Variable Selection is Hard

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Abstract

Variable selection for sparse linear regression is the problem of finding, given an \( m \times p \) matrix \( B \) and a target vector \( y \), a sparse vector \( x \) such that \( Bx \) approximately equals \( y \). Assuming a standard complexity hypothesis, we show that no polynomial-time algorithm can find a \( k' \)-sparse \( x \) with \( \|Bx - y\|^2 \leq h(m, p) \), where \( k' = k \cdot 2^{\log^{1-\delta} p} \) and \( h(m, p) \leq p^{C_1} m^{1-C_2} \), where \( \delta > 0, C_1 > 0, C_2 > 0 \) are arbitrary.

This is true even under the promise that there is an unknown \( k \)-sparse vector \( x^* \) satisfying \( Bx^* = y \). We prove a similar result for a statistical version of the problem in which the data are corrupted by noise.

To the authors’ knowledge, these are the first hardness results for sparse regression that apply when the algorithm simultaneously has \( k' > k \) and \( h(m, p) > 0 \).

1 Introduction

Consider a linear regression problem in which one is given an \( m \times p \) matrix \( B \) and a target vector \( y \in \mathbb{R}^m \). The goal is to approximately represent \( y \) as a linear combination of as few columns of \( B \) as possible. If a polynomial-time algorithm \( A \) is presented with \( B \) and \( y \), and it is known that \( y \) is an exact linear combination of some \( k \) columns of \( B \), and \( A \) is allowed to choose more than \( k \) columns of \( B \), how many columns must \( A \) choose in order to generate a linear combination which is close to \( y \)?

Note that we have allowed \( A \) to “cheat” both on the number of columns and on the accuracy of the resulting linear combination. In this paper, we show the problem is intractable despite the allowed cheating.

Formally, we define \((g, h)\)-SPARSE REGRESSION as follows. Let \( e^{(m)} \) denote the \( m \)-dimensional vector of 1’s, and for any vector \( z \), and let \( \|z\|_0 \) denote the number of nonzeros in \( z \). Let \( g(\cdot) \) and \( h(\cdot, \cdot) \) denote arbitrary functions. An algorithm for \((g, h)\)-SPARSE REGRESSION satisfies the following.

- Given: An \( m \times p \) Boolean matrix \( B \) and a positive integer \( k \) such that there is a real \( p \)-dimensional vector \( x^* \), \( \|x^*\|_0 \leq k \), such that \( Bx^* = e^{(m)} \). (Call such an input valid.)

- Goal: Output a (possibly random) \( p \)-dimensional vector \( x \) with \( \|x\|_0 \leq k \cdot g(p) \) such that \( \|Bx - e^{(m)}\|^2 \leq h(m, p) \) with high probability over the algorithm’s internal randomness.

Since we are focused on hardness, restricting \( B \) to have entries in \( \{0, 1\} \) and the target vector to be \( e^{(m)} \) only makes the result stronger.

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There is, of course, an obvious exponential-time deterministic algorithm for \((g, h)\)-\textsc{Sparse Regression}, for \(g(p) = 1\) for all \(p\), and \(h(m, p) = 0\): enumerate all subsets \(S\) of \(\{1, 2, \ldots, p\}\) of size \(k\). For each \(S\), use Gaussian elimination to determine if there is an \(x\) with \(x_j = 0\) for all \(j \not\in S\) such that \(Bx = e^{(m)}\), and if there is one, to find it. Since \(Bx^* = e^{(m)}\) and \(\|x^*\|_0 \leq k\), for at least one \(S\) the algorithm must return an \(x\) with \(\|x\|_0 \leq k\), \(Bx = e^{(m)}\).

Before getting too technical, we warm up with a simple hardness proof for \((g, h)\)-\textsc{Sparse Regression}. The short proof can be understood on its own. This theorem is just a warmup; later we will prove a stronger result (relying, unfortunately, on a stronger complexity assumption). To the authors’ knowledge, this simple proof was not known, though similar arguments had been used previously to show weaker hardness results for related problems (cf. [2] Proposition 6 and [6]).

**Theorem 1** Let \(0 < \delta < 1\). If there is a deterministic polynomial-time algorithm \(A\) for \((g, h)\)-\textsc{Sparse Regression}, for which \(g(p) = (1 - \delta) \ln p\) and \(h(m, p) = m^{1-\delta}\), then \(\text{SAT} \in \text{DTime}(n^{O(\log \log n)})\).

**Proof.** Feige [9] gives a reduction from SAT, running in deterministic time \(N^{O(\log \log N)}\) on SAT instances of size \(N\), to \textsc{Set Cover}, in which the resulting, say, \(m \times p\), incidence matrix \(B\) (whose rows are elements and columns are sets) has the following properties. There is a (known) \(k\) such that (1) if a formula \(\phi \in \text{SAT}\), then there is a collection of \(k\) disjoint sets which covers the universe (that is, there is a collection of \(k\) columns of \(B\) whose sum is \(e^{(m)}\)), and (2) if \(\phi \not\in \text{SAT}\), then no collection of at most \(k \cdot [(1 - \delta) \ln p]\) columns covers the universe (in other words, no set of at most \(k \cdot [(1 - \delta) \ln p]\) columns of \(B\) has a sum which is coordinate-wise at least \(e^{(m)}\)). This is already enough to establish that any polynomial-time algorithm for \((g(p), 0)\)-\textsc{Sparse Regression} implies an \(N^{O(\log \log N)}\), time algorithm for \text{SAT}. The remainder of the proof is devoted to establishing the analogous statement for \((g(p), m^{1-\delta})\)-\textsc{Sparse Regression}.

Build a matrix \(B'\) by stacking \(r\) copies of \(B\) atop one another, \(r\) to be determined later. Let \(M = rm\). If \(\phi \in \text{SAT}\), then there is a collection of \(k\) columns summing to \(e^{(M)}\). This is a linear combination of sparsity at most \(k\). If \(\phi \not\in \text{SAT}\), then for any linear combination of at most \(k \cdot [(1 - \delta) \ln p]\) column vectors, in each of the \(r\) copies of \(B\), there is squared error at least \(1\) (since the best one could hope for, in the absence of a set cover, is \(m - 1\) ’s and one \(0\)). This means that the squared error overall is at least \(r\). We want \(r > M^{1-\delta} = (rm)^{1-\delta}\), i.e., \(r^\delta > m^{1-\delta}\), and hence we define \(r = \lceil m^{1/\delta - 1} \rceil + 1\).

Construct an algorithm \(A'\) for \text{SAT} as follows. Run \(A\) on instance \((B', k)\) of \((g, h)\)-\textsc{Sparse Regression} for \(T(N)\) time steps, where \(T(N)\) is the time \(A\) would need on an \(rm \times p\) matrix if \((B', k)\) were a valid input for \((g, h)\)-\textsc{Sparse Regression} (which it may not be); since in the valid case, \(A\) runs in time polynomial in the size of the input matrix \(B'\) (which is \(N^{O(\log \log N)}\)), \(T(N)\) is also \(N^{O(\log \log N)}\). If \(A\) outputs a vector \(x\) such that \(\|B'x - e^{(M)}\|^2 \leq r\) and \(\|x\|_0 \leq k(1 - \delta) \ln p\) within this time bound, then \(A'\) outputs “Satisfiable.” Otherwise \((A\) doesn’t terminate in the allotted time or it terminates and outputs an inappropriate vector), \(A'\) outputs “Unsatisfiable.”

Clearly \(A'\) runs in time \(N^{O(\log \log N)}\) on inputs \(\phi\) of size \(N\). It remains to show that \(A'\) is a correct algorithm for \text{SAT}.

To this end, suppose that \(\phi \in \text{SAT}\). In this case, there is a solution \(x^*\) of sparsity at most \(k\) with \(B'x^* = e^{(M)}\), and since \(A\) is a correct algorithm for \((g, h)\)-\textsc{Sparse Regression}, \(A\) will find such a solution, causing \(A'\) to output “Satisfiable” when run on \(\phi\). On the other hand, if \(\phi \not\in \text{SAT}\), then there is no vector \(x^*\) with \(\|x^*\|_0 \leq k \cdot [(1 - \delta) \ln p]\) such that \(\|B'x^* - e^{(M)}\|^2 \leq r\). Hence, \(A'\) must output “Unsatisfiable” when run on \(\phi\). We conclude that \(A'\) is a correct algorithm for \text{SAT} running in time \(N^{O(\log \log N)}\) on instances of size \(N\). \(\square\)

One can combine Dinur and Steurer’s PCP [8] with Feige’s construction, or the earlier construction of Lund and Yannakakis [12], to strengthen the conclusion of Theorem [1] to \text{SAT} \(\in \text{P}\)
Throughout, BPTIME(T) will denote the set of all languages decidable by randomized algorithms, with two-sided error, running in expected time T. Our main result is that unless NP $\subseteq$ BPTIME($n^{\text{polylog}(n)}$), then even if $g(p)$ grows at a “nearly polynomial” rate, and $h(m, p) \leq p^{C_1 \cdot m^{1-C_2}}$ for any positive constants $C_1, C_2$, there is no quasipolynomial-time (randomized) algorithm for $(g, h)$-SPARSE REGRESSION:

**Theorem 2** Assume that NP $\not\subseteq$ BPTIME($n^{\text{polylog}(n)}$). For any positive constants $\delta, C_1, C_2$, there exist a $g(p)$ in $2^{\Omega(\log^{1-\delta}(p))}$ and an $h(m, p)$ in $\Omega(p^{C_1 \cdot m^{1-C_2}})$ such that there is no quasipolynomial-time randomized algorithm for $(g, h)$-SPARSE REGRESSION.

Note that the “−$C_2$” cannot be removed from the condition $h(m, p) \leq p^{C_1 \cdot m^{1-C_2}}$ in Theorem 2 the algorithm that always outputs the all-zeros vector solves $(g, h)$-SPARSE REGRESSION for $g(p) = 0$ and $h(m, p) = m$.

We also show, assuming a slight strengthening of a standard conjecture—known as the projection games conjecture (PGC) [14]—about the existence of probabilistically checkable proofs with small soundness error, that $(g, h)$-SPARSE REGRESSION is hard even if $g$ grows as a constant power. We refer to our slight strengthening of the PGC as the “Biregular PGC,” and state it formally in Section 2.3.

**Theorem 3** Assuming the Biregular PGC, the following holds: If NP $\not\subseteq$ BPP, then for any positive constants $C_1, C_2$, there exist a $g(p)$ in $p^{\Omega(1)}$ and an $h(m, p)$ in $\Omega(p^{C_1 \cdot m^{1-C_2}})$ such that there is no polynomial-time randomized algorithm for $(g, h)$-SPARSE REGRESSION.

We might consider $(g, h)$-SPARSE REGRESSION to be a “computer science” version of Sparse Regression, in the sense that the data are deterministic and fully specified. In an alternative, “statistics” version of Sparse Regression, the data are corrupted by random noise unknown to the algorithm. Specifically we consider the following problem, which we call $(g, h)$-NOISY SPARSE REGRESSION.

- There are a positive integer $k$ and an $m \times p$ Boolean matrix $B$, such that there exists an unknown $p$-dimensional vector $x^*$ with $\|x^*\|_0 \leq k$ such that $Bx^* = e^{(m)}$. An $m$-dimensional vector $\epsilon$ of i.i.d. $N(0, 1)$ “noise” components $\epsilon_i$ is generated and $y$ is set to $Bx^* + \epsilon = e^{(m)} + \epsilon$. $B, k,$ and $y$ (but not $\epsilon$ or $x^*$) are revealed to the algorithm.

- Goal: Output a (possibly random) $x \in \mathbb{R}^p$ such that $E[\|B(x-x^*)\|^2] \leq h(m, p)$ and $\|x\|_0 \leq k \cdot g(p)$. Here, the expectation is taken over both the internal randomness of the algorithm and of the $\epsilon_i$’s.

We give a simple reduction from $(g, h)$-SPARSE REGRESSION to $(g, h)$-NOISY SPARSE REGRESSION that proves the following theorems.

**Theorem 4** Assume that NP $\not\subseteq$ BPTIME($n^{\text{polylog}(n)}$). For any positive constants $\delta, C_1, C_2$, there exist a $g(p)$ in $2^{\Omega(\log^{1-\delta}(p))}$ and an $h(m, p)$ in $\Omega(p^{C_1 \cdot m^{1-C_2}})$ such that there is no quasipolynomial-time randomized algorithm for $(g, h)$-NOISY SPARSE REGRESSION.

**Theorem 5** Assuming the Biregular PGC, the following holds. If NP $\not\subseteq$ BPP, then for any positive constants $C_1, C_2$, there exist a $g(p)$ in $p^{\Omega(1)}$ and an $h(m, p)$ in $\Omega(p^{C_1 m^{1-C_2}})$ such that there is no polynomial-time randomized algorithm for $(g, h)$-NOISY SPARSE REGRESSION.
Importance and Prior Work. Variable selection is a crucial part of model design in statistics. People want a model with a small number of variables partially for simplicity and partially because models with fewer variables tend to have smaller generalization error, that is, they give better predictions on test data (data not used in generating the model). Standard greedy statistical algorithms to choose features for linear regression include forward stepwise selection (also known as stepwise regression or orthogonal least squares), backward elimination, and least angle regression. Standard non-greedy feature selection algorithms include LASSO and ridge regression.

There are algorithmic results for sparse linear regression that guarantee good performance under certain conditions on the matrix $B$. For example, equation (6) of [18] states that a “restricted eigenvalue condition” implies that the LASSO algorithm will give good performance for the statistical version of the problem. A paper by Natarajan [16] presents an algorithm, also known as forward stepwise selection or orthogonal least squares (see also [4]), which achieves good performance provided that the $L_2$-norm of the pseudoinverse of the matrix obtained from $B$ by normalizing each column is small [16]. It appears that no upper bound for stepwise regression under reasonable assumptions on the matrix was known prior to the appearance of Natarajan’s algorithm in 1995 (and the equivalence of Natarajan’s algorithm with forward stepwise selection appears not to have been noticed until recently). In the appendix, we include an example proving that Natarajan’s algorithm can perform badly when the $L_2$-norm of that pseudoinverse is large, or in other words, that Natarajan’s analysis of his algorithm is close to tight. Another example proving necessity of the factor involving the pseudoinverse appeared in [5, p. 10].

There have been several prior works establishing hardness results for variants of the sparse regression problem. Natarajan [16] used a reduction from EXACT COVER BY 3-SETS to prove that, given an $n \times p$ matrix $A$, a vector $b \in \mathbb{R}^m$, and $\epsilon > 0$, it is NP-hard to compute a vector $x$ satisfying $\|Ax - b\| < \epsilon$ if such an $x$ exists, such that $x$ has the fewest nonzero entries over all such vectors. Davis et al. [7] proved a similar NP-hardness result. The hardness results of [7,16] only establish hardness if the algorithm is not allowed to “cheat” simultaneously on both the sparsity and accuracy of the resulting linear combination.

Arora et al. [2] showed that, for any $\delta > 0$, a problem called MIN-UNSATISFY does not have any polynomial-time algorithm achieving an approximation factor of $2^{\log^{1+\delta}(n)}$, assuming NP $\not\subseteq$ DTIME($n^{\text{polylog}(n)}$). In this problem, the algorithm is given a system $Ax = b$ of linear equations over the rationals, and the cost of a solution $x^*$ is the number of equalities that are violated by $x^*$. Amaldi and Kann [1] built directly on the result of Arora et al. [2] to show, in our terminology, that $(2^{\log^{1+\delta}(n)}, 0)$-SPARSE-REGRESSION also has no polynomial-time algorithm under the same assumption. Also see a result by S. Muthukrishnan [15, pp. 20-21].

Finally, Zhang et al. [18] showed a hardness result for $(g, h)$-NOISY SPARSE REGRESSION. We defer a discussion of the result of [18] to Section 3.1. For now, we just note that their hardness only applies to algorithms which cannot “cheat” on the sparsity, that is, to algorithms that must generate a solution with at most $k$ nonzeros.

In summary, to the best of our knowledge, our work is the first to establish that sparse linear regression is hard to approximate, even when the algorithm is allowed to “cheat” on both the sparsity of the solution output and on the accuracy of the resulting linear combination.
2 Proofs of Theorems 2 and 3

2.1 Notation and Proof Outline

Throughout, we will use lower-case boldface letters to denote vectors. For any vector $y \in \mathbb{R}^m$, $\|y\|$ will denote the Euclidean norm of $y$, while $\|y\|_0$ will denote the sparsity (i.e., the number of nonzeros) of $y$. Let $e^{(m)} \in \mathbb{R}^m$ denote the all-ones vector, and for any vector $b$, let $\text{Ball}_\Delta(b) = \{b' : \|b - b'\|^2 \leq \Delta\}$ denote the ball of radius $\Delta$ around $b$ in the square of the Euclidean norm.

If $y = \sum_i c_i w_i$ represents a vector $y$ as a linear combination of vectors $w_i$, we say that $w_i$ participates in the linear combination if $c_i \neq 0$.

We will use the symbol $N$ to denote the input size of SAT instances used in our reductions.

The first step in our proof of Theorem 2 is to establish Proposition 6 below.

**Proposition 6** If SAT $\not\in \text{BPTIME}(N^{\text{polylog}(N)})$, then for any constant $\delta < 1$, there are a polynomial $m = m(p)$, a $k = k(p)$, and a pair $\sigma = \sigma(p), \Delta = \Delta(p)$ of values both in $2^{\Omega(\log^{1-\delta}(p))}$, such that no quasipolynomial-time randomized algorithm distinguishes the following two cases, given an $m \times p$ Boolean matrix $B$:

1. There is an $x \in \{0,1\}^p$ such that $Bx = e^{(m)}$ and $\|x\|_0 \leq k$.
2. For all $x \in \mathbb{R}^p$ such that $Bx \in \text{Ball}_\Delta(e^{(m)})$, $\|x\|_0 \geq k \cdot \sigma$.

(For the purpose of proving Theorem 2 having $\Delta = 1$ in Proposition 6 would actually suffice.)

The second step of the proof describes a simple transformation of any $(g,h)$-Sparse Regression algorithm for a “fast-growing” function $h$ into a $(g,h)$-Sparse Regression algorithm for $h(m,p) = 1$.

The proof appears in Section 2.4.

**Proposition 7** Let $C_1, C_2$ be any positive constants. Let $A$ be an algorithm for $(g,h)$-Sparse Regression running in time $T(m,p)$, for some function $g(\cdot) \geq 1$, and for $h(m,p) = p^{C_1 m^{1-C_2}}$. Then there is an algorithm $A'$ for $(g,1)$-Sparse Regression that runs in time $\text{poly}(T(\text{poly}(m,p),p))$.

**Proof.** (Proof of Theorem 2 assuming Propositions 6 and 7) Suppose by way of contradiction that there are positive constants $\delta, C_1, C_2$ such that there is a quasipolynomial-time randomized algorithm $A$ for $(g,h)$-Sparse Regression where $g(p) = 2^{\Omega(\log^{1-\delta}(p))}$ and $h(m,p) = p^{C_1 m^{1-C_2}}$. By Proposition 7, $A$ can be transformed into a randomized quasipolynomial-time algorithm $A'$ for $(g,1)$-Sparse Regression.

Clearly $A'$ is capable of distinguishing the following two cases, for any $\Delta \geq 1$:

1. There is an $x \in \{0,1\}^p$ such that $Bx = e^{(m)}$ and $\|x\|_0 \leq k$.
2. For all $x \in \mathbb{R}^p$ such that $Bx \in \text{Ball}_\Delta(e^{(m)})$, $\|x\|_0 \geq k \cdot g(p)$.

In particular, the above holds for $\Delta = \Delta(p)$ in $2^{\Omega(\log^{1-\delta}(p))} > 1$, which contradicts Proposition 6.

**Proof Outline for Proposition 6.** Lund and Yannakakis showed, assuming SAT cannot be solved by algorithms running in time $O(N^{\text{polylog}(N)})$, that SET-COVER cannot be approximated within a factor of $c \cdot \log_2 N$ for any constant $c < 1/4$ [12]. Here, an instance of SET-COVER consists of a set $D$ of size $N$ and a family $\{D_1, \ldots, D_M\}$ of subsets of $D$, and the goal is to find a minimal collection of the $D_i$’s whose union equals $D$. 


Lund and Yannakakis’ transformation from an instance of \( \phi \) of SAT to an instance of SET-COVER has a (known) remarkable property: if \( \phi \) is satisfiable, then the generated instance of SET-COVER does not just have a small cover \( C \) of the base set \( D \), it has a small partition of \( D \). That is, if \( c_i \) denotes the indicator vector of set \( C_i \), then \( \sum_{C_i \in C} c_i = e(S) \). This is a stronger property than \( \sum_{C_i \in C} c_i \geq e(|S|) \), which is the condition required to show hardness of SET-COVER. This observation naturally allows us to define a corresponding instance of the Linear Regression problem with a sparse solution; the columns of the matrix \( B \) in the regression problem are simply the indicator vectors \( c_i \).

A central ingredient in Lund and Yannakakis’ transformation is a certain kind of set system \((V_1, \ldots, V_M)\) over a base set \( S \). Their set system naturally requires that any union of fewer than \( \ell \) \( V \)'s or \( \overline{V} \)'s cannot cover \( S \), unless there is some \( i \) such that both \( V_i \) and \( \overline{V}_i \) participate in the union. As a result, they needed \( |S| \) to be polynomial in \( M \) and exponential in \( \ell \). Since we are studying Sparse Regression rather than SET-COVER, we can impose a weaker condition on our set systems. Specifically, we need that:

any linear combination of \( \ell \) indicator vectors of the sets or their complements is “far” from \( e(m) \), unless the linear combination includes both the indicator vector of a set and the indicator vector of its complement.

(See Definition \[1\] below.) As a result, we are able to take \( |S| \) to be much smaller relative to \( M \) and \( \ell \) than can Lund and Yannakakis. Specifically, we can take \( |S| \) to be polynomial in \( \ell \) and logarithmic in \( M \). This is the key to establishing hardness for super-logarithmic approximation ratios, and to obtaining hardness results even when we only require an approximate solution to the system of linear equations.

2.2 Proof of Proposition \[6\]

2.2.1 Preliminaries

A basic concept in our proofs is that of \( \Delta \)-useful set systems, defined below.

**Definition 1** Let \( M \) and \( \ell \) be positive integers, and let \( S \) be any finite set. A set system \( S_{M, \ell} = \{V_1, \ldots, V_M\} \) of size \( M \) over \( S \) is any collection of \( M \) distinct subsets of \( S \). \( S_{M, \ell} \) is called \( \Delta \)-useful, for \( \Delta \geq 0 \), if the following properties are satisfied.

Let \( v_i \in \{0, 1\}^{|S|} \) denote the indicator vector of \( V_i \), that is, \( v_{i,j} = 1 \) if \( j \in V_i \), and 0 otherwise. Let \( \overline{v}_i \) denote the indicator vector of the complement of \( V_i \). Then no \( \ell \)-sparse linear combination of the vectors \( v_1, v_1, \ldots, v_M, \overline{v}_M \) is in \( \text{Ball}_{\Delta}(e(|S|)) \), unless there is some \( i \) such that \( v_i \) and \( \overline{v}_i \) both participate in the linear combination.

(Note that there is a 2-sparse linear combination involving \( v_i \) and \( \overline{v}_i \) that exactly equals \( e(|S|) \), namely, \( v_i + \overline{v}_i = e(|S|) \).)

**Lemma 8** For any pair \( M, \ell, M \geq 2 \), of positive integers, there exists a set \( S = \{1, 2, \ldots, |S|\} \) of size \( O(\ell^2 \log M) \) such that there is a \( \Delta \)-useful set system \( S_{M, \ell} \) over \( S \), for some \( \Delta = \Omega(|S|) \). Moreover, there is a polynomial-time randomized algorithm that takes \( M \) and \( \ell \) and generates a \( \Delta \)-useful set system \( S_{M, \ell} \) over \( S \) with probability at least .99.

(It seems likely that a deterministic construction that appears in \[2\] could be modified to generate a \( \Delta \)-useful set system.)

**Proof.** Throughout the proof, we set

\[ |S| = \lceil 256 \ell^2 \ln M \rceil. \]  

(1)
To avoid notational clutter, we denote $e^{(|S|)}$ simply as $e$ throughout the proof of Lemma 8. The core of our argument is the following technical lemma bounding the probability that $e$ is “close” to the span of a “small” number of randomly chosen vectors from $\{0,1\}^{|S|}$.

**Sublemma 9** Given $M \geq 2$ and $\ell$, define $|S|$ as above. Let $v_1, \ldots, v_\ell \in \{0,1\}^{|S|}$ be chosen independently and uniformly at random from $\{0,1\}^{|S|}$. Let $E$ denote the event that there exist coefficients $c_1, \ldots, c_{\ell} \in \mathbb{R}$ such that $\|e - \left( \sum_{i=1}^{\ell} c_i v_i \right)\|^2 \leq |S|/32$. Then the probability of $E$ is at most $M^{-32\ell}$.

**Proof.** Rather than reasoning directly about the Boolean vectors $v_1, \ldots, v_\ell$ in the statement of the sublemma, it will be convenient to reason about vectors in $\{-1,1\}^\ell$. Accordingly, for any vector $v \in \{0,1\}^{|S|}$, define $v^* = 2v - 1$. Notice that if $v$ is a uniform random vector in $\{0,1\}^{|S|}$, then $v^*$ is a uniform random vector in $\{-1,1\}^{|S|}$. The primary reason we choose to work with vectors over $\{-1,1\}^{|S|}$ rather than $\{0,1\}^{|S|}$ is that vectors in $\{-1,1\}^{|S|}$ always have squared Euclidean norm exactly equal to $|S|$, in contrast to Boolean vectors, which can have squared Euclidean norm as low as 0 and as large as $|S|$. Throughout, given a set $T$ of vectors in $\mathbb{R}^{|S|}$, and another vector $v$ in $\mathbb{R}^{|S|}$, we let $\Pi_T(v)$ denote the projection of $v$ onto the linear span of the vectors in $T$.

Note that for any Boolean vectors $w, v \in \{0,1\}^{|S|}$, $\|w^* - v^*\|^2 = 4\|w - v\|^2$. Hence, event $E$ from the statement of Sublemma 9 occurs if and only if there exist coefficients $c_1, \ldots, c_{\ell} \in \mathbb{R}$ such that $\|e^* - \left( \sum_{i=1}^{\ell} c_i v_i^* \right)\|^2 \leq |S|/8$. We bound the probability of this occurrence as follows.

Let $T = \{v_1^*, \ldots, v_\ell^*\}$. Note that

$$\min_{c_1, \ldots, c_\ell \in \mathbb{R}} \|e^* - \left( \sum_{i=1}^{\ell} c_i v_i^* \right)\|^2 = \|e^*\|^2 - \|\Pi_T(e^*)\|^2 = |S| - \|\Pi_T(e^*)\|^2.$$  

Combined with the previous paragraph, we see that event $E$ occurs if and only if

$$\|\Pi_T(e^*)\|^2 \geq 7|S|/8.$$  

By equation (1), $7|S|/8 > 128\ell^2 \ln M$, so it suffices to bound from above the probability that

$$\|\Pi_T(e^*)\|^2 > 128\ell^2 \ln M.$$  

**Bounding the probability that** $\|\Pi_T(e^*)\|^2 > 128\ell^2 \ln M$. Our analysis consists of two steps. In step 1, we bound the probability that $\|\Pi_T(w^*)\|^2 > 128\ell^2 \ln M$, where $w^*$ is a vector chosen uniformly at random from $\{-1,1\}^{|S|}$, independently of $v_1^*, \ldots, v_\ell^*$. In step 2, we argue that, even though $e^*$ is a fixed vector (not chosen uniformly at random from $\{-1,1\}^{|S|}$), it still holds that $\|\Pi_T(e^*)\|^2 > 128\ell^2 \ln M$ with exactly the same probability (where the probability is now only over the random choice of vectors $v_1^*, \ldots, v_\ell^* \in \{-1,1\}^{|S|}$).

**Step 1.** Let $\ell' \leq \ell$ denote the dimension of the linear subspace $V := \text{span} \{v_1^*, \ldots, v_\ell^*\}$. Let $\{z_1^*, \ldots, z_{\ell'}^*\}$ denote an arbitrary orthonormal basis for $V$. We have

$$\|\Pi_T(w^*)\|^2 = \sum_{i=1}^{\ell'} (w^*, z_i)^2 = \sum_{i=1}^{\ell'} \left( \sum_{j=1}^{|S|} w_j^* \cdot z_{i,j} \right)^2,$$  

(2)
where \( w_j^* \) denotes the \( j \)th entry of \( w^* \), and \( z_{i,j} \) denotes the \( j \)th entry of \( z_i \). For any \( i \), let \( w_i \) denote the value of the sum \( w_i = \sum_{j=1}^{[S]} w_j^* \cdot z_{i,j} \). Since \( z_i \) is a vector in an orthonormal basis, we know that \( \sum_{j=1}^{[S]} z_{i,j}^2 = 1 \).

Meanwhile, each \( w_j^* \) is a Rademacher random variable. Because we can view \( \ell_1, \ldots, \ell_\ell \) and hence \( z_{i,1}, \ldots, z_{i,\ell} \) as fixed while \( w^* \) is random, and because \( \sum_{j=1}^{[S]} z_{i,j}^2 = 1 \), standard concentration results for Rademacher sequences [13] imply that for \( \ell > 0 \) and for all \( i \), \( \Pr[w_i > \ell] = \Pr[\sum_{j=1}^{[S]} w_j^* \cdot z_{i,j} > \ell] \leq e^{-\ell^2/2} \), for all fixed \( \ell_1, \ldots, \ell_\ell \), and hence even if \( \ell_1, \ldots, \ell_\ell \) is random. In particular, \( \Pr[w_i > \sqrt{128 \ell \ln M}] \leq e^{-64\ell \ln M} \). A union bound over all \( i \) implies that \( \Pr[w_i > \sqrt{128 \ell \ln M} \text{ for all } i \in [\ell']] \leq \ell' \cdot e^{-64\ell \ln M} \leq e^{-32\ell \ln M} = M^{-32\ell} \). In this event, i.e., for all \( i \sum_{j=1}^{[S]} w_j^* \cdot z_{i,j} \leq \sqrt{128 \ell \ln M} \), we can bound the right-hand side of equation (2) by

\[
\ell' \left( \sum_{j=1}^{[S]} w_j^* \cdot z_{i,j} \right)^2 \leq \ell' \cdot 128 \ell \ln M \leq 128 \ell^2 \ln M.
\]

**Step 2.** For vectors \( \ell_1, \ldots, \ell_\ell, w^* \in \{-1, 1\}^{[S]} \), let \( \Pi(\ell_1, \ldots, \ell_\ell; w^*) \) equal 1 if \( ||\Pi_T(w^*)||^2 > 128 \ell^2 \ln M \), and 0 otherwise, where \( T = \{\ell_1, \ldots, \ell_\ell\} \). We claim that \( \mathbb{E}[\Pi(\ell_1, \ldots, \ell_\ell; w^*)] = \mathbb{E}[\Pi(\ell_1, \ldots, \ell_\ell; e^*)] \), where the first expectation is over random choice of \( \ell_1, \ldots, \ell_\ell, w^* \in \{-1, 1\}^{[S]} \), and the second expectation is only over the random choice of \( \ell_1, \ldots, \ell_\ell \in \{-1, 1\}^{[S]} \).

For two vectors \( x, y \in \{-1, 1\}^{[S]} \), let \( x \otimes y \) denote the component-wise product of \( x \) and \( y \), i.e., \((x \otimes y)_i = x_i \cdot y_i \). Then we can write:

\[
\mathbb{E}[\Pi(\ell_1, \ldots, \ell_\ell; w^*)] = 2^{-[S] \cdot (\ell + 1)} \sum_{\ell_1, \ldots, \ell_\ell, w^* \in \{-1, 1\}^{[S]}} \Pi(\ell_1, \ldots, \ell_\ell; w^*)
\]

\[
= 2^{-[S] \cdot (\ell + 1)} \sum_{w^* \in \{-1, 1\}^{[S]}} \left[ \sum_{\ell_1, \ldots, \ell_\ell \in \{-1, 1\}^{[S]}} \Pi(\ell_1, \ldots, \ell_\ell; w^*) \right]
\]

\[
= 2^{-[S] \cdot (\ell + 1)} \sum_{w^* \in \{-1, 1\}^{[S]}} \left[ \sum_{\ell_1, \ldots, \ell_\ell \in \{-1, 1\}^{[S]}} \Pi(\ell_1 \otimes w^*, \ldots, \ell_\ell \otimes w^*; w^* \otimes w^*) \right]
\]

\[
= 2^{-[S] \cdot \ell} \sum_{\ell_1, \ldots, \ell_\ell \in \{-1, 1\}^{[S]}} \Pi(\ell_1, \ldots, \ell_\ell; e^*)
\]

\[
= \mathbb{E}[\Pi(\ell_1, \ldots, \ell_\ell; e^*)].
\]

Here, the first equality is the definition of expectation; the second follows by rearranging the sum. The third equality holds because multiplying each of \( \ell_1, \ldots, \ell_\ell \) and \( w^* \) component-wise by \( w^* \) is the same as premultiplying each vector by the unitary matrix \( U = \text{diag}(w^*) \), and premultiplying by a unitary matrix just rotates the space, which doesn’t change lengths of projections. The fourth equality holds because, for any fixed \( w^* \), if the vectors \( \ell_1, \ldots, \ell_\ell \) are uniformly distributed on \( \{-1, 1\}^{[S]} \), then so are the vectors \( \ell_1 := \ell_1 \otimes w^*, \ldots, \ell_\ell := \ell_\ell \otimes w^* \). The final equality is the definition of expectation. \( \square \)

Let \( V_1, \ldots, V_M \) be random subsets of \( S \), and \( v_i \) be the indicator vector of \( V_i \). Consider any subset \( Z \) of \( \ell \) of the vectors \( v_1, v_1, \ldots, v_M, V_M \) in which there is no \( i \) such that both \( v_i \) and \( v_i \) are in \( Z \). (There are exactly \( \binom{M}{\ell} 2^\ell \) such subsets \( Z \).) Then the vectors in \( Z \) are all uniformly random vectors in \( \{0, 1\}^{[S]} \) that are
chosen independently of each other. Sublemma 9 implies that the probability that there exist coefficients \( c_1, \ldots, c_\ell \in \mathbb{R} \) such that \( \|e - \left( \sum_{i=1}^\ell c_i v_i \right) \| \leq |S|/32 \) is at most \( M^{-32\ell} \).

We can take a union bound over all \( \binom{M}{\ell} \cdot 2^\ell \leq M^{2\ell} \) possible choices of \( Z \) to conclude that \( \|e - \left( \sum_{i=1}^\ell c_i v_i \right) \|^2 > |S|/32 \) for all possible choices of the subset \( Z \) with probability at least \( 1 - M^{2\ell} \cdot M^{-32\ell} = 1 - M^{-30\ell} \geq 0.99 \). In this event, there is no \( \ell \)-sparse linear combination of the \( v_i \) and \( \bar{v}_i \) vectors in \( \text{Ball}_\Delta(e) \) for \( \Delta = |S|/32 \), unless \( v_i \) and \( \bar{v}_i \) both participate in the linear combination for some \( i \). That is, \( \{V_1, \ldots, V_M\} \) is a \( \Delta \)-useful set system, as desired.

This completes the proof of Lemma 8. \( \square \)

2.2.2 Full Proof of Proposition 6

Suppose that \( A^* \) is a BPP algorithm that distinguishes the two cases appearing in the statement of Proposition 6. We show how to use \( A^* \) to design an efficient randomized algorithm for SAT.

MIP Notation. Our construction of \( A^* \) assumes the existence of a perfectly complete multiprover interactive proof (MIP) with two provers and soundness error \( \epsilon_{\text{sound}} < 1 \). Let \( R \) denote the possible values of the MIP verifier’s private randomness, \( Q_i \) denote the set of possible queries that may be posed to the \( i \)th prover, and \( A_i \) denote the set of possible responses from the \( i \)th prover. Denote by \( V(r, a_1, a_2) \in \{0,1\} \) the verifier’s output given internal randomness \( r \) and answers \( (a_1, a_2) \) from the two provers, with an output of 1 being interpreted as the verifier’s accepting the provers’ answers as valid proofs that the input is satisfiable.

As in prior work extending Lund and Yannakakis’ methodology [3], we require the MIP to satisfy four additional properties, and we call an MIP canonical if all four additional properties hold. Specifically:

**Definition 2** An MIP is said to be canonical if it satisfies the following four properties. (These properties are, of course, independent of any provers.)

- **Functionality:** for each \( r \in R \) and each \( a_1 \in A_1 \), there is at most one \( a_2 \in A_2 \) with the property that the verifier, after choosing \( r \in R \), accepts answer vector \( (a_1, a_2) \).
- **Uniformity:** For each \( i \in \{1,2\} \), the verifier’s queries to \( P_i \) are distributed uniformly over \( Q_i \).
- **Equality of Question-Space Sizes:** the sets \( Q_1 \) and \( Q_2 \) are of the same size.
- **Disjointness of Question and Answer Spaces:** \( A_1 \) and \( A_2 \) are disjoint, as are \( Q_1 \) and \( Q_2 \).

We will show how to turn any canonical MIP for SAT into a randomized algorithm \( A'' \) that accomplishes the following. Fix any function \( \ell : \mathbb{N} \rightarrow \mathbb{N} \). Given any SAT instance \( \phi \) of size \( N \), \( A'' \) outputs an \( m \times p \) Boolean matrix \( B \), where \( m \) and \( p \) are polynomial in the parameters \( |R|, |Q_1|, |Q_2|, |A_1|, |A_2| \). Moreover, \( B \) satisfies the following two properties with high probability over the internal randomness of \( A'' \).

**Property 1:** If \( \phi \in \text{SAT} \), then there is a vector \( x^* \in \{0,1\}^p \) such that \( B x^* = e^{(m)} \) and \( \|x^*\|_0 \leq |Q_1| + |Q_2| \).

**Property 2:** If \( \phi \notin \text{SAT} \), then any \( x \in \mathbb{R}^p \) such that \( Bx \in \text{Ball}_\Delta(e^{(m)}) \) satisfies \( \|x\|_0 \geq \left(1 - \epsilon_{\text{sound}} \cdot \ell(N)^2 \cdot \frac{\ell(N)}{2} \right) \cdot (|Q_1| + |Q_2|) \), for some \( \Delta = \Omega \left( \ell^2(N) \cdot \log (|A_1| + |A_2|) \right) \).

Before describing \( A'' \), we complete the proof of Proposition 6 assuming the existence of \( A'' \).

**Proof.** It is well-known that the standard PCP theorem can be combined with Raz’s parallel repetition theorem [17] to yield a canonical two-prover MIP for a SAT instance of size \( N \) with the following two properties, for any constant \( c > 0 \):
• the soundness error is \( \varepsilon_{\text{sound}}(N) \leq 2^{-\log^c(N)} \), and

• \(|R|, |A_1|, |A_2|, |Q_1|, |Q_2|\) are all of size \( N^{O(\log^c(N))} \).

See, e.g., [7, Section 2.2]. Throughout the remainder of the proof, we choose \( c \) to satisfy \( \frac{c - 1}{1 - c} = 1 - \delta \).

Hence, setting \( \ell(N) = \left(10 \cdot \varepsilon_{\text{sound}}(N)\right)^{-1/2} \) and \( k = |Q_1| + |Q_2| \), properties 1 and 2 above imply the following. Given an instance \( \phi \) of SAT of size \( N \), \( A'' \) outputs an \( m \times p \) Boolean matrix \( B \), where \( m \) and \( p \) are \( N^{O(\log^c(N))} \), and \( B \) satisfies the following properties.

• If \( \phi \in \text{SAT} \), then there is a vector \( x^* \in \{0, 1\}^p \) such that \( Bx^* = e^m \) and \( \|x^*\|_0 \leq k \).

• If \( \phi \notin \text{SAT} \), then any \( x \in \mathbb{R}^p \) such that \( Bx \in \text{Ball}_{\Delta}(e^m) \) satisfies \( \|x\|_0 \geq \sigma \cdot k \), where \( \ell = \ell(N) \) and \( \Delta = \Omega\left(\ell^2 \cdot \log(|A_1| + |A_2|)\right) \geq \Omega(\ell) \) and \( \sigma = (1 - \varepsilon_{\text{sound}} \cdot \ell^2) \cdot \frac{\ell}{2} \geq 0.45\ell \).

Suppose that there is a randomized algorithm \( \mathcal{A}' \) that runs in \( \text{BPTIME}(\text{polylog}(N)) \) and distinguishes the above two cases. By running \( \mathcal{A}' \) on the matrix \( B \) output by \( A'' \), we determine with high probability whether \( \phi \) is satisfiable. Thus, if \( \text{SAT} \notin \text{BPTIME}(N^{\text{polylog}(N)}) \), no such algorithm \( \mathcal{A}' \) can exist.

Proposition 6 follows by recalling that \( p = N^{O(\log^c(N))} \), where \( c \) is chosen such that \( 1 - \delta = \frac{c - 1}{1 - c} \); thus \( \ell = 2^{O(\log^c(N))} = 2^{O(\log^{\frac{1}{1-c}}(p))} = 2^{O(\log^{1-\delta}(p))} \).

We now turn to the description of the algorithm \( A'' \) that generates the matrix \( B \). Our description borrows many ideas from Lund and Yannakakis [12]; accordingly, our presentation closely follows that of [12].

**Construction of \( B \).** Let \( M = |A_2| \) and let \( \ell > 2 \) be arbitrary. Let \( S_{M, \ell} = \{V_{a_2} | a_2 \in A_2\} \) be a \( \Delta \)-useful set system over a set \( S \), where \( |S| = O(\ell^2 \cdot \log |A_2|) \), \( \Delta = \Omega(|S|) \). Lemma 8 guarantees \( S_{|A_2|, \ell} \) can be output by an efficient randomized algorithm with high probability.

The matrix \( B \) will have \( m = |R| \times |S| \) rows, and \( p = |Q_1| \cdot |A_1| + |Q_2| \cdot |A_2| \) (which is quasipolynomial in \( N \)) columns. We associate each of the \( m \) rows of \( B \) with a point \( (r, s) \), where \( r \) is a random seed to the verifier and \( s \) is a point in the set \( S \). Likewise, we associate each of the \( p \) columns of \( B \) with a pair \((q_i, a_i)\) \( \in Q_i \times A_i \), for \( i \in \{1, 2\} \).

For \( r \in R \) and \( i \in \{1, 2\} \), let \( q[r, i] \) be the query that the verifier asks the \( i \)th prover when using the random seed \( r \). Likewise, for each \( r \in R \) and each \( a_1 \in A_1 \), let \( UA[r, a_1] \) denote the unique answer such that \( V(r, a_1, UA[r, a_1]) = 1 \), if such an answer exists, and be undefined otherwise. By functionality of the MIP, there exists at most one such answer.

Now we define the following sets, following [12]. For each pair \((q_1, a_1) \in Q_1 \times A_1\), define:

\[
W_{q_1, a_1} = \{ (r, s) | q_1 = q[r, 1], UA[r, a_1] \text{ is defined, and } s \notin V_{UA[r, a_1]} \}. 
\]

Similarly, for each pair \((q_2, a_2) \in Q_2 \times A_2\), define:

\[
W_{q_2, a_2} = \{ (r, s) | q_2 = q[r, 2] \text{ and } s \in V_{a_2} \}. 
\] (3)

(4)

Let \( w_{q_i, a_i} \in \{0, 1\}^m \) denote the indicator vector of \( W_{q_i, a_i} \). We define \( B \) to be the \( m \times p \) matrix whose columns are the \( w_{q_i, a_i} \) vectors over \( i = 1, 2, q_i \in Q_i, a_i \in A_i \).

Here is an equivalent way to construct \( B \). Start with the zero matrix. For each \( r \in R \), \( q_1 \in Q_1 \), let the \((r, q_1)\) block of \( B \) be the \(|S| \times |A_1|\) submatrix corresponding to \( r \) and \( q_1 \). Similarly, for each \( r \in R \), \( q_2 \in Q_2 \), let the \((r, q_2)\) block of \( B \) be the \(|S| \times |A_2|\) submatrix corresponding to \( r \) and \( q_2 \).

Then, for each \( r \in R \), do the following.
1. For each \( a_2 \in A_2 \), replace the 0-column corresponding to \( a_2 \) in the \((r, q[r, 2])\) block by \( v_{a_2} \).

2. For each \( a_1 \in A_1 \) such that \( UA[r, a_1] \) is defined, replace the 0-column corresponding to \( a_1 \) in the \((r, q[r, 1])\) block by \( \bar{v}_{UA[r,a_1]} \).

Note that for each \( r \), there are only one value of \( q \in Q_1 \), namely, \( q = q[r, 1] \), and only one value of \( q \in Q_2 \), namely, \( q = q[r, 2] \), such that the \((r, q)\) block of \( B \) can be nonzero. Note also that a prover strategy corresponds to choosing, for each \( i = 1, 2 \) and each \( q_i \in Q_i \), exactly one column from the block of columns corresponding to \( q_i \).

**Analysis of \( B \).**

**Proposition 10** \( B \) satisfies property 1 above.

**Proof.** Assume \( \phi \in \text{SAT} \) and consider a strategy of the provers that makes the verifier accept with probability 1. Such a strategy must exist, because the MIP has perfect completeness. For this strategy, for \( i = 1, 2 \), let \( P_i(q_i) \in A_i \) be the answer \( P_i \) gives when given query \( q_i \).

**Lemma 11**

\[
e^{(m)} = \sum_{i=1}^{2} \sum_{q_i \in Q_i} w_{q_i, P_i(q_i)}. \tag{5}
\]

**Proof.** Consider any \( r \in R \). Because on query \( r \), the responses from provers \( P_1 \) and \( P_2 \) induce acceptance, we have \( V(r, P_1(q[r, 1]), P_2(q[r, 2])) = 1 \), and therefore

\[
UA(r, P_1(q[r, 1])) = P_2(q[r, 2]). \tag{6}
\]

Furthermore, the only nonzero \((r, q)\) blocks corresponding to \( r \in R \) are those for \( q[r, 1] \) and \( q[r, 2] \). It suffices to show that

\[
w_{q[r,1], P_1(q[r,1])} + w_{q[r,2], P_2(q[r,2])} = e^{(|S|)},
\]

since any other columns appearing in the sum, in the rows corresponding to \( r \), are all zero. (Here, for an \( |R| \times |S| \)-vector \( z \), \( z_r \) denotes the \( |S| \)-vector corresponding to block \( r \).) By the definition of \( B \),

\[
w_{q[r,2], P_2(q[r,2])} = v_{P_2(q[r,2])}. \]

In addition, \( w_{q[r,1], P_1(q[r,1])} = \bar{v}_{UA(r, P_1(q[r,1]))} \). But by (6), we know that \( UA(r, P_1(q[r,1])) = P_2(q[r, 2]) \), and therefore \( w_{q[r,1], P_1(q[r,1])} = v_{P_2(q[r,2])} + w_{q[r,2], P_2(q[r,2])} = e^{(|S|)}. \)

Thus, equation (5) represents \( e^{(m)} \) as a linear combination (with coefficients in \( \{0, 1\} \)) of exactly \( |Q_1| + |Q_2| \) columns of \( B \), proving property 1. □

**Proposition 12** \( B \) satisfies property 2 above.

**Proof.** Let \( x \in \mathbb{R}^P \) be any vector satisfying \( Bx \in \text{Ball}_\Delta(e^{(m)}) \). We claim that if \( x \) is sufficiently sparse, then we can transform \( x \) into a prover strategy causing the verifier to accept with probability larger than \( \epsilon_{\text{sound}} \), a contradiction.

We say that \( x \) involves \( w_{q_i, a_i} \) if \( x_{q_i, a_i} \neq 0 \). We will write \( w_{q_i, a_i} \in x \) to mean the same thing.

Given \( x \), and recalling that \( Q_1 \cap Q_2 = \emptyset \), we define a cost function \( \text{cost} : Q_1 \cup Q_2 \to \mathbb{N} \) via \( \text{cost}(q_i) = |\{a_i \in A_i : w_{q_i, a_i} \in x\}| \) for \( i = 1, 2 \) and \( q_i \in Q_i \). Similarly, we define the cost of \( r \in R \) as \( \text{cost}(r) = \text{cost}(q[r, 1]) + \text{cost}(q[r, 2]) \). We say that \( r \in R \) is good if \( \text{cost}(r) \leq \ell \). Let \( \gamma \) denote the fraction of values \( r \in R \) that are good. We claim that we can transform \( x \) into a prover strategy causing the verifier to accept with probability at least \( \gamma / \ell^2 \). To establish this, we require several lemmas.
SUBLEMMA 13 If \( r \) is good, then \( \mathbf{x} \) must involve vectors \( \mathbf{w}_{q[r,1],a_1}^* \) and \( \mathbf{w}_{q[r,2],a_2}^* \) for some pair \((a_1^*, a_2^*)\) of answers, respectively, such that \( \mathcal{V}(r, a_1^*, a_2^*) = 1 \).

PROOF. For \( r \in R \), let \( B|_r \) denote the \( |S| \times p \) matrix obtained by restricting \( B \) to rows corresponding to pairs of the form \((r, s)\) for some \( s \in S \). Similarly, let \( (B\mathbf{x})|_r \in \mathbb{R}^{|S|} \) denote the subvector of \( B\mathbf{x} \) whose \( s \)th entry is \((B\mathbf{x})|_{r,s}\), for each \( s \in S \). Note that since \( \|B\mathbf{x} - \mathbf{e}(\mathbf{m})\|^2 \leq \Delta \), it also holds that

\[
\|(B\mathbf{x})|_r - \mathbf{e}(|S|)\|^2 \leq \Delta. \tag{7}
\]

Since \( r \) is good, the vector \((B\mathbf{x})|_r \in \mathbb{R}^{|S|}\) is a linear combination of at most \( \ell \) columns of \( B|_r \). To see this, observe that the definition of \( B \) implies that the only non-zero columns of \( B|_r \) correspond to vectors of the form \( \mathbf{w}_{q[r,1],a_1} \) or \( \mathbf{w}_{q[r,2],a_2} \) for some \( a_1 \in A_1 \) or \( a_2 \in A_2 \). Since \( r \) is good, \( \text{cost}(q[r,1]) + \text{cost}(q[r,2]) \leq \ell \), i.e., \(|\{a_1 \in A_1 : \mathbf{w}_{q[r,1],a_1} \in \mathbf{x}\}| + |\{a_2 \in A_2 : \mathbf{w}_{q[r,2],a_2} \in \mathbf{x}\}| \leq \ell \), and hence there are at most \( \ell \) vectors of the form \( \mathbf{w}_{q[r,1],a_1} \) or \( \mathbf{w}_{q[r,2],a_2} \) such that \( \mathbf{x}_{q[r,1],a_1} \) or \( \mathbf{x}_{q[r,2],a_2} \) are non-zero. Hence, \( B|_r \cdot \mathbf{x} \) is a linear combination of at most \( \ell \) non-zero columns of \( B|_r \). Since \((B\mathbf{x})|_r = B|_r \cdot \mathbf{x} \), it follows that \((B\mathbf{x})|_r \) is a linear combination of at most \( \ell \) columns of \( B|_r \), as claimed.

Moreover, each column of \( B|_r \) is equal to either \( \mathbf{v}_{a_2} \) (see equation (4)) or \( \bar{\mathbf{v}}_{a_2} \) (see equation (5)) for some \( a_2 \in A_2 \) (recall that \( \mathbf{v}_{a_2} \) denotes the indicator vector of set \( V_{a_2} \) in the set system \( \mathcal{B}_{M,\ell} \)). Since \( \mathcal{B}_{M,\ell} \) is \( \Delta \)-useful, the definition of \( \Delta \)-useful set systems (Definition 11 and equation (7)) together imply that there is some \( a_2^* \) such that both \( \mathbf{v}_{a_2^*} \) and \( z := \bar{\mathbf{v}}_{a_2^*} \) are involved in the linear combination representing \((B\mathbf{x})|_r \) as described in the previous paragraph. (Because \( \ell > 2 \), the definition of \( \Delta \)-useful implies that there is no \( a \in A_2 \) with \( a \neq a_2^* \) such that \( z = \mathbf{v}_a \) and that there is no \( a \in A_2 \) such that \( z = \bar{\mathbf{v}}_a \).) Now equation (5) implies that there exists a (possibly not unique) \( a_1^* \in A_1 \) such that \( \mathbf{U}A|r,a_1^* = a_2^* \).

In summary, we have established that \( \mathbf{x} \) involves \( \mathbf{w}_{q[r,1],a_1^*} \) and \( \mathbf{w}_{q[r,2],a_2^*} \) for some \((a_1^*, a_2^*)\) such that \( \mathcal{V}(r, a_1^*, a_2^*) = 1 \), proving the sublemma. \( \Box \)

SUBLEMMA 14 There exists a pair of provers such that the verifier accepts the input \( \phi \) with probability at least \( \gamma/\ell^2 \).

PROOF. For each prover \( i \in \{1, 2\} \), define \( \ell \) strategies \( \mathcal{P}_{i,1}, \ldots, \mathcal{P}_{i,\ell} \) as follows: For each query \( q_i \in Q_i \), order the elements of the set \( \{a_i \in A_i : \mathbf{w}_{q_i,a_i} \in \mathbf{x}\} \) (which can have size \( \text{cost}(q_i) \) exceeding \( \ell \)) arbitrarily and, for \( 1 \leq j \leq \ell \), make the \( j \)th strategy’s answer to \( q_i \) be the \( j \)th element of the set, if such an element exists; otherwise, assign an arbitrary answer.

If \( r \) is good, then \(|\{a_i \in A_i : \mathbf{w}_{q[r,i],a_i} \in \mathbf{x}\}| \leq \ell \) for all \( i \), because \( \text{cost}(q[r,1]) \) and \( \text{cost}(q[r,2]) \) are both at most \( \ell \). Sublemma 13 implies that \( \mathbf{x} \) must involve vectors \( \mathbf{w}_{q[r,1],a_1^*} \) and \( \mathbf{w}_{q[r,2],a_2^*} \) that correspond to answers \((a_1^*, a_2^*)\) that make the verifier accept when using random choice \( r \). It follows, since \( \gamma \) is the fraction of \( r \in R \) that are good, that if the two provers execute the following (randomized) strategy, the verifier will accept with probability at least \( \gamma/\ell^2 \): on query \( q_i \), the \( i \)th prover chooses \( j \in \{1, \ldots, \ell\} \) at random, and replies according to strategy \( \mathcal{P}_{i,j} \). By averaging, this implies that there is some fixed pair \((j_1, j_2) \in \{1, \ldots, \ell\}^2 \) such that \( (\mathcal{P}_{1,j_1}, \mathcal{P}_{2,j_2}) \) makes the verifier accept with probability at least \( \gamma/\ell^2 \). \( \Box \)

SUBLEMMA 15 \( \|\mathbf{x}\|_0 \geq (1 - \gamma) \frac{\ell}{2} (|Q_1| + |Q_2|) \).

PROOF. Note that

\[
\sum_{r \in R} \text{cost}(r) \geq \sum_{r \in R: r \text{ is not good}} \text{cost}(r) \geq (1 - \gamma)|R| \cdot \ell. \tag{8}
\]
The second inequality follows from
\[ |\{ r \in R : r \text{ is not good}\} | = (1 - \gamma)|R| \]
and \( \text{cost}(r) > \ell \) if \( r \) is not good.

We can also write:
\[ \sum_{r \in R} \text{cost}(r) = \sum_{r \in R} \sum_{i=1}^{2} \text{cost}(q[r, i]) = \sum_{i=1}^{2} \left[ \sum_{r \in R} \text{cost}(q[r, i]) \right]. \tag{9} \]

By uniformity (see Definition 2), for each \( i \in \{1, 2\}, \sum_{r \in R} \text{cost}(q[r, i]) = |\sum_{q_i \in Q_i} \text{cost}(q_i)| \frac{|R|}{|Q_1|}. \]
Hence, the right-hand side of equation (9) equals
\[ \sum_{i=1}^{2} \sum_{q_i \in Q_i} \text{cost}(q_i) \cdot \frac{|R|}{|Q_1|}. \tag{10} \]
Since \( |Q_1| = |Q_2| \) by equality of question-space sizes, expression (10) implies that
\[ \sum_{r \in R} \text{cost}(r) = \frac{|R|}{|Q_1|} \cdot \sum_{i=1}^{2} \sum_{q_i \in Q_i} \text{cost}(q_i) = \frac{|R|}{|Q_1|} \|\mathbf{x}\|_0, \tag{11} \]
where the final equality holds because \( \sum_{i=1}^{2} \sum_{q_i \in Q_i} \text{cost}(q_i) = \|\mathbf{x}\|_0. \)

Combining inequality (8) and equation (11), we conclude that \( \|\mathbf{x}\|_0 \sum_{q_i \in Q_i} \text{cost}(q_i) \geq (1 - \gamma)\frac{\ell}{|Q_1|} \), from which we conclude that \( \|\mathbf{x}\|_0 \geq (1 - \gamma)^{\frac{\ell}{2}} \left(|Q_1| + |Q_2|\right) \) as claimed. \( \square \)

Sublemma 14 implies that if \( \phi \not\in \text{SAT} \), then \( \gamma/\ell^2 \leq \varepsilon_{\text{sound}} \) and hence \( 1 - \gamma \geq 1 - \ell^2\varepsilon_{\text{sound}} \). Hence, using Sublemma 15
\[ \|\mathbf{x}\|_0 \geq (1 - \gamma)^{\frac{\ell}{2}} \left(|Q_1| + |Q_2|\right) \geq (1 - \ell^2 \cdot \varepsilon_{\text{sound}})^{\frac{\ell}{2}} \left(|Q_1| + |Q_2|\right). \]

This completes the proof of Proposition 12. \( \square \)

2.3 Proof of Theorem 3

Let \( \text{PCP}_{\text{sound}}[b, k][\Sigma] \) denote the class of languages that have a PCP verifier with perfect completeness, soundness \( \varepsilon_{\text{sound}} \), \( b \) bits of randomness used by the verifier, and \( k \) queries to a proof over alphabet \( \Sigma \). The \textit{sliding scale conjecture} of Bellare et al. \cite{bellare1997} postulates that for every \( 0 \leq \delta \leq 1 \), \( \text{SAT} \in \text{PCP}_{\text{sound}}[O(\log(N)), 2]_{\text{poly}(1/\epsilon)} \), where \( \epsilon = 2^{-O(\log(N))} \). In words, the sliding scale conjecture asserts that SAT has a polynomial-length PCP in which the verifier makes two queries to the proof (note that each query reads an entire symbol from the alphabet \( \Sigma \), and hence corresponds to approximately \( \log |\Sigma| \) consecutive bits of the proof), and in which the soundness error falls exponentially with the number of bits of the proof that the verifier accesses.

Note that any \( k \)-query PCP can be transformed into a \( k \)-prover MIP by posing each of the PCP verifier’s \( k \) queries to a different prover. Thus, the sliding scale conjecture (at \( \delta = 0 \)) implies the existence of a 2-prover MIP for SAT in which \( |R|, |Q_1|, |A_1|, |Q_2|, |A_2| \) are all at most \( \text{poly}(N) \), and whose soundness error is polynomially small in \( N \).
In [14], Moshkovitz formulates a slight strengthening of the sliding scale conjecture that she calls the Projection Games Conjecture (PGC). Here, the term projection games refers to a certain class of two-player one-round games defined over bipartite graphs. Projection games are equivalent, in a formal sense, to two-prover MIP’s that satisfy functionality. This equivalence is standard; we provide the details in the appendix for completeness.

Just like the sliding scale conjecture, the PGC postulates the existence of a 2-prover MIP for SAT in which \(|R|, |Q_1|, |A_1|, |Q_2|, |A_2|\) are all at most \(\text{poly}(N)\), and in which the soundness error is inversely polynomially small in \(N\). But the PGC additionally postulates that there is such an MIP that satisfies the functionality property used in the proof of Proposition 6 (see Definition 2). Formally, the PGC states the following [14].

**Conjecture 16 (Reformulation of the Projection Games Conjecture (PGC) [14])** There exists \(c > 0\) such that for all \(N\) and all \(\epsilon \geq 1/N^c\), there is a 2-prover MIP for SAT instances of size \(N\), with perfect completeness and soundness error at most \(\epsilon\), which satisfies the following two properties:

- \(|R|, |Q_1|, |A_1|, |Q_2|, |A_2|\) are all at most \(\text{poly}(N, 1/\epsilon)\).
- The MIP satisfies functionality.

We will need a slight strengthening of the PGC that requires that SAT be efficiently reduced to a 2-prover MIP that satisfies uniformity in addition to functionality. We note that in the language of projection games, the uniformity condition is equivalent to requiring the bipartite graph underlying the projection game be biregular. Moreover, the most efficient known reductions from SAT to projection games do produce biregular graphs (and hence 2-prover MIP’s satisfying uniformity) [8].

**Conjecture 17 (Biregular PGC)** Conjecture 16 holds in which the MIP satisfies uniformity as well.

Proposition 18 below describes a strengthening of Proposition 6 that holds under the Biregular PGC. Theorem 3 then follows by the argument of Section 2.1 using Proposition 7 in place of Proposition 18.

**Proposition 18** Assuming Conjecture 17, the following holds for some pair of values \(\sigma = \sigma(p), \Delta = \Delta(p)\), both in \(p^{\Omega(1)}\).

If \(\text{SAT} \not\subseteq \text{BPP}\), then there are a polynomial \(m = m(p)\) and \(k = k(p)\) such that no BPP algorithm distinguishes the following two cases, given an \(m \times p\) Boolean matrix \(B\):

1. There is an \(x \in \{0, 1\}^p\) such that \(Bx = e^{(m)}\) and \(\|x\|_0 < k\).
2. For all \(x \in \mathbb{R}^p\) such that \(Bx \in \text{Ball}_\Delta(e^{(m)}), \|x\|_0 \geq k \cdot \sigma\).

**Proof.** Conjecture 17 implies the existence of a perfectly complete two-prover MIP for SAT with soundness error \(\epsilon_{\text{sound}} = 1/N^{\Omega(1)}\) that satisfies functionality and uniformity, with \(|R|, |A_1|, |A_2|, |Q_1|, |Q_2|\) all bounded by \(\text{poly}(N)\). Moreover, this MIP can be transformed into an equivalent MIP that satisfies both equality of question space sizes and disjointness of answer spaces, while keeping \(|R|, |A_1|, |A_2|, |Q_1|, |Q_2|\) all bounded by \(\text{poly}(N)\). Disjointness of answer spaces can be ensured by simple padding (only one bit of

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1Technically, Moshkovitz’s PGC is slightly stronger than our Conjecture 21 in that \(|R|, |Q_1|,\) and \(|Q_2|\) are all required to be bounded by \(N^{1+\omega(1)}\text{poly}(1/\epsilon)\), rather than \(\text{poly}(N, 1/\epsilon)\). She states the weaker version in a footnote, and the weaker version suffices for our purposes.
padding is required). Equality of question space sizes can be ensured via a technique of Lund and Nanakakis [12], with at most a quadratic blowup in the size of $Q_1$ and $Q_2$. Namely, the verifier generates two independent queries $(q_1, q_2)$ and $(q_1', q_2')$ in $Q_1 \times Q_2$. Then, the verifier asks the first prover the query $(q_1, q_2)$ and the second prover the query $(q_2, q_1')$. The provers ignore the second component, and answer the queries by answering the first component of the query according to the original MIP. The verifier accepts if and only if the verifier in the original MIP would have accepted the answers to $(q_1, q_2)$.

Thus, we have established that Conjecture [7] implies the existence of a perfectly complete canonical MIP for SAT with soundness error $\epsilon_{\text{sound}} = 1/\sqrt{N}$ and for which $|R|, |A_1|, |A_2|, |Q_1|, |Q_2|$ are all bounded by $\text{poly}(N)$. The remainder of the proof is now identical to that of Proposition [6] and we provide the details for completeness.

The proof of Proposition [6] specified an algorithm $A''$ that output a Boolean $m \times p$ matrix $B$ of size polynomial in the parameters $|R|, |A_1|, |A_2|, |Q_1|, |Q_2|$, which are themselves polynomial in $N$ (so that $mp$ is polynomial in $N$), such that:

Property 1: If $\phi \in \text{SAT}$, then there is a vector $x^* \in \{0,1\}^p$ such that $Bx^* = e^{(m)}$ and $\|x^*\|_0 \leq |Q_1| + |Q_2|$.

Property 2: Abusing notation, let $\ell := \ell(N)$. If $\phi \not\in \text{SAT}$, then any $x \in \mathbb{R}^p$ such that $Bx \in \text{Ball}_\Delta(e^{(m)})$ satisfies $\|x\|_0 \geq (1 - \epsilon_{\text{sound}} \cdot \ell^2) \cdot \frac{\ell}{2} \cdot (|Q_1| + |Q_2|)$, for some $\Delta = \Omega(\ell^2 \cdot \log(|A_1| + |A_2|))$.

Setting $\ell = (10 \cdot \epsilon_{\text{sound}})^{-1/2}$ and $k = |Q_1| + |Q_2|$, properties 1 and 2 imply that

- If $\phi \in \text{SAT}$, then there is a vector $x^* \in \{0,1\}^p$ such that $Bx^* = e^{(m)}$ and $\|x^*\|_0 \leq k$.
- If $\phi \not\in \text{SAT}$, then any $x \in \mathbb{R}^p$ such that $Bx \in \text{Ball}_\Delta(e^{(m)})$ satisfies $\|x\|_0 \geq \sigma \cdot k$, where $\Delta(p) = \Omega(\ell^2 \cdot \log(|A_1| + |A_2|)) \geq \Omega(\ell) = p^{\Omega(1)}$, and $\sigma(p) = (1 - \epsilon_{\text{sound}} \cdot \ell^2) \cdot \frac{\ell}{2} \geq \Omega(\ell) = p^{\Omega(1)}$. (Recall that $p$ is polynomial in $N$.)

Suppose there is a randomized algorithm $A'$ that runs in $\text{BPTIME}(\text{poly}(m, p)) = \text{BPTIME}(\text{poly}(N))$ and distinguishes between the above two cases. By running $A'$ on the matrix $B$ output by $A''$, we determine with high probability whether $\phi$ is satisfiable. Thus, if $\text{SAT} \not\subseteq \text{BPTIME}(\text{poly}(N))$, no such algorithm $A'$ can exist. \qed

### 2.4 Proof of Proposition [7]

**Proof.** Let $g(\cdot)$ be any function and let $h(m, p) = p^{C_1} \cdot m^{1-C_2}$ for some positive constants $C_1, C_2$. Let $A$ be any algorithm for $(g, h)$-\textsc{Sparse Regression} that runs in randomized time $T(m, p)$. Then on any $m \times p$ input matrix $B$ such that there exists a vector $x^* \in \mathbb{R}^p$ with $\|x^*\|_0 \leq k$, $A$ outputs a vector $x$ with $\|x\|_0 \leq g(p) \cdot k$ satisfying $\|Bx - e^{(m)}\|_2^2 \leq p^{C_1} \cdot m^{1-C_2}$.

We show how to transform $A$ into an algorithm $A'$ for the $(g, 1)$-\textsc{Sparse Regression} problem, such that $A'$ runs in time $T(p^{C_1/C_2} \cdot m^{1/C_2}, p)$.

Let $r = \left\lceil (p^{C_1} \cdot m^{1-C_2})^{1/C_2} \right\rceil$. Let $B'$ denote the $(mr) \times p$ matrix obtained by stacking $r$ copies of $B$ atop each other. The algorithm $A'$ simply runs $A$ on $B'$, to obtain an approximate solution $x$ to the system of linear equations given by $B'x = e^{(rm)}$, and outputs $x$. Notice that the running time of $A'$ is indeed $O(T(p^{C_1/C_2} \cdot m^{1/C_2}, p) = O(T(p^{poly}(m, p), p))$.

**Analysis of $x$:** Since $Bx^* = e^{(m)}$, it also holds that $B'x^* = e^{(rm)}$. That is, the system $B'x = e^{(rm)}$ has an exact solution of sparsity $k$. Hence, $A$ must output a vector $x$ with $\|x\|_0 \leq g(p) \cdot k$, satisfying
\[ \|B'x - e^{(rm)}\|^2 \leq p^{C_1} \cdot (rm)^{1-C_2}. \] Notice that:
\[ \|B'x - e^{(rm)}\|^2 = r \cdot \|Bx - e^{(m)}\|^2. \]

Thus,
\[ \|Bx - e^{(m)}\|^2 = (1/r) \cdot \left[ \|B'x - e^{(rm)}\|^2 \right] \leq (1/r) \left[ p^{C_1} \cdot (rm)^{1-C_2} \right] = p^{C_1} \cdot m^{1-C_2} \cdot r^{-C_2} \leq 1. \]

This completes the proof. □

3 The Statistical Problem

3.1 Motivation and Prior Work

In this section we motivate the \((g, h)\)-Noisy Sparse Regression problem introduced in Section[1]. The problem \((g, h)\)-Noisy Sparse Regression is a sparse variant of a less challenging problem known as Noisy Regression that is popular in the statistics literature. Noisy Regression, instead of being given a fixed matrix \(B\) for which we seek a sparse vector \(x\) such that \(Bx \approx e\), we assume corrupting random noise injected into the problem. Specifically, we assume that there are (1) a known \(m \times p\) matrix \(X\) (which plays the role of \(B\)) and (2) an unknown real \(p\)-dimensional vector \(\theta\). An \(m\)-dimensional vector \(\epsilon\) of random noise is generated, with the \(\epsilon_i\)'s being i.i.d. \(N(0, 1)\) random variables. The vector \(y = X\theta + \epsilon\) (but not \(\epsilon\) itself) is then revealed to the algorithm (along with \(X\), which it already knew). On an instance specified by \((X, \theta)\) with \(y\) revealed to the algorithm, an algorithm will produce \(\hat{\theta} = \hat{\theta}(X, y)\). We define \(\hat{y} = X\hat{\theta}\). The instance's prediction loss is defined to be \(\|\hat{y} - E[y]\|^2 = \|X(\hat{\theta} - \theta)\|^2\). (By “\(E[y]\)” here, since \(y = X\theta + \epsilon\), we mean the vector \(X\theta\).) Its expected value \(E[\|X(\hat{\theta} - \theta)\|^2]\), over the random vector \(\epsilon\), is defined to be the risk.

The goal is to make the risk as small as possible. However, as \(\theta\) is unknown, it is hard to measure the performance of an algorithm at a single \(\theta\). After all, an algorithm which luckily guesses \(\theta\) and then sets \(\hat{\theta} = \theta\) will have zero risk for that \(\theta\). For this reason, one usually seeks an algorithm that minimizes the supremum of the risk over all \(\theta\). In other words, this minimax estimator finds \(\hat{\theta}\) to minimize the supremum over \(\theta\) of \(E[\|X(\hat{\theta} - \theta)\|^2]\). Noisy Regression refers to the problem of minimizing this last supremum.

Note here that we are using random noise whose components have variance 1, regardless of the magnitudes of the entries of \(X\). For our hard instances, the entries of \(X\) are 0-1.

It is well-known that ordinary least squares is a minimax estimator: least squares achieves risk \(p\), independently of \(m\) [10] bottom of page 1950. Moreover, no estimator, polynomial-time or not, achieves smaller maximum risk. (It’s not even \textit{a priori} obvious that the risk can be made independent of \(m\).)

The problem \((g, h)\)-Noisy Sparse Regression defined in Section[1] is a “sparse version” of Noisy Regression which is identical to Noisy Regression except that there is a positive integer \(k\), known to the algorithm, such that \(\|\theta\|_0 \leq k\). The goal of the algorithm is to find an estimate \(\hat{\theta}\) of \(\theta\) for which both the supremum over \(\theta\) of the sparsity \(\|\hat{\theta}\|_0\) and the supremum over \(\theta\) of the risk \(E[\|X(\hat{\theta} - \theta)\|^2]\) are as small as possible.

As was the case for \((g, h)\)-Sparse Regression, there is a naive exponential-time algorithm for Sparse Noisy Regression. Namely, try all subsets \(S\) of \(\{1, 2, ..., p\}\) of size \(k\). For each \(S\), use ordinary least squares to find the \(\hat{\theta}_j\)'s, except require that \(\hat{\theta}_j = 0\) for all \(j \notin S\). An upper bound on the risk of this algorithm is known:
Theorem 19 [10, page 1962] For any $\theta$, the risk of the naive exponential-time algorithm is bounded by $4k \ln p$.

The question we are interested in is, how small can a polynomial-time algorithm make the risk, if it is allowed to “cheat” somewhat on the sparsity? More formally, if a polynomial-time algorithm is allowed to output a vector $\hat{\theta}$ with $\|\theta\|_0 \leq k \cdot g(k)$ (where $g(k) \geq 1$ is an arbitrary function), how small can its risk be?

Theorem 3 (respectively, Theorem 5) from Section 1 asserts that the risk cannot be bounded above by $p^{C_1 \cdot m^{1-C_2}}$ for any positive constants $C_1, C_2$, even if one is allowed to cheat by a nearly polynomial (respectively, polynomial) factor on the sparsity of the returned vector. This is in stark contrast to the NOISY REGRESSION, where there is a polynomial-time algorithm (least-squares regression) that achieves risk $O$ independently of sparsity at most $N_{NP}$ where $N_{NP}$ has not halted after $\lceil 20 \rceil$ iterations of the above procedure.

Now we can describe the result of Zhang, Wainwright, and Jordan [18]. There is a standard (non-greedy) algorithm known as LASSO which is known to achieve risk bounded by $O((1/\gamma^2(B))(k \log p))$, where $\gamma(B)$ is $B$’s “restricted eigenvalue constant” [18]. Zhang, Wainwright, and Jordan prove that, unless $NP \subseteq P/poly$, for any $\delta > 0$ any polynomial-time algorithm has risk $\Omega((1/\gamma^2(B))(k^{1-\delta} \log p))$, for some matrices $B$ (for which $\gamma(B)$ can be made arbitrarily small). To contrast their results with ours, [18] requires a different complexity assumption and proves a different lower bound (involving $\gamma(B)$, which ours does not), but most importantly, their bound only applies to algorithms that return linear combinations of $k$ columns, whereas ours allows linear combinations of $2^{O((1-\delta) \log p)} \cdot k$ columns.

3.2 Proof of Theorems 4 and 5

Proposition 20 For every pair $g(\cdot), h(\cdot, \cdot)$ of polynomials, if there is an algorithm for $(g, h/2)$-NOISY SPARSE REGRESSION that runs in time $T(m, p)$, then there is an algorithm for $(g, h)$-SPARSE REGRESSION that runs in time $O(T(m, p))$.

Combining Proposition 20 with Theorem 2 proves Theorem 4 and combining Proposition 20 with Theorem 3 proves Theorem 5.

Proof. Let $\mathcal{A}$ be an algorithm for $(g, h/2)$-NOISY SPARSE REGRESSION. The following algorithm $\mathcal{A}'$ solves $(g, h)$-SPARSE REGRESSION, with failure probability at most $\delta$.

$\mathcal{A}'$ does the following $\lceil \log(1/\delta) \rceil$ times: it generates an $m$-dimensional vector $\epsilon$ whose components are (approximately) i.i.d. $N(0, 1)$ random variables and sets $y = e^{(m)} + \epsilon$. (We will assume that the algorithm generates $N(0, 1)$ random variables exactly.) It then runs $\mathcal{A}$ on input $(B, k, y)$ to obtain a vector $x$ of sparsity at most $k \cdot g(p)$, and checks whether $\|Bx - e^{(m)}\|^2 \leq h(m, p)$. If so, it halts and outputs $x$. If $\mathcal{A}'$ has not halted after $\lceil \log(1/\delta) \rceil$ iterations of the above procedure, $\mathcal{A}'$ outputs a special failure symbol $\bot$.

We must argue, given an $m \times p$ input matrix $B$ such that there is a $p$-dimensional vector $x^\star$ satisfying $\|x^\star\|_0 \leq k$ and $B x^\star = e^{(m)}$, that $\mathcal{A}'$ outputs a vector $x$ of sparsity at most $k \cdot g(p)$ satisfying $\|Bx - e^{(m)}\|^2 \leq h(m, p)$, with probability at least $1 - \delta$. Clearly if $\mathcal{A}'$ does not output $\bot$, then the vector $x$ that it outputs satisfies the required conditions. We argue that the probability $\mathcal{A}'$ outputs $\bot$ is at most $1/2^{\lceil \log(1/\delta) \rceil} \leq \delta$.

Since $e^{(m)} = B x^\star$, each invocation of $\mathcal{A}$ must return a vector $x$ which must satisfy (a) $\|x\|_0 \leq k \cdot g(p)$ and (b) $E[\|B(x - x^\star)\|^2] = E[\|Bx - e^{(m)}\|^2] \leq h(m, p)/2$, where the expectation is taken over the internal randomness of $\mathcal{A}$ and the random choice of $\epsilon$. By Markov’s inequality, the probability that $\|Bx - e^{(m)}\|^2 > h(m, p)$ is at most $1/2$. Hence, the probability that $\mathcal{A}$ returns a vector $x$ such that $\|B(x - x^\star)\|^2 > h(m, p)$ during $\lceil \log(1/\delta) \rceil$ consecutive invocations is bounded above by $1/2^{\lceil \log(1/\delta) \rceil}$. This completes the proof of the theorem. □
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Formally, a projection game consists of (a) a bipartite multigraph $G = (Q_1,Q_2,R)$, (b) finite alphabets $A_1, A_2$, and (c) projection constraints, specified via a partial function $\pi_r: A_1 \rightarrow A_2$ for every edge $r \in R$. The goal is to find assignments $P_1: Q_1 \rightarrow A_1$ and $P_2: Q_2 \rightarrow A_2$ that satisfy as many edges as possible, where $(P_1, P_2)$ is said to satisfy an edge $r = (q_1,q_2)$ if $\pi_r(P_1(q_1)) = P_2(q_2)$. The size of the projection game is defined to be $n = |Q_1| + |Q_2| + |R|$, and the size of the alphabet of the projection game is defined to be $\max\{|A_1|,|A_2|\}$. The acceptance probability of the projection game is the maximum, over $P_1: Q_1 \rightarrow A_1$ and $P_2: Q_2 \rightarrow A_2$, of the fraction of edges that are satisfied by the pair $(P_1, P_2)$ of provers.

Equivalence between projection games and 2-prover MIP’s satisfying functionality should be clear, except possibly for the $\pi_r$’s, in light of the letters used in the definition of a projection game. In an MIP satisfying functionality, for each $r$ and $a_1$, there is at most one $a_2$ such that $V(r,a_1,a_2) = 1$. Given $r$ and $a_1$, if $a_2$ exists with $V(r,a_1,a_2) = 1$, then $a_2$ is unique; define $\pi_r(a_1) = a_2$ in this case. If no such $a_2$ exists, leave $\pi_r(a_1)$ undefined. Conversely, if $\pi_r(a_1) = a_2$, let $V(r,a_1,a_2) = 1$ and let $V(r,a_1,a) = 0$ for all other $a \in A_2$, and if $\pi_r(a_1)$ is undefined, let $V(r,a_1,a) = 0$ for all $a \in A_2$. In this way we have defined $V(r,a_1,a_2)$ for all $r \in R$, $a_1 \in A_1$, and $a_2 \in A_2$.

In [14], Moshkovitz states the Projection Games Conjecture as follows:

\[ \mathsf{PGC} \]

In [14], Moshkovitz defines $\pi_r$ to be a total function. We allow $\pi_r$ to be a partial function in accordance with standard definitions, and to ensure a precise equivalence between projection games and 2-prover MIP’s satisfying functionality.

\[ \mathsf{PGC} \]

Technically, Moshkovitz’s PGC is slightly stronger than our Conjecture, in that size of the projection game is required to be $n = N^{1+\epsilon/[2]}\text{poly}(1/\epsilon)$, rather than $\text{poly}(N,1/\epsilon)$. She states the weaker version in a footnote, and the weaker version suffices for our purposes.
**Conjecture 21 (Projection Games Conjecture (PGC) [14])** There exists a $c > 0$ such that for all $N$ and $\epsilon \geq 1/n^c$, SAT on inputs of size $N$ can be efficiently reduced to projection games of size $n = \text{poly}(N, 1/\epsilon)$ over an alphabet of size $\text{poly}(1/\epsilon)$, such that a SAT input which is satisfiable maps to a projection game of acceptance probability 1 and a SAT input which is not satisfiable maps to a projection game of acceptance probability at most $\epsilon$.

The equivalence of projection games and 2-prover MIP’s with functionality implies that the PGC is equivalent to Conjecture [16] from Section 2.3

**B A Bad Example for Natarajan’s Algorithm**

First we give the problem solved by Natarajan’s algorithm. We are given a collection $C$ of vectors (the columns of a matrix $B$) and a target vector $b$. The algorithm successively picks columns from $C$ in an attempt to represent $b$ approximately as a linear combination of the selected columns. The algorithm guarantees that the number of columns selected by the algorithm is at most $\lceil \text{Opt}(\epsilon / 2) \rceil$, where $\text{Opt}(\epsilon / 2)$ denotes the minimum sparsity of a linear combination with $\epsilon / 2$ error at most $\epsilon / 2$ and $B^+$ is the pseudoinverse of the matrix obtained from $B$ by normalizing each column. Our goal is to show that the $||B^+||^2$ factor is necessary.

The algorithm is as follows. Start with $S = \emptyset$, $V = C$, and $b' = b$. At the end of a generic iteration of the algorithm, we know that the vectors in $S$ are orthogonal to each other and to all $v \in V - S$. In the $i$th iteration, the algorithm chooses the $v \notin S$ which has maximum value of $(b' \cdot v)/||v||$ (in other words, has smallest angle with $b'$). That vector $v^+$ is added to $S$. Now we make $b'$ and the vectors in $V - S$ orthogonal to $v^+$ by transforming those vectors, by setting $b' := b' - (b' \cdot v^+)/(v^+ \cdot v^+)$ and $\hat{v} := \hat{v} - (\hat{v} \cdot v^+)/(v^+ \cdot v^+)$ for each $v \in V - S$. This vector-selection phase continues as long as $||b'|| > \epsilon$.

The span of the vectors in $S$ is the same as that of the “corresponding” vectors in $C$, because the linear operations made during the execution of the algorithm do not change the spans. When we are finished selecting vectors, we write $b - b'$ as a linear combination of the vectors in $S$ or, equivalently, of the vectors in $C$ corresponding to the vectors in $S$.

The example is simple. We simply have to show that the vector-selection phase selects a large number of vectors, relative to $\text{Opt}(\epsilon / 2)$. The vector $b = \epsilon^{(m)}$. There are $m + 2$ vectors. The first $m / 2$, $v_1, v_2, \ldots, v_{m/2}$, are binary and have two 1’s each, $v_i$, having ones in positions $2i - 1$ and $2i$. There are two additional vectors, which we call $v^+$ and $v^-$. We start by setting $v^+ = \gamma := (1, -1, 1, -1, 1, -1, \ldots, 1, -1)^T \in \mathbb{R}^m$ and $v^- = -\gamma$, and then we add $\delta \epsilon^{(m)}$ to $v^+$ and $v^-$, where $\delta = 1/\sqrt{m}$. Set $v'^+ = v^+$ and $v'^- = v^-$. It is clear that $(v^+ + v^-)/(2\delta) = b$, so that there is an exact way to write $b$ as a linear combination of two vectors. Let $\epsilon = 0.5\sqrt{2}$. Therefore $\text{Opt}(\epsilon / 2) = 2$. We will show that the algorithm may add the vectors in the order $v_1, v_2, \ldots, v_{m/2}$, not terminating until it’s added $m/2$ vectors. Since $\ln(||b||/\epsilon) = \ln((\sqrt{m})/(1/2)) = \ln(2\sqrt{m}) \leq 1 + 0.5 \cdot \ln m$, the number of vectors produced by the algorithm should be at most $2 \cdot 2^{\ln(\sqrt{m})/\ln(\sqrt{m})} = \text{Opt}(m/\log m)$, and that the $||B^+||^2$ factor is necessary in the analysis.

Let $p_j$ denote the $m$-vector which starts with $m - j$ 0’s and ends with $j$ 1’s. We will prove inductively that at the beginning of the $h$th iteration, $h = 0, 1, 2, \ldots, m/2 - 1$, $S = \{v_1, v_2, \ldots, v_h\}$; $V - S$ consists of $v_{h+1}, \ldots, v_{m/2}$, together with $v'^+$ and $v'^-$ (both of which change over time), where $v'^+ = \gamma + \delta \cdot p_{m-2h}$ and $v'^- = -\gamma + \delta \cdot p_{m-2h}$; and that the vector $b'$ (which changes over time) equals $p_{m-2h}$.

It is easy to see that the invariant is true at the beginning of the 0th iteration.
Now we start the inductive proof. Choose any \( h \geq 0 \) and assume that the invariant holds. To decide which vector to add, we compare \((b' \cdot v_i)/||v_i|| = 2/\sqrt{2} = \sqrt{2} \) (for any \( i \geq h + 1 \)) to \((b' \cdot v^{+i})/||v^{+i}|| = (b' \cdot v^{-i})/||v^{-i}|| = ((m - 2h)\delta)/\sqrt{(m - 2h)(1 + \delta^2)} = \sqrt{m - 2h} \left[ \frac{\delta}{\sqrt{1 + \delta^2}} \right] \). Because \( \delta < \sqrt{2}/\sqrt{m} \), the former is larger, and hence we can assume that the algorithm adds \( v_{h+1} \) to \( \mathcal{S} \). Vectors \( v_{h+2}, v_{h+3}, ..., v_{m/2} \) are all orthogonal to \( v_h \) so do not have to change. We update \( v^{+i} \) by

\[
v^{+i} := v^{+i} - v_{h+1} \left( \frac{v_{h+1} \cdot v^{+i}}{v_{h+1} \cdot v_{h+1}} \right).
\]

Since \( v_{h+1} \cdot v^{+i} = 2\delta \) and \( v_{h+1} \cdot v_{h+1} = 2 \), this formula is

\[
v^{+i} := v^{+i} - \delta v_{h+1}.
\]

Since before modification \( v^{+i} = \gamma + \delta \cdot p_{m-2h} \), the new \( v^{+i} = \gamma + \delta \cdot p_{m-2h} \). A similar computation shows that the new \( v^{-i} = -\gamma + \delta \cdot p_{m-2h} \). Yet another similar computation shows that \( b' \) is now updated to \( p_{m-2h} \). This completes the inductive step.

Since until the end of the last iteration, \( ||b'|| \geq \sqrt{2} > \epsilon \), the algorithm continues running for \( m/2 \) iterations, when \( ||b'|| \) drops to 0 and the algorithm terminates. \( \blacksquare \)

The authors have a proof, to be included in a future version of this paper, that the same example thwarts the LASSO algorithm.