Aspects of multimetric gravity

M Hohmann
Teoreetilise Füüsika Labor, Füüsika Instituut, Tartu Ülikool, Ravila 4c, 50411 Tartu, Estonia

Abstract. We present a class of gravity theories containing \( N \geq 2 \) metric tensors and a corresponding number of standard model copies. In the Newtonian limit gravity is attractive within each standard model copy, but different standard model copies mutually repel each other. We discuss several aspects of these multimetric gravity theories, including cosmology, structure formation, the post-Newtonian limit and gravitational waves. The most interesting feature we find is an accelerating expansion of the universe that naturally becomes small at late times.

1. Motivation

Among the most important open problems of modern physics are the observed accelerating expansion of the universe [1, 2] and the formation of large scale structures [3]. Their most widely accepted explanation is given by the \( \Lambda \)CDM model, which claims that, in addition to visible matter, the universe must contain dark matter and dark energy in order to match cosmological observations. Measurements of the cosmic microwave background by Planck indicate that their amounts are given by 68.3% dark energy, 26.8% dark matter and only 4.9% visible matter [Ade P A R et al arXiv:1303.5076 Planck Collaboration]. However, despite its success in modeling cosmological observations, the \( \Lambda \)CDM model is purely empirical and does not provide explanations for the constituents of dark matter and dark energy.

This shortcoming of the standard model of cosmology invites for various supplemental theories from particle physics and alternative theories from gravitational physics. The model we discuss here belongs to the second class and is based on the idea to introduce a second copy of the standard model, which is governed by a second metric. Both copies are assumed to interact only gravitationally, mediated by an interaction of their respective metrics, so that they mutually appear dark. The gravitational interaction is further assumed to be attractive within each matter sector, but repulsive of equal strength between the two standard model copies. If one finally assumes the universe to be constituted by equal amounts of both matter types, the mutual repulsion may act as the driving force for the observed accelerating expansion, and mimic the presence of dark matter effects by pushing light and visible matter towards visible galaxies.

However, it turns out that repulsive gravity cannot be achieved in the bimetric case [4], but can indeed account for an accelerating cosmology for \( N \geq 3 \) standard model copies and metrics [5]. In the following we will discuss various aspects of these multimetric gravity theories [5, 6, 7, 8].

We will give a concise definition of the class of theories we consider here in section 2, where we also provide a concrete family of multimetric gravity theories. We then enter a detailed discussion of their properties in the remaining sections. In section 3 we derive their post-Newtonian limit and obtain constraints from high-precision solar system experiments. For this constrained class of theories we discuss cosmology in section 4, the formation of large scale structures in section 5 and gravitational waves in section 6. We end with a conclusion in section 7.
2. Multimetric gravity

We begin our discussion with a brief description of the class of multimetric gravity theories we consider. These theories can be characterized by a simple set of assumptions. In this section we first list these assumptions, before we present a generic example theory. Various aspects of this example theory will then be discussed in the remaining sections.

In the following we will assume:

(i) The field content is given by $N \geq 2$ copies $\varphi^1, \ldots, \varphi^N$ of standard model matter and a corresponding number of metric tensors $g^1, \ldots, g^N$.

(ii) The dynamics are governed by a diffeomorphism invariant action of the type

$$S = S_G[g^1, \ldots, g^N] + \sum_{I=1}^{N} S_{M}[g^I, \varphi^I]$$

where $S_M$ denotes the standard model action.

(iii) The field equations are obtained by variation with respect to the metrics $g_{ab}^1, \ldots, g_{ab}^N$, and so are a set of symmetric two-tensor equations of the form $K_{ab}^I = 8\pi G N T_{ab}^I$.  

(iv) The geometry tensor $K_{ab}^I$ contains at most second derivatives of the metric, which can be achieved by a suitable choice of the gravitational action (2.1).

(v) The field equations are symmetric with respect to arbitrary permutations of the sectors $(g^I, \varphi^I)$, which can be understood as a generalized Copernican principle.

(vi) The vacuum solution is given by a set of flat metrics $g_{ab}^I = \eta_{ab}$. (Poincaré symmetry for all metrics simultaneously implies $g_{ab}^I = \lambda^I \eta_{ab}$ for constants $\lambda^I$ that, invoking the Copernican principle for the vacuum, should be equal and can be set to $\lambda^I = 1$.)

Assumption (ii) implies that each standard model copy $\varphi^I$ couples only to its corresponding metric tensor $g_{ab}^I$. This ensures that the dynamics and causality of each standard model copy are governed by a single metric. It further ensures that the interaction between the different standard model copies is mediated only through gravity, so that they appear mutually dark.

Variation of the gravitational action $S_G$ with respect to the metrics $g_{ab}^I$ then yields the geometry tensors $K_{ab}^I$, while variation of the matter action $S_{M}$ yields the usual energy-momentum tensors $T_{ab}^I$, as stated in assumption (iii). Assumption (iv) is a technical requirement, which we use here in order to restrict the possible terms that may appear in the linearized theory and the post-Newtonian limit. In assumption (v) we employ the Copernican principle in the sense that, in addition to the matter sector symmetry imposed by choosing $N$ identical standard model actions, there is no distinction between the different standard model copies by gravitational effects. We finally exclude cosmological constants by assumption (vi).

For the calculations we present here we use a concrete multimetric gravity theory satisfying the assumptions listed above. The theory we consider here is defined by the action

$$S_G = \frac{1}{16\pi} \int d^4x \sqrt{g_0} \left[ \sum_{I=1}^{N} \left( c_1 R^I + g^{ij} \left( c_3 S_{ij}^I S_{ij}^I + c_5 S_{ik}^I S_{jk}^I + c_7 S_{ik}^{II} S_{jk}^{II} + c_{10} S_{ik}^{I} S_{jk}^{IJ} + c_{12} S_{ij}^{I} S_{kl}^{IJ} \right) \right) 
+ g^{ij} g^{kl} g_{mn} \left( c_2 S_{mn}^I S_{kl}^I + c_{11} S_{mn}^I S_{kl}^{II} \right) 
+ \sum_{I,J=1}^{N} \left( c_2 g^{ij} R_{ij}^I + g^{ij} \left( c_4 S_{ij}^{I} S_{ij}^{I} + c_6 S_{ij}^{IJ} S_{ij}^{IJ} + c_8 S_{ij}^{I} S_{ij}^{IJ} \right) \right) 
+ g^{ij} g^{kl} \left( c_{10} S_{ij}^{I} S_{kl}^{IJ} + c_{12} S_{ij}^{I} S_{kl}^{IJ} \right) \right]$$

(2.2)
where $c_1, \ldots, c_{12}$ are constant parameters and we used the connection difference tensors

$$S^{Ij}{}_{jk} = \Gamma^{Ij}{}_{jk} - \Gamma^{Ij}{}_{jk} \quad S^{Ij} = S^{Ijk}{}_{jk} \quad \tilde{S}^{Ij}{}_{jk} = \frac{1}{N} \sum_{I=1}^{N} S^{Ij}{}_{jk} \quad \tilde{S}^{Ij} = \tilde{S}^{Ijk}{}_{jk}$$

(2.3)

and the mixed density

$$g_0 = \prod_{I=1}^{N} (g^I)^{\frac{1}{2}}$$

(2.4)

Here we have chosen units in which the Newtonian gravitational constant takes the value $G_N = 1$. This action is both general and simple, in the sense that it contains a large number of different terms, each of which has a simple form. In the following sections we will discuss several aspects of this particular multimetric gravity model.

3. Post-Newtonian consistency

In this section we discuss the experimental consistency of our multimetric gravity theory. For this purpose we make use of the parameterized post-Newtonian (PPN) formalism [9, 10, 11, 12]. In its standard form the PPN formalism applies to gravity theories in which a single metric tensor determines the motion of test bodies, and characterizes each gravity theory by a set of ten parameters. The values of these parameters have been measured with high precision in the solar system. Here we present an extension of the PPN formalism to multimetric gravity theories [6, 8] and use the measured values of the PPN parameters to derive constraints on the concrete theory displayed in the preceding section.

The starting point of our derivation is a perturbative expansion of the metrics $g^I_{ab}$ in orders of the velocity $\vec{v}$ of the source matter in a given frame of reference. Using assumption (vi) this is a weak field approximation around the flat vacuum metric $\eta_{ab}$ in Cartesian coordinates $(x^a) = (t, \vec{x})$.

$$g^I_{ab} = \eta_{ab} + h^I_{ab} = \eta_{ab} + h^{I(1)}_{ab} + h^{I(2)}_{ab} + h^{I(3)}_{ab} + h^{I(4)}_{ab}$$

(3.1)

where each term $h^{I(n)}_{ab}$ is of order $|\vec{v}|^n \equiv O(n)$. However, not all of these terms are necessary in order to describe the motion of test bodies, while others vanish due to Newtonian energy conservation or time reversal symmetry. Here we consider only the relevant, non-vanishing components. Using the Laplacian $\Delta = \partial_a \partial^a$ we can express them in the form

$$h^{I(2)}_{00} = -\sum_{J=1}^{N} a^{Ij} \Delta \chi^J$$

(3.2a)

$$h^{I(2)}_{\alpha\beta} = \sum_{J=1}^{N} (2\theta^{IJ} \chi^J_{,\alpha\beta} - (\gamma^{IJ} + \theta^{IJ}) \Delta \chi^J \delta_{\alpha\beta})$$

(3.2b)

$$h^{I(3)}_{0a} = \sum_{J=1}^{N} (\sigma^{I}{}_{J} W^J{}_{a} + \sigma^{I}{}_{J} W^J{}_{-a})$$

(3.2c)

$$h^{I(4)}_{00} = \sum_{J=1}^{N} \left( \phi^{I}{}_{J} \Phi^{J}{}_{-} + \phi^{I}{}_{J} \Phi^{J}{}_{+} + \sum_{A=1}^{2} \omega^{I}{}_{J} \Omega^{A} \right) + \sum_{J,K=1}^{N} \sum_{A=1}^{7} \psi^{I}{}_{J}^{K} \Psi^{J}{}_{K}$$

(3.2d)

The so-called PPN potentials $\chi^I, W^I, \Phi^I, \Phi^I, \Phi^{I}{}_{1}, \Phi^{I}{}_{2}, \Omega^{I}{}_{1}, \Omega^{I}{}_{2}, \Psi^{I}{}_{1}, \ldots, \Psi^{I}{}_{7}$ are Poisson-like integrals over the source matter distribution, which is assumed to be a perfect fluid with rest
energy density $\rho^I \sim O(2)$, velocity $v^I_\alpha \sim O(1)$, internal energy density $\rho^I \Pi^I \sim O(4)$ and pressure $p^I \sim O(4)$ for each matter type $I = 1, \ldots, N$. The PPN parameters $\alpha^{IJ}, \gamma^{IJ}, \theta^{IJ}, \sigma^I_+^J, \phi_p^I, \phi^I_I, \omega^I_1, \omega^I_2, \psi^I_1, \ldots, \psi^{IJK}_7$ are constants which are characteristic for each multimetric gravity theory. They can be determined from a perturbative solution of the gravitational field equations

$$K^I_{ab} = 8\pi T^I_{ab}$$

starting from the lowest velocity order $O(2)$ and up to the highest velocity order $O(4)$.

The most important asset of the standard PPN formalism for a single metric is the possibility to measure the values of the PPN parameters at high precision using solar system experiments. A comparison between the multimetric PPN formalism displayed here and the standard PPN formalism shows [8] that a multimetric gravity theory is consistent with solar system observations at the post-Newtonian level if its PPN parameters satisfy the relations

$$\alpha^{11} = \gamma^{11} = 1, \quad \sigma^I_+^1 = \psi^I_3^{111} = \psi^I_6^{111} = -2, \quad \sigma^-_1 = -\frac{3}{2}, \quad \phi^I_1 = 2, \quad \phi_p^1 = 6$$

$$\omega^I_1 = 4, \quad \theta^{11} = \omega^I_2 = \psi^I_1^{111} = \psi^I_2^{111} = \psi^I_3^{111} = \psi^I_5^{111} = \psi^I_7^{111} = 0$$

(3.4)

where we have identified the standard model copy $\varphi^1$ with the visible standard model, which is the only matter source and the only type of test masses in the solar system. Hence, these are the only PPN parameters which are accessible to solar system experiments. A thorough analysis shows [8] that among these sixteen numbers are two gauge degrees of freedom and one rescaling freedom. This leaves us with thirteen physical parameters, including the ten known parameters from the standard PPN formalism.

We finally apply our version of the PPN formalism to the multimetric gravity theory defined by the action (2.2). From the requirement that its PPN parameters satisfy the consistency conditions (3.4) and that in the Newtonian limit the gravitational interaction between the different standard model copies becomes repulsive of equal strength we derive the restrictions

$$c_1 = c_4 - \frac{1}{2}c_6 + \frac{1}{8}c_8 - \frac{3}{2}c_{10} - \frac{3N + 2}{8(N - 2)}$$

(3.5a)

$$c_2 = -c_4 + \frac{1}{2}c_6 - \frac{1}{2}c_8 + \frac{3}{2}c_{10} + \frac{3}{8}$$

(3.5b)

$$c_3 = -\frac{7}{3}c_4 + \frac{1}{6}c_6 - \frac{1}{6}c_8 - \frac{3}{2}c_{10} - \frac{N + 30}{24(N - 2)}$$

(3.5c)

$$c_5 = -\frac{5}{3}c_4 - \frac{7}{6}c_6 - \frac{5}{6}c_8 - \frac{3}{2}c_{10} - \frac{5N + 6}{24(N - 2)}$$

(3.5d)

$$c_7 = 2c_4 - c_6 - c_8 + 3c_{10} + \frac{N + 6}{4(N - 2)}$$

(3.5e)

$$c_9 = \frac{4}{3}c_4 - \frac{2}{3}c_6 + \frac{2}{3}c_8 - 2c_{10} - \frac{N - 3}{3(N - 2)}$$

(3.5f)

on the constant coefficients $c_1, \ldots, c_{12}$. This further yields us the PPN parameters

$$\alpha^{IJ} = 2\delta^{IJ} - 1 \quad \gamma^{IJ} = 2\delta^{IJ} - 1 \quad \theta^{IJ} = 0$$

$$\sigma^I_+^J = 2 - 4\delta^{IJ} \quad \sigma^-_1 = \frac{5}{2} - 4\delta^{IJ} \quad \omega^I_1 = 10\delta^{IJ} - 6$$

(3.6)

$$\omega^I_2 = 2 - 2\delta^{IJ} \quad \phi^I_p = 16\delta^{IJ} - 10 \quad \phi^I_I = 4\delta^{IJ} - 2$$

where $\delta^{IJ}$ denotes the Kronecker symbol, and lengthy expressions for $\psi^I_1^{JK}, \ldots, \psi^{IJK}_7$; see [8] for a complete list. In the following sections we will restrict ourselves to a discussion of this experimentally consistent repulsive gravity model.
4. Cosmology and accelerating expansion

We will now discuss the cosmological properties of multimetric gravity [5]. For this purpose we will start from the action (2.2), where we restrict the constant input parameters using the experimental consistency conditions (3.5). Using standard assumptions on cosmological symmetries we derive the cosmological equations of motion for a universe filled with general perfect fluid matter. We provide explicit solutions of these equations for both radiation and dust matter.

The standard assumption on homogeneous and isotropic cosmologies comprises the existence of six Killing vector fields responsible for spatial translations and rotations. The requirement that these fields are symmetry generators for all metrics simultaneously restricts their form to be of Robertson–Walker type,

\[ g^I = -(n^I)^2(t) dt \otimes dt + (a^I)^2(t) \gamma_{\alpha\beta} dx^\alpha \otimes dx^\beta \]  

(4.1)

with lapse functions \( n^I(t) \), scale factors \( a^I(t) \), and a common purely spatial metric \( \gamma_{\alpha\beta} \) of constant curvature \( k \in \{-1, 0, 1\} \) and Riemann tensor \( R(\gamma)_{\alpha\beta\gamma\delta} = 2k \gamma_{[\alpha\gamma} \gamma_{\delta]\beta} \). The matter content consistent with the cosmological symmetries is given by a set of \( N \) homogeneous fluids with density \( \rho^I(t) \) and pressure \( p^I(t) \). Their energy-momentum tensors can be written as

\[ T^{Iab} = (\rho^I + p^I) u^I_a u^I_b + p^I g^{Iab} \]  

(4.2)

with velocities normalized by the relevant metrics from their sector so that \( g^{Iab} u^I_a u^I_b = -1 \). These tensors can be decomposed into the components \( T^I_{ab} = \rho^I(n^I)^2 \) and \( T^I_{\alpha\beta} = p^I(a^I)^2 \gamma_{\alpha\beta} \).

As another assumption on cosmological symmetries we employ the Copernican principle in the sense that we assume equal amounts of matter to be present in all matter sectors. This means we can omit the sector index \( I \) from all matter densities and pressure functions. From the symmetry assumption \((v)\) it then follows that also the metrics should be equal, hence also the lapse functions and scale factors. We may now rescale the cosmological time so that \( n(t) \equiv 1 \). It follows that the connection difference tensors in the action (2.2) vanish and the gravitational field equations take the simple form

\[ (c_1 + c_2) \left( R_{ab} - \frac{1}{2} R g_{ab} \right) = 8\pi T_{ab} \]  

(4.3)

Using the conditions (3.5) on the parameters \( c_1, \ldots, c_{12} \) we find the simple expression

\[ c_1 + c_2 = 1/(2 - N) \]  

(4.4)

which becomes singular for \( N = 2 \) and negative for \( N \geq 3 \). Inserting the Robertson–Walker form (4.1) of the metric we then find the cosmological equations of motion

\[ 8\pi \rho = \frac{3}{2 - N} \left( \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \]  

(4.5a)

\[ 8\pi p = \frac{1}{N - 2} \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \]  

(4.5b)

Positive matter densities require \( k = -1 \) and \(|\dot{a}| < 1\). We further derive the acceleration equation

\[ \frac{\ddot{a}}{a} = \frac{4\pi(N - 2)}{3} (\rho + 3p) . \]  

(4.6)

We thus find that the acceleration becomes positive for \( N \geq 3 \) and standard model matter with \( \rho + 3p > 0 \).
In order to solve the cosmological equations of motion we further introduce the equation of state \( p = w \rho \) with constant barotropic parameter \( w \). The physical reasoning for this choice is the assumption that the matter content of the universe can be approximated by radiation, or relativistic particles, with \( w = 1/3 \) in the very early universe, and by dust, or non-relativistic particles, with \( w = 0 \) in the late universe. Using the conformal time \( \eta \) defined by \( dt = a \, d\eta \) we then find the solutions

\[
a = a_{\text{min}} \left[ \cosh \left( \frac{3w + 1}{2} (\eta - \eta_{\text{bounce}}) \right) \right]^{\frac{2}{3w+1}}
\]

\[
\rho = \rho_{\text{max}} \left[ \cosh \left( \frac{3w + 1}{2} (\eta - \eta_{\text{bounce}}) \right) \right]^{-\frac{6w+6}{3w+1}}
\]

Here \( a_{\text{min}} > 0 \) and \( \eta_{\text{bounce}} \) are integration constants and \( \rho_{\text{max}} \) is given by

\[
\rho_{\text{max}} = \frac{3}{8\pi(N-2)a_{\text{min}}^2}
\]

From these solutions we find that our simple cosmological model features a big bounce, i.e., a positive minimal value of the scale factor at finite time, as a consequence of the negative effective gravitational constant in equation (4.3). In order to study its late time behavior we must transform the solutions from conformal time \( \eta \) to cosmological time \( t \). By integration we obtain

\[
t = -\frac{a_{\text{min}}}{4^{3w+1}e^{\eta-\eta_{\text{bounce}}} 2F_1 \left( \frac{1}{3w+1}, \frac{2}{3w+1}; \frac{3w}{3w+1}; -e^{(3w+1)(\eta-\eta_{\text{bounce}})} \right)}
\]

in terms of the hypergeometric function \( 2F_1 \). Now the big bounce at \( \eta = \eta_{\text{bounce}} \) corresponds to \( t = 0 \). We then find that for late times the acceleration becomes small, \( \ddot{a} \to 0 \), and the expansion rate approaches the asymptotic value \( \dot{a} \to 1 \). This also becomes evident when we plot our solutions. Figure 1 shows the evolution of the scale factor \( a(t) \). Figure 2 shows the matter density \( \rho(t) \). Finally, figure 3 shows the Hubble parameter \( H = \dot{a}/a \). In all plots the solution for radiation is displayed by a dashed line, while a solid line shows the dust matter solution.

![Figure 1](image-url)

**Figure 1.** Scale factor \( a \) of the radiation-filled universe (dashed line) and the dust-filled universe (solid line) plotted over cosmological time \( t \).
Figure 2. Matter density $\rho$ of the radiation-filled universe (dashed line) and the dust-filled universe (solid line) plotted over cosmological time $t$.

Figure 3. Hubble parameter $H = \dot{a}/a$ of the radiation-filled universe (dashed line) and the dust-filled universe (solid line) plotted over cosmological time $t$.

5. Formation of large scale structures

In this section we take the cosmological solution shown in the previous section as a basic ingredient for the evolution of large scale structures in the universe. This allows us to construct a simple model of structure formation in multimetric gravity. Our model will be based on a set of well-motivated assumptions, which we display in the first part of this section. We then discuss the resulting equations of motion which govern the dynamics of structure formation. We finally present the result of a simple computer simulation.

We start from the following set of assumptions:

(i) The metrics $g^{I}_{ab}$ can be written as a small perturbation $g^{I}_{ab} = g^{0}_{ab} + h^{I}_{ab}$ around the cosmological solution

$$
g^{0} = -dt \otimes dt + a^2(t)\gamma_{\alpha\beta}dx^{\alpha} \otimes dx^{\beta}
$$

(ii) The relevant scale for structure formation is small compared to the curvature radius of the universe, so that we can restrict ourselves to a cube $0 \leq x^{\alpha} \leq \ell$ of edge length $\ell$, for
which we impose periodic boundary conditions, and approximate the spatial part $\gamma_{\alpha\beta}$ of the metric (5.1) by a flat metric $\delta_{\alpha\beta}$ within this cube.

(iii) The matter content of the universe is constituted by an equal number of $n$ point masses of equal mass $M$ per unit volume $(a\ell)^3$ for each of the $N$ standard model copies.

(iv) The mean distance $a\ell/\sqrt{Nn}$ between two point masses is large compared to their Schwarzschild radius $2GM$, so that we can treat their gravitational interaction in a weak field approximation.

(v) The spatial velocities $v^\alpha_{Ii} = a\dot{x}^\alpha_{Ii} \ll 1$ of the point masses measured in co-moving coordinates are small.

(vi) The initial state is given by a uniform matter distribution which is at rest with respect to our chosen coordinates, $\dot{x}^\alpha_{Ii}(t = 0) = 0$.

It follows from these assumptions that the matter content of our model corresponds to dust with vanishing pressure $p = 0$ and a density

$$\rho = Mn/(a\ell)^3 \quad (5.2)$$

which enters the cosmological equations of motion (4.5). These govern the global dynamics of our model. The local dynamics is governed by the components $h^I_{00}$ of the metric perturbation in the Newtonian limit, which we write in the form

$$h^I_{00} = 2\sum_{J=1}^{N} \alpha^{IJ}U^J \quad (5.3)$$

where the PPN parameter $\alpha^{IJ} = 2\delta^{IJ} - 1$ corresponds to the repulsive Newtonian limit. We further define the Newtonian potentials

$$U^I(t, \vec{x}) = \frac{M}{a} \sum_{i=1}^{n} \frac{1}{d(\vec{x}, \vec{x}_{Ii}(t))} \quad (5.4)$$

using the distance function

$$d(\vec{x}, \vec{x'}) = \min_{\vec{n} \in \mathbb{Z}^3} |\vec{x} - \vec{x'} + \ell\vec{n}| \quad (5.5)$$

which respects the periodic boundary conditions we imposed. Finally, the dynamics of the point masses is governed by the geodesic equation, which takes the form

$$\ddot{x}^\alpha_{Ii} = \frac{\partial_\alpha h^I_{00}}{2a^2} - 2\frac{\dot{a}}{a} \dot{x}^\alpha_{Ii} \quad (5.6)$$

in the Newtonian limit. These equations provide the full dynamics of our simple model of structure formation. They can be solved using the techniques of $n$-body simulations. The result of a simple implementation using only direct $n$-body interactions visualized using Slootch [http://www.mpa-garching.mpg.de/~kdolag/Slootch] is shown in figure 4. In this calculation $n = 2^{16} = 65536$ point masses for each of $N = 16$ matter types have been simulated. The simulation has been performed with the aid of GPU computing.

We can see in this simulation that visible matter forms clusters, which are connected by filament-like structures, in agreement with observations. We further find large, “empty” voids, in the sense that visible matter is absent. In our model it turns out that these are not truly empty, but contain clusters of dark galaxies formed by the additional standard model copies. This result is of particular interest in combination with the calculated value (3.6) of the PPN parameter $\gamma^{IJ}$, from which follows that these clusters should have a negative gravitational lensing effect on visible light.
Figure 4. Stereoscopic image from a simulation of structure formation with \( N = 16 \) matter types and \( n = 65536 \) point masses for each matter type. Only visible matter is shown. For three-dimensional vision, view the two images from a short distance, so that each eye sees a different image, until they appear to merge. Then slowly increase the viewing distance to achieve focus.

6. Gravitational waves

We finally discuss the propagation of gravitational waves in the context of multimetric gravity [7]. The starting point of our discussion will be the most general linearized field equations of multimetric gravity. We will consider planar wave solution and discuss their propagation velocity. We will then make use of the Newman-Penrose formalism and derive the possible polarizations.

The starting point for our derivation is given by the most general linearized vacuum field equations compatible with our assumptions on multimetric gravity theories. Using the perturbation ansatz (3.1) they take the form

\[
0 = K^I_{ab} = \sum_{J=1}^{N} \left( P^{IJ} \partial^p \partial_{(a} h^J_{b)p} + Q^{IJ} \Box h^J_{ab} + R^{IJ} \partial_a \partial_b h^J + M^{IJ} \partial^p \partial^q h^J_{pq} \eta_{ab} + N^{IJ} \Box h^J \eta_{ab} \right)
\]  

(6.1)

where indices are raised with the flat metric \( \eta \) and \( \Box = \eta^{ab} \partial_a \partial_b \). The coefficients \( P^{IJ}, Q^{IJ}, R^{IJ}, M^{IJ}, N^{IJ} \) are constant parameters which are determined by the concrete gravity theory under consideration. Due to the the symmetry assumption (v) they take the form

\[
O^{IJ} = O^{-} + (O^{+} - O^{-}) \delta^{IJ}
\]

(6.2)

with diagonal entries \( O^{+} \) and off-diagonal entries \( O^{-} \) for \( O = P, Q, R, M, N \). From these expressions it further follows that the linearized vacuum field equations (6.1) decouple,

\[
0 = R^I_{ab} = P_1 \partial^p \partial_{(a} h^I_{b)p} + Q_1 \Box h^I_{ab} + R_1 \partial_a \partial_b h^I + M_1 \partial^p \partial^q h^I_{pq} \eta_{ab} + N_1 \Box h^I \eta_{ab}
\]  

(6.3a)

\[
0 = R^I_{ab} = P_0 \partial^p \partial_{(a} h^I_{b)p} + Q_0 \Box h^I_{ab} + R_0 \partial_a \partial_b h^I + M_0 \partial^p \partial^q h^I_{pq} \eta_{ab} + N_0 \Box h^I \eta_{ab}
\]  

(6.3b)

where \( O_0 = O^{+} - O^{-} \) and \( O_1 = O^{+} + (N - 1)O^{-} \) are the eigenvalues of the corresponding parameter matrix \( O^{IJ} \). The diffeomorphism invariance assumption (ii) restricts their values to

\[
P_1 = -2Q_1 = -2R_1 = -2M_1 = 2N_1
\]

(6.4)
The quantities $h^1_{ab}$ and $h^3_{ab}$ for $i = 2, \ldots, N$ are linear combinations of the metric perturbations $h^l_{ab}$. Using the gauge invariant linear perturbation formalism they can be decomposed into physical degrees of freedom and pure gauge quantities. From the planar wave ansatz

$$h^l_{ab}(x) = \hat{h}^l_{ab}e^{ika x^a}$$

(6.5)

with constant amplitude $\hat{h}^l_{ab}$ and wave vector $k_a$ for a single Fourier mode one can then show [7] that solutions for the physical degrees of freedom exist only if $k_a$ is null, $k_a k^a = 0$. Hence, gravitational waves must propagate at the speed of light.

We now turn our attention to the possible polarizations of gravitational waves. For this purpose we employ the Newman-Penrose formalism [13] and make use of a double null basis $(l^a, n^a, m^a, \tilde{m}^a)$ of the tangent space, which can be expressed as

$$l^a = (1, 0, 0, 1), \quad n^a = \frac{1}{2} (1, 0, 0, -1), \quad m^a = \frac{1}{\sqrt{2}} (0, 1, i, 0), \quad \tilde{m}^a = \frac{1}{\sqrt{2}} (0, 1, -i, 0)$$

(6.6)

in the $(x^0, x^\alpha)$ basis. We consider a plane wave propagating in the positive $x^3$ direction,

$$h_{ab} = \hat{h}_{ab} e^{i\omega (x^a - x^3)} = \hat{h}_{ab} e^{i\omega u}$$

(6.7)

where $h_{ab}$ denotes one of the metric components $h_{ab}$ in the decoupled field equations (6.3) and we introduced the retarded time $u = x^0 - x^3$. The effect of this wave on test masses depends only on its Riemann tensor $\mathcal{R}$, which is determined completely by the six electric components

$$\mathcal{R}_{nlml} = -\frac{1}{2} \mathcal{R}_{ll} \quad \mathcal{R}_{nlmm} = \mathcal{R}_{nmnl} = -\frac{1}{2} \mathcal{R}_{nlm} = -\frac{1}{2} \mathcal{R}_{nm}$$

$$\mathcal{R}_{mmnm} = -\frac{1}{2} \mathcal{R}_{nm} = -\frac{1}{2} \mathcal{R}_{mm} \quad \mathcal{R}_{nmnm} = -\frac{1}{2} \mathcal{R}_{mnm}$$

(6.8)

where dots denote derivatives with respect to $u$. Their presence is restricted by the gravitational field equations. In the Newman-Penrose basis (6.6) we find that the five component equations

$$0 = \mathcal{R}_{ll} = \mathcal{R}_{mm} = \mathcal{R}_{\tilde{m}m} = \mathcal{R}_{lm} = \mathcal{R}_{\tilde{m}l}$$

(6.9)

are satisfied identically, while the remaining five component equations take the form

$$0 = \mathcal{R}_{ln} = 2 \mathcal{R}_{\tilde{m}ln} - (P + 2R) \mathcal{R}_{ln}$$

(6.10a)

$$0 = \mathcal{R}_{ln} = -\frac{1}{2} (P + 2M) \mathcal{R}_{ln}$$

(6.10b)

$$0 = \mathcal{R}_{nm} = -\frac{1}{2} P \mathcal{R}_{nm}$$

(6.10c)

$$0 = \mathcal{R}_{\tilde{m}n} = -\frac{1}{2} P \mathcal{R}_{\tilde{m}n}$$

(6.10d)

$$0 = \mathcal{R}_{\tilde{m}m} = M \mathcal{R}_{\tilde{m}l}$$

(6.10e)

where the curvature tensor $\mathcal{R}_{ab}$ and the eigenvalues $P, R, M$ of the parameter matrices are taken from the same field equation (6.3) as the metric perturbation $h_{ab}$ in the plane wave ansatz (6.7).

We now follow the scheme developed in [14, 15] to classify multimetric gravity theories by the allowed polarizations (6.8) of gravitational waves, labeled by representations of the little group $E(2)$. For $P \neq 0$ and $P + 2R = 0$ we find that only the two tensor modes $h_{mm}, h_{\tilde{m}\tilde{m}}$ are allowed by the component equations (6.10), the $E(2)$ class is thus $N_2$. For $P \neq 0$ and $P + 2R \neq 0$ we find an additional scalar mode $\mathcal{R}_{\tilde{m}m}$, corresponding to the $E(2)$ class $N_3$. In the case $P = 0$ and
Figure 5. Classification of the different E(2) classes depending on the eigenvalues of the coefficient matrices in the linearized field equations.

\[ P_{1} = -2Q_{1} = -2R_{1} = -2M_{1} = 2N_{1} = \frac{1}{2-N}, \quad Q_{0} = -\frac{1}{4} \]
\[ P_{0} = -2R_{0} = -2M_{0} = \frac{6-N}{6(2-N)} + 4y, \quad N_{0} = \frac{N}{12(N-2)} + y \]  

(6.11)

where \( y \) is given by

\[ y = \frac{2}{3}c_{4} - \frac{1}{3}c_{6} + \frac{1}{3}c_{8} + \frac{1}{2}c_{11} + c_{12} \] 

(6.12)

From this we immediately read off that for the metric perturbation \( h_{ab}^{1} \), which is governed by the eigenvalues \( P_{1}, Q_{1}, R_{1}, M_{1}, N_{1} \), the E(2) class is always \( N_{2} \), i.e., the only allowed polarizations are the two tensor modes \( h_{mm}, h_{\bar{m}\bar{m}} \). The same applies to the remaining perturbations \( h_{ab}^{i} \) governed by the eigenvalues \( P_{0}, Q_{0}, R_{0}, M_{0}, N_{0} \), unless \( y \) takes the special value

\[ y = \frac{6-N}{24(2-N)} \]  

(6.13)

in which case all six polarizations of gravitational waves are allowed and the E(2) class is \( \Pi_{6} \).

7. Conclusion

We have presented a class of gravity theories with \( N \geq 2 \) metric tensors and a corresponding number of standard model copies and discussed various aspects of multimetric gravity. In particular we considered theories in which gravity is attractive within each matter sector, but repulsive between the different standard model copies in the Newtonian limit.

As the first aspect we have studied the post-Newtonian limit of multimetric gravity. For this purpose we constructed an extension of the parameterized post-Newtonian (PPN) formalism. Our extended formalism characterizes multimetric gravity theories with a set of constant parameters, thirteen of which can be measured by visible matter experiments in the solar system. Our findings have shown that multimetric gravity is fully consistent with current measurements of the ten standard PPN parameters in solar system experiments.

We continued our discussion in the field of cosmology, where we started from the standard assumptions of isotropy and homogeneity. We showed that under these assumptions multimetric
gravity can be described by a single common metric, which obeys the Einstein equations with a negative effective gravitational constant. As a consequence we obtained an accelerating expansion of the universe which naturally becomes small at late times and a big bounce.

Using the exact cosmological solution for dust matter as a background geometry, we constructed a simple model for the formation of large scale structures. From a numerical simulation we found that voids, in which the density of visible matter is low, should not be empty, but contain clusters of the additional, dark, standard model copies. From our post-Newtonian analysis it further follows that these would act as negative gravitational lenses.

Finally we discussed the linearized gravitational field equations in vacuum under the aspect of gravitational waves. We presented a wave ansatz, from which follows that gravitational waves propagate at the speed of light. This allowed us to apply the Newman-Penrose formalism and to calculate the allowed polarizations and the E(2) class of multimetric gravity. We found that in general these can be $N_2$, $N_3$, III$_5$ and II$_6$, and displayed example theories for $N_2$ and II$_6$.

Naturally the question arises whether the number $N$ of metrics can be fixed or bounded by observations. An immediate restriction follows from our no-go theorem for bimetric repulsive gravity, which states that $N = 2$ is excluded in the case that different standard model copies repel each other with equal strength [4]. This assumption appears natural, since it does not introduce a second, undetermined gravitational coupling strength, and has also entered our discussions of cosmology and structure formation. However, we do not find other restrictions on $N$ from the presented calculations or from current observational data.

The work we presented here is still fundamental and further studies are required to test multimetric gravity using observations. Of particular interest would be tests in strongly coupled systems, such as double pulsars, for which high-precision timing measurements of radio pulses exist and observations of gravitational waves are expected for the near future. The cosmic microwave background and the distribution of large scale structures could provide another testbed for multimetric gravity theories, using cosmological perturbation analysis and further simulations of structure formation. Finally, our findings motivate new experimental tests of gravity, such as solar system measurements of the three newly introduced PPN parameters, and the search for negative gravitational lenses in the galactic voids.

Acknowledgments
The author gratefully acknowledges full financial support from the Estonian Research Council through the Postdoctoral Research Grant ERMOS115.

References
[1] Riess A G et al 1998 [Supernova Search Team Collaboration] Astron. J. 116 1009
[2] Perlmutter S et al 1999 [Supernova Cosmology Project Collaboration] Astrophys. J. 517 565
[3] Davis M, Efstathiou G, Frenk C S and White S D M 1985 Astrophys. J. 292 371
[4] Hohmann M and Wohlfarth M N R 2009 Phys. Rev. D 80 104011
[5] Hohmann M and Wohlfarth M N R 2010 Phys. Rev. D 81 104006
[6] Hohmann M and Wohlfarth M N R 2010 Phys. Rev. D 82 084028
[7] Hohmann M 2012 Phys. Rev. D 85 084024
[8] Hohmann M 2013 Classical Quantum Gravity 31 135003
[9] Nordtvedt K 1968 Phys. Rev. 169 1017
[10] Thorne K S and Will C M 1971 Astrophys. J. 163 595; 163 611; 169 125
[11] Will C M 1993 Theory and Experiment in Gravitational Physics (Cambridge Univ. Press)
[12] Will C M 2014 Living Rev. Rel. 17 4
[13] Newman E and Penrose R 1962 J. Math. Phys. 3 566
[14] Eardley D M, Lee D L, Lightman A P et al 1973 Phys. Rev. Lett. 30 884
[15] Eardley D M, Lee D L and Lightman A P 1973 Phys. Rev. D 8 3308