Abstract. In set theory without the Axiom of Choice (AC), we observe new relations of the following statements with weak choice principles.

- If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.
- If in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$ for any regular $\aleph_\alpha$.
- CS (Every partially ordered set without a maximal element has two disjoint cofinal subsets).
- CWF (Every partially ordered set has a cofinal well-founded subset).
- DT (Dilworth’s decomposition theorem for infinite p.o.sets of finite width).

We also study a graph homomorphism problem and a problem due to András Hajnal without AC. Further, we study a few statements restricted to linearly-ordered structures without AC.

1. Introduction

Firstly, the first author observes the following in ZFA (Zermelo-Fraenkel set theory with atoms).

(1) In Problem 15, Chapter 11 of [KT06], applying Zorn’s lemma, Komjáth and Totik proved the statement “Every partially ordered set without a maximal element has two disjoint cofinal subsets” (CS). In Theorem 3.26 of [THS16], Tachtsis, Howard and Saveliev proved that CS does not imply ‘there are no amorphous sets’ in ZFA. We observe that $\text{CS} \not\rightarrow AC_{\text{fin}}^\omega$ (the axiom of choice for countably infinite families of non-empty finite sets), $\text{CS} \not\rightarrow AC_n^\omega$ (‘Every infinite family of $n$-element sets has a partial choice function’) for every $2 \leq n < \omega$ and $\text{CS} \not\rightarrow LOKW_{4}^\omega$ (Every infinite linearly orderable family $\mathcal{A}$ of 4-element sets has a partial Kinna–Wegner selection function) in ZFA.

(2) In Problem 14, Chapter 11 of [KT06], applying the well-ordering theorem, Komjáth and Totik proved the statement “Every partially ordered set has a cofinal well-founded subset” (CWF). In Theorem 10(ii) of [Tacs17], Tachtsis proved that CWF holds in the basic Fraenkel model. Moreover, in Lemma 5 of [Tacs17], Tachtsis proved that CWF is equivalent to AC in ZF. We observe that $\text{CWF} \not\rightarrow AC_{\text{fin}}^\omega$, $\text{CWF} \not\rightarrow AC_n^\omega$ for every $2 \leq n < \omega$ and $\text{CWF} \not\rightarrow LOKW_{4}^\omega$ in ZFA.

(3) In Problem 7, Chapter 11 of [KT06], applying Zorn’s lemma, Komjáth and Totik proved that if in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable. We observe that ‘If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable’ $\not\rightarrow AC_{\text{fin}}^\omega$ ($\forall n \geq 2$), $\not\rightarrow \text{‘There are no amorphous sets’}$ in ZFA which are new results. Moreover, we prove that ‘For any regular $\aleph_\alpha$, if in a partially ordered set, all chains are finite and all..."
antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$, ' $\not\rightarrow AC^*_n (\forall n \geq 2)$, 'There are no amorphous sets' in ZFA.

(4) Dilworth [Di50] proved the following statement: 'If $\mathbb{P}$ is an arbitrary p.o. set, and $k$ is a natural number such that $\mathbb{P}$ has no antichains of size $k + 1$ while at least one $k$-element subset of $\mathbb{P}$ is an antichain, then $\mathbb{P}$ can be partitioned into $k$ chains', we abbreviate by DT (see Problem 4, Chapter 11 of [KT06] also). Tachtsis [Tac19] investigated the possible placement of DT in the hierarchy of weak choice principles. He proved that DT does not imply $AC^*_{\aleph_1}$ as well as $AC_2$ (Every family of pairs has a choice function).

We observe that DT does not imply $AC^*_n$ for any $2 \leq n < \omega$ in ZFA. In particular, we observe that DT holds in the permutation model of Theorem 8 of [HT19], due to Halbeisen and Tachtsis. We also observe that a weaker form of Loš’s lemma (Form 253 of [HR98]) fails in the permutation model of Theorem 8 of [HT19].

(5) In Theorem 4.5.2 of [Kom], Komjáth sketched the following generalization of the $n$-coloring theorem (For every graph $G = (V, E)$ if every finite subgraph of $G$ is $n$-colorable then $G$ is $n$-colorable) applying the Boolean prime ideal theorem (BPI): 'For an infinite graph $G = (V_G, E_G)$ and a finite graph $H = (V_H, E_H)$, if every finite subgraph of $G$ has a homomorphism into $H$, then so has $G$' we abbreviate by $\mathcal{P}_{G,H}$. We observe that if $X \in \{AC_3, AC^*_{\aleph_1}\}$, then $\mathcal{P}_{G,H}$ restricted to finite graph $H$ with 2 vertices does not imply $X$ in ZFA.

Secondly, we study a weaker formulation of a problem due to András Hajnal in ZFA.

(1) In Theorem 2 of [Haj85], Hajnal proved that if the chromatic number of a graph $G_1$ is finite (say $k < \omega$), and the chromatic number of another graph $G_2$ is infinite, then the chromatic number of $G_1 \times G_2$ is $k$ using the Gödel’s Compactness theorem. In the solution of Problem 12, Chapter 23 of [KT06], Komjáth provided another argument using the Ultrafilter lemma. For a natural number $k < \omega$, we denote by $\mathcal{P}_k$ the following statement.

'\[\chi(G_1) = k < \omega \text{ and } \chi(G_2) \geq \omega \text{ implies } \chi(G_1 \times G_2) = k.\]

We observe that if $X \in \{AC_3, AC^*_{\aleph_1}\}$, then $\mathcal{P}_k \not\rightarrow X$ in ZFA when $k = 3$.

Lastly, we study a few algebraic and graph-theoretic statements restricted to linearly-ordered structures without AC. We abbreviate the statement ‘The union of a well-orderable family of finite sets is well-orderable’ by $UT(WO, fin, WO)$. In Theorem 3.1 (i) of [Tac19], Tachtsis proved DT for well-ordered infinite p.o. sets with finite width in ZF applying the following theorem.

Theorem 1.1. (Theorem 1 of [Loe65]). Let $\{X_i\}_{i \in I}$ be a family of compact spaces which is indexed by a set $I$ on which there is a well-ordering $\leq$. If $I$ is an infinite set and there is a choice function $F$ on the collection $\{C : C$ is closed, $C \neq \emptyset, C \subset X_i$ for some $i \in I\}$, then the product space $\Pi_{i \in I} X_i$ is compact in the product topology.

Using the same technique from Theorem 3.1 of [Tac19], we prove a few algebraic and graph-theoretic statements restricted to well-ordered sets, either in ZF or in ZF + $UT(WO, fin, WO)$. Consequently, those statements restricted to linearly ordered sets are true, in permutation models where LW (Every linearly ordered set can be well-ordered) holds. In particular, we observe the following.

(1) In Theorem 18 of [HT13], Howard and Tachtsis obtained that for every finite field $\mathcal{F} = \{F, \ldots\}$, for every nontrivial vector space $V$ over $\mathcal{F}$, there exists a non-zero linear functional $f : V \rightarrow F$ applying BPI. Fix an arbitrary $2 \leq n < \omega$. We observe that ‘For every finite field $\mathcal{F} = \{F, \ldots\}$, for every nontrivial linearly-ordered vector space $V$ over $\mathcal{F}$, there exists a non-zero linear functional $f : V \rightarrow F$ ‘ $\not\rightarrow AC^*_{\aleph_n}$, ‘ $\not\rightarrow LOKW_4$, and ‘$\not\rightarrow AC^*_4$ in ZFA.

(2) Fix an arbitrary $2 \leq n < \omega$. We observe that ‘For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices $V_G$ and a finite graph $H = (V_H, E_H)$, if every finite
subgraph of $G$ has a homomorphism into $H$, then so has $G'$ \(\not\rightarrow AC\_\text{fin}, \not\rightarrow LOKW\_\alpha\), and
\(\not\rightarrow AC\_\alpha\) in ZFA.
(3) Fix an arbitrary 2 \(\leq n < \omega\). We prove that for every 3 \(\leq k < \omega\), the statement ‘$P_k$ if the graph $G_1$ is on some linearly-orderable set of vertices’ \(\not\rightarrow AC\_\text{fin}, \not\rightarrow LOKW\_\alpha\), and
\(\not\rightarrow AC\_\alpha\) in ZFA.
(4) Marshall Hall [Hal18] proved that if $S$ is a set and \(\{S_i\}_{i \in I}\) is an indexed family of finite subsets of $S$, then if the following property holds,

\[(P) \text{ for every finite } F \subseteq I, \text{ there is an injective choice function for } \{S_i\}_{i \in F}.\]

then there is an injective choice function for \(\{S_i\}_{i \in F}\). We abbreviate the above assertion by MHT. We recall that BPI implies MHT and MHT implies the Axiom of choice for finite sets (\(AC\_\text{fin}\)) in ZF (c.f. [HR98]). Fix an arbitrary 2 \(\leq n < \omega\). We prove that MHT restricted to a linearly-ordered collection of finite subsets of a set does not imply $AC\_\alpha$ in ZFA.

2. A List of Forms and Definitions

(1) The Axiom of Choice, $AC$ (Form 1 in [HR98]): Every family of nonempty sets has a choice function.
(2) The Axiom of Choice for Finite Sets, $AC\_n$ (Form 62 in [HR98]): Every family of non-empty nite sets has a choice function.
(3) $AC\_2$ (Form 88 in [HR98]): Every family of pairs has a choice function.
(4) $AC\_n$ for each $n \in \omega, n \geq 2$ (Form 61 in [HR98]): Every family of $n$ element sets has a choice function. We denote by $AC\_n$ the statement ‘Every infinite family of $n$-element sets has a partial choice function’ (Form $342(n)$ in [HR98], denoted by $C\_n$ in Definition 1 (2) of [HT19]). We denote by $LOKW\_n$ the statement ‘Every infinite linearly orderable family $A$ of $n$-element sets has a partial Kinna–Wegner selection function’ (c.f. Definition 1 (2) of [HT19]).
(5) $AC\_n$ (Form 10 in [HR98]): Every countably infinite family of non-empty nite sets has a choice function. We denote by $PAC\_\text{fin}$ the statement ‘Every countably infinite family of non-empty nite sets has a partial choice function’.
(6) The Principle of Dependent Choice, DC (Form 43 in [HR98]): If $S$ is a relation on a non-empty set $A$ and $(\forall x \in A)(\exists y \in A)(xSy)$ then there is a sequence $a_0, a_1, \ldots$ of elements of $A$ such that $(\forall n \in \omega)(a(n)Sa(n + 1))$.
(7) LW (Form 90 in [HR98]): Every linearly-ordered set can be well-ordered.
(8) UT(WO, WO, WO) (Form 231 in [HR98]): The union of a well-ordered collection of well-ordered sets is well-ordered.
(9) UT(WO, fin, WO) (Form 10N in [HR98]): The union of a well-ordered family of finite sets is well-ordered.
(10) $(\forall \alpha)UT(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$ (Form 23 in [HR98]): For every ordinal $\alpha$, if $A$ and every member of $A$ has cardinality $\aleph_\alpha$, then $| \cup A | = \aleph_\alpha$.
(11) UT($\aleph_\emptyset$, fin, $\aleph_\emptyset$) (Form 10A in [HR98]): The union of a denumerable collection of finite sets is countable.
(12) The Boolean Prime Ideal Theorem, BPI (Form 14 in [HR98]): Every Boolean algebra has a prime ideal. We recall the following equivalent formulations of BPI.
- Form 14AW in [HR98]: The Compactness theorem for propositional logic.
- The Ultrafilter lemma, UL (Form 14A in [HR98]): Every proper filter over a set $S$ in $\mathcal{P}(S)$ can be extended to an ultrafilter.
- The n-coloring theorem for $n \geq 3$, (Form 14G(n)$(n \in \omega, n \geq 3$ in [HR98]): For every graph $G = (V, E)$ if every finite subgraph of $G$ is $n$-colorable then $G$ is $n$-colorable. This is De Bruijn–Erdős theorem for $n \geq 3$ colorings.
(13) The Principle of consistent choice, PCC (Form 14AH in [HR98]): Let $\mathcal{A} = \{A\_i\}_{i \in I}$ be a family of finite sets and $\mathcal{R}$ is a symmetric binary relation on $\cup_{i \in I} A\_i$. Suppose that for every finite $W \subseteq I$, there is an $\mathcal{R}$-consistent choice function for $\{A\_i\}_{i \in W}$, then there is an $\mathcal{R}$-consistent choice function for $\{A\_i\}_{i \in I}$. 
We note that Form 14AH in [HR98] is different than the above formulation. Łoś/Ryll-Nardzewski [LN53] introduced both the formulations where it was noted that they are equivalent. Let $n \in \omega \setminus \{0, 1\}$. We recall the notation $F_n$ introduced by Cowen in [Cow77], which is PCC restricted to families $A = \{A_i : i \in I\}$, where $|A_i| \leq n$ for all $i \in I$.

(14) **Marshall Hall’s theorem, MHT** (Form 107 in [HR98]): If $S$ is a set and $\{S_i\}_{i \in I}$ is an indexed family of finite subsets of $S$, then if the following property holds,

(P) for every finite $F \subseteq I$, there is an injective choice function for $\{S_i\}_{i \in F}$, then there is an injective choice function for $\{S_i\}_{i \in I}$.
Philip Hall’s theorem states that the property $(P)$ is equivalent to the Hall’s condition which states that \( \forall F \in [I]\leq \omega, |\bigcup_{i \in F} S_i| \geq |F| \). We recall that Philip Hall’s theorem or finite Hall’s theorem can be proved in ZF without using any choice principles.

(15) **A weaker form of Łos’s lemma, LT (Form 253 in [HR98]):** If \( A = \langle A, R^A \rangle \) is a non-trivial relational \( \mathcal{L} \)-structure over some language \( \mathcal{L} \) and \( \mathcal{U} \) be an ultrafilter on a non-empty set \( I \), then the ultrapower \( A^I/\mathcal{U} \) and \( A \) are elementarily equivalent.

(16) **MCC (c.f. Definition 5 and Definition 6 of [Tac17]):** Every topological space with the minimal cover property is compact.

(17) **Bounded and unbounded amorphous sets:** An infinite set \( X \) is called amorphous if \( X \) cannot be written as a disjoint union of two infinite subsets. There are two types of amorphous sets, namely bounded amorphous sets and unbounded amorphous sets. Let \( \mathcal{U} \) be a finitary partition of an amorphous set \( X \). Then all but finitely many elements of \( \mathcal{U} \) have the same cardinality, say \( n(\mathcal{U}) \). Let \( \Pi(X) \) be the set of all finitary partitions of \( X \) and \( n(X) = \sup\{n(\mathcal{U}) : \mathcal{U} \in \Pi(X)\} \). If \( n(X) \) is finite, then \( X \) is called bounded amorphous and if \( n(X) \) is infinite, then \( X \) is called unbounded amorphous. We recall **Theorem 6 of [Tac17]** which states that MCC \( \rightarrow \) “there are no bounded amorphous sets”.

(18) **(Form 64 in [HR98]):** There are no amorphous sets.

(19) **Martin’s Axiom (c.f. [Tac16b]):** If \( \kappa \) is a well-ordered cardinal, we denote by \( MA(\kappa) \) the principle ‘If \( (P, \prec) \) is a nonempty, c.c.c. quasi order and \( \mathcal{D} \) is a family of \( \leq \kappa \) dense sets in \( P \), then there is a filter \( \mathcal{F} \) of \( P \) such that \( \mathcal{F} \cap D \neq \emptyset \) for all \( D \in \mathcal{D} \)’. We recall from **Remark 2.7 of [Tac16b]**: Every partially ordered set without a maximal element has two disjoint cofinal subsets. We abbreviate the above formulation as DT. We recall **Theorem 3.1(i) of [Tac19]**, which states that DT for well-ordered infinite p.o. sets with finite width is provable in ZF.

(21) **The Chain/Antichain Principle, CAC (Form 217 in [HR98]):** Every infinite p.o. set has an infinite chain or an infinite antichain. We recall that CAC implies \( AC_{fin}^{\omega} \) from **Lemma 4.4 of [Tac15a]**.

(22) **CS (c.f. [THS16]):** Every partially ordered set without a maximal element has two disjoint cofinal subsets.

(23) **CWF (c.f. Definition 6 (11) of [Tac17]):** Every partially ordered set has a cofinal well-founded subset.

(24) **Chromatic number of the product of graphs:** We recall a few basic terminologies of graphs. An **independent set** is a set of vertices in a graph, no two of which are connected by an edge. A **good coloring** of a graph \( G = (V_G, E_G) \) with a color set \( C \) is a mapping \( f : V_G \rightarrow C \) such that for every \( \{x, y\} \in E_G \), \( f(x) \neq f(y) \). The **chromatic number** \( \chi(G) \) of a graph \( G = (V_G, E_G) \) is the smallest cardinal \( \kappa \) such that the graph \( G \) can be colored by \( \kappa \) colors. We define the cartesian product of two graphs \( G_1 = (V_{G_1}, E_{G_1}) \) and \( G_2 = (V_{G_2}, E_{G_2}) \) as the graph \( G_1 \times G_2 = (V_{G_1 \times G_2}, E_{G_1 \times G_2}) = (V_{G_1} \times V_{G_2}, \{(x_0, y_0), (y_0, y_1), \ldots \} : \{x_0, y_0\} \in E_{G_1}, \{x_1, y_1\} \in E_{G_2}\} \) where \( V_{G_1} \times V_{G_2} \) is the cartesian product of the vertex sets \( V_{G_1} \) and \( V_{G_2} \). It can be seen that \( \chi(G_1 \times G_2) \leq \min(\chi(G_1), \chi(G_2)) \). In particular, if \( \chi(G_{E_1}) = k < \omega \) then \( \chi(G_{E_1 \times G_2}) = k \), since if \( f : V_{G_1} \rightarrow \{1, \ldots, k\} \) is a good \( k \)-coloring of \( G_1 \), then \( f(\langle x, y \rangle) = f(x) \) is a good \( k \)-coloring of \( G_1 \times G_2 \). In **Theorem 2 of [Haj85]**, Hajnal proved that if \( \chi(G_{E_1}) \) is finite (say \( k < \omega \)), and \( \chi(G_{E_2}) \) is infinite, then \( \chi(G_{E_1 \times G_2}) = k \).

2.1. **Permutation models.** Let \( M \) be a model of ZFA + AC where \( A \) is a set of atoms or ur-elements. Each permutation \( \pi : A \rightarrow A \) extends uniquely to a permutation of \( \pi' : M \rightarrow M \) by \( \epsilon \)-induction. Let \( G \) be a group of permutations of \( A \) and \( \mathcal{F} \) be a normal filter of subgroups of \( G \).
For \( x \in M \), we denote the symmetric group with respect to \( \mathcal{G} \) by \( \text{sym}_\mathcal{G}(x) = \{ g \in \mathcal{G} \mid g(x) = x \} \).

We say \( x \) is \( \mathcal{F} \)-symmetric if \( \text{sym}_\mathcal{G}(x) \in \mathcal{F} \) and \( x \) is hereditarily \( \mathcal{F} \)-symmetric if \( x \) is \( \mathcal{F} \)-symmetric and each element of transitive closure of \( x \) is symmetric. We define the permutation model \( \mathcal{N} \) with respect to \( \mathcal{G} \) and \( \mathcal{F} \), to be the class of all hereditarily \( \mathcal{F} \)-symmetric sets. It is well-known that \( \mathcal{N} \) is a model of ZFA (see Theorem 4.1 of [Kec73]). If \( \mathcal{I} \subseteq \mathcal{P}(A) \) is a normal ideal, then the set \( \{ \text{fix}_\mathcal{G} E : E \in \mathcal{I} \} \) generates a normal filter over \( \mathcal{G} \). Let \( \mathcal{I} \) be a normal ideal generating a normal filter \( \mathcal{F}_\mathcal{I} \) over \( \mathcal{G} \). Let \( \mathcal{N} \) be the permutation model determined by \( \mathcal{M}, \mathcal{G}, \) and \( \mathcal{F}_\mathcal{I} \). We say \( E \in \mathcal{I} \) supports a set \( \sigma \in \mathcal{N} \) if \( \text{fix}_\mathcal{G} E \subseteq \text{sym}_\mathcal{G}(\sigma) \).

3. Well-ordered structures in ZF

3.1. Applications of Loeb’s theorem. We recall the following fact from [Ker00].

**Lemma 3.1.** (ZF). If \( X \) is well-orderable, then \( 2^X \) is compact.

**Remark.** We can also prove Lemma 3.1 applying Theorem 1 of [Loe65].

**Observation 3.2.** \( \text{UT}(WO, fin, WO) \) implies Marshall Hall’s theorem for any well-ordered collection of finite subsets of a set.

**Proof.** Let \( S \) be a set and \( \{ S_i \}_{i \in I} \) be a well-ordered indexed family of finite subsets of \( S \) such that the following property holds,

\[
(P) \text{ for every finite } F \subseteq I, \text{ there is an injective choice function for } \{ S_i \}_{i \in F}. \]

We work with the propositional language \( \mathcal{L} \) with the following sentence symbols.

\[
A_{i,j} \text{ where } j \in S_i \text{ and } i \in I. 
\]

Let \( \mathcal{F} \) be the set of all formulae of \( \mathcal{L} \) and \( \Sigma \subseteq \mathcal{F} \) be the collection of the following formulae.

\[
\begin{align*}
(1) \quad & \neg(A'_{i,m} \land A'_{j,m}) \text{ for } i \neq j, m \in S_i \cap S_j, \\
(2) \quad & \neg(A'_{i,j} \land A'_{l,j}) \text{ for any } l \neq j \in S_i \text{ where } i \in I, \\
(3) \quad & A'_{i,y_1} \lor A'_{i,y_2} \lor \ldots \lor A'_{i,y_k} \text{ for each } i \in I \text{ where } S_i = \{ y_1, \ldots, y_k \}. 
\end{align*}
\]

We enumerate \( \text{Var} = \{ A'_{i,j} : i \in I, j \in S_i \} \) since each \( S_i \) is finite, \( I \) is well-orderable and \( \text{UT}(WO, fin, WO) \) is assumed. For every \( W \in [I]^{<\omega} \setminus \{ \} \), we let \( \Sigma_W \) be the subset of \( \mathcal{F} \), which is defined as \( \Sigma \) except that the subscripts in the formulae are from the set \( W \cup \bigcup_{i \in W} S_i \). Endow the discrete 2-element space \( \{ 0, 1 \} \) with the discrete topology and consider the product space \( 2^{\text{var}} \) with the product topology. Let \( F_W = \{ f \in 2^{\text{var}} : \forall \phi \in \Sigma_W (f'(\phi) = 1) \} \) where for \( f \in 2^{\text{var}} \), the element \( f' \) of \( 2^X \) denotes the valuation mapping determined by \( f \). By Philip Hall’s theorem which is provable in ZF without using any choice principles, each \( F_W \) is non-empty and the family \( \mathcal{X} = \{ F_W : W \in [I]^{<\omega} \setminus \{ \} \} \) has the finite intersection property. Also for each \( W \in [I]^{<\omega} \setminus \{ \} \), \( F_W \) is closed in the topological space \( 2^{\text{var}} \). By Lemma 3.1 since \( 2^{\text{var}} \) is compact in ZF, \( \cap \mathcal{X} \) is non-empty. Pick an \( f \in \cap \mathcal{X} \) and let \( f' \in 2^X \) be the unique valuation mapping that extends \( f \). Clearly, \( f'(\phi) = 1 \) for all \( \phi \in \Sigma \). Consequently, we obtain an injective choice function for \( \{ S_i \}_{i \in I} \) by the following claim.

**Claim 3.3.** If \( v \) is a truth assignment which satisfies \( \Sigma \), then we can define a system of distinct representatives by

\[ y \in S_i \text{ if and only if } v(A'_{i,y}) = T. \]

**Proof.** By (2) and (3) for each \( i \in I \), each collection \( S_i \) gets assigned a unique representative. By (1), distinct sets \( S_i \) and \( S_j \) gets assigned distinct representatives. \( \square \)

\( \square \)
Banachewski [Bana92] proved the uniqueness of the algebraic closure of an arbitrary field applying BPI[1].

**Observation 3.4.** UT(WO, fin, WO) implies ‘If a field $K$ has an algebraic closure, and the ring of polynomials $K[x]$ is well-orderable, then the algebraic closure is unique’.

**Proof.** Let $K$ be a field, and suppose $E$ and $F$ be two algebraic closures of $K$. We prove that there is an isomorphism from $E$ onto $F$ which fixes $K$ pointwise. Let $E_u$ and $F_u$ be the splitting fields of $u \in K[x]$ inside $E$ and $F$ respectively. Let $H_u$ be the set of all isomorphisms from $E_u$ onto $F_u$ which fix $K$. Clearly, $H_u$ is a non-empty, finite set. Also, we can see that $\bigcup_u E_u = E$ and $\bigcup_u F_u = F$. Let $H = \Pi_{u \in K[x]} H_u$, and if $v \mid w$ define $H_{v,w} = \{ (h_u) \in H : h_v = h_w \upharpoonright E_v \}$. Clearly, $H_{v,w}$ has finite intersection property and they are closed in the product topology of $H$, where each $H_u$ is discrete. Since $K[x]$ is well-orderable as assumed and for each $u \in K[x]$, $H_u$ is finite, we have that $\bigcap_{u \in K[x]} H_u$ is well-orderable by UT(WO, fin, WO). By Theorem 1 of [Loeb65], $H$ is compact. Consequently, $\bigcap_{v \mid w} H_{v,w} \neq \emptyset$ and each $(h_u)$ in this intersection determines a unique embedding $h : \bigcup_u E_u \to \bigcup_u F_u$ which is onto and fixes $K$.

**Observation 3.5.** The statement ‘For an infinite graph $G = (V_G, E_G)$ on a well-ordered set of vertices $V_G$ and a finite graph $H = (V_H, E_H)$, if every finite subgraph of $G$ has a homomorphism into $H$, then so has $G$’ is provable in ZF.

**Proof.** Fix a finite graph $H = (V_H, E_H)$ and a graph $G = (V_G, E_G)$ on a well-ordered set of vertices $V_G$. We consider $V_H = \{ v_1, \ldots, v_k \}$ for some $k < \omega$. We work with the propositional language $L$ with the following sentence symbols.

$$A'_{x_i, v_j} \text{ where } v_j \in V_H \text{ and } x_i \in V_G.$$  

Let $\mathcal{F}$ be the set of all formulae of $L$ and $\Sigma \subset \mathcal{F}$ be the collection of the following formulae.

1. $A'_{x_i, v_m} \land A'_{x_j, v_l}$ if and only if $\{x_i, x_j\} \in E_G$ implies $(v_m, v_l) \in E_H$.
2. $\neg (A'_{x_i, v_j} \land A'_{x_i, v_k})$ for any $v_i, v_j, v_k \in V_H$ such that $v_i \neq v_j$ and each $x_i \in V_G$.
3. $A'_{x_i, v_j} \lor A'_{x_i, v_j} \lor \ldots \lor A'_{x_i, v_k}$ for each $x_i \in V_G$.

By our assumption $V_G$ is well-orderable and $V_H$ is finite. So $V_H$ is well-orderable. Consequently, $V_G \times V_H$ is well-orderable in ZF. We enumerate $\text{Var} = \{ A'_{x_i, v_j} : x_i \in V_G, v_j \in V_H \}$. By assumption, for every $s \in |G|^\omega$ there is a homomorphism $f_s : G \upharpoonright s \to H$ of $G \upharpoonright s$ into $H$. Following the methods used in the proof of Observation 3.2, we may obtain a $f' \in 2^F$ such that $f'(\phi) = 1$ for all $\phi \in \Sigma$. Consequently, we can obtain a homomorphism $h$ from $G$ to $H$.

**Observation 3.6.** The statement ‘For every finite field $F = \{ f_i \}$, for every nontrivial well-ordered vector space $V$ over $F$, there exists a non-zero linear functional $f : V \to F^*$ is provable in ZF.

**Proof.** We follow the proof of Theorem 18 of [HT13] and modify it in the context of well-ordered vector space. Fix a finite field $F = \{ f_i \}$ where $F = \{ v_1, \ldots, v_k \}$ and a nontrivial well-ordered vector space $V$ over $F$. We work with the propositional language $L$ with the following sentence symbols.

$$A'_{x_i, v_j} \text{ where } v_j \in F \text{ and } x_i \in V.$$  

Let $\mathcal{F}'$ be the set of all formulae of $L$ and $\Sigma \subset \mathcal{F}'$ be the collection of the following formulae.

1. $A'_{x_i, v_j}$.
2. $A'_{x_i, v_j} \to A'_{v_k, x_i, v_k, v_j}$ for $v_k, v_j \in F$ and $x_i \in V$.
3. $A'_{x_i, v_j} \land A'_{x_i, v_j} \land A'_{x_i, v_j} \land A'_{x_i, v_j}$ for $x_i, x'_i \in V$ and $v_j, v'_j \in F$.

\[\text{c.f. the last paragraph of page 384 and page 385 of [Bana92].}\]
(4) \( \neg (A'_{x_i, v_j} \land A'_{x_i, v_k}) \) for any \( v_i, v_j \in F \) such that \( v_i \neq v_j \) and each \( x_i \in V \).
(5) \( A'_{x_i, v_1} \lor A'_{x_i, v_2} \ldots \lor A'_{x_i, v_k} \) for each \( x_i \in V \).

By our assumption \( V \) is well-orderable and \( F \) is finite. So \( F \) is well-orderable. Consequently, \( V \times F \) is well-orderable. We enumerate \( \text{Var} = \{ A'_{x_i, v_j} : x_i \in V, v_j \in F \} \). Fix \( V' \subseteq [V]^{<\omega} \). Let \( W \) be the subspace of \( V \) generated by the finite set \( V' \cup \{ a \} \). We can see that \( W \) is finite since \( F \) is finite. Consequently, a linear functional \( f : W \to F \) with \( f(a) = 1 \) can be constructed in ZF.

Following the methods used in the proof of Observation 3.2, we can obtain a non-zero linear functional \( f : V \to F \).

\[ \square \]

Observation 3.7. For every \( 3 \leq k < \omega \), the statement \( \mathcal{P}_k \) for the graph \( G_1 \) on some well-orderable set of vertices is provable under ZF.

**Proof.** Fix \( 3 \leq k < \omega \). Suppose \( \chi(G_1) = k \), \( \chi(G_2) \geq \omega \) and \( G_1 \) is a graph on some well-orderable set of vertices. First we observe that if \( g : V_{G_1} \to \{ 1, \ldots, k \} \) is a good \( k \)-coloring of \( G_1 \), then \( G(\langle x, y \rangle) = g(x) \) is a good \( k \)-coloring of \( G_1 \times G_2 \). So, \( \chi(G_1 \times G_2) \leq k \). For the sake of contradiction assume that \( F : V_{G_1} \times V_{G_2} \to \{ 1, \ldots, k - 1 \} \) is a good coloring of \( G_1 \times G_2 \). For each color \( c \in \{ 1, \ldots, k - 1 \} \) and each vertex \( x \in V_{G_1} \) we let \( A_{x,c} = \{ y \in V_{G_2} : F(x, y) = c \} \).

**Claim 3.8.** (ZF). For all finite \( F \subseteq V_{G_1} \), there exists a mapping \( i_F : F \to \{ 1, \ldots, k - 1 \} \) such that for any \( x, x' \in F \), \( A_{x,i_F(x)} \cap A_{x',i_F(x')} \) is not independent.

**Proof.** Since any superset of non-independent set is non-independent, it is enough to show that for all finite \( F \subseteq V_{G_1} \), there exists an \( i_F : F \to \{ 1, \ldots, k - 1 \} \) such that \( \cap_{x \in F} A_{x,i_F(x)} \) is not independent. For the sake of contradiction assume that there exist a finite \( F \subseteq V_{G_1} \) such that for all \( i_F : F \to \{ 1, \ldots, k - 1 \} \), \( \cap_{x \in F} A_{x,i_F(x)} \) is independent. Now, \( V_{G_2} = \cup_{i_F : F \to \{ 1, \ldots, k - 1 \}} \cap_{x \in F} A_{x,i_F(x)} \). Thus \( V_{G_2} \) can be written as a finite union of independent sets which contradicts the fact that \( \chi(G_2) = \omega \) is infinite. Thus for all finite \( F \subseteq V_{G_1} \), we can obtain a mapping \( i_F : F \to \{ 1, \ldots, k - 1 \} \) such that \( \cap_{x \in F} A_{x,i_F(x)} \) is not independent.

\[ \square \]

**Figure 2.** A map \( f : V_{G_1} \to \{ 1, \ldots, k - 1 \} \) such that intersection of any two elements in \( \{ A_{x,i_F(x)} : x \in V_{G_1} \} \) is not independent.

Endow \( \{ 1, 2, \ldots, k - 1 \} \) with the discrete topology. Since \( V_{G_1} \) is well-orderable, \( \{ 1, 2, \ldots, k - 1 \} \times V_{G_1} \) is well-orderable under ZF. Applying Theorem 1 of [Loe65], \( \{ 1, 2, \ldots, k - 1 \}^{V_{G_1}} \) is compact. For \( s \in [V_{G_1}]^{<\omega} \), define \( F_s = \{ f \in \{ 1, 2, \ldots, k - 1 \}^{V_{G_1}} : x, y \in s, x \neq y \to A_{x,f(x)} \cap A_{y,f(y)} \) is not independent \}. By Claim 3.8, for each \( s \in [V_{G_1}]^{<\omega} \) we have that \( F_s \) is non-empty. We can see that \( F_s : s \in [V_{G_1}]^{<\omega} \) has finite intersection property as \( F_{s_1 \cup \ldots \cup s_k} \subseteq F_{s_1} \cap \ldots \cap F_{s_k} \). Thus by compactness of \( \{ 1, 2, \ldots, k - 1 \}^{V_{G_1}} \), there is a \( f \in \cap \{ F_s : s \in [V_{G_1}]^{<\omega} \} \). Clearly, for any \( x, x' \in V_{G_1} \), \( A_{x,f(x)} \cap A_{x',f(x')} \) is not independent (see Figure 2). Since \( x \to f(x) \) is not a good coloring in \( G_1 \) as \( \chi(G_1) = k \), there are \( x, x' \in V_{G_1} \) with \( f(x) = f(x') = j \) and \( \{ x, x' \} \in E_{G_1} \).
Consequently, $A' = A_{x,f(x)} \cap A_{x',f(x')}$ is not independent. Pick $y, y' \in A'$ joined by an edge in $E_G$. Then $(x, y)$ and $(x', y')$ are joined in $E_{G_1 \times G_2}$ and get the same color $j$ which is a contradiction to the fact that $F$ is a good coloring of $G_1 \times G_2$. \hfill \Box

3.2. On partially ordered sets based on a well-ordered set of elements. The first author modifies the arguments from Claim 5 of Tac16 and observes the following.

Observation 3.9. The following holds.

1. $UT(\aleph_0, \aleph_0, \aleph_0)$ implies ‘If in a partially ordered set based on a well-ordered set of elements, all chains are finite and all antichains are countable, then the set is countable’.

2. $UT(\aleph_0, \aleph_0, \aleph_0)$ implies ‘If in a partially ordered set based on a well-ordered set of elements, all chains are finite and all antichains have size $\aleph_0$, then the set has size $\aleph_0$’ for any regular $\aleph_0$.

Proof. We prove Observation 3.9 (1). For the sake of contradiction, assume that all antichains of $(P, \leq)$ are countable, all chains of $(P, \leq)$ are finite, but the set $P$ is uncountable and well-ordered. We construct an infinite chain in $(P, \leq)$ using $UT(\aleph_0, \aleph_0, \aleph_0)$ and obtain the desired contradiction.

claim 3.10. $\leq$ is a well-founded relation on $P$ i.e., every non-empty subset of $P$ has a $\leq$-minimal element.

Proof. Let $P$ is well-orderable, say by $\leq$. We claim that $\leq$ is a well-founded relation on $P$. Otherwise, there is a non-empty subset $P_1 \subseteq P$ with no minimal elements. Consequently, using the fact that $\leq$ is a well-ordering in $P$, we can obtain a strictly $\leq$-decreasing sequence of elements of $P_1$. This contradicts the assumption that $P$ has no infinite chains. \hfill \Box

Without loss of generality we may assume $P = \cup\{P_\alpha : \alpha < \kappa\}$ where $\kappa$ is a well-ordered cardinal, $P_0$ is the set of minimal elements of $P$ and for each $\alpha < \kappa$, $P_\alpha$ is the set of minimal elements of $P\setminus \cup\{P_\beta : \beta < \alpha\}$. For each $\alpha < \kappa$, $P_\alpha$ is countable since $P_\alpha$ is an antichain.

- We note that $P = \cup\{P_\alpha : p \in P_\alpha\}$ where $P_\alpha = \{q \in P : p \leq q\}$. Since $P$ is uncountable and $P_0$ is countable, $P_p$ is uncountable for some $p \in P_0$. Otherwise for all $p \in P_0$, $P_p$ is either countable or finite and $UT(\aleph_0, \aleph_0, \aleph_0) + UT(\aleph_0, fin, \aleph_0)$ implies $P$ is countable which is a contradiction. Now $UT(\aleph_0, \aleph_0, \aleph_0)$ implies $UT(\aleph_0, fin, \aleph_0)$ in ZF, thus $UT(\aleph_0, \aleph_0, \aleph_0)$ suffices. Since $\{q \in P_0 : P_\alpha$ is uncountable$\}$ is a non-empty subset of $P$, we can find a least $p_0 \in P_0$ with respect to $\leq$ such that $P_{p_0}$ is uncountable.

- Let us consider $P' = P_{p_0}\setminus\{p_0\}$. So, $P'$ is uncountable. Again if $P'_1$ is the set of minimal elements of $P'$, we can write $P' = \cup\{P_p : p \in P'_1\}$ where $P_p = \{q \in P : p \leq q\}$. Since $P'$ is uncountable and $P'_1$ is countable (since all antichains of $(P, \leq)$ are countable by assumption), once again applying $UT(\aleph_0, \aleph_0, \aleph_0)$ as in the previous paragraph, $P_p$ is uncountable for some $p \in P'_1$. Since $\{q \in P'_1 : P_q$ is uncountable$\}$ is a non-empty subset of $P$, we can find a least $p_1 \in P'_1$ with respect to $\leq$ such that $P_{p_1}$ is uncountable. We can see that $p_0 < p_1$.

Continuing this process step by step we obtain a sequence $\langle p_n : n \in \omega\rangle$ of elements of $P$ such that $p_n < p_{n+1}$ for each $n \in \omega$. Consequently, we obtain an infinite chain.

Remark. Similarly for any regular $\aleph_0$, assuming $UT(\aleph_0, \aleph_0, \aleph_0)$ we can prove the following since alephs are well-ordered. ‘If in a partially ordered set based on well-ordered set of elements, all chains are finite and all antichains have size $\aleph_0$, then the set has size $\aleph_0$’. Consequently, we can prove Observation 3.9(2). \hfill \Box

4. Consistency results

Theorem 4.1. For every natural number $n \geq 2$, there is a permutation model $\mathcal{N}$ of ZFA where CAC holds and $AC^n_\kappa$ fails. Moreover, we can observe the following in the model.
Proof. Recall the model constructed in the proof of Theorem 8 of [HT19], Halbeisen and Tachtsis constructed a permutation model $\mathcal{N}$ where for arbitrary $n \geq 2$, $\text{AC}^{-\alpha}_n$ fails but CAC holds. We fix an arbitrary integer $n \geq 2$ and recall the model constructed in the proof of Theorem 8 of [HT19] as follows.

- **Defining the ground model $M$.** We start with a ground model $M$ of ZFA + $\text{AC}$ where $A$ is a countably infinite set of atoms written as a disjoint union $\bigcup \{ A_i : i \in \omega \}$ where for each $i \in \omega$, $A_i = \{ a_{i1}, a_{i2}, ..., a_{in} \}$.

- **Defining the group $G$ of permutations and the filter $F$ of subgroups of $G$.**
  - Defining $G$. $G$ is defined in [HT19] in a way so that if $\eta \in G$, then $\eta$ only moves finitely many atoms and for all $i \in \omega$, $\eta(A_i) = A_k$ for some $k \in \omega$.
  - Defining $F$. Let $F$ be the filter generated by $\{ \text{fix}_G(E) : E \in [A]^{<\omega} \}$.

- **Defining the permutation model.** Consider the permutation model $\mathcal{N}$ determined by $M$, $G$ and $F$.

Following point 1 in the proof of Theorem 8 of [HT19], both $A$ and $A = \{ A_i \}_{i \in \omega}$ are amorphous in $\mathcal{N}$ and no infinite subfamily $\mathcal{B}$ of $A$ has a Kimna–Wegner selection function. Consequently, $\text{AC}^{-\alpha}_\omega$ fails. The first author observes the following.

**Lemma 4.2.** In $\mathcal{N}$, DT, CS as well as CWF holds. Moreover the following holds in $\mathcal{N}$.

- If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.
- If in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$ for any regular $\aleph_\alpha$.

Proof. We follow the steps below.

1. If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable. Moreover, if in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$ for any regular $\aleph_\alpha$.

2. Since $\eta \in G$, then $\eta$ only moves finitely many atoms. $\text{Orb}_E(p)$ is an antichain in $P$ for each $p \in P$. Otherwise there is a $p \in P$, such that $\text{Orb}_E(p)$ is not an antichain in $(P, \leq)$. Thus, for some $\phi, \psi \in \text{fix}_G(E)$, $\phi(p)$ and $\psi(p)$ are comparable. Without loss of
generality we may assume $\phi(p) < \psi(p)$. Since if $\eta \in G$, then $\eta$ only moves finitely many atoms, there exists some $k < \omega$ such that $\phi^k = 1_A$. Let $\pi = \psi^{-1}\phi$. Consequently, $\pi(p) < p$ and $\pi^k = 1_A$ for some $k \in \omega$. Thus, $p = \pi^k(p) < \pi^{k-1}(p) < ... < \pi(p) < p$. By transitivity of $<, p < p$, which is a contradiction.

(3) We prove that in $N$, DT holds. Let $E \subseteq A$ be a finite support of an infinite p.o.set $E = (P,<)$ with finite width. Then $P = \bigcup \{ \text{Orb}_E(p) : p \in P \}$. Following (2), $\text{Orb}_E(p)$ is an antichain in $P$. Consequently, $\text{Orb}_E(p)$ is finite for each $p \in P$ since the width of $P$ is finite. Following (1), $\{ \text{Orb}_E(p) : p \in P \}$ is well-orderable in $N$. Following point 4 in the proof of Theorem 8 of [HTT19] and Lemma 3 of [Tac16], $\text{UT}(WO,WO,WO)$ holds in $N$, and so $P$ is well-orderable in $N$. Applying Theorem 3.1(i) of [Tac19], DT holds in $N$.

(4) To see that CS as well as CWF holds in $N$ we follow Theorem 3.26 of [THS16] and Theorem 10(ii) of [Tac17] respectively. We sketch the important steps below.

(a) We follow Theorem 3.26 of [THS16] to see that CS holds in $N$ as follows. Let $(P,\leq)$ be a poset without maximal elements supported by $E$. Following (1), $O = \{ \text{Orb}_E(p) : p \in P \}$ is a well-ordered partition of $P$. Define $\leq_0$ on $O$, as $X \leq_0 Y \iff \exists x \in X, \exists y \in Y$ such that $x \leq y$. Since $(P,\leq)$ has no maximal element, $(O,\leq_0)$ has no maximal element following (2). Since $O$ is well-ordered there exists a partition $U_P = \{ \mathcal{Q}, \mathcal{R} \}$ of $O$ in 2 cofinal subsets. Consequently, $U_P = \{ \mathcal{Q}, \mathcal{R} \}$ is a partition of $P$ in 2 cofinal subsets.

(b) We follow Theorem 10 (ii) of [Tac17] to see that CWF holds in $N$ as follows. Let $(P,\leq)$ be a poset supported by $N$. Since $O = \{ \text{Orb}_E(p) : p \in P \}$ is well-orderable, it has a cofinal well-founded subset $W = \{ W_\alpha : \alpha < \gamma \}$ such that for $\beta < \alpha$, $W_\alpha \not\subseteq W_\beta$ for all $\beta, \alpha < \gamma$. Consequently, $C = \cup W$ is a cofinal well-founded subset of $P$.

(5) We show the following in $N$.

‘If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.’

It is known that in every FM-model $UT(WO,WO,WO)$ implies $(\forall\alpha)UT(\aleph_\alpha,\aleph_\alpha,\aleph_\alpha)$ (c.f. page 176 of [HR98]). Consequently, $UT(\aleph_0,\aleph_0,\aleph_0)$ holds in $N$. Let $(P,<)$ be an uncountable p.o.set in $N$ where all antichains are countable and $E \in [A]^{<\omega}$ be a support of $(P,<)$. Following (1), $O = \{ \text{Orb}_E(p) : p \in P \}$ is a well-ordered partition of $P$ since for all $p \in P$, $E$ is a support of $\text{Orb}_E(p)$.

Following (2), $\text{Orb}_E(p)$ is an antichain and hence countable. Consequently, $\text{Orb}_E(p)$ is well-orderable. Since $UT(WO,WO,WO)$ holds in $N$, $P$ is well-orderable. By Observation 3.9(1), since $UT(\aleph_0,\aleph_0,\aleph_0)$ holds in $N$, there is an infinite chain in $N$.

(6) Following (5) and Observation 3.9(2), we can prove the following in $N$.

‘If in a partially ordered set $(P,<)$, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$’.

Remark. The referee pointed out that the statements If in a partially ordered set based on a well-ordered set of all chains are finite and all antichains are countable then the set is countable and If in a partially ordered set based on a well-ordered set of elements all chains are finite and all antichains have size $\aleph_\alpha$ then the set has size $\aleph_\alpha$ are true in all Fraenkel-Mostowski permutation models. So Observation 3.9(1) and Observation 3.9(2) are not needed in the proofs of parts (5) and (6) of the proof of Lemma 4.2.

Lemma 4.3. In $N$, $MA(\aleph_0)$ fails.

Proof. Since $A$ is amorphous, the statement ‘for all infinite $X$, $2^X$ is Baire’ is false following Remark 2.7 of [Tac16b]. Since CAC holds in $N$, $AC^*_\text{fin}$ holds as well (c.f. Lemma 4.4 of [Tac19a]). Consequently, $MA(\aleph_0)$ fails following Remark 2.7 of [Tac16b].

Lemma 4.4. In $N$, MCC fails.
Proof. Modifying the proof of Theorem 8 (ii) of [Tac17], we can see that $n(A) = n$. Thus there is a bounded amorphous set $A$. Consequently, MCC fails by Theorem 6 of [Tac17]. □

Lemma 4.5. In $\mathcal{N}$, LT fails.

Proof. Since $\mathcal{A}$ is an amorphous set of non-empty sets which has no choice function in $\mathcal{N}$, following Lemma 4.1(i) [Tac19a], LT fails in $\mathcal{N}$.

Lemma 4.6. In $\mathcal{N}$, the following statements hold for linearly-ordered structures.

1. Marshall Hall’s theorem for linearly-ordered collection of finite subsets of a set.
2. For every $3 \leq k < \omega$, $\mathcal{P}_k$ holds for any graph $G_1$ on some linearly-orderable set of vertices.
3. For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices $V_G$ and a finite graph $H = (V_H, E_H)$, if every finite subgraph of $G$ has a homomorphism into $H$, then so has $G$.
4. For every finite field $\mathcal{F} = \langle F, \ldots \rangle$, for every nontrivial linearly-orderable vector space $V$ over $\mathcal{F}$, there exists a non-zero linear functional $f : V \to F$.

Proof. Since $UT(WO, WO, WO)$ and LW holds in $\mathcal{N}$ (c.f. pt 4 and pt 3 in the proof of Theorem 8 in [HT19]), (1), (2), (3) and (4) hold in $\mathcal{N}$ following the observations in section 3.

Remark 1. In Theorem 7 of [Tac19a], Tachtsis generalized the above construction and proved that $AC^{LO} + LW \not\Rightarrow LT$ by constructing a permutation model $\mathcal{N}$. Since $AC^{WO}$ holds in $\mathcal{N}$, DC holds in $\mathcal{N}$ as well (c.f. Theorem 8.2 of [Jec73]). We observe another standard argument to see that DC holds in $\mathcal{N}$. Since $\mathcal{I}$ is closed under countable unions, we can see that DC holds in $\mathcal{N}$. Let $\mathcal{R}$ be a relation in $\mathcal{N}$ such that if $x \in dom(\mathcal{R})$, there exists a $y$ such that $xRy$. Consequently, there is a sequence $(x_n : n \in \omega)$ in the ground model $\mathcal{M}$ such that for each $n \in \omega$, $x_nRx_{n+1}$. If $x_n$ is supported by $E_n$ for every $n \in \omega$, then $(x_n : n \in \omega)$ is supported by $\cup_{n \in \omega} E_n$. Since $\mathcal{I}$ is closed under countable unions, the sequence $(x_n : n \in \omega)$ is in $\mathcal{N}$.

A class of models $\mathcal{M}_{\aleph_\alpha}$ for any regular cardinal $\aleph_\alpha$ (similar to the model $\mathcal{M}_{\aleph_1}$ constructed in Theorem 7 of [Tac19a]) can be defined where $AC^{LO}$ and LW holds but LT fails, by replacing $\aleph_1$ by $\aleph_\alpha$. Moreover in $\mathcal{M}_{\aleph_\alpha}$, DC$\&\&\aleph_\alpha$ holds since $\mathcal{I}$ is closed under $\aleph_\alpha$ unions.

Remark 2. In the permutation model $\mathcal{N}$ of [Tac17], CS, as well as CWF, holds following the work in this section. Moreover the following statement holds in $\mathcal{N}$, following the work in this section.

‘If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.’

Theorem 4.7. There is a permutation model $\mathcal{N}$ of ZFA, where there is an amorphous set. Moreover, the following holds in $\mathcal{N}$.

1. If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.
2. If in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$ for any regular $\aleph_\alpha$.

Proof. We consider the basic Fraenkel model (labeled as Model $\mathcal{N}_1$ in [HR08]) where ‘there are no amorphous sets’ is false and $UT(WO, WO, WO)$ holds (c.f. [HR08]). Let $(P, \leq)$ be a p.o.set in $\mathcal{N}_1$, and $E$ be a nitie support of $(P, \leq)$. By (1) in the proof of Lemma 4.2, $\mathcal{O} = \{Orb_E(p) : p \in P\}$ is a well-ordered partition of $P$. Now for each $p \in P$, $Orb_E(p)$ is an antichain (c.f. the proof of Lemma 9.3 in [Jec73]). Thus, by methods of Lemma 4.2, (1) and (2) hold in $\mathcal{N}_1$. □
Theorem 4.8. There is a permutation model of ZFA where CS, as well as CWF, holds, but AC_{fin} fails. Moreover, the following statements hold in the model.

1. For every $3 \leq k < \omega$, $P_k$ holds for any graph $G_1$ on some linearly-orderable set of vertices.
2. For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices $V_G$ and a finite graph $H = (V_H, E_H)$, if every finite subgraph of $G$ has a homomorphism into $H$, then so has $G$.
3. For every finite field $F = \langle F, \ldots \rangle$, for every nontrivial linearly-orderable vector space $V$ over $F$, there exists a non-zero linear functional $f : V \rightarrow F$.

Proof. We recall the Lévy’s permutation model (labeled as Model $N_6$ in [HR98]).

- **Defining the ground model $M$.** We start with a ground model $M$ of ZFA + AC where $A$ is a countably infinite set of atoms written as a disjoint union $\cup\{P_n : n \in \omega\}$, where $P_n = \{a^n_1, \ldots, a^n_{p_n}\}$ such that $p_n$ is the $n$th-prime number.

- **Defining the group $G$ of permutations and the filter $F$ of subgroups of $G$.**
  - Defining $G$, $G$ be the group generated by the following permutations $\pi_n$ of $A$.
    \[
    \pi_n : a^n_1 \mapsto a^n_2 \mapsto \ldots \mapsto a^n_{p_n} \mapsto a^n_1 \quad \text{and} \quad \pi_n(x) = x \quad \text{for all} \quad x \in A \setminus P_n.
    \]
  - Defining $F$, $F$ be the filter of subgroups of $G$ generated by $\{\text{fix}_\pi(E) : E \in [A]^{<\omega}\}$.

- **Defining the permutation model.** Consider the permutation model $N_6$ determined by $M$, $G$ and $F$.

It is well-known that in $N_6$, AC_{fin} fails since $\{P_i : i \in \omega\}$ has no (partial) choice function (c.f. [Jec73]). Consequently, following Lemma 4.4 of [Jec95], CAC fails in $N_6$. Since every permutation $\phi \in G$ moves only finitely many atoms, following the arguments in Lemma 4.2, we can observe that CS, as well as CWF, holds in $N_6$.

Lemma 4.9. In $N_6$, LW holds.

Proof. Let $(X, \leq)$ be a linearly ordered set in $N_6$ supported by $E$. We show $\text{fix}_G E \subseteq \text{fix}_G X$ which implies that $X$ is well-orderable in $N_6$. For the sake of contrary assume $\text{fix}_G E \not\subseteq \text{fix}_G X$. So there is an element $y \in X$ which is not supported by $E$ and there is a $\phi \in \text{fix}_G E$ such that $\phi(y) \neq y$. Since $\phi(y) \neq y$ and $\leq$ is a linear order on $X$, we obtain either $\phi(y) < y$ or $y < \phi(y)$. Let $\phi(y) < y$. Since every permutation $\phi \in G$ moves only finitely many atoms there exists some $k < \omega$ such that $\phi^k = 1_A$. Thus, $p = \phi^k(p) < \phi^{k-1}(p) < \ldots < \phi(p) < p$ which is a contradiction. Similarly we can arrive at a contradiction if we assume $y < \phi(y)$.

Since LW holds in $N_6$, we can observe (1), (2) and (3) in $N_6$ by observations in section 3.

Theorem 4.10. There is a permutation model of ZFA where CS, as well as CWF, holds, but LOKW fails. Moreover, the following statements hold in the model.

1. For every $3 \leq k < \omega$, $P_k$ holds for any graph $G_1$ on some linearly-orderable set of vertices.
2. For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices $V_G$ and a finite graph $H = (V_H, E_H)$, if every finite subgraph of $G$ has a homomorphism into $H$, then so has $G$.
3. For every finite field $F = \langle F, \ldots \rangle$, for every nontrivial linearly-orderable vector space $V$ over $F$, there exists a non-zero linear functional $f : V \rightarrow F$.

Proof. We recall the permutation model $M$ from the second assertion of Theorem 10(ii) of [HT19].

- **Defining the ground model $M$.** Let $\kappa$ be any infinite well-ordered cardinal number.

  We start with a ground model $M$ of ZFA + AC where $A$ is a $\kappa$-sized set of atoms written as a disjoint union $\cup\{A_\alpha : \alpha < \kappa\}$, where $A_\alpha = \{a_{\alpha,1}, a_{\alpha,2}, a_{\alpha,3}, a_{\alpha,4}\}$ such that $|A_\alpha| = 4$ for all $\alpha < \kappa$.  

• Defining the group $G$ of permutations and the filter $F$ of subgroups of $G$.
  
  – Defining $G$. Let $G$ be the weak direct product of $G_\alpha$’s where $G_\alpha$ is the alternating group on $A_\alpha$ for each $\alpha < \kappa$.
  
  – Defining $F$. Let $F$ be the normal filter of subgroups of $G$ generated by $\{ \text{fix}_G(E) : E \in [A]^{<\omega} \}$.

• Defining the permutation model. Consider the permutation model $M$ determined by $M$, $G$ and $F$.

In $M$, $\text{LKW}_d^-$ fails (c.f. Theorem 10(ii) of [HT19]). Since every permutation, $\phi \in G$ moves only finitely many atoms, following the arguments in Lemma 4.2 we can observe that CS, as well as CWF, holds in $M$. Since LW holds in $M$ (c.f. Theorem 10(ii) of [HT19]), we can observe (1), (2) and (3) in $M$ by observations in section 3.

5. Observations in Howard’s model

Theorem 5.1. For any $3 \leq k < \omega$, $\mathcal{P}_k$ follows from $\mathcal{F}_{k-1}$ in ZF. Moreover, if $X \in \{ AC_3, AC_{f'in}^\omega \}$, then the statement $\mathcal{P}_k$ does not imply $X$ in ZFA when $k = 3$.

Proof. Fix $3 \leq k < \omega$. Suppose $\chi(E_{G_1}) = k$, $\chi(E_{G_2}) \geq \omega$ and $G_1$ is a graph on some well-orderable set of vertices. First we observe that if $g : V_{G_1} \to \{1,...,k\}$ is a good $k$-coloring of $G_1$, then $G((x,y)) = g(x)$ is a good $k$-coloring of $G_1 \times G_2$. So, $\chi(E_{G_1 \times G_2}) \leq k$. For the sake of contradiction assume that $F : V_{G_1} \times V_{G_2} \to \{1,...,k-1\}$ is a good coloring of $G_1 \times G_2$. For each color $c \in \{1,...,k-1\}$ and each vertex $x \in V_{G_1}$ we let $A_{x,c} = \{ y \in V_{G_2} : F(x,y) = c \}$. Define a relation $R$ on $\{1,...,k-1\}$ as $(v_1, i) R (v_2, j)$ if and only if $v_1 \neq v_2$ implies $A_{v_1,i} \cap A_{v_2,j}$ is not independent for $v_1, v_2 \in V_{G_1}$. By $F_{k-1}$ and claim 3.8 there exist a choice function $f$ such that for any $x, x' \in V_{G_1}$, $A_{x,f(x)} \cap A_{x',f(x')}$ is not independent. Since $x \to f(x)$ is not a good coloring in $G_1$ as $\chi(E_{G_1}) = k$, there are $x, x' \in V_{G_1}$ with $f(x) = f(x') = j$ and $\{ x, x' \} \in E_{G_1}$. Consequently, $A' = A_{x,f(x)} \cap A_{x',f(x')}$ is not independent. Pick $y, y' \in A'$ joined by an edge in $E_{G_2}$. Then $(x,y)$ and $(x',y')$ are joined in $E_{G_1} \times E_{G_2}$ and get the same color $j$ which is a contradiction to the fact that $F$ is a good coloring of $G_1 \times G_2$.

For the second assertion, we consider the permutation model $\mathcal{N}$ from section 3 of [How84] where $AC_3$ fails, and $F_2$ holds. Consequently, $\mathcal{P}_3$ holds in $\mathcal{N}$. In $\mathcal{N}$, there is a countable family $A = \{ A_i : i \in \omega \}$ which has no partial choice function. Consequently, $\mathcal{P}_{AC_{f'in}^\omega}$ fails. Since $\mathcal{P}_{AC_{f'in}^\omega}$ is equivalent to $\mathcal{P}_{AC_{f'in}^\omega}$ (see the proof of Lemma 4.4 of [Tak19a]), $\mathcal{P}_{AC_{f'in}^\omega}$ fails in $\mathcal{N}$.

Question 5.2. If $k > 3$, does UL follow from $\mathcal{P}_k$? Otherwise is there any model of ZF or ZFA, where $\mathcal{P}_k$ holds for $k > 3$, but UL fails?

Theorem 5.3. For any $2 \leq k < \omega$, $\mathcal{P}_{G,H}$ restricted to finite graph $H$ with $k$ vertices follows from $\mathcal{F}_{k-1}$ in ZF. Moreover, if $X \in \{ AC_3, AC_{f'in}^\omega \}$, then $\mathcal{P}_{G,H}$ restricted to finite graph $H$ with $2$ vertices does not imply $X$ in ZFA.

Proof. Fix $2 \leq k < \omega$. Let $V_H = \{ v_1, ..., v_k \}$. For each $x \in V_G$, let $A_x = \{ (x, v_1), ..., (x, v_k) \}$. Define a relation $R$ on $\cup_{x \in V_G} A_x$ by $(x,v_1) R (x',v_2)$ if and only if $\{ x, x' \} \in E_G$ implies $(v_1, v_2) \in E_H$ for $(x, v_1) \in A_x, (x', v_2) \in A_{x'}$. By assumption, for all finite $F \subset V_G$, there exists a homomorphism $h_F : G \to F \to H$. For any finite $F \subset V_G$, and an homomorphism $h_F$ of $F$, let $h_F^*(j) = (j, h_F(j))$ for $j \in F$. Clearly, $h_F^*$ is an $R$-consistent choice function for $\{ A_x \}_{x \in F}$. By $F_k$, there is a $R$-consistent choice function $h_F^*$ for $\{ A_x \}_{x \in V_G}$. Define $h_{V_G}$ on $V_G$ by $h_{V_G}(j) = (j, h_{V_G}(j))$ for $j \in V_G$. Let $(j, j') \in E_G$ such that $j, j' \in V_G$. Since $i^*_F$ is $R$-consistent, $(j, h_{V_G}(j)) R (j', h_{V_G}(j'))$. By the definition of $R$, $(h_{V_G}(j), h_{V_G}(j')) \in E_H$.

For the second assertion, we once more consider the permutation model $\mathcal{N}$ from section 3 of [How84] where $AC_3$ and $AC_{f'in}^\omega$ fails, and $F_2$ holds. Consequently, $\mathcal{P}_{G,H}$ for a finite graph $H$ with $2$ vertices’ holds in $\mathcal{N}$.
Acknowledgement. We would like to thank the reviewer for reading the manuscript carefully and providing suggestions for improvement.

References

[Bana92] B. Banaschewski, Algebraic closure without choice, MLQ Math. Log. Q. 38 (1992), no. 1, pp. 383-385.
[Cow77] R. H. Cowen, Generalizing König’s infinity lemma, Notre Dame J. Formal Logic 18 (1977), no. 2, pp. 243-247.
[Diil50] R. P. Dilworth, A decomposition theorem for partially ordered sets, Ann. of Math. 51 (1950), pp. 161166.
[Haj85] A. Hajnal, The chromatic number of the product of two ℵ1-chromatic graphs can be countable, Combinatorica 5 (1985), pp. 137140.
[Hal48] M. Hall Jr., Distinct representatives of subsets, Bull. Amer. Math. Soc. 54 (1948), pp. 922-926.
[HR98] P. Howard and J. E. Rubin, Consequences of the Axiom of Choice, Mathematical Surveys and Monographs 59 (1998), DOI: http://dx.doi.org/10.1090/surv/059 MR 1637107
[HT19] L. Halbeisen and E. Tachtsis, On Ramsey Choice and Partial Choice for infinite families of n-element sets, Arch. Math. Logic (2019). DOI: https://doi.org/10.1007/s00153-019-00705-7.
[HT13] P. Howard and E. Tachtsis, On vector spaces over specific fields without choice, MLQ Math. Log. Q. 59 (2013), no. 3.
[How84] P. Howard, Binary consistent choice on pairs and a generalization of König’s infinity lemma, Fund. Math. 121 (1984), pp. 17-23.
[Jec73] T. J. Jech, The axiom of choice. North-Holland Publishing Co., Amsterdam, 1973, Studies in Logic and the Foundations of Mathematics, Vol. 75. MR 0396271
[Ker00] K. Keremedis, The compactness of 2R and the axiom of choice, MLQ Math. Log. Q. 46 (2000), pp. 569-571.
[KT06] P. Komjáth and V. Totik, Problems in Classical Set Theory, Springer (2006).
[Kom] P. Komjáth, Infinite graphs, Manuscript under preparation.
[Loeb65] P. Loeb, A New Proof of the Tychonoff Theorem, The American Mathematical Monthly 72 (1965), no. 7, pp. 711-717.
[LN51] J. Łoś and Czesław Ryll-Nardzewski, On the application of Tychonoff’s theorem in mathematical proofs, Fund. Math. 38 (1951), no. 1, pp. 233-237. ISSN: 0016-2736.
[Tac19] E. Tachtsis, Dilworth’s decomposition theorem for posets in ZF, Acta Math. Hungar. 159 (2019), pp. 603-617, DOI: 10.1007/s10879-019-00667-w.
[Tac19a] Loś’s theorem and the axiom of choice, MLQ Math. Log. Q. 65 (2019), no. 3, pp. 280-292, DOI: 10.1002/malq.201700074.
[Tac17] On the Minimal Cover Property and Certain Notions of Finite, Arch. Math. Logic. 57 (2018), no. 5-6, pp. 655-686.
[Tac17a] On variants of the principle of consistent choices, the minimal cover property and the 2-compactness of generalized Cantor cubes, Topology and its Applications (Elsevier). 219 (2017), pp. 122-140.
[Tac16] On Ramsey’s Theorem and the existence of Infinite Chains or Infinite Anti-Chains in Infinite Posets, The Journal of Symbolic Logic. 81 (2016), no. 1, pp. 384-394, DOI: 10.1017/jsl.2015.47.
[Tac16b] On Martin’s Axiom and Forms of Choice, MLQ Math. Log. Q. 62 (2016), no. 3, pp. 190-203.
[THS16] E. Tachtsis, P. Howard, and D.I. Saveliev, On the set-theoretic strength of the existence of disjoint cofinal sets in posets without maximal elements, MLQ Math. Log. Q. 62 (2016), no. 3, pp. 155-176. DOI: 10.1002/malq.201400089.

Department of Logic, Institute of Philosophy, Eötvös Loránd University, Múzeum krt. 4/1 Budapest, H-1088 Hungary
E-mail address: banerjee.amitayu@gmail.com

Institute of Philosophy, Department of Logic, Jagiellonian University, Grodzka 52, 33-332, Kraków, Poland
E-mail address: zalan.gyenis@gmail.com