The symmetry, inferable from Bogoliubov transformation, between the processes induced by the mirror in two-dimentional and the charge in four-dimentional space-time

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Abstract

The symmetry between the creation of pairs of massless bosons or fermions by accelerated mirror in 1+1 space and the emission of single photons or scalar quanta by electric or scalar charge in 3+1 space is deepened in this paper. The relation of Bogoliubov coefficients, describing the processes generated by the mirror, with Fourier’s components of current or charge density leads to the coincidence of the spin of any disturbances bilinear in scalar or spinor field with the spin of quanta emitted by the electric or scalar charge. The mass and invariant momentum transfer of these disturbances are essential for the relation of Bogoliubov coefficients with invariant singular solutions and Green’s functions of wave equations both for 1+1 and 3+1 spaces and especially for the integral relations (20) and (100) between these solutions. Namely the relation (20) leads to the coincidence of the self-action changes and vacuum-vacuum amplitudes for the accelerated mirror in two-dimentional space-time and charge in four-dimentional space-time. Thus, both invariants of the Lorentz group, spin and mass, perform intrinsic role in established symmetry. The symmetry embraces not only the processes of real quanta radiation. It extends also to the processes of the mirror and the charge interactions with the fields carrying spacelike momenta. These fields accompany their sources and define the Bogoliubov matrix coefficients $\alpha^{B,F}_{\omega'}$. It is shown that the Lorentz-invariant traces $\pm \text{tr} \alpha^{B,F}_{\omega'}$ describe the vector and scalar interactions of accelerated mirror with uniformly moving detector. This interpretation rests essentially on the relation (100) between the propagators of the waves with spacelike momenta in 2- and 4-dimentional spaces. The traces $\pm \text{tr} \alpha^{B,F}_{\omega'}$ coincide actually with the products of the mass shift $\Delta m_{1,0}$ of accelerated electric or scalar charge and the proper time of the shift formation. The symmetry fixes the value of the bare fine structure constant $\alpha_0 = 1/4\pi$.

1 Introduction

The Hawking’s mechanism for particle production at the black hole formation is analogous to the emission from an ideal mirror accelerated in vacuum [1]. In its turn there is a close analogy between the radiation of pairs of scalar (spinor) quanta from accelerated mirror in 1+1 space and the radiation of photons (scalar quanta) by an accelerated electric (scalar) charge in 3+1 space [2,3]. Thus all these processes turn out to be mutually related. In
problems with moving mirrors the in- and out-sets of the wave equation solutions are usually used which for massless scalar field look as follows

$$\phi_{\text{in}, \omega} \sim e^{-i\omega' v} - e^{-i\omega' f(u)}, \quad \phi^{*}_{\text{in}, \omega'}, \quad \phi_{\text{out}, \omega} \sim e^{-i\omega g(v)} - e^{-i\omega u}, \quad \phi^{*}_{\text{out}, \omega};$$

(1)

with zero boundary condition $\phi|_{\text{traj}} = 0$ on the mirror’s trajectory. Here the variables $u = t - x$, $v = t + x$ are used and the mirror’s (or charge’s) trajectory on the $u, v$ plane is given by any of the two mutually inverse functions $u^{\text{mir}} = f(u), \; u^{\text{mir}} = g(v)$.

For the in- and out-sets of massless Dirac equation solutions see [3]. Dirac solutions differ from (1) by the presence of bispinor coefficients at $u$- and $v$-plane waves. The current densities corresponding to these solutions have only tangential components on the boundary. So, the boundary condition both for scalar and spinor field is purely geometrical, it does not contain any dimensional parameters.

The Bogoliubov coefficients $\alpha_{\omega' \omega}, \beta_{\omega' \omega}'$ appear as the coefficients of the expansion of the solutions of the out-set in the solutions of the in-set and $\alpha^{*}_{\omega' \omega}, \mp \beta_{\omega' \omega}$ as the coefficients of the inverse expansion. The upper and lower signs correspond to scalar (Bose) and spinor (Fermi) field. Then the mean number of quanta with frequency $\omega$ and wave vector $\omega > 0$ radiated by accelerated mirror to the right semispace is given by the integral

$$d\bar{n}_{\omega} = \frac{d\omega}{2\pi} \int_{0}^{\infty} \frac{d\omega'}{2\pi} |\beta_{\omega' \omega}|^2.$$

(2)

At the same time the spectra of photons and scalar quanta emitted by electric and scalar charges moving along the trajectory $x_{\alpha}(\tau)$ in 3+1 space are defined by the Fourier transforms of the electric current density 4-vector $j_{\alpha}(k)$ and the scalar charge density $\rho(k)$,

$$s = 1, \quad j_{\alpha}(k) = e \int d\tau \dot{x}_{\alpha}(\tau) e^{-ik^{a} x_{a}(\tau)},$$

(3)

$$d\bar{n}_{k}^{(1)} = |j_{\alpha}(k_{+}, k_{-})|^2 \frac{dk_{+}dk_{-}}{(4\pi)^2},$$

(4)

$$s = 0, \quad \rho(k) = e \int d\tau e^{-ik^{a} x_{a}(\tau)},$$

(5)

$$d\bar{n}_{k}^{(0)} = |ho(k_{+}, k_{-})|^2 \frac{dk_{+}dk_{-}}{(4\pi)^2},$$

(6)

where $s$ and $k^{a}$ are the spin and 4-momentum of quanta,

$$k^{2} = k_{1}^{2} + k_{2}^{2} - k_{0}^{2} = 0, \quad k_{-}^{2} = k_{0}^{2} - k_{2}^{2} = k_{+} k_{-}, \quad k_{\pm} = k_{0} \pm k_{1},$$

and in (4) and (6) it is supposed that the trajectory $x_{\alpha}(\tau)$ has only $x^{0}$ and $x^{1}$ nontrivial components.

The symmetry between the creation of Bose or Fermi pairs by accelerated mirror in 1+1 space and the emission of single photons or scalar quanta by electric or scalar charge in 3+1 space consists, first of all, in the coincidence of the spectra. If one puts $2\omega = k_{+}, \; 2\omega' = k_{-}$, then

$$|\beta_{\omega \omega'}^{B}|^2 = \frac{1}{e^{2}}|j_{\alpha}(k_{+}, k_{-})|^2, \quad |\beta_{\omega \omega'}^{F}|^2 = \frac{1}{e^{2}}|\rho(k_{+}, k_{-})|^2.$$

(7)
More refined assertion for the Bose case:

$$\beta_{\omega\omega}^{B*} = -\sqrt{\frac{k_+}{k_-}} \frac{j_-(k)}{e} = \sqrt{\frac{k_-}{k_+}} \frac{j_+(k)}{e} = \frac{\varepsilon_{\alpha\beta} k^\alpha j^\beta(k)}{e \sqrt{k_+ k_-}},$$

(8)

$$j_-(k) = e \int du e^{i\frac{1}{2}(k_+ u + k_+ f(u))}, \quad j_+(k) = e \int dv e^{i\frac{1}{2}(k_- v + k_- g(v))}.$$  

(9)

The 2-vectors $j_\alpha(k)$ and $a_\beta(k) = \varepsilon_{\alpha\beta} k^\alpha / \sqrt{k_+ k_-}$ are spacelike for timelike $k^\alpha$ and in a system where $k_+ = k_-$ or $\omega = \omega'$ they have only spatial components precisely equal to $e^{B*}_{\omega\omega}$ and 1 correspondingly.

And for the Fermi case:

$$\beta_{\omega\omega}^{F*} = \frac{1}{e} \rho(k).$$

(10)

In the present paper in Section 2 we underline the symmetry of analytical expressions for the Bogoliubov coefficients $\alpha$, $\beta^*$ and, at the same time, the physical distinction between them: $\beta^{B,F*}$ is the amplitude of a source of the waves which are bilinear in massless Bose or Fermi fields and carry the timelike momenta, whereas $\alpha^{B,F}$ is the amplitude of a source of the similar waves but which carry the spacelike momenta, see (14), (15). In Sections 3 and 4 it is shown that the waves with timelike momenta emitted and absorbed by the source are involved in the forming of the imaginary part of the source selfaction. This physical picture is naturally embodied in the integral relation (20) between propagators $\Delta_2(z, m)$ of virtual pairs with masses $m, \mu \leq m < \infty$, in two-dimentional spacetime and propagator $\Delta_4(z, \mu)$ of the particle in four-dimentional spacetime. The analytical properties of the expressions appeared allow to define too the real part of the selfactions and, hence, of the vacuum-vacuum amplitudes of the mirror and the charge if one considers that $e^2 = 1$. In Section 5 the fields of perturbations carried the spacelike momenta are considered. These fields are defined by the matrixes $\alpha^{B,F}$. Their Lorentz-invariant traces $\pm \mathrm{tr} \alpha^{B,F}$ are considered in Section 6. They describe correspondingly the vector and scalar interactions of the accelerated mirror with the uniformly moving detector in the neighbourhood of the point of contact of their trajectories. In Sections 7 and 8 the traces $\pm \mathrm{tr} \alpha^{B,F}$ are found for the three concrete trajectories permitting the analytical solutions. At the same place the general expressions for the traces are given and their ultraviolet and infrared singularities are considered. In these Sections the comparison of the found traces $\pm \mathrm{tr} \alpha^{B,F}$ with the mass shifts $\Delta m_{1,0}$ of the electric and scalar charges, moving along the same trajectory as the mirror’s one but in 3+1-space, is carried out. In this connection, in Sector 9 the mass shifts $\Delta m_{1,0}$ of the charges moving along the exponential trajectory are found. In Conclusions we discuss the connection of the traces $\pm \mathrm{tr} \alpha^{B,F}$ with the general definition of the selfaction accounting for the interference effects, and call attention to the fact that the symmetry fixes the value of the bare charge squared, $e_0^2 = 1$, that corresponds to bare fine structure constant $\alpha_0 = 1/4\pi$. The smallness and geometrical origin of this value may be interesting in quantum electrodynamics. In Appendix the even singular solutions of inhomogeneous wave equations with mass and momentum transfer parameters are considered. The integral relations (20) and (100) between these solutions for 1+1- and 3+1-spaces are very important for the symmetry considered.
2 The physical interpretation of $\beta^*_{\omega'}\omega$

The absolute pair production amplitude and single-particle scattering amplitude are connected by the relation [4]

$$\langle \text{out}\omega''\omega|\text{in}\rangle = -\sum_{\omega'} \langle \text{out}\omega''|\omega'|\text{in}\rangle \beta^*_{\omega'}\omega.$$  \hspace{1cm} (11)

The $\beta^*_{\omega'}\omega$ was interpreted as the amplitude of a source of a pair of the massless particles potentially emitted to the right and to the left with frequencies $\omega$ and $\omega'$ respectively. While the particle with frequency $\omega$ actually escaped to the right the particle with frequency $\omega'$ propagates some time then is reflected by the mirror and is actually emitted to the right with altered frequency $\omega''$, see Fig. 1.

On the time interval between pair creation and reflection of the left particle we have the virtual pair with energy $k^0$, momentum $k^1$, and mass $m$:

$$k^0 = \omega + \omega', \quad k^1 = \omega - \omega', \quad m = \sqrt{-k^2} = 2\sqrt{\omega\omega'}.$$  \hspace{1cm} (12)

Apart from the polar timelike 2-vector $k^\alpha$, very important is the axial spacelike 2-vector $q^\alpha$,

$$q_\alpha = \varepsilon_{\alpha\beta}k^\beta, \quad q^0 = -k^1 = -\omega + \omega', \quad q^1 = -k^0 = -\omega - \omega' < 0.$$  \hspace{1cm} (13)

With the help of $k^\alpha$ and $q^\alpha$ the symmetry between $\alpha$ and $\beta$ coefficients becomes clearly expressed:

$$s = 1, \quad e_{\beta^*_{\omega'}\omega} = -\frac{q_\alpha j^\alpha(k)}{\sqrt{k_+k_-}}, \quad e_{\alpha_{\omega'}\omega} = -\frac{k_\alpha j^\alpha(q)}{\sqrt{k_+k_-}},$$  \hspace{1cm} (14)

$$s = 0, \quad e_{\beta^*_{\omega'}\omega} = \rho(k), \quad e_{\alpha_{\omega'}\omega} = \rho(q).$$  \hspace{1cm} (15)

Note that the equations (3) and (5) define the current density $j^\alpha(k)$ and the charge density $\rho(k)$ as the functionals of the trajectory $x^\alpha(\tau)$ and the functions of any 2- or 4-vector $k^\alpha$. It can be shown that in 1+1-space $j^\alpha(k)$ and $j^\alpha(q)$ are the spacelike and timelike polar vectors correspondingly if $k^\alpha$ and $q^\alpha$ are the timelike and spacelike vectors.

The boundary condition on the mirror evokes in the vacuum of massless scalar or spinor field the appearance of vector or scalar disturbance waves bilinear in massless fields. There are two types of these waves:

1) The waves with amplitude $\alpha_{\omega'}\omega$ ($\alpha^*_{\omega'}\omega$) which carry the spacelike momentum directed to the left (right), and

2) The waves with amplitude $\beta^*_{\omega'}\omega$ ($\beta_{\omega'}\omega$) which carry the timelike momentum with positive (negative) frequency.

The waves with the spacelike momenta appear even if the mirror is in rest or moves uniformly (Casimir effect), while the waves with the timelike momenta appear only in the case of accelerated mirror.

The pair of Bose (Fermi) particles has spin 1 (0) because its source is the current density vector (charge density scalar), see [5] or the problem 12.15 in [6].
3 The appearance of the mass in massless theory and of the invariant singular solutions of wave equation with mass

The bilinear in massless bose-field disturbances, defined by the amplitudes $\beta B^* \omega$, form, according to (8), the positive-frequency current density vector. Its minus-component in the point $U, V$ can be represented as

$$
\int_0^{\infty} \int_0^{\infty} \frac{d\omega d\omega'}{(2\pi)^2} e^{-j(k)} e^{-i\omega U - i\omega' V} = \frac{1}{8\pi^2} \int_0^{\infty} d\rho \rho \int_{-\infty}^{\infty} d\theta e^{-i\rho (z^0 \text{ch} \theta - z^1 \text{sh} \theta)},
$$

if instead of $\omega, \omega'$ the hyperbolic variables $\rho, \theta$ are used,

$$
d\omega d\omega' = \frac{1}{2} \rho d\rho d\theta, \quad \omega = \frac{1}{2} \rho e^{\theta}, \quad \omega' = \frac{1}{2} \rho e^{-\theta}, \quad \rho = 2\sqrt{\omega \omega'}, \quad \theta = \ln \frac{\omega}{\omega'},
$$

and $z^\alpha = x^\alpha - x^\alpha(\tau)$, see Fig. 2.

As it is seen from (12) $\rho = m$ is the mass of the pair and $\theta$ is the rapidity. The integral over rapidity in (16) is the well known invariant positive-frequency singular function of wave equation for 2-dimentional spacetime:

$$
\int_{-\infty}^{\infty} d\theta e^{-im(z^0 \text{ch} \theta - z^1 \text{sh} \theta)} = -4\pi i \Delta_2^+ (z, m) = 2\theta(-z^2)K_0(i\varepsilon(z^0)m\sqrt{-z^2}) + 2\theta(z^2)K_0(m\sqrt{z^2}),
$$

$$
(\partial^2_t - \partial^2_x + m^2) \Delta_2^+ (z, m) = 0.
$$

This function describes the wave field of pairs with mass $m$ and any possible positive-frequency momenta. According to it the pairs are created, propagated and absorbed near the mirror within spacelike interval of the order of $m^{-1}$.

By using the very important integral relation between the singular functions of wave equations for $d$- and $d+2$-dimentional spacetimes,

$$
\Delta_{d+2}^f (z, \mu) = \frac{1}{4\pi} \int_{\mu^2}^{\infty} dm^2 \Delta_d^f (z, m),
$$

the right-hand side of (16) can be represented in the form

$$
-\frac{i}{4\pi} \int_{\mu^2}^{\infty} \int_{-\infty}^{\infty} dm^2 \Delta_2^+ (z, m) = -i \int_{-\infty}^{\infty} du \Delta_1^+ (z, \mu).
$$

The small mass $\mu$ is retained to eliminate the infrared divergence in the following.
Analogously, the positive-frequency plus-component of the current density in the point \( U, V \) can be represented as

\[
\int_0^\infty \frac{d\omega d\omega'}{(2\pi)^2} e^{ij(k)} e^{-i\omega U - i\omega' V} = -i \int_{-\infty}^\infty dv \Delta_4^+(z, \mu). \tag{22}
\]

The differentials \( du \) and \( dv \) in (21) and (22) may be replaced by \( d\tau \dot{x}_-(\tau) \) and \( d\tau \dot{x}_+(\tau) \).

The bilinear in massless fermi-field disturbances defined by the amplitudes \( \beta^{F*}_{\omega\omega} \) form the positive-frequency charge density scalar. At the point \( U, V \) it can be represented by

\[
\int_0^\infty \frac{d\omega d\omega'}{(2\pi)^2} e^{i\omega U - i\omega' V} = -i \int_{-\infty}^\infty d\tau \Delta_4^+(z, \mu). \tag{23}
\]

If we put the point \( U, V \) on the trajectory, so that \( U = x_-(\tau') \), \( V = x_+(\tau') \) and \( z^\alpha = x^\alpha(\tau') - x^\alpha(\tau) \), and integrate (21) over \( V \) and (22) over \( U \) then their half of the sum will differ from the \( \text{tr} \beta^+\beta \) only by the multiplier \( i \):

\[
\text{tr} \beta^B + \beta^B \equiv \int_0^\infty \frac{d\omega d\omega'}{(2\pi)^2} |\beta^B_{\omega\omega}|^2 = \\
\frac{i}{2} \int (du dV + dv dU) \Delta_4^+(z, \mu) = -i \int d\tau d\tau' \dot{x}_\alpha(\tau) \dot{x}^\alpha(\tau') \Delta_4^+(z, \mu). \tag{24}
\]

As the function \( \Delta^+ \) has odd in \( z \) real part and even in \( z \) imaginary part which are connected with the even in \( z \) causal (Feynman) function \( \Delta^f \),

\[
\Delta^+(z, \mu) = \frac{1}{2} \Delta(z, \mu) + \frac{i}{2} \Delta^1(z, \mu), \quad \text{Re} \Delta^+ = \varepsilon(z^0) \text{Re} \Delta^f, \quad \text{Im} \Delta^+ = \text{Im} \Delta^f, \tag{25}
\]

the \( \text{tr} \beta^B + \beta^B \) may be written in the form

\[
\text{tr} (\beta^+ \beta)^B = \text{Im} \int d\tau d\tau' \dot{x}_\alpha(\tau) \dot{x}^\alpha(\tau') \Delta^f_4(z, \mu). \tag{26}
\]

The \( \text{tr} \beta^F + \beta^F \) can be obtained from the right-hand side of (26) by the substitution \( \dot{x}_\alpha(\tau) \dot{x}^\alpha(\tau') \rightarrow 1 \).

\section{Vacuum-vacuum amplitude \( \langle \text{out} | \text{in} \rangle = e^{iW} \)}

According to DeWitt [7], Wald [8] and others (including myself [4])

\[
2 \text{Im} W^{B,F} = \pm \frac{1}{2} \text{tr} \ln(1 \pm \beta^+\beta) \quad \text{or} \quad \pm \text{tr} \ln(1 \pm \beta^+\beta) \tag{27}
\]

correspondingly to the cases when particle is identical or nonidentical to antiparticle. We confine ourselves by the last case and \( \text{tr} \beta^+\beta \ll 1 \). Then

\[
2 \text{Im} W^{B,F} = \text{Im} \int d\tau d\tau' \left\{ \frac{\dot{x}_\alpha(\tau) \dot{x}^\alpha(\tau')}{1} \right\} \Delta^f_4(z, \mu). \tag{28}
\]
We may omit the Im-signs from both of sides of this equation and define the actions for bose- and fermi-mirrors in 1+1-space as

\[ W_{B,F} = \frac{1}{2} \int \! \int d\tau d\tau' \left\{ \frac{\dot{x}_\alpha(\tau)\dot{x}_\alpha(\tau')}{1} \right\} \Delta^f_4(z,\mu). \] (29)

Compare this with the well known actions for electric and scalar charges in 3+1-space:

\[ W_{1,0} = \frac{1}{2} e^2 \int \! \int d\tau d\tau' \left\{ \frac{\dot{x}_\alpha(\tau)\dot{x}_\alpha(\tau')}{1} \right\} \Delta^f_4(z,\mu). \] (30)

The symmetry would be complete if \( e^2 = 1 \), i.e. if the fine structure constant were \( \alpha = 1/4\pi \). This ”ideal” value of fine structure constant for the charges would correspond to the ideal, geometrical boundary condition on the mirror.

For the mirror trajectory with nonzero relative velocity \( \beta_{21} \) of its ends (nonzero relative rapidity \( \theta = \text{Arth} \beta_{21} \)) the changes of actions due to acceleration are given by the formulae

\[ \text{Re} \Delta W_B = \frac{1}{8\pi} (\frac{\theta}{\text{th} \theta} - 1), \quad \text{Re} \Delta W_F = \frac{1}{8\pi} (1 - \frac{\theta}{\text{sh} \theta}). \] (31)

For uniformly accelerated mirror with proper acceleration \( a \) its velocity \( \beta(\tau) = \text{th} a \tau, \tau \) is the proper time. Then \( \theta = a(\tau_2 - \tau_1) \) and for \( \tau_2 - \tau_1 \to \infty \)

\[ \text{Re} \Delta W_B = \frac{|a|}{8\pi} (\tau_2 - \tau_1). \] (32)

By definition, the

\[ \text{Re} \Delta m_B = -\frac{\partial \text{Re} \Delta W_B}{\partial \tau_2} = -\frac{|a|}{8\pi} \]

is the self-energy shift of bose-mirror at acceleration. It differs from the mass shift of uniformly accelerated electron only by the absence of the multiplier \( e^2 = 4\pi \alpha \). The self-energy shift of uniformly accelerated fermi-mirror \( \text{Re} \Delta m^F = 0 \).

There are two arguments in favour of definition of action by means of the causal function \( \Delta^f_4(z,\mu) \).

1. The action must represent not only the radiation of real quanta but also the self-energy and polarization effects. While the first effects are described by the solutions of homogeneous wave equation the second ones require the inhomogeneous wave equation solutions which contain information about proper field of a source. Namely such solutions of homogeneous and inhomogeneous wave equations are the functions \( (1/2)\Delta^1 = \text{Im} \Delta^f \) and \( \Delta = \text{Re} \Delta^f \).

2. While the appearance of \( (1/2)\Delta^1 \equiv \text{Im} \Delta^f \) in the imaginary part of the action is the consequence of the mathematical transformations of the integral

\[ \int_0^\infty \int_0^\infty \frac{d\omega d\omega'}{(2\pi)^2} |\beta_{\omega \omega'}|^2, \]

transformations similar to the Plancherel theorem, the function \( \Delta \equiv \text{Re} \Delta^f \) in the real part of the action will be unique if it appears as the real part of that analytical continuation of the function \( (i/2)\Delta^1(z,\mu) \) to the negative \( z^2 \) which is even in \( z \) as \( \Delta^1 \) itself.
In conclusion of the third and fourth sections it should be noted that due to the transition from variables $\omega, \omega'$ to hyperbolic variables $\rho, \theta$, which reflect the Lorentzian symmetry of the problem, both the function $\Delta_2(z, m)$, describing the propagation of virtual pair with mass $m = \rho = 2\sqrt{\omega\omega'}$ in two-dimensional space-time, and the mass spectrum of these pairs arise. Further integration over the mass leads to the function $\Delta_4(z, \mu)$, which coincides with the propagator of a particle moving in four-dimensional space-time with the mass $\mu$ equal to the least mass of virtual pairs. Thus, the relation (20) appears in the framework of the present method and is immanent to the symmetry, connecting the processes in two- and four-dimensional space-times.

In the paper [9] on the same topic as this one the relation (20) was obtained by the author as independent of the processes considered and was in need of proof that the integration variable figuring in it coincides with the pair mass $m = 2\sqrt{\omega\omega'}$ indeed.

5 Formation of tachyon disturbances with invariant momentum transfer

The bilinear in massless bose-field perturbations, which are defined by the amplitudes $a_{\omega^B,\omega}$ and carry to the left the spacelike momenta, can be represented in the point $U, V$ by the two current density components

$$\int_{-\infty}^{\infty} d\rho \int_{0}^{\infty} d\omega \int_{0}^{2\pi} d\theta e^{i\rho(z_0^0 \sin \theta - z_1^0 \cosh \theta)} = \frac{i}{4\pi} \int d\tau \int_{-\infty}^{\infty} d\rho \int_{0}^{\infty} d\omega \int_{0}^{2\pi} d\theta e^{i\rho(z_0^0 \sin \theta - z_1^0 \cosh \theta)}$$  (34)

if one again uses the change of variables (17) and the notation $z^{a} = x^{a} - x^{a}(\tau)$. Now the integral over $\theta$ is equal to

$$\int_{-\infty}^{\infty} d\theta e^{i\rho(z_0^0 \sin \theta - z_1^0 \cosh \theta)} = 4\pi i \Delta_2^L(z, \rho) = 2\theta(-z^2)K_0(\rho\sqrt{-z^2}) + 2\theta(z^2)K_0(i\varepsilon(z^1)\rho\sqrt{z^2}).$$  (35)

The integrand in the left-hand side of (35) is the wave with spacelike 2-momentum $q^a = -\omega - \omega' = -\rho \sin \theta$, $q^0 = -\omega + \omega' = -\rho \sin \theta$, $\rho = \sqrt{q^2}$. The function $\Delta_2^L(z, \rho)$ is the superposition of plane waves with spacelike momenta directed to the left and with fixed invariant momentum transfer $\rho = 2\sqrt{\omega\omega'}$. It satisfies the wave equation with negative mass squared:

$$\left(\partial_{\tau}^2 - \partial_x^2 - \rho^2\right)\Delta_2^L(z, \rho) = 0.$$  (36)

By using the integral relation analogous to relation (20) (see Appendix) the right-hand side of (34) can be represented in the form

$$\int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} d\rho \int_{0}^{\infty} d\rho e^{i\rho(z_0^0 \sin \theta - z_1^0 \cosh \theta)} = -i \int_{0}^{\infty} d\tau \int_{0}^{\infty} d\rho \int_{0}^{\infty} d\rho e^{i\rho(z_0^0 \sin \theta - z_1^0 \cosh \theta)}$$  (37)

The small momentum transfer $\nu$ is retained to eliminate the infrared divergence in what follows.
Analogously, the bilinear in fermi-field disturbances, which are defined by the amplitudes $\alpha_{F,\omega}^\omega$ and carry the left-directed spacelike momenta, form the charge density scalar. It can be represented in the point $U, V$ by the integral

$$
\int_0^\infty \int_0 d\omega d\omega' \frac{1}{(2\pi)^2} \rho(q) e^{i\omega U - i\omega' V} = -i \int d\tau \Delta^T_4(z, \nu). \tag{38}
$$

These representations can be useful in the problems close to static ones where apart from acceleration or instead of it another characteristic length enters.

6 Interpretation of the traces $\pm \text{tr} \alpha^{B,F}$ of Bogoliubov coefficients

The invariant description of the mirror’s trajectory on the $u, v$-plane demands that the function $u_{\text{mir}} = g(v)$ contains the two positive parameters $\kappa, \kappa'$, transforming as $x_+ = v, x_- = u$, and actually connects the invariant variables $\kappa u, \kappa' v$ between themselves:

$$
u_{\text{mir}} = g(v) = \frac{1}{\kappa} G(\kappa' v). \tag{39}
$$

Its expansion near the origin of coordinates $u = v = 0$, being on the trajectory, has the form

$$g(v) = \frac{1}{\kappa}(\kappa' v + b\kappa'^2 v^2 + \frac{1}{3} c\kappa'^3 v^3 + \ldots), \tag{40}
$$

where $b, c, \ldots$-some numbers. Since the mirror’s velocity $\beta(v)$ and proper acceleration $a(v)$ are defined by the formulae

$$
\beta(v) = \frac{1 - g'(v)}{1 + g'(v)}, \quad a(v) = -\frac{g''(v)}{2g'^2(v)}, \tag{41}
$$

the two first coefficients of the expansion (40) define the mirror’s velocity $\beta_0$ and acceleration $a_0$ at zero point:

$$
\beta_0 = \frac{1 - \kappa'/\kappa}{1 + \kappa'/\kappa}, \quad a_0 = -b \sqrt{\kappa \kappa'}. \tag{42}
$$

The absolute value of acceleration at zero point will be denoted as $w_0 = |b| \sqrt{\kappa \kappa'}$.

Let us define the Lorentz-invariant $\text{tr} \alpha$ by the formula

$$
\text{tr} \alpha = \int_0^\infty \int_0 d\omega d\omega' \frac{1}{(2\pi)^2} \alpha_{\omega,\omega'} 2\pi \delta \left( \sqrt{\kappa' \omega} - \sqrt{\kappa' \omega'} \right), \tag{43}
$$

in which the Lorentz-invariant argument of $\delta$-function is the difference of frequencies

$$
\Omega = \sqrt{\kappa' \omega}, \quad \Omega' = \sqrt{\kappa' \omega'} \tag{44}
$$
of reflected and incident waves in the proper system of the mirror at the moment \( u = v = 0 \). According to (42) the multipliers \( \sqrt{\omega'/\omega}, \sqrt{\omega/\omega'} \) figuring in formula (44) are the Doppler factors connecting the frequencies in the laboratory and proper systems. In proper system of the mirror \( \Omega = \Omega' = \sqrt{\omega/\omega'} \).

According to (43) the \( \text{tr} \alpha \) is Lorentz-invariant and dimensionless quantity or, perhaps, has dimensionality of action, since \( \hbar = 1 \). Let us consider its physical meaning. For this let us turn to equality of expressions (34) and (37)

\[
\int_0^\infty \int_0^\infty \frac{d\omega d\omega'}{(2\pi)^2} e^{i\omega U - i\omega' V} = -i \int d\tau \dot{x}_\pm(\tau) \Delta^L_1(z, \nu),
\]

where \( z^\alpha = x^\alpha - x^\alpha(\tau) \), \( x_- = U \), \( x_+ = V \). Let us put the point \( U, V \) on the tangent line to the mirror’s trajectory at zero point, so that

\[
U = X_-(\tau') = \sqrt{\frac{\kappa'}{\kappa}} \tau', \quad V = X_+(\tau') = \sqrt{\frac{\kappa}{\kappa'}} \tau'
\]

\( \tau' \)-is the proper time of the point on the tangent line, and integrate the both sides of equation (45) over \( dU = \dot{X}_- d\tau' \) or \( dV = \dot{X}_+ d\tau' \) correspondingly the upper or lower sign in (45). Then accounting for the formula (14) and current conservation we obtain on the left the \( \text{tr} \alpha \) both for upper and lower signs in (45). On the right we obtain the integral

\[
-i \int d\tau d\tau' \dot{x}_\pm(\tau) \dot{X}_\pm(\tau') \Delta^L_1(z, \nu), \quad z^\alpha = X^\alpha(\tau') - x^\alpha(\tau),
\]

in which according to the result for the left part one may left instead of

\[
\dot{x}_\pm(\tau) \dot{X}_\pm(\tau') = -\dot{x}_\alpha(\tau) \dot{X}_\alpha(\tau') \mp \epsilon_{\alpha\beta} \dot{x}_\beta(\tau) \dot{X}_\beta(\tau')
\]

only the first term symmetrical with respect to permutation \( \dot{x}_\alpha(\tau) \rightleftharpoons \dot{X}_\alpha(\tau') \). Thus we obtain

\[
\text{tr} \alpha^B = i \int d\tau d\tau' \dot{x}_\alpha(\tau) \dot{X}_\alpha(\tau') \Delta^L_1(z, \nu), \quad z^\alpha = X^\alpha(\tau') - x^\alpha(\tau).
\]

Integrating similarly the both parts of equation (38) along tangent line (46) and taking into account the formulae (15) and (43) we obtain

\[
\text{tr} \alpha^F = -i \int d\tau d\tau' \Delta^L_1(z, \nu), \quad z^\alpha = X^\alpha(\tau') - x^\alpha(\tau).
\]

For the trajectories situated on the Minkowsky plane on the left from their tangent line at zero point the coordinate \( z^1 \geq 0 \). In this case the \( \Delta^L_1(z, \nu) \) can be replaced by the function

\[
\Delta^L_{1R}(z, \nu) = \frac{1}{4\pi} \delta(z^2) - \frac{\nu}{8\pi \sqrt{z^2}} \theta(z^2) [J_1(\nu \sqrt{z^2}) - iN_1(\nu \sqrt{z^2})] + i \frac{\nu}{4\pi^2 \sqrt{-z^2}} \theta(-z^2) K_1(\nu \sqrt{-z^2}),
\]

(51)
which differs from the causal function $\Delta_f(z, \mu)$ by complex conjugation and replacement $\mu \to i\nu$ (or by the replacement $z^2 \to -z^2$, $\mu \to \nu$). More detail about this function see in Appendix.

Thus for the mentioned trajectories

$$\pm \text{tr} \alpha^{B,F} = i \int d\tau d\tau' \left\{ \frac{\dot{x}_\alpha(\tau)\dot{X}_\alpha(\tau')}{1} \right\} \Delta^{LR}_4(z, \nu), \quad z^\alpha = X^\alpha(\tau') - x^\alpha(\tau). \quad (52)$$

The expression obtained allows to interpret $\pm \text{tr} \alpha^{B,F}$ as a functional describing the interaction of two vector or scalar sources by means of exchange by vector or scalar quanta with spacelike momenta. At the same time one of the sources moves along the mirror’s trajectory while another one moves along the tangent line to it at zero point. The last source can be considered as a probe or detector of excitation created by the accelerated mirror in vacuum.

7 The traces of Bogoliubov’s coefficients for hyperbolic and exponential trajectories

Let us consider the $\text{tr} \alpha^{B,F}$ for hyperbolic mirror’s trajectory

$$u_{\text{mir}} = g(v) = \frac{\mathcal{K} v}{\mathcal{K}(1 - \mathcal{K} v)}. \quad (53)$$

Using the formulae (14) and (4) of the paper [3] it is not difficult to represent $\alpha^{B,F}_{\omega'\omega}$ via Macdonald’s functions $K_{1,0}$:

$$\alpha^{B,F}_{\omega'\omega} = \frac{2}{\sqrt{\mathcal{K}\mathcal{K}'}} e^{i(\mathcal{K} + \mathcal{K}')/2} K_{1,0} \left(2i \sqrt{\mathcal{K}\mathcal{K}'}\right). \quad (54)$$

Then according to the formula (43)

$$\text{tr} \alpha^{B,F} = \frac{1}{\pi} \int_0^\infty d\left(\frac{\omega}{\mathcal{K}}\right) e^{2i\mathcal{K}} K_{1,0} \left(2i \frac{\omega}{\mathcal{K}}\right) = \frac{1}{2\pi} \int_0^\infty dz e^{iz} K_{1,0}(iz). \quad (55)$$

The variable $z$ in this integral has a simple physical meaning: it is equal to the ratio of invariant momentum transfer to invariant proper acceleration at zero point (but for hyperbolic motion the acceleration is the same on the whole trajectory):

$$z = \frac{\rho}{w_0}, \quad \rho = 2\sqrt{\mathcal{K}\mathcal{K}'}, \quad w_0 = \sqrt{\mathcal{K}\mathcal{K}'} \quad (56)$$

The ultraviolet divergency of the integral (55) is removed by subtraction from the integrand its asymptotics for $z \to \infty$. The infrared divergency (for the Bose-case) is removed by introducing the nonzero lower limit $\varepsilon = \nu/w_0 \ll 1$, defined by the minimal momentum transfer $\nu$. As a result we obtain the integral

$$\text{tr} \alpha^{B,F} = \left. \frac{1}{2\pi} \int_{s\varepsilon}^\infty dz \left[ e^{iz} K_s(iz) - \sqrt{\frac{\pi}{2iz}} \right] \right|_{s = 1, 0, \varepsilon \ll 1}.$$
Now the integration contour can be turned on the negative imaginary semiaxis going round (in Bose-case) the singularity at zero along the arc of a circle with small radius $\varepsilon$. The further calculation leads to the simple expressions

$$\text{tr} \alpha^B = \frac{1}{2\pi} \left[ -\frac{\pi}{2} - i \left( \ln \frac{2w_0}{\gamma \nu} - 1 \right) \right], \quad \nu \ll w_0, \quad \gamma = 1, 781 \ldots, \quad (58)$$

$$\text{tr} \alpha^F = \frac{1}{2\pi} i. \quad (59)$$

For the exponential mirror’s motion

$$u_{\text{mir}}^{\prime} = -\frac{1}{\kappa} \ln(1 - \kappa v), \quad v_{\text{mir}}^{\prime} = \frac{1}{\kappa} \frac{1}{\kappa} e^{-\kappa u}, \quad (60)$$

the same formulae (14) and (4) from [3] leads to the Bogoliubov’s coefficients

$$\alpha_{\omega,\omega'}^{B} = \frac{1}{\kappa} \sqrt{\frac{\omega}{\omega'}} \left[ \Gamma \left( i \frac{\omega}{\kappa} \right) e^{i \frac{\omega'}{\kappa} \ln \frac{\omega'}{\omega}} - \frac{2\pi}{ix} \right], \quad (61)$$

$$\alpha_{\omega,\omega'}^{F} = \frac{1}{\sqrt{i \kappa \omega'}} \left[ \Gamma \left( \frac{1}{2} + i \frac{\omega}{\kappa} \right) e^{i \frac{\omega'}{\kappa} \ln \frac{\omega'}{\omega}} - \frac{2\pi}{ix} \right]. \quad (62)$$

The traces $\text{tr} \alpha^{B,F}$ which divergences were removed by the above mentioned prescription are such

$$\text{tr} \alpha^B = \frac{1}{2\pi} \int_{\varepsilon}^{\infty} dx \left[ \Gamma(i x) e^{ix - ix \ln ix} - \sqrt{\frac{2\pi}{ix}} \right], \quad (63)$$

$$\text{tr} \alpha^F = \frac{1}{2\pi} \int_{0}^{\infty} dx \left[ \Gamma \left( \frac{1}{2} + ix \right) e^{ix - ix \ln \frac{ix}{\sqrt{ix}}} - \sqrt{\frac{2\pi}{ix}} \right]. \quad (64)$$

In these integrals the variable $x$ is equal to one fourth of the variable $z$ which has, as well as in (56), the sense of momentum transfer in units of $w_0$:

$$x = \frac{1}{4} z, \quad z = \frac{\rho}{w_0}, \quad \rho = 2\sqrt{\omega \omega'}, \quad w_0 = \frac{1}{2} \sqrt{\omega \omega'}. \quad (65)$$

Analogously, $\varepsilon = \nu/w_0 \ll 1$. Note that at exponential motion (60) the proper acceleration increases from zero to infinity and as a function of proper time $\tau$ is given by the formula

$$a(\tau) = -\frac{w_0}{1 - w_0 \tau}. \quad (66)$$

Now it is not difficult to see that the subtractive terms in integrals (63),(64) exactly coincide with the similar terms in integrals (57) if one expresses them via physical variable $z$. In other words, up to removing ultraviolet divergency from the integrals defining $\text{tr} \alpha$ the asymptotical behaviour of the integrands in the variable $z = \rho/w_0 \to \infty$ is described by the universal formula

$$\frac{1}{2\pi} \sqrt{\frac{\pi}{2iz}}. \quad (67)$$
It will be shown in the next section that this assertion is correct for any timelike trajectory in the expansion (40) for which the coefficient \( b > 0 \).

The integration contour in integrals (63),(64) can be turned on the negative imaginary axis going around the infrared singularity at zero (in Bose-case) along the arc with radius \( \varepsilon \). Then we obtain

\[
\operatorname{tr} \alpha^B = \frac{1}{2\pi} \left[ -\frac{\pi}{2} - i \left( \ln \frac{4w_0}{\nu} - \int_0^\infty dt \ln t B'(t) \right) \right], \quad \nu \ll w_0, \quad (68)
\]

\[
\operatorname{tr} \alpha^F = -\frac{1}{2\pi} i \int_0^\infty \frac{dt}{\sqrt{t}} \left( \Gamma\left(\frac{1}{2} + t\right) e^{t - t\ln t} - \sqrt{2\pi} \right) = \frac{1}{2\pi} i \cdot 0, 8843 \ldots . \quad (69)
\]

In the integral of the formula (68) the function \( B'(t) \) is the derivative of the function \( B(t) = \Gamma(1 + t) e^{t - t\ln t} - \sqrt{2\pi} \). Numerical value of this integral is 2,194 \ldots . If one transforms the imaginary part of (68) to the form of the imaginary part of (58), then we obtain

\[
\ln \frac{4w_0}{\nu} - 2,194 \ldots = \ln \frac{2w_0}{\gamma\nu} - 0,9491 \ldots .
\]

So, the \( \operatorname{tr} \alpha^{B,F} \) for the exponential and hyperbolic motions rather close to each other.

## 8 Ultraviolet and infrared singularities of \( \operatorname{tr} \alpha^{B,F} \)

It is not difficult to obtain the general expression for the \( \operatorname{tr} \alpha^{B,F} \) in the form of a double integral which is a functional of the mirror’s trajectory and the tangent to it at the point \( u = v = 0 \). Indeed, after substituting the Bogoliubov coefficients

\[
\alpha^B_{\omega'\omega} = \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{\infty} dv e^{i\omega'v - i\omega g(v)}, \quad \alpha^F_{\omega'\omega} = \int_{-\infty}^{\infty} dv \sqrt{g'(v)} e^{i\omega'v - i\omega g(v)} \quad (70)
\]

to the formula (43) and trivial integration over the frequency \( \omega' \), we obtain

\[
\operatorname{tr} \alpha^{B,F} = \frac{1}{2\pi} \int_0^\infty d\left(\frac{\omega}{\varkappa}\right) \int_{-\infty}^{\infty} dx \left\{ 1, \sqrt{G'(x)} \right\} e^{-i\frac{\pi}{2}(G(x)-x)}, \quad (71)
\]

where 1 and \( \sqrt{G'(x)} \) in the braces refer to the Bose- and Fermi-cases respectively. The Lorentz-invariance of these expressions is evident. However, the integral over \( (\omega/\varkappa) \) diverges on the upper limit since its integrand behaves as \( \sqrt{\varkappa/\omega} \) at \( \omega/\varkappa \to \infty \). Indeed, for \( \omega/\varkappa \to \infty \) in the integral over \( x \) the \( |x| \ll 1 \) will be essential. Then the functions \( G(x) - x \) and \( G'(x) \) can be replaced by the first terms of their expansions near zero, that is by \( bx^2 \) and 1, see (40). Consequently, at \( \omega/\varkappa \to \infty \) the integral over \( x \) is reduced to

\[
\int_{-\infty}^{\infty} dx \ e^{-i\frac{\pi}{2}bx^2} = \sqrt{\frac{\pi\varkappa}{ib\omega}}, \quad (72)
\]

both in the Bose- and Fermi-case.

It is easy to show that the next term of the asymptotical expansion of the integral over \( x \) behaves as \( (\varkappa/\omega)^{3/2} \). Then, after subtraction from the integral over \( x \) of the first term
its asymptotical expansion in the parameter $\omega/\kappa \to \infty$, we make the integral over $\omega/\kappa$ convergent on the upper limit. If one goes from the variable $\omega/\kappa$ to the variable $z$,

$$
\frac{\omega}{\kappa} = \sqrt{\frac{\omega\omega'}{\kappa\kappa'}} = \frac{b\rho}{2w_0} = \frac{1}{2}bz,
$$

(73)

the subtractive term in $\text{tr} \alpha^{B,F}$ acquires the universal form

$$
\frac{1}{2\pi} \int_0^\infty dz \sqrt{\frac{\pi}{2iz}}.
$$

(74)

Remember that $z = \rho/w_0$ has the sense of the invariant momentum transfer in units of proper acceleration.

Though the expressions

$$
\text{tr} \alpha^{B,F} = \frac{1}{2\pi} \int_{0}^{\infty} ds \left[ \int_{-\infty}^{\infty} dx \left\{ 1, \sqrt{G'(x)} \right\} e^{-is(G(x) - x)} - \sqrt{\frac{\pi}{ibs}} \right], \quad s = \frac{\omega}{\kappa},
$$

(75)

do not contain the ultraviolet divergences, they can contain infrared divergences, if the spectral function (the function of $s$ in square brackets of (75)) has the singlar behaviour $\propto 1/s$ for $s \to 0$. It is clear that the behaviour of the spectral function near $s = \omega/\kappa = 0$ and also in the main forming region of the integral over $s$ is defined by the behaviour of the trajectory $G(x)$ far from the point of contact, where the expansion (40) can not be applied, i.e. at the distances $|x| \gtrsim 1$.

Let us demonstrate the working of the formula (75) on an example of another one trajectory

$$
u^{mir} = -\frac{1}{\kappa} \ln (2 - e^{\kappa'v}), \quad G(x) = - \ln (2 - e^x),
$$

(76)

for which the spectral function can be expressed in terms of the well known transcendental functions. This trajectory, as the hyperbolic one (53), has two asymptoties but snuggle up to them not as a power but an exponential manner. Therefore, on the both ends of the trajectory the proper acceleration

$$
a(v) = -\sqrt{\frac{\kappa\kappa'}{e^{\kappa'v}(2 - e^{\kappa'v})}}
$$

(77)

tends to $-\infty$ and at zero point attains the minimal in modulus value $a_0 = -\sqrt{\kappa\kappa'}$.

The integral over $x$ in (75), in which the upper limit for the trajectory (76) is equal to $\ln 2$, after changing the variable $x$ on $t = 1 - e^x$ is reduced to the tabular integral 2.25.1 of the reference book [10]. As a result we obtain

$$
\text{tr} \alpha^B = \frac{1}{2\pi} \int_{\varepsilon}^{\infty} ds \left\{ \frac{\sqrt{\pi \Gamma(is)}}{\Gamma(\frac{1}{2} + is)} - \sqrt{\frac{\pi}{is}} \right\},
$$

(78)

$$
\text{tr} \alpha^F = \frac{1}{2\pi} \int_{0}^{\infty} ds \left\{ \frac{\sqrt{\pi \Gamma(\frac{1}{2} + is)}}{\Gamma(1 + is)} - \sqrt{\frac{\pi}{is}} \right\},
$$

(79)

14
Since the spectral function has in Bose-case the infrared singularity the corresponding divergency of the integral over s for the $\text{tr} \alpha^B$ is removed by introducing the small but finite lower limit $\varepsilon = \nu/2w_0$. Its physical meaning is the minimal momentum transfer in units of acceleration at zero point.

After the turning of the integration contour over $s$ on the negative imaginary semiaxis with the detour (in Bose-case) of the singularity at zero along the arc of a circle with radius $\varepsilon = \nu/2w_0$. Its physical meaning is the minimal momentum transfer in units of acceleration at zero point.

We obtain

$$\text{tr} \alpha^B = \frac{1}{2\pi} \left[ -\frac{\pi}{2} - i(\ln \frac{2w_0}{\nu} - B) \right],$$

(80)

$$\text{tr} \alpha^F = \frac{1}{2\pi} i \cdot F,$$

(81)

where the positive constants $B$, $F$ are defined by the integrals

$$B = \int_0^\infty dt \ln t B'(t) = 1,887789 \ldots, \quad B(t) = \frac{\sqrt{\pi} \Gamma(1 + t)}{\Gamma(1 + t)} - \sqrt{\pi} t,$$

(82)

$$F = - \int_0^\infty dt \left[ \frac{\sqrt{\pi} \Gamma(1/2 + t)}{\Gamma(1 + t)} - \frac{\sqrt{\pi}}{t} \right] = 1,869957 \ldots.$$

(83)

The imaginary part of (80) can be transformed to the form of the imaginary part of (57):

$$\ln \frac{2w_0}{\nu} - 1,887789 \ldots = \ln \frac{2w_0}{\gamma \mu} - 1,310574 \ldots.$$

The expressions for $\pm \text{tr} \alpha^B,F$ obtained for the three different mirror’s trajectories are close to each other qualitatively and quantitatively, see (58-59), (68-69) and (80-81). All of them have negative imaginary part which in the Bose-case has infrared logarithmic singularity. This singularity is accompanied by the appearance of the real negative part for $\text{tr} \alpha^B$, namely, $\text{Re} \text{tr} \alpha^B = -1/4$, whereas $\text{Re} \text{tr} \alpha^F = 0$. Similar expressions for $\pm \text{tr} \alpha^B,F$ are typical for the trajectories $G(x)$-function of which increases stronger (falls weaker) than $x$ when $x$ tends to upper (lower) limit.

Since the functionals $\pm \text{tr} \alpha^B,F$ have, according to (52), the meaning of action, compare them with the changes $\Delta W_{1,0}$ of selfactions of electric and scalar charges at hyperbolic motion [11,12]:

$$\Delta W_{1,0} = -(\tau_2 - \tau_1) \cdot \Delta m_{1,0},$$

(84)

$$\Delta m_1 = \frac{e^2 w_0}{4\pi^2} \left[ -\frac{\pi}{2} - i \left( \ln \frac{2w_0}{\gamma \mu} - \frac{1}{2} \right) \right], \quad \Delta m_0 = -i \frac{e^2 w_0}{8\pi^2}.$$

(85)

In such a motion the proper acceleration of a charge is constant and the square of the interval between two points on the trajectory is the function only of the length of the arc connecting them:

$$(x_\alpha(\tau) - x_\alpha(\tau'))^2 = f(\tau - \tau').$$

(86)

Therefore, the change of the charge’s selfinteraction is proportional to the duration $\tau_2 - \tau_1$ of the charge’s stay at hyperbolic motion multiplied by the mass shift $\Delta m_{1,0}$ of the charge. The mass shift owes its origin to the change of the interaction of a charge with its own field,
which essentially modified at the distances $\sim w_0^{-1}$ from the charge due to acceleration. In other words, the shift is formed on the arclength $|\tau - \tau'| \sim w_0^{-1}$ with the center $\tau_e$ in any point of the trajectory inside the interval $(\tau_1, \tau_2)$ of acceleration. The independence of the shift from $\tau_e$ means that it is a constant of motion. This is not so for the trajectories with variable acceleration, see Section 9.

As distinct from $\Delta W_{1,0}$ describing the change of interaction of the charge with itself due to acceleration, the functionals $\pm \text{tr } \alpha^{B,F}$ describe the interaction of accelerated mirror with the probe executing uniform motion along the tangent to the mirror’s trajectory at the point where mirror has acceleration $w_0$. This interaction is transmitted by the vector or scalar perturbations created by the mirror in the vacuum of Bose- or Fermi-field and carrying the spacelike momentum of the order of $w_0$. According to (51), the field of these perturbations decreases at the distances of the order of $w_0^{-1}$ while the charge all the time interacts with itself and feels the change of interaction over the all time of acceleration. Therefore, it is not surprising that the expressions for $\pm \text{tr } \alpha^{B,F}$ coincide in essence with $\Delta W_{1,0}$ if in these latter one puts $\tau_2 - \tau_1 = 2\pi/w_0$, $e^2 = 1$ and changes the sign on the opposite one. In other words, the $\pm \text{tr } \alpha^{B,F}$ are the mass shifts of the mirror’s proper field multiplied by characteristic proper time of their formation.

9 Mass shifts of electric and scalar charges at exponential motion

For calculation of the self-actions of electric and scalar charges at exponential motion let us make use of the formula (30). The charge’s trajectory (60) is convenient to use in a form of a function of proper time:

$$u^{\text{mir}}(\tau) = -\frac{2}{\kappa} \ln (1 - w_0 \tau), \quad v^{\text{mir}}(\tau) = \frac{1}{\kappa'} (2w_0 \tau - w_0^2 \tau^2).$$

Then

$$\dot{x}_\alpha(\tau) \dot{x}^\alpha(\tau') = \frac{1 + z^2}{1 - z^2}, \quad (x - x')^2 = -(\tau - \tau')^2 \text{Arth} \frac{z}{\tau}, \quad z = \frac{w_0 (\tau - \tau')}{2 - w_0 (\tau + \tau')}. \quad (88)$$

Introduce now the new variables $\xi = (\tau + \tau')/2$, $z$ instead of $\tau$, $\tau'$. At fixed $\xi$, lying in the interval $-\infty < \xi < w_0^{-1}$, the variable $z$ changes in the interval $-1 < z < 1$. By using the causal function $\Delta_4^f$ expressed via Macdonald’s function, we obtain

$$\Delta W_1 = e^2 \int_{-\infty}^{w_0^{-1}} d\xi \left( \frac{1}{w_0} - \xi \right) \int_1^{w_0} dz \dot{x}_\alpha(\tau) \dot{x}^\alpha(\tau') \Delta_4^f x - x', \mu) =$$

$$-e^2 \int_{-\infty}^{w_0^{-1}} d\xi \int_0^\infty du \frac{\mu}{\text{sh} 2u} \left\{ \text{ch} 2u \frac{\mu}{u} K_1(i\lambda \sqrt{u} \text{th} u) - K_1(i\lambda \text{th} u) \right\}. \quad (89)$$

In the last expression instead of $z$ the variable $u = \text{Arth} z$ is used, and $\lambda$ is the function of $\xi$, $\lambda(\xi) = 2\mu(w_0^{-1} - \xi)$.
Our problem now is to find the integral over $u$ in that region of the variable $\xi$ where $\lambda(\xi) \ll 1$, supposing, of course, that the infrared parameter $\mu/w_0 \ll 1$. This integral coincides, in essence, with the mass shift of an electric charge

$$\Delta m_1 = \frac{e^2}{2\pi^2} \int_0^\infty \frac{du}{\sinh u} \left\{ \chi 2u \sqrt{\frac{\ln u}{u}} K_1(i\lambda\sqrt{u}) - K_1(i\lambda u) \right\}. \quad (90)$$

To calculate $\Delta m_1$ when $\lambda(\xi) \ll 1$ let us divide the integration interval into two intervals, $0 \leq u \leq u_1$ and $u_1 \leq u < \infty$, by the point $u_1$ where $u_1 \gg 1$, but $\lambda u_1 \ll 1$. Then, using the expansion of Macdonald’s function at small argument, we obtain

$$\Delta m_1 \approx \frac{e^2 w_0}{4\pi^2(1-w_0\xi)} \left\{ \frac{1}{i} \int_0^{u_1} du \left( \frac{\text{csch} \, 2u}{u} - \frac{1}{2\sinh^2 u} \right) + \int_{u_1}^\infty \frac{du}{\sqrt{u}} \lambda K_1(i\lambda u) \right\} =$$

$$= \frac{e^2 w_0}{4\pi^2(1-w_0\xi)} \left\{ -\pi - i \left( 2\ln \frac{w_0}{\gamma\mu(1-w_0\xi)} + \ln \frac{2\gamma}{\pi} + \frac{1}{2} \right) \right\}. \quad (91)$$

The mass shift $\Delta m_0$ of the scalar charge differs from (90) by the replacement $\chi 2u \to -1$ in the first term of the braces and by the change of the sign of the second term. Then at the same condition $\lambda(\xi) \ll 1$ we get

$$\Delta m_0 = -i \frac{e^2 w_0}{4\pi^2(1-w_0\xi)} (\ln 2 - \frac{1}{2}). \quad (92)$$

As is seen from (91), (92) and (66), the mass shift depends on the absolute value $w(\xi) = w_0/(1-w_0\xi)$ of the proper acceleration of the mirror at the instant $\xi$, which may be considered as a center of forming region of the shift. As the acceleration essentially changes on such an interval, the mass shifts (91), (92) do not coincide with the mass shifts (84), (85) of uniformly-accelerated charges if one replaces $w(\xi)$ by $w_0$. Nevertheless, rather close coincidence arises at the replaces $w(\xi) \to 0, 5 w_0$ and $w(\xi) \to 2, 6 w_0$ for the $\Delta m_1$ and $\Delta m_0$ correspondingly.

## 10 Conclusion

The basis for the symmetry between the processes induced by the mirror in two-dimensional and by the charge in four-dimensional space-time is the relation (14), (15) between the Bogoliubov’s coefficients $\beta^{B,F}_{\omega'}$ and the current density $j^\alpha(k)$ or charge density $\rho(k)$ depending on the timelike momentum $k^\alpha$. The squares of these quantities represent the spectra of real pairs and particles radiated by accelerated mirror and charge.

In the present paper the symmetry is extended to the selfactions of the mirror and the charge and to the corresponding vacuum-vacuum amplitudes, cf. (29) and (30). In essence, it is embodied in the discovered relation (20) between propagators of a massive pair in two-dimensional space and of a single particle in four-dimensional space.

The formula (29) for $W^{B,F}$ was obtained provided that the mean number $\text{tr} \beta^+ \beta$ of pairs created is small and the interference of two or more pairs is negligible. In the general case the $W^{B,F}$ is given by the formula (27), which can be written also in the form

$$2 \text{Im} W^{B,F} = \pm \text{tr} \ln(\alpha^+ \alpha)^{B,F}, \quad (93)$$
since $\alpha^+ \alpha \mp \beta^+ \beta = 1$, see [7], [4]. As is seen from (27) or (93) the imaginary part of the action differs from zero and then is positive only if $\beta \neq 0$, i.e. if the radiation of real particles is happened indeed.

Formula (93) allows to choose for $W^{B,F}$ the expression

$$W^{B,F} = \pm i \text{tr} \ln \alpha^{B,F},$$

(94)

that was named natural by DeWitt [7]. However, this expression is by no means unique. The expressions $W^{B,F} = \pm i \text{tr} \ln(a e^{i\gamma})^{B,F}$ and $W^{B,F} = \pm i \text{tr} \ln \alpha^{B,F+}$ have the same imaginary part. Nevertheless, the formula (94) is interesting as the definition both the real and imaginary parts of the selfactions $W^{B,F}$ by means of the Bogoliubov’s coefficients $\alpha^{B,F}_\omega$ only, which, according to the formulae (14), (15), reduce to the current density $j^\alpha(q)$ or to the charge density $\rho(q)$ dependent on the spacelike momentum $q^\alpha$. This means that the field of the corresponding perturbations propagates in vacuum together with the mirror, comoves it, and, at the same time, it contains the information about the radiation of the real quanta.

Unfortunately, the author failed to find a simple integral representation for the matrix $\ln \alpha$. Nevertheless, if one again assumes that the mean number of emitted particles is small, then one may consider $\alpha$ or $i\alpha$ close to 1. Then, expanding the $\ln i\alpha$ near $i\alpha = 1$ and confine ourselves by the first term we obtain

$$W^{B,F} = \pm i \text{tr} \ln i\alpha^{B,F} \approx \pm i \text{tr} (i\alpha^{B,F} - 1) = \mp \text{tr} \alpha^{B,F} + \ldots .$$

(95)

These qualitative arguments allow to state that the functionals $\pm \text{tr} \alpha^{B,F}$ are similar to the corresponding selfactions with opposite sign and therefore must have the negative imaginary parts. This is confirmed by all examples considered in Sections 7 and 8. Nevertheless, the exact physical meaning of the quantities $\pm \text{tr} \alpha^{B,F}$ is clearly defined by the formula (52).

Here we want also to concentrate attention on one prediction followed from the symmetry between processes induced by the mirror in two-dimensional and by the charge in four-dimensional space-times. The symmetry predicts the value $e_0^2 = 1$ for the charge squared (in Heviside’s units), that corresponds to the fine structure constant $\alpha_0 = 1/4\pi$. Since the radiation corrections are not taken into account in both spaces, and the processes in 1+1-space are due to the purely geometrical boundary condition, it is natural to think that the above mentioned values for the charge squared and for the fine structure constant are the nonrenormalized bare values of these constants. Therefore they are marked by index 0.

It is very interesting that the bare fine structure constant has the purely geometrical origin and, also, that its value is small: $\alpha_0 = 1/4\pi \ll 1$. The smallness of $\alpha_0$ has the essential meaning for the quantum electrodynamics where it justifies a priori the applicability of the perturbation theory and where the radiative corrections in accordance with the well known formula [13, 14]

$$\alpha = \frac{\alpha_0}{1 + \frac{\alpha_0}{3\pi} N \ln \frac{\Lambda^2}{m^2}}$$

(96)

decrease the renormalized value of $\alpha$ in comparison with unrenormalized one. Here $N$ is the number of charged particles with masses in the interval $(m, \Lambda)$, and $\Lambda$ is the upper limit of the particle energy up to which the quantum electrodynamics is correct.
11 Appendix

The singular function $\Delta^{LR}_d(z, \nu)$ in $d$-dimensional space-time, as well as the causal function $\Delta^f_d(z, \mu)$, is convenient to define by Fourier representation

$$\Delta^{LR}_d(z, \nu) = \int \frac{d^d q}{(2\pi)^d} \frac{e^{i q z}}{q^2 - \nu^2 + i\varepsilon}, \quad \Delta^f_d(z, \mu) = \int \frac{d^d q}{(2\pi)^d} \frac{e^{i q z}}{q^2 + \mu^2 - i\varepsilon}.$$  \hspace{1cm} (97)

These functions are the even singular solutions of the inhomogeneous wave equations

$$(-\partial^2 - \nu^2) \Delta^{LR}_d(z, \nu) = \delta(z), \quad (-\partial^2 + \mu^2) \Delta^f_d(z, \mu) = \delta(z),$$  \hspace{1cm} (98)

with opposite signs before the parameters $\nu^2$ and $\mu^2$, where $\nu$ and $\mu$ are the momentum transfer and mass. Their proper time representations (in particular, for $d=4$)

$$\Delta^{LR}_4(z, \nu) = \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{s^2} e^{-i\nu^2 s - iz^2/4s}, \quad \Delta^f_4(z, \mu) = \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{s^2} e^{-i\mu^2 s + iz^2/4s},$$  \hspace{1cm} (99)

as well as the explicit expressions in terms of Macdonald’s function, differ by the complex conjugation and the replacement $\mu \rightarrow i \nu$ or by the replacement $z^2 \rightarrow -z^2, \mu \rightarrow \nu$.

For the symmetry being discussed in this paper the integral relation

$$\Delta^{LR}_{d+2}(z, \nu) = -\frac{1}{4\pi} \int_{\nu^2}^{\infty} d\rho^2 \Delta^{LR}_d(z, \rho)$$  \hspace{1cm} (100)

is very important. It differs from the similar relation (20) for the causal functions not only by the sign. Being written for $z^2 < 0$, it is understood for $z^2 > 0$ in a sense of analytical continuation in the lower half-plane of complex $z^2$. Whereas the relation (20), being written for $z^2 > 0$, is understood for $z^2 < 0$ as the analytical continuation in the upper half-plane of complex $z^2$. For the $\Delta^+$-functions such a continuation must be carry out in the upper half-plane if $z^0 > 0$, and in the lower one if $z^0 < 0$.

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