A novel linearized and momentum-preserving Fourier pseudo-spectral scheme for the Rosenau-Korteweg de Vries equation

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Abstract

In this paper, we design a novel linearized and momentum-preserving Fourier pseudo-spectral scheme to solve the Rosenau-Korteweg de Vries equation. With the aid of a new semi-norm equivalence between the Fourier pseudo-spectral method and the finite difference method, a prior bound of the numerical solution in discrete $L^\infty$-norm is obtained from the discrete momentum conservation law. Subsequently, based on the energy method and the bound of the numerical solution, we show that, without any restriction on the mesh ratio, the scheme is convergent with order $O(N^{-s} + \tau^2)$ in discrete $L^\infty$-norm, where $N$ is the number of collocation points used in the spectral method and $\tau$ is the time step. Numerical results are addressed to confirm our theoretical analysis.

AMS subject classification: 65M12, 65M15, 65M70

Keywords: Rosenau-KdV equation, Fourier pseudo-spectral method, priori estimate, momentum-preserving scheme.

1 Introduction

To describe the dynamics of dense discrete systems, Rosenau \textsuperscript{24, 25} derived the so-called Rosenau equation, as follows:

$$u_t + uu_x + u_{xxx} + u_{xxxx} = 0,$$

where the existence and the uniqueness of the solution for \textsuperscript{21} were proved by Park \textsuperscript{22}. On the other hand, for the further consideration of the nonlinear wave, the viscous term $+u_{xxxx}$ needs to be included \textsuperscript{29}

$$u_t + uu_x + u_{xxx} + u_{xxxx} = 0.$$  \hspace{1cm} (1.2)

The resulting equation is usually called the Rosenau-KdV equation and the generalized case is \textsuperscript{8}

$$u_t + uu_x + u_{xxx} + u_{xxxx} + (u^p)_x = 0.$$  \hspace{1cm} (1.3)

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The Rosenau-KdV equation has been studied theoretically and numerically in the literature. For the existence and uniqueness of the solution of the Rosenau-KdV equation, please refer to Refs. [8, 29]. Known strategies to solve the Rosenau-KdV equation numerically include the finite difference method [1, 4, 15, 20], the Fourier pseudo-spectral method [5], etc. However, there are few works on the Rosenau-KdV equation in high dimensions. Thus, in this paper, we focus on developing a numerical method for the following generalized Rosenau-KdV (GR-KdV) equation in two dimensions (2D) [1]

\[ u_t + \Delta^2 u_t + \Delta u_x + (1 + u^p) \Delta u = 0, \quad (x, y) \in \Omega, \quad 0 < t \leq T, \tag{1.4} \]

with \((l_1, l_2)\)-periodic boundary conditions

\[ u(x, y, t) = u(x + l_1, y, t), \quad u(x, y, t) = u(x, y + l_2, t), \quad (x, y) \in \Omega, \quad 0 < t \leq T, \]

and initial condition

\[ u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega, \]

where \(\Delta = \partial_{xx} + \partial_{yy}, \quad \mathbb{L} = \partial_x + \partial_y, \quad p \geq 1\) is a given integer, \(\Omega = [x_L, x_R] \times [y_L, y_R] \subset \mathbb{R}^2\), \(l_1 = x_R - x_L, \quad l_2 = y_R - y_L\), and \(u_0(x, y)\) is a given real-valued function. Under the periodic boundary conditions, the system (1.4) has the following momentum conservation law

\[ \mathcal{P}(t) = \int_{\Omega} \left( u^2 + |\Delta u|^2 \right) dx \, dy = \mathcal{P}(0). \tag{1.5} \]

In recent years, there has been growing interest in geometric methods or structure-preserving methods, which can preserve as much as possible the intrinsic properties of the given dynamical system. It has been shown that, compared with traditional numerical methods, structure-preserving methods have excellent stability and superior performance in long time simulations. For more details, please refer to Refs. [3, 9, 13] and references therein. With the aid of the variational formulation [19], Cai et al. first derived some multi-symplectic schemes for the Rosenau-type equation [5]. More recently, based on the multi-symplectic Hamiltonian formula [3], a new multi-symplectic scheme has been proposed for the Rosenau-type equation with the power law nonlinearity in Ref. [4]. Besides the multi-symplectic structure, the Rosenau-KdV equation also admits some invariants, such as the momentum conservation law (1.5). In many significant cases, the ability to preserve some invariant properties of the original differential equation is a criterion to judge the success of a numerical simulation. Thus, when discretizing such a conservative system in space and time, it is a natural idea to design numerical schemes that preserve rigorously a discrete invariant. In Ref. [15], the authors proposed a three-level linear finite difference scheme, which can preserve the conservation law (1.5), for the Rosenau-KdV equation in 1D. In Ref. [1], Atouani and Omrani constructed two conservative schemes for the GR-KdV equation in 2D. However, most of the existing conservative schemes have only second order accuracy in space. To construct high order schemes in space, the Fourier spectral method [26] was employed to discrete the Rosenau-KdV and two high order structure-preserving schemes were constructed in Ref. [4]. However, the resulting schemes are fully implicit, which implies that one has to solve a system of nonlinear equations, at each time step. Thus, the first purpose of this paper is to develop a novel linearized and momentum-preserving Fourier pseudo-spectral scheme to solve the GR-KdV equation in 2D. In addition, to the best of our knowledge, there has been no reference considering an error estimate of the Fourier spectral scheme of the GR-KdV equation. Thus, another purpose of this paper is to establish an a priori estimate for the proposed Fourier pseudo-spectral scheme in discrete \(L^\infty\)-norm.
The outline of this paper is organized as follows. In Section 2, we first establish a new semi-norm equivalence between the Fourier pseudo-spectral method and the finite difference method. Then, with the help of the Fourier pseudo-spectral approximation for spatial derivatives, a semi-discrete scheme of the GR-KdV equation in 2D is presented. Finally, we show the semi-discrete scheme can preserve the semi-discrete momentum conservation law exactly. In Section 3, a fully discrete scheme is further obtained from the semi-discrete scheme by using a linear Crank-Nicolson scheme in time. We show that the resulting scheme can preserve the discrete momentum conservation law and that it is uniquely solvable. An a priori estimate is established for the proposed scheme in discrete $L^\infty$-norm in Section 4. Some numerical experiments are presented in Section 5. We draw some conclusions in Section 6.

2 Structure-preserving spatial discretization

Let $\Omega_h = \{(x_{j_1}, y_{j_2}) | x_{j_1} = j_1 h_1, y_{j_2} = j_2 h_2; 0 \leq j_r \leq N_r, r = 1, 2\}$ be a partition of $\Omega$ with mesh sizes $h_1 = \frac{b_1}{N_1}$ and $h_2 = \frac{b_2}{N_2}$, respectively, where $N_1$ and $N_2$ are two even numbers. Denote

$$J_h = \{(j_1, j_2) | 0 \leq j_r \leq N_r, r = 1, 2\}, \quad J'_h = \{(j_1, j_2) | 0 \leq j_r \leq N_r - 1, r = 1, 2\}.$$

A discrete mesh function $U_{j_1,j_2}$, $(j_1,j_2) \in \mathbb{Z} \times \mathbb{Z}$ is said to satisfy periodic boundary conditions if and only if

$$x - \text{periodic} : U_{j_1,j_2} = U_{j_1 + N_1,j_2} \quad \text{and} \quad y - \text{periodic} : U_{j_1,j_2} = U_{j_1,j_2+N_2}.$$

Let

$$\mathbb{V}_h = \{U | U = (U_{0,0}, U_{1,0}, \ldots, U_{N_1-1,0}, U_{0,1}, U_{1,1}, \ldots, U_{N_1-1,1}, \ldots, U_{0,N_2-1}, U_{1,N_2-1}, \ldots, U_{N_1-1,N_2-1})^T\}$$

be the space of mesh functions defined on $\Omega_h$ and satisfy the periodic boundary conditions \cite{21}. Subsequently, the discrete difference operators, and norms will be defined in an appropriate way. We introduce some discrete difference operators for any mesh function $U \in \mathbb{V}_h$, as follows:

$$\delta_x^+ U_{j_1,j_2} = \frac{U_{j_1+1,j_2} - U_{j_1,j_2}}{h_1}, \quad \delta_y^+ U_{j_1,j_2} = \frac{U_{j_1,j_2+1} - U_{j_1,j_2}}{h_2}, \quad \delta_x^- U_{j_1,j_2} = \frac{U_{j_1,j_2} - U_{j_1-1,j_2}}{h_1},$$

$$\delta_y^- U_{j_1,j_2} = \frac{U_{j_1,j_2} - U_{j_1,j_2-1}}{h_2}, \quad \nabla_h U_{j_1,j_2} = \left(\delta_x^+ U_{j_1,j_2}, \delta_y^+ U_{j_1,j_2}\right)^T,$$

$$\Delta_h U_{j_1,j_2} = (\delta_x^+ \delta_x^- + \delta_y^+ \delta_y^-) U_{j_1,j_2}.$$

We define discrete inner product as follows:

$$\langle U, V \rangle_h = h_1 h_2 \sum_{j_1=0}^{N_1-1} \sum_{j_2=0}^{N_2-1} U_{j_1,j_2} V_{j_1,j_2}, \quad \forall \ U, V \in \mathbb{V}_h.$$

The discrete $L^2$-norm of $U \in \mathbb{V}_h$ and its difference quotients are defined, respectively, as

$$||U||_h = \sqrt{\langle U, U \rangle_h}, \quad ||\delta_x^+ U||_h = \sqrt{\langle \delta_x^+ U, \delta_x^+ U \rangle_h}, \quad ||\delta_y^+ U||_h = \sqrt{\langle \delta_y^+ U, \delta_y^+ U \rangle_h},$$

$$||\nabla_h U||_h = \sqrt{\langle \delta_x^+ U, \delta_x^+ U \rangle_h + \langle \delta_y^+ U, \delta_y^+ U \rangle_h}, \quad ||\Delta_h U||_h = \sqrt{\langle \Delta_h U, \Delta_h U \rangle_h}.$$
In fact, it is easy to show that
\[
\|\nabla_h U\|_h = \sqrt{\langle - (I_{N_2} \otimes B_1 + B_2 \otimes I_{N_1}) U, U \rangle_h} := \sqrt{\langle - \Delta_h U, U \rangle_h},
\]
\[
\|\Delta_h U\|_h = \sqrt{\langle (I_{N_2} \otimes B_1^2 + 2B_2 \otimes B_1 + B_2^2 \otimes I_{N_1}) U, U \rangle_h} := \sqrt{\langle \Delta_h^2 U, U \rangle_h},
\]
where $\otimes$ means the Kronecker product, $I_{N_r}$ is an $N_r \times N_r$ identity matrix, and
\[
B_r = \frac{1}{h_r^r} \begin{pmatrix}
-2 & 1 & 0 & \cdots & 1 \\
1 & -2 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -2 & 1 \\
1 & 0 & \cdots & 1 & -2
\end{pmatrix}_{N_r \times N_r}, \quad r = 1, 2.
\]
Here, $B_r, r = 1, 2$ is the usual finite difference discretization of the second derivative, by taking into account of the periodic boundary conditions. We also define discrete $H^2_h$ and $L^\infty$-norms as
\[
\|U\|_{H^2_h} = \sqrt{\|U\|^2_h + \|\nabla_h U\|^2_h + \|\Delta_h U\|^2_h}, \quad \|U\|_{L^\infty} = \max_{(j_1, j_2) \in J_h} |U_{j_1, j_2}|.
\]
We note that the discrete norms $\|\delta_x^+ U\|_h$, $\|\delta_y^+ U\|_h$, $\|\nabla_h U\|_h$ and $\|\Delta_h U\|_h$ defined above are semi-norms. In addition, we denote `·' as the element product of vectors $U, V \in \mathbb{V}_h$, that is,
\[
U \cdot V = (U_{0,0}V_{0,0}, \ldots, U_{N_1-1,0}V_{N_1-1,0}, \ldots, U_{0,N_2-1}V_{0,N_2-1}, \ldots, U_{N_1-1,N_2-1}V_{N_1-1,N_2-1})^T.
\]
For brevity, we denote $\underbrace{U \cdot \ldots \cdot U}_p$ as $U^p$.

### 2.1 Fourier pseudo-spectral method and some lemmas

Let
\[
S'_N = \text{span}\{g_{j_1}(x)g_{j_2}(y), \ 0 \leq j_r \leq N_r - 1, r = 1, 2\},
\]
be the interpolation space, where $g_{j_1}(x)$ and $g_{j_2}(y)$ are trigonometric polynomials of degree $N_1/2$ and $N_2/2$, given, respectively, by
\[
g_{j_1}(x) = \frac{1}{N_1} \sum_{l=-N_1/2}^{N_1/2} \frac{1}{a_l} e^{i j_1 \pi x} e^{i j_1 \pi x}, \quad g_{j_2}(y) = \frac{1}{N_2} \sum_{q=-N_2/2}^{N_2/2} \frac{1}{b_q} e^{i j_2 \pi y} e^{i j_2 \pi y},
\]
with $a_l = \begin{cases} 1, & |l| < \frac{N_1}{2} \\ \frac{N_1}{2}, & |l| = \frac{N_1}{2} \end{cases}$, $b_q = \begin{cases} 1, & |q| < \frac{N_2}{2} \\ \frac{N_2}{2}, & |q| = \frac{N_2}{2} \end{cases}$, and $0 \leq j_r \leq N_r - 1, \ r = 1, 2$.

We define the interpolation operator $I_N : C(\Omega) \to S'_N$ as follows \cite{7}:
\[
I_N U(x, y, t) = \sum_{j_1=0}^{N_1-1} \sum_{j_2=0}^{N_2-1} U_{j_1, j_2}(t) g_{j_1}(x) g_{j_2}(y),
\]
where $U_{j_1, j_2}(t) = U(x_{j_1}, y_{j_2}, t)$.
Taking the derivative with respect to $x$, and then evaluating the resulting expressions at the collocation points $(x_{j_1}, y_{j_2})$, we can obtain

$$\frac{\partial^s I_N U(x_{j_1}, y_{j_2}, t)}{\partial x^s} = \sum_{j=0}^{N_1-1} U_{j_1 j_2}(t) \frac{d^{s_1} g_j(x_{j_1})}{dx^{s_1}} = [(I_N \otimes D^x_{s_1})U]_{j_1 j_2}, \quad U(t) \in \mathbb{V}_h,$$

where $D^x_{s_1}$ is an $N_1 \times N_1$ matrix, with elements given by

$$(D^x_{s_1})_{j_1 j} = \frac{d^{s_1} g_j(x_{j_1})}{dx^{s_1}},$$

and $[(I_N \otimes D^x_{s_1})U]_{j_1 j_2}$ represents the $(N_1(j_2 - 1) + j_1)$-th component of the vector $(I_N \otimes D^x_{s_1})U$. For brevity, the notation is still be adopted in subsequent sections. In a similar way, we can obtain

$$\frac{\partial^s I_N U(x_{j_1}, y_{j_2}, t)}{\partial y^s} = \sum_{k=0}^{N_2-1} U_{j_1 k}(t) \frac{d^{s_2} g_k(y_{j_2})}{dy^{s_2}} = [(D^y_{s_2} \otimes I_N)U]_{j_1 j_2}, \quad U \in \mathbb{V}_h,$$

where $D^y_{s_2}$ is an $N_2 \times N_2$ matrix with elements given by

$$(D^y_{s_2})_{j_2 k} = \frac{d^{s_2} g_k(y_{j_2})}{dy^{s_2}}.$$

For first and second derivatives, we have, respectively,

$$\frac{\partial I_N U(x_{j_1}, y_{j_2}, t)}{\partial x} = [(I_N \otimes D^x_{1})U]_{j_1 j_2}, \quad \frac{\partial I_N U(x_{j_1}, y_{j_2}, t)}{\partial y} = [(D^y_{1} \otimes I_N)U]_{j_1 j_2},$$

and

$$\frac{\partial^2 I_N U(x_{j_1}, y_{j_2}, t)}{\partial x^2} = [(I_N \otimes D^x_{2})U]_{j_1 j_2}, \quad \frac{\partial^2 I_N U(x_{j_1}, y_{j_2}, t)}{\partial y^2} = [(D^y_{2} \otimes I_N)U]_{j_1 j_2},$$

where $D^x_{1}$ and $D^y_{1}$ are real skew-symmetric matrices, and $D^x_{2}$ and $D^y_{2}$ are real symmetric matrices, respectively. Now, for $U \in \mathbb{V}_h$, we introduce three new semi-norms induced by the spectral differential matrices, as follows:

$$|U|_{1,h} = \sqrt{\langle -(I_N \otimes (D^x_{1})^2) + (D^y_{1})^2 \otimes I_N \rangle U, U \rangle_h},$$

$$||L_h U||_h = \sqrt{\langle L_h U, L_h U \rangle} = \sqrt{\langle -(I_N \otimes (D^x_{1})^2) + 2D^y_{1} \otimes D^x_{1} + (D^y_{1})^2 \otimes I_N \rangle U, U \rangle_h},$$

and

$$|U|_{2,h} = \sqrt{\langle (I_N \otimes (D^x_{2})^2) + 2D^y_{2} \otimes D^x_{2} + (D^y_{2})^2 \otimes I_N \rangle U, U \rangle_h},$$

where $L_h = I_N \otimes D^x_{2} + D^y_{1} \otimes I_N$.

**Lemma 2.1.** For any mesh function $U \in \mathbb{V}_h$, we have

$$||L_h U||_h \leq \sqrt{2} |U|_{1,h}.$$

**Proof.** With the Cauchy-Schwarz inequality, it is clear to see

$$||L_h U||_h^2 = |U|_{1,h}^2 + 2\langle (I_N \otimes D^x_{1})U, (D^y_{1} \otimes I_N)U \rangle_h \leq |U|_{1,h}^2 + ||(I_N \otimes D^x_{1})U||_h^2 + ||(D^y_{1} \otimes I_N)U||_h^2.$$
which further implies
\[
\|L_h U\|_h \leq \sqrt{2} \|U\|_{1,h}.
\]
This completes the proof.

\[\square\]

Lemma 2.2. \cite{10, 11} For the matrices \(B_r, r = 1, 2\) and \(D^w_2, w = x, y\), the following results hold
\[
\begin{align*}
B_r &= F^H_{N_r} \Lambda_r F_{N_r}, \\
D^x_1 &= F^H_{N_1} \Lambda_3 F_{N_1}, \\
D^y_1 &= F^H_{N_2} \Lambda_4 F_{N_2}, \\
D^x_2 &= F^H_{N_1} \Lambda_5 F_{N_1}, \\
D^y_2 &= F^H_{N_2} \Lambda_6 F_{N_2},
\end{align*}
\]
where \(F_{N_r}, r = 1, 2\), is the discrete Fourier transform matrix with elements \((F_{N_r})_{j,k} = \frac{1}{\sqrt{N_r}} e^{-ijk 2\pi/N_r}\), \(F^H_{N_r}\) is the conjugate transpose matrix of \(F_{N_r}\) and
\[
\begin{align*}
\Lambda_r &= \text{diag} \left[ \lambda_{B_{r,0}}, \lambda_{B_{r,1}}, \ldots, \lambda_{B_{r,N_r-1}} \right], \\
\lambda_{B_{r,j}} &= -\frac{4}{h_r^2} \sin^2 \frac{j\pi}{N_r}, \\
\Lambda_3 &= \text{diag} \left[ \lambda_{D^x_1,0}, \lambda_{D^x_1,1}, \ldots, \lambda_{D^x_1,N_1-1} \right], \\
\lambda_{D^x_1,j} &= \begin{cases} 
ij\mu_1, & 0 \leq j \leq N_1/2 - 1, \\
0, & j = N_1/2, \\
i(j - N_1)\mu_1, & N_1/2 < j < N_1,
\end{cases} \\
\lambda_{D^x_2,j} &= \begin{cases} 
ij\mu_2, & 0 \leq j \leq N_2/2 - 1, \\
0, & j = N_2/2, \\
i(j - N_2)\mu_2, & N_2/2 < j < N_2,
\end{cases} \\
\Lambda_5 &= \text{diag} \left[ \lambda_{D^y_1,0}, \lambda_{D^y_1,1}, \ldots, \lambda_{D^y_1,N_1-1} \right], \\
\lambda_{D^y_1,j} &= \begin{cases} 
-(j\mu_1)^2, & 0 \leq j \leq N_1/2, \\
-(j(N_1)\mu_1)^2, & N_1/2 < j < N_1,
\end{cases} \\
\lambda_{D^y_2,j} &= \begin{cases} 
-(j\mu_2)^2, & 0 \leq j \leq N_2/2, \\
-(j(N_2)\mu_2)^2, & N_2/2 < j < N_2.
\end{cases}
\end{align*}
\]
In addition, the following inequalities hold \cite{11}
\[
\begin{align*}
0 &\leq -\frac{4}{\pi^2} \lambda_{(D^x_1)^2,j} \leq -\frac{4}{\pi^2} \lambda_{D^x_1,j} \leq -\lambda_{B_1,j} \leq -\lambda_{D^x_1,j}, & 0 \leq j \leq N_1 - 1, \\
0 &\leq -\frac{4}{\pi^2} \lambda_{(D^y_1)^2,j} \leq -\frac{4}{\pi^2} \lambda_{D^y_1,j} \leq -\lambda_{B_2,j} \leq -\lambda_{D^y_1,j}, & 0 \leq j \leq N_2 - 1, \\
0 &\leq \frac{16}{\pi^4} \lambda_{D^x_1,j} \leq \lambda_{B_1,j} \leq \lambda_{D^x_1,j}, & 0 \leq j \leq N_1 - 1, \\
0 &\leq \frac{16}{\pi^4} \lambda_{D^y_1,j} \leq \lambda_{B_2,j} \leq \lambda_{D^y_1,j}, & 0 \leq j \leq N_2 - 1.
\end{align*}
\]

Lemma 2.3. For any mesh function \(U \in \mathbb{V}_h\), we have
\[
\|\Delta_h U\|_h \leq \|U\|_{2,h} \leq \frac{\pi^2}{4} \|\Delta_h U\|_h.
\]

Proof. We denote
\[
I^2 := \|U\|_{2,h}^2
\]
which implies that

\[
\frac{16}{\pi^4} I^2 \leq J^2 \leq I^2, \quad r = 1, 2, 3,
\]

which implies that

\[
J_1 \leq I < J_2 < \frac{\pi^4}{16} J, \quad r = 1, 2, 3.
\]

With (2.6), we can get

\[
J^2 \leq I^2 \leq \frac{\pi^4}{16} J^2,
\]

that is,

\[
\|\Delta_h U\|_h \leq \|U\|_{2,h} \leq \frac{\pi^4}{4}\|\Delta_h U\|_h.
\]

We complete the proof. \(\square\)
Lemma 2.4. For any mesh function $U \in \mathbb{V}_h$, we have

$$|U|_{1,h} \leq \frac{\pi}{2} \|\nabla_h U\|_h.$$ 

The proof is similar to the Lemma 2.3. For brevity, we omit it.

Lemma 2.5. For any mesh function $U \in \mathbb{V}_h$, we have

$$\|\nabla_h U\|_h^2 \leq \|U\|_h \|\Delta_h U\|_h,$$
$$\|U\|_{h,\infty}^2 \leq C\|U\|_h (\|\Delta_h U\|_h + \|U\|_h).$$

2.2 Momentum-preserving spatial semi-discretization

Eq. (1.4) can be rewritten as the following equivalent form

$$u_t + \Delta^2 u_t + \Delta u_x + Lu + \frac{1}{p+2} (u^p L + Lu^p) u = 0,$$

where the operator $(u^p L + Lu^p)$ operates to a function $u$ in such a way that $(u^p L + Lu^p) u = u^p L u + L(u^p u)$. The same convection applies to the discrete version.

Applying the Fourier pseudo-spectral method to the system (2.7) in space, we have

$$(I + A^2) \frac{d}{dt} U + D(U) U = 0, \quad U \in \mathbb{V}_h,$$

with

$$D(U) = B + \mathbb{L}_h + \frac{1}{p+2} (\text{diag}(U^p) \mathbb{L}_h + \mathbb{L}_h \text{diag}(U^p)).$$

where $A = I_{N_2} \otimes D_x^2 + D_y^2 \otimes I_{N_1}$ and $B = I_{N_2} \otimes D_x^2 + D_y^2 \otimes D_y^2$. Note that we have used the equality $D_w^w = (D_w^2)^2$, $w = x, y$ in the above equation. For more details, please refer to Ref. [10]. In addition, with noting the anti-symmetric property of $B$ and $\mathbb{L}_h$, we can prove that the matrix $D(U)$ is anti-symmetric for $U$.

Lemma 2.6. The semi-discrete system (2.8) possesses the following semi-discrete momentum conservation law

$$P(t) = P(0), \quad P(t) = \|U\|_{h}^2 + \|U\|_{2,h}^2, \quad U \in \mathbb{V}_h.$$ 

Proof. Making the discrete inner product of (2.8) with $U$, we then have

$$\frac{d}{dt} (\|U\|_h^2 + \|U\|_{2,h}^2) = 0,$$

which further shows

$$P(t) = P(0),$$

where the anti-symmetric property of $D(U)$ is used. This completes the proof. \qed
3 Construction of the linearized Crank-Nicolson momentum-preserving (LCN-MP) scheme

In this section, we will propose a LCN-MP scheme by using the linearized Crank-Nicolson method to the semi-discrete system (2.8).

For a positive integer \( M \), let
\[
\Omega_{\tau} = \{ t_n | t_n = n\tau; 0 \leq n \leq M \}
\]
be a uniform partition of \([0, T]\) with time step \( \tau = T/M \). Let \( \Omega_{h\tau} = \Omega_h \times \Omega_{\tau} \), and denote \( u^n_{j_1,j_2} = u(x_{j_1}, y_{j_2}, t_n) \), \((j_1, j_2) \in J'_h\) and \( U^n_{j_1,j_2} \) be its numerical solution. For any mesh function \( U^n \in V_h \) defined on \( \Omega_{h\tau} \), we define
\[
\delta^+_t U^n = \frac{U^{n+1} - U^n}{\tau}, \quad U^{n+\frac{1}{2}} = \frac{U^{n+1} + U^n}{2}, \quad U^{n+\frac{3}{2}} = \frac{3U^n - U^{n-1}}{2}.
\]

Applying the linear Crank-Nicolson method to the semi-discrete system (2.8) in time, then we can obtain
\[
(I + A^2)\delta^+_t U^n + D(U^{n+\frac{1}{2}})U^{n+\frac{3}{2}} = 0, \quad U^n \in V_h, \quad n = 1, \ldots, M, \quad (3.1)
\]
where \( U^1 \in V_h \) is the solution of the following equation
\[
(I + A^2)\delta^+_t U^0 + D(U^0)U^{\frac{3}{2}} = 0, \quad U^0 \in V_h, \quad (3.2)
\]
which comprises our linearized Crank-Nicolson momentum-preserving (LCN-MP) scheme for the GR-KdV equation. In this paper, for simplicity, we denote \( C \) a positive constant which may be dependent on the regularity of exact solution and may be different in different case.

**Theorem 3.1.** The scheme (3.1)-(3.2) possesses the following discrete global momentum conservation law
\[
P^n = \cdots = P^0, \quad P^n = \| U^n \|_{h}^2 + \| \nabla_h U^n \|_{2,h}^2, \quad U^n \in V_h.
\]

**Proof.** We first show
\[
P^1 = P^0.
\]
By nothing
\[
\langle D(U^0)U^{\frac{3}{2}}, U^{\frac{3}{2}} \rangle_h = 0,
\]
we make the discrete inner product of (3.2) with \( U^{\frac{3}{2}} \) and obtain
\[
\delta^+_t P^0 = 0,
\]
that is,
\[
P^1 = P^0.
\]
Similarly, by taking the discrete inner product of (3.1) with \( U^{n+\frac{1}{2}} \), we have
\[
P^n = \cdots = P^1.
\]
This completes the proof.

**Lemma 3.1.** Supposing \( P^0 \leq C \), the solution \( U^n \in V_h \) of the scheme (3.1)-(3.2) satisfies
\[
\| U^n \|_h \leq C, \quad \| \nabla_h U^n \|_h \leq C, \quad \| \Delta_h U^n \|_h \leq C, \quad \| U^n \|_{h,\infty} \leq C, \quad 1 \leq n \leq M.
\]
This completes the proof.

**Proof.** According to $P^h \leq C$ and Theorem 3.1, we have

$$\|U^n\|_h^2 + |U^n|_{2,h}^2 \leq C, \ 1 \leq n \leq M,$$

which implies

$$\|U^n\|_h \leq C, \ |U^n|_{2,h} \leq C.$$  \hspace{1cm} (3.3)

With the aid of Lemmas 2.3 and 2.5, we can obtain

$$\|\nabla_h U^n\|_h \leq C, \ |\Delta_h U^n|_h \leq C, \ |U^n|_{h,\infty} \leq C.$$  \hspace{1cm} (3.4)

This completes the proof. \hfill \Box

**Theorem 3.2.** The LCN-MP scheme is uniquely solvable.

**Proof.** For a fixed $n$, the LCN-MP scheme can be rewritten as the following linear equation system

$$A U^{n+\frac{1}{2}} = b, \ U^n \in \mathbb{V}_h,$$  \hspace{1cm} (3.6)

where $A = \left(I + \mathbb{A}^2 + \frac{h}{2} \mathbb{D}(\hat{U}^{n+\frac{1}{2}})\right)$ and $b = \left(I + \mathbb{A}^2\right) U^n$. In order to obtain the unique solvability of the scheme, we need to prove that the matrix $A$ is invertible.

If $A x = 0, \ x \in \mathbb{V}_h$, we have

$$0 = x^T A x = x^T (I + \mathbb{A}^2) x,$$  \hspace{1cm} (3.7)

where the anti-symmetry of $\mathbb{D}(U)$ is used. Note that $I + \mathbb{A}^2$ is symmetric positive definite, thus, $x = 0$, that is, $A x = 0$ has only zero solution. Therefore, $A$ is invertible. This completes the proof. \hfill \Box

**Lemma 3.2.** Let

$$\|U^n\|_h \leq C, \ |\nabla_h U^n|_h \leq C, \ |\Delta_h U^n|_h \leq C, \ |U^n|_{h,\infty} \leq C, \ |V^n|_{h,\infty} \leq C, \ U, \ V \in \mathbb{V}_h,$$  \hspace{1cm} (3.8)

we have

$$\left\langle \mathbb{D}(U^n) U^{\frac{n+1}{2}} - \mathbb{D}(V^n) V^{\frac{n+1}{2}}, \eta^{\frac{n+1}{2}} \right\rangle_h \leq C \left(|\eta^n|^2_h + |\Delta_h \eta^n|^2_h + |\eta^{n+1}|^2_h + |\Delta_h \eta^{n+1}|^2_h\right),$$

and

$$\left\langle \mathbb{D}(U^{n+\frac{1}{2}}) U^{n+\frac{1}{2}} - \mathbb{D}(V^{n+\frac{1}{2}}) V^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}} \right\rangle_h$$

$$\leq C \left(|\eta^{n-1}|^2_h + |\eta^n|^2_h + |\eta^{n+1}|^2_h + |\Delta_h \eta^n|^2_h + |\Delta_h \eta^{n+1}|^2_h\right),$$

where $\eta = U - V \in \mathbb{V}_h$ and $n = 1, \cdots, M$.

**Proof.** Denoting

$$F(x_1, x_2, x_3) = \frac{1}{p+2} \left(\frac{3x_2 - x_1}{2}\right)^p \cdot L_h \left(\frac{x_3 + x_2}{2}\right) + \frac{1}{p+2} L_h \left(\frac{3x_2 - x_1}{2}\right)^p \cdot \frac{x_3 + x_2}{2}$$

$$:= F_1(x_1, x_2, x_3) + F_2(x_1, x_2, x_3), \ x_i \in \mathbb{V}_h, \ i = 1, 2, 3,$$

we then have

$$f := F_1(U^{n-1}, U^n, U^{n+1}) - F_1(V^{n-1}, V^n, V^{n+1})$$
\[
\begin{align*}
    &= \frac{1}{p+2} \left( U^{n+\frac{1}{2}} \right)^{p} \cdot \mathbb{L}_{h} U^{n+\frac{1}{2}} - \frac{1}{p+2} \left( V^{n+\frac{1}{2}} \right)^{p} \cdot \mathbb{L}_{h} V^{n+\frac{1}{2}} \\
    &= \frac{1}{p+2} \left( \left( U^{n+\frac{1}{2}} \right)^{p} - \left( V^{n+\frac{1}{2}} \right)^{p} \right) \cdot \mathbb{L}_{h} U^{n+\frac{1}{2}} + \frac{1}{p+2} \left( \left( V^{n+\frac{1}{2}} \right)^{p} \cdot \eta^{n+\frac{1}{2}} \right) \\
    &= \frac{1}{p+2} \left[ \eta^{n+\frac{1}{2}} \cdot \sum_{l=0}^{p-1} \left( \left( U^{n+\frac{1}{2}} \right)^{p-l-1} \cdot \left( V^{n+\frac{1}{2}} \right)^{l} \right) \right] \cdot \mathbb{L}_{h} U^{n+\frac{1}{2}} \\
    &\quad + \frac{1}{p+2} \left( \left( V^{n+\frac{1}{2}} \right)^{p} \cdot \eta^{n+\frac{1}{2}} \right), \\
\end{align*}
\]

and
\[
\begin{align*}
    g := F_{2}(U^{n-1}, U^{n}, U^{n+1}) - F_{2}(V^{n-1}, V^{n}, V^{n+1}) \\
    &= \frac{1}{p+2} \left( \left( U^{n+\frac{1}{2}} \right)^{p} \cdot U^{n+\frac{1}{2}} \right) - \frac{1}{p+2} \left( \left( V^{n+\frac{1}{2}} \right)^{p} \cdot U^{n+\frac{1}{2}} \right) \\
    &= \frac{1}{p+2} \left[ \left( \left( U^{n+\frac{1}{2}} \right)^{p} - \left( V^{n+\frac{1}{2}} \right)^{p} \right) \cdot U^{n+\frac{1}{2}} \right] + \frac{1}{p+2} \left( \left( \left( V^{n+\frac{1}{2}} \right)^{p} \cdot \eta^{n+\frac{1}{2}} \right) \right) \\
    &= \frac{1}{p+2} \left[ \eta^{n+\frac{1}{2}} \cdot \sum_{l=0}^{p-1} \left( \left( U^{n+\frac{1}{2}} \right)^{p-l-1} \cdot \left( V^{n+\frac{1}{2}} \right)^{l} \right) \cdot U^{n+\frac{1}{2}} \right] \\
    &\quad + \frac{1}{p+2} \left( \left( V^{n+\frac{1}{2}} \right)^{p} \cdot \eta^{n+\frac{1}{2}} \right). \\
\end{align*}
\]

With noting
\[
\left\langle (\mathbb{B} + \mathbb{L}_{h}) \eta^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}} \right\rangle_{h} = 0,
\]
we then have
\[
\left\langle \mathbb{D}(\overline{U}^{n+\frac{1}{2}}) U^{n+\frac{1}{2}} - \mathbb{D}(\overline{V}^{n+\frac{1}{2}}) V^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}} \right\rangle_{h} = \left\langle f, \eta^{n+\frac{1}{2}} \right\rangle_{h} + \left\langle g, \eta^{n+\frac{1}{2}} \right\rangle_{h}. \tag{3.12}
\]

By using Lemmas 2.1 and 2.3 and Eqs. 3.3–3.9, we can deduce from (3.10)–(3.11) that
\[
\left\langle f, \eta^{n+\frac{1}{2}} \right\rangle_{h} = \frac{h_{1} h_{2}}{p+2} \sum_{j_{1}=0}^{N_{1}-1} \sum_{j_{2}=0}^{N_{2}-1} \left\{ \left( \eta^{n+\frac{1}{2}} \right)^{p} \cdot \mathbb{L}_{h_{1}} U^{n+\frac{1}{2}} \right\}_{j_{1}, j_{2}} \left( \left( \left( U^{n+\frac{1}{2}} \right)^{p-l-1} \cdot \left( V^{n+\frac{1}{2}} \right)^{l} \right) \right)_{j_{1}, j_{2}} \}
\]
\[
\leq C(\|\eta^{n+\frac{1}{2}}\|^{2}_{h} + \|\eta^{n+\frac{1}{2}}\|\cdot \|\eta^{n+\frac{1}{2}}\|_{\infty}) \|\mathbb{L}_{h} U^{n+\frac{1}{2}}\|^{2}_{h} + \|\mathbb{L}_{h} \eta^{n+\frac{1}{2}}\|^{2}_{h} + \|\eta^{n+\frac{1}{2}}\|^{2}_{h})
\]
\[
\leq C(\|\eta^{n+\frac{1}{2}}\|^{2}_{h} + \|\eta^{n+\frac{1}{2}}\|^{2}_{h} + \|\nabla_{h} \eta^{n+\frac{1}{2}}\|^{2}_{h} + \|\eta^{n+\frac{1}{2}}\|^{2}_{h})
\]
\[
\leq C(\|\eta^{n-1}\|^{2}_{h} + \|\eta^{n+1}\|^{2}_{h} + \|\Delta_{h} \eta^{n}\|^{2}_{h} + \|\Delta_{h} \eta^{n+1}\|^{2}_{h}), \tag{3.13}
\]
and
\[
\left\langle g, \eta^{n+\frac{1}{2}} \right\rangle_{h} = \frac{1}{p+2} \left( \left( \eta^{n+\frac{1}{2}} \right)^{p} \cdot \sum_{l=0}^{p-1} \left( \left( U^{n+\frac{1}{2}} \right)^{p-l-1} \cdot \left( V^{n+\frac{1}{2}} \right)^{l} \right) \cdot U^{n+\frac{1}{2}}, -\mathbb{L}_{h} \eta^{n+\frac{1}{2}} \right)_{h} \]
\[
\leq C(\|\nabla_{h} \eta^{n+\frac{1}{2}}\|^{2}_{h} + \|\eta^{n+\frac{1}{2}}\|^{2}_{h} + \|\eta^{n+\frac{1}{2}}\|_{\infty})
\]
\[
\leq C(\|\Delta_{h} \eta^{n+\frac{1}{2}}\|^{2}_{h} + \|\eta^{n+\frac{1}{2}}\|^{2}_{h} + \|\eta^{n+\frac{1}{2}}\|^{2}_{h})
\]
\[
\leq C(\|\eta^{n-1}\|^{2}_{h} + \|\eta^{n+1}\|^{2}_{h} + \|\Delta_{h} \eta^{n}\|^{2}_{h} + \|\Delta_{h} \eta^{n+1}\|^{2}_{h}). \tag{3.14}
\]

Substituting (3.13) and (3.14) into (3.12), we have
\[
\left\langle \mathbb{D}(\overline{U}^{n+\frac{1}{2}}) U^{n+\frac{1}{2}} - \mathbb{D}(\overline{V}^{n+\frac{1}{2}}) V^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}} \right\rangle_{h}
\]
\[ \leq C(||\eta^{n-1}\|_h^2 + ||\eta^n\|_h^2 + ||\eta^{n+1}\|_h^2 + ||\Delta_h\eta^n\|_h^2 + ||\Delta_h\eta^{n+1}\|_h^2). \]

In a same way, we can obtain
\[ \left\langle D(U^0)U^{1/2} - D(V^0)V^{1/2}, \eta^1 \right\rangle_h \leq C(||\eta^0\|_h^2 + ||\Delta_h\eta^0\|_h^2 + ||\eta^1\|_h^2 + ||\Delta_h\eta^1\|_h^2). \]

\[ \square \]

\section{An a priori estimate}

In this section, we will establish an a priori estimate for the proposed scheme \([3.1]-[3.2]\) in discrete \(L^\infty\)-norm. For simplicity, we let \(\Omega = [0, 2\pi]^2\). More general cuboid domain can be translated into \(\Omega\). We assume that \(C_p^\infty(\Omega)\) be a set of infinitely differentiable functions with the period \(2\pi\) defined on \(\Omega\) for all variables. \(H_p^s(\Omega)\) is the closure of \(C_p^\infty(\Omega)\) in \(H^s(\Omega)\). The semi-norm and the norm of \(H_p^s(\Omega)\) are denoted by \(| \cdot |_r\) and \(\parallel \cdot \parallel_r\), respectively. \(\parallel \cdot \parallel_0\) is denoted by \(\parallel \cdot \parallel\) for simplicity.

Let \(N_1 = N_2 = N\), the interpolation space \(S_N^r\) can be rewritten as
\[ S_N^r = \left\{ u | u = \sum_{|j_1; j_2| \leq \frac{N}{2}} \tilde{u}_{j_1, j_2} e^{i(j_1 x + j_2 y)} : \tilde{u}_{j_1, j_2} = \tilde{u}_{-j_1, -j_2}, \tilde{u}_{j_1, \frac{N}{2}} = \tilde{u}_{j_1, -\frac{N}{2}} \right\}, \]
where \(c_1 = 1\), \(|l| < \frac{N}{2}\), \(c = \frac{N}{2} = 2\). The projection space is defined as
\[ S_N = \left\{ u | u = \sum_{|j_1; j_2| \leq \frac{N}{2}} \tilde{u}_{j_1, j_2} e^{i(j_1 x + j_2 y)} \right\}. \]

It is clear that \(S_{N-2} \subseteq S_N^r \subseteq S_N\). We denote by \(P_N : L^2(\Omega) \to S_N\) as the orthogonal projection operator and recall the interpolation operator \(I_N : C(\Omega) \to S_N^r\). Further, \(P_N\) and \(I_N\) satisfy \([11]\):

1. \(P_N \partial_u u = \partial_u P_N u, I_N \partial_u u \neq \partial_u I_N u, u = x, y.\)
2. \(P_N u = u, \forall u \in S_N, I_N u = u, \forall u \in S_N^r.\)

\textbf{Lemma 4.1.} \([11]\) For \(u \in S_N^r\), \(\parallel u \parallel \leq \parallel u \parallel_h \leq 2\parallel u \parallel\).

\textbf{Lemma 4.2.} \([6]\) If \(0 \leq l \leq s\) and \(u \in H_p^s(\Omega)\), then
\[ \parallel P_N u - u \parallel_l \leq C N^{l-s} |u|_s, \]
\[ \parallel P_N u \parallel_l \leq C |u|_l, \]
and in addition if \(s > 1\) then
\[ \parallel I_N u - u \parallel_l \leq C N^{l-s} |u|_s. \]

\textbf{Lemma 4.3.} \([11]\) For \(u \in H_p^s(\Omega), s > 1\), let \(u^* = P_{N-2} u, N > 2\). Then, we have
\[ \parallel u^* - u^\prime \parallel_h \leq C N^{-s} |u|_s, u^*, u \in \mathbb{V}_h. \]

\textbf{Lemma 4.4.} \([12]\) For any \(u \in S_q N\), we have
\[ \parallel I_N u \parallel_l \leq q \parallel u \parallel_l. \]

\[ 12 \]
Lemma 4.5. For $u \in H^{s+1}_p(\Omega)$, $s > 1$, let $u^* = P_{N-2}u$, $N > 2$. Then, we have
\[
\|\nabla_h(u^* - u)\|_h \leq C N^{-s}\|u^*\|_{s+1}, \quad u^*, u \in \mathbb{V}_h.
\]
The proof is similar to Lemma 4.4 in Ref. [10]. For brevity, we omit it.

Lemma 4.6. For $u \in H^{s+2}_p(\Omega)$, $s > 1$, let $u^* = P_{N-2}u$, $N > 2$. Then, we have
\[
\|\Delta_h(u^* - u)\|_h \leq C N^{-s}\|u^*\|_{s+2}, \quad u^*, u \in \mathbb{V}_h.
\]

**Proof.** According to Lemmas 2.3 and 4.1, we have
\[
\|\Delta_h(u^* - u)\|_h \leq \|u^* - u\|_{2,h} = \|A(u^* - u)\|_h \\
\leq 2\|I_N(\Delta(I_N(u^* - u)))\|,
\]
(4.1)
where we have used the fact $[A(u^* - u)]_{j_1,j_2} = [I_N(\Delta(I_N(u^* - u)))](x_{j_1}, y_{j_2})$. By noting $\Delta(I_N(u^* - u)) \in S_{2N}$, we can deduce from Lemmas 4.2 and 4.4 that
\[
\|I_N(\Delta(I_N(u^* - u)))\| \leq 2\|\Delta(I_N(u^* - u))\| \\
\leq 2\|\Delta(u^* - u)\| + \|\Delta(u - I_Nu)\| \\
\leq C(\|u^* - u\|_2 + \|u - I_Nu\|_2) \\
\leq C N^{-s}\|u\|_{s+2}.
\]
(4.2)
Substituting (4.2) into (4.1), we finish the proof. □

Lemma 4.7. Let $u^* = P_{N-2}u$ and
\[
\begin{align*}
\Delta^2 u^*_t + \Delta u^*_x + Lu^* + \frac{1}{p+2}((u^*)^p Lu^* + L(I_N(u^*)^{p+1})) = \\
u_t + \Delta^2 u_t + \Delta u_x + Lu + \frac{1}{p+2}(u^p Lu + L(u^{p+1})) + \xi_1.
\end{align*}
\]
If $u \in C^1(0,T;H^{s+4}(\Omega))$, $s > 1$, we have
\[
\|\xi_1\|_h \leq C N^{-s}, \quad \xi_1 \in \mathbb{V}_h.
\]

**Proof.** With Lemma 4.2, we can obtain the following approximation estimate
\[
\|\partial_t^k(u^* - u)\|_l = \|P_{N-2}(\partial_t^k u) - \partial_t^k u\|_l \leq C N^{-s}\|\partial_t^k u\|_{s+l}.
\]
(4.3)
Let
\[
\tilde{\xi} = ((u^*)^p Lu^* + L(I_N(u^*)^{p+1})) - (u^p Lu + L(u^{p+1}))
\]
(4.4)
\[
= [(u^*)^p - u^p]L(u^* - u) + [L(I_N(u^*)^{p+1} - (u^*)^{p+1}) + L((u^*)^{p+1} - u^{p+1})]
\]
\[
= \tilde{\xi}_1 + \tilde{\xi}_2.
\]
(4.5)
With Lemma 4.2, we can obtain
\[
\|\tilde{\xi}_1\| \leq \|(u^*)^p - u^p\|L u^*\| + \|u^p L(u^* - u)\|
\leq \|L u^*\|_{L^\infty}\|(u^*)^p - u^p\| + \|u\|_{L^\infty}\|L(u^* - u)\|
\leq \|L u^*\|_{L^\infty}\left(\left|\sum_{l=0}^{p-1}(u^*)^{p-l-1}u^l\right|_{L^\infty}\|u^* - u\| + \|u\|_{L^\infty}\|L(u^* - u)\|ight)
\]
\[
\leq \|L u^*\|_{L^\infty}\left(\left|\sum_{l=0}^{p-1}(u^*)^{p-l-1}u^l\right|_{L^\infty}\|u^* - u\| + \|u\|_{L^\infty}\left|\sum_{l=0}^{p-1}(u^*)^{p-l-1}u^l\right|_{L^\infty}\|u^* - u\|ight).
\]
(4.6)
Further, by noting \( \xi \)

With (4.3) and (4.8), we have

Thus, we can deduce from (4.6) and (4.7) that

where Lemmas 4.1 and 4.4 are used.

Proof. Let \( \xi_1 \in \mathcal{S}_{p+1} \), we then have

Thus, we can deduce from (4.6) and (4.7) that

With (4.3) and (4.8), we have

Further, by noting \( \xi_1 \in \mathcal{S}_{p+1} \), we can obtain

where Lemmas 4.1 and 4.4 are used. \( \square \)

Lemma 4.8. Let

and

where \( n = 1, 2, \cdots, M - 1 \). If \( u \in C^3\left(0, T; H^p_{p+1}(\Omega)\right), \ s > 1 \), we then have

Proof. We denote

and

By the Taylor expansion, we have

\[ \delta_t^+(u^*)^0 = \partial_t u^*(x, y, 0) + \tau c_1 \partial_t u^*(x, y, \zeta_1), \ \zeta_1 \in (0, \tau), \] (4.13)
\[(u^*)^{\frac{1}{2}} = u^*(x, y, 0) + \tau c_2 \partial_t u^*(x, y, c_2), \; \zeta_2 \in (0, \tau), \quad (4.14)\]

where \(c_1\) and \(c_2\) are constants. With noting \(u \in C^3 \left(0, T; H^{s+4}_p(\Omega) \right), \; s > 1\), we have

\[
\|\xi_2^n\| \leq C_T \left( \| (1 + \Delta^2) \partial_t u^*(x, y, \zeta_1) \| + \| (\Delta \partial_x + L) \partial_t u^*(x, y, \zeta_2) \| \right)
\]

\[
\leq C_T (\| \partial_t u^*(x, y, \zeta_1) \|_4 + \| \partial_t u^*(x, y, \zeta_2) \|_3)
\]

\[
\leq C_T (\| \partial_t u(x, y, \zeta_1) \|_4 + \| \partial_t u(x, y, \zeta_2) \|_3)
\]

\[
\leq C_T,
\]

and

\[
\|\xi_3^n\| \leq C_T \left( \| (u^*(x, y, 0))^p \cdot L \partial_t u^*(x, y, \zeta_2) \| + \| I_N \left( (u^*(x, y, 0))^p \cdot \partial_t u^*(x, y, \zeta_2) \right) \| \right)
\]

\[
\leq C_T \left( \| (u^*(x, y, 0))^p \cdot L \partial_t u^*(x, y, \zeta_2) \| \right)
\]

\[
\leq C_T \left( \| (u^*(x, y, 0))^p \|_{L_\infty} \cdot \| I_N (u^*(x, y, \zeta_2))^p \|_{L_\infty} \right)
\]

\[
\leq C_T.
\]

(4.15)

Denoting

\[
(1 + \Delta^2) \delta_t^n (u^*)^n + \Delta \partial_x (u^*)^{n+\frac{1}{2}} + L (u^*)^{n+\frac{1}{2}}
\]

\[
= (1 + \Delta^2) \partial_t u^*(x, y, t_{n+\frac{1}{2}}) + (\Delta \partial_x + L) u^*(x, y, t_{n+\frac{1}{2}}) + \xi_2^{n+\frac{1}{2}},
\]

(4.16)

and

\[
\frac{1}{p+2} \left( (u^*)^{n+\frac{1}{2}} \right)^p \| L \partial_t u^*(x, y, \zeta_2) \| + I_N \left( (u^*)^{n+\frac{1}{2}} \right)^p \| u^*(x, y, t_{n+\frac{1}{2}}) \|
\]

\[
= \frac{1}{p+2} \left( (u^*(x, y, t_{n+\frac{1}{2}}))^p \cdot L u^*(x, y, t_{n+\frac{1}{2}}) \right)
\]

\[
+ I_N \left( (u^*(x, y, t_{n+\frac{1}{2}}))^p \cdot u^*(x, y, t_{n+\frac{1}{2}}) \right) + \xi_3^{n+\frac{1}{2}}.
\]

(4.17)

By the Taylor expansion, we have

\[
\delta_t^n (u^*)^n = \partial_t u^*(x, y, t_{n+\frac{1}{2}}) + c_3 \tau^2 \partial_{tt} u^*(x, y, \zeta_3), \; \zeta_3 \in (t_n, t_n + \tau),
\]

(4.18)

\[
(u^*)^{n+\frac{1}{2}} = u^*(x, y, t_{n+\frac{1}{2}}) + c_4 \tau^2 \partial_t u^*(x, y, \zeta_4), \; \zeta_4 \in (t_n, t_n + \tau),
\]

(4.19)

where \(c_3\) and \(c_4\) are constants. An argument similar to (4.16) and (4.17) used in (4.15) shows that

\[
\|\xi_2^{n+\frac{1}{2}}\| \leq C_T^2,
\]

(4.20)

and

\[
\|\xi_3^{n+\frac{1}{2}}\| \leq C_T^2.
\]

(4.21)

Noting \(\xi_2^n, \xi_2^{n+\frac{1}{2}} \in S_{2N}\) and \(\xi_3^n, \xi_3^{n+\frac{1}{2}} \in S_{(p+1)N}\), then by using Lemma 4.3, we can prove

\[
\|\xi_2^n\|_h + \|\xi_2^n\|_h \leq 2(\| I_N \xi_2^n \| + \| I_N \xi_3^n \|) \leq C(\|\xi_2^n\| + \|\xi_3^n\|) \leq C_T,
\]

(4.22)
This completes the proof. By Lemmas 2.3 and 4.3-4.6, we can deduce that
\[ w \]
and
\[ \|\xi_2^{n+\frac{1}{2}}\|_h + \|\xi_3^{n+\frac{1}{2}}\|_h \leq C(\|\xi_2^{n+\frac{1}{2}}\|_h + \|\xi_3^{n+\frac{1}{2}}\|_h) \]
\[ \leq C\tau^2. \quad (4.23) \]

It is clear to see that
\[ \xi^0 = \xi_1(0) + \xi_2^0 + \xi_3^0, \quad \xi^{n+\frac{1}{2}} = \xi_1(t_{n+\frac{1}{2}}) + \xi_2^{n+\frac{1}{2}} + \xi_3^{n+\frac{1}{2}}. \]
Thus, with Lemma 4.7, we can deduce from (4.22) and (4.23) that
\[ \|\xi^0\|_h \leq \|\xi_1(0)\|_h + \|\xi_2^0\|_h + \|\xi_3^0\|_h \leq C(N^{-s} + \tau), \]
\[ \|\xi^{n+\frac{1}{2}}\|_h \leq \|\xi_1(t_{n+\frac{1}{2}})\|_h + \|\xi_2^{n+\frac{1}{2}}\|_h + \|\xi_3^{n+\frac{1}{2}}\|_h \leq C(N^{-s} + \tau^2). \]
This completes the proof. \qed

We define the error function by
\[ e_{j_1,j_2} = (u^*)_{j_1,j_2} - U^n_{j_1,j_2}, \quad (j_1, j_2) \in J'_h, \quad 1 \leq n \leq M. \]
Subtracting (3.11) and (3.2) from (4.12) and (4.11), respectively, we can get
\[ (I + A^2)\delta^i e^0 + \mathbb{D}((u^*)^0)(u^*)^\frac{1}{2} - \mathbb{D}(U^0)U^\frac{1}{2} = \xi^0, \quad (4.24) \]
and
\[ (I + A^2)\delta^i e^n + \mathbb{D}((u^*)^{n+\frac{1}{2}})(u^*)^{n+\frac{1}{2}} - \mathbb{D}(U^{n+\frac{1}{2}})U^{n+\frac{1}{2}} = \xi^{n+\frac{1}{2}}, \quad (4.25) \]
where \( e^n = (u^*)^n - U^n \in V_h, \quad n = 0, 1, \cdots, M. \)

**Theorem 4.1.** We assume \( u \in C^2\left(0, T; H^{s+4}_p(\Omega)\right), \quad s > 1. \) Then, there exists a constant \( \tau_0 > 0 \) sufficiently small, such that, when \( 0 < \tau \leq \tau_0 \), we have
\[ \|u^1 - U^1\|_h + \|\Delta_h(u^1 - U^1)\|_h \leq C(N^{-s} + \tau^2), \quad u^1, U^1 \in V_h. \]

**Proof.** Making the discrete inner product of (4.24) with \( e^\frac{1}{2} \), when \( 0 < \tau \leq \tau_0 \), we have
\[ F^1 - F^0 \leq C\tau(F^1 + F^0) + C(N^{-s} + \tau^2)^2, \quad (4.26) \]
with
\[ F^n = \|e^n\|_h^2 + |e^n|_{2,h}^2, \quad n = 0, 1, \]
where Lemmas 3.2 and 4.8 are used. By Lemmas 2.3 and 4.9, we can deduce that
\[ F^0 = \|e^0\|^2_h + |e^0|^2_{2,h} \leq \|e^0\|^2_h + \pi^4 \|\Delta_h e^0\|^2_h \]
\[ = \|((u^*)^0 - u^0)^2\|^2_h + \pi^4 \|\Delta_h((u^*)^0 - u^0)\|^2_h \]
\[ \leq CN^{-2s}. \quad (4.27) \]
With (4.27), when \( \tau \) is sufficiently small, such that, when \( 0 < \tau \leq \tau_0 \), we can get from (4.26) that
\[ \|e^1\|^2_h + |e^1|^2_{2,h} \leq C(N^{-s} + \tau^2)^2. \quad (4.28) \]
Noting Lemma 2.3, we get
\[ \|e^1\|_h^2 + \|\Delta_h e^1\|_h^2 \leq C(N^{-s} + \tau^2)^2, \] (4.29)
which implies that
\[ \|e^1\|_h + \|\Delta_h e^1\|_h \leq C(N^{-s} + \tau^2). \] (4.30)

By using Lemmas 4.3 and 4.6, and Eq. (4.30), we have
\[ \|u^1 - U^1\|_h + \|\Delta_h (u^1 - U^1)\|_h \leq (\|u^1 - (u^*)^1\|_h + \|\Delta_h (u^1 - (u^*)^1)\|_h + \|\Delta_h e^1\|_h) \leq C(N^{-s} + \tau^2). \] (4.31)

With Lemma 2.5, we can deduce from (4.31) that
\[ \|u^1 - U^1\|_{h, \infty} \leq C(N^{-s} + \tau^2). \]
This completes the proof. \(\square\)

**Theorem 4.2.** We assume \(u \in C^3\left(0, T; H^{s+1}_p(\Omega)\right), \ s > 1.\) Then, when \(\tau\) is sufficiently small, such that, \(C\tau \leq \frac{1}{2},\) we have
\[ \|u^n - U^n\|_h + \|\Delta_h (u^n - U^n)\|_h \leq C(N^{-s} + \tau^2), \|U^n\|_{h, \infty} \leq C(N^{-s} + \tau^2), \]
where \(u^n, U^n \in \mathcal{V}_h, \ n = 2, 3, \ldots, M.\)

**Proof.** Making the discrete inner product of (4.25) with \(e^{n+\frac{1}{2}},\) we then have
\[ F^{n+1} - F^n \leq C\tau \left(F^{n+1} + F^n\right) + C\tau \|e^{n-1}\|_h^2 + C\tau (N^{-s} + \tau^2)^2, \] (4.32)
with
\[ F^n = \|e^n\|_h^2 + \|e^n\|_{2,h}^2, \ 1 \leq n \leq M, \]
where Lemmas 3.2 and 4.8 are used. Summing up for \(n\) from 1 to \(m\) and then replacing \(m\) by \(n - 1,\) we can get from (4.32) that
\[ F^n \leq F^1 + C\tau \sum_{l=1}^n F^l + C\tau \|e^0\|_h^2 + CT(N^{-s} + \tau^2)^2 \]
\[ \leq C\tau \sum_{l=1}^n F^l + CT(N^{-s} + \tau^2)^2, \] (4.33)
where (4.28) and (4.30) are used. Applying the Gronwall inequality \(28\) to (4.33), then we have
\[ \|e^n\|_h^2 + \|e^n\|_{2,h}^2 \leq C(N^{-s} + \tau^2)^2. \] (4.34)
where \(\tau\) is sufficiently small, such that \(C\tau \leq \frac{1}{2}.\) With the aid of Lemma 2.3, we have
\[ \|e^n\|_h^2 + \|\Delta_h e^n\|_h^2 \leq C(N^{-s} + \tau^2)^2, \]
that is,
\[ \|e^n\|_h + \|\Delta_h e^n\|_h \leq C(N^{-s} + \tau^2). \] (4.35)
By using Lemmas 4.3 and 4.6 and Eq. (4.35), we have
\[
||u^n - U^n||_h + ||\Delta_h(u^n - U^n)||_h \\
\leq (||u^n - (u^*)^n||_h + ||\Delta_h(u^n - (u^*)^n)||_h + ||e^n||_h + ||\Delta_h e^n||_h) \\
\leq C(N^{-s} + \tau^2).
\]
(4.36)

With Lemma 2.5 we can deduce from (4.36) that
\[
||u^n - U^n||_{h,\infty} \leq C(N^{-s} + \tau^2).
\]

This completes the proof.

5 Numerical examples

In this section, we will investigate the numerical behavior of the LCN-MP scheme (3.1)-(3.2) for the GR-KdV equation in 1D and 2D. Also, the results are compared with some existing conservative finite difference schemes for the convergence rate and the discrete momentum conservation law. For the LCN-MP scheme (3.1)-(3.2), we use the following iteration method to solve the linear equation:

\[
(I + \kappa^2 + \frac{\tau}{2} B + \frac{\tau}{2} L_h)U^{n+\frac{1}{2},s+1} = (I + \kappa^2)U^n \\
- \frac{\tau}{2(p+2)} \left[ \text{diag}((U^{n+\frac{1}{2}})^p) L_h U^{n+\frac{1}{2},s} + L_h \left[ \text{diag}((U^{n+\frac{1}{2}})^p) \cdot U^{n+\frac{1}{2},s} \right] \right].
\]

We take the initial iteration vector \(U^{n+\frac{1}{2},0} = U^n\) and each iteration will terminate if the infinity norm of the error between two adjacent iterative steps is less than \(10^{-14}\).

Further, for a fixed iteration step \(s\), the fast solver presented in Ref. [17] is applied to solve the linear equations efficiently. For the convergence rate, we use the formula
\[
\text{Rate} = \frac{\ln(\text{error}_1/\text{error}_2)}{\ln(\tau_1/\tau_2)},
\]
where \(\tau_l, \text{error}_l, (l=1,2)\) are step sizes and errors with the step size \(\tau_l\) respectively.

5.1 Simulations of the R-KdV equation in 1D

In this section, we consider the following R-KdV equation in 1D
\[
u_t + u_{xxxx} + u_{xx} + u_x + uu_x = 0,
\]
with initial condition
\[
u(x,0) = \left( -\frac{35}{24} + \frac{35}{312} \sqrt{313} \right) \text{sech}^4 \left[ \frac{1}{24} \sqrt{-26 + 2\sqrt{313}} x \right].
\]

Eq. (5.1) possesses the following exact solution [29]
\[
u(x,t) = \left( -\frac{35}{24} + \frac{35}{312} \sqrt{313} \right) \text{sech}^4 \left[ \frac{1}{24} \sqrt{-26 + 2\sqrt{313}} \left( x - \frac{1}{2} + \frac{1}{26} \sqrt{313} t \right) \right].
\]

In our computation, we take the computational domain \(\Omega = [-50,50]\) and the periodic boundary condition. Table 1 shows numerical error and convergence rate of the proposed scheme with \(N = 1024\) and different time steps at \(T = 1\). As illustrated in Table 1 the LCN-MP scheme has second-order convergence rate in time. In Table 2 we display the spatial numerical error and convergence rate of the proposed scheme with
\( \tau = 10^{-4} \) and different mesh points at \( T = 1 \), which implies that the scheme has spectral accuracy in space. We should note that, after \( N = 64 \), the spatial error of LCN-MP scheme does not decrease and is dominated by the time discretization error. It conforms that, for sufficiently smooth problems, the Fourier pseudo-spectral method is of arbitrary order in space. The numerical error and the CPU time of different scheme with different mesh points and time steps at \( T = 1 \) are shown in Table 3. Compared with the linearized and conservative finite difference (LC-FD) scheme presented in Ref. [15], our scheme provides smaller numerical error. Further, it is clear to see that, for a given \( L^\infty \)-error, the LCN-MP scheme is computationally cheaper than the LC-FD scheme.

In Fig. 1 (a), we display the propagation of the soliton by the LCN-MP scheme over the time interval \( t \in [0, 200] \), which shows that shapes of the soliton is preserved accurately in long time computation. The energy errors over the time interval \( t \in [0, 200] \) are investigated in Fig. 1 (b). As illustrated in the figure, the momentum error provided by our scheme is much smaller than the one provided by the LC-FD scheme.

**Table.** 1: The temporal numerical error and convergence rate of the proposed scheme with \( N = 1024 \) and different time steps at \( T = 1 \).

| \( \tau \) | \( L^2 \)  | Rate | \( L^\infty \) | Rate |
|----------|---------|------|--------------|------|
| 0.1      | 7.2992e-05 | -    | 2.8001e-05   | -    |
| 0.05     | 1.8132e-05 | 2.01 | 6.9584e-06   | 2.01 |
| 0.025    | 4.5184e-06 | 2.00 | 1.7341e-06   | 2.00 |
| 0.0125   | 1.1277e-06 | 2.00 | 4.3281e-07   | 2.00 |

**Table.** 2: The spatial numerical error and convergence rate of the proposed scheme with \( \tau = 10^{-4} \) and different mesh points at \( T = 1 \).

| \( N \)  | \( L^2 \)    | Rate | \( L^\infty \)   | Rate |
|---------|--------------|------|------------------|------|
| 16      | 8.2999e-02   | -    | 1.7538e-02      | -    |
| 32      | 1.7211e-03   | 5.6  | 4.2655e-04      | 5.4  |
| 64      | 9.4160e-08   | 14.2 | 2.3647e-08      | 14.1 |

**Table.** 3: The numerical error and the CPU time of different schemes with different mesh points and time steps at \( T = 1 \).

| Scheme    | \((N, \tau)\)     | \( L^\infty \) | CPU (s) |
|-----------|-------------------|--------------|--------|
| LCN-MP    | (1000,0.01)       | 2.7688e-07   | 1.49   |
|           | (2000,0.005)      | 6.9176e-08   | 5.61   |
|           | (4000,0.0025)     | 1.7289e-08   | 7.42   |
| LC-FD [15]| (1000,0.01)       | 1.8893e-05   | 1.41   |
|           | (2000,0.005)      | 4.7232e-06   | 3.44   |
|           | (4000,0.0025)     | 1.1808e-06   | 3.78   |
5.2 Simulations of the GR-KdV equation in 2D

In this section, we consider the following GR-KdV equation in 2D, as follows [1]:
\[ u_t + \Delta^2 u_t + \Delta u_x + (1 + u^p)\Delta u = 0, \quad (x, y) \in \Omega, \quad 0 < t \leq T, \tag{5.2} \]
with initial condition
\[ u(x, y, 0) = 0.1(1 + \sin(3x) \sin(5y)), \quad (x, y) \in \Omega. \]

In our computation, we take \( \Omega = [0, 2\pi]^2 \), \( p = 2 \) and the periodic boundary conditions. The exact solution is obtained numerically by the LCN-MP scheme under a very small time step \( \tau = 0.0001 \) and spatial steps \( h_1 = h_2 = \frac{2\pi}{128} \). Temporal and spatial numerical errors and convergence rates of the LCN-MP scheme at \( T = 1 \) are shown in Tables 4 and 5, respectively. It can be observed from those tables that the LCN-MP scheme has second-order convergence rate in time and spectral accuracy in space, respectively, which confirms the theoretical analysis. The numerical error and the CPU time of different schemes with different mesh sizes and time steps at \( T = 1 \) are shown in Table 6. From the table, we find that, compared with the existing linearized and conservative finite difference (LC-FD) scheme in Ref. [1], our scheme admits much smaller numerical errors. In addition, we observe that, for a given \( L^\infty \)-error, the LCN-MP scheme is computationally cheaper than the LC-FD scheme in Ref. [1]. In Fig. 2, we show the momentum errors provided by the LCN-MP scheme and LC-FD scheme, respectively, with \( h_1 = h_1 = \frac{2\pi}{50} \) and \( \tau = 0.1 \) over the time interval \( t \in [0, 200] \). It is clear to see from the figure that our scheme provides much smaller error than the one provided by the LC-FD scheme.

Table. 4: The temporal numerical error and convergence rate of the proposed scheme with \( N_1 = N_2 = 100 \) and different time steps at \( T = 1 \).

| \( \tau \) | \( L^2 \) | Rate | \( L^\infty \) | Rate |
|---|---|---|---|---|
| 0.1 | 1.1046e-06 | - | 2.2706e-07 | - |
| 0.05 | 2.7610e-07 | 2.00 | 5.6758e-08 | 2.00 |
| 0.025 | 6.9017e-08 | 2.00 | 1.4189e-08 | 2.00 |
| 0.0125 | 1.7253e-08 | 2.00 | 3.5469e-09 | 2.00 |
Table. 5: The spatial numerical error and convergence rate of the proposed scheme with $\tau = 10^{-4}$ and different mesh sizes at $T = 1$.

| $N$  | $L^2$       | Rate | $L^\infty$    | Rate |
|------|-------------|------|---------------|------|
| 4    | 1.0668e-01 | -    | 2.7783e-02    | -    |
| 8    | 2.2779e-02 | 2.23 | 7.0849e-03    | 1.97 |
| 16   | 8.7270e-06 | 11.35| 2.6313e-06    | 11.39|
| 32   | 8.6541e-013| 23.27| 4.6810e-013   | 22.42|

Table. 6: The numerical error and the CPU time of different schemes with different mesh points and time steps at $T = 1$.

| Scheme       | $(N, \tau)$ | $L^\infty$    | CPU (s) |
|--------------|-------------|---------------|---------|
| LCN-MP       | (32, 0.02)  | 9.0743e-09    | 0.88    |
|              | (64, 0.01)  | 2.2699e-09    | 3.29    |
|              | (128, 0.005)| 5.6734e-010   | 39.96   |
| LC-FD [1]    | (32, 0.02)  | 5.2880e-05    | 0.93    |
|              | (64, 0.01)  | 1.4105e-05    | 3.55    |
|              | (128, 0.005)| 3.5763e-06    | 30.08   |

Fig. 2: Momentum errors ($P^0 = 114.59$) with $h_1 = h_1 = \frac{2\pi}{50}$ and $\tau = 0.1$ over the time interval $t \in [0, 200]$.

6 Concluding remarks

In this paper, we propose a new linearized and momentum-preserving Fourier pseudo-spectral method for the GR-KdV equation. With the help of the new semi-norm equivalence and the discrete momentum conservation law, we obtain the bound of the numerical solution in $L^\infty$-norm. Subsequently, based on the energy method and the bound of the numerical solution, an a priori estimate in discrete $L^\infty$-norm for the scheme is established without any restriction on the mesh ratio. Numerical results verify the theoretical analysis. Compared with the existing conservative schemes, our scheme is more accurate and has the significant advantage in computational efficiency and preserving the discrete momentum conservation law. Furthermore, the technique presented in this paper can also be used to establish an optimal $L^\infty$-error estimate for the linearized and momentum-preserving Fourier pseudo-spectral schemes of other Rosenau-type equation, such as the Rosenau-RLW equation [21], the Rosenau-Kawahara equation [2, 29], the Rosenau-KdV-RLW equation [27], etc.
Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant Nos. 11771213, 41504078), the National Key Research and Development Project of China (Grant Nos. 2016YFC0600310, 2018YFC0603500), the Major Projects of Natural Sciences of University in Jiangsu Province of China (Grant No. 15KJA110002) and the Priority Academic Program Development of Jiangsu Higher Education Institutions.

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