A NONLINEAR STEFAN PROBLEM WITH VARIABLE EXPONENT AND DIFFERENT MOVING PARAMETERS

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ABSTRACT. In this paper, we consider a nonlinear diffusion problem with variable exponent, accompanied by double free boundaries possessing different moving parameters, where the variable exponent function $m(x)$ satisfies that $m(x) - 1$ can change its sign. Local existence and uniqueness of solution are established firstly, and then, some sufficient conditions are achieved for finite time blowup, and as well for global existence. Asymptotic behavior is further investigated for global solution, and existences of fast solution and slow solution are presented by making use of upper-sub solutions, energy and scaling arguments.

1. Introduction. Let us consider nonnegative solutions to the following nonlinear diffusive problem with free boundary conditions and variable exponent:

$$
\begin{align*}
&u_t - u_{xx} = u^{m(x)}, \quad t > 0, \quad g(t) < x < h(t), \\
&u(t, g(t)) = 0, \quad g'(t) = -\mu_1 u_x(t, g(t)), \quad t > 0, \\
&u(t, h(t)) = 0, \quad h'(t) = -\mu_2 u_x(t, h(t)), \quad t > 0, \\
&-g(0) = h(0) = h_0, \quad u(0, x) = u_0(x), \quad -h_0 \leq x \leq h_0,
\end{align*}
$$

where moving parameters $\mu_1, \mu_2 \geq 0, h(t)$ and $g(t)$ are so called free boundary, $h_0$ denotes the size of initial habitat, function $m(x) \in C(\mathbb{R})$ is bounded and $m(x) > 0$ in $\mathbb{R}$. The initial function $u_0(x)$ satisfies

$$
\begin{align*}
&u_0(x) \in W^2_p(-h_0, h_0) \text{ with } p > 3, \\
&u_0(x) > 0 \text{ in } (-h_0, h_0), \quad u_0(-h_0) = u_0(h_0) = 0.
\end{align*}
$$

Here, what is worthy mentioning is that if assuming $(u, h, g)$ satisfies problem (1) in $[0, T) \times [g(t), h(t)]$, then $(u, h, g)$ is called by a nonnegative solution if $u \geq 0$ in $[0, T) \times [g(t), h(t)]$. In addition, if $(u, h, g)$ is a nonnegative solution of problem (1) with $u_0(x) > 0$ in $(-h_0, h_0)$, then it holds that $u > 0$ in $[0, T) \times (g(t), h(t))$ by the strong maximum principle.

The issue of free boundary has long been deeply concerned. They have been suggested as mathematical schemes of several processes, such as change of phase,
chemical reactions, problem in statistics, biomechanics, etc. Please refer to [6, 7, 15, 25, 28, 32] and the references therein for more details.

Let $\alpha, a$ and $b$ be constants with $a, b \geq 0$ and $a^2 + b^2 \neq 0$. For the following one-phase Stefan problem with a power-type reaction term:

$$\begin{cases}
u_t = u_{xx} + u^{1+\alpha}, & t > 0, \ 0 < x < s(t), \\
u u(t, 0) + bu_x(t, 0) = 0, & t > 0, \\
u t, s(t) = 0, & t > 0, \\
u s(0) = s_0, & 0 \leq x \leq s_0, \\
u s(t) = 0, & t > 0, \\
u up(t, s(t)) = -u_x(t, s(t)), & t > 0, \\
u s(t) = 0, & t \geq 0,
\end{cases} \quad (2)$$

Then it holds that

$$\lim_{t \to \infty} \|u(t, \cdot)\|_{L^\infty(0, s(t))} = 0.$$ 

Moreover, if we let $s_\infty = \lim_{t \to \infty} s(t) \leq \infty$, then one of the following two possibilities occurs:

(i) (fast solution) $s_\infty < \infty$ and there exist constants $C, \beta > 0$ (depending on $u$) such that

$$\|u(t, \cdot)\|_{L^\infty(0, s(t))} \leq Ce^{-\beta t}, \quad \forall \ t \geq 0;$$

(ii) (slow solution) $s_\infty = \infty$ and one has the estimates

$$s(t) = O(t^{2/3}) \text{ as } t \to \infty, \quad \liminf_{t \to \infty} s^{2/\alpha}(t)/u(t, \cdot)\|_{L^\infty(0, s(t))} > 0,$$

hence, in particular,

$$\liminf_{t \to \infty} t^{4/(3\alpha)}\|u(t, \cdot)\|_{L^\infty(0, s(t))} > 0.$$

During recent decades, there are more and more interesting results available about diffusion equation (or system) with one free boundary condition or double free boundary conditions, which are formulated to study the spread of an invasive species, please refer to, for example, [8]–[13], [20, 21, 23, 25, 27, 29], [34]–[46] and the references therein.

When dealing with a parabolic problem with non-free boundary value, there have been some results about the influence of variable exponent function on properties of solutions. For initial-boundary value problem of equation ($\Omega \subset \mathbb{R}^N$ is bounded):

$$u_t = \Delta u + u^{m(x)}, \quad (t, x) \in (0, T) \times \Omega,$$ 

Pinsaco([30]) proved existence of blowup solution when $\inf_{x \in \Omega} m(x) > 1$, and for nonnegative continuous and bounded functions $m(x)$, Ferreira, Pablo, Perez-Llanos and Rossi ([16]) explored existence or nonexistence of global solution and/or blowup solution in much more details. It should be remarked here that paper [16] not only
focuses on initial-boundary value problem of (3), but also discusses initial value problem of (3) ($\Omega = \mathbb{R}^N$). Since then, many scholars in succession generalize results of [16], and we refer the reader to [5, 26] for more references.

In contrast to the number of papers available for the case when $m(x) = m$ is a constant and $\mu_1 = \mu_2$ and for the case when the domain $\Omega$ does not change over time, much less is known when $m(x)$ is a nonnegative function and $\mu_1 \neq \mu_2$ for problem (1). The main purpose of the present paper is to generalize these results to the case where $m(x)$ is any positive function and $\mu_1 \neq \mu_2$. Analogous to the above [16] and [17, 19], we will find conditions on the variable exponent function $m(x)$ and the initial datum $u_0(x)$ to ensure existence or nonexistence of global solution and/or blowup solution. In addition, we will also analyze long-time behavior of global solution, and find conditions for existence of fast solution and/or slow solution.

The paper is organized as follows: in the following section we deal with questions of existence, uniqueness and regularity for solution; in Section 3 we establish comparison principle and perform the study of finite time blowup and global existence; in Section 4 we explore long-time behavior of global solution for problem (1).

2. Auxiliary results. For the convenience of discussions that follow, in this section we consider properties of nonnegative solution to problem

$$\begin{cases}
  u_t - u_{xx} = f(t, x, u), & t > 0, \ g(t) < x < h(t), \\
  u(t, g(t)) = 0, \ g'(t) = -\mu_1 u_x(t, g(t)), & t > 0, \\
  u(t, h(t)) = 0, \ h'(t) = -\mu_2 u_x(t, h(t)), & t > 0, \\
  g(0) = g_0, \ h(0) = h_0, & u(0, x) = u_0(x), \ g_0 \leq x \leq h_0,
\end{cases}$$

where constants $\mu_1, \mu_2$ and function $u_0(x)$ are determined as above, and constants $h_0, g_0$ satisfy $h_0 - g_0 > 0$. We make some assumptions about function $f$.

(H$_1$) $f(t, x, 0) \geq 0$. For any given constants $\tau, a, b, k$ with $b - a > 0$, $f \in L^\infty([0, \tau] \times (a, b) \times (0, k))$ and there exists a constant $L_1(\tau, a, b, k)$ such that for any $t \in [0, \tau], \ x \in [a, b]$, and $u, v \in [0, k]$,

$$|f(t, x, u) - f(t, x, v)| \leq L_1(\tau, a, b, k)|u - v|.$$

(H$_2$) For any given constants $\tau, a, b, k$ with $b - a > 0$, there exists a constant $L_2(\tau, a, b, k)$ such that for any $t \in [0, \tau], \ x, y \in [a, b]$ and $u \in [0, k]$,

$$|f(t, x, u) - f(t, y, u)| \leq L_2(\tau, a, b, k)|x - y|.$$

(H$_3$) For any given constants $\tau, a, b, k$ with $b - a > 0$, there exists a constant $L_3(\tau, a, b, k)$ such that for some constant $0 < \mu < 1 - 3/p$,

$$\|f(\cdot, x, u)\|_{C^\frac{3}{p}(0, \tau)} \leq L_3(\tau, a, b, k), \quad \forall \ x \in [a, b], \ u \in [0, k].$$

Denote $g^* = -\mu_1 u_0^*(g_0), \ h^* = -\mu_2 u_0^*(h_0)$ and

$$\Lambda = \{g_0, h_0, g^*, h^*, \|u_0\|_{w_{\infty}^2([g_0, h_0])}\}.$$

For any given constant $T > 0$, define

$$D_T = (0, T) \times (g(t), h(t)).$$

Throughout the paper, we only focus on properties of nonnegative solution, and notations like $C(\cdot), C(\cdot, \cdot)$ and so on represent positive constants with their values depending only on all arguments in the bracket.
For local existence, uniqueness, regularity and extension of solution, we have

**Proposition 1.** Let assumption \((H_1)\) be satisfied and \(0 < \mu < 1 - 3/p\) be any given constant.

(i) There exists a constant \(0 < T \ll 1\) depending only on \(\Lambda\), such that problem (4) has at least one solution \((u, h, g)\) and \((u, h, g)\) possesses the following properties:

\[
(P_T) \quad u > 0 \text{ in } D_T, \quad h'(t) > 0 \quad \text{and} \quad g'(t) < 0 \text{ in } (0, T),
\]

\[
u \in W^{1, 2}_p(D_T) \cap C^{1+\mu, 1+p}(\bar{D}_T), \quad h, g \in C^{1+\mu}(0, T).
\]

(ii) Let constant \(\tau > 0\) and \((u, h, g)\) be a nonnegative solution of problem (4) over \(D_\tau\). If \(u \in L^\infty(D_\tau)\) and \((h, g) \in C^1([0, \tau]) \times C^1([0, \tau])\). Then \((u, h, g)\) satisfies Property \((P_T)\) in \(D_T\). Moreover, there exists a positive constant \(C(\Lambda)\), such that

\[
\|u\|_{W^{1, 2}_p(D_\tau)} + \|u\|_{C^{1+\mu, 1+p}(D_\tau)} + \|g\|_{C^{1+\mu}(0, \tau)} \leq C(\Lambda)
\]

provided the constant \(0 < \sigma \ll 1\).

(iii) In addition, if assumption \((H_2)\) also holds, then nonnegative solution \((u, h, g)\) of problem (4) is unique in the class \(L^\infty(D_T) \times C^1([0, T]) \times C^1([0, T])\) as long as \(0 < T \leq \tilde{T}\), where \(\tilde{T}\) relies only on \(\Lambda\).

(iv) Let all the above hypotheses be verified. If further assumption \((H_3)\) is verified, then the unique nonnegative solution \((u, h, g)\) obtained in the above satisfies \(u \in C^{1+\frac{\mu}{2}, 2+\mu}(0, T) \times [g(t), h(t)]\), and for any given constant \(\varepsilon \in (0, T)\), it holds that

\[
\|u\|_{C^{1+\frac{\mu}{2}, 2+\mu}(\varepsilon, T) \times [g(t), h(t)]} \leq C(\varepsilon, T, \Lambda).
\]

(v) Define

\[
T_{\max} = \sup \{T > 0 \mid (u, h, g) \in L^\infty(D_T) \times C^1([0, T]) \times C^1([0, T])
\]

is a nonnegative solution of problem (4). Assume that \(f\) satisfies all requirements claimed by above (iv), and \(f\) is bounded in \(\mathbb{R}_+ \times \mathbb{R} \times (0, k)\) for any given constant \(k > 0\). Then either \(T_{\max} = \infty\) or \(T_{\max} < \infty\). Moreover, when \(T_{\max} < \infty\), it holds that

\[
\limsup_{t \to T_{\max}} \max_{x \in [g(t), h(t)]} u(t, x) = \infty.
\]

Such \(T_{\max}\) is called the maximal existence time of solution \((u, h, g)\).

**Proof.** We remark here that the author of [36] has discussed one-phase free boundary problem (4), and achieved existence and uniqueness of solution. Proposition 1 can be concluded by carrying out discussions similar as in [36], and however, we should better point out that the transformation in [36] is not suitable for problem (4). In this moment, we introduce the transformation

\[
y = \frac{2x - g(t) - h(t)}{h(t) - g(t)}, \quad v(t, y) = u(t, x), \quad \tilde{f}(t, y, v) = f(t, x, u)
\]

(9)

to straighten free boundaries \(x = g(t)\) and \(x = h(t)\). The resulted problems are

\[
\begin{align*}
\frac{dv}{dt} - Av_{yy} + Bv_y &= \tilde{f}(t, y, v), & t > 0, & -1 < y < 1, \\
v(t, -1) &= v(t, 1) = 0, & t > 0, \\
v(0, y) &= v_0(y) = u_0\left(\frac{(h_0 - g_0)y + h_0 + g_0}{2}\right), & -1 \leq y \leq 1
\end{align*}
\]

(10)
and
\[
\begin{align*}
g'(t) &= -\frac{2\mu_1}{h(t) - g(t)} v_y(t, -1) \quad \text{for } t > 0, \quad g(0) = g_0, \\
h'(t) &= -\frac{2\mu_2}{h(t) - g(t)} v_y(t, 1) \quad \text{for } t > 0, \quad h(0) = h_0,
\end{align*}
\]  
(11)
where
\[
\begin{align*}
A(g(t), h(t)) &= \left(\frac{\partial y}{\partial x}\right)^2 = \frac{4}{(h(t) - g(t))^2}, \\
B(g(t), h(t), y) &= \frac{\partial y}{\partial t} = -\frac{y[h'(t) - g'(t)] + g'(t) + h'(t)}{h(t) - g(t)},
\end{align*}
\]  
(12)
and \(v(y)\) satisfies the compatibility assumption: \(v_0(-1) = v_0(1) = 0\).

The remaining proof is similar as in [36], and we omit it in details. \(\square\)

Assumption (H_3) is essential when deducing (8) (refer to [36, Theorem 1.2] for details). When assumption (H_3) does not hold, we denote
\[
T_{\max} = \sup \left\{ T > 0 \mid u \in L^\infty(D_T) \cap W_p^{1,2}(D_T), \ h, g \in C^1([0,T]) \right\}
\]
and obtain a result below.

**Lemma 2.1.** Let assumptions (H_1) and (H_2) be verified, and let \((u, h, g)\) be a nonnegative solution of problem (4) such that \(u \in W_p^{1,2}(D_{T_{\max}}) \cap L^\infty(D_{T_{\max}})\) and
\[
(u, h, g) \text{ is a nonnegative solution of problem (4)},
\]
(13)
and obtain a result below.

**Lemma 2.2.** Let constant \(p > 3\). Assume that nonnegative function \(u \in W_p^{1,2}(D_T) \cap L^\infty(D_T)\) and functions \(h, g \in C^1([0,T])\) satisfy problem (4), such that \(g'(t) \leq 0\) and \(h'(t) \geq 0\) in \((0,T)\), and \(u \leq M_1\) in \(D_T\) for a constant \(M_1 > 0\). Let \(\|f\|_{L^\infty([0,T] \times (g(t), h(t)) \times (0, M_1))} \leq M_2\) for some constant \(M_2\). Then there is a constant \(K > 0\) depending only on \(h_0, M_1\) and \(M_2\) but not on \(T\), such that
\[-2\mu_1M_1K \leq g'(t) \leq 0, \quad 0 \leq h'(t) \leq 2\mu_2M_1K, \quad \forall 0 < t < T.
\]

**Proof.** Put
\[
K = \max \left\{ \sqrt{\frac{M_2}{2M_1}}, \frac{1}{h_0}, \frac{1}{|g_0|}, \frac{1}{M_1 \cdot \min_{[-h_0, h_0]} u'_0(x)} \right\}.
\]
Similar as in [11, Lemma 2.2], by defining
\[
\Omega_h = \left\{ (t, x) \in \mathbb{R}^2 \mid 0 < t < T, \ h(t) - 1/K < x < h(t) \right\},
\]\[
w(t, x) = M_1 \left[ 2K(h(t) - x) - K^2(h(t) - x)^2 \right], \quad (t, x) \in \Omega_h,
\]\and by comparing with \(u\) and \(w\) over \(\Omega_h\), it can be verified that
\[w \geq u, \quad (t, x) \in \Omega_h.
\]
From which follows
\[ u_x(t, h(t)) \geq w_x(t, h(t)) = -2M_1K, \quad 0 < t < T, \]
and thus,
\[ h'(t) = -\mu_2u_x(t, h(t)) \leq 2\mu_2M_1K, \quad 0 < t < T. \]
The desired assertion for \( h' \) is concluded.

Likewise, by introducing
\[ \Omega_g = \{(t, x) \in \mathbb{R}^2 | 0 < t < T, \ g(t) < x < g(t) + 1/K \}, \]
\[ \tilde{w}(t, x) = M_1[2K(x - g(t))^2 - K^2(x - g(t))^2], \quad (t, x) \in \Omega_g, \]
we can examine that \( g'(t) \geq -2\mu_1M_1K \) for every \( 0 \leq t < T \).

Now we are able to justify the reasonability of Lemma 2.1.

**Proof of Lemma 2.1.** Conclusion (i) of Proposition 1 asserts \( T_{\max} > 0 \). Hence, by definition (7), if \( T_{\max} \neq \infty \) then \( T_{\max} < \infty \). When \( T_{\max} < \infty \), we assume that \( u \leq M_1 \) on \([0, T_{\max}) \times [g(t), h(t)] \) for a constant \( M_1 > 0 \), and we are going to demonstrate
\[
\lim_{t \to T_{\max}} \|u(t, \cdot)\|_{W^{2,p}(g(t), h(t))} = \infty. \tag{15}
\]

By Proposition 1 and Lemma 2.2, there exists a constant \( C(M_1) > 0 \) such that

\[ 0 > g'(t) \geq -C(M_1), \quad 0 < h'(t) \leq C(M_1), \quad \forall \ 0 < t < T_{\max}, \]

which follows
\[ g_0 - C(M_1)T_{\max} \leq g(t) \leq g_0, \quad h_0 \leq h(t) \leq h_0 + C(M_1)T_{\max}, \quad 0 \leq t \leq T_{\max}. \]

Define \( y, v(t, y) \) and \( f(t, y, v) \) as in (9), then \( v \) and \( (h, g) \) verify problems (10) and (11). For any given \( T < T_{\max} \), an application of the \( L^p \) theory to problem (10) yields existence of a constant \( C_1(\Lambda, M_1, T_{\max}) \) independent of \( T \), such that
\[ \|v\|_{W^{1,2}(\Delta T)} \leq C_1(\Lambda, M_1, T_{\max}), \]
where \( \Delta T = [0, T] \times [-1, 1] \). Hence, \( v \in W^{1,2}(\Delta T_{\max}) \) and
\[ \|v\|_{W^{1,2}(\Delta T_{\max})} \leq C_1(\Lambda, M_1, T_{\max}). \tag{16} \]

Thereafter, (11) supplies us with \( h, g \in C^{1+\frac{2}{p}}([0, T_{\max}]) \) and
\[ \|g\|_{C^{1+\frac{2}{p}}([0, T_{\max})} \leq C_1(\Lambda, M_1, T_{\max}), \quad \|h\|_{C^{1+\frac{2}{p}}([0, T_{\max})} \leq C_1(\Lambda, M_1, T_{\max}). \tag{17} \]
The Fubini theorem tells us that \( v(t, \cdot) \in W^{2,p}_p(-1, 1) \) for almost \( t \in [0, T] \), which implies \( u(t, \cdot) \in W^{2,p}_p(g(t), h(t)) \) for almost \( t \in [0, T_{\max}] \), and by (16),
\[ \|u\|_{W^{2,p}_p(DT_{\max})} \leq C_1(\Lambda, M_1, T_{\max}). \tag{18} \]

If (15) was not true, there would exist a constant \( K \) and a sequence \( \{t_n\} \) satisfying that \( t_n \in (0, T_{\max}) \) and \( t_n \nearrow T_{\max} \) as \( n \nearrow \infty \), such that
\[ \|u(t_n, \cdot)\|_{W^{2,p}_p(g(t_n), h(t_n))} \leq K, \quad \forall \ n. \]

Now we respectively use \( t_n \) and \( (u(t_n, x), h(t_n), g(t_n)) \) as initial time and initial datum, similar to the proof of Proposition 1(i) one can find a constant \( 0 < T_0 < 1 \) which depends only on \( g(t_n), h(t_n), g'(t_n), h'(t_n) \) and \( \|u(t_n, \cdot)\|_{W^{2,p}_p(g(t_n), h(t_n))} \), such that problem (4) has a unique solution \( (u_n, h_n, g_n) \) over \([t_n, t_n+T_0] \). The uniqueness shows that \( u(t, h, g) = (u_n, h_n, g_n) \) for every \( t_n \leq t \leq \min\{t_n + T_0, T_{\max}\} \). It shows that solution \((u, h, g)\) of problem (4) can be extended uniquely to \([0, t_n+T_0]\).
Recall that $0 < t_n < T_{max}$. By (17) and (18) we know that
\[|g(t_n)|, |g'(t_n)|, |h(t_n)|, |h'(t_n)| \leq C_1(\Lambda, M_1, T_{max}),\]
\[\|u(t_n, \cdot)\|_{L^\infty((g(t_n), h(t_n)))} \leq M_1, \quad \|u(t, \cdot)\|_{W^2_2((g(t_n), h(t_n)))} \leq C_1(\Lambda, M_1, T_{max}).\]
Hence, we can choose a $T_0$ independent of $n$. Fix such $T_0$. On account of $t_n \not\to T_{max}$ as $n \not\to \infty$, it is obvious that $t_n + T_0 > T_{max}$ for all suitable large $n$, which contradicts the definition of $T_{max}$. The desired conclusions of this lemma is achieved.

3. Existence and uniqueness of solution to problem (1). We note that $m(x)$ is a variable function, which could be larger than 1 or smaller than 1, and it results that $u^{m(x)}$ does not satisfy Lipschitz continuous condition. We divide our discussion into two cases: $m(x) - 1 \geq 0$ in $\mathbb{R}$ and $m(x) - 1$ changes its sign in $\mathbb{R}$.

3.1. The case when $m(x) \geq 1$ in $\mathbb{R}$.

**Theorem 3.1.** Let $m(x) \geq 1$ in $\mathbb{R}$, and for any given constants $a, b$ with $b - a > 0$,
\[|m(x) - m(y)| \leq L(a, b)|x - y|, \quad \forall x, y \in [a, b]\] (19)
hold for some constant $L(a, b) > 0$. Then there exists a positive constant $T_{max} \leq \infty$, such that problem (1) owns a unique nonnegative solution $(u, h, g)$ over $[0, T_{max}]$ with $T_{max}$ determined by (7), and $(u, h, g)$ satisfying property $(P_{T_{max}})$, estimate (5) and estimate (6). In addition, if $T_{max} < \infty$, then assertion (8) is also valid.

**Proof.** Take $f(t, x, u) = u^{m(x)}$, then by $m(x) \geq 1$ in $\mathbb{R}$, it is not difficult to check that such $f$ satisfies both assumption $(H_1)$ and assumption $(H_3)$. Moreover, it is obvious that $f$ is bounded by $\max\{1, k^{m+}\}$ in $\mathbb{R}_+ \times \mathbb{R} \times (0, k)$ for any given constant $k > 0$, where $m_+ = \sup_{\mathbb{R}} m(x)$.

We next to demonstrate that assumption $(H_2)$ holds for above $f$. As $m(x) > 0$ in $\mathbb{R}$, it follows that for such given positive constant $k$,
\[f(t, x, u) - f(t, y, u) = 0, \quad \text{if } u = 0,\]
\[f(t, x, u) - f(t, y, u) = (u^\xi \ln u)(m(x) - m(y)), \quad \text{if } u \in (0, k],\]
(20)
here $\xi$ is a function between $m(x)$ and $m(y)$, which implies that $\xi > 0$ for all $x, y \in \mathbb{R}$. From
\[
\lim_{u \to 0^+} u^\xi \ln u = 0,
\]

it can be deduced that there exists a constant $0 < \delta < \min\{1, k\}$ small enough, such that
\[|u^\xi \ln u| \leq 1, \quad \forall u \in (0, \delta).\] (21)
We employ assumptions on $m(x)$ to find that $0 < m_+ < \infty$ and
\[|u^\xi \ln u| \leq \max\{|\ln \delta|, |\ln k|, k^{m+}|\ln k|\}, \quad \forall u \in [\delta, k].\] (22)
Consequently, from (19)-(22) it yields that for any given constants $\tau, a, b$ satisfying $\tau > 0$ and $b - a > 0$,
\[|f(t, x, u) - f(t, y, u)| \leq \max\{1, |\ln \delta|, |\ln k|, k^{m+}|\ln k|\}|m(x) - m(y)|
\leq L(a, b) \max\{1, |\ln \delta|, |\ln k|, k^{m+}|\ln k|\}|x - y|,
\quad \forall t \in [0, \tau], \quad x, y \in [a, b], \quad u \in (0, k],\]

and assumption $(H_2)$ follows immediately.

Therefore, all assumed conditions in Proposition 1 are satisfied, and thus, conclusions of this theorem directly follow from Proposition 1. \qed
3.2. The case when \( m(x) - 1 \) changes its sign in \( \mathbb{R} \). In this subsection, we consider local existence, regularity and extension of nonnegative solution to problem (1) when \( m(x) - 1 \) changes its sign by approximation arguments. We begin with some supplementary definitions and lemmas.

**Definition 3.2.** For any given \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \), we say that \( a \leq b \) if \( a_i \leq b_i \) and \( a_n \geq b_n \) for all \( 1 \leq i \leq n - 1 \); \( a < b \) if \( a_i < b_i \) and \( a_n > b_n \) for all \( 1 \leq i \leq n - 1 \).

Let \( D_T \) be defined as before. By a positive solution \( (u, h, g) \) of problem (1) in \( D_T \), it means that \( g'(t) \leq 0, h'(t) \geq 0 \) in \( [0, T) \) and \( u > 0 \) in \( D_T \) such that problem (1) holds over \( D_T \) with \( u \in W^{1,2}_p(D_T) \cap C^{1+\mu,1+\mu}(D_T) \) and \( h, g \in C^{1+\frac{\mu}{2}}(0, T) \) for some constants \( p > 3 \) and \( 0 < \mu < 1 \).

\((u, h, g)\) is named by a maximal positive solution of problem (1) in \( D_T \), if \( (u, h, g) \) is a positive solution of problem (1) in \( D_T \), and for any given positive constant \( \hat{T} \leq T \) and any positive solution \( (\bar{u}, \bar{h}, \bar{g}) \) of problem (1) in \( D_{\hat{T}} \), here must hold that

\[
(\bar{u}, \bar{h}, \bar{g}) \leq (u, h, g), \quad \forall \ (t, x) \in [0, \hat{T}) \times (\hat{g}(t), \hat{h}(t)).
\]

By definition, the maximal positive solution is unique when it exists.

**Lemma 3.3.** Let constant \( T \in (0, \infty) \), and let \( a \) and \( \bar{a} \) be nonnegative constants. Assume that functions \( \bar{h}, \bar{h}, \bar{g}, \bar{g} \in C^1([0, T]) \). Define

\[
Q_T = \{(t, x) \in \mathbb{R}^2 \mid t \in (0, T), \ x \in (\bar{g}(t), \bar{h}(t))\},
\]

\[
Q^*_T = \{(t, x) \in \mathbb{R}^2 \mid t \in (0, T), \ x \in (\bar{g}(t), \bar{h}(t))\}.
\]

Suppose that positive functions \( u \in W^{1,2}_p(Q_T) \cap C^{1+\mu,1+\mu}(Q_T) \) and \( \bar{u} \in W^{1,2}_p(Q^*_T) \cap C^{1+\mu,1+\mu}(Q^*_T) \) for some constants \( p > 3 \) and \( 0 < \mu < 1 \), satisfying

\[
\begin{align*}
\{ & u_t - u_{xx} \leq (u + a)^m(x), \quad (t, x) \in Q_T, \\
& u'(t) \geq -\mu_1 u_x(t, \bar{g}(t)), \quad u(t, \bar{g}(t)) = 0, \quad t \in (0, T), \\
& \bar{u}'(t) \geq -\mu_2 \bar{u}_x(t, \bar{h}(t)), \quad \bar{u}(t, \bar{h}(t)) = 0, \quad t \in (0, T)
\end{align*}
\]

and

\[
\begin{align*}
\{ & \bar{u}_t - \bar{u}_{xx} \geq (\bar{u} + \bar{a})^m(x), \quad (t, x) \in Q^*_T, \\
& \bar{g}'(t) \leq -\mu_1 \bar{u}_x(t, \bar{g}(t)), \quad \bar{u}(t, \bar{g}(t)) = 0, \quad t \in (0, T), \\
& \bar{h}'(t) \geq -\mu_2 \bar{u}_x(t, \bar{h}(t)), \quad \bar{u}(t, \bar{h}(t)) = 0, \quad t \in (0, T).
\end{align*}
\]

If \( a \leq \bar{a}, \ (\bar{h}(0), \bar{g}(0)) < (\bar{h}(0), \bar{g}(0)), \ \bar{u}(0, \bar{g}(0)) \geq 0, \ \bar{u}(0, \bar{h}(0)) > 0 \) and

\[
0 < u(0, x) \leq \bar{u}(0, x), \quad \forall \ x \in [\bar{g}(0), \bar{h}(0)],
\]

then

\[
(u, h, g) < (\bar{u}, \bar{h}, \bar{g}), \quad \forall \ (t, x) \in Q_T.
\]

**Proof.** The ideas of [42, Lemma 4.1] enlighten us to carry out our argument. We first claim that \( (h(t), g(t)) < (\bar{h}(t), \bar{g}(t)) \) for all \( t \in [0, \hat{T}) \). Set

\[
T^* = \sup \{ \tau \in [0, T] \mid \ (h(t), g(t)) < (\bar{h}(t), \bar{g}(t)), \forall \ t \in [0, \tau] \}.
\]

Noting that \( (h(0), g(0)) < (\bar{h}(0), \bar{g}(0)) \), the continuity makes us certain that \( T^* > 0 \). Suppose on the contrary that \( T^* < T \), then \( (h, g) < (\bar{h}, \bar{g}) \) for all \( t \in [0, T^*] \), and
\(h(T^*) = \bar{h}(T^*)\) or \(g(T^*) = \bar{g}(T^*)\). One may think without loss of generality that \(\bar{g}(T^*) = \bar{g}(T^*)\). Then from \(\bar{g}(t) > \bar{g}(t)\) for any \(t \in [0, T^*)\) it follows that
\[
\bar{g}'(T^*) \geq \bar{g}'(T^*). \tag{24}
\]

We next consider \(u\) and \(\bar{u}\) over \(\Omega_{T^*}\) with
\[
\Omega_{T^*} = \{(t, x) \in \mathbb{R}^2 \mid t \in (0, T^*), \ x \in (\bar{g}(t), \bar{h}(t))\}.
\]
Since \(\bar{u}(t, \bar{g}(t)) > 0 = u(t, g(t))\) and \(\bar{u}(t, \bar{h}(t)) > 0 = u(t, h(t))\) on \([0, T^*]\), we again use continuity to find that for any given constant \(\varepsilon \in (0, 1)\) small enough, there exists a constant \(0 < \delta_0 < 1\), such that for any \(\delta \in (0, \delta_0]\),
\[
\bar{u}(t, \bar{g}(t) + \delta) > u(t, g(t) + \delta), \ \forall \ t \in [0, T^* - \varepsilon],
\]
\[
\bar{u}(t, \bar{h}(t) - \delta) > u(t, h(t) - \delta), \ \forall \ t \in [0, T^* - \varepsilon].
\]

Denote
\[
\Omega^*_\varepsilon := \{(t, x) \in \mathbb{R}^2 \mid t \in [0, T^* - \varepsilon], \ x \in [\bar{g}(t) + \delta, \bar{h}(t) - \delta]\}
\]
then \(\bar{u}, u > 0\) in \(\Omega^*_\varepsilon\), and there exists a constant \(b > 0\) such that \(\bar{u}, u > b\) in \(\Omega^*_\varepsilon\), which implies that \((\bar{u} + a)m(x)\) and \((\bar{u} + \bar{a})m(x)\) are Lipschitz continuous about \(u\) and \(\bar{u}\) for every \((t, x) \in \Omega^*_\varepsilon\). Application of the comparison principle in \(\Omega^*_\varepsilon\) yields that \(\bar{u} \geq u\) in \(\Omega^*_\varepsilon\). By letting \(\delta \to 0\) we obtain \(\bar{u} \geq u\) in \(\Omega_{T^*}\), where
\[
\Omega_{T^*} := \{(t, x) \in \mathbb{R}^2 \mid t \in [0, T^* - \varepsilon], \ x \in (\bar{g}(t), \bar{h}(t))\}.
\]
The arbitrariness of \(\varepsilon\) shows that \(u \leq \bar{u}\) in \(\Omega_{T^*}\).

Let \(w = \bar{u} - u\), then by \(\bar{a} \leq a\) and \(m(x) > 0\) it can be inferred that
\[
\begin{align*}
\begin{cases}
\frac{\partial}{\partial t} w - w_{xx} \geq (\bar{u} + \bar{a})m(x) - (u + a)m(x) \geq 0, & (t, x) \in (0, T^*) \times (g(t), h(t)), \\
w(t, g(t)) = \bar{u}(t, g(t)) > 0, & t \in (0, T^*), \\
w(t, h(t)) = u(t, h(t)) > 0, & t \in (0, T^*), \\
w(0, x) \geq 0, & x \in [\bar{g}(0), \bar{h}(0)].
\end{cases}
\end{align*}
\]
By the strong maximum principle,
\[
w > 0, \ \forall \ (t, x) \in \Omega_{T^*}. \tag{25}
\]
Furthermore, \(w(T^*, g(T^*)) = \bar{u}(T^*, g(T^*)) - u(T^*, g(T^*)) = \bar{u}(T^*, g(T^*)) - u(T^*, g(T^*)) = 0\) by \(g(T^*) = \bar{g}(T^*)\) and \(T^* < T\).

Replace \(u, h(t)\) and \(g(t)\) respectively by \(w, h(t)\) and \(g(t)\) in both (9) and (12), then from the resulted transform (9) and \(w \geq 0\), it follows that
\[
\begin{align*}
\begin{cases}
\frac{\partial}{\partial t} v - Av_{yy} + Bv_y \geq 0, & (t, x) \in (0, T^*) \times (-1, 1), \\
v(t, -1) > 0, v(t, 1) > 0, & t \in (0, T^*), \\
v(0, y) = w(0, h_0y) \geq 0, & y \in [-1, 1].
\end{cases}
\end{align*}
\]
The strong maximum principle shows that \(v > 0\) for all \((t, y) \in (0, T^*) \times (-1, 1)\). Noting that \(v(T^*, -1) = 0\), application of the Hopf boundary lemma yields \(v_y(T^*, -1) > 0\), and thereafter, \(w_x(T^*, g(T^*)) > 0\), which illustrates that \(\bar{u}_x(T^*, g(T^*)) > u_x(T^*, g(T^*))\). Hence, by \(\bar{g}(T^*) = \bar{g}(T^*)\),
\[
\bar{g}'(T^*) \geq -\mu_1 \bar{u}_x(T^*, g(T^*)) > -\mu_1 \bar{u}_x(T^*, g(T^*)) = -\mu_1 \bar{u}_x(T^*, g(T^*)) \geq \bar{g}'(T^*).\]
\[
\text{It is opposed to (24), and thus, we conclude that } (h(t), g(t)) < (h(t), \bar{g}(t)) \text{ for every } t \in [0, T). \text{ Proceeding analysis as in the induction of (25) one easily achieves that } w = \bar{u} - u \geq 0 \text{ in } (0, T) \times (g(t), h(t)). \text{ Therefore, } \bar{u} > u \text{ in } (0, T) \times (g(t), h(t)), \text{ and the proof of this lemma is finished.} \]
\]
\[
\]
Let 0 < \varepsilon \ll 1 be any constant. Define
\[ h_0^\varepsilon = h_0 + \varepsilon, \quad g_0^\varepsilon = g_0 - \varepsilon. \]
Choose \( u_0^\varepsilon(x) \in C^2([g_0^\varepsilon, h_0^\varepsilon]) \) satisfying that \( u_0^\varepsilon(x) \) is increasing in \( \varepsilon \), and
\[ u_0^\varepsilon(x) > 0 \text{ in } (g_0^\varepsilon, h_0^\varepsilon), \quad u_0^\varepsilon(g_0^\varepsilon) = u_0^\varepsilon(h_0^\varepsilon) = 0, \]

\[ u_0^\varepsilon(x) \geq \bar{u}_0(x) \text{ on } [g_0, h_0], \quad u_0^\varepsilon(x) \rightarrow \bar{u}_0(x) \text{ in } W_p^2(g_0, h_0) \text{ as } \varepsilon \to 0. \]

For any given constant \( \bar{a} > 0 \), by (19) it follows that function \((u + \bar{a})^{m(x)}\) satisfies assumptions \((H_1)\) and \((H_2)\) for any \( u \geq 0 \). Moreover, \((u + \bar{a})^{m(x)}\) is bounded for all \((t, x, u) \in \mathbb{R}_+ \times \mathbb{R} \times (0, k), \) where \( k > 0 \) is any chosen constant. Then, similar as proofs of Proposition 1, Lemmas 2.2 and 2.1, we know problem

\[ u_t - u_{xx} = (u + \bar{a})^{m(x)}, \quad t > 0, \quad g(t) < x < h(t), \]

\[ g'(t) = -\mu_1 u_x(t, g(t)), \quad u(t, g(t)) = 0, \quad t > 0, \]

\[ h'(t) = -\mu_2 u_x(t, h(t)), \quad u(t, h(t)) = 0, \quad t > 0, \]

\[ u(0, x) = u_0^\varepsilon(x), \quad g(0) = g_0^\varepsilon, \quad h(0) = h_0^\varepsilon, \quad g_0^\varepsilon \leq x \leq h_0^\varepsilon \]

has a unique positive solution \((u^\varepsilon, h^\varepsilon, g^\varepsilon)\) in \([0, T^\varepsilon]\) with its maximal existence time \( T^\varepsilon \), where \( T^\varepsilon \) relies only on \( \mu_1, \mu_2, h_0^\varepsilon, g_0^\varepsilon, (u_0^\varepsilon)'(g_0^\varepsilon), (u_0^\varepsilon)'(h_0^\varepsilon) \) and \( \|u_0^\varepsilon(x)\|_{W_p^2(g_0^\varepsilon, h_0^\varepsilon)} \).

On account of constructions of \( h_0^\varepsilon, g_0^\varepsilon \) and \( u_0^\varepsilon(x) \), it is not difficult to see that \( T^\varepsilon \) relies only on \( \mu_1, \mu_2, g_0, h_0 \) and \( \|\bar{u}_0(x)\|_{W_p^2([g_0, h_0])} \). Lemma 3.3 asserts that

\[ (u^\varepsilon, h^\varepsilon, g^\varepsilon) < (u^\varepsilon, h^\varepsilon, g^\varepsilon) \text{ in } (0, \min\{T^\varepsilon, T\}) \times (g(t), h(t)), \]

and for \( \varepsilon_1 < \varepsilon_2, \)

\[ (u^{\varepsilon_1}, h^{\varepsilon_1}, g^{\varepsilon_1}) < (u^{\varepsilon_2}, h^{\varepsilon_2}, g^{\varepsilon_2}) \text{ in } (0, \min\{T^{\varepsilon_1}, T^{\varepsilon_2}\}) \times (g^{\varepsilon_1}(t), h^{\varepsilon_1}(t)). \]

Hence, \( T^{\varepsilon_2} \leq T^{\varepsilon_1} \), and there exist \((\hat{u}, \hat{h}, \hat{g})\) and \( \hat{T} > 0 \) such that for each \((t, x) \in [0, \hat{T}] \times (\hat{g}(t), \hat{h}(t))\),

\[ T^\varepsilon \not> \hat{T} \text{ and } (u^\varepsilon, h^\varepsilon, g^\varepsilon) \rightarrow (\hat{u}, \hat{h}, \hat{g}) \text{ as } \varepsilon \searrow 0. \]

From the above, we find that \( \hat{T} \) also depends only on \( \mu_1, \mu_2, g_0, h_0 \) and \( \|\bar{u}_0(x)\|_{W_p^2([g_0, h_0])} \). In addition, (27) illustrates that for any \( \varepsilon > 0, \)

\[ (\hat{u}, \hat{h}, \hat{g}) < (u^\varepsilon, h^\varepsilon, g^\varepsilon), \quad \forall (t, x) \in (0, T^\varepsilon) \times (g(t), h(t)), \]

and for \( \varepsilon_1 < \varepsilon_2, \)

\[ (u^{\varepsilon_1}, h^{\varepsilon_1}, g^{\varepsilon_1}) < (u^{\varepsilon_2}, h^{\varepsilon_2}, g^{\varepsilon_2}) \text{ in } (0, \min\{T^{\varepsilon_1}, T^{\varepsilon_2}\}) \times (g^{\varepsilon_1}(t), h^{\varepsilon_1}(t)). \]
and (26) demonstrates that
\[(u, h, g) \leq (\bar{u}, \bar{h}, \bar{g}), \quad \forall (t, x) \in [0, \min\{\bar{T}, T\}) \times (g(t), h(t)). \quad (29)\]

For any constant \(\delta > 0\), denote
\[Q^\delta = \{(t, x) \in \mathbb{R}^2 \mid t \in [0, T^\varepsilon - \delta], \quad x \in [g^\varepsilon(t), h^\varepsilon(t)]\}.\]

Recalling that \(T^\varepsilon\) depends only on \(\mu_1, \mu_2, \bar{g}_0, \bar{h}_0\) and \(\|\bar{u}_0(x)\|_{W^2_2([\bar{g}_0, \bar{h}_0])}\), by the \(L^p\) theory and the Sobolev embedding theorem we find that for any given constant \(\delta, \mu_1, \mu_2, \bar{g}_0, \bar{h}_0\) and \(\|\bar{u}_0(x)\|_{W^2_2([\bar{g}_0, \bar{h}_0])}\), such that
\[
\|u^\varepsilon\|_{W^{1,2}_2(Q^\varepsilon_\delta)} + \|u^\varepsilon\|_{C^{1+\alpha}(Q^\varepsilon_\delta)} + \|h^\varepsilon\|_{C^{1+\alpha}(0, T^\varepsilon - \delta)} + \|g^\varepsilon\|_{C^{1+\alpha}(0, T^\varepsilon - \delta)} \leq C.
\]

Combining the above with (27) and (28) illustrates that for some constant \(\alpha \in (0, \mu)\), \(\bar{u} \rightarrow \hat{u}\) in \(C^{1+\alpha}(\hat{Q}^\delta)\) and \((\bar{h}, \bar{g}) \rightarrow (\hat{h}, \hat{g})\) in \(C^1(\delta\min\{\bar{T}, T\})\) as \(\varepsilon \to 0\), where
\[Q^\delta = \{(t, x) \in \mathbb{R}^2 \mid t \in (0, \hat{T} - \delta), \quad x \in (g(t), h(t))\}.\]

Hence, arbitrariness of \(\delta > 0\) shows that \((\hat{u}, \hat{h}, \hat{g})\) is a positive solution to problem
\[
\begin{aligned}
\hat{u}_t - \hat{u}_{xx} &= (\hat{u} + \bar{a})^m(x), \quad (t, x) \in (0, \hat{T}) \times (\hat{g}(t), \hat{h}(t)), \\
\hat{g}'(t) &= -\mu_1 \hat{u}_x(t, \hat{g}(t)), \quad \hat{u}(t, \hat{g}(t)) = 0, \quad t \in (0, \hat{T}), \\
\hat{h}'(t) &= -\mu_2 \hat{u}_x(t, \hat{h}(t)), \quad \hat{u}(t, \hat{h}(t)) = 0, \quad t \in (0, \hat{T}), \\
\hat{u}(0, x) &= \bar{u}_0(x), \quad \hat{g}(0) = \bar{g}_0, \quad \hat{h}(0) = \bar{h}_0, \quad x \in [\bar{g}(0), \bar{h}(0)].
\end{aligned}
\quad (30)
\]

By Proposition 1, we are acknowledged that positive solution of problem (30) is unique. On the other hand, by assumptions of Lemma 3.4 it is clear that \((\hat{u}, \hat{h}, \hat{g})\) is also a positive solution of problem (30), and thereafter, \((\hat{u}, \hat{h}, \hat{g}) = (\hat{u}, \hat{h}, \hat{g})\) for all \(0 \leq t \leq \min\{\bar{T}, T\}\). In view of (29),
\[(u, h, g) \leq (\bar{u}, \bar{h}, \bar{g}), \quad \forall (t, x) \in [0, \min\{\bar{T}, T\}) \times (g(t), h(t)). \quad (31)\]

When \(\hat{T} > T\), the validity of conclusion in this lemma is confirmed directly by (31). We now assume \(T > \hat{T}\). In the same way as above, it can be illustrated that, for any given constant \(\delta \in (0, \hat{T})\), there exists a constant \(\tau > 0\) which depends only on \(\mu_1, \mu_2, \bar{g}_0, \bar{h}_0\) and \(\|\bar{u}_0(x)\|_{W^2_2([\bar{g}_0, \bar{h}_0])}\) but not on \(\delta\), such that problem
\[
\begin{aligned}
u_t - \nu_{xx} &= (\nu + \bar{a})^m(x), \quad t > \delta, \quad g(t) < x < h(t), \\
g'(t) &= -\mu_1 \nu_x(t, g(t)), \quad \nu(t, g(t)) = 0, \quad t > \delta, \\
h'(t) &= -\mu_2 \nu_x(t, h(t)), \quad \nu(t, h(t)) = 0, \quad t > \delta, \\
(\nu, \bar{\nu}, \bar{\nu}) &= \bar{u}(\delta, x), \quad g(\delta) = \bar{g}(\delta), \quad h(\delta) = \bar{h}(\delta), \quad x \in [\bar{g}(\delta), \bar{h}(\delta)].
\end{aligned}
\]
has a unique solution \((\nu, \bar{\nu}, \bar{\nu})\) on \([\delta, \delta + 2\tau]\), and
\[(u, h, g) \leq (u, \bar{h}, \bar{g}) \in [\delta, \min\{\delta + 2\tau, \bar{T}\}) \times (g(t), h(t)). \quad (32)\]

Put \(\delta = \hat{T} - \tau\) in (32), then (31) and (32) tell us that
\[(u, h, g) \leq (\bar{u}, \bar{h}, \bar{g}) \in [0, \min\{\bar{T} + \tau, T\}) \times (g(t), h(t)). \quad (33)\]

If \(\hat{T} + \tau > T\), conclusion of this lemma follows immediately from (33). If \(T > \hat{T} + \tau\), remembering that the constant \(\tau > 0\) depends only on \(\mu_1, \mu_2, \bar{g}_0, \bar{h}_0\) and
\[ \|\tilde{u}_0(x)\|_{W^2_1([g_0,ho])} \] but not on \( \delta \), we thus can repeat such above discussion step by step (for example, taking \( \delta_1 = \tilde{T} \) in the second time, \( \delta_2 = \tilde{T} + \tau \) in the third time if necessary, and so forth), and this lemma will ultimately be concluded. \( \square \)

We are able to state existence of positive solution for problem (1) for the case when \( m(x) - 1 \) changes its sign in \( \mathbb{R} \).

**Theorem 3.5.** Let \( D_T \) be defined as above. Assume that \( m(x) - 1 \) changes its sign in \( \mathbb{R} \) and satisfies (19). Then there is a positive constant \( T_{\text{max}} \leq \infty \) defined by (13), such that problem (1) admits a maximal positive solution \((u,h,g)\) over \([0,T_{\text{max}}]\), and \((u,h,g)\) satisfies (PT\(_{\text{max}}\)) and estimate (5). Furthermore, If \( T_{\text{max}} < \infty \), then assertion (14) also holds.

**Proof:** The proof is almost exactly the same as that of [42, Theorem 5.1]. For sake of completeness, in the following we still state it in a rough outline.

**Step 1. Local existence of solution by approximation.** Define \( f_n(t,x,u) = (u + 1/n)^{m(x)} \) for any \( n \geq 1 \), then \( f_n \) satisfies assumptions (H\(_1\))-(H\(_2\)) for every \( u \geq 0 \), and \( f_n \) is bounded in \( \mathbb{R}_+ \times \mathbb{R} \times (0,k) \), here \( k \) is any given positive constant. Thus, From Proposition 1 and Lemma 2.1 it follows that problem (4) with \( f = f_n \) and \( g_0 = -h_0 \) has a unique positive solution \((u_n,h_n,g_n)\) over \([0,T_n]\) with its maximal existence time \( T_n \), such that both conclusions of (i) and (ii) in Proposition 1 and that of Lemma 2.1 are valid in \( D_n = (0,T_n) \times (g_n(t),h_n(t)) \) with a constant \( C(\Lambda) \), where \( \Lambda = \{m_+, h_0, g^*, h^*, \|u_0(x)\|_{W^2_1([-ho,ho])}\} \) and \( g^*, h^* \) are determined in Section 2. Lemma 3.4 illustrates that

\[
(u_{n+1},h_{n+1},g_{n+1}) \leq (u_n,h_n,g_n) \in [0,\min\{T_n,T_{n+1}\}] \times (g_{n+1}(t),h_{n+1}(t)),
\]

and hence, \( T_n = \min\{T_n,T_{n+1}\} \).

Employing similar arguments as in the proof Lemma 3.3 confirms us existence of \( \tilde{T} \leq \infty \) and \((\tilde{u},\tilde{h},\tilde{g})\) with \( \tilde{u} \in L^\infty((0,\tilde{T}) \times (\tilde{g}(t),\tilde{h}(t))) \cap C^{1+\alpha}_t((0,\tilde{T}) \times (\tilde{g}(t),\tilde{h}(t))) \) and \( \tilde{h},\tilde{g} \in C^{1+\frac{\alpha}{2}}(0,\tilde{T}) \), such that \( T_n \not\nearrow \tilde{T} \) as \( n \not\nearrow \infty \), and for any given suitable small constant \( \delta > 0 \), \( u_n \to \tilde{u} \) in \( L^\infty(Q^\delta) \cap C^{1+\alpha}_t(Q^\delta) \) and \( (h_n,g_n) \to (\tilde{h},\tilde{g}) \) in \( C^{1+\frac{\alpha}{2}}(0,\tilde{T} - \delta) \) as \( n \to \infty \), where

\[
Q^\delta = \{(t,x) \in \mathbb{R}^2 | t \in (0,\tilde{T} - \delta), x \in (\tilde{g}(t),\tilde{h}(t))\}.
\]

In consideration of arbitrariness of \( \delta \), we find that \((\tilde{u},\tilde{h},\tilde{g})\) is a positive solution of problem (1). In addition, from the above we also have that for all \( n \geq 1 \),

\[
(\tilde{u},\tilde{h},\tilde{g}) \leq (u_n,h_n,g_n), \ \forall (t,x) \in [0,T_n] \times (\tilde{g}(t),\tilde{h}(t)).
\]

**Step 2. To demonstrate that \((\tilde{u},\tilde{h},\tilde{g})\) is the maximal positive solution over \([0,\tilde{T})\).** Results of Step 1 and the strong maximum principle show that \( \tilde{u} > 0 \) in \((0,\tilde{T}) \times (\tilde{g}(t),\tilde{h}(t))\), and hence, \((\tilde{u},\tilde{h},\tilde{g})\) is a positive solution of problem (1) in \([0,\tilde{T})\).

Choose any constant \( T \in (0,\tilde{T}) \), and let \((u,h,g)\) be any positive solution of problem (1) in \([0,T)\). Since \( T_n \not\nearrow \tilde{T} \) as \( n \not\nearrow \infty \), there is a constant \( n_0 \gg 1 \) so that \( T < T_n \) for all \( n \geq n_0 \). By utilizing Lemma 3.4 to compare \((u,h,g)\) and \((u_n,h_n,g_n)\) over \([0,T]\), and then by letting \( n \to \infty \), it follows that \((\tilde{u},\tilde{h},\tilde{g})\) is the maximal positive solution of problem (1) over \([0,\tilde{T})\).

**Step 3. Extension \((\tilde{u},\tilde{h},\tilde{g})\) to the right of \([0,\tilde{T})\).** Suppose \( \tilde{u} \leq M_1 \) in \([0,\tilde{T}) \times (\tilde{g}(t),\tilde{h}(t))\) for some positive constant \( M_1 \). Then from Step 1 and Lemma 2.2, and...
Following the proof of Step 1, we know that for any given constant \( \delta \in (\varepsilon, \bar{T}) \), there exists a constant \( \tau > 0 \) depending only on \( C(\varepsilon, M_1, m_+, \Lambda) \) but not on \( \delta \), such that problem

\[
\begin{aligned}
\hat{h}, \hat{g} &\in C^{1+\frac{\sigma}{p}}([\varepsilon, \bar{T}]), \quad \hat{u} \in W^{1,2}_p([\varepsilon, \bar{T}] \times (\hat{g}(t), \hat{h}(t))), \\
\|\hat{g}\|_{C^{1+\frac{\sigma}{p}}([\varepsilon, \bar{T}])} &\leq C(\varepsilon, M_1, m_+, \Lambda), \quad \|\hat{h}\|_{C^{1+\frac{\sigma}{p}}([\varepsilon, \bar{T}])} \leq C(\varepsilon, M_1, m_+, \Lambda), \\
\|\hat{u}\|_{W^{1,2}_p([\varepsilon, T] \times (\hat{g}(t), \hat{h}(t)))} &\leq \|\hat{u}\|_{C^{1+\frac{\sigma}{p}}([\varepsilon, \bar{T}] \times (\hat{g}(t), \hat{h}(t)))} \leq C(\varepsilon, M_1, m_+, \Lambda).
\end{aligned}
\]

has a maximal positive solution on \([\delta, \delta + 2\tau]\). Put \( \delta = \bar{T} - \tau \) and write \((u^*, h^*, g^*)\) as the resulted maximal positive solution of problem (34) over \([\delta, \bar{T} + \tau]\). Since \( \tau \) relies only on \( C(\varepsilon, M_1, m_+, \Lambda) \) but not upon \( \delta \), similar as in step 3 of proof of [42, Theorem 5.1] one can achieve the following conclusions: for any given positive constant \( \sigma \) with \( \sigma + \delta < \bar{T} \), it holds that

\[
(u^*, h^*, g^*) = (\hat{u}, \hat{h}, \hat{g}), \quad \forall (t,x) \in [\delta, \delta + \sigma] \times (g^*(t), h^*(t)),
\]

and by defining

\[
(\tilde{u}, \tilde{h}, \tilde{g}) = (\hat{u}, \hat{h}, \hat{g}) \text{ when } (t,x) \in [0, \delta + \sigma] \times [g^*(t), h^*(t)],
\]

\[
(\tilde{u}, \tilde{h}, \tilde{g}) = (u^*, h^*, g^*) \text{ when } (t,x) \in [\delta, \delta + 2\tau] \times [g^*(t), h^*(t)],
\]

\((\tilde{u}, \tilde{h}, \tilde{g})\) is shown to be a positive maximal solution of problem (1) on \([0, \delta + 2\tau] = [0, \bar{T} + \tau]\).

**Step 4. To conclude this theorem.** As doing in Step 3, one can make extension of \((\tilde{u}, \tilde{h}, \tilde{g})\) step by step, and ultimately gain a maximal existence time \( T_{\text{max}} \) such that positive maximal solution \((\hat{u}, \hat{h}, \hat{g})\) of problem (1) is defined over \([0, T_{\text{max}}]\).

Looking through the proof of (14), uniqueness, estimate (5) and results of Lemma 2.2 are crucial. Note that \((\hat{u}, \hat{h}, \hat{g})\) is unique, and meanwhile, from the above we know that \((\tilde{u}, \tilde{h}, \tilde{g})\) satisfies estimate (5). Moreover, Lemma 2.2 is also valid in this time. Therefore, applying arguments exactly the same as in the proof of Lemma 2.1 we fulfill the proof.

4. **Blowup and global existence for problem (1).** In this section, we are going to explore global existence and finite time blowup of nonnegative solution for problem (1).

In the section and the sequel section we always suppose that

\[
u_0(x) \in C^2([-h_0, h_0]), \quad u_0(-h_0) = u_0(h_0) = 0, \quad u_0(x) > 0 \text{ in } (-h_0, h_0).
\]

It is not difficult to see that if \( m(x) \geq 1 \) in \( \mathbb{R} \) and (19) holds, then the unique nonnegative solution \((u, h, g)\) of problem (1) in \([0, T)\) satisfies that \( u > 0 \) in \([0, T) \times (g(t), h(t)), u \in C^{0,1}([0, T) \times [g(t), h(t)]) \cap C^{1,2}([0, T) \times (g(t), h(t))].\)
If a nonnegative solution \((u, h, g)\) of problem (1) possesses the above properties, then we call it a classical positive solution of problem (1). A classical nonnegative solution \((u, h, g)\) of problem (1) is said to blow up in finite time \(T < \infty\), if
\[
\limsup_{t \to T} \max_{x \in \{g(t), h(t)\}} u(t, x) = \infty.
\]

The following lemma can be verified by similar way as in [11] (see also, for example, [42], or proofs of Lemmas 3.3 and 3.4 in this paper), and we omit details here.

**Lemma 4.1.** Let \(T > 0\) and \(m(x) \geq 1\) satisfying (19). Suppose that functions \(g, h \in C^{1}([0, T])\) and positive function \(\bar{u} \in C^{0,1}([0, T] \times \{\bar{g}(t), \bar{h}(t)\}) \cap C^{1,2}((0, T) \times (\bar{g}(t), \bar{h}(t)))\) such that
\[
\bar{u}_t - \bar{u}_{xx} \geq \bar{u}^{m(x)}, \quad 0 < t < T, \quad \bar{g}(t) < x < \bar{h}(t),
\]
\[
\bar{u}(t, \bar{g}(t)) = \bar{u}(t, \bar{h}(t)) = 0, \quad 0 < t < T,
\]
\[
\bar{g}'(t) \leq -\bar{\mu}_1 \bar{u}_x(t, \bar{g}(t)), \quad 0 < t < T,
\]
\[
\bar{h}'(t) \geq -\bar{\mu}_2 \bar{u}_x(t, \bar{h}(t)), \quad 0 < t < T.
\]

Let \((u, h, g)\) be a nonnegative solution of problem (1) in \([0, T]\). If \(\mu_1 \leq \bar{\mu}_1, \mu_2 \leq \bar{\mu}_2, (h_0, g_0) \leq (\bar{h}(0), \bar{g}(0))\) and \(u_0(x) \leq \bar{u}(0, x)\) on \([g_0, h_0]\), then
\[
(u, h, g) \leq (\bar{u}, \bar{h}, \bar{g}), \quad \forall (t, x) \in [0, T] \times [g(t), h(t)].
\]

Let \(\lambda_\Omega\) and \(\varphi_\Omega\) be the first eigenvalue and the corresponding eigenfunction of problem
\[
\begin{cases}
- \varphi_{xx} = \lambda \varphi, & x \in \Omega, \\
\varphi = 0, & x \in \partial \Omega
\end{cases}
\]
such that \(\int_\Omega \varphi_\Omega(x) dx = 1\).

**Theorem 4.2.** Let \((u, g, h)\) be a nonnegative solution of problem (1) with the maximal existence time \(T\).

(i) Assume that there exist a set \(B \subset (-h_0, h_0)\) and constants \(\sigma, \gamma\) with \(1 < \sigma \leq m(x) \leq \gamma\) in \(B\). Then, if \(u_0(x)\) satisfies
\[
\int_B u_0(x) \varphi_B(x) dx > \max\{1, (2^\gamma \lambda_B)^{1/(\sigma-1)}\},
\]
then \(u\) blows up in finite time \(T\) with
\[
T \leq \frac{\ln \left\{1 - 2^\gamma \lambda_B \left(\int_B u_0(x) \varphi_B(x) dx\right)^{1-\sigma}\right\}}{(1 - \sigma) \lambda_B}.
\]

(ii) If \(m(x) \leq 1\) everywhere, then \(T = \infty\), and
\[
u \leq C \sigma^\gamma, \quad \forall (t, x) \in [0, \infty) \times [g(t), h(t)]
\]
for a positive constant \(C\) depending only on \(\|u_0(x)\|_{L^\infty((-h_0, h_0))}\). In other words, \(u\) exists globally for any initial datum \(u_0(x)\).

**Proof.** (i) The following proof is an application of the classical Kaplan argument for blowup ([22]). From Theorems 3.1 and 3.5 we know that \(g'(t) < 0\) and \(h'(t) > 0\) in \((0, T)\). As a result, both \(g_{\min} = \lim_{t \to T} g(t)\) and \(h_{\max} = \lim_{t \to T} h(t)\) exist with
\(-\infty \leq g_{\min} < -h_0 < h_{\text{max}} \leq +\infty\). We extend \(\varphi_B\) by 0 outside \(B\) (for simplicity, we still denote it as \(\varphi_B\)). Set

\[
J(t) = \int_B u\varphi_B dx,
\]

then from (1) and results of [14] (one may directly use Lemma 3.1 of [16] and its proof), it yields that for each \(t \in (0, T)\),

\[
J'(t) = \int_B u_t \varphi_B dx = \int_{g(t)}^{h(t)} u_t \varphi_B dx = \int_{g(t)}^{h(t)} \left[ u_{xx} \varphi_B + u^{m(x)} \varphi_B \right] dx
\]

\[
= \int_{g(t)}^{h(t)} \left[ u(\varphi_B)_{xx} + u^{m(x)} \varphi_B \right] dx = \int_B \left[ u(\varphi_B)_{xx} + u^{m(x)} \varphi_B \right] dx
\]

\[
\geq -\lambda_B J(t) + 2^{-\gamma} \left( \int_B u \varphi_B dx \right)^{\sigma} = -\lambda_B J(t) + 2^{-\gamma} J^{\sigma}(t)
\]

provided that

\[
J(t) \geq 1, \ 0 < t < T.
\]  

(38)

By virtue of (36),

\[
J(0) = \int_B u_0(x) \varphi_B(x) dx > 1 \text{ and } -\lambda_B J(0) + 2^{-\gamma} J^{\sigma}(0) > 0,
\]

thereafter (37) and (38) follow. From (37) we have

\[
T \leq \frac{\ln \left( 1 - 2^{\gamma} \lambda_B J^{1-\sigma}(0) \right)}{1 - \sigma} \lambda_B < \infty \text{ and } J(t) \to \infty \text{ as } t \to T.
\]

Definitions of \(J(t)\) and \(\varphi_B\) show that

\[
\lim_{t \to T} \max_{x \in [g(t), h(t)]} u(t, x) = \infty.
\]

Therefore, we arrive at the conclusion (i).

(ii) Let \((\hat{u}, \hat{h}, \hat{g})\) be the maximal positive solution of problem (1) in \([0, T_{\text{max}}]\), and let

\[
\hat{u}(t, x) = (1 + \|u_0(x)\|_{L^\infty([-h_0, h_0])})^{e^t}, \ t \geq 0, \ x \in \mathbb{R}.
\]

It follows that \(\hat{u} > 1\) in \([0, \infty) \times \mathbb{R}\). Joining with \(0 < m(x) \leq 1\) we have

\[
\left\{
\begin{array}{l}
\hat{u}_t - \hat{u}_{xx} = \hat{u} \geq \hat{u}^{m(x)}, \ t > 0, \ \hat{g}(t) < x < \hat{h}(t), \\
\hat{u}(t, \hat{g}(t)) > 0, \ \hat{u}(t, \hat{h}(t)) > 0, \ t > 0, \\
\hat{u}(0, x) > u_0(x), \ -h_0 \leq x \leq h_0.
\end{array}
\right.
\]

Hence, by the usual comparison principle we have that \(T_{\text{max}} = \infty\) and

\[
\hat{u} \geq \hat{u}, \ \forall (t, x) \in [0, \infty) \times (\hat{g}(t), \hat{h}(t)).
\]

Both the above inequality and definition of the maximal positive solution guarantee the validity of the conclusion (ii) in this theorem.

On basis of results of [16, Theorem 3.9], we immediately obtain below Theorem 4.3, which gives a sufficient condition such that any solution \((u, h, g)\) of problem (1) blows up in finite time. Just as what has been pointed out by authors of [16], it is an important difference with respect to the problem with a constant exponent.
Theorem 4.3. Assume that $m^*(x) \in C(-1, 1)$ and $m^*(x) - 1$ changes sign in $(-1, 1)$. Let $(u, h, g)$ be a nonnegative solution of problem (1) with the maximal existence time $T$. Then if there exist two suitable large constants $M > L > 0$ such that $(x_0 - M, x_0 + M) \subset (g(t), h(t))$ for some $x_0$, then $u$ corresponding to problem (1) with $m(x) = m^*(x - x_0)/L$ for every $x \in (x_0 - L, x_0 + L)$ blows up in finite time $T$.

Remark 1. (i) Note that $g'(t) < 0$ and $h'(t) > 0$ in $(0, T)$ which are proven in Theorems 3.1 and 3.5. It is clear that $h(t) - g(t) > h(0) - g(0) = 2h_0$. Thanks to this fact, the assumption that $(x_0 - M, x_0 + M) \subset (g(t), h(t))$ must be available for suitable large $h_0$.

(ii) Limited by arguments and techniques, it is still quite difficult for us to explore sufficient conditions for global existence of solution when $m(x) - 1$ changes its sign, because of the present of free boundary. We expect ourself or/and someone to solve it in the coming future.

When it comes to the case $m_+ > 1$, a sufficient condition in energy form is to be explored for finite time blowup, and it will supply us a key estimate (45), which will play an important role in the discussion of long-time behavior for global solution (see Section 5). To this aim, we define

$$E(u) = \int_{g(t)}^{h(t)} \left[ \frac{u_x^2}{2} - \frac{u^{1+m(x)}}{1 + m(x)} \right] dx, \quad |u(t)|_1 = \int_{g(t)}^{h(t)} u dx,$$

$$E(u_0) = \int_{-h_0}^{h_0} \left[ \frac{u_0^2}{2} - \frac{u_0^{1+m(x)}}{1 + m(x)} \right] dx, \quad |u_0|_1 = \int_{-h_0}^{h_0} u_0(x) dx.$$

Lemma 4.4. Let $(u, h, g)$ be a nonnegative solution of problem (1) with the maximal existence time $T$. Then for every $t \in [0, T)$,

$$\frac{dE(u)}{dt} = \frac{(g'(t))^3}{2\mu_1^2} - \frac{(h'(t))^3}{2\mu_2^2} - \int_{g(t)}^{h(t)} u_1^2 dx,$$

$$|u(t)|_1 - |u_0|_1 = \frac{h_0 - h(t)}{\mu_2} + \frac{g(t) - h_0}{\mu_1} + \int_0^t \int_{g(\tau)}^{h(\tau)} u^{m(x)} dx d\tau.$$

Proof. From equations and boundary conditions of problem (1), it is not difficult to verify the results. We omit the proof here.

Lemma 4.5. Let $A = \int_0^\infty \left[ \frac{(h'(t))^3}{\mu_2^2} - \frac{(g'(t))^3}{\mu_1^2} \right] dt$ and

$$Q(u_0) = \min\{\mu_1^3, \mu_2^3\} \left(\frac{|u_0|_1}{\text{max}\{\mu_1^2, \mu_2^2\} (2h_0 + \max\{\mu_1, \mu_2\}) |u_0|_1}ight)^3.$$

If $(u, h, g)$ is a global nonnegative solution of problem (1), then $A \geq \frac{\pi^2}{32} Q(u_0)$.

Proof. As is well known from [18, Chapter 8] that problem

$$\begin{cases} v_t - v_{xx} = 0, & t > 0, \quad s(t) < x < \sigma(t), \\
v(t, s(t)) = 0, & s'(t) = -\mu_1 v_x(t, s(t)), \quad t > 0, \\
v(t, \sigma(t)) = 0, & \sigma'(t) = -\mu_2 v_x(t, \sigma(t)), \quad t > 0, \\
v(0, x) = u_0(x), & -h_0 < x < h_0, \quad s(0) = \sigma(0) = h_0, \quad -h_0 < x < h_0 \end{cases}$$

has a unique nonnegative solution $v(t, x)$ in $(0, T) \times (-h_0, h_0)$ such that

$$\int_0^\infty \int_{-h_0}^{h_0} v(t, x) dx dt = \int_0^\infty \int_{-h_0}^{h_0} u_0(x) dx dt.$$
has a unique nonnegative solution \( v \) which exists over \([0, \infty)\). Furthermore, the Hopf boundary Lemma asserts that \( s'(t) < 0 \) and \( \sigma'(t) > 0 \) in \([0, \infty)\). Hence, by Lemma 4.1 we know that \( u \geq v \geq 0 \), \( -h_0 \geq s(t) \geq g(t) \) and \( h(t) \geq \sigma(t) \geq h_0 \) for every \((t, x) \in [0, \infty) \times |s(t), \sigma(t)|\).

Similar as the proof of Lemma 4.4 we easily have
\[
|u_0|_1 - |v(t)|_1 = \frac{\sigma(t) - h_0 - s(t) + h_0}{\mu_2} - \frac{\mu_1}{\mu_1} \leq \frac{\sigma(t) - s(t) - 2h_0}{\min\{\mu_1, \mu_2\}} \leq \frac{h(t) - g(t) - 2h_0}{\min\{\mu_1, \mu_2\}} \quad \forall \ t > 0, \tag{39}
\]
\[
|u_0|_1 - |v(t)|_1 \geq \frac{\sigma(t) - s(t) - 2h_0}{\max\{\mu_1, \mu_2\}} \quad \forall \ t > 0. \tag{40}
\]

Remember that \( h'(t) > 0 \) and \( g'(t) < 0 \) for all \( t \geq 0 \). In view of the elementary inequality and Hölder’s inequality, by (39) we deduce that for each \( t > 0 \),
\[
A \geq \frac{1}{4} \int_0^t \left( \frac{h'(\tau)}{\mu_2^{2/3}} - \frac{g'(\tau)}{\mu_1^{2/3}} \right)^3 d\tau \geq \frac{1}{4t^2} \left[ \int_0^t \left( \frac{h'(\tau)}{\mu_2^{2/3}} - \frac{g'(\tau)}{\mu_1^{2/3}} \right) d\tau \right] \geq \frac{4t^2}{\max\{\mu_1^{2/3}, \mu_2^{2/3}\}} \geq \frac{\min\{\mu_1, \mu_2\} \left( |u_0|_1 - |v(t)|_1 \right)^3}{4t^2} \tag{41}
\]

As the above function \( v \) is a subsolution of Cauchy problem
\[
\begin{align*}
w_t - w_{xx} &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\
w(0, x) &= w_0(x) = \begin{cases} u_0(x), & -h_0 \leq x \leq h_0, \\
0, & x \in \mathbb{R} \setminus [-h_0, h_0], \end{cases}
\end{align*}
\]
and such Cauchy problem has a unique solution \( w \) which satisfies
\[
\|w(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq (4\pi t)^{-1/2} |u_0|_1 = (4\pi t)^{-1/2} |u_0|_1, \quad \forall \ t > 0.
\]

By the comparison principle it follows that \( 0 \leq v \leq w \), and thereafter,
\[
\|v(t, \cdot)\|_{L^\infty(s(t), \sigma(t))} \leq (4\pi t)^{-1/2} |u_0|_1, \quad \forall \ t > 0. \tag{42}
\]

Hence, joining (40) and (42) asserts that for every \( t > 0 \),
\[
|v(t)|_1 \leq \frac{(\sigma(t) - s(t)) \|v(t, \cdot)\|_{L^\infty(s(t), \sigma(t))}}{\sqrt{4\pi t}} \leq \frac{(\sigma(t) - s(t)) |u_0|_1}{\sqrt{4\pi t}} \leq \frac{(2h_0 + \max\{\mu_1, \mu_2\} |u_0|_1) |u_0|_1}{\sqrt{4\pi t}}.
\]

Take
\[
t_0 = \frac{(2h_0 + \max\{\mu_1, \mu_2\} |u_0|_1)^2}{\pi},
\]
then from the above inequality we see
\[
|v(t)|_1 \leq \frac{|u_0|_1}{2}, \quad \forall \ t \geq t_0. \tag{43}
\]

Putting \( t_0 \) in (41), the desired conclusion is drawn by (43). \( \square \)
Theorem 4.6. Let \( m(x) \) meet with \( m_\geq 1 \) and (19). Assume that \((u,h,g)\) is a nonnegative solution of problem (1) with the maximal existence time \( T_{\text{max}} \). If
\[
E(u_0) < \frac{\pi^2}{64} Q(u_0),
\]
where \( Q(u_0) \) is defined by Lemma 4.5, then \( T_{\text{max}} < \infty \), and thereupon, \( u \) blows up in \( T_{\text{max}} \) by Theorem 3.1.

Proof. We suppose inversely that \( T_{\text{max}} = \infty \). Set
\[
F(t) = \int_0^t \int_{g(\tau)} h(\tau) u^2 \, dx, \quad t \geq 0,
\]
then from problem (1), definition of \( E(u) \) and Lemma 4.4 we have

\[
F'(t) = \int_{g(t)}^h h(\tau) u^2 \, dx, \quad t > 0,
\]

\[
F''(t) = 2 \int_{g(t)}^h u^{1+m(x)} \, dx - 2 \int_{g(t)}^h u_\tau^2 \, dx \\
\geq 2(1 + m_-) \int_{g(t)}^h \frac{u^{1+m(x)}}{1 + m(x)} \, dx - 2 \int_{g(t)}^h u_\tau^2 \, dx \\
= (m_- - 1) \int_{g(t)}^h u_\tau^2 \, dx - 2(1 + m_-) E(u) \\
= 2(1 + m_-) \int_0^t \int_{g(\tau)}^{h(\tau)} u_\tau^2 \, dx + (m_- - 1) \int_{g(t)}^h u_\tau^2 \, dx \\
\geq \frac{(h'(\tau))^3}{\mu^2_2} \frac{(g'(\tau))^3}{\mu^2_1} d\tau - 2 E(u_0), \quad t > 0.
\]

By assumptions of this lemma together with Lemma 4.5, there exists \( t_1 > t_0 \) so large that
\[
2E(u_0) < \int_0^t \left[ \frac{(h'(\tau))^3}{\mu^2_2} - \frac{(g'(\tau))^3}{\mu^2_1} \right] d\tau, \quad \forall \ t \geq t_1.
\]
Thus, thanks to (44), it can be inferred that
\[
F''(t) > 2(1 + m_-) \int_0^t \int_{g(\tau)}^{h(\tau)} u_\tau^2 \, dx, \quad \forall \ t \geq t_1.
\]
From the above inequality, one could employ the classical concavity argument of [24], doing exactly the same as in [19], to deduce a contradiction. Therefore, the proof is complete.

Remark 2. Theorem 4.6 gives us a efficient condition for global solution: if \( u \) is a nonnegative global solution of problem (1), then it must be held up that
\[
E(u_0) \geq \frac{\pi^2}{64} Q(u_0) > 0.
\]
Since nonnegative solution is unique by \( m_\geq 1 \) and (19), it is not difficult to see that under \( m_\geq 1 \) and (19) assumed, if \( u \) is a global nonnegative solution of
Hence, Lemma 4.1 asserts that $E(u) \geq 0$ for all $t \geq 0$. Consequently, from Lemma 4.4, we see that
\begin{equation}
\int_0^t \int_{g(\tau)}^{h(\tau)} u^2 dx d\tau + \int_0^t \left[ \frac{(h'(\tau))^3}{2\mu_2^2} - \frac{(g'(\tau))^3}{2\mu_1^2} \right] d\tau = E(u_0) - E(u) \leq E(u_0), \quad \forall \ t \geq 0.
\end{equation}

5. Fast solution and slow solution for problem (1). This section is devoted to investigating long-time behavior of global solution for problem (1). According to [17], we define fast solution and slow solution of problem (1) as follows.

**Definition 5.1.** Let $(u, h, g)$ be a nonnegative solution of problem (1) globally in time (i.e. $T_{\text{max}} = \infty$), and let $g_\infty = \lim_{t \to \infty} g(t)$ and $h_\infty = \lim_{t \to \infty} h(t)$.

(i) **(fast solution)** $g_\infty > -\infty, h_\infty < +\infty$, and there exist constants $C, \alpha > 0$ (depending on $u_0(x)$) such that
\[ \|u(t, \cdot)\|_{L^\infty(g(t), h(t))} \leq Ce^{-\alpha t}, \quad \forall \ t \geq 0; \]

(ii) **(slow solution)** $g_\infty = -\infty, h_\infty = +\infty$, and
\[ \lim_{t \to \infty} \|u(t, \cdot)\|_{L^\infty(g(t), h(t))} = 0. \]

Denote $m_- = \inf_{\mathbb{R}} m(x)$. We first check existence of fast solution.

**Theorem 5.2.** Assume that $m_- > 1$ and (19) is satisfied. Let $(u, h, g)$ be a nonnegative solution of problem (1) with the maximal existence time $T$. Then $T = \infty$ for suitable small $u_0(x)$. Moreover, there exist positive constants $\alpha, C_1$ and $C_2$ which depend only on $h_0, \mu_1$ and $\mu_2$, such that
\[ u \leq C_1 e^{-\alpha t}, \quad -g_\infty, h_\infty \leq C_2, \quad \forall \ (t, x) \in [0, \infty) \times [g(t), h(t)]. \]

**Proof.** As $m_- > 1$ and (19) holds, $(u, h, g)$ is the unique nonnegative solution of problem (1). Inspired by [31], define
\[ p(t) = -q(t) = 2h_0(2 - e^{-\alpha t}), \quad y = \frac{x}{p(t)}, \quad (t, x) \in [0, \infty) \times [g(t), p(t)], \]
\[ V(y) = 1 - y^2, \quad w(t, x) = e^{\alpha t} V(y), \quad (t, y) \in [0, \infty) \times [-1, 1], \]
where positive constants $\alpha$ and $\varepsilon$ verify
\[ 0 < \alpha < (8h_0^2)^{-1}, \quad \varepsilon < \min \{1, \left[(8h_0^2)^{-1} - \alpha\right]^{\frac{1}{2(1-\mu_-^2)}}, 2\alpha \mu_2^{-1} h_0^2, 2\alpha \mu_1^{-1} h_0^2\}. \]

Then, for such $\alpha$ and $\varepsilon$, simple computation gives that
\[ w_t - w_{xx} - w^{m(x)} \geq w_t - w_{xx} - w^{m_-} \geq ce^{-\alpha t} \left[(8h_0^2)^{-1} - \alpha - \varepsilon^{m_- - 1}\right] \geq 0, \quad t > 0, \quad x \in (q(t), p(t)), \]
\[ w(t, q(t)) = w(t, p(t)) = 0, \quad t > 0, \]
\[ p'(t) = 2\alpha h_0 e^{-\alpha t} \geq \frac{\mu_2 \varepsilon}{h_0} e^{-\alpha t} \geq -\mu_2 w_x(t, p(t)), \quad t > 0, \]
\[ q'(t) \leq -\mu_1 w_x(t, q(t)), \quad t > 0. \]

Hence, Lemma 4.1 asserts
\[ (u, h, g) \leq (w, p, q) \quad \text{for all} \ (t, x) \times [0, \infty) \times [g(t), h(t)], \tag{46} \]
provided that
\[ u_0(x) \leq (3\varepsilon)/4, \quad \forall \ x \in [-h_0, h_0], \]
which implies
\[ u_0(x) \leq w(0, x) = \varepsilon V \left( \frac{x}{2h_0} \right), \quad \forall \ x \in [-h_0, h_0]. \]

Remember that \( g'(t) < 0 \) and \( h'(t) > 0 \) in \([0, \infty)\). By (46) and definitions of \( p(t) \) and \( w(t, x) \), we draw desired conclusions with \( C_1 = \varepsilon \) and \( C_2 = 4h_0 \).

**Remark 3.** Taking of \( u \) as a species, for example, results in Theorem 5.2 illustrate that such species will be extinct in the long run \( \left( \lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} u(t, x) = 0 \right) \), and could not spread successfully, because of \( g_\infty > -\infty \) and \( h_\infty < +\infty \).

We next discuss slow solution for problem (1). In the following we assume that \((u, h, g)\) is a nonnegative global solution of problem (1), \( m_- > 1 \), and (19) is verified. We will employ scaling methods to deduce some estimates under assumption
\[ \ell = \limsup_{t \to \infty} \|u(t, \cdot)\|_{L^\infty(g(t), h(t))} > 0. \]

As in [17], we can find a sequence \( \{t_n\} \) with \( t_n \to \infty \) as \( n \to \infty \) and a positive constant \( \varepsilon < 1 \) such that
\[ \sigma_n := \|u(t_n, \cdot)\|_{L^\infty(g(t_n), h(t_n))} \geq \frac{1}{2} \sup_{[t_0, t_n]} \|u(t, \cdot)\|_{L^\infty(g(t), h(t))} \geq \varepsilon \]
for some \( t_0 > 0 \) such that \( \sup_{[t_0, \infty]} \|u(t, \cdot)\|_{L^\infty(g(t), h(t))} \leq \ell/2 \). Select \( x_n \in (g(t_n), h(t_n)) \) such that \( u(t_n, x_n) = \sigma_n \), and set
\[ \lambda_n = \sigma_n^{-(m_- - 1)/2} \leq \varepsilon^{-(m_- - 1)/2}, \quad \forall \ n. \tag{47} \]

We extend \( u \) by 0 on both \((-\infty, g(t))\) and \((h(t), \infty)\). Define
\[ \tilde{D}_n := \{(\tau, y) \in \mathbb{R}^2 \mid \lambda_n^{-2}(t_0 - t_n) \leq \tau \leq 0, \ y \in \mathbb{R}\}, \]
\[ v_n(\tau, y) = \lambda_n^{2/(m_- - 1)}u(t_n + \lambda_n^2 \tau, x_n + \lambda_n y), \quad (\tau, y) \in \tilde{D}_n. \tag{48} \]

Denote
\[ y_1(\tau) = \lambda_n^{-1}(g(t_n + \lambda_n^2 \tau) - x_n), \quad y_2(\tau) = \lambda_n^{-1}(h(t_n + \lambda_n^2 \tau) - x_n), \]
\[ D_n = \{(\tau, y) \in \mathbb{R}^2 \mid \lambda_n^{-2}(t_0 - t_n) \leq \tau \leq 0, \ y_1(\tau) < y < y_2(\tau)\}. \]
A simply calculation shows that \( v_n(\tau, y) \) satisfies
\[ (v_n)_\tau - (v_n)_{yy} = \lambda_n^{\theta_n} v_n \left( m_n(y) \right), \quad (\tau, y) \in D_n, \]
\[ 0 \leq v_n(\tau, y) \leq 2, \quad v_n(0, 0) = 1, \quad (\tau, y) \in \tilde{D}_n, \tag{49} \]
where \( \theta_n = \frac{2(m_- - m_n(y))}{m_+ - 1} \) and \( m_n(y) = m(x_n + \lambda_n y) \) for all \( (\tau, y) \in \tilde{D}_n \). Taking account of \( 0 < \varepsilon < 1 \), (47) and (49) illustrate that
\[ 0 \leq \lambda_n^{\theta_n} v_n \left( m_n(y) \right) \leq \varepsilon^{-(m_- - m_+)} 2^{m_-}, \quad \forall \ (\tau, y) \in \tilde{D}_n. \tag{50} \]

We next demonstrate local estimates of \( v_n \) in compact subsets of the form \( Q_k := [-k, 0] \times [-k, k] \) for all \( k \geq 1 \).

**Lemma 5.3.** For all \( k \geq 1 \) and \( n \geq N(k) \) large, there holds that
\[ \int_{-k}^{k} (v_{ny})^2 dy \leq C(k), \quad \forall \ \tau \in [-k, 0], \]
\[ \int_{Q_k} \left[ \lambda_n^{\theta_n} v_n \left( m_n(y) \right) \right]^2 d\tau d\tau \leq C(k), \quad \int_{Q_k} \left[ (v_n)^2 + (v_{ny})^2 \right] d\tau d\tau \leq C(k). \]
Proof. The second inequality comes directly from (50).

For other assertions, choose \( \phi(y) \in C_0^\infty(\mathbb{R}) \) such that \( \phi(y) = 1 \) on \([-k, k]\). We write \( v(\tau, y) = v_n(\tau, y) \) for convenience. As \( v(\tau, y_1(\tau)) = v(\tau, y_2(\tau)) = 0 \), by (49) and integration by parts it follows that for enough large \( n \),
\[
\frac{1}{2} \frac{d}{d\tau} \int_{y_1(\tau)}^{y_2(\tau)} v^2 \phi^2 \, dy = \int_{y_1(\tau)}^{y_2(\tau)} vv_\tau \phi^2 \, dy
\]
\[
= \int_{y_1(\tau)}^{y_2(\tau)} \lambda_n^\theta y_1^{1+m_n} \phi^2 \, dy - \int_{y_1(\tau)}^{y_2(\tau)} v^2_y \phi^2 \, dy - 2 \int_{y_1(\tau)}^{y_2(\tau)} vv_y \phi_y \, dy
\]
\[
\leq \int_{y_1(\tau)}^{y_2(\tau)} \lambda_n^\theta y_1^{1+m_n} \phi^2 \, dy - \frac{1}{2} \int_{y_1(\tau)}^{y_2(\tau)} v^2_y \phi^2 \, dy + 4 \int_{y_1(\tau)}^{y_2(\tau)} \phi_y^2 \, dy
\]
holds for every \( \tau \in [-k, 0] \). Taking advantage of \( 0 \leq v \leq 2 \), we obtain
\[
\int_{-k}^{0} \int_{y_1(\tau)}^{y_2(\tau)} v^2_y \phi^2 \, dy \, d\tau \leq C \int_{-k}^{0} \int_{y_1(\tau)}^{y_2(\tau)} \left( \lambda_n^\theta y_1^{1+m_n} \phi^2 + v^2 \phi_y^2 \right) \, dy \, d\tau
\]
\[
+ \int_{y_1(-k)}^{y_2(-k)} v^2(-k, y) \phi_y^2 \, dy \leq C(k).
\]

Likewise, by considering \( \frac{d}{d\tau} \int_{y_1(\tau)}^{y_2(\tau)} v^2 \phi_y^2 \, dy \), we have
\[
\int_{-k}^{0} \int_{y_1(\tau)}^{y_2(\tau)} v^2_y \phi_y^2 \, dy \, d\tau \leq C \int_{-k}^{0} \int_{y_1(\tau)}^{y_2(\tau)} \left( \lambda_n^\theta y_1^{1+m_n} \phi^2_y + v^2_y \phi_y^2 \right) \, dy \, d\tau
\]
\[
+ \int_{y_1(-k)}^{y_2(-k)} v^2(-k, y) \phi_y^2 \, dy \leq C(k).
\]

By defining
\[
E_\phi(\tau) = \int_{y_1(\tau)}^{y_2(\tau)} \left( \frac{v^2}{2} - \frac{\lambda_n^\theta y_1^{1+m_n}(y)}{1+m_n(y)} \right) \phi^2 \, dy,
\]
a direct calculation yields that for every \(-k \leq \tau \leq 0\),
\[
\frac{dE_\phi(\tau)}{d\tau} = \int_{y_1(\tau)}^{y_2(\tau)} \left( v_y v_y - \lambda_n^\theta y_1^{m_n(y)} v_y \right) \phi^2 \, dy
\]
\[
+ \frac{v^2_y(\tau, y_2(\tau)) \phi^2(y_2(\tau)) y'_2(\tau) - v^2_y(\tau, y_1(\tau)) \phi^2(y_1(\tau)) y'_1(\tau)}{2}
\]
\[
=(v_y v_y \phi^2)(\tau, y_2(\tau)) - (v_y v_y \phi^2)(\tau, y_1(\tau)) - \int_{y_1(\tau)}^{y_2(\tau)} v_y v_y \phi^2 \, dy
\]
\[
- 2 \int_{y_1(\tau)}^{y_2(\tau)} v_y v_y \phi_y \, dy - \int_{y_1(\tau)}^{y_2(\tau)} \lambda_n^\theta v_y m_n(y) v_y \phi^2 \, dy
\]
\[
+ \frac{v_y^2(\tau, y_2(\tau)) \phi^2(y_2(\tau)) y'_2(\tau) - v_y^2(\tau, y_1(\tau)) \phi^2(y_1(\tau)) y'_1(\tau)}{2}.
\]

Note that for \( i = 1, 2 \),
\[
0 = \frac{d}{d\tau} (\tau, y_i(\tau)) = v_\tau(\tau, y_i(\tau)) + v_y(\tau, y_i(\tau)) y'_i(\tau), \quad -k \leq \tau \leq 0.
\]
It follows that for any $\tau \in [-k, 0]$ and for $i = 1, 2$,

$$(-1)^i(v_r y^i \phi^2 + \frac{1}{2} v_r^2 \phi^2 y^i_\tau)(\tau, y_i(\tau)) = \frac{(-1)^{i+1}}{2}(v_r^2 \phi^2)(\tau, y_i(\tau))y_i(\tau) \leq 0,$$

and then joining with (53) we have

$$\frac{dE_\phi(\tau)}{d\tau} \leq - \int_{y_1(\tau)}^{y_2(\tau)} v_r^2 \phi^2 dy - 2 \int_{y_1(\tau)}^{y_2(\tau)} v_r y \phi \phi_y dy$$

$$\leq - \frac{1}{2} \int_{y_1(\tau)}^{y_2(\tau)} v_r^2 \phi^2 dy + \int_{y_1(\tau)}^{y_2(\tau)} v_r^2 \phi^2 dy, \quad -k \leq \tau \leq 0.$$  

For any given $\alpha, \beta \in [-k, 0]$ with $\alpha \leq \beta$, by integrating the above inequality over $[\alpha, \beta]$, in view of (49)–(53) we know

$$\int_{\alpha}^{\beta} \int_{y_1(\tau)}^{y_2(\tau)} v_r^2 \phi^2 dy d\tau + \int_{\alpha}^{\beta} (v_r^2 \phi^2)(\beta, y) dy$$

$$\leq 4 \int_{\alpha}^{\beta} \int_{y_1(\tau)}^{y_2(\tau)} v_r^2 \phi^2 dy \tau + \int_{\alpha}^{\beta} (v_r^2 \phi^2)(\alpha, y) dy$$

$$+ \int_{\alpha}^{\beta} 2 v_r^2 \phi^2 \frac{2 \lambda_n \nu^2 + 1 + m_n(y)}{1 + m_n(y)} dy$$

$$\leq \int_{\alpha}^{\beta} (v_r^2 \phi^2)(\alpha, y) dy + C(k).$$

Integrating (54) in $\alpha$ over $[\beta - 1, \beta]$, and using (51), it leads to

$$\int_{y_1(\beta)}^{y_2(\beta)} (v_r^2 \phi^2)(\beta, y) dy \leq \int_{\beta - 1}^{\beta} \int_{y_1(\alpha)}^{y_2(\alpha)} (v_r^2 \phi^2)(\alpha, y) dy d\alpha + C(k) \leq C_1(k).$$

Consequently,

$$\int_{-k}^{k} v_r^2(\tau, y) dy \leq C(k), \quad -k \leq \tau \leq 0,$$

which is the first inequality in this lemma.

Putting $\alpha = -k$ and $\beta = 0$ in (54), one could obtain

$$\int_{-k}^{0} \int_{-k}^{k} v_r^2 dy d\tau \leq C(k).$$

Therefore, combining this estimate with the first inequality of this lemma, we arrive at the last inequality in this lemma. \qed

By (49) and results of Lemma 5.3,

$$\int_{Q_k} [(v_{n,y})^2] dy d\tau \leq C(k).$$

Using the compact embedding $H^1(Q_k) \hookrightarrow L^2(Q_k)$, $H^1([-k, k]) \hookrightarrow C([-k, k])$ and properties of the space $L^2$, there exist functions $w, \psi, \theta, \varrho$ and $z$ with $z(0) = 1,$
such that (maybe some subsequences of) \( \{v_n\} \) and \( \{\lambda_n v_n^{m_n(y)}\} \) satisfy
\[
v_n \to w \text{ in } L^2_{loc}((\infty, 0) \times \mathbb{R}), \quad 0 \leq w \leq 2, \]
\[
v_n(0, \cdot) \to z \text{ in } C_{loc}(\mathbb{R}), \quad z(0) = 1, \]
\[
\lambda_n v_n^{m_n(y)} \to \psi \text{ in } L^2_{loc}((\infty, 0) \times \mathbb{R}), \quad 0 \leq \psi \leq \varepsilon^{-(m_+ - m_-)} 2^{m_+},
\]
\[
(v_n)_{yy} \to \vartheta \text{ in } L^2_{loc}((\infty, 0) \times \mathbb{R}), \quad (v_n)_{\tau} \to \varphi \text{ in } L^2_{loc}((\infty, 0) \times \mathbb{R}).
\]

**Lemma 5.4.** Write \( Q = (\infty, 0) \times \mathbb{R} \) and let functions \( w \) and \( \psi \) be defined as above. Then
\[
-w_{yy} = \psi \text{ in } D'(Q).
\]

**Proof.** By (45),
\[
\int_0^\infty \int_0^t u^2_x dxdt < \infty.
\]
Subsequently, from the fact that \( \{\lambda_n\} \) is bounded and \((m_+^2 + 6m_+ - 3)/(m_+ - 1)^2 > 0\) due to \( m_+ > m_- > 1 \), it can be deduced that
\[
\int_{t_n-k\lambda_n}^{t_n} \int_{g(t)}^{1/(m+1)} (v_n)^2 dyd\tau = \lambda_n^{m_+} \int_{t_n-k\lambda_n}^{t_n} u^2_x(t_n + \lambda_n \tau, x_n + \lambda_n y) dyd\tau
\]
\[
\leq \lambda_n^{m_+ - 1} \|\phi\| \int_{t_n-k\lambda_n}^{t_n} \left[ \frac{h'(t_n + \lambda_n \tau)}{\mu_2} - \frac{g'(t_n + \lambda_n \tau)}{\mu_1} \right] d\tau + \int_Q (v_n - w) \phi dyd\tau
\]
\[
\leq (k\lambda_n)^{2/3} \lambda_n^{m_+ - 1} \|\phi\| \int_{t_n-k\lambda_n}^{t_n} \left[ \frac{h'(t_n)}{\mu_2} - \frac{g'(t_n)}{\mu_1} \right]^3 d\tau
\]
\[
\leq 2k^{2/3} \lambda_n^{2/(m_+ - 1)} \|\phi\| \int_{t_n-k\lambda_n}^{t_n} \left[ \frac{h'(t_n)^3}{\mu_2^3} - \frac{g'(t_n)^3}{\mu_1^3} \right] d\tau
\]
\[
+ \int_Q (v_n - w) \phi dyd\tau,
\]
and hence,
\[
(v_n)_\tau \to 0 \text{ in } L^2_{loc}((\infty, 0) \times \mathbb{R}).
\]
where \( \| \phi \|_{\infty} = \| \phi \|_{L^\infty(Q)} \). From (45) it follows that
\[
\int_0^\infty \left[ \frac{(h'(t))^3}{\mu_2^3} - \frac{(g'(t))^3}{\mu_1^3} \right] dt < E(u_0).
\]
Note that \( \{ \lambda_n \} \) is bounded and \( t_n \to \infty \) as \( n \to \infty \). Hence, in view of (55) and all above inequalities,
\[
\left| \int_Q [(v_n)_{yy} \phi - w \phi_{yy}] \, dy \, dz \right| \to 0 \quad \text{as} \quad n \to \infty,
\]
which joining with (55) gives that as \( n \to \infty \),
\[
\left| \int_Q (\partial \phi - w \phi_{yy}) \, dy \, dz \right| \leq \left| \int_Q [(\partial - (v_n)_{yy}) \phi] \, dy \, dz \right| + \left| \int_Q [(v_n)_{yy} \phi - w \phi_{yy}] \, dy \, dz \right| \to 0.
\]
As a result,
\[
\int_Q w \phi_{yy} \, dy \, dz = \int_Q \partial \phi \, dy \, dz, \quad \forall \phi \in C_0^\infty(Q),
\]
and (57) holds. Therefore, by (49) and (55)–(57), we fulfill the lemma. \( \square \)

**Theorem 5.5.** Assume that \( m_+ > 1 \) and (19) holds. Let \((u, h, g)\) be a nonnegative solution to problem (1) with \( T_{\max} = \infty \). Then
\[
\sup_{t \geq 0} \| u(t, \cdot) \|_{L^\infty((g(t), h(t)))} \leq C,
\]
where the constant \( C \) depends only upon \( \mu_1, \mu_2, \| u_0(x) \|_{C^2([-h_0, h_0])}, h_0 \) and \( h_0^{-1} \), it remains bounded whenever \( \mu_1, \mu_2, \| u_0(x) \|_{C^2([-h_0, h_0])}, h_0 \) and \( h_0^{-1} \) are bounded.

**Proof.** From Theorem 3.1, for each constant \( M > 1 \) there exists a constant \( \delta > 0 \) such that if \( \| u_0(x) \|_{C^2([-h_0, h_0])} < M \) and \( 1/M < -g(0) = h(0) < M \), then \( T_{\max} > \delta \) and \( \| u(t, \cdot) \|_{L^\infty((g(t), h(t)))} < 2M \) for all \( t \in [0, \delta] \).

Assume conversely that the conclusion of this theorem was not true. From the above there would exist a constant \( M > 1 \) and then a constant \( \delta > 0 \) and a sequence of initial values \( \{ (u_n(0, x), h_n(0), g_n(0)) \} \) verifying
\[
1/M < -g_n(0) = h_n(0) < M, \quad \| u_n(0, \cdot) \|_{C^2([-h_0, h_0])} < M
\]
such that the corresponding sequence of global solutions \( \{ (u_n, h_n, g_n) \} \) of problem (1) possessing properties as follows: \( \| u_n(t, \cdot) \|_{L^\infty((g_n(t), h_n(t)))} < 2M \) for all \( t \in [0, \delta] \), and
\[
\sup_{t \geq 0} \| u_n(t, \cdot) \|_{L^\infty((g_n(t), h_n(t)))} \to \infty \quad \text{as} \quad n \to \infty.
\]
Here should be pointed out that the above constant \( \delta \) relies only on \( M \) but not on \( n \). Fix such constant \( \delta \). For every large \( n \), there are \( t_n > \delta \) and \( x_n \in (g_n(t_n), h_n(t_n)) \) such that
\[
\sup_{t \geq 0} \| u_n(t, \cdot) \|_{L^\infty((g_n(t), h_n(t)))} = u_n(t_n, x_n) =: \sigma_n.
\]
Writing \( \lambda_n = \sigma_n^{(m-1)/2} \), it is clear that \( \lambda_n \to 0 \) as \( n \to \infty \). We extend \( u_n(t, \cdot) \) by 0 on both \((\infty, g_n(t))\) and \((h_n(t), \infty)\), and then rescale the resulted \( u_n \) as in (48) with \( t_0 = 0 \) and \( u_n \) instead of \( u \). Recall that \( t_n \geq \delta \) and \( \delta \) is independent of \( n \). It follows that \( \lambda_n^{-2} t_n > \lambda_n^{-2} \delta \to \infty \) as \( n \to \infty \). The function \( v_n \) now satisfies (49) with \( 0 \leq v_n \leq 1 = v_n(0, 0) \) on \( \tilde{D}_n \) and \( 0 \leq \lambda_n^{m-1} v_n^{m-1}(0) \leq 1 \) on \( \tilde{D}_n \) for all suitable large \( n \).
Employing analysis as above, there exist a subsequence of \( \{ v_n \} \) and a subsequence of \( \{ \lambda_{m}(y) \} \) such that (55) holds with \( \psi = 0 \) in this time, and \( w \) satisfies the conclusion of Lemma 5.4 such that \( w(0) = 1 \) and \( \lim_{|y| \to \infty} w = 0 \), all of which lead to a contradiction. Therefore, the proof is complete. \( \square \)

**Theorem 5.6.** Under hypotheses of Theorem 5.5, there holds that
\[
\lim_{t \to \infty} \| u(t, \cdot) \|_{L^{\infty}(g(t), h(t))} = 0.
\]

Proof. Assume \( \limsup_{t \to \infty} \| u(t, \cdot) \|_{L^{\infty}(g(t), h(t))} = \ell > 0 \) to the contrary. Then by defining \( \lambda_n \) and \( v_n \) respectively as in (47) and (48), and by employing above arguments, we finally obtain two bounded functions \( w, \psi \geq 0 \) such that conclusion of Lemma 5.4 holds. Moreover, \( w \) is continuous in \( \mathbb{R} \), \( w(0) = 1 \) and \( \lim_{|y| \to \infty} w = 0 \). From conclusion of Lemma 5.4 and \( \psi \geq 0 \) we know that \( w \) is concave, and thus, \( w \equiv 0 \). This is a contradiction with \( w(0) = 1 \). Therefore, this theorem is accomplished. \( \square \)

**Theorem 5.7.** Let \( \mu_1 = \mu_2 \) and \( m_2 > 1 \). Assume that (19) holds and \( m(-x) = m(x) \) in \( \mathbb{R} \). Suppose that \( (u, h, g) \) is a nonnegative solution of problem (1) with \( u_0(x) = \lambda \varphi(x) \), where \( \lambda \) and \( \varphi(x) \) is the first eigenvalue and the corresponding eigenfunction of problem (35) in \( \Omega = (-h_0, h_0) \). Then there exists \( \lambda > 0 \) such that
\[
\lim_{t \to \infty} \| u(t, \cdot) \|_{L^{\infty}(g(t), h(t))} = 0, \quad g_\infty := \lim_{t \to \infty} g(t) = -\infty, \quad h_\infty := \lim_{t \to \infty} h(t) = +\infty,
\]
which illustrates that \( u \) is a slow solution.

Proof. Write \( u, g, h \) and the maximal existence time \( T \) as \( u(u_0; \cdot), g(u_0; \cdot), h(u_0; \cdot) \) and \( T(u_0) \), and define
\[
\lambda^* = \sup \{ \lambda > 0 \mid T(\sigma \varphi) = \infty, \quad g_\infty(\sigma \varphi) > -\infty \text{ and } h_\infty(\sigma \varphi) < \infty, \quad \forall \sigma \in [0, \lambda] \}.
\]

Theorem 5.2 shows that \( \lambda^* > 0 \), and \( \lambda^* < \infty \) by Theorem 4.2. Namely, \( \lambda^* \in (0, \infty) \).

Set
\[
v(t, x) = u(\lambda^* \varphi; t, x), \quad g^*(t) = g(\lambda^* \varphi; t), \quad h^*(t) = h(\lambda^* \varphi; t), \quad \tau = T(\lambda^* \varphi).
\]

We extend \( u(t, x) \) by 0 on \((-\infty, g(t)) \cup (h(t), \infty)\), and we claim \( \tau = \infty \). In fact, by continuous dependence, for each chosen \( t \in [0, \tau] \), \( u(\lambda \varphi; t, x) \) approaches \( v(t, x) \) in \( L^\infty(\mathbb{R}) \), \( g(\lambda \varphi; t) \to g^*(t) \) and \( h(\lambda \varphi; t) \to h^*(t) \) as \( \lambda \to \lambda^* \). As \( T(\lambda \varphi) = \infty \) for every \( \lambda \in (0, \lambda^*) \), by Theorem 5.5 we have
\[
\| v(t, \cdot) \|_{L^{\infty}(g^*(t), h^*(t))} \leq C \quad \text{for all } t \in [0, \tau).
\]

If \( \tau < \infty \), then from Theorem 3.1 it follows that
\[
\limsup_{t \to \tau} \max_{x \in [g^*(t), h^*(t)]} v(t, x) = \infty,
\]
and it is a contradiction. Hence, \( \tau = \infty \).

Since both the initial datum \( u_0(x) = \lambda \varphi(x) \) and \( m(x) \) are symmetric, \( u_0(x) \) is concave on \([-h_0, h_0] \) and \( m_1 = m_2 \), by applying standard arguments and uniqueness of solution we have \(-g(t) = h(t) \) on \([0, \infty) \). Thus, \( g(t) \) and \( h(t) \) are both finite or infinite at the same time, and so do \( g_\infty^* \) and \( h_\infty^* \). We assume conversely that \(-g_\infty^* = h_\infty^* < \infty \). As \( \tau = \infty \), by Theorem 5.6 we see that \( \| v(t, \cdot) \|_{L^\infty(-h^*(t), h^*(t))} \to 0 \) as \( t \to \infty \). It yields existence of a point \( t_0 \) so large that
\[
\| v(t_0, \cdot) \|_{L^\infty(-h^*(t_0), h^*(t_0))} \leq \frac{3}{4} \min \left\{ \frac{1}{(8 \mu_1)}, \left(4h_\infty^* \right)^{\frac{1}{\tau - \tau}} \right\}
\]
because of $h_0 < h^*(t) < h^*_\infty$ in $(0, \infty)$. Continuous dependence asserts that for all $\lambda > \lambda^*$, 

$$
\|u(\lambda \varphi, t_0, \cdot)\|_{L^\infty(-h(t_0), h(t_0))} \leq \frac{3}{4} \min \left\{ \frac{1}{8\mu_1}, \left[ 4h^*(t_0) \right]^{-\frac{2}{m+2}} \right\},
$$

(58)

On the other hand, if in the proof of Theorem 5.2 we put $\lambda > \lambda^*$ close to $\lambda^*$, then by (58),

$$
\|u(\lambda \varphi, t_0, \cdot)\|_{L^\infty(-h(t_0), h(t_0))} < \left( \frac{3}{4} \min \left\{ \frac{1}{8\mu_1}, \left[ 4h^*(t_0) \right]^{-\frac{2}{m+2}} \right\} \right) / 4,
$$

and $\lambda > \lambda^*$ close to $\lambda^*$.

Consequently, from the proof of Theorem 5.2 and the uniqueness of positive solution, one can arrive at that $T(\lambda \varphi) = \infty$ and $-g^*_\infty(\lambda \varphi) = h^*_\infty(\lambda \varphi) < \infty$ for all $\lambda > \lambda^*$ close to $\lambda^*$, which is a contradiction to the definition of $\lambda^*$. Therefore, we finish the proof of this theorem.

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