Quantum Optical Channels that are ‘Classical’

Krishna Kumar Sabapathy

Física Teòrica: Informació i Fenòmens Quàntics, Universitat Autònoma de Barcelona, ES-08193 Bellaterra (Barcelona), Spain

The Glauber-Sudarshan diagonal weight function provides a natural divide between classical and nonclassical states of continuous variables systems. Based on this division, a channel is said to be classical (or nonclassicality breaking) if the output of the channel is always classical for any input state. We classify all multimode Bosonic Gaussian channels that are classical. This is achieved by introducing a new criterion that needs to be satisfied by the matrices representing the Bosonic Gaussian channels, which can be interpreted as a kind of quantum benchmark for Bosonic Gaussian channels. We then prove a striking duality between classical and entanglement breaking Bosonic Gaussian channels, namely, we show that every classical Bosonic Gaussian channel is entanglement breaking, whereas every entanglement breaking Bosonic Gaussian channel can be rendered classical by the action of a Gaussian unitary whose active component consists only of parallel single-mode canonical squeezing elements. Consequently, Bosonic Gaussian channels that are classical have additive classical capacity and zero quantum capacity.

PACS numbers: 03.67.Mn, 42.50.-p, 03.65.Yz, 42.50.Dv, 03.67.-a

I. INTRODUCTION

The quantum-optical notion of classical states is well-established based on the Glauber-Sudarshan diagonal weight function for continuous variables systems [1,2]. A state is said to be classical if it can be written as a convex mixture of coherent states, the most elementary quantum states displaying classical behaviour [3]. For the class of Gaussian states, there is a necessary and sufficient condition on the associated covariance matrix of the state for it to be classical [4].

Inspired by the definition of classical states, the notion of nonclassicality breaking channels was introduced in [5], though some examples were observed earlier in [6,7]. A channel is said to nonclassicality breaking if its output is classical for every input, i.e., the channel breaks the nonclassicality of all input states. Due to this property, we choose to interchangeably call such channels ‘classical’ channels, channels that only produce classical states at the output. Necessary and sufficient conditions for single-mode Bosonic Gaussian channels (BGCs) to be classical were derived in [5]. In this work we settle the question of when a multimode BGC is classical. It turns out that this new criterion can be written in a manner analogous to that for which Gaussian states are classical.

A state of a quantum mechanical system specified by a density operator $\hat{\rho}$ can be faithfully described by any member of the one-parameter family of $s$-ordered quasi-probability distributions or, equivalently, by the corresponding $s$-ordered characteristic functions [8]. For $n$-modes of an electromagnetic field with quadrature operators $(\hat{x}_i, \hat{p}_i), i = 1, \cdots, n$ satisfying the commutation relation $[\hat{x}_i, \hat{p}_j] = i\Omega_{ij}$, the $s$-ordered characteristic function associated with a state $\hat{\rho}$ is defined as

$$\chi_s(\xi; \hat{\rho}) = \exp \left[ \frac{s}{2} |\xi|^2 \right] \text{Tr} [\hat{\rho} \mathcal{D}(\xi)], \quad -1 \leq s \leq 1. \quad (1)$$

Here $\xi = (\xi_1, \xi_2, \cdots, \xi_{2n})^T \in \mathbb{R}^{2n}$, $\mathcal{D}(\xi) = \exp[-i\sqrt{2} \xi^T \mathbf{R}]$ is the unitary multimode phase space Weyl-Heisenberg displacement operator, $\mathbf{R} = (\hat{x}_1, \hat{p}_1, \cdots, \hat{x}_n, \hat{p}_n)^T$, $\Omega = \oplus_{i=1}^n i\sigma_2$ is the $n$-mode symplectic metric where $\sigma_2$ is the antisymmetric Pauli matrix, and $s \in [-1, 1]$ is the order parameter. The special cases of $s = 1, 0, -1$ correspond to the normal-ordering $N$, symmetric-ordering $W$, and antinormal-ordering $A$ of the mode operators, respectively. Further, it immediately follows from (1) that the characteristic functions of a state $\hat{\rho}$ for two different values $s_1, s_2$ of $s$ are related as

$$\chi_{s_1}(\xi; \hat{\rho}) = \exp \left[ -\frac{(s_2 - s_1) |\xi|^2}{2} \right] \chi_{s_2}(\xi; \hat{\rho}). \quad (2)$$
By Fourier transforming the $s$-ordered characteristic function $\chi_s(\xi; \hat{\rho})$, we obtain

$$W_s(\alpha; \hat{\rho}) = \frac{1}{(2\pi)^{n}} \int d^{2n} \xi \exp[i\sqrt{2} \alpha^T \xi] \chi_s(\xi; \hat{\rho}).$$  \hspace{1cm} (3)

The quasiprobabilities corresponding to $s = -1, 0, 1$ are commonly known as the $Q$ function ($W_{-1}$), the Wigner function ($W_0$), and the diagonal ‘weight’ function ($W_1$) (also called the Glauber-Sudarshan $P$ or $\phi$ function), respectively. The $Q$ function $Q(\alpha; \hat{\rho}) = \langle \alpha | \hat{\rho} | \alpha \rangle$, which by definition is manifestly pointwise nonnegative over the phase-space $\mathbb{R}^{2n} \cong \mathbb{C}^n$, is a genuine probability distribution. This fact will be of crucial importance in the following sections.

Any density operator $\hat{\rho}$ representing some state of an $n$-mode of radiation field can always be expanded as $\rho = \int d^{2n} \phi(\alpha; \hat{\rho}) |\alpha\rangle \langle \alpha|$, where $\phi(\alpha; \hat{\rho}) = W_1(\alpha; \hat{\rho})$ is the diagonal ‘weight’ function, and $\{|\alpha\rangle\}$ being the over-complete set of coherent states. The diagonal representation plays an important role from an experimental point of view in quantum optics experiments [3].

This diagonal representation lends a natural quantum-optical notion of classical states. A state $\hat{\rho}$ is said to be classical if it can be expressed as a convex mixture of coherent states. We have,

$$\hat{\rho} \text{ is classical} \iff \phi(\alpha; \hat{\rho}) \geq 0 \text{ for all } \alpha \in \mathbb{R}^{2n}.\hspace{1cm} (5)$$

We now recall some properties of Gaussian states which will be of use to us. A state $\hat{\rho}$ is said to be Gaussian if its $s$-ordered quasi-probability or, equivalently, its $s$-ordered characteristic function is Gaussian. The symmetric or Weyl-ordered characteristic function ($s = 0$) corresponding to a Gaussian state has the form [2, 11]

$$\chi_s(\xi; \hat{\rho}) = \exp \left[ -\frac{\xi^T V \xi}{2} \right].$$ \hspace{1cm} (6)

where $V$ is the covariance matrix of the state $\hat{\rho}$ and is defined as $V_{ij} = \langle \{ \xi_i, \xi_j \} \rangle$. Here we have assumed the state to have vanishing first moments. The covariance matrix of any state is real, symmetric, positive definite, and specifies a zero-mean Gaussian state completely, and $V$ necessarily obeys the multimode uncertainty relation [4]

$$V + i \Omega \succeq 0, \quad \Omega = \oplus^n_{j=1} i \sigma_2.$$ \hspace{1cm} (7)

Note that the chosen convention is such that the covariance matrix corresponding to the vacuum state is $\mathbb{I}_{2n}$. Henceforth, we shall drop the subscript $2n$ on $\mathbb{I}_{2n}$. A necessary and sufficient condition for a Gaussian state to be classical is given in terms of its covariance matrix as [4]

$$V \succeq \mathbb{I}.$$ \hspace{1cm} (8)

II. MULTIMODE BOSONIC GAUSSIAN CHANNELS

A Gaussian channel is one that maps every Gaussian state to a Gaussian state. Under the action of a Bosonic Gaussian channel (BGC) described by real matrices $(X, Y)$ the covariance matrix $V$ corresponding to an input Gaussian state transforms as [12]

$$V \rightarrow V' = X^T V X + Y,$$ \hspace{1cm} (9)

$Y$ being symmetric positive semidefinite.

For an arbitrary input state $\hat{\rho}_m$, the action of a BGC represented by $(X, Y)$ at the level of the symmetric-ordered characteristic function $\chi_s$ is given by

$$\chi_s(\xi; \hat{\rho}_m) \rightarrow \chi_s(\xi; \hat{\rho}_{out}) = \chi_s(X \xi; \hat{\rho}_m) \exp \left[ -\frac{\xi^T Y \xi}{2} \right].$$ \hspace{1cm} (10)

We now recall a few definitions regarding BGCs that will be useful for the rest of the paper. Let $\Lambda_A : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_A)$.
be a positive map and $\hat{\rho}_{AE} = [A_A \otimes I_E] [\hat{\rho}_{AE}]$. We assume that all maps under consideration are trace-preserving unless otherwise stated.

**Definition 1**: A positive map $\Lambda_A$ is said to be completely positive (CP) if and only if $\hat{\rho}_{AE}$ is positive for all $\hat{\rho}_{AE}$ [3]. It is well known that a Gaussian map described by $(X, Y)$ is completely positive if and only if $(X, Y)$ satisfies the condition [14]

$$Y + i\Omega \geq iX^T\Omega X.$$ \hfill (11)

**Definition 2**: A channel $\Lambda_A$ is said to be positive under partial transpose (PPT) if and only if $\hat{\rho}_{AE}$ is PPT [15] for every $\hat{\rho}_{AE}$. A BGC is known to be PPT if and only if the pair $(X, Y)$ of matrices satisfies the condition [16]

$$Y - i\Omega \geq iX^T\Omega X.$$ \hfill (12)

**Definition 3**: A channel $\Lambda_A$ is said to be entanglement breaking (EB) if and only if $\hat{\rho}_{AE}$ is separable for every $\hat{\rho}_{AE}$ [17]. It was shown in [18] that a BGC is EB if and only if the defining pair of matrices $(X, Y)$ is such that the noise matrix $Y$ can be decomposed into $Y = Y_1 + Y_2$ such that

$$Y_1 + i\Omega \geq 0, \quad Y_2 \geq iX^T\Omega X.$$ \hfill (13)

**Definition 4**: A channel $\Lambda_A$ is said to be classical or nonclassicality breaking (NB) if and only if the output of the channel $\hat{\rho}_A = \Lambda_A(\hat{\rho}_A)$ is classical for every input state $\hat{\rho}_A$ [3]. We note that this is a single party notion unlike the previous definitions.

In the present paper we give a characterization of BGCs that are classical in terms of the corresponding $X$ and $Y$ matrices (Theorem 5 in Section III), analogous to the above for PPT and EB channels, and then show a close “dual” relationship between classical and entanglement breaking BGCs (Theorem 10 in Section IV), after which we conclude.

### III. MAIN RESULTS

**Theorem 5**: A Gaussian channel described by real $2n \times 2n$ matrices $(X, Y)$ with $Y = Y^T$, $Y \geq 0$ is NB or classical if and only if the pair of matrices $(X, Y)$ satisfies the following inequality:

$$Y - I \geq iX^T\Omega X.$$ \hfill (14)

**Necessary**: We first begin with the action of the channel on Gaussian states and this will give us a necessary condition. Consider an input Gaussian state with covariance matrix denoted by $V_{in}$. By Eq. (9), the output Gaussian state has covariance matrix $V_{out} = X^TV_{in}X + Y$. By Eq. (15), we have that the output covariance matrix corresponds to a classical state if and only if

$$V_{out} \geq I, \text{ i.e., } X^TV_{in}X + Y \geq I.$$ \hfill (15)

The second inequality of Eq. (15) can be rewritten as

$$X^TV_{in}X + (iX^T\Omega X - iX^T\Omega X) + Y - I \geq 0.$$ \hfill (16)

The terms of Eq. (16) can be rearranged as

$$[X^T(V_{in} + i\Omega) X] + [Y - I - iX^T\Omega X] \geq 0.$$ \hfill (17)

By Eq. (17), we note that the operator in the first square bracket of Eq. (17) is always positive semidefinite since the covariance matrix corresponding to the input Gaussian state satisfies the multimode uncertainty relation. For convenience we write $T := V_{in} + i\Omega$ and $N := Y - I - iX^T\Omega X$. We wish to emphasize that given a $(X, Y)$ pair, the corresponding matrix $N$ is fixed. One can now resolve the domain of the operator $T$ into $\text{Ker}T \oplus (\text{Ker}T)\perp$. We note that for any $|v\rangle$ in $\text{Range}X \cap \text{Ker}T$ we have that $\langle v|N|v\rangle \geq 0$. This is true for any Gaussian state. So we have that $\langle v|N|v\rangle \geq 0$ for $|v\rangle \in \{\text{Ker}T \cap \text{Range}X\}$ for all valid $V_{in}$. Let us denote $K_i = \{\text{Ker}T \cap \text{Range}X\}$ corresponding to $V_{in} = V_{i}$, for $i = 1, 2, \cdots$, which runs over all valid covariance matrices. Then we see that $\text{span}\{K_1, K_2, \cdots\} = \text{Range}X$. For $|u\rangle \in \text{Ker}X$ we have $\langle u|N|u\rangle \geq 0$. Collecting these facts together, we have that the inequality $N \geq 0$ becomes a necessary condition for the set of Gaussian states. Therefore, it is a necessary
condition for the entire state space. We note however that this condition is manifestly sufficient as well for the set of Gaussian states.

**Sufficient:** To prove the sufficiency of Eq. (14), we have to make use of general phase space techniques. Consider the action of the Gaussian channel on the symmetric-ordered characteristic function associated with an input state \( \hat{\rho}_{\text{in}} \). We have

\[
\chi_W(\xi; \hat{\rho}_{\text{out}}) = \chi_W(X\xi; \hat{\rho}_{\text{in}}) \exp \left[ -\frac{\xi^T Y \xi}{2} \right].
\]  

(18)

Since we know that \( Y \) has to necessarily be greater than \( \mathbb{I} \) from the discussion above, we write \( Y = \mathbb{I} + Y_0 \), where \( Y_0 \geq iX^T \Omega X \). We first consider the case of additive classical noise channels.

**Lemma 6:** For an additive classical noise channel, i.e., \( X = \mathbb{I} \) and \( Y \geq 0 \), \( Y_0 + i\Omega \geq 0 \) is a sufficient condition to render the channel classical. In other words, \( Y_0 \) is a valid covariance matrix.

**Proof:** The above Eq. (18) can be equivalently written as

\[
\chi_N(\xi; \hat{\rho}_{\text{out}}) = \chi_N(\xi; \hat{\rho}_{\text{in}}) \exp \left[ -\frac{\xi^T Y \xi}{2} \right] = \chi_W(\xi; \hat{\rho}_{\text{in}}) \exp \left[ -\frac{\xi^T Y_0 \xi}{2} \right].
\]  

(19)

We now apply a symplectic transformation \( S \in \text{Sp}(2n, \mathbb{R}) : \xi \rightarrow S\xi \), such that \( S^T Y_0 S \) is rendered diagonal, as guaranteed by Williamson’s theorem [4, 19, 20]. Further since \( Y_0 \) is a valid covariance matrix, we have that all the diagonal entries are \( \geq 1 \). Let us write \( S^T Y_0 S = \mathbb{I} + \Delta \), with \( \Delta \geq 0 \). So Eq. (18) now reads

\[
\chi_N(S\xi; \hat{\rho}_{\text{out}}) = \chi_W(S\xi; \hat{\rho}_{\text{in}}) \exp \left[ -\frac{\xi^T \Delta \xi}{2} \right].
\]  

(20)

Let \( U_S \) be the unitary (metaplectic) operator that induces the symplectic transformation \( S \), and we have

\[
\chi_N(S\xi; \hat{\rho}_{\text{out}}) = \chi_W(\xi; U_S \hat{\rho}_{\text{in}} U_S^\dagger) \exp \left[ -\frac{\xi^T \Delta \xi}{2} \right]
\times \exp \left[ -\frac{\xi^T \Delta \xi}{2} \right].
\]  

(21)

Let us denote \( U_S \hat{\rho}_{\text{in}} U_S^\dagger \) by \( \hat{\rho}' \). Then by Eq. (21) we have

\[
\chi_N(S\xi; \hat{\rho}_{\text{out}}) = \chi_W(\xi; \hat{\rho}') \exp \left[ -\frac{\xi^T \Delta \xi}{2} \right]
= \chi_A(\xi; \hat{\rho}') \exp \left[ -\frac{\xi^T \Delta \xi}{2} \right].
\]  

(22)

Now we apply the Fourier transform (10) to the above Eq. (22) and we have that \( \phi(S^T \alpha; \hat{\rho}_{\text{out}}) \) is the convolution of \( Q(\alpha; \hat{\rho}') \) with a Gaussian of the correct signature. We see that the diagonal weight function of the output state evaluated at the point \( S^T \alpha \) is always positive since the \( Q \) function is always positive. Therefore the condition in Eq. (14) is sufficient for the additive classical noise channel to be a nonclassicality breaking channel. 

Next we consider the case when the matrix \( X \) is non-singular. The action of the channel at the level of the symmetric-ordered characteristic function is given by

\[
\chi_W(\xi; \hat{\rho}_{\text{out}}) = \chi_W(X\xi; \hat{\rho}_{\text{in}}) \exp \left[ -\frac{\xi^T (\mathbb{I} + Y_0) \xi}{2} \right],
\]  

(23)

where as before we denote \( Y = \mathbb{I} + Y_0 \). Eq. (23) can now be rewritten as

\[
\chi_N(\xi; \hat{\rho}_{\text{out}}) = \chi_W(X\xi; \hat{\rho}_{\text{in}}) \exp \left[ -\frac{\xi^T Y_0 \xi}{2} \right].
\]  

(24)
Condition on $V + i\Omega \geq 0$  

| Property of a | Condition on $V$ | Property of a | Condition on $\mathcal{V}(X,Y)$ |
|---------------|------------------|---------------|-----------------------------|
| Gaussian state|                  | Gaussian channel|                             |
| Uncertainty relation | $V \Delta \in \Gamma$, $\Delta \geq 0$ | CP | $\mathcal{V}(X,Y) + i\Omega \geq 0$ |
| Pure (Extremals in state space) | $V \Delta \in \Gamma$, $\Delta \geq 0$ | Quantum-limited (Extremals in convex hull [CP]) | $\mathcal{V}(X,Y) - \Delta \in \mathcal{G}$, $\Delta \geq 0$ |
| Mixed | $V \Delta \in \Gamma$, $\Delta \geq 0$ | Noisy | $\mathcal{V}(X,Y) - \Delta \in \mathcal{G}$, $\Delta \geq 0$ |
| PPT | $V_{AB} + i[\Omega_A \pm \Omega_B] \geq 0$ | PPT | $\mathcal{V}(X,Y) + i\Omega \geq 0$ |
| Separable | $V_{AB} \geq V_A \oplus V_B$, $V_{AB} \in \Gamma$ | EB | $\mathcal{V}(X,Y) \geq Y_2$, $Y_2 \in \mathcal{G}$ |
| Classical | $V \geq \mathbb{1}$ | Classical (or NB) | $\mathcal{V}(X,Y) \geq \mathbb{1}$ |

TABLE I: Showing a comparison of the various fundamental notions for Gaussian states and Gaussian channels. Here $\mathcal{V}(X,Y) = Y - iX^T\Omega X$ is the characteristic matrix of a BGC described by $(X,Y)$. $\Gamma = \{ V \mid V + i\Omega \geq 0, V \text{ real}, V = V^T \}$ is the set of all valid covariance matrices in $n$-modes, and $\mathcal{G} = \{ \mathcal{V}(X,Y) \mid \mathcal{V}(X,Y) + i\Omega \geq 0, Y = Y^T, Y \geq 0, X,Y \text{ real} \}$ is the set of all valid $(X,Y)$ pairs corresponding to BGCs on $n$-modes. Property 5 for Gaussian states was shown in [21] and extremality for quantum-limited BGCs was shown in [6,22].

Now we relabel $X\xi$ with $\xi$. Then by Eq. (24) we have

$$\chi_N(X^{-1}\xi; \rho_{\text{out}}) = \chi_N(\xi; \hat{\rho}_{\text{in}}) \exp \left[ -\frac{\xi^T (X^T Y_0 X^{-1}) \xi}{2} \right].$$

By Lemma 6 (and Eq. [15]), we know that the channel is nonclassicality breaking if $X^{-T}Y_0X^{-1}$ is a valid covariance matrix. In other words $X^{-T}Y_0X^{-1} + i\Omega \geq 0$ or $Y_0 \geq iX^T\Omega X$, where we have applied the complex conjugation to the inequality. By introducing the $\mathbb{1}$, we recover the condition in Theorem 5. So we see that the criterion in Theorem 5 is also sufficient for non-singular $X$.

The final case left is that of singular $X$. We now introduce $X_\epsilon = X + \epsilon \mathbb{1}$. Then in the limit $\epsilon \to 0$, we recover the original $X$ matrix. We now apply the criterion to the non-singular $X_\epsilon$ and we have that $Y - \mathbb{1} \geq iX^T\Omega X_\epsilon$. Now we expand $X_\epsilon$ and take the limit $\epsilon \to 0$. We recover the condition in Theorem 5. Hence we see that sufficiency of the inequality in Theorem 5 is proved for arbitrary $X$.

Having characterised all multimode BGCs that are classical, it is both instructive and transparent to introduce a new operator $\mathcal{V}(X,Y)$ we call the characteristic matrix of a BGC described by the pair $(X,Y)$. We define it as

$$\mathcal{V}(X,Y) := Y - iX^T\Omega X.$$  

In terms of the characteristic matrix, the condition of Theorem 5 for a BGC described by the pair $(X,Y)$ to be classical is succinctly rewritten as

$$\mathcal{V}(X,Y) \geq \mathbb{1},$$

analogous to the condition in Eq. (8) for a Gaussian state to be classical. We now rewrite the fundamental properties of BGCs in terms of its corresponding characteristic matrices and list them alongside the analogous notions for Gaussian states in Table II. The essence of Theorem 5 is provided in entry 6 of Table II. This close relation between Gaussian states and BGCs in their basic structure and properties is anticipated in lieu of the Choi-Jamiołkowski channel-state isomorphism [13,23].

Remark 7: We see that the criterion in Theorem 5 subsumes the CP and PPT condition. We have by Eq. (14)

$$Y - \mathbb{1} + (i\Omega - i\Omega) \geq iX^T\Omega X$$

$$\Rightarrow Y + i\Omega \geq iX^T\Omega X + (\mathbb{1} + i\Omega)$$

$$\Rightarrow Y + i\Omega \geq iX^T\Omega X,$$

which is the CP condition of Eq. (11). A similar argument shows that the condition in Theorem 5 also subsumes the PPT condition. This fact is rendered further interesting for the following reason. We reiterate that while the notions of CP, PPT, and EB all require the one-sided action of the channel on a suitable bipartite state, the notion
of a channel to be classical requires only a single party. In other words, the entire derivation of Theorem 5 did not make use of any bipartite system but rather general phase-space techniques of multimode systems of a single party.

**Remark 8**: In passing we note that for single-mode BGCs and multimode gauge-covariant BGCs, PPT is equivalent to EB. The former statement follows from Simon’s criterion \[24\] whereas the latter statement was proved in \[25\].

**Remark 9**: The criterion in Theorem 5 has the appropriate symmetry properties. When dealing with channels in general, we deem two channels to be equivalent if they differ up to pre- and post-processing by unitaries. This double-cosetting, for example, is used to obtain canonical forms for channels. When we consider Bosonic Gaussian channels, this unitary freedom is appropriately taken to be Gaussianity-preserving unitaries \[26\]. These are unitaries that are generated by quadratic Hamiltonians and Weyl-displacements \[4, 11\]. When we consider nonclassicality breaking Bosonic Gaussian channels, we have to further restrict this freedom. We allow for arbitrary Gaussian unitaries during pre-processing while considering only those unitaries which induce passive transformations (also known as phase space rotations) \[4, 27\] for post-processing. Let us go back to the criterion in Theorem 5. We consider a Bosonic Gaussian channel which is nonclassicality breaking. We now apply an arbitrary symplectic transformation before the channel action and a symplectic rotation after the channel action. Then the matrix pair \((X, Y)\) transforms as \((SXR, R^T Y R)\). We see that the criterion is manifestly invariant under this special transformation. To summarise we present the various kinds of BGCs and their corresponding covariance properties in Table II.

| Type of BGC | Equivalence under pre/post-processing | Transformation on \((X, Y)\) |
|-------------|---------------------------------------|-----------------------------|
| CP, PPT, EB | \(U[S_1], U[S_2]\)                     | \((S_1XS_2, S_2^TYS_2)\)    |
| Classical   | \(U[R_1], U[R_2]\)                     | \((S_1XR_2, R_2^TYR_2)\)    |
| GCo, GCn    | \(U[R_1], U[R_2]\)                     | \((R_1XR_2, R_2^TYR_2)\)    |

**IV. DUALITY BETWEEN ENTANGLEMENT BREAKING AND CLASSICAL BGCs**

**Theorem 10**: Every classical BGC is entanglement breaking. Every entanglement breaking BGC can be rendered classical by composition with a suitable Gaussian unitary whose active component only consists of parallel single-mode canonical squeezing elements.

**Proof**: The first part of the theorem is easy to show. The noise matrix of every classical BGC can be expressed as \(Y = \mathbb{1} + Y_0\), where \(Y_0 \geq iX^T \Omega X\) and trivially \(\mathbb{1} + i\Omega \geq 0\). By Eq. (13), this is exactly a decomposition of the noisy matrix \(Y\) so that the pair \((X, Y)\) is an entanglement breaking BGC. Hence every classical BGC is automatically EB as well.

The second part of the proof is also straightforward. Let \((X, Y)\) with \(Y = Y_1 + Y_2\) be an entanglement breaking channel with \(Y_1 + i\Omega \geq 0\) (being a valid covariance matrix) and \(Y_2 \geq iX^T \Omega X\). Let us apply the symplectic transformation that diagonalises \(Y_1\) as guaranteed by the Williamson’s theorem \[4\]. We have \((X, Y) \rightarrow (XS, S^TYS)\). This transformation does not change the entanglement breaking property of the original channel. Since \(Y_1\) is a valid covariance matrix its symplectic eigenvalues are \(\geq 1\). Since \(S\) has diagonalised \(Y_1\) we now have \(Y_1 \geq \mathbb{1}\) and \(Y_2 \geq iX^T \Omega X\), where the second inequality involving \(Y_2\) is covariant under the symplectic transformation. But this is precisely the criterion in Theorem 5 for a BGC to be nonclassicality breaking. Hence we see that every entanglement breaking BGC can be rendered nonclassicality breaking by the action of a suitable active transformation. The symplectic matrix \(S\) that brings about this conversion of an EB channel to a NB channel is non-unique as explained below in Lemma 11.

This striking duality between nonclassicality breaking and entanglement breaking channels is presented in Fig. I.
FIG. 1: A schematic diagram depicting the duality between EB and NB (classical) BGCs. The figure on the left denotes the definitions of EB and NB channels. The diagram on the right brings out the following notion: Every NB channel is EB whereas every EB channel can be rendered classical by following the channel action by a suitable active Gaussian unitary transformation $U[S]$, $S \in \text{Sp}(2n, \mathbb{R})$. This aspect is further elaborated in Fig. 2. Here $\text{Id}$ denotes the identity channel.

FIG. 2: A schematic diagram depicting the conversion of an EB channel to a NB channel by the action of a passive transformation followed by parallel single-mode canonical squeezing elements, and the final application of another suitable passive transformation. Here $R_1$, $R_2$ are symplectic rotations, and $U[\nu_i]$, $i = 1, \cdots, n$ are single-mode canonical squeezing elements with squeeze parameters $\nu_i$ respectively.

One can see the apparent role reversal of the two types of channels when the two notions are compared side-by-side in light of Theorem 10.

**Lemma 11**: The active component of the symplectic matrix that converts every EB channel into a classical channel can be constructed from parallel single-mode canonical squeezing elements.

**Proof**: This is a direct consequence of the Euler decomposition of arbitrary symplectic transformations [27]. Every symplectic matrix $S$ can be decomposed as $S = R_1 D(\nu) R_2$, where $R_1, R_2$ are symplectic rotations and $D(\nu) = \text{diag}(\nu_1, \nu_1^{-1}, \nu_2, \nu_2^{-1}, \cdots, \nu_n, \nu_n^{-1})$ is a positive diagonal matrix. Hence we see that the active part of the symplectic transformation consists of single-mode canonical squeezing and is depicted schematically in Fig. 2. This is to be compared with the single-mode case where this structure was explicitly demonstrated [3]. Further, this decomposition is inherently non-unique [27].

**Corollary 12**: The classical capacity of every multimode NB BGC is additive and its quantum capacity is zero.

**Proof**: This is a straightforward consequence of Theorem 10. It is well known that for every EB channel the classical capacity is additive and the quantum capacity is zero [13, 29]. The statement of Corollary 16 follows from the fact that every NB BGC is automatically EB. Hence the classical capacity is additive and the quantum capacity is zero for every classical BGC.
FIG. 3: A schematic diagram depicting the (non-convex) sets of BGCs that are completely positive (CP), entanglement breaking (EB), positive under partial transpose (PPT), and classical (NB).

V. CONCLUSIONS

We have classified all multimode BGCs that are classical. We depict the set-theoretic nature of the various notions of a BGC in Fig. 3. We know that the set of classical channels and the set of extremal channels have a non-trivial intersection due to the existence of single-mode quantum-limited phase conjugation channel which is also classical and extremal. The conditions for a Gaussian map to be CP, PPT, and EB were previously known. The criterion for BGCs to be classical was derived in this article and listed in Table I. Further, we proved an interesting duality between classical and entanglement breaking BGCs. We find that every classical BGC is entanglement breaking and that every entanglement breaking BGC can be rendered classical by the action of a Gaussian unitary whose active component consists only of single-mode canonical squeezing elements. We find that, in general, too much noise could be detrimental as it can render the channel classical. In such a case the channel is not only rendered EB but it also only produces classical outputs and hence is ineffective for protocols aiming to exploit the power of nonclassical states. In effect, the condition in Theorem 5 can be interpreted as a kind of quantum benchmark for BGCs in the sense that every BGC satisfying this criterion is guaranteed to only produce classical states at the output irrespective of the input. We believe that the results presented here have far-reaching implications for both theoretical and experimental aspects of realization of quantum optical networks, quantum benchmarking, continuous variable QKD, and for various other quantum optical protocols.

Acknowledgements: The author is very grateful to Prof. Andreas Winter for motivating him to work on this problem and for many insightful comments. He would also like to thank Prof. Rajiah Simon and Dr. J. Solomon Ivan for numerous discussions on the subject matter. The author also acknowledges the facilities at ‘Biblioteca Jaume Fuster’. The author is supported by the ERC, Advanced Grant “IRQUAT”, contract no. ERC-2010-AdG-267386.

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