A NOTE ON EFFECTIVE DESCENT FOR OVERCONVERGENT ISOCRYSTALS

CHRISTOPHER LAZDA

ABSTRACT. In this short note we explain the proof that proper surjective and faithfully flat maps are morphisms of effective descent for overconvergent isocrystals. We then show how to deduce the folklore theorem that for an arbitrary variety over a perfect field of characteristic $p$, the Frobenius pull-back functor is an equivalence on the overconvergent category.

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INTRODUCTION

Let $k$ be a field of characteristic $p > 0$. One of the major problems in arithmetic geometry over the last 50 years or so has been that of describing a ‘good’ category of coefficients for $p$-adic cohomology of varieties over $k$, with behaviour mirroring that of the category of $\ell$-adic étale sheaves for $\ell \neq p$. The first attempt at doing so was the category of crystals on a variety introduced by Berthelot in his thesis, following an idea of Grothendieck. However, this category fails one of the basic requirements that one expects of such a ‘good’ category of coefficients, namely topological invariance. This manifests itself in the fact that the Frobenius pull-back functor

$$F^* : \text{Crys}(X/W)_\mathbb{Q} \to \text{Crys}(X/W)_\mathbb{Q}$$

on isocrystals is not necessarily an equivalence of categories, even if $k$ is perfect and $X$ is smooth and proper. This problem was rectified by the introduction of the category of convergent isocrystals in [Ogu84], which when $k$ is perfect turns out to be the largest full sub-category of $\text{Crys}(X/W)_\mathbb{Q}$ on which $F^*$ is an equivalence. This characterisation is deduced in part from the fact that the category of convergent isocrystals satisfies descent under proper and surjective morphisms of varieties, which in turn implies the required topological invariance.

When $X$ is not proper, Berthelot introduced in [Ber96] a refinement of the category of convergent isocrystals on $X$, by considering ‘overconvergence conditions’ (on both objects and morphisms) along the boundary.
of some compactification $X \hookrightarrow \overline{X}$. A natural question then arises of whether or not this category of overconvergent isocrystals satisfies proper descent, and this was originally proved by Shiho [Shi07, Proposition 7.3] in rather greater generality. Specifically, Shiho works relative to a general $p$-adic formal scheme of finite type over a DVR, and with partially overconvergent cohomology of pairs. The purpose of this note is to explain the proof of the following version of descent for overconvergent isocrystals, using flat descent in adic geometry.

**Theorem (5.1).** Let $f : X \to Z$ be a proper surjective or faithfully flat morphism of varieties over $k$. Then $f$ is a morphism of effective descent for overconvergent isocrystals.

Note also that the ‘full faithfulness’ part of descent also follows from the more general results on cohomological descent proved in [Tsu03, ZB14]. In the expected manner, one can then use this result to obtain invariance of $\text{Isoc}^\dagger(X/K)$ under universal homeomorphisms, and in particular the following folklore theorem.

**Corollary (6.2).** Assume that $k$ is perfect, and let $X$ be a variety over $k$. Then the Frobenius pull-back functor

$$F^* : \text{Isoc}^\dagger(X/K) \to \text{Isoc}^\dagger(X/K)$$

is an equivalence of categories.

For smooth varieties with good compactifications, this follows from Berthelot’s theorem on Frobenius descent for arithmetic $\mathcal{D}$-modules [Ber00]. A more general version of this result (and therefore the deduction of Corollary 6.2) forms part of current work in progress of Crew (see the introduction to [Cre17] for details). The proof via Theorem 5.1, however, is reasonably direct (i.e. does not depend on any results on arithmetic $\mathcal{D}$-modules), and Theorem 5.1 itself is potentially of independent interest.

The key input into the proof of Theorem 5.1 is the following version of flat descent in analytic geometry.

**Theorem (2.8).** Let $f : X \to Y$ be a faithfully flat morphism of adic spaces locally of finite type over a complete, discretely valued field. Then $f$ is a morphism of effective descent for coherent sheaves.

This is essentially just a rephrasing of the descent results of [Con06, §4]; our modest contribution is the rather satisfying observation that Conrad’s condition that a flat map of rigid analytic spaces (in the sense of Tate) ‘admits local fpqc sections’ translates exactly into the surjectivity of the associated map on adic spaces. We would therefore like to view this result as yet more evidence (if it were needed) that Huber’s theory of adic spaces really is the correct setting in which to do non-archimedean analytic geometry.

Given this analytic descent result the proof of Theorem 5.1 proceeds more or less as expected, the proper surjective case being entirely similar to the proof of the corresponding result in [Shi07]. The point is that a projective surjective map of varieties can, locally on the base, be extended to a proper flat morphism of frames. One can then show that the induced morphism on suitably small neighbourhoods of the respective tubes is faithfully flat (in the sense of adic geometry) and therefore is a morphism of effective descent for coherent sheaves. Applying this universally in frames mapping to the base $Z$ and using Le Stum’s ‘site-theoretic’ interpretation of $\text{Isoc}^\dagger(X/K)$ given in [LS07, §8] completes the proof. The faithfully flat case can then easily be deduced.

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Notations and conventions. We will let \( \mathcal{V} \) be a complete, discrete valuation ring with fraction field \( K \) of characteristic 0 and residue field \( k \) of characteristic \( p > 0 \). We will let \( \mathfrak{p} \) be a uniformiser of \( \mathcal{V} \). A variety over \( k \) will be a separated scheme of finite type, and the category of these objects will be denoted \( \text{Var}_k \). All formal schemes over \( \mathcal{V} \) will be assumed to be of finite type.

An analytic space over \( K \) will be an adic space locally of finite type over \( \text{Spa}(K, \mathcal{V}) \). It will be called an analytic variety over \( K \) if in addition the structure morphism to \( \text{Spa}(K, \mathcal{V}) \) is separated. Similarly, we will refer to a rigid analytic space in the sense of Tate [BGR84, §9.3.1] as a ‘rigid space’, and when it is separated over \( \text{Sp}(K) \) we will call it a rigid variety. Thus by [Hub96, §1.1.11] there is a fully faithful functor from rigid spaces over \( K \) to analytic spaces over \( K \), which induces an equivalence on the full-subcategories of quasi-separated objects. We will denote the category of analytic spaces over \( K \) by \( \text{An}_K \). We will abbreviate ‘quasi-compact and quasi-separated’ as ‘qcqs’.

1. The formalism of descent

In this section we will very briefly recall the formalism of descent, and introduce the three examples that particularly interest us, namely coherent sheaves on analytic spaces over \( K \), \( \mathbb{J} \)-modules on frames over \( \mathcal{V} \), and overconvergent isocrystals on algebraic varieties over \( k \). So suppose that we have a fibred category

\[ \mathcal{F} \to \mathcal{C} \]

over some base category \( \mathcal{C} \). That is, for every object \( X \in \mathcal{C} \) we have a category \( \mathcal{F}_X \), and for every morphism \( f : X \to Y \) a pull-back functor \( f^* : \mathcal{F}_Y \to \mathcal{F}_X \), which are compatible under composition. Then for any morphism \( f : X \to Y \) in \( \mathcal{C} \) we have two pull-back functors

\[ \pi_0^1, \pi_1^1 : \mathcal{F}_X \to \mathcal{F}_{X \times_Y X} \]

associated to the two projections \( \pi_i : X \times_Y X \to X \). Similarly, we have three projections

\[ \pi_{01}, \pi_{12}, \pi_{02} : X \times_Y X \times_Y X \to X \times_Y X \]

giving rise to corresponding pull-back functors.

**Definition 1.1.** If \( E \in \mathcal{F}_X \) then descent data on \( E \) relative to \( f \) is an isomorphism \( \alpha : \pi_0^1 E \cong \pi_1^1 E \) such that

\[ \pi_{02}^1(\alpha) = \pi_{12}^1(\alpha) \circ \pi_{01}^1(\alpha). \]

The category of objects in \( \mathcal{F}_X \) equipped with descent data is denoted \( \mathcal{F}_{X \times_Y X \Rightarrow X} \), pull-back by \( f \) induces

\[ f^* : \mathcal{F}_Y \to \mathcal{F}_{X \times_Y X \Rightarrow X}. \]

**Definition 1.2.** We say that \( f \) is a morphism of descent for \( \mathcal{F} \) if the functor

\[ f^* : \mathcal{F}_Y \to \mathcal{F}_{X \times_Y X \Rightarrow X} \]

is fully faithful. We say that \( f \) is a morphism of effective descent for \( \mathcal{F} \) if \( f^* \) is an equivalence of categories.

The three key examples of fibred categories we will consider in this note are the following.

**Example 1.3.** (1) As in [Cre92, §1] (but fixing the ground field \( K \)) we will view the category of overconvergent isocrystals \( \text{Isoc}^\dagger \) as a fibred category

\[ \text{Isoc}^\dagger \to \text{Var}_k \]
over the category of \( k \)-varieties. That is, for every \( X \in \text{Var}_k \) we have the category \( \text{Isoc}^\dagger(\mathcal{X}/\mathcal{K}) \) of overconvergent isocrystals on \( \mathcal{X}/\mathcal{K} \), and for every morphism \( f : X \rightarrow Y \) there is a pull-back functor \( f^* : \text{Isoc}^\dagger(\mathcal{Y}/\mathcal{K}) \rightarrow \text{Isoc}^\dagger(\mathcal{X}/\mathcal{K}) \), compatibly with composition.

(2) Similarly, we may view the category \( \text{Coh} \) of coherent sheaves as a fibred category

\[
\text{Coh} \rightarrow \text{An}_K
\]

over the category of analytic spaces over \( K \).

(3) Let \( \text{Frame}_\mathcal{Y} \) denote the category of frames over \( \mathcal{Y} \), that is triples \( (X, \overline{X}, \mathcal{X}) \) consisting of an open immersion of \( k \)-varieties \( X \rightarrow \overline{X} \) and a closed immersion \( \overline{X} \rightarrow \mathcal{X} \) of separated formal \( \mathcal{V} \)-schemes. Then taking \( (X, \overline{X}, \mathcal{X}) \) to the category of coherent \( j_x^\dagger \mathcal{O}_{\overline{X} \times \mathcal{X}} \)-modules (in the sense of adic geometry) gives rise to a fibred category

\[
\text{Coh}^\dagger \rightarrow \text{Frame}_\mathcal{Y}.
\]

We will explain this example in great detail in §3 below.

2. FLAT DESCENT FOR ANALYTIC SPACES

The purpose of this section is to give a careful discussion of flat descent for analytic spaces, and in particular rephrasing the results of [Con06] in terms of adic spaces. One particularly pleasing aspect of this reformulation is that it gives a very natural interpretation of the condition appearing in [Con06, §4] that a flat map of rigid spaces ‘admits local fpqc sections’ - it simply means that the induced map on adic spaces is surjective. This will then let us deduce a simple-to-state version of flat descent for analytic spaces over \( K \).

**Definition 2.1.** Let \( f : X \rightarrow Y \) be a morphism of analytic spaces over \( K \).

(1) We say that \( f \) is flat if for all \( x \in X \) the ring homomorphism \( \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x} \) is flat.

(2) We say that \( f \) is faithfully flat if in addition \( f \) is surjective.

(3) We say that \( f \) is fpqc if it is faithfully flat and quasi-compact.

Note that the second condition is stronger than simply requiring surjectivity on rigid points, as the following example shows.

**Example 2.2.** Let \( X \) be the disjoint union of the open unit disc and the closed annulus of radius 1. Let \( Y \) be the closed unit disc, and \( f : X \rightarrow Y \) the obvious map. Then \( f \) is flat and surjective on rigid points, but not faithfully flat in our sense.

Since we will be comparing with the situation of rigid spaces, let us recall the following definitions.

**Definition 2.3.** Let \( f_0 : X_0 \rightarrow Y_0 \) be a morphism of rigid spaces over \( K \).

(1) We say that \( f_0 \) is flat if for all \( x \in X_0 \) the ring homomorphism \( \mathcal{O}_{Y_0,f_0(x)} \rightarrow \mathcal{O}_{X_0,x} \) is flat.

(2) We say that \( f_0 \) is fpqc if it is flat, quasi-compact and surjective.

Note that we have deliberately avoided giving the definition of a faithfully flat map of rigid spaces without additional quasi-compactness hypotheses. We will first need to check various compatibilities of these notions. Note that it follows immediately from the definitions that a morphism \( f_0 : X_0 \rightarrow Y_0 \) of rigid spaces over \( K \) is flat if the associated morphism \( f : X \rightarrow Y \) of analytic spaces over \( K \) is so, and in fact the converse is also true.
Proposition 2.4. Let $f_0 : X_0 \to Y_0$ be a morphism of qcqs rigid spaces over $K$, with induced morphism $f : X \to Y$ of analytic spaces over $K$. Then the following are equivalent.

1. $f_0$ is flat (resp. fpqc);
2. $f$ is flat (resp. fpqc);
3. there exists a flat (resp. fpqc) morphism $X \to Y'$ of admissible formal schemes over $\mathcal{V}'$ whose induced morphism on rigid generic fibres is $f_0$, and on adic generic fibres is $f$.

Proof. We clearly have (2)$\Rightarrow$(1), let us show that (3)$\Rightarrow$(2). In the flat case this is straightforward - since flatness is local, and on the level of formal schemes is equivalent to locally being of the form $\text{Spf}(B^\circ) \to \text{Spf}(A^\circ)$ with $A^\circ \to B^\circ$ a flat morphism of topologically finite type $\mathcal{O}$-adic $\mathcal{V}'$-algebras. It therefore suffices to show that if $A \to B$ is a flat morphism of affinoid $K$-algebras, then $\text{Spa}(B,B^+) \to \text{Spa}(A,A^+)$ is flat. But this simply follows from the fact that analytic localisations are flat morphisms.

In the fpqc case this is a little more involved, the key claim is that if $X \to Y$ is fpqc, then the induced map $X \to Y$ on adic generic fibres is surjective. Applying [Sch12, Theorem 2.22], we will divide the map $X \to Y$ into two parts. First of all, let $\mathcal{E}$ denote the category of admissible blow-ups of $Y$, and $\mathcal{D}$ that of $X$. By [Stacks, Tag 080F] there is therefore a canonical functor $\mathcal{E} \to \mathcal{D}$ taking $Y' \to Y$ to its base change $X' \to X$, the induced map $X' \to Y'$ is therefore faithfully flat. We now consider the maps

$$X = \lim_{\leftarrow \mathcal{E}} X' \to \lim_{\leftarrow \mathcal{D}} Y' = Y,$$

and can conclude by applying [FK13, Theorem 0.2.2.13] twice.

Finally, we note that (1)$\Rightarrow$(3) in the flat case follows from [Bos09, Theorem 7.1], for the fpqc case, we can argue as follows. Suppose we have a flat formal model $X \to Y$ of $X_0 \to Y_0$ which is not fpqc. Then we get a factorisation $X \to U \to Y$ where $U \subset Y$ is a proper open formal sub-scheme. Thus we obtain a factorisation $X_0 \to U_0 \to Y_0$ where $U_0 \subset Y_0$ is a proper open sub-variety, contradicting surjectivity of $f_0$. □

Corollary 2.5. Let $f : X \to Y$ be a flat morphism of analytic spaces over $K$. Then $f$ is open.

Proof. The question is local on both $Y$ and $X$, we may therefore assume them both to be qcqs. It moreover suffices to show that the image $f(X)$ is open. We know that there exists a flat formal model $X \to Y$ of $f$, and arguing as in [Bos09, Corollary 7.2] we can see that this map has to factor as a fpqc map $X \to U$ followed by an open immersion $U \to Y$. We can now apply Proposition 2.4 above. □

This then begs the question of what the `rigid’ analogue of faithful flatness is, and rather pleasingly this turns out to be exactly the descent condition appearing in [Con06, Theorem 4.2.8].

Definition 2.6. We say that a flat map $f_0 : X_0 \to Y_0$ of rigid spaces ‘admits local fpqc sections’ if there exists an admissible cover $Y_0 = \bigcup_{0,i} Y_0,i$, fpqc maps $Z_{0,i} \to Y_0$, and for each $i$ factorisations $Z_{0,j} \to X_0 \to Y_0$ of $Z_{0,i} \to Y_0$.

The proof of the following is similar to that of [Con06, Theorem 4.2.8].

Theorem 2.7. Let $f_0 : X_0 \to Y_0$ be a flat morphism of rigid spaces over $K$, with $f : X \to Y$ the induced morphism of analytic spaces over $K$. Then $f_0$ admits local fpqc sections if and only if $f$ is faithfully flat.

Proof. First suppose that $f_0$ admits fpqc local sections, we must show that $f$ is surjective. This is clearly local for an admissible covering of $Y_0$, hence we may assume that $Y_0$ is affinoid, and that there exists an fpqc
map $Z_0 \to Y_0$ and a commutative diagram

\[
\begin{array}{ccc}
X_0 & \searrow & Z_0 \\
& & \searrow \downarrow \\
& & Z_0 \to Y_0.
\end{array}
\]

Hence by simple functoriality, it suffices to show that if $Z_0 \to Y_0$ is fpqc, then the induced map $Z \to Y$ is surjective. This follows from Corollary 2.4 above.

Conversely, let us suppose that $f$ is faithfully flat, we wish to show that $f_0$ admits local fpqc sections.

This question is clearly local for an admissible cover of $Y_0$, which we may therefore assume to be affinoid, and in particular qcqs. Thus for any qcqs open $V \subset X$ we know that $f(V)$ is a qcqs open in $Y$. Hence the fpqc map $V \to f(V)$ between qcqs analytic spaces over $K$ has to come from an fpqc map $V_0 \to f_0(V_0)$ of Tate spaces over $K$. As $V$ ranges over an open cover of $X$ by qcqs opens, the images $f(V)$ form an open cover of $Y$. Hence the images $f_0(V_0)$ form an admissible open cover of $Y_0$, and $f_0$ admits fpqc local sections.

By following the proof of this theorem, that is arguing as in the proof of [Con06, Theorem 4.2.8], we can deduce the required descent result from [BG98, Theorem 3.1].

**Theorem 2.8.** Let $f : X \to Y$ be a faithfully flat morphism of analytic spaces over $K$. Then $f$ is a morphism of effective descent for coherent sheaves.

**Proof.** We may assume that $Y$ is affinoid. Let $\{V_i\}$ be an open cover of $X$ by qcqs opens, and let $U_i = f(V_i)$. Then we have a commutative diagram

\[
\begin{array}{ccc}
\bigsqcup V_i & \searrow & X \\
& & \searrow \downarrow \\
& & \bigsqcup U_i \to Y
\end{array}
\]

and since we know effective descent for open covers, it suffices to show that each $f_i : V_i \to U_i$ is of effective descent for coherent sheaves. But now $f_i$ is an fpqc morphism between qcqs analytic spaces, in particular it comes from an fpqc morphism $V_{0,i} \to U_{0,i}$ of rigid spaces. Hence we may apply [BG98, Theorem 3.1] to conclude.

**Remark 2.9.** Of course, [BG98, Theorem 3.1] applies to coherent module on rigid spaces, not analytic spaces. However, if $X_0$ is a qcqs rigid space, with associated analytic space $X$, then there is a canonical equivalence of categories $\text{Coh}(\mathcal{O}_{X_0}) \cong \text{Coh}(\mathcal{O}_X)$, functorial in $X$.

### 3. Tubes and Isocrystals in Adic Geometry

In this section, we will give a quick review of Berthelot’s theorem of tubes and overconvergence, from the point of view of adic geometry.

**Definition 3.1.** A frame $(X, Y, \mathfrak{F})$ over $\mathcal{V}$ is a triple consisting of an open immersion $X \hookrightarrow Y$ of $k$-varieties and a closed immersion $Y \hookrightarrow \mathfrak{F}$ of formal $\mathcal{V}$-schemes.

Let $\mathfrak{F}_X$ denote the adic generic fibre of $\mathfrak{F}$. Then we have a *continuous* specialisation map

$$\text{sp} : \mathfrak{F}_k \to \mathfrak{F}_k$$
and we define $|Y|_{\mathfrak{Q}} := \text{sp}^{-1}(Y)^{\circ}$ to be the topological interior of the inverse image of $Y$ under sp. Thus we have a continuous specialisation map

$$\text{sp}_Y : |Y|_{\mathfrak{Q}} \to Y$$

and we define $|X|_{\mathfrak{Q}} := \text{sp}^{-1}(X)$ to be the topological closure of the inverse image of $X$ under $\text{sp}_Y$. This only depends on the embedding $X \hookrightarrow \mathfrak{Q}$ and not on $Y$. Thus we have a closed immersion

$$j : |X|_{\mathfrak{Q}} \to |Y|_{\mathfrak{Q}}$$

and we define, for any sheaf $\mathcal{F}$ on $|Y|_{\mathfrak{Q}}$, the sheaf $j_{X}^{\ast}\mathcal{F} := j_{\ast}j^{-1}\mathcal{F}$. In particular, we may consider the category $\text{Coh}(j_{X}^{\ast}\mathcal{O}_{|Y|_{\mathfrak{Q}}})$ of coherent $j_{X}^{\ast}\mathcal{O}_{|Y|_{\mathfrak{Q}}}$-modules on $|Y|_{\mathfrak{Q}}$, and this category is canonically equivalent to its rigid analogue as considered in [LS07, §5.4].

If $f : (W, Z, \mathfrak{Q}) \to (X, Y, \mathfrak{Q})$ is a morphism of frames there is an obvious pull-back functor

$$f^{\ast} : \text{Coh}(j_{X}^{\ast}\mathcal{O}_{|Y|_{\mathfrak{Q}}}) \to \text{Coh}(j_{W}^{\ast}\mathcal{O}_{|Z|_{\mathfrak{Q}}})$$

which again can be identified with its rigid analogue. In particular, we may apply [LS07, §8] to obtain the following description of the category $\text{Isoc}^r(X/K)$ of overconvergent isocrystals.

**Theorem 3.2.** The category $\text{Isoc}^r(X/K)$ is canonically equivalent to the following category.

- **Objects:**
  - for every frame $(T, \overline{T}, \Sigma)$ equipped with a map $T \to X$, a coherent $j^{\ast}_{T}\mathcal{O}_{T}^{\Sigma}$-module $E_{T}$;
  - for every map $g : (T', \overline{T}', \Sigma') \to (T, \overline{T}, \Sigma)$ over $X$ an isomorphism $u_{g} : g^{\ast}E_{T} \to E_{T'}$;
  - such that $u_{g} \circ g'^{\ast}(u_{g'}) = u_{g'\circ g}$ for composable morphisms of frames $g, g'$ over $X$.

- **Morphisms:**
  - for every frame $(T, \overline{T}, \Sigma)$ equipped with a map $T \to X$, a morphism $\psi_{T} : E_{1, T} \to E_{2, T}$;
  - such that $u_{g}^{\ast}(\psi_{T}) = \psi_{T'}$ for every morphism of frames $g : (T', \overline{T}', \Sigma') \to (T, \overline{T}, \Sigma)$ over $X$.

4. **Descent for coherent $j^{\ast}$-modules**

The strategy of proof of Theorem 5.1 below will be to follow that of Ogus [Ogu84, Theorem 4.6.1] in the convergent case, and just as the key component of the proof there is a version of flat descent for coherent sheaves on rigid spaces, so we will need a version of flat descent for coherent $j^{\ast}$-modules on frames.

**Theorem 4.1.** Let $f : (X, \overline{X}, \overline{X}) \to (T, \overline{T}, \Sigma)$ be a morphism of frames such that:

1. $X \to T$ is proper surjective;
2. $\overline{X} \to \overline{T}$ is proper;
3. $\overline{X}$ is flat.

Then $f$ is a morphism of effective descent for coherent $j^{\ast}$-modules (taken in the sense of adic geometry).

Unsurprisingly, the idea will be to reduce to flat descent for rigid analytic varieties, and the key lemma that will enable us to do so is the following.

**Lemma 4.2.** Let $f : (X, \overline{X}, \overline{X}) \to (T, \overline{T}, \Sigma)$ be a morphism of frames. Then for every neighbourhood

$$|X \times_{T} X|_{\overline{X} \times_{\overline{T}} \overline{X}} \subseteq W \subseteq |\overline{X} \times_{\overline{T}} \overline{X}|_{X \times_{T} X}$$

of $|X \times_{T} X|_{|X \times_{T} X|}$ in $|\overline{X} \times_{\overline{T}} \overline{X}|_{|X \times_{T} X|}$ there exists a neighbourhood

$$|X|_{X} \subseteq V \subseteq |\overline{X}|_{X}$$

of $|X|_{X}$ in $|\overline{X}|_{X}$ such that $V \times_{X} V \subseteq W$. 

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Proof. This is similar to [LS07, Proposition 3.2.12]. The question is local on both $\mathcal{S}$ and $\mathcal{X}$, hence we may assume that they are both affine. In particular, we may choose functions $f_1, \ldots, f_r, g_1, \ldots, g_s \in \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ such that $\overline{X} = \bigcap_i V(f_i) \subset X_k$, and $X = \bigcup_j D(g_j) \subset \overline{X}$. Hence we have

$$\overline{X} \times_T \overline{X} = \bigcap_{i,j} V(f_i \otimes f_j) \subset X_k \times_{X_k} X_k$$

and

$$X \times_T X = \bigcup_{j,j'} D(g_j \otimes g_{j'}) \subset \overline{X} \times_T \overline{X}.$$ 

By [Ber96, §1.2.4] we may assume that there exists an increasing sequence $m_n$ of integers such that

$$W = \bigcup_n \left\{ x \in \mathcal{X}_X \times_{\mathcal{S}_X} \mathcal{X}_X \mid v_x(\sigma^{-1} f_i^n \otimes f_j^n) \leq 1 \forall i, i', \exists j, j' \text{ s.t. } v_x(\sigma^{-1} g_j^{m_n} \otimes g_{j'}^{m_n}) \geq 1 \right\}.$$

Again applying [Ber96, §1.2.4] we may construct the neighbourhood

$$V = \bigcup_n \left\{ x \in \mathcal{X}_X \mid v_x(\sigma^{-1} f_i^n) \leq 1 \forall i, i', \exists j \text{ s.t. } v_x(\sigma^{-1} g_j^{m_n}) \geq 1 \right\}$$

of $]X[\mathcal{X}$ inside $]X[\mathcal{X}$ which clearly satisfies $V \times_{\mathcal{S}_X} V \subset W$. \hfill \square

We can now prove Theorem 4.1.

Proof of Theorem 4.1. We may assume that $X$ and $T$ are dense in $\mathcal{X}$ and $\mathcal{T}$ respectively, from which we deduce that $\overline{X} \rightarrow \mathcal{T}$ is also proper and surjective. In particular, we can see that the square

$$\begin{array}{ccc}
X & \rightarrow & \overline{X} \\
\downarrow & & \downarrow \\
T & \rightarrow & \mathcal{T}
\end{array}$$

is Cartesian. Thus we have $f(]X[\mathcal{X}) = ]T[\mathcal{T}$ and hence using Lemma 2.5 we can deduce that if

$$V \subset ]X[\mathcal{X}$$

is a neighbourhood of $]X[\mathcal{X}$, then

$$f(V) \subset ]T[\mathcal{T}$$

must be a neighbourhood of $]T[\mathcal{T}$. Moreover, if $\{V\}$ forms a cofinal system of neighbourhoods of $]X[\mathcal{X}$ in $]X[\mathcal{X}$, then $\{f(V)\}$ forms a cofinal system of neighbourhoods of $]T[\mathcal{T}$ in $]T[\mathcal{T}$.

In particular, for any such $V$ we may consider the category

$$\text{Coh}(V \times_{f(V)} V \Rightarrow V)$$

of coherent $\mathcal{O}_V$-modules together with descent data relative to $V \rightarrow f(V)$. Since $V \times_{f(V)} V$ is a neighbourhood of $]X[\mathcal{T} \times_{\mathcal{S} \times \mathcal{T}} ]X[\mathcal{X} \times_{\mathcal{S} \times \mathcal{T}} ]X[\mathcal{X}$, we therefore obtain a pull-back functor

$$\text{Coh}(V \times_{f(V)} V \Rightarrow V) \rightarrow \text{Coh}(j^!_X \mathcal{O}_{]X[\mathcal{X}} \Rightarrow j_X \mathcal{T} \times_{\mathcal{T}} ]X[\mathcal{X} \times_{\mathcal{T}} ]X[\mathcal{X})$$

and hence a functor

$$2\text{-colim} \text{Coh}(V \times_{f(V)} V \Rightarrow V) \rightarrow \text{Coh}(j^!_X \mathcal{O}_{]X[\mathcal{X}} \Rightarrow j_X \mathcal{T} \times_{\mathcal{T}} ]X[\mathcal{X} \times_{\mathcal{T}} ]X[\mathcal{X}).$$

It follows from Lemma 4.2 together with [LS07, Proposition 6.1.15] that this functor is an equivalence of categories. By Theorem 2.8 above we have an equivalence of categories

$$\text{Coh}(V \times_{f(V)} V \Rightarrow V) \cong \text{Coh}(f(V)),$$

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5. Effective descent for isocrystals

We can now give the proof of the first main descent result.

**Theorem 5.1.** Let \( f : X \to Z \) be a proper surjective map of \( k \)-varieties. Then \( f \) is a morphism of effective descent for overconvergent isocrystals.

Throughout the proof, we will use Le Stum’s ‘site-theoretic’ characterisation of overconvergent isocrystals [LS07, §8] recalled above.

*Proof.* Let us first treat the case of a proper surjective map \( f : X \to Z \). As usual, we may use Chow’s lemma to assume that \( f \) is projective, and since the question is also local on \( Z \), we may assume that we have some closed immersion \( X \hookrightarrow \mathbb{P}_Z^n \). The basic construction we will use is the following.

Let \((T, \overline{T}, \Xi)\) be a frame equipped with a map to \( Z \). Then we may base change \( X \hookrightarrow \mathbb{P}_Z^n \) by \( T \to Z \) to obtain \( X_T \hookrightarrow \mathbb{P}_T^n \) and hence we may extend \( X_T \to T \) to a morphism of frames

\[
\begin{array}{ccc}
X_T & \longrightarrow & \mathbb{P}_T^n \\
\downarrow & & \downarrow \\
T & \longrightarrow & \Xi
\end{array}
\]

where \( \overline{X}_T \) is simply the closure of \( X_T \) inside \( \mathbb{P}_T^n \). Since proper surjective maps are stable by base change, this morphism satisfies the conditions of Theorem 4.1.

Let us first consider the ‘fully faithful’ part of descent, the functor

\[ f^* : \text{Isoc}^\dagger(Z/K) \to \text{Isoc}^\dagger(X \times_Z X \rightrightarrows X/K) \]

is clearly faithful, since \( f^* : \text{Isoc}^\dagger(Z/K) \to \text{Isoc}^\dagger(X/K) \) is, it therefore suffices to show fullness. So suppose that we have objects \( E_1, E_2 \in \text{Isoc}^\dagger(Z/K) \) and a morphism \( \psi : f^* E_1 \to f^* E_2 \) compatible with the canonical descent data. Then for any frame \((T, \overline{T}, \Xi)\) equipped with a map \( T \to Z \) we may form \( f_T : (X_T, \overline{X}_T, \mathbb{P}_T^n) \to (T, \overline{T}, \Xi) \) as above, and realise \( E_i \) on \((T, \overline{T}, \Xi)\) to obtain coherent \( j^! \mathcal{O}_{\mathbb{P}_T^n} \)-modules \( E_{i,T} \). Moreover, we may realise \( \psi \) on \((X_T, \overline{X}_T, \mathbb{P}_T^n)\) to obtain a morphism

\[ \psi_T : f_T^! E_{1,T} \to f_T^! E_{2,T}, \]

which has to be compatible with the natural descent data on \( E_{i,T} \) relative to \( f_T : (X_T, \overline{X}_T, \mathbb{P}_T^n) \to (T, \overline{T}, \Xi) \). Hence by Theorem 4.1 it has to come from a unique morphism \( E_{1,T} \to E_{2,T} \) of coherent \( j^! \mathcal{O}_{\mathbb{P}_T^n} \)-modules. By uniqueness these are then compatible as \((T, \overline{T}, \Xi)\) varies, and hence give rise to a morphism \( E_1 \to E_2 \) of overconvergent isocrystals on \( Z/K \).

Next, let us treat the effectiveness part of descent. So let \( E \in \text{Isoc}^\dagger(X/K) \) be equipped with descent data relative to \( f \); we wish to produce an overconvergent isocrystal on \( Z/K \), and we will do so by constructing its realisations on any frame \((T, \overline{T}, \Xi)\) equipped with a map \( T \to Z \). In this situation, we may form \((X_T, \overline{X}_T, \mathbb{P}_T^n) \to (T, \overline{T}, \Xi)\) as above, and realise \( E \) on \((X_T, \overline{X}_T, \mathbb{P}_T^n)\) to obtain a coherent \( j^!_{X_T} \mathcal{O}_{\overline{X}_T/\mathbb{P}_T^n} \)-module \( E_{X_T} \). Moreover, the descent data for \( E \) relative to \( X \to Z \) gives rise to descent data for \( E_{X_T} \) relative to \( (X_T, \overline{X}_T, \mathbb{P}_T^n) \to (T, \overline{T}, \Xi) \).
Hence by Theorem 4.1 we obtain a coherent $j^* \mathcal{O}_{\mathcal{T}}$-module $F_T$ whose pullback to $(X_T, \overline{X}_T, \mathbb{P}^n_T)$ is $E_X$. Note that once our original embedding $X \to \mathbb{P}^n_Z$ was fixed the construction of $F_T$ is completely canonical, and does not depend on any further choices. Thus one easily checks using the corresponding properties of $E$ together with Theorem 4.1 that if $g : (T', \mathcal{T}', \mathcal{X}') \to (T, \mathcal{T}, \mathcal{X})$ is a morphism of frames over $Z$, then there is a corresponding isomorphism $g^! F_T \cong F_{T'}$, and these moreover satisfy the cocycle condition. Hence there is a unique overconvergent isocrystal $F$ on $Z/K$ whose realisation on each $(T, \mathcal{T}, \mathcal{X})$ is exactly $F_T$. This completes the proof in the proper surjective case.

Now using [Stacks, Tag 05WN] we may deduce the faithfully flat case.

**Theorem 5.2.** Let $f : X \to Z$ be a faithfully flat map of $k$-varieties. Then $f$ is a morphism of effective descent for overconvergent isocrystals.

**Proof.** By [Stacks, Tag 05WN] we know that if $f : X \to Z$ is faithfully flat, then there exists a composite $Z' \to Z$ of Zariski covers and finite faithfully flat maps such that $X' := X \times_Z Z'$ admits a section. By the usual arguments, together with the fact that we know effective descent for Zariski covers, we may therefore reduce the faithfully flat case to the finite faithfully flat case, and hence to the proper surjective case already handled.

6. **Topological invariance and equivalence of Frobenius pull-back**

The main application of Theorem 5.1 we have in mind is the following.

**Theorem 6.1.** Let $f : X \to Z$ be a universal homeomorphism (i.e. $f$ is finite, surjective and radicial). Then $f^* : \text{Isoc}^1(Z/K) \to \text{Isoc}^1(X/K)$ is an equivalence of categories.

**Proof.** We follow the proof of [Ogu84, Corollary 4.10]. Under the given assumptions on $f$, the diagonal

$$X \to X \times_Z X$$

is a nilpotent immersion. Since the tube of a $k$-sub-scheme $T$ inside some formal $\mathcal{F}$-scheme $\mathcal{X}$ only depends on the underlying set of $T$, we can thus deduce directly from the definitions that we have an equivalence of categories

$$\text{Isoc}^1(X \times_Z X/K) \cong \text{Isoc}^1(X/K)$$

and hence an equivalence

$$\text{Isoc}^1(X/K) \cong \text{Isoc}^1(X \times_Z X \Rightarrow X/K).$$

In other words, every $E \in \text{Isoc}^1(X/K)$ is equipped with a canonical descent data relative to $f : X \to Z$. Since $f$ is finite and surjective, we may therefore apply Theorem 5.1.

Now let us suppose that we have chosen a lift $\sigma$ to $K$ of the $q$-power Frobenius on $k$. Thus we obtain a $q$-power Frobenius pull-back functor

$$F^* : \text{Isoc}^1(X/K) \to \text{Isoc}^1(X/K).$$

**Corollary 6.2.** Assume that $k$ is perfect, and let $X$ be any $k$-variety. Then $F^*$ is an equivalence of categories.
Proof. Let $X^{(q)}$ be the pull-back of $X$ by the $q$-power Frobenius of $k$. Since $k$ is perfect, the corresponding (semi-linear) pull-back functor

$$\text{Isoc}^s(X/K) \to \text{Isoc}^s(X^{(q)}/K)$$

is an equivalence of categories. It therefore suffices to observe that the relative Frobenius $F_{X/k}: X \to X^{(q)}$ is a universal homeomorphism and apply Theorem 6.1. □

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