On the distance observable in the Moyal plane and in a novel two-dimensional space with string-theory pregeometry

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Abstract: Motivation for the study of spacetime noncommutativity comes primarily from its possible use in investigations of (Planck-scale) spacetime fuzziness, but most work focuses on S-matrix/field-theory observables and still very little has been established for geometric observables. We argue that it might be useful to exploit the ”pregeometric” formulation of spacetime noncommutativity, which in particular describes the coordinates of the Moyal plane in terms of the phase-space coordinates of a point particle ”living” in a auxiliary/fictitious spacetime \(\{x_1, x_2\}_\text{Moyal} = \{q, p\}_\text{particle}\). This leads us straightforwardly to a distance operator for the Moyal plane, and allows us to expose some limitations of a previous attempt to describe the ”area of a disc” in the Moyal plane. We also observe that from our pregeometric perspective it is rather natural to contemplate a spacetime whose pregeometric picture is based on the phase-space coordinates (fields) of a string. The fact that such ”stringspaces” essentially provide spacetime points with extendedness is relevant for the fuzziness of geometric spacetime observables in ways that we preliminarily characterize through an analysis of the distance observable on a two-dimensional stringspace and through the observation that by implementing the Amati-Ciafaloni-Veneziano/Gross-Mende uncertainty relation in the phase space of the pregeometric string one could have stringspaces with a novel type of fuzzy coordinates.
1. Introduction and summary

The hypothesis that the fundamental description of spacetime structure should involve one form or another of quantization has been explored extensively, mostly (but not exclusively) within the literature devoted to the study of the quantum-gravity problem. One of the most studied forms of quantization introduces noncommutativity of the coordinates of a Minkowski-type (flat) spacetime in a sense that attempts to reproduce the success of the Heisenberg noncommutativity of the phase-space position coordinates of point particles. A simple and much studied example is the (Groenewold-)Moyal plane \([1, 2, 3]\), which is a two-dimensional flat space with noncommutativity of the coordinates \(x_1\) and \(x_2\) given by:

\[
[x_1, x_2] = i\theta .
\]  

(1.1)

For four-dimensional spacetime noncommutativity the simplest possibility is still a “canonical spacetime” \([4]\), with coordinates such that \(\{\mu, \nu\} \in \{0, 1, 2, 3\}\)

\[
[x_\mu, x_\nu] = i\theta_{\mu\nu} .
\]  

(1.2)

[The noncommutativity parameters \(\theta\) in (1.1) and \(\theta_{\mu\nu}\) in (1.2) commute\(^1\) with the coordinates.]

\(^1\)While over the last few years canonical spacetimes are indeed mostly studied assuming that \(\theta_{\mu\nu}\) commutes with the coordinates, the research programme that first led to the proposal of canonical noncommutativity \([4]\) is contemplating the possibility \([5]\) of a \(\theta_{\mu\nu}\) that does not commute with the coordinates.
Most of the interest in these studies, especially from the quantum-gravity side, originates primarily from heuristic arguments suggesting that when gravitational and quantum-mechanical effects are both taken into account it should not be possible to determine sharply the coordinates of a spacetime point/event. However, the “fuzziness” of noncommutative spacetimes has not been extensively investigated. Instead most studies focus on the structure of field theories introduced in such spacetimes and on “S-matrix observables” (such as the implications of noncommutativity for the probability of occurrence of certain particle-physics processes).

We here argue that a valuable tool for the exploration of the fuzziness could be provided by the pregeometric perspective on spacetime noncommutativity, which essentially is the idea of exploiting fully the correspondence between certain types of spacetime noncommutativity and the noncommutativity of the Heisenberg phase space. For example, certain properties of functions of the coordinates of the Moyal plane can be obtained describing them as operators acting on the Hilbert space of a particle “living” in a pregeometric/auxiliary/fictitious spacetime. The availability of such a pregeometric picture is rather obvious and has already been stressed by several authors, but so far its potentialities of providing intuition (and tools) for computations that one might want to do in the noncommutative-spacetime setting have been explored only in a very limited way.

In the next section we report our analysis of the distance operator for the Moyal plane, for which the pregeometric perspective proves to be valuable. Interestingly our distance operator has a discrete spectrum and a (non-zero) minimum eigenvalue. While we are not aware of any previous published studies of the distance operator in the Moyal plane, our comments on the area operator are offered (in the short Section 3) also in relation to the analysis of the area of a disc in the Moyal plane reported in Ref. [12]. We shall attempt a detailed analysis of the area-observable issue in a forthcoming paper, but the observations we report here suffice to expose some limitations of the proposal put forward in Ref. [12].

In section 4 we discuss briefly other applications (beyond Moyal) of the conventional pregeometric picture for spacetime noncommutativity, and we also propose a possible new way to make use of the pregeometric perspective: while it usually intervenes to provide intuition on how to derive some characterizations of a preexisting (noncommutative) spacetime geometry, we argue that it might be useful to adopt a “pregeometric perspective” for the task of devising spacetime geometries of certain desired structures.

In Section 5 we give a first example of the new way to make use of the pregeometric perspective which we advocate. Specifically we observe that a way to devise a spacetime geometry with coordinates each affected by irreducible uncertainties (whereas in standard noncommutative geometries irreducible uncertainties are found only for combinations of coordinates) can perhaps be inspired by the string-theory Amati-Ciafaloni-Veneziano/Gross-Mende results on uncertainty relations for the phase space of strings: if the pregeometric picture of a point of spacetime is given by a (functional) point in the phase-space of such strings then the coordinates of this (“extended”) spacetime points should indeed be individually affected by irreducible uncertainties. On the technical side we are at present not ready to develop in detail such a fully articulated string-theory-based pregeometric picture, but we do attempt to provide a preliminary intuition for its structure through an exploratory
analysis of some properties of the distance operator in such a “stringspace” (for the case of a two-dimensional stringspace and relying only on a rudimentary characterization of the string-theoretic pregeometry).

In the closing Section 6 we offer some brief considerations on the outlook of application of the “pregeometric strategy” proposed in this paper (both for the Moyal plane and other conventional noncommutative spaces, and for our stringspaces).

2. Pregeometry and distance operator for the Moyal plane

A standard pregeometric description of a point in the Moyal plane is obtained in terms of the phase-space coordinates of a fictitious nonrelativistic quantum particle “living” on a 1+1-dimensional pregeometric spacetime. Denoting by $\xi$ the spatial coordinate of the pregeometric spacetime, the states of the fictitious particle are described by “wave functions” $\psi(\xi)$ and its phase-space “observable” coordinates $\hat{\xi}$ and $\hat{\pi}$ are noncommutative operators acting on the space of $\psi(\xi)$’s, subject to the Heisenberg commutation relation

$$\left[\hat{\xi}, \hat{\pi}\right] = i\hbar_{\text{pregeom}}. \quad (2.1)$$

Clearly one obtains a faithful characterization of the Moyal plane by taking

$$x_1 \sim \hat{\xi}, \quad x_2 \sim \hat{\pi}, \quad \theta = \hbar_{\text{pregeom}}. \quad (2.2)$$

On the basis of this description of the Moyal plane it is possible to make use of well-established properties of the phase-space observables of nonrelativistic quantum particles on a one-dimensional space to characterize at least some aspects of the geometric structure of the Moyal plane.

Note that our choice of conventions attempts to provide easy-to-recognize characterizations of the variables that appear in our analysis. A variable with a hat (say, $\hat{q}$) is always an observable (be it physical or pregeometric), while variables without a hat (say, $q$) are always coordinates on the geometry of the spacetime framework in which observables (such as $\hat{q}$) are introduced. In particular here $\xi$ is the (algebraically trivial) spatial coordinate on the fictitious spacetime where the pregeometric theory is formulated, $\hat{\xi}$ is the pregeometric observable that gives the position of a particle in the space with coordinates $\xi$, and $x_1, x_2$ are (noncommutative, Moyal) spatial coordinates of the spacetime which should be considered as the “physical” spacetime of our picture. We would denote by $\hat{x}_1, \hat{x}_2$ observables describing the position of a nonrelativistic particle in the Moyal plane (they should be operators acting on some wave functions, $\psi(x_1, x_2)$, whose arguments are noncommutative coordinates).

Both for its intrinsic interest and because it is a valuable illustrative example of Moyal-plane analysis that can be based on the pregeometric perspective, we report here a study of the distance observable in the Moyal plane. In light of the characterization of points

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Footnote: In this paper we reserve the symbol “$\sim$”, which ordinarily describes the near equality of two quantities, to characterize the link between structures in the “physical” space and their pregeometric description in terms of observables in the pregeometry.
in the Moyal plane given above, one should evidently be able to describe the distance between two points of the Moyal plane as some appropriate operator acting on two-particle pregeometric states. Let us denote the Moyal-plane (noncommutative) coordinates of one of the points with \( x^{(1)}_1, x^{(1)}_2 \) and use for the other point \( x^{(2)}_1, x^{(2)}_2 \). These can indeed be described (pregeometrically) in terms of the following operators acting on two-particle pregeometric states:

\[
x^{(1)}_1 \sim \hat{\xi} \otimes 1, \quad x^{(1)}_2 \sim \hat{\pi} \otimes 1, \quad x^{(2)}_1 \sim 1 \otimes \hat{\xi}, \quad x^{(2)}_2 \sim 1 \otimes \hat{\pi}.
\] (2.3)

As the observable (squared-)distance between two points in the Moyal plane it is natural to take the standard formula:\[3:\]

\[
d^2 = (x^{(1)}_1 - x^{(2)}_1)^2 + (x^{(1)}_2 - x^{(2)}_2)^2.
\] (2.4)

Valuable insight on the properties of \( d^2 \) is obtained, within the pregeometric setup, by straightforward analysis of the spectrum of a would-be-Hamiltonian operator \( \hat{H} \) for two-particle pregeometric states. One in fact can take (using again the symbol \( \sim \) to link a structure in the Moyal space to an observable in the pregeometry)

\[
d^2 \sim \hat{H} \equiv \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{q}^2,
\] (2.5)

with \( \hat{q} \equiv \hat{\xi} \otimes 1 - 1 \otimes \hat{\xi}, \hat{p} \equiv \hat{\pi} \otimes 1 - 1 \otimes \hat{\pi}, m = \frac{1}{2}, \omega = 2 \). Then, also noticing that from (2.1) it follows that \([\hat{q}, \hat{p}] = 2i\hbar_{\text{pregeom}} = 2i\theta\), one easily concludes that the spectrum of \( d^2 \) is

\[
d^2_n = E_n = 4\theta \left( n + \frac{1}{2} \right),
\] (2.6)

with \( n \) integer and nonnegative.

Since one of the motivations for investigating the idea of a quantum spacetime originates from interest (see, e.g., Ref. [14]) in the possibility of discrete spectra for spacetime geometric observables, it is noteworthy that the (squared-)distance observable in the Moyal plane is discrete, with separations of \( 4\theta \) between the eigenvalues, and has a minimum value: \( d^2_{\text{min}} = 2\theta \).

3. On the area observable in the Moyal plane

We are not aware of any previous published studies of the distance observable in the Moyal plane, and our original result concerning discretization and a minimum value for this observable might well prove to be an insightful characterization of Moyal noncommutativity. Having figured out a few things on the distance observable the next natural tasks should concern the area observable, and in the previous literature one does find a study, in Ref. [12], that might be relevant for the area observable in the Moyal plane. In a quantum-spacetime

\[3\] Beside the familiarity of its structure, as we shall discuss elsewhere [13], the operator \( d^2 \) introduced in (2.4) appears to be a natural choice also because it turns out to be compatible with the (rather trivial) symmetries of the Moyal plane.
setup the concept of area of a surface is much more challenging than the one of distance between two points, and we postpone a detailed discussion of the area observable to a forthcoming paper \[13\], but we do want to offer here a few remarks characterizing the complications that the description of the area of a surface entails in a noncommutative space, also observing that the pioneering attempt of analysis of areas reported in Ref. \[12\] might require rather substantial revisions.

Let us indeed start with a brief characterization of the results reported in Ref. \[12\], which focused on an attempt to describe the area of a disc in the Moyal plane. A first potentially worrisome aspect of the analysis reported in Ref. \[12\] originates from the fact that the disc is specified through a single point in the Moyal plane, with noncommutative coordinates $x_1, x_2$. The area $A_{\text{disc}}$ of the disc was then described as follows \[12\]

$$A_{\text{disc}} = \pi \left[ x_1^2 + x_2^2 \right].$$

(3.1)

in terms of the noncommutative coordinates $x_1, x_2$ of that single point. The analysis ends up proposing a spectrum for $A_{\text{disc}}$ which is discrete and has a minimum (nonzero) value.

First of all let us notice that, in a plane characterized by uncertainty relations of the type

$$\delta x_1 \delta x_2 \geq \theta$$

(3.2)

it is to be expected that, as we found in the previous section, the distance observable has a nonzero minimum value, but it would be rather paradoxical to have a nonzero minimum allowed value of areas. A surface is identified by a network of points and it suffices to have the case of all points with the same value of, say, $x_1$ to have then zero area. And the requirement $\delta x_1 \delta x_2 \geq \theta$ of course does not exclude the possibility of a network of points all with the same value of $x_1$ (the $x_2$ coordinates of these points may well be affected by strong uncertainties, but the area would be zero exactly/sharply). For the distance between two points to be zero instead one should demand that all coordinates of the two points coincide, which is very clearly incompatible with $\delta x_1 \delta x_2 \geq \theta$, and indeed we did report a minimum-distance result in the previous section.

So it seems that we should conclude that the minimum-area result of Ref. \[12\] must be an artifact, the result of some unwarranted assumptions made in the derivation. The idea of characterizing a surface through a single point on the boundary of the surface, as done in Ref. \[12\], should have been subject to further scrutiny in any case, and must then be viewed even with more skepticism in light of the puzzling mismatch between the derived properties of areas and the uncertainty relations $\delta x_1 \delta x_2 \geq \theta$.

4. Pregeometry: some standard applications and a new strategy

In physics and mathematics the characterization as “pregeometry” has been introduced in many different ways, from several different perspectives. It is clearly beyond the scopes of this paper to propose a “historical route” through this broad literature. We do need to characterize our own perspective on “pregeometry”, and for this we find useful to link it to two of the concepts of pregeometry introduced in the physics literature. Our first influence
is perhaps the most classical concept of pregeometry contemplated in the quantum-gravity literature, which was championed by Wheeler already in the 1960s and 1970s. This is the idea that perhaps a solution to the quantum-gravity problem might be more easily reached if, rather than working in ways that refer directly to spacetime, the formalisms developed looking for candidate solutions of the quantum-gravity problem were articulated in terms of some deeper underlying rules of physics from which then in the end spacetime would be in some appropriate sense derived. Our second influence is the concept of pregeometry as sometimes used in studies of spacetime noncommutativity, where by describing the noncommutative spacetime coordinates in terms of Heisenberg phase-space observables of an “underlying” (but possibly only fictitious, mathematically auxiliary) particle theory one finds ways to perform more intuitively certain computations that spacetime noncommutativity invites one to contemplate.

The perspective we advocate here is roughly a hybrid of these two influences, further elaborating on a “pregeometric perspective” of similar hybrid nature advocated in Ref. [15] (which in turn was elaborating on earlier “pregeometric perspectives” proposed by Majid [16]). The ways in which we would contemplate formalizing the pregeometric level make use of particle-physics (and string-theory, see later) intuition, in a sense that is closely related to the pregeometric techniques sometimes used in the noncommutative spacetime literature. But the physical picture (and therefore the entities introduced at the pregeometric level and the interpretation of the symbolism used at the pregeometric level) is guided by the physical-problem-solving spirit of the Wheeler perspective. And, while in the noncommutative-spacetime literature usually a formulation of the properties of spacetime comes first and then a pregeometric description is later identified, mainly as a convenient tool for computations, here we see reasoning at the pregeometric level as an opportunity to devise new pictures of spacetime structure. By limiting ourselves, in the search of alternative ways to describe the structure of spacetime, to the use of intuition that is indigenous to spacetime itself we might be missing on opportunities provided by our experience with the large variety of physical systems Nature confronts with: physics has encountered many “spaces” along the way to characterize the diversity of Nature and each of these spaces is in principle a possible source of intuition for the structure of the space we most care about, which is spacetime. Besides the example of the phase space of a particle, usually adopted in pregeometric pictures of spacetime noncommutativity, one can (in a sense that will be roughly illustrated in the next section) consider the phase space of a string, and (in a sense that we hope to be in a position to illustrate in the not-so-distant future) even spaces that at first do not appear to look like spacetime (or even phase space) could well come into play, and a particularly amusing possibility would be the one of associating spacetime to some space of solutions of a dynamical system (amusing because it would link fully dynamical entities in the pregeometry to the kinematic/structural properties of the physical spacetime).

This would open strikingly novel (and, admittedly, perhaps “too novel”) possibilities, as illustrated by the fact that for example for world lines in some pregeometry one could introduce the concept of braiding, while a concept of “braiding of points” (points of spacetime) is something we might struggle to come to terms with (and perhaps indeed we might prudently choose not to).
In a sense that perhaps is close to the one intended by Wheeler, we envision that it might be in some cases advantageous/fruitful to adopt as laws of dynamics of spacetime some laws that are primitively introduced (and most naturally viewed) at the pregeometric level, and that would amount to spacetime dynamics only as a result of the link established between spacetime structures and pregeometric entities.

The tentative characterization of our work in progress offered in this section is relevant to the present paper only inasmuch as it provides some tools of interpretation for our readers of the motivation for the observations reported in the second part of this paper. While the observations reported in the previous two sections reflect a type of use of pregeometry that is standard in the noncommutative-spacetime literature, the proposal sketched out in the next section can be perhaps more intelligible to our readers in light of the perspective we are offering in this section.

In closing this section let us stress that, while both in the first part and in the second part of this paper the starting point is the Moyal plane (investigated in the first part, and generalized, in appropriate sense, in the second part), some of the techniques that are available in the noncommutative-spacetime literature should easily allow to generalize all of our results to richer examples of spacetime noncommutativity. For the much-studied four-dimensional canonical spacetimes, which (as already reported in Eq. (1.2)) are characterized by \[ [x_\mu, x_\nu] = i\theta_{\mu\nu}, \]
an example of pregeometric description is obtained by posing (see, e.g., Ref. [17])

\[
x_\mu \sim \hat{\xi}_\mu - \frac{1}{2\hbar_{\text{pregeom}}} \delta^{\rho\sigma} \theta_{\mu\sigma} \hat{\pi}_\rho,
\]

where \( \delta^{\mu\nu} \) is the Kroenecker delta, and \( \hat{\xi}_\mu \) and \( \hat{\pi}_\nu \) are phase-space coordinates (observables) of a pregeometric particle “living” in a 5-dimensional auxiliary spacetime:

\[
[\hat{\xi}_\mu, \hat{\pi}_\nu] = i\delta_{\mu\nu} \hbar_{\text{pregeom}}, \quad [\hat{\xi}_\mu, \hat{\xi}_\nu] = [\hat{\pi}_\mu, \hat{\pi}_\nu] = 0.
\]

And for the four-dimensional “\( \kappa \)-Minkowski spacetime”, characterized by

\[
[x_j, t] = i\lambda x_j \quad [x_j, x_k] = 0,
\]
on which there is also a rather sizeable literature, several alternative formulations in terms of the phase spaces of particles “living” in a four-dimensional auxiliary/fictitious spacetime have been contemplated, including the possibility

\[
x_j \sim \hat{\xi}_j, \quad t \sim e^{\lambda(\hat{\pi}_1 + \hat{\pi}_2 + \hat{\pi}_3)/\hbar_{\text{pregeom}}},
\]

and the possibility [16, 15] of the limit \( \rho, \hbar_{\text{pregeom}} \to \infty \) (with \( \hbar_{\text{pregeom}}/\rho = \lambda \)) of \( \delta^{\mu\nu} \)

\[
[\hat{\xi}_j, \hat{\pi}_k] = i\delta_{jk} \hbar_{\text{pregeom}}, \quad [\hat{\xi}_j, \hat{\xi}_k] = [\hat{\pi}_j, \hat{\pi}_k] = 0.
\]

But in this case there would be strong mathematical motivation [16, 15] for introducing in the pregeometric picture also the following nontrivial coalgebraic property: \( \Delta(\hat{\xi}_j) = \hat{\xi}_j \otimes 1 + 1 \otimes \hat{\xi}_j, \quad \Delta(\hat{\pi}_j) = \hat{\pi}_j \otimes 1 + e^{-\hat{\xi}_j/\rho} \otimes \hat{\pi}_j. \)
5. Introducing stringspaces

5.1 Fuzzy spacetime points with noncommutativity and extendedness

Also as an illustrative example of how the new perspective on pregeometry here proposed might be exploited, in this section we introduce a new candidate for the structure of spacetime, which would not be naturally contemplated by using intuitive reasoning directly at the level of spacetime itself, but is indeed a rather natural candidate from our pregeometric perspective. We have discussed in the previous sections how the Moyal plane and other noncommutative spaces (and spacetimes) can be described in terms of the phase spaces of particles "living" in a classical pregeometric/fictitious spacetime. With so much to be learned on possible spacetime geometries by studying the phase spaces of particles it is indeed natural to wonder whether something meaningful/useful could be found by exploring the possibility to describe spacetime, one way or another, in terms of the phase spaces of strings "living" in a classical pregeometric spacetime.

At first this hypothesis might appear to be rather peculiar, because the coordinates of a string in phase space are fields, and field coordinates do not look like natural candidates to describe the coordinates of points. However, the physics community has already well established that our present picture, with point particles propagating in a spacetime of points, can be meaningfully generalized by allowing for "string particles" propagating in a spacetime of points. In particular the string-theory literature (and by now even string-theory textbooks) shows very clearly how such extended objects would appear as point particles to all measuring devices enabled with presently-available sensitivities (but could add new physics in a high-energy "stringy regime"). It is therefore not inconceivable that a "deformation" of similar (but different, see below) type might be achieved by instead describing particles in terms of close-to-ordinary point-particle fields propagating however in a spacetime with "extended points".

While we are unprepared to offer a fully articulated hypothesis of formulation of such a "stringspace", we hope that the exploratory formulation and observations reported in this section will suffice to illustrate the empowerment that can come from our new perspective on pregeometry, and perhaps also illustrate some reasons for our stringspaces to be further explored.

Just because our formalization of the idea of stringspaces is at present very rudimentary it is perhaps useful, even before getting to it, to offer some intuition for the type of novel features for spacetime geometry that more refined formulations of stringspaces might end up providing. The simplest example we can propose is based on the renowned Amati-Ciafaloni-Veneziano/Gross-Mende Generalized Uncertainty Principle (GUP) for string theory. Investigations of this GUP establish that the Heisenberg phase-space uncertainty principle, \( \delta X \delta P \geq 1 \), is modified in the string-theory setting in such a way that (in natural units)

\[
\delta X \geq \frac{1}{\delta P} + L_s^2 \delta P
\]

and as a result \( \min(\delta X) \approx L_s \) (with \( L_s \) the string length). If for these string-theoretic phase-space coordinates \( X \) and \( P \) we now contemplate, according to our novel perspective,
a pregeometric reinterpretation as coordinates of space we would have a picture that is somewhat similar to the one of the Moyal plane, but with more stringent uncertainty principles for the localization of points on the plane: on the Moyal plane a single coordinate can be established with arbitrarily high accuracy (at the price of losing all information on the other coordinate) but on a stringspace whose pregeometry is governed by the GUP even attempts to determine a single coordinate would be affected by an unavoidable minimum-uncertainty constraint. Several arguments based on the quantum-gravity problem have advocated exactly this type of fuzziness, and perhaps for these intuitions some suitable formulation of stringspaces might prove to be a valuable conceptual tool.

5.2 Exploratory formulation of a two-dimensional stringspace

While we envision the possibility to introduce some rather rich formulations of stringspaces, at present we are only able to offer a very rudimentary formulation. The pregeometric picture of our exploratory formulation of a two-dimensional stringspace is “stringy” only in the sense that it involves a pair of fields \( \phi(\sigma, \tau) \), \( \pi(\sigma, \tau) \), defined on a pregeometric world-sheet of coordinates \( \sigma \) and \( \tau \) \((0 \leq \sigma \leq \ell, -\infty \leq \tau \leq \infty) \), with \( \phi(\sigma, \tau) \) playing the role of a coordinate (-field) of the pregeometric string in the pregeometric target space and \( \pi(\sigma, \tau) \) playing the role of “momentum field”, conjugate to \( \phi(\sigma, \tau) \).

In our present rudimentary formulation of the two-dimensional stringspace the quantization of the pregeometric picture is introduced for (conceptual) simplicity through the functional Schroedinger picture, so that quantum states of the fields are codified in pregeometric “wave” functionals \( \Psi[\phi(\sigma)] \), with evolution in \( \tau \) described by

\[
i \frac{\partial}{\partial \tau} \Psi[\phi] = H[\hat{\phi}, \hat{\pi}] \Psi[\phi] ,
\]

in terms of some Hamiltonian \( H[\hat{\phi}, \hat{\pi}] \), and with the (pregeometric) “observables” \( \hat{\phi}(\sigma) \) and \( \hat{\pi}(\sigma) \) acting respectively by simple multiplication and by a functional derivative:

\[
\hat{\phi}(\sigma) \triangleright \Psi[\phi] = \phi(\sigma)\Psi[\phi] ,
\]

\[
\hat{\pi}(\sigma) \triangleright \Psi[\phi] = -i\hbar_{\text{pregeom}} \frac{\delta \Psi[\phi]}{\delta \phi(\sigma)} .
\]

These of course ensure that

\[
[\hat{\phi}(\sigma), \hat{\pi}(\sigma')] = i\hbar_{\text{pregeom}} \delta(\sigma - \sigma') .
\]

In analogy with what discussed before for the pregeometric picture of the Moyal plane, we associate to this pregeometric picture a description of the (would-be-physical) space geometry in which the fundamental entities are “extended points” with non-commutative (field-) coordinates \( x_1(\sigma) \) and \( x_2(\sigma) \) describable in terms of the phase-space fields of the pregeometric string: \( x_1(\sigma) \sim \phi(\sigma) \) and \( x_2(\sigma) \sim \pi(\sigma) \). But of course\(^6\) the primary characterization

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\(^6\)The extendedness of spacetime points, here conceptualized, would have to be mostly undetectable, if at all present, since we must of course reproduce the indications of our observations, so far all providing no evidence of such an extendedness.
of the stringspace should be given in terms of $\overline{x_1}$ and $\overline{x_2}$,

$$\overline{x_1} \sim \frac{1}{\ell} \int_0^\ell d\sigma \hat{\phi}(\sigma)$$ \hspace{1cm} (5.6)

$$\overline{x_2} \sim \frac{1}{\ell} \int_0^\ell d\sigma \hat{\pi}(\sigma)$$ \hspace{1cm} (5.7)

which essentially give the average values of the fields $x_1(\sigma)$ and $x_2(\sigma)$.

It is straightforward to notice that

$$[\overline{x_1}, \overline{x_2}] \sim \frac{1}{\ell^2} \int_0^\ell \int_0^\ell d\sigma d\sigma' [\hat{\phi}(\sigma), \hat{\pi}(\sigma')] = i \frac{\hbar_{\text{pregeom}}}{\ell}$$ \hspace{1cm} (5.8)

and therefore $\overline{x_1, x_2}$ (which however codify only part, the non-extendedness part, of the structure of the two-dimensional stringspace) are governed by a commutation relation that is exactly of Moyal-plane type (see Eq. (1.1)), with $\theta = \hbar_{\text{pregeom}}/\ell$. As a way to stress this point we shall in this section use interchangeably the notations $\hbar_{\text{pregeom}}/\ell$ and $\theta$.

### 5.3 Some remarks on the distance observable

This rudimentary formulation of stringspaces, which at present is the best we are able to offer, still allows us to at least articulate some issues and some observations that should be relevant for the analysis of the distance observable even in more structured stringspaces. Naturally we would like to treat distances in stringspace making profit at least in part of the simple technology introduced in Section 2 in the simpler context of the Moyal plane. To some extent we find that this can be done, and a clear way to illustrate this point is offered by the simplicity on a special class of Schroedinger-picture functionals in the pregeometric picture, the “gaussian wavefunctionals”,

$$\Psi_{[\phi]} = \det^{-\frac{i}{4}} \left[ \frac{\Omega}{\pi} \right] \exp \left( \int d\sigma d\sigma' \phi(\sigma) \Omega(\sigma, \sigma') \phi(\sigma') \right)$$ \hspace{1cm} (5.9)

which usually play a rather special role in the Schroedinger functional picture\(^7\).

We shall work with them assuming that the function $\Omega(\sigma, \sigma')$ (the “covariance” of the gaussian functional) is real and positive. And the reader should also notice that our functional-determinant pre-factor $\det^{-\frac{i}{4}} \left[ \frac{\Omega}{\pi} \right]$ ensures normalization $\int \mathcal{D}[\phi] \Psi^*_{[\phi]} \Psi_{[\phi]} = 1$.

These Schroedinger-picture gaussian functionals are rather obviously of interest for us since they could be to stringspaces what the “pregeometric gaussian wavefunctions” are to the Moyal plane: these might contain the pregeometric description of the quantum state of the geometry of spacetime in which the distance between two points is minimal.

A proper full characterization of the concept of distance between two (extendedness-endowed) points of stringspace, of (field-) coordinates $\{x_1^{(1)}(\sigma), x_2^{(1)}(\sigma)\}$ and $\{x_1^{(2)}(\sigma), x_2^{(2)}(\sigma)\}$, should probably be multilayered, but clearly a key first level of characterization should be describable in terms of $\{\overline{x_1^{(1)}}, \overline{x_2^{(1)}}\}$ and $\{\overline{x_1^{(2)}}, \overline{x_2^{(2)}}\}$. And for this level of characterization we do have a natural candidate for the distance observable, which is of course based on the

\(^7\)In particular, one can form Fock bases starting from them and applying creation operators $\hat{b}$. 

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\[ \text{Eq. (1.1)} \]

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\[ \text{Det} \]

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\[ \text{Schroedinger functional picture} \]

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\[ \text{Pregeometric gaussian wavefunctions} \]
correspondence between \( \{ \vec{x}_1, \vec{x}_2 \} \) and coordinates in the Moyal plane discussed at the end of the previous section:

\[
\vec{d}^2 \equiv \left( \frac{x_1^{(1)} - x_2^{(1)}}{\ell} \right)^2 + \left( \frac{x_1^{(2)} - x_2^{(2)}}{\ell} \right)^2.
\]  

(5.10)

We shall of course characterize the properties of \( \vec{d}^2 \) pregeometrically, in terms of properties of the corresponding phase-space observables acting on states of two strings (one string for each point in stringspace that appears in the geometric spacetime observable, just like in the case of the Moyal plane two pregeometric particles intervened in the description of geometric observables for pairs of points of the Moyal plane). Let us first investigate the properties of \( \vec{d}^2 \) in the case in which our gaussian wave functionals have the following simple form

\[
\Psi_\omega[\phi_1, \phi_2] = \det^{-1/2} \left[ \frac{\omega^2}{\pi} \right] \exp \left[ \int d\sigma (\phi_1^2(\sigma) + \phi_2^2(\sigma)) \omega(\sigma) \right],
\]  

(5.11)

which essentially amounts to placing each of the two pregeometric strings in a state described by a gaussian functional with covariance function \( \Omega \) of the special form \( \Omega(\sigma, \sigma') = \omega(\sigma) \delta(\sigma - \sigma') \) (with positive \( \omega(\sigma) \), as a result of the assumed positivity of \( \Omega(\sigma, \sigma') \)).

In order to evaluate \( \vec{d}^2 |_{\Psi_\omega} \) (which is our notation for the expected value of \( \vec{d}^2 \) in a state of the geometry of stringspace whose pregeometric description is given by \( \Psi_\omega[\phi_1, \phi_2] \)) it is useful to first notice, using (5.3), (5.4) and elementary steps of functional integration and functional differentiation, that

\[
\langle \Psi_\omega | \hat{d}^2(\sigma, \sigma') | \Psi_\omega \rangle = \int \mathcal{D}[\phi] \Psi_\omega^*[\phi] \hat{d}^2(\sigma, \sigma') \Psi_\omega[\phi] = \left( \frac{1}{\omega(\sigma)} + \ell^2 \theta^2 \omega(\sigma) \right) \delta(\sigma - \sigma'),
\]  

(5.12)

for

\[
\hat{d}^2(\sigma, \sigma') \equiv \left[ \hat{\phi}_1(\sigma) - \hat{\phi}_2(\sigma) \right] \left[ \hat{\phi}_1(\sigma') - \hat{\phi}_2(\sigma') \right] + \left[ \hat{\pi}_1(\sigma) - \hat{\pi}_2(\sigma) \right] \left[ \hat{\pi}_1(\sigma') - \hat{\pi}_2(\sigma') \right].
\]  

(5.13)

From the result (5.12) one then easily concludes that

\[
\vec{d}^2 |_{\Psi_\omega} = \frac{1}{\ell^2} \int_0^\ell \int_0^\ell d\sigma d\sigma' \langle \Psi_\omega | \hat{d}^2(\sigma, \sigma') | \Psi_\omega \rangle = \int d\sigma \left( \frac{1}{\ell^2 \omega(\sigma)} + \theta^2 \omega(\sigma) \right).
\]  

(5.14)

Interestingly this (taking into account the positivity of \( \omega(\sigma) \)) allows us to conclude that, at least when restricted on states of the type \( \Psi_\omega \), the expectation values of \( \vec{d}^2 \) are never smaller than \( 2\theta \): \( \min(\vec{d}^2 |_{\Psi_\omega}) = 2\theta \). We conjecture that it should be possible to verify (or easily find a relatively minor reformulation such) that on our rudimentary two-dimensional stringspace this property holds for all admissible states and not only for the restricted class \( \Psi_\omega \):

\[
\vec{d}^2 |_{\Psi} \geq 2\theta \quad \forall \Psi.
\]  

(5.15)

And among the states \( \Psi_\omega \) we do find an “eigenstate of \( \vec{d}^2 \)” that saturates the bound. This is the following state

\[
\Psi_{\omega\text{min}} = \det^{-1/2} \left[ \frac{\delta}{\pi \ell^2 \theta} \right] \exp \left[ \frac{1}{\ell^2 \theta} \int d\sigma (\phi_1^2(\sigma) + \phi_2^2(\sigma)) \right],
\]  

(5.16)
which actually is such that
\[ \tilde{d}^2(\sigma, \sigma')\Psi_{\omega_{\text{min}}} [\phi_1, \phi_2] = 2\theta \ell \delta(\sigma - \sigma')\Psi_{\omega_{\text{min}}} [\phi_1, \phi_2]. \] (5.17)

In light of this result (5.17), it is then evident that
\[ \frac{1}{\ell^2} \int_0^\ell \int_0^\ell d\sigma d\sigma' \{ \tilde{d}^2(\sigma, \sigma')\Psi_{\omega_{\text{min}}} [\phi_1, \phi_2] \} = 2\theta \Psi_{\omega_{\text{min}}} [\phi_1, \phi_2] \] (5.18)
which indeed amounts to the fact that \( \tilde{d}^2 = 2\theta \) sharply in the state described pregeometrically by \( \Psi_{\omega_{\text{min}}}. \)

Other noteworthy properties of \( \Psi_{\omega_{\text{min}}} \) that are easily verified are
\[ (x_1(1) - x_1(2))|_{\Psi_{\omega}} = 0 = (x_2(1) - x_2(2))|_{\Psi_{\omega}} \] (5.19)
and
\[ \int_0^\ell d\sigma \int_0^\ell d\sigma' \langle \Psi_{\omega_{\text{min}}} | [\hat{\phi}_1(\sigma) - \hat{\phi}_2(\sigma)][\hat{\phi}_1(\sigma') - \hat{\phi}_2(\sigma')] | \Psi_{\omega_{\text{min}}} \rangle = \theta = \int_0^\ell d\sigma \int_0^\ell d\sigma' \langle \Psi_{\omega_{\text{min}}} | [\hat{\pi}_1(\sigma) - \hat{\pi}_2(\sigma)][\hat{\pi}_1(\sigma') - \hat{\pi}_2(\sigma')] | \Psi_{\omega_{\text{min}}} \rangle. \] (5.20)

In particular this result (5.20), in light of (5.19), should be viewed as the pregeometric characterization of the fact that both \( x_1(1) - x_1(2) \) and \( x_2(1) - x_2(2) \) have uncertainty \( \sqrt{\theta} \) when the pregeometric state is \( \Psi_{\omega_{\text{min}}}. \) This may invite a characterization of \( \Psi_{\omega_{\text{min}}} \) as a minimum-uncertainty (pregeometric) state.

In closing this subsection exploring the concept of distance in a two-dimensional stringspace we find appropriate to stress that, while the study of \( \tilde{d}^2 \) provides the most intelligible (because most familiar looking) characterization of distances in (our rudimentary) two-dimensional stringspace, it exposes only marginally the implications of the “extendedness” of the points of stringspace. The little insight we are gaining in our preliminary (and presently unpublishable) investigations of this extendedness appears to suggest that its meaningful characterization may prove to be a formidable challenge. The only tangible contribution we can presently offer is a sort of “probe” of this fuzziness, which we characterize through the action of the following pregeometric-phase-space observable
\[ \hat{D}^2 \equiv \frac{1}{\ell^2} \int_0^\ell \int_0^\ell d\sigma d\sigma' \hat{d}^2(\sigma, \sigma') \delta(\sigma - \sigma'), \] (5.21)
on the following gaussian wave functional
\[ \Psi_{\Omega}[\phi_1, \phi_2] = \Psi_{\Omega}[\phi_1]\Psi_{\Omega}[\phi_2] = \det^{-1/2} \left[ \frac{\Omega}{\pi} \right] \exp \left( \int d\sigma d\sigma' \{ \phi_1(\sigma)\Omega(\sigma, \sigma')\phi_1(\sigma') + \phi_2(\sigma)\Omega(\sigma, \sigma')\phi_2(\sigma') \} \right). \] (5.22)

In order to compute the expectation of \( \hat{D}^2 \) in \( \Psi_{\Omega}[\phi_1, \phi_2] \) it is useful to first observe that\(^8\)
\[ \langle \Psi_{\Omega}[\phi_1, \phi_2] | \hat{d}^2(\sigma, \sigma') | \Psi_{\Omega}[\phi_1, \phi_2] \rangle = \Omega^{-1}(\sigma, \sigma') + \ell^2 \theta^2 \Omega(\sigma, \sigma'), \] (5.23)
\(^8\)Eq. (5.23) can be verified with calculations that are not much different from the ones needed for Eq. (5.12), although slightly more tedious.
where $\Omega^{-1}(\sigma, \sigma')$ is the inverse kernel of the covariance, defined by $\int d\sigma' \Omega^{-1}(\sigma, \sigma')\Omega(\sigma', \sigma'') = \int d\sigma'\Omega(\sigma, \sigma')\Omega^{-1}(\sigma', \sigma'') = \delta(\sigma - \sigma')$.

On the basis of (5.23) it is then easy to conclude that

$$\langle \Psi^{\Omega} [\phi_1, \phi_2] | \hat{D}^2 | \Psi^{\Omega} [\phi_1, \phi_2] \rangle = \frac{1}{\ell} \int_0^\ell d\sigma \left[ \Omega^{-1}(\sigma, \sigma) + \ell^2 \theta^2 \Omega(\sigma, \sigma) \right] , \quad (5.24)$$

and this can be interestingly compared with $\overline{d}^2_{\Psi^{\Omega}}$, for which one easily finds

$$\overline{d}^2_{\Psi^{\Omega}} = \frac{1}{\ell^2} \int_0^\ell \int_0^\ell d\sigma d\sigma' \left[ \langle \Psi^{\Omega} | \hat{d}^2(\sigma, \sigma') | \Psi^{\Omega} \rangle \right] = \frac{1}{\ell^2} \int_0^\ell \int_0^\ell d\sigma d\sigma' \left[ \Omega^{-1}(\sigma, \sigma') + \ell^2 \theta^2 \Omega(\sigma, \sigma') \right] . \quad (5.25)$$

By contemplating both (5.24) and (5.25) one can gain, in spite of their limited role in the overall characterization of a stringspace, some valuable insight. The most important point is that whereas $\overline{d}^2$ can be expressed (see (5.10)) in terms of the “average coordinates”, $x^{(1)}_1$, $x^{(2)}_1$, $x^{(1)}_2$, $x^{(2)}_2$, no association with the average coordinates is possible for $\hat{D}^2$. The average coordinates are the trait d’union between the Moyal plane and our two-dimensional stringspace: the stringspace has much more structure then the Moyal plane but if we restricted our investigations of stringspace only to entities describable in terms of the average coordinates (which, as stressed above, satisfy a standard Moyal-algebra commutator) the differences between our two-dimensional stringspace and the Moyal plane would probably not be very significant. The features of stringspace that are more novel are probed by tools like $\hat{D}^2$, which have no Moyal-plane counterpart, and therefore are likely to characterize the additional fuzziness\(^9\) of stringspace.

### 5.4 Some candidate additional structures for stringspaces

The concept of stringspace is potentially rather broad: a space whose points are described in terms of the (possibly functional) positions in phase space of some strings “living” in an auxiliary spacetime can be many things, reflecting the many possible ways to characterize the pregeometric strings. The rudimentary exploratory description of a two-dimensional stringspace we offered in the previous two subsections is formulated at a level such that essentially one only takes into account a few features of the kinematical Hilbert space [20, 21, 22] of the pregeometric string. For some of the most interesting spacetime features that one might envisage introducing through a stringspace construction it might be beneficial to enrich the pregeometric picture with at least some of the structures that a full string theory in the pregeometry could provide. This in particular appears to be necessary in order to realize the “spacetime GUP”, which we discussed in Subsection 5.1 as one of the most appealing opportunities for stringspace modelling, since the derivations [18] of the string-theoretic phase-space GUP appear to require a fully structured string theory. What exactly would be strictly needed as structure of the pregeometric string theory in order to get the GUP stringspace is not completely clear, as a result of the fact that the role of

\(^9\)The Moyal plane is already “fuzzy” in the sense of ordinary spacetime noncommutativity, but on top of that our two-dimension stringspace has other sources of fuzziness associated with its peculiar feature of having points endowed with extendedness.
the standard GUP in the foundational conceptual ingredients of string theory is still not well understood \[23\]. But in general (and perhaps in particular for this GUP-stringspace issue) it appears to us, consistently with the observation we reported in Section 4, that one should not exclude \textit{a priori} the possibility of a fully dynamical formulation of the pregeometry, and that one should at least contemplate the possibility to associate some kinematical/structural aspects of the stringspace to genuinely dynamical entities of the pregeometric level.

Another key issue which was not considered in the preliminary exploration of stringspaces reported in this section is the one of symmetries. In a stringspace setup there are at least three levels at which symmetry issues should be contemplated: symmetries of the pregeometric world-sheet of the string, symmetries of the pregeometric target space of the string, and, most importantly, symmetries of the phase space of the pregeometric string, which will affect directly the symmetry structure of the stringspace. It is perhaps amusing to observe that, even when the stringspace is intended as a nonclassical description of a quasi-Minkowski spacetime, one might well have a valuable formulation of the stringspace in cases in which the pregeometric target space is not Poincaré invariant, not even approximately. In such instances one would have to insist on (at least approximate) Poincaré symmetry of the stringspace, but this translates into a demand for the pregeometric phase space and not necessarily the pregeometric target space. In some cases however this might amount to a rather involved technical issues for the level of the pregeometric phase space, as one would notice immediately in the analysis of stringspaces of 3 or more dimensions. In fact, while the Moyal plane is essentially trivial \[13\] from a symmetry perspective (and at least some of that triviality should be reflected in the pregeometric phase space), the generalizations to 3 or more spacetime dimension (see e.g. our Section 4) are often described in terms of Hopf-algebra symmetries \[24, 25, 26, 27, 28, 29, 30\] and stringspaces that in the “average coordinate limit” resemble such noncommutative spacetimes may well be affected by this feature.

Of course, if stringspaces were ever to be used to do physics, we would not want just theories of stringspace but actual theories formulated in stringspace. And in this respect it should be noticed that, for example, a stringspace version of an ordinary theory of “classical fields in commutative Riemannian spacetime” in our rudimentary two-dimensional stringspace would have to involve functionals of noncommutative (quantum) fields. For example, a metric field should be of the type $G_{ij}[x_1(\sigma), x_2(\sigma)] \sim G_{ij}[\hat{\phi}(\sigma), \hat{\pi}(\sigma)]$. It is worth noticing how different this is from the D-dimensional target-space metric field that one introduces in writing certain actions in the standard string-theory framework, which is of the type $G_{\mu,\nu}[\phi_1(\sigma, \tau), \phi_2(\sigma, \tau), ..., \phi_D(\sigma, \tau)]$. In particular $G_{ij}[\hat{\phi}(\sigma), \hat{\pi}(\sigma)]$ must reflect the noncommutativity between $\hat{\phi}(\sigma)$ and $\hat{\pi}(\sigma)$, whereas $G_{\mu,\nu}[\phi_1(\sigma, \tau), \phi_2(\sigma, \tau), ..., \phi_D(\sigma, \tau)]$ depends exclusively on the $\phi_\mu$ coordinate fields on the world-sheet which of course all commute with one another. It is also worth stressing that in the study of theories in stringspace a natural fourth level of symmetry analysis would arise (in addition to the three natural levels of symmetry analysis mentioned above), since one would need to establish the symmetries of these theories introduced in stringspace.

In closing this section, after mentioning so many structures that could (and probably
should) be added to our pregeometric-string picture, let us also mention that we might however have introduced at least one more feature than needed: guided by the analogy with the use of pregeometry in the study of noncommutative spacetimes we immediately assumed that the pregeometric string should be quantized, rendering its phase space noncommutative, so that indeed the stringspace coordinates would be described by noncommutative quantum fields. It is however at least not obvious that one should necessarily proceed in this way: classical pregeometric strings may well lead to meaningful scenarios, some “commutative stringspaces”, and probably non-trivial scenarios, since, even without the noncommutativity feature, the corresponding stringspaces would still be affected by fuzziness induced by the extendedness provided to the points. And it is amusing to wonder whether there could be a a theory appropriately formulated in a “commutative stringspace” that would reproduce string theory: it is not inconceivable that in some specific setups the extendedness of particles (replacing a particle by a string) could be traded for an extendedness of points (replacing classical Minkowski space with a Minkowskian commutative stringspace).

6. Outlook

The techniques used in our Sections 2 and 3, within a conventional setup of exploitation of the pregeometry perspective, should be applicable to a rather broad collection of geometric observables of the Moyal plane, and of other noncommutative spacetimes. Our observations on the area observable (which shall be extended in Ref. [13]) and on the distance observable appear to suggest that there is no universal behaviour of this type of observables: for distances we derived the presence of a minimum nonzero value, while zero area is possible. And this (as stressed in Section 3) appears to reflect correctly what one should expect on the basis of the form of the Moyal-plane commutator of coordinates. Besides providing a meaningful characterization of Moyal-plane noncommutativity, this observation should prompt a careful reexamination of some of the semi-heuristic arguments that appeared in the quantum-gravity literature, especially during the last decade. Several of these arguments motivate rather rigorously (as rigorously as possible using heuristic arguments) the existence of a minimum-length bound, and then with somewhat loose use of logics assume that one would reach similar conclusions for areas and volumes, as if in a quantum spacetime the presence of a minimum-length bound would necessarily imply corresponding bounds for areas and volumes. We believe that the Moyal plane, in spite of its relative simplicity, may provide a significant counter-example for the assumptions that guide this type of reasoning.

For what concerns our proposal of string-theoretic pregeometries, and the associated “stringspaces”, at present the only natural target which appears to be in sight is the implementation of the “GUP in spacetime” (here described in Subsection 5.1), possibly exploiting some of the observations we offered in Subsection 5.4. If that is accomplished then one could attempt to formulate quantum field theories in the relevant stringspace, and in particular we conjecture that such quantum field theories would be immune from the peculiar “IR/UV mixing” [31, 32, 33] which instead appears to affect quantum field theories
in Moyal/canonical noncommutative spaces. IR/UV mixing is a result of the connection between short distances in one direction and large distances in another direction which are generated by Moyal-type uncertainty relations (the type $\delta x_1 \delta x_2 \geq \theta$) and we believe it should not be present if instead the uncertainty relations are such that for all coordinates $x_i$ one has $\delta x_i \geq \sqrt{\theta}$ (which is what we expect of the “GUP in spacetime”). But our proposal of stringspaces is perhaps even more interesting as an illustrative example of a novel way to make use of the pregeometry perspective (here advocated in Section 4) which transforms the whole ensemble of results of theoretical physics into a reservoir of ideas/intuitions for what could be the structure of spacetime at short (possibly Planckian) distances.

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