(2+1)-dimensional static cyclic symmetric traversable wormhole: quasinormal modes and causality

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Abstract
In this paper we study a static cyclic symmetric traversable wormhole (WH) in (2+1)-dimensional gravity coupled to nonlinear electrodynamics in anti-de Sitter spacetime. The solution is characterized by three parameters: mass $M$, cosmological constant $\Lambda$ and one electromagnetic parameter, $q_{\alpha}$. The causality of this spacetime is studied, determining its maximal extension and constructing then the Penrose diagram. The quasinormal modes (QNMs) that result from considering a massive scalar test field in the WH background are determined by solving in exact form the Klein–Gordon equation; the effective potential resembles the one of a harmonic oscillator shifted from its equilibrium position and, consequently, the QNMs have a pure point spectrum.

Keywords: wormhole, (2+1)-gravity, quasinormal modes, nonlinear electrodynamics

(Some figures may appear in colour only in the online journal)

1. Introduction

Anti-de Sitter (AdS) gravity in (2+1)-dimensions has attracted a lot of attention due to its connection to a Yang–Mills theory with the Chern–Simons term [1, 2]. Moreover, taking advantage of simplifications due to the dimensional reduction, three dimensional Einstein theory of gravity has turned out a good model from which extract relevant insights regarding the quantum nature of gravity [3]. In three spacetime dimensions, general relativity becomes a topological field theory without propagating degrees of freedom. Additionally, in string theory, there are near extremal black holes (BHs) whose entropy can be calculated and have a near-horizon geometry containing the Bañados–Teitelboim–Zanelli (BTZ) solution [4, 5].
Particularly for the (2+1)-dimensional BTZ black hole (BH), the two-dimensional conformal description has by now well established [6]: the BTZ-BH provides a precise mathematical model of a holographic manifold. For these reasons systems where the conformal description can be carried out all the way through are very valuable.

On the other hand nonlinear electrodynamics (NLED) has gained interest for a number of reasons. Nonlinear electrodynamics consists of theories derived from Lagrangians that depend arbitrarily on the two electromagnetic invariants, \( F = 2(E^2 - B^2) \) and \( G = E \cdot B \), i.e. \( L(F, G) \). The ways in which \( L(F, G) \) may be chosen are many, but two of them are outstanding: the Euler–Heisenberg theory [7], derived from quantum electrodynamics assumptions, takes into account some nonlinear features like the interaction of light by light. And the Born–Infeld theory [8, 9], proposed originally with the aim of avoiding the singularity in the electric field and the self-energy due to a point charge, it is a classical effective theory that describes nonlinear features arising from the interaction of very strong electromagnetic fields, where Maxwell linear superposition principle is not valid anymore. Interesting solutions have been derived from the Einstein gravity coupled to NLED, like regular BHs, wormholes (WHs) sustained with NLED, among others, see for instance [10]. It is also worth to mention that some NLED arise from the spontaneous Lorentz symmetry breaking (LSB), triggered by a non-zero vacuum expectation value of the field strength [11].

WHs in the AdS gravity are interesting objects to study. For instance, regarding the transmission of information through the throat, the understanding of the details of the traversable wormhole (WH) and its quantum information implications would shed light on the lost information problem [12]. The thermodynamics of a WH and its trapped surfaces was addressed in [13], establishing that the accretion of phantom energy, considered as thermal radiation coming out from the WH, can significantly widen the radius of the throat. In [14] it is shown that Euclidean geometries with two boundaries that are connected through the bulk are similar to WH in the sense that they connect two well understood asymptotic regions. In [15] it is constructed a WH via a double trace deformation. Alternatively, WH solutions are constructed by gluing two spacetimes at null hypersurfaces, [16, 17]. Contrasting this procedure, in a recent paper the authors derived exact solutions of the Einstein equations coupled to NLED that can be interpreted as WHs and for certain values of the parameters such solutions become the BTZ-BH [18]. Which has become an excellent laboratory for studying quantum effects since the seminal paper [19]. And regarding LSB, it can be mentioned as well that WH solutions have been derived in the context of the bumblebee gravity [20], their QNMs have been studied in [21], and the corresponding gravitational lensing in [22].

Moreover, WHs are related to BHs; BH and WH spacetimes are obtained by identifying points in (2+1)-dimensional AdS space by means of a discrete group of isometries, some of them resulting in non-eternal BHs with collapsing WH topologies [23].

In this paper we present an exact solution of the Einstein equations in (2+1)-dimensions with a negative cosmological constant (AdS) coupled to NLED. The solution can be interpreted as a WH sourced by the NLED field with a Lagrangian of the form \( F^{1/2} \). This solution is a particular case of a broader family of solutions previously presented in [18]. The solution is characterized by three parameters: mass \( M \), cosmological constant \( -\Lambda = 1/l^2 \) and the electromagnetic parameter \( q_\alpha \). The causality is investigated by means of the Penrose diagram, showing that the light trajectories traverse the WH. The WH Penrose diagram resembles the AdS one with the WH embedded in it.

A massive scalar test field is considered in the WH background; the corresponding Klein–Gordon (KG) equation, when written in terms of the tortoise coordinate, acquires a Schrödinger-like form and it is solved in exact form determining the frequencies of the massive scalar field; the boundary conditions are of purely ingoing waves at the throat and zero
outgoing waves at infinity. The effective potential in the KG equation is a confining one and, accordingly, we found that the spectrum is real, showing then that the WH does not swallow the field as a BH would, but the field goes through the throat passing then to the continuation of the WH, and preserving the energy of the test field. This also shows the stability of the scalar field in this WH background.

The outline of the paper is as follows. In the next section we present the metric for the WH and the field that sources it as well as a brief review on its derivation. In section 3 we find the maximal extension and then the Penrose diagram is constructed. In section 4 the KG equation for a massive scalar field is considered in the WH background and is exactly solved, obtaining the QNMs by imposing the appropriate WH boundary conditions. Final remarks are given in the last section. Details on the derivation of the QNMs as well as the setting of the boundary conditions are presented as an appendix.

2. The WH sourced by nonlinear electrodynamics

The action of the (2+1) Einstein theory with cosmological constant, coupled to NLED is given by

$$S_{[g_{ab},A_a]} = \int d^3x \sqrt{-g} \left( \frac{1}{16\pi} (R - 2\Lambda) + L(F) \right),$$

(1)

where $R$ is the Ricci scalar and $\Lambda$ is the cosmological constant; $L(F)$ is the NLED characteristic Lagrangian. Varying this action with respect to gravitational field gives the Einstein equations,

$$G_{ab} + \Lambda g_{ab} = 8\pi E_{ab},$$

(2)

where $E_{ab}$ is the electromagnetic energy–momentum tensor,

$$4\pi E_{ab} = g_{ab}L(F) - f_{ac}f_{b}^{\delta \phi}F,$$

(3)

where $L_F$ stands for the derivative of $L(F)$ with respect to $F$ and $f_{ab}$ are the components of the electromagnetic field tensor. The variation with respect to the electromagnetic potential $A_a$ entering in $f_{ab} = 2\partial_a A_{b\phi}$ yields the electromagnetic field equations,

$$\nabla_a(L_F f_{ab}) = 0 = \nabla_a((\mathcal{J})^a),$$

(4)

where $(\mathcal{J})^a$ is the dual electromagnetic field tensor which, for (2+1)-dimensional gravity, in terms of $f_{ab}$, is defined by $(\mathcal{J})^a = \sqrt{-g} \left( f^{\mu} \delta_\mu^a + f^{\phi} \delta_\phi^a + f^{\rho}\delta_\rho^a \right)$ with $(a = t, r, \phi)$. We shall consider the particular nonlinear Lagrangian, $L(F) = \sqrt{-sF}$; these kind of Lagrangians have been called Einstein-power-Maxwell theories [24, 25]. On the other hand, in [18] was shown that in (2+1) Einstein theory coupled to NLED the most general form of the electromagnetic fields for stationary cyclic symmetric (2+1) spacetimes, i.e. the general solution to equation (4), is given by $\mathcal{J} = (g_r c/\sqrt{-g}) d\tau + (a/3L_F) dt + (b/3L_\phi) d\phi$, where $a$, $b$ and $c$ are constant, that by virtue of the Ricci circularity conditions, are subjected to the restriction that $ac = 0 = bc$. Therefore, in this geometry, in order to describe the electromagnetic field tensor, we have two disjoint branches; $[a = 0 = b, c \neq 0]$ and $[(a \neq 0 \vee b \neq 0), c = 0]$. Here we are considering the branch $c \neq 0$, and thus the only non-null electromagnetic field tensor component and the electromagnetic invariant are given, respectively, by

$$f^{\phi t} = \frac{3g_{\phi t} c}{(\sqrt{-g})^2}, \quad F = \frac{1}{2} f^{\phi t} f_{\phi t} = \frac{9}{2} \frac{c^2}{g_{\phi \phi}}.$$

(5)
With these assumptions a five-parameter family of solutions with a charged rotating WH interpretation was previously presented in [18]. In this work we shall address in detail the (2+1)-dimensional static cyclic symmetric WH.

For the sake of completeness, we give a brief review on the derivation of the solution. The field equations of general relativity (with cosmological constant) coupled to NLED for a static cyclic symmetric (2+1)-dimensional static cyclic symmetric WH.

Now, by substituting \( F(1) \) from equation (12) into (8), we arrive at

\[
\frac{f^2 N_f}{r N} = 2L - \Lambda \Rightarrow \left( \frac{M - \Lambda r^2}{r N} \right) N_f = 2 \sqrt{\frac{s}{2}} \left( \frac{3c}{r N} \right)^2 - \Lambda \Rightarrow \left( -M - \Lambda r^2 \right) N_f + \Lambda r N = 3\sqrt{2} c.
\] (13)

Now, by substituting \( c = \sqrt{2} M^2 q_\alpha / (6\sqrt{3}) \) into the previous equation, yields

\[
\left( -M - \Lambda r^2 \right) N_f + \Lambda r N = M^2 q_\alpha,
\] (14)

whose general solution is

\[
N(r) = -q_\alpha M r + q_\beta \sqrt{-M - \Lambda r^2},
\] (15)
where $q_\beta$ is an integration constant. Finally, by substituting (11) and (15) into (9), one finds
\[
\frac{f(N_{\alpha})_\epsilon}{N} = \sqrt{-M - \Lambda r^2} \left( q_\alpha M - \frac{q_\beta}{\sqrt{-M - \Lambda r^2}} - q_\beta \Lambda \right) = \left( q_\alpha M r - q_\beta \sqrt{-M - \Lambda r^2} \right) \Lambda \\
- q_\alpha M r + q_\beta \sqrt{-M - \Lambda r^2} = - \Lambda,
\]
\[(16)\]
such that equation (9) is trivially satisfied by the Lagrangian $L = \sqrt{-sF}$, the structural functions $f^2(r)$, $N^2(r)$ given by (11) and (15), and the electromagnetic field given by (5).

2.1. WH properties

Let us show that the solution (10) allows a traversable WH interpretation.

The canonical metric for a $(2+1)$-dimensional static cyclic symmetric WH [26] is given by
\[
ds^2 = - \sqrt{s} e^{2\phi(r)} \, dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\phi^2.
\]\[(17)\]

By comparison with (10) we see that $e^{2\phi(r)} = -q_\alpha M r + q_\beta \sqrt{-M - \Lambda r^2}$ and $b(r) = r(1 + M + \Lambda r^2)$, where $-\Lambda = 1/f^2$. In this paper the case $q_\beta = 0$ will be the subject of our study, with metric given by
\[
ds^2 = -(q_\alpha M r)^2 \, dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\phi^2 \quad \text{with} \quad M > 0.
\]\[(18)\]

Then we can check the WH properties of the metric (18):

(i) The existence of a throat $r_0$ where $b(r_0) = r_0$. Such a throat is located at $r_0 = \sqrt{F/M}$. The range of the $r$-coordinate is the interval $r \in [r_0, \infty)$.

(ii) The absence of horizons. It is fulfilled since $e^{2\phi(r)} = (-q_\alpha M r)^2$ is nonzero for all $r \in [r_0, \infty)$.

(iii) The fulfilment of the flaring out condition that is related to the traversability of the WH.

We shall see that traversability has a consequence on the form of the QNMs. This condition is guaranteed if the derivative of $b(r)$ when evaluated at the throat is less than one, $b'(r_0) < 1$; in our case, with $M > 0$, $b'(r_0) = 1 - 2M < 1$.

The nonlinear field in our case is generated by the Lagrangian $L(F) = \sqrt{-sF}$, where $F$, the electromagnetic invariant, and the only non-vanishing electromagnetic component, $f_{\phi\phi}$, are given, respectively, by
\[
F = -\frac{M^2}{4s} \frac{M^2}{4s} \quad \text{and} \quad f_{\phi\phi} = -\partial_\phi A_\phi = \frac{q_\alpha M^2}{\sqrt{2s}}.
\]\[(19)\]

Moreover, it is well known that in GR matter obeying the standard energy conditions is not worth to open a throat and so create a traversable WH. In the case we are analyzing, the NLED energy–momentum tensor does not satisfy the null energy condition (NEC), rendering this into a traversable WH. To check the violation of the NEC due to NLED, let us consider the null vector in the orthonormal frame, $n = (1, 1, 0)$, and calculate $E_{(\alpha)(\beta)n^{(\alpha)}n^{(\beta)}} = E_{(0)(0)} + E_{(1)(1)} = L(F)/(4\pi)$, then, using (8) to determine $L(F)$, we obtain that
\[
E_{(\alpha)(\beta)n^{(\alpha)}n^{(\beta)}} = -\frac{M}{8\pi r^2} < 0,
\]\[(20)\]
from which we see that NEC is violated; particularly, evaluating at the throat $r_0^2 = M^2$, $E_{(\alpha)(\beta)} n^{(\alpha)} n^{(\beta)} = -1/(8\pi^2)$.  

3. The maximal extension and causality: Penrose diagram

In order to understand the causal structure and the structure at infinity of the WH with metric (18), we will construct its Penrose diagram. To start with, since the causal structure is defined by the light cones, we need to consider the radial null geodesics which by definition satisfy the null condition $0 = ds^2(k^\alpha, k^\beta)$, $k^\alpha$ being a null vector; that implies\[\frac{dr}{dr} = \pm \frac{1}{\sqrt{(r^2/P^2 - M) q^2 M r^2}}.\] (21)

Since the metric (18) has a coordinate singularity at $r = \sqrt{-M/\Lambda} = \sqrt{PM}$, we shall use the tortoise coordinate $r_*$ defined by\[dr_* = \sqrt{\frac{g^{tt}}{g^{rr}}} = \frac{1}{\sqrt{(r^2/P^2 - M) (q_M r)^2}}.\] (22)

Integrating equation (22) for the tortoise coordinate, $r_*$, we obtain\[r_* = -\frac{i}{2\sqrt{q_M^2 M^2}} \ln \left(\frac{\sqrt{M - r^2/P^2} + \sqrt{M}}{\sqrt{M - r^2/P^2} - \sqrt{M}}\right).\] (23)

We should remark that $r_*$ is real, in spite of how it looks equation (23). It turns out that $r_*$ in the previous form is very convenient when applying the WH boundary conditions to the KG equation. It can be shown that $r_*$ can be written equivalently as\[r_* = -\frac{1}{\sqrt{q_M^2 M^2}} \tan^{-1} \left(\frac{M}{r^2/P^2 - M}\right).\] (24)

Since the function $\tan(x)$ is periodic, then $r_*$ is not uniquely defined in terms of $r$, i.e. for each value of $r$ there are multiple values of $r_*$, $r_* + \frac{1}{\sqrt{q_M^2 M^2}}\pi\xi$, with $\xi \in \mathbb{Z}$. The range of $r_*$ is determined by its values at the throat, $r_0$, and at infinity: at the throat $r_*(r_0) = \frac{1}{\sqrt{q_M^2 M^2}}\left(-\frac{\pi}{2} + n\pi\xi\right)$, while at the AdS infinity, $r \sim \infty$, $r_* \sim \frac{\pi\xi}{\sqrt{q_M^2 M^2}}$, where $\xi$ is the integer defining each particular branch. Since all these branches are equivalent, we select $\xi = 1$; consequently, the range of the tortoise coordinate is $-\frac{\pi}{2\sqrt{q_M^2 M^2}} \leq r_* < \frac{\pi}{2\sqrt{q_M^2 M^2}}$. From equation (24) we can obtain $r(r_*)$,
\[r^2 = M^2 \left[1 + \cot^2 \left(\sqrt{q_M^2 M^2} r_*\right)\right] = M^2 \csc^2 \left(\sqrt{q_M^2 M^2} r_*\right).\] (25)

The tortoise coordinate as a function of $r$ as well as its inverse are shown in figure 1.

In terms of the coordinates $(t, r_*, \phi)$ the line element (18) becomes\[ds^2 = q_M^2 M^2 r^2 \left(-dt^2 + dr_*^2\right) + r^2 d\phi^2.\] (26)

In terms of these coordinates the radial null geodesics satisfy $t = \pm r_* + \text{constant}$. The metric (26) is well defined in both sides of the throat, including the throat.
We introduce now the rescaled coordinates $\lambda := \sqrt{q^2 \alpha M^3 t}$, $\rho := \sqrt{q^2 \alpha M^3 \rho^*}$, noting that $\rho$ can be extended to $(0, \pi)$ on which $r$ is well defined; the two sides of the WH are the strips $(0, \pi/2]$ and $[\pi/2, \pi)$, separated by the throat at $\rho = \pi/2$.

Then, using that $r^2 = \ell^2 M \csc^2 \rho$, the WH metric in terms of $(\lambda, \rho)$, yields
\begin{equation}
\text{d}s^2 = \ell^2 \csc^2 \rho \left(-d\lambda^2 + d\rho^2 + M d\tilde{\phi}^2\right) = \frac{\ell^2}{\sin^2 \rho} \left(-d\lambda^2 + d\rho^2 + d\tilde{\phi}^2\right), \quad \text{with} \quad \tilde{\phi} = M\phi. \tag{28}
\end{equation}

Now in order to draw the Penrose diagram, by using $r^2 = \ell^2 M \csc^2 \rho$, we can see that for the range of $\rho$, i.e. $\rho \in (0, \pi)$, the regions that define the WH spacetime correspond to

\begin{align}
\text{Wormhole throat} : \quad & (r = r_0 = \sqrt{\ell^2 M}) \equiv \left(\rho = \frac{\pi}{2}\right), \tag{29} \\
\text{Asymptotic AdS regions} : \quad & (r \sim \infty) \equiv (\rho = 0, \pi). \tag{30}
\end{align}

Accordingly, the Penrose diagram for the metric (18) is the strip $\rho \in (0, \pi)$, that is shown in figure 2. In a similar way than the BTZ Penrose diagram, the WH Penrose diagram can be embedded into the Einstein Universe. Thus, in figure 2, every light ray coming from infinity $\rho \sim \pi$ (or $\rho \sim 0$) will pass through the WH throat $\rho = \pi/2$, and reach infinity $\rho \sim 0$ (or $\rho \sim \pi$). According to the Penrose diagram, anything that crosses the throat is not lost, but passes to the other part of the WH, in the extended manifold.

4. QNMs of a massive scalar test field in the WH spacetime

The QNMs encode the information on how a perturbing field behaves in certain spacetime; they depend on the type of perturbation and on the geometry of the background system. The QNMs of the BTZ-BH have been determined for a number of perturbing fields, namely scalar, massive scalar, electromagnetic, etc [27–29]. In this section we address the perturbation of the previously introduced WH, (18), by a massive scalar field $\Psi(t, \vec{r})$. The effect is described by the solutions of the KG equation,
\[
\left( \nabla^\alpha \nabla_\alpha - \mu^2 \right) \Psi(t, \vec{r}) = 0,
\]
where \( \mu \) is the mass of the scalar field; equivalently, the KG equation is,

\[
\partial_\alpha \left( \sqrt{-g} g^{\alpha\beta} \partial_\beta \Psi(t, \vec{r}) \right) - \sqrt{-g} \mu^2 \Psi(t, \vec{r}) = 0.
\]

The scalar field is suggested of the form

\[
\Psi(t, \vec{r}) = e^{-i\omega t} e^{i\ell \phi} R(r),
\]
where \( \omega \) is the frequency of the perturbation and \( \ell \) its azimuthal angular momentum. Substituting (33) into the KG equation, we arrive at a second order equation for \( R(r) \),

\[
R'' + \frac{2M \ell^2 - 3r^2}{(M^2 - r^2)r} R' - \left( \frac{\omega^2 - q_\alpha M^2 (\ell^2 + \mu^2 r^2)}{q_\alpha M^2 (M - r^2/I^2) r^2} \right) R = 0.
\]

\( R(r) \) is completely determined once the appropriate boundary conditions are imposed. For the WH, with \( r_0 \) being the throat, the boundary conditions for the QNMs consist in assuming purely ingoing waves at the throat of the WH, \( r = r_0 \), that in terms of the tortoise coordinate,
defined in equation (22), is \( R(r) \sim e^{-i\omega r} \). While at infinity, \( r \to \infty \), the AdS boundary demands the vanishing of the solution; i.e. the boundary conditions are

\[
\begin{align*}
  r \sim r_0 & \Rightarrow R(r) \sim e^{-i\omega r}, \\
  r \sim \infty & \Rightarrow R(r) \sim 0.
\end{align*}
\]

Let us return to the radial part of the KG equation. Since equation (34) diverges at the throat, \( r_0^2 = -M/\Lambda = M^2 \), it is useful to put it in terms of the tortoise coordinate, \( r_* \). By transforming \( R(r_*) = \psi(r_*)/\sqrt{r} \) (considering \( r \) as a function of \( r_* \)), we arrive at

\[
\ddot{\psi}(r_*) + [\omega^2 - V_{\text{eff}}(r_*)] \psi(r_*) = 0,
\]

where \( \ddot{f} = df/dr_* \). In terms of \( r \), the effective potential \( V_{\text{eff}} \) in the Schrödinger-like equation (37), is

\[
V_{\text{eff}}(r) = q^2 M^2 \left[ \left( \mu^2 + \frac{3}{4} \right) r^2 + \ell^2 - \frac{M}{4} \right], \quad \text{for} \quad r \in [r_0, \infty)
\]

that we identify as the potential of a displaced harmonic oscillator, with frequency \( \omega^2 = q^2 M^2 (\mu^2 + \frac{3}{4}) \). Note that the displacement is proportional to the angular momentum, \( \ell^2 \).

Notice that in terms of \( r_* \), the throat is a regular point of the Schrödinger problem, then, in principle, no boundary condition should be specified there. So the boundary condition (36) might well be changed to \( r_* \sim \infty \Rightarrow R(r) \sim 0 \). The effective potential can be written in terms of the tortoise coordinate as

\[
V_{\text{eff}}(r_*) = q^2 M^2 \left\{ \left( \mu^2 + \frac{3}{4} \right) \csc^2 \left( \sqrt{q^2 M^2 r_*} \right) + \frac{\ell^2}{M} - \frac{1}{4} \right\}, \quad \text{for} \quad \frac{\pi}{2\sqrt{q^2 M^2}} \leq r_* < \frac{\pi}{\sqrt{q^2 M^2}},
\]

that can be written in terms of the coordinate \( \rho \), introduced in section 3 as \( r^2 = \ell^2 M \csc^2 \rho \); in contrast to the coordinates \( r \) and \( r_* \), \( \rho \) covers both sides of the WH spacetime, side I: \( \rho \in (0, \pi/2] \), and side II: \( \rho \in [\pi/2, \pi) \), connected by the WH throat located at \( \rho = \frac{\pi}{2} \). The effective potential in terms of \( \rho \) is

\[
V_{\text{eff}}(\rho) = q^2 M^2 \left\{ \left( \mu^2 \ell^2 + \frac{3}{4} \right) \csc^2(\rho) + \frac{\ell^2}{M} - \frac{1}{4} \right\}, \quad \text{for} \quad \rho \in (0, \pi).
\]

The effective potential is depicted in figure 3, both, as a function of \( r \) and of \( \rho \). It diverges at infinity, being, as a function of \( r \), a confining harmonic oscillator-type potential; while as a function of \( \rho \) it is a potential of the Rosen–Morse type [30].

4.1. The solution for the QNMs of the WH

In general, the spectrum of the Schrödinger-like operator \( \hat{H} = -\frac{d^2}{dr_*^2} + V_{\text{eff}}(r_*) \), can be decomposed into three parts: point spectrum (often called discrete spectrum); continuous spectrum and residual spectrum. In the case of our interest it turns out that the QNMs correspond to a point spectrum and it could be foreseen from the shape of the effective potential.

In [31] Weyl showed that if \( V(x) \) is a real valued continuous function on the real line \( \mathcal{R} = (-L, L), \ L \in \mathbb{R} \), and such that \( \lim_{|x| \to L} V(x) \to \infty \), and that \( V(x) \) is monotonic in \( |x| \in (-L, L) \), then the unbounded operator \( -\frac{d^2}{dx^2} + V(x) \), acting on \( L^2(\mathcal{R}) \), has pure point
spectrum. Moreover, since $V(x)$ has the structure of an infinite well, it implies that all the eigenvalues of the operator $\hat{H} = -\frac{d^2}{dx^2} + V(x)$ will be real, necessarily. Subsequently, in [32], this result was extended for the case in which $V(x)$ is not necessarily monotonic in $|x| = (-L, L)$.

Specifically in our case, the potential $V_{\text{eff}}(\rho)$, in equation (40), is a real valued continuous function in $(0, \pi)$, and since $\lim_{|\rho| \to \pi} V_{\text{eff}}(\rho) = \lim_{|\rho| \to 0} V_{\text{eff}}(\rho) \to \infty$, then according to [31, 32], the Schrödinger-like operator has a pure point spectrum. Thus we can conclude that the QNMs of the scalar field in the WH background (18) are purely real; i.e. these QNMs are in fact normal modes (NMs) of oscillations. In agreement with the previous argument, the general solution of equation (37) with $V_{\text{eff}}(r_\ast)$ in equation (40), is given by

$$\psi(\rho) = B_1 P_{V}^{\ast} \left( \sqrt{1 - \csc^2 \rho} \right) + B_2 Q_{V}^{\ast} \left( \sqrt{1 - \csc^2 \rho} \right) = B_1 P_{V}^{\ast} (i \cot \rho) + B_2 Q_{V}^{\ast} (i \cot \rho),$$

where $B_1$ and $B_2$ are integration constants, $P_{V}^{\ast}(x)$ are the associated Legendre functions of the first kind, and $Q_{V}^{\ast}(x)$ are the associated Legendre functions of the second kind; while the parameters $V$ and $Z$ are given, respectively, by

$$V = \sqrt{1 - \mu^2 r^2} - \frac{1}{2}, \quad Z = \frac{i}{\sqrt{M}} \sqrt{\ell^2 - M^2 - \omega^2 q_\alpha^2 M^2}. \quad (42)$$

For the sake of fluency in the text we skip the details on imposing the boundary conditions and include them in the appendix. When working in regular coordinates $(\rho)$ there is no need to introduce a condition at the throat, namely, it automatically is satisfied since the effective potential is approximately constant at the throat. While related to the AdS asymptotics it shall be required the vanishing of the solution at infinity, i.e. at $\rho \to 0$ or $\rho \to \pi$ ($r \to \infty$) $\psi(\rho) \to 0$. These conditions imply restrictions in the values of the arguments of the Gamma functions related to the hypergeometric functions. Joining both conditions we arrive at the following restrictions for the WH parameters $(M, \Lambda, q_\alpha)$ that combined with restrictions on the parameters of the perturbing field $(\ell, \mu, \omega)$ amount to

$$1 - Z + V = -n, \quad n \in \mathbb{N} + \{0\}, \quad (43)$$
\[ \frac{1}{2} - \sqrt{\frac{\omega^2}{q_\alpha^2 M^2} + \frac{\ell^2}{M} + \frac{1}{4} + \sqrt{1 + \mu^2 l^2}} = -n, \]  
\[ \Rightarrow \omega^2 = q_\alpha^2 M^2 \left[ \left( n + \frac{1}{2} + \sqrt{\mu^2 l^2 + 1} \right)^2 + \frac{\ell^2}{M} - \frac{1}{4} \right]. \]  

Besides, the condition (43), when applied to the solution (41), renders that the second term becomes a multiple of the first one, and then the solution (41), in terms of the hypergeometric function, takes the form

\[ \psi(\rho) = \tilde{B}_1 \left( \frac{i \cot \rho + 1}{i \cot \rho - 1} \right)^{\frac{3}{2}} \tilde{F}_1 \left( -V, V + 1; 1 - Z; \frac{1 - i \cot \rho}{2} \right), \]  

being \( \tilde{B}_1 \) a constant, while \( \tilde{F}_1(a, b; c; x) \) is a regularized Gauss (or ordinary) hypergeometric function, related to the Gauss hypergeometric function \( \mathcal{F}_1(a, b; c; x) \) through \( \tilde{F}_1(a, b; c; x) = \mathcal{F}_1(a, b; c; x)/\Gamma(c) \), see appendix for details.

Clearly from the expression for \( \omega^2 \), equation (45), we see that it is real and always positive. It may be a surprise that \( \omega \) is not complex, as corresponds to an open system, as deceptively may appear a WH. However a clue that this is not the case (WH as an open system) came from the form of the effective potential. The throat is not similar to the event horizon in a BH, where amounts of fields are lost once penetrating the horizon. In the case of a WH it is supposed that the waves that penetrate the throat are passing to the continuation of the WH. This image is in concordance with the Penrose diagram. Moreover the fact that the frequency of the QNMs has not an imaginary part, tells us that the system will remain the same, i.e. massive scalar field solutions are stable. From equation (45) we deduce that the tendency is of growing \( \omega \) as \( M \) or \( q_\alpha \) increase as well as a very slow increase of \( \omega \) when \( l \) or \( \ell \) increase. In figure 4 are plotted the effective potential and the frequencies \( n = 0, 1, 2 \) for fixed values of the parameters. Finally we just note that for the hypergeometric function, if the first or the second argument is a non-positive integer, then the function reduces to a polynomial. In our problem it is possible to
impose that condition, by making $V = n$, being $n$ an integer; however, this is not our aim and we will not go further in this direction.

A remarkable particular case is $\ell = \pm \frac{\sqrt{M}}{2}$,

$$\pm \omega = \left( n + \frac{1}{2} \right) \omega_0 + \left( \sqrt{\mu^2 \ell^2 + 1} \right) \omega_0, \text{ with } \omega_0 = |q_\alpha| \sqrt{M_3}. \quad (47)$$

This spectrum resembles the corresponding to a quantum harmonic oscillator under the influence of an electric field $E$, $E_n = (n + \frac{1}{2}) \hbar \omega - \frac{q^2 E^2}{2m \omega}$, with a frequency given by $\omega = \sqrt{\frac{C}{m}}$ i.e. for this particular frequency, $\omega_0 = \sqrt{M_3 q_\alpha^2}$, the massive scalar field, confined by the WH-AdS spacetime, will oscillate harmonically.

5. Final remarks

We have determined in exact form the QNMs of a massive scalar field in the background of a charged, static, cyclic symmetric $(2+1)$-dimensional traversable WH, determining that the characteristic frequencies are real and discrete (point spectrum), showing that as far as the test scalar field is concerned, the potential is a confining one. The WH Penrose diagram agrees with this interpretation, since light trajectories passing through the WH throat arrive to the extended manifold, i.e. the other side of the WH.

Since there are no propagating degrees of freedom in the purely $(2+1)$-dimensional gravity, it is important to couple $(2+1)$-gravity with other fields as well as probe $(2+1)$-systems with test fields such as scalar fields. The BTZ-BH has been of great relevance providing a mathematical model of a holographic manifold. Then $(2+1)$-systems in which quasinormal modes are exactly calculated, are encouraging examples for trying to go all the way through and find the correspondence with a holography theory [33]. Holographic principle, roughly speaking, consists in finding a lower-dimensional dual field theory that contains the same information as gravity. In the system worked out in the present paper, we consider two fields, an electromagnetic field, characterized by a gauge, $A^\mu$ that could be the starting point to try a quantization scheme. We also show that the KG equation for a massive scalar field can be exactly solvable, providing then a scalar field, that could be used in searching for a correspondence in the AdS boundary. In other words, the system worked out here stimulates to explore the possibility of obtaining the conformal field associated to this AdS-WH solution in the bulk, according to the AdS/CFT correspondence.

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Appendix

In this appendix, we present the details in setting the boundary conditions for the massive scalar field in the WH spacetime with metric (18).
A.1. Behavior of the scalar field at the throat \( r = r_0 \) and boundary conditions at infinity \( r \sim \infty \)

The condition at the throat (incoming waves) is satisfied automatically: very close to the throat \( r \sim r_0 = \sqrt{-M} = \sqrt{\ell^2 M} \) we shall require that \( R(r_0) \sim e^{-i\omega r_0} \). For simplicity, the description will be presented in terms of the tortoise coordinate, \( r_+ \). Then, the asymptotic form of \( e^{-i\omega r} \) near the throat \( r_+ \sim \frac{\pi}{2\sqrt{q_+ M^3}} \), is given by

\[
r_+ \sim \frac{\pi}{2\sqrt{q_+ M^3}} \Rightarrow e^{-i\omega r} \sim (e^{i\pi})^{-\frac{\omega}{\sqrt{q_+ M^3}}} = (-1)^{-\frac{\omega}{\sqrt{q_+ M^3}}}.
\]  

(A.1)

In such a way that the condition (35) goes like

\[
r_+ \sim \frac{\pi}{2\sqrt{q_+ M^3}} \Rightarrow R(r_+) \sim \frac{\psi(r_+)}{r^2(r_+)} \sim (-1)^{-\frac{\omega}{\sqrt{q_+ M^3}}} \sim \text{constant}
\]  

(A.2)

To implement this asymptotic behavior in the solution \( R(r_+) = \psi(r_+)/r^2(r_+) \), with \( \psi \) given in equation (41), with \( \rho \) and \( r_+ \) related by \( \rho = \sqrt{q_+ M^3} r_+ \), recalling that the ranges are different, we shall write it as

\[
R(r_+) = \frac{B_1}{r^2(r_+)} P_c^2 \left( \text{cot}(\sqrt{q_+ M^3} r_+) \right) + \frac{B_2}{r^2(r_+)} Q_c^2 \left( \text{cot}(\sqrt{q_+ M^3} r_+) \right) = B_1 R_1(r_+) + B_2 R_0(r_+),
\]  

(A.3)

the two terms, \( R_1(r_+) = \frac{1}{r^2(r_+)} P_c^2 \left( \text{cot}(\sqrt{q_+ M^3} r_+) \right) \) and \( R_0(r_+) = \frac{1}{r^2(r_+)} Q_c^2 \left( \text{cot}(\sqrt{q_+ M^3} r_+) \right) \) shall be analyzed separately. Moreover, in terms of the hypergeometric functions \( R_i(r_+) \) becomes

\[
R_i(r_+) = \frac{1}{r^2(r_+)} \left( \frac{\text{cot}(\sqrt{q_+ M^3} r_+) + 1}{\text{cot}(\sqrt{q_+ M^3} r_+)} - \frac{1}{2} \right) 2F1 \left( -V, V + 1; 1 - \frac{1 - \text{cot}(\sqrt{q_+ M^3} r_+)}{2} \right).
\]  

(A.4)

The behavior of \( R(r_+) \) at the WH throat \( r_+ \sim \frac{\pi}{2\sqrt{q_+ M^3}} \) (in this neighborhood \( \ell^2 M - r^2 \sim 0 \)) is

\[
R_i(r_+) \sim \frac{1}{\sqrt{\ell^2 M}} \left( \frac{\text{cot}(\sqrt{q_+ M^3} r_+) + 1}{\text{cot}(\sqrt{q_+ M^3} r_+)} - \frac{1}{2} \right) 2F1 \left( -V, V + 1; 1 - \frac{1}{2} \right).
\]  

(A.5)

Now, using the Bailey’s summation theorem

\[
2F1 \left( a, 1 - a; c; \frac{1}{2} \right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1 + a}{2})}{\Gamma(\frac{1}{2} + a)\Gamma(\frac{1}{2} - a)},
\]  

(A.6)

the equation (A.5) takes the form

\[
R_i(r_+) \sim \frac{1}{\sqrt{\ell^2 M}} \left( \frac{\text{cot}(\sqrt{q_+ M^3} r_+) + 1}{\text{cot}(\sqrt{q_+ M^3} r_+)} - \frac{1}{2} \right) \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1 + a}{2})}{\Gamma(\frac{1}{2} + a)\Gamma(\frac{1}{2} - a)} \sim R_i \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2} + a)}{\Gamma(\frac{1}{2} - a)\Gamma(\frac{1}{2} + a)}.
\]  

(A.7)
\[ r_s \sim \frac{\pi}{2\sqrt{q_0^2 M^3}} \Rightarrow R_l(r_s) \sim \text{constant } \in \mathbb{C}. \quad (A.8) \]

Having then accomplished that for \( r \sim r_0 = \sqrt{F/M} \Rightarrow R_l(r_s) \sim e^{-i\omega r_s} \) (condition (35)).

In order to get the function that describes the asymptotic behavior of the second term in \( R_l(r_s) \) as \( r \sim r_0 = \sqrt{F/M} \), \( R_l(r_0) = \frac{1}{r}\sqrt{\frac{a - b}{c\Gamma(1 - Z)}} e^{i\omega r_0} \), we write it in terms of the hypergeometric functions,

\[ R_0(r_0) = \frac{\pi}{2\sqrt{q_0^2 M^3}}. \]

Now we can analyze the behavior of \( R_0 \) as \( r_s \sim \frac{\pi}{2\sqrt{q_0^2 M^3}} \).

\[ R_0(r_0) \sim \frac{\pi}{2}\hat{R}(r_0) - \frac{\Gamma(Z + V + 1)}{\Gamma(-Z + V + 1)} \frac{\pi \csc (Z\pi)}{2\sqrt{F/M}} \left( \frac{i\cot(\sqrt{q_0^2 M^3 r_0}) - 1}{i\cot(\sqrt{q_0^2 M^3 r_0}) + 1} \right)^{\frac{\sqrt{q_0^2 M^3}}{\pi}} \frac{\sqrt{q_0^2 M^3}}{\pi} \] \( \Rightarrow F_i = \frac{1}{2\sqrt{q_0^2 M^3}} \left( -V, V + 1; Z, -V, V + 1 \right) \). \]

Therefore we have determined the asymptotic behavior of \( R_0(r_0) \) at the throat. We considered that as \( r_s \sim \frac{\pi}{2\sqrt{q_0^2 M^3}} \), i.e. \( r \sim r_0 \), then \( F - M - r^2 \sim 0 \), and we also used equation (A.6).

### A.2. Boundary condition at infinity \((r \sim \infty)\)

In what follows we shall impose the second boundary condition at infinity, \( r \sim \infty \); in terms of the tortoise coordinate is equivalent to \( r_s \sim \frac{\pi}{\sqrt{q_0^2 M^3}} \), then \( R_l(r_s) \rightarrow 0 \) (condition (36)). It will be done separately for \( R_l(r_s) \) and \( R_0(r_s) \).

It shall be considered first the term \( R_l(r_s) \) written in terms of the hypergeometric function, equation (A.4). Since the last argument \( x = (1 - i\cot(\sqrt{q_0^2 M^3 r_0})) / 2 \) of the hypergeometric function diverges when \( r_s \sim \frac{\pi}{\sqrt{q_0^2 M^3}} \), the following identity can be used,

\[ \frac{\Gamma(a, b; c; x)}{\Gamma(a, b; c; x)} = \frac{\Gamma(c) \Gamma(b - a)}{\Gamma(b) \Gamma(c - a)} \left( 1 - x \right)^{-a} xF_1 \left( a, a - c + 1; a - b + 1; \frac{1}{x} \right) \]

that allows us to write the asymptotic expression for \( R_l(r_s) \) as

\[ R_l(r_s) \sim \frac{\Gamma(2V + 1) C_1}{\Gamma(V + 1) \Gamma(1 - Z + V)} r^{V - 1} \left( r_s \right) - F_i (-V, Z; -2V; 0) \]

\[ + \frac{\Gamma(-2V - 1) C_2}{\Gamma(-V) \Gamma(-Z - V)} r^{-V - 1} \left( r_s \right) F_i (V + 1, Z + V + 1; 2V + 2; 0), \quad (A.13) \]
where $C_1$ and $C_2$ are complex constants. Using now that $\, _2F_1(a, b; c; 0) = \Gamma(c) \, _2F_1(a, b; c; 0) = 1/\Gamma(c)$, the previous equation can be written as

$$R_i(r_e) \sim \frac{\Gamma(2V + 1)C_1}{\Gamma(V + 1)\Gamma(1 - Z + V)} \frac{1}{\Gamma(-2V)} V^{-\frac{1}{2}}(r_e) + \frac{\Gamma(-2V - 1)C_2}{\Gamma(-V)\Gamma(-Z - V)} \frac{1}{\Gamma(2V + 2)} V^{-\frac{1}{2}}(r_e).$$

(A.14)

On the other hand, given that $V = \sqrt{1 + \mu^2l^2} - \frac{1}{2}$, and since $\mu, l \in \mathbb{R}$, $\Rightarrow \sqrt{1 + \mu^2l^2} > 1 \Rightarrow V > \frac{1}{2}$. Then the behavior of $R_i(r)$ goes like

$$\lim_{r \to \infty} r^{-V^{-\frac{1}{2}}} \to \infty, \quad \text{and} \quad \lim_{r \to \infty} r^{-V^{-\frac{1}{2}}} \to 0. \quad (A.15)$$

Therefore, the fulfilment of the proper behavior, $R_i(r) \sim 0$ when $r \to \infty$, and keeping the convergence of the second term in $(A.14)$, imposes the condition that $(-2V - 1) \neq 0, -1, -2, -3, \ldots - n, \ldots$ with $n \in \mathbb{N}$, guaranteeing then that $\Gamma(-2V - 1)$ be finite. In other words, $(-2V - 1) \neq 0, -1, -2, -3, \ldots - n, \ldots$ implying that $1/\Gamma(-2V) \neq 0$. Moreover, $V + 1 > 0$ implies that $1/\Gamma(V + 1) \neq 0$, but the fulfilment of the boundary condition that $R_i(r)$ vanishes at infinity requires that $1/\Gamma(1 - Z + V) = 0$; this condition imposes that $(1 - Z + V) = -n$ with $n \in \mathbb{N}$. This guarantees the vanishing of the first term in $(A.14)$, accomplishing then the desired behavior at infinity.

A.3. Behavior of $R_i(r_e)$ at infinity

Substituting the previously derived condition $(1 - Z + V) = -n$ into $R_i(r_e)$, equation (A.9), leads to the vanishing of the second term of $R_i(r_e)$ since $1/\Gamma(1 - Z + V) = 0$. This will occur whenever (i) $Z$ is not an integer, otherwise $\csc(Z\pi)$ will diverge; and (ii) $1 + Z + V \neq -n$ with $n \in \mathbb{N} + 0$, otherwise $\Gamma(1 + Z + V)$ diverges.

Summarizing, the fulfilment of the condition $(1 - Z + V) = -n$, along with $Z \neq \pm n$ and $1 + Z + V \neq -n$, $\ n \in \mathbb{N} + 0$, leads to the following simplifications,

$$R_i(r_e) = \frac{n}{2} R_i(r_e) \Rightarrow R(r_e) = B_i R_i(r_e) + B_j R_i(r_e) = B_i R_i(r_e), \quad \text{i.e.} \ R(r_e) = B_i R_i(r_e), \quad \forall r_e.$$  

(A.16)

Being then achieved the fulfilment of the two boundary conditions, at the throat and at infinity, for the solution $R(r_e)$ for the QNMs of the scalar test field coming from the WH.

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