Speed of convergence towards attracting sets for endomorphisms of $\mathbb{P}^k$

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Abstract

Let $f$ be a non-invertible holomorphic endomorphism of $\mathbb{P}^k$ having an attracting set $A$. We show that, under some natural assumptions, $A$ supports a unique invariant positive closed current $\tau$, of the right bidegree and of mass $1$. Moreover, if $R$ is a current supported in a small neighborhood of $A$ then its push-forwards by $f^n$ converge to $\tau$ exponentially fast. We also prove that the equilibrium measure on $A$ is hyperbolic.

1 Introduction

Let $f$ be a holomorphic endomorphism of algebraic degree $d \geq 2$ on the complex projective space $\mathbb{P}^k$. A compact subset $A$ of $\mathbb{P}^k$ is called an attracting set if it has a trapping neighborhood $U$ i.e. $f(U) \subset U$ and $A = \cap_{n \geq 0} f^n(U)$ where $f^n := f \circ \cdots \circ f$, $n$ times. It follows that $A$ is invariant, $f(A) = A$. Furthermore, if $A$ contains a dense orbit then $A$ is a trapped attractor. Typical examples of such objects are fractal and their underlying dynamics are hard to study. We refer to [Mil85], [Rue89] for general discussions on attractors and to [FW99], [JW00], [FS01], [BDM07], [Taf10] and references therein for examples of different types of attractors in $\mathbb{P}^2$.

Attracting sets are stable under small perturbations. Indeed, if $f$ has an attracting set $A = \cap_{n \geq 0} f^n(U)$ then any small perturbation $f_\epsilon$ of $f$ has an attracting set defined by $A_\epsilon = \cap_{n \geq 0} f_\epsilon^n(U)$. For example, when $f$ restricted to $\mathbb{C}^k$ defines a polynomial self-map then the hyperplane at infinity $\mathbb{P}^k \setminus \mathbb{C}^k$ is an attracting set. In the same way, it is easy to create examples where the attracting set is a projective subspace of arbitrary dimension. In this paper, we consider a family of endomorphisms, stable under small perturbations,
which contains these examples. It was introduced by Dinh in [Dim07] and we briefly recall the context.

In the sequel, we always assume that \( f \) possesses an attracting set \( A \) which has a trapped neighborhood \( U \) satisfying the following properties. There exist an integer \( 1 \leq p \leq k - 1 \) and two projective subspaces \( I \) and \( L \) of dimension \( p - 1 \) and \( k - p \) respectively such that \( I \cap U = \emptyset \) and \( L \subset U \). We do not assume that \( L \) and \( I \) are invariant. Since \( I \cap L = \emptyset \), for each \( x \in L \) there exists a unique projective subspace \( I(x) \) of dimension \( p \) which contains \( I \) and such that \( L \cap I(x) = \{ x \} \). Furthermore, for each \( x \in L \) we ask that \( U \cap I(x) \) is strictly convex as a subset of \( I(x) \setminus I \simeq \mathbb{C}^p \). All these assumptions are stable under small perturbations of \( f \). The geometric assumption on \( U \) is slightly stronger than the one of Dinh, who only requires \( U \cap I(x) \) to be star-shaped at \( x \). We need convexity in order the solve the \( \bar{\partial} \)-equation on \( U \). Indeed, under our assumption \( U \) is a \((p-1)\)-convex domain in the sense of [HL88].

If \( E \) is a subset of \( \mathbb{P}^k \), let \( \mathcal{C}_q(E) \) denote the set of all positive closed currents of bidegree \((q,q)\), supported in \( E \) and of mass 1. It is well known that for any integer \( 1 \leq q \leq k \) and any smooth form \( S \) in \( \mathcal{C}_q(\mathbb{P}^k) \), the sequence \( d^{-q \cdot n}(f^n)^*(S) \) converges to a positive closed current \( T^q \) of bidegree \((q,q)\) called the Green current of order \( q \) of \( f \). We refer to [DS10] for a detailed exposition on these currents and their effectiveness in holomorphic dynamics.

When \( q = k \), it gives the equilibrium measure of \( f, \mu := T^k \). It is exponentially mixing and it is the unique measure of maximal entropy \( k \log d \) on \( \mathbb{P}^k \). Moreover, it is hyperbolic and all its Lyapunov exponents are larger or equal to \((\log d)/2 \). The dynamics outside the support of \( \mu \) is not very well understood. The aim of this paper is to continue the investigation started in [Dim07] on the attracting sets described above, which do not intersect \( \text{supp}(\mu) \). Indeed, since \( I \cap U = \emptyset \), by regularization there exists a smooth form \( S \in \mathcal{C}_q(\mathbb{P}^k \setminus \overline{U}) \), where \( \Omega := \mathbb{P}^k \setminus \overline{U} \). As \( f^{-1}(\Omega) \subset \Omega \), it follows that \( \text{supp}(T^{k-p+1}) \cap U = \emptyset \), and hence \( \text{supp}(T^q) \cap U = \emptyset \) if \( q \geq k-p+1 \).

The set \( \mathcal{C}_p(U) \) is non-empty since it contains the current \([L] \) of integration on \( L \) and its regularizations in \( U \). In the situation described above, Dinh proved that if \( R \) is a continuous element of \( \mathcal{C}_p(U) \) then its normalized push-forwards by \( f^n, d^{-(k-p)n}(f^n)_*(R) \), converge to a current \( \tau \) which is independent of the choice of \( R \). Moreover, the current \( \tau \) gives us information on the geometry of \( A \) and on the dynamics of \( f_A \): it is supported, in \( A \) and invariant i.e. \( f_A(\tau) = d^{k-p}\tau \). Our main result is that, with a natural additional assumption on \( f_U \), stable under small perturbations, we obtain an explicit exponential speed of the above convergence for any \( R \) in \( \mathcal{C}_p(U) \).

**Theorem 1.1.** Let \( f \) and \( \tau \) be as above and assume that \( \| \wedge^{k-p+1} D f(z) \| < 1 \) on \( U \). There is a constant \( 0 < \lambda < 1 \) such that for each \( 0 < \alpha \leq 2 \) the
following property holds. There exists $C > 0$ such that for any element $R$ of $\mathcal{C}_p(U)$ and any $(k-p,k-p)$-form $\varphi$ of class $C^\alpha$ on $\mathbb{P}^k$ we have

$$|\langle d^{- (k-p)n}(f^n)_*(R) - \tau, \varphi \rangle| \leq C \lambda^{n/2} \|\varphi\|_{C^\alpha}. \quad (1.1)$$

In particular, $\tau$ is the unique invariant current in $\mathcal{C}_p(U)$ and $d^{- (k-p)n}(f^n)_*(R)$ converge to $\tau$ uniformly on $R \in \mathcal{C}_p(U)$.

Recall that $f$ induces a self-map $Df$ on the tangent bundle $T\mathbb{P}^k$ which also gives a self-map $\wedge^q Df$ on the exterior power $\wedge^q T\mathbb{P}^k$, $1 \leq q \leq k$. In the sequel, all the norms on $C^\alpha$, $L^r$, etc. are with respect to the Fubini-Study metric on $\mathbb{P}^k$. It also gives a uniform norm which induces an operator norm for $\wedge^q Df$.

In the same spirit as Theorem 1.1, we proved in [Taf11] that for a generic current $S$ in $\mathcal{C}_1(\mathbb{P}^k)$, the sequence $d^{-n}(f^n)^*(S)$ converges to $T$ exponentially fast. However, the contexts are quite different. Here, we consider currents of arbitrary bidegree which are in general much harder to handle. Moreover, in [Taf11] we deeply use that $T$ has Hölder continuous local potentials. In the present situation, we can expect that the attracting current $\tau$ is always more singular. The idea to prove Theorem 1.1 is to use Henkin-Leiterer’s solution with estimates of the $dd^c$-equation on $U$ in order to study separately the harmonic and non-harmonic parts of the left hand side term of (1.1). When $dd^c \varphi = 0$ on $U$, we use the “geometry” of $\mathcal{C}_p(U)$, introduced in [Din07] and [DS06], and Harnack’s inequality to obtain exponential estimates. In order to deal with the non-harmonic part, we use the assumption on $\|\wedge^{k-p+1} Df\|$. This assumption comes naturally in several basic examples and their perturbations.

In [Din07], Dinh also showed that the equilibrium measure associated to $A$, defined by $\nu := \tau \wedge T^{k-p}$, is invariant, mixing and of maximal entropy $(k-p) \log d$ on $A$. Theorem 1.1 is a first step in order to obtain other ergodic and stochastic properties on $\nu$ as exponential mixing or central limit theorem. We postpone this question in a future work.

Under the same assumptions, we deduce from the work of de Thélin [dT08], see also [Dup09], the following result on $\nu$.

**Theorem 1.2.** If $f$ is as in Theorem 1.1 then the measure $\nu$ is hyperbolic. More precisely, counting with multiplicity it has $k-p$ Lyapunov exponents larger than or equal to $(\log d)/2$ and $p$ Lyapunov exponents strictly smaller than $-(k-p)(\log d)/2$.  

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2 Structural discs of currents

In this section we recall the notion of structural varieties of currents. It was introduced by Dinh and Sibony in order to put a geometry on the space $\mathcal{C}_p(U)$ which is of infinite dimension, see [DS06] and [Din07]. The definition of structural varieties is based on slicing theory and they can be seen as complex subvarieties inside $\mathcal{C}_p(U)$. In [DS09], the authors developed the notion of super-potential which involves more deeply this geometry.

Slicing theory can be seen as a generalization to currents of restriction of smooth forms to submanifolds. We will briefly explain it in our context and refer to [Fed69] for a more complete account.

Let $U$ be an open subset of $\mathbb{P}^k$ satisfying the geometric hypothesis as above. Let $V$ be a complex manifold of dimension $l$. We denote by $\pi_U$ and $\pi_V$ the canonical projections of $U \times V$ to $U$ and $V$ respectively. If $\mathcal{R}$ is a positive closed current of bidegree $(p,p)$ on $U \times V$ with $\pi_U(\text{supp}(\mathcal{R})) \subseteq U$ then, for all $\theta$ in $V$, the slice $\langle \mathcal{R}, \pi_V, \theta \rangle$ is well defined. For any $(k-p,k-p)$-form $\phi$ on $U \times V$ we have

$$\langle \mathcal{R}, \pi_V, \theta \rangle(\phi) = \lim_{\epsilon \to 0} \langle \mathcal{R} \wedge \pi_V^*(\psi_{\theta,\epsilon}), \phi \rangle,$$

where $\psi_{\theta,\epsilon}$ is an appropriate approximation in $V$ of the Dirac mass at $\theta$. It is a $(p+l,p+l)$-current on $U \times V$ supported on $U \times \{\theta\}$ which can be identified to a $(p,p)$-current on $U$. A family of currents $(R_{\theta})_{\theta \in V}$ in $\mathcal{C}_p(U)$ is a structural variety if there exists a positive closed current $\mathcal{R}$ in $U \times V$ such that $R_{\theta} = \langle \mathcal{R}, \pi_V, \theta \rangle$. When $V$ is isomorphic to the unit disc of $\mathbb{C}$, we call $(R_{\theta})_{\theta \in V}$ a structural disc.

Recall that in our situation $f(U) \subseteq U$. Under the geometrical assumption on $U$, Dinh constructed in [Din07] p.233 a family of structural discs in $\mathcal{C}_p(U)$. He uses that for each $x \in L$ the set $I(x) \cap U$ is star-shaped at $x$ to obtain a property similar to star-sharpness for $\mathcal{C}_p(U)$.

More precisely, up to an automorphism, we can assume that

$$I = \{x \in \mathbb{P}^k \mid x_i = 0, \ 0 \leq i \leq k-p\}, \quad L = \{x \in \mathbb{P}^k \mid x_i = 0, \ k-p+1 \leq i \leq k\},$$

where $x = [x_0 : \cdots : x_k]$ are the homogeneous coordinates of $\mathbb{P}^k$. For $\theta \in \mathbb{C}$, $A_{\theta}(x) := [x_0 : \cdots : x_{k-p} : \theta x_{k-p+1} : \cdots : \theta x_k]$ is an automorphism of $\mathbb{P}^k$ except for $\theta = 0$ where it is the projection of $\mathbb{P}^k \setminus I$ on $L$. Let set $U' := f(U)$. As $I(x) \cap U$ is star-shaped at $x \in L$, there exists a simply connected open neighborhood $V \subseteq \mathbb{C}$ of $[0,1]$ such that $A_{\theta}(U') \subseteq U$ for all $\theta$ in $V$. If $S$ is in $\mathcal{C}_p(U')$ then the family $(S_{\theta})_{\theta \in V}$ with $S_{\theta} := (A_{\theta})^*(S)$ defined a structural disc. Indeed, if $\Lambda : \mathbb{P}^k \times V \to \mathbb{P}^k$ is the meromorphic map defined by $\Lambda(x, \theta) = (A_{\theta})^{-1}(x)$ and $\mathcal{S} := \Lambda^* S$ then $S_{\theta} = \langle \mathcal{S}, \pi_V, \theta \rangle$, see [Din07] for
more details. For any $S$ in $\mathcal{C}_p(U')$, we have that $S_1 = S$ and $S_0 = [L]$ which is independent of $S$. In other words, any current in $\mathcal{C}_p(U')$ is linked to $[L]$ by a structural disc in $\mathcal{C}_p(U)$. Moreover, $S_{\theta}$ depends continuously on $\theta$ and we have the following important property.

**Lemma 2.1** ([Dim07]). Let $S$ be in $\mathcal{C}_p(U')$ and $(S_{\theta})_{\theta \in \mathcal{V}}$ be the structural disc described above. If $\phi$ is a real continuous $(k - p, k - p)$-form with $dd^c \phi = 0$ on $U$ then the function $\theta \mapsto \langle S_{\theta}, \phi \rangle$ is harmonic on $\mathcal{V}$.

### 3 q-Convex set and d-bar equation

The concept of $q$-convexity generalizes both Stein and compact manifolds. Andreotti and Grauert [AG62] obtained vanishing or finiteness theorems for $q$-convex manifolds and, in [HL88], Henkin and Leiterer developed a similar theory using integral representations. In particular, they obtained solutions of the $\bar{\partial}$-equation with explicit estimates, which play a key role in our proof. For this reason, we use the conventions of [HL88].

If $1 \leq q \leq k$ is an integer then a real $C^2$ function $\rho$ on an open subset $V \subset \mathbb{P}^k$ is called $q$-convex if, in any holomorphic local coordinates, the Hermitian form

$$L_{\rho}(x) t = \sum_{i,j=1}^{k} \frac{\partial^2 \rho(x)}{\partial z_i \overline{\partial z_j}} t_i \overline{t_j}$$

has at least $q$ strictly positive eigenvalues at any point $x \in V$.

Let $0 \leq q \leq k - 1$. We say that an open subset $D$ of $\mathbb{P}^k$ is strictly $q$-convex if there exists a $(q + 1)$-convex function $\rho$ in a neighborhood $V$ of $\partial D$ such that

$$D \cap V = \{ x \in V \mid \rho(x) < 0 \}.$$ 

Moreover, if the same condition is satisfied with $V$ a neighborhood of $\overline{D}$ then $D$ is called completely strictly $q$-convex.

The strict $q$-convexity has the following important consequence, see [HL88, Theorem 11.2].

**Theorem 3.1.** Let $D$ be a strictly $q$-convex open subset of $\mathbb{P}^k$ with $C^2$ boundary. If $\phi$ is a continuous $\bar{\partial}$-exact form of bidegree $(r, s)$ in a neighborhood of $\overline{D}$ with $0 \leq s \leq k$, $k - q \leq r \leq k$, then there exists a continuous $(s, r - 1)$-form $\psi$ on $D$ such that $\bar{\partial} \psi = \phi$ and

$$\| \psi \|_{\infty, D} \leq C \| \phi \|_{\infty, D}$$

for some $C > 0$ independent of $\phi$. 

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Furthermore, Andreotti and Grauert proved the following vanishing theorem, see [AG62] and [HL88, Theorem 12.7].

**Theorem 3.2.** If \( D \) is a completely strictly \( q \)-convex open subset of \( \mathbb{P}^k \) with \( C^2 \) boundary then \( H^{s,r}(D, \mathbb{C}) = 0 \) for any \( 0 \leq s \leq k \) and \( k - q \leq r \leq k \).

Henkin and Leiterer [HL88, Theorem 5.13] give the following criteria of \( q \)-convexity, which is closely related to our geometric assumption on \( U \) with \( q = p - 1 \).

**Theorem 3.3.** Let \( D \) be an open subset of \( \mathbb{P}^k \) with \( C^2 \) boundary. If for each \( x \in \partial D \) there exists a complex submanifold \( Y \subset \mathbb{P}^k \) of dimension \( q + 1 \), transverse to \( \partial D \) and such that \( Y \cap D \) is a strictly pseudoconvex domain in \( Y \), then \( D \) is strictly \( q \)-convex.

This result applies to our trapping neighborhood \( U \) with \( q = p - 1 \). Indeed, observe that, possibly by exchanging \( U \) by a slightly smaller open set which contains \( f(U) \), we can assume that \( \partial U \) is smooth and the intersection of \( \partial U \) with \( I(x) \) is transverse for all \( x \in L \). The projective space \( I(x) \) has dimension \( p = q + 1 \) and \( U \cap I(x) \) is strictly convex in \( I(x) \setminus I \simeq \mathbb{C}^p \), so in particular strictly pseudoconvex in \( I(x) \). Therefore, by Theorem 3.3, \( U \) is strictly \((p - 1)\)-convex. In the sequel, we always choose such an attracting neighborhood \( U \).

Up to an automorphism of \( \mathbb{P}^k \), \( I \) is defined in homogeneous coordinates by \( I = \{ x \in \mathbb{P}^k \mid x_i = 0, \ 0 \leq i \leq k - p \} \). The function

\[
\eta(x) = \frac{|x_{k-p+1}|^2 + \cdots + |x_k|^2}{|x_0|^2 + \cdots + |x_{k-p}|^2},
\]

is a \((q + 1)\)-convex exhausting function of \( \mathbb{P}^k \setminus I \), i.e. \( \mathbb{P}^k \setminus I \) is completely \( q \)-convex. In general, strictly \( q \)-convex subsets of a completely \( q \)-convex domain are not completely strictly \( q \)-convex. However, in our case it is easy to construct from a \( q \)-convex function \( \rho \) such that

\[
U \cap V = \{ x \in V \mid \rho(x) < 0 \}
\]

for some neighborhood \( V \) of \( \partial U \), a \( q \)-convex defining function defined in a neighborhood of \( \overline{U} \). Indeed, it is enough to compose \((\eta, \rho)\) with a good approximation of the maximum function (see [HL88, Definition 4.12]). It will give a \((q + 1)\)-convex function since the positive eigenvalues of the complex Hessians of \( \rho \) and \( \eta \) are in the same directions. Therefore, \( U \) is completely strictly \((p - 1)\)-convex and we have the following solution for the \( dd^c \)-equation in symmetric bidegrees.
Theorem 3.4. Let $U$ be as above. If $\varphi$ is a $C^2(r,r)$-form in a neighborhood of $U$ with $k - p \leq r \leq k$, then there exists a continuous $(r,r)$-form $\psi$ on $U$ such that $dd^c \psi = dd^c \varphi$ and

$$\|\psi\|_{\infty,U} \leq C \|dd^c \varphi\|_{\infty,U}$$

for some $C > 0$ independent of $\varphi$.

Proof. The proof follows closely the proof of Theorem 2.7 in [DNS08].

Without loss of generality, we can assume that $\varphi$ is real and therefore $dd^c \varphi$ is also real. First, we solve the equation $d\xi = dd^c \varphi$ with estimates. Let $W$ be a small neighborhood of $U$, with the same geometric property and such that $\varphi$ is defined on $W$. The maps $A_\theta$ defined in Section 2 give a homotopy $A_\theta : [0,1] \times W \to W$, $A_\theta(x) = A_\theta(x)$, between $A_1 = \text{Id}$ and the projection $A_0$ of $W$ on $L$. Since $L$ has dimension $k - p$, $A_0^* \varphi$ vanish identically on $(r + 1, r + 1)$-forms if $r \geq k - p$. Therefore, by homotopy formula (see e.g. [BT82, p38]), there exists a form $\xi$ on $W$ such that $d\xi = dd^c \varphi$ and $\|\xi\|_{\infty,U} \lesssim \|dd^c \varphi\|_{\infty,U}$.

Moreover, possibly by exchanging $\xi$ by $(\xi + \xi)/2$, we can assume that $\xi = \Xi + \Xi$ where $\Xi$ is a $(r,r+1)$-form. As $d\xi$ is a $(r+1, r+1)$-form, it follows that $\partial \Xi = 0$ and $d\Xi = \partial \Xi + \partial \Xi$. Therefore, by Theorem 3.2, $\Xi$ is $\partial$-exact and by Theorem 3.1, there exists a continuous $(r,r)$-form $\Psi$ such that $\partial \Psi = \Xi$ and $\|\Psi\|_{\infty,U} \lesssim \|\Xi\|_{\infty,U}$.

Finally, if $\psi = -i\pi(\Psi - \Psi)$ we have

$$dd^c \psi = \partial \partial(\Psi - \Psi) = \partial \Xi + \partial \Xi = dd^c \varphi,$$

and

$$\|\psi\|_{\infty,U} \lesssim \|\Xi\|_{\infty,U} \lesssim \|dd^c \varphi\|_{\infty,U}.$$
real continuous form with \( d\mathbf{c}(f^*(\phi)) = 0 \). The set \( X \) is a truncated convex cone and the first part of the proof of Theorem 1.1 is to show that \( d^{-1}(k-p)f^* \) acts by contraction on it. This result is available without any assumption on \( \|\wedge^{k-p+1}Df\| \). It is based on Lemma 2.1 and Harnack’s inequality for harmonic functions.

**Lemma 4.1.** There exists a constant \( 0 < \lambda_1 < 1 \) such that for any \( R \) in \( \mathcal{C}_p(U) \), \( \phi \) in \( X \) and \( n \) in \( \mathbb{N} \) we have

\[
|\langle R_n - \tau, \phi \rangle| \leq \lambda_1^n.
\]

**Proof.** If \( R \) is in \( \mathcal{C}_p(U) \) and \( \phi \) in \( X \), \( R_1 := d^{-(k-p)}f_*(R) \) is in \( \mathcal{C}_p(U') \) and we define the function \( h_{R,\phi} \) on \( V \) by \( h_{R,\phi}(\theta) := (R_{1,\theta} - \tau, \phi) \), where \( \theta \mapsto R_{1,\theta} \) is the structural disc described in Section 2. The definition of \( X \) implies that \( |h_{R,\phi}| \leq 1 \) on \( V \), for all \( R \in \mathcal{C}_p(U) \) and \( \phi \in X \). Moreover, since \( R_1 \) is in \( \mathcal{C}_p(U') \), it follows from Lemma 2.1 that all these functions are harmonic on \( V \).

Now, observe that if we take \( R = \tau \) then \( h_{R,\phi}(1) = 0 \) for all \( \phi \in X \), since \( d^{p-k}f_*\tau = \tau \). Hence, as \( |h_{R,\phi}| \leq 1 \) on \( V \), Harnack’s inequality says that there exists \( 0 \leq a < 1 \) such that \( |h_{R,\phi}(0)| \leq a \) for all \( \phi \) in \( X \). On the other hand, \( R_{1,0} \) is a current independent of \( R \). So, for all \( R \in \mathcal{C}_p(U) \) and \( \phi \in X \) we have \( h_{R,\phi}(0) = h_{R,\phi}(0) \) and therefore \( |h_{R,\phi}(0)| \leq a \). Once again, we deduce from Harnack’s inequality there exists \( 0 < \lambda_1 < 1 \), independent of \( R \) and \( \phi \), such that \( |h_{R,\phi}(1)| \leq \lambda_1 \) or equivalently

\[
\left| \langle R_1 - \tau, \frac{\phi}{\lambda_1} \rangle \right| = |\langle R - \tau, \phi_1 \rangle| \leq 1,
\]

where \( \phi_1 = d^{-(k-p)}f^*(\phi/\lambda_1) \). Moreover, \( \phi_1 \) is defined on \( U \) and \( d\mathbf{c}\phi_1 = 0 \). It follows that \( \phi_1 \) is in \( X \). Using the same arguments with \( \phi_1 \) instead of \( \phi \) gives that \( |\langle R_1 - \tau, \phi_1 \rangle| \leq \lambda_1 \) which can be rewrite \( |\langle R_2 - \tau, \phi \rangle| \leq \lambda_1^2 \). Inductively, we obtain that \( |\langle R_n - \tau, \phi \rangle| \leq \lambda_1^n \). \( \square \)

**Remark 4.2.** The constant \( \lambda_1 \) is not directly related to \( f \). Indeed, it only depends on \( V \) i.e. on the size of \( U \) and the distance between \( \partial U \) and \( \partial f(U) \).

If \( h \) is the unique biholomorphism between \( V \) and the unit disc in \( \mathbb{C} \) such that \( h(0) = 0 \) and \( h(1) = \alpha \in [0,1] \) then Harnack’s inequality gives explicitly that we can take \( a = 2\alpha/(1+\alpha) \) and \( \lambda_1 = 4\alpha/(1+\alpha)^2 \).

In order to prove Theorem 1.1 we use Theorem 3.4 together with the assumption on \( \|\wedge^{k-p+1}Df\| \) and Lemma 4.1.
If \( \| \Lambda^{k-p+1} Df(z) \| < 1 \) on \( \overline{U} \) then by continuity, there exists a constant 0 < \( \lambda_2 < 1 \) such that \( \| \Lambda^{k-p+1} Df(z) \| < \lambda_2 \) on \( U \). Hence, if \( \varphi \) is a \((k-p,k-p)\)-form of class \( C^2 \), we have for \( \varphi_i := d^{-i(k-p)}(f^*)^i(\varphi) \) with \( i \in \mathbb{N} \)

\[
\|dd^c \varphi_i\|_{\infty,U} \lesssim \frac{\lambda_2^i}{d^{i(k-p)}} \| \varphi \|_{C^2}.
\]

Here, the symbol \( \lesssim \) means inequality up to a constant which only depends on our conventions and on \( U \). By Theorem 3.4 with \( r = k-p \), there exists a continuous \((k-p,k-p)\)-form \( \psi_i \) on \( U \) such that

\[
\begin{align*}
\langle R - \tau, \varphi_i \rangle &= \langle R_i - \tau, \varphi_i \rangle = \langle R_i - \tau, \varphi_i - \psi_i \rangle + \langle R_i - \tau, \psi_i \rangle,
\end{align*}
\]

since \( \tau \) is invariant. On the one hand,

\[
\langle R_i - \tau, \psi_i \rangle \lesssim \| \psi_i \|_{\infty,U} \lesssim \frac{\lambda_2^i}{d^{i(k-p)}} \| \varphi \|_{C^2},
\]

since \( R_i \) and \( \tau \) are supported on \( U \). On the other hand, observe that there exists a constant \( M \geq 1 \) independent of \( \varphi \) such that \( \| d^{-(k-p)} f^*(\varphi) \|_{\infty} \leq M \| \varphi \|_{\infty} \). Therefore,

\[
\begin{align*}
\| \varphi_i - \psi_i \|_{\infty,U} &\leq M^i \| \varphi \|_{\infty} + \| \psi_i \|_{\infty,U} \leq M^i \| \varphi \|_{\infty} + C \frac{\lambda_2^i}{d^{i(k-p)}} \| \varphi \|_{C^2} \\
\lesssim M^i \| \varphi \|_{C^2},
\end{align*}
\]

and in particular

\[
|\langle S - \tau, \varphi_i - \psi_i \rangle| \lesssim M^i \| \varphi \|_{C^2},
\]

for any \( S \) in \( \mathcal{C}_r(U) \).

Moreover, \( \varphi_i - \psi_i \) is a real continuous \((k-p,k-p)\)-form on \( U \) and \( d\bar{d} \varphi_i = d\bar{d} \psi_i = 0 \). Hence, \( (\varphi_i - \psi_i)/(CM^i \| \varphi \|_{C^2}) \) belongs to \( X \) where \( C > 0 \)

\[
\text{End of the proof of Theorem 1.1.}
\]
is a constant depending only on $U$ and on our conventions. It follows by Lemma 4.1 that

$$|\langle R_l - \tau, \varphi \rangle| \leq CM_l^2 \|\varphi\|_{C^2} \lambda_1^l. \quad (4.2)$$

To summarize, equations (4.1) and (4.2) imply that there are constants $0 < \lambda_1, \lambda_2 < 1$, and $M \geq 1$ such that

$$|\langle R_n - \tau, \varphi \rangle| \lesssim \|\varphi\|_{C^2} \left( M^l \lambda_1^l + \frac{\lambda_2^l}{d^{(k-p)}} \right).$$

If $q \in \mathbb{N}$ is large enough then $M \lambda_q^q < 1$. Therefore, if we choose $n \simeq (q + 1)i$, we obtain $l \simeq iq$ and

$$|\langle R_n - \tau, \varphi \rangle| \lesssim \|\varphi\|_{C^2} \lambda^n,$$

where $\lambda := \max(\lambda_2^2 d^{-(k-p)}, M \lambda_1^q)^{1/(q+1)} < 1$. This estimate holds for arbitrary $n$ in $\mathbb{N}$ and is uniform on $\varphi$ and $R$. \hfill \Box

**Remark 4.3.** In Theorem 1.1, it is enough to assume that $\|\wedge^{k-p+1} Df(z)\| < d^{(k-p)/2}$ on $\overline{U}$. Moreover, it is easy using small perturbations of a suitable polynomial map to construct examples with $\|\wedge^{k-p+1} Df(z)\|$ as small as we want on $\overline{U}$.

## 5 Hyperbolicity of the equilibrium measure

In this section, we prove Theorem 1.2. Recall that the equilibrium measure associated to $A$ is given by $\nu := \tau \wedge T^{k-p}$. It has maximal entropy on $A$ equal to $(k-p) \log d$. On the other hand, we have the following powerful result, see [dT08] and [Dup09].

**Theorem 5.1.** If the Lyapunov exponents of $\nu$ are ordered so that

$$\chi_1 \geq \cdots \geq \chi_{a-1} > \chi_a \geq \cdots \geq \chi_k,$$

then

$$h(\nu) \leq (a - 1) \log d + 2 \sum_{i=a}^{k} \chi_i^+, \quad (5.1)$$

where $h(\nu)$ denotes the entropy of $\nu$ and $\chi_i^+ := \max(\chi_i, 0)$.

Now, let $1 \leq c \leq k$ be such that

$$\chi_1 \geq \cdots \geq \chi_c > 0 \geq \chi_{c+1} \geq \cdots \geq \chi_k.$$

If we take $a = c + 1$ in Theorem 5.1, we obtain $h(\nu) \leq c \log d$. Since $h(\nu) = (k-p) \log d$, it follows that $c \geq (k-p)$. It means there are at least $k-p$ strictly
positive Lyapunov exponents. Moreover, if we have equality, \( c = k - p \), Theorem \[5.1\] applied to the smallest \( a \) such that \( \chi_a = \chi_c \) gives
\[
(k - p) \log d = h(\nu) \leq (a - 1) \log d + 2(k - p - a + 1)\chi_c.
\]
Hence, \( \chi_c \geq (\log d) / 2. \) Note that in this part we do not need the assumption on \( \| \wedge^{k-p+1} Df \| \).

It remains to prove that the assumptions of Theorem \[1.1\] imply that \( c \leq k - p \) and \( \chi_{c+1} < -(k - p)(\log d) / 2. \) It is not hard to deduce form Oseledec theorem \[Ose68\] that the sum of the \( q \) largest Lyapunov exponents verifies
\[
\chi_1 + \cdots + \chi_q = \lim_{n \to \infty} \frac{1}{n} \log \| \wedge^q Df^n(z) \|
\]
for \( \nu \)-almost all \( z. \) Moreover, we have
\[
\| \wedge^q Df^{n+m}(z) \| \leq \| \wedge^q Df^n(z) \| \| \wedge^q Df^m(f^n(z)) \|.
\]
Therefore, it follows that
\[
\| \wedge^q Df^n(z) \| \leq (\max_{z \in U} \| \wedge^q Df(z) \|)^n
\]
and
\[
\chi_1 + \cdots + \chi_q \leq \log \max_{z \in U} \| \wedge^q Df(z) \| =: \gamma.
\]
Hence, if \( \| \wedge^{k-p+1} Df(z) \| < 1 \) on \( \overline{U} \) then
\[
\chi_1 + \cdots + \chi_{k-p+1} \leq \gamma < 0.
\]
Therefore, \( c \leq k - p \) and we have seen above that in this case \( c = k - p \) and \( \chi_c \geq (\log d) / 2. \) Finally, we have
\[
\gamma \geq \chi_1 + \cdots + \chi_{k-p} + \chi_{k-p+1} \geq \frac{k - p}{2} \log d + \chi_{k-p+1},
\]
which implies
\[
\chi_{k-p+1} \leq \gamma - \frac{k - p}{2} \log d.
\]

**Remark 5.2.** Theorem \[5.1\] with \( a = 1 \) implies the Ruelle inequality, i.e.
\[
\chi_1 + \cdots + \chi_c \geq \frac{k - p}{2} \log d.
\]

Therefore, it is enough to assume that \( \| \wedge^{k-p+1} Df(z) \| < d^{(k-p)/(k-p+1)} \) on \( \overline{U} \) since
\[
\chi_1 + \cdots + \chi_{k-p+1} \geq \frac{k - p + 1}{c} (\chi_1 + \cdots + \chi_c),
\]
if \( c \geq k - p + 1. \)
References

[AG62] A. Andreotti and H. Grauert. Théorème de finitude pour la cohomologie des espaces complexes. *Bull. Soc. Math. France*, 90:193–259, 1962.

[BDM07] A. Bonifant, M. Dabija, and J. Mihor. Elliptic curves as attractors in $\mathbb{P}^2$. I. Dynamics. *Experiment. Math.*, 16(4):385–420, 2007.

[BT82] R. Bott and L. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[Din07] T.-C. Dinh. Attracting current and equilibrium measure for attractors on $\mathbb{P}^k$. *J. Geom. Anal.*, 17(2):227–244, 2007.

[DNS08] T.-C. Dinh, V.-A. Nguyên, and N. Sibony. Dynamics of horizontal-like maps in higher dimension. *Adv. Math.*, 219(5):1689–1721, 2008.

[DS06] T.-C. Dinh and N. Sibony. Geometry of currents, intersection theory and dynamics of horizontal-like maps. *Ann. Inst. Fourier (Grenoble)*, 56(2):423–457, 2006.

[DS09] T.-C. Dinh and N. Sibony. Super-potentials of positive closed currents, intersection theory and dynamics. *Acta Math.*, 203(1):1–82, 2009.

[DS10] T.-C. Dinh and N. Sibony. Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings. In *Holomorphic dynamical systems*, volume 1998 of *Lecture Notes in Math.*, pages 165–294. Springer, Berlin, 2010.

[dT08] H. de Thélin. Sur les exposants de Lyapounov des applications méromorphes. *Invent. Math.*, 172(1):89–116, 2008.

[Dup09] C. Dupont. Large entropy measures for endomorphisms of CP(k). *arXiv/0911.4675*, 2009.

[Fed69] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.
J.-E. Fornæss and N. Sibony. Dynamics of $\mathbb{P}^2$ (examples). In Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998), volume 269 of Contemp. Math., pages 47–85. Amer. Math. Soc., Providence, RI, 2001.

J.-E. Fornæss and B. Weickert. Attractors in $\mathbb{P}^2$. In Several complex variables (Berkeley, CA, 1995–1996), volume 37 of Math. Sci. Res. Inst. Publ., pages 297–307. Cambridge Univ. Press, Cambridge, 1999.

G. Henkin and J. Leiterer. Andreotti-Grauert theory by integral formulas, volume 74 of Progress in Mathematics. Birkhäuser Boston Inc., Boston, MA, 1988.

M. Jonsson and B. Weickert. A nonalgebraic attractor in $\mathbb{P}^2$. Proc. Amer. Math. Soc., 128(10):2999–3002, 2000.

J. Milnor. On the concept of attractor. Comm. Math. Phys., 99(2):177–195, 1985.

V. I. Oseledec. A multiplicative ergodic theorem. Characteristic Lyapunov, exponents of dynamical systems. Trudy Moskov. Mat. Obšč., 19:179–210, 1968.

D. Ruelle. Elements of differentiable dynamics and bifurcation theory. Academic Press Inc., Boston, MA, 1989.

J. Taflin. Invariant elliptic curves as attractors in the projective plane. J. Geom. Anal., 20(1):219–225, 2010.

J. Taflin. Equidistribution speed towards the Green current for endomorphisms of $\mathbb{P}^k$. Adv. Math., 227:2059–2081, 2011.

H. Triebel. Interpolation theory, function spaces, differential operators. Johann Ambrosius Barth, Heidelberg, second edition, 1995.

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