On Internal and External Alignment of Dust Grains in Protostellar Environments

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Abstract

Multiwavelength observations toward protostars reveal complex properties of dust polarization, which are challenging to interpret. Here we study the physical processes inducing the alignment of the grain axis of the maximum inertia moment with the angular momentum (\(J\); i.e., internal alignment) and of \(J\) with the magnetic field (i.e., external alignment) of very large grains (VLGs; of radius \(a > 10 \mu\text{m}\)) using the alignment framework based on radiative torques (RATs) and mechanical torques (METs). We derive analytical formulae for critical sizes of grain alignment, assuming grains aligned at low-\(J\) and high-\(J\) attractors by RATs (METs). For protostellar cores, we find that super-Barnett relaxation induces efficient internal alignment for VLGs with large iron inclusions, but inelastic relaxation is inefficient for VLGs regardless of composition aligned at high-\(J\) attractors by RATs (METs). For external alignment, VLGs with iron inclusions aligned at high-\(J\) attractors have magnetic alignment by RATs (B-RAT) or METs (B-MET), enabling dust polarization as a reliable tracer of magnetic fields in dense regions. Still, grains at low-\(J\) attractors or without iron inclusions have alignment with \(J\) along the radiation direction (\(k\)-RAT) or gas flow (\(v\)-MET). For protostellar disks, we find that super-Barnett relaxation is efficient for grains with large iron inclusions in the outer disk thanks to spin-up by METs, but inelastic relaxation is inefficient. VLGs aligned at low-\(J\) attractors can have \(k\)-RAT (\(v\)-MET) alignment, but grains aligned at high-\(J\) attractors likely exhibit B-RAT (B-MET) alignment. We also find that grain alignment by METs is more important than that by RATs in protostellar disks.

Unified Astronomy Thesaurus concepts: Starlight polarization (1571); Interstellar dust (836); Interstellar magnetic fields (845); Protostars (1302); Protoplanetary disks (1300); Interstellar dust processes (838)

1. Introduction

Dust and magnetic fields are ubiquitous in the universe and play important roles in many astrophysical processes, including star formation and evolution of the interstellar medium (ISM). Dust grains are the building blocks of planets and catalytic surfaces for the formation of water and complex molecules. Dust absorbs optical-UV starlight and reemits in the infrared, which is a powerful tracer of modern astrophysics (Draine 2011). Interstellar dust grains are known to have nonspherical shapes and are aligned with interstellar magnetic fields, as demonstrated through the polarization of starlight (Hall 1949; Hiltner 1949) as well as polarized thermal dust emission (Hildebrand 1988; Planck Collaboration et al. 2015). Therefore, dust polarization induced by grain alignment becomes the popular technique to observe interstellar magnetic fields (see Pattie & Fissel 2019 for a recent review) and extragalactic magnetism (Lopez-Rodriguez et al. 2022).

The process of grain alignment in general includes (1) the alignment of the axis of the maximum moment of inertia (\(a_1\)) with the angular momentum (\(J\); so-called internal alignment) and (2) the alignment of \(J\) with a preferred direction in space (the magnetic field, the anisotropic radiation, or the gas flow; so-called external alignment, see Andersson et al. 2015; Lazarian et al. 2015).

The leading process for the internal alignment of interstellar grains is the Barnett relaxation effect (Purcell 1979, hereafter P79). The Barnett relaxation arises from the dissipation of grain rotational energy into heat due to rotating Barnett magnetization within a paramagnetic grain rotating around a nonprincipal axis. For the ISM (e.g., the diffuse ISM and molecular clouds (MCs)) where grains are essentially small (of radius of \(a < 1 \mu\text{m}\), the Barnett relaxation is usually faster than the randomization of grain orientation by gas collisions, resulting in the perfect internal grain alignment with \(a_1 \parallel J\) (e.g., Hoang 2022). For external alignment, the Larmor precession of the grain magnetic moment aligned with \(J\) around the magnetic field (\(B\)) is much faster than the gas randomization, so that \(B\) is the axis of grain alignment (e.g., Hoang 2022). Moreover, the angular momentum \(J\) can be aligned with \(B\) by radiative torques (RATs; Dolginov & Mitrofanov 1976; Draine & Weingartner 1997; Lazarian & Hoang 2007a; Hoang & Lazarian 2008), mechanical torques (METs; Lazarian & Hoang 2007b; Hoang et al. 2018), and/or enhanced paramagnetic relaxation (Davis & Greenstein 1951; Lazarian & Hoang 2008). The first two mechanisms are the most efficient, which are known as the B-RAT and B-MET alignments.

In this paper, small grains are defined by their sizes of \(a < 1 \mu\text{m}\), large grains of \(a \sim 1-10 \mu\text{m}\), and very large grains of sizes \(a \geq 10 \mu\text{m}\).
whereas the third one is to enhance the efficiency of B-RAT and B-MET alignment (Hoang & Lazarian 2016a). Within the RAT or MET paradigm, the effect of paramagnetic relaxation can enhance the alignment degree beyond the level induced by RATs or METs alone. Thus, this effect is called magnetically enhanced RAT alignment (MRAT; Hoang & Lazarian 2016a) or magnetically enhanced MET alignment (Hoang et al. 2018).

The resulting polarization of starlight and of thermal dust emission is parallel and perpendicular to the magnetic field, respectively. Therefore, dust polarization is a reliable tool for tracing interstellar magnetic fields.

Magnetic fields are usually thought to play an important role in the formation of stars and protostellar disks (see Crutcher 2012; Pattle & Fissel 2019 for reviews). Recent advances in interferometric polarimetry allow us to conduct high-spatial resolution observations of polarized thermal dust emission toward very dense regions where young stars are forming (hereafter star-forming regions or SFRs; Hull & Zhang 2019). For instance, dust polarization observations toward a large sample of protostars (Class 0 young stellar objects; YSOs) using Atacama Large Millimeter Array (ALMA) are reported by Cox et al. (2018; for Perseus MC), Sadavoy et al. (2019; for Ophiuchus cloud), and Liu (2021; for OMC-3). ALMA dust polarization observations toward small scales of ~100 au within protostellar disks have also been reported (Kataoka et al. 2017; Stephens et al. 2017). We note that on disk scales of ~100 au, dust polarization can arise from scattering of radiation emitted by dust itself, a process known as self-scattering (Kataoka et al. 2015). The key features of submillimeter dust polarization observed toward protostars include the complex variation of dust polarization (in both pattern and fraction) from the protostellar envelope to the inner disk region (≤100 au) and the variation of the polarization pattern with the wavelength (e.g., Kataoka et al. 2017; Stephens et al. 2017; Liu 2021). As a result, the question of whether dust polarization can trace magnetic fields in protostellar environments is crucially important, yet remains unclear. To answer this question, we need first to understand how and where dust grains can align and what are the polarization properties produced by aligned grains in these environments.

Incidentally, very large grains (VLGs) of radius $a > 10\,\mu m$ are often reported in protostellar environments, including the envelope (Miotello et al. 2014), protostellar disks around Class 0 protostars (Kwon et al. 2009; Galametz et al. 2019), and protoplanetary disks (e.g., Testi et al. 2014; Maupoil et al. 2021). Therefore, the key difference in the physical properties of the ISM and protostellar environments is the huge difference in grain sizes and the gas density by several orders of magnitudes. This indicates that the leading mechanisms for grain alignment in the ISM cannot be directly applied to the protostellar conditions.

Extensive efforts have been done to model and interpret submillimeter dust polarization from protostellar environments (see, e.g., Valdivia et al. 2022). Nevertheless, previous studies usually make simplified assumptions on grain alignment physics for modeling dust polarization. For example, grains are usually assumed to be aligned with the shortest axis ($\hat{a}_1$) perfectly aligned with the magnetic field or radiation direction (Ohashi et al. 2018, 2020; Yang et al. 2018; Harrison et al. 2019; Kataoka et al. 2019). This assumption disregarded the important effect of internal alignment (e.g., Lin et al. 2021; Mori & Kataoka 2021; Tatsuuma & Kataoka 2021), which does not ensure that the inferred magnetic fields are accurate.

A detailed modeling of thermal dust polarization by aligned grains due to RATs in a protoplanetary disk was presented in Tazaki et al. (2017). They found that VLGs can be aligned with $\mathbf{J}$ along the radiation direction ($\mathbf{k}$) instead of along the $\mathbf{B}$, which is known as k-RAT, as predicted in Lazarian & Hoang (2007a). However, they assumed the perfect internal alignment of all grains rotating suprathermally at an attractor point with angular momentum much greater than the thermal value (i.e., high-J attractors; Hoang & Lazarian 2008, 2014). Such an assumption is not always justified because the perfect internal alignment is only achieved when internal relaxation is faster than gas randomization. Note that internal relaxation depends not only on the rotation rate, but also on the grain size, the magnetic susceptibility (for Barnett relaxation; Hoang 2022), and shear modulus of grain material (for inelastic relaxation; P79; Lazarian & Efroimsky 1999 (LE99)). Moreover, before grains reaching a high-J attractor by RATs/METs, they experience a period of thermal rotation during which efficient internal alignment is not satisfied (Hoang & Lazarian 2008, 2016a). As the net dust polarization is dominated by grains at high-J attractors (Hoang & Lazarian 2014, 2016a), a detailed study of the dependence of internal relaxation on the grain angular velocity is required to accurately determine how internal alignment occurs at high-J attractors. Here we refer to the k-RAT with the perfect internal alignment of $\hat{a}_1|| \mathbf{J}$ as right k-RAT. Subsequent studies for HL Tau (Stephens et al. 2017; Kataoka et al. 2017) interpret the azimuthal polarization patterns as the evidence of right k-RAT, but the details of grain alignment are not discussed. Yang et al. (2018) attempted to interpret dust polarization observed toward HL Tau and suggested that the Gold mechanism (Gold 1952) can produce the elliptical polarization pattern, compared to the circular pattern predicted by the right k-RAT. As shown by Lazarian & Hoang (2007a) and Hoang et al. (2018), mechanical torques are far more efficient than the Gold mechanism. Later, Kataoka et al. (2019) discussed dust polarization produced by grain alignment by mechanical torques, but the perfect internal alignment is also assumed. Recently, Draine (2022) suggested that circular polarization of dust emission could be detected in protoplanetary disks, assuming that the grain has efficient internal alignment so that the shortest axis is parallel to $\mathbf{J}$.

We note that for paramagnetic and superparamagnetic grains (i.e., grains with iron inclusions) in the ISM, which are small of radius $a \lesssim 1\,\mu m$ (see, e.g., Hoang & Lazarian 2016a), internal alignment is expected to be efficient thanks to fast Barnett relaxation and nuclear relaxation compared to the randomization of grain orientation by gas collisions (Hoang & Lazarian 2009a; Hoang 2022). Recently, Hoang (2022) studied the internal alignment by Barnett and nuclear relaxation effects for dust grains in dense clouds (DCs) and found that micron-sized grains can have efficient internal alignment if grains have embedded iron inclusions (see also Hoang & Lazarian 2016a). Indeed, observations reveal that about 90% of Fe is locked in dust (Jenkins 2009; Dwye 2016). Although the form of Fe in dust is unclear, one expect that a fraction of Fe is in the forms of metallic Fe or iron oxides (FeO, Fe$_2$O$_3$, Fe$_3$O$_4$) nanoparticles. Thus, the existence of iron inclusions is expected for large dust grains in protostellar environments, which are grown by grain–grain collisions, during which iron nanoparticles are most likely incorporated into large dust grains. Moreover, iron
nanoparticles are reported to be present in local interstellar dust by the Cassini mission (Altobelli et al. 2016) and in primitive interplanetary dust (Hu & Winarski 2021). Because physical conditions (i.e., gas density, radiation field, and gas kinematic) of protostellar environments are much different from those of DCs, the question whether VLGs can have efficient internal alignment remains uncertain and will be addressed in this paper. For our study, we consider composite grains with iron inclusions, which is the most likely form of dust in protostellar environments.

Moreover, the problem of the external alignment of VLGs in protostellar environments is not yet studied in detail. External alignment of grains in protostellar environments is expected to be complicated due to the dynamical nature of star-forming regions. While grains in the diffuse ISM and MCs are most likely aligned with $\mathbf{J}$ along the magnetic field via RATs/METs (the so-called $B$-RAT/$B$-MET alignment), grains in protostellar environments may align with $\mathbf{J}$ along the radiation direction/gas flow (the so-called $k$-RAT/$v$-MET alignment; Lazarian & Hoang 2019). Understanding in what conditions $B$-RAT/$B$-MET alignment can still occur is crucially important for interpreting dust polarization and tracing magnetic fields using dust polarization. Therefore, the main challenge for modeling and interpreting dust polarization and for inferring magnetic fields in protostellar environments lies in the lack of detailed studies of grain alignment for these complex regions. The physics of grain alignment (both internal and external alignment processes) previously developed and tested for the ISM (see Andersson et al. 2015; Lazarian et al. 2015 for reviews) must be revised for protostellar environments.

For the DC conditions, as shown in Hoang (2022), Barnett relaxation can be efficient only when grains have iron inclusions, but the maximum size for internal alignment is less than $10 \mu m$. The efficiency of internal relaxation can be enhanced by increasing its grain angular momentum to above the thermal value (i.e., grains having suprathermal rotation; Hoang 2022). Suprathermal rotation can be achieved by RATs due to protostellar radiation (Hoang et al. 2021) or METs due to grain drift relative to the gas (Lazarian & Hoang 2007b; Hoang et al. 2018). Therefore, the most promising mechanism that can induce internal alignment is inelastic relaxation due to its rapid increase with the angular velocity ($P79$; LE99), which will be quantified in this study.

The main goal of this paper is to discuss in detail the main processes of internal and external alignment for very large grains in protostellar environments and determine the range of grain sizes with efficient alignment using the framework of the RAT (MET) alignment. Such a range of grain sizes with efficient grain alignment is required for the physical forward modeling of dust polarization and interpreting observational polarization data. Grain alignment is also found to be important for grain growth in protostellar environments and leaves imprints in the internal structure of cometary dust aggregates that can be observed via in situ spacecraft (Hoang 2022).

The paper is structured as follows. In Section 2, we describe the grain model and magnetic properties. In Section 3, we review the essential components of the modern paradigm of interstellar grain alignment based on RATs and METs. In Sections 4 and 5, we study the leading physical processes that induce internal alignment and external alignment, and then derive the basic formulae for quantifying grain alignment in protostellar environments. In Section 6, we apply our theoretical framework obtained from the previous section to study grain alignment within a protostellar disk. An extended discussion on implications of our results for dust polarization observations is presented in Section 7. We summarize our main findings in Section 8.

2. Grain Model and Assumptions

2.1. Grain Geometry

Astrophysical dust grains are expected to have an irregular (nonspherical) shape to produce the polarization of starlight and thermal dust emission. A triaxial irregular shape is described by the principal axes $\hat{a}_1, \hat{a}_2, \hat{a}_3$. Let $I_1 > I_2 > I_3$ be the principal moments of inertia along the principal axes, respectively. However, for the sake of convenience of numerical estimates, throughout this paper, we assume an oblate spheroidal shape for dust grains, such that the principal moments of inertia along the principal axes are $I_1 > I_2 = I_3$. Let us denote $I_\parallel = I_1$ and $I_\perp = I_2$ for simplicity, and $h = I_\parallel / I_\perp$ be the ratio of the principal moments of inertia. The length of the symmetry (semiminor) axis is denoted by $c$, and the lengths of the semimajor axes are denoted by $a$ and $b$ with $a = b$. The assumed grain model is illustrated in Figure 1.

For a general case, the grain angular momentum ($\mathbf{J}$) and angular velocity ($\Omega$) are not parallel to the axis of maximum inertia, $\hat{a}_1$. So let $\theta$ be the angle between $\mathbf{J}$ and $\hat{a}_1$. The angular velocity component projected onto $\hat{a}_1$ is $\Omega_1 = J_1 / I_1 = J \cos \theta / I_1 \equiv \Omega_1 \cos \theta$ with $\Omega_0 = J / I_1$ (see Figure 1). In the grain’s body frame, the tips of $\mathbf{J}$ and $\Omega$ precess around $\hat{a}_1$ with the same angular rate of $\omega = (h - 1)\Omega_1 = (h - 1)\Omega_0 \cos \theta$, which is evaluated at $\theta = \pi/4$ for numerical estimates (e.g., Purcell 1979; Hoang et al. 2010).

Figure 1. Illustration of torque-free motion of an oblate spheroidal grain with semimajor and semiminor axes of $a$ and $c$. In the grain’s body frame, the angular momentum ($\mathbf{J}$) and angular velocity ($\Omega$) are both precessing around the axis of maximum moment of inertia $\hat{a}_1$ with an angular rate $\omega$. 
The principal moments of inertia for the rotation parallel and perpendicular to the grain symmetry axis are given by
\[ I_\parallel = \frac{8\pi}{15} \rho a^4 c = \frac{8\pi}{15} \rho a^5 s, \]
\[ I_\perp = \frac{4\pi}{15} \rho a^2 c (a^2 + c^2) = \frac{4\pi}{15} \rho a^5 (1 + s^2), \]
where \( \rho \) is the mass density of the grain, \( s = c/a < 1 \) is the axial ratio, and \( h = I_\parallel/I_\perp = 2/(1 + s^2) > 1 \) for \( s < 1 \) (see also Hoang & Lazarian 2014). The effective grain size \( a_{\text{eff}} \) is defined as the equivalent sphere of the same volume, so \( a_{\text{eff}} = (\pi c^2 a)/3 = s^3/a. \) For our formula and numerical estimates, we use the semimajor length \( a \) and adopt a fixed axial ratio of \( s = 1/2. \)

2.2. Grain Rotation and Rotational Damping

The rotational kinetic energy of an axisymmetric grain is given by
\[ E_{\text{rot}}(J, \Theta) = \sum_{i=1}^{3} \frac{I_0_{\Omega_i}^2}{2} = \sum_{i=1}^{3} J_i^2 = \frac{J_{\parallel}^2}{2h} + \frac{J_{\perp}^2}{2J_{\parallel}}, \]
\[ = \frac{J_{\parallel}^2}{2h} [1 + (h - 1) \sin^2 \Theta], \]
where \( J_{\parallel} = J_3 = J_2 \) and \( J_{\perp} = J_2^2 \sin^2 \Theta. \)

In the ISM, dust grains interact with gas species through random collisions and can achieve energy balance if we disregard other interaction processes such as with interstellar random collisions and can achieve energy balance if we disregard other interaction processes such as with interstellar radiation and cosmic rays. Let \( \Omega_T \) be the thermal angular velocity of the grain rotation along the symmetry axis in the gas of temperature \( T_{\text{gas}} \), which is defined by \( I_\parallel \Omega_T^2/2 = kT_{\text{gas}}/2. \) Then, \( \Omega_T = (kT_{\text{gas}}/I_\parallel)^{1/2} = 5.2 \times 10^4 \rho^{1/2} T_{\text{gas}}^{1/2} a^{-1/2} \text{rad s}^{-1} \) with \( T_{\text{gas}} = T_{\text{gas}}/10 \text{ K} \) and \( \rho = \rho/(3 \text{ cm}^{-3}). \) Throughout this paper, all values are given in cgs units, unless stated otherwise. The grain thermal angular momentum can be defined as \( J_T = I_0 \Omega_T. \)

Dust grains can be spun up to an angular momentum greater than its thermal value (i.e., superthermal rotation) by various grain surface processes (Purcell 1979), interaction with radiation (RATs), and gas flow (METs; see Hoang 2020 for a review). To describe the grain suprathermal rotation, we introduce a dimensionless parameter, \( \St = J/J_T = \Omega_\text{eff}/\Omega_T, \) which is referred to as the suprathermal rotation number.

Evaporation of gas species from the grain surface (e.g., those species that stick to the grain surface upon collisions) carries away some of the grain’s angular momentum and induces the damping of the grain rotation. The characteristic timescale of the gas rotational damping is (Roberge et al. 1993)
\[ \tau_{\text{gas}} = \frac{3}{4 \sqrt{\pi}} \frac{I_0}{1.22 m_{11} n_{11} v_T a^4} = \frac{\sqrt{\pi} \rho a s}{3 m_{11} n_{11} v_T I_0 \Gamma_\parallel}, \]
\[ \approx 0.083 \rho \left( \frac{a_{-5}}{n_{s, 23} T_{\text{gas, 1}}^{1/2}} \right) \text{ yr}, \]
where \( a_{-5} = a/(10^{-5} \text{ cm}), \) \( n_{s} = n_{11}/(10^8 \text{ cm}^{-3}), \) \( v_T = (2kT_{\text{gas}}/m_{11})^{1/2}, \) and the gas density is normalized using the typical value of the protostellar core with \( n_{11} = 10^8 \text{ cm}^{-3}. \)

Above, \( \Gamma_{\parallel} \) is the geometrical factor given by (Roberge et al. 1993)
\[ \Gamma_{\parallel}(e_m) = \frac{3}{16} (3 + 4(1 - e_m^2)) g(e_m) - e_m^2 [1 - (1 - e_m^2)^2 g(e_m)], \]
where \( e_m = \sqrt{1 - s^2} \) is the grain shape eccentricity, and \( g(e_m) \) is given by
\[ g(e_m) = \frac{1}{2e_m} \ln \left( \frac{1 + e_m}{1 - e_m} \right). \]

One has \( \Gamma_{\parallel} = 1 \) for spherical grains of \( s = 1, \) and \( \Gamma_{\parallel} = 0.51, \) 0.62, 0.74 for \( s = 1/3, 1/2, 2/3, \) respectively.

Subject to a radiation field of energy density \( u_{\text{rad}} \), dust grains are heated to high temperatures and subsequently cool down by infrared (IR) emission. The IR emission also results in the rotational damping of the grain due to the loss of angular momentum carried away by photons (see Draine & Lazarian 1998). For a grain in thermal equilibrium of equilibrhum temperature \( T_{\text{ir}} \), the IR damping rate is \( \tau_{\text{IR}} = F_{\text{IR}}/s_{\text{IR}} \) with \( F_{\text{IR}} \) being the dimensionless IR damping parameter,
\[ F_{\text{IR}} \approx \left( 3.8 \times 10^{-11} \right) \left( \frac{a_{10}}{n_{s, 23} T_{\text{gas, 1}}^{1/2}} \right), \]
where \( U_6 = U/10^6 \) with \( U = u_{\text{rad}}/u_{\text{tMMP63}} \) the strength of the radiation field; \( u_{\text{tMMP63}} \) is the energy density of the radiation field in the solar neighborhood from Mathis et al. (1983). For VLGs in protostellar environments of high gas density, \( F_{\text{IR}} \ll 1, \) so the rotational damping is dominated by gas collisions.

2.3. Magnetic Susceptibility and Magnetic Moment

Iron is among the most abundant elements in the universe. Observations reveal that about 90% of Fe is locked in dust (Jenkins 2009; Dwek 2016). The presence of Fe atoms with unpaired electrons makes dust a natural paramagnetic material (PM). Let \( f_p \) be the fraction of atoms that are paramagnetic (e.g., Fe) in the dust. The number density of paramagnetic atoms in the dust grain is then \( n_p = f_p n_p \) with \( n_p \) being the total number density of atoms.

The zero-frequency susceptibility of a paramagnetic grain at rest with a dust temperature \( T_d \) is described by Curie’s law:
\[ \chi_p(0) = \frac{n_p \mu_p^2}{3k_B T_d}, \]
which corresponds to
\[ \chi_p(0) \approx 0.06 f_p n_{23} \rho^{2} \left( \frac{10 K}{T_d} \right). \]

where \( n_{23} = n/10^{23} \text{ cm}^{-3}, \) \( \mu_p = \mu_B \) is the effective magnetic moment per iron atom with \( g \) the \( g \) factor and \( \rho = p/5.5, \) \( \mu_B = e \hbar/(2m_e) \) is the Bohr magneton. For silicate of MgFeSiO4 structure, one has \( f_p = 1/7 \) and \( p \approx 5.5 \) (see Hoang & Lazarian 2016a). A smaller value of \( f_p \) is expected because iron may be present in other structures.

Fe atoms are likely incorporated in the dust in the form of iron clusters, which makes dust a superparamagnetic material (SPM; e.g., Jones & Spitzer 1967). Iron nanoparticles are an
essential constituent in the composite Astrodust model (Draine & Hensley 2021). Let \( N_{cl} \) be the number of iron atoms per cluster and \( \phi_{sp} \) be the volume filling factor of iron clusters. Following Hoang & Lazarian (2016a), the zero-frequency superparamagnetic susceptibility is described by

\[
\chi_{sp}(0) \approx 0.052 N_{cl} \phi_{sp} \frac{10 \text{ K}}{T_d} \frac{10^{11}}{T_{d,1}},
\]

(10)

where the possible value of \( N_{cl} \) spans from \( \sim 20 \) to \( 10^5 \) (Jones & Spitzer 1967), and \( \phi_{sp} \sim 0.3 \) if 100% of the Fe abundance present in the dust is in the form of iron clusters (Hoang & Lazarian 2016a). For a given \( \phi_{sp} \), the superparamagnetic susceptibility is larger for grains with larger iron clusters.

Comparing Equations (10) and (9), one can see that \( \chi_{sp}(0) \sim \chi(0) \) for \( N_{cl} \sim 1 \), assuming \( \phi_{sp} \approx f_p \). Therefore, in the following, we adopt \( \chi_{sp} \) to describe the magnetic susceptibility of magnetic dust for a general case, and \( N_{cl} \sim 1 \) is implicitly intended for the PM case.

The imaginary part of the complex magnetic susceptibility is a function of the frequency and usually represented as \( \chi(\omega) = \omega K(\omega) \), where \( K(\omega) \) is the function obtained from solving the magnetization dynamics equation. For a superparamagnetic material, \( K_{sp}(\omega) \) is given by (see Hoang & Lazarian 2016a):

\[
K_{sp}(\omega) = \frac{\chi_{sp}(0) \tau_{sp}}{[1 + (\omega \tau_{sp}/2)^2]^2},
\]

\[
\approx 5.2 \times 10^{-11} N_{cl} \phi_{sp} \frac{10^{11}}{T_{d,1}} \frac{k_{sp}(\omega)}{T_d},
\]

(11)

where \( T_{d,1} = T_d/10 \text{ K} \), \( \tau_{sp} \) is the timescale of remagnetization by thermal fluctuations given by

\[
\tau_{sp} \approx \nu_0^{-1} \exp \left( \frac{N_{cl} T_{act}}{T_d} \right)
\]

(12)

with \( \nu_0 \) being the characteristic frequency of thermal fluctuations of iron clusters, and \( T_{act} \) the activation temperature of superparamagnetism (see, e.g., Jones & Spitzer 1967), which have the typical values of \( \nu_0 \approx 10^8 \text{ s}^{-1} \) and \( T_{act} \approx 0.011 \text{ K} \) from experiments (see Morrish 2001). Above,

\[
k_{sp}(\omega) = \exp \left( \frac{N_{cl} T_{act}}{T_d} \right) \left[ 1 + \left( \frac{\omega \tau_{sp}}{2} \right)^2 \right]^{-2},
\]

(13)

which increases with \( N_{cl} \) but decreases with increasing \( T_d \) and \( \omega \).

The susceptibility \( K_{sp}(\omega) \) is constant for low frequency but decreases rapidly at high frequency for \( \omega > 2/\tau_{sp} \sim 2/\nu_0 \sim 10^9 \text{ rad s}^{-1} \).

A magnetic grain of zero-frequency susceptibility, \( \chi(0) \), rotating with an angular velocity \( \Omega \) becomes magnetized via the Barnett effect (Barnett 1915) and acquires an instantaneous magnetic moment,

\[
\mu_{Bar} = \frac{\chi(0) V}{\gamma_e} \Omega,
\]

(14)

where \( V \) is the grain volume, \( \gamma_e = -e \mu_B / h \) the electron gyromagnetic ratio, where \( e \gamma_e \approx 2, \chi(0) = \chi_{sp}(0) \) and \( \chi_{sp}(0) \) for paramagnetic and superparamagnetic grains, respectively.

3. Review of the Interstellar Grain Alignment Paradigm Driven by RATs/METs

Here we first review the essential components (framework) of the modern paradigm of grain alignment for interstellar grains driven by RATs and METs. We then discuss that the RAT (MET) paradigm can be used to describe grain alignment in protostellar environments.

3.1. RATs (METs) and Their Basic Effects on Grain Dynamics: Spin-up/spin-down, Precession, and Alignment

Dolginoi & Mitrofanov (1976) first suggested that the interaction of an anisotropic radiation with a helical grain can induce RATs due to the differential scattering/absorption of left- and right-handed circularly polarized photons. Later, RATs were numerically demonstrated in Draine & Weingartner (1996) for three irregular grain shapes. Lazarian & Hoang (2007a) introduced an analytical model (AMO) of RATs, which is based on a helical grain consisting of an oblate spheroid and a weightless mirror. The AMO is shown to reproduce the basic properties of RATs obtained from numerical calculations for realistically irregular grain shapes (Lazarian & Hoang 2007a; Hoang & Lazarian 2008; Herranen et al. 2021), and enables us to make quantitative predictions for various conditions (Hoang & Lazarian 2014) and dust compositions (Lazarian & Hoang 2008; Hoang & Lazarian 2009b, 2016a). Many predictions were observationally tested (see Andersson et al. 2015). As shown in previous studies (Draine & Weingartner 1997; Lazarian & Hoang 2007a; Lazarian & Hoang 2008), RATs in general can induce three fundamental effects on grain rotational dynamics, including (1) the grain precession around the radiation direction, (2) spin-up of the grain to suprathermal rotation as well as spin-down to thermal rotation, (3) and align the grain with \( \mathbf{J} \) along the radiation \( \mathbf{k} \) (see Figure 2).

The interaction of an irregular grain with a gas flow results in METs on the dust grain (Lazarian & Hoang 2007b; Hoang et al. 2018). Due to the equivalence of the frame of reference, METs arising from the grain at rest bombarded by a gas (mechanical) flow of speed \( v_d \) are the same as those induced by the grain drifting through the ambient gas with the same speed. Similar to RATs, METs have a component that causes the spin-up and spin-down, a torque component that aligns the grain with the gas flow, and another component that causes the grain to precess around the flow direction, \( v_d \) (Lazarian & Hoang 2007b).
3.2. Spin-up and Suprathermal Rotation by RATs

Grain suprathermal rotation due to the spin-up effect of RATs is crucially important for modeling grain alignment and rotational disruption (see Hoang 2020 for a review).

Let $\gamma_{\text{rad}}$ and $\lambda$ be the anisotropy degree and the mean wavelength of the radiation field. Following Hoang (2021; see also Hoang et al. 2021), an irregular grain of effective size $a_{\text{eff}}$ subject to a protostellar radiation field can be spun up by RATs to a maximum angular velocity given by,

$$\Omega_{\text{RAT}} = \frac{3\gamma_{\text{rad}}u_{\text{rad}}a_{\text{eff}}}{1.6\mu_{\text{H}}^2/2\pi m_{\text{H}}kT_{\text{gas}}} \left( \frac{1}{1 + F_{\text{IR}}} \right)$$

$$\approx 9.4 \times 10^3 \gamma_{\text{rad}} U_6 \left( \frac{\lambda}{2 \mu_{\text{m}}} \right) \left( \frac{\gamma_{\text{rad}} U_6}{n_8 T_{\text{gas},1}^2} \right) \left( \frac{1}{1 + F_{\text{IR}}} \right) \text{rad s}^{-1},$$

for $a_{\text{eff}} \ll a_{\text{trans}}$ with $a_{\text{trans}} = \lambda/2.5$ being the transition size at which the average RAT efficiency changes the slope.$^8$

For large grains with $a_{\text{eff}} > a_{\text{trans}}$, one has

$$\Omega_{\text{RAT}} = \frac{1.5\gamma_{\text{rad}}u_{\text{rad}}a_{\text{trans}}^2}{12\mu_{\text{H}}^2/2\pi m_{\text{H}}kT_{\text{gas}}} \left( \frac{1}{1 + F_{\text{IR}}} \right)$$

$$\approx 8.1 \times 10^3 \gamma_{\text{rad}} U_6 \left( \frac{\lambda}{2 \mu_{\text{m}}} \right) \left( \frac{\gamma_{\text{rad}} U_6}{n_8 T_{\text{gas},1}^2} \right) \left( \frac{1}{1 + F_{\text{IR}}} \right) \text{rad s}^{-1}.$$

The suprathermal rotation number for the grain spin-up by RATs is then,

$$s_{\text{RAT}} = \frac{\Omega_{\text{RAT}}}{\Omega_{\text{f}}},$$

$$\approx 1800 \gamma_{\text{rad}} U_6 \left( \frac{\lambda}{2 \mu_{\text{m}}} \right) \left( \frac{\gamma_{\text{rad}} U_6}{n_8 T_{\text{gas},1}} \right) \left( \frac{1}{1 + F_{\text{IR}}} \right),$$

and

$$s_{\text{RAT}} \approx 2.1 \times 10^5 \gamma_{\text{rad}} U_6 \left( \frac{\lambda}{2 \mu_{\text{m}}} \right) \left( \frac{\gamma_{\text{rad}} U_6}{n_8 T_{\text{gas},1}} \right) \left( \frac{1}{1 + F_{\text{IR}}} \right),$$

which reveal that the suprathermal rotation number increases rapidly with the grain size as $s^{7/2}$ for $s^{1/2} a < a_{\text{trans}}$ and as $s^{1/2}$ as $s^{1/2} a > a_{\text{trans}}$.

We note that for protostellar conditions of high gas density, $1 + F_{\text{IR}} \approx 1$ because $F_{\text{IR}} \ll 1$ (Equation (7)), so that Equations (17) and (18) yield $s_{\text{RAT}} \propto U/(n_{\text{H}} T_{\text{gas}})$. However, under the condition of strong radiation fields and low gas density, $F_{\text{IR}} \gg 1$ (see Equation (7)); using $1 + F_{\text{IR}} \approx F_{\text{IR}}$, one obtains $s_{\text{RAT}} \propto U^{1/3}$.

$^8$ Hoang et al. (2021) omitted the subscript in the effective size $a_{\text{eff}}$, which is written explicitly in this paper.

3.3. Radiative Precession around the Radiation Direction and k-RAT Alignment

In the presence of an anisotropic radiation field, the irregular grain experiences radiative precession due to RATs, with the grain angular momentum precessing around the radiation direction ($k$). For a grain with angular momentum $J$, the radiative precession time is given by (Lazarian & Hoang 2007a; Hoang & Lazarian 2014),

$$\tau_\gamma = \frac{2\pi}{|d\phi/dt|} \approx \frac{2\pi J}{\gamma_{\text{rad}} u_{\text{rad}} a_{\text{eff}}^3 Q_{e3}},$$

$$\simeq 56.8 \gamma_{\text{rad}}^{1/2} T_{\text{gas},1}^{1/2} s^{-1/6} a_{\text{eff}}^{1/2} \frac{1.2 \mu_{\text{m}}}{n_8 T_{\text{gas},1}} \frac{U_6}{\gamma_{\text{rad}}^{1/2} \lambda_{\text{rad}} g_{\text{a3}} U_{\text{eff}}},$$

(19)

where $Q_{e3} = Q_{e3}/10^{-2}$ with $Q_{e3}$ being the third component of RATs that induces the grain precession around $k$ and the normalization is done using the typical value of $Q_{e3}$ (see Lazarian & Hoang 2007a).

Equation (19) shows the linear dependence of radiative precession on the instantaneous angular momentum (or suprathermal rotation number $s_{\text{RAT}}$) of the grain. This is an important feature that needs to be taken into account when studying the RAT alignment.

When the radiative precession is faster than gas damping, then RATs can cause the grain angular momentum ($J$) to be coupled with $k$. In this case, $k$ becomes an axis of grain alignment by RATs, which is referred to as $k$-RAT alignment (Lazarian & Hoang 2007a).

3.4. Spin-up and Suprathermal Rotation by METs

As RATs, the suprathermal rotation by METs determines the efficient grain alignment and rotational disruption (see Hoang 2020 for a review). Following Hoang et al. (2018), the magnitude of METs induced by the grain drift through the gas of density $n_{\text{H}}$ with velocity $v_{\text{d}}$ that acts to spin-up the irregular grain is given by

$$\Gamma_{\text{MET}} = n_{\text{H}} m_{\text{H}} \gamma_{\text{eff}} \pi a_{\text{eff}}^2 Q_{\text{spinup}},$$

(20)

where $Q_{\text{spinup}}$ is the spin-up efficiency, which physically depends on the grain shape, structure, and scattering properties (Hoang et al. 2018). For a spherical shape, $Q_{\text{spinup}}$ is zero. For irregular shapes, numerical calculations in Hoang et al. (2018) report the range of $Q_{\text{spinup}} \sim 10^{-6} - 10^{-3}$ (see the spin-up component in their Figure 5). Recent Monte Carlo simulations in Reissl et al. (2022) show that the MET efficiency $Q_{\text{spinup}}$ changes significantly, up to 3 orders of magnitude, with the grain shape. Therefore, for an arbitrary shape, we treat $Q_{\text{spinup}}$ as a free parameter, with a conservative value of $Q_{\text{spinup}} = 10^{-3}$.

The maximum angular velocity of the dust grain spun up by METs is given by (Hoang et al. 2018)

$$\Omega_{\text{MET}} = \frac{\Gamma_{\text{MET}} T_{\text{gas}}}{l_{\|}}$$

$$= n_{\text{H}} m_{\text{H}} \gamma_{\text{eff}} \pi a_{\text{eff}}^3 \gamma_{\text{rad}}^{1/2} T_{\text{gas},1}^{1/2} \left( \frac{\sqrt{\pi} \rho_{\text{g}} a_{\text{eff}}}{3 n_{\text{H}} m_{\text{H}} v_{\text{d}} \Gamma_{\|}} \right) \left( \frac{s_{\|} v_{\text{d}}}{a_{\text{eff}}} \right) \left( \frac{5 \sqrt{\pi} s_{\|} Q_{\text{spinup}}}{8 l_{\|}} \right)$$

$$= 4.5 \times 10^4 s_{\|}^{1/2} a_{\text{eff}}^{1/2} Q_{\text{spinup}}^{-3} \Gamma_{\|}^{-1} \frac{\gamma_{\text{rad}}^{1/2} T_{\text{gas},1}^{1/2} \gamma_{\text{eff}}^{1/2} \pi a_{\text{eff}}^3}{s_{\|} v_{\text{d}} a_{\text{eff}}} \text{ rad s}^{-1},$$

(21)
gas and dust with horizontal line indicates StRAT decreases rapidly with the gas density because the solid lines for the different radiation strength, RATs and rotational damping by the gas. This is the basic which only depends on the grain properties and the drift speed, but does not depend on the gas density. The suprathermal rotation number for the grain spin-up by METs is then,

\[
\text{St}_{\text{MET}} = \frac{\Omega_{\text{MET}}}{\Omega_f} = \left( \frac{s_d v_T}{a} \right) \left( \frac{5s_d \sqrt[3]{Q_{\text{spinup}}}}{8 \Gamma_{\text{gas}}} \right) \left( \frac{l}{K T_{\text{gas}}} \right)^{1/2} = \frac{5s_d \sqrt[3]{Q_{\text{spinup}}}}{8 \Gamma_{\text{gas}}} \left( \frac{16 \pi \sigma a^3}{15 m_H} \right)^{1/2} \simeq 0.86 \beta^{1/2} \frac{s_d}{a} \Omega_{\text{MET}}^{1/2} \frac{Q_{\text{spinup}}}{\Gamma_{\text{gas}}},
\]

which only depends on the grain properties and the drift speed, but does not depend on the gas density. This is the basic property of METs, which is different from RATs in which StRAT decreases rapidly with the gas density because the suprathermal rotation number is determined by the spin-up by RATs and rotational damping by the gas.

Figure 3 shows the suprathermal rotation numbers for grains spun up by RATs and METs for the different radiation strengths and drift velocities, assuming thermal equilibrium of gas and dust with \( T_{\text{gas}} = T = 16.4 \times 10^4 \text{ cm}^{-3} \). For this particular condition, METs are more important than RATs for the smallest and largest grains. For the radiation field of \( U = 10^2 \), METs dominate for \( s_d \gg 0.5 \).

\[ u_{\text{rad}} \text{ by the kinetic energy density of the gas flow, } n_H m_H v_T^2: \]

\[
\Gamma_{\text{MET, prec}} = n_H m_H v_T^2 a^3 s (e^2 - 1) K (\Theta, e) \sin 2\Theta = n_H m_H v_T^2 a^3 Q_{\text{prec}},
\]

where \( \Theta \) is the angle between the grain symmetry axis and the drift direction, \( e \) is the eccentricity of the oblate grain, and \( K(\Theta, e) \) is a fitted function of order unity (see also Lazarian & Hoang 2019). To account for various shapes, we have introduced \( Q_{\text{prec}} = se(e^2 - 1) K(\Theta, e)\sin 2\Theta \) to describe the MET efficiency of precession. For numerical estimates of the precession torque, we take \( Q_{\text{prec}} = 0.1 \) as a typical value.\(^9\)

For a grain rotating with angular velocity \( \Omega \), the MET induces the precession around the drift direction at the precession frequency:

\[
\Omega_{\text{MET, prec}} = \frac{d\phi}{dt} = \frac{\Gamma_{\text{MET, prec}}}{I_1 \Omega},
\]

so the mechanical precession timescale is

\[
\tau_\nu = \frac{2\pi}{\Omega_{\text{MET, prec}}} = \frac{2\pi a^3 n_H m_H v_T^2 Q_{\text{prec}}}{Q_{\text{prec}} - 1} \Phi \frac{1}{R_{\text{gas}}},
\]

\[
\simeq 1.7 \times 10^{-2} \frac{\text{St}}{a^1/2} Q_{\text{prec}}, \Phi \frac{1}{R_{\text{gas}}}, \sim 10^{-4} \left( \frac{1}{n_H m_H v_T^2 a^3 Q_{\text{prec}} - 1} \right) \text{ yr},
\]

where \( \text{St} = \Omega / \Omega_f \) and \( Q_{\text{prec}, -1} = Q_{\text{prec}} / 0.1. \)\(^{10}\) Equation (25) implies a decrease in the precession timescale with the gas density.

When radiative precession is faster than gas damping, grain alignment occurs with \( \nu \) along \( \nu \), which is referred to as \( \nu \)-MET alignment (Lazarian & Hoang 2007b).

3.6. Larmor Precession and B-RAT (B-MET) Alignment

The interaction of the grain magnetic moment produced by the Barnett effect with an external magnetic field causes Larmor precession of the grain angular momentum \( (J) \) around the magnetic field \( (B) \) direction (see Figure 4). For superparamagnetic grains, the rate of the Larmor precession is given by

\[
\tau_B = \frac{2\pi}{|d\phi / dt|} = \frac{2\pi a^3}{|\mu_B B|} = \frac{2\pi a^3 |\mu_B|}{\chi_{sp}(0) V B},
\]

\[
= \frac{2\pi a^3 |\mu_B|}{\hbar} \frac{2 \rho a^2}{5 \chi_{sp}(0) B} \simeq 8.1 \times 10^{-4} \frac{\rho T_d a^2}{N a \phi_{sp} \gamma^2 B_3^2} \text{ yr},
\]

where \( B_3 = B/(10^3 \mu G) \) is the normalized magnetic field strength.

Comparing Equations (26) and (4), one can see that \( \tau_B \ll \tau_{\text{gas}} \) for the conditions where the gas density is not very high (i.e., \( n_H \ll 1 \)). Moreover, comparing with Equations (19) and (25), one can also see that \( \tau_B \ll \tau_\nu \) and \( \tau_B \ll \tau_r \) for the typical parameters of the diffusive ISM to DCs with a gas density of \( n_H < 1 \), grain size of \( a < 10 \mu m \), radiation field of \( U \sim 1 \), and drift velocity of \( s_{d-1} \sim 1 \). Therefore, the magnetic field is the

\[ \text{The prefactor } n_H \text{ is missing in Equation (15) of Lazarian & Hoang (2019), and we denote } a \text{ to be the semimajor length where Lazarian & Hoang (2019) uses } a \text{ for the semiminor axis.} \]

\[ \text{The term } a_5^{1/2} \text{ is missing in Equation (16) of Lazarian & Hoang (2019).} \]

\[ \text{The prefactor } n_H \text{ is missing in Equation (15) of Lazarian & Hoang (2019), and we denote } a \text{ to be the semimajor length where Lazarian & Hoang (2019) uses } a \text{ for the semiminor axis.} \]
preferred axis of grain alignment for PM and SPM dust in the ISM and MCs, which is referred to as B-RAT (B-MET) alignment.

3.7. A General Model of Grain Alignment by RATs and METs

In addition to the spin-up, spin-down, and precession effects, RATs (METs) have an aligning torque component that acts to align the grain angular momentum with $k$ (or $v$) at a high-J attractor and a low-J attractor (Lazarian & Hoang 2007a; Hoang & Lazarian 2008, 2016a). Therefore, a fraction of grain ensemble can be aligned at the high-J attractor, denoted by $f_{\text{high-J}}$, and the $1 - f_{\text{high-J}}$ fraction of grains are aligned at the low-J attractor when the grain randomization by gas collisions is disregarded (Hoang & Lazarian 2014). This process can occur on a timescale shorter than the gas damping, which is the so-called fast alignment (Lazarian & Hoang 2007a, 2021).

Grains aligned at the high-J attractor rotate suprathermally with $\Omega \sim \Omega_{\text{RAT}}$ for RATs or $\Omega_{\text{MET}}$ for METs, corresponding to the suprathermal number of $S_{\text{high-J}} > 1$. Grains at the low-J attractor rotate with $\Omega \sim \Omega_{T}$ or $S_{\text{low-J}} \sim 1$. When the gas randomization effect is taken into account, grains at low-J attractors are randomized, and they are eventually transported to the high-J attractor after a timescale greater than the gas damping time, which is usually called slow alignment (Hoang & Lazarian 2008; Lazarian & Hoang 2021). Without high-J attractors, grains are cycling between the low-J attractor and high-J repellor point and have an average low degree of alignment (Hoang & Lazarian 2008, 2016a; Lazarian & Hoang 2021). The exact value of $f_{\text{high-J}}$ depends on the grain properties (shape and size) and magnetic susceptibility. For grains with ordinary paramagnetic material (e.g., silicate), Herranen et al. (2021) found that $f_{\text{high-J}}$ can be about $10\%-70\%$ based on calculations of RATs for an ensemble of Gaussian random shapes. The presence of iron inclusions embedded in the grains increases grain magnetic susceptibility and superparamagnetic relaxation, which can produce universal high-J attractors (i.e., $f_{\text{high-J}} \sim 100\%$; Hoang & Lazarian 2016a; Lazarian & Hoang 2021).

The above model of grain alignment with low-J and high-J attractors established for the ISM can be applied to protostellar environments where strong RATs/METs can rapidly drive the grain angular momentum to align with a preferred direction in space. Indeed, for grains with efficient internal relaxation, internal alignment occurs rapidly so $\hat{a}_{\|} \parallel \hat{J}$; thus, the situation is similar to the case of interstellar grains. On the other hand, for VLGs with slow internal relaxation, internal alignment occurs much slower than external alignment (Hoang & Lazarian 2009a). Therefore, in the following, we will use this idealized model of external alignment by RATs/METs to evaluate the efficiency of internal relaxation and grain alignment for protostellar environments.

4. Internal Alignment by Internal Relaxation

Internal alignment of the grain axis with the grain angular momentum is essential for modeling and interpreting dust polarization. Here, we study two leading processes that can induce internal alignment in protostellar conditions, including Barnett relaxation and inelastic relaxation (P79; LE99). We determine the physical parameters for which internal alignment is faster than gas randomization that induces efficient internal alignment for grains rotating thermally at the low-J attractor and suprathermally at the high-J attractor due to RATs (METs), as implied by the RAT (MET) paradigm.

4.1. Barnett Relaxation

A spinning paramagnetic grain acquires an instantaneous magnetic moment, $\mu \propto \hat{\Omega}$, via the Barnett effect. In the grain’s body frame, the magnetization vector has a component perpendicular to $\hat{a}_{\|}$, rotating with respect to $\hat{a}_{\|}$ at an angular rate $\omega$ (see Figure 1). The rotating magnetization component has some lag behind the grain material and induces the dissipation of the grain rotational energy, leading to the internal alignment of $\hat{a}_{\|}$ with $\hat{\Omega}$ and $\hat{J}$ that correspond to the minimum rotational energy state (Purcell 1979).

As shown in Hoang (2022), Barnett relaxation for paramagnetic grains is inefficient for large grains in dense regions like protostellar cores and disks of density $n_{H} \gtrsim 10^{6} \text{ cm}^{-3}$. For superparamagnetic grains, the relaxation time by the Barnett effect (the so-called super-Barnett relaxation) is given by

$$
\tau_{BR, sp} = \frac{2}{V_{K sp}(\omega) h^2 (h - 1) J^2} \approx 0.16 \frac{\hat{\sigma}^3 f(\hat{\sigma})}{N_{d} \hat{\phi}_{sp, 2} g^2 \hat{\phi}_{sp, 2}} \left( \frac{J_{d}}{J} \right)^2 \times \left( \frac{T_{d, 1}}{k_{sp}(\omega)} \right) \text{yr},
$$

(27)

where $\hat{\sigma} = s/0.5$, $f(\hat{\sigma}) = \hat{\sigma}(1 + \hat{\sigma}^2)/2^2$, $J_{d} = \sqrt{k_{B} T_{d} / (h - 1)}$ is the dust thermal angular momentum (see also Hoang & Lazarian 2014), and $\phi_{sp, 2} = \phi_{sp}/0.01$.

4.2. Inelastic Relaxation

Atoms and molecules within a precessing dust grain experience centrifugal acceleration (stress). The centrifugal force makes the material stretch out (strain), while the mutual attractive force between atoms tends to pull it. The stress causes material deformation, the so-called strain. Due to the inelasticity of the dust material, the deformation stress–strain process induces the dissipation energy of the rotation energy into the heat, resulting in a decrease in the rotational energy to

11 We disregard the situation where the internal alignment and external alignment processes occur on similar timescales.
its minimum state corresponding to the internal alignment of the axis of maximum moment of inertia \((\mathbf{a}_1)\) with \(\mathbf{J}\). This effect, frequently called inelastic relaxation, was first studied for asteroids (Prendergast 1958; Burns et al. 1973) and was then applied to interstellar dust (P79). Later, LE99 revisited inelastic relaxation by taking the double-frequency contribution for the squared-prism grain shape into account. Molina et al. (2003) studied inelastic relaxation for the oblate spheroidal shape using LE99’s approach (see Appendix A for more details).

The strain caused by centrifugal stress produces a potential energy stored within the grain material, denoted by \(W\). Following Molina et al. (2003), the total strain energy for an oblate spheroidal grain is given by

\[
W = [W(\omega) + 2W(2\omega)] = \frac{32\pi\rho^2\Omega_0^4\sin^4\theta + a_2^2\cot\theta + a_3^2c/(1 + \sigma)}{105\mu} \times (1 + s^2)^4.
\]

where \(W(\omega)\) and \(W(2\omega)\) are the amplitudes of the strain energies associated with the principal oscillation frequency \(\omega\) and double frequency \(2\omega\), respectively (see Efroimsky & Lazarian 2000). Here, \(\mu\) is the shear modulus (i.e., modulus of rigidity), and \(\sigma\) is the Poisson ratio taken to be \(\sigma = 0.25\) (Molina et al. 2003).

In principle, the rate of energy loss by inelastic relaxation is proportional to the potential energy and precession rate \(\omega\). Thus, one can write

\[
\frac{dE_{\text{loss}}}{dt} = \left(\frac{2\omega W}{Q}\right),
\]

where \(Q\) is the quality factor of the grain material, which is assumed to be independent of the frequency, and the factor of 2 accounts for the average elastic energy (Efroimsky & Lazarian 2000).

Using the grain rotational energy from Equation (3) and the energy conservation law, one obtains

\[
\frac{dE_{\text{rot}}}{dt} = \frac{dE_{\text{loss}}}{dt} = (h - 1) \frac{64\pi\rho^2\Omega_0^2\sin^4\theta \cos\theta}{105\mu Q} \times a_2^2\cot\theta + a_3^2c/(1 + \sigma) \times (1 + s^2)^4.
\]

Using \(dE_{\text{rot}}/dt = I_1\Omega_0^2(h - 1)\sin\theta \cos\theta 0\theta/dt\) and the above equation, one obtains

\[
\frac{d\theta}{dt} = \frac{8}{7} \frac{a_2^2\rho^2\Omega_0^3\sin^3\theta \cot\theta}{\mu Q} \times \frac{2\cot\theta + 1/(1 + \sigma)}{(1 + s^2)^4}.
\]

The characteristic time of inelastic relaxation can be estimated as

\[
\tau_{\text{IER}} = \frac{1}{d\theta/dt} \Big|_{\theta = \pi/4} = \frac{\mu Q}{\rho^2\Omega_0^2} g(s),
\]

where \(g(s)\) is a geometrical factor that depends on the axial ratio as

\[
g(s) = \frac{2^{3/2}}{3} \frac{(1 + s^2)^4}{s^3 + 1/(1 + \sigma)},
\]

which corresponds to \(g(s) = 7.0\) and 4.6 for \(s = 1/2\) and 1/3, respectively.

Representing the grain angular velocity as the suprathermal rotation number, \(\Omega_0 = \Omega_{\text{RAT}}\), Equation (32) becomes

\[
\tau_{\text{IER}} \approx \frac{\mu}{(kT_{\text{gas}}/s)^{3/2}} \mu_s Q_3 g'(s),
\]

where \(g'(s) \approx 2.2 s^3 g(s)\). Using the same normalization throughout this paper, one obtains

\[
\tau_{\text{IER}} = 0.034 \mu^{1/2} a_2^{11/2} \mu_s Q_3 g'(s) / T_{\text{gas}}^{3/2} \text{ yr},
\]

where \(\mu_s = \mu/(10^3 \text{ erg cm}^{-3})\), and \(Q_3 = Q/10^3\). The typical value \(Q\) is about 100 for silicate rocks (see Efroimsky & Lazarian 2000) and between 400 and 2000 for vitreous silica (see Purcell 1979). The value of \(\mu\) depends on the grain structure, which is high for compact and low for composite/porous grains or dust aggregates (Seizinger et al. 2013). Studies of cometary dust reveal \(\mu \sim 3.6 \times 10^{-3} – 3.46 \times 10^{-1} \text{ erg cm}^{-3}\) (Knapmeyer et al. 2018). Throughout this paper, we assume \(\mu_s = 1\) and \(Q_3 = 1\) for numerical results, unless stated otherwise.

For the thermal rotation of \(\Omega = 1\), inelastic relaxation is slower than Barnett relaxation, but it increases slower with the grain size as \(\tau_{\text{IER}} \sim a^{11/2}\), compared to \(\tau_{\text{BR},\text{sup}} \sim a^2\). Moreover, the inelastic relaxation time decreases faster with increasing suprathermal rotation number \(\Omega\) than the Barnett relaxation.

Note that inelastic relaxation does not depend on the magnetic susceptibility of dust, so it can work for nonmagnetic (or diamagnetic) grains such as carbonaceous dust.

### 4.3. Timescales of Internal Relaxation

The efficient internal alignment is only achieved when internal relaxation occurs faster than the grain’s orientation randomization by gas collisions, which is described by the gas damping time \((\tau_{\text{gas}})\). Here we calculate the internal relaxation timescales for grains that are suprathermally rotating by RATs with \(\Omega = \Omega_{\text{RAT}}\) and by METs with \(\Omega = \Omega_{\text{MET}}\) and compare them with the gas damping time.

We consider a range of the parameter space with density from \(n_1 \sim 10^3 – 10^6 \text{ cm}^{-3}\) for protostellar cores to \(n_1 \sim 10^{10} – 10^{12} \text{ cm}^{-3}\) typical in protostellar disks (see, e.g., Figures 2 and 3 in Tung & Hoang 2020). For a given local condition of the gas density, temperature, and radiation field, we compute the suprathermal rotation numbers \(\Omega_{\text{RAT}}\) and \(\Omega_{\text{MET}}\), and plug them into Equations (27) and (34) to obtain the timescales of super-Barnett relaxation and inelastic relaxation. The dust temperature is taken as the equilibrium temperature of silicate with \(T_{\text{d}} = 16.4a_{-5}^{-1/2}U_{1/6} \text{ K}\) (see Draine 2011), which is valid for large grains. Here, we assume the dust and gas are in thermal equilibrium, i.e., \(T_{\text{gas}} = T_{\text{d}}\), which is valid for dense regions like protostellar environments.

Figure 5 compares the different timescales by super-Barnett relaxation and inelastic relaxation with the gas damping for the protostellar core of typical density of \(n_1 = 10^5 \text{ cm}^{-3}\), assuming two values of the radiation strength \((U)\). For the lower radiation...
strength, the super-Barnett relaxation is faster than the gas damping for VLGs of size \( a \sim 1–20 \mu m \) when iron impurities are large of \( N_{\text{cl}} \gtrsim 5000 \), but inelastic relaxation (dashed line) is faster for grains of \( a \sim 1–100 \mu m \) (see left panel). For the higher radiation strength, both super-Barnett and inelastic relaxation become more efficient. Inelastic relaxation is faster than gas damping for \( a < 0.1–10^3 \mu m \), while super-Barnett relaxation is faster for \( a < 10–50 \mu m \) for \( N_{\text{cl}} \sim 100–10^4 \) (see right panel).

Figure 6 shows the different timescales by inelastic relaxation and super-Barnett relaxation for suprathermal rotation by METs and internal alignment by RATs and METs. Using Equations (35) and (4), we calculate the ratio of the Barnett relaxation to the gas damping times,

\[
\frac{\tau_{\text{BR,sp}}}{\tau_{\text{gas}}} = \frac{\gamma_1^2 J_T^4 \pi \nu \cdot 1.2n_{\text{H}} m_{\text{H}} v_T |a|^{1/2} \cdot 1 + \left( \frac{\omega \tau_{\text{SP}}}{2} \right)^2}{3 V h^2 (h - 1) J_T^2 \chi_{\text{SP}}(0) \tau_{\text{SP}}},
\]

\[
\approx 10^{-3} \frac{m_{\text{sp}} T_{\text{gas}}^{1/2} a_{\text{sp}}^{6/3} I_0}{h^2 N_{\text{cl}} a_{\text{sp}} - \phi_{\text{sp}}^{-2} k_{\text{sp}}(\omega)} \left( \frac{J_T}{J} \right)^2.
\]

(36)

Replacing \( J/J_T = (J/J_T)_{\text{br}} = \text{St}(h - 1) T_{\text{gas}}/T_{\text{d}}, \) where \( \text{St} = J/J_T = \Omega_{\text{d}}/\Omega_{\text{SP}} \), one can determine the critical suprathermal rotation for efficient internal alignment by Barnett relaxation using the criterion \( \tau_{\text{BR}}/\tau_{\text{gas}} < 1 \) (Lazarian & Roberge 1997), yielding

\[
\text{St} > \text{St}_{\text{crit,sp}}(\text{BR}) = 0.03 a_{\text{sp}}^{3} \left( \frac{n_{\text{H}} T_{\text{gas}}^{1/2} I_0}{h^2} \right)^{1/2} \times \left( \frac{1}{N_{\text{cl}} a_{\text{sp}} - \phi_{\text{sp}}^{-2} k_{\text{sp}}(\omega)} \right)^{1/2} \left( \frac{1}{(h - 1) T_{\text{gas}}} \right)^{1/4}.
\]

(37)

which implies that, for small grains, even thermal rotation of \( \text{St} < 1 \) can have efficient Barnett relaxation. However, to align \( a = 1, 10 \mu m \) grains, one needs \( \text{St}_{\text{crit,sp}}(\text{BR}) \sim 31 \) and 31,622, assuming \( n_{\text{H}} = 1 \) and \( N_{\text{cl}} = 1 \).

Equation (36) implies a steep increase with the grain size as \( a^6 \), so that sufficiently large grains may not have internal alignment. Let \( a_{\text{max,sp}} \) be the maximum grain size for internal alignment between \( \tilde{a}_{1} \) and \( J \). The maximum size for efficient internal alignment by Barnett relaxation is given by \( \tau_{\text{BR,sp}}/\tau_{\text{gas}} = 1 \), yielding

\[
a_{\text{max,sp}}(\text{BR}) \approx 0.32 h^{1/3} \text{St}^{1/3} \left( \frac{N_{\text{cl}} a_{\text{sp}} - \phi_{\text{sp}}^{-2}}{m_{\text{sp}} T_{\text{gas}}^{1/2} I_0} \right)^{1/6} \times \left( \frac{1}{k_{\text{sp}}(\omega)} \right)^{1/6} \times \left( \frac{(h - 1) T_{\text{gas}}}{T_{\text{d}}} \right)^{1/6} \mu m.
\]

(38)

4.4. Critical Sizes of Internal Alignment by Barnett Relaxation

We now determine the critical sizes for efficient internal alignment by super-Barnett relaxation for grains aligned at high-\( J \) attractors by RATs and METs. Using Equations (4), (27), and (11), we calculate the ratio of the Barnett relaxation to

- Figure 5. Internal relaxation times by inelastic and super-Barnett relaxations for grains spun up by RATs in a radiation field of \( \lambda = 10 \mu m \). \( \gamma_{\text{rad}} = 0.5 \) and gas density of \( n_{\text{H}} = 10^5 \text{ cm}^{-3} \) for \( U = 10^7 \) (left panel) and \( U = 10^9 \) (right panel). A sharp rise in the timescale at \( a \sim \lambda/2 \) is due to the change in the slope of RATs. Higher radiation strength increases the suprathermal rotation number \( n_{\text{cl,RT}} \) and reduces the relaxation times, extending the range of grains with fast internal relaxation.

- Figure 6 shows the different timescales by inelastic relaxation and super-Barnett relaxation for suprathermal rotation by METs for two values of the drift parameters \( |a| = 0.5 \) (left panel) and \( |a| = 0.5 \) (right panel). Super-Barnett relaxation is faster than inelastic relaxation and gas damping for small grains, but it becomes slower for VLGs of \( a > 100 \mu m \). Both the Barnett and inelastic relaxation processes are faster for increasing \( |a| \). Note that inelastic relaxation has a slightly steeper slope than gas damping due to its steeper scaling with the gas temperature \( T_{\text{gas}} \), which decreases with the grain size (see Equation (35) versus Equation (4)). For \( |a| = 0.25 \) (left panel), inelastic relaxation is faster than gas damping for \( n_{\text{H}} = 10^{6} \text{ cm}^{-3} \) for all grain sizes. However, super-Barnett relaxation is faster than gas damping only for small grains of \( a < 0.1–10 \mu m \) at \( n_{\text{H}} = 10^{10} \text{ cm}^{-3} \). For \( |a| = 0.5 \) (right panel), thanks to the increase in the suprathermal rotation number, inelastic relaxation becomes faster than gas damping for all three considered densities. Super-Barnett relaxation becomes also more efficient than gas damping for grains of \( a < 1–50 \mu m \) at \( n_{\text{H}} = 10^6 \text{ cm}^{-3} \) due to the higher suprathermal rotation number. At \( n_{\text{H}} = 10^8 \text{ cm}^{-3} \), super-Barnett relaxation is only efficient for small grains of \( a < 1 \mu m \) for \( |a| = 0.1 \) (left) and \( a < 10 \mu m \) for \( |a| = 0.5 \) (right), assuming \( N_{\text{cl}} = 10^4 \).

The total rate of internal relaxation by Barnett and inelastic effects is then \( \tau_{\text{INR}} = \tau_{\text{BR,sp}} + \tau_{\text{IER}} \).
which implies $a_{\text{max},\text{al}}(\text{BR}) \approx 0.32 \mu m$ for $N_{\text{cl}} \sim 10^4$ and $J/J_d = 1$ and $n_g = 1$. One can see that $a_{\text{max},\text{al}}$ increases with iron inclusions as $N_{\text{cl}}^{1/6}$ and with suprathermal rotation as $\hat{\eta}_{\text{cl}}^{1/3}$, but decreases with the gas density as $n_{\text{H}}^{-1/6}$.

Plugging $\hat{\eta} = \hat{\eta}_{\text{RAT}}$ (Equation (18)) into Equation (38) one can get $a_{\text{max},\text{al}}(\text{BR})$ due to Barnett relaxation for grains spun up by RATs. Figure 7 shows the critical size of internal alignment by Barnett and inelastic relaxation for grains for a mean wavelength of radiation field of $\lambda = 10 \mu m$ and different radiation strengths. The assumed value of $\hat{\eta} = 10 \mu m$ is typical for the expected radiation field in the protostellar environment, which includes the attenuated radiation field from the central protostar (Hoang et al. 2021) and infrared radiation emitted by the hot dust shell near the protostar (Hoang 2021). For a low radiation strength of $U = 1, 100$, super-Barnett relaxation is more efficient than inelastic relaxation, but both effects are only effective at $n_{\text{H}} < 10^3 \text{ cm}^{-3}$. Increasing the radiation strength to $U = 10^4, 10^6$, inelastic relaxation becomes faster than super-Barnett relaxation, as expected for suprathermal rotation due to its faster increase with the angular momentum and slower variation with the grain size (see Equation (34) versus Equation (27)), which induces the efficient internal alignment of large grains even in the region of $n_{\text{H}} \sim 10^3-10^5 \text{ cm}^{-3}$.

Similarly, plugging $\hat{\eta} = \hat{\eta}_{\text{MET}}$ (Equation (22)) one can solve for the range of grain sizes with efficient internal relaxation, $a_{\text{max,al}}$, $a_{\text{max,al}}(\text{MET})$, for grains spun up by METs.

### 4.5. Critical Size for Internal Alignment by Inelastic Relaxation

To estimate the maximum size for internal alignment by inelastic relaxation, we use Equations (35) and (4),

$$
\frac{\tau_{\text{IER}}}{\tau_{\text{gas}}} = 0.41 a_{\frac{5}{2}} \frac{\mu_b Q_3}{\sqrt{\rho}} \frac{n_g}{T_{\text{gas}}^{1/3}} \frac{g'(s)}{s} \frac{1}{\hat{\eta}_{\text{cl}}^{1/3} \text{St}^{3/3}}.
$$

For a given grain size, the efficient internal alignment occurs when $\tau_{\text{IER}}/\tau_{\text{gas}} < 1$, which requires the maximum grain size of

$$
a_{\text{max,al}}(\text{iER}) = 0.12 \left(\frac{\mu_b Q_3}{\sqrt{\rho}}\right)^{2/9} \left(\frac{T_{\text{gas}}^{1/3}}{n_g}\right)^{2/9} \left(\frac{g'(s)}{s}\right)^{2/9} \hat{\eta}_{\text{cl}}^{2/3} \mu m,
$$

which implies that a suprathermal rotation number of $St = 10^3$ can align large grains of $a_{\text{max,al}} = 12 \mu m$, assuming the normalized parameters.

Comparing $a_{\text{max,al}}(\text{iER})$ with $a_{\text{max,al}}(\text{BR})$ (Equation (38)), one can see that at thermal rotation (e.g., low-$J$ attractor), inelastic relaxation is less efficient than super-Barnett relaxation for internal alignment. However, for suprathermal rotation, inelastic relaxation becomes more efficient due to its steeper increase with the suprathermal number as $St^{2/3}$, compared to $St^{1/3}$ of super-Barnett relaxation.

For a given grain size, the internal alignment requires the suprathermal rotation of the number,$$
St > S_{\text{cri,al}}(\text{iER}) = 0.74 a_{\frac{5}{2}}^{1/3} \left(\frac{\mu_b Q_3}{\sqrt{\rho}}\right)^{1/3} \left(\frac{g'(s)}{s}\right)^{1/3} \frac{n_g}{T_{\text{gas}}^{1/3}} \hat{\eta}_{\text{cl}}^{1/3},
$$

which increases rapidly with the grain size and gas density. Here the critical suprathermal rotation number $S_{\text{cri,al}}$ corresponds to $\tau_{\text{IER}} = \tau_{\text{gas}}$.

Equation (41) implies that inelastic relaxation can be efficient for thermal rotation if $a_{\text{max,al}} < 1$, but for large grains of $a > 1 \mu m$ suprathermal rotation is required at a level of $S_{\text{gas}} > 3.5$, assuming $n_g = 1$. For a protostellar core of $n_g = 10^4 \text{ cm}^{-3}$, one requires $S_{\text{cri,al}}(\text{iER}) \sim 35$ and 1100 to align $a = 1 \mu m$ and $10 \mu m$ grains, respectively. Thus, large grains in denser regions require high suprathermality to be efficiently aligned by inelastic relaxation.

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13 Note that efficient internal alignment is only achieved when $\tau_{\text{BR,sp}} < \tau_{\text{gas}}$, but here we derive the critical limit for convenience.
4.5.1. Suprathermal Rotation by RATs

For suprathermally rotating grains at high-$J$ attractors by RATs with sizes $a < a_{\text{trans}} = \bar{\lambda}/2.5$, using $S_{\text{RAT}}$ from Equation (17) for Equation (39), one obtains

\[
\frac{\tau_{\text{ER, high-J}}}{\tau_{\text{gas}}} \simeq 7.1 \times 10^{-11} \alpha a^{-6} \frac{\mu_{\text{g}}}{\bar{\rho}^2} \frac{Q_{3}}{T_{\text{gas,1}}} n_{8} \frac{g^{4/3}}{s^{5/2}} \left( \frac{\bar{\lambda}}{1.2 \, \mu m} \right)^{6} \times \left( \frac{1}{1 + F_{\text{IR}}} \right)^{-3}
\]

(42)

which decreases rapidly with the grain size as $1/a^6$.

For $a > a_{\text{trans}}$, using $S_{\text{RAT}}$ from Equation (18), one obtains

\[
\frac{\tau_{\text{ER, high-J}}}{\tau_{\text{gas}}} \simeq 1.1 \times 10^{-16} \alpha a^{-5} \frac{\mu_{\text{g}}}{\bar{\rho}^2} \frac{Q_{3}}{T_{\text{gas,1}}} n_{8} \frac{g^{4/3}}{s^{5/2}} \left( \frac{\bar{\lambda}}{1.2 \, \mu m} \right)^{5} \times \left( \frac{1}{1 + F_{\text{IR}}} \right)^{-3}
\]

(43)

which increases rapidly with the grain size as $a^5$.

Using the condition for efficient internal alignment as $\tau_{\text{ER}}/\tau_{\text{gas}} < 1$, Equation (42) yields the range of grain sizes with internal alignment,

\[
a > a_{\text{min, al}}(\text{iER}) = \frac{2.3 \times 10^{-3} \left( \frac{\mu_{\text{g}} Q_{3} g^{4/3}}{\bar{\rho}^2 s^{5/2}} \right)^{1/6} \left( \frac{n_{8}}{T_{\text{gas,1}}} \right)^{1/6} \left( \frac{\bar{\lambda}}{1.2 \, \mu m} \right)}{n_{8} T_{\text{gas,1}}} \left( \frac{1}{1 + F_{\text{IR}}} \right)^{-1/2} \mu m,
\]

(44)

where $a_{\text{min}}$ is the minimum grain size for internal alignment, which is the solution of the equation $\tau_{\text{ER}} = \tau_{\text{gas}}$.

Similarly, Equation (43) yields the grain sizes for internal alignment as

\[
a < a_{\text{max, al}}(\text{iER}) = \frac{2.1 \times 10^{-4} \left( \frac{\mu_{\text{g}} Q_{3} g^{4/3}}{\bar{\rho}^2 s^{5/2}} \right)^{1/3} \left( \frac{n_{8}}{T_{\text{gas,1}}} \right)^{1/3} \left( \frac{\bar{\lambda}}{1.2 \, \mu m} \right)}{n_{8} T_{\text{gas,1}}} \left( \frac{1}{1 + F_{\text{IR}}} \right) \mu m,
\]

(45)

where $a_{\text{max}}$ is the maximum grain size for internal alignment, which is the solution of the equation $\tau_{\text{ER}} = \tau_{\text{gas}}$. The equation above implies $a_{\text{max, al}}(\text{iER}) \sim 30 \, \mu m$ for $U = 10^{6}$, $n_{H} \sim 10^{10} \, \text{cm}^{-3}$. Therefore, under the condition $a$ of strong radiation...
field around a protostar, RATs can induce efficient internal alignment by inelastic relaxation.

Therefore, suprathermal rotation by RATs produces a range of grains that can have efficient (right) internal alignment by inelastic relaxation between \( a^{\text{RAT}}_{\text{max,al}} = a^{\text{RAT}}_{\text{max}} \). Equation (45) reveals the dependence of \( a^{\text{RAT}}_{\text{max,al}} \) on the values of \( \mu_{g} \) and \( Q_{3} \), which are determined by the mechanical properties of dust grains. One expects the value of \( \mu \) to decrease with increasing grain size due to grain coagulation in protostellar environments, which increases the porosity (or increases the filling factor; Seizinger et al. 2013). Therefore, for the same local conditions, larger (more porous) grains would have more efficient internal alignment by inelastic relaxation than smaller ones.

Figure 7 shows the range of internal alignment sizes by BR and inelastic relaxation due to spin-up by RATs. Inelastic relaxation is more efficient than super-Barnett relaxation for \( U \gtrsim 10^{4} \). Both effects become inefficient for increasing \( n_{h} \) due to stronger gas damping and lower \( S_{0}\text{RAT} \).

4.5.2. Suprathermal Rotation by METs

For suprathermally rotating grains aligned at high-J attractors by METs, the ratio between the inelastic relation and gas damping timescales becomes

\[
\frac{\tau_{\text{ER},\text{MET},\text{high}-J}}{\tau_{\text{gas}}} \simeq 0.65s_{0}^{-6}Q_{3}^{-1}Q_{\text{spinup,-3}}^{-3} \left( \frac{\mu_{g}Q}{\rho_{g}} \right)^{\frac{8}{3}} \left( \frac{T_{\text{gas},1}}{s_{0}} \right)^{\frac{5}{2}},
\]

which does not depend on the grain size, but the ratio decreases rapidly with increasing drift speed as \( s_{d}^{-6} \). Thus, one can determine the critical drift parameter required for internal alignment by inelastic relaxation,

\[
s_{\text{cri,al}}(\text{IER}) = 0.09Q_{\text{spinup,-3}}^{-1/2} \left( \frac{\mu_{g}Q_{3}}{\rho_{g}} \right)^{1/6} \left( \frac{g^{l}T_{l}^{4}}{s_{0}^{3/2}} \right)^{1/6} \left( \frac{n_{8}}{T_{\text{gas},1}} \right)^{1/6},
\]

\[
\simeq 0.22Q_{\text{spinup,-3}}^{-1/2} \left( \frac{\mu_{g}Q_{3}}{\rho_{g}} \right)^{1/6} \left( \frac{n_{8}}{T_{\text{gas},1}} \right)^{1/6} \times \left( \frac{g^{l}T_{l}^{4}}{s_{0}^{3/2}} \right)^{1/6},
\]

which implies \( s_{d} > 0.22 \) for \( s = 0.5 \), assuming the normalized parameters. The critical value \( s_{\text{cri,al}}(\text{IER}) \) increases in denser/cooler regions of \( n_{8}/T_{\text{gas},1} > 1 \) and decreases in the less dense/hotter regions of \( n_{8}/T_{\text{gas},1} < 1 \) (see also Figure 6). For a very dense region of \( n_{h} = 10^{12} \text{ cm}^{-3} \), the critical drift parameter becomes \( s_{\text{cri,al}} \approx 1.02 \).

Figures 8 and 9 show the maximum grain size of internal alignment by sup-Barnett relaxation due to spin-up by METs, assuming a range of density relevant for star-forming regions from \( 10^{6} \) to \( 10^{12} \text{ cm}^{-3} \), and a local radiation strength of \( U = 1 \) and \( U = 10^{4} \), respectively. Two values of the drift parameter of \( s_{d} = 0.25 \) (left) and \( 0.5 \) (right) are considered. The alignment size increases with \( s_{d} \) due to increasing suprathermal rotation.

5. External Alignment in the Presence of Magnetic Fields

Here, we discuss in detail the main physical processes involved in the external alignment of the grain angular momentum with a preferred direction by RATs (METs) in the presence of an ambient magnetic field. We will derive general formulae for critical grain sizes for external alignment for protostellar conditions with a high gas density of \( n_{h} \gtrsim 10^{6} \text{ cm}^{-3} \).

5.1. Larmor Precession and Magnetic Alignment

If the Larmor precession of the grain angular momentum around the ambient magnetic field is very fast compared to the gas damping time, then the magnetic field becomes an axis of grain alignment, which is called magnetic alignment. From Equations (4) and (26), one derives the ratio of the Larmor precession time to the gas damping time for superparamagnetic grains as (see also Hoang 2022):

\[
\frac{\tau_{\text{R}}}{\tau_{\text{gas}}} = \frac{2\pi g_{s} \mu_{B}}{h} \frac{1}{\chi_{\text{sp}}(0) B} \left( \frac{1.2n_{h} m_{H} v T_{l} a}{\sqrt{\pi} s} \right),
\]

\[
\simeq 0.01a_{5} \left( \frac{n_{8} T_{\text{gas},1}^{1/2} T_{d,1}}{N_{\text{cl}} \phi_{\text{sp},-2} \rho_{g}^{2} B_{5}^{4}} \right) \mu_{m}, \tag{48}
\]

The maximum size for grain alignment with \( J \) aligned with the magnetic field (B) via Larmor precession, denoted by \( a_{\text{max,JB}} \) (Lar), is then determined by \( \tau_{\text{B}}/\tau_{\text{gas}} = 1 \), yielding

\[
a_{\text{max,JB}} \simeq 5.1 \times 10^{-5} \left( \frac{N_{\text{cl}} \phi_{\text{sp},-2} \rho_{g}^{2} B_{5}^{4}}{n_{8} T_{\text{gas},1}^{1/2} T_{d,1}} \right) \mu_{m}, \tag{49}
\]

which implies that VLGs of \( a \sim 5.1 \text{ mm} \) can still be aligned with the magnetic field against gas randomization at \( n_{h} \sim 10^{6} \), but \( a_{\text{max,JB}} \sim 51 \mu_{m} \) for \( n_{h} = 10^{4} \), assuming large iron inclusions of \( N_{\text{cl},4} = 1 \). However, for ordinary paramagnetic grains of \( N_{\text{cl}} \sim 1 \), only grains smaller than \( a_{\text{max,JB}} \lesssim 51 \mu_{m} \) can be aligned with \( B \) in very dense regions of \( n_{h} \gtrsim 1 \).

5.2. Effect of Magnetic Relaxation on Magnetic Alignment

Following Davis & Greenstein (1951), a rotating paramagnetic grain with the angular momentum \( J \) making an angle relative to the ambient magnetic field experiences magnetic dissipation of the grain rotational energy due to the existence of the rotating magnetization with respect to the grain body. This paramagnetic relaxation process eventually leads to the alignment of \( J \) with the magnetic field (see Figure 4), which is known as the Davis–Greenstein (DG) alignment mechanism. For grains with iron inclusions, the characteristic time of superparamagnetic relaxation is given by (see, e.g., Hoang & Lazarian 2016a)

\[
\tau_{\text{mag,sp}} = \frac{I_{l}}{VK_{sp}(\Omega) B^{2}} = \frac{2 \mu a_{2}^{2}}{5 K_{sp}(\Omega) B^{2}},
\]

\[
\simeq 0.15 \frac{\mu a_{2}^{2}}{N_{\text{cl}} \phi_{\text{sp},-2} \rho_{g}^{2} B_{5}^{4}} \frac{T_{d,1}}{k_{sp}(\Omega)} \text{ yr}, \tag{50}
\]

where \( K_{sp}(\Omega) \) from Equation (11) has been used.
To see if superparamagnetic relaxation can help align grains against gas collisions, it is convenient to introduce a dimensionless parameter

\[ \delta_{\text{mag,sp}} = \frac{\tau_{\text{gas}}}{\tau_{\text{mag,sp}}} = 56a^{-5}N_{\text{cl}}\phi_{\text{sp}}(\Omega) \beta_{\text{gas}}^{1/2} B_z^2 k_{\text{sp}}(\Omega) T_{d,1}, \]  

which implies \( \delta_{\text{mag,sp}} = 5.6 \) and 0.56 for \( a = 1 \) and \( 10 \mu m \) with \( n_g = 1 \) and \( N_{\text{cl,4}} = 1 \).

As shown in Hoang & Lazarian (2016b), grain alignment by superparamagnetic relaxation is inefficient for thermally rotating grains because of internal thermal fluctuations within the dust grain. The joint action of superparamagnetic relaxation and RATs (METs) enhances the alignment degree so that SPM grains can achieve perfect alignment if \( \delta_{\text{mag,sp}} > 10 \) (Hoang & Lazarian 2016a). Therefore, one can determine the maximum size for which superparamagnetic relaxation is important for external alignment using the condition \( \delta_{\text{mag,sp}} > 1 \), yielding

\[ a_{\text{max,JB}}^{\text{mag}} \simeq \frac{5.6 N_{\text{cl}}\phi_{\text{sp}}(\Omega) k_{\text{sp}}(\Omega)}{\beta_{\text{gas}}^{1/2} T_{d,1}} \mu m. \]  

For a protostellar core with a typical magnetic field of \( B = 10^3 \mu G \) (see Hull & Zhang 2019; Pattle et al. 2022 for reviews), one has \( a_{\text{max,JB}}^{\text{mag}} \approx 55.3 \mu m \) and 0.553 \( \mu m \) for \( n_H = 10^3 \) and \( 10^7 \) cm\(^{-3} \), assuming \( N_{\text{cl}} = 10^7 \). Therefore, superparamagnetic relaxation can be important for the external alignment of VLGs in prestellar cores or protostellar envelopes of density \( n_H \lesssim 10^6 \) cm\(^{-3} \). However, in protostellar cores of very high density of \( n_H \gtrsim 10^7 \) cm\(^{-3} \), superparamagnetic relaxation is negligible for VLGs.

### 5.3. The k-RAT versus B-RAT Alignment

As discussed in Section 3, the magnetic field is usually the axis of grain alignment in the ISM due to the fact that Larmor precession is much faster than radiative precession and gas collisions. In protostellar environments, grain sizes and
radiation fields are much larger than the ISM, so that Larmor precession is not likely faster than radiative precession. To determine which direction (k or B) acts as the axis of grain alignment (see Figure 4), we need to compare their relative timescales. k-RAT alignment occurs when \( \tau_k < \tau_B \), and B-RAT alignment occurs when \( \tau_k > \tau_B \).

Using Equation (19) one calculates the radiative precession timescale for grains aligned at low-J attractors with \( St = 1 \),

\[
\tau_k^{low-J} \approx 56.8 \mu m 1/\gamma \rho^{1/2} \frac{1}{\gamma_{rad} \hat{\tilde{Q}}_e U} \frac{1.2 \mu m}{\gamma_{rad} \hat{\tilde{Q}}_e U} \text{yr},
\]

and for grains aligned at high-J attractors with \( St = St_{RAT} \) from Equation (18), one has

\[
\tau_k^{high-J} \approx 11.7 \frac{\rho a_{-5}}{n_{gas, 1}} \left( \frac{1}{1 + F IR} \right) \frac{1.2 \mu m}{\gamma_{rad} \hat{\tilde{Q}}_e U} \text{yr}.
\]

Comparing \( \tau_k^{low-J} \) and \( \tau_k^{high-J} \) with \( \tau_B \) (Equation (26)) yields

\[
\frac{\tau_k^{low-J}}{\tau_B} \approx 6.96 \times 10^4 \frac{\gamma_{rad} \hat{\tilde{Q}}_e U}{\rho a_{-5}} \frac{1.2 \mu m}{\gamma_{rad} \hat{\tilde{Q}}_e U} \frac{1}{1 + F IR} \frac{T_d,1}{n_{gas, 1}},
\]

for grains at low-J attractors, and

\[
\frac{\tau_k^{high-J}}{\tau_B} \approx 1.4 \times 10^4 \frac{\gamma_{rad} \hat{\tilde{Q}}_e U}{\rho a_{-5}} \frac{1.2 \mu m}{\gamma_{rad} \hat{\tilde{Q}}_e U} \frac{1}{1 + F IR} \frac{T_d,1}{n_{gas, 1}},
\]

for grains aligned at high-J attractors.

For grains aligned at low-J attractors, the k-RAT alignment occurs when \( \tau_k^{low-J} < \tau_B \). The minimum size for the k-RAT alignment, which is also the maximum size for B-RAT alignment, is then determined by \( \tau_k^{low-J}/\tau_B = 1 \), yielding

\[
a_{min, low-J}^{RAT} = a_{max, JB}^{RAT, low-J} \approx 7.85 \frac{\gamma_{rad} \hat{\tilde{Q}}_e U}{\rho a_{-5}} \left( \frac{1.2 \mu m}{\gamma_{rad} \hat{\tilde{Q}}_e U} \right) \left( \frac{T_d,1}{n_{gas, 1}} \right) \mu m,
\]

which decreases with increasing radiation strength but increases with \( N_{Cl} \). Thus, VLGs aligned at low-J attractors can experience k-RAT alignment even grains have strong iron inclusions of \( N_{Cl} = 1 \).

For grains aligned at high-J attractors, k-RAT alignment occurs when \( \tau_k^{high-J} < \tau_B \), and one obtains the minimum size for k-RAT (maximum size for B-RAT) alignment,

\[
a_{min, high-J}^{RAT} = a_{max, JB}^{RAT, high-J} \approx 1.4 \times 10^7 \left( \frac{\gamma_{rad} \hat{\tilde{Q}}_e U}{\rho a_{-5}} \right) \left( \frac{T_d,1}{n_{gas, 1}} \right) \mu m.
\]

5.4. Mechanical Precession and v-MET versus B-MET Alignment

For grains to be aligned with the gas flow, one must have \( \tau_v < \tau_{gas} \). For rotating grains with a suprathermal rotation number \( St \), one obtains

\[
a_{v, low,J}^{MET} > 0.035 \left( \frac{St}{Q_{prec, -1} \hat{\tilde{Q}}_{gas, 1}} \right)^{2/3} \mu m,
\]

which implies that even for a moderate suprathermal number of \( St \sim 3 \), mechanical precession is faster than gas randomization.

For grains aligned at high-J attractors by METs, i.e., \( St = St_{RAT} \) (Equation (22)), the v-MET alignment requires \( \tau_v < \tau_{gas} \), which yields

\[
\frac{Q_{v, low,J}}{Q_{spinup, -3}} > 0.2,
\]

which implies that a small magnitude of the precession component of METs can establish v-MET alignment.

Similar to RATs, in the presence of magnetic fields, the grain experiences both Larmor precession and mechanical precession. To determine which direction (B versus \( \nu_d \)) is the axis of grain alignment, one needs to compare the timescales of mechanical precession and Larmor precession. Using \( \tau_v \) and \( \tau_B \) from Equations (25) and (26), one obtains

\[
\frac{\tau_v}{\tau_B} \approx 0.21 \frac{St}{Q_{prec, -1} \hat{\tilde{Q}}_{gas, 1}} \left( \frac{T_d,1}{n_{gas, 1}} \right) \left( \frac{N_{Cl} \psi_{sp, -2} \hat{\tilde{R}}^2 B_3}{\rho a_{-5}} \right). \]

For grains aligned at low-J attractors, \( St = 1 \), the minimum size for the v-MET alignment is determined by \( \tau_v^{low-J}/\tau_B < 1 \), yielding

\[
a_{min, low,J}^{MET} \approx 1.6 \left( \frac{N_{Cl} \psi_{sp, -2} \hat{\tilde{R}}^2 B_3}{\rho Q_{prec, -1} \hat{\tilde{Q}}_{gas, 1}} \right)^{2/5} \mu m,
\]

which implies that VLGs can experience v-MET alignment.
For grains aligned at high-\(J\) attractors by METs, by plugging \(\text{St} = \text{St}_{\text{MET}}\) into Equation (61) one obtains

\[
\frac{\tau_{\text{high}-J}}{\tau_B} \simeq 2.1 \times 10^4 \frac{Q_{\text{spinup}, -3}}{Q_{\text{prec}, -1}} \frac{N_{\text{cl}} \phi_{\text{sp}, -2} \beta^3 B_3}{\hat{\rho}_{0}^{1/2} a_{-5}},
\]

such that the condition \(\tau_{\text{high}-J}/\tau_{B, \text{sp}} < 1\) yields

\[
a > a_{\text{MIN, high-J}} \simeq 2.1 \times 10^4 \left(\frac{Q_{\text{spinup}, -3}}{Q_{\text{prec}, -1}}\right) \left(\frac{1}{n_B \hat{\rho}_{0}^{1/2}}\right) \times \left(\frac{N_{\text{cl}} \phi_{\text{sp}, -2} \beta^3 B_3}{\hat{\rho}_{0}^{1/2}}\right) \mu m,
\]

which decreases with increasing \(n_B\) and decreasing \(N_{\text{cl}}\). For the protostellar core of density \(n_B = 10^8 \text{ cm}^{-3}\) and assuming the typical parameters, mechanical precession is faster than Larmor precession for VLGs with \(a > a_{\text{MIN, low-J}} \sim 0.21 \text{ cm}\). At a higher density of \(n_B = 10^{12} \text{ cm}^{-3}\), \(\nu\)-MET alignment can occur for smaller sizes of \(a > 2.1 \mu m\) for \(N_{\text{cl}} a = 1\). In the case of grains with small iron inclusions of \(N_{\text{cl}} \leq 10\), small grains with \(a > 0.21 \mu m\) can experience \(\nu\)-MET alignment in protostellar cores.

In summary, similar to RATs, large grains at low-\(J\) attractors can experience \(\nu\)-MET alignment, whereas large grains at high-\(J\) attractors likely experience \(B\)-MET alignment. As the net degree of grain alignment at low-\(J\) attractors is rather low, its contribution to the observed polarization is subdominant if \(f_{\text{high}-J}\) is considerable (e.g., \(f_{\text{high}-J} \sim 0.1-0.2\)).

Table 1 summarizes different regimes of internal and external alignment for VLGs in protostellar environments by RATs due to stellar radiation and METs due to gas flows grain drift for two cases of fast internal relaxation (i.e., \(\tau_{\text{INR}} < \tau_{\text{gas}}\)) and slow internal relaxation (\(\tau_{\text{INR}} > \tau_{\text{gas}}\)). For internal alignment, grains aligned at high-\(J\) attractors have right IA in both cases of fast and slow relaxation, whereas grains aligned at low-\(J\) attractors have two possibilities of right IA and wrong IA. For external alignment, superparamagnetic grains can be aligned with \(\nu \| B\) (\(B\)-RAT/\(B\)-MET) at high-\(J\) attractors, while paramagnetic grains are likely aligned along the radiation or gas flow (\(k\)-RAT/\(\nu\)-MET). Both superparamagnetic and paramagnetic grains aligned at low-\(J\) attractors likely experience alignment along the radiation or gas flow.

### 5.5. Minimum Size for Grain Alignment by RATs and METs

Regardless of the axis of grain alignment (\(B, k\) or \(\nu\)), efficient alignment of dust grains with a preferred direction in space is only achieved when grains rotate suprathermally because the grain orientation is randomized by gas collisions when they rotate thermally (Hoang & Lazarian 2016). Therefore, one can determine the minimum size for grain alignment by RATs by using the suprathermal condition \(\text{St}_{\text{RAT}} = 3\) (e.g., Hoang & Lazarian 2014). Using Equation (17) yields

\[
a_{\text{align}}^{\text{RAT}} \simeq 0.016 \mu m \left(\frac{\chi}{1.2 \mu m}\right)^{1/3} \left(\frac{\dot{\chi}}{1.2 \mu m}\right)^{2/3} \left(\frac{\text{St}_{\text{RAT}}}{\text{St}_{\text{MET}}\text{,low-J}}\right)^{2/7} \mu m,
\]

which implies that small grains of \(a > 0.017 \mu m\) can be aligned in high-density regions of \(n_B = 1\) but in a strong radiation field of \(U_{\nu} = 1\).

Similarly, the minimum size for grain alignment by METs is defined by \(\text{St}_{\text{MET}} = 3\) (Hoang et al. 2018), which yields

\[
a_{\text{align}}^{MET} = 0.013 \mu m \left(\frac{r}{a_{-3}}\right)^{2/3} \left(\frac{Q_{\text{spinup}, -3}}{Q_{\text{prec}, -1}}\right)\mu m,
\]

which implies that small grains of \(a > 0.013 \mu m\) can be aligned by METs with \(\dot{\chi} \sim 0.1\), assuming \(Q_{\text{spinup}, -3} = 1\).

### 6. Application for a Protostellar Disk

Now, we apply our general analysis from the previous section for a typical case of a protostellar disk and study the various grain alignment mechanisms within the disk midplane.

#### 6.1. Disk Model Assumption

We consider a flared disk model with the midplane surface density at the disk radius \(r\) from the central star given by (see, e.g., Tung & Hoang 2020)

\[
\Sigma(r) = \Sigma_0 \left(\frac{r}{a_{-1}}\right)^{-\alpha},
\]

where \(\Sigma_0\) is the mass surface density at \(r = 1 \text{ au}\), and we assume \(\alpha = 3/2\). The typical values are \(\Sigma_0 \sim 100-1000 \text{ g cm}^{-2}\) for different disks (see Williams & Cieza 2011).

We adopt the flared disk model from Chiang & Goldreich (1997) with the pressure-scale height increasing with the radius as

\[
\frac{H_p}{r} = 0.19 \left(\frac{r}{a_{-1}}\right)^{1/7}.
\]

The gas number density at radius \(r\) is calculated as

\[
n_{H}(r) = \frac{1}{2 m_{H}} \frac{\Sigma(r)}{\sqrt{2\pi} H_p} = \frac{1}{2 m_{H}} \frac{\Sigma_0}{\sqrt{2\pi} H_p} \left(\frac{r}{a_{-1}}\right)^{-\alpha} \left(\frac{r}{a_{-1}}\right)^{-8/7} \text{ cm}^{-3},
\]

and we denote \(n_0 = 4 \times 10^{13} \Sigma_0 / 10^3 \text{ g cm}^{-2}\) and \(\alpha_0 = \alpha + 8/7\).

The gas temperature in the disk plane can be described by Chiang & Goldreich (1997)

\[
T_{\text{gas}}(r) = T_0 \left(\frac{r}{a_{-1}}\right)^{-\beta},
\]
where $\beta = 3/7$ and $T_0 \approx 150$ K for $r < 84$ au and $T_{\text{gas}}(r) = 21$ K for $84$ au $< r < 209$ au (see, e.g., Tung & Hoang 2020).

The magnetic field in the disk consists of the toroidal and poloidal components (see, e.g., Tazaki et al. 2017). Here, we assume that the strength of the magnetic field can be approximated as

$$B(r) = B_0 \left(\frac{\dot{M}}{10^{-8} M_{\odot} \text{ yr}^{-1}}\right) \left(\frac{r}{\text{au}}\right)^{11/8},$$  

(71)

where $\dot{M}$ is the mass accretion rate and $B_0 = 10^6 \mu$G (see Yang 2021).

As noted above, the gas temperature has a piecewise profile with $T_{\text{gas}} \sim r^{-\beta}$ for $r < 84$ au, while $T_{\text{gas}} = 21$ K for $r > 84$ au. In the following sections, we only show our relevant variables with $T_{\text{gas}} \sim r^{-\beta}$ as a general expression. These variables are indeed easily converted to the case of $r > 84$ au by replacing $T_0 = 21$ K and $\beta = 0$.

### 6.2. Magnetic Alignment by Superparamagnetic Relaxation and Fast Larmor Precession

Plugging $n_\text{d}(r)$, $T_{\text{gas}}(r)$, and $B(r)$ into Equation (52), one obtains the maximum size for alignment by superparamagnetic relaxation,

$$a_{\text{max,JB}}(r) \approx 17.7 \frac{N_{\text{cl},4} q_{\text{sp}}^{-2} B_0^2}{\dot{\rho} n_{13} T_{0,2}^{1/2}} \left(\frac{\dot{M}}{10^{-8} M_{\odot} \text{ yr}^{-1}}\right)^2 \times \left(\frac{r}{\text{au}}\right)^{\alpha + \beta/2 - 11/4} \mu\text{m},$$  

(72)

where $B_{0,6} = B_0/(10^6 \mu$G), $n_{0,13} = n_0/(10^{13} \text{ cm}^{-3})$, $T_{0,2} = T_0/(100$ K), and $\alpha + \beta/2 - 11/4 = 0.107$, which implies $a_{\text{max,JB}} \sim 17.7, 24.6, 30.1, \text{ and } 34.2 \mu$m at $r = 1, 10, 50, \text{ and } 100$ au, respectively, assuming $N_{\text{cl},4} = 1$.

Similarly, using Equation (49), one obtains the maximum size for the magnetic alignment of $J$ with $B$ at disk radius $r$,

$$a_{\text{max,JB}}(r) \approx 7.9 \frac{N_{\text{cl},4} q_{\text{sp}}^{-2} B_0^2}{\dot{\rho} n_{13} T_{0,2}^{1/2}} \left(\frac{\dot{M}}{10^{-8} M_{\odot} \text{ yr}^{-1}}\right) \times \left(\frac{r}{\text{au}}\right)^{\alpha + \beta/2 - 11/8} \mu\text{m},$$  

(73)

which yields $a_{\text{max,JB}} \sim 25, 59, 108, \text{ and } 139 \mu$m at $r = 1, 10, 50, \text{ and } 100$ au, respectively, assuming the high level of iron inclusions with $N_{\text{cl},4} = 1$.

Figure 10 shows the maximum size for the external alignment of $J$ with $B$ as a function of the disk radius for grains with different iron inclusions of $N_{\text{cl}} \geq 100$ and for the two disk models. For $\Sigma_0 = 100$ g cm$^{-2}$ (left panel), VLGs with the maximum number of iron atoms per cluster of $N_{\text{cl}} = 10^9$ can be magnetically aligned throughout the disk. Beyond the disk radius $r > 10$ au, VLGs of sizes $a > 10$ $\mu$m can be aligned with the magnetic field. As $a_{\text{max,JB}} \sim \Sigma_0^{-1} \cdot t$, a 10 times higher gas surface density ($\Sigma_0 = 10^7$ g cm$^{-2}$) results in a magnitude lower in $a_{\text{max,JB}}$ (right panel).

### 6.3. Radiation Field, RATs, and Grain Alignment

Dust grains in the disk midplane are illuminated by attenuated stellar radiation and thermal emission from hot dust in the surface layer. The radiative transfer modeling of

Tazaki et al. (2017) obtains the mean wavelength of the radiation field in the disk interior, which is $\lambda \sim 50–150$ $\mu$m, and the anisotropy degree $\gamma_{\text{rad}} \approx 0.3 – 1$. Assuming gas–dust thermal equilibrium (i.e., $T_\gamma(r) = T_{\text{gas}}(r)$), the radiation strength in the disk plane can be estimated using the gas temperature from Equation (70):

$$U(r) = \left(\frac{T_{\text{gas}}(r)}{16.4 \text{ K}}\right)^6 = U_0 \left(\frac{r}{\text{au}}\right)^{-6/\beta},$$  

(74)

where $U_0 = (T_0/16.4 \text{ K})^6 = 5.9 \times 10^5 (T_0/150 \text{ K})^6$ for silicate grains, and the weak dependence of the grain temperature on its size is disregarded for simplicity (see Draine 2011).

Plugging $n_\text{d}(r)$ and $U(r)$ into Equation (44), one obtains the minimum size for efficient internal alignment by inelastic relaxation for grains at high-$J$ attractors by RATs,

$$a_{\text{min,al}}(\text{iER}) \approx \frac{889.6}{\left(\frac{\mu g Q_1 g}{\beta^2 s^{3/2}}\right)^{1/6}} \left(\frac{n_{0,13}}{T_{0,2}}\right)^{1/6} \times \left(\frac{\lambda}{100 \mu\text{m}}\right)^{1/3} \left(\frac{T_{0,2}}{n_{0,13}}\right)^{1/3} \times \left(\frac{r}{\text{au}}\right)^{2(-\alpha + 4\beta)/3} \mu\text{m},$$  

(75)

where $2(-\alpha + 4\beta)/3 = -0.62$ for $\alpha_0 = 3/2 + 8/7$ and $\beta = 1/2$, which decreases with increasing disk radius.

Similarly, the maximum size of internal alignment by inelastic relaxation is (see Equation (45))

$$a_{\text{max,al}}(\text{iER}) \approx 0.081 \left(\frac{\beta g^{1/2}}{\mu g Q_1 g^{1/2}}\right)^{1/3} \left(\frac{T_{0,2}}{n_{0,13}}\right)^{1/3} \times \left(\frac{\lambda}{100 \mu\text{m}}\right)^{1/3} \left(\frac{T_{0,2}}{n_{0,13}}\right)^{1/3} \times \left(\frac{r}{\text{au}}\right)^{4(-\alpha + 4\beta)/3} \mu\text{m},$$  

(76)

where $4(-\alpha_0 + 4\beta)/3 = 1.24$, indicating an increase with the disk radius $r$.

The above equations reveal that there exists no satisfactory range of grain sizes that can have fast inelastic relaxation at $r = 1$ au because $a_{\text{max,al}} < a_{\text{min,al}}$, assuming the normalized parameters. Reducing the values of $\mu Q$ can increase the efficiency of inelastic relaxation, as implied by Equations (75) and (76).

We can now calculate the maximum size for internal alignment by super-Barnett relaxation using Equation (38). For a more massive disk of $\Sigma_0 = 10^3$ g cm$^{-2}$, we found that both super-Barnett relaxation and internal relaxation is inefficient in the whole disk. For a thinner disk with the surface mass density $\Sigma_0 = 100$ g cm$^{-3}$ and low values of $\mu Q = 10^9$ erg cm$^{-3}$, inelastic relaxation can be efficient for VLGs beyond $r > 100$ au, as shown in Figure 11, but super-Barnett relaxation is still inefficient.

For external alignment, plugging $U$ and $n_\text{d}(r)$ into Equation (65), one obtains the minimum alignment size by
RATs,

\[
a_{\text{align}}(r) \approx 5.4 \left( \frac{\hat{\lambda}}{100 \, \mu m} \right)^{2/7} \left( \frac{\gamma_{\text{rad}} U_{0.6}}{n_{0.13} T_{0.2}} \right)^{-2/7} \times \left( \frac{r}{\text{au}} \right)^{-2(\alpha - 5\beta)/7} \mu m,
\]

where \(-2(\alpha - 5\beta)/7 = -0.14\), which implies a decrease in the grain alignment by RATs with increasing radius.

To understand if grain alignment occurs via \(k\)-RAT or \(B\)-RAT, plugging \(n_{11}(r), T_{\text{gas}}(r)\) and \(B(r)\) into Equation (57), one can determine the minimum size for the \(k\)-RAT (or maximum size for \(B\)-RAT) alignment at low-\(J\) attractors,

\[
a_{\text{min, low-}J} \equiv a_{\text{max, high-}J}
\]

\[
\approx 49.2^{3/5} \left( \frac{100 \, \mu m}{\gamma_{\text{rad}} \hat{Q}_{0.6}} \right)^{4/15} \times \left( \frac{N_{\text{cl}}}{{\phi}_{sp} - 2\hat{p}B_{0.6}} \right)^{2/3} \times \left( \frac{r}{\text{au}} \right)^{-2(\alpha - 11/12)} \mu m,
\]

which follows that VLGs can experience \(k\)-RAT alignment, especially grains with small iron clusters of \(N_{\text{cl}} \lesssim 10\).

Similarly, using Equation (78), one obtains the minimum size for \(k\)-RAT (or maximum size for \(B\)-RAT) alignment at high-\(J\) attractors,

\[
a_{\text{min, high-}J} \equiv a_{\text{max, low-}J}
\]

\[
\approx 4.3 \times 10^{5^{3/6}} \left( \frac{1}{{n}_{0.13} \hat{Q}_{0.2}^{-1/2}} \right) \times \left( \frac{N_{\text{cl}}}{{\phi}_{sp} - 2\hat{p}B_{0.6}} \right)^{2/3} \times \left( \frac{r}{\text{au}} \right)^{\alpha + \beta/2 - 11/8} \mu m,
\]

where \(\alpha + \beta/2 - 11/8 = 1.48\), which increases rapidly with increasing disk radius.

The above equations suggest that, for grains with iron inclusions aligned at a low-\(J\) attractor, \(k\)-RAT alignment can occur for \(a > a_{\text{min, low-}J}^{\text{low-}J \sim 10.6, 49.4 \, \mu m}\), even for large iron clusters of \(N_{\text{cl}} \approx 10^3\) and \(10^4\), respectively. However, \(k\)-RAT alignment at high-\(J\) attractors requires huge grains of \(a > 40 \, \text{cm}\) for \(N_{\text{cl}} = 10^4\) and \(a > 430 \, \mu m\) for \(N_{\text{cl}} \sim 10\). Large paramagnetic (i.e., \(N_{\text{cl}} \sim 1\)) grains of \(a > 43 \, \mu m\) can experience \(k\)-RAT alignment. The difference arises from the fact that radiation precession is much slower when the grain rotates fast (see Equation (19)). Therefore, VLGs with large iron inclusions aligned at high-\(J\) attractors can experience \(B\)-RAT alignment, while those at low-\(J\) attractors experience \(k\)-RAT alignment.

6.4. Grain Drift, METs, and Grain Alignment

In protostellar disks, dust grains follow a Keplerian azimuthal motion, while gas follows the sub-Keplerian pressured-supported disk because the gas experiences the gas.
pressure but the dust does not (Takeuchi & Lin 2002). Therefore, the dust moves faster than the gas and experiences the head wind.

For a disk with a central protostar of mass $M_*$, the Keplerian angular velocity is $\Omega_K = (GM_*/r^3)^{1/2}$, and the Keplerian tangential velocity becomes

$$v_K = \Omega_K r = \left( \frac{GM_*}{r} \right)^{1/2} = 2.98 \times 10^9 \left( \frac{M_*}{M_\odot} \right)^{1/2} \left( \frac{r}{\text{au}} \right)^{-1/2} \text{ cm s}^{-1}. \quad (80)$$

Dust grains orbit the central protostar at the Keplerian velocity $v_K$. The gas moves at a sub-Keplerian velocity so that its velocity relative to the dust can be written as $v_{\text{gas}} = v_K - \eta v_K$ with a coefficient of $\eta < 1$. The gas relative velocity has radial and azimuthal components,

$$v_r = \frac{2\text{Stk}}{1 + \text{Stk}^2} \eta v_K, \quad (81)$$

$$v_\phi = \frac{\text{Stk}^2}{1 + \text{Stk}^2} \eta v_K, \quad (82)$$

where Stk is the Stokes number (Tatsuuma & Kataoka 2021), and the total dust–gas relative velocity is $v_d = (v_r^2 + v_\phi^2)^{1/2}$.

For high Stokes numbers of Stk $\gg 1$, the azimuthal component dominates, and $v_d \approx v_\phi \approx \eta v_K = v_0(r/\text{au})^{-1/2}$. The grain drift parameter is then

$$s_d(r) = \frac{v_0}{v_K} \left( \frac{r}{\text{au}} \right)^{-1/2} = s_d(0) \left( \frac{r}{\text{au}} \right)^{(\beta - 1)/2}, \quad (83)$$

where $s_d(0)$ is the drift parameter at $r = 1$ au, and $s_d$ decreases with $r$. Comparing $s_d(r)$ with the critical drift for inelastic relaxation from Equation (47),

$$s_{\text{crit,al}} \approx 1.02 Q^{-1/2} \left( \frac{\mu g Q}{\rho^3} \right)^{1/6} \left( \frac{\eta_0}{T_2} \right)^{1/6} \times \left( \frac{g^2(\Theta_\parallel \parallel \Gamma^4)}{g^2(\Theta_\parallel \parallel \Gamma^4)} \right)^{1/6}, \quad (84)$$

which implies that the grain drift is not enough to induce inelastic relaxation.

Plugging $s_d(r)$ into Equation (46) one obtains

$$\frac{\tau_{\text{ER}}}{\tau_{\text{gas}}} \approx 6500 s_{\text{d,0}}^{-6} Q_{\text{spinup,} -3}^{-3} \left( \frac{\mu g Q_3}{\rho^3} \right) \left( \frac{\eta_0}{T_2} \right)^{1/6} \times \left( \frac{r}{\text{au}} \right)^{3 - 2\beta - \alpha_n}, \quad (85)$$

which yields

$$r < r_{\text{max,al}}(\text{ER}) \approx 3 \times 10^{-6} Q_{\text{spinup,} -3}^6 \left( \frac{s_{\text{d,0}}}{0.5} \right)^{12} \left( \frac{\rho^2}{T_2} \right) \left( \frac{\mu g Q_3}{\rho^3} \right)^{11/2} \left( \frac{n_{10}}{T_2} \right)^{1/2} \left( \frac{\eta_0}{T_2} \right)^{1/6} \left( \frac{\eta_0}{T_2} \right)^{1/6}, \quad (86)$$

where $3 - 2\beta - \alpha_n = -0.5$, which implies that METs cannot induce the inelastic relaxation for $s_{\text{d,0}} = 0.5$. Therefore, inelastic relaxation by METs is inefficient to induce efficient internal alignment of grains in the disk.

Now, we calculate the maximum size of internal alignment by the super-Barnett relaxation effect by first plugging $n_{\text{d}}(r), T_{\text{gas}}(r), s_d(r)$ into Equation (22) to obtain $N_{\text{d,MET}}$. Then, plugging $N_{\text{d,MET}}$ and $B(r)$ into Equation (36) and solve for the solution of the equation $r_{\text{BR}}/r_{\text{gas}} = 1$. The results are shown in Figure 12 for two values of the disk surface mass $\Sigma_0$. The maximum size of internal alignment is larger for lower $\Sigma_0$, but due to weaker gas damping. VLGs of $a > 10$ µm can be internally aligned at $r > 50$ au, and only smaller grains can be aligned in the inner disk, assuming grains of large iron inclusions with $N_{\text{d,4}} = 1$.

Figure 13 shows the same results as Figure 12 but for a larger drift parameter ($s_d = 0.5$). Stronger METs increase the maximum size of internal alignment because of the increase in the super-Barnett relaxation with $N_{\text{d,MET}}$. VLGs of large iron clusters with size $a > 10$ µm can have efficient internal relaxation beyond $r > 20$ au, and VLGS of $a > 100$ µm can be aligned at $r > 100$ µm. However, millimeter-sized grains with large iron inclusions still have slow internal relaxation, although they can be spun up by METs to suprathermal rotation.

To understand whether grain alignment occurs with $J$ along the magnetic field ($B$-MET) or the drift direction ($\nu$-MET), we plug $n_{\text{d,1}}, T_{\text{gas}},$ and $B$ into Equation (64) and obtain the critical size for $\nu$-MET alignment,

$$a_{\text{min,Jr}}(r) = 4230 \left( \frac{r}{\text{au}} \right)^{-1/2} \left( \frac{\mu g Q_3}{\rho^3} \right) \left( \frac{\eta_0}{T_2} \right)^{1/6} \left( \frac{n_{10}}{T_2} \right)^{1/6} \left( \frac{\eta_0}{T_2} \right)^{1/6} \times \left( \frac{r}{\text{au}} \right)^{\alpha_n + \beta/2 - 11/8} \mu m, \quad (87)$$

where $\alpha_n + \beta/2 - 11/8 = 1.48$, so that VLGS of $a > 423$, 4230 µm can experience the alignment with $J$ along the drift direction for $N_{\text{d,1}} = 10^4, 10^5$, respectively. For grains with small iron inclusions of $N_{\text{d,1}} < 10$, $\nu$-MET alignment can occur for $a > 4.23$ µm. Comparing to $a_{\text{min,Jr}}$, one can see that $\nu$-MET can occur for smaller grains compared to $k$-RAT.

We now estimate the minimum alignment size by METs using Equation (66),

$$a_{\text{align}}(r) \approx 0.013 \left( \frac{r}{\text{au}} \right)^{3 - 2\beta - \alpha_n} \left( \frac{\mu g Q_3}{\rho^3} \right) \left( \frac{\eta_0}{T_2} \right)^{1/6} \left( \frac{n_{10}}{T_2} \right)^{1/6} \left( \frac{\eta_0}{T_2} \right)^{1/6} \mu m. \quad (88)$$

Comparing with $a_{\text{align}}$ (Equation (77)) one can see that METs are more efficient than RATs in aligning grains in the disk midplane for the typical parameters of $Q_{\text{spinup,} -3} = 1$.

7. Discussion

7.1. Comparison of RATs and METs in Protostellar Environments

Modern grain alignment theory implies that both RATs (METs) can induce three fundamental effects on the grain dynamics, including grain alignment, spin-up/spin-down, and precession of the grain angular momentum around the radiation (gas flow; Lazarian & Hoang 2007a, 2007b; Hoang et al. 2018). For protostellar environments containing large grains and strong radiation field (gas flow), strong RATs (METS) can
rapidly align the grain angular momentum along the radiation direction, \( \mathbf{k} \) (gas flow \( \mathbf{v} \)), i.e., axis of grain alignment, with a fraction of grains at a high-\( J \) attractor and the rest at the lower-\( J \) attractor (Hoang & Lazarian 2008; Hoang et al. 2018). In the presence of an ambient magnetic field, the grain experiences also Larmor precession around \( \mathbf{B} \). Therefore, the axis of grain alignment is then determined by the direction \( (\mathbf{k}, \mathbf{v} \) or \( \mathbf{B} \)) around which the grain precession rate is the greatest (Lazarian & Hoang 2019). We have derived the general formulae for the critical sizes for grain alignment with the magnetic field, radiation direction, and gas flow, for grains with different iron inclusions in protostellar environments (see Section 5). The notations used in this paper are summarized in Table 2.

When the grain is aligned at a low-\( J \) attractor, the radiation (mechanical) precession is then determined by the radiation intensity (gas flux) and the precession component of RATs (METs) efficiency. However, when the grain is rotating at a high-\( J \) attractor, the radiation (mechanical) precession rate is independent of the radiation intensity (flow speed) and only determined by the ratio of the magnitude of precession component to spin-up components, \( Q_{\text{prec}}/Q_{\text{spinup}} \), depends on the grain properties and radiation field. Following the Analytical MOdel (AMO), component \( Q_{e3} \) is determined by the interaction of the photon/gas flow with the massive spheroidal body, which then weakly depends on the whole grain properties. However, component \( Q_{\text{spinup}} \) depends on the grain properties. For RATs, the variation of \( Q_{\text{spinup}} \) with the grain shapes is rather small, within an order of magnitude. However, for METs, \( Q_{\text{spinup}} \) can vary by 3–4 orders of magnitudes, and it is about 2 orders of magnitude lower than the precession component (Das & Weingartner 2016; Reissl et al. 2022). Therefore, the ratio of the MET components is \( Q_{\text{prec}}/Q_{\text{spinup}} \gg 1 \). This has an important implication for the external alignment of \( J \) along the drift direction instead of along the \( B \) field because the axis of alignment depends on the mechanical precession rate, which is proportional to \( Q_{\text{prec}}/Q_{\text{spinup}} \) (see Equation (61)) when grains are rotating suprathermally by METs. Moreover, the radiative suprathermal number increases with the grain size as \( S_{\text{RAT}} \sim a^{7/2} \) for large grains of \( a > \bar{\lambda}/2 \), whereas the mechanical suprathermal number increases rapidly with the grain size as \( S_{\text{MET}} \propto a^{3/2} \). Furthermore, the radiative suprathermal number decreases with
the gas density (Equations (17) and (18)), but the mechanical suprathermal number does not depend on the gas density (Equation (22)). As a result, METs can be more important than RATs and dominate the inelastic relaxation and internal alignment of VLGs in very dense regions like protostellar disks. Detailed modeling of dust polarization by MET alignment will be presented elsewhere.

7.2. Effects of Iron Inclusions on Internal and External Alignment

Observations reveal that about 90% of the Fe abundance is locked in dust (Jenkins 2009; Dwek 2016). Although the form of Fe in dust is unclear, one expects that a fraction of Fe is in the form of metallic (Fe) and iron oxide (FeO, Fe₂O₃, Fe₃O₄) nanoparticles. The existence of iron inclusions is expected for dust in protostellar environments. Interestingly, iron nanoparticles are reported to be present in the local interstellar dust grains captured by the Cassini mission (Altobelli et al. 2016) and in primitive interplanetary dust (Hu & Winterski 2021).

Iron inclusions in dust increase significantly the grain magnetic susceptibility, which increases the rate of Barnett relaxation (see Equation (27)) and IA. Therefore, the range of grain sizes with efficient IA is determined by the level of iron inclusions locked in the dust (see Equation (38)).

Iron inclusions also increase the efficiency of the external alignment of J with the magnetic field through two following effects. First, the enhanced magnetic susceptibility increases Larmor precession (with respect to radiative precession) and thus extends the range of grain sizes that experience B-RAT (B-MET) alignment. Second, iron inclusions increase the rate of magnetic relaxation, which enhances the fraction of grains that can be aligned at high-J attractors, \( f_{\text{hig}h-J} \) (Hoang & Lazarian 2016a). Therefore, the degree and pattern of thermal dust polarization depends sensitively on the level of iron locked in the dust.

Hoang & Lazarian (2016a) quantified the effect of iron inclusions embedded in dust grains on external alignment and found that iron inclusions can increase \( f_{\text{hig}h-J} \) to 100% for \( \delta_{\text{mag},\text{sp}} \gtrsim 10 \), which is known as super-RAT (SRAT) and super-MET (SMET) alignment (see also Lazarian & Hoang 2021). For the ISM, the gas damping rate is low due to a low gas density \( (n_{\text{H}} < 10^2 \text{ cm}^{-3}) \), so that a small level of iron inclusions can help grains reach \( \delta_{\text{mag},\text{sp}} > 10 \) and produce universal high-J attractors (Hoang & Lazarian 2016a). However, for protostellar environments of higher gas densities, the value of \( \delta_{\text{mag},\text{sp}} \) is greatly reduced, assuming the same level of iron inclusions as in the ISM.

As a result, the maximum size for which grain alignment is affected by superparamagnetic relaxation is reduced to \( \delta_{\text{max}} \approx 0.56 (N_{\text{d}, \text{Fe}, \text{sp}} B_{\text{d}}^2 n_{\text{H}}^{-1} T_{\text{gas}}^{-1/2})/\langle \delta_{\text{mag},\text{sp}} \rangle \) for \( n_{\text{H}} = 10^8 \text{ cm}^{-3} \) (see Equation (52)). Therefore, the value of \( f_{\text{hig}h-J} \) for VLGs is only determined by RATs/METs, as opposed to SRAT or SMET mechanisms. Based on calculations of RATs for an ensemble of Gaussian random shapes, Hermann et al. (2021) found that the value of \( f_{\text{hig}h-J} \) spans from 10% to 70% for grains aligned only by RATs, in the absence of magnetic relaxation.

7.3. Internal Alignment of VLGs with Slow Internal Relaxation

Our detailed calculations for protostellar environments show that large grains aligned at low-J attractors likely have slow internal relaxation. VLGs aligned at high-J attractors have high probability of having fast internal relaxation due to the increase in the relaxation rate with the angular momentum. However, some VLGs may have slow internal relaxation, depending on the grain magnetic susceptibility and the angular momentum. The question now is how the axis of grains with slow internal relaxation (\( \tau_{\text{INR}} > \tau_{\text{gas}} \)) align? Addressing this question is important for the modeling of dust polarization and interpretation of observational data.

Hoang & Lazarian (2009a) first studied the alignment of grains without internal relaxation by RATs and found that strong RATs can rapidly bring J to be aligned with B at high-J attractors on a timescale much shorter than the gas damping \( \tau_{\text{gas}} \) (fast alignment; Lazarian & Hoang 2007a, 2021), and the IA occurs with \( \hat{a}_i \parallel J \) (right IA). However, for grains aligned at low-J attractors, they found that IA can occur with two possible configurations of \( \hat{a}_i \parallel J \) (right IA) or \( \hat{a}_i \perp J \) (wrong IA). The discussion is then extended for both RATs and METs by Lazarian (2020) for carbonaceous grains that have no internal relaxation. However, Hoang & Lazarian (2009a)'s study did not account for the effect of gas random collisions on the grain orientation. Here, we discuss the potential effect of gas randomization on alignment of VLGs with slow internal relaxation.

Grains aligned at low-J attractors with (sub)thermal rotation are obviously sensitive to strong randomization by gas collisions (Hoang & Lazarian 2008, 2016a). In the absence of external torques (e.g., RATs, METs, or paramagnetic torques), the grain axis is only affected by gas collisions. Physically, thermal dust–gas collisions induce energy equipartition with the grain rotational energy \( \langle J^2 \rangle /2l = \langle J^2 \rangle /2l = kT_{\text{av}}/2 \) with the average temperature \( T_{\text{av}} = (T_{\text{d}} + T_{\text{gas}})/2 \), following the equipartition theorem. This corresponds to \( \langle J^2 \rangle = h^2 /2 \) with \( h > 1 \), revealing that the angular momentum tends to align with \( \hat{a}_i \), i.e., the degree of alignment of the grain axis of maximum inertia (\( \hat{a}_i \)) with \( J \) is of \( Q_{\text{r}} > 0 \). Quantitatively, thermal gas collisions will establish the Maxwellian distribution for the angle \( \theta \) between \( \hat{a}_i \) and \( J \), \( f_{\text{MW}}(\theta) = h/(4\pi) \times (\cos^2 \theta + h \sin^2 \theta)^{-1/2} \) (Jones & Spitzer 1967; Hoang et al. 2010). The net degree of IA is \( Q_{\text{X,IA}} = \int f_{\text{MW}}(\theta) 1/(3 \cos^2 \theta − 1) \sin \theta d\theta \) and is still positive due to “inertial nonsphericity,” i.e., on average, \( \hat{a}_i \) is still parallel to \( J \) (right IA), although the IA degree is rather small (<10%) for grains of axial ratios of \( s > 0.4 \) (Lazarian & Roberge 1997).

However, the effect of external torques (RATs or METs) can change dramatically the distribution of the angle \( \theta \) from the Maxwellian distribution. As shown in Hoang & Lazarian (2009a), if the driving RATs are strong enough, grains can be stably aligned at low-J attractors with two possible configurations of the right or wrong IA. The wrong IA can also be sustained with the help of pinwheel torques (Hoang & Lazarian 2009b; Lazarian & Hoang 2021). On the other hand, for grains at high-J attractors, RATs can rapidly stabilize the alignment of right IA, even if the grains have no internal relaxation (Hoang & Lazarian 2009a) if the RAT alignment is faster than the gas randomization (fast alignment case). Let \( f_{\text{hig}h-J,\text{IA}} \) be the fraction of grains at low-J attractors that have the right IA. Therefore, if RATs/METs are very strong, the torques can maintain a fraction \( f_{\text{hig}h-J} \) of grains with right IA at
high-\(J\) attractors and a fraction \((1 - f_{\text{high-J}}) f_{\text{right-IA}}\) with the right IA, and \((1 - f_{\text{high-J}}) (1 - f_{\text{right-IA}})\) with the wrong IA.

Nevertheless, in protostellar environments with a high gas density, the case of slow alignment is more common than fast alignment, which implies that grains must go through a long period of low-\(J\) rotation before reaching high-\(J\) attractors (Hoang & Lazarian 2008, 2016a; Lazarian & Hoang 2021). Therefore, by the time grains reach a high-\(J\) attractor, gas collisions have already significantly altered the IA due to the inefficient internal relaxation. Thus, perfect (right) IA at high-\(J\) attractors is only satisfied when the internal relaxation is much faster than the gas damping, which was used in this paper to derive the range of grain sizes that have efficient IA. Grains with slow internal relaxation are expected to have right IA, but its degree of IA is low due to the gas collisions. The exact value of IA degree depends on the timescale required for slow alignment, which can be from several to \(\sim 100\) years (Hoang & Lazarian 2008, 2016a; Lazarian & Hoang 2021). Comparison of dust polarization from detailed numerical modeling that take the above effects of IA into account to observational data would help constrain the physics of IA.

7.4. Grain Alignment and Tracing Magnetic Fields with Dust Polarization Toward Protostellar Cores

Grains in protostellar cores are subject to both protostellar radiation field and gas–grain drift due to ambipolar diffusion as well as strong rotational damping by gas collisions. We here summarize our main results of grain alignment by RATs and METs for protostellar cores and discuss the implications for tracing magnetic fields via dust polarization.

7.4.1. k-RAT versus B-RAT Alignment and B-field Tracing

Assuming that grains can be aligned at the low-\(J\) and high-\(J\) attractors by RATs, we derived the analytical formulae for the critical sizes for IA and external alignment (including the k-RAT and B-RAT alignment) as functions of the local physical parameters (see Section 5.3).

For a protostellar core of typical density \(n_H \sim 10^2 - 10^3 \text{ cm}^{-3}\), the interstellar radiation field is heavily reddened (Hoang et al. 2021). For the typical reddened interstellar radiation field strength of \(U \lesssim 1\) and \(\lambda = 10 \mu m\), our results in Figure 11 show that RATs are insufficient to spin grains up to highly suprathermal rotation and induce efficient inelastic relaxation. However, grains with iron inclusions can have fast super-Barnett relaxation and achieve efficient IA for grain sizes of \(a \lesssim 10 \mu m\) (see left panel of Figure 11).

The presence of a central protostar can increase the radiation strength to \(U \gtrsim 100\), increasing the suprathermal rotation number \(s_{\text{RAT}} \gg 1\) (see Equation (17)). As a result, inelastic relaxation becomes faster, inducing the efficient IA of VLGs up to \(a \sim 10^5 \mu m\), whereas super-Barnett relaxation is also efficient for large grains of \(a \lesssim 50 \mu m\) (see the right panels of Figure 11), as found in Hoang (2022). Because millimeter-sized grains are not expected to be present in protostellar cores, the efficient IA with \(\hat{a} \perp \mathbf{J}\) is expected for all dust grains due to the combination of the super-Barnett and inelastic relaxation effects.

For external alignment, we found that grains at low-\(J\) attractors can experience k-RAT alignment in the region with strong protostellar radiation fields (e.g., near the protostar). As grains aligned at low-\(J\) attractors are expected to have a low degree of alignment due to thermal fluctuations, the dust polarization by the k-RAT alignment is low. However, large grains aligned at high-\(J\) attractors have radiative precession much slower than the Larmor precession because of the increase in \(\tau_J\) with \(J\), which results in two possibilities of k-RAT and B-RAT alignment, but grains with iron inclusions most likely experience B-RAT due to enhanced magnetic susceptibility (see Equation (58)). As grains at high-\(J\) attractors have a higher degree of alignment due to fast rotation, dust polarization produced by grain alignment at high-\(J\) attractors is dominant. For an ensemble of grains with \(f_{\text{high-J}}\), the dust polarization properties are mostly determined by grains at high-\(J\) attractors due to their dominance. Therefore, thermal dust polarization toward protostellar cores can reliably trace magnetic fields thanks to its efficient IA and B-RAT alignment.

7.4.2. v-MET versus B-MET Alignment and B-field Tracing

It is known that ambipolar diffusion is ubiquitous in dense prestellar cores due to the low ionization fraction of the gas. This effect can produce a drift velocity of \(v_\text{drift} \sim 0.2-0.3 \text{ km s}^{-1}\) calculated in Robere et al. (1995), corresponding to \(s_d \sim 0.4-0.6\) for \(v_T \sim 0.47^{1/2} \text{ km s}^{-1}\), for which METs can spin-up grains to suprathermal rotation. One can see that the range of the available drift parameter \(s_d\) is greater than the critical value required for the efficient inelastic relaxation of dense cores (see Equation (47)). Note that Barnett relaxation is still important for grains of \(a < 10 \mu m\). As a result, thanks to METs, both super-Barnett and inelastic relaxation become efficient and induce the efficient IA for grains in protostellar cores.

For external alignment, in the protostellar cores, METs can align small grains with sizes \(a_{\text{align}}^\text{MET} \sim 0.01 \mu m\) as given by Equation (66). Grains aligned at high-\(J\) attractors can still be efficiently aligned with the magnetic field due to the Larmor precession, i.e., B-MET alignment (see Equation (49)). Same as RATs, grains at low-\(J\) attractors can experience v-MET alignment. Therefore, the net alignment configuration is the right IA of \(\hat{a} / J\) and external alignment of \(J / B\), which implies the dominance of B-MET alignment over v-MET. As a result, dust polarization from grains aligned by METs can most likely trace B fields in protostellar cores.

Figure 14 summarizes our understanding of grain alignment for superparamagnetic grains in protostellar environments (see Sections 4 and 5 for the relevant formulae). In general, small grains of sizes \(a < 10 \mu m\) tend to align with the magnetic fields by RATs/METs due to fast Larmor precession, whereas VLGs of \(a > 10 \mu m\) can align along the radiation direction or the gas flow due to fast radiative/mechanical precession. In such dense protostellar environments, paramagnetic grains however cannot align with the magnetic field.

7.5. Alignment of VLGs in Protostellar Disks and Implications for Dust Polarization

Our calculations of critical sizes for grain alignment in Sections 6 do not require knowledge of the grain size distribution. However, to discuss the implications of our analysis for observations, we here assume that grains in protostellar disks can be very large, with sizes of \(a \gtrsim 100 \mu m\), due to grain growth, as revealed by numerous observations (Kwon et al. 2009; Testi et al. 2014; Galametz et al. 2019).
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**Figure 14.** Schematic illustration of internal alignment (IA) and external alignment of grains as a function of the grain size expected for protostellar environments, assuming grains are aligned by RATs (METs). The largest grains can experience $k$-RAT alignment (orange area) with two possible states of right and wrong IA. Smaller grains tend to experience $B$-RAT (-MET) alignment due to fast Larmor precession, with right IA (gray area) by inelastic relaxation (pink bar) and Barnett relaxation (green).

**Figure 15.** Expected polarization pattern of thermal dust emission from aligned grains with $J$ along the radiation direction $k$ for the case of right IA (IA, upper) and wrong IA (lower panel).

### 7.5.1. $k$-RAT versus $B$-RAT Alignment and Dust Polarization

Assuming that grains in the protostellar disk are spun up by RATs, our results in Section 6 show that both inelastic relaxation and super-Barnett relaxation are inefficient due to the weak radiation field and high gas density for a typical disk of density $\Sigma_0 = 10^5 \, \text{g cm}^{-2}$ (see Equation (76)). For a thin disk of $\Sigma_0 = 100 \, \text{g cm}^{-2}$ and low values of $Q_\mu$, inelastic relaxation can be efficient only in the outer disk at distances $r > 100 \, \text{au}$ (see Figure 11). Therefore, VLGs in protostellar disks do not have efficient internal relaxation, so both types of right and wrong IA are possible.

For grains at low-$J$ attractors, RAT alignment can occur with $J$ along $k$ for grain sizes of $a > a_{\text{RAT, low-IA}}$ (Equation (78)), even when grains contain large iron clusters (i.e., superparamagnetic material). Large paramagnetic grains of $a > 40 \, \mu \text{m}$ at high-$J$ attractors can experience $k$-RAT alignment (see Equation (79)). However, superparamagnetic grains at high-$J$ attractors cannot align with $k$ due to the slow radiative precession, leading to $B$-RAT alignment. When high-$J$ attractors are absent (i.e., only low-$J$ attractors exist; see Section 3.7), the grains experience $k$-RAT alignment and dust polarization does not trace the magnetic field. Note that the degree of grain alignment at low-$J$ attractors is rather low, leading to a low polarization degree. If high-$J$ attractors are present, the high degree of alignment at high-$J$ attractors dominates the net polarization of dust emission. In this case, dust polarization can trace magnetic fields.

**Figure 15** shows the polarization pattern expected from grain alignment in a protostellar disk with $J$ along the radiation direction $k$ for two cases of right IA ($\langle \hat{a} \parallel J \rangle$, upper panel) and wrong IA ($\langle \hat{a} \perp J \rangle$, lower panel) for a disk. The polarization patterns are azimuthal for the right IA and radial for the wrong IA case.

In Figure 16, we illustrate the polarization pattern of thermal dust emission induced by grain alignment with $J \parallel B$ for two cases of right IA (upper panel) and wrong IA (lower panel) for a protostellar disk with the toroidal magnetic field. The polarization patterns are then illustrated in the upper and lower panels, respectively. Note that the polarization pattern observed at a given wavelength will be determined by the grain size distribution, iron inclusions, magnetic fields, and radiation and gas flow. If the polarization is dominated by small grains with efficient IA, the polarization degree is high, and the pattern is radial (upper right panel). On the other hand, if the dust polarization is dominated by VLGs with inefficient IA, the polarization degree is rather low, and the pattern is azimuthal if the wrong IA is present (lower right panel). Detailed modeling of multiwavelength dust polarization from protostellar disks will be presented elsewhere.

### 7.5.2. $v$-MET versus $B$-MET Alignment and Dust Polarization

Assuming grains in the disk are spun up by METs, our results in Section 6 suggest that inelastic relaxation is slower than gas damping because the gas–grain drift parameter $s_d$ is
below the critical value required for METs (see Equation (84)). However, VLGs of size $a \sim 10-50 \mu m$ aligned at high-$J$ attractors can have fast super-Barnett relaxation and thus efficient IA at a disk radius of $r > 50$ au, provided $N_{elA} \sim 1$. However, VLGs of $a > 100 \mu m$ have slow internal relaxation throughout the disk, leading to inefficient IA (see Figures 8 and 9). For grains aligned at low-$J$ attractors with thermal rotation, internal relaxation is much slower than gas damping, leading to inefficient IA and possible alignment with the long axis parallel to the spinning axis (i.e., wrong IA). As the net alignment degree for thermal rotation at low-$J$ attractors is rather small, the decisive parameter for describing the dust polarization is the fraction of grains aligned at high-$J$ attractors, $f_{hi-RAT}^J$ (Hoang & Lazarian 2016a; Lazarian & Hoang 2019; Herranen et al. 2021).

For external alignment, our results reveal that grains at high-$J$ attractors can align with $J$ along the drift velocity $v_d$ if grains have a low level of iron inclusions or ordinary paramagnetic material. On the other hand, grains with large iron inclusions experience $B$-MET alignment due to the fast Larmor precession. Grains at low-$J$ attractors can experience $\nu$-MET alignment. Therefore, dust polarization can trace magnetic fields if $f_{hi-RAT}^J$ is considerable (e.g., $f_{hi-RAT}^J \sim 0.1-0.2$).

In Figure 17, we show the polarization pattern for $J$ aligned with the drift direction ($v_d$) assumed to be azimuthal. For the case of right IA, the polarization pattern is radial (upper right panel). For the case of wrong IA, the polarization vector is parallel to the drift $v_d$, producing the azimuthal polarization pattern. One can see that, in the disk, the polarization pattern by mechanical alignment is similar to magnetic alignment because both the drift direction and magnetic field are azimuthal. The contribution of the inward drifting will cause the drift direction to be spiral, leading to the difference in the polarization pattern from the magnetic alignment.

Those grains can produce the polarization map as seen in the upper panel of Figure 17. Grains of thermal rotation (grains at low-$J$ attractors) experience wrong IA, and the polarization map is like the lower panel of Figure 17.

We note that the magnetic field in the disk is very uncertain, but it is expected to be dominantly toroidal because of the disk rotation (see, e.g., Flock et al. 2015). Thus, the magnetic field aligns with the drift direction if the azimuthal drift component is dominant. In this case, $B$-MET and $\nu$-MET are the same, and the alignment with $J$ is directed along the azimuthal direction. The net polarization angle of dust emission is only characterized by internal relaxation.

Our results in this paper reveal that dust polarization can still trace magnetic fields in the protostellar disk due to the reduction of radiative precession at high-$J$ attractors.

7.5.3. Previous Numerical Modeling of Grain Alignment and Dust Polarization from Disks

The first detailed modeling of polarized dust emission from circumstellar disks using the early RAT theory (Draine & Weingartner 1996, 1997) is presented in Cho & Lazarian (2007), where a perfect IA and external alignment of grains with the magnetic field are assumed for all grains larger than $d_{align}$. Later, using the modern RAT alignment theory (Lazarian & Hoang 2007a; Hoang & Lazarian 2008, 2009a, 2014, 2016a), Tazaki et al. (2017) performed numerical modeling of dust polarization and demonstrated the importance of the azimuthal polarization due to the $k$-RAT alignment, assuming the perfect IA (i.e., efficient internal relaxation) for grains at high-$J$ attractors (i.e., right $k$-RAT) and the right IA for grains at low-$J$ attractors. However, we find that Barnett relaxation is inefficient for large grains without iron inclusions that can experience $k$-RAT alignment. Inelastic relaxation is also inefficient due to weak radiation fields in the disk. Therefore, the assumption of perfect IA for $k$-RAT alignment is not valid for all grains, but only valid for small grains. Moreover, Tazaki et al. (2017) calculated the radiative precession time for grains at low-$J$ attractors with a thermal rotation of $J = J_T$ (i.e., $\tau_k^{-1}$) and then compared that timescale with that of Larmor precession to determine the critical size for $k$-RAT alignment even for grains aligned at high-$J$ attractors. However, as shown in this paper, the radiation precession rate at high-$J$ attractors is much slower due to faster rotation, so that large grains with iron inclusions can still exhibit $B$-RAT alignment. This results in an overestimate of the importance of $k$-RAT alignment in protostellar disks because $a_{min,J}^{RAT,low-J} \ll a_{min,J}^{RAT,high-J}$. Therefore, numerical modeling of dust polarization taking our detailed physics presented in this paper is required to accurately predict the polarization map and spectrum and reliably interpret polarization data. A detailed modeling of dust polarization from grains with iron inclusions in protostellar environments will be presented elsewhere (Giang et al. 2022).

7.6. Implications for Interferometric Polarization Observations to Protostellar Disks

The physical condition of protostellar disks is suitable for multiple alignment mechanisms to be active. However, due to the rapid decrease in the gas density with increasing disk radius, each alignment mechanism can be efficient in some specific region.

In the outer region of radius $r > 100$ au, the gas density is lower, so efficient IA can be achieved by super-Barnett relaxation due to spin-up by METs. Moreover, $B$-RAT
alignment can occur in this outer region, as given by Figure 10. In this case, the polarization pattern is like in the upper panel of Figure 16. In the inner region of $r < 100$ au, IA can be efficient for small grains by super-Barnett relaxation, and MET alignment can occur due to high densities. As a result, the polarization pattern would not change from the outer region to the inner region. However, in the inner disk, the polarization does not trace $B$ fields, but it traces the infalling motion of the gas flow. Therefore, transition from outer to inner disks can occur. In the intermediate disk radius of $10$ au $< r < 100$ au, grain alignment may be the contribution of both RATs and METs.

In the disk condition where the gas–dust drift is azimuthal, which is similar to the magnetic toroidal field, the joint action of METs and Larmor precession can most likely induce the right IA and external alignment with the shortest axis parallel to $J \parallel B$ for small grains. For VLG, one can have the wrong IA due to inefficient internal relaxation, such that the alignment can be along the long axis parallel to $J \parallel B$ (see Figure 14).

Submillimeter Array (SMA) observations of a number of high-mass protostars on subparsec scales of $< 0.1$ pc are also reported (Zhang et al. 2014). ALMA polarization observations have been conducted toward a large sample of low-mass protostars (Class 0 YSOs) in Perseus (Cox et al. 2018) and Ophiuchus clouds (Sadavoy et al. 2019), showing the different polarization patterns from the envelope to the central region (disk of $\sim 100$ au scales). In particular, ALMA observations toward class I/II YSOs (protoplanetary disks) show different polarization patterns at different wavelengths (Harrison et al. 2019; Aso et al. 2021). For example, multiwavelength ALMA observations toward HD 142527 by Kataoka et al. (2017, 2019) revealed the variation of the polarization pattern from the envelope to the central region (disk of $\sim 100$ au scales). In particular, ALMA observations toward class I/II YSOs (protoplanetary disks) show different polarization patterns at different wavelengths (Harrison et al. 2019; Aso et al. 2021). For example, multiwavelength ALMA observations toward HD 142527 by Kataoka et al. (2016) revealed the variation of the polarization pattern from the north to the south, which is suggested due to the difference in the grain size distributions (Ohashi et al. 2018).

Our detailed analysis here indicates that the alignment mechanisms are far more complicated in protostellar disks because the case of $B$-RAT alignment with the wrong IA can also reproduce an azimuthal polarization pattern (see Figure 15). Moreover, Guillet et al. (2020) noticed that the effect of scattering on the Rayleigh regime for VLGs of $a > 100$ $\mu$m can cause polarization parallel to the $B$ field. Therefore, detailed modeling of dust polarization accounting for detailed dust physics discussed in this paper and comparison with multiwavelength observational data would help clarify the main polarization mechanisms. Moreover, circular polarization of dust emission from protostellar disks would be valuable to disentangle the different grain alignment mechanisms, as previously suggested (Lazarian 2007; Hoang & Lazarian 2016a; Lazarian & Hoang 2019) and modeled in Draine (2022).

7.7. Constraining Iron Locked in Dust and Magnetic Fields using $k$-RAT versus $B$-RAT Dust Polarization

Our detailed analysis show that the critical sizes of aligned grains by $k$-RAT depends on the iron level and magnetic field strength. The observed polarization pattern and fraction then will depend on these parameters. Observations of dust polarization can help us constrain the magnetic properties using $k$-RAT and $B$-RAT (Lazarian & Hoang 2019).

Our results also show that the minimum size for $k$-RAT at low-$J$ attractors depends on the radiation strength $U_0$. Therefore, the spike in the protostar luminosity due to episodic accretion (see Francis et al. 2022) can change the polarization pattern from $B$-RAT to $k$-RAT if the polarization is dominated by grains aligned at low-$J$ attractors. Moreover, the increase in the radiation strength can change the range of grain sizes with efficient IA at high-$J$ attractors and facilitate $B$-RAT alignment. Observations toward episodic protostars would be valuable to test this transient transition from the RAT alignment paradigm and constrain the fraction of grains with high-$J$ attractors $f_{\text{high}}$ and grain irregularity.

7.8. Implications for Grain Growth of Aligned Grains

Our results show that VLGs in protostellar disks can still have efficient IA by inelastic relaxation if they rotate suprathermally by METs. This has an important effect on grain growth and the internal structure of dust aggregates (Hoang 2022). If grains are aligned with the short axis along the drift velocity, grain growth by grain–grain collisions would form dust aggregates containing embedded individual grains of parallel short axes. If the alignment is with the long axis parallel to the drift velocity, resulting dust aggregates will contain individual grains with parallel long axes. Such features may leave imprints in the structure of cometary dust and interplanetary dust.

We note that, in addition to grain alignment, RATs and METs can induce rotational disruption when grains are spun up to extremely fast rotation (Hoang et al. 2019, 2021; Hoang 2020). A detailed study on rotational disruption by RATs is presented in Hoang et al. (2021) for protostars and in Tung & Hoang (2020) for protoplanetary disks. A detailed study on the rotational disruption by METs for protostellar environments is presented elsewhere.

7.9. Selective Effect of RAT-D and Increase in the $k$-RAT Alignment

Following the RAT-D mechanism (Hoang et al. 2019), in the vicinity of a protostar, grains can be spun up to extremely fast rotation and disrupted into small fragments when the resulting centrifugal stress exceeds the grain tensile strength, $S_{\text{max}}$. Following Hoang et al. (2021), the minimum size of rotational disruption by the RAT-D mechanism is given by

$$a_{\text{disr}} = \left( \frac{0.8 n_{\text{H}}}{{\gamma}_{\text{rad}} U_0 \lambda^2} \right)^{1/2} \left( \frac{S_{\text{max}}}{{\rho}} \right)^{1/4} (1 + f_{\text{IR}})^{1/2} \approx 0.17 \left( \frac{\gamma_{\text{rad}} U_0}{n_s T_1^2} \right)^{-1/2} \left( \frac{\lambda}{1.2 \mu\text{m}} \right)^{1/4} \left( \frac{S_{\text{max}}}{\rho} \right)^{1/4} \times (1 + f_{\text{IR}})^{1/2} \mu\text{m},$$

(89)

14 The timescale of episodic accretion is much longer than the various timescales involved in grain alignment (see Equations (4), (53), and (26)).
with $S_{\text{max},J} = S_{\text{max}}/10^7 \text{erg cm}^{-3}$, which implies that large grains can be disrupted in the region of densities of $n_g \lesssim 1$ near the protostar (see Hoang et al. 2021 for more details).

We now discuss the effect of the selective disruption of RAT-D on the dust polarization observed toward a protostar. For an ensemble of grain shapes, a fraction $f_{\text{high},-1}$ of grains can have high-$J$ attractors points. Thus, the fraction $f_{\text{high},-1}$ of grains on high-$J$ attractors that have large sizes of $a > a_{\text{dist}}$ can be disrupted into smaller ones by RAT-D. However, the fraction $1 - f_{\text{high},-1}$ of grains that are aligned at low-$J$ attractors do not experience RAT-D. Therefore, in the presence of RAT-D, large grains at $a > a_{\text{dist}}$ now can only rotate thermally at low-$J$ attractors. This selective disruption by RAT-D decreases the degree of dust polarization (Lee et al. 2020) and may change the polarization pattern. The former was observationally tested (Tram et al. 2021a, 2021b, 2021c).

The potential change in the polarization pattern due to RAT-D arises from the fact that grains at low-$J$ attractors tend to align via $k$-RAT instead of $B$-RAT. Indeed, if $a_{\text{dist}} < a_{\text{min},J,k}$, then all grains above $a_{\text{dist}}$ will experience $k$-RAT. For grains with slow internal relaxation of sizes $a > a_{\text{max},J,f} \sim 0.1 \mu$m for $n_g \sim 1$ (see Equation (40), $\alpha = 1$), there may exist a fraction $f_{\text{right,IA}}$ of grains with the right IA and $(1-f_{\text{right,IA}})$ with the wrong IA. The dust polarization at the longest wavelengths would be dominated by grains at low-$J$ attractors because the population of large grains at high-$J$ attractors is reduced by RAT-D. As a result, the polarization pattern now depends on $f_{\text{right,IA}}$. One have the radial polarization pattern if the wrong IA dominates, and the azimuthal pattern if the right IA dominates. Detailed modeling with both RAT-A and RAT-D combined with multiwavelength dust polarization observations would be useful for testing the selective effect of RAT-D.

8. Summary

Using the RAT (MET) alignment framework developed for interstellar grains, we study in detail the alignment of very large grains (VLGs) in protostellar environments due to various physical effects. Our main findings are summarized as follows:

1. For grains aligned at low-$J$ attractors by RATs (METs), both super-Barnett and inelastic relaxation are inefficient for large grains in protostellar cores. The rate of internal relaxation is significantly increased for grains aligned at high-$J$ attractors that have suprathermal rotation. The increase in inelastic relaxation is higher than super-Barnett relaxation due to its faster increase with the angular momentum and slower variation with the grain size.

2. We derived analytical formulae for estimating the range of grain sizes that have the efficient IA by super-Barnett and inelastic relaxation effects for grains aligned at low-$J$ and high-$J$ attractors by RATs and METs. VLGs at high-$J$ attractors can have the efficient IA due to inelastic relaxation and super-Barnett relaxation.

3. For external alignment, we revisit the condition for $k$-RAT (v-MET) versus $B$-RAT (B-MET) by comparing the radiative/mechanical precession time at both low-$J$ and high-$J$ attractors with Larmor precession. We find that large grains aligned at low-$J$ attractors can experience $k$-RAT, v-MET alignment, but large grains with large iron inclusions aligned at high-$J$ attractors experience $B$-RAT, v-MET alignment because of the suppression of radiative/mechanical precession at a higher angular momentum. The parameter space for $B$-RAT ($B$-MET) is extended due to the decrease in the radiative/mechanical precession rate at high-$J$ attractors, enabling a greater parameter space for tracing $B$ fields via dust polarization in protostellar cores.

4. We studied in detail the internal and external alignment of large grains by RATs and METs in a protostellar disk. We find that RATs cannot induce fast internal relaxation, but METs can induce fast super-Barnett relaxation for grains of $a \sim 10-50 \mu$m in the outer disk of $r > 50$ au. In the inner disk of $r < 50$ au, VLGs have the inefficient IA due to slow internal relaxation, which results in a low degree of dust polarization at submillimeter wavelengths and decreases toward the central protostar.

5. External alignment with the radiation field ($k$-RAT) occurs for VLGs aligned at low-$J$ attractors or grains without iron inclusions. Grains with large iron inclusions aligned at high-$J$ attractors experience $B$-RAT alignment due to the decrease in the radiative precession time with $J$ and increased Larmor precession, enhancing the parameter space for $B$-RAT and tracing the magnetic field via dust polarization.

6. METs are found to dominate over RATs in both internal alignment and external alignment in the disk midplane due to their stronger magnitudes. Grains with iron inclusions at high-$J$ attractors can be aligned with the magnetic field ($B$-MET) in the outer regions of the disk. The regions with magnetic alignment increase with the decreasing surface mass density of the disk. Therefore, depending on the disk mass, dust polarization can trace the magnetic field.

7. Thermal dust polarization from aligned grains is expected to change across the disk due to the combination of different alignment mechanisms at different radii. Magnetic alignment by RATs is expected to be effective in the outer disk, whereas MET alignment is efficient in the inner disk.

8. The diverse polarization properties observed toward protostellar disks by ALMA may be explained by multiple alignment mechanisms. Detailed modeling of dust polarization based on the grain alignment physics presented in this paper would help interpret the observational data and disentangle different mechanisms and constrain dust properties in protostellar disks.

9. The polarization pattern produced by $k$-RAT alignment with the right IA is similarly azimuthal as those produced by $B$-RAT or v-MET alignment with the wrong IA. Multiwavelength polarimetric observations can disentangle this degeneracy.

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Appendix A
Review on Inelastic Relaxation

A.1. Acceleration within a Rotating Grain

Let \( \mathbf{X} \) be a vector connecting two atoms in the grain. The deformation can induce a change in \( \mathbf{X} \) with time. Taking both the intrinsic displacement and rotation into account, the time variation in \( \mathbf{X} \) with respect to the lab frame can be described by equation (see Goldstein et al. 2002)

\[
\frac{d\mathbf{X}}{dt}_{\text{LF}} = \left( \frac{d\mathbf{X}}{dt}_{\text{GF}} \right) + [\boldsymbol{\Omega} \times \mathbf{X}],
\]

(A1)

which returns to the rigid rotation when \( (d\mathbf{r}/dt)_{\text{LF}} = 0 \).

Acceleration of material within a rotating grain includes the intrinsic acceleration and inertial acceleration arising from the grain rotation. To find the total acceleration, we consider the grain angular velocity \( \boldsymbol{\Omega} \). Let \( \mathbf{u} \) be the displacement vector in the body frame. The new position of the atom at the moment \( t \) is then

\[
\mathbf{r}(t) = \mathbf{r}_0(t) + \mathbf{u}(t),
\]

(A2)

where \( \mathbf{u} = 0 \) and \( \mathbf{R} = \mathbf{r} = \mathbf{r}_0 \) assuming no deformation.

The position of the atom at the moment \( t + dt \) is then

\[
\mathbf{r}(t + dt) = \mathbf{r}_0(t + dt) + \mathbf{u}(t + dt).
\]

(A3)

Note that even if there is no further deformation during \( dt \) (e.g., deformation ceases), the vector \( \mathbf{u}(t + dt) \) still changes in the lab frame due to rotation.

The total velocity of the atom in the lab frame is

\[
\mathbf{v}_{\text{LF}} = \frac{d\mathbf{r}}{dt}_{\text{LF}} = \frac{d\mathbf{r}_0}{dt} + \frac{d\mathbf{u}}{dt}_{\text{GF}} = \left( \frac{d\mathbf{X}}{dt}_{\text{GF}} \right) + [\boldsymbol{\Omega} \times \mathbf{r}_0] + [\mathbf{u} \times \boldsymbol{\Omega}] + [\mathbf{u} \times [\mathbf{u} \times \boldsymbol{\Omega}]],
\]

(A4)

where the first term is the angular velocity of the lattice atoms in the absence of deformation and the atom velocity changes just because of rotation, and \( \mathbf{v}_{\text{GF}} = \mathbf{u} \) is the velocity of atoms measured in the GF, and the \( du/dt \) is rewritten using Equation (A1).

The acceleration of the atoms in the lab frame is then

\[
\mathbf{a}_{\text{LF}} = \frac{d\mathbf{v}_{\text{LF}}}{dt} = \frac{d}{dt} \left( \frac{d\mathbf{X}}{dt}_{\text{GF}} \right) + \frac{d}{dt} \mathbf{v}_{\text{GF}} + \frac{d}{dt} ([\mathbf{u} \times \boldsymbol{\Omega}]).
\]

(A5)

Using Equation (A1) for the time derivative in the lab frame, one obtains

\[
\frac{d}{dt} \mathbf{v}_{\text{GF}} = \mathbf{a}_{\text{GF}} + [\boldsymbol{\Omega} \times \mathbf{v}_{\text{GF}}],
\]

(A6)

\[
\frac{d}{dt} ([\mathbf{X} \times \mathbf{r}_0]) = \left( \frac{d}{dt} \mathbf{X} \right) \times \mathbf{r}_0 + [\mathbf{X} \times \mathbf{r}_0] = \boldsymbol{\Omega} \times \mathbf{r}_0 + [\mathbf{X} \times \mathbf{r}_0],
\]

(A7)

\[
\frac{d}{dt} ([\mathbf{u} \times \mathbf{u}]) = \left( \frac{d}{dt} \mathbf{u} \right) \times \mathbf{u} + [\mathbf{u} \times \mathbf{u}]
\]

\[
= \boldsymbol{\Omega} \times \mathbf{u} + [\mathbf{u} \times [\mathbf{u} \times \boldsymbol{\Omega}]] + [\mathbf{u} \times \mathbf{v}_{\text{GF}}],
\]

(A8)

where \( \mathbf{r}_0 = \mathbf{r}_0(t) = 0 \).

Combining the above equations, one then obtains

\[
\mathbf{a}_{\text{LF}} = \mathbf{a}_{\text{GF}} + [2(\mathbf{X} \times \mathbf{v}_{\text{GF}}) + [\boldsymbol{\Omega} \times \mathbf{r}]] + [\boldsymbol{\Omega} \times [\mathbf{X} \times \mathbf{r}]],
\]

(A9)

where \( \mathbf{r} = \mathbf{r}_0 + \mathbf{u} \).

As suggested by LE99, the acceleration of the atoms within the grain is \( \mathbf{a}_{\text{GF}} \sim (\delta t)^2/\tau \) with \( \delta t \) being the deformation of atoms within the grain, which can be disregarded due to small \( \delta t \). The acceleration of the atom in the LF is then just

\[
\mathbf{a}_{\text{LF}} = \left( \boldsymbol{\Omega} \times [\mathbf{X} \times \mathbf{r}] + [\mathbf{X} \times \mathbf{r}] + 2[\mathbf{X} \times \mathbf{v}_{\text{GF}}] \right),
\]

(A10)

which implies that even an atom at rest in the GF still has acceleration with respect to the lab \( \mathbf{a}_{\text{LF}} = \boldsymbol{\Omega} \times [\mathbf{X} \times \mathbf{r}] \) due to rotation (i.e., the centripetal acceleration).

According to the equivalence principle, in the rotating frame, atoms will experience a centrifugal force that is opposite to the centripetal acceleration,

\[
\mathbf{F}_{\text{cent}} = -m \mathbf{a}_{\text{LF}} = \left( \boldsymbol{\Omega} \times [\mathbf{X} \times \mathbf{r}] + [\mathbf{X} \times \mathbf{r}] + 2[\mathbf{X} \times \mathbf{v}_{\text{GF}}] \right) + [\mathbf{X} \times \mathbf{r}] + 2[\mathbf{X} \times \mathbf{v}_{\text{GF}}] + \mathbf{g}.
\]

(A11)

Therefore, atoms moving in the rotating frame can be considered to move relative to an inertial frame under a centrifugal force \( \mathbf{F}_{\text{cent}} \). In other words, the rotating frame can be considered as the inertial frame if the object is subject to the centrifugal force; thus all physical laws have the same form. Thus, Equation (A10) describes the acceleration of an atom within the rotating grain, which is comoving with the grain. If we take the gravity acting on the atom into account, the total acceleration is

\[
\mathbf{a}_{\text{LF}} = \left( \boldsymbol{\Omega} \times [\mathbf{X} \times \mathbf{r}] + [\mathbf{X} \times \mathbf{r}] + 2[\mathbf{X} \times \mathbf{v}_{\text{GF}}] + \mathbf{g} \right),
\]

(A12)

where the first term is due to the nonuniform rotation, the second term is the centrifugal acceleration, and the third term is the Coriolis acceleration exerted on the particle moving in a rotating system (see also Frouard & Efroimsky 2018).

A.2. Stress Tensor and Strain Potential Energy

When the acceleration field in the grain is known, one can calculate the components of the stress tensor using the differential equations

\[
\frac{\partial}{\partial x} \sigma_{ij} = \frac{\partial}{\partial y} \rho a_i \partial x_j + C, \quad \text{or}
\]

\[
\sigma_{xy}(x, y) = \frac{1}{2} \left( \int_x^y \rho a_i \partial x_j + \int_x^y \rho a_j \partial x_i \right) + C,
\]

(A13)

where \( C \) is the integral constant determined by the boundary conditions (see Efroimsky 2001 for a review).

The stress causes the deformation of the material structure within the grain. When the stress tensor is known, one can use the strain–stress relationship to find the strain \( \sigma_{ij} = \sum_k \sigma_{ijkl} C_{ijkl} \), where \( C_{ijkl} \) are the constants that depend on the grain’s elasticity. The strain potential energy is calculated as (see LE99)

\[
W = \frac{1}{2} \sum_{ij} \epsilon_{ij} \sigma_{jk},
\]

(A14)

which acts to bring the grain to the original state when the stress/external force is turned off.

Due to the time variation of the centrifugal acceleration induced by the grain’s precession, the deformation causes some dissipation of strain energy if the material is inelastic. The
alternating stress caused by the precession of $\Omega$ with $\hat{a}_1$ lags behind the grain material and induces the dissipation of the grain rotation energy into the heat, resulting in the IA of $\Omega$ and $J$ with $\hat{a}_1$ (Burns et al. 1973).

A.3. Inelastic Energy Dissipation and Relaxation Time

The energy loss by inelastic relaxation can occur on a timescale on the order of the precession period $\tau \sim 2\pi/\omega$, which can be written as $\tau = Q/\omega$; larger $Q$ implies a slower dissipation and better material quality. So, $Q$ is the quality factor of the grain material. As a result, one can write the energy loss as

$$\frac{dE_{\text{loss}}}{dt} = \frac{W}{\tau} = \frac{2\omega W}{Q},$$

(A15)

where $W = W(\omega) + 2W(2\omega)$ is the total strain energy associated with the principal frequency ($\omega$) and double frequency ($2\omega$) of oscillations, respectively (Efroimsky & Lazarian 2000).

Appendix B

Here we summarize our notations used in the paper.
| Category         | Notation | Description                                           | Typical Value (cgs units) | Normalized Notation |
|------------------|----------|------------------------------------------------------|---------------------------|---------------------|
| **Physical constants** | $k$      | Boltzmann constant                                   |                           |                     |
|                   | $h$      | Reduced Planck constant                              |                           |                     |
|                   | $\mu_B = e\hbar/2m_e$ | Bohr magneton                                      |                           |                     |
|                   | $m_H$    | Hydrogen atom mass                                   |                           |                     |
|                   | $m_e$    | Electron mass                                         |                           |                     |
|                   | $g_e$    | Electron spin $g$ factor                              |                           |                     |
|                   | $\gamma_e = -g_e\mu_B/h$ | Electron gyromagnetic ratio |                           |                     |
|                   | $T_{\text{gas}}$ | Gas temperature                                      | $\geq 10^6$ cm$^{-3}$ for protostellar cores | $n_{\text{H}} = n_{\text{H}}/10^6$ cm$^{-3}$ |
| **Environment parameters** | $n_{\text{H}}$ | Hydrogen number density                              |                           |                     |
|                   | $B$      | Magnetic field                                        |                           |                     |
|                   | $u_{\text{rad}}$ | Energy density of the radiation field               |                           |                     |
|                   | $U = n_{\text{rad}}/n_{\text{MMP83}}$ | Ratio between the local radiation to that in the solar neighborhood $n_{\text{MMP83}}$ | $10^6$ near the protostar | $U_{\text{rad}} = U/10^6$ |
|                   | $\gamma_{\text{rad}}$ | Anisotropy degree of the radiation field | 0.3                      |                     |
|                   | $\lambda$ | Mean wavelength of the radiation field              | 1.2 $\mu$m                |                     |
| **Grain mechanical parameters** | $a$      | Semimajor length of grains                           | $10^{-5}$ cm              | $a_{\text{eff}} = a_{\text{eff}}/10^{-5}$ cm |
|                   | $s = c/a$ | Ratio between semiminor and semimajor axis of grains | 0.5                       | $s = s/0.5$         |
|                   | $a_{\text{eff}}$ | Effective grain size defined as the equivalent sphere of radius $a_{\text{eff}}$ with the same volume as irregular grains |                           |                     |
|                   | $\rho$ | Mass density of grain material                        | 3 g cm$^{-3}$             | $\rho = \rho/(3$ g cm$^{-3}$) |
|                   | $I_1$ (Equation (1)) | Principal moment of inertia for the rotation parallel to the grain symmetry axis |                           |                     |
|                   | $I_\perp$ (Equation (2)) | Principal moment of inertia for the rotation perpendicular to the grain symmetry axis |                           |                     |
|                   | $k = 2/(1 + x^2)$ | Ratio between $I_1$ and $I_\perp$            |                           |                     |
|                   | $T_d$ | Grain temperature                                     | $10^{15}$ cm$^{-3}$      | $n_{23} = n/10^{23}$ cm$^{-3}$ |
|                   | $n$ | Number density of atoms within the dust material      | $10^{23}$ cm$^{-3}$      |                     |
|                   | $\hat{a}_1$, $\hat{a}_3$, $\hat{a}_5$ | Unit vectors for the grain principal axes; $\hat{a}_1$ denotes the axis of maximum inertia |                           |                     |
|                   | $\Omega$ | Angular velocity of grains                           |                           |                     |
|                   | $J$ | Angular momentum of grains                            |                           |                     |
|                   | $J = J_x$ and $J_y = J_z = J_\perp$ | Components of $J$ projected onto the grain principal axes $\hat{a}_x$, $\hat{a}_3$, and $\hat{a}_5$ |                           |                     |
| Category                             | Notation | Description                                                                 | Typical Value (cgs units) | Normalized Notation |
|-------------------------------------|----------|------------------------------------------------------------------------------|---------------------------|---------------------|
| Category                            |          |                                                                              |                           |                     |
|                                    | $\Omega_h = J/I_h$ | Angular momentum defined by the ratio between $J$ and $I_h$ |                           |                     |
|                                    | $v_T = \sqrt{2k_B T/m_W}$ | Thermal speed of gas                                                        |                           |                     |
|                                    | $\Omega_T = \sqrt{k_B T/m_W}$ | Grain thermal angular speed                                                  |                           |                     |
|                                    | $I_T = I_\parallel \Omega_T$ | Grain thermal angular momentum                                             |                           |                     |
|                                    | $S_t = J/J_T = \Omega_0/\Omega_T$ | Suprathermal rotation number                                               |                           |                     |
|                                    | $v_d$ | Grain drift speed                                                            |                           |                     |
|                                    | $s_d = v_d/v_T$ | Grain drift parameter                                                        |                           |                     |
|                                    | $Q_s$ | Degree of internal alignment of the grain axis of maximum inertia with $J$ |                           |                     |
|                                    | $\theta$ | Angle between $\hat{a}_1$ and $J$                                           |                           |                     |
|                                    | $f_{MW}(\theta)$ | Maxwellian distribution for the angle $\theta$ established by thermal gas collisions |                           |                     |
|                                    | $Q_{s,MW}$ | Degree of internal alignment assuming that $\theta$ follows $f_{MW}(\theta)$ |                           |                     |
|                                    | $j_p$ | Fraction of paramagnetic (Fe) atoms in the dust grain                        | 1/7 for silicate of MgFeSiO$_4$ structure |                     |
|                                    | $n_p = j_p n$ | Number density of paramagnetic atoms                                        |                           |                     |
|                                    | $p$ | Coefficient for the effective magnetic moment per paramagnetic atom         | 5.5                       | $\tilde{p} = p/5.5$ |
|                                    | $\mu_e = p \mu_B$ | Effective magnetic moment per paramagnetic atom                            |                           |                     |
|                                    | $\chi_p(0)$ (Equations (8) and (9)) | Zero-frequency susceptibility of paramagnetic grains |                           |                     |
|                                    | $N_{cl}$ | Number of iron atoms per cluster                                             | 20-10$^6$                 | $N_{cl} = N_{cl}/10^4$ |
|                                    | $\phi_{cl}$ | Volume filling factor of iron clusters                                       | 0.01-0.3                  | $\phi_{cl} = \phi_{cl}/10^{-2}$ |
|                                    | $\chi_{sp}(0)$ (Equation (10)) | Zero-frequency superparamagnetic susceptibility                             |                           |                     |
|                                    | $\omega \approx (\hbar - 1)J/(2I_h) \Omega \Omega_T$ | Precession frequency of the angular momentum $J$ around the axis of maximum inertia $\hat{a}_1$ |                           |                     |
|                                    | $\nu_0$ | Characteristic frequency of thermal fluctuations of iron clusters           | $10^3$ s                  |                     |
|                                    | $T_{act}$ | Characteristic temperature required for thermal remagnetization             | 0.011 K                   |                     |
|                                    | $\tau_{sp}$ (Equation (12)) | Timescale of thermally activated remagnetization                           |                           |                     |
|                                    | $\chi_2(\omega)$ | Imaginary part of the complex magnetic susceptibility                      |                           |                     |
|                                    | $K_{sp} = \chi_2(\omega)/\omega$ (Equation (11)) | Ratio between $\chi_2(\omega)$ and $\omega$                              |                           |                     |
|                                    | $k_{sp} = K_{sp} \mu_B/\mu_e \chi_p(0)$ (Equation (13)) | Describing the dependence of $K_{sp}(\omega)$ on dust temperature and frequency |                           |                     |
|                                    | $\Gamma_3$ (Equation (5)) | Geometrical factor adopted in the rotational damping time due to gas collisions | 0.62                      |                     |
|                                    | $\tau_{gas}$ (Equation (4)) | Damping time due to gas collisions                                           |                           |                     |
|                                    | $F_{IR}$ (Equation (7)) | Coefficient for the rotational damping due to IR emission                   |                           |                     |
|                                    | $\tau_{IR} = \tau_{gas}/F_{IR}$ | Damping time due to IR emission                                             |                           |                     |
|                                    | $\mu_{B Barnett}$ (Equation (14)) | Grain magnetization obtained via the Barnett effect                         |                           |                     |
|                                    | $\tau_B$ (Equation (26)) | Larmor precession time of $J$ around $B$                                   |                           |                     |
| Category                | Notation | Description                                                                 | Typical Value (cgs units) | Normalized Notation |
|-------------------------|----------|------------------------------------------------------------------------------|---------------------------|---------------------|
| External alignment      | $k$      | Radiation direction                                                          |                           |                     |
|                         | $d_{\text{trans}} \simeq \lambda / 2.5$ | Transition size at which the average RAT efficiency changes the slope       |                           |                     |
|                         | $\Omega_{\text{RAT}}$ (Equations (15) and (16)) | Maximum angular rotation speed due to the spin-up effect of RATs            |                           |                     |
|                         | $S_{\text{RAT}} = \Omega_{\text{RAT}} / \Omega_{T}$ (Equations (17) and (18)) | Suprathermal rotation number for grains spun up by RATs                    |                           |                     |
| External alignment by   | $Q_{\text{RAT}}$ | Third component of RATs that induces the precession of $J$ around $k$        | $10^{-2}$                 | $\hat{Q}_{\text{RAT}} = Q_{\text{RAT}} / 10^{-2}$ |
|                         | $f_{\text{high}}$ | Fraction of grains aligned at high-$J$ attractors                           |                           |                     |
|                         | $\tau_{\text{R}}$ (Equation (19)) | Precession time of $J$ around $k$                                           |                           |                     |
|                         | $\tau_{\text{low}} = \tau_{\text{R}} (\Omega = \Omega_{T})$ (Equation (53)) | Precession time of $J$ around $k$ at low-$J$ attractors                    |                           |                     |
|                         | $\tau_{\text{high}} = \tau_{\text{R}} (\Omega = \Omega_{\text{RAT}})$ (Equation (54)) | Precession time of $J$ around $k$ at high-$J$ attractors                  |                           |                     |
| External alignment by   | $\Gamma_{\text{MET}}$ (Equation (20)) | Magnitude of METs induced by the grain drift through ISM gas               |                           |                     |
|                         | $\Omega_{\text{MET}}$ (Equation (21)) | Maximum angular rotation velocity of grains spun up METs                   |                           |                     |
|                         | $S_{\text{MET}} = \Omega_{\text{MET}} / \Omega_{T}$ (Equation (22)) | Suprathermal rotation number for grains spun up by METs                    |                           |                     |
|                         | $Q_{\text{prec}}$ | Coefficient to describe the MET efficiency to induce the grain precession around the gas flow | $0.1$                     | $Q_{\text{prec},-1} = Q_{\text{prec}} / 10^{-1}$ |
|                         | $\Gamma_{\text{MET,prec}}$ (Equation (23)) | Component of METs inducing the precession of $J$ around the drift direction |                           |                     |
|                         | $\Omega_{\text{MET,prec}}$ (Equation (24)) | Precession frequency of $J$ around the drift direction                     |                           |                     |
|                         | $\tau_{\text{v}}$ (Equation (25)) | Precession time of $J$ around the drift direction                          |                           |                     |
| External alignment via  | $\tau_{\text{mag,sp}}$ (Equation (50)) | Superparamagnetic relaxation time                                           |                           |                     |
| magnetic relaxation     | $\delta_{\text{mag,sp}}$ (Equation (51)) | Ratio between the gas damping time and the superparamagnetic relaxation time |                           |                     |
|                         | $a_{\text{mag,sp}}$ (Equation (52)) | Maximum size for which superparamagnetic relaxation is important for external alignment given by $\delta_{\text{mag,sp}} = 1$ |                           |                     |
| Super-Barnett relaxation| $L_{T} = \sqrt{kT_{0}/(k - 1)}$ | Characteristic thermal angular momentum                                     |                           |                     |
| for internal alignment  | $f(x) = x(1 + x^{2}) / 2$ | Geometric factor adopted in the super-Barnett relaxation time              |                           |                     |
|                         | $\tau_{\text{BR,sp}}$ | Barnett relaxation time for superparamagnetic grains                       |                           |                     |
| Inelastic relaxation    | $\sigma$ | Poisson ratio taken from Molina et al. (2003)                                | $0.5$                     |                     |
| for internal alignment  | $\mu$ | Shear modulus                                                                | $10^{8}$ erg cm$^{-1}$    | $\mu_{s} = \mu / (10^{8}$ erg cm$^{-1}$) |
|                         | $Q$ | Quality factor of grain material                                             | $100$ (for silicate rocks) | $Q_{s} = Q / 10^{5}$ |
|                         | $g'(s) = 2.2s^{3/2}g(s)$ (see Equation (33) for $g(s)$) | Geometrical factor adopted in the inelastic relaxation time |                           |                     |
| Category | Notation | Description | Typical Value (cgs units) | Normalized Notation |
|----------|----------|-------------|--------------------------|---------------------|
| Inelastic relaxation time | \( \tau_{\text{iER}} \) | \( \tau_{\text{iER}} \) | | |
| Critical sizes and drift parameters for internal alignment | \( a_{\text{max,al}}(\text{BR}) \) (Equation (38)) | Maximum grain size for efficient internal alignment by Barnett relaxation (BR) given by \( \tau_{\text{BR,al}} = \tau_{\text{g}\text{as}} \) | | |
| | \( a_{\text{max,al}}(\text{ER}) \) (Equation (40)) | Critical grain size for efficient internal alignment by inelastic relaxation given by \( \tau_{\text{ER}} = \tau_{\text{g}\text{as}} \) | | |
| | \( a_{\text{RAT,high J}} \) (ER) (Equation (44)) | Minimum grain size for efficient internal alignment by inelastic relaxation given by \( \tau_{\text{ER}} = \tau_{\text{g}\text{as}} \) and assuming \( \Omega = \Omega_{\text{RAT}} \) | | |
| | \( a_{\text{RAT,high J}} \) (ER) (Equation (45)) | Maximum grain size for efficient internal alignment by inelastic relaxation for grains aligned at high-\( J \) given by \( \tau_{\text{ER}} = \tau_{\text{g}\text{as}} \) | | |
| Critical sizes for external alignment | \( \kappa_{\text{RAT}}(\text{ER}) \) | Critical drift parameter required for internal alignment by inelastic relaxation given by \( \tau_{\text{ER}} = \tau_{\text{g}\text{as}} \) | | |
| Critical sizes for external alignment | \( a_{\text{Lar}} \) (Equation (49)) | Maximum size for the grain alignment of \( J \) with \( B \) constrained by Larmor precession | | |
| Critical sizes for external alignment | \( a_{\text{RAT,low J}} \) (Equation (57)) | Minimum size for the \( k \)-RAT alignment at the low-\( J \) attractor given by \( \tau_{\text{RAT}} = \tau_{\text{g}\text{as}} \) | | |
| Critical sizes for external alignment | \( a_{\text{MET,low J}} \) (Equation (58)) | Minimum size for the \( k \)-RAT alignment at the high-\( J \) attractor given by \( \tau_{\text{RAT}} = \tau_{\text{g}\text{as}} \) | | |
| Critical sizes for external alignment | \( a_{\text{MET,low J}} \) (Equation (59)) | Minimum size for the \( \nu \)-MET alignment given by \( \tau_{\text{MET}} = \tau_{\text{g}\text{as}} \) | | |
| Critical sizes for external alignment | \( a_{\text{MET,high J}} \) (Equation (62)) | Minimum size for the \( \nu \)-MET alignment at the low-\( J \) attractor given by \( \tau_{\text{MET}} = \tau_{\text{g}\text{as}} \) | | |
| Critical sizes for external alignment | \( a_{\text{MET,high J}} \) (Equation (64)) | Minimum size for the \( \nu \)-MET alignment at the high-\( J \) attractor given by \( \tau_{\text{MET}} = \tau_{\text{g}\text{as}} \) | | |
| Critical sizes for grain alignment by RATs and METs | \( a_{\text{RAT}} \) (Equation (65)) | Minimum size for grain alignment by RATs given by \( \Omega_{\text{RAT}} = 3 \Omega_{\jmath} \) (or \( \theta_{\text{RAT}} = 3 \)) | | |
| Critical sizes for grain alignment by RATs and METs | \( a_{\text{MET}} \) (Equation (66)) | Minimum size for grain alignment by METs given by \( \Omega_{\text{MET}} = 3 \Omega_{\jmath} \) (or \( \theta_{\text{MET}} = 3 \)) | | |
| Concentric region of protostellar disk | \( r \) | Radial distance from the central star | | |
| Protostellar disk parameters | \( \Sigma_0 \) | Surface mass density in the midplane of the disk at \( r = 1 \) au | 100–1000 \ g \ cm^{-2} | |
| Protostellar disk parameters | \( \Sigma(r) \) (Equation (67)) | Surface mass density of the disk as a function of \( r \) | | |
| Protostellar disk parameters | \( H_p \) (Equation (68)) | Pressure-scale height for the profile of gas perpendicular to the midplane of the disk | | |
| Protostellar disk parameters | \( n_0 = 4.0 \times 10^{3}(\Sigma_0/10^3 \ g \ cm^{-2}) \ cm^{-3} \) | Gas density in the midplane of the disk at \( r = 1 \) au | \( n_0/10^{13} \ cm^{-2} \) | |
| Protostellar disk parameters | \( \alpha_r \) | Index for the gas density profile as a function of \( r \) | 37/14 | |
| Protostellar disk parameters | \( n_0(r) = n_0(r/au)^{-\alpha_r} \) (Equation (69)) | Gas density in the midplane of the disk as a function of \( r \) | | |
| Protostellar disk parameters | \( T_{\text{gas}}(r) = T_0(r/au)^{-3} \) (Equation (70)) | Gas temperature in the midplane of the disk as a function of \( r \) | | |
| Protostellar disk parameters | \( M \) | Mass accretion rate of the central star | \( 10^{-8} M_\odot \ yr^{-1} \) | |
| Protostellar disk parameters | \( B_0 \) | Magnetic field strength in the midplane of the disk at \( r = 1 \) au normalized for \( M = 10^{-8} M_\odot \ yr^{-1} \) | \( 10^8 \mu G \) | |
| Protostellar disk parameters | \( T_{\text{gas}}(r) = T_0(r/au)^{-3} \) (Equation (70)) | Gas temperature in the midplane of the disk as a function of \( r \) | | |
| Category | Notation | Description | Typical Value (cgs units) | Normalized Notation |
|----------|----------|-------------|---------------------------|---------------------|
|          | $B(r)$ (Equation (71)) | Magnetic field strength in the midplane of the disk as a function of $r$ | | |
|          | $M_*$ | Mass of the central protostar | | |
|          | $\Omega_*$ | Keplerian angular velocity | | |
|          | $v_K$ (Equation (80)) | Keplerian velocity | | |
|          | $St_k$ | Stokes number | | |
|          | $v_{gas}$ | Gas velocity | | |
|          | $\eta = v_K/v_{gas}$ | Ratio between $v_{gas}$ and $v_K$ | | |
|          | $v_r$ (Equation (81)) | Radial component of the gas velocity | | |
|          | $v_{\phi}$ (Equation (82)) | Radial component of the gas velocity | | |
|          | $\psi = (v_r^2 + v_\phi^2)^{1/2}$ | Dust–gas relative velocity | | |
|          | $s_d(r) = v_{\phi}/v_r$ | Grain drift parameter in the midplane of the disk as a function of $r$ | | |
|          | $s_A1$ | Grain drift parameter at $r = 1$ au | | $s_{A0.1} = s_{A1}/0.1$ |
