Dynamics of quantum Fisher information in a squeezed thermal bath

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Abstract

We study the dynamics of quantum Fisher information of a qubit interacting with a squeezed thermal environment. We obtain the optimal initial state of the qubit, the optimal temperature of the environment and the optimal time of interaction, which maximize quantum Fisher information. Based on the ohmicity of the environment, we compare the dynamics of quantum Fisher information in ohmic, sub-ohmic and super-ohmic regimes of the environment. The dynamics shows that the sub-ohmic regime may be useful for precise measurements in quantum thermometry.

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Quantum Fisher Information (QFI) is an important concept of Quantum estimation theory (QET) [12], a direct extension of the classical estimation theory [3]. In QET, it establishes a lower bound for the estimation of parameters, which are not directly accessible to measurements and potentially affect the dynamics of different physical quantities of interest [4–7]. Besides QET, it has numerous uses in other fields based on the applications of quantum mechanics. For example, in quantum information theory, it is used as a tool for measuring non-Markovianity of environments, quantum entanglement of many body systems, quantum phase transition and distinguishability of quantum states on Hilbert spaces [8–12].

A common strategy for studying the dynamics of QFI and using it as a tool for either estimating some unknown parameters or other characteristics of environment is to monitor the evolution of an open quantum probe. Generally, a small controlled quantum system, usually a qubit, is allowed to interact with a bigger uncontrolled system (environment). During the interaction between the two, the information about the unknown parameter of the environment is encoded onto the space of the qubit. This encoded piece of information is then revealed through studying the dynamics of QFI which, in turn, enables for employing the estimation procedure. This strategy is very successful in studying the behavior of QFI under various setups. For instance, its dynamics provides useful information about the effects of single bosonic environment [12–13], multiple bosonic environments [14–15] and about the cutoff frequency of Ohmic type bosonic environments [16].

In quantum thermodynamics, QFI has pivotal role in estimating optimal temperature for different quantum mechanical processes [17–20]. Many recent studies have focused on the measurement of temperature at very small scales in which nano sized thermal environments are considered. The temperature of such environments become very sensitive to the disturbances caused by the presence of quantum probes [21–25]. Quantum harmonic oscillators [26], atomic condensates [27–28] and nano mechanical resonators [29] are some interesting paradigms of nano scale thermometry. In modern science and technology, precise estimation of optimal temperature has great significance on experimental realizations of many devices [30–32].

In this paper, we study the dynamics of QFI as a function of temperature and time.
of interaction for an exactly solvable model of a squeezed thermal environment interacting with a single qubit. In addition to the effect of temperature and time of interaction, we also investigate the effects of squeezing strength and relative phase of the environment on its behavior. Our results show that QFI can be maximized for certain values of both temperature of the environment and the time of interaction where the maximum varies with the choices of squeezing and phase parameters of the environment. Our findings, regarding the maximization of QFI with respect to temperature, can be used to characterize thermal environment for practical purposes in the field of quantum thermometry.

The paper is organized as follows: In section II, we introduce the Hamiltonian describing the dynamics of our system and briefly review the measure of QFI. In section III we present our analytical results, followed by their numerical simulations and the relevant discussion. We summarize our work in section IV.

II. PHYSICAL MODEL

We consider a qubit interacting with a squeezed bosonic thermal environment. The Hamiltonian describing the dynamics of the composite system can be expressed as

\[
H = \frac{1}{2}\omega_0\sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sigma_z \sum_k (g_k b_k^\dagger + g_k^* b_k),
\]

where on the right, the parts from left to right, represent the Hamiltonian of the qubit, the reservoir and the interaction between the two components. In the first part, $\omega_0 (\hbar = 1)$ is the transition energy of the qubit and $\sigma_z$ is the Pauli matrix acting on the space of the qubit. In the second part, $\omega_k$ is the energy corresponding to the frequency of $k$th mode of the reservoir and $b_k$ ($b_k^\dagger$) is the annihilation (creation) operator obeying the usual commutation relations of bosonic operators. In the third part, $g_k$ represents the coupling strength of the corresponding mode with the qubit. Since the Hamiltonian of the qubit commutes with the interaction part, the overall effect of $H$ on the qubit’s space is to decohere it.

The squeezed thermal state of the reservoir can be written as

\[
\rho_R(0) = S_\zeta \rho_{Th} S_\zeta^\dagger,
\]

where $\rho_{Th} = \exp[-H_R/T]/Z$ is the thermal state, $Z = \text{Tr}(\exp[-H_R/T])$ is the partition function, $H_R$ is the reservoir Hamiltonian (second part in Eq.(1)) and $T$ represents the
temperature of the reservoir. The squeeze operator $S_\zeta$ in Eq.(2) can be expressed as

$$S_\zeta = \exp \left[ \frac{1}{2} (\zeta^* b_k^2 - \frac{1}{2} \zeta_k (b_k^\dagger)^2) \right], \quad (3)$$

where $\zeta_k = r_k \exp[i\theta_k]$ with the parameters $r_k \geq 0$ and $\theta_k \in [0, 2\pi]$ denote the squeezing and phase parameters, respectively. The evolved density matrix of the qubit can be written as

$$\rho_s(t) = \text{Tr}_R[U_I(t)\rho_{SR}(0)U_I^\dagger(t)], \quad (4)$$

where $\rho_{SR}(0) = \rho_S(0) \otimes \rho_R(0)$ is the initial state of the composite system with $\rho_S(0)$ being the initial state of the qubit. The unitary operator $U_I(t)$ in the interaction picture is given by

$$U_I(t) = \exp \left[ \frac{\sigma_z}{2} (\alpha_k b_k^\dagger - \alpha^* b_k) \right], \quad (5)$$

with $\alpha_k = 2g_k (1 - e^{i\omega_k t})/\omega_k$.

For a spectrally decomposed density matrix $\rho(\eta) = \sum_i \lambda_i(\eta) |\phi_i(\eta)\rangle \langle \phi_i(\eta)|$, the QFI with respect to the parameter $\eta$ can be expressed as \[16\]

$$I(T) = \sum_i \left( \frac{\partial_\eta \lambda_i(\eta)}{\lambda_i(\eta)} \right)^2 + 2 \sum_{i \neq j} \frac{(\lambda_i(\eta) - \lambda_j(\eta))^2}{\lambda_i(\eta) + \lambda_j(\eta)} |\langle \phi_i(\eta)| \partial_\eta \phi_j(\eta) \rangle|^2, \quad (6)$$

where the first part depends only on the eigenvalues and corresponds to classical Fisher information \[33\]. The second part, in addition to eigenvalues, also depends on eigenvectors and is thus quantum mechanical in nature. A measurement on the final density matrix is said to be optimal for which the QFI reduces to classical Fisher Information. Moreover, being used as an estimation tool for different purposes \[4, 33\], it is required to search for the parameters of the system’s space that maximizes QFI.

### III. RESULTS AND DISCUSSION

In this section, we present our results by using the mathematical machinery developed in the previous section. We begin from a generic superposition state of the qubit in the form $|\psi(0)\rangle = \cos(\alpha/2)|0\rangle + \sin(\alpha/2)|1\rangle$. Using Eqs.(1) and (3) in Eq.(2), combining the result with $\rho_S(0) = |\psi(0)\rangle \langle \psi(0)|$ for $\rho_{SR}(0)$ and putting it along with Eq.(5) in Eq.(4) leads to the evolved density matrix of qubit, which can be written as

$$\rho_s(T, t) = \cos^2(\alpha/2)|0\rangle \langle 0| + \sin^2(\alpha/2)|1\rangle \langle 1| + \frac{1}{2} \sin \alpha \exp[-\Gamma(T, t)](|0\rangle \langle 1| + |1\rangle \langle 0|), \quad (7)$$
where $\Gamma(T, t)$ carries the information of environment, embedded onto the space of the qubit with its exponent written as

$$\exp[-\Gamma(T, t)] = \sum_k \langle \exp[\eta_k b^\dagger - \eta^* b_k]\rangle. \quad (8)$$

This, in fact, is the characteristic function of the Wigner representation of $\rho_R(0)$, a squeezed thermal state of all modes [13, 16]. In terms of the environment characteristic parameters, the function $\Gamma(T, t)$ can be expressed as

$$\Gamma(T, t) = \sum_k \frac{1}{2} |\eta_k(t)|^2 \coth \left( \frac{\omega_k}{2T} \right), \quad (9)$$

with

$$\eta_k(t) = \alpha_k \cosh r_k + \alpha^*_k e^{i\theta_k} \sinh r_k. \quad (10)$$

For an environment of continuous mode distribution the summation over $|g_k|^2$ can be replaced with integral over the spectral density function $J(\omega)$ of the environmental modes, that is, $\sum_k |g_k|^2 \to \int J(\omega) d\omega$. Using the value of $\alpha_k$ from Eq.(5), and applying continues mode approximation, it can explicitly be expressed as

$$\Gamma(T, t) = \int J(\omega) \left( \frac{1 - \cos \omega t}{\omega^2} \right) [\cosh 2r - \cos(\theta - \omega t) \sinh 2r] \coth \left( \frac{\omega}{2T} \right) d\omega. \quad (11)$$

For an environment belongs to ohmic family, the spectral density function is given by

$$J(\omega, \omega_c) = \frac{\omega^s}{\omega^s_c - 1} \exp[-\omega/\omega_c], \quad (12)$$

where $\omega_c$ is the cut off frequency, which describes a natural boundary in frequency response of the system and $s$ is the ohmicity dimensionless parameter, which grades the reservoir into three classes, ohmic ($s = 1$), sub-ohmic ($s < 1$) and super-ohmic ($s > 1$).

The sets of eigenvalues and eigenvectors of the final density matrix in Eq.(7) can be expressed as

$$\lambda^\pm(T, t) = \frac{1}{2} \left( 1 \pm \exp[-\Gamma(T, t)] \chi(\Gamma, \alpha) \right), \quad (13)$$

and

$$\phi^\pm(T, t) = [\exp[\Gamma(T, t)] \cot \theta \pm \csc \theta \chi(\Gamma, \alpha)] |0\rangle + |1\rangle, \quad (14)$$
FIG. 1: (Colour online) QFI verses temperature $T$ (upper panel) and QFI verses time $t$ (lower panel) at different values of squeezing parameter and phase parameter for sub ohmic ($s = 0.5$) case. Here we used (a) $t = 1, \theta = 1$ (b) $t = 1, r = 0.1$ (c) $T = 0.5, \theta = 1$ and (d) $T = 0.5, r = 0.1$.

with function $\chi(\Gamma, \alpha) = [\exp[2\Gamma(T, t)] \cos^2 \alpha + \sin^2 \alpha]^{1/2}$. Substitution of Eqs. (13) and (14) into Eq.(6) leads to the following form for QFI

$$I(T, t) = \frac{\sin^2 \alpha [\partial_T \Gamma(T, t)]^2}{\exp[2\Gamma(T, t)] - 1}. \tag{15}$$

It can be seen that the initial state parameter $\alpha$ maximizes Eq.(15) for $\alpha = \pi/2$, which corresponds to the equatorial state $|\psi(0)\rangle = |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ of the qubit. In other words, the maximal superposition of the computational bases is the optimal initial state of the qubit. With this choice of the initial state of the qubit, the function $\chi(\Gamma, \alpha) = 1$ and the eigenfunctions in Eq.(14) becomes independent of temperature $T$. This, in turn, reduces the second part of Eq.(6) to zero, which means that the projectors corresponding to the computational bases of the qubit constitute the set of optimal measurement. Now, due to the complex form of $\Gamma(T, t)$ it is not easy to analytically observe the effects of the different parameters of the environment on QFI. To this end, we resort to numerical simulation of Eq.(15) for the optimal initial state of the qubit.

Figure (1) shows the dynamics of QFI as a function of temperature $T$ of the environment.
FIG. 2: (Colour online) QFI verses temperature T (upper panel) and QFI verses time t (lower panel) at different values of squeezing parameter and phase parameter for ohmic ($s = 1$) case. Here we used (a) $t = 1$, $\theta = 1$ (b) $t = 1$, $r = 0.1$ (c) $T = 0.5$, $\theta = 1$ and (d) $T = 0.5$, $r = 0.1$.

(upper panel) and the time of interaction $t$ (lower panel) for three different choices of the squeezing parameter $r$ (first column) and the phase parameter $\theta$ (second column) for the sub-ohmic regime with $s = 0.5$. The qualitative behavior of QFI is not considerably affected by the choices of the parameters $r$ and $\theta$ when plotted as a function of temperature. In each case, it first rapidly grows to a maximum and then monotonically falls to a vanishing value with increasing temperature. The peak shifts minutely to lower temperature with increasing $r$ (figure (1a)). Also, in both cases the peak value and the rate of decrease, beyond the maximum, varies as per the choice of $r$ and $\theta$. The peak values are highest for $r = 1.5$ and $\theta = \pi$ in the two figures, respectively. On the other hand, the rate of increase and decrease at either side of the peak value is relatively gradual and the peaks are broadened when plotted as a function of the time of interaction. In fact, maximum information about the environment is encoded onto the space of the qubit at the peaks and thus they increase the optimality of the projective measurements over the qubit state. This type of behavior of QFI is very beneficial in different estimation procedures [4, 13].
The behavior of QFI in the ohmic regime \((s = 1)\) of the environment for the same set of values of the rest parameters as in figure \((1)\) is shown in figure \((2)\). One can see that in each case the peak values are highly reduced as compared to the case of sub-ohmic regime, however, the peaks are relatively broadened and shifted to the right in each case. On one hand, the reduced values of the peaks confine the utility of QFI both in quantum estimation procedures and quantum information theory. On the other hand, the broadened peaks help in increasing the precision of estimability in different quantum estimation processes. The shifting of peaks to the regime of higher temperature increases the easiness of its experimental realizations. Another salient feature, in comparison to figure \((1)\), is the order of the peaks which is reversed with respect to the values of the squeezing parameter \(r\).

In figure \((3)\), QFI is plotted for super-Ohmic regime for \(s = 3\) with the rest parameters set to the same values as before. It can be read from the figures that the peak values are further reduced and become negligibly small for the highest choice of \(r\). Unlike its behavior in figures \((1)\) and \((2)\) as function of time, it saturates for each value of the parameters \(r\) and \(\theta\). The highly reduced maximum values of QFI against temperature reduces its utility in the super-Ohmic case.

IV. CONCLUSION

In this paper we have studied the effect of different parameters of squeezed thermal environment on the dynamics of QFI by using a qubit as a probe. We have analytically found the optimal initial state of the qubit that maximizes QFI. We have further investigated the optimal temperature and optimal time of interaction through numerical simulation of the analytical result for QFI. The dynamics of QFI in the three different regimes of environment, namely, sub-Ohmic, Ohmic and super-Ohmic, are compared with each other. In each case, the qualitative behavior of QFI as a function of temperature is similar, it increases initially reaching a maximum followed by a gradual monotonic decay and reach a vanishing value in the asymptotic limit. However, quantitatively the dynamics strongly depends on the ohmicity parameter, characterizing the three regimes of the environment. The existence of optimal temperature and optimal initial time in each regime means that their values can precisely be estimated through the estimation procedures, which may be beneficial in
FIG. 3: (Colour online) QFI verses temperature $T$ (upper panel) and QFI verses time $t$ (lower panel) at different values of squeezing parameter and phase parameter for super-ohmic ($s = 3$) case. Here we used (a) $t = 1, \theta = 1$ (b) $t = 1, \ r = 0.1$ (c) $T = 0.5, \ \theta = 1$ and (d) $T = 0.5, \ r = 0.1$.

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