Gravothermal oscillations in two-component models of star clusters

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ABSTRACT

In this paper, gravothermal oscillations are investigated in two-component clusters with a range of different stellar mass ratios and total component mass ratios. The critical number of stars at which gravothermal oscillations first appeared is found using a gas code. The nature of the oscillations is investigated and it is shown that the oscillations can be understood by focusing on the behaviour of the heavier component because of mass segregation. It is argued that, during each oscillation, the recollapse of the cluster begins at larger radii while the core is still expanding. This recollapse can halt and reverse a gravothermally driven expansion. This material outside the core contracts because it is losing energy both to the cool expanding core and to the material at larger radii. The core-collapse times for each model are also found and discussed. For an appropriately chosen case, direct N-body runs were carried out, in order to check the results obtained from the gas model, including evidence of the gravothermal nature of the oscillations and the temperature inversion that drives the expansion.

Key words: methods: numerical – globular clusters: general.

1 INTRODUCTION

Gravothermal oscillations are one of the most interesting phenomena which may arise in the post-collapse evolution of a star cluster. The inner regions of a post-collapse cluster are approximately isothermal and are subject to a similar instability as the one found in an isothermal sphere in a spherical container, as studied by Antonov (1962) and Lynden-Bell & Wood (1968). Gravothermal oscillations, which are thought to be a manifestation of this instability, were discovered by Bettwieser & Sugimoto (1984) whilst studying the post-collapse evolution of star clusters using a gas model. For a gas model of a one-component cluster it was found that gravothermal oscillations first appear when the number of stars $N$ is greater than 7000 (Goodman 1987). This value of $N$ has also been found with Fokker–Planck calculations (Cohn, Hut & Wise 1989) and by direct N-body simulations (Makino 1996). However, in a multicomponent cluster the situation is more complicated. The presence of different mass components introduces different dynamical processes to the system such as mass stratification. Multicomponent systems try to achieve kinetic energy equipartition between the components, which causes the heavier stars to move more slowly and sink towards the centre. This can lead to the Spitzer instability (Spitzer 1987) in which the heavier stars continuously lose energy to the lighter stars without ever being able to reach equipartition. Murphy, Cohn & Hut (1990) found that the post-collapse evolution for multicomponent models was stable to much higher values of $N$ than in the case of the one-component system and that the value of $N$ at which gravothermal oscillations appeared varied with different mass functions.

In order to gain a deeper understanding of gravothermal oscillations, it is desirable to work with simpler models in which some of the effects which are present in real star clusters are ignored or simplified. For example, real star clusters have a range of stellar masses present, but in the current paper, the stellar masses are limited to two. Gaseous models are often used in this kind of research (Bettwieser & Sugimoto 1984; Goodman 1987; Heggie & Aarseth 1992) because they are computationally efficient. Kim, Lee & Goodman (1998) have already completed research in this area using Fokker–Planck models. However, their research was limited to mostly Spitzer stable models and only a small range of stellar mass ratios. The study in the present paper looks at the more general Spitzer unstable models using various stellar mass and total mass ratios.

There is also evidence of gravothermal oscillations in real star clusters. Giersz & Heggie (2009) modelled the cluster NGC 6397 using Monte Carlo models and found fluctuation in the core radius. Their time-scale suggests that they are gravothermal. Subsequently, they confirmed these fluctuations using direct N-body methods with initial conditions generated from the Monte Carlo model (Heggie & Giersz 2009).

Two-component clusters may seem very unrealistic but there is reason to believe that they may be a good approximation to multicomponent systems. Kim & Lee (1997) were able to find good approximate matches for half-mass radius $r_h$, central velocity dispersion $v_c$, core density $\rho_c$ and core-collapse time $t_{cc}$ between two-component models and 11-component models which were designed to approximate a power-law initial mass function. Also see

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Kim et al. (1998) for a discussion of the realism of two-component models.

This paper is structured as follows. In Section 2, we describe the models which are used. This is followed by Section 3, in which the results concerning gravothermal oscillations are given. Section 4 is concerned with the results of the core-collapse times. In Section 5, the results of $N$-body simulations are given. Finally, Section 6 consists of the conclusions and a discussion.

## 2 MODELS

### 2.1 Gas model

#### 2.1.1 Basic equations and notation

In our model, we ignore primordial binaries and stellar evolution, and assume that the energy-generating mechanism is the formation of binary stars in three-body encounters and subsequent encounters of binaries with single stars. In a one-component model the rate of energy generation per unit mass is approximately

$$\epsilon = 85 \frac{G^2 m^2}{\sigma_c^2} \frac{n^2}{\rho} \tag{1}$$

(Heggie & Hut 2003), where $m$ is the stellar mass, $n$ is the number density, $\sigma_c$ is the one-dimensional velocity dispersion of the core and $G$ is the gravitational constant. Goodman (1987), whose results on the one-component model we shall occasionally refer to, used a similar formula, with a coefficient which is, in effect, in the range 140–170 (depending on the value of $N$).

The equations of the two-component gas model (Heggie & Aarseth 1992) are

$$\frac{\partial M_i}{\partial r} = 4\pi \rho_i r^2, \quad i = 1, 2 \tag{2}$$

$$\frac{\partial \rho_i}{\partial r} = -\frac{G(M_1 + M_2)}{r^2} \rho_i, \quad i = 1, 2 \tag{3}$$

$$\frac{\partial \sigma_i}{\partial r} = -\frac{\sigma_i L_i}{12\pi C m_i \rho i r^3 \ln \Lambda}, \quad i = 1, 2 \tag{4}$$

$$\frac{\partial L_i}{\partial r} = -4\pi \rho_i \left[ \sigma_i \left( \frac{D \rho_i}{Dr} \right) \ln \left( \frac{\sigma_i^2}{\rho_i} \right) + \frac{\rho_{i-1} - \rho_i}{r^2} \right] \left[ m_{i-1} \sigma_{i-1}^2 - m_i \sigma_i^2 \right] + 4(2\pi)^{1/2} G^2 \ln \Lambda \left[ \frac{\rho_{i-1}}{\left( \sigma_{i-1}^2 + \sigma_i^2 \right)^{3/2}} \right] \left( m_{3-i} \sigma_{3-i}^2 - m_i \sigma_i^2 \right), \quad i = 1, 2 \tag{5}$$

where $i = 1, 2$. This model in turn is ultimately inspired by the one-component model of Lynden-Bell & Eggleton (1980).

The meaning of the symbols can be found in Table 1. The major difference between the above equations and those for the one-component model is the last term of equation (5), which involves the exchange of kinetic energy between the two components. See Spitzer (1987, p. 39) for information on this term. As the heavier component dominates in the core of the cluster it is assumed that all of the energy is that generated from the second component. This is why the energy generation term (E) is multiplied by the Kronecker delta $\delta_{i2}$ in the last equation. There are two constants in the gas code which can be adjusted: $C$ and the coefficient $\lambda$ of $N$ in $\Delta = \lambda N$. The value of $\lambda = 0.02$ was used as it was found to provide a good fit for multicomponent models (Giersz & Heggie 1996).

The value of $C$ was used was 0.104 (Heggie & Ramamani 1989). This value of $C$ results from the comparison of core collapse between gas and Fokker–Planck models of single-component systems and it is not clear if it applies accurately to post-collapse two-component models.

#### 2.1.2 The role of $N$ in the gas code

This paper places emphasis on the role of $N$ in evolution, but it is not clear what role $N$ plays in equations (2)–(5). For fixed structure [i.e. $\rho_i(r, \text{etc.})$, $N$ appears explicitly in $\Lambda$ (where its role is rather insignificant) and in the individual masses $m_i$. These appear in equations (4) and (5). In a system with fixed structure, equation (4) shows

$$L_i \propto m \ln \Lambda \propto \frac{\ln \lambda N}{N},$$

reflecting the fact that the flux $L$ is caused by two-body relaxation, and its time-scale is proportional to $N/(\ln \lambda N)$. In equation (5) $N$ plays a similar role in the last term on the right, which governs the approach to equipartition. It also appears implicitly through $\epsilon$ because of the $m$ dependence in equation (1). For a system of given structure, its contribution to $L$ in equation (5) is proportional to $N^{-1}$ [as we are assuming that $\rho = mn$ is fixed and so $\epsilon$ in equation (1) is proportional to $m^1$]. It would seem as though this term is insignificant for large $N$. In practice, however, the system compensates by increasing the central density so that $\epsilon$ plays a comparable role to the relaxation terms (see Section 3.2.1).

### 2.2 Direct $N$-body

Direct $N$-body simulations were conducted using the NBODY6 code (Aarseth 2003) enabled for use with Graphical Processing Units (GPUs). NBODY6 has a range of features and options such as individual time steps which make it an excellent direct $N$-body code. NBODY6 is written in FORTRAN and is publicly available for download from www.ast.cam.ac.uk/~sverre/web/pages/nbody.htm.

## 3 CRITICAL VALUE OF $N$

If the value of $N$ is not too large, then, after core collapse, the cluster expands at a steady rate (Fig. 1, top). However, at a larger value of $N$ the central density ($\rho_c$) was found to oscillate (Fig. 1, bottom). Goodman (1987) showed that for one-component models the steady expansion is unstable for large values of $N$ and found that the value at which oscillations first appeared is $N = 7000$. In this present paper, the case of two-component models is investigated.
Gravothermal oscillations

Figure 1. Logarithm of the central density versus time (in units of the initial value of $t_{rh}$) for a two-component gas model, $m_2/m_1 = 2$, $M_2/M_1 = 1$, top: $N = 1.5 \times 10^4$ (stable), bottom: $2.5 \times 10^4$ (unstable). For initial conditions see Section 3.1.

3.1 Results of the gas code

In all cases, the initial conditions used were Plummer models (Plummer 1911; Heggie & Hut 2003). The initial velocity dispersions of both components were equal and the initial ratio of density of each component was equal at all locations. The initial conditions were constructed with different stellar mass ratios $m_2/m_1 = 2, 3, 4, 5, 10, 20$ and $50$ and for each of these mass ratios, a model with total mass ratios $M_2/M_1 = 0.1, 0.2, 0.3, 0.4, 0.5$ and $1$ was constructed. A PYTHON script was used to run the gas model code over a range of values of $N$ for each of the pairs of mass ratios. Each run terminated when the time value reached $30$ initial relaxation times ($t_{i, rh}$). The value of the central density was checked for an increase in value of $5$ per cent or more in any interval over the time period between $20t_{i, rh}$ and $30t_{i, rh}$. If an increase was found, the run was deemed to be unstable and the range of $N$ was refined. This process continued until the critical value of $N (N_{crit})$ at which oscillations first appeared was determined (correct to $10$ per cent). The values of $N_{crit}$ were also visually confirmed from the output of the gas code. The obtained values of $N_{crit}$ in units of $10^4$ are given in Table 2. Fig. 2 shows a contour plot of $\log_{10} N_{crit}$.

Table 2. Critical value of $N (N_{crit})$ in units of $10^4$.

| $M_2/M_1$ | 1.0  | 1.7  | 2.0  | 2.4  | 2.8  | 5.0  | 8.5  | 18   |
|------------|------|------|------|------|------|------|------|------|
| 0.5        | 2.2  | 2.8  | 3.5  | 4.0  | 4.0  | 7.2  | 13   | 30   |
| 0.4        | 2.3  | 3.2  | 3.8  | 4.6  | 8.2  | 15   | 33   |      |
| 0.3        | 2.6  | 3.6  | 4.6  | 5.4  | 10   | 18   | 42   |      |
| 0.2        | 3.0  | 4.4  | 5.5  | 7.0  | 12   | 22   | 55   |      |
| 0.1        | 3.8  | 6.0  | 8.5  | 10   | 22   | 36   | 100  |      |

3.2 Interpretation of the results

In order to attempt to interpret the results in the previous subsection, it is helpful to illustrate the mass density distribution of each component within the cluster and this is done in Fig. 3.

First, let us consider models in which $m_2/m_1 \gg 1$. In a region where both components are present at comparable densities, there is a strong tendency towards mass segregation. Therefore, in the region at which $\rho_2/r_0 \gg 1$, the ratio $\rho_2/r_0$ is a rapidly decreasing function of the radius, i.e. the transition region is narrow. Inside this region, $m_2$ dominates, and $m_1$ dominates outside. Clearly the radius at which this region is located increases with $M_2/M_1$, and must be near $r_0$ when $M_2/M_1 = 1$ (Fig. 4). Finally, for models in which $m_2/m_1 \ll 1$, the tendency towards mass segregation decreases, the decrease of $\rho_2/r_0$ with $r$ is more gradual, and the transition region is more extensive (Fig. 5). For the same reason the regions dominated by a single component are more restricted than when $m_2/m_1 \gg 1$. 

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arguments we have presented are consistent with the results in the uppermost rows of Table 3.

Secondly, consider the case $M_2/M_1 \lesssim 1$ (Fig. 4, right). If the system is Spitzer unstable, the heavy component decouples from the light component and forms its own subsystem. This heavy sub-system can itself become gravothermally unstable and exhibit a temperature inversion in the same way as a one-component model. In this case, however, there is not enough mass in the heavy component to dominate throughout the region within $r_h$, and so it is not quite as easy to relate this to the one-component case. Rather, we assume that the heavy component behaves like a detached one-component model. However, the basic conclusion is still the same, the stability of the model is determined by the heavy component. Since the heavy component is again sitting in the potential well of the lighter stars, it is easier for a nearly isothermal region to be set up in the heavier stars than if the entire system consisted of heavy stars, and we again expect $N_{crit}$ to correspond to a lower value of $N_2$ than in the one-component case.

There is also a noticeable increase in the values of $N_2$ with decreasing $m_2/m_1$ in the top rows of Table 3. There is currently no clear interpretation of this effect but it may possibly be related to the effect of mass segregation, as the region dominated by the heavy component is large for larger $m_2/m_1$ (see Fig. 5).

3.2.2 Goodman’s stability parameter

Goodman (1993) suggested that the quantity

$$\epsilon = \frac{E_{tot}/t_r}{E_c/t_c}$$

(6)

should indicate the stability universally, where $\log_{10} \epsilon \sim -2$ is the stability limit below which the cluster would become unstable. Here, $E_{tot}$ is the total energy, $E_c$ is the energy of the core, $t_r$ is the core relaxation time and $t_h$ is the half-mass relaxation time. Kim et al. (1998) carried out research using a Fokker–Planck model which seemed to support the condition, although the models they studied were all Spitzer stable.

We have compared the values of $\epsilon$ found by Kim et al. (1998) to results obtained from the gas code (Table 4). All the models compared in Table 4, which are the same as those studied by Kim et al. (1998), are stable in the post-collapse expansion as well as being Spitzer stable. An important difference between the Fokker–Planck model used by Kim et al. (1998) and the gas code used in this paper is that Kim et al. (1998) included an energy generation term in both components, whereas the gas code only contains an energy generation term in the heavier component. Therefore, it would be expected, in the case of the gas code, that the core would have to collapse further in order to generate the required amount of energy (from Hénon’s principle, see Section 3.3). This could explain the

Table 3. Number of heavy stars ($N_2$) at $N_{crit}$ in units of $10^4$.

| $M_2/M_1$ | $N_2$ | 10 | 20 | 50 |
|-----------|-------|----|----|----|
| 1.0       | 0.57  | 0.48| 0.47| 0.44| 0.41| 0.35|
| 0.5       | 0.44  | 0.39| 0.36| 0.34| 0.32| 0.30|
| 0.4       | 0.38  | 0.35| 0.34| 0.32| 0.29| 0.26|
| 0.3       | 0.34  | 0.32| 0.31| 0.29| 0.27| 0.25|
| 0.2       | 0.27  | 0.26| 0.25| 0.24| 0.22| 0.22|
| 0.1       | 0.18  | 0.21| 0.20| 0.19| 0.18| 0.20|

Table 4. Comparison of values of $\epsilon$ and $r_c/t_h$.

| $m_2/m_1$ | $N$ | Kim et al. log $\epsilon$ | Gas model log $\epsilon$
|-----------|-----|--------------------------|--------------------------|
| 2         | 0.02| $3 \times 10^4$          | $-1.620$                | $-1.553$
| 3         | 0.03| $3 \times 10^4$          | $-1.224$                | $-1.167$
| 3         | 0.03| $10^5$                   | $-1.597$                | $-1.544$

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differences in the values of \( r_c/r_h \) in Table 4. However, as \( M_2/M_1 \) increases, the heavier component will dominate in the core and the energy generation of the lighter component will then become negligible. As can be seen in Table 4, there is good agreement between the two results for \( \log_{10} \epsilon \) even though there are only small values of \( M_2/M_1 \). Also, it is possible that Kim et al. (1998) used a definition of \( t_{ec} \) which is different from the one used in this paper (see equation 7). However, as there is such good agreement between the values in Table 4, it is unlikely that Kim et al. (1998) used a significantly different definition. Unlike the other models in the present paper, the models in Table 4 are Spitzer stable. These runs have only been carried out in order to make a comparison of the calculation of \( \epsilon \) and \( r_c/r_h \) using the gas code with the results of Kim et al. (1998). Next we will test the use of epsilon as a stability criterion for Spitzer unstable cases.

We tested the stability criterion based on equation (6) for the subset of models given in Table 5. For each fixed \( M_2/M_1 \) and \( m_2/m_1 \), the values of \( \epsilon \) were found to decrease with increased \( N \) until the post-collapse evolution became unstable. The values of \( \log_{10} \epsilon \) given in Table 5 are the values for the run with the highest stable \( N \). As can be seen from Table 5, the value of \( \log_{10} \epsilon \) is indeed in the region of \(-2 \). However, the limiting value of stable \( \epsilon \) varies with \( m_2/m_1 \) and to a much lesser extent with \( M_2/M_1 \).

Now we shall try to improve upon the definition of \( \epsilon \). In equation (6), \( t_{ec} \) and \( t_{nh} \) are defined by

\[
t_{ec} = \frac{0.34 \rho_c^3}{G^2 m^2 \rho_c \ln \Lambda} \quad (7)
\]

and

\[
t_{nh} = \frac{0.138 N^{1/2} r_h^{3/2}}{(GM_1)^{3/2} \ln \Lambda} \quad (8)
\]

Note that \( t_{ec} \) was defined using the properties of the heavy component in the core rather than the averages of both components, as the heavy component dominates in the core. However, \( t_{nh} \) (equation 8) depends on \( N \) and \( \rho \), which can vary dramatically with \( N \) and \( m_2/m_1 \). The half-mass relaxation time for fixed \( N \) whereas, as argued in Section 3.2, the important criterion is the number of heavy stars. We suggest that a modified version of the Goodman stability parameter could be constructed using a relaxation time based on the heavy component in place of \( t_{nh} \). For example, if \( M_2/M_1 \gtrsim 1 \) and we assume that the heavier component dominates within \( r_h \), then the properties of the system within \( r_h \) would be roughly similar to that of a one-component system with the same total mass. We can attempt to treat the system as if it consist entirely of the heavy component with an effective number of stars \( N_{ef} = M/m_2 \). The half-mass relaxation time of this one-component system would be

\[
t_{nh,ef} = \frac{0.138 N_{ef}^{1/2} r_h^{3/2}}{(GM_{2,ef})^{3/2} \ln \lambda N_{ef}} = \left( \frac{1 + M_2/M_1}{M_2/M_1 + m_2/m_1} \right) \left( \frac{\ln \lambda N_{ef}}{\ln \lambda N_{ef}} \right) t_{nh}
\]

We can define a modified stability condition by replacing \( t_{nh} \) with \( t_{nh,ef} \) in the definition of \( \epsilon \) which would then give the following condition:

\[
\epsilon_2 \equiv \frac{E_{\text{st}}}{E_{\text{eh}}}
\]

The values of \( \log_{10} \epsilon \) and \( \log_{10} \epsilon_2 \) are compared in Table 6 for the case \( M_2/M_1 \). The values of \( \log_{10} \epsilon_2 \) are in much better agreement with each other than those of \( \log_{10} \epsilon \) and suggest that a better stability condition is \( \log_{10} \epsilon_2 \approx -1.5 \) rather than \( \log_{10} \epsilon \approx -2 \). For the cases with \( M_2/M_1 \leq 1 \) it is unclear how to define an appropriate relaxation time, and so we will not consider the modified stability condition for those cases.

To summarize, the values of \( \epsilon \) (and especially \( \epsilon_2 \)) seem to give an indication of stability for the two-component models but the values of \( \epsilon \) were found to change with different conditions (e.g. \( m_2/m_1 \)). The critical value of \( \epsilon_2 \) is much less variable. The critical value of \( \log_{10} \epsilon_2 \) is still to be tested for multicomponent models, and this would be an interesting topic for further research.

### 3.3 Weak oscillations

Hénon (1975) suggested that the energy generation rate of the core is determined by the requirement that it meets the energy demands of the rest of the cluster. This demand is normally thought of in terms of the energy flux at the half-mass radius. We shall refer to this as Hénon’s principle. This principle, together with the notion of gravothermal instability, is the basis of the usual qualitative picture of gravothermal oscillations (Bettwieser & Sugimoto 1984), which we now recap.

In a situation with very large \( N \), the core has to collapse to a small size in order to meet the required energy generation. The steady state is gravothermally unstable, as there would be a large density contrast in a nearly isothermal region. If the core generates more energy than can be conducted away, this would cause the core to expand, cool and reduce its rate of energy generation. If there is sufficient expansion, then the core would be cooler than its surroundings. This would result in the core starting to absorb heat. Since the core has a negative specific heat capacity, this would cause the core to expand further and become even cooler than before (Bettwieser & Sugimoto 1984). Ultimately, however, the core must collapse again to meet the energy requirements of the rest of the cluster. Here, we adapt this explanation of gravothermal oscillations to the case of two-component clusters.

In one-component gas models, as \( N \) increases the instability first appears in the form of periodic oscillations\(^{1}\) (Heggie & Ramamani 1989). In order to study the instability for the case of weak or low-amplitude oscillations in our two-component model, a model was chosen which demonstrated periodic oscillation with parameters \( m_2/m_1 = 2, M_2/M_1 = 1 \) and \( N = 2.0 \times 10^4 \) (the value of \( N_{\text{crit}} \) for \( m_2/m_1 = 2, M_2/M_1 = 1 \) is \( 1.7 \times 10^3 \) from Table 2). Fig. 6 plots in \( p \) at various fixed values of \( \log r \) for this model. The total energy flux \( L \) is shown in Fig. 7 over the particular expansion phase from 24.54\( t_{nh} \) to 25.18\( t_{nh} \) and the contraction phase from 25.18\( t_{nh} \) to 24.54\( t_{nh} \).

\(^{1}\) Strictly, only periodic if one scales out the steady expansion.
26.52\,t_{i,\text{rh}}. Fig. 8 shows the profiles of \( \log \rho \) and \( \log \sigma^2 \) over the expansion phase from 24.54\,t_{i,\text{rh}} to 25.18\,t_{i,\text{rh}}.

During the expansion of the core, the flux in the inner region (between \( r_i \) and \( r_h \)) drops and eventually becomes negative (Fig. 7, top) in a small range of the radius. At this point there is an inward flux of energy to the core. Since the core has a negative heat capacity, it would be expected that this would enhance the negative flux and therefore the expansion. However, the expansion stops at this point. This is similar to behaviour observed by McMillan & Engle (1996).

Now we explain why this happens. Hénon (1975) argues that the flux at \( r_h \) must be maintained, and we note that there is always a positive flux at the half-mass radius \( r_h \). Since the flux from the core becomes negative at some radius between \( r_i \) and \( r_h \), there must be a positive flux gradient in some region between the core and half-mass radius. This can be seen in Fig. 7 (top) towards the end of the expansion and it continues into the early part of the contraction phase (Fig. 7, bottom).

The flux gradient can be related to density via equation (5). As the heavier component dominates in the inner regions (see Fig. 8), the main contribution to the flux is from the heavier component (i.e. \( L \sim L_2 \)). Outside the core the energy generation will be negligible. Finally, the temporal change in \( \ln \rho \) is greater than that in \( \ln \sigma^3 \). Taking all of this into account and rearranging equation (5) will result in the following:

\[
\frac{1}{r^2 \rho^3 \sigma^2} \frac{\partial L}{\partial r} \simeq \left( \frac{D}{Dt} \right) \ln \left( \frac{\rho_1}{\rho_2} \right) \simeq \left( \frac{D}{Dt} \right) \ln \left( \frac{\sigma_1}{\sigma_2} \right).
\]  

(9)

Since all of the coefficients of the flux gradient are positive the sign of flux gradient must be the same as that of the Lagrangian derivative of the density. Thus a positive radial flux gradient in space implies that the density increases with time. This can be seen in Fig. 6, where the dashed lines mark the moment when the contraction begins an expansion, and the solid lines mark the time when contraction resumes. It is clear that the contraction begins at large radii (\( \log r \gtrsim -1.6 \)) while the core is expanding, and that this region of contraction propagates inwards at later times. This can be related to the position of the positive gradient in Fig. 7 via the above equation (as long as the density is low enough that energy generation is negligible). Therefore, the collapse of the parts of the cluster between the core and \( r_h \) starts while the core is expanding and brings the expansion to a halt. Note that Fig. 6 plots density at fixed radius whereas time derivatives in equation (9) are at fixed mass. Nevertheless in Fig. 6 we can also see that there are intermediate radii in which the density evolves in the opposite way from the core.

Although we have constructed the details of this description in the context of two-component models, nothing we have said depends entirely on this, and it seems likely that similar ideas will apply to one-component and multicomponent models.
While it may seem that the study of core collapse times is inappropriate in the context of gravothermal oscillations, it can be argued that the collapse phase of a gravothermal oscillation is not essentially different from the phenomenon of core collapse. Furthermore, another reason for its inclusion is that the evolution of isolated two-component models is an interesting research topic in its own right, and with the aim of constructing a comprehensive approximate theory of these models, studying the core collapse time is an appropriate first step.

The core collapse time $t_{cc}$ for a one-component cluster with Plummer model initial conditions has been found to be approximately $15.5t_{i,\text{rh}}$ (Heggie & Hut 2003; Binney & Tremaine 2008) using various methods. Takahashi (1995) found a longer $t_{cc}$ of $17.6t_{i,\text{rh}}$ with a one-component anisotropic Fokker–Planck code. However, the presence of a range of stellar masses can have a dramatic effect on the collapse time because of the process of mass segregation. The effect of mass segregation in multicomponent models has been studied using Fokker–Planck calculations (Chernoff & Weinberg 1990; Murphy et al. 1990) and Monte Carlo methods (Gürkan, Freitag & Rasio 2004). The effect of mass segregation in two-component models has already been studied extensively using direct $N$-body methods (Khalisi, Amaro-Seoane & Spurzem 2007).

For the gas model runs discussed in Section 3, Table 7 gives the values of the collapse time in units of the initial half-mass relaxation time. Fig. 9 shows a contour plot of $\log(t_{cc}/t_{i,\text{rh}})$ as a function of $M_2/M_1$ and $m_2/m_1$.

For two-component systems, the time-scale of mass segregation varies as $(m_2/m_1)^{-1}$ (Fregeau et al. 2002, and references therein). As mass segregation enhances the central density, it is expected that the mass-segregation time-scale is comparable with the time-scale of core collapse. Fig. 10 compares the variation of the time-scale of core collapse with the expected time-scale of mass segregation. For the case of $M_2/M_1 = 1.0$ (top line in Fig. 10) the collapse time indeed appears to vary as $(m_2/m_1)^{-1}$. However, for lower values of $M_2/M_1$, the core collapse time decreases more quickly than for $(m_2/m_1)^{-1}$. Khalisi et al. (2007) also found a steeper decrease of the

| $M_2/M_1$ | $t_{cc}/t_{i,\text{rh}}$ |
|----------|----------------------|
| 1.0      | 8.95 7.80 4.78 3.87 2.0 1.1 0.5 |
| 0.5      | 7.80 4.78 3.45 2.75 1.38 0.72 0.35 |
| 0.4      | 7.58 4.43 2.89 2.49 1.23 0.66 0.31 |
| 0.3      | 7.44 4.17 2.88 2.24 1.1 0.55 0.20 |
| 0.2      | 7.42 3.89 2.65 1.97 0.91 0.47 0.16 |
| 0.1      | 8.1 3.95 2.4 1.7 0.75 0.38 0.13 |

For the case of $M_2/M_1 = 1.0$, the collapse time appears to vary as $(m_2/m_1)^{-1}$. However, for lower values of $M_2/M_1$, the core collapse time decreases more quickly than for $(m_2/m_1)^{-1}$. Khalisi et al. (2007) also found a steeper decrease of the

**Figure 8.** Profiles of $\log(\rho)$ (top) and $\log(\sigma^2)$ (bottom) for each component at maximum (dashed line) and minimum (solid line) expansion over times shown in Fig. 7. The heavy (light) curves refer to the more (less) massive component.

**Figure 9.** Contours of $\log(t_{cc}/t_{i,\text{rh}})$ as a function of $M_2/M_1$ and $m_2/m_1$.

**Figure 10.** Solid lines are $\log(t_{cc}/t_{i,\text{rh}})$ versus $\log(m_2/m_1)$: from top to bottom $M_2/M_1 = 1.0, 0.5, 0.4, 0.3, 0.2$ and $0.1$. Dashed lines are $\log(M_2/M_1)$ versus $\log(m_2/m_1)$ for various values of $k$. 

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core collapse time in their study for the case $M_2/M_{\text{L, 0.1}} = 0.1$, where $M_{\text{L, 0.1}}$ is the initial total cluster mass.

We can attempt to improve on these ideas at least qualitatively by considering in more detail a Spitzer unstable model. In that case, we can separate the pre-collapse evolution of the cluster into an initial mass-segregation-dominated stage and a later gravothermal collapse-dominated stage, in which the centrally concentrated heavy component behaves almost as a one-component system thermally detached from the lighter component. We propose that this separation can be located via the minimum of the rate of change of central density ratio $\left[\text{min} \left( \frac{d}{dt} \left( \rho_2 / \rho_1 \right) \right) \right]$. The reasoning behind this is as follows: as time passes, the increase in the density ratio caused by mass segregation starts to slow due to a combination of decreasing relative density and increasing temperature of the lighter component in the central regions. We assume that it is at this point that the gravothermal collapse of the heavier component becomes the dominant behaviour of the system. The gravothermal collapse in the heavy component increases the temperature of the heavy component, and because the light component absorbs energy from the heavy component, the collapse of the heavy component causes a deceleration in the collapse of the light component. This in turn enhances the rate of increase in the density ratio.

Fig. 11 shows the density ratio $\rho_2/\rho_1$ versus time for $N = 10000$ and $M_2/M_1 = 1, 0.1$. For the case of $m_2/m_1 = 2$ (the lowest curve) there is a clear distinction between the part before the point of inflection at about $t_{\text{exp}} = 5$ (i.e. the initial mass-segregation phase) and the part after the point of inflection (i.e. the gravothermal collapse phase). As $m_2/m_1$ increases, the initial phase dominated by mass segregation becomes more substantial and eventually the initial mass-segregation phase brings the system all the way to core bounce. However, as $N$ increases, binary energy generation becomes less efficient relative to the energy demands of the cluster (Goodman 1987). Therefore, the core needs to reach a higher density at core bounce for larger $N$. As the initial phase of mass segregation is self-limiting for the reason given above, mass segregation cannot increase the central density beyond a certain point. Therefore, it would be expected that the gravothermal collapse-dominated phase must eventually return with increasing $N$ for any given $M_2/M_1$ and $m_2/m_1$.

5 DIRECT N-BODY

Bettwieser & Sugimoto (1985) compared $N$-body systems to gaseous models using a direct $N = 1000$ model. Even though the value of $N$ is small by today’s standards there was still fair agreement during the pre-collapse phase. There were large statistical fluctuations in the post-collapse phase and this was most likely due to the small particle number. However, it is still important to confirm a sample of the results of the gas model by using a direct $N$-body code.

The case of $m_2/m_1 = 2$ and $M_2/M_1 = 1$ was chosen because it had the smallest value of $N_{\text{crit}}$. The values of $N$ used for these runs were 8$k$, 16$k$, 32$k$ and 64$k$. The collapse times of the runs in $N$-body units (see Heggie & Mathieu 1986) and units of $t_{\text{exp}}$ are given in Table 8. The average collapse time measured in units of $t_{\text{exp}}$ is about 7.5 which is lower than the predicted value of 8.95 given in Table 7. The difference in collapse time could be because of the approximate treatment of two-body relaxation in the gas model, the neglect of escape or parameter choices in the gas code (Section 2.1).

For the case of the runs with $N$ equal to 8$k$ and 16$k$, no behaviour was found which could be described as gravothermal oscillation. This is in agreement with the gas code, which gave $N_{\text{crit}} = 17000$. However, the 32$k$ case does show a cycle of expansion and contraction of the core over the time interval 4500–5500 $N$-body units (see Fig. 12). In order to check that the expansion was not driven by sustained binary energy generation, we consider the evolution of the relative binding energy $E_b/E$, where $E_b$ is the total binding energy of the binaries and $E$ is the absolute value of the total energy of the cluster over this time period. This is plotted in Fig. 13 along with the core radius. There are small changes in the binding energy of binaries over this period, decreases as well as increases, but this cannot fully account for the expansion phase that is observed, as there are other periods with similar binary activity in which no sustained expansion occurs. Also, the time-scale of the expansion is much longer than the relaxation time in the core (about 0.5 in $N$-body time units). Therefore, we assume that the expansion must be driven by phenomena outside the core, and gravothermal behaviour is a plausible explanation.

Several other pieces of evidence point to this conclusion. Fig. 14 shows the density in Lagrangian shells of the heavier component. As discussed in Section 3.3 (e.g. Fig. 6, top) the region further away

| Table 8. Collapse time $t_{\text{exp}}$. |
|---|---|---|---|---|
| $N$ | 8$k$ | 16$k$ | 32$k$ | 64$k$ |
| $N$-body units | 1160 | 1990 | 3480 | 6380 |
| $t_{\text{exp}}$ | 7.76 | 7.56 | 7.41 | 7.51 |

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Figure 11. $\log_{10}(\rho_2/\rho_1)$ versus time (in units of $t_{\text{exp}}$) for the case of $M_2/M_1 = 1$ (top) and $M_2/M_1 = 0.1$ (bottom) Curves from bottom to top are $m_2/m_1 = 2, 3, 4, 5, 10, 20$ and 50.
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Figure 12. $\log_{10} r_c$ versus time, where $r_c$ is defined as in \textsc{nbbody6}, $m_2/m_1 = 2, M_2/M_1 = 1$ and $N = 32k$.

Figure 13. 32k N-body results. Top: relative binding energy $E_b/E$ compared to $\log r_c$ over time 4500–5500 N-body units. Bottom: $\log r_c$ versus $\log v_c^2$ over the same time period, where • and ■ represent the starting and finishing points.

Figure 14. 32k N-body results. $\log \rho$ in Lagrangian shells of 1, 2, 5, 10, 20, 30, 40, 50, 62.5, 75 and 90 per cent mass in the heavier component. It can be seen that the collapse at 5400 starts further out (while the core is still expanding) and propagates towards the core.

Figure 15. Evolution of the central properties of the heavy component in a 32k gas model. Top: $\log \rho_{2,c}$ versus time (units $t_{\odot,he}$), bottom: $\log \rho_{2,c}$ versus $\log v_{2,c}^2$. All cycles are clockwise. The initial drop in $\log v_{2,c}^2$ results from the two components trying to achieve thermal equilibrium.
from the core is seen to contract while the core expands. Also, in the cycle of $\ln \rho_c$ versus the core velocity dispersion $\log v_c^2$, the temperature is low during the expansion where heat is absorbed and high during the collapse where heat is released (Fig. 13, bottom). This is similar to the cycles found by Makino (1996) for one-component models and is another sign of gravothermal behaviour. The results from the 32$^k$ gas run are shown in Fig. 15 for comparison.

The 64$^k$ run shown in Figs 16 and 17 has large amplitude oscillations. The results from the 64$^k$ gas run are shown in Fig. 18 for comparison. There is a part of the expansion which is shown in Fig. 17 between 7353 and 7390 in which the relative binding energy of binaries is nearly constant. Therefore binary activity cannot be what drives the expansion. Fig. 17 (bottom) shows the evolution of the profile of $\log v_c^2$ over part of the expansion. A negative temperature gradient is visible towards the end of this expansion and this is what drives the expansion. From the results of the 32$^k$ and 64$^k$ runs it seems that the value of $N_{\text{crit}} = 17\,000$ obtained by the gas code is a reasonable indicator of stability for the $N$-body case in the sense that none of the signs of gravothermal behaviour was found for $N \lesssim 16^k$.

### 6 CONCLUSIONS AND DISCUSSION

The main focus of this paper has been on the gravothermal oscillations of two-component systems. The critical value of $N$ for the onset of instability has been found for a range of stellar mass ratios and total mass ratios using a gas model. The case of $M_2/M_1 = 1$ and $m_2/m_1 = 2$ was further investigated using the direct $N$-body code $\text{NBody6}$. The value of $N_{\text{crit}}$ obtained from the gas code seems to be a good indicator for stability in $N$-body runs for this case. Based on this, it is a reasonable assumption that the other $N_{\text{crit}}$ values would give an indication of the stability for direct $N$-body systems. The values of $N_{\text{crit}}$ for the two-component model were found to be much higher than for the one-component case and were found to vary with $m_2/m_1$ and $M_2/M_1$. However, the value of $N_{\text{crit}}$ at the stability limit was found to vary much less than $N$ itself. This seems to suggest that instability depends on the properties of the heavy component (see Section 3.2). A possible explanation of this is given in Section 3.2.

The physical manifestation of the oscillations was investigated for the case of small-amplitude periodic oscillations in the gas model. It has been pointed out that the collapse of the region between $r_c$ and $r_h$ is an important mechanism which can halt the expansion phase of a gravothermal oscillation. This mechanism should also be present in one-component models and it would be an interesting topic for future work to see how this mechanism would behave with different stellar mass functions.

Kim et al. (1998) argued that two-component clusters may be realistic approximations of multicomponent clusters, where the two components are neutron stars and main-sequence stars, and the
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Effect of white dwarfs (heavier than the turnoff mass) was assumed to be negligible. They also studied cases that were only Spitzer stable, which means that the components were able to achieve equipartition of kinetic energy. For the two-component case, it is only possible for it to be Spitzer stable if there is only a small amount of the heavier component present. As there is a significant range of stellar masses in a real star cluster, it is likely that some form of the Spitzer instability will be present.

To apply our ideas to a multicomponent system, it may be possible to group the heavier components together if they are able to achieve approximate thermal equilibrium. This could be considered as a single heavier component which is Spitzer unstable with respect to the remaining components. This would help to reduce a multicomponent system to the two-component case studied in this paper.

Nevertheless, it is not clear quantitatively how the considerations of this research are to be applied to a multicomponent cluster. Furthermore, we have ignored many things such as primordial binaries, tidal fields and stellar evolution, and these are important in the evolution of a real star cluster. Further study is needed in order to understand the phenomenon that is gravothermal oscillation.

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