SPECTRAL AND ASYMPTOTIC PROPERTIES OF CONTRACTIVE SEMIGROUPS ON NON-HILBERT SPACES

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Abstract. We analyse $C_0$-semigroups of contractive operators on real-valued $L^p$-spaces for $p \neq 2$ and on other classes of non-Hilbert spaces. We show that, under some regularity assumptions on the semigroup, the geometry of the unit ball of those spaces forces the semigroup's generator to have only trivial (point) spectrum on the imaginary axis. This has interesting consequences for the asymptotic behaviour as $t \to \infty$. For example, we can show that a contractive and eventually norm continuous $C_0$-semigroup on a real-valued $L^p$-space automatically converges strongly if $p \notin \{1, 2, \infty\}$.

1. Introduction and Preliminaries

If we try to analyse the asymptotic behaviour of a bounded $C_0$-semigroup $(e^{tA})_{t \geq 0}$ on a Banach space $X$, then an important question to ask is whether the spectrum of $A$ on the imaginary axis is trivial, i.e. contained in the set $\{0\}$. If this is the case, then we can often conclude that $e^{tA}$ converges strongly as $t \to \infty$ (see e.g. [1 Corollary 2.6]). Therefore, one is interested in simple criteria to ensure that $\sigma(A) \cap i\mathbb{R} \subset \{0\}$. A typical example for such a criterion is positivity of the semigroup with respect to a Banach lattice cone (in combination with an additional regularity assumption on the semigroup, see [3 Corollary C-III.2.13]).

In this article, we consider another geometric condition on the semigroup, namely contractivity of each operator $e^{tA}$. We will see that on many important function spaces, this condition ensures that $\sigma(A) \cap i\mathbb{R} \subset \{0\}$. To give the reader a more concrete idea of what we are going to do, let us state the following two results which will be proved (in a more general form) in the subsequent sections:

**Theorem 1.1.** Let $(\Omega, \Sigma, \mu)$ be a measure space and let $1 < p < \infty$, $p \neq 2$. Let $(e^{tA})_{t \geq 0}$ be an eventually norm continuous, contractive $C_0$-semigroup on the real-valued $L^p$-space $L^p(\Omega, \Sigma, \mu; \mathbb{R})$. Then $\sigma(A) \cap i\mathbb{R} \subset \{0\}$. In particular, $e^{tA}$ converges strongly as $t \to \infty$.

In this theorem, $\sigma(A)$ denotes the spectrum of a complex extension of $A$ to a complexification of the real space $L^p(\Omega, \Sigma, \mu; \mathbb{R})$, see Appendix A for details. The above theorem (in fact, a generalization of it) is proved in Subsection 4.2 below (Corollaries 4.6 and 4.7).

**Theorem 1.2.** Let $E$ be one of the real-valued sequence spaces $c(\mathbb{N}; \mathbb{R})$ (the space of convergent sequences) or $l^p(\mathbb{N}; \mathbb{R})$ for $1 \leq p < \infty$, $p \neq 2$. If $(e^{tA})_{t \geq 0}$ is a contractive $C_0$-semigroup on $E$, then $\sigma_{pnt}(A) \cap i\mathbb{R} \subset \{0\}$. If, in addition, $A$ has compact resolvent, then $e^{tA}$ converges strongly as $t \to \infty$.

Here, $\sigma_{pnt}(A)$ denotes the point spectrum of a complex extension of $A$ to a complexification of the real space $E$; we say that $A$ has compact resolvent, if any complex extension of $A$ has compact resolvent (see again Appendix A for details).

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The above theorem follows from results that we are going to prove in Subsection 2.2 (see Corollary 2.3 for the case $E = l^p$; for the case $E = c_0(N;\mathbb{R})$, see Theorem 2.5; Corollary 2.7) and the assertion preceding to Example 2.3.

Both of the above theorems make use of the fact that the underlying Banach space behaves in some manner oppositely to a Hilbert space (which explains the condition $p \neq 2$ in the assumptions). The idea to employ this observation for the spectral analysis of contractions is not new. A similar approach was used by Krasnosel’ski˘ı in [13] and by Lyubich in [16] to analyse the peripheral spectrum of compact and of finite dimensional contractions. Furthermore, in [9], Goldstein used similar ideas to analyse $C_0$-groups of isometries on Orlicz spaces. Moreover, we note that in [17], Lyubich presented a unified approach to the spectral analysis of operators with invariant convex sets in finite dimensions; in particular, this approach allows the analysis of contractions and the analysis of positive operators within the same framework.

Let us give a short outline of the current article: We mainly focus on contractive $C_0$-semigroups $(e^{tA})_{t \geq 0}$ (and a slight generalization of them, see Definition 2.2), but we also give several results for single operators which are derived from the semigroup results. In Section 2 we consider Banach spaces that do not isometrically contain a two-dimensional Hilbert space. This is a very strong condition, which enables us to prove strong results on the point spectrum of semigroups and operators. Some classical sequence spaces will turn out to be typical examples of such Banach spaces. In Section 3 we weaken the geometric condition on the Banach space to allow for a wider range of spaces. Instead we will impose additional assumptions on our semigroups to prove similar results on the point spectrum. In Section 4 we employ an ultra power technique to deduce results on the spectrum rather than on the point spectrum. To make this approach work we need geometrical assumptions not only on the Banach space $X$, but also on an ultra power $X_U$ of $X$. It turns out that those assumptions are fulfilled for $L^p$-spaces if $p \notin \{1, 2, \infty\}$.

Throughout the paper, we consider semigroups and operators on real Banach spaces. To employ spectral theoretic methods to their study, we need complexifications of such Banach spaces. Those are often used tacitly in the literature, because they mostly do not cause notable difficulties. However, since many of our arguments heavily depend on the relation between real Banach spaces and their complexification, we prefer a more explicit treatment of them, which is given in Appendix A. In Appendix B we give a few results on the boundary spectrum of linear operators, which are needed on some occasions in the article.

Before starting our analysis, let us fix some notation that will be used throughout the article: Whenever $X$ is a vector space over a field $\mathbb{F}$ and $M \subset X$, then we denote by $\text{span}_\mathbb{F}(M)$ the linear span of $M$ in $X$ over $\mathbb{F}$. Whenever $X$ is a real or complex Banach space, the set of bounded linear operators on $X$ will be denoted by $\mathcal{L}(X)$. For a complex Banach space $X$ and a closed linear operator $A : X \supset D(A) \to X$, its spectrum, point spectrum and approximate point spectrum are denoted respectively by $\sigma(A)$, $\sigma_{\text{pnt}}(A)$ and $\sigma_{\text{appr}}(A)$. If even $A \in \mathcal{L}(X)$, then $r(A)$ denotes the spectral radius of $A$. Whenever $\lambda \notin \sigma(A)$, then $R(\lambda, A) = (\lambda - A)^{-1}$ is the resolvent of $A$ in $\lambda$. For each $\lambda \in \mathbb{C}$, we denote by $\text{Eig}(\lambda, A) := \ker(\lambda - A)$ the eigenspace of $A$ for the number $\lambda$; this notation will be used even if $\lambda \notin \sigma_{\text{pnt}}(A)$, and in this case we have $\text{Eig}(\lambda, A) = \{0\}$, of course. Whenever $X$ is a real Banach space, then we can consider a complexification $X_\mathbb{C}$ of $X$; this is discussed in Definition A.1 and in the paragraphs subsequent to this definition. For a real Banach space $X$ and a linear operator $A : X \supset D(A) \to X$, the spectrum and point spectrum of $A$ are defined by considering the complex extension $A_\mathbb{C}$ to a complexification $X_\mathbb{C}$ of $X$, see Definitions A.3 and A.5 for details. If $X$ is a real or complex Banach space,
then we denote by $X'$ its dual space, and if $A : X \supset D(A) \to X$ is a linear operator, then we denote by $A'$ the adjoint operator of $A$. We will also need that a linear functional $x' \in X'$ on a real Banach space $X$ has a complex extension $(x')_C$ to any complexification $X_C$ of $X$ (cf. Definition A.3). Finally, we will always use the symbol $T$ to denote the complex unit circle $T := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$.  

2. The point spectrum on extremely non-Hilbert spaces

2.1. Extremely non-Hilbert spaces. When we consider contractive $C_0$-semi-groups $(e^{tA})_{t \geq 0}$ on a real Banach space $X$, it turns out that the existence of purely imaginary eigenvalues of (the complex extension of) $A$ is related to the existence of two-dimensional Hilbert spaces in $X$, see Theorem 2.5 below. This motivates the following notion.

**Definition 2.1.** A real Banach space $X$ is called extremely non-Hilbert if it does not isometrically contain a two-dimensional Hilbert space.

Clearly, any closed subspace of an extremely non-Hilbert space is extremely non-Hilbert itself. Before we analyse the spectral properties of contractive semigroups on extremely non-Hilbert spaces, we want to provide some examples of those spaces to give the reader an idea of when our subsequent results are applicable.

**Example 2.2.** (a) Let $1 \leq p < \infty$, $p \notin 2\mathbb{N}$. Then the real sequence space $l^p := l^p(\mathbb{N}; \mathbb{R})$ is extremely non-Hilbert (see [3] Corollary 1.8).

(b) Let $1 \leq p < \infty$, $p \notin 2\mathbb{N}$. If we endow $\mathbb{R}^d$ with the $p$-norm $\| \cdot \|_p$, then $(\mathbb{R}^n, \| \cdot \|_p)$ is extremely non-Hilbert as well. This is for example shown in [15].

(c) If $p \in 2\mathbb{N}$, then $l^p$ is not extremely non-Hilbert. This is obvious for $p = 2$, and for $p \in \{ 4, 6, ... \}$ it can be shown that even a finite dimensional $(\mathbb{R}^n, \| \cdot \|_p)$ exists which isometrically contains a two-dimensional Hilbert space. This is proved for example in [22] p. 283 - 284 (note however that the necessary dimension $n$ increases with $p$). See also [16] the proof of Proposition 2], [18] and [12] for a discussion of this and related topics.

In contrast to $l^p$-sequence spaces, real-valued $L^p$-spaces on non-discrete measure spaces are not extremely non-Hilbert, in general. For $L^p$-spaces on the torus we will give an explicit example of an isometrically embedded two-dimensional Hilbert space in Example 2.11. Moreover, it can be shown by probabilistic methods that $L^p([0, 1]; \mathbb{R})$ even isometrically contains the infinite dimensional sequence space $l^2$, see [11] p. 16.

A further example of an extremely non-Hilbert space is the space $c := c(\mathbb{N}; \mathbb{R})$ of real-valued convergent sequences on $\mathbb{N}$. This follows from the next, more general example:

**Example 2.3.** Let $K$ be a compact Hausdorff space and let $C(K; \mathbb{R})$ be the space of continuous, real-valued functions on $K$ which is endowed with the supremum norm. If $K$ is countable, then $C(K, \mathbb{R})$ is extremely non-Hilbert.

**Proof.** Assume for a contradiction that $V \subset C(K; \mathbb{R})$ is a two-dimensional Hilbert space. Since $V$ is isometrically isomorphic to $(\mathbb{R}^2, \| \cdot \|_2)$, there are two vectors $x, y \in V$ such that

$$\varphi : [0, 2\pi] \to V, \quad t \mapsto \cos(t)x + \sin(t)y$$

is a mapping into the surface of the unit ball of $V$. Hence, the mapping $t \mapsto \|\varphi(t)\| = \max_{k \in K} |\varphi(t)(k)|$ is constantly 1.

Now, for each $k \in K$, let $A_k := \{ t \in [0, 2\pi] : |\varphi(t)(k)| = 1 \}$. Each set $A_k$ is closed and we have $\bigcup_{k \in K} A_k = [0, 2\pi]$. Since $K$ is countable, Baire’s Theorem
implies that at least one of the sets \(A_k\), say \(A_{k_0}\), has non-empty interior and thus contains a non-empty open interval \(I\). For continuity reasons, the mapping

\[ t \mapsto \varphi(t)(k_0) = \cos(t)x(k_0) + \sin(t)y(k_0) \]

is either constantly 1 or constantly \(-1\) on \(I\). Hence, it is constantly 1 or constantly \(-1\) on the whole interval \([0, 2\pi]\), due to the Identity Theorem for analytic functions. This is a contradiction since the functions \(\varphi\), \(\cos\) and \(\sin\) are linearly independent elements of the vector space \(\mathbb{R}^{[0,2\pi]}\).

\(\square\)

Preceding to the above example, we claimed that the sequence space \(c\) is extremely non-Hilbert. This follows indeed from Example 2.3 since \(c\) is isometrically isomorphic to the space \(C(\mathbb{N} \cup \{\infty\}; \mathbb{R})\), where \(\mathbb{N} \cup \{\infty\}\) is the one-point-compactification of the discrete space \(\mathbb{N}\).

2.2. \(C_0\)-semigroups on extremely non-Hilbert spaces. Since we have now several examples of extremely non-Hilbert spaces at hand, let us turn to contractive semigroups on them. In fact, we will not really need our semigroups to be contractive. In each of our theorems, one of the following asymptotic properties will suffice.

**Definition 2.4.** Let \((e^{tA})_{t \geq 0}\) be a \(C_0\)-semigroup on a real Banach space \(X\) and let \(T\) be a bounded linear operator in \(X\). The semigroup \((e^{tA})_{t \geq 0}\) is called

(a) uniformly asymptotically contractive if \(\limsup_{t \to \infty} ||e^{tA}|| \leq 1\).

(b) strongly asymptotically contractive if \(\limsup_{t \to \infty} ||e^{tA}x|| \leq 1\) for each \(x \in X\) with \(||x|| = 1\).

(c) weakly asymptotically contractive if \(\limsup_{t \to \infty} ||e^{tA}x, x'|| \leq 1\) for all \(x \in X\) and all \(x' \in X'\) with \(||x|| = ||x'|| = 1\).

For the operator \(T \in \mathcal{L}(X)\), the same notions are defined by replacing the semigroup by the powers \(T^n\).

If a \(C_0\)-semigroup \((e^{tA})_{t \geq 0}\) on \(X\) is weakly asymptotically contractive, then it is bounded, and if an operator \(T \in \mathcal{L}(X)\) is weakly asymptotically contractive, then it is power-bounded; this follows from the Uniform Boundedness Principle.

We now begin our analysis with a result on the purely imaginary eigenvalues of the semigroup generator. Recall that for an operator \(A\) on a real Banach space \(X\), the point spectrum \(\sigma_{\text{pnt}}(A)\) is defined to be the point spectrum \(\sigma_{\text{pnt}}(A_C)\) of a complex extension \(A_C\) of \(A\) (see Appendix \(A\) for details).

**Theorem 2.5.** Let \(X\) be a real Banach space which is extremely non-Hilbert and let \((e^{tA})_{t \geq 0}\) be a weakly asymptotically contractive \(C_0\)-semigroup on \(X\). Then \(\sigma_{\text{pnt}}(A) \cap i\mathbb{R} \subset \{0\}\).

**Proof.** Let \(X_C\) be a complexification of \(X\). According to Remark \(A.8\) the complex extension \(A_C\) of \(A\) generates a \(C_0\)-semigroup \((e^{tA_C})_{t \geq 0}\) on \(X_C\), and each operator \(e^{tA_C}\) is the complex extension of the operator \(e^{tA}\). By Definition \(A.3\) we have \(\sigma_{\text{pnt}}(A) = \sigma_{\text{pnt}}(A_C)\).

Assume for a contradiction that \(ia\) is an eigenvalue of \(A_C\), where \(a \in \mathbb{R} \setminus \{0\}\). Then we have \(A_Cz = iz\) for some \(0 \neq z = x + iy \in X_C\), where \(x, y \in X\). In Proposition \(A.6\) it is shown that \(A_Cz = -ia\bar{z}\) and that \(x\) and \(y\) are linearly independent over \(\mathbb{R}\). In particular, \(x \neq 0\) and thus we may assume that \(||x|| = 1\).

Let \(V\) be the linear span of \(x\) and \(y\) over \(\mathbb{R}\). We show that \(V\) a Hilbert space with respect to the norm induced by \(X\). For each \(t \geq 0\) we have

\[ e^{tA}x = \frac{1}{2}e^{tA_C}(z + \bar{z}) = \frac{1}{2}(e^{iat}z + e^{-iat}\bar{z}) = \Re(e^{iat}z) = \cos(at)x - \sin(at)y. \]
This shows that the orbit \((e^{tA}x)_{t \geq 0}\) is periodic. Together with the weak asymptotic contractivity of \((e^{tA})_{t \geq 0}\) this implies that \(\|e^{tA}x\| = 1\) for each \(t \geq 0\). Hence, the vectors \(\cos(\alpha t)x - \sin(\alpha t)y\) are contained in the surface of the unit ball of \(V\). Now, endow \(\mathbb{R}^2\) with the euclidean norm \(\|\cdot\|_2\) and consider the linear bijection \(\phi : \mathbb{R}^2 \to V, w \mapsto w_1x - w_2y\). Whenever \(\|w\|_2 = 1\), the vector \(w\) can be written as \(w = (\cos(\alpha t), \sin(\alpha t))\) for some \(t \geq 0\). Hence, \(\phi(w) = \cos(\alpha t)x - \sin(\alpha t)y\) and thus, \(\phi\) maps the surface of the unit ball of \((\mathbb{R}^2, \|\cdot\|_2)\) into the surface of the unit ball of \(V\). This shows that \(\phi\) is isometric, i.e. \(V\) is a Hilbert space.

It follows from Example 2.2 that the dual or the pre-dual of an extremely non-Hilbert space need not be extremely non-Hilbert, in general. For example, the sequence space \(\ell^4\) is extremely non-Hilbert, while its dual and pre-dual space \(l^4\) is not. Nevertheless, the assertion of the above Theorem 2.5 can be extended to reflexive Banach spaces with an extremely non-Hilbert dual space:

**Corollary 2.6.** Let \(X\) be a reflexive Banach space over the real field and suppose that its dual space \(X'\) is extremely non-Hilbert. Let \((e^{tA})_{t \geq 0}\) be a weakly asymptotically contractive \(C_0\)-semigroup on \(X\). Then \(\sigma_{\text{pat}}(A) \cap i\mathbb{R} \subset \{0\}\).

**Proof.** Since \(X\) is reflexive, the semigroup of adjoint operators \((e^{tA'})_{t \geq 0}\) is a \(C_0\)-semigroup on \(X'\) and its generator is the adjoint \(A'\) of \(A\). The reflexivity of \(X\) also implies that the adjoint semigroup \((e^{tA'})_{t \geq 0}\) is weakly asymptotically contractive, and thus we conclude that \(\sigma_{\text{pat}}(A') \cap i\mathbb{R} \subset \{0\}\) by Theorem 2.5. However, if \(X_C\) is a complexification of \(X\), then we have

\[
\sigma_{\text{pat}}(A') \cap i\mathbb{R} = \sigma_{\text{pat}}((A_C')' \cap i\mathbb{R} \supset \sigma_{\text{pat}}(A_C) \cap i\mathbb{R} = \sigma_{\text{pat}}(A) \cap i\mathbb{R},
\]

where the first equality follows from Proposition A.7 (b) and the inclusion from Proposition B.1 (b). This proves the corollary. □

We say that an operator \(A\) on a real Banach space \(X\) has compact resolvent if its complex extension \(A_C\) to a complexification \(X_C\) of \(X\) has compact resolvent. For semigroups whose generator has compact resolvent, Theorem 2.5 and Corollary 2.6 yield the following convergence result.

**Corollary 2.7.** Suppose the assumptions of Theorem 2.5 or of Corollary 2.6 are fulfilled. If \(A\) has compact resolvent, then \(e^{tA}\) strongly converges as \(t \to \infty\).

**Proof.** Let \(A_C\) be the complex extension of \(A\) to a complexification \(X_C\) of \(X\). The spectrum \(\sigma(A_C)\) consists only of eigenvalues, since \(A_C\) has compact resolvent. So we conclude from Theorem 2.5 respectively from Corollary 2.6 that \(\sigma(A_C) \cap i\mathbb{R} \subset \{0\}\). Therefore, \(e^{tA_C}\) strongly converges as \(t \to \infty\) (see 6. Theorem V.2.14 and Corollary V.2.15) and so does \(e^{tA}\). □

Let us mention the particular case of \(l^p\)-spaces in an extra corollary:

**Corollary 2.8.** Let \((e^{tA})_{t \geq 0}\) be a strongly continuous semigroup on \(l^p = l^p(\mathbb{N}; \mathbb{R})\) for \(1 \leq p < \infty\) and \(p \neq 2\). If \((e^{tA})_{t \geq 0}\) is weakly asymptotically contractive, then \(\sigma_{\text{pat}}(A) \cap i\mathbb{R} \subset \{0\}\).

If, in addition, \(A\) has compact resolvent, then \(e^{tA}\) strongly converges as \(t \to \infty\).

**Proof.** If \(p\) is not an even integer, then \(l^p\) is an extremely non-Hilbert space by Example 2.2 and thus the assumptions of Theorem 2.5 are fulfilled. If \(p > 2\) is an even integer instead, then the conjugate index \(p'\), which is defined by \(\frac{1}{p} + \frac{1}{p'} = 1\), is not an even integer. Thus, the dual space \((l^p)' = l^{p'}\) is extremely non-Hilbert and so the assumptions of Corollary 2.6 are fulfilled. □
The following connection of Corollary 2.8 to the spectral theory of positive semigroups is interesting:

**Remark 2.9.** It is shown in [4, Theorem 9] that a contractive positive $C_0$-semigroup on $L^p$ for $1 \leq p < \infty$ always fulfills $\sigma_{\text{pot}}(A) \cap i\mathbb{R} \subset \{0\}$. Corollary 2.8 shows that the assumption of positivity is not needed for this result whenever $p \neq 2$. Note however, that the result for positive semigroups can be generalized to a large class of semigroups on other Banach lattices, as well (see e.g. the current article [8] and the references therein for details).

Now, let us demonstrate by a couple of examples what goes wrong with our above results on spaces which are not extremely non-Hilbert. First, we consider an example of a contraction semigroup on a two-dimensional Hilbert-space:

**Example 2.10.** Consider the two-dimensional euclidean space $(\mathbb{R}^2, \| \cdot \|_2)$ and the $C_0$-semigroup $(e^{tA})_{t \geq 0}$, where the matrices of $A$ and $e^{tA}$ are given by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. $$

Then $e^{tA}$ acts a rotation with angle $t$ on $\mathbb{R}^2$, and $(e^{tA})_{t \geq 0}$ is clearly a contraction semigroup with respect to the euclidean norm $\| \cdot \|_2$. Since $(\mathbb{R}^2, \| \cdot \|_2)$ is a Hilbert space, we cannot apply Theorem 2.5 and indeed the spectrum of the generator $A$ is given by $\sigma(A) = \{i, -i\}$, i.e. the assertion of Theorem 2.8 fails in our example.

It is also very instructive to observe what happens on $(\mathbb{R}^2, \| \cdot \|_{p})$ for $p \neq 2$. If we consider the same semigroup as above on such a space, then we still have $\sigma(A) = \{i, -i\}$, but the space is extremely non-Hilbert now (for $p \notin 2\mathbb{N}$ this follows from Example 2.2, but it is true even for $p \in 2\mathbb{N}\setminus\{2\}$ since our space is two-dimensional). Hence, another assumption of Theorem 2.8 must fail here, and indeed, it is easily seen that our rotation semigroup is not contractive with respect to the $\| \cdot \|_p$-norm due to the low symmetry of the unit ball of this norm.

Our next example shows that Theorem 2.8 does in general not hold true on $L^p$-spaces on non-discrete measure spaces.

**Example 2.11.** Let $1 \leq p < \infty$, let the complex unit circle $\mathbb{T}$ be equipped with a non-zero Haar measure $\mu$ and let $(e^{tA})_{t \geq 0}$ be the shift semigroup on $L^p(\mathbb{T}; \mathbb{R})$, i.e. $e^{tA}f(z) = f(e^{it}z)$ for all $f \in L^p(\mathbb{T}; \mathbb{R})$. Then $(e^{tA})$ is contractive, but the point spectrum of $A$ is given by $\sigma_{\text{pot}}(A) = i\mathbb{Z}$.

By virtue of Theorem 2.8 this implies that $L^p(\mathbb{T}; \mathbb{R})$ is not projectively non-Hilbert; however we can also see this explicitly. In fact, let $f_1 : \mathbb{T} \to \mathbb{R}$, $f_1(z) = \text{Re} z$ and $f_2 : \mathbb{T} \to \mathbb{R}$, $f_2(z) = \text{Im} z$. Let $\alpha, \beta \in \mathbb{R}$ and choose $\varphi \in [0, 2\pi)$ and $r \geq 0$ such that $\alpha - i\beta = re^{i\varphi}$. Then we obtain

$$
\|\alpha f_1 + \beta f_2\|_p = \left( \int_\mathbb{T} |\text{Re}((\alpha - i\beta)z)|^p \, dz \right)^{\frac{1}{p}} = \left( \int_\mathbb{T} |\text{Re}(re^{i\varphi}z)|^p \, dz \right)^{\frac{1}{p}} = \\
= r \cdot \left( \int_\mathbb{T} |\text{Re}z|^p \, dz \right)^{\frac{1}{p}} = (\alpha^2 + \beta^2)^{\frac{1}{2}} \cdot ||f_1||_p.
$$

Hence, $(\alpha, \beta) \mapsto \frac{||\alpha f_1 + \beta f_2||_p}{||f_1||_p}$ yields an isometric isomorphism between the two-dimensional euclidean space $(\mathbb{R}^2, \| \cdot \|_2)$ and the subspace $\text{span}_\mathbb{R}\{f_1, f_2\}$ of $L^p(\mathbb{T}; \mathbb{R})$. The reader might also compare [9, Remark 5] for a slightly different presentation of this example.

Finally, we want to demonstrate that it is essential for our above results (and also for the results in the subsequent sections) that we consider $C_0$-semigroups of real operators:
Example 2.12. Consider the space \((\mathbb{C}^n, || \cdot ||_p)\) for some arbitrary \(p \in [1, \infty]\) and let \(A \in \mathbb{C}^{n \times n}\) be the diagonal matrix whose diagonal entries are all equal to \(i\). Then \((e^{tA})_{t \geq 0}\) is clearly a contractive \(C_0\)-semigroup on \((\mathbb{C}^n, || \cdot ||_p)\), but we obtain \(\sigma(A) = \{i\}\) for the spectrum of its generator.

2.3. Single operators on extremely non-Hilbert spaces. Next, we show how Theorem 2.5 can be applied to yield a result on the point spectrum of a single operator.

Theorem 2.13. Let \(X\) be a real Banach space which is extremely non-Hilbert and let \(T \in \mathcal{L}(X)\) be weakly asymptotically contractive. Then \(\sigma_{\text{pnt}}(T) \cap \mathbb{T}\) consists only of roots of unity.

Proof. Let \(T_C\) be the complex extension of \(T\) to a complexification \(X_C\) of \(X\). Assume for a contradiction that \(e^{\alpha t} \in \sigma_{\text{pnt}}(T_C) \cap \mathbb{T}\) (where \(\alpha \in \mathbb{R}\)) is not a root of unity and let \(T_C z = e^{\alpha t} z\) for some \(0 \neq z = x + iy \in X_C\), where \(x, y \in X\). By Proposition A.6 \(x\) and \(y\) are linearly independent over \(\mathbb{R}\) and we have \(T_C z = e^{-it} \overline{z}\).

Let \(V = \text{span}_\mathbb{R} \{x, y\}\) and \(V_C = \text{span}_\mathbb{C} \{z, \overline{z}\}\). Then we have \(V_C = V \oplus iV\), and therefore \(V_C\) is a complexification of \(V\). We define a \(C_0\)-semigroup \((e^{tA_C})_{t \geq 0}\) on \(V_C\) by means of

\[
e^{tA_C} z = e^{it} z, \quad e^{tA_C} \overline{z} = e^{-it} \overline{z}.
\]

Note that \(e^{tA_C}\) is well-defined since \(z\) and \(\overline{z}\) are linearly independent over \(\mathbb{C}\) (as they are eigenvectors of \(T_C\) for two different eigenvalues). Clearly, this semigroup is strongly (in fact uniformly) continuous and and its generator \(A_C\) is the complexification of an operator \(A \in \mathcal{L}(V)\), given by

\[
Ax = -y \quad \text{and} \quad Ay = x.
\]

We have \(\sigma_{\text{pnt}}(A) = \sigma_{\text{pnt}}(A_C) = \{-i, i\}\), and the semigroup \((e^{tA_C})_{t \geq 0}\) is the complexification of the semigroup \((e^{tA})_{t \geq 0}\) due to Remark A.8. We now show that \((e^{tA})_{t \geq 0}\) is a contraction semigroup on \(V\), so that we can apply Theorem 2.5.

Let \(t \geq 0\). Since \(e^{\alpha t}\) is not a root of unity, we find a sequence of real numbers \(N_n \to \infty\) such that \(e^{iN_n \alpha} \to e^{it}\). Since \(V \subset V_C\), each element \(v \in V\) can be written in the form \(v = \lambda_1 z + \lambda_2 \overline{z}\) with complex scalars \(\lambda_1, \lambda_2\). We thus obtain for each \(x' \in X'\) that

\[
||e^{tA}v, x'|| = ||e^{tA}v, (x')_C|| = ||(\lambda_1 e^{it} z + \lambda_2 e^{-it} \overline{z}, (x')_C)|| = \lim_n ||(\lambda_1 T_{C}^{N_n} z + \lambda_2 T_{C}^{N_n} \overline{z}, (x')_C)|| = \lim_n ||(T^{N_n}v, x')|| \leq ||(v, x')||.
\]

Thus, each operator \(e^{tA}\) is contractive. Since \(X\) is extremely non-Hilbert, so is its subspace \(V\), and we conclude from Theorem 2.5 that \(\sigma_{\text{pnt}}(A) \cap i\mathbb{R} \subset \{0\}\), which is a contradiction. \(\square\)

Corollary 2.14. Let \(X\) be a reflexive Banach space over the real field and suppose that its dual space \(X'\) is extremely non-Hilbert. Let \(T \in \mathcal{L}(X)\) be weakly asymptotically contractive. Then \(\sigma_{\text{pnt}}(T) \cap \mathbb{T}\) consists only of roots of unity.

Proof. Since \(X\) is reflexive, the adjoint operator \(T'\) is also weakly asymptotically contractive, so \(\sigma_{\text{pnt}}(T') \cap \mathbb{T}\) only consists of roots of unity by Theorem 2.13. Moreover, for a complexification \(X_C\) of \(X\), we have

\[
\sigma_{\text{pnt}}(T') \cap \mathbb{T} = \sigma_{\text{pnt}}(T_C)' \cap \mathbb{T} \supset \sigma_{\text{pnt}}(T_C) \cap \mathbb{T} = \sigma_{\text{pnt}}(T) \cap \mathbb{T},
\]

where the first equality is due to Proposition A.7(b), and the inclusion follows from Proposition B.1(a). This proves the corollary. \(\square\)
Again, we note that the sequence spaces $l^p$ for $1 \leq p < \infty$ and $p \neq 2$ fulfil the assumption of either Theorem 2.13 or Corollary 2.14. If we imposed some additional assumptions on $T$, then we could use Theorem 2.13 and Corollary 2.14 to derive results on the asymptotic behaviour of the powers $T^n$. However, we delay this to subsection 3.3. There we first prove additional spectral results on single operators on another type of Banach spaces; then we describe the asymptotic behaviour of $T^n$ in Corollary 3.11.

3. The point spectrum on projectively non-Hilbert spaces

3.1. Projectively non-Hilbert spaces. The results of Section 2 all have the major flaw that they are applicable only on the small range of Banach spaces which are extremely non-Hilbert or have an extremely non-Hilbert dual space. We have seen in Example 2.11 that $L^p$-spaces on non-discrete measure spaces do not fulfil this property. This indicates that we should consider Banach spaces which only fulfil a weaker geometric condition (at the cost of extra conditions on our semigroup, of course). The geometric condition in the subsequent Definition 3.1 seems to be the appropriate condition to do this. Before stating this definition, let us recall that a subspace $V$ of a (real or complex) Banach space $X$ is said to admit a contractive linear projection in $X$ if there is a projection $P \in \mathcal{L}(X)$ such that $PX = V$ and such that $\|P\| \leq 1$.

Definition 3.1. A real Banach space $X$ is called projectively non-Hilbert if it does not isometrically contain a two-dimensional Hilbert space $V \subset X$ which admits a contractive linear projection in $X$.

Note that if $X$ is a projectively non-Hilbert space and $P \in \mathcal{L}(X)$ is a contractive projection, then the range $PX$ is a projectively non-Hilbert space, too. It was shown by Lyubich in [10] that projectively non-Hilbert spaces provide an appropriate setting for the spectral analysis of finite dimensional (and more generally, for compact) contractive operators. Preceding to his work, Krasnosel’skiı achieved similar results in [13], but on a more special class of Banach space which he called completely non-Hilbert spaces.

Of course, a Banach space which is extremely non-Hilbert, is also projectively non-Hilbert, but in fact the class of projectively non-Hilbert spaces is much larger. Again, we start with several examples for those spaces.

Example 3.2. Let $(\Omega, \Sigma, \mu)$ be an arbitrary measure space (not necessarily $\sigma$-finite) and let $1 \leq p < \infty$. If $p \neq 2$, then the real Banach space $L^p := L^p(\Omega, \Sigma, \mu; \mathbb{R})$ is projectively non-Hilbert.

To see this, let $V \subset L^p$ be a two-dimensional subspace and assume that $P$ is a contractive linear projection on $L^p$ with range $PL^p = V$. Then $V$ is isometrically isomorphic to $L^p(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu}; \mathbb{R})$ for another measure space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$, see [26, Theorem 6]. Since $p \neq 2$, $V$ cannot be a Hilbert space.

We saw in Example 2.11 that the space $L^p(\mathbb{T}; \mathbb{R})$ on the complex unit circle $\mathbb{T}$ isometrically contains a two-dimensional Hilbert space $V$. The above Example 3.2 shows that this subspace cannot admit a contractive projection (although it of course admits a bounded projection since it is finite dimensional). To obtain further examples of projectively non-Hilbert spaces, we now show how they behave with respect to duality.

Proposition 3.3. Let $X$ be a real Banach space. If its dual space $X'$ is projectively non-Hilbert, then so is $X$ itself.
This proposition shows that projectively non-Hilbert spaces behave better with respect to duality than extremely non-Hilbert spaces do (compare the comments before Corollary 2.6). The proof of Proposition 3.3 relies on the following elementary observation:

**Lemma 3.4.** Let $P$ be a bounded linear projection on a (real or complex) Banach space $X$. Then the mapping

$$i : P'X' \to (PX)', \quad x' \mapsto x'|_{PX}$$

is a contractive Banach space isomorphism. If $P$ is contractive, then $i$ is even isometric.

**Proof.** Clearly, $i$ is linear and contractive. If $x' \in P'X'$ and $i(x') = x'|_{PX} = 0$, then we obtain for every $x \in X$ that

$$\langle x', x \rangle = \langle P'x', x \rangle = \langle x', Px \rangle = 0,$$

hence $x' = 0$. This shows that $i$ is injective. Surjectivity of $i$ follows from the Hahn-Banach Theorem: A given functional $\tilde{x}' \in (PX)'$ can be extended to a functional $x' \in X'$ and we obtain for each $x \in PX$ that

$$\langle P'x', x \rangle = \langle x', Px \rangle = \langle x', x \rangle = (\tilde{x}', x).$$

Thus, $i(P'x') = (P'x'|_{PX} = \tilde{x}'$, i.e. $i$ is surjective and hence a Banach space isomorphism.

Finally, assume that the projection $P$ is a contraction and let $x' \in P'X'$. For each $\varepsilon > 0$, we can find a normalized vector $x \in X$ such that $|\langle x', x \rangle| \geq ||x'|| - \varepsilon$. Since $||Px|| \leq ||x|| = 1$, we obtain

$$||i(x')|| = ||x'|_{PX}|| \geq ||\langle x', x \rangle|| = ||x'|| - \varepsilon.$$

Thus, $||i(x')|| \geq ||x'||$. Since $i$ is a contraction, we conclude that it is in fact isometric. □

**Proof of Proposition 3.3.** If $X$ is not projectively non-Hilbert, then there is a two-dimensional Hilbert space $V \subset X$ and a contractive projection $P$ from $X$ onto $V$. The adjoint operator $P'$ is also a contractive projection and by Lemma 3.4 its range $P'X'$ is isometrically isomorphic to $(PX)' = V'$ which is a two-dimensional Hilbert space. Thus, $X'$ is not projectively non-Hilbert, either. □

From Proposition 3.3 we obtain another class of examples of projectively non-Hilbert spaces:

**Example 3.5.** Let $L$ be a locally compact Hausdorff space and let $C_0(L; \mathbb{R})$ be the space of real-valued continuous functions on $L$ which vanish it infinity, endowed with the supremum norm. Then $C_0(L; \mathbb{R})$ is projectively non-Hilbert.

To see this, note that the dual space $C_0(L; \mathbb{R})'$ is isometrically isomorphic to $L^1(\Omega, \Sigma, \mu)$ for some measure space $(\Omega, \Sigma, \mu)$. This well-known result follows from Kakutani's Representation Theorem for abstract $L$-spaces, see [21] Proposition 1.4.7 (i) and Theorem 2.7.1. Since we know from Example 3.2 that $L^1(\Omega, \Sigma, \mu)$ is projectively non-Hilbert, Proposition 3.3 implies that $C_0(L; \mathbb{R})$ is projectively non-Hilbert as well.

If $K$ is a compact Hausdorff space, then we of course have $C_0(K; \mathbb{R}) = C(K; \mathbb{R})$, and so the space $C(K; \mathbb{R})$ of continuous, real-valued functions on $K$ is projectively non-Hilbert. This also implies that the space $L^\infty(\Omega, \Sigma, \mu; \mathbb{R})$, for an arbitrary measure space $(\Omega, \Sigma, \mu)$, is projectively non-Hilbert. In fact, each $L^\infty$-space is isometrically isomorphic to a $C(K; \mathbb{R})$-space for some compact Hausdorff space $K$, due
3.2. $C_0$-semigroups on projectively non-Hilbert spaces. We are now going to study the point spectrum of contractive $C_0$-semigroups on projectively non-Hilbert spaces. Before stating the main result of this section, we recall that a $C_0$-semigroup $(e^{tA})_{t \geq 0}$ on a (real or complex) Banach space $X$ is called weakly almost periodic, if for each $x \in X$, the orbit $\{e^{tA}x : t \geq 0\}$ is weakly pre-compact in $X$.

**Theorem 3.6.** Let $X$ be real Banach space which is projectively non-Hilbert and let $(e^{tA})_{t \geq 0}$ be a weakly almost periodic $C_0$-semigroup on $X$. If $(e^{tA}x)_{t \geq 0}$ is weakly asymptotically contractive and if $\sigma_{\text{pat}}(A) \cap i\mathbb{R}$ is bounded, then we actually have $\sigma_{\text{pat}}(A) \cap i\mathbb{R} \subset \{0\}$.

Before proving Theorem 3.6, let us make a few remarks on the assumptions and consequences of the theorem: The condition on $\sigma_{\text{pat}}(A) \cap i\mathbb{R}$ to be bounded is for example fulfilled if $(e^{tA})_{t \geq 0}$ is eventually norm continuous, since in this case the intersection of the entire spectrum $\sigma(A)$ with every right half plane $\{\lambda \in \mathbb{C} : \text{Re}\lambda \geq \alpha\}$ (where $\alpha \in \mathbb{R}$) is automatically bounded (see [6, Theorem II.4.18]). The condition that $(e^{tA})_{t \geq 0}$ be weakly almost periodic is automatically fulfilled if the space $X$ is reflexive. It is also automatically fulfilled if the semigroup is eventually compact (i.e. if $e^{tA}$ is compact for sufficiently large $t$) or if its generator has compact resolvent; in the latter two cases the orbits of the semigroup are even pre-compact in the norm topology on $X$, see [6, Corollary V.2.15].

Moreover, for eventually compact semigroups we even obtain the following convergence result as a corollary:

**Corollary 3.7.** Let $X$ be a real Banach space which is projectively non-Hilbert and let $(e^{tA})_{t \geq 0}$ be an eventually compact $C_0$-semigroup on $X$ which is weakly asymptotically contractive. Then $e^{tA}$ converges uniformly as $t \to \infty$.

**Proof.** Let $A_{\mathbb{C}}$ be the complex extension of $A$ to a complexification $X_{\mathbb{C}}$ of $X$. Since the semigroup $(e^{tA_{\mathbb{C}}})_{t \geq 0}$ is eventually compact, it is weakly almost periodic (in fact, its orbits are even pre-compact in norm, see [6, Corollary V.2.15 (ii)]). Moreover, as an eventually compact semigroup, $(e^{tA_{\mathbb{C}}})_{t \geq 0}$ is eventually norm-continuous (cf. [6, Lemma II.4.22]) and therefore, the intersection of $\sigma(A_{\mathbb{C}})$ with any right half plane $\{z \in \mathbb{C} : \text{Re}z \geq \alpha\}$, $\alpha \in \mathbb{R}$, is bounded. We thus can apply Theorem 3.6 to conclude that $\sigma_{\text{pat}}(A_{\mathbb{C}}) \cap i\mathbb{R} \subset \{0\}$. Since the spectrum of $A_{\mathbb{C}}$ consists only of eigenvalues (see [6, Corollary V.3.2 (i), and assertion (ii) at the end of paragraph IV.1.17]), this implies that $\sigma(A_{\mathbb{C}}) \cap i\mathbb{R} \subset \{0\}$. If $0 \notin \sigma(A_{\mathbb{C}})$, then $s(A_{\mathbb{C}}) < 0$, and thus $e^{tA_{\mathbb{C}}}$ uniformly converges to 0 as $t \to \infty$. On the other hand, if 0 is a spectral value of $A_{\mathbb{C}}$, then it is a first order pole of the resolvent, since $(e^{tA_{\mathbb{C}}})_{t \geq 0}$ is bounded. Thus, $e^{tA_{\mathbb{C}}}$ uniformly converges as $t \to \infty$, see [6, Corollary V.3.3].

Next, we want to prove Theorem 3.6. Since we are going to reuse several parts of the proof in Section 3, we extract those parts into two lemmas. The first lemma is based on an idea used by Lyubich in the proof of [16, Theorem 1].

**Lemma 3.8.** Let $X_{\mathbb{C}}$ be a complexification of a real Banach space $X$ and let $(e^{tA})_{t \geq 0}$ be a $C_0$-semigroup of isometries on $X$. Furthermore, suppose that we have $\sigma_{\text{pat}}(A_{\mathbb{C}}) = \{i, -i\}$ and that $X_{\mathbb{C}} = Y_1 \oplus Y_2$, where $Y_1 := \text{Eig}(i, A_{\mathbb{C}})$ and $Y_2 := \text{Eig}(-i, A_{\mathbb{C}})$. Then $X$ cannot be projectively non-Hilbert.

**Proof.** We define the following binary operation $\circ : \mathbb{C} \times X_{\mathbb{C}} \to X_{\mathbb{C}}$ by

$$\lambda \circ (y_1 + y_2) = \lambda y_1 + \overline{\lambda} y_2 \quad \text{for all} \quad y_1 \in Y_1, \ y_2 \in Y_2.$$
The mapping $\circ$ is continuous and moreover it is immediately checked that $(X_C, +, \circ)$ is a complex vector space again. For each $t \geq 0$, each $r \geq 0$ and each $x = y_1 + y_2 \in X_C$ (where $y_1 \in Y_1$, $y_2 \in Y_2$) we have

$$(re^{it}) \circ x = re^{it}y_1 + re^{-it}y_2 = r \cdot (e^{iAc}y_1 + e^{iAc}y_2) = re^{iAc}x.$$ 

This implies that $X$ is invariant under the complex multiplication $\circ$, i.e. $X$ is a complex vector subspace of $(X_C, +, \circ)$.

Moreover, we have for each $t \geq 0$, each $r \geq 0$ and each $x \in X$ that $||(re^{it}) \circ x|| = ||re^{iAc}x|| = ||re^{it}x|| = r ||x||$ since $e^{iA}$ is isometric. Hence, $||\lambda \circ x|| = ||\lambda|| ||x||$ for each $\lambda \in \mathbb{C}$ and each $x \in X$. Therefore, the norm $|| \cdot ||$ on our real Banach space $X$ is also a norm on the complex vector space $(X, +, \circ)$.

Now, choose an element $x_0 \in X$ such that $||x_0|| = 1$. The one-dimensional complex subspace $\mathbb{C} \circ x_0 \subset X$ admits a contractive $\mathbb{C}$-linear projection $P : X \to \mathbb{C} \circ x_0$ due to the Hahn-Banach Theorem. Moreover, $P$ is also a linear mapping over the real field with respect to the original multiplication on $X$ since we have $r \circ x = rx$ for each $r \in \mathbb{R}$ and each vector $x \in X$. Also note that $(\mathbb{C} \circ x_0, +, \circ)$ is two-dimensional over the real field, and since the restriction of $\circ$ to real scalars coincides with the original multiplication on $X$, $\mathbb{C} \circ x_0$ is a two-dimensional real vector subspace of the original space $X$.

Finally, we show that $\mathbb{C} \circ x_0$ is a real Hilbert space. Since $||x_0|| = 1$, the mapping $\psi : \mathbb{C} \to \mathbb{C} \circ x_0$, $\lambda \mapsto \lambda \circ x_0$ is a $\mathbb{C}$-linear and isometric bijection. If we restrict all scalars to the real field, $\psi$ thus becomes an isometric and linear bijection between the two-dimensional normed real spaces $(\mathbb{R}^2, || \cdot ||_2)$ and $(\mathbb{C} \circ x_0, || \cdot ||)$. This proves the lemma.

The next lemma is an elementary fact from linear algebra.

**Lemma 3.9.** Let $T$ be a linear operator on a complex vector space $X$ and let $\lambda \in \mathbb{C} \setminus \{0\}$. Then $\text{Eig}(\lambda, T^2) = \text{Eig}(\mu, T) \oplus \text{Eig}(-\mu, T)$, where $\mu$ and $-\mu$ denote the complex square roots of $\lambda$.

**Proof.** First note, that the sum of the two eigenspaces is indeed algebraically direct, since $\mu \neq -\mu$.

The inclusion $\supseteq$ is obvious. To show the other inclusion, let $x \in \ker(\lambda - T^2)$; we may assume $x \neq 0$. Since $T^2x = \lambda x$, the linear span $V = \text{span}_\mathbb{C} \{x, Tx\}$ is a $T$-invariant vector subspace of $X$. Since that $T^2x = \lambda x$ and $T^2Tx = \lambda^2Tx$, we have $(T|_V)^2 = T^2|_V = \lambda^21_V$. Since $\lambda \neq 0$, the operator $T|_V$ cannot be similar to a non-trivial Jordan block; this is obvious if $\dim V = 1$ and it follows from $(T|_V)^2 = \lambda^21_V$ and a short matrix computation if $\dim V = 2$. Hence $V$ contains a basis of eigenvectors of $T|_V$. From the spectral mapping theorem we know that $\sigma(T|_V) \subset \{\pm \mu\}$ and thus $V = \text{span}_\mathbb{C} \{x, Tx\}$ is spanned by eigenvectors of $T$ with corresponding eigenvalues $-\mu$ and/or $\mu$. In particular, $x$ is a linear combination of such eigenvectors. \qed

Note that the above lemma is false for $\lambda = 0$; of course, the sum is no longer direct in this case, but the assertions even fails if we do not require the sum to be direct. Simply choose $T$ as a two-dimensional Jordan block with eigenvalue 0 to see this.

We are now ready to prove Theorem 3.6.

**Proof of Theorem 3.6.** (a) Let $X_C$ be a complexification of $X$ and let $(e^{iAc})_{t \geq 0}$ be the complex extension of the semigroup $(e^{iA})_{t \geq 0}$, see Remark 3.8. Assume for a contradiction that $\sigma_{\text{pnt}}(A_C) \cap i \mathbb{R} \not\subset \{0\}$. Replacing $A$ by $cA$ for some $c > 0$, we may assume that $i, -i \in \sigma_{\text{pnt}}(A_C)$ and that $\sigma_{\text{pnt}}(A_C) \subset i \cdot (-3, 3)$. 


(b) We proceed with some observations about several eigenspaces that will be used throughout the proof. For each \( t \in (0, \frac{\pi}{2}] \) we have that

\[
(1) \quad \text{Eig}(e^{i t}, e^{t A_C}) = \text{span}_{n \in \mathbb{Z}} \text{Eig}(i + \frac{2 \pi in}{t}, A_C) = \text{Eig}(i, A_C) =: Y_1
\]

\[
(2) \quad \text{Eig}(e^{-i t}, e^{t A_C}) = \text{span}_{n \in \mathbb{Z}} \text{Eig}(-i + \frac{2 \pi in}{t}, A_C) = \text{Eig}(-i, A_C) =: Y_2.
\]

The equalities on the left follow from a general relationship between the eigenspaces of \( A_C \) and \( e^{t A_C} \) (see [2] Corollary IV.3.8) and hold for all \( t > 0 \); the equalities on the right hold for all \( t \in (0, \frac{\pi}{2}] \) due to the condition \( \sigma_{\text{part}}(A_C) \subseteq (0, \infty) \).

While the equalities (1) and (2) hold true only for \( t \in (0, \frac{\pi}{2}] \), they imply that for all other \( t \geq 0 \) at least the inclusions

\[
(3) \quad Y_1 \subseteq \text{Eig}(e^{i t}, e^{t A_C}) \quad \text{and} \quad Y_2 \subseteq \text{Eig}(e^{-i t}, e^{t A_C})
\]

hold true. For the time \( t = \pi \) we can make a more precise observation: In fact, we have

\[
\text{Eig}(-1, e^{\pi A_C}) = \text{Eig}(e^{i \frac{\pi}{2}}, e^{\frac{\pi}{2} A_C}) \oplus \text{Eig}(e^{-i \frac{\pi}{2}}, e^{-\frac{\pi}{2} A_C}) = Y_1 \oplus Y_2 =: Z_C.
\]

The first of these equalities follows if we apply Lemma 5.9 to the operator \( T = e^{\frac{\pi}{2} A_C} \) and to the complex number \( \lambda = -1 \). The second equality follows from (1) and (2).

(c) Let us analyse the space \( Z_C \). Since the operator \( -e^{\pi A_C} \) is almost weakly periodic, it is mean ergodic. Therefore, the Cesàro means \( \frac{1}{n} \sum_{k=1}^{n} (-e^{\pi A_C})^k \) converge to a projection \( P_C : X_C \rightarrow Z_C \), because \( Z_C \) is the fixed space of \( -e^{\pi A_C} \). Clearly, \( P_C \) leaves the real space \( X \) invariant and therefore, the range \( Z_C = P_C X_C \) is the complexification of the real space \( Z := P_C X = X \cap Z_C \), see Proposition (4). Moreover, the restriction \( P_C |_X \) is contractive, since the operator \( -e^{\pi A} = -e^{i A_C} |_X \) is weakly asymptotically contractive. Thus, the real Banach space \( Z \) is the range of the contractive projection \( P_C |_X \in \mathcal{L}(X) \) and is therefore projectively non-Hilbert.

(d) Finally, we want to apply Lemma 5.8 to the space \( Z_C = Y_1 \oplus Y_2 \) and to the restricted semigroup \( (e^{A_C} |_{Z_C})_{t \geq 0} \) in order to obtain a contradiction. First, note that \( (e^{A_C} |_{Z_C})_{t \geq 0} \) is indeed a semigroup on \( Z_C \), since \( e^{A_C} \) leaves \( Z_C \) invariant due to (3). Moreover, the spectral assumptions of Lemma 5.8 are clearly fulfilled. Since the semigroup \( (e^{A_C} |_{Z_C})_{t \geq 0} \) is periodic and since it is weakly asymptotically contractive on \( Z \subseteq X \), we conclude that it acts in fact isometrically on \( Z \). Thus, the assumptions of Lemma 5.8 are fulfilled and we can conclude from this lemma that \( Z \) is not projectively non-Hilbert. This contradicts (c).

A crucial argument in step (c) of the preceding proof is the use of a mean ergodic theorem to obtain a contractive projection onto the real part of \( Y_1 \oplus Y_2 \). This argument stems from the proof of [10] Theorem 1. In this context, we should also mention that the condition on \( (e^{A_C})_{t \geq 0} \) to be weakly almost periodic can be slightly relaxed in Theorem 5.8. Indeed, the proof of the theorem shows that we only need that each negative operator \( -e^{A_C} \) is mean ergodic. However, we preferred to state the theorem with the condition that \( (e^{A_C})_{t \geq 0} \) be weakly almost periodic, since this seems to be more natural then a condition on the negative operators \( -e^{A_C} \).

Note that Theorem 3.3 fails if we do not require the set \( \sigma_{\text{part}}(A) \cap i \mathbb{R} \) a priori to be bounded. A counter example is again provided by the shift semigroup on \( L^p(T; \mathbb{R}) \).

### 3.3. Single operators on projectively non-Hilbert spaces

As in Section 2 we now apply our semigroup result to study the single operator case. Recall that an operator \( T \) on a (real or complex) Banach space \( X \) is called weakly almost periodic if for each \( x \in X \), the orbit \( \{T^n x : n \in \mathbb{N}_0\} \) is weakly pre-compact.
Theorem 3.10. Let \( X \) be a real Banach space which is projectively non-Hilbert and let \( T \in \mathcal{L}(X) \) be weakly almost periodic. If \( T \) is weakly asymptotically contractive and if \( \sigma_{\text{put}}(T) \cap \mathbb{T} \) is finite, then \( \sigma_{\text{pup}}(T) \cap \mathbb{T} \) in fact only consists of roots of unity.

Proof. (a) By means of Theorem \([6, 7]\) we may assume that \( T \) is an isometric bijection with \( \sigma_{\text{pup}}(T) \subset \mathbb{T} \) and that any complexification \( X_C \) of \( X \) is the closed linear span of eigenvectors of the complex extension \( T_C \) of \( T \). Since the point spectrum of \( T_C \) is finite and \( T_C \) is power bounded, we obtain from Proposition \([6, 3]\) that actually \( X_C = \bigoplus_{\lambda \in \sigma_{\text{pup}}(T_C)} \text{Eig}(\lambda, T_C) \).

(b) We may choose a finite set \( A \subset \mathbb{R} \) such the exponential function maps \( 2\pi i A \) bijectively to \( \sigma_{\text{pup}}(T_C) \). Let \( B \subset A \cup \{1\} \) be a basis of \( \text{span}_2 \{A \cup \{1\}\} \) over the field of rational numbers \( \mathbb{Q} \) which fulfills \( 1 \in B \). For each \( \beta \in B \) and each \( \alpha \in A \), denote by \( \alpha \beta \in \mathbb{Q} \) the uniquely determined rational number such that \( \alpha = \sum_{\beta \in B} \alpha \beta \). We can find an integer \( k \neq 0 \) such that \( k\alpha \beta \in \mathbb{Z} \) for each \( \alpha \in A \) and each \( \beta \in B \). Now, assume for a contradiction that at least one element of \( A \) is not contained in \( \mathbb{Q} \).

Then \( B \) also contains a irrational number \( \beta_0 \).

(c) Now, we define a strongly (in fact uniformly) continuous semigroup \( (e^{tA})_{t \geq 0} \) on \( X_C \) by means of

\[
e^{tAC}x_\alpha = e^{ikt_\alpha \beta_0} x_\alpha \quad \text{for} \quad x_\alpha \in \text{Eig}(e^{2\pi i \alpha}, T_C).
\]

We show that this semigroup leaves \( X \) invariant and that its restriction to \( X \) is contractive: Let \( t \geq 0 \). Since \( B \) is linearly independent over \( \mathbb{Q} \) and since \( 1 \in B \), we conclude from Kronecker’s Theorem that the powers of the tuple \( (e^{2\pi i \beta})_{\beta \in B \cup \{1\}} \) are dense in \( \mathbb{T}^{\mathbb{B} \cup \{1\}} \). Hence, we can find a sequence of natural numbers \( (N_\alpha)_{\alpha \in \mathbb{N}} \) such that \( e^{2\pi i N_\alpha \beta_0} \to e^{it} \) and such that \( e^{2\pi i N_\alpha \beta \beta_0} \to 1 \) for each \( \beta \in B \setminus \{ \beta_0, 1 \} \). Moreover, we clearly have \( e^{2\pi i N_\alpha -1} \to 1 \), so that \( e^{2\pi i N_\alpha \beta} \to 1 \) holds actually true for each \( \beta \in B \setminus \{ \beta_0 \} \). Therefore, we obtain for each \( \alpha \in A \) and \( x_\alpha \in \text{Eig}(e^{2\pi i \alpha}, T_C) \) that

\[
T_C^{N_\alpha}x_\alpha = e^{2\pi i k N_\alpha \alpha} x_\alpha = \prod_{\beta \in B} e^{2\pi i k N_\alpha \alpha \beta} x_\alpha = \prod_{\beta \in B} \left( e^{2\pi i N_\alpha \beta} \right)^{k N_\alpha \alpha \beta} \to e^{ikt_\alpha \beta_0} x_\alpha = e^{tAC}x_\alpha,
\]

which implies \( e^{tAC}x = \lim_n T_C^{N_\alpha} x \) for each \( x \in X_C \). This shows that \( e^{tAC} \) leaves \( X \) invariant and that it is contractive on \( X \). We conclude that \( A_C \) is the complex extension of an operator \( A \) on \( X \) and that \( A \) generates a contractive \( C_0 \)-semigroup \( (e^{tA})_{t \geq 0} \) on \( X \) whose complex extension is given by \( (e^{tAC})_{t \geq 0} \).

(d) For the point spectrum of \( A \) we have \( \sigma_{\text{pup}}(A) = \sigma_{\text{pup}}(AC) = \{ ik \alpha_\beta : \alpha \in A \} \). In particular, the element \( 0 \neq i k = ik(\beta_0) \beta_0 \) is contained in \( \sigma_{\text{pup}}(A) \). However, we can apply Theorem 5.6 to the semigroup \( (e^{tA})_{t \geq 0} \): Indeed, the semigroup \( (e^{tAC})_{t \geq 0} \) is weakly almost periodic, since each trajectory \( \{ e^{tAC} x : t \geq 0 \} \) is bounded and contained in a finite-dimensional space. Hence, \( (e^{tA})_{t \geq 0} \) is almost weakly periodic, and Theorem 5.6 implies that \( \sigma_{\text{pup}}(A) \cap i \mathbb{R} \subset \{0\} \). This is a contradiction. \( \square \)

For compact operators, Theorem 3.10 was proved by Lyubich in [16] by similar methods (but more directly, since the paper [14] focussed on single operators rather then on \( C_0 \)-semigroups). Moreover, on some important functions space, Theorem 3.10 was proved for compact operators even earlier by Krasnosel’skiı in [13].

Under some additional a-priori assumptions on the spectrum, we obtain the following simple corollary concerning the asymptotics of \( (T^n)_{n \in \mathbb{N}_0} \):

Corollary 3.11. Let \( X \) be a real Banach space which is projectively non-Hilbert and suppose that \( T \in \mathcal{L}(X) \) is weakly asymptotically contractive. Assume furthermore that \( \sigma(T) \cap \mathbb{T} \) is finite, isolated from the rest of the spectrum and consists only of
poles of the resolvent $R(\cdot, T_C)$. Then $T$ can be decomposed into two linear operators $T = T_{\text{per}} + T_0$ such that $\|T_0\| \to 0$ as $n \to \infty$ and $T_{\text{per}}^{n_0+1} = T_{\text{per}}$ for some $n_0 \geq 1$.

Proof. Let $T_C$ denote the complex extension of $T$ to a complexification $X_C$ of $X$ and let $P_C$ be the spectral projection belonging to the part $\sigma(T_C) \cap T$ of the spectrum of $T_C$. Then the powers of the operator $T_{C,0} := T_C(1 - P_C)$ uniformly converge to 0.

Moreover, for each $x \in X_C$, the vector $P_C x$ can be written as a finite sum $P_C x = x_1 + \ldots + x_m$, where $x_1, \ldots, x_m$ are all eigenvectors of $T$; this follows from the fact that $T_C$ is power-bounded, so that all spectral values in $\sigma(T_C) \cap T$ are in fact simple poles of the resolvent. Hence, the bounded set $\{T_C^n x : n \in \mathbb{N}_0\}$ is contained in the finite dimensional space span$_C\{x_1, \ldots, x_m\}$ and is thus pre-compact. Since for each $\varepsilon > 0$, the set

$$\{T_C^n(1 - P_C)x : n \in \mathbb{N}_0\} = \{(T_C(1 - P_C))^n x : n \in \mathbb{N}_0\}.$$ 

contains only finitely many elements whose norm is large then $\varepsilon$, we conclude that this set is pre-compact, as well. This implies, that the trajectory $\{T_C^n x : n \in \mathbb{N}_0\}$ is pre-compact. In particular, $T_C$ is weakly almost periodic, and so is $T$.

Hence, we can apply Theorem 3.10 to conclude that $\sigma_{\text{apt}}(T) \cap T$ only consists of roots of unity. Therefore, the powers of the operator $T_{C,\text{per}} := T_C P_C$ are periodic. Finally, note that the spectral projection $P_C$ leaves $X$ invariant (this easily follows from the representation of $P_C$ by Cauchy’s Integral Formula), and hence the operators $T_{C,0}$ and $T_{C,\text{per}}$ leave $X$ invariant, as well. Thus, the assertion follows with $T_0 := T_{C,0}|_X$ and $T_{\text{per}} := T_{C,\text{per}}|_X$. 

Note that the assumptions on $\sigma(T) \cap T$ in Corollary 3.11 are for example fulfilled if $T$ is compact or, more generally, if the essential spectral radius $r_{\text{ess}}(T_C) := \sup\{|\lambda| : \lambda - T_C$ is not Fredholm} is strictly smaller then 1.

Besides this, we want to mention that a result very similar to Corollary 3.11 holds true even for non-linear operators if they satisfy an additional assumption on their $\omega$-limit sets; this was shown by Lemmens and van Gaans in [15, Theorem 2.8], employing a result on projectively non-Hilbert spaces from [16, Theorem 4]. For the special case of $L_p$-spaces, $1 < p < \infty$, $p \neq 2$, such a non-linear result had already been shown by Sine in [25, Theorem 3].

4. The spectrum on $L^p$-spaces

In the preceding sections, we considered $C_0$-semigroups $(e^{tA})_{t \geq 0}$ and focussed on results that ensure that $A$ has no purely imaginary eigenvalues. However, to turn those results into convergence results, we always needed rather strong compactness conditions.

In this final section, we will employ an ultra-power technique to ensure that the intersection of the entire spectrum with $i\mathbb{R}$ is trivial. This will allow us to derive a much more general convergence result. To make our method work, we have to ensure that an ultra-power of our space $X$ is still a projectively non-Hilbert space. This seems to be rather difficult in general, but $L^p$-spaces are particularly well-suited for this task since an ultra-power of an $L^p$-space is again an $L^p$-space.

4.1. Ultra powers of Banach spaces. We shortly recall the most important facts about ultra powers of Banach spaces that will be used in the next subsection. For a more detailed treatment, we refer for example to [10, 21, p. 251 - 253] and [24, Section V.1]. Let $X$ be a real or complex Banach space. By $l^\infty(X) := l^\infty(\mathbb{N}, X)$ we denote the space of all bounded sequences in $X$. We endow this space with the supremum norm $\|x\|_\infty = \sup_{n \in \mathbb{N}} \|x_n\|$ for $x = (x_n)_{n \in \mathbb{N}} \in l^\infty(X)$. 


Let $\mathcal{U}$ be a free ultra filter on $\mathbb{N}$ and define $c_{0,\mathcal{U}}(X) := \{ x \in l^\infty(X) : \lim_{n \to \infty} x_n = 0 \}$. The space $c_{0,\mathcal{U}}(X)$ is a closed vector subspace of $l^\infty(X)$, and the quotient space $X_{\mathcal{U}} := l^\infty(X)/c_{0,\mathcal{U}}(X)$ is called the $\mathcal{U}$-ultra power of $X$. For each $x = (x_n)_{n \in \mathbb{N}} \in l^\infty(X)$, we denote by $x_{\mathcal{U}}$ the equivalence class $x_{\mathcal{U}} := x + c_{0,\mathcal{U}}(X) \in X_{\mathcal{U}}$. It turns out that we can compute the quotient norm on the space $X_{\mathcal{U}}$ rather easily: For each $x_{\mathcal{U}} \in X_{\mathcal{U}}$, we have $\| x_{\mathcal{U}} \|_{X_{\mathcal{U}}} = \lim_{n \to \infty} \| x_n \|$ (for complex Banach spaces $X$, this can be found e.g. in $[21$, Theorem 4.1.6$]$ or $[24$, Proposition V.1.2$]$; for real Banach spaces, the proof is the same). Note that this limit always exists since $\mathcal{U}$ is an ultra filter and since the set $\{ \| x_n \| : n \in \mathbb{N} \}$ is pre-compact in $\mathbb{R}$.

If $T \in L(X)$, we can define an operator $\tilde{T} \in L(l^\infty(X))$ by the pointwise operation

$$\tilde{T} x = \tilde{T}(x_n)_{n \in \mathbb{N}} := (Tx_n)_{n \in \mathbb{N}} \text{ for all } x = (x_n)_{n \in \mathbb{N}} \in l^\infty(X).$$

The bounded linear operator $\tilde{T}$ leaves the subspace $c_{0,\mathcal{U}}(X) \subset l^\infty(X)$ invariant and thus induces another bounded linear operator $T_{\mathcal{U}} \in L(X_{\mathcal{U}})$ via

$$T_{\mathcal{U}} x_{\mathcal{U}} := (\tilde{T} x)_{\mathcal{U}} \text{ for all } x_{\mathcal{U}} \in X_{\mathcal{U}}.$$ One reason for the usefulness of ultra products in operator theory is the fact, that the induced operator $T_{\mathcal{U}}$ shares a lot of spectral properties with the original operator $T$ and even improves some of them:

**Proposition 4.1.** Let $T \in L(X)$ for a complex Banach space $X$ and let $\mathcal{U}$ be a free ultra filter on $\mathbb{N}$.

(a) For the spectra of $T_{\mathcal{U}}$ and $T$ we have $\sigma(T_{\mathcal{U}}) = \sigma(T)$.

(b) For the approximate point spectra $\sigma_{\text{appr}}$ we have

$$\sigma_{\text{appr}}(T_{\mathcal{U}}) = \sigma_{\text{appr}}(T) = \sigma_{\text{appr}}(T).$$

**Proof.** See $[21$, Theorem 4.1.6$]$ or $[24$, Proposition V.1.3 and Theorem V.1.4$]$. □

If $X$ is a real or complex Banach space and $S, T \in L(X)$, then we have $\| T_{\mathcal{U}} \| = \| T \|$ and $(ST)_{\mathcal{U}} = S_{\mathcal{U}} T_{\mathcal{U}}$. This is very easy to see and will be used tacitly below. For our application of ultra powers in the next subsection, it is important to know how they behave in connection with complexifications of real Banach spaces (see Definition A.1). This is described by the following proposition.

**Proposition 4.2.** Let $X_\mathbb{C}$ be a complexification of a real Banach space $X$ and let $T_\mathbb{C}$ be the complex extension of an operator $T \in L(X)$. Let $\mathcal{U}$ be a free ultra filter on $\mathbb{N}$. Then $(X_\mathbb{C})_{\mathcal{U}}$ is a complexification of $(X)_{\mathcal{U}}$ via the embedding

$$(X)_{\mathcal{U}} \to (X_\mathbb{C})_{\mathcal{U}}: x + c_{0,\mathcal{U}}(X) \mapsto x + c_{0,\mathcal{U}}(X_\mathbb{C})$$

and the operator $(T_\mathbb{C})_{\mathcal{U}}$ is the complex extension of the operator $T_{\mathcal{U}}$.

**Proof.** The proof is straightforward and therefore left to the reader. □

The last result that we need on ultra powers of Banach spaces is the stability of $L^p$-spaces with respect to the construction of ultra powers:

**Proposition 4.3.** Let $(\Omega, \Sigma, \mu)$ be an arbitrary measure space, let $1 \leq p < \infty$ and let $X$ be the real-valued $L^p$-space $X := L^p(\Omega, \Sigma, \mu; \mathbb{R})$. If $\mathcal{U}$ is a free ultra filter on $\mathbb{N}$, then there is a measure space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ such that the ultra power $X_{\mathcal{U}}$ is isometrically isomorphic to $L^p(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu}; \mathbb{R})$.

**Proof.** See [10], Theorem 3.3 (ii)]. □
4.2. Contraction semigroups on $L^p$-spaces. Now, we will use the ultra power technique described above together with the ideas of Section 3 to analyse the spectrum of certain semigroups on $L^p$-spaces. To this end, consider a $C_0$-semigroup $(e^{tA})_{t \geq 0}$ on a (real or complex) Banach space and suppose that the growth bound $\omega(A)$ is larger than $-\infty$; the semigroup $(e^{tA})_{t \geq 0}$ is called norm continuous at infinity, if it fulfils the condition

$$\lim_{t \to \infty} \limsup_{h \to 0} \|e^{(t+h)(A-\omega(A))} - e^{t(A-\omega(A))}\| = 0.$$

Of course a $C_0$-semigroups on a real Banach space $X$ is norm continuous at infinity if and only if its complex extension to any complexification $X_\mathbb{C}$ of $X$ is so. The class of $C_0$-semigroups which are norm continuous at infinity contains the class of all $C_0$-semigroups which are eventually norm continuous and fulfil $\omega(A) > -\infty$.

The notion of norm continuity at infinity was introduced by Martinez and Maizon in [20, Definition 1.1], where they showed several spectral properties of those semigroups. We will need the following of those properties in the sequel:

**Proposition 4.4.** Let $(e^{tA})_{t \geq 0}$ be a $C_0$-semigroup on a complex Banach space. Suppose that $\omega(A) > -\infty$ and that $(e^{tA})_{t \geq 0}$ is norm-continuous at infinity.

(a) Let $\Gamma_t := \{ \lambda \in \mathbb{C} : |\lambda| = r(e^{tA}) \}$. Then the following partial spectral mapping theorem holds true for each $t \geq 0$:

$$\sigma(e^{tA}) \cap \Gamma_t = e^{t\sigma(A)} \cap \Gamma_t.$$

(b) There is an $\varepsilon > 0$ such that the set

$$\{ \lambda \in \sigma(A) : \text{Re} \lambda \geq s(A) - \varepsilon \}$$

is bounded.

**Proof.** See [20, Theorem 1.2] for (a) and [20, Theorem 1.9] for (b). □

The following theorem is the main result of this section.

**Theorem 4.5.** Let $(\Omega, \Sigma, \mu)$ be an arbitrary measure space and let $1 < p < \infty$, $p \neq 2$. Suppose that $(e^{tA})_{t \geq 0}$ is a $C_0$-semigroup on $X = L^p(\Omega, \Sigma, \mu; \mathbb{R})$ which fulfils $\omega(A) > -\infty$, is norm continuous at infinity and uniformly asymptotically contractive. Then $\sigma(A) \cap i\mathbb{R} \subset \{0\}$.

Before we prove the theorem, let us briefly discuss its assumptions and its consequences: For some comments on the validity of Theorem 4.5 on other spaces, we refer to Remark 4.3 below. The assumption $\omega(A) > -\infty$ in the theorem is of course only a technical condition to ensure that the notion “norm-continuous at infinity” is well-defined. For eventually norm continuous semigroups, the following formulation of Theorem 4.5 might be more convenient:

**Corollary 4.6.** Let $(\Omega, \Sigma, \mu)$ be an arbitrary measure space and let $1 < p < \infty$, $p \neq 2$. Suppose that $(e^{tA})_{t \geq 0}$ is a $C_0$-semigroup on $X = L^p(\Omega, \Sigma, \mu; \mathbb{R})$ which is eventually norm continuous and uniformly asymptotically contractive. Then $\sigma(A) \cap i\mathbb{R} \subset \{0\}$.

**Proof.** If $\omega(A) = -\infty$, the assertion is trivial. If $\omega(A) > -\infty$, our semigroup is norm-continuous at infinity and thus the assertion follows from Theorem 4.5. □

For the asymptotic behaviour of our semigroups, we obtain the following corollary:

**Corollary 4.7.** Suppose that the assumptions of Theorem 4.5 or Corollary 4.6 are fulfilled. Then $e^{tA}$ strongly converges as $t \to \infty$.

**Proof.** This follows from [11, Corollary 2.6] or from [6, Exercise V.2.25 (4) (ii)]. □
Similarly, we can see the continuity for \( \varepsilon > 0 \) norm continuity at infinity of \((e^TA)\). Except for the ultra-power technique involved, the major steps of proof are rather similar to those in the proof of Theorem 3.6.

(a) We have \( \omega(A) \leq 0 \) and for \( \omega(A) < 0 \) the assertion is trivial. So let \( \omega(A) = 0 \), and assume for a contradiction that the assertion of the theorem fails. Replacing \( A \) by \( cA \) for some \( c > 0 \), we may assume that \( i, -i \in \sigma(A) \) and that \( \sigma(A) \cap i\mathbb{R} \subset i \cdot [-1, 1] \). Let \( X_C \) be a complexification of \( X \) and let \((e^{tA})_{t \geq 0}\) be the complex extension of the semigroup \((e^A)_{t \geq 0}\). We then have for each \( t \geq 0 \) that

\[
\sigma(e^{tA_C}) \cap \mathbb{T} = e^{it\sigma(A_C)} \cap \mathbb{T} \subset \{ e^{i\varphi} : |\varphi| \leq t \};
\]

the equality on the left follows from Proposition 4.1(b) since \( r(e^{tA_C}) = e^{t\omega(A_C)} = 1 \).

(b) Let \( \mathcal{U} \) be a free ultra filter on \( \mathbb{N} \) and denote by \( X_{\mathcal{U}} \) and \((X_C)_{\mathcal{U}}\) the corresponding ultra powers of \( X \) and \( X_C \). According to Proposition 3.2, \((X_C)_{\mathcal{U}}\) is a complexification of \( X_{\mathcal{U}} \) and the operator \((e^{tA})_{\mathcal{U}}\) is the complex extension of \((e^A)_{\mathcal{U}}\). Moreover, it follows from Proposition 4.1 that \( \sigma_{\mathcal{U}}((e^{tA})_{\mathcal{U}}) \cap \mathbb{T} = \sigma((e^{tA})_{\mathcal{U}}) \cap \mathbb{T} \) for each \( t \geq 0 \), since the boundary of the spectrum of an operator is automatically contained in the approximate point spectrum. Besides that, we still have \( \limsup_{t \to \infty} ||(e^A)_{\mathcal{U}}|| \leq 1 \).

However, note that the operator family \((e^{tA})_{\mathcal{U}}\) might not be strongly continuous on \((X_C)_{\mathcal{U}}\), although it of course still fulfils the semigroup law.

(c) We proceed to analyse some properties of several eigenspaces that will be used in the rest of the proof. Define

\[
Y_1 := \text{Eig}(e^{\pi d}, (e^{\pi d}A)_{\mathcal{U}}) \quad \text{and} \quad Y_2 := \text{Eig}(e^{-\pi d}, (e^{\pi d}A)_{\mathcal{U}}).
\]

By virtue of Lemma 3.3 it follows that \( \text{Eig}(-1, (e^{\pi d}A)_{\mathcal{U}}) = Y_1 \oplus Y_2 =: Z_C \). From the same Lemma and from the fact that \( \sigma((e^{itA})_{\mathcal{U}}) \cap \mathbb{T} = \sigma((e^{itA})_{\mathcal{U}}) \cap \mathbb{T} \subset \{ e^{i\varphi} : |\varphi| \leq t \} \), we also conclude that \( Y_1 = \text{Eig}(e^{i\pi d}, (e^{i\pi d}A)_{\mathcal{U}}) \) and \( Y_2 = \text{Eig}(e^{-i\pi d}, (e^{i\pi d}A)_{\mathcal{U}}) \) for all \( n \in \mathbb{N} \). Thus,

\[
(4) \quad Y_1 \subset \text{Eig}(e^{\pi d}, (e^{i\pi d}A)_{\mathcal{U}}) \quad \text{and} \quad Y_2 \subset \text{Eig}(e^{-i\pi d}, (e^{i\pi d}A)_{\mathcal{U}})
\]

for each dyadic number \( d \geq 0 \). Next, note that the mapping \([0, \infty) \to (X_C)_{\mathcal{U}}, t \mapsto (e^{tA})_{\mathcal{U}}x_{\mathcal{U}}\) is continuous for each \( x_{\mathcal{U}} \in Y_1 \) and each \( x_{\mathcal{U}} \in Y_2 \): Indeed, let \( x_{\mathcal{U}} \in Y_1 \) such that \( ||x_{\mathcal{U}}|| = 1 \), let \( t \geq 0 \) and \( \varepsilon > 0 \). For sufficiently large \( n \in \mathbb{N} \), the norm continuity at infinity of \((e^{tA})_{t \geq 0}\) implies that

\[
\limsup_{h \to 0} ||(e^{\pi d+ih}A)_{\mathcal{U}} - (e^{\pi d+ih}A)_{\mathcal{U}}|| = \limsup_{h \to 0} ||(e^{\pi d+ih}A)_{\mathcal{U}} - (e^{\pi d+ih}A)_{\mathcal{U}}|| \leq \varepsilon.
\]

Hence, we have

\[
\varepsilon \geq \limsup_{h \to 0} ||(e^{\pi d+ih}A)_{\mathcal{U}}x_{\mathcal{U}} - (e^{\pi d+ih}A)_{\mathcal{U}}x_{\mathcal{U}}|| = \limsup_{h \to 0} ||(e^{ih}A)_{\mathcal{U}}x_{\mathcal{U}} - (e^{ih}A)_{\mathcal{U}}x_{\mathcal{U}}||,
\]

where the last equality follows from the fact that \( (e^{\pi d}A)_{\mathcal{U}}x_{\mathcal{U}} = e^{\pi d}x_{\mathcal{U}} \). Since \( \varepsilon > 0 \) was an arbitrary number, we conclude that

\[
\lim_{h \to 0} ||(e^{ih}A)_{\mathcal{U}}x_{\mathcal{U}} - (e^{ih}A)_{\mathcal{U}}x_{\mathcal{U}}|| = 0.
\]

Similarly, we can see the continuity for \( x_{\mathcal{U}} \in Y_2 \). The strong continuity of \( t \mapsto (e^{tA})_{\mathcal{U}} \) on \( Y_1 \) and \( Y_2 \) together with (4) implies that

\[
\text{Eig}(e^{it}, (e^{tA})_{\mathcal{U}}) \supset Y_1 \quad \text{and} \quad \text{Eig}(e^{-it}, (e^{tA})_{\mathcal{U}}) \supset Y_2
\]

for all \( t \geq 0 \).

(d) Let us now analyse the subspace \( Z_C \). This space coincides with the fixed space of the power-bounded operator \(-((e^{\pi d}A)_{\mathcal{U}})\). Since \( X_{\mathcal{U}} \) is an \( L^p \)-space (see Proposition 3.3 and \( 1 < p < \infty \), the space \( X_{\mathcal{U}} \) is reflexive and so is its complexification \((X_C)_{\mathcal{U}}\).
Thus, \( -(e^{\sigma_{AC}})_{t\mu} \) is mean ergodic. The mean ergodic projection \( P : (X_{\Sigma})_{t\mu} \to (X_{\Sigma})_{t\mu} \) has \( Z_{\Sigma} \) as its range. Moreover, \( P \) leaves \( X_{\mu} \) invariant and the restriction \( P|_{X_{\mu}} \) is contractive, since \( \limsup_{n \to \infty} \|(-e^{\sigma_{AC}})_{t\mu})_{n}|_{X_{\mu}} \| = \limsup_{n \to \infty} \|((-e^{\sigma_{AC}})_{t\mu})_{n}|_{X_{\mu}} \| \leq 1 \).

It now follows from Proposition 4.4 that \( Z_{\Sigma} \) is the complexification of the space \( Z := P_{\Sigma}(X_{\mu}) = X_{\mu} \cap Z_{\Sigma} \). The space \( Z \) is projectively non-Hilbert since it is the image of the projectively non-Hilbert space \( X_{\mu} \) under the contractive projection \( P_{\Sigma}|_{X_{\mu}} \).

(c) Finally, we want to apply Lemma 3.8 to the space \( Z_{\Sigma} = Y_1 \oplus Y_2 \) and to the restricted semigroup \( (e^{\sigma_{AC}})_{t\mu}|_{Z_{\Sigma}} \geq 0 \). It follows from (5), that the space \( Z_{\Sigma} \) is indeed invariant with respect to the operators \( (e^{\sigma_{AC}})_{t\mu} \), and the strong continuity assertion shown in (c) implies that the restricted semigroup \( (e^{\sigma_{AC}})_{t\mu}|_{Z_{\Sigma}} \geq 0 \) is indeed a \( C_0 \)-semigroup. Due to (5), this \( C_0 \)-semigroup satisfies the spectral assumptions in Lemma 3.8. Since it is also periodic (again due to (5)) and uniformly asymptotically contractive on \( Z \subset X_{\mu} \), the semigroup acts in fact isometrically on \( Z \). Thus, the conditions in Lemma 3.8 are fulfilled and hence, the lemma yields that \( Z \) is not projectively non-Hilbert. This is a contradiction to (d).

After this proof, some remarks on the validity of Theorem 4.5 on other spaces are in order.

**Remarks 4.8.** (a) For contractive \( C_0 \)-semigroups which are norm-continuous at infinity, the assertion of Theorem 4.5 remains true on \( E := L^1(\Omega, \Sigma, \mu; \mathbb{R}) \). The proof of this assertion relies on the theory of Banach lattices and is therefore rather different from our proof above. The precise argument works as follows:

First note that \( E_{\Sigma} := L^1(\Omega, \Sigma, \mu; \mathbb{R}) \) is a complexification of our space \( E \). Moreover, if the operator \( e^{it\lambda} \) is contractive, then its complex extension \( e^{it\lambda} : E_{\Sigma} \to E_{\Sigma} \) is contractive as well; this follows from [7, Proposition 2.1.1]. Now, assume for a contradiction that \( 0 \neq i\beta \in \sigma(A_{\Sigma}) \), where \( \beta \in \mathbb{R} \). Then \( e^{i\beta t} \in \sigma(e^{it\lambda}) \) for all \( t \geq 0 \). Moreover, it follows from Proposition 4.4 that \( \sigma(e^{it\lambda}) \ni T \) is contained in a sector of small angle in the right half plane, if we choose \( t > 0 \) sufficiently small. However, since \( e^{it\beta} \in \sigma(e^{it\lambda}) \), it follows from [24, Corollary 2 to Theorem V.7.5] that \( e^{i(2n+1)\beta} \in \sigma(e^{it\lambda}) \) for each \( n \in \mathbb{Z} \). This is a contradiction.

The result from [24, Corollary 2 to Theorem V.7.5] that we used in the last step is based on the spectral theory of positive operators on Banach lattices and on the fact that real operators on \( L^1 \)-spaces always have a modulus. Besides that, we note that the above argument strongly depends on the fact that \( e^{it\lambda} \) is contractive for small \( t \). Currently, the author does not know whether the assertion \( \sigma(A) \cap i\mathbb{R} \subset \{0\} \) remains true on \( L^1 \)-spaces if we consider uniformly asymptotically contractive instead of contractive semigroups.

(b) In Theorem 4.5 we do not really need to consider \( L^p \)-spaces. Instead, we can replace the \( L^p \)-space in the theorem by a real Banach space \( E \) which fulfills that some ultra power \( E_{\mu} \) of \( E \) is still reflexive and projectively non-Hilbert. The theorem remains true for those spaces, since its proof only uses that \( E_{\mu} \) is reflexive and projectively non-Hilbert. However, currently the author does not know any example of a Banach space \( E \) which is not an \( L^p \)-space, but has a reflexive and projectively non-Hilbert ultra power \( E_{\mu} \).

One might wonder why we required the semigroup to be uniformly asymptotically contractive in Theorem 4.5 which is slightly stronger than the weak asymptotic contractivity that we required in the Theorems 2.3 and 3.8. The problem with weak (and even with strong) asymptotic contractivity is that it does not carry over to the lifted operators on an ultra power. The uniform condition \( \limsup_{n \to \infty} \|e^{it\lambda} \| \leq 1 \) however holds still true for the lifted operators, since their norm coincides with the
norm of the original operators. In fact, the assertion of Theorem 4.6 may fail for strongly (and in particular for weakly) asymptotically contractive semigroups:

**Example 4.9.** Let $1 < p < \infty$, $p \neq 2$, and endow $\mathbb{R}^2$ with the $p$-Norm. For each $n \in \mathbb{N}$, let $A_n$ be the operator on $\mathbb{R}^2$ whose representation matrix with respect to the canonical basis is given by

$$
\begin{pmatrix}
-\frac{1}{n} & -1 \\
1 & -\frac{1}{n}
\end{pmatrix}.
$$

We have $\sigma(A_n) = \{ i - \frac{1}{n}, -i - \frac{1}{n} \}$, and the representation matrix of $e^{tA_n}$ with respect to the canonical basis is given by

$$
e^{-\frac{1}{n}t} \begin{pmatrix} 
\cos t & -\sin t \\
\sin t & \cos t
\end{pmatrix}.$$

We clearly have $\|e^{tA_n}\| \to 0$ as $t \to \infty$. Now, consider the vector-valued $L^p$-space $X = L^p(\mathbb{N}; \mathbb{R}^2)$ (which is isometrically isomorphic to the space $L^p(\mathbb{N}; \mathbb{R})$) and let $A := \bigoplus_{n=1}^{\infty} A_n \in \mathcal{L}(X)$. Then $A$ generates a $C_0$-semigroup $(e^{tA})_{t \geq 0}$ on $X$ and it is easy to see that this semigroup fulfills $\limsup_{n \to \infty} \|e^{tA}x\| \leq \|x\|$ (and in fact even $e^{tA}x \to 0$ as $t \to \infty$) for each $x \in X$, but that $\limsup_{n \to \infty} \|e^{tA}\| > 1$. Thus, the semigroup is strongly, but not uniformly asymptotically contractive. Furthermore, we have $\sigma(A) \cap i\mathbb{R} = \{ i, -i \} \subset \{ 0 \}$, so the assertion of Theorem 4.6 does not hold true for the semigroup $(e^{tA})_{t \geq 0}$. However, note that $i$ and $-i$ are not eigenvalues of $A$, which is in accordance with Theorem 3.6.

Example 4.9 is based on an idea that was used in [2] to construct a counter example in the theory of eventually positive semigroups. Note that we can easily modify Example 4.9 to obtain a semigroup which is strongly asymptotically contractive, but even fulfills $\sigma(A) \cap i\mathbb{R} = \{ i, -i \} \subset \{ 0 \}$: We simply have to multiply the off-diagonal entries of the representation matrices of $A_n$ by numbers $\alpha_n$, where $(\alpha_n)_{n \in \mathbb{N}} \subset [0, 1]$ is a sequence which is dense in $[0, 1]$.

4.3. **Contractive single operators on $L^p$-spaces.** Using the ultra power technique described at the beginning of this section, we can also derive a result on single operators which is similar to Theorem 3.10 but now for the spectrum instead of the point spectrum. Actually the proof for the single operator case as much easier then for the semigroup case, since we can simply lift a single operator to an ultra product of $X$ and apply Theorem 3.10

**Theorem 4.10.** Let $(\Omega, \Sigma, \mu)$ be an arbitrary measure space, let $1 < p < \infty$, $p \neq 2$ and set $X := L^p(\Omega, \Sigma, \mu; \mathbb{R})$. Suppose that $T \in \mathcal{L}(X)$ is uniformly asymptotically contractive. If $\sigma(T) \cap \mathbb{T}$ is finite, then it only consists of roots of unity.

**Proof.** Let $T_C$ be the complex extension of $T$ to a complexification $X_C$ of $X$. Let $\mathcal{U}$ be a free ultra filter on $\mathbb{N}$ and denote by $X_{\mathcal{U}}$ and $(X_C)_{\mathcal{U}}$ the ultra powers of $X$ and $X_C$. Then $X_{\mathcal{U}}$ is an $L^p$-space again and $(X_C)_{\mathcal{U}}$ is a complexification of $X_{\mathcal{U}}$. Denote by $T_{\mathcal{U}}$ and $(T_C)_{\mathcal{U}}$ the liftings of $T$ and $T_C$ to the ultra powers $X_{\mathcal{U}}$ and $(X_C)_{\mathcal{U}}$. Then $(T_C)_{\mathcal{U}}$ is the complex extension of $T_{\mathcal{U}}$ and we have $\sigma(T_C) \cap \mathbb{T} = \sigma_{\text{pnt}}((T_C)_{\mathcal{U}}) \cap \mathbb{T} = \sigma_{\text{pnt}}(T_{\mathcal{U}}) \cap \mathbb{T}$.

Since the operator $T_{\mathcal{U}}$ is defined on a reflexive $L^p$-space and fulfills the condition $\limsup_{n \to \infty} \|T_{\mathcal{U}}^n\| \leq 1$, we can apply Theorem 3.10 to conclude that $\sigma_{\text{pnt}}(T_{\mathcal{U}}) \cap \mathbb{T}$ consists only of roots of unity.

Comparing our results on $C_0$-semigroups with our results on single operators, it seems that the semigroup results are much more satisfying. For example, the boundedness condition on $\sigma_{\text{pnt}}(A) \cap i\mathbb{R}$ from Theorem 3.6 and the norm continuity assumption from Theorem 4.6 will be satisfied on many occasions, even without
any compactness assumptions. By contrast, the condition from Theorems 4.10 and 6.10 that the peripheral (point) spectrum \( \sigma(T) \cap \mathbb{T} \) (respectively \( \sigma_{pnt}(T) \cap \mathbb{T} \)) be finite seems to be rather strong. It would be interesting to know whether the same conclusions for single operators hold true under weaker a-priori assumptions on the spectrum.

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**Appendix A. Complexification of real Banach spaces**

In order to analyse linear operators on a real Banach space \( X \) by spectral theoretic methods, we need the concept of a *complexification* of \( X \). In this appendix, we give a short overview of those notions and facts that are needed in the paper.

**Definition A.1.** Let \( X \) be a real Banach space. A *complexification* of \( X \) is a tuple \((X_C, J)\), where \( X_C \) is a complex Banach space and \( J : X \to X_C \) is an isometric \( \mathbb{R} \)-linear mapping such that the following properties are fulfilled:

(a) \( X_C = J(X) \oplus iJ(X) \), where \( \oplus \) denotes the algebraically direct product of two real vector subspaces of \( X_C \).

(b) The (\( \mathbb{R} \)-linear) projection from \( X_C \) onto \( J(X) \) along \( iJ(X) \) is contractive.

Note that the algebraically direct product in assertion (a) of the above definition is in fact topologically direct, since \( J(X) \) and \( iJ(X) \) are both closed. It is often convenient to identify \( X \) with its image \( J(X) \) and to shorty say that \( X_C \supset X \) is a complexification of \( X \), thereby suppressing the mapping \( J \) in the notation. We shall mostly use this slightly imprecise notation; however, it is important that we can always reformulate the subsequent definitions and results in a formal way, which explicitly involves the embedding \( J \).

If \( X_C \supset X \) is a complexification of a real Banach space \( X \), then we can decompose each element \( z \in X_C \) into a sum \( z = x + iy \), where \( x \) and \( y \) are uniquely determined elements of \( X \). The elements \( x \) and \( y \) are referred to as the real and the imaginary part of \( z \) and they are denoted by \( x =: \text{Re} z \) and \( y =: \text{Im} z \). Moreover, we denote by \( \overline{z} := x - iy \) the complex *conjugate vector* of \( z \). By assertion (b) of the above definition, the operator \( \text{Re} \) is contractive, and so is the operator \( \text{Im} \) since we have \( \text{Im} z = -\text{Re}(iz) \) for each \( z \in X_C \).

It is not difficult to see that each real Banach space \( X \) has a complexification \( X_C \). For example, we can simply endow an algebraic complexification \( X_C \) of \( X \) with the norm \( \| x + iy \| := \sup_{\theta \in [0, 2\pi]} | x \cos \theta + y \sin \theta | \). However, the reader should be warned that in many cases this special norm does not coincide with the “natural norm” which one would usually endow an algebraic complexification of \( X \) with. For instance, if \( X = \mathbb{R}^2 \) is endowed with the \( \| \cdot \|_1 \)-norm, then the norm on \( \mathbb{C}^2 \) given by \( \| x + iy \| := \sup_{\theta \in [0, 2\pi]} | x \cos \theta + y \sin \theta | \) does not coincide with the \( \| \cdot \|_1 \)-norm on \( \mathbb{C}^2 \). Nevertheless, \((\mathbb{C}^2, \| \cdot \|_1)\) is also a complexification of \((\mathbb{R}^2, \| \cdot \|_1)\), which in turn indicates, that two complexifications of a real Banach space \( X \) might not be (canonically) isometric, in general. More precisely, we can make the following observation concerning uniqueness of the complexification:

**Remark A.2.** If \( X_{C,1} \supset X \) and \( X_{C,2} \supset X \) are two complexifications of \( X \), then there is a unique (complex) Banach space isomorphism \( \Phi : X_{C,1} \to X_{C,2} \) which acts as the identity when restricted to \( X \). We call the mapping \( \Phi \) the *canonical isomorphism* between \( X_{C,1} \) and \( X_{C,2} \). However, the above example on \( \mathbb{C}^2 \) shows that \( \Phi \) is not isometric in general.
We now come to the topic of extending $\mathbb{R}$-linear operators on $X$ to a complexification $X_C$ of $X$.

**Definition A.3.** Let $D(A)$ be a vector subspace of a real Banach space $X$ and let $A : D(A) \to X$ be a linear operator. Let $X_C \supset X$ be a complexification of $X$. Then $D(A_C) := D(A) + iD(A)$ is a complex vector subspace of $X_C$ and the operator $A$ has a unique $\mathbb{C}$-linear extension $A_C : D(A_C) \to X_C$. This extension is given by $A_C(x + iy) = Ax + iAy$ for $x, y \in D(A)$. The operator $A_C$ is called the **complex extension** of $A$.

Similarly, a bounded $\mathbb{R}$-linear functional $x' \in X'$ has a unique $\mathbb{C}$-linear extension $(x')_C \in (X_C)'$, which is given by $\langle (x')_C, x + iy \rangle = \langle x', x \rangle + i\langle x', y \rangle$ for all $x, y \in X$.

Let $X_C$ be a complexification of a real Banach space $X$ and let $A : X \supset D(A) \to X$ be a linear operator. Many properties of $A$ are in direct correspondence to the properties of its complex extension $A_C$. For example, $A$ is closed if and only if $A_C$ is closed, $A$ is bounded if and only if $A_C$ is bounded, and so on. However, one should be warned that the complex extension $A_C$ of a bounded linear operator $A \in \mathcal{L}(X)$ might in general not have the same norm as $A_C$. In fact, we always have $\|A\| \leq \|A_C\| \leq 2\|A\|$, where the second inequality holds true since the operators $\text{Re}$ and $\text{Im}$ are contractive.

The following elementary observation is easily verified; however, since we use it on many occasions throughout the article, we state it explicitly here.

**Proposition A.4.** Let $X_C$ be a complexification of a real Banach space $X$ and let $P_C \in \mathcal{L}(X_C)$ be a projection which leaves $X$ invariant.

Then $P_C$ is the complex extension of the real projection $P := P_C|_X \in \mathcal{L}(X)$ and we have $P_CX_C = PX \oplus iPX$, i.e. $P_CX_C$ is a complexification of the real Banach space $PX$. Moreover, the spaces $PX$ and $X \cap P_CX_C$ coincide.

Since our original goal was to do spectral theory, it is important to note that the spectral properties of the complex extension of an operator $A$ do not depend on the choice of the complexification $X_C$. Indeed, if $X_{C,1}$ and $X_{C,2}$ are two complexifications of $X$ and if $A_{C,1}$ and $A_{C,2}$ denote the corresponding complex extensions of $A$, then $A_{C,1}$ and $A_{C,2}$ are intertwined by the canonical isomorphism $\Phi : X_{C,1} \to X_{C,2}$, and therefore the spectra and point spectra of $A_{C,1}$ and $A_{C,2}$ coincide. Thus, the following definition makes sense:

**Definition A.5.** Let $A : X \supset D(A) \to X$ be a linear operator on a real Banach space $X$. The spectrum and the point spectrum of $A$ are defined to be the sets

$$\sigma(A) := \sigma(A_C) \quad \text{and} \quad \sigma_{\text{pnt}}(A) := \sigma_{\text{pnt}}(A_C),$$

where $A_C$ denotes the complex extension of $A$ to any complexification $X_C$ of $X$.

If $A$ is a linear operator on a real Banach space $X$, then its spectrum enjoys several symmetry properties which are listed in the next proposition:

**Proposition A.6.** Let $X$ be a real Banach space and let $A : X \supset D(A) \to X$ be a linear operator. Let $A_C$ be the complex extension of $A$ to a complexification $X_C$ of $X$.

(a) We have $\lambda \in \sigma(A)$ whenever $\lambda \in \sigma(A_C)$.

(b) If $\lambda \in \mathbb{C}$ is an eigenvalue of $A_C$ with eigenvector $z = x + iy \in D(A_C)$, then $\overline{\lambda}$ is an eigenvalue of $A_C$ with eigenvector $\overline{z} = x - iy \in D(A_C)$.

(c) If $z = x + iy \in X_C$ (where $x, y \in X$) is an eigenvector of $A_C$ for a non-real eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $x, y$ are linearly independent over $\mathbb{R}$.

**Proof.** (b) By Definition A.3 the domain $D(A_C)$ is the sum $D(A_C) = D(A) + iD(A)$. Thus $x + iy \in D(A_C)$ implies $x, y \in D(A)$ and therefore $x - iy \in D(A_C)$.
Using the fact that $\overline{\mu w} = \overline{\mu} \overline{w}$ for all $\mu \in \mathbb{C}$ and all $w \in X_C$, we immediately obtain $A_C \overline{z} = Ax - iAy = \overline{Ax + iAy} = \overline{x} - i\overline{y}$.

(a) It is easily seen that $\lambda - A_C$ is surjective whenever $\overline{X} - A_C$ is so. Thus the assertion follows from (b).

(c) Assume that $\alpha x + \beta y = 0$ for $\alpha, \beta \in \mathbb{R}$. We then obtain $\frac{\alpha - i\beta}{2} z + \frac{\alpha + i\beta}{2} \overline{z} = \alpha x + \beta y = 0$. Since $z$ and $\overline{z}$ are eigenvectors of $A_C$ for two distinct eigenvalues $\lambda \neq \overline{\lambda}$, we know that $z, \overline{z}$ are linearly independent over $\mathbb{C}$. Hence, $\frac{\alpha + i\beta}{2} = 0$, which finally yields $\alpha = \beta = 0$. $\square$

On a few occasions in the article, we need the adjoint operator $A'$ of an operator $A : X \supset D(A) \rightarrow X$, where $X$ is a real Banach space. Therefore, we should clarify have dual spaces and adjoints behave with respect to complexifications:

**Proposition A.7.** Let $X_C$ be a complexification of a real Banach space $X$ and let $A : X \supset D(A) \rightarrow X$ be a linear operator on $X$.

(a) The dual space $(X_C)'$ is a complexification of the dual space $X'$ via the embedding $$J : X' \rightarrow (X_C)' : x' \mapsto (x')_C,$$

where $(x')_C$ denotes the complex extension of the linear functional $x'$.

(b) The operator $(A_C)'$ is the complex extension of $A'$ to the complexification $(X_C)'$ of $X'$.

**Proof.** (a) Let us first show that the map $J$ is isometric: Clearly, we have $||x'|| \leq ||(x')_C||$ for each $x \in X'$. Now, let $z \in X_C$. There is a complex number $\zeta \in \mathbb{T}$ such that $||(x')_C, z|| = \zeta ||(x')_C, z|| \in [0, \infty)$. Hence, we have $||(x')_C, z|| = \langle (x')_C, \zeta z \rangle = \langle x', \text{Re}(\zeta z) \rangle + i \langle x', \text{Im}(\zeta z) \rangle = \langle x', \text{Re}(\zeta z) \rangle \leq ||x'|| \cdot ||\text{Re}(\zeta z)|| \leq ||x'|| \cdot ||\zeta|| \cdot ||z|| = ||x'|| \cdot ||z||.$

For the last inequality we used that the operator Re on $X_C$ is contractive. We showed that $||(x')_C|| = ||x'||$, so $J$ is isometric.

We clearly have $J(X') \cap iJ(X') = \{0\}$. Moreover, each element $z' \in (X_C)'$ can be decomposed as $z' = x' + iy'$, where the two elements $x', y' \in J(X')$ are given by

$$\langle x', x + iy \rangle = \text{Re}(z', x) + i \text{Re}(z', y)$$

and

$$\langle y', x + iy \rangle = \text{Im}(z', x) + i \text{Im}(z', y)$$

for all $x, y \in X$. Hence, $(X_C)' = J(X') \oplus iJ(X')$. Finally, we show that $||x'|| \leq ||z'||$.

To this end, let $z \in X_C$ and choose $\zeta \in \mathbb{T}$ such that $||\langle x', z \rangle|| = \zeta ||\langle x', z \rangle|| \in [0, \infty)$. We then obtain

$$||\langle x', z \rangle|| = ||(x', \zeta z) = \text{Re}(z', \text{Re}(\zeta z)) + i \text{Re}(z', \text{Im}(\zeta z)) = \text{Re}(z', \text{Re}(\zeta z)) \leq ||z'|| \cdot ||\text{Re}(\zeta z)|| \leq ||z'|| \cdot ||\zeta|| \cdot ||z|| = ||z'|| \cdot ||z||,$$

where we again used for the last inequality that the operator Re on $X_C$ is contractive. We showed that $||x'|| \leq ||z'||$, and hence the projection from $(X_C)'$ onto $J(X')$ along $iJ(X')$ is contractive. This proves (a).

(b) This is a straightforward exercise which we leave to the reader. $\square$

We close this section with the following observation on the complex extension of a $C_0$-semigroup.

**Remark A.8.** Let $X_C$ be a complexification of a real Banach space $X$ and let $(e^{tA})_{t \geq 0}$ be a $C_0$-semigroup on $X$ with generator $A$. Then the complex extension $A_C$ of $A$ generates a $C_0$-semigroup $(e^{tA_C})_{t \geq 0}$ on $X_C$ and for each $t \geq 0$, the operator $e^{tA_C}$ is the complex extension of $e^{tA}$. 
The $C_0$-semigroup $(e^{tA})_{t \geq 0}$ is called the complex extension of the $C_0$-semigroup $(e^{tA})_{t \geq 0}$.

APPENDIX B. A FEW RESULTS ON THE BOUNDARY SPECTRUM

In this appendix we give a few results on the peripheral spectrum of linear operators. More precisely, let $X$ be a complex Banach space; we are concerned with those spectral values of an operator $T \in \mathcal{L}(X)$ which are located on the spectral circle $\{ \lambda \in \mathbb{C} : |\lambda| = r(T) \}$ and with those spectral values of a semigroup generator $A$ on $X$ which are located on the line $\{ \lambda \in \mathbb{C} : \Re \lambda = s(A) \}$ (where $s(A) := \sup \{ \Re \lambda : \lambda \in \sigma(A) \}$ denotes the spectral bound of $A$). Let us start with a result relating the point spectrum of an operator to the point spectrum of its adjoint:

**Proposition B.1.** Let $X$ be a complex Banach space. Let $T \in \mathcal{L}(X)$ be power-bounded and let $A : X \supset D(A) \to X$ be the generator of a bounded $C_0$-semigroup on $X$.

(a) We have $\sigma_{\text{pnt}}(T) \cap \mathbb{T} \subset \sigma_{\text{pnt}}(T^*) \cap \mathbb{T}$.

(b) We have $\sigma_{\text{pnt}}(A) \cap i\mathbb{R} \subset \sigma_{\text{pnt}}(A^*) \cap i\mathbb{R}$.

**Proof.** (a) Let $\lambda \in \sigma_{\text{pnt}}(T) \cap \mathbb{T}$. Multiplying $T$ by $\lambda^{-1}$ if necessary, we may assume that $\lambda = 1$. Thus, we have $Tx = x$ for some $x \in X \setminus \{0\}$. Choose $x' \in X'$ such that $\langle x', x \rangle = 1$ and define a sequence $(y'_n) \subset X'$ by $y'_n = \frac{1}{n} \sum_{k=1}^{n} (T^k)x'$. Since $T$ is power-bounded, the sequence $(y_n)$ is bounded and hence, it has a subnet $(y'_{n_j})$ which converges to an element $y_0'$ with respect to the weak* topology on $Y'$. The limit element $y_0'$ fulfills 

$$\langle y_0', x \rangle = \lim_{j} \langle y'_{n_j}, x \rangle = \langle x', x \rangle = 1,$$

so we conclude that $y_0' \neq 0$. Moreover, $T'y_0' = y_0'$, since we have for each $y \in X$ that

$$\langle T'y_0', y \rangle = \lim_{j} \frac{1}{n_j} \sum_{k=1}^{n_j} \langle (T^k)x', y \rangle = \lim_{j} \langle y_{n_j} + \frac{(T^k)x' - x'}{n_j}, y \rangle = \langle y_0', y \rangle.$$

(b) Let $i\beta \in \sigma_{\text{pnt}}(A) \cap i\mathbb{R}$. Replacing $A$ by $A - i\beta$ if necessary, we may assume that $i\beta = 0 = s(A)$. If $Ax = 0$ for some non-trivial $x \in X$, then $R(1, A)x = x$, and since $A$ generates a bounded $C_0$-semigroup, we know that $R(1, A)$ is power-bounded. Hence, 1 is contained in the point spectrum of $R(1, A)' = R(1, A^*)$ by (a), and this shows that $0 \in \sigma_{\text{pnt}}(A^*)$. Alternatively, one could also show (b) in a similar way as (a), see e.g. [1] Lemma 2.3. \qed

Recall that an operator $T \in \mathcal{L}(X)$ on a real or complex Banach space $X$ is said to be weakly almost periodic, if for each $x \in X$, the orbit $\{ T^n x : n \in \mathbb{N}_0 \}$ is weakly pre-compact. Equivalently, the set $\{ T^n : n \in \mathbb{N}_0 \}$ is pre-compact in $\mathcal{L}(X)$ with respect to the weak operator topology. Clearly, an operator on a real Banach space $X$ is weakly almost periodic if and only if its complex extension to any complexification $X_C$ of $X$ is.

For weakly almost periodic operators, the point spectrum on the unit circle can be split off by a certain projection. We state a version of this famous Jacobs-deLeeuw-Glicksberg decomposition in the setting of real Banach spaces:

**Theorem B.2.** Let $X$ be a real Banach space and suppose that $T \in \mathcal{L}(X)$ is weakly almost periodic and weakly asymptotically contractive. Then there is a contractive projection $P \in \mathcal{L}(X)$ with the following properties:
(a) $PT = TP$, i.e. the range $PX$ and the kernel $\ker P$ are invariant with respect to $T$.
(b) The restricted operator $T|_{PX}$ is bijective and isometric on $PX$.
(c) $\sigma_{\text{pat}}(T|_{PX}) = \sigma_{\text{pat}}(T) \cap \mathbb{T}$.
(d) If $Y_C$ is a complexification of $PX$, then $Y_C$ is the closed linear span of all eigenvectors of the complex extension of $T|_{PX}$.

Proof. Let $X_C$ be a complexification of $X$ and let $T_C$ be the complex extension of $T$. We construct a projection $P_C$ in the following, usual way: Let $\mathcal{S} := \{T_C^n : n \in \mathbb{N}_0\}^\omega$, where $\overline{\mathcal{M}}^\omega$ denotes the closure of any set $\mathcal{M} \subset \mathcal{L}(X_C)$ with respect to the weak operator topology. Since $T$ is weakly almost periodic, so is $T_C$, and hence $\mathcal{S}$ is compact in the weak operator topology, and in fact $\mathcal{S}$ is a compact semitopological abelian semigroup with respect to the usual operator multiplication. Therefore, the Sushkevich kernel

$$\mathcal{K} := \bigcap_{S \in \mathcal{S}} S \cdot \mathcal{S} \subset \bigcap_{n \in \mathbb{N}_0} \{T_C^k : k \geq n\}^\omega$$

is a closed ideal in the semigroup $\mathcal{S}$ and even more, it is a compact topological group, see [8, Theorems V.2.3 and V.2.4] and [14, Theorem 4.1 on p. 104 and the subsequent discussion]. Denote by $P_C$ the neutral element of $\mathcal{K}$. Then $P_C$ is idempotent, and thus a projection on the space $X_C$. Moreover, $P_C$ leaves $X$ invariant, since it is an element of $\mathcal{S}$. Define $P := P_C|_X$.

Before showing that $P$ fulfills the asserted properties, we observe that all elements $S_C \in \mathcal{K}$ act contractively on $X$: Indeed, let $x \in X$ and $x' \in X'$ such that $\|x\| = \|x'\| = 1$ and let $\varepsilon > 0$. Since $T$ is weakly asymptotically contractive, there is a $n \in \mathbb{N}$ such that $\|\langle T^n x, x' \rangle\| \leq 1 + \varepsilon$ for each $k \geq n$. Therefore, $\|\langle S_C x, x' \rangle\| \leq 1 + \varepsilon$, since $S_C \in \{T_C^k : k \geq n\}$.

This shows that $\|S_C x\| \leq 1$. As a first conclusion of this observation, we note that $P = P_C|_X$ is contractive as claimed. Now, let us show the claimed properties (a) - (d) of the projection $P$:

(a) Since $P_C$ is an element of $\mathcal{S}$, it commutes with $T_C$. Hence, $P$ and $T$ commute, too.

(b) Since $\mathcal{K}$ is an ideal in $\mathcal{S}$, the operator $P_C T_C$ is an element of the group $\mathcal{K}$ and thus there is an element $R_C \in \mathcal{K}$ such that $(T_C P_C) R_C = R_C (T_C P_C) = P_C$. The operator $R_C$ leaves $X$ invariant, since it is an element of $\mathcal{S}$; moreover, it commutes with $P_C$, so it also leaves $P_C X_C$ invariant. Since we know from Proposition [14, Theorem 4.4] that $PX = X \cap P_C X_C$, we conclude that $R_C$ also leaves $PX$ invariant. We have $T|_{PX} R_C|_{PX} = R_C|_{PX} T|_{PX} = P_C|_{PX} = \text{id}_{PX}$, so we conclude that $T|_{PX}$ is bijective on $PX$. To see that $T|_{PX}$ is isometric, recall from above that each element $S_C \in \mathcal{K}$ acts contractively on $X$. Since $T_C P_C \in \mathcal{K}$, we conclude that $\|T|_{PX}\| \leq 1$. Moreover, we have $R_C \in \mathcal{K}$ and thus, $\|(T|_{PX})^{-1}\| = \|R_C|_{PX}\| \leq 1$. Hence, $T|_{PX}$ is isometric.

(c) It can be shown that the range $P_C X_C$ of $P_C$ coincides with the closed linear span of all eigenvectors of $T_C$ belonging to eigenvalues of modulus 1, see [14, Theorems 4.4 on p. 105 and Theorem 4.5 on p. 106]. Hence, we have $\sigma_{\text{pat}}(T_C) \cap \mathbb{T} \subset \sigma_{\text{pat}}(T_C|_{P_C X_C})$. However, the operator $T_C|_{P_C X_C}$ cannot have any eigenvalue $\lambda$ with $|\lambda| < 1$: Indeed, suppose that $x \in P_C X_C$ is an eigenvector for such an eigenvalue $\lambda$, and let $\varepsilon > 0$. Then there is an $n \in \mathbb{N}_0$ such that $\|T_C^n x\| \leq \varepsilon$ for each $k \geq n$. Since $P_C \in \{T_C^k : k \geq n\}$, we can conclude that $\|\langle P_C x, x' \rangle\| \leq \varepsilon$ for each $k \geq n$ and for each normalized $x' \in X$. Hence, $x = P_C x = 0$, which is a contradiction.

Thus, we conclude that $\sigma_{\text{pat}}(T_C) \cap \mathbb{T} = \sigma_{\text{pat}}(T_C|_{P_C X_C})$. Note that $P_C X_C$ is a complexification of $PX$ according to Proposition [14, Theorem 4.4] and clearly, $T_C|_{P_C X_C}$ is a complex extension of $T|_{PX}$. This implies $\sigma_{\text{pat}}(T) \cap \mathbb{T} = \sigma_{\text{pat}}(T|_{PX})$.

(d) We have noted above that (d) holds true for one complexification $Y_C := P_C X_C$ of $PX$, and hence it holds true for every complexification. \qed
If $X$ is a (real or complex) Banach space and $V_1, V_2 \subset X$ are two closed subspaces with $V_1 \cap V_2 = \{0\}$, then the algebraically direct sum $V_1 \oplus V_2$ need not be closed (and thus not topologically direct) in general. However, the situation changes if we consider eigenspaces belonging to unimodular eigenvalues of a power bounded operator:

**Proposition B.3.** Let $X$ be a complex Banach space and let $T \in \mathcal{L}(X)$ be power bounded, i.e. $\sup_{n \in \mathbb{N}_0} \|T^n\| < \infty$. Furthermore, let $\lambda_1, \ldots, \lambda_n \in \sigma_{\text{pnt}}(T) \cap \Gamma$. Then for each $k = 1, \ldots, n$ the projection

$$P_k : \bigoplus_{j=1}^n \text{Eig}(\lambda_j) \to \text{Eig}(\lambda_k), \quad x_1 + \ldots + x_n \mapsto x_k$$

is continuous. In particular, the vector space $\bigoplus_{j=1}^n \text{Eig}(\lambda_j)$ is closed in $X$ and the algebraically direct sum is also topologically direct.

**Proof.** Let $M := \sup_{n \in \mathbb{N}_0} \|T^n\|$. For each $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, the Neumann series representation of the resolvent yields $\|R(\lambda, T)\| \leq \frac{M}{|\lambda| - 1}$. For $x_1 \in \text{Eig}(\lambda_1)$, ..., $x_n \in \text{Eig}(\lambda_n)$ and $k \in \{1, \ldots, n\}$ we have

$$P_k(x_1 + \ldots + x_n) = x_k = \sum_{j=1}^n \lim_{r \to 1} (r - 1)\lambda_k R(r\lambda_k, T)x_j$$

and hence we obtain

$$\|P_k(x_1 + \ldots + x_n)\| \leq \lim_{r \to 1} \|(r - 1)\lambda_k R(r\lambda_k, T) (x_1 + \ldots + x_n)\| \leq M \|x_1 + \ldots + x_n\| \quad \Box$$

The same technique as in the preceding proof was used e.g. in [19, Theorem 1 and Corollary 1] to analyse the eigenspaces belonging to unimodular eigenvalues of a contraction (see also [23] for an English translation of this article).

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