LONG–TIME ASYMPTOTICS FOR SOLUTIONS OF THE NLS EQUATION
WITH INITIAL DATA IN A WEIGHTED SOBOLEV SPACE

Percy Deift
New York University

Xin Zhou
Duke University

§1. Introduction

Long–time asymptotics for solutions \( q(x,t) \) of the defocusing Nonlinear Schrödinger (NLS) Equation

\begin{align}
\begin{cases}
  iq_t + q_{xx} - 2|q|^2q = 0 \\
  q(t = 0, x) = q_0(x) \to 0 \text{ as } |x| \to \infty
\end{cases}
\end{align}

have been obtained in [ZaMa][DIZ][DZ2][DZ4] for initial data with sufficient smoothness and decay: as \( t \to \infty \)

\begin{align}

q(x,t) = t^{-1/2} \alpha(z_0) e^{ix^2/(4t) - in(z_0) \log 2t} + O(\log t/t),
\end{align}

where

\begin{align}

\nu(z) = -\frac{1}{2\pi} \log(1 - |r(z)|^2), \quad |\alpha(z)|^2 = \nu(z)/2,
\end{align}

and

\begin{align}

\arg \alpha(z) = \frac{1}{\pi} \int_{-\infty}^z \log(z - s) d(\log(1 - |r(s)|^2)) + \frac{\pi}{4} + \arg \Gamma(i\nu(z)) + \arg r(z).
\end{align}

Here \( \Gamma \) is the gamma function and the function \( r \) is the so-called reflection coefficient for the potential \( q_0(x) = q(x,t = 0) \), as described below. The error term \( O(\log t/t) \) is uniform for all \( x \in \mathbb{R} \). The above asymptotic form was first obtained in [ZaMa], but without the error estimate. Based on the nonlinear steepest descent method introduced in [DZ1], the error estimate in (1.1) was derived in [DIZ] (see also [DZ2] for a pedagogic presentation). As noted above, some high orders of decay and smoothness are required for the initial data. In this paper we describe a new method that produces an error estimate of order \( O(t^{-\frac{1}{2} - \kappa}) \) for any \( 0 < \kappa < \frac{1}{4} \), just under the assumption that the initial data \( q_0 \) lies in the weighted Sobolev space

\begin{align}

H^{1.1} = \{ f \in L^2(\mathbb{R}) : x f, x f' \in L^2(\mathbb{R}) \}.
\end{align}

Such an estimate is needed, for example, in [DZ3][DZ5] where the authors obtain long-time asymptotics for solutions of the perturbed NLS equation, \( iq_t + q_{xx} - 2|q|^2q - \epsilon |q|^4 q = 0 \), for \( l > 2 \) and \( \epsilon > 0 \). As we will see (cf. Section 4 below), the estimate \( O(t^{-\frac{1}{2} - \kappa}) \) in fact depends only on the weighted \( L^2 \) norm of the initial data, \( (\int_{\mathbb{R}} (1 + x^2)|q_0(x)|^2 dx)^{1/2} < \infty \). The estimate is (essentially) optimal even in the linear case where the standard Fourier method produces an error of order \( O(t^{-3/4}) \). Our new method is a further development of the steepest descent method of [DZ1], and replaces certain key absolute type estimates in [DZ1] with cancellations from oscillations.
The NLS equation can be integrated by using the familiar scattering theory/inverse scattering theory for the ZS–AKNS system [ZS][AKNS] associated to NLS,

\[
\partial_x \psi = \left( i z \sigma + \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix} \right) \psi
\]

(1.3)

where \( \sigma = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \). As is well known [ZS], the NLS equation is equivalent to an isospectral deformation of the operator \( \partial_x - \left( i z \sigma + \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix} \right) \). As described in Section 3 below, for each \( z \in \mathbb{C} \setminus \mathbb{R} \), one constructs solutions \( \psi(x, z) \) of (1.3) of the type considered in [BC] with the following properties:

(a) \( m(x, z) \equiv \psi(x, z) e^{-izx} \to I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) as \( x \to -\infty \),

(b) \( m(x, z) \) is bounded as \( x \to +\infty \).

For each fixed \( x \), the \( 2 \times 2 \) matrix function \( m(x, z) \) solves the following Riemann-Hilbert problem (RHP) in \( z \):

\[
\begin{align*}
\text{(i)} & \quad m(x, z) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}, \\
\text{(ii)} & \quad m_+(x, z) = m_-(x, z) v_x(z), \quad z \in \mathbb{R},
\end{align*}
\]

where \( m_\pm(x, z) = \lim_{\epsilon \to 0} m(x, z \pm i \epsilon) \) and \( v_x(z) = \begin{pmatrix} 1 - |r(z)|^2 & r(z)e^{izx} \\ -r(z)e^{-izx} & 1 \end{pmatrix} \) where \( r(z) \) is the reflection coefficient of \( q \).

(iii) \( \lim_{z \to \infty} m(x, z) = I \).

The sense in which the limits in (ii) and (iii) are achieved will be made precise in Sections 2 and 3. The reflection coefficient satisfies the important \textit{a priori} bound \( \|r\|_{L^\infty(dx)} < 1 \).

If we expand out the limit in (iii),

\[
m(x, z) = I + \frac{m_1(x)}{z} + o \left( \frac{1}{z} \right),
\]

(1.5)

then we obtain an expression for \( q \),

\[
q(x) = -i (m_1(x))_{12}.
\]

(1.6)

The direct scattering map \( \mathcal{R} \) is obtained by mapping \( q \mapsto r \),

\[
q \mapsto m(x, z) = m(x, z; q) \mapsto v_x(z) \mapsto r = \mathcal{R}(q).
\]

Given \( r \), the inverse scattering map \( \mathcal{R}^{-1} \) is obtained by solving the RHP (1.4) and mapping to \( q \) via (1.6),

\[
r \mapsto \text{RHP} \mapsto m(x, z) = m(x, z; r) \mapsto m_1(x) \mapsto q = \mathcal{R}^{-1}(r).
\]

The remarkable fact discovered in [ZS] is that if \( q(t) = q(x, t) \) solves the NLS equation, then \( r(t) = \mathcal{R}(q(t)) \) evolves simply as

\[
r(t) = r(z, t) = e^{-itz^2} r_0(z),
\]

(1.7)

where \( r_0 = \mathcal{R}(q_0 = q(t = 0)) \). Using \( \Diamond \) to denote a dummy variable, we may rewrite (1.7) as a formula

\[
q(t) = \mathcal{R}^{-1}(e^{-i\Diamond^2t} r(\Diamond, t = 0)),
\]

(1.8)

for the solution of NLS. This formula shows that the problem of computing the asymptotics of \( q(t) \) as \( t \to \infty \), reduces to the problem of analyzing the map \( \mathcal{R}^{-1} \), and it was precisely for this purpose that the nonlinear steepest descent method was introduced in [DZ1] in the context of the modified Korteweg deVries equation.

In order to proceed, we must specify the domain and range of \( \mathcal{R} \). Many authors have commented that \( \mathcal{R} \) is a nonlinear Fourier-type map and indeed the results in [BC] imply that \( \mathcal{R} \) is a bijection from Schwartz
Remark on Notation (1.11). If \( A = (a_{ij}) \) is an \( l \times m \) matrix, we define \( |A| \equiv \left( \sum_{i,j} |a_{ij}|^2 \right)^{1/2} = (\text{tr} \ A^* A)^{1/2} \). We say \( A(z) = (a_{ij}(z)) \) is in \( L^p(\Sigma) \) for some contour \( \Sigma \subset \mathbb{C} \) if each of the entries \( a_{ij}(z) \in L^p(\Sigma) \) and we define \( \|A\|_{L^p(\Sigma)} \equiv \|A\|_{L^p(\Sigma)}. \)

Throughout the text constants \( c > 0 \) are used generically. Statements such as \( \|f\| \leq 2(c + ec') \leq c \), for example, should not cause any confusion.
Acknowledgements. The authors would like to thank Jalal Shatah and Jean Bourgain for useful discussions on the well-posedness of equation (1.1) in $L^2$. The work of the authors was supported in part by NSF Grants DMS 0003268 and DMS 0071398 respectively.

§2 INHOMOGENEOUS RIEMANN-HILBERT PROBLEMS

We begin by summarizing the basic properties of the Cauchy operator

\begin{equation}
Ch(z) = C_T h(z) \equiv \int_{\Gamma} \frac{h(s)}{s-z} \frac{ds}{2\pi i}, \quad z \in \mathbb{C} \setminus \Gamma,
\end{equation}

for an oriented contour $\Gamma$ in the plane. To fix notation, if we move along the contour in the direction of the orientation, we say that the (+)-side (resp. (−)-side) lies to the left (resp. right). We have, for example, the following figure:

![Figure 2.2](image)

The properties and estimates that we present below are true for a very general class of contours, which certainly includes contours that are finite unions of smooth curves in $\mathbb{C}$. In particular, the results are true for contours which are finite unions of straight lines, which is all that is needed for this paper. We refer the reader to standard texts on the subject, such as [Dur], for the proofs of the results that follow.

Suppose $\Gamma$ is given.

(2.3) Let $h \in L^p(\Gamma, |dz|), 1 \leq p < \infty$. Then

$$C^\pm h(z) \equiv \lim_{z' \to z} (Ch)(z')$$

exists as a non-tangential limit for a.e. $z \in \Gamma$. As usual, “non-tangential” means that $z' \to z$ in any fixed (truncated) cone $C$ based at $z$ with $\overline{C} \setminus \{z\}$ lying entirely on $\mathbb{C} \setminus \Gamma$.

(2.4) Let $h \in L^p(\Gamma, |dz|), 1 < p < \infty$. Then

$$\|C^\pm h\|_{L^p(\Gamma)} \leq c_p \|h\|_{L^p(\Gamma)}$$

for some constant $c_p$. Moreover

$$C^\pm h = \pm \frac{1}{2} h - \frac{1}{2} (Hf)$$
where \[ Hh(z) \equiv \lim_{\epsilon \downarrow 0} \int_{\Gamma} \frac{h(s)}{z-s} \frac{ds}{i\pi}, \quad z \in \Gamma, \]
is the Hilbert transform. The above limit exists a.e. on \( \Gamma \) and also in \( L^p(\Gamma) \), and we have
\[ \|Hh\|_{L^p(\Gamma)} \leq c'_p \|h\|_{L^p(\Gamma)} \]
for some constant \( c'_p \). Clearly
\[ C^+ - C^- = 1 \quad \text{and} \quad C^+ + C^- = -H. \]

(2.5) If \( \gamma \) is a smooth open segment lying in \( \Gamma \),

\[
\gamma
\]

and \( h \in H^1_p(\gamma) = \{ f : f \in L^p(\gamma,|dz|), \partial_z f \in L^p(\gamma,|dz|) \}, \ 1 < p < \infty \), then \( Ch(z) \) is continuous up to \( \gamma \). Thus the limits \( C^\pm h(z) \) in (2.3) exists for all \( z \in \gamma \) (and without the restriction of non-tangential convergence). Furthermore, for \( h \in H^1_p(\Gamma) \), \( Ch(z) \) is bounded in \( C \setminus \Gamma \) and goes to zero uniformly as \( z \to \infty \).

(2.7) It follows, in particular, from (9.3) that if \( \Gamma_1 \) and \( \Gamma_2 \) are two contours in \( \mathbb{C} \), then for \( 1 < p < \infty \)
\[ \|C^\pm h\|_{L^p(\Gamma_2)} \leq c_p \|h\|_{L^p(\Gamma_1)} \]
for \( h \in L^p(\Gamma_1) \). For example, if \( \Gamma_1 = \mathbb{R} \) and \( \Gamma_2 = z_0 + \mathbb{R}_+ e^{i\theta} \) for \( z_0 \in \mathbb{R}, \ 0 < \theta < \pi \). Then
\[ \|C^\pm h\|_{L^p(z_0+\mathbb{R}_+ e^{i\theta})} \leq c_p \|h\|_{L^p(\mathbb{R})} \]
for \( h \in L^p(\mathbb{R}) \). Moreover, it is not hard to see that the constant \( c_p \) can be chosen independent of \( z_0 \in \mathbb{R} \) and \( 0 < \theta < \pi \).

Throughout this section we will assume, without further comment, that \( \Gamma \) is a contour for which the above estimates are true. We now present some of the basic facts about inhomogeneous RHP’s. Many of the results are well known and can be found, implicitly or explicitly, in standard texts on the subject (see, for example, [CG]). Other results are new and are tailored specifically to the analysis of RHP’s with external parameters, of the kind that will arise further on in the paper. For the convenience of the reader who is interested in applying the theory of inhomogeneous RHP’s in different situations, we present the theory in greater generality than is needed in the present text.
Consider an oriented contour $\Sigma \subset \mathbb{C}$ as above with a $k \times k$ jump matrix $v$: as a standing assumption throughout the text, we always assume that $v, v^{-1} \in L^\infty(\Sigma \to GL(k, \mathbb{C}))$. We associate to the pair $\Sigma, v$ two inhomogeneous RHP’s as follows.

For $1 < p < \infty$, let $Ch(z) = C_\Sigma h(z) = \int_{\Sigma} \frac{h(s) \, ds}{z - s}$ be the Cauchy operator on $\Sigma$. Then we say that a pair of $L^p(\Sigma)$-functions $f_\pm \in \partial C(L^p)$ if there exists a (unique) function $h \in L^p(\Sigma)$ such that

\begin{equation}
    f_\pm(z) = (C^\pm h)(z), \quad z \in \Sigma.
\end{equation}

In turn we will call $f(z) = Ch(z), z \in \mathbb{C} \setminus \Sigma$, the extension of $f_\pm = C^\pm h \in \partial C(L^p)$ off $\Sigma$.

**Inhomogeneous RHP of the first kind IRHP1**

Fix $1 < p < \infty$. Given $\Sigma, v$ and a function $f$, we say that $m_\pm \in f + \partial C(L^p)$ solves an IRHP1$_{L^p}$ if

\[ m_+(z) = m_-(z)v(z), \quad z \in \Sigma. \]

**Inhomogeneous RHP of the second kind IRHP2**

Fix $1 < p < \infty$. Given $\Sigma, v$ and a function $F \in L^p(\Sigma)$, we say that $M_\pm \in \partial C(L^p)$ solves an IRHP2$_{L^p}$ if

\[ M_+(z) = M_-(z)v(z) + F(z), \quad z \in \Sigma. \]

Recall that $m$ solves the normalized RHP $(\Sigma, v)$ if, at least formally (cf. 1.4) above,

- $m(z)$ is analytic in $\mathbb{C} \setminus \Sigma$
- $m_+(z) = m_-(z)v(z), z \in \Sigma$
- $m(z) \to I$ as $z \to \infty$.

More precisely, we make the following definition.

**Definition 2.9.** Fix $1 < p < \infty$. We say that $m_\pm$ solves the normalized RHP $(\Sigma, v)\_L^p$ if $m_\pm$ solves the IRHP1$_{L^p}$ with $f \equiv I$.

In the above definition, if $m_\pm - I = C^\pm h$, then clearly the extension $m = I + Ch$ of $m_\pm$ off $\Sigma$ solves the above RHP in the formal sense. If $p = 2$, which is mostly the case of interest, we will drop the subscript and simply write IRHP1, IRHP2 and $(\Sigma, v)$. Let $v = (v^-)^{-1}v^+ = (I - w^-)^{-1}(I + w^+)$ be a factorization of $v$ with $v^\pm, (v^\pm)^{-1} \in L^\infty$, and let $C_w, w = (w^-, w^+)$, denote the basic, associated singular integral operator

\begin{equation}
    C_wh = C^+(hw^-) + C^- (hw^+)
\end{equation}

acting on $L^p(\Sigma)$-matrix-valued functions $h$. As $w^\pm \in L^\infty$, $C_w$ is clearly bounded from $L^p \to L^p$ for all $1 < p < \infty$. The utility of IRHP1$_{L^p}$ and IRHP2$_{L^p}$ lies in the following circle of ideas. As we shall shortly see, the solution $m_\pm$ of an IRHP1$_{L^p}$ is directly related to the inverse of $1 - C_w$. However, for general $f$ (as opposed to the case $f \equiv I$ in the normalized RHP above), $m_\pm$ does not have an analytic continuation to $\mathbb{C} \setminus \Sigma$. On the other hand, the solution $M_\pm$ of an IRHP2$_{L^p}$ corresponds to the boundary values of an analytic function. The key point, as we shall see, is that under appropriate assumptions on $v$ and $f$, IRHP1$_{L^p}$ and IRHP2$_{L^p}$ are equivalent, and hence steepest-descent type methods applied directly to $M_\pm$ can be used to control the resolvent $(1 - C_w)^{-1}$.

Now suppose $f$ and $v$ are such that $f(v - I) \in L^p(\Sigma)$ for some $1 < p < \infty$. Let $M_\pm$ solve the IRHP2$_{L^p}$ with $F = f(v - I)$. Then simple algebra shows that $m_\pm = M_\pm + f$ solves the IRHP1$_{L^p}$, $m_+ = m_- v$, $m_\pm - f \in \partial C(L^p)$. If we reverse the argument, we see that if $m_\pm$ solves the IRHP1$_{L^p}$ with any given $f$, then $M_\pm = m_\pm - f$ solves IRHP2$_{L^p}$ with $F$ of the form $f(v - I)$ which, however, is not the general case as $v - I$ need not be invertible. To prove the equivalence of IRHP1$_{L^p}$ and IRHP2$_{L^p}$, we must proceed in a different way. Given $F \in L^p(\Sigma)$, let $m_\pm$ solve IRHP1$_{L^p}$ with $f = C^- F$. By assumption, $m_\pm - f = C^\pm h$ for some $h \in L^p(\Sigma)$, and hence

\[ m_- = f + C^- h = C^- (F + h), \]
\[ m_+ + F = f + C^+ h + (C^+ - C^-)F = C^+(F + h). \]

It follows that $M_+ = m_+ + F, M_- = m_-$ solve IRHP2$_{L^p}$ with the given $F$. We have proved the following result.
Proposition 2.11 (Equivalence of IRHP1_{L^p} and IRHP2_{L^p}). Suppose f and v are such that f(v - I) ∈ L^p(Σ) for some 1 < p < ∞. Then

\begin{equation}
1 - C_w \text{ is a bijection in } L^p(Σ)
\end{equation}

\[ m_± = M_± + f \]
solves IRHP1_{L^p} with the given f, if M_± solve IRHP2_{L^p} with F = f(v - I). Conversely, if F ∈ L^p(Σ), then

\begin{equation}
M_+ = m_+ + F, \quad M_- = m_-
\end{equation}
solve IRHP2_{L^p} with the given F, if m_± solve IRHP1_{L^p} with f = C^{-1}F.

We will be interested principally in the situation where f ∈ L^∞(Σ) and v - I ∈ L^p(Σ) for some 1 < p < ∞. To establish the connection with the inverse of 1 - C_w, let m_± solve IRHP1_{L^p} with f ∈ L^p(Σ). Thus m_± = f + C^±h for some h ∈ L^p(Σ). Also m_+ = m_-v ≡ m_-(v^-)^{1/2}v^+. Set μ = m_+(v^+) = m_-(v^-)^{-1} ∈ L^p(Σ), and define H(z) = (C(μ(w^+ + w^-))(z), z ∈ C \ Σ. Then a simple calculation shows that H_±(z) = (C_w - 1)μ_m_±, or m_± - f - H_± = (1 - C_w)μ - f. But m_± - f - H_± ∈ ∂C(L^p), and it follows that (1 - C_w)μ = f. Conversely, if μ ∈ L^p(Σ) satisfies (1 - C_w)μ = f, then H = C(μ(w^+ + w^-)) satisfies H_± = -f + μv^±. Thus setting m_± = μv^±, we see that m_+ = m_-v and m_± - f ∈ ∂C(L^p). In particular, μ ∈ L^p solves (1 - C_w)μ = 0 if and only if m_± = μv^± solves the homogeneous RHP

\[ m_+ = m_-v, \quad m_± ∈ ∂C(L^p). \]

We summarize the above calculations as follows.

Proposition 2.14. Let 1 < p < ∞. Then

\begin{equation}
1 - C_w \text{ is a bijection in } L^p(Σ)
\end{equation}

\[ IRHP1_{L^p} \text{ has a unique solution for all } f ∈ L^p(Σ) \]

\[ IRHP2_{L^p} \text{ has a unique solution for all } F ∈ L^p(Σ). \]

Moreover, if (1 - C_w)^{-1} exists, then for f ∈ L^p(Σ),

\begin{equation}
(1 - C_w)^{-1}f = m_+(v^+)^{-1} = m_-(v^-)^{-1}
\end{equation}

\[ = (M_+ + f)(v^+)^{-1} = (M_- + f)(v^-)^{-1} \]

where m_± solve IRHP1_{L^p} with the given f, and M_± solve IRHP2_{L^p} with F = f(v - I). Conversely, if M_± solves IRHP2_{L^p} with F ∈ L^p(Σ), then

\begin{equation}
M_+ = ((1 - C_w)^{-1}C^{-1}F)v^+ + F \quad \text{and} \quad M_- = ((1 - C_w)^{-1}C^{-1}F)v^-.
\end{equation}

Finally, if f ∈ L^∞(Σ) and v^± - I ∈ L^p(Σ), then (2.15) remains valid provided we interpret

\begin{equation}
(1 - C_w)^{-1}f = f + (1 - C_w)^{-1}(C_w f).
\end{equation}

This is true in particular for the normalized RHP (Σ, v)_p where f ≡ I.

The above Proposition implies, in particular, that if μ ∈ I + L^p(Σ) solves

\begin{equation}
(1 - C_w)μ = I
\end{equation}

as in (2.17), then m_± = μv^± solves the normalized RHP (2.9).

The fact that IRHP1_{L^p} and IRHP2_{L^p} depend on v, and not on the particular factorization v = (v^-)^{-1}v^+, has the following immediate consequence.
Corollary to Proposition 2.14. Let as $1 < p < \infty$. The operator $1 - C_w$ is bijective in $L^p(\Sigma)$ for all factorizations $v = (v^-)^{-1}v^+ = (I - w)^{-1}(I + w^+)$, if and only if $1 - C_{w'}$ is bijective for at least one factorization $v = (v')^{-1}v' = (I - w')^{-1}(I - w'^+)$. Moreover, for $f \in L^p(\Sigma)$

$$(1 - C_w)^{-1}f = [(1 - C_{w'})^{-1}f]b$$

where $b = v'^+(v^+)^{-1} = v''(v^-)^{-1}$.

We now describe a useful and simple fact about the operator $C_w$. Given $\Sigma$ and $v$, suppose we reverse the orientation of $\Sigma$ on some subset $\gamma \subset \Sigma$. For example, consider the following figure.

![Figure 2.19](image)

If we denote the new contour by $\hat{\Sigma}$ and set $\hat{v} \equiv v$ on $\Sigma \setminus \gamma$, $\hat{v} \equiv v^{-1}$ on $\gamma$, then we clearly have equivalent RHP’s on $\Sigma, v$ and on $\hat{\Sigma}, \hat{v}$. Furthermore, a factorization $v = (v^-)^{-1}v^+$, leads to the natural factorization $\hat{v} = (v^-)^{-1}\hat{v}^+$ on $\Sigma \setminus \gamma$ and $\hat{\delta} = (v^+)^{-1}v^-$ on $\gamma$. Thus, where $v^\pm = I \pm w^\pm$, $\hat{v}^\pm = I \pm \hat{w}^\pm$, we have $\hat{w}^\pm = w^\mp$ on $\Sigma \setminus \gamma$, $\hat{w}^\pm = -w^\mp$ on $\gamma$.

**Proposition 2.20.** Let $\hat{\Sigma}, \hat{v}$ be defined as above. Then

$$(2.21) \quad C_{\hat{\Sigma}} = C_w$$

in $L^p(\Sigma) = L^p(\hat{\Sigma})$ for any $1 < p < \infty$.

**Proof.** Let $h \in L^p(\Sigma) = L^p(\hat{\Sigma})$. Then it follows directly from the above definitions that $C_{\Sigma}(h(w^+ + w^-)) = C_{\hat{\Sigma}}(h(\hat{w}^+ + \hat{w}^-))$, where $C_{\Sigma}, C_{\Sigma}$ denote the Cauchy operators on the (oriented) contours $\Sigma, \hat{\Sigma}$ respectively. Now

$$(2.22) \quad C_{\Sigma}^+(h(w^+ + w^-)) = C_{\Sigma}^- h(w^+ + w^-) = C_{\Sigma} h(w^+ + w^-).$$

On $\Sigma \setminus \gamma$, $C_{\Sigma}^+(h(w^+ + w^-)) = C_{\Sigma}^-(h(\hat{w}^+ + \hat{w}^-))$ and $w^\pm = \hat{w}^\mp$, and it follows that $C_{\Sigma} h = C_{\Sigma} h$. But on $\gamma$, $C_{\Sigma}^+(h(w^+ + w^-)) = C_{\Sigma}^-(h(\hat{w}^+ + \hat{w}^-))$ and $w^\pm = -\hat{w}^\mp$, and it follows again that $C_{\Sigma} h = C_{\Sigma} h$. The proof is done. \(\square\)

Finally, we consider uniqueness for the solution of the normalized RHP $(\Sigma, v)_p$ as given in Definition 2.9. Observe first that if $F(z) = (Cf)(z)$ for $f \in L^p(\Sigma)$ and $G(z) = (Cg)(z)$ for $g \in L^q(\Sigma)$, $\frac{1}{p} + \frac{1}{q} \leq 1$, then a simple computation shows that

$$(2.22) \quad (FG)(z) = (Ch)(z)$$

where

$$(2.23) \quad h(s) = -\frac{1}{2}(g(s)(Hf)(s) + f(s)(Hg)(s))$$

and $Hf(s) = \lim_{\epsilon \rightarrow 0^+} \int_{|s - s'| > \epsilon} \frac{f(s')}{s - s'} ds'$, $Hg(s) = \lim_{\epsilon \rightarrow 0^+} \int_{|s - s'| > \epsilon} \frac{g(s')}{s - s'} ds'$ denote the Hilbert transforms of $f$ and $g$ respectively (see (2.4)). As $h$ clearly lies in $L^r(\Sigma)$, it follows that

$$(2.24) \quad F_+(z)G_+(z) - F_-(z)G_-(z) = h(z)$$

for a.e. $z \in \Sigma$. It is now easy to prove uniqueness for the normalized RHP in the following form.
Theorem 2.25. Fix $1 < p < \infty$. Suppose $m(z)$ solves the normalized RHP $(\Sigma, v)_{L^p}$ in the sense of Definition 2.9, and suppose in addition that $(m(z))^{-1}$ exists for all $z \in \mathbb{C}\setminus\Sigma$ and that $(m^{-1})_\pm - I \in \mathcal{O}(L^q)$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} \leq 1$. Then $m$ is unique.

Proof. Suppose $\hat{m}$ is a second solution of the normalized RHP. Then arguing as above, $\hat{m}m^{-1} - I = (\hat{m} - I)(m^{-1} - I) + (\hat{m} - I) + (m^{-1} - I) = Ch$ where $h \in L^p(\Sigma) + L^q(\Sigma) + L^q(\Sigma)$, and $\hat{m}m^{-1} - \hat{m}m^{-1} = h$. But $\hat{m}m^{-1} = (\hat{m} - v)(m - v)^{-1} = \hat{m}m^{-1}$ a.e. on $\Sigma$, and hence $h = 0$. Thus $\hat{m}m^{-1} - I = 0$, or $\hat{m} = m$. \hfill $\square$

Remark. Observe that in the above proof we did not need to assume that also $(\hat{m}^{-1})_\pm - I \in \mathcal{O}(L^q)$.

In the special case in which $k = 2, p = 2$ and $\det v(z) = 1$ a.e. on $\Sigma$, there is the following stronger result.

Theorem 2.26. If $k = 2, p = 2$ and $\det v(z) = 1$ a.e. on $\Sigma$, then the solution $m$ of the normalized RHP $(\Sigma, v) = (\Sigma, v)_{L^2}$ is unique.

Proof. As $k = 2$ and $p = 2$, (2.22) and (2.23) imply that $m(z) - 1 = (Ch)(z)$ where $h \in L^1(\Sigma) + L^2(\Sigma)$, and $dt m_+(z) - dt m_-(z) = h(z)$ a.e. on $\Sigma$. But $dt m_+(z) = dt m_-(z) dt v(z) = dt m_-(z)$, and hence $h = 0$ and so $dt m(z) = 1$. But if $m = (m_{ij})_{1 \leq i, j \leq 2}$, then $m^{-1} = \left( \begin{array}{c} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{array} \right)$ and clearly $(m^{-1})_\pm - I \in \mathcal{O}(L^2)$. The result then follows from Theorem 2.25. \hfill $\square$

The above result is of considerable general interest in practice, and applies, in particular, to the RHP (1.4) for the NLS equation. But in the case $\Sigma = \mathbb{R}$ with the special jump matrix $v_x$ in (1.4) uniqueness and existence for the normalized RHP follows immediately from the following observation. Write

\begin{equation}
(1 - r e^{i2\pi x})^{-1} = (I - w_x^*)^{-1}(I + w_x^+)
\end{equation}

so that

\begin{equation}
(w_x^-, w_x^+) = \left( \begin{array}{c} 0 & r e^{i2\pi x} \\ 0 & 0 \end{array} \right), \left( \begin{array}{c} 0 & 0 \\ -r e^{-i2\pi x} & 0 \end{array} \right).
\end{equation}

Then for $C_{w_x}$ in (2.10), and $h = (h_{ij})_{1 \leq i, j \leq 2}$, $C_{w_x} h = \left( \begin{array}{c} C^{-}(h_{12} r e^{-i2\pi x}) \\ C^{-}(h_{21} r e^{-i2\pi x}) \end{array} \right)$ and hence

\begin{equation}
\|C_{w_x} \|_{L^2} \leq \|r\|_{L^\infty} \|h\|_{L^2}
\end{equation}

But, under the Fourier transform, $C^+$ (resp. $-C^-$) is just multiplication by the characteristic function of $(0, \infty)$ (resp. $(-\infty, 0)$) and hence $\|C^\pm\|_{L^2(\mathbb{R})} = 1$. Thus (recall Remark (1.10)), $\|C_{w_x} \|_{L^2} \leq \|r\|_{L^\infty} \|h\|_{L^2}$ and hence

\begin{equation}
\|C_{w_x} \|_{L^2} \leq \|r\|_{L^\infty}.
\end{equation}

As noted in the Introduction, we always have $\|r\|_{L^\infty} < 1$, and it follows that $(1 - C_{w_x})^{-1}$ exists and is uniformly bounded on $L^2$ for all $x \in \mathbb{R}$,

\begin{equation}
\|(1 - C_{w_x})^{-1}\|_{L^2} \leq \frac{1}{1 - \|r\|_{L^\infty}}
\end{equation}

By Section 3, we always have $r \in L^2(\mathbb{R})$. It follows then by Proposition 2.14 that we have proved the following result.
Proposition 2.31. The normalized RHP \((\Sigma = \mathbb{R}, v_x)\) has a unique solution \(m_{\pm} \in I + \partial C(L^2)\) for each \(x \in \mathbb{R}\). Moreover

\[
m_{\pm} = I + C^\pm(\mu(w_x^+ + w_x^-)),
\]

where \(\mu \in I + L^2(\mathbb{R})\) is the unique solution of

\[
(1 - C_w)\mu = I.
\]

In the remainder of this section, we use the relationships between various inhomogeneous RHP’s to establish a uniform bound (see Proposition 2.49 below) of type (2.30) for an associated model RHP which plays a key role in proving Theorem 1.10. We refer the reader to §4 for an explanation of the origin and relevance of the model.

By (1.7)(1.8) we need to consider the normalized RHP (1.4) with a time-dependent reflection coefficient \(r(t) = e^{-iz^2}r\) and hence a space and time dependent jump matrix

\[
v_\theta = \begin{pmatrix}
1 - |r(z)|^2 & r(z)e^{i\theta} \\
-r(z)e^{-i\theta} & 1
\end{pmatrix},
\]

where

\[
\theta = xz - tz^2.
\]

Throughout the paper, if \(A\) is a \(2 \times 2\) matrix, then \(A_\theta\) denotes \(e^{i\theta} ad A = e^{i\theta} A e^{-i\theta}\), where \(\sigma = \text{diag}(1/2, -1/2)\) as before. In this notation, \(v_\theta = e^{i\theta} ad v\), where \(v = \begin{pmatrix}
1 - |r(z)|^2 & r(z) \\
-r(z) & 1
\end{pmatrix} \) on \(\partial C\).

Let \(z_o = x/(2t)\) denote the stationary phase point for \(e^{i\theta}\). For a function \(f\) on \(\mathbb{R}\) we introduce the notation

\[
[f](z) = \frac{f(z_0)}{(1 + i(z - z_0))^2}, \quad z \in \mathbb{R}.
\]

Clearly \([f](z_0) = f(z_0)\).

Let \(\delta_{\pm}\) be the solution of the scalar, normalized RHP \((\mathbb{R}_- + z_0, 1 - |r|^2)\),

\[
\begin{cases}
\delta_+ = \delta_- (1 - |r|^2), & z \in \mathbb{R}_- + z_0, \\
\delta_+ - 1 \in \partial C(L^2),
\end{cases}
\]

where the contour \(\mathbb{R}_- + z_0\) is oriented from \(-\infty\) to \(z_0\). The properties of \(\delta\) can be read off from the following elementary proposition, whose proof is left to the reader.

Proposition 2.38. Suppose \(r \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})\) and \(\|r\|_{L^\infty} \leq \rho < 1\). Then the solution \(\delta_{\pm}\) of the scalar normalized RHP (2.37) exists and is unique and is given by the formula

\[
\delta_{\pm}(z) = e^{C_{\mathbb{R}_+ + z_0} \log(1 - |r|^2)} = e^{\frac{i\pi}{4\gamma} \int_{-\infty}^{z_0} \frac{\log(1 - |r(s)|^2)}{s - z} ds}, \quad z \in \mathbb{R}.
\]

The extension \(\delta\) of \(\delta_{\pm}\) off \(\mathbb{R}_- + z_0\) is given by

\[
\delta(z) = e^{C_{\mathbb{R}_+ + z_0} \log(1 - |r|^2)} = e^{\frac{i\pi}{4\gamma} \int_{-\infty}^{z_0} \frac{\log(1 - |r(s)|^2)}{s - z} ds}, \quad z \in \mathbb{C} \setminus (\mathbb{R}_- + z_0),
\]

and satisfies for \(z \in \mathbb{C} \setminus (\mathbb{R}_- + z_0),\)

\[
\delta(z)\overline{\delta(z)} = 1,
\]

\[
(1 - \rho)^{\frac{1}{2}} \leq (1 - \rho^2)^{\frac{1}{2}} \leq |\delta(z)|, |\delta^{-1}(z)| \leq (1 - \rho)^{-\frac{1}{2}} \leq (1 - \rho)^{-\frac{1}{2}};
\]
and
\begin{equation}
|\delta^\pm(z)| \leq 1 \quad \text{for} \quad \pm \text{Im} \ z > 0.
\end{equation}

For real \( z \),
\begin{equation}
|\delta_+(z)\delta_-(z)| = 1 \quad \text{and, in particular,} \quad |\delta(z)| = 1 \quad \text{for} \quad z > z_0,
\end{equation}
and
\begin{equation}
|\delta_+(z)| = |\delta_-(z)| = (1 - |r(z)|^2)^{\frac{1}{2}}, \quad z < z_0,
\end{equation}
and
\begin{equation}
\Delta \equiv \delta_+\delta_- = e^{P.V. \int_{-\infty}^{z_0} \frac{\log(1 - |r(s)|^2)}{s - z} ds}, \quad \text{where P.V. denotes the principal value.}
\end{equation}

Also \( |\Delta| = |\delta_+\delta_-| = 1 \).

\begin{equation}
\|\delta_+ - 1\|_{L^2(dz)} \leq c\|r\|_{L^2} \frac{1}{1 - \rho}.
\end{equation}

For a reflection coefficient \( r(z) \), \( \|r\|_{L^\infty} < 1 \), set
\begin{equation}
\hat{w} = (\hat{w}^-, \hat{w}^+) = \begin{cases} 
\left( \begin{array}{cc} 0 & [r] \delta^2 \\ 0 & 0 \\ -[\bar{r}] \delta^{-2} & 0 \\ 0 & 0 \\ 0 & [1 - |r|^2] \delta^2 \\ 0 & 0 \\
\end{array} \right), & z > z_0, \\
\left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \\ -[\bar{r}] \delta^{-2} & 0 \\ 0 & 0 \\
\end{array} \right), & z < z_0,
\end{cases}
\end{equation}

and as above, set \( \hat{w}_\theta = e^{i\theta \text{ad} \sigma} \hat{w} = (e^{i\theta \text{ad} \sigma} \hat{w}^-, e^{i\theta \text{ad} \sigma} \hat{w}^+) \). We consider the model normalized RHP (\( \Sigma = \mathbb{R}, \bar{\sigma}_0 \)), where \( \bar{\sigma}_0 = (I - \hat{w}_\theta)^{-1}(I + \hat{w}_\theta^+) \). Our goal is to prove the following analog of the resolvent bound in (2.30).

**Proposition 2.49.** For \( t \) sufficiently large, say \( t \geq t_0 \), \((1 - C\hat{w}_\theta)^{-1} \) exists in \( L^2(\mathbb{R}) \) and for some constant \( c \),
\begin{equation}
\| (1 - C\hat{w}_\theta)^{-1} \|_{L^2} \leq c
\end{equation}
for all \( x \in \mathbb{R} \) and for all \( t \geq t_0 \).

In order to understand the origin of this bound, consider \( z \) close to \( z_0 \), so that \([r](z) \sim r(z)\) etc., a direct calculation using the jump relation (2.37) shows that \( \hat{w}_\theta(z) \sim \delta^\pm v_\theta \delta^\mp \), where \( v_\theta \) is given in (2.34) and \( \sigma_3 = 2\sigma = \text{diag}(1, -1) \) is the third Pauli matrix. Now by (2.30), \((1 - C\hat{w}_\theta)^{-1} \) exists and \( \| (1 - C\hat{w}_\theta)^{-1} \|_{L^2} \leq \frac{1}{1 - \|r(t)\|_{L^\infty}} \) for all \( x, t \) and \( r(t) = e^{-iz^2t}r \). It follows then from Proposition 2.14 that the IRHP2,
\[ M_+ = M_- v_\theta + F, \quad M_\pm \in \partial C(L^2) \]
is solvable and \( \| M_\pm \|_{L^2} \leq \frac{c}{1 - \|r\|_{L^\infty}} \| F \|_{L^2} \) for all \( F \in L^2 \), uniformly for all \( x, t \). But then \( M^\delta_\pm \equiv M_\pm \delta_+^{-\sigma_3} \) solves
\[ M^\delta_+ = M_+^\delta (\delta^\pm v_\theta \delta^\mp \delta_+^{-\sigma_3}) + F^\delta \]
with \( F^\delta = F \delta_+^{-\sigma_3} \). By the analyticity and boundedness properties of \( \delta(z) \) in Proposition 2.38, it is easy to see (cf. the approximation argument \( h^\varepsilon \to h \) preceding (2.56) below) that \( M^\delta_+ \in \partial C(L^2) \) and \( F^\delta \in L^2 \). As \( \delta_+^{-1}, \delta^{-1}_+ \) bound, what the above calculation in fact shows is that the IRHP2 for \( \delta^\pm v_\theta \delta^\mp \delta_+^{-\sigma_3} \) is uniquely solvable for all \( F^\delta \in L^2 \) and \( \| M^\delta_\pm \| \leq \frac{c}{1 - \|r\|_{L^\infty}} \| F^\delta \|_{L^2} \) uniformly for all \( x, t \). Again by Proposition 2.14 we infer finally the analog of the uniform bound (2.50) for \( \delta^\pm v_\theta \delta^\mp \delta_+^{-\sigma_3} \). The thrust of the argument that
follows is that as \( t \to \infty \) the RHP \((\R, \hat{\sigma}_0)\) indeed “localizes” near \( z = z_0 \) and \( \delta^{x_2} v_0 \delta^{x_3} \) is a good enough approximation to \( \hat{\sigma}_0(z) \) to infer the desired bound on \((1 - C_{\hat{w}_0})^{-1}\).

Consider the extended problem on \( \Sigma^c = \R \cup (z_0 + e^{i\pi/4} \R) \cup (z_0 + e^{-i\pi/4} \R) \) oriented as in Figure 2.51

![Figure 2.51 \( \Sigma^c \)](image)

with \( v^c \equiv \hat{\sigma}_0 \) on \( \R \) and \( v^c \equiv I \) on \( \Sigma^c \setminus \R \). The opening angle \( \pi/4 \) is chosen only for convenience; any angle between 0 and \( \pi/2 \) would do.

We now show that the bound (2.50) on \((1 - C_{\hat{w}_0})^{-1}\) is equivalent to the bound

\[
\|(1 - C_{\hat{w}_0})^{-1}\|_{L^2(\Sigma^c)} \leq c
\]

for all \( x \in \R \) and \( t \geq t_0 \) for some \( t_0 \). Here \( w^c = (\hat{w}_0^+, \hat{w}_0^-) \) on \( \R \) and \( w^c = 0 \) on \( \Sigma^c \setminus \R \). Suppose \((1 - C_{\hat{w}_0})^{-1}\) exists in \( L^2(\R) \) and consider the equation \((1 - C_{w^c}) \mu^c = F^c \) for \( F^c \in L^2(\Sigma^c) \). Then as \( w^{c,k} = 0 \) on \( \Sigma^c \setminus \R \), we obtain a decoupled equation for \( \mu \mid \R \) alone,

\[
(1 - C_{\hat{w}_0})(\mu^c \restriction \R) = F^c \restriction \R
\]

on \( \R \), and hence

\[
(\mu^c \restriction \R) = (1 - C_{\hat{w}_0})^{-1}(F^c \restriction \R),
\]

and on \( \Sigma^c \setminus \R \),

\[
\mu^c = F^c + C((\mu^c \restriction \R)(\hat{w}_0^+ + \hat{w}_0^-))
\]

and therefore, using the properties of the Cauchy operator and the fact that \( \|r\|_{L^\infty} < 1 \),

\[
\|\mu^c\|_{L^2(\Sigma^c)} \leq c\|(1 - C_{\hat{w}_0})^{-1}\|_{L^2(\R)}\|F^c\|_{L^2(\Sigma^c)}.
\]

Conversely, suppose \((1 - C_{w^c})^{-1}\) exists in \( L^2(\Sigma^c) \). Then if \( \hat{F} \in L^2(\R) \), we set \( F^c = \hat{F} \) on \( \R \) and zero on \( \Sigma^c \setminus \R \). Let \( \mu^c \) solve \((1 - C_{w^c}) \mu^c = F^c \). Then

\[
\|\mu^c\|_{L^2(\Sigma^c)} \leq c\|(1 - C_{w^c})^{-1}\|_{L^2(\R)}\|\hat{F}\|_{L^2(\Sigma^c)}.
\]

But again on \( \R \) we have \((1 - C_{\hat{w}_0})(\mu^c \restriction \R) = \hat{F} \) i.e. \( \hat{\mu} \equiv \mu^c \restriction \R \) solves \((1 - C_{\hat{w}_0})\hat{\mu} = \hat{F} \) and \( \|\hat{\mu}\|_{L^2(\R)} \leq \|\mu^c\|_{L^2(\Sigma^c)} \leq \|(1 - C_{w^c})^{-1}\|_{L^2(\R)}\|\hat{F}\|_{L^2(\Sigma^c)} \). Also it is clear that \( \hat{\mu} \) is the unique solution of \((1 - C_{\hat{w}_0})\hat{\mu} = \hat{F} \).

For if \((1 - C_{\hat{w}_0})\tilde{\mu} = 0 \), \( \tilde{\mu} \in L^2(\R) \), then \((1 - C_{w^c})\tilde{\mu}^c = 0 \) where \( \tilde{\mu}^c \in L^2(\Sigma^c) \) is given by \( \hat{\mu} \) on \( \R \) and by \( C(\tilde{\mu}(\hat{w}_0^+ + \hat{w}_0^-)) \) on \( \Sigma^c \setminus \R \). But then as \((1 - C_{w^c})^{-1}\) exists, \( \tilde{\mu}^c = 0 \) and hence \( \hat{\mu} = 0 \). It follows that we must prove (2.52) on \( \Sigma^c \).

In view of the preceding remarks we must show that the RHP \((\Sigma^c, v^c)\) localizes near \( z_0 \) and by the steepest descent method introduced in [DZ1], this is done by taking into account the signature table of \( \text{Re} i\theta = \text{Re} i(tz_0^2 - t(z - z_0)^2) = t \text{Im}(z - z_0)^2 \).
is solvable for all $\hat{c}$

\[ (2.54) \]

for all $t > 0$ sufficiently large and for all $x \in \mathbb{R}$.

Define $\Phi_+, \Phi_-$ respectively on $\Sigma^+$ as follows:

\[
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 & [r] \delta^2 e^{i\theta} \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right)
\]

on $z_0 + \mathbb{R}_+$

\[
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 & [r] \delta^2 e^{i\theta} \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right)
\]

on $z_0 + e^{i\pi/4}\mathbb{R}_+$

\[
\left( \begin{array}{cc}
1 - \left[ \frac{r}{1-|r|^2} \right] \delta^2 e^{i\theta} & 0 \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 - \left[ \frac{r}{1-|r|^2} \right] \delta^2 e^{i\theta} & 0 \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right)
\]

on $z_0 + e^{-i\pi/4}\mathbb{R}_-$

\[
\left( \begin{array}{cc}
1 - \left[ \frac{r}{1-|r|^2} \right] \delta^2 e^{i\theta} & 0 \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 - \left[ \frac{r}{1-|r|^2} \right] \delta^2 e^{i\theta} & 0 \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right)
\]

on $z_0 + e^{i\pi/4}\mathbb{R}_+$

\[
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right)
\]

on $z_0 + e^{-i\pi/4}\mathbb{R}_-$

\[
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right)
\]

on $z_0 + e^{i\pi/4}\mathbb{R}_+$

\[
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right)
\]

on $z_0 + e^{-i\pi/4}\mathbb{R}_-$

\[
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right)
\]

on $\mathbb{R}$

and set $m_0^0 = m_\pm^0 \Phi_\pm$. Using the properties of $r, \delta$ and the signature table for Re $i\theta$, observe that $\Phi_\pm$ and $\Phi_\pm^{-1}$ are uniformly bounded for all $x \in \mathbb{R}, t \geq 0$. A direct calculation shows that $m_\pm^0$ satisfy the relations

\[ m_+^0 = m_-^0 v^0 + F^0 \]

on $\Sigma^+$ where $F^0 = F^0 \Phi_+ and$

\[ v^0 = \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right),
\left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right)
\]

on $\mathbb{R}$

\[ m_+^0 = m_-^0 v^0 + F^0 \]

on $z_0 + e^{i\pi/4}\mathbb{R}_+$

\[ m_+^0 = m_-^0 v^0 + F^0 \]

on $z_0 + e^{-i\pi/4}\mathbb{R}_-$

\[ m_+^0 = m_-^0 v^0 + F^0 \]

on $z_0 + e^{i\pi/4}\mathbb{R}_-$

\[ m_+^0 = m_-^0 v^0 + F^0 \]

on $z_0 + e^{-i\pi/4}\mathbb{R}_+$

Figure 2.53. Signature table for Re $i\theta$
Moreover, \( m_0^\pm \in \partial C(L^2) \). To verify this, observe that \( \Phi_{\pm} \) are the boundary values of a bounded, analytic function \( \Phi \), say, in \( \mathbb{C} \setminus \Sigma^e \). For example, in the sector \( 0 < \arg z < \pi/4 \), \( \Phi(z) = \left( \begin{array}{cc} 1 & 0 \\ \delta e^{-i\theta} & 1 \end{array} \right) \), etc.

Writing \( m_0^\pm = C_{X\mp}^\pm h \) for some \( h \in L^2(\Sigma^e) \), we choose \( h^\epsilon \in L^2(\Sigma^e) \) such that \( h^\epsilon \to h \) in \( L^2(\Sigma^e) \) as \( \epsilon \downarrow 0 \), and set \( m^\epsilon(z) = (C_{X\mp} h^\epsilon)(z) \), \( z \in \mathbb{C} \setminus \Sigma^e \). Applying Cauchy’s theorem to \( m^\epsilon(z) \Phi(z) \), we find \( m^\epsilon(z) \Phi(z) = (C_{X\mp} (m_+^\epsilon \Phi_+ - m_-^\epsilon \Phi_-))(z) \), \( z \in \mathbb{C} \setminus \Sigma^e \). Evaluating this relation on \( \Sigma^e \), and then letting \( \epsilon \downarrow 0 \), we obtain

\[
m_0^\pm = m_0^\epsilon(z) \Phi_\pm(z) = (C_{X\mp} (m_+^\epsilon \Phi_+ - m_-^\epsilon \Phi_-))(z) \in \partial C(L^2)
\]
as desired. As \( \Phi_{\pm} \) and \( \Phi_{\pm}^{-1} \) are uniformly bounded in \( x \) and \( t \), we see that (2.54) is equivalent to showing that the IRHP2

\[
M_0^\epsilon = M_0^\epsilon v^0 + F^0, \quad M_0^\epsilon \in \partial C(L^2),
\]
is solvable for all \( F^0 \in L^2 \) and

\[
\|M_0^\epsilon\|_{L^2(\Sigma^e)} \leq c\|F^0\|_{\Sigma^e}
\]
for all \( t > 0 \) sufficiently large and for all \( x \in \mathbb{R} \). The key point to note is that the IRHP2 for \( v^\epsilon \) has been deformed to a problem with jump matrix \( v^0 \) where all the exponential factors \( e^{\pm i\theta} \) are now exponentially decreasing.

We need to prove some estimates on \( \delta(z) \). Let \( \|r\|_{L^\infty(\mathbb{R})} \leq \rho < 1 \). From Proposition 2.38, we have

\[
\delta(z) = e^{(C_{X\epsilon} + i\theta) \log(1 - |r(z)|^2)}, \quad z \in \mathbb{C} \setminus (\mathbb{R}_- + z_0).
\]

Now where \( \chi^0(s) \) denotes the characteristics function of the interval \( (z_0 - 1, z_0) \) we have \( z \in \mathbb{C} \setminus (\mathbb{R}_- + z_0) \)

\[
(C_{\mathbb{R}_- + z_0} \log(1 - |r|^2))(z) = \int_{-\infty}^{z_0} \{ \log(1 - |r(z)|^2) - \log(1 - |r(z_0)|^2) \} \chi^0(s - z_0 + 1) \frac{ds}{2\pi i(s - z)} \\
+ \log(1 - |r(z)|^2) \int_{z_0 - 1}^{z_0} \frac{s - z_0 + 1}{s - z} \frac{ds}{2\pi i} \\
= \beta(z, z_0) + i\nu(z_0)(1 - (z - z_0 + 1) \log(z - z_0 + 1) \\
+ (z - z_0) \log(z - z_0) + \log(z - z_0))
\]

where \( \nu(z_0) = -\frac{1}{2\pi} \log(1 - |r(z)|^2) \) as in (1.2), and

\[
\beta(z, z_0) = \int_{-\infty}^{z_0} \{ \log(1 - |r(z)|^2) - \log(1 - |r(z_0)|^2) \} \chi^0(s - z_0 + 1) \frac{ds}{2\pi i(s - z)}.
\]

Clearly \( \beta_\pm(z, z_0) \) lies in the Sobolev space \( H^1(\mathbb{R}) = H^{1,0} \), and a direct calculation shows that

\[
\|\beta_\pm\|_{H^1(\mathbb{R})} \leq c\|r\|_{H^{1,0}} \frac{1}{1 - \rho}.
\]

Focusing on the ray \( L_{-\pi/4} = z_0 + e^{-i\pi/4}u \in \{ z = z_0 + u e^{-i\pi/4}, u \geq 0 \} \), we find similarly

\[
\|\beta\|_{H^1(L_{-\pi/4})} \leq c\|r\|_{H^{1,0}} \frac{1}{1 - \rho}.
\]

It follows then by standard Sobolev estimates that on \( L_{-\pi/4} \)

\[
\begin{align*}
(2.57) \quad & \begin{cases}
(i) & \beta(z, z_0) \text{ is continuous up to } z = z_0 \\
(ii) & \|\beta(z, z_0)\|_{L^\infty(L_{-\pi/4})} \leq c\|r\|_{H^{1,0}} / (1 - \rho) \\
(iii) & |\beta(z, z_0) - \beta(z_0, z_0)| \leq c\|r\|_{H^{1,0}} (z - z_0)^{1/2} \end{cases}
\end{align*}
\]

where the constant \( c \) is independent of \( x \in \mathbb{R} \) and \( t > 0 \). Write

\[
(2.58) \quad \delta = \delta_0 \delta_1
\]
where
\[ \delta_0 = e^{\beta(z_0, z_0) + iv(z_0)}(z - z_0)^{iv(z_0)}, \quad \delta_1 = e^{\zeta(z, z_0)} \]
and
\[ \zeta(z, z_0) = (\beta(z, z_0) - \beta(z_0, z_0)) + iv(z_0)((z - z_0) \log(z - z_0) - (z - z_0 + 1) \log(z - z_0 + 1)). \]
We want to estimate
\[ (2.59) \]
\[ D(z) \equiv [r] \delta^2 e^{i\theta} - r(z_0) \delta_0^2 e^{i\theta} \]
\[ = \delta_0^2 r(z_0) e^{i\theta} \left( \frac{\delta_1^2}{(1 + i(z - z_0))^2} - 1 \right) \]
on \( L_{-\pi/4} \) (cf. (2.55)). Write \( D(z) = D_1(z) + D_2(z) \) where
\[ D_1(z) = \delta^2 r(z_0) e^{i\theta} \frac{-2i(z - z_0) + (z - z_0)^2}{(1 + i(z - z_0))^2} \]
and
\[ D_2(z) = \delta_0^2 r(z_0) e^{i\theta} (\delta_1^2(z) - 1). \]
Now as \( \beta(z_0, z_0) \in \mathbb{R}, \delta_0 \mid = |e^{iv(z_0) log(z - z_0)}| = e^{(v(z_0))/\pi} \) and so as \( \delta(z) \mid \leq (1 - \rho)^{-1/2} \), we learn that \( \delta_1(z) \mid \leq \frac{e^{-\pi/2}u(1 - \rho)/2}{(1 - \rho)/2}. \) On \( L_{-\pi/4} = \{ z = z_0 + u e^{-i\pi/4}, u \geq 0 \}, |D_1(z)| \leq \frac{\rho}{(1 - \rho/2)u^2} |e^{-\pi/2}u(2u + w^2)| \leq \frac{e^{-\pi/2}u(2u + w^2)}{(1 - \rho/2)u^2} \)
for \( t \geq 1 \), where \( c = \sup e^{-w^2/2}(2w^2 + w^2) \). Also \( \left\| \frac{d\zeta}{ds} \right\|_{L^2(L_{-\pi/4})} \leq c \) where \( c \) is independent of \( x \) and \( t \geq 0 \), by the \( H^1 \) property of \( \beta(z, z_0) \) and the properties of the logarithm: hence \( |\zeta(z, z_0)| \leq c|z - z_0|^{1/2} \), \( z \in L_{-\pi/4} \).

It follows that for \( z \in L_{-\pi/4} \),
\[ |\delta_1^2(z) - 1| = \left| \int_0^1 \frac{d}{ds} e^{2s\zeta(z, z_0)} ds \right| \leq c|z - z_0|^{1/2} \max_{0 \leq s \leq 1} |e^{2\zeta(z, z_0)}| |s \]
\[ \leq c|z - z_0|^{1/2} \max_{0 \leq s \leq 1} \delta_1(z)^2 \leq \frac{c}{1 - \rho} |z - z_0|^{1/2}, \]
and we conclude that on \( L_{-\pi/4}, |D_2(z)| \leq c e^{-\pi/2} \frac{\rho}{(1 - \rho/2)u^2} u^{1/2} \leq \frac{ce^{-\pi/2}u^2}{u^{1/4}}. \) Assembling the above estimates we have for \( t \geq 1 \) and \( z \in L_{-\pi/4}, \)
\[ (2.60) \]
\[ |D(z)| \leq \frac{ce^{(-t/2)|z - z_0|^2}}{t^{1/4}} \]
The situation on the other rays in \( \Sigma^c \setminus \mathbb{R} \) is similar. Set
\[ (2.61) \]
\[
\begin{aligned}
\quad v^# &= I & & \text{on } \mathbb{R} \\
\quad v^# &= \begin{pmatrix} 1 & 0 \\ -r(z_0)\delta_0^2 e^{-i\theta} & 1 \end{pmatrix} & & \text{on } z_0 + e^{i\pi/4}\mathbb{R}_+ \\
\quad v^# &= \begin{pmatrix} 1 & 0 \\ r(z_0)/(1 - |r(z_0)|^2)\delta_0^2 e^{i\theta} & 1 \end{pmatrix} & & \text{on } z_0 + e^{-i\pi/4}\mathbb{R}_- \\
\quad v^# &= \begin{pmatrix} 1 & 0 \\ -r(z_0)/(1 - |r(z_0)|^2)\delta_0^2 e^{-i\theta} & 1 \end{pmatrix} & & \text{on } z_0 + e^{i\pi/4}\mathbb{R}_- \\
\quad v^# &= \begin{pmatrix} 1 & 0 \\ r(z_0)\delta_0^2 e^{i\theta} & 1 \end{pmatrix} & & \text{on } z_0 + e^{-i\pi/4}\mathbb{R}_+ \\
\end{aligned}
\]
We have proved the following.
Lemma 2.62. For $1 \leq p \leq \infty$,

\[(2.63) \quad \|v^\# - v^0\|_{L^p(\Sigma^c)} \leq \frac{c}{t^{1 + \frac{1}{2p}}} \]

uniformly for $t \geq 1$ and all $x \in \mathbb{R}$.

Remarks

1. If we knew that $r$ had more smoothness, say $r'' \in L^2$, then we could replace estimate $|\beta(z, z_0) - \beta(z_0, z)| \leq c|z - z_0|^{1/2}$ by $|\beta(z, z_0) - \beta(z_0, z)| \leq c|z - z_0|$, which would lead eventually to the bound $\frac{c}{t^{1 + \frac{1}{4p}}}$ in (2.63).

2. If $r$ had more decay, say $r \in H^{-1}_1$, then one can easily show that $\beta(z_0, z_0) + r(z_0) = -\frac{1}{2\pi i} \int_{-\infty}^{z_0} \log(z_0 - s) \log(1 - |r(s)|^2)$. We will need this fact later.

Set $w^# = (0, w^{#+} = v^# - I), w^0 = (0, w^{0+} = v^0 - I)$ on $\Sigma^c$. We will show shortly that $(1 - C_{w^0})^{-1}$ exists on $L^2(\Sigma^c)$ for all $x, t$ and

\[(2.64) \quad \|(1 - C_{w^0})^{-1}\|_{L^2(\Sigma^c)} \leq c \]

where $c$ is independent of space and time. By (2.63), for $t \geq 1$,

$$\|C_{w^0} - C_{w^#}\|_{L^2} = \|C^{-1}(v^0 - v^#)\|_{L^2} \leq c\|v^0 - v^#\|_{L^\infty} \leq \frac{c}{t^{1/4}},$$

and we conclude by the resolvent identity that $(1 - C_{w^0})^{-1}$ exists in $L^2(\Sigma^c)$ for $t$ sufficiently large and that

\[(2.65) \quad \|(1 - C_{w^0})^{-1}\|_{L^2(\Sigma^c)} \leq c, \quad (1 - C_{w^0})^{-1} - (1 - C_{w^#})^{-1}\|_{L^2} \leq \frac{c}{t^{1/4}} \]

uniformly for all $x \in \mathbb{R}$ and $t \geq t_0$, say. But then again by Proposition (2.14), and the Corollary to Proposition (2.14), this proves (2.56) and hence, retracing the argument, we obtain the desired bound in (2.50).

Now in order to prove (2.64), observe that if we retrace the steps for $v^#$ rather than $v^0$, then instead of obtaining $\hat{v}_\theta$ on $\mathbb{R}$ we would obtain $\hat{v}_\theta^#$, where

\[
\hat{v}_\theta^# = \begin{pmatrix} 1 & r(z_0) \delta_0^e i^\theta \\ 0 & 1 \end{pmatrix} \left( \begin{pmatrix} 1 & 1 \\ -r(z_0) \delta_0^2 e^{-i\theta} & 0 \end{pmatrix} \right), \quad z > z_0,
\]

\[
\hat{v}_\theta^# = \begin{pmatrix} 1 & r(z_0) \delta_0^e i^\theta \\ 0 & 1 \end{pmatrix} \left( \begin{pmatrix} 1 & 1 \\ -r(z_0) \delta_0^2 e^{-i\theta} & 0 \end{pmatrix} \right), \quad z < z_0,
\]

(cf. (2.48)). Multiplying out the matrices, we find

\[(2.66) \quad \hat{v}_\theta^#(z) = \delta_0^{\sigma_2} \left( \begin{pmatrix} 1 & r(z_0) e^{i\theta(z)} \\ -r(z_0) e^{-i\theta(z)} & 1 \end{pmatrix} \right) \delta_0^{\sigma_3}, \quad z \in \mathbb{R},
\]

where again $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Set $\hat{w}_\theta^# = (0, \hat{w}_\theta^{#+} = \hat{v}_\theta^# - I)$. By Proposition (2.14), and its Corollary, in order to complete the proof of (2.64), and hence the proof of Proposition (2.49), it is enough to show that $(1 - C_{\hat{w}_\theta^#})^{-1}$ exists in $L^2(\mathbb{R})$ for all $x, t$ and

\[(2.67) \quad \|(1 - C_{\hat{w}_\theta^#})^{-1}\|_{L^2} \leq c \]

uniformly for all $x, t$. But by the analyticity and boundedness properties of $\delta_0 = \delta_0(z)$, it follows that the IRHP2 for $\hat{v}_\theta^#$ is equivalent to the IRHP2 for $r_1 = \begin{pmatrix} 1 & -|r_1|^2 \\ -\bar{r}_1 & 1 \end{pmatrix}$ where $r_1(z) = r(z_0) e^{i\theta(z)}$ (cf. with the situation above for $v^c$ and $v^0$). But as $\|r_1\|_{L^\infty} = |r(z_0)| \leq \|r\|_{L^\infty} < 1$, it follows from (2.30) that for $w_1 = (w_1^+, w_1^-) = \left( \begin{pmatrix} 0 & r_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -\bar{r}_1 & 0 \end{pmatrix} \right)$, $(1 - C_{w_1})^{-1}$ exists in $L^2$ and

\[(2.68) \quad \|(1 - C_{w_1})^{-1}\|_{L^2} \leq \frac{1}{1 - \|r\|_{L^\infty}} \]

for all $x, t$. The bound (2.67) then follows by two applications of Proposition 2.14. This completes the proof of Proposition (2.49).

Remark 2.69. The above proof of (2.50) should be contrasted with the proof of the same fact in [DZ2], which relies in turn on various intricate formulae taken from [DZ1].
3. Scattering and Inverse Scattering for $q \in H^{1,1}$

The scattering/inverse scattering theory for the ZS–AKNS operator $T = \partial_x - \left( i \sigma + \begin{pmatrix} 0 & q(x) \\ \bar{q}(x) & 0 \end{pmatrix} \right)$ is well known. An account of the classical theory is contained in [ZS], [AKNS], and the Riemann–Hilbert point of view can be inferred, as a special case, from the general theory in [BC].

In this section, we present a somewhat complete, mostly self-contained, sketch of the general theory for the operator $T$. Our main focus is on a detailed analysis (following [Z1]) of the mapping properties of the scattering map $R$, showing, in particular, that $R$ is a bijection from $H^{1,1}$ onto $H^{1,1}$. Enough of the general theory is described, however, so that the interested reader should be able to fill in the missing details without too much effort.

The principal objects of study in the theory are eigensolutions $\psi = \psi(x, z)$ of the ZS–AKNS operator

\begin{equation}
(\partial_x - \left( i \sigma + \begin{pmatrix} 0 & q(x) \\ \bar{q}(x) & 0 \end{pmatrix} \right) ) \psi = 0. \tag{3.1}
\end{equation}

Setting

\begin{equation}
m = \psi e^{-ixz\sigma}, \tag{3.2}
\end{equation}

equation (3.1) takes the form

\begin{equation}
\partial_x m = iz \text{ad} \sigma(m) + Qm, \quad Q = \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}, \tag{3.3}
\end{equation}

where $\text{ad} A(B) = [A, B] = AB - BA$. Under exponentiation we have $e^{\text{ad} A(B)} = \sum_{n=0}^{\infty} \frac{(\text{ad} A)^n(B)}{n!} = e^A B e^{-A}$ (cf. $A_0$ following (2.35)). The theory of ZS–AKNS [ZS], [AKNS] is based on the following two Volterra integral equations for real $z$,

\begin{equation}
m^{(\pm)}(x, z) = I + \int_{\pm\infty}^{x} e^{i(z-y)z} \text{ad} \sigma Q(y)m^{(\pm)}(y, z)dy \equiv I + K_{q, z, \pm}m^{(\pm)}. \tag{3.4}
\end{equation}

By iteration, one sees that these equations have bounded solutions continuous for both $x$ and real $z$ when $q \in L^1(\mathbb{R})$. The matrices $m^{(\pm)}(x, z)$ are the unique solutions of (3.4) normalized to the identity as $x \to \pm\infty$.

The following are some relevant results of ZS–AKNS theory:

\begin{enumerate}
\item[(3.5a)] There is a continuous matrix function $A(z)$ for real $z$, $\det A(z) = 1$, defined by $\psi^{(+)} = \psi^{(-)} A(z)$, where $\psi^{(\pm)} = m^{(\pm)} e^{ixz\sigma}$ and $A$ has the form $A(z) = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$.
\item[(3.5b)] $a$ is the boundary value of an analytic function, also denoted by $a$, in the upper half-plane $\mathbb{C}_+$: $a$ is continuous and non-vanishing in $\overline{\mathbb{C}_+}$, and $\lim_{z \to \infty} a(z) = 1$.
\end{enumerate}

\begin{equation}
\begin{cases}
a(z) = \det (m_1^{(\pm)}, m_2^{(-)}) = 1 - \int_{\mathbb{R}} q(y)m_1^{(+)}(y, z)dy + \int_{\mathbb{R}} q(y)m_2^{(-)}(y, z)dy, \\
b(z) = e^{ixz} \det (m_1^{(-)}, m_1^{(+)}) = - \int_{\mathbb{R}} \bar{q}(y)e^{iyz}m_1^{(+)}(y, z)dy - \int_{\mathbb{R}} \bar{q}(y)e^{iyz}m_1^{(-)}(y, z)dy,
\end{cases} \tag{3.5c}
\end{equation}

where $m^{(\pm)} = (m_1^{(\pm)}, m_2^{(\pm)}) = \begin{pmatrix} m_1^{(\pm)} & m_2^{(\pm)} \\ m_1^{(\pm)} & m_2^{(\pm)} \end{pmatrix}$.

\begin{enumerate}
\item[(3.5d)] The reflection coefficient $r$ is defined by $\bar{b}/\bar{a}$. As $\det A = 1$, $|a|^2 - |b|^2 = 1$ so that $|a| \geq 1$ and $|r|^2 = 1 - |a|^2 < 1$. Together with (3.5b), this implies $\|r\|_{L^\infty(\mathbb{R})} < 1$. The transmission coefficient $t(z)$ is defined by $1/a(z)$. Thus $\|t\|_{L^\infty(\mathbb{R})} \leq 1$ and $|r(z)|^2 + |t(z)|^2 = 1$.
\end{enumerate}

Remark 3.6. It is easy to verify that if we set $z_0 = +\infty$ in (2.39), then $\delta(z) = t(z)$. 
Now $m_1^{(+)}(x, z)$ and $m_2^{(-)}(x, s)$, the first and second columns of $m^{(+)}$ and $m^{(-)}$ respectively, have analytic continuations to $\mathbb{C}_+$. Moreover if we set

$$m = m(x, z) = \left( \frac{m_1^{(+)}(x, z)}{a(z)}, m_2^{(-)}(x, z) \right), \quad z \in \mathbb{C}_+,$$

then $m(x, z)$ is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}_+}$ and

\[
\begin{cases}
  m(x, z) \to I \quad \text{as} \quad x \to -\infty \\
  m(x, z) \quad \text{is bounded as} \quad x \to +\infty.
\end{cases}
\]

Using the fact that the substitution $m(x, z) \to \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \overline{m(x, z)} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$ preserves solutions of (3.3), we define

$$m(x, z) \equiv \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \overline{m(x, z)} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$$

for $z \in \mathbb{C}_-$, and by continuity in $\overline{\mathbb{C}_-}$. With these definitions $\psi(x, z) = m(x, z)e^{iz\sigma}$, $z \in \mathbb{C}\setminus\mathbb{R}$, is the (unique) solution of $[\mathcal{B}C]$–type mentioned in the Introduction. Orienting $\mathbb{R}$ from $-\infty$ to $+\infty$, we let $\psi_\pm(x, z)$, $m_\pm(x, z)$ denote the boundary values of $\psi(x, z) = m(x, z)e^{iz\sigma}$, $m(x, z)$ from $\mathbb{C}_\pm$ respectively. By uniqueness of solutions of (3.1), we must have $\psi_\pm(x, z) = \psi_\pm(x)v$ for some $v = \psi(z)$ independent of $x$. Direct calculation shows that $v(z) = \left( \begin{array}{cc} 1 - |r(z)|^2 & r(z) \\ -\bar{r}(z) & 1 \end{array} \right)$ where $r = -\bar{b}/\bar{a}$ is the reflection coefficient in (3.5d). For fixed $x, m(x, z)$ solves the normalized RHP (1.4) in the following precise sense

\[
\begin{cases}
  \bullet \quad m(x, z) \text{ is analytic in } \mathbb{C}\setminus\mathbb{R} \text{ and continuous up to the boundary} \\
  \bullet \quad m_+(x, z) = m_-(x, z)v_x(z) \quad \text{for } z \in \mathbb{R}, \text{ where } v_x = e^{iz\text{ad } \sigma}v \\
  \bullet \quad m(x, z) \to I \text{ uniformly as } z \to \infty \text{ in } \mathbb{C}_+ \cup \overline{\mathbb{C}_-}.
\end{cases}
\]

The above considerations require only that $q \in L^1(\mathbb{R})$. We are interested in the case that $q \in H^{0,1} \subset L^1$, and for such $q$ we will see $m_\pm$ also solves the normalized RHP in the sense (2.9),

$$m_\pm = m_- v_x \quad \text{on } \mathbb{R}, \quad m_\pm - I \in \partial C(L^2).$$

In fact, even more is true, viz. $m_\pm - I \in \partial C(H^1)$. In other words $m_\pm = C^\pm h$, where $h$ lies in the Sobolev space $H^1 = H^{1,0}$.

As noted in the Introduction, in [Z1], the author proved that $q \mapsto r = \mathcal{R}(q)$ is a bi-Lipschitz map from $H^{k,j}$ to $H^{k,j}_1$ for $k \geq 0, j \geq 1$. We now prove this fact in the special case $k = j = 1$, which is, of course, the main case of interest in this paper. For simplicity, we will only prove that the map is bijective. Our method is based on the analysis of certain rational forms of some linear operators which depend linearly on $q, q$ or $r, \bar{r}$ and it will be clear to the reader that the maps $\mathcal{R}$ and $\mathcal{R}^{-1}$ are indeed Lipschitz.

The bijectivity of $\mathcal{R}$ is covered by the following four theorems. In the following if $M$ is a measure space and $B$ is a Banach space, then $B \otimes L^p(M) \equiv L^p(M \to B)$ denotes the space of $B$-valued $L^p$ functions with norm $\|f\|_{B \otimes L^p(M)} = \|f\|_B \|f\|_{L^p(M)}$.

**Theorem 3.9.** $\mathcal{R}$ maps $H^{0,1}$ into $H^{1,0}$.

**Proof.** By the Wronskian formula (3.5c) and the fact that $|a(z)| \geq 1$, we only need to prove that $m^{(\pm)}(x, \cdot) - I \in H^{1,0}$ for any fixed $x$, say $x = 0$. We will provide the proof only for $m^{(+)}$. Clearly, $\|K_{q,z,+}\|_{L^\infty(dx) \to L^\infty(dx)} \leq \|Q\|_{L^1}$. The operator $K_{q,z,+}$ can also be lifted to a bounded operator $K_{q,+}$ from $L^2(dx) \otimes L^\infty(dx)$ to $L^2(dx) \otimes L^\infty(dx)$:

$$\|K_{q,+}f\|_{L^2(dx) \otimes L^\infty(dx)} \leq \|Q\|_{L^1} \|f\|_{L^2(dx) \otimes L^\infty(dx)}.$$

The standard iteration method for Volterra integral equation gives the estimate

$$\|(1 - K_{q,+})^{-1}\|_{L^2(dx) \otimes L^\infty(dx) \to L^2(dx) \otimes L^\infty(dx)} \leq e\|Q\|_{L^1}.$$
By Fourier theory and Hardy’s inequality,
\[
\|K_{q,+}I\|_{L^2(dx)\otimes L^\infty(dx)} = \sqrt{2\pi} \left\| \left( \int_{-\infty}^{\infty} e^{i(x-y)z} dy \right)^{\frac{1}{2}} \right\|_{L^\infty(dx)} \leq \sqrt{2\pi} \|Q\|_{L^2}
\]
\[
\|K_{q,+}I\|_{L^2(dx)\otimes L^2_{\mathbb{R}^+}(dx)} = \sqrt{2\pi} \left\| \left( \int_{-\infty}^{\infty} e^{i(x-y)z} dy \right)^{\frac{1}{2}} \right\|_{L^2_{\mathbb{R}^+}(dx)} \leq \sqrt{2\pi} \|Q\|_{H^0.5} \leq \sqrt{2\pi} \|Q\|_{H^{0,1}}.
\]
Thus
\[
\|m^{(+)} - I\|_{L^2(dx)\otimes L^\infty(dx)} = \|(1 - K_{q,+})^{-1}K_{q,+}I\|_{L^2(dx)\otimes L^\infty(dx)} \leq ce\|Q\|_{H^1} \|Q\|_{L^2}.
\]
Estimating the RHS of (3.4), and using Hardy’s inequality [HLP], we see that in fact
\[
m^{(+)} - I \in L^2(dx) \otimes (L^\infty \cap L^2_{\mathbb{R}^+}(dx)).
\]
Since \(\partial_z m^{(+)}\) satisfies the equation
\[
\partial_z m^{(+)}(x, z) = \int_{+\infty}^{x} i(x-y) \sigma e^{i(x-y)z} \sigma Q(y)m^{(+)}(y, z)dy
\]
\[
+ \int_{+\infty}^{x} e^{i(x-y)z} \sigma Q(y)\partial_z m^{(+)}(y, z)dy,
\]
the function \(\hat{m} = (\partial_z - ix \sigma) m^{(+)}\) satisfies the equation
\[
\hat{m}(x, z) = -i \int_{+\infty}^{x} ye^{i(x-y)z} \sigma (\sigma Q(y))m^{(+)}(y, z)dy
\]
\[
+ \int_{+\infty}^{x} e^{i(x-y)z} \sigma Q(y)\hat{m}(y, z)dy
\]
\[
\equiv h_1 + h_2 + K_{q,+}\hat{m},
\]
where
\[
h_1 = -i \int_{+\infty}^{x} ye^{i((x-y)) - (z)} \sigma (\sigma Q(y))dy \in L^2(dz) \otimes L^\infty(dx),
\]
and
\[
h_2 = -i \int_{+\infty}^{x} ye^{i(x-y) - (z)} \sigma (\sigma Q(y))(m^{(+)}(y, z) - I)dy \in L^2(dz) \otimes L^\infty(dx)
\]
by (3.11) and the fact that \(Q \in H^{0,1}\). Hence \(h_1 + h_2 \in L^2(dx) \otimes L^\infty(dx)\) and
\[
\hat{m} = (1 - K_{q,+})^{-1}(h_1 + h_2) \in L^2(dx) \otimes L^\infty(dx).
\]
In particular \(\partial_z m^{(+)}(x = 0, z) = \partial_y \hat{m}(0, z) \in L^2(dx)\) and by (3.10), \(m^{(+)}(0, z) - I \in L^2(dx)\). Analogous estimates for \(m^{(-)}\) show that \(\partial_z m^{(-)}(0, z)\) and \(m^{(-)}(0, z) - I\) lie in \(L^2(dx)\), and hence \(r \in H^{1,0}\) by (3.5bc).

Note that the above calculations show that \(m^{(+)}(x, z) - I\) and \(\partial_z m^{(+)}(x, z) \in L^2(dx) \otimes L^\infty(dx)\). There is a similar calculation for \(m^{(-)}(x, z)\). It is then easy to show that for each \(x \in \mathbb{R}\), \(m_{\pm}(x, \cdot) - I \in \partial C(H^1)\), as advertised above.

We are now ready to consider the regularity of \(q\).
Theorem 3.12. If \( q \in H^{1,1} \), then \( r \in H^{1,1}_1 \).

Proof. Since \( r \in H^{1,0} \), we only need to study its decay behavior as \( |z| \to \infty \). As \( |a| \geq 1 \), we only need to study the decay behavior of \( b \) at \( \infty \). We use the formulae (3.5c),

\[
b = - \int q e^{iyz} m_{11}^{(-)} = - \int q e^{iyz} (m_{11}^{(-)} - 1) - \int q e^{iyz}.
\]

Since the second term on the RHS is clearly in \( H^{1,1} \), we only need to show that \( \int q e^{iyz} (m^{(-)} - I) \) is in \( H^{0,1} \).

Denote \( K_{q,z,-} \) by \( K \). Noting that \( \text{ad} \sigma \) is invertible on off-diagonal matrices we integrate by parts to obtain

\[
(KI)(x) = -(iz \text{ ad} \sigma)^{-1} Q + (iz \text{ ad} \sigma)^{-1} \int e^{i(x-y)z} \text{ ad} \sigma Q(y) dy \equiv h_1 + h_2.
\]

Note that we are only interested in the decay at \( z = \infty \); the singularity at \( z = 0 \) is irrelevant.

We write

\[
m^{(-)} - I = (1 - K)^{-1} K I = h_1 + Kh_1 + (1 - K)^{-1} K^2 h_1 + (1 - K)^{-1} h_2 \equiv h_1 + g_1 + g_2 + g_3.
\]

Thus we need to show that

\[
\int q e^{iyz} (h_1 + g_1 + g_2 + g_3)
\]

is in \( H^{0,1} \). It is obvious that \( \int q e^{iyz} h_1 \) is in \( H^{0,2} \).

Since \( Q \) ad \( \sigma Q \) is diagonal,

\[
(K(z^{-1} \text{ ad} \sigma Q)(x) = z^{-1} \int_{-\infty}^{x} e^{i(x-y)z} \text{ ad} \sigma Q \text{ ad} Q(y) dy = z^{-1} \int_{-\infty}^{x} Q \text{ ad} Q dy.
\]

Now \( \int q e^{iyz} g_1 \) is in \( H^{0,2} \) because \( \bar{q} \int_{-\infty}^{x} Q \text{ ad} Q \in H^{1,0}(dx) \).

Since \( Q \) ad \( Q \) is diagonal, and \( Q \int_{-\infty}^{x} Q \text{ ad} Q \) is off-diagonal,

\[
(K^2(z^{-1} \text{ ad} \sigma Q)(x) = z^{-1} \int_{-\infty}^{x} e^{i(x-y)z} \text{ ad} \sigma Q \int_{-\infty}^{y} Q \text{ ad} Q(u) du
\]

is in \( H^{0,1} \otimes L^{\infty}(dx) \) and so is \( g_2 \). We see that \( \int q e^{iyz} g_2 \) is also in \( H^{0,1} \).

But, \( g_2 \) is in \( H^{0,1} \otimes L^{\infty}(dx) \) and so is \( g_3 \). Thus \( \int q e^{iyz} g_3 \) is also in \( H^{0,1} \).

Finally we indeed have \( r \in H^{1,1}_1 \). As \( b, \partial_x b \in L^2(dx), b(z) \to 0 \) as \( z \to \infty \) and hence \( |a(z)|^2 = 1 + |b(z)|^2 \to 1 \). Clearly \( |a(z)| \geq 1 \) and hence the bound \( \|r\|_{L^{\infty}(dx)} < 1 \) follows from the formula \( |r(z)|^2 = 1 - |a(z)|^2 \).

We now construct the inverse scattering transform from \( H^{1,1}_1 \) to \( H^{1,1} \). Let \( r \in H^{1,1}_1 \) be given.

The inverse problem is formulated as a normalized RH problem \( (\mathbb{R}, v_x) = (\mathbb{R}, v_x)_L^2 \) where \( v = v_+ v_- = \begin{pmatrix} 1 - |r|^2 & r \\ -r & 1 \end{pmatrix} \). This problem is adapted for studying the decay behavior of \( q \) as \( x \to -\infty \). To obtain an equivalent RH problem adapted for studying the decay behavior of \( q \) as \( x \to \infty \), we set \( \hat{m} = m \delta^{-\sigma_3}, \delta = \delta_{\infty = \infty} = \exp C_{\delta} \log(1 - |r|^2), \) and \( \hat{v} = \delta^{\sigma_3} v \delta^{-\sigma_3} \) [BC]. In the following Lemma and Theorem, we prove the decay behavior of \( q \) as \( x \to -\infty \). The decay behavior of \( q \) as \( x \to \infty \) can be obtained in a similar manner using the RH problem for \( \hat{v} \).

The normalized RHP \( (\mathbb{R}, v_x) \),

\[
m_+ = m_- v_x \text{ on } \mathbb{R}, \quad m_\pm \in I + \partial C(L^2)
\]

is solvable for each \( x \) by Proposition 2.31, and moreover, by (2.32)–(2.33), \( m_\pm = I + C^\pm(\mu(w_x^+ + w_x^-)) \) where \( \mu \in I + L^2 \) is the unique solution of \( (1 - C_{w_x})\mu = I \).

But more is true: if \( r \in H^{1,0} \), then \( \mu - I \in H^1(dx) \). Consider first the case where \( r \in C^\infty_0(\mathbb{R}) \) and for any \( \epsilon \in \mathbb{R} \) write \( \mu^\epsilon(z) = \mu(z + \epsilon) \). Then the equation for \( \mu \) implies \( (\mu^\epsilon - \mu)/\epsilon = (1 - C_{w_x})^{-1} C_{((w_x^- - w_x^+)/\epsilon)} \mu \).
and using the fact that $\mu \in I + L^2$ and

$$\frac{d^2w_+}{dx^2} \to \frac{4}{\pi^2} f(x) \mu_+ \text{ in } L^2 \cap L^\infty(dz)$$

as $\epsilon \downarrow 0$, we infer that $\mu'$ exists in $L^2(dz)$ and $\mu' = (1 - C_{w_x})^{-1} C_{w'} \mu$. Then as $\mu - I \in L^2$, $\mu' \in L^2$ we have by Sobolev,

$$\|\mu - I\|_{L^\infty} \leq \gamma^{-1}\|\mu - I\|_{L^2}$$

for any $\gamma > 0$. Inserting this bound in $\|\mu'\|_{L^2} \leq \frac{\epsilon\|r\|_{L^2} + \|\mu - I\|_{L^\infty}}{\|r\|_{H^{1,0}}}$, we obtain the a priori bound, $\|\mu'\|_{L^2} \leq \frac{\epsilon}{1 - \frac{\epsilon}{\gamma}}(1 + \frac{1}{2})\|r\|_{H^{1,0}}$ for $\gamma$ sufficiently small. Note that the constant $c = c(x)$ grows at most linearly with $x$ and therefore, for given $0 < \rho < 1$, $\lambda > 0$, $\gamma$ may be chosen uniformly for $\|r\|_{H^{1,0}} \leq \lambda$, $\|r\|_{L^\infty} \leq \rho$ and all $x$ in compacta. Now given $r \in H^{1,0}_1$, $\|r\|_{H^{1,0}} < \lambda$, $\|r\|_{L^\infty} < \rho < 1$, we may choose $r_n \in C_0^\infty(\mathbb{R})$, $\|r_n\|_{H^{1,0}} \leq \lambda$, $\|r_n\|_{L^\infty} \leq \rho$ such that $r_n \to r$ in $H^{1,0}$. By the above calculations and remarks it follows that $\mu - I \in H^1(dz)$ as desired. Moreover it is clear that $\|\mu(x, \xi) - I\|_{H^1(dz)}$ is bounded for $x$ in compact sets.

Knowing that $\mu \in I + H^1(dz) \subset I + L^\infty(dz)$, we may now differentiate $(1 - C_{w_x})\mu = I$ with respect to $x$ to conclude that $\frac{d}{dx}\mu = (1 - C_{w_x})^{-1}(C_{(dwx/dz)}\mu) \in L^2(dz)$. But then

$$\frac{d}{dx}(\mathcal{M}_\pm - I) = C^\pm \left[ \left( \frac{d}{dx} \right) (w_+^x + w_-^x) + \mu \left( \frac{d}{dx} (w_+^x + w_-^x) \right) \right] \in \partial C(L^2),$$

and differentiating the jump relation $m_+ = m_- v_x$, we obtain $\frac{d}{dx} m_+ + i\xi m_+ + \xi m_+ \sigma = (\frac{d}{dx} m_- + i\xi m_- \sigma) v_x$.

A simple calculation shows that $i\xi[m_+ \sigma] = Q + C^\pm(i\xi(w_+^x + w_-^x), \sigma)$, where

$$(3.13) \quad Q = Q(x) = \frac{ad\sigma}{2\pi} \int_\mathbb{R} \mu(w_+^x + w_-^x)dz.$$ 

It follows that $M_\pm \equiv \frac{d}{dx} m_+ - i\xi[m_+ \sigma] - Q m_+ \in \partial C(L^2)$ and $M_+ = M_- v_x$. But then $M_\pm = 0$ by Proposition 2.14. Thus $m_\pm$ solve the differential equation $\frac{d}{dx} m_\pm + i\xi[m_\pm \sigma] + Q m_\pm$. By the symmetry property of $v_x$, $v_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\tilde{v}_x)^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, it follows that $Q(x)$ is of the form

$$\begin{pmatrix} 0 & q(x) \\ q(x) & 0 \end{pmatrix}$$

for some potential $q(x)$. Set $I(r) \equiv q$. The following results show that $I$ maps $H^{-1,1}$ into $H^{-1,1}(\mathbb{R}_-)$.

Lemma 3.14. For $x \leq 0$,

$$\|C^+(I - v_x^-)\|_{L^2}, \|C^-(v_x^- - I)\|_{L^2} \leq \frac{1}{(1 + x^2)^{1/2}} \|r\|_{H^{1,0}}.$$ 

Proof. We will only estimate $u \equiv C^-(v_x^- - I)$ in which $u_{21}$ is the only nonzero entry; $C^+(I - v_x^-)$ may be estimated in a similar manner. By Fourier transform

$$r(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iz\xi} \hat{f}(\xi) d\xi,$$

and hence

$$(3.15) \quad u_{21}(x, \xi) = C^- e^{-iz\xi} (-\bar{\hat{f}}) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i(\xi - x)\xi} (-\bar{\hat{f}})(\xi) d\xi,$$

where $f(\xi)$ denotes the function $z \mapsto f(z)$. Thus we have

$$(3.16) \quad \|u_{21}(x, \cdot)\|_{L^2(dz)} = \left( \int_{-\infty}^{+\infty} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \leq (1 + x^2)^{-1/2} \left( \int_{-\infty}^{+\infty} (1 + \xi^) |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \leq (1 + x^2)^{-1/2} \|r\|_{H^{1,0}}.$$
Theorem 3.17. I maps $H^{1,0}_1$ into $H^{0,1}$.

Proof. As above, we only consider $x \leq 0$. Write

$$\int_x ((1 - C_{v_{x\pm}})^{-1}I)(v_{x+} - v_{x-})dz \equiv \int_1 + \int_2 + \int_3,$$

where

$$\int_1 = \int (v_{x+} - v_{x-}),$$
$$\int_2 = \int (C_{v_{x\pm}}I)(v_{x+} - v_{x-})$$
$$\int_3 = \int (C_{v_{x\pm}}(1 - C_{v_{x\pm}})^{-1}C_{v_{x\pm}}I)(v_{x+} - v_{x-})$$
$$= \int (C_{v_{x\pm}}(\mu - I))(v_{x+} - v_{x-}).$$

We remark that for calculating $q$, the estimate of $\int_2$ is not needed because it is diagonal and $\text{ad} \sigma \int_2 = 0$. But the estimate is useful for other problems. Clearly, $\int (v_{x+} - v_{x-}) \in H^{0,1}(dx)$ by the Fourier transform. Using the triangularity of $v_{x\pm}$, the fact that $C^+ - C^- = 1$, Cauchy’s theorem, and Lemma 3.14, we have for $x \leq 0$,

$$\int_2 = \left| \int [(C^+(I - v_{x-}))(v_{x+} - I) + (C^-(I + I))(I - v_{x-})] \right|$$
$$\leq c(1 + x^2)^{-1},$$

for some $c > 0$.

Finally, by Lemma 3.14 and the uniform $L^2$-boundedness of $(1 - C_{v_{x\pm}})^{-1}$,

$$\|\mu - I\|_{L^2} = \|(1 - C_{v_{x\pm}})^{-1}C_{v_{x\pm}}I\|_{L^2} \leq c(1 + x^2)^{-1/2}$$

for some $c > 0$,

we have

$$\int_3 = \left| \int [(C^+(\mu - I))(v_{x+} - I) + (C^-(\mu - I))(v_{x+} - I)] \right|$$
$$\leq c(1 + x^2)^{-1},$$

for some $c > 0$.

Theorem 3.22. If $r \in H^{1,1}_1$, then $q \in H^{1,0}$.

Proof. Using the relation $\partial_{x\mu} = (iz \text{ ad} \sigma + Q)\mu$ and the fact that ad $\sigma$ is a derivation, we have

$$\partial_{x}(\mu e^{ixz} \text{ ad} \sigma (v_+ - v_-)) = (iz \text{ ad} \sigma + Q)(\mu e^{ixz} \text{ ad} \sigma (v_+ - v_-)).$$

Thus

$$Q' = \frac{\text{ad} \sigma}{2\pi} Q \int \mu(x_{x+} - v_{x-}) + \frac{\text{ad} \sigma}{2\pi} \int i \text{ ad} \sigma \mu(zv_{x+} - zv_{x-}).$$

The first term still gives rise to an $H^{0,1}$ function. On the other hand, the estimate (3.20) shows the second term in (3.23) gives rise to an $L^2$ function. □
Now consider the normalized RHP \((\mathcal{R}, \check{\nu}_x = \delta_{\pm}^3 v_x \delta_{\pm}^3)\) for \(m = \nu \delta^{-3}\) as above. By the properties of \(\delta = \delta_{\infty} = +\infty\), one sees that \(\check{\nu}_x\) has the form \(\begin{pmatrix} 1 & -\hat{\check{r}} e^{i\sigma x} \\ -\hat{\check{r}} e^{-i\sigma x} & 1 - |\hat{\check{r}}(z)|^2 \end{pmatrix}\), where \(\hat{r} = r \delta, ||\hat{r}||_{L\infty} = ||r||_{L\infty} < 1\).

There is a completely parallel theory for this RHP, with the difference that the problem is now adapted for studying the decay at \(x \to +\infty\). In particular one finds that \(\check{m}_\pm\) solve the differential equation \(\dot{\check{m}}_\pm = iz(\sigma, \check{m}_\pm) + \check{Q}\check{m}_\pm\), where \(\check{Q}(x) = \begin{pmatrix} 0 & \check{q}(x) \\ \check{q}(x) & 0 \end{pmatrix}\) \(\in H^{1,1}(\mathbb{R}_+).\) But as \(\delta = \delta(z)\) is independent of \(x\), it follows that the differential equations for \(m_\pm\) and \(\check{m}_\pm\) can only be compatible if \(Q = \check{Q}\) i.e. \(q(x) = \check{q}(x)\): we conclude in particular that \(q \in H^{1,1}(\mathbb{R})\). (More precisely, we have shown that \(\check{q}\) extends \(q(x)\) defined on \(\mathbb{R}_-\) to an \(H^{1,1}\) function on all of \(\mathbb{R}\).)

Let \(m(x,z) = I + (C(\mu(w^+_x + w^-_x)))(z)\) be the extension of \(m_\pm = I\) off \(\Sigma = \mathbb{R}\). Then by analytic continuation, for each \(z \in \mathbb{C}\backslash \mathbb{R}, m = m(x,z)\) solves the differential equation \(\frac{\partial m}{\partial x} = iz[\sigma, m] + Qm\) and by the Riemann–Lebesgue lemma and \((3.20)\), we conclude that

- \(m(x,z) \to I\) as \(x \to -\infty\).

Similarly \(\check{m}(x,z) \to I\) as \(x \to +\infty\) and hence

- \(m(x,z) = \check{m}(x,z)\delta(z)^{\nu_3}\) is bounded as \(x \to +\infty\).

This shows that \(m(x,z)\) is the (unique) \([BC]\)-type solution for \(Q = \begin{pmatrix} 0 & q(x) \\ \check{q}(x) & 0 \end{pmatrix}\) for all \(z \in \mathbb{C}\backslash \mathbb{R}\). As \(m_+ = m_- v_x = m_- \begin{pmatrix} 1 - |r|^2 & re^{i\sigma x} \\ -re^{-i\sigma x} & 1 \end{pmatrix}\), this means by definition that \(\mathcal{R}(q) = r\) i.e. \(\mathcal{R}(\mathcal{I}(r)) = r\). On the other hand \(\mathcal{R}\) is one-to-one, for if \(r = \mathcal{R}(q) = \mathcal{R}(q^\#)\) then we have two solutions \(m\) and \(m^\#\) of the normalized RHP \((\mathcal{R}, v_x)\), and hence \(\Delta m_+ = \Delta m_- v_x\) on \(\mathcal{R}, \Delta m_\pm \in \partial C(L^2)\), where \(\Delta m = m - m^\#\). But again by Proposition 2.14, if \(\Delta m \neq 0\), this implies that \(\text{Ker}(1 - C_{w_x}) \neq \{0\}\), which is a contradiction. Hence \(m_+(x,z) = m^\#(x,z)\) for all \(x\) and so \(q(x) = q^\#(x)\). Thus \(\mathcal{R}\) is a bijection with inverse \(\mathcal{R}^{-1} = \mathcal{I}\). This completes the proof that \(\mathcal{R}\) is bijective from \(H_1^{1,1}\) onto \(H_1^{1,1}\).

Finally, as noted above, the solution \(m_\pm\) of \((\Sigma, v_x)\) has the form \(m_\pm = I + C^\pm(\mu(w^+_x + w^-_x))\). Let \(m(x,z) = I + C(\mu(w^+_x + w^-_x))(z)\) again be the extension of \(m_\pm\) off \(\mathbb{R}\). As \(r \in H_1^{1,1}\), we see that

\[
(3.24) \quad m(x,z) = I + \frac{m_1(x)}{z} + o\left(\frac{1}{z}\right)
\]

as \(z \to \infty\) in any proper subsector of \(\mathbb{C}^+\) or \(\mathbb{C}^-\), where the residue \(m_1(x)\) of \(m(x,z)\) is given by

\[
(3.25) \quad m_1(x) = -\frac{1}{2\pi i} \int_\mathbb{R} \mu(w^+_x + w^-_x) ds.
\]

We conclude in particular that

\[
(3.26) \quad Q(x) = -i \text{ ad } \sigma(m_1(x))
\]

which implies

\[
(3.27) \quad q(x) = -i(m_1(x))_{12}
\]

as in \((1.6)\) above.

**Remark 3.28.** The above calculations also show that \(\mathcal{R}\) is a bijection from \(H^{0,1}\) onto \(H^{1,0}_1\).

### 4. Smoothing estimates and the proof of Theorem 1.10

Throughout this section we always assume that \(r \in H_1^{1,1}\), which corresponds to initial data \(q_0 = q(t = 0) = R^{-1}(r)\) in \(H_1^{1,1}\) for NLS, by the results of §3. The (unique, weak) solution of NLS in \(H_1^{1,1}\) with initial data \(q_0\) is given by \((1.8), q(t) = R^{-1}(e^{-i\phi^2 tr(\delta)}))\). In terms of the normalized RHP \((\Sigma = \mathbb{R}, v_0)\),

\[
(4.1) \quad m_+ = m_- v_0 \quad \text{on} \quad \mathbb{R}, \quad m_\pm \in I + \partial C(L^2),
\]
with $v_\theta = \begin{pmatrix} 1 - |r|^2 & r e^{i\theta} \\ -\bar{r} e^{-i\theta} & 1 \end{pmatrix} = (I - w_\theta^-)^{-1}(I + w_\theta^+)$ and

$$w_\theta = (w_\theta^-, w_\theta^+) = \begin{pmatrix} 0 & r e^{i\theta} \\ 0 & -\bar{r} e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -\bar{r} e^{-i\theta} \end{pmatrix},$$

the solution $q(t)$ is given by (3.13)

$$Q(t) = \begin{pmatrix} 0 \\ q(t) \end{pmatrix} = \frac{a d\sigma}{2\pi} \int_{\mathbb{R}} \mu(w_\theta^+ + w_\theta^-)dz$$

where $\mu \in I + L^2$ is the unique solution of $(1 - C w_\theta)\mu = I$.

As noted before, the steepest descent method proceeds by taking advantage of the signature table for $\mathbb{R}^{2}i\theta$ (see Figure 2.53). The key step (cf. [DIZ], [DZ2]) is to separate the factor $e^{i\theta}$ algebraically from the factor $e^{-i\theta}$. For $z > z_0$, we use the upper/lower factorization

$$v_\theta = \begin{pmatrix} 1 & r e^{i\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar{r} e^{-i\theta} & 1 \end{pmatrix},$$

and for $z < z_0$, we use the lower/diagonal/upper factorization

$$v_\theta = \begin{pmatrix} \frac{1 - r e^{-i\theta}}{1 - |r|^2} & 0 \\ 0 & \frac{1}{1 - |r|^2} \end{pmatrix} \begin{pmatrix} 1 - |r|^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r e^{i\theta} & 1 \end{pmatrix}. $$

The diagonal factors $(1 - |r|^2)^{\pm 1}$ can be removed by conjugating $v_\theta$ by $\delta_\pm^3$, where $\delta_\pm = e^{-i\theta} e_{-\theta}^\pm \log |r|^2$ as in (2.39). We obtain for $z > z_0$,

$$\tilde{v}_\theta \equiv \delta_+^3 v_\theta \delta_-^3 = \begin{pmatrix} 1 & r \delta_+^2 e^{i\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar{r} \delta_-^2 e^{i\theta} & 1 \end{pmatrix}$$

and for $z < z_0$

$$\tilde{v}_\theta \equiv \delta_+^3 v_\theta \delta_-^3 = \begin{pmatrix} 1 & r \delta_-^2 e^{i\theta} \\ -\bar{r} \delta_+^2 e^{-i\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(cf. (R, $\tilde{v}_\theta$) in §3). Note that the multipliers $e^{i\theta}$ and $e^{-i\theta}$ have now been separated in the sense that they lie in different factors of $\tilde{v}_\theta$. If $\tilde{m}_\pm$ is the solution of the normalized RHP $(\mathbb{R}, \tilde{v}_\theta)$, then $\tilde{m}_\pm \equiv \tilde{m}_\pm \delta_\pm^3$ clearly solves the normalized RHP $(\mathbb{R}, \tilde{v}_\theta)$. Rewriting the jump relation for $\tilde{v}_\theta$ for $z > z_0$ in the form

$$\tilde{m}_+ \begin{pmatrix} 1 & 0 \\ -\bar{r} \delta_+^2 e^{i\theta} & 1 \end{pmatrix}^{-1} = \tilde{m}_- \begin{pmatrix} 1 & r \delta_-^2 e^{i\theta} \\ 0 & 1 \end{pmatrix},$$

and neglecting, for the moment, analyticity issues for the coefficients $r$, $-\bar{r}$, we see that $\tilde{m}_1 \begin{pmatrix} 1 & 0 \\ -\bar{r} \delta_+^2 e^{i\theta} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ can be continued to a sector above $\mathbb{R}_+ + z_0$ and $\tilde{m}_- \begin{pmatrix} 1 & r \delta_-^2 e^{-i\theta} \\ 0 & 1 \end{pmatrix}$ can be continued to the sector below $\mathbb{R}_+ + z_0$. The same is true for the appropriate factors on $\mathbb{R}_- + z_0$. We then obtain a RHP on a cross $(z_0 + e^{i\pi/4} \mathbb{R}) \cup (z_0 + e^{-i\pi/4} \mathbb{R})$, say, and things are so arranged so that all the factors $e^{\pm i\theta}$ are now exponentially decreasing. As $t \to \infty$, the RHP problem then localizes at $z_0$.

The main analytical task in the method is to handle the lack of analyticity of the coefficients $r$, $-\bar{r}$ etc. In [DIZ], [DZ2], this is done by approximating these coefficients by rational functions to high enough order at $z = z_0$, and this requires a high order of smoothness and decay for $r$. In this paper we show that a suitable approximation can still be done when $r$ is just in $H^{1,1}_1$, but now we must utilize cancellations from oscillations, and not just absolute type estimates. As mentioned in §1, Theorem 1.10 with $r = \mathcal{R}(q_0) \in H^{1,1}_1$ is precisely what we need for the perturbation theory of NLS.
For reasons that will become clear further on we reverse the orientation on $\mathbb{R}_- + z_0$ to obtain a contour $\tilde{R}_{z_0} = e^{i\pi} (\mathbb{R}_+ + z_0) \cup (\mathbb{R}_+ + z_0)$ with associated jump matrix $\tilde{v}_\theta = \tilde{v}_\theta$ for $z > z_0$ and $\tilde{v}_\theta = \tilde{v}_\theta$ for $z < z_0$ (see discussion preceding Proposition 2.20). Observe that if $\tilde{m}_\pm = I + C_{\mathbb{R}_-} \tilde{h} \in I + \partial C(L^2)$ is the solution of the normalized RHP ($\mathbb{R}, \tilde{v}_\theta$), then $\tilde{m}_\pm = I + C_{\tilde{R}_{z_0}} \tilde{h} \in I + \partial C(L^2)$ is the solution of the normalized RHP ($\tilde{R}_{z_0}, \tilde{v}_\theta$) if $\tilde{h} \equiv \tilde{h}$ for $z > z_0$ and $\tilde{h} \equiv \tilde{h}$ for $z < z_0$, and vice versa. Moreover, the extension $\tilde{m}(z)$ of $\tilde{m}_\pm$ of $\tilde{R}_{z_0}$ is the same as the extension $\tilde{m}(z)$ of $\tilde{m}_\pm$ off $\mathbb{R}$, and is clearly given by $m(z)\delta(z)^{-\sigma_3}$, $z \in \mathbb{C} \setminus \mathbb{R}$. If $m_1 = m_1(x,t)$ and $\delta_1$ are the residues (cf. (3.24)) of $m(z) = m(x,t,z)$ and $\delta(z)^{-\sigma_3}$ respectively,

$$\begin{align*}
m(x,t,z) &= I + \frac{m_1(x,t)}{z} + o\left(\frac{1}{z}\right), \\
\delta(z)^{-\sigma_3} &= I + \frac{\delta_1(z)}{z} + o\left(\frac{1}{z}\right) \equiv I + \left(\frac{1}{2\pi i} \int_{\mathbb{R}_- + z_0} \log(1 - |r|^2) \delta_1^{-\sigma_3} \right) \frac{1}{z} + o\left(\frac{1}{z}\right), \end{align*}$$

then we see that the residue $\tilde{m}_1(x,t)$ of $\tilde{m}(x,t,z)$ is given by $\tilde{m}_1(x,t) = m_1(x,t) + \delta_1$. But then by (3.27), as $\delta_1$ is diagonal,

$$q(x,t) = -i (m_1(x,t))_{12} = -i (\tilde{m}_1(x,t))_{12}.$$

From the form of $\tilde{v}_\theta$, we see that $\tilde{v}_\theta (I - \tilde{w}_\theta^{-1})^2 (I + \tilde{w}_\theta^{-1})$ where

$$\tilde{w}_\theta = (\tilde{w}_\theta^-, \tilde{w}_\theta^+) = \begin{pmatrix} \begin{pmatrix} 0 & r \delta_+ e^{i\theta} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ -r \delta_+ e^{-i\theta} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -r \delta_+ \delta_+ e^{i\theta} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ r \delta_+^{-1} \delta_+^{-1} e^{-i\theta} & 0 \end{pmatrix} \end{pmatrix},$$

which can also be written as

$$\tilde{w}_\theta = \begin{pmatrix} \begin{pmatrix} 0 & -r \delta_+ e^{i\theta} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ r \delta_+^{-1} \delta_+^{-1} e^{-i\theta} & 0 \end{pmatrix} \end{pmatrix},$$

where $\delta_\pm(z)$ denotes the boundary values of $\delta(z)$ on $\tilde{R}_{z_0}$. Thus $\tilde{\delta}_\pm(z) = \delta_\pm(z)$ for $z > z_0$ and $\tilde{\delta}_\pm(z) = \delta_\mp(z)$ for $z < z_0$. Observe that if we reverse the orientation as above for the normalized RHP ($\tilde{R}, \tilde{v}_\theta$) (see (2.48) et seq), we obtain a normalized RHP ($\tilde{R}_{z_0}, \tilde{v}_\theta$) where $\tilde{v}_\theta = (\tilde{w}_\theta^-, \tilde{w}_\theta^+) = \begin{pmatrix} \begin{pmatrix} 0 & -r \delta_+ e^{i\theta} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ r \delta_+^{-1} \delta_+^{-1} e^{-i\theta} & 0 \end{pmatrix} \end{pmatrix}$, $z < z_0$.

Our goal is to prove that the solution of ($\tilde{R}_{z_0}, \tilde{v}_\theta$) is close to the solution of ($\tilde{R}_{z_0}, \tilde{v}_\theta$) as $t \to \infty$. For later purposes, note that what the reversal of orientation on $\mathbb{R}_+ + z_0$ achieves, is that the triangularity of $\tilde{w}_\theta^+$ is the same for $z > z_0$ as for $z < z_0$, and similarly for $\tilde{w}_\theta^+$. Also note that by Proposition 2.20, $C_{\tilde{w}_\theta} = C_{\tilde{w}_\theta}$, and hence by Proposition 2.49, $(1 - C_{\tilde{w}_\theta})^{-1}$ exists for large $t$, say $t \geq t_0$, and

$$\|(1 - C_{\tilde{w}_\theta})^{-1}\|_{L^2(\tilde{R}_{z_0})} \leq c$$

for all $x \in \mathbb{R}$ and for all $t \geq t_0$.

Also, by (2.30), $\|(1 - C_{\tilde{w}_\theta})^{-1}\|_{L^2(\mathbb{R})} \leq \frac{1}{1 - \|r\|_{L^\infty}}$, and again using Proposition (2.20) it is easy to see that $(1 - C_{\tilde{w}_\theta})^{-1}$ exists in $L^2(\tilde{R}_{z_0})$ and we have the bound

$$\|(1 - C_{\tilde{w}_\theta})^{-1}\|_{L^2(\tilde{R}_{z_0})} \leq c.$$

First we need some technical lemmas. We use $\rho < 1, \lambda$ and $\eta$ to denote $L^\infty$, $H^{1,0}$ and $H^{1,1}$ bounds for $r$ respectively. Thus

$$\|r\|_{L^\infty} \leq \rho, \quad \|r\|_{H^{1,0}} \leq \lambda, \quad \|r\|_{H^{1,1}} \leq \eta.$$

Extend $\tilde{R}_{z_0}$ to a contour $\Gamma_{z_0} = \tilde{R}_{z_0} \cup (z_0 + e^{i\pi/2 R_+}) \cup (z_0 + e^{-i\pi/2 R_-})$. 

LONG–TIME ASYMPTOTICS 25
As a complete contour (see e.g. [Z1]), $\Gamma_{z_0}$ has the important property

\begin{equation}
C^{+}\Gamma_{z_0}C^{-}\Gamma_{z_0} = C^{-}\Gamma_{z_0}C^{+}\Gamma_{z_0} = 0.
\end{equation}

In the following Lemmas we will assume for convenience that $x = 0$. Thus $z_0 = 0$, $\delta = \delta_{z_0} = 0$, $\Delta = \Delta_{z_0} = 0$, $\overline{R} \equiv \overline{R}_{z_0} = 0$, $\Gamma \equiv \Gamma_{z_0} = 0$, and $\theta = -tz^2$. Observe that the bounds in the Lemmas depend only on $\rho$ and $\lambda$, but not on $\eta$.

**Lemma 4.12.** For $z \in \mathbb{R}\setminus 0$,

\begin{equation}
|\Delta'(z)| \leq I + II,
\end{equation}

where

\begin{align}
||I||_{L^2} &\leq \frac{c\rho \lambda}{1 - \rho}, \\
II &\leq \frac{c\rho^2}{1 - \rho} \frac{1}{|z|}.
\end{align}

**Proof.** We have

$$\Delta'(z) = -\Delta(z)\frac{d}{dz}H((\log(1 - |r|)\chi_R)$$

$$= \Delta(z)H\left(\frac{|r|^2}{1 - |r|^2}\chi_R\right) - \frac{i\Delta \log(1 - |r(0)|^2)}{z}.$$ 

The result now follows from the $L^2$ mapping properties of $H$, the identity $|\Delta| = 1$, and the elementary bound $|\log(1 - |r(0)|^2)| \leq \frac{|r(0)|^2}{1 - |r(0)|^2}$. \[\square\]

**Lemma 4.16.** Suppose $f \in H^{1,0}$. Then for all $t > 1$,

\begin{equation}
\left| \int_{\mathbb{R}} f \Delta^{\pm} e^{\mp itz^2} \, dz \right| \leq \frac{c}{t^{3/2}} \frac{1 + \lambda}{(1 - \rho)} \|f\|_{H^{1,0}}.
\end{equation}

In addition, if $f(0) = 0$, then for all $t > 1$,

\begin{equation}
\left| \int_{\mathbb{R}} f \Delta^{\pm} e^{\mp itz^2} \, dz \right| \leq \frac{c}{t^{3/4}} \frac{1 + \lambda}{(1 - \rho)} \|f\|_{H^{1,0}}.
\end{equation}

**Proof.** We only consider the case $\Delta = \Delta^+$ above. The other case is similar. Decompose the integral as follows:

$$\int_{-\infty}^{\infty} f \Delta e^{-itz^2} \, dz = \int_{|z| < \frac{1}{t^2}} f \Delta e^{-itz^2} \, dz + \int_{|z| > \frac{1}{t^2}} f \Delta e^{-itz^2} \, dz \equiv I + II.$$

Figure 4.10  $\Gamma_{z_0}$
Changing variables, we obtain
\[ |I| = \frac{1}{\sqrt{t}} \left| \int_{-1}^{1} f \left( \frac{z}{\sqrt{t}} \right) \Delta \left( \frac{z}{\sqrt{t}} \right) e^{-iz^2} \, dz \right| \leq \frac{1}{\sqrt{t}} \|f\|_{L^\infty} \leq \frac{c}{\sqrt{t}} \|f\|_{H^{1,0}}. \]

We consider \( z < -t^{-1/2} \). The case \( z > t^{1/2} \) is similar. Integration by parts leads to
\[
\int_{z < -t^{-1/2}} f \Delta e^{-itz^2} \, dz = \frac{e^{-i}}{2i\sqrt{t}} f \left( -\frac{1}{\sqrt{t}} \right) \Delta \left( -\frac{1}{\sqrt{t}} \right) + \frac{1}{2it} \int_{-\infty}^{-t^{1/2}} e^{-itz^2} \left( \frac{f' \Delta}{z} + \frac{f' \Delta'}{z} - \frac{f \Delta}{z} \right) \, dz \\
\equiv \Gamma + \Pi'' + \Pi' + IV'.
\]

Clearly
\[ |\Gamma| \leq \frac{c\|f\|_{H^{1,0}}}{\sqrt{t}}, \quad |\Pi'| \leq \frac{c}{t^{3/4}} \|f\|_{H^{1,0}}, \quad \text{and} \quad |IV'| \leq \frac{c\|f\|_{H^{1,0}}}{t^{1/2}}. \]

Finally using Lemma (4.12), we obtain
\[
|\Pi'| \leq \frac{\|f\|_{L^\infty}}{2t} \int_{-\infty}^{-t^{1/2}} \left| \frac{\Delta'}{z} \right| \, dz \leq \frac{\|f\|_{H^{1,0}}}{2t} \left( \frac{\rho \lambda}{1 - \rho} t^{1/4} + \frac{c\rho^2}{1 - \rho} t^{1/2} \right)
\]

which now leads directly to (4.17). If \( f(0) = 0 \), then the same arguments together with the bound \( |f(z)| \leq |z|^{1/2}\|f\|_{H^{1,0}} \) for \( |z| \leq 1 \), say, in I, I', III', and IV', yield (4.18).

Let \( D_j, j = 1, \ldots, 4 \), be the \( j \)th quadrant in \( \mathbb{C}\setminus\Gamma \)

\[
\begin{array}{c|c|c}
D_2 & D_3 & D_4 \\
\hline
D_3 & D_4 & \Gamma
\end{array}
\]

\[ \text{Figure 4.19} \]

In the Lemma below \( H^q \) denotes Hardy space. A general reference for Hardy spaces is, for example, [Dur].

**Lemma 4.20.** Suppose \( f \in H^{1,0} \), then for \( 2 \leq p < \infty \) and for all \( t \geq 0 \),
\[
\begin{align*}
\|C_{e^{it\Delta}} f e^{it\varphi^2} \|_{L^p} &\leq \frac{c}{(1 + t)^{1/2p}} \|\varphi^2\|_{L^\infty(D_1)} \|f\|_{H^{1,0}} \leq \frac{c}{(1 + t)^{1/2p}} \|f\|_{H^{1,0}}, \\
\|C_{e^{it\Delta}} f e^{it\varphi^2} \|_{L^p} &\leq \frac{c}{(1 + t)^{1/2p}} \|\varphi^2\|_{L^\infty(D_1)} \|f\|_{H^{1,0}} \leq \frac{c}{(1 + t)^{1/2p}} \|f\|_{H^{1,0}}, \\
\|C_{e^{it\Delta}} f e^{it\varphi^2} \|_{L^p} &\leq \frac{c}{(1 + t)^{1/2p}} \|\varphi^2\|_{L^\infty(D_1)} \|f\|_{H^{1,0}} \leq \frac{c}{(1 + t)^{1/2p}} \|f\|_{H^{1,0}}, \\
\|C_{e^{it\Delta}} f e^{it\varphi^2} \|_{L^p} &\leq \frac{c}{(1 + t)^{1/2p}} \|\varphi^2\|_{L^\infty(D_1)} \|f\|_{H^{1,0}} \leq \frac{c}{(1 + t)^{1/2p}} \|f\|_{H^{1,0}}.
\end{align*}
\]

Suppose in addition that \( f(0) = 0 \) and that \( g \) is a function in the Hardy space \( H^q(\mathbb{C}\setminus\mathbb{R}) \) for some \( 2 \leq q \leq \infty \). Then for all \( t \geq 0 \),
\[
\begin{align*}
\|C_{e^{it\Delta}} f e^{it\varphi^2} \|_{L^2} &\leq \frac{c}{(1 + t)^{1/4}} \|g\|_{H^q(\mathbb{C}\setminus\mathbb{R})} \|f\|_{H^{1,0}}, \\
\|C_{e^{it\Delta}} f e^{it\varphi^2} \|_{L^2} &\leq \frac{c}{(1 + t)^{1/4}} \|g\|_{H^q(\mathbb{C}\setminus\mathbb{R})} \|f\|_{H^{1,0}}, \\
\|C_{e^{it\Delta}} f e^{it\varphi^2} \|_{L^2} &\leq \frac{c}{(1 + t)^{1/4}} \|g\|_{H^q(\mathbb{C}\setminus\mathbb{R})} \|f\|_{H^{1,0}}, \\
\|C_{e^{it\Delta}} f e^{it\varphi^2} \|_{L^2} &\leq \frac{c}{(1 + t)^{1/4}} \|g\|_{H^q(\mathbb{C}\setminus\mathbb{R})} \|f\|_{H^{1,0}},
\end{align*}
\]
where \( g_{\pm} \) are the boundary values of \( g \) on \( \mathbb{R} \) and \( \tilde{g}_{\pm} = g^{\mp} \) on \( e^{i\pi} \mathbb{R}_+ \).

**Proof.** Consider the first inequality in (4.21). The other cases in (4.21) are similar. By Fourier theory,
\[
f(z) = \frac{1}{\sqrt{2\pi}} \int e^{-iyz} \hat{f}(y) dy.
\]
We have for any \( \epsilon > 0 \),
\[
C_{\mathbb{R}_+}^{-1} \delta^{-2} e^{-\epsilon \hat{\phi}} e^{i \hat{\phi}^2} = \frac{1}{\sqrt{2\pi}} \int \hat{f}(y) e^{-iyz} \hat{F}_1(y) dy + \frac{1}{\sqrt{2\pi}} \int \hat{f}(y) e^{-iyz} \hat{F}_2(y) dy
\]
where
\[
F_1(y) = C_{\mathbb{R}_+}^{-1} \left( \delta^{-2} e^{-\epsilon \hat{\phi}} \chi_{(0,a)} e^{i t(\hat{\phi}^2 - y/2t)} \right)
\]
and
\[
F_2(y) = C_{\mathbb{R}_+}^{-1} \left( \delta^{-2} e^{-\epsilon \hat{\phi}} \chi_{(a,\infty)} e^{i t(\hat{\phi}^2 - y/2t)} \right)
\]
and \( a = \max(0,y/2t) \). (The factor \( e^{-\epsilon \hat{\phi}} \) is included just to ensure that \( F_2(y) \) exists in \( L^p \).) Clearly \( F_1(y) \) is supported on \( \mathbb{R}_+ \). Assume first that \( p > 2 \). Then for \( y > 0 \),
\[
\|F_1\|_{L^p(\Gamma)} \leq c \|\delta^{-2} \|_{L^\infty(\mathbb{R}_+)} \|\chi_{(0,a)}\|_{L^p} \leq c \|\delta^{-2}\|_{L^\infty(\mathbb{R}_+)} \left| \frac{y^{1/p}}{\Gamma^1/p} \right|
\]
and hence
\[
\left\| \frac{1}{\sqrt{2\pi}} \int dy \hat{f}(y) e^{-i y^2/4t} \hat{F}_1 \right\|_{L^p(\Gamma)} \leq c \left| \frac{\|\delta^{-2}\|_{L^\infty(\mathbb{R}_+)}}{\Gamma^{1/p}} \right| \int_0^\infty \hat{f}(y) |y|^p dy \leq c \left| \frac{\|\delta^{-2}\|_{L^\infty(\mathbb{R}_+)}}{\Gamma^{1/p}} \right| \hat{f} \|_{H^{1,0}}.
\]
For \( p = 2 \), rewrite the integral as
\[
\frac{1}{\sqrt{2\pi}} C_{\mathbb{R}_+}^{-1} \left( \int_0^\infty \delta^{-2} e^{-\epsilon \hat{\phi}} \hat{f}(y) e^{-i y^2/4t} e^{i \hat{\phi}^2} dy \right).
\]
Using Hardy’s inequality [HLP]
\[
\left\| \int_0^\infty |\hat{f}(y)| dy \right\|_{L^2(\mathbb{R}_+)} \leq \frac{2}{\sqrt{\pi}} \|\hat{f}\|_{L^2(\mathbb{R}_+)}
\]
and hence
\[
\left\| \frac{1}{\sqrt{2\pi}} \int \hat{f} e^{-\epsilon \hat{\phi}} e^{-i y^2/4t} \hat{F}_1 \right\|_{L^2(\Gamma)} \leq c \left| \frac{\|\delta^{-2}\|_{L^\infty(D_1)}}{\Gamma^{1/2}} \right| \hat{f} \|_{H^{1,0}}.
\]
For \( F_2 \), first consider the case when \( y < 0 \), and hence \( a = 0 \). Then for \( p \geq 2 \),
\[
\|C_{\mathbb{R}_+}^{-1} \delta^{-2} e^{-\epsilon \hat{\phi}} e^{i t(\hat{\phi}^2 - y/2t)} \|_{L^p(\Gamma)} = \|C_{\mathbb{R}_+}^{-1} \delta^{-2} e^{-\epsilon \hat{\phi}} e^{i t(\hat{\phi}^2 - y/2t)} \|_{L^p(\Gamma)}
\]
by Cauchy’s Theorem,
\[
\leq c \|\delta^{-2}\|_{L^\infty(D_1)} \|e^{i t(\hat{\phi}^2 - y/2t)}\|_{L^p(e^{i \pi/4 R_+})}
\]
\[
\leq c \|\delta^{-2}\|_{L^\infty(D_1)} \|e^{i \hat{\phi}^2} e^{-i y^2/4t} \|_{L^p(e^{i \pi/4 R_+})}
\]
\[
\leq c \|\delta^{-2}\|_{L^\infty(D_1)} \|e^{i \hat{\phi}^2}\|_{L^2(e^{i \pi/4 R_+})}, \quad \text{as } y < 0
\]
\[
\leq \frac{c}{\Gamma^{1/2p}} \|\delta^{-2}\|_{L^\infty(D_1)}, \quad \text{by scaling}.
\]
If \( y > 0 \), then \( a = y/2t \) and for \( p \geq 2 \),
\[
\|C_{\mathbb{R}_+}^{-1} \delta^{-2} e^{-\epsilon \hat{\phi}} \chi_{(a,\infty)} e^{i t(\hat{\phi}^2 - y/2t)} \|_{L^p(\Gamma)} = \|C_{(y/2t,\infty)}^{-1} \delta^{-2} e^{-\epsilon \hat{\phi}} e^{i t(\hat{\phi}^2 - y/2t)} \|_{L^p(\Gamma)}
\]
\[
= \|C_{(y/2t+e^{i \pi/4 R_+})^{-1}} \delta^{-2} e^{-\epsilon \hat{\phi}} e^{i t(\hat{\phi}^2 - y/2t)} \|_{L^p(\Gamma)}, \quad \text{again by Cauchy},
\]
\[
\leq c \|\delta^{-2}\|_{L^\infty(D_1)} \|e^{i t(\hat{\phi}^2 - y/2t)}\|_{L^p(y/2t+e^{i \pi/4 R_+})}
\]
\[
\leq \frac{c}{\Gamma^{1/2p}} \|\delta^{-2}\|_{L^\infty(D_1)}, \quad \text{again by scaling}.
\]
Thus for \( p \geq 2 \),
\[
(4.27) \quad \left\| \frac{1}{\sqrt{2\pi}} \int dy \tilde{f}(y)e^{-iy^2/4t}F_2 \right\|_{L^p(\Gamma)} \leq \frac{c}{t^{\frac{2}{p}}} \|f\|_{H^{1.0}}.
\]

Letting \( \epsilon \downarrow 0 \) in (4.25)–(4.27), we obtain for \( p \geq 2 \) and \( t \geq 1 \),
\[
\|C^{-}_{\mathbb{R}_+} \rightarrow \Gamma (\delta^{-2} f e^{it\mathcal{O}^2})\|_{L^p(\Gamma)} \leq \frac{c}{t^{\frac{2}{p}}} \|\delta^{-2} f \|_{L^\infty(\mathcal{D}_1)} \|f\|_{H^{1.0}}.
\]
As
\[
\|C^{-}_{\mathbb{R}_+} \rightarrow \Gamma (\delta^{-2} f e^{it\mathcal{O}^2})\|_{L^p(\Gamma)} \leq c \|\delta^{-2} f \|_{L^\infty(\mathcal{D}_1)} \|f\|_{H^{1.0}} \quad \text{for all} \quad t > 0,
\]
we obtain (4.21).

Now we prove the first inequality in (4.22). Again, the remaining inequalities are similar. As before, we have the representation,
\[
C^{-}_{\mathbb{R}_+} \rightarrow \Gamma g_+ e^{-\epsilon_0} f e^{it\mathcal{O}^2} = \frac{1}{\sqrt{2\pi}} \int f(y)e^{-iy^2/4t}F_1 \ dy + \frac{1}{\sqrt{2\pi}} \int f(y)e^{-iy^2}F_2 \ dy
\]
but now as \( \int f(y)dy = \sqrt{2\pi} f(0) = 0 \),
\[
F_1(y) = C^{-}_{\mathbb{R}_+} \rightarrow \Gamma (g_+ e^{-\epsilon_0} \chi_{(0,a)} e^{i(\tilde{h} - y/2t)^2} (1 - e^{iy\mathcal{O}}))
\]
\[
F_2(y) = C^{-}_{\mathbb{R}_+} \rightarrow \Gamma (g_+ e^{-\epsilon_0} \chi_{(a,\infty)} e^{i(\tilde{h} - y/2t)^2} (1 - e^{iy\mathcal{O}})).
\]
As before, for \( y > 0 \), the integral involving \( F_1 \) can be rewritten as
\[
\frac{1}{\sqrt{2\pi}} C^{-}_{\mathbb{R}_+} \rightarrow \Gamma \left( \int_{2t\tilde{h}}^\infty g_+ e^{-\epsilon_0} \tilde{f}(y)(e^{-iy\mathcal{O}} - 1)e^{i\tilde{h}^2 \ dy} \right).
\]
For \( q' \geq 2 \), using the inequality
\[
\left( \int_{2t\tilde{h}}^\infty |\tilde{f}(y)|dy \right)^{q'} \leq \left( \int_0^\infty |\tilde{f}(y)|dy \right)^{q'} \left( \int_{2t\tilde{h}}^\infty |\tilde{f}(y)|dy \right)^2
\]
together with the above Hardy inequality, we obtain for \( t > 0 \)
\[
\left\| \int_{2t\tilde{h}}^\infty |\tilde{f}(y)|dy \right\|_{L^{q'}(\mathbb{R}_+)} \leq \frac{c}{t^{\frac{1}{q'}}} \|f\|_{H^{1.0}}.
\]
Hence for \( \frac{1}{q} + \frac{1}{q'} = \frac{1}{2} \),
\[
\left\| \frac{1}{\sqrt{2\pi}} \int f e^{-\epsilon_0} e^{-iy^2/4t}F_1 \ dy \right\|_{L^2(\Gamma)} \leq \frac{c\|g_+\|_{L^q(\mathbb{R}_+)} \|\tilde{f}\|_{H^{1.0}}}{t^{\frac{1}{q}} \|f\|_{H^{1.0}}} \leq \frac{c\|g_+\|_{H^\alpha(\mathcal{C}\backslash \mathbb{R})}}{t^{\frac{1}{2} - \frac{1}{q}}} \|f\|_{H^{1.0}}.
\]
Now for \( y < 0 \), as above,
\[
\|F_2(y)\|_{L^2} \leq c\|g e^{i(\tilde{h} - y/2t)^2} (1 - e^{iy\mathcal{O}})\|_{L^2(\mathbb{R}_{\epsilon^2/4}\mathbb{R}_+)} = c\|g e^{i\tilde{h}^2} (e^{-iy\mathcal{O}} - 1)\|_{L^2(\mathbb{R}_{\epsilon^2/4}\mathbb{R}_+)}. \]
Hence for \( \alpha > 0 \) and \( t > 0 \),
\[
\left\| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 f e^{-\epsilon_0} e^{-iy^2/4t}F_2 \ dy \right\|_{L^2(\Gamma)} \leq c \int_{-\infty}^0 \ dy |\tilde{f}(y)| \left( \int_{-\infty}^\infty d\gamma |g(e^{i\pi/4}\mathcal{O})|^2 e^{-2\gamma^2} |e^{-ie\pi/4\gamma y} - 1|^2 \right)^{\frac{1}{2}}
\]
\[
\leq c \int_{-\infty}^0 \ dy |\tilde{f}(y)| \left( \int_{-\infty}^\infty d\gamma \gamma^2 |g(e^{i\pi/4}\mathcal{O})|^2 e^{-2\gamma^2} \right)^{\frac{1}{2}}
\]
\[
+ \int_{-\infty}^{-t} \ dy |\tilde{f}(y)| \left( \int_{-\infty}^\infty d\gamma |g(e^{i\pi/4}\mathcal{O})|^2 e^{-2\gamma^2} \right)^{\frac{1}{2}}
\]
\[
\leq c(t^{-3/4+\alpha/2} + t^{-\alpha/2-1/4})\|f\|_{H^{1.0}} \|g(\sqrt{-\tilde{h}})\|_{H^{\alpha}(\mathcal{C}\backslash \mathbb{R})}.
\]
Taking $\alpha = 1/2$, we obtain
\[
\left\| \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{f} e^{-it\phi} e^{-iy^2/4t} F_2 \ dy \right\|_{L^2(\Gamma)} \leq \frac{c}{L^{1/2 - 1/2q}} \|f\|_{H^{1,0}} \|g\|_{H^q(C \setminus \mathbb{R})}.
\]
Finally for $y > 0$,
\[
\|F_2(y)\|_{L^2} \leq c \|ge^{it(\phi-y/2t)}(1-e^{iy\phi})\|_{L^2(y/2t+e^{i\pi/4}\mathbb{R}_+)},
\]
and hence for $\alpha > 0$ and $t > 0$,
\[
\left\| \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{f} e^{-it\phi} e^{-iy^2/4t} F_2 \ dy \right\|_{L^2(\Gamma)}
\leq c \int_0^\infty dy|\hat{f}(y)| \left( \int_0^\infty d\gamma |g(y/2t + e^{i\pi/4}\gamma)|^2 e^{-2t\gamma^2} |1 - e^{iy(y/2t+e^{i\pi/4}\gamma)|^2} \right)^{1/2}
\leq c \int_0^\infty dy|\hat{f}(y)| \left( \int_0^\infty d\gamma |g(y/2t + e^{i\pi/4}\gamma)|^2 e^{-2t\gamma^2} \right)^{1/2}
\]
\[
+c \int_0^\infty dy|\hat{f}(y)| \left( \int_0^\infty d\gamma |g(y/2t + e^{i\pi/4}\gamma)|^2 e^{-2t\gamma^2} \right)^{1/2}
\leq c(t^{-5/4+3\alpha/2} + t^{-3/4+\alpha/2} + t^{-\alpha/2-1/4}) \|f\|_{H^{1,0}} \|g(\phi/\sqrt{t})\|_{H^q(C \setminus \mathbb{R})}.
\]
Again, setting $\alpha = 1/2$, we find for $t > 0$,
\[
\left\| \frac{1}{\sqrt{2\pi}} \int_0^\infty f e^{-it\phi} e^{-iy^2/4t} F_2 \ dy \right\|_{L^2(\Gamma)} \leq \frac{c}{L^{1/2 - 1/2q}} \|f\|_{H^{1,0}} \|g\|_{H^q(C \setminus \mathbb{R})}.
\]
On the other hand, as before,
\[
\|C_{\mathbb{R}_+}^-g + f e^{it\phi}\|_{L^2(\Gamma)} \leq c \|g\|_{L^\infty(\mathbb{R}_+)} \|f\|_{H^{1,0}} \leq c \|g\|_{H^q(C \setminus \mathbb{R})} \|f\|_{H^{1,0}} \quad \text{for all} \quad t > 0,
\]
and (4.22) follows. \qed

**Corollary 4.28 to Lemma 4.20.** For any $2 \leq p < \infty$, and for all $t \geq 0$
\[
(4.29) \quad \|C_{\mathbb{R}_+}^- g + \tilde{w}_t \|_{L^p(C \setminus \mathbb{R})} \leq \frac{c}{(1 + t)^{1/2p}} \lambda \frac{1}{(1 - \rho)^{2p}}.
\]

**Proof.** Translate $z \to z_0 + z$, and then apply Lemma (4.20) to appropriate choices of $t$ (see $\tilde{w}_t$ in (4.6)-(4.7)). The inequality then follows as the $H^{1,0}$ and $L^\infty$ norm of $r$ is invariant under translation, $r(\cdot) \to r(\cdot + z_0)$. \qed

We need the following $L^p$ bound on solutions of RHP’s of type (1.4).

**Proposition 4.30.** Suppose that $r$ is a continuous function on $\mathbb{R}, \lim_{z \to \infty} r(z) = 0$ and $\|r\|_{L^\infty(\mathbb{R})} < \rho < 1$. Then for any $p \geq 2$, there exists $t_0 = t_0(r,p)$ such that for $t \geq t_0$ and all $x \in \mathbb{R}$, $(1 - C_{w_x})^{-1}$ exists in $L^p(\mathbb{R})$ and
\[
(4.31) \quad \|(1 - C_{w_x})^{-1}\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leq c
\]
for $t \geq t_0$ and all $x \in \mathbb{R}$.

**Remark.** In [DZ6] (see also WEBPAGE to [DZ5]) the authors prove the following stronger, \textit{a priori} estimate, which is needed for the perturbation theory in [DZ5]: Suppose $r \in H^{1,0}_1, \|r\|_{H^{1,0}} \leq \lambda, \|r\|_{L^\infty} \leq \rho < 1$. Then for any $2 < p < \infty$, there exists $l_1 = l_1(p), l_2 = l_2(p) > 0$, and a constant $c = c_p$, such that
\[
\|(1 - C_{w_x})^{-1}\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leq c \left( \frac{1 + \lambda}{1 - \rho} \right)^{l_1}
\]
for all $x, t \in \mathbb{R}$. Inequality (4.31, however, is sufficient for the result in this paper.

In the linear case, the Cauchy operator $C^+$, for example, maps $L^p \to L^p$, $\|C^+ f\|_{L^p(\mathbb{R})} \leq c_p \|f\|_{L^p(\mathbb{R})}$, for $1 < p < \infty$. In particular, if $\phi = \phi(z)$ is a real valued function, we see that $\|C^+ e^{iz\theta}\|_{L^p(\mathbb{R})} \leq c_p \|e^{iz\theta}\|_{L^p(\mathbb{R})} =$ $c_p \|f\|_{L^p(\mathbb{R})}$, and the bound is independent of $\phi$. The above bounds on $(1 - C_{\omega_\theta})^{-1}$ in $L^p$, which are uniform in the multiplier $e^{i\theta}$, should be viewed as nonlinear versions of such estimates.

**Proof of Proposition 4.30.**

For $\epsilon_1 > 0$, to be determined below, choose a rational function $R = R(z)$ such that

$$\|r-R\|_{L^\infty} \leq \epsilon_1$$

and

$$\|R\|_{L^\infty} < \rho, \quad \lim_{z \to \pm \infty} R(z) = 0.$$  

Associated with $R$ is the model RHP with jump matrix $v^# = v_R^#$ on $\Sigma^\epsilon$ as in (2.61), except $r(z_0)$ is replaced by $R(z_0)$ etc., and in the definition of $\delta_0$ (see (2.58)) one must use $R(z)$ in place of $r(z)$. Now the solution $m^#_{\pm} \in I + \partial \mathcal{C}(L^2)$ of the normalized RHP $(\Sigma^\epsilon, v^# = v^#_R)$ can be computed explicitly in terms of parabolic cylinder functions (see [DZ1],[DIZ],[DZ2]) and one learns that $m^#_{\pm}(m^#_{\pm})^{-1} \in L^\infty(\Sigma^\epsilon)$ with bounds which depend only on $\rho < 1$. Now we know from §2 that the IRHP2$_{L^2}(\Sigma)$, $M_+ = M_-v^# + F$, has a unique solution $M_{\pm} \in \partial \mathcal{C}(L^2(\Sigma^\epsilon))$, for each $F \in L^2$. But as $v^# = (m^#_{\pm})^{-1}m^#_{\pm}$, we have $M_{\pm}(m^#_{\pm})^{-1} = M_- (m^#_{\pm})^{-1} + F(m^#_{\pm})^{-1}$ and hence $M_{\pm} = \left(C^+(F(m^#_{\pm})^{-1})m^#_{\pm}\right)^{-1}$. But as $m^#_{\pm}, (m^#_{\pm})^{-1} \in L^\infty(\Sigma^\epsilon)$, it follows that if $F \in L^2 \cap L^p(\Sigma)$, then $M_{\pm} \in \partial \mathcal{C}(L^p)$ for any $F \in L^p$, by density. Moreover, $\|M_{\pm}\|_{L^p} \leq c\|F\|_{L^p}$, where the constant depends only on $\rho$ and $p$. Again by Proposition 2.14, this implies $\|(1 - C_{\omega_\theta})^{-1}\|_{L^p(\Sigma^\epsilon)} \leq \frac{c}{(1-\rho)^2}$ for all $x, t \in \mathbb{R}$.

Now let $W_\theta = (W^-_\theta, W^+_\theta) = \left(\begin{array}{cc} 0 & Re^{i\theta} \\ 0 & 0 \end{array}\right), \left(\begin{array}{cc} 0 & 0 \\ -Re^{-i\theta} & 0 \end{array}\right)$ as in (4.2), but with $r$ replaced by $R$. Then it follows by arguments similar to the deformation arguments in §2, that the above bound on $(1 - C_{\omega_\theta})^{-1}$ implies a similar bound on $(1 - C_{W_\theta})^{-1}$ on $L^p(\mathbb{R})$, $\|(1 - C_{W_\theta})^{-1}\|_{L^p(\mathbb{R})} \leq \frac{c}{(1-\rho)^2}$, for $t$ sufficiently large, say $t \geq t_0$, and all $x \in \mathbb{R}$. Again $c'$ depends only on $\rho$ and $p$.

The Proposition follows by making the following choices. First choose $\epsilon_1 > 0$ sufficiently small and $R$ as above so that $\|C_{\omega_\theta} - C_{W_\theta}\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leq \frac{1}{2(1-\rho)^2}$. Then choose $t_0$ sufficiently large to ensure the a bound $\|(1 - C_{W_\theta})^{-1}\|_{L^p(\mathbb{R})} \leq \frac{c'}{(1-\rho)^2}$ for $t \geq t_0$ and all $x \in \mathbb{R}$. Then use the second resolvent identity to obtain the bound on $\|(1 - C_{\omega_\theta})^{-1}\|_{L^p(\mathbb{R})}$. \hfill \Box

**Remark.**

The time $t_0$ depends on the full function $r = r(z)$ and not just on $\|r\|_{L^\infty}$. In particular it depends on the location of the poles of the approximating rational function $R(z)$. If one assumes that $r \in H_1^{1,0}$, $\|r\|_{L^\infty} \leq \rho < 1$, as in the previous Remark, then we may approximate $r$ via the Poisson formula $R(z) = \int_{\mathbb{R}^2} \frac{r(s)}{(s-z)^2 + \gamma^2} \pi$, leading to bounds $\|r - R\|_{L^\infty} \leq c\sqrt{\gamma} \|r\|_{L^2}, \|R\|_{H^{1,0}} \leq \|r\|_{H^{1,0}} \leq \lambda$. Choosing $\gamma$ sufficiently small, depending only on $\rho$ and $\lambda$, the reader may check that the proof of the Proposition now implies that $t_0$ may be chosen to depend only on $\rho, \lambda$ and $p$, $t_0 = t_0(\rho, \lambda, p)$ (for more details, see [DZ6]).

Reversing the orientation on $\mathbb{R}$ as above, we conclude that for $r \in H_1^{1,0}$ and $2 \leq p < \infty$,  

$$\|(1 - C_{\omega_\theta})^{-1}\|_{L^p(\tilde{R}_{\omega_\theta}) \to L^p(\tilde{R}_{\omega_\theta})} \leq c_p$$  

where $c_p$ is uniform for all $x \in \mathbb{R}$ and all $t \geq t_0$.

**Corollary 4.33 to Lemma 4.20.** For any $2 \leq p < \infty$ and for all $t \geq 0$,

$$\|\hat{\mu} - I\|_{L^p} \leq \frac{c_p}{(1 + t)^{1/2p}} \frac{\lambda}{(1 - \rho)^2}.$$

**Proof.** The result follows from the formula

$$\hat{\mu} - I = (1 - C_{\omega_\theta})^{-1}(C_{\omega_\theta} I)$$
together with (4.29) and (4.31). \qed

Of course, in order to prove (4.33) all we need are the mapping properties of $C_{\tilde{\mathbb{R}}_{z_0} \to \tilde{\mathbb{R}}_{z_0}}$, etc. The full estimates in (4.29) for $C_{\tilde{\mathbb{R}}_{z_0} \to \Gamma_{z_0}}$ are needed below.

**Lemma 4.35.** For any $2 \leq p < \infty$ and for $t$ sufficiently large, $\|C^\pm \tilde{\mu}(w_{\theta}^+ - \tilde{w}_\theta^+)\|_{L^2} \leq ct^{-\frac{1}{4} + \frac{1}{p}}$.

**Proof.** By triangularity,

$$C^\pm \tilde{\mu}(w_{\theta}^+ - \tilde{w}_\theta^+) = C^\pm(w_{\theta}^+ - \tilde{w}_\theta^+) + C^\pm(C_w \tilde{\mu})(w_{\theta}^+ - \tilde{w}_\theta^+)$$

$$= C^\pm(w_{\theta}^+ - \tilde{w}_\theta^+) + C^\pm(C^\tau \tilde{\mu} w_{\theta}^+)(w_{\theta}^+ - \tilde{w}_\theta^+) \equiv I + II.$$

After translation $z \to z_0 + z$, we obtain by (4.22), $\|I\|_{L^2} \leq ct^{-1/2}$. For II, $C^\tau \tilde{\mu} w_{\theta}^+ = C^\tau(\tilde{\mu} - I)w_{\theta}^+ + C^\tau \tilde{w}_\theta^+ \equiv g_1 + g_2$. By (4.34), $\|g_1\|_{L^p} \leq ct^{-1/2p}$, and by Lemma 4.20, $\|g_2\|_{L^p} \leq ct^{-1/2p}$. It now follows from (4.22) that $\|II\|_{L^p} \leq ct^{-\frac{1}{4} + \frac{1}{p}}$ and the lemma is proved. \qed

For convenience we abuse the notation by writing $\tilde{\mu} = \tilde{w}_\theta^+ + \tilde{w}_\theta^-$ and $w = w_{\theta}^+ + w_{\theta}^-$. 

**Lemma 4.36.** For any $2 \leq p < \infty$ and for $t$ sufficiently large, $\|\tilde{\mu} w - \tilde{\mu} w\| \leq ct^{-\frac{1}{4}}$. 

**Proof.** Write $\int \tilde{\mu} \tilde{w} - \int \mu w = \int \tilde{\mu} \tilde{w} - \int \mu w + \int (\mu - I)(\tilde{w} - w) + \int (\tilde{\mu} - \mu)\tilde{w} \equiv I + II + III$. Again, after translation $z \to z_0 + z$, we obtain by (4.18), $\|I\| \leq ct^{-3/4}$. Again by triangularity, $II = \int \{C_{w} \mu\}(\tilde{w} - w) = \int (C^\tau \mu w_{\theta}^+)(\tilde{w}_\theta^+ - w_{\theta}^+) + (C^\tau \mu w_{\theta}^+)(\tilde{w}_\theta^+ - w_{\theta}^-) \equiv II_+ + II_-.$ By Cauchy, (4.33), and the $z \to z_0 + z$ translation of (4.21)-(4.22), $\|II_+\| = |\int (C^\tau \mu w_{\theta}^+)(\tilde{w}_\theta^+ - w_{\theta}^+)| \leq \|C^\tau \mu w_{\theta}^+\|_{L^2} \|C^\tau(\tilde{w}_\theta^+ - w_{\theta}^+)|_{L^2} \leq ct^{-1/4 + \frac{1}{p}} = ct^{-\frac{1}{4}}$. 

The estimate for $II_-$ is similar and therefore $\|II\| \leq ct^{-\frac{1}{4}}$. We compute

$$\tilde{\mu} - \mu = (1 - C_{w})^{-1}C_{\tilde{w} - w}\tilde{\mu} = C_{\tilde{w} - w}\tilde{\mu} + C_{w}\phi,$$

where $\phi = (1 - C_{w})^{-1} C_{\tilde{w} - w}\tilde{\mu}$ Thus by triangularity and Cauchy,

$$III = \int (C_{\tilde{w} - w}\tilde{\mu})\tilde{w} + \int (C_{w}\phi)\tilde{w}$$

$$= -\int (C^\tau \tilde{\mu}(\tilde{w}_\theta^- - w_{\theta}^-))(C^{-1} \tilde{w}_\theta^+) + \int (C^{-1} \tilde{\mu}(\tilde{w}_\theta^+ - w_{\theta}^+))(C^\tau \tilde{w}_\theta^-)$$

$$- \int (C^\tau \phi w_{\theta}^-)C^{-1} \tilde{w}_\theta^+ + \int (C^{-1} \phi w_{\theta}^+)C^\tau \tilde{w}_\theta^-$$

$$\equiv A_+ + A_- + B_+ + B_-.$$

By Lemma 4.35 and (the translate of) (4.21), $|A_+| \leq \|C^\tau \tilde{\mu}(w_{\theta}^- - \tilde{w}_\theta^-)\|_{L^2} \|C^{-1} \tilde{w}_\theta^+\|_{L^2} \leq ct^{-\frac{1}{4} + \frac{1}{p}}$, with a similar estimate for $A_-$. 

Now consider $B_\pm$. Again by Lemma 4.35, $\|C_{\tilde{w} - w}\tilde{\mu}\|_{L^2} \leq ct^{-\frac{1}{4} + \frac{1}{p}}$ and hence $\|\phi\|_{L^2} \leq ct^{-\frac{1}{4} + \frac{1}{p}}$. Together with (4.21), this implies $\|B_\pm\|_{L^2} \leq ct^{-\frac{1}{4} + \frac{1}{p}}$, and hence $\|III\| \leq ct^{-\frac{1}{4} + \frac{1}{p}}$. Adding up the estimates for I, II and III, the result follows. \qed

By (4.3) and the lemma above, $q = \frac{1}{2}\pi \int (\tilde{\mu} \tilde{w})_{12} = \frac{1}{2}\pi \int (\mu w)_{12} + O(t^{-\frac{1}{4} + \frac{1}{p}})$. By the results of §2 and the residue calculation at the end of §3, we have $\int \mu w = \int_{\Sigma^*} \mu^0 w^0$ where $\mu^0 \in I + L^2(\Sigma^*)$ is the solution of $(1 - C_{w^0})\mu^0 = I$. Let $\mu^0 \in I + L^2(\Sigma^*)$ be the solution of $(1 - C_w)\mu^0 = I$. Write

$$\int_{\Sigma^*} \mu^0 w^0 - \int_{\Sigma^*} \mu^0 w^0 = \int_{\Sigma^*} (w^0 - w^0) + \int_{\Sigma^*} (\mu^0 - I)(w^0 - w^0) + \int_{\Sigma^*} (\mu^0 - \mu^0)w^0$$

$$= I + II + III.$$
By (2.63), $|I| \leq \frac{c}{t^{3/4}}$. From the formula $\mu^# - I = (1 - C_{w^#})^{-1}(C_{w^#}I)$, we have by (2.65) and the explicit form of $v^#$ in (2.61), $\|\mu^# - I\|_{L^2} \leq \|(1 - C_{w^#})^{-1}\|_{L^2}\|C^- (v^# - I)\|_{L^2} \leq \frac{c}{t^{3/4}}$. Again by (2.63) conclude $|I| \leq \frac{c}{t^{3/4}} \cdot \frac{c}{t^{3/4}} = \frac{c^2}{t^{3/2}}$. Finally $\mu^# - \mu^0 = (1 - C_{w^0})^{-1}C^- (\mu^# (v^# - v^0))$ and so

$$\|\mu^# - \mu^0\|_{L^2} \leq c \|(\mu^# - I)\|_{L^2}\|v^# - v^0\|_{L^\infty} + \|v^# - v^0\|_{L^2}$$

$$\leq c \left( \frac{1}{1/4} \frac{1}{t^{3/4}} + \frac{1}{t+\frac{1}{4}} \right) \leq \frac{c}{t^{1/2}}.$$

As $\|w^0\|_{L^2} \leq \frac{c}{t^{3/4}}$, we conclude that $|III| \leq \frac{c}{t^{3/4}}$. Hence as $t \to \infty$,

$$\int_{\Sigma^t} \mu^# w^# - \int_{\Sigma^t} \mu^0 w^0 = O \left( \frac{1}{t^{3/4}} \right).$$

Assembling the above results we conclude that for any $p \geq 2$

$$q(x,t) = \frac{1}{2\pi} \int_{\Sigma^t} (\mu^# w^#)_{12} + O(t^{-3/4+1/(2p)}).$$

But the normalized RHP $(\Sigma^t, v^#)$ can be solved explicitly in terms of parabolic cylinder functions as in [DIZ], [DZ2] and we find $\frac{1}{2\pi} \int_{\Sigma^t} (\mu^# w^#)_{12} = q_{as}(x,t)$ where $q_{as}(x,t)$ is precisely the form $t^{-1/2} \alpha(z_0) e^{ix^2/4t - i\nu(z_0)} \log 2t$ in (1.2). This completes the proof of Theorem 1.10 with $q(t=0) \in H^{1,1}$.

Finally, we observe that the error term only depends on the $H^{1,0}$ norm of $r = R^{-1}(q(t=0))$. However, as mentioned in the Introduction, equation (1.1) is not well-posed in $H^{0,1}$. Indeed, for $q(t=0) = q_0 \in H^{0,1}$, if $q(t) \in H^{0,1}$ for $t > 0$, then $r(t) = R(q(t)) \in H^{1,0}$. But for $t > 0$, $r(t) = e^{-ix^2/4t} r(t=0)$, which does not lie in $H^{1,0}$ for general $r(t=0) = R(q_0) \in H^{0,1}$. However, standard contraction-mapping methods using the Strichartz-type estimate

(4.37)\[ \|e^{-itH_0} f\|_{L^\infty(dx) \otimes L_2^p(dt)} = \left( \int_{-\infty}^{\infty} (\|u(t)\|_{L^\infty(dx)}^4)^{1/4} \right)^{1/4} \leq \|f\|_{L^2(dx)}, \]

together with the standard estimate

(4.38)\[ \|e^{-itH_0} f\|_{L^q(dx)} \leq (4\pi|t|)^{1/2}(\|f\|_{L^p(dx)}, \quad 1/p + 1/q = 1, \quad 1 \leq p \leq 2, \]

yields the following well-known result (see, for example, [CW][GV][CSU] and the references therein).

**Theorem 4.39.** Let $q_0 \in L^2(\mathbb{R})$ be given. Then there exists a unique solution $q$ of (1.9) with $q \in C(\mathbb{R}_+, L^2(dx)) \cap (L^\infty(dx) \otimes L^1_{loc}(dt))$, and $q(t=0) = q_0$. Also $\|q(t)\|_{L^1(dx)}$ is conserved.

**Remark.** Theorem 4.39 is commonly referred to as well-posedness in $L^2$. This is, of course, something of a misnomer as we need to place additional restrictions on the solution in order to prove uniqueness. This nomenclature, however, is well-established in the literature.

We show that our solution $q = q(x,t)$ obtained from the RHP (4.1) via (4.3) solves (1.9) in the sense of Theorem 4.39. In view of the preceding remarks (recall also Remark 3.28) this means that we have in fact proved the following extended version of Theorem 1.10.

**Theorem 4.40.** Let $q(t)$, $t \geq 0$, solve (1.9) with $q_0 = q(t=0) \in H^{0,1} \subset L^2$, as in Theorem 4.39. Fix $0 < \kappa < 1/4$. Then as $t \to \infty$,

$$q(x,t) = t^{-1/2} \alpha(z_0) e^{ix^2/(4t)} e^{iz^2/4t - i\nu(z_0)} \log 2t + O \left( t^{-(1/2 + \kappa)} \right),$$

where $\alpha$ and $\nu$ are given in terms of $r = R(q_0)$ as before. The error term $O \left( t^{-(1/2 + \kappa)} \right)$ is uniform for all $x \in \mathbb{R}$.

Note first that $\int \mu(w^+_\partial + w^-_\partial) = \int (\mu - I)(w^+_\partial + w^-_\partial) + \int (w^+_\partial + w^-_\partial) = I + II$. By (2.30), we have, $|I| \leq c \|x\|_{L^2}^2 / (1 - \rho)$. As the non-zero entries of $\int (w^+_\partial + w^-_\partial)$ are proportional to $e^{-itH_0 f}$ and its complete conjugate, we have the bound $\|II\|_{L^\infty(dx) \otimes L_2^1(dt)} \leq c \|r\|_{L^2}$ by (4.37). This implies the bound

(4.41)\[ \|q\|_{L^\infty(dx) \otimes L_2^0(dx)} \leq c \|r\|_{L^2} T^{1/4} \left( \frac{1}{1 - \rho} + c \|r\|_{L^2} \right) \]
for any \( T > 0 \). Now if \( q_{n0} \in H^{1,1} \), say, \( n=1,2,\ldots \), let \( q_n(t) \in C(\mathbb{R}_+, H^{1,1}) \) be the global solution of (1.9) in \( H^{1,1} \) described in the Introduction. For such solutions we have the conservation law

\[
\|q_n(\hat{\cdot}, t)\|_{L^2(dx)} = - \frac{1}{2\pi} \int \log(1 - |r_n(z)|^2)dz,
\]

where \( r_n = R(q_{n0}) \). For \( T > 0 \), define the norm

\[
\|u\|_{L^2_{0,T}} = \max_{0 \leq t \leq T} \|u(t)\|_{L^2(dx)} + \|u\|_{L^{\infty}(dx) \otimes L^1_{[0,T]}(dt)}.
\]

Then a standard calculation applied to (1.9) using (4.38)(4.39) implies

\[
\|q_n - q_m\|_{L^2_{0,T}} \leq c\|q_{n0} - q_{m0}\|_{L^2(dx)} + cT^{1/2}\|q_n\|_{L^{\infty}(dx) \otimes L^1_{[0,T]}(dt)} + \|q_m\|_{L^{\infty}(dx) \otimes L^1_{[0,T]}(dt)}\|q_n - q_m\|_{L^2_{0,T}}
\]

for \( n, m \geq 1 \). If \( q_0 \in H^{0,1} \) is given, we can choose \( q_{n0} \in H^{1,1} \) such that \( q_{n0} \to q_0 \) in \( H^{0,1} \). But then \( r_n = R(q_{n0}) \to r = R(q_0) \) in \( H^{1,0} \) and hence \( \|q_n\|_{L^{\infty}(dx) \otimes L^1_{[0,T]}(dt)} \leq c \) for all \( n \geq 1 \) by (4.41). It follows then by (4.43) that the \( q_n \)'s converge in \( \|\cdot\|_{L^2_{0,T}} \), say \( q_n \to q^0 \), for some (sufficiently small) \( T_0 > 0 \). Taking the limit as \( n \to \infty \) in (1.9), we see, in particular, that \( q^0 \) solves (1.9) in the sense of Theorem 4.39, at least for \( 0 \leq t \leq T_0 \). But then we can repeat the argument starting at \( t = T_0 \), and we conclude that \( q^0 \) solves (1.9) for all \( t \geq 0 \). Finally, as \( r_n \to r \) in \( H^{1,0} \), we can take the limit \( n \to \infty \) in I+II to conclude that for \( t > 0 \), \( \|q_n(t) - q(t)\|_{L^2(dx)} \to 0 \). But for \( t > 0 \), \( \|q_n(t) - q^0(t)\|_{L^2(dx)} \leq \|q_n - q^0\|_{L^2_{[0,t]}} \to 0 \) and we conclude that \( q(t) = q^0(t) \), and hence \( q \) solves (1.9) in the sense of Theorem 4.39. This concludes the proof of Theorem 4.40.

Remark.

As \( q_n(t) \to q^0(t) = q(t) \) in \( L^2(dx) \), we learn from (4.42) that

\[
\|q(\hat{\cdot}, t)\|_{L^2(dx)} = - \frac{1}{2\pi} \int \log(1 - |r(z)|^2)dz = \text{const}.
\]

It is an interesting fact that we do not seem able to derive this conservation law for \( q_0 \in H^{0,1} \) directly from the inverse scattering formalism.

References

[A] M. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, The inverse scattering transform - Fourier analysis for nonlinear problems, Stud. Appl. Math. 53 (1974), 249–315.

[BC] R. Beals and R. Fedoroff, Scattering and inverse scattering for first order systems, Comm. Pure Appl. Math 37 (1984), 39–90.

[CSU] N. H. Chang, J. Shatah and K. Uhlenbeck, Schrödinger maps, Comm. Pure Appl. Math. 53 (2000), 590–602.

[CW] T. Casasnovas and F. B. Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation in \( H^1 \); Nonlinear Analysis, Theory, Methods and Applications 14 (1990), 807–836.

[DZ] P. Deift, A. Its and X. Zhou, Long-time Asymptotics for Integrable Nonlinear Wave Equations, Important Developments in Soliton Theory 1980–1990 (A.S. Fokas and V.E. Zakharyev, eds.), Springer-Verlag, 1993, pp. 181–204.

[DZ1] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation, Ann. of Math. 137 (1993), 295–368.

[DZ2] Long-time behavior of the non-focusing nonlinear Schrödinger equation – a case study, New Series: Lectures in Math. Sciences, vol. 5, University of Tokyo, 1994.

[DZ3] Near integrable systems on the line. A case study–perturbation theory of the defocusing nonlinear Schrödinger equation, Math. Res. Lett. 4 (1997), 761–772.

[DZ4] Long-time asymptotics for integrable systems. Higher order theory, Comm. Math. Phys. 165 (1994), 175–191.

[DZ5] Perturbations theory for infinite dimensional integrable systems on the line. A case study (to appear in Acta Mathematica with an attached WEBPAGE).

[DZ6] Uniform \( L^p \) estimates for solutions of Riemann–Hilbert problems depending on external parameters, In preparation.

[Dur] P. Duren, Theory of \( H^p \) Spaces, Academic Press, New York, 1970.

[FT] L. Faddeev and L. Takhtajan, Hamiltonian Methods in the Theory of Solitons, Springer-Verlag, Berlin, Heidelberg, 1987.
[GV] J. Ginibre and G. Velo, *The global Cauchy problem for the nonlinear Schrödinger equation revisited*, Ann. Inst. H. Poincaré 2 (1985), 309–327.

[HLP] G. Hardy, J.E. Littlewood and G. Pólya, *Inequalities, Second Edition*, Cambridge Univ. Press, Cambridge, 1952.

[Z1] X. Zhou, *The $L^2$-Sobolev space bijectivity of the scattering and inverse scattering transforms*, Comm. Pure Appl. Math. 51 (1998), 697–731.

[ZS] V.E. Zakharov and A.B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Sov. Phys. JETP 34 (1972), 62–69.

[ZaMa] V.E. Zakharov and S.V. Manakov, *Asymptotic behavior of nonlinear wave systems integrated by the inverse method*, Sov. Phys. JETP 44 (1976), 106–112.