THE SPECTRAL SLOPE AND KOLMOGOROV CONSTANT OF MHD TURBULENCE

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ABSTRACT

The spectral slope of strong MHD turbulence has recently been a matter of controversy. While Goldreich-Sridhar model (1995) predicts Kolmogorov’s -5/3 slope of turbulence, shallower slopes were often reported by numerical studies. We argue that earlier numerics was affected by driving due to a diffuse locality of energy transfer in MHD case. Our highest-resolution simulation (3072^3 x 1024) has been able to reach the asymptotic -5/3 regime of the energy slope. Additionally, we found that so-called dynamic alignment, proposed in the model with -3/2 slope, saturates and therefore can not affect asymptotic slope. The observation of the asymptotic regime allowed us to measure Kolmogorov constant C_{KA} = 3.2 ± 0.2 for purely Alfvénic turbulence and C_K = 4.1 ± 0.3 for full MHD turbulence. These values are much higher than the hydrodynamic value of 1.64. The larger value of Kolmogorov constant is an indication of a fairly inefficient energy transfer and, as we show in this Letter, is in theoretical agreement with our observation of diffuse locality. We also explain what has been missing in numerical studies that reported shallower slopes.

1. INTRODUCTION

The equations of incompressible ideal magnetohydrodynamics, written in terms of Elsasser variables,

\[ \partial_t w^\pm + \hat{S}(w^\mp \cdot \nabla)w^\pm = 0, \]

where \( \hat{S} = (1 - \nabla \Delta^{-1} \nabla) \) is a solenoidal projection, and \( w^\pm \) (Elsasser variables) are \( w^+ = v + b \) and \( w^- = v - b = B/(4\pi_0)^{1/2} \) are remarkably similar to the Euler’s equation. However, \( w^\pm \) have different transformation properties than \( v \). While in hydrodynamic turbulence the local average velocity can always be excluded by the choice of reference frame, in MHD \( w^\pm \) always contain the average local mean magnetic field that can not be excluded. This leads to a situation when large scale magnetic field is much stronger than small-scale turbulent perturbations and dynamics is dominated by this local mean magnetic field [Iroshnikov 1963; Kraichnan 1965]. But turbulence does not fluctuate weaker down the cascade as was proposed in aforementioned models. A proper perturbation theory [Galtier et al 2000] revealed that MHD turbulence has a tendency of becoming stronger on smaller scales, rather than weaker, due to the fact that the cascade increases perpendicular wavenumber \( k_\perp \), keeping parallel wavenumber \( k_\parallel \) constant. The “strength” of turbulence, as the ratio of the mean-field term to the nonlinear term can be approximated as \( \xi = w k_\perp/v_A k_\parallel \) and can increase due to the increasing anisotropy of perturbations \( k_\perp/k_\parallel \) down the cascade. As turbulence becomes marginally strong (\( \xi \sim 1 \)), i.e., the linear term is comparable to the nonlinear term, the cascading timescales become close to the dynamical timescales \( \tau_{\text{cascade}} \sim \tau_{\text{dyn}} = 1/(w k_\perp) \). However, as was argued in [Goldreich & Sridhar 1993], the perturbation frequency \( \omega \) has a lower bound due to an uncertainty relation \( \omega \tau_{\text{cascade}} > 1 \), therefore the combination of this lower bound, that limits the strength of turbulence \( \xi \) and the tendency of turbulence to become stronger will make it “critically balanced” with \( \xi \sim 1 \). This critically balanced cascade is strong in a sense that cascading timescale is always of the order of the dynamical timescale and will, therefore, have a Kolmogorov -5/3 spectrum. These critically balanced perturbations will be strongly anisotropic with respect to the local mean magnetic field. Using \( \xi \sim 1 \) and \( w \sim k_\perp^{-1/3} \) one obtains \( k_\parallel \sim k_\perp^{-2/3} \), i.e. the anisotropy will increase towards small scales without limit. One can further simplify Eq. 1 by neglecting the term \( (\delta w^\pm \cdot \nabla)\delta w^\pm \) which is much smaller than the mean field term \( (v_A \nabla)\delta w^\pm \). After this Eq. 1 splits into two equations, one for \( \delta w^+ \), which, in this strongly anisotropic case, \( k_\parallel \ll k_\perp \) represents slow (or pseudo-Alfvén) mode and the equation for \( \delta w^- \) which represent Alfvénic mode. The equation for slow mode is passive and does not provide any back-reaction for the Alfvénic equation which could be written in the following form:

\[ \partial_t \delta w^\pm + (v_A \cdot \nabla) \delta w^\pm + \hat{S}(\delta w^\mp \cdot \nabla)\delta w^\pm = 0, \]

One can study this purely Alfvénic dynamics and assume that the omitted slow mode has similar cascade and similar statistical properties. Eq. 2 is known as reduced MHD approximation or RMHD, see, e.g., [Kadomtsev & Pogutse 1974; Strauss 1976]. RMHD equations provide further support towards Goldreich-Sridhar model, as it has a precise symmetry with respect to anisotropy and the strength of the mean field. Indeed, as long as one increases \( v_A \) and stretches the fields in the parallel direction, decreasing \( \nabla \parallel \), by the same factor, Eq. 2 will be unchanged. Furthermore, a Kolmogorov argument of universality of nonlinear dynamics at each

[1] Another bound on \( w \) that follow from the uncertainty in the direction of the \( v_A \) vector [Beresnyak & Lazarian 2008], give the same estimate in the case of balanced turbulence considered in

[Goldreich & Sridhar 1993]. In the imbalanced case it could lead to a modified relation.
scale, which is based on a two-parametric scaling symmetry, could be amended it with a proper scaling for the anisotropy:

$$w \to wA, \quad \lambda \to \lambda B, \quad t \to tB/A, \quad \Lambda \to \Lambda B/A, \quad (3)$$

where $\lambda$ is a perpendicular scale, $A$ is a parallel scale, $A$ and $B$ are arbitrary parameters. Due to this precise symmetry one can hypothesize that strong Alfvénic turbulence has a universal regime, utilizing the same argumentation as Kolmogorov (1941). In nature, this universal regime can only be achieved as long as $\delta w^\perp \ll v_A$.

In numerical simulations, we can directly solve the reduced Eq. (2), which has precise symmetry already built in. From practical viewpoint, the statistics from the full MHD simulation with $\delta u^\perp \sim 0.1v_A$ is virtually indistinguishable from RMHD statistics and even $\delta w^\perp \sim v_A$ are fairly similar to the former (Beresnyak & Lazarian 2009a,b, 2010). Table 1 shows the parameters of the simulations. The Kolmogorov scale is defined as $\eta = (v^2/\epsilon)^{1/(3n-2)}$, the integral scale $L = 3\pi/4E\int_0^\infty k^{-1}E(k)\,dk$ (which was approximately 0.79 for R1-3). Dimensionless ratio $\Lambda/\eta$ could serve as a “length of the spectrum”, although usually spectrum is around an order of magnitude shorter.

The resolution in the direction parallel to the mean magnetic field, $n_x$, was reduced by a factor of 3 for simulations R1-3. This was possible due to an empirically known lack of energy in the parallel direction in $k$-space. We ran a simulation R2.5 which has full resolution in $n_x$ to compare with R2 and check the influence of this resolution reduction on the power spectrum. Although the bottleneck effect was slightly less pronounced in R2.5 compared to R2, there was only a small influence in the inertial range. We concluded that using $n_x$ reduced by a factor of 2 or 3 is possible.

For the purpose of this paper we used driving that had a constant energy injection rate. In RMHD simulations R1-3 we drove turbulence to the amplitude that it will be strong on the outer scale. R1-3 were started from lower-resolution simulation that reached stationary state and were further evolved in high resolution for approximately 12 Alfvénic times, which, for strong MHD turbulence also correspond to about 12 dynamical times. The averaged quantities were obtained for the last 6 Alfvénic times. In all magnetic simulations M1, R1-3, we were using hyperviscosity ($n > 2$) instead of normal viscosity. This is possible due to the fact that bottleneck effect is much less pronounced in the MHD case, compared to hydro.

3. SPECTRA AND UNIVERSALITY

Much of the study of hydrodynamic turbulence was dedicated to Kolmogorov model which assumes a universal cascade of energy through scales (Kolmogorov 1941). This model predicts that the power spectrum of turbulence, $E(k)$, will be a power-law function of scale,

$$E(k) = C_K k^{-5/3}. \quad (4)$$

where $C_K$ is a Kolmogorov constant. It is well-known that this scaling is not precisely correct and typically has an intermittency correction $(kL)^\alpha$, where $L$ is an outer scale and $\alpha$ is a small number, around 0.035 (She & Leveque 1994). However, in simulations or mea-

| Run | Type | $n_x \cdot n_y \cdot n_z$ | Dissipation $\langle \epsilon \rangle$ | $L/\eta$ |
|-----|------|--------------------------|----------------|---------|
| H1  | hydro | 512$^3$                  | $-3.02 \cdot 10^{-6} k^2$ | 0.091 | 190 |
| H2  | hydro | 1024$^3$                 | $-1.20 \cdot 10^{-4} k^2$ | 0.091 | 370 |
| M1  | MHD  | 1024$^3$                 | $-1.65 \cdot 10^{-4} k^4$ | 0.150 | 280 |
| R1  | RMHD | 256$^3$ - 768$^2$        | $-6.82 \cdot 10^{-14} k^6$ | 0.073 | 280 |
| R2  | RMHD | 512$^3$ - 1536$^2$       | $-1.51 \cdot 10^{-15} k^6$ | 0.073 | 570 |
| R2.5 | RMHD | 1536$^3$                | $-1.51 \cdot 10^{-15} k^6$ | 0.073 | 570 |
| R3  | RMHD | 1024$^3$ - 3072$^2$      | $-3.33 \cdot 10^{-17} k^6$ | 0.073 | 1100 |
measurements with small inertial range this correction can often be neglected. In particular, a compilation of experimental results for hydrodynamic turbulence (Sreenivasan 1995) suggests that a Kolmogorov constant is universal for a wide variety of flows. High-resolution numerical simulations of isotropic incompressible hydrodynamic turbulence (Gotoh et al. 2002) suggest the same value for the Kolmogorov constant.

A robust method for determining the spectral slope and the Kolmogorov constant from simulations is a resolution study (see, e.g., Gotoh et al. 2002), when a number of numerical experiments are performed with different resolution and the spectra are plotted with respect to the dimensionless wavevector, \( k \eta \). A physical meaning of such a comparison is based on an assumption that a simulation with higher numerical resolution can be considered both as a simulation resolving smaller physical scales and as a simulation of a larger volume of turbulence (see Fig. 1). This assumption is true as long as turbulence can be considered local, i.e. the effects of driving can be neglected in the inertial range. Our hydrodynamic simulations reveal a good convergence of spectra with numerical resolution and show a universal Kolmogorov constant consistent with the one obtained in Gotoh et al. (2002). Also the shape of the dissipation range is similar to the one in aforementioned paper, showing a typical “bump” due to a bottleneck effect. Despite moderate resolution, the inertial ranges converge, which is due to locality of hydrodynamic cascade in spacial scales, making it possible to consider higher and lower resolution simulations on a common ground, neglecting the influence of large scales, where energy is provided by driving.

Fig. 2 presents a resolution study for simulations R1-3 determining the spectral slope and Kolmogorov constant for Alfvénic turbulence. If the spectrum \(-3/2\) was universal, the outer scale point, corresponding to \( k = 3 \) which is marked by a cross will go down from R1 to R3 by a factor of around 1.26, instead it stays at about the same level, indicating that deviations from \(-5/3\) slope are small (note that the outer-scale point moves horizontally in Fig. 1 as well). The flat part of Fig. 2 in R3 simulation between \( k = 54 \) (\( k \eta \approx 0.037 \)) and \( k = 91 \) (\( k \eta \approx 0.063 \)) with central frequency \( k = 70 \) was fit to obtain Kolmogorov constant. The value obtained in this fit was \( C_K = 3.2 \pm 0.2 \) where the error was mostly due to fluctuation of spectrum in time.

4. DYNAMIC ALIGNMENT

It was suggested that the spectral slope of MHD turbulence is modified by so-called “dynamic alignment” that increases indefinitely towards small scales. Although the tentative correspondence with theoretical scaling from Boldyrev model has been obtained with only one particular measure of alignment, this was interpreted by some studies as a confirmation of the aforementioned model. In this paper we refer to our earlier studies Beresnyak & Lazarian (2006, 2009b) that measured several types of alignment and their dependence on scale. In these studies there were no conclusive evidence that all alignment measures follow the same scaling. In this paper we confirm this finding with higher-resolution simulations, in addition we found evidence that all alignment measures saturate, i.e. approach an asymptotic constant value on small scales. Fig. 3 shows the alignment measures in R3, where AA, AA2, DA and PI are different alignment measures:

\[
AA = \langle |\delta w_\perp^x| |\delta w_\perp^y| |\delta w_\perp^z| \rangle / \langle |\delta w_\perp^\lambda| \rangle,
\]

\[
AA2 = \langle |\delta v_\perp^x| \times |\delta b_\perp^x| \times |\delta b_\perp^y| \rangle / \langle |\delta w_\perp^\lambda| \rangle,
\]

\[
PI = \langle |\delta w_\perp^x| \times |\delta w_\perp^y| \times |\delta w_\perp^z| \rangle / \langle |\delta w_\perp^\lambda| \rangle,
\]

\[
DA = \langle |\delta v_\perp^x| \times |\delta b_\perp^x| \times |\delta b_\perp^y| \rangle / \langle |\delta b_\perp^\lambda| \rangle.
\]

Having an inertial range of around two orders of magnitude in scale, if alignment was proportional to \( \lambda^{1/4} \) as in Boldyrev (2005, 2006), we would expect alignment increase by a factor of 3.2, while in reality the polarization intermittency PI increases only by a factor of 1.3, and dynamic alignment DA by a factor of 1.8. This is consistent with the range of \( \lambda \) between 3 and 10.

We are not aware of any convincing physical argument explaining why alignment should be a power-law of scale. Boldyrev (2006) provides an explanation arguing that alignment will tend to increase indefinitely, but will be bounded by field wandering, i.e. the alignment on each scale will be created independently of other scales (hence the term “dynamic alignment”) and will be proportional to the relative perturbation amplitude \( \delta B/B \). But this directly violates precise symmetry of Eq. 2 i.e. Eq. 3 which states that nothing should depend on \( \delta B/B \) as long as other quantities are scaled properly. Phys-
ically, this means that field wandering cannot destroy alignment or imbalance. Indeed a perfectly aligned state, e.g., with \( \delta B = 0 \) is a precise solution of Eq. \( \dots \) and it is not destroyed by its own field wandering. Additionally, Beresnyak & Lazarian (2009b) measured alignment in simulations of strong MHD turbulence with different values of \( \delta B_k / B_0 \) and found very little or no dependence on this parameter. Fig. 3 also compares alignment measure with a first-order structure function of the perturbation amplitude, i.e. \( \langle |\delta B_k| \rangle \). According to Boldyrev (2006) they should scale the same way, but this is not observed.

To summarize, our numerical data are consistent with alignment measures becoming constant in the inertial range and inconsistent with the hypothesis that they depend as \( \lambda^{1/4} \) on scale. This finding is important, because if alignment is constant on scale in the asymptotic regime, there is no reason to expect that the power-law scaling of turbulence will deviate from its \(-5/3\) value for strong Goldreich-Sridhar turbulence. This result is further supported by the results of the previous section where a steeper asymptotic spectra has been observed.

5. THE AMOUNT OF SLOW MODE AND THE TOTAL KOLMOGOROV CONSTANT FOR MHD TURBULENCE

Full incompressible MHD turbulence have a cascade of slow mode, which was not included in our reduced MHD simulations R1-3. Although in nature slow mode is often damped, it is normally present in full MHD incompressible simulations, e.g. the ones presented in Biskamp (2003). The passive cascade of slow mode will have the same energy spectral slope as a Alfvénic mode, and, assuming that the ratio of slow to Alfvénic energies is \( C_s \), the total Kolmogorov constant for MHD turbulence will be expressed as

\[
C_K = C_{KA} (1 + C_s)^{1/3}.
\]

The ratio \( C_s \) is supposedly depend on how the MHD turbulence is driven. However, historically, previous studies simulated MHD turbulence with zero mean field, either decaying or driven with statistically isotropic forcing, e.g. Muller & Grappin (2003). In this idealized case Kolmogorov constant has been measured, although with fairly limited resolution (Biskamp 2003). In this paper we will use a less straightforward approach, by measuring \( C_s \) from a simulation with zero mean field and substituting it into Eq. 5. This approach is motivated by our finding that MHD turbulence is less local and therefore it is much harder to achieve an asymptotic universal cascade if one uses zero-mean field simulation. Indeed, one has to observe the transition to the strong local mean field case, which will require at least a couple of order of magnitude in scale and subsequently a transition to universal cascade, which as we observed in previous section, takes about two orders of magnitude in scale, as long as the power-law scaling and Kolmogorov constants are concerned. It is, therefore, impossible to directly measure the properties of universal cascade in zero mean field simulations of currently available simulations. The “natural” value of \( C_s \) is unity, because the incompressible MHD equations have four degrees of freedom, out of which Alfvénic mode uses two and slow mode also uses two. Having the same amount of degrees of freedom and the isotropic driving that does not prefer any direction we would expect that the energy will be distributed equally between the modes. We measured how energy is partitioned on small scales of simulation M1 by making a local Fourier transform of smaller cubes and decomposing into modes with respect to the local mean field. The actual partition of energy shows \( C_s \) being around 1.3. Although statistical errors in this measurement are small, it is hard to claim a particular value of \( C_s \) based on a numerical simulation with a finite resolution. Conservatively, we will assume that \( C_s \) is between 1, which is equipartition, and 1.3, which is observed in our simulation M1. The total Kolmogorov constant will be estimated as \( 4.1 \pm 0.3 \).

6. SCALE LOCALITY AND KOLMOGOROV CONSTANT

The energy flux through scales in both MHD and hydrodynamic turbulence can be expressed as a certain third order \textit{signed} structure function divided by scale and has to be scale-local due to an upper analytical bound on contributions from different \( k \) wavebands (see, e.g. Ahie & Eivin 2010). This upper bound, however, is well applicable to similar third order \textit{unsigned} structure function. This \textit{unsigned} third order structure function is related by self-similarity hypothesis to second order structure function, which is a measure of energy. Therefore, we would expect that the ratio of \textit{unsigned} third order structure function to the signed one will scale approximately as \( C_K^{3/2} \). This seriously limits the bound on scale locality from practical standpoint as long as \( C_K \) becomes large, i.e., the energy transfer becomes less efficient (Beresnyak & Lazarian 2010). Indeed, if the define “scale locality” as a ratio of largest to smallest wavevectors \( k_1/k_2 \) which still significantly contribute to energy flux through some central wavevector \( k_0 \), then this ratio will have an upper bound that scale asymptotically with Kolmogorov constant as \( C_K^{9/4} \). In other words, inefficient energy transfer can still be very local, but it is also possible that it is less local than efficient energy transfer. A nonlocal or diffuse energy transfer \textit{must} be inefficient and \textit{must} have a high value of Kolmogorov constant.

A comparative study of energy spectra in MHD and hydro turbulence in Beresnyak & Lazarian (2009b) revealed that the bottleneck effect is less pronounced or altogether absent in MHD simulations, while in hydro it is always present, both in simulations with normal \((n = 2)\) and hyperviscosity \((n > 2)\). This was interpreted as an indication that MHD cascade is less local. Now, our measurement of Kolmogorov constant revealed that MHD energy transfer is less efficient, therefore MHD cascade may be less local than hydro cascade. In view of all numerical evidence available today, MHD cascade is most likely less local than hydro cascade.

7. DISCUSSION

Previous measurements of the slope usually relied on the highest-resolution simulation and fitted the slope in the fixed \( k \)-range close to driving scale typically between \( k = 5 \) and \( k = 20 \). In this paper we argue that such a fit is unphysical and instead one should fit a fixed \( k \eta \) range. In the former case the result would be a shallower spectral slope due to proximity to the outer scale and driving. In the latter case the effect of the driving will diminish with increasing resolution and one will observe...
shallower spectra at small resolutions that will become steeper with increasing resolution.

Earlier measurements of Kolmogorov constant in MHD turbulence reported lower values than this study, e.g. $C_K = 2.2$ in Biskamp (2003). We believe this is due to insufficient resolution in those simulations, which prevented the observation of the asymptotic regime. In particular, in the case of statistically isotropic simulations like the ones in Biskamp (2003) a transition to small scale sub-alfvenic regime precede the transition to asymptotic regime. These two transitions require numerical resolution that is even higher than the highest resolution presented in this paper and for now seems computationally impossible. Our own statistically isotropic simulation M1 shows Kolmogorov constant of 3.5, which is still only a lower limit, consistent with 4.1 derived in this paper. For M1 and similar lower-resolution simulations the estimate of $C_K$ continues to grow with increasing resolution, which supports argumentation above.

In this paper we treated so called balanced case, where the rms amplitudes of the $w^\pm$ were statistically the same. A number of attempts to generalize the GS95 model has been made recently (Lithwick et al 2007, Beresnyak & Lazarian 2008, Chandran 2008, Perez & Boldyrev 2009). Some of these models can be rejected by numerics, in particular the model based on alignment (Perez & Boldyrev 2009) is grossly inconsistent with the dissipation rates measured in im-balanced numerical simulations (Beresnyak & Lazarian 2009a, 2010).

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