A stochastic SICA epidemic model for HIV transmission

Jasmina Djordjevic, Cristiana J. Silva, Delfim F. M. Torres

Abstract

We propose a stochastic SICA epidemic model for HIV transmission, described by stochastic ordinary differential equations, and discuss its perturbation by environmental white noise. Existence and uniqueness of the global positive solution to the stochastic HIV system is proven, and conditions under which extinction and persistence in mean hold, are given. The theoretical results are illustrated via numerical simulations.

Key words: SICA epidemic model, HIV infection, stochastic differential equations, Brownian motion, extinction and persistence.

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1. Introduction

Epidemics are, inevitably, affected by environmental white noise, which is an important component to be taken into account by mathematical models, providing an additional degree of realism in comparison to their deterministic counterparts [1]. Here, our aim is to improve the deterministic SICA epidemic model for HIV transmission recently proposed in [2, 3], by considering environmental interactions. For that, we follow [1, 4, 5, 6, 7, 8] and introduce stochastic noise in the form of a Brownian motion with positive intensity. The advantage of our model with respect to previous ones in [2, 3] is that we assume fluctuations in the environment, manifesting in the transmission coefficient rate, thus making it more biologically realistic for the transmission dynamics of HIV/AIDS in a homogeneously mixing population of variable size.

The model subdivides human population into four mutually-exclusive compartments: susceptible individuals ($S$); HIV-infected individuals with no clinical symptoms of AIDS (the virus is living or developing in the individuals but without producing symptoms or only mild ones) but able to transmit HIV to other individuals ($I$); HIV-infected individuals under ART treatment (the so-called chronic stage) with a viral load remaining low ($C$); and HIV-infected individuals with AIDS clinical symptoms ($A$). The total population at time $t$, denoted by $N(t)$, is given by $N(t) = S(t) + I(t) + C(t) + A(t)$. For sake of simplicity, we assume that the associated AIDS-induced mortality is negligible. Using the same arguments as in [3], we consider a force of infection given by $\beta(I + \eta_C C + \eta_A A)$ with $\beta = \frac{\beta_0 \mu}{\Lambda}$, where $\beta_0$ is the effective contact rate for HIV transmission. The modification parameter $\eta_A \geq 1$ accounts for the relative infectiousness of individuals with AIDS symptoms, in comparison to those infected with HIV with no AIDS symptoms. Individuals with AIDS symptoms are more infectious than HIV-infected individuals (pre-AIDS) because they have a higher viral load and there is a positive correlation between viral load and infectiousness [9]. On the other hand, $\eta_C \leq 1$ translates the partial restoration of immune function of individuals with HIV infection that use ART correctly [10]. All individuals suffer from natural death, at a constant rate $\mu$. We also assume that HIV-infected individuals, with and without AIDS symptoms, have access to ART treatment. HIV-infected individuals with no AIDS symptoms $I$ progress to the class of individuals with HIV infection under ART treatment $C$ at a rate $\phi$, and HIV-infected individuals with AIDS symptoms are treated for HIV at rate $\alpha$. 

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Individuals in the class $C$ leave to the class $I$ at a rate $\omega$. Moreover, an HIV-infected individual with AIDS symptoms $A$ that starts treatment moves to the class of HIV-infected individuals $I$, moving only to the chronic class $C$ if the treatment is maintained. HIV-infected individuals $I$ with no AIDS symptoms, which do not take ART treatment, progress to the AIDS class $A$ at rate $\rho$. Precisely, we consider the model

$$
\begin{align*}
(1)
\frac{dS(t)}{dt} &= \left[ \Lambda - \beta (I(t) + \eta_c C(t) + \eta_A A(t)) S(t) - \mu S(t) \right] dt, \\
\frac{dI(t)}{dt} &= \left[ \beta (I(t) + \eta_c C(t) + \eta_A A(t)) S(t) - \xi_2 I(t) + \alpha A(t) + \omega C(t) \right] dt, \\
\frac{dC(t)}{dt} &= \left[ \phi I(t) - \xi_2 C(t) \right] dt, \\
\frac{dA(t)}{dt} &= \left[ \rho I(t) - \xi_1 A(t) \right] dt,
\end{align*}
$$

where $\xi_1 = \alpha + \mu + d$, $\xi_2 = \omega + \mu$ and $\xi_3 = \rho + \phi + \mu$. Existence and uniqueness of solution to the deterministic model (1) is proved in [2, 3], where it is shown that the system has one disease free equilibrium when the basic reproduction number is less than one and one endemic equilibrium when the basic reproduction number is greater than one. Local and global stability of the equilibrium points of (1) is also proved in [2, 3]. Motivated by [5], we consider here fluctuations in the environment, which are assumed to manifest themselves as fluctuations in the parameter $\beta$, so that $\beta \to \beta + \sigma B(t)$, where $B(t)$ is a standard Brownian motion with intensity $\sigma^2 > 0$. Our stochastic model takes then the following form:

$$
\begin{align*}
\frac{dS(t)}{dt} &= \left[ \Lambda - \beta (I(t) + \eta_c C(t) + \eta_A A(t)) S(t) - \mu S(t) \right] dt - \sigma (I(t) + \eta_c C(t) + \eta_A A(t)) S(t) dB(t), \\
\frac{dI(t)}{dt} &= \left[ \beta (I(t) + \eta_c C(t) + \eta_A A(t)) S(t) - \xi_2 I(t) + \alpha A(t) + \omega C(t) \right] dt + \sigma (I(t) + \eta_c C(t) + \eta_A A(t)) S(t) dB(t), \\
\frac{dC(t)}{dt} &= \left[ \phi I(t) - \xi_2 C(t) \right] dt, \\
\frac{dA(t)}{dt} &= \left[ \rho I(t) - \xi_1 A(t) \right] dt.
\end{align*}
$$

(2)

The paper is organized as follows: Section 2 is devoted to existence and uniqueness of a global positive solution to the Stochastic Differential Equation (SDE) (2) (cf. Theorem 2.1); Section 3 to conditions for the extinction of HIV within the population (cf. Theorem 3.1); and Section 4 to conditions for the persistence in mean of the disease (cf. Theorem 4.1). We end with Section 5, illustrating both theoretical results of extinction and persistence with numerical simulations.

2. Existence and uniqueness of a positive global solution

Throughout the paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}\}_{t \geq 0}$, which is right continuous and such that $\mathcal{F}$ contains all $\mathbb{P}$-null sets. The scalar Brownian motion $B(t)$ of (2) is defined on the given probability space. Also, we denote $\mathbb{R}^4_+ = (x_1, x_2, x_3, x_4), x_i > 0, i = 1, 2, 3, 4$.

**Theorem 2.1.** For any $t \geq 0$ and any initial value $(S(0), I(0), C(0), A(0)) \in \mathbb{R}^4_+$, there is a unique solution $(S(t), I(t), C(t), A(t))$ to the SDE (2) and the solution remains in $\mathbb{R}^4_+$ with probability one. Moreover,

$$
N(t) \to \frac{\Lambda}{\mu} \text{ as } t \to \infty,
$$

where $N(t) = S(t) + I(t) + C(t) + A(t)$.

**Proof.** Having in mind that $N(t) = S(t) + I(t) + C(t) + A(t)$, we know that $A(t) = N(t) - S(t) - I(t) - C(t) \geq 0$, $t \geq 0$. It also follows that we can eliminate $A(t)$ from our SICA model (2), reducing it to a system of three equations:

$$
\begin{align*}
\frac{dS(t)}{dt} &= \left[ \Lambda - \beta (I(t) + \eta_c C(t) + \eta_A (N(t) - S(t) - I(t) - C(t))) S(t) - \mu S(t) \right] dt \\
&\quad - \sigma (I(t) + \eta_c C(t) + \eta_A (N(t) - S(t) - I(t) - C(t))) S(t) dB(t), \\
\frac{dI(t)}{dt} &= \left[ \beta (I(t) + \eta_c C(t) + \eta_A (N(t) - S(t) - I(t) - C(t))) S(t) - \xi_2 I(t) + \alpha (N(t) - S(t) - I(t) - C(t)) + \omega C(t) \right] dt \\
&\quad + \sigma (I(t) + \eta_c C(t) + \eta_A (N(t) - S(t) - I(t) - C(t))) S(t) dB(t), \\
\frac{dC(t)}{dt} &= \left[ \phi I(t) - \xi_2 C(t) \right] dt.
\end{align*}
$$

(4)
If we prove that there exists a unique positive solution \((S(t), I(t), C(t))\) of system (4) for \(t \geq 0\), then we can replace process \(I(t)\) in the last equation of system (2) and solve it explicitly. From this fact, the existence of a unique positive solution for system (2) is obtained. For a given \((S(0), I(0), C(0), A(0)) \in \mathbb{R}^4_+\), we prove that there exists a unique positive solution of system (4) for every \(t \geq 0\). Because the coefficients of system (4) are locally Lipschitz continuous, there is a unique local solution on \([0, \tau_0)\) for any initial value \((S(0), I(0), C(0))\), where \(\tau_0\) is known in the literature as the explosion time. It is necessary to prove that the solution is global, i.e., that \(\tau_0 = +\infty\) almost surely (a.s., for brevity). Let us define

\[
\tau^* = \inf \{t \in [0, \tau_0) : S(t) \leq 0 \text{ or } I(t) \leq 0 \text{ or } R(t) \leq 0\}. 
\]

Because the infimum of an empty set is \(\infty\) and \(\tau^* \leq \tau_0\), if we prove that \(\tau^* = +\infty\) a.s., then the proof of our theorem is complete. Indeed, if \(\tau^* = +\infty\) a.s., then \(\tau_0 = \infty\), which means that \((S(t), I(t), C(t)) \in \mathbb{R}^3_+\) for \(t \geq 0\) a.s. Let us assume that \(\tau^* < \infty\). Then there exists \(T > 0\) such that \(P(\tau^* < T) > 0\). Define the function \(V(t) = \ln(S(t)I(t)C(t))\), which is twice differentiable and defined on positive values. By Ito’s formula,

\[
d(V(t)) \geq K(S(t), I(t), C(t)) + \sigma \left( \frac{S(t)}{I(t)} - 1 \right) \left( I(t) + \eta_C C(t) + \eta_A (N(t) - S(t) - I(t) - C(t)) \right) dW(t),
\]

where

\[
K(S(t), I(t), C(t)) = \beta (I(t) + \eta_C C(t) + \eta_A (N(t) - S(t) - I(t) - C(t))) - \mu - \frac{1}{2} \sigma^2 (I(t) + \eta_C C(t) + \eta_A (N(t) - S(t) - I(t) - C(t))) (S(t) - \xi_3 - \xi_4 - \xi_5).
\]

By definition of our model, we are considering the case when infection is high. Thus, \(\frac{S(t)}{I(t)} - 1 \leq 0\) and

\[
V(t) \geq V(0) + \int_0^t K(S(s), I(s), C(s)) ds + \int_0^t \sigma \left( \frac{S(s)}{I(s)} - 1 \right) \left( I(s) + \eta_C C(s) + \eta_A (N(s) - S(s) - I(s) - C(s)) \right) dB(s).
\]

It follows that \(\lim_{t \to \tau^*} V(t) = -\infty\). Letting \(t \to \tau^*\) in (5), we have

\[
-\infty \geq V(t) \geq V(0) + \int_0^\tau K(S(s), I(s), C(s)) ds + \int_0^\tau \sigma \left( \frac{S(s)}{I(s)} - 1 \right) \left( I(s) + \eta_C C(s) + \eta_A (N(s) - S(s) - I(s) - C(s)) \right) dB(s) > -\infty,
\]

which is in contradiction with the assumptions. We conclude that \(\tau^* = +\infty\) a.s.

It remains to prove (3). If we sum all equations from system (2), then

\[
\frac{d(S(t) + I(t) + C(t) + A(t))}{dt} = \left[ \Lambda - \mu S(t) + (\phi - \xi_3 + \rho) I(t) + (\alpha - \xi_1) A(t) + (\omega - \xi_2) C(t) \right] dt = \left[ \Lambda - \mu S(t) + (\phi - \rho - \phi - \mu + \rho) I(t) + (\alpha + \mu) A(t) + (\omega - \omega - \mu) C(t) \right] dt
\]

So we obtain

\[
S(t) + I(t) + C(t) + A(t) = e^{-\mu t} \left[ S(0) + I(0) + C(0) + A(0) + \int_0^t (\Lambda - \mu (S(t) + I(t) + C(t) + A(t)) - d \cdot A(t)) \right]
\]

Applying L’Hospital’s rule, it follows that \(\lim_{t \to +\infty} (S(t) + I(t) + C(t) + A(t)) = \frac{\Lambda}{\mu}\). The proof is complete. \(\square\)
3. Extinction

In this section, we prove a condition for the extinction of the disease.

**Theorem 3.1.** Let \( Y(t) = (S(t), I(t), C(t), A(t)) \) be the solution of system (2) with positive initial value. Assume that \( \sigma^2 > \frac{\beta}{2\alpha} \). Then,

\[
I(t), C(t), A(t) \to 0 \text{ a.s. and } S(t) \to \frac{\Lambda}{\mu} \text{ a.s.,}
\]
as \( t \to +\infty \).

**Proof.** By Theorem 2.1, the solution of system (2) is positive for every \( t \geq 0 \). Applying Itô’s formula on the second equation of system (2), we have

\[
d(\log I(t)) = \left\{ -\frac{\sigma^2}{2} \left( \frac{S(t)}{I(t)} \right)^2 + \frac{\beta^2}{2\alpha^2} \left( \int_{0}^{t} \sigma \, d\xi_t + \beta A(t) + \omega C(t) \right) \right\} dt + \frac{\sigma \sigma^2}{I(t)} d\xi_t.
\]

Integrating both sides of (6) from 0 to \( t \), and then dividing by \( t \), we obtain that

\[
\frac{\log I(t)}{t} \leq \frac{\log I(0)}{t} + \frac{\beta^2}{2\alpha^2} - \frac{\xi_0}{t} t + \frac{J(t)}{t} + \frac{M(t)}{t},
\]

where we define

\[
J(t) = \int_{0}^{t} \frac{\alpha A(s) + \omega C(s)}{I(s)} ds, \quad M(t) = \int_{0}^{t} \frac{\sigma S(s)}{I(t)} \left( I(s) + C(s) - A(s) \right) dW(s).
\]

We need to estimate functions \( J(t) \) and \( M(t) \). As \( N(t) = S(t) + I(t) + C(t) + A(t) \), then each coordinate of the population \((S, I, C, A)\) is less or equal than the number of the whole population \( N(t) \). We have

\[
J(t) = \int_{0}^{t} \frac{\alpha A(s) + \omega C(s)}{I(s)} ds \leq (\alpha + \omega) \frac{\Lambda}{\mu} t.
\]

Because \( M(t) \) is an integral with respect to the Brownian motion, it is local continuous martingale. Also, if we replace the upper bound with \( t = 0 \) in \( M(t) \), then we have \( M(0) = 0 \). Further, we can find the quadratic variation and obtain the following limits:

\[
\limsup_{t \to +\infty} \frac{M(t)}{t} \leq \frac{\sigma^2 \Lambda^2 (1 + \eta C + \eta A)^2}{\mu^2} < +\infty.
\]

Applying the large number theorem for martingales [5], we have that

\[
\lim_{t \to +\infty} \frac{M(t)}{t} = 0 \text{ a.s.}
\]

If we use (8) into estimates in (7), and the fact that \( \sigma^2 > \frac{\beta}{2\alpha} \), then

\[
\limsup_{t \to +\infty} \frac{\log I(t)}{t} \leq \frac{\beta^2}{2\alpha^2} - \frac{\xi_0}{t} < 0, \text{ a.s.}
\]

This implies that \( \lim_{t \to +\infty} I(t) = 0 \text{ a.s.} \). Solving explicitly the ordinary differential equation for process \( C(t) \), from system (2) we have

\[
C(t) = e^{-\xi t} \left[ C(0) + \int_{0}^{t} \phi I(s) e^{-\xi s} \, ds \right] \leq e^{-\xi t} C(0) + \phi \int_{0}^{t} I(s) \, ds.
\]

As \( \lim_{t \to +\infty} I(t) = 0 \text{ a.s.}, \) we also have that \( \lim_{t \to +\infty} C(t) = 0 \text{ a.s.} \). Similarly, we obtain that \( \lim_{t \to +\infty} A(t) = 0 \text{ a.s.} \). Estimation of \( S(t) \) is easy: since \( N(t) = S(t) + I(t) + C(t) + A(t) \) and from (3) we know that \( N(t) \to \frac{\Lambda}{\mu} \text{ a.s.} \) as \( t \to +\infty \), replacing \( I(t), C(t), A(t) \to 0 \text{ a.s.} \), \( t \to +\infty \), we obtain that \( S(t) \to \frac{\Lambda}{\mu} \text{ a.s.}, t \to +\infty \). \( \square \)
4. Persistence in mean

We begin by recalling the notion of persistence in mean.

**Definition 4.1.** System (2) is said to be persistent in mean if \( \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} I(s)ds > 0 \) a.s.

Let us introduce the notation \([x(t)] = \frac{1}{t} \int_{0}^{t} x(s)ds > 0\).

**Theorem 4.1.** Let

\[
K_1 = \frac{\beta}{\mu} \left( \frac{\alpha \rho}{\xi_1} - \xi_3 - \frac{\omega \phi}{\xi_2} + \frac{\mu (\xi_1 \xi_2 - \mu)}{\Lambda (1 + \eta_c + \eta_A)} \right).
\]

For any initial value \((S(0), I(0), C(0), A(0)) \in \mathbb{R}_+^4\) such that

\[
S(t) + I(t) + C(t) + A(t) = N(t) \to \frac{\Lambda}{\mu} \text{ as } t \to \infty,
\]

if \( K_1 \neq 0 \), \( \frac{\alpha \rho}{\mu} \left( \frac{\Lambda - \beta}{\mu} - \xi_1 \xi_2 - \frac{\omega \phi}{2 \mu} \right) > 0 \) and \( \xi_1, \xi_2 > 1 \), then the solution \((S(t), I(t), C(t), A(t))\) satisfies

\[
\liminf_{t \to \infty} [I(t)] \geq \frac{1}{K_1} \left( \frac{\Lambda \beta}{\mu} - \xi_1 \xi_2 - \frac{\sigma^2 \Lambda^2}{2 \mu^2} \right).
\]

**Proof.** An integration of system (2) yields

\[
\frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} + \frac{\alpha A(t) - A(0)}{t} + \frac{\omega C(t) - C(0)}{t} = \Lambda - \mu [S(t)] + \left( \frac{\alpha \rho}{\xi_1} - \xi_3 - \frac{\omega \phi}{\xi_2} \right) [I(t)].
\]

From here, one has

\[
[S(t)] = \frac{\Lambda}{\mu} + \frac{1}{\mu} \left( \frac{\alpha \rho}{\xi_1} - \xi_3 - \frac{\omega \phi}{\xi_2} \right) [I(t)] - \frac{K(t)}{\mu},
\]

where

\[
K(t) = \frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} + \frac{\alpha A(t) - A(0)}{t} + \frac{\omega C(t) - C(0)}{t}.
\]

By Itô’s formula, we obtain

\[
d \log \left( I(t) + \eta_c C(t) + \eta_A A(t) \right) \\
\geq \left\{ \beta S(t) - \frac{\mu I(t) + \xi_1 \eta_c A(t) + \xi_2 \eta_c C(t)}{I(t) + \eta_c C(t) + \eta_A A(t)} - \frac{\sigma^2 S^2(t)}{2(I(t) + \eta_c C(t) + \eta_A A(t))} \right\} dt + \sigma S(t) dB(t).
\]

We assume \( \xi_1, \xi_2 > 1 \). Then,

\[
d \log (I(t) + \eta_c C(t) + \eta_A A(t)) \\
\geq \left\{ \beta S(t) - \frac{\mu I(t) + \xi_1 \xi_2 \eta_c A(t) + \xi_1 \xi_2 \eta_c C(t)}{I(t) + \eta_c C(t) + \eta_A A(t)} - \frac{\sigma^2 \Lambda^2}{2 \mu^2} \right\} dt + \sigma S(t) dB(t)
\]

Integrating the last inequality from 0 to \( t \), and dividing it with \( t \), we have

\[
\frac{\log(I(t) + \eta_c C(t) + \eta_A A(t)) - \log(I(0) + \eta_c C(0) + \eta_A A(0))}{t} \\
\geq \beta [S(t)] + \frac{\mu (\xi_1 \xi_2 - \mu)}{\Lambda (1 + \eta_c + \eta_A)} [I(t)] - \xi_1 \xi_2 - \frac{\sigma^2 \Lambda^2}{2 \mu^2} + \frac{\sigma}{t} \int_{0}^{t} S(s) dB(s)
\]

\[
\geq \frac{\Lambda \beta}{\mu} + \frac{\beta}{\mu} \left( \frac{\alpha \rho}{\xi_1} - \xi_3 - \frac{\omega \phi}{\xi_2} \right) + \frac{\mu (\xi_1 \xi_2 - \mu)}{\Lambda (1 + \eta_c + \eta_A)} [I(t)] - \xi_1 \xi_2 - \frac{\sigma^2 \Lambda^2}{2 \mu^2} - \frac{\beta K(t)}{\mu} + \frac{M(t)}{t}.
\]
It follows that

\[
[I(t)] \geq \frac{1}{K_1} \left[ \frac{\Lambda \beta}{\mu} - \xi_1 \xi_2 - \frac{\sigma^2 A^2}{2 \mu^2} - \frac{\beta K(t)}{\mu} + \frac{M(t)}{t} \right] \log \left( \frac{I(t) + \eta_C C(t) + \eta_A A(t)}{\log(0 + \eta_C C(0) + \eta_A A(0))} \right),
\]

where \( K_1 \) is given by (9) and \( M(t) = \sigma \int_0^t S(t) dB(t) \). Process \( M(t) \) is a local continuous martingale with value 0 for \( t = 0 \), and it has the property that \( \limsup_{t \to \infty} \frac{M(t)}{t} \leq \frac{\sigma^2 A^2}{\mu^2} < +\infty \), a.s. Applying the large number theorem for martingales [5], it follows that \( \lim_{t \to \infty} \frac{M(t)}{t} = 0 \), a.s. As \( S(t) + I(t) + C(t) + A(t) \leq K \),

\[-\infty < \log(I(t) + \eta_C C(t) + \eta_A A(t)) < \log \left( \frac{\Lambda}{\mu}(1 + \eta_C + \eta_A) \right) \]

and \( \lim_{t \to \infty} K(t) = 0 \). By taking the limit inferior of both sides in Eq. (11), one has

\[
\liminf_{t \to \infty} [I(t)] \geq \frac{1}{K_1} \left( \frac{\Lambda \beta}{\mu} - \xi_1 \xi_2 - \frac{\sigma^2 A^2}{2 \mu^2} \right),
\]

which completes the proof. \( \square \)

5. Numerical simulations

In this section we consider the following initial conditions:

\[
S(0) = 5, \quad I(0) = 1, \quad C(0) = 0.1, \quad I(0) = 0.1.
\]

We start by illustrating the extinction result proved in Theorem 3.1. The transmission coefficient is assumed to take the value \( \beta_0 = 0.005 \) and the intensity of the Brownian motion to take the value \( \sigma^2 = 0.01 \). The inequality \( \sigma^2 > \frac{\beta}{2 \mu^2} \) from Theorem 3.1 is satisfied: \( \sigma^2 - \frac{\beta}{2 \mu^2} = 0.009 > 0 \). The rest of the parameters take the following values based on [3] and references cited therein: \( \mu = 1/69.54, \Lambda = 2.1 \mu, \eta_C = 0.015, \eta_A = 1.3, \phi = 1, \rho = 0.1, \alpha = 0.33, \) and \( \omega = 0.09 \). The extinction of the stochastic model is observed numerically in Figure 1. To illustrate the persistence result proved in Theorem 4.1, we consider the parameter values

\[
\mu = 1/69.54, \Lambda = 2.1 \mu, \eta_C = 0.015, \eta_A = 1.3, \phi = 1, \rho = 0.1, \omega = 0.99 \text{ and } \alpha = 0.99. \text{ We take } \beta = 0.4 \text{ and } \sigma^2 = 0.01. \text{ For these parameter values, one has } \frac{1}{K_1} \left( \frac{\Lambda \beta}{\mu} - \xi_1 \xi_2 - \frac{\sigma^2 A^2}{2 \mu^2} \right) = 0.3012 > 0. \text{ In Figure 2, we observe the persistence of the disease.}
\]

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Figure 2: Infected individuals $I$: stochastic (dashed line) and deterministic (continuous line) cases.

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