Errors and ambiguity in transition from Fourier series to Fourier integrals

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Abstract
Transition from Fourier series to Fourier integrals is considered and error introduced by ordinary substitution of integration for summing is estimated. Ambiguity caused by transition from discrete function to continuous one is examined and conditions under which this ambiguity does not arise are suggested.

1 Introduction
Quite often it appears to be appropriate to solve some physical problems at first in a finite size box (see e.g. [1, 2, 3]) and to increase the volume infinitely only at the final stage. In particular, such approach provides a reasonable regularization. As a rule periodic border conditions are imposed, so Fourier transform in spatial variables becomes a discrete function (Fourier coefficients), which is assumed to convert into a continuous one, when borders are pulled apart and period $2\pi \tau$ is infinitely increased. Ordinary substitution integration for summing is acceptable for 'smooth enough' and 'fast enough' decreasing functions, but being applied to distributions it may introduce errors which we estimate in this paper. Moreover, a transition from a discrete function to a continuous one may introduce ambiguity. We estimate possible ambiguity and consider conditions under which such ambiguity doesn’t arise. It should be noted that even if the original function is smooth, after transition $\tau \to \infty$ it may convert into a distribution.

In this paper we confine ourselves to the case of tempered distributions $F(x) \in S'$, which are defined as functionals $(F(x), \phi(x))$ on fast decreasing test functions $\phi(x) \in S$ [4, 5, 6]. Recall that $F(x)$ is a tempered distribution, if and only if it is a finite order derivative of some continuous tempered function.

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\( G(x) \), i.e. \( F(x) = G^{(n)}(x), n < \infty \). Function \( G(x) \) is called tempered, if there exists some constant \( \sigma \) such that

\[ |G(x)| < |x|^{\sigma}; \quad x \to \pm \infty. \]  

(1)

Fourier transform of any tempered distribution is a tempered distribution as well [4, 5, 6].

The standard procedure of transition from Fourier series to Fourier integral is based on infinite increasing of \( \tau \) (see e.g. [7, 8]). Usually one starts from locally summable function \( f(x) \) and constructs a truncated function \( \tilde{F}(x) \) as

\[ \tilde{F}(x) \equiv \begin{cases} \tilde{\phi}(x) & \text{if } -\pi \tau < x < \pi \tau \\ 0 & \text{if } |x| > \pi \tau \end{cases} \]  

(2)

Then \( \tilde{F}(x) \) is extended periodically (with the period \( 2\pi \tau \)) on the whole real axis \( x \), i.e. \( \tilde{F}(x) \to \tilde{F}(x/\tau) \) where\(^1\)

\[ \tilde{F}(\varphi) = \tilde{F}(\varphi + 2\pi) \]  

(3)

and

\[ \tilde{F}(x/\tau) = \tilde{F}(x) = \tilde{F}(x); \]  

(4)

for \(-\pi \tau < x < \pi \tau\).

This procedure may be easily generalized for a case when \( f(x), \tilde{F}(x) \), and \( \tilde{F}(x/\tau) \) are distributions. It is clear, that since generally distributions are not defined pointwise, the periodicity condition (3) should be interpreted as:

\[ \langle \tilde{F}\phi_\varphi \rangle = \langle \tilde{F}\phi_{\varphi+2\pi} \rangle \]  

(5)

where \( \phi_\varphi \) is arbitrary test function \( \phi_\varphi \in S \) with support located in the interval \((\varphi - \epsilon, \varphi + \epsilon)\) with arbitrary small, but finite \( \epsilon \).

It is evident that the periodic distribution \( \tilde{F}(x/\tau) \) may be obtained as the result of the cyclization procedure [10] defined as

\[ \tilde{F}(x/\tau) = \Sigma \tilde{F}(x) = \sum_{m=-\infty}^{\infty} \tilde{F}(x + 2\pi m \tau) \]  

(6)

Periodic distribution may be represented as the Fourier series

\[ \tilde{F}(x/\tau) = \sum_{n=-\infty}^{\infty} F_n \exp \{inx/\tau\} \]  

(7)

where Fourier coefficients \( F_n \) are given by the standard expression

\[ F_n = \frac{1}{2\pi \tau} \int_{-\pi \tau}^{\pi \tau} \tilde{F}(x/\tau) \exp \{-inx/\tau\} \, dx \]  

(8)

\(^1\)Below the tilde always marks periodic functions with the period \( 2\pi \).
Fourier coefficients can be also computed as

\[ F_n = \frac{1}{2\pi \tau} \int_{-\pi \tau}^{\pi \tau} \bar{F}(x) \exp\{-inx/\tau\} \, dx, \]  

(9)

Although correspondence between \( F_n \) and \( \tilde{F}(x/\tau) \) is biunique, there are no biunique correspondence between \( F_n \) and \( F(x) \) in a general case (see Appendix I).

Lastly, we wish to stress that all mentioned error and ambiguity problems are of no particular interest, if the considered transition is used only to define Fourier integral, as it is e.g. in [7, 8]. When such definition is done, even for some restricted class of functions, further extension of Fourier transform on a wide class of functions and even on distributions needs no reference to the original Fourier series. Bijection is needed only for the function and its transform, given by the Fourier integral. However, there are cases when the transition from Fourier series to Fourier integral is of particular interest, e.g. when \( \tau \to \infty \) corresponds to the lifting of regularization. Hence the degree of ambiguity in such transition deserves a more detailed consideration.

2 Transition from Fourier series to Fourier integral

After [8] we split the region of summation \(-\infty < n < \infty\) into 'bursts', i.e. intervals \( \tau k - \tau \delta k/2 < n < \tau k + \tau \delta k/2 \) and approximately\(^2\) replace the sum by its average value in each interval

\[ \frac{1}{\delta k} \sum_{n=\tau k-\tau \delta k/2}^{\tau k+\tau \delta k/2} \exp\{inx/\tau\} F_n \simeq \tau \left[ \exp\{inx/\tau\} F_n \right]_{n=\tau k} = \exp\{ikx\} \tau F_{k\tau} \]  

(10)

Some details of partition into 'bursts' may be clarified, if we note that after formal change \( n \to k \tau \) we get from [8]

\[ \frac{1}{2\pi} \int_{-\pi \tau}^{\pi \tau} \tilde{F}(x/\tau) \exp\{-ikx\} \, dx = f_k \tau \delta k \]  

(11)

with definition

\[ f_k \equiv \tau F_{k\tau} \]  

(12)

It is assumed in [8], that when proceeding to limit \( \tau \to \infty \), the expression \[^{11}\] was converted into the inverse Fourier transform

\[ \frac{1}{2\pi} \lim_{\tau \to \infty} \int_{-\pi \tau}^{\pi \tau} \tilde{F}(x/\tau) \exp\{-ikx\} \, dx = f_k \]  

(13)

\[^2\] Although the term 'approximately' isn’t specified in [8], one may interpret it so that the approximate equality is assumed to become an exact one with the vanishing \( \delta k \) and \( 1/\tau \).
From (11) we conclude, that it may happen only if for large enough \( \tau \), we impose the condition

\[
\tau \delta k = 1 \quad (14)
\]

it means that 'bursts' are defined as \( \tau k - 1/2 < n < \tau k + 1/2 \), so that each 'burst' contains only the sole term \( \exp \{ikx\} F_{k\tau} \), where \( k\tau \) is still an integer number.

Relation (12) is nothing but a new definition so it gives nothing new concerning the conditions under which right-hand member in (10) may be treated as an integrand of the Fourier integral in variable \( k \).

For lack of something better we suggest to claim the fulfillment of approximate relation\(^3\)

\[
F_n \exp \{in\varphi\} \simeq \int_{n+\epsilon-1}^{n+\epsilon} F_t \exp \{it\varphi\} dt; \quad 0 < \epsilon < 1 \quad (15)
\]

which directly provides the desired result

\[
\sum_{n=-\infty}^{\infty} F_n \exp \{in\varphi\} \simeq \int_{-\infty}^{\infty} F_t \exp \{it\varphi\} dt = \int_{-\infty}^{\infty} \tau F_{k\tau} \exp \{ikx\} dk \quad (16)
\]

or

\[
\tilde{F}(x/\tau) \simeq \int_{-\infty}^{\infty} f_k \exp \{ikx\} dk \quad (17)
\]

If we define

\[
F(x) \equiv \int_{-\infty}^{\infty} f_k \exp \{ixk\} dk \quad (18)
\]

then (17) may be rewritten as

\[
\tilde{F}(x/\tau) \simeq F(x) \quad (19)
\]

With the definition (12) the approximate relation (15) acquires the form

\[
f_k \exp \{ikx\} \simeq \frac{1}{\delta k} \int_{k-(1-\epsilon)\delta k}^{k+\epsilon\delta k} f_t \exp \{ixt\} dt \quad (20)
\]

Relation (20) resembles the result of the mean value theorem, however \( f_t \) should not obligatory obey the restrictive conditions of such theorem because (20) must be satisfied only for infinitesimal \( \delta k \) and only approximately.

Making allowance for (18), the inverse Fourier transform may be written with the same accuracy as

\[
f_k = \frac{1}{2\pi} \int_{-\pi\tau}^{\pi\tau} F(x) \exp \{-ikx\} dx \quad (21)
\]

In other words, ordinary transition may be done by the formal substitution

\[
n \to k\tau; \quad \sum_{n=-\infty}^{\infty} \to \tau \int_{-\infty}^{\infty} dk; \quad F_n \to F_{k\tau} \equiv \frac{1}{\tau} f_k. \quad (22)
\]

\(^3\)The error of such approximation will be estimated later.
In this paper we try to clarify under what conditions substitution \(2 \rightarrow 2\) leads to
\[
\lim_{\tau \to \infty} \tilde{F} \left( \frac{x}{\tau} \right) = F(x)
\] (23)
which we treat as an exact result.

It is clear that nontrivial result appears after proceeding to limit \(\tau \to \infty\) in (12) and (23), only if \(F_t\) and \(\tilde{F}(x/\tau)\) have specific properties. Some of them are discussed in Appendix II.

3 Ambiguity in choosing of extension

In the distribution theory Fourier series exist even for such distributions that obey neither Riemann-Lebesgue lemma nor the Dirichlet condition. As it is proved in \([8]\), \(F_n\) exist and are called Fourier coefficients, if \(\tilde{F}(\varphi)\) is locally summable (i.e. at any finite interval). Distribution theory allows to sum series in which \(F_n\) is a tempered function of discrete argument \(n\). Therefore, an appropriate tempered distribution may be taken, as an extension \(F_n\) for noninteger \(n\). For instance, if Fourier coefficients are given by
\[
F_n = \begin{cases} 
  n^\lambda; & \text{if } n > 0 \\
  0; & \text{if } n \leq 0 
\end{cases} 
\]
\(\lambda = \Re \lambda = \text{const}\) (24)
straightforward extension \(n \to t\) gives the distribution
\[
F_t = \begin{cases} 
  t^\lambda; & \text{if } t > 0 \\
  0; & \text{if } t \leq 0 
\end{cases} \equiv t^\lambda_+. 
\] (25)
If extension \(F_t^{[1]}\) is allowable, then \(F_t^{[2]}\) is allowable too under the condition that \(F_t^{[1]} - F_t^{[2]} = \omega_t\) turns into zero for all integer \(t = n\), i.e. \(\omega_t \in \Omega_{[10]}\). However, when a class of the sought distribution is specified, the choice of extension may be essentially restricted. If, for example, it is required that the target distribution must have continuous derivative up to \(m\)-th order in some specified area (e.g. in the vicinity \(t = 0\)), then straightforward extension (25) is unsuitable if \(m > \lambda\).

There exists an extension of \(F_n\) with exceptional analytical properties, namely the entire function
\[
F_t = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{F}(x/\tau) \exp\{-itx/\tau\} \, dx 
\] (26)
which may be obtained from expression (8). When according to problem situation the extension (26) is prescribed, then even if \(F_t\) breaks the conditions of Carlson theorem (see e.g. \([9]\)), it may be shown that such extension (10) is unique.

On the other hand, if \(\tilde{F}(x/\tau)\) is known, \(F(x)\) may be simply computed with (23). In many interesting and more common situations technical difficulties prevent summarizing series in (7).
Formally, an extension (26) may be reconstructed from any other, without direct computation of series in (7). Indeed, the substitution of (7) in (26) gives

$$F_t = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\{-it\varphi\} \sum_{n=-\infty}^{\infty} F_n \exp\{in\varphi\} d\varphi = \sum_{n=-\infty}^{\infty} \frac{F_n}{\pi} \frac{\sin \pi (n-t)}{\pi (n-t)}$$  \(27\)

where \(F_t\) is an entire function. Unfortunately, the formal expression (27) adds nothing to computation of \(f_k\) or \(F(x)\), since (22) with (11) 5(58) gives

$$f_k = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} F_n \frac{\sin \pi (k-n)}{k-n} = \sum_{n=-\infty}^{\infty} F_n \delta \left( k - \frac{n}{\tau} \right)$$  \(28\)

and Fourier transform of (28) brings us back to (7).

If we take an exact summation of series in (27) we, undoubtedly, obtain a practicable expression. Unfortunately, as a rule the sum computation in (27) is as difficult as the summation of series in (7), so we consider the simple example

$$F_n = \frac{i \sinh \nu \pi}{\pi} \cos \frac{\pi n}{n+i\nu}$$  \(29\)

where \(\nu\) is positive noninteger number. From (27) we find that the corresponding entire function is

$$F_t = \frac{\sin \pi (t+i\nu)}{\pi (t+i\nu)}$$  \(30\)

As it should be, the difference of \(F_t\) obtained from (29) by the straightforward extension \(n \rightarrow t\) and \(F_t\) from (30)

$$\omega^{(\nu)}_t = \frac{\cosh \nu \pi \sin t\pi}{\pi} \in \Omega$$  \(31\)

For sure, Fourier series either with the coefficients given in (29) or those given in (30) converge to the same periodic function \(\exp \{\nu \varphi\}\), which is equal \(\exp \{\nu \varphi\}\) in the interval \(-\pi < \varphi < \pi\).

4 Error and ambiguity introduced by formal transition

Although procedure proposed in (7, 8) is entirely appropriate to define the Fourier integral, being applied to transition from Fourier series to Fourier integral, for a wider class of functions it introduces appreciable difference between \(F(x)\) obtained after formal transition (22) and an exact one: \(F(x) = \lim_{\tau \rightarrow \infty} \tilde{F}(x/\tau)\). To estimate the error we apply the Poisson formula

$$\sum_{n=-\infty}^{\infty} \Phi_n \exp\{in\varphi\} = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\varphi-2\pi m)t} \Phi_t dt$$  \(32\)

The Abel-Poisson regularization is implied, so the change of the order of summation and integration is legal.
to the right-hand member in (7). It gives
\[ \tilde{F}(x/\tau) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} F_n \exp \left\{ i \frac{m}{\tau} - 2\pi imn \right\} dn \] (33)
that with (12) may be rewritten as
\[ \tilde{F}(x/\tau) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f_k \exp \{ ik (x + 2\pi n\tau) \} dk \] (34)
and, taking into account (18), we finally get
\[ \tilde{F}(x/\tau) = \sum_{n=-\infty}^{\infty} F(x - 2\pi n\tau) \] (35)
Since both (12) and (18) are simply definitions and include no approximations, (35) is an exact expression. Comparing it with (19) we see that \( \tilde{F}(x/\tau) \) differs from \( F(x) \) in (18) by
\[ \Delta(x) = \sum_{n \neq 0} F(x - 2\pi n\tau) \] (36)
With \( \tau \to \infty \) correction \( \Delta(x) \) must disappear for any physical value \( F(x) \). It is not true, however, for auxiliary values, e.g. for gauge fields.

For instance, from
\[ F_t = \omega_t = (1 - \exp \{ 2it \}) R_t \in \Omega \] (37)
which turns into zero\(^5\) for all integer \( t \), we obtain after the formal substitution (22)
\[ f_k = (1 - \exp \{ i2\pi k\tau \}) r_k \] (38)
where
\[ r_k = \tau R_k \tau \] (39)
so Fourier transform of (38) gives
\[ F(x) = r(x) + r(x + 2\pi k\tau) \] (40)
hence if
\[ r(x) \equiv \int_{-\infty}^{\infty} r_k e^{ikx} dk \] (41)
decreases with \( x \to \infty \), then for \( \tau \to \infty \) and any finite \( x \) we get \( F(x) = r(x) \), despite the fact that corresponding Fourier series is identically zero.

Nonetheless, in a case when extension (26) is prescribed, no ambiguity arises. Indeed, if we apply the formal substitution to extension (26), we get for \( \tau \to \infty \)
\[ f_k = \tau F_k \tau = \frac{1}{2\pi} \int_{-\pi \tau}^{\pi \tau} \tilde{F}(x/\tau) e^{-i k x} dx \] (42)
\(^5\)The distribution \( R_t \) is assumed not too singular at integer \( t \) to spoil that condition.
On the other hand, from the definition (13) we obtain

\[ f_k = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{\pi\tau+2\pi n\tau}^{\pi\tau+2\pi n\tau} F(x) e^{-ikx} \, dx \]  

(43)

that with (42) evidently leads to the exact result (35).

5 Asymptotic series expansion

As it can be seen from (34), one may estimate the correction \( \Delta (x) \) by computing the limit of \( \exp \{ i k x / \tau \} f_k \) for \( \tau \to \infty \). In [12, 13] the method of asymptotic series expansion of \( \exp \{ i k \tau \} e^{ikx}f_k \) for \( \tau \to \pm \infty \) was suggested. In particular, it is shown that \( e^{ikx}f_k \) vanish in such limit, if \( f_k \) is analytical in the vicinity of \( k = 0 \). In our case it means the vanishing of \( \Delta (x) \). An exhaustive investigation of the power type distributions, i.e. distributions of a form \( f (x) \sim (x \pm i 0)^{\lambda} \ln^m (x \pm i 0) \) has been undertaken in [13, 11]. In [14] this method was extended over a family of distributions and the case when the regularization term in \( f_k \) depends on \( \tau \) was considered. It was shown that

\[ e^{ikx}f_k = e^{-ikx} \left( f_k^{(+)} - f_k^{(-)} \right) = \sum_{n=0}^{\infty} C_n (\tau, \nu) \delta^{(n)} (k) \]  

(44)

where

\[ f_k^{(+)} = \int_{0}^{\infty} F(t) \exp \{ itk \} \, dt; \quad f_k^{(-)} = - \int_{-\infty}^{0} F(t) \exp \{ itk \} \, dt \]  

(45)

and

\[ C_n (\tau, \nu) = \frac{i^{-n}}{2\pi} \sum_{m=0}^{\infty} \frac{F^{(n-m)} (\tau) (-\nu / |\tau|)^m}{(n-m)! m!} \]  

(46)

where \( F^{(n)} (t) \) is n-th derivative of the Fourier transform of \( f_k \).

In particular, for \( \lambda = -1 \) we get

\[ e^{-ikx} \left( k + \frac{i \nu}{\tau} \right)^{-1} = \mp i e^{\pm i \nu} \sum_{n=0}^{\infty} \nu^n |\tau|^{-n} i^{-n} \delta^{(n)} (k) \quad \text{if} \quad \tau \to \pm \infty \]  

(47)

and

\[ e^{ikx} \left( k + \frac{i \nu}{\tau} \right)^{-1} = 0 \quad \text{if} \quad \tau \to \pm \infty \]  

(48)

so returning to the example considered above, we see that after formal substitution [22] from [24] we get for \( \tau \to \infty \)

\[ F_n = \frac{i \sinh \nu \pi \cos n \pi \nu}{\pi n + i \nu} \to \tau F_k = f_k^{(c)} = \sinh (\nu \pi) e^{-\pi \nu} \delta (k) + O (1/\tau) \]  

(49)
and from \(31\) we obtain
\[
\omega_n^{(\nu)} = \frac{\cosh \nu \pi \sin n \pi}{n + i \nu} \rightarrow \tau \omega_n^{(\nu)} = f_k^{(s)} = \cosh (\nu \pi) e^{-\pi \nu} \delta (k) + O (1/\tau)
\] (50)
so
\[
f_k = f_k^{(c)} + f_k^{(s)} = \delta (k)
\] (51)
and, consequently,
\[
F (x) = 1
\] (52)
that, indeed, coincides with the exact result
\[
F (x) = \lim_{\tau \to \infty} \tilde{F} (x/\tau) = \lim_{\tau \to \infty} \exp \{i \nu \tilde{x}/\tau\} = 1.
\] (53)

6 Conclusions

The transition from Fourier series to Fourier integrals is considered and errors introduced by the formal substitution \(22\) are estimated. Also the ambiguity caused by transition from discrete function to continuous one is considered. Conditions under which such ambiguity disappears are suggested.

7 Appendix I.

As it is shown in \[10\], the distribution
\[
\Delta F (x) = \rho \left( \frac{x}{\tau} - \pi \right) - \rho \left( \frac{x}{\tau} + \pi \right) \in \Omega'
\] (54)
have only trivial periodical extension
\[
\Delta \tilde{F} (x/\tau) \triangleq \hat{\Sigma} (\Delta F (x)) = 0,
\] (55)
so both \(F (x)\) and \(F_{\rho} (x) \equiv F (x) + \Delta F (x)\) after the cyclization procedure \(6\) give the same \(\tilde{F} (x/\tau)\). It also means that inverse procedure \(F (x) \triangleq \hat{\Sigma}^{-1} \tilde{F} (x/\tau)\) is ambiguous, i.e. \(F (x)\) cannot be reconstructed uniquely for a given \(\tilde{F} (x/\tau)\). Since \(F_{\rho} (x)\) should vanish outside \(-\pi \tau \leq x \leq \pi \tau\), distributions \(\rho \left( \frac{x}{\tau} \pm \pi \right)\) may differ from zero only at the points \(x = \mp \pi \tau\), i.e. \(\rho \left( \frac{x}{\tau} \pm \pi \right)\) have point support. As it is known (see e.g. \[11 \, 13\]), distribution with point support is finite linear combinations of Dirac \(\delta\)-functions and its derivatives, so one may choose
\[
\Delta F (x) = \sum_{m=0}^{M} c_m \left[ \delta^{(m)} \left( \frac{x}{\tau} - \pi \right) - \delta^{(m)} \left( \frac{x}{\tau} + \pi \right) \right],
\] (56)
with arbitrary finite constants \(c_m\) and \(M\).

As it can be seen from \[56\] no contribution may come from \(\Delta F (x)\) to Fourier coefficients, in other words
\[
\Delta F_t \equiv \frac{1}{2 \pi \tau} \int_{-\pi \tau}^{\pi \tau} \Delta F (x) e^{-ix/\tau} dx
\] (57)
must turn into zero for all integer $t = n$. To guarantee this, we must remove some uncertainty in integration $\delta^{(m)}(\frac{x}{\varepsilon} \pm \pi)$ on $[-\pi \varepsilon, \pi \varepsilon]$. Recall that Dirac $\delta$-function is defined as the functional $\int_{-\infty}^{\infty} \phi(y) \delta(y) dy = \phi(0)$, for any test function $\phi(y)$ continuous in the vicinity $y = 0$. However, there exists no such $\phi(y)$ with support located in the interval $0 \leq y < \infty$, i.e.

$$<\phi, \delta> = \int_{-\infty}^{\infty} \phi(y) \delta(y) dy = \int_{0}^{\infty} \phi(y) \delta(y) dy = \phi(0)$$

with $\phi(0) \neq 0$. Certainly $<\phi, \delta> = 0$ does not look winning at all. Even if we put $\int_{-y}^{y} \delta(y') dy' = \theta(y)$, then

$$\Delta F_t = \lim_{\varepsilon \to 0} \left[ e^{i\pi t} \theta(\varepsilon) - e^{-i\pi t} \theta(-\varepsilon) \right] \frac{1}{2\pi} \sum_{m=0}^{M} (it)^m c_m$$

so we only shifted the trouble to $\theta(x)$, which is undefined at $x = 0$. To be more exact, value $\theta(0)$ may be chosen arbitrarily.

Indeed, let us consider some function $D(x)$ which possesses arbitrary values on a countable set of $x$ and is equal to zero on a complement of such set. We assume also that $D(x)$ does not include linear combinations of Dirac $\delta$-functions and its derivatives. In this case the functional $<D, \phi> = 0$ for any of the test functions $\phi$. The distribution theory expressly makes no distinction between distributions, which differ by $D(x)$, if $D(x)$ satisfies the above condition. In other words, on a countable set of points one may arbitrarily choose values of any distribution, if at these points such distribution can not be presented as Dirac $\delta$-functions and its derivatives.

In particular, one may choose so called 'symmetric' definition

$$\theta(x) = \lim_{\varepsilon \to 0} \left( \frac{\theta(x + \varepsilon) + \theta(x - \varepsilon)}{2} \right)$$

that gives $\theta(0) = 1/2$. The 'symmetric' definition is especially convenient, if distribution is represented as the Fourier series, because, according to Fejér theorem (see e.g. [15]), if Fourier series is convergent at $\varphi + 0$ and $\varphi - 0$, it converges to $\tilde{F}(\varphi) = \left( \tilde{F}(\varphi + 0) + \tilde{F}(\varphi - 0) \right)/2$. The 'symmetric' definition allows to get for $\Delta F_t$ an acceptable result.

$$\Delta F_t = \frac{i \sin(\pi t)}{\pi} \sum_{m=0}^{M} (it)^m c_m$$

Nevertheless, that or another choice is merely a matter of convenience, because this problem lies outside the scope of the distribution theory. For example, in [11]8.1(732) another definition is taken

$$\int_{0}^{\infty} e^{-px} \delta(x) dx = \int_{0}^{\infty} \delta(x) dx = 1$$

i.e. $\theta(0) = 1$. 

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8 Appendix II. Some remarks about analytical properties of $F_t$ and $\tilde{F}(x/\tau)$

As it already mentioned, after proceeding to limit $\tau \to \infty$ the nontrivial result may appear, only if $F_t$ and $\tilde{F}(x/\tau)$ have specific properties. Indeed, if $\tilde{F}(x/\tau)$ depends on $x$ and $\tau$ only through $x/\tau$, then $\lim_{\tau \to \infty} \tilde{F}(x/\tau) = \tilde{F}(0)$ for any $x$ where such limit exists, unless $x$ is the infinite point. In particular for any finite Fourier series, one may write

$$\tilde{F}(x/\tau) = \sum_{n=-N_-}^{N_+} F_n \exp \{i n x / \tau\} \to \sum_{n=-N_-}^{N_+} F_n = \tilde{F}(0) = \text{const} \quad (63)$$

for any fixed $N_\pm$.

In proceeding to limit $\tau \to \infty$ the positions of singularity of $\tau F_{k\tau}$ and $\tilde{F}(x/\tau)$ are essentially rearranged. For instance, if all singularities of $F_t$ are located in the finite area $|t - t_0| < R$, where $t_0$ and $R$ may be arbitrarily large, but finite, then for $\tau \to \infty$ all corresponding singularities of $f_k$ are moved to the infinitely small vicinity of $k = 0$. Indeed, in our case $F_t$ may be expanded in Laurent series

$$F_t = \sum_{m=1}^{\infty} \lambda_m (t - t_0)^m \quad (64)$$

for any $t$ which obeys $|t - t_0| > R$. Therefore, after the formal substitution we get

$$F_t \to f_k = \lim_{\tau \to \infty} \tau F_{k\tau} = \lim_{\tau \to \infty} \sum_{m=1}^{\infty} \frac{\tau^{1-m} \lambda_m}{(k - t_0/\tau)^m} = \frac{\lambda_1}{k - i \varepsilon \text{signum}(\text{Im} t_0)} \quad (65)$$

Less trivial and more realistic result may be obtained only when $\tau$-dependence of $\lambda_m$ is allowed. We exclude the case when $\lambda_m$ increase faster than $\tau^{m-1}$ with $\tau \to \infty$, because it makes $f_k$ infinite for all $k$. Therefore, either $\lambda_m \to c_m \tau^{m-1}$ or corresponding terms vanish. Thereby, we may write

$$f_k = \lim_{\tau \to \infty} \sum_{m=1}^{\infty} \frac{c_m}{(k - t_0/\tau)^m} = \lim_{\varepsilon \to +0} \sum_{m=1}^{\infty} \frac{c_m}{(k - i \varepsilon \text{signum}(\text{Im} t_0))^m} \quad (66)$$

So if series converges, $f_k$ became an analytic function in a whole complex plane $k$, except the point $k = 0$.

Let now $\tilde{F}(\varphi)$ be regular in the area $|\varphi| \leq c$, where $c$ is arbitrary small, but finite, then $\tilde{F}(x/\tau)$, and consequently $F(x)$, may have singularity only at $|x| > c \tau$, so after $\tau \to \infty$ distribution $\tilde{F}(x/\tau)$ converts into entire function. When $\tilde{F}(\varphi)$ depends not only on $\varphi$, but on $\tau$ as well, i.e. $\tilde{F} = \tilde{F}(\varphi, \tau)$, and if
\(\tilde{F}(\varphi, \tau)\) is analytical in a ring \(r_- < |\varphi| < r_+\), i.e.

\[
\tilde{F}(\varphi, \tau) = \sum_{n=-\infty}^{\infty} \tilde{F}_n(0, \tau) \varphi^{-n-1}
\]

with \(r_{\pm} = \lim_{n \to \pm \infty} \left| \tilde{F}_n(0, \tau) \right|^{-\frac{1}{n}}\), such \(\tilde{F}(\varphi, \tau)\) may have nontrivial behavior for infinitely increasing period under the conditions which may be easily found. Indeed, within such ring \(\tilde{F}(\varphi, \tau)\) may be presented as Laurent series and for a reason similar to the one mentioned above we obtain for \(\tau \to \infty\)

\[
\tilde{F}_n(0, \tau) \to \tau^n \Phi_n; \quad \Phi_n = \text{const}
\]

so the distribution

\[
\Phi(x) = \lim_{\tau \to \infty} \tilde{F}\left(\frac{x}{\tau}, \tau\right) = \sum_{n=-\infty}^{\infty} \Phi_n x^{-n-1}
\]

is analytic in a ring \(R_- < |x| < R_+\) with \(R_{\pm} = \lim_{n \to \pm \infty} |\Phi_n|^{-\frac{1}{n}}\).

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