UNIFORM BOUNDS FOR RUIN PROBABILITY IN MULTIDIMENSIONAL RISK MODEL

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Abstract: In this paper we consider some generalizations of the classical $d$-dimensional Brownian risk model. This contribution derives some non-asymptotic bounds for simultaneous ruin probabilities of interest. In addition, we obtain non-asymptotic bounds also for the case of general trend functions and convolutions of our original risk model.

Key Words: Brownian risk model; Brownian motion; simultaneous ruin probability; uniform bounds.

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1. Introduction and first Result

Let $B(t), t \geq 0$ be a $d$-dimensional Brownian motion with independent standard Brownian motion components and set $Z(t) = AB(t), t \geq 0$ with $A$ a $d \times d$ real non-singular matrix. The recent contribution [1] derived the following remarkable inequality

\[ 1 \leq \frac{\mathbb{P}\{\exists t \in [0, T] : Z(t) \geq b\}}{\mathbb{P}\{Z(T) \geq b\} } \leq K(T), \quad K(T) = \frac{1}{\mathbb{P}\{Z(T) \geq 0\} } \tag{1} \]

valid for all $b \in \mathbb{R}^d, T > 0$. In our notation bold symbols are column vectors with $d$ rows and all operations are meant component-wise, for instance $x \geq 0$ means $x_i \geq 0$ for all $i \leq d$ with $0 = (0, \ldots, 0) \in \mathbb{R}^d$.

The special and crucial feature of (1) is that the bounds are uniform with respect to $b$. Moreover, if at least one component of $b$ tends to infinity, then $\mathbb{P}\{\exists t \in [0, T] : Z(t) \geq b\}$ can be accurately approximated (up to some constant) by the survival probability $\mathbb{P}\{Z(T) \geq b\}$.

Inequality (1) has been crucial in the context of Shepp-statistics investigated in [1]. It is also of great importance in the investigation of simultaneous ruin probabilities in vector-valued risk models (see [2–4]).

Specifically, consider the multidimensional risk model

\[ R(t, u) = au - X(t), \quad X(t) = Z(t) - ct \]

for some vectors $a, c \in \mathbb{R}^d$ and $Z(t), t \geq 0$ defined above. Typically, $R$ models the surplus of all $d$-portfolios of an insurance company, where $a_i u > 0$ plays the role of the initial capital. Here the component $Z_i$ models the accumulated claim amount up to time $t$ and $c_i t$ is the premium income for the $i$th portfolio.

Given a positive integer $k \leq d$, of interest is the calculation of the $k$-th simultaneous ruin probability, i.e., at least $k$ out of $d$ portfolios are ruined on a given time interval $[0, T]$ with $T$ possibly also infinite. That ruin probability can be written as

\[ \mathbb{P}\{\exists t \in [0, T] : Z(t) - ct \in uS\}, \quad u > 0, \]
where
\[ S := \bigcup_{I \subseteq \{1, \ldots, d\} \atop |I| = k} S_I, \quad S_I = \{ x \in \mathbb{R}^d : \forall i \in I, x_i > a_i \}. \]

The particular case \( Z(t) = AB(t), t \geq 0 \) with \( A \) a \( d \times d \) non-singular matrix is of special importance for insurance risk models, see e.g., [5]. Clearly, this instance is also of great importance in statistics and probability given the central role of the \( \mathbb{R}^d \)-valued Brownian motion as a natural limiting process in [6] it has been shown that (1) can be extended for this risk model, i.e., for all \( u, T > 0 \)
\[ 1 \leq \frac{\mathbb{P} \{ \exists t \in [0, T] : X(t) \in uS \}}{\mathbb{P} \{ X(T) \in uS \}} \leq K_S(T), \quad X(t) = Z(t) - c(t), \]
with \( c(t) = ct, t \geq 0 \) and some known constant \( K_S(T) > 0 \). Again the bounds are uniform with respect to \( u \).

It is clear that the inequality (2) does not hold for an arbitrary set \( S \subset \mathbb{R}^d \). Since Brownian motion has almost surely continuous sample paths, , if it hits some closed set \( S \), it definitely hits its boundary. Hence in the following special case, for all \( u \) positive we have
\[ \{ \exists t \in [0, T] : Z(t) \geq u \} = \{ \exists t \in [0, T] : Z(t) = u \}. \]
Hence, taking \( S = \{ x \in \mathbb{R}^d : x_1 = 1 \} \) and \( c(t) = 0 \) we have that
\[ \mathbb{P} \{ \exists t \in [0, T] : X(t) \in uS \} = \mathbb{P} \{ \exists t \in [0, T] : X_1(t) > u \} \geq \mathbb{P} \{ X_1(T) > u \} > 0, \]
\[ \mathbb{P} \{ X(T) \in uS \} = 0, \]
and (2) does not hold.

Therefore hereafter we shall consider only closed sets \( S \) described as follows:

**Definition 1.1.** Let \( X \) and \( Z \) are as defined above. The closed Borel set \( S \subset \mathbb{R}^d \) satisfies the cone condition with respect to the the vector-valued process \( X \) if there exists a strictly positive function \( \varepsilon_S(t), t > 0 \) such that for any point \( x \in S \) and any \( t > 0 \) there exists a Borel set \( V_x \subset S \) that contains \( x \) and not depending on \( t \), satisfying \( V_x - x \subset C(V_x - x) \) for all \( C > 1 \) and \( \mathbb{P} \{ Z(t) \in V_x - x \} \geq \varepsilon_S(t) \).

It is of interest to consider a general trend function in (2). We consider below a large class of trend functions which is tractable if \( Z \) has self-similar coordinates with index \( \alpha > 0 \). This is in particular the case when \( Z = AB \).

**Definition 1.2.** A continuous measurable vector-valued function \( c : [0, +\infty) \to \mathbb{R}^d \) belongs to \( RV_{t_0}(\alpha) \) for some \( \alpha > 0 \), \( t_0 \in [0, T] \) if for some \( M > 0 \), all \( i \in \{1 \ldots d\} \), \( t \in [0, T] \)
\[ |c_i(t) - c_i(t_0)| \leq M|t - t_0|^\alpha. \]

We state next our first result. Below \( F : \mathbb{R}^d \to \mathbb{R}^d \) growing means that for any \( x, y \in \mathbb{R}^d \) such that for all \( i \in \{1, \ldots, d\} \) \( x_i \geq y_i \) we have that \( F_i(x) \geq F_i(y) \) holds for all \( i \in \{1, \ldots, d\} \).

**Theorem 1.3.** If \( S \subset \mathbb{R}^d \) satisfies the cone condition with respect to the process \( Z = AB \) such that \( 0 \not\in S \) and \( c \in RV_T(1/2) \), then for all constants \( T > 0, u > 1 \) the inequality (2) holds with
\[ K(T) = \frac{2^{d/2}}{c(T)\varepsilon_S(T)}, \quad c(T) = \inf_{t \in [0, T]} e^{-T \left( \frac{c(T) - c(t)}{\sqrt{T}} \right)^T \Sigma^{-1} \left( \frac{c(T) - c(t)}{\sqrt{T}} \right)} > 0, \]
where $\Sigma$ is the covariance matrix of $Z(T)$. In particular, for any growing function $F : \mathbb{R}^d \to \mathbb{R}$
\[
P\{ \exists t \in [0, T] : F(Z(t) - c(t)) > ua \} \leq C_T P \{ F(Z(T) - c(T)) > ua \}
\]
we have $a \in \mathbb{R}^d \setminus (-\infty, 0]^d$, $u > 1$ and some constant $C_T$ which does not depend on $u$.

If $Z$ is a given separable random field, it is of interest to determine conditions such that (2) can be extended to
\[
1 \leq \frac{P \{ \exists t \in T : Z(t) - c(t) \in uS \}}{P \{ Z(T) - c(T) \in uS \}} \leq K_S(T),
\]
where $T = [0, T_1] \times \ldots \times [0, T_n]$ and $T = (T_1, \ldots, T_n)$ has positive components. For the case $Z(t) = \sum_{i=1}^n Z_i(t_i)$ where $Z_i$'s are independent copies of $Z$, and $c(t) = 0$ the result (3) was shown in [1][Thm 1.1] for some special set $S$. For more general set $S$ we have the following result:

**Theorem 1.4.** If $S \subset \mathbb{R}^d$ satisfies the cone condition with respect to $Z$, $0 \notin S$ and all $c_i \in RV_T(1/2)$, then for all constants $T_1, \ldots, T_n > 0, u > 1$ the inequality (3) holds with $Z(t) = \sum_{k=1}^n Z_k(t_k)$ and $c(t) = \sum_{k=1}^n c_k(t_k)$ with
\[
K_S(T) = \prod_{k=1}^n \frac{2^{d/2}}{c_k(T_k) \varepsilon_S(T_k)}, \quad c_k(T_k) = \inf_{t \in [0, T_k]} e^{-T_k \left( \frac{c_k(t_k) - c_k(t_0)}{\sqrt{T_k}} \right) ^T \Sigma^{-1} T_k \left( \frac{c_k(t_k) - c_k(t_0)}{\sqrt{T_k}} \right)} > 0,
\]
where $\varepsilon_S$ is any function satisfies the claims of Definition 1.1.

**2. Discussion**

In this section as in Introduction we consider first
\[
Z(t) = AB(t), \quad t \geq 0
\]
with $A$ non-singular and $B$ a $d$-dimensional Brownian motion with independent components. We shall discuss next the generalisation of the upper bound (2) for various special cases.

**2.1. Order statistics.** The classical multidimensional Brownian motion risk model (see [5]) is formulated in terms of some risk process $R$ specified by
\[
R(t, u) = au - Z(t) + ct
\]
for some vectors $a, c \in \mathbb{R}^d$. We are interested in the finite-time simultaneous ruin probability for $k$ out of $d$ portfolios, i.e. the probability that at least $k$ portfolios are ruined. In other words, we are investigating the probability
\[
P \{ \exists t \in [0, T] : Z(t) - c(t) \in uS \}, \quad u > 0,
\]
where
\[
S_u := \bigcup_{\mathcal{I} \subset \{1, \ldots, d\} \atop |\mathcal{I}| = k} S_{I, u}, \quad S_{I, u} = \{ x \in \mathbb{R}^d : \forall i \in I \ x_i \geq a_i u \}.
\]
Asymptotic approximations of such probability was already obtained in [6]. Now we want to derive a uniform non-asymptotic bound based on our previous findings. It is clear that all sets $S_{I, u}$ satisfy the
cone condition with respect to the process $Z$. Thus, $S_u$ also satisfies the cone condition with respect to the process $Z$, hence we can use Theorem 1.3 and write for some positive constant $C$
\[ \mathbb{P} \{ Z(T) - cT \in S_u \} \leq \mathbb{P} \{ \exists t\in[0,T] : Z(t) - ct \in S_u \} \leq C \mathbb{P} \{ Z(T) - cT \in S_u \} . \]

2.2. Fractional Brownian motion. Consider next the 1-dimensional risk model
\[ R(u, t) = u - B_H(t) + ct, \quad t > 0, \]
where $B_H(t)$ is a standard fractional Brownian motion with zero mean and variance function $|t|^{2H}$, $H \in (0, 1]$. We are interested in the calculation of the finite-time ruin probability for given $T > 0$. The inequalities below have already been shown in [2]. We retrieve them using our findings. Namely, by the Slepian inequality, we can write for $H > \frac{1}{2}$ and $W$ a standard Brownian motion
\[ \mathbb{P} \{ \exists \in [0,T] R(u, t) \leq 0 \} \leq \mathbb{P} \{ \exists \in [0,T] W (t^{2H}) - ct \geq u \} = \mathbb{P} \{ \exists \in [0,2u] W (t) - ct^{1/2H} \geq u \} = \mathbb{P} \{ \exists \in [0,1] W (t) - ct^{1-H}t^{1/2H} \geq u/T^H \} . \]

Since $ct^{1-H}t^{1/2H} \in RV_1(1/2)$, using Theorem 1.3, for some positive constant $C$ we can write
\[ \mathbb{P} \{ \exists \in [0,1] W (t) - ct^{1-H}t^{1/2H} \geq u/T^H \} \leq CP \{ W (1) - ct^{1-H} \geq u/T^H \} = CP \{ W (t^{2H}) - ct \geq u \} = CP \{ R(u, T) \leq 0 \} . \]

The above can be extended considering the convolution of $n$ independent one-dimensional fractional Brownian motions $B_{H_i}(t), t > 0, i \leq n$. Let $H_i > 1/2$ and define the risk processes
\[ R_i(u, t) = u/n - B_{H_i}(t) + c_i t, \quad i \leq n. \]

Consider the convolution of processes $R_i(u, t)$. Using Slepian inequality, for all $H_i > \frac{1}{2}$ we can write
\[ \mathbb{P} \left\{ \exists \in \prod_{i=1}^{n} [0,T_i] \sum_{i=1}^{n} R_i(u, t_i) \leq 0 \right\} \leq \mathbb{P} \left\{ \exists \in \prod_{i=1}^{n} [0,T_i] \sum_{i=1}^{n} W_i (t^{2H_i}) - c_i t_i \geq u \right\} = \mathbb{P} \left\{ \exists \in \prod_{i=1}^{n} [0,T_i^{2H_i}] \sum_{i=1}^{n} W_i (t) - c_i t_i^{1/2H_i} \geq u \right\} . \]

Here $W_i$ stands for an independent copy of Brownian motion. As $c_i t_i^{1/2H_i} \in RV_{T_i}(1/2, 1)$, using Theorem 1.4, for some positive constant $C$ we can write
\[ \mathbb{P} \left\{ \exists \in \prod_{i=1}^{n} [0,T_i^{2H_i}] \sum_{i=1}^{n} W_i (t) - c_i t_i^{1/2H_i} \geq u \right\} \leq CP \left\{ \sum_{i=1}^{n} W_i (T_i^{2H_i}) - c_i T_i \geq u \right\} = CP \left\{ \sum_{i=1}^{n} B_{H_i} (T_i) - c_i T_i \geq u \right\} . \]
3. Vector-valued time-transformation

Finally, we discuss some extensions of (2) under different time transformations. We use the notation from Section 2 and define the following time transform. Let \( f(t) : [0, +\infty) \in \mathbb{R}^d \) be a growing vector-valued function and define

\[
Z(f(t)) = (Z_1(f_1(t)), \ldots, Z_d(f_d(t)))^\top.
\]

Hence \( f(t) \) can be considered as a generalised transformation of time.

**Theorem 3.1.** Let \( c(t), f(t) : [0, T] \rightarrow \mathbb{R}^d \) be given. Suppose that all \( f_i(t) \)'s are continuous, strictly growing and for all \( i \in \{1, \ldots, d\} \) we have \( f_i(0) = 0 \) and function \( \delta_i(t) = \frac{f_i(t) - f_i(T)}{f_1(T) - f_1(t)} \) has a positive finite limit as \( t \rightarrow T \). Let also \( |c_i(T) - c_i(t)| < M \sqrt{f_1(T) - f_1(t)} \) for all \( t \in [0, T] \), all \( i \in \{1, \ldots, d\} \), some \( M > 0 \), and \( S \) satisfies the cone condition with respect to the process \( Z \). If \( 0 \notin S \), then for all constants \( T > 0, u > 1 \) the inequality (2) holds with \( X(t) = Z(f(t)) \) and

\[
K^*(T) = \frac{(2f_1(T))^{d/2}}{\mathcal{C}(T)\bar{\epsilon} S}, \quad \mathcal{C}(T) = \inf_{t \in [0, T]} e^{-\left(\frac{c(T) - c(t)}{\sqrt{f_1(T) - f_1(t)}}\right)^\top \Sigma^{-1}(\delta(t))\left(\frac{c(T) - c(t)}{\sqrt{f_1(T) - f_1(t)}}\right)} > 0,
\]

where

\[
\bar{\epsilon} S = \left(\inf_{i \in \{1, \ldots, d\}} \sup_{t \in [0, T]} \delta_i(t)\right)^{d/2} \epsilon S \left(\inf_{i \in \{1, \ldots, d\}} \delta_i(t)\right) > 0.
\]

**Remark 3.2.** The function \( f \) in Theorem 3.1 may also be an almost surely growing stochastic process, independent of \( Z \), satisfying

\[
\max_{i \in \{1, \ldots, d\}} f_i(T) < F, \quad \max_{i \in \{1, \ldots, d\}} \sup_{t \in [0, T]} \left| c_i(T_k) - c_i(t) \right| < M,
\]

\[
\delta < \inf_{i \in \{1, \ldots, d\}} \sup_{t \in [0, T]} \delta_i(t) < \Delta,
\]

almost surely with some positive constants \( F, M, \delta, \Delta \). In this case the inequality (2) holds with

\[
K^*(T) = \frac{(2F)^{d/2}}{\mathcal{C}(T)\bar{\epsilon} S}, \quad \mathcal{C}(T) = \min_{x \in [-M, M]^d} \epsilon S (x) > 0,
\]

and

\[
\bar{\epsilon} S = \left(\frac{\delta}{\Delta}\right)^{d/2} \epsilon S (\delta) > 0.
\]

We illustrate the above findings considering again \( d \) independent one-dimensional fractional Brownian motions \( B_{H_i}(t), t > 0 \) with Hurst parameters \( H_i > \frac{1}{2}, i \leq d \). Define \( d \) ruin portfolios

\[
R_i(u, t) = u - B_{H_i}(t) + c_i t,
\]
and we are interested in probability that all of them will be simultaneously ruined in \([0, T]\).

Using Gordon inequality (see [7, page 55]), we obtain

\[
\mathbb{P}\left\{ \exists t \in [0,T] \forall i \in \{1,\ldots,d\} R_i(u, t) < 0 \right\} \leq \mathbb{P}\left\{ \exists t \in [0,T] \forall i \in \{1,\ldots,d\} W_i \left( t^{2H_i} \right) - c_i t > u \right\}.
\]

Where \(B_i(t)\) are independent Brownian motions. Since

\[
\lim_{t \to T} \frac{T^{2H_i} - t^{2H_i}}{T^{2H_i - 1}} = \frac{2H_i}{2H_i - 1} > 0,
\]

using Theorem 3.1, for some positive constant \(C\), which does not depend on \(u\) we can write

\[
\mathbb{P}\left\{ \exists t \in [0,T] \forall i \in \{1,\ldots,d\} W_i \left( t^{2H_i} \right) - c_i t > u \right\} \leq C \mathbb{P}\left\{ \forall i \in \{1,\ldots,d\} W_i \left( T^{2H_i} \right) - c_i T > u \right\}
= C \mathbb{P}\left\{ \forall i \in \{1,\ldots,d\} B_{H_i} (T) - c T > u \right\}
= C \mathbb{P}\left\{ \forall i \in \{1,\ldots,d\} R_i(u, T) < 0 \right\}.
\]

4. Proofs

Let us note the following property of the function \(\varepsilon_S(t)\).

**Lemma 4.1.** If set \(S\) satisfies the cone condition with respect to the process \(Z(t)\) with some function \(\varepsilon_S(t)\), then for any constant \(u > 1\) set \(uS\) also satisfies the cone condition with respect to the process \(Z(t)\), and for any function \(\varepsilon_S(t)\) exists a function \(\varepsilon_uS(t)\) such that

\[
\varepsilon_uS(t) \geq \varepsilon_S(t).
\]

**Proof of Lemma 4.1:** Fix some \(x \in uS\). Then we know that \(y = x/u \in S\). As \(S\) satisfies the cone condition with respect to the process \(Z(t)\), there exists some cone \(V_y \subset S\) with vertex \(y\) such that

\[
\mathbb{P}\{Z(t) \in V_y - y\} \geq \varepsilon_S(t).
\]

Hence, \(uV_y \subset uS\) for all \(u > 1\). Note that using the properties of cone

\[
uV_y = u(y + (V_y - y)) = x + u(V_y - y) \supset x + (V_y - y).
\]

Hence, \(x + (V_y - y) \subset uS\) is some cone with vertex \(x\), and

\[
\mathbb{P}\{Z(t) \in uV_y - x\} \geq \mathbb{P}\{Z(t) \in V_y - y\} \geq \varepsilon_S(t).
\]

\qed

**Proof of Theorem 1.3:** Consider the first inequality. Define the following stopping moment

\[
\tau = \inf\{t \in [0, T] : Z(t) - c(t) \in uS\}.
\]

According to the strong Markov property

\[
\mathbb{P}\{Z(T) - c(T) \in uS\} = \int_0^T \int_{u\partial S} \mathbb{P}\{Z(\tau) - c(\tau) \in dx, \tau \in dt\} \mathbb{P}\{Z(T) - c(T) \in uS|Z(t) - c(t) = x\}.
\]

Using Lemma 4.1, \(uS\) satisfies the cone condition with respect to the process \(Z(t)\). Hence for all \(x \in uS, t \in [0, T]\)

\[
\mathbb{P}\{Z(T) - c(T) \in uS|Z(t) - c(t) = x\} \geq \mathbb{P}\{Z(T) - c(T) \in V_x|Z(t) - c(t) = x\}
= \mathbb{P}\{Z(T - t) - (c(T) - c(t)) \in V_x - x\}.
\]
Proof of Theorem 1.1. Consequently,

\[ \{ \mathbb{P} \{ \mathbf{Z}(T) \in uS \} \geq \mathbb{E}(T) \mathbb{E}(S(T)) - \varepsilon(T) \mathbb{E}(S(T)) \mathbb{P} \{ \mathbf{Z}(T) \in uS \} \}
\]

where \( \mathbf{V}_x \) is the cone from Definition 1.1. As the right part does not depend on \( x \) and \( t \), we can write

\[ \mathbb{P} \{ \mathbf{Z}(T) - c(T) \in uS \} \geq \mathbb{E}(T) \mathbb{E}(S(T)) - \varepsilon(T) \mathbb{E}(S(T)) \mathbb{P} \{ \mathbf{Z}(T) > uS \} \]

Hence, the first inequality holds. Consider the second one. Define a set

\[ a^+ = \{ x \in \mathbb{R}^d : x \geq a \} \]

and

\[ S_u = \{ x \in \mathbb{R}^d : F(x) \in ua^+ \}. \]

Set \( S_u \) satisfies the cone condition with respect to the process \( \mathbf{Z}(t) \) for \( \mathbf{Z}_x = x^+ \), as for any \( y \geq x \in S_u \)

\[ F(uy) \geq F(ux) \geq ua^+. \]

Consequently, \( y \in S_u \), and

\[ \varepsilon_s(t) = \mathbb{P} \{ \mathbf{X}(t) \in a^+ - x \} = \mathbb{P} \{ \mathbf{X}(t) \in [0, +\infty)^d \} \]

does not depend on \( u \). Applying the result above for the set \( S_u \) we obtain

\[ \mathbb{P} \{ \exists t \in [0, T] : \mathbf{X}(t) \in uS_u \} \leq \frac{2^{d/2} \mathbb{P} \{ \mathbf{X}(T) \in uS \} \mathbb{E}(T) \mathbb{E}(S(T))}{2^{d/2} \mathbb{P} \{ \mathbf{X}(T) \in uS \} \mathbb{E}(T) \mathbb{E}(S(T))}. \]

As the event \( \{ \mathbf{X}(t) \in uS_u \} \) is equal to the event \( \{ \mathbf{F}(\mathbf{X}(t) - c(t)) > ua \} \), this completes the proof. \( \square \)

**Proof of Theorem 1.4:** Define

\[ \psi_k(S) := \mathbb{P} \{ \exists t \in T_k : \sum_{i=1}^k (Z_i(t_i) - c_i(t_i)) + \sum_{i=k+1}^n (Z_i(T_i) - c_i(T_i)) \in S \} \]

where \( T_k = [0, T_1] \times \ldots \times [0, T_k] \). As in the previous section we are going to prove that the inequality

\[ \psi_k(uS) \leq \frac{2^{d/2} \psi_{k-1}(uS)}{\varepsilon_s(T_k) \mathbb{E}(S(T_k))} \]

takes place for any \( k \in \{1, \ldots, n\} \). We can fix the trajectories of processes \( Z_i(t) \) called \( x_i(t) \), fix random vectors \( Z_i(T_i) \) called \( x_i \), and define the process

\[
Z^{sk}(t, t^k) = Z_k(t) - c_k(t) + \sum_{i=1}^{k-1} (x_i(t_i) - c_i(t_i)) + \sum_{i=k+1}^{n} (x_i - c_i(T_i))
\]

where \( t^k = (t_1, \ldots, t_{k-1}) \in T_{k-1} \).

Since \( Z_i \) are independent, it is enough to show that for every set of trajectories \( x_i(t) \) and points \( x_j \), the inequality

\[
\psi^*(uS) \leq \frac{2^{d/2} \nu(uS)}{\varepsilon S(T_k) \mathcal{C}_k(T_k)}
\]

takes place, where

\[
\psi^*(S) = \mathbb{P} \left\{ \exists t \in [0, T_k] : Z^sk(t, t^k) \in S \text{ for some } t^k \in T_{k-1} \right\},
\]

\[
\nu(S) = \mathbb{P} \left\{ Z^sk(T_k, t^k) \in S \text{ for some } t^k \in T_{k-1} \right\}.
\]

Define the following stopping time:

\[
\tau_k = \inf \left\{ t : Z^sk(t, t^k) \in uS \text{ for some } t^k \in T^k \right\},
\]

and the random vector

\[
\tilde{x}_k = \begin{cases} x^*, & \tau_k \leq T_k, \\ 0, & \text{otherwise,} \end{cases}
\]

where \( x^* \) is any point from the following set:

\[
\bigcup_{t^k \in T^k} \left\{ Z^sk(\tau_k, t^k) \right\} \bigcap uS.
\]

Using the total probability formula we obtain

\[
\nu(uS) = \int_0^{T_k} \int_{u \partial S} \mathbb{P} \{ \tilde{x}_k \in dx_0, \tau_k \in dt \} \mathbb{P} \left\{ Z^sk(T_k, t^k) \in uS \text{ for some } t^k \in T_{k-1} | \tau_k = t, \tilde{x}_k = x_0 \right\}.
\]

For any \( t^k \in T_{k-1} \) we have

\[
Z^sk(T_k, t^k) - Z^sk(t, t^k) = Z_k(T_k) - Z_k(t) - (c_k(T_k) - c_k(t)).
\]

Thus, using the same chain of inequalities as in Theorem 1.3 we obtain

\[
\mathbb{P} \left\{ Z^sk(T_k, t^k) \in uS \text{ for some } t^k \in T^k | \tau_k = t, \tilde{x}_k = x_0 \right\} \geq \mathbb{P} \left\{ Z_k(T_k) - Z_k(t) - (c_k(T_k) - c_k(t)) \in uS - x_0 \right\} \geq \frac{\mathcal{C}_k(T_k) \varepsilon S(T_k)}{2^{d/2}}
\]

which completes the proof.

\( \square \)

**Remark 4.2.** The random variable \( \tau_k \) is measurable, because it can be represented as

\[
\tau_k = \inf \left\{ t : Z_k(t) - c_k(t) \in S_k^\circ \right\},
\]
Proof of Theorem 3.1: Define a stopping time
\[ \tau = \inf \{ t \in [0, T] : Z(f(t)) - c(t) \in uS \} \]
According to the strong Markov property
\[
\mathbb{P}\{Z(f(T)) - c(T) \in uS\} = \int_0^T \int_{u\partial S} \mathbb{P}\{Z(f(\tau)) - c(\tau) \in dx, \tau \in dt\}
\times \mathbb{P}\{Z(f(T)) - c(T) \in uS|Z(f(t)) - c(t) = x, \tau = t\}.
\]
In view of Lemma 4.1, \( uS \) satisfies the cone condition with respect to the process \( Z(t) \). Consequently, we have

\[
\mathbb{P}\{Z(f(T)) - c(T) \in uS|Z(f(t)) - c(t) = x, \tau = t\}
= \mathbb{P}\{Z(f(T)) - c(T) \in uS|Z(f(t)) - c(t) = x\}
\geq \mathbb{P}\{Z(f(T)) - Z(f(t)) - c(T) + c(t) \in V_x - x\}
= \mathbb{P}\{Z(f(T) - f(t)) - (c(T) - c(t)) \in V_x - x\}
\geq \mathbb{P}\left\{Z(\delta(t)) - \frac{c(T) - c(t)}{\sqrt{f_1(T) - f_1(t)}} \in \frac{V_x - x}{\sqrt{f_1(T) - f_1(t)}}\right\}
\geq \int_{y \in \frac{V_x - x}{\sqrt{f_1(T)}}} \varphi_{\delta(t)} \left( \frac{y + \frac{c(T) - c(t)}{\sqrt{f_1(T) - f_1(t)}}}{\sqrt{f_1(T) - f_1(t)}} \right) dy
\geq \int_{y \in \frac{V_x - x}{\sqrt{f_1(T)}}} \mathcal{C}(T) \varphi_{\delta(t)} (\sqrt{2y}) dy
\geq \frac{\mathcal{C}(T)}{2^{d/2}} \mathbb{P}\left\{Z(\delta(t)) \in \frac{V_x - x}{\sqrt{f_1(T)}}\right\}
= \frac{\mathcal{C}(T)}{(2f_1(T))^{d/2}} \mathbb{P}\left\{Z(\delta(t)) \in V_x - x\right\}
= \frac{\mathcal{C}(T)}{(2f_1(T))^{d/2}} \mathbb{P}\left\{B(\delta(t)) \in A^{-1}(V_x - x)\right\}
= \frac{\mathcal{C}(T)}{(2f_1(T))^{d/2}} \frac{1}{\sqrt{2\pi} \prod_{i=1}^d \delta_i(t)} \int_{y \in A^{-1}(V_x - x)} e^{-\frac{1}{2} \sum_{i=1}^d \delta_i(t)^2} dy,
\]
where \( \varphi_{\delta(t)} \) is the pdf of \( Z(\delta(t)) \). Using that all the functions \( \delta_i(t) \) are bounded and separated from zero for \( t \in [0, T] \), there exists some constants \( \delta, \Delta > 0 \), such that for all \( i \in \{1, \ldots, d\} \) and all \( t \in [0, T] \)
\[ \delta \leq \delta_i(t) \leq \Delta. \]
Hence we obtain
\[
\frac{1}{\sqrt{2\pi \prod_{i=1}^d \delta_i(t)}} \geq \frac{1}{\sqrt{2\pi \prod_{i=1}^d \Delta}}, \quad e^{-\frac{1}{2} \sum_{i=1}^d \frac{x_i^2}{\Delta}} \geq e^{-\frac{1}{2} \sum_{i=1}^d \frac{x_i^2}{\delta_i(t)}},
\]
and finally
\[
\mathbb{P}\{Z(f(T)) - c(T) \in uS|Z(f(t)) - c(t) = x, \tau = t\} \geq \frac{c(T)}{(2f_1(T))^{d/2}} \frac{1}{\sqrt{2\pi \prod_{i=1}^d \Delta}} \int_{y \in A^{-1}(V_x - x)} e^{-\frac{1}{2} \sum_{i=1}^d \frac{y_i^2}{\delta_i(t)}} \, dy \\
= \frac{c(T)}{(2f_1(T))^{d/2}} \frac{\sqrt{\prod_{i=1}^d \delta_i(t)}}{\sqrt{\prod_{i=1}^d \Delta}} \mathbb{P}\{B(\delta) \in A^{-1}(V_x - x)\} \\
\geq \frac{c(T)}{(2f_1(T))^{d/2}} \frac{\sqrt{\prod_{i=1}^d \delta_i(t)}}{\sqrt{\prod_{i=1}^d \Delta}} S(\delta).
\]
Hence the claim follows. \qed

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