Optimal decay rates of a nonconservative compressible two-phase fluid model

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We are concerned with the time decay rates of strong solutions to a nonconservative compressible viscous two-phase fluid model in the whole space $\mathbb{R}^3$. Compared to the previous related works, the main novelty of this paper lies in the fact that it provides a general framework that can be used to extract the optimal decay rates of the solution as well as its all-order spatial derivatives from one-order to the highest-order, which are the same as those of the heat equation. Furthermore, for well-chosen initial data, we also show the lower bounds on the decay rates. Our methods mainly consist of Hodge decomposition, low- and high-frequency decomposition, delicate spectral analysis, and energy method based on finite induction.

1 INTRODUCTION

As is well-known, most of the flows encountered in nature are multifluid flows. They are widely used in nuclear power, chemical processing, oil and gas manufacturing, and so on. The classic approach to simplify the complexity of multiphase flows and satisfy the engineer’s need of some modeling tools is the well-known volume-averaging method. This approach leads to so-called averaged multiphase models, see Refs. [1, 2, 38] for details. As a result of such a procedure, one can obtain
the following generic compressible two-phase fluid model:

\[
\begin{align*}
\alpha^+ + \alpha^- &= 1, \\
\partial_t (\alpha^\pm \rho^\pm) + \text{div} (\alpha^\pm \rho^\pm u^\pm) &= 0, \\
\partial_t (\alpha^\pm \rho^\pm u^\pm) + \text{div} (\alpha^\pm \rho^\pm u^\pm \otimes u^\pm) + \alpha^\pm \nabla P^\pm (\rho^\pm) &= \text{div} (\alpha^\pm \tau^\pm), \\
P^+ (\rho^+) - P^- (\rho^-) &= f(\alpha^- \rho^-),
\end{align*}
\]

(1.1)

where the variable \(0 \leq \alpha^+(x, t) \leq 1\) is the volume fraction of fluid + (the liquid), and \(0 \leq \alpha^-(x, t) \leq 1\) that of fluid—(the gas). \(\rho^\pm(x, t) \geq 0, u^\pm(x, t)\) and \(P^\pm(\rho^\pm) = A^\pm(\rho^\pm)^{\bar{\gamma}^\pm}\) denote the densities, the velocity of each phase, and the two pressure functions, respectively. \(\bar{\gamma}^\pm \geq 1, A^\pm > 0\) are positive constants. In what follows, we set \(A^+ = A^- = 1\) without loss of any generality. The main purpose of this article is to get the optimal decay rates of the model (1.1) in the whole space \(\mathbb{R}^3\).

As in Evje et al. [21], we also assume that \(f(\cdot) \in C^3([0, \infty))\) and \(f\) is a strictly decreasing function near the equilibrium. Moreover, \(\tau^\pm\) are the viscous stress tensors defined by

\[
\tau^\pm := \mu^\pm (\nabla u^\pm + \nabla^T u^\pm) + \lambda^\pm \text{div} u^\pm \text{Id},
\]

(1.2)

where the constants \(\mu^\pm\) and \(\lambda^\pm\) are shear and bulk viscosity coefficients satisfying the physical condition: \(\mu^\pm > 0\) and \(2\mu^\pm + 3\lambda^\pm \geq 0\), which implies that \(\mu^\pm + \lambda^\pm > 0\). This system is known as a two-fluid flow system with algebraic closure.

For more information about this model, we refer to Refs. [3, 4, 21, 24, 31, 37, 38] and references therein. However, it is well-known that for the well-posedness and stability of this type of model (1.1), there are several new mathematical challenges, which can be listed as follows:

- The combination of different density–pressure laws corresponding to the different phases giving rise to nonconventional, nonlinear pressure functions, which presents a challenge that need to be addressed;
- Transition to single-phase regions, that is, regions where the mass \(\alpha^+ \rho^+\) or \(\alpha^- \rho^-\) becomes zero, can happen because the volume fractions \(\alpha^\pm\) become zero and/or because densities \(\rho^\pm\) vanish (formation of vacuum);
- The appearances of the nonconservative pressure terms \(\alpha^\pm \nabla P^\pm\) typically prevent one from applying ideas used for the compressible Navier–Stokes equations.

In turn, it is also highly nontrivial to derive reliable and accurate methods for computing numerical approximations. For more information, we refer to Refs. [15, 16, 24] and references therein. They studied the models similar to Equation (1.1), however, where the viscous terms \(\tau^\pm\) are absence and a common pressure \(P^+ = P^- = P\) is assumed.

Recently, Bretsch et al. in the seminal work [3] considered a model similar to Equation (1.1). More specifically, they made the following assumptions:

- a common pressure \(P^+ = P^- = P\);
- inclusion of viscous terms of the form (1.2) where \(\mu^\pm\) depends on densities \(\rho^\pm\) and \(\lambda^\pm = 0\);
- inclusion of a third-order derivative of \(\alpha^\pm \rho^\pm\), which are so-called internal capillary forces represented by the well-known Korteweg model on each phase.

They obtained the global weak solutions in the periodic domain with \(1 < \bar{\gamma}^\pm < 6\). Later, Bresch et al. [4] established the global existence of weak solutions in one space dimension without the internal capillary forces when \(\bar{\gamma}^\pm > 1\), and Cui et al. [6] obtained the time-decay rates of classical solutions for the three-dimensional Cauchy problem by combining detailed analysis of the Green’s function to the linearized system with energy estimates to the nonlinear system.

The relation between the pressures of Equation (1.1) implies the differential identity

\[
dP^+ - dP^- = df(\alpha^- \rho^-),
\]

(1.3)

where \(P^\pm := P^\pm(\rho^\pm)\). It is clear that

\[
dP^+ = \bar{s}^+ d\rho^+, \quad dP^- = \bar{s}^- d\rho^-,
\]

where \(\bar{s}^\pm := \frac{dP^\pm}{d\rho^\pm}(\rho^\pm) = \bar{\gamma}^\pm \frac{P^\pm(\rho^\pm)}{\rho^\pm}.
\]

(1.4)
Here $s_{±}$ represent the sound speed of each phase, respectively. As in Bresch et al. [3], we introduce the variables

$$ R_{±} = α_{±}ρ_{±}, $$

(1.5)

which together with Equation (1.1) gives

$$ dρ_{±} = \frac{1}{α_{±}} (dR_{±} - ρ_{±} dα_{±}), $$

(1.6)

By virtue of Equations (1.3) and (1.5), we finally get

$$ dα_{±} = \frac{α_{±} s_{±}^2}{α_{±} ρ_{±} s_{±}^2 + α_{±} ρ_{±} s_{±}^2} dR_{±} - \frac{α_{±} α_{±}}{α_{±} ρ_{±} s_{±}^2 + α_{±} ρ_{±} s_{±}^2} \left( \frac{s_{±}^2}{α_{±}} + f' \right) dR_{±}. $$

(1.7)

Substituting Equation (1.7) into Equation (1.6), we deduce the following expressions:

$$ dρ_{±} = \frac{ρ_{±} ρ_{±} s_{±}^2}{R_{±}(ρ_{±})^2 s_{±}^2 + R_{±}(ρ_{±})^2 s_{±}^2} \left( ρ_{±} dR_{±} + \left( ρ_{±} + ρ_{±} \frac{α_{±} f'}{s_{±}^2} \right) dR_{±} \right), $$

(1.8)

and

$$ dρ_{±} = \frac{ρ_{±} ρ_{±} s_{±}^2}{R_{±}(ρ_{±})^2 s_{±}^2 + R_{±}(ρ_{±})^2 s_{±}^2} \left( ρ_{±} dR_{±} + \left( ρ_{±} - ρ_{±} \frac{α_{±} f'}{s_{±}^2} \right) dR_{±} \right), $$

(1.9)

which together with Equation (1.3) give the pressure differential $dP_{±}$:

$$ dP_{±} = \left( ρ_{±} dR_{±} + \left( ρ_{±} + ρ_{±} \frac{α_{±} f'}{s_{±}^2} \right) dR_{±} \right), $$

(1.10)

and

$$ dP_{±} = \left( ρ_{±} dR_{±} + \left( ρ_{±} - ρ_{±} \frac{α_{±} f'}{s_{±}^2} \right) dR_{±} \right), $$

(1.11)

where

$$ \phi^2 := \frac{s_{±}^2 s_{±}^2}{α_{±} ρ_{±} s_{±}^2 + α_{±} ρ_{±} s_{±}^2}. $$

(1.12)

Next, by noting the fundamental relation: $α_± + α_± = 1$, we can get the following equality:

$$ \frac{R_{±}}{ρ_{±}} + \frac{R_{±}}{ρ_{±}} = 1, \text{ and thus } \frac{R_{±}}{ρ_{±}} = \frac{R_{±}-ρ_{±}}{ρ_{±}-ρ_{±}}. $$

(1.13)

Then, we have from the pressure relation (1.1) that

$$ \varphi(ρ_{±}, R_{±}, R_{±}) := P_{±}(ρ_{±}) - P_{±} \left( \frac{R_{±}}{ρ_{±}-R_{±}} \right) - f(R_{±}) = 0. $$

(1.14)

Thus, we can employ the implicit function theorem to define $ρ_{±}$. To see this, by differentiating the above equation with respect to $ρ_{±}$ for given $R_{±}$ and $R_{±}$, we get

$$ \frac{∂φ}{∂ρ_{±}} (ρ_{±}, R_{±}, R_{±}) = s_{±}^2 + s_{±}^2 \frac{R_{±} R_{±}}{(ρ_{±}-R_{±})^2}. $$

(1.15)
which is positive for any $\rho^+ \in (R^+, +\infty)$ and $R^\pm > 0$. This together with the implicit function theorem implies that $\rho^+ = \rho^+(R^+, R^-) \in (R^+, +\infty)$ is the unique solution of Equation (1.14). By virtue of Equations (1.5), (1.14), and (1.1), $\rho^-$ and $\alpha^\pm$ can be defined by

$$
\rho^-(R^+, R^-) = \frac{R^-\rho^+(R^+, R^-)}{\rho^+(R^+, R^-) - R^+},
$$
$$
\alpha^+(R^+, R^-) = \frac{R^+}{\rho^+(R^+, R^-)},
$$
$$
\alpha^-(R^+, R^-) = 1 - \frac{R^+}{\rho^+(R^+, R^-)} = \frac{R^-}{\rho^-(R^+, R^-)}.
$$

We refer the readers to Bresch et al. [4, pp. 614] for more details.

Therefore, we can rewrite system (1.1) into the following equivalent form:

$$
\begin{cases}
\partial_t R^\pm + \text{div} (R^\pm u^\pm) = 0, \\
\partial_t (R^+ u^+) + \text{div} (R^+ u^+ \otimes u^+) + \alpha^+ \rho^+ \left[ \rho^- \nabla R^+ + \left( \rho^+ + \frac{\rho^+}{\rho^-} \frac{\partial f}{\partial s} \right) \nabla R^- \right] \\
= \text{div} \left\{ \alpha^+ \left[ \mu^+ (\nabla u^+ + \nabla u^+)^T + \lambda^+ \text{div} u^+ I \right] \right\}, \\
\partial_t (R^- u^-) + \text{div} (R^- u^- \otimes u^-) + \alpha^- \rho^- \left[ \rho^- \nabla R^+ + \left( \rho^+ - \rho^- \frac{\alpha^+ f^1}{s^2} \right) \nabla R^- \right] \\
= \text{div} \left\{ \alpha^- \left[ \mu^- (\nabla u^- + \nabla u^-)^T + \lambda^- \text{div} u^- I \right] \right\}.
\end{cases}
$$

In the present paper, we consider the initial value problem to Equation (1.17) in the whole space $\mathbb{R}^3$ with the initial data

$$
(R^+, u^+, R^-, u^-)(x, 0) = (R^+_0, u^+_0, R^-_0, u^-_0)(x) \rightarrow (R^+_\infty, 0, R^-\infty, 0) \quad \text{as} \quad |x| \rightarrow \infty \in \mathbb{R}^3,
$$

where $R^\pm_\infty > 0$ denote the background doping profiles, and for simplicity, are taken as 1 in this paper.

To put our results into context, let us highlight some recent progress on the topics of nonconservative compressible viscous two-phase fluid model and related model. By taking the following simplifications:

- Due to the fact that the liquid phase is much heavier than the gas phase, typically to the order $\rho^+/\rho^- \sim 10^3$, we can neglect the gas phase in the momentum equation corresponding to the gas;
- A nonslip condition is assumed, that is, $u_+ = u_- = u$,

and setting $m = \alpha^+ \rho^+$ and $n = \alpha^- \rho^-$, one can get the simplified version of Equation (1.1):

$$
\begin{cases}
m_t + \text{div}(mu) = 0, \\
n_t + \text{div}(nu) = 0, \\
(mu)_t + \text{div}(mu \otimes u) + \nabla P(m, n) = \text{div}(\mu (\nabla u + \nabla u^T) + \lambda (\text{div} u) I).
\end{cases}
$$

For the simplified model (1.19) and related model that are subject to various initial and initial-boundary conditions have been explored thoroughly during the past decades, the global existence and asymptotic behavior of the solutions (weak, strong, classic) were proved. We refer the readers to Refs. [13–20, 22, 23, 26–28, 34, 39, 40, 42–53] and references therein. When there is no liquid ($\alpha^+ \equiv 0$), the system (1.1) reduces to the compressible Navier–Stokes system, which is one of the most important systems in fluid dynamics, the readers can refer to Refs. [9–11, 29, 30, 33] and the references therein for studies on decay rates of the compressible Navier–Stokes equations.

However, as mentioned before, due to difficulties coming from different nonlinear density–pressure laws corresponding to the different phases and the appearance of the nonconservative pressure terms $\alpha^\pm \nabla P^\pm$, so far, there is few results on the nonconservative compressible viscous two-phase fluid model (1.17). Recently, Evje et al. [21] studied the global
well-posedness and decay rates of the Cauchy problem (1.17)–(1.18). More precisely, under the assumptions that

\[ \frac{s^+ - s^-}{\alpha^+ - \alpha^-} < f'(1) < \frac{\eta - s^+}{\alpha^+ - \alpha^-} < 0, \]  

(1.20)

and

\[ \| (R^+_0 - 1, u^+_0, R^-_0 - 1, u^-_0) \|_{H^2(\mathbb{R}^3)} \leq \varepsilon, \quad \text{and} \quad \| (R^+_0 - 1, u^+_0, R^-_0 - 1, u^-_0) \|_{L^1} < \infty, \]  

(1.21)

where \( \eta \) and \( \varepsilon \) are two sufficiently small positive constants, then the following decay-in-time estimates hold:

\[ \| (R^+_0 - 1, u^+_0, R^-_0 - 1, u^-_0)(t) \|_{L^2(\mathbb{R}^3)} \leq C(1 + t)^{-3/4} \text{ for all } t \geq 0, \]  

(1.22)

\[ \| \nabla (R^+_0, u^+_0, R^-_0, u^-_0)(t) \|_{H^1(\mathbb{R}^3)} \leq C(1 + t)^{-5/4} \text{ for all } t \geq 0, \]  

(1.23)

Noting the decay rate in Equation (1.23), for the second-order (i.e., the highest-order) spatial derivative of the solution, it holds that

\[ \| \nabla^2 (R^+_0, u^+_0, R^-_0, u^-_0)(t) \|_{L^2(\mathbb{R}^3)} \leq C(1 + t)^{-7/4}. \]  

(1.24)

On the other hand, let us revisit the following classical result of the heat equation:

\[ \begin{cases} \partial_t u - \Delta u = 0, & \text{in } \mathbb{R}^3, \\ u|_{t=0} = u_0. \end{cases} \]  

(1.25)

If \( u_0 \in H^N(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) \) with \( N \geq 0 \) be an integer, then for any \( 0 \leq \ell \leq N \), the solution of the heat equation (1.25) has the following decay rate:

\[ \| \nabla^\ell u(t) \|_{L^2(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{1}{4} - \frac{\ell}{2}}. \]  

(1.26)

Particularly, taking \( N = 2 \) and \( \ell = 2 \), one has

\[ \| \nabla^2 u(t) \|_{L^2(\mathbb{R}^3)} \leq C(1 + t)^{-7/4}. \]  

(1.27)

Therefore, in view of Equations (1.24) and (1.27), it is clear that the decay rate of the second-order spatial derivative of the solution in Equation (1.24) is slower than that of the heat equation as in Equation (1.27). So, the decay rate of the second-order spatial derivative of the solution in Equation (1.23) is not optimal in this sense.

The main motivation of this paper is to provide a general framework that can be used to extract the optimal decay rates of the solution to the Cauchy problem (1.17)–(1.18) as well as its all-order spatial derivatives from one order to the highest order. More precisely, we obtain the optimal decay rates of the solution to the Cauchy problem (1.17)–(1.18) as well as its all-order spatial derivatives from one order to the highest order, which are the same as those of the heat equation. Moreover, for well-chosen initial data, we also show the lower bounds on the decay rates. Our methods mainly include Hodge decomposition, low- and high-frequency decomposition, delicate spectral analysis and energy method based on finite induction.

Before stating our main result, let us first introduce the notations and conventions used throughout this paper. We use \( H^k(\mathbb{R}^3) \) to denote the usual Sobolev spaces with norm \( \| \cdot \|_{H^k} \) and \( L^p \), \( 1 \leq p \leq \infty \) to denote the usual \( L^p(\mathbb{R}^3) \) spaces with norm \( \| \cdot \|_{L^p} \). For the sake of conciseness, we do not precise in functional space names when they are concerned with scalar- or vector-valued functions, \( \| (f, g) \|_X \) denotes \( \| f \|_X + \| g \|_X \). We will employ the notation \( a \lesssim b \) to mean that \( a \leq Cb \) for a universal constant \( C > 0 \) that only depends on the parameters coming from the problem. We denote \( \nabla = \partial_x = (\partial_1, \partial_2, \partial_3) \), where \( \partial_i = \partial_{x_i} \), \( \nabla_i = \partial_i \) and put \( \partial^\ell_x f = \nabla^\ell f = \nabla(\nabla^{\ell - 1} f) \). Let \( \Lambda^s \) be the pseudo differential operator defined by

\[ \Lambda^s f = \mathbf{\mathcal{F}}^{-1}(|\xi|^{2s} \hat{f}), \quad \text{for } s \in \mathbb{R}, \]  

(1.28)
where $\hat{f}$ and $\mathcal{F}(f)$ are the Fourier transform of $f$. The homogenous Sobolev space $H^s(\mathbb{R}^3)$ with norm given by $\|f\|_{H^s} \triangleq \|A^s f\|_{L^2}$.

For a radial function $\phi \in C_0^\infty(\mathbb{R}^3)$ such that $\phi(\xi) = 1$ when $|\xi| \leq \eta_0/2$ and $\phi(\xi) = 0$ when $|\xi| \geq \eta_0$ with a given positive constant $\eta_0$, we define the low-frequency part of $f$ by

$$ f^l = \mathcal{F}^{-1}[\phi(\xi) \hat{f}] $$

and the high-frequency part of $f$ by

$$ f^h = \mathcal{F}^{-1}[(1 - \phi(\xi)) \hat{f}]. $$

It is direct to check that $f = f^l + f^h$ if the Fourier transform of $f$ exists.

Now, we are in a position to state our main result.

**Theorem 1.1.** Assume that $R^+_0 - 1, u^+_0, R^-_0 - 1, u^-_0 \in H^N(\mathbb{R}^3)$ for an integer $N \geq 2$ and Equation (1.20) holds for a given small positive constant $\eta$. There exists a constant $\delta_0$ such that if

$$ \|(R^+_0 - 1, u^+_0, R^-_0 - 1, u^-_0)\|_{H^2} \leq \delta_0, $$

then the Cauchy problem (1.17)–(1.18) admits a unique solution $(R^+, u^+, R^-, u^-)$ globally in time in the sense that

$$ R^+ - 1, R^- - 1 \in C^0([0, \infty); H^N(\mathbb{R}^3)) \cap C^1([0, \infty); H^{N-1}(\mathbb{R}^3)), $$
$$ u^+, u^- \in C^0([0, \infty); H^N(\mathbb{R}^3)) \cap C^1([0, \infty); H^{N-2}(\mathbb{R}^3)). $$

Moreover, the following convergence rates hold true.

- **Upper bounds.** If additionally

$$ N_0 = \|(R^+_0 - 1, u^+_0, R^-_0 - 1, u^-_0)\|_{L^1} < +\infty, $$

then for all $t \geq 0$,

$$ \|\nabla^\ell (R^+ - 1, u^+, R^- - 1, u^-)(t)\|_{L^2} \leq C(N_0)(1 + t)^{-\frac{3}{4} - \frac{\ell}{2}}, $$

and

$$ \|(R^+ - 1, u^+, R^- - 1, u^-)(t)\|_{L^p} \leq C(N_0)(1 + t)^{-\frac{3}{2} \left(1 - \frac{1}{p}\right)}, $$

for $0 \leq \ell \leq N$ and $2 \leq p \leq \infty$.

- **Lower bounds.** Let $(n^+_0, u^+_0, R^-_0, u^-_0) = (\alpha_1(R^+_0 - 1), \sqrt{\alpha_1} u^+_0, \alpha_4(R^-_0 - 1), \sqrt{\alpha_4} u^-_0)$ where the definitions of the positive constants $\alpha_1$ and $\alpha_4$ are given in Section 2 and assume that the Fourier transform of functions $(n^+_0, u^+_0, n^-_0, u^-_0)$ satisfy

$$ \hat{n}^-_0(\xi) = 0, \wedge^{-1} \text{div} \hat{n}^+_0(\xi) = \wedge^{-1} \text{div} \hat{u}^-_0(\xi) = 0, \text{ and } |\hat{n}^+_0(\xi)| \geq N_0 \sqrt{\delta_0}, $$

for any $|\xi| \leq \eta_1$. Then there is a positive constant $c_0$ independent of time such that for any large enough $t$,

$$ \min \left\{ \|\nabla^\ell (R^+ - 1)(t)\|_{L^2}, \|\nabla^\ell u^+(t)\|_{L^2}, \|\nabla^\ell (R^- - 1)(t)\|_{L^2}, \|\nabla^\ell u^-(t)\|_{L^2} \right\} $$

$$ \geq c_0(1 + t)^{-\frac{3}{4} - \frac{\ell}{2}}, $$

for $0 \leq \ell \leq N$.

**Remark 1.2.** Compared to Evje et al. [21], the main new contribution of Theorem 1.1 lies in that it provides a general framework that can be used to derive the optimal decay rates of the solution as well as its all-order spatial derivatives from one order to the highest order, which are the same as those of the heat equation. More specifically, under the assumptions...
that the initial data belongs to $H^N$ with any integer $N \geq 2$. $H^2$-norm of the initial data is sufficiently small and $L^1$-norm of the initial data is bounded, but the higher-order norms can be arbitrarily large, our approach shows that the optimal decay rates of the solution as well as its all-order spatial derivatives from one order to the highest order ($N$ order) are the same as those of the heat equation. Particularly, by taking $N = 2$ in Theorem 1.1, it is clear that main theorem of Evje et al. [21] is a direct corollary of Theorem 1.1. Moreover, it is clear that in Equation (1.34), the second-order (the highest order) spatial derivative of the solution decays at the $L^2$-rate $\left(1 + t\right)^{-\frac{7}{4}}$, which is faster than the $L^2$-rate $\left(1 + t\right)^{-\frac{5}{4}}$ in Equation (1.23). On the other hand, for the general case $N > 2$, under the assumption that $H^2$-norm of the initial data is small but its higher-order norm can be arbitrarily large, we obtain the optimal decay rates of the solution as well as its all-order spatial derivatives from one order to the highest order ($N$ order), which are the same as those of the heat equation. Finally, for well-chosen initial data, we also show the lower bounds on the decay rates. It is worth mentioning that it seems impossible to obtain the lower bounds on the decay rates by following the method of Evje et al. [21].

Remark 1.3. It is interesting to make a comparison between Theorem 1.1 and those of Hoff and Zumbrun [29], where the authors derived the decay rates of the solutions for the compressible Navier–Stokes system ($\alpha^+ = 0$ in Equation 1.1). Their main results can be listed as follows:

Let $U_0 = (\rho_0 - 1, m_0)$ with $m_0 = \rho_0 u_0$ and $l \geq 3$ be an integer, if $E_0 = ||U_0||_{L^1} + ||U_0||_{H^{l+1}}$ is sufficiently small, the authors in Ref. [29] proved that the compressible Navier–Stokes equations has a small smooth solutions satisfying the following decay rate with $k \leq (l - 3)/2$:

$$
\|\nabla^k U\|_{L^p} \leq C(1 + t)^{-n/(1 - 1/p) - k/2}, \text{ for } 2 \leq p \leq \infty.
$$

(1.38)

Taking $n = 3$ in Equation (1.38), the main results of the Ref. [29] imply that if the $L^1$-norm and $H^{l+1}$-norm of the initial data are sufficiently small, the small smooth solution has the following decay rate with $k \leq (l - 3)/2$:

$$
\|\nabla^k U\|_{L^p} \leq C(1 + t)^{-3/(1 - 1/p) - k/2}, \text{ for } 2 \leq p \leq \infty,
$$

(1.39)

which is a direct consequences of the decay rate (1.34) by noticing the simple fact that

$$
\|\nabla^k U\|_{L^p} \lesssim \|\nabla^k U\|_{L^p}^{2/p} \|\nabla^k U\|_{L^\infty}^{1-2/p} \lesssim \|\nabla^k U\|_{L^p}^{2/p} (\|\nabla^{k+1} U\|_{L^2} \|\nabla^{k+2} U\|_{L^2})^{1 - 1/p}.
$$

(1.40)

It should be mentioned that we only need the smallness of $H^2$-norm of the initial data, but its higher-order Sobolev norm and $L^1$-norm can be arbitrarily large. Moreover, we also give the optimal decay rates for the higher-order spatial derivatives ($k > (l - 3)/2$) of the solutions. Finally, for well-chosen initial data, we also get the lower bound on the decay rates. Therefore, our results can be regarded as a generalization of the results in Hoff and Zumbrun [29] from compressible Navier–Stokes equations (single-phase gas model) to the nonconservative two-phase gas–liquid model.

Remark 1.4. In Refs. [3, 6], $\Delta P = P^+ - P^- = 0$ and a third-order derivative of $\alpha^+ \rho^+$ accounting for internal capillary pressure forces are involved. In Evje et al. [21], they considered the capillary pressure $\Delta P = P^+ - P^- = f(\alpha^- \rho^-)$, where $f$ satisfies the assumption (1.20). It should be mentioned that both in Refs. [3, 6] and in Ref. [21], the capillary term plays a key role in the proofs of their main results. Therefore, a natural and important problem is that what will happen if $\Delta P = P^+ - P^- = 0$ and no capillary term is involved. In fact, in this case, the linear system of the model has zero eigenvalue, which makes the problem become much more difficult and complicated. This will be our future work.

Remark 1.5. Employing the pure energy method developed in Refs. [25, 42], one can obtain the optimal decay rates but except the highest-order one and low bounds on the optimal decay rates. However, it seems impossible to get the optimal decay rate of the highest-order spatial derivative of the solution and the low bounds on the decay rates due to the strong “degenerate” and “nonlinear” structure of the system.

Now, let us sketch the strategy of proving Theorem 1.1 and explain some of the main difficulties and techniques involved in the process. As mentioned before, the main purpose of the present paper is to put forward a general framework to derive the optimal decay rates of the solution as well as its all-order spatial derivatives from one order to the highest order, which are the same as those of the heat equation. Therefore, compared to Evje et al. [21], we need to develop new ingredients in the proof to handle the optimal decay rate of the highest-order spatial derivative of the solution and the lower bound on the
optimal decay rate of the solution, which requires some new ideas. More precisely, we will employ Hodge decomposition, the low- and high-frequency decomposition, delicate spectral analysis, and energy methods based on finite induction. Roughly speaking, our proof mainly involves the following five steps.

First, we rewrite the system (1.17)–(1.18) in perturbation form and analyze the spectral of the solution semigroup to the corresponding linear system. We, therefore, encounter a fundamental obstacle that the matrix \( \mathcal{A}(\xi) \) in Equation (2.22) is an eight-order matrix and is not self-adjoint. Particularly, it is easy to check that the matrix \( \mathcal{A}(\xi) \) cannot be diagonalizable (see Sideris et al. [41] pp. 807 for example). Therefore, it seems impossible to apply the usual time decay investigation through spectral analysis. To overcome this difficulty, we will employ the Hodge decomposition technique developed in Refs. [5, 7, 8] to split the linear system into three systems. One has four distinct eigenvalues and the other two are classic heat equations. By making careful pointwise estimates on the Fourier transform of Green’s function to the linearized equations, we can obtain the desired linear \( L^2 \)-estimates.

Second, we deduce the optimal convergence rate on \( \| \nabla^j (n^+, u^+, n^-, u^-) \|_{L^2} \) with \( 0 \leq j \leq N \). Our method is to use Duhamel’s principle, linear \( L^2 \)-estimates, nonlinear energy estimates. However, we encounter a difficulty from the fact that when we estimate the highest-order term \( \| \nabla^N (n^+, u^+, n^-, u^-) \|_{L^2} \), it requires us to control the terms involving \( \nabla^N (u^+, u^-) \), which, however, do not belong to the solution space. To get around this difficulty, we separate the time interval into two parts and make full use of the benefit of the low- and high-frequency decomposition to get our desired convergence rates (see the proof of Equation 3.71 for details).

Third, we prove the optimal decay rates for Case I: \( N = 2 \). By virtue of Equations (1.22) and (1.23), it suffices to prove Equation (3.3). To illustrate the main idea of our approach, one may consider the linear system of Equation (2.15) and note that the corresponding linear energy equality reads as: for \( j = 0, 1, 2 \),

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \beta_1 (|\nabla n^+|^2 + |\nabla u^+|^2) + \beta_4 (|\nabla n^-|^2 + |\nabla u^-|^2) dx \\
+ \int_{\mathbb{R}^3} \frac{\beta_1}{\beta_2} \left( v_1^+ |\nabla^j u^+|^2 + v_2^+ |\nabla^j \text{div } u^+|^2 \right) + \frac{\beta_4}{\beta_3} \left( v_1^- |\nabla^j u^-|^2 + v_2^- |\nabla^j \text{div } u^-|^2 \right) dx \\
+ \left\langle \nabla \nabla n^-, \beta_1 \nabla u^+ \right\rangle + \left\langle \nabla \nabla n^+, \beta_2 \nabla u^- \right\rangle = 0.
\]

Notice that the energy equality (1.41) only gives the dissipative estimates for \( u^+ \) and \( u^- \). In order to explore the dissipative estimates for the variables \( n^+ \) and \( n^- \), one may follow the method of Evje et al. [21] by constructing the interactive energy functionals between \( u^+ \) and \( n^+ \), and \( u^- \) and \( n^- \), respectively to get for \( j = 0, 1 \)

\[
\frac{d}{dt} \left\{ \frac{v_1^+ + v_2^+}{2\beta_1\beta_2} \int_{\mathbb{R}^3} |\nabla^j n^+|^2 dx + \left\langle \nabla u^+, \frac{1}{\beta_2} \nabla \nabla n^+ \right\rangle \right\} \\
+ \frac{\beta_1}{\beta_2} \int_{\mathbb{R}^3} |\nabla n^+|^2 dx + \left\langle \nabla n^+, \nabla \nabla n^+ \right\rangle \leq C \left( \frac{\beta_1}{\beta_2} \| \nabla \text{div } u^+ \|_{L^2}^2 + \delta_0 \left( \| \nabla \nabla (n^+, n^-) \|_{L^2}^2 + \| \nabla u^+ \|_{H^1}^2 \right) \right),
\]

\[
\frac{d}{dt} \left\{ \frac{v_1^- + v_2^-}{2\beta_3\beta_4} \int_{\mathbb{R}^3} |\nabla^j n^-|^2 dx + \left\langle \nabla u^-, \frac{1}{\beta_4} \nabla \nabla n^- \right\rangle \right\} \\
+ \frac{\beta_4}{\beta_3} \int_{\mathbb{R}^3} |\nabla n^-|^2 dx + \left\langle \nabla n^+, \nabla \nabla n^- \right\rangle \leq C \left( \frac{\beta_4}{\beta_3} \| \nabla \text{div } u^- \|_{L^2}^2 + \delta_0 \left( \| \nabla \nabla (n^+, n^-) \|_{L^2}^2 + \| \nabla u^- \|_{H^1}^2 \right) \right).
\]

This means that to obtain the dissipative estimates for \( \nabla^2 (n^+, n^-) \), one needs to do the energy estimate (1.41) at both the first and second levels. Consequently, one can only show that the convergence rate of the second-order (highest order) spatial derivative of solution, that is, \( \| \nabla^2 (n^+, u^+, n^-, u^-) \|_{L^2} \) is the same as that of \( \| \nabla (n^+, u^+, n^-, u^-) \|_{L^2} \). In particular, this implies that the convergence rate of the highest-order spatial derivative of the solution is not optimal. To tackle with this difficulty, we will make full use of the benefit of the low- and high-frequency decomposition and the key linear
More precisely, instead of using Equations (1.42)–(1.43), we will employ the new interactive energy functionals between \( u^+, h \) and \( n^+, h \), and \( n^-, h \) and \( u^-, h \), respectively to obtain
\[
\frac{d}{dt} \left\{ \frac{\nu^+ + \nu^-}{2 \beta_1 \beta_2} \left\| \nabla^2 n^+, h \right\|_{L^2}^2 + \left\langle \nabla u^+, h, \frac{1}{\beta_2} \nabla^2 n^+, h \right\rangle \right\} + \frac{\beta_1}{\beta_2} \left\| \nabla^2 n^+, h \right\|_{L^2}^2 + \left\langle \nabla^2 n^-, h, \nabla^2 n^+, h \right\rangle \\
\leq \left\| \nabla \text{div} u^+ \right\|_{L^2}^2 + \delta_0 \left( \left\| \nabla^2 (n^+, n^-) \right\|_{L^2}^2 + \left\| \nabla^2 u^+ \right\|_{H^1}^2 \right),
\]

(1.44)

Next, we choose three sufficiently small positive constants \( C_1, C_3, \) and \( C_4 \), and define the temporal energy functional
\[
\mathcal{E}(t) = \frac{\beta_1}{2 \beta_2} \left\| \nabla^2 n^+ \right\|_{L^2}^2 + \left( \frac{\nu^+ + \nu^-}{2 \beta_1 \beta_2} \right) \left\| \nabla^2 n^+, h \right\|_{L^2}^2 + \frac{\beta_1}{2 \beta_2} \left\| \nabla^2 u^+ \right\|_{L^2}^2 + \frac{C_1}{\beta_2} \left\langle \nabla u^+, h, \nabla^2 n^+ \right\rangle \\
+ \frac{C_3 \beta_4}{2 \beta_3} \left\| \nabla^2 n^- \right\|_{L^2}^2 + \left( \frac{\nu^+ + \nu^-}{2 \beta_1 \beta_2} \right) \left\| \nabla^2 n^-, h \right\|_{L^2}^2 + \frac{C_4 \beta_4}{2 \beta_3} \left\| \nabla^2 u^- \right\|_{L^2}^2 + \frac{C_4}{\beta_3} \left\langle \nabla u^-, h, \nabla^2 n^- \right\rangle,
\]

(1.46)

which is equivalent to \( \left\| \nabla^2 (n^+, u^+, n^-, u^-) \right\|_{L^2}^2 \). Taking \( j = 2 \) in Equation (1.41) and combining Equations (1.44)–(1.45), we deduce that
\[
\frac{d}{dt} \mathcal{E}(t) + C \mathcal{E}(t) \leq \left\| \nabla^2 (n^{+,i}, u^{+,i}, n^{-,i}, u^{-,i})(t) \right\|_{L^2}^2.
\]

(1.47)

Consequently, by combining Equation (1.47) with the optimal convergence rate of \( \left\| \nabla^2 (n^{+,i}, u^{+,i}, n^{-,i}, u^{-,i}) \right\|_{L^2} \) obtained in Step 2, we can prove Equation (3.3) and thus complete the proof of the optimal decay rates for Case I: \( N = 2 \). However, for the nonlinear problem (2.15), it is much more complicated due to the nonlinear estimates. Compared to Evje et al. [21], our nonlinear energy estimate is new and different. This is another point of this article. Indeed, in Lemmas 3.2 and 3.3 of Ref. [21], the terms \( \left\| \nabla u^\pm \right\|_{L^2} \) are involved in the right-hand side of energy inequality. Therefore, we can not follow the method of Evje et al. [21] directly to close the energy estimate of \( \nabla^2 (n^+, u^+, n^-, u^-) \). To get around this problem, one main observation of this paper is to take full advantage of the smallness of \( \left\| \nabla u^\pm \right\|_{L^\infty} \). More precisely, by Sobolev inequality, one has
\[
\left\| \nabla u^\pm \right\|_{L^\infty} \lesssim \left\| \nabla^2 u^\pm \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^3 u^\pm \right\|_{L^2}^{\frac{1}{2}}.
\]

(1.48)

Noticing Equation (3.2) and the fact that the term involving \( \nabla^3 u^\pm \) can be absorbed by the left-hand side of energy inequality, it is clear that the term \( \left\| \nabla u^\pm(t) \right\|_{L^\infty} \) can provide a smallness of the order \( \delta_0^\frac{3}{2} \). In view of this key observation, we can close our desired energy estimates (see the proofs of Equations (3.12)–(3.13) and Equations (3.27)–(3.28) for details).

Fourth, we prove the optimal decay rates for Case II: \( N > 2 \). The main novelty of the present paper in this step is to apply finite mathematical induction to close our energy estimates. More specifically, for any \( 0 \leq \ell \leq N \), we define the time-weighted energy functional as
\[
\mathcal{E}_\ell^N(t) = \sup_{0 \leq \tau \leq t} \left\{ (1 + \tau) \frac{3 + \ell}{2} \left\| \nabla^\ell (n^+, u^+, n^-, u^-) \right\|_{H^{N-\ell}} \right\}. \tag{1.49}
\]
Therefore, it suffices to show that for any $0 \leq \ell \leq N$, $\phi_\ell(t)$ has an uniform time-independent bound. We will employ mathematical induction to achieve this goal (see the proofs of Lemmas 3.4 and 3.5). We remark that the benefit of the low- and high-frequency decomposition and the key linear $L^2$-estimates obtained in Step 2 play a crucial role in the process. Moreover, compared to Case I: $N = 2$, the energy estimates are much complicated and subtle. For example, as in Case I: $N = 2$, we also need the smallness of the term $\|\nabla u^\pm\|_{L^\infty}$. However, we can not follow the method used in Case I: $N = 2$ since the term involving $V^3 u^\pm$ can not be absorbed by the left-hand side of the energy inequality. Instead, we will apply Equation (3.2) with $N = 3$ and Equation (3.3) to get

$$\|\nabla u^\pm\|_{L^\infty} \lesssim \|\nabla^2 u^\pm\|_{L^2}^{1/2} \|\nabla^3 u^\pm\|_{L^2}^{1/2} \lesssim (1 + t)^{-7/8},$$

which implies that the term $\|\nabla u^\pm\|_{L^\infty}$ can provide a smallness of the order $(1 + t)^{-7/8}$ if $t$ large enough. We remark that the smallness of $\|\nabla u^\pm\|_{L^\infty}$ plays a crucial role in closure of our energy estimates.

In the last step, we show the lower bounds on the convergence rates of solutions. To do this, we will employ Plancherel theorem and careful analysis on the solution semigroup. First, we derive the convergence rate in $H^{-1}$. Then, we can prove the lower bound on the convergence rates by an interpolation trick.

The rest of this paper is organized as follows. In the next section, we first rewrite the Cauchy problem (1.17)–(1.18). Then, we take Hodge decomposition to the corresponding linear system, and get desired linear estimates by making careful spectral analysis on the linear system. In Section 3, we first prove the optimal convergence rates of the solutions for Case I: $N = 2$. Then, we deal with Case II: $N > 2$ by induction, and then show the lower bounds on the convergence rates by an interpolation trick.

# Spectral Analysis and Linear $L^2$-Estimates

## Reformulation

In this subsection, we first reformulate the system. Setting

$$n^\pm = R^\pm - 1,$$

system (1.17) can be rewritten as

$$\begin{aligned}
\partial_t n^+ + \text{div } u^+ &= F_1, \\
\partial_t u^+ + \alpha_1 \nabla n^+ + \alpha_2 \nabla n^- - \nu_1^+ \Delta u^+ - \nu_2^+ \text{div } u^+ &= F_2, \\
\partial_t n^- + \text{div } u^- &= F_3, \\
\partial_t u^- + \alpha_3 \nabla n^+ + \alpha_4 \nabla n^- - \nu_1^- \Delta u^- - \nu_2^- \text{div } u^- &= F_4,
\end{aligned}$$

where $\nu_1^\pm = \frac{\mu^\pm}{\rho^\pm(1,1)}$, $\nu_2^\pm = \frac{\mu^\pm + \lambda^\pm}{\rho^\pm(1,1)} > 0$, $\alpha_1 = \frac{\nu^2(1,1)\rho^-(1,1)}{\rho^+(1,1)}$, $\alpha_2 = \nu^2(1,1) + \frac{\nu^2(1,1)\rho^-(1,1)}{s^2(1,1)}$, $\alpha_3 = \nu^2(1,1)$, $\alpha_4 = \nu^2(1,1)$, and the nonlinear terms are given by

$$F_1 = - \text{div } (n^+ u^+),$$

$$F_2^i = - g_+ (n^+, n^-) \partial_i n^+ - g_+ (n^+, n^-) \partial_i n^- - (u^+ \cdot \nabla) u^+_i,$$

$$+ \mu^+ h_+ (n^+, n^-) \partial_j n^+ \partial_j u^+_i + \mu^+ k_+ (n^+, n^-) \partial_j n^- \partial_j u^+_i,$$

$$+ \mu^+ h_+ (n^+, n^-) \partial_j n^+ \partial_j u^+_j + \mu^+ k_+ (n^+, n^-) \partial_j n^- \partial_j u^+_j$$

$$+ \lambda^+ h_+ (n^+, n^-) \partial_j n^+ \partial_j u^+_j + \lambda^+ k_+ (n^+, n^-) \partial_j n^- \partial_j u^+_j,$$

$$+ \mu^+ l_+ (n^+, n^-) \partial^2 u^+_i + (\mu^+ + \lambda^+) l_+ (n^+, n^-) \partial_i \partial_j u^+_j,$$

$$F_3 = - \text{div } (n^- u^-),$$

$$F_4 = - g_+ (n^+, n^-) \partial_i n^- - g_+ (n^+, n^-) \partial_i n^+ - (u^- \cdot \nabla) u^-_i.$$
\[ F_4^i = - \mathcal{G}(n^+, n^-) \partial n^- - \mathcal{G}(n^+, n^-) \partial n^+ - (u^- \cdot \nabla)u_i^- \]
\[ + \mu^- h_-(n^+, n^-) \partial n^+ \partial u_i^- + \mu^- k_-(n^+, n^-) \partial n^- \partial u_i^- \]
\[ + \lambda^- h_-(n^+, n^-) \partial n^+ \partial u_j^- + \lambda^- k_-(n^+, n^-) \partial n^- \partial u_j^- \]
\[ + \mu^- \lambda_-(n^+, n^-) \partial_n^\mu u_i^- + (\mu^- + \lambda^-) \partial_n^\mu u_j^- \]
\[ + \mu^- \lambda_-(n^+, n^-) \partial_j u_i^- + (\mu^- + \lambda^-) \partial_j u_j^- , \]
\[ \text{where} \]
\[ \begin{aligned}
 g_+(n^+, n^-) &= \frac{\varepsilon^2 \rho^-(n^+, n^-)}{\rho^+(n^+, n^-)} - \frac{\varepsilon^2 \rho^-(1,1)}{\rho^+(1,1)} \\
 g_-(n^+, n^-) &= \frac{\varepsilon^2 \rho^+(n^+, n^-)}{\rho^-(n^+, n^-)} - \frac{\varepsilon^2 \rho^+(1,1)}{\rho^-(1,1)} - f'(n^-) \frac{\varepsilon^2 (\rho^+)}{s^2(n^+, n^-)} \\
 \bar{g}^- &= \frac{\varepsilon^2 (\rho^+)}{s^2(n^+, n^-)} \\
 \bar{g}^+ &= \frac{\varepsilon^2 (\rho^-)}{s^2(n^+, n^-)} \\
 h_+(n^+, n^-) &= \frac{\varepsilon^2 (\rho^-)}{(n^+, n^-)} \\
 h_-(n^+, n^-) &= -\frac{\varepsilon^2 (\rho^+)}{(n^+, n^-)} \\
 k_+(n^+, n^-) &= -\frac{\varepsilon^2 (\rho^-)}{(n^+, n^-)} + \frac{f'(n^-) \varepsilon^2 (\rho^+)}{(n^+, n^-)} \\
 k_-(n^+, n^-) &= -\frac{\varepsilon^2 (\rho^-)}{(n^+, n^-)} + \frac{f'(n^-) \varepsilon^2 (\rho^+)}{(n^+, n^-)} \\
 l_\pm(n^+, n^-) &= \frac{1}{\rho_\pm(n^+, n^-)} - 1 \frac{1}{\rho_\pm(1,1)} \\
 \end{aligned} \]

Taking change of variables
\[ n^+ \to \alpha_1 n^+, \ u^+ \to \sqrt{\alpha_1} u^+, \ n^- \to \alpha_4 n^-, \ u^- \to \sqrt{\alpha_4} u^-, \]

and setting
\[ \beta_1 = \sqrt{\alpha_1}, \ \beta_2 = \frac{\alpha_2 \sqrt{\alpha_1}}{\alpha_4}, \ \beta_3 = \frac{\alpha_3 \sqrt{\alpha_4}}{\alpha_1}, \ \beta_4 = \sqrt{\alpha_4} \]

and
\[ \beta_1 = \sqrt{\beta_1}, \ \beta_2 = \frac{\beta_1}{\beta_3}, \ \beta_4 = \sqrt{\beta_4} \]

the Cauchy problem (2.2) and (1.18) can be reformulated as
\[ \begin{cases}
 \partial_t u^+ + \beta_1 \partial_1 u^+ = \mathcal{F}_1, \\
 \partial_t u^+ + \beta_1 \nabla v^+ + \beta_2 v n^- - v_1^+ \Delta u^+ - v_2^+ \nabla u^+ = \mathcal{F}_2, \\
 \partial_t n^+ + \beta_4 \partial_4 u^- = \mathcal{F}_3, \\
 \partial_t u^- + \beta_3 \partial_3 v^+ + \beta_4 \nabla n^- - v_1^- \Delta u^- - v_2^- \nabla u^- = \mathcal{F}_4,
\end{cases} \]
with the initial data
\[
(n^+, u^+, n^-, u^-)(x, 0) = (n_0^+, u_0^+, n_0^-, u_0^-)(x) \to (0, 0, 0, 0), \quad \text{as } |x| \to +\infty,
\]
where the nonlinear terms are given by
\[
\mathcal{F}_1 = \alpha_1 F_1 \left( n^+ \alpha_1, u^+ \sqrt{\alpha_1} \right), \quad \mathcal{F}_2 = \sqrt{\alpha_1} F_2 \left( n^+ \alpha_1, u^+ \sqrt{\alpha_1}, n^- \alpha_4, u^- \sqrt{\alpha_4} \right),
\]
and
\[
\mathcal{F}_3 = \alpha_4 F_3 \left( n^- \alpha_4, u^- \sqrt{\alpha_4} \right), \quad \mathcal{F}_4 = \sqrt{\alpha_4} F_4 \left( n^+ \alpha_1, u^+ \sqrt{\alpha_1}, n^- \alpha_4, u^- \sqrt{\alpha_4} \right).
\]
Noticing that
\[
\beta_1 \beta_4 - \beta_2 \beta_3 = -\frac{\epsilon^2(1, 1)f'(1)}{\sqrt{\alpha_1 \alpha_4 \rho^+(1, 1)}} > 0,
\]
it is clear that \(\beta^+ \beta^- > 1\). It should be mentioned that the relation (2.19) is possible, since the representation of capillary pressure includes \(f \neq 0\), which is a strictly decreasing function near 1. We remark that the relation (2.19) plays a fundamental role in the proof of Theorem 1.1.

Define \(\bar{U} = (\bar{n}^+, \bar{u}^+, \bar{n}^-, \bar{u}^-)'\). In terms of the semigroup theory for evolutionary equation, we will investigate the following initial value problem for the corresponding linearized system of Equation (2.15):
\[
\begin{cases}
\bar{U}_t = \mathcal{B} \bar{U}, \\
\bar{U}|_{t=0} = \bar{U}_0,
\end{cases}
\]
where the operator \(\mathcal{B}\) is given by
\[
\mathcal{B} = \begin{pmatrix}
0 & -\beta_1 \text{div} & 0 & 0 \\
-\beta_1 \nabla & \nu_1^+ \Delta + \nu_1^+ \nabla \otimes \nabla & -\beta_2 \nabla & 0 \\
0 & 0 & 0 & -\beta_4 \text{div} \\
-\beta_3 \nabla & 0 & -\beta_4 \nabla & \nu_1^- \Delta + \nu_2^- \nabla \otimes \nabla
\end{pmatrix}.
\]
Applying the Fourier transform to the system (2.20), one has
\[
\begin{cases}
\hat{\bar{U}}_t = \mathcal{A}(\xi) \hat{\bar{U}}, \\
\hat{\bar{U}}|_{t=0} = \hat{\bar{U}}_0,
\end{cases}
\]
where \(\hat{\bar{U}}(\xi, t) = \mathcal{F}(\bar{U}(x, t))\), \(\xi = (\xi^1, \xi^2, \xi^3)'\), and \(\mathcal{A}(\xi)\) is defined by
\[
\mathcal{A}(\xi) = \begin{pmatrix}
0 & -i\beta_1 \xi^i & 0 & 0 \\
-i\beta_1 \xi & -\nu_1^+ |\xi|^2 I_{3\times 3} - \nu_1^+ \xi \otimes \xi & -i\beta_2 \xi & 0 \\
0 & 0 & 0 & -i\beta_4 \xi \\
-i\beta_3 \xi & 0 & -i\beta_4 \xi & -\nu_1^- |\xi|^2 I_{3\times 3} - \nu_2^- \xi \otimes \xi
\end{pmatrix}.
\]
To derive the linear time-decay estimates, by using a real method as in Refs. [32, 35], one need to make a detailed analysis on the properties of the semigroup. Unfortunately, it seems untractable, since the system (2.22) has eight equations and the matrix \(\mathcal{A}(\xi)\) cannot be diagonalizable (see Sideris et al. [41] pp. 807 for example). To tackle this issue, we employ the Hodge decomposition of the system (2.20) such that the system (2.20) can be decoupled into three systems. One has four equations whose characteristic polynomial possesses four distinct roots, and the other two are classic heat equations. This key observation allows us to derive the optimal linear convergence rates.
Let $\varphi^\pm = \Lambda^{-1} \text{div} \hat{u}^\pm$ be the “compressible part” of the velocities $\hat{u}^\pm$, and denote $\phi^\pm = \Lambda^{-1} \text{curl} \hat{u}^\pm$ (with \((\text{curl} z)^j_i = \partial x_j z^i - \partial x_i z^j\)) by the “incompressible part” of the velocities $\hat{u}^\pm$. Then, we can rewrite the system (2.20) as follows:

$$
\begin{align*}
\frac{\partial}{\partial t} \hat{n}^+ + \beta_1 \Lambda \varphi^+ &= 0, \\
\frac{\partial}{\partial t} \varphi^+ - \beta_1 \Lambda \hat{n}^+ - \beta_2 \Lambda \hat{n}^- + \nu^+ \Lambda^2 \varphi^+ &= 0, \\
\frac{\partial}{\partial t} \hat{n}^- + \beta_4 \Lambda \varphi^- &= 0, \\
\frac{\partial}{\partial t} \varphi^- - \beta_2 \Lambda \hat{n}^+ - \beta_4 \Lambda \hat{n}^- + \nu^- \Lambda^2 \varphi^- &= 0,
\end{align*}
$$

(2.24) and

$$
\begin{align*}
\frac{\partial}{\partial t} \phi^+ + \nu^+ \Lambda^2 \phi^+ &= 0, \\
\frac{\partial}{\partial t} \phi^- + \nu^- \Lambda^2 \phi^- &= 0,
\end{align*}
$$

(2.25)

where $\nu^\pm = \nu_1^\pm + \nu_2^\pm$.

### 2.2 Spectral analysis for IVP (2.24)

In terms of the semigroup theory, we may represent the IVP (2.24) for $\mathcal{U} = (\hat{n}^+, \varphi^+, \hat{n}^-, \varphi^-)^t$ as

$$
\begin{align*}
\mathcal{U}_t &= \mathcal{B}_1 \mathcal{U}, \\
\mathcal{U}|_{t=0} &= \mathcal{U}_0,
\end{align*}
$$

(2.26)

where the operator $\mathcal{B}_1$ is defined by

$$
\mathcal{B}_1 = \begin{pmatrix}
0 & -\beta_1 \Lambda & 0 & 0 \\
\beta_1 \Lambda & -\nu^+ \Lambda^2 & \beta_2 \Lambda & 0 \\
0 & 0 & 0 & -\beta_4 \Lambda \\
\beta_3 \Lambda & 0 & \beta_4 \Lambda & -\nu^- \Lambda^2
\end{pmatrix}
$$

(2.27)

Taking the Fourier transform to the system (2.26), we obtain

$$
\begin{align*}
\hat{\mathcal{U}}_t &= \mathcal{A}_1(\xi) \hat{\mathcal{U}}, \\
\hat{\mathcal{U}}|_{t=0} &= \hat{\mathcal{U}}_0,
\end{align*}
$$

(2.28)

where $\hat{\mathcal{U}}(\xi, t) = \mathcal{F}(\mathcal{U}(x, t))$ and $\mathcal{A}_1(\xi)$ is given by

$$
\mathcal{A}_1(\xi) = \begin{pmatrix}
0 & -\beta_1 |\xi| & 0 & 0 \\
\beta_1 |\xi| & -\nu^+ |\xi|^2 & \beta_2 |\xi| & 0 \\
0 & 0 & 0 & -\beta_4 |\xi| \\
\beta_3 |\xi| & 0 & \beta_4 |\xi| & -\nu^- |\xi|^2
\end{pmatrix}
$$

(2.29)

We compute the eigenvalues of the matrix $\mathcal{A}_1(\xi)$ from the determinant

$$
\begin{align*}
\det(\lambda I - \mathcal{A}_1(\xi)) &= \lambda^4 + (\nu^+ |\xi|^2 + \nu^- |\xi|^2)\lambda^3 + (\beta_2^2 |\xi|^2 + \beta_4^2 |\xi|^2 + \nu^+ \nu^- |\xi|^4)\lambda^2 + (\nu^+ \beta_2^2 |\xi|^4 + \nu^- \beta_4^2 |\xi|^4)\lambda \\
&\quad + \beta_3^2 \beta_4^2 |\xi|^4 - \beta_1 \beta_2 \beta_3 \beta_4 |\xi|^4 \\
&= 0,
\end{align*}
$$

(2.30)
which implies that the matrix $\mathcal{A}(\xi)$ possesses four different eigenvalues:

$$\lambda_1 = \lambda_1(|\xi|), \quad \lambda_2 = \lambda_2(|\xi|), \quad \lambda_3 = \lambda_3(|\xi|), \quad \lambda_4 = \lambda_4(|\xi|).$$  

Consequently, the semigroup $e^{t \mathcal{A}(\xi)}$ can be decomposed into

$$e^{t \mathcal{A}(\xi)} = \sum_{i=1}^{4} e^{\lambda_i t} P_i(\xi),$$  

where the projector $P_i(\xi)$ is defined by

$$P_i(\xi) = \prod_{j \neq i} \frac{\mathcal{A}(\xi) - \lambda_j I}{\lambda_i - \lambda_j}, \quad i, j = 1, 2, 3, 4.$$  

Thus, the solution of IVP (2.28) can be expressed as

$$\tilde{U}(\xi, t) = e^{t \mathcal{A}(\xi)} \tilde{U}_0(\xi) = \left(\sum_{i=1}^{4} e^{\lambda_i t} P_i(\xi)\right) \tilde{U}_0(\xi).$$  

To derive long time properties of the semigroup $e^{t \mathcal{A}(\xi)}$ in $L^2$-framework, one need to analyze the asymptotical expansions of $\lambda_i$, $P_i$ ($i = 1, 2, 3, 4$) and $e^{t \mathcal{A}(\xi)}$ in the low-frequency part. Employing the similar argument of Taylor series expansion as in Refs. [32, 35], we have the following lemma from tedious calculations.

**Lemma 2.1.** There exists a positive constants $\eta_1 \ll 1$ such that, for $|\xi| \leq \eta_1$, the spectral has the following Taylor series expansion:

$$\begin{align*}
\lambda_1 &= \left[-\frac{v^+ + v^-}{4} - \frac{v^+(\beta_1^2 - \beta_2^2) + v^-(\beta_3^2 - \beta_4^2)}{8\kappa_1}\right]|\xi|^2 + \sqrt{\kappa_2 - \kappa_1}|\xi| + O(|\xi|^3), \\
\lambda_2 &= \left[-\frac{v^+ + v^-}{4} - \frac{v^+(\beta_1^2 - \beta_2^2) + v^-(\beta_3^2 - \beta_4^2)}{8\kappa_1}\right]|\xi|^2 - \sqrt{\kappa_2 - \kappa_1}|\xi| + O(|\xi|^3), \\
\lambda_3 &= \left[-\frac{v^+ + v^-}{4} + \frac{v^+(\beta_1^2 - \beta_2^2) + v^-(\beta_3^2 - \beta_4^2)}{8\kappa_1}\right]|\xi|^2 + \sqrt{\kappa_2 + \kappa_1}|\xi| + O(|\xi|^3), \\
\lambda_4 &= \left[-\frac{v^+ + v^-}{4} + \frac{v^+(\beta_1^2 - \beta_2^2) + v^-(\beta_3^2 - \beta_4^2)}{8\kappa_1}\right]|\xi|^2 - \sqrt{\kappa_2 + \kappa_1}|\xi| + O(|\xi|^3),
\end{align*}$$  

where $\kappa_1 = \sqrt{\frac{\beta_2^2 - \beta_1^2}{4}} + \beta_1\beta_2\beta_3\beta_4$ and $\kappa_2 = \frac{\beta_2^2 + \beta_4^2}{2}$.

By virtue of Equations (2.33)–(2.35), we can establish the following estimates for the low-frequency part of the solutions $\tilde{U}(t, \xi)$ to the IVP (2.28):

**Lemma 2.2.** Let $\bar{\nu}_1 = \frac{v^+ + v^-}{4} - \frac{v^+(\beta_1^2 - \beta_2^2) + v^-(\beta_3^2 - \beta_4^2)}{8\kappa_1} > 0$, we have

$$|\tilde{n}^+|, |\tilde{\varphi}^+|, |\tilde{n}^-|, |\tilde{\varphi}^-| \leq e^{-\bar{\nu}_1|\xi|^2}(|\tilde{n}_0^+| + |\tilde{\varphi}_0^+| + |\tilde{n}_0^-| + |\tilde{\varphi}_0^-|),$$  

for any $|\xi| \leq \eta_1$.

**Proof.** By virtue of formula (2.34) and Taylor series expansion of $\lambda_i$ ($1 \leq i \leq 4$) in Equation (2.35), we can represent $P_i$ ($1 \leq i \leq 4$) as follows:
DuetoEquation (2.36)andPlancherel theorem, we have

\begin{align*}
P_1(\xi) &= \begin{pmatrix}
\frac{2x + \beta^2_1 - \beta^2_2}{8k_1} & \frac{\beta_3(2x + \beta^2_1 - \beta^2_2)}{8k_1 \sqrt{k_2 - k_1}} & \frac{\beta_2x}{4k_1} & \frac{\beta_3}{4k_1} \\
\frac{\beta_1(\beta_1^2 - 2x) + 2\beta_2\beta_3}{8k_1 \sqrt{k_2 - k_1}} & \frac{-\beta_1x}{4k_1} & \frac{\beta_1}{4k_1} & \frac{-\beta_2\beta_3}{4k_1} \\
\frac{\beta_3}{4k_1} & \frac{\beta_2x}{8k_1} & \frac{\beta_1^2 - \beta^2_1 - 2x}{8k_1 \sqrt{k_2 - k_1}} & \frac{\beta_1x}{8k_1} \\
\frac{\beta_1\sqrt{k_2 - k_1}}{4k_1} & \frac{-\beta_1x}{8k_1} & \frac{\beta_2x}{8k_1} & \frac{\beta_1^2 - \beta^2_1}{8k_1 \sqrt{k_2 - k_1}} \end{pmatrix} + o(\|\xi\|),
\end{align*}

(2.37)

\begin{align*}
P_2(\xi) &= \begin{pmatrix}
\frac{2x + \beta^2_1 - \beta^2_2}{8k_1} & \frac{-\beta_1x}{8k_1 \sqrt{k_2 - k_1}} & \frac{-\beta_1^2 x}{8k_1} & \frac{\beta_2\beta_3}{8k_1} \\
\frac{\beta_1(\beta_1^2 - 2x) + 2\beta_2\beta_3}{8k_1 \sqrt{k_2 - k_1}} & \frac{-\beta_1^2 x}{4k_1} & \frac{\beta_1x}{4k_1} & \frac{-\beta_2\beta_3}{4k_1} \\
\frac{-\beta_1x}{8k_1} & \frac{\beta_1^2 - \beta^2_1 - 2x}{8k_1 \sqrt{k_2 - k_1}} & \frac{\beta_1^2}{8k_1} & \frac{-\beta_2\beta_3}{8k_1} \\
\frac{-\beta_1\sqrt{k_2 - k_1}}{4k_1} & \frac{-\beta_1x}{8k_1} & \frac{-\beta_1^2 x}{8k_1} & \frac{\beta_1^2 - \beta^2_1}{8k_1 \sqrt{k_2 - k_1}} \end{pmatrix} + o(\|\xi\|),
\end{align*}

(2.38)

\begin{align*}
P_3(\xi) &= \begin{pmatrix}
\frac{2x + \beta^2_1 - \beta^2_2}{8k_1} & \frac{-\beta_1x}{8k_1 \sqrt{k_2 + k_1}} & \frac{-\beta_1^2 x}{8k_1 \sqrt{k_2 + k_1}} & \frac{\beta_2\beta_3}{8k_1} \\
\frac{\beta_1(\beta_1^2 + 2x) + 2\beta_2\beta_3}{8k_1 \sqrt{k_2 + k_1}} & \frac{-\beta_1^2 x}{4k_1} & \frac{\beta_1x}{4k_1} & \frac{-\beta_2\beta_3}{4k_1} \\
\frac{-\beta_1\sqrt{k_2 + k_1}}{4k_1} & \frac{-\beta_1x}{8k_1} & \frac{-\beta_1^2 x}{8k_1 \sqrt{k_2 + k_1}} & \frac{\beta_1^2}{8k_1 \sqrt{k_2 + k_1}} \\
\frac{\beta_1\sqrt{k_2 + k_1}}{4k_1} & \frac{\beta_1x}{8k_1} & \frac{\beta_1^2 - \beta^2_1}{8k_1 \sqrt{k_2 + k_1}} & \frac{-\beta_2\beta_3}{8k_1} \end{pmatrix} + o(\|\xi\|),
\end{align*}

(2.39)

\begin{align*}
P_4(\xi) &= \begin{pmatrix}
\frac{2x + \beta^2_1 - \beta^2_2}{8k_1} & \frac{-\beta_1x}{8k_1 \sqrt{k_2 + k_1}} & \frac{-\beta_1^2 x}{8k_1 \sqrt{k_2 + k_1}} & \frac{\beta_2\beta_3}{8k_1} \\
\frac{\beta_1(\beta_1^2 + 2x) + 2\beta_2\beta_3}{8k_1 \sqrt{k_2 + k_1}} & \frac{-\beta_1^2 x}{4k_1} & \frac{\beta_1x}{4k_1} & \frac{-\beta_2\beta_3}{4k_1} \\
\frac{-\beta_1\sqrt{k_2 + k_1}}{4k_1} & \frac{-\beta_1x}{8k_1} & \frac{-\beta_1^2 x}{8k_1 \sqrt{k_2 + k_1}} & \frac{\beta_1^2}{8k_1 \sqrt{k_2 + k_1}} \\
\frac{\beta_1\sqrt{k_2 + k_1}}{4k_1} & \frac{\beta_1x}{8k_1} & \frac{\beta_1^2 - \beta^2_1}{8k_1 \sqrt{k_2 + k_1}} & \frac{-\beta_2\beta_3}{8k_1} \end{pmatrix} + o(\|\xi\|),
\end{align*}

(2.40)

and

\begin{align*}
\| \nabla^k e^{\alpha t} \|_{L^2} &\lesssim (1 + t)^{-\frac{k}{2}} \| \hat{u}(0) \|_{L^\infty},
\end{align*}

(2.41)

for any \( |\xi| \leq \eta_1 \). Therefore, Equation (2.36) follows from Equations (2.34)–(2.35) and Equations (2.37)–(2.40) immediately.

With the help of Equation (2.36), we can get the following proposition, which is concerned with the optimal \( L^2 \)-convergence rate on the low-frequency part of the solution.

**Proposition 2.3 (\( L^2 \) theory).** For any \( k > -\frac{3}{2} \), it holds that

\begin{align*}
\| \nabla^k e^{\alpha t} \|_{L^2} &\lesssim (1 + t)^{-\frac{k}{2}} \| \hat{u}(0) \|_{L^\infty},
\end{align*}

(2.41)

for any \( t > 0 \).

**Proof.** Due to Equation (2.36) and Plancherel theorem, we have
\[ \|\nabla^k e^{i\xi x} \ast \varphi^j(0)\|_{L^2}^2 = \|\xi^k e^{i\xi x}(\xi) \ast \varphi^j(0)\|_{L^2}^2 \]
\[ \lesssim \int_{|\xi| \lesssim \eta_1} e^{-2\eta_1|\xi|^2}
\|\varphi^j(0)\|_{L^2}^2 d\xi \]
\[ \lesssim (1 + t)^{-\frac{1}{4} - \frac{k}{2}} \|\varphi^j(0)\|_{L^\infty}^2, \]
(2.42)

which implies Equation (2.41). Therefore, the proof of Proposition 2.3 has been completed.

It is worth mentioning that the $L^2$-convergence rates derived above are optimal. Indeed, we have the lower bound on the convergence rates, which is stated in the following proposition.

**Proposition 2.4.** Assume that $(n_0^+, \varphi_0^+, n_0^-, \varphi_0^-) \in L^1$ satisfies
\[ \hat{\varphi}_0^+(\xi) = \overline{n}_0^-(\xi) = \hat{\varphi}_0^-(\xi) = 0 \quad \text{and} \quad |\hat{n}_0^+(\xi)| \geq c_0, \]
(2.43)
for any $|\xi| \leq \eta_1$. Then the global solution $(\hat{n}^+, \varphi^+, \hat{n}^-, \varphi^-)$ of the IVP (2.28) satisfies
\[ \min \left\{ \|\hat{n}^+\|_{L^2}, \|\varphi^+\|_{L^2}, \|\hat{n}^-\|_{L^2}, \|\varphi^-\|_{L^2} \right\} \geq c_0(1 + t)^{-\frac{1}{4}}. \]
(2.44)
for large enough $t$.

**Proof.** Let $\overline{\varphi}_2 = \frac{\varphi^+ + \varphi^-}{4} + \frac{\varphi^+ (\beta_1^2 - \beta_2^2) + \varphi^- (\beta_1^2 - \beta_2^2)}{8\kappa_1} > 0$. Due to Equation (2.43), it follows from Equations (2.34)–(2.35) and Equations (2.37)–(2.40) that
\[ \overline{\varphi}^+ \sim \frac{2\kappa_1 + \beta_3^2 - \beta_1^2}{4\kappa_1} e^{-\eta_1|\xi|^2 t} \left( \sqrt{\kappa_2 - \kappa_1|\xi|} + O(|\xi|^3) \right) \overline{n}_0^+ \]
(2.45)

This together with Plancherel theorem and the double angle formula gives that
\[ \|\hat{n}^+\|_{L^2}^2 = \|\overline{\varphi}^+\|_{L^2}^2 \]
\[ \geq \frac{(2\kappa_1 + \beta_3^2 - \beta_1^2)^2 c_0}{16\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-2\eta_1|\xi|^2 t} \cos \left( \sqrt{\kappa_2 - \kappa_1|\xi|} + O(|\xi|^3) \right) d\xi \]
\[ + \frac{(2\kappa_1 + \beta_3^2 - \beta_1^2)^2 c_0}{16\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-2\eta_1|\xi|^2 t} \cos \left( \sqrt{\kappa_2 + \kappa_1|\xi|} + O(|\xi|^3) \right) d\xi \]
\[ + \frac{[4\kappa_1^2 + (\beta_1^2 - \beta_2^2)^2] c_0}{8\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{\frac{\varphi^+ + \varphi^-}{2} - |\xi|^2 t} \cos \left( \sqrt{\kappa_2 - \kappa_1|\xi|} + O(|\xi|^3) \right) t \]
\[ \cdot \cos \left( \sqrt{\kappa_2 + \kappa_1|\xi|} + O(|\xi|^3) \right) d\xi \]
\[ = \frac{(2\kappa_1 + \beta_3^2 - \beta_1^2)^2 c_0}{32\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-2\eta_1|\xi|^2 t} \] 
\[ + \frac{(2\kappa_1 + \beta_3^2 - \beta_1^2)^2 c_0}{32\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-2\eta_1|\xi|^2 t} \cos \left( 2\sqrt{\kappa_2 - \kappa_1|\xi|} + O(|\xi|^3) \right) d\xi \]
\[ + \frac{(2\kappa_1 + \beta_3^2 - \beta_1^2)^2 c_0}{32\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-2\eta_1|\xi|^2 t} \cos \left( 2\sqrt{\kappa_2 + \kappa_1|\xi|} + O(|\xi|^3) \right) d\xi \]
\[ \geq \frac{(2\kappa_1 + \beta_3^2 - \beta_1^2)^2 c_0}{32\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-2\eta_1|\xi|^2 t} d\xi \]
\[ + \frac{(2\kappa_1 + \beta_3^2 - \beta_1^2)^2 c_0}{32\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-2\eta_1|\xi|^2 t} d\xi \]
\[ + \frac{(2\kappa_1 + \beta_3^2 - \beta_1^2)^2 c_0}{32\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-2\eta_1|\xi|^2 t} d\xi \]
\[ + \frac{(2\kappa_1 + \beta_3^2 - \beta_1^2)^2 c_0}{32\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-2\eta_1|\xi|^2 t} d\xi \]
\[ + \frac{(2\kappa_1 + \beta_3^2 - \beta_1^2)^2 c_0}{32\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-2\eta_1|\xi|^2 t} d\xi \]
\[ + \frac{(2\kappa_1 + \beta_3^2 - \beta_1^2)^2 c_0}{32\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-2\eta_1|\xi|^2 t} d\xi \]
\[ + \frac{(2\kappa_1 + \beta_3^2 - \beta_1^2)^2 c_0}{32\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-2\eta_1|\xi|^2 t} d\xi \]
\[ + \frac{(2\kappa_1 + \beta_3^2 - \beta_1^2)^2 c_0}{32\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-2\eta_1|\xi|^2 t} d\xi \]
\[ + \frac{(2\kappa_1 + \beta_3^2 - \beta_1^2)^2 c_0}{32\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-2\eta_1|\xi|^2 t} d\xi \]
\[ + \frac{[4\kappa_1^2 + (\beta_1^2 - \beta_2^2)]c_0^2}{16\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-\frac{\nu^2 + \kappa_1^2}{2}|\xi|^2} \cos \left[ \sqrt{\kappa_2 - \kappa_1 |\xi|} + \sqrt{\kappa_2 + \kappa_1 |\xi|} + O(|\xi|^3) \right] td\xi \]

\[ + \frac{[4\beta_1^2 + (\beta_2^2 - \beta_4^2)]c_0^2}{16\kappa_1^2} \int_{|\xi| \leq \frac{\eta_1}{2}} e^{-\frac{\nu^2 + \beta_1^2}{2}|\xi|^2} \cos \left[ \sqrt{\kappa_2 - \kappa_1 |\xi|} - \sqrt{\kappa_2 + \kappa_1 |\xi|} + O(|\xi|^3) \right] td\xi \]

\[ \geq c_0^2 (1 + t)^{-\frac{3}{2}}, \quad (2.46) \]

if \( t \) large enough. Using a similar procedure as in Equation (2.46) to handle \( \| (\varphi^+, \bar{\eta}^-, \varphi^-) \|_{L^2} \), one has Equation (2.44). Therefore, we have completed the proof of Proposition 2.4. \( \square \)

From the classic theory of the heat equation, it is clear that the solution \( V = (\varphi^+, \varphi^-) \) to the IVP (2.25) satisfies the following decay estimates.

**Proposition 2.5** \((L^2 \text{ theory}).\) For any \( k > -\frac{3}{2} \), there exists a positive constant \( C \), which is independent of \( t \) such that

\[ \| \nabla^k e^{tA} * \varphi^l(0) \|_{L^2} \leq C (1 + t)^{-\frac{3}{4} - \frac{k}{2}} \| \varphi^l(0) \|_{L^\infty}, \quad (2.47) \]

for any \( t \geq 0 \).

By virtue of the definition of \( \varphi^\pm \) and \( \phi^\pm \), and the fact that the relations

\[ \bar{u}^\pm = -\Lambda^{-1} \nabla \varphi^\pm - \Lambda^{-1} \text{div} \phi^\pm \quad (2.48) \]

involve pseudo-differential operators of degree zero, the estimates in space \( H^k(\mathbb{R}^3) \) for the original function \( \bar{u}^\pm \) will be the same as for \( (\varphi^\pm, \phi^\pm) \). Combining Propositions 2.3–2.5, we have the following result concerning long time properties for the solution semigroup \( e^{-tA} \).

**Proposition 2.6.** For any \( k > -\frac{3}{2} \) and \( 2 \leq r \leq \infty \). Assume that the initial data \( U_0 \in L^1(\mathbb{R}^3) \), then for any \( t \geq 0 \), the global solution \( \bar{U} = (\bar{n}^+, \bar{u}^+, \bar{n}^-, \bar{u}^-)^t \) of the IVP (2.20) satisfies

\[ \| \nabla^k e^{tA} \bar{U}(0) \|_{L^2} \leq C (1 + t)^{-\frac{3}{4} - \frac{k}{2}} \| \bar{U}(0) \|_{L^1}, \quad (2.49) \]

If additionally the initial data satisfies (1.36), we also have the following lower bound on convergence rate:

\[ \min \{ \| \dot{\bar{n}}^+(t) \|_{L^2}, \| \dot{\bar{u}}^+(t) \|_{L^2}, \| \dot{\bar{n}}^-(t) \|_{L^2}, \| \dot{\bar{u}}^-(t) \|_{L^2} \} \geq C_1 N_0 \delta_0 (1 + t)^{-\frac{3}{2}}, \quad (2.50) \]

if \( t \) is large enough.

## 3 | OPTIMAL CONVERGENCE RATE

In this section, we shall show the optimal convergence rate of the solution stated in Theorem 1.1. The global existence and uniqueness of the solution in \( H^2 \) to the Cauchy problem (2.15)–(2.16) have been proven in Evje et al. [21] based on the classical energy method developed in Refs. [32, 35]. Thus, we can follow the proof of Evje et al. [21] step by step to obtain the global existence and uniqueness of the solution in \( H^N \) with an integer \( N \geq 2 \), and thus we omit the details for the sake of simplicity.

**Theorem 3.1.** Assume that \( (n_0^+, u_0^+, n_0^-, u_0^-) \in H^N(\mathbb{R}^3) \) for an integer \( N \geq 2 \) and Equation (1.20) holds. There exists a constant \( \delta_0 > 0 \) such that if

\[ \| (n_0^+, u_0^+, n_0^-, u_0^-) \|_{H^2} \leq \delta_0, \quad (3.1) \]
then the Cauchy problem (2.15)–(2.16) admits a unique globally classical solution \((n^+, u^+, n^-, u^-)\) such that for any \(t \in [0, \infty)\),

\[
\|(n^+, u^+, n^-, u^-)(t)\|_{H^N}^2 + \int_0^t \left( \|\nabla(n^+, n^-)(\tau)\|_{H^{N-1}}^2 + \|\nabla(u^+, u^-)(\tau)\|_{H^N}^2 \right) d\tau \leq \|(n_0^+, u_0^+, n_0^-, u_0^-)\|_{H^N}^2.
\]

(3.2)

In what follows, we focus our attention on the proof of the optimal convergence rate of the solution stated in Theorem 1.1. We first prove the upper bound on the optimal convergence rate of the solution stated in Equations (1.34)–(1.35). We will split the proof into two cases: **Case I:** \(N = 2\) and **Case II:** \(N > 2\). To begin with, we deal with **Case I:** \(N = 2\). Owing to Equations (1.22) and (1.23), it suffices to prove the following theorem.

**Theorem 3.2 (Case I: \(N = 2\)).** Assume that the hypotheses of Theorem 3.1 and Equation (1.33) are in force. Then there exists a positive constant \(C\), which is independent of \(t\), such that

\[
\|\nabla^2(n^+, u^+, n^-, u^-)(t)\|_{L^2} \leq C(1 + t)^{-\frac{1}{2}},
\]

(3.3)

for all \(t \geq 0\).

**Proof.** We will make full use of the benefit of the low- and high-frequency decomposition to prove Theorem 3.2. The process involves the following three steps.

**Step 1.** \(L^2\) estimate of \(\nabla^2(n^+, u^+, n^-, u^-)\). Multiplying \(\nabla^2(2.15)_1, \nabla^2(2.15)_2, \nabla^2(2.15)_3\), and \(\nabla^2(2.15)_4\) by \(\frac{\beta_1}{\beta_2}\nabla^2 n^+, \frac{\beta_1}{\beta_2}\nabla^2 u^+, \frac{\beta_1}{\beta_2}\nabla^2 n^-, \frac{\beta_1}{\beta_2}\nabla^2 u^-, \frac{\beta_4}{\beta_3}\nabla^2 u^+, \frac{\beta_4}{\beta_3}\nabla^2 n^-, \frac{\beta_4}{\beta_3}\nabla^2 u^-, \) respectively, and then integrating over \(\mathbb{R}^3\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{\beta_1}{\beta_2} \|\nabla^2 n^+\|^2 + \frac{\beta_1}{\beta_2} \|\nabla^2 u^+\|^2 \right\} + \frac{\beta_1}{\beta_2} \left( v_1^+ \|\nabla^3 u^+\|^2 + v_2^+ \|\nabla^2 \text{div } u^+\|^2 \right) = \left< \nabla^2 \mathcal{F}_1, \frac{\beta_1}{\beta_2} \nabla^2 n^+ \right> + \left< \nabla^2 \mathcal{F}_2, \frac{\beta_1}{\beta_2} \nabla^2 u^+ \right> - \left< \nabla^2 \nabla \nabla n^+, \beta_1 \nabla^2 u^+ \right> =: I_1 + I_2 + I_3,
\]

(3.4)

and

\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{\beta_4}{\beta_3} \|\nabla^2 n^-\|^2 + \frac{\beta_4}{\beta_3} \|\nabla^2 u^-\|^2 \right\} + \frac{\beta_4}{\beta_3} \left( v_1^- \|\nabla^3 u^-\|^2 + v_2^- \|\nabla^2 \text{div } u^-\|^2 \right) = \left< \nabla^2 \mathcal{F}_3, \frac{\beta_4}{\beta_3} \nabla^2 n^- \right> + \left< \nabla^2 \mathcal{F}_4, \frac{\beta_4}{\beta_3} \nabla^2 u^- \right> - \left< \nabla^2 \nabla \nabla n^-, \beta_4 \nabla^2 u^- \right> =: I_4 + I_5 + I_6.
\]

(3.5)

By virtue of Equation (3.2) and Lemmas A.1 and A.2, we have from integration by parts and Young’s inequality that

\[
|I_1| \leq \left| \left< \nabla^2 n^+, \nabla^2 (n^+ \text{ div } u^+) \right> \right| + \left| \left< \nabla^2 n^+, \nabla^2 (\nabla n^+ \cdot u^+) \right> \right| \leq \|\nabla^2 n^+\|_{L^2} \|\nabla^2 n^+\|_{L^2} \|\nabla^2 u^+\|_{L^2} \|\nabla^3 u^+\|_{L^2} \leq \|\nabla^2 n^+\|_{L^2} \left( \|\nabla^2 n^+\|_{L^2} \|\nabla^2 u^+\|_{L^2} \|\nabla^3 u^+\|_{L^2} \right) \leq \frac{1}{2} \beta_1 \frac{1}{\beta_2} \|\nabla^3 u^+\|_{L^2} \|\nabla^2 u^+\|_{L^2} \|\nabla^2 n^+\|_{L^2} \|\nabla^2 u^+\|_{L^2} \|\nabla^3 u^+\|_{L^2} \leq \frac{1}{2} \beta_1 \frac{1}{\beta_2} \|\nabla^3 u^+\|_{L^2} \|\nabla^2 u^+\|_{L^2} \|\nabla^2 n^+\|_{L^2} \|\nabla^2 u^+\|_{L^2} \|\nabla^3 u^+\|_{L^2} \leq \delta_0 \left( \|\nabla^2 n^+\|_{L^2} \|\nabla^2 u^+\|_{L^2} \|\nabla^2 n^+\|_{L^2} \|\nabla^2 u^+\|_{L^2} \right),
\]

(3.6)
Similarly, for the term $I_4$, we have

$$|I_4| \lesssim \delta_0 \left( \| \nabla^2 n^- \|_{H^1}^2 + \| \nabla^3 u^- \|_{L^2}^2 \right).$$

Employing the similar arguments used in Equation (3.6), we also have

$$|I_2| \lesssim \left| \langle \nabla \left[ g_+ (n^+, n^-) \nabla n^+ \right], \nabla^3 u^+ \rangle \right|$$

$$+ \left| \langle \nabla \left[ g_+ (n^+, n^-) \nabla n^- \right], \nabla^3 u^+ \rangle \right|$$

$$+ \left| \langle \nabla^2 \left[ (u^+ \cdot \nabla) u^+ \right], \nabla^2 u^+ \rangle \right|$$

$$+ \left| \langle \nabla \left[ h_+ (n^+, n^-) (\nabla n^+ \cdot \nabla) u^+ \right], \nabla^3 u^+ \rangle \right|$$

$$+ \left| \langle \nabla \left[ k_+ (n^+, n^-) \nabla n^+ \cdot \nabla' u^+ \right], \nabla^3 u^+ \rangle \right|$$

$$+ \left| \langle \nabla \left[ h_+ (n^+, n^-) \nabla n^+ \cdot \nabla' u^+ \right], \nabla^3 u^+ \rangle \right|$$

$$+ \left| \langle \nabla \left[ k_+ (n^+, n^-) \nabla n^- \cdot \nabla' u^+ \right], \nabla^3 u^+ \rangle \right|$$

$$+ \left| \langle \nabla \left[ h_+ (n^+, n^-) \nabla n^+ \cdot \nabla' u^+ \right], \nabla^3 u^+ \rangle \right|$$

$$+ \left| \langle \nabla \left[ k_+ (n^+, n^-) \nabla n^- \cdot \nabla' u^+ \right], \nabla^3 u^+ \rangle \right|$$

$$+ \left| \langle \nabla \left[ l_+ (n^+, n^-) \Delta u^+ \right], \nabla^3 u^+ \rangle \right|$$

$$+ \left| \langle \nabla \left[ l_+ (n^+, n^-) \nabla \nabla u^+ \right], \nabla^3 u^+ \rangle \right|$$

$$\lesssim \| \nabla^3 u^+ \|_{L^2} \left( \| g_+ (n^+, n^-) \|_{L^\infty} + \| g_+ (n^+, n^-) \|_{L^6} \| \nabla n^+ \|_{L^3} \right)

+ \| \nabla^3 u^+ \|_{L^2} \left( \| g_+ (n^+, n^-) \|_{L^\infty} + \| \nabla n^- \|_{L^3} \right)

+ \| \nabla^2 u^+ \|_{L^6} \left( \| u^+ \|_{L^\infty} \| \nabla^3 u^+ \|_{L^2} \right)

+ \| \nabla^2 u^+ \|_{L^2} \left( \| \nabla n^+ \|_{L^3} \right)

+ \| k_+ (n^+, n^-) \nabla u^+ \|_{L^\infty} \| \nabla^2 n^- \|_{L^2}

+ \| l_+ (n^+, n^-) \|_{L^\infty} \| \nabla^3 u^+ \|_{L^2}

+ \| \nabla l_+ (n^+, n^-) \|_{L^2} \| \nabla^2 u^+ \|_{L^3}

\lesssim \delta_0 \left( \| \nabla^2 (n^+, n^-) \|_{L^2}^2 + \| \nabla^2 u^- \|_{H^1}^2 \right).$$

Similarly, for the term $I_5$, we have

$$|I_5| \lesssim \delta_0 \left( \| \nabla^2 (n^+, n^-) \|_{L^2}^2 + \| \nabla^2 u^- \|_{H^1}^2 \right).$$
For the terms \( I_3 \) and \( I_6 \), we have from Young's inequality that

\[
|I_3| \lesssim \frac{\beta_1 \beta_2}{\nu_1^+} \| \nabla^2 n^- \|_{L^2}^2 + \frac{\beta_1}{4 \beta_2} \nu_1^+ \| \nabla^3 u^+ \|_{L^2}^2,
\]

(3.10)

and

\[
|I_6| \lesssim \frac{\beta_3 \beta_4}{\nu_1^-} \| \nabla^2 n^+ \|_{L^2}^2 + \frac{\beta_4}{4 \beta_3} \nu_1^- \| \nabla^3 u^- \|_{L^2}^2.
\]

(3.11)

Combining the estimates (3.4)–(3.11) and noting the smallness of \( \delta_0 \), we conclude that

\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{\beta_1}{\beta_2} \| \nabla^2 n^+ \|_{L^2}^2 + \frac{\beta_1}{\beta_2} \| \nabla^2 u^+ \|_{L^2}^2 \right\} + \frac{\beta_1}{2 \beta_2} \left( \nu_1^+ \| \nabla^3 u^+ \|_{L^2}^2 + \nu_2^+ \| \nabla^2 \text{div} u^+ \|_{L^2}^2 \right)
\]

\[
\leq C \delta_0 \left( \| \nabla^2 (n^+, n^-) \|_{L^2}^2 + \| \nabla^2 u^+ \|_{L^2}^2 \right) + \frac{\beta_1 \beta_2}{\nu_1^+} \| \nabla^2 n^- \|_{L^2}^2,
\]

(3.12)

and

\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{\beta_4}{\beta_3} \| \nabla^2 n^- \|_{L^2}^2 + \frac{\beta_4}{\beta_3} \| \nabla^2 u^- \|_{L^2}^2 \right\} + \frac{\beta_4}{2 \beta_3} \left( \nu_1^- \| \nabla^3 u^- \|_{L^2}^2 + \nu_2^- \| \nabla^2 \text{div} u^- \|_{L^2}^2 \right)
\]

\[
\leq C \delta_0 \left( \| \nabla^2 (n^+, n^-) \|_{L^2}^2 + \| \nabla^2 u^- \|_{L^2}^2 \right) + \frac{\beta_3 \beta_4}{\nu_1^-} \| \nabla^2 n^+ \|_{L^2}^2,
\]

(3.13)

for some positive constant \( C \) independent of \( \delta_0 \).

**Step 2. Dissipation of \( \nabla^2 (n^{+,h}, n^{-,h}) \).** Applying the operator \( \nabla \mathcal{F}^{-1}(1 - \phi(\xi)) \) to Equations (2.15)\textsubscript{2} and (2.15)\textsubscript{4} and then multiplying the resulting equations by \( \frac{1}{\beta_2} \nabla^2 n^{+,h} \) and \( \frac{1}{\beta_3} \nabla^2 n^{-,h} \) respectively, integrating over \( \mathbb{R}^3 \), we get

\[
\frac{d}{dt} \left\langle \nabla u^{+,h} \frac{1}{\beta_2} \nabla^2 n^{+,h} \right\rangle + \frac{\beta_1}{\beta_2} \| \nabla^2 n^{+,h} \|_{L^2}^2 + \left\langle \nabla^2 n^{-,h}, \nabla^2 n^{+,h} \right\rangle
\]

\[
= \frac{1}{\beta_2} \left\langle \nabla u^{+,h}, \partial_t \nabla^2 n^{+,h} \right\rangle + \frac{\nu_1^+}{\beta_2} \left\langle \nabla \Delta u^{+,h}, \nabla^2 n^{+,h} \right\rangle + \frac{\nu_2^+}{\beta_2} \left\langle \nabla^2 \text{div} u^{+,h}, \nabla^2 n^{+,h} \right\rangle
\]

\[
+ \frac{1}{\beta_2} \left\langle \nabla \mathcal{F}^h, \nabla^2 n^{+,h} \right\rangle
\]

\[
= J_1 + J_2 + J_3 + J_4,
\]

(3.14)

and

\[
\frac{d}{dt} \left\langle \nabla u^{-,h} \frac{1}{\beta_3} \nabla^2 n^{-,h} \right\rangle + \frac{\beta_4}{\beta_3} \| \nabla^2 n^{-,h} \|_{L^2}^2 + \left\langle \nabla^2 n^{+,h}, \nabla^2 n^{-,h} \right\rangle
\]

\[
= \frac{1}{\beta_3} \left\langle \nabla u^{-,h}, \partial_t \nabla^2 n^{-,h} \right\rangle + \frac{\nu_1^-}{\beta_3} \left\langle \nabla \Delta u^{-,h}, \nabla^2 n^{-,h} \right\rangle + \frac{\nu_2^-}{\beta_3} \left\langle \nabla^2 \text{div} u^{-,h}, \nabla^2 n^{-,h} \right\rangle
\]

\[
+ \frac{1}{\beta_3} \left\langle \nabla \mathcal{F}^h, \nabla^2 n^{-,h} \right\rangle
\]

\[
= J_5 + J_6 + J_7 + J_8.
\]

(3.15)
Due to Equation (3.2), Lemmas A.1–A.4, we can use integration by parts, and Young’s inequality to deduce that

\[
|J_1| = \left| -\frac{1}{\beta^2_2} \langle \nabla u^+, \beta_1 \nabla^2 \text{div } u^+ \rangle + \frac{1}{\beta^2_2} \langle \nabla u^+, \nabla^2 \mathcal{F}^h_1 \rangle \right|
\]

\[
= \left| \frac{\beta_1}{\beta_2} \left\| \nabla u^+ \right\|_{L^2}^2 - \frac{1}{\beta^2_2} \langle \nabla \text{div } u^+, \nabla \mathcal{F}^h \rangle \right|
\]

\[
\leq \frac{\beta_1}{\beta_2} \left\| \nabla u^+ \right\|_{L^2}^2 + C \left\| \nabla u^+ \right\|_{L^2} \left( \| u^+ \|_{L^3} \| \nabla^2 u^+ \|_{L^6} + \| u^+ \|_{L^\infty} \| \nabla^2 n^+ \|_{L^3} \right)
\]

\[
\leq \frac{\beta_1}{\beta_2} \left\| \nabla u^+ \right\|_{L^2}^2 + C \left\| \nabla^3 u^+ \right\|_{L^2} \left( \| u^+ \|_{H^3} \| \nabla^3 u^+ \|_{L^2} + \| u^+ \|_{L^\infty} \| \nabla^2 n^+ \|_{L^3} \right)
\]

\[
\leq \frac{\beta_1}{\beta_2} \left\| \nabla u^+ \right\|_{L^2}^2 + C \delta_0 \left( \| \nabla^2 n^+ \|_{L^2}^2 + \| \nabla^3 u^+ \|_{L^2}^2 \right).
\]

Similarly, for the term \( J_5 \), we have

\[
|J_5| \leq \frac{\beta_4}{\beta_3} \left\| \nabla \text{div } u^{-h} \right\|_{L^2}^2 + C \delta_0 \left( \| \nabla^2 n^- \|_{L^2}^2 + \| \nabla^3 u^- \|_{L^2}^2 \right).
\]

By virtue of Equation (2.15), we can rewrite \( J_2 + J_3 \) as

\[
J_2 + J_3 = \frac{\nu_1^+ + \nu_2^+}{\beta_2} \langle \nabla^2 \text{div } u^+, \nabla^2 n^+ \rangle
\]

\[
= -\frac{\nu_1^++\nu_2^+}{2\beta_1 \beta_2} \text{d} \left\| \nabla^2 n^+ \right\|_{L^2}^2 - \frac{\nu_1^+ + \nu_2^+}{2\beta_1 \beta_2 \sqrt{\alpha_1}} \langle \nabla^2 \text{div } (n^+ u^+)^h, \nabla^2 n^+ \rangle.
\]

On the other hand, we have

\[
\langle \nabla^2 \text{div } (n^+ u^+)^h, \nabla^2 n^+ \rangle = \langle \nabla^2 \text{div } (n^+ u^+), \nabla^2 n^+ \rangle - \langle \nabla^2 \text{div } (n^+ u^+)^l, \nabla^2 n^+ \rangle + \langle \nabla^2 \text{div } (n^+ u^+)^l, \nabla^2 n^+ \rangle
\]

\[
= J_{2,3}^1 + J_{2,3}^2 + J_{2,3}^3.
\]

For the term \( J_{2,3}^1 \), by using integration by parts, (3.2), Lemmas A.1–A.4 and Young’s inequality, we have

\[
\left| J_{2,3}^1 \right|
\]

\[
\leq \left| \langle \text{div } u^+, \nabla^2 n^+ \rangle \right| + \left| \langle \nabla^2 (u^+ \cdot \nabla n^+), \nabla^2 n^+ \rangle \right| + \left| \langle \nabla^2 \left( n^+ \cdot \text{div } u^+ \right), \nabla^2 n^+ \rangle \right|
\]

\[
\leq \| \text{div } u^+ \|_{L^\infty} \| \nabla^2 n^+ \|_{L^2}^2 + \| \nabla^2 n^+ \|_{L^2} \left( \| \nabla u^+ \|_{L^\infty} \| \nabla^2 n^+ \|_{L^2} + \| \nabla^2 u^+ \|_{L^6} \| \nabla n^+ \|_{L^3} \right)
\]

\[
+ \| \nabla^2 n^+ \|_{L^2} \left( \| \nabla^3 u^+ \|_{L^6} \| n^+ \|_{L^3} + \| \text{div } u^+ \|_{L^\infty} \| \nabla^2 n^+ \|_{L^2} \right)
\]

\[
\leq \| \nabla^2 u^+ \|_{L^2} \| \nabla^3 u^+ \|_{L^6} \| n^+ \|_{L^3} + \| \nabla^2 n^+ \|_{L^2} \left( \| \nabla^3 u^+ \|_{L^2} + \| \nabla u^+ \|_{L^\infty} \| \nabla n^+ \|_{H^3} \| \nabla^2 n^+ \|_{L^2} \right)
\]

\[
\leq \delta_0 \left( \| \nabla^2 n^+ \|_{L^2}^2 + \| \nabla^3 u^+ \|_{L^2}^2 \right).
\]

By employing similar arguments, for the terms \( J_{2,3}^2 \) and \( J_{2,3}^3 \), we have
\begin{align}
|J_{2,3}^2| & \lesssim \|\nabla^2 n^{+,h}\|_{L^2} \left( \|\nabla^3 n^{+,l}\|_{L^2} \|u^+\|_{L^\infty} + \|n^{+,l}\|_{L^\infty} \|\nabla^3 u^+\|_{L^2} \right) \\
& \lesssim \|\nabla^2 n^+\|_{L^2} \left( \|\nabla^2 n^+\|_{L^2} \|u^+\|_{H^2} + \|n^+\|_{H^2} \|\nabla^3 u^+\|_{L^2} \right) \\
& \lesssim \delta_0 \left( \|\nabla^2 n^+\|^2_{L^2} + \|\nabla^3 u^+\|^2_{L^2} \right).
\end{align}

(3.21)

\begin{align}
|J_{3,2}^3| & \lesssim \|\nabla^2 n^{+,h}\|_{L^2} \|\nabla^2 \text{div} (n^+ u^+)^{1/2}\|_{L^2} \\
& \lesssim \|\nabla^2 n^+\|_{L^2} \|\nabla^2 (n^+ u^+)\|_{L^2} \\
& \lesssim \|\nabla^2 n^+\|_{L^2} \left( \|\nabla^2 n^+\|_{L^2} \|u^+\|_{L^\infty} + \|n^+\|_{H^2} \|\nabla^3 u^+\|_{L^2} \right) \\
& \lesssim \delta_0 \left( \|\nabla^2 n^+\|^2_{L^2} + \|\nabla^3 u^+\|^2_{L^2} \right).
\end{align}

(3.22)

Combining the estimates (3.18)–(3.22), we get

\begin{align}
|J_2 + J_3| & \lesssim -\frac{\nu_1^+ + \nu_1^+}{2\beta_1 \beta_2} \frac{d}{dt} \|\nabla^2 n^{+,h}\|_{L^2}^2 + \delta_0 \left( \|\nabla^2 n^+\|^2_{L^2} + \|\nabla^3 u^+\|^2_{L^2} \right). \\
(3.23)
\end{align}

Similarly for the terms \(J_6\) and \(J_7\), we have

\begin{align}
|J_6 + J_7| & \lesssim -\frac{\nu_1^- + \nu_1^-}{2\beta_3 \beta_4} \frac{d}{dt} \|\nabla^2 n^{-,h}\|_{L^2}^2 + \delta_0 \left( \|\nabla^2 n^-\|^2_{L^2} + \|\nabla^3 u^-\|^2_{L^2} \right). \\
(3.24)
\end{align}

Applying the similar arguments used in Equation (3.8), for the terms \(J_4\) and \(J_8\), we have

\begin{align}
|J_4| & \lesssim \delta_0 \left( \|\nabla^2 (n^+, n^-)\|_{L^2}^2 + \|\nabla^2 u^+\|_{H^2}^2 \right), \\
(3.25)

|J_8| & \lesssim \delta_0 \left( \|\nabla^2 (n^+, n^-)\|_{L^2}^2 + \|\nabla^2 u^-\|_{H^2}^2 \right). \\
(3.26)
\end{align}

Combining the estimates (3.14)–(3.17) and (3.23)–(3.26), we finally conclude that

\begin{align}
\frac{d}{dt} \left\{ \frac{\nu_1^+ + \nu_1^+}{2\beta_1 \beta_2} \|\nabla^2 n^{+,h}\|_{L^2}^2 + \left\langle \nabla u^{+,h}, \frac{1}{\beta_2} \nabla^2 n^{+,h} \right\rangle \right\} \\
& \quad + \frac{\beta_1}{\beta_2} \|\nabla^2 n^{+,h}\|_{L^2}^2 + \left\langle \nabla^2 u^{+,h}, \nabla^2 n^{+,h} \right\rangle \\
& \leq C \left( \frac{\beta_1}{\beta_2} \|\nabla^2 \text{div} u^+\|_{L^2}^2 + \delta_0 \left( \|\nabla^2 (n^+, n^-)\|_{L^2}^2 + \|\nabla^2 u^+\|_{H^2}^2 \right) \right),
\end{align}

(3.27)

and

\begin{align}
\frac{d}{dt} \left\{ \frac{\nu_1^- + \nu_1^-}{2\beta_3 \beta_4} \|\nabla^2 n^{-,h}\|_{L^2}^2 + \left\langle \nabla u^{-,h}, \frac{1}{\beta_3} \nabla^2 n^{-,h} \right\rangle \right\} \\
& \quad + \frac{\beta_4}{\beta_3} \|\nabla^2 n^{-,h}\|_{L^2}^2 + \left\langle \nabla^2 u^{-,h}, \nabla^2 n^{-,h} \right\rangle \\
& \leq C \left( \frac{\beta_4}{\beta_3} \|\nabla^2 \text{div} u^-\|_{L^2}^2 + \delta_0 \left( \|\nabla^2 (n^+, n^-)\|_{L^2}^2 + \|\nabla^2 u^-\|_{H^2}^2 \right) \right).
\end{align}

(3.28)
Step 3. Proof of Theorem 3.2. In this step, we are in a position to prove Theorem 3.2. To begin with, noticing that

\[ \frac{s^2(1,1)}{\alpha^2(1,1)} < f'(1) < \eta - \frac{s^2(1,1)}{\alpha^2(1,1)} < 0, \]

(3.29)

where \( \eta \) is a positive, small fixed constant, it is easy to see that \( \alpha_2^2 = C_2(1,1) + C_2(1,1) \alpha(1,1) f'(1) / s^2(1,1) < C_2(1,1) \) and \( \alpha_4 \) is bounded. Therefore, \( \beta_2 = \frac{\alpha^2(1,1)}{\alpha_4} \) is a small positive constant, which will be determined later.

Computing (3.12) + \( C_1 \times (3.27) \), we have

\[
\frac{d}{dt} \left\{ \frac{\beta_1}{2 \beta_2} \| \nabla^2 n^+ \|^2_{L^2} + \frac{(v_1^+ + v_2^+) C_1}{2 \beta_1 \beta_2} \| \nabla^2 u^+ \|^2_{L^2} + \frac{\beta_1}{2 \beta_2} \| \nabla^2 u^+ \|^2_{L^2} + \frac{C_1}{\beta_2} \langle \nabla u^+ h, \nabla^2 n^+ h \rangle \right\} 
\]

(3.30)

\[
+ \frac{\beta_1}{2 \beta_2} \left( v_1^+ \| \nabla^2 u^+ \|^2_{L^2} + v_2^+ \| \nabla^2 \text{div} u^+ \|^2_{L^2} \right) + C_1 \left( \frac{\beta_1 \beta_2}{2 \beta_2} \| \nabla^2 n^+ h \|^2_{L^2} + \langle \nabla^2 n^+ h, \nabla^2 n^+ h \rangle \right) 
\]

(3.31)

\[
\leq C \delta_0 \left( \| \nabla^2 (n^+, n^-) \|^2_{L^2} + \| \nabla^2 u^+ \|^2_{L^2} \right) + C C_1 \beta_1 \beta_2 \| \nabla^2 (n^+, n^-) \|^2_{L^2} + C C_1 \beta_1 \beta_2 \| \nabla^2 \text{div} u^+ \|^2_{L^2}. 
\]

(3.32)

Choosing \( C_1 \) as a fixed positive constant with \( C_1 \leq \min \left\{ \frac{v_2^+}{4 C}, \frac{v_1^+ v_2^+}{2 C^2_0} \right\} \) and making use of the smallness of \( \delta_0 \), we get

\[
\frac{d}{dt} \beta_1(t) + \frac{\beta_1}{6 \beta_2} \left( v_1^+ \| \nabla^2 u^+ \|^2_{L^2} + v_2^+ \| \nabla^2 \text{div} u^+ \|^2_{L^2} \right) 
\]

(3.33)

\[
+ C_1 \beta_1 \beta_2 \left( \| \nabla^2 n^+ h \|^2_{L^2} + \langle \nabla^2 n^+ h, \nabla^2 n^+ h \rangle \right) 
\]

(3.34)

By virtue of \( C_1 \leq \frac{v_1^+ v_2^+}{2 C_0^2} \), Lemma A.4 and Young’s inequality, there exists a positive constant \( C_2 \) independent of \( \delta_0 \) and \( \beta_2 \), such that

\[
\frac{1}{C_2 \beta_2} \| \nabla^2 (n^+, u^+)(t) \|^2_{L^2} \leq \beta_1(t) \leq \frac{C_2}{\beta_2} \| \nabla^2 (n^+, u^+)(t) \|^2_{L^2}. 
\]

(3.35)

Similarly, by calculating (3.33) + \( C_3 \times (3.13) \), we have

\[
\frac{d}{dt} \beta_1(t) + \frac{\beta_1}{6 \beta_2} \left( v_1^+ \| \nabla^2 u^- \|^2_{L^2} + v_2^+ \| \nabla^2 \text{div} u^- \|^2_{L^2} \right) 
\]

(3.36)

\[
+ C_1 \beta_1 \beta_2 \left( \| \nabla^2 n^- h \|^2_{L^2} + \langle \nabla^2 n^- h, \nabla^2 n^- h \rangle \right) + C C_1 \beta_1 \beta_2 \| \nabla^2 u^- \|^2_{L^2} + C_1 \beta_1 \beta_2 \| \nabla^2 \text{div} u^- \|^2_{L^2} 
\]

where \( 0 < C_3 \leq \min \left\{ 1, \frac{C_1 v_1^+ v_2^+}{\beta_2 \beta_4} \right\} \).
Next, from (3.36) + C_3 \times (3.28), we have
\[
\frac{d}{dt} \left\{ \phi(t) + C_3 \beta_4 \frac{1}{2 \beta_3} \left\| \nabla^2 n^- \right\|_{L^2} + \frac{1}{2 \beta_3} \left\| \nabla^2 u^- \right\|_{L^2} + C_4 \left\langle \nabla u^-, \nabla^2 n^- \right\rangle \right\}
\]
\[+ \frac{\beta_1}{6 \beta_2} \left( \nu_1^+ \left\| \nabla^3 u^+ \right\|_{L^2} + \nu_2^+ \left\| \nabla^2 \text{div } u^+ \right\|_{L^2} \right) + C_1 \beta_1 \left( \frac{1}{\beta_2^2} - 1 \right) \left\| \nabla^2 n^- \right\|_{L^2}^2
\]
\[+ C_1 \left\langle \nabla^2 n^- \right. \left\langle \nabla_2 n^- \right\rangle \right\rangle + C_3 \beta_4 \frac{1}{2 \beta_3} \left( \nu_1 \left\| \nabla^3 u^- \right\|_{L^2} + \nu_2 \left\| \nabla^2 \text{div } u^- \right\|_{L^2} \right)
\]
\[+ C_4 \left( \frac{1}{3 \beta_3} \left\| \nabla^2 n^- \right\|_{L^2} + \left\| \nabla^2 n^- \right\|_{L^2} \right) \leq C_0 \left( \left\| \nabla^2 (n^+, n^-) \right\|_{L^2}^2 + \left\| \nabla^2 (u^+, u^-) \right\|_{L^2}^2 \right)
\]
\[+ \frac{\beta_1 \beta_2}{\nu_1^+} \left\| \nabla^2 n^- \right\|_{L^2} + C_3 \beta_4 \frac{1}{2 \beta_3} \left\| \nabla^2 n^+ \right\|_{L^2}^2,
\]
where \( C_4 \leq \min \left\{ \frac{C_2 v_1^-}{24}, \frac{C_4 (\nu_1^- + \nu_2^-)}{2 C_0^2} \right\} \) and we have used the fact that \( \delta_0 \) is sufficiently small. Taking \( \beta_2 \leq \min \left\{ \frac{1}{2}, \frac{C_4 \nu_1^+}{4 \beta_3^4} \right\} \), we get
\[
\frac{d}{dt} \phi'(t) + \frac{\beta_1}{6 \beta_2} \left( \nu_1^+ \left\| \nabla^3 u^+ \right\|_{L^2} + \nu_2^+ \left\| \nabla^2 \text{div } u^+ \right\|_{L^2} \right) + C_1 \beta_1 \left( \frac{1}{\beta_2^2} - 1 \right) \left\| \nabla^2 n^- \right\|_{L^2}^2
\]
\[+ C_3 \beta_4 \frac{1}{2 \beta_3} \left( \nu_1 \left\| \nabla^3 u^- \right\|_{L^2} + \nu_2 \left\| \nabla^2 \text{div } u^- \right\|_{L^2} \right)
\]
\[+ C_4 \left( \frac{1}{3 \beta_3} \left\| \nabla^2 n^- \right\|_{L^2} + \left\| \nabla^2 n^- \right\|_{L^2} \right) \leq C_0 \left( \left\| \nabla^2 (n^+, n^-) \right\|_{L^2}^2 + \left\| \nabla^2 (u^+, u^-) \right\|_{L^2}^2 \right)
\]
where
\[
\phi'(t) = \phi_1(t) + \phi_2(t),
\]
and
\[
\phi_2(t) = C_3 \beta_4 \frac{1}{2 \beta_3} \left\| \nabla^2 n^- \right\|_{L^2}^2 + \frac{1}{2 \beta_3} \left\| \nabla^2 u^- \right\|_{L^2}^2 + C_4 \left\langle \nabla u^-, \nabla^2 n^- \right\rangle.
\]
Due to \( C_4 \leq \frac{C_4 (\nu_1^- + \nu_2^-)}{2 C_0^2} \), we have from Lemma A.4 and Young’s inequality that
\[
\frac{1}{C_3} \left\| \nabla^2 (n^-, u^-) \right\|_{L^2}^2 \leq \phi_2(t) \leq C_5 \left\| \nabla^2 (n^-, u^-) \right\|_{L^2}^2
\]
for some positive constant \( C_5 \) independent of \( \delta_0 \) and \( \beta_2 \). Therefore, we have
\[
\frac{1}{C_6} \left( \frac{1}{\beta_2^2} \left\| \nabla^2 (n^+, u^+) \right\|_{L^2}^2 + \left\| \nabla^2 (n^-, u^-) \right\|_{L^2}^2 \right)
\]
\[\leq \phi'(t) \leq C_6 \left( \frac{1}{\beta_2^2} \left\| \nabla^2 (n^+, u^+) \right\|_{L^2}^2 + \left\| \nabla^2 (n^-, u^-) \right\|_{L^2}^2 \right),
\]
for some positive constant \( C_6 \) independent of \( \delta_0 \) and \( \beta_2 \). Choosing \( \beta_2 \) sufficiently small, such that
\[
\frac{C_1 \beta_1}{2} \left( \frac{1}{\beta_2} - 1 \right) \cdot \frac{C_4 \beta_4}{2 \beta_3} \geq 4(C_1 + C_4)^2,
\]
that is,

\[
\beta_2 \leq \min\left\{ \frac{1}{2}, \frac{C_4 v_1^4 \beta_2}{4 \beta_1^3}, \frac{1}{1 + 16 \beta_2^2 (C_1 + C_4)^2} \right\},
\] (3.45)

we have

\[
\frac{C_1 \beta_1}{2} \left( \frac{1}{\beta_2} - 1 \right) \left\| \nabla^2 n^{+, h} \right\|_{L^2}^2 + C_4 \beta_2 \frac{\beta_4}{2 \beta_1^3} \left\| \nabla^2 n^{-, h} \right\|_{L^2}^2 + (C_1 + C_4) \left\langle \nabla V n^{-, h}, \nabla V n^{+, h} \right\rangle
\geq - \frac{C_4 \beta_1}{4} \left( \frac{1}{\beta_2} - 1 \right) \left\| \nabla^2 n^{+, h} \right\|_{L^2}^2 + C_3 \frac{\beta_4}{4 \beta_3^3} \left\| \nabla^2 n^{-, h} \right\|_{L^2}^2
\geq \frac{C_1 \beta_1}{8} \left\| \nabla^2 n^{+, h} \right\|_{L^2}^2 \] + \frac{C_4 \beta_4}{4 \beta_3^3} \left\| \nabla^2 n^{-, h} \right\|_{L^2}^2.
\] (3.46)

As a result, combining Equation (3.38) with Equation (3.46) and using Lemma A.4 and the smallness of \( \delta_0 \), we have

\[
\frac{d}{dt} E(t) + C_7 \left\{ \frac{1}{\beta_2} \left( \left\| \nabla^3 u^+ \right\|^2_{L^2} + \left\| \nabla^2 n^{+, h} \right\|^2_{L^2} \right) \right. \\
+ \left. \left( \left\| \nabla^3 u^- \right\|^2_{L^2} + \left\| \nabla^2 n^{-, h} \right\|^2_{L^2} \right) \right\} \leq \left\| \nabla^2 (u^{+, i}, u^{+, j}, n^{-, i}, u^{-, j}) \right\|^2_{L^2},
\] (3.47)

for some positive constant \( C_7 \) independent of \( \delta_0 \) and \( \beta_2 \). Plugging \( C_7 \left( \frac{1}{\beta_2} \left\| \nabla^2 n^{+, h} \right\|^2_{L^2} + \left\| \nabla^2 n^{-, h} \right\|^2_{L^2} \) \) into two sides of Equation (3.47), we have

\[
\frac{d}{dt} \mathcal{E}(t) + C_7 \left\{ \frac{1}{\beta_2} \left( \left\| \nabla^3 u^+ \right\|^2_{L^2} + \left\| \nabla^2 n^+ \right\|^2_{L^2} \right) \right. \\
+ \left. \left( \left\| \nabla^3 u^- \right\|^2_{L^2} + \left\| \nabla^2 n^- \right\|^2_{L^2} \right) \right\} \leq \left\| \nabla^2 (u^{+, i}, u^{+, j}, n^{-, i}, u^{-, j}) \right\|^2_{L^2}.
\] (3.48)

Using Lemma A.4 again, we have

\[
\left\| \nabla^2 (u^+, u^-) \right\|^2_{L^2} \leq C_0 \left[ \left\| \nabla^3 (u^+, u^-) \right\|^2_{L^2} + \left\| \nabla^2 (u^{+, i}, u^{-, j}) \right\|^2_{L^2},
\] (3.49)

which together with Equation (3.48) implies

\[
\frac{d}{dt} \mathcal{E}(t) + C_8 \mathcal{E}(t) \leq \left\| \nabla^2 (u^{+, i}, u^{+, j}, n^{-, i}, u^{-, j}) \right\|^2_{L^2},
\] (3.50)

for some positive constant \( C_8 \) independent of \( \delta_0 \) and \( \beta_2 \).

Next, we employ Proposition (2.6) to deduce the optimal decay rate of \( \left\| \nabla^2 (u^{+, i}, u^{+, j}, n^{-, i}, u^{-, j}) \right\|_{L^2} \). To begin with, by defining \( U = (n^+, u^+, n^-, u^-)' \) and \( \mathcal{F} = (\mathcal{F}^1, \mathcal{F}^2, \mathcal{F}^3, \mathcal{F}^4)' \), we have from Duhamel’s principle that

\[
U = e^{\mathcal{F}(t)} U(0) + \int_0^t e^{(\mathcal{F} - \mathcal{F}(\tau))} \mathcal{F}(\tau) d\tau.
\] (3.51)
By virtue of Proposition 2.6, Equation (3.51), Plancherel theorem, and Hölder’s inequality, we have

\[ \| \nabla^2 (n^+, l, u^+, l, n^-, l, u^-, l)(t) \|_{L^2} \leq \| |\xi|^2 (\hat{n}^+, l, \hat{u}^+, l, \hat{n}^-, l, \hat{u}^-, l)(t) \|_{L^2} \lesssim (1 + t)^{-\frac{7}{4}} \| (n^+, u^+, n^-, u^-)(0) \|_{L^1} + \int_0^t (1 + t - \tau)^{-\frac{7}{4}} \| \mathcal{F}(\tau) \|_{L^1} d\tau. \] (3.52)

Next, we shall estimate the second term on the right-hand side of Equation (3.52). To do this, by virtue of the definition of \( \mathcal{F} \) and Equations (1.22)–(1.23), we can bound the term \( \| \mathcal{F}(t) \|_{L^1} \) by

\[ \| \mathcal{F}(t) \|_{L^1} \lesssim \| (\text{div}(n^\pm u^\pm), n^\pm \nabla n^\pm, n^\pm \nabla n^+, n^\pm \nabla u^+, n^\pm \nabla u^-, n^\pm \nabla^2 u^\pm, n^\pm \nabla^2 u^\pm)(t) \|_{L^1} \lesssim \| (n^+, u^+, n^-, u^-)(t) \|_{L^2}^2 + \| (n^+, u^+, n^-, u^-)(t) \|_{L^2} + \| (n^+, n^-)(t) \|_{L^2} \| \nabla^2 (u^+, u^-)(t) \|_{L^2} \lesssim C(N_0)(1 + t)^{-2}. \] (3.53)

Plugging Equation (3.53) into Equation (3.52) gives that

\[ \| \nabla^2 (n^+, l, u^+, l, n^-, l, u^-, l)(t) \|_{L^2} \lesssim C(N_0)(1 + t)^{-\frac{7}{2}}. \] (3.54)

Finally, substituting Equation (3.54) into Equation (3.50) and using Gronwall’s inequality, we conclude that

\[ \mathcal{E}(t) \lesssim (1 + t)^{-\frac{7}{2}}, \] (3.55)

which implies Equation (3.3).

Therefore, the proof of Theorem 3.2 is completed. \( \square \)

In what follows, we will devote ourselves to dealing with Case II: \( N > 2 \). To begin with, for any \( 0 \leq \ell \leq N \), we define the time-weighted energy functional as

\[ \mathcal{E}^N_\ell(t) = \sup_{0 \leq \tau \leq t} \left\{ (1 + \tau)^{\frac{3}{2} + \frac{\ell}{2}} \| \nabla^\ell (n^+, u^+, n^-, u^-)(\tau) \|_{H^{N-\ell}} \right\}. \] (3.56)

Therefore, it suffices to prove that for any \( 0 \leq \ell \leq N \), \( \mathcal{E}^N_\ell(t) \) has an uniform time-independent bound for \( t \geq t_0 \) with a sufficiently large positive constant \( t_0 \). In what follows, we always assume \( t \geq t_0 \). We will take advantage of the low- and high-frequency decomposition and use the key linear convergence estimates obtained in Section 2 to achieve this goal by induction.

**Theorem 3.3** (Case II: \( N > 2 \)). Assume that the hypotheses of Theorem 3.1 and Equation (1.33) are in force. Then there exists a positive constant \( C \), which is independent of \( t \), such that

\[ \mathcal{E}^N_\ell(t) \leq C(N_0), \] (3.57)

for \( 0 \leq \ell \leq N \).

**Proof.** We will employ finite mathematical induction to prove Theorem 3.3. Therefore, it suffices to prove the following Lemmas 3.4 and 3.5. Thus, the proof Theorem 3.3 is completed. \( \square \)

The first lemma is concerned with the estimate on \( \mathcal{E}^N_0(t) \).
**Lemma 3.4.** Assume that the hypotheses of Theorem 3.1 and Equation (1.33) are in force. Then there exists a positive constant $C$, which is independent of $t$, such that

$$
\delta^N_0(t) \leq C N_0. 
$$

**Proof.** Using the similar argument of Evje–Wang–Wen [21] for the a priori estimates on $(n^+, u^+, n^-, u^-)$, it is straightforward to deduce that there exists a temporal energy functional $\phi^N_0(t)$, which is equivalent to $\|(n^+, u^+, n^-, u^-)(t)\|_{H^N}^2$ and satisfies

$$
\frac{d}{dt} \phi^N_0(t) + \|\nabla (n^+, n^-)(t)\|_{H^{N-1}}^2 + \|\nabla (u^+, u^-)(t)\|_{H^N}^2 \leq 0, 
$$

which implies that there exists a positive constant $D_1$ such that

$$
\frac{d}{dt} \phi^N_0(t) + \frac{1}{D_1} \phi^N_0(t) \leq C \|(n^+, u^+, n^-, u^-)(t)\|_{L^2}^2, 
$$

where we have used Lemma A.4.

Similar to the proof of Equation (3.54), we have

$$
\left\| (n^+, u^+, n^-, u^-)(t) \right\|_{L^2} \lesssim C (1 + t)^{-\frac{3}{2}} (N_0 + \delta_0 \phi^N_0(t)). 
$$

Substituting Equation (3.61) into Equation (3.60) yields that

$$
\frac{d}{dt} \phi^N_0(t) + \frac{1}{D_1} \phi^N_0(t) \leq C (1 + t)^{-\frac{3}{2}} (N_0 + \delta_0 \phi^N_0(t))^2, 
$$

which together with Gronwall’s argument gives

$$
\phi^N_0(t) \leq e^{-\frac{1}{D_1} t} \phi^N_0(0) + C \int_0^t e^{-\frac{1}{D_1} (t-\tau)} (1 + t - \tau)^{-\frac{3}{2}} (N_0 + \delta_0 \phi^N_0(t))^2 \, d\tau 
$$

$$
\leq C (1 + t)^{-\frac{3}{2}} (N_0^2 + \delta_0^2 \phi^N_0(t))^2, 
$$

which implies that

$$
(1 + t)^{\frac{3}{2}} \| (n^+, u^+, n^-, u^-)(t) \|_{H^N}^2 \leq C (N_0^2 + \delta_0^2 (\phi^N_0(t))^2). 
$$

Since $\phi^N_0(t)$ is nondecreasing, it follows from Equation (3.64) that

$$
(\phi^N_0(t))^2 \leq C N_0^2 + \delta_0^2 (\phi^N_0(t))^2, 
$$

which implies

$$
\phi^N_0(t) \leq C N_0, 
$$

if $\delta_0$ is sufficiently small. Therefore, the proof of Lemma 3.4 has been completed.

The next lemma is devoted to closing the estimates $\phi^N_\ell(t)$, $1 \leq \ell \leq N$.

**Lemma 3.5.** Assume that the hypotheses of Theorem 3.1 and Equation (1.33) are satisfied. If additionally

$$
\phi^N_\ell(t) \leq C(N_0), 
$$

then it holds that

$$
\phi^N_\ell(t) \leq C(N_0), 
$$

for $1 \leq \ell \leq N$ and $t \geq t_0$. 

Proof. We will combine the key linear estimates with delicate nonlinear energy analysis based on good properties of the low- and high-frequency decomposition to prove Lemma 3.5, and the process involves the following four steps.

Step 1. $L^2$ estimate of $(\nabla^j n^+, \nabla^j u^+, \nabla^j n^-, \nabla^j u^-)$ with $\ell \leq j \leq N$. First, similar to the proof of Equation (3.54), we also have

\[
\| \nabla^j (n^+, u^+, n^-, u^-)(t) \|_{L^2} \leq (1 + t)^{-\frac{3}{4} - \frac{j}{2}} \| \nabla^j (n^+, u^+, n^-, u^-)(0) \|_{L^1} \leq (1 + t)^{-\frac{3}{4} - \frac{j}{2}} \| \nabla^j (n^+, u^+, n^-, u^-)(0) \|_{L^1} + \int_0^t (1 + \tau)^{-\frac{3}{4} - \frac{j}{2}} \| \nabla^j \mathcal{F}(\tau) \|_{L^1} d\tau
\]

(3.69)

Next, we shall estimate the second term on the right-hand of Equation (3.69). The main idea of our approach is to make full use of the benefit of the low- and high-frequency decomposition. To see this, by virtue of the assumption (3.67), Lemma A.4, we can bound the term $\| \nabla^j (\nabla^j \psi)^{\ell}(t) \|_{L^1}$ by

\[
\| \nabla^j (\nabla^j \psi)^{\ell}(t) \|_{L^1} \leq \| \nabla^j (\nabla^j \psi)^{\ell}(t) \|_{L^2} + \| \nabla^j (\nabla^j \psi)^{\ell}(t) \|_{L^1} - \| \nabla^j (\nabla^j \psi)^{\ell}(t) \|_{L^2}
\]

(3.70)

Substituting Equation (3.70) into Equation (3.69) yields that

\[
\| \nabla^j (n^+, u^+, n^-, u^-)(t) \|_{L^2} \leq C(N_0)(1 + t)^{-\frac{3}{4} - \frac{j}{2}}.
\]

(3.71)

It should be mentioned that the low-frequency convergence estimate (3.71) plays a critical role in proving the optimal convergence rates of the highest-order spatial derivatives of solutions.

Step 2. $L^2$ estimate of $(\nabla^j n^+, \nabla^j u^+, \nabla^j n^-, \nabla^j u^-)$ with $\ell \leq j \leq N$. Multiplying $\nabla^j (2.15)_1, \nabla^j (2.15)_2, \nabla^j (2.15)_3$, and $\nabla^j (2.15)_4$ by $\frac{\beta_1}{\beta_2} \nabla^j n^+, \frac{\beta_1}{\beta_2} \nabla^j u^+, \frac{\beta_1}{\beta_2} \nabla^j n^-, \text{ and } \frac{\beta_1}{\beta_2} \nabla^j u^-$, respectively, and then integrating over $\mathbb{R}^3$, we have

\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{\beta_1}{\beta_2} \| \nabla^j n^+ \|_2^2 + \frac{\beta_1}{\beta_2} \| \nabla^j u^+ \|_2^2 \right\} + \frac{\beta_1}{\beta_2} \left\{ \| \nabla^j+1 u^+ \|_2^2 + \| \nabla^j u^- \|_2^2 \right\}
\]

(3.72)
and

\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{\beta_4}{\beta_3} \|\nabla^j n^-\|^2 + \frac{\beta_4}{\beta_3} \|\nabla^j u^-\|^2 \right\} + \frac{\beta_4}{\beta_3} \left( \nu^-_1 \|\nabla^{j+1} u^-\|^2 + \nu^-_2 \|\nabla^j \div u^-\|^2 \right)
= \left\langle \nabla^j \mathcal{F}_3, \frac{\beta_4}{\beta_3} \nabla^j n^- \right\rangle + \left\langle \nabla^j \mathcal{F}_4, \frac{\beta_4}{\beta_3} \nabla^j u^- \right\rangle - \left\langle \nabla^j \nabla^j u^+, \frac{\beta_4}{\beta_3} \nabla^j u^- \right\rangle
= : K^j_4 + K^j_5 + K^j_6.
\] (3.73)

From Lemmas A.1 and A.2, integration by parts and Young’s inequality, we have

\[
\|K^j_1\| \lesssim \left( \|\nabla^j n^+, \nabla^j (n^+ \div u^+)\| + \|\nabla^j (n^+ \cdot u^+)\| \right)
\lesssim \|\nabla^j n^+\|_L^2 \left( \|\nabla^j n^+\|_L^2 \|\nabla u^+\|_L^\infty + \|n^+\|_L^\infty \|\nabla^{j+1} u^+\|_L^2 \right)
+ \|\nabla^j n^+\|_L^2 \|\nabla u^+\|_L^\infty + \|\nabla^j n^+\|_L^2 \|\nabla (n^+ \cdot u^+) - \nabla^{j+1} n^+ \cdot u^+\|_L^2
\lesssim \|\nabla^j n^+\|_L^2 \left( \|\nabla^j n^+\|_L^2 \|\nabla^2 u^+\|_L^2 \|\nabla^3 u^+\|_L^2 + \|n^+\|_L^\infty \|\nabla^{j+1} u^+\|_L^2 \right)
+ \|\nabla^j n^+\|_L^2 \left( \|\nabla^2 n^+\|_L^2 \|\nabla^2 u^+\|_L^2 \|\nabla^3 u^+\|_L^2 + \|n^+\|_L^2 \|\nabla u^+\|_L^\infty \|\nabla^{j+1} u^+\|_L^2 \right)
\lesssim (1 + t_0)^{-7/8} \left( \|\nabla^j n^+\|_L^2 + \|\nabla^{j+1} u^+\|_L^2 \right),
\] (3.74)

where we have used Equation (3.2) for \( N = 3 \) and Equation (3.3). Similarly, for the term \( K^j_4 \), we have

\[
\|K^j_4\| \lesssim (1 + t_0)^{-7/8} \left( \|\nabla^j n^+\|_L^2 + \|\nabla^{j+1} u^+\|_L^2 \right).
\] (3.75)

Employing the similar arguments used in Equation (3.75), we also have

\[
\|K^j_2\| \lesssim \left| \left\langle \nabla^{j-1} \left[ g_+ (n^+, n^-) \nabla n^+ \right], \nabla^{j+1} u^+ \right\rangle \right|
+ \left| \left\langle \nabla^{j-1} \left[ g_+ (n^+, n^-) \nabla n^- \right], \nabla^{j+1} u^+ \right\rangle \right|
+ \left| \left\langle \nabla^{j-1} \left[ (u^+ \cdot \nabla) u^+ \right], \nabla^{j+1} u^+ \right\rangle \right|
+ \left| \left\langle \nabla^{j-1} \left[ h_+ (n^+, n^-) (\nabla n^+ \cdot \nabla) u^+ \right], \nabla^{j+1} u^+ \right\rangle \right|
+ \left| \left\langle \nabla^{j-1} \left[ k_+ (n^+, n^-) (\nabla n^- \cdot \nabla) u^+ \right], \nabla^{j+1} u^+ \right\rangle \right|
+ \left| \left\langle \nabla^{j-1} \left[ h_+ (n^+, n^-) \nabla n^+ \cdot \nabla^j u^+ \right], \nabla^{j+1} u^+ \right\rangle \right|
+ \left| \left\langle \nabla^{j-1} \left[ k_+ (n^+, n^-) \nabla n^- \cdot \nabla^j u^+ \right], \nabla^{j+1} u^+ \right\rangle \right|
+ \left| \left\langle \nabla^{j-1} \left[ h_+ (n^+, n^-) \nabla n^+ \div u^+ \right], \nabla^{j+1} u^+ \right\rangle \right|
+ \left| \left\langle \nabla^{j-1} \left[ k_+ (n^+, n^-) \nabla n^- \div u^+ \right], \nabla^{j+1} u^+ \right\rangle \right|
+ \left| \left\langle \nabla^{j-1} \left[ l_+ (n^+, n^-) \Delta u^+ \right], \nabla^{j+1} u^+ \right\rangle \right|
+ \left| \left\langle \nabla^{j-1} \left[ l_+ (n^+, n^-) \div u^+ \right], \nabla^{j+1} u^+ \right\rangle \right|
\]
\[
\begin{align*}
\lesssim & \|\nabla^{j+1}u\|_{L^2}^2 \left( \|g_+(n^+, n^-)\|_{L^\infty}^2 + \|\nabla^{j-1}g_+(n^+, n^-)\|_{L^6} \|\nabla n^+\|_{L^3} \right) \\
+ & \|\nabla^{j+1}u\|_{L^2}^2 \left( \|\bar{g}_+(n^+, n^-)\|_{L^\infty}^2 + \|\nabla^{j-1}\bar{g}_+(n^+, n^-)\|_{L^6} \|\nabla n^-\|_{L^3} \right) \\
+ & \|\nabla^{j+1}u\|_{L^2} \|u^+\|_{L^3} \|\nabla^{j+1}u^+\|_{L^2} + \|\nabla^{j-1}u^+\|_{L^3} \|\nabla u^+\|_{L^3} \\
+ & \|\nabla^{j+1}u\|_{L^2} \|\nabla^{j-1} \left[ h_+(n^+, n^-) \nabla u^+ \right]\|_{L^6} \|\nabla n^+\|_{L^3} \\
+ & \|\nabla^{j+1}u\|_{L^2} \|\nabla^{j-1} h_+(n^+, n^-)\|_{L^6} \|\nabla^2 u^+\|_{L^3} \\
\leq & (1 + t_0)^{-\frac{7}{8}} \left( \|\nabla^j (n^+, n^-)\|_{L^2}^2 + \|\nabla^j u^+\|_{H^1}^2 \right) .
\end{align*}
\]

Similarly, for the term \(K_5^j\), we have
\[
|K_5^j| \lesssim (1 + t_0)^{-\frac{7}{8}} \left( \|\nabla^j (n^+, n^-)\|_{L^2}^2 + \|\nabla^j u^-\|_{H^1}^2 \right) .
\]

For the terms \(K_3^j\) and \(K_6^j\), we have from Young’s inequality that
\[
|K_3^j| \lesssim \frac{\beta_1 \beta_2}{\nu_1} \|\nabla^j n^-\|_{L^2}^2 + \frac{\beta_1}{4\beta_2} \nu_1^+ \|\nabla^j u^+\|_{L^2}^2 ,
\]
and
\[
|K_6^j| \lesssim \frac{\beta_3 \beta_4}{\nu_1} \|\nabla^j n^+\|_{L^2}^2 + \frac{\beta_4}{4\beta_3} \nu_1^- \|\nabla^j u^-\|_{L^2}^2 .
\]

Combining the relations (3.72)–(3.79) and using the fact that \(t_0\) is sufficiently large, we finally conclude that
\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{\beta_1}{\beta_2} \|\nabla^j n^-\|_{L^2}^2 + \frac{\beta_1}{\beta_2} \|\nabla^j u^+\|_{L^2}^2 \right\} + \frac{\beta_1}{2\beta_2} \left( \nu_1^+ \|\nabla^j u^+\|_{L^2}^2 + \nu_2^- \|\nabla^j \text{div } u^+\|_{L^2}^2 \right) \\
\leq C (1 + t_0)^{-\frac{7}{8}} \left( \|\nabla^j (n^+, n^-)\|_{L^2}^2 + \|\nabla^j u^+\|_{L^2}^2 \right) + \frac{\beta_1 \beta_2}{\nu_1^+} \|\nabla^j n^-\|_{L^2}^2 ,
\]
and
\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{\beta_4}{\beta_3} \|\nabla^j n^+\|_{L^2}^2 + \frac{\beta_4}{\beta_3} \|\nabla^j u^-\|_{L^2}^2 \right\} + \frac{\beta_4}{2\beta_3} \left( \nu_1^- \|\nabla^j u^-\|_{L^2}^2 + \nu_2^- \|\nabla^j \text{div } u^-\|_{L^2}^2 \right) \\
\leq C (1 + t_0)^{-\frac{7}{8}} \left( \|\nabla^j (n^+, n^-)\|_{L^2}^2 + \|\nabla^j u^-\|_{L^2}^2 \right) + \frac{\beta_3 \beta_4}{\nu_1^-} \|\nabla^j n^+\|_{L^2}^2 ,
\]
for some positive constant \(C\) independent of \(t_0\).
Step 3. Dissipation of $\nabla^j (n^{+h}, n^{+h})$ with $\ell \leq j \leq N$. Applying the operator $\nabla^{j-1} \mathcal{F}^{-1} (1 - \phi(x))$ to Equation (2.15) and then multiplying the resultant equation by $\frac{1}{\beta_2^j} \nabla^{j-1} u^{+h}$, integrating over $\mathbb{R}^3$, we have

$$
\frac{d}{dt} \left( \nabla^{j-1} u^{+h}, \frac{1}{\beta_2^j} \nabla^{j-1} u^{+h} \right) + \frac{\beta_1^j}{\beta_2^j} \left\| \nabla^{j-1} u^{+h} \right\|_{L^2}^2 + \left( \nabla^{j-1} n^{-h}, \nabla^{j-1} n^{+h} \right) \\
= \frac{1}{\beta_2^j} \left( \nabla^{j-1} u^{+h}, \partial_t \nabla^{j-1} n^{+h} \right) + \frac{\nu^+}{\beta_2^j} \left( \nabla^{j-1} \Delta u^{+h}, \nabla^{j-1} n^{+h} \right) + \frac{\nu^+}{\beta_2^j} \left( \nabla^{j} \text{div} u^{+h}, \nabla^{j} n^{+h} \right) \\
+ \frac{1}{\beta_2^j} \left( \nabla^{j-1} \mathcal{F} h^{1}, \nabla^{j} n^{+h} \right) \\
:= L_1^{j} + L_2^{j} + L_3^{j} + L_4^{j}. 
$$

(3.82)

Due to Equation (3.2), Lemmas A.1–A.4, we have from integration by parts, and Young’s inequality that

$$
|L_1^{j}| = \left| - \frac{1}{\beta_2^j} \left( \nabla^{j-1} u^{+h}, \beta_1^j \nabla^{j} \text{div} u^{+h} \right) + \frac{1}{\beta_2^j} \left( \nabla^{j-1} u^{+h}, \nabla^{j} \mathcal{F} h^{1} \right) \right| \\
= \frac{\beta_1^j}{\beta_2^j} \left\| \nabla^{j-1} u^{+h} \right\|_{L^2}^2 - \frac{1}{\beta_2^j} \left( \nabla^{j-1} \text{div} u^{+h}, \nabla^{j-1} \mathcal{F} h^{1} \right) \\
\leq \frac{\beta_1^j}{\beta_2^j} \left\| \nabla^{j-1} u^{+h} \right\|_{L^2}^2 + C \left\| \nabla^{j-1} \text{div} u^{+h} \right\|_{L^2} \left( \left\| u^{+h} \right\|_{L^6} \left\| \nabla^{j} u^{+h} \right\|_{L^3} + \left\| u^{+h} \right\|_{H^1} \left\| \nabla^{j+1} u^{+h} \right\|_{L^2} \right) \\
\leq \frac{\beta_1^j}{\beta_2^j} \left\| \nabla^{j-1} u^{+h} \right\|_{L^2}^2 + C \delta_0 \left( \left\| \nabla^{j} u^{+h} \right\|_{L^2}^2 + \left\| \nabla^{j+1} u^{+h} \right\|_{L^2}^2 \right). 
$$

(3.83)

From Equation (2.15), $L_2^{j} + L_3^{j}$ can be rewritten as

$$
L_2^{j} + L_3^{j} = \frac{\nu^+}{\beta_2^j} + \frac{\nu^+}{\beta_2^j} \left( \nabla^{j} \text{div} u^{+h}, \nabla^{j} n^{+h} \right) \\
= - \frac{\nu^+ + \nu^+}{2\beta_1^j \beta_2^j} \frac{d}{dt} \left\| \nabla^{j} n^{+h} \right\|_{L^2}^2 - \frac{\nu^+ + \nu^+}{2\beta_1^j \beta_2^j \sqrt{\alpha_1}} \left( \nabla^{j} \text{div}(u^{+h}), \nabla^{j} n^{+h} \right). 
$$

(3.84)

On the other hand, we have

$$
\left\langle \nabla^{j} \text{div}(u^{+h} n^{+h}), \nabla^{j} n^{+h} \right\rangle = \left\langle \nabla^{j} \text{div}(u^{+h}) - \nabla^{j} \text{div}(u^{+h})^{1}, \nabla^{j} n^{+h} \right\rangle \\
= \left\langle \nabla^{j} \text{div}(u^{+h} n^{+h}), \nabla^{j} n^{+h} \right\rangle + \left\langle \nabla^{j} \text{div}(u^{+h} n^{+h}), \nabla^{j} n^{+h} \right\rangle \\
- \left\langle \nabla^{j} \text{div}(u^{+h})^{1}, \nabla^{j} n^{+h} \right\rangle \\
=: L_{1,2}^{j} + L_{2,3}^{j} + L_{3,3}^{j}. 
$$

(3.85)

For the term $L_{2,1}^{j}$, by using integration by parts, Lemmas A.1–A.4 and Young’s inequality, we have

$$
\left| L_{1,2}^{j} \right| = \left| - \left\langle \text{div} u^{+}, \nabla^{j} n^{+h} \right\rangle \right| + \left\langle \nabla^{j} (u^{+} \cdot \nabla^{j} n^{+h}) - u^{+} \cdot \nabla^{j+1} n^{+h}, \nabla^{j} n^{+h} \right\rangle + \left\langle \nabla^{j} (n^{+h} \text{div} u^{+}), \nabla^{j} n^{+h} \right\rangle \\
\leq \left\| \text{div} u^{+} \right\|_{L^{\infty}} \left\| \nabla^{j} n^{+h} \right\|_{L^2} + \left\| \nabla^{j} n^{+h} \right\|_{L^2} \left( \left\| u^{+} \right\|_{L^{\infty}} \left\| \nabla^{j} u^{+} \right\|_{L^3} + \left\| \nabla^{j} u^{+} \right\|_{L^6} \left\| \nabla^{j+1} u^{+} \right\|_{L^2} \right) \\
+ \left\| \nabla^{j} n^{+h} \right\|_{L^2} \left( \left\| \nabla^{j+1} u^{+} \right\|_{L^2} \left\| n^{+h} \right\|_{L^6} + \left\| \text{div} u^{+} \right\|_{L^{\infty}} \left\| \nabla^{j} n^{+h} \right\|_{L^2} \right). 
$$
\[ \begin{align*}
\|\nabla^2 u^+\|_{L^2} + \|\nabla^3 u^+\|_{L^2} + \|\nabla n^+\|_{H^1} + \|\nabla n^{+h}\|_{L^2} + \|\nabla^j u^+\|_{L^2} \\
\lesssim \left( \delta_0 + (1 + t_0)^{-\frac{7}{8}} \right) \left( \|\nabla n^{+h}\|_{L^2}^2 + \|\nabla^j u^+\|_{L^2}^2 \right),
\end{align*} \]

(3.86)

where we have used Equation (3.2) for \( N = 3 \) and Equation (3.3). By using similar arguments, for the terms \( L_{2,3}^{j,2} \) and \( L_{2,3}^{j,3} \), we have

\[ \begin{align*}
|L_{2,3}^{j,2}| &\lesssim \|\nabla n^{+h}\|_{L^2} \left( \|\nabla^{j+1} n^+\|_{L^2} + \|n^{+h}\|_{L^\infty} \right) + \|\nabla^{j+1} u^+\|_{L^2}, \\
|L_{2,3}^{j,3}| &\lesssim \|\nabla n^{+h}\|_{L^2} \left( \|\nabla^{j+1} n^+\|_{L^2} + \|\nabla^{j+1} u^+\|_{L^2} \right) \left( \delta_0 + (1 + t_0)^{-\frac{7}{8}} \right)
\end{align*} \]

(3.87)

Combining the relations (3.84)–(3.88), we have

\[ |L_2^j + L_3^j| \lesssim -\frac{\nu^+ + \nu^+}{2\beta_1 \beta_2} \frac{d}{dt} \|\nabla n^{+h}\|_{L^2}^2 + \left( \delta_0 + (1 + t_0)^{-\frac{7}{8}} \right) \left( \|\nabla n^+\|_{L^2}^2 + \|\nabla^{j+1} u^+\|_{L^2}^2 \right). \]

(3.89)

Applying the similar arguments used in Equation (3.8), for the term \( L_4^j \), we have

\[ |L_4^j| \lesssim \left( \delta_0 + (1 + t_0)^{-\frac{7}{8}} \right) \left( \|\nabla (n^+, n^-)\|_{L^2}^2 + \|\nabla^j u^+\|_{H^1}^2 \right). \]

(3.90)

Combining the relations (3.82)–(3.83) and (3.89)–(3.90), we finally conclude that

\[ \begin{align*}
\frac{d}{dt} \left\{ \frac{\nu^+ + \nu^+}{2\beta_1 \beta_2} \|\nabla n^{+h}\|_{L^2}^2 + \left( \frac{\beta_1}{\beta_2} \right) \|\nabla^{j-1} u^{+h} + \frac{1}{\beta_2} \nabla n^{+h}\|_{L^2}^2 \right\} \\
+ \frac{\beta_1}{\beta_2} \|\nabla^{j-1} u^{+h} + \frac{1}{\beta_2} \nabla n^{+h}\|_{L^2}^2 + \left( \delta_0 + (1 + t_0)^{-\frac{7}{8}} \right) \left( \|\nabla (n^+, n^-)\|_{L^2}^2 + \|\nabla^j u^+\|_{H^1}^2 \right) \\
\leq C \left( \frac{\beta_1}{\beta_2} \|\nabla^j u^+\|_{L^2}^2 + \left( \delta_0 + (1 + t_0)^{-\frac{7}{8}} \right) \left( \|\nabla (n^+, n^-)\|_{L^2}^2 + \|\nabla^j u^+\|_{H^1}^2 \right) \right).
\end{align*} \]

(3.91)

Similarly, for the dissipation estimate of \( \nabla n^{-j, h} \), we also have

\[ \begin{align*}
\frac{d}{dt} \left\{ \frac{\nu^- + \nu^-}{2\beta_3 \beta_4} \|\nabla n^{-j, h}\|_{L^2}^2 + \left( \frac{\beta_3}{\beta_4} \right) \|\nabla^{j-1} u^{-j, h} + \frac{1}{\beta_4} \nabla n^{-j, h}\|_{L^2}^2 \right\} \\
+ \frac{\beta_3}{\beta_4} \|\nabla^{j-1} u^{-j, h} + \frac{1}{\beta_4} \nabla n^{-j, h}\|_{L^2}^2 + \left( \delta_0 + (1 + t_0)^{-\frac{7}{8}} \right) \left( \|\nabla (n^+, n^-)\|_{L^2}^2 + \|\nabla^j u^-\|_{H^1}^2 \right) \\
\leq C \left( \frac{\beta_3}{\beta_4} \|\nabla^j u^-\|_{L^2}^2 + \left( \delta_0 + (1 + t_0)^{-\frac{7}{8}} \right) \left( \|\nabla (n^+, n^-)\|_{L^2}^2 + \|\nabla^j u^-\|_{H^1}^2 \right) \right).
\end{align*} \]

(3.92)
Step 4: Closing the estimates. Now, we are in a position to close the estimates. Due to the key energy estimates (3.80)–(3.81) and (3.91)–(3.92), for any \(\ell \leq j \leq N\), we can follow the proof of Equation (3.50) step by step to get

\[
\frac{d}{dt} \delta_j(t) + D_2 \delta_j(t) \leq \left\| \nabla^j (n^{+,l}, u^{+,l}, n^{-,l}, u^{-,l}) \right\|_{L^2}^2,
\]

where \(D_2 > 0\) is a given positive constant and \(\delta_j(t)\) is equivalent to \(\|\nabla^j(n^+, u^+, n^-, u^-)\|_{L^2}^2\). Then, summing up the estimate (3.93) from \(j = \ell\) to \(N\), one has

\[
\frac{d}{dt} \tilde{\delta}_\ell^N(t) + D_2 \tilde{\delta}_\ell^N(t) \leq \left\| \nabla^\ell (n^{+,l}, u^{+,l}, n^{-,l}, u^{-,l}) \right\|_{H_N-\ell}^2,
\]

where \(\tilde{\delta}_\ell^N(t)\) is equivalent to \(\|\nabla^\ell(n^+, u^+, n^-, u^-)\|_{H_N-\ell}^2\). Next, applying Gronwall’s inequality to Equation (3.94) and using Equation (3.71) with \(\ell \leq j \leq N\), we obtain

\[
\tilde{\delta}_\ell^N(t) \leq C(N_0)(1 + t)^{-\frac{3}{4} - \frac{\ell}{2}},
\]

which together with the definition of \(\tilde{\delta}_\ell^N(t)\) in Equation (3.56) implies Equation (3.68).

Therefore, the proof of Lemma 3.5 has been completed.

In rest of this section, we devote ourselves to deducing the lower bound on the convergence rate of the global solution to complete the proof of Theorem 1.1.

**Theorem 3.6.** Assume that the hypotheses of Theorem 3.1 and Equation (1.36) are in force, then there is a positive constant \(c_2\) independent of time such that for any large enough \(t\),

\[
\min\{\|\nabla^\ell n^+(t)\|_{L^2}, \|\nabla^\ell u^+(t)\|_{L^2}, \|\nabla^\ell n^-(t)\|_{L^2}, \|\nabla^\ell u^-(t)\|_{L^2}\}
\geq c_2(1 + t)^{-\frac{3}{4} - \frac{\ell}{2}},
\]

for \(0 \leq \ell \leq N\).

**Proof.** If \(t\) is large enough, it follows from Equation (3.51), Proposition 2.4, Lemmas 3.4 and A.3 that

\[
\|\Lambda^{-1}(n^+, u^+, n^-, u^-)(t)\|_{L^2}
\leq \|\Lambda^{-1}(n^{+,l}, u^{+,l}, n^{-,l}, u^{-,l})(t)\|_{L^2} + \|\Lambda^{-1}(n^{+,h}, u^{+,h}, n^{-,h}, u^{-,h})(t)\|_{L^2}
\leq C(N_0)(1 + t)^{-\frac{1}{4}} + \int_0^t (1 + t - \tau)^{-\frac{1}{2}} \|\mathcal{F}(\tau)\|_{L^1} d\tau + \|(n^{+,h}, u^{+,h}, n^{-,h}, u^{-,h})(t)\|_{L^2}
\leq C(N_0)(1 + t)^{-\frac{1}{4}} + \int_0^t (1 + t - \tau)^{-\frac{1}{2}}(1 + \tau)^{-\frac{3}{2}} d\tau
\leq C(N_0)(1 + t)^{-\frac{1}{4}},
\]

and

\[
\min \{\|n^+(t)\|_{L^2}, \|u^+(t)\|_{L^2}, \|n^-(t)\|_{L^2}, \|u^-(t)\|_{L^2}\}
\geq \min \{\|n^{+,l}(t)\|_{L^2}, \|u^{+,l}(t)\|_{L^2}, \|n^{-,l}(t)\|_{L^2}, \|u^{-,l}(t)\|_{L^2}\}
\geq \sqrt{\delta_0} N_0(1 + t)^{-\frac{1}{4}} - \int_0^t (1 + t - \tau)^{-\frac{1}{2}} \|\mathcal{F}(\tau)\|_{L^1} d\tau
\geq C_1 N_0 \sqrt{\delta_0}(1 + t)^{-\frac{1}{4}} - C\delta_0 N_0(1 + t)^{-\frac{3}{4}}
\geq c_3(1 + t)^{-\frac{3}{4}},
\]
since $\delta_0$ is sufficiently small. These together with the interpolations

$$\| f \|_{L^2} \leq C \| f \|_{L^2} \| \nabla f \|_{L^2}^{1/\ell} \| \nabla^{\ell+1} f \|_{L^2}^{1/\ell}$$  \hspace{1cm} (3.99)

imply Equation (3.96) immediately, and thus the proof of Theorem 3.6 is completed.

Therefore, we have completed the proof of Theorem 1.1.

\[ \square \]

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APPENDIX: ANALYTIC TOOLS

We recall the Sobolev interpolation of the Gagliardo–Nirenberg inequality.

**Lemma A.1.** Let $0 \leq i, j \leq k$, then we have

$$
\| \nabla^i f \|_{L^p} \lesssim \| \nabla^j f \|_{L^q}^{1-a} \| \nabla^k f \|_{L^r}^a
$$

(A.1)

where $a$ satisfies

$$
i - \frac{1}{p} = \left( \frac{j}{3} - \frac{1}{q} \right) (1 - a) + \left( \frac{k}{3} - \frac{1}{r} \right) a.
$$

(A.2)

**Proof.** This is a special case of Nirenberg [36, pp. 125, THEOREM].

**Lemma A.2.** For any integer $k \geq 1$, we have

$$
\| \nabla^k (fg) \|_{L^p} \lesssim \| f \|_{L^{p_1}} \| \nabla^k g \|_{L^{p_2}} + \| \nabla^k f \|_{L^{p_3}} \| g \|_{L^{p_4}},
$$

(A.4)

and

$$
\| \nabla^k (fg) - f \nabla^k g \|_{L^p} \lesssim \| f \|_{L^{p_1}} \| \nabla^{k-1} g \|_{L^{p_2}} + \| \nabla^k f \|_{L^{p_3}} \| g \|_{L^{p_4}},
$$

(A.5)

where $p, p_1, p_2, p_3, p_4 \in [1, \infty]$ and

$$
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
$$

(A.6)

**Proof.** See Duan et al. [12].

Finally, the following two lemmas concern the estimate for the low- and high-frequency parts of $f$.

**Lemma A.3.** If $f \in L^r(\mathbb{R}^3)$ for any $2 \leq r \leq \infty$, then we have

$$
\| f^l \|_{L^r} + \| f^h \|_{L^r} \lesssim \| f \|_{L^r}.
$$

(A.7)

**Proof.** For $2 \leq r \leq \infty$, by Young’s inequality for convolutions, for the low frequency, it holds

$$
\| f^l \|_{L^r} \lesssim \| \mathcal{F}^{-1} \phi \|_{L^1} \| f \|_{L^r} \lesssim \| f \|_{L^r},
$$

(A.8)

and hence

$$
\| f^h \|_{L^r} \lesssim \| f \|_{L^r} + \| f^l \|_{L^r} \lesssim \| f \|_{L^r}.
$$

(A.9)

**Lemma A.4.** Let $f \in H^k(\mathbb{R}^3)$ for any integer $k \geq 2$. Then there exists a positive constant $C_0$ such that

$$
\| \nabla^\ell f^h \|_{L^2} \leq C_0 \| \nabla^{\ell+1} f \|_{L^2},
$$

(A.10)

and

$$
\| \nabla^{\ell+1} f^l \|_{L^2} \leq C_0 \| \nabla^\ell f \|_{L^2},
$$

(A.11)

for any $0 \leq \ell \leq k - 1$.

**Proof.** This lemma can be shown directly by the definitions of the low frequency and high frequency of $f$ and the Plancherel theorem, and thus we omit the details.