COMPLETE $\lambda$-HYPERSURFACES OF WEIGHTED VOLUME-PRESERVING MEAN CURVATURE FLOW

QING-MING CHENG AND GUOXIN WEI

Abstract. In this paper, we introduce a definition of $\lambda$-hypersurfaces of weighted volume-preserving mean curvature flow in Euclidean space. We prove that $\lambda$-hypersurfaces are critical points of the weighted area functional for the weighted volume-preserving variations. Furthermore, we classify complete $\lambda$-hypersurfaces with polynomial area growth and $H-\lambda \geq 0$, which are generalizations of the results due to Huisken [18], Colding-Minicozzi [10]. We also define a $\mathcal{F}$-functional and study $\mathcal{F}$-stability of $\lambda$-hypersurfaces, which extend a result of Colding-Minicozzi [10]. Lower bound growth and upper bound growth of the area for complete and non-compact $\lambda$-hypersurfaces are also studied.

1. Introduction

Let $X: M \rightarrow \mathbb{R}^{n+1}$ be a smooth $n$-dimensional immersed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. A family $X(\cdot, t)$ of smooth immersions:

$$X(\cdot, t): M \rightarrow \mathbb{R}^{n+1}$$

with $X(\cdot, 0) = X(\cdot)$ is called a mean curvature flow if they satisfy

$$\left(\frac{\partial X(p, t)}{\partial t}\right)^\perp = H(p, t),$$

where $H(t)$ denotes the mean curvature vector of hypersurface $M_t = X(M^n, t)$ at point $X(p, t)$. Huisken [16] proved that the mean curvature flow $M_t$ remains smooth and convex until it becomes extinct at a point in the finite time. If we rescale the flow about the point, the rescaling converges to the round sphere. An immersed hypersurface $X: M \rightarrow \mathbb{R}^{n+1}$ is called a self-shrinker if

$$H + \langle X, N \rangle = 0,$$

where $H$ and $N$ denote the mean curvature and the unit normal vector of $X: M \rightarrow \mathbb{R}^{n+1}$, respectively. $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^{n+1}$. It is known that self-shrinkers play an important role in the study of the mean curvature flow because they describe all possible blow ups at a given singularity of the mean curvature flow.

For $n = 1$, Abresch and Langer [1] classified all smooth closed self-shrinker curves in $\mathbb{R}^2$ and showed that the round circle is the only embedded self-shrinker. For

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n ≥ 2, Huisken [18] studied compact self-shrinkers. He proved that if \( M \) is an \( n \)-dimensional compact self-shrinker with non-negative mean curvature in \( \mathbb{R}^{n+1} \), then \( X(M) = S^n(\sqrt{n}) \). In the remarkable paper [10], Colding and Minicozzi have classified complete self-shrinkers with non-negative mean curvature and polynomial area growth (which is called polynomial volume growth in [10] and [19]) in \( \mathbb{R}^{n+1} \). We should remark that Huisken [19] proved the same results if the squared norm of the second fundamental form is bounded. Colding and Minicozzi [10] have introduced a notation of \( F \)-functional and computed the first and the second variation formulas of the \( F \)-functional. They have proved that an immersed hypersurface \( X : M \to \mathbb{R}^{n+1} \) is a self-shrinker if and only if it is a critical point of the \( F \)-functional. Furthermore, they have given a complete classification of the \( F \)-stable complete self-shrinkers with polynomial area growth.

On the other hand, Huisken [17] studied the volume-preserving mean curvature flow

\[
\left( \frac{\partial X(t)}{\partial t} \right)^\perp = (-h(t)N(t) + H(t)),
\]

where \( X(t) = X(\cdot, t) \), \( h(t) = \frac{\int_M H(t) d\mu}{\int_M d\mu} \) and \( N(t) \) is the unit normal vector of \( X(t) : M \to \mathbb{R}^{n+1} \). He proved that if the initial hypersurface is uniformly convex, then the above volume-preserving mean curvature flow has a smooth solution and it converges to a round sphere. Furthermore, by making use of the Minkowski formulas, Guan and Li [15] have studied the following type of mean curvature flow

\[
\left( \frac{\partial X(t)}{\partial t} \right)^\perp = (-N(t) + H(t)),
\]

which is also a volume-preserving mean curvature flow. They have gotten that the flow converges to a solution of the isoperimetric problem if the initial hypersurface is a smooth compact, star-shaped hypersurface.

In this paper, we consider a new type of mean curvature flow:

\[
(\frac{\partial X(t)}{\partial t})^\perp = (-\alpha(t)N(t) + H(t)),
\]

with

\[
\alpha(t) = \frac{\int_M H(t) \langle N(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu}{\int_M \langle N(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu},
\]

where \( N \) is the unit normal vector of \( X : M \to \mathbb{R}^{n+1} \). We define a weighted volume of \( M_t \) (see, section 2) by

\[
V(t) = \int_M \langle X(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu.
\]

We can prove that the flow (1.1) preserves the weighted volume \( V(t) \). Hence, we call the flow (1.1) a weighted volume-preserving mean curvature flow.

The properties of solutions of the weighted volume-preserving mean curvature flow (1.1) will be studied in Cheng and Wei [8].

This paper is organized as follows. In section 2, we give a definition of the weighted volume and the first variation formula of the weighted area functional for all weighted volume-preserving variations is given. As critical points of it, \( \lambda \)-hypersurface is
defined. Self-similar solutions of the weighted volume-preserving mean curvature flow is considered. In section 3, the basic properties of \( \lambda \)-hypersurfaces are studied. In section 4, we give a classification for compact \( \lambda \)-hypersurfaces with \( H - \lambda \geq 0 \). In sections 5 and 6, we define \( F \)-functional. The first and second variation formulas of \( F \)-functional are proved. Notation of \( F \)-stability and \( F \)-unstability of \( \lambda \)-hypersurfaces are introduced. We prove that spheres \( S^n(r) \) with \( r \leq \sqrt{n} \) or \( r > \sqrt{n + 1} \) are \( F \)-stable and spheres \( S^n(r) \) with \( \sqrt{n} < r \leq \sqrt{n + 1} \) are \( F \)-unstable. In section 7, we study the weak stability of the weighted area functional for the weighted volume-preserving variations. In section 8, a classification for complete and non-compact \( \lambda \)-hypersurfaces with polynomial area growth and \( H - \lambda \geq 0 \) is given. In sections 9 and 10, the area growth of complete and non-compact \( \lambda \)-hypersurfaces are studied.

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2. The first variation formula and \( \lambda \)-hypersurfaces

Let \( X : M^n \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional connected hypersurface of the \((n + 1)\)-dimensional Euclidean space \( \mathbb{R}^{n+1} \). We choose a local orthonormal frame field \( \{e_A\}_{A=1}^{n+1} \) in \( \mathbb{R}^{n+1} \) with dual coframe field \( \{\omega_A\}_{A=1}^{n+1} \), such that, restricted to \( M^n \), \( e_1, \cdots, e_n \) are tangent to \( M^n \). Then we have

\[
\begin{aligned}
dX &= \sum_i \omega_i e_i, \\
d e_i &= \sum_j \omega_{ij} e_j + \omega_{\text{n+1}} e_{\text{n+1}}
\end{aligned}
\]

and

\[
d e_{\text{n+1}} = \sum_i \omega_{\text{n+1}i} e_i.
\]

We restrict these forms to \( M^n \), then

\[
\omega_{\text{n+1}} = 0, \quad \omega_{\text{n+1}i} = -\sum_{j=1}^n h_{ij} \omega_j, \quad h_{ij} = h_{ji},
\]

where \( h_{ij} \) denotes the component of the second fundamental form of \( X : M^n \to \mathbb{R}^{n+1} \). \( H = \sum_{j=1}^n h_{jj} e_{\text{n+1}} \) is the mean curvature vector field, \( H = |H| = \sum_{j=1}^n h_{jj} \) is the mean curvature and \( II = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j e_{\text{n+1}} \) is the second fundamental form of \( X : M^n \to \mathbb{R}^{n+1} \). Let

\[
\begin{aligned}
f_i &= \nabla_i f, \quad f_{ij} = \nabla_j \nabla_i f, \quad h_{ijk} = \nabla_k h_{ij} \quad \text{and} \quad h_{ijkl} = \nabla_i \nabla_k h_{ij},
\end{aligned}
\]

where \( \nabla_j \) is the covariant differentiation operator. The Gauss equations and Codazzi equations are given by

\[
R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk},
\]

(2.1)

\[
h_{ijk} = h_{ikj},
\]

(2.2)
where $R_{ijkl}$ and $h_{ijk}$ denote components of curvature tensor and components of the covariant derivative of $h_{ij}$, respectively. Furthermore, we have the Ricci formula:

\begin{equation}
(2.3) \quad h_{ijkl} - h_{ijlk} = \sum_{m=1}^{n} h_{im} R_{m j k l} + \sum_{m=1}^{n} h_{mj} R_{m i k l}.
\end{equation}

For a constant vector $a \in \mathbb{R}^{n+1}$, one has

\[ \langle X, a \rangle_i = \langle e_i, a \rangle, \quad \langle N, a \rangle_i = -\sum_{j} h_{ij} \langle e_j, a \rangle, \]

\[ \langle X, a \rangle_{ij} = h_{ij} \langle N, a \rangle, \]

\[ \langle N, a \rangle_{ij} = -\sum_{k} h_{ijk} \langle e_k, a \rangle - \sum_{k} h_{ik} h_{jk} \langle N, a \rangle. \]

We call $X(t)$ a variation of $X$ if $X(t) : M \rightarrow \mathbb{R}^{n+1}$, $t \in (-\varepsilon, \varepsilon)$ is a family of immersions with $X(0) = X$. For $X_0 \in \mathbb{R}^{n+1}$ and a real number $t_0$, we define a weighted area function $A : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ by

\[ A(t) = \int_{M} e^{-\frac{|X(t) - X_0|^2}{2t_0}} d\mu, \]

where $d\mu$ is the area element of $M$ in the metric induced by $X(t)$. The weighted volume function $V : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ is defined by

\[ V(t) = \int_{M} \langle X(t) - X_0, N \rangle e^{-\frac{|X - X_0|^2}{2t_0}} d\mu. \]

Then we have the following first variation formulas of $A(t)$ and $V(t)$:

**Lemma 2.1.**

\begin{equation}
(2.4) \quad \frac{dA(t)}{dt} = \int_{M} \left( -\frac{\langle X(t) - X_0, \frac{\partial X(t)}{\partial t} \rangle}{t_0} - H(t) \langle \frac{\partial X(t)}{\partial t}, N(t) \rangle \right) e^{-\frac{|X - X_0|^2}{2t_0}} d\mu,
\end{equation}

\begin{equation}
(2.5) \quad \frac{dV(t)}{dt} = \int_{M} \langle \frac{\partial X(t)}{\partial t}, N \rangle e^{-\frac{|X - X_0|^2}{2t_0}} d\mu.
\end{equation}

Let $\frac{\partial X(t)}{\partial t} = W(t)$. Then the vector field $\frac{\partial X(t)}{\partial t}|_{t=0} = W(0) = W$ is called a variation vector field. Set $f(t) = \langle W(t), N(t) \rangle$, where $N(t)$ is the normal vector of $M$, $N(0) = N$. In this paper, we only consider the normal variation vector field, which can be expressed as $\frac{\partial X(t)}{\partial t}|_{t=0} = fN$. We say a variation of $X$ is a weighted volume-preserving variation if $V(t) = V(0)$ for all $t$, that is

\begin{equation}
(2.6) \quad 0 = \frac{dV(t)}{dt} = \int_{M} \langle \frac{\partial X(t)}{\partial t}, N \rangle e^{-\frac{|X - X_0|^2}{2t_0}} d\mu
= \int_{M} f(t) \langle N(t), N \rangle e^{-\frac{|X - X_0|^2}{2t_0}} d\mu.
\end{equation}

We can prove the following lemma using the same method as that of Lemma (2.4) of [3].
Lemma 2.2. Given a smooth function $f : M \to \mathbb{R}$ with $\int_M f e^{-\frac{|X-X_0|^2}{2t_0}} \, d\mu = 0$, there exists a weighted volume-preserving normal variation such that its variation vector field is $fN$.

Let
\[
\lambda = \frac{1}{A} \int_M \left( \frac{X - X_0}{t_0}, N \right) + H \right) e^{-\frac{|X-X_0|^2}{2t_0}} \, d\mu,
\]
with
\[
A = \int_M e^{-\frac{|X-X_0|^2}{2t_0}} \, d\mu
\]
and define $J : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by
\[
J(t) = A(t) + \lambda V(t),
\]
for constant $\lambda$. Then, one has

Proposition 2.1. Let $X : M \to \mathbb{R}^{n+1}$ be an immersion. The following statements are equivalent with each other:

1. $\left< \frac{X - X_0}{t_0}, N \right> + H = \lambda$.
2. For all weighted volume-preserving variations, $A'(0) = 0$.
3. For all arbitrary variations, $J'(0) = 0$.

Proof. From Lemma 2.1 we have (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). We next prove (2) $\Rightarrow$ (1). Assume that at a point $p \in M$, we have $\left( \left< \frac{X - X_0}{t_0}, N \right> + H - \lambda \right)(p) \neq 0$. We can assume that $\left( \left< \frac{X - X_0}{t_0}, N \right> + H - \lambda \right)(p) > 0$. Let
\[
M^+ = \{ q \in M : \left( \left< \frac{X - X_0}{t_0}, N \right> + H - \lambda \right)(q) > 0 \},
\]
\[
M^- = \{ q \in M : \left( \left< \frac{X - X_0}{t_0}, N \right> + H - \lambda \right)(q) < 0 \}.
\]
Let $\varphi$ and $\psi$ be non-negative real smooth functions on $M$ such that
\[
p \in \text{supp} \varphi \subset M^+, \quad \text{supp} \psi \subset M^-,
\]
and
\[
\int_M (\varphi + \psi) \left( \left< \frac{X - X_0}{t_0}, N \right> + H - \lambda \right) e^{-\frac{|X-X_0|^2}{2t_0}} \, d\mu = 0.
\]
Since $\int_M \left( \left< \frac{X - X_0}{t_0}, N \right> + H - \lambda \right) e^{-\frac{|X-X_0|^2}{2t_0}} \, d\mu = 0$, we know that such a choice is possible.

Let $f = (\varphi + \psi) \left( \left< \frac{X - X_0}{t_0}, N \right> + H - \lambda \right)$, then $\int_M f e^{-\frac{|X-X_0|^2}{2t_0}} \, d\mu = 0$. By Lemma 2.2 we get a weighted volume-preserving variation such that its variation vector field is $fN$. From our assumption,
\[
A'(0) = \int_M \left( -\left< \frac{X - X_0}{t_0}, N \right> - H \right) f e^{-\frac{|X-X_0|^2}{2t_0}} \, d\mu = 0.
\]
Hence, we have
\[ 0 = \int_M f\left(\frac{X - X_0}{t_0}, N\right) + H - \lambda)e^{-\frac{|X-X_0|^2}{2\sigma_0}} d\mu \]
\[ = \int_M (\varphi + \psi)\left(\frac{X - X_0}{t_0}, N\right) + H - \lambda)^2 e^{-\frac{|X-X_0|^2}{2\sigma_0}} d\mu \]
\[ > 0. \]

It is a contradiction. It follows that \( \langle \frac{X-X_0}{t_0}, N \rangle + H = \lambda. \)

**Definition 2.1.** Let \( X : M \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional immersed hypersurface in the Euclidean space \( \mathbb{R}^{n+1} \). If \( \langle \frac{X-X_0}{t_0}, N \rangle + H = \lambda \) holds, we call \( X : M \to \mathbb{R}^{n+1} \) a \( \lambda \)-hypersurface of the weighted volume-preserving mean curvature flow.

**Remark 2.1.** If \( \lambda = 0 \), then the \( \lambda \)-hypersurface is a self-shrinker of the mean curvature flow. Hence, we know that the notation of the \( \lambda \)-hypersurface is a generalization of the self-shrinker.

**Theorem 2.1.** Let \( X : M \to \mathbb{R}^{n+1} \) be an immersed hypersurface. The following statements are equivalent with each other:

1. \( X : M \to \mathbb{R}^{n+1} \) is a \( \lambda \)-hypersurface.
2. \( X : M \to \mathbb{R}^{n+1} \) is a critical point of the weighted area functional \( A(t) \) for all weighted volume-preserving variations.
3. \( X : M \to \mathbb{R}^{n+1} \) is a hypersurface with constant mean curvature \( \lambda \) in \( \mathbb{R}^{n+1} \) with respect to the metric \( g_{ij} = e^{-\frac{|X-X_0|^2}{n\sigma_0}} \delta_{ij} \).

**Example 2.1.** The \( n \)-dimensional sphere \( S^n(r) \) with radius \( r > 0 \) is a compact \( \lambda \)-hypersurface in \( \mathbb{R}^{n+1} \) with \( \lambda = \frac{n}{r} - r \). It should be remarked that the sphere \( S^n(\sqrt{n}) \) is the only self-shrinker sphere in \( \mathbb{R}^{n+1} \).

**Example 2.2.** For \( 1 \leq k \leq n - 1 \), the \( n \)-dimensional cylinder \( S^k(r) \times \mathbb{R}^{n-k} \) with radius \( r > 0 \) is a complete and non-compact \( \lambda \)-hypersurface in \( \mathbb{R}^{n+1} \) with \( \lambda = \frac{k}{r} - r \). We should notice that the cylinder \( S^k(\sqrt{k}) \times \mathbb{R}^{n-k} \) is the only self-shrinker cylinder in \( \mathbb{R}^{n+1} \).

**Proposition 2.2.** Let \( X : M \to \mathbb{R}^{n+1} \) be a \( \lambda \)-hypersurface in the Euclidean space \( \mathbb{R}^{n+1} \). If the mean curvature \( H \) is constant, then \( X : M \to \mathbb{R}^{n+1} \) is isometric to \( S^k(r) \times \mathbb{R}^{n-k} \), \( 0 \leq k \leq n \), locally.

**Proof.** Since \( X : M \to \mathbb{R}^{n+1} \) is a \( \lambda \)-hypersurface, we have \( \langle X, N \rangle + H = \lambda \). If \( H \) is constant, we get, for any \( 1 \leq i \leq n \),
\[ \nabla_i \langle X, N \rangle = -\lambda_i \langle X, e_i \rangle = 0, \]
where \( \lambda_i \) is the principal curvature of the \( \lambda \)-hypersurface. If \( \lambda_{i_0} \neq 0 \) at a point \( p \) for some \( i_0 \), there exists a neighborhood \( U \) of \( p \) such that \( \lambda_{i_0} \neq 0 \) in \( U \). Hence, we know \( \langle X, e_{i_0} \rangle = 0 \) in \( U \). Thus,
\[ X = \sum_{j \neq i_0} \langle X, e_j \rangle e_j + \langle X, N \rangle N. \]
We obtain

\[ e_{i_0} = \nabla_{i_0} X = -\langle X, N \rangle \lambda_{i_0} e_{i_0}, \]

that is, \( \lambda_{i_0}(H - \lambda) = 1 \) is constant. Thus, on \( U \), \( \lambda_{i_0} \) is constant. Therefore, the \( \lambda \)-hypersurface is isoparametric. We obtain that \( X : M \to \mathbb{R}^{n+1} \) is isometric to \( S^k(r) \times \mathbb{R}^{n-k}, 0 \leq k \leq n \), locally. \( \square \)

**Definition 2.2.** A family of \( n \)-dimensional immersed hypersurfaces \( X(t) : M \to \mathbb{R}^{n+1} \) in the Euclidean space \( \mathbb{R}^{n+1} \) is called a self-similar solution of the weighted volume-preserving mean curvature flow if \( X(t) = \beta(t)X \) holds, where \( \beta(t) > 0 \).

**Proposition 2.3.** A family of \( n \)-dimensional immersed hypersurfaces \( X(t) : M \to \mathbb{R}^{n+1} \) in the Euclidean space \( \mathbb{R}^{n+1} \) is a self-similar solution of the weighted volume-preserving mean curvature flow if and only if \( X(t) = \sqrt{1 + \beta_0 t}X \), where \( \beta_0 \) is a constant.

**Proof.** If \( X(t) : M \to \mathbb{R}^{n+1} \) is a self-similar solution of the weighted volume-preserving mean curvature flow, we have \( X(t) = \beta(t)X \). Hence, the mean curvature \( H(t) \) of \( X(t) \) satisfies

\[ H(t) = \frac{H}{\beta(t)}. \]

Thus,

\[ \alpha(t) = \frac{\int_M H(t) \langle N(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu}{\int_M \langle N(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu} = \frac{\int_M H e^{-\frac{|X|^2}{2}} d\mu}{\beta(t) \int_M e^{-\frac{|X|^2}{2}} d\mu}. \]

From the equation of the weighted volume-preserving mean curvature flow, we have

\[ \frac{\partial \beta(t)}{\partial t} X^+, = \frac{1}{\beta(t)} \left(-\alpha(0)N + H\right). \tag{2.8} \]

We obtain \( \frac{\partial \beta(t)}{\partial t} = \beta_0 = \text{constant} \). Since \( \beta(0) = 1 \), we have \( \beta(t) = \sqrt{1 + \beta_0 t} \).

The inverse is obvious. \( \square \)

**Proposition 2.4.** Let \( X : M \to \mathbb{R}^{n+1} \) be a \( \lambda \)-hypersurface in the Euclidean space \( \mathbb{R}^{n+1} \). If \( X(t) = \sqrt{1 + \beta_0 t}X \) is a self-similar solution of the weighted volume-preserving mean curvature flow, then \( X : M \to \mathbb{R}^{n+1} \) is isometric to \( S^k(r) \times \mathbb{R}^{n-k}, 0 \leq k \leq n \), locally or \( V(0) = 0 \) and \( \beta_0 = -2 \).

**Proof.** Since \( X : M \to \mathbb{R}^{n+1} \) is a \( \lambda \)-hypersurface, we have \( \langle X, N \rangle + H = \lambda \) and

\[ V(t) = \int_M \langle X(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu = \sqrt{1 + \beta_0 t}V(0). \]

Since \( X(t) = \sqrt{1 + \beta_0 t}X \) is a self-similar solution of the weighted volume-preserving mean curvature flow, then \( \beta_0 = 0 \) or \( V(0) = 0 \). If \( \beta_0 = 0 \), then \( H \) is constant from \( (2.8) \). According to the proposition 2.2 we know that \( X : M \to \mathbb{R}^{n+1} \) is isometric to \( S^k(r) \times \mathbb{R}^{n-k}, 0 \leq k \leq n \), locally. If \( \beta_0 \neq 0 \), we have \( V(0) = 0 \) since \( V(t) \) is constant. The \( (2.8) \) gives \( \beta_0 = -2 \). \( \square \)
Definition 2.3. If $X : M \to \mathbb{R}^{n+1}$ is an $n$-dimensional hypersurface in $\mathbb{R}^{n+1}$, we say that $M$ has polynomial area growth if there exist constant $C$ and $d$ such that for all $r \geq 1$,
\[
\text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0) \cap X(M)} \, d\mu \leq Cr^d,
\]
where $B_r(0)$ is a standard ball in $\mathbb{R}^{n+1}$ with radius $r$ and centered at the origin.

3. Properties of $\lambda$-hypersurfaces

In this section, we give several properties of $\lambda$-hypersurfaces. We define an elliptic operator $\mathcal{L}$ by
\[
\mathcal{L}f = \Delta f - \langle X, \nabla f \rangle,
\]
where $\Delta$ and $\nabla$ denote the Laplacian and the gradient operator of the $\lambda$-hypersurface, respectively. We should notice that the $\mathcal{L}$ operator was introduced by Colding and Minicozzi in [10] for self-shrinkers.

By a direct calculation, for a constant vector $a \in \mathbb{R}^{n+1}$, we have
\[
\mathcal{L}\langle X, a \rangle = \Delta \langle X, a \rangle - \langle X, \nabla \langle X, a \rangle \rangle \\
= \sum_i \langle X, a \rangle,_{ii} - \sum_i \langle X, a \rangle,_{i} \langle X, e_i \rangle \\
= \langle HN, a \rangle - \sum_i \langle e_i, a \rangle \langle X, e_i \rangle \\
= \langle HN, a \rangle - \langle X, a \rangle + \langle X, N \rangle \langle N, a \rangle \\
= \lambda \langle N, a \rangle - \langle X, a \rangle,
\]
\[
\mathcal{L}\langle N, a \rangle = \sum_i \langle N, a \rangle,_{ii} - \sum_i \langle N, a \rangle,_{i} \langle X, e_i \rangle \\
= \langle -H_i e_i - SN, a \rangle + \sum_i \langle X, e_i \rangle \langle \sum_j h_{ij} e_j, a \rangle \\
= \langle X, N \rangle,_{i} \langle e_i, a \rangle - \langle SN, a \rangle + \sum_i \langle X, e_i \rangle \langle \sum_j h_{ij} e_j, a \rangle \\
= -S \langle N, a \rangle,
\]
where $S = \sum_{i,j} h_{ij}^2$ is the squared norm of the second fundamental form.

\[
\frac{1}{2} \mathcal{L}(|X|^2) = \langle \Delta X, X \rangle + \sum_i \langle X, X,_{i} \rangle - \sum_i \langle X, e_i \rangle \langle X, e_i \rangle \\
= n - |X|^2 + \lambda \langle X, N \rangle.
\]

Hence, we have the following

Lemma 3.1. If $X : M \to \mathbb{R}^{n+1}$ is a $\lambda$-hypersurface, then we have
\[
\mathcal{L}\langle X, a \rangle = \lambda \langle N, a \rangle - \langle X, a \rangle,
\]
\[ (3.3) \quad \mathcal{L}\langle N, a \rangle = -S\langle N, a \rangle, \]

\[ (3.4) \quad \frac{1}{2} \mathcal{L}(|X|^2) = n - |X|^2 + \lambda \langle X, N \rangle. \]

The following lemma due to Colding and Minicozzi [10] is needed in order to prove our results.

**Lemma 3.2.** If \( X : M \to \mathbb{R}^{n+1} \) is a hypersurface, \( u \) is a \( C^1 \)-function with compact support and \( v \) is a \( C^2 \)-function, then

\[ (3.5) \quad \int_M u(\mathcal{L}v)e^{-\frac{|X|^2}{2}}d\mu = -\int_M \langle \nabla u, \nabla v \rangle e^{-\frac{|X|^2}{2}}d\mu. \]

**Corollary 3.1.** Let \( X : M \to \mathbb{R}^{n+1} \) be a complete hypersurface. If \( u, v \) are \( C^2 \) functions satisfying

\[ (3.6) \quad \int_M (|u\nabla v| + |\nabla u| |\nabla v| + |u\mathcal{L}v|) e^{-\frac{|X|^2}{2}}d\mu < +\infty, \]

then we have

\[ (3.7) \quad \int_M u(\mathcal{L}v)e^{-\frac{|X|^2}{2}}d\mu = -\int_M \langle \nabla u, \nabla v \rangle e^{-\frac{|X|^2}{2}}d\mu. \]

**Lemma 3.3.** Let \( X : M \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional complete \( \lambda \)-hypersurface with polynomial area growth, then

\[ (3.8) \quad \int_M (\langle X, a \rangle - \lambda \langle N, a \rangle) e^{-\frac{|X|^2}{2}}d\mu = 0, \]

\[ (3.9) \quad \int_M (n - |X|^2 + \lambda \langle X, N \rangle) e^{-\frac{|X|^2}{2}}d\mu = 0, \]

\[ (3.10) \quad \int_M \langle X, a \rangle |X|^2 e^{-\frac{|X|^2}{2}}d\mu \]

\[ = \int_M \left( 2n\lambda \langle N, a \rangle + 2\lambda \langle X, a \rangle (\lambda - H) - \lambda \langle N, a \rangle |X|^2 \right) e^{-\frac{|X|^2}{2}}d\mu, \]

\[ (3.11) \quad \int_M \langle X, a \rangle^2 e^{-\frac{|X|^2}{2}}d\mu = \int_M \left( |a^T|^2 + \lambda \langle N, a \rangle \langle X, a \rangle \right) e^{-\frac{|X|^2}{2}}d\mu, \]

where \( a^T = \sum_i < a, e_i > e_i \).

\[ (3.12) \quad \int_M \left( |X|^2 - n - \frac{\lambda(\lambda - H)}{2} \right)^2 e^{-\frac{|X|^2}{2}}d\mu \]

\[ = \int_M \left\{ \left( \frac{\lambda^2}{4} - 1 \right)(\lambda - H)^2 + 2n - H^2 + \lambda^2 \right\} e^{-\frac{|X|^2}{2}}d\mu. \]
Proof. Equations (3.8) and (3.9) just follow from the corollary 3.1 and equations (3.2), and (3.4). Since $X : M \to \mathbb{R}^{n+1}$ is an $n$-dimensional complete $\lambda$-hypersurface with polynomial area growth, by making use of $u = |X|^2$, $v = \langle X, a \rangle$ in the lemma 3.2, we have

$$\int_M \langle X, a \rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu$$

$$= -\int_M \mathcal{L}\langle X, a \rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu + \int_M \lambda \langle N, a \rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu$$

$$= -\int_M \langle X, a \rangle \mathcal{L}|X|^2 e^{-\frac{|X|^2}{2}} d\mu + \int_M \lambda \langle N, a \rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu$$

$$= -\int_M 2\langle X, a \rangle [n + \lambda \langle X, N \rangle - |X|^2] e^{-\frac{|X|^2}{2}} d\mu + \int_M \lambda \langle N, a \rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu$$

$$= 2 \int_M \langle X, a \rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu - 2n \int_M \langle X, a \rangle - 2\lambda \langle X, a \rangle (\lambda - H) e^{-\frac{|X|^2}{2}} d\mu$$

$$+ \int_M \lambda \langle N, a \rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu.$$

Hence, it follows that

$$\int_M \langle X, a \rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu$$

$$= \int_M \left( 2n\lambda \langle N, a \rangle + 2\lambda \langle X, a \rangle (\lambda - H) - \lambda \langle N, a \rangle |X|^2 \right) e^{-\frac{|X|^2}{2}} d\mu.$$

Taking $u = v = \langle X, a \rangle$ in Lemma 3.2 we can get (3.11). Putting $u = v = |X|^2$ in Lemma 3.2 we can have

$$\int_M \lambda (\lambda - H)|X|^2 e^{-\frac{|X|^2}{2}} d\mu$$

$$= \int_M (|X|^4 - n|X|^2 + \frac{1}{2}|X|^2 \mathcal{L}|X|^2) e^{-\frac{|X|^2}{2}} d\mu$$

$$= \int_M (|X|^4 - n|X|^2) e^{-\frac{|X|^2}{2}} d\mu - \int_M \frac{1}{2} \langle \nabla |x|^2, \nabla |x|^2 \rangle e^{-\frac{|X|^2}{2}} d\mu$$

$$= \int_M (|X|^4 - (n + 2)|X|^2 + 2(\lambda - H)^2) e^{-\frac{|X|^2}{2}} d\mu,$$

that is,

$$\int_M \left( |X|^4 - [n + \lambda (\lambda - H)]|X|^2 - 2|X|^2 + 2(\lambda - H)^2 \right) e^{-\frac{|X|^2}{2}} d\mu = 0.$$

Thus, we have
\begin{align*}
0 &= \int_M \left\{ |X|^4 - 2[n + \frac{(\lambda - H)\lambda}{2}]|X|^2 + n^2 + n\lambda(\lambda - H) \\
&\quad - 2n - 2\lambda(\lambda - H) + 2(\lambda - H)^2 \right\} e^{-\frac{|X|^2}{2}} d\mu \\
&= \int_M \left\{ \left( |X|^2 - (n + \frac{\lambda(\lambda - H)}{2}) \right)^2 - \frac{\lambda^2(\lambda - H)^2}{4} + 2(\lambda - H)^2 \\
&\quad - 2n - 2\lambda(\lambda - H) \right\} e^{-\frac{|X|^2}{2}} d\mu \\
&= \int_M \left\{ \left( |X|^2 - n - \frac{\lambda(\lambda - H)}{2} \right)^2 - \left( \frac{\lambda^2}{4} - 1 \right)(\lambda - H)^2 - 2n + H^2 - \lambda^2 \right\} e^{-\frac{|X|^2}{2}} d\mu,
\end{align*}

namely,
\begin{align*}
\int_M \left( |X|^2 - n - \frac{\lambda(\lambda - H)}{2} \right)^2 e^{-\frac{|X|^2}{2}} d\mu \\
= \int_M \left\{ \left( \frac{\lambda^2}{4} - 1 \right)(\lambda - H)^2 + 2n - H^2 + \lambda^2 \right\} e^{-\frac{|X|^2}{2}} d\mu.
\end{align*}

\[\Box\]

4. A classification of compact \(\lambda\)-hypersurfaces

In this section, we will give a classification of compact \(\lambda\)-hypersurfaces. First of all, we give some lemmas.

**Lemma 4.1.** Let \(X : M \to \mathbb{R}^{n+1}\) be an \(n\)-dimensional \(\lambda\)-hypersurface. Then, the following holds.

\[(4.1)\quad \mathcal{L}H = H + S(\lambda - H),\]

\[(4.2)\quad \frac{1}{2}\mathcal{L}S = \sum_{i,j,k} h_{ijk}^2 + (1 - S)S + \lambda f_3,\]

\[(4.3)\quad \mathcal{L}\sqrt{S} = \frac{1}{\sqrt{S}} \left( \sum_{i,j,k} h_{ijk}^2 - |\nabla \sqrt{S}|^2 \right) + \sqrt{S}(1 - S) + \frac{1}{\sqrt{S}} \lambda f_3,\]

\[(4.4)\quad \mathcal{L}\log(H - \lambda) = 1 - S + \frac{\lambda}{H - \lambda} - |\nabla \log(H - \lambda)|^2, \quad \text{if } H - \lambda > 0,\]

where \(f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki}.\)

**Proof.** Since \(\langle X, N \rangle + H = \lambda\), one has

\[(4.5)\quad H_i = \sum_j h_{ij} \langle X, e_j \rangle,\]
\[ H_{ik} = \sum_j h_{ijk} \langle X, e_j \rangle + h_{ik} + \sum_j h_{ij} h_{jk} (\lambda - H). \]

Hence,
\[(4.6) \quad \Delta H = \sum_i H_{ii} = \sum_i H_{i} \langle X, e_i \rangle + H + S(\lambda - H)\]

and
\[ \mathcal{L}H = \Delta H - \sum_i \langle X, e_i \rangle H_{ii} = H + S(\lambda - H). \]

By a direct calculation, we have from (2.3)
\[ \mathcal{L}h_{ij} = \Delta h_{ij} - \sum_k \langle X, e_k \rangle h_{ijk} = (1 - S)h_{ij} + \lambda \sum_k h_{ik} h_{kj}. \]

Then it follows that
\[ \frac{1}{2} \mathcal{L}S = \frac{1}{2} \left( \Delta \sum_{i,j} h_{ij}^2 - \sum_k \langle X, e_k \rangle \left( \sum_{i,j} h_{ij}^2 \right)_k \right) \]
\[ = \sum_{i,j,k} h_{ijk}^2 + (1 - S)S + \lambda f_3. \]

Since
\[(4.7) \quad \mathcal{L}S = 2 |\nabla \sqrt{S}|^2 + 2 \sqrt{S} \mathcal{L} \sqrt{S}, \]

we have
\[ \mathcal{L} \sqrt{S} = \frac{1}{2 \sqrt{S}} \mathcal{L}S - \frac{|\nabla \sqrt{S}|^2}{\sqrt{S}} \]
\[ = \frac{1}{\sqrt{S}} \left( \sum_{i,j,k} h_{ijk}^2 - |\nabla \sqrt{S}|^2 \right) + \sqrt{S}(1 - S) + \frac{1}{\sqrt{S}} \lambda f_3. \]

\[ \mathcal{L} \log(H - \lambda) = \Delta \log(H - \lambda) - \sum_i \langle X, e_i \rangle (\log(H - \lambda))_i \]
\[ = \frac{1}{H - \lambda} \mathcal{L}H - |\nabla \log(H - \lambda)|^2 \]
\[ = 1 - S + \frac{\lambda}{H - \lambda} - |\nabla \log(H - \lambda)|^2. \]

We complete the proof of the lemma.

\[ \square \]

**Theorem 4.1.** Let \( X : M \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional compact \( \lambda \)-hypersurface in \( \mathbb{R}^{n+1} \). If \( H - \lambda \geq 0 \) and \( \lambda (f_3(H - \lambda) - S) \geq 0 \), then \( X : M \to \mathbb{R}^{n+1} \) is isometric to a round sphere \( S^n(r) \) with \( \lambda = \frac{n}{r} - r \).
Proof. Since
\[ \mathcal{L}H = H + S(\lambda - H) \]
and
\[ H - \lambda \geq 0, \]
we have
\[ \mathcal{L}H - H \leq 0. \]
If \( \lambda \leq 0 \), we conclude from the maximum principle that either \( H \equiv \lambda \) or \( H - \lambda > 0 \). If \( H \equiv \lambda \), (4.6) gives that \( H = \lambda = 0 \) and \( H - \lambda \geq 0 \). In this case, if \( H - \lambda = 0 \) at some point \( p \in M \), then \( S = 0 \) and \( H = \lambda = 0 \) at \( p \), that is \( \lambda \equiv 0 \) and \( M \) is self-shrinker, it is also impossible since \( M \) is compact. Hence for any \( \lambda \), we have \( H - \lambda > 0 \).
From the lemma 4.1, we can get
\[ \mathcal{L} \frac{1}{(H - \lambda)^2} = \Delta \frac{1}{(H - \lambda)^2} - \sum_i \langle X, e_i \rangle \left( \frac{1}{(H - \lambda)^2} \right)_i \]
\[ = \frac{6}{(H - \lambda)^4} |\nabla (H - \lambda)|^2 - \frac{2}{(H - \lambda)^3} [H - S(H - \lambda)] \]
and
\[ \mathcal{L} \frac{S}{(H - \lambda)^2} = \Delta \frac{S}{(H - \lambda)^2} - \sum_i \langle X, e_i \rangle \left( \frac{S}{(H - \lambda)^2} \right)_i \]
\[ = \frac{1}{(H - \lambda)^2} \mathcal{L}S + 2 \langle \nabla S, \nabla \left( \frac{1}{(H - \lambda)^2} \right) \rangle + S \mathcal{L} \left( \frac{1}{(H - \lambda)^2} \right) \]
\[ = \frac{2}{(H - \lambda)^2} \left( \sum_{i,j,k} h^2_{ijk} + (1 - S)S + \lambda f_3 \right) + 2 \langle \nabla S, \nabla \left( \frac{1}{(H - \lambda)^2} \right) \rangle \]
\[ + S \left( \frac{6}{(H - \lambda)^4} |\nabla (H - \lambda)|^2 - \frac{2}{(H - \lambda)^3} [H - S(H - \lambda)] \right). \]
By multiplying \( Se^{-\frac{|x|^2}{2}} \) in the above equation and using
\[ \int_M \mathcal{L} \frac{S}{(H - \lambda)^2} e^{-\frac{|x|^2}{2}} d\mu = - \int_M \langle \nabla S, \nabla \left( \frac{S}{(H - \lambda)^2} \right) \rangle e^{-\frac{|x|^2}{2}} d\mu, \]
one has
\[ 2 \int_M \frac{S}{(H - \lambda)^4} \sum_{i,j,k} |h_{ijk}(H - \lambda) - h_{ij} H_k|^2 e^{-\frac{|x|^2}{2}} d\mu \]
\[ + \int_M |\nabla \left( \frac{S}{(H - \lambda)^2} \right)|^2 (H - \lambda)^2 e^{-\frac{|x|^2}{2}} d\mu \]
\[ + 2 \int_M \frac{S}{(H - \lambda)^2} \lambda \left( f_3 - \frac{S}{H - \lambda} \right) e^{-\frac{|x|^2}{2}} d\mu = 0. \]
Then it follows from \( \lambda(f_3(H - \lambda) - S) \geq 0 \) that
\[
\lambda(f_3 - \frac{S}{H - \lambda}) = 0, \quad \frac{S}{(H - \lambda)^2} = \text{constant}, \quad h_{ijk}(H - \lambda) = h_{ij}H_k.
\]

We next consider two cases.

**Case 1:** \( \lambda = 0 \)

In this case, we know \( M \) is isometric to \( S^n(\sqrt{n}) \) from Huisken’s result [18].

**Case 2:** \( \lambda \neq 0 \)

In this case, one gets
\[
f_3 - \frac{S}{H - \lambda} = 0, \quad h_{ijk}(H - \lambda) = h_{ij}H_k.
\]

If \( H \) is constant, then \( h_{ijk} = 0 \), thus \( M \) is \( S^n(r) \) by the result of Lawson [21].

If \( H \) is not constant, then there exists a neighborhood \( U \) such that \( |\nabla H| \neq 0 \) on \( U \). We can choose \( e_1, \ldots, e_n \) such that \( e_1 = \frac{\nabla H}{|\nabla H|} \). It follows from \( h_{ijk} = h_{ikj} \) that
\[
0 = \sum_{i,j,k} |h_{ij}H_k - h_{ik}H_j|^2
= 2S|\nabla H|^2 - 2 \sum_{i} h_{1i}^2|\nabla H|^2
= 2|\nabla H|^2(S - \sum_{i} h_{1i}^2),
\]

that is,
\[
\sum_{i=1}^n h_{1i}^2 = S = h_{11}^2 + 2 \sum_{j \neq 1} h_{1j}^2 + \sum_{k,l \geq 2} h_{kl}^2.
\]

Therefore, \( S = h_{11}^2 = H^2 \) on \( U \). On the other hand, we see from \( \frac{S}{(H - \lambda)^2} = \text{constant} \) that \( H \) is constant on \( U \). It is a contradiction. The proof of the theorem 4.1 is completed.

\[\square\]

**Remark 4.1.** In the theorem 4.1 we assume \( \lambda(f_3(H - \lambda) - S) \geq 0 \), which is satisfied for self-shrinkers of the mean curvature flow, automatically. We do not know whether the assumption is essential. In particular, for case of complete and non-compact \( \lambda \)-hypersurfaces, this condition will be used for many times in section 8.

## 5. The First Variation of \( \mathcal{F} \)-Functional

In this section, we will give another variational characterization of \( \lambda \)-hypersurfaces. Let \( X(s) : M \to \mathbb{R}^{n+1} \) be immersions with \( X(0) = X \). The variation vector field \( \frac{\partial}{\partial s} X(s) \big|_{s=0} \) is the normal variation vector field \( fN \).

For \( X_0 \in \mathbb{R}^{n+1} \) and a real number \( t_0 \), the \( \mathcal{F} \)-functional is defined by
\[
\mathcal{F}_{X_0,t_0}(s) = \mathcal{F}_{X_0,t_0}(X(s))
= (4\pi t_s)^{-\frac{n}{2}} \int_M e^{-\frac{|X(s) - X|}{2t_s}} d\mu_s + \lambda(4\pi t_0)^{-\frac{n}{2}}(t_0)^\frac{1}{2} \int_M \langle X(s) - X_0, N \rangle e^{-\frac{|X(s) - X_0|^2}{2t_0}} d\mu,
\]
where $X_s$ and $t_s$ denote the variations of $X_0$ and $t_0$. Let

$$
\frac{\partial t_s}{\partial s} = h(s), \quad \frac{\partial X_s}{\partial s} = y(s), \quad \frac{\partial X(s)}{\partial s} = f(s)N(s),
$$

one calls that $X : M \to \mathbb{R}^{n+1}$ is a critical point of $F_{X,s,t_s}(s)$ if it is critical with respect to all normal variations and all variations in $X_0$ and $t_0$.

**Lemma 5.1.** Let $X(s)$ be a variation of $X$ with normal variation vector field $\frac{\partial X(s)}{\partial s}\big|_{s=0} = fN$. If $X_s$ and $t_s$ are variations of $X_0$ and $t_0$ with $\frac{\partial X(s)}{\partial s}\big|_{s=0} = y$ and $\frac{\partial t_s}{\partial s}\big|_{s=0} = h$, then the first variation formula of $F_{X,s,t_s}(s)$ is given by

$$
F'_{X_0,t_0}(0) = (4\pi t_0)^{-\frac{n}{2}} \int_M \left( \lambda - (H + \langle \frac{X - X_0}{t_0}, N \rangle) \right) e^{-\frac{|X - X_0|^2}{2t_0}} d\mu
$$

$$
+ (4\pi t_0)^{-\frac{n}{2}} \int_M \left( \langle \frac{X - X_0}{t_0}, y \rangle - \lambda \langle N, y \rangle \right) e^{-\frac{|X - X_0|^2}{2t_0}} d\mu
$$

$$
+ (4\pi t_0)^{-\frac{n}{2}} \int_M \left( \|X - X_0\|^2 - n - \lambda \langle X - X_0, N \rangle \right) \frac{h}{2t_0} e^{-\frac{|X - X_0|^2}{2t_0}} d\mu.
$$

**Proof.** Defining

$$
A(s) = \int_M e^{-\frac{|X(s) - X_0|^2}{4t_s}} d\mu_s, \quad V(s) = \int_M \langle X(s) - X_0, N \rangle e^{-\frac{|X - X_0|^2}{2t_0}} d\mu,
$$

then

$$
F'_{X,s,t_s}(s) = (4\pi t_0)^{-\frac{n}{2}} A'(s) + \lambda (4\pi t_0)^{-\frac{n}{2}} \left( \frac{t_0}{t_s} \right)^{\frac{1}{2}} V'(s)
$$

$$
- (4\pi t_0)^{-\frac{n}{2}} \frac{n}{2t_s} hA(s) - \lambda (4\pi t_0)^{-\frac{n}{2}} \left( \frac{t_0}{t_s} \right)^{\frac{1}{2}} \frac{h}{2t_s} V(s).
$$

Since

$$
A'(s) = \int_M \left\{ -\left( \frac{X(s) - X_0}{t_s}, \frac{\partial X(s)}{\partial s} - \frac{\partial X}{\partial s} \right) + \frac{|X(s) - X_0|^2}{2t_s^2} h
$$

$$
- H(s) \left( \frac{\partial X(s)}{\partial s}, N(s) \right) \right\} e^{-\frac{|X(s) - X_0|^2}{4t_s}} d\mu_s,
$$

$$
V'(s) = \int_M \left( \frac{\partial X(s)}{\partial s} - \frac{\partial X}{\partial s}, N \right) e^{-\frac{|X - X_0|^2}{2t_0}} d\mu,
$$
we have
\[
\begin{align*}
F'_{X_s, t_s}(s) &= (4\pi t_s)^{-\frac{n}{2}} \int_M -\left( H_s + \left( \frac{X(s) - X_s}{t_s}, N(s) \right) \right) \left( \frac{1}{2} \right)^{\frac{X(s) - X_s}{t}} d\mu_s \\
&+ (4\pi t_s)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t}} \int_M \lambda f(N(s), N) e^{-\frac{|X - X_0|^2}{2 t_0}} d\mu \\
&+ (4\pi t_s)^{-\frac{n}{2}} \int_M \left( \frac{X(s) - X_s}{t_s}, y \right) e^{-\frac{|X(s) - X_s|^2}{2 t_s}} d\mu_s \\
&+ (4\pi t_s)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t}} \int_M \lambda (-y, N) e^{-\frac{|X - X_0|^2}{2 t_0}} d\mu \\
&+ (4\pi t_s)^{-\frac{n}{2}} \int_M \left( \frac{|X - X_0|^2}{2 t_s} - n - \lambda \langle X - X_0, N \rangle \right) \left( \frac{1}{2} \right)^{\frac{|X - X_0|^2}{2 t_0}} d\mu.
\end{align*}
\]

If \( s = 0 \), then \( X(0) = X, X_s = X_0, t_s = t_0 \) and
\[
\begin{align*}
F'_{X_0, t_0}(0) &= (4\pi t_0)^{-\frac{n}{2}} \int_M \left( \lambda - \left( H + \left( \frac{X - X_0}{t_0}, N \right) \right) \right) \left( \frac{1}{2} \right)^{\frac{|X - X_0|^2}{2 t_0}} d\mu \\
&+ (4\pi t_0)^{-\frac{n}{2}} \int_M \left( \frac{X - X_0}{t_0}, y \right) - \lambda \langle N, y \rangle \left( \frac{1}{2} \right)^{\frac{|X - X_0|^2}{2 t_0}} d\mu \\
&+ (4\pi t_0)^{-\frac{n}{2}} \int_M \left( \frac{|X - X_0|^2}{2 t_0} - n - \lambda \langle X - X_0, N \rangle \right) \left( \frac{1}{2} \right)^{\frac{|X - X_0|^2}{2 t_0}} d\mu.
\end{align*}
\]

From the lemma 5.1, we know that if \( X : M \to \mathbb{R}^{n+1} \) is a critical point of \( F \)-functional \( F_{X_s, t_s}(s) \), then
\[
H + \langle \frac{X - X_0}{t_0}, N \rangle = \lambda.
\]

We next prove that if \( H + \langle \frac{X - X_0}{t_0}, N \rangle = \lambda \), then \( X : M \to \mathbb{R}^{n+1} \) must be a critical point of \( F \)-functional \( F_{X_s, t_s}(s) \). For simplicity, we only consider the case of \( X_0 = 0 \) and \( t_0 = 1 \). In this case, \( H + \langle \frac{X - X_0}{t_0}, N \rangle = \lambda \) becomes
\[
(5.3) \quad H + \langle X, N \rangle = \lambda.
\]

Furthermore, we know that \((M, X_0, t_0)\) is the critical point of the \( F \)-functional if and only if \( M \) is the critical point of \( F \)-functional with respect to fixed \( X_0 \) and \( t_0 \).

**Theorem 5.1.** \( X : M \to \mathbb{R}^{n+1} \) is a critical point of \( F_{X_s, t_s}(s) \) if and only if
\[
H + \langle \frac{X - X_0}{t_0}, N \rangle = \lambda.
\]
Proof. We only prove the result for $X_0 = 0$ and $t_0 = 1$. In this case, the first variation formula (5.1) becomes

$$F'_0,1(0) = (4\pi)^{-\frac{n}{2}} \int_M \left( \lambda - (H + \langle X, N \rangle) \right) f e^{-\frac{|X|^2}{2}} d\mu$$

(5.4)

$$+ (4\pi)^{-\frac{n}{2}} \int_M \left( \langle X, y \rangle - \lambda \langle N, y \rangle \right) e^{-\frac{|X|^2}{2}} d\mu$$

$$+ (4\pi)^{-\frac{n}{2}} \int_M \left( |X|^2 - n - \lambda \langle X, N \rangle \right) \frac{h}{2} e^{-\frac{|X|^2}{2}} d\mu.$$

If $X : M \to \mathbb{R}^{n+1}$ is a critical point of $F_{0,1}$, then $X : M \to \mathbb{R}^{n+1}$ should satisfy $H + \langle X, N \rangle = \lambda$. Conversely, if $H + \langle X, N \rangle = \lambda$ is satisfied, then we know that $X : M \to \mathbb{R}^{n+1}$ is a $\lambda$-hypersurface. Therefore, the last two terms in (5.4) vanish for any $h$ and any $y$ from (3.8) and (3.9) of the lemma 3.3. Therefore $X : M \to \mathbb{R}^{n+1}$ is a critical point of $F_{0,1}$. □

Corollary 5.1. $X : M \to \mathbb{R}^{n+1}$ is a critical point of $F_{X,s,t}(s)$ if and only if $M$ is the critical point of $F$-functional with respect to fixed $X_0$ and $t_0$.

6. The second variation of $F$-functional

In this section, we shall give the second variation formula of $F$-functional.

Theorem 6.1. Let $X : M \to \mathbb{R}^{n+1}$ be a critical point of the functional $F(s) = F_{X,s,t}(s)$. The second variation formula of $F(s)$ for $X_0 = 0$ and $t_0 = 1$ is given by

$$(4\pi)^{\frac{n}{2}} F''(0) = - \int_M f L f e^{-\frac{|X|^2}{2}} d\mu + \int_M \left( -|y|^2 + \langle X, y \rangle^2 \right) e^{-\frac{|X|^2}{2}} d\mu$$

$$+ \int_M \left\{ 2 \langle N, y \rangle + (n+1-|X|^2)\lambda h - 2hH - 2\lambda \langle X, y \rangle \right\} f e^{-\frac{|X|^2}{2}} d\mu$$

$$+ \int_M \left\{ \lambda \langle N, y \rangle - (n+2)\langle X, y \rangle + \langle X, y \rangle |X|^2 \right\} h e^{-\frac{|X|^2}{2}} d\mu$$

$$+ \int_M \left\{ \frac{n^2 + 2n}{4} - \frac{n+2}{2} |X|^2 + \frac{|X|^4}{4} + \frac{3\lambda}{4} (\lambda - H) \right\} h^2 e^{-\frac{|X|^2}{2}} d\mu,$$

where the operator $L$ is defined by

$L = \mathcal{L} + S + 1 - \lambda^2$. 
\[ \mathcal{F}''(s) = \left(4\pi t_s\right)^{-\frac{n}{2}} \int_M \left( -\frac{\partial^2}{\partial s^2} + \frac{1}{2} \left( \frac{X(s) - X_s}{t_s} \right) \right) \rho(s) e^{-\frac{|X(s) - X_s|^2}{2t_s^2}} d\mu_s \\
+ \left(4\pi t_0\right)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \int_M \lambda f(N(s), N) \rho(s) e^{-\frac{|X(s) - X_0|^2}{2t_0^2}} d\mu \\
+ \left(4\pi t_s\right)^{-\frac{n}{2}} \int_M \left( \frac{\partial}{\partial s} + \frac{1}{2} \left( \frac{X(s) - X_s}{t_s} \right) \rho(s) e^{-\frac{|X(s) - X_s|^2}{2t_s^2}} d\mu_s \\
+ \left(4\pi t_0\right)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \int_M -\frac{h}{2t_s} \lambda(N(s), N) \rho(s) e^{-\frac{|X(s) - X_0|^2}{2t_0^2}} d\mu \\
+ \left(4\pi t_s\right)^{-\frac{n}{2}} \int_M \left( \frac{\partial}{\partial s} + \frac{1}{2} \left( \frac{X(s) - X_s}{t_s} \right) \right) \rho(s) e^{-\frac{|X(s) - X_s|^2}{2t_s^2}} d\mu_s \\
+ \left(4\pi t_0\right)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \int_M -\frac{h}{2t_s} \lambda(N, y) \rho(s) e^{-\frac{|X(s) - X_0|^2}{2t_0^2}} d\mu \\
+ \left(4\pi t_s\right)^{-\frac{n}{2}} \int_M \left( \frac{\partial}{\partial s} + \frac{1}{2} \left( \frac{X(s) - X_s}{t_s} \right) \right) \rho(s) e^{-\frac{|X(s) - X_s|^2}{2t_s^2}} d\mu_s \\
+ \left(4\pi t_0\right)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \int_M -\frac{h}{2t_s} \lambda(N, y) \rho(s) e^{-\frac{|X(s) - X_0|^2}{2t_0^2}} d\mu \\
+ \left(4\pi t_s\right)^{-\frac{n}{2}} \int_M \left( \frac{\partial}{\partial s} + \frac{1}{2} \left( \frac{X(s) - X_s}{t_s} \right) \right) \rho(s) e^{-\frac{|X(s) - X_s|^2}{2t_s^2}} d\mu_s \\
+ \left(4\pi t_0\right)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \int_M -\frac{h}{2t_s} \lambda(N, y) \rho(s) e^{-\frac{|X(s) - X_0|^2}{2t_0^2}} d\mu \]
\[ + (4\pi t_s)^{-\frac{n}{2}} \left( -\frac{nh}{2t_s} \int_M \left( -\frac{n}{2t_s} + \frac{|X(s) - X_s|^2}{2t_s^2} \right) he^{-\frac{|X(s) - X_s|^2}{2t_s^2}} \right) d\mu_s \]
\[ + (4\pi t_0)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \left( -\frac{1}{2t_s} \left( \frac{|X(s) - X_s|^2}{2t_s^2} \right) h(-H_s f) \right. \]
\[ - \left( \frac{X(s) - X_s}{t_s}, \frac{\partial X(s)}{\partial s} - \frac{\partial X(s)}{\partial s} \right) e^{-\frac{|X(s) - X_s|^2}{2t_s^2}} d\mu_s \]
\[ + (4\pi t_s)^{-\frac{n}{2}} \int_M (-\frac{n}{2t_s} + \frac{|X(s) - X_s|^2}{2t_s^2}) h(-H_s f) \]
\[ \times \left( \frac{X(s) - X_s}{t_s}, \frac{\partial X(s)}{\partial s} - \frac{\partial X(s)}{\partial s} \right) e^{-\frac{|X(s) - X_s|^2}{2t_s^2}} d\mu_s \]
\[ + (4\pi t_s)^{-\frac{n}{2}} \int_M \left( \frac{X(s) - X_s}{t_s}, y \right) \frac{|X(s) - X_s|^2}{2t_s^2} he^{-\frac{|X(s) - X_s|^2}{2t_s^2}} d\mu_s \]
\[ + (4\pi t_s)^{-\frac{n}{2}} \int_M (-\frac{n}{2t_s} + \frac{|X(s) - X_s|^2}{2t_s^2}) h\left( \frac{|X(s) - X_s|^2}{2t_s^2} he^{-\frac{|X(s) - X_s|^2}{2t_s^2}} \right) d\mu_s. \]

Since \( X : M \to \mathbb{R}^{n+1} \) is a critical point, we get
\[
H + \left( \frac{X - X_0}{t_0}, N \right) = \lambda,
\]
\[
\int_M \left( n + \lambda(X - X_0, N) - \frac{|X - X_0|^2}{t_0} \right) e^{-\frac{|X - X_0|^2}{2t_0}} d\mu = 0,
\]
\[
\int_M (\lambda(N, a) - \left( \frac{X - X_0}{t_0}, a \right)) e^{-\frac{|X - X_0|^2}{2t_0}} d\mu = 0.
\]

On the other hand,
\[
H' = \Delta f + Sf, \quad N' = -\nabla f.
\]
Using of the above equations and letting \( s = 0 \), we obtain

\[
(4\pi t_0)^\frac{n}{2} \mathcal{F}''(0) = \int_M -fLfe^{-\frac{|X-X_0|^2}{2t_0}} d\mu
\]

\[
+ \int_M \left( \frac{2}{t_0} \langle N, y \rangle + \frac{2h}{t_0} \frac{X - X_0}{t_0}, N \rangle + \frac{n - 1}{t_0} \lambda h - \frac{|X - X_0|^2}{t_0} \lambda h - 2\lambda \frac{X - X_0}{t_0}, y \rangle \right) fe^{-\frac{|X-X_0|^2}{2t_0}} d\mu
\]

\[
+ \int_M \left( \frac{-n + 2}{t_0} \frac{X - X_0}{t_0}, y \rangle + \frac{\lambda}{t_0} \langle N, y \rangle + \langle \frac{X - X_0}{t_0}, y \rangle \frac{|X - X_0|^2}{t_0} \right) he^{-\frac{|X-X_0|^2}{2t_0}} d\mu
\]

\[
+ \int_M \left( \frac{n^2}{4t_0^2} + \frac{n}{2t_0^2} \frac{|X - X_0|^2}{t_0^2} + \frac{|X - X_0|^4}{4t_0^4} \right)
\]

\[
+ \frac{3\lambda}{4t_0} \langle \frac{X - X_0}{t_0}, N \rangle h^2 e^{-\frac{|X-X_0|^2}{2t_0}} d\mu
\]

\[
+ \int_M \left( \frac{-1}{t_0} \langle y, y \rangle + \langle \frac{X - X_0}{t_0}, y \rangle \frac{|X - X_0|^2}{t_0} \right) e^{-\frac{|X-X_0|^2}{2t_0}} d\mu,
\]

where the operator \( L \) is defined by \( L = \Delta + S + \frac{1}{t_0} - \langle \frac{X - X_0}{t_0}, \nabla \rangle - \lambda^2 \). When \( t_0 = 1 \), \( X_0 = 0 \), then \( L = \mathcal{L} + S + 1 \).
\[
\begin{align*}
&= \int_M -fLfe^{-\frac{|x|^2}{2}} d\mu \\
&+ \int_M [2\langle N, y \rangle + (n + 1 - |X|^2)\lambda - 2hH - 2\lambda \langle X, y \rangle]f e^{-\frac{|x|^2}{2}} d\mu \\
&+ \int_M (\lambda \langle N, y \rangle - (n + 2)\langle X, y \rangle + \langle X, y \rangle |X|^2)he^{-\frac{|x|^2}{2}} d\mu \\
&+ \int_M \left( \frac{n^2 + 2n}{4} - \frac{n + 2}{2} |X|^2 + \frac{|X|^4}{4} + \frac{3\lambda}{4} (\lambda - H) \right)h^2 e^{-\frac{|x|^2}{2}} d\mu \\
&+ \int_M (-|y|^2 + \langle X, y \rangle^2)e^{-\frac{|x|^2}{2}} d\mu.
\end{align*}
\]

\[\square\]

**Definition 6.1.** One calls that a critical point \(X : M \to \mathbb{R}^{n+1}\) of the \(\mathcal{F}\)-functional \(\mathcal{F}_{X_0, t_0}(s)\) is \(\mathcal{F}\)-stable if, for every normal variation \(fN\), there exist variations of \(X_0\) and \(t_0\) such that \(\mathcal{F}''_{X_0, t_0}(0) \geq 0\);

One calls that a critical point \(X : M \to \mathbb{R}^{n+1}\) of the \(\mathcal{F}\)-functional \(\mathcal{F}_{X_0, t_0}(s)\) is \(\mathcal{F}\)-unstable if there exist a normal variation \(fN\) such that for all variations of \(X_0\) and \(t_0\), \(\mathcal{F}''_{X_0, t_0}(0) < 0\).

**Theorem 6.2.** If \(r \leq \sqrt{n}\) or \(r > \sqrt{n + 1}\), the \(n\)-dimensional round sphere \(X : S^n(r) \to \mathbb{R}^{n+1}\) is \(\mathcal{F}\)-stable; If \(\sqrt{n} < r \leq \sqrt{n + 1}\), the \(n\)-dimensional round sphere \(X : S^n(r) \to \mathbb{R}^{n+1}\) is \(\mathcal{F}\)-unstable.

**Proof.** For the sphere \(S^n(r)\), we have

\[X = -rN, \quad H = \frac{n}{r}, \quad S = \frac{H^2}{n} = \frac{n}{r^2}, \quad \lambda = H - r = \frac{n}{r} - r\]

and

\[Lf = \mathcal{L}f + (S + 1 - \lambda^2)f = \Delta f + \left(\frac{n}{r^2} + 1 - \lambda^2\right)f.\]

Since we know that eigenvalues \(\mu_k\) of \(\Delta\) on the sphere \(S^n(r)\) are given by

\[\mu_k = \frac{k^2 + (n - 1)k}{r^2},\]

and constant functions are eigenfunctions corresponding to eigenvalue \(\mu_0 = 0\). For any constant vector \(z \in \mathbb{R}^{n+1}\), we get

\[-\Delta \langle z, N \rangle = \Delta \langle z, \frac{X}{r} \rangle = \langle z, \frac{1}{r}HN \rangle = \frac{n}{r^2} \langle z, N \rangle,\]

that is, \(\langle z, N \rangle\) is an eigenfunction of \(\Delta\) corresponding to the first eigenvalue \(\mu_1 = \frac{n}{r^2}\). Hence, for any normal variation with the variation vector field \(fN\), we can choose a real number \(a \in \mathbb{R}\) and a constant vector \(z \in \mathbb{R}^{n+1}\) such that

\[f = f_0 + a + \langle z, N \rangle,\]
and $f_0$ is in the space spanned by all eigenfunctions corresponding to eigenvalues $\mu_k$ ($k \geq 2$) of $\Delta$ on $S^n$. Using the lemma 3.3, we get

$$
(4\pi)^\frac{n}{2} e^{-\frac{r^2}{2}} \mathcal{F}''(0)
= \int_{S^n} -(f_0 + a + \langle z, N \rangle)L(f_0 + a + \langle z, N \rangle) d\mu
+ \int_{S^n} [2\langle N, y \rangle + (n + 1 - r^2)\lambda h - 2\frac{n}{r} h + 2\lambda \langle rN, y \rangle](f_0 + a + \langle z, N \rangle) d\mu
+ \int_{S^n} (-r)\langle N, y \rangle (r^2 - n - 1)hd\mu
+ \int_{S^n} (\frac{n^2 + n + 2}{4} - \frac{n + 2}{2} r^2 + \frac{r^4}{4} + \frac{3}{4} r^2 - \frac{3}{4} n)h^2 d\mu
+ \int_{S^n} (-|y|^2 + \langle X, y \rangle^2) d\mu
\geq \int_{S^n} \left\{ \left( \frac{n + 2}{r^2} - 1 + \lambda^2 \right)f_0^2 - \left( \frac{n}{r^2} + 1 - \lambda^2 \right)a^2 + (\lambda^2 - 1)\langle z, N \rangle^2 \right\} d\mu
+ \int_{S^n} \left\{ 2(1 + \lambda r)\langle N, y \rangle \langle N, z \rangle + [(n + 1 - r^2)\lambda - 2\frac{n}{r} a]h \right\} d\mu
+ \int_{S^n} \frac{1}{4} [r^4 - (2n + 1)r^2 + n(n - 1)]h^2 d\mu
+ \int_{S^n} (-|y|^2 + \langle X, y \rangle^2) d\mu.
$$
(6.5)

From the lemma 3.3, we have

$$
\int_{S^n} (-|y|^2 + \langle X, y \rangle^2) d\mu = -\int_{S^n} (1 + \lambda r)\langle N, y \rangle^2 d\mu.
$$
(6.6)

Putting (6.6) and $\lambda = \frac{n}{r} - r$ into (6.5), we obtain

$$
(4\pi)^\frac{n}{2} e^{-\frac{r^2}{2}} \mathcal{F}''(0)
\geq \int_{S^n} \frac{1}{r^2} \left\{ (r^2 - n - \frac{1}{2})^2 + \frac{7}{4} \right\} f_0^2 d\mu
+ \int_{S^n} \frac{1}{r^2} [r^4 - (2n + 1)r^2 + n(n - 1)]\left( \frac{a}{r} + \frac{h}{2} \right)^2 d\mu
\geq \int_{S^n} \frac{1}{r^2} [r^4 - (2n + 1)r^2 + n^2] \langle z, N \rangle^2 d\mu
+ \int_{S^n} 2(1 + n - r^2)\langle N, y \rangle \langle N, z \rangle d\mu
+ \int_{S^n} -(1 + n - r^2)\langle N, y \rangle^2 d\mu.
$$
(6.7)
If we choose $h = -\frac{2a}{r}$, then we have

\[
\begin{align*}
(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} F''(0) \\
\geq \int_{S^2} \frac{1}{r^2} \left\{ \left( r^2 - n - \frac{1}{2} \right)^2 + \frac{7}{4} \right\} f_0^2 d\mu \\
+ \int_{S^2} \left( \lambda^2 - 1 \right) \langle z, N \rangle^2 d\mu \\
+ \int_{S^2} 2(1 + \lambda r) \langle N, y \rangle \langle N, z \rangle d\mu \\
+ \int_{S^2} - (1 + \lambda r) \langle N, y \rangle^2 d\mu.
\end{align*}
\]

(6.8)

Let $y = k z$, then we have

\[
\begin{align*}
(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} F''(0) \\
\geq \int_{S^2} \frac{1}{r^2} \left\{ \left( r^2 - n - \frac{1}{2} \right)^2 + \frac{7}{4} \right\} f_0^2 d\mu \\
+ \int_{S^2} \left\{ \lambda^2 - 1 + 2(1 + \lambda r) k - (1 + \lambda r) k^2 \right\} \langle z, N \rangle^2 d\mu \\
= \int_{S^2} \frac{1}{r^2} \left\{ \left( r^2 - n - \frac{1}{2} \right)^2 + \frac{7}{4} \right\} f_0^2 d\mu \\
+ \int_{S^2} \left\{ \lambda^2 + \lambda r - (1 + \lambda r)(1 - k)^2 \right\} \langle z, N \rangle^2 d\mu.
\end{align*}
\]

(6.9)

We next consider three cases:

**Case 1:** $r \leq \sqrt{n}$

In this case, $\lambda \geq 0$. Taking $k = 1$, then we get

$F''(0) \geq 0$.

**Case 2:** $r \geq \frac{1 + \sqrt{1 + 4n}}{2}$.

In this case, $\lambda \leq -1$. Taking $k = 2$, we can get

$F''(0) \geq 0$.

**Case 3:** $\sqrt{n + 1} < r < \frac{1 + \sqrt{1 + 4n}}{2}$.

In this case, $-1 < \lambda < 0$, $1 + \lambda r < 0$, we can take $k$ such that $(1 - k)^2 \geq \frac{\lambda(\lambda + r)}{1 + \lambda r}$, then we have

$F''(0) \geq 0$. 

Thus, if $r \leq \sqrt{n}$ or $r > \sqrt{n+1}$, the $n$-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is $\mathcal{F}$-stable;

If $\sqrt{n} < r \leq \sqrt{n+1}$, the $n$-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is $\mathcal{F}$-unstable. In fact, in this case, $-1 < \lambda < 0$, $1 + \lambda r \geq 0$. We can choose $f$ such that $f_0 = 0$, then we have

\[
(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{F}''(0) = \int_{S^n(r)} (\lambda^2 - 1) \langle z, N \rangle^2 d\mu \\
+ \int_{S^n(r)} 2(1 + \lambda r) \langle N, y \rangle \langle N, z \rangle d\mu \\
+ \int_{S^n(r)} -(1 + \lambda r) \langle N, y \rangle^2 d\mu \\
= (\lambda^2 + \lambda r) \int_{S^n(r)} \langle z, N \rangle^2 d\mu \\
- (1 + \lambda r) \int_{S^n(r)} (\langle z, N \rangle - \langle y, N \rangle)^2 d\mu \\
< 0.
\]

This completes the proof of the theorem 6.2.

According to our theorem 6.2, we would like to propose the following:

**Problem 6.1.** Is it possible to prove that spheres $S^n(r)$ with $r \leq \sqrt{n}$ or $r > \sqrt{n+1}$ are the only $\mathcal{F}$-stable compact $\lambda$-hypersurfaces.

**Remark 6.1.** Colding and Minicozzi [9] have proved that the sphere $S^n(\sqrt{n})$ is the only $\mathcal{F}$-stable compact self-shrinkers. In order to prove this result, the property that the mean curvature $H$ is an eigenfunction of $L$-operator plays a very important role. But for $\lambda$-hypersurfaces, the mean curvature $H$ is not an eigenfunction of $L$-operator in general.

7. The weak stability of the weighted area functional for weighted volume-preserving variations

Define

\[
(7.1) \quad \mathcal{T}(s) = (4\pi t_s)^{-\frac{n}{2}} \int_M e^{-\frac{|X(s) - X_0|^2}{2t_s}} d\mu_s.
\]
We compute the first and the second variation formulas of the general $\mathcal{T}$-functional for weighted volume-preserving variations. By a direct calculation, we have

$$\mathcal{T}'(s)$$

$$= (4\pi t_s)^{-\frac{n}{2}} \int_M \left( -H_s + \left\langle \frac{X(s) - X_s}{t_s}, N(s) \right\rangle f e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s \right)$$

$$+ (4\pi t_s)^{-\frac{n}{2}} \int_M \frac{\partial H_s}{\partial s} \left\langle \frac{X(s) - X_s}{t_s}, N(s) \right\rangle f e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s$$

$$+ (4\pi t_s)^{-\frac{n}{2}} \int_M \eta \left\langle \frac{X(s) - X_s}{t_s}, N(s) \right\rangle h e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s$$

$$+ (4\pi t_s)^{-\frac{n}{2}} \int_M \left( \frac{\partial X(s)}{\partial s} - \frac{\partial X_s}{\partial s} \right) \left\langle \frac{X(s) - X_s}{t_s}, N(s) \right\rangle h e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s$$

$$+ (4\pi t_s)^{-\frac{n}{2}} \int_M \left( \left\langle \frac{X(s) - X_s}{t_s}, N(s) \right\rangle \frac{dN}{ds} \right) e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s$$

$$+ (4\pi t_s)^{-\frac{n}{2}} \left( -\frac{n}{2t_s} + \frac{|X(s) - X_s|^2}{2t_s^2} \right) \eta e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s$$

$$+ (4\pi t_s)^{-\frac{n}{2}} \left( -\frac{n}{2t_s} + \frac{|X(s) - X_s|^2}{2t_s^2} \right) \eta' e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s$$

$$+ (4\pi t_s)^{-\frac{n}{2}} \left( -\frac{n}{2t_s} + \frac{|X(s) - X_s|^2}{2t_s^2} \right) \eta \gamma e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s$$

$$+ (4\pi t_s)^{-\frac{n}{2}} \left( -\frac{n}{2t_s} + \frac{|X(s) - X_s|^2}{2t_s^2} \right) \eta \gamma' e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s$$
\[ + (4\pi t_s)^{-\frac{n}{2}} \int_M \left( \frac{nh}{2t_s} - \frac{|X(s) - X_s|^2}{t_s^3} + \frac{\langle X(s) - X_s, \frac{\partial X(s)}{\partial s} - \frac{\partial X_s}{\partial s} \rangle}{t_s^2} \right) \times \\
\quad \times e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s \\
+ (4\pi t_s)^{-\frac{n}{2}} \int_M \left( -\frac{n}{2t_s} + \frac{|X(s) - X|^2}{2t_s^2} \right) \right) h(-H_s f \\
- \frac{|X(s) - X_s|^2}{t_s} \right) \right) e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s \\
+ (4\pi t_s)^{-\frac{n}{2}} \int_M \left( \frac{X(s) - X_s, y}{t_s} \right) \frac{|X(s) - X_s|^2}{2t_s^2} \right) h \right) e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s \\
+ (4\pi t_s)^{-\frac{n}{2}} \int_M \left( -\frac{n}{2t_s} + \frac{|X(s) - X_s|^2}{2t_s^2} \right) h \right) e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s. \]

**Lemma 7.1.**

\[ \int_M f'(0) e^{-\frac{|X-X_0|^2}{2t_0}} d\mu = 0. \]

**Proof.** Since \( V(t) = \int_M \langle X(t) - X_0, N \rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu = V(0) \) for any \( t \), we have

\[ \int_M f(t) \langle N(t), N \rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu = 0. \]

Hence, we get

\[
0 = \frac{d}{dt} \bigg|_{t=0} \int_M f(t) \langle N(t), N \rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu \\
= \int_M f'(0) e^{-\frac{|X-X_0|^2}{2t_0}} d\mu. \]

\[ \square \]

Since \( M \) is a critical point of \( T(s) \), we have

\[ H + \langle \frac{X - X_0}{t_0}, N \rangle = \lambda. \]

On the other hand, we have

(7.2) \quad H' = \Delta f + S f, \quad N' = -\nabla f.
Then for \( t_0 = 1 \) and \( X_0 = 0 \), the second variation formula becomes

\[
(4\pi)^\frac{2}{n} T''(0) = \int_M \langle X, y' \rangle e^{-\frac{|X|^2}{2}} d\mu + \int_M \left( \frac{|X|^2}{2} - \frac{n}{2} \right) h' e^{-\frac{|X|^2}{2}} d\mu \\
+ \int_M f \left( \mathcal{L} f + (S + 1 - \lambda^2) f \right) e^{-\frac{|X|^2}{2}} d\mu \\
+ \int_M \left( 2\langle N, y \rangle + (n - |X|^2) \lambda h + 2\langle X, N \rangle h \\
+ 2\langle N, y \rangle - 2\lambda\langle X, y \rangle \right) f e^{-\frac{|X|^2}{2}} d\mu \\
+ \int_M \left( -n + 2\langle X, y \rangle + \langle X, y \rangle |X|^2 \right) h e^{-\frac{|X|^2}{2}} d\mu \\
+ \int_M \left( n^2 + 2n - \frac{n + 2}{2} |X|^2 + \frac{|X|^4}{4} \right) h^2 e^{-\frac{|X|^2}{2}} d\mu \\
+ \int_M \left( -|y|^2 + \langle X, y \rangle^2 \right) e^{-\frac{|X|^2}{2}} d\mu.
\]

**Theorem 7.1.** Let \( X : M \to \mathbb{R}^{n+1} \) be a critical point of the functional \( T(s) \) for the weighted volume-preserving variations with fixed \( X_0 = 0 \) and \( t_0 = 1 \). The second variation formula of \( T(s) \) is given by

\[
(4\pi)^\frac{2}{n} T''(0) = \int_M -f \left( \mathcal{L} f + (S + 1 - \lambda^2) f \right) e^{-\frac{|X|^2}{2}} d\mu.
\]

**Definition 7.1.** A critical point \( X : M \to \mathbb{R}^{n+1} \) of the functional \( T(s) \) is called weakly stable if, for any weighted volume-preserving normal variation, \( T''(0) \geq 0 \); a critical point \( X : M \to \mathbb{R}^{n+1} \) of the functional \( T(s) \) is called weakly unstable if there exists a weighted volume-preserving normal variation, such that \( T''(0) < 0 \).

**Theorem 7.2.** If \( r \leq \frac{-1 + \sqrt{1 + 4n}}{2} \) or \( r \geq \frac{1 + \sqrt{1 + 4n}}{2} \), the \( n \)-dimensional round sphere \( X : S^n(r) \to \mathbb{R}^{n+1} \) is weakly stable; if \( \frac{-1 + \sqrt{1 + 4n}}{2} < r < \frac{1 + \sqrt{1 + 4n}}{2} \), the \( n \)-dimensional round sphere \( X : S^n(r) \to \mathbb{R}^{n+1} \) is weakly unstable.

**Proof.** For the sphere \( S^n(r) \), we have

\[
X = -r N, \quad H = \frac{n}{r}, \quad S = \frac{n}{r^2}, \quad \lambda = H - r = \frac{n}{r} - r
\]

and

\[
L f = \mathcal{L} f + (S + 1 - \lambda^2) f = \Delta f + \left( \frac{n}{r^2} + 1 - \lambda^2 \right) f.
\]

Since we know that eigenvalues \( \mu_k \) of \( \Delta \) on the sphere \( S^n(r) \) are given by

\[
\mu_k = \frac{k^2 + (n - 1)k}{r^2},
\]
and constant functions are eigenfunctions corresponding to eigenvalue $\mu_0 = 0$. For any constant vector $z \in \mathbb{R}^{n+1}$, we get

\[(7.6) \quad -\Delta \langle z, N \rangle = \frac{n}{r^2} \langle z, N \rangle,\]

that is, $\langle z, N \rangle$ is an eigenfunction of $\Delta$ corresponding to the first eigenvalue $\mu_1 = \frac{n}{r}$. Hence, for any weighted volume-preserving normal variation with the variation vector field $fN$ satisfying

\[\int_{S^n(r)} f e^{-\frac{r^2}{s}} d\mu = 0,\]

we can choose a constant vector $z \in \mathbb{R}^{n+1}$ such that

\[(7.7) \quad f = f_0 + \langle z, N \rangle,\]

and $f_0$ is in the space spanned by all eigenfunctions corresponding to eigenvalues $\mu_k (k \geq 2)$ of $\Delta$ on $S^n(r)$. By making use of the theorem 7.1, we have

\[(4\pi)^\frac{n}{2} e^{\frac{s^2}{2r^2}} T''(0) \geq \int_{S^n(r)} \left\{ \left( \frac{n+2}{r^2} - 1 + \lambda^2 \right)f_0^2 + (\lambda^2 - 1)\langle z, N \rangle^2 \right\} d\mu.\]

According to $\lambda = \frac{\sqrt{4n+1}}{2} - r$, we obtain

\[(4\pi)^\frac{n}{2} e^{\frac{s^2}{2r^2}} T''(0) \geq \int_{S^n(r)} \left\{ \left( r^2 - n - \frac{1}{2} \right)^2 + \frac{7}{4} \right\} f_0^2 d\mu + \int_{S^n(r)} \left( \frac{n}{r} - r - 1 \right) \left( \frac{n}{r} - r + 1 \right) \langle z, N \rangle^2 d\mu \geq 0\]

if

\[r \leq -\frac{1 + \sqrt{4n+1}}{2} \quad \text{or} \quad r \geq 1 + \frac{\sqrt{4n+1}}{2}.\]

Thus, the $n$-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is weakly stable. If

\[-1 + \frac{\sqrt{4n+1}}{2} < r < 1 + \frac{\sqrt{4n+1}}{2},\]

choosing $f = \langle z, N \rangle$, we have

\[\int_{S^n(r)} f e^{-\frac{r^2}{s}} d\mu = 0.\]

Hence, there exists a weighted volume-preserving normal variation with the variation vector field $fN$ such that

\[(4\pi)^\frac{n}{2} e^{\frac{s^2}{2r^2}} T''(0) = \int_{S^n(r)} \left( \frac{n}{r} - r - 1 \right) \left( \frac{n}{r} - r + 1 \right) \langle z, N \rangle^2 d\mu < 0.\]

Thus, the $n$-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is weakly unstable. It finishes the proof. \qed
Remark 7.1. From the theorem 6.2 and theorem 7.2, we know the $\mathcal{F}$-stability and the weak stability are different. The $\mathcal{F}$-stability is a weaker notation than the weak stability.

Remark 7.2. Is it possible to prove that spheres $S^n(r)$ with $r \leq \frac{-1+\sqrt{1+4n^2}}{2}$ or $r \geq \frac{1+\sqrt{1+4n^2}}{2}$ are the only weak stable compact $\lambda$-hypersurfaces.

8. Complete and non-compact $\lambda$-hypersurfaces

In this section, we will give a classification of complete and non-compact $\lambda$-hypersurfaces.

Theorem 8.1. $S^k(r) \times \mathbb{R}^{n-k}$, $0 \leq k \leq n$, are the only complete embedded $\lambda$-hypersurfaces with polynomial area growth in $\mathbb{R}^{n+1}$ if $H - \lambda \geq 0$ and $\lambda(f_3(H - \lambda) - S) \geq 0$.

At first, we prepare the following lemmas and propositions.

Lemma 8.1. Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional immersed hypersurface in the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. At any point $p \in M$, we have

\begin{equation}
|\nabla \sqrt{S}|^2 \leq \sum_{i,k} h_{iik}^2 \leq \sum_{i,j,k} h_{ijk}^2,
\end{equation}

\begin{equation}
\frac{n+3}{n+1} |\nabla \sqrt{S}|^2 \leq \sum_{i,j,k} h_{ijk}^2 + \frac{2n}{n+1} |\nabla H|^2.
\end{equation}

Its proof is standard. See Schoen, Simon and Yau [27] and Colding and Minicozzi [10].

Proposition 8.1. Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional complete $\lambda$-hypersurface with $H - \lambda > 0$ and $\lambda(f_3(H - \lambda) - S) \geq 0$. If $\eta$ is a function with compact support, then

\begin{equation}
\int_M \eta^2(S + |\nabla \log(H - \lambda)|^2)e^{-|x|^2/2}d\mu \leq c(n, \lambda) \int_M (|\nabla \eta|^2 + \eta^2)e^{-|x|^2/2}d\mu,
\end{equation}

where $c(n, \lambda)$ is constant depending on $n$ and $\lambda$.

Proof. Since $H - \lambda > 0$, $\log(H - \lambda)$ is well-defined. Suppose $\eta$ is a function with compact support, the lemma 4.1 and the corollary 3.1 give

\begin{equation}
\int_M \langle \nabla \eta^2, \nabla \log(H - \lambda) \rangle e^{-|x|^2/2}d\mu
\end{equation}

\begin{equation}
= -\int_M \eta^2(\mathcal{L} \log(H - \lambda))e^{-|x|^2/2}d\mu
\end{equation}

\begin{equation}
= \int_M \eta^2 \left(S - 1 - \frac{\lambda}{H - \lambda} + |\nabla \log(H - \lambda)|^2 \right)e^{-|x|^2/2}d\mu.
\end{equation}

Combining this with inequality:

\begin{equation}
\langle \nabla \eta^2, \nabla \log(H - \lambda) \rangle \leq \varepsilon |\nabla \eta|^2 + \frac{1}{\varepsilon} \eta^2 |\nabla \log(H - \lambda)|^2
\end{equation}
gives that
\[
\int_M (\eta^2 S + \eta^2 (1 - \frac{1}{\varepsilon})|\nabla \log(H - \lambda)|^2)e^{-\frac{|X|^2}{2}} d\mu
\]
\[
\leq \int_M (\varepsilon|\nabla \eta|^2 + \eta^2 + \frac{\lambda}{H - \lambda}\eta^2)e^{-\frac{|X|^2}{2}} d\mu,
\]
for \(\varepsilon > 0\). Since
\[
\frac{\lambda}{H - \lambda} \leq \frac{\lambda f_3}{S} \leq |\lambda| \sqrt{S} \leq |\lambda|(\frac{S}{2\delta} + \frac{\delta}{2})
\]
for \(\delta > 0\), we have from (8.6) and (8.7)
\[
\int_M \left\{ (1 - \frac{|\lambda|}{2\delta})\eta^2 S + \eta^2 (1 - \frac{1}{\varepsilon})|\nabla \log(H - \lambda)|^2 \right\}e^{-\frac{|X|^2}{2}} d\mu
\]
\[
\leq \int_M \left( \varepsilon|\nabla \eta|^2 + (1 + \frac{|\lambda|}{2\delta})\eta^2 \right)e^{-\frac{|X|^2}{2}} d\mu.
\]
By choosing \(\varepsilon, \delta\) and constant \(c(n, \lambda)\), we get
\[
\int_M \eta^2 (S + |\nabla \log(H - \lambda)|^2)e^{-\frac{|X|^2}{2}} d\mu \leq c(n, \lambda) \int_M (|\nabla \eta|^2 + \eta^2)e^{-\frac{|X|^2}{2}} d\mu.
\]

**Proposition 8.2.** Let \(X : M \to \mathbb{R}^{n+1}\) be an \(n\)-dimensional complete \(\lambda\)-hypersurface with \(H - \lambda > 0\) and \(\lambda(f_3 - \frac{S}{H - \lambda}) \geq 0\). If \(M\) has polynomial area growth, then
\[
\int_M \langle \nabla S, \nabla \log(H - \lambda) \rangle e^{-\frac{|X|^2}{2}} d\mu
\]
\[
= -\int_M S \mathcal{L} \log(H - \lambda)e^{-\frac{|X|^2}{2}} d\mu
\]
\[
= \int_M S \left( S - 1 - \frac{\lambda}{H - \lambda} + |\nabla \log(H - \lambda)|^2 \right)e^{-\frac{|X|^2}{2}} d\mu,
\]
and
\[
\int_M |\nabla \sqrt{S}|^2 e^{-\frac{|X|^2}{2}} d\mu
\]
\[
= -\int_M \sqrt{S} \mathcal{L} \sqrt{S} e^{-\frac{|X|^2}{2}} d\mu
\]
\[
\leq \int_M (S^2 - S - \lambda f_3)e^{-\frac{|X|^2}{2}} d\mu.
\]

**Proof.** Taking \(\eta = \phi\) in (8.4), we have
\[
\int_M \langle \nabla \phi^2, \nabla \log(H - \lambda) \rangle e^{-\frac{|X|^2}{2}} d\mu
\]
\[
= -\int_M \phi^2 (\mathcal{L} \log(H - \lambda))e^{-\frac{|X|^2}{2}} d\mu
\]
\[
= \int_M \phi^2 \left( S - 1 - \frac{\lambda}{H - \lambda} + |\nabla \log(H - \lambda)|^2 \right)e^{-\frac{|X|^2}{2}} d\mu.
\]
Since

\begin{equation}
(8.13) \quad \langle \nabla \phi^2, \nabla \log(H - \lambda) \rangle \leq |\nabla \phi|^2 + \phi^2 |\nabla \log(H - \lambda)|^2,
\end{equation}

we derive

\begin{equation}
(8.14) \quad \int_M \phi^2 S e^{-\frac{|x|^2}{2}} d\mu \leq \int_M (|\nabla \phi|^2 + \phi^2 + \frac{\lambda}{H - \lambda} \phi^2) e^{-\frac{|x|^2}{2}} d\mu.
\end{equation}

Let \( \phi = \eta \sqrt{S} \), where \( \eta \geq 0 \) has a compact support, for \( \alpha > 0 \), we have

\begin{equation}
(8.15) \quad \int_M \eta^2 S^2 e^{-\frac{|x|^2}{2}} d\mu
\leq \int_M \left\{ \eta^2 |\nabla \sqrt{S}|^2 + 2\eta \sqrt{S} |\nabla \eta| |\nabla \sqrt{S}| + S |\nabla \eta|^2 + (1 + \frac{\lambda}{H - \lambda}) \eta^2 S \right\} e^{-\frac{|x|^2}{2}} d\mu
\end{equation}

The lemma \[4.1\] and lemma \[8.1\] give the following inequality

\begin{equation}
\mathcal{LS} = 2 \sum_{i,j,k} h_{ij} h_{jk} + 2(1 - S) S + 2\lambda f_3
\geq \frac{2(n + 3)}{n + 1} |\nabla \sqrt{S}|^2 - \frac{4n}{n + 1} |\nabla H|^2 + 2S - 2S^2 + 2\lambda f_3,
\end{equation}

Integrating this with \( \frac{1}{2} \eta^2 \) and using the lemma \[3.2\] we obtain

\begin{equation}
-2 \int_M \eta \sqrt{S} (\nabla \eta, \nabla \sqrt{S}) e^{-\frac{|x|^2}{2}} d\mu
\geq \int_M \left\{ \eta^2 \frac{(n + 3)}{n + 1} |\nabla \sqrt{S}|^2 - \frac{2n}{n + 1} \eta^2 |\nabla H|^2 + S \eta^2 - S^2 \eta^2 + \lambda f_3 \eta^2 \right\} e^{-\frac{|x|^2}{2}} d\mu.
\end{equation}

Since \( 2ab \leq \epsilon a^2 + \frac{\epsilon b^2}{\epsilon} \) for \( \epsilon > 0 \), we infer

\begin{equation}
(8.17) \quad \int_M \left\{ \eta^2 S^2 + \frac{2n}{n + 1} \eta^2 |\nabla H|^2 + \frac{1}{\epsilon} S |\nabla \eta|^2 \right\} e^{-\frac{|x|^2}{2}} d\mu
\geq \int_M \left\{ \left( \frac{n + 3}{n + 1} - \epsilon \right) \eta^2 |\nabla \sqrt{S}|^2 + S \eta^2 + \lambda f_3 \eta^2 \right\} e^{-\frac{|x|^2}{2}} d\mu.
\end{equation}

From \( (8.15) \), \( (8.17) \) and \( \lambda \frac{S}{H - \lambda} \leq \lambda f_3 \), by taking \( \alpha \) and \( \epsilon \) such that \( \frac{1 + \alpha}{n + 1 - \epsilon} > 0 \), we have
\[
\int_M \eta^2 S^2 e^{-\frac{|x|^2}{2}} d\mu \\
\leq \frac{1 + \alpha}{n+3} - \epsilon \int_M \eta^2 S^2 e^{-\frac{|x|^2}{2}} d\mu + \frac{2n}{n+1} \cdot \frac{1 + \alpha}{n+3} - \epsilon \int_M \eta^2 |\nabla H|^2 e^{-\frac{|x|^2}{2}} d\mu \\
+ \int_M \left[ \frac{1 + \alpha}{n+3} - \epsilon \left( \frac{1}{\epsilon} |\nabla \eta|^2 - \eta^2 \right) + (1 + \frac{1}{\alpha})|\nabla \eta|^2 + (1 + \frac{\lambda}{H - \lambda})\eta^2 \right] Se^{-\frac{|x|^2}{2}} d\mu \\
+ \frac{1 + \alpha}{n+3} - \epsilon \int_M (-\lambda f_3 \eta^2) e^{-\frac{|x|^2}{2}} d\mu \\
\leq \frac{1 + \alpha}{n+3} - \epsilon \int_M \eta^2 S^2 e^{-\frac{|x|^2}{2}} d\mu + \frac{2n}{n+1} \cdot \frac{1 + \alpha}{n+3} - \epsilon \int_M \eta^2 |\nabla H|^2 e^{-\frac{|x|^2}{2}} d\mu \\
+ \int_M \left\{ \left[ \frac{1 + \alpha}{n+3} - \epsilon \frac{1}{\epsilon} + 1 + \frac{1}{\alpha} \right]|\nabla \eta|^2 + (1 - \frac{1 + \alpha}{n+3} - \epsilon)\eta^2 \\
+ \frac{1 + \alpha}{H - \lambda} \left( 1 - \frac{1 + \alpha}{n+3} - \epsilon \right) \right\} Se^{-\frac{|x|^2}{2}} d\mu.
\]

Using
\[
\lambda \frac{S}{H - \lambda} \leq \lambda f_3 \leq |\lambda| S \sqrt{S} \leq \frac{1}{2\delta} |\lambda| S^2 + \frac{\delta}{2} |\lambda| S,
\]
for \(\delta > 0\), we obtain, by taking \(\alpha\) and \(\epsilon\) such that \(1 - \frac{1 + \alpha}{n+3} - \epsilon > 0\)

\[
\left( 1 - \frac{1 + \alpha}{n+3} - \epsilon \right) \left( 1 - \frac{|\lambda|}{2\delta} \right) \int_M \eta^2 S^2 e^{-\frac{|x|^2}{2}} d\mu
\leq \frac{2n}{n+1} \frac{1 + \alpha}{n+3} - \epsilon \int_M \eta^2 |\nabla H|^2 e^{-\frac{|x|^2}{2}} d\mu \\
+ \int_M \left\{ \left( \frac{1 + \alpha}{n+3} - \epsilon \frac{1}{\epsilon} + 1 + \frac{1}{\alpha} \right)|\nabla \eta|^2 + \left( 1 - \frac{1 + \alpha}{n+3} - \epsilon \right)\eta^2 \\
+ \left( 1 - \frac{1 + \alpha}{n+3} - \epsilon \right) \right\} Se^{-\frac{|x|^2}{2}} d\mu.
\]

Assuming \(|\eta| \leq 1\) and \(|\nabla \eta| \leq 1\), choosing \(\delta\) such that \(\frac{|\lambda|}{2\delta} < 1\), we have

\[
\int_M \eta^2 S^2 e^{-\frac{|x|^2}{2}} d\mu \leq C(n, \lambda) \int_M (|\nabla H|^2 + S) e^{-\frac{|x|^2}{2}} d\mu
\]

for some constant \(C(n, \lambda)\) depending on \(n\) and \(\lambda\). Since \(|\nabla H| \leq \sqrt{S}|X|\) holds from (4.5), one has from (8.19)

\[
\int_M \eta^2 S^2 e^{-\frac{|x|^2}{2}} d\mu \leq C(n, \lambda) \int_M S(1 + |X|^2) e^{-\frac{|x|^2}{2}} d\mu.
\]

Since \(H - \lambda > 0\) and \(\lambda f_3 \geq \lambda \frac{S}{H - \lambda}\), let \(\eta_j\) be one on \(B_j\) and cut off linearly to zero from \(\partial B_j\) to \(\partial B_{j+1}\), where \(B_j = X(M) \cap B_j(0)\) with \(B_j(0)\) is the Euclidean ball of radius \(j\) centered at the origin. Applying the proposition 8.1 with \(\eta = \eta_j|X|\), letting
$j \to \infty$, the dominated convergence theorem and the polynomial area growth give that $\int_{M} S(1 + |X|^2) e^{-\frac{|X|^2}{2}} d\mu < +\infty$. Thus \[(8.20)\] and the dominated convergence theorem give that
\[
\int_{M} S^2 e^{-\frac{|X|^2}{2}} d\mu < +\infty.
\]
Hence, from \[(8.17)\], we also have
\[
\int_{M} |\nabla \sqrt{S}|^2 e^{-\frac{|X|^2}{2}} d\mu < +\infty.
\]
We next prove $\int_{M} \sum_{i,j,k} h^{2}_{ijk} e^{-\frac{|X|^2}{2}} d\mu < +\infty$. From \[(4.2)\], one has
\[
\int_{M} \eta^2 \sum_{i,j,k} h^{2}_{ijk} e^{-\frac{|X|^2}{2}} d\mu = \int_{M} (S^2 - S) e^{-\frac{|X|^2}{2}} d\mu - \int_{M} \lambda \eta^2 e^{-\frac{|X|^2}{2}} d\mu \]
\[
- \int_{M} 2\eta \sqrt{S} (\nabla \eta, \nabla \sqrt{S}) e^{-\frac{|X|^2}{2}} d\mu \]
\[
\leq C_0(n, \lambda) \int_{M} (\eta^2 S^2 + \eta S + |\nabla \eta|^2 |\nabla \sqrt{S}|^2) e^{-\frac{|X|^2}{2}} d\mu \]
\[
< +\infty,
\]
where $C_0(n, \lambda)$ is constant depending on $n$ and $\lambda$. The dominated convergence theorem gives that
\[
\int_{M} \sum_{i,j,k} h^{2}_{ijk} e^{-\frac{|X|^2}{2}} d\mu < +\infty.
\]
This shows that
\[
\int_{M} (S + S^2 + |\nabla \sqrt{S}|^2 + \sum_{i,j,k} h^{2}_{ijk}) e^{-\frac{|X|^2}{2}} d\mu < +\infty.
\]
From \[(8.23)\], we have
\[
\int_{M} (S^2 + |\nabla \sqrt{S}|^2) e^{-\frac{|X|^2}{2}} d\mu < +\infty,
\]
that is, $\sqrt{S}$ is in the weighted $W^{1,2}$ space. Applying the proposition \[8.1\] with $\eta = \eta_j \sqrt{S}$, letting $j \to \infty$, using the dominated convergence theorem, one has
\[
\int_{M} S |\nabla \log(H - \lambda)|^2 e^{-\frac{|X|^2}{2}} d\mu < +\infty.
\]
It follows that
\[
\int_{M} |\nabla S| |\nabla \log(H - \lambda)| e^{-\frac{|X|^2}{2}} d\mu \leq \int_{M} (|\nabla \sqrt{S}|^2 + S |\nabla \log(H - \lambda)|^2) e^{-\frac{|X|^2}{2}} d\mu < +\infty.
\]
\[(4.4)\] gives that
\[
\int_M S|L \log(H - \lambda)|e^{-\frac{|X|^2}{2}} d\mu
\]
\[
= \int_M S \left| 1 - S + \frac{\lambda}{H - \lambda} - |\nabla \log(H - \lambda)|^2 \right| e^{-\frac{|X|^2}{2}} d\mu
\]
\[
\leq C_1(n, \lambda) \int_M \left\{ S^2 + S + S|\nabla \log(H - \lambda)|^2 \right\} e^{-\frac{|X|^2}{2}} d\mu
\]
\[
< +\infty,
\]
where \(C_1(n, \lambda)\) is constant. Thus, we obtain
\[
(8.27) \int_M \left\{ S|\nabla \log(H - \lambda)| + |\nabla S||\nabla \log(H - \lambda)| + S|L \log(H - \lambda)| \right\} e^{-\frac{|X|^2}{2}} d\mu < +\infty.
\]
By applying the corollary 3.1 to \(S\) and \(\log(H - \lambda)\), we get
\[
(8.28) \int_M \langle \nabla S, \nabla \log(H - \lambda) \rangle e^{-\frac{|X|^2}{2}} d\mu
\]
\[
= - \int_M S|L \log(H - \lambda)| e^{-\frac{|X|^2}{2}} d\mu
\]
\[
= \int_M S \left( S - 1 - \frac{\lambda}{H - \lambda} + |\nabla \log(H - \lambda)|^2 \right) e^{-\frac{|X|^2}{2}} d\mu.
\]
On one hand, \((4.3)\) gives
\[
(8.29) \int_M \sqrt{S}|L \sqrt{S}| e^{-\frac{|X|^2}{2}} d\mu
\]
\[
= \int_M \left| \sum_{i,j,k} h_{i,j,k}^2 - |\nabla \sqrt{S}|^2 + S(1 - S) + \lambda f_3 \right| e^{-\frac{|X|^2}{2}} d\mu
\]
\[
\leq C_2(n, \lambda) \int_M \left( \sum_{i,j,k} h_{i,j,k}^2 + |\nabla \sqrt{S}|^2 + S + S^2 \right) e^{-\frac{|X|^2}{2}} d\mu
\]
\[
< +\infty.
\]
Hence
\[
(8.30) \int_M \left( \sqrt{S}|\nabla \sqrt{S}| + |\nabla \sqrt{S}|^2 + \sqrt{S}|L \sqrt{S}| \right) e^{-\frac{|X|^2}{2}} d\mu < +\infty.
\]
On the other hand, we have from \((4.3)\) and the lemma 8.1
\[
(8.31) \quad L \sqrt{S} \geq \sqrt{S} - \sqrt{SS} + \frac{\lambda f_3}{\sqrt{S}}.
\]
Then we can apply the corollary 3.1 to $\sqrt{S}$ and $\sqrt{S}$ and obtain
\[
\int_M |\nabla \sqrt{S}|^2 e^{-\frac{|X|^2}{2}} \, d\mu = -\int_M \sqrt{S} L \sqrt{S} e^{-\frac{|X|^2}{2}} \, d\mu \leq \int_M (S^2 - S - \lambda f_3) e^{-\frac{|X|^2}{2}} \, d\mu.
\] (8.32)

Proof of Theorem 8.1. Since $H - \lambda \geq 0$ and $L H - H \leq 0$, if $\lambda \leq 0$, we have from the maximum principle that either $H - \lambda \equiv 0$ or $H - \lambda > 0$. If $H - \lambda \equiv 0$, (4.5) and (4.6) give that $\lambda = 0 = H$, then $M$ is a self-shrinker of the mean curvature flow. According to the results of Colding and Minicozzi 10, $M$ is $\mathbb{R}^n$. If $\lambda > 0$ and $H - \lambda = 0$ at some point $p \in M$, then we see from $\lambda (f_3 (H - \lambda) - S) \geq 0$ that $S = 0$ and $H = 0$ at $p$, then $\lambda \equiv 0$, according to the results of Colding and Minicozzi, we know that $M$ is $\mathbb{R}^n$. Hence, for any $\lambda$, we have either $M$ is $\mathbb{R}^n$ or $H - \lambda > 0$.

Next, we assume that $H - \lambda > 0$. From the proposition 8.2, we have
\[
\int_M \langle \nabla S, \nabla \log(H - \lambda) \rangle e^{-\frac{|X|^2}{2}} \, d\mu = -\int_M S L \log(H - \lambda) e^{-\frac{|X|^2}{2}} \, d\mu = \int_M S \left( S - 1 - \frac{\lambda}{H - \lambda} + |\nabla \log(H - \lambda)|^2 \right) e^{-\frac{|X|^2}{2}} \, d\mu,
\] (8.33)

and
\[
\int_M |\nabla \sqrt{S}|^2 e^{-\frac{|X|^2}{2}} \, d\mu = -\int_M \sqrt{S} L \sqrt{S} e^{-\frac{|X|^2}{2}} \, d\mu \leq \int_M (S^2 - S - \lambda f_3) e^{-\frac{|X|^2}{2}} \, d\mu.
\] (8.34)

Substituting (8.34) into (8.33) and using $\lambda f_3 \geq \lambda \frac{S}{H - \lambda}$, one has
\[
0 \geq \int_M \left\{ |\nabla \sqrt{S}|^2 - 2 \sqrt{S} \langle \nabla \sqrt{S}, \nabla \log(H - \lambda) \rangle + S |\nabla \log(H - \lambda)|^2 
+ \lambda f_3 - \lambda \frac{S}{H - \lambda} \right\} e^{-\frac{|X|^2}{2}} \, d\mu
\geq \int_M |\nabla \sqrt{S} - \sqrt{S} \nabla \log(H - \lambda)|^2 e^{-\frac{|X|^2}{2}} \, d\mu.
\] (8.35)

Hence we conclude that $\nabla \sqrt{S} = \sqrt{S} \nabla \log(H - \lambda)$. Thus, we obtain
\[
\sqrt{S} = \beta (H - \lambda)
\] (8.36)
for a constant $\beta > 0$. Since all inequalities in above equations become equalities, we
obtain

$$(8.37) \quad \sum_{i,j,k} h_{ijk}^2 = |\nabla \sqrt{S}|^2, \quad \lambda f_3 = \lambda \frac{S}{H - \lambda}. $$

From the lemma 8.1 and (8.37), we know

1. There is a constant $C_k$ such that $h_{iik} = C_k \lambda_i$ for every $i$ and $k$.

2. If $i \neq j$, then $h_{ijk} = 0$, that is, $h_{ijk} = 0$ unless $i = j = k$ since $h_{ijk} = h_{ikj}$.

If $\lambda_i \neq 0$ and $j \neq i$, then $0 = h_{iij} = C_j \lambda_i$. It follows that $C_j = 0$. If the rank of
matrix $(h_{ij})$ is at least two at $p$, then $C_j = 0$ for $j \in \{1, 2, \ldots, n\}$. Hence, we have
$h_{ijk}(p) = 0$.

We next consider two cases.

**Case 1:** The rank of matrix $(h_{ij})$ is greater than one at $p$.

In this case, we will prove that the rank of $(h_{ij})$ is at least two everywhere. In fact, for $q \in M$, let $\lambda_1(q)$ and $\lambda_2(q)$ be the two eigenvalues of $(h_{ij})(q)$ that are largest in
absolute value and define the set

$$(8.38) \quad \Omega = \{q \in M| \lambda_1(q) = \lambda_1(p), \lambda_2(q) = \lambda_2(p)\}. $$

Then $p \in \Omega$, since $\lambda_i$’s are continuous, so $\Omega$ is closed. Given any point $q \in \Omega$, it
follows that the rank of $(h_{ij})$ is at least two at $q$. Hence there is an open set $U$, $q \in U$, where the rank of $(h_{ij})$ is at least two. On $U$, we have $h_{ijk} = 0$ and the eigenvalues
of $(h_{ij})$ are constant on $U$. Thus, $U \subset \Omega$, $\Omega$ is open. Since $M$ is connected, we have
$\Omega = M$ and $h_{ijk} \equiv 0$ on $M$. We know that $M = S^k(r) \times \mathbb{R}^{n-k}$, where $k > 1$.

**Case 2:** The rank of matrix $(h_{ij})$ is one.

From Case 1, we know that the rank of $(h_{ij})$ is one everywhere. Hence $S = H^2$. On
the other hand, $S = \beta^2(H - \lambda)^2$, hence $H^2 = \beta^2(H - \lambda)^2$. If $\lambda = 0$, then $M$ is
a self-shrinker of the mean curvature flow. If $\lambda \neq 0$, then we have $H$ is constant. $M$ is
$S^1(r) \times \mathbb{R}^{n-1}$ from the proposition 2.2. This completes the proof of Theorem 8.1.

---

9. **Properness and polynomial area growth for $\lambda$-hypersurfaces**

For $n$-dimensional complete and non-compact Riemannian manifolds with nonnegative Ricci curvature, the well-known theorem of Bishop and Gromov says that geodesic balls have at most polynomial area growth:

$$\text{Area}(B_r(x_0)) \leq Cr^n. $$

For $n$-dimensional complete and non-compact gradient shrinking Ricci solutions, Cao and Zhou [5] have proved geodesic balls have at most polynomial area growth. It is our purposes in this section to study the area growth for $\lambda$-hypersurfaces. First of all, we study the equivalence of properness and polynomial area growth for $\lambda$-hypersurfaces. If $X: M \rightarrow \mathbb{R}^{n+1}$ is an $n$-dimensional hypersurface in $\mathbb{R}^{n+1}$, we say
\(M\) has polynomial area growth if there exist constant \(C\) and \(d\) such that for all \(r \geq 1\),

\[
\text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0) \cap X(M)} d\mu \leq Cr^d,
\]

where \(B_r(0)\) is a round ball in \(\mathbb{R}^{n+1}\) with radius \(r\) and centered at the origin.

**Theorem 9.1.** Let \(X : M \to \mathbb{R}^{n+1}\) be a complete and non-compact properly immersed \(\lambda\)-hypersurface in the Euclidean space \(\mathbb{R}^{n+1}\). Then, there is a positive constant \(C\) such that for \(r \geq 1\),

\[
\text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0) \cap X(M)} d\mu \leq Cr^{n+\frac{\lambda^2}{4} - 2\beta - \inf H^2},
\]

where \(\beta = \frac{1}{4} \inf(\lambda - H)^2\).

**Proof.** Since \(X : M \to \mathbb{R}^{n+1}\) is a complete and non-compact properly immersed \(\lambda\)-hypersurface in the Euclidean space \(\mathbb{R}^{n+1}\), we have

\[
\langle X, N \rangle + H = \lambda.
\]

Defining \(f = \frac{|X|^2}{4}\), we have

\[
f - |\nabla f|^2 = \frac{|X|^2}{4} - \frac{|X^T|^2}{4} = \frac{|X_\perp|^2}{4} = \frac{1}{4}(\lambda - H)^2,
\]

\[
\Delta f = \frac{1}{2}(n + H\langle N, X \rangle)
\]

\[
= \frac{1}{2}(n + \lambda\langle N, X \rangle - \langle N, X \rangle^2)
\]

\[
= \frac{1}{2}n + \frac{\lambda^2}{4} - \frac{H^2}{4} - f + |\nabla f|^2.
\]

Hence, we obtain

\[
|\nabla(f - \beta)|^2 \leq (f - \beta),
\]

\[
\Delta(f - \beta) - |\nabla(f - \beta)|^2 + (f - \beta) \leq \left(\frac{n}{2} + \frac{\lambda^2}{4} - \beta - \frac{\inf H^2}{4}\right).
\]

Since the immersion \(X\) is proper, we know that \(\overline{f} = f - \beta\) is proper. Applying the theorem 2.1 of X. Cheng and Zhou [9] to \(\overline{f} = f - \beta\) with \(k = \left(\frac{n}{2} + \frac{\lambda^2}{4} - \beta - \frac{\inf H^2}{4}\right)\), we obtain

\[
\text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0) \cap X(M)} d\mu \leq Cr^{n+\frac{\lambda^2}{4} - 2\beta - \inf H^2},
\]

where \(\beta = \frac{1}{4} \inf(\lambda - H)^2\) and \(C\) is a constant. \(\Box\)

**Remark 9.1.** The estimate in our theorem 9.1 is best possible because the cylinders \(S^k(r_0) \times \mathbb{R}^{n-k}\) satisfy the equality.

**Remark 9.2.** By making use of the same assertions as in X. Cheng and Zhou [9] for self-shrinkers, we can prove the weighted area of a complete and non-compact properly immersed \(\lambda\)-hypersurface in the Euclidean space \(\mathbb{R}^{n+1}\) is bounded.
By making use of the same assertions as in X. Cheng and Zhou [9] for self-shrinkers, we can prove the following theorem. We will leave it for readers.

**Theorem 9.2.** If $X : M \to \mathbb{R}^{n+1}$ is an $n$-dimensional complete immersed $\lambda$-hypersurface with polynomial area growth, then $X : M \to \mathbb{R}^{n+1}$ is proper.

10. A LOWER BOUND GROWTH OF THE AREA FOR $\lambda$-HYPERSONFACES

For $n$-dimensional complete and non-compact Riemannian manifolds with nonnegative Ricci curvature, the well-known theorem of Calabi and Yau says that geodesic balls have at least linear area growth:

$$\text{Area}(B_r(x_0)) \geq Cr.$$  

Cao and Zhu [6] have proved that $n$-dimensional complete and non-compact gradient shrinking Ricci solutions must have infinite volume. Furthermore, Munteanu and Wang [25] have proved that areas of geodesic balls for $n$-dimensional complete and non-compact gradient shrinking Ricci solutions have at least linear growth. In this section, we study the lower bound growth of the area for $\lambda$-hypersurfaces. The following lemmas play a very important role in order to prove our results.

**Lemma 10.1.** Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional complete noncompact proper $\lambda$-hypersurface, then there exist constants $C_1(n,\lambda)$ and $c(n,\lambda)$ such that for all $t \geq C_1(n,\lambda)$,

$$\text{Area}(B_{t+1}(0) \cap X(M)) - \text{Area}(B_t(0) \cap X(M)) \leq c(n,\lambda) \frac{\text{Area}(B_t(0) \cap X(M))}{t}$$

and

$$\text{Area}(B_{t+1}(0) \cap X(M)) \leq 2\text{Area}(B_t(0) \cap X(M)).$$

**Proof.** Since $X : M \to \mathbb{R}^{n+1}$ is a complete $\lambda$-hypersurface, one has

$$\frac{1}{2} \Delta |X|^2 = n + H \langle N, X \rangle = n + H\lambda - H^2.$$  

Integrating (10.3) over $B_r(0) \cap X(M)$, we obtain

$$\frac{n \text{Area}(B_r(0) \cap X(M))}{2} + \int_{B_r(0) \cap X(M)} H\lambda d\mu - \int_{B_r(0) \cap X(M)} H^2 d\mu$$

$$= \frac{1}{2} \int_{B_r(0) \cap X(M)} \Delta |X|^2 d\mu$$

$$= \frac{1}{2} \int_{\partial(B_r(0) \cap X(M))} \nabla |X|^2 \cdot \nabla \rho |\nabla \rho| d\sigma$$

$$= \int_{\partial(B_r(0) \cap X(M))} |X^T| d\sigma$$

$$= \int_{\partial(B_r(0) \cap X(M))} \frac{|X|^2 - (\lambda - H)^2}{|X^T|} d\sigma$$

$$= r(\text{Area}(B_r(0) \cap X(M))) - \int_{\partial(B_r(0) \cap X(M))} \frac{(\lambda - H)^2}{|X^T|} d\sigma,$$
where $\rho(x) := |X(x)|$, $\nabla \rho = \frac{xT}{|x|^2}$. Here we used, from the co-area formula,

$$\int_{\partial(B_r(0) \cap X(M))} \frac{1}{|X'|} d\sigma. $$

Hence, we obtain

$$(n + \frac{\lambda^2}{4}) \text{Area}(B_r(0) \cap X(M)) - r (\text{Area}(B_r(0) \cap X(M)))'$$

$$= \int_{B_r(0) \cap X(M)} (H - \frac{\lambda}{2})^2 d\mu - \frac{(\lambda - H)^2}{|X'|} d\mu. $$

From (10.5), $(H - \lambda)^2 = \langle N, X \rangle^2 \leq |X|^2 = r^2$ on $\partial(B_r(0) \cap X(M))$ and (10.6), we conclude

$$\int_{B_r(0) \cap X(M)} (H - \frac{\lambda}{2})^2 d\mu \leq (n + \frac{\lambda^2}{4}) \text{Area}(B_r(0) \cap X(M)). $$

Furthermore, we have

$$\int_{B_r(0) \cap X(M)} (H - \lambda)^2 d\mu \leq \int_{B_r(0) \cap X(M)} 2[(H - \frac{\lambda}{2})^2 + \frac{\lambda^2}{4}] d\mu$$

$$\leq (2n + \lambda^2) \text{Area}(B_r(0) \cap X(M)). $$

(10.6) implies that

$$(r^{-n-\frac{\lambda^2}{4}} \text{Area}(B_r(0) \cap X(M)))'$$

$$= r^{-n-1-\frac{\lambda^2}{4}} \left( r (\text{Area}(B_r(0) \cap X(M)))' - (n + \frac{\lambda^2}{4}) \text{Area}(B_r(0) \cap X(M)) \right)$$

$$= r^{-n-1-\frac{\lambda^2}{4}} \int_{\partial(B_r(0) \cap X(M))} \frac{(H - \lambda)^2}{|X'|} d\sigma - r^{-n-1-\frac{\lambda^2}{4}} \int_{B_r(0) \cap X(M)} (H - \frac{\lambda}{2})^2 d\mu.$$

Integrating (10.9) from $r_2$ to $r_1$ ($r_1 > r_2$), one has

$$r_1^{-n-\frac{\lambda^2}{4}} \text{Area}(B_{r_1}(0) \cap X(M)) - r_2^{-n-\frac{\lambda^2}{4}} \text{Area}(B_{r_2}(0) \cap X(M))$$

$$= r_1^{-n-2-\frac{\lambda^2}{4}} \int_{B_{r_1}(0) \cap X(M)} (H - \lambda)^2 d\mu - r_2^{-n-2-\frac{\lambda^2}{4}} \int_{B_{r_2}(0) \cap X(M)} (H - \lambda)^2 d\mu$$

$$+ (n + 2 + \frac{\lambda^2}{4}) \int_{r_2}^{r_1} s^{-n-3-\frac{\lambda^2}{4}} (\int_{B_s(0) \cap X(M)} (H - \lambda)^2 d\mu) ds$$

$$- \int_{r_2}^{r_1} s^{-n-1-\frac{\lambda^2}{4}} (\int_{B_s(0) \cap X(M)} (H - \frac{\lambda}{2})^2 d\mu) ds$$

$$\leq (r_1^{-n-2-\frac{\lambda^2}{4}} + r_2^{-n-2-\frac{\lambda^2}{4}}) \int_{B_{r_1}(0) \cap X(M)} (H - \lambda)^2 d\mu.$$
and $\text{Area}(B_r(0) \cap X(M))$ is non-decreasing in $r$ from (10.5). Combining (10.10) with (10.8), we have

$$
\text{Area}(B_{r_1}(0) \cap X(M)) - \text{Area}(B_{r_2}(0) \cap X(M)) \\
\leq (2n + \lambda^2)\left(\frac{1}{r_1^{n+\lambda^2}} + \frac{1}{r_2^{n+\lambda^2}}\right) \text{Area}(B_{r_1}(0) \cap X(M)).
$$

(10.11)

Putting $r_1 = t + 1$, $r_2 = t > 0$, we get

$$
\left(1 - \frac{2(2n + \lambda^2)(t + 1)^{n+\lambda^2}}{t^{n+2+\lambda^2}}\right) \text{Area}(B_{t+1}(0) \cap X(M)) \\
\leq \text{Area}(B_t(0) \cap X(M))\left(\frac{t + 1}{t}\right)^{n+\lambda^2/4}.
$$

(10.12)

For $t$ sufficiently large, one has, from (10.12),

$$
\text{Area}(B_{t+1}(0) \cap X(M)) - \text{Area}(B_t(0) \cap X(M)) \\
\leq \text{Area}(B_t(0) \cap X(M))\left(1 + \frac{1}{t}\right)^n - 1 + \frac{C(t + 1)^{2n+\lambda^2/4}}{t^{2n+2+\lambda^2}})
$$

(10.13)

where $C$ is constant only depended on $n$, $\lambda$. Therefore, there exists some constant $C_1(n, \lambda)$ such that for all $t \geq C_1(n, \lambda)$,

$$
\text{Area}(B_{t+1}(0) \cap X(M)) - \text{Area}(B_t(0) \cap X(M)) \\
\leq c(n, \lambda) \frac{\text{Area}(B_t(0) \cap X(M))}{t},
$$

(10.14)

and

$$
\text{Area}(B_{t+1}(0) \cap X(M)) \leq 2\text{Area}(B_t(0) \cap X(M)),
$$

(10.15)

where $c(n, \lambda)$ depends only on $n$ and $\lambda$. This completes the proof of the lemma 10.1.

□

Using Logarithmic Sobolev inequality for hypersurfaces in Euclidean space due to Ecker [13] and conformal theory, we can show

**Lemma 10.2.** Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional hypersurface with measure $d\mu$. Then the following inequality

$$
\int_M f^2(\ln f^2)e^{-|X|^2/2}d\mu - \int_M f^2e^{-|X|^2/2}d\mu \ln(\int_M f^2e^{-|X|^2/2}d\mu) \\
\leq \int_M |\nabla f|^2e^{-|X|^2/2}d\mu + \frac{1}{4}\int_M |H + \langle X, N \rangle|^2f^2e^{-|X|^2/2}d\mu + C(n)\int_M f^2e^{-|X|^2/2}d\mu
$$

(10.16)

holds for any nonnegative function $f$ for which all integrals are well-defined and finite, where $C(n)$ is a positive constant depending on $n$. 

Corollary 10.1. For an $n$-dimensional $\lambda$-hypersurface $X : M \to \mathbb{R}^{n+1}$, we have the following inequality

$$
\int_M f^2(\ln f)e^{-\frac{\lambda^2}{2}}d\mu \leq \frac{1}{2}\int_M |\nabla f|^2 e^{-\frac{\lambda^2}{2}}d\mu + \left(\frac{1}{2}C(n) + \frac{1}{8}\lambda^2\right)2^{-\frac{n}{2}}
$$

for any nonnegative function $f$ which satisfies

$$
\int_M f^2e^{-\frac{\lambda^2}{2}}2^\frac{n}{2}d\mu = 1.
$$

Corollary 10.2. If $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional $\lambda$-hypersurface, then the following inequality

$$
\int_M u^2(\ln u^2)d\mu - \int_M u^2d\mu \ln(\int_M u^2d\mu)
$$

holds for any nonnegative function $f$ which satisfies

$$
f = ue^{\frac{|x|^2}{2}}.
$$

Lemma 10.3. (23) Let $X : M \to \mathbb{R}^{n+1}$ be a complete properly immersed hypersurface. For any $x_0 \in M$, $r \leq 1$, if $|H| \leq \frac{\rho}{r}$ in $B_r(X(x_0)) \cap X(M)$ for some constant $C > 0$. Then

$$
\text{Area}(B_r(X(x_0)) \cap X(M)) \geq \kappa r^n,
$$

where $\kappa = \omega_n e^{-C}$.

Lemma 10.4. If $X : M \to \mathbb{R}^{n+1}$ is an $n$-dimensional complete and non-compact proper $\lambda$-hypersurface, then it has infinite area.

Proof. Let

$$
\Omega(k_1, k_2) = \{ x \in M : 2^{k_1-\frac{1}{2}} \leq \rho(x) \leq 2^{k_2-\frac{1}{2}} \},
$$

where $\rho(x) = |X(x)|$. Since $X : M \to \mathbb{R}^{n+1}$ is a complete and non-compact proper immersion, $X(M)$ can not be contained in a compact Euclidean ball. Then, for $k$ large enough, $\Omega(k, k+1)$ contains at least $2^{2k-1}$ disjoint balls

$$
B_r(x_i) = \{ x \in M : \rho_{x_i}(x) < 2^{-\frac{k}{2}}r \}, \quad x_i \in M, \quad r = 2^{-k}
$$

where $\rho_{x_i}(x) = |X(x) - X(x_i)|$. Since, in $\Omega(k, k+1)$,

$$
|H| \leq |H - \lambda| + |\lambda| = |\langle X, N \rangle| + |\lambda| \leq |X| + |\lambda| \leq 2^k \sqrt{2} + |\lambda| \leq \frac{\sqrt{2} + |\lambda|}{r},
$$

by using of the lemma [10.3] we get

$$
A(k, k+1) \geq \kappa_1 2^{2k-1-kn},
$$

with $\kappa_1 = \omega_n e^{-\frac{1}{2}(\sqrt{2} + |\lambda|)2^{-\frac{k}{2}}2^{-\frac{k}{2}}}$.

Claim: If $\text{Area}(X(M)) < \infty$, then, for every $\varepsilon > 0$, there exists a large constant $k_0 > 0$ such that,

$$
A(k_1, k_2) \leq \varepsilon \quad \text{and} \quad A(k_1, k_2) \leq 2^m A(k_1 + 2, k_2 - 2), \quad \text{if} \quad k_2 > k_1 > k_0.
$$
In fact, we may choose $K > 0$ sufficiently large such that $k_1 \approx \frac{K}{2}$, $k_2 \approx \frac{3K}{2}$. Assume (10.24) does not hold, that is,

$$A(k_1, k_2) \geq 2^{4n} A(k_1 + 2, k_2 - 2).$$

If

$$A(k_1 + 2, k_2 - 2) \leq 2^{4n} A(k_1 + 4, k_2 - 4),$$

then we complete the proof of the claim. Otherwise, we can repeat the procedure for $j$ times, we have

$$A(k_1, k_2) \geq 2^{4nj} A(k_1 + 2j, k_2 - 2j).$$

When $j \approx \frac{K}{4}$, we have from (10.23)

$$\text{Area}(X(M)) \geq A(k_1, k_2) \geq 2^{nK} A(K, K + 1) \geq \kappa_1 2^{2K - 1}.$$

Thus, (10.24) must hold for some $k_2 > k_1$ because $\text{Area}(M) < \infty$. Hence for any $\varepsilon > 0$, we can choose $k_1$ and $k_2 \approx 3k_1$ such that (10.24) holds.

We define a smooth cut-off function $\psi(t)$ by

$$(10.25) \quad \psi(t) = \begin{cases} 1, & 2^{k_1 + \frac{1}{4}} \leq t \leq 2^{k_2 - \frac{5}{2}}, \\ 0, & \text{outside } [2^{k_1 - \frac{1}{2}}, 2^{k_2 - \frac{1}{2}}]. \end{cases} \quad 0 \leq \psi(t) \leq 1, \quad |\psi'(t)| \leq 1.$$

Letting

$$(10.26) \quad f(x) = e^{L + \frac{|X|^2}{4}} \psi(\rho(x)),$$

we choose $L$ satisfying

$$(10.27) \quad 1 = \int_M f^2 e^{-\frac{|X|^2}{2}} 2^{\frac{n}{2}} d\mu = e^{2L} \int_{\Omega(k_1, k_2)} \psi^2(\rho(x)) 2^{\frac{n}{2}} d\mu.$$

We obtain from the corollary 10.1 and $t \ln t \geq -\frac{1}{\varepsilon}$ for $0 \leq t \leq 1$

$$(\frac{1}{2} C(n) + \frac{1}{8} \lambda^2) 2^{-\frac{n}{2}} \geq \int_{\Omega(k_1, k_2)} e^{2L} \psi^2 (L + \frac{|X|^2}{4} + \ln \psi) d\mu$$

$$- \frac{1}{2} \int_{\Omega(k_1, k_2)} e^{2L} |\psi' \nabla \rho + \frac{X^T}{2}|^2 d\mu$$

$$\geq \int_{\Omega(k_1, k_2)} e^{2L} \psi^2 (L + \frac{|X|^2}{4} + \ln \psi) d\mu$$

$$- \int_{\Omega(k_1, k_2)} e^{2L} |\psi'|^2 d\mu - \frac{1}{4} \int_{\Omega(k_1, k_2)} e^{2L} \psi^2 |X|^2 d\mu$$

$$(10.28) \quad = 2^{-\frac{n}{2}} L + \int_{\Omega(k_1, k_2)} e^{2L} \psi^2 \ln \psi d\mu - \int_{\Omega(k_1, k_2)} e^{2L} |\psi'|^2 d\mu$$

$$\geq 2^{-\frac{n}{2}} L - \left( \frac{1}{2e} + 1 \right) e^{2L} A(k_1, k_2).$$
Therefore, it follows from (10.24) that
\[
\left( \frac{1}{2} C(n) + \frac{1}{8} \lambda^2 \right) 2^{-\frac{n}{2}} \geq 2^{-\frac{n}{2}} L - \left( \frac{1}{2e} + 1 \right) e^{2L} 2^{4n} A(k_1 + 2, k_2 - 2) \\
\geq 2^{-\frac{n}{2}} L - \left( \frac{1}{2e} + 1 \right) e^{2L} 2^{4n} \int_{\Omega(k_1, k_2)} \psi^2(\rho(x)) d\mu \\
= 2^{-\frac{n}{2}} L - \left( \frac{1}{2e} + 1 \right) 2^{4n} 2^{-\frac{n}{2}},
\]
On the other hand, we have, from (10.24) and definition of \( f(x) \),
\[
(10.30) \quad 1 \leq e^{2L} \varepsilon 2^{\frac{n}{2}}.
\]
Letting \( \varepsilon > 0 \) sufficiently small, then \( L \) can be arbitrary large, which contradicts (10.29). Hence, \( M \) has infinite area. \( \square \)

**Theorem 10.1.** Let \( X : M \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional complete proper \( \lambda \)-hypersurface. Then, for any \( p \in M \), there exists a constant \( C > 0 \) such that
\[
\text{Area}(B_r(X(x_0)) \cap X(M)) \geq Cr,
\]
for all \( r > 1 \).

**Proof.** We can choose \( r_0 > 0 \) such that \( \text{Area}(B_r(0) \cap X(M)) > 0 \) for \( r \geq r_0 \). It is sufficient to prove there exists a constant \( C > 0 \) such that
\[
(10.31) \quad \text{Area}(B_r(0) \cap X(M)) \geq Cr
\]
holds for all \( r \geq r_0 \). In fact, if (10.31) holds, then for any \( x_0 \in M \) and \( r > |X(x_0)| \),
\[
B_r(X(x_0)) \supset B_{r-|X(x_0)|}(0),
\]
and
\[
(10.33) \quad \text{Area}(B_r(X(x_0)) \cap X(M)) \geq \text{Area}(B_{r-|X(x_0)|}(0) \cap X(M)) \geq \frac{C}{2} r,
\]
for \( r \geq 2|X(x_0)| \).

We next prove (10.31) by contradiction. Assume for any \( \varepsilon > 0 \), there exists \( r \geq r_0 \) such that
\[
(10.34) \quad \text{Area}(B_r(0) \cap X(M)) \leq \varepsilon r.
\]
Without loss of generality, we assume \( r \in \mathbb{N} \) and consider a set:
\[
D := \{ k \in \mathbb{N} : \text{Area}(B_t(0) \cap X(M)) \leq 2\varepsilon t \text{ for any integer } t \text{ satisfying } r \leq t \leq k \}.
\]
Next, we will show that \( k \in D \) for any integer \( k \) satisfying \( k \geq r \). For \( t \geq r_0 \), we define a function \( u \) by
\[
(10.35) \quad u(x) = \begin{cases} 
  t + 2 - \rho(x), & \text{in } B_{t+2}(0) \cap X(M) \setminus B_{t+1}(0) \cap X(M), \\
  1, & \text{in } B_{t+1}(0) \cap X(M) \setminus B_{t}(0) \cap X(M), \\
  \rho(x) - (t - 1), & \text{in } B_{t}(0) \cap X(M) \setminus B_{t-1}(0) \cap X(M), \\
  0, & \text{otherwise}.
\end{cases}
\]
Using the corollary \(10.2\) \(|\nabla \rho| \leq 1\) and \(t \ln t \geq -\frac{1}{e}\) for \(0 \leq t \leq 1\), we have

\[
\begin{align*}
&- \left( \int_M u^2 d\mu \right) \ln \left\{ \left( \text{Area}(B_{t+2}(0) \cap X(M)) - \text{Area}(B_{t-1}(0) \cap X(M)) \right) 2^\frac{t}{\ln 2} \right\} \\
&\leq C_0 \left( \text{Area}(B_{t+2}(0) \cap X(M)) - \text{Area}(B_{t-1}(0) \cap X(M)) \right),
\end{align*}
\]

(10.36)

where \(C_0 = 2 + \frac{1}{e} + \frac{\lambda^2}{4} + \frac{n}{2} \ln 2 + C(n)\), \(C(n)\) is the constant of the corollary \(10.2\).

For all \(t \geq C_1(n, \lambda) + 1\), we have from the lemma \(10.1\)

\[
\begin{align*}
&\text{Area}(B_{t+2}(0) \cap X(M)) - \text{Area}(B_{t-1}(0) \cap X(M)) \\
&\leq c(n, \lambda) \left( \frac{\text{Area}(B_{t+1}(0) \cap X(M))}{t+1} + \frac{\text{Area}(B_{t}(0) \cap X(M))}{t} + \frac{\text{Area}(B_{t-1}(0) \cap X(M))}{t-1} \right) \\
&\leq c(n, \lambda) \left( \frac{2}{t+1} + \frac{1}{t} + \frac{1}{t} \left( 1 + \frac{1}{C_1(n, \lambda)} \right) \right) \text{Area}(B_{t}(0) \cap X(M)) \\
&\leq C_2(n, \lambda) \frac{\text{Area}(B_{t}(0) \cap X(M))}{t},
\end{align*}
\]

(10.37)

where \(C_2(n, \lambda)\) is constant depended only on \(n\) and \(\lambda\). Note that we can assume \(r \geq C_1(n, \lambda) + 1\) for the \(r\) satisfying \(10.34\). In fact, if for any given \(\varepsilon > 0\), all the \(r\) which satisfies \(10.34\) is bounded above by \(C_1(n, \lambda) + 1\), then \(\text{Area}(B_{r}(0) \cap X(M)) \geq Cr\) holds for any \(r > C_1(n, \lambda) + 1\). Thus, we know that \(M\) has at least linear area growth. Hence, for any \(k \in D\) and any \(t\) satisfying \(r \leq t \leq k\), we have

\[
\text{Area}(B_{t+2}(0) \cap X(M)) - \text{Area}(B_{t-1}(0) \cap X(M)) \leq 2C_2(n, \lambda)\varepsilon.
\]

(10.38)

Since

\[
\int_M u^2 d\mu \geq \text{Area}(B_{t+1}(0) \cap X(M)) - \text{Area}(B_{t}(0) \cap X(M)),
\]

holds, if we choose \(\varepsilon\) such that \(2C_2(n, \lambda)\varepsilon 2^\frac{t}{\ln 2} < 1\), from \(10.36\), we obtain

\[
\begin{align*}
&\left( \text{Area}(B_{t+1}(0) \cap X(M)) - \text{Area}(B_{t}(0) \cap X(M)) \right) \ln \left( 2^\frac{t}{\ln 2} + 1 \right) C_2(n, \lambda)\varepsilon^{-1} \\
&\leq C_0 \left( \text{Area}(B_{t+2}(0) \cap X(M)) - \text{Area}(B_{t-1}(0) \cap X(M)) \right).
\end{align*}
\]

(10.40)

Iterating from \(t = r\) to \(t = k\) and taking summation on \(t\), we infer, from the lemma \(10.1\)

\[
\begin{align*}
&\left( \text{Area}(B_{k+1}(0) \cap X(M)) - \text{Area}(B_{r}(0) \cap X(M)) \right) \ln \left( 2^\frac{t}{\ln 2} + 1 \right) C_2(n, \lambda)\varepsilon^{-1} \\
&\leq 3C_0 \text{Area}(B_{k+2}(0) \cap X(M)) \leq 6C_0 \text{Area}(B_{k+1}(0) \cap X(M)).
\end{align*}
\]

(10.41)
Hence, we get
\[
\text{Area}(B_{k+1}(0) \cap X(M)) \leq \frac{\ln(2^{2+1}C_2(n, \lambda)\varepsilon)^{-1}}{\ln(2^{2+1}C_2(n, \lambda)\varepsilon)^{-1} - 6C_0} \text{Area}(B_r(0) \cap X(M))
\]
\[
\leq \frac{\ln(2^{2+1}C_2(n, \lambda)\varepsilon)^{-1}}{\ln(2^{2+1}C_2(n, \lambda)\varepsilon)^{-1} - 6C_0} \varepsilon r.
\]
We can choose \(\varepsilon\) small enough such that
\[
\frac{\ln(2^{2+1}C_2(n, \lambda)\varepsilon)^{-1}}{\ln(2^{2+1}C_2(n, \lambda)\varepsilon)^{-1} - 6C_0} \leq 2.
\]
Therefore, it follows from (10.42) that
\[
\text{Area}(B_{k+1}(0) \cap X(M)) \leq 2\varepsilon r,
\]
for any \(k \in D\). Since \(k + 1 \geq r\), we have, from (10.44) and the definition of \(D\), that \(k + 1 \in D\). Thus, by induction, we know that \(D\) contains all of integers \(k \geq r\) and
\[
\text{Area}(B_k(0) \cap X(M)) \leq 2\varepsilon r,
\]
for any integer \(k \geq r\). This implies that \(M\) has finite volume, which contradicts with the lemma 10.4. Hence, there exist constants \(C\) and \(r_0\) such that \(\text{Area}(B_r(0) \cap X(M)) \geq Cr\) for \(r > r_0\). It completes the proof of the theorem 10.1.

**Remark 10.1.** The estimate in our theorem is best possible because the cylinders \(S^{n-1}(r_0) \times \mathbb{R}\) satisfy the equality.

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QING-MING CHENG, DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF SCIENCES, FUKUOKA UNIVERSITY, 814-0180, FUKUOKA, JAPAN, cheng@fukuoka-u.ac.jp

GUOXIN WEI, SCHOOL OF MATHEMATICAL SCIENCES, SOUTH CHINA NORMAL UNIVERSITY, 510631, GUANGZHOU, CHINA, weiguoxin@tsinghua.org.cn