Matrix models, emergent gravity and gauge theory

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Abstract. Matrix models of Yang-Mills type describe noncommutative spaces, which are considered as models for dynamical space-time. Gravity arises on these branes through an intrinsically noncommutative mechanism. We review recent progress in the understanding of this emergent gravity. The metric is not fundamental but arises effectively in the semi-classical limit, along with nonabelian gauge fields. This leads to a mechanism for protecting certain geometries from quantum corrections, including flat space-time.

1. Background and motivation
Quantum field theory and general relativity (GR) provide the basis of our present understanding of fundamental matter and interactions. In spite of the success of these two theories, there is up to now no satisfactory way to reconcile them in a consistent quantum theory. In particular, quantum mechanics combined with GR strongly suggests a “foam-like” or quantum structure at the Planck scale $10^{-33}$ cm, where quantum fluctuations of space-time are expected to be important. While some kind of quantum structure of space-time indeed arises e.g. in string theory, a satisfactory understanding is still missing. The cosmological constant problem should be seen in the same context: the observed tiny (or zero) value of the cosmological constant is in striking contradiction with quantum mechanical expectations, which are off by a factor of order $10^{120}$. Reconciling quantum mechanics with gravity is therefore of utmost importance in theoretical physics.

In view of these problems, it is natural to consider noncommutative (NC) or quantum spaces as models for space-time. For fixed backgrounds, considerable work has been done in this context, leading to NC field theory [4]. Recently, it was understood that gravity emerges naturally from NC gauge theory, without having to introduce an explicit dynamical metric. Earlier forms of this idea [6, 5] can be cast in concise form for matrix models of Yang-Mills type [8], which describe dynamical quantum spaces. We discuss basic results of this approach. The IKKT model [10] is singled out as a prime candidate for a quantum theory of space-time and matter.

2. The quantization of Poisson manifolds
Space-time in GR is modeled by a 4-dimensional manifold $\mathcal{M}$ with metric $G_{\mu\nu}(x)$. The basic assumption of the present approach is that space-time carries an additional Poisson structure $\{x^\mu, x^\nu\} = \theta^{\mu\nu}(x)$ (which will be related to the metric in (23)), more precisely that space-time is the quantization $\mathcal{M}_\theta$ of such a Poisson manifold.

In principle, a Poisson structure breaks (local) Lorentz invariance, which may seem incompatible with observation. However, it turns out that $\theta^{\mu\nu}$ does not enter explicitly the
effective action of the models discussed here, to leading order in an expansion in $\theta^{\mu\nu}$. If we assume that the scale of noncommutativity $\Lambda_{NC}$ defined by $\det \theta^{\mu\nu} = \Lambda_{NC}^8$, is at or near the Planck scale, it is then quite conceivable that its presence through higher-order terms in the effective action has not been detected up to now.

It is well-known that a Poisson-manifold $(\mathcal{M}, \theta^{\mu\nu}(x))$ can be quantized [12]. This means that there exists a quantization map

$$
\mathcal{C}(\mathcal{M}) \rightarrow \mathcal{A} \subset L(\mathcal{H})
$$

$$
f(x) \rightarrow \hat{f}(X)
$$

(1)

such that $i\{f, g\} \rightarrow [\hat{f}, \hat{g}] + O(\theta^2)$. Here $\mathcal{C}(\mathcal{M})$ denotes a suitable space of functions on $\mathcal{M}$, and $\mathcal{A}$ is interpreted as quantized algebra of functions on $\mathcal{M}$. The quantization of the coordinate functions $x^\mu$ will be denoted by $X^\mu$. The matrix model will indeed provide preferred coordinates $X^\mu \sim x^\mu$, where $\sim$ indicates the leading contribution in a semi-classical expansion in powers of $\theta^{\mu\nu}$. These $x^\mu$ are not observable and thus not in conflict with any experimental constraints. The integral is replaced by the trace, more precisely by the volume of the symplectic volume form

$$
(2\pi)^2 \text{Tr} \hat{f} \sim \int d^4 x \rho(x) f
$$

$$
\rho(x) = (\det \theta^{-1})^{1/2}
$$

(2)

assuming that $\theta^{\mu\nu}$ is non-degenerate. This is essentially the Bohr-Sommerfeld quantization law.

A simple example of a Poisson manifold is given by $\mathbb{R}^4$ together with a constant antisymmetric matrix $\{x^\mu, x^\nu\} = \tilde{\theta}^{\mu\nu}$. Its quantization gives the Moyal-Weyl quantum plane $\mathbb{R}^4$, where

$$
[X^\mu, X^\nu] = i\tilde{\theta}^{\mu\nu}.
$$

(3)

This is formally the (doubled) Heisenberg algebra, i.e. the usual quantum-mechanical phase space. However, we need to consider the quantization of generic Poisson manifold here. We only consider the semi-classical or geometrical limit of such a quantum space in this paper. This means that the space is described in terms of functions on $\mathcal{M}$ using (1), keeping only the Poisson bracket on the rhs of (1) and dropping all higher-order terms in $\theta$. Accordingly, we will always replace $[\hat{f}(X), \hat{g}(X)] \rightarrow i\{f(x), g(x)\}$ and $[X^\mu, X^\nu] \rightarrow i\theta^{\mu\nu}(x)$. In particular,

$$
[X^\mu, f(X)] \sim i\theta^{\mu\nu}(x) \frac{\partial}{\partial x^\nu} f(x).
$$

(4)

3. Yang-Mills matrix models and their effective geometry

Consider a matrix model with action

$$
S_{YM} = -\text{Tr}[X^a, X^b][X^c, X^d]_{\eta_{ad}}\eta_{bc},
$$

(5)

for hermitian matrices or operators $X^a$, $a = 1, \ldots, D$ acting on some Hilbert space $\mathcal{H}$. Here

$$
\eta_{a'b'} = \text{diag}(1,1,\ldots,1) \quad \text{or} \quad \eta_{ab} = \text{diag}(-1,1,\ldots,1)
$$

(6)

in the Euclidean resp. Minkowski case. The above action is invariant under the following gauge symmetry

$$
X^\mu \rightarrow U^{-1} X^\mu U, \quad U \in U(\mathcal{H}).
$$

(7)

The equations of motion are

$$
[X^a, [X^b, X^c]]_{\eta_{ad}} = 0.
$$

(8)
The matrix model should be considered as background-independent, since no geometry whatsoever is present a priori; rather, the geometry will arise dynamically. In particular, the model admits as solutions 4-dimensional noncommutative spaces $\mathcal{M}_\theta \subset \mathbb{R}^D$, interpreted as space-time embedded in $D$ dimensions. To see this, we split the matrices as

$$X^a = (X^\mu, \phi^i), \quad \mu = 1, \ldots, 4, \ i = 1, \ldots, D - 4$$

where the “scalar fields” $\phi^i = \phi^i(X^\mu)$ are assumed to be functions of $X^\mu$. The prototype of such a solution is a flat embedding of a 4-dimensional quantum space

$$[X^\mu, X^\nu] = i \theta^{\mu\nu}, \quad \mu, \nu = 1, \ldots, 4,$$

$$\phi^i = 0, \quad i = 1, \ldots, D - 4$$

where $X^a$ generates a 4-dimensional NC space $\mathcal{M}_\theta$; for example, $\mathbb{R}^4_\theta \subset \mathbb{R}^D$ is realized by $[X^\mu, X^\nu] = i \theta^{\mu\nu} 1$.

In general, $\phi^i(X) \sim \phi^i(x)$ will be nontrivial. One could interpret $\phi^i(x)$ as scalar fields on $\mathbb{R}^4_\theta$; however, it is more appropriate to interpret $\phi^i(x)$ as purely geometrical degrees of freedom, defining the embedding of a submanifold $\mathcal{M} \subset \mathbb{R}^D$. In the semi-classical limit, suitable “optimally localized states” of $\langle X^a \rangle \sim x^a$ will then be located on $\mathcal{M} \subset \mathbb{R}^D$. This $\mathcal{M}$ carries the induced metric

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \partial_\mu \phi^i \partial_\nu \delta_{ij} = \partial_\mu x^a \partial_\nu x^b \eta_{ab}$$

via pull-back of $\eta_{ab}$. Note that $g_{\mu\nu}(x)$ is not the metric responsible for gravity, and will enter the action only implicitly. Rather, it turns out that all fields arising from the matrix model will only live on the brane $\mathcal{M}$, and couple to the effective metric $G_{\mu\nu}$ given below (17). As opposed to standard braneworld- scenarios, there really is no higher-dimensional “bulk” which could carry physical degrees of freedom here, not even gravitons.

**Effective metric and Poisson structure** Expressing the $\phi^i$ in terms of $X^a$, we obtain

$$[\phi^i, f(X^\mu)] \sim i \theta^{\mu\nu} \partial_\mu \phi^i \partial_\nu f$$

in the semi-classical limit. This involves only the components $\mu = 1, \ldots, 4$ of the antisymmetric “tensor” $[X^a, X^b] \sim i \theta^{ab}(x)$, which has rank 4 in the semi-classical limit. Here the derivations

$$-i [X^\mu, \cdot] \sim \theta^{\mu\nu} \partial_\nu$$

span the 4-dimensional tangent space of $\mathcal{M} \subset \mathbb{R}^D$, and define a preferred frame. We can interpret

$$[X^\mu, X^\nu] \sim i \theta^{\mu\nu}(x)$$

as Poisson structure on $\mathcal{M}$, noting that the Jacobi identity is trivially satisfied. This is the Poisson structure on $\mathcal{M}$ whose quantization is given by the matrices $X^\mu, \ \mu = 1, \ldots, 4$, interpreted as quantization of the coordinate functions $x^\mu$ on $\mathcal{M}$. In particular, the rank of $\theta^{\mu\nu}(x)$ coincides with the dimension of $\mathcal{M}$. Its inverse $\theta^{\mu\nu}(x)$ defines a symplectic form on $\mathcal{M}$.

We can now extract the semi-classical limit of the matrix model and its physical interpretation. To understand the effective geometry on $\mathcal{M}$, consider a (test-) particle on $\mathcal{M}$, modeled by an additional scalar field $\varphi$. Due to gauge invariance, the only reasonable kinetic term is

$$S[\varphi] \equiv -\text{Tr}[X^a, \varphi][X^b, \varphi] \eta_{ab} = -\text{Tr} ([X^\mu, \varphi][X^\nu, \varphi] \eta_{\mu\nu} + [\phi^i, \varphi][\phi^j, \varphi] \delta_{ij})$$

(15)
(for example, $\varphi$ could be an $su(n)$ component of $\phi$). In the semi-classical limit, this becomes

$$S[\varphi] \sim \frac{1}{(2\pi)^2} \int d^4x |G_{\mu\nu}|^{1/2} G_{\mu\nu} \partial_{\mu}\varphi \partial_{\nu}\varphi$$

(16)

which has the correct covariant form, where\footnote{$G^{\mu\nu}$ corresponds to $\tilde{G}^{\mu\nu}$ in \cite{1, 2}.} [2]

$$G^{\mu\nu}(x) = e^{-\sigma} \theta^{\mu'\nu'}(x) \theta^{\nu'\mu'}(x) g_{\mu'\nu'}(x).$$

(17)

Here $g_{\mu\nu}(x)$ is the metric (11) induced on $\mathcal{M} \subset \mathbb{R}^D$ via pull-back of $\eta_{ab}$, and

$$\rho = (\det \theta^{-1})^{1/2}, \quad e^{-\sigma} = \rho |g_{\mu\nu}|^{-\frac{1}{2}},$$

(18)

$$\eta(x) = \frac{1}{4} e^\sigma G_{\mu\nu} g_{\mu\nu}.$$  

(19)

Therefore the kinetic term on $\mathcal{M}_\theta$ is governed by the metric $G_{\mu\nu}(x)$, which depends on the Poisson tensor $\theta^{\mu\nu}$ and the embedding metric $g_{\mu\nu}$. We note that

$$|G_{\mu\nu}(x)| = |g_{\mu\nu}(x)|,$$

(20)

hence $G_{\mu\nu}$ is unimodular for trivially embedded branes (in the preferred matrix coordinates $x^\mu$) \cite{8}. Since $\theta^{\mu\nu}$ does not enter the Riemannian volume at all, this leads to a very interesting mechanism for stabilizing flat space, which may hold the key for the cosmological constant problem as discussed below.

4. Covariant equations of motion

We are now in a position to rewrite the basic matrix e.o.m. (8) in a covariant way. For the scalar fields $\phi^i$, the e.o.m. is

$$0 = [X^a, [X^b, \phi^i]] \eta_{ab} = [X^a, [X^b, \phi^i]] \eta_{ab} + [\phi^j, [\phi^j, \phi^i]] \delta_{ij}$$

$$= i[X^a, \theta^{\beta\gamma} \partial_\beta \phi^i] \eta_{ab} + i[\phi^j, \theta^{\gamma\delta} \partial_\gamma \phi^i] \delta_{ij}$$

$$\sim -\theta^{\beta\gamma} \partial_\beta (\theta^{\gamma\delta} \partial_\gamma \phi^i) \eta_{ab} - \theta^{\gamma\delta} \partial_\gamma (\theta^{\beta\gamma} \partial_\beta \phi^i) \delta_{ij}$$

$$= e^\sigma (\Gamma^\eta \partial_\eta \phi^i - G^{\eta\gamma} \partial_\gamma \phi^i) = -e^\sigma \Delta G \phi^i;$$

(21)

for a detailed derivation see \cite{2}. The same computation gives

$$\Delta G x^\mu = 0 = \Gamma^\mu,$$

(22)

consistent with the ambiguity of the splitting $X^a = (X^\mu, \phi^i)$ into coordinates and scalar fields. Together with (21) this implies \cite{2}

$$G^{\gamma\eta} \nabla_\gamma (e^\sigma \theta_{\eta\nu}^{-1}) = e^{-\sigma} G_{\mu\nu} \theta^{\eta\gamma} \partial_\gamma \eta.$$  

(23)

Here $\nabla$ denotes the Levi-Civita connection with respect to $G_{\mu\nu}$. Remarkably, this equation is also a consequence of a “matrix” Noether theorem due to the symmetry $X^a \rightarrow X^a + e^\sigma \mathbb{I}$, as shown in \cite{1}. This means that equation (23) is protected from quantum corrections, and should be taken serious at the quantum level. It has the structure of covariant Maxwell equations, and gives the relation between the noncommutativity $\theta^{\mu\nu}(x)$ and the effective metric $G^{\mu\nu}$. For
given asymptotical behavior, $\theta^{\mu\nu}(x)$ should therefore be completely determined (apart from gravitational waves, see below) by the scalar functions $\eta(x)$ and $\epsilon^\rho$.

We note that (21) and (22) have a simple interpretation: the embedding $\mathcal{M} \subset \mathbb{R}^D$ is harmonic w.r.t. $G_{\mu\nu}$. In particular, the dynamical matrices $X^\mu \sim x^\mu$ define harmonic coordinates for on-shell geometries, which in general relativity would be interpreted as gauge condition. However, (21) and (22) individually might be subject to quantum corrections as indicated below.

The main message is that the kinetic term in the matrix model necessarily involves the metric $G_{\mu\nu}(x)$. This also holds for gauge fields and fermions, possibly up to density factors. Therefore $G_{\mu\nu}$ must be interpreted as gravitational metric. It is dynamical, because it depends on the embedding fields $\phi^i$ and the Poisson structure $\theta^{\mu\nu}$, both of which are dynamical. Using a standard embedding theorem [11], it follows that $G_{\mu\nu}$ can describe in principle the most general metric in 4 dimensions for the case $D \geq 10$ (ignoring the e.o.m. for now). Therefore the matrix model provides a non-perturbative definition of a theory of space-time and gravity, which naturally includes also gauge fields and matter. The quantization can also be performed relatively easily in a perturbative manner, taking advantage of an alternative interpretation as noncommutative gauge theory: the dynamical Poisson structure can be parametrized in terms of a $u(1)$ gauge field as indicated below. This leads to the hope that this type of matrix model may serve as a model for quantum gravity, quite possibly more accessible than other, more conventional approaches.

5. Dynamical emergent gravity

Before making contact with real physics, we need to understand the dynamics of emergent gravity. Notice that we have not encountered anything like the Einstein-Hilbert action, and in fact it seems impossible to simply add such a term in the matrix model. This is of course extremely interesting: either the model will fail hopelessly, or it will provide a very remarkable and predictive new mechanism for gravity, free of theoretical prejudice.

We can give at least 2 arguments which suggest that the dynamics of the geometry resp. gravity in the matrix model should be quite close to GR:

(i) the dynamics of $\theta^{\mu\nu}(x)$ as given by the matrix model indeed implies $R_{\mu\nu} = 0$ at least for linearized metric fluctuations (30) around flat Moyal-Weyl space without matter. Moreover, it provides the 2 physical degrees of freedom for gravitons (30). This is a remarkable observation, essentially due to Rivelles [5]; see [1] for more details.

(ii) as soon as the matrix model is quantized, the effective action will contain an induced Einstein-Hilbert action. This is discussed below in more detail.

5.1. Quantization and induced gravity

The quantization of the matrix model (5) is defined by

$$Z = \int dX^a e^{-S_{YM}[X]},$$

which is easily generalized to include fermions, notably in the IKKT model [10] for $D = 10$. As shown in [15], this integral is well-defined for finite $N$ (in $D \geq 3$ dimensions), apart from the flat directions corresponding to $X^a \to X^a + e^a\eta$ which could be handled using standard methods. This provides a non-perturbative definition of the model at the quantum level. To gain some insight, we consider a perturbative treatment around a given background as discussed above. For simplicity, consider the quantization of an additional scalar field $\phi$ coupled to the matrix model as in (15), which upon integration leads to an effective action

$$e^{-\Gamma_\phi} = \int d\phi \ e^{-S[\phi]}, \text{ where } \Gamma_\phi = \frac{1}{2} \text{Tr log } \Delta_G.$$  (25)
A standard argument using the heat kernel expansion of $\Delta G$ gives

$$\Gamma_\varphi = \frac{1}{16\pi^2} \int d^4 x \sqrt{|G_{\mu \nu}|} \left( c_1 \Lambda_1^4 + c_4 R[G] \Lambda_1^2 + O(\log \Lambda) \right), \quad (26)$$

which is also the general structure of the 1-loop effective action for the geometrical sector due to other fields. The coefficients $c_i$ as well as the effective cutoffs $\Lambda_i$ depend on the detailed field content of the model, and can be obtained essentially from Seeley-de Witt coefficients\(^2\). This is essentially the mechanism of induced gravity \([14]\).

The action (26) suggests to relate the cutoff $\Lambda_4^2$ with the gravitational constant $\frac{1}{16\pi G}$. On the other hand, $\Lambda_4$ is expected to be the scale of $N = 4$ SUSY breaking, since only $N = 4$ supersymmetric models do not induce this term \([7, 9]\); this is related to the lack of UV/IR mixing in these models as discussed below. This suggests that in order to have a finite gravitational coupling constant, the model should have $N = 4$ supersymmetry above a certain scale. This is realized in the IKKT model, where $D = 10$. The scale $\Lambda_4$ may be different from $\Lambda_4$.

Note that a term $\int d^4 x \sqrt{G} \Lambda_4^4$ is usually interpreted as cosmological constant, and its scaling with $\Lambda_4$ usually presents a major problem. This is nothing but the cosmological constant problem, which is arguably the biggest challenge to our understanding of gravity. We claim that this problem may be resolved here. To see this, recall $|G_{\mu \nu}| = |g_{\mu \nu}|$ (20), independent of $\theta^{\mu \nu}(x)$. Moreover,

$$\delta \int d^4 x \sqrt{G} \sim \int d^4 x \sqrt{g} \delta \theta^{\mu \nu} \sim \int d^4 x \sqrt{g} \phi^{ij} \delta \phi^{ij} \delta_{ij}$$

vanishes for harmonic embeddings $\Delta_g \phi^i = 0$. Therefore for such backgrounds, this “would-be cosmological constant term” is irrelevant and does not enter the equations of motion for the geometry. For example, flat (Moyal-Weyl) space is a solution even at one loop\(^3\), without fine-tuning $\Lambda$. Therefore minimally or harmonically embedded branes (w.r.t. $g_{\mu \nu}$) are protected from the cosmological constant problem \([2]\); the term $\int d^4 x \sqrt{G} \Lambda_4^4$ is present but simply does not imply any cosmological constant. This is due to the particular parametrization of the geometry in terms of $\theta^{\mu \nu}$ and $\phi^i$ rather than a fundamental metric. Notice that $g_{\mu \nu}$ does not necessarily (but quite possibly) coincide with the effective metric $G_{\mu \nu}$. This makes the full analysis of the gravitational sector in the matrix model quite non-trivial, and more work is required. Once more realistic solutions are found which satisfy this stability condition, the present model would arguably be in much better agreement with observation than GR without fine-tuning.

6. Gauge fields

We now relate the above discussion with noncommutative gauge theory. The main result is that the matrix model naturally includes $su(n)$ gauge theory coupled to gravity. If the same model is (improperly) viewed as $u(n)$ gauge theory on $\mathbb{R}^4$, then the trace-$u(1)$ sector is afflicted by the notorious UV/IR mixing. This is nothing but a reflection of the induced gravitational action.

6.1. Would-be $u(1)$ gauge fields

Recall that a particular solution of the matrix equations of motion (8) is given by the generators $X^\mu$ of the Moyal-Weyl quantum plane $\mathbb{R}_{NC}^4$. Its effective geometry is indeed flat, given by

$$\tilde{G}^{\mu \nu} = \tilde{\rho} \tilde{g}^{\mu \nu} \tilde{g}_{\mu \nu} \quad \tilde{\rho} = (\det \tilde{g})^{-1/2} \equiv \Lambda_{NC}^4 \quad (27)$$

\(^2\) However, the contributions from fermions and the “would-be $U(1)$ fields” are non-standard as discussed in \([9]\).  
\(^3\) And presumably to all loops, this can also be seen from the point of view of $U(1)$ gauge theory discussed below.
assuming trivial embedding. Fluctuations of $X^a$ can be parametrized in terms of $u(1)$ gauge fields $A_\mu$ and scalar fields $\phi^i$ on $\mathbb{R}^4$, using

$$X^a = (\tilde{X}^\mu + A_\mu, \phi^i)$$

$$[\tilde{X}^\mu + A_\mu, f] = i \tilde{g}^{\mu\nu} \frac{\partial}{\partial x^\nu} f \equiv i \tilde{g}^{\mu\nu} D_\nu f$$

where $A^a \sim A^a(\vec{x}) \equiv -\tilde{\sigma}^{ab} A_b(\vec{x})$. The actions (5) could then be written as

$$S_{YM} = \int d^4x \left( \tilde{\rho}^{-1} \tilde{G}^{\mu\nu} G_{\mu\nu} F_{\mu\nu} F_{\rho\sigma} + 2 \tilde{G}^{\mu\nu} D_\mu \phi^i D_\nu \phi^j \delta_{ij} - \tilde{\rho}[\phi^i, \phi^j][\phi^i, \phi^j]/\delta_{ij}\delta_{ji} + \tilde{G}^{\mu\nu} g_{\mu\nu} \right)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu]$ is the $u(1)$ field strength. This might suggest an interpretation as noncommutative $u(1)$ gauge theory on $\mathbb{R}^4$. However, $A_\mu(\vec{x})$ is completely absorbed in the metric $G^{\mu\nu}(x)$ resp. $\theta^{\mu\nu}(x)$ in the proper geometrical interpretation, via

$$\theta^{\mu\nu}(x) = \tilde{\theta}^{\mu\nu}(\vec{x}) - \tilde{\theta}^{\mu\nu} \tilde{\theta}^{\rho\sigma} F_{\mu\nu} F_{\rho\sigma}.$$ (29)

This explains why these would-be $u(1)$ gauge fields cannot be disentangled e.g. from the $su(n)$ components: all fields couple to the metric $G_{\mu\nu}$, which is a function of $F_{\mu\nu}$ via (29). In particular, an induced gravitational action (26) arises upon quantization, which does not simply renormalize the tree-level action. This is precisely the “strange” UV/IR contribution [16] to the effective action for the would-be $u(1)$ gauge fields in the IR limit [3].

Expanding the effective metric $G_{\mu\nu} = \tilde{G}_{\mu\nu} + h_{\mu\nu}$ around the flat Moyal case to leading order in $F_{\mu\nu}$ leads to an expression for the linearized metric fluctuations

$$h_{\mu\nu} = -\tilde{G}_{\mu\nu} \tilde{\theta}^{\rho\sigma} \tilde{F}_{\rho\sigma} - \tilde{G}_{\mu\nu} \tilde{\theta}^{\rho\sigma} \tilde{F}_{\rho\sigma}.$$ (30)

Remarkably, the equations of motion of the matrix model imply that the vacuum geometries are Ricci-flat [5],

$$R_{\mu\nu} [G + h] = 0 \quad O(\theta^2), \quad \text{while} \quad R_{\mu\nu \rho\sigma} \neq 0.$$ (31)

6.2. Nonabelian gauge fields

Finally we show how $su(n)$-valued gauge fields arise in the same matrix model, on a suitable background. To avoid confusion we denote such a “nonabelian” background with

$$Y^a = \left( \begin{array}{c} Y^\mu \\ Y^i \end{array} \right) = \left\{ \begin{array}{c} X^\mu \otimes I_n, \quad a = \mu = 1, \ldots, 4, \\
\phi^i \otimes I_n, \quad a = 4 + i, \quad i = 1, \ldots, D - 4. \end{array} \right.$$

This can be understood as $n$ copies of the brane configurations considered above. We want to understand general fluctuations around this new background. Since the $u(1)$ components describe the geometry, we expect to find $su(n)$-valued gauge fields as well as scalar fields in the adjoint. It turns out that the following gives an appropriate parametrization of these general fluctuations:

$$\left( \begin{array}{c} Y^\mu \\ Y^i \end{array} \right) = \left( \begin{array}{c} X^\mu \otimes I_n + A^\mu \\
\phi^i \otimes I_n + A_i \Phi^i + A^\rho \partial_\rho (\phi^i \otimes I_n + \Phi^i) \end{array} \right)$$

where

$$A^\mu = A^\mu_\alpha \otimes \lambda^\alpha = -\theta^{\mu\nu} A_{\nu\alpha} \otimes \lambda^\alpha,$$

$$\Phi^i = \Phi^i_\alpha \otimes \lambda^\alpha,$$

(34)
parametrize the $su(n)$-valued gauge fields resp. scalar fields, and $\lambda^a$ denotes the generators of $su(n)$. This amounts to the leading term in a Seiberg-Witten map [13], relating noncommutative and commutative $su(n)$ gauge fields.

One can now show that the semi-classical limit of the matrix model action (5) for these $su(n)$-valued gauge fields $A_\mu$ on general 4-dimensional $\mathcal{M}_\partial \subset \mathbb{R}^4$ is given by

$$ S_{YM}[A] \sim \int d^4x \sqrt{|G_{\mu\nu}|} e^\sigma G^\mu_{\mu'} G^{\nu}_{\nu'} \text{tr}(F_{\mu\nu} F^\mu_{\mu'} F_{\nu\nu'}) + 2 \int \eta(x) \text{tr} F \wedge F. \quad (35) $$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the $su(n)$-valued field strength. This was shown in [8] for $D = 4$ using a direct but long computation, and in [1] for the general case via the equations of motion. A similar result applies for higher-dimensional branes. Remarkably, the corresponding Yang-Mills equations of motion are a direct consequence of a matrix Noether theorem corresponding to the symmetry $X^a \rightarrow X^a + c^a I$ of the matrix model [1].

Finally, fermions are naturally included in these models [10, 2] and turn out to couple to the same metric $G_{\mu\nu}$, albeit possibly (depending on the geometry) with a non-standard spin connection; see [9] for results on the $D = 4$ case.

We conclude that Yang-Mills matrix models are strong candidates for a unified theory of fundamental interactions, and promise advantages over GR for quantization and the cosmological constant problem. A simple and intrinsically noncommutative mechanism for gravity has been identified. However, more work is required to obtain a thorough understanding and judgment.

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