COMPARISONS BETWEEN FOURIER MULTIPLIERS AND STFT MULTIPLIERS: THE SMOOTHING EFFECT OF THE SHORT-TIME FOURIER TRANSFORM

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Abstract. We study the connection between STFT multipliers $A_{g_1,g_2}^{1\otimes m}$ having windows $g_1, g_2$, symbols $a(x,\omega) = (1 \otimes m)(x,\omega) = m(\omega), (x,\omega) \in \mathbb{R}^{2d}$, and the Fourier multipliers $T_{m_2}$ with symbol $m_2$ on $\mathbb{R}^d$. We find sufficient and necessary conditions on symbols $m, m_2$ and windows $g_1, g_2$ for the equality $T_{m_2} = A_{g_1,g_2}^{1\otimes m}$. For $m = m_2$ the former equality holds only for particular choices of window functions in modulation spaces, whereas it never occurs in the realm of Lebesgue spaces. In general, the STFT multiplier $A_{g_1,g_2}^{1\otimes m}$, also called localization operator, presents a smoothing effect due to the so-called two-window short-time Fourier transform which enters in the definition of $A_{g_1,g_2}^{1\otimes m}$. As a by-product we prove necessary conditions for the continuity of anti-Wick operators $A_{g,g}^{1\otimes m} : L^p \rightarrow L^q$ having multiplier $m$ in weak $L^r$ spaces. Finally, we exhibit the related results for their discrete counterpart: in this setting STFT multipliers are called Gabor multipliers whereas Fourier multiplier are better known as linear time invariant (LTI) filters.

1. Introduction

STFT multipliers, also called localization operators, have been introduced by Daubechies [15] and investigated by Ramanathan and Topiwala [41] as a mathematical tool to restrict functions to a region in the time-frequency plane and to extract time-frequency features. For this reason, they have been widely studied in signal analysis and other applications [28, 51]. Their discrete versions are known as Gabor multipliers. Motivated by the overlap-add implementation of convolution [39], in some signal processing software system linear filtering is implemented by using Gabor multipliers instead of working out a convolution in time domain like in e.g. STx [5, 52]. There are several reasons for doing that. First, they are easy to implement. Then, quasi real time processing with finite support windows is possible. Moreover, in order to implement a filter in a linear setting it is straightforward to do this by masking unwanted frequency components. In many chosen settings the results also

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closely match the expectations (see Example 6.4). The problem of representation and approximation of linear operators by means of Gabor multipliers (and suitable modifications) was studied by Dörfler and Torrésani in [18], further investigations are contained in [31, 42]. More generally, approximating problems for pseudodifferential operators via STFT multipliers (“wave packets” were exhibited in the work by Cordoba and Fefferman [16], see also Folland [30] and the PhD thesis [19]).

Special instances of localization operators are the so-called “Anti-Wick operators”, introduced earlier by Berezin [9] in the framework of quantum mechanics, details can be found in Shubin’s book [44].

STFT multipliers can be introduced by a time-frequency representation, the short-time Fourier transform (STFT), as follows. First, recall the modulation $M_{\omega}$ and translation $T_{x}$ operators of a function $f$ on $\mathbb{R}^d$:

$$M_{\omega} f(t) = e^{2\pi i t \omega} f(t), \quad T_{x} f(t) = f(t - x), \quad \omega, x \in \mathbb{R}^d.$$ 

Fix a non-zero Schwartz function $g \in S(\mathbb{R}^d) \setminus \{0\}$. We define the short-time Fourier transform of a tempered distribution $f \in S'(\mathbb{R}^d)$ as

$$V_g f(x, \omega) = \langle f, M_{\omega} T_{x} g \rangle = \mathcal{F}(f \cdot T_{x} g)(\omega) = \int_{\mathbb{R}^d} f(y) g(y - x) e^{-2\pi i y \omega} dy.$$ 

The STFT multiplier $A_{a}^{g_1, g_2}$ with symbol $a$, analysis window $g_1$, and synthesis window $g_2$ is formally defined to be

$$A_{a}^{g_1, g_2} f(t) = \int_{\mathbb{R}^{2d}} a(x, \omega) V_{g_1} f(x, \omega) M_{\omega} T_{x} g_2(t) dxd\omega.$$ 

If $g_1(t) = g_2(t) = e^{-\pi t^2}$, then the STFT multiplier $A_{a} = A_{a}^{g_1, g_2}$ becomes the classical Anti-Wick operator. We recall that the mapping $a \mapsto A_{a}^{g_1, g_2}$ is a quantization rule [9, 17, 44, 51].

For $\alpha, \beta > 0$, consider the lattice $\Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$, then a Gabor multiplier with windows $g_1, g_2 \in L^2(\mathbb{R}^d)$ can formally be defined as

$$G_{a}^{g_1, g_2} f = \sum_{k, n \in \mathbb{Z}^{2d}} a(\alpha k, \beta n) V_{g_1} f(\alpha k, \beta n) T_{\alpha k} M_{\beta n} g_2, \quad f \in L^2(\mathbb{R}^d),$$ 

see also [43]. Observe that a Gabor multiplier is the discrete version of a STFT multiplier; in fact it can be obtained from (2) by replacing the Lebesgue measure $dxd\omega$ with the discrete measure $\nu = \sum_{k, n \in \mathbb{Z}^{2d}} \delta_{\alpha k, \beta n}$; the integration with respect to $\nu$ becomes the summation

$$\int_{\mathbb{R}^{2d}} F(x, \omega) d\nu(x, \omega) = \sum_{k, n \in \mathbb{Z}^{2d}} F(\alpha k, \beta n).$$ 

Hence the main results for STFT multipliers (cf. Theorem 3.1) can be easily adapted to the framework of Gabor multipliers. Note that this is a particular instance of
a continuous frame multiplier, a (discrete) frame multiplier and their relation, see [3, 4, 6].

Fourier multipliers, also named linear time invariant (LTI) filters, are well known in both partial differential equations and signal analysis. They can be viewed as a special instance of Kohn-Nirenberg (KN) operators $T_m$ with KN symbol $m$ which depends only on the frequency variables $\omega \in \mathbb{R}^d$. Precisely, a Fourier multiplier with multiplier $m \in S'(\mathbb{R}^d)$ is defined by

$$T_m f(t) = \mathcal{F}^{-1}(m \mathcal{F} f)(t) = (\mathcal{F}^{-1} m * f)(t), \quad f \in \mathcal{S}(\mathbb{R}^d).$$

In Section 6 we shall adopt the notation

$$h = \mathcal{F}^{-1} m$$

both for the continuous and finite discrete setting. This is called the transfer function in signal processing [39].

Such operator is a well-defined linear mapping from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. Boundedness properties of Fourier multipliers $T_m$:

$$L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$$

are studied in the classical paper by Hörmander [35]. The most important examples of Fourier multipliers can be obtained by taking $p = q = 2$. Then $T_m$ is bounded if and only if the multiplier $m \in L^\infty(\mathbb{R}^d)$ and $\|T_m\|_{B(L^2)} = \|m\|_\infty$. For $p = q = 1$ and $p = q = \infty$ the only bounded Fourier multipliers are Fourier transforms of bounded measures. For the cases $p = q \in (1, \infty) \setminus \{2\}$ only sufficient conditions on $m$ are known. The assumptions $m \in L^\infty$ is necessary, though. The main result by Hörmander in [35, Theorem 1.11] (see also its generalization to locally compact groups [1]) states:

**Theorem 1.1.** If $1 < p \leq 2 \leq q < \infty$, $m \in L^{r,\infty}(\mathbb{R}^d)$ with

$$\frac{1}{q} = \frac{1}{r} + \frac{1}{p},$$

then $T_m$ is bounded $T_m : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$.

Here $L^{r,\infty}(\mathbb{R}^d)$ is the weak $L^r$-space, see (14) in the Preliminaries below. For example, every $m$ on $\mathbb{R}^d$ with $|m(\omega)| \leq C|\omega|^{-d/r}$, $C > 0$, satisfies $m \in L^{r,\infty}(\mathbb{R}^d)$. For simplicity, we define $L^{\infty,\infty}(\mathbb{R}^d) := L^\infty(\mathbb{R}^d)$, so that, inserting $r = \infty$ in Theorem 1.1 we recapture the boundedness of the multiplier $T_m$ on $L^2(\mathbb{R}^d)$.

In signal analysis, both Gabor and Fourier multipliers are introduced to localize a signal; the former in the time-frequency space, the latter only in the frequency space. Since, as we said above, Gabor multipliers are easy to be implemented numerically, the mathematical question is to determine under which conditions a Gabor multiplier is equivalent to a linear time invariant filter. More generally, we aim at answering the following question:

Given a STFT multiplier $A_{g_1,g_2}^a$ with symbol $a(x,\omega) = (1 \otimes m)(x,\omega) = m(\omega)$, $x,\omega \in \mathbb{R}^d$ (a depends only on the frequency variable $\omega \in \mathbb{R}^d$), is it possible to write it in the form of a Fourier multiplier?
We study the equality
\[ A_{g_1,g_2}^m = T_{m_2} \text{ on } \mathcal{S}(\mathbb{R}^d). \]

In order to give a flavour of our results, we need to introduce a new window function which correlates \( g_1 \) and \( g_2 \). Recall the reflection operator \( I \) of a function \( f \) on \( \mathbb{R}^d \)
\[ I f(t) = f(-t), \quad t \in \mathbb{R}^d. \]
For \( g_1, g_2 \in L^2(\mathbb{R}^d) \), the window correlation function of the pair \( (g_1, g_2) \) is defined by
\[ C_{g_1,g_2}(y) = (I g_2 \ast \bar{g}_1)(y), \quad y \in \mathbb{R}^d \]
(observe that \( C_{g_1,g_2} \) is a continuous function on \( \mathbb{R}^d \)). The window correlation function enjoys several properties depending on the function/distribution space of the windows \( g_1, g_2 \), cf. Proposition 2.6 in the sequel.

The equality (7) is possible if and only if
\[ m_2 = m \ast F^{-1}(C_{g_1,g_2}), \]
with \( m, m_2 \in \mathcal{S}'(\mathbb{R}^d), g_1, g_2 \in \mathcal{S}(\mathbb{R}^d) \) or other suitable function spaces, see Theorem 3.1.

In particular, if we choose \( m = m_2 \) the equality (7) holds for any multiplier \( m \in \mathcal{S}(\mathbb{R}^d) \) if and only if
\[ C_{g_1,g_2} = 1 \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d). \]
Condition (11) above is very restrictive, so that (7) never holds for classical anti-Wick operators, whose Gaussian windows provide a smoothing effect we shall explain presently.

First, we recall that the Hörmander’s condition \( p \leq 2 \leq q \) in Theorem 1.1 is sharp. More precisely, if there exists a function \( F \) such that \( \{ F > 0 \} \) has non-zero measure and for all \( m : \mathbb{R}^d \to \mathbb{R} \) with \( |m| \leq |F|, T_m : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d) \) is bounded, then \( p \leq 2 \leq q \) (cf. [35, Theorem 1.12]). Moreover, also (6) is necessary by the \( L^p \) inequalities for potentials (see [46, pag. 119]). We shall present a direct proof by rescaling arguments of the following necessary condition (see Section 4):

**Proposition 1.2.** For \( p, q, r \in (1, \infty] \) we assume that the Fourier multiplier \( T_m \) satisfies
\[ \|T_mf\|_q \leq C\|m\|_{L^{r,\infty}}\|f\|_p, \quad \text{for every } f, m \in \mathcal{S}(\mathbb{R}^d), \]
then we must have the indices’ relation:
\[ \frac{1}{q} \leq \frac{1}{r} + \frac{1}{p}. \]
In this paper we also investigate the smoothing effects of the anti-Wick operator $A_{1 \otimes m}^{g,g}$ with respect to the corresponding Fourier multiplier $T_m$. It can be stated as follows (see the proof in Section 5). Please note the similarity (and differences) to Hörmander’s result, Theorem 1.1.

**Theorem 1.3.** If $1 < p \leq 2 \leq q < \infty$, $m \in L^{r,\infty}(\mathbb{R}^d)$ with indices satisfying (13), then the anti-Wick operator $A_{1 \otimes m}^{g,g}$ is bounded from $L^p(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$.

The previous result holds true for more general STFT multipliers $A_{1 \otimes m}^{g_1,g_2}$ with $g_1, g_2 \in S'(\mathbb{R}^d)$ such that the window correlation function satisfies $C_1,g_2 \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, cf. Theorem 5.1 in Section 5 below. For $p = 2$, the boundedness of the Fourier multiplier $T_m$ in Theorem 1.1 forces the indices’ choice: $q = 2$ and $r = \infty$, whereas condition in (13) is more flexible, allowing to choose $q \geq 2$ and $r \leq \infty$.

The necessity of condition (13) for anti-Wick operators is proved in Theorem 5.3.

We conjecture that other possible smoothing effects could be shown by replacing $L^p$ and $L^{r,\infty}$ with Wiener amalgam spaces (cf. [23]). This will be the subject of future investigations.

The connection between Fourier and Gabor multipliers was studied earlier by Weisz in [50]. The focus is different and can be viewed in our framework as follows: if a symbol $m$ gives rise to a Fourier multiplier $T_m$ which is bounded on $L^p(\mathbb{R}^d)$, then the STFT multiplier $A_{1 \otimes m}^{g_1,g_2}$ is also bounded on $L^p$ (and more generally, on Wiener amalgam spaces cf. [50, Theorem 8]), provided the windows are smooth enough, that is, are included in suitable Wiener amalgam spaces containing the modulation space $M^1$.

For applications, we will study the finite dimensional discrete setting, considering signals $f \in \mathbb{C}^N$. The problems under investigation are similar to the ones for the continuous setting, the tools at hand however are sometimes different. Where necessary, for a better understanding, we will also investigate an in-between step and prove results for the infinite-dimensional discrete setting, i.e. $\ell^2$.

### 2. Preliminaries

**Notations.** In this paper $\hookrightarrow$ denotes the continuous embeddings of function spaces. The conjugate exponent $p'$ of $p \in [1, \infty]$ is defined by $1/p + 1/p' = 1$.

For $r \in [1, \infty)$, the **weak** $L^r$ space $L^{r,\infty}(\mathbb{R}^d)$ is the space of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|f\|_{L^{r,\infty}} := \sup_{\alpha > 0} \lambda_f(\alpha)^{\frac{1}{r}} < \infty,$$

where $\lambda_f(\alpha) := \mu(\{t \in \mathbb{R}^d : |f(t)| > \alpha\})$, $\alpha > 0$, with $\mu$ being the Lebesgue measure (see, e.g., [48]).

Note that the quantity in (14) is only a quasi-norm.
For convenience, we write $L^{\infty,\infty}(\mathbb{R}^d) := L^\infty(\mathbb{R}^d)$. Observe that weak $L^r$ spaces are special instances of Lorentz spaces and $L^r(\mathbb{R}^d) \subseteq L^{r,\infty}(\mathbb{R}^d)$, $1 \leq r \leq \infty$.

For $t = (t_1, \ldots, t_d), \omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$, the inner product is denoted by $t \omega = t \cdot \omega = t_1 \omega_1 + \ldots + t_d \omega_d$. So that we adopt the notation $t^2 = |t|^2 = t_1^2 + \ldots + t_d^2$.

For $f \in L^1(\mathbb{R}^d)$ the Fourier transform is normalized to be

$$\mathcal{F}f(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^d} e^{-2\pi i t \omega} f(t) \, dt.$$  

**Weight functions.** We denote by $v$ a continuous, positive, even, submultiplicative weight function on $\mathbb{R}^d$, i.e., $v(z_1 + z_2) \leq v(z_1)v(z_2)$, for all $z_1, z_2 \in \mathbb{R}^d$. We say that $w \in \mathcal{M}_v(\mathbb{R}^d)$ if $w$ is a positive, continuous, even weight function on $\mathbb{R}^d$ which is $v$-moderate: $w(z_1 + z_2) \leq C v(z_1)w(z_2)$, for all $z_1, z_2 \in \mathbb{R}^d$ and some $C > 0$. We will mainly work with polynomial weights of the type

$$v_s(z) = \langle z \rangle^s = (1 + |z|^2)^{s/2}, \quad s \in \mathbb{R}$$

(for $s < 0$, $v_s$ is $v_{|s|}$-moderate).

Given two weight functions $w_1, w_2$ on $\mathbb{R}^d$, we write

$$(w_1 \otimes w_2)(x, \omega) := w_1(x)w_2(\omega), \quad x, \omega \in \mathbb{R}^d.$$  

**Modulation spaces.** These spaces were introduced by one of the authors in [23], where many of their properties were already investigated. Nowadays, they are treated in many textbooks, see, e.g. [8, 14]. For a general extension to the quasi-Banach setting on locally compact Abelian groups we mention the recent [7].

Fix a non-zero window $g$ in the Schwartz class $S(\mathbb{R}^d)$, a weight $w \in \mathcal{M}_v$ and $1 \leq p, q \leq \infty$. The modulation space $M^{p,q}_w(\mathbb{R}^d)$ consists of all tempered distributions $f \in S'(\mathbb{R}^d)$ such that the norm

$$\|f\|_{M^{p,q}_w} = \|V_g f\|_{L^{p,q}_w} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \omega)|^p w(x, \omega)^q \, dx \right)^{\frac{2}{p}} \, d\omega \right)^{\frac{1}{q}}$$

(natural changes with $p = \infty$ or $q = \infty$) is finite. If $p = q$, we write $M^p_w(\mathbb{R}^d)$ instead of $M^{p,p}_w(\mathbb{R}^d)$; if $w \equiv 1$, we write $M^p(\mathbb{R}^d)$ in place of $M^{1,q}_1(\mathbb{R}^d)$.

The space $M^{p,q}(\mathbb{R}^d)$ is a Banach space whose definition is independent of the choice of the window $g \in S(\mathbb{R}^d)$, that is, different non-zero window functions in the Schwartz class yield equivalent norms. Furthermore, the window class can be extended to the modulation space $M^1(\mathbb{R}^d)$, also known as Feichtinger’s algebra, see [37]. The modulation space $M^{\infty,1}(\mathbb{R}^d)$ is also called Sjöstrand’s class [15].

For any $p, q \in [1, \infty]$, the inner product $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d)} = \langle \cdot, \cdot \rangle$ restricted to $S(\mathbb{R}^d) \times S(\mathbb{R}^d)$ extends to a continuous sesquilinear map $M^{p,q}(\mathbb{R}^d) \times M^{p',q'}(\mathbb{R}^d) \to \mathbb{C}$.

For $1 \leq p, q < \infty$, the duality property for $M^{p,q}(\mathbb{R}^d)$ is given by

$$(M^{p,q}_w)'(\mathbb{R}^d) = M^{p',q'}_{1/w}(\mathbb{R}^d),$$
with \( p', q' \) being the conjugate exponents and

\[
\langle f, g \rangle = \int_{\mathbb{R}^d} V_h f(z) \overline{V_h g(z)} dz, \quad f \in M_{w}^{p,q}(\mathbb{R}^d), g \in M_{1/w}^{p',q'}(\mathbb{R}^d),
\]

\( w \in \mathcal{M}_w, \) for any fixed \( h \in M^1_{\mathbb{R}}(\mathbb{R}^d) \setminus \{0\}. \) Observe that Hölder’s inequality for \( L^p_m \) spaces satisfies the following inclusion properties:

\[
\text{(17)} \quad \langle f, g \rangle \leq \|f\|_{M_{w}^{p,q}} \|g\|_{M_{1/w}^{p',q'}}, \quad f \in M_{w}^{p,q}(\mathbb{R}^d), g \in M_{1/w}^{p',q'}(\mathbb{R}^d).
\]

Proposition 2.1. Let \( \nu(\omega) > 0 \) be an even weight function on \( \mathbb{R}^d. \) Furthermore let \( 1 \leq p, q, r, t, u, \gamma \leq \infty, \) with

\[
\text{(20)} \quad \frac{1}{u} + \frac{1}{t} \geq \frac{1}{\gamma},
\]

and

\[
\text{(21)} \quad \frac{1}{p} + \frac{1}{q} \geq 1 + \frac{1}{r}.
\]

For given \( w \in \mathcal{M}_w(\mathbb{R}^{2d}), \) let \( w_1 \) and \( w_2 \) be the restriction to \( \mathbb{R}^d \times \{0\} \) and \( \{0\} \times \mathbb{R}^d, \) respectively, i.e \( w_1(x) := w(x,0) \) and \( w_2(\omega) := w(0,\omega). \) Define \( v_1 \) and \( v_2 \) in a similar way. If \( f \in M_{w_1 \otimes v_1}^{p,u}(\mathbb{R}^d), h \in M_{v_1 \otimes v_2 - 1}^{q,t}(\mathbb{R}^d) \) then \( f \ast h \in M_{w}^{r,\gamma}(\mathbb{R}^d) \) with norm inequality

\[
\text{(22)} \quad \|f \ast h\|_{M_{w}^{r,\gamma}} \lesssim \|f\|_{M_{w_1 \otimes v_1}^{p,u}} \|h\|_{M_{v_1 \otimes v_2 - 1}^{q,t}}.
\]

Proposition 2.2. Consider \( 1 \leq p, q \leq \infty, \) with \( p', q' \) being conjugate exponents of \( p, q, \) respectively.

(i) For \( 1 \leq p, q \leq \infty, \) \( f \in M_{w_1}^{p,q}(\mathbb{R}^d), \) \( h \in M_{1/(w_1 \otimes v_2)}^{p',q'}(\mathbb{R}^d), \) we have that \( f \ast h \in C_0(\mathbb{R}^d). \)

(ii) For \( 1 < p, q < \infty, \) \( f \in M_{w_1}^{p,q}(\mathbb{R}^d), \) \( h \in M_{1/(w_1 \otimes v_2)}^{p',q'}(\mathbb{R}^d), \) we have that \( f \ast h \in C_0(\mathbb{R}^d). \)

(iii) If either \( f \in M^{\infty,1}_{w_1}(\mathbb{R}^d) \) and \( h \in M^{1,\infty}(\mathbb{R}^d) \) or \( f \in M_{w_1}^{1,\infty}(\mathbb{R}^d) \) and \( h \in M^{\infty,1}(\mathbb{R}^d), \) then \( f \ast h \in C_0(\mathbb{R}^d). \)
Proof. This result is well known, see [21] and [22]. For sake of clarity we provide a direct proof.

(i) Using the density of $S(\mathbb{R}^d)$ in both spaces we can find sequences $\{f_n\}, \{h_n\} \subset S(\mathbb{R}^d)$ such that $\|f_n - f\|_{M^{p,q}} \to 0$ and $\|h_n - h\|_{M^{p',q'}} \to 0$, now $f_n * h_n \in S(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d)$ so that, using

$$|f * h(t)| = |\langle f, T_t I(h) \rangle| \leq \|f\|_{M^{p,q}} \|T_t I(h)\|_{M^{p',q'}} = \|f\|_{M^{p,q}} \|h\|_{M^{p',q'}} , \quad \forall t \in \mathbb{R}^d,$$

$$\|f_n * h_n - f * h\|_{L^\infty} \leq \|f_n * (h_n - h)\|_{L^\infty} + \|(f_n - f) * h\|_{L^\infty} \leq \|f_n\|_{M^{p,q}} \|h_n - h\|_{M^{p',q'}} + \|f_n - f\|_{M^{p,q}} \|h\|_{M^{p',q'}}.$$  

Hence $f * h \in C_0(\mathbb{R}^d)$. Item (ii) is obtained by the same argument as in (i).

(iii) Using the convolution relations of Proposition 2.1 we infer

$$M^{\infty,1}(\mathbb{R}^d) * M^{1,\infty}(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d) \quad \text{and} \quad M^1(\mathbb{R}^d) * M^{\infty}(\mathbb{R}^d) \subset M^{1,\infty}(\mathbb{R}^d).$$

It follows immediately from the definition of the modulation space $M^{\infty,1}(\mathbb{R}^d)$ that

$$M^{\infty,1}(\mathbb{R}^d) \subset (\mathcal{F} L^1(\mathbb{R}^d))_{\text{loc}} \cap L^\infty(\mathbb{R}^d) \subset C_b(\mathbb{R}^d).$$

and we are done. \hfill \Box

Remark 2.3. We observe that the convolution relations

$$M^1(\mathbb{R}^d) * M^{\infty}(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$$

where already shown in [27], Lemma 8.

Here we show an optimal result for $M^{p,q}$-boundedness (and in particular $L^2$-boundedness) of STFT multipliers. We extend Theorem 5.2 in [13] and Theorem 1.1 in [12].

Theorem 2.4. Consider $s \geq 0, p_1, p_2, q_1, q_2 \in [1, \infty]$, with $1/p_1 + 1/p_2 \geq 1, 1/q_1 + 1/q_2 \geq 1$. If $g_1 \in M^{p_1,q_1}_{v_s}(\mathbb{R}^d)$, $g_2 \in M^{p_2,q_2}_{v_s}(\mathbb{R}^d)$, and $a \in M^{\infty,1}(\mathbb{R}^{2d})$, then $A_t^{a,g_2}$ is bounded on every $M^{p,q}_{v_s}(\mathbb{R}^d)$, $p,q \in [1, \infty]$. In particular, the operator $A_t^{a,g_2}$ is bounded on the Shubin-Sobolev space $Q_s := M^{2}_v$. (In particular, for $s = 0$, $A_t^{a,g_2}$ is bounded on $L^2(\mathbb{R}^d)$.)

Proof. If $g_1 \in M^{p_1,q_1}_{v_s}(\mathbb{R}^d)$, $g_2 \in M^{p_2,q_2}_{v_s}(\mathbb{R}^d)$ with $1/p_1 + 1/p_2 \geq 1, 1/q_1 + 1/q_2 \geq 1$, by [11], Theorem 4] we infer that their cross-Wigner distribution

$$W(g_2, g_1)(x, \omega) = \int_{\mathbb{R}^d} g_2 \left( x + \frac{t}{2} \right) g_1 \left( x - \frac{t}{2} \right) e^{-2 \pi i t \omega} dt$$

is in $M^{1,\infty}_{1\otimes v_s}(\mathbb{R}^{2d})$. Rewriting the STFT multiplier $A_t^{a,g_2}$ as a Weyl operator $L_a$ with $\sigma = a * W(g_2, g_1)$, the convolution relations for modulation spaces in Proposition 2.1 give

$$\sigma \in M^{\infty,1}(\mathbb{R}^{2d}) * M^{1,\infty}_{1\otimes v_s}(\mathbb{R}^{2d}) \subset M^{1,\infty}_{1\otimes v_s}(\mathbb{R}^{2d}).$$
The result follows by the continuity properties of Weyl operators in [13, Theorem 5.2]. □

For sake of completeness let us recall [12, Corollary 4.2]:

**Proposition 2.5.** If $a \in M^{\infty}(\mathbb{R}^d)$ and $g_1, g_2 \in M_1^1(\mathbb{R}^d)$, $w \in M_v$, then $A^{g_1, g_2}_w$ is bounded on $M_p^q(\mathbb{R}^d)$ for $1 \leq p, q \leq \infty$. In particular, it is bounded on $L^2(\mathbb{R}^d)$.

2.1. **Correlation functions.** For $g_1, g_2 \in L^2(\mathbb{R}^d)$, let us introduce the so-called **shifted window correlation function** of the pair $(g_1, g_2)$:

$$G_{g_1, g_2}(t, y) = \int_{\mathbb{R}^d} g_2(t - u) \overline{g_1(y - u)} \, du.$$  

It is straightforward to show that $G_{g_1, g_2} \in L^\infty(\mathbb{R}^{2d})$. Observe that the function $G_{g_1, g_2}$ is well defined also for windows $g_1, g_2$ belonging to function/distributions spaces other than $L^2(\mathbb{R}^d)$ (it can be proved by using Proposition 2.6).

We can rewrite the shifted window correlation function $G_{g_1, g_2}$ on $\mathbb{R}^{2d}$ as a **time shift** of the mapping $C_{g_1, g_2}$ on $\mathbb{R}^d$ defined in (9).

In fact, a straightforward computation shows that

$$G_{g_1, g_2}(t, y) = C_{g_1, g_2}(y - t) = T_t C_{g_1, g_2}(y), \quad t, y \in \mathbb{R}^d.$$  

Let us study the properties of $C_{g_1, g_2}$.

**Proposition 2.6.** The window correlation function $C_{g_1, g_2}$ enjoys the following properties.

(i) If $g_1, g_2 \in S(\mathbb{R}^d)$, then $C_{g_1, g_2} \in S(\mathbb{R}^d)$.

(ii) If either $g_1$ is in $S'(\mathbb{R}^d)$ and $g_2 \in S(\mathbb{R}^d)$ or $g_1 \in S(\mathbb{R}^d)$ and $g_2 \in S'(\mathbb{R}^d)$ then $C_{g_1, g_2} \in C(\mathbb{R}^d)$ with at most polynomial growth.

(iii) If $g_1 \in L^p(\mathbb{R}^d)$, $g_2 \in L^{p'}(\mathbb{R}^d)$, with $1 < p < \infty$, $1/p + 1/p' = 1$, then $C_{g_1, g_2} \in C_b(\mathbb{R}^d)$. If either $p = 1$ ($p' = \infty$) or $p = \infty$ ($p' = 1$) then $C_{g_1, g_2} \in C_b(\mathbb{R}^d)$. The same statements hold if we replace the Lebesgue space $L^p(\mathbb{R}^d)$ (resp. $L^{p'}(\mathbb{R}^d)$) with the modulation space $M^p(\mathbb{R}^d)$ (resp. $M^{p'}(\mathbb{R}^d)$).

(iv) If $g_1 \in M^p_{\mu, u} L^q_{\nu, v^\gamma}(\mathbb{R}^d)$, $g_2 \in M^{q', t}_{\nu, v^\gamma - 1}(\mathbb{R}^d)$, with $1 \leq p, q, r, t, u, \gamma \leq \infty$ satisfying [20] and [21], and the weights as in the assumptions of Proposition 2.1, then $C_{g_1, g_2}$ is in $M^r_{\nu, v^\gamma}(\mathbb{R}^d)$, with norm inequality

$$\|C_{g_1, g_2}\|_{M^r_{\nu, v^\gamma}} \leq \|g_1\|_{M^p_{\mu, u}} \|g_2\|_{M^{q', t}_{\nu, v^\gamma - 1}}.$$  

Proof. The proofs of items (i), (ii) follow by the convolution properties for the Schwartz class $S$, its dual $S'$ respectively, see, e.g., the textbooks [29;34]. Item (iii) is a consequence of the convolution properties for $L^p(\mathbb{R}^d)$ spaces which can be found e.g., in [29;34]. For modulation spaces $M^p$ we use the convolution properties in Proposition 2.2.
(iv). By assumption all the weights under consideration are even, so that \( I g_2 \in M^{q,t}_{v_1 \otimes v_2 - 1}(\mathbb{R}^d) \) whenever \( g_2 \in M^{q,t}_{v_1 \otimes v_2 - 1}(\mathbb{R}^d) \). Moreover modulation spaces are closed under complex conjugation, hence the result immediately follows by applying the convolution relations in Proposition 2.1. \( \square \)

Example 2.7. In what follows we exhibit examples of window correlation functions.

(i) Consider two \( L^2 \)-normalized Gaussian functions \( g_1(t) = g_2(t) = 2^{d/4} e^{-\pi t^2}, \ t \in \mathbb{R}^d \). In this case, the window correlation function \( C_{g_1,g_2} \) in \((9)\) is a Gaussian as well

\[
C_{g_1,g_2}(t) = I(g_1 * I(g_2))(t) = 2^{d/2} (e^{\pi(t)^2} * e^{\pi(t)^2})(-t) = e^{-\pi t^2}, \ t \in \mathbb{R}^d.
\]

(ii) Consider \( g_1 = \chi_{[0,1]^d}; \ g_2(t) = 1, \) for every \( t \in \mathbb{R}^d \). Observe \( g_1 \in L^1(\mathbb{R}^d), g_2 \in L^\infty(\mathbb{R}^d) \). Then the window correlation function becomes

\[
C_{g_1,g_2}(t) = g_1 * I(g_2)(-t) = \int_{[0,1]^d} dy = 1, \ \forall t \in \mathbb{R}^d.
\]

2.2. \( L^{r,\infty} \) quasi-norms of rescaled Gaussians.

Lemma 2.8. For \( r \in [1, \infty), \lambda > 0 \) and \( g(t) = e^{-\pi t^2}, \ t \in \mathbb{R}^d \), we consider the rescaled Gaussians \( g_\lambda(t) := e^{-\pi \lambda t^2} \). Then we have

\[
\|g_\lambda\|_{L^{r,\infty}(\mathbb{R}^d)} = \frac{\left(\frac{d}{2r}\right)^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right) \lambda^{d/2}} e^{-\frac{d}{2r}}.
\]

Hence,

\[
\|g_\lambda\|_{L^{r,\infty}(\mathbb{R}^d)} = C(d, r) \lambda^{-\frac{d}{2r}},
\]

with \( C(d, r) = e^{-\frac{d}{2r}} \left(\frac{d}{2r}\right)^{d/2} \Gamma\left(\frac{d}{2} + 1\right)^{-1} \).

Proof. Observe that for \( \alpha \geq 1 \) we have \( \{t : |g_\lambda(t)| > \alpha\} = \emptyset \). For \( 0 < \alpha < 1 \), \( \{t : |g_\lambda(t)| > \alpha\} = \{t : |t| < \pi^{-1/2} \lambda^{-1/2} (\log(1/\alpha))^{1/2}\} \). The Lebesgue measure of the set is given by

\[
A_\lambda := \mu(\{t : |t| < \pi^{-1/2} \lambda^{-1/2} (\log(1/\alpha))^{1/2}\}) = \frac{\log(1/\alpha)^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right) \lambda^{d/2}}.
\]

Now, using the definition of the quasi-norm in \((14)\),

\[
\|g_\lambda\|_{L^{r,\infty}(\mathbb{R}^d)} = \sup_{\alpha > 0} \alpha \mu(\{t : |g_\lambda(t)| > \alpha\})^{1/r}
\]

\[
= \sup_{0 < \alpha < 1} \alpha A^{1/r}_\lambda
\]

\[
= \frac{1}{\Gamma\left(\frac{d}{2} + 1\right) \lambda^{d/2r}} \sup_{0 < \alpha < 1} \alpha (\log(1/\alpha))^{d/2r}.
\]
An easy computation shows that the function \( y(\alpha) := \alpha (\log(1/\alpha))^{\frac{d}{2}} \) on \((0, 1)\) admits the maximum point \( t_M := e^{-\frac{d}{2}} \) and the maximum is \( y(t_M) = (d/(2r))^{2/(2r)} e^{-2/(2r)} \), so that we obtain the claim. \( \square \)

We observe that in the \( L^{r,\infty} \) spaces the rescaled Gaussians behave like in the usual \( L^r \) spaces, meaning \( \|g_{\lambda}\|_r \leq \|g_{\lambda}\|_{L^{r,\infty}} \approx \lambda^{d/(2r)} \).

3. Study the equality \( A_{1\otimes m}^{g_1,g_2} = T_{m_2} \).

The following issue can be viewed as the answer of the question raised in the introduction.

**Theorem 3.1.** Fix multiplier symbols \( m, m_2 \in \mathcal{S}'(\mathbb{R}^d) \) (resp. \( m, m_2 \in M^\infty(\mathbb{R}^d) \)) and windows \( g_1, g_2 \in \mathcal{S}(\mathbb{R}^d) \) (resp. in \( M^1(\mathbb{R}^d) \)). Then the equality

\[
A_{1\otimes m}^{g_1,g_2} = T_{m_2} \quad \text{on} \quad \mathcal{S}(\mathbb{R}^d) \quad \text{(resp.} \quad M^1(\mathbb{R}^d))
\]

holds if and only if

\[
m_2 = m \ast \mathcal{F}^{-1}(C_{g_1,g_2}) \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d) \quad \text{(resp.} \quad M^\infty(\mathbb{R}^d)).
\]

The same conclusions hold under the following assumptions:

(i) The symbols \( m, m_2 \) in \( \mathcal{S}(\mathbb{R}^d) \) (resp. in \( M^1(\mathbb{R}^d) \)) and the window functions \( (g_1, g_2) \) in \( \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \) (resp. \( M^\infty(\mathbb{R}^d) \times M^1(\mathbb{R}^d) \));

(ii) The symbols \( m, m_2 \) in \( \mathcal{S}(\mathbb{R}^d) \) (resp. in \( M^1(\mathbb{R}^d) \)) and the window functions \( (g_1, g_2) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d) \) (resp. \( M^1(\mathbb{R}^d) \times M^\infty(\mathbb{R}^d) \)).

**Proof.** Assume \( m, m_2 \in \mathcal{S}'(\mathbb{R}^d) \) and \( (g_1, g_2) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \). First, we show that the operators \( A_{1\otimes m}^{g_1,g_2} \) and \( T_{m_2} \) are well defined and continuous from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \).

For every \( f, g \in \mathcal{S}(\mathbb{R}^d) \), the weak definition of STFT multiplier \([35]\) and the standard properties of the STFT give the result, since \( V_{g_1}f \in \mathcal{S}(\mathbb{R}^d) \) and \( V_{g_2}g \in \mathcal{S}(\mathbb{R}^d) \) and the mappings \( V_{g_1}, V_{g_2} \) are continuous on \( \mathcal{S}(\mathbb{R}^d) \), see for example \([13, \text{Chapter 1}]\). For the Fourier multiplier we use the continuity of \( \mathcal{F} \) (resp. \( \mathcal{F}^{-1} \)) on \( \mathcal{S}(\mathbb{R}^d) \) (resp. \( \mathcal{S}'(\mathbb{R}^d) \)) and of the product \( \mathcal{S}(\mathbb{R}^d) \cdot \mathcal{S}'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \).

Writing them as integral operators we obtain

\[
A_{1\otimes m}^{g_1,g_2} f(t) = \int_{\mathbb{R}^d} K_A(t,y)f(y)dy,
\]

with kernel

\[
K_A(t,y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i(t-x)\omega} \hat{m}(\omega) g_2(t-x) \bar{g_1(y-x)} \, dx \, d\omega
\]

\[
= \hat{m}(y-t) \mathcal{C}_{g_1,g_2}(y-t) = T_1(\hat{m} \mathcal{C}_{g_1,g_2})(y),
\]

and

\[
T_{m_2} f(t) = \int_{\mathbb{R}^d} K_B(t,y)f(y)dy,
\]

where

\[
K_B(t,y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i(t-x)\omega} m(\omega) g_2(t-x) \bar{g_1(y-x)} \, dx \, d\omega.
\]
with kernel

\[(33) \quad K_B(t, y) = \int_{\mathbb{R}^d} e^{2\pi i (t-y) \omega} m_2(\omega) d\omega = \hat{m}_2(y-t) = T_t \hat{m}_2(y).\]

By the Schwartz’ kernel theorem the operators \(A_{g_1, g_2}^{q_1, q_2}\) and \(T_{m_2}\) coincide if and only if their kernels \(K_A\) and \(K_B\) coincide in \(\mathcal{S}'(\mathbb{R}^{2d})\). Equating the kernels we obtain \((41)\).

Consider now case (i): \(m, m_2 \in \mathcal{S}(\mathbb{R}^d)\) and \((g_1, g_2) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)\). We use similar arguments as above, observing that the STFT \(V_{g_1} f \in \mathcal{S}'(\mathbb{R}^{2d})\) for every \(f \in \mathcal{S}(\mathbb{R}^d)\) (cf. [14, Chapter 1]). The case (ii) is analogous and left to the reader.

Second, assume \(m, m_2 \in M^\infty(\mathbb{R}^d), g_1, g_2 \in M^1(\mathbb{R}^d)\). We use the same arguments as in the first step, simply replacing \(S\) with \(M^1\) and its dual \(S'\) with \((M^1)' = M^\infty\). Hence we obtain that \(T_{m_2}\) and the STFT multiplier \(A_{g_1, g_2}^{q_1, q_2}\) are well-defined linear and bounded operators from \(M^1(\mathbb{R}^d)\) into \(M^\infty(\mathbb{R}^d)\). Rewriting them as integral operators and using the kernel theorem in the framework of modulation spaces [20, 24] we come up to the result. The cases: (i) \(m, m_2 \in M^1(\mathbb{R}^d), g_1 \in M^1(\mathbb{R}^d) g_2 \in M^\infty(\mathbb{R}^d)\), (ii) \(m, m_2 \in M^1(\mathbb{R}^d), g_1 \in M^\infty(\mathbb{R}^d) g_2 \in M^1(\mathbb{R}^d)\) are similar. \(\square\)

In this case the symbol \(m\) of the STFT multiplier is smoothed by the convolution with the Fourier transform of the window correlation function \(\mathcal{C}_{g_1, g_2}\) and the result is a multiplier symbol \(m_2\) of \(T_{m_2}\) smoother than \(m\). For example, if you consider \(m \in M^\infty(\mathbb{R}^d), g_1, g_2 \in M^1(\mathbb{R}^d)\), as explained in Proposition 2.6 (iv), then we have

\[m_2 = m * \mathcal{F}^{-1}(\mathcal{C}_{g_1, g_2}) \in M^\infty(\mathbb{R}^d) * \mathcal{F}^{-1} M^1(\mathbb{R}^d).\]

Using the convolution property in Proposition 2.1

\[(34) \quad m_2 \in M^\infty(\mathbb{R}^d) * \mathcal{F}^{-1} M^1(\mathbb{R}^d) = M^\infty(\mathbb{R}^d) \ast M^1(\mathbb{R}^d) \subset M^{\infty-1}(\mathbb{R}^d) \subset \mathcal{C}_b(\mathbb{R}^d)\]

and we infer that the multiplier symbol \(m_2\) belongs to \(\mathcal{C}_b(\mathbb{R}^d)\). Then one can play with the convolution properties for modulation (and other function) spaces to obtain a Fourier multipliers’ symbol \(m_2\) in different function spaces.

For applications it is often useful to consider windows \(g_1, g_2 \in L^2(\mathbb{R}^d)\) and multiplier \(m \in L^\infty(\mathbb{R}^d)\). In this case the multiplier \(m_2\) enjoys the smoothing below.

**Lemma 3.2.** Assume \(g_1, g_2 \in L^2(\mathbb{R}^d), m \in L^\infty(\mathbb{R}^d)\). Then \(m_2\) as in \((30)\) belongs to \(\mathcal{C}_b(\mathbb{R}^d)\).

**Proof.** For \(g_1, g_2 \in L^2(\mathbb{R}^d), \) the window correlation function satisfies \(\mathcal{F}^{-1} \mathcal{C}_{g_1, g_2} \in L^1(\mathbb{R}^d)\), since \(\mathcal{I} g_2, \mathcal{I} g_1 \in L^2(\mathbb{R}^d)\) and

\[\mathcal{F}^{-1}(\mathcal{C}_{g_1, g_2}) \in \mathcal{F}^{-1}(L^2(\mathbb{R}^d) * L^2(\mathbb{R}^d)) = \mathcal{F}^{-1} L^2(\mathbb{R}^d) * \mathcal{F}^{-1} L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d) * L^2(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)\]

Hence, by Proposition 2.6 (iii) we obtain

\[m_2 \in L^\infty(\mathbb{R}^d) * L^1(\mathbb{R}^d) \subset \mathcal{C}_b(\mathbb{R}^d),\]

as desired. \(\square\)
4. Study the equality $A_{1,g_2}^{g_1,g_2} = T_m$

We first prove by rescaling arguments the necessary condition in Proposition 1.2.

Proof of Proposition 1.2. Let us choose the multiplier $m(t) = m_\lambda(t) := g_\lambda(t)$ and the function $f(t) = g_\lambda(t)$ as well. Observe that $\hat{g}_\lambda(\xi) = \lambda^{-d/2} e^{-\pi \lambda^{-1} \xi^2}$, so that we compute

$$T_{m_\lambda} g_\lambda(t) = \lambda^{-d/2} \mathcal{F}^{-1}(e^{-\pi \lambda^{-1} \xi^2})(t) = (\lambda^2 + 1)^{-d/2} e^{-\frac{\lambda^2}{\lambda^2 + 1} t^2}.$$ 

The $L^q$ norm of the function above is given by

$$\|T_{m_\lambda} g_\lambda\|_q \asymp \lambda^{-\frac{d}{2q}} (\lambda^2 + 1)^{-\frac{d}{2q}},$$

with $q'$ being the conjugate exponent of $q$. We have $\|g_\lambda\|_p \asymp \lambda^{-d/(2p)}$. Assuming now (12) in our context

$$\|T_{m_\lambda} g_\lambda\|_q \leq C \|m_\lambda\|_{L^r,\infty} \|g_\lambda\|_p$$

we get

$$\lambda^{-\frac{d}{2q}} (\lambda^2 + 1)^{-\frac{d}{2q}} \leq C \lambda^{-\frac{d}{2p}} \lambda^{-\frac{d}{2p}}.$$ 

Letting $\lambda \to 0^+$ we obtain the desired estimate [13].

In this section we shall use the weak definition of a STFT multiplier. Namely, for $a \in \mathcal{S}'(\mathbb{R}^{2d})$, $g_1, g_2 \in \mathcal{S}(\mathbb{R}^d)$, the STFT multiplier $A^{g_1,g_2}_a$ can be defined weakly as follows

$$\langle A^{g_1,g_2}_a f, g \rangle = \langle a V_{g_1} f, V_{g_2} g \rangle = \langle a, V_{g_1}^* V_{g_2} g \rangle,$$

where the brackets $\langle \cdot, \cdot \rangle$, linear in the first component and conjugate-linear in the second one, denote the duality between $\mathcal{S}'$ and $\mathcal{S}$ (or any other suitable pair of dual spaces).

For any symbol $a(x, \omega) = (1 \otimes m)(x, \omega) = m(\omega)$, $x, \omega \in \mathbb{R}^d$, the STFT multiplier $A^{g_1,g_2}_{1 \otimes m}$ can be formally re-written in terms of the related correlation function. Assume for simplicity that the windows $g_1, g_2$ and multiplier $m = m(\omega)$ are in $\mathcal{S}(\mathbb{R}^d)$. We start with $f \in \mathcal{S}(\mathbb{R}^d)$; for every fixed $t \in \mathbb{R}^d$, the integrals below are absolutely convergent and we are allowed to use Fubini’s Theorem. Moreover it is straightforward
Corollary 4.2. Fix a multiplier symbol $m \in \mathcal{S}'(\mathbb{R}^d)$. Simple computations give

$$A_{1 \hat{\otimes} m}^{g_1,g_2} f(t) = \int_{\mathbb{R}^d} e^{2\pi i \omega t} m(\omega) \int_{\mathbb{R}^d} f(y) e^{-2\pi i \omega y} \mathcal{G}_{g_1,g_2}(t,y) dy d\omega$$

(36)

$$= \int_{\mathbb{R}^d} e^{2\pi i \omega t} m(\omega) \int_{\mathbb{R}^d} f(y) e^{-2\pi i \omega y} T_t \mathcal{C}_{g_1,g_2}(y) dy d\omega$$

(37)

$$= \int_{\mathbb{R}^d} e^{2\pi i \omega t} m(\omega) \mathcal{F}(f T_t \mathcal{C}_{g_1,g_2})(\omega) d\omega.$$

Note that, if we assume condition (11), then $T_t \mathcal{C}_{g_1,g_2} = 1$ for every $t \in \mathbb{R}^d$ and $A_{1 \hat{\otimes} m}^{g_1,g_2} = T_m$, as desired.

The equality (37) suggests the introduction of a new time-frequency representation closely related to the STFT.

Definition 4.1. For $g_1 \in L^1(\mathbb{R}^d), g_2 \in L^2(\mathbb{R}^d)$, we define the two-window short-time Fourier transform of a signal $f \in L^2(\mathbb{R}^d)$ by

$$\int_{\mathbb{R}^d} e^{-2\pi i \omega y} f(y) T_t \mathcal{C}_{g_1,g_2}(y) dy = \langle f, M_\omega T_t \mathcal{C}_{g_1,g_2} \rangle = V_{\mathcal{C}_{g_1,g_2}} f(t, \omega), \quad (t, \omega) \in \mathbb{R}^{2d}.$$ (38)

For $g_1 \in L^1(\mathbb{R}^d), g_2 \in L^2(\mathbb{R}^d)$, Young’s Inequality gives $\mathcal{C}_{g_1,g_2} \in L^2(\mathbb{R}^d)$. Thus the integral above is an absolutely convergent for every $f \in L^2(\mathbb{R}^d)$. The same argument applies if we replace the condition $g_1 \in L^1(\mathbb{R}^d), g_2 \in L^2(\mathbb{R}^d)$ with the more general one $g_1 \in L^p(\mathbb{R}^d), g_2 \in L^q(\mathbb{R}^d)$ such that $1/p + 1/q = 3/2$.

Using (37), the action of the STFT multiplier $A_{1 \hat{\otimes} m}^{g_1,g_2}$ can be rewritten as

$$A_{1 \hat{\otimes} m}^{g_1,g_2} f(t) = \int_{\mathbb{R}^d} e^{2\pi i \omega t} m(\omega) V_{\mathcal{C}_{g_1,g_2}} f(t, \omega) d\omega = \mathcal{F}^{-1}_2 [m V_{\mathcal{C}_{g_1,g_2}} f(t, \cdot)](t), \quad t \in \mathbb{R}^d$$

(39)

where $\mathcal{F}^{-1}_2$ denotes the partial Fourier transform w.r.t. the second coordinate $\omega$. The formal equality above can be made rigorous by studying the properties of the two-window short-time Fourier transform $V_{\mathcal{C}_{g_1,g_2}}$ and the multiplier symbol $m(\omega)$.

The following issue stems from Theorem 3.1 with $m = m_2$.

Corollary 4.2. Fix a multiplier symbol $m \in \mathcal{S}'(\mathbb{R}^d)$ (resp. $m \in M^\infty(\mathbb{R}^d)$) and windows $g_1, g_2$ in $\mathcal{S}(\mathbb{R}^d)$ (resp. in $M^1(\mathbb{R}^d)$). Then the equality

$$A_{1 \hat{\otimes} m}^{g_1,g_2} = T_m \quad \text{on} \quad \mathcal{S}(\mathbb{R}^d) \quad (\text{resp.} \quad M^1(\mathbb{R}^d))$$

(40)

holds if and only if

$$\hat{m} \mathcal{C}_{g_1,g_2} = \hat{m} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d) \quad (\text{resp.} \quad M^\infty(\mathbb{R}^d)).$$

(41)

The same conclusions hold under the following assumptions:

(i) The symbol $m$ in $\mathcal{S}(\mathbb{R}^d)$ (resp. in $M^1(\mathbb{R}^d)$) and the window functions $(g_1, g_2)$ in $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ (resp. $M^\infty(\mathbb{R}^d) \times M^1(\mathbb{R}^d)$).
(ii) The symbol $m \in \mathcal{S}(\mathbb{R}^d)$ (resp. in $M^1(\mathbb{R}^d)$) and the window functions $(g_1, g_2) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ (resp. $M^1(\mathbb{R}^d) \times M^\infty(\mathbb{R}^d)$).

Straightforward consequences of the result above are the following.

**Corollary 4.3.** Consider either $(g_1, g_2) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ or $(g_1, g_2) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$. Then the equality (41) holds for every symbol $m \in \mathcal{S}(\mathbb{R}^d)$ if and only if condition (11) is satisfied.

**Proof.** The condition (11) immediately follows if we take $m(\omega) = e^{-\pi \omega^2} \in \mathcal{S}(\mathbb{R}^d)$ in the equality (41). □

**Corollary 4.4.** It is not possible to find $g_1, g_2 \in \mathcal{S}(\mathbb{R}^d)$ such that the equality (40) holds for every multiplier $m \in \mathcal{S}'(\mathbb{R}^d)$.

**Proof.** Taking $m(\omega) = e^{-\pi \omega^2} \in \mathcal{S}(\mathbb{R}^d)$ in the equality (41) we obtain condition (11). Since $g_1, g_2 \in \mathcal{S}(\mathbb{R}^d)$, by Proposition 2.6 we infer $C_{g_1, g_2} \in \mathcal{S}(\mathbb{R}^d)$, thus condition (11) is never satisfied. □

Let us try to understand better the condition (11) for operators having windows/symbols in modulation spaces.

Notice that under the assumption $g_1, g_2 \in M^1(\mathbb{R}^d)$ the window correlation function $C_{g_1, g_2}$ is in $M^1(\mathbb{R}^d)$ (use Proposition 2.6 (iv) or the well-known fact that $M^1$ is an algebra under convolution). As a consequence of Theorem 3.1, if we want condition (41) to be satisfied for every multiplier $m \in M^\infty(\mathbb{R}^d)$, the window correlation function $C_{g_1, g_2}$ must satisfy

$$C_{g_1, g_2}(t) = 1, \quad t \in \mathbb{R}^d.$$  \hspace{1cm} (42)

But this is not possible since $C_{g_1, g_2} \in M^1(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$.

To overcome this issue, we look for rougher windows that could guarantee condition (42). This requires smoother symbols.

**Theorem 4.5.** Consider $p_1, p_2, q_1, q_2 \in [1, \infty]$, with $1/p_1 + 1/p_2 \geq 1$, $1/q_1 + 1/q_2 \geq 1$, $g_1 \in M^{p_1,q_1}(\mathbb{R}^d)$, $g_2 \in M^{p_2,q_2}(\mathbb{R}^d)$, and $m \in M^\infty(\mathbb{R}^d)$. Then both the Fourier multiplier $T_m$ and the STFT multiplier $A_{1\otimes m}^{q_1,q_2}$ are well-defined linear and bounded operators on $L^2(\mathbb{R}^d)$ and the equality (40) holds on $M^\infty(\mathbb{R}^d)$ if and only if condition (11) is satisfied on $M^\infty(\mathbb{R}^d)$. As a consequence, if we want (11) to be fulfilled for every symbol $m \in M^{\infty,1}(\mathbb{R}^d)$, the window correlation function $C_{g_1, g_2}$ must satisfy (42).

**Proof.** We start with $g_1, g_2 \in M^1(\mathbb{R}^d) \hookrightarrow M^{p,q}(\mathbb{R}^d)$, for every $p, q \in [1, \infty]$. Notice that, if the multiplier $m \in M^{\infty,1}(\mathbb{R}^d)$, then the localization symbol $(1 \otimes m)$ is in $M^{\infty,1}(\mathbb{R}^{2d})$, since

$$(1 \otimes m) \in M^{\infty,1}(\mathbb{R}^d) \otimes M^{\infty,1}(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^{2d})$$
and we have \(1 \in M^{\infty,1}(\mathbb{R}^d)\). In fact, for any fixed non-zero \(g \in S(\mathbb{R}^d)\), we work out
\[
V_g 1(x, \omega) = \mathcal{F}(T_x \hat{g})(\omega) = M_{-x} \hat{g}(\omega), \quad (x, \omega) \in \mathbb{R}^{2d},
\]
so that
\[
\|1\|_{M^{\infty,1}(\mathbb{R}^d)} \asymp \|V_g 1\|_{L^{\infty,1}(\mathbb{R}^{2d})} = \|\hat{g}\|_{L^{1}(\mathbb{R}^d)} = \|g\|_{\mathcal{F}L^{1}(\mathbb{R}^d)} < \infty.
\]
Hence by Theorem 2.4 the STFT multiplier \(A_{1 \otimes m}^{g_1, g_2}\) is bounded on any \(M^{p,q}(\mathbb{R}^d)\) and in particular on \(L^2(\mathbb{R}^d)\). This is also the case for the Fourier multiplier \(T_m\) with \(m \in M^{\infty,1}(\mathbb{R}^d)\), since the inclusion relation in \((23)\) gives in particular \(m \in L^\infty(\mathbb{R}^d)\) and hence \(T_m \in B(L^2)\) \cite{35}. Using Theorem 3.1, such operators coincide whenever condition \((11)\) is satisfied.

Next, consider \(g_1 \in M^{p_1,q_1}(\mathbb{R}^d), g_2 \in M^{p_2,q_2}(\mathbb{R}^d)\) satisfying the assumptions. We shall show that the related kernel \(K_A\) of \(A_{1 \otimes m}^{g_1, g_2}\) is in \(M^\infty(\mathbb{R}^{2d})\). In fact, Proposition 2.6 \((iv)\) gives the window correlation function \(C_{g_1, g_2} \in M^{\infty,1}(\mathbb{R}^d)\). If the multiplier \(m \in M^{\infty,1}(\mathbb{R}^d)\), then \(\hat{m} \in W(\mathcal{F}L^\infty, L^1)(\mathbb{R}^d) \hookrightarrow W(\mathcal{F}L^\infty, L^\infty)(\mathbb{R}^d) = M^{\infty}(\mathbb{R}^d)\) \(\text{cf.}, \text{ e.g.}, \cite{14} \text{Chapter 2})\) and the multiplication relations for modulation spaces \cite{14} Prop. 2.4.23
\[
\|\hat{m} C_{g_1, g_2}\|_{M^{\infty}} \lesssim \|\hat{m}\|_{M^{\infty}} \|C_{g_1, g_2}\|_{M^{\infty,1}} \lesssim \|m\|_{M^{\infty,1}} \|g_1\|_{M^{p_1, q_1}} \|g_2\|_{M^{p_2, q_2}} < \infty.
\]
Hence we obtain condition \((11)\). \(\square\)

Thanks to the results above, if the window functions \(g_1\) and \(g_2\) are rough enough, they can satisfy condition \((11)\), as in the following issue.

**Example 4.6.** An example of window correlation functions \(C_{g_1, g_2}\) satisfying \((11)\). Consider \(g_2 = 1 \in M^{\infty,1}(\mathbb{R}^d)\) and any \(g_1 \in M^{1,\infty}(\mathbb{R}^d)\) satisfying
\[
\int_{\mathbb{R}^d} g_1(y) \, dy = 1.
\]
This gives \((42)\). In particular, observe that \((43)\) is fulfilled if we consider \(g_2(t) = e^{-\pi t^2} \in S(\mathbb{R}^d) \subset M^{1,\infty}(\mathbb{R}^d)\). Hence the operators \(A_{1 \otimes m}^{g_1, g_2}\) and \(T_m\) coincide for every multiplier \(m \in M^{\infty,1}(\mathbb{R}^d)\).

The realm of modulation spaces seems the only possible environment to get the equality \(A_{1 \otimes m}^{g_1, g_2} = T_m\). Also for the standard case of \(L^2\)-window functions the equality fails, as shown below.

**Theorem 4.7.** Consider \(g_1, g_2 \in L^2(\mathbb{R}^d), \text{ and the multiplier } m \in L^\infty(\mathbb{R}^d)\). Then both the Fourier multiplier \(T_m\) and the STFT multiplier \(A_{1 \otimes m}^{g_1, g_2}\) are well-defined linear and bounded operators on \(L^2(\mathbb{R}^d)\) and the equality
\[
A_{1 \otimes m}^{g_1, g_2} = T_m \quad \text{on} \quad L^2(\mathbb{R}^d)
\]
holds if and only if condition \((11)\) is satisfied. As a consequence, if we want \((11)\) to be fulfilled for every multiplier \(m \in L^\infty(\mathbb{R}^d)\), the window correlation function \(C_{g_1, g_2}\) must satisfy \((12)\), and this is never the case.
Proof. The boundedness of $A_{1,\odot m}^{g_1,g_2}$ on $L^2(\mathbb{R}^d)$ is shown in [49]. For the Fourier multiplier we recall that $T_m$ is bounded on $L^2(\mathbb{R}^d)$ since $m$ is in $L^\infty(\mathbb{R}^d)$ [35]. Condition (41) then follows by Theorem 3.1. The window correlation function we recall that $C_{g_1,g_2}$ never satisfies (41) because $g_1, g_2 \in L^2(\mathbb{R}^d)$ implies $C_{g_1,g_2} \in C_0(\mathbb{R}^d)$, by Proposition 2.6 (iii).

A natural question is whether we can consider windows $g_1 \in M^p(\mathbb{R}^d)$, $g_2 \in M^p(\mathbb{R}^d)$, $1 \leq p, p' \leq \infty$, $1/p + 1/p' = 1$, and multiplier $m \in L^\infty(\mathbb{R}^d)$. This is the case explained below.

**Proposition 4.8.** If we consider $g_1 \in M^p(\mathbb{R}^d)$, $g_2 \in M^p(\mathbb{R}^d)$, $1 \leq p, p' \leq \infty$, $1/p + 1/p' = 1$, and multiplier $m \in L^\infty(\mathbb{R}^d)$, then the thesis of Theorem 3.1 holds true. In particular, the equality in (41) is fulfilled if and only if Condition (42) is satisfied.

Proof. The Fourier multiplier $T_m$ is obviously well-defined, linear and bounded from $S(\mathbb{R}^d)$ to $S'(\mathbb{R}^d)$, since $T_m$ is bounded on $L^2(\mathbb{R}^d)$. For the STFT multiplier $A_{1,\odot m}^{g_1,g_2}$, we use its weak definition in [35], Hölder’s inequality, the switching property of the STFT [14], Lemma 1.2.3] and the change of window in [14], Lemma 1.2.29, where we take $\gamma \in S(\mathbb{R}^d)$, with $\|\gamma\|_{L^2} = 1$; for every $f, g \in S(\mathbb{R}^d)$,

$$|\langle A_{a}^{g_1,g_2} f, g \rangle| = |\langle a, V_{g_1} f \rangle V_{g_2} g \rangle|$$

$$\leq \langle a \rangle_{L^\infty(\mathbb{R}^{2d})} \parallel V_{g_1} f \parallel_{L^1(\mathbb{R}^{2d})} \parallel V_{g_2} g \parallel_{L^1(\mathbb{R}^{2d})}$$

$$\leq \langle m \rangle_{L^\infty(\mathbb{R}^d)} \parallel V_{g_1} f \parallel_{L^p(\mathbb{R}^{2d})} \parallel V_{g_2} g \parallel_{L^{p'}(\mathbb{R}^{2d})}$$

$$= \langle m \rangle_{L^\infty(\mathbb{R}^d)} \parallel V_{g_1} f \parallel_{L^p(\mathbb{R}^{2d})} \parallel V_{g_2} g \parallel_{L^{p'}(\mathbb{R}^{2d})}$$

$$\leq \langle m \rangle_{L^\infty(\mathbb{R}^d)} \parallel V_{g_1} f \parallel_{L^1(\mathbb{R}^{2d})} \parallel V_{g_2} g \parallel_{L^1(\mathbb{R}^{2d})}$$

$$= \langle m \rangle_{L^\infty(\mathbb{R}^d)} \parallel f \parallel_{M^1(\mathbb{R}^{2d})} \parallel V_{g_1} f \parallel_{L^p(\mathbb{R}^{2d})} \parallel g \parallel_{M^1(\mathbb{R}^{2d})} \parallel V_{g_2} g \parallel_{L^{p'}(\mathbb{R}^{2d})}$$

$$= \langle m \rangle_{L^\infty(\mathbb{R}^d)} \parallel f \parallel_{M^1(\mathbb{R}^{2d})} \parallel V_{g_1} f \parallel_{L^p(\mathbb{R}^{2d})} \parallel g \parallel_{M^1(\mathbb{R}^{2d})} \parallel V_{g_2} g \parallel_{L^{p'}(\mathbb{R}^{2d})}$$

Since $S(\mathbb{R}^d) \hookrightarrow M^1(\mathbb{R}^d)$, the estimate above gives the continuity of $A_{1,\odot m}^{g_1,g_2}$ from $S(\mathbb{R}^d)$ into $S'(\mathbb{R}^d)$. Then, arguing as in the proof of Theorem 3.1 we obtain the claim.

Considering $g_2(t) = 1$ for every $t \in \mathbb{R}^d$, hence $g_2 \in L^\infty(\mathbb{R}^d) \subset M^\infty(\mathbb{R}^d)$, and any $g_1 \in M^1(\mathbb{R}^d)$ satisfying (43), we provide examples for Condition (42) being satisfied.

5. **Smoothing effects of STFT multipliers**

Thanks to the smoothing effect of the two-window STFT we obtain boundedness results for STFT multipliers which extend the case of Fourier multipliers. The main tool is to use the representation of $A_{a}^{g_1,g_2}$ in (37), that is

$$A_{a}^{g_1,g_2} f(t) = \int_{\mathbb{R}^d} e^{2\pi i \omega t} m(\omega) \mathcal{F}(f T_t C_{g_1,g_2}) (\omega) d\omega = \mathcal{F}^{-1} [m \mathcal{V}_{g_1,g_2} f (t, \cdot)].$$
Theorem 5.1. Assume $1 < p \leq 2 \leq q < \infty$, $m \in L^{r,\infty}(\mathbb{R}^d)$ such that condition \([13]\) is satisfied. Consider windows $g_1, g_2 \in S'(\mathbb{R}^d)$ such that the correlation function satisfies

$$C_{g_1, g_2} \in L^p(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d),$$

with $1/p + 1/p' = 1$ (conjugate exponents). Then the STFT operator $A_{1\otimes m}^{g_1, g_2}$ is bounded from $L^p(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$.

Proof. Consider a function $f$ in $L^p(\mathbb{R}^d)$, $p \leq 2$, then

$$\|f T_t C_{g_1, g_2}\|_1 \leq \|f\|_p \|T_t C_{g_1, g_2}\|_{p'} = \|f\|_p \|C_{g_1, g_2}\|_{p'}, \quad \forall t \in \mathbb{R}^d$$

and

$$\|f T_t C_{g_1, g_2}\|_p \leq \|f\|_p \|T_t C_{g_1, g_2}\|_\infty \leq \|f\|_p \|C_{g_1, g_2}\|_\infty, \quad \forall t \in \mathbb{R}^d.$$ 

So that by complex interpolation, $f T_t C_{g_1, g_2} \in L^s(\mathbb{R}^d)$, for every $1 \leq s \leq p$ (hence $1/s \geq 1/p$) $\forall t \in \mathbb{R}^d$, with

$$\|f T_t C_{g_1, g_2}\|_{L^s(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)},$$

for a constant $C > 0$ independent of $t$.

By Theorem 1.1 if $m \in L^{r,\infty}(\mathbb{R}^d)$, then the Fourier multiplier

$$T_m f = \mathcal{F}_2^{-1}[m V_{\mathbb{C}^1_{g_1, g_2}} f(t, \cdot)] = \mathcal{F}_2^{-1}[m \mathcal{F}_2(f T_t C_{g_1, g_2})]$$

acts continuously from $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$, with $q \geq 2$ satisfying the index condition in \([13]\). \qed

Remark 5.2. If $g_1 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $g_2 \in L^2(\mathbb{R}^d)$ (or vice versa) then the window correlation function satisfies condition \([45]\). In fact, by Proposition 2.6 it follows that $C_{g_1, g_2} \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subset L^{p'}(\mathbb{R}^d)$, for every $2 \leq p' \leq \infty$.

This shows the smoothing effect of the two-window STFT $V_{\mathbb{C}^1_{g_1, g_2}} f$. For simplicity, let us consider $f \in L^2(\mathbb{R}^d)$. The Fourier multiplier $T_m$ takes the function $f \in L^2(\mathbb{R}^d)$ and consider is Fourier transform $\hat{f}$ that lives in $L^2(\mathbb{R}^d)$ by Plancherel theorem, but we cannot infer any other further property for $f$. Instead, in the STFT multiplier $A_{1\otimes m}^{g_1, g_2}$ we replace $\hat{f}$ with the two-window STFT $V_{\mathbb{C}^1_{g_1, g_2}} f$. Assuming the condition \([45]\), we obtain that $V_{\mathbb{C}^1_{g_1, g_2}} f \in C_b(\mathbb{R}^{2d}) \cap L^2(\mathbb{R}^{2d})$ and uniformly continuous on $\mathbb{R}^{2d}$ (cf. \([14]\) Proposition 1.2.10, Corollary 1.2.12), and this implies $V_{\mathbb{C}^1_{g_1, g_2}} f(t, \cdot) \in C_b(\mathbb{R}^{2d}) \cap L^2(\mathbb{R}^{2d})$ for every fixed $t \in \mathbb{R}^d$, so that the related multiplier $\mathcal{F}_2^{-1}[m V_{\mathbb{C}^1_{g_1, g_2}} f(t, \cdot)]$ can enjoy the smoothing effect above, uniformly with respect to $t \in \mathbb{R}^d$. 
5.1. The anti-Wick case. Thanks to the discussions above, we can state that an anti-Wick operator $A_{g,g}^{\varphi,m}$, with Gaussian windows $g(t) = 2^{d/4}e^{-\pi t^2}$ and multiplier symbol $m \in S'(\mathbb{R}^d)$, can never be written in the Fourier multiplier form. In fact, recalling that the window correlation function in this case is given by $C_{g,g}(t) = e^{-\frac{\pi}{2} t^2}$, cf. formula (26), we infer that condition (11) is never satisfied.

Let us better understand the smoothing effects for such operators. Using the expression in (37), we can write

$$A_{g,g}^{\varphi,m}f(t) = \int_{\mathbb{R}^d} e^{2\pi i \omega t} m(\omega) \mathcal{F}(fT_t(e^{-\frac{\pi}{2} t^2}))(\omega)d\omega.$$  

The anti-Wick operator in terms of the two-window STFT defined in (38) can be written as

$$A_{g,g}^{\varphi,m}f(t) = \mathcal{F}_2^{-1}[mV_{c_{g,g}} f(t, \cdot)], \quad t \in \mathbb{R}^d.$$  

Roughly speaking, here the signal $f$ is first smoothed by multiplying with the shifted Gaussian $T_t(e^{-\frac{\pi}{2} t^2})$, that is

$$g_t(y) := f(y)T_t(e^{-\frac{\pi}{2} t^2})(y),$$

then the multiplier $T_m$ is applied to the modified signal $g_t$. In other words,

$$A_{g,g}^{\varphi,m}f(t) = T_m(g_t)(t), \quad f \in L^2(\mathbb{R}^d).$$

From the equality above it is clear the smoothing effect of the anti-Wick operator $A_{g,g}^{\varphi,m}$ with respect to the Fourier multiplier $T_m$, stated in Theorem 1.3, that we are going to prove very easily.

Proof of Theorem 1.3. Since the window correlation function $C_{g,g}(t) = e^{-\frac{\pi}{2} t^2}$ is in $S(\mathbb{R}^d) \hookrightarrow L^{p'}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, for any $2 \leq p' < \infty$ condition in (45) is satisfied and the thesis follows by Theorem 5.1. □

We end up this section by showing the necessity of the indices’ relation in (13).

Theorem 5.3. If there exists a $C > 0$ such that the anti-Wick operator satisfies

$$\|A_{g,g}^{\varphi,m}f\|_q \leq C\|m\|_{L^{p'}\cap L^\infty} \|f\|_p, \quad \forall f,m \in S(\mathbb{R}^d),$$

then condition (13) holds true.

Proof. We write condition (48) for the multipliers $m_\lambda(\xi) = g_\lambda(\xi) = e^{-\pi \lambda \xi^2}$, $\lambda > 0$, and functions $f_\lambda(t) = g_\lambda(t)$ as well. Then we compute the anti-Wick operator $A_{g,g}^{\varphi,m_\lambda}f_\lambda$. A tedious computation shows

$$A_{g,g}^{\varphi,m_\lambda}f_\lambda(t) = c_\lambda e^{-\pi b_\lambda t^2},$$

with

$$c_\lambda := \frac{2^{d/2}}{(6\lambda^2 + 4\lambda + 1)^{d/2}}, \quad b_\lambda := \frac{2\lambda(6\lambda^3 + 10\lambda^2 + 9\lambda + 1)}{(6\lambda^2 + 4\lambda + 1)(2\lambda + 1)^2}.$$
This yields the norm estimate
\[ \| A_{1_{x \otimes m}} f \|_q \approx c \lambda b_{\lambda}^{-\frac{d}{q}} \leq \frac{(2\lambda + 1)^{\frac{d}{q}}}{\lambda^{\frac{d}{q}} (6\lambda^2 + 4\lambda + 1)^{\frac{d}{q}} (6\lambda^3 + 10\lambda^2 + 9\lambda + 1)^{\frac{d}{q}}}. \]

Letting \( \lambda \to 0^+ \) we infer the inequality in (13). \( \square \)

6. Gabor multipliers

The spreading representation of an integral operator \( L \) is useful for applications (see, e.g., [18] or [33, Chapter 14]):

\[ L f(t) = \int_{\mathbb{R}^d} \eta_L(x, \omega)(M_\omega T_x f)(t) dx d\omega, \quad f \in S(\mathbb{R}^d), \]

where \( \eta_L \), also denoted by \( \eta(L) \), is called the spreading function and is related to the kernel \( K_L = K(L) \) of the operator \( L \) by the following transform

\[ \eta_L(x, \omega) = \int_{\mathbb{R}^d} K_L(y, y - x) e^{-2\pi i y \omega} dy. \]

In addition, the spreading function \( \eta_L \) and Kohn-Nirenberg symbol \( \sigma_L \) of an operator are connected via the symplectic Fourier Transform \( \mathcal{F}_s \) as

\[ \eta_L(x, \omega) = \mathcal{F}_s(\sigma_L)(x, \omega) = \int_{\mathbb{R}^{2d}} e^{-2\pi i (y \omega - x \eta)} \sigma_L(y, \eta) dy d\eta. \]

Whenever clear, we may suppress the lower indexes in \( \eta_L, K_L \) and \( \sigma_L \).

There is an exact analogue of these objects in the finite discrete case, see [26]. In fact, if \( L : \mathbb{C}^N \to \mathbb{C}^N \) is a linear operator then we denote its matrix representation by \( K_L = K(L) \) and define its spreading function as

\[ \eta_L(u, v) = \sum_{k=0}^{N-1} K_L(k, k - u) e^{-2\pi i ku} N. \]

So that \( L \) can be seen as a finite superposition of TF-shifts:

\[ L = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \eta_L(k, l) \pi(k, l). \]

6.1. Finite discrete setting. In the finite discrete setting, we will always identify \( \mathbb{C}^N \) with \( l^2(\mathbb{Z}_N) \). We shall denote by \( 1 \in \mathbb{C}^N \) the constant function equal to 1. From now on, we will always consider a rectangular lattice of the form

\[ \Lambda = \alpha \mathbb{Z}_N \times \beta \mathbb{Z}_N, \quad \alpha, \beta \in \mathbb{N}_+, \quad \alpha := \frac{N}{N}, \quad \beta := \frac{N}{\beta} \in \mathbb{N}_+. \]
which can be thought also as $\Lambda = \alpha \mathbb{Z}^A \times \beta \mathbb{Z}^B$. Since $\alpha, \beta$ are divisors of $N$, $\Lambda$ is a subgroup. Therefore in this case translation and modulation operator take the form:

$$T_k f(t) = f(t - k), \quad M_l f(t) = e^{\frac{2\pi i lt}{N}} f(t), \quad f \in \mathbb{C}^N, \ t = 0, \ldots, N - 1, \ k, l \in \mathbb{Z}.$$ 

Of course we put $\pi(k, l) = M_l T_k$ and define the STFT of a signal $f \in \mathbb{C}^N$ w.r.t. the window $g \in \mathbb{C}^N$ as the matrix in $\mathbb{C}^{N \times N}$

$$V_g f(u, v) = \langle f, \pi(u, v) g \rangle = \sum_{k=0}^{N-1} f(k) \overline{g(k-u)} e^{-\frac{2\pi i ku}{N}}.$$ 

The Gabor system generated by a window $g \in \mathbb{C}^N$ and lattice $\Lambda$ as in (50) is defined as

$$G(g, \alpha, \beta) := \{ \pi(k, l) g, \ (k, l) \in \Lambda \} = \{ \pi(\alpha k, \beta l) g, \ k = 0, \ldots, A - 1, \ l = 0, \ldots, B - 1 \}.$$ 

A Gabor system $G(g, \alpha, \beta)$ is said to be a Gabor frame for $\mathbb{C}^N$ if there exist $C_1, C_2 > 0$ such that

$$C_1 \| f \|_2^2 \leq \sum_{k=0}^{A-1} \sum_{l=0}^{B-1} |\langle f, \pi(\alpha k, \beta l) g \rangle|^2 \leq C_2 \| f \|_2^2 \quad \forall f \in \mathbb{C}^N,$$

where $\langle f, \pi(\alpha k, \beta l) g \rangle = \sum_{u=0}^{N-1} f(u) \overline{\pi(\alpha k, \beta l) g(u)}$ and $\|\cdot\|_2$ is the induced norm. Since we are in finite-dimension, this is equivalent to ask that $G(g, \alpha, \beta)$ spans $\mathbb{C}^N$.

The discrete Fourier transform (DFT) on $\mathbb{C}^N$ is the linear operator represented by the following $N \times N$ complex matrix

$$(\mathcal{F}_N)_{k,l} := e^{-\frac{2\pi i k l}{N}},$$

which inverse if given by

$$(\mathcal{F}_N)^{-1} = \frac{1}{N} e^{\frac{2\pi i k l}{N}}.$$ 

We shall denote sometimes by $\hat{f}$ the vector $\mathcal{F}_N f$, $f \in \mathbb{C}^N$. Therefore, the discrete two-dimensional Fourier transform of a matrix $a \in \mathbb{C}^N \times \mathbb{C}^N$ and its inverse are defined as

$$\mathcal{F}_2 a(u, v) := \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a(k, l) e^{-\frac{2\pi i uk}{N}} e^{-\frac{2\pi i vl}{N}}, \quad \mathcal{F}_2^{-1} a(u, v) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a(k, l) e^{\frac{2\pi i uk}{N}} e^{\frac{2\pi i vl}{N}}.$$
The action of $F_2$ on the (pointwise) product of $a$ and $b$ in $\mathbb{C}^{N \times N}$ is well-known and we mention it for sake of completeness:

$$F_2(a \cdot b) = \frac{1}{N^2} (F_2 a \ast F_2 b),$$

(54)

where the (two-dimensional discrete) convolution on the right-hand side is defined similarly to (61). The Kronecker delta function $\delta \in \mathbb{C}^N$ is defined as

$$\delta(u) = \begin{cases} 
1 & \text{for } u = 0, \\
0 & \text{for } u = 1, \ldots, N - 1.
\end{cases}$$

We recall also the following identity:

$$F_N \left( \frac{1}{N} \right) (u) = \delta(u).$$

The so-called impulse train, or Dirac comb, will be useful in some of the subsequent computations:

$$\mathbb{I}_{(\alpha, \beta)}(u, v) := \sum_{p=0}^{A-1} \sum_{q=1}^{B-1} \delta(u - \alpha p) \delta(v - \beta q)$$

$$= \chi_{\alpha Z_N}(u) \cdot \chi_{\beta Z_N}(v)$$

$$= \frac{1}{\alpha \beta} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \delta(u - \alpha k) \delta(v - \beta l),$$

(55)

for $u, v = 0, \ldots, N - 1$.

For sake of the reader, we recall the Poisson summation formula (56) and its two-dimensional analogue in the following lemma, see [32] and [38, Theorem 3.2.1].

**Lemma 6.1.** Under the assumptions in (50):

$$F_N \chi_{\alpha Z_N} = A \chi_{A Z_N},$$

(56)

$$F_2 \mathbb{I}_{(\alpha, \beta)} = AB \mathbb{I}_{(A, B)}.$$ 

(57)

The discrete symplectic Fourier transform of a matrix $a \in \mathbb{C}^{N \times N}$ is defined as

$$F_s a(u, v) := \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a(k, l) e^{2\pi i (ku - lv)/N}$$

(58)

with $u, v = 0, \ldots, N - 1$. Hence the relation between $F_2$ and $F_s$ is as follows:

$$F_s a(u, v) = \frac{1}{N} F_2 (a^T)(-u, v) = \frac{1}{N} F_2 a(v, -u),$$

(59)
\(a^T\) being the transpose of \(a\). Recall that given two vectors \(f, g \in \mathbb{C}^N\), the tensor product \(f \otimes g \in \mathbb{C}^{N \times N}\) is the matrix
\[
f \otimes g(u, v) = f(u)g(v), \quad u, v = 0, \ldots, N - 1.
\]
We mention also that
\[
F_s(f \otimes \hat{g}) = g \otimes \hat{f}.
\]

**Definition 6.2.** A Fourier multiplier, or linear time invariant (LTI) filter, or convolution operator \(H: \mathbb{C}^N \rightarrow \mathbb{C}^N\) is uniquely determined by the so called impulse response \(h \in \mathbb{C}^N\) via
\[
Hf(u) := h * f(u) := \sum_{k=0}^{N-1} h(u - k)f(k), \quad f \in \mathbb{C}^N, \ u = 0, \ldots, N - 1.
\]
Clearly
\[
Hf(u) = h * f(u) = (\mathcal{F}_N^{-1} \mathcal{F}_N h * \mathcal{F}_N^{-1} \mathcal{F}_N f)(u) = \mathcal{F}_N^{-1} \left( \hat{h} \cdot \hat{f} \right)(u),
\]
see (5), \(\hat{h}\) is also called frequency response. It is straightforward to see that a LTI filter \(H\) on \(\mathbb{C}^N\) has matrix representation
\[
K_H(u, v) = h(u - v), \quad u, v = 0, \ldots, N - 1.
\]
We can define the associated discrete spreading function \(\eta_H \in \mathbb{C}^{N \times N}\) as
\[
\eta_H(u, v) = h \otimes \delta(u, v).
\]
Given a rectangular lattice \(\Lambda\) as in (50), windows \(g_1, g_2 \in \mathbb{C}^N\), mask or symbol \(a \in \mathbb{C}^{N \times N}\), we define the (finite) Gabor multiplier applied to \(f \in \mathbb{C}^N\) as follows:
\[
G_{g_1, g_2}^a f = \sum_{k=0}^{A-1} \sum_{l=0}^{B-1} a(\alpha k, \beta l) V_{g_1}f(\alpha k, \beta l) \pi(\alpha k, \beta l) g_2.
\]
Whenever clear, we shall write \(G_{g_1, g_2}^a\) in place of \(G_{g_1, g_2}^{a, \Lambda}\). We mention that in the finite discrete setting Gabor multipliers coincide with STFT multipliers. It is straightforward to obtain the matrix representation of \(G_{g_1, g_2}^{a, \Lambda}\):
\[
K(G_{g_1, g_2}^{a, \Lambda})(u, v) = \sum_{k=0}^{A-1} \sum_{l=0}^{B-1} a(\alpha k, \beta l) g_1(v - \alpha k) g_2(u - \alpha k) e^{2\pi i \beta l (u-v) / N}.
\]
Let us introduce the notation
\[
\mathcal{S} := F_s a,
\]
\(a \in \mathbb{C}^{N \times N}\) symbol of a Gabor multiplier.
Proposition 6.3. The spreading function of a (finite) Gabor multiplier $G_{a,\Lambda}^{g_1, g_2}$ is given by

$$\eta(G_{a,\Lambda}^{g_1, g_2})(u, v) = \frac{N}{\alpha \beta} \sum_{t=0}^{\alpha-1} \sum_{k=0}^{\beta-1} S(u + Bk, v - Al) V_{g_1} g_2(u, v).$$

Proof. A direct computation gives

$$\eta(G_{a,\Lambda}^{g_1, g_2})(u, v) = \sum_{t=0}^{N-1} K(G_{a,\Lambda}^{g_1, g_2})(t, t-u) e^{-2\pi i tv}$$

$$= \sum_{t=0}^{N-1} \sum_{k=0}^{A-1} \sum_{l=0}^{B-1} a(\alpha k, \beta l) g_1(t-u-\alpha k) g_2(t-\alpha k) e^{2\pi i \beta lu} e^{-2\pi i tv} e^{-2\pi i \alpha kv}$$

$$= \sum_{k=0}^{A-1} \sum_{l=0}^{B-1} a(\alpha k, \beta l) e^{2\pi i \beta lu}$$

$$\times \sum_{t=0}^{N-1} g_1(t-u-\alpha k) g_2(t-\alpha k) e^{-2\pi i tv}.$$  

(68)

Performing the substitution $t' = t - \alpha k$ in (68) gives

$$\sum_{t=0}^{N-1} g_1(t-u-\alpha k) g_2(t-\alpha k) e^{-2\pi i tv} = \sum_{t'=0}^{N-1} g_2(t') g_1(t'-u) e^{-2\pi i (t'+\alpha k) v}$$

$$= \sum_{t'=0}^{N-1} g_2(t') g_1(t'-u) e^{-2\pi i v} e^{-2\pi i \alpha k v}$$

$$= V_{g_1} g_2(u, v) e^{-2\pi i \alpha k v}.$$
Hence, recalling the definition of $\mathbf{III}_{(\alpha, \beta)}$, $F_s$, $F_2$, and using Lemma 6.1 together with (54), we get

\[
\eta(G^{g_1, g_2}(u, v)) = \sum_{k=0}^{A-1} \sum_{l=0}^{B-1} a(\alpha k, \beta l) e^{\frac{2\pi i \alpha k u}{N}} e^{\frac{-2\pi i \beta l v}{N}} V_{g_1} g_2(u, v)
\]

\[
= NF_s \left( a \cdot \mathbf{III}_{(\alpha, \beta)} \right)(u, v) V_{g_1} g_2(u, v)
\]

\[
= F_2 \left( a^T \cdot \mathbf{III}_{(\alpha, \beta)}^T \right)(-u, v) V_{g_1} g_2(u, v)
\]

\[
= F_2 \left( a^T \cdot \mathbf{III}_{(\beta, \alpha)} \right)(-u, v) V_{g_1} g_2(u, v)
\]

\[
= \frac{1}{N^2} \left( F_2 a^T * F_2 \mathbf{III}_{(\beta, \alpha)} \right)(-u, v) V_{g_1} g_2(u, v)
\]

\[
= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \frac{1}{N} F_2 a^T(-u - k, v - l) F_2 \mathbf{III}_{(\beta, \alpha)}(k, l) V_{g_1} g_2(u, v)
\]

\[
= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} F_s a(u + k, v - l) A B \mathbf{III}_{(B, A)}(k, l) V_{g_1} g_2(u, v)
\]

\[
= \frac{AB}{N} \sum_{k=0}^{\alpha-1} \sum_{l=0}^{\beta-1} F_s a(u + B k, v - A l) V_{g_1} g_2(u, v).
\]

This concludes the proof. \[\square\]

We shall frequently denote the periodization of $S$ by $S^{BA}_p$:

\[
S^{BA}_p(u, v) := \sum_{l=0}^{\alpha-1} \sum_{k=0}^{\beta-1} S(u + B k, v - A l),
\]

the periodicity is meant in the sense that

\[
S^{BA}_p(u, v) = S^{BA}_p(u + B k, v + A l)
\]

for $u, v = 0, \ldots, N - 1$ and $k = 0, \ldots, \beta - 1$, $l = 0, \ldots, \alpha - 1$. So that (67) can be written as

\[
\eta(G^{g_1, g_2}_a)(u, v) = \frac{N}{\alpha \beta} S^{BA}_p(u, v) V_{g_1} g_2(u, v).
\]

The factor $N/\alpha \beta$ is also called redundancy.

By using the convolution theorem for $F_s$, cf. (54) and (59) and see [26, Theorem 4.3], and Lemma 6.1 we get

\[
F_s \left( a \cdot \mathbf{III}_{(\alpha, \beta)} \right)(u, v) = \frac{1}{\alpha \beta} \sum_{l=0}^{\alpha-1} \sum_{k=0}^{\beta-1} S(u + B k, v - A l).
\]
Figure 1. The left part shows the (Gaussian) spectrogram of the output of an “ideal band-pass filter” with cut-off frequency $R = 80$, applied to a random signal in $\mathbb{C}^N$, $N = 480$. The right hand side represents the spectrogram of the output of the corresponding STFT multiplier $G_{a^{1.9}}^g$.

Therefore

\[(72)\quad S_{PA}^B(u, v) = \alpha \beta F_s (a \cdot \sqrt[3]{(\alpha, \beta)}) (u, v).\]

Example 6.4. As example, we consider a low pass filter as it is often implemented in practice. We choose the frequency response $\hat{h} \in \mathbb{C}^N$ equal to the characteristic function, which is 1 on $[-R, R]$ and zero elsewhere. The resulting convolution operator $H$ is compared to the filter generated by a Gabor multiplier $G_{a^{1.9}}^{g_1 g_2}$ with symbol $a = 1 \otimes \hat{h}$. As analysis and synthesis window for $G_{a^{1.9}}^{g_1 g_2}$ we choose the Gaussian window normalized by the factor $1/N$, which is the redundancy since we take $\alpha = \beta = 1$. Both operations are applied to a low frequency random signal $f_0$, e.g. a random signal with Fourier transform $\hat{f}_0$ supported in an interval $[-c, c]$.

A graphical comparison of the LTI filter approach and of the Gabor multiplier one is shown in Figure 1.
6.2. Representation of LTI filter as Gabor Multiplier. Using intuition and visual comparison as an indication that the implementation of a LTI filter by a Gabor multiplier seems to work quite well, we are now going to analyze under which conditions it is analytically possible to have equivalence between a LTI filter and a Gabor multiplier. We will see immediately in the first theorem that exactly the most interesting class of perfect filters with characteristic function as frequency response does not qualify as suited candidates.

The result below is a consequence of the general setting in Theorem 4.7, but the estimate \((73)\) is new.

**Theorem 6.5.** Let \(T_{m_2}: L^2(\mathbb{R}) \to L^2(\mathbb{R})\) be a LTI filter with frequency response \(m_2 = \hat{h} = \chi_{\Omega}, \Omega \subseteq \mathbb{R}\) interval. Then \(T_{m_2}\) can never be represented exactly as Gabor multiplier with symbol \(a = 1 \otimes m, m \in L^\infty(\mathbb{R})\), and

\[
\|T_{m_2} - G_{a}^{g_1,g_2}\|_{Op} \geq \frac{1}{2} \tag{73}
\]

for every and \(g_1, g_2 \in L^2(\mathbb{R})\).

**Proof.** The spreading function is a Banach Gelfand Triple isomorphism between \((\mathcal{B}, \mathcal{H}, \mathcal{B}')\) and \((S_0, L^2, S_0')((\mathbb{R}^2),\) see [25] for notations. Therefore two operators are identical in \((\mathcal{B}, \mathcal{H}, \mathcal{B}')\) if and only if their spreading functions are identical. The integral kernel of a Fourier multiplier with symbol \(m_2\) was calculated in \((33)\) and it is related to the spreading function as follows:

\[
\eta(T_{m_2})(x, \omega) = \int_{\mathbb{R}^d} K(T_{m_2})(y, y - x)e^{-2\pi i \omega y} dy
\]

\[
= \int_{\mathbb{R}^d} T_y \hat{m}_2(y - x)e^{-2\pi i \omega y} dy
\]

\[
= \int_{\mathbb{R}^d} \hat{m}_2(-x)e^{-2\pi i \omega y} dy
\]

\[
= (I \circ \mathcal{F}(m_2) \otimes \delta)(x, \omega)
\]

\[
= (h \otimes \delta) (x, \omega). \tag{74}
\]

The spreading function of a Gabor multiplier \(G_{a}^{g_1,g_2}\) defined trough a lattice \(\Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d\) is given by

\[
\eta(G_{a}^{g_1,g_2})(x, \omega) = \mathcal{F}_a(a)(x, \omega) \cdot V_{g_1} g_2(x, \omega) =: \mathcal{A}(x, \omega) \cdot V_{g_1} g_2(x, \omega),
\]

where \(\mathcal{A} = \mathcal{F}_a a = \mathcal{F}^{-1}(m) \otimes \mathcal{F}(1) = \mathcal{F}^{-1}(m) \otimes \delta\) is the \((\frac{1}{\beta}, \frac{1}{\alpha})\)-periodic symplectic Fourier Transform of the symbol \(a = 1 \otimes m\) (compare [18]). Therefore a Gabor multiplier is equivalent to a convolution operator if and only if

\[
(h \otimes \delta) (x, \omega) = \left(\mathcal{F}^{-1}(m) \otimes \delta\right) V_{g_1} g_2(x, \omega) \quad \forall (x, \omega) \in \mathbb{R}^{2d}. \tag{75}
\]
Let us calculate symbol $a$

**Theorem 6.6.**

observe however the same behavior as we have in the continuous case.

If the value \( \hat{h} \) implies

\[
(76) \quad h(x) = \mathcal{F}^{-1}(m)V_{g_1}g_2(x,0) \iff \hat{h} = m * \mathcal{F}(V_{g_1}g_2(\cdot,0)).
\]

Let us calculate

\[
\mathcal{F}(V_{g_1}g_2(\cdot,0))(\omega) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_2(t)g_1(x)dt e^{2\pi i \omega x} dx
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_2(t)g_1(x')e^{-2\pi i (x-y)} dx' dt
\]

\[
= \mathcal{F}(g_2)(\omega)\mathcal{F}^{-1}(\mathcal{F}(g_1))(\omega).
\]

Therefore

\[
(77) \quad \hat{h} = m * \mathcal{F}(g_2)\mathcal{F}^{-1}(\mathcal{F}(g_1)).
\]

Since the windows \( g_1, g_2 \) belong to \( L^2(\mathbb{R}^d) \), we have \( s(\omega) := \mathcal{F}(g_2)\mathcal{F}^{-1}(\mathcal{F}(g_1)) \in L^1(\mathbb{R}^d) \) and the right-hand side of (77) is bounded and uniformly continuous. Since we are assuming \( \hat{h} \) to be the characteristic function of an interval \( \Omega \subseteq \mathbb{R} \), we obtained the first assertion of the thesis.

About estimate (73) we distinguish two cases. If there is \( \omega_0 \in \mathbb{R} \) such that \( |s(\omega_0)| = 1/2 \), being the image of \( \hat{h} \) the set \( \{0,1\} \), then

\[
|\hat{h}(\omega_0) - s(\omega_0)| \geq ||\hat{h}(\omega_0)| - |s(\omega_0)|| \geq \frac{1}{2}
\]

which implies

\[
\sup_{\omega \in \mathbb{R}} |\hat{h}(\omega) - s(\omega)| = \|T_{m_2} - G_{g_a}^{g_1,g_2}\|_{op} \geq \frac{1}{2}.
\]

If the value 1/2 is never attained by \( |s(\omega)| \) the argument is identical. This concludes the proof.

In the finite discrete case the problem presents itself in a similar way. In the next theorem we state the necessary conditions on the window functions in order to get perfect equivalence. It can be seen that without subsampling perfect equivalence would in theory be always possible if \( \text{supp}(h) \subseteq \text{supp}(\mathcal{F}g_1 * g_2) \). Numerically, we observe however the same behavior as we have in the continuous case.

**Theorem 6.6.** Let us fix a LTI filter \( H : \mathbb{C}^N \rightarrow \mathbb{C}^N \) with impulse response \( h \in \mathbb{C}^N \) and lattice constants \( \alpha, \beta \geq 1 \).

If \( H \) can be written as a Gabor multiplier \( G_{g_a}^{g_1,g_2} \) with lattice constants \( \alpha, \beta \), for some symbol \( a \in \mathbb{C}^{N \times N} \) and window functions \( g_1, g_2 \in \mathbb{C}^N \), then the following hold for every \( \forall u \in \text{supp}(h) \):

1) \( V_{g_1}g_2(u,0) = (\mathcal{F}g_1 * g_2)(u) \neq 0 \);
2) \( V_{g_1}g_2(u + Bk,LA) = 0, \forall k = 0, \ldots, \beta - 1, \forall l = 1, \ldots, \alpha - 1 \);
3) \( V_{g_1}g_2(u + Bk,0) = 0, \forall k = 1, \ldots, \beta - 1 \text{ s.t. } (u + Bk) \notin \text{supp}(h) \);
4) \( V_{g_1,g_2}(u + Bk, 0) = \frac{h(u + Bk)}{h(u)} V_{g_1,g_2}(u, 0), \forall k = 1, \ldots, \beta - 1 \) s.t. \((u + Bk) \in \text{supp}(h)\).

Vice versa, if there are window functions \( g_1, g_2 \in \mathbb{C}^N \) fulfilling 1)–4), then there exists a symbol \( a \in \mathbb{C}^{N \times N} \) such that \( H = G_{g_1,g_2}^a \).

**Proof.** Let us assume that \( H = G_{g_1,g_2}^a \) for some \( a \in \mathbb{C}^{N \times N}, g_1, g_2 \in \mathbb{C}^N \). Two operators are identical if and only if their spreading functions are identical. From (63) and (71), \( H = G_{g_1,g_2}^a \) if and only if

\[
(78) \quad h \otimes \delta(u, v) = \frac{N}{\alpha \beta} S_{BA}^A(u, v) V_{g_1,g_2}(u, v)
\]

This, in turn is equivalent to

\[
(79) \quad h(u) = N (\alpha \beta)^{-1} S_{BA}^A(u, 0) V_{g_1,g_2}(u, 0), \quad u = 0, \ldots, N - 1;
\]

\[
(80) \quad 0 = N (\alpha \beta)^{-1} S_{BA}^A(u, v) V_{g_1,g_2}(u, v), \quad u, v = 0, \ldots, N - 1, \quad v \neq 0.
\]

From equation (79), condition 1) follows. Note that for \( S_{BA}^A(u, 0) = 0 \) we have from equation (78) \( h(u) = 0 \) and therefore \( u \notin \text{supp}(h) \). Hence by equation (80) together with the periodicity of \( S_{BA}^A \) it follows condition 2). The periodicity of \( S_{BA}^A \) in the time domain together with equation (79) gives condition 3). Finally through (79) and (70) we compute

\[
(81) \quad \frac{\alpha \beta}{N} \frac{h(u)}{V_{g_1,g_2}(u, 0)} = S_{BA}^A(u, 0) = S_{BA}^A(u + Bk, 0) = \frac{\alpha \beta}{N} \frac{h(u + Bk)}{V_{g_1,g_2}(u + Bk, 0)}
\]

for \( k = 1, \ldots, \beta - 1, \quad (u + kB) \in \text{supp}(h) \), hence we get condition 4).

On the other hand, let us consider \( g_1, g_2 \in \mathbb{C}^N \) fulfilling conditions 1) – 4). Let us define for \( u = 0, \ldots, N - 1 \)

\[
(82) \quad V(u) := \begin{cases} 
V_{g_1,g_2}(u, 0) & \text{if } u \in \text{supp}(h) \\
1 & \text{otherwise}
\end{cases}
\]

and

\[
(83) \quad C(u) := \begin{cases} 
\#\{u + B\mathbb{Z}_N\} \cap \text{supp}(h) & \text{if } \{u + B\mathbb{Z}_N\} \cap \text{supp}(h) \neq \emptyset \\
1 & \text{otherwise},
\end{cases}
\]

we notice that \( C(u + Bk) = C(u) \) for any \( k = 0, \ldots, \beta - 1 \), since \( u + B\mathbb{Z}_N = u + Bk + B\mathbb{Z}_N \).
Let us observe that
\[
\frac{h}{C \cdot V} \ast \chi_{BZ_N}(u) = \sum_{k=0}^{N-1} \frac{h(u-k)}{C(u-k)V(u-k)} \chi_{BZ_N}(k) = \frac{1}{C(u)} \sum_{k=0}^{N-1} \frac{h(u-k)}{V(u-k)} \chi_{-BZ_N}(k)
\]
\[
= \frac{1}{C(u)} \sum_{k=0}^{\beta-1} \frac{h(u+Bk)}{V(u+Bk)} = \frac{1}{C(u)} \left( \frac{h}{V} \ast \chi_{BZ_N} \right)(u).
\]

We define
\[
S_{BA}^P(u, v) := \frac{\alpha \beta}{N} \left( \frac{h}{C \cdot V} \ast \chi_{BZ_N} \right) \otimes \chi_{AZ_N}(u, v),
\]
which is periodic in the sense of (70) since \(C(u+Bk) = C(u)\) for any \(k = 0, \ldots, \beta-1\) and
\[
\left( \frac{h}{V} \ast \chi_{BZ_N} \right)(u+Bk) = \frac{1}{C(u)} \sum_{j=0}^{N-1} \frac{h(j)}{V(j)} \chi_{BZ_N}(u+Bk-j)
\]
\[
= \frac{1}{C(u)} \sum_{j=0}^{N-1} \frac{h(j)}{V(j)} \chi_{BZ_N-Bk}(u-j)
\]
\[
= \frac{1}{C(u)} \sum_{j=0}^{N-1} \frac{h(j)}{V(j)} \chi_{BZ_N}(u-j)
\]
\[
= \left( \frac{h}{V} \ast \chi_{BZ_N} \right)(u).
\]

In order to verify (79), fix \(u \in \{0, \ldots, N-1\}\) and let us write the partition
\[
\{0, \ldots, \beta - 1\} = S_{in}(u) \cup S_{out}(u),
\]
where
\[
S_{in}(u) := \{ k \in \{0, \ldots, \beta - 1\} \mid u + Bk \in \text{supp}(h) \},
\]
\[
S_{out}(u) := \{ k \in \{0, \ldots, \beta - 1\} \mid u + Bk \notin \text{supp}(h) \}.
\]
Therefore if \( u \in \text{supp}(h) \) we have \( 0 \in S_{in}(u) \neq \emptyset \), starting from the right-hand side of \((79)\) and using \(4\) we get

\[
N(\alpha\beta)^{-1}S_{BA}^P(u,0)V_{g_1g_2}(u,0) = \frac{1}{C(u)} \sum_{k=0}^{\beta-1} \frac{h(u+Bk)}{V(u+Bk)} V_{g_1g_2}(u,0)
= \frac{1}{C(u)} \sum_{k \in S_{in}(u)} \frac{h(u+Bk)}{V_{g_1g_2}(u+Bk,0)} V_{g_1g_2}(u,0)
= \frac{1}{C(u)} \sum_{k \in S_{in}(u)} \frac{h(u)}{V_{g_1g_2}(u,0)} V_{g_1g_2}(u,0)
= \frac{1}{C(u)} C(u) h(u).
\]

If \( u \notin \text{supp}(h) \) and \( S_{in}(u) = \emptyset \), then \( S_{BA}^P(u,0) = 0 \) and \((79)\) is fulfilled. If \( u \notin \text{supp}(h) \) and \( S_{in}(u) \neq \emptyset \), then \( u + Bj = z \in \text{supp}(h) \) for some \( j \in S_{in}(u) \). Hence we can write \( u = z - Bj = z + Bs \) for a certain \( s \in \{0, \ldots, \beta - 1\} \) and from \(3\) we get \( V_{g_1g_2}(u,0) = V_{g_1g_2}(z + Bs,0) = 0 \), which guarantees \((79)\).

Equation \((80)\) is fulfilled if \( v \notin A\mathbb{Z}_N \setminus \{0\} \). Let us fix \( v \in A\mathbb{Z}_N \setminus \{0\} \) and distinguish two cases: if \( u \) appearing in \((80)\) belongs to \( \text{supp}(h) + B\mathbb{Z}_N \), then \( V_{g_1g_2}(u,v) = 0 \) due to \(2\) and we are done; if \( u \) does not belong to \( \text{supp}(h) + B\mathbb{Z}_N \), then \( S_{BA}^P(u,v) = 0 \) and \((80)\) if verified once more.

Eventually, in order to find a symbol \( a \) which gives the function \( S_{BA}^P \) defined above, we use \((72)\):

\[
(85) \quad \alpha\beta F_s(a \cdot \mathbf{III}_{\alpha,\beta})(u,v) = \frac{\alpha\beta}{N} \left( \frac{h}{C \cdot V} \ast \chi_{A\mathbb{Z}_N} \right) \otimes \chi_{A\mathbb{Z}_N}(u,v).
\]

Being \( F_s^{-1} = F_s \) and for \((56)\) we derive

\[
a(u,v)\mathbf{III}_{(\alpha,\beta)}(u,v) = \frac{1}{N} F_s \left( \left( \frac{h}{C \cdot V} \ast \chi_{A\mathbb{Z}_N} \right) \otimes \chi_{A\mathbb{Z}_N} \right)(u,v)
= \frac{1}{N} A^{-1} \chi_{A\mathbb{Z}_N}(u) F_N \left( \frac{h}{C \cdot V} \ast \chi_{B\mathbb{Z}_N} \right)(v)
= \alpha \frac{1}{N^2} \chi_{A\mathbb{Z}_N}(u) F_N \left( \frac{h}{C \cdot V} \right)(v) \beta \chi_{B\mathbb{Z}_N}(v)
= \frac{\alpha\beta}{N^2} F_N \left( \frac{h}{C \cdot V} \right) (v) \mathbf{III}_{(\alpha,\beta)}(u,v).
\]

So that a possible choice for the symbol is

\[
(86) \quad a(u,v) = \frac{\alpha\beta}{N^2} \left( 1 \otimes F_N \left( \frac{h}{C \cdot V} \right) \right)(u,v).
\]
This concludes the proof. □

**Figure 2.** This figure gives a visual outline of Theorem 6.6 on the representation of a LTI filter by a Gabor multiplier. The conditions on the support of \( V_{g_1g_2}(u,v) \) are shown once for \( \text{supp}(h) \subseteq [-B,B] \) and once for \( \text{supp}(h) \not\subseteq [-B,B] \). Black lines indicate the regions where \( V_{g_1g_2} \) has to be zero.

**Remark 6.7.** Theorem 6.6 can be seen as a special result on the reproducing property, compare [36] or [47] equation (4). If \( \text{supp}(h) \subseteq (-B,B) \) we would get perfect reproduction for \( V_{g_1g_2}(u,0) = 1 \) on \( u \in \text{supp}(h) \) and \( V_{g_1g_2}(u,v) = 0 \) outside the fundamental region of the adjoint lattice for \( (u,v) \not\in (-B,B) \times (-A,A) \). If \( \text{supp}(h) \subset (-B,B) \), the region \( X \) with \( \text{supp}(h) \subset X \subset (-B,B) \) introduces the freedom to choose \( (I_{g_1} * g_2)(u) \) having smooth decay on \( X \). As for an LTI filter \( \eta_H(u,v) = 0 \ \forall v \neq 0 \), see (63), we have this freedom in the frequency domain for \( Y := \{(x,y) : 0 < |y| < A\} \) irrespective of the choice of \( h \).

The conditions given in Theorem 6.6 will be central for the remaining part of this section. Therefore a visual outline of them is shown in Figure 2. The next theorem can be seen as a special case of the last result, having no subsampling, i.e. \( \alpha = \beta = 1 \).
Theorem 6.8. Consider a LTI filter \( H : \mathbb{C}^N \rightarrow \mathbb{C}^N \) with impulse response \( h \in \mathbb{C}^N \) and \( g_1, g_2 \in \mathbb{C}^N \) with \( (\mathcal{I}g_1 \ast g_2)(u) \neq 0 \) for every \( u = 0, \ldots, N-1 \). Then the \( H \) can be represented as Gabor multiplier \( G_{g_1,g_2}^a \) with \( \alpha = \beta = 1 \) and lower symbol

\[
(87) \quad a = \frac{1}{N^2} \left( 1 \otimes \mathcal{F}_N \left( \frac{h}{\mathcal{I}g_1 \ast g_2} \right) \right).
\]

Proof. Let us observe that, since \( \alpha = \beta = 1 \), we have \( S = S_{P}^{BA} \), see (66) and (72). Taking \( a \) as in (87), recalling \( \mathcal{I}g_1 \ast g_2(\cdot) = V_{g_1}g_2(\cdot,0) \) and \( \mathcal{F}_N(N^{-1}1)(v) = \delta(v) \), we compute

\[
S(u,v) = F_s a(u,v) = \frac{1}{N} F_s \left( \frac{1}{N} 1 \otimes \mathcal{F}_N \left( \frac{h(\cdot)}{V_{g_1}g_2(\cdot,0)} \right) \right)(u,v) = \frac{1}{N V_{g_1}g_2(u,0)} \cdot \delta(v).
\]

Similarly to what done in the proof of Theorem 6.6, \( H \) and \( G_{g_1,g_2}^a \) coincide if their spreading functions do; on account of the previous computation we get

\[
h \otimes \delta(u,v) = NS(u,v)V_{g_1}g_2(u,v) = \frac{h(u)}{V_{g_1}g_2(u,0)} \delta(v)V_{g_1}g_2(u,v)
\]

which is true since \( V_{g_1}g_2(u,0) = \mathcal{I}g_1 \ast g_2(u) \neq 0 \) for every \( u \). This concludes the proof. \( \square \)

This means, given window functions, for which the convolution (up to \( \mathcal{I} \) and a conjugation) is non-zero on the support of the impulse response \( h \), a LTI filter \( H \) can always be represented exactly as Gabor multiplier \( G_{g_1,g_2}^a \). The error between the LTI filter and the Gabor multiplier is the error introduced through subsampling of the mask \( a \). The representation is always possible if we allow for the degenerate case of \( g_1 = g_2 = 1 \). We should, however, keep in mind that if we want to have a meaningful parameter set for applications this is after all a very strong condition on the smoothness of \( \hat{h} \). Even if met, for applications, the exact representation is not too well suited due to poor calculation efficiency and bad numerical behaviour for \( \mathcal{I}g_1 \ast g_2 \) close to zero.

Knowing from Theorem 6.8 that every LTI filter with bandlimited impulse response \( h \) can be represented as Gabor multiplier, we are now turning the focus to the opposite direction, asking whether it is clear that a Gabor multiplier having a mask constant in time is equivalent to a LTI filter. Reading equation (87) the other way round, we see implicitly that a Gabor multiplier with time invariant symbol is a convolution operator. The frequency response of this convolution operator, however, is not exactly equal to the frequency mask of the Gabor multiplier but smoothed by a convolution with the Fourier transform of the window functions. In Figure 3 a visual representation can be found. Smooth window functions have the advantage of preserving the edges of the frequency mask rather well at the cost of a longer time delay needed in return. Theorem 6.9 formalizes this fact.
Figure 3. The figure shows the effect of implementing a Gabor (STFT) multiplier with mask $a = 1 \otimes \chi_{\Omega}$, $\Omega = [-R, R]$, with $R = 80$ and $N = 480$. The resulting operator is still an LTI operator as long as no subsampling is performed ($\alpha = \beta = 1$), but now looking at the difference of the spectrograms given in the first plot, which is strongly concentrated around the cut-off frequency. The central plot shows the 20 largest singular values of the difference between the implemented STFT multiplier and the perfect low pass filter. In the last plot, we show only a segment of the first singular vector, because otherwise, we would not be able to see well the high regular oscillations.

**Theorem 6.9.** Consider a Gabor multiplier $G_{g_1,g_2}^{\alpha,\beta}$ with no time subsampling, i.e. $\alpha = 1$, symmetric windows $g_1, g_2 \in \mathbb{C}^N$ and symbol

$$a = 1 \otimes \hat{h}$$
for some $\hat{h} \in \mathbb{C}^N$. Then it is also a LTI filter with impulse response

$$
(89) \quad \frac{1}{\beta} \sum_{k=0}^{\beta-1} h(\cdot + Bk)(\overline{f_1} * g_2)(\cdot).
$$

**Proof.** We start from the kernel representation of the Gabor multiplier (65) with $\alpha = 1$

$$
K(G_{a \cdot}^{g_1,g_2})(u,v) = \sum_{k=0}^{N-1} \sum_{l=0}^{B-1} a(k,\beta l) \overline{g_1}(v-k)g_2(u-k) e^{\frac{2\pi i \beta l(u-v)}{N}}
$$

$$
= \sum_{k=0}^{N-1} \sum_{l=0}^{B-1} \hat{h}(\beta l) \overline{g_1}(v-k)g_2(u-k) e^{\frac{2\pi i \beta l(u-v)}{N}}
$$

$$
= \sum_{l=0}^{B-1} \hat{h}(\beta l) e^{\frac{2\pi i \beta l(u-v)}{N}} \sum_{k=0}^{N-1} \overline{g_1}(v-k)g_2(u-k).
$$

$$
(90)
$$

Fixing $v \in \{0, \ldots, N-1\}$, performing the change of variable $t = v - k$ and using the symmetry of $g_1$ and $g_2$, we write the second factor as

$$
\sum_{k=0}^{N-1} \overline{g_1}(v-k)g_2(u-k) = \sum_{t=0}^{N-1} \overline{g_1}(t)g_2(u-v+t)
$$

$$
= \sum_{t=0}^{N-1} \overline{g_1}(t)g_2(u-v-t)
$$

$$
= (\overline{f_1} * g_2)(u-v).
$$

For the first factor in (90), using (56):

$$
\sum_{l=0}^{B-1} \hat{h}(\beta l) e^{\frac{2\pi i \beta l(u-v)}{N}} = \mathcal{F}_N^{-1}(\hat{h} \cdot \chi_{\beta Z_N})(u-v)
$$

$$
= \mathcal{F}_N^{-1} \hat{h} \mathcal{F}_N^{-1} \chi_{\beta Z_N}(u-v)
$$

$$
= \sum_{k=0}^{N-1} h(u-v-k) \frac{1}{N} \mathcal{F}_N \chi_{\beta Z_N}(-k)
$$

$$
= \sum_{k=0}^{N-1} h(u-v-k) \frac{N}{B} \chi_{BZ_N}(-k)
$$

$$
= \frac{B}{\beta} \sum_{k=0}^{\beta-1} h(u-v+Bk).
$$
Eventually we get

$$K(g_1^{g_2})(u,v) = \frac{1}{\beta} \sum_{k=0}^{\beta-1} h(u-v+Bk)(\mathcal{F}_1 \ast g_2)(u-v)$$

and the thesis follows by \((62)\).

We observe that the convolution in \((89)\) is the restriction of \(V_{g_1}g_2\) to the time-axis, since we are considering symmetric windows \(g_1\) and \(g_2\).

It is important to note that the LTI property is only valid in case of no time subsampling. In case of a common Gabor multiplier with \(\alpha > 1\), in contrast, the second sum in equation \((90)\) would depend on \(u\) and be \(\alpha\)-periodic, explicitly:

$$\sum_{k=0}^{A-1} g_1(u-\alpha k)g_2(u-\alpha k).$$

Therefore as soon as we have time domain subsampling of the signal, the LTI property of the operator is lost even though the mask being constant in time.

As already mentioned, it becomes apparent that an LTI filter can be considered as a special case of a Gabor multiplier with degenerated window functions \(g_1 = g_2 = 1\).

We want to put emphasis also on the interconnection between sharp frequency cut off of the filter and smoothness of the window functions corresponding to a time delay in filtering. Condition 1) of Theorem 6.6 requires the impulse response \(h\) to have a faster decay than \(\mathcal{F}_1 \ast g_2\). This means that in case we want to have a sharp cut off in the frequency filter \(\hat{h}\), which corresponds to a slow decay in \(h\), we have to choose a smooth window function which corresponds to a large time lag.

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