THE HAAGERUP PROPERTY FOR TWISTED GROUPOID 
DYNAMICAL SYSTEMS

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Abstract. We introduce the Haagerup property for twisted groupoid $C^*$-dynamical systems in terms of naturally defined positive-definite operator-valued multipliers. By developing a version of ‘the Haagerup trick’ we prove that this property is equivalent to the Haagerup property of the reduced crossed product $C^*$-algebra with respect to the canonical conditional expectation $E$. This extends a theorem of Dong and Ruan for discrete group actions, and implies that a given Cartan inclusion of separable $C^*$-algebras has the Haagerup property if and only if the associated Weyl groupoid has the Haagerup property in the sense of Tu. We use the latter statement to prove that every separable $C^*$-algebra which has the Haagerup property with respect to some Cartan subalgebra satisfies the Universal Coefficient Theorem. This generalises a recent result of Barlak and Li on the UCT for nuclear Cartan pairs.

1. Introduction

One of the most active areas of research at the intersection of geometric group theory and the theory of operator algebras in recent years is the study of interactions between the geometric properties of a group and analytic properties of the associated $C^*$- or von Neumann algebras. The prototypical example is the equivalence of amenability of a discrete group $\Gamma$ and the nuclearity of the group $C^*$-algebra $C^*_r(\Gamma)$ or injectivity of the group von Neumann algebra $VN(\Gamma)$, see e.g. [BrO]. Similarly, the Haagerup property for $\Gamma$ can be characterised by approximation properties of $C^*_r(\Gamma)$ or $VN(\Gamma)$, see [BrO, CCJJV, Cho, Don]. The aforementioned correspondences, at a technical level, are achieved in two stages: first one encodes some property of $\Gamma$ in terms of the existence of certain functions on $\Gamma$, and then one shows that an abstract approximation property of the associated operator algebra can be achieved with the approximating maps being Herz-Schur multipliers, i.e. maps determined by functions on $\Gamma$. This second step is sometimes called ‘averaging maps into Herz-Schur multipliers’, and is essentially based on a simple but very efficient trick attributed to Haagerup (see [Haa2]).

With the increased interest in various constructions generalising the group operator algebras there is a need to develop and apply appropriate versions of the above procedure in other contexts. This has been done, for example, for (unimodular) discrete quantum groups ([KrR], [DFSW]) and for $C^*$-algebraic crossed products ([MSTT], [DoR]). The
latter case shows an interesting dichotomy: if one is interested directly in the Haagerup property of the crossed product by an action on a $C^*$-algebra $A$, one has to allow a broader class of analogues of Herz-Schur multipliers, sometimes called mapping multipliers ([MTT], [BC2]), whereas if one wants to consider only the $A$-valued multipliers, one arrives rather at a relative Haagerup property of the crossed product with respect to $A$ viewed as a subalgebra, as defined in [DoR].

Recent years have also brought a renewed interest in $C^*$-algebras associated to twisted étale groupoids and their actions. This is partially due to the fact that such structures lead to many new interesting examples of $C^*$-algebras, but also due to a certain universality: the celebrated result of Renault from [Ren5] says that any Cartan pair $(B,A)$, i.e. a $C^*$-algebra $B$ and a regular maximal abelian subalgebra $A \subseteq B$ that admits a faithful conditional expectation, comes from a (unique) twisted-groupoid model. Also Cartan pairs are ubiquitous. For example Li proves in [Li] that any simple classifiable $C^*$-algebra admits a Cartan subalgebra, so also a twisted groupoid model. On the other hand Barlak and Li show in [BalL] that nuclear algebras admitting Cartan subalgebras satisfy the UCT. They exploit results of Tu ([Tu]) that connect the UCT property for étale groupoid $C^*$-algebras with what we call the Haagerup property of a groupoid. Our first initial motivation was to generalise the result of [BalL] by replacing nuclearity with a weaker property of a Cartan pair, equivalent to the Haagerup property of the associated Weyl groupoid. A relevant variant of the weaker property mentioned above was introduced by Dong and Ruan [DoR]. The authors of [DoR] defined the relative Haagerup property for a unital $C^*$-inclusion $A \subseteq B$ and characterised it when $B = A \rtimes \Gamma$ is the reduced crossed product, of a discrete group action $\Gamma$ on a unital $C^*$-algebra $A$, via $Z(A)$-valued positive-definite multipliers defined on $\Gamma$. Thus our second key motivation was to extend this result to the twisted actions of locally compact étale groupoids on not necessarily unital $C^*$-algebras. The necessity to consider the twist comes from the fact that Renault’s theorem involves twisted groupoids. However, it suffices to consider only Kumjian’s twists by a circle bundle [Kum1], and in the present paper we restrict ourselves to such twists. Again a central role in the proofs is based on a suitable version of the Haagerup Trick.

In particular we establish the following two results (see Theorems 5.17, 6.1) that are far reaching generalisations of the main results of [DoR] and [BalL], and apply to $C^*$-algebras admitting Cartan subalgebras (see Corollary 6.7). Pertinent definitions of the Haagerup property can be found in Definitions 5.6 and 5.12 below.

**Theorem A.** Let $B$ be the reduced crossed product of a twisted action $(A,G,\Sigma,\alpha)$ of an étale locally compact Hausdorff groupoid $G$ on the $C^*$-algebra $A := C_0(\mathcal{A})$ where $\mathcal{A}$ has a continuous unit section. Denote the canonical expectation from $B$ to $A$ by $E$. Then $B$ has the $E$-Haagerup property, if and only if $(A,G,\Sigma,\alpha)$ has the Haagerup property.

**Theorem B.** For any continuous separable saturated Fell bundle $B$ over a second countable locally compact Hausdorff étale groupoid $G$ with the Haagerup property such that $C^*(B|_{G(0)})$ is of type I, both the reduced $C^*$-algebra $C^*_r(B)$ and the full $C^*$-algebra $C^*(B)$ satisfy the UCT.
Corollary C. Let $B$ be a separable $C^*$-algebra which admits a Cartan subalgebra $A$. If the inclusion $A \subseteq B$ has the Haagerup property, then $B$ satisfies the UCT.

An important fact which distinguishes our work is that we treat groupoids with not necessarily compact unit spaces. This forces us to work with non-unital algebras and has significant consequences even at the level of the relevant definitions. Since the proof of Theorem [E] relies heavily on the work of Tu (in fact it is a combination of the arguments of [BaL] and the Stabilization Theorem from [IKSW]), our model initial condition is existence of a locally proper function of negative type in the sense of [Tu]. This local properness, when translated to an approximation property of positive-definite maps, means that the maps in question are only locally $C_0$ (see Proposition 5.4). Accordingly, we adapt the relative Haagerup property of [DoR] to not-necessarily unital $C^*$-inclusions $A \subseteq B$, mimicking the ‘locally $C_0$ condition’ by using the Pedersen’s ideal of $A$ (see Definition 5.12).

As alluded to above, the key step in proving Theorem A is a version of the Haagerup Trick that gives a systematic way of passing from general maps acting on the crossed product to certain functions on $G$, which in turn lead to analogues of Herz-Schur multipliers. In the context of operator algebras associated to groupoids, generalisations of Herz-Schur multipliers were studied for example in [RaW], [Ren4], [Tak] and [BrO, Subsection 5.6]. We assume in Theorem A that $A$ has a continuous unit section only to facilitate the Haagerup trick. This is enough for all of our applications and allows us to consider very concrete and natural operator-valued multipliers. The genuinely new element is that we are dealing with the multipliers of twisted groupoid $C^*$-dynamical systems, with twisting in the sense of Green-Renault [Gre], [Ren2], [Ren3]. In the group case this brings us close to some considerations in [BC1]. The authors of [BC1] employ twists by cocycles in the spirit of [Z-M], [BuS], rather than that of Green [Gre], but as shown in [PaR] cocycle twists cover those coming from the group extension by a choice of a global continuous section. This can not be directly generalised to groupoids, as groupoid extensions admit non-trivial continuous sections only locally. Nevertheless, it seems that we need a variant of this result to make the technicalities related to our framework tractable. We found a way out by describing twisted groupoid dynamical systems using suitable pictures of inverse semigroup actions, that exploit the ideas of [BuE2], [BuM], [BEM] and [KwM1]. We establish the relevant equivalences, describing them in detail, as we believe they are of independent interest and could be used in other contexts. In particular, for étale groupoids twisted crossed products in the sense of Green-Kumjian-Renault can be replaced by inverse semigroup crossed products twisted by a cocycle, as introduced in [BuE2]. This is the content of the following theorem (see Theorem 2.16).

Theorem D. Let $(A, G, \Sigma, \alpha)$ be a twisted groupoid $C^*$-dynamical system with $G$ étale locally compact and Hausdorff. There is a natural twisted inverse semigroup action $(A, S, \beta, \omega)$ (described in Lemma 2.3) where $A = C^*(A)$ and $S$ is an inverse semigroup consisting of open bisections, and we have natural isomorphisms for the corresponding crossed products: $C^*(A, G, \Sigma, \alpha) \cong A \rtimes_{\beta, \omega} S$, $C^r(A, G, \Sigma, \alpha) \cong A \rtimes_{\beta, r} S$.

We use the $S$-grading of the reduced crossed product from Theorem D to establish our version of the Haagerup Trick and study its properties (see Propositions 4.1, 4.4).
believe this tool will be useful in the study of other approximation properties of groupoid crossed products.

The detailed plan of the paper is as follows: after discussing preliminary facts and notations concerning groupoid Fell bundles in the beginning of Section 2, we provide in the same section an extended discussion of various equivalent pictures of twisted groupoid actions and associated crossed product operator algebras, focusing on the interpretation in terms of inverse semigroup actions on Hilbert bimodules. In Section 3 we introduce various classes of Herz-Schur type multipliers for such twisted crossed products and show that positive-definite \( A \)-valued functions correspond to completely positive multipliers. Section 4 contains a version of the Haagerup Trick in our framework: every map \( \Phi \) on the resulting crossed product algebra which is a bimodule map (with respect to the subalgebra given by the \( C^* \)-bundle \( A \) on which our groupoid acts) is shown to determine in a canonical way a function \( h^\Phi \) on the groupoid with values in the associated bundle. There we also establish the interplay between properties of \( \Phi \) and \( h^\Phi \), which are key for the following results. In Section 5 we discuss the Haagerup property for groupoids, as defined in [Tu], and its variant (equivalent in the \( \sigma \)-compact setting) which provides the right notion for twisted groupoid dynamical systems. Then we exploit the content of the previous sections to show one of the main results of the paper, namely the equivalence between the Haagerup property for such a system and a relative Haagerup property of the pair of associated \( C^* \)-algebras. We also record in this section certain immediate corollaries for Cartan pairs. Finally in Section 6 we present some consequences of the theorem mentioned above for the UCT property of Cartan pairs, based on the observation that the key arguments of [BaL] require only the Haagerup property of the associated groupoid, and not necessarily amenability, as was used in that paper.

2. Groupoid Fell bundles, twisted groupoid actions and their \( C^* \)-algebras

In this section we establish the notation and conventions regarding groupoids and their Fell bundles. We also discuss the concept of twisted groupoid actions on \( C^* \)-algebras and relate them to certain twisted inverse semigroup actions. Further we describe the relevant convolution operator algebras and prove their isomorphism (Theorem D of the introduction).

2.1. Banach bundles. Throughout this paper \( X \) is a locally compact Hausdorff space. We consider Fell bundles that are upper semicontinuous, see for instance [DuQi], [BuE2]. So by a Banach bundle over \( X \) we mean a topological space \( \mathcal{B} = \bigsqcup_{x \in X} B_x \) such that the canonical projection \( p : \mathcal{B} \twoheadrightarrow X \) is continuous and open, each fiber \( B_x \) is a complex Banach space, and the Banach space structure is consistent with the topology of \( \mathcal{B} \) in the sense that the maps \( \mathcal{B} \times \mathcal{B} \supseteq \bigsqcup_{x \in X} B_x \times B_x \ni (v, w) \to v + w \in \mathcal{B} \) and \( \mathbb{C} \times \mathcal{B} \ni (\lambda, v) \to \lambda v \in \mathcal{B} \) are continuous, but we require the map \( \mathcal{B} \ni v \mapsto \|v\| \in \mathbb{R} \) only to be upper semicontinuous.

One also requires another technical condition: if \( \{v_i\}_{i \in I} \) is a net in \( \mathcal{B} \) such that \( \|v_i\| \xrightarrow{i \in I} 0 \) and \( p(v_i) \xrightarrow{i \in I} x \) in \( X \), then \( v_i \) converges to \( 0_x \), where \( 0_x \) is the zero element in \( B_x \). If the map \( \mathcal{B} \ni v \mapsto \|v\| \in \mathbb{R} \) is continuous, then we say the bundle is continuous. We will also
use an analogous notion of (upper semicontinuous) \( C^* \)-bundles. The bundle as above is said to be **locally trivial** if for any point \( x \in X \) there is an open neighbourhood \( U \) of \( x \) in \( X \) and a Banach space \( \mathcal{Y} \) such that \( \mathcal{B}|_U \cong U \times \mathcal{Y} \), where the isomorphism respects the bundle structure.

In a number of sources, including the original work of Fell, Banach bundles are continuous by definition, see, for instance, [FeD], [Kum], [Tak]. However, nowadays it seems more standard to consider upper-semicontinuous bundles. Firstly, this more general setting is often very useful. Secondly, upper semicontinuity is usually sufficient to prove important theorems for Banach bundles. For instance, the reconstruction theorem [FeD] Theorem II.13.18 is generalised (to upper semicontinuous bundles) in [BuE] Proposition 2.4, and the Douady-dal Soglio-Herault theorem [FeD, Appendix C] is generalised in [Laz, Corollary 2.10]. The reconstruction theorem says that whenever we have a disjoint sum of Banach spaces \( \mathcal{B} = \bigsqcup_{x \in X} B_x \) and a set \( \Gamma \) of sections of \( \mathcal{B} \) such that (a) for each \( f \in \Gamma \) the function \( X \ni x \mapsto \|f(x)\| \in \mathbb{R} \) is upper semicontinuous, and (b) for each \( x \in X \) the set \( \{f(x) : f \in \Gamma \} \) is dense in \( B_x \), then there is a unique topology on \( \mathcal{B} \) making it into a Banach bundle such that \( \Gamma \subseteq C(\mathcal{B}) \), i.e. all the sections in \( \Gamma \) become continuous. The Douady-dal Soglio-Herault theorem states that for every Banach bundle \( \mathcal{B} = \bigsqcup_{x \in X} B_x \) and every \( v \in B_x \) there is a continuous section \( f \in C(\mathcal{B}) \) such that \( f(x) = v \). In fact, one can choose \( f \) to be vanishing at infinity in the sense that for every \( \varepsilon > 0 \) the set \( \{x \in X : \|f(x)\| \geq \varepsilon \} \) is compact. In particular, the space of continuous sections vanishing at infinity, which we denote by \( C_0(\mathcal{B}) \), is a Banach \( C_0(X) \)-module (with the norm \( \|f\| = \sup_{x \in X} \|f(x)\| \)), and we can recover \( \mathcal{B} = \bigsqcup_{x \in X} B_x \) from this module.

A **line bundle** is a Banach bundle \( \mathcal{B} = \bigsqcup_{x \in X} B_x \) where each space \( B_x \), \( x \in X \), is one-dimensional (we will usually use \( \mathcal{L} \) to denote line bundles). It is well known that every **continuous line bundle is locally trivial**. More specifically, if \( \mathcal{B} \) is a continuous line bundle, then for every point \( x_0 \in X \) there is a section \( f \in C_0(\mathcal{B}) \) with \( |f(x_0)| = 1 \) so that \( U := \{x \in X : |f(x)| > 1/2\} \) is an open neighborhood of \( x_0 \). Then \( C_0(\mathcal{B}|_U) \cong C_0(U) \) because for every \( g \in C_0(\mathcal{B}|_U) \) there is a unique function \( h : U \to \mathbb{C} \) such that \( g(x) = h(x)f(x) \) for every \( x \in U \). Continuity of \( g \) and \( f \) forces continuity of \( h \) and since \( g \in C_0(\mathcal{B}|_U) \) and \( \|f|_U\| \geq 1/2 \) we get \( h \in C_0(U) \). Thus \( \mathcal{B}|_U = \bigsqcup_{x \in U} B_x \) is isomorphic to the trivial bundle. We stress that general (upper semicontinuous) Banach bundles need not be locally trivial. For instance, we may consider a bundle \( \mathcal{B} = \bigsqcup_{x \in X} \mathbb{C} \) with topology in which all upper semicontinuous functions are continuous sections.

### 2.2. Fell bundles over groupoids and groupoid dynamical systems.

In this paper \( G \) will always denote an étale locally compact Hausdorff groupoid. Equivalently, \( G \) is a topological groupoid such that the unit space \( X := G^{(0)} \) is locally compact and Hausdorff, the source and range maps \( s, r : G \to X \) are local homeomorphisms and \( X \) is a clopen subset of \( G \) (note that it is in fact automatically open as a range of a local homeomorphism). Yet another equivalent phrasing is that \( G \) is a locally compact Hausdorff groupoid with a topological basis consisting of bisections. A bisection for \( G \) is a subset (usually assumed to be open) \( U \subseteq G \) such that the restrictions \( s : U \to s(U) \), \( r : U \to r(U) \) are homeomorphisms. The set of all open bisections of \( G \) will be denoted \( \text{Bis}(G) \).
A Fell bundle over the groupoid $G$ is (an upper-semicontinuous) Banach bundle $B = \bigcup_{\gamma \in G} B_\gamma$ equipped with a continuous involution $*: B \to B$ and a continuous multiplication $\cdot: \{(a, b) \in B \times B : a \in B_{\gamma_1}, b \in B_{\gamma_2}, (\gamma_1, \gamma_2) \in G(2)\} \to B$, that satisfy the standard set of axioms, for details see, for instance, [BuE2, Section 2] or [Kum2, Tak]. Accordingly, this structure turns the fibres $B_x$ for $x \in X$ into $C^*$-algebras and the fibres $B_\gamma$ for $\gamma \in G$ into Hilbert $B_{r(\gamma)} \cdot B_{s(\gamma)}$-bimodules, such that the multiplication map yields isometric Hilbert bimodule maps $B_{\gamma_1} \otimes B_{s(\gamma_1)} B_{\gamma_2} \to B_{\gamma_1 \gamma_2}$ for all $(\gamma_1, \gamma_2) \in G(2)$. If these maps are surjective, the Fell bundle $B$ is called saturated. This is equivalent to assuming that all Hilbert $B_{s(\gamma)}$-modules $B_\gamma$ are full, i.e. for each $\gamma \in G$ the linear span of $\langle B_{\gamma}, B_{\gamma}\rangle_{B_s(\gamma)} := \{\langle \xi, \eta \rangle_{B_s(\gamma)} : \xi, \eta \in B_{\gamma}\}$ is dense in $B_{s(\gamma)}$. Obviously, all Fell line bundles are saturated.

For every Fell bundle $B = \bigcup_{\gamma \in G} B_\gamma$ its restriction $B|_X := \bigcup_{x \in X} B_x$ to the unit space is an upper semicontinuous $C^*$-bundle and thus $A := C_0(B|_X)$ is a $C_0(X)$-$C^*$-algebra, i.e. $A$ is a $C^*$-algebra and a $C_0(X)$-module with the module map taking values in $ZM(A)$ - the center of the multiplier algebra. Every $C_0(X)$-$C^*$-algebra $A$ arises as the algebra of continuous sections for some $C^*$-bundle $B|_X$, cf. [Will, Appendix C]; further $A$ is called a continuous $C_0(X)$-$C^*$-algebra if the corresponding bundle is continuous.

(Twisted) groupoid actions on (continuous) $C_0(X)$-$C^*$-algebras were introduced in [Ren2].

**Definition 2.1.** A groupoid $C^*$-dynamical system is a triple $(A, G, \alpha)$ where $A = \bigcup_{x \in X} A_x$ is an upper semicontinuous $C^*$-bundle, $G$ is a topological groupoid with unit space $X$ and $\alpha$ is a continuous groupoid homomorphism, i.e. a family $\{\alpha_\gamma\}_{\gamma \in G}$ of $*$-isomorphisms $\alpha_\gamma : A_{s(\gamma)} \to A_{r(\gamma)}$, $\gamma \in G$, such that $\alpha_{\gamma \eta} = \alpha_\gamma \circ \alpha_\eta$ for all $(\gamma, \eta) \in G(2)$ and the map $(\gamma, a) \to \alpha_\gamma(a)$ from $G \ast_s A := \{(\gamma, a) \in G \times A : a \in A_{s(\gamma)}\}$ to $A$ is continuous.

**Remark 2.2.** Sometimes groupoid $C^*$-dynamical systems are written as $(A, G, \alpha)$ where $A = C_0(A)$ is the $C_0(X)$-$C^*$-algebra corresponding to the $C^*$-bundle $A$. If $A$ is (isomorphic to) $C_0(X)$, that is $A = X \times C$ is a trivial line bundle, then there is a unique action $\alpha$ of $G$ on $A$ consisting of identities on $C$. Thus the triple $(C_0(X), G, \alpha)$ might be then identified with $G$. Note also that for every groupoid $C^*$-dynamical system the ‘unit’ automorphisms $\alpha_x \in Aut(A_x)$ for $x \in G^{(0)}$ are all trivial.

**Example 2.3 ([Kum2]).** Every groupoid $C^*$-dynamical system $(A, G, \alpha)$ yields a saturated Fell bundle $A \rtimes_\alpha G$ over $G$ that extends the $C^*$-bundle $A$. As a topological space $A \rtimes_\alpha G$ is the pullback bundle subspace $A \rtimes_r G := \{(a, g) \in A \times G : a \in A_{r(\gamma)}, \gamma \in G\}$. We may also write $A \rtimes_\alpha G = \bigcup_{\gamma \in G} B_\gamma$ where $B_\gamma := A_{r(\gamma)}$ for $\gamma \in G$. The multiplication maps and involutions are defined by $B_\eta \times B_\gamma \to B_{\eta \gamma}, (a, b) \mapsto a \cdot \alpha_\eta(b)$, and $B_\gamma \to B_{\gamma^{-1}}, a \mapsto \alpha_{\gamma}^{-1}(a^*)$.

2.3. **Twisted groupoid $C^*$-dynamical systems vs twisted inverse semigroup actions.** In most of this paper (with a brief exception in Section 6) we will only consider Kumjian twists ([Kum1], i.e. twists by the trivial circle bundle $X \times \mathbb{T}$). Thus by a *twist of a groupoid $G$* we mean a topological groupoid $\Sigma$ such that we have a central groupoid extension

$$X \times \mathbb{T} \hookrightarrow \Sigma \xrightarrow{\pi} G,$$
where \( X \times \mathbb{T} \) is the group bundle over \( X \). More specifically, cf. \cite[Definition 5.1.1]{Sim}, \( i \) and \( \pi \) are groupoid homomorphisms that restrict to the identity on the unit spaces \( X = \Sigma^{(0)} \cong G^{(0)}, i \) is injective, \( \pi \) is surjective and \( \pi^{-1}(X) = i(X \times \mathbb{T}) \). So suppressing the map \( i \) we may assume that \( X \times \mathbb{T} \) is a subgroupoid of \( \Sigma \). This subgroupoid is assumed to be central in the sense that for every \( \sigma \in \Sigma \) and \( z \in \mathbb{T} \) we have \((r(\sigma), z)\sigma = \sigma(s(\sigma), z)\); we denote this common element by \( z \cdot \sigma \). Moreover, one also requires that \( \Sigma \) is a locally trivial \( G \)-bundle in the sense that every point \( \gamma \in G \) has a bisection neighbourhood \( U \) on which there exists a continuous section \( c : U \to \Sigma \) satisfying \( \pi \circ c = id_U \), and such that the map \( U \times \mathbb{T} \ni (\gamma, z) \mapsto z \cdot c(\gamma) \in \pi^{-1}(U) \) is a homeomorphism \( U \times \mathbb{T} \cong \pi^{-1}(U) \). Thus \( \Sigma \) is a principal \( \mathbb{T} \)-space and \( \Sigma/\mathbb{T} \cong G \). We will also call the pair \((G, \Sigma)\) as above simply a twisted groupoid.

Renault \cite{Ren}, \cite{Ren2} generalised Green’s notion of twisted group actions \cite{Gre} to groupoid actions as follows. We denote by \( UM(A) \) the group of unitaries in the multiplier algebra of a \( C^* \)-algebra \( A \).

**Definition 2.4.** A twisted groupoid \( C^\ast \)-dynamical system, usually denoted by a quadruple \((A, G, \Sigma, \alpha)\), consists of a twisted groupoid \((G, \Sigma)\) and a groupoid \( C^\ast \)-dynamical system \((A, \Sigma, \alpha)\) that are combined by a “twisting” continuous groupoid homomorphism \( u : X \times \mathbb{T} \to \bigsqcup_{x \in X} UM(A_x) \). This means that putting \( u_x(z) := u(x, z), x \in X, z \in \mathbb{T} \), we have a family \( \{u_x\}_{x \in X} \) of group homomorphisms \( u_x : \mathbb{T} \to UM(A_x), x \in X \), such that

1. the map \( A \times \mathbb{T} \to A \), where \( A \times \mathbb{T} \ni (a, x, z) \mapsto a_x u_x(z) \in A_x \), is continuous;
2. \( \alpha_{(x, z)} : A_x \to A_x \) is given by \( u_x(z)(\cdot)u_x(z)^{-1} \), for every \( (x, z) \in X \times \mathbb{T} \);
3. \( u_{r(\sigma)}(z) = \alpha(\Sigma(u_{s(\sigma)}(z))) \) for every \( \sigma \in \Sigma \) and \( z \in \mathbb{T} \), where \( \alpha_{\sigma} : M(A_{s(\sigma)}) \to M(A_{r(\sigma)}) \) is the unique strictly continuous extension of \( \alpha_{\sigma} : A_{s(\sigma)} \to A_{r(\sigma)} \).

**Example 2.5.** Every twisted groupoid \( C^\ast \)-dynamical system \((A, G, \Sigma, \alpha)\) yields a saturated Fell bundle over \( G \), cf. \cite[Section 3.1]{Lal}. We define it as the quotient \( A \rtimes_{\alpha} \Sigma/\mathbb{T} \) of the Fell bundle \( A \rtimes_{\alpha} \Sigma \) associated to \((A, \Sigma, \alpha)\) as in Example 2.3 by the \( \mathbb{T} \)-action defined by

\[
(2.1) \quad z \cdot (a, \sigma) := (au_{r(\sigma)}(z)^{-1}, z \cdot \sigma), \quad (a, \sigma) \in A \rtimes_{\alpha} \Sigma, z \in \mathbb{T}.
\]

We denote by \([a, \sigma]\) the corresponding class - the \( \mathbb{T} \)-orbit of \((a, \sigma)\). We also write \( \hat{\sigma} \) for the image of \( \sigma \in \Sigma \) in \( G = \Sigma/\mathbb{T} \). Then the fibers of the corresponding bundle are \( B_{\hat{\sigma}} = \{[a, \sigma] : a \in A_{r(\sigma)}\} \). The multiplication maps and involutions on \( A \rtimes_{\alpha} \Sigma \) factor through to well defined operations on the quotient \( A \rtimes_{\alpha} \Sigma/\mathbb{T} \). So the formulas \([a, \sigma] \cdot [b, \tau] := [a\alpha_{\sigma}(b), \sigma\tau]\), and \([a, \sigma]^* := [a^{-1}(a^*), \sigma^{-1}]\), \( a \in A_{r(\sigma)}, b \in A_{r(\tau)} \), yield the structure of the Fell bundle on \( B := \bigsqcup_{\gamma \in G} B_{\hat{\gamma}} \). We will refer to it as the Fell bundle associated to the twisted groupoid \( C^\ast \)-dynamical system \((A, G, \Sigma, \alpha)\).

**Remark 2.6.** If \( A = X \times \mathbb{C} \) is a trivial line bundle (and \( u_x \) are identities), then the construction in Example 2.3 specializes to the standard construction of a continuous line bundle \( L = \bigsqcup_{\gamma \in G} L_{\gamma} \) from a twisted groupoid \((G, \Sigma)\), see \cite[25.iv]{Kum} or \cite{Ren2}. This gives a part of the well known equivalence between continuous Fell line bundles over \( G \) and twists of \( G \). For the other part, recall that if \( L = \bigsqcup_{\gamma \in G} L_{\gamma} \) is a continuous Fell line
bundle then the circle bundle $\Sigma := \{ \xi \in \mathcal{L} : |\xi| = 1 \}$ with the topology and multiplication inherited from $\mathcal{L}$ is a twist of $G$. Note that in this case both the line bundle $\mathcal{L}$ and the extension $\mathbb{T} \times X \rightarrow \Sigma \rightarrow G$ are locally trivial.

Busby and Smith [BuS] introduced twisted actions by groups that involve cocycles rather than group extensions. As shown by Packer and Raeburn [PaR] this formalism covers Green’s approach to twisted crossed products. Also the original Renault’s twisted groupoid $C^*$-algebras [Ren1] were defined in the spirit of Busby and Smith (dating back in fact already to Zeller-Meier [Z-M]). Then Renault switched to Green’s approach in [Ren2], [Ren3], as it was not clear whether the Packer-Raeburn result generalises to groupoids. This question is raised, for instance, in [Sim, Remark 5.1.6], and we were informed by Tristan Bice that the answer to it is negative. Bice’s example of a groupoid twist that does admit a continuous global section comes from the Pedersen-Petersen $C^*$-Algebras, see [BiF]. The relevant $C^*$-algebra $\mathcal{B}_1$ is a cross-sectional algebra of a continuous line bundle over the principal groupoid $G = \mathbb{C}P^1 \times \{0,1\}^2$ whose every continuous section has to vanish by the Borsuk-Ulam Theorem.

Nevertheless, we will show that Kumjian-Renault twisted actions are covered by a version of Busby-Smith twisted actions adapted to inverse semigroups by Buss and Exel [BuE1].

**Definition 2.7.** A twisted action of an inverse semigroup $S$ on a $C^*$-algebra $A$ consists of partial automorphisms $\beta_t : D_t \rightarrow D_t$ of $A$ for $t \in S$ – that is, $D_t$ is an ideal in $A$ and $\beta_t$ is a $*$-isomorphism – and unitary multipliers $\omega(t,u) \in UM(D_{tu})$ for $t,u \in S$, such that $D_1 = A$ and the following conditions hold for $r,t,u \in S$ and $e,f \in E(S)$, where $E(S)$ is the set of units in $S$:

1. $D_{(rt)t^*} = D_{t^*} \cap \beta_t^{-1}(D_{rt})$ and $\beta_r \circ \beta_t = Ad_{\omega(r,t)}\beta_{rt}$;
2. $\beta_r(a\omega(t,u))\omega(rt,ru) = \beta_r(a)\omega(r,t)\omega(rt,ru)$ for $a \in D_{rt} \cap D_{tu}$;
3. $\omega(e,f) = 1_{ef}$ and $\omega(r,r^*r) = \omega(rr^*,r) = 1_r$, where $1_r$ is the unit of $M(D_r)$;
4. $\omega(t^*e,tf) = \omega(t^*e,t)a = \omega(t^*,t)a$ for all $a \in D_{t^*}$.

Given an étale groupoid $G$, the set of its open bisections $\text{Bis}(G)$ forms naturally an inverse semigroup with operations

$$U \cdot V := \{ \gamma \eta : \gamma \in U, \eta \in V, s(\gamma) = r(\eta) \}, \quad U^* = \{ \gamma^{-1} : \gamma \in U \}, \quad U,V \in \text{Bis}(G).$$

An inverse semigroup $S \subseteq \text{Bis}(G)$ defines an inverse semigroup action $\{ h_U \}_{U \in S}$ on the unit space $X = G^{(0)}$, where $h_U := r \circ s_U^{-1} : s(U) \rightarrow r(U)$. The transformation groupoid $X \rtimes h S$ for this action is naturally isomorphic to $G$ if and only if the semigroup $S$ is wide, that is $\bigcup S = G$ and $U \cap V$ is a union of bisections in $S$ for all $U,V \in S$; the corresponding notions and facts mentioned above are described for example in [KwM1, Section 2].

Note that the conclusion of the next lemma does not depend on the choice of the section $c_U$; the resulting actions for different choices of sections need not explicitly coincide, but the resulting crossed products are isomorphic (see Theorem 2.16 below).

**Lemma 2.8.** Let $(\mathcal{A}, G, \Sigma, \alpha)$ be a twisted groupoid $C^*$-dynamical system with $G$ étale. Let $S$ be the family of open bisections $U$ of $G$ on which the $\mathbb{T}$-bundle $\pi : \Sigma \rightarrow G$ is trivial, and for each $U \in S$ choose a continuous section $c_U : U \rightarrow \Sigma$ that induces the homeomorphism
\[ U \times T \cong \pi^{-1}(U). \] If \( U \subseteq X \) we choose \( c_U \) to be given by \( c_U(x) := (x, 1) \subseteq X \times T \subseteq \Sigma, \) for \( x \in U. \) Then the following statements hold.

1. \( S \) is a wide unital inverse subsemigroup of \( \text{Bis}(G). \)
2. For each \( U \in S \) we may treat \( D_U := C_0(\mathcal{A}|_{r(U)}) \) as an ideal in \( A := C_0(\mathcal{A}). \) The formula
   \[ \beta_U(a)(r(\gamma)) = \alpha_{c_U(\gamma)}(a(s(\gamma))), \quad a \in D_{U^*} = C_0(\mathcal{A}|_{s(U)}), \quad \gamma \in U, \]
   gives a well defined \(*\)-isomorphism \( \beta_U : D_{U^*} \to D_U. \)
3. For every \( U, V \in S \) the formula
   \[ \omega(U, V)(r(\gamma \eta)) = u(c_U(\gamma)c_V(\eta)c_{UV}(\gamma \eta)^{-1}), \quad \gamma \in U, \eta \in V \]
   gives a well defined element in \( UM(D_{UV}). \)

The quadruple \((A, S, \beta, \omega)\) described above yields a twisted action of an inverse semigroup in the sense of Definition 2.7.

**Proof.**

1. Let \( U, V \in S \) and \( c_U : U \to \Sigma \) and \( c_V : V \to \Sigma \) be continuous sections that trivialise \( \pi^{-1}(U) \) and \( \pi^{-1}(V) \). Then putting \( d_{U^*}(\gamma) := c_U(\gamma)^{-1}, \) for \( \gamma \in U^* \), and \( d_{UV}(\gamma \eta) := c_U(\gamma)c_V(\eta) \) for \( \gamma \in U, \eta \in V \), we get continuous sections that trivializes \( \pi^{-1}(U^*) \) and \( \pi^{-1}(UV) \), respectively. Thus \( S \) is an inverse subsemigroup of \( \text{Bis}(G). \) It is unital because \( X \in S \) and it is wide as it is closed under inclusions and every \( \gamma \in G \) has a neighbourhood \( U \in S. \)

2. Identifying sections of \( \mathcal{A}|_{r(U)} \) with sections of \( \mathcal{A} \) that vanish outside the open set \( r(U) \) we have that \( D_U := C_0(\mathcal{A}|_{r(U)}) \subseteq C_0(\mathcal{A}) = A \) is an ideal. Using continuity of the map \( \Sigma \times_s \mathcal{A} \ni (\sigma, a) \to \alpha_{\sigma}(a) \in \mathcal{A} \) and that \( r : U \to r(U), \) \( s : U \to s(U) \) and \( c_U : U \to c_U(U) \subseteq \Sigma \) are homeomorphisms, we see that the formula \( \beta_U(f)(x) := \alpha_{c_U(r U^{-1}(x))}(f(s(r U^{-1}(x)))) \) for \( f \in C_0(\mathcal{A}|_{s(U)}), \) \( x \in r(U), \) gives a well defined element \( \beta_U(f) \) in \( C_0(\mathcal{A}|_{r(U)}) \). This is exactly the formula in 2. The map \( \beta_U : D_{U^*} \to D_U \) is a \(*\)-isomorphism with the inverse given by \( \beta_U^{-1}(f)(s(\gamma)) = \alpha_{c_U(\gamma)^{-1}}(f(r(\gamma))) \) for \( f \in D_U = C_0(\mathcal{A}|_{r(U)}), \gamma \in U. \)

3. Note first that we have \( UV \ni \gamma \eta \to c_U(\gamma)c_V(\eta)c_{UV}(\gamma \eta)^{-1} \in r(UV) \times T \subseteq X \times T \) because \( \pi(c_U(\gamma)c_V(\eta)c_{UV}(\gamma \eta)^{-1}) = \pi(r(\gamma \eta)). \) Moreover this function is continuous, as so is the natural map \( UV \ni \gamma \eta \to (\gamma, \eta) \in U \times V. \) Since \( r : UV \to r(UV) \) is a homeomorphism and \( u : X \times T \to \bigcup_{x \in X} UM(A_x) \) is continuous, we get that for every \( a \in D_{UV} = C_0(\mathcal{A}|_{r(UV)}) \) the formula
   \[ \omega(U, V)a(r(\gamma \eta)) = u(c_U(\gamma)c_V(\eta)c_{UV}(\gamma \eta)^{-1})a(r(\gamma \eta)) \]
   defines an element of \( C_0(\mathcal{A}|_{r(UV)}). \) Now one readily infers that \( \omega(U, V) \) is a multiplier of \( C_0(\mathcal{A}|_{r(UV)}) \) with the adjoint given by \( \omega(U, V)^* = u(c_U(\gamma)c_V(\eta)c_{UV}(\gamma \eta)^{-1}). \) Hence \( \omega(U, V) \in UM(D_{UV}). \) Note that in terms of the notation introduced in Definition 2.4 we have \( (\gamma \in U, \eta \in V, \gamma \eta \in UV) \)
   \[ \omega(U, V)(r(\gamma \eta)) = u_{r(\gamma)}(p(c_U(\gamma)c_V(\eta)c_{UV}(\gamma \eta)^{-1})), \]
   where \( p : X \times T \to T \) is the projection onto the second coordinate. Now we need to check conditions (T1)-(T4) in Definition 2.7. To simplify the notation in what follows we will
sometimes write $\omega_{U,V}$ instead of $\omega(U,V)$. Choose $\gamma \in U \subseteq S$, $\eta \in V \subseteq S$ so that $\gamma \eta \in UV$. Then for $x := r(\gamma \eta) = r(\gamma)$ and $a \in D_{UV}^i$, using Definition 2.4(2), we have

$$
(Ad_{\omega_{U,V}} \circ \beta_{UV})(a)(x) = \alpha_{c_U}(\gamma)c_V(\eta)c_{UV}(\gamma \eta)^{-1}(\alpha_{c_U}(\gamma \eta)(a(x)) = \alpha_{c_U}(\gamma)c_V(\eta)(a(x)) = ((\beta_U \circ \beta_V)(a))(x).
$$

This proves (T1).

Now take $a \in D_{Z^*} \cap D_{UV} = C_0(\mathcal{A}|_{s(Z) \cap r(UV)})$ where $U, V, Z \subseteq S$. Take any $x \in r(ZUV)$ and choose $\gamma \in U$, $\eta \in V$ and $\sigma \in Z$ so that $x = r(\sigma \gamma \eta)$. Denoting by $p$ again the projection onto the second coordinate and using Definition 2.4(3) we get

$$
\beta_Z(a \cdot \omega_{U,V})(z) = \alpha_{cz(\sigma)}(a(s(\sigma))u(c_U(\gamma)c_V(\eta)c_{UV}(\gamma \eta)^{-1}))
$$

$$
= \alpha_{cz(\sigma)}(a(s(\sigma)) \cdot u_r(\gamma)\Big(p(c_U(\gamma)c_V(\eta)c_{UV}(\gamma \eta)^{-1})\Big))
$$

$$
= \beta_Z(a)(x) \cdot u_r(\gamma)\Big(p(c_U(\gamma)c_V(\eta)c_{UV}(\gamma \eta)^{-1})\Big).
$$

Using that the extension is central, for the pointwise multiplication on $r(ZUV)$ we have

$$
p(c_Uc_Vc_{UV}^{-1})p(c_Zc_Uc_{ZUV}^{-1}) = p(c_Zp(c_Uc_Vc_{UV}^{-1})c_Uc_{ZUV}^{-1}) = p(c_Zc_Uc_Vc_{ZUV}^{-1})
$$

$$
= p(c_Zc_Uc_{ZUV}^{-1}) = p(c_Zc_Uc_{ZUV}c_{ZUV}^{-1}) = p(c_Zc_Uc_{ZUV}^{-1})p(c_Zc_Uc_{ZUV}^{-1}).
$$

Combining the above and using that $\omega_{Z,UV}(x) = u_r(\gamma)(p(c_Zc_Uc_{UV}(\gamma \eta)c_{ZUV}^{-1}))(\sigma \gamma \eta)^{-1}$, we get

$$
\beta_Z(a \cdot \omega_{U,V}) \cdot \omega_{Z,UV} = \beta_Z(a) \cdot u\Big(p(c_U(\gamma)c_V(\eta)c_{UV}(\gamma \eta)^{-1})\big(c_Zc_Uc_{UV}(\gamma \eta)c_{ZUV}^{-1}\big)\Big)
$$

$$
= \beta_Z(a) \cdot u(c_Zc_Uc_{UV}(\gamma \eta)c_{ZUV}(\gamma \eta)^{-1})u(c_Zc_Uc_{UV}(\gamma \eta)c_{ZUV}(\gamma \eta)^{-1})
$$

$$
= \beta_Z(a)\omega_{Z,UV} \omega_{Z,UV}
$$

on $r(ZUV)$. This proves (T2).

Idempotents in $S$ correspond to open subsets of $X$. Thus axiom (T3) follows readily from our choice of $c_U$ for $U \subseteq X$. To check (T4) let $U, V \in S$ where $V \subseteq X$, so that $c_V(V) = V \times \{1\} \subseteq V \times T \subseteq \Sigma$. Then on the set $r(U^*VU)$ we have $\omega(U^*VU) = u(c_U^*c_Vc_{UV}^*c_{UV}^{-1}) = u(c_U^*c_Uc_{UV}^*c_{UV}^{-1}) = \omega(U^*, U)$, where we for example use the fact that $U^*VU$ is an idempotent, so that $c_U^*VU = c_U^*|_{U^*VU}$.

**Definition 2.9** (Definition 4.7, [BuM]). Let $S$ be a unital inverse semigroup and $A$ a $C^*$-algebra. An action of $S$ on $A$ by Hilbert bimodules is a semigroup $C = \bigcup_{t \in S} C_t$ where each fiber $C_t$ is a Hilbert $A$-bimodule, $C_1 = A$ is a trivial bimodule, and the semigroup product in $C$ induces isomorphisms $C_t \otimes C_s \cong C_{ts}$ of Hilbert $A$-bimodules, for all $t, s \in S$. 

*Saturated Fell bundles over inverse semigroups* ([Exe], [BuE1], [BuE2]) are equivalent to inverse semigroup actions by Hilbert bimodules ([BuM], [BEM]), whose definition is relatively concise.
Remark 2.10. Any inverse semigroup action by Hilbert bimodules $\mathcal{C} = \bigsqcup_{t \in S} C_t$ induces uniquely defined canonical involutions $C_t^* \to C_t, x \mapsto x^*$, and inclusion maps $j_{s,t} : C_t \to C_s$ for $t, s \in S$, $t \leq s$ that satisfy the conditions required for a saturated Fell bundle, as studied in [Exel, BuE] and [BuE], see [BuM, Theorem 4.8]. Note further that the structure of $\mathcal{C} = \bigsqcup_{t \in S} C_t$ is also uniquely determined by the multiplication and involution in $\mathcal{C}$ (these operations determine the Hilbert $A$-module operations on each fiber $C_t$).

Example 2.11. Any twisted action $(A,S,\beta,\omega)$ of an inverse semigroup $S$ on an $C^*$-algebra $A$ yields a natural action of $S$ on $A$ by Hilbert bimodules given by a family of the standard bimodules associated to partial automorphisms $\{\beta_t\}_{t \in S}$ of $A$. Namely, we put $\mathcal{C} := \bigsqcup_{t \in S} D_t$, where each $D_t$ is a Hilbert $A$-bimodule with operations

$$a \cdot \xi := a\xi, \quad \xi \cdot a := \beta_t^{-1}(\xi)a, \quad \langle \xi, \psi \rangle_A := \beta_t^{-1}(\xi^*\psi).$$

where $a \in A$, $\xi, \psi \in D_t$. Multiplication between different fibers is given by

$$D_s \times D_t \ni (\xi_s, \xi_t) \mapsto \beta_s(\beta_t^{-1}(\xi_s)\xi_t)\omega(s,t) \in D_{st}.$$  

The induced involution is given by $D_s \ni \xi_s \mapsto \beta_s^{-1}(\xi_s^*)\omega(s,s)^* \in D_{ss^*}$, cf. [BuE, page 250]. In particular, every twisted groupoid $C^*$-dynamical system $(A, G, \Sigma, \alpha)$ yields a saturated Fell bundle $\bigsqcup_{t \in S} C_0(A|_{\gamma(t)})$ with operations coming from the twisted action $(A,S,\beta,\omega)$ of an inverse semigroup $S$ on $A := C_0(A)$ as described in Lemma 2.8.

2.4. The associated convolution $C^*$-algebras. Let $\mathcal{B} = \bigsqcup_{\gamma \in G} B_{\gamma}$ be a Fell bundle over an étale locally compact Hausdorff groupoid $G$. The space of compactly supported continuous sections $C_c(\mathcal{B})$ is naturally a $*$-algebra with operations

$$(f * g)(\gamma) := \sum_{r(\eta) = r(\gamma)} f(\eta) \cdot g(\eta^{-1} \cdot \gamma), \quad (f^*)(\gamma) := f(\gamma^{-1})^*,$$

$f, g \in C_c(\mathcal{B})$, $\gamma \in G$. The full section $C^*$-algebra $C^*_{r}(\mathcal{B})$ is the maximal $C^*$-completion of the $*$-algebra $C_c(\mathcal{B})$. Let $X := G^{(0)}$ be the unit space. The $C_0(X)$-$C^*$-algebra $A := C_0(\mathcal{B}|_X)$ embeds naturally into $C^*_{r}(\mathcal{B})$, and the restriction map $C_c(\mathcal{B}) \ni f \to f|_X \in A$ extends to a conditional expectation $C^*_{r}(\mathcal{B}) \to A$. The reduced section $C^*$-algebra $C^*_{r}(\mathcal{B})$ can be defined as the completion of the $*$-algebra $C_c(\mathcal{B})$ with respect to the minimal $C^*$-norm such that the restriction map $C_c(\mathcal{B}) \to A$ is contractive. Thus the restriction extends to a faithful conditional expectation $E : C^*_{r}(\mathcal{B}) \to A$. Note that if this conditional expectation is tracial, then $A$ is commutative; but this condition is itself not sufficient. We will also need an explicit faithful representation of $C^*_{r}(\mathcal{B})$, as described for example in [Kum2]. Given $x \in G^{(0)}$ put $G_x := \{\gamma \in G : s(\gamma) = x\}$ and consider the Hilbert $B_{\gamma}$-module $V_x := \bigoplus_{\gamma \in G_x} B_{\gamma}$ (recall that the scalar product comes from the bundle operations: if $a, b \in B_\gamma$, then $ab \in B_{\gamma^{-1}\gamma} \subseteq B_{\gamma^{-1} \gamma} = B_{s(\gamma)}$). Then we have a representation $\pi_x : C^*_{r}(\mathcal{B}) \to B(V_x)$ determined by the formula

$$(\pi_x(f)(\bigoplus_{\gamma \in G_x} \xi_\gamma) = \bigoplus_{\gamma \in G_x} \left(\sum_{\beta \in G_x} f(\gamma\beta^{-1})\xi_\beta\right),$$

\(11\)
where \( f \in C_c(B) \), \( \xi_\gamma \in B_\gamma \) for each \( \gamma \in G_x \), and the series \( \sum_{\gamma \in G_x} \langle \xi_\gamma, \xi_\gamma \rangle \) converges in \( B_x \).

The direct sum of the representations \( \pi_x \) over \( x \in G^{(0)} \) is faithful.

**Remark 2.12.** When equipped with the supremum norm, \( C_0(B) \) is a Banach space. The embedding \( C_0(B) \to C_0(B) \) extends uniquely to an injective and contractive linear map \( C^*_r(B) \to C_0(B) \) which allows us to assume the identification

\[
C^*_r(B) \subseteq C_0(B).
\]

Under this identification the \(*\)-algebraic operations in \( C^*_r(B) \) are still given by \( (2.2) \), see [KwM] Proposition 7.10. Moreover, for any \( C^\ast\)-completion \( B = C_c(B) \) in a \( C^\ast\)-norm \( \| \cdot \| \) that extends the supremum norm on \( C_c(B|_{G^{(0)}}) \) we have natural isometric embeddings \( C_0(B|_U) \hookrightarrow B \) for all open bisections \( U \subseteq G \). Indeed, if \( f \in C_c(B|_U) \), then \( \| f \|^2 = \| f \ast f \|_{C_0(B|_{G^{(0)}})} = \max_{x \in G^{(0)}} \| (f \ast f)(x) \| = \max_{\gamma \in U} \| f^\gamma f(\gamma) \| = \max_{\gamma \in U} \| f(\gamma) \|^2 \).

**Example 2.13.** Let \( (A,G,\alpha) \) be a groupoid \( C^\ast\)-dynamical system and let \( A \rtimes_\alpha G \) be the associated Fell bundle, cf. Example 2.3. The \(*\)-algebra \( C_0(A \rtimes_\alpha G) \) consists of compactly supported continuous sections of \( A \rtimes_\alpha G = \bigcup_{\gamma \in G} A_{\gamma(\gamma)} \) with operations given by

\[
(f \ast g)(\gamma) := \sum_{\eta \in G} f(\eta) \cdot \alpha_\eta(g(\eta^{-1} \cdot \gamma)), \quad (f^\gamma)(\gamma) := \alpha_\gamma(f(\gamma^{-1}))^*, \quad \gamma \in G.
\]

The **full and reduced crossed product**, denoted by \( C^*(A,G,\alpha) \) and \( C^*_r(A,G,\alpha) \), is by definition \( C^*(A \rtimes_\alpha G) \) and \( C^*_r(A \rtimes_\alpha G) \), respectively.

Let now \( (A,G,\Sigma,\alpha) \) be a twisted groupoid \( C^\ast\)-dynamical system and let \( B \) be the associated Fell bundle over \( G \), cf. Example 2.5. By definition, the **full and reduced crossed product**, denoted by \( C^*(A,G,\Sigma,\alpha) \) and \( C^*_r(A,G,\Sigma,\alpha) \), is \( C^*(B) \) and \( C^*_r(B) \), respectively. Using the fact that \( \pi^{-1}(X) = X \times T \subseteq \Sigma \) is a trivial bundle, we get that \( A \supseteq A_x \ni a \mapsto [a,(x,1)] \in B_x \subseteq B|_X \) is the isomorphism of \( C^\ast\)-bundles \( A \cong B|_X \). Therefore we may treat

\[
A := C_0(A) \cong C_0(B|_X)
\]

as a \( C^\ast\)-subalgebra of both \( C^*(A,G,\Sigma,\alpha) \) and \( C^*_r(A,G,\Sigma,\alpha) \). In particular, we have a faithful conditional expectation \( \pi : C^*_r(A,G,\Sigma,\alpha) \to A \). In a similar manner we may identify the spaces of sections of \( B \) restricted to open bisections on which the twist \( \Sigma \rightarrow G \) is trivial.

**Lemma 2.14.** Let \( (A,G,\Sigma,\alpha) \) be a twisted groupoid \( C^\ast\)-dynamical system. Let \( B \) be the associated Fell bundle over \( G \) (see Example 2.7), and let \( (A,S,\beta,\omega) \) be the associated twisted action of an inverse semigroup \( S \) on \( A := C_0(A) \) described in Lemma 2.8. Put \( B := C^*_r(A,G,\Sigma,\alpha) \). We have the following natural actions of the inverse semigroup \( S \) by Hilbert bimodules on \( A \), which are all naturally isomorphic:

1. \( \bigcup_{U \in S} B_U \) where \( B_U := \overline{C_c(B|_U)} \subseteq B \) and all the operations are inherited from \( B \) (the closure is with respect to the supremum norm).
2. \( \bigcup_{U \in S} C_0(B|_U) \) with the multiplication \( C_0(B|_U) \times C_0(B|_V) \ni (f,g) \mapsto f \ast g \in C_0(B|_{U \cap V}) \) and the involution \( C_0(B|_U) \ni f \mapsto f^* \in C_0(B|_{U^\circ}) \) given by

\[
f \ast g(\gamma \eta) := f(\gamma) \cdot g(\eta), \quad f^*(\gamma^{-1}) = f(\gamma)^*, \quad \text{for } \gamma \in U \text{ and } \eta \in V.
\]
(3) $\bigcup_{U \in S} C_0(A \ast_r G|_U)$ with the multiplication $C_0(A \ast_r G|_U) \times C_0(A \ast_r G|_U) \ni (f, g) \mapsto f \ast g \in C_0(A \ast_r G|_U)$ and the involution $C_0(A \ast_r G|_U) \ni f \mapsto f^* \in C_0(A \ast_r G|_U^*)$ given by

$$f \ast g (\gamma \eta) := f(\gamma)\alpha_{c_U(\gamma)}(g(\eta))u(c_U(\gamma)c_V(\eta)c_{UV}(\gamma \eta)^{-1}),$$

$$f^*(\gamma^{-1}) := \alpha_{c_U(\gamma)}^{-1}(f(\gamma)^*)u(c_U^*(\gamma^{-1})c_U(\gamma))^*,$$

where $\gamma \in U$ and $\eta \in V$.

(4) $\bigcup_{U \in S} D_U$ associated to $(A, S, \beta, \omega)$ as in Example 2.11 so that the Hilbert $A$-bimodule structure on $D_U = C_0(A|_{r(U)})$ is given by

$$a \cdot f := af, \quad f \cdot a := \beta_U(\beta_U^{-1}(f)a), \quad \lambda(f, g) := fg^*, \quad \langle f, g \rangle_A := \beta_U^{-1}(fg),$$

and the multiplication $D_U \times D_U \ni (f, g) \mapsto \beta_U(\beta_U^{-1}(f)g)\omega(U, V) \in D_{UV}$. The isomorphisms are given by $B_U \supseteq C_c(B|_U) \ni f \mapsto f \circ r \in C_0(A \ast_r G|_U)$ and $C_c(A \ast_r G|_U) \ni f \mapsto [f] \in C_c(B|_U) \subseteq B_U$ where $[f](\gamma) := [f(\gamma), c_U(\gamma)]$.

Proof. For the isomorphism between (1) and (2) recall that the identity map $B_U \supseteq C_c(B|_U)$ is isometric and hence extends to the isomorphism $B_U \simeq C_0(B|_U)$, see Remark 2.12. That this isomorphism is consistent with the multiplication and the involution is obvious.

The isomorphism between the actions in (3) and (4) is straightforward, since for each $U \in S$ the homeomorphism $r : U \to r(U)$ induces an isomorphism $A \ast_r G|_U \simeq A|_{r(U)}$ of bundles (recall also that $c_U(x) = (x, 1)$ for $U \subseteq X$, by definition). We will show below that these actions are isomorphic to the action (2).

Let $U \in S$. Recall that $U \times T \supseteq (\gamma, z) \mapsto z \cdot c_U(\gamma) \in \pi^{-1}(U)$ is a homeomorphism $U \times T \simeq \pi^{-1}(U)$. This homeomorphism transfers $A \ast_r \Sigma|_{\pi^{-1}(U)}$ into a bundle over $U \times T$ whose quotient under the $T$-action (2.11) is isomorphic to $A \ast_r G|_U$. Thus $B|_U = A \ast_r \Sigma|_{\pi^{-1}(U)}/T \simeq A \ast_r G|_U$ and the isomorphism between algebras presented in items (3) and (2) of the lemma is given by $C_c(A \ast_r G|_U) \ni f \mapsto [f] \in C_c(B|_U) \subseteq B_U$ where $[f](\gamma) := [f(\gamma), c_U(\gamma)]$. In particular, for any $f \in C_c(A \ast_r G|_U)$, $g \in C_c(A \ast_r G|_U)$, $\gamma \in U$ and $\eta \in V$ we have

$$[f][g](\gamma \eta) = [f(\gamma)[g(\eta)] = [f(\gamma), c_U(\gamma)][g(\eta), c_U(\gamma)] = [f(\gamma)\alpha_{c_U(\gamma)}(g(\eta)), c_U(\gamma)c_U(\gamma)]$$

$$= [f(\gamma)\alpha_{c_U(\gamma)}(g(\eta))u(c_{UV}(\gamma \eta)c_{UV}(\gamma \eta)^{-1}c_U(\gamma)^{-1})^{-1}, c_{UV}(\gamma \eta)]$$

$$= [f(\gamma)\alpha_{c_U(\gamma)}(g(\eta))u(c_U(\gamma)c_V(\eta)c_{UV}(\gamma \eta)^{-1}), c_{UV}(\gamma \eta)]$$

$$= [f \ast g(\gamma \eta), c_{UV}(\gamma \eta)].$$

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Thus we have a semigroup isomorphism $\bigsqcup_{U \in S} C_c(A \ast_r G|_U) \to C_c(B|_U)$. It preserves the involution as we have

$$[f]^*(\gamma^{-1}) = [f](\gamma^*) = [f(\gamma), c_U(\gamma)]^* = [\alpha_{c_U(\gamma)^{-1}}(f(\gamma)^*), c_U(\gamma)^{-1}]$$

$$= [\alpha_{c_U(\gamma)}^{-1}(f(\gamma)^*), (c_U(\gamma^{-1})c_U(\gamma))^{-1}c_U(\gamma^{-1})]$$

$$= [\alpha_{c_U(\gamma)}^{-1}(f(\gamma)^*)u(c_U(\gamma^{-1})c_U(\gamma))^{-1}, c_U(\gamma^{-1})]$$

$$= [f^*](\gamma^{-1}).$$

\[\square\]

**Remark 2.15.** The last lemma could be equally well formulated with $B = C^*(A, G, \Sigma, \alpha)$ or more generally with $B$ being any quotient of $C^*(A, G, \Sigma, \alpha)$ by an ideal that intersects $A$ trivially, cf. Remark 2.12.

If $\mathcal{C} = \bigsqcup_{t \in S} C_t$ is an action of an inverse semigroup $S$ by Hilbert bimodules on $A = C_1$, then $\bigoplus_{t \in S} C_t$ with operations from $\mathcal{C}$ is naturally a $*$-algebra and using the induced inclusion maps one can define a quotient $*$-algebra of $\bigoplus_{t \in S} C_t$ whose full completion and a completion allowing a faithful weak conditional expectation are respectively the full and reduced $C^*$-algebras of $\mathcal{C}$, see [BEM]. They coincide with the $C^*$-algebras associated to $\mathcal{C}$ treated as a Fell bundle over $S$ as defined in [Exe], [BuE1], or [BuE2]. The full $A \rtimes_\beta^w S$ and reduced crossed product $A \rtimes_\beta^r S$ of an inverse semigroup twisted action $(A, S, \beta, \omega)$ are by definition the full and reduced $C^*$-algebra of the associated Fell bundle over $S$, described in Example 2.11, see [BuE1].

As explained below Remark 2.6, the following result can be viewed as a generalisation of [Par] Proposition 5.1] from group actions to groupoid actions. It also generalises a part of [BEM] Theorem 7.2] that characterises $C^*$-algebras of groupoid twists as crossed products by twisted actions.

**Theorem 2.16.** Let $(\mathcal{A}, G, \Sigma, \alpha)$ be a twisted groupoid $C^*$-dynamical system with $G$ étale and let $(A, S, \beta, \omega)$ be the twisted inverse semigroup action defined in Lemma 2.8 so that $S$ denotes the family of open bisections $U$ of $G$ on which the $\mathbb{T}$-bundle $\pi : \Sigma \to G$ is trivial. Let $S_0 \subseteq S$ be any inverse semigroup which is wide in $G$, as defined before Lemma 2.8 (one may take $S_0 = S$). Then the corresponding twisted crossed products are canonically isomorphic:

$$C^*(\mathcal{A}, G, \Sigma, \alpha) \cong A \rtimes_\beta^w S_0, \quad C^*_r(\mathcal{A}, G, \Sigma, \alpha) \cong A \rtimes_\beta^r S_0.$$   

*Proof.* Let $\mathcal{C} = \bigsqcup_{U \in S} C_0(B|_U)$ be the inverse semigroup action by Hilbert bimodules on $A$ described in Lemma 2.14[2]. This is a standard inverse semigroup action associated to the Fell bundle $\mathcal{B}$ over $G$, see [BuE2, Example 2.9] or [KwM] section 7.1. Thus restricting $\mathcal{C}$ to any inverse subsemigroup $S_0 \subseteq S$ which is wide in $G$ we get $C^*(\bigsqcup_{U \in S_0} C_0(B|_U)) \cong C^*(\mathcal{B})$ and $C^*_r(\bigsqcup_{U \in S_0} C_0(B|_U)) \cong C^*_r(\mathcal{B})$ by [BuE2], Theorem 2.13], or [BuM] Corollary 5.6], and [BEM Theorem 8.11], cf. also [KwM] Propositions 7.6, 7.9]. This gives the assertion as $\mathcal{B}$ is isomorphic to the bundle associated to $(\mathcal{A}, G, \Sigma, \alpha)$ by Lemma 2.14. \[\square\]
3. Multipliers for Twisted Groupoid Dynamical Systems

In this section we discuss the notion of mapping multipliers for Fell bundles over groupoids, and we characterise a natural class of completely positive multipliers for Fell bundles coming from twisted groupoid dynamical systems. We consider the maps acting on the reduced $C^*$-algebras, as opposed to the universal ones, studied in detail for example in [RaW], [Ren] or [Ren6] (albeit only in the case of trivial line bundles).

Definition 3.1. Let $\mathcal{B}$ be a Fell bundle over the groupoid $G$. A mapping $\mathcal{B}$-multiplier is a function $G \ni \gamma \mapsto h(\gamma) \in B(B_\gamma)$ which is continuous in the sense that the formula

$$ (m_h f)(\gamma) = h(\gamma) f(\gamma) \quad f \in C_c(\mathcal{B}), \gamma \in G, $$

defines a map (a priori with no continuity requirement) $m_h : C_c(\mathcal{B}) \to C_c(\mathcal{B})$. We say that a mapping $\mathcal{B}$-multiplier $h$ is bounded (resp. completely bounded or completely positive) if the map $m_h$ extends to to a bounded (resp. completely bounded, completely positive) map on $C^*_r(\mathcal{B})$.

Example 3.2 (Scalar multipliers). Every continuous function $h : G \to \mathbb{C}$ can be treated as a mapping $\mathcal{B}$-multiplier for every Fell bundle $\mathcal{B}$ over $G$, as then $[\mathcal{A}]$ defines $m_h : C_c(\mathcal{B}) \to C_c(\mathcal{B})$. We call such multipliers scalar $\mathcal{B}$-multipliers. Recall that a function $h : G \to \mathbb{C}$ is called positive-definite if for any $x \in G^{(0)}$ and a finite set $F \subseteq G_x$ the matrix

$$ [h(\gamma^{-1} \eta)]_{\eta, \gamma \in F} $$

is positive-definite. Takeishi showed in [Tak, Lemma 4.2] that every compactly supported positive-definite function is a completely positive scalar $\mathcal{B}$-multiplier for any $G$-bundle $\mathcal{B}$ and remarked that the compact support condition can be dropped; we record the corresponding fact, together with its converse, in Proposition 3.7.

Example 3.3 ($M(\mathcal{B}|_{G^{(0)}})$-valued multipliers). Let again $\mathcal{B}$ be a Fell bundle over $G$. Then $\mathcal{B}|_{G^{(0)}}$ is a continuous field of $C^*$-algebras and so we have a pullback bundle $\mathcal{B}|_{G^{(0)}} *_r G = \bigsqcup_{\gamma \in G} B_r(\gamma)$. Every fiber $B_\gamma$ is naturally a left $B_r(\gamma)$-Hilbert module, and hence also a left $M(B_r(\gamma))$-module. Hence every strictly continuous section $h$ of the multiplier bundle $M(\mathcal{B}|_{G^{(0)}} *_r G) := \bigsqcup_{\gamma \in G} M(B_r(\gamma))$ can be treated as a mapping $\mathcal{B}$-multiplier. Strict continuity here means that for any continuous section $a$ of $\mathcal{B}|_{G^{(0)}} *_r G$ the pointwise product $ha$ is also continuous as a section of $\mathcal{B}|_{G^{(0)}} *_r G$. If $\mathcal{B}$ is saturated then such multipliers coincide with mapping $\mathcal{B}$-multipliers with the values in adjointable operators on the fibers $B_\gamma$, $\gamma \in G$. Sometimes we will call mapping multipliers of this type simply $\mathcal{B}$-multipliers.

Recall that if $\mathcal{B}$ is a Fell bundle associated to a twisted groupoid $C^*$-dynamical system $(\mathcal{A}, G, \Sigma, \alpha)$, then $\mathcal{B}|_{G^{(0)}} \cong \mathcal{A}$ with the isomorphism given by $\mathcal{A} \ni a_r \mapsto [a_\gamma, 1]$. So every strictly continuous section $h$ of $M(\mathcal{A} *_r G) = \bigsqcup_{\gamma \in G} M(A_r(\gamma))$ is a $\mathcal{B}$-multiplier; we have $(m_h f)(\tilde{\sigma}) = [h(\tilde{\sigma}) a, \sigma]$ for each $\sigma \in \Sigma$, $f \in C_c(\mathcal{B})$ and $a \in A_r(\sigma)$ such that $f(\tilde{\sigma}) = [a, \sigma]$.

In general it is not clear how to describe explicitly completely positive mapping $\mathcal{B}$-multipliers. But we can do it for $\mathcal{B}$-multipliers on Fell bundles coming from twisted groupoid $C^*$-dynamical systems.
**Definition 3.4.** Let \((\mathcal{A}, G, \Sigma, \alpha)\) be a twisted groupoid \(C^*\)-dynamical system. We say that a section \(h\) of \(M(\mathcal{A} \ast, G)\) is **positive-definite** if for any \(x \in G^{(0)}\) and any finite set \(F \subseteq G_x\), for some section \(c : F \to \Sigma\) the matrix
\[
\left[ \alpha_{c(\gamma)}^{-1}(h(\gamma \eta^{-1})) \right]_{\gamma, \eta \in F} \in M_F(M(A_x))
\]
is positive.

We will now show that the definition above in fact does not depend on the choice of the section, and discuss the central \(\mathcal{B}\)-multipliers.

**Lemma 3.5.** Let \((\mathcal{A}, G, \Sigma, \alpha)\) be a twisted groupoid \(C^*\)-dynamical system and let \(h\) be a section of \(M(\mathcal{A} \ast, G)\).

1. Let \(x \in G^{(0)}\) and let \(F \subseteq G_x\) be a finite set. If there is a section \(c : F \to \Sigma\) such that the matrix \(\left[ \alpha_{c(\gamma)}^{-1}(h(\gamma \eta^{-1})) \right]_{\gamma, \eta \in F}\) is positive, then the corresponding matrix is positive for any section \(c' : F \to \Sigma\).
2. The map \(m_h\) is a \(C_c(\mathcal{A})\)-bimodule map if and only if \(h(\gamma) \in ZM(A_{r(\gamma)})\), for all \(\gamma \in G\).

**Proof.** \[1\] Consider two sections \(c, c' : F \to \Sigma\) for \(x, F\) as above. Then there is a function \(f : F \to \mathbb{T}\) such that \(c = fc'\). Note that using Definition 2.4(2) we obtain for any element \(z \in \mathbb{T}\) and \(\sigma \in \Sigma\) the following equality: \(\alpha_{z\sigma}(\gamma) \gamma = u_{r(\sigma)}(z) \alpha_{\sigma}(\gamma)\). By Definition 2.4(3) we obtain \(\alpha_{c(\gamma)}^{-1}(\gamma) = \alpha_{\sigma}(u_{\gamma}(\sigma) \cdot u_{\gamma}(\sigma)^{-1})\). This is in turn equivalent to \(\alpha_{c(\gamma)}^{-1}(\gamma) = u_{\gamma}(\sigma) \cdot u_{\gamma}(\sigma)^{-1}\). Thus for any \(\{a_{\gamma}\}_{\gamma \in F} \subseteq M(A_x)\) putting \(b_{\gamma} := a_{\gamma} u_{\gamma}(f(\gamma))\), we get
\[
\sum_{\gamma, \eta \in F} a_{\gamma} \alpha_{c(\gamma)}^{-1}(h(\gamma \eta^{-1})) a_{\eta}^* = \sum_{\gamma, \eta \in F} b_{\gamma} \alpha_{c(\gamma)}^{-1}(h(\gamma \eta^{-1})) b_{\eta}^*.
\]
This proves that positivity of \([\alpha_{c(\gamma)}^{-1}(h(\gamma \eta^{-1})))_{\gamma, \eta \in F}\) implies that \(\chi_{\gamma, \eta \in F}\).

\[2\] The formula (3.4) implies that each \(m_h\) is a right \(C_c(\mathcal{A})\)-module map. Hence it is also a \(C_c(\mathcal{A})\)-bimodule map if \(h(\gamma) \in ZM(A_{r(\gamma)})\), for all \(\gamma \in G\). Conversely, assume \(m_h\) is a left \(A\)-module map. For arbitrary \(a, b \in A_{r(\sigma)}\), \(\sigma \in \Sigma\), there are functions \(f \in C_c(\mathcal{A})\) and \(g \in C_c(\mathcal{B})\) supported on open bisections and such that \(f(r(\sigma)) = a, g(\sigma) = [b, \sigma]\). Then, denoting the canonical image of \(f\) in \(C_c(\mathcal{B})\) by \(\hat{f}\) (so that \(\hat{f}(\sigma) = [f(\sigma), r(\sigma)]\)), we have
\[
[h(\sigma)ab, \sigma] = m_h(\hat{f} \cdot g)(\hat{\sigma}) = (\hat{f} \cdot m_h(g)(\hat{\sigma})) = [a, r(\sigma)] \cdot [h(\hat{\sigma}) b, \sigma] = [ah(\hat{\sigma}) b, \sigma].
\]
Hence \(h(\hat{\sigma})ab = ah(\hat{\sigma}) b\), for every \(a, b \in A_{r(\sigma)}\). This implies \(h(\hat{\sigma}) \in ZM(A_{r(\sigma)})\).

Following the classical argument from the context of Herz-Schur multipliers (see for example [BrO] Subsection 5.6) we obtain the next result.

**Proposition 3.6.** Let \(\mathcal{B}\) be a Fell bundle associated to a twisted groupoid \(C^*\)-dynamical system \((\mathcal{A}, G, \Sigma, \alpha)\), and let \(h\) be a section of \(M(\mathcal{A} \ast, G)\). The following conditions are equivalent:
(1) If these equivalent conditions hold then $h$ is a completely positive $B$-multiplier, i.e. (3.4) defines a completely positive map $m_h : C^*_r(B) \to C^*_r(B)$.

(2) $h$ is strictly continuous, bounded and positive-definite.

If these equivalent conditions hold then $h$ is central (i.e. $h(\gamma) \in ZM(A_{r(\gamma)})$, for all $\gamma \in G$), $m_h$ is a $C_0(A)$-bimodule map, and $\|m_h\| = \sup_{\gamma \in G} \|h(\gamma)\| = \sup_{x \in G(\gamma)} \|h(x)\|$.

**Proof.** (1) $\Rightarrow$ (2) Clearly, $h$ is a mapping $B$-multiplier, i.e. it defines $m_h : C^*_c(B) \to C^*_c(B)$, if and only if $h$ is strictly continuous. Since $m_h : C^*_r(B) \to C^*_r(B)$ is completely positive, it is automatically bounded and $\|m_h\| = \sup_\lambda \|m_h(\mu_\lambda)\|$ for an approximate unit $(\mu_\lambda)$ in $A := C_0(A)$. This readily implies that $\|m_h\| = \sup_{x \in G(\gamma)} \|h(x)\| \leq \sup_{\gamma \in G} \|h(\gamma)\|$. Moreover, for any $\gamma \in G$ and $\varepsilon > 0$ we may find an open bisection $U$ containing $\gamma$ and a section $a \in C^*_c(B|_U) \subseteq C^*_c(B) \subseteq C^*_r(B)$ such that $\|h(\gamma)\| \leq \|h(\gamma)\| + \varepsilon$ and $\|a\| = 1$ (cf. the second part of Remark 2.12). Hence $\|h(\gamma)\| \leq \|m_h(a(\gamma))\| + \varepsilon \leq \|m_h\| + \varepsilon$. This implies that $\sup_{\gamma \in G} \|h(\gamma)\| = \|m_h\|$, and so $h$ is bounded. Furthermore, since $m_h : C^*_r(B) \to C^*_r(B)$ is (completely) positive it is $*$-preserving, and by its very definition (3.4), $m_h$ is a right $C_0(A)$-module map. Hence $m_h$ is a $C_0(A)$-bimodule map: $m_h(ab) = m_h(b^*a^*) = am_h(b)$, $a \in C^*_c(A)$, $b \in C^*_c(B)$. Thus $h$ is central by Lemma (3.3) (2).

To see that $h$ is positive-definite, fix $x \in G(\gamma)$ and a finite set $F \subseteq G_x$. For every $\gamma \in F$ let $f_\gamma \in C^*_c(B)$ be a section supported on an open bisection. The matrix $(f_\gamma f_\eta^*)_{\gamma,\eta \in F} \in M_F(C^*_r(B))$ is positive (recall that the representation $\pi_x$ and the module $V_X$ were introduced in the formula (2.3) and the discussion preceding it). So also the operator $B := \pi_x^{(|F|)} \circ m_h^{(|F|)}((f_\gamma f_\eta^*)_{\gamma,\eta \in F}) \in B(V_x^{[|F|]}$)

is positive. Let us choose a section $c : F \to \Sigma$ and write $f_\gamma(\gamma) = [a_\gamma, c(\gamma)]$ for $a_\gamma \in A_{r(\gamma)}$ and $\gamma \in F$. Note that

$$(f_\gamma f_\eta^*)(\gamma^{-1}) = f_\gamma(\gamma) f_\eta(\eta)^* = [a_\gamma, c(\gamma)] [a_\gamma c(\eta)^{-1}(a_\eta^*), c(\eta)^{-1}] = [a_\gamma, a_\gamma c(\gamma)c(\eta)^{-1}] = \alpha_{c(\gamma),c(\eta)^{-1}}(a_\eta^*) c(\gamma) c(\eta)^{-1}].$$

Let then $\xi = \bigoplus_{\eta \in F} \xi_\eta = \bigoplus_{\eta \in F} \bigoplus_{\beta \in G_x} \xi_\eta(\beta) \in V_x^{[|F|]}$ be such that $\xi_\eta(\beta) \in B_\beta$ is zero unless $\beta = \eta$. Write $\xi_\eta(\eta) = [b_\eta, c(\eta)]$ for $b_\eta \in A_{r(\eta)}$ and $\eta \in F$. We have

$$0 \leq \langle \xi, B \xi \rangle = \sum_{\gamma,\eta \in F} \langle \xi_\gamma, \pi_x(m_h((f_\gamma f_\eta^*) (\gamma^{-1})) (\eta)) \rangle = \sum_{\gamma,\eta \in F} \langle \xi_\gamma(\gamma), m_h(f_\gamma f_\eta^*) (\gamma^{-1}) \rangle \xi_\eta(\eta)$$

$$= \sum_{\gamma,\eta \in F} \langle [b_\eta, c(\eta)], [h(\gamma^{-1}) a_\gamma c(\gamma)c(\eta)^{-1}(a_\eta^*), c(\gamma) c(\eta)^{-1}] \rangle [b_\eta, c(\eta)]$$

$$= \sum_{\alpha_{c(\gamma),c(\eta)^{-1}}(a_\eta^*)} \alpha_{c(\gamma),c(\eta)^{-1}}^{-1} \langle h(\gamma^{-1}) a_\gamma c(\gamma)c(\eta)^{-1}(a_\eta^*), c(\gamma) c(\eta)^{-1} \rangle \alpha_{c(\gamma),c(\eta)^{-1}}^{-1} (a_\eta^*)$$

$$= \sum_{\gamma,\eta \in F} \alpha_{c(\gamma),c(\eta)^{-1}}^{-1} (h(\gamma^{-1})(a_\gamma c(\gamma)c(\eta)^{-1}(a_\eta^*)) a_\gamma c(\gamma)c(\eta)^{-1}(a_\eta^*) b_\eta)$$

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It now remains to note that we can choose the functions \( f^i \) so that the corresponding \( a^i \) approximate \( 1_{M(\mathcal{A}_{\mathcal{H}(\gamma)})} \) in the strict topology. Then taking the limit over \( i \) we obtain

\[
0 \leq \sum_{\gamma, \eta \in F} \alpha^{-1}_{c(\gamma)}(b^\gamma)\alpha^{-1}_{c(\gamma)}(h(\gamma\eta^{-1}))\alpha^{-1}_{c(\eta)}(b^\eta).
\]

Now as the choice of elements \( b^\gamma \) was arbitrary, the matrix \( \left[ \alpha^{-1}_{c(\gamma)}(h(\gamma\eta^{-1})) \right]_{\gamma, \eta \in F} \) is positive in \( M_{|F|}(M(\mathcal{A}_x)) \).

(2) \( \Rightarrow \) (1) Since \( h \) is bounded we may normalize it so that it is a strictly continuous section of \( M(\mathcal{A} \ast_r G) \) bounded by 1. Fix \( x \in G^{(0)} \) and choose a section \( c : G_x \to \Sigma_x \subseteq \Sigma \). By Lemma 3.5 the formula \( k_x(\gamma, \eta) = \alpha_{c(\gamma)}^{-1}h(\gamma\eta^{-1}) \), \( \gamma, \eta \in G_x \) defines a positive-definite kernel \( k_x : G_x \times G_x \to M(\mathcal{A}_x) \). Hence by [Mur, Theorem 2.3] there exists a Hilbert \( \mathcal{A}_x \)-module \( H_x \) and a map \( \zeta : G_x \to \mathcal{L}(\mathcal{A}_x, H_x) \) (with values in the unit ball) such that for each \( \kappa, \beta \in G_x \) we have

\[
\alpha^{-1}_{c(\kappa)}(h(\kappa\beta^{-1})) = \zeta(\kappa)^*\zeta(\beta).
\]

Note that we can view \( \mathcal{L}(\mathcal{A}_x, H_x) \) as a Hilbert module \( \tilde{H}_x \) over \( M(\mathcal{A}_x) \simeq \mathcal{L}(\mathcal{A}_x) \), cf. [RaT, Proposition 1.2].

The isomorphisms \( A_{\mathcal{H}(\gamma)} \ni a_\gamma \mapsto [a_\gamma, c(\gamma)] \in B_\gamma \), extended to direct sums, allow us to identify the right \( \mathcal{A}_x \)-module \( V_x = \bigoplus_{\gamma \in G_x} B_\gamma \) (described in the construction of the regular representation) with the \( \mathcal{A}_x \)-module \( \bigoplus_{\gamma \in G_x} A_{\mathcal{H}(\gamma)} \) with operations given by

\[
\left( \bigoplus_{\gamma \in G_x} a_\gamma \right) \cdot a = \bigoplus_{\gamma \in G_x} a_\gamma \alpha_{c(\gamma)}(a), \quad \left( \bigoplus_{\gamma \in G_x} b_\gamma, \bigoplus_{\gamma \in G_x} d_\gamma \right)_{\mathcal{A}_x} = \sum_{\gamma \in G_x} \alpha^{-1}_{c(\gamma)}(b^\gamma d_\gamma).
\]

We equip \( V_x \) with a natural left action \( \phi_x : M(\mathcal{A}_x) \to \mathcal{L}(V_x) \), given by the formula:

\[
\phi_x(m) \cdot \bigoplus_{\gamma \in G_x} a_\gamma := \bigoplus_{\gamma \in G_x} \alpha_{c(\gamma)}(m) a_\gamma, \quad m \in M(\mathcal{A}_x), \bigoplus_{\gamma \in G_x} a_\gamma \in V_x.
\]

Thus we can form a new \( \mathcal{A}_x \)-Hilbert module \( W_x = \tilde{H}_x \otimes_{M(\mathcal{A}_x)} V_x \). Consider the map \( \Theta_x : V_x \to W_x \) given by

\[
\Theta_x \left( \bigoplus_{\gamma \in G_x} b_\gamma \right) = \bigoplus_{\gamma \in G_x} \zeta(\gamma) \otimes b_\gamma.
\]

One readily verifies that \( \Theta_x \) is a contraction and we can define a completely positive contractive map \( \Psi_x : B(V_x) \to B(V_x) \) by the formula

\[
\Psi_x(T) = \Theta_x^*(1_{\mathcal{H}_x} \otimes T)\Theta_x, \quad T \in B(V_x).
\]

The formula (2.3) allows us to check that we have

\[
\pi_x(m_h f) = \Psi_x(\pi_x(f)), \quad f \in C_c(\mathcal{B}).
\]

Indeed, by linearity and continuity, to establish the above equality it suffices to verify that for each \( \eta, \gamma \in G_x \) and \( b_\gamma \in B_\gamma, d_\eta \in B_\eta \) we have \( \langle b_\gamma, \pi_x(m_h f)d_\eta \rangle_{\mathcal{A}_x} = \langle b_\gamma, \Psi_x(\pi_x(f))d_\eta \rangle_{\mathcal{A}_x} \).

Using (3.5) we get \( \langle m_h f(\gamma\eta^{-1})d_\eta = \phi_x((\zeta(\gamma), \zeta(\eta))_{\mathcal{A}_x})f(\gamma\eta^{-1})d_\eta \). Therefore, using the
definition of the internal tensor product of Hilbert modules and of the representation \( \pi_x \) we obtain

\[
\langle b_\gamma, (m_h f)(\gamma \eta^{-1}) d_\eta \rangle_{A_x} = \langle b_\gamma, \phi_x((\zeta(\gamma), \zeta(\eta))_{M(A_x)}) \cdot (f(\gamma \eta^{-1}) \cdot d_\eta) \rangle_{A_x}
\]

\[
= \langle \zeta(\gamma) \otimes b_\gamma, \zeta(\eta) \otimes \pi_x(f) d_\eta \rangle_{A_x}
\]

\[
= \langle \Theta_x(b_\gamma), (I \otimes \pi_x(f))\Theta_x(d_\eta) \rangle_{A_x}
\]

\[
= \langle b_\gamma, \Psi_x(\pi_x(f))d_\eta \rangle_{A_x}.
\]

Now, as the direct sum of the representations \( \pi_x \) is faithful on \( C^*_r(B) \), (3.6) implies that

the map \( m_h \) is contractive and completely positive. \( \square \)

A very similar proof gives also the following result: note that we allow below arbitrary \( G \)-bundles, but consider only scalar multipliers.

**Proposition 3.7.** Let \( h : G \rightarrow \mathbb{C} \) be a bounded continuous function and let \( \mathcal{B} \) be a \( G \)-bundle. Consider the following conditions:

(i) \( h \) is positive-definite;

(ii) \( h \) is a completely positive scalar \( \mathcal{B} \)-multiplier.

Then (i) implies (ii), and if \( \mathcal{B} \) is “strongly saturated” in the sense that for every \( \gamma \in G \) there is an approximate unit for \( B_{s(\gamma)} \) contained in \( (B_\gamma, B_\gamma)_{B_{s(\gamma)}} \), then (ii) implies (i). Thus

(i) \( \Leftrightarrow \) (ii) for continuous line bundles and for bundles coming from twisted groupoid actions.

**Remark 3.8.** If we consider a transformation groupoid \( G := \Gamma \times X \) (where a discrete group \( \Gamma \) acts on a compact space \( X \)), and consider the trivial line bundle over \( G \), then a function \( h \in C(G) \) can be canonically identified with a map \( \tilde{h} : \Gamma \rightarrow C(X) \). The positive-definiteness condition for \( h \) reads then as follows: for any \( x \in X \) and a finite set \( F \subseteq \Gamma \) we have that the matrix

\[
\left[ \tilde{h}(f^{-1}g)(f \cdot) \right]_{f,g \in F} \in M_{|F|}(C(X))
\]

is positive. This is the condition appearing for example in [DoR, Equation (3.2)] (up to a change of convention) or in [BC2, Definition 4.2]; see also a discussion in [MSTT, Section 2].

4. **The Haagerup trick for twisted groupoid dynamical systems**

Throughout the section we fix a twisted groupoid \( C^* \)-dynamical system \((\mathcal{A}, G, \Sigma, \alpha)\) where \( G \) is an étale locally compact Hausdorff groupoid. We let \( \mathcal{B} \) be the Fell bundle associated to it and consider the following \( C^* \)-inclusion

\[
A := C_0(\mathcal{A}) \subseteq B = C^*_r(\mathcal{A}, G, \Sigma, \alpha)
\]
equipped with the canonical faithful conditional expectation \( E : B \rightarrow A \). In Proposition 3.6 we have seen that strictly continuous positive-definite sections of \( M(\mathcal{A}^* \mathcal{B}) \) give rise to completely positive maps on \( B \). In this section we prove the converse. To this end, we develop a generalisation of ‘the Haagerup Trick’, cf. [Haa2, Lemma 2.5]. To avoid technicalities, and also because the approximation properties we want to study in sequel sections seem to be designed for the unital context, from now on we will assume here that
the bundle $\mathcal{A}$ has a continuous unital section, i.e. each fiber $A_x$, $x \in G^{(0)}$ is unital and the unit section $G^{(0)} \ni x \mapsto 1_x \in A_x \subseteq \mathcal{A}$ is continuous. Here and below $1_x$ denotes the unit in the algebra $A_x$, $x \in G^{(0)}$. Our standing assumption implies that $A$ is unital if and only if $G^{(0)}$ is compact. It also implies that $M(\mathcal{A}_*, G) = \mathcal{A}_*, G$ and for sections of this bundle strict continuity is just continuity. Further we may assume that $C_0(G) \subseteq Z(A)$ by identifying functions in $C_0(G)$ with their product by the unit section.

In the following proposition we use the notation introduced in Lemma 2.8 (see also Lemma 2.14), we denote by $\mathcal{B}$ the Fell bundle associated to $(\mathcal{A}, G, \Sigma, \alpha)$ in Example 2.5, and use the identification $C^*_r(\mathcal{B}) \subseteq C_0(\mathcal{B})$ from Remark 2.12.

**Proposition 4.1** (Haagerup Trick). Suppose $\Phi : B \to B$ is a bounded right $A$-module map. Fix an element $a \in A$. For each $\gamma \in G$ choose a bisection $U \in \Sigma$ containing $\gamma$ and a section $f \in C_c(\mathcal{A}|_{r(U)})$ which attains units at a neighborhood of $r(\gamma)$, and define

$$f_\gamma(\eta) := [f(r(\eta)), c_U(\eta)], \quad \eta \in U.$$  

Then $f_\gamma \in C_c(\mathcal{B}|_U) \subseteq B$ and we define

$$h^\Phi(\gamma) := \Phi(f_\gamma)(\gamma)f_\gamma(\gamma)^* = E(\Phi(f_\gamma)f^*_\gamma)(r(\gamma)) \in A_{r(\gamma)}.$$  

The formula above does not depend on the choice of $U$, $f$ and $c_U$, and defines a bounded continuous section $h^\Phi$ of $A_*, G$ (we have $\|h^\Phi\|_\infty \leq \|\Phi\|$). Moreover, if $\Phi$ is completely positive then $h^\Phi$ is positive-definite.

**Proof.** The element $f_\gamma$ corresponds to the section $f$ under the isomorphism $C_0(\mathcal{B}|_U) \cong C_0(\mathcal{A}_*, G|_U) \cong C_0(\mathcal{A}|_{r(U)})$ from Lemma 2.14. In particular, $f_\gamma \in C_c(\mathcal{B}|_U) \subseteq B$. To see that the definition of $h^\Phi(\gamma)$ does not depend on the choice of the section $c_U : U \to \Sigma$, let $\gamma'_U : U \to \Sigma$ be another continuous section that trivializes the $\mathbb{T}$-bundle, and put $f'_\gamma(\eta) := [f(r(\eta)), c'_U(\eta)], \eta \in G$. Then $c(r(\eta)) := c'_U(\eta)c_U(\eta)^{-1}, \eta \in U$, defines a continuous section of $\Sigma|_{r(U)} \cong r(U) \times \mathbb{T} \to r(U)$. Thus identifying $c(x)$ with the uniquely determined element of $\mathbb{T}$ we see that $s(U) \ni s(\eta) \mapsto u(s(\eta))(c(r(\eta)))) \in A_x \subseteq A$ is also a continuous section. Let $h \in C_0(s(U))$ be any function which is equal to one on $s(s(f_\gamma))$. Then $u(s(\eta)) := h(s(\eta))u(s(\eta))(c(r(\eta))))$, $\eta \in U$, defines an element of $C_0(\mathcal{A}|_{s(U)}) \subseteq A$. For any $\eta \in \text{supp}(f_\gamma) \subseteq U$ we have

$$f'_\gamma(\eta) = [f(r(\eta)), c'_U(\eta)] = [f(r(\eta)), c(r(\eta))c_U(\eta)] = [f(r(\eta))u(r(\eta))(c(r(\eta))), c_U(\eta)] = [f(r(\eta))\alpha_{c_U(\eta)}(u(s(\eta)(c(r(\eta)))), c_U(\eta)] = (f, u(\eta)).$$

Hence $f'_\gamma = f_\gamma u$ where $u \in A$ is such that $uu^* \eta$ is the unit section on $s(s(f_\gamma))$. As $\Phi$ is a right $A$-module map and $(uu^*)(x) = 1_x$ for all $x \in \text{supp}(f_\gamma)$, we obtain

$$\Phi(f'_\gamma)(f'_\gamma)^* = \Phi(f_\gamma)uu^*f^*_\gamma = \Phi(f_\gamma)f^*_\gamma.$$  

Independence of $h^\Phi(\gamma)$ from the choice of $U$ will follow once we show its independence from the choice of $f$. To show the latter, let $f' \in C_c(\mathcal{A}|_{r(U')}), U' \in \Sigma$, be another section which is unital at a neighborhood of $r(\gamma)$, and put $(f'_\gamma)(\eta) := [f'(r(\eta)), c_U(\eta)]$. Then
\( f_t|_V = f'_t|_V \) for some neighborhood \( V \) of \( \gamma \) (find \( V \) such that both \( f \) and \( f' \) are unital on \( r(V) \subseteq r(U) \cap r(U') \)). Let \( h \in C_v(s(V)) \) be such that \( h(s(\gamma)) = 1 \). We may treat \( h \) as an element of \( A \) by multiplying it by the unit section of \( A \). Then \( f_t h = f'_t h \) and using that \( \Phi \) is a right \( A \)-module map we get

\[
\Phi(f_\gamma)(\gamma)(f_\gamma)^*(\gamma) = \Phi(f_\gamma)(\gamma)h(s(\gamma))h(s(\gamma))(f_\gamma^*(\gamma)) = \Phi(f'_\gamma h)(\gamma)(f'_\gamma h)^*(\gamma)
\]

\[
= \Phi(f'_\gamma h)(\gamma)(f'_\gamma h)^*(\gamma) = \Phi(f'_\gamma)(\gamma)h(s(\gamma))h(s(\gamma))(f'_\gamma)^*(\gamma)
\]

\[
= \Phi(f'_\gamma)(\gamma)(f'_\gamma)^*(\gamma).
\]

Hence the formula (4.7) gives a well defined section \( h^\Phi \) of \( A \ast_r G \); the fact that we also have the expression involving the conditional expectation follows as \( f_\gamma \) is supported on a bisection. For any \( \gamma \in G \) the section \( f_\gamma \) is given by \( f \) with \( f(r(\eta)) = 1_{r(\eta)} \) for all \( \eta \in V \) for some open neighbourhood \( V \) of \( \gamma \). Hence by the independence on the choice of a right \( f \) established above we have \( h^\Phi(\eta) = E(\Phi(f_\gamma)f_\gamma^*)(r(\eta)) \) for any \( \eta \in V \). As \( E(\Phi(f_\gamma)f_\gamma^*) \in \gamma c(0(A)) \) is continuous, this implies that \( h^\Phi \) is continuous at \( \gamma \). Also we may choose \( f \) so that \( \|f\| = 1 \), equivalently \( \|f_\gamma\| = 1 \), and this implies that \( \|h^\Phi(\gamma)\| = \|E(\Phi(f_\gamma)f_\gamma^*)(r(\gamma))\| \leq \|\Phi\| \). Thus the section \( h^\Phi \) is bounded by \( \|\Phi\| \).

Now suppose that \( \Phi \) is completely positive. To see that \( h^\Phi \) is positive-definite fix \( x \in G^{(0)} \) and a finite set \( F \subseteq G \). For each \( \gamma \in F \) choose \( f_\gamma \) as in (I.7) supported on \( U_\gamma \subseteq S \). Define a section \( c : F \to \Sigma \) by \( c(\gamma) := c_{U_\gamma}(\gamma), \gamma \in F \). Note that for any \( \eta, \gamma \in F \), the product \( f_\gamma f_\eta^* \) is supported on an open bisection in \( S \) and it is the unit section on some neighbourhood of \( \gamma \eta^{-1} \). Hence

\[
h^\Phi(\gamma \eta^{-1}) = \Phi(f_\gamma f_\eta^*)(\gamma \eta^{-1})(f_\gamma f_\eta^*)(\gamma \eta^{-1})^* = \Phi(f_\gamma f_\eta^*)(\gamma \eta^{-1})f_\eta(\gamma) f_\gamma(\gamma)^*.
\]

A simple computation using the formulas from Example 2.5 and the fact that \( f_\gamma(\gamma) = [1_{A_r(\gamma)}, c_{U}(\gamma)] \) shows that \( f_\gamma(\gamma)^* h^\Phi(\gamma \eta^{-1}) f_\gamma(\gamma) = \alpha_{c(\gamma)^{-1}}(h^\Phi(\gamma \eta^{-1})) \) and \( f_\gamma(\gamma)^* f_\gamma(\gamma) = 1_{s(\gamma)} \). Thus we get

\[
\alpha_{c(\gamma)^{-1}}(h^\Phi(\gamma \eta^{-1})) = f_\gamma(\gamma)^* \Phi(f_\gamma f_\eta^*)(\gamma \eta^{-1}) f_\eta(\gamma) = E (f_\gamma^* \Phi(f_\gamma f_\eta^*) f_\eta)(x),
\]

where in the last equality we use again the fact that both \( f_\gamma \) and \( f_\eta \) are supported on bisections. Thus the matrix

\[
\left[ \alpha_{c(\gamma)^{-1}}(h^\Phi(\gamma \eta^{-1})) \right]_{\eta, \gamma \in F} = \left[ E (f_\gamma^* \Phi(f_\gamma f_\eta^*) f_\eta)(x) \right]_{\eta, \gamma \in F}
\]

is positive as \( \Phi \) and \( E \) are completely positive. \( \square \)

**Remark 4.2.** In the case of a groupoid dynamical system \((A, G, \alpha)\) (without a twist) we may replace \( B \) by \( A \ast_r G \) and then the map (4.7) induced by \( \Phi : B \to B \) could be defined by \( h^\Phi(\gamma) = \Phi(f)(\gamma) \) where \( f \in C_v(A \ast_r G | \gamma) \) is supported on an open bisection \( U \) and such that \( f(\eta) = 1_{r(\eta)} \) on some neighbourhood of \( \gamma \). We also invite the reader to check that when the crossed product in question is just the usual group \( C^* \)-algebra (i.e. \( G \) is a discrete group and the bundle \( A \) is trivial) the formula (4.7) reduces to the one from the classical Haagerup trick.
Remark 4.3. Given any bounded mapping $\mathcal{B}$-multiplier $h \in C_b(\mathcal{A} \ast_r G)$ the map $m_h : B \to B$ is a bounded right $A$-module map and $h^{m_h} = h$. But for a general bounded right $A$-module $\Phi : B \to B$ it may happen that $m_h \Phi \neq \Phi$.

We now wish to characterise when a section obtained by the Haagerup Trick is of $C_0$ or $C_c$ class. To this end, we use the right Hilbert $A$-module $B_E$ associated to the canonical expectation $E : B \to A$. Thus

$$B_E := \mathcal{B}^{\| \cdot \|_E}, \quad \langle x, y \rangle_E := E(x^* y), \quad x, y \in B \subseteq B_E.$$ 

As is common, we will write $\mathcal{L}(B_E)$ for the algebra of all adjointable operators on $B_E$, for $\xi, \eta \in B_E$ the symbol $\Theta_{\xi, \eta} \in \mathcal{L}(B_E)$ will denote the obvious ‘rank-one operator’ and $\mathcal{K}(B_E)$ will denote the closed linear span of rank-one operators inside $\mathcal{L}(B_E)$ (see for example [Lan]). Denoting by $\| \cdot \|_E$ the norm in $B_E$, by $\| \cdot \|_\infty$ the supremum norm on $C_0(\mathcal{B})$ and by $\| \cdot \|_r$ the norm in $B = C^*_r(\mathcal{B})$ we have $\| f \|_\infty \leq \| f \|_E \leq \| f \|_r$ for all $f \in C_c(\mathcal{B})$. In particular, we may assume the identifications

$$C_c(\mathcal{B}) \subseteq B \subseteq B_E = \{ f \in C_0(\mathcal{B}) : \sup_{x \in G^{(0)}} \| \sum_{\gamma \in G_x} f^*(\gamma) f(\gamma) \|_{A x} < \infty \} \subseteq C_0(\mathcal{B}),$$

cf. Remark 2.12.

In general, for an element $h \in C_b(\mathcal{A} \ast_r G)$ the formula (3.4), does not define a bounded map $m_h$ on $B = C^*_r(\mathcal{B})$, and a bounded right $A$-module map $\Phi : B \to B$ does not need to extend to a map $\tilde{\Phi} \in \mathcal{L}(B_E)$. This can be seen already if one considers $A$ to be trivial and $G$ to be a group, so that we are in the context of Herz-Schur multipliers. However, we have the following result.

Proposition 4.4. The space $C_b(\mathcal{A} \ast_r G)$ of continuous bounded sections equipped with pointwise operations and supremum norm is a $C^*$-algebra. We have an injective $*$-homomorphism $C_b(\mathcal{A} \ast_r G) \ni h \mapsto \tilde{m}_h \in \mathcal{L}(B_E)$ where

$$\tilde{m}_h(a)(\gamma) := h(\gamma) \cdot a(\gamma), \quad a \in C_c(\mathcal{B}) \subseteq B_E, \gamma \in G.$$ 

Moreover, if $h \in C_b(\mathcal{A} \ast_r G)$ then

1. $h \in C_0(\mathcal{A} \ast_r G)$ if and only if $\tilde{m}_h \in \mathcal{F}(C_c(\mathcal{B})) := \text{span} \{ \Theta_{f,g} : f, g \in C_c(\mathcal{B}) \}$.
2. $h \in C_0(\mathcal{A} \ast_r G)$ if and only if $\tilde{m}_h \in \mathcal{K}(B_E)$.

Similarly, if $\Phi : B \to B$ is a bounded right $A$-module map that extends to a bounded map $\tilde{\Phi} : B_E \to B_E$, then we have

1. $h^\# \in C_c(\mathcal{A} \ast_r G)$ if and only if $\tilde{\Phi} \in \mathcal{F}(C_c(\mathcal{B}))$.
2. $h^\# \in C_0(\mathcal{A} \ast_r G)$ if and only if $\tilde{\Phi} \in \mathcal{K}(B_E)$.

Proof. The first claim is immediate. Let $h \in C_b(\mathcal{A} \ast_r G)$. Clearly, (4.9) defines a map $\tilde{m}_h : C_c(\mathcal{B}) \to C_c(\mathcal{B})$. Let $a, b \in C_c(\mathcal{B})$ and $x \in G^{(0)}$. Since $E(a^* b)(x) = (a^* b)(x) = \sum_{\gamma \in G_x} a(\gamma)^* b(\gamma)$, we get

$$E(\tilde{m}_h(a)^* b)(x) = \sum_{\gamma \in G_x} a(\gamma)^* h(\gamma)^* b(\gamma) = E(a^* \tilde{m}_h^*(b))(x).$$

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That is, \( \langle \tilde{m}_h(a), b \rangle_E = \langle a, \tilde{m}_h(b) \rangle_E \). Similarly,

\[
E(\tilde{m}_h(a)^*\tilde{m}_h(a))(x) = \sum_{\gamma \in G_x} a(\gamma)^* |h(\gamma)|^2 a(\gamma) \leq \|h\|^2 \sum_{\gamma \in G_x} a(\gamma)^* a(\gamma) = \|h\|^2 E(a^* a)(x).
\]

This implies that \( \langle \tilde{m}_h(a), \tilde{m}_h(b) \rangle_E \leq \|h\| \langle a, a \rangle_E \) in \( A \), and hence \( \|\tilde{m}_h(a)\|^2_E \leq \|h\|^2 \|a\|^2_E \). So \( \tilde{m}_h \) is bounded, and \( \|\tilde{m}_h\| \leq \|h\| \). Accordingly, \( \tilde{m}_h : C_c(B) \to C_c(B) \) extends to a bounded map on \( B_E \). By \( \{1, 4\} \) this map is adjointable, with the adjoint equal to \( \tilde{m}_h \).

Thus the map \( C_b(A*, G) \ni h \to \tilde{m}_h \in L(B_E) \) is well defined and \(*\)-preserving. By \( \{4, 9\} \) it is immediate that this map is also linear, multiplicative and injective.

\( \{1\} \) and \( \{3\} \): To show that \( h \in C_c(A*, G) \) implies \( \tilde{m}_h \in \mathcal{F}(C_c(B)) \) it suffices to consider the case when \( h \) is supported on an open bisection \( U \in S \) where \( S \) is as in Lemma 2.8 cf. also Lemma 2.14. Using that \( C_0(A^r|U) \) is a \( C^* \)-algebra and \( r \) induces isomorphisms \( B|U \cong A|U \) and \( C_c(B|U) \cong C_c(A|U) \), we can find elements \( f, g \in C_c(A^r G|U) \) such that \( h(\gamma) = f(\gamma) g(\gamma)^* \) in \( A_r(\gamma) \) for every \( \gamma \in G \). Then for any \( a \in C_c(A^r G|V), V \subset S \), and \( \gamma \in U \cap V \) we get

\[
h(\gamma)[a](\gamma) = [f(\gamma) g(\gamma)^* a(\gamma), c_U(\gamma)] = [f(\gamma), c_U(\gamma)] [a_{c_U(\gamma)}^{-1} g(\gamma)^* a(\gamma), s(\gamma)] = [f(\gamma)[g, [a]](\gamma), a(s(\gamma))] = (\Theta_{[f], [g]}[a])(\gamma).
\]

Hence \( \tilde{m}_h \) is \( \Theta_{[f], [g]} \in \mathcal{F}(C_c(G)) \).

Suppose now that \( h \in C_b(A^r, G) \) is any section such that \( \tilde{m}_h \in \mathcal{F}(C_c(B)) \) or that \( h = h^\Phi \) where \( \Phi \in \mathcal{F}(C_c(B)) \). Then either \( \tilde{m}_h \) or \( \Phi \) is of the form \( T = \sum_{i=1}^n \Theta_{f_i, g_i} \) for \( f_i \in C_c(B|U_i), g_i \in C_c(B|V_i) \), where \( U_i, V_i \subset G \) are open bisections, \( i = 1, \ldots, n \). For each \( \gamma \in G \) choose \( f_\gamma \) as in Proposition 4.4. Then

\[
(T f_\gamma)(\gamma) = \left( \sum_{i=1}^n f_i : E(g_i^* \cdot f_\gamma) \right)(\gamma) = \sum_{i=1}^n f_i(\gamma) g_i(\gamma)^* f_\gamma(\gamma) = \left( \sum_{i=1}^n f_i(\gamma) g_i(\gamma)^* \right) f_\gamma(\gamma).
\]

If \( T = \tilde{m}_h \) we get \( h(\gamma) f_\gamma(\gamma) = (\sum_{i=1}^n f_i(\gamma) g_i(\gamma)^*) f_\gamma(\gamma) \). And if \( T = \tilde{\Phi} \) we get \( h(\gamma) = h^\Phi(\gamma) = (\sum_{i=1}^n f_i(\gamma) g_i(\gamma)^*) \). In both cases, we conclude that \( h(\gamma) \neq 0 \) if and only if \( \sum_{i=1}^n f_i(\gamma) g_i(\gamma)^* \neq 0 \), so the support of \( h \) is compact, that is \( h \in C_c(A^r, G) \).

\( \{2\} \) and \( \{4\} \): By what we have proved \( C_c(A^r G) \ni h \to \tilde{m}_h \in \mathcal{F}(C_c(B)) \subset \mathcal{K}(B_E) \) is a contractive linear map. Hence it extends uniquely to a contractive map \( C_0(A^r, G) \ni h \to \tilde{m}_h \in \mathcal{K}(B_E) \), and it is easy to check that this extension is compatible with the formula defining the map in Proposition 4.4. Thus if \( h \in C_0(A^r, G) \), then \( \tilde{m}_h \in \mathcal{K}(B_E) \).

Conversely, assume that \( h \in C_b(A^r, G) \) is such that \( \tilde{m}_h \in \mathcal{K}(B_E) \) or that \( h = h^\Phi \) where \( \Phi \in \mathcal{K}(B_E) \). Let \( T \in \mathcal{K}(B_E) \) stand either for \( \tilde{m}_h \) or for \( \tilde{\Phi} \) and let \( \varepsilon > 0 \). Since \( \mathcal{F}(C_c(B)) = \mathcal{K}(B_E) \), there is \( R = \sum_{i=1}^n \Theta_{f_i, g_i} \) with \( f_i \in C_c(B|U_i), g_i \in C_c(B|V_i) \), where \( U_i, V_i \subset S, i = 1, \ldots, n \), and \( \|T - R\|_{\mathcal{L}(B_E)} < \varepsilon/2 \). It suffices to show that the closed set \( \{ \gamma \in G : \|h(\gamma)\| \geq \varepsilon \} \) is contained in the following compact subset of \( G \):

\[
K := \bigcup_{i=1}^n (s|V_i)^{-1}(s(\text{supp } f_i)).
\]
To see this, note that for any $\gamma \notin K$ we may find $U$ and $f_\gamma \in C_c(B|U)$ as in Proposition 4.4 such that $U \subseteq G \setminus K$ and $\|f_\gamma\|_\infty = 1$. Then $\|f_\gamma\|_E^2 = \|E(f_\gamma^* f_\gamma)\|_E = \|f_\gamma^* f_\gamma\|_E = 1$. Moreover, $R(f_\gamma) = 0$, as for any $\eta \in G$,

$$(R(f_\gamma))(\eta) = \sum_{i=1}^n f_i(\eta)g_i(s|_{V_i}(s(\eta))) = 0.$$ 

If $T = \Phi$ then $\|h^B(\gamma)\| = \|\Phi(f_\gamma(\gamma)f_\gamma(\gamma)^*)\| \leq \|\Phi(f_\gamma(\gamma))\| \leq \|\Phi(f_\gamma(\gamma))\|_E = \|\Phi(f_\gamma) - R(f_\gamma)\|_E < \varepsilon$. If $T = \tilde{m}_h$, then $\|h(\gamma)\| = \|h(\gamma)^* h(\gamma)\| \leq \|\tilde{m}_h(f_\gamma)\|_E = \|\tilde{m}_h(f_\gamma) - R(f_\gamma)\|_E < \varepsilon$. Thus in both cases we get that $\{\gamma \in G : \|h^B(\gamma)\| \geq \varepsilon\} \subseteq K$. Hence $h \in C_0(B)$.

The uniform convergence on compact subsets of $G$ can be rephrased in terms of pointwise convergence in the $\| \cdot \|_E$ norm.

**Lemma 4.5.** A net $(h_i)_{i \in I} \subseteq C_b(A^*,G)$ converges to the unit section uniformly on compact sets, i.e. $\sup_{\gamma \in K} \|h_i(\gamma) - 1_{r(\gamma)}\| \to 0$ for every compact $K \subseteq G$, if and only if $(\tilde{m}_{h_i})_{i \in I}$ converges pointwise on $C_c(B) \subseteq B_E$ to the identity operator (so if $(\tilde{m}_{h_i})_{i \in I}$ is bounded it converges pointwise to the identity operator on $B_E$). Also if $(\Phi_i)_{i \in I}$ is a net of bounded $A$-bimodule maps on $C^*_r(B)$ which converges to identity pointwise on $C_c(B) \subseteq C^*_r(B)$ in the $\| \cdot \|_E$ norm, then $(\Phi_i^*)_{i \in I}$ converges to the unit section uniformly on compact subsets.

**Proof.** Suppose that the net $(h_i)_{i \in I}$ as above converges to the unit section uniformly on compact sets. To show that $\lim_{i \in I} \|\tilde{m}_{h_i}(b) - b\|_E = 0$ for $b \in C_c(B)$ we may assume that $b \in C_c(B|U)$ where $U \subseteq S$ is an open bisection. Thus there is $a \in C_c(A^*,G|U)$ such that $b(\gamma) = [a(\gamma), c_U(\gamma)]$ for all $\gamma \in G$. Since $K := \{\gamma : b(\gamma) \neq 0\}$ is compact we then get

$$\|\tilde{m}_{h_i}(b) - b\|_E^2 = \sup_{x \in G^{(0)}} \|((\tilde{m}_{h_i}(b) - b)^*(\tilde{m}_{h_i}(b) - b))(x)\|$$

$$= \sup_{\gamma \in K} \|a(\gamma)^* h_i(\gamma) - 1_{r(\gamma)}(h_i(\gamma) - 1_{r(\gamma)}) a(\gamma)\|$$

$$\leq \sup_{\gamma \in K} \|a(\gamma)^* a(\gamma)\| \|h_i(\gamma) - 1_{r(\gamma)}\|^2$$

$$\leq \|a\|^2_\infty \sup_{\gamma \in K} \|h_i(\gamma) - 1_{r(\gamma)}\|^2 \to 0.$$ 

Now suppose conversely that $(\tilde{m}_{h_i})_{i \in I}$ converges pointwise on $C_c(B) \subseteq B_E$ to the identity operator, or that $(h_i)_{i \in I} = (h^B_i)_{i \in I}$ where $(\Phi_i)_{i \in I}$ converges to identity pointwise on $C_c(B) \subseteq C^*_r(B)$ in the $\| \cdot \|_E$ norm. Let $K \subseteq G$ be a compact set. We need to show that $\sup_{\gamma \in K} \|h(\gamma) - 1_{r(\gamma)}\| \to 0$. There is a finite open cover $\{V_j\}_{j=1}^n$ of $K$ such that $\overline{V}_j$ is compact and contained in an open bisection $U_j \subseteq S$, for each $j = 1, \ldots, n$. Accordingly, there are sections $f_j \in C_c(B|U_j)$ such that $f_j(\gamma) = [1_{r(\gamma)}, c_{U_j}(\gamma)]$ for $\gamma \in \overline{V}_j$ and $j = 1, \ldots, n$. If $(h_i)_{i \in I} = (h^B_i)_{i \in I}$, then

$$\sup_{\gamma \in \overline{V}_j} \|h^B(\gamma) - 1_{r(\gamma)}\| = \sup_{\gamma \in \overline{V}_j} \|\Phi_i(f_j)(\gamma)f_j(\gamma)^* - f_j(\gamma)f_j(\gamma)^*\|$$

$$\leq \sup_{\gamma \in \overline{V}_j} \|\Phi_i(f_j(\gamma)) - f_j(\gamma)\| \leq \|\Phi_i(f_j) - f_j\|_E \to 0.$$
In the remaining case we have \( \sup_{\gamma \in V_j} \| h_i(\gamma) - 1_{r(\gamma)} \| = \sup_{\gamma \in V_j} \| \tilde{m}_h(f_j)(\gamma) - f_j(\gamma) \| \leq \| \tilde{m}_h(f_j) - f_j \|_E \to 0 \). Thus in both cases \( \sup_{\gamma \in V_j} \| h_i(\gamma) - 1_{r(\gamma)} \| \leq \max_{j=1,\ldots,n} \sup_{\gamma \in V_j} \| h(\gamma) - 1_{r(\gamma)} \| \to 0 \). □

5. The Haagerup property for twisted groupoid dynamical systems

In this section we analyse the relations between the geometric Haagerup property for groupoids, based on conditions introduced in [Tu], and its \( C^* \)-algebraic counterpart from [DoR], working in the context of general twisted groupoid \( C^* \)-dynamical systems. For second countable groups the Haagerup property can be formulated using various equivalent conditions, see [CCJJV, Definition 1.1.1]. These conditions can be naturally generalised to \( \text{étale} \) topological groupoids. However, as we explain below, the relationship between some of them is subtle when the space of units is not compact.

**Definition 5.1** ([Tu]). A **locally proper negative type function** on an \( \text{étale} \) groupoid \( G \) is a continuous function \( \psi : G \to \mathbb{R} \) which is

(i) normalised: \( \psi|_{G^{(0)}} = 0 \);

(ii) symmetric: \( \psi(g) = \psi(g^{-1}) \) for all \( g \in G \);

(iii) conditionally negative-definite, i.e. for each \( x \in G^{(0)}, \, n \in \mathbb{N}, \, g_1, \ldots, g_n \in G_x \) and \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) such that \( \sum_{i=1}^{n} \lambda_i = 0 \) we have

\[
\sum_{i,j=1}^{n} \lambda_i \psi(g_j g_i^{-1}) \lambda_j \leq 0,
\]

(iv) locally proper: the function \( (\psi, r, s) : G \to \mathbb{R} \times G^{(0)} \times G^{(0)} \) is proper.

**Remark 5.2.** Suppose that the groupoid \( G \) is \( \sigma \)-compact. If \( G \) acts properly on a continuous field \( H = (H_x)_{x \in G^{(0)}} \) of affine Euclidean spaces, see [Tu] Definition 3.2, 3.3], then the field admits a continuous section that allows us to write the action as \( \gamma \xi = \alpha_{\gamma}(\xi) + b(\gamma) \) where \( \alpha \) is the linear part and \( \gamma \to b(\gamma) \in H_{r(\gamma)} \) is a cocycle. Putting \( \psi(\gamma) := \| b(\gamma) \|^2 \) we obtain a locally proper negative type function. Moreover, every locally proper function of negative type has this form, see [Tu] Proposition 3.8]. Hence, \( G \) admits a locally proper negative type function if and only if \( G \) acts properly on a continuous field of affine Euclidean spaces. In particular, this holds when \( G \) is amenable, by [Tu] Lemma 3.5].

It is standard that if \( h : G \to \mathbb{C} \) is positive-definite then for each \( g \in G \) we have \( h(g) = \overline{h(g^{-1})}, |h(g)| \leq h(r(g)) \). Further if the function as above is real-valued, then to verify positive-definiteness it suffices to show the corresponding condition working only with real coefficients. Normalised symmetric conditionally negative-definite functions take only non-negative values. Finally if \( G^{(0)} \) is compact then a function \( \psi : G \to \mathbb{R} \) as above is locally proper if and only if it is proper in the usual sense (as a function from \( G \) to \( \mathbb{R} \)). We will make an extensive use of countable exhaustions by compact sets, as discussed in the lemma below (which is likely well-known, but we could not locate a reference).
Lemma 5.3. A locally compact Hausdorff space is σ-compact if and only if it admits a countable exhaustion by compact sets, i.e. there is a sequence \((L_n)_{n \in \mathbb{N}}\) of compact sets whose union is the whole space and \(L_n\) is contained in the interior of \(L_{n+1}\), for each \(n \in \mathbb{N}\).

Proof. The ‘if’ part is clear. To show the ‘only if’ part let \((K_n)_{n \in \mathbb{N}}\) be a sequence of compact sets that cover the space. Put \(L_1 := K_1\). By [Rud Rein 2.7], for any open \(U\) containing \(L_1 \cup K_2\) there is an open set \(V\) with compact closure such that \(L_1 \cup K_2 \subseteq V \subseteq \overline{V} \subseteq U\). Put \(L_2 = \overline{V}\). Apply the same fact to \(L_2 \cup K_3\) to find \(L_3\), and proceed by induction. \(\Box\)

The proof of the next fact uses standard ideas, appearing in different setups for example in [A-D] in the setting of measured groupoids, in [Jol] for measured equivalence relations.

Proposition 5.4. Let \(G\) be an étale locally compact Hausdorff groupoid. Consider the following conditions:

(1) \(G\) admits a locally proper negative type function;
(2) there exists a sequence of continuous positive-definite functions \((h_n)_{n \in \mathbb{N}}\) such that
    (a) each \(h_n\) is normalised: \(h_n|_{G^{(0)}} = 1\),
    (b) each \(h_n\) is locally \(C_0\): for each compact set \(K \subseteq G^{(0)}\) we have \(h_n|_{G^K} \in C_0(G^K)
    \) where \(G^K := r^{-1}(K) \cap s^{-1}(K)\),
    (c) the sequence converges to 1 uniformly on compact sets;
(3) there is a net \((h_i)_{i \in I} \subseteq C_0(G)\) of positive-definite functions, with values in the unit disk and converging to 1 uniformly on compact subsets of \(G\).

Then \((1) \Rightarrow (2) \Rightarrow (3)\). If \(G\) is σ-compact, then \((1) \Leftrightarrow (2)\) and a net in \((3)\) may be replaced by a sequence. If \(G\) is σ-compact and \(G^{(0)}\) is a union of compact open sets, then all the conditions \((1), (2), (3)\) are equivalent.

Proof. \((1) \Rightarrow (2)\): Assume that \(\psi\) is a locally proper negative type function, and define

\[ h_n(\gamma) = \exp \left( -\frac{1}{n} \psi(\gamma) \right), \quad n \in \mathbb{N}, \gamma \in G. \]

Then \(h_n\) is continuous and normalised. It is positive-definite by well-known arguments (here we use the fact it is real-valued). Moreover the sequence \((h_n)_{n \in \mathbb{N}}\) converges to 1 (uniformly on compacts, as \(\psi\) is continuous, so bounded on compact sets). It remains to note that each \(h_n\) is locally \(C_0\): without loss of generality we consider \(n = 1\). Fix \(\epsilon \in (0, 1)\) and note that the pre-image of the set \([0, -\log \epsilon] \times K \times K\) with respect to \((\psi, r, s)\) is compact – denote it by \(L\). Then for \(\gamma \in G\) such that \(r(\gamma), s(\gamma) \in K\) we have that \(|h_1(\gamma)| < \epsilon\) if and only if \(\gamma \notin L\).

\((2) \Rightarrow (3)\): Let \((h_n)_{n \in \mathbb{N}}\) be as described in \((2)\). Then it is bounded by 1 as using positive-definiteness we have \(|h(\gamma)| \leq |h(r(\gamma))| = 1\) for \(\gamma \in G\). We can modify \((h_n)_{n \in \mathbb{N}}\), dropping the normalisation condition, so that its members are \(C_0\), and not just locally \(C_0\). To this end, we fix an approximate unit \((f_i)_{i \in I} \subseteq C_0(G^{(0)})^+\) for \(C_0(G^{(0)})\). Then the net \(((f_i \circ r)h_n(f_i \circ s))_{(m,i) \in \mathbb{N} \times I}\) satisfies the conditions in \((3)\); in particular each of the functions is positive-definite and is bounded by 1.

From now on we assume \(G\) is σ-compact. By Lemma 5.3 \(G\) admits a countable exhaustion \((L_n)_{n \in \mathbb{N}}\) by compact sets. In particular, if \((h_i)_{i \in I}\) is a net as in \((3)\) for each \(n \in \mathbb{N}\)
we may choose \( i_n \in I \) so that \( \| h_{i_n} - 1_{L_n} \|_\infty < 1/n \). Then the sequence \( h_{i_n} \) converges to 1 uniformly on compact subsets of \( G \). Hence the net \( (h_i)_{i \in I} \) might be replaced by a sequence.

(2) \( \Rightarrow \) (1): Fix two sequences of strictly positive numbers \( (\alpha_n)_{n \in \mathbb{N}} \) and \( (\epsilon_n)_{n \in \mathbb{N}} \) such that the first one is increasing to infinity, and the series \( \sum_{n \in \mathbb{N}} \alpha_n \epsilon_n \) is convergent. Then proceed as follows: for each \( n \in \mathbb{N} \) find a function \( \phi_n : G \rightarrow \mathbb{C} \) which is positive-definite, normalised, vanishes at infinity, and such that \( |1 - \phi_n(\gamma)| < \epsilon_n \) for \( \gamma \in L_n \) (these are naturally just chosen by picking a subsequence of \( (h_n)_{n \in \mathbb{N}} \)). Then define the function \( \psi : G \rightarrow \mathbb{R} \) by the formula:

\[
\psi(\gamma) = \sum_{n=1}^{\infty} \alpha_n \Re(1 - \phi_n(\gamma)), \quad \gamma \in G.
\]

The series above converges uniformly on compact subsets (as any of these is contained in almost all \( L_n \)), so yields a continuous function, which is conditionally negative-definite as a sum of conditionally negative-definite functions. It is obviously normalised.

It remains to check that it is locally proper. Another exhaustion argument (for example) guarantees that any compact subset of \( \mathbb{R} \times G(0) \times G(0) \) is contained in a set of the form \([-M, M] \times K \times K \) for some \( M > 0 \) and \( K \subseteq G(0) \) compact. Fix then such \( M \) and \( K \). Let \( k \in \mathbb{N} \) be such that \( \sum_{n=1}^{k} \alpha_n > 2M \) and for each \( n = 1, \ldots, k \) find a compact set \( L_n \subseteq G \) such that for \( \gamma \in G \setminus L_n \) such that \( r(\gamma), s(\gamma) \in K \) we have \( |\phi_n(\gamma)| \leq \frac{1}{2} \). Then put \( L = \bigcup_{n=1}^{k} L_n \). If \( \gamma \in G \setminus L \), \( r(\gamma), s(\gamma) \in K \), then

\[
\psi(\gamma) \geq \sum_{n=1}^{k} \alpha_n \Re(1 - \phi_n(\gamma)) \geq \sum_{n=1}^{k} \alpha_n (1 - |\phi_n(\gamma)|) \geq M.
\]

Thus the pre-image of the set \([-M, M] \times K \times K \) with respect to \( (\psi, r, s) \) is contained in \( L \). This ends the proof of (2) \( \Rightarrow \) (1).

Assume now that \( G \) is \( \sigma \)-compact and \( G(0) \) is a union of compact open sets. We will then show that (3) \( \Rightarrow \) (1). There is an exhaustion \( (L_n)_{n \in \mathbb{N}} \) of \( G \) such that \( r(L_n) = s(L_n) \) and this set, denoted by \( K_n \), is compact open in \( G(0) \), for all \( n \in \mathbb{N} \). Indeed, our assumptions clearly imply that there is an exhaustion \( (K_n)_{n \in \mathbb{N}} \) of \( G(0) \) by compact open sets and an exhaustion \( (\tilde{L}_n)_{n \in \mathbb{N}} \) of \( G \) by compact sets. Then it suffices to find an increasing sequence \( (m_n)_{n \in \mathbb{N}} \) such that \( K_n \subseteq \tilde{L}_{m_n} \) for each \( n \in \mathbb{N} \) and put \( L_n := \tilde{L}_{m_n} \cap s^{-1}(K_n) \cap r^{-1}(K_n) \).

Let \( (h_n)_{n=1}^{\infty} \subseteq C_0(G) \) be an approximate unit consisting of positive-definite functions. We can modify this sequence so that for each \( n \in \mathbb{N} \) the \( n \)-th element of the sequence is normalised on \( K_n \). Indeed, we may find an increasing sequence of natural numbers \( (k_n)_{n \in \mathbb{N}} \) such that \( \| h_{k_n} |_{K_n} - 1_{K_n} \| < \frac{1}{2} \). Consider the new functions defined by the formula

\[
\tilde{h}_n(\gamma) = \begin{cases} (h_{k_n}(r(\gamma)) - \frac{1}{2}) h_{k_n}(\gamma)(h_{k_n}(s(\gamma)) - \frac{1}{2}), & \gamma \in s^{-1}(K_n) \cap r^{-1}(K_n), \\ 0, & \gamma \notin s^{-1}(K_n) \cap r^{-1}(K_n). \end{cases}
\]

The function \( \tilde{h}_n \) is continuous, as \( K_n \) is clopen in \( G \), and explicit verification shows it is positive-definite. Thus the sequence \( (\tilde{h}_n)_{n=1}^{\infty} \) satisfies all the requirements, including the normalization condition: \( \tilde{h}_n |_{K_n} = 1, n \in \mathbb{N} \). Choose sequences \( (\alpha_n)_{n \in \mathbb{N}} \) and \( (\epsilon_n)_{n \in \mathbb{N}} \) as
Remark 5.5. It follows from the proof above that if $G$ is $\sigma$-compact and $G^{(0)}$ is a union of compact open sets, then the net in (3), if it exists, can be arranged so that $\sup_{i \in I, x \in G^{(0)}} |h_i(x)| \leq 1$.

Let $G$ be a $\sigma$-compact étale Hausdorff groupoid. Then $G^{(0)}$ is a union of compact open sets if and only if the $C^*$-inclusion $C_0(G^{(0)}) \subseteq C^*_r(G)$ is relative $\sigma$-unital in the sense of [Matl. Definition 2.1]. This condition is automatically satisfied when $G^{(0)}$ is compact or totally disconnected (equivalently, the inclusion is unital or the groupoid $G$ is ample).

We do not know whether this condition is necessary to deduce the equivalence between conditions (3) and (1) in Proposition 5.4. On the other hand, we would like to view the Haagerup property as an approximation property. Therefore in this paper we choose the condition (2) as the most convenient one. It generalises to groupoid actions as follows.

Definition 5.6. Let $(\mathcal{A}, G, \Sigma, \alpha)$ be a twisted groupoid $C^*$-dynamical system such that the bundle $\mathcal{A}$ has a continuous unit section. We say that $(\mathcal{A}, G, \Sigma, \alpha)$ has the Haagerup property if if there is a net $(h_i)_{i \in I}$ of continuous positive-definite sections of $\mathcal{A} \ast_r G$ such that

(i) each $h_i$ is normalised: $h_i|_{G^{(0)}}$ is the unit section of $\mathcal{A}$,
(ii) each $h_i$ is locally $C_0$: for each compact set $K \subseteq G^{(0)}$ we have $h_i|_{G^{(0)}_K} \in C_0(\mathcal{A}|_{K \ast_r G^{(0)}_K})$,
(iii) the net converges to the unit section uniformly on compact sets: for every compact $K \subseteq G$ we have $\sup_{\gamma \in K} \|h_i(\gamma) - 1_{r(\gamma)}\| \xrightarrow{i \in I} 0$.

The groupoid $G$ itself has the Haagerup property if the above holds for the trivial bundle $\mathcal{A} = G^{(0)} \times \mathbb{C}$ (i.e. there is a net with properties listed in Proposition 5.4 (2)).

Remark 5.7. We could define the Haagerup property above without the assumption that $\mathcal{A}$ has a continuous unit section by replacing the last condition by the following one: for every compact $a \in C_c(\mathcal{A} \ast_r G)$ we have $\|h_i a - a\| \xrightarrow{i \in I} 0$. However, this would itself imply that $A := C_0(\mathcal{A})$ has a central approximate unit. Indeed, by Proposition 3.6 positive-definite sections $(h_i)_{i \in I}$ are necessarily central, that is we necessarily have $h_i(\gamma) \in Z(M(A_{r(\gamma)})$ for each $\gamma$ and $i \in I$. Thus $(m h_i|_A)_{i \in I}$ would be a central approximate unit in $A$. 28
One of the benefits from the continuous unit section assumption on $\mathcal{A}$ is that we immediately obtain the desirable fact that if $G$ has the Haagerup property then every twisted $C^*$-dynamical systems $(\mathcal{A}, G, \Sigma, \alpha)$ the Haagerup property.

**Proposition 5.8.** A $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$ with $G(0)$ compact and $A := C_0(\mathcal{A})$ unital has the Haagerup property if and only if there is a net $(h_i)_{i \in I} \subseteq C_0(\mathcal{A} \ast_r G)$ of positive-definite sections that converge to the unit section uniformly on compact sets.

**Proof.** Since $G(0)$ is compact, being locally $C_0$ is the same as being $C_0$. Hence the “only if” part is trivial. So assume $(h_i)_{i \in I} \subseteq C_0(\mathcal{A} \ast_r G)$ is a net of positive-definite sections that converge to the unit section uniformly on compact sets. We need to modify the elements of this net so that they are normalised. Since $G(0)$ is compact this is easy to observe: for big enough $i \in I$, say $i \geq i_0$, we have $\|h_i|_{G(0)} - 1_{G(0)}\| < \frac{1}{2}$. Consider for $i \geq i_0$ the new functions defined by the formula

$$\tilde{h}_i(\gamma) = (h_i(r(\gamma)))^{-\frac{1}{2}}h_i(\gamma)\alpha_{\gamma}\left((h_i(s(\gamma)))^{-\frac{1}{2}}\right), \quad \gamma \in G.$$ 

The function $\tilde{h}_i$ is continuous, as $G(0)$ is clopen in $G$, and in fact $\tilde{h}_i \in C_0(\mathcal{A} \ast_r G)$. By construction $\tilde{h}_i|_{G(0)} = 1_{G(0)}$. Since $h_i$ is positive-definite for any finite set $F \subseteq G_x$, $x \in G(0)$, we have $[\alpha^{-1}_\gamma (h_i(\gamma \eta^{-1}))]_{\eta, \gamma \in F} \geq 0$ in $M_{|F|}(A_x)$. Putting $a_{\gamma} := \alpha^{-1}_\gamma ((h_i(r(\gamma)))^{-\frac{1}{2}})$, $\gamma \in F$ we get positive elements in $A_x$ and

$$\left[\alpha^{-1}_{\gamma}(\tilde{h}_i(\gamma \eta^{-1}))\right]_{\eta, \gamma \in F} = [a_{\gamma} \alpha^{-1}_\gamma (h_i(\gamma \eta^{-1})))a_{\eta}]_{\eta, \gamma \in F} \geq 0.$$

Hence $\tilde{h}_i$ is positive-definite. Let $K \subseteq G$ be a compact set. The net $(\tilde{h}_i|_K)_{i \in I}$ converges uniformly to the unit section $1_K$, as the net $h_i$ converges uniformly to the unit section on $K \cup r(K) \cup s(K)$, and therefore the nets of sections $K \in \gamma \mapsto h_i(r(\gamma))^{-\frac{1}{2}}$ and $K \in \gamma \mapsto \alpha_{\gamma}(h_i(s(\gamma)))^{-\frac{1}{2}}$ converge uniformly to the unit section on $K$.

**Remark 5.9.** By Proposition 5.8, Definition 5.6 generalises the Haagerup property introduced in [DoR, Section 3] for discrete group actions on unital $C^*$-algebras. Indeed, if $G = \Gamma$ is a discrete group and the twist is trivial, then the groupoid action $(\mathcal{A}, G, \alpha)$ is nothing but a group action $\alpha : \Gamma \curvearrowright A$ on a unital $C^*$-algebra $A = \mathcal{A}_e$. This action has the the Haagerup property if and only if there exists a net $(h_i)_{i \in I}$ of functions $h_i : \Gamma \rightarrow Z(A)$ which are positive-definite, in the sense of Section 3 of [DoR], and such that $h_i \rightarrow 1$ pointwise on $\Gamma$. A similar interpretation is true for actions of discrete groups twisted by $\mathbb{T}$-valued cocycles: again our definition means that there exists a net $(h_i)_{i \in I}$ of functions $h_i : \Gamma \rightarrow Z(A)$ which are positive-definite in a twisted sense (see for example Definition 4.2 of [BC2]) and converge pointwise to 1. Note however that even for non-twisted actions our notion differs from the Haagerup property of actions of discrete groups on $C^*$-algebras as defined in [MSTT]. Roughly speaking, in [MSTT] the authors allowed arbitrary mapping $\mathcal{B}$-multipliers.
Lemma 5.10. Let \((G, \Sigma)\) be a twisted groupoid where \(G\) is étale locally compact Hausdorff. Treated as a twisted action on \(C_0(G)\), \((G, \Sigma)\) has the Haagerup property if and only if \(G\) has the Haagerup property.

Proof. We identify \((G, \Sigma)\) with \((\mathcal{A}, G, \Sigma, \alpha)\) where \(\mathcal{A} = G^{(0)} \times \mathbb{C}\), \(\alpha_\gamma = \text{id}\) and \(u_x(z) = 1\) for \(\gamma \in G\), \(x \in G^{(0)}\) and \(z \in \mathbb{T}\). Hence \(\mathcal{A} \ast_r G \cong G \times \mathbb{C}\) and therefore \(C_0(\mathcal{A} \ast_r G) \cong C_0(G)\).

It is immediate that the respective notions of positive-definiteness coincide. \(\square\)

We will now modify the definition of the Haagerup property for a \(C^*\)-inclusion introduced in [DoR]. Recall that if \(E : B \to A \subseteq B\) is a faithful conditional expectation from a \(C^*\)-algebra \(B\) onto \(A\), then \(B_E\) denotes the right Hilbert \(A\)-module associated to \(E\) via (4.8). In fact \(B_E\) is a \(C^*\)-correspondence over \(A\) with natural left action. To define “locally compact” maps on \(B_E\) we will use the notion of a Pedersen’s ideal, i.e. a minimal hereditary dense ideal (see [Ped, 5.6]). Recall that every \(C^*\)-algebra \(A\) contains such an ideal; we denote it by \(K(A)\). The ideal \(K(A)\) is a linear span of \(K(A)_+ := \{a \in A_+ : a \leq \sum_{i=1}^{n} a_k, a_k \in K(A)_0\}\) where \(K(A)_0 := \{f(a) : a \in A_+, f \in C_c(0, \infty), f \geq 0\}\).

Lemma 5.11. Let \(A\) be a \(C^*\)-bundle over a locally compact Hausdorff space \(X\) that admits a continuous unit section. The Pedersen’s ideal of \(A = C^*(A)\) is \(C_c(A)\).

Proof. Recall that Since \(A\) is unital, we may identify \(C_c(X)\) with a subalgebra of \(C_c(A) \subseteq A\). Then obviously \(C_c(X)_+ \subseteq K(A)_0\) and every \(a \in C_c(A)_+\) is dominated by some element in \(C_c(X)_+\). Hence \(C_c(A) \subseteq K(A)\). The reverse inclusion follows because \(C_c(A)\) is clearly a hereditary ideal in \(A\). \(\square\)

Definition 5.12. Let \(A \subseteq B\) be a nondegenerate \(C^*\)-inclusion equipped with a faithful conditional expectation \(E : B \to A\). We say that \(B\) has the \(E\)-Haagerup property if there exists a net \((\Phi_i)_{i \in I}\) of completely positive maps on \(B\) such that

(i) \(\Phi_i|_A = \text{id}|_A\) for each \(i \in I\);

(ii) for each \(i \in I\), \(\Phi_i\) extends to a bounded map \(\widetilde{\Phi}_i : B_E \to B_E\) which is “locally compact”, that is \(k \widetilde{\Phi}_i k \in K(B_E)\) for every \(k \in K(A)\) (here \((k \widetilde{\Phi}_i k)(b) := k \Phi_i(b) k\) for all \(b \in B_E\));

(iii) \(\lim_{i \in I} \|\Phi_i(b) - b\|_{B_E} = 0\) for every \(b \in B\).

We will say that a \(C^*\)-inclusion \(A \subseteq B\) has the Haagerup property if the inclusion has the unique conditional expectation, denoted \(E\), and \(B\) has the \(E\)-Haagerup property.

Remark 5.13. Condition (i) above implies that \(\Phi_i\) is a contractive \(A\)-bimodule map. Indeed, since we assume that \(A\) and \(B\) have a common (contractive) approximate unit \((\mu_\lambda)_{\lambda \in \Lambda}\) we get \(\|\Phi\| = \lim_{\lambda \in \Lambda} \|\Phi_i(\mu_\lambda)\| = \lim_{\lambda \in \Lambda} \|\mu_\lambda\| = 1\). Hence the multiplicative domain of \(\Phi_i\) is equal to \(\{b \in B : \Phi_i(bb^*) = \Phi_i(b)\Phi_i(b^*), \Phi_i(b^*b) = \Phi_i(b^*)\Phi_i(b)\}\). This together with (i) implies that \(\Phi_i\) is an \(A\)-bimodule map.

Remark 5.14. A sufficient condition for a completely positive map \(\Phi : B \to B\) to extend to a bounded map \(\tilde{\Phi} : B_E \to B_E\) with \(\|\tilde{\Phi}\| \leq m\) (where \(m > 0\)), is that \(E \circ \Phi \leq mE\).

Indeed, for every \(b \in B\) we then have
\[
E(\Phi(b^* \Phi(b))) \leq \|\Phi\| E(\Phi(b^* b)) \leq m\|\Phi\| E(b^* b),
\]
for every $i$.

**Remark 5.15.** Dong and Ruan speak simply of the $A$-Haagerup property, but this is not quite precise, as shown by Suzuki in [Suz]: in general the property depends not only on the pair $(A,B)$ but also on the choice of a conditional expectation $E$. On the other hand, there are natural situations when the inclusion $A \subseteq B$ has a unique conditional expectation. This holds for instance when $B$ is the reduced cross-sectional $C^*$-algebra of a Fell bundle $\mathcal{B}$ over a topologically free étale locally compact Hausdorff groupoid $G$, and more generally for noncommutative Cartan subalgebras $A \subseteq B$, cf. [Exe], [KwM].

It is immediate that our definition is consistent with the one in [DoR]. It is also consistent with the one given by Dong and Ruan [DoR], concerning unital inclusions, if we assume further that the conditional expectation is tracial – as is the case for the canonical expectation related to crossed products of actions of discrete groups on commutative algebras, being the main focus of [DoR].

**Proposition 5.16.** Let $A \subseteq B$ be a unital inclusion of $C^*$-algebras and let $E : B \to A \subseteq B$ be a faithful tracial conditional expectation. Then $B$ has the $E$-Haagerup property if and only if there exists a net $(\Phi_i)_{i \in \mathcal{I}}$ of completely positive $A$-bimodule maps on $B$ such that, for every $i \in \mathcal{I}$, $\Phi_i$ extends to a compact map $\tilde{\Phi}_i \in K(B_E)$ and $\lim_{i \in \mathcal{I}} \| \Phi_i(b) - b \|_{B_E} = 0$ for every $b \in B$.

**Proof.** Since $1 \in A \subseteq B$ “locally compact” and “compact” on $B_E$ is the same. Hence the “only if” part is clear by Remark 5.13. Let then $(\Phi_i)_{i \in \mathcal{I}}$ be completely positive $A$-bimodule maps on $B$ such that for every $i \in \mathcal{I}$ we have $\tilde{\Phi}_i \in K(B_E)$ and $\lim_{i \in \mathcal{I}} \| \Phi_i(b) - b \|_{B_E} = 0$ for $b \in B$. Find $i_0 \in \mathcal{I}$ such that $\| \Phi_i(1) - 1 \| < \frac{1}{2}$ for $i \geq i_0$. Note that $\Phi_i(1)$ is a positive element commuting with $A$ (as $\Phi$ is an $A$-bimodule map). Thus putting, for $i \geq i_0$,

$$\Psi_i(b) := \Phi_i(1) - \frac{1}{2} \Phi_i(b) \Phi_i(1) - \frac{1}{2}, \quad b \in B,$$

we get a completely positive map $\Psi_i$ on $B$ such that $\Psi_i|_A = id|_A$ and $\Psi_i$ extends to a compact map $\tilde{\Psi}_i = \Phi_i(1) - \frac{1}{2} \tilde{\Phi}_i \Phi_i(1) - \frac{1}{2}$ on $B_E$. Moreover, for $b \in B$ we have $\lim_{i \geq i_0} \| \Psi_i(b) - b \|_{B_E} = 0$ by a 3e-argument. Note that this is the place where we use the tracial property of $E$, so that we have the ‘right’ estimate $\| bc \|_{B_E} \leq \| c \|_{\infty} \| b \|_{B_E}$ for $b, c \in B$. \hfill $\square$

Using the facts from previous sections we get the following far reaching generalisation of [DoR] Theorem 3.6, one of the main results of our paper.

**Theorem 5.17.** Let $B := C^*_\tau(A,G,\Sigma,\alpha)$ be the reduced crossed product of the twisted action $(A,G,\Sigma,\alpha)$ of an étale locally compact Hausdorff groupoid $G$ on the $C^*$-algebra $A := C_0(A)$, and let $E : B \to A$ be the canonical conditional expectation. Then $B$ has the $E$-Haagerup property if and only if $(A,G,\Sigma,\alpha)$ has the Haagerup property.
Proof. Suppose that \((\mathcal{A}, G, \Sigma, \alpha)\) has the Haagerup property. Let \((h_i)_{i \in I}\) be a net as in Definition 5.6. By Proposition 3.7 we have the corresponding completely positive contractive multiplier maps \(m_{h_i} : C_r^*(B) \to C_r^*(B)\). By this proposition \(\Phi_i := m_{h_i}\) is an \(A\)-bimodule map. By Definition 5.6 (i) we have \(\Phi_i|_A = id|_A\) and the form of \(E\) implies that \(E \circ \Phi_i = E\). Hence \((\Phi_i)_{i \in I}\) is uniformly bounded by 1, and therefore it converges pointwise to the identity operator on \(B_E\), by Lemma 4.5. Finally, let \(k \in K(A) = C_c(A)\). Take any positive function \(h \in C_c(G(0))\) which is equal to 1 on the support of \(k\). Then \(\tilde{h}_i(\gamma) := h(r(\gamma))h_i(\gamma)h(s(\gamma)), \gamma \in G\), is a positive-definite section of \(A \ast_r G\). Denote the support of \(h\) by \(K\). Since \(h_i\) vanishes outside \(G_K^r\) and \(h_i|_K \in C_0(A|_K \ast_r G_K^r)\) (by Definition 5.6 (ii)) we get \(\tilde{h}_i \in C_0(A \ast_r G)\). Therefore \(\tilde{m}_{h_i} \in K(B_E)\) by Proposition 4.4 (2). Hence \(k\tilde{\Phi}_i k = k\tilde{m}_{h_i} k \in K(B_E)\).

Now let \((\Phi_i)_{i \in I}\) be any net as in Definition 5.12. Let \(h^{\Phi_i}, i \in I\), be given by Proposition 4.1. The formula (4.7) and the equality \(\Phi_i|_A = id|_A\) imply that \(h^{\Phi_i}|_{G(0)} = 1_{G(0)}\). By Lemma 4.5 \((h^{\Phi_i})_{i \in I}\) converges to the unit section uniformly on compact sets. To see that \(h_i|_{G_K^r} \in C_0(|A|_K \ast_r G_K^r)\), for any compact set \(K \subseteq G(0)\), fix such a \(K\) and take a positive function \(h \in C_c(G(0))\) equal to 1 on \(K\). Multiplying it by the unit section of \(A\) we get an element \(k \in K(A)\). It follows from (4.7) that \(h^{k\Phi_i,k}(\gamma) = h(r(\gamma))h^{\Phi_i}(\gamma)h(s(\gamma)), \gamma \in G\). Since \(k\Phi_i k \in K(B_E)\), we get \(h^{k\Phi_i,k} \in C_0(|A|_K \ast_r G(0))\) by Proposition 4.4 (4). Therefore \(h^{\Phi_i}|_{G_K^r} = h^{k\Phi_i,k}|_{G_K^r} \in C_0(|A|_K \ast_r G_K^r)\). This verifies that \((\mathcal{A}, G, \Sigma, \alpha)\) has the Haagerup property. \(\Box\)

Remark 5.18. It follows from the proof above that \(B := C_r^*(\mathcal{A}, G, \Sigma, \alpha)\) has the \(E\)-Haagerup property if and only if we can find a net \((\Phi_i)_{i \in I}\) as in Definition 5.12 with the additional property that \(E \circ \Phi_i = E\) for each \(i \in I\).

Theorem 5.17 apart from generalising the main result of [DoR], has an immediate consequence for the relative Haagerup property with respect to a Cartan subalgebra.

Corollary 5.19. Let \(A \subseteq B\) be a (commutative) Cartan subalgebra of a \(C^*\)-algebra \(B\). Then \(A \subseteq B\) has the Haagerup property if and only if the associated Weyl groupoid has the Haagerup property.

Proof. We may assume identifications \(A = C_0(G(0))\) and \(B = C_r^*(G, \Sigma)\) for a twisted groupoid \((G, \Sigma)\) (this is well known when \(B\) is separable [Ren] but holds in general, see [KwM2] or [Raa]). The twisted groupoid \((G, \Sigma)\) is uniquely determined by the inclusion \(A \subseteq B\), and \(G\) is called the Weyl groupoid of \(A \subseteq B\). Thus it suffices to combine Theorem 5.17 and Lemma 5.10. \(\Box\)

As another immediate consequence, we obtain a generalisation of [DoR, Theorem 4.2] from countable groups to metric spaces with bounded geometry.

Corollary 5.20. Let \(X\) be a metric space with bounded geometry and \(G(X)\) be its coarse groupoid. Then \(X\) coarsely embeds into a Hilbert space if and only if there is a net \((h_i)_{i \in I} \subseteq C_0(G(X))\) of positive-definite functions, converging to 1 uniformly on compact
subsets of $G(X)$ if and only if for all (equivalently, some) twist $\Sigma$ over $G(X)$, the $C^*$-inclusion $C(G(X)^{(0)}) \subseteq C^r_r(G(X), \Sigma)$ has the Haagerup property.

**Proof.** By [STY, Proposition 3.2], $G(X)$ is a locally compact Hausdorff, principal étale groupoid. Moreover, $G(X)$ is $\sigma$-compact and has a compact unit space. By [STY, Theorem 5.4], $X$ coarsely embeds into a Hilbert space if and only if $G(X)$ admits a proper negative type function. Hence the assertion follows from Proposition 5.4, Lemma 5.10 and Corollary 5.19. □

**Remark 5.21.** Recall that the coarse groupoid $G(X)$ of a bounded geometry metric space $X$ is amenable if and only if $X$ has property A if and only if $C^r_r(G(X), \Sigma)$ is nuclear for all (equivalently, some) twist $\Sigma$ over $G(X)$ (see [STY, Theorem 5.3] and [Tak, Theorem 5.4]). On the other hand, there exist metric spaces with bounded geometry which do not have property A but coarsely embed into a Hilbert space (see e.g. [AGS, Os]).

### 6. The Haagerup Property and the Universal Coefficient Theorem

In this section we discuss the consequences of the results of the last section for the questions related to the Universal Coefficient Theorem. We recall that the Universal Coefficient Theorem (UCT) for $C^*$-algebras was introduced by Rosenberg and Schochet in [RoS]. A separable $C^*$-algebra $A$ is said to satisfy the UCT if for every separable $C^*$-algebra $B$ the following natural sequence

$$0 \to \text{Ext}(K_*(A), K_{*-1}(B)) \to KK_*(A, B) \to \text{Hom}(K_*(A), K_*(B)) \to 0$$

is exact. A separable $C^*$-algebra satisfies the UCT if and only if it $KK$-equivalent to a commutative $C^*$-algebra (see [Bla, Theorem 23.10.5]). Although there exist exact non-nuclear $C^*$-algebras that do not satisfy the UCT (see [Ska]), it is still open whether all separable nuclear $C^*$-algebras satisfy the UCT. This question is often refereed to as the UCT problem, and it is receiving renewed interest due to the recent breakthrough results in the classification program of separable simple nuclear $C^*$-algebras satisfying the UCT (see e.g. [Win1, Win2]).

It follows from Tu’s remarkable paper [Tu] that all $C^*$-algebras associated to groupoids with the Haagerup property satisfy the UCT. Building on Tu’s techniques, Barlak and Li have proved that this also holds for twisted amenable étale groupoids (see [BaL]), using recent articles of Takeishi [Tak] and of van Erp and Williams [ErW]. In this section, we generalise the key results of [BaL] concerning UCT, following the same idea. But instead of [ErW] we exploit the Stabilization Theorem [IKSW] of Ionescu, Kumjian, Sims and Williams, and instead of amenability we assume the Haagerup property. This enables us to obtain a stronger assertion under weaker assumptions.

Recall that a Fell bundle is said to be separable if each of its fibers is separable.

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1It is also worth noting that there exist separable non-exact $C^*$-algebras which satisfy the UCT. Indeed, this is the case for the reduced and full group $C^*$-algebras of $\Gamma$, where $\Gamma$ is a finitely generated non-exact group with the Haagerup property (as exhibited in [Osa, Theorem 2]).
Theorem 6.1. Let $\mathcal{B}$ be a continuous saturated and separable Fell bundle over a second countable locally compact Hausdorff étale groupoid $G$. Assume that $C^*(\mathcal{B}|_{G^{(0)}})$ is of type I and $G$ has the Haagerup property. Then both $C^*_r(\mathcal{B})$ and $C^*(\mathcal{B})$ satisfy the UCT.

Proof. We claim that we may reduce the situation to the case where $\mathcal{B}$ is the Fell bundle associated to a groupoid dynamical system $(\mathcal{A}, G, \alpha)$, using the Stabilization Theorem [IKSW, Theorem 3.7], see also [Lal]. Indeed, the Stabilization Theorem states that $\mathcal{B}$ is Morita equivalent to a groupoid $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$. Then $\mathcal{A} = C^*(\mathcal{A})$ is Morita equivalent to $C^*_r(\mathcal{B}|_{G^{(0)}})$ and hence is of type I. Since $\mathcal{B}$ is separable and continuous, $\mathcal{A}$ is separable and continuous as well (this is not stated explicitly in [IKSW] but it follows immediately from the construction of the corresponding groupoid dynamical system). By Renault’s equivalence theorems for Fell bundles, see [SiW, Theorem 14], and [MuW, Theorem 6.4], $C^*_r(\mathcal{B})$ (resp. $C^*(\mathcal{B})$) is Morita equivalent to $C^*_r(\mathcal{A}, G, \alpha)$ (resp. $C^*(\mathcal{A}, G, \alpha)$), cf. also [Lal]. Since the UCT is stable under Morita equivalence this shows our claim.

Moreover, it follows from [Tu, Proposition 4.12 and Theorem 9.3] that $C^*_r(\mathcal{A}, G, \alpha)$ and $C^*(\mathcal{A}, G, \alpha)$ are $KK$-equivalent. Since the UCT is stable under $KK$-equivalences [Bla, Theorem 23.10.5], it suffices to show that $C^*_r(\mathcal{A}, G, \alpha)$ has the UCT.

So let us consider a groupoid dynamical system $(\mathcal{A}, G, \alpha)$ such that $G$ is a second countable locally compact Hausdorff étale groupoid, $\mathcal{A}$ is a continuous bundle and $\mathcal{A} := C_0(\mathcal{A})$ is separable and type I. By Proposition 5.4 and Remark 5.2 (which is essentially [Tu, Proposition 3.8]), $G$ acts properly on a continuous field $H$ of affine Euclidean spaces. From this point on one can continue as in the proof of [BaL, Theorem 3.1].

In the next few results we consider twists for an action $\alpha$ of a discrete group $\Gamma$ on a $C^*$-algebra $\mathcal{A}$, associated to cocycles $\omega : \Gamma \times \Gamma \to UM(\mathcal{A})$. We refer to [PaR] for relevant definitions, and denote the respective reduced/universal crossed products by $\mathcal{A} \rtimes^{\omega, r}_\alpha \Gamma$ and $\mathcal{A} \rtimes^{\omega}_\alpha \Gamma$. Similarly, we write $C^*_r(\Gamma, \omega)$ and $C^*(\Gamma, \omega)$ for twisted reduced/universal group $C^*$-algebras (which are crossed products by an action of $\Gamma$ on $\mathbb{C}$). In fact, the twisted group actions are particular examples of twisted inverse semigroup actions, as described in Definition 2.7, and the corresponding $C^*$-algebras coincide, cf. page 14. In particular, twisted group actions can be viewed as saturated Fell bundles over groups. Hence Theorem 6.1 gives immediately the following generalisation of [ELPW, Proposition 6.1].

Corollary 6.2. Let $(\mathcal{A}, \Gamma, \alpha, \omega)$ be a twisted $C^*$-dynamical system such that the $C^*$-algebra $\mathcal{A}$ is separable and type I, and $\Gamma$ is a countable discrete group with the Haagerup property. Then $\mathcal{A} \rtimes^{\omega, r}_\alpha \Gamma$ and $\mathcal{A} \rtimes^{\omega}_\alpha \Gamma$ satisfy the UCT.

We list some more concrete applications.

Corollary 6.3. Let $\Gamma$ be a countable discrete group. If $N$ is a virtually abelian normal subgroup of $\Gamma$ such that $\Gamma/N$ has the Haagerup property, then $C^*_r(\Gamma)$ and $C^*(\Gamma)$ satisfy the UCT.

Proof. It is well-known that $C^*_r(\Gamma) \cong C^*_r(N) \rtimes^{\omega, r}_\alpha (\Gamma/N)$ for some twisted action $(\alpha, \omega)$ of the quotient group $\Gamma/N$ on $C^*_r(N)$ (see e.g. [PaR, Theorem 4.1] and [Béd, Theorem 2.1]).
Since $N$ is virtually abelian, $C^*_r(N)$ is type I and we complete the proof by the previous proposition. The same argument is valid in the universal case. 

**Corollary 6.4.** Suppose that $H, N$ are countable discrete groups such that $H$ acts on $N$ by automorphisms and assume that $\omega : N \times N \to \mathbb{T}$ is a 2-cocycle invariant under the $H$-action (i.e. $\omega(h \cdot n, h \cdot m) = \omega(n,m)$ for all $n,m \in N$ and $h \in H$), so that we have natural actions of $H$ on $C^*_r(N,\omega)$ and on $C^*(N,\omega)$.

If both $N$ and $H$ have the Haagerup property, then both $C^*_r(N,\omega) \rtimes_r H$ and $C^*(N,\omega) \rtimes H$ satisfy the UCT.

**Proof.** Since $H$ has the Haagerup property, $C^*_r(N,\omega) \rtimes H$ and $C^*_r(N,\omega) \rtimes_r H$ are $KK$-equivalent (as are $C^*(N,\omega) \rtimes H$ and $C^*(N,\omega) \rtimes_r H$). Hence, we only have to show the UCT for $C^*_r(N,\omega) \rtimes_r H$ (the proof for $C^*(N,\omega) \rtimes_r H$ would be identical from this point).

For this it suffices to show the UCT of $C^*_r(N,\omega) \rtimes_r F$ for every finite subgroup $F$ of $H$ (see [MeN, Corollary 9.4] and [HiK]). On the other hand, it follows from [ELPW, Lemma 2.1] that

$$C^*_r(N,\omega) \rtimes_r F \cong C^*_r(N \rtimes F,\tilde{\omega}),$$

where $\tilde{\omega}$ is a 2-cocycle on $N \rtimes F$ given by $\tilde{\omega}((n,h),(n',h')) = \omega(n,h \cdot n'), h \in H, n, n' \in N$.

Since $N \rtimes F$ has the Haagerup property as well, $C^*_r(N \rtimes F,\tilde{\omega})$ satisfies the UCT for every finite subgroup $F$ by Corollary [6.2]. This ends the proof.

**Example 6.5.** Let $\theta$ be an irrational number. We will identify $\theta$ with the real $2 \times 2$ skew-symmetric matrix $\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$. Define a 2-cocycle $\omega_\theta : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$ by $\omega_\theta(x,y) := e^{-\pi i (\theta x, y)}$, $x, y \in \mathbb{Z}^2$. Then the reduced twisted group $C^*$-algebra $C^*_r(\mathbb{Z}^2,\omega_\theta)$ is known to be isomorphic to $A_\theta$, the irrational rotation algebra. Moreover, $\omega_\theta$ is invariant under the action of $SL(2,\mathbb{Z})$ on $\mathbb{Z}^2$ via matrix multiplication (see [ELPW] Section 2 for details). From Corollary 6.4 we see that $A_\theta \rtimes_r SL(2,\mathbb{Z}) = C^*_r(\mathbb{Z}^2,\omega_\theta) \rtimes_r SL(2,\mathbb{Z})$ satisfies the UCT (as does its universal counterpart). It is worth noticing that we can not apply Corollary [6.2] directly here, as $A_\theta$ is not a type I $C^*$-algebra, and $A_\theta \rtimes_r SL(2,\mathbb{Z}) = C^*_r(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}),\tilde{\omega}_\theta)$ but the group $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ does not have the Haagerup property.

We will now combine Theorem 6.1 with the key results of Section 5. We start with a generalisation of [BaL, Theorem 1.1].

**Proposition 6.6.** Assume that $(G, \Sigma)$ is a twisted étale Hausdorff locally compact second countable groupoid. If $C^*_r(G,\Sigma)$ has the $E$-Haagerup property with respect to the canonical conditional expectation $E : C^*_r(G,\Sigma) \to C_0(G^{(0)})$ (which is automatic when $C^*_r(G,\Sigma)$ is nuclear), then both $C^*_r(G,\Sigma)$ and $C^*(G,\Sigma)$ satisfy the UCT.

**Proof.** Theorem [5.17] and Lemma [5.10] imply that $C^*_r(G,\Sigma)$ has the $E$-Haagerup property if and only if $G$ has the Haagerup property. When $C^*_r(G,\Sigma)$ is nuclear, then $G$ is amenable by [Tak, Theorem 5.4] and hence has the Haagerup property by Remark [5.2]. If $G$ has the Haagerup property, then treating $(G, \Sigma)$ as a continuous Fell line bundle over $G$, cf. Remark [2.6] get that $C^*_r(G,\Sigma)$ and $C^*(G,\Sigma)$ satisfy the UCT by Theorem 6.1. 

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We now present some applications of Proposition 6.6 involving Cartan subalgebras. The first one generalises Proposition 3.4 and Corollary 3.2.

**Corollary 6.7.** Let \((B, A)\) be a separable Cartan pair, and let \(\Gamma\) be a countable group acting on \(B\) such that \(\gamma(A) = A\) for every \(\gamma \in \Gamma\). If \(B \rtimes_r \Gamma\) has the \(E\)-Haagerup property with respect to the canonical conditional expectation \(E : B \rtimes_r \Gamma \to A\) factorising via \(B\) (which is automatic when \(B \rtimes_r \Gamma\) is nuclear), then \(B \rtimes_r \Gamma\) satisfies the UCT.

In particular, if \(B\) is a separable \(C^*\)-algebra which admits a Cartan subalgebra \(A\) and the inclusion \(A \subseteq B\) has the Haagerup property, then \(B\) satisfies the UCT.

**Proof.** By Theorem 5.9 the pair \((B, A)\) is isomorphic to a pair \((C^*_r(G, \Sigma), C_0(G^{(0)}))\) where \((G, \Sigma)\) is a twisted étale Hausdorff locally compact second countable groupoid. In the proof of Proposition 3.4 it is shown that \((\Gamma \rtimes G, \Gamma \rtimes \Sigma)\) is also a twisted étale Hausdorff locally compact second countable groupoid and \((C^*_r(\Gamma \rtimes G, \Gamma \rtimes \Sigma), C_0(G^{(0)})) \cong (B \rtimes_r \Gamma, A)\). Hence, \(B \rtimes_r \Gamma\) satisfies the UCT by Proposition 6.6. The second statement follows by putting \(\Gamma = \{e\}\).

The second corollary extends Corollary 7.4.

**Corollary 6.8.** Let \(B\) be a separable \(C^*\)-algebra and let \(A\) be an abelian subalgebra of \(B\). If there exists a discrete abelian group \(\Gamma\) such that \((B, A)\) is a \(\Gamma\)-Cartan pair as defined in Definition 3.10 and \(B\) has the \(\Delta\)-Haagerup property with respect to the canonical conditional expectation \(\Delta : B \to A\), then \(B\) satisfies the UCT.

**Proof.** It follows from Theorem 4.36 that \((B, A) \cong (C^*_r(G, \Sigma), C_0(G^{(0)}))\) for a twisted étale Hausdorff locally compact second countable groupoid \((G, \Sigma)\). Hence, \(B \cong C^*_r(G, \Sigma)\) satisfies the UCT by Proposition 6.6.

We finish by a remark concerning the case where we can deduce the UCT property for all the quotients of a given \(C^*\)-algebra. The following corollary applies to the reduced groupoid \(C^*\)-algebra of a second countable locally compact Hausdorff étale groupoid which is residually topologically principal and inner exact, see for instance Corollary 3.12.

**Corollary 6.9.** Let \(A \subseteq B\) be a Cartan inclusion where \(B\) is separable and \(A\) separates ideals in \(B\), i.e. the map \(J \mapsto J \cap A\) defined on ideals of \(B\) is injective. If the inclusion \(A \subseteq B\) has the Haagerup property then all quotients of \(B\) satisfy the UCT.

**Proof.** By and Remark 2.6 we may identify \((B, A)\) with \((C^*_r(\mathcal{L}), C_0(G^{(0)}))\) for a continuous Fell line bundle \(\mathcal{L}\) over \(G\). If \(C_0(G^{(0)})\) separates the ideals of \(C^*_r(\mathcal{L})\), then every ideal in \(C^*_r(\mathcal{L})\) is generated by a \(G\)-invariant ideal in \(C_0(G^{(0)})\), see for instance Propositions 2.11, 6.9. Hence every ideal in \(C^*_r(\mathcal{L})\) is generated by \(C_0(G^{(0)}) \setminus D\) where \(D\) is a closed \(G\)-invariant set. This implies that the sequence

\[0 \to C_0(\mathcal{L}|_{X \setminus D}) \to C^*_r(\mathcal{L}) \to C^*_r(\mathcal{L}|_D) \to 0,\]

which exists by Propositions 4.2, 4.3, is exact. Thus every quotient of \(C^*_r(\mathcal{L})\) is of the form \(C^*_r(\mathcal{L}|_D)\). If \(C_0(G^{(0)}) \subseteq C^*_r(\mathcal{L})\) has the Haagerup property, then by Theorem 5.17, Lemma 5.10 and Proposition 5.4 \(G\) has the Haagerup property. So does the closed subgroupoid \(G|_D\) and the UCT of \(C^*_r(G|_D, \Sigma|_D)\) follows from Theorem 6.4. \(\square\)


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References

[A-D] C. Anantharaman-Delaroche, The Haagerup property for discrete measured groupoids, Operator algebra and dynamics, 1—30, Springer Proc. Math. Stat., 58, Springer, Heidelberg, 2013.

[AGS] G. Arzhantseva, E. Guentner and J. Špakula, Coarse non-amenability and coarse embeddings, Geom. Funct. Anal. 22 (2012), no. 1, 22–36.

[BaLi] S. Barlak and X. Li, Cartan subalgebras and the UCT problem, Adv. Math. 316 (2017), 748–769.

[Bla] B. Blackadar, “K-theory for operator algebras”, Cambridge University Press, Second edition (1998).

[Béd] E. Bédos, Discrete groups and simple C∗-algebras, Math. Proc. Cambridge Philos. Soc. 109 (1991), no. 3, 521–537.

[BC1] E. Bédos and R. Conti, Fourier series and twisted C∗-crossed products, J. Fourier Anal. Appl. 21 (2015), no. 1, 32–75.

[BC2] É. Bédos and R. Conti, The Fourier-Stieltjes algebra of a C∗-dynamical system, Internat. J. Math. 27 (2016), no. 6, 1650050, 50 pp.

[BiF] T. Bice and I. Farah, Traces, ultrapowers, and the Pedersen-Petersen C∗-algebras, Houston J. Math. 41 (2015), 1175–1190.

[BöLi] C. Bönicke and K. Li, Ideal structure and pure infiniteness of ample groupoid C∗-algebras, Ergodic Theory Dynam. Systems 40 (2020), no. 1, 34–63.

[BFPR] J. H. Brown, A.H. Fuller, D. R. Pitts and S. A. Reznikoff, Graded C∗-algebras and twisted groupoid C∗-algebras, New York J. Math. 27 (2021), 205–252.

[BrO] N. P. Brown and N. Ozawa, “C∗-algebras and finite dimensional approximations", American Mathematical Society, 2008.

[BuS] R. C. Busby and H. A. Smith, Representations of twisted group algebras, Trans. Amer. Math. Soc. 149 (1970), 503–537.

[BuE1] A. Buss and R. Exel, Twisted actions and regular Fell bundles over inverse semigroups, Proc. Lond. Math. Soc. (3) 103 (2011), no. 2, 235–270.

[BuE2] A. Buss and R. Exel, Fell bundles over inverse semigroups and twisted étale groupoids, J. Operator Theory 67 (2012), no. 1, 153–205.

[BEM] A. Buss, R. Exel and R. Meyer, Reduced C∗-algebras of Fell bundles over inverse semigroups, Israel J. Math. 220 (2017), no. 1, 225–274.

[BuM] A. Buss and R. Meyer, Inverse semigroup actions on groupoids, Rocky Mountain J. Math. 47 (2017), no. 1, 53–159.

[CCJJV] P.A. Cherix, M. Cowling, P. Jolissaint, P. Julg and A. Valette, “Groups with the Haagerup property. Gromov’s a-T-menability”, Progress in Mathematics, 197, Basel, 2001.

[Cho] M. Choda, Group factors of the Haagerup type, Proc. Japan Acad. Ser. A Math. Sci. 59 (1983), 174–177.

[DFS] M. Daws, P. Fima, A. Skalski and S. White, The Haagerup property for locally compact quantum groups, J. Reine Angew. Math. 711 (2016), 189–229.

[Dou] Z. Dong, Haagerup property for C∗-algebras, J. Math. Anal. Appl. 377 (2011), no. 2, 631–644.

[DoR] Z. Dong and Z.J. Ruan, A Hilbert module approach to the Haagerup property, Integr. Equ. Oper. Theory, 73 (2012), 431–454.
[DuG] M. J. Dupré and R. M. Gillette, “Banach Bundles, Banach Modules and Automorphisms of C∗-Algebras”, Res. Notes Math., vol. 92, Pitman (Adv. Publ. Program), Boston 1983.

[ELPW] S. Echterhoff, W. Lück, N.C. Phillips and S. Walters, The structure of crossed products of irrational rotation algebras by finite subgroups of SL_{2}(Z), J. Reine Angew. Math. 639 (2010), 173-221.

[ErW] E. van Erp and D. P. Williams, Groupoid crossed products of continuous-trace C∗-algebras, J. Operator Theory 72 (2014), no. 2, 557-576.

[Exe] R. Exel, Noncommutative Cartan subalgebras of C∗-algebras, New York J. Math. 17 (2011), 331–382.

[FeD] J. M. G. Fell and R. S. Doran, “Representations of ∗-Algebras, Locally Compact Groups, and Banach ∗-Algebraic Bundles”, Vol. 1. Basic Representation Theory of Groups and Algebras, Pure Appl. Math., vol. 126, Academic Press Inc., Boston, MA 1988.

[Gre] P. Green, The local structure of twisted covariance algebras, Acta Math. 140 (1978), no. 3-4, 191–250.

[Haa1] U. Haagerup, An example of a non nuclear C∗-algebra, which has the metric approximation property, Invent. Math. 50 (1978), no. 3, 279–293.

[Haa2] U. Haagerup, Group C∗-algebras without the completely bounded approximation property, J. of Lie Theory 26 (2016), no. 3, 861–887.

[HiK] N. Higson and G. Kasparov, E-theory and KK-theory for groups which act properly and isometrically on Hilbert space, Invent. Math. 144 (2001), no. 1, 23–74.

[IKSW] M. Ionescu, A. Kumjian, A. Sims and D. P. Williams, A stabilization theorem for Fell bundles over groupoids, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 148 (2018), no. 1, 79–100.

[KrR] J. Kraus and Z.-J. Ruan, Approximation properties for Kac algebras, Indiana Univ. Math. J. 48 (1999), no. 2, 469–535.

[Jol] P. Jolissaint, The Haagerup property for measure-preserving standard equivalence relations, Ergodic Theory Dynam. Systems 25 (2005), no. 1, 161–174.

[Kum1] A. Kumjian, On C∗-diagonals, Canad. J. Math. 38 (1986), 969–1008.

[Kum2] A. Kumjian, Fell bundles over groupoids, Proc. Amer. Math. Soc. 126 (1998), no. 4, 1115–1125.

[KwM1] B. K. Kwasniewski and R. Meyer, Essential crossed products by inverse semigroup actions: simplicity and pure infiniteness, Doc. Math. 26 (2021), 271–335.

[KwM2] B. K. Kwasniewski and R. Meyer, Noncommutative Cartan C∗-subalgebras, Trans. Amer. Math. Soc. 373 (2020), no. 12, 8697–8724.

[KwM3] B. K. Kwasniewski and R. Meyer, Stone duality and quasi-orbit spaces for generalised C∗-inclusions, Proc. Lond. Math. Soc. 121 (2020), no. 4, 788–827.

[Lal] S. Lalonde, Some consequences of stabilization theorem for Fell bundles over exact groupoids, J. Operator Theory 81 (2019), no. 2, 335–369.

[Lan] E.C. Lance, “Hilbert C∗-modules. A toolkit for operator algebraists”, London Mathematical Society Lecture Note Series, 210. Cambridge University Press, Cambridge, 1995.

[Laz] A. J. Lazar, A selection theorem for Banach bundles and applications, J. Math. Anal. Appl. 462 (2018), 448–470.

[Li] X. Li, Every classifiable simple C∗-algebra has a Cartan subalgebra, Invent. Math. 219 (2020), no. 2, 653–699.

[Mat] K. Matsumoto, Relative Morita equivalence of Cuntz-Krieger algebras and flow equivalence of topological Markov shifts, Trans. Amer. Math. Soc. 370 (2018), 7011–7050.

[MSSTT] A. McKee, A. Skalski, I. G. Todorov and L. Turowska, Positive Herz-Schur multipliers and approximation properties of crossed products, Math. Proc. Camb. Phil. Soc. 165 (2018), no. 3, 511–532.

[MTT] A. McKee, I. G. Todorov and L. Turowska, Herz-Schur multipliers of dynamical systems, Adv. Math. 331 (2018), 387–438.
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