A POSITIVE SOLUTION TO A CONJECTURE OF A. KATOK
FOR DIFFEOMORPHISM CASE

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Abstract. A. Katok has conjectured that a $C^{1+\alpha}$ map $g: M^n \to M^n, n \geq 2$, which is H"older conjugated to an Anosov diffeomorphism is also an Anosov diffeomorphism. Using Pesin stable manifold theorem and Liao spectrum theorem, we show that under the hypothesis of such a conjecture, $g$ is an Axiom A diffeomorphism having no cycles. Particularly, if $g$ is H"older conjugated to a hyperbolic toral automorphism, then $g$ is Anosov.

1. Introduction

Let $M^n$ be a connected, compact, smooth, and closed Riemannian manifold of dimension $n \geq 2$. A. Katok has conjectured that if $g \in \text{Diff}^{1+\alpha}(M)$ is H"older conjugated to an Anosov $f \in \text{Diff}^1(M)$; i.e., there is a H"older-homeomorphism $h$ of $M$ such that $f = h \circ g \circ h^{-1}$, then $g$ is also an Anosov diffeomorphism. Here “H"older-homeomorphism $h$” means that $h$ and its inverse $h^{-1}$ both are H"older continuous. And $\text{Diff}^{1+\alpha}(M)$ is the set of all $C^1$ diffeomorphisms with $\alpha$-H"older derivatives for some H"older exponent $\alpha$ with $0 < \alpha \leq 1$.

For convenience, if $g \in \text{Diff}^{1+\alpha}(M)$ is H"older conjugated to an Anosov diffeomorphism, then $g$ is temporally said to be Katok. Under the hypothesis of such a conjecture, in [7] the authors proved the following.

Theorem A ([7]). If $g$ is Katok, then all periodic points of $g$ have only non-zero Lyapunov exponents, and such exponents are uniformly bounded away from zero.

In this paper, using Pesin stable manifold theorem, Liao spectrum theorem and Liao reordering theorem, and shadowing property, based on Theorem A above we obtain a positive solution to Katok’s conjecture as follows.

Theorem B. If $g \in \text{Diff}^{1+\alpha}(M)$ is Katok, then $g$ is an Axiom A diffeomorphism having no cycles.

Consequently, if $g$ is H"older conjugated to an Anosov diffeomorphism $f$ that satisfies $\Omega(f) = M$ such as a hyperbolic toral automorphism, then $g$ is also Anosov. In addition, if $g$ is volume-preserving, then $g$ is Anosov too. Theorem B shows that Anosov diffeomorphisms have strong rigidity.

This paper is organized as follows. In [2] we will introduce the Liao spectrum theorem and reordering theorem for $C^1$ differential systems on Euclidean spaces. Then we will prove a semi-uniform ergodic theorem which provides us with a criterion from nonuniform hyperbolicity to uniform hyperbolicity. In [3] we will prove

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an approximation theorem of ergodic measure by periodic measures. In [11] we will first prove that a Katok diffeomorphism is nonuniformly hyperbolic for any invariant measures and then show that it is uniformly hyperbolic. We will consider a volume-preserving Katok diffeomorphism in the last section.

2. Liao spectrum and reordering theorem

In this section, we will introduce the Liao spectrum theorem and Liao reordering theorem, which are basic in Liao theory. Then, applying the Liao spectrum theorem, we will provide with a criterion of uniform contraction.

For simplicity, let us consider throughout this section a nonsingular autonomous system of $C^1$-differential equations in an $(n+1)$-dimensional Euclidean $w$-space $E^{n+1}$, $n \geq 2$

\[ \dot{w} = S(w) \quad w \in E^{n+1}, \; S(w) \in \mathbb{R}^{n+1} - \{0\}, \]

where we write $T_w E^{n+1} = \mathbb{R}^{n+1}$ for all $w$ to distinguish the $w$-state-space $E^{n+1}$ from its tangent $x$-space $\mathbb{R}^{n+1}$, which then naturally gives rise to a $C^1$-flow on the state-space $E^{n+1}$

\[ \phi: \mathbb{R} \times E^{n+1} \to E^{n+1}; \; (t, w) \mapsto t \cdot w. \]

It further induces, on the tangent bundle $TE^{n+1} = E^{n+1} \times \mathbb{R}^{n+1}$, a smooth linear skew-product flow

\[ \Phi: \mathbb{R} \times E^{n+1} \times \mathbb{R}^{n+1} \to E^{n+1} \times \mathbb{R}^{n+1}; \; (t, (w, x)) \mapsto (t \cdot w, \Phi_{t,w}(x)) \]

where $\Phi_{t,w}: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}; \; x \mapsto \frac{\partial \phi(t, w)}{\partial w} x$, corresponding to the extended system

\[ \dot{w} = S(w), \quad \dot{x} = S'(w)x \]

on the extended $(w, x)$-phase-space $E^{n+1} \times \mathbb{R}^{n+1}$.

2.1. Let $T = \bigcup_{w \in E^{n+1}} T_w$, where $T_w = \{ x \in \mathbb{R}^{n+1} | \langle S(w), x \rangle = 0 \}$, denote the subbundle of the tangent bundle $E^{n+1} \times \mathbb{R}^{n+1}$ transversal to $S$ over $E^{n+1}$, called the transversal tangent bundle to $S$ over $E^{n+1}$. Then there is another naturally induced smooth linear skew-product flow

\[ \Psi: \mathbb{R} \times T \to T; \; (t, (w, x)) \mapsto (t \cdot w, \Psi_{t,w}(x)) \]

where along the fiber direction, $\Psi_{t,w}: T_w \to T_{t \cdot w}$ is defined as the component of $\Phi_{t,w}(x)$ transversal to $S(t \cdot w)$ for any $(w, x) \in T$; that is, $\Phi_{t,w}(x) = r S(t \cdot w) + \psi_{t,w}(x)$, $\psi_{t,w}(x) \in T_{t \cdot w}$, for some $r \in \mathbb{R}$. Particularly, let

\[ \mathcal{F}_1^{*} = \{ (w, x) \in T; \| x \| = 1 \} \]

be the unit transversal tangent bundle to $S$ over $E^{n+1}$. Then, there is a natural skew-product flow

\[ \psi^*_1: \mathbb{R} \times \mathcal{F}_1^{*} \to \mathcal{F}_1^{*}; \; (t, (w, x)) \mapsto (t \cdot w, \psi^*_1(x)) \]

where

\[ \psi^*_1(x) = \psi_{t,w}(x)/\| \psi_{t,w}(x) \|. \]
2.4. Let the bundle of transversal orthonormal $n$-frames over $\mathbb{E}^{n+1}$, it means $\mathcal{F}_n^*$, where the fiber at $w$ is given by

$$\mathcal{F}_{n,w}^* = \left\{ \gamma \in \mathbb{T}_w \setminus \{0\} \times \cdots \times \mathbb{T}_w \setminus \{0\} \mid (\operatorname{col}_i \gamma, \operatorname{col}_j \gamma) = 0, \ 1 \leq i \neq j \leq n \right\}.$$  

Here and in the future, for $1 \leq i \leq n$

$$\operatorname{col}_i \colon (v_1, \ldots, v_n) \mapsto v_i.$$  

Using the well-known Gram-Schmidt orthogonalizing process, based on $(\mathbb{E}^{n+1}, \phi)$, we can obtain from $\Psi$ the well-defined skew-product flow on $\mathcal{F}_n^*$ as follows

$$\chi^* \colon \mathbb{R} \times \mathcal{F}_n^* \to \mathcal{F}_n^* ; \ (t, (w, \gamma)) \mapsto (t.w, \chi^*_t \gamma).$$  

The bundle of transversal orthonormal $n$-frames over $\mathbb{E}^{n+1}$ is written as $\mathcal{F}_n^*$, where the fiber at $w$ is defined as

$$\mathcal{F}_{n,w} = \{ \gamma \in \mathcal{F}_n^* : \|\operatorname{col}_j \gamma\| = 1, \ j = 1, \ldots, n \}.$$  

Furthermore, there is a natural skew-product flow based on $(\mathbb{E}^{n+1}, \phi)$

$$\chi_{\phi}^* : \mathbb{R} \times \mathcal{F}_n^* \to \mathcal{F}_n^* ; \ (t, (w, \gamma)) \mapsto (t.w, \chi_{\phi}^*_t \gamma)$$

called the Liao transversal orthonormal $n$-frame flow of $S$. For convenience, let 

$$\pi : \mathcal{F}_n^* \to \mathbb{E}^{n+1} ; \ (w, \gamma) \mapsto w$$

be the bundle projection. Then, the following commutativity holds:

$$t.w = \phi(t, \pi(w, \gamma)) = \pi(\chi_{\phi}^*(t, (w, \gamma))) \quad \forall (t, (w, \gamma)) \in \mathbb{R} \times \mathcal{F}_n^*.$$  

2.3. Now, the so-called Liao qualitative functions of $S$ are the following

$$\omega_i^* : \mathcal{F}_n^* \to \mathbb{R}$$

for $i = 1, \ldots, n$, given by

$$\omega_i^*(w, \gamma) = \frac{d}{dt}\bigg|_{t=0}\|\operatorname{col}_i \circ \chi_{\phi}^*_t \gamma\|$$

where $\chi_{\phi}^*_t : \mathcal{F}_{n,w} \to \mathcal{F}_{n,w}$ is as in (2.2) for any $(t, w) \in \mathbb{R} \times \mathbb{E}^{n+1}$. Particularly, let

$$\omega^* : \mathcal{F}_{1,n+1}^* \to \mathbb{R} ; \ (w, x) \mapsto \frac{d}{dt}\bigg|_{t=0}\|\Psi_{t,w} x\|.$$  

Since $S$ is of class $C^1$, these functions $\omega^*, \omega_i^*, 1 \leq i \leq n$, all are well defined and continuous; see [14,10].

2.4. Let $\mathcal{M}_{\text{inv}}$ and $\mathcal{M}_{\text{erg}}$ denote the set of all invariant Borel probability measures and ergodic Borel probability measures of a dynamical system, respectively. The following theorem equivalently describes the Lyapunov characteristic spectrum of $(\mathbb{E}^{n+1}, S)$.

**Theorem 2.1** (Liao spectrum theorem [14,10]). Let $S(w), S'(w)$ be bounded on $\mathbb{E}^{n+1}$ and assume that there is an ergodic $\phi$-invariant Borel probability measure $\mu$ on $\mathbb{E}^{n+1}$. Then, there exists a $\phi$-invariant Borel subset $L(\mu)$ of $\mathbb{E}^{n+1}$ such that:

1. $\mu(L(\mu)) = 1$;
2. every point $w$ in $L(\mu)$ is Oseledets regular for $\Psi$ based on $(\mathbb{E}^{n+1}, \phi)$;
(3) given any \( P \in \mathcal{M}_\text{erg}(\mathcal{F}_n^\Psi, \chi^\Psi) \) with marginal \( \mu \); i.e., \( \mu = P \circ \pi^{-1} \),

\[
\mathcal{S}_\text{la}(S, \mu) := \{ \vartheta_i^*(P) | i = 1, \ldots, n \}
\]

is just the Lyapunov spectrum of \( \Psi \) based on \((\phi, \mu)\), counting with multiplicity and ignoring the order, which is called the “spectrum of transversal Lyapunov exponents” of \((S, \mu)\) and is independent of the choices of \( P \), where

\[
\vartheta_i^*(P) := \int_{\mathcal{F}_n^\Psi} \omega_i^*(w, \gamma)\,dP(w, \gamma) \quad \text{for} \quad i = 1, \ldots, n.
\]

Notice that under the hypotheses of Theorem 2.1 above, there \( P \in \mathcal{M}_\text{erg}(\mathcal{F}_n^\Psi, \chi^\Psi) \) with marginal \( \mu \) is always existent from \cite{10}.

**Theorem 2.2** (Liao reordering theorem \cite{14, 10}). Under the hypotheses of Theorem 2.1, let \( \mathcal{S}_\text{la}(S, \mu) = \{ \lambda_i^*(\mu) | i = 1, \ldots, n \} \) be the spectrum of transversal Lyapunov exponents of \((S, \mu)\). If \( i \rightarrow \vartheta(i) \) is any given permutation of \{1, \ldots, n\}, then there is some \( P_\vartheta \in \mathcal{M}_\text{erg}(\mathcal{F}_n^\Psi, \chi^\Psi) \) with marginal \( \mu \) such that

\[
\vartheta_i^*(P_\vartheta) = \lambda_{\vartheta(i)}^*(\mu) \quad \text{for} \quad i = 1, \ldots, n.
\]

The above spectrum theorem 2.1 and reordering theorem 2.2 will play an important role for the proof of Theorem B stated in \cite{11}.

2.5. Based on Theorem 2.1, for any \( P \in \mathcal{M}_\text{erg}(\mathcal{F}_1^\Psi, \Psi^\Psi) \) with marginal \( \mu \) we have

\[
\vartheta^*(P) := \int_{\mathcal{F}_1^\Psi} \omega^*(w, x)\,dP(w, x) \in \mathcal{S}_\text{la}(S, \mu).
\]

Now, the following semi-uniform ergodic theorem will play an important role for the proof of our main result.

**Theorem 2.3.** Let \( \Lambda \) be an \( \phi \)-invariant compact subset of \( \mathbb{E}^{n+1} \) and \( \Delta \) an \( \phi \)-invariant Borel subset of \( \Lambda \) with total measure \( 1 \); that is to say, \( \mu(\Delta) = 1 \) for all \( \mu \in \mathcal{M}_\text{inv}(\Lambda, \phi|\Lambda) \). Let

\[
\mathbb{D}: \Delta \ni w \mapsto D(w) \subset \mathbb{T}_w
\]

be an \( 1 \)-dimensional \( \Psi \)-invariant measurable distribution for some integer \( 1 \leq i < n \). If \( \Psi|\mathbb{D} \) has only negative Lyapunov exponents at almost every \( w \in \Delta \) and \( \mathbb{D} \) is such that \( \lim_{t \to -\infty} D(w_t) = D(w) \) provided, of course, that this limit exists for \( w_t \to w \) in \( \Delta \), then \( \Psi|\mathbb{D} \) is uniformly contracting.

**Proof.** Without any loss of generality, assume \( \Lambda = \overline{\Delta} \). Let

\[
\mathcal{F}_1^\Psi(\Delta) = \left\{ (w, x) \in \mathcal{F}_1^\Psi | w \in \Delta \text{ and } x \in D(w) \right\},
\]

which is \( \Psi^\Psi \)-invariant. Then, it is easily seen that \( Y := \overline{\mathcal{F}_1^\Psi(\Delta)} \) is an \( \Psi^\Psi \)-invariant compact subbundle of \( \mathcal{F}_1^\Psi \) over \( \Lambda \), such that

\[
P(Y - \mathcal{F}_1^\Psi(\Delta)) = 0 \quad \text{for all} \quad P \in \mathcal{M}_\text{inv}(Y, \Psi^\Psi|Y).
\]

Moreover, according to Theorem 2.1 we obtain

\[
\int_{Y} \omega^*\,dP < 0 \quad \forall \ P \in \mathcal{M}_\text{erg}(Y, \Psi^\Psi|Y),
\]
since $\Psi|D$ is nonuniformly contracting for any $\mu \in \mathcal{M}_{\text{erg}}(\Lambda, \phi|\Lambda)$. From the continuous-time version of a semi-uniform theorem of [19] ([9, Lemma 3.1]), it follows that there exist constants $\sigma > 0$ and $T_0 > 0$ such that

$$\int_Y \omega^* dP \leq -\sigma \quad \forall \ P \in \mathcal{M}_{\text{erg}}(Y, \Psi^1|Y)$$

and uniformly, for all $T \geq T_0$

$$\frac{1}{T} \int_0^T \omega^*(\Psi^\mu(t+y)) dt \leq -\sigma/2$$

for all $y \in Y$ and for any $s \in \mathbb{R}$. Next, by the identity

$$\frac{1}{T} \log \|\Psi_{t,w}x\| = \frac{1}{T} \int_0^T \omega^*(\Psi^\mu(t,(w,x)))) dt$$

for all $(w,x) \in \mathcal{F}_1^\mu$ and for any $T \neq 0$, we can easily obtain that $\Psi|D$ is uniformly contracting.

This proves the theorem. \hfill \Box

Notice here that if the distribution $D$ is continuous; that is, $D(w_\ell) \to D(w)$ provided that $w_\ell \to w$ in $\Lambda$, then from [9] one can directly obtain the uniform contraction of $\Psi|D$.

3. Shadowing property and approximation of ergodic measures

If $g \in \text{Diff}^1(M)$ is conjugated to an Anosov diffeomorphism, then $\text{Per}(g) = \Omega(g)$ and $g$ has the shadowing property (see [17, Proposition 8.5]). Naturally, we ask if every ergodic measure of $g$ can be approximated by periodic measures or not. Under our context, this is the case.

Let $S$ be a $C^1$-differential system on $\mathbb{E}^{n+1}$ as in [2] and $\Lambda$ an $\phi$-invariant nonempty compact subset of $\mathbb{E}^{n+1}$. We say that $(\phi, \Lambda)$ has the shadowing by periodic points property provided that to any $\epsilon > 0$, there corresponds to some $\alpha > 0$, such that for any orbit arc $\phi([0,\tau],w) \subset \Lambda, \tau \geq 2$ with $\|w - \tau.w\| < \alpha$, there exists a periodic point $p \in \Lambda$ with period $\tau$ satisfying $\|t.w - t.p\| < \epsilon$ for all $t \in [0,\tau]$. In this case, we say that the orbit $\phi(t,p)$ $\epsilon$-shadows the orbit arc $\phi([0,\tau],w)$. Notice here that we require $p \in \Lambda$.

An $\phi$-invariant Borel probability measure $\mu$ on $\mathbb{E}^{n+1}$ is called a periodic measure if it is supported on a periodic orbit of $\phi$; that is, $\text{supp}(\mu) = \phi(\mathbb{E},p)$ for some periodic point $p$. The following result shows that periodic measures are dense in $\mathcal{M}_{\text{erg}}(\Lambda, \phi)$ under the weak *-topology.

**Theorem 3.1.** If the compact subsystem $(\Lambda, \phi)$ has the shadowing by periodic points property, then periodic measures are dense in $\mathcal{M}_{\text{erg}}(\Lambda, \phi)$; that is, for any $\mu$ in $\mathcal{M}_{\text{erg}}(\Lambda, \phi)$ there is a sequence of periodic measures $(\mu_k)$ on $\Lambda$ such that $\mu_k \to \mu$ as $k$ tends to $\infty$.

**Proof.** Let $\mu \in \mathcal{M}_{\text{erg}}(\Lambda, \phi)$ be non-periodic, and let $Q_\mu(\Lambda, \phi)$ be the quasi-regular point set of $(\Lambda, \mu, \phi)$; that is, $w \in Q_\mu(\Lambda, \phi)$ if and only if

$$\lim_{T \to \infty} T^{-1} \int_0^T \phi(t,w) dt = \int_\Lambda \varphi d\mu$$

for all $\varphi \in C(\Lambda)$. 

\hfill \Box
For any \( w \in \Lambda \) and \( T > 0 \), using the Riesz representation theorem, we define the empirical measure \( \mu_{w,T} \) on \( \Lambda \) by

\[
\mu_{w,T}(\varphi) = \frac{1}{T} \int_0^T \varphi(t.w) \, dt \quad \forall \varphi \in C(\Lambda).
\]

Then, given any Poisson stable (recurrent) point \( \hat{w} \in Q_\mu(\Lambda, \phi) \) we have \( \mu_{\hat{w},T} \to \mu \) as \( T \to \infty \) in the sense of weak \( * \)-topology; that is, \( \mu_{\hat{w},T}(\varphi) \to \mu(\varphi) \) for all \( \varphi \in C(\Lambda) \).

For any \( \varphi \in C(\Lambda) \), let

\[
\|\varphi\|_\infty = \sup_{x \in \Lambda} |\varphi(x)|
\]

and

\[
\|\varphi\|_L = \sup_{x,y \in \Lambda, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\|x - y\|}.
\]

Then \( BL(\Lambda) = \{ \varphi \in C(\Lambda); \|\varphi\|_\infty + \|\varphi\|_L < \infty \} \) is dense in \( (C(\Lambda), \| \cdot \|_\infty) \); see [12, Theorem 11.2.4].

Now, by the shadowing by periodic points property and the recurrence of the motion \( \phi(t, \hat{w}) \), we can choose a sequence of periodic points \( p_i \) with period \( T_i \to \infty \) such that for any \( i \), \( |t.\hat{w} - t.p_i| < 1/i \) for all \( t \in [0, T_i] \). Then, it is easily seen that \( \mu_{p_i} \), defined by

\[
\mu_{p_i}(\varphi) = \frac{1}{T_i} \int_0^{T_i} \varphi(t.p_i) \, dt \quad \forall \varphi \in C(\Lambda),
\]

is an ergodic periodic measure of the subsystem \( (\Lambda, \phi) \).

Next, for any \( \varphi \in BL(\Lambda) \) we have

\[
\lim_{i \to \infty} \left| \int \varphi \, d\mu - \int \varphi \, d\mu_{p_i} \right| \leq \lim_{i \to \infty} \left| \int \varphi \, d\mu - \int \varphi \, d\mu_{\hat{w},T_i} \right| + \limsup_{i \to \infty} \left| \int \varphi \, d\mu_{\hat{w},T_i} - \int \varphi \, d\mu_{p_i} \right|
\]

\[
\leq \limsup_{i \to \infty} \frac{1}{T_i} \int_0^{T_i} |\varphi(t.\hat{w}) - \varphi(t.p_i)| \, dt
\]

\[
\leq \limsup_{i \to \infty} \frac{1}{T_i} \int_0^{T_i} \|\varphi\|_L |t.\hat{w} - t.p_i| \, dt
\]

\[
= \limsup_{i \to \infty} \frac{\|\varphi\|_L}{i} \to 0,
\]

which implies that \( \mu_{p_i} \to \mu \) by the density of \( BL(\Lambda) \) in \( (C(\Lambda), \| \cdot \|_\infty) \), as required. This proves the theorem. \( \square \)

4. Hyperbolicity of Katok maps

In this section, we will finish the proof of our main result Theorem B stated in the Introduction, using the theorems introduced in §§2 and 3.

4.1. Let \( S \) be a \( C^1 \)-differential system on \( E^{n+1} \) as in [2] and \( \Lambda \) an \( \phi \)-invariant nonempty compact subset of \( E^{n+1} \).

**Theorem 4.1.** Assume that the subsystem \( (\phi, \Lambda) \) has the shadowing by periodic points property, and, for each periodic point \( p \) in \( \Lambda \), let \( \lambda_1^*(p) \leq \cdots \leq \lambda_n^*(p) \) be the spectrum of transversal Lyapunov exponents of \( S \) at \( p \), counting with multiplicity. If there are some \( \sigma < \varsigma \) such that \( \lambda_1^*(p) \leq \sigma \) and \( \lambda_n^*(p) \geq \varsigma \) for all \( p \in \text{Per}(\Lambda, \phi) \),
then for all \( \mu \in \mathcal{M}_{\text{erg}}(\Lambda, \phi) \), \((S, \mu)\) has at least two transversal Lyapunov exponents \( \lambda^- (\mu) \leq \sigma \) and \( \lambda^+ (\mu) \geq \varsigma \).

Proof. Let \( \mu \in \mathcal{M}_{\text{erg}}(\Lambda, \phi) \) be non-periodic. To prove the statement, it is enough to show that \((S, \mu)\) has at least two transversal Lyapunov exponents \( \lambda^- (\mu) \) and \( \lambda^+ (\mu) \) such that \( \lambda^- (\mu) \leq \sigma \) and \( \lambda^+ (\mu) \geq \varsigma \).

Let \( \mathcal{F}^1_n (\Lambda) = \{(w, \gamma) \in \mathcal{F}^1_n | w \in \Lambda\} \). From Theorem 3.1 we can take a sequence of periodic measures, say \( \{\mu_p\}, \) in \( \mathcal{M}_{\text{erg}}(\Lambda, \phi) \) with \( \mu_p \to \mu \). By using Theorems 2.1 and 2.2 we can choose some

\[
P_i \in \mathcal{M}_{\text{erg}}(\mathcal{F}^1_n (\Lambda), \chi^{\nu}) \quad \text{with marginal } \mu_p,
\]

for all \( i, \) such that

\[
\lambda^+_i (p_i) = \vartheta^+_i (P_i) \leq \cdots \leq \vartheta^+_n (P_i) = \lambda^+_i (p_i).
\]

Since \( \mathcal{M}_{\text{ins}}(\mathcal{F}^1_n (\Lambda), \chi^{\nu}) \) is compact under the weak *-topology, there is no loss of generality in assuming that \( P_i \to P \) for some \( P \in \mathcal{M}_{\text{ins}}(\mathcal{F}^1_n (\Lambda), \chi^{\nu}) \) with marginal \( \mu \); i.e., \( P \circ \pi^{-1} = \mu \). Thus,

\[
\lim_{i \to \infty} \int_{\mathcal{F}^1_n (\Lambda)} \omega^n_i dP_i = \int_{\mathcal{F}^1_n (\Lambda)} \omega^n dP \leq \sigma
\]

and

\[
\lim_{i \to \infty} \int_{\mathcal{F}^1_n (\Lambda)} \omega^n dP_i = \int_{\mathcal{F}^1_n (\Lambda)} \omega^n dP \geq \varsigma
\]

for \( \omega^+_i \) and \( \omega^n \) both are continuous on \( \mathcal{F}^1_n (\Lambda) \). Then, by the classical ergodic decomposition theorem we can choose at least two \( P_- \) and \( P_+ \) in \( \mathcal{M}_{\text{erg}}(\mathcal{F}^1_n (\Lambda), \chi^{\nu}) \) with marginal \( \mu \) such that

\[
\lambda^- (\mu) := \int_{\mathcal{F}^1_n (\Lambda)} \omega^n dP_- \leq \sigma \quad \text{and} \quad \lambda^+ (\mu) := \int_{\mathcal{F}^1_n (\Lambda)} \omega^n dP_+ \geq \varsigma.
\]

By Theorem 2.1 again, \( \lambda^- (\mu) \) and \( \lambda^+ (\mu) \) both lie in \( \text{Sp}_{\text{ins}}(S, \mu) \), as required.

This proves the theorem. \( \square \)

4.2. To prove Theorem B, we need a further remark on Theorem A stated in [11]. Let \( g \in \text{Diff}^1 (M) \) be Hölder conjugated to an Anosov \( f \in \text{Diff}^1 (M) \). Theorem A asserts that all periodic points of \( g \) have only non-zero Lyapunov exponents. However, there is no information that if there are any contracting or expanding periodic orbits. The following lemma shows that a Katok map has no any contracting or expanding periodic orbits.

Recall \( f \in \text{Diff}^1 (M) \) is said to be Anosov if there is a continuous splitting of \( T_x M = E^s (x) \oplus E^u (x) \) for every \( x \in M \) and constants \( C > 0, \lambda > 1 \) such that

\[
D_x f (E^s (x)) = E^s (f (x)) \quad \text{and} \quad D_x f (E^u (x)) = E^u (f (x)),
\]

\[
\| (D_x f^n) v \| \leq C \lambda^{-n} \| v \| \quad \forall v \in E^s (x), \ n \in \mathbb{N},
\]

\[
\| (D_x f^{-n}) u \| \leq C \lambda^{-n} \| u \| \quad \forall u \in E^u (x), \ n \in \mathbb{N}.
\]

Then, the nonnegative integer

\[
\text{Ind} (x) := \dim E^s (x)
\]

for all \( x \in M \) is called the index of \( f \) at \( x \).
For any \( f \in \text{Diff}^1(M) \) and \( \delta > 0 \), as usual, for \( x \in M \) let
\[
W^s(x) = \{ y \in M \mid \text{dist}(f^k x, f^k y) \to 0 \text{ as } k \to \infty \}
\]
and
\[
W^s_\delta(x) = \{ y \in M \mid \text{dist}(f^k x, f^k y) \leq \delta \text{ and } \lim_{k \to \infty} \text{dist}(f^k x, f^k y) = 0 \}
\]
be the stable set and the local stable set of \( f \) at \( x \), respectively. If \( f \) is partially hyperbolic or \( f \) is a nonuniformly partially hyperbolic \( C^{1+\alpha} \) diffeomorphism, then \( W^s(x) \) has local smooth manifold structure with \( T_x W^s(x) = E^s(x) \) a.e. (13) [15]. Similarly, one can define the unstable manifolds \( W^u(x) \) and \( W^u_\delta(x) \).

**Lemma 4.2.** If \( g \in \text{Diff}^{1+}(M) \) is Hölder conjugated to an Anosov \( f \in \text{Diff}^1(M) \), then all periodic points of \( g \) have only non-zero Lyapunov exponents, and such exponents are uniformly bounded away from zero, and
\[
\text{Ind}(p) \equiv i \quad \forall p \in \text{Per}(g)
\]
for some integer \( i \) with \( 1 \leq i < n \).

**Proof.** We first assert that for an Anosov \( f : M^n \to M^n \), there is an integer \( i \) with \( 1 \leq i < n \) such that
\[
\text{Ind}(\hat{x}, f) = i \quad \text{for all } \hat{x} \in M.
\]
In fact, let \( \Lambda_i = \{ \hat{x} \in M : \text{Ind}(\hat{x}, f) = i \} \) for \( i = 0, 1, \ldots, n \). Since the splitting \( T_x M = E^s(\hat{x}, f) \oplus E^u(\hat{x}, f) \) is continuous with respect to \( \hat{x} \in M \), \( \Lambda_i \) is closed and further open in \( M \) for all \( i = 0, 1, \ldots, n \). Thus, every \( \Lambda_i \) is either equal to \( \emptyset \) or to \( M \). Clearly, \( \Lambda_0 = \Lambda_n = \emptyset \). This shows the assertion.

Let \( h : M \to M \) be a Hölder conjugacy from \( g \) to \( f \). Given any \( p \in \text{Per}(g) \) and let \( \hat{p} = h(p) \). From Theorem A it follows that \( h(W^s(p; g)) \subset W^s(\hat{p}; f) \) and \( h(W^u(p; g)) \subset W^u(\hat{p}; f) \). Since \( h \) is Hölder homeomorphic, we have \( \dim W^s_{\text{loc}}(p; g) \leq \dim W^s_{\text{loc}}(\hat{p}; f) \) and \( \dim W^u_{\text{loc}}(p; g) \leq \dim W^u_{\text{loc}}(\hat{p}; f) \). Then, we can obtain that \( \dim W^s_{\text{loc}}(p; g) = \dim W^s_{\text{loc}}(\hat{p}; f) \) and so \( \text{Ind}(p) = i \) constant with \( 1 \leq \text{Ind}(p) < n \).

This proves the lemma. \( \square \)

4.3. Before proving Theorem B, we first prove that a Katok map is non-uniformly hyperbolic.

**Theorem 4.3.** Let \( g \in \text{Diff}^{1+}(M) \) be Katok. Then

1. there is a \( \sigma > 0 \) such that to any \( \mu \in \mathcal{M}_{\text{erg}}(M, g) \), \( (g, \mu) \) is non-uniformly hyperbolic and has at least two Lyapunov exponents, say \( \lambda_-(\mu) \) and \( \lambda_+(\mu) \), with \( \lambda_-(\mu) \leq -\sigma \) and \( \lambda_+(\mu) \geq \sigma \);

2. there is an invariant subset \( \Gamma \) and a measurable function \( \sigma : \Gamma \to (0, \infty) \) such that \( \mu|_{\Gamma} = 1 \) for all \( \mu \in \mathcal{M}_{\text{erg}}(M, g) \) and \( x \mapsto W^s_{\delta(x)}(x), \ x \mapsto W^u_{\delta(x)}(x) \) both are well defined and continuous for \( x \) in \( \Gamma \).

Here \( W^s_\delta(x) \) and \( W^u_\delta(x) \) mean the local stable and unstable manifolds of \( g \) at \( x \), respectively.

**Proof.** Let \( g \in \text{Diff}^{1+}(M) \) be Hölder conjugated to an Anosov diffeomorphism \( f : M \to M \). Given any \( \mu \in \mathcal{M}_{\text{erg}}(M, g) \).

By the so-called suspension technique, from Lemma 4.2 and Theorem 4.1 it follows immediately that \( (g, \mu) \) has at least two Lyapunov exponents, say \( \lambda_-(\mu) \) and
\(\lambda_+(\mu),\) such that \(\lambda_-(\mu) \leq -\sigma\) and \(\lambda_+(\mu) \geq \sigma,\) where \(\sigma\) is some positive constant which is independent of \(\mu.\)

We next proceed to prove that \((g, \mu)\) is non-uniformly hyperbolic. Let
\[
T_{\delta} M = E^s(x, g) \oplus E^c(x, g) \oplus E^u(x, g)
\]
for \(\mu\)-a.e. \(x \in M,\) where \(E^s(x, g), E^c(x, g),\) and \(E^u(x, g)\) stand for the stable direction, central direction and unstable direction, respectively, associated to the Oseledets splitting of \(Dg\) at \(x.\)

We have \(\dim E^s(x, g) \geq 1\) and \(\dim E^u(x, g) \geq 1\) for \(\mu\)-a.e. \(x \in M.\) Since \(g\) is of class \(C^{1+\alpha}\) for some Hölder exponent \(0 < \alpha < 1,\) according to Pesin theory there are local stable manifold \(W^s_{\text{loc}}(x; g)\) and local unstable manifold \(W^u_{\text{loc}}(x; g)\) with \(\dim W^s_{\text{loc}}(x; g) = \dim E^s(x, g)\) and \(\dim W^u_{\text{loc}}(x; g) = \dim E^u(x, g)\) for \(\mu\)-a.e. \(x \in M.\) On the other hand, by the \(C^\alpha\)-conjugation \(h^{-1}: M \to M\) from the Anosov diffeomorphism \(f\) to \(g,\) we obtain that for \(\tilde{x} = h(x)\)
\[
\dim W^s_{\text{loc}}(x; g) \geq \dim W^s_{\text{loc}}(\tilde{x}; f) \quad \text{and} \quad \dim W^u_{\text{loc}}(x; g) \geq \dim W^u_{\text{loc}}(\tilde{x}; f),
\]
which implies \(\dim W^s_{\text{loc}}(x; g) + \dim W^u_{\text{loc}}(x; g) = n\) and so \(E^c(x, g) = \emptyset\) for \(\mu\)-a.e. \(x \in M.\) Thus, \((g, \mu)\) is non-uniformly hyperbolic. This proves the statement (1).

Next, we are going to prove the statement (2). Since \(f\) is Anosov, there is some constant \(\delta > 0\) such that the local stable foliation \(\mathcal{W}^s = (W^s_\delta(\tilde{x}; f))_{\tilde{x} \in M}\) is continuous in \(\tilde{x} \in M.\) Let \(\Gamma\) be the non-uniformly hyperbolic Pesin regular set of \(g.\)

Noticing that \(h, h^{-1}\) both are Hölder and \(h^{-1}(W^s_\delta(\tilde{x}; f)) \subset W^s(x; g)\) where \(h(x) = \tilde{x}\) for all \(x \in \Gamma,\) we can easily find some measurable function \(\delta: \Gamma \to (0, \infty),\) which satisfies the requirements of the statement (2). This proves the statement (2).

Thus, Theorem 4.3 is proved.

4.4. In [6], the authors exhibit an example of a non-hyperbolic horseshoe such that all Lyapunov exponents are non-zero and uniformly bounded away from zero for all invariant measures. This phenomenon is named “completely nonuniformly hyperbolic.” Theorem 4.3 implies that a Katok map \(g \in M^2\) has just two Lyapunov exponents \(\lambda_-(\mu) < 0 < \lambda_+(\mu),\) uniformly bounded away from zero, for all \(\mu\) in \(\mathcal{M}_{\text{erg}}(M^2, g).\)

Nevertheless, there is still an essential gap from Theorem 4.3 to Katok’s conjecture even though in the 2-dimensional case, the continuity of the foliation \((W^s_\delta(x))_{x \in \Gamma}\) guaranteed by Theorem 4.3(2), can avoid the occurrence of the completely nonuniformly hyperbolic phenomenon.

Now we can finish the proof of Theorem B using the semi-uniform ergodic theorem Theorem 2.3.

Proof of Theorem B. Let \(\Gamma\) be defined by Theorem 4.3(2) and let
\[
T_{\delta} M = E^s(x) \oplus E^u(x)
\]
for all \(x \in \Gamma\) be the Oseledets splitting of \(g\) according to the multiplicative ergodic theorem. Let \(i\) be the index of \(g\) and \(\mathcal{G}(T_{\Gamma} M)\) the Grassmannian of i-dimensional linear subspaces of \(T_{\Gamma} M.\) Then
\[
\mathbb{D}: \Gamma \to \mathcal{G}(T_{\Gamma} M); \ x \mapsto E^s(x)
\]
is a \(Dg\)-invariant measurable distribution over \(\Gamma\) such that \(T_{\delta} W^s_{\delta(x)}(x) = E^s(x)\) and \(\dim E^s(x) = i\) for all \(x \in \Gamma.\)

Let \(x_\ell \to x\) in \(\Gamma\) and \(\lim_{\ell \to \infty} E^s(x_\ell) = E(x)\) for some i-dimensional linear subspace \(E(x) \subset T_{\delta} M.\) As \(W^s_{\delta(x_\ell)}(x_\ell) \to W^s_{\delta(x)}(x)\) as \(\ell \to \infty\) by Theorem 4.3 it follows
from $T_x W^s_\delta(x) = E_s(x)$ that $T_x W^s(x) = E(x)$. Thus, $E(x) = E^s(x)$. Then, by the discretization of Theorem 2.3 using the terms introduced in [8], we obtain that $Dg: \bigcup_{x \in \Gamma} E^s(x) \to \bigcup_{x \in \Gamma} E^s(x)$ is uniformly contracting.

Similarly, we can show that $Dg: \bigcup_{x \in \Gamma} E^u(x) \to \bigcup_{x \in \Gamma} E^u(x)$ is uniformly expanding. Therefore, $g$ is uniformly hyperbolic on $\Gamma$. Since $\Omega(g) = \text{Per}(g)$, we have that $\Gamma = \Omega(g)$. Thus, $g$ is of Axiom A. Clearly, $g$ has no cycles. This proves the theorem.

5. Volume-preserving Katok maps

Let $\text{Leb}$ denote the standard volume measure of $\mathbb{M}^n$. A Borel probability measure $\mu$ on $M$ is called a smooth probability measure if $\mu$ is absolutely continuous with respect to $\text{Leb}$ such that

$$C \leq \frac{d\mu}{d\text{Leb}} \leq K$$

for some constants $C, K > 0$. If $g$ is a $C^{1+\alpha}$ volume-preserving Anosov diffeomorphism, $g$ is ergodic due to Anosov [1, 2]. For other proof of Anosov’s ergodicity theorem, see [20]. However, we are going to prove that if a Katok map preserves a smooth probability measure, then it is Anosov and thus also ergodic.

Using different approaches, it was proved independently by Bochi & Viana [4] and Xia [20] that the uniformly hyperbolic closed sets of every $C^{1+\alpha}$ volume-preserving diffeomorphisms have zero Lebesgue measure, unless they coincide with the whole ambient compact manifold (Anosov case). This result was generalized by using Pesin theory as follows:

**Lemma 5.1** ([11]). Let $f$ be a diffeomorphism preserving a smooth probability measure $\mu$ on a compact, connected, and closed Riemannian manifold $M$. Let $\Lambda \subset M$ be a uniformly hyperbolic invariant Borel set (not necessarily closed). If $\mu(\Lambda) > 0$, then $f$ is Anosov and $\Lambda = M$ (mod 0).

Now we can prove the ergodicity of a Katok map.

**Corollary 5.2.** Let $g$ be a $C^{1+\alpha}$ Katok diffeomorphism of $M$ which preserves a smooth probability measure $\mu$. Then $g$ is Anosov and ergodic with $\Omega(g) = M$.

**Proof.** Theorem B and Lemma 5.1 imply that $g$ is Anosov with $\Omega(g) = M$. This proves Corollary 5.2.

**Remark 5.3.** We noted that it was recently announced by Zhizhong Xia [18] that an Anosov diffeomorphism must be topologically transitive. Then, our Theorem B implies that every Katok diffeomorphism must be Anosov.

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