Hall coefficient and magnetoresistance in boson+fermion dimer models for the Pseudogap phase of high Tc superconductors

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We show that the Hall coefficient of the boson+fermion dimer model for the pseudogap phase of high temperature superconductivity introduced in [13] changes sign from negative at low temperatures to positive at high temperatures at a characteristic temperature scale of $\kappa_0 T \sim \hbar \omega_C$ (the cyclotron frequency of the fermionic dimers, here $\kappa \sim 0.7 O(1)$ fits the experimental data well [16, 20, 21]). We show that this is an effect of the changing of the sign of the coupling between the fermionic dimer and the magnetic field from negative coupling $\sim -e$ at low temperatures to positive coupling $\sim +e$ at high temperature, with the Hall coefficient being proportional to $R_H \sim eB_{\text{EE}}G$ (the product of the magnetic charge, electric charge and current charge all of which we carefully define). We relate the Hall conductivity to the coefficient in Kohler’s like rule for magnetoconductivity and calculate some corrections which are relevant near the intermediate temperature range $\sim 50K$ (typical values for $k_BT_0 \sim \hbar \omega_C$). Furthermore we make a sharp prediction that the magnetoresistance effect vanishes to order $B^2$ at the temperature and magnetic field where the Hall coefficient vanishes.

I. INTRODUCTION

The pseudogap phase is one of the most enigmatic phases of high temperature cuprate superconductivity. Recently there has been considerable experimental evidence that the pseudogap phase of the cuprates has a description in terms of a vanilla Fermi liquid with nearly free fermionic quasiparticles. Indeed transport experiments on the cuprates show that the quasiparticle lifetime $\tau(T, \omega)$ follows conventional Fermi liquid behavior with $\tau^{-1}(T, \omega) \sim \omega^2 + c^2 T^2$ [12] (where $c$ is some constant). Furthermore the pseudogap phase, at high temperatures $\sim 100K - 200K$, obeys Kohler’s rule for in plane magnetoconductance with the longitudinal resistance being proportional to $\rho_{xx} \sim \tau^{-1}(1 + bH^2 \tau^2)$ [8], where $b$ is again some constant. Even more evidence of the existence on nearly free fermionic quasiparticles obeying Fermi Dirac statistics comes from the observation of quantum oscillations for the underdoped cuprates [9].

The frequency of the oscillations being between 500 and 600 $T$ showing that there is a very small Fermi surface with a Fermi area $\sim p/8$ (where $p$ is the doping) indicating there are $2 \times 4$ pockets of area $p/8$ each (where the factor of 2 comes from spin degeneracy [17]), furthermore the amplitude of these magnetic oscillations follows very well the Lifshitz-Kosevich formula for the amplitude of quantum oscillations of free fermions [9] [17]. A description of the pseudogap phase in terms of nearly free fermions has been achieved in ref. [6], which we follow in this work. A good way to compare the properties of the Fermi liquid introduced in ref. [6] and the Fermi liquid for the pseudogap is the study of the Hall coefficient, which is a powerful probe of the Fermi surface of most Fermi liquids. The sign of the Hall coefficient allows one to determine if the charge carries are particles or holes [15]. One of the most enigmatic aspects of the underdoped cuprates, the pseudogap phase, is that the Hall coefficient switches signs from negative to positive as a function of temperature and magnetic field [16, 20, 21]. The transition from negative Hall coefficient at low temperatures to positive Hall coefficient at high temperatures moves to progressively higher and higher temperature with increasing magnetic field [16, 20, 21], furthermore as we shall see below the transition temperature where the Hall coefficient goes to zero is proportional to the cyclotron frequency of quantum oscillations. Any faithful model of the pseudogap of the high temperature superconductors must reproduce this qualitative feature. Building on previous work [8] this is the main thrust of this research. We also study the magnetoresistance and show that it obeys Kohler’s rule like effect with a temperature and field dependent constant $b(T, B)$, (the field dependence is a slight deviation from the conventional Kohler’s rule behavior of the magnetoconductivity, though the field dependence may be neglected at both low and high temperature and is only relevant at an intermediate temperature range $\sim 50K$ or the cyclotron frequency relevant to that particular doping), we further relate the coefficient $b(T, B)$ in the magnetoconductivity $\sigma_{xx}^{-1} \sim \tau^{-1} \left(1 + \tilde{b}(T, B) H^2 \tau^2\right)$ to the Hall conductivity $\sigma_{xy}(B, T)$ and the effective mass of quantum oscillations $m^*$ leading to a relation that involves only directly measurable quantities [9, 17].

The Rokhsar-Kivelson quantum dimer model (QDM) was introduced to describe a magnetically disordered phase (the resonating valence bond (RVB) phase) of the underdoped cuprate materials [14]. Recently QDMs have been once again revisited as models of high-temperature superconductivity [1, 3]. This was motivated by the need to reconcile transport experiments [12] and photoemission data [13] in the underdoped region of cuprate superconductors. Photoemission data shows Fermi arcs enclosing an area of $1 + p$ (with $p$ being the doping), while transport measurements indicate plain Fermi-liquid properties consistent with an area $p$. The authors of Refs.
 introduced a model for the pseudogap region of the cuprate superconductors which consists of two types of dimers: one spinless bosonic dimer (representing a valence bond between two neighboring spins) and one spin 1/2 fermionic dimer representing a hole delocalized between two sites. Using a slave-boson/slave-fermion approach [6], we were previously able to confirm the numerical results of refs. [1–3] analytically supporting the existence of a fractionalized Fermi liquid enclosing an area p and to extend this model to show that, in fact, it describes a larger portion of the phase diagram and captures well the emergence of d-wave superconductivity [6]. Indeed using a meanfield approach we presented a model of the pseudogap where the fermionic dimers are nearly free quasiparticles obeying Fermi Dirac statistics which can condense to form superconductivity [6].

In this work, within the meanfield introduced in reference [6] we will show how the Hall coefficient switches from negative to positive values as a function of temperature. We will show that the effective coupling to a static external magnetic field changes sign from $-e$ to $+e$ in a crossover with a transition temperature $T_0 (B)$ (where $e_B (T_0 (B), B) = 0$) by $\kappa e_B T_0 (B) \approx \hbar \omega_C = \frac{e B}{m^*}$ (the cyclotron frequency of the fermionic dimers [9] with $\kappa \approx 0.7, O(1)$ fitting the experimental data) leading to a change of sign of the Hall coefficient. Indeed we show that $R_{xy} \sim e_B e \epsilon_f 3 \hbar^2$ (the product of the magnetic charge, electric charge and current charge which we carefully define with $e_f = e_{2g} = +e$ independent of temperature). Over all this matches well with experimental data on the underdoped region of the cuprates [16,20,21] (the cyclotron frequency is highly doping dependent and its dependence on doping is well reproduced in the doping dependence of $\kappa T_0 (B)$, see also Appendix A). We also compute explicit formulas for the magnetoresistance and the Hall coefficient (see Eq. (22)). We make a sharp prediction that the magnetoresistance effect vanishes to order $B^2$ when the Hall resistivity goes through zero, or in other words the coefficient in Kohler’s rule of magnetoresistance vanishes at the temperature and magnetic field where the Hall coefficient is zero, we further relate the coefficient $B (T, B)$ for magnetoresistivity to the Hall conductivity $\sigma_{xy}$ and the effective mass of quantum oscillations $m^*$ all directly efficiently experimentally measurable [9,17].

In Section II we review the form of the main Hamiltonian used in the text. In section III we show how to project the $t - J$ model Hamiltonian with the external magnetic and electric fields onto the dimer subspace. Section IV is our main result which shows the Hall coefficient and magnetoresistance as a function of temperature and field. In the appendices we review the semiclassical equations of motion needed to derive our key results.

II. MAIN HAMILTONIAN

We will consider a system of dimers as described in reference [6]. The total Hamiltonian for our system is given in [6], we will also use the notation introduced in ref. [6]. As pointed out in reference [6] we can make substantial progress in understanding the fermionic component of the theory without detailed analysis of the bosonic component. Indeed, any translationally invariant (liquid-like) ansatz for the bosonic dimers introduced in refs. [1–3] that does not break the symmetry between the $x$ and $y$ axis, yields similar fermionic effective theories. The effective fermionic mean-field Hamiltonian reads [6]:

$$H_{FB} = \sum_{\sigma} \sum_i \left\{ -t_1 c_{i+\gamma,y,\sigma}^\dagger c_{i,x,\sigma}^\dagger (b_{i+\gamma,y} + b_{i+\gamma,y}) + 1 \text{ term} ight.$$  

$$- t_1 c_{i+\gamma,y,\sigma}^\dagger c_{i,x,\sigma}^\dagger (b_{i+x,y} + b_{i+\gamma,y}) + 1 \text{ term} $$

$$- t_2 c_{i,\gamma,y,\sigma}^\dagger c_{i+x,\sigma}^\dagger (b_{i+x,y} + b_{i+\gamma,y}) + 7 \text{ terms} $$

$$- t_3 c_{i,\gamma,y,\sigma}^\dagger c_{i+\gamma+y,x,\sigma}^\dagger (b_{i+x+y,x} + b_{i+\gamma,y}) + 7 \text{ terms} $$

$$- t_4 c_{i,\gamma,y,\sigma}^\dagger c_{i+2\gamma,x,\sigma}^\dagger (b_{i+2\gamma,x} + b_{i+\gamma,y}) + 7 \text{ terms} \right\}$$

$$(-2\lambda - \mu) \sum_{i,\sigma} c_{i,x,\sigma}^\dagger c_{i,\sigma} + 7 \text{ terms}$$

which is effectively a tight-biding model with renormalized hoppings $T_1 = t_1 (b_{i+\gamma,y} + b_{i+\gamma,y})$, $T_2 = t_2 (b_{i+\gamma,y} + b_{i+\gamma,y})$ and $T_3 = t_3 (b_{i+x+y,x} + b_{i+\gamma,y})$. Here $c_{i,\gamma,y,\sigma}^\dagger$ refers to a fermionic creation operator (representing a fermionic dimer on the link connecting vertices $i$ and $i + y$ with spin $\sigma$), while $b_{i,\gamma}$ refers to spinless bosonic dimers [6]. $\lambda$ is a Lagrange multiplier used to enforce the constraint that there is exactly one dimer per site and $\mu$ is the chemical potential of the electrons in the pseudogap phase. The coefficients $t_1/2/3$ were introduced in refs. [1–3]. Furthermore we assume that there is no time reversal symmetry breaking so that all expectation values for the bosons are real at zero external field. We note that changes in the phases of $(b_{i+\gamma,y} + b_{i+\gamma,y})$, $(b_{i+x+y,x} + b_{i+\gamma,y})$ and $(b_{i+x+y,x} + b_{i+\gamma,y})$ due to the external magnetic field will play a crucial role in the dynamics of the fermionic dimers, see the discussion below Eq. (19) below.

The resulting model is defined on a square lattice with a two point basis. The horizontal $(x)$ and vertical $(y)$ links make up the two sublattices where the fermions reside. We define (in momentum space) the spinor that encodes these two flavors of fermions as $\psi_{k,\sigma} = (c_{k,\gamma,y,\sigma}^\dagger, c_{k,\gamma,x,\sigma}^\dagger)$ and the Hamiltonian in momentum space is given by [6]:

$$H_{FB} = \sum_{k,\sigma} \psi_{k,\sigma}^\dagger \left( \begin{array}{cc} \gamma_k^x & \gamma_k^y \\ \gamma_k^y & \gamma_k^x \end{array} \right) \psi_{k,\sigma} \ ,$$

(2)
t − J Hamiltonian on the square lattice:

\[ H_{tJ} = - \sum_{\alpha} t_{ij} d_{i,\alpha}^\dagger d_{j,\alpha} + J \sum_{\langle i,j \rangle} \left( \vec{S}_i \cdot \vec{S}_j - \frac{1}{4} n_i n_j \right) \]  \( (4) \)

subject to the constraint that \( n_i \leq 1 \). Here \( d_{i,\alpha} \) and \( d_{i,\alpha}^\dagger \) are the electron creation and annihilation operators (\( \alpha = \uparrow, \downarrow \)) of the \( t − J \) model, \( \vec{S}_i = \sum_{\alpha,\beta} \sigma_{\alpha,\beta} d_{i,\alpha} d_{i,\beta}^\dagger \), and \( n_i = d_{i,\uparrow}^\dagger d_{i,\uparrow} + d_{i,\downarrow}^\dagger d_{i,\downarrow} \). Under projection, described below, it is not too hard to see that the term proportional to \( J \) does not contribute to the part of the Hamiltonian that is biquadratic in the fermions and the bosons (see Eq. (1) but only contributes to the fermion fermion interaction and the boson boson interaction terms in the Hamiltonian ((which leads to superconductivity [6] which is suppressed by the large magnetic field) and produces the RK Hamiltonian [10] that acts only on the bosons thereby providing the expectation values \( \langle b_{i,\alpha}^\dagger b_{i,\beta} \rangle \), \( \langle b_{i,\alpha}^\dagger b_{i,\beta}^\dagger \rangle \) and \( \langle b_{i,\alpha}^\dagger b_{i,\beta} b_{i,\beta} \rangle \) and therefore will be dropped from now on. We will first include a vector potential, magnetic field, and scalar potential electric field into the \( t − J \) model using the Pierls substitution:

\[ H_{tJ} = - \sum_{\alpha} t_{ij} \exp \left( i \frac{\varphi_i}{\hbar} n_i \right) d_{i,\alpha}^\dagger d_{j,\alpha} - e \sum_i \varphi_i n_i \] \( (5) \)

Where \( A_{ij} \equiv \int_x \vec{A} \cdot d\vec{r} \) and \( \varphi_i \) is the scalar potential at site \( i \). We would like to project this more general Hamiltonian onto the dimer subspace (the projection without any Pierls substitutions for \( E = B = 0 \) was done in refs. [1][3]). To do so we can identify the dimer Hilbert space with a subspace of the Hilbert space for the \( t − J \) model, where the zero dimers state (which is outside the physical dimer space) corresponds to the state with zero electrons, and the rest of the Hilbert space can be introduced via the operators \( \tilde{b}_{i,\eta}^\dagger \equiv \Upsilon_{i,\eta} (d_{i,\eta}^\dagger - d_{i,\eta})/\sqrt{2} \) and \( \epsilon_{i,\eta,\sigma} \equiv \Upsilon_{i,\eta}(d_{i,\sigma} + d_{i,\sigma}^\dagger)/\sqrt{2} \). The phases \( \Upsilon_{i,\eta} \) represent a gauge choice and we shall follow the one by Rokhsar and Kivelson [10] and define \( \Upsilon_{i,\eta} = 1 \) and \( \Upsilon_{i,\eta} = (-1)^{i_y} \), where \( i_y \) is the \( y \)-component of the 2D square lattice site index \( i \). The projection procedure can be described as:

\[ H_D = \begin{pmatrix} H_{DD} & H_{DO} \\ H_{OD} & H_{OO} \end{pmatrix} \] \( (6) \)

Where we divide the \( t − J \) model Hilbert space into the dimer Hilbert space and its orthogonal complement. We can then write the \( t − J \) Hamiltonian in block diagonal form as shown in Eq. (6) and keep only the terms \( H_{DD} \). We will not do the projection calculation explicitly but instead we would give qualitative arguments about the form of the effective Hamiltonian in Section [V]. We will
find it convenient to work in the gauge where \( E = -\nabla \varphi \) and \( B = \nabla \times A \) with \( \varphi \) and \( A \) time independent. In this gauge we note that for a time independent electric field the projection can be carried out directly, and that the electric field couples to the dimers minimally, e.g.,

\[
\varepsilon \rightarrow \varepsilon + \varepsilon_E \varphi (r) .
\]

Where \( E = -\nabla \varphi \) and \( \varepsilon_E = E_0 \) is a periodic Hamiltonian. We note that \( \varepsilon_E = +e \) indicating that under an electric field the dimer acts as a positively charged object (indeed a bare fermionic dimer has charge \( e \)) however when a fermionic dimer moves a bosonic dimer carrying charge \(-2e\) moves in the opposite direction leading to an electric charge \( 2e - e = +e \), alternative under projection the electric field energy is equal to \( +e \sum \varphi_i \) where the sum is taken over the unoccupied electrons (holes) in the \( t-J \) model or equivalently the positions of the fermionic dimers.

IV. PHASES AND CHARGES

We would like to carefully discuss the phases and charges of the fermionic and bosonic dimers. There are two relevant gauge groups for the dimers the internal local \( U(1) \) gauge symmetry:

\[
\begin{align*}
&b_{i,n} \rightarrow e^{i \theta_i} b_{i,n} e^{i \theta_{i+n}} , \\
c_{i,n,\sigma} \rightarrow e^{i \theta_i} c_{i,n,\sigma} e^{i \theta_{i+n}} ,
\end{align*}
\]

with a phase \( \theta_i \) associate to each vertex \( i \). There is also the \( U(1) \) due to its coupling to electromagnetism. We now determine the charges of the bosonic and fermionic dimers under the electromagnetic gauge field. Under electromagnetism the \( t-J \) model electrons transform as

\[
A \rightarrow A - \nabla \alpha , \\
d_i \rightarrow e^{i \bar{\alpha}} d_i
\]

Now a fermionic dimer is made of a single electron operator while a bosonic dimer is made of two. This means that under an electromagnetic gauge transformation the fermionic dimer has charge \(-e\) while a bosonic dimer has charge \(-2e\). Indeed or the dimer model we have the following operator equivalences:

\[
\begin{align*}
&b_{i,n}^\dagger \sim d_{i,\sigma}^\dagger d_{i+\eta,\sigma}^\dagger - d_{i,\sigma} d_{i+\eta,\sigma} , \\
&b_{i,n} \sim e^{i \bar{\alpha}}(\alpha_{i+n}) , \\
b_{r} \sim e^{2i \bar{\alpha}(r)} b_{r}
\end{align*}
\]

If \( d_i \rightarrow e^{i \bar{\alpha}} d_i \) under a gauge transformation, then

\[
\begin{align*}
&b_{i,n} \rightarrow e^{i \bar{\alpha}(\alpha_{i+n})} , \\
b_{r} \rightarrow e^{2i \bar{\alpha}(r)} b_{r}
\end{align*}
\]

Therefore \( b \) has gauge charge \(-2e\) under electromagnetism. Where we have assumed a long wavelength limit description. Similarly:

\[
\begin{align*}
&c_{i,n,\sigma}^\dagger \sim d_{\sigma}^\dagger + d_{i+\eta,\sigma}
\end{align*}
\]

Then for \( d_i \rightarrow e^{i \bar{\alpha}} d_i \) under a gauge transformation

\[
c_{r,\sigma} \rightarrow e^{i \bar{\alpha}(r)} c_{r,\sigma}
\]

We now need to calculate the phases \( \langle b^\dagger_{i,n} b^\dagger_{j,\nu} \rangle \). To do so we introduce the electromagnetically gauge invariant greens functions \[13]:

\[
\hat{G}(r,r') \equiv \left\langle b^\dagger(r) \exp \left( 2i \frac{e}{c} \int_{r}^{r'} A(r'') dr'' \right) b((r')) \right\rangle
\]

These greens functions are gauge invariant \[13\], see also Appendix [D]. As such they are rotationally and translationally invariant (indeed for a constant magnetic field the system is translationally and rotationally invariant as such a translation or a rotation is a gauge transformation which does not change the gauge invariant greens functions),

\[
\hat{G}(r,r') = \hat{G}(|r-r'|) = \hat{G}(r',r) = \hat{G}^*(r,r')
\]

As such \( \hat{G}(r,r') \) has zero phase so

\[
\langle b^\dagger(r) b(r') \rangle \sim \exp \left( -2i \frac{e}{c} \int_{r}^{r'} A(r'') dr'' \right)
\]

This derivation however ignores Elitzur’s theorem which says that a gauge symmetry (given in Eq. \[8\]) cannot be spontaneously broken. Indeed

\[
\langle b^\dagger(r) b(r') \rangle = 0 ,
\]

\[
\langle b^\dagger((r,\tau)) b((r',\tau)) b^\dagger((r',0)) b((r,0)) \rangle \sim \exp (-T \tau / N)
\]

Where \( N \) is the number of dimer flavors [6][19] (\( N = 1 \) for our case). This means that for time scales bigger then the inverse temperature we have that the dimer expectation value \( \langle b^\dagger((r)b((r')) \rangle \) has no phase.

V. HALL COEFFICIENT AND MAGNETORESISTANCE (MAIN EQUATIONS)

In Appendix [C2] we obtained that the semiclassical equations of motion for the fermionic dimers at an arbitrary temperature in time independent electric and magnetic fields, these are given by:

\[
\dot{k}_c \equiv i \times e_B (T, B) B + e_E \vec{E}
\]

\[
\dot{\varepsilon}_c \equiv \frac{\partial \varepsilon_p}{\partial k_c}
\]

Where \( \varepsilon_p (k) = E_+ (k) \) was introduced below Eq. \[3\]. Furthermore \( e_B (T, B) \) depends on temperature and the magnetic field, being negative \(-e\) at low temperatures and positive \(+e\) at high temperatures with a crossover temperature given by \( \kappa k_B T \sim \hbar \omega_C = \frac{eB}{m} \sim 50K \) (the
cyclootron frequency of the bosonic dimers with mass \( m_B \sim (1 - 3) \cdot m_e \) at \( B \sim 50T \) and \( \kappa \sim 0.7O(1) \) fits the experimental data well, see Appendix [A]. We now compute \( e_B (T) \), the magnetic charge as a function of temperature and field. First we claim that at zero temperature \( e_B = - e \). Indeed when the dimer hops under the effect of the Hamiltonian in Eq. [1] there is charge \( + e \) moving with the dimer. To understand this note that before projection, an electron of charge \(- e\) must move in the opposite direction as the motion of a fermionic dimer for the dimers to hop. This leads to a contribution to the phase picked up by a fermionic dimer under hopping of \( + e \vec{A} \cdot \Delta \vec{r} / c \) (for a dimer hopping a distance \( \Delta r \) coming from the term \( t_{ij} \rightarrow t_{ij} \exp (i \vec{A} \cdot \vec{r} / c) \)). Furthermore the expectation value \( \langle b_{i,\eta}^\dagger b_{j,\nu} \rangle \) contributes at any temperature to the phase which a fermionic dimer picks up under hopping. At zero temperature it is given by Eq. [16] or \( \langle b_{i,\eta}^\dagger b_{j,\nu} \rangle \) contributes a phase of \(- 2 e \vec{A} \cdot \Delta \vec{r} / c \) to a fermionic dimer hopping a distance \( \Delta r \) (indeed \( T_{ij/2} \sim \langle b_{i,\eta}^\dagger b_{j,\nu} \rangle \) and \( \langle b_{i,z}^\dagger b_{j,\nu} \rangle \) respectively so pick up the phases of the bosonic expectation values). This leads to a total for the hopping of a fermionic dimer of \(- 2 e \vec{A} \cdot \Delta \vec{r} / c + e \vec{A} \cdot \Delta \vec{r} / c = - e \vec{A} \cdot \Delta \vec{r} / c \) so \( e_B = - e \). We note that this result, that the phase of \( \langle b_{i,\eta}^\dagger b_{j,\nu} \rangle \) has a phase of \(- 2 e \vec{A} \cdot \Delta \vec{r} / c \) is only true ignoring Elitzur’s theorem (which can be done in the ground state, zero temperature, where \( \langle b_{i,\eta}^\dagger b_{j,\nu} \rangle \) is power law correlated). In thermal states \( \langle b_{i,\eta}^\dagger b_{j,\nu} \rangle \) does not have a phase for processes longer then the the inverse temperature, see Eq. [17]. For such process \( \langle b_{i,\eta}^\dagger b_{j,\nu} \rangle \) effectively has no phase so the fermionic dimer transform only with phase \( + e \vec{A} \cdot \Delta \vec{r} / c \) leading to positive charge at large temperature, \( e_B = + e \). The crossover temperature is given by the cyclotron frequency of fermionic dimers \( \hbar k_B T_0 \sim \hbar c = \frac{\hbar c}{m_B} \sim 50K \) (the cyclotron frequency at \( B \sim 50T \) and \( \kappa \sim 0.7 \) fits the experimental data well, see Appendix [A] which is the time it takes for a dimer to go around a fermionic pocket and as such “feel” the magnetic field. By dimensional analysis \( e_B (B, T) = e_B (\frac{\hbar c}{\hbar c}) \). Furthermore the fermionic dimers have an effective charge for current of \( e_J = + 2 e - e = + e \), as the motion of a fermionic dimer is anti-correlated with the motion of a bosonic dimer of charge \(- 2 e\). We note that this procedure automatically counts the current of the bosonic dimers so we don’t need to add it to the current of the electronic dimers. We recall that \( e_E = + e \) indicating that under an electric field the dimer acts as a positively charged object (since when a dimer moves a bosonic dimer carrying charge \(- 2 e \) moves in the opposite direction). Applying the semiclassical equations of motion to the dimers and obtaining the Boltzmann equation analogously to ref. [15], the Hall response at low temperatures within the relaxation time approximation for the linearized Boltzmann equation approximation is given by [15]:

$$
\sigma_{xy} = 2 (e E e_J) \frac{1}{\tau} \int d^2 k \cdot m_{\nu \mu}^{-1} (k) (k_{\nu \mu}) \left( \frac{d f}{d \epsilon} \right) \times \left( \frac{\tau}{m_{\nu \mu}^{-1}} \right) \times \left( \frac{\epsilon e_B (B, T) \hbar c}{m_{\nu \mu}^{-1}} \right)
$$

$$
\sigma_{xy} = 2 (e E e_J) \frac{1}{\tau} \int d^2 k \cdot m_{\nu \mu}^{-1} (k) (k_{\nu \mu}) \left( \frac{d f}{d \epsilon} \right) \times \left( \frac{\tau}{m_{\nu \mu}^{-1}} \right) \times \left( \frac{\epsilon e_B (B, T) \hbar c}{m_{\nu \mu}^{-1}} \right)
$$

(20)

Where \( g^i (\epsilon) \) are the density of states for the four pockets and \( m_{\alpha \beta}^{-1} (k/\epsilon) \) are the local effective masses for the four pockets (and \( \epsilon \) is measured from the bottom of the band). In the last equation we divided the contribution to the Hall conductivity into four terms for each of the four fermion pockets see Fig. 1. For simplicity assuming a uniform effective mass for each pocket with two principle axis along the \( x' \) and \( y' \) axis for each of the four pockets (here by symmetry \( x' \) is along the diagonal of the Brillouin zone going through the pocket and \( y' \) is the axis perpendicular to it) we get that for each pocket [14]

$$
\sigma_{xy} (B) = \frac{p}{4} \left( \frac{e E e_J \hbar c}{m_{\nu \mu}^{-1}} \right) \left( \frac{1}{m_{\nu \mu}^{-1}} \right)
$$

(21)

Where \( \alpha, \beta = x', y' \). Now summing over the four pockets and switching to original co-ordinates we get that [15]:

$$
\sigma_{xy} (B) = \frac{p}{4} \left( \frac{e E e_J \hbar c}{m_{\nu \mu}^{-1}} \right) \left( \frac{1}{m_{\nu \mu}^{-1}} \right)
$$

(22)

This, Eq. (22) is the main result of this work. We don’t need to count a boson Hall or longitudinal magnetotransport coefficients since we already counted the motion of the bosons, in other words when \( p = 0 \) the model introduced in [13] predicts zero conductivity as the bosonic dimers cannot move but merely exchange positions. In particular the Hall coefficient is negative for zero temperature since \( e_B \rightarrow - e \) when \( T \rightarrow 0 \). In the high temperature limit we have that the dimers couple with charge \( e_B = + e \) to the magnetic field, leading to a positive Hall coefficient with the crossover temperature being given by
\[ k_B T \sim \hbar \omega_C \sim 50K \] see the discussion below Eq. (19).

We now note that the magnetoresistance is given by the matrix:

\[
\rho_{\alpha\beta} = \left( \frac{1 + \left( \frac{eB(T,B)B}{m_x m_y'} \right)^2}{\frac{ev_e e_j}{p e \epsilon_j} \left( \frac{eB(T,B)B}{m_x m_y'} \right)^2 + \frac{1}{2} \left( \frac{1}{m_x'} + \frac{1}{m_y'} \right)^2} \right) \times \left( \frac{1}{2} \left( \frac{1}{m_x'} + \frac{1}{m_y'} \right) - \frac{eB(T,B)B}{m_x m_y'} \right) \] \quad (23)

Where \( \alpha, \beta = x, y \). We get that the Kohler’s coefficient is given by:

\[
\frac{\rho_{xx}(B,T) - \rho_{xx}(0,T)}{\rho_{xx}(0,T)} \sim \left( \frac{eB(T,B)}{\sqrt{m_x m_y'}} \right)^2 (24)
\]

The dependence of \( e(B,T) \) on \( B \) is a slight deviation from Kohler’s rule but it is only important for intermediate temperatures \( \sim 50K \). We note that we can extract the coefficient \( \tilde{b}(T,B) = \left( \frac{eB(T,B)}{\sqrt{m_x m_y'}} \right)^2 \) of \( \sigma_{xx}^{-1} \sim \tau^{-1} \left( 1 + \tilde{b}(T,B) H^2 \tau^2 \right) \) from the Hall coefficient \( \sigma_{xy} \sim \frac{eB(T,B)}{m_x m_y'} \) and the effective mass for quantum oscillations of the cuprates \[9, 17] \( m^* \approx \sqrt{m_x m_y'} \) through the relation:

\[
\sigma_{xy}(T,B) = p \cdot \left( 1 + b(T,B) B^2 \tau^2 \right) \frac{eB(T,B)}{m^*} B \tau. \quad (26)
\]

Note that there is a slight deviation from Kohler’s law at low temperatures at \( \sim 50K \) as \( b(T,B) \) explicitly depends on the magnetic field through \( eB(T,B) \). Furthermore as a sharp qualitative test we note that the magnetoresistance effect vanishes when the Hall conductivity goes to zero as \( e_B(T) \rightarrow 0 \). This provides a clear test of our theory.

**VI. CONCLUSIONS**

We have shown that the Hall coefficient of the underdoped cuprates changes sign as a function of the temperature. We did so by showing that for static fields the coupling to the magnetic field changes sign as a function of temperature, while the coupling to the electric field and the current charge are given by \( e_E = e_j = +e \). The crossover temperature for the transition between positive and negative \( e_B \) is given by \( k_B T \sim \hbar \omega_C \sim 50K \) (the cyclotron frequency of the fermionic dimers at \( B \sim 50T \) and \( \kappa \approx 0.7 \) fits the experimental data well, see Appendix A). This result matches well with experimental data on the Hall coefficient of the underdoped cuprates \[16\]. This result confirms further that the model introduced in refs. \[13\] is a good effective model for the pseudogap and that the meanfield introduced in ref. \[9\] captures most of the qualitative features of the pseudogap. We also predict that the magnetoresistance effect vanishes when the Hall conductivity goes to zero, e.g. when \( e_B \rightarrow 0 \).

Furthermore we find a relation between the coefficient in Kohler’s like rule for magnetococonductivity and the Hall conductivity which can be used to further experimentally test the validity of the theory. We postulate that relation in Eq. (26) can be generalized to other materials with quasiparticle descriptions.

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**Appendix A: Comparing with experiments**

The main qualitative output of our work is that \( kT_0 = \hbar \omega_C = \frac{eB}{m_j} \) for \( \kappa = O(1) \), (here \( T_0 \) is the temperature the Hall coefficient vanishes). By comparing with the experimental data we get an excellent fit with \( \kappa \approx 0.7 \) see Fig. 2.

**Appendix B: Background on semiclassical equations of motion for electrons under general perturbations**

We would like to review the theory of semiclassical electron motion under general weak slowly time and position dependent perturbations. this would help us derive Eq. (19) in the main text, we will closely follow the presentation in refs. [4, 5]. We will assume that the Hamiltonian can be written as [14, 5]:

\[
H (r, p; \beta_1 (r, t), \ldots, \beta_\delta (r, t)) \quad (B1)
\]

Where \( \beta_i \) are some small perturbations. We will assume that the fermion is a wave packet centered around the momentum \( q_e \) and position \( r_c \). We will assume that the fermion is sufficiently localized that it is safe to Taylor expand the Hamiltonian [4, 5]:

\[
H = H_c + \Delta H
\]

\[
H_c = H (r, p; \{ \beta_i (r_c, t) \})
\]

\[
\Delta H = \frac{1}{2} \sum_i \nabla_{r_c} \beta_i (r_c, t) \cdot \left\{ (r - r_c), \frac{\partial H}{\partial \beta_i} \right\} \quad (B2)
\]
We see that the Hamiltonian $H_c$ has the same periodicity as $H$ as $H_c$ is simply shifted by a constant term with respect to $H$. Therefore it is possible to choose Bloch eigenvalues for the Hamiltonian:

$$H_c |\psi_q (r_c, t)\rangle = \varepsilon_c (r_c, q, t) |\psi_q (r_c, t)\rangle \quad (B3)$$

Now introducing the Fourier space version of $H_c$ we have that $H_c (q, r_c, t) = e^{-iq \cdot r} H_c (r_c, t) e^{iq \cdot r}$ and whose eigenstates are the periodic part of the Bloch functions $|u (q, r_c, t)\rangle = e^{-iq \cdot r} |\psi_q (r_c, t)\rangle$. We then get a Berry potential defined as [4, 5]:

$$\Lambda_{t, q, r} = \langle u | \frac{\partial}{\partial t, q, r} | u \rangle \quad (B4)$$

We have the effective Lagrangian [4, 5]:

$$L = -\varepsilon + q_c \cdot \dot{r}_c + \dot{q}_c \cdot \Lambda_q + \dot{r}_c \cdot \Lambda_r + \Lambda_t \quad (B5)$$

Here $\varepsilon = \varepsilon_c + \Delta \varepsilon$ where

$$\Delta \varepsilon = (\Delta H) = -Im \left( \frac{\partial u}{\partial \varepsilon_c} \right) \cdot \left( \varepsilon_c - H_c \right) \left( \frac{\partial u}{\partial q} \right) \quad (B6)$$

Here the dot product is taken by identifying $c \in R^2$ and $q_c \in R^2$. From Euler-Lagrange equations for the Lagrangian in Eq. (B5) we obtain that [4, 5]:

$$\dot{r}_c = \frac{\partial \varepsilon}{\partial q_c} - \left( \Omega_{q, r} \cdot \dot{r}_c + \Omega_{q, q} \cdot \dot{q}_c \right) - \Omega_{q, t} \quad (C1)$$

$$\dot{q}_c = -\frac{\partial \varepsilon}{\partial r_c} + \left( \Omega_{r, r} \cdot \dot{r}_c + \Omega_{r, q} \cdot \dot{q}_c \right) + \Omega_{r, t} \quad (C2)$$

Where for example

$$\Omega_{\alpha, \beta}_{q, r} = \partial_{\alpha q} \Lambda_{r \beta} - \partial_{\alpha r} \Lambda_{r \beta} \quad (C3)$$

**Appendix C: Equations of Motion**

1. **Simplifying the Berry curvatures**

We now specialize to the dimer model used in the main text we will assume that the magnetic and electric fields don’t depend on time, e.g. $\Omega_{q t} = \Omega_{r t} = 0$. We would like to simplify the Berry curvatures that enter Eq. (B7) above for the dimer system. The key formula we will use is that for a two level system, with a Hamiltonian of the form $n (\bar{x}) \cdot \sigma + \epsilon (x) \cdot Id$, for the lower band the Berry curvature is given by:

$$\Omega_{x, x_j} = -\frac{1}{2} \hat{n} \cdot (\partial_{x_i} \hat{n} \times \partial_{x_j} \hat{n}) \quad (C1)$$
where we have ignored a term $\vec{\Omega}_{qr}$ which is small for small $B$. We have dropped the difference between $q$ and $k_c$ in $\vec{\Omega}_{q,r}$ and then dropped the $r_c$ dependence which is zero by gauge invariance. Which further simplifies to a single equation:

$$\dot{k}_c = \left(1 + \vec{\Omega}_{k_c,r}(k_c)\right)^{-1}(\vec{r}_c \times e_B B + e_E E)$$

Furthermore for small $B$ we have that $\vec{\Omega}_{k_c,r}(k_c) \approx 0$ and

$$\dot{k}_c \approx \vec{r}_c \times e_B B + e_E E$$

$$\dot{r}_c \approx \frac{\partial \varepsilon_p}{\partial k_c}$$

Within the same approximation $\Delta \varepsilon \approx 0$ see Eq. \((C2)\). Where as discussed previously below Eq. \((C4)\) $e_B(T,B)$ depends on temperature and magnetic field. As such we obtain Eq. \((19)\) in the main text.

**Appendix D: Gauge Invariance of the gauge invariant green’s functions**

Under a gauge transformation the gauge invariant green’s functions do not transform, indeed under a gauge transformation in Eq \((9)\) we have that:

$$\left< b^{\dagger}(r) \exp \left( 2ie \int_{r'} A(r'') \, dr'' \right) b(r') \right> \rightarrow \left< b^{\dagger}(r) e^{-2i\xi\alpha(r)} \exp \left( 2ie \int_{r'} \{ A(r'') - \nabla \alpha(r'') \} \, dr'' \right) b(r') e^{2i\xi\alpha(r')} \right>$$

$$= \left< b^{\dagger}(r) \exp \left( 2ie \int_{r'} A(r'') \, dr'' \right) b(r') \right>$$

\((D1)\)

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