A QUILLEN MODEL CATEGORY STRUCTURE ON SOME CATEGORIES OF COMONOIDs

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Abstract. We prove that for certain monoidal (Quillen) model categories, the category of comonoids therein also admits a model structure.

1. Introduction

A monoidal model category is a closed symmetric monoidal category which admits a Quillen model category structure compatible in a certain sense with the monoidal product [6], [9]. The majority of the natural occurring examples of model categories are monoidal model categories. In [9], the authors gave a sufficient condition which ensured that the category of monoids in a monoidal model category admits a model structure, extended in an appropriate sense from the base category. This condition was called the monoid axiom, and it is satisfied in many examples.

Dually, one can consider comonoids in a monoidal category which has a model structure and ask for a model structure for comonoids, somehow inherited from the base category. We were not able to find in the literature a general result along these lines. The situation turns out to be more complicated than with monoids. In this note we give a (very) partial answer to this problem. We prove

Theorem 1.1. Let E be a symmetric monoidal category with unit I and let Comon(E) be the category of (coassociative and counital) comonoids in E. We assume that

(i) E is locally presentable, abelian and the monoidal product preserves colimits and finite limits in each variable;

(ii) E has two classes of maps W and Cof such that Cof and the class of monomorphisms of E are the cofibrations of two model structures on E with the same class W of weak equivalences; furthermore, either of the two model structures is cofibrantly generated;

(iii) the pushout-product axiom between the two model structures holds: if i : K → L belongs to Cof and i' : X → Y is a monomorphism, then the canonical map

\[ K \otimes Y \bigcup_{K \otimes X} L \otimes X \longrightarrow L \otimes Y \]

is a monomorphism, which is a weak equivalence if either one of i or i' is;

(iv) I is Cof-cofibrant and E has a coalgebra interval, by which we mean a factorisation of the codiagonal

\[ I \sqcup I \quad \nabla \quad I \]

\[ i_0 \sqcup i_1 \quad \alpha \quad Cyl(I) \]

\[ p \]

such that \( i_0 \sqcup i_1 \) belongs to Cof, \( p \) is a weak equivalence and the whole diagram lives in Comon(E).
Then $\text{Comon}(E)$ admits a model category structure in which a map is a weak equivalence (resp. cofibration) if and only if the underlying map is a weak equivalence (resp. monomorphism) in $E$.

An analogue of 1.1 for the category of comodules over a comonoid in $E$ is presented in section 3.

One of the motivations for writing this note was the paper [5]. In [2], the authors extended the main result of [5] to the category of cooperads, or $F_2$-comonoids, in the category of non-negatively graded chain complexes of vector spaces. We do not know whether the technique used in this paper would provide a model structure on the category of cooperads in $E$.

2. Proof of Theorem 1.1

In order to prove theorem 1.1 we shall use two results of J.H. Smith, recalled below.

Theorem 2.1. ([3, Thm. 1.7]) Let $E$ be a locally presentable category, $W$ a full accessible subcategory of $\text{Mor}(E)$, and $I$ a set of morphisms of $E$. Suppose they satisfy:

- $c_0$: $W$ has the three-for-two property.
- $c_1$: $\text{inj}(I) \subseteq W$.
- $c_2$: The class $\text{cof}(I) \cap W$ is closed under transfinite composition and under pushout.

Then setting weak equivalences $=: W$, cofibrations $=: \text{cof}(I)$ and fibrations $=: \text{inj}(\text{cof}(I) \cap W)$, one obtains a cofibrantly generated model structure on $E$.

Theorem 2.2. The class of weak equivalences of a combinatorial model category is accessible.

Proofs of the preceding theorem have been given in [4] and [5]. By general arguments the forgetful functor $U: \text{Comon}(E) \to E$ has a right adjoint and the category $\text{Comon}(E)$ is locally presentable, see e.g. ([1], Remark below Lemma 2.76 and the dual of Corollary 2.75). We shall define a set $I$ which will generate the class of cofibrations and then check condition $c_1$ of 2.1.

Let $C \in \text{Comon}(E)$. We say that $(D, i) \in \text{Comon}(E)/C$ is an $E$-subobject of $C$ if $U(i): U(D) \to U(C)$ is a monomorphism. As pointed out to us by Steve Lack, the $E$-subobjects are precisely the strong subobjects in $\text{Comon}(E)$. This can be seen using the left exactness of the monoidal product.

For example, if $f: C \to D$ is a map of comonoids, then the subobject $m: \text{Im}(f) \to U(D)$ is an $E$-subobject of $D$ and the canonical epi $e: U(C) \to \text{Im}(f)$ is a map of comonoids. To see this, one uses the fact that $\text{Com}(f) \cong \text{Im}(f)$ and again the left exactness of the monoidal product. For $C$ and $D$ comonoids we write $C \subseteq D$ if $C$ is an $E$-subobject of $D$.

Lemma 2.3. There is a regular cardinal $\kappa$ such that every comonoid $C$ is a $\kappa$-filtered colimit $C = \text{colim}C_i$, with $C_i \subseteq C$.

Proof. The functor $U$ preserves and reflects epimorphisms. Let $\lambda$ be a regular cardinal such that $\text{Comon}(E)$ is locally $\lambda$-presentable and let $C$ be a comonoid. Write $C = \text{colim}D_i$, with canonical arrows $\varphi_i: D_i \to C$ and with $D_i$ $\lambda$-presentable. Factor $U(\varphi_i)$ as $U(D_i) \xrightarrow{m_i} U(C_i) \xrightarrow{e_i} U(C)$, with $m_i$ mono and $e_i$ epi. By the above, $C_i \subseteq D$ and one clearly has $C = \text{colim}C_i$. Since $\text{Comon}(E)$ is co-well-powered, there is a set (up to isomorphism) $Q$ of all quotients of all $\lambda$-presentable objects. Therefore there is a regular cardinal $\kappa$ such that $Q$ is contained in the set of all $\kappa$-presentable objects of $\text{Comon}(E)$. \qed
We define $I$ to be the set of all isomorphism classes of cofibrations $A \to B$ with $B$ $\kappa$-presentable.

**Lemma 2.4.** A map has the right lifting property with respect to the cofibrations iff it has the right lifting property with respect to the maps in $I$.

To prove this lemma we need the following general result.

**Lemma 2.5.** Let $E$ be an abelian and monoidal category with monoidal product $\otimes$ which is left exact in each variable. If $A \to X$ and $B \to Y$ are subobjects, then $A \otimes B = (A \otimes Y) \cap (X \otimes B)$. As a consequence, if $i : D \to C$ and $j : E \to C$ are maps of comonoids in $E$ such that $U(i)$ and $U(j)$ are monomorphisms, then $D \cap E$ is a comonoid in $E$.

**Proof.** For the first part, start with the short exact sequences $0 \to A \to X \to X/A \to 0$ and $0 \to B \to Y \to Y/B \to 0$. By tensoring them one produces a $3 \times 3$ diagram all whose rows and columns are exact. The assertion follows from the nine-lemma. For the second part, consider the cube diagram in $E$

\[
\begin{array}{ccc}
D \cap E & \to & E \\
\downarrow & & \downarrow j \\
D & \to & C \\
\downarrow & & \downarrow \\
P & \to & E \otimes E \\
\downarrow & & \downarrow j \otimes j \\
D \otimes D & \to & C \otimes C.
\end{array}
\]

in which the top and bottom faces are pullbacks. The bottom face can be calculated as an iterated pullback

\[
\begin{array}{ccc}
P & \to & (D \cap E) \otimes E \\
\downarrow & & j \otimes E \\
D \otimes (D \cap E) & \to & D \otimes E \\
\downarrow & & \downarrow C \otimes j \\
D \otimes D & \to & C \otimes C
\end{array}
\]

therefore $P$ is $(D \cap E) \otimes (D \cap E)$ by the first part. This provides $D \cap E$ a comultiplication. The counit of $D \cap E$ is the counit of $C$ restricted to $D \cap E$. \hfill $\Box$

**Proof.** (of lemma 2.4) The proof is standard. Let

\[
\begin{array}{ccc}
C & \overset{f}{\to} & X \\
i & & \downarrow p \\
D & \overset{i}{\to} & Y
\end{array}
\]

be a commutative diagram with $i$ a cofibration and $p$ having the right lifting property with respect to the maps in $I$. Let $S$ be the set consisting of pairs $(E, l)$, where
$C \preceq E \preceq D$ and $l : E \to X$ is a morphism making the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{l} & X \\
\downarrow & & \\
E & \xrightarrow{p} & D \\
\end{array}
\]

commutative. We order $S$ by $(E, l) \preceq (E', l')$ iff $E \preceq E'$ and $l'$ is an extension of $l$. Then $S$ is nonempty, as it contains $(C, f)$. Let $C$ be any chain in $S$ and let $\kappa'$ be a regular cardinal such that both $E$ and $\text{Comon}(E)$ are locally $\kappa'$-presentable. Then $C$ is $\kappa'$-directed and therefore $	ext{colim} C$ is defined in $\text{Comon}(E)$, and $U(\text{colim} C)$ is the colimit of the $U(F)$, $(F, m) \in C$. Hence colim$C \to D$ is a cofibration. Also, we have a unique $l : \text{colim} C \to X$ extending each $m$, and clearly (colim$C$, $l$) is an element of $S$. This shows that Zorn’s lemma is applicable, therefore the set $S$ has a maximal element $(E, l)$. We are going to show that $E \sim D$ by showing that for each $\kappa$-presentable comonoid $B \preceq D$, one has $B \preceq E$. This suffices since $D$, being the $\kappa$-filtered colimit of all of its $E$-subobjects, is the least upper bound of its $\kappa$-presentable $E$-subobjects.

Take $B \preceq D$ with $B \kappa$-presentable. Using lemma 2.5 and the hypothesis we have a diagonal filler $d$ in the commutative diagram

\[
\begin{array}{ccc}
E \cap B & \xrightarrow{d} & E \\
\downarrow & & \downarrow \\
B & \xrightarrow{p} & D \\
\end{array}
\]

Therefore in the diagram

\[
\begin{array}{ccc}
E \cap B & \xrightarrow{l'} & B \\
\downarrow & & \downarrow \\
E & \xrightarrow{d} & E \cup B \\
\downarrow & & \downarrow \\
X & \xrightarrow{p} & X \\
\end{array}
\]

in which the square is a pushout, there is a map $l' : E \cup B \to X$ extending $l$, and so $(E \cup B, l') \in S$. This shows that $(E \cup B, l') \preceq (E, l)$ since $(E, l)$ was maximal. It follows that $B \preceq E$. □

By performing the small object argument it follows from lemma 2.4 and a retract argument that the class of cofibrations is the class $\text{Cof}(I)$. It remains to check condition c1 of 2.1. For this we shall use

2.6. The dual of Quillen path-object argument. Let $\mathcal{E}$ be a model category and let

\[F : \mathcal{C} \rightleftarrows \mathcal{E} : G\]

be an adjoint pair ($F : \mathcal{C} \to \mathcal{E}$ is the left adjoint). We define a map $f$ of $\mathcal{C}$ to be a cofibration (resp. weak equivalence) if $F(f)$ is such in $\mathcal{E}$. Suppose that $\mathcal{C}$ is finitely cocomplete, it has a cofibrant replacement functor and a functorial cylinder object for cofibrant objects. Then a map of $\mathcal{C}$ that has the right lifting property with respect to all cofibrations is a weak equivalence.
Proof. We recall its proof for the sake of completeness. Let \( f : X \to Y \) be map of \( C \) which has the right lifting property with respect to all cofibrations. Let

\[
\begin{array}{ccc}
\hat{C}X & \xrightarrow{i_X} & X \\
\downarrow \hat{C}(f) & & \downarrow f \\
\hat{C}Y & \xrightarrow{i_Y} & Y \\
\end{array}
\]

be the cofibrant replacement of \( f \). Then the diagram

\[
\begin{array}{ccc}
\emptyset & \xrightarrow{i_X} & \hat{C}X \\
\downarrow & & \downarrow f \\
\hat{C}Y & \xrightarrow{i_Y} & Y \\
\end{array}
\]

has a diagonal filler \( d \). Let \( \hat{C}X \sqcup \hat{C}X \xrightarrow{\psi} \text{Cyl}(\hat{C}X) \xrightarrow{\rho} \hat{C}X \) be the cylinder object for \( \hat{C}X \). Consider the commutative diagram

\[
\begin{array}{ccc}
\hat{C}X \sqcup \hat{C}X & \xrightarrow{(d\hat{C}(f),i_X)} & X \\
\downarrow i_0 \sqcup i_1 & & \downarrow f \\
\text{Cyl}(\hat{C}X) & \xrightarrow{i_X \rho} & Y \\
\end{array}
\]

By hypothesis it has a diagonal filler \( H \), and so \( d\hat{C}(f) \) is a weak equivalence. Since the weak equivalences of \( \mathcal{E} \) satisfy the two out of six property, it follows that \( d \) is a weak equivalence.

We return to the proof of 1.1. By 2.6 it suffices to show that there is a functorial cylinder object for comonoids. This is guaranteed by hypotheses (iii) and (iv). The proof of theorem 1.1 is complete.

Remark 2.7. Let \( \mathcal{E} \) be as in the statement of theorem 1.1. If, moreover, the cylinder object \( \text{Cyl}(I) \) for \( I \) is a cocommutative comonoid, then the category \( \text{CC}omon(\mathcal{E}) \) of cocommutative comonoids in \( \mathcal{E} \) admits a model category structure in which a map is a weak equivalence (resp. cofibration) if and only if the underlying map is a weak equivalence (resp. monomorphism) in \( \mathcal{E} \).

Examples. (a) Let \( R \) be a commutative von Neumann regular ring and let \( \text{Ch}(R) \) be the category of unbounded chain complexes of \( R \)-modules. We consider on \( \text{Ch}(R) \) the projective and injective model structures [6]. \( \text{Ch}(R) \) has a well-known coalgebra interval given by

\[
\cdots \to 0 \to Re \xrightarrow{\partial} Ra \oplus Rb \to 0 \to \cdots,
\]

where \( \partial(e) = b - a \) and \( Ra \oplus Rb \) is in degree 0. The maps \( i_0 \) and \( i_1 \) are the inclusions and the map \( p \) is \( a, b \mapsto 1 \), see e.g. ([8], section 5). The last part of (i) is shown in ([10], Proof of Prop. 3.3 for \( \text{Ch} \)).

(b) The above considerations apply to the category of non-negatively graded chain complexes as well.
3. Comodules

Let $\mathcal{E}$ be a monoidal category with monoidal product $\otimes$. Given a (coassociative and counital) comonoid $C$ in $\mathcal{E}$, we denote by $\mathbf{Mod}^C$ the category of right $C$-comodules in $\mathcal{E}$. There is a forgetful-cofree adjunction

$$U : \mathbf{Mod}^C \rightleftarrows \mathcal{E} : - \otimes C$$

(1)

**Theorem 3.1.** Let $\mathcal{E}$ be a cofibrantly generated monoidal model category and let $C$ be a (coassociative and counital) comonoid in $\mathcal{E}$. Suppose that

(i) the cofibrations of the model structure are precisely the monomorphisms;

(ii) $\mathcal{E}$ is locally presentable, abelian, and for each object $X$ of $\mathcal{E}$ the functor $- \otimes X$ is left exact, where $\otimes$ denotes the monoidal product of $\mathcal{E}$.

Then $\mathbf{Mod}^C$ admits a cofibrantly generated model structure in which a map $f$ is a weak equivalence (resp. cofibration) if and only if the underlying map is a weak equivalence (resp. monomorphism) in $\mathcal{E}$.

The proof of the above theorem follows the same steps as the proof of 1.1, except that condition c1 of 2.1 will be a consequence of lemma 3.2 below.

We say that a map of $C$-comodules is a **fibration** if it has the right lifting property with respect to the maps which are both cofibrations and weak equivalences. We say that a map of $C$-comodules is a **trivial fibration** if it is both a fibration and a weak equivalence.

**Lemma 3.2.** The category $\mathbf{Mod}^C$ has a weak factorisation system (cofibrations, trivial fibrations).

*Proof.* We follow an idea from [5]. Let $f : M \to N$ be a map of $C$-comodules. We factor the map $U(M) \to 0$ as a monomorphism followed by a trivial fibration $U(M) \xrightarrow{j} X \to 0$. Then $f$ factors as

$$M \xrightarrow{j} N \times (X \otimes C) \xrightarrow{p_1} N$$

where $j = (i^*, f)$, $i^*$ is the adjoint transpose of $i$ and $p_1 : N \times (X \otimes C) \to N$ is the projection. The map $p_1$ is a weak equivalence since it is the map $N \oplus (X \otimes C) \to N \oplus (0 \otimes C) \cong N$, which is a weak equivalence. We show that the underlying map of $j$ is a monomorphism. One has $i = \epsilon_X U(p_2 j)$, where $\epsilon_X$ is the counit of the adjunction (1) and $p_2 : N \times (X \otimes C) \to (X \otimes C)$ is the projection. Therefore $j$ is a cofibration. Next we show that $p_1 : N \times (X \otimes C) \to N$ has the right lifting property with respect to all cofibrations. Let

$$
\begin{array}{ccc}
M' & \to & N \times (X \otimes C) \\
\downarrow k & & \downarrow p_1 \\
N' & \to & N
\end{array}
$$

be a commutative diagram with $k$ a cofibration. This diagram has a diagonal filler if and only if the diagram

$$
\begin{array}{ccc}
U(M') & \to & X \\
\downarrow U(k) & & \downarrow \ \\
U(N') & \to & 0
\end{array}
$$

has one. The latter is true by the assumption on $X$. Therefore $p_1$ is a fibration. Let now $f : M \to N$ be a trivial fibration. Factor it as above $M \xrightarrow{j} N \times (X \otimes C) \xrightarrow{p_1} N$. 


Since \( j \) is a weak equivalence, there is a diagonal filler in the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{j} & M \\
\downarrow & & \downarrow \\
N \times (X \otimes C) & \to & N
\end{array}
\]

hence \( f \) is a (domain) retract of a map which has the right lifting property with respect to all cofibrations, therefore \( f \) has the right lifting property with respect to all cofibrations. Conversely, let \( f : M \to N \) have the right lifting property with respect to all cofibrations. The same argument shows that \( f \) is a (domain) retract of a trivial fibration, hence \( f \) is a trivial fibration.

\[\square\]

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