Power transfer efficiency formulation for reciprocal and non-reciprocal linear passive two-port systems

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Abstract: This paper formulates the power transfer efficiency of linear passive two-port systems. Focusing on the port voltages and currents, we express the efficiency in terms of impedance matrix components. Algebraic manipulation derives an explicit formula that opens up the vista on how the efficiency behaves on the complex load impedance plane. As a result, we reach a rigorous formula of the maximum efficiency without resorting to reciprocity assumption. This study originates in wireless power transfer engineering, but the hereby deduced formulas are even applicable to performance evaluation of non-reciprocal circuit networks such as ferrite isolators.

Keywords: circuit theory, impedance matrix, $kQ$ product

Classification: Microwave and millimeter-wave devices, circuits, and modules

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1 Introduction

One decade ago in Massachusetts, an experimental demonstration was carried out for wireless power transfer, which drew wide attention from electromagnetics engineers across the world [1]. They were surprised as to how the remote coils could exchange magnetic-field energy at a considerable efficiency. To address this wonderful question, there have been numerous studies from physical and mathematical aspects [2, 3, 4, 5, 6, 7]. Among them, a two-port circuit theory culminated in three key laws on power transfer: 1) the maximum power transfer efficiency is exclusively dominated by a single figure of merit that is called $kQ$; 2) $kQ$ is not just the product of coupling coefficient $k$ and quality factor $Q$ of coils but is defined for any scheme of wireless couplers; 3) $kQ$ can be rigorously calculated from the two-port impedance matrix. These laws fully characterize the relationship between the maximum power transfer efficiency and the port parameter matrix. In recent progress, Duong and Okada presented a theory for power transfer systems to solve a problem involving more than two ports [8]. This was a major step for the possible application deployment of wireless power. We notice, however, that all those theories seen in Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] assume reciprocity between the two ports, namely $Z_{12} = Z_{21}$. Indeed reciprocity makes the resultant formulas simple and easy to remember, it still confines the versatility of the theory. Therefore, restarting back from the fundamental voltage-to-current relations, this paper deduces general formulas even available for power transfer systems that involve both reciprocal and non-reciprocal elements. We also show that the general formulas can be reduced by assuming reciprocity down to the ones consistent with predecessor papers.

2 Two-port linear system

Consider a black box having two ports for input and output as shown in Fig. 1. The box is excited by a sinusoidal-wave power supply (transmitter) at port 1, and transfers the power to a load (receiver) at port 2. Since the system consists of linear elements, the port voltages are expressed as linear combinations of port currents as

\[
V_1 = Z_{11}I_1 - Z_{12}I_2 \\
V_2 = Z_{21}I_1 - Z_{22}I_2
\]

(1)

where the negative signs in front of $Z_{12}$ and $Z_{22}$ stem from the outgoing direction of $I_2$. The four complex constants

\[
Z_{mn} = R_{mn} + jX_{mn}, \quad mn = 11, 12, 21, 22
\]

(2)
are known as two-port Z parameters. In a similar way, the voltage and current at the load $Z$ are formulated as

$$V_2 = Z I_2, \quad Z = R + jX$$

which is called Ohm’s law for complex impedance. Equations (1), (2), and (3) are all we need to start the formulation. Provided that the box’s two-port $Z$ parameters and the load impedance $Z$ are given, the three equations uniquely yield voltage transfer ratio $V_2/V_1$, current transfer ratio $I_2/I_1$, and input impedance $V_1/I_1$.

![Fig. 1. Power transfer system block diagram](image)

### 3 Power transfer efficiency

For linear systems, either reciprocal or non-reciprocal in general, output power $P_2$ increases in direct proportion to input power $P_1$. The output-to-input power ratio

$$\eta = \frac{P_2}{P_1} = \frac{\Re \{ V_2 I_2^* \}}{\Re \{ V_1 I_1^* \}}$$

is called power transfer efficiency, where symbol $\Re$ denotes the real part of a complex. Substituting Eqs. (1), (2), and (3) into Eq. (4), the voltages and currents are eliminated and thus the efficiency results in

$$\eta = \frac{R \left| \frac{Z_{21}}{Z + Z_{22}} \right|^2}{\Re \left\{ \frac{Z_{11} - Z_{12}Z_{21}}{Z + Z_{22}} \right\}}$$

This can be rewritten as

$$\eta = \frac{|Z_{21}|^2}{\Sigma + \Delta + \frac{A}{R}}$$

by introducing two quadratics

$$\Sigma = R_{11}R_{22} + X_{12}X_{21} \quad \Delta = R_{11}R_{22} - R_{12}R_{21} \quad (7)$$

and three polynomial substitutes

$$A = R_{11}X^2 + BX + C$$

$$B = 2R_{11}X_{22} - R_{12}X_{21} - R_{21}X_{12}$$

$$C = R_{22}(R_{11}R_{22} - R_{12}R_{21} + X_{12}X_{21})$$

$$+ X_{22}(R_{11}X_{22} - R_{12}X_{21} - R_{21}X_{12})$$

for algebraic convenience.
4 Resonance

Consider an efficiency maximization problem on the condition that the black box is given and we can adjust the load impedance. The efficiency is formulated in Eq. (6) as a function of $R$ and $X$. They are independently adjustable, so we can begin with either. We focus on $X$ first as it appears only in polynomial $A$, which is found in the denominator of the right-hand side of Eq. (6). Since the efficiency cannot exceed unity in nature, the denominator must be equal to or greater than $|Z_{21}|$. Therefore, we should minimize $A$ in order to maximize the efficiency.

Polynomial $A$ can be transformed into a complete square as

$$A = R_{11}\left(X + \frac{B}{2R_{11}}\right)^2 - \frac{B^2}{4R_{11}} + C$$

Passive systems always exhibit positive $R_{11}$, this formula draws a concave parabola on $X$-$A$ plane. We thus find that $A$ is minimized by tuning $X$ into its symmetric axis

$$X_{opt} = -\frac{B}{2R_{11}} = \frac{R_{12}X_{21} + R_{21}X_{12}}{2R_{11}} - X_{22}$$

This exactly specifies the load reactance that leads series resonance to port 2. Especially for systems where either trans-resistance or trans-reactance vanishes, the right-hand side leaves $X_{22}$ alone. In accordance with the polarity of $X_{22}$, the load should be capacitive or inductive to compensate for the imaginary part of the box’s output impedance.

From Eqs. (8), (9), and (10), the minimum of $A$ is found as

$$A_{min} = -\frac{B^2}{4R_{11}} + C = \frac{\Sigma \Delta - \Theta^2}{R_{11}}$$

On this right-hand side, all the terms meet reciprocity except for the numerator’s final one, i.e.

$$\Theta = \frac{1}{2}(R_{12}X_{21} - R_{21}X_{12})$$

which is named impedance exchange term in this paper. It appears when the box includes non-reciprocal material such as magnetized ferrite.

Applying Eq. (11) to Eq. (6), we get the efficiency in series resonance as

$$\eta_{res} = \frac{|Z_{21}|^2}{\Sigma + \Delta + R_{11}R + \frac{A_{min}}{R}}$$

For any passive system, none of the four terms in the denominator can be negative. Particularly, we should be aware that $A_{min} \geq 0$. If it were negative, the denominator could be nullified for some positive $R$, and thus the efficiency would go to infinite. Such behavior might be seen in oscillators, but never takes place in passive systems like wireless couplers.

5 Load resistance tuning

As the optimization of $X$ is completed in the previous section, we move our focus onto the load resistance $R$. To find optimum $R$, the denominator of Eq. (13) is transformed into a complete square with respect to $R$ as
\[ \eta_{\text{dc}} = \frac{|Z_{21}|^2}{\Sigma + \Delta + 2\sqrt{R_{11}A_{\text{min}}} + \left(\sqrt{R_{11}}R - \sqrt{A_{\text{min}}R}\right)^2} \]  

One may think that it looks more complicated than Eq. (13), but the point is that \( R \) appears only in the squared parenthesis. Other terms are all constants coming from the box’s two-port parameters not dependent on the load. The denominator is minimized when the squared parenthesis vanishes. This can be done by tuning \( R \) into

\[ R_{\text{opt}} = \sqrt{\frac{A_{\text{min}}}{R_{11}}} \]  

This is called optimum load resistance for the efficiency maximization. Putting it back into Eq. (14) with the help of Eq. (11), we finally reach

\[ \eta_{\text{max}} = \frac{|Z_{21}|^2}{\Sigma + \Delta + 2\sqrt{\Sigma \Delta - \Theta^2}} \]  

This is the general formula for maximum power transfer efficiency in terms of two-port \( Z \) parameters taking non-reciprocal effects into account.

### 6 Reciprocal subset

This section gives a supplement to the formulas described in previous sections. Assume a special case where the box consists of only reciprocal elements. In the impedance domain, reciprocity is expressed as

\[ R_{12} = R_{21}, \quad X_{12} = X_{21} \]  

Under this condition, the impedance exchange term \( \Theta \) becomes zero from Eq. (12). Accordingly, Eqs. (10), (15), and (16) also reduce to

\[ R_{\text{opt}} = \frac{\sqrt{\Sigma \Delta}}{R_{11}} \]  

\[ X_{\text{opt}} = \frac{R_{12}X_{21}}{R_{11}} - X_{22} \]  

\[ \eta_{\text{max}} = \frac{\sqrt{\Sigma - \Delta}}{\sqrt{\Sigma + \Delta}} \]  

To make this formula even more elegant, we introduce the \( kQ \) product defined as

\[ kQ = \frac{|Z_{21}|}{\sqrt{\Delta}} \]  

The right-hand side physically signifies the magnitude of impedance-domain transfer function denominated by the equivalent scalar resistance when we regard the box as a linear two-port network system. See Refs. [6, 7, 9, 10] for further details and Appendix for example.

Thanks to Eq. (21), we can translate Eq. (20) into an ultimate elegance, i.e.

\[ \eta_{\text{max}} = \frac{\rho - 1}{\rho + 1} \]  

associated with the hypotenuse length
\[
\rho = \sqrt{1 + (kQ)^2}
\]  

(23)

of a right triangle orthogonally spanned by unity and \(kQ\).

7 Conclusion

Starting from the voltage-current relation of two-port linear systems, we successfully reached the power transfer efficiency formula. It covers not only reciprocal but also non-reciprocal systems. A key point is the impedance exchange term that appears when the system involves non-reciprocal elements. The deduced formulas will contribute to future power transfer engineering suffering from, or even positively exploiting, non-reciprocal materials or devices such as ferrite isolators.

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Appendix

To help comprehend Eq. (21), imagine a symmetrical magnetic coupler with self inductance \(L\), mutual inductance \(M\), and parasitic series resistance \(R\). As is well known, it exhibits two-port \(Z\) parameters

\[
Z_{11} = Z_{22} = R + j\omega L \\
Z_{12} = Z_{21} = j\omega M
\]

at angular frequency \(\omega = 2\pi f\). Putting them into Eq. (21), we find

\[
kQ = \frac{\omega M}{R}
\]

which is exactly identical to the product of coupling coefficient \(M/L\) and quality factor \(\omega L/R\).