 0)}(\hat{R}|_W(k_2 e_2, x))$$

$$= R^{(r_x^{(1)}(k_1), 0)}(\hat{R}|_W(k_2 e_2, x)) = \hat{R}|_W(k_1 e_1, \hat{R}|_W(k_2 e_2, x)).$$

Therefore, for every $k, n \in \mathbb{Z}^2$, we have

$$(\hat{R}|_W)^{k+n}(x) = (\hat{R}|_W)^{(k_1+n_1)e_1}(\hat{R}|_W)^{(k_2+n_2)e_2}(x)$$

$$= (\hat{R}|_W)^{k_1e_1}(\hat{R}|_W)^{n_1e_1}(\hat{R}|_W)^{k_2e_2}(\hat{R}|_W)^{n_2e_2}(x)$$

$$= (\hat{R}|_W)^{k_1e_1}(\hat{R}|_W)^{n_1,k_2}(\hat{R}|_W)^{n_2e_2}(x)$$

$$= (\hat{R}|_W)^{k_1e_1}(\hat{R}|_W)^{k_2e_2}(\hat{R}|_W)^{n_1e_1}(\hat{R}|_W)^{n_2e_2}(x)$$

$$= (\hat{R}|_W)^k(\hat{R}|_W)^n(x),$$

which shows that $\hat{R}|_W$ is a $\mathbb{Z}^2$-action on $W$. $\square$

A consequence of the lemma is that the induced $\mathbb{Z}^2$-action $\hat{R}|_W$ is generated by two commutative maps

$$(\hat{R}|_W)^{e_1}(x) = R^{(r_x^{(1)}(1), 0)}(x) \quad \text{and} \quad (\hat{R}|_W)^{e_2}(x) = R^{(0, r_x^{(2)}(1))}(x)$$

which are the first return maps of $R^{e_1}$ and $R^{e_2}$ to $W$:

$$(\hat{R}|_W)^{e_1}(x) = \tilde{r}_1|x_W(x) \quad \text{and} \quad (\hat{R}|_W)^{e_2}(x) = \tilde{r}_2|x_W(x).$$

Recall that the first return map $\tilde{T}|_W$ of a dynamical system $(X, T)$ maps a point $x \in W \subset X$ to the first point in the forward orbit of $T$ lying in $W$, i.e.

$$\tilde{T}|_W(x) = T^r(x)(x) \quad \text{where } r(x) = \min\{k \in \mathbb{Z}_{>0} : T^k(x) \in W\}.$$ 

From Poincaré’s recurrence theorem, if $\mu$ is a finite $T$-invariant measure on $X$, then the first return map $\tilde{T}|_W$ is well defined for $\mu$-almost all $x \in W$. When $T$ is a translation on a torus, if the subset $W$ is open, then the first return map is well-defined for every point $x \in W$. Moreover if $W$ is a.
polygon, then the first return map $\widehat{T}|_W$ is a polygon exchange transformation. An algorithm to compute the induced transformation $\widehat{T}|_W = \widehat{R}|_W^e_i$ of the sub-action $R^e_i$ is provided in Section 9.

7. Toral partitions induced by toral $\mathbb{Z}^2$-rotations

As for IETs, the domain on which we define the Rauzy induction of $\mathbb{Z}^2$-rotations is given by a union of atoms from the partition which defines the IET itself or its inverse. But in the examples that follow, we code the orbits of toral $\mathbb{Z}^2$-rotations by partitions which carries more information than the natural partition expressing a $\mathbb{Z}^2$-rotation as a pair of polygon exchange transformations on a fundamental domain. The partition that we use are non-trivial refinements of the natural partition involving well-chosen vertices and sloped lines. Thus we also need to define the Rauzy induction of the involved coding partitions and not only of the $\mathbb{Z}^2$-rotations.

Let $\Gamma$ be a lattice in $\mathbb{R}^2$ and $T = \mathbb{R}^2/\Gamma$ be a 2-dimensional torus. Let $(T, \mathbb{Z}^2, R)$ be the dynamical system given by a $\mathbb{Z}^2$-rotation $R$ on $T$. Assuming the $\mathbb{Z}^2$-rotation $R$ is Cartesian on a window $W \subset T$, then there exist two strictly increasing sequences $r_x^{(1)} , r_x^{(2)} : \mathbb{Z} \to \mathbb{Z}$ are such that

$$(r, s) = \left( r_x^{(1)}(1), r_x^{(2)}(1) \right)$$

is the first return time of a starting point $x \in T$ to the window $W$ under the action $R$, see Equation (4). It allows to define the return word map as

$$\text{ReturnWord} : W \to \mathcal{A}^2$$

$$x \mapsto \begin{pmatrix} \text{Code}(R^{(0,s-1)}x) & \cdots & \text{Code}(R^{(r-1,s-1)}x) \\ \vdots & \ddots & \vdots \\ \text{Code}(R^{(0,0)}x) & \cdots & \text{Code}(R^{(r-1,0)}x) \end{pmatrix},$$

where $r, s \geq 1$ both obviously depend on $x$.

The image $L = \text{ReturnWord}(W) \subset \mathcal{A}^* \mathcal{A}$ is a language called the set of return words. When the return time to $W$ is bounded, the set of return words $L$ is finite. Let $L = \{w_b\}_{b \in \mathcal{B}}$ be an enumeration of $L$ for some finite set $\mathcal{B}$. The way the enumeration of $L$ is done influences the substitutions which are obtained afterward. To obtain a canonical ordering when the words in $L$ are 1-dimensional, we use the radix order on $L$. It is used at Line 14 of Algorithm 1.

**Definition 7.1 (radix order).** The total order $(\mathcal{A}^*, \prec)$ is defined by $u \prec v$ if $|u| < |v|$ or $|u| = |v|$ and $u <_{\text{lex}} v$ for every $u, v \in \mathcal{A}^*$.

The induced partition of $\mathcal{P}$ by the action of $R$ on the sub-domain $W$ is a topological partition of $W$ defined as the set of preimage sets under ReturnWord:

$$\widehat{\mathcal{P}}|_W = \{\text{ReturnWord}^{-1}(w_b)\}_{b \in \mathcal{B}}.$$ 

This yields the induced coding on $W$

$$\text{Code}|_W : W \to \mathcal{B}$$

$$y \mapsto b \quad \text{if and only if} \quad y \in \text{ReturnWord}^{-1}(w_b).$$

A natural substitution comes out of this induction procedure:

$$\omega : \mathcal{B} \to \mathcal{A}^2$$

$$b \mapsto w_b.$$

The partition $\widehat{\mathcal{P}}|_W$ of $W$ can be effectively computed by the refinement of the partition $\mathcal{P}$ with translated copies of the sub-domain $W$ under the action of $R$. In Section 9 we propose Algorithm 1 to compute the induced partition $\widehat{\mathcal{P}}|_W$ and substitution $\omega$. The next result shows that the coding of the orbit under the $\mathbb{Z}^2$-rotation $R$ is the image under the 2-dimensional substitution $\omega$ of the coding of the orbit under the $\mathbb{Z}^2$-action $\widehat{R}|_W$. 


Lemma 7.2. If the \(\mathbb{Z}^2\)-action \(R\) is Cartesian on a window \(W \subset T\), then \(\omega\) is a 2-dimensional morphism, and for every \(x \in W\) we have

\[
\text{CONFIG}_{x}^{P,R} = \omega \left( \text{CONFIG}_{x}^{\widehat{R}[W,\widehat{R}[W]]} \right).
\]

Proof. Let \(x \in W\). By hypothesis, there exists \(P, Q \subset \mathbb{Z}\) such that the set of returns times satisfies \(\delta_W(x) = P \times Q\) and we may write \(P = \{r_i\}_{i \in \mathbb{Z}}\) and \(Q = \{s_j\}_{j \in \mathbb{Z}}\) as increasing sequences such that \(r_0 = s_0 = 0\). Therefore, \(\text{CONFIG}_{x}^{P,R}\) may be decomposed into a lattice of rectangular blocks. More precisely, for every \(i, j \in \mathbb{Z}\), the following block is the image of a letter under \(\omega\):

\[
\begin{pmatrix}
\text{CODE}(R^{(r_i, s_j+1-1)}x) & \cdots & \text{CODE}(R^{(r_i+1, s_j+1-1)}x) \\
\vdots & \ddots & \vdots \\
\text{CODE}(R^{(r_i, s_j)}x) & \cdots & \text{CODE}(R^{(r_i+1, s_j)}x)
\end{pmatrix} = \text{RETURNWORD}(R^{(r_i, s_j)}x) = w_{b_{ij}} = \omega(b_{ij})
\]

for some letter \(b_{ij} \in \mathcal{B}\). Moreover,

\[
b_{ij} = \text{CODE}|_W(R^{(r_i, s_j)}x) = \text{CODE}|_W((\widehat{R}[W])^{(i,j)}x).
\]

Since the adjacent blocks have matching dimensions, for every \(i, j \in \mathbb{Z}\), the following concatenations

\[
\omega \left( b_{ij} \odot^1 b_{(i+1)j} \right) = \omega \left( b_{ij} \right) \odot^1 \omega \left( b_{(i+1)j} \right)
\]

and

\[
\omega \left( b_{ij} \odot^2 b_{(j+1)i} \right) = \omega \left( b_{ij} \right) \odot^2 \omega \left( b_{(j+1)i} \right)
\]

are well defined. Thus \(\omega\) is a 2-dimensional morphism on the set \(\{\text{CONFIG}_{x}^{\widehat{R}[W,\widehat{R}[W]]} | x \in W\}\) and we have

\[
\text{CONFIG}_{x}^{P,R} = \omega \left( \text{CONFIG}_{x}^{\widehat{R}[W,\widehat{R}[W]]} \right)
\]

which ends the proof. Note that the domain of \(\omega\) can be extended to its topological closure. \(\Box\)

Proposition 7.3. Let \(\mathcal{P}\) be a topological partition of \(T\). If the \(\mathbb{Z}^2\)-action \(R\) is Cartesian on a window \(W \subset T\), then \(X_{\mathcal{P},R} = \overline{\omega(X_{\mathcal{P}}^{\widehat{R}[W,\widehat{R}[W]])}\).\)

Proof. Let \(Y = \{\text{CONFIG}_{x}^{P,R} | x \in T\}\) and \(Z = \{\text{CONFIG}_{x}^{\widehat{R}[W,\widehat{R}[W]]} | x \in W\}\),

\((\supseteq)\). Let \(x \in W\). From Lemma 7.2, \(\omega \left( \text{CONFIG}_{x}^{\widehat{R}[W,\widehat{R}[W]]} \right) = \text{CONFIG}_{x}^{P,R}\) with \(x \in W \subset T\).

\((\subseteq)\). Let \(x \in T\). There exists \(k_1, k_2 \in \mathbb{N}\) such that \(x' = R^{-(k_1, k_2)}(x) \in W\). Therefore, we have \(x = R^{(k_1, k_2)}(x')\) where \(0 \leq k_1 < r(x')\) and \(0 \leq k_2 < s(x')\). Thus the shift \(k = (k_1, k_2) \in \mathbb{Z}^2\) is bounded by the maximal return time of \(R^{e_1}\) and \(R^{e_2}\) to \(W\). We have

\[
\text{CONFIG}_{x}^{P,R} = \text{CONFIG}_{x'}^{R^{(k_1, k_2)}} = \sigma^k \circ \text{CONFIG}_{x'}^{P,R}
\]

\[
= \sigma^k \circ \omega \left( \text{CONFIG}_{x'}^{\widehat{R}[W,\widehat{R}[W]]} \right)
\]

where we used Lemma 7.2 with \(x' \in W\). We conclude that \(Y = \overline{\omega(Z)^\mathcal{P}}\). The result follows from Lemma 5.3 by taking the topological closure on both sides. \(\Box\)
Including the first vs the last letter. We finish with a remark on the definition of the return word map. There is a choice to be made in its definition whether we include the first or the last letter of the orbit that comes back to the window (vertically and horizontally). For example, another option is to define return words that includes the last letter instead of the first letter. In general, for each value of \( \varepsilon_1, \varepsilon_2 \in \{0, 1\} \), we may define

\[
\text{ReturnWord}_{\varepsilon_1, \varepsilon_2} : W \to \mathcal{A}^2
\]

\[
x \mapsto \begin{pmatrix}
\text{Code}(R^{(\varepsilon_1, \varepsilon_2+s-1)}x) & \cdots & \text{Code}(R^{(\varepsilon_1+r-1, \varepsilon_2+s-1)}x)
\end{pmatrix}
\]

where \( r = r^{(1)}_x(x) \) and \( s = r^{(2)}_x(x) \). An alternative is to consider ReturnWord for the \( \mathbb{Z}^2 \)-actions \( R^{(-n_1, n_2)} \), \( R^{(n_1, -n_2)} \) or \( R^{(-n_1, -n_2)} \) and taking horizontal and/or vertical mirror images of the obtained return words.

8. Renormalization schemes

In what follows, we define a renormalization scheme on a topological partition of the domain \( X \) together with a dynamical systems \( (X, \mathbb{Z}^2, R) \) defined by a \( \mathbb{Z}^2 \)-action \( R \) on \( X \).

Sometimes, when \( R \) is a \( \mathbb{Z}^2 \)-action on \( X \), the induced \( \mathbb{Z}^2 \)-action \( \hat{R}|_W \) can be renormalized as a \( \mathbb{Z}^2 \)-action on the original domain \( X \). This leads to the definition of renormalization schemes. The following definition is inspired from [AKY19] and adapted to the case dynamical systems defined by \( \mathbb{Z}^2 \)-actions.

**Definition 8.1.** A dynamical system \( (X, \mathbb{Z}^2, R) \) has a renormalization scheme if there exists a proper subset \( W \subset X \), a homeomorphism \( \phi : W \to X \), a renormalized dynamical system \( (X, \mathbb{Z}^2, R') \) such that

\[
(R')^n = \phi \circ (\hat{R}|_W)^n \circ \phi^{-1}
\]

for all \( n \in \mathbb{Z}^2 \), meaning that the following diagram commute:

\[
\begin{array}{ccc}
W & \xrightarrow{\phi} & X \\
\downarrow{(\hat{R}|_W)^n} & & \downarrow{(R')^n} \\
W & \xrightarrow{\phi} & X \\
\end{array}
\]

The dynamical system \( (X, \mathbb{Z}^2, R) \) is self-induced if it is equal to the renormalized one, that is, if \( R = R' \).

Now we define the renormalization of a topological partition \( \mathcal{P} = \{ P_a \}_{a \in \mathcal{A}} \) of \( X \). If a dynamical system \( (X, \mathbb{Z}^2, R) \) has a renormalization scheme with the same notations as above, then the induced partition \( \mathcal{P}|_W = \{ Q_b \}_{b \in \mathcal{B}} \) of \( W \) can be renormalized as a partition

\[
\phi(\mathcal{P}|_W) = \{ \phi(Q_b) \}_{b \in \mathcal{B}}
\]

of \( X \). If the dynamical system \( (X, \mathbb{Z}^2, R) \) is self-induced and there exists a bijection \( \pi : \mathcal{B} \to \mathcal{A} \) such that \( P_{\pi(b)} = \phi(Q_b) \) for all \( b \in \mathcal{B} \), then we say that the topological partition \( \mathcal{P} \) of \( X \) is self-induced.

**Proposition 8.2.** Let \( \mathcal{P} \) be a topological partition of \( \mathbb{T} \) and suppose that the \( \mathbb{Z}^2 \)-action \( R \) is Cartesian on a window \( W \subset \mathbb{T} \). If \( \mathcal{P} \) is self-induced with bijection \( \pi : \mathcal{B} \to \mathcal{A} \), then \( X_{\mathcal{P}, R} \) is self-similar satisfying \( X_{\mathcal{P}, R} = \omega \pi^{-1}(X_{\mathcal{P}, R})^\sigma \) where \( \omega \) is the 2-dimensional morphism defined in Eq. (5).
Proof. From Proposition 7.3, we have $\mathcal{X}_{P,R} = \omega(\mathcal{X}_{\widehat{P}_{aw},\widehat{R}_{aw}})^\sigma$. It remains to show $\mathcal{X}_{\widehat{P}_{aw},\widehat{R}_{aw}} = \pi^{-1}(\mathcal{X}_{P,R})$. Since $\mathcal{P}$ is self-induced, we have $\pi \circ \text{CODE}|_W = \text{CODE} \circ \phi$. Thus for every $m,n \in \mathbb{Z}$ and every $x \in W$, we have

$$\pi \left( \text{CONFIG}_{x}^{\widehat{P}_{aw},\widehat{R}_{aw}}(n) \right) = \pi \circ \text{CODE}|_W \circ (\widehat{R}_{aw})^n(x) = \text{CODE} \circ \phi \circ (\widehat{R}_{aw})^n(x) = \text{CODE} \circ R^n \circ \phi(x) = \text{CONFIG}_{\phi(x)}^{P,R}(n).$$

Since $\phi : W \to \mathbf{T}$ is a homeomorphism, we obtain

$$\mathcal{X}_{P,R} = \left\{ \text{CONFIG}_{x}^{R} \mid x \in \mathbf{T} \right\} = \left\{ \text{CONFIG}_{\phi(x)}^{P,R} \mid x \in W \right\}$$

and the conclusion follows. $\square$

9. Algorithms

Algorithm 1 Compute the induced partition $\widehat{P}_{aw}$ and substitution $\omega$ associated to the induced transformation $\widehat{T}_{aw}$ (we use it when $T = R^e_i$ for some $i$.)

Precondition: $T$ is a polytope exchange transformation (PET) on a convex domain $D \subset \mathbb{R}^d$ and $\mathcal{G}$ is a partition of $D$ into convex polytopes such that the restriction of $T$ on each atom of $\mathcal{G}$ is continuous; $v \in \mathbb{R}^{d+1}$ defines a half space $H_v = \{ x \in \mathbb{R}^d \mid v_0 + \sum_{i=1}^d v_ix_i \geq 0 \}$ such that $D \cap H_v = W$; $\mathcal{P}$ is a list of pairs $(a,p)$ such that $\{ p \mid (a,p) \in \mathcal{P} \}$ is a partition of $D$ into convex polytopes indexed by the alphabet $\mathcal{A} = \{ a \mid (a,p) \in \mathcal{P} \}$.

1: function INDUCEDPARTITION($T$, $v$, $\mathcal{P}$)
2: $\mathcal{Q} \leftarrow \{ (\varepsilon, W) \}$ \texttt{\text{$\bowtie$ $\varepsilon \in \mathcal{A}^*$ is the empty word and $W = D \cap H_v$}}
3: $\mathcal{K} \leftarrow T(\mathcal{P} \land \mathcal{G}) = \{ (a, T(p \cap g)) \mid (a, p) \in \mathcal{P} \text{ and } g \in \mathcal{G} \text{ and } p \cap g \neq \emptyset \}$
4: $S \leftarrow \text{EMPTYLIST}()$
5: \textbf{while} $\mathcal{Q}$ not empty \textbf{do}
6: $\mathcal{Q} \leftarrow T^{-1}(\mathcal{Q} \cap \mathcal{K}) = \{ (au, T^{-1}(q \cap k)) \mid (u,q) \in \mathcal{Q} \text{ and } (a,k) \in \mathcal{K} \text{ and } q \cap k \neq \emptyset \}$
7: $S \leftarrow S \cup (\mathcal{Q} \cap H_v) = S \cup \{ (u,q \cap H_v) \mid (u,q) \in \mathcal{Q} \text{ and } q \cap H_v \neq \emptyset \}$
8: $\mathcal{Q} \leftarrow \mathcal{Q} \land \left( \mathbb{R}^d \setminus H_v \right) = \{ (u,q \cap \left( \mathbb{R}^d \setminus H_v \right)) \mid (u,q) \in \mathcal{Q} \text{ and } q \cap \left( \mathbb{R}^d \setminus H_v \right) \neq \emptyset \}$
9: $\mathcal{L} \leftarrow \{ u \in \mathcal{A}^* \mid (u,q) \in S \}$ \texttt{\text{$\bowtie$ the set of return words}}
10: $\mathcal{B} \leftarrow \{ 0, 1, \ldots, \#\mathcal{L} - 1 \}$ \texttt{\text{$\bowtie$ the new alphabet}}
11: $\omega : \text{bijection } \mathcal{B} \to \mathcal{L}$ \texttt{\text{$\bowtie$ s.t. } i < j \text{ if and only if } \omega(i) < \omega(j), \text{ see Definition 7.1}}$
12: $\mathcal{P}' \leftarrow \{ (\omega^{-1}(u),q) \mid (u,q) \in S \}$ \texttt{\text{$\bowtie$ the induced partition labeled by $\mathcal{B}$}}
13: \textbf{return} $\mathcal{P}'$, $\omega$

Postcondition: $\{ q \mid (b,q) \in \mathcal{P}' \} = \widehat{P}_{aw}$ is the induced partition of $W$ into convex polytopes; $\mathcal{P}'$ is a list of pairs $(b,q)$ such that $\text{CODE}|_W(x) = b$ for every $x \in q$; the map $\omega : \mathcal{B} \to \mathcal{A}^*$ extends to a morphism of monoid $\omega : \mathcal{B}^* \to \mathcal{A}^*$ satisfying the following equation for one-dimensional subshifts: $\mathcal{X}_{P,T} = \omega(\mathcal{X}_{\widehat{P}_{aw},\widehat{R}_{aw}})^\sigma$; if $T = R^{e_i}$ for some $i$ and the $\mathbb{Z}^2$-action $R$ is Cartesian on the window $W$, then $\omega : \mathcal{B} \to \mathcal{A}^*$ defines a $d$-dimensional morphism in the direction $e_i$ satisfying $\mathcal{X}_{P,R} = \omega(\mathcal{X}_{\widehat{P}_{aw},\widehat{R}_{aw}})^\sigma$. 


Algorithm 2 Compute the induced transformation $\hat{T}|_W$

**Precondition:** $T$ is a polytope exchange transformation (PET) on a convex domain $D \subset \mathbb{R}^d$ given as a pair $(\mathcal{P}, h)$ where $\mathcal{P}$ is a list of pairs $(a, p)$ such that $\{p \mid (a, p) \in \mathcal{P}\}$ is a partition of $D$ into convex polytopes, $\mathcal{A} = \{a \mid (a, p) \in \mathcal{P}\}$ is some alphabet and $h : \mathcal{A} \rightarrow \mathbb{R}^d$ is a map such that $(a, p) \in \mathcal{P}$ implies that $T(x) = x + h(a)$ for all $x \in p$, the map $h$ extends to a morphism of monoids $h : \mathcal{A}^* \rightarrow \mathbb{R}^d$ satisfying $h(a \cdot v) = h(a) + h(v)$; $v \in \mathbb{R}^{d+1}$ defines a half space $H_v = \{x \in \mathbb{R}^d \mid v_0 + \sum_{i=1}^d v_i x_i \geq 0\}$ such that $D \cap H_v = W$.

1: function INDUCEDTRANSFORMATION($T, v$)
2: $(\mathcal{P}, h) \leftarrow T$
3: $(\mathcal{P}', \omega) \leftarrow INDUCEDPARTITION($T, v, \mathcal{P}$)
4: $T' \leftarrow (\mathcal{P}', h \circ \omega)$ ▷ the induced transformation
5: return $(T', \omega)$

**Postcondition:** $T'$ is a PET equal to the induced transformation $\hat{T}|_W$ given as a pair $(\mathcal{P}', h \circ \omega)$ where $\{q \mid (b, q) \in \mathcal{P}'\}$ is a partition of $W$ into convex polytopes, $\mathcal{B} = \{b \mid (b, q) \in \mathcal{P}'\}$ is some alphabet, $h \circ \omega : \mathcal{B}^* \rightarrow \mathbb{R}^d$ is a morphism of monoids such that $(b, q) \in \mathcal{P}'$ implies that $T'(x) = x + h \circ \omega(b)$ for all $x \in q$; $\omega : \mathcal{B}^* \rightarrow \mathcal{A}^*$ is a morphism of monoid satisfying the following equation for one-dimensional subshifts: $X_{\mathcal{P}, T} = \omega(X_{\mathcal{P}'|_W, \hat{T}|_W})^\sigma$.

In this section, we provide two algorithms to compute the induced partition and induced transformation of a polytope exchange transformation on a sub-domain. More precisely, Algorithm [1] computes the induced partition $\mathcal{P}'|_W$ of a partition $\mathcal{P}$ of $D$ by a polytope exchange transformation $T : D \rightarrow D$ on a sub-domain $W \subset D$. It also computes the substitution $\omega$ allowing to express configuration coded by the partition $\mathcal{P}$ as the image of configurations coded by the induced partition $\mathcal{P}'|_W$. Algorithm [2] computes the induced transformation $\hat{T}|_W$ of a polytope exchange transformation $T : D \rightarrow D$ on a sub-domain $W \subset D$. We first present the algorithm computing the induced partition since the other one can be deduced from it.

Algorithms are written for domain $D \subset \mathbb{R}^d$, with $d \geq 1$, as they work in arbitrary dimension. All polytopes manipulated in the algorithms are assumed to be open so that the intersection of two polytopes is nonempty if and only if it is of positive volume and convex so that they can be represented as a list of linear inequalities. Partitions are represented as a list of pairs $(a, p)$ where $a$ is the index associated to the convex polytope $p$. The same index can be used for different polytopes, thus allowing to deal with polytope partitions where some atoms are not convex. Non-convex atoms must be split into many convex polytopes with the same index. For example, the atom with index 5 in Figure 3 is not convex.

We assume that the polytope exchange transformation $T : D \rightarrow D$ is defined on domain $D \subset \mathbb{R}^d$ which is a convex polytope and that the sub-domain $W$ on which the induction is being done is such that $W = D \cap H_v$ where $v \in \mathbb{R}^{d+1}$ and $H_v$ is a half-space given by the part of $\mathbb{R}^d$ on one side of a hyperplane:

$$H_v = \left\{ x \in \mathbb{R}^d \mid v_0 + \sum_{i=1}^d v_i x_i \geq 0 \right\}.$$

The reason is that in the algorithms, polytopes are intersected sometimes with $H_v$ and sometimes with the complement $\mathbb{R}^d \setminus H_v$, that is, both sides of the hyperplane given by $v$ and we want the result to be convex in both cases. Inducing on more general polytope $W$ must be done in many steps (once for each inequalities defining $W$).

When considering $\mathbb{Z}^d$-action $R$ on a polytope $D \subset \mathbb{R}^d$ that is Cartesian on a window $W \subset D$, it is possible to compute the induced $\mathbb{Z}^d$-action $\hat{R}|_W$ by considering each subaction $\hat{R}|_{e_i}|_W$ individually.
for \( i \in \{1, \ldots, d \} \). In the next sections, we use the algorithms when the return times to \( W \subset D \) under \( R^{e_j} \) is 1 for all \( x \in W \) and for every direction \( e_j \) except for some \( j = i \). In that case, the set of return words \( L = \text{RETURNWORD}(W) \) is a set of one-dimensional words in the direction \( e_i \) for some \( i \) and the substitution \( \omega \) is a \( d \)-dimensional morphism.

When the return time to the window \( W \) is not bounded, Algorithm 1 does not halt. In this case, an approximation of the induced partition can be obtained by replacing line 5 by \textbf{while} \( \sum_{(a,q) \in Q} \text{volume}(q) < \varepsilon \) \textbf{do} for some small \( \varepsilon > 0 \) or by a fixed number of iterations. In the examples considered in this article, the return time is always bounded.

Algorithm 1 and Algorithm 2 are implemented in the module on PETs of the optional package \texttt{slabbe} \cite{Lab20} for SageMath \cite{Sag20}.

Part 3. Substitutive structure of \( X_{P_0,R_0} \)

In this part, we start with the partition introduced in \cite{Lab21a} shown in Figure 3 and construct a sequence of induced partition until the induction process loops. The proofs can be verified with a computer as we provide the SageMath code to reproduce the computations of all induced \( \mathbb{Z}^2 \)-actions, induced partitions and 2-dimensional morphisms.

10. Inducing the \( \mathbb{Z}^2 \)-action \( R_0 \) and the partition \( \mathcal{P}_0 \) to get \( X_{P_1,R_1} \)

Let \( \Gamma_0 = \langle (\varphi,0),(1,\varphi + 3) \rangle_\mathbb{Z} \) be a lattice with \( \varphi = \frac{1+\sqrt{5}}{2} \). We consider the dynamical system \((\mathbb{R}^2/\Gamma_0,\mathbb{Z}^2,R_0)\) defined by the \( \mathbb{Z}^2 \)-action

\[
R_0 : \mathbb{Z}^2 \times \mathbb{R}^2/\Gamma_0 \to \mathbb{R}^2/\Gamma_0
\]

\[
(n,x) \mapsto x + n.
\]

We consider the topological partition \( \mathcal{P}_0 \) of \( \mathbb{R}^2/\Gamma_0 \) illustrated in Figure 6 on a fundamental domain where each atom is associated to a letter in the alphabet \( A_0 = [0, 10] \). The partition \( \mathcal{P}_0 \) defines a coding map \( \text{Code}_0 : \mathbb{R}^2/\Gamma_0 \to A_0 \) and it was proved in \cite{Lab21a} that the coding of orbits under the \( \mathbb{Z}^2 \)-action \( R_0 \) are valid Wang configurations using the 11 Wang tiles from Jeandel-Rao tile set.

![Figure 6](image_url)

**Figure 6.** The partition \( \mathcal{P}_0 \) of \( \mathbb{R}^2/\Gamma_0 \) into atoms associated to letters in \( A_0 \) illustrated on a rectangular fundamental domain.
Proposition 10.1. Let \((\mathbb{R}^2/\Gamma_1,\mathbb{Z}^2,R_1)\) be a dynamical system with lattice \(\Gamma_1 = \varphi\mathbb{Z} \times \mathbb{Z}\) and \(\mathbb{Z}^2\)-action

\[
R_1 : \mathbb{Z}^2 \times \mathbb{R}^2/\Gamma_1 \to \mathbb{R}^2/\Gamma_1
\]

\[
(n, x) \mapsto x + n_1 e_1 + n_2(\varphi^{-1}, \varphi^{-2}).
\]

Let \(P_1\) be the topological partition illustrated in Figure 9 where each atom is associated to a letter in \(A_1 = [0,27]\). Then \(X_{P_0,R_0} = \beta_0(X_{P_1,R_1})\) where \(\beta_0 : A_1^2 \to A_0^2\) is the 2-dimensional morphism defined by

\[
\beta_0 : \begin{cases}
0 \mapsto 7, & 1 \mapsto 5, & 2 \mapsto 9, & 3 \mapsto 3, & 4 \mapsto 6, & 5 \mapsto 8, & 6 \mapsto 2, & 7 \\
7 \mapsto 5, & 8 \mapsto 6, & 9 \mapsto 7, & 10 \mapsto 10, & 11 \mapsto 5, & 12 \mapsto 6, & 13 \mapsto 3, & 14 \\
14 \mapsto 3, & 15 \mapsto 3, & 16 \mapsto 7, & 17 \mapsto 7, & 18 \mapsto 7, & 19 \mapsto 7, & 20 \mapsto 8, & 21 \\
21 \mapsto 8, & 22 \mapsto 10, & 23 \mapsto 3, & 24 \mapsto 8, & 25 \mapsto 2, & 26 \mapsto 10, & 27 \mapsto 4
\end{cases}
\]

Proof. Let \(W = (0, \varphi) \times (0, 1)\). The \(\mathbb{Z}^2\)-action \(R_0\) is Cartesian on the window \(W\). Thus from Lemma 6.2 the induced \(\mathbb{Z}^2\)-action \(\tilde{R}_0|_W : \mathbb{Z}^2 \times W \to W\) given by

\[
(\tilde{R}_0|_W)^n(x) = (\tilde{R}_0^{e_1}|_W)^{n_1} \circ (\tilde{R}_0^{e_2}|_W)^{n_2}(x),
\]

for every \(n = (n_1, n_2) \in \mathbb{Z}^2\), is well-defined. The first return time to \(W\) under \(R_0\) is \((1,4)\) or \((1,5)\). The orbits of points in \(W\) until the first return to \(W\) under \(\tilde{R}_0^{e_2}\) is shown in Figure 7.

The maximal subsets of \(W\) on which the first return map \(\tilde{R}_0^{e_2}|_W\) is continuous yield a partition \(\{A, B, C, D\}\) of \(W\) defined as

\[
A = (0, 1) \times (\varphi^{-1}, 1), \quad B = (1, \varphi) \times (\varphi^{-1}, 1), \\
C = (0, 1) \times (0, \varphi^{-1}), \quad D = (1, \varphi) \times (0, \varphi^{-1}).
\]

The return time of \(A \cup B\) to \(W\) under \(\tilde{R}_0^{e_2}\) is 4. The return time of \(C \cup D\) to \(W\) under \(\tilde{R}_0^{e_2}\) is 5. The first return maps \(\tilde{R}_0^{e_2}|_W\) is defined as exchanges of rectangles \(\{A, B, C, D\}\) which, from Lemma 4.3 is equivalent to a toral translation by \((\varphi^{-1}, \varphi^{-2})\) on \(\mathbb{R}^2/\Gamma_1\).

The first return map \(\tilde{R}_0^{e_2}|_W\) is defined as an exchange of two rectangles, see Figure 8 which, from Lemma 4.3 is equivalent to a toral translation by \(e_1\) on \(\mathbb{R}^2/\Gamma_1\). Thus, we may identify \(W\) with the torus \(\mathbb{R}^2/\Gamma_1\) and define the induced \(\mathbb{Z}^2\)-action as toral rotations on \(\mathbb{R}^2/\Gamma_1\):

\[
(\tilde{R}_0|_W)^n(x) = x + n_1 e_1 + n_2(\varphi^{-1}, \varphi^{-2}) \mod \Gamma_1.
\]

We observe that \(R_1 = \tilde{R}_0|_W\).
Figure 7. The set \( \{A, B, C, D\} \) forms a partition of the rectangle \( W = (0, \varphi) \times (0, 1) \). The return time to \( W \) under the map \( R_0^{e_2} \) is 4 or 5. The orbit of \( A, B, C \) and \( D \) under \( R_0^{e_2} \) before it returns to \( W \) yields a partitions of \( \mathbb{R}^2/\Gamma_0 \). The first return map \( R_0^{e_2}|_W \) is equivalent to a toral translation by the vector \((\frac{1}{\varphi}, \frac{1}{\varphi^2})\) on \( \mathbb{R}^2/\Gamma_1 \).

Figure 8. The return time to \( W \) under map \( R_0^{e_1} \) is always 1. The first return map \( R_0^{e_1}|_W \) is equivalent to a toral translation by the vector \((1, 0)\) on \( \mathbb{R}^2/\Gamma_1 \).

Thus we have that the return word function \( \text{ReturnWord} : W \to [0, 10]^{*2} \) is

\[
\begin{cases}
\text{CODE}_0(R_0^{(0,3)}(x)) & \text{if } x \in A \cup B \\
\text{CODE}_0(R_0^{(0,2)}(x)) & \text{if } x \in B \cup C \\
\text{CODE}_0(R_0^{(0,1)}(x)) & \text{if } x \in C \cup D \\
\text{CODE}_0(R_0^{(0,0)}(x)) & \text{if } x \in D \\
\end{cases}
\]
We get the set of return words $L = \{w_b \}_{b \in [0,27]} = \text{returnWord}(W)$ as listed in (6). The partition induced by $R_0$ on $W$ is

$$P_1 := \hat{P}_0|_W = \{\text{returnWord}^{-1}(w_b)\}_{b \in [0,27]}$$

which is a topological partition of $W$ made of 28 atoms (the atoms $\text{returnWord}^{-1}(w_{19})$ and $\text{returnWord}^{-1}(w_{22})$ are both the union of two triangles), see Figure 9.

![Partition P1](image)

**Figure 9.** The partition $P_1 := \hat{P}_0|_W$ of $\mathbb{R}^2/\Gamma_1$ into 30 convex atoms each associated to one of the 28 letters in $A_1$ (indices 19 and 22 are both used twice).

The induced coding is

$$\text{Code}_1 := \text{Code}_0|_W : W \rightarrow [0,27]$$

$$y \mapsto b \quad \text{if and only if} \quad y \in \text{returnWord}^{-1}(w_b).$$

A natural substitution comes out of this construction:

$$\beta_0 : [0,27] \rightarrow [0,10]^*$$

$$b \mapsto w_b$$

which corresponds to the one defined in Equation (6). From Proposition 7.3 we have $X_{P_0,R_0} = \beta_0(X_{P_1,R_1})^\sigma$ which ends the proof.

**Remark 10.2 (SageMath Code).** The following allows to reproduce the proof of Proposition 10.1 using SageMath with the optional package `slabbe`. First we construct the golden mean as a element of a quadratic number field because it is more efficient for arithmetic operations and comparisons:

```sage
sage: z = polygen(QQ, "z")
sage: K = NumberField(z**2-z-1, "phi", embedding=RR(1.6))
sage: phi = K.gen()
```

We import the polygon partition $P_0$ of $\mathbb{R}^2/\Gamma_0$ which is predefined in slabbe:

```sage
sage: from slabbe.arXiv_1903_06137 import jeandel_rao_wang_shift_partition
sage: P0 = jeandel_rao_wang_shift_partition()
```

We import polyhedron exchange transformations from the package:

```sage
sage: from slabbe import PolyhedronExchangeTransformation as PET
```

We define the lattice $\Gamma_0$ and the maps $R_0^1$, $R_0^2$ which can be seen as a polygon exchange transformations on a rectangular fundamental domain of $\mathbb{R}^2/\Gamma_0$:

```sage
sage: Gamma0 = matrix.column([(phi,0), (1,phi+3)])
sage: fundamental_domain = polytopes.parallelotope([(phi,0), (0,phi+3)])
sage: R0e1 = PET.toral_translation(Gamma0, vector((1,0)), fundamental_domain)
sage: R0e2 = PET.toral_translation(Gamma0, vector((0,1)), fundamental_domain)
```
We compute the induced partition $P_1$ of $\mathbb{R}^2/\Gamma_1$, the substitution $\beta_0$ and the induced $\mathbb{Z}^2$-action $R_1$:

sage: y_le_1 = [1, 0, -1]  # syntax for the inequality $y \leq 1$

sage: P1, beta0 = R0e2.induced_partition(y_le_1, P0, substitution_type="column")

sage: R1e1, _ = R0e1.induced_transformation(y_le_1)

sage: R1e2, _ = R0e2.induced_transformation(y_le_1)

Remark 10.3 (SageMath Code). To observe and verify the computed induced partitions, substitutions and induced $\mathbb{Z}^2$-action, one can do:

sage: P1.plot()  # or P1.tikz().pdf()

sage: show(beta0)

sage: R1e1

Polyhedron Exchange Transformation of Polyhedron partition of 2 atoms with 2 letters
with translations {0: (1, 0), 1: (-phi + 1, 0)}

sage: R1e2

Polyhedron Exchange Transformation of Polyhedron partition of 4 atoms with 4 letters
with translations {0: (phi - 1, -phi + 1), 1: (-1, -phi + 1),
2: (phi - 1, -phi + 2), 3: (-1, -phi + 2)}

sage: R1e2.plot()

11. Changing the base of the $\mathbb{Z}^2$-action to get $X_{P_2, R_2}$

Recall that $\Gamma_1 = \varphi \mathbb{Z} \times \mathbb{Z}$ and that the coding on the torus $\mathbb{R}^2/\Gamma_1$ is given by the $\mathbb{Z}^2$-action $R_1$ of $(n, x) \mapsto x + n_1 e_1 + n_2 (\varphi^{-1}, \varphi^{-2})$. We remark that $R_1^{e_1}$ is a horizontal translation while $R_1^{e_2}$ is not. But we observe that $R_1^{e_1+e_2} = R_1^{e_1} \circ R_1^{e_2}$ is a vertical translation $x \mapsto x + (0, \varphi^{-2})$. So it is natural to change the base of the $\mathbb{Z}^2$-action $R_1$ by using $R_1^{e_1+e_2}$ instead of $R_1^{e_2}$. Therefore we let

$$R_2^{(n_1,n_2)}(x) = R_1^{(n_1+n_2,n_2)}(x) = x + n_1 e_1 + \varphi^{-2} n_2 e_2$$

for every $n = (n_1, n_2) \in \mathbb{Z}^2$ which defines an action $\mathbb{Z}^2 \times \mathbb{R}^2/\Gamma_2 \rightarrow \mathbb{R}^2/\Gamma_2$ where $\Gamma_2 = \Gamma_1$. We keep the partition as is, that is $A_2 = A_1$, $P_2 = P_1$ and $\text{CODE}_2 = \text{CODE}_1$. We get the setup illustrated in Figure 10. We show in this section that this base change corresponds to a shear conjugacy $\beta_1 : X_{P_2, R_2} \rightarrow X_{P_1, R_1}$.

Remark 11.1 (SageMath Code). The following defines $P_2$, $R_2^{e_1}$ and $R_2^{e_2}$ in SageMath:

sage: P2 = P1

sage: R2e1 = R1e1

sage: R2e2 = (R1e1 * R1e2).merge_atoms_with_same_translation()

In general we have the following result.
Lemma 11.2. Let $A$ be some finite alphabet. For every $M \in \text{GL}_d(\mathbb{Z})$, the map

$$\theta_M : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$$

$$w \mapsto (n \mapsto w(M^{-1}n))$$

is a $\text{GL}_d(\mathbb{Z})$-conjugacy of the full shift $(A^{\mathbb{Z}^d}, \mathbb{Z}^d, \sigma)$ satisfying $\sigma^{Mk} \circ \theta_M = \theta_M \circ \sigma^k$ for all $k \in \mathbb{Z}^d$.

Proof. The map $\theta_M$ is continuous and admits an inverse $\theta_{M^{-1}}$ which is also continuous. If $w \in A^{\mathbb{Z}^d}$ and $k, n \in \mathbb{Z}^d$, then

$$\left[\sigma^{Mk} \circ \theta_M(w)\right](n) = \left[\theta_M(w)\right](n + Mk) = w(M^{-1}(n + Mk)) = w(M^{-1}n + k)$$

$$= \left[\sigma^k w\right](M^{-1}n) = \left[\theta_M(\sigma^k w)\right](n) = \left[\theta_M \circ \sigma^k(w)\right](n).$$

Hence $\theta_M$ is a $\text{GL}_d(\mathbb{Z})$-conjugacy of $(A^{\mathbb{Z}^d}, \mathbb{Z}^d, \sigma)$. \qed

Let $\beta_1 = \theta_M : A_2^{\mathbb{Z}^2} \to A_1^{\mathbb{Z}^2}$ be the map defined by the shear matrix $M = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ where $A_1 = A_2 = [0, 27]$.

Lemma 11.3. The map $\beta_1$ is a shear conjugacy $\beta_1 : \mathcal{X}_{p_2, r_2} \to \mathcal{X}_{p_1, r_1}$ satisfying $\sigma^{Mk} \circ \beta_1 = \beta_1 \circ \sigma^k$ for every $k \in \mathbb{Z}^d$ with $M = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Proof. From Lemma 11.2, $\beta_1$ is a shear-conjugacy $(A_2^{\mathbb{Z}^2}, \mathbb{Z}^2, \sigma) \to (A_1^{\mathbb{Z}^2}, \mathbb{Z}^2, \sigma)$ satisfying $\sigma^{Mk} \circ \beta_1 = \beta_1 \circ \sigma^k$ for all $k \in \mathbb{Z}^d$. The restriction of the map $\beta_1$ on the domain $\mathcal{X}_{p_2, r_2}$ is a shear conjugacy onto its image. It remains to show that the image satisfies $\beta_1(\mathcal{X}_{p_2, r_2}) = \mathcal{X}_{p_1, r_1}$.

For every $x \in \mathbb{R}^2/\Gamma_1 = \mathbb{R}^2/\Gamma_2$ and $m, n \in \mathbb{Z}$, we have

$$\beta_1\left(\text{CONFIG}_{x, p_2, r_2}(m, n)\right) = \text{CONFIG}_{x, p_2, r_2}(m - n, n) = \text{CODE}_2(R_2^{m-n,n}(x))$$

$$= \text{CODE}_1(R_1^{m,n}(x)) = \text{CONFIG}_{x, p_1, r_1}(m, n)$$

Therefore

$$\beta_1(\mathcal{X}_{p_2, r_2}) = \beta_1\left(\{\text{CONFIG}_{x, p_2, r_2} \mid x \in \mathbb{R}^2/\Gamma_2 \setminus \Delta_{p_2, r_2}\}\right)$$

$$= \left\{\beta_1\left(\text{CONFIG}_{x, p_2, r_2}\right) \mid x \in \mathbb{R}^2/\Gamma_1 \setminus \Delta_{p_2, r_2}\right\}$$

$$= \{\text{CONFIG}_{x, p_1, r_1} \mid x \in \mathbb{R}^2/\Gamma_1 \setminus \Delta_{p_1, r_1}\} = \mathcal{X}_{p_1, r_1}.$$ 

Thus $\beta_1$ is a shear conjugacy $\mathcal{X}_{p_2, r_2} \to \mathcal{X}_{p_1, r_1}$. \qed

12. Inducing the symbolic dynamical system $\mathcal{X}_{p_2, r_2}$ to get $\mathcal{X}_{p_8, r_8}$ and $\mathcal{X}_{p_{14}, r_{14}}$

In this section, we induce the topological partition $\mathcal{P}_2$ until the process loops. We need six induction steps before obtaining a topological partition $\mathcal{P}_8$ which is self-induced, see Figure 11.

![Figure 11. The substitutive structure of $\mathcal{X}_{p_2, r_2}$](image-url)
Proposition 12.1. For every $i$ with $2 \leq i \leq 9$, there exist a lattice $\Gamma_{i+1}$, an alphabet $\mathcal{A}_{i+1}$, a $\mathbb{Z}^2$-action $R_{i+1}: \mathbb{Z}^2 \times \mathbb{R}^2 / \Gamma_{i+1} \to \mathbb{R}^2 / \Gamma_{i+1}$, a topological partition $P_{i+1}$ of $\mathbb{R}^2 / \Gamma_{i+1}$ and substitutions $\beta_i: \mathcal{A}_{i+1} \to \mathcal{A}_i^2$ such that

$$X_{P_i,R_i} = \beta_i(\mathcal{X}_{P_{i+1},R_{i+1}})''. $$

Moreover, there exist a bijection $\tau: \mathcal{A}_8 \to \mathcal{A}_{10}$ such that $X_{P_{10},R_{10}} = \tau(X_{P_8,R_8})$.

The partition $P_8$ is self-induced and $X_{P_8,R_8}$ is self-similar satisfying $X_{P_8,R_8} = \beta_8 \beta_0 \tau(X_{P_8,R_8})''$ where $\beta_8 \beta_0 \tau$ is an expansive and primitive self-similarity.

The SageMath code allowing to reproduce the proof of Proposition 12.1 that is, to construct the partitions $P_2$, $P_3$, $P_4$, $P_5$, $P_6$, $P_7$, $P_8$, $P_9$, $P_{10}$ and show that $P_8$ and $P_{10}$ are equivalent is embedded in the proof below. To visualize the induced partitions, substitutions and induced PETs that are computed, do as in Remark 10.3.

Proof. We proceed as in the proof of Proposition 10.1 by making use of Proposition 7.3 many times. We use again and again the same induction process by producing at each step a new partition until we reach a partition which is self-induced.

We start with the lattice $\Gamma_2 = \langle (\varphi, 0), (0, 1) \rangle_\mathbb{Z}$, the partition $P_2$, the coding map $\text{CODE}_2: \mathbb{R}^2 / \Gamma_2 \to \mathcal{A}_2$, the alphabet $\mathcal{A}_2 = \{0, 27\}$ and $\mathbb{Z}^2$-action $R_2$ defined on $\mathbb{R}^2 / \Gamma_2$ as in Figure 10.

We consider the window $W_2 = (0, 1) \times (0, 1)$ as a subset of $\mathbb{R}^2 / \Gamma_2$. The action $R_2$ is Cartesian on $W_2$. Thus from Lemma 6.2, $R_3 := \tilde{R}_2|_{W_2}: \mathbb{Z}^2 \times W_2 \to W_2$ is a well-defined $\mathbb{Z}^2$-action. From Lemma 4.3, the $\mathbb{Z}^2$-action $R_3$ can be seen as toral translation on $\mathbb{R}^2 / \Gamma_3$ with $\Gamma_3 = \mathbb{Z}^2$. Let $P_3 = \tilde{P}_2|_{W_2}$ be the induced partition. From Proposition 7.3, then $X_{P_3,R_3} = \beta_2(X_{P_3,R_3})''$. The partition $P_3$, the action $R_3$ and substitution $\beta_2$ are given below with alphabet $\mathcal{A}_3 = \{0, 19\}$.

\[
\beta_2: \mathcal{A}_3 \to \mathcal{A}_2^2
\]

\[
\begin{array}{cccc}
0 & \mapsto (2), & 1 & \mapsto (9), \\
2 & \mapsto (10), & 3 & \mapsto (20), \\
4 & \mapsto (21), & 5 & \mapsto (22), \\
6 & \mapsto (26), & 7 & \mapsto (1, 0), \\
8 & \mapsto (6, 5), & 9 & \mapsto (7, 4), \\
10 & \mapsto (8, 4), & 11 & \mapsto (11, 3), \\
12 & \mapsto (12, 3), & 13 & \mapsto (16, 15), \\
14 & \mapsto (17, 15), & 15 & \mapsto (18, 14), \\
16 & \mapsto (19, 14). & 17 & \mapsto (22, 13, \\
18 & \mapsto (25, 24). & 19 & \mapsto (27, 23). \\
\end{array}
\]

We consider the window $W_3 = (0, \varphi^{-1}) \times (0, 1)$ as a subset of $\mathbb{R}^2 / \Gamma_3$. The action $R_3$ is Cartesian on $W_3$. Thus from Lemma 6.2, $R_4 := \tilde{R}_3|_{W_3}: \mathbb{Z}^2 \times W_3 \to W_3$ is a well-defined $\mathbb{Z}^2$-action. From Lemma 4.3, the $\mathbb{Z}^2$-action $R_4$ can be seen as toral translation on $\mathbb{R}^2 / \Gamma_4$ with $\Gamma_4 = (\varphi^{-1}\mathbb{Z}) \times \mathbb{Z}$. Let $P_4 = \tilde{P}_3|_{W_3}$ be the induced partition. From Proposition 7.3, then $X_{P_3,R_4} = \beta_3(X_{P_3,R_4})''$. The partition $P_4$, the action $R_4$ and substitution $\beta_3$ are given below with alphabet $\mathcal{A}_4 = \{0, 19\}$.
Now we consider the window \( W_4 = (0, \varphi^{-1}) \times (0, \varphi^{-1}) \) as a subset of \( \mathbb{R}^2/\Gamma_4 \). The action \( R_4 \) is Cartesian on \( W_4 \). Thus from Lemma 6.2, \( R_5 := R_4|_{W_4} : \mathbb{Z}^2 \times W_4 \to W_4 \) is a well-defined \( \mathbb{Z}^2 \)-action. From Lemma 4.3, the \( \mathbb{Z}^2 \)-action \( R_5 \) can be seen as toral translation on \( \mathbb{R}^2/\Gamma_5 \) with \( \Gamma_5 = (\varphi^{-1}\mathbb{Z}) \times (\varphi^{-1}\mathbb{Z}) \). Let \( \mathcal{P}_5 = \tilde{\mathcal{P}}_4|_{W_4} \) be the induced partition. From Proposition 7.3, then \( x_{\mathcal{P}_4, R_4} = \beta_4(x_{\mathcal{P}_5, R_5}) \). The partition \( \mathcal{P}_5 \), the action \( R_5 \) and substitution \( \beta_4 \) are given below with alphabet \( \mathcal{A}_5 = \{0, 21\} \):

\[
\beta_3 : \mathcal{A}_4 \to \mathcal{A}_4^2
\]

\[
\beta_4 : \mathcal{A}_5 \to \mathcal{A}_4^2
\]

Now it is appropriate to rescale the partition \( \mathcal{P}_5 \) by the factor \(-\varphi\). Doing so, the new obtained action \( R'_5 \) is the same as two steps before, that is, \( R_3 \) on \( \mathbb{R}^2/\mathbb{Z}^2 \). More formally, let \( h : (\mathbb{R}/\varphi^{-1}\mathbb{Z})^2 \to (\mathbb{R}/\mathbb{Z})^2 \) be the homeomorphism defined by \( h(x) = -\varphi x \). We define \( \mathcal{P}'_5 = h(\mathcal{P}_5) \), \( \text{CODE}'_5 = \text{CODE}_5 \circ h^{-1} \), \( (R'_5)^n = h \circ (R_3)^n \circ h^{-1} \) as shown below:

\[
\beta_3 : \mathcal{A}_4 \to \mathcal{A}_4^2
\]

\[
\beta_4 : \mathcal{A}_5 \to \mathcal{A}_4^2
\]

\[
R'_4(x) = x - \varphi^{-3}n
\]

\[
R'_5(x) = \begin{cases} 
0 \mapsto (3), & 1 \mapsto (4), & 2 \mapsto (5), \\
3 \mapsto (6), & 4 \mapsto (12), & 5 \mapsto (13), \\
6 \mapsto (14), & 7 \mapsto (15), & 8 \mapsto (18), \\
9 \mapsto \left( \begin{array}{c} 0 \\ 4 \end{array} \right), & 10 \mapsto \left( \begin{array}{c} 0 \\ 5 \end{array} \right), & 11 \mapsto \left( \begin{array}{c} 1 \\ 5 \end{array} \right), \\
12 \mapsto \left( \begin{array}{c} 2 \\ 5 \end{array} \right), & 13 \mapsto \left( \begin{array}{c} 0 \\ 6 \end{array} \right), & 14 \mapsto \left( \begin{array}{c} 8 \\ 13 \end{array} \right), \\
15 \mapsto \left( \begin{array}{c} 10 \\ 14 \end{array} \right), & 16 \mapsto \left( \begin{array}{c} 10 \\ 15 \end{array} \right), & 17 \mapsto \left( \begin{array}{c} 11 \\ 16 \end{array} \right), \\
18 \mapsto \left( \begin{array}{c} 9 \\ 17 \end{array} \right), & 19 \mapsto \left( \begin{array}{c} 11 \\ 17 \end{array} \right), & 20 \mapsto \left( \begin{array}{c} 7 \\ 18 \end{array} \right), \\
21 \mapsto \left( \begin{array}{c} 9 \\ 19 \end{array} \right). 
\end{cases}
\]
The action $R$ using the above commutative properties, for every $y \in \mathbb{R}^2/\Gamma_5$ and $m, n \in \mathbb{Z}$, we have

$$\text{CONFIG}_{P_5,R_5}^y(n) = \text{CODE}_5(R_5^m(y)) = \text{CODE}_5^* \circ h(R_5^m(y)) = \text{CONFIG}_{P_5,R_5}^y(n).$$

Thus $X_{P_5,R_5} = X_{P_5,R_5}^*$. Now we consider the window $W_5 = (0, \varphi^{-1}) \times (0, 1)$ as a subset of $\mathbb{R}^2/\mathbb{Z}$.

By construction, the following diagrams commute:

$$(\mathbb{R}/\varphi^{-1}\mathbb{Z})^2 \xrightarrow{h} (\mathbb{R}/\mathbb{Z})^2 \xrightarrow{(R_5^6)^n} (\mathbb{R}/\varphi^{-1}\mathbb{Z})^2 \xrightarrow{h} (\mathbb{R}/\mathbb{Z})^2$$

Using the above commutative properties, for every $y \in \mathbb{R}^2/\Gamma_5$ and $m, n \in \mathbb{Z}$, we have

$$\text{CONFIG}_{P_5,R_5}^y(n) = \text{CODE}_5(R_5^m(y)) = \text{CODE}_5^* \circ h(R_5^m(y)) = \text{CONFIG}_{P_5,R_5}^y(n).$$

Thus $X_{P_5,R_5} = X_{P_5,R_5}^*$. Now we consider the window $W_5 = (0, \varphi^{-1}) \times (0, 1)$ as a subset of $\mathbb{R}^2/\mathbb{Z}$.

The action $R_6^*$ is Cartesian on $W_5$. Thus from Lemma 6.2, $R_6 := \hat{R}_6|_{W_5} : \mathbb{Z}^2 \times W_5 \to W_5$ is a well-defined $\mathbb{Z}^2$-action. From Lemma 4.3, the $\mathbb{Z}^2$-action $R_6$ can be seen as toral translation on $\mathbb{R}^2/\Gamma_6$ with $\Gamma_6 = (\varphi^{-1}\mathbb{Z}) \times \mathbb{Z}$. Let $P_6 = \hat{P}_6|_{W_5}$ be the induced partition. From Proposition 7.3 then $X_{P_5,R_5} = \beta_5(X_{P_5,R_5})$. The partition $P_6$, the action $R_6$ and substitution $\beta_5$ are given below with alphabet $A_6 = \{0, 17\}$.

Now we consider the window $W_6 = (0, \varphi^{-1}) \times (0, \varphi^{-1})$ as a subset of $\mathbb{R}^2/\Gamma_6$. The action $R_6$ is Cartesian on $W_6$. Thus from Lemma 6.2, $R_7 := \hat{R}_6|_{W_6} : \mathbb{Z}^2 \times W_6 \to W_6$ is a well-defined $\mathbb{Z}^2$-action. From Lemma 4.3, the $\mathbb{Z}^2$-action $R_7$ can be seen as toral translation on $\mathbb{R}^2/\Gamma_7$ with $\Gamma_7 = (\varphi^{-1}\mathbb{Z}) \times (\varphi^{-1}\mathbb{Z})$. Let $P_7 = \hat{P}_6|_{W_6}$ be the induced partition. From Proposition 7.3 then
\( \mathcal{X}_{\beta_6, R_6} = \overline{\beta_6(\mathcal{X}_{\beta_6, R_6})^\sigma} \). The partition \( \mathcal{P}_7 \), the action \( R_7 \) and substitution \( \beta_6 \) are given below with alphabet \( \mathcal{A}_7 = [0, 20] \):

```python
sage: P7, beta6 = R6e2.induced_partition(y_le_phi_inv, P6, substitution_type="column")
sage: R7e1, _ = R6e1.induced_transformation(y_le_phi_inv)
sage: P7, beta6 = R6e2.induced_partition(y_le_phi_inv, P6, substitution_type="row")
sage: R7e2, _ = R6e2.induced_transformation(y_le_phi_inv)

\[ \beta_6 : \mathcal{A}_7 \to \mathcal{A}_6^2 \]
\[
\begin{align*}
0 &\mapsto (1), & 1 &\mapsto (2), & 2 &\mapsto (3), \\
3 &\mapsto (12), & 4 &\mapsto (13), & 5 &\mapsto (14), \\
6 &\mapsto (15), & 7 &\mapsto (16), & 8 &\mapsto (17), \\
9 &\mapsto (0, 1), & 10 &\mapsto (0, 2), & 11 &\mapsto (0, 3), \\
12 &\mapsto (8, 9), & 13 &\mapsto (10, 14), & 14 &\mapsto (11), \\
15 &\mapsto (6, 12), & 16 &\mapsto (5, 13), & 17 &\mapsto (8), \\
18 &\mapsto (7, 14), & 19 &\mapsto (5, 15), & 20 &\mapsto (7, 17).
\end{align*}
\]

Again it is appropriate to rescale the partition \( \mathcal{P}_7 \) by the factor \(-\phi\). We use the homeomorphism \( h : (\mathbb{R}/\phi^{-1}\mathbb{Z})^2 \to (\mathbb{R}/\mathbb{Z})^2 \) defined previously by \( h(x) = -\phi x \). We define \( \mathcal{P}'_7 = h(\mathcal{P}_7) \), \( \text{CODE}'_7 = \text{CODE}_7 \circ h^{-1}, (R'_7)^n = h \circ (R_7)^n \circ h^{-1} \) as shown below:

```python
sage: P7_scaled = (-phi*P7).translate((1,1))
sage: R7e1_scaled = (-phi*R7e1).translate_domain((1,1))
sage: P7_scaled = (-phi*P7).translate_domain((1,1))
sage: R7e2_scaled = (-phi*R7e2).translate_domain((1,1))
```

\[ (R'_7)^n(x) = x + \phi^{-2}n \]

Using the above commutative properties, for every \( y \in \mathbb{R}^2/\Gamma_7 \) and \( m, n \in \mathbb{Z} \), we have

\[ \text{CONFIG}_{\mathcal{P}_7, R_7}(\mathcal{N}) = \text{CODE}_7(R_7^n(y)) = \text{CODE}'_7 \circ h(R_7^n(y)) = \text{CODE}'_7 \circ (R'_7)^n(h(y)) = \text{CONFIG}_{\mathcal{P}_7', R'_7}(\mathcal{N}). \]

Thus \( \mathcal{X}_{\beta_7, R_7} = \mathcal{X}_{\beta_7', R'_7} \). Now we consider the window \( W_7 = (0, \phi^{-1}) \times (0, 1) \) as a subset of \( \mathbb{R}^2/\mathbb{Z}^2 \). The action \( R'_7 \) is Cartesian on \( W_7 \). Thus from Lemma 6.2, \( R_8 := \overline{\mathcal{P}_7|_{W_7}} : \mathbb{Z}^2 \times W_7 \to W_7 \) is a well-defined \( \mathbb{Z}^2 \)-action. From Lemma 4.3, the \( \mathbb{Z}^2 \)-action \( R_8 \) can be seen as toral translation on \( \mathbb{R}^2/\Gamma_8 \) with \( \Gamma_8 = (\phi^{-1}\mathbb{Z}) \times \mathbb{Z} \). Let \( \mathcal{P}_8 = \overline{\mathcal{P}_7|_{W_7}} \) be the induced partition. From Proposition 7.3 then \( \mathcal{X}_{\beta_7, R_7} = \overline{\beta_7(\mathcal{X}_{\beta_7, R_7})^\sigma} \). The partition \( \mathcal{P}_8 \), the action \( R_8 \) and substitution \( \beta_7 \) are given below with alphabet \( \mathcal{A}_8 = [0, 18] \):

```python
sage: P8, beta7 = R7e1_scaled.induced_partition(x_le_phi_inv, P7_scaled, substitution_type="row")
sage: R8e1, _ = R7e1_scaled.induced_transformation(x_le_phi_inv)
sage: R8e2, _ = R7e2_scaled.induced_transformation(x_le_phi_inv)
```
We need two more steps before the induction procedure loops. We obtain the partitions and substitutions below.

```
sage: P9, beta8 = R8e2.induced_partition(y_le_phi_inv, P8, substitution_type="column")
sage: R9e1, _ = R8e1.induced_transformation(y_le_phi_inv)
sage: R9e2, _ = R8e2.induced_transformation(y_le_phi_inv)
```

\[ R_9^n(x) = x + (-\varphi^{-3},0)n_1 + (0,\varphi^{-2})n_2 \]

```
sage: P9_scaled = (-phi*P9).translate((1,1))
sage: R9e1_scaled = (-phi*R9e1).translate_domain((1,1))
sage: R9e2_scaled = (-phi*R9e2).translate_domain((1,1))
```

\[ \beta_7 : A_8 \to A_8^2 \]
\[ 0 \mapsto (5), \quad 1 \mapsto (8), \]
\[ 2 \mapsto (14), \quad 3 \mapsto (15), \]
\[ 4 \mapsto (18), \quad 5 \mapsto (20), \]
\[ 6 \mapsto (3,1), \quad 7 \mapsto (4,2), \]
\[ 8 \mapsto (5,1), \quad 9 \mapsto (6,0), \]
\[ 10 \mapsto (7,1), \quad 11 \mapsto (8,1), \]
\[ 12 \mapsto (12,11), 13 \mapsto (13,11), \]
\[ 14 \mapsto (14,9), \quad 15 \mapsto (15,10), \]
\[ 16 \mapsto (16,11), 17 \mapsto (17,11), \]
\[ 18 \mapsto (19,9). \]

```
sage: P10, beta9 = R9e1_scaled.induced_partition(x_le_phi_inv, P9_scaled, substitution_type="row")
sage: R10e1, _ = R9e1_scaled.induced_transformation(x_le_phi_inv)
sage: R10e2, _ = R9e2_scaled.induced_transformation(x_le_phi_inv)
```

\[ (R'_9)^n(x) = x + \varphi^{-2}n \]
We may observe that $R_{10} = R_8$ and the partition $\mathcal{P}_{10}$ is the same as $\mathcal{P}_8$ up to a permutation $\tau$ of the indices of the atoms:

\[
\begin{align*}
sage: \ P8.is_equal_up_to_relabeling(P10) \\
True
sage: from slabbe import Substitution2d
sage: P8.is_equal_up_to_relabeling(P10)

\tau : A_8 \rightarrow A_{10} \\
\begin{cases}
0 \mapsto (1), & 1 \mapsto (0), & 2 \mapsto (4), & 3 \mapsto (3), & 4 \mapsto (5), & 5 \mapsto (2), & 6 \mapsto (10), \\
7 \mapsto (9), & 8 \mapsto (11), & 9 \mapsto (8), & 10 \mapsto (7), & 11 \mapsto (6), & 12 \mapsto (15), & 13 \mapsto (18), \\
14 \mapsto (17), & 15 \mapsto (16), & 16 \mapsto (13), & 17 \mapsto (14), & 18 \mapsto (12).
\end{cases}
\]

Thus the induction procedure loops as we have

\[
\mathcal{X}_{\mathcal{P}_{8}, R_8} = \beta_8(\mathcal{X}_{\mathcal{P}_{8}, R_8})^\sigma = \beta_8 \beta_9(\mathcal{X}_{\mathcal{P}_{10}, R_{10}})^\sigma = \overline{\beta_8 \beta_9 \tau(\mathcal{X}_{\mathcal{P}_{8}, R_8})^\sigma}
\]

The self-similarity $\beta_8 \beta_9 \tau$ is

\[
sage: show(beta8*beta9*tau)
\]

\[
\beta_8 \beta_9 \tau : A_8 \rightarrow A_{8}^1 \\
\begin{cases}
0 \mapsto (17), & 1 \mapsto (12), & 2 \mapsto \left(\frac{10}{16}\right), & 3 \mapsto \left(\frac{9}{16}\right), & 4 \mapsto \left(\frac{7}{17}\right), \\
5 \mapsto \left(\frac{12}{17}\right), & 6 \mapsto \left(\frac{16}{2}\right), & 7 \mapsto \left(\frac{14}{4}\right), & 8 \mapsto \left(\frac{17}{2}\right), & 9 \mapsto \left(\frac{13}{3}\right), \\
10 \mapsto \left(\frac{13}{2}\right), & 11 \mapsto \left(\frac{12}{2}\right), & 12 \mapsto \left(\frac{11}{15}\right), & 13 \mapsto \left(\frac{10}{18}\right), & 14 \mapsto \left(\frac{10}{16}\right), \\
15 \mapsto \left(\frac{9}{14}\right), & 16 \mapsto \left(\frac{6}{14}\right), & 17 \mapsto \left(\frac{8}{14}\right), & 18 \mapsto \left(\frac{6}{13}\right).
\end{cases}
\]

which is primitive and expansive. □

We now want to make the link between the partition $\mathcal{P}_8$ and dynamical system $\mathcal{X}_{\mathcal{P}_8,R_8}$ computed above from the induction process and the partition $\mathcal{P}_U$ and dynamical system $\mathcal{X}_{\mathcal{P}_U,R_U}$ that was introduced in \textup{[Lab21a]}. Recall that on the torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$, the dynamical system $(\mathbb{T}^2, \mathbb{Z}^2, R_U)$ was defined by the $\mathbb{Z}^2$-action

\[
R_U : \mathbb{Z}^2 \times \mathbb{T}^2 \rightarrow \mathbb{T}^2 \\
(n, x) \mapsto x + \varphi^{-2}n
\]

where $\varphi = \frac{1 + \sqrt{5}}{2}$. In \textup{[Lab21a]}, we proved that $\mathcal{P}_U = \{P_a\}_{a \in [0,18]}$ is a Markov partition for the dynamical system $(\mathbb{T}^2, \mathbb{Z}^2, R_U)$. It is illustrated in Figure \textup{[12]}. 

Lemma 12.2. \( \mathcal{X}_{\mathcal{P}_x,R_s} \) and \( \mathcal{X}_{\mathcal{P}_y,R_t} \) are topologically conjugate. More precisely, \( \mathcal{X}_{\mathcal{P}_x,R_s} = \zeta(\mathcal{X}_{\mathcal{P}_y,R_t}) \)
where \( \zeta : [0,18]^2 \to [0,18]^2 \) is the 2-dimensional morphism defined by the following letter-to-letter bijection:

\[
\zeta : \begin{cases} 
0 \mapsto 0, & 1 \mapsto 1, & 2 \mapsto 9, & 3 \mapsto 7, & 4 \mapsto 8, \\
5 \mapsto 11, & 6 \mapsto 10, & 7 \mapsto 6, & 8 \mapsto 2, & 9 \mapsto 4, \\
10 \mapsto 5, & 11 \mapsto 3, & 12 \mapsto 18, & 13 \mapsto 14, & 14 \mapsto 16, \\
15 \mapsto 13, & 16 \mapsto 12, & 17 \mapsto 17, & 18 \mapsto 15.
\end{cases}
\]

Proof. The topological partition \( \mathcal{P}_d \) defines a coding map \( \text{CODE}_{\mathcal{P}_d} : \mathbb{T}^2 \to [0,18] \). We have \( R_n^{\alpha}(x) = x + \varphi^{-2}n \). Let \( g : \mathbb{R}^2/\Gamma_8 \to \mathbb{T}^2 \) be the homeomorphism defined by \( (x,y) \mapsto (x,y) \). We observe that \( g(\mathcal{P}_d) \) is equal to \( \mathcal{P}_d \) after permuting the coding alphabet of \( \mathcal{P}_d \) by \( \zeta \). In other words, \( \text{CODE}_8 = \zeta \circ \text{CODE}_{\mathcal{P}_d} \circ g \). Also \( g \circ R_n^\alpha = R_n^\alpha \circ g \). Thus for every \( x \in \mathbb{R}^2/\Gamma_8 \) and \( n \in \mathbb{Z}^2 \), we have

\[
\text{CONFIG}_{\mathcal{P}_x,R_s}(n) = \text{CODE}_8 \circ R_n^\alpha(x) = \zeta \circ \text{CODE}_{\mathcal{P}_d} \circ g \circ R_n^\alpha(x) = \zeta(\text{CONFIG}_{\mathcal{P}_y,R_t}(n)).
\]

Thus we conclude \( \mathcal{X}_{\mathcal{P}_x,R_s} = \zeta(\mathcal{X}_{\mathcal{P}_y,R_t}) \).

\[ \square \]

13. PROOF OF MAIN RESULTS

We may now complete the proof of the substitutive structure of \( \mathcal{X}_{\mathcal{P}_0,R_0} \).

Proof of Theorem 1.2. (i) The existence of \( \beta_0 \) was proved in Proposition 10.1. (ii) The existence of the shear conjugacy \( \beta_1 \) was proved in Lemma 11.3. (iii) The existence of \( \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \) and \( \beta_7 \) was proved in Proposition 12.1. (iv) The existence of \( \beta_8, \beta_9 \) and \( \tau \) was proved in Proposition 12.1. (v) The topological conjugacy \( \zeta : \mathcal{X}_{\mathcal{P}_d,R_t} \to \mathcal{X}_{\mathcal{P}_x,R_s} \) was proved in Lemma 12.2. \[ \square \]

For comparison with the statement of Theorem 1.2 we now recall the main result proved in \text{[Lab21b]} about the substitutive structure of the minimal subshift \( X_0 \subset \Omega_0 \) of the Jeandel-Rao Wang shift.

Theorem 13.1. \text{[Lab21b]} Let \( \Omega_0 \) be the Jeandel-Rao Wang shift. There exist sets of Wang tiles \( \{\mathcal{T}_i\}_{1 \leq i \leq 12} \) together with their associated Wang shifts \( \{\Omega_i\}_{1 \leq i \leq 12} \) that provide the substitutive structure of Jeandel-Rao Wang shift. More precisely,

(i) There exists a sequence of recognizable 2-dimensional morphisms:

\[
\Omega_0 \xrightarrow{\omega_0} \Omega_1 \xrightarrow{\omega_1} \Omega_2 \xrightarrow{\omega_2} \Omega_3 \xrightarrow{\omega_3} \Omega_4
\]

that are onto up to a shift, i.e., \( \overline{\omega_i(\Omega_{i+1})} = \Omega_i \) for each \( i \in \{0, 1, 2, 3\} \).
(ii) There exists an embedding \( j : \Omega_5 \to \Omega_4 \) which is a topological conjugacy onto its image.

(iii) There exists a shear conjugacy \( \eta : \Omega_6 \to \Omega_5 \) which shears configurations by the action of the matrix \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

(iv) There exists a sequence of recognizable 2-dimensional morphisms:

\[
\begin{align*}
\Omega_6 & \xrightarrow{\omega_6} \Omega_7 & \xrightarrow{\omega_7} \Omega_8 & \xrightarrow{\omega_8} \Omega_9 & \xrightarrow{\omega_9} \Omega_{10} & \xrightarrow{\omega_{10}} \Omega_{11} & \xrightarrow{\omega_{11}} \Omega_{12} \\
\end{align*}
\]

that are onto up to a shift, i.e., \( \omega_i(\Omega_{i+1}) = \Omega_i \) for each \( i \in \{6, 7, 8, 9, 10, 11\} \).

(v) The Wang shift \( \Omega_{12} \) is equivalent to \( \Omega_U \) for some topological conjugacy \( \rho : \Omega_U \to \Omega_{12} \):

\[
\Omega_{12} \leftarrow \rho \Omega_U
\]

thus \( \Omega_{12} \) is self-similar, aperiodic and minimal.

(vi) \( X_0 = \omega_0 \omega_1 \omega_2 \omega_3 \beta(\Omega_5) \subseteq \Omega_0 \) is an aperiodic minimal subshift of Jeandel-Rao Wang shift \( \Omega_0 \).

The Wang shift \( \Omega_U \) given by a set of 19 Wang tiles describing the internal self-similar structure hidden in the Jeandel-Rao Wang shift was studied separately in \Lab{19}.

**Theorem 13.2.** \Lab{19} The Wang shift \( \Omega_U \) is aperiodic, minimal and self-similar satisfying \( \Omega_U = \omega_U(\Omega_U)^\sigma = \chi_{\omega_U} \) for some primitive and expansive 2-dimensional morphism \( \omega_U \).

The substitutions \( \omega_i, j \) and \( \eta \) were computed from algorithms on Wang tiles based on marker tiles. Surprisingly, they are the same as the substitutions constructed here in previous sections using 2-dimensional Rauzy induction on toral partitions! Thus, we can now prove that the symbolic dynamical system \( X_{P_{u_0}, R_0} \) and the subshift \( X_0 \subset \Omega_0 \) have the same substitutive structure.

**Proof of Theorem 13.3.** The reader may check that the equality \( \beta_0 = \omega_0 \omega_1 \omega_2 \omega_3 \) holds by comparing \( \beta_0 \) computed in Proposition 10.1 with the product \( \omega_0 \omega_1 \omega_2 \omega_3 \) shown in \Lab{21} end of §5.

The maps \( \eta : \Omega_6 \to \Omega_5 \) and \( \beta_1 : \chi_{P_2, R_2} \to \chi_{P_1, R_1} \) are shear conjugacies both shearing configurations by the action of the matrix \( M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). The reader may check that the effect of the map \( \beta_1 \beta_2 \) on the alphabet \( A_3 = [0, 19] \) is the same as \( j \eta \omega_6 \).

The reader may check that the equalities \( \beta_3 = \omega_7, \beta_4 = \omega_8, \beta_5 = \omega_9, \beta_6 = \omega_{10} \) and \( \beta_7 = \omega_{11} \) hold by comparing the \( \beta_i \) computed in Proposition 12.1 with the \( \omega_{i+4} \) computed in \Lab{21} Prop. 8.1.

The reader may verify the equality \( \zeta = \rho \beta_8 \beta_9 \tau \) by comparing \( \zeta \) computed in Lemma 12.2 with \( \rho \) computed in \Lab{21} Prop. 9.1. Both maps are 2-dimensional morphism defined by a permutation of the alphabet.

To prove that \( \beta_8 \beta_9 \tau = \rho \omega_U \rho^{-1} \) the reader may compare the product \( \beta_8 \beta_9 \tau \) shown at the end of proof of Proposition 12.1 but unfortunately the product \( \rho \omega_U \rho^{-1} \) is not computed explicitly in \Lab{21}. An alternative is to compute \( \rho^{-1} \beta_8 \beta_9 \tau \rho = \zeta^{-1} \beta_8 \beta_9 \tau \zeta \) and prove that it is equal to \( \omega_U \). We obtain

\[
\zeta^{-1} \beta_8 \beta_9 \tau \zeta : [0, 18] \to [0, 18]^{\ast 2}
\]

\[
\begin{align*}
0 \mapsto (17), & \quad 1 \mapsto (16), & \quad 2 \mapsto (15, 11), & \quad 3 \mapsto (13, 9), & \quad 4 \mapsto (17, 8), \\
5 \mapsto (16, 8), & \quad 6 \mapsto (15, 8), & \quad 7 \mapsto (14, 8), & \quad 8 \mapsto \left( \begin{array}{c} 6 \\ 14 \end{array} \right), & \quad 9 \mapsto \left( \begin{array}{c} 3 \\ 17 \end{array} \right), \\
10 \mapsto \left( \begin{array}{c} 6 \\ 1 \\ 12 \\ 9 \end{array} \right), & \quad 11 \mapsto \left( \begin{array}{c} 2 \\ 14 \\ 15 \\ 11 \end{array} \right), & \quad 12 \mapsto \left( \begin{array}{c} 7 \\ 1 \\ 11 \\ 14 \\ 9 \\ 13 \end{array} \right), & \quad 13 \mapsto \left( \begin{array}{c} 6 \\ 1 \\ 14 \\ 11 \end{array} \right), & \quad 14 \mapsto \left( \begin{array}{c} 7 \\ 1 \\ 13 \\ 9 \end{array} \right), \\
15 \mapsto \left( \begin{array}{c} 6 \\ 1 \\ 12 \\ 9 \end{array} \right), & \quad 16 \mapsto \left( \begin{array}{c} 5 \\ 1 \\ 18 \\ 10 \end{array} \right), & \quad 17 \mapsto \left( \begin{array}{c} 4 \\ 1 \\ 13 \\ 9 \end{array} \right), & \quad 18 \mapsto \left( \begin{array}{c} 2 \\ 0 \\ 14 \end{array} \right) \\
\end{align*}
\]

which is equal to the 2-dimensional morphism \( \omega_U \) computed in \Lab{19}. \( \square \)
Similarly for intermediate subshifts, we have proved that $X_{\mathcal{P}_u,R_t}$ is self-similar satisfying $X_{\mathcal{P}_u,R_t} = \beta_8 \beta_9 \tau (X_{\mathcal{P}_u,R_t})^\sigma$, where the product $\beta_8 \beta_9 \tau$ is an expansive and primitive self-similarity and $X_{\mathcal{P}_u,R_t} = \zeta (X_{\mathcal{P}_u,R_t})$ where $\zeta$ is a topological conjugacy. Therefore,

$$X_{\mathcal{P}_u,R_t} = \beta_8 \beta_9 \tau (X_{\mathcal{P}_u,R_t})^\sigma = \omega_{\mathcal{T}} (X_{\mathcal{P}_u,R_t})^\sigma.$$

As a consequence, the symbolic dynamical system $X_{\mathcal{P}_u,R_t}$ contains the substitutive subshift $X_{\mathcal{P}_u,R_t}$. From Lemma 5.3, $X_{\mathcal{P}_u,R_t}$ is minimal from which we conclude that $X_{\mathcal{P}_u,R_t} = X_{\mathcal{P}_u,R_t}$. From Theorem 13.2, we have $\Omega_t = \omega_{\mathcal{T}} (\Omega_t)^\sigma = X_{\mathcal{P}_u,R_t}$ so that $X_{\mathcal{P}_u,R_t} = X_{\mathcal{P}_u,R_t} = \Omega_t$. From Theorem 13.1, we have $\rho (\Omega_t) = \Omega_{12}$. From Theorem 13.3, the equality $\rho = \zeta$ holds. Thus

$$X_{\mathcal{P}_u,R_t} = \zeta (X_{\mathcal{P}_u,R_t}) = \rho (\Omega_t) = \Omega_{12}.$$

From Theorem 13.2, $X_{\mathcal{P}_u,R_t} = \beta_8 (X_{\mathcal{P}_{i+1},R_{i+1}}) = \beta_8 (X_{\mathcal{P}_{i+1},R_{i+1}})$ for each $i \in \{0, 1, \ldots, 7\}$. Thus using Theorem 13.1 and Theorem 13.3, we obtain

$$X_{\mathcal{P}_u,R_t} = \beta_8 (X_{\mathcal{P}_{i+1},R_{i+1}}) = \beta_8 (X_{\mathcal{P}_{i+1},R_{i+1}}),$$

which ends the proof.

We may now prove the main result.

Proof of Theorem 1.1. In Corollary 1.5, we proved that $X_{\mathcal{P}_0,R_0} = X_0$. In [Lab21b, Lemma 6.2], we proved that $X_0$ is a shift of finite type. In [Lab21a], we proved that the partition $\mathcal{P}_0$ gives a symbolic representation of $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$. Thus $\mathcal{P}_0$ is a Markov partition for the dynamical system $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$.

Proof of Corollary 1.6. The same statement was proved in [Lab21a] for $X_{\mathcal{P}_0,R_0}$ and we proved in Corollary 1.5 that $X_0 = X_{\mathcal{P}_0,R_0}$. 

\square
Conclusion

14. Concluding remarks

This concludes a study of an aperiodic subshift of finite type defined by 11 Wang tiles which was discovered by Jeandel and Rao [JR21]. Jeandel-Rao proved that 11 is minimal as Wang shifts defined with fewer Wang tiles are either empty or contain a periodic configuration. Our work ended up being split into four articles in a publication ordering which does not correspond to the order in which the discoveries were made. Before discussing possible extensions, it may be worth to take a step back and explain the research process leading to this work and the relations between the articles.

Indeed, the partition $P_0$ and $\mathbb{Z}^2$-rotation $R_0$ shown in Figure 3 discovered after few months of experimentations in 2017 and published in [Lab21a] was the starting point. Surprisingly, the weird polygonal partition $P_0$ and toral $\mathbb{Z}^2$-rotations $R_0$ allows to easily build valid configurations with the 11 Jeandel-Rao tiles, i.e, $X_{P_0,R_0} \subseteq \Omega_0$. The next question was whether $\Omega_0 \subseteq X_{P_0,R_0}$. To answer this question, it was natural to study the substitutive structure of the symbolic dynamical system $X_{P_0,R_0}$ using a higher-dimensional version of Rauzy induction as done in the current article through computations of the $\beta_i$’s. Then, we needed to check whether the Wang shift $\Omega_0$ also has the same substitutive structure given by the $\beta_i$’s, including the shear-conjugacy $\beta_1$. Polygonal partitions and the $\beta_i$’s are not mentioned in [Lab21b], but they served as beacons for the author to describe the substitutive structure of the Jeandel-Rao Wang shift independently of the polygonal partitions and uniquely from the Wang tiles themselves and the desubstitution of Wang shifts with the notion of marker tiles. This process leads to the self-similar structure hidden in the Jeandel-Rao Wang shift which was considered separately as a first step [Lab19].

The idea of writing the substitutions in a canonical way independent of the partitions and of the Wang tiles from which they are computed, as mentioned in Remark 1.4 and Definition 7.1, came after the publication of [lab19]. This is why the bijections $\rho$ and $\zeta$ are not the identity map because the ordering of the tiles defining $\Omega_U$ was chosen according to another convention.

It turns out that $\Omega_0 \setminus X_{P_0,R_0}$ is non-empty due to the existence of horizontal fault lines in some configurations of $\Omega_0$ ($X_{P_0,R_0}$ is minimal, but $\Omega_0$ is not). We believe that the difference is a null set, see Conjecture 1.7. Some time was spent on the question with Jennifer McLoud-Mann and Casey Mann during their sabbatical year 2019–2020 in Bordeaux, but unfortunately, proving it seems a challenge beyond reach.

Some open questions. This article and the three others were dedicated to the study of a single Wang shift. The next step is to find other examples or even families of examples hoping that the tools developed here will simplify their description. It seems reasonable that there is a characterization of the toral $\mathbb{Z}^2$-rotations which admit symbolic dynamical systems that are subshifts of finite type or more generally sofic subshifts. In the spirit of $\beta$-expansion of real numbers in real bases, the condition could be expressed algebraically involving for instance Parry numbers, see [BR10, Theorem 2.3.15].

Of course, some impossibility results are expected. Since there are only countably many $\mathbb{Z}^2$-SFTs, we can not encode all toral $\mathbb{Z}^2$-rotations into SFTs. For instance, a $\mathbb{Z}^2$-rotation given by non-computable real numbers can not be encoded into a finite set of Wang tiles.

Observe that the $19 \times 19$ incidence matrix of $\beta_8 \beta_9 \tau$ is not hyperbolic but, as shown by its characteristic polynomial, it is hyperbolic on a 8-dimensional subspace:

```sage
sage: (beta8*beta9*tau).incidence_matrix().charpoly().factor()
x^3 * (x - 1)^4 * (x + 1)^4 * (x^2 - 3*x + 1) * (x^2 + x - 1)^3
```

One question is whether the polygonal Markov partition $P_U$ for the $\mathbb{Z}^2$-rotation $R_U$ on $\mathbb{T}^2$ is related or is the projection of some 8-dimensional Markov partition of the hyperbolic automorphism on
\( \mathbb{T}^3 \) associated to the restriction of the action of the incidence matrix of the self-similarity \( \omega_U \) to a subspace.

It is known from the work of Bowen \cite{Bow78} that for hyperbolic automorphisms of \( \mathbb{T}^3 \), the boundary of the atoms of the Markov partitions are typically fractal. Later, Cawley \cite{Caw91} proved that the only hyperbolic toral automorphisms \( f \) for which there exist Markov partitions with piecewise smooth boundary are those for which a power \( f^k \) is linearly covered by a direct product of automorphisms of the 2-torus. These results give the impression that smooth polyhedral Markov partitions for \( \mathbb{Z}^d \)-rotations on \( \mathbb{T}^d \) exist only in the case of \( \mathbb{Z}^d \)-rotations involving quadratic integers. Also it would be interesting to find an example of a fractal Markov partition of \( \mathbb{T}^2 \) for a \( \mathbb{Z}^2 \)-rotation involving algebraic integers of degree \( \geq 3 \), or maybe it does not exist?

As pointed out by the referee, the examples provided in this article are such that the lattice \( \Gamma_0 \) and \( \mathbb{Z}^2 \) (the orbit of zero under the \( \mathbb{Z}^2 \)-action \( R_0 \)) have parallel elements (e.g., \((1, 0)\) is parallel to \((\varphi, 0)\)). We do not know if similar results could be proven in the case when \( \Gamma_0 \) has no vectors parallel to a vector of \( \mathbb{Z}^2 \).

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