1/N corrections to $F_1$ and $F_2$ structure functions of vector mesons from holography

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Abstract

The structure functions $F_1$ and $F_2$ of the hadronic tensor of vector mesons are obtained at order $1/N$ and strong coupling using the gauge/gravity duality. We find that the large $N$ limit and the high energy one do not commute. Thus, by considering the high energy limit first, our results of the first moments of $F_1$ for the rho meson agree well with those from lattice QCD, with an important improvement of the accuracy with respect to the holographic dual calculation in the planar limit.
Contents

1 Introduction 2

2 DIS in the D3D7-brane system 5
  2.1 One-particle exchange: the $N \to \infty$ limit .................. 7

3 DIS in the $1/N$ expansion 8
  3.1 Leading diagram for vector mesons at order $1/N$ ............... 8
  3.2 Structure functions for vector mesons .......................... 11

4 Discussion and conclusions 13

Appendix: Angular integrals 18
1 Introduction

The idea of the present work is to investigate the leading $1/N$ corrections to the structure functions $F_1$ and $F_2$ of the hadronic tensor of unpolarized vector mesons at strong 't Hooft coupling $\lambda$, using the gauge/gravity duality. For this purpose we consider vector mesons from the D3D7-brane system in type IIB string theory [1].

We are interested in the electromagnetic deep inelastic scattering (DIS) of a charged lepton from a vector meson. The DIS cross section is given by the contraction of a leptonic tensor, $l^{\mu\nu}$, with a hadronic one, $W_{\mu\nu}$. The process involves an incoming charged lepton interacting with a hadron with momentum $P$ through the exchange of a virtual photon with momentum $q$, with the condition $q^2 \gg -P^2$. We consider the definitions given in [2], however we use the mostly-plus signature. Thus, the DIS differential cross section is given by

$$\frac{d^2\sigma}{dx \, dy \, d\phi} = \frac{e^4}{16\pi^2q^4} \, y \, l^{\mu\nu} \, W_{\mu\nu},$$

where $y$ is the lepton fractional energy loss and $e$ denotes the electron charge. The hadronic tensor depends on the hadron structure, where there are important contributions from soft QCD processes. For this reason the gauge/string theory duality becomes a suitable tool for the calculation of this tensor, and therefore the structure functions.

In this work we focus on the structure functions associated with unpolarized vector mesons$^4$. The corresponding hadronic tensor has the form

$$W_{\mu\nu} = F_1(x, q^2) \eta_{\mu\nu} - \frac{F_2(x, q^2)}{P \cdot q} P_{\mu} P_{\nu}. \quad (2)$$

DIS is related to the forward Compton scattering (FCS) through the optical theorem, which is an special case of the Cutkosky rules, based on the fact that the $S$-matrix is unitary. It relates the imaginary part of the FCS amplitude to the DIS amplitude. Then, the tensor $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} = i \langle P, Q | \hat{T}(\hat{J}_{\mu}(q) J_{\nu}(0)) | P, Q \rangle,$$

where $J_{\mu}$ and $J_{\nu}$ are the electromagnetic current operators, $Q$ denotes the charge of the hadron, while $\hat{T}$ represents time-ordered product. Tildes indicate the Fourier transform. In terms of the optical theorem we can write

$$\text{Im} \bar{F}_j = 2\pi F_j,$$

being $\bar{F}_j$ the $j$-th structure function of the $T_{\mu\nu}$ tensor, while $F_j$ is the one corresponding to the $W_{\mu\nu}$ tensor.

$^4$The polarized structure functions $b_{1,2,3,4}$ and $g_{1,2}$ [2] have been obtained at strong coupling and in the planar limit in [3, 4] from supergravity and in [5] from superstring theory scattering amplitudes. The latter gives the relevant behavior for small values of the Bjorken parameter $x$. 

2
At this point it is convenient to define the Bjorken parameter \( x = -q^2 / (2P \cdot q) \) for \( q^2 > 0 \), being its physical kinematic range \( 0 \leq x \leq 1 \). On the other hand, in the unphysical region for \( 1 \ll x \) the product of the two electromagnetic currents in the hadron can be written as an operator product expansion (OPE), in terms of operators \( \mathcal{O}_{n,k} \) multiplied by powers of \( (\Lambda^2/q^2)^\gamma_{n,k}/2 \), where \( n \) is the spin of \( \mathcal{O}_{n,k} \), while \( \delta_{n,k} \), \( \gamma_{n,k} \), and \( \Delta_{n,k} = \delta_{n,k} + \gamma_{n,k} \), represent the engineering, the anomalous and the total scaling dimensions of the operator, respectively [6]. Then, we can define the twist of each operator as \( \tau_{n,k} = \Delta_{n,k} - n \). The relation with the physical parametric region \( 0 \leq x \leq 1 \) is given by a contour argument, which allows to connect the OPE with the moments of the structure functions in the DIS process. Thus, the \( n \)-moment of the \( j \)-th structure function can be expressed as the sum of three contributions\(^5\)

\[
M^j_n(q^2) \approx \frac{1}{4} \sum_k C^j_{n,k} A_{n,k} \left( \frac{\Lambda^2}{q^2} \right)^{\frac{1}{2} \tau_{n,k} - 1} + \frac{1}{4} \sum_{Q_p=Q} C^j_{n,p} A_{n,p} \left( \frac{\Lambda^2}{q^2} \right)^{\tau_{p} - 1} + \frac{1}{4} \frac{1}{N} \sum_{Q_p \neq Q} C^j_{n,p} a_{n,p} \left( \frac{\Lambda^2}{q^2} \right)^{\tau_{p} - 1},
\]

where the coefficients \( C^j_{n,p} \) are dimensionless, while \( A_{n,p} \) and \( a_{n,p} \) depend on the matrix elements of the operators \( < P, Q | \mathcal{O}_{n,k} | P, Q > \) for a hadron state with four-momentum \( P \) and charge \( Q \).

Let us briefly explain how different contributions behave in equation (5). We can study this equation for the photon virtuality \( q \) to be large, intermediate or small, in comparison with the confinement scale \( \Lambda \). At weak coupling the Feynman’s parton model gives a suitable description of hadrons, thus the leading contribution comes from the first term. This contribution only sums over terms associated with operators with the lowest twist \( \tau_{n,k} \approx 2 \) at large \( q^2 \). In this case perturbative methods of quantum field theory are suitable. On the other hand, the second sum dominates at strong coupling and in the planar limit, i.e. \( 1 \ll \lambda \ll N \). In this case protected double-trace operators constructed from the protected single-trace ones have the smaller twist at strong coupling. Therefore, the calculations can be done by using the gauge/gravity duality, considering a forward Compton scattering with the exchange of a single on-shell particle between incoming and outgoing states. Within this regime exchange of more than one intermediate state is suppressed by \( 1/N \) powers. The third sum in equation (5) becomes the leading one when \( q^2 \geq \Lambda^2 N^{1/(\tau_Q - \tau_c)} \). In this case \( \tau_Q \) is the minimum twist of protected operators with charge \( Q \), while \( \tau_c \) is the minimum twist of all electrically charged protected operators. The \( 1/N \) suppression of the third sum is expected for mesons, while for glueballs there is a \( 1/N^2 \) suppression.

In addition, we should comment on the different parametric regions in terms of \( x \) and the ’t Hooft coupling. For \( 1/\sqrt{\lambda} \ll x \ll 1 \) only supergravity states contribute since in

\(^5\)We use the notation of [6], in particular equation (5) is similar to equation (27) of that reference but for mesons, i.e. the second term has a factor \( 1/N \).
this region the ten-dimensional $s$-channel Mandelstam variable satisfies $\tilde{s} \ll 1/\alpha'$, where $\alpha'$ is the string constant. When $\exp(-\sqrt{\lambda}) \ll x \ll 1/\sqrt{\lambda}$ excited strings are produced and their dynamics becomes important. The holographic dual calculation is derived from four-point superstring theory scattering amplitudes. Finally, for the exponentially small region the size of the excited string becomes comparable with the AdS radius. In this case dual Pomeron techniques are useful [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. In previous papers we have calculated $F_1$ and $F_2$ by considering the FCS process with the exchange of a single intermediate state for scalar and vector mesons [3, 4, 5, 18]. Then, we have also calculated these functions by considering the exchange of two intermediate states for glueballs [19] and for scalar mesons [20] in the D3D7-brane system of reference [1]. In both cases we found that the large $N$ limit does not commute with the high energy one. By considering the high energy limit first, which corresponds to the physical situation, in the case of the pion we have obtained the first moments of the structure function $F_2$ and compared them with lattice QCD calculations [21, 22, 23], obtaining a substantial improvement of the accuracy, namely: from 10.8% for a single intermediate state [18] to 1.27% in the case of two intermediate states [20]. Then, a natural question is whether or not this effect also occurs in the case of vector mesons. The present work answers it positively as we shall explain in detail in the next sections.

For finite values of $N$ we can expand the structure functions of mesons as follows

$$F_j = f_j^{(0)} \left( \frac{\Lambda^2}{q^2} \right)^{\tau_{in}^{-1}} + \frac{1}{N} f_j^{(1)} \left( \frac{\Lambda^2}{q^2} \right) + \frac{1}{N^2} f_j^{(2)} \left( \frac{\Lambda^2}{q^2} \right) + \cdots$$  \hspace{1cm} (6)

where $\tau_{in}$ is the twist of the incident dual vector meson state in type IIB supergravity, $f_j^{(n)}$'s indicate the structure functions at order in $1/N^n$, with $j = 1, 2$ and $n = 1, \ldots$. Notice that in the definitions of $f_j^{(n)}$'s powers of $\Lambda^2/q^2$ have been factored out. The index $n$ indicates the number of exchanged intermediate on-shell states of the FCS Feynman diagram. This corresponds to the number of hadrons in the final state of the DIS process. From expression (6) one can easily see that the high energy ($q^2 \gg \Lambda^2$) and the large $N$ limits do not commute. Moreover, by taking first the high energy limit, since the power of $\Lambda^2/q^2$ in the first term is larger than for the rest, it vanishes, and then in the $1/N$ expansion the second term dominates. We would like to emphasize that $1/N$ corrections to the $F_1$ and $F_2$ structure functions for scalar mesons were studied in [20], but not for vector mesons. Therefore, we consider this calculation to be important for the investigation of such limits beyond scalar hadrons, since there are also lattice QCD results of the first moments of $F_1$ for the rho meson to compare with [21].

The work is organized as follows. In section 2 we discuss generalities of DIS in the context of the D3D7-brane system at large $N$. In section 3 we consider the $1/N$ expansion and obtain the relevant Feynman-Witten diagram in the bulk theory. Also in this section we develop the calculation of the $F_1$ and $F_2$ structure functions for vector mesons. In section 4
we carry out the analysis of our results. We focus on the calculation of the first moments of the structure function $F_1$ of the rho meson and compare them with the available results from lattice QCD.

2 DIS in the D3D7-brane system

DIS processes of charged leptons from scalar and vector mesons in the D3D7-brane system have been studied in several papers [3, 4, 5], by considering the large $N$ limit, which means that the final state has only a single hadron. A more realistic calculation for vector mesons must include $1/N$ corrections. This corresponds to final multi-particle states. In this work we consider $1/N$ corrections of DIS of charged leptons from unpolarized vector mesons.

Firstly, we give a brief description of the D3D7-brane system. Let us consider $N$ coincident D3-branes in type IIB superstring theory. The corresponding near-horizon geometry is the AdS$_5 \times S^5$ spacetime, with the metric

$$ ds^2 = \frac{r^2}{R^2} \eta_{\mu \nu} dx^\mu dx^\nu + \frac{R^2}{r^2} d\vec{Z} \cdot d\vec{Z}, $$

(7)

where $\vec{Z}$ are coordinates of the directions perpendicular to the D3-branes, being the radial coordinate $r = |\vec{Z}|$. The radius of AdS$_5$ is $R = (4\pi g_s N\alpha'^2)^{1/4}$, where $g_s$ is the string coupling. Now, one can add a D7-brane in the probe approximation, at a distance $L = |\vec{Z}|$ in the (8,9) plane. The induced metric on the D7-brane is given by

$$ ds^2 = \frac{\rho^2 + L^2}{R^2} \eta_{\mu \nu} dx^\mu dx^\nu + \frac{R^2}{\rho^2 + L^2} d\rho^2 + \frac{R^2 \rho^2}{\rho^2 + L^2} d\Omega_3^2, $$

(8)

where $\rho^2 = r^2 - L^2$ and the angles contained in $\Omega_3$ span a three-sphere. For $L = 0$ equation (8) gives the AdS$_5 \times S^3$ metric, otherwise the metric is only asymptotically AdS$_5 \times S^3$. This is the situation where the conformal symmetry is preserved.

For $L > 0$ the 3-7 quarks become massive, and meson type excitations are energetically favored. Mesons correspond to excitations of open strings ending on the D7-brane. The dynamics of these fluctuations is described by the action

$$ S_{D7} = -\mu_7 \int d^8 \xi \sqrt{-\det (P [g]_{ab} + 2\pi \alpha' F_{ab})} + \frac{(2\pi \alpha')^2}{2} \mu_7 \int P [C^{(4)}] \wedge F \wedge F, $$

(9)

where $\mu_7 = [(2\pi)^7 g_s \alpha'^4]^{-1}$ is the D7-brane tension, $\xi^a$ denotes the D7-brane coordinates, $g_{ab}$ stands for the metric (8), and $P$ is the pullback of the background fields on the D7-brane. The second term is the Wess-Zumino term.

It is possible to induce excitations in the transverse directions to the D7-brane. These are two types of scalar excitations $\phi$ and $\chi$, related to the $Z^5$ and $Z^6$ coordinates, respectively.
On the other hand, it is also possible to perturb the gauge fields $F_{ab} = \partial_a B_b - \partial_b B_a$ on the Dirac-Born-Infeld (DBI) action. In this case, there are three types of solutions for the $B_a$ modes, related to the expansion the solutions in scalar or vector spherical harmonics on $S^3$. The three classes of solutions are [1]

$$\text{type I : } B_\mu = 0, \ B_\rho = 0, \ B_i = \phi^I_+(\rho) \ e^{i k \cdot x} Y^I_{i,l}(\Omega),$$

$$\text{type II : } B_\mu = \epsilon_\mu \phi_{II}(\rho) \ e^{i k \cdot x} Y^I_{i,l}(\Omega), \ k \cdot \epsilon = 0, \ B_\rho = 0, \ B_i = 0,$$

$$\text{type III : } B_\mu = 0, \ B_\rho = \phi_{III}(\rho) \ e^{i k \cdot x} Y^I_{i,l}(\Omega), \ B_i = \tilde{\phi}_{III}(\rho) e^{i k \cdot x} \nabla_i Y^I_{i,l}(\Omega).$$

$Y^I_{i,l}(\Omega)$ and $Y^I_{i,l}(\Omega)$ are scalar and vector spherical harmonics, respectively. Some of their properties are described in the appendix and in references therein. Note that in this case type I and III are scalar modes, while type II modes represent vector fields from the (asymptotically) AdS perspective. The different modes of the scalar and vector perturbations are shown in table 1, together with their relevant quantum numbers.

| Field | Type of field in 5D | Built from | $\Delta(l)$ | $SU(2) \times SU(2)$ irrep |
|-------|--------------------|------------|-------------|----------------------------|
| $\phi, \chi$ | scalars | $\phi, \chi$ | $l + 3, l \geq 0$ | $(l \frac{1}{2}, l \frac{1}{2})$ |
| $B_\mu$ | vector | $B^{II}_\mu$ | $l + 3, l \geq 0$ | $(l \frac{1}{2}, l \frac{1}{2})$ |
| $\phi^I_-$ | scalar | $B^I_1$ | $l + 1, l \geq 1$ | $(l + \frac{1}{2}, l \frac{1}{2})$ |
| $\phi^I_+$ | scalar | $B^I_1$ | $l + 5, l \geq 1$ | $(l \frac{1}{2}, l + \frac{1}{2})$ |
| $\phi_{III}$ | scalar | $B^{III}_{i,z}$ | $l + 3, l \geq 1$ | $(l \frac{1}{2}, l \frac{1}{2})$ |

Table 1: Some features of D7-brane fluctuations on the AdS$_5 \times S^3$ background relevant to this work. The integer $l$ indicates the $SO(4) \sim SU(2) \times SU(2)$ irreducible representation (irrep) and it defines the corresponding Kaluza-Klein mass. The relation between the scaling dimension of the associated operator $\Delta$ and $l$ is also presented.

Beyond the quadratic order, the interaction Lagrangian for these modes has been derived in reference [20].

Up to this point we have described the D3D7-brane system presented in [1], where the solutions were computed in terms of hypergeometric functions. In the context of DIS from mesons, one identifies the parameter that controls the separation between the D7 and the D3-branes in the $(Z^5, Z^6)$ plane with the IR scale $\Lambda$ introduced as a cutoff in the radial direction to induce confinement [6]. Thus, we take $L \sim \Lambda R^2$. Therefore, the relevant interactions take place at values of $\rho$ considerably larger than $L$, and in this region the solutions are well approximated by the typical AdS$_5$ expressions in terms of Bessel functions, which we write in section 3. The AdS masses can only take discrete values. The presence of a small but non-zero value of $L$ is important for the vertices we will need to consider.
2.1 One-particle exchange: the $N \to \infty$ limit

For unpolarized vector mesons we shall study only the contributions to the $F_i$ structure functions. These functions can be written in terms of $W^{\mu \nu}$ and the vector $v^{\mu} = \frac{1}{q} (P^{\mu} + \frac{q^{\mu}}{2x})$ as

$$F_1(x, q^2) = \frac{1}{2} \left( g^{\mu \nu} - \frac{q^{\mu} q^{\nu}}{q^2} \right) W_{\mu \nu} + 2x^2 v^{\mu} v^{\nu} W_{\mu \nu}, \quad (13)$$

$$F_2(x, q^2) = x \left( g^{\mu \nu} - \frac{q^{\mu} q^{\nu}}{q^2} \right) W_{\mu \nu} + 12x^3 v^{\mu} v^{\nu} W_{\mu \nu}. \quad (14)$$

The FCS amplitude can be derived by using the gauge/string theory duality, by studying a four-point interaction with vector modes on the D7-brane and graviphotons related to current insertion on the boundary as external states. This gauge field arises from a particular decomposition of the graviton mode in ten dimensions:

$$\delta g_{m_j} = A_m (\rho, x) v_j (\Omega_3), \quad (15)$$

where $v_j$ are the Killing vectors on $S^3$, and $m = (\mu, \rho)$.

The structure functions have been calculated in this context by considering a single intermediate hadron state in [3, 4], obtaining the following results at leading order

$$F_1(x, q^2) = A(x) \frac{1}{12x^3} (1 - x), \quad (16)$$

$$F_2(x, q^2) = A(x) \frac{1}{6x^3} (1 - x), \quad (17)$$

with

$$A(x) = A_0 Q^2 \left( \frac{\mu_0^2 (\alpha')^4}{\Lambda^8} \right) \left( \frac{\Lambda^2}{q^2} \right)^{l+1} x^{l+6} (1 - x)^l, \quad (18)$$

and $A_0 = |c_i|^2 |c_X|^2 2^{6+2l} [\Gamma(3 + l)]^2 \pi^5$ is a dimensionless constant. Also, $c_i$ and $c_X$ are the normalization constants of the incident and intermediate dual hadron states, respectively, while $Q$ is the charge of the hadron, associated to a $U(1)$ subgroup of the $S^3$ isometries.

These results are valid for DIS from mesons considered in the context of the D3D7-brane system. However, it is important to keep in mind that [3, 4] showed that completely analogous formulas hold in the context of different Dp-brane models, such as the D4D8D8-brane system [24] and the D4D6D6-brane system [25], both in type IIA superstring theory. These models are very different to each other, and each of them shares certain phenomenological features with large-$N$ QCD. In consequence, it is reasonable to expect that the qualitative form of the structure functions we just described is universal, in the sense that it would hold in any holographic Dp-brane model for mesons at strong coupling and in the planar limit. Although in the present work we focus on the D3D7-model, we expect this universality to hold also for the leading one-loop correction, at least at the qualitative level.
3 DIS in the $1/N$ expansion

The non-planar $1/N$ corrections to $F_1$ and $F_2$ structure functions for scalar mesons were studied in [20], and also for the $\mathcal{N} = 4$ SYM theory glueball in reference [19]. From the point of view of DIS, they correspond to processes with multi-particle final states. The first non-trivial contribution comes from considering a two-hadron final state in the hadronic tensor $W_{\mu\nu}^2$, which can be related to a FCS process with two intermediate on-shell states, denoted as $T_{2}^{\mu\nu}$. Writing this tensor in terms of the $U(1)$ conserved current $J^\mu$ we obtain [19]

$$\text{Im} \left( T_{2}^{\mu\nu} \right) = \pi \sum_{X_1, X_2} \langle P, Q \left| \tilde{J}^{\mu}(q) \right| X_1, X_2 \rangle \langle X_1, X_2 \left| J^{\nu}(0) \right| P, Q \rangle$$

$$= \pi \sum_{M_2, M_3} \int \frac{d^3q'}{2E_{q'}(2\pi)^3} \frac{d^3p'}{2E_{p'}(2\pi)^3} \langle P, Q \left| \tilde{J}^{\mu}(q) \right| X_1, X_2 \rangle \langle X_1, X_2 \left| J^{\nu}(0) \right| P, Q \rangle$$

$$= 4\pi^3 \sum_{M_2, M_3} \int \frac{d^4q'}{(2\pi)^4} \delta \left( M_2^2 - q'^2 \right) \delta \left( M_3^2 - (P + q - q')^2 \right) \times \langle P, Q \left| J^{\mu}(0) \right| X_1, X_2 \rangle \langle X_1, X_2 \left| J^{\nu}(0) \right| P, Q \rangle,$$

where $X_1$ and $X_2$ are the intermediate states with momenta $p'$ and $q'$ respectively, as shown in figure 1. The current $J^\mu$ matrix element is related to its Fourier transform as

$$\langle P, Q \left| \tilde{J}^{\mu}(q) \right| X_1, X_2 \rangle = (2\pi)^4 \delta^{(4)}(P + q - p' - q') \langle P, Q \left| J^{\mu}(0) \right| X_1, X_2 \rangle.$$

Using the AdS/CFT duality, this current can be related to an specific field in the bulk theory. In [19], considering $1/N$ corrections to DIS from a scalar meson we have shown that the leading contribution to the DIS process with two hadron final states is given by a specific Feynman diagram where one of these two outgoing hadrons has the lowest twist. In the next subsection we will explain the amplitude we need to calculate in the case of a spin-1 meson.

### 3.1 Leading diagram for vector mesons at order $1/N$

We want to study the $1/N$ corrections to the DIS process from a vector meson (for instance a rho meson), associated to a type II vector mode $B_{\mu}^{II}$ on the D7-brane. Based on the results of [20], the leading diagram is the $s$-channel one, where the exchanged particle is the one with the lowest twist, $\tau = \Delta - n$. This can be done by looking at table 1, which gives the relevant quantum numbers of the different solutions.

Since the lowest $\tau$ is associated to the lowest $\Delta$, the exchanged field should be the $\phi_{\Delta}$ mode with $\tau = \Delta = 2$. This is what has been done in [20]. However, the interaction term between $B_{\mu}^{II}$ and $\phi_{\Delta}$ modes given in [20]

$$L_3 = -\mu_\gamma(2\pi\alpha')^3 \sqrt{-g} \frac{L}{\rho^2 + L^2} \phi \left( F_{aJ}^{\alpha \alpha} - F_{a\mu}^{\alpha \mu} \right),$$

8
vanishes, due to the nature of the field solutions. It can be seen from equations (10) and (11), that a type I scalar has only angular components, while type II vectors have only $\mu$ components. Therefore, the interaction between these two modes vanishes. The vector mode with $\tau = 2$ does not contribute to the DIS process either. This can be easily seen by analyzing the charge of this vector associated with the 3-sphere. For $\tau = 2$ we need to consider $\Delta = 3$, but this implies that $l = 0$, meaning that the vector mode has no charge over the 3-sphere. Thus, there is no interaction with the $A_m$ photon. Another possible interaction could arise from the Wess-Zumino term in the low-energy action of the D7-brane. This is because the gauge field is actually a particular linear combination of the ten-dimensional graviton and RR 4-form perturbations. This is described in detail in [16, 17], where this type of vertex has been used to study the antisymmetric contributions to the hadronic tensor for glueballs and spin-1/2 hadrons. However, it can be seen that the relevant angular integrals vanish in the present case.

The next step is to consider the exchange of $\tau = 3$ modes. There are two possibilities:

1. $\phi, \chi$ scalars, with $\Delta = 3$.
2. $B_\mu$ vector, with $\Delta = 4$.

In the former case the perturbations have $l = 0$, thus they are not charged with respect to the $U(1)$ we are considering. Therefore, the only possibility is the exchange of a type II vector mode with $l = 1$.

In the IR region, the relevant interaction includes two type II modes (one associated to the $\tau = 3$ mode we just discussed and the other one with the incoming vector meson) and a scalar field, and from (21) we see that it is described by [1, 20]

$$ S_{\text{IR}} \approx \mu_7 (2\pi\alpha')^3 \frac{L}{R^4} \int d^8 \xi \sqrt{g} \, z^2 \phi F_{\mu\nu} F^{\mu\nu}. $$

(22)

Note that we only keep the first term of the $L \ll \rho$ expansion, where $\rho \simeq R^2/z$, being $z$ the Poincaré radial coordinate of AdS$_5$.

For the UV vertex we have to consider the interaction between the $A_m$ gauge field and two $B_\mu$ modes. The standard interaction is [4]

$$ S_{\text{UV}} = -\mu_7 (\pi \alpha')^2 iQ \int d^5 x \sqrt{g} \, A_\mu (B_\nu^* F^{\mu\nu} - B_\nu (F^{\mu\nu})^*) , $$

(23)

where we have already integrated over the $S^3$, being $Q$ the charge of the vector mode, given by the eigenvalue equation $\nu^j \partial_i B_\mu = iQ B_\mu$. Note that $Q$ does not have to coincide with $Q$, the charge of the original hadron. In fact, one has to sum over all possible intermediate states, and in particular all possible values of $Q$ (see appendix).
Figure 1: Feynman-Witten diagram associated to the DIS process with two intermediate hadron states. The momentum and the twist of each field are indicated, while the solutions are given in section 2. The incident hadron is represented by a double line, the intermediate spin-one modes are indicated by dashed lines, while the solid line denotes a scalar mode. The vertical dotted line represents the cut of the Cutkosky rule.

The diagram we need to calculate is shown in figure 1, where there is an incoming photon with momentum \( q_\mu \), which interacts with a massive \( B_\mu \) vector with momentum \( q'_\mu (q'^2 = -M_2^2) \) and \( \tau_{\text{min}} = 3 \). This vector interacts with an incident rho meson of momentum \( P_\mu \), and with an outgoing scalar field with momentum \( p'_m (p'^2 = -M_3^2) \) and conformal dimension \( \tau' \).

In order to calculate the diagram we need the AdS\(_5\) solutions of the fields. In the axial gauge the gravi-photon solution is

\[
A_\mu(x, z) = n_\mu e^{iq_x(qz)K_1(qz)} , \quad A_z(x, z) = 0 ,
\]

while the fields on the probe D7-brane are given at small \( L \) by the following approximate expressions [4]

\[
B_\mu^*(x, z) = \frac{1}{\sqrt{N}} c_B^* \zeta_\mu e^{-iq_x} \zeta \sqrt{\frac{\Lambda}{M_2}} z J_2(M_2 z) ,
\]

\[
\rho_\mu(x', z') = \frac{1}{\sqrt{N}} e_\rho e_\mu e^{iP_x} \sqrt{\frac{\Lambda}{M_1}} z J_{\tau-1}(M_1 z') ,
\]

\[
\phi^*(x', z') = \frac{1}{\sqrt{N}} \epsilon^* e^{-iq'_x} \epsilon' \sqrt{\Lambda M_3} z J_{\tau'-1}(M_3 z') ,
\]  

where \( c \)'s are numerical constants, and we have also included the polarization vectors. We have only written the AdS\(_5\) solution, the full ten-dimensional solution includes the 3-sphere contribution that only have the product of the scalar spherical harmonics \( Y^l(\Omega_3) \). On the
other hand, the propagator of the type II vector mode is given by [28]

\[ G_{\mu\nu}(x, x', z, z') = -\frac{i}{N} \int \frac{d^4k}{(2\pi)^4} G^{(k)}(z, z') \mathcal{T}_{\mu\nu} e^{ik \cdot (x-x')}, \]

where \( \mathcal{T}_{\mu\nu} = \eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \). Recall that in our case of interest the exchanged mode has conformal dimension \( \tau_{\text{min}} = 3 \).

Finally, we redefine the D7-brane fields as \( \Phi \to \frac{\Phi}{\sqrt{N}} \), such that they are canonically normalized in terms of \( N \). The \( 1/N \)-power counting shows that the interaction terms now scale as

\[ S_{\text{UV}} \to \frac{1}{N} S_{\text{UV}}, \quad S_{\text{IR}} \to \frac{1}{N^{3/2}} S_{\text{IR}}. \] (27)

### 3.2 Structure functions for vector mesons

We now derive the FCS amplitude related to the one-point function \( n_\mu \langle X_1, X_2 | J^\mu(0) | P, Q \rangle \).

Looking at the diagram of figure 1, the associated amplitude is

\[
A = -8i \mu^2 (\pi \alpha')^5 Q \frac{L}{R^3} (2\pi)^4 \delta^{(4)}(P + q - p' - q') \mathcal{I} \int dz dz' \sqrt{g} \sqrt{g'} z^2 \omega^2 \partial'^\alpha \partial'^\beta (z') \phi^*(z') \times 
\left[ B^{*\nu}(z) \left( \partial'_\alpha \partial_\mu G_{\nu\beta}(z, z') - \partial'_\alpha \partial_\beta G_{\mu\beta}(z, z') \right) - \partial'_\alpha G_{\nu\beta}(z, z') \left( \partial_\mu B^{*\nu}(z) - \partial'^\nu B^*_\mu(z) \right) \right] A^\mu(z),
\]

where \( \partial'_\alpha \) and \( \partial'_\beta \) are derivatives with respect to the primed coordinates. Also, we have already performed the integrals in the variables \( x \) and \( x' \), which lead to the momentum conservation relations

\[ k_\mu = q_\mu - q_\mu = P_\mu - p'_\mu. \] (29)

In equation (28) \( \mathcal{I} \) represents the integral of the scalar spherical harmonics over the 3-sphere, which is given in the appendix. Replacing the solutions (24), (25) and (26) in the amplitude, and using the relations (13), (14) and (19), the structure functions \( F_i \) \( (i = 1, 2, L) \) can be written as

\[ F_i(q^2, x) = |\tilde{c}|^2 Q^2 \mu^4 L^2 \alpha^{10} R^8 \sum_{M_2, M_3} \int \frac{d^3q'}{2E_{q'}(2\pi)^3} \frac{d^3p'}{2E_{p'}(2\pi)^3} \delta^{(4)}(P + q - q' - p') \times \Lambda^3 \frac{M_3}{M_1 M_2} q^2 |C_i|^2 F_i^T(x, P, Q, Q'), \] (30)
where $|\vec{c}|^2 = |c_\phi|^2 |c_B|^2 |c_\rho|^2$, $C_t$ contains the radial coordinate integrals, given by

$$C_t = \int d\omega d\omega' \int \frac{d\omega}{(q - q')^2 + \omega^2} \times \left[ z^4 J_{\tau - 1}(M_1 z') J_{\tau - 1}(M_3 z') J_2(\omega') \right] \left[ z^2 K_1(qz) J_2(M_2 z) J_2(\omega z) \right]. \quad (31)$$

In order to obtain $F_i^T$, the factor in (30) which depends only of the tensor contractions, we need to calculate $J^T_{\mu} J^T_{\nu}$, with

$$J^T_{\mu} = [\zeta^{\nu'}(k_{\mu} T_{\nu \beta} - k_{\nu} T_{\mu \beta}) + T_{\nu \beta}(q_{\mu} \zeta^{\nu'} - q^{\nu'} \zeta_{\mu}')] k_{\alpha} P^{[\alpha \beta]}]. \quad (32)$$

Recall that $B^*_{\mu}$ is an intermediate state, thus we need to sum over the outgoing vector polarizations $\zeta_{\mu}$

$$\sum_{\lambda} \zeta_{\mu} \zeta^*_{\nu} = -q'^2 \eta_{\mu \nu} + q_{\mu}' q_{\nu}'. \quad (33)$$

Since we are only interested in the unpolarized structure functions, we also average over the polarization vector of the incoming hadron

$$\frac{\epsilon_{\mu} \epsilon_{\nu}'}{3} = \frac{1}{3} (-P^2 \eta_{\mu \nu} + P_{\mu} P_{\nu}). \quad (34)$$

By comparing $J^T_{\mu} J^T_{\nu}$ with equations (13) and (14) we obtain the following expressions

$$F_1^T = \frac{1}{24 q^2 x^2} P^2 \left[ 128 x^4 (q')^2 (P \cdot q')^4 + 128 x^3 (q')^2 q \cdot q' (P \cdot q')^3 + 8q^6 x (q')^2 P \cdot q' \right. \left. + 40 q^4 x^2 (P \cdot q')^2 - 8q^4 x^2 (q \cdot q') (P \cdot q')^2 + 32 q^4 x (q')^4 P \cdot q' \right. \right.$$

$$-16 q^4 x (q \cdot q')^2 P \cdot q' + 28 q^2 x (q')^2 (q \cdot q') (P \cdot q') - 16 q^2 x^4 (P \cdot q')^4 \right. \right. \right. \right. \right.$$

$$+ 128 q^2 x^3 (q')^2 (P \cdot q')^3 - 48 q^2 x^3 (q \cdot q') (P \cdot q')^3 + 32 q^2 x^2 (q')^4 (P \cdot q')^2 \right. \right. \right. \right. \right.$$

$$- 40 q^2 x^2 (q \cdot q')^2 (P \cdot q')^2 + 128 q^2 x^2 (q')^2 (q \cdot q') (P \cdot q')^2 + 2q^8 (q')^2 \right. \right. \right. \right. \right.$$

$$+ 9q^6 (q')^4 - 2q^6 (q \cdot q')^2 - 2q^6 (q')^2 q \cdot q' \right. \right. \right. \right. \right.$$

(35)

$$F_2^T = \frac{1}{12 q^2 x} P^2 \left[ 384 x^4 (q')^2 (P \cdot q')^4 + 384 x^3 (q')^2 (q \cdot q') (P \cdot q')^3 \right. \right.$$

$$+ 64 x^2 (q')^2 (q \cdot q')^2 (P \cdot q')^2 + 8q^6 x (q')^2 P \cdot q' + 96 q^4 x^2 (q')^2 (P \cdot q')^2 \right. \right. \right. \right. \right.$$

$$- 8q^4 x^2 (q \cdot q') (P \cdot q')^2 + 32 q^4 x (q')^4 (P \cdot q') - 16 q^4 x (q \cdot q')^2 (P \cdot q') \right. \right. \right. \right. \right.$$

$$+ 76 q^4 x (q')^2 (q \cdot q') (P \cdot q') - 48 q^2 x^4 (P \cdot q')^4 + 384 q^2 x^3 (q')^2 (P \cdot q')^3 \right. \right. \right. \right. \right.$$

$$- 144 q^2 x^3 (q \cdot q') (P \cdot q')^3 + 32 q^2 x^2 (q')^4 (P \cdot q')^2 - 136 q^2 x^2 (q \cdot q')^2 (P \cdot q')^2 \right. \right. \right. \right. \right.$$

$$+ 384 q^2 x^2 (q')^2 (q \cdot q') (P \cdot q')^2 - 32 q^2 x (q \cdot q')^3 (P \cdot q') + 64 q^2 x (q')^2 (q \cdot q')^2 (P \cdot q') \right. \right. \right. \right. \right.$$

$$+ 2q^8 (q')^2 + 9q^6 (q')^4 - 2q^6 (q \cdot q')^2 - 2q^6 (q')^2 q \cdot q' + 8q^4 (q')^2 (q \cdot q')^2 \right. \right. \right. \right. \right.$$

(36)
In order to calculate the integrals in (31) we need to use a few reasonable approximations, in a similar way as in \[20, 19\]. The main assumption is that $\Lambda$ and the masses of the hadrons are small in comparison with the momentum of the virtual photon. The IR integral selects the mass of the exchanged field as follows \[19\]

$$
\int_0^{\Lambda^{-1}} dz' z'^4 J_{\tau-1}(M_1 z') J_{\tau-1}(M_3 z') J_2(\omega z') \approx \frac{1}{\Lambda^3} \frac{1}{\sqrt{M_1 M_3}} \left[ (-1)^\alpha \delta(\omega - (M_1 + M_3)) + (-1)^\beta \delta(\omega - (M_1 - M_3)) \right],
$$

for some integers $\alpha$ and $\beta$. The integral leads to $\omega = |M_1 \pm M_3|$. Then, the UV integral can be obtained by expanding $J_2(\omega z) \approx \omega^2 z^2/8$ for $\omega \ll q$, and taking the upper limit as infinity since $K_1$ decays quickly in the bulk. We obtain

$$
\int_0^\infty dz \frac{\omega^2}{8} z^4 K_1(qz) J_2(M_2 z) \approx \frac{6 M_2^2 q \omega^2}{(M_2^2 + q^2)^4}.
$$

With these two equations we can obtain $C_t$, after noticing that the leading term comes from $\omega = M_1 - M_3$ \[20\], we obtain

$$
|C_t|^2 = \frac{q^2 36 M_2^4}{(M_2^2 + q^2)^8} \frac{1}{\Lambda^6} \frac{1}{M_1 M_3} \frac{1}{((q - q')^2 + (M_1 - M_3)^2)^2}.
$$

The next step is to integrate over the on-shell momenta $\vec{p}'$ and $\vec{q}'$, and sum over the corresponding masses\(^6\). The final results for the structure functions are

$$
F_1(x, q^2) = \frac{1}{\lambda N} C \left( \frac{M_1}{\Lambda} \right)^6 \frac{\Lambda^2}{q^2} \frac{1}{2} x^3 (1 - x)^3
$$

$$
F_2(x, q^2) = \frac{1}{\lambda N} C \left( \frac{M_1}{\Lambda} \right)^6 \frac{\Lambda^2}{q^2} x^3 (1 - x)^3 (4 - 3x)
$$

$$
F_L(x, q^2) = \frac{1}{\lambda N} C \left( \frac{M_1}{\Lambda} \right)^6 \frac{\Lambda^2}{q^2} 4 x^3 (1 - x)^4,
$$

where $C$ is a numerical constant.

We expect qualitatively similar results to hold in the context of different Dp-brane models.

4 Discussion and conclusions

We have obtained the $1/N$ corrections to the $F_1$, $F_2$ and $F_L$ structure functions corresponding to vector mesons, using the gauge/gravity duality. Motivated by previous work for $N = 4$

\(^6\)For more details we refer the reader to reference \[20\].
SYM theory glueballs, and particularly for scalar mesons in the D3D7-brane system, the idea is to investigate how two very different limits behave, namely: the large $N$ limit in comparison with the high energy limit ($\Lambda^2 / q^2 \to 0$). Our first result is that they do not commute for the vector mesons. Then, since the physical way to consider these limits implies to take first the high energy one, followed by the large $N$ limit, we find that in this situation the third term in the expansion of equation (5) dominates the moments of the structure functions. This is a very interesting result which says that at strong coupling this third term becomes the leading one for $q^2 \geq \Lambda^2 N^{1/(\tau_Q - \tau_c)}$, where $\tau_Q$ and $\tau_c$ are the minimum twist of protected operators with charge $Q$ and the minimum twist of all electrically charged protected operators, respectively.

This is similar to what happens for the glueball in the IR deformed version of $\mathcal{N} = 4$ SYM [6, 19], where the process is given in terms of closed string modes and at strong coupling one finds that the $1/N$ result dominates for $q^2 \geq \Lambda^2 N^{2/(\tau_Q - \tau_c)}$. However, note that in the present case, i.e. for mesons, the correction is proportional to the $1/N$ instead of $1/N^2$, as expected. Thus, at large $N$ the critical value for the photon virtuality $q$ where this happens is much smaller. The same occurs for the results presented in [20] for the case of scalar mesons.

The physical implication of this result is that, at strong coupling, for the above energy range DIS is dominated by a two-hadron final state. From the viewpoint of the FCS process it corresponds, through the optical theorem, to a situation where there are two intermediate hadron states. The structure functions $F_1$ and $F_2$ behave as $(1 - x)^3$ as $x$ approaches 1. We have also considered the longitudinal structure function $F_L$ which behaves as $(1 - x)^4$ as $x$ approaches 1. We should notice that, although the states as well as the interactions for the vector and scalar mesons are different, all these structure functions have the same dependence on $\Lambda^2 / q^2$, $1/N$, $1/\lambda$ and $M_1/\Lambda$ as in the case of scalar mesons in the D3D7-brane system.

It is important to consider the moments of the structure functions defined as

$$M_n[F_i] = \int_0^1 dx \, x^{n-1} F_i(x, q^2),$$

(43)

where $F_i$ can be $F_1$ and $F_2$ in this case.

Several moments of these structure functions have been calculated in reference [18] in the large $N$ limit, i.e. by considering a single intermediate hadron state in the FCS process. This has been done for the first moments of the structure function $F_2$ in the case of the pion as well as for $F_1$ of the rho meson. In [18] we have compared these results with lattice QCD data from references [21, 22, 23] for the pion, associated with the lightest pseudoscalar mode. In addition, in the case of the rho meson associated to the $l = 2$ spin-1 mode of the type II solutions, the comparison has been made with respect to results from lattice QCD of [21]. The best fit for the case of the pion leads to results with an accuracy of 10.8%...
or better, while in the case of the rho meson the accuracy is of 18.5% or better \(^7\), for the D3D7-brane system. In [18] also the Sakai-Sugimoto model of the D4D8D8-brane system and the D4D6D6-brane system, both in type IIA string theory, have been considered for FCS with a single intermediate exchanged state. The next step has been done in [20] where we have considered the leading 1/N corrections to the structure functions. The accuracy is notoriously enhanced to 1.27\% for the scalar mesons in the D3D7-brane system in this case. It leads to a natural question which is whether for vector mesons the accuracy of the fit can also be substantially improved by considering 1/N corrections.

In order to investigate this point we have carried out the best fit of the structure \(F_1\) including 1/N corrections in comparison with lattice QCD data from [21]. Recall that the results of the present work have been obtained in the type IIB supergravity regime, i.e. where \(1/\sqrt{\lambda} \ll x < 1\), which means that for the calculation of the moments we have integrated our result for the functions between \(x = 0.1\) and \(x = 1\). On the other hand, we also need to carry out the integration over the range \(\exp(-\sqrt{\lambda}) \ll x \ll 1/\sqrt{\lambda}\), where we assume that the behavior of the structure functions is similar to the behavior shown in [5] and used in [18], i.e. \(F_L^{\text{small}-x} \propto x^{-1}\). We support this assumption on the fact that, in the \(1/\sqrt{\lambda} \ll x < 1\) range the difference between the large-N calculation (where there is only a single on-shell hadron state exchanged in the FCS process) and the 1/N calculation (where the leading Feynman-Witten diagram has two on-shell hadron states exchanged) is that in the former the dependence with the photon virtuality and the Bjorken parameter is given by \(\tau_{\text{in}}\) corresponding to the incident meson, while in the later the \(q^2\) and \(x\) dependence is determined by \(\tau_{\text{min}}\). This corresponds to one of the lowest conformal dimension from the supergravity excitations. However, for \(\exp(-\sqrt{\lambda}) \ll x \ll 1/\sqrt{\lambda}\) things are different, namely: the calculation from the two-open and two-closed strings scattering amplitudes is independent of \(\tau_{\text{in}}\). Thus, we assume that in this low-x regime the genus-zero result from type IIB superstring theory should not be very different with respect to a much more complicated calculation on the torus. Then, for this string theory regime of the holographic dual calculation we use the expressions for the structure functions at tree level from [18]. For our numerical calculation at low-x we consider that the integration for the moments is performed between \(x = 0.0001\) and \(x = 0.1\) as before [18, 20]. Then, we split each structure function in two parts, each of one having a dimensionless constant to be fixed by fitting with respect to lattice QCD data [21]. There is a constant \(C_1\) multiplying the low-x \(F_1\) function. In addition, there is a second constant \(C_2\) on the large-x \(F_1\) function.

Results of the first moments of \(F_1\) of the rho meson are presented in table 2. The values we obtain for the constants are \(C_1 = 0.0087\) and \(C_2 = 32.1939\). Note that they are of the same order as the ones found in our previous work [18] in the large \(N\) limit, for which the constants associated with the small-x \(F_1\) and with the large-x \(F_1\) of the rho meson are 0.012

\(^7\)There is a mistake in table 4 of reference [18] that we have corrected here. The original errors presented in that reference were overestimated.
Table 2: Comparison of our results for the first moments of the structure function $F_1$ of the lightest vector meson for a suitable choice (best fitting) of the normalization constants with respect to the results of the lattice QCD simulations in [21] and in comparison with previous results presented in [18]. Uncertainties in the lattice QCD computations are omitted.

| Model / Moment | $M_2(F_1)$ | $M_3(F_1)$ | $M_4(F_1)$ |
|----------------|------------|------------|------------|
| Lattice QCD    | 0.1743     | 0.074      | 0.035      |
| D3D7 ($N \to \infty$) | 0.1753 | 0.060 | 0.039 |
| Percentage error | -0.6        | 18.5       | -12.8      |
| D3D7 (1/$N$)    | 0.1750     | 0.065      | 0.038      |
| Percentage error | -0.39       | 12.5       | -9.6       |

Figure 2 shows the structure function $F_1$ as a function of $x$. The blue bell-shaped curve indicates the $1/N$ calculation of this work. The black dashed bell-shaped line corresponds to the case obtained in reference [18] for the large $N$ limit. For small-$x$ values we use the result from [5], leading to the monotonically decreasing curves. The difference between the two curves at small values of the Bjorken parameter comes from the slightly different constants which correspond to the best fit developed in each situation. There is an important improvement with the inclusion of the leading $1/N$ correction. As it happened in the scalar

Figure 2: $F_1$ as a function of $x$. We display the leading results at low and moderate ($0.1 < x \leq 1$) values of $x$. Dashed curves represent the $F_1$ computed for $N \to \infty$, while the blue ones correspond to the $1/N$ corrections. The constants $C_1$ and $C_2$ are those which give the moments of $F_1$ shown in table 2.
case, the location of the maximum is shifted to the left with respect to the results obtained in the planar limit, matching better the phenomenological expectations.

From table 2 we can appreciate the enhancement of the accuracy of the moments of $F_1$ for the case of the rho meson which goes from 18.5% for one particle exchange in the FCS, down to 12.5% in the leading $1/N$ contribution, i.e. for the exchange of two intermediate on-shell states. This is very important because it confirms the trend found previously for glueballs in $\mathcal{N} = 4$ SYM theory [19] and for the scalar mesons of the $\mathcal{N} = 2$ SYM theory from the D3D7-brane system [20]. Thus, it indicates that in order to infer realistic conclusions for physical systems it is crucial to consider the $1/N$ expansion of the observables, and consider first the large momentum transfer limit. Possibly, this behavior can be identified in other physical process, and although we have restricted our investigation to strongly coupled gauge theories it would be very interesting to investigate the relations between the large $N$ limit and the high energy one for gauge field theories at perturbative level.

Acknowledgments

N.K. acknowledges kind hospitality at the Institut de Physique Théorique, CEA Saclay, and at the Institute for Theoretical Physics, University of Amsterdam, during the completion of this work. G.M. acknowledges kind hospitality at the International Center for Theoretical Physics, Trieste, where part of this work has been done. This work has been supported by the National Scientific Research Council of Argentina (CONICET), the National Agency for the Promotion of Science and Technology of Argentina (ANPCyT-FONCyT) Grants PICT-2015-1525 and PICT-2017-1647, and the CONICET Grant PIP-UE Búsqueda de nueva física.

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See for example references [26, 27].
Appendix: Angular integrals

Scalar spherical harmonics on the 3-sphere belong to the \((l/2, l/2)\) representation of \(SU(2) \times SU(2) \equiv SO(4)\), where \(l\) is a non-negative integer, while \(-l/2 \leq m, n \leq l/2\). They form a basis of eigenfunctions of the Laplace operator on the sphere,

\[
\nabla^2 Y_l^{m,n} = -l(l+2)Y_l^{m,n},
\]

and satisfy the orthogonality relation

\[
\int_{S^3} (Y_l^{m,n})^* Y_{l'}^{m',n'} = \delta_{ll'} \delta_{mm'} \delta_{nn'}.
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\]

where

\[
(Y_l^{m,n})^* = (-1)^{m+n} Y_{l}^{-m,-n}.
\]

The integral we want to calculate has three of scalar spherical harmonics. An analytic expression for this type of integrals can be found in [29, 30], and it reads

\[
\int_{S^3} Y_l^{m,n} Y_{l'}^{m',n'} Y_{l''}^{m'',n''} = R_1(l, l', l'') \left( \begin{array}{ccc} \frac{l}{2} & \frac{l'}{2} & \frac{l''}{2} \\ m & m' & m'' \end{array} \right) \left( \begin{array}{ccc} \frac{l}{2} & \frac{l'}{2} & \frac{l''}{2} \\ n & n' & n'' \end{array} \right),
\]

where we have included the 3j-symbols, and the function \(R_1\) is defined as

\[
R_1(x, y, z) = \frac{(-1)^\sigma}{\pi} \sqrt{\frac{(x+1)(y+1)(z+1)}{2}}, \quad \text{with } \sigma = \frac{(x+y+z)}{2}
\]

and must satisfy the triangle inequality \(|x - z| \leq y \leq x + z\) in order to be non-vanishing. Also, the 3j-symbols must satisfy the physical condition: \(m + m' + m'' = n + n' + n'' = 0\).

In our case of interest, i.e. the UV vertex of the diagram shown in figure 1, the incoming state is a rho meson with quantum numbers \((l, m, n)\), the exchanged particle is a vector mode with \((l', m', n')\), and the outgoing state is the scalar \(\phi\) with \((l'', m'', n'')\). The leading diagram is the one where \(\Delta' = 4\), fixing \(l' = 1\) and \(-1/2 \leq m', n' \leq 1/2\). The rho meson is a hadron which can have positive, null or negative charge. In the D3D7-brane system, the charge is given by the \(R\)-symmetry associated to the \(R\) sector of the \(SU(2)_L \times SU(2)_R\). This implies that \(l = 2\), while \(m\) can take three values 1, 0, -1 for a positive, null or negative charged meson respectively. Therefore, we have the following product of 3j-symbols

\[
R_1(2, 1, l'') \left( \begin{array}{ccc} 1 & \frac{1}{2} & \frac{l''}{2} \\ m & \pm \frac{1}{2} & -m \mp \frac{1}{2} \end{array} \right) \left( \begin{array}{ccc} 1 & \frac{1}{2} & \frac{l''}{2} \\ n & \pm \frac{1}{2} & -n \mp \frac{1}{2} \end{array} \right),
\]

18
where \( m \) can be 1, 0, −1. For any of these values, equation (49) does no vanish only if \( l'' = 1, 2, 3 \). We want to sum over all possible exchanged and outgoing states, this is a sum over \( m', n' = \pm 1/2 \), and over \( l'' \). The final result is the same for the three values of \( m \), and it reads

\[
\frac{1}{2} \sum_{m',n'=-1/2}^{1/2} \sum_{l''=1}^{3} \left[ R_l(2,1,l'') \begin{pmatrix} 1 & \frac{1}{2} & \frac{l''}{2} \\ m & m' & -m-m' \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{l''}{2} \\ n & n' & -n-n' \end{pmatrix} \right]^2 = \frac{6}{\pi^2},
\]

which is the value of \( I \) present in equation (28).
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