Local derivative estimates for the heat equation coupled to the Ricci flow

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Abstract

In this note we obtain local derivative estimates of Shi-type for the heat equation coupled to the Ricci flow. As applications, in part combining with Kuang’s work, we extend some results of Zhang and Bamler-Zhang including distance distortion estimates and a backward pseudolocality theorem for Ricci flow on compact manifolds to the noncompact case.

Key words: local derivative estimates; heat equation; Ricci flow; distance distortion estimates; backward pseudolocality

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1 Introduction

The Bernstein method is a strategy for obtaining derivative estimates for a solution to some PDE via applying the maximum principle to some partial differential inequality satisfying by a suitable combination of the solution and its derivatives. It is very useful in PDE and geometric analysis. In [B87]/[S89] Bando/Shi adapted this method to obtain global/local derivative estimates for the Ricci flow. Shi’s local derivative estimates are fundamental for the study of the Ricci flow. For expositions and/or alternative proofs of Shi’s local derivative estimates see for example, Hamilton [H95], Cao-Zhu [CZ], Chow-Lu-Ni [CLN], Chow et al [C+] and Tao [T]. With bounds on some derivatives of curvatures of the initial metrics Lu (see [LT] and [C+]) got a modified version of Shi’s local derivative estimates. Ecker-Huisken [EH] got Shi-type estimates for the mean curvature flow. Grayson-Hamilton [GH] derived Shi-type estimates for the harmonic map heat flow. For the heat equation on a Riemannian manifold, Kotschwar [K] obtained a Shi-type local gradient estimate, while the author [Hu] obtained local higher derivative estimates. Recently the Shi-type estimates are also derived for some other geometric evolution equations. See for example [LW] and [Ch].

In his lectures at Tsinghua University in 2012/13, Hamilton constructed a new comparison function, which is related to Lemma 8.3 in Perelman [P], and used it to simplify Shi’s proof of the local derivative estimates for the Ricci flow. In this note we use Hamilton’s comparison function to obtain local derivative estimates of Shi-type for the heat equation coupled to the Ricci flow. To state our results we
first introduce some notations. Fix $T > 0$. Let $(M, (g(t))_{t \in [0,T]})$ be a solution (not necessarily complete) to Hamilton’s Ricci flow

$$\frac{\partial g(t)}{\partial t} = -2\text{Ric}(g(t))$$

on a manifold $M$ (without boundary) of dimension $n$. For $x \in M$, $t \in [0,T]$ and $r > 0$, let $B(x, t, r)$ be the open metric ball with center $x$ and of radius $r$ w.r.t. the metric $g(t)$, and for $T' \in (0,T]$, let $PB_r(x, T')$ denote a parabolic ball (as called in [H13]) that is the set of all points $(x', t)$ with $x' \in B(x, t, r)$ (the closure of $B(x, t, r)$) and $t \in [0,T']$. For any points $x, y \in M$ let $d_t(x, y)$ be the distance between $x$ and $y$ w.r.t. $g(t)$.

We have the following gradient estimate.

**Theorem 1.1.** Fix $T > 0$. Let $(M, (g(t))_{t \in [0,T]})$ be a solution (not necessarily complete) to the Ricci flow on a manifold $M$ (without boundary) of dimension $n$. Fix $x_0 \in M$ and $r > 0$. Assume that for any $t \in [0,T]$ the closure of the open metric ball $B(x_0, t, r)$ is compact, and $\text{Ric} \leq \frac{n-1}{r}$ on the parabolic ball $PB_r(x_0, T)$. Let $u$ be a smooth solution to the heat equation $(\partial_t - \Delta_{g(t)})u = 0$ coupled to the Ricci flow on $M \times [0,T]$. Suppose $|u| \leq a$ on $PB_r(x_0, T)$, where $a$ is a positive constant. Then

$$|\nabla u| \leq C_1a\left(\frac{1}{r} + \frac{1}{\sqrt{t}}\right) \text{ on } PB_r(x_0, T) \setminus \{(x,0)|x \in M\},$$

where the constant $C_1$ depends only on the dimension.

Compare Theorem 2.2 in Bailesteanu-Cao-Pulemotov [BCP], where a two-sided bound on the Ricci curvature is assumed. See also Exercise 2.19 in Chow-Lu-Ni [CLN] for a related global estimate.

We also get a Hessian estimate.

**Theorem 1.2.** Fix $T > 0$. Let $M$ be a manifold (without boundary) of dimension $n$. Fix $x_0 \in M$ and $r > 0$. Suppose $g(t)$ is a solution (not necessarily complete) to the Ricci flow on $M \times [0,T]$. Assume that for any $t \in [0,T]$ the closure of the open metric ball $B(x_0, t, r)$ is compact, and $|\text{Rm}| \leq \frac{1}{r^2}$ on the parabolic ball $PB_r(x_0, T)$. Let $u$ be a smooth solution to the heat equation $(\partial_t - \Delta_{g(t)})u = 0$ coupled to the Ricci flow on $M \times [0,T]$. Suppose $|u| \leq a$ on $PB_r(x_0, T)$, where $a$ is a positive constant. Then

$$|\nabla^2 u| \leq C_2a\left(\frac{1}{r^2} + \frac{1}{t}\right) \text{ on } PB_r(x_0, T) \setminus \{(x,0)|x \in M\},$$

where the constant $C_2$ depends only on the dimension.

Compare Theorem 1.3 (b) in Han-Zhang [HZ], where an upper bound for the Hessian matrix of $u$ is obtained at points with certain distances away from the parabolic boundary.
In Section 2 we prove Theorems 1.1 and 1.2, and derive similar estimates for higher derivatives. In Section 3 we extend some derivative estimates in Zhang [Z06], Cao-Hamilton [CH] and Bamler-Zhang [BZ] on compact manifolds to the noncompact case using Theorems 1.1 and 1.2. In Section 4 we get a slight improvement of some results in Kuang [Ku1] and [Ku2], and point out that combining this and results in Section 3 one can extend some results in Zhang [Z12a] and Bamler-Zhang [BZ] including distance distortion estimates and a backward pseudolocality theorem for Ricci flow on compact manifolds to certain noncompact manifolds.

2 Shi-type estimates

Fix $T > 0$. Let $(M, (g(t)), t \in [0, T])$ be a solution (not necessarily complete) to the Ricci flow on a manifold $M$ (without boundary) of dimension $n$. Let $u$ be a smooth solution to the heat equation $(\partial_t - \Delta g(t))u = 0$ coupled to the Ricci flow. Recall that (compare for example [CLN] and [To])

$$(\partial_t - \Delta)|\nabla u|^2 = -2|\nabla^2 u|^2$$

(2.1)

$$(\partial_t - \Delta)^2 u = Rm \ast \nabla^2 u,$$

$$(\partial_t - \Delta)|\nabla^2 u|^2 = -2|\nabla^3 u|^2 + Rm \ast \nabla^2 u \ast \nabla u,$$

(2.2)

$$(\partial_t - \Delta)\nabla^k u = \sum_{i=0}^{k-2} \nabla^i Rm \ast \nabla^{k-i} u, \quad k \geq 2,$$

and

$$(\partial_t - \Delta)|\nabla^k u|^2 = -2|\nabla^{k+1} u|^2 + \sum_{i=0}^{k-2} \nabla^i Rm \ast \nabla^{k-i} u \ast \nabla^k u, \quad k \geq 2,$$

(2.3)

where, as usual, for tensors $A$ and $B$, $A \ast B$ denotes a linear combination of contractions of the tensor product $A \otimes B$.

Proof of Theorem 1.1

Let $G_1 = (A_1 a^2 + u^2)|\nabla u|^2$, where $A_1$ is a positive constant to be chosen depending only on the dimension. Using (2.1) we get

$$(\partial_t - \Delta)G_1 = -2(A_1 a^2 + u^2)|\nabla^2 u|^2 - 2|\nabla u|^4 + u \nabla u \ast \nabla u \ast \nabla^2 u.$$
where $C$ is a constant depending only on the dimension, so

$$|u \nabla u \ast \nabla u \ast \nabla^2 u| \leq A_1 a^2 |\nabla^2 u|^2 + |\nabla u|^4$$

for $A_1 \geq \frac{1}{4} C^2$, and

$$(\partial_t - \Delta) G_1 \leq -|\nabla u|^4.$$  

Choose $b_1 = \frac{1}{(A_1+1)^2 a^4}$, and let $F_1 = b_1 G_1$. Then

$$(\partial_t - \Delta) F_1 \leq -F_1^2.$$  

Since $\text{Ric} \leq \frac{n-1}{n^2}$ on $PB_r(x_0, T)$, as in Hamilton [H13], we can construct a function $\Psi_1$ on $\{(x, t)| x \in B(x_0, t, r), t \in [0, T]\}$ of the form

$$\Psi_1 = \frac{\alpha_1 r^2}{(r^2 - d_t(x, x_0)^2)^2}$$

(where $\alpha_1$ is a positive constant depending only on the dimension) satisfying

$$(\partial_t - \Delta) \Psi_1 \geq -\Psi_1^2$$

everywhere on $\{(x, t)| x \in B(x_0, t, r), t \in [0, T]\}$ in the constructive comparison sense. Let

$$\Phi_1 = \Psi_1 + \frac{1}{t} = \frac{\alpha_1 r^2}{(r^2 - d_t(x, x_0)^2)^2} + \frac{1}{t}$$

on $\{(x, t)| x \in B(x_0, t, r), t \in (0, T]\}$. Then

$$(\partial_t - \Delta) \Phi_1 \geq -\Phi_1^2$$

everywhere on $\{(x, t)| x \in B(x_0, t, r), t \in (0, T]\}$ in the constructive comparison sense.

Note that near the parabolic boundary of $PB_r(x_0, T)$ we have $F_1 < \Phi_1$.

Using the maximum principle we get that $F_1 \leq \Phi_1$ everywhere on $\{(x, t)| x \in B(x_0, t, r), t \in (0, T]\}$, and in particular,

$$b_1 A_1 a^2 |\nabla u|^2 \leq \frac{\alpha_1 r^2}{(r^2 - d_t(x, x_0)^2)^2} + \frac{1}{t}.$$  

On $PB_r(x_0, T) \setminus \{(x, 0)| x \in M\}$ we have $d_t(x, x_0) \leq \frac{r}{2}$ and $r^2 - d_t(x, x_0)^2 \geq \frac{3}{4} r^2$, and the result follows by our choice of $b_1$.  

\textbf{Remark} In the statement of Theorem 1.1, if we assume in addition $|\nabla u| \leq \frac{a}{r}$ at $t = 0$ in $B(x_0, 0, r)$, then we have $|\nabla u| \leq C_1 a r$ on $PB_r(x_0, T)$, because in this case we can choose $\Psi_1$ instead of $\Phi_1$ as the (space-time) comparison function.

\textbf{Proof of Theorem 1.2}
On \( PB_r(x_0, T) \), using (2.2) and our assumption, we have
\[
(\partial_t - \Delta)|\nabla^2 u|^2 \leq -2|\nabla^3 u|^2 + \frac{C}{r^2}|\nabla^2 u|^2,
\]
where \( C \) depends only on the dimension.

Let
\[
G_2 = (A_2 a^2 (\frac{1}{r^2} + \frac{1}{t}) + |\nabla u|^2)|\nabla^2 u|^2,
\]
where \( A_2 \) is a positive constant to be chosen depending only on the dimension. We have
\[
(\partial_t - \Delta)G_2
\]
\[
\leq -\frac{A_2 a^2}{t^2} |\nabla^2 u|^2 - 2|\nabla^2 u|^4 + (A_2 a^2 (\frac{1}{r^2} + \frac{1}{t}) + |\nabla u|^2)(-2|\nabla^3 u|^2 + \frac{C}{r^2}|\nabla^2 u|^2)
\]
\[
+ C|\nabla u||\nabla^2 u|^2|\nabla^3 u|.
\]
On \( PB_\frac{r}{2}(x_0, T) \backslash \{(x, 0)|x \in M\} \) we have
\[
|\nabla u| \leq C_1 a(\frac{1}{r} + \frac{1}{\sqrt{t}})
\]
by Theorem 1.1, so
\[
(A_2 a^2 (\frac{1}{r^2} + \frac{1}{t}) + |\nabla u|^2)\frac{C}{r^2}|\nabla^2 u|^2 \leq \frac{1}{2}|\nabla^2 u|^4 + C^2 (A_2 + 2C_1^2) a^4 \frac{1}{r^4}(\frac{1}{r^2} + \frac{1}{t})^2,
\]
and
\[
C|\nabla u||\nabla^2 u|^2|\nabla^3 u| \leq \frac{1}{2}|\nabla^2 u|^4 + A_2 a^2 (\frac{1}{r^2} + \frac{1}{t})|\nabla^3 u|^2
\]
by choosing \( A_2 \geq C^2 C_1^2 \).

Then
\[
(\partial_t - \Delta)G_2
\]
\[
\leq -|\nabla^2 u|^4 + C^2 (A_2 + 2C_1^2) a^4 \frac{1}{r^4}(\frac{1}{r^2} + \frac{1}{t})^2
\]
\[
\leq -\frac{G_2^2}{(A_2 + 2C_1^2) a^2 (\frac{1}{r^2} + \frac{1}{t})^2} + C^2 (A_2 + 2C_1^2) a^4 \frac{1}{r^4}(\frac{1}{r^2} + \frac{1}{t})^2.
\]

Let \( v = \frac{1}{r^2} + \frac{1}{t} \) and \( F_2 = \frac{b_2 G_2}{v^2} \) (compare the proof of Theorem 1.4.2 in [CZ]), where \( b_2 \) is a positive constant to be chosen later. We have
\[
(\partial_t - \Delta)F_2
\]
\[
\leq -\frac{F_2^2}{b_2 (A_2 + 2C_1^2) a^4 v} + b_2 C^2 (A_2 + 2C_1^2) a^4 \frac{v}{r^4} + F_2 \frac{1}{vt^2}
\]
\[
\leq -\frac{F_2^2}{b_2 (A_2 + 2C_1^2) a^4 v} + b_2 C^2 (A_2 + 2C_1^2) a^4 v^3 + F_2 v
\]
\[
\leq -\frac{F_2^2}{2b_2 (A_2 + 2C_1^2) a^4 v} + b_2 (A_2 + 2C_1^2) a^4 (\frac{1}{2} + C^2) v^3,
\]
where in the last inequality we use
\[ F_2v \leq \frac{F^2}{2b_2(A_2 + 2C_1^2)^2a^4v} + \frac{b_2}{2}(A_2 + 2C_1^2)^2a^4v^3. \]
Choose \( b_2 = \left((A_2 + 2C_1^2)^2a^4(2 + C_2^2)\right)^{-1}. \) Then we have
\[ (\partial_t - \Delta)F_2 \leq -\frac{F^2}{v} + v^3. \]
Write \( s = s(x,t) = d_t(x,x_0), \) and let
\[ \Psi_2(s) = \frac{\alpha_2 r^2}{(\frac{r^2}{4} - s^2)^2} \]
on \( \{(x,t) | x \in B(x_0, t, \frac{r}{2}), t \in [0, T]\}. \) Again as in [H13] we can choose positive constant \( \alpha_2 \) depending only on the dimension such that
\[ (\partial_t - \Delta)\Psi_2 \geq -\Psi^2 \]
everywhere on \( \{(x,t) | x \in B(x_0, t, \frac{r}{2}), t \in (0, T]\} \) in the constructive comparison sense. Let
\[ \Phi_2 = \beta \Psi_2^2 + \frac{1}{t^2} = \beta \frac{\alpha_2^2 r^4}{(\frac{r^2}{4} - s^2)^4} + \frac{1}{t^2} \]
on \( \{(x,t) | x \in B(x_0, t, \frac{r}{2}), t \in (0, T]\}, \) where \( \beta \) and \( \gamma \) are positive constants to be chosen later. We have
\[ (\partial_t - \Delta)\Phi_2 \]
\[ = 2\beta \Psi_2(\partial_t - \Delta)\Psi_2 - \frac{2\gamma}{t^3} - 2\beta |\nabla \Psi_2|^2 \]
\[ \geq -2\beta \Psi_2^3 - \frac{2\gamma}{t^3} - 2\beta \Psi_2'(s)^2. \]
We will choose constants \( \beta \) and \( \gamma \) such that
\[ -2\beta \Psi_2^3 - \frac{2\gamma}{t^3} - 2\beta \Psi_2'(s)^2 \geq -\frac{\Phi^2}{v} + v^3 \]
that is,
\[ \Phi_2^2 \geq 2\beta(\Psi_2^3 + \Psi_2'(s)^2)v + \frac{2\gamma v}{t^3} + v^4. \]
On \( \{(x,t) | x \in B(x_0, t, \frac{r}{2}), t \in (0, T]\} \) we have \( s^2 < \frac{r^2}{4}, \) and
\[ \Psi_2'(s)^2 = \frac{16\alpha_2^2 r^4 s^2}{(\frac{r^2}{4} - s^2)^6} < \frac{4\alpha_2^2 r^6}{(\frac{r^2}{4} - s^2)^6}, \]
so it suffices to have
\[ \beta^2 \alpha_2^4 \frac{r^8}{(\frac{r^2}{4} - s^2)^8} + \frac{\gamma^2}{t^4} \geq 2\beta(\alpha_2^3 + 4\alpha_2^2) \frac{r^6}{(\frac{r^2}{4} - s^2)^6} v + \frac{2\gamma v}{t^3} + v^4. \]
Note that
\[ v^4 \leq 8 \left( \frac{1}{r^8} + \frac{1}{t^4} \right). \]

Using the elementary inequality
\[ y^3 z \leq \frac{3}{4} (y^3)^{\frac{4}{3}} + \frac{1}{4} z^4 = \frac{3}{4} y^4 + \frac{1}{4} z^4 \]
for \( y, z \in \mathbb{R} \), we get
\[ 2\beta (\alpha_2^3 + 4\alpha_2^2) \frac{r^6}{(r^2 - s^2)^6} \frac{1}{t} \leq \frac{3}{4} (2\beta (\alpha_2^3 + 4\alpha_2^2))^{\frac{4}{3}} (r^2 - s^2)^8 + \frac{1}{4t^4}, \]
and
\[ \frac{1}{t^5} \frac{1}{r^2} \leq \frac{3}{4t^4} + \frac{1}{4r^8}. \]
We also have
\[ \frac{r^6}{(r^2 - s^2)^6} r^2 \leq \frac{r^8}{16(r^2 - s^2)^8} \]
and
\[ \frac{1}{r^8} \leq \frac{r^8}{4^8(r^2 - s^2)^8}. \]

First choose \( \gamma > 0 \) such that
\[ \gamma^2 \geq \frac{7}{2} + \frac{33}{4}. \]
Then choose \( \beta > 0 \) depending only on the dimension such that
\[ \beta^2 \alpha_2^4 \geq \frac{3}{4} (2\beta (\alpha_2^3 + 4\alpha_2^2))^{\frac{4}{3}} + \frac{1}{8} \beta (\alpha_2^3 + 4\alpha_2^2) + \frac{1}{4^8} (\gamma^2 + 8). \]
With \( \beta \) and \( \gamma \) chosen this way we have
\[ (\partial_t - \Delta) \Phi_2 \geq - \frac{\Phi_2^2}{v} + v^3 \]
everywhere on \( \{(x,t)|x \in B(x_0,t,\frac{r}{2}), t \in (0,T]\} \) in the constructive comparison sense.

Note that near the parabolic boundary of \( PB_{\frac{r}{2}}(x_0,T) \) we have \( F_2 < \Phi_2 \).
Using the maximum principle we get that \( F_2 \leq \Phi_2 \) everywhere on \( \{(x,t)|x \in B(x_0,t,\frac{r}{2}), t \in (0,T]\} \), and in particular,
\[ b_2 A_2 a^2 |\nabla^2 u|^2 \leq \beta \frac{\alpha_2^2 r^4}{(r^2 - s^2)^4} + \frac{1}{t^2}. \]
On \( PB_{\frac{r}{2}}(x_0,T) \setminus \{(x,0)|x \in M\} \) we have \( s \leq \frac{r}{4} \) and \( r^2 - s^2 \geq \frac{3}{16} r^2 \), and the result follows by our choice of \( b_2 \).

\[ \square \]

**Remark** In the statement of Theorem 1.2, if we assume in addition \( |\nabla u| \leq \frac{a}{r} \) and \( |\nabla^2 u| \leq \frac{a}{r^2} \) at \( t = 0 \) in \( B(x_0,0,r) \), then we have \( |\nabla^2 u| \leq C_2 \frac{a}{r^2} \) on \( PB_{\frac{r}{2}}(x_0,T) \).
because in this case we can choose $\beta \Psi^2$ instead of $\Phi^2$ as the (space-time) comparison function.

Similar to Theorems 1.1 and 1.2 we have

**Theorem 2.1.** Fix $T > 0$. Let $M$ be a manifold (without boundary) of dimension $n$. Fix $x_0 \in M$ and $r > 0$. Suppose $g(t)$ is a solution (not necessarily complete) to the Ricci flow on $M \times [0, T]$. Assume that for any $t \in [0, T]$ the closure of the open metric ball $B(x_0, t, r)$ is compact, and $|Rm| \leq \frac{1}{r^2}$ on the parabolic ball $PB_r(x_0, T)$. Let $u$ be a smooth solution to the heat equation $(\partial_t - \Delta_{g(t)})u = 0$ coupled to the Ricci flow on $M \times [0, T]$. Suppose $|u| \leq a$ on $PB_r(x_0, T)$, where $a$ is a positive constant. Then for any $k \geq 2$,

$$|\nabla^k u| \leq C_ka\left(\frac{1}{r^k} + \frac{1}{t^{k/2}}\right)$$

on $PB_{\frac{r}{2k}}(x_0, T) \setminus \{(x, 0) | x \in M\}$, where the constant $C_k$ depends only on $k$ and the dimension.

**Proof** The proof is by induction. Below we will use $C$ to denote various constants depending only on $k$ and the dimension, which may be different from line to line. On $PB_{\frac{r}{2k}}(x_0, T) \setminus \{(x, 0) | x \in M\}$ we have

$$|\nabla u| \leq C_1a\left(\frac{1}{r} + \frac{1}{\sqrt{t}}\right)$$

by Theorem 1.1. For $k = 2$, the result is exactly Theorem 1.2. Suppose on $PB_{\frac{r}{2i}}(x_0, T) \setminus \{(x, 0) | x \in M\}$ ($2 \leq i \leq k$) we have

$$|\nabla^i u| \leq C_ia\left(\frac{1}{r^i} + \frac{1}{t^{i/2}}\right),$$

where $C_i$ depends only on $i$ and the dimension. Let

$$G_{k+1} = (A_{k+1}a^2\left(\frac{1}{r^{2k}} + \frac{1}{t^k}\right) + |\nabla^k u|^2) |\nabla^{k+1} u|^2,$$

where $A_{k+1} > 1$ is a constant to be chosen depending only on $k$ and the dimension. Using (2.3) we have

$$(\partial_t - \Delta)G_{k+1} = (-A_{k+1}a^2\frac{k}{t^{k+1}} - 2|\nabla^{k+1} u|^2 + \sum_{i=0}^{k-2} \nabla^i Rm * \nabla^{k-i} u * \nabla^k u)|\nabla^{k+1} u|^2$$

$$+ (A_{k+1}a^2\left(\frac{1}{r^{2k}} + \frac{1}{t^k}\right) + |\nabla^k u|^2) (-2|\nabla^{k+2} u|^2 + \sum_{i=0}^{k-1} \nabla^i Rm * \nabla^{k+1-i} u * \nabla^{k+1} u)$$

$$+ \nabla^k u * \nabla^{k+1} u * \nabla^{k+1} u * \nabla^{k+2} u.$$
Since $|Rm| \leq \frac{1}{r}$ on $PB_{r}(x_0, T)$ by assumption, we have

$$|\nabla^i Rm| \leq C'_i r^{i} (\frac{1}{r^2} + \frac{1}{t^{i/2}})$$

on $PB_{\frac{r}{2k}}(x_0, T) \setminus \{(x, 0)|x \in M\}$ ($1 \leq i \leq k - 1$) by Shi’s local derivative estimates (actually the version of Shi estimates that we need is in [H13], where only the details for the gradient estimate are given, but the higher derivative estimates can be obtained similarly to that of our Theorem 2.1), where $C'_i$ depends only on $i$ and the dimension.

On $PB_{\frac{r}{2k}}(x_0, T) \setminus \{(x, 0)|x \in M\}$, we have

$$|\sum_{i=0}^{k-2} \nabla^i Rm \star \nabla^{k-i} u \star \nabla^k u| \leq C_2 a^2 (\frac{1}{r^{2k}} + \frac{1}{t^{k}}),$$

$$|\sum_{i=0}^{k-2} \nabla^i Rm \star \nabla^{k-i} u \star \nabla^k u| \leq \frac{1}{3} |\nabla^{k+1} u|^4 + C_4 a^4 (\frac{1}{r^{2k}} + \frac{1}{t^{k}})^2,$$

$$|\sum_{i=0}^{k-1} \nabla^i Rm \star \nabla^{k+1-i} u \star \nabla^{k+1} u| \leq C (\frac{1}{r^{2}} |\nabla^{k+1} u|^2 + a^2 (\frac{1}{r^{2(k+1)}} + \frac{1}{t^{k+1}})),$$

$$|(A_{k+1} a^2 (\frac{1}{r^{2k}} + \frac{1}{t^{k}}) + |\nabla^k u|^2) \sum_{i=0}^{k-1} \nabla^i Rm \star \nabla^{k+1-i} u \star \nabla^{k+1} u|$$

$$\leq \frac{1}{3} |\nabla^{k+1} u|^4 + C (A_{k+1} + 2C_2 a^4 (\frac{1}{r^{2k}} + \frac{1}{t^{k}})^2$$

$$+ C (A_{k+1} + 2C_2 a^4 (\frac{1}{r^{2k}} + \frac{1}{t^{k}})(\frac{1}{r^{2(k+1)}} + \frac{1}{t^{k+1}})),$$

and

$$|\nabla^k u \star \nabla^{k+1} u \star \nabla^{k+1} u \star \nabla^{k+2} u| \leq \frac{1}{3} |\nabla^{k+1} u|^4 + A_{k+1} a^2 (\frac{1}{r^{2k}} + \frac{1}{t^{k}}) |\nabla^{k+2} u|^2$$

by choosing $A_{k+1}$ sufficiently large (compared to $C_2^2$).
So
\[
(\partial_t - \Delta)G_{k+1} \\
\leq -|\nabla^{k+1}u|^4 + C(A_{k+1} + 2C^2_k)\alpha^4 \left(\frac{1}{r^{2k}} + \frac{1}{r^2}\right)^2 \\
+ C(A_{k+1} + 2C^2_k)\alpha^4 \left(\frac{1}{r^{2k+1}} + \frac{1}{r^{2k+1}}\right) \\
\leq -|\nabla^{k+1}u|^4 + C(A_{k+1} + 2C^2_k)\alpha^4 \left(\frac{1}{r^{2k}} + \frac{1}{r^{2k+1}}\right) \\
+ C(A_{k+1} + 2C^2_k)\alpha^4 \left(\frac{1}{r^{2k+1}} + \frac{1}{r^{2k+1}}\right) \\
\leq -|\nabla^{k+1}u|^4 + C(A_{k+1} + 2C^2_k)\alpha^4 \left(\frac{1}{r^{2k+1}} + \frac{1}{r^{2k+1}}\right) \\
\leq -\frac{G^2_{k+1}}{(A_{k+1} + 2C^2_k)\alpha^4 (r^{2k} + r^2)^2} + C(A_{k+1} + 2C^2_k)^2\alpha^4 \left(\frac{1}{r^{2(2k+1)}} + \frac{1}{r^{2k+1}}\right).
\]

Let \( v = \frac{1}{r^2} + \frac{1}{t} \) and \( F_{k+1} = \frac{b_{k+1}G_{k+1}}{r^2} \) as in the proof of Theorem 1.4.2 in [CZ], where \( b_{k+1} \) is a positive constant to be chosen later. Then
\[
(\partial_t - \Delta)F_{k+1} \\
\leq -\frac{F^2_{k+1}}{b_{k+1}(A_{k+1} + 2C^2_k)^2\alpha^4 v_k} + b_{k+1}C(A_{k+1} + 2C^2_k)^2\alpha^4 v_{k+1} + kF_{k+1}v \\
\leq -\frac{F^2_{k+1}}{2b_{k+1}(A_{k+1} + 2C^2_k)^2\alpha^4 v_k} + b_{k+1}(C + 2k^2)(A_{k+1} + 2C^2_k)^2\alpha^4 v_{k+2}.
\]
Choosing \( b_{k+1} = \frac{1}{(C+2k^2)(A_{k+1} + 2C^2_k)^2\alpha^4} \), we get
\[
(\partial_t - \Delta)F_{k+1} \leq -\frac{1}{v_k}F^2_{k+1} + v_{k+2}.
\]

Write \( s = s(x,t) = d_t(x,x_0) \), and let
\[
\Psi_{k+1}(s) = \frac{\alpha_{k+1}r^2}{(r^2 - s^2)^2}
\]
on \( \{(x,t)|x \in B(x_0, t, \frac{\alpha_{k+1}}{r^2}), t \in [0,T]\} \). Again as in [H13] we can choose constant \( \alpha_{k+1} > 0 \) depending only on the dimension and \( k \) such that
\[
(\partial_t - \Delta)\Psi_{k+1} \geq -\Psi^2_{k+1}
\]
everywhere on \( \{(x,t)|x \in B(x_0, t, \frac{\alpha_{k+1}}{r^2}), t \in [0,T]\} \) in the constructive comparison sense. Let
\[
\Phi_{k+1} = \beta_{k+1}\Psi_{k+1} + \gamma_{k+1}\frac{1}{t^{k+1}} = \beta_{k+1}r^2\alpha_{k+1}r^{2(k+1)} + \gamma_{k+1}\frac{1}{t^{k+1}}
\]

on \( \{x, t\} \mid x \in B(x_0, t, \frac{r}{4\gamma}), t \in (0, T]\)\), where \(\beta_{k+1}\) and \(\gamma_{k+1}\) are positive constants to be chosen later.

We have

\[
(\partial_t - \Delta) \Phi_{k+1} = \beta_{k+1}(k + 1) \Psi_k^k(\partial_t - \Delta) \Psi_k - (k + 1) \frac{\gamma_{k+1}}{t^{k+2}} - k(k + 1) \beta_{k+1} \Psi_{k+1}^{k+1} |\nabla \Psi_{k+1}|^2
\]

\[
\geq - \beta_{k+1}(k + 1) \Psi_{k+1}^{k+2} - (k + 1) \frac{\gamma_{k+1}}{t^{k+2}} - k(k + 1) \beta_{k+1} \Psi_{k+1}^{k+1} \Psi_{k+1}'(s)^2.
\]

We will choose constants \(\beta_{k+1}\) and \(\gamma_{k+1}\) such that

\[
- \beta_{k+1}(k + 1) \Psi_{k+1}^{k+2} - (k + 1) \frac{\gamma_{k+1}}{t^{k+2}} - k(k + 1) \beta_{k+1} \Psi_{k+1}^{k+1} \Psi_{k+1}'(s)^2 \\
\geq - \frac{\Phi_{k+1}^2}{v^k} + v^{k+2},
\]

that is,

\[
\Phi_{k+1}^2 \geq \beta_{k+1}(k + 1)(\Psi_{k+1}^{k+2} + k\Psi_{k+1}^{k+1} \Psi_{k+1}'(s)^2)v^k + (k + 1) \frac{\gamma_{k+1}v^k}{t^{k+2}} + v^{2(k+1)}.
\]

On \(\{x, t\} \mid x \in B(x_0, t, \frac{r}{4\gamma}), t \in (0, T]\) we have \(s^2 < \frac{r^2}{4\gamma}\), and

\[
\Psi_{k+1}'(s)^2 = \frac{16\alpha_{k+1}^2 r^4 s^2}{(\frac{r^2}{4\gamma} - s^2)^6} < \frac{\alpha_{k+1}^2 r^6}{4^{k-2}(\frac{r^2}{4\gamma} - s^2)^6},
\]

so it suffices to have

\[
\beta_{k+1}^2 \alpha_{k+1}^{2(k+1)} \left(\frac{r^2}{4\gamma} - s^2\right)^{4(k+1)} + \frac{\gamma_{k+1}^2}{t^{2(k+1)}} \geq \beta_{k+1}(k + 1)(\alpha_{k+1}^{k+2} + \frac{k}{4^{2(k-2)}} \alpha_{k+1}^{k+1} r^{2(k+2)}(\frac{r^2}{4\gamma} - s^2)^2 v^k + (k + 1) \frac{\gamma_{k+1}v^k}{t^{k+2}} + v^{2(k+1)}.
\]

Note that

\[
v^k \leq 2^{k-1}(\frac{1}{r^{2k}} + \frac{1}{t_k})
\]

and

\[
v^{2(k+1)} \leq 2^{2k+1}(\frac{1}{r^{4(k+1)}} + \frac{1}{t^{2(k+1)}}).
\]

Using the elementary inequality

\[
y^{k+2} z^k \leq \frac{k + 2}{2(k + 1)} (y^{k+2})^{2(k+1)} + \frac{k}{2(k + 1)} (z^k)^{2(k+1)}
\]

\[
= \frac{k + 2}{2(k + 1)} y^{2(k+1)} + \frac{k}{2(k + 1)} z^{2(k+1)}
\]
for $y, z \in \mathbb{R}$, we get

$$2^{k-1} \beta_{k+1} (k+1) \left( \alpha_{k+1}^{k+2} + \frac{k}{4k-2} \alpha_{k+1}^{k+1} \right) \frac{r^{2(k+2)}}{(\frac{r^2}{4} - s^2)^{2(k+2)}} t^k \leq \frac{k + 2}{2(k+1)} \left[ 2^{k-1} \beta_{k+1} (k+1) \left( \alpha_{k+1}^{k+2} + \frac{k}{4k-2} \alpha_{k+1}^{k+1} \right) \frac{r^{2(k+1)}}{(\frac{r^2}{4} - s^2)^{2(k+1)}} \right]$$

and

$$\frac{1}{r^{k+2} t^{2k}} \leq \frac{k + 2}{2(k+1)} \frac{1}{r^{2(k+1)}} + \frac{k}{2(k+1)} \frac{1}{r^{4(k+1)}} .$$

We also have

$$\frac{r^{2(k+2)}}{(\frac{r^2}{4} - s^2)^{2(k+2)}} t^{2k} \leq \frac{r^{4(k+1)}}{4^{2k+1} (\frac{r^2}{4} - s^2)^{4(k+1)}}$$

and

$$\frac{1}{r^{4(k+1)}} \leq \frac{r^{4(k+1)}}{4^{2k} (\frac{r^2}{4} - s^2)^{4(k+1)}} .$$

First choose $\gamma_{k+1} > 0$ depending only on $k$ such that

$$\gamma_{k+1}^2 \geq 2^{k-1} (k+1) \left( 1 + \frac{k + 2}{2(k+1)} \right) \gamma_{k+1} + 2^{k+1} + \frac{k}{2(k+1)} .$$

Then choose $\beta_{k+1} > 0$ depending only on the dimension and $k$ such that

$$\beta_{k+1}^2 \alpha_{k+1}^{2(k+1)} \geq \frac{k + 2}{2(k+1)} \left[ 2^{k-1} \beta_{k+1} (k+1) \left( \alpha_{k+1}^{k+2} + \frac{k}{4k-2} \alpha_{k+1}^{k+1} \right) \frac{r^{2(k+1)}}{(\frac{r^2}{4} - s^2)^{2(k+1)}} \right]$$

and

$$\beta_{k+1} \frac{k + 1}{2^{k+1} k^{k+1}} \left( \alpha_{k+1}^{k+2} + \frac{k}{4k-2} \alpha_{k+1}^{k+1} \right) + \frac{1}{2^{k+1} k^{k+1}} (\gamma \gamma_{k+1} + 2^{k+3}) .$$

With $\beta_{k+1}$ and $\gamma_{k+1}$ chosen this way we have

$$(\partial_t - \Delta) \Phi_{k+1} \geq - \frac{\Phi_{k+1}^2}{v_k} + v^{k+2}$$

everywhere on $\{(x, t) | x \in B(x_0, t, \frac{r}{2^k}), t \in (0, T) \}$ in the constructive comparison sense.

Note that near the parabolic boundary of $PB_{\frac{r}{2k}}(x_0, T)$ we have $F_{k+1} < \Phi_{k+1}$. Using the maximum principle we get that $F_{k+1} \leq \Phi_{k+1}$ everywhere on $\{(x, t) | x \in B(x_0, t, \frac{r}{2^k}), t \in (0, T) \}$, and in particular,

$$\frac{1}{2^{k-1} b_{k+1} A_{k+1} a^2} |\nabla^{k+1} u|^2 \leq \beta_{k+1} \frac{\alpha_{k+1}^{k+1} r^{2(k+1)}}{(\frac{r^2}{4} - s^2)^{2(k+1)}} + \gamma_{k+1} \frac{1}{\ell_{k+1}} .$$
On $PB_{\frac{r}{2}}(x_0, T) \setminus \{(x, 0) | x \in M\}$ we have $s \leq \frac{r^2}{2r + 1}$ and $\frac{r^2}{4} - s^2 \geq \frac{3}{4r + 1}r^2$, and the result follows by our choice of $b_{k+1}$.

Remark In the statement of Theorem 2.1, if we assume in addition $|\nabla_i Rm| \leq \frac{1}{r^2}$ for $1 \leq i \leq k - 2$ and $|\nabla^i u| \leq \frac{r}{r^i}$ for $1 \leq i \leq k$ at $t = 0$ in $B(x_0, 0, r)$, then we have $|\nabla^k u| \leq C_k \frac{a}{t}$ on $PB_{\frac{r}{2}}(x_0, T)$, because in this case we can choose $\beta_{k+1}^{k+1} \Psi_{k+1}$ instead of $\Phi_{k+1}$ as the (space-time) comparison function with the aid of Lu's modified Shi estimates (actually the version of Lu's estimates we need is in [H13]).

Of course we can state Theorem 2.1 for all $k \geq 1$ and use Theorem 1.1 instead of Theorem 1.2 as the beginning of the induction, so the proof of Theorem 1.2 can be omitted. But we prefer to reserve it since it serves as a guide for the proof of Theorem 2.1. Note also that in the conclusion of Theorem 2.1 we can replace $PB_{\frac{r}{2}}(x_0, T)$ by $PB_{\frac{r}{2}}(x_0, T)$.

3 Some applications of Theorems 1.1 and 1.2

Using Theorem 1.1 we can extend an estimate in Theorem 3.3 in Zhang [Z06] and Theorem 5.1 in Cao-Hamilton [CH] to a more general situation.

Proposition 3.1. (cf. Zhang [Z06], Cao-Hamilton [CH]) Let $(M, (g(t)))_{t \in (0, T)}$ be a complete solution to the Ricci flow with bounded Ricci curvature on any compact time subinterval. Let $0 < u \leq a$ be a solution to the heat equation $\partial_t u = \Delta_{g(t)} u$ coupled to the Ricci flow on $M \times (0, T)$, where $a$ is a positive constant. Then

$$\frac{|\nabla u(x, t)|}{u(x, t)} \leq \frac{1}{t} \sqrt{\ln \frac{a}{u(x, t)}} \text{ on } M \times (0, T).$$

Proof. Compare the proof of Lemma 6.3 in [CTY]. Clearly we can assume that $T < \infty$; otherwise we only need to restrict to every finite time subinterval $(0, T')$. Then using Theorem 1.1 and a standard trick (cf [H93], [B]) we can reduce the proof in the general case to the case that $g(t)$ extends smoothly up to $t = 0$ with $\sup_{(x, t) \in M \times [0, T]} |\text{Ric}| < \infty$ and $\sup_{(x, t) \in M \times [0, T]} |\nabla u| < \infty$. The reason is as follows: Fix $(x_0, t_0) \in M \times (0, T)$. Choose a small $\varepsilon > 0$ such that $t_0 \in (\varepsilon, T - 2\varepsilon)$. By assumption

$$\sup_{(x, t) \in M \times [\frac{T}{2}, T - \varepsilon]} |\text{Ric}| < \infty$$

and $0 < u \leq a$. By Theorem 1.1 we have

$$\sup_{(x, t) \in M \times [\varepsilon, T - \varepsilon]} |\nabla u| < \infty.$$  

Let $\tilde{g}(t) = g(t + \varepsilon)$ and $\tilde{u}(t) = u(t + \varepsilon)$, $t \in [0, T - 2\varepsilon]$. Note that $\tilde{g}(t)$ is also a solution to the Ricci flow, and $\tilde{u}(t)$ is also a solution to the heat equation coupled
to the Ricci flow \( \tilde{g}(t) \). Now \( \sup_{(x,t) \in M \times [0,T-2\varepsilon]} |\tilde{Ric}|_{\tilde{g}(t)} < \infty \), \( 0 < u \leq a \), and \( \sup_{(x,t) \in M \times [0,T-2\varepsilon]} |\nabla \tilde{g}(t)|_{\tilde{g}(t)} < \infty \). Suppose in this case we have

\[
\frac{|\nabla \tilde{g}(t)|_{\tilde{g}(t)}(x,t)}{\tilde{u}(x,t)} \leq \sqrt{\frac{1}{t} \ln \frac{a}{\tilde{u}(x,t)}} , \quad t \in (0, T-2\varepsilon).
\]

In particular the above inequality holds at \((x_0,t_0)\). Now letting \( \varepsilon \to 0 \) we get the desired inequality.

Now let \( v = u + \delta \), where \( \delta \) is a positive constant. Then \( \delta \leq v \leq a + \delta \) is a solution to the heat equation coupled to the Ricci flow with bounded gradient. Now as in [Z06] and [CH] we have

\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right)(t \frac{|\nabla v|^2}{v} - v \ln \frac{a + \delta}{v}) \leq 0, \quad t \in [0, T].
\]

Since \( \delta \leq v \leq a + \delta \) and \( \sup_{M \times [0,T]} |\nabla v| < \infty \), we have \( \sup_{(x,t) \in M \times [0,T]} t |\nabla v|^2 < \infty \) and \( \sup_{M \times [0,T]} v \ln \frac{a + \delta}{v} < \infty \). We also have \( \sup_{(x,t) \in M \times [0,T]} |\tilde{Ric}| < \infty \). So with the help of Bishop volume comparison theorem, by the maximum principle (Theorem 12.22 in [C+]) we have

\[
t \frac{|\nabla v|^2}{v} - v \ln \frac{a + \delta}{v} \leq 0
\]

everywhere, since it is true when \( t = 0 \). Now letting \( \delta \to 0 \) and we are done. \( \square \)

**Remark** Theorem 3.3 in [Z06] is stated for complete manifolds and does not impose any curvature bound, but Theorem 6.5.1 in [Z11] assumes the curvature is uniformly bounded. In both places the details on justifying the use of the maximum principle are not supplied. Moreover, our statement is slightly more general than that in Theorem 6.5.1 of [Z11] in that we do not assume the Ricci flow is defined at \( t = 0 \), so in our case the curvature is not necessarily uniformly bounded. Actually we only assume the Ricci curvature is bounded on compact time interval.

Using Theorems 1.1 and 1.2 we also extend Lemma 3.1 in Bamler-Zhang [BZ] to the noncompact case.

**Proposition 3.2.** Let \((M,(g(t))_{t \in (0,T)})\) be a complete solution to the Ricci flow with bounded curvature on any compact time subinterval. Let \( 0 < u \leq a \) be a solution to the heat equation \( \partial_t u = \Delta_{g(t)} u \) coupled to the Ricci flow on \( M \times (0,T) \), where \( a \) is a positive constant. Then

\[
(|\Delta u| + \frac{|\nabla u|^2}{u} - aR)(x,t) \leq \frac{Ba}{t} \text{ on } M \times (0,T),
\]

where the constant \( B \) depends only on the dimension.

**Proof.** As before we can assume that \( T < \infty \). Then as in the proof of Proposition 3.1 we can reduce the proof in the general case to the case that \( g(t) \) extends smoothly up to \( t = 0 \) with \( \sup_{(x,t) \in M \times [0,T]} |Rm| < \infty \) and \( \sup_{(x,t) \in M \times [0,T]} |\nabla^k u| < \infty \) for all \( k \).
\( \infty, k = 1, 2: \) Fix \( (x_0, t_0) \in M \times (0, T) \). Choose a small \( \varepsilon > 0 \) such that \( t_0 \in (\varepsilon, T - 2\varepsilon) \). By Theorem 1.1 and Theorem 1.2 we have \( \sup_{(x,t) \in M \times (\varepsilon, T - \varepsilon)} |\nabla^k u| < \infty \) for \( k = 1, 2 \). Let \( \tilde{g}(t) = g(t + \varepsilon) \) and \( \tilde{u}(t) = u(t + \varepsilon), t \in [0, T - 2\varepsilon] \). Then \( \sup_{(x,t) \in M \times [0, T - 2\varepsilon]} |Rm|_{\tilde{g}(t)} < \infty, \sup_{(x,t) \in M \times [0, T - 2\varepsilon]} |\nabla^k \tilde{u}|_{\tilde{g}(t)} < \infty \) for \( k = 1, 2 \).

Suppose in this case we have
\[
\left| \Delta_{\tilde{g}(t)} \tilde{u} \right| + \frac{\left| \nabla_{\tilde{g}(t)} \tilde{u} \right|^2_{\tilde{g}(t)}}{\tilde{u}} - a\tilde{R}(x,t) \leq \frac{Ba}{t}, \quad t \in (0, T - 2\varepsilon),
\]
where the constant \( B \) depends only on the dimension. In particular the above inequality holds at \((x_0, t_0)\). Now letting \( \varepsilon \to 0 \) we get the desired inequality.

Also note that by the same trick of replacing \( u \) by \( u + \delta \) and letting \( \delta \to 0 \) as in the proof of Proposition 3.1 we can assume that \( u \geq \delta > 0 \).

By rescaling we may assume that \( a = 1 \). Let \( L_1 = -\Delta u + \frac{|\nabla u|^2}{u^2} - R \), and choose \( B > 0 \) with \( \frac{B + e^{-2}}{B^2} = \frac{1}{n} \); then as in the proof of Lemma 3.1 in [BZ], we have
\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right)(L_1 - \frac{B}{t}) \leq -\frac{1}{n}(L_1 + \frac{B}{t})(L_1 - \frac{B}{t})
\]
for \( t \in (0, T] \).

Now given any \( \varepsilon > 0, C > 0 \), let \( \varphi(x,t) = \varepsilon e^{At} f(x) \) be a positive function as in the proof of Lemma 5.2 in [H93], which satisfies \( f(x) \to \infty \) as \( x \to \infty \) and \( \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \varphi > C \varphi \). So we have
\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right)(L_1 - \frac{B}{t} - \varphi) \leq -\frac{1}{n}(L_1 + \frac{B}{t})(L_1 - \frac{B}{t}) - C \varphi \tag{3.1}
\]
for \( t \in (0, T] \). We claim \( L_1 - \frac{B}{t} - \varphi < 0 \) for \( t \in (0, T] \). Note that this is true for \( t > 0 \) sufficiently small by our assumption on \( |Rm|, u \) and \( |\nabla^k u|, k = 1, 2 \). Suppose it is not true for some large \( t \). Then there exist the first time \( t_0 \) and a point \( x_0 \) such that \( L_1(x_0, t_0) - \frac{B}{t_0} - \varphi(x_0, t_0) = 0 \) since \( f(x) \to \infty \) as \( x \to \infty \). Now at \( (x_0, t_0) \) we have \( \frac{\partial}{\partial t}(L_1 - \frac{B}{t} - \varphi) \geq 0 \), and \( \Delta (L_1 - \frac{B}{t} - \varphi) \leq 0 \). This contradicts (3.1), since at \( (x_0, t_0) \) the RHS of (3.1) < 0. Now letting \( \varepsilon \to 0 \) we get \( L_1 \leq \frac{B}{t} \).

Let \( L_2 = \Delta u + \frac{|\nabla u|^2}{u^2} - R \), and choose \( B > 0 \) with \( B^{-1} + \frac{1 + \frac{4}{n}}{e^2} B^{-2} = \frac{1}{2n} \). As in the proof of Lemma 3.1 in [BZ] we have
\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right)(L_2 - \frac{B}{t}) \leq -\frac{1}{2n}(L_2 + \frac{B}{t})(L_2 - \frac{B}{t})
\]
for \( t \in (0, T] \). Arguing as above we get the desired inequality for \( L_2 \). \( \square \)

4 Perelman’s W-entropy on noncompact manifolds

Now as in for example [Ku1], [Z12b], [RV] and [L], we consider Perelman’s W-entropy (see [P])
\[
W(g, v, \tau) = \int_M |\tau(4|\nabla v|^2 + Rv^2) - v^2 \ln v^2 - \frac{n}{2}(\ln 4\pi \tau)v^2 - nv^2| dg
\]
on a complete noncompact Riemannian manifold \((M, g)\), where \(v \in W^{1,2}(M, g)\), and \(\tau > 0\) is a parameter. Note that by Theorem 3.1 in [He], \(W^{1,2}(M, g) = W_0^{1,2}(M, g)\). Let
\[
\mu(g, \tau) = \inf \{W(g, v, \tau) \mid v \in C_0^\infty(M), \|v\|_{L^2(M, g)} = 1\}.
\]
For \((M, g)\) with Ricci curvature bounded below and injectivity radius bounded away from 0, we have that \(\mu(g, \tau)\) is finite; for a proof see for example [RV].

The following proposition is a very slight improvement of some results in [Ku1], [Z12b] and [L] which in turn extend the entropy formula in Perelman [P] to the noncompact case. The improvement is on lowering the order of derivatives of the curvature tensor which are required to be uniformly bounded to guarantee the inequality (4.1) below.

**Proposition 4.1.** (cf. [Ku1], [Z12b] and [L]) Let \((M, g_0)\) be a complete noncompact Riemannian manifold with bounded curvature such that the injectivity radius is bounded away from 0 and \(\sup_M |\nabla Rm_{g_0}| < \infty\). Let \((M, (g(t))_{t \in [0,T]}\) be a complete solution to the Ricci flow with \(\sup_{M \times [0,T]} |Rm| < \infty\) and with \(g(0) = g_0\). Let \(v_T \in C^\infty(M)\) or \(v_T \in C^\infty(M)\) with \(|v_T(x)| \leq Ae^{-a d(x, x_0)^2}\) for any \(x \in M\), where \(A\) and \(a\) are positive constants, and \(x_0\) is some point in \(M\). Assume that \(u\) is a solution to the conjugate heat equation coupled to the Ricci flow, \(u_t + \Delta g(t) u - Ru = 0\), with \(u(x, T) = v_T(x)^2\). Let \(v(x, t) = \sqrt{u(x, t)}\). Fix \(L > T\) and let \(\tau(t) = L - t\). Then
\[
\frac{d}{dt} W(g(t), v(\cdot, t), \tau(t)) = 2\tau(t) \int_M |Ric - Hess\ln u - \frac{1}{2\tau(t)} g(t)|^2 u dg(t) \tag{4.1}
\]
for \(t \in [0, T]\). Consequently
\[
\mu(g(t_1), \tau(t_1)) \leq \mu(g(t_2), \tau(t_2))
\]
for \(0 \leq t_1 \leq t_2 \leq T\).

**Proof** In the proof of Theorem 16 in Kuang [Ku1], the extra assumption
\[
\sup_{M \times [0,T]} |\nabla \Delta Rm| < \infty \quad \text{or more precisely} \quad \sup_{M \times [0,T]} |\Delta R| < \infty
\]
comes from Lemma 6.3 in [CTY] which is cited there. But if we use Theorem 10 in [EKNT] (with \(\rho \to \infty\)) instead of Lemma 6.3 in [CTY], we only need to assume that \(\sup_{M \times [0,T]} |\nabla R| < \infty\) instead of \(\sup_{M \times [0,T]} |\Delta R| < \infty\). Note that \(\sup_{M \times [0,T]} |\nabla R| < \infty\) is implied by the condition \(\sup_{M \times [0,T]} |Rm| < \infty\) and \(\sup_M |\nabla Rm_{g_0}| < \infty\) by Lu’s modified version (see [LT]) of Shi’s derivative estimates. Also note that in the proof of Theorem 16 in [Ku1] Kuang only considers \(v_T\) with compact support; see Remark 17 there. But as in Step 1 of the proof of Corollary 4.1 in [Z12b] one can also allow \(v_T\) with quadratic exponential decay.

The monotonicity of the \(\mu\)-functional is stated on p. 1847 in [L], but the condition under which it holds is not stated explicitly there. (Note that [L] cites Theorem 7.1 (ii) in [CTY] which needs assumption (a1) there.) By taking a minimizing sequence (with compact support) for \(W(g(t_2), \cdot, L - t_2)\) and evolving it
backward under the conjugate heat equation one can derive the monotonicity of the $\mu$-functional from that of the $W$-entropy as in the compact case.

Using Proposition 4.1 one sees that in the main result of [Ku1][Ku2], which is an extension of the uniform Sobolev inequality along the Ricci flow on compact manifolds proved by Zhang [Z07] (see also Ye [Y]) to the noncompact case, the condition that the Laplacian of the scalar curvature is uniformly bounded along the Ricci flow can be removed. (This condition is not stated explicitly in Theorem 20 in [Ku1], but the proof of Theorem 20 in [Ku1] requires it since the proof use Theorem 16 in [Ku1] whose proof as written there needs it.) By the way, [Ku2] is not available to me; but see the reviews in Math. Review and zbMATH. Note that on p. 31 of [Ku1] Kuang used the minimizer of the $W$-entropy to derive the monotonicity of the $\mu$-functional. As pointed out in Zhang [Z12b] the minimizer of the $W$-entropy on a noncompact manifold does not always exist. However the monotonicity of the $\mu$-functional in the situation in [Ku1] holds true.

Combining this uniform Sobolev inequality along the Ricci flow on noncompact manifolds (and adapting Step 3 in the proof of Theorem 20 in [Ku1]) and results in Section 3 one can extend some results in Zhang [Z12a] and Bamler-Zhang [BZ] including distance distortion estimates, construction of a cutoff function, heat kernel estimates, a backward pseudolocality theorem and a strong $\varepsilon$-regularity theorem for Ricci flow on compact manifolds to the following situation: complete solutions to the Ricci flow with $\sup_{M \times [0,T]} |Rm| < \infty$, where the injectivity radius of the initial metric $g_0$ is bounded away from 0, and $\sup_M |\nabla Rm_{g_0}| < \infty$. (Note that on p. 411 of [BZ], the equality $\int \Delta K(\cdot, t)dg_t = 0$ for the heat kernel is used. In the noncompact case, this needs justification; but this can be done by adapting the argument used by Kuang in deriving (3.18) in [Ku1].)

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