A Jacobi theta series and its transformation laws

Matthew Krauel

Abstract
We consider a generalization of Jacobi theta series and show that every such function is a quasi-Jacobi form. Under certain conditions we establish transformation laws for these functions with respect to the Jacobi group and prove such functions are Jacobi forms. In establishing these results we construct other functions which are also Jacobi forms. These results are motivated by applications in the theory of vertex operator algebras.

1 Introduction

Let $Q$ be a positive definite integral quadratic form, and $B$ be the associated bilinear form so that $2Q(x) = B(x, x)$. Fix $h \in \mathbb{Z}^f$ and let $A$ be the matrix of $Q$ with even rank $f = 2r$. It is well known (page 81 of [4], for example) that the Jacobi theta series

$$
\sum_{m \in \mathbb{Z}^f} q^{Q(m)} \zeta^{B(m, h)}
$$

is a Jacobi form of weight $r$ and index $Q(h)$ on the full Jacobi group $J_1 = \text{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z} \times \mathbb{Z})$. Here, $\zeta = e^{2\pi i z}$ with $z \in \mathbb{C}$, and $q = e^{2\pi i \tau}$ with $\tau$ in the complex upper-half plane $\mathbb{H}$.

Fix an element $v \in \mathbb{C}^f$. Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, and set $h = (h_1, \ldots, h_n)$ for linearly independent elements $h_1, \ldots, h_n \in \mathbb{Z}^f$. In this paper we consider functions of the form

$$
\theta_k(Q, v, k, \tau, z) := \sum_{m \in \mathbb{Z}^f} B(v, m)^k q^{Q(m)} \zeta_1^{B(m, h_1)} \cdots \zeta_n^{B(m, h_n)},
$$

for any natural number $k$. We show such functions are quasi-Jacobi forms of weight $k$ for all $v \in \mathbb{C}^f$. In the case $Q(v) = 0$ and $B(v, h_j) = 0$ for all $j$, we develop transformation laws with respect to a subgroup of the Jacobi group $J_n = \text{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^n \times \mathbb{Z}^n)$ and show these functions are Jacobi forms on this subgroup. Precise definitions of Jacobi forms and quasi-Jacobi forms are given in Section 2.

In the special case $\underline{z} = 0$, such functions are considered in [2] and transformation laws with respect to $\text{SL}_2(\mathbb{Z})$ are developed which expands on work of [11], [10], [8], and [5] for similar functions. Many of the techniques used in the proofs below are attributed
to these authors. We will reference them often, and attempt to maintain the notation
developed in these works (especially [2]). Other techniques have been developed to
establish transformation laws for theta series of higher degree [1].

Holomorphic Jacobi forms of higher degree have been considered by Ziegler in [12],
where Jacobi theta series consisting of two complex matrix variables are shown to be
examples. Richter generalizations some of this work in [9]. However, the literature
lacks such results for Jacobi theta series as defined in (1). This paper fills this gap
and is motivated by the occurrence of functions of the form (1) and (6) in the theory
of vertex operator algebras in work similar to those of [3] and [7]. This work will be
appear elsewhere. The author would like to thank Geoffrey Mason and Olav Richter
for helpful discussions. In particular, Richer pointed out that there are other possible
approaches to some of the results that we prove.

Let \( \Gamma := \text{SL}_2(\mathbb{Z}) \) and \( \Gamma_0(N) \) be the subgroup of \( \Gamma \) defined by
\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \mod N \right\}.
\]
We denote the subgroup \( \Gamma_0(N) \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) \) of \( \mathcal{J}_n \) by \( \mathcal{J}_n^0(N) \) and the spaces of Jacobi
forms and quasi-Jacobi forms of weight \( k \) and index \( F \) on the group \( \mathcal{J}_n^0(N) \) by
\( \mathcal{J}_n^{k,F}^0(N) \) and \( \mathcal{Q}_n^{k,F}^0(N) \), respectively. Let \( G \) be the Gram matrix associated to the bilinear form
\( B(\cdot, \cdot) \) and elements \( h_1, \ldots, h_n \), so that \( G = (B(h_i, h_j)) \). For a vector \( \alpha = (\alpha_1, \ldots, \alpha_n) \),
set \( G[\alpha] := \alpha^t G \alpha \). We define \( \epsilon(n) \) for \( n > 0 \) by \( \epsilon(n) = (-1)^{r \det(A) n} \), and \( \epsilon(-n) = (-1)^r \epsilon(n) \). \( \epsilon \) is a Dirichlet character (see page 216 in [11] for example). For a matrix
\( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), we often write \( \gamma \tau \) and \( \gamma z \) to denote
\( a \tau + b c \tau + d \) and \( c \tau + d \), respectively.

We are now in position to state our first result.

**Theorem 1.1** For any \( v, h \), and \( k \) as defined above, we have
\[
\theta_h(Q, v, k, \tau, z) \in \mathcal{Q}_0^{n,k,F}(N)
\]
and \( \theta_h \) has a Fourier expansion of the form
\[
\theta_h(Q, v, k, \tau, z) = \sum_{\ell \in \mathbb{Z}^n, \ell \in \mathbb{Q}, 4\ell \cdot (G/2)^{-1}[\ell] \geq 0} c(\ell, z) q^{\ell} \exp\left(2\pi i (\ell^t z)\right)
\]
where \( \ell \geq 0 \) and \( c(\ell, z) \) are scalars. In particular, \( \ell = Q(m) \in \mathbb{Z} \) and \( \tau = (B(m, h_1), \ldots, B(m, h_n)) \).

If in addition \( Q(v) = 0 \) and \( B(v, h_j) = 0 \) for all \( 1 \leq j \leq n \), then for any
\( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \) and \( (\lambda, \mu) \in \mathbb{Z}^n \times \mathbb{Z}^n \) we have
\[
\theta_h(Q, v, k, \gamma \tau, \gamma z) = \epsilon(d) \exp\left(\frac{\pi i cG[z]}{c \tau + d}\right) (c \tau + d)^{k+r} \theta_h(Q, v, k, \tau, z),
\]
\[ \theta_h(Q, v, k, \tau, z + \lambda \tau + \mu) = \exp \left( -\pi i (G[\lambda] \tau + 2z^t G \lambda) \right) \theta_h(Q, v, k, \tau, z). \]  

(5)

The expansion (3) along with the transformation laws (4) and (5) imply that if \( Q(v) = 0 \) and \( B(v, h_j) = 0 \) for all \( 1 \leq j \leq n \), then (11) is a Jacobi form of weight \( k + r \), index \( G/2 \), and character \( \epsilon \). Theorem 1.1 is proved in Section 3.

In proving Theorem 1.1 we consider functions of the form

\[ \Psi_h(Q, v, k, \tau, z) := \sum_{t=0}^{k} \rho(t, k) (2Q(v)E_2(\tau))^t \theta_h(Q, v, k - 2t, \tau, z), \]  

(6)

where \( E_2(\tau) \) is the usual modular Eisenstein series of weight 2 and \( \rho(t, k) \) is defined by

\[ \rho(t, k) = \frac{k!}{2^t!(k-2t)!}. \]  

(7)

We establish the following theorem in Section 3.

**Theorem 1.2** Suppose \( B(v, h_j) = 0 \) for all \( 1 \leq j \leq n \). Then for any \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N) \) and \( [\lambda, \mu] \in (\mathbb{Z}^n \times \mathbb{Z}^n) \) we have

\[ \Psi_h(Q, v, k, \gamma \tau, \gamma z) = \epsilon(d)(c\tau + d)^{k+r} \exp \left( \frac{\pi i cG[\lambda]}{c\tau + d} \right) \Psi_h(Q, v, k, \tau, z) \]  

(8)

and

\[ \Psi_h(Q, v, k, \tau, z + \lambda \tau + \mu) = \exp \left( -\pi i (G[\lambda] \tau + 2z^t G \lambda) \right) \Psi_h(Q, v, k, \tau, z). \]  

(9)

In particular, \( \Psi_h(Q, v, k, \tau, z) \) is a Jacobi form of weight \( k+r \), index \( G/2 \), and character \( \epsilon \) on the subgroup \( J^0_0(N) \).

To obtain proofs for the previous theorems we consider functions of the form

\[ \Theta_h(Q, v, \tau, z, X) := \sum_{n \geq 0} \frac{2^{n/2}\theta_h(Q, v, n, \tau, z)}{n!} (2\pi i X)^n, \]  

(10)

which are similar to Jacobi-like forms. The above theorems follow from manipulation of the following result which is proved in Section 4.

**Theorem 1.3** Suppose \( B(v, h_j) = 0 \) for all \( 1 \leq j \leq n \). Then for any \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N) \) we have

\[ \Theta_h(Q, v, \gamma \tau, \gamma z, X) \left( \frac{X}{c\tau + d} \right)^2 \]  

\[ = \epsilon(d)e^{\frac{2\pi i G[\lambda]}{c\tau + d}} (c\tau + d)^r \exp \left( \frac{2\pi i [2Q(v)]cX^2}{(c\tau + d)^3} \right) \Theta_h(Q, v, \tau, z, X (c\tau + d)). \]  

(11)
2 Preliminaries

Let $\text{Hol}_{\mathbb{H} \times \mathbb{C}^n}$ denote the space of holomorphic functions from $\mathbb{H} \times \mathbb{C}^n$ to $\mathbb{C}$ and $F$ be a real symmetric positive definite $n \times n$ matrix. We say a function $\phi \in \text{Hol}_{\mathbb{H} \times \mathbb{C}^n}$ is a Jacobi form of weight $k$, index $F$, and character $\chi$ ($\chi : \Gamma' \to \mathbb{C}^*$) on a subgroup $\Gamma'$ of $\Gamma$ if $\phi$ has an expansion of the form

$$\phi(\tau, z) = \sum_{\ell \in \mathbb{Z}^n, \ell \in \mathbb{Q}, \frac{\ell}{4\ell - F^{-1}d} \geq 0} c(\ell, z) q^\ell \exp \left( 2\pi i (\ell, \tau, z) \right),$$

(12)

where $\ell \geq \ell_0$ for some $\ell_0$, $c(\ell, z)$ are scalars, and for all $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma'$ and $(\Delta, \mu) \in \mathbb{Z}^n \times \mathbb{Z}^n$ we have

$$\phi(\gamma \tau, \gamma z) = \chi(\gamma)(c\tau + d)^k \exp \left( 2\pi i \frac{cF[z]}{c\tau + d} \right) \phi(\tau, z),$$

(13)

and

$$\phi(\tau, z + \Delta \tau + \mu) = \exp \left( -2\pi i (\tau F[\Delta] + 2z^t F\Delta) \right) \phi(\tau, z).$$

(14)

In the case $\ell_0 \geq 0$, $\phi$ is called holomorphic, otherwise it is meromorphic.

A function $\phi \in \text{Hol}_{\mathbb{H} \times \mathbb{C}^n}$ is a quasi-Jacobi form of weight $k$ and index $F$ on $\Gamma'$ if for fixed $\tau \in \mathbb{H}$, $z \in \mathbb{C}^n$, $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma'$, and $(\Delta, \mu) \in \mathbb{Z}^n \times \mathbb{Z}^n$, we have

1. $(c\tau + d)^{-k} \exp \left( -\frac{2\pi i c F[z]}{c\tau + d} \right) \phi(\gamma \tau, \gamma z) \in \text{Hol}_{\mathbb{H} \times \mathbb{C}^n} \left[ \frac{cz_1}{c\tau + d}, \ldots, \frac{cz_n}{c\tau + d}, \frac{c}{c\tau + d} \right]$ with coefficients dependent only on $\phi$, and

2. $\exp \left( 2\pi i (\tau F[\Delta] + 2z^t F\Delta) \right) \phi(\tau, z + \Delta \tau + \mu) \in \text{Hol}_{\mathbb{H} \times \mathbb{C}^n} \left[ \lambda_1, \ldots, \lambda_n \right]$ with coefficients dependent only on $\phi$.

In other words, there are natural numbers $s_1, \ldots, s_n, t$, and holomorphic functions $S_{i_1, \ldots, i_n}(\phi)$ and $T_{i_1, \ldots, i_n}(\phi)$ on $\mathbb{H} \times \mathbb{C}^n$ determined only by $\phi$ such that

$$(c\tau + d)^{-k} \exp \left( -\frac{2\pi i c F[z]}{c\tau + d} \right) \phi(\gamma \tau, \gamma z) = \sum_{i_1 \leq s_1, \ldots, i_n \leq s_n} S_{i_1, \ldots, i_n}(\phi)(\tau, z) \left( \frac{cz_1}{c\tau + d} \right)^{i_1} \cdots \left( \frac{cz_n}{c\tau + d} \right)^{i_n} \left( \frac{c}{c\tau + d} \right)^t,$$

(15)

and

$$\exp \left( 2\pi i (\tau F[\Delta] + 2z^t F\Delta) \right) \phi(\tau, z + \Delta \tau + \mu) = \sum_{i_1 \leq s_1, \ldots, i_n \leq s_n} T_{i_1, \ldots, i_n}(\phi)(\tau, z) \lambda_1^{i_1} \cdots \lambda_n^{i_n}. $$

(16)
If \( \phi \neq 0 \), we may take \( S_{s_1, \ldots, s_n, t}(\phi) \neq 0 \) and \( T_{s_1, \ldots, s_n}(\phi) \neq 0 \) and say that \( \phi \) is a quasi-Jacobi form of depth \((s_1, \ldots, s_n, t)\). Direct calculation shows the space of quasi-Jacobi forms is invariant under applications of \( \frac{d}{dx} \), \( \frac{d}{d\tau} \), and \( E_2(\tau) \). In particular, \( \frac{d}{dz} \) and \( E_2(\tau) \) applied to a quasi-Jacobi form of weight \( k \) increases the weight to \( k + 1 \) and \( k + 2 \), respectively.

3 Proofs of Theorems 1.1 and 1.2

In this section we assume Theorem 1.3 and use this to prove Theorems 1.1 and 1.2. We begin by proving the \( \Gamma_0(N) \) transformations (4) and (8). Take \( Q(v) \neq 0 \) and consider the function

\[
\hat{E}_2(Q, v, \tau, X) := \exp\left(2Q(v)E_2(\tau)(2\pi i X)^2\right).
\]

Using the transformation \((c\tau + d)^{-2}E_2(\gamma\tau) = E_2(\tau) - \frac{c}{2\pi i (c\tau + d)}\), we find

\[
\hat{E}_2(Q, v, \gamma\tau, \frac{X}{(c\tau + d)^2}) = \hat{E}_2(Q, v, \tau, \frac{X}{c\tau + d}) \exp\left(-\frac{2\pi ic[2Q(v)]X^2}{(c\tau + d)^3}\right). \tag{17}
\]

Combining (17) and Theorem 1.3 we find

\[
\hat{E}_2(Q, v, \gamma\tau, \frac{X}{(c\tau + d)^2}) \Theta_h(Q, v, \gamma\tau, \gamma z, \frac{X}{c\tau + d}) = \hat{E}_2(Q, v, \tau, \frac{X}{c\tau + d}) \Theta_h(Q, v, \tau, z, \frac{X}{c\tau + d}) \exp\left(-\frac{2\pi ic[2Q(v)]X^2}{(c\tau + d)^3}\right). \tag{18}
\]

Expanding the left hand side of (18), we find

\[
\hat{E}_2(Q, v, \gamma\tau, \frac{X}{(c\tau + d)^2}) \Theta_h(Q, v, \gamma\tau, \gamma z, \frac{X}{c\tau + d})
= \sum_{\ell \geq 0} \frac{1}{\ell!} (2Q(v)E_2(\gamma\tau))^\ell \left(\frac{2\pi i X}{(c\tau + d)^2}\right)^{2\ell} \sum_{n \geq 0} \frac{2^{n/2}}{n!} \theta_h(Q, v, n, \gamma\tau, \gamma z) \left(\frac{2\pi i X}{(c\tau + d)^2}\right)^n
= \sum_{\ell \geq 0} \sum_{n \geq 0} \frac{2^{n/2}}{\ell! \cdot n!} (2Q(v)E_2(\gamma\tau))^\ell \theta_h(Q, v, n, \gamma\tau, \gamma z) (c\tau + d)^{-2(\ell + n)} (2\pi i X)^{2\ell + n}
= \sum_{k \geq 0} \sum_{\ell = 0}^{k} \frac{2^{k/2}}{k!} \delta(\ell, k) (2Q(v)E_2(\gamma\tau))^\ell \theta_h(Q, v, k - 2\ell, \gamma\tau, \gamma z) (c\tau + d)^{-2k} (2\pi i X)^k, \tag{19}
\]
where we set \( k = 2\ell + n \) and use the fact \( 2^{k-2\ell}/\ell!(k-2\ell)! = 2^{k/2}\delta(\ell, k)/k! \). Expanding the right hand side of (15) shows

\[
\mathcal{E}_2 \left( Q, v, \tau, \frac{X}{c\tau + d} \right) \epsilon(d)e^{2\pi i G[\lambda]}(c\tau + d)^\tau \Theta_\lambda \left( Q, v, \tau, z, \frac{X}{c\tau + d} \right)
\]

\[
= \epsilon(d)(c\tau + d)^\tau e^{2\pi i G[\lambda]} \sum_{\ell \geq 0} \sum_{n \geq 0} \frac{2^{n/2}}{\ell! n!} (2Q(v)E_2(\tau))^\ell \]

\[
\cdot \theta_\lambda(Q, v, n, \tau, \bar{z})(c\tau + d)^{-2\ell-n}(2\pi i X)^{2\ell+n}
\]

\[
= \epsilon(d)(c\tau + d)^\tau e^{2\pi i G[\lambda]} \sum_{k \geq 0} \sum_{\ell = 0}^{2^{k/2}k!} \frac{2^{k/2}}{\ell!} \delta(\ell, k) (2Q(v)E_2(\tau))^\ell \]

\[
\cdot \theta_\lambda(Q, v, k-2\ell, \tau, \bar{z})(c\tau + d)^{-k}(2\pi i X)^{k},
\]

where we again set \( k = 2\ell + n \). Using (15) to combine (19) and (20) and then comparing the coefficients of \( X^k \), we find

\[
\frac{2^{k/2}k!}{\ell!} \sum_{\ell = 0}^{k} \delta(\ell, k) (2Q(v)E_2(\gamma\tau))^\ell \theta_\lambda(Q, v, k-2\ell, \gamma\tau, \gamma\bar{z})
\]

\[
= \epsilon(d)(c\tau + d)^{\tau+k} e^{2\pi i G[\lambda]} 2^{k/2}k! \sum_{\ell = 0}^{k} \delta(\ell, k) (2Q(v)E_2(\tau))^\ell \theta_\lambda(Q, v, k-2\ell, \tau, \bar{z}).
\]

That is,

\[
\Psi_\lambda(Q, v, k, \gamma\tau, \gamma\bar{z}) = \epsilon(d)(c\tau + d)^{\tau+k} e^{2\pi i G[\lambda]} \Psi_\lambda(Q, v, \tau, \bar{z}),
\]

as desired. This establishes (8). Taking \( Q(v) = 0 \) gives (4).

We will now prove the \( \mathbb{Z}^n \times \mathbb{Z}^n \) transformations (5) and (9). Since \( \mu_j B(m, h_j) \in \mathbb{Z} \) for each \( j \), we have \( \theta_\lambda(Q, v, k, \tau, \bar{z} + \lambda\tau + \mu) = \theta_\lambda(Q, v, k, \tau, \bar{z} + \lambda\tau) \). Using that

\[
Q(m + \lambda_1 h_1 + \cdots + \lambda_n h_n) = Q(m) + \sum_{j=1}^{n} \lambda_j B(m, h_j) + \frac{1}{2} G[\lambda],
\]

we find

\[
\theta_\lambda(Q, v, k, \tau, \bar{z} + \lambda\tau + \mu)
\]

\[
e^{-\pi i G[\lambda]} \sum_{m \in \mathbb{Z}^l} B(v, m) e^{2\pi i Q(m) + \sum_{j=1}^{n} \lambda_j h_j} e^{2\pi i \sum_{j=1}^{n} z_j B(m, h_j)}
\]

\[
e^{-\pi i (G[\lambda] - 2\pi i G[\lambda])} \sum_{m \in \mathbb{Z}^l} B \left( v, m - \sum_{j=1}^{n} \lambda_j h_j \right) e^{2\pi i Q(m)} e^{2\pi i \sum_{j=1}^{n} z_j B(m, h_j)},
\]

6
where we replaced $m$ with $m - \sum_{j=1}^{n} \lambda_j h_j$ in the last equality. Using the assumption $B(v, h_j) = 0$ for $1 \leq j \leq n$ establishes (5). Equation (9) follows immediately from (5). The proof of Theorem 1.2 is complete.

We now turn our attention to proving (2) and (3), and therefore no longer assume $B(v, h_j) = 0$ for all $j$. First we establish the following lemma.

**Lemma 3.1** For any $v \in \mathbb{C}^f$ there exists an element $u \in \mathbb{C}^f$ satisfying $B(u, h_j) = 0$ for $1 \leq j \leq n$ such that

$$
\theta_h(Q, v, k, \tau, \bar{z}) = \left(\sum_{p_1 + \cdots + p_n + k_1 = k} \left(\frac{k}{p_1, \ldots, p_n, k_1}\right) \prod_{i=1}^{n} \alpha_i^{p_i} \partial_{z_i}^{p_i}\right) \theta_h(Q, u, k_1, \tau, \bar{z}).
$$

Here $\sum_{p_1 + \cdots + p_n + k_1 = k}$ denotes summing over the positive integers $p_1, \ldots, p_n$, and $k_1$ which sum to $k$, $\alpha_i$ are scalars, $\partial_{z_i} = \frac{1}{2\pi i} \frac{d}{dz_i}$, and $(\frac{k}{p_1, \ldots, p_n, k_1})$ are the multinomial coefficients $(\frac{k}{p_1, \ldots, p_n, k_1}) = \frac{k!}{p_1! \cdots p_n! k_1!}$.

**Proof** Extend the set $\{h_1, \ldots, h_n\}$ into a basis $\{h_1, \ldots, h_n, u_{n+1}, \ldots, u_f\}$ of $\mathbb{C}^f$ such that $B(h_i, m) = 0$ for all $i, j$. Then there are scalars $\alpha_i, \beta_j \in \mathbb{C}$ such that $v = \sum_i \alpha_i h_i + \sum_j \beta_j u_j$. Set $u = \sum_j \beta_j u_j$. We have

$$
\theta_h(Q, v, k, \tau, \bar{z}) = \sum_{m \in \mathbb{Z}^f} B \left(\sum_i \alpha_i h_i + u, m\right) q^{Q(m)} \zeta_1^{B(m, h_1)} \cdots \zeta_n^{B(m, h_n)}.
$$

The lemma follows by use of the multinomial theorem and replacing each $B(h_i, m)$ with $\partial_{z_i}$. \[\square\]

We now establish the convergence of the functions $\theta_{h, u}(Q, v, k, \tau, \bar{z})$. If we again extend the set $\{h_1, \ldots, h_n\}$ into a basis $\{h_1, \ldots, h_n, u_{n+1}, \ldots, u_f\}$ for $\mathbb{C}^f$ and write $v = \sum_i \alpha_i h_i + \sum_j \alpha_j u_j$, we may consider functions of the form

$$
\theta_{h, u}(Q, v, k, \tau, \bar{z}) = \sum_{m \in \mathbb{Z}^f} B(v, m) q^{Q(m)} \zeta_1^{B(m, h_1)} \cdots \zeta_n^{B(m, h_n)} \zeta_{n+1}^{B(m, u_{n+1})} \cdots \zeta_f^{B(m, u_f)}.
$$

By the previous lemma we have

$$
\theta_{h, u}(Q, v, k, \tau, \bar{z}) = \left(\sum_{p_1 + \cdots + p_f = k} \left(\frac{k}{p_1, \ldots, p_f}\right) \prod_{i=1}^{f} \alpha_i^{p_i} \partial_{z_i}^{p_i}\right) \theta_{h, u}(Q, 0, \tau, \bar{z}).
$$
where \( \theta_{\mathbb{H},m}(Q,0,\tau,\mathbf{z}) = \sum_{m \in \mathbb{Z}^f} q^{Q(m)} \zeta_1^{B(m,h_1)} \cdots \zeta_n^{B(m,h_n)} \zeta_{n+1}^{B(m,u_{n+1})} \cdots \zeta_d^{B(m,u_f)} \) converges for each \( z_i \) when we fix the remaining \( z_j, j \neq i \). Therefore, by Hartogs' Theorem the function converges for all \( \mathbf{z} \) on \( \mathbb{H} \times \mathbb{C}^f \). It follows that \( \theta_{\mathbb{H},m}(Q, v, k, \tau, \mathbf{z}) \) converges on \( \mathbb{H} \times \mathbb{C}^f \). Setting \( z_{n+1} = \cdots = z_f = 0 \), shows that \( \theta_{\mathbb{H},m}(Q, v, k, \tau, \mathbf{z}) \) is convergent on \( \mathbb{H} \times \mathbb{C}^n \).

We turn to proving the inequality

\[
4Q(m) - (B(m,h_1), \ldots, B(m,h_n))(G/2)^{-1}(B(m,h_1), \ldots, B(m,h_n))^t \leq 0
\]

holds. Setting \( \beta = (B(m,h_1), \ldots, B(m,h_n)) \), we rewrite this expression as

\[
\beta G^{-1} \beta^t \leq B(m,m).
\]  

(21)

Since \( G \) is a real symmetric matrix it has the decomposition \( G = Q D Q^{-1} = Q D Q^t \), where \( D \) is a diagonal matrix and \( Q = (q_{ij}) \) is orthogonal. Therefore, \( G^{-1} = Q D^{-1} Q^t \) and we rewrite (21) as \( (\beta Q) D^{-1} (\beta Q)^t \leq B(m,m) \). Setting \( v_i = \sum_j q_{ij} h_j \), this becomes

\[
(B(m,v_1), \ldots, B(m,v_n)) D^{-1} (B(m,v_1), \ldots, B(m,v_n))^t \leq B(m,m).
\]  

(22)

So long as \( D = (B(v_i,v_j)) \), establishing (21) is equivalent to proving (22). However, this is the case since

\[
B(v_i,v_j) = B(v_j,v_i) = \left( \sum_r q_{rj} h_r, \sum_s q_{si} h_s \right) = \sum_{r,s} q_{rj} h_r h_s q_{si}
\]

is the \( i,j \)-th component of the matrix \( D = Q^t G Q \). It is therefore sufficient to verify (22).

First we assume that \( h_1, \ldots, h_n \) span \( \mathbb{Z}^f \). Then \( m = \sum_j \lambda_j v_j \) for some \( \lambda_j \). In this case we have

\[
(B(m,v_1), \ldots, B(m,v_n)) D^{-1} (B(m,v_1), \ldots, B(m,v_n))^t
= \sum_{i,j} \lambda_j^2 B(v_j,v_i)^2 B(v_i,v_i)^{-1} = \sum_i \lambda_i^2 B(v_i,v_i) = B(m,m)
\]

since \( B(v_j,v_i) = 0 \) for \( i \neq j \) being the off-diagonal components of the diagonal matrix \( D \). This establishes (22) when \( h_1, \ldots, h_n \) span \( \mathbb{Z}^f \).

Suppose next that \( h_1, \ldots, h_n \) do not span \( \mathbb{Z}^f \). Since they are linear independent they form an \( \mathbb{R} \)-basis for \( \mathbb{R}^n \). We consider the orthogonal semi-direct product \( \mathbb{R}^f = \mathbb{R}^n \perp \mathbb{R}^{f-n} \). Let \( h_{n+1}, \ldots, h_f \) be a basis for \( \mathbb{R}^{f-n} \) and write \( m = m' + m'' \), where
in Theorem 1.2 are Jacobi forms. In particular, since
is a quasi-Jacobi form of weight
is a quasi-Jacobi form of weight
satisfies (21), it must be that \( \Psi \) is a Jacobi form of weight
satisfies (12).

However, we find
by our arguments above and the fact \( B(m'', m'') \geq 0 \). This proves (21) for all linearly independent \( h_1, \ldots, h_n \).

Note that it is the expansion (3) which allows us to claim the functions \( \Psi_h(Q, v, k, \tau, \bar{z}) \) in Theorem 1.2 are Jacobi forms. In particular, since \( E_2(\tau) \) has a Fourier expansion with positive powers of \( q \) and the Fourier expansion of \( \theta_h(Q, v, k - 2t, \tau, \bar{z}) \) satisfies (21), it must be that \( \Psi_h(Q, v, k, \tau, \bar{z}) \) satisfies (12).

It remains to prove (2). As mentioned before, \( \partial_z \) maps quasi-Jacobi forms of weight \( k \) to weight \( k + 1 \). In the case \( Q(v) = 0 \), (4), (5), and (6), along with Lemma 3.1 establish that \( \theta_h(Q, v, k, \tau, \bar{z}) \) is a quasi-Jacobi form of weight \( k + r \) and index \( G/2 \).

Assume now that \( Q(v) \neq 0 \). Recalling Lemma 3.1 (and its notation) we have
\[
\theta_h(Q, v, k, \tau, \bar{z}) = \left( \sum_{p_1 + \cdots + p_n + k_1 = k} \left( \prod_{i=1}^{k_1} \alpha_i \partial_{z_i} \right) \theta_h(Q, u, k_1, \tau, \bar{z}) \right) \tag{23}
\]
We will prove by induction on \( k_1 \) that \( \theta_h(Q, u, k_1, \tau, \bar{z}) \) is a quasi-Jacobi form of weight \( k_1 + r \) and index \( G/2 \). Clearly \( \theta_h(Q, u, 0, \tau, \bar{z}) \) is a Jacobi form of weight \( r \) and index \( G/2 \). Suppose that for each \( \ell < k_1 \), \( \theta_h(Q, u, \ell, \tau, \bar{z}) \) is a quasi-Jacobi form of weight \( \ell + r \) and index \( G/2 \). Since \( \rho(0, k_1) = 1 \) and \( B(u, h_j) = 0 \) for each \( j \), Theorem 1.2 gives
\[
\Psi_h(Q, u, k_1, \tau, \bar{z}) = \theta_h(Q, u, k_1, \tau, \bar{z}) + \sum_{t=0}^{k_1-1} \rho(t, k_1)(2Q(u)E_2(\tau))^t \theta_h(Q, u, k_1 - 2t, \tau, \bar{z})
\]
is a Jacobi form of weight \( k_1 + r \). Using our induction hypothesis we find \( \theta_h(Q, u, k_1, \tau, \bar{z}) \) is a quasi-Jacobi form of weight \( k_1 + r \) and index \( G/2 \). Combining this with (23) completes the proof of (2).

4 Proof of Theorem 1.3

Let \( A \) be a matrix of level \( N \) such that \( 2Q(x) = x^tAx \) and \( B(x, y) = x^tAy \). Take \( p \) so that \( Ap \equiv 0 \mod N \) and denote the greatest integer less than or equal to \( k/2 \) by
For $\ell^t = (\ell_1, \ldots, \ell_f) \in \mathbb{C}^f$ we set
\[
\theta(A, p, \ell, k, z) = \frac{1}{N^k} \sum_{m \in \mathbb{Z}^f} (\ell^t A m)^k e^{2\pi i (\tau Q(m)/N^2 + z_1 B(m, h_1)/N + \cdots + z_n B(m, h_n)/N)}.
\]
(24)

The following theorem is analogous to Theorem 3.4 in [2] but includes the complex variables $z$ and vectors $h_1, \ldots, h_n$. It is important for the proof of Theorem 1.3.

**Theorem 4.1** Let the notation be as before and assume $\ell^t A h_j = 0$ for all $1 \leq j \leq n$ (that is, $B(\ell, h_j) = 0$). If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $d > 0$, then
\[
e^{-\frac{2\pi i c G[\tilde{z}]}{c\tau + d}} (c\tau + d)^{-(r+k)} \theta(A, p, \ell, k, \gamma \tau, \gamma \tilde{z}) = \epsilon(d) \exp(2\pi i Q(p) ab/N^2) \sum_{j=0}^{[k/2]} \left( \frac{Q(\ell) c}{\pi i (c\tau + d)} \right)^j \gamma(j, k) \theta(A, bp, \ell, k - 2j, \tau, \tilde{z}).
\]
(25)

In particular, if we take $p = 0$, then
\[
e^{-\frac{2\pi i c G[\tilde{z}]}{c\tau + d}} (c\tau + d)^{-(r+k)} \theta(A, \ell, k, \gamma \tau, \gamma \tilde{z}) = \epsilon(d) \sum_{j=0}^{[k/2]} \left( \frac{Q(\ell) c}{\pi i (c\tau + d)} \right)^j \delta(j, k) \theta(A, \ell, k - 2j, \tau, \tilde{z}).
\]
(26)

**Proof** The ideas and techniques parallel those in [2], which in turn are based on [11]. We begin with a function of the form $\Theta_h(Q, x) := \sum_{m \in \mathbb{Z}^f} e^{-2Q(m+x)} e^{-2B(m+x, h)}$ and consider its Fourier coefficients. The analysis is similar to that of [2] and [11] with occasional changes and we omit the details. One key difference, however, is that we must replace $h$ with $z_1 h_1 + \cdots + z_n h_n$ during the proof. The reader may also consult [6] for a detailed proof of this theorem.

\[\square\]

Define the functions
\[
\Theta_h^{\text{even}}(Q, v, \tau, \tilde{z}) := \sum_{n \geq 0} \frac{2^n \theta_h(Q, v, 2n, \tau, \tilde{z})}{(2n)!} (2\pi i X)^{2n},
\]
and
\[
\Theta_h^{\text{odd}}(Q, v, \tau, \tilde{z}) := \sum_{n \geq 0} \frac{2^{(n+1)/2} \theta_h(Q, v, 2n + 1, \tau, \tilde{z})}{(2n + 1)!} (2\pi i X)^{2n+1}.
\]
Note that $\Theta_h(Q, v, \tau, z) = \Theta_h^{\text{even}}(Q, v, \tau, z) + \Theta_h^{\text{odd}}(Q, v, \tau, z)$ and $\theta_h(Q, v, k, \tau, z) = \theta(Q, 0, v, k, \tau, z)$. Using Theorem 4.1, we find for functions of the form (10) that

$$
\Theta_h \left( Q, v, \gamma \tau, \gamma \frac{X}{(ct + d)^2} \right) = \sum_{n \geq 0} \frac{2^{n/2} \theta_h(Q, v, n, \gamma \tau, \gamma \frac{z}{ct + d})}{n!} \left( \frac{2\pi i X}{(ct + d)^2} \right)^n 
$$

$$
= \epsilon(d) e^{\frac{2\pi iQ}{ct + d}} (ct + d)^r \sum_{n \geq 0} \sum_{j=0}^{[n/2]} \frac{2^{n/2}}{n!} \left( \frac{2Q(v)c}{2\pi i (ct + d)} \right)^j 
$$

$$
\cdot \delta(j, n)(ct + d)^n \Theta_h(Q, v, n - 2j, \tau, z) \left( \frac{2\pi i X}{(ct + d)^2} \right)^n 
$$

$$
= \epsilon(d) e^{\frac{2\pi iQ}{ct + d}} (ct + d)^r \sum_{n \geq 0} \sum_{j=0}^{[n/2]} \frac{2^{n/2}(n-2j)}{(n-2j)!} 
$$

$$
\cdot \Theta_h(Q, v, n - 2j, \tau, z) \left( \frac{2\pi i X}{ct + d} \right)^{n-2j} \left( \frac{2Q(v)c}{2\pi i (ct + d)^2} \right)^j 
$$

$$
= \epsilon(d) e^{\frac{2\pi iQ}{ct + d}} (ct + d)^r \left[ \sum_{n \geq 0} \sum_{j=0}^{[n/2]} \frac{2^{n/2}(n-2j)}{(n-2j)!} \right] 
$$

$$
\cdot \Theta_h(Q, v, 2n - 2j, \tau, z) \frac{1}{(ct + d)^n} (2\pi i X)^{2n-2j} \left( \frac{2Q(v)c}{ct + d} \right)^j \frac{X^j}{j!} 
$$

$$
+ \sum_{n \geq 0} \sum_{j=0}^{[n/2]} \frac{2^{n/2}(2n-2j+1)}{(2n-2j+1)!} \Theta_h(Q, v, 2n - 2j + 1, \tau, z) 
$$

$$
\cdot \frac{1}{(ct + d)^n} (2\pi i X)^{2n-2j+1} \left( \frac{2Q(v)c}{ct + d} \right)^j \frac{X^j}{j!} 
$$

$$
= \epsilon(d) e^{\frac{2\pi iQ}{ct + d}} (ct + d)^r \exp \left( \frac{2\pi i [2Q(v)] c X^2}{(ct + d)^3} \right) 
$$

$$
\cdot \left[ \Theta_h^{\text{even}}(Q, v, \tau, z) + \Theta_h^{\text{odd}}(Q, v, \tau, z) \right] 
$$

$$
= \epsilon(d) e^{\frac{2\pi iQ}{ct + d}} (ct + d)^r \exp \left( \frac{2\pi i [2Q(v)] c X^2}{(ct + d)^3} \right) \Theta_h \left( Q, v, \tau, z, \frac{X}{ct + d} \right), 
$$

11
where we used $\frac{2^{n/2}}{n!} \delta(j, n) = \frac{2^{(n-2j)/2}}{(n-2j)!} \frac{1}{j!}$. We have now proved Theorem 1.3 for matrices $\gamma_1 = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N)$ with $d > 0$. Let $\gamma_2 = -\gamma_1$. Note that $\gamma_2 \tau = \gamma_1 \tau$ and $\gamma_2 z = -\gamma_1 z$. We have

$$\theta_{\Delta}(Q, v, k, \gamma_2 \tau, \gamma_2 z) = \theta_{\Delta}(Q, v, k, \gamma_1 \tau, -\gamma_1 z) = (-1)^k \theta_{\Delta}(Q, v, k, \gamma_1 \tau, \gamma_1 z).$$

It follows that

$$\Theta_{\Delta} \left( Q, v, \gamma_2 \tau, \gamma_2 z, \frac{X}{(c\tau + d)^2} \right) = \Theta_{\Delta} \left( Q, v, \gamma_1 \tau, \gamma_1 z, \frac{-X}{(c\tau + d)^2} \right).$$

Using (11) which we have already established for $d > 0$, we find

$$\Theta_{\Delta} \left( Q, v, \gamma_2 \tau, \gamma_2 z, \frac{X}{(-c\tau - d)^2} \right) = \epsilon(-d) e^{\frac{2\pi i[Q(v)]cX^2}{(c\tau + d)^2}} \exp \left( \frac{2\pi i[Q(v)]cX^2}{(c\tau + d)^2} \right) \Theta_{\Delta} \left( Q, v, \tau, z, \frac{X}{-c\tau - d} \right).$$

However the same expression is obtained if we replace $c$ and $d$ on the right hand side of (11) with $-c$ and $-d$, respectively. It follows that (11) also holds for $\gamma_2$, and so for all $\gamma \in \Gamma_0(N)$. The proof of Theorem 1.3 is now complete. \hfill \square

References

[1] A. Andriaov and G. Malolekin, Behavior of theta series of degree $N$ under modular substitutions, in Math. USSR-Izv, 9(2) (1975) 227–241.

[2] C. Dong and G. Mason, Transformation laws for theta functions, in Proceedings on Moonshine and related topics (Montréal, QC, 1999), ed. J. McKay and A. Sebbar, CRM Proc. Lecture Notes, Vol. 30, (Amer. Math. Soc., 2001) 15–26.

[3] C. Dong, G. Mason, and K. Nagatomo, Quasi-modular forms and trace functions associated to free boson and lattice vertex operator algebras, in Internat. Math. Res. Notices, (8) (2001) 409–427.

[4] M. Eichler and D. Zagier, The theory of Jacobi forms, Vol. 55 of Progress in Mathematics, Birkhäuser Boston Inc., Boston, MA, (1985).

[5] E. Hecke, Analytische Arithmetik der positiven quadratischen Formen, in Danske Vid. Selsk. Math.-Fys. Medd., 17(12) (1940) 134.

[6] M. Krauel, Vertex operator algebras and Jacobi forms, (PhD Dissertation), in ProQuest Dissertations and Theses (1039264015). ISBN: 9781267533739, (2012).
[7] M. Krauel and G. Mason, Vertex operator algebras and weak Jacobi forms, in *Int. Journ. Math.*, 23(6) (2012).

[8] A. Ogg, *Modular forms and Dirichlet series*, W. A. Benjamin, Inc., New York-Amsterdam, (1969).

[9] O. Richter, On transformation laws for theta functions, in *Rocky Mountain J. Math.*, 34(4) (2004) 1473–1481.

[10] B. Schoeneberg, Das Verhalten von mehrfachen Thetareihen bei Modulsubstitutionen, in *Math. Ann.*, 116(1) (1936) 511–523.

[11] B. Schoeneberg, *Elliptic modular functions: an introduction*, Springer-Verlag, New York, (1974). Translated from the German by J. R. Smart and E. A. Schwandt, Die Grundlehren der mathematischen Wissenschaften, Band 203.

[12] C. Ziegler, Jacobi forms of higher degree, in *Abh. Math. Sem. Univ. Hamburg*, (59) (1989) 191–224.