ON COHOMOLOGICAL INVARIANTS OF LOCAL RINGS IN POSITIVE CHARACTERISTIC

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Abstract. The Frobenius depth denoted by F-depth defined by Hartshorne-Speiser in 1977 and later by Lyubeznik in 2006, in a different way, for rings of positive characteristic. The aim of the present paper is to compare the F-depth with formal grade, and depth to shed more light on the notion of Frobenius depth from a different point of view.

1. Introduction

Let \( Y \) be a closed subscheme of \( \mathbb{P}^n_k \), the projective space over a field \( k \) of characteristic \( p > 0 \). Vanishing of \( H^i(\mathbb{P}^n - Y, \mathcal{F}) \) for all coherent sheaves \( \mathcal{F} \) was asked by Grothendieck ([6]). Among the attempts to answer the mentioned question Hartshorne and Speicer in [8] used the notion of Frobenius depth of \( Y \) to give an essentially complete solution to this problem.

To be more precise, Let \( Y \) be a Noetherian scheme of finite dimension, whose local rings are all of characteristic \( p > 0 \). Let \( y \in Y \) be a (not necessarily closed) point. Let \( d(y) \) be the dimension of the closure \( \{ y \}^- \) of the point \( y \). Let \( \mathcal{O}_y \) be the local ring of \( y \), let \( k_0 \) be its residue field, let \( k \) be a perfect closure of \( k_0 \), and let \( \hat{\mathcal{O}}_y \), be the completion of \( \mathcal{O}_y \). Choose a field of representatives for \( k_0 \) in \( \hat{\mathcal{O}}_y \). Then we can consider \( \hat{\mathcal{O}}_y \) as a \( k_0 \)-algebra, and we let \( A_y \) be the local ring \( \hat{\mathcal{O}}_y \otimes_{k_0} k \) obtained by base extension to \( k \). Let \( Y_y = \text{Spec} A_y \) and let \( P \) denote its closed point. So, the Frobenius depth of \( Y \) is denoted by F-depth \( Y \) is the largest integer \( r \) (or \( +\infty \)) such that for all points \( y \in Y \), one has \( H^i_p(Y_y, \mathcal{O}_{Y_y}) = 0 \) (the stable part of \( H^i_p(Y_y, \mathcal{O}_{Y_y}) \)) for all \( i < r - d(y) \).

From the local algebra point of view, Grothendieck’s problem is stated to find conditions under which \( H^i_I(M) = 0 \) for all \( i > n \) \((n \in \mathbb{Z})\) and all \( A \)-modules \( M \), where \( A \) is a commutative Noetherian local ring and \( I \subset A \) is an ideal. For an \( A \)-module \( M \), we denote by \( H^i_I(M) \) the \( i \)th local cohomology module of \( M \) with respect to \( I \). For more details the reader may consult [5] and [2]. From the celebrated result of Hartshorne (cf. [4] pp. 413), it is enough to find conditions

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for the vanishing of \( H^i_I(A) \). In this direction, for a local ring \((A, \mathfrak{m})\), Lyubeznik in [11] using the Frobenius map from \( H^{i_{\mathfrak{m}}}I(A) \) to itself defined the Frobenius depth of \( A \) denoted by \( F-depth \) \( A \) as the smallest \( i \) such that for every iteration of Frobenius map, \( H^{i_{\mathfrak{m}}}I(A) \) does not go to zero. It is noteworthy to say that the Lyubeznik’s \( F-depth \) coincides with the notion of \( F-depth \) defined by Hartshorne and Speiser, whenever \( A \) admits a surjection from a regular local ring and \( Y = \text{Spec} \ A \) (cf. [11, Corollary 6.3]).

Consider the family of local cohomology modules \( \{ H^i_{\mathfrak{m}}(M/\mathfrak{m}^nM) \}_{n \in \mathbb{N}} \) where, \((A, \mathfrak{m})\) is not necessarily of characteristic \( p > 0 \). For every \( n \in \mathbb{N} \) there is a natural homomorphism

\[
H^i_{\mathfrak{m}}(M/\mathfrak{m}^{n+1}M) \to H^i_{\mathfrak{m}}(M/\mathfrak{m}^nM)
\]

(induced from the natural projections \( M/\mathfrak{m}^{n+1}M \to M/\mathfrak{m}^nM \)) such that the family forms a projective system. The projective limit \( \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{m}^nM) \) is called the \( i \)th formal local cohomology of \( M \) with respect to \( I \) (cf. [15]). Formal local cohomology modules were used by Peskine and Szpiro in [14] when \( A \) is a regular ring of prime characteristic. It is noteworthy to mention that if \( U = \text{Spec}(A) \setminus \{ \mathfrak{m} \} \) and \((\widehat{U}, \mathcal{O}_{\widehat{U}})\) denote the formal completion of \( U \) along \( V(I) \setminus \{ \mathfrak{m} \} \) and also \( \widehat{F} \) denotes the \( \mathcal{O}_{\widehat{U}} \)-sheaf associated to \( \varprojlim_n M/\mathfrak{m}^nM \), they have described the formal cohomology modules \( H^i(\widehat{U}, \mathcal{O}_{\widehat{U}}) \) via the isomorphisms \( H^i(\widehat{U}, \mathcal{O}_{\widehat{U}}) \cong \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{m}^nM), i \geq 1 \). See also [13, proposition (2.2)] when \( A \) is a Gorenstein ring.

The formal grade, \( \text{fgrade}(I, M) \), is defined as the index of the minimal nonvanishing formal cohomology module, i.e., \( \text{fgrade}(I, M) = \inf \{ i \in \mathbb{Z} \mid \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{m}^nM) \neq 0 \} \). One way to check out vanishing of local cohomology modules is the following duality

\[
(1.1) \quad \varprojlim_n H^i_{\mathfrak{m}}(A/I^n) \cong \text{Hom}_A(H^i_{\mathfrak{m}}(A), E(A/\mathfrak{m})),
\]

where \((A, \mathfrak{m})\) is a Gorenstein local ring and \( E(A/\mathfrak{m}) \) denotes the injective hull of the residue field (cf. [15, Remark 3.6]). To be more precise, in this case the last non vanishing amount of \( H^i_I(A) \) may be described with the \( \text{fgrade}(I, A) \). Thus, it motivates us to consider the invariants \( F-depth \) and \( \text{fgrade} \) to shed more light on the notion of Frobenius depth from a different point of view. For this reason, in Section 2, we bring some auxiliary results and among them we examine the structure of \( \varprojlim_n H^i_{\mathfrak{m}}(A/I^n) \) as a unit \( A[F^\infty] \)-module (Theorem 2.8). Moreover, it has a structure of \( D \)-modules. In Section 3, we concentrate on \( F-depth \) and reprove some known results and then compare it with the formal grade and depth, (cf. Theorem 3.8 and Corollary 3.9).
2. Auxiliary Results

Throughout this section all rings are assumed to contain a field of positive characteristic. The symbol $A$ will always denote a commutative Noetherian ring of finite characteristic. We adapt the notation from [1] and except for notation we mostly follow Lyubeznik [10]. We let $F = F_A$ the Frobenius map on $A$, that is $F : A \to A$, with $a \mapsto a^p$, $a \in A$. We denote the $e$th iterate of the Frobenius map by $A^e$ which is the $A - A$-bimodule. As a left $A$-module it is $A$ and as a right $A$-module we have $m.a = a^{p^e}m$ for $m \in A^e$. We say $A$ is $F$-finite, whenever $A^e$ is a finitely generated right $A$-module.

Remark 2.1. Let us recall from [8, Proposition 1.1(a)] that for a ring $A$ which is either a localization of an algebra of finite type over a perfect field $k$, or a complete local ring containing a perfect field $k$ as its residue field, then $A$ is $F$-finite.

In the present section, among our results we recall various results due to Hartshorne-Speiser [8], Peskine-Szpiro [14], Lyubeznik [10] and Blickle [1].

Peskine and Szpiro in [14] defined the Frobenius functor as follows:

Definition 2.2. The Frobenius functor is the right exact functor from $A$-modules to $A$-modules given by

$$F_A^* M := A^1 \otimes_A M.$$ 

Its $e$th power is $F_A^{e*} M = A^e \otimes_A M$. For brevity we often write $F_A^*$ for $F_A^{e*}$ when there is no ambiguity about the ring $A$.

It follows from the definition that $F_A^*$ commutes with direct sum, direct limit and localization. By a theorem of Kunz [9] the Frobenius functor is flat whenever $A$ is a regular ring, hence in this case $F_A^*$ will be exact and immediately one has $F_A^{e*} A = A^e \otimes_A A \cong A$ and $F_A^{e*} I = I[p^e]$ an ideal of $A$ generated by $p^e$th powers of the elements of $I$.

Definition 2.3. An $A[F^e]$-module is an $A$-module $M$ together with an $A$-linear map

$$\mathcal{V}_M^e : F_A^{e*} M \to M.$$ 

A morphism between two $A[F^e]$-modules $(M, \mathcal{V}_M^e)$ and $(N, \mathcal{V}_N^e)$ is an $A$-linear map $\varphi : M \to N$ such that the following diagram commutes:

$$\begin{diagram}
F_A^{e*} M & \xrightarrow{F_A^* (\varphi)} & F_A^{e*} N \\
\mathcal{V}_M^e \downarrow & & \mathcal{V}_N^e \\
M \xrightarrow{\varphi} & & N
\end{diagram}$$
In fact, we can consider $F^e M$ as a $p^e$-linear map from $M \to M$; as such it is not $A$-linear but we have $F^e(am) = a[p^e]F^e(m)$, $a \in A^e, m \in M$. Furthermore, an $A[F^e]$-module $(M, \mathcal{V}_M)$ is called a unit $A[F^e]$-module if $\mathcal{V}_M$ is an isomorphism (cf. \cite{b} page 16).

**Remark 2.4.** As we have seen above, $A$ is a unit $A[F^e]$-module, whenever $A$ is regular but $I$ is not in general. For a multiplicatively closed subset $S$ of $A$, the structural map $\mathcal{V}^e_{S^{-1}A} : A^e \otimes S^{-1}A \to S^{-1}A$ is an isomorphism (see \cite{b}).

The following definition introduced in \cite{b}:

**Definition 2.5.** Let $(M, \mathcal{V})$ be an $A[F^e]$-module. We define $G(M)$ as the inverse limit generated by the structural map $\mathcal{V}^e$, i.e.

$$G(M) := \lim_{\rightarrow}(\cdots \to F^{3e}M \overset{F^{2e}\mathcal{V}^e}{\to} F^{2e}M \overset{F^{e}\mathcal{V}^e}{\to} F^{e}M \overset{\mathcal{V}^e}{\to} M).$$

Note that there are natural maps $\pi_e : G(M) \to F^{e}M$. Moreover, the maps $F^{e}\pi_e : F^{e}G(M) \to F^{e(r+1)}M$ are compatible with the maps defining $G(M)$ and thus by the universal property of inverse limits, the map $F^{e}G(M) \to G(M)$ defines the natural $A[F^e]$-module structure on $G(M)$.

**Proposition 2.6.** (\cite{b} Proposition 1.2 and \cite{b} Proposition 4.1) Let $A$ be regular and $F$-finite and $M$ an $A[F^e]$-module. Then $G(M)$ is a unit $A[F^e]$-module.

In order to extend the Matlis duality functor $D(-) = \text{Hom}(-, E_A)$ where, $E_A$ is the injective hull of the residue field, the functor $\mathcal{D}$ from $A[F^e]$-modules to $A[F^e]$-modules is defined as follows (\cite{b} Section 4):

Let $(M, \mathcal{V})$ be an $A[F^e]$-module. We define

$$\mathcal{D}(M) = \lim_{\leftarrow}(D(M) \overset{D(\mathcal{V}^e)}{\longrightarrow} D(F^{e}M) \overset{D(F^{e}\mathcal{V}^e)}{\longrightarrow} D(F^{2e}M) \longrightarrow \cdots).$$

An element $m \in M$ is called $F$-nilpotent if $F^{re}(m) = 0$ for some $r$. Then $M$ is called $F$-nilpotent if $F^{e}(M) = 0$ for some $r \geq 0$. It is possible that every element of $M$ is $F$-nilpotent but $M$ itself is not.

Below, we recall some properties of the functor $\mathcal{D}$.

**Proposition 2.7.** Let $A$ be a complete regular local ring.

(a) On the subcategory of $A[F^e]$-modules which are cofinite (i.e. satisfy the descending chain condition for submodules) as $A$-modules $\mathcal{D}$ is exact and its values are finitely generated unit $A[F^e]$-modules (cf. \cite{b} Theorem 4.2(i) and \cite{b} Proposition 4.16).

(b) Let $M$ be an $A[F^e]$-module that is finitely generated or cofinite as an $A$-module. Then $\mathcal{D}(\mathcal{D}(M)) \cong G(M)$ (cf. \cite{b} Proposition 4.17).
(c) Let $M$ be an $A[F^e]$-module which is a cofinite $A$-module. Then $M$ is $F$-nilpotent if and only if $\mathcal{D}(M) = 0$ (cf. [10] Theorem 4.2(ii) and [11] Proposition 4.20).

Let $A$ be a regular local ring and let $I$ be an ideal of $A$. As we have seen (following [11] page 19)) that $A$ is an $A[F^e]$-module and come down this structure to its localizations. The local cohomology modules $H^i_I(A)$ of $A$ with support in $I$ can be calculated as the cohomology modules of the Čech complex

$$\tilde{C}(A; x_1, \ldots, x_n) = A \to A_{x_i} \to A_{x_ix_j} \to A_{x_1x_2\cdots x_n}$$

where $I$ is generated by $x_1, x_2, \ldots, x_n$. Thus, the modules $H^i_I(A)$ are $A[F^e]$-modules as the category of $A[F^e]$-modules is Abelian. Moreover, the $H^i_I(A)$ are unite $A[F^e]$-modules for all $i \in \mathbb{Z}$, but the modules $H^i_I(A)$ are not unit in general. For formal local cohomology modules, the situation is a bit more complicated, however, we show that these kind of modules have unite $A[F^e]$-modules structure, where $A$ is $F$-finite.

**Theorem 2.8.** Let $(A, \mathfrak{m})$ be a regular $F$-finite local ring. Then

$$G(H^i_m(A/I)) \cong \lim_{\leftarrow n} H^i_m(A/I^n), \quad i \in \mathbb{Z}$$

which is a unit $A[F^e]$-module. In particular,

$$\lim_{\leftarrow n} H^i_m(A/I^n) \cong H^i_m(\mathring{A}), \quad i \in \mathbb{Z}$$

as $A[F^e]$-modules, where $\mathring{A}$ is the completion of $A$ along $I$.

**Proof.** By what we have seen above, $H^i_m(A/I)$ is an $A[F^e]$-module for all $i \in \mathbb{N}$. Now, we may apply functor $G(-)$ to $H^i_m(A/I)$:

$$G(H^i_m(A/I)) = \lim_{\leftarrow n} H^i_m(A/I^{[e^n]} \mathbin{\overset{r \cdot \psi}{\rightarrow}} H^i_m(A/I^{[e^n]}) \mathbin{\overset{F^e \cdot \psi}{\rightarrow}} H^i_m(A/I^{[e^n]})) \mathbin{\overset{\psi}{\rightarrow}} H^i_m(A/I)).$$

By virtue of Proposition 2.6, $G(H^i_m(A/I))$ is a unite $A[F^e]$-module. Notice that, the right hand side is nothing but $\lim_{\leftarrow n} H^i_m(A/I^{[e^n]})$.

On the other hand, one has $\lim_{\leftarrow n} H^i_m(A/I^n) \cong \text{Hom}_A(H^i_I(A, \mathfrak{m}), E)$, where $E := E(A/\mathfrak{m})$ is the injective hull of the residue field. The natural map

$$F^{e^n} \text{Hom}_A(H^i_I(A, \mathfrak{m}), E) \to \text{Hom}_A(H^i_I(A, \mathfrak{m}), E)$$

by sending $r \otimes \varphi$ to $rF^{e^n}(\varphi)$ is an isomorphism of $A[F^e]$-modules ($r \in A$ and $\varphi \in \text{Hom}_A(H^i_I(A, \mathfrak{m}), E)$). To this end note that $H^i_I(A, \mathfrak{m})$ and $E \cong H^i_m(A)$ carry natural unite $A[F^e]$-structure. Thus, $\lim_{\leftarrow n} H^i_m(A/I^n)$ is a unite $A[F^e]$-module for each $i \in \mathbb{Z}$.

In order to complete the proof, it is enough to show that $\lim_{\leftarrow n} H^i_m(A/I^{[e^n]}) \cong \lim_{\leftarrow n} H^i_m(A/I^n)$. For this reason, consider the decreasing family of ideals $\{I^{[e^n]}\}_e$. 
Clearly, its topology is equivalent to the $I$-adic topology on $A$. Thus, by [15, Lemma 3.8] there exists a natural isomorphism

$$
\lim_{\leftarrow} e H^i_m(A/I[\rho^e]) \cong \lim_{\leftarrow} n H^i_m(A/I^n)
$$

for all $i \in \mathbb{Z}$.

The last part follows by [8, Proposition 2.1].

**Remark 2.9.** It should be noted that with the assumptions of Theorem [2.8] for all $i \in \mathbb{Z}$, the module $\lim_{\leftarrow} n H^i_m(A/I^n)$ carries a natural $D_A$-module and $\mathcal{V}^e$ (cf. Definition [2.9]) is a map of $D_A$-modules. The interested reader may consult [10, Section 5] and [11, Chapter 3].

### 3. Frobenius depth

Let $A$ be a regular local $F$-finite ring of characteristic $p > 0$ and let $I$ be an ideal of $A$. As we have seen in the previous section the formal local cohomology modules, are unite $A[F^e]$-modules. In the present section we use the unit $A[F^e]$ structure of $\lim_{\leftarrow} n H^i_m(A/I^n)$ in order to prove our results.

**Proposition 3.1.** Let $(A, m)$ be a regular local and $F$-finite ring. Then $\lim_{\leftarrow} n H^i_m(A/I^n) = 0$ if and only if $H^i_m(A/I)$ is $F$-nilpotent.

**Proof.** By the assumptions $A$ is $F$-finite that is $A^e$ is a finitely generated $A$-module. Then tensoring with $A^e$ commutes with the inverse limit, as $A^e$ is a free right $A$-module (cf. [8, Proposition 1.1(b)]). Thus, we have

$$
A^e \otimes_A \tilde{A} = A^e \otimes_A (\lim_{\leftarrow} n A/I^n) \cong \lim_{\leftarrow} n A^e/A^n I^n = \lim_{\leftarrow} n A^e/I^n[\rho^e] A^e \cong \tilde{A}^e.
$$

On the other hand, since the Frobenius action is the same in both $H^i_m(A/I)$ and $H^i_m(\tilde{A}/I\tilde{A})$, so we may assume that $A$ is a complete regular local $F$-finite ring.

As $H^i_m(A/I)$ is an $A[F^e]$-module which is a cofinite $A$-module, then $D(H^i_m(A/I))$ is a finitely generated unit $A[F^e]$-module (cf. [2.7(a)]) and therefore

$$
D(D(H^i_m(A/I))) \cong D(D(H^i_m(A/I))).
$$

As the functor $D(-)$ transforms direct limits to inverse limits, then

$$
D(D(H^i_m(A/I))) = D(\lim_{\leftarrow} (D(H^i_m(A/I)) \to D(F^{e*}H^i_m(A/I)) \to D(F^{2e*}H^i_m(A/I)) \to \cdots)
$$

$$
\cong \lim_{\leftarrow} (\cdots \to D(D(F^{e*}H^i_m(A/I))) \to D(D(F^{e*}H^i_m(A/I))) \to D(D(H^i_m(A/I))))
$$

$$
\cong \lim_{\leftarrow} e H^i_m(A/I[\rho^e]).
$$

As we have seen in the proof of Theorem [2.8] $\lim_{\leftarrow} e H^i_m(A/I[\rho^e]) \cong \lim_{\leftarrow} n H^i_m(A/I^n)$. Therefore, $H^i_m(A/I)$ is $F$-nilpotent if and only if $D(H^i_m(A/I)) = 0$ ([2.7(c)]) if and only if $\lim_{\leftarrow} n H^i_m(A/I^n) = 0$. \[\square\]
Remark 3.2. Notice that in the Proposition 3.1 the $F$-finiteness of $A$ is vital, because it guarantees the $A[F^r]$ structure of the modules $\lim_{n \to \infty} H^i_{m}(A/I^n)$. However, in the light of [11] Lemma 4.12 if $H^{\dim A-i}_{m}(A)$ is cofinite, then the module $\lim_{n \to \infty} H^i_{m}(A/I^n)$ is $A[F^r]$-module.

Corollary 3.3. ([11] Corollary 3.2) Let $(A, m)$ be a regular local ring and $I$ an ideal of $A$. Then $H^{\dim A-i}_{m}(A) = 0$ if and only if $F^r : H^i_{m}(A/I) \to H^i_{m}(A/I)$ is the zero map for some $r > 0$.

Proof. As $\widehat{A}$ is a faithful flat $A$-module then by passing to the completion we may assume that $A$ is complete regular local ring. Let $H^{\dim A-i}_{m}(A) = 0$, so it is cofinite. Then by the duality (1.1) in the introduction, one has $\lim_{n \to \infty} H^i_{m}(A/I^n) = 0$. Hence, Proposition 3.1 implies that $H^i_{m}(A/I)$ is $F$-nilpotent.

Conversely, assume that $H^i_{m}(A/I)$ is $F$-nilpotent. Then by Proposition 2.7(c) $D(H^i_{m}(A/I)) = 0$ and therefore one has $H^{\dim A-i}_{m}(A) = 0$. To this end note that, $D(H^i_{m}(A/I^{[i]})) \cong \text{Ext}^i_{A}(A/I^{[i]}, A)$ for all $i \geq 0$ and the Frobenius powers of $I$ are cofinal with its ordinary powers.

In the light of Corollary 3.3 Lyubeznik [11] defined the $F$-depth of a local ring in order to give a solution to Grothendieck’s Problem.

Definition 3.4. Let $(A, m)$ be a local ring. The $F$-depth of $A$ is the smallest $i$ such that $F^r$ does not send $H^i_{m}(A)$ to zero for any $r$.

One of elementary properties of $F$-depth shows that $F$-depth $A$ is equal to the $F$-depth of its $m$-adic completion, $\widehat{A}$ (cf. [11] Proposition 4.4) because $H^i_{m}(A) \cong H^i_{m}(\widehat{A})$. In the next result we give an alternative proof of [11] Lemma 4.2 to emphasize that $F$-depth of $A$ is bounded above by its Krull dimension.

Proposition 3.5. Let $(A, m)$ be a local ring and $I$ an ideal of $A$. Then $F^r$ does not send $H^{\dim A}_{m}(A)$ to zero for any $r$. In particular, $0 \leq F$-depth $A \leq \dim A$.

Proof. Since the Frobenius action is the same in both $H^i_{m}(A)$ and $H^i_{m}(\widehat{A})$, so we may assume that $A$ is a complete local ring. Thus, by the Cohen’s Structure Theorem $A \cong R/J$, where $R$ is a complete regular local ring and $J \subset R$ and ideal. In the contrary, assume that $H^{\dim A}_{m}(R/J)$ is $F$-nilpotent. Then, $D(H^{\dim A}_{m}(R/J)) = 0$ (cf. 2.7(c)) and with a similar argument given in the proof of Corollary 3.3 one can deduce the vanishing of $H^0_{m}(R)$. Hence, by virtue of [11], in the introduction, one has $\lim_{n \to \infty} H^{\dim A}_{m}(R/J^n) = 0$ which is contradiction (cf. [15] Theorem 4.5).

To investigate the other properties of $F$-depth, in the next Theorem we compare the Frobenius depth of $A$ and $A^{sh}$. For this reason, let us recall some preliminaries. For a local ring $A$ we denote by $A^{sh}$ the strict Henselization of $A$. A local ring
\((A, \mathfrak{m}, k)\) is said to be strictly Henselian if and only if every monic polynomial \(f(T) \in A[T]\) for which \(f(T) \in k[T]\) is separable splits into linear factors in \(A[T]\). For more advanced expositions on this topic we refer the interested reader to [12].

**Proposition 3.6.** Let \((A, \mathfrak{m})\) be a complete local ring and \(I\) be an ideal of \(A\). Then \(F\text{-depth } A = F\text{-depth } A^{sh}\).

**Proof.** First assume that \(A\) is a regular local ring. We show that \(F\text{-depth } A/I = F\text{-depth } (A/I)^{sh}\). Put \(F\text{-depth } A/I = t\). Then, by virtue of Corollary 3.3, one has \(H^i(A) = 0\) for all \(i > \text{dim } A - t\). Due to the faithfully flatness of the inclusion \(A \to A^{sh}\) and the fact that \(A^{sh}\) is a regular local ring, it implies that \(H^i(A^{sh}) = 0\) for all \(i > \text{dim } A - t\). Again, using Corollary 3.3, it follows that \(F\text{-depth } (A/I)^{sh} \geq t\).

To this end note that \(\text{dim } A = \text{dim } A^{sh}\) and \((A/I)^{sh} = A^{sh}/IA^{sh}\). With the similar argument one has \(F\text{-depth } A/I \geq F\text{-depth } (A/I)^{sh}\). This completes the assertion.

Since \(A\) is a complete local ring, then by virtue of Cohen’s Structure Theorem, \(A\) is a homomorphic image of a regular local ring \(R\), i.e. \(A = R/J\) for some ideal \(J\) of \(R\). Now, we are done by the previous paragraph. To this end note that

\[
F\text{-depth } A = F\text{-depth } R/J = F\text{-depth } (R/J)^{sh} = F\text{-depth } A^{sh}.
\]

\(\square\)

**Remark 3.7.** From Proposition 3.6 and [11, Proposition 4.4] one may deduce that

\[
F\text{-depth } A = F\text{-depth } (\hat{A})^{sh},
\]

where, \(A\) is a local ring.

In the following, we compare the invariants depth, \(F\text{-depth}\) and \(f\text{grade}\).

**Theorem 3.8.** Let \((A, \mathfrak{m})\) be a local \(F\text{-finite}\) ring which is a homomorphic image of a regular local \(F\text{-finite}\) ring and let \(I\) be an ideal of \(A\). Then

\[
f\text{grade}(I, A) \leq \text{depth } A \leq F\text{-depth } A.
\]

**Proof.** We have \(\lim_{\leftarrow n} H^i_{\mathfrak{m}, A} (\hat{A}/I^n\hat{A}) \cong \lim_{\leftarrow n} H^i_{\mathfrak{m}} (A/I^n A)\) (cf. [15, Proposition 3.3]) and \(F\text{-depth } A \cong F\text{-depth } \hat{A}\) (cf. [11, Proposition 4.4]). Thus, we may assume that \(A\) is a complete local ring. Suppose that \((R, n)\) is a regular local \(F\text{-finite}\) ring with \(A \cong R/J\) where, \(J\) is an ideal of \(R\).

Put \(f\text{grade}(J, R) = t\). Then by definition \(\lim_{\leftarrow n} H^i_n (R/J^n) = 0\) for all \(i < t\). It follows from the Proposition 3.1 that \(H^i_n (R/J)\) is \(F\)-nilpotent for all \(i < t\), i.e. \(F\text{-depth } R/J \geq t\). With a similar argument and again using Proposition 3.1 we have \(f\text{grade}(J, R) \geq F\text{-depth } R/J\). Thus, \(f\text{grade}(J, R) = F\text{-depth } R/J\).
Now, we are done by \cite{15 Lemma 4.8(b)}, \cite{3 Remark 3.1} and the previous paragraph. To this end note that
\[ f\text{grade}(I, A) \leq \text{depth } A \leq f\text{grade}(J, R) = \text{F-depth } R/J = \text{F-depth } A. \]

Let $A$ be a complete local ring containing a perfect field $k$ as its residue field, then $A$ satisfies the condition of Theorem \cite{5 Corollary 3.8}. To this end, note that by Remark \cite{24 A is F-finite}. Furthermore, by virtue of Cohen’s Structure Theorem $A \cong R/J$ for some ideal $J \subset R$, where $R = k[[x_1, \ldots, x_n]]$ is a regular $F$-finite ring.

**Corollary 3.9.** Let $(A, \mathfrak{m})$ be a regular local and $F$-finite ring. Then we have
\[ \text{depth } A/I \leq f\text{grade}(I, A) = \text{F-depth } A/I \leq \text{dim } A/I. \]

**Proof.** The assertion follows from what we have seen in the proof of Theorem \cite{3.8} and \cite{1 Remark 3.1].

**Remark 3.10.** (a) The necessary and sufficient conditions for small values of the F-depth of $A$ is given in \cite{11 Corollary 4.6].

(1) F-depth $A > 0$ if and only if dim $A > 0$.

(2) F-depth $A > 1$ if and only if dim $A \geq 2$ and the punctured spectrum of $A$ is formally geometrically connected.

Now, let $A$ be $F$-finite and depth $A = 0 < \text{dim } A$. Then, one has $f\text{grade}(I, A) = 0 < \text{F-depth } A$. It shows that the inequality in Theorem \cite{3.8} can be strict.

(b) Keep the assumptions in Corollary \cite{3.9} if F-depth $A/I > 1$, then by \cite{15 Lemma 5.4], one has $\text{Supp}_{\hat{A}}(\hat{A}/I\hat{A}) \setminus \{\hat{m}\}$ is connected. To this end, note that $\hat{A}$ is a local ring so it is indecomposable.

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