Spreading of ultrarelativistically expanding shell: an application to GRBs

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Abstract
Optically thick energy dominated plasma created in the source of Gamma-Ray Bursts (GRBs) expands radially with acceleration and forms a shell with constant width measured in the laboratory frame. When strong Lorentz factor gradients are present within the shell it is supposed to spread at sufficiently large radii. There are two possible mechanisms of spreading: hydrodynamical and thermal ones. We consider both mechanisms evaluating the amount of spreading that occurs during expansion up to the moment when the expanding shell becomes transparent for photons. We compute the hydrodynamical spreading of an ultrarelativistically expanding shell. In the case of thermal spreading we compute the velocity spread as a function of two parameters: comoving temperature and bulk Lorentz factor of relativistic Maxwellian distribution. Based on this result we determine the value of thermal spreading of relativistically expanding shell. We found that thermal spreading is negligible for typical GRB parameters. Instead hydrodynamical spreading appears to be significant, with the shell width reaching \( \sim 10^{10} \) cm for total energy \( E = 10^{54} \) erg and baryonic loading \( B = 10^{-2} \). Within the fireshell model such spreading will result in the duration of Proper Gamma-Ray Bursts up to several seconds.

Keywords: gamma rays: bursts; hydrodynamics; radiative transfer; relativity

1. Introduction
Optically thick pair plasma with baryon loading is assumed to power GRBs in many models considered in the literature, see, e.g., [Piran (1999), Meszaros (2000), Ruffini et al. (2009)]. Such plasma is self accelerated to large bulk Lorentz factors due to initial energy dominance. Due to relativistic kinematics in laboratory reference frame it forms a shell with width approximately equal to

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its initial size. When the plasma becomes transparent to Compton scattering a flash of radiation is emitted. In the literature such emission is often associated with so-called precursors (Lazzati, 2005). In the fireshell model (Ruffini et al., 2009) this emission is called the Proper GRB (P-GRB).

For typical baryonic loading parameters of observed GRBs $10^{-3} < B < 10^{-2}$, where $B = M c^2 / E_0$, $M c^2$ is total baryonic rest mass energy, $E_0$ is total energy released in the source of GRB, the fraction of energy emitted at transparency can reach several percents of the total energy. Considering that typical duration of long GRBs is of the order of hundreds of seconds, and P-GRB typically lasts for less than few seconds (Ruffini et al., 2009) luminosities of both events are comparable in magnitude.

Early treatments assuming the absence of baryons and thin shell approximation (Bianco et al., 2001), and non instantaneous energy release (Ruffini et al., 2005) produced estimations of P-GRBs duration $\sim 10^{-2} - 10^{-1}$ sec for the progenitor mass range $10 - 10^3 M_\odot$, much larger than the naive estimation of the gravitational collapse time $\sim GM/c^3 \simeq 5 \cdot 10^{-6} M/M_\odot$ sec, where $G$ is Newton’s constant, $M$ is black hole mass, $c$ is the speed of light. Observed durations of P-GRBs (Ruffini et al., 2009) of most energetic GRBs is of the order of several seconds. Main purpose of this paper is to outline a possibility of resolution of this tension considering different mechanisms of spreading of ultrarelativistically expanding shell.

The paper is organized as follows. In Sec. 2 the hydrodynamic phase of GRB is discussed, hydrodynamical spreading mechanism is recalled, and results of its application to GRBs are shown. In Sec. 3 the concept of thermal spreading is introduced, evaluation of spreading for relativistic Maxwellian distribution function is performed, and the results of thermal spreading computation are applied to GRBs. Discussion of the spreading effects on the duration of P-GRB and conclusions follow in the last section.

2. Hydrodynamic phase and optical depth of GRBs

Energy, entropy and baryonic number conservation laws for each differential subshell of ultrarelativistically expanding shell imply (Ruffini and Vereshchagin, 2013)

$$\left( p + \rho \right) \Gamma^2 r^2 dr = \text{const}, \quad (1)$$
$$\sigma \Gamma r^2 dr = \text{const}, \quad (2)$$
$$n \Gamma r^2 dr = \text{const}, \quad (3)$$

where $\rho$ and $p$ are respectively comoving energy density and pressure, $\sigma$ and $n$ are respectively comoving entropy and comoving baryon number density, $\Gamma$ is Lorentz factor, and $r$ is laboratory radial coordinate. Assuming polytropic equation of state

$$p = (\gamma - 1) \rho, \quad \sigma = \rho^\gamma, \quad (4)$$

$$\sigma$$
where γ is the thermal index, equations (1) can be solved for each differential subshell in ultrarelativistic and nonrelativistic asymptotic cases, respectively. Recall that expanding plasma consists of two components: nonrelativistic baryons and ultrarelativistic electron-positron pairs and photons. Hence, in analogy with cosmology, see e.g. Ruffini and Vereshchagin (2013), these two limiting cases are characterized by two adiabatic indices, respectively: γ = 4/3 when comoving temperature is relativistic (energy dominated phase), and γ ≃ 1 when comoving temperature is nonrelativistic (matter dominated phase). In the latter case the pressure is usually neglected, see e.g. Piran et al. (1993). In what follows we assume \( p = 0 \) at the matter dominated phase. Therefore one has

\[
\Gamma \propto r, \quad \rho \propto r^{-4}, \quad n \propto r^{-3}
\]  

(5)

during the energy dominated phase and

\[
\Gamma \simeq B^{-1} = \text{const}, \quad \rho \propto r^{-2}, \quad n \propto r^{-2}
\]  

(6)

during the matter dominated phase. These scaling laws are different from the ones in cosmology, see Ruffini and Vereshchagin (2013) for details and comparison. The transition from energy to matter domination occurs at \( R_{eq} \simeq R_0/B \), where \( R_0 \) is initial size of the energy dominated region.

In Ruffini et al. (2013) we computed the optical depth and dynamics of P-GRB for the case when Γ is constant across the shell. General expression for the optical depth of the shell is

\[
\tau = \int_R^{R^*} \sigma_T n \Gamma (1 - \beta) dr
\]  

(7)

where \( \sigma_T \) is Thomson’s cross section and \( R^* \) is the radius at which light ray emitted from the inner boundary of the shell at radius \( R \) crosses its outer boundary. For large baryonic loading with

\[
B \gg 1 \times 10^{-3} E_54^{-1/5} R_8^{2/5},
\]  

(8)

where \( R_0 = R_8 10^8 \text{ cm}, E_0 = E_{54} 10^{54} \text{ erg} \), optical depth of the shell is

\[
\tau = \tau_0 \left( \frac{R_0}{R} \right)^2,
\]  

(9)

where \( \tau_0 = \sigma_T n_0 R_0 \), \( n_0 \) is given by equation (13) with \( r_0 = R_0 \). Equation (9) correspond to the photon thin asymptotics (Ruffini et al., 2013) of equation (7) defined by the condition \( R^* - R = 2 \Gamma^2 R_0 \ll R \).

The observed duration of P-GRB is determined in this asymptotics by the process of radiative diffusion (Ruffini et al., 2013)

\[
\Delta t_a = \frac{t_D}{2 \Gamma^2} \simeq 0.12 E_54^{1/3} B_{-2}^{5/3} R_8^{1/3} \text{ s},
\]  

(10)

where \( B = B_{-2} 10^{-2} \) and time of diffusion \( t_D \) was found by solution of radiative transfer equation.
All these results are derived assuming constant Lorentz factor across the shell. Hence the width of the shell remains constant during its expansion. This fact has been used within the fireshell model and termed as the “constant thickness approximation”, provided that the baryon loading is not too heavy \( B \leq 10^{-2} \) (Ruffini et al., 2000). Hydrodynamic analytical (Shemi and Piran, 1990, Bisnovatyi-Kogan and Murzina, 1995) and numerical (Piran et al, 1993, Mészáros et al, 1993) calculations show that Lorentz factor gradient is developing during shell expansion that leads to spreading of the shell at sufficiently large radii at the matter dominated phase.

At this phase of expansion each differential subshell is expanding with almost constant speed \( v = \beta c \approx c(1 - 1/2\Gamma^2) \), so the spreading of the shell is determined by the radial dependence of the Lorentz factor \( \Gamma(r) \). In a variable shell there can be regions with \( \Gamma(r) \) decreasing with radius and \( \Gamma(r) \) increasing with radius. At sufficiently large radii only the regions with increasing \( \Gamma \) contribute to the spreading of the shell. From equations of motion of external and internal boundaries of this region we obtain (Piran et al, 1993) the thickness of the region as function of radial position of the region

\[
l(R) = R_0 + \frac{R}{2} \left( \frac{1}{\Gamma_e^2} - \frac{1}{\Gamma_i^2} \right),
\]

where \( \Gamma_e \) and \( \Gamma_i \) are Lorentz factors at external and internal boundaries, \( R_0 \) is the width of the region at small \( R \), \( R \) being the radial position of inner boundary.

Let us consider such a region in two limiting cases:

(a) when relative Lorentz factor difference is strong,
\[
\Delta \Gamma = \Gamma_e - \Gamma_i \gg \Gamma_i;
\]
(b) when relative Lorentz factor difference is weak,
\[
\Delta \Gamma = \Gamma_e - \Gamma_i \ll \Gamma_i.
\]

In the case (a) the second term in parenthesis in equation (11) can be neglected, and we obtain that the spreading becomes efficient at \( R > R_b = 2\Gamma_i^2 R_0 \), see Mészáros et al (1993) and Piran et al (1993). In the case (b) we find the corresponding critical radius of hydrodynamical spreading \( R_b = (\Gamma_i/\Delta \Gamma)\Gamma_i^2 l \gg \Gamma_i^2 R_0 \). From equation (11) one can see that in both cases for \( R \gg R_b \) width of the shell is increasing linearly with radius \( l(R) \approx (\Delta \Gamma/\Gamma_i)R/\Gamma_i^2 \).

The discussion above corresponds to the case (b) since for weak Lorentz factor difference \( R_b \gg R_{tr} \). In what follows we focus on the case (a) and derive corresponding relations assuming strong relative Lorentz factor difference across the shell. Let us take an element of fluid with constant number of particles \( dN \) in the part of the shell with gradient of \( \Gamma \). Internal boundary of the element is moving with velocity \( v \), and external one is moving with velocity \( v + dv = v + \frac{dv}{dt} dr \), where \( dr \) is the differential thickness at some fixed laboratory time \( t = 0 \) and derivative \( dv/dr \) is taken at the same laboratory time. Then at time \( t \) the width of the element is \( dl = dr + tdv \), its radial position is \( R(t) = r_0 + vt \), where \( r_0 \) is initial radial position of the element, and corresponding laboratory
density is
\[
n_i = \frac{dN}{dV} = \frac{dN}{4\pi R^2 \left(1 + \frac{t}{\Gamma^2} \right) dr} = n_0 \frac{r_0^2}{R^2 \left(1 + \frac{t}{\Gamma^2} \right)},
\]
(12)

where
\[
n_0 = \frac{dN}{dV_0} = \frac{dN}{4\pi r_0^2 dr}.
\]
(13)

At large enough \(t\) using \(R \simeq c t\) we have in contrast with (6)
\[
\Gamma \simeq \text{const}, \quad \rho \propto r^{-4}, \quad n \propto r^{-3},
\]
(14)

\[
R \gg R_b = \frac{1}{\Gamma^3 \left(\frac{d\Gamma}{dr}\right)^{-1}}.
\]

In order to compute the integral (7) we need to find the expression for baryonic number density along the light ray. Taking into account hydrodynamical spreading (11) we obtain
\[
n = n_0 \frac{R_0}{\Gamma} \left(\frac{R}{r}\right)^2 \frac{1}{1 + \frac{r}{\Gamma \frac{d\Gamma}{dr}}},
\]
(15)

that is exact in ultrarelativistic limit. Notice the difference between equations (15) and (12): in the former case \(d\Gamma/\,dr\) is computed along the light ray, while in the latter case \(dv/\,dr\) is computed along the radial coordinate at fixed laboratory time. The expression (15) reduces to equation (6) when \(d\Gamma/\,dr = 0\). Instead when the second term in the denominator of the expression (15) dominates, namely when \(r \gg \Gamma(d\Gamma/\,dr)^{-1}\), density radial dependence coincides with the one given by relations (14).

An estimate for \(d\Gamma/\,dr\) can be given for strong relative Lorentz factor difference \(\Delta\Gamma \sim \Gamma\) in the shell
\[
\frac{d\Gamma}{dr} \sim \frac{\Delta\Gamma}{\Delta r} \sim \frac{\Gamma}{2\Gamma^2 R_0} = \frac{1}{2\Gamma R_0},
\]
(16)

where \(\Delta r \sim 2\Gamma^2 R_0\) is the distance inside the shell along the light ray. Numerical results from Piran et al. (1993), Mészáros et al. (1993) and analytical ones from Bisnovatyi-Kogan and Murzina (1995) support this estimate.

Integrating expression (16) we obtain Lorentz factor dependence on radial coordinate along the light ray
\[
\Gamma(r) \sim \sqrt{\frac{r - R}{R_0}}.
\]
(17)

Since we are interested in the asymptotics when the hydrodynamical spreading is essential, we can assume in the integral (7) \(r \gg R\) and \(R^* \gg R\). Under these conditions the optical depth is
\[
\tau = \tau_0 \left(\frac{R_0}{R}\right)^2.
\]
(18)
This result coincides with equation (9) up to a numerical factor. However its physical meaning is different. It represents photon thick asymptotics of equation (7), since $R^* \gg R$ (Ruffini et al., 2013).

Transparency radius is defined by equating (18) to unity and is given by

$$R_{tr} \simeq \left( \frac{\sigma_T B E_0}{32 \pi m_p c^2} \right)^{1/2} = 2 \times 10^{14} B_{-2}^{1/2} E_{54}^{1/2} \text{ cm},$$

(19)

where $m_p$ is proton mass. At the radius of transparency the width of the shell (11) spreads up to

$$\Delta l_{hydr} \simeq 10^{10} B_{-2}^{1/2} E_{54}^{1/2} \Gamma_2^{-2} \text{ cm},$$

(20)

where $\Gamma_i = 100 \Gamma_2$.

In this case radiative diffusion is irrelevant and duration of P-GRB is then given by the time of arrival of photons emitted from the shell all the way up to $R_{tr}$

$$\Delta t_a \simeq \frac{R_{tr}}{2 \Gamma_i c} = 0.3 B_{-2}^{1/2} E_{54}^{1/2} \Gamma_2^{-2} \text{ s}.$$  

(21)

In contrast to equation (10) in the case of strong Lorentz factor difference, namely with $\Gamma_i < (2B)^{-1}$, this duration can be of the order of several seconds with agreement with observations.

3. Thermal spreading

We now determine the velocity spread of particles as a function of comoving temperature $T$ and bulk Lorentz factor $\Gamma$ for relativistic Maxwellian distribution. Based on this result we compute the value of thermal spreading of expanding shell.

The distribution of particles in the momentum space in the laboratory frame is a Lorentz-boosted Maxwellian

$$f(p_x, p_y, p_z) = A \exp \left( -\frac{1}{mc^2} \left[ m^2 c^2 + p_y^2 + p_z^2 \right. \right. $$

$$\left. \left. + \left( \Gamma p_x - \sqrt{\Gamma^2 - 1} (m^2 c^2 + p_y^2 + p_z^2) \right)^{1/2} \right] \right),$$

(22)

where we assumed that the relative motion of the frames is along their $x$-axes and $\theta = kT/mc^2$ is dimensionless comoving temperature.

Velocity dispersion in the $x$-direction is

$$D(v_x) = M(v_x^2) - M^2(v_x),$$

(23)

where $M(\chi)$ denotes the average value of $\chi$, which is defined by the convolution with the distribution function (22)

$$M(\chi) = \frac{\int d^3 p \chi(p) f(p)}{\int d^3 p f(p)}.$$

(24)
The above written integrals cannot be computed analytically, but their numerical approximations can be found.

Numerical issues in the velocity dispersion calculations by (23) arise from the fact that for high $\Gamma$ we need to subtract two numbers $M(v^2)$ and $M^2(v)$ which are very close to each other and to $c^2$. This leads to substantial reduction of accuracy. A different formula for dispersion

$$D(v_x) = M([v_x - M(v_x)]^2)$$

(25)

proves to be more convenient. The spread of particle velocities is then $(\Delta v)_{therm} = \sqrt{D(v)}$.

For nonrelativistic comoving temperatures the correct asymptotics is

$$\left(\frac{\Delta v}{c}\right)_{\theta \ll 1} = \Gamma^{-2}\theta^{1/2}.$$  

(26)

The case of ultrarelativistic comoving temperature ($\theta \gg 1$) is more interesting. Starting close to the maximal value $1/\sqrt{2}$, the velocity spread for $10 \lesssim \Gamma \lesssim \theta$ reaches approximately

$$\left(\frac{\Delta v}{c}\right)_{10 \lesssim \Gamma \lesssim \theta} \simeq \Gamma^{-3/2},$$

(27)

which means that the dispersion is independent on the temperature. For $\Gamma \gg \theta$ the asymptotics (26) is restored just up to a multiplier close to unity

$$\left(\frac{\Delta v}{c}\right)_{1 \lesssim \theta \ll \Gamma} \simeq 1.16 \Gamma^{-2}\theta^{1/2}.$$  

(28)

Our results suggest that (27) gives absolute upper limit for the velocity spread, and temperature dependence of (26) and (28) reduce the spread even further.

Initial temperature of the plasma formed in the source of GRB can be estimated from its initial size $R_0 \sim 10^8$ cm and total energy released

$$10^{48} \text{ erg} < E_0 < 10^{55} \text{ erg}.$$  

Assuming that the temperature is determined by $e^+e^-$ pairs only

$$\frac{E_0}{V_0} = \frac{3E_0}{4\pi R_0^3} = aT_0^4$$

(29)

we get for initial temperature

$$0.40 < \frac{kT_0}{m_e c^2} < 22.$$  

(30)

At radiation dominated phase the comoving temperature of the plasma decreases as $T \simeq T_0 R_0/r$, at matter dominated phase as $T \simeq T_0 B^{1/3}(R_0/v)^{2/3}$. Now we compute the thermal spreading at both phases.
For the first phase the reasonable approximation of Lorentz factor is

$$\Gamma(t) \simeq \sqrt{1 + \left(\frac{ct}{R_0}\right)^2}.$$  

Due to the nature of Lorentz transformations in constantly accelerated frame the final spreading of the shell \(\Delta l_1 = \int_0^t \Delta v \, dt\) appears to be finite even if we extend this phase infinitely in time, and the main part of the spreading is connected with initial part of motion with relatively small \(\Gamma\).

While the temperature of protons in the source of GRB is nonrelativistic \(\theta \ll 1\), velocity spread is given by \((26)\) which in energy dominated phase leads to the spreading

$$\frac{\Delta l_1}{R_0} \lesssim 2.2 \sqrt{\frac{kT_0}{m_p c^2}} = 0.18 E_{54}^{1/8} R_8^{-3/8}. \quad (31)$$

In the matter dominated phase the additional spreading of the shell is

$$\frac{\Delta l_2}{R_0} \simeq 3 B_{7/3} \sqrt{\frac{kT_0}{m_p c^2}} \left(\frac{R_{tr}}{R_0}\right)^{1/3} = 6.7 \cdot 10^{-4} E_{54}^{7/24} R_8^{-17/24} B_{5/2}^{5/2}, \quad (32)$$

when \(R_{tr} \gg R_0\). Comparing to the hydrodynamical spreading for reasonable GRB parameters the spreading coming from both \((31)\) and \((32)\) is negligible.

Note that velocity dispersion in any case does not exceed the value given by equation \((27)\) with \(\theta \gg 1\)

$$\frac{\Delta l_1}{R_0} = \frac{1}{R_0} \int_0^{t_1} \Delta v(t) \, dt \lesssim \frac{1}{R_0} \int_0^{\infty} c\Gamma(t)^{-3/2} \, dt \simeq 2.6, \quad (33)$$

which gives an absolute maximum of the thermal spreading on the energy dominated phase.

### 4. Conclusion

In this paper we considered two mechanisms of spreading of relativistically expanding plasma shell. We also discuss their implications for the duration of electromagnetic signal from transparency of the plasma.

Firstly, following the proposal of Piran et al. (1993) hydrodynamical spreading of relativistically expanding shell is estimated. Secondly, thermal spreading is considered. Assuming relativistic Maxwellian distribution function we determined the velocity dispersion depending on temperature and the Lorentz factor of the bulk motion.

We then applied these results to GRBs within the framework of the fireshell model. It is shown that thermal spreading provides negligible spreading for typical parameters of GRBs. Instead, hydrodynamical spreading results in the increase of the duration of P-GRB. For nonspreading shells characterized by
almost constant Lorentz factor distribution within the shell the duration of P-GRB is determined by the time of diffusion, see equation (10). If strong Lorentz factor difference is present within the shell, hydrodynamical spreading prevents occurrence of photon thin asymptotics and leads to duration of P-GRB given by equation (21).

Our results imply that for high enough baryon loading and energy of the burst the duration of the P-GRB is not determined by the initial size of the plasma $R_0$, but by the value of the hydrodynamical spreading, reaching up to several seconds.

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