CHAPTER 1

NON-SINGULAR COSMOLOGY AND GAUGE THEORIES OF GRAVITATION

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The resolution of the problem of cosmological singularity in the framework of gauge theories of gravitation is discussed. Generalized cosmological Friedmann equations for homogeneous isotropic models filled by interacting scalar fields and usual gravitating matter are deduced. It is shown that generic feature of cosmological models of flat, open and closed type is their regular bouncing character.

1.1. Introduction

As it is well known the problem of cosmological singularity (PCS) is one of the most principal problems of general relativity theory (GR) and relativistic cosmology. There were many attempts to resolve PCS in the frame of GR as well as various generalizations of Einstein's theory of gravitation. A number of regular cosmological solutions was obtained in the frame of metric theories of gravitation and also other theories, in the frame of which gravitation is described by using more general geometry than the Riemannian one (see 1, 2, 3 and Refs given herein). In connection with this note that the resolution of PCS means not only obtaining regular cosmological solutions, but also excluding singular solutions of cosmological equations, as result generic feature of cosmological solutions has to be their regular character. Moreover, gravitation theory and cosmological equations have to satisfy the correspondence principle with Newton's theory of gravitation and GR in the case of usual gravitating systems with sufficiently small energy densities and weak gravitational fields excluding nonphysical solutions.
The greatest part of existent attempts to resolve the PCS does not satisfy indicated conditions. Let’s take for instance some examples. Although GR permits to build regular closed inflationary models, the greatest part of cosmological inflationary solutions of GR is singular. Metric theories of gravitation based on gravitational Lagrangians including terms quadratic in the curvature tensor lead to cosmological equations with high derivatives, and although these theories permit to obtain regular cosmological solutions with Friedmann and de Sitter asymptotics, however, they possess non-physical solutions also. At last time single regular cosmological solutions were found in the frame of superstring theory (brane cosmology) by using specific suppositions (see for example). To build cyclic model of the Universe specific negative scalar field potential was introduced in Refs.

As it was shown in a number of our papers, the gauge theories of gravitation (GTG) – the Poincare GTG, metric-affine GTG – possess important regularizing properties and permit to resolve the PCS. Satisfying the correspondence principle with GR in the case of usual gravitating systems with rather small energy densities and pressures, the GTG lead to essentially different consequences in the case of gravitating systems at extreme conditions with extremely high energy densities and pressures, namely the GTG lead to conclusion on possible existence of limiting energy density for gravitating systems, nearby of which gravitation has the character of repulsion, but not attraction. The investigation of homogeneous isotropic cosmological models in GTG shows, that by certain restrictions on equation of state of gravitating matter at extreme conditions and indefinite parameter of cosmological equations of GTG generic feature of cosmological models including inflationary models is their regular bouncing character. Such conclusions were obtained, in particular, in our recent papers for inflationary cosmological models filled by noninteracting scalar fields and gravitating matter with linear equation of state.

Present paper is devoted to further study of regular cosmology in the frame of GTG. In Section 2 cosmological equations for homogeneous isotropic models filled by interacting scalar fields and gravitating matter with equation of state in general form are deduced. In Section 3 general mathematical properties of deduced equations and their solutions are analyzed. As illustration of discussed theory in Section 4 particular solution for regular inflationary cosmological model is given.
1.2. Generalized cosmological Friedmann equations in GTG

Homogeneous isotropic models in GTG are described by the following generalized cosmological Friedmann equations (GCFE)

\[
\frac{k}{R^2} + \left\{ \frac{d}{dt} \ln \left[ R \sqrt{|1 - \beta (\rho - 3p)|} \right] \right\}^2 = \frac{8\pi}{3M_p^2} \frac{\rho - \frac{\beta}{3} (\rho - 3p)^2}{1 - \beta (\rho - 3p)}, \tag{1}
\]

\[
\frac{\dot{R} + R \left[ \ln \sqrt{|1 - \beta (\rho - 3p)|} \right]}{R} = -\frac{4\pi}{3M_p^2} \frac{\rho + 3p + \frac{\beta}{2} (\rho - 3p)^2}{1 - \beta (\rho - 3p)}, \tag{2}
\]

where \( R(t) \) is the scale factor of Robertson-Walker metrics, \( k = +1, 0, -1 \) for closed, flat, open models respectively, \( \beta \) is indefinite parameter with inverse dimension of energy density, \( M_p \) is Planckian mass, a dot denotes differentiation with respect to time. (The system of units with \( \hbar = c = 1 \) is used). At first the GCFE were deduced in Poincare GTG [14], and later it was shown that Eqs.(1)–(2) take place also in metric-affine GTG [25, 26]. In the frame of considered theory the conservation law in usual form takes place

\[
\dot{\rho} + 3H (\rho + p) = 0, \tag{3}
\]

where \( H = \frac{\dot{R}}{R} \) is the Hubble parameter. Besides cosmological equations (1)–(2) gravitational equations of GTG lead to the following relation for torsion function \( S \) and nonmetricity function \( Q \)

\[
S - \frac{1}{4} Q = -\frac{1}{4} \frac{d}{dt} \ln |1 - \beta (\rho - 3p)|. \tag{4}
\]

In Poincare GTG \( Q = 0 \) and Eq. (4) determines the torsion function. In metric-affine GTG there are three kinds of model [20] in the Riemann-Cartan space-time \( (Q = 0) \), in the Weyl space-time \( (S = 0) \), in the Weyl-Cartan space-time \( (S \neq 0, Q \neq 0) \), the function \( S \) is proportional to the function \( Q \). The value of \( |\beta|^{-1} \) determines the scale of extremely high energy densities. The GCFE (1)–(2) coincide practically with Friedmann cosmological equations of GR if the energy density is small \( |\beta (\rho - 3p)| \ll 1 \) \( (p \neq \frac{1}{3}\rho) \). The difference between GR and GTG can be essential at extremely high energy densities \( |\beta (\rho - 3p)| \gtrsim 1 \). Ultrarelativistic matter \( (p = \frac{1}{3}\rho) \) and gravitating vacuum \( (p = -\rho) \) with constant energy density are exceptional systems because Eqs. (1)–(2) are identical to Friedmann cosmological equations of GR in these cases independently on values of energy density and \( S = Q = 0 \). The behaviour of solutions of Eqs. (1)–(2) depends essentially

\[\text{Parameter } \beta \text{ is defined as } \beta = -\frac{1}{3} (16\pi)^2 f M_p^{-4}, \text{ where } f \text{ is linear combination of coefficients at terms of gravitational Lagrangian quadratic in the curvature tensor.}\]
on equation of state of gravitating matter at extreme conditions and on sign of parameter $\beta$. In the case of models filled by gravitating matter without scalar fields the GCFE (1)–(2) lead to regular in metrics cosmological solutions by the following restrictions: 1) $\beta > 0$ and at extreme conditions $p < \frac{1}{3} \rho$, 2) $\beta < 0$ and $p > \frac{1}{3} \rho$.[1] The investigation of models including scalar fields on the base of Eqs (1)–(2) shows that the choice $\beta < 0$ permits to exclude the divergence of time derivative of scalar fields.[2] In connection with this we put below that parameter $\beta$ is negative and in the frame of our classical description $|\beta|^{-1} < 1 \cdot M_p^4$.

By using GCFE (1)–(2) we will study homogeneous isotropic models filled by interacting scalar field $\phi$ minimally coupled with gravitation and gravitating matter with equation of state in general form $p_m = p_m(\rho_m)^b$. Then the energy density $\rho$ and pressure $p$ take the form

$$\rho = \frac{1}{2} \dot{\phi}^2 + V + \rho_m, \quad p = \frac{1}{2} \dot{\phi}^2 - V + p_m,$$

where scalar field potential $V = V(\phi, \rho_m)$ includes the interaction between scalar field and gravitating matter. In the most important particular case of radiation (ultrarelativistic matter) the expressions of $V(\phi, \rho_m)$ can be obtained by taking into account temperature corrections for given scalar field potentials[27] and the following relation for energy density $\rho_m \sim T^4$ ($T$ is absolute temperature). By using the scalar field equation in homogeneous isotropic space

$$\ddot{\phi} + 3H \dot{\phi} = -\frac{\partial V}{\partial \phi}$$

we obtain from Eqs. (3), (5), (6) the conservation law for gravitating matter

$$\dot{\rho}_m \left(1 + \frac{\partial V}{\partial \rho_m}\right) + 3H (\rho_m + p_m) = 0.$$  

(7)

By virtue of Eqs.(5)–(7) the GCFE (1)–(2) can be transformed to the following form

$$\left\{ H \left[ Z - 3\beta \dot{\phi}^2 - \frac{3\beta}{2} \rho_m + p_m \left(1 + \frac{\partial V}{\partial \rho_m}\right) \left(3\rho_m - 4V\right) - 1 \right] \right\}^2 - 3\beta \frac{\partial V}{\partial \phi} \dot{\phi}^2$$

$$+ \frac{k}{R^2} Z^2 = \frac{8\pi}{3M_p^2} \left[ \rho_m + \frac{1}{2} \dot{\phi}^2 + V - \frac{1}{4} \beta \left(4V - \dot{\phi}^2 + \rho_m - 3p_m\right)^2 \right] Z$$

(8)

[6]The generalization for the case with several scalar fields can be made directly.
Non-Singular Cosmology...

\[ \dot{H} \left\{ Z - 3\beta \left[ \dot{\phi}^2 + \frac{1}{2} \rho_m + p_m \left( \frac{d}{d\rho_m} (3p_m - 4V) - 1 \right) \right] \right\} Z \]

\[ + H^2 \left\{ Z + 15\beta \dot{\phi}^2 - 3\beta \rho_m + p_m \left( \frac{d}{d\rho_m} (3p_m - 4V) - 1 \right) \right\} \]

\[ + \frac{9\beta}{2} \frac{\rho_m + p_m}{(1 + \frac{dV}{d\rho_m})^2} \left( 1 + \frac{d\rho_m}{d\rho_m} \right) \left( \frac{d}{d\rho_m} (3p_m - 4V) - 1 \right) \]

\[ + 3(\rho_m + p_m) \left( \frac{d^2p_m}{d\rho_m^2} \right) \left( Z - \frac{9}{2} \beta^2 \left[ \rho_m + p_m \frac{d^2p_m}{d\rho_m^2} \left( \frac{d}{d\rho_m} (3p_m - 4V) - 1 \right) + 2\dot{\phi}^2 \right] \right) \]

\[ - 12\beta H \dot{\phi} \left\{ \frac{3\beta}{2} \frac{\partial V}{\partial \phi} \left[ \rho_m + p_m \frac{d^2p_m}{d\rho_m^2} \left( \frac{d}{d\rho_m} (3p_m - 4V) - 1 \right) + 2\dot{\phi}^2 \right] \right\} \]

\[ - \frac{\partial V}{\partial \phi} + \frac{9}{8} \frac{\rho_m + p_m}{\left( 1 + \frac{dV}{d\rho_m} \right)^2} \frac{\partial^2V}{\partial \phi \partial \rho_m} \left( 1 + \frac{1}{3} \frac{d}{d\rho_m} (p_m + 2V) \right) \]

\[ - 3\beta \left[ \frac{\partial^2V}{\partial \phi^2} \dot{\phi}^2 - \left( \frac{\partial V}{\partial \phi} \right)^2 \right] Z - 18\beta^2 \left( \frac{\partial V}{\partial \phi} \right)^2 \dot{\phi}^2 \]

\[ = \frac{8\pi}{3M_p^2} \left[ V - \dot{\phi}^2 - \frac{1}{2} (\rho_m + 3p_m) - \frac{1}{4} \beta \left( 4V - \dot{\phi}^2 + \rho_m - 3p_m \right) \right] Z, \quad (9) \]

where \( Z = 1 - \beta \left( 4V - \dot{\phi}^2 + \rho_m - 3p_m \right) \). Relation (4) takes the form

\[ S - \frac{1}{4} Q = \frac{3\beta}{2Z} \left\{ H \left[ \dot{\phi}^2 + \frac{1}{2} \rho_m + p_m \left( \frac{d}{d\rho_m} (3p_m - 4V) - 1 \right) \right] + \frac{\partial V}{\partial \phi} \dot{\phi} \right\}. \quad (10) \]

Unlike GR the cosmological equation (8) leads to essential restrictions on admissible values of scalar field and gravitating matter. Imposing \( \beta < 0 \), we obtain from Eq. (8) in the case \( k = 0, +1 \)

\[ Z \geq 0, \quad \text{or} \quad \dot{\phi}^2 \leq 4V + |\beta|^{-1} + \rho_m - 3p_m. \quad (11) \]

The condition (11) is valid also for open models discussed below. This means that relation (11) is fulfilled for all cosmological models independently on their type \( (k = 0, +1, -1) \). Now let us introduce the 3-dimensional space \( P \) with axes \( (\phi, \dot{\phi}, \rho_m) \). The domain of admissible values of scalar field \( \phi \), time derivative \( \dot{\phi} \) and energy density \( \rho_m \) in space \( P \) determined by (11) is
limited by bound $L$ defined as

$$Z = 0 \quad \text{or} \quad \dot{\phi} = \pm \left(4V + |\beta|^{-1} + \rho_m - 3p_m \right)^{\frac{3}{2}}.$$  (12)

From Eq. (8) the Hubble parameter on the bound $L$ is equal to

$$H_L = -\frac{\partial V}{\partial \phi} \frac{\dot{\phi}}{\dot{\phi}^2 + \frac{1}{2} \frac{\rho_m + p_m}{1 + \frac{\partial V}{\partial p_m}}} \left(1 + \frac{\partial V}{\partial \rho_m} \left(3p_m - 4V\right) - 1\right).$$  (13)

The right-hand part of Eq. (10) with the Hubble parameter determined by (13) is equal to $\frac{1}{2}H$, this means that the torsion (nonmetricity) in this case will be regular, if the Hubble parameter is regular.

1.3. Solutions properties of GCFE

Let us consider the most important general properties of cosmological solutions of GCFE (8)–(9). At first note, by given initial conditions for variables $(\phi, \dot{\phi}, \rho_m)$ and value of $R$ there are two different solutions corresponding to two values of the Hubble parameter following from Eq. (8):

$$H_\pm = \frac{3\beta \frac{\partial V}{\partial \phi} \dot{\phi} \pm \sqrt{D}}{Z - 3\beta \left[\dot{\phi}^2 + \frac{1}{2} \frac{\rho_m + p_m}{1 + \frac{\partial V}{\partial p_m}} \left(1 + \frac{\partial V}{\partial \rho_m} \left(3p_m - 4V\right) - 1\right)\right]},$$  (14)

where

$$D = \frac{8\pi}{3M_p^2} \left[\rho_m + \frac{1}{2} \dot{\phi}^2 + V - \frac{1}{4} \beta \left(4V - \dot{\phi}^2 + \rho_m - 3p_m\right)^2\right] Z - \frac{k}{R^2} Z^2.$$  (15)

Unlike GR, the values of $H_+$ and $H_-$ in GTG are sign-variable and, hence, both solutions corresponding to $H_+$ and $H_-$ can describe the expansion as well as the compression in dependence on their sign. Below we will call solutions of GCFE corresponding to $H_+$ and $H_-$ as $H_+$-solutions and $H_-$-solutions respectively. Any cosmological solution contains both $H_-$- and $H_+$-solutions. In points of bound $L$ we have $D = 0, H_+ = H_-$ and the Hubble parameter is determined by (13). The GCFE (8)–(9) are satisfied on the bound $L$, corresponding solutions of GCFE – $L$-solutions – are their particular solutions. The scalar field $\phi$ and energy density $\rho_m$ for $L$-solutions can be found by integration of Eqs. (6)–(7), where the Hubble parameter is defined by (13), and by using the constrain relation $Z = 0$. $L$-solutions can
be regular as well singular in dependence on equation of state of gravitating matter, scalar field potential $V$ and initial conditions. Trajectories of particular solutions situated on the bound $L$ have with $H_{\pm}$-solutions common points, $H_{-}$-solutions reach the bound $L$ and $H_{+}$-solutions originate from them, the surface $Z = 0$ containing trajectories of particular solutions is envelope in space $P$ for cosmological solutions. By using the following formula obtained for $H_{\pm}$-solutions

$$
\dot{Z} = 3\beta \left[ \dot{\phi}^2 + \frac{1}{2} \frac{\rho_m + p_m}{1 + \frac{\partial V}{\partial \rho_m}} \left( \frac{d}{d\rho_m} \left( 3p_m - 4V \right) - 1 \right) \right]
$$

(16)

it is easy to show that

$$
\lim_{Z \to 0} \dot{H}_{\pm} = \dot{H}_L - \frac{8\pi}{9\beta M_p^2} \rho_m + \frac{1}{2} \dot{\phi}^2 + V - \frac{1}{4} \beta \left( 4V - \dot{\phi}^2 + \rho_m - 3p_m \right)^2
$$

(17)

From (17) follows that in points of bound $L$ the derivatives $\dot{H}_+$ and $\dot{H}_-$ are equal and their values do not depend on the model type, as result we have the smooth transition from $H_{-}$-solution to $H_{+}$-solution on bound $L$, and corresponding cosmological solutions for all types models are regular in metrics, Hubble parameter and its time derivative. At the same time according to (17) the value of the time derivative $\dot{H}_L$ for $L$-solutions is not equal to $\lim_{Z \to 0} \dot{H}_{\pm}$ and smooth transition from $H_{-}$-solutions to $L$-solutions and from $L$-solutions to $H_{+}$-solutions without jump of the derivative $\dot{H}$ is impossible. Note, that according to Eq. (16) the functions $S$ and $Q$ have the following asymptotics for $H_{+}$- and $H_{-}$-solutions at $Z \to 0$:

$$
\lim_{Z \to 0} (S - \frac{1}{4} Q) = -\frac{1}{4} \lim_{Z \to 0} \frac{\dot{Z}}{Z} \sim \mp Z^{-\frac{1}{2}}
$$

(18)

Unlike flat and open models, for which $H_+ = H_-$ only in points of bound $L$ and regular inflationary models include $H_{+}$- and $H_{-}$-solutions reaching bound $L$, in the case of closed models the regular transition from $H_{-}$-solution to $H_{+}$-solution is possible without reaching the bound $L$. It is because by certain value of $R$ according to (14)–(15) we have $H_+ = H_-$ in the case $Z \neq 0$. Such models are regular also in torsion and/or nonmetricity. Regular inflationary solution of such type was considered in Ref. [6].
In order to study the behaviour of cosmological models at the beginning of cosmological expansion, let us analyze extreme points for the scale factor $R(t)$: $R_0 = R(0)$, $H_0 = H(0) = 0$. (This means that in the case of $H_+$--solutions $H_+ = 0$ and in the case of $H_-$--solutions $H_- = 0$). Denoting values of quantities at $t = 0$ by means of index "0", we obtain from (8)–(9):

$$\frac{k}{R_0^2} Z_0^2 + 9\beta^2 \left( \frac{\partial V}{\partial \phi} \right)_0^2 \dot{\phi}_0^2 = \frac{8\pi}{3M_p^2} \left[ \rho_{m0} + \frac{1}{2} \dot{\phi}_0^2 + V_0 - \frac{1}{4} \beta \left( 4V_0 - \dot{\phi}_0^2 + \rho_{m0} - 3p_{m0} \right) + \frac{9}{2} \beta \left( 4V_0 - \dot{\phi}_0^2 + \rho_{m0} - 3p_{m0} \right)^2 \right] Z_0, \quad (19)$$

$$\dot{H}_0 = \frac{8\pi}{3M_p^2} \left[ V_0 - \dot{\phi}_0^2 - \frac{1}{2} \left( \rho_{m0} + 3p_{m0} \right) - \frac{1}{4} \beta \left( 4V_0 - \dot{\phi}_0^2 + \rho_{m0} - 3p_{m0} \right)^2 \right] + 3\beta \left( \left( \frac{\partial^2 V}{\partial \phi^2} \right)_0 \dot{\phi}_0^2 - \left( \frac{\partial V}{\partial \phi} \right)_0^2 \right) + 18\beta^2 \left( \frac{\partial V}{\partial \phi} \right)_0^2 \dot{\phi}_0^2 Z_0^{-1} \right) \times \left\{ Z_0 - 3\beta \left[ \dot{\phi}_0^2 + \frac{1}{2} \rho_{m0} + p_{m0} \right] + \beta \left( \frac{d}{d\rho_m} \left( 3p_m - 4V_0 \right) - 1 \right) \right\}^{-1}, \quad (20)$$

where $Z_0 = 1 - \beta \left( 4V_0 - \dot{\phi}_0^2 + \rho_{m0} - 3p_{m0} \right)$. A bounce point is described by Eq. (19), if the value of $\dot{H}_0$ is positive. By using Eq.(19) we can rewrite the expression of $\dot{H}_0$ in the form

$$\dot{H}_0 = \frac{8\pi}{M_p^2} \left[ V_0 + \frac{1}{2} \left( \rho_{m0} - 3p_{m0} \right) - \frac{1}{4} \beta \left( 4V_0 - \dot{\phi}_0^2 + \rho_{m0} - 3p_{m0} \right)^2 \right] + 3\beta \left[ \left( \frac{\partial^2 V}{\partial \phi^2} \right)_0 \dot{\phi}_0^2 - \left( \frac{\partial V}{\partial \phi} \right)_0^2 \right] - \frac{2k}{R_0^2} Z_0 \right) \times \left\{ Z_0 - 3\beta \left[ \dot{\phi}_0^2 + \frac{1}{2} \rho_{m0} + p_{m0} \right] + \beta \left( \frac{d}{d\rho_m} \left( 3p_m - 4V_0 \right) - 1 \right) \right\}^{-1}. \quad (21)$$

We see from (21) unlike GR the presence of gravitating matter (with $p_m \leq \rho_m$) does not prevent from the bounce realization: Eq. (19) determines in space $P$ extremum surfaces depending on the value of $\beta$ and in the

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*In GR a bounce is possible only in closed models if the following condition $V_0 - \dot{\phi}_0^2 - \frac{1}{4} \left( \rho_{m0} + 3p_{m0} \right) > 0$ takes place.*
case of closed and open models also parametrically on the scale factor $R_0$. In the case of various scalar field potentials applying in inflationary cosmology the value of $\dot{H}_{+0}$ or $\dot{H}_{-0}$ is positive on the greatest part of extremum surfaces, which can be called "bounce surfaces". By giving concrete form of potential $V$ and choosing values of $R_0$, $\phi_0$, $\dot{\phi}_0$ and $\rho_{m0}$ at a bounce, we can obtain numerically particular bouncing solutions of GCFE for various values of parameter $\beta$.

The analysis of GCFE shows, that properties of cosmological solutions depend essentially on parameter $\beta$, i.e. on the scale of extremely high energy densities. From physical point of view interesting results can be obtained, if the value of $|\beta|^{-1}$ is much less than the Planckian energy density $6$, i.e. in the case of large in module values of parameter $\beta$ (by imposing $M_p = 1$). In order to investigate cosmological solutions at the beginning of cosmological expansion in this case, let us consider the GCFE by supposing that

$$|\beta|^{-1} \approx \frac{1}{M_p^4},$$

if $|\beta|^{-1} \ll M_p^4$ the derivative $\dot{H}_0$ is negative in the neighborhood of origin of coordinates in space $P$ that leads to appearance of oscillating solutions of GCFE.

$$\dot{H} = 4V_0 + 2\dot{\phi}^2 + \rho_0 - 3\rho_m \left( 1 + \frac{\partial V}{\partial \rho_m} \right) \left( \frac{d}{d\rho_m} (3\rho_m - 4V) - 1 \right)$$

$$\rho_m + \frac{1}{2} \dot{\phi}^2 + V \ll |\beta| \left( 4V_0 - \dot{\phi}^2 + \rho_0 - 3\rho_m \right)^2.$$  \hspace{1cm} (22)

Note that the second condition (22) does not exclude that ultrarelativistic matter energy density can dominate at a bounce. We obtain:

$$\frac{k}{R^2} \left( 4V - \dot{\phi}^2 + \rho_0 - 3\rho_m \right)^2 +$$

$$\left\{ H \left[ 4V + 2\dot{\phi}^2 + \rho_0 - 3\rho_m + \frac{3}{2} \frac{\rho_m + \rho_m}{1 + \frac{\partial V}{\partial \rho_m}} \left( \frac{d}{d\rho_m} (3\rho_m - 4V) - 1 \right) \right] \right.$$

$$\left. + 3 \frac{\partial V}{\partial \phi} \right\}^2 = \frac{2\pi}{3M_p^2} \left( 4V - \dot{\phi}^2 + \rho_0 - 3\rho_m \right)^3$$ \hspace{1cm} (23)
Equations (23)–(24) do not contain the parameter $\beta$. According to Eq. (23) the Hubble parameter in considered approximation is equal to

\[
H_{\pm} = \left[ 4V + 2\dot{\phi}^2 + \rho_m - 3p_m + \frac{3}{2} \frac{\rho_m + p_m}{1 + \frac{\partial V}{\partial \rho_m}} \left( \frac{d}{d\rho_m} (3p_m - 4V) - 1 \right) \right]^{-1} \times \left[ -3 \frac{\partial V}{\partial \phi} \dot{\phi} \mp 4V - \dot{\phi}^2 + \rho_m - 3p_m \right] \sqrt{\frac{2\pi}{3M_p^2} \left( 4V - \dot{\phi}^2 + \rho_m - 3p_m \right) - \frac{k}{R^2}}
\]

and extreme points of the scale factor are determined by the following condition

\[
\frac{k}{R_0^2} + 9 \left[ \frac{\left( \frac{\partial V}{\partial \phi} \right) \dot{\phi}_0}{4V_0 - \dot{\phi}_0^2 + \rho_{m0} - 3p_{m0}} \right]^2 = \frac{2\pi}{3M_p^2} \left( 4V_0 - \dot{\phi}_0^2 + \rho_{m0} - 3p_{m0} \right).
\]

From Eq. (24) the time derivative of the Hubble parameter at extreme points is
\[
\dot{H}_0 = \left\{ \frac{2\pi}{3M_p^2} \left( 4V_0 - \dot{\phi}_0^2 + \rho_{m0} - 3p_{m0} \right) \right\}^2 \\
-3 \left[ \left( \frac{\partial^2 V}{\partial \phi^2} \right)_0 \dot{\phi}_0^2 - \left( \frac{\partial V}{\partial \phi} \right)_0^2 \right] + \frac{18 \left( \frac{\partial V}{\partial \phi} \right)_0^2 \dot{\phi}_0^2}{4V_0 - \dot{\phi}_0^2 + \rho_{m0} - 3p_{m0}} \\
\times \left[ 4V_0 + 2\dot{\phi}_0^2 + \rho_{m0} - 3p_{m0} - \frac{3 \rho_{m0} + p_{m0}}{2} + \left( \frac{\partial V}{\partial \rho_m} \right)_0 \left( \frac{d}{d\rho_m} (3p_m - 4V) - 1 \right) \right]^{-1}
\]

(27)

Obviously Eqs. (26)–(27) correspond to (19)–(20) in considered approximation.

The analysis given in this Section confirms conclusions obtained in our previous papers. The existence of limiting bound \( L \) and bounce surface in space \( P \) in the case of models including scalar fields and gravitating matter ensures regular character of cosmological solutions in metrics, Hubble parameter and its time derivative; corresponding restriction on equation of state of gravitating matter and potential \( V \) at extreme conditions is:

\[
\left( 1 + \frac{\partial V}{\partial \rho_m} \right)^{-1} \left( \frac{d}{d\rho_m} (3p_m - 4V) - 1 \right) > 0 \quad \text{(see (14))}
\]

In absence of scalar fields the bound \( L \) is reduced to a point, particular solution of GCFE is stationary (\( H=0 \)), and the condition \( Z = 0 \) determines a bounce point and the value of limiting energy density simultaneously if at extreme conditions \( p_m > \frac{1}{3} \rho_m \).

1.4. Regular inflationary cosmological models

As illustration of discussed theory regular cosmological models in the simplest particular case will be considered in this Section. We will consider models including noninteracting scalar field with potential \( V(\phi) \) and ultra-relativistic matter (\( p_m = \frac{1}{3} \rho_m \)). In this case the bound \( L \) in space \( P \) is reduced to two cylindric surfaces \( \dot{\phi} = \pm \left( 4V + |\beta|^{-1} \right)^{\frac{1}{2}} \). Bounce surface is reduced also to cylindric surfaces in the case under consideration, when the scale of extremely high energy densities is much smaller than the Planckian energy density (see (26)). In connection with this we will consider instead of space \( P \) the plane of variables (\( \phi, \dot{\phi} \)) and intersections of the bound \( L \) and bounce surface with this plane. We have in this plane two bound \( L_{\pm} \)-curves and in the case of flat models two bounce curves \( B_1 \) and \( B_2 \) determined by
equation
t
$$4V_0 - \dot{\phi}_0^2 = 3 \left( \frac{M^2}{2\pi} V_0'' \dot{\phi}_0^2 \right)^{\frac{1}{3}}.$$  

Each of two curves $B_{1,2}$ contains two parts corresponding to vanishing of $H_+$ or $H_-$ and denoting by $(B_{1+}, B_{2+})$ and $(B_{1-}, B_{2-})$ respectively. If $V''$ is positive (negative) in quadrants 1 and 4 (2 and 3) on the plane $(\phi, \dot{\phi})$, the bounce will take place in points of bounce curves $B_{1+}$ and $B_{2+}$ ($B_{1-}$ and $B_{2-}$) in quadrants 1 and 3 (2 and 4) for $H_+$-solutions ($H_-$-solutions) (see Fig. 1.1). To analyze flat bouncing models we have to take into account that besides regions lying between curves $L_{\pm}$ and corresponding bounce curves the sign of values $H_+$ and $H_-$ for applying potentials is normal: $H_+ > 0$, $H_- < 0$. The Hubble parameter $H_+$ is negative in regions between curves $(L_+ \text{ and } B_{1+})$, $(L_- \text{ and } B_{2+})$, and the value of $H_-$ is positive in regions between curves $(L_+ \text{ and } B_{1-})$, $(L_- \text{ and } B_{2-})$. As it was noted above any cosmological solution has to contain both $H_-$- and $H_+$-solution. The regular transition from $H_-$-solution to $H_+$-solution takes place in points of $L_\pm$ where $H_+ = H_-$. In the case of open and closed models Eq.(26) determines 1-parametric family of bounce curves with parameter $R_0$. Bounce curves

\begin{align*}
\text{Fig. 1.1. Bound } L_{\pm}\text{-curves and bounce curves for flat models in the case of potential } V = \frac{1}{2} \lambda \phi^4. 
\end{align*}
curves of closed models are situated in region between two bounce curves $B_1$ and $B_2$ of flat models, and in the case of open models bounce curves are situated in two regions between the curves: $L_+$ and $B_1$, $L_-$ and $B_2$. In general case, when approximation (22) is not valid, bounce surface in space $P$ of cosmological models including scalar field and ultrarelativistic matter determined by Eq.(19) in space $P$ depends on parameter $\beta$ and it is not more cylindric surface. The situation concerning cosmological solutions of Eqs. (8)–(9) does not change.

Note that the analysis of inflationary solutions reaching the bound $L$ by numerical integration of Eqs. (9) and (6) is difficult, because the coefficient at $\dot{H}$ in Eq. (9) tends to zero at $Z \to 0$.

Below particular bouncing cosmological inflationary solution for flat model by using scalar field potential in the form $V = \frac{1}{4} \lambda \phi^4$ ($\lambda = 10^{-12}$) is given. The solution was obtained by numerical integration of Eqs. (6), (9) and by choosing in accordance with Eq.(19) (or (26)) the following initial conditions at a bounce: $\phi_0 = \sqrt{2} \cdot 10^{3/2} M_p$, $\dot{\phi}_0 = 1.998 \cdot 10^{-3} M_p^2$ ($\beta = -10^{18} M_p^{-4}$); initial value of $R_0$ can be arbitrary. As was noted above, the radiation energy density does not have influence on the dynamics of cosmological model, if the value of $\rho_m$ satisfies the condition (22). A bouncing solution includes: quasi-de-Sitter stage of compression, the stage of transition from compression to expansion, quasi-de-Sitter inflationary stage, stage after inflation. The dynamics of the Hubble parameter and scalar field is presented for different stages of obtained bouncing solution in Figures 1.2–1.4 (by choosing $M_p = 1$). The transition stage from compression to expansion (Fig. 1.2) is essentially asymmetric with respect to the point $t = 0$ because of $\dot{\phi}_0 \neq 0$. In course of transition stage the Hubble parameter changes from maximum in module negative value at the end of compression stage to maximum positive value at the beginning of expansion stage. The
scalar field changes linearly at most part of transition stage, the derivative $\dot{\phi}$ grows at first being positive to maximum value $\dot{\phi} \sim \dot{\phi}_0$ and then the value of $\dot{\phi}$ decreases and becomes negative. Quasi-de-Sitter inflationary stage and quasi-de-Sitter compression stage are presented in Fig. 1.3. The amplitude and frequency of oscillating scalar field after inflation (Fig. 1.4) are different than that of GR, this means that approximation of small energy densities $\left| \beta \left( 4V - \dot{\phi}^2 - 2\rho_1 \right) \right| \ll 1$ at the beginning of this stage is not valid; however, the approximation (23)–(24) is not valid also because of dependence on parameter $\beta$ of oscillations characteristics. The behaviour of the Hubble parameter after inflation is also noneinsteinian, at first the Hubble parameter oscillates near the value $H = 0$, and later the Hubble parameter becomes positive and decreases with the time like in GR. Before quasi-de-Sitter compression stage there are also oscillations of the Hubble parameter and scalar field not presented in Figures 1.2–1.4. Ultrarelativistic matter, which could dominate at a bounce has negligibly small energy densities at quasi-de Sitter stages. At the same time the gravitating matter could be at compression stage in more realistic bouncing models, and scalar fields could appear only at certain stage of cosmological compression. As it follows from our consideration regular character of such inflationary cosmological models has to be ensured by cosmological equations of GTG.

The interaction between scalar fields and radiation leads to quantitative corrections of considered inflationary cosmological models. It is necessary to note that corrections for scalar field potentials quadratic in the temperature $T^2 \sim \rho_1^{1/2}$ can play the important role for more late stages of cosmological evolution, when energy densities are sufficiently small and consequences of GTG and GR coincide. In accordance with Eq. (7) these terms change essentially the connection between scale factor $R$ and radiation energy density. By certain restrictions on parameters of potentials $V$ indicated terms can be greater than radiation energy density and scalar field energy dens-
Note that bouncing character have solutions not only in classical region, where scalar field potential, kinetic energy density of scalar field and energy density of gravitating matter do not exceed the Planckian energy density, but also in regions, where classical restrictions are not fulfilled and according to accepted opinion quantum gravitational effects can be essential.

1.5. Conclusion

As it is shown in our paper, GTG permit to build non-singular cosmology and at first of all regular inflationary cosmology, if gravitating matter and scalar fields satisfy certain restrictions at extreme conditions and indefinite parameter $\beta$ is negative. All cosmological solutions for flat, open and closed models are regular in metrics, Hubble parameter, its time derivative and, hence, not limited in the time. The presence of scalar fields leading to appearance of inflation in cosmological models changes essentially the structure of GCFE, as result a family of closed models regular also in torsion and/or nonmetricity appears. To build realistic cosmological models we have to know properties of matter filling the evolving Universe, and at first of all the change of equation of state of gravitating matter and properties of scalar fields by evolution of the Universe.

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References

1. A. V. Minkevich, *Acta Phys. Polon.* **B29**, 949 (1998).
2. A. V. Minkevich, *Nonlinear Phenomena in Complex Systems*, **6**, N4, 824 (2003); gr-qc/0307026
3. A. V. Minkevich, gr-qc/0312068 to be published in *Annals of European Acad. Sci.* (2004).
4. A. A. Starobinsky. *Pis’ma v Astron. Zhurn.* **4**, 155 (1978).
5. V. A. Belinsky, L. P. Grishchuk, Ya. B. Zeldovich, I. M. Khalatnikov, *Zhurn. Exp. Teor. Fiz.* **89**, 346 (1985).
6. A. V. Minkevich, gr-qc/0303022
7. R. Brandenberger, V. Mukhanov, A. Sornborger, *Phys. Rev.*, **D48**, N 9, 1629 (1993).
8. Varun Sahni, gr-qc/0208047
9. M. Gasperini, G. Veneziano, *Phys. Rep.*, **373**, 1 (2003); hep-th/0207130
10. F. Quevedo, Lectures on String/Brane Cosmology, hep-th/0210292
11. A. A. Coley and S. Hernik, gr-qc/0303003
12. P. J. Steinhardt and N. Turok, *Phys. Rev.*, **D65**, 126003 (2002).
13. N. Turok and P. J. Steinhardt, hep-th/0403020
14. A. V. Minkevich, *Vestsi Akad. Nauk BSSR*. Ser. fiz.-mat., no. 2, 87 (1980); *Phys. Lett*. **A80**, 232 (1980).
15. A. V. Minkevich, *Dokl. Akad. Nauk BSSR*. **29**, 998 (1985).
16. A. V. Minkevich, *Dokl. Akad. Nauk BSSR*. **30**, 311 (1986).
17. A. V. Minkevich, N. H. Chuong, F. I. Fedorov, *Dokl. Akad. Nauk BSSR*. **29**, 697 (1985); *Class. Quantum Grav.* **5**, 515 (1988).
18. A. V. Minkevich, N. H. Chuong, N. V. Hoang, In: *Gravitation and Electromagnetism*, Minsk, BSU, p. 103 (1987).
19. A. V. Minkevich, *Vestsi Akad. Nauk Belarus*. Ser. fiz.-mat., no. 5, 123 (1995); in *Proc. of the Third Alexander Friedmann Intern. Seminar on Gravitation and Cosmology*, St.-Petersburg, July 4–12, 1995, p. 12–22.
20. A. V. Minkevich, N. H. Chuong, *Vestsi Akad Nauk BSSR*. Ser. fiz.-mat., no. 5, 63 (1989).
21. A. V. Minkevich, I. M. Nemenman, *Class. Quantum Grav.* **12**, 1259 (1995).
22. A. V. Minkevich, A. S. Garkun, *Gravitation & Cosmology*. **5**, 115 (1999).
23. A. V. Minkevich, *Int. J. Mod. Phys.* **A17**, 4442 (2002).
24. A. V. Minkevich, *Phys. Lett*. **95A**, 422 (1983).
25. A. V. Minkevich, *Dokl. Akad. Nauk Belarus*. **37**, 33 (1993).
26. A. V. Minkevich and A. S. Garkun, *Class. Quantum Grav.* **17**, 3045 (2000).
27. A. D. Linde, *Physics of Elementary Particles and Inflationary Cosmology*. (Nauka, Moscow, 1990).
28. A. V. Minkevich, A. S. Garkun, Yu. G. Vasilevski, *Nonlinear Phenomena in Complex Systems*, **7**, N1, 78 (2004); gr-qc/0310060
29. A. V. Minkevich, in preparation (2004).