Dual Gauged Supergravities

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Abstract

We shall review a novel formulation of four dimensional gauged supergravity which is manifestly covariant with respect to the non-perturbative electric-magnetic duality symmetry transformations of the ungauged theory, at the level of the equations of motion and Bianchi identities. We shall also discuss the application of this formalism to the description of M-theory compactified on a twisted torus in the presence of fluxes and to the interpretation from a M/Type IIA theory perspective of the $D = 5 \rightarrow D = 4$ generalized Scherk-Schwarz reduction. This latter analysis will bring up the issue of non-geometric fluxes.
1 Introduction

There are reasons to believe that superstring theory in ten dimensions or the dual M-theory in eleven dimensions may have an important role in the definition of the fundamental quantum theory of gravity. Since we live in a four dimensional universe, the first requirement for any predictable model, based on these theories, is to encode a mechanism of dimensional reduction from ten or eleven dimensions to four. The simplest mechanism of this type is ordinary Kaluza-Klein compactification of string/M-theory on solutions with geometry of the form \( M^{(1,3)} \times \mathcal{M} \), where \( M^{(1,3)} \) is a maximally symmetric four dimensional space–time with Lorentzian signature and \( \mathcal{M} \) is a compact internal manifold. It is known that the low–energy dynamics of string/M–theory realized on these backgrounds, which involve only the massless modes on \( M^{(1,3)} \), is captured (in some cases only in part) by a four dimensional supergravity theory. In our discussion we shall focus on compactifications which yield theories in four dimensions with \( N \geq 2 \) supersymmetries (extended supergravities) on a Minkowski space-time. Supergravity models on four dimensional Minkowski vacua, obtained through ordinary Kaluza–Klein reduction on a Ricci-flat manifold (for instance superstring theory compactified on a Calabi–Yau 3–fold), are far from being phenomenologically interesting, since they are typically plagued, at the classical level, by a plethora of massless scalar fields \( \phi \) (associated for instance with the geometric moduli of the internal manifold which describe its shape and size) whose vacuum expectation values define a continuum of degenerate vacua. In fact there is no dynamics, encoded in some effective scalar potential \( V(\phi) \), which can lift this degeneracy and thus, apart from predicting massless scalars which are not observed in our real world, these models also suffer from an intrinsic lack of predictiveness. An other feature of these supergravities is the absence of a local internal symmetry gauged by the vector fields. In other words the vector fields are not minimally coupled to any other field in the theory. For this reason these models are also called ungauged.

Realistic string/M-theory-based models in four dimensions need to feature a non trivial scalar potential which could on the one hand lift the moduli degeneracy, thus making the theory more predictive, and on the other hand select a vacuum state for our universe with some interesting physical properties such as for instance spontaneous supersymmetry breaking, a hierarchy of scales for gravitational and Standard Model interactions, a (small) positive cosmological constant etc. From a higher dimensional perspective the presence of fluxes (for recent reviews on flux compactifications see [1]) in the space–time background seems to do the job by inducing a scalar potential in the four dimensional theory which can fix part or even all the moduli. By fluxes here we mean not just the v.e.v. of higher dimensional \( p \)-form field strengths across non–trivial cycles of the internal manifold (form fluxes) [2, 3, 4], but also background quantities associated with the geometry of the manifold itself (geometric fluxes) [5]-[17]. Recently the meaning of fluxes has been extended to include background quantities associated with a new class of manifolds with no definite global or even local geometry (non–geometric fluxes) [19, 20, 21, 22]. From a four dimensional perspective, the only known way for introducing a non–trivial scalar potential without explicitly breaking supersymmetry is
through the *gauging* procedure [23]-[30], which consists in promoting a suitable global symmetry group of the Lagrangian to local symmetry by introducing minimal couplings for the vector fields, mass deformations and a scalar potential. Flux compactifications, as opposed to ordinary Kaluza–Klein compactifications to Minkowski space-time, typically yield gauged supergravities in four dimensions and a precise statement can be made about the correspondence between the internal fluxes and the local symmetry of the lower–dimensional field theory. In some cases the effective scalar potential $V(\phi)$, at the classical level, is non–negative and defines vacua with vanishing cosmological constant in which supersymmetry is spontaneously broken and part of the moduli are fixed. Models of this type are generalizations of the so called “no–scale” models [31] which were subject to intense study during the eighties. 

Special care is needed in defining a limit in which (gauged) supergravity is reliable as a low–energy description of a flux compactification. In many cases it suffices for the back reaction of the fluxes on the background geometry to be negligible. This regime, for instance in the case of form-fluxes, can be attained if the size of the internal manifold is much larger than the string scale, so that the following hierarchy of scales is realized: (flux-induced masses) ≪ Kaluza–Klein masses ≪ mass of string excitations. In other cases, the gauged supergravity only describes a consistent truncation of the lowest lying modes, as for the compactification of M-theory on a seven–sphere [23].

The recent study of this new kind of string/M–theory compactifications had opened a Pandora’s box containing an enormous number of possible microscopic settings and thus of vacua. We are left with the hope that on the one hand this picture may simplify by the presence of some duality symmetry underlying the landscape of vacua, as a consequence of which several of these solutions can be thought of as different descriptions of a single microscopic one. On the other hand that there could exist some, yet unknown, dynamic mechanism which can select one vacuum out of the many.

The paper is organized as follows. In section 2, after recalling the main facts about global symmetries in extended four dimensional supergravities and their relation to string dualities, we shall introduce the concept of embedding tensor [25, 27, 28, 29, 30] in terms of which we shall make a precise statement about the correspondence between flux compactifications and gauged supergravity. In section 3 we shall give a formal discussion of gauged supergravities, focusing mainly on the maximally supersymmetric theory. In section 4 a novel formulation of these models [30] is reviewed, in which the global symmetries of the ungauged theory are restored at the level of field equations and Bianchi identities. In section 5 we shall discuss an application of this machinery to the specific example of M–theory compactified on a “twisted” torus in the presence of fluxes and show how two different four dimensional gauged supergravities can actually be considered as “dual” descriptions of the same compactification. In section 6 we shall identify the components of the embedding tensor corresponding to the “non–geometric” fluxes which are T-dual to the NS-NS 3-form flux. We refer the reader to appendix A for a detailed, group theoretical discussion of the flux-embedding tensor correspondence. In section 7 the parameters of the so called “generalized Scherk-Schwarz reduction” from $D = 5$ to $D = 4$ [32, 26] will be described in terms of M–theory/Type IIA
form-, geometric and non-geometric fluxes. In particular we will show that one of these parameters can be described in terms of U-dual fluxes. We shall end with some final comments and outlook.

2 The role of global symmetries

An important role in understanding several non-perturbative aspects of superstring theory has been played by the global symmetries of the lower dimensional supergravity. Behind the concept of string duality there is the idea that superstring theories or M-theory on various backgrounds, are just different realizations of a unique fundamental quantum theory and the correspondence among them is called duality. Upon ordinary dimensional reduction to four dimensional Minkowski space–time, these dualities are conjectured to be encoded in the global symmetries of the resulting (ungauged) supergravity [33]. A wide class of ungauged extended supergravities feature at the classical level a continuous group of global symmetries which act, as we shall see, as generalized electric–magnetic dualities. Already at the field theory level, Dirac-Zwanziger quantization condition on electric and magnetic charges causes this global symmetry group to break to a suitable discrete subgroup. It is the latter which is conjectured to describe the string/M–theory dualities. Here we shall restrict our analysis to classical supergravity only. There are two important features of these global symmetries.

• In four dimensional ungauged supergravity, antisymmetric tensor fields and scalar fields are related by Poincaré duality. The amount of global symmetry of the theory depends on the number of antisymmetric tensor fields which have been dualized into scalar fields. It is maximal, and we shall denote it by $G$, when all antisymmetric tensors are dualized into scalar fields. This phenomenon is called Dualization of Dualities and was studied in [34].

• In the presence of internal fluxes, the lower-dimensional supergravity is no longer ungauged but features minimal couplings and, by consistency of the theory, local symmetries. Since minimal couplings involve only electric vector fields the electric-magnetic duality symmetry of the ungauged theory, obtained in the limit of zero fluxes, is manifestly broken by the background quantities.

In fact, in all known instances of compactification, fluxes enter the four dimensional gauged Lagrangian $L_g$ not just in the minimal couplings, but they also determine mass terms and a scalar potential. These background quantities, as it was shown in [6] in the heterotic theory, can be associated with representations of the global symmetry group $G_e \subset G$ of the ungauged Lagrangian $L_{u-g}$ so that the original $G_e$–invariance is restored at the level of $L_g$ if the fluxes are transformed under $G_e$ as well. It turns out that we can make an even stronger statement:

Fluxes can be associated with representations of the whole global symmetry group $G$, so that, if transformed accordingly, the original $G$–invariance is restored at the level of the gauged equations of motion and the Bianchi identities.
However $G$ can no longer be regarded as a symmetry of the gauged theory, since it has a non-trivial action on the background quantities (coupling constants), but rather it should be thought of as a mapping (duality) between different theories, i.e. different compactifications.

How can fluxes be associated with representations of the full electric-magnetic duality group $G$? It was found in [27] that the most general gauge group which can be introduced in an extended supergravity can be described in terms of a $G$–covariant tensor called the embedding tensor. It turns out that in all known instances of flux compactifications, fluxes enter the lower dimensional gauged supergravity as components of the embedding tensor:

$$\text{internal flux} \leftrightarrow \text{embedding tensor } \Theta. \hspace{1cm} (1)$$

This identification represents a precise statement on the correspondence between flux compactifications and local symmetry of the lower–dimensional supergravity.

Since $G$ acts, in extended supergravities, as a generalized electric-magnetic duality, in order to achieve full $G$ covariance of the gauged field equations and Bianchi identities, it is necessary to introduce magnetic couplings besides the electric ones. As we shall see this can be done in a local field theory provided the embedding tensor satisfies certain locality conditions and moreover it requires the introduction in the theory of antisymmetric tensor fields. We shall illustrate this new construction of gauged extended supergravities in the maximal case, namely for the $N = 8$ theory in four dimensions. The ungauged version of this theory, with no antisymmetric tensor fields, was constructed in [35] by dimensionally reducing eleven dimensional supergravity [36] (which describes the low-energy limit of M-theory) on a seven torus $T^7$ and then dualizing seven antisymmetric tensors to scalar fields. This theory was shown to exhibit a duality symmetry $G = E_{7(7)}$ at the level of field equations and Bianchi identities.

### 3 The $N = 8$, $D = 4$ Supergravity

The four dimensional maximal supergravity is characterized by having $N = 8$ supersymmetry (that is 32 supercharges), which is the maximal amount of supersymmetry allowed in order for the theory to be local. As anticipated we shall restrict ourselves to the ungauged $N = 8$ theory with no antisymmetric tensor field. The theory describes a single massless graviton supermultiplet consisting of the graviton $g_{\mu\nu}$, eight spin $3/2$ gravitini $\psi^A_\mu$ ($A = 1, \ldots, 8$) transforming in the fundamental representation of the R–symmetry group $\text{SU}(8)$, 28 vector fields $A^A_\mu$, $A = 0, \ldots, 27$, 56 spin $1/2$ “dilatini” $\chi_{ABC}$ transforming in the $\mathbf{56}$ of $\text{SU}(8)$ and finally 70 real scalar fields $\phi^r$. A common feature of supergravity theories is that the scalar fields are described by a non–linear $\sigma$–model on a target space which is a Riemannian manifold $M_{\text{scal}}$. In other words the scalar fields are local coordinates on $M_{\text{scal}}$ and the scalar action is invariant under the global action of the isometry group $\text{Isom}(M_{\text{scal}})$ of $M_{\text{scal}}$ on the scalars. For $N > 2$ supersymmetry requires $M_{\text{scal}}$ to be a homogeneous symmetric manifold, namely a manifold of the form $G/H$ where the isometry group $G = \text{Isom}(M_{\text{scal}})$ is a semisimple Lie group and $H \subset G$ its maximal compact subgroup. In the $N = 8$ model, the scalar manifold
has the form
\[ M_{\text{scal}} = \frac{G}{H} = \frac{E_{7(7)}}{SU(8)}, \] (2)

the isometry group being \( G = E_{7(7)} \) and \( H = SU(8) \) is the R–symmetry group. Although we shall mainly be interested in the maximal theory, most of our treatment will hold also for extended supergravities with lower supersymmetry, describing \( n_v \) vector fields and scalar fields \( \phi^r \) spanning a homogeneous manifold of the form \( M_{\text{scal}} = G/H \). For non-maximal theories the subgroup \( H \) will have the general form \( H = H_R \times H_{\text{matter}} \), where \( H_R \) is the R–symmetry group and \( H_{\text{matter}} \) is a compact group acting on the matter fields. The gravitino and fermion fields will transform in representations of \( H \).

The bosonic action of an ungauged supergravity model has the following general form
\[ S_{u-g} = \int L_{u-g} = \int d^4x \left( -\frac{e}{2} R + \frac{e}{4} \text{Im} N_{\Lambda\Gamma} F^\Lambda_{\mu\nu} F_{\Gamma|\mu\nu} + \frac{1}{8} \text{Re} N_{\Lambda\Gamma} \epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Gamma_{\rho\sigma} + \right. \]
\[ \left. + \frac{e}{2} g_{rs}(\phi) \partial_\mu \phi^r \partial^\mu \phi^s \right), \] (3)

where \( F^\Lambda_{\mu\nu} = 2 \partial_\mu A^\Lambda_\nu \) (\( \Lambda = 0, \ldots, n_v - 1 \)) are the vector field strengths and \( g_{rs}(\phi) \) is the metric on the scalar manifold. We can associate with the electric field strengths \( F^\Lambda_{\mu\nu} \) their magnetic “duals” \( G_{\mu\nu\Lambda} \) defined as follows
\[ * G_{\Lambda|\mu\nu} \equiv \frac{\partial L}{\partial F^\Lambda_{\mu\nu}}, \] (4)

where \( * \) denotes the Hodge duality operation: \( * F^\mu_{\nu\rho\sigma} = \frac{2}{e} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \). In terms of \( F^\Lambda \) and \( G_{\Lambda} \) the Maxwell equations read
\[ \nabla^\mu * F^\Lambda_{\mu\nu} = 0 ; \ \nabla^\mu * G_{\Lambda|\mu\nu} = 0. \] (5)

A general feature of (3) is that the scalar fields enter the kinetic terms of the vector fields through the complex symmetric matrix \( N_{\Lambda\Sigma}(\phi) \) whose negative definite imaginary part generalizes the inverse of the squared coupling constant appearing in ordinary gauge theories while its real part is instead a generalization of the \emph{theta}-angle of quantum chromodynamics. As a consequence of this, a symmetry transformation of the scalar field part of the action, will not in general leave the vector field part invariant. In extended supergravities however, as it was shown in \[37\], the global symmetry group \( G \) of the scalar action can be promoted to global invariance of at least the field equations and the Bianchi identities, provided its (non–linear) action on the scalar fields can be associated with a linear transformation on the vector field strengths \( F^\Lambda_{\mu\nu} \) and their magnetic duals \( G_{\mu\nu\Lambda} \).

\[ g \in G : \begin{cases} \phi^r \rightarrow \phi^r_{\Lambda}(\phi) \\ F^\Lambda_{\mu\nu} \rightarrow \iota(g) \cdot \left( \begin{array}{c} A^\Lambda \\ B^\Lambda \end{array} \right) = \left( \begin{array}{cc} A_{\Lambda\Sigma} & B_{\Lambda\Sigma} \\ C_{\Lambda\Sigma} & D_{\Lambda\Sigma} \end{array} \right) \left( \begin{array}{c} F^\Sigma \\ G^\Sigma \end{array} \right) \end{cases} \] (non–linear action)

\[ \begin{array}{c} \phi^r \rightarrow \phi^r_{\Lambda}(\phi) \\ F^\Lambda_{\mu\nu} \rightarrow \iota(g) \cdot \left( \begin{array}{c} A^\Lambda \\ B^\Lambda \end{array} \right) = \left( \begin{array}{cc} A_{\Lambda\Sigma} & B_{\Lambda\Sigma} \\ C_{\Lambda\Sigma} & D_{\Lambda\Sigma} \end{array} \right) \left( \begin{array}{c} F^\Sigma \\ G^\Sigma \end{array} \right) \end{array} \] (linear transformation)

\[ \text{(6)} \]

\footnote{In most cases this linear action can be associated with all the isometries of the scalar manifold. However this is not a general rule. We are grateful to A. Van Proeyen for making this point.}
The linear transformation \( \iota(g) \) associated with the element \( g \) of \( G \) mixes electric and magnetic field strengths and therefore acts as a generalized electric–magnetic duality. Consistency of this transformation with the definition \( (14) \) of \( G_\Lambda \) requires \( \iota(g) \) to be a symplectic transformation, namely it should leave the following \( 2n_v \times 2n_v \) antisymmetric matrix

\[
\Omega = \begin{pmatrix}
0_{n_v} & \mathbb{1}_{n_v} \\
-\mathbb{1}_{n_v} & 0_{n_v}
\end{pmatrix},
\]

invariant: \( \iota(g)^T \cdot \Omega \cdot \iota(g) = \Omega \). Summarizing, if \( n_v \) is the number of vector fields of the model \( (n_v = 28 \) in the maximal case) then the global symmetry group of the field equations and Bianchi identities acts as a duality transformation defined by the embedding \( \iota \) of the isometry group \( G \) of the scalar manifold into the symplectic group \( \text{Sp}(2n_v, \mathbb{R}) \)

\[
G \overset{\iota}{\hookrightarrow} \text{Sp}(2n_v, \mathbb{R}).
\]

As a consequence of this, the electric field strengths and their magnetic duals transform in the \( 2 \mathbf{n}_v \) symplectic representation of \( G \). In the maximal case \( G = E_{7(7)} \) and the electric and magnetic charges fill the \( 56 \) representation, which is symplectic. The duality action \( \iota(G) \) of \( G \) is not unique, since it depends on which elements of the basis of the \( 2 \mathbf{n}_v \) representation are chosen to be the \( n_v \) elementary vector fields, to be described locally in the Lagrangian, and which their magnetic duals. This amounts to choosing the symplectic frame and determines the embedding of \( G \) inside \( \text{Sp}(2n_v, \mathbb{R}) \), which is not unique. Different choices of the symplectic frame may yield inequivalent Lagrangians (namely Lagrangians not related by local field redefinitions) with different global symmetries. Indeed the global symmetry group of the Lagrangian is defined as the subgroup \( G_e \) of \( G \) whose duality action is linear on the electric field strengths

\[
\iota(G_e) = \begin{pmatrix}
A^\Lambda_\Sigma & 0 \\
C_{\Lambda\Sigma} & D_\Lambda^\Sigma
\end{pmatrix} \Rightarrow \begin{cases}
F^\Lambda \to A^\Lambda_\Sigma F^\Sigma \\
G_\Lambda \to C_{\Lambda\Sigma} F^\Sigma + D_\Lambda^\Sigma G_\Sigma
\end{cases}.
\]

3.1 The gauging

As anticipated in the introduction, the gauging procedure consists in promoting a suitable global symmetry group \( \mathcal{G} \subset G_e \) of the Lagrangian to local symmetry gauged by the vector fields of the theory, and therefore we must have \( \text{dim}(\mathcal{G}) \leq n_v \). The first condition for a global symmetry group \( \mathcal{G} \) to be a viable gauge group is that there should exist a subset of the vector fields \( A^\Lambda_\mu \) which transform under the duality action of \( \mathcal{G} \) in its co–adjoint representation. These fields will become the gauge vectors associated with the generators \( X_\Lambda \) of \( \mathcal{G} \). If we denote by \( \Omega_\mu \) the gauge connection, it will have the form

\[
\Omega_\mu = A^\Lambda_\mu X_\Lambda.
\]

The gauge algebra, which need not be compact or even semisimple, is characterized by the structure constants \( f_{\Lambda\Sigma}^\Gamma \) which define the commutation relations

\[
[X_\Lambda, X_\Sigma] = f_{\Lambda\Sigma}^\Gamma X_\Gamma.
\]
Being \( \mathcal{G} \) a subgroup of \( G \), its generators \( X_\Lambda \) will have an electric-magnetic duality action which is represented by a symplectic matrix of the form (9)

\[
X_\Lambda \equiv \begin{pmatrix} X_\Lambda^\Sigma \Gamma & 0 \\ X_{\Lambda \Sigma} \Gamma & X_\Lambda^\Sigma \Gamma \end{pmatrix}.
\] (12)

The symplectic condition on the matrix form of \( X_\Lambda \) implies the following properties: \( X_\Lambda^\Sigma \Gamma = -X_{\Lambda \Gamma}^\Sigma \), \( X_{\Lambda \Sigma} \Gamma = X_{\Lambda \Sigma \Gamma} \). The first step in the gauging procedure is to associate the fields with representations of \( \mathcal{G} \) and to covariantize the derivatives acting on them accordingly (thus introducing the minimal couplings)

\[
\partial_\mu \rightarrow \nabla_\mu = \partial_\mu - g A_\mu^\Lambda X_\Lambda,
\] (13)

\( g \) being the coupling constant. If the symplectic duality action (12) of \( X_\Lambda \) has a non-vanishing off diagonal block \( X_{\Lambda \Sigma} \Gamma \), gauge invariance further requires the addition to the Lagrangian of a topological term \( [38] \) of the form

\[
\mathcal{L}_{\text{top}} = -\frac{1}{3} g \varepsilon^{\mu \nu \rho \sigma} X_{\Lambda \Sigma} \Gamma A_\mu^\Lambda A_\nu^\Sigma \left( \partial_\rho A_\sigma^\Gamma + \frac{3}{8} g X_{\Xi \Omega}^\Gamma A_\Xi^\Sigma A_\Omega^\Gamma \right),
\] (14)

provided the following condition holds

\[
X_{(\Lambda \Sigma \Gamma)} = 0,
\] (15)

where total symmetrization of the indices within the round brackets is understood. As we shall see below, condition (15) is a consequence of the constraints on the gauge algebra which are required by supersymmetry. Indeed, although the resulting Lagrangian will be locally \( \mathcal{G} \)-invariant, the minimal couplings will explicitly break both supersymmetry and the duality symmetry \( G \). In order to restore supersymmetry one needs to further deform the Lagrangian in the following way.

- Add order–\( g \) terms to the supersymmetry transformation rules for the gravitino \( (\psi_{\mu A}) \) and the fermion fields \( (\chi^I) \), which are characterized by some scalar-dependent matrices \( S_{AB}, N^{TA} \), called the fermion shift matrices, in appropriate representations of the \( H \)-group \((A, B) \) are the indices labelling the supersymmetry generators and \( I, J \) are the generic indices labelling the fermion fields)

\[
\delta_\epsilon \psi_{\mu A} = g S_{AB} \gamma_\mu \epsilon^B + \ldots,
\]

\[
\delta_\epsilon \chi^T = g N^{TA} \epsilon_A + \epsilon.
\] (16)

- Add gravitino and fermion mass terms to the Lagrangian defined by the shift matrices

\[
e^{-1} \mathcal{L} = \ldots + g \bar{\psi}_{\mu}^A \gamma^{\mu \nu} S_{AB} \psi_{\nu}^B + g \bar{\chi}^I \gamma_\mu N^{TA} \psi_{\mu A}.
\] (17)

- Finally add an order \( g^2 \) scalar potential \( V(\phi) \) whose expression is totally fixed as a bilinear in the shift matrices by supersymmetry

\[
\delta_\epsilon^B V(\phi) \sim g^2 \left( N^{TA} N_{TB} - 3 S^{AC} S_{BC} \right).
\] (18)
• Not for all choices of the gauge group supersymmetry can be restored by the above prescription. There are further constraints on the Lie algebra of \( \mathcal{G} \) (supersymmetry constraints) which need to be satisfied. These constraints are linear and quadratic in the gauge generators \( X_\Lambda \) and we shall discuss them below in a general context.

It is useful to encode all the information about the gauge algebra in a single \( \mathcal{G} \)–covariant tensor \( \theta^\Lambda_\sigma \) \((\Lambda = 1, \ldots, n_v, \sigma = 1, \ldots, \dim(\mathcal{G})\)) called the embedding tensor, which expresses the gauge generators as linear combinations of the global symmetry generators \( t_\sigma \) of \( \mathcal{G} \)

\[
X_\Lambda = \theta^\Lambda_\sigma t_\sigma ; \quad \theta^\Lambda_\sigma \in n_v \times \text{Adj}(\mathcal{G}).
\]

The advantage of this description is that the \( \mathcal{G} \) invariance of the original ungauged Lagrangian \( \mathcal{L}_{u-g} \) is restored at the level of the gauged Lagrangian \( \mathcal{L}_g \) provided \( \theta^\Lambda_\sigma \) is transformed under \( \mathcal{G} \) as well. However the full global symmetry group \( \mathcal{G} \) of the field equations and Bianchi identities is still broken since the parameters \( \theta^\Lambda_\sigma \) can be viewed as \( \dim(\mathcal{G}) \) electric charges whose presence manifestly break electric-magnetic duality invariance.

In order to restore the original \( \mathcal{G} \)–invariance, we would need to introduce magnetic components of the embedding tensor as well. The natural way of doing this is by extending the definition of \( \theta \) to a \( \mathcal{G} \)–covariant tensor \( \theta^\Lambda_\sigma \rightarrow \theta^{n\alpha} \equiv (\theta^{\Lambda \alpha}, \theta^{\Lambda \alpha}) \in 2n_v \times \text{Adj}(\mathcal{G}) \),

\[
\theta^\Lambda_\sigma \rightarrow \theta^{n\alpha} \equiv (\theta^{\Lambda \alpha}, \theta^{\Lambda \alpha}) \in 2n_v \times \text{Adj}(\mathcal{G}),
\]

where the index \( n = 1, \ldots, 2n_v \) labels the \( 2n_v \) representation of \( \mathcal{G} \) (the 56 representation of \( E_7(7) \) in the maximally supersymmetric case) and we have expressed a generic vector in this representation by \( W^\alpha = (W^\Lambda, W_\Lambda), \alpha = 1, \ldots, \dim(\mathcal{G}) \) labels the generators \( t_\alpha \) of \( \mathcal{G} \). In the maximally supersymmetric case, for example, this generalized embedding tensor is associated with the 56 \( \times \) 133 representation of \( E_7(7) \). Consistency of this definition requires that \( \text{rank}(\theta) \leq n_v \) since no more than the available vector fields can be involved in the minimal couplings. The linear supersymmetry constraint amounts to a condition on the \( \mathcal{G} \)–representation of \( \theta \). For instance, in the maximally supersymmetric case it requires \( \theta \) to transform in the 912 representation of \( E_7(7) \) contained in the decomposition of the 56 \( \times \) 133. This condition has been solved explicitly in [27]. The quadratic constraint, on the other hand, can be viewed as the condition that \( \theta \) be invariant under the action of the gauge group itself.

Let us now write the \( \mathcal{G} \) generators \( t_\alpha \), in the \( 2n_v \) representation, as \( 2n_v \times 2n_v \) matrices \( (t_\alpha)^m_n \). These matrices belong to the algebra of \( \text{Sp}(2n_v, \mathbb{R}) \) and therefore they satisfy the following property

\[
(t_\alpha)^m_n \Omega_{mp} = (t_\alpha)^p_m \Omega_{mn},
\]

where the symplectic invariant matrix \( \Omega = \{\Omega_{mn}\} \) was defined in [7]. It is convenient for what follows to introduce the following \( \mathcal{G} \)–tensor \( X_{mn}^p = \theta_m^a (t_\alpha)^n_p \). This tensor can be viewed as the matrix representation in the \( 2n_v \) of \( 2n_v \) gauge generators \( X_m \equiv \{(X_m)^n_p\} \). Of course this is just a symplectic covariant notation, and, as we shall see below, the properties of \( \theta \) guarantee that only at most \( n_v \) out of the \( X_n \) generators are actually independent. However
if the embedding tensor has magnetic components, the symplectic representation of $X_n$ will also feature a non-vanishing upper off diagonal block $X_n^{\Sigma\Gamma}$, besides the blocks $X_n^{\Sigma\Gamma}$, $X_n^{\Sigma\Gamma}$, and $X_n^{\Sigma\Gamma}$. It was shown in [28] that the linear and quadratic supersymmetry constraints can be recast in the following $G$–covariant form:

**Linear constraint:**

$$X_{(mnp)} = 0, \quad (21)$$

**Quadratic constraint:**

$$X_{mn}^p X_{qp}^\ell - X_{qn}^p X_{mp}^\ell + X_{mq}^p X_{pn}^\ell = 0, \quad (22)$$

where $X_{mnp} \equiv X_{mn}^q \Omega_{qp}$. Note that equation (21) implies condition (15). The $X_{mn}^p$ tensor provides a symplectic covariant description of the structure constants. Indeed equations (22) can also be written as a $G$–covariant closure condition of the gauge algebra

$$[X_m, X_n] = -X_{mn}^p X_p. \quad (23)$$

The supersymmetry constraints (21) and (22) also imply the following quadratic condition on the embedding tensor

$$\Omega^{mn} \theta_m^\alpha \theta_n^\beta = 0 \iff \theta^{\Lambda[\alpha} \theta^\beta] = 0, \quad (24)$$

which ensures that rank ($\theta$) $\leq n_v$, namely that no more than $n_v$ vector fields will be involved in the minimal couplings and, as we shall see, guarantees also locality of the resulting theory with magnetic couplings. In fact, in virtue of eq. (24), $\theta_n^\alpha$ can always be brought, by means of a symplectic transformation $E_n^{\alpha}$, into an “electric” form in which all the magnetic components are zero

$$(\theta^{\Lambda\alpha}, \theta_\Lambda^\alpha) \xrightarrow{E} (0, \theta_{\alpha}^\alpha). \quad (25)$$

In this symplectic frame the gauged Lagrangian therefore features electric couplings only.

In [28] this formalism was applied to the study of Type IIB toroidal compactification to four dimensions in the presence of internal RR and NS-NS fluxes: $F^{(3)}$, $H^{(3)}$. These background quantities were consistently identified with components of $\theta$ and the condition (24) was equivalent to the tadpole cancellation condition $F^{(3)} \wedge H^{(3)} = 0$ in the absence of localized sources.

### 4 Duality covariant gauged supergravities

In [30] a formulation of gauged extended supergravity was given in which the Lagrangian $L_g$ accommodates both the “electric” ($\theta_\Lambda^\alpha$) and “magnetic” ($\theta^{\Lambda\alpha}$) charges coupled in a symplectic invariant way to electric and magnetic gauge fields. The ingredients for this construction are:

- Antisymmetric tensor fields $B_{\mu
u\alpha}$ transforming in the adjoint representation of $G$ (133 tensor fields in the maximally supersymmetric theory). The presence of these fields is related to the magnetic components of the embedding tensor since they enter the action
only in the combinations $\theta^{\Lambda \alpha} B_{\mu \nu \alpha}$. Therefore in the “electric” symplectic frame in which $\theta^{\Lambda \alpha} = 0$ the antisymmetric tensor fields will disappear all together and we are back to the standard formulation of gauged supergravity;

- $n_v$ magnetic vector fields $A_{\mu \Lambda}$ which, together with the existing electric ones $A^\Lambda_{\mu}$, define $2 n_v$ vector fields $a^\mu_{\Lambda}$ transforming in the $2 n_v$ of $G$;

- Additional tensor and vector gauge invariance which guarantees the correct counting of degrees of freedom.

This result generalizes previous work in [39]. Let us review the main features of this model. The electric and magnetic gauge fields enter the gauge connection in a symplectic invariant fashion

$$\Omega_\mu = g A^\mu_n X_n = g A^\Lambda_{\mu} X_\Lambda + g A_{\mu \Lambda} X^\Lambda ,$$

and the corresponding gauge field strengths are defined by the following $G$–covariant expression

$$F^\mu_{\nu \rho} = \partial_\mu A^\nu_\rho - \partial_\rho A^\nu_\mu + g X_{[m\ell]} A^\nu_\mu A^\ell_m .$$

The gauge covariant quantities however are not $F^\mu_{\nu \rho}$ but rather the following combinations

$$H^\mu_{\nu \rho} = (H^\Lambda_{\mu \nu}, H^{\mu \mu}_{\nu \rho})$$

of $F^\mu_{\nu \rho}$ and the antisymmetric tensors $B_{\mu \nu \alpha}$

$$H^\Lambda_{\mu \nu} = F^\Lambda_{\mu \nu} + \frac{g}{2} \theta^{\Lambda \alpha} B_{\alpha \mu \nu} ,$$

$$H^{\mu \mu}_{\nu \rho} = F^{\mu \mu}_{\nu \rho} - \frac{g}{2} \theta^{\mu \alpha} B_{\alpha \mu \nu} .$$

Off shell the magnetic quantities $H_{\mu \nu \Lambda}$ are not identified with the “dual” field strengths $G_{\mu \nu \Lambda}$ associated with $H^\Lambda_{\mu \nu}$: $H_{\mu \nu \Lambda} \neq G_{\mu \nu \Lambda} \equiv -e \varepsilon_{\mu \rho \sigma} \frac{\partial L_{\Lambda}}{\partial H^\rho_{\sigma}}$. The bosonic Lagrangian reads [30]

$$L_g = -\frac{e}{2} R + \frac{e}{4} \text{Im}(N)_{\Lambda \Sigma} H^\Lambda_{\mu \nu} H^{\mu \nu \Sigma} + \frac{1}{8} \text{Re}(N)_{\Lambda \Sigma} \varepsilon^{\mu \nu \rho \sigma} H^\Lambda_{\mu \nu} H^{\rho \sigma \Sigma} -$$

$$- \frac{1}{8} g \varepsilon^{\mu \nu \rho \sigma} \Theta^{\Lambda \alpha} B_{\mu \nu \alpha} \left(2 \partial_\rho A_{\sigma \Lambda} + g X_{m\ell \Lambda} A^m_{\rho} A^{\ell \sigma} - \frac{1}{4} g \Theta^{\rho \beta} B_{\rho \sigma \beta} \right)$$

$$- \frac{1}{3} g \varepsilon^{\mu \nu \rho \sigma} X_{m\ell \Lambda} A_{\mu}^m A_{\nu}^\ell \left(\partial_\rho A_{\sigma \Lambda} + \frac{1}{4} g X^{\sigma \Lambda} A_{\rho} A_{\sigma}^q \right)$$

$$- \frac{1}{6} g \varepsilon^{\mu \nu \rho \sigma} X_{m\ell \Lambda} A_{\mu}^m A_{\nu}^\ell \left(\partial_\rho A_{\sigma \Lambda} + \frac{1}{4} g X^{\sigma \Lambda} A_{\rho} A_{\sigma}^q \right) + L_{\text{matter}},$$

where $L_{\text{matter}}$ is the gauge invariant scalar action. The topological terms in the above Lagrangian generalize to the presence of antisymmetric tensors, the Chern-Simons-like term in (14). The Lagrangian (28) is invariant with respect to the following vector gauge transformations, parametrized by $2 n_v$ local parameters $\Lambda^n = (\Lambda^\Sigma, \Lambda_{\Sigma})$, and tensor gauge transformations, parametrized by 1–forms $\Xi_\mu^\alpha$,

$$\delta A^\mu_n = D_\mu \Lambda^n - \frac{g}{2} \theta^{\mu \alpha} \Xi^\alpha ,$$

$$\delta B_{\mu \nu \alpha} = 2 \theta^{\mu \alpha} \left[ D_{[\mu} \Xi_{\nu] \alpha} + t_{\alpha \beta \gamma} A^m_{[\mu} \delta A^n_{\nu] \beta} \right] - 2 \Lambda^n \left[ X^{\Sigma \Lambda} H^{\Sigma \nu}_{\mu} - X^{\Lambda \Sigma} G_{\nu \mu} \right] .$$

10
where $\theta^{\alpha n} = \Omega^{\alpha m} \theta^{\alpha m}$ and $D_\mu \Lambda^n = \partial_\mu \Lambda^n + g X_{\mu \nu} A_\mu^{\alpha n} \Lambda^\alpha$.

The field equations obtained by varying the Lagrangian with respect to the tensor fields and the vector fields have the following $G$-covariant form

$$\frac{\delta L_g}{\delta B_{\mu \nu \alpha}} = 0 \iff \theta^{\alpha n} (G^{\alpha n}_{\mu \nu} - H^{\alpha n}_{\mu \nu}) = 0,$$

$$\frac{\delta L_g}{\delta A^\alpha \Lambda_\mu} = 0 \iff \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} D_\nu G^{\alpha \rho \sigma}_{\mu \sigma} = \Omega^{\alpha m n} \frac{\delta L_{\text{matter}}}{\delta A^\alpha \Lambda_\mu} \equiv \Omega^{\alpha m n} g j^n_\mu,$$

where $G^\Lambda = H^\Lambda$. Equations (29) imply that on shell $H^\Lambda$ are dual to $H^\Lambda$. Since the vector fields enter the scalar action only through the symplectic invariant minimal couplings, the current $j^n_\mu$ will be proportional to $\theta^{\alpha n}$, and therefore, in virtue of eq. (24), it will satisfy the property

$$\theta^{\alpha n} \Omega^{\alpha m n} j^n_\mu = 0.$$

(31)

Gauge invariance further requires that $D_\mu j^n_\mu = 0$ on shell. Let us discuss now the issue of locality. Although both electric and magnetic vector fields take part to the action, the combinations of them which are actually involved in the minimal couplings are well defined since the corresponding magnetic currents vanish. Indeed let $r = \text{rank}(\theta^{\alpha n}) \leq n_v$ and let us rewrite the gauge connection in the following form, by using the fact that $X_{\mu \nu} = \theta^{\alpha n} t_\alpha$.

$$\theta^{\alpha n} \equiv A_\mu^{\alpha n} \theta^{\alpha n}.$$

(32)

We see that only $A_\mu^{\alpha n}$ are actually involved in the minimal couplings and moreover that there are only $r$ independent of them. If we contract both sides of eq. (30) by $\theta^{\alpha n}$ and use eq. (31) we see that the $A_\mu^{\alpha n}$ fields are well defined since the corresponding field strengths $G^{\alpha n}_{\mu \nu} = \theta^{\alpha n} G^{\alpha n}_{\mu \nu}$ satisfy the Bianchi identities

$$\epsilon^{\mu \nu \rho \sigma} D_\nu G^{\alpha \rho \sigma}_{\mu \sigma} = 0.$$

(33)

The field equations derived from the variation of the tensor fields $B_{\mu \nu \alpha}$ and the magnetic vector fields $A_\mu \Lambda$ are non dynamic. This ensures the right number of propagating degrees of freedom. For instance eqs. (23) can be solved to eliminate all the $B_{\mu \nu \alpha}$ in favor of their scalar duals. The propagating degrees of freedom can be “distributed” among the fields in different ways by performing different gauge fixings and then solving the non-dynamic field equations. Until the gauge fixing is performed, the theory is manifestly $G$-covariant.

The embedding tensor totally determines the form of the gauged Lagrangian. In particular it determines the fermion shift matrices $S_{AB}$, $N^{IA}$ which enter the mass terms for gravitino and the fermion fields and the scalar potential. In order to compute $S_{AB}$, $N^{IA}$ we need first to introduce a scalar dependent $H$–tensor $T_{\hat{m} \hat{n} \hat{p}}(\phi)$, called the $T$–tensor [23] (the hatted indices label the $2 n_v$ as a reducible representation of $H$). This tensor is obtained by “dressing” $X_{\mu \nu} \rho, \sigma$ by means of the vielbein $V_{\hat{m} \hat{n} \hat{p}}(\phi)$ of the coset manifold $M_{\text{scal}}$

$$T_{\hat{m} \hat{n} \hat{p}}(\phi) = (V^{-1})^{\hat{m} \hat{n} \hat{p}} (V^{-1})^{\hat{m} \hat{n} \hat{p}} V_{\hat{m} \hat{n} \hat{p}} X_{\mu \nu} \rho, \sigma.$$

(34)

The shift matrices $S_{AB}$, $N^{IA}$ are then obtained by projecting $T$ into the relevant representations of $H$. 

11
5 M–theory compactified on a twisted seven–torus with fluxes

Let us now discuss an application of the formalism discussed in the previous section to the study of a specific compactification. This will allow us to understand the relation between two dual descriptions of the same theory. We shall consider M–theory \[8, 10, 11\] compactified on a twisted seven-torus in the presence of fluxes. We start from the low-energy \(D = 11\) supergravity, which has 32 supercharges and whose bosonic field content consists of a graviton field \(V^a\) and a 3–form field \(A^{(3)}_{\mu\nu}\), where \(\mu, \nu = 0, \ldots, 10\) and \(a, b\) are the corresponding rigid indices. The compactification on a twisted torus proceeds as follows. Let us denote by \(x^\mu (\mu = 0, \ldots, 3)\) the four dimensional space–time coordinates, by \(y^I (I = 4, \ldots, 10)\) the coordinates on the internal seven–torus and by \(a, b\)… the rigid indices on the torus. The twisted seven–torus can be locally described as a seven dimensional Lie group manifold described by a basis of 1–forms \(\sigma^I (y) = U(y) J^I dy^J\) satisfying the following Cartan–Maurer equations

\[
d\sigma^I = \frac{1}{2} T^{JK I} \sigma^J \wedge \sigma^K ,
\]

where the structure constants of the group \(T^{JK I}\) define the so called “twist tensor”. This tensor is an instance of geometric flux. We shall restrict to “volume preserving” groups defined by the condition \(T^{IJ J} = 0\). In \[9\] the compactification on a twisted torus was alternatively described as an ordinary toroidal compactification in the presence of an internal torsion \(T^{JK I}\).

The dimensional reduction on this manifold is effected using for the various fields the same ansatz as for the toroidal reduction except that \(dy^I\) are replaced by \(\sigma^I\). In particular the ansatz for the metric and the 3–form reads

\[
V^a = \begin{cases} 
V^\mu x^\mu \\
V^a = \phi^a_I (\sigma^I + A^I_I d\sigma^I) 
\end{cases}
\]

\[
A^{(3)} = A^{(3)} + B_I \wedge V^I + A_{IJ} \wedge V^I \wedge V^J + C_{JK} \wedge V^I \wedge V^J \wedge V^K .
\]

where \(A^I_I(x)\) are the seven Kaluza–Klein vectors and \(\phi^a_I(x)\) are the 28 moduli of the internal metric which span the coset manifold \(\text{GL}(7, \mathbb{R})/\text{SO}(7)\). The field content of the resulting four dimensional theory consists in: The graviton field \(V^\mu\), 28 vector fields \(A^I_I, A_{IJ}\), 7 antisymmetric tensors \(B_{\mu\nu I}\) and 63 scalar fields, 35 of which are the axions \(C_{IJK}\) originating from the eleven dimensional 3–form, while the remaining 28 are the \(\phi^a_I\) fields. In the limit \(T^{JK I} \rightarrow 0\) we are back to the ordinary toroidal reduction which yields a version of the four dimensional ungauged maximal supergravity featuring seven antisymmetric tensor fields. As anticipated in section 2, in order for the global symmetry group \(G = \text{E}_{7(7)}\) of the ungauged theory to be manifest at the level of field equations and Bianchi identities, the antisymmetric tensors need to be dualized into scalar fields. The global symmetry group of the ungauged Lagrangian is \(G_e = \text{GL}(7, \mathbb{R})\) and all fields and fluxes come in representations of \(G_e\), except for the metric moduli \(\phi^a_I\) on which the action of \(G_e\) is non-linear. For instance the twist tensor can be naturally associated with the representation \(140 + 3\) of \(G_e\), where the subscript refers to the grading with respect to the \(O(1, 1)\) subgroup of \(G_e\) acting as a rescaling of the internal
volume. We can also switch on form fluxes described by the v.e.v. of the eleven dimensional 4–form field strength along the internal directions \(g_{IJKL}\) and along the four dimensional space–time directions \(\tilde{g}\),

\[
F^{(4)} = dA^{(3)} + \tilde{g} \epsilon_{\mu \nu \rho \sigma} \, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma - g_{IJKL} \sigma^I \wedge \sigma^J \wedge \sigma^K \wedge \sigma^L .
\]

The background quantities \(g_{IJKL}\) and \(\tilde{g}\) are naturally associated with the representations \(35' + 5\) and \(1 + 7\) of \(G_e\) respectively. The \(G_e\)–representation of the fields and fluxes is summarized in the Table below.

| Fields-fluxes | \(V^a_\mu\) | \(A^I_\mu\) | \(A_{\mu IJ}\) | \(B_{\mu \nu I}\) | \(C_{IJK}\) | \(T_{IJ K}\) | \(g_{IJKL}\) | \(\tilde{g}\) |
|---------------|------------|-------------|-------------|-------------|-------------|-------------|-------------|------------|
| GL(7, \mathbb{R})–reps. | 10 | 7'–3 | 21–1 | 7–4 | 35+2 | 140+3 | 35'+5 | 1+7 |

We can find the above representations in the branching of the relevant \(E_{7(7)}\) representations with respect to \(G_e\).

\[
56 \rightarrow 7'–3 + 21–1 + 7+3 + 21'+1, \quad \text{(38)}
\]

\[
133 \rightarrow 48 + 10 + 35+2 + 7'+4 + 35'–2 + 7–4, \quad \text{(39)}
\]

\[
912 \rightarrow 1–7 + 1+7 + 35–5 + 35'+5 + (140' + 7')–3 + (140 + 7)+3 + 21–1 + 21'+1 + 28–1 + 28'+1 + 224–1 + 224'+1 . \quad \text{(40)}
\]

In the branching \(38\) the representations \(7+3\) and \(21'+1\) describe the vector fields \(\tilde{A}^I_\mu\), \(\tilde{A}^{IJ}_\mu\) dual to \(A^I_\mu\) and \(A_{\mu IJ}\) respectively. Recall that in the formulation discussed in the previous section, the theory features all 70 scalar fields together with 133 tensor fields, which include their “duals”. The branching \(39\) can be used to identify the scalar fields transforming linearly under \(G_e\), which are \(C_{IJK}\) and the scalars \(\tilde{B}^I\) in the \(7'+4\), dual to \(B_{\mu \nu I}\). It can also be used to identify the tensor fields \(B_{\mu \nu I}\) in the \(7–4\), which, as we shall see, are the only tensor fields entering the Lagrangian. Finally the branching of the \(912\) is useful in order to identify the fluxes which are present in the compactification under consideration. Indeed the most general gauging of maximal supergravity is described by an embedding tensor transforming in the \(912\) representation of \(E_{7(7)}\). In the right hand side of eq. \(40\) we find, among the various representations, those defining the fluxes which characterize the compactification we are considering, thus confirming that fluxes enter the lower–dimensional gauged supergravity as components of the embedding tensor which define the gauge group. Therefore in order to construct the gauge algebra of the theory we just need to restrict the embedding tensor to the representations \(140+3, 35'+5\) and \(1+7\). Group theory will do the rest by completely determining the gauge algebra and then, through the gauging procedure, the whole \(\mathcal{L}_g\). The gauge connection has the form

\[
\Omega_\mu = \tilde{A}_\mu^{MN} W_{MN} + A_{MN \mu} W^{MN} + A_\mu^M Z_M . \quad \text{(41)}
\]
where the gauge generators $W_{MN}, W^{MN}$ and $Z_N$ close the following algebra

$$[X_n, X_m] = -X_{nm}^p X_p \Leftrightarrow \begin{cases} [Z_M, Z_N] = T_{MN}^P Z_P + g_{MNPQ} W^{PQ} + \tilde{g} W_{MN}, \\ [Z_M, W_{PQ}] = 2 T_{MR}^P [W^Q]^R + g_{MM_1M_2M_3} \epsilon^{M_1M_2M_3PQRS} W_{RS}, \\ [Z_M, W_{PQ}] = T_{PQ}^L W_{ML}, \\ [W_{IJ}, W_{KL}] = -3 T_{I_1I_2}^K W_{I_3I_4} \epsilon^L_{IJ1...I_4} \end{cases}$$

all other commutators being zero. The locality condition (24), which is also the condition for the above algebra to close inside the algebra of $E_7(7)$, amounts to requiring that

$$T_{[MN}^P T_{Q]P}^L = 0, \quad T_{[MN}^P g_{QLR}P = 0. \quad (42)$$

The first condition is nothing but the Jacobi identity for the algebra (35).

We still have a redundancy of fields and of gauge invariance. Let us now discuss two relevant gauge fixings. The magnetic components of the embedding tensor are given by the twist tensor: $\theta^{\Lambda\alpha} \equiv \theta_{MN}^\alpha N = T_{MN}^N$. They contract only the tensor fields $B_{\mu\nu}$, out of the $B_{\mu\nu\alpha}$, which correspond to the $E_7(7)$ isometries $t_I$ acting as translations on the Peccei-Quinn scalars $\tilde{B}^I$. Suppose that $T_{MN}^N$, as a $21 \times 7$ matrix, has maximal rank 7. Note then that the “magnetic” vector fields $\tilde{A}_{\mu}^M$ enter the Lagrangian only in the seven independent combinations $\tilde{A}_{\mu}^M \equiv T_{NP}^M \tilde{A}_{\mu}^{NP}$. We can write the Stueckelberg term in the covariant derivative of $\tilde{B}^M$ as follows

$$D_\mu \tilde{B}^M = \partial_\mu \tilde{B}^M + T_{NP}^M \tilde{A}_{\mu}^{NP} + \cdots = \partial_\mu \tilde{B}^M + \tilde{A}_{\mu}^M + \cdots, \quad (44)$$

and eliminate $\tilde{B}^M$ by fixing the gauge freedom on $\tilde{A}_{\mu}^M$. After doing so, we can use one of the non–dynamic field equations, which has the following form

$$\epsilon^{\mu\nu\rho\sigma} \partial_\nu B_{\rho\sigma I} \propto g_{IJ} \tilde{A}^{\mu J} + \cdots, \quad (45)$$

to eliminate $\tilde{A}_{\mu}^M$ in favor of $B_{\rho\sigma I}$. The resulting gauge fixed theory is the one obtained in (8) by direct dimensional reduction and contains 7 antisymmetric tensor fields and 63 scalar fields. This model features only the electric vector fields $A_{\mu\mu M}, A_{\mu}^M$.

We can also perform a different gauge fixing. Let us split the 21 vector fields $A_{\mu MN}$ into seven vector fields $A_{\mu M}$ and 14 orthogonal components $A'_{\mu MN}$

$$A_{\mu MN} = T_{MN}^P A_{\mu P} + A'_{\mu MN}. \quad (46)$$

The field strength $H_{\mu\nu MN}$ contains $dA_M$ and $B_M$ in the following combination

$$H_{\mu\nu MN} = T_{MN}^P (\partial_\mu A_{\nu P} - \partial_\nu A_{\mu P} - B_{\mu\nu P}) + \cdots. \quad (47)$$

We can therefore fix the tensor gauge transformation associated with $B_{\mu\nu M}$ by “eating” the seven $A_{\mu P}$. Then, by using the non-dynamic equations (29), which read

$$F_{\mu\nu MN} + \epsilon^{MNL_1...L_4} g_{L_1...L_4} B_{\mu\nu P} \propto \epsilon_{\mu\nu\rho\sigma} \frac{\delta L}{\delta F_{\rho\sigma MN}}, \quad (48)$$

14
the tensor fields $B_{\mu \nu \rho}$ can be eliminated in favor of $A^M_{\mu N}$ (or equivalently of $\tilde{A}^M_{\mu}$). The resulting gauge fixed theory, constructed in [10, 11], features 70 scalar fields, no antisymmetric tensor fields, and 28 electric vector fields consisting of the seven $A^M_{\mu}$, the 14 independent vectors out of the $A^I_{\mu MN}$ and the seven $\tilde{A}^M_{\mu}$, which where originally described as magnetic. This gauge fixing has therefore implied a symplectic rotation [40]. This is precisely the transformation $E$ in (25) which is required to set the magnetic components of the embedding tensor to zero. It is straightforward to generalize the above discussion to the case in which the rank of $T_{MN N}$, as a $21 \times 7$ matrix, is not maximal.

Let us finally comment on the vacua of the theory. These are defined as the points in the scalar manifold which extremize the scalar potential $V(\phi)$. The explicit expression of $V(\phi)$ was found in [11, 12]. It consists of three terms

$$V = V_E + V_K + V_{C-S},$$

$$V_E = \frac{1}{V_7} \left( 2 G_{KL} T_{KJ}^{\ i} T_{LI}^{\ j} + G_{IJ} G_{J}^{\ i} G_{K}^{\ j} T_{JK}^{\ i} T_{J}^{\ k} T_{K}^{\ k} \right),$$

$$V_K = \frac{3}{16} \frac{1}{7!} \frac{1}{V_7} (g_{IJKL} + \frac{3}{2} T_{[IJ}^{\ R} C_{KL]R})(g_{MNQP} + \frac{3}{2} T_{[MN}^{\ R} C_{PQ]R}) G^{IM} G^{JN} G^{KP} G^{LQ},$$

$$V_{C-S} = \frac{1}{6} \frac{1}{V_7^3} \left( C_{IJK} (g_{LPQR} + \frac{3}{4} T_{[LP}^{\ N} C_{QR]N}) \epsilon_{IJKLPQ} + \tilde{g} \right)^2. \quad (49)$$

which come from the ten dimensional Einstein–Hilbert term, the kinetic term of the 3–form and the Chern–Simons term respectively ($V_7$ being the volume of the internal manifold). Note that $V_K$ and $V_{C-S}$ are always positive definite while $V_E$ is not. Extremizing $V$ with respect to $C_{IJK}$ we find the following conditions

$$g_{IJKL} + \frac{3}{2} T_{[IJ}^{\ R} C_{KL]R} = 0, \quad (50)$$

which admit a solution $C_{IJK} \equiv C_{IJK}^{(0)}$ in the axions only for certain choices of $g_{IJKL}$. It is straightforward to show that this potential is always non–positive at its critical points. Vacua with positive cosmological constant are thus ruled out. In [13] it was shown that $V$ can at most vanish at its critical points, thus ruling out also vacua with anti–de Sitter geometry. It is interesting to consider the choices of fluxes which allow for vacua with vanishing cosmological constant (called “flat” vacua). They define instances of the so called “flat” models which were extensively studied in the literature. It was shown in [11] that Minkowski vacua correspond to critical points at which the three terms in $V$ vanish separately

$$\text{Minkowski vacua} \Rightarrow V_E = V_K = V_{C-S} = 0. \quad (51)$$

The vanishing of $V_{C-S}$, in particular, implies the following condition on $\tilde{g}$

$$\tilde{g} = \frac{3}{4} C_{IJK}^{(0)} T_{[LP}^{\ N} C_{QR]N}^{(0)} \epsilon_{IJKLPQR}. \quad (52)$$

The effect of the form–fluxes in these models is thus only to fix the vacuum value of the axions, while the mass spectrum is determined by $V_E$ and the mass parameters are encoded in $T_{MN P}$. 

15
To make a concrete example, let us consider the case in which $I = 4, i (i = 5, \ldots, 10)$, with $T_{IJK} = T_{4i}$, zero otherwise, and $g_{IJKL} = g_{4ijkl}$, zero otherwise. In this case $T_{4i} = M_j^i$ is chosen to be an antisymmetric matrix of rank 3 which can be set in the form:

$$M_j^i = \begin{pmatrix} m_1 \epsilon & 0 & 0 \\ 0 & m_2 \epsilon & 0 \\ 0 & 0 & m_3 \epsilon \end{pmatrix}; \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (53)$$

In this context the equation (50) fixes all $C_{ijk}$ fields but not the $C_{4ij}$ scalars. The $C_{4ij}$ fields give masses to the $A_{i\mu}$ vector fields with the exception of the three entries $(ij) = (5,6), (7,8), (9,10)$. Therefore three of the $C_{4ij}$ scalar remain massless moduli. The $G_{IJK}$ sector gives, as discussed in reference [5], four additional massless scalars, two of which are the volume $V_7$ and $G_{44}$ and two other come from internal components of the metric.

If one further discusses the spectrum of the remaining fields, the six vectors $A_{\mu4i}$ are eaten by the six antisymmetric tensors $B_{\mu\nu i}$. An additional massless scalar comes from the massless 2–form $B_{\mu\nu4}$ and finally an additional massless vector come from the $A_{4i}^I$ Kaluza–Klein vector. The other six $A_{4i}^I$ vectors become massive because of the twisting of the torus. We conclude that in this theory there are always eight massless scalars and four massless vectors, in agreement with [5].

The fact that in all the models featuring Minkowski vacua the form–fluxes never contribute to the physical spectrum can be understood from a different perspective. The fluxes $g_{IJKL}$ and $\tilde{g}$ for which eqs. (50) and (52) admit a solution $C_{IJK}^{(0)}$, can be seen as generated by acting on $T_{MN P}$ by means of an $E_7(7)$ duality transformation in the $35 + 2$ representation on the right hand side of (39), parametrized by $C_{IJK}^{(0)}$. All these models therefore lie in the same $E_7(7)$ duality orbit as the model with $g_{IJKL} = \tilde{g} = 0$. If a suitable discrete form of $E_7(7)$ (the $U$–duality) were an exact symmetry of the fundamental quantum theory of gravity, as conjectured in [33], then all the flat models arising from the class of flux compactifications considered here should be interpreted as different descriptions of the same microscopic degrees of freedom.

6 The issue of “non–geometric” fluxes

The background quantities $g_{IJKL}, \tilde{g}, T_{IJK}$ seem to exhaust all possible form– and geometric fluxes which can be switched on in an M–theory compactification on a torus. Since all the other components of the embedding tensor, defined by the representations on the right hand side of (10), lead to a consistent four dimensional gauged theory, we may wonder if they have any interpretation in terms of higher dimensional fluxes. All the representations in the decomposition of the $912$ are connected to each other by the action of $E_7(7)$, which is conjectured to encode all known string dualities. Therefore we can interpret the $GL(7, \mathbb{R})$ representations in the $912$, which do not correspond to $g_{IJKL}, \tilde{g}, T_{IJK}$, as the “duality image” of the known form– and geometric fluxes. Most of these new background quantities can not be described in the context of dimensional reduction on some compact manifold with
some global geometrical structure and therefore are called “non–geometric” fluxes. To give an example let us perform the toroidal compactification of M–theory to four dimensions in two steps: First compactify it on an $S^1$ along the eleventh dimension $x^{10}$ and then compactify the resulting theory, which is Type IIA superstring theory in ten dimensions, on a six torus down to four dimensions

\[ \text{M–theory } \xrightarrow{S^1(x^{10})} \text{Type IIA superstring } (D = 10) \xrightarrow{T^6} D = 4 \text{ supergravity}. \] (54)

The manifest symmetry of the resulting four dimensional Lagrangian will be the subgroup $\text{SL}(2, \mathbb{R}) \times \text{GL}(6, \mathbb{R})$ of $E_{7(7)}$, in particular the original $\text{GL}(7, \mathbb{R})$ of the seven–torus is broken to $O(1, 1) \times \text{GL}(6, \mathbb{R})$, where $\text{GL}(6, \mathbb{R})$ acts on the metric moduli of the six–torus. The geometric flux $T_{IJ}^{K}$ gives rise, upon reduction on $S^1$, to the following quantities (for the sake of simplicity we shall write, together with the $\text{SL}(6, \mathbb{R})$–representation, only the grading with respect to the $O(1, 1)$ in $\text{GL}(7, \mathbb{R})$) :

\[ T_{IJ}^{K} \rightarrow T_{uvw}^{(84+3)}, \quad T_{w10}^{10} (15+3), \quad T_{10u}^{v} \ (35+3), \quad (T_{uv}^{v} - T_{u10}^{10}) \ (6+3), \quad (55) \]

where $u, v = 4, \ldots, 9$, $T_{uvw}^{w}$ is the twist tensor defining a “twisted” six–torus and $T_{w10}^{10}$ can be viewed as the form–flux associated with the field strength of the R-R 1–form field in Type IIA superstring theory. Each flux, being identified with components of the $912$, is associated with a definite $E_{7(7)}$ weight [9], see appendix A, so now we can study the effect of dualities on them. $T$-duality is the perturbative equivalence between two string theories: One compactified on a circle of radius $R$ and the other compactified on a circle of radius $R' = 1/R$ ($\alpha' = 1$). The most general $T$–duality transformation in superstring theory compactified on a six–torus is described by the discrete group $O(6, 6; \mathbb{Z})$, where the restriction to the integer numbers is required by the boundary conditions on the coordinates of the torus. In particular $T$-duality transformations along an odd number of directions are represented by elements of $O(6, 6; \mathbb{Z})$ with negative determinant. Let us consider the effect of $T$-duality transformations on the geometric flux $T_{uvw}^{w}$. If we apply on $T_{uvw}^{w}$ first a $T$–duality along $y^{v}$ ($T^{(v)}$) and then one along $y^{u}$ ($T^{(u)}$) we obtain the following quantities

\[ T_{uvw}^{w} (84+3) \xrightarrow{T^{(v)}} Q_{u}^{vw} (84'_+1) \xrightarrow{T^{(u)}} R^{uvw} (20_-1). \] (56)

As observed in [21] the new quantities $Q_{u}^{vw}$ and $R^{uvw}$ are instances of “non–geoemtric” fluxes. However the corresponding representations $84'_+1$ and $20_-1$ are contained in the $\text{GL}(7, \mathbb{R})$–representations $224'_+1$ and $224_-1$ in the branching (40) of the $912$. By restricting the embedding tensor to these representations one can construct the whole four dimensional supergravity originating from this generalized flux-compactification. This is an example of how the representations appearing the branching (40) can be related to each other by the action of string dualities.
7 The $D = 5 \rightarrow D = 4$ Scherk-Schwarz reduction and “non–geometric” fluxes

The Scherk–Schwarz (S-S) reduction on a circle from five to four dimensions was originally studied in [5, 32] as a possible mechanism for producing an effective four dimensional supergravity featuring spontaneous supersymmetry breaking at various scales. It represents a generalized type of dimensional reduction in which the ansatz for the five dimensional fields, on a space-time of the form $\mathbb{R}^{1,3} \times S^1$, contains a dependence on the internal $S^1$ coordinate $y$ through a global symmetry transformation of the five dimensional Lagrangian, called Scherk–Schwarz “twist”

$$\Phi(x^\mu, y) = e^M y \cdot \phi(x^\mu),$$

(57)

where $e^M y$ is the twist matrix and $M$ is a global symmetry generator of the five dimensional theory which has a non–trivial action on the field $\Phi$. This property of $M$ guarantees that the dependence on $y$ ultimately disappears in the four dimensional theory. However since $y$ has the dimension of an inverse mass, $M$ has the dimension of a mass and will induce mass deformations in the lower dimensional theory. They originate from terms, in the $D = 5$ Lagrangian, containing derivatives with respect to $y$: $\partial^2_y \Phi(x, y) = M^2 \cdot \Phi(x, y)$. If we start from five dimensional maximal (ungauged) supergravity, whose Lagrangian has an $E_6(6)$ global symmetry group, we can perform a S-S reduction by taking as $M$ any generator of $E_{6(6)}$. The resulting four dimensional supergravity is a gauged supergravity, as was first shown in [26]. This model is an instance of a “no–scale” supergravity as it features a non–negative scalar potential. The only possible vacua are of Minkowski type and are defined by the points in the scalar manifold in which the potential vanishes. These points exist only if $M^T = -M$, namely if $M$ is a generator of the maximal compact subgroup $USp(8)$ of $E_{6(6)}$, in which case the gauge group is called “flat” group. The embedding tensor for this theory was constructed in [27]. It is defined by the $78_{+3}$ representation in the branching of the $912$ with respect to the subgroup $E_{6(6)} \times O(1,1)$ of $E_{7(7)}$. In the basis of the $56$ in which the 28 electric vector fields are $A^A_\mu = \{A^0_\mu, A^A_\mu\}$, where $A^A_\mu$, $\lambda = 1, \ldots, 27$, are the dimensionally reduced five–dimensional vectors in the $27^-_1$ of $E_{6(6)} \times O(1,1)$ and $A^0_\mu$ is the Kaluza–Klein vector in the $1^-_3$ of the same group, the embedding tensor has just electric components $\theta^\sigma_\lambda$ and the gauge generators $X_\lambda$ read:

$$X_\lambda = \begin{cases}
X_0 = \theta_{0,\lambda} \delta t_\lambda \\
X_0 = \theta_{0,\delta} t_\lambda
\end{cases} \quad ; \quad \theta_{0,\lambda} = \theta_{0,\delta} = M_{\lambda \delta} \in E_{6(6)}. $$

(58)

where $M_{\lambda \delta}$ is the twist matrix depending in general on 78 parameters, $t_\lambda$ are the $E_{6(6)}$ generators, and $t_\delta$ are $E_{7(7)}$ generators in the $27_{+2}'$, according to the following branching of

---

In the literature distinction is made between reductions in which the twist is taken in the symmetry group $SL(n,\mathbb{R})$ of the five dimensional theory, once the latter is interpreted as originating from a dimensional reduction on an $n$–torus [3], and reductions in which the twist is a generic global symmetry of the theory itself [32]. In the former case the reduction is referred to as “Scherk–Schwarz reduction”, while in the latter case as “generalized Scherk–Schwarz reduction” or “duality twist”. Here we shall not use this distinction.
the $E_7(7)$ generators with respect to $E_6(6) \times O(1,1)$:

$$133 \rightarrow 78_0 + 1_0 + 27'_{+2} + 27_{-2}. \quad (59)$$

In this case the relevant components of the gauge generators $X_{\Lambda m}^n$ are:

$$X_{0\lambda}^\delta = -X_{\lambda 0}^\delta = X_0^\delta \lambda = X_\lambda^\delta 0 = -M_\lambda^\delta \delta; \quad X_{\lambda\delta\gamma} = M_\lambda^\nu d_{\nu\delta\gamma}, \quad (60)$$

where $d_{\lambda\delta\gamma}$ denotes the three times symmetric invariant tensor of the $27$ of $E_6(6)$. To obtain eqs. (60) we have used the properties $(t^\lambda_\gamma)_\delta^\gamma = -(t^\delta_\gamma)_\lambda^\gamma = \delta^\gamma_\delta \delta_{\lambda\gamma} = (1/27) \delta^\gamma_\delta \delta_{\lambda\gamma}$. The gauge algebra has the following structure:

$$[X_0, X_\lambda] = M_\lambda^\delta X_\delta, \quad (61)$$

all other commutators vanishing.

If $M$ is non–compact the corresponding theory depends effectively only on six parameters and the potential is of run–away type, namely there is no vacuum solution. If, on the other hand, $M$ is compact, the theory has Minkowski vacua and depends effectively on four mass parameters $m_1, m_2, m_3, m_4$, since $M$ can always be reduced to an element of the maximal torus of $USp(8)$. These mass parameters fix the scale of spontaneous supersymmetry breaking, which can yield an $N = 6, 4, 2$ or $N = 0$ effective theory.

Can we interpret this four dimensional spontaneously broken supergravity as originating from an eleven dimensional flux compactification? To this end let us describe the five dimensional theory as originating from the compactification of $M$–theory on a six–torus. This is done by branching the relevant $E_6(6)$–representations with respect to the $SL(2,\mathbb{R}) \times SL(6,\mathbb{R})$ subgroup of $E_6(6)$, $SL(6,\mathbb{R})$ being, as usual, the group acting on the metric moduli of the six–torus. In particular we are interested in interpreting the embedding tensor of the model in terms of higher dimensional fluxes. This is done by decomposing the representation $78_{+3}$ of $\theta$ with respect to $SL(6,\mathbb{R}) \times SL(2,\mathbb{R})$

$$78 \to _{\frac{SL(6,\mathbb{R}) \times SL(2,\mathbb{R})}{} (35, 1) + (1, 3) + (20, 2)}, \quad (62)$$

From the detailed analysis in [9], it follows that the $(35, 1)$ can be interpreted as the $T_{i,j}^K$ components of the twist tensor $T_{i,j}^K$. Indeed $T_{i,j}^K$ is in general a generator of $SL(6,\mathbb{R})$, and flat vacua occur only if this matrix is antisymmetric, namely if it is an $SO(6)$ generator. In this case it can always be brought to the skew–diagonal form (53) and will contribute three mass parameters to the theory. How about the remaining fourth mass parameter? It will come from a compact $E_6(6)$ twist which is not in an $SL(6,\mathbb{R})$-generator. Let us have a closer look at the branching (62). The representation $20$ of $SL(6,\mathbb{R})$ appears in a doublet with respect to $SL(2,\mathbb{R})$. One component of the doublet, which we shall call $20_+$, as it was shown in [9], can be interpreted as the components $g_{4ijkl}$ of $g_{IJKL}$. It corresponds to a nilpotent generator of $E_6(6)$ and thus, alone, it cannot contribute any mass parameter. In fact, as we have seen in section 5, its only effect is to fix the axions to certain values. Similarly the highest grading component of the triplet $(1, 3)$, which we shall denote by $1_+$, can be identified
with \( \tilde{g} \). It also singles out a nilpotent twist matrix \( M \), this time corresponding to the positive root \( \alpha \) of \( SL(2, \mathbb{R}) \). In order to get a compact matrix \( M \), namely antisymmetric since we are always working with real representations, it should be the combination of an upper and a lower triangular matrix, namely of two shift matrices with opposite gradings, corresponding to positive \( E_{6(6)} \) roots and their negative respectively. Therefore the fourth parameter can only arise from a combination of the \( 20_+ \) with \( 20_- \) components of the \( (20, 2) \), or from a combination of the \( 1_+ \) and \( 1_- \) components of the \( (1, 3) \). The \( 20_- \) and the \( 1_- \) components do not have an interpretation in terms of form–or geometric fluxes, nevertheless they are obtained by acting on the known \( 20_+ \) and \( 1_+ \) respectively by means of the Weyl transformation [41] associated with the root \( \alpha \) of \( SL(2, \mathbb{R}) \). This is a proper \( U \)–duality transformation which mixes the internal radii of the torus (in the ten dimensional string frame) \( R_i \), together with the ten dimensional dilaton field \( \phi \). Its action can be deduced using the analysis of [9]

\[
\begin{align*}
R'_4 &= e^{-\frac{\phi}{2}} V_6^{\frac{1}{2}} R_4^{\frac{1}{2}}, \\
R'_i &= e^{\frac{\phi}{2}} V_6^{-\frac{1}{2}} R_4^{\frac{1}{2}} R_i; \quad i = 5, \ldots, 9, \\
e^{\phi'} &= e^{\frac{3}{2} \phi} V_6^{-\frac{1}{2}} R_4^{\frac{1}{2}},
\end{align*}
\]

where \( V_6 = R_4 R_5 \ldots R_9 \) is the volume of the six torus, see appendix A. To be able to interpret the fourth S-S parameter therefore we need to include, apart from the form–fluxes, also their \( U \)-duality image, in a single picture! This unifying picture of compactification, accommodating at the same time \( U \)-dual background quantities, could be provided by the so called “\( U \)-folds” [20], namely non–geometric manifolds having \( U \)–duality transformations as transition functions. A similar analysis, from a different perspective, was carried out in [20].

An other interesting duality between different flux compactifications was found in [3] using the embedding tensor description of gauged supergravity. In this paper it was observed that if we restricted the embedding tensor to the representation \( (20, 2) \) in (62), upon an \( N = 4 \) truncation of the \( N = 8 \) supergravity, the resulting theory coincides with the \( N = 4 \) gauged supergravity describing Type IIB superstring compactified on a \( T^6/\mathbb{Z}_2 \)-orientifold in the presence of R-R and NS-NS 3–form fluxes \( F^{(3)}, H^{(3)} \). Indeed these fluxes transform in the \( (20, 2) \) representation of \( SL(6, \mathbb{R}) \times SL(2, \mathbb{R}) \), though in the Type IIB setting the \( SL(2, \mathbb{R}) \) symmetry group is interpreted as the global symmetry of the ten dimensional theory. This is an instance of a duality between two seemingly different theories: Type IIB superstring reduced on a \( T^6/\mathbb{Z}_2 \)-orientifold in the presence of 3–form fluxes and a (truncation of a) Scherk–Schwarz reduction from five dimensions, which has a more natural description as originating from M–theory or Type IIA superstring.

8 Conclusions and outlook

In the present paper have reviewed a new description of gauged supergravity in which the whole global (non–perturbative) symmetry group \( G \) of the ungauged Lagrangian is preserved at the level of field equations and Bianchi identities. This description includes the scalar
fields together with their dual tensor fields, the electric vector fields and their magnetic duals. All the information about the gauge group is encoded in the \(G\)-covariant embedding tensor. We have also discussed some applications of this description. The relevance of this formulation is apparent if we consider flux compactifications, since the internal fluxes naturally enter the lower dimensional effective supergravity as components of the embedding tensor. Form- and geometric fluxes, however, fit a restricted number of components. All the remaining components, which are consistent with the supersymmetry constraints, can be obtained from the known fluxes by means of \(U\)-duality transformations, which include the \(T\) and \(S\)-dualities. Lifting the corresponding gauged models to ten or eleven dimensions is still an open problem. Nevertheless the embedding tensor formulation of gauged supergravity represents an ideal laboratory in which to study the web of dualities connecting the known flux compactifications with generalizations thereof. For example the duality covariant formulation reviewed in section 4 can be applied to the construction of the complete mirror-symmetry-covariant four dimensional supergravity description of Type II superstring compactified on \(SU(3) \times SU(3)\)-structure manifold in the presence of general fluxes [22, 42]. In this case the (electric) embedding tensor, defined in [43], which gauges (an abelian subalgebra of) the Heisenberg isometry algebra of the \(N = 2\) quaternionic manifold, and which reproduces the form- and geometric fluxes, has to be extended to include magnetic components as well, in order to account for the non-geometric fluxes. This has been done in [44], were the low-energy gauged \(N = 2\) supergravity reproducing the general flux compactification described in [22] was constructed. It is still an open question whether such compactifications can be further generalized to yield a non-abelian gauging in four dimensions.

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A Fluxes and \(E_{7(7)}\) weights

In this appendix we illustrate is some detail how to associate the known fluxes, in ten dimensional Type II superstring compactified on a six-torus, with components of the embedding tensor which defines the corresponding four dimensional gauged supergravity, namely with elements of the \(912\) representation of \(E_{7(7)}\). An element of a Lie group \(G\) representation is characterized by its transformation property under the action of the maximal torus of \(G\), generated by its Cartan subalgebra (CSA). This property is encoded in the \(weights\) of the representation. Let \(G\) be the global symmetry group of an extended supergravity theory. Its maximal torus has a diagonal action on the electric field strengths and their duals, and therefore it is a symmetry of the ungauged Lagrangian, namely it is contained in \(G_e\). If
fluxes are to be assigned, by identification with components of the embedding tensor, to \( G \)-representations, so as to restore on shell global \( G \) invariance, they should couple in the action to the dilatonic fields parametrizing the CSA of \( G \) according to their weights. In the maximal theory the CSA of \( E_7(7) \) is seven dimensional and is parametrized, from the Type II superstring point of view, by the six radial moduli of the internal torus \( R_i = e^{\sigma_i} \) \((i = 4, \ldots, 9)\) and by the ten dimensional dilaton \( \phi \). The bosonic zero-modes of Type II superstring theory consist in the ten dimensional dilaton \( \phi \), the metric \( \hat{V}_\mu^r = e^{\phi_4} V_\mu^r \); \( \hat{V}^u = \phi_u \hat{\phi}^v (dy^v + A_u^\mu dx^\mu) \),

\[ (64) \]

where \( u, v = 4, \ldots, 9 \) and \( \hat{u}, \hat{v} = 4, \ldots, 9 \) are the curved and rigid indices on the six torus respectively, \( V_\mu^r \) is the four dimensional metric in the four dimensional Einstein frame, \( A_u^\mu \) are the six Kaluza Klein vectors, \( \phi_4 = \phi - \frac{1}{2} \sum_u \sigma_u \) is the four dimensional dilaton and \( \phi_u \hat{\phi}^v \) are the metric moduli of the internal torus, which can be identified with the coset representative of \( \text{GL}(6,\mathbb{R})/\text{SO}(6) \). By suitably fixing the \( \text{SO}(6) \) symmetry we can adopt the solvable Lie algebra representation of the manifold \( \text{GL}(6,\mathbb{R})/\text{SO}(6) \) [35, 45, 34] and write \( \phi_u \hat{\phi}^v \) in the form

\[ \phi_u \hat{\phi}^v \equiv U e^{\sum_{u=4}^9 \hat{\sigma}_u H_{u\hat{u}}} , \]

\[ U = \prod_{u<v} e^{\gamma_{uv} E_u^v} \text{ (no summation)} , \]

(65)

where \( \epsilon_u \) is an orthonormal basis of vectors, \( E_u^v \) are the \( \text{SL}(6,\mathbb{R}) \) shift generators corresponding to the positive root \( \epsilon_u - \epsilon_v \) and \( \gamma_{uv} \) are the moduli parametrizing the off diagonal components of the internal metric. The internal metric will read \( G_{uv} = -\sum_{\hat{u}\hat{v}} \phi_u \hat{\phi}^v \phi_{\hat{u}} \hat{\phi}^{\hat{v}} \). Let us define a representative of the maximal torus of \( E_7(7) \) to have the form \( \exp(H_{\vec{h}}) \), where \( \vec{h} \) is defined as

\[ \vec{h}(\sigma, \phi) = \sum_{u=4}^9 \sigma_u \epsilon_u - \sqrt{2} \phi_4 \epsilon_{10} = \sum_{u=4}^9 \hat{\sigma}_u \left( \epsilon_u + \frac{1}{\sqrt{2}} \epsilon_{10} \right) - \frac{1}{2} \phi \hat{a} , \]

\[ a = -\frac{1}{2} \sum_{u=4}^9 \sigma_u \epsilon_u + \frac{1}{\sqrt{2}} \epsilon_{10} , \]

(66)

where \( \epsilon_I = (\epsilon_u, \epsilon_{10}) \) is an orthonormal basis of seven dimensional vectors and \( \hat{\sigma}_u = \sigma_u - \phi/4 \) are the radial moduli in the ten dimensional Einstein frame. The four dimensional Lagrangian,
resulting from the dualization of the 2-forms to scalar fields, will contain the following terms

\[ e^{-1} \mathcal{L}_{\text{scal}} = \frac{1}{2} \partial_{\mu} \tilde{h} \cdot \partial^{\mu} \tilde{h} + \frac{1}{4} \sum_{u,v} e^{-2 \alpha_u^B \cdot \tilde{h}} \partial_{\mu} B_{uv} \partial^{\mu} B_{uv} + \frac{1}{2} \sum_{u<v} e^{-2 \alpha_u^V \cdot \tilde{h}} \partial_{\mu} \gamma_u^V \partial^{\mu} \gamma_v + \]

\[ + \sum_k \frac{1}{2k!} \sum_{u_1, \ldots, u_k} e^{-2 \alpha^C_{u_1 \ldots u_k} \cdot \tilde{h}} \partial_{\mu} C_{u_1 \ldots u_k} \partial^{\mu} C_{u_1 \ldots u_k} + \frac{1}{2} e^{-2 \alpha^B \cdot \tilde{h}} \partial_{\mu} B \partial^{\mu} B \ldots, \]

\[ (67) \]

\[ e^{-1} \mathcal{L}_{\text{vec}} = - \sum_u e^{-2 W^u \cdot \tilde{h}} \partial_{[\mu} A^v_{\nu]} \partial^{[\mu} A^{\nu]}_u - \sum_u e^{-2 W^B_{\mu \nu} \cdot \tilde{h}} \partial_{[\mu} B_{\nu]}_u \partial^{[\mu} B^{\nu]}_u - \]

\[ - \sum_k \frac{1}{(k-1)!} \sum_{u_1, \ldots, u_{k-1}} e^{-2 W^C_{u_1 \ldots u_{k-1}} \cdot \tilde{h}} \partial_{[\mu} C_{v]}_{u_1 \ldots u_{k-1}} \partial^{[\mu} C^{v]}_{u_1 \ldots u_{k-1}} + \ldots. \]

\[ (68) \]

where the internal indices of the scalar and vector fields are “dressed” with the matrix \( U \) in eq. (65) and the ellipses comprise the non linear couplings deriving from the Chern-Simons terms in the definition of the ten dimensional field strengths. The 2-form \( B_{\mu\nu} \) has been dualized to the axion \( B \) while in the Type IIA theory the tensors \( C_{\mu\nu} \) were dualized to \( \epsilon_{u_1u_2u_3} C_{u_1u_2u_3} \) and in the Type IIB theory \( C_{\mu\nu} \) was dualized to the scalar \( C_{\alpha \beta \gamma} \). The range of values of \( k \) in the summations in (67) and (68) is: \( k = 1, 3, 5 \) in Type IIA and \( k = 0, 2, 4, 6 \) in Type IIB.

The seven dimensional vectors \( \alpha \) and \( W \) in the exponential factors of (67) and (68) have the form

\[ \alpha_u^V = \epsilon_u - \epsilon_v; \quad \alpha^B_{uv} = \epsilon_u + \epsilon_v; \quad \alpha^B = \sqrt{2} \epsilon_{10}; \quad \alpha^C_{u_1 \ldots u_k} = a + \epsilon_{u_1} + \cdots + \epsilon_{u_k}, \]

\[ W^u = -\epsilon_u - \frac{1}{\sqrt{2}} \epsilon_{10}; \quad W^B_u = \epsilon_u - \frac{1}{\sqrt{2}} \epsilon_{10}; \quad W^C_{u_1 \ldots u_{k-1}} = a + \epsilon_{u_1} + \cdots + \epsilon_{u_{k-1}} - \frac{1}{\sqrt{2}} \epsilon_{10}, \]

\[ (69) \]

\[ (70) \]

If we define the simple roots of the \( \epsilon_{7(7)} \) algebra to be of the form

\[ \alpha_{u-3} = \epsilon_u - \epsilon_{u+1} \quad (u = 4, \ldots, 8); \quad \alpha_6 = \epsilon_8 + \epsilon_9; \quad \alpha_7 = \begin{cases} a & \text{Type IIB} \\ a + \epsilon_9 & \text{Type IIA} \end{cases}, \]

\[ (71) \]

the vectors in (69) are the \( \epsilon_{7(7)} \) positive roots while those in (70), together with their opposite \(-W\) (corresponding to the magnetic vector fields), are the weights of the 56 representation (in the Type IIB description, the weights \( W^C_{u_1u_2u_3} \) are 20 and correspond to the vectors \( C_{\mu\nu u_1u_2u_3} \) originating from the 4-form; in virtue of the property of the 5-form field strength of being self dual, these 20 weights already include 10 weights corresponding to electric vector fields and their opposite associated with the magnetic duals). We may follow a similar strategy in order to associate fluxes with \( \epsilon_{7(7)} \) weights, namely read off the weight from the dilaton dependence of the term in the action of the form (flux)^2. The relevant terms in the Lagrangian are

\[ e^{-1} \mathcal{L} = \sum_{u,v,w} e^{-2 W^T_{uv} \cdot \tilde{h}} (T_{uv}^w)^2 - \frac{1}{12} e^{-2 W^H_{uvw} \cdot \tilde{h}} (H_{uvw}^{(3)})^2 + \]

\[ + \sum_k (-1)^{(k+1)} \frac{1}{2(k+1)!} \sum_{u_1, \ldots, u_{k+1}} e^{-2 W^F_{u_1 \ldots u_{k+1}} \cdot \tilde{h}} (F_{u_1 \ldots u_{k+1}}^{(k+1)})^2 + \ldots, \]

\[ (72) \]
where the term containing \((T_{uvw})^2\) is part of the Scherk-Schwarz potential \([5]\) which, in the case of M-theory compactification on a twisted torus, is described by \(V_E\) in (49). The rank \(k\) in the summation over the R-R fluxes have the values \(k = -1, 1, 3, 5\) in Type IIA theory, corresponding to the internal components of the forms \(F^{(0)}, F^{(2)}, F^{(4)}, F^{(6)}\), and \(k = 0, 2, 4\) in Type IIB theory, corresponding to the field strengths \(F^{(1)}, F^{(3)}, F^{(5)}\). We may also consider R-R fluxes with four space-time indices which do not explicitly break Lorentz invariance. By performing the dimensional reduction, we find the general field-strength–weight correspondence:

\[
H^{(3)}_{u_1u_2u_3} \leftrightarrow W^H_{u_1u_2u_3} = \epsilon_{u_1} + \epsilon_{u_2} + \epsilon_{u_3} + \frac{1}{\sqrt{2}} \epsilon_{10},
\]

\[
T^{u_3}_{u_1u_2} \leftrightarrow W^T_{u_1u_2} = \epsilon_{u_1} + \epsilon_{u_2} - \epsilon_{u_3} + \frac{1}{\sqrt{2}} \epsilon_{10},
\]

\[
F^{(k+1)}_{\mu_1...\mu_\ell u_1...u_s} \leftrightarrow W^F_{\mu_1...\mu_\ell u_1...u_s} = -\frac{1}{2} \sum_u \epsilon_u + \epsilon_{u_1} + \ldots + \epsilon_{u_s} + \frac{2 - \ell}{\sqrt{2}} \epsilon_{10} \quad (\ell + s = k + 1).
\]

(73)

In the M-theory reduction on a torus the \(O(1,1)\) factor in \(G_c = GL(7, \mathbb{R})\) is generated by the Cartan operator \(H_\lambda\) where

\[
\lambda = \sum_u \epsilon_u + 2\sqrt{2} \epsilon_{10},
\]

and the \(O(1,1)\) weight associated with the field strengths in (73) are simply computed as the scalar product of \(\lambda\) with the corresponding weight \(W\): \(\lambda \cdot W\). From the embedding of the \(SL(6, \mathbb{R})\) group, corresponding to the six torus in the compactification of Type II theories, inside \(E_7(7)\), we may deduce the \(SL(6, \mathbb{R})\)-representation of each of the weights in (73) and identify it, together with the relevant \(O(1,1)\) gradings, with representations in the branching of the embedding tensor representation \(912\). The embedding of \(SL(6, \mathbb{R})\) inside \(E_7(7)\) is defined by identifying its simple roots with \(\alpha_1 \ldots \alpha_5\).

Let us now consider the effect of dualities. An important role is played by those dualities which are effected as Weyl transformations \(\sigma_\Delta\) with respect to a given weight \(\Delta\), whose action on a weight \(W\) is defines as follows

\[
W \rightarrow \sigma_\Delta(W) = W - 2 \left(\frac{W \cdot \Delta}{\Delta \cdot \Delta}\right) \Delta.
\]

(75)

If \(\Delta\) is a root of \(e_7(7)\) then this transformation is an \(E_7(7)\) transformation and therefore a symmetry of the theory. This is not always the case for the known string dualities. Let us see what the effect of these transformations is on the dilatonic scalars. Since \(\sigma_\Delta\) is an orthogonal transformation in the Euclidean vector space of the CSA of \(e_7(7)\), we can rewrite the generic exponent in (67), (68) and (72) as follows

\[
e^{-2W \cdot \hat{\eta}} = e^{-2\sigma_\Delta(W) \cdot \sigma_\Delta(\hat{\eta})},
\]

(76)

namely the field which is described by the weight \(W\) (which, for a scalar field, is a positive root) in the original theory, corresponds to the new weight \(\sigma_\Delta(W)\) in the dual theory, which
features a new set of dilatonic scalars $\sigma'_u$, $\phi'$, entering the dilatonic vector $\sigma_\Delta(\vec{h})$. The relation between $\sigma'_u$, $\phi'$ and $\sigma_u$, $\phi$ can be deduced by the following condition

$$\vec{h}(\sigma', \phi') \equiv \sigma_\Delta(\vec{h}(\sigma, \phi)) \Rightarrow \sigma' = \sigma'(\sigma, \phi), \quad \phi' = \phi'(\phi). \tag{77}$$

Let us consider as an example the $T$–duality $T^{(u)}$ along the internal direction $y^u$. It is implemented by the Weyl transformation $\sigma_u \ [\hat{16}]$, corresponding to the vector $\Delta = \epsilon_u$. Since this vector is not an $\epsilon_{7(7)}$ root, in general $T$–duality along an odd number of direction is not an $E_{7(7)}$ transformation. If we compute the corresponding transformation property of the dilatonic scalars, using the procedure illustrated in (77), we find

$$\sigma'_{v \neq u} = -\sigma_u \quad \iff \quad R'_{v \neq u} = R_{v \neq u} \quad ; \quad R_u = \frac{1}{R_u},$$

which is the known effect of a $T$–duality along a single direction. We can verify that under the effect of $T^{(w)}$ the weight of $H_{uvw}$ is mapped into the weight of $T_{uwv}$ and moreover subsequent actions of $T^{(v)}$ and $T^{(u)}$ allow to define the weights $W^{Q}_{u vw}$ and $W^{R}_{uvw}$, associated with the non-geometric fluxes $Q_{u vw}$ and $R_{uvw}$ in eq. (55) respectively

$$W^{H}_{uvw} \xrightarrow{T^{(w)}} W^{T}_{uw} \xrightarrow{T^{(v)}} W^{Q}_{u vw} \xrightarrow{T^{(u)}} W^{R}_{uvw}$$

$$W^{Q}_{u vw} = \epsilon_u - \epsilon_v - \epsilon_w + \frac{1}{\sqrt{2}} \epsilon_{10} \quad ; \quad W^{R}_{uvw} = -\epsilon_u - \epsilon_v - \epsilon_w + \frac{1}{\sqrt{2}} \epsilon_{10}. \tag{79}$$

The non perturbative $S$–duality is implemented as a Weyl transformation with respect to the vector $\Delta = \alpha$ in eq. (56), which is an $\epsilon_{7(7)}$ root in Type IIB theory, but not in Type IIA theory, see equation (71). This represents the known fact that $S$–duality is a symmetry of Type IIB theory (it corresponds to an $E_{7(7)}$ transformation) but not of Type IIA theory (it maps Type IIA superstring into M-theory). If we compute its action on the dilatonic scalars, using (77), we find

$$\hat{\sigma}_u' = \hat{\sigma}_u \quad ; \quad \hat{\phi}' = -\phi, \tag{80}$$

where $\hat{\sigma}_u$ are the radial moduli in the ten dimensional Einstein frame. One can verify, using the weight representation in eq. (74), that in Type IIB theory

$$\sigma_\alpha(W^{H}_{u_1 u_2 u_3}) = W^{F}_{u_1 u_2 u_3} \tag{81}$$

which is the known $S$–duality correspondence between the NS-NS and the R-R 3–form fluxes $H^{(3)}_{a_1 u_2 u_3}$, $F^{(3)}_{a_1 u_2 u_3}$. One can also verify that the torsion $T_{uwv}$ is inert under $S$–duality, while the action of $S$–duality on the non–geometric fluxes gives rise to more general fluxes which we can identify with components of the embedding tensor, knowing the corresponding weights.

Finally let us consider the $U$–duality transformation introduced at the end of section 7 to describe the relation between $20_+$ and $20_-$ and between the $1_+$ and the $1_-$ representations in the branching of the $78$ of $E_{6(6)}$ with respect to $SL(6, \mathbb{R}) \times SL(2, \mathbb{R})$. The embedding of $E_{6(6)}$ inside $E_{7(7)}$ is defined by identifying the simple roots of the former with $\alpha_2, \ldots, \alpha_7$. 25
This duality transformation is implemented by the Weyl transformation with respect to the root \( \alpha \) of \( \text{SL}(2, \mathbb{R}) \), which has the form

\[
\alpha = a + \epsilon_5 + \cdots + \epsilon_9.
\]  

(82)

From the transformation property of \( \vec{h} \) under \( \sigma_\alpha \) in (77) we deduce the transformation rules for the dilatonic scalars in eqs. (63).

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