The sum of two measurable functions

Jan Pachl
pachl@acm.org

December 22, 2005*

Summary

Following Weizsäcker [3], we use this notation: For a complete probability space $(\Omega, \Sigma, P)$ and a locally convex space $E$, denote by $L^0(\Omega, \Sigma, P, E)$ the set of all Borel-measurable functions $f : \Omega \to E$ for which the image measure $f[P]$ on $E$ is Radon.

In 1976 E. Thomas asked, in a conversation with the author, whether $L^0(\Omega, \Sigma, P, E)$ is always closed under addition. The question is motivated by the observation that some of the results in [2] can be proved for functions in $L^0(\Omega, \Sigma, P, E)$.

This note presents an example where $L^0(\Omega, \Sigma, P, E)$ is not closed under addition. However, Weizsäcker [3] showed that this obstacle is not as serious as would seem.

Terminology

All measures will be probability measures, i.e. positive and with total mass 1. Say that $(X, A, \mu)$ is a compact Radon measure space if $X$ is a compact Hausdorff space, $A$ is a sigma-algebra on $X$ containing all Borel subsets of $X$ and $\mu$ is a complete measure on $A$ such that

$$
\mu B = \sup \{ \mu K \mid K \subseteq B \text{ and } K \text{ is compact} \}
$$

for every $B \in A$.

When $(X, A)$ and $(B, B)$ are two measurable spaces (sets with sigma-algebras), denote by $A \otimes B$ the product sigma-algebra on $X \times Y$; this is the smallest sigma-algebra on $X \times Y$ making both projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ measurable.

*Transcribed from the author’s manuscript dated April 1980.
Example

The example will be constructed in three steps.

**Step 1** Construct a compact Radon measure space \((X, \mathcal{A}, \mu)\) such that \(\mu B = 0\) whenever \(B \in \mathcal{A}\) has cardinality less than or equal to \(2^{\aleph_0}\).

**Construction** Let \(I\) be a set of cardinality \(2^{\aleph_0}\), let \(X\) be the compact space \(\{0, 1\}^I\), and let \(\mu\) be the standard product measure on \(X\) (defined on \(\mathcal{A}\), the \(\mu\)-completion of the Borel sigma-algebra in \(X\)). That is, \(\mu\) is the product of measures on \(\{0, 1\}\) each of which gives measure \(\frac{1}{2}\) to \(\{0\}\) and \(\frac{1}{2}\) to \(\{1\}\).

For every subset \(J\) of \(I\), define an automorphism \(T_J\) of \((X, \mathcal{A}, \mu)\) by
\[
T_J(\{x_i\}_{i \in I}) = \{y_i\}_{i \in I}
\]
where \(y_i = x_i\) for \(i \in J\) and \(y_i = 1 - x_i\) for \(i \in I\setminus J\).

If \(B \in \mathcal{A}\) has cardinality \(\leq 2^{\aleph_0}\) then there is a set \(J \subseteq I\) such that \(B \cap T_J(B) = \emptyset\). Indeed, choose an injective map \(\alpha : B \times B \to I\) and define
\[
J = \{ j \in I \mid j = \alpha(\{x_i\}_{i \in I}, \{y_i\}_{i \in I}) \text{ for } \{x_i\}, \{y_i\} \in B \text{ and } x_j \neq y_j \}.
\]

It follows that for each \(B \in \mathcal{A}\) of cardinality \(\leq 2^{\aleph_0}\) there is a sequence of \(\mu\)-automorphisms \(S_1, S_2, S_3, \ldots\) such that
\[
B_k \cap S_k(B_k) = \emptyset, \quad k = 1, 2, 3, \ldots,
\]
where
\[
B_1 = B, \quad B_{k+1} = B_k \cup S_k(B_k).
\]

Hence there are infinitely many disjoint sets of the same measure as \(B\). Therefore \(\mu B = 0\).

**Step 2** Construct a compact Radon measure space \((X, \mathcal{A}, \mu)\) and a measure \(\nu\) on the product sigma-algebra \(\mathcal{A} \otimes \mathcal{A}\) such that for the “diagonal” \(D = \{(x, x) \mid x \in X\}\) and the projections \(\pi_1 : X \times X \to X\) and \(\pi_2 : X \times X \to X\) we have

(i) \(\nu G = 1\) for every \(G \in \mathcal{A} \otimes \mathcal{A}\) such that \(G \cup D = X \times X\),
and \(\nu H = 1\) for every \(H \in \mathcal{A} \otimes \mathcal{A}\) such that \(H \supseteq D\);

(ii) \(\pi_1[\nu] = \mu = \pi_2[\nu]\).
Construction Take the \((X, \mathcal{A}, \mu)\) constructed in Step 1. Denote by \(\beta : X \to X \times X\) the map defined by \(\beta(x) = (x, x)\). We have \(\beta^{-1}(G) \in \mathcal{A}\) for each \(G \in \mathcal{A} \otimes \mathcal{A}\); let \(\nu G = \mu(\beta^{-1}(G))\) for \(G \in \mathcal{A} \otimes \mathcal{A}\) (that is, \(\nu = \beta[\mu]\)). Since both \(\pi_1 \circ \beta\) and \(\pi_2 \circ \beta\) are the identity map on \(X\), (ii) follows.

If \(H \in \mathcal{A} \otimes \mathcal{A}\) and \(H \supseteq D\) then, by the definition of \(\nu\), we have \(\nu H = 1\).

Take \(G \in \mathcal{A} \otimes \mathcal{A}\) such that \(G \cup D = X \times X\). We have
\[
\nu G = \inf \left\{ \sum_{n=1}^{\infty} \nu(B_n \times C_n) \mid B_n, C_n \in \mathcal{A} \text{ and } \bigcup_{n=1}^{\infty} (B_n \times C_n) \supseteq G \right\}
\]
(see e.g. \([1]\), 13.A); thus it suffices to show that
\[
\sum_{n=1}^{\infty} \nu(B_n \times C_n) \geq 1 \text{ whenever } \bigcup_{n=1}^{\infty} (B_n \times C_n) \supseteq G, \quad B_n, C_n \in \mathcal{A}.
\]
Fix such \(B_n, C_n\) and let \(V = \bigcup_{n=1}^{\infty} (B_n \times C_n)\). Then \((X \times X) \setminus V \subseteq D\). We show that the cardinality of \((X \times X) \setminus V\) is at most \(2^{8_0}\): If \((x, x), (y, y) \in (X \times X) \setminus V\) and \(x \neq y\) then there is \(n\) such that \((x, y) \in B_n \times C_n\) and \((x, x) \notin B_n \times C_n\); hence \(x\) and \(y\) are separated by \(C_n\). It follows that \((X \times X) \setminus V\) has at most \(2^{8_0}\) points. Consequently, \(\beta^{-1}(X \times X) \setminus V\) has at most \(2^{8_0}\) points and
\[
\nu((X \times X) \setminus V) = \mu(\beta^{-1}((X \times X) \setminus V)) = 0
\]
by the property of \(\mu\). Thus
\[
\sum_{n=1}^{\infty} \nu(B_n \times C_n) \geq \nu V = \nu V + \nu((X \times X) \setminus V) = \nu(X \times X) = 1.
\]
It follows that \(\nu G = 1\).

Step 3 Construct a complete probability space \((\Omega, \Sigma, \mathbf{P})\), a locally convex space \(E\) and two functions \(f, g : \Omega \to E\) such that

(a) \(f^{-1}(B), g^{-1}(B) \in \Sigma\) for every Borel set \(B \subseteq E\);

(b) the image measures \(f[\mathbf{P}]\) and \(g[\mathbf{P}]\) are Radon;

(c) the function \(h = f + g\) has the property \(h^{-1}(0) \notin \Sigma\).

Thus, in this example, \(L^0(\Omega, \Sigma, \mathbf{P}, E)\) is not closed under addition.
Construction Take the $(X, \mathcal{A}, \mu), \nu, D, \pi_1$ and $\pi_2$ as in Step 2.

Every compact Hausdorff space $Y$ is a topological subspace of a locally convex space (e.g. let $C(Y)$ be the Banach space of real-valued continuous functions on $Y$; then $Y$ embeds canonically into the dual of $C(Y)$ endowed with the $w^*$ topology). Fix such an embedding $e : X \hookrightarrow E$ of $X$ into a suitable locally convex space $E$. Let $(\Omega, \Sigma, \mathbf{P})$ be the completion of $(X \times X, \mathcal{A} \otimes \mathcal{A}, \nu)$, and let $f = e \circ \pi_1$, $g = -e \circ \pi_2$.

Now (a) is obvious, and (b) is true because the measures $f[P]$ and $g[P]$ are continuous images of the Radon measure $\mu$ (by (ii) in Step 2). Finally, $h^{-1}(0) = D$ and $D \notin \Sigma$ in view of (i) in Step 2; that proves (c).

References

[1] P.R. Halmos. Measure Theory. Springer-Verlag 1974.

[2] G.E.F. Thomas. Integration of functions with values in locally convex Suslin spaces. Trans. Amer. Math. Soc. 212 (1975) 61-81.

[3] H. v. Weizsäcker. Strong measurability, liftings and the Choquet-Edgar theorem. Vector Space Measures and Applications II. Springer-Verlag Lecture Notes in Mathematics Vol. 645 (1977) 209-218.