NONCOMMUTATIVE BESSEL SYMMETRIC FUNCTIONS

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Abstract. The consideration of tensor products of 0-Hecke algebra modules leads to natural analogs of the Bessel $J$-functions in the algebra of noncommutative symmetric functions. This provides a simple explanation of various combinatorial properties of Bessel functions.

1. Introduction

It is known that the theory of noncommutative symmetric functions and quasi-symmetric functions is related to 0-Hecke algebras in the same way as ordinary symmetric functions are related to symmetric groups. Thus, one may expect that natural questions about representations of 0-Hecke algebras lead to the introduction of interesting families of noncommutative symmetric functions. By “interesting”, one may mean “noncommutative analogs” of the Frobenius characteristics of representations of symmetric groups based on combinatorial objects, which may themselves give back various identities for the ordinary, exponential, or $q$-exponential generating functions of these objects. This amounts to specialize the complete noncommutative symmetric functions $S_n(A)$ to $h_n(X)$, $1$, $1/n!$ or $1/(q)_n$, respectively.

Examples of this situation can be found in [17], where the analysis of the representation of $H_n(0)$ on parking functions leads naturally to the combinatorics of the noncommutative Lagrange inversion formula, and to the introduction of noncommutative analogs of various special functions, such as the Abel polynomials, the Lambert binomial series or the Eisenstein exponential, and allows one to recover in a straightforward and unified way a number of enumerative formulas.

The present paper addresses the following question. The 0-Hecke algebra is the algebra of a monoid, hence admits a natural coproduct for which the monoid elements are grouplike. This allows one to define the tensor product of 0-Hecke modules, which induces on quasi-symmetric functions an analog of the internal product of symmetric functions. What are the properties of this operation, and of the dual coproduct on noncommutative symmetric functions?

It turns out that the second part of the question is the most interesting. Basically, the answer is: the dual coproduct governs the combinatorics of Bessel functions. Indeed, its explicitation leads to the introduction of noncommutative analogs $J_n(A,B)$ of the $J$-functions of integer index, of which a few basic properties are readily established. Then, the above mentioned specializations (and other more complicated ones) give back various classical enumerative formulas.
2. Background

2.1. Notations. Our notations for noncommutative symmetric functions are as in [12, 14]. The Hopf algebra of noncommutative symmetric functions is denoted by Sym, or by Sym\(_A\) if we consider the realization in terms of an auxiliary alphabet. Bases of Sym\(_n\) are labelled by compositions \(I\) of \(n\). The noncommutative complete and elementary functions are denoted by \(S_n\) and \(\Lambda_n\), and the notation \(S^I\) means \(S_{i_1} \cdots S_{i_r}\). The ribbon basis is denoted by \(R_I\). The notation \(I \vdash n\) means that \(I\) is a composition of \(n\). The conjugate composition is denoted by \(I^\sim\).

The graded dual of Sym is QSym (quasi-symmetric functions). The dual basis of \((S^I)\) is \((M_I)\) (monomial), and that of \((R_I)\) is \((F_I)\).

The Hecke algebra \(H_n(q)\) \((q \in \mathbb{C})\) is the \(\mathbb{C}\)-algebra generated by \(n-1\) elements \(T_1, \ldots, T_{n-1}\) satisfying the braid relations and \((T_i - 1)(T_i + q) = 0\). We are interested in the case \(q = 0\), whose representation theory can be described in terms of quasi-symmetric functions and noncommutative symmetric functions [15, 8].

The Hopf structures on Sym and QSym allows one to extend the \(\lambda\)-ring notation of ordinary symmetric functions (see [14], and [16] for background on the original commutative version). If \(A\) and \(X\) totally ordered sets of noncommuting and commuting variables respectively, the noncommutative symmetric functions of \(XA\) are defined by

\[
\sigma_I(XA) = \sum_{n \geq 0} t^n S_n(XA) = \prod_{x \in X} \left( \sum_{I \vdash n} t^{\vert I \vert} M_I(X) S^I(A) \right).
\]

Thanks to the commutative image homomorphism Sym \(\rightarrow\) Sym, noncommutative symmetric functions can be evaluated on any element \(x\) of a \(\lambda\)-ring, \(S_n(x)\) being \(S^n(x)\), the \(n\)-th symmetric power. Recall that \(x\) is said of rank one (resp. binomial) if \(\sigma_1(x) = (1 - tx)^{-1}\) (resp. \(\sigma_1(x) = (1 - t)^{-x}\)). The scalar \(x = 1\) is the only element having both properties. We usually consider that our auxiliary variable \(t\) is of rank one, so that \(\sigma(tA) = \sigma_1(tA)\).

The argument \(A\) of the noncommutative symmetric functions can be a “virtual alphabet”. This means that, being algebraically independent, the \(S_n\) can be specialized to any sequence \(\alpha_n \in \mathcal{A}\) of elements of any associative algebra \(\mathcal{A}\). Writing \(\alpha_n = S_n(A)\) defines all the symmetric functions of \(A\), and allows one to use the powerful notations \(F(nA)\), \(F((1 - q)A)\), etc., for more or less complicated transformations of the specialized functions.

The (commutative) specializations \(A = \mathbb{E}\), defined by

\[
S_n(\mathbb{E}) = \frac{1}{n!}
\]

and \(A = \frac{1}{1 - q}\), for which

\[
S_n \left( \frac{1}{1 - q} \right) = \frac{1}{(q)_n}
\]

are of special importance.
2.2. **Noncommutative analogs of special functions.** Since the discovery by D. André of a combinatorial interpretation of tangent and secant numbers, several classical generating functions have been lifted to the algebra of symmetric functions, and more recently, to noncommutative symmetric functions. The general idea is as follows. Given the exponential generating function

\[ f(t) = \sum_{n \geq 0} c_n \frac{t^n}{n!} \]

of a combinatorial sequence \( a_n \in \mathbb{N} \), one looks for a noncommutative symmetric function \( F(A) \) such that \( F(tE) = f(t) \). The noncommutative analog is interesting when \( F_n(A) \) can be directly interpreted as the formal sum of the combinatorial objects counted by \( c_n \), under the embedding of \( \text{Sym} \) into some larger algebra. For example, in the case of tangent and secant numbers, the series

\[ \left( \sum_{n \geq 0} (-1)^n S_{2n}(A) \right)^{-1} \left( 1 + \sum_{n \geq 0} (-1)^n S_{2n+1}(A) \right) \]

becomes the formal sum of the alternating permutations (shapes \((2^n)\) and \((2^n1)\)) under the embedding of \( \text{Sym} \) in \( \text{FQSym} \) [12]. One can also find in [12] the noncommutative Eulerian polynomials, and in [17], analogs of the Abel polynomials and of the Lambert and Eisenstein functions.

In general, \( F_n \) turns out to be the characteristic of some projective 0-Hecke module. Projective modules are always specializations of generic modules, thus also representations of the symmetric group, whose Frobenius characteristic are then the commutative images \( F_n(X) \). In general, setting \( X = \frac{t^r}{1-q} \) gives back an interesting \( q \)-analog of \( f(t) \).

In this note, we shall show that the consideration of 0-Hecke modules obtained from a natural notion of tensor products leads immediately to noncommutative analogs of the Bessel \( J \) (or \( I \)) functions. Here, we need two alphabets \( A \) and \( B \), and we are led to the combinaorics of bi-exponential generating functions.

### 3. Tensor products of 0-Hecke modules

3.1. The 0-Hecke algebra \( H_n(0) \) is the algebra \( \mathbb{C}[\Pi_n] \), where the monoid \( \Pi_n \) is generated by elements \( \pi_1, \ldots, \pi_{n-1} \) (\( \pi_i = 1 + T_i \)) satisfying the braid relations

\[ \pi_i \pi_j = \pi_j \pi_i \quad |i - j| > 1 \]

\[ \pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1} \]

and the idempotency condition

\[ \pi_i^2 = \pi_i \]

There is a canonical coproduct on \( H_n(0) \) defined by

\[ \delta \wedge \pi = \pi \otimes \pi \quad \text{for} \quad \pi \in \Pi_n \]
Hence, tensor products of $H_n(0)$-modules can be defined, and it is obvious from the definition of the simple module $S_I$ that
\[
S_H \otimes S_K = S_I \quad \text{where } \text{Des}(I) = \text{Des}(H) \cap \text{Des}(K).
\]
This induces an internal product $\wedge$ on $QSym_n = G_0(H_n(0))$, similar to the internal product of symmetric functions, such that
\[
F_H \wedge F_K = F_I \quad \text{where } I = H \wedge K, \text{ that is, } \text{Des}(I) = \text{Des}(H) \cap \text{Des}(K).
\]
By duality, this defines a coproduct on $\text{Sym}_n$, given by
\[
\gamma_\wedge R_I = \sum_{\text{Des}(I) = \text{Des}(H) \cap \text{Des}(K)} R_H \otimes R_K.
\]

3.2. There is a canonical involution $\iota$ on $H_n(0)$, defined by
\[
\iota(\pi) = \bar{\pi}_i = 1 - \pi_i,
\]
so that we can regard $H_n(0)$ as $\mathbb{C}[\bar{\Pi}_n]$ as well. Hence, we have another tensor product, defined from the coproduct
\[
\delta_\vee \bar{\pi} = \bar{\pi} \otimes \bar{\pi} \quad \text{for } \pi \in \Pi_n,
\]
which induces a second internal product $\vee$ on $QSym$,
\[
F_H \vee F_K = F_I \quad \text{where } I = H \vee K, \text{ that is, } \text{Des}(I) = \text{Des}(H) \cup \text{Des}(K).
\]
It is of course sufficient to study one of them. However, it is interesting to observe that this second product appears in another guise in [8], in the process of calculating a basis of primitive elements of $FQSym$. Let us recall this construction. Let $p_n$ denote the projection onto the homogeneous component $FQSym_n$ of $FQSym$, and let $\mu_q : F_\alpha \otimes F_\beta \mapsto F_{\alpha \mu q \beta}[k]$ be the multiplication map of $FQSym_q$. The $q$-convolution of two graded linear endomorphisms $f, g$ of $FQSym$ is defined by
\[
f \circ_q g = \mu_q \circ (f \otimes g) \circ \Delta.
\]
For $q = 1$, this reduces to ordinary convolution. We are interested in the case $q = 0$. For a composition $I = (i_1, \ldots, i_m)$, let
\[
p_I = p_{i_1} \circ_0 \cdots \circ_0 p_{i_m}.
\]
It is proved in [8] that the $p_I$ are mutually commuting projectors, and more precisely that
\[
p_I \circ p_J = \begin{cases} 
0 & \text{if } |I| \neq |J|. \\
p_{I \vee J} & \text{otherwise}.
\end{cases}
\]
Hence, $j : F_I \mapsto p_I$ defines an embedding of $(QSym, \vee)$ in the composition algebra of graded endomorphisms of $FQSym$. Moreover,
\[
\pi = \sum_{|I| \geq 1} (-1)^{|I|-1} p_I
\]
which is a projector onto the primitive Lie algebra of $\text{FQSym}$, is the image of the primitive element $\sum_n M_n$ of $\text{QSym}$ under $j$, and it easy to see that more generally, for any $f \in \text{QSym}$

\begin{equation}
(j \otimes j)(\Delta_{\text{QSym}} f) = \Delta_{\text{FQSym}} \circ j(f).
\end{equation}

However, $j$ does not map the usual (external) product of $\text{QSym}$ to the ordinary convolution of endomorphisms. It is nevertheless interesting to pull back the 0-convolution to $\text{QSym}$, by defining

\begin{equation}
F_I \odot_0 F_J = F_I \cdot J,
\end{equation}

where $I \cdot J$ means as usual concatenation of the compositions. Then, we have a splitting formula

\begin{equation}
(f_1 \odot_0 f_2 \odot_0 \cdots \odot_0 f_r) \vee g = \mu_0[(f_1 \otimes \cdots \otimes f_r) \vee \Delta_{\text{QSym}}(g)]
\end{equation}

analogous to the one satisfied in $\text{Sym}$.

It can be shown that the involution $\iota$ maps the simple module $S_I$ and the indecomposable projective module $P_I$ to $S_{\overline{I}}$ and $P_{\overline{I}}$, respectively.

3.3. Identifying as usual a tensor product $F \otimes G$ with $F(A)G(B)$, where $A$ and $B$ are two mutually commuting alphabets, we have

\begin{equation}
\sigma_1(XA) \land \sigma_1(XB) = \sum_K F_K(X)\gamma_\land(R_K) = \gamma_\land \sigma_1(XA),
\end{equation}

which may be compared with the following identity relating the internal product $*$ of $\text{Sym}$ and its dual coproduct $\delta F = F(XY)$ on $\text{QSym}$:

\begin{equation}
\sigma_1(XA) * \sigma_1(YA) = \sigma_1(XYA) = \delta \sigma_1(XA).
\end{equation}

**Theorem 3.1.** The coproduct $\gamma_\land$ is a morphism for the ordinary (outer) product of noncommutative symmetric functions, that is

\begin{equation}
\gamma_\land(FG) = \gamma_\land(F)\gamma_\land(G).
\end{equation}

In particular, it is completely determined by the images of the elementary functions, $\gamma_\land \Lambda_n = \Lambda_n \otimes \Lambda_n$, which implies the combinatorial inversion formula

\begin{equation}
\left(\sum_{n \geq 0} (-1)^n \Lambda_n \otimes \Lambda_n\right)^{-1} = \sum_{\text{Des}(H) \cap \text{Des}(K) = \emptyset} R_H \otimes R_K.
\end{equation}

**Proof** – This is equivalent to Theorem 4.1 below.

As we will see, this simple identity has many interesting enumerative corollaries. Applying the involution $\omega$ on the second factor gives the inverse of

\begin{equation}
\left(\sum_{n \geq 0} (-1)^n \Lambda_n \otimes S_n\right)^{-1} = \sum_{\text{Des}(H) \cap \text{Des}(K) = \emptyset} R_H \otimes R_{K^\sim}.
\end{equation}

The right hand side of this equality occurs in [13], where it is interpreted as the decomposition of the algebra $\text{H Sym}_n$ as a bimodule over itself. The inverse of the left hand side can legitimately be considered as a noncommutative analog of the
Bessel function $J_0$, as if we specialize both sides to $x\mathcal{E}$, we recover $J_0(2x)$. Moreover, specializing $A$ to $x/(1 - q)$ gives a classical $q$-analog of $J_0$, and the other ones are obtained by simple transformations. This first step being granted, it is not difficult to guess the correct definition of the noncommutative analogues of the other $J_\nu$. This will be done in the forthcoming section.

4. **Noncommutative Bessel functions**

4.1. Let $A$ and $B$ be two mutually commuting alphabets. The noncommutative Bessel functions $J_n(A, B)$ are defined by the generating series

$$
\sum_{n \in \mathbb{Z}} z^n J_n(A, B) = \lambda_{-1/z}(A)\sigma_z(B),
$$

that is,

$$
J_n(A, B) = \sum_{m \geq 0} (-1)^m \Lambda_{m-n}(A) S_m(B).
$$

For $A = B = x\mathcal{E}$, this is the usual Bessel function $J_n(2x)$. In particular,

$$
J_0(A, B) = \sum_{m \geq 0} (-1)^m \Lambda_m(A) S_m(B)
$$

can be regarded as $\lambda_{-1}(\mathcal{J})$, for the virtual alphabet $\mathcal{J} = (A, B)$ such that

$$
\Lambda_n(\mathcal{J}) = \Lambda_n(A) S_n(B).
$$

This defines an embedding of algebras

$$
j : \text{Sym} \to \text{Sym}(A, B) = \text{Sym} \otimes \text{Sym}
$$

$$
\Lambda_n(A) \mapsto \Lambda_n(\mathcal{J}) = \Lambda_n \otimes S_n.
$$

It is not difficult to describe the image of the ribbon basis under this embedding. We need the following piece of notation. For two compositions $I$ and $J$ of the same integer $n$, we define the composition $K = I \setminus J$ of $n$ by the condition

$$
\text{Des}(K) = \text{Des}(I) \setminus \text{Des}(J) \quad (\text{set difference}).
$$

Then, we can state:

**Theorem 4.1.** The image of $R_K$ by $j$ is

$$
R_K(\mathcal{J}) = \sum_{I \setminus J = K} R_I(A) R_J(B).
$$

**Proof** – The formula is true for $K = (1^n)$ by definition. The general case follows by induction on $l(K^\sim)$, the number of columns of the ribbon diagram of $K$. Indeed, it suffices to prove that

$$
R_K(\mathcal{J}) R_{1^m}(\mathcal{J}) = R_{K \triangleright 1^m}(\mathcal{J}) + R_{K \triangleright 1^m}(\mathcal{J}),
$$

which follows easily from the usual multiplication rule of ribbon functions. \qed
Corollary 4.2 ([3]). Let \( a_n \) be defined by

\[
\frac{1}{J_0(2\sqrt{t})} = \sum_{n \geq 0} a_n \frac{t^n}{(n!)^2}.
\]

Then, \( a_n \) is equal to the number of pairs of permutations \( (\sigma, \tau) \in S_n \times S_n \) such that \( \text{Des}(\sigma) \subseteq \text{Des}(\tau) \).

Let \( \partial \) be the linear operator on \( \text{Sym} \) (acting on the right) defined by

\[
S^{(i_1, \ldots, i_r)} \xleftarrow{\partial} = S^{(i_1, \ldots, i_r - 1)}.
\]

It has the following properties (see [14], Prop. 9.1):

\[
(FG) \xleftarrow{\partial} = F \cdot (G \xleftarrow{\partial}) + (F \xleftarrow{\partial}) \cdot G_0,
\]

where \( G_0 \) denotes the constant term of \( G \), and

\[
R_I \xleftarrow{\partial} = \begin{cases} R_{i_1, \ldots, i_r - 1} & \text{if } i_r > 1, \\ 0 & \text{if } i_r = 1. \end{cases}
\]

In particular, if \( G_0 = 0 \),

\[
(1 - G)^{-1} \xleftarrow{\partial} = (1 - G)^{-1}(G \xleftarrow{\partial}).
\]

Let us apply this with \( \xleftarrow{\partial} = \xleftarrow{\partial}_B \) acting only on \( \text{Sym}(B) \) to

\[
J_0(A, B)^{-1} = \left( 1 - \sum_{n \geq 1} (-1)^{n-1} \Lambda_n(B) S_n(A) \right)^{-1} = \sum_I S_I(A) R_I(B).
\]

We obtain

\[
J_0(A, B)^{-1} J_{-1}(A, B) = \sum_I S_I(A)(R_I \xleftarrow{\partial}(B))
\]

Corollary 4.3 ([3]). The coefficient \( c_n \) in

\[
\frac{J_1(2x)}{J_0(2x)} = \sum_{n \geq 1} c_n \frac{x^{2n-1}}{(n-1)!n!}
\]

is equal to the number of pairs of permutations \( (\alpha, \beta) \in S^2_n \) such that \( \text{Des}(\alpha) \subseteq \text{Des}(\beta) \) and \( \beta(n) = n \).

4.2. Bessel-Carlitz functors. Let \( F \) be the functor which associates with a pair of vector spaces \( (V, W) \) the graded subalgebra of the exterior algebra \( \Lambda(V \oplus W) \)

\[
F(V, W) = \bigoplus_{n \geq 0} \Lambda_n(V) \otimes \Lambda_n(W).
\]
This is a quadratic algebra (see [18]). If \((v_i), (w_j)\) are bases of \(V\) and of \(W\), the relations are as follows. For \(i < k\) and \(j < l\),

\[
\begin{align*}
\begin{bmatrix} i & k \\ j & l \end{bmatrix} + \begin{bmatrix} k & i \\ j & l \end{bmatrix} &= 0, \\
\begin{bmatrix} i & k \\ j & l \end{bmatrix} + \begin{bmatrix} i & k \\ l & j \end{bmatrix} &= 0, \\
\begin{bmatrix} i & i \\ j & l \end{bmatrix} &= 0, \\
\begin{bmatrix} i & k \\ j & j \end{bmatrix} &= 0,
\end{align*}
\]

(45) (46) (47) (48)

where \(\begin{bmatrix} i & k \\ j & l \end{bmatrix} = v_i \otimes w_k\).

Hence, the Koszul dual \(G(V, W) = F(V, W)^\dagger\) is the quadratic algebra on \(V^* \otimes W^*\) presented by

\[
\begin{bmatrix} i & k \\ j & l \end{bmatrix} = \begin{bmatrix} k & i \\ j & l \end{bmatrix} = \begin{bmatrix} i & k \\ l & j \end{bmatrix} \quad \text{for} \quad i < k \quad \text{and} \quad j < l.
\]

The combinatorial investigation of Bessel functions has been initiated by Carlitz [2]. Hence, the polynomial bi-functors defined by \(F\) and \(G\) can appropriately be called Bessel-Carlitz functors. One or two occurrences of \(\Lambda\) can be replaced by \(S\) in the definition of \(F\). In the mixed case \(\Lambda \otimes S\), the best interpretation is probably as functors defined on super (i.e., \(\mathbb{Z}_2\)-graded) vector spaces \(V = V_0 \oplus V_1\).

5. THE \(\theta\)-SPECIALIZATION

This section is devoted to the interpretation of a few formulas from [4, 10, 11] in terms of noncommutative symmetric functions.

5.1. Let \(\theta \subseteq A \times A\) be any binary relation. We denote by \(\overline{\theta}\) the complement of \(\theta\) in \(A \times A\) and set

\[
\begin{align*}
X &= X(A; \theta) = \{w = a_1 \cdots a_n \in A^* | a_1 \theta a_2 \theta \cdots \theta a_n\}, \\
Y &= Y(A; \theta) = X(A; \overline{\theta}),
\end{align*}
\]

(50)

where we write \(a \theta b\) for \((a, b) \in \theta\). Note that the empty word \(1\) and the letters belong to both \(X\) and \(Y\).

The \(\theta\)-specialization \(\text{Sym}(A; \theta)\) is then defined by specifying the elementary symmetric functions

\[
\Lambda_n(A; \theta) = \sum_{w \in X \cap A^n} w.
\]

(51)

The following basic lemma, implicit in [4], generalizes the case \(\theta = \{(a, b) | a > b\}\).

**Lemma 5.1** (Carlitz-Koszul duality for alphabets). The complete symmetric functions \(S_n(A; \theta)\) are given by

\[
S_n(A; \theta) = \Lambda_n(A; \overline{\theta}).
\]

(52)
More generally, if one denotes by $\theta \text{Adj}(w) = \{i | a_i \theta a_{i+1}\}$ the $\theta$-adjacency set of $w = a_1a_2 \cdots a_n$, and by $C_\theta(w)$ the associated composition of $n$, one has
\[
R_I(A; \theta) = \sum_{C_\theta(w) = I} w.
\]

Proof – We need to prove that
\[
\sum_{k=0}^{n} (-1)^k \Lambda_k(A, \theta) \Lambda_{n-k}(A, \bar{\theta}) = 0
\]
for $n > 0$. Let $w = uv$ be such that $u \in \Lambda_k(A, \theta)$ and $v \in \Lambda_{n-k}(A, \bar{\theta})$. Then if $\text{last}(u) \theta \text{first}(v)$, $w$ appears in $\Lambda_{k+1}(A, \theta) \Lambda_{n-k-1}(A, \bar{\theta})$, and similarly, if $\text{last}(u) \bar{\theta} \text{first}(v)$, then $w$ appears in $\Lambda_{k-1}(A, \theta) \Lambda_{n-k+1}(A, \bar{\theta})$. Moreover, $w$ cannot appear in any other product, so that its coefficient in the sum is 0. \qed

5.2. The $\theta$-Eulerian polynomials. Recall from [12] that the noncommutative Eulerian polynomials
\[
A_n(t; A) = \sum_{I \supseteq n} t^{l(I)} R_I(A)
\]
admit the generating function
\[
A(t, A) = \sum_{n \geq 0} A_n(t, A) = \frac{1 - t}{1 - t \sigma_{1-t}(A)}
\]
(see [7] for the commutative version of this identity), and since $l(C_\theta(w)) = \theta \text{adj}(w) + 1$, we have immediately
\[
\sum_{w \in A^*} t^{\theta \text{adj}(w) + 1} w = \frac{1 - t}{1 - t \sigma_{1-t}(A; \theta)}.
\]
Note that $\theta \text{adj}(w) + \bar{\theta} \text{adj}(w) = n - 1$. Replacing $\theta$ by $\bar{\theta}$, $A$ by $t^{-1} A$, then $t$ by $t^{-1}$, and simplifying by $(1 - t)$ the resulting expression, we obtain
\[
\sum_{w \in A^*} t^{\theta \text{adj}(w)} w = \frac{1}{1 - \sum_{w \in X(A; \theta)} (t - 1)^{l(w)-1} w},
\]
which is Theorem 2 of [11].

For a letter $c \in A$, denote by $\partial_c$ the linear operator defined by
\[
w \partial_c := \begin{cases} u & \text{if } w = uc \text{ for some } u, \\ 0 & \text{otherwise}. \end{cases}
\]
Then, as in [37], for any series $F$ without constant term,
\[
(1 - F)^{-1} \partial_c = (1 - F)^{-1} \cdot (F \partial_c).
\]
The same is true for the operators
\[
D_C = \sum_{c \in C} \partial_c \cdot c.
\]
where $C$ is a subset of $A$. Applying this to (58), we obtain

$$\sum_{w \in A^*C} t^{\theta\text{adj}(w)} w = -\frac{\sum_{w \in X_C} (t-1)^{l(w)} - 1}{1 - \sum_{w \in X \theta}(t-1)^{l(w)} - 1}$$

which is Theorem 3 of [11].

5.3. The $\theta$-Major index. If one defines the $\theta$-Major index by

$$\theta \text{maj}(w) = \sum_{i \in \theta \text{Adj}(w)} i$$

one has clearly

$$\sum_{w \in A^n} q^{\theta \text{maj}(w)} w = \sum_{I \vdash n} q^{\text{maj}(I)} R_I(A; \theta) = (q)_n S_n \left( \frac{A}{1 - q}; \theta \right),$$

where as usual

$$\sigma_z \left( \frac{A}{1 - q}; \theta \right) = \prod_{n \geq 0} \sigma_{zq^n}(A; \theta).$$

6. Double Eulerian polynomials and Bessel functions

6.1. The noncommutative Bessel function $J_0(A, B)$ can now be properly interpreted as a generating series of $\theta$ elementary symmetric functions, if we interpret $\mathcal{J}$ as the product alphabet $A \times B$, endowed with the relation

$$(a, b)(a', b') \Leftrightarrow a > a' \text{ and } b \leq b'.$$

As is customary, we denote words over $A \times B$ by biwords

$$w = [u, v] = \begin{bmatrix} u \\ v \end{bmatrix} \quad u \in A^n, \ v \in B^n.$$

Observing that

$$\theta \text{Adj} \left( \begin{bmatrix} u \\ v \end{bmatrix} \right) = \text{Des}(u) \cap \overline{\text{Des}(v)} = \text{Des}(u) \setminus \text{Des}(v),$$

we can now write

$$\sum_{w = (u, v) \in (A \times B)^*} t^{\theta\text{adj}(w)} z^{l(w)} w = \frac{1 - t}{J_0((1-t)z; A, B) - t}$$

$$= \sum_K z^{\left| K \right|} t^{l(K)} R_K(A, B; \theta)$$

where from now on we shall use the notation

$$J_0(x; A, B) = \lambda_{-x}(\mathcal{J}) = \lambda_{-x}(A, B; \theta).$$

The coefficient of $z^n$ is the $n$th double $\theta$-Eulerian polynomial, denoted by $A_n(t; A, B; \theta)$. Setting $A = B = \mathcal{E}$, we recover the enumeration of pairs of permutations $(\alpha, \beta) \in \mathcal{S}_n \times \mathcal{S}_n$ by the cardinality of $\text{Des}(\alpha) \cap \overline{\text{Des}(\beta)}$ (cf. [3]).
7. The Fédou-Rawlings polynomials

By considering simultaneously the specializations of (69) to all positive \(q\) and \(p\)-integers, \(A_i = [i + 1]_q\) and \(B_j = [j + 1]_p\), one arrives at the five parameter generalizations of the double Eulerian polynomials introduced by Fédou and Rawlings [11].

For \(w \in A^n\), where \(A\) is the infinite chain \(A = \{a_1 < a_2 < \ldots\}\), let \(q^w\) be the image of \(w\) by the multiplicative homomorphism \(a_i \mapsto q^{i-1}\). Writing, for a composition \(I\) of \(n\)

\[
R_I(A) = \sum_{C(\sigma) = I} \sum_{\text{Std}(w) = \sigma} w
\]

and taking into account the identity

\[
\sum_{\text{Std}(w) = \sigma} x^{\text{max}(w)} q^w = \frac{x^{\text{des}(\sigma^{-1})} q^{\text{coimaj}(\sigma)}}{(xq; q)_n}
\]

where \(\text{coimaj}(\sigma)\) denotes the co-major index of \(\sigma^{-1}\),

\[
\text{coimaj}(\sigma) = \sum_{d \in \text{Des}(\sigma^{-1})} (n - d),
\]

(Indeed, it is easily checked that the minimal word \(v\) for the lexicographic order such that \(\text{Std}(v) = \sigma\) satisfies \(q^v = q^{\text{coimaj}(\sigma)}\)), we find

\[
\sum_{i \geq 0} x^i R_I(1, q, \ldots, q^i) = \frac{1}{1 - x} \sum_{C(w) = I} x^{\text{max}(w)} q^w = \frac{1}{(xq; q)_{n+1}} \sum_{C(\sigma) = I} x^{\text{des}(\sigma^{-1})} q^{\text{coimaj}(\sigma)}
\]

so that finally, we recover the double generating series of [11]

\[
\sum_{i,j \geq 0} x^i y^j \frac{1 - t}{J_0((1 - t)z; A_i, B_j)} - t
\]

\[= \sum_{n \geq 0} \left(\frac{z^n}{(xq; q)_{n+1}} \frac{y^n}{(yp; p)_{n+1}} \right) \sum_{\alpha, \beta \in S_n} t^{\text{desis}(\alpha, \beta)} x^{\text{des}(\alpha^{-1})} y^{\text{des}(\beta^{-1})} q^{\text{coimaj}(\alpha)} p^{\text{coimaj}(\beta)} ,
\]

where \(\text{desis}(\alpha, \beta) = |\text{Des}(\alpha) \setminus \text{Des}(\beta)|\).

The second generating series of [11] is recovered in the same way. If we denote by \(b_j\) the greatest letter of \(B_j\), then, on the one hand,

\[
S_n(B_j) \partial_{b_j} = S_{n-1}(B_j).
\]
On the other hand,

\[ \sum_{j \geq 0} y^j R_J(B_j) \leftarrow \partial_{b_j} \cdot b_j = \frac{1}{1 - y} \sum_{C(\sigma) = J, \max(\nu) = \text{last}(\nu)} y^{\max(\nu) \nu} \]

\[ = \frac{1}{1 - y} \sum_{C(\sigma) = J, \text{Std}(\nu) = \sigma} \sum_{\sigma(n) = n} y^{\max(\nu) \nu}, \]

so that, applying the operator \( \leftarrow \partial_{b_j} \cdot b_j \) to the coefficient of \( x^i y^j \) in (75), we obtain

\[ \sum_{K \setminus J = K} t^{l(K) - 1} \sum_{i,j \geq 0} x^i y^j R_1(A_i) R_J(B_j) \cdot \leftarrow \partial_{b_j} \cdot b_j \]

\[ = \sum_{i,j \geq 0} x^i y^j \left( 1 - \sum_{n \geq 1} z^n (t - 1)^{n-1} A_n(A_i) S_n(B_j) \right)^{-1} \leftarrow \partial_{b_j} \cdot b_j \]

\[ = \sum_{i,j \geq 0} x^i y^j \left( -\sum_{n \geq 1} z^n (t - 1)^{n-1} A_n(A_i) S_{n-1}(B_j) b_j \right) (1 - t) \]

\[ \leftarrow \partial_{b_j} \cdot b_j \]

\[ \sum_{i,j \geq 0} x^i y^j J_z((1 - t)z; A_i, B_j) - t. \]

Specializing \( A_i = [i + 1]_q, B_j = [j + 1]_p \), this becomes, in the notation of [11],

\[ \sum_{i,j \geq 0} x^i y^j \frac{J_z^1((1 - t)z; q, p)}{J^1_0((1 - t)z; q, p)} - t \]

\[ = \sum_{n \geq 0} \sum_{\alpha, \beta \in S_n} \frac{z^n}{(x; q)_n(y; p)_n} \sum_{\alpha, \beta \in S_n} t^{\text{des}(\alpha, \beta)} x^{\text{des}(\alpha^{-1})} y^{\text{des}(\beta^{-1})} q^{\text{coimaj}(\alpha)} p^{\text{coimaj}(\beta)} \]

which is equivalent to [11] (3)]. Here,

\[ J_z^{(i,j)}(z; q, p) := (-1)^i J_z^i(z[i + 1]_q, [j + 1]_p). \]

The other results of [11] can be rederived in the same way, by changing the specializations of \( A_i \) and \( B_j \).

8. Heaps of segments and polyominos

Bessel functions and their multiparameter analogs play a crucial role in the enumerative theory of polyominos [11, 6]. Elegant combinatorial proofs of such enumerative results can be achieved by means of Viennot’s theory of heaps of segments [19, 1]. As we shall see, this can also be conveniently formulated in terms of \( \theta \)-noncommutative symmetric functions.
8.1. A parallelogram (or staircase) polyomino $P$, which is also the same as a connected skew Young diagram, can be encoded as a biword

\[(81)\]
\[w = a_{i_1 j_1} \cdots a_{i_n j_n} = \begin{bmatrix} i_1 \cdots i_n \\ j_1 \cdots j_n \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}\]

where $j_k$ is the height of the $k$th column $C_k$, and $i_k$ is the number of common rows between $C_k$ and $C_{k+1}$ (with a conventional value $i_n = 1$ for the last column). For example, the following polyomino

\[
\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

is encoded by the biword

\[(82)\]
\[\begin{bmatrix} 2122111 \\ 2323212 \end{bmatrix}\]

The biwords corresponding to polyominos are the words over the alphabet

\[(83)\]
\[A = \{a_{ij} | i \leq j\}\]

satisfying the $\theta$-adjacency conditions

\[(84)\]
\[a_{i_k j_k} \theta a_{i_{k+1} j_{k+1}} \iff i_k \leq j_{k+1}\]

and the ending condition

\[(85)\]
\[i_n = 1.\]

Hence, the generating series (by length) of all biwords satisfying (84) is

\[(86)\]
\[\lambda_t(A, \theta) = [\lambda_{-t}(A, \bar{\theta})]^{-1} = \left(1 - \sum_{n \geq 1} (-1)^{n-1} t^n \sum_{i_k > j_{k+1}} \begin{bmatrix} i_1 \cdots i_n \\ j_1 \cdots j_n \end{bmatrix}\right)^{-1}\]

and restriction of the series to the biwords satisfying (85) is achieved as above by applying the operator

\[(87)\]
\[D = \sum_{j \geq 1} \delta_{a_{1j}}\]

so that we end up once more with a series of the form

\[(88)\]
\[\left(1 - \sum_{n \geq 1} (-1)^{n-1} t^n \sum_{i_k > j_{k+1}} \begin{bmatrix} i_1 \cdots i_n \\ j_1 \cdots j_n \end{bmatrix}\right)^{-1} \left(- \sum_{n \geq 1} (-1)^{n-1} t^n \sum_{i_k > j_{k+1}; i_n = 1} \begin{bmatrix} i_1 \cdots i_n \\ j_1 \cdots j_n \end{bmatrix}\right)\]
which acquires the structure \( J_1/J_0 \) once \( A \) is specialized to
\[
a_{ij} = xy^{j-i}q^i,
\]
the generating series by width, height and area.

8.2. This can of course be interpreted in terms of heaps of segments. A segment is an interval \([i, j]\) of \( \mathbb{N}^* \). To each segment, we associate a variable
\[
a_{ij} = [i \ j],
\]
in our \( A = \{a_{ij} | i \leq j\} \). The monoid of heaps is the quotient of the free monoid \( A^* \) by the commutation relations
\[
a_{ij}a_{kl} \equiv a_{kl}a_{ij} \text{ if } j < k
\]
which means that the segments do not overlap and can be vertically slid independently of each other.

The first basic lemma of the theory (which is also a special case of the Cartier-Foata formula for the Moebius functions of free partially commutative monoids \[5\]) amounts to the calculation of \( S_n(A, \theta) \) for the relation defined by
\[
a_{ij} \theta a_{kl} \iff i \leq l.
\]
Indeed, with this choice, \( \Lambda_n(A, \bar{\theta}) \) is the formal sum of trivial heaps (products of mutually commuting segments arranged in decreasing order), and \( \Lambda_n(A, \theta) \) is the sum of all biwords
\[
w = a_{i_1 j_1} \cdots a_{i_n j_n} = [i_1 \cdots i_n \ j_1 \cdots j_n]
\]
such that \( i_k \leq j_{k+1} \) for all \( k \), those encoding polyominos. We have therefore shown that each heap, or, equivalently, each element of \( A^*/\equiv \) has a unique representative of this form.

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