COHOMOLOGICAL LOCALIZATION FOR TRANSVERSE LIE ALGEBRA ACTIONS ON RIEMANNIAN FOLIATIONS

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ABSTRACT. We prove localization and integration formulas for the equivariant basic cohomology of Riemannian foliations. As a corollary we obtain a Duistermaat-Heckman theorem for transversely symplectic foliations.

Contents

1. Introduction 1
2. Preliminaries 2
3. Molino theory and some applications 9
4. The orbit type stratification 19
5. Borel-Atiyah-Segal localization 23
6. Atiyah-Bott-Berline-Vergne localization 24
Notation Index 33
References 34

1. Introduction

Most of the recent advances in the area of Riemannian foliations, such as the index theorems of Brüning et al. [8] and Gorokhovsky and Lott [16], and the cohomological localization formulas of Töben [30] and Goertsches et al. [14], build on the structure theory developed by Molino in his monograph [27]. The central feature of this structure theory is that the leaf closures of a Riemannian foliation, in sharp contrast to those of a general foliation, form a singular foliation. This leaf closure foliation possesses a type of “internal” symmetry in the shape of a locally constant sheaf of Lie algebras of vector fields, known as the Molino structure sheaf or centralizer sheaf, which acts transversely on each leaf in such a way that the orbit of the leaf is the closure of the leaf. A Killing foliation is a Riemannian foliation whose centralizer sheaf is globally constant, in which case it is automatically abelian, as shown by Molino. The Goertsches-Nozawa-Töben cohomological localization formulas hold in the setting of Killing foliations. They are integration formulas for differential forms that are basic with respect to the foliation and equivariant with respect to the Molino centralizer algebra.

Many Riemannian foliations possess additional, “external” Lie algebras of symmetries that commute with the Molino centralizer sheaf, and the goal of this paper is to extend the cohomological localization formulas to this wider class of symmetries. Our main results are two localization theorems in equivariant basic de Rham cohomology. The first is a contravariant Borel-Atiyah-Segal version, which resolves a problem posed in [14].

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Theorem. Let $M$ be a compact manifold equipped with a Riemannian foliation $\mathcal{F}$ and an isometric transverse action of an abelian Lie algebra $\mathfrak{g}$. Then the inclusion of the fixed leaf set $M^0$ into $M$ induces an isomorphism in localized equivariant basic cohomology $S^{-1}H_0(M, \mathcal{F}) \cong S^{-1}H_0(M^0, \mathcal{F})$.

Here $S$ is the multiplicative subset $S_\mathfrak{g}^* \setminus \{0\}$ of the symmetric algebra $S_\mathfrak{g}^*$. See Theorem 5.1 for a more precise version. Our second main result is a covariant Atiyah-Bott-Berline-Vergne localization theorem.

Theorem. Let $M$ be a compact manifold equipped with a Riemannian foliation $\mathcal{F}$ and an isometric transverse action of an abelian Lie algebra $\mathfrak{g}$. Suppose that the foliation $\mathcal{F}$ is transversely oriented and that the Molino structure bundle has trivial determinant. Let $\alpha$ be a $d_\mathfrak{g}$-closed equivariant basic form on $M$. Then

$$\int_M \alpha = \sum_X \int_X \eta_X^{-1} \wedge i_X^* \alpha,$$

where the sum is over the connected components $X$ of the fixed-leaf manifold $M^0$.

Here $\int$ denotes transverse integration and $\eta_X$ is the equivariant basic Euler form of the component $X$. See Theorem 6.4.2 for a fuller statement. These results rely on a version of the equivariant Thom isomorphism theorem which we established in [24]. An immediate consequence of these theorems is a Duistermaat-Heckman theorem for the transversely symplectic case, Theorem 6.4.4. Another application is the computation of the basic Betti numbers of toric quasifolds in [25, §6]. We postpone further applications of our theorems to a sequel to this paper. See also [10] for related recent work.

Lie algebras in this paper act on foliated manifolds not as true vector fields, but as transverse vector fields, which are equivalence classes of vector fields. Because of this one cannot form an action Lie algebroid in the usual way. However, there is a way of amalgamating the Lie algebra with the foliation to create what we call the transverse action Lie algebroid. Just like action algebroids and foliation algebroids, transverse action algebroids are always integrable (Proposition 2.2.5). Although this fact plays no further role in the paper, we have included it because it may offer an approach to cohomological localization formulas for foliations that are not Riemannian and to which the Molino structure theory does not apply. Namely, the equivariant basic de Rham complex is a subcomplex of the much larger simplicial de Rham complex of any Lie groupoid integrating the transverse action algebroid, the properties of which remain to be explored.

Another subsidiary result that we hope may be useful elsewhere is an orbit type stratification theorem for isometric transverse Lie algebra actions on Riemannian foliations, Theorems 4.7 and 4.10, which is almost the same as for proper Lie group actions, even though the slice theorem fails in our context.

A notation index is provided at the end of the paper.

2. Preliminaries

All manifolds in this paper will be smooth ($C^\infty$) and paracompact and all vector fields and foliations will be smooth. In this preliminary section we start by reviewing some elementary foliation theory, largely for the purpose of introducing notation. See also the notation index in the back. Then we recall the notion of a transverse Lie algebra action on a foliated manifold. This is not a Lie algebra action on the manifold, but should be thought of as a model for a Lie algebra action on the leaf space of the foliation. We introduce a certain integrable Lie algebroid associated with a transverse action, which we call the transverse
action Lie algebroid, and which does not seem to have appeared in the literature before. We show that it is integrable to a not necessarily Hausdorff Lie groupoid. We finish by reviewing Goertsches and Töben’s equivariant basic de Rham complex.

2.1. Foliations. Let $M$ be a manifold and let $\mathcal{F}$ be a foliation of $M$. We denote by $\mathcal{F}(x)$ the leaf through $x \in M$, by $T_x \mathcal{F}$ the tangent space to the leaf at $x$, and by $T \mathcal{F}$ the integrable subbundle of $TM$ tangent to $\mathcal{F}$. We call a subset $X$ of the manifold $\mathcal{F}$-invariant or $\mathcal{F}$-saturated if for every $x \in X$ the leaf $\mathcal{F}(x)$ is contained in $X$. The normal bundle of the foliation is the vector bundle $N \mathcal{F} = TM/T \mathcal{F}$ over $M$. A smooth map $f : M \to M'$ to a second foliated manifold $(M', \mathcal{F}')$ is said to be foliate if it maps each leaf of $\mathcal{F}$ to a leaf of $\mathcal{F}'$, i.e. $f(\mathcal{F}(x)) \subseteq \mathcal{F}'(f(x))$ for all $x \in M$. Thus $f$ is foliate if and only if the tangent map $T f : TM \to T M'$ maps $T \mathcal{F}$ to $T \mathcal{F}'$. The tangent map then descends to a bundle map $N f : N \mathcal{F} \to N \mathcal{F}'$, which we refer to as the normal derivative.

Let $\mathfrak{x}(M) = \Gamma(TM)$ be the Lie algebra of all vector fields on $M$ and let $\mathfrak{x}(\mathcal{F}) = \Gamma(T \mathcal{F})$ be the space of sections of $T \mathcal{F}$. Then $\mathfrak{x}(\mathcal{F})$ is the Lie subalgebra of $\mathfrak{x}(M)$ consisting of all vector fields tangent to the leaves of the foliation. We denote the normalizer $N_{\mathfrak{x}(M)}(\mathfrak{x}(\mathcal{F}))$ of $\mathfrak{x}(\mathcal{F})$ in $\mathfrak{x}(M)$ by $\mathfrak{r}(\mathcal{F})$ and we call elements of $\mathfrak{r}(\mathcal{F})$ foliate vector fields. A vector field $w$ on $M$ is foliate if and only if the flow of $w$ consists of foliate maps if and only if for every vector field $v$ tangent to $\mathcal{F}$ the commutator $[v, w]$ is also tangent to $\mathcal{F}$. We call elements of the quotient Lie algebra $\mathfrak{x}(M, \mathcal{F}) = \mathfrak{r}(\mathcal{F})/\mathfrak{x}(\mathcal{F})$ transverse vector fields. A transverse vector field is not a vector field, but an equivalence class of foliate vector fields modulo vector fields tangent to $\mathcal{F}$. These Lie algebras, which go by various different names in the standard references [20], [26], [27], and [31], form a short exact sequence

\[(2.1.1) \quad \mathfrak{x}(\mathcal{F}) \longhookrightarrow \mathfrak{r}(\mathcal{F}) \longrightarrow \mathfrak{x}(M, \mathcal{F}).\]

The transverse vector fields form a subspace of the space of sections of the normal bundle,

$\mathfrak{x}(M, \mathcal{F}) = \mathfrak{r}(\mathcal{F})/\mathfrak{x}(\mathcal{F}) \subseteq \Gamma(N \mathcal{F}) = \mathfrak{x}(M)/\mathfrak{x}(\mathcal{F})$.

Being the quotient of a Lie algebra by a subalgebra, the space $\Gamma(N \mathcal{F})$ is an $\mathfrak{x}(\mathcal{F})$-module. The subspace $\mathfrak{x}(M, \mathcal{F})$ is equal to $\Gamma(N \mathcal{F})^{\mathfrak{x}(\mathcal{F})}$, the space of $\mathfrak{x}(\mathcal{F})$-fixed sections of $N \mathcal{F}$. Thus transverse vector fields are $\mathfrak{x}(\mathcal{F})$-fixed sections of the normal bundle $N \mathcal{F}$.

If the foliation $\mathcal{F}$ is strictly simple, i.e. given by the fibres of a surjective submersion $M \to P$ with connected fibres, then the leaf space $M/\mathcal{F}$ is a manifold diffeomorphic to $P$. In this case the Lie algebra of transverse vector fields $\mathfrak{x}(M, \mathcal{F})$ is nothing but the Lie algebra of vector fields on the leaf space. If the foliation is not strictly simple, we regard transverse vector fields as a substitute for vector fields on the leaf space. We can also regard the leaf space as an étale stack and $\mathfrak{x}(M, \mathcal{F})$, following [18, § 5.7], as the Lie algebra of vector fields on this stack.

Given a foliate map $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$, we say that two transverse vector fields $v \in \mathfrak{x}(M, \mathcal{F})$ and $v' \in \mathfrak{x}(M', \mathcal{F}')$ are $f$-related, and we write

\[(2.1.2) \quad v \sim_f v',\]

if the normal derivative of $f$ satisfies $N_x f(v_x) = v'_{f(x)}$ for all $x \in M$.

The $\mathfrak{x}(\mathcal{F})$-module structure on the space of sections $\Gamma(N \mathcal{F})$,

$\nabla^\mathcal{F} : \mathfrak{x}(\mathcal{F}) \times \Gamma(N \mathcal{F}) \to \Gamma(N \mathcal{F})$,

is induced by the Lie bracket $\mathfrak{x}(\mathcal{F}) \times \mathfrak{x}(M) \to \mathfrak{x}(M)$ and is called the Bott connection or partial connection of the foliation. It satisfies the usual rules

$\nabla^\mathcal{F}_{f u} v = f \nabla_u v, \quad \nabla^\mathcal{F}_u (f v) = L(u)(f)v + f \nabla^\mathcal{F}_u v$. 

for all smooth functions $f$ and sections $u \in \mathfrak{X}(\mathcal{F})$, $v \in \Gamma(N\mathcal{F})$. The partial connection provides a partial horizontal lifting map $T\mathcal{F} \to T(N\mathcal{F})$, i.e. a splitting over $T\mathcal{F}$ of the surjection $T(N\mathcal{F}) \to TM$. For every leaf $L$ of $\mathcal{F}$ the partial connection induces a genuine flat connection on the restricted bundle $N\mathcal{F}|_L$.

2.2. Transverse Lie algebra actions. Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra and let $(M,\mathcal{F})$ be a foliated manifold. A foliate action of $\mathfrak{g}$ on $(M,\mathcal{F})$ is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(\mathcal{F})$. By the Lie-Palais theorem [28, Ch. 4], if the vector fields induced by $\mathfrak{g}$ are complete, a foliate $\mathfrak{g}$-action integrates to a smooth $G$-action, where $G$ is a suitable Lie group with Lie algebra $\mathfrak{g}$. The group $G$ then acts by foliate diffeomorphisms and we refer to this as a foliate $G$-action.

A transverse action of $\mathfrak{g}$ on $(M,\mathcal{F})$ is a Lie algebra homomorphism $\alpha : \mathfrak{g} \to \mathfrak{X}(M,\mathcal{F})$. For $\xi \in \mathfrak{g}$ we denote the transverse vector field $\alpha(\xi)$ by $\xi_M$. Each of these is represented by a foliate vector field $\tilde{\xi}_M \in \mathfrak{X}(\mathcal{F})$. However, there is in general no way of choosing the representatives $\tilde{\xi}_M$ in such a way that they generate a foliate $\mathfrak{g}$-action. In other words a transverse $\mathfrak{g}$-action does not necessarily lift to a foliate $\mathfrak{g}$-action, and in particular there is no guarantee that a transverse $\mathfrak{g}$-action “integrates” to a group action. Nevertheless, Proposition 2.2.5 below tells us that a transverse Lie algebra action always integrates to the action of a certain Lie groupoid. We regard transverse Lie algebra actions, which are the focus of our interest, as a substitute for Lie group actions on the leaf space $M/\mathcal{F}$.

For the remainder of § 2.2 we fix a transverse $\mathfrak{g}$-action $\alpha : \mathfrak{g} \to \mathfrak{X}(M,\mathcal{F})$.

If the foliation is strictly simple, then the transverse action amounts to an ordinary $\mathfrak{g}$-action on the manifold $M/\mathcal{F}$. Similarly, if $S$ is a transversal to the foliation, the transverse action on $M$ restricts to an ordinary $\mathfrak{g}$-action on $S$, because every $x \in S$ has an open neighbourhood $U$ such that the foliation $\mathcal{F}|_U$ is strictly simple with leaf space $S \cap U$. Indeed, since transverse vector fields on $M$ are $\mathfrak{X}(\mathcal{F})$-invariant sections of $N\mathcal{F}$, the transverse $\mathfrak{g}$-action on $U$ is uniquely determined by the $\mathfrak{g}$-action on $S \cap U$.

A Lie algebroid over $M$ is a real vector bundle $\mathfrak{a} \to M$ equipped with a vector bundle map $t : \mathfrak{a} \to TM$ called the anchor and a real Lie algebra structure

\[
[-,-] : \Gamma(\mathfrak{a}) \times \Gamma(\mathfrak{a}) \to \Gamma(\mathfrak{a})
\]

on the space of smooth sections which satisfies the Leibniz rule

\[ [s_1, f s_2] = f[s_1, s_2] + L(t(s_1))(f)s_2 \]

for all sections $s_1, s_2 \in \Gamma(\mathfrak{a})$ and functions $f \in C^\infty(M)$. Here $L(t(s_1))(f)$ denotes the Lie derivative of $f$ along the vector field $t(s_1)$.

If the transverse $\mathfrak{g}$-action on $M$ lifts to a true Lie algebra action $\mathfrak{g} \to \mathfrak{X}(M)$, we can form the action Lie algebroid $\mathfrak{g} \ltimes M$ as in [26, § 6.2]. For a general transverse $\mathfrak{g}$-action the definition of an action Lie algebroid does not make sense, but must be modified as follows.

Let $\mathfrak{g}_M$ be the trivial bundle with fibre $\mathfrak{g}$ over $M$. We define the transverse action Lie algebroid $\mathfrak{g} \ltimes \mathcal{F}$ of the transverse $\mathfrak{g}$-action to be the fibre product

\[
\mathfrak{g} \ltimes \mathcal{F} = \mathfrak{g}_M \times_{\mathcal{F} \cup M} TM = \{ (x, \xi, v) \in M \times \mathfrak{g} \times TM \mid v \in T_x M \text{ and } \xi_{M,x} = v \mod T_x \mathcal{F} \},
\]

which is a smooth subbundle of the bundle $\mathfrak{g}_M \times TM$ over $M$. The projection $\mathfrak{g} \ltimes \mathcal{F} \to \mathfrak{g}_M$ defined by $(x, \xi, v) \mapsto (x, \xi)$ has kernel $T\mathcal{F}$, so we have a short exact sequence

\[
T\mathcal{F} \hookrightarrow \mathfrak{g} \ltimes \mathcal{F} \twoheadrightarrow \mathfrak{g}_M
\]

of vector bundles over $M$. Define the bundle map

\[ t : \mathfrak{g} \ltimes \mathcal{F} \to TM \]
by \( t(\xi, v) = v \). For every smooth map \( \xi : M \to \mathfrak{g} \) and every \( x \in M \) we define \( \xi_x \in T_x \mathcal{F} \) to be the value of the transverse vector field \( a(\xi(x)) \in \mathfrak{X}(M, \mathcal{F}) \) at \( x \). A smooth section of \( \mathfrak{g} \rhd \mathcal{F} \) is a pair \((\xi, v) \in C^\infty(M, \mathfrak{g}) \times \mathfrak{X}(M)\) satisfying \( \xi_x = v_x \mod T_x \mathcal{F} \) for all \( x \in M \). We define the bracket of two sections \((\xi, v)\) and \((\eta, w)\) by

\[
[\xi, \eta] \ni (\xi, v)(x) = ([\xi(x), \eta(x)] + L(v)(\eta)(x) - L(w)(\xi)(x), [v, w](x))
\]

for \( x \in M \). We assert that this bracket makes \( \mathfrak{g} \rhd \mathcal{F} \) a Lie algebroid over \( M \).

**2.2.4. Proposition.** The bracket (2.2.3) is the unique skew symmetric \( \mathbb{R} \)-bilinear operation on \( \Gamma(\mathfrak{g} \rhd \mathcal{F}) \) that satisfies the Leibniz rule (2.2.1) with respect to the vector bundle map (2.2.2) and has the property \([\xi, \eta], [\xi, w] = ([\xi, \eta], [v, w])\) for all constant maps \( \xi, \eta \in C^\infty(M, \mathfrak{g}) \). This operation is a Lie bracket on the space of sections \( \Gamma(\mathfrak{g} \rhd \mathcal{F}) \) and hence makes \( \mathfrak{g} \rhd \mathcal{F} \) a Lie algebroid over \( M \) with anchor \( t \).

**Proof.** We omit the verification that (2.2.3) satisfies the Leibniz rule. The product

\[
\mathfrak{L} = \{ (\xi, v) \in \mathfrak{g} \times \mathcal{R}(\mathcal{F}) \mid \xi_M = v \mod \mathfrak{X}(\mathcal{F}) \}.
\]

The Lie algebra \( \mathfrak{L} \) naturally identifies with a subalgebra of \( \Gamma(\mathfrak{g} \rhd \mathcal{F}) \), namely by regarding an element \( \xi \in \mathfrak{g} \) as a constant map \( \xi \in C^\infty(M, \mathfrak{g}) \). On the subspace \( \mathfrak{L} \) the bracket (2.2.3) coincides with the Lie bracket coming from \( \mathfrak{g} \times \mathcal{R}(\mathcal{F}) \), namely \([\xi, \eta], [v, w] = ([\xi, \eta], [v, w])\). Every section of \( \mathfrak{g} \rhd \mathcal{F} \) can be written as a finite sum \( \sum_i f_i(\xi_i, v_i) \) with \( f_i \in C^\infty(M) \) and \( (\xi_i, v_i) \in \mathfrak{L} \). In other words, the space of sections \( \Gamma(\mathfrak{g} \rhd \mathcal{F}) \) is generated by \( \mathfrak{L} \) as a \( C^\infty(M) \)-module. Armed with these facts one shows that \( \Gamma(\mathfrak{g} \rhd \mathcal{F}) \) is closed under the bracket (2.2.3), that (2.2.3) is the unique skew symmetric bilinear map that satisfies the Leibniz rule and coincides with the Lie bracket on \( \mathfrak{L} \), and that it satisfies the Jacobi identity.

QED

**A Lie groupoid** over \( M \) is a (not necessarily Hausdorff) manifold \( G \) equipped with surjective submersions \( s, t : G \to M \), smooth maps \( m : G_2 \to G \) (where \( G_2 \) is the fibre product \( \{ (f, g) \in G \times G \mid s(f) = t(g) \} \)), \( i : G \to G \), and \( u : M \to G \), which make \( G \) an abstract groupoid over \( M \) with source map \( s \), target map \( t \), multiplication law \( m \), inversion law \( i \), and unit map \( u \). If \( G \) is a Lie groupoid over \( M \), the normal bundle \( \alpha \to M \) of the unit bisection \( u(M) \cong M \) is equipped with a natural Lie algebroid structure. We say that a Lie algebroid \( \alpha \) obtained in this way is *integrable*, and that \( G \) *integrates* \( \alpha \); cf. [26, Ch. 6].

**2.2.5. Proposition.** The transverse action Lie algebroid \( \mathfrak{g} \rhd \mathcal{F} \) is integrable.

**Proof.** The following method is adapted from [11, Théorème 2.1]; see also [26, §6.2]. Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \) and let \( p : G \times M \to M \) be the projection onto the second factor. For every section \( (\xi, v) \in C^\infty(M, \mathfrak{g}) \times \mathfrak{X}(M) \) of \( \mathfrak{g} \rhd \mathcal{F} \) define a vector field \( a_{G \times M}(\xi, v) \in \mathfrak{X}(G \times M) \) by

\[
a_{G \times M}(\xi, v)_{(g, x)} = (T_g L_g(\xi_x), v_x),
\]

where \( L_g : G \to G \) denotes left multiplication by \( g \). The map \( a_{G \times M} : \Gamma(\mathfrak{g} \rhd \mathcal{F}) \to \mathfrak{X}(G \times M) \) defines a free action of the Lie algebroid \( \mathfrak{g} \rhd \mathcal{F} \) on \( G \times M \) in the sense of [22, Definition 3.1], so the vector fields \( a_{G \times M}(\xi, v) \) span a foliation \( \mathcal{F} \) of \( G \times M \). The left multiplication action of \( G \) on the first factor of \( G \times M \) is \( \mathcal{F} \)-foliate, and therefore lifts naturally to a \( G \)-action on the monodromy groupoid \( \text{Mon}(\mathcal{F}) \), which is free and proper. The quotient manifold \( \text{Mon}(\mathcal{F})/G \) is a Lie groupoid over \((G \times M)/G = M\), whose Lie algebroid is isomorphic to \( \mathfrak{g} \rhd \mathcal{F} \).

QED
Let $\mathfrak{a}$ be a Lie algebroid over $M$ with anchor $\tau: \mathfrak{a} \to TM$ and let $x$ be a point of $M$. We denote by $\alpha_x$ the fibre of $\mathfrak{a}$ at $x$, by $\operatorname{stab}(\alpha, x) = \ker(t_x: \alpha_x \to T_x M)$ the isotropy or stabilizer of $x$, and by $\alpha(x)$ the $\alpha$-orbit of $x$, i.e. the leaf through $x$ of the singular foliation $t(\alpha) \subseteq TM$. The stabilizer $\operatorname{stab}(\alpha, x) \subseteq \alpha_x$ inherits a Lie algebra structure from $\mathfrak{a}$. The orbit $\alpha(x)$ is an immersed submanifold of $M$ with tangent space $T_x \alpha(x) = t(\alpha_x)$ at $x$.

The stabilizer of $x$ for the transverse action Lie algebroid $\mathfrak{a} = g \ltimes \mathcal{F}$ is equal to

$$\operatorname{stab}(g \ltimes \mathcal{F}, x) = \{ \xi \in g \mid \xi_{M,x} = 0 \}.$$  

The image of the anchor map $\tau$ is a singular subbundle of $TM$, which generates a singular foliation denoted by $g \ltimes \mathcal{F}$ in [15, §2]. The orbit $g \ltimes \mathcal{F}(x)$ of $x$ under the transverse action Lie algebroid is a leaf of this singular foliation, which we also refer to as the $g$-orbit of the leaf $\mathcal{F}(x)$. The next statement shows that the stabilizers of different elements in the same $g \ltimes \mathcal{F}$-orbit are conjugate under the adjoint group of $g$.

2.2.7. Lemma. (i) Suppose $x$ and $y \in M$ are in the same leaf of $\mathcal{F}$. Then $\operatorname{stab}(g \ltimes \mathcal{F}, x) = \operatorname{stab}(g \ltimes \mathcal{F}, y)$.

(ii) Let $x \in M$ and $\xi \in g$. Choose a foliate vector field $\xi_M \in \mathfrak{X}(\mathcal{F})$ representing $\xi \in \mathfrak{X}(M, \mathcal{F})$. Let $\phi_t$ denote the flow of $\xi_M$ and suppose that $\phi_t(x)$ exists for all $t \in [0, 1]$. Let $y = \phi_1(x)$ and let $e^{\text{ad}_\xi}$ denote the exponential of $\text{ad}_\xi$ in the adjoint group $\text{Ad}(g)$. Then $\operatorname{stab}(g \ltimes \mathcal{F}) = e^{\text{ad}_\xi}(\operatorname{stab}(x \ltimes \mathcal{F}, y))$.

Proof. (i) Let $\xi \in g$. Then $\xi_M$ is a transverse vector field, so $\nabla_v \xi_M = 0$ for all $v \in \mathfrak{X}(\mathcal{F})$. Therefore $\xi_{M, \phi_t(x)} = \phi_t(\xi_{M,x})$, where $\phi_t$ denotes the flow of $v$. Now suppose $\xi \in \operatorname{stab}(g \ltimes \mathcal{F}, x)$, i.e. $\xi_{M,x} = 0$. Then $\xi_{M, \phi_t(x)} = 0$. Since this holds for all $v \in \mathfrak{X}(\mathcal{F})$, we have $\xi_{M,y} = 0$, i.e. $\xi \in \operatorname{stab}(g \ltimes \mathcal{F}, y)$.

(ii) Let $G$ be a Lie group with Lie algebra $g$. The path $\gamma(t) = (\exp_G(t\xi), \phi_t(x)) \in G \times M$ is the trajectory through $(1, x)$ of the vector field $a(\xi, \xi_M) = (\xi_L, \xi_M)$ defined in (2.2.6), where $\xi_L$ is the left-invariant vector field associated with $\xi$. It is therefore tangent to the foliation $\mathcal{G}$ defined in the proof of Proposition 2.2.5. Its homotopy class $[\gamma]$ is an element of the monodromy groupoid $\operatorname{Mon}(\mathcal{G})$ with source $\gamma(0) = (1, x)$ and target $\gamma(1) = (g, y)$, where $g = \exp_G(\xi)$. Therefore the image of $[\gamma]$ in the quotient groupoid $\operatorname{Mon}(\mathcal{G})/G$ is an arrow $f$ with source $x$ and target $y$. It follows that $\operatorname{Stab}(y,f) = f \cdot \operatorname{Stab}(x) \cdot f^{-1}$, where $\operatorname{Stab}(x)$ is the stabilizer group of $x$ in the quotient groupoid. Passing to the Lie algebras we get

$$\operatorname{stab}(y, g \ltimes \mathcal{F}) = \text{Ad}_g(\operatorname{stab}(x, g \ltimes \mathcal{F})) = e^{\text{ad}_\xi}(\operatorname{stab}(x, g \ltimes \mathcal{F})).$$  

QED

The transverse $g$-action is called free at $x$ if $\operatorname{stab}(g \ltimes \mathcal{F}, x) = 0$, and free if it is free at all $x \in M$. We say that $x \in M$ is $g \ltimes \mathcal{F}$-fixed if $\operatorname{stab}(g \ltimes \mathcal{F}, x) = g$. This means that for all $\xi \in g$ the transverse vector field $\xi_M$ vanishes at $x$, or equivalently, by Lemma 2.2.7(i), that every foliate representative of $\xi_M$ is tangent to the leaf $\mathcal{F}(x)$. (So if the $g$-action integrates to a foliate $G$-action, the leaf $\mathcal{F}(x)$, viewed as a point in the leaf space, is $G$-fixed.) We call the set

$$M^g = \{ x \in M \mid \operatorname{stab}(g \ltimes \mathcal{F}, x) = g \}$$  

the fixed-leaf set of $M$.

2.3. Basic differential forms. Let $(M, \mathcal{F})$ be a foliated manifold. A differential form $\alpha$ on $M$ is $\mathcal{F}$-basic if its Lie derivatives $L(u)\alpha$ and contractions $\iota(u)\alpha$ vanish for all vector fields $u \in \mathfrak{X}(\mathcal{F})$. The set of $\mathcal{F}$-basic forms is a differential graded subalgebra of the de Rham complex $\Omega(M)$, which we denote by $\Omega(M, \mathcal{F})$. Its cohomology is a graded commutative algebra called the $\mathcal{F}$-basic de Rham cohomology and denoted by $H(M, \mathcal{F})$. A foliate
map \( f : (M, \mathcal{F}) \to (M', \mathcal{F}') \) induces a pullback morphism of differential graded algebras \( f^* : \Omega(M', \mathcal{F}') \to \Omega(M, \mathcal{F}) \) and hence a morphism of graded algebras \( f^* : H(M', \mathcal{F}') \to H(M, \mathcal{F}) \). If the foliation is strictly simple, the basic de Rham complex is just the de Rham complex of the manifold \( M/\mathcal{F} \).

### 2.4. \( g \)-differential graded algebras

Let \( g \) be a finite-dimensional Lie algebra. A \( g \)-differential graded module (or \( g \)-dgm) is a graded vector space \( M \) equipped with a linear map \( d \) of degree 1 and, for each \( \xi \in g \), a linear map \( \iota(\xi) \) of degree \(-1\) and a linear map \( L(\xi) \) of degree 0 that depend linearly on \( \xi \) and satisfy the Cartan commutation rules

\[
\begin{align*}
[\iota(\xi), \iota(\eta)] &= 0, \\
[L(\xi), \iota(\eta)] &= L([\xi, \eta]), \\
[d, \iota(\xi)] &= 0, \\
[L(\xi), d] &= 0, \\
\iota(\xi), d &= L(\xi), \\
\{L(\xi), \iota(\eta)\} &= \iota([\xi, \eta]),
\end{align*}
\]

where the brackets denote graded commutators. This notion was introduced by Cartan \([g]\) to axiomatize the Chern-Weil theory of connections and curvature. See also \([29, \text{Appendix}], [17, [1], or [24, Appendix A]. A morphism \( \phi : M \to M' \) of \( g \)-dgm \( M \) and \( M' \) is a degree 0 map of graded vector spaces that satisfies \([\phi, d] = [\phi, \iota(\xi)] = [\phi, L(\xi)] = 0 \) for all \( \xi \in g \). A homotopy between two morphisms \( \phi_0, \phi_1 : M \to M' \) of \( g \)-dgm is a degree \(-1\) map of graded vector spaces \( \kappa : M \to M'[-1] \) that satisfies \([\kappa, d] = \phi_1 - \phi_0 \) and \([\kappa, \iota(\xi)] = [\kappa, L(\xi)] = 0 \) for all \( \xi \in g \).

An element \( m \) of a \( g \)-dgm \( M \) is \( g \)-invariant if \( L(\xi)m = 0 \) for all \( \xi \in g \), \( g \)-horizontal if \( \iota(\xi)m = 0 \) for all \( \xi \in g \), and \( g \)-basic if it is \( g \)-invariant and \( g \)-horizontal. The subspace of \( g \)-basic elements

\[
M_{g-bas} = \{ m \in M \mid L(\xi)m = \iota(\xi)m = 0 \ \text{for all} \ \xi \in g \}
\]

is a subcomplex of \( M \). The basic cohomology of \( M \) is defined as the cohomology of the complex \( M_{g-bas} \) and is denoted by \( H_{g-bas}(M) \).

A \( g \)-differential graded algebra (or \( g \)-dga) is a graded algebra \( A \) that is also a \( g \)-dgm and for which the structure maps \( d, \iota(\xi), L(\xi) \) are graded derivations. We say that a \( g \)-dga \( A \) is locally free if it admits a connection, i.e. a linear map \( \theta : g^* \to A^1 \) satisfying

\[
\iota(\xi)(\theta(x)) = (\xi, x) \in A^0 \quad \text{and} \quad L(\xi)(\theta(x)) = -\theta(\text{ad}^*(\xi)x)
\]

for all \( \xi \in g \) and \( x \in g^* \).

The Weil algebra \( W_g \) is a locally free commutative \( g \)-differential graded algebra that is universal in the following sense: it is equipped with a connection \( \theta_{uni} \), and for every locally free commutative \( g \)-dga \( A \) and every connection \( \theta \) on \( A \) there is a unique morphism of \( g \)-dga \( c_\theta : W_g \to A \) such that the diagram

\[
\begin{array}{ccc}
g^* & \xrightarrow{\theta} & A \\
\downarrow{\theta_{uni}} & & \downarrow{c_\theta} \\
W_g & & \\
\end{array}
\]

commutes. The basic complex \( (W_g)_{g-bas} \) is isomorphic to \((Sg^*)^g\), the algebra of \( g \)-invariant polynomials on \( g \), equipped with the zero differential and with the generating subspace \( g^* \) placed in degree 2. The morphism \( c_\theta \) induces a morphism of graded algebras

\[
c_\theta : (Sg^*)^g \to A_{g-bas}
\]

known as the characteristic or Cartan-Chern-Weil homomorphism of the connection \( \theta \).

The Weil complex \( M_g \) of a \( g \)-differential graded module \( M \) is the basic complex of the \( g \)-dgm \( W_g \otimes M \),

\[
M_g = (W_g \otimes M)_{g-bas}.
\]
The equivariant cohomology of $M$ is defined as the cohomology of the complex $M_\alpha$ and is denoted by $H^\alpha(M)$. The equivariant cohomology $H^\alpha(M)$ is a module over $(W^0)_{q\text{-bas}} = (\delta q)^0$. A morphism $\phi: M \to M'$ of $g\text{-dgm}$ induces a morphism $H(\phi): H^\alpha(M) \to H^\alpha(M')$, and if two morphisms $\phi_0$ and $\phi_1$ are homotopic, then $H(\phi_0) = H(\phi_1)$.

For our purposes the main point of $g\text{-differential graded algebras is the next result, which contains as a special case the well-known fact that the $G\text{-equivariant de Rham complex of a principal bundle with structure group } G$ is homotopy equivalent to the de Rham complex of the base manifold. (The latter fact is true in general only if $G$ is compact, but the following statement holds for arbitrary $g$.) See [24, Theorem A.5.1] for a proof.

### 2.4.3. Theorem

Let $A$ be a locally free commutative $g\text{-differential graded algebra. The morphism } j: A \to W^0 \otimes A \text{ defined by } j(a) = 1 \otimes a \text{ is a homotopy equivalence with homotopy inverse } W^0 \otimes A \xrightarrow{\sim} A \text{ defined by } x \otimes a \mapsto c_0(x)a, \text{ where } \theta \text{ is a connection on } A \text{ and } c_0 \text{ is as in (2.4.1). Hence } j \text{ induces a homotopy equivalence } A_{g\text{-bas}} \xrightarrow{\sim} (W^0 \otimes A)_{g\text{-bas}} = A_g \text{ and an isomorphism } H^\alpha_{g\text{-bas}}(A) \xrightarrow{\sim} H^\alpha_g(A).

We will require a relative version of this theorem. Suppose that $g = \mathfrak{t} \times \mathfrak{h}$ is the product of two Lie subalgebras $\mathfrak{t}$ and $\mathfrak{h}$, and let $A$ be a $g\text{-dga}$. Suppose that $A$ is locally free as a $\mathfrak{t}\text{-dga}$ and let $\theta: \mathfrak{t}^* \to \mathfrak{a}^1$ be a $\mathfrak{t}$-connection. We say that the connection $\theta$ is $\mathfrak{h}$-invariant if $\theta(\eta) \in \mathfrak{a}^1$ is $\mathfrak{h}$-invariant for all $\eta \in \mathfrak{t}^*$. The analogue of the map (2.4.1) in this situation is a homomorphism
\begin{equation}
(2.4.4) \quad c_{\mathfrak{h},\theta}: \mathfrak{w}^\mathfrak{t} \longrightarrow \mathfrak{w}^\mathfrak{h} \otimes A,
\end{equation}
which on basic elements gives the $\mathfrak{h}$-equivariant Cartan-Chern-Weil homomorphism
\begin{equation}
(2.4.5) \quad c_{\mathfrak{h},\theta}: (S^*\mathfrak{t}^\mathfrak{t})^\mathfrak{t} \longrightarrow (A_{\mathfrak{t}\text{-bas}})_\mathfrak{h}.
\end{equation}
See [24, Theorem A.6.3] for the next result.

### 2.4.6. Theorem

Let $\mathfrak{g} = \mathfrak{t} \times \mathfrak{h}$ be the product of two Lie algebras $\mathfrak{t}$ and $\mathfrak{h}$. Let $A$ be a commutative $g\text{-differential graded algebra. Suppose that } A \text{ is } \mathfrak{t}\text{-locally free and admits an } \mathfrak{h}\text{-invariant } \mathfrak{t}\text{-connection } \theta: \mathfrak{t}^* \to \mathfrak{a}^1. \text{ The morphism } j: \mathfrak{w}^\mathfrak{h} \otimes A \to \mathfrak{w}^\mathfrak{h} \otimes A \text{ given by the inclusion } \mathfrak{w}^\mathfrak{h} \to \mathfrak{w}^\mathfrak{h} \text{ is a homotopy equivalence with homotopy inverse }
\begin{equation}
(2.4.6) \quad \mathfrak{w}^\mathfrak{h} \otimes A \cong \mathfrak{w}^\mathfrak{t} \otimes \mathfrak{w}^\mathfrak{h} \otimes A \xrightarrow{\sim} \mathfrak{w}^\mathfrak{h} \otimes A
\end{equation}
defined by $x \otimes y \otimes a \mapsto c_0(x)y \otimes a, \text{ where } c_0 \text{ is as in (2.4.4). Hence } j \text{ induces a homotopy equivalence}
\begin{equation}
(2.4.7) \quad (\mathfrak{w}^\mathfrak{h} \otimes A)_{g\text{-bas}} = (A_{g\text{-bas}})_{\mathfrak{h}} \xrightarrow{\sim} (\mathfrak{w}^\mathfrak{h} \otimes A)_{g\text{-bas}} = A_g
\end{equation}
and an isomorphism $H^\alpha_{g\text{-bas}}(A_{g\text{-bas}}) \xrightarrow{\sim} H^\alpha_g(A)$.

### 2.5. Equivariant basic differential forms.

Let $(M, \mathcal{F})$ be a foliated manifold, $\mathfrak{g}$ a finite-dimensional Lie algebra, and $\alpha: \mathfrak{g} \to \chi(M, \mathcal{F})$ a transverse $\mathfrak{g}$-action on $M$. For $\xi \in \mathfrak{g}$ let $\xi_M = \alpha(\xi) \in \chi(M, \mathcal{F})$ denote the transverse vector field on $M$ defined by the $\mathfrak{g}$-action. For $\alpha \in \Omega(M, \mathcal{F})$ define
\begin{equation}
\iota(\xi) \alpha = \iota(\tilde{\xi}_M) \alpha, \quad L(\xi) \alpha = L(\tilde{\xi}_M) \alpha,
\end{equation}
where $\tilde{\xi}_M$ is a foliate vector field that represents $\xi_M$. Since $\alpha$ is $\mathcal{F}$-basic, these contractions and derivatives are independent of the choice of the representative $\tilde{\xi}_M$ of $\xi_M$. Goertsches and Töben [15, Proposition 3.1.2] observed that, equipped with these operations, the basic de Rham complex $\Omega(M, \mathcal{F})$ a $\mathfrak{g}$-differential graded algebra, and they called elements of the Weil complex
\begin{equation}
\Omega_g(M, \mathcal{F}) = (\mathfrak{w}^\mathfrak{g} \otimes \Omega(M, \mathcal{F}))_{g\text{-bas}}
\end{equation}
$\mathfrak{g}$-equivariant $\mathcal{F}$-basic differential forms. The cohomology

$$H_\mathfrak{g}(M, \mathcal{F}) = H(\Omega_\mathfrak{g}(M, \mathcal{F}))$$

is the $\mathfrak{g}$-equivariant $\mathcal{F}$-basic de Rham cohomology of the foliated manifold with respect to the transverse action. We will usually say “equivariant basic” instead of “$\mathfrak{g}$-equivariant $\mathcal{F}$-basic”.

Let $(M', \mathcal{F}')$ be another foliated manifold equipped with a transverse $\mathfrak{g}$-action. We say that a foliate map $f : M \to M'$ is $\mathfrak{g}$-equivariant if the transverse vector fields $\xi_M$ and $\xi_M'$ are $f$-related as in (2.1.2). A $\mathfrak{g}$-equivariant foliate map $f$ induces a morphism of $\mathfrak{g}$-differential graded algebras $\Omega(M', \mathcal{F}') \to \Omega(M, \mathcal{F})$. We call a smooth map $f : [0,1] \times M \to M'$ a $\mathfrak{g}$-equivariant foliate homotopy if the map $f_t : M \to M'$ defined by $f_t(x) = f(t, x)$ is $\mathfrak{g}$-equivariant foliate for all $t \in [0,1]$. We quote the following equivariant basic homotopy lemma from [24, Lemma 4.2.1].

2.5.1. Lemma. Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be foliated manifolds equipped with transverse actions of a Lie algebra $\mathfrak{g}$. Let $f : [0,1] \times M \to M'$ be a $\mathfrak{g}$-equivariant foliate homotopy. Then the pullback morphisms $f_0$ and $f_1 : \Omega(M', \mathcal{F}') \to \Omega(M, \mathcal{F})$ are homotopic as morphisms of $\mathfrak{g}$-differential graded algebras. In particular they induce the same homomorphisms in equivariant basic cohomology: $f_0^* = f_1^* : H_\mathfrak{g}(M', \mathcal{F}') \to H_\mathfrak{g}(M, \mathcal{F})$.

3. Molino theory and some applications

This section is a précis of Molino’s structure theory of Riemannian foliations, partly based on [15, § 4]. A full account of these results can be found in [27] and also in [26, Ch. 4]. We restate them slightly in terms of transverse action Lie algebroids, and to Molino’s two structure theorems we add a third structure theorem, which concerns a dual pair of bundles of Lie algebras and a generalized morphism of groupoids associated to a Riemannian foliation. At the end we state several corollaries that will be used in later sections.

A transverse Riemannian metric on a foliated manifold $(M, \mathcal{F})$ is a positive definite symmetric bilinear form $g$ on the normal bundle $N\mathcal{F}$ with the property that $L(v)g = 0$ for all vector fields $v \in \mathfrak{X}(\mathcal{F})$ tangent to the leaves. The pair $(\mathcal{F}, g)$ is then called a Riemannian foliation. A bundle-like Riemannian metric on $M$ is a Riemannian metric $g_{TM}$ on $M$ with the property that the function $g(v, w)$ is basic for all foliate vector fields $v$ and $w \in \mathfrak{X}(\mathcal{F})$ that are perpendicular to the leaves of $\mathcal{F}$. A bundle-like metric $g$ gives rise to a transverse metric $\bar{g}$ by identifying $N\mathcal{F}$ with the $g_{TM}$-orthogonal complement of $\mathcal{F}$ and then restricting $g_{TM}$ to $N\mathcal{F}$. Conversely, for every transverse metric $g$ there is a bundle-like metric $g_{TM}$ which induces $g$; see [27, § 3.2]. Following [14, Definition 3.1] or [15, § 2.1] we call a Riemannian foliation $(\mathcal{F}, g)$ on $M$ (metrically) complete if there exists a bundle-like metric $g_{TM}$ which induces $g$ and which is complete. A foliate map $f$ from $M$ to a second manifold $M'$ equipped with a Riemannian foliation $(\mathcal{F}', g')$ is a transverse Riemannian submersion if it is a submersion and if the normal derivative $N_x f : N_x \mathcal{F} \to N_{f(x)} \mathcal{F}'$ preserves scalar products of vectors perpendicular to $\ker(N_x f)$.

3.1. Notation and conventions. In the rest of this section $M$ denotes a connected manifold equipped with a metrically complete Riemannian foliation $(\mathcal{F}, g)$. We denote the (constant) codimension $\dim(M) - \dim(\mathcal{F}(x))$ of the foliation by $q$, the orthogonal group $O(q)$ by $K$, and its Lie algebra by $\mathfrak{f}$. We define $g_K$ to be the bi-invariant Riemannian metric on $K$ with respect to which $K$ has volume 1 and we let $g_t$ be the associated inner product on $\mathfrak{f}$. We denote by $V_M$ the trivial vector bundle over $M$ with fibre a vector space $V$. See § 2.1 and the notation index in the back for general notational conventions.
3.2. The Molino diagram. Define a relation among points of $M$ as follows: $x \sim y$ if the closures of the leaves $\mathcal{F}(x)$ and $\mathcal{F}(y)$ intersect. The gist of Molino’s theory is that, unlike for general foliations, this is an equivalence relation, and that $x \sim y$ if and only if the closure of the leaf $\mathcal{F}(x)$ is *equal* to the closure of the leaf $\mathcal{F}(y)$. The leaf closures are the leaves of a singular foliation $\mathcal{F}$, we can form the *leaf closure space* $M/\mathcal{F}$, which is a Hausdorff topological space, and we have a continuous map $M/\mathcal{F} \to M/\mathcal{F}$. The kicker is that this leaf closure space is the quotient space of a Riemannian manifold $W$, which we call the Molino manifold and whose definition we review below, by an isometric action of the orthogonal group $K = \text{O}(q)$. This enables us to reduce certain questions about the foliation $\mathcal{F}$ to usually much easier questions about the $K$-action on $W$. The situation can be summarized by the Molino diagram,

$$
\begin{array}{ccc}
(P, \mathcal{F}, g_P) & \xrightarrow{\pi} & (M, \mathcal{F}, g) \\
\downarrow & & \downarrow \downarrow \\
(W, g_W) & \xrightarrow{\theta} & M/\mathcal{F} \\
\end{array}
$$

The manifold $P$, which we call the *Molino bibundle*, is the bundle of orthonormal frames of the normal bundle $N\mathcal{F}$, and is a foliated principal $K$-bundle over $M$ with structure group $K = \text{O}(q)$. (In case $\mathcal{F}$ is transversely orientable, we choose a transverse orientation, we take $P$ to consist of *oriented* orthonormal frames, and we replace $K$ by $\text{SO}(q)$.) We describe the foliation $\mathcal{F}_P$ by specifying a partial connection on $P$ as follows. A *Killing vector field* on $(M, \mathcal{F})$ is a vector field $\nu \in \mathfrak{x}(M)$ that satisfies $L(\nu)g = 0$ (not $L(\nu)g_{TM} = 0$). By [27, Lemma 3.5] all Killing vector fields are foliate and therefore they form a Lie subalgebra of $\mathfrak{h}(\mathcal{F})$, which we denote by $\mathfrak{h}(\mathcal{F}, g)$. Vector fields in $\mathfrak{x}(\mathcal{F})$ are by definition Killing, so $\mathfrak{x}(\mathcal{F})$ is an ideal of $\mathfrak{h}(\mathcal{F}, g)$. We denote the quotient Lie algebra, which is a subalgebra of $\mathfrak{x}(M, \mathcal{F})$, by $\mathfrak{x}(M, \mathcal{F}, g)$ and call its elements *transverse Killing vector fields*. The short exact sequence (2.1.1) restricts to a short exact sequence

$$
\mathfrak{x}(\mathcal{F}) \hookrightarrow \mathfrak{h}(\mathcal{F}, g) \twoheadrightarrow \mathfrak{x}(M, \mathcal{F}, g).
$$

The flow $\phi_t$ of a Killing vector field $\nu$ is a 1-parameter group of foliate diffeomorphisms, so the normal derivative $N\phi_t$ is a 1-parameter group of vector bundle automorphisms of $N\mathcal{F}$. The flow $N\phi_t$ preserves the metric $g$, so it maps orthonormal frames to orthonormal frames and therefore lifts naturally to a $K$-equivariant flow $\phi_{P,t}$ of bundle automorphisms of $P$. The infinitesimal generator $\pi^\nu(\nu)$ of $\phi_{P,t}$ is a $K$-invariant vector field, and the map $\nu \mapsto \pi^\nu(\nu)$ is a homomorphism of Lie algebras

$$
\pi^\uparrow: \mathfrak{h}(\mathcal{F}, g) \rightarrow \mathfrak{x}(P)^K,
$$

which we call the *natural lifting homomorphism*. The restriction of $\pi^\uparrow$ to $\mathfrak{x}(\mathcal{F})$ gives a vector bundle map $\pi^\uparrow T\mathcal{F} \rightarrow TP$. This is the partial connection on $P$. Its image is the tangent bundle to the foliation $\mathcal{F}_P$ that makes $P$ a foliated principal $K$-bundle.

The partial connection extends to a unique torsion-free connection, the *transverse Levi-Civita connection*, which is $\mathcal{F}_P$-basic; see [27, §3.3]. We denote this connection, and its associated connection 1-form in $\Omega^1(P, \mathcal{F}_P; T)^K$, by $\theta_{\text{LC}}$. The $\theta_{\text{LC}}$-horizontal lift of a vector field $\nu \in \mathfrak{x}(M)$ agrees with the canonical lift $\pi^\uparrow(\nu)$ if $\nu$ is tangent to $\mathcal{F}$, but may differ from $\pi^\uparrow(\nu)$ if $\nu$ is an arbitrary Killing vector field. Since $\theta_{\text{LC}}$ maps $\pi^\uparrow T\mathcal{F}$ to $T\mathcal{F}_P$, it induces a
vector bundle map \( \bar{\theta}_{LC} : \pi^* \mathcal{N} \to \mathcal{N} \mathcal{F}_P \), which is a splitting (right inverse) of the normal derivative \( N\pi : \mathcal{N} \mathcal{F}_P \to \pi^* \mathcal{N} \mathcal{F} \) of \( \pi \).

The solder form or fundamental form of \( P \) is the \( \mathbb{R}^q \)-valued 1-form \( \sigma \in \Omega^1(P, \mathcal{F}_P; \mathbb{R}^q)^K \) defined as follows: let \( p \in P, v \in T_pP \), and \( x = \pi(p) \), view the orthonormal frame \( p \) as an isometry \( p : \mathbb{R}^q \to \mathcal{N} \mathcal{F}_P \), and put
\[
\sigma_p(v) = p^{-1}(T_p\pi(v) \mod T_x\mathcal{F}) \in \mathbb{R}^q.
\]
The solder form is equivariant with respect to the \( K \)-actions on \( P \) and \( \mathbb{R}^q \). For every \( p \) the linear map \( \sigma_p : T_pP \to \mathbb{R}^q \) is surjective and its kernel is equal to the sum \( T_p(K.p) \oplus T_p\mathcal{F}_P \). Therefore \( \sigma \) descends to a surjective vector bundle map \( \bar{\sigma} : \mathcal{N} \mathcal{F}_P \to \mathbb{R}^q \) with kernel \( \ker \pi \).

The connection form \( \theta_{LC} \) and the solder form \( \sigma \) give us the soldering diagram of the foliated bundle \( P \),
\[
\begin{array}{c}
\pi^*T\mathcal{F} & \xleftarrow{\theta_{LC}} & \pi^*TM & \xrightarrow{\bar{\sigma}} & \mathbb{R}^q_P \\
N\mathcal{F}_P & \xleftarrow{\theta_{LC}} & \mathcal{N} \mathcal{F}_P & \xrightarrow{\pi^*N\mathcal{F}_P} & \mathbb{R}^q_P \\
T\mathcal{F}_P & \xleftarrow{\theta_{LC}} & TP & \xrightarrow{\pi^*N\mathcal{F}_P} & \mathbb{R}^q_P \\
\end{array}
\]
The diagram commutes, its rows and columns are exact, and \( \theta_{LC} \) and \( \bar{\theta}_{LC} \) are splittings, so we get isomorphisms \( \bar{\sigma} \circ \bar{\theta}_{LC} : \pi^*N\mathcal{F}_P \cong \mathbb{R}^q_P \) and
\[
(3.2.4) \quad N\mathcal{F}_P \cong \mathfrak{t} \oplus \mathbb{R}^q_P.
\]

Thus a choice of bases of \( \mathfrak{t} \) and \( \mathbb{R}^q \) gives rise to a transverse parallelism of \( P \), i.e. a global frame of the normal bundle \( N\mathcal{F}_P \) given by sections that are constant with respect to the trivialization \((3.2.4)\). Constant sections of \( N\mathcal{F}_P \) are transverse vector fields on \( P \).

The following theorem, which is an excerpt of Molino’s results, is a list of the main features of the Molino diagram \((3.2.1)\). The notation is as in § 3.1 and as in the diagrams \((3.2.1)\) and \((3.2.3)\).

3.2.5. Theorem (first structure theorem). Let \( M \) be a connected manifold equipped with a complete Riemannian foliation \((\mathcal{F}, g)\).

(i) There is a unique transverse Riemannian metric \( g_P \) on the Molino birelief \((P, \mathcal{F}_P)\) with respect to which the trivialization \((3.2.4)\) is an isometry. This transverse metric \( g_P \) is \( K \)-invariant and complete, and the projection \( \pi : P \to M \) is a transverse Riemannian submersion. The natural lift of a Killing vector field on \((M, \mathcal{F}, g)\) is a \( K \)-invariant Killing vector field on \((P, \mathcal{F}_P, g_P)\), so the natural lifting homomorphism \( \pi^* : \mathfrak{X}(M, \mathcal{F}, g) \to \mathfrak{X}(P, \mathcal{F}_P, g_P)^K \).

(ii) The foliation \( \mathcal{F}_P \) is transversely parallelizable and homogeneous. Hence the leaf closure foliation \( \mathcal{F}_P \) is strictly simple and the space of leaf closures is a manifold \( P/\mathcal{F}_P = W \). The \( K \)-action on \((P, \mathcal{F}_P)\) is foliate and therefore descends uniquely to a smooth \( K \)-action on \( W \). The quotient map \( \bar{\varphi} : P \to W \) is a \( K \)-equivariant locally trivial fibre bundle. There exists a unique Riemannian metric \( g_W \) on \( W \) with respect to which \( \bar{\varphi} \) is a transverse Riemannian submersion. The metric \( g_W \) is complete. Every transverse vector field \( v \) on \( P \) is \( \bar{\varphi} \)-related to a unique vector field \( \varphi_1(v) \) on \( W \). The map
\[
\varphi_1 : \mathfrak{X}(P, \mathcal{F}_P) \to \mathfrak{X}(W)
\]
is a Lie algebra homomorphism. If \( v \in \mathfrak{x}(P, \mathcal{F}_P) \) is Killing, then so is \( \varphi(v) \).

(iii) Let \( L \) be a leaf of \( \mathcal{F} \) and let \( L_P \) be a leaf of \( \mathcal{F}_P \) mapping to \( L \). Then \( \bar{L} = \pi(\bar{L}_P) \) and \( \bar{L} \) is an embedded submanifold of \( M \). The \( K \)-equivariant map \( \varphi : P \to W \) induces a continuous map \( M = P/K \to W/K \), which descends to a homeomorphism \( \overline{\mathcal{F}^*} = \mathcal{F}^* / \mathcal{F}_P \to W/K \).

Molino also showed that each leaf closure \( \overline{\mathcal{F}}(x) \) of \( M \) is the orbit of the leaf \( \mathcal{F}(x) \) under the action of a certain flat bundle of Lie algebras over \( M \). He did not point out, but it follows immediately from his results, that \( \overline{\mathcal{F}}(x) \) is actually the orbit of \( x \) under the action of a certain Lie algebroid. We reformulate Molino’s second structural theorem in terms of this Lie algebroid as follows. The \textit{transverse tangent sheaf} \( \mathcal{F}_M = \mathcal{F} \) of \( (M, \mathcal{F}) \) is the sheaf of Lie algebras on \( M \) associated to the presheaf \( U \mapsto \mathfrak{x}(U, \mathcal{F}|_U) \), where \( U \) ranges over all open subsets of \( M \). In other words, \( \mathcal{F} \) is the quotient of the sheaves \( U \mapsto \mathfrak{K}(\mathcal{F}|_U) \) and \( U \mapsto \mathfrak{x}(\mathcal{F}|_U) \). The \textit{transverse Killing sheaf} \( \mathcal{K}_M = \mathcal{K} \) is the subsheaf of \( \mathcal{F} \) associated to the presheaf \( U \mapsto \mathfrak{x}(U, \mathcal{F}|_U, g) \). We have a similar sheaves \( \mathcal{F}_p \) and \( \mathcal{K}_p \) on \( P \). The Lie algebra of global transverse vector fields \( \mathfrak{x}(P, \mathcal{F}_P) \) is a subspace of the space of global sections of \( \mathcal{F}_P \). The sheaf \( \mathcal{F}_P \) is constant, because \( P \) has a transverse parallelism. Molino’s \textit{centralizer sheaf} \( \mathcal{C}_P \) of \( P \) is the subsheaf \( \mathcal{C}_P \) of \( \mathcal{F}_P \) consisting of all sections that centralize the global transverse vector fields:

\[
\mathcal{C}_P(V) = \{ v \in \mathcal{F}_P(V) \mid \forall w \in \mathfrak{x}(P, \mathcal{F}_P) \text{ with } [v, w] = 0 \}.
\]

The natural lifting homomorphism provides a homomorphism of sheaves of Lie algebras \( \pi^\dagger : \mathcal{K} \to \pi_* \mathcal{F}_P \). The \textit{centralizer sheaf} \( \mathcal{C} = \mathcal{C}_M \) of \( M \) is the inverse image of \( \mathcal{C}_P \) under this homomorphism. In other words, \( \mathcal{C} \) is the subsheaf of \( \mathcal{K} \) consisting of all sections \( u \in \mathfrak{K}(U) \) with the property that \( \pi^\dagger(u) \in \mathfrak{C}_P(\pi^\dagger(U)) \).

The next result, which again is excerpted from Molino’s book [27], explains that the closures of the leaves of \( \mathcal{F} \) are the orbits of the sheaf of Lie algebras \( \mathcal{C} \), and therefore constitute a singular foliation \( \overline{\mathcal{F}} \) of \( M \). The notation is as in Theorem 3.2.5. See also the table at the end of the notation index.

3.2.6. \textbf{Theorem} (second structure theorem). \textit{Let} \( M \) \textit{be a connected manifold equipped with a complete Riemannian foliation} \( (\mathcal{F}, g) \).

(i) The sheaves \( \mathcal{C}_P \) and \( \mathcal{C} \) are locally constant. The natural lifting homomorphism \( \pi^\dagger \) induces an isomorphism \( \mathcal{C} \cong (\pi_* \mathcal{C}_P)^K \). Let \( p \in P \) and let \( x = \pi(p) \in M \). Let \( \Lambda \) be the closure of the leaf \( \mathcal{F}_P(p) \), let \( \mathcal{F}_\Lambda \) be the restriction of the foliation \( \mathcal{F}_P \) to \( \Lambda \), and let \( \mathcal{F}_\Lambda \) be the transverse tangent sheaf of \( (\Lambda, \mathcal{F}_\Lambda) \). The stalk \( \mathcal{C}_{P,p} \) is a Lie algebra equal to the centralizer of \( \mathfrak{x}(\Lambda, \mathcal{F}_\Lambda) \) in \( \mathcal{F}_\Lambda \). The dimension of \( \mathcal{C}_{P,p} \) is

\[
\dim(\mathcal{C}_{P,p}) = \dim(\overline{\mathcal{F}_P}(p)) - \dim(\mathcal{F}_P(p)).
\]

(ii) The sheaf \( \mathcal{C}_P \) is the sheaf of flat sections of a \( K \)-equivariant flat bundle of Lie algebras \( c_P \), whose fibre at \( p \) is equal to the stalk \( \mathcal{C}_{P,p} \). The evaluation maps \( \mathcal{C}_{P,p} \to N_p \mathcal{F}_P \) give rise to a bundle map \( \mathcal{C}_P \to N \mathcal{F}_P \). Similarly, the sheaf \( \mathcal{C}_M = \mathcal{C} \) is the sheaf of flat sections of a flat bundle of Lie algebras \( c_M = c \), whose fibre at \( x \) is equal to the stalk \( c_x \); and \( c \) is equipped with a bundle map \( c \to N \mathcal{F} \).

We have isomorphisms \( \pi^\dagger c \cong c_P \) and \( c = c_P / K \).

(iii) The fibred product

\[
\mathcal{F}_P \ltimes \mathcal{F}_P = \mathcal{F}_P \times_{N \mathcal{F}_P} TP
\]

is a Lie algebroid over \( P \), which acts freely on \( P \) and whose orbits are the closures of the leaves of \( \mathcal{F}_P \). The decomposition into leaf closures \( \overline{\mathcal{F}}_P \) is a foliation of \( P \)
of (constant) codimension \( q + \frac{1}{2}q(q - 1) - \dim(\mathfrak{g}_{P,p}) \). Similarly,

\[ \zeta \rtimes \mathcal{F} = \zeta \times_{N_{\mathcal{F}}} TM \]

is a Lie algebroid over \( M \), whose orbits are the closures of the leaves of \( \mathcal{F} \). The decomposition into leaf closures \( \mathcal{F} \) is a singular foliation of \( M \). The codimension of \( \mathcal{F}(x) \) is \( q + \dim(\text{Stab}(K,w)) - \dim(\mathfrak{g}_{P,p}) \), where \( x = \pi(p) \) and \( \text{Stab}(K,w) \) denotes the stabilizer of \( w = \mathfrak{g}(p) \in W \) under the \( K \)-action.

We call \( \mathfrak{g} \) and \( \zeta \) the centralizer bundles of \( P \), resp. \( M \). We call \( \zeta \rtimes \mathcal{F} \) and \( \zeta \rtimes \mathcal{F} \) the centralizer Lie algebroids of \( P \), resp. \( M \).

A transverse parallelism of \( P \) is the same thing as a basis of the space of transverse vector fields \( \mathcal{X}(P, \mathcal{F}_P) \) considered as a module over the ring of basic functions \( \Omega^0(P, \mathcal{F}_P) \approx C^\infty(W) \). In particular, this module is free over \( C^\infty(W) \) (of rank \( q + \frac{1}{2}q(q - 1) \)) and is therefore the module of sections of a (trivial) vector bundle \( \mathfrak{b} \) over \( W \). The Lie bracket on \( \mathcal{X}(P/\mathcal{F}_P) \) and the surjective homomorphism \( Q_W : \mathcal{X}(P/\mathcal{F}_P) \to \mathcal{X}(W) \) make \( \mathfrak{b} \) a transitive Lie algebroid over \( W \), called the basic Lie algebroid in [26, § 6.4]. The anchor of \( \mathfrak{b} \) being surjective, the stabilizers \( \mathfrak{s}_{\mathcal{F},w} = \text{Stab}(b, w) \) for \( w \in W \) form a locally trivial Lie algebroid \( \mathfrak{s}_{\mathcal{F},w} \) (which is denoted by \( \mathfrak{g}_{W} \) on [27, p. 119]).

The first three items of the next result are a reformulation of further results of Molino. The last item, which is not in Molino, but which we state without proof as we won't use it here, justifies our usage of the term “bibundle” for the transverse frame bundle \( P \), and it places Molino’s structure theory in the context of Lie groupoids. See [23, § 3.2] for an explanation of bibundles.

3.2.7. **Theorem** (third structure theorem). Let \( M \) be a connected manifold equipped with a complete Riemannian foliation \( (\mathcal{F}, \mathfrak{g}) \).

(i) Let \( w \in W \). Let \( \Lambda_w = \mathfrak{g}^{-1}(w) \) and let \( \mathcal{F}_w \) be the restriction of the foliation \( \mathcal{F}_P \) to \( \Lambda_w \). Then \( \mathfrak{s}_{\mathcal{F},w} = \mathfrak{g}^{-1}(w) \) and \( \mathcal{F}_w \) over \( P \) form a dual pair in the sense that for each \( p \in P \) the fibres \( \mathfrak{s}_{\mathcal{F},w} \) and \( \mathcal{F}_w \) are each other’s centralizer in the Lie algebra \( \mathcal{X}(\mathcal{F}_P, \mathcal{F}_P) \).

(ii) The Lie algebra bundle \( \mathfrak{s}_{\mathcal{F},w} \) integrates to a locally trivial Lie group bundle \( \mathcal{S}_{\mathcal{F},w} \) over \( \mathcal{F}_w \) with simply connected fibres. For \( w \in W \) let \( \mathcal{P}_w \) be the Darboux cover of the leaf closure \( \mathfrak{g}^{-1}(w) \). The union \( \mathcal{P}_w = \bigcup_{w \in W} \mathcal{P}_w \) is a fibre bundle over \( W \) equipped with a bundle map \( \mathcal{P} \to \mathcal{S}_{\mathcal{F},w} \). For each \( w \in W \) the fibre \( \mathfrak{s}_{\mathcal{F},w} \) is the Lie algebra of left-invariant vector fields on \( S_w \) and for every \( p \in \mathfrak{g}^{-1}(w) \) the fibre \( \mathfrak{c}_{\mathcal{F},w} \) of \( \mathfrak{c}_w \) is the Lie algebra of right-invariant vector fields on \( S_w \).

(iii) The manifold \( P \) is a bibundle for the holonomy groupoid \( \text{Hol}(M, \mathcal{F}) \) and the action groupoid \( K \times W \). Therefore \( P \) defines a generalized groupoid morphism from \( \text{Hol}(M, \mathcal{F}) \) to \( K \times W \), as well as a morphism from the leaf stack \( [M/\mathcal{F}] \) to the quotient stack \( [W/K] \).

3.3. **Some corollaries of Molino theory.** The Molino structure theorems, Theorems 3.2.5–3.2.7, have useful things to say about transverse Lie algebra actions. A transverse action \( \alpha : \mathfrak{g} \to \mathcal{X}(M, \mathcal{F}) \) of a finite-dimensional Lie algebra \( \mathfrak{g} \) on a Riemannian foliated manifold \((M, \mathcal{F}, \mathfrak{g})\) is isometric if the homomorphism \( \alpha \) takes values in the transverse Killing vector fields \( \mathcal{X}(M, \mathcal{F}, \mathfrak{g}) \). For the remainder of § 3.3 we fix an isometric transverse \( \mathfrak{g} \)-action \( \alpha \) on \( M \). We denote by \( \mathfrak{X}(W, \mathfrak{g}_W) \) the Lie algebra of Killing vector fields of the Molino manifold \((W, \mathfrak{g}_W)\).
Our first application of Molino theory is that isometric transverse Lie algebra actions correspond to isometric Lie group actions on the Molino manifold.

3.3.1. Proposition. Let $M$ be a connected manifold equipped with a complete Riemannian foliation $(\mathcal{F}, g)$ and let $\alpha: \mathfrak{g} \to \mathfrak{X}(M, \mathcal{F}, g)$ be an isometric transverse action. The homomorphism $\alpha_P = \pi^1 \circ \alpha: \mathfrak{g} \to \mathfrak{X}(P, \mathcal{F}_P, g_P)$ is an isometric transverse $\mathfrak{g}$-action on $P$. The homomorphism $\alpha_W = \alpha_1 \circ \alpha_P: \mathfrak{g} \to \mathfrak{X}(W, g_W)$ is an isometric $\mathfrak{g}$-action on $W$. The bibundle projections $\pi: P \to M$ and $\eta: P \to W$ are equivariant with respect to these actions. Let $G$ be a simply connected Lie group with Lie algebra $\mathfrak{g}$. The action $\alpha_W$ integrates to an isometric $G$-action, which commutes with the $K$-action.

Proof. The first two assertions follow immediately from Theorem 3.2.5(i)–(ii). Also by Theorem 3.2.5, our hypothesis that $(M, \mathcal{F}, g)$ is complete implies that the Riemannian manifold $(W, g_W)$ is complete, so the Killing vector field $\xi_W$ is complete for each $\xi \in \mathfrak{g}$, and therefore the Lie algebra action $\alpha_W$ integrates to a $G$-action by the Lie-Palais theorem. The projection $\pi: P \to M$ is $K$-invariant, so the transverse vector fields $\xi_P$ commute with the vector fields $\eta_P$ for all $\xi \in \mathfrak{g}$ and $\eta \in 1$, and so the $G$-action on $W$ commutes with the $K$-action.

QED

It follows from Lemma 2.2.7(i) that if $y$ is in the leaf closure of a point $x$, then we have an inclusion of stabilizers $\text{stab}(\mathfrak{g} \ltimes \mathcal{F}, x) \subseteq \text{stab}(\mathfrak{g} \ltimes \mathcal{F}, y)$. The next observation shows that in the situation of Riemannian foliations this inclusion is an equality.

3.3.2. Proposition. Let $M$ be a connected manifold equipped with a complete Riemannian foliation $(\mathcal{F}, g)$ and let $\alpha: \mathfrak{g} \to \mathfrak{X}(M, \mathcal{F}, g)$ be an isometric transverse action.

(i) The transverse $\mathfrak{g}$-action $\alpha$ commutes with the transverse action of the centralizer Lie algebroid $\mathcal{C}$ defined in Theorem 3.2.6.

(ii) Let $x$ and $y \in M$. If $y \in \mathcal{F}(x)$, then $\text{stab}(\mathfrak{g} \ltimes \mathcal{F}, x) = \text{stab}(\mathfrak{g} \ltimes \mathcal{F}, y)$.

(iii) The transverse action Lie algebroid $(\mathfrak{g} \ltimes \mathcal{C}) \ltimes \mathcal{F}$ is well-defined. Its orbit through a point $x \in M$ is equal to

$$(\mathfrak{g} \ltimes \mathcal{C}) \ltimes \mathcal{F}(x) = \bigcup_{y \in \mathcal{F}(x)} \mathfrak{g} \ltimes \mathcal{F}(y) = \bigcup_{z \in (\mathfrak{g} \ltimes \mathcal{F}) (x)} \mathcal{F}(z).$$

(iv) Every locally closed $\mathfrak{g} \ltimes \mathcal{F}$-invariant subset of $M$ is $(\mathfrak{g} \ltimes \mathcal{C}) \ltimes \mathcal{F}$-invariant.

Proof. (i) Let $\xi \in \mathfrak{g}$. By definition the transverse vector field $\pi^1(\xi_M)$ commutes with the sections of the centralizer subsheaf $\mathcal{C}_P$. Since the lifting homomorphism $\pi^1$ induces an isomorphism $\mathcal{C} \cong \pi^*_P \mathcal{C}_P$ (Theorem 3.2.6(i)), it follows that $\xi_M$ commutes with the sections of $\mathcal{C}$.

(ii) The leaf closure $\mathcal{F}(x)$ is the orbit of $x$ under the centralizer Lie algebroid (Theorem 3.2.6(iii)). By (i) the action of this Lie algebroid commutes with that of $\mathfrak{g}$, so $y$ has the same stabilizer as $x$.

(iii) It follows from (i) that the bundle of Lie algebras $\mathfrak{g} \ltimes \mathcal{C}$ acts transversely on $M$, so we have a well-defined transverse action $\alpha$ on $M$ defined in Theorem 3.2.6. The orbit $(\mathfrak{g} \ltimes \mathcal{C}) \ltimes \mathcal{F}(x)$ can be alternatively described as the $\mathfrak{g}$-orbit of the leaf closure $\mathcal{F}(x) = \mathcal{C}(x)$ or as the $\mathcal{C}$-orbit of the leaf $\mathcal{F}(x)$.

(iv) It follows from (iii) that every closed $\mathfrak{g} \ltimes \mathcal{F}$-invariant subset of $M$ is $(\mathfrak{g} \ltimes \mathcal{C}) \ltimes \mathcal{F}$-invariant. Now let $X \subseteq M$ be locally closed and $\mathfrak{g} \ltimes \mathcal{F}$-invariant. Then $X$ and $X \setminus X$ are closed and $\mathfrak{g} \ltimes \mathcal{F}$-invariant, hence $(\mathfrak{g} \ltimes \mathcal{C}) \ltimes \mathcal{F}$-invariant, so $X$ itself is $(\mathfrak{g} \ltimes \mathcal{C}) \ltimes \mathcal{F}$-invariant. QED

3.3.3. Remark. Item (ii) of this proposition is true even if the transverse $\mathfrak{g}$-action is not isometric. The reason is that a foliate vector field on $M$ that is tangent to $\mathcal{F}(x)$ for some
Let \( x \in M \) is tangent to \( \mathcal{F}(y) \) for every \( y \) in the leaf closure of \( x \). (Cf. remark on [3, p. 325].) Item (iv) shows that every \( g \ltimes \mathcal{F} \)-invariant embedded submanifold of \( M \) is \((g \ltimes \iota) \ltimes \mathcal{F} \)-invariant.

Another desirable item is the existence of suitable tubular neighbourhoods. We call a subset \( X \) of \( M \) invariant or saturated with respect to a Lie algebroid \( \mathcal{F} \) over \( M \) if for every \( x \in X \) the \( a \)-orbit of \( x \) is contained in \( X \). Let \( X \) be a \( g \ltimes \mathcal{F} \)-invariant embedded submanifold of \( M \) and let \( \mathcal{F}_X = \mathcal{F}|_X \) be the restriction of the foliation to \( X \). Then the normal bundle

\[
N_M X = N_X = TM|_X/TX \cong N\mathcal{F}/N\mathcal{F}_X
\]

is a foliated vector bundle over \((X, \mathcal{F}_X)\) and is equipped with a natural transverse \( g \)-action with the property that the bundle projection \( N_X \to X \) is equivariant. A \( g \ltimes \mathcal{F} \)-invariant tubular neighbourhood of \( X \) is a \( g \)-equivariant foliate embedding \( f : N_X \to M \) with the following properties: the image \( f(N_X) \) is a \( g \ltimes \mathcal{F} \)-invariant open subset of \( M \); \( f|_X = \text{id}_X \); and \( T_x f = \text{id}_{N_X} \) for all \( x \in X \). (Here we identify \( X \) with the zero section of \( N_X \) and the normal bundle of \( X \) in \( N_X \) with \( N_X \).)

We cannot guarantee that every \( g \ltimes \mathcal{F} \)-invariant \( X \) has a \( g \ltimes \mathcal{F} \)-invariant tubular neighbourhood: we must require in addition that \( X \) be closed. (By Remark (3.3.3) \( X \) and its tubular neighbourhood are then automatically invariant under the bigger Lie algebroid \((g \ltimes \iota) \ltimes \mathcal{F} \).)

3.3.4. Proposition. Let \( M \) be a connected manifold equipped with a complete Riemannian foliation \( (\mathcal{F}, g) \) and let \( \alpha : g \to \mathcal{F}(M, g) \) be an isometric transverse action. Every \( g \ltimes \mathcal{F} \)-invariant closed embedded submanifold \( X \) of \( M \) has a \( g \ltimes \mathcal{F} \)-invariant tubular neighbourhood. For every pair of \( g \ltimes \mathcal{F} \)-invariant tubular neighbourhoods \( f_0, f_1 : N_X \to M \) there exists a \( g \)-equivariant foliate isotopy from \( f_0 \) to \( f_1 \).

Proof. Let \( X_P = \pi^{-1}(X) \) and \( X_W = g(X_P) \). Then \( X_P \) is a \( K \)-invariant and \( g \ltimes \mathcal{F}_P \)-invariant closed embedded submanifold of \( P \), and \( X_W \) is a \( G \times K \)-invariant closed embedded submanifold of \( W \), where \( G \) is as in Proposition 3.3.1. The \( G \times K \)-action on \( W \) preserves the metric \( g_W \). Let \( H \) be the closure of the image of \( G \times K \) in the isometry group of \( W \). Then \( X_W \), being closed, is \( H \)-invariant, and \( H \) acts properly on \( W \), so the \( H \)-equivariant version of the standard tubular neighbourhood theorem holds. Let \( f_W : N_XW \to W \) be a \( G \times K \)-invariant tubular neighbourhood of \( X_W \). Pulling back through \( g \) we obtain an embedding \( f_P : g^* N_XW \cong N_XP \to P \), which is a \( g \)-equivariant and \( K \)-invariant tubular neighbourhood of \( X_P \). Hence the quotient by \( K \) is an embedding \( f : N_X \to M \), which is a \( g \ltimes \mathcal{F} \)-invariant foliate tubular neighbourhood of \( X \). Given two such tubular neighbourhoods \( f_0, f_1 : N_X \to M \), we have a corresponding pair of tubular neighbourhoods \( f_{W,0}, f_{W,1} : N_XW \to W \) of \( X_W \). There exists a \( G \times K \)-equivariant isotopy \( F_W : [0,1] \times N_XW \to W \) from \( f_{W,0} \) to \( f_{W,1} \), which gives rise to a \( g \)-equivariant foliate isotopy \( F : [0,1] \times N_X \to M \) from \( f_0 \) to \( f_1 \).

3.3.5. Remark. For later use we mention some further properties of the normal bundle \( N_X \). The transverse metric \( g \) on \((M, \mathcal{F})\) restricts to a transverse metric on \((X, \mathcal{F}_X)\) and to a fibre metric on \( N_X \cong N\mathcal{F}/N\mathcal{F}_X \). The transverse Levi-Civita connection descends to a \( g \)-invariant metric connection on \( N_X \). Similarly, the normal bundle \( N_XW \) is equipped with a \( G \times K \)-invariant metric and connection coming from the Riemannian metric and Levi-Civita connection on \( W \). Given a pair of embeddings \( f : N_X \to M \) and \( f_W : N_XW \to W \) as in the proof of the proposition, the isomorphism \( \pi^* N_X \cong g^* N_XW \) is an isomorphism of metric vector bundles with connection.
Equally useful is the existence of appropriate partitions of unity. We call an open cover \( \mathcal{U} \) of \( M \) invariant with respect to a Lie algebroid \( \mathfrak{a} \) over \( M \) if every \( U \in \mathcal{U} \) is \( \mathfrak{a} \)-invariant. We say that a partition of unity \( (\chi_U)_{U \in \mathcal{U}} \) subordinate to an \( \mathfrak{a} \)-invariant open cover \( \mathcal{U} \) is \( \mathfrak{a} \)-invariant if each \( \chi_U \) is constant along every \( \mathfrak{a} \)-orbit. The existence of \( \mathcal{F} \)-invariant partitions of unity was established in [3, Lemma 2.2]. Here is a \( \mathfrak{g} \ltimes \mathcal{F} \)-invariant extension of that result.

3.3.6. Proposition. Let \( M \) be a connected manifold equipped with a complete Riemannian foliation \((\mathcal{F}, \mathfrak{g})\) and let \( \alpha : \mathfrak{g} \to \mathfrak{X}(M, \mathcal{F}, \mathfrak{g}) \) be an isometric transverse action. Let \( \mathcal{U} \) be a \( \mathfrak{g} \ltimes \mathcal{F} \)-invariant open cover of \( M \). There exists a \( \mathfrak{g} \ltimes \mathcal{F} \)-invariant partition of unity subordinate to \( \mathcal{U} \).

Proof. Let \( U \in \mathcal{U} \). The complement \( M \setminus U \) is closed and \( \mathcal{F} \)-invariant, and therefore it is \( \mathcal{F} \)-invariant. It follows that \( U \) itself is \( \mathcal{F} \)-invariant, because the leaf closures \( \mathcal{F}(x) \) decompose \( M \) into disjoint subsets. Therefore the open subset \( U_p = \pi^{-1}(U) \) of \( P \) is \( K \)-invariant and \( \mathfrak{g} \ltimes P \)-invariant. Hence \( U_p = g^{-1}(U_p) \) for a unique open subset \( U_p \) of \( W \), namely \( U_W = g(U_p) \). The set \( U_W \) is \( G \times K \)-invariant, where \( G \) is as in Proposition 3.3.1. The correspondence \( U \leftrightarrow U_W \) gives us a cover \( \mathcal{U}_W \) of \( W \) by \( G \times K \)-invariant open subsets. Since \( G \) acts isometrically, the closure \( \overline{G} \) acts properly, so there exists a \( G \times K \)-invariant partition of unity subordinate to \( \mathcal{U}_W \), which transports back to a \( \mathfrak{g} \ltimes \mathcal{F} \)-invariant partition of unity subordinate to \( \mathcal{U} \).

QED

We obtain from this a Mayer-Vietoris theorem for equivariant basic de Rham theory, the non-equivariant version of which is due to [3, Theorem 2.3]. For a \( \mathfrak{g} \ltimes \mathcal{F} \)-invariant open cover \( \mathcal{U} = \{U_i\}_{i \in I} \) of \( M \) we have the basic Čech-de Rham complex \( \check{C}(\mathcal{U}, \delta) \) associated with \( \mathcal{U} \), which is defined by
\[
\check{C}^p(\mathcal{U}) = \bigoplus_{i \in I^{p+1}} \Omega(U_i, \mathcal{F}) \quad \text{and} \quad (\delta \alpha)_i = \sum_{q=0}^{p+1} (-1)^q \alpha_{(i_0, i_1, \ldots, i_q, \ldots, i_{p+1})} |U_i|.
\]
Here \( U_i \) denotes the intersection \( U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p} \) for a multi-index \( i = (i_0, i_1, \ldots, i_p) \in I^{p+1} \). We obtain an augmentation \( \Omega(M, \mathcal{F}) \to \check{C}^0(\mathcal{U}) \) by sending the form \( \alpha \) to the tuple \( (\alpha|_{U_i})_{i \in I} \). The basic Čech-de Rham complex has a second grading coming from the differential form degree, and a second differential, namely the exterior derivative, which makes it a double complex \( (\check{C}(\mathcal{U}), \delta, d) \). We denote the associated total complex by \( \check{C}^{(\delta, d)}(\mathcal{U}) \). Since \( U_i \) is \( \mathfrak{g} \ltimes \mathcal{F} \)-invariant for each \( i \in I^p \), the complex \( \Omega(U_i, \mathcal{F}) \) is a \( \mathfrak{g} \)-differential graded algebra (as defined in Section 2.5), and therefore \( \check{C}^p(\mathcal{U}) \) is a \( \mathfrak{g} \)-differential graded algebra. It follows that the associated Weil complex
\[
\check{C}^p_\mathfrak{g}(\mathcal{U}) = (W_\mathfrak{g} \otimes \check{C}^p(\mathcal{U}))_{p \text{-bas}}
\]
is a double complex \( (\check{C}^{(\delta, d)}_\mathfrak{g}(\mathcal{U}), \delta, d) \), which we call the \( \mathfrak{g} \)-equivariant \( \mathcal{F} \)-basic Čech-de Rham complex. The following Mayer-Vietoris principle states that the basic and the equivariant basic Čech-de Rham complexes are homotopically trivial with respect to the Čech differential \( \delta \).

3.3.7. Proposition (equivariant basic Mayer-Vietoris principle). Let \( M \) be a connected manifold equipped with a complete Riemannian foliation \((\mathcal{F}, \mathfrak{g})\) and let \( \alpha : \mathfrak{g} \to \mathfrak{X}(M, \mathcal{F}, \mathfrak{g}) \) be an isometric transverse action. Let \( \mathcal{U} = \{U_i\}_{i \in I} \) be a \( \mathfrak{g} \ltimes \mathcal{F} \)-invariant open cover of \( M \). The augmented basic Čech-de Rham complex
\[
0 \longrightarrow \Omega(M, \mathcal{F}) \longrightarrow \check{C}^0(\mathcal{U}) \longrightarrow \check{C}^1(\mathcal{U}) \longrightarrow \cdots
\]
is a complex of \( g \)-differential graded modules and is homotopically equivalent to zero. The augmented equivariant basic Čech-de Rham complex
\[
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^0_g(M, \mathcal{F}) & \longrightarrow & \mathcal{C}^0_g(\mathcal{U}) & \longrightarrow & \mathcal{C}^1_g(\mathcal{U}) & \longrightarrow & \cdots \\
\end{array}
\]
is likewise homotopically equivalent to zero. Therefore the augmentation induces homotopy equivalences
\[
\begin{array}{ccc}
\Omega(M, \mathcal{F}) & \overset{\sim}{\longrightarrow} & \mathcal{C}_\text{tot}(\mathcal{U}), \\
\Omega^0_g(M, \mathcal{F}) & \overset{\sim}{\longrightarrow} & \mathcal{C}^0_{g,\text{tot}}(\mathcal{U}),
\end{array}
\]
and isomorphisms
\[
H(M, \mathcal{F}) \overset{\sim}{\longrightarrow} H(\mathcal{C}_\text{tot}(\mathcal{U})), \quad H^0_g(M, \mathcal{F}) \overset{\sim}{\longrightarrow} H(\mathcal{C}^0_{g,\text{tot}}(\mathcal{U})).
\]

Proof. The Čech differential \( \delta \) is induced by inclusions of \( g \times \mathcal{F} \)-invariant open subsets, and is therefore a morphism of \( g \)-differential graded modules. The usual construction of a null-homotopy \( \kappa \) as in [5, Proposition 8.5] works: let \( (\chi_i)_i \) be a \( g \times \mathcal{F} \)-invariant partition of unity subordinate to \( \mathcal{U} \), the existence of which is guaranteed by Proposition 3.3.6, and let \( \alpha \in \mathcal{C}^p(\mathcal{U}) \). For every \( j \in I \) and \( i \in I^j \) the form \( \chi_j \alpha_{i,j_1 \cdots j_{p-1}} \in \Omega(U_j \cap U_i, \mathcal{F}) \) is supported on \( \text{supp}(\chi_j) \cap U_i \), so it extends by zero to a unique form \( \beta_{j,i} \in \Omega(U_i, \mathcal{F}) \). Define \( \kappa(\alpha) \in \mathcal{C}^{p-1}(\mathcal{U}) \) by \( (\kappa(\alpha))_i = \sum_{j \in I} \beta_{j,i} \). This sum is locally finite as the family \( (\text{supp}(\chi_j))_{i \in I} \) is locally finite; so \( \kappa \) is well-defined and we have \( [\delta, \kappa] = 0 \). Since the \( \chi_i \) are \( g \times \mathcal{F} \)-invariant, we have \( [\chi(\xi), \kappa] = [L(\xi), \kappa] = 0 \) for all \( \xi \in g \). This proves that (3.3.8) is null-homotopic as a complex of \( g \)-differential graded modules. Being a homotopy of \( g \)-dgm, the map \( \kappa \) extends to a map of Weil complexes
\[
k_\kappa: \mathcal{C}^p_g(\mathcal{U}) \longrightarrow \mathcal{C}^{p-1}_g(\mathcal{U}),
\]
which satisfies \( [\delta, k_\kappa] = 0 \), thus showing that (3.3.9) is null-homotopic as well. The last assertion is a formal consequence of the \( \delta \)-exactness; see [5, Proposition 8.8]. QED

The next corollary is a variation on a result of Molino [27, Proposition 3.7] and is mentioned without proof in [15, § 3.5].

3.3.10. Proposition. Let \( M \) be a connected manifold equipped with a complete Riemannian foliation \( (\mathcal{F}, g) \) and let \( a: g \to \mathfrak{X}(M, \mathcal{F}, g) \) be an isometric transverse action. If the leaves of \( \mathcal{F} \) are closed, the \( K \)-action on \( W \) is of constant infinitesimal orbit type, and therefore \( M/\mathcal{F} \) is an orbifold.

Proof. Let \( w \in W \). Choose \( p \in g^{-1}(w) \) and let \( x = \pi(p) \). Let \( \text{Stab}(K, w) \) be the stabilizer of \( w \) under the \( K \)-action and let \( \text{stab}(t, w) \) be its Lie algebra. Then
\[
\text{stab}(t, w) = \{ \eta \in \mathfrak{t} \mid \eta_{p,p} \in T_{p,\mathcal{F}}(p) \},
\]
and therefore the map \( \eta \mapsto \eta_{p,p} \) is an isomorphism of vector spaces
\[
\text{stab}(t, w) \cong T_{p}(K\cdot p) \cap T_{p,\mathcal{F}}(p).
\]
Let \( F_x = \pi^{-1}(\mathcal{F}(x)) \) and let \( \pi_x : F_x \to \mathcal{F}(x) \) be the restriction of \( \pi \) to \( F_x \). Then \( F_x = K\cdot F_x(p)(p) \) is the \( K \)-orbit of the leaf closure \( \mathcal{F}(p) \), so
\[
T_x F_x = T_p(K\cdot p) + T_{p,\mathcal{F}}(p), \quad \ker(T_p \pi_x) = T_p(K\cdot p) \cap T_{p,\mathcal{F}}(p) \cong \text{stab}(t, w).
\]
The hypothesis that the leaves of \( \mathcal{F} \) are closed tells us that \( \dim(\mathcal{F}(x)) = \dim(\mathcal{F}(x)) \) is constant, which yields that \( \dim(F_x) \) and \( \dim(\text{stab}(t, w)) \) are constant. This implies that the Lie subalgebras \( \text{stab}(t, w) \) are all conjugate to one another, i.e. all \( w \in W \) are of the same infinitesimal orbit type. Therefore, by Theorem 3.2.5(iii), \( M/\mathcal{F} = M/\mathcal{F} = W/K \) is an
orbifold. (More precisely, in the language of Theorem 3.2.7(iii) the generalised morphism \( H_0(M, \mathcal{F}) \to K \times W \) defined by the bibundle \( P \) is a weak equivalence of groupoids, and therefore induces an equivalence of étale stacks \([M/\mathcal{F}] \cong [W/K]\).)

QED

The final result of this section, which extends [15, Proposition 4.5], describes the stabilizer Lie algebras of the Lie algebroid \((g \times \mathcal{E}) \ltimes \mathcal{F}\) over \(M\) in terms of the infinitesimal stabilizers of the \(G \times K\)-action on the Molino manifold \(W\). Recall that \(\theta_L\) denotes the transverse Levi-Civita connection of \(P\). Recall also that for \(x \in M\) the fibre \(\zeta_x\) of the centralizer bundle \(\mathcal{E}\) consists of all germs at \(x\) of transverse Killing vector fields \(\eta\) with the property that the natural lift \(\pi^!(\eta)\) commutes with all global transverse vector fields of \(P\). Let us denote the value of \(\eta\) at \(x\) by \(\eta_{M,x} \in \mathcal{F}_x\) and the value of \(\pi^!(\eta)\) at \(p \in P\) by \(\eta_{P,p} \in \mathcal{F}_p\). Since \(\theta_L\) is \(\mathcal{F}_p\)-basic, for all \(\xi \in g\) and \(\eta \in \mathcal{E}\), the expression \(\iota(\xi_{P,p} + \eta_{P,p})\theta_L\) is a well-defined element of \(T_p(K \cdot p) \cong \mathfrak{l}\).

3.3.11 Proposition (stabilizer correspondence). Let \(M\) be a connected manifold equipped with a complete Riemannian foliation \((\mathcal{F}, g)\) and let \(a : \mathcal{E} \to \mathcal{X}(M, \mathcal{F}, g)\) be an isometric transverse action. Let \(p \in P\), \(x = \pi(p) \in M\) and \(w = \varphi(p) \in W\). Define \(\phi : g \times \mathcal{E} \to g \times \mathfrak{l}\) by

\[
\phi(\xi, \eta) = (\xi, -\iota(\xi_{P,p} + \eta_{P,p})\theta_L).
\]

Let

\[
l_x = \text{stab}(g \times \mathcal{E}, x) \subseteq g \times \mathcal{E}
\]

be the stabilizer of \(x\) with respect to the \(g \times \mathcal{E}\)-action on \(M\), and let

\[
l_w = \text{stab}(g \times \mathfrak{l}, w) \subseteq g \times \mathfrak{l}
\]

be the stabilizer of \(w\) with respect to the \(g \times \mathfrak{l}\)-action on \(W\). The restriction of \(\phi\) to \(l_x\) is a Lie algebra isomorphism from \(l_x\) onto \(l_w\).

Proof. Let \(\nu\) be a transverse Killing vector field defined on an open neighbourhood \(U\) of \(x\). Its natural lift \(\pi^!(\nu)\) is defined on \(\pi^{-1}(U) \subseteq P\) and decomposes into a vertical part \(\pi^!(\nu)_{\text{vert}}\) and a horizontal part \(\pi^!(\nu)_{\text{hor}}\) with respect to the transverse Levi-Civita connection. Thus \(\nu\) vanishes at \(x\) if and only if \(\pi^!(\nu)_{\text{hor}} = 0\). Let us apply this observation to the transverse Killing vector field \(\nu = \xi_M + \eta_M\), where \((\xi, \eta) \in g \times \mathcal{E}\). Then \(\nu\) is defined in a neighbourhood of \(x\) and vanishes at \(x\) precisely when the pair \((\xi, \eta)\) is in \(l_x\). Thus

\[
(3.3.12) \quad (\xi, \eta) \in l_x \Longleftrightarrow (\xi_{P,p} + \eta_{P,p})_{\text{hor}} = 0.
\]

Let \(\phi(\xi, \eta)_{W}\) denote the vector field on \(W\) induced by \(\phi(\xi, \eta)\). Any \(u \in T_P P\) defines an element \(\iota(u)\theta_L \in \mathfrak{l}\), which induces a vector field \(u'\) on \(P\), whose value at \(p\) is equal to \(u' = u_{\text{vert}}\). Taking \(u = \xi_{P,p} + \eta_{P,p}\) we get \(u' = (\xi_{P,p} + \eta_{P,p})_{\text{vert}}\), so that \(\phi(\xi, \eta)_{W,w} = \xi_{W,w} - T_p \varphi((\xi_{P,p} + \eta_{P,p})_{\text{vert}})\). So if \((\xi, \eta)\) is in \(l_x\), it follows from (3.3.12) that

\[
\phi(\xi, \eta)_{W,w} = \xi_{W,w} - T_p \varphi((\xi_{P,p} + \eta_{P,p})_{\text{vert}}) = \xi_{W,w} - \xi_{W,w} = 0,
\]

i.e. \(\phi(\xi, \eta)\) is in \(l_w\). This shows that \(\phi(l_x) \subseteq l_w\). Conversely, let \((\xi, \zeta) \in l_w\), i.e. \(\xi_{W,w} + \zeta_{W,w} = 0\). Then the transverse vector field \(\xi_{P,p} + \zeta_{P,p}\) on \(P\) is tangent to the fibre \(\varphi^{-1}(w)\). It follows from Theorem 3.2.6 that \(\xi_{P,p} + \zeta_{P,p} \in T_p \mathcal{F}_p \cap T_p \mathcal{F}_p = c_{P,p}\), so \(\xi_{P,p} + \zeta_{P,p} = -\eta_{P,p}\) for some \(\eta \in c_{P,p} \cong \mathcal{E}\). Since \(\zeta \in \mathfrak{l}\), the vector field \(\zeta_{P,p}\) is vertical, so

\[
(\xi_{P,p} + \eta_{P,p})_{\text{vert}} = -(\xi_{P,p})_{\text{vert}} = -\xi_{P,p} = \xi_{P,p} + \eta_{P,p}.
\]

Therefore \((\xi, \eta)\) is in \(l_x\) by (3.3.12), and also \(\phi(\xi, \eta) = (\xi, \iota(\xi_{P,p})\theta_L) = (\xi, \zeta)\). This shows that \(\phi(l_x) \supseteq l_w\). Next suppose \((\xi, \eta) \in l_x\) satisfies \(\phi(\xi, \eta) = 0\). Then \(\xi = 0\), so \(\iota(\eta_{P,p})\theta_L = 0\), so \((\eta_{P,p})_{\text{hor}} = 0\), so \(\eta_{P,p} = 0\) by (3.3.12). It follows that \(\eta = 0\), because by Theorem 3.2.6(iii)
the centralizer Lie algebroid \( \mathfrak{c} \ltimes \mathcal{F} \) acts on \( P \). This shows that the restriction of \( \phi \) to \( \mathfrak{l}_x \) is injective. We finish by showing that the restriction of \( \phi \) to \( \mathfrak{l}_x \) is a Lie algebra homomorphism. The map \( \phi_0 : \mathfrak{l}_x \to \mathfrak{t} \) defined by \( \phi_0(\xi, \eta) = u(\xi_{P,p} + \eta_{P,p})\theta_{kI} \) is the composition of three maps

\[
\mathfrak{l}_x \xrightarrow{\phi_1} \mathcal{X}(M, \mathcal{F}, g) \xrightarrow{\pi^1} \mathcal{X}(K \cdot p)^K \xrightarrow{\phi_2} \mathfrak{t},
\]

where \( \mathcal{X}(M, \mathcal{F}, g) \) denotes the Lie algebra of Killing vector fields on \( M \) that vanish at \( x \), and \( \phi_1(\xi, \eta) = \xi_{M,x} + \eta_{M,x} \), and \( \phi_2 \) is the bijective map that sends a \( K \)-invariant vector field \( u \) tangent to \( K \cdot p \) to \( u(\theta_{kI}(u)) \). Note that \( \pi^1 \) takes values in the \( K \)-invariant vector fields (see (3.2.2)) and maps vector fields vanishing at \( x \) to vector fields tangent to \( K \cdot p \) because of (3.3.12). The maps \( \phi_1 \) and \( \pi^1 \) are homomorphisms and \( \phi_2 \) is an anti-homomorphism, so \( \phi(\xi, \eta) = (\xi, -\phi_0(\xi, \eta)) \) is a homomorphism. \( \square \)

4. The orbit type stratification

In this section we show that an isometric transverse Lie algebra action on a Riemannian foliated manifold can be locally linearized and gives rise to a tidy stratification of the manifold by orbit type. These facts were partly known to Kobayashi [21], and have been long established for smooth proper Lie group actions, as set forth e.g. in [6, § IX.9]. A little caution is needed: orbits of Lie algebra actions may not have tubular neighbourhoods (e.g. Kronecker’s dense line in the torus), so the “slice theorem” is valid only in a weak form, Proposition 4.6 below.

Notation and conventions. See §§ 2.1–2.2 and the notation index in the back for our general conventions regarding foliations and transverse Lie algebra actions. We denote the Lie algebra of Killing vector fields of a Riemannian manifold \((S, g_S)\) by \( \mathcal{X}(S, g_S) \). Throughout § 4 \( \mathfrak{g} \) denotes a finite-dimensional Lie algebra and \( M \) denotes a manifold equipped with a Riemannian foliation \((\mathcal{F}, g)\) and an isometric transverse action \( a : \mathfrak{g} \to \mathcal{X}(M, \mathcal{F}, g) \). We will assume that \((M, \mathcal{F}, g)\) is metrically complete in the sense of § 3, so that the Molino structure theory applies, although many of the results of this section hold without this assumption. For \( \xi \in \mathfrak{g} \) we denote the transverse vector field \( a(\xi) \) by \( \xi_M \). By a submanifold of \( M \) we mean an embedded submanifold. We will allow our submanifolds to be “impure”, meaning that they may have connected components of different dimensions.

The stratification. We start with the simple observation that every foliation chart of \((M, \mathcal{F})\) is \( \mathfrak{g} \)-equivariant in a suitable sense. This will allow us for most purposes to disregard the foliation and restrict our attention to the transverse directions. Recall that the transverse metric \( g \) restricts to an ordinary Riemannian metric \( g_S \) on any transversal \( S \) to the foliation, and that the transverse action \( a \) restricts to an ordinary Lie algebra action \( a_S : \mathfrak{g} \to \mathcal{X}(S, g_S) \) on \( S \) by Killing vector fields.

4.1. Lemma. Let \( x \in M \). Put \( T = T_x M \) and \( F = T_x \mathcal{F} \). Choose a transversal \( S \) to the foliation \( \mathcal{F} \) at \( x \). Let \( g_S \) be the Riemannian metric on \( S \) induced by \( g \) and let \( a_S : \mathfrak{g} \to \mathcal{X}(S, g_S) \) be the isometric \( \mathfrak{g} \)-action on \( S \) induced by \( a \). Equip the product \( S \times F \) with the foliation given by the fibres of the projection \( S \times F \to S \) and with the transverse \( \mathfrak{g} \)-action defined by the action \( a_S \) on the first factor. Let \( B \) be an open neighbourhood of \( x \) in \( S \) and let \( \gamma : B \times F \to M \) be a foliate open embedding satisfying \( \gamma(y, 0) = y \) for all \( y \in B \), and \( T_x \gamma = \text{id}_F \). Then \( \gamma \) is \( \mathfrak{g} \)-equivariant.

Proof. This follows from the fact that the transverse \( \mathfrak{g} \)-action on the foliation chart \( \gamma(B \times F) \) is determined by the \( \mathfrak{g} \)-action on the transversal \( B \), as discussed in § 2.2. \( \square \)
The following result says that the $g$-action can be linearized at a fixed leaf, which is the analogue of the Bochner linearization theorem. A subpace $W$ of a vector space $V$ defines a foliation of $V$, whose leaves are the affine subspaces parallel to $W$ and whose leaf space is the quotient $V/W$. We call this the linear foliation of $V$ defined by $W$.

4.2. Proposition. Let $x\in M$ and $\mathfrak{h}=\text{stab}(x,g\ltimes\mathcal{F})$. Let $\mathcal{F}_T$ be the linear foliation of the tangent space $T=T_xM$ defined by the linear subspace $F=T_x\mathcal{F}$.

(i) The inner product $g_x$ on $T/F$ defines a transverse Riemannian metric on $(T,\mathcal{F}_T)$. The Lie algebra $\mathfrak{h}$ acts transversely on $(T,\mathcal{F}_T)$ by linear infinitesimal isometries.

(ii) There is an $\mathfrak{h}$-equivariant foliate open embedding $\psi:T\rightarrow M$ with the properties $\psi(0)=x$ and $T_0\psi=\text{id}_T$.

Proof. (i) The quotient vector space $T/F$ is the fibre at $x$ of the normal bundle $N\mathcal{F}$ of the foliation, so it carries the inner product $g_x$. The normal bundle of the linear foliation $\mathcal{F}_T$ is the trivial bundle over $T$ with fibre $T/F$, so the inner product $g_x$ defines a translation-invariant transverse Riemannian metric on the foliated manifold $(T,\mathcal{F}_T)$. The Lie algebra of isometric linear transverse vector fields on $(T,\mathcal{F}_T)$ is equal to $\mathfrak{v}(T/F)$, the Lie algebra of skew-symmetric linear endomorphisms of the inner-product space $T/F$. For $\xi\in\mathfrak{h}$ the transverse vector field $\xi_M$ vanishes at $x$, so $\xi_T=\text{ad}\xiMx$ is a well-defined linear endomorphism of the fibre $N_x\mathcal{F}=T/F$, which is skew-symmetric because $\xi_M$ preserves $g$. The map $a_x:\mathfrak{h}\rightarrow\mathfrak{v}(T/F)$ that sends $\xi$ to $\xi_T$ defines a linear isometric transverse $\mathfrak{h}$-action $(T,\mathcal{F}_T)$.

(ii) Choose a transversal $S$ to the foliation $\mathcal{F}$ at $x$. Let $g_S$ be the metric on $S$ induced by the transverse metric $g$ and $a_S:g\rightarrow\mathfrak{x}(S,g_S)$ the $g$-action on $M$ induced by the transverse $g$-action $a$. Since $x\in S$ is $\mathfrak{h}$-fixed, for $\xi\in\mathfrak{h}$ each trajectory of the vector field $\xi_S=a_S(\xi)$ stays at a fixed distance from $x$, which shows that the $\xi_S$ are complete on a sufficiently small ball $B$ about $x$. Let $T$ be the image of $\mathfrak{h}$ in $\mathfrak{x}(S,g_S)$. By the Lie-Palais theorem the $T$-action exponentiates to an action of $K$, a connected immersed Lie subgroup of the isometry group of $(S,g_S)$ with Lie algebra $T$. Then the closure $\hat{K}$ of $K$ in the isometry group fixes $x$ and is therefore compact, and we have Lie algebra homomorphisms $\mathfrak{h}\rightarrow T\rightarrow\hat{t}$, where $\hat{t}=\text{Lie}(\hat{K})$. Let $E=T_xS$. Then $E$ is a subspace of $T$ complementary to $F$, so it is an inner product space isomorphic to $T/F$, and the linearization of the $K$-action at $x$ is a homomorphism $\hat{t}\rightarrow\mathfrak{v}(E)$. By the usual Bochner linearization theorem (see e.g. [6, §IX.9, Proposition 5]) there exists (after replacing $B$ by a smaller ball if necessary) a $\hat{K}$-equivariant diffeomorphism $\phi:B\rightarrow E$ mapping $x$ to 0 and with derivative $T_0\phi=\text{id}_E$. Since the map $\phi$ is $\hat{K}$-equivariant, it is also $\mathfrak{h}$-equivariant. After again shrinking $B$ if necessary, we have a foliation chart $\gamma:B\times F\rightarrow M$ as in Lemma 4.1, and we define $\psi:T=E\oplus F\rightarrow M$ by $\psi(e,f)=\gamma(\phi^{-1}(e),f)$. The map $\psi$ satisfies our requirements. QED
algebra \( \mathfrak{g} \). Define an action \( \mathfrak{g} \ltimes \mathfrak{h} \rightarrow \mathfrak{X}(G \times V) \) by

\[
(\xi, \eta) \mapsto (\xi_R - \eta_L, \eta_V),
\]

where \( \xi_L \) (resp. \( \xi_R \)) denotes the left-invariant (resp. right-invariant) vector field on \( G \) induced by \( \xi \in \mathfrak{g} \), and \( \eta_V \) denotes the linear vector field on \( V \) induced by \( \eta \in \mathfrak{h} \). The action of the second factor \( \mathfrak{h} \) is free, so the \( \mathfrak{h} \)-orbits form a foliation \( \mathcal{H} \) of \( G \times V \). The \( \mathfrak{g} \)-action preserves the foliation \( \mathcal{H} \) and therefore descends to a transverse action \( \mathfrak{g} \rightarrow \mathfrak{X}((G \times V)/\mathcal{H}) \).

The map

\[
\beta: B_G \times B_V \rightarrow S, \quad (\exp_G(\xi), v) \mapsto \exp(\xi_S)(v)
\]

is well-defined for sufficiently small open neighbourhoods \( B_G \) of \( 1 \in G \) and \( B_V \) of \( 0 \in V \). The map \( \beta \) is \( \mathfrak{g} \)-equivariant and constant along the leaves of \( \mathcal{H} \). Its tangent map

\[
T_{(1,0)}\beta: \mathfrak{g} \times V \rightarrow E \text{ is given by }
\]

\[
T_{(1,0)}\beta(\xi, v) = \xi_S + v
\]

for \( \xi \in \mathfrak{g} \) and \( v \in V \), so \( \beta \) is a submersion at \( (1,0) \). Hence, after replacing \( B_G \) and \( B_V \) by smaller open subsets if necessary, the image \( B = \beta(B_G \times B_V) \) is an open neighbourhood of \( x \) in \( S \). Let us take \( B_G \) and \( B_V \) so small that we have a foliation chart \( \gamma: B \times F \rightarrow M \) as in Lemma 4.4, and define

\[
\alpha: B_G \times B_V \times F \rightarrow M, \quad (\exp_G(\xi), v, f) \mapsto \gamma(\beta(\exp(\xi_S)(\psi(v))), f).
\]

4.6. Proposition. Let \( x \in M \). Put \( \mathfrak{h} = \text{stab}(x, \mathfrak{g} \ltimes \mathcal{F}) \) and \( F = T_x\mathcal{F} \). Let \( G \) be a Lie group with \( \mathfrak{g} \)-action on \( G \times V \) by \((4.3)\). Let \( \mathcal{H} \) be the foliation of \( G \times V \) defined by the \( \mathfrak{h} \)-action, and let \( X \subseteq B_G \times B_V \) be a transversal at \( (1,0) \) to \( \mathcal{H} \). Let \( \alpha \) be the map defined in \((4.5)\). There exists an open neighbourhood \( B_X \) of \( (1,0) \) in \( X \) with the property that \( \alpha: B_X \times F \rightarrow M \) is a \( \mathfrak{g} \)-equivariant foliate open embedding which maps \((1,0,0) \) to \( x \).

\textbf{Proof.} The property \( \alpha(1,0,0) = x \) is evident from \((4.5)\). Since \( X \) is a transversal to the foliation \( \mathcal{H} \), the transverse \( \mathfrak{g} \times V \)-action on \( G \times V \) restricts to a \( \mathfrak{g} \)-action on \( X \), and \( \beta: X \rightarrow S \) is \( \mathfrak{g} \)-equivariant. It follows from \((4.4)\) that the kernel of \( T_{(1,0)}\beta: \mathfrak{g} \times V \rightarrow E \) is equal to \( \mathfrak{h} \times \{0\} = T_{(1,0)}\mathcal{H} \). Therefore \( \beta: X \times S \) is étale at \( (1,0) \), so \( \beta \) restricts to a \( \mathfrak{g} \)-equivariant open embedding \( \beta: B_X \times S \rightarrow M \) on a sufficiently small open neighbourhood \( B_X \subseteq X \) of \( x \). Lemma 4.1 now shows that \( \alpha: B_X \times F \rightarrow M \) is a \( \mathfrak{g} \)-equivariant open embedding. QED

Let \( \mathfrak{h} \) be a Lie subalgebra of \( \mathfrak{g} \). We say that \( x \in M \) is of \textit{symmetry type} \( \mathfrak{h} \) if the stabilizer \( \text{stab}(x, \mathfrak{g} \ltimes \mathcal{F}) \) is equal to \( \mathfrak{h} \), and we define

\[
M_{\mathfrak{h}} = \{ x \in M \mid \text{is of symmetry type } \mathfrak{h} \}.
\]

We denote by \( (\mathfrak{h}) \) the collection of all \( \mathfrak{h} \)-symmetry types of \( \mathfrak{g} \) that are conjugate to \( \mathfrak{h} \) under the adjoint group \( \text{Ad}(\mathfrak{g}) \). We say that \( x \in M \) is of \textit{orbit type} \( (\mathfrak{h}) \) if the stabilizer \( \text{stab}(x, \mathfrak{g} \ltimes \mathcal{F}) \) is an element of \( (\mathfrak{h}) \), and we define the \textit{stratum of orbit type} \( (\mathfrak{h}) \) to be

\[
M_{(\mathfrak{h})} = \{ x \in M \mid \text{is of orbit type } (\mathfrak{h}) \}.
\]

There are obvious inclusions \( M_{(\mathfrak{h})} \subseteq M^0 \) and \( M_{\mathfrak{h}} \subseteq M_{(\mathfrak{h})} \). We will now deduce from the local model theorem, Proposition 4.6, that the sets \( M^0, M_{\mathfrak{h}}, \) and \( M_{(\mathfrak{h})} \) are submanifolds of \( M \). Since they are \( \mathcal{F} \)-invariant, it then follows from Remark 3.3.3 that they are automatically invariant under the centralizer Lie algebroid \( c \ltimes \mathcal{F} \), i.e. they are unions of leaf closures.

4.7. Theorem. Let \( \mathfrak{h} \) be a Lie subalgebra of \( \mathfrak{g} \) with normalizer \( \mathfrak{n} = n_{\mathfrak{g}}(\mathfrak{h}) \).

(i) The fixed-leaf set \( M^0 \) is an \( \mathfrak{n} \ltimes \mathcal{F} \)-invariant closed submanifold of \( M \).

(ii) The symmetry type set \( M_{\mathfrak{h}} \) is an \( \mathfrak{n} \ltimes \mathcal{F} \)-invariant open submanifold of \( M^0 \).
(iii) The orbit type stratum \( M_{(b)} \) is a \( g \ltimes \mathcal{F} \)-invariant submanifold of \( M \).

**Proof.** (i) The \( n \ltimes \mathcal{F} \)-invariance follows from Lemma 2.2.7 and Proposition 3.3.2(ii). Every subalgebra \( \mathfrak{h} \) acts isometrically transversely on \( M \), so it is enough to prove the rest of the assertion for \( \mathfrak{h} = \mathfrak{g} \). That \( M^\mathfrak{h} \) is closed is evident from the smoothness of the transverse vector fields \( \xi_M \). Let \( x \in M^\mathfrak{g} \). We compute \( M^\mathfrak{g} \) in the chart at \( x \) given by Proposition 4.2. The tangent space \( T = T_xM \) is a direct sum \( T = E \oplus F \), where \( E \) is an inner-product space with a linear isometric action and the foliation of \( T \) is the linear foliation defined by \( F \). Thus \( T^\mathfrak{g} = E^\mathfrak{g} \oplus F \), where \( E^\mathfrak{g} \) is the \( \mathfrak{g} \)-fixed subspace of \( E \). It follows that \( M^\mathfrak{g} \) is a submanifold.

(ii) The \( n \ltimes \mathcal{F} \)-invariance follows from Lemma 2.2.7 and Proposition 3.3.2(ii). Let \( x \in M_h \). To show that \( M_h \) is open in \( M^\mathfrak{h} \) it suffices to show that \( M_h \cap U = M^\mathfrak{h} \cap U \) for some open neighbourhood of \( x \). We take \( U \) to be of the form \( U = \alpha(B_X \times F) \) as in Proposition 4.6, which allows us to replace \( M \) with \( X \times F \). We have \( (X \times F)_h = X_h \times F \) and \( (X \times F)^\mathfrak{h} = X^\mathfrak{h} \times F \), so it is enough to show that \( X_h = X^\mathfrak{h} \). The manifold \( X \) is a transversal to the foliation \( \mathcal{H} \) of \( G \times V \), so the \( \mathfrak{g} \)-stabilizer of a point \((k, v)\) in the \( \mathfrak{g} \)-manifold \( X \) is the same as the stabilizer of \((k, v)\) viewed as a point in \( G \times V \) with respect to the transverse action Lie algebroid \( \mathfrak{g} \ltimes \mathcal{H} \) over \( G \times V \). But this Lie algebroid can be integrated as follows. Let \( \mathcal{H} \) be the immersed connected subgroup of \( G \) corresponding to the Lie subalgebra \( \mathfrak{h} \subseteq \mathfrak{g} \), and let \( \zeta : \mathcal{H} \to G \) be a covering group with the property that the action \( \mathfrak{h} \to \mathfrak{v}(V) \) lifts to an action \( \mathcal{H} \to \text{SO}(V) \). Then \( G \times \mathcal{H} \) acts on \( G \times V \) by \((g, \mathfrak{h}) \cdot (k, v) = (gk\zeta(\mathfrak{h})^{-1}, hv)\). The corresponding action groupoid \( (G \times \mathcal{H}) \ltimes (G \times V) \) has source and target maps defined by \( s(g, \mathfrak{h}, k, v) = (k, v) \) and \( t(g, \mathfrak{h}, k, v) = (gk\zeta(\mathfrak{h})^{-1}, hv) \), and its Lie algebroid is isomorphic to \( \mathfrak{g} \ltimes \mathcal{H} \). The stabilizer of \((k, v)\) relative to this groupoid is
\[
\text{Stab}(k, v) = k \text{Stab}(v, \mathcal{H}) k^{-1},
\]
where \( \text{Stab}(v, \mathcal{H}) \) denotes the stabilizer of \( v \in V \) with respect to the linear \( \mathcal{H} \)-action, and where \( G \) acts on \( \mathcal{H} \) by conjugation. Infinitesimally we have
\[
(4.8) \quad \text{stb}(k, v), \mathfrak{g} \ltimes \mathcal{H}) = \text{Ad}_k(\text{stab}(v, \mathfrak{h}))
\]
for \( k \in G \) and \( v \in V \). In particular \((k, v)\) is of symmetry type \( \mathfrak{h} \) if and only if \( k \) is in the normalizer \( N_G(\mathfrak{h}) \) and \( v \in V^\mathfrak{h} \). Thus \( (G \times V)_h = (G \times V)^\mathfrak{h} = N_G(\mathfrak{h}) \times V^\mathfrak{h} \), which proves that \( X_h = X^\mathfrak{h} \).

(iii) The \( \mathfrak{g} \ltimes \mathcal{F} \)-invariance follows from Lemma 2.2.7 and Proposition 3.3.2(ii). In the local model \( X \times F \) we have \( (X \times F)^\mathfrak{h} = X^\mathfrak{f} \times F \). From (4.8) we get \( X^\mathfrak{h} = X \cap (G \times V)^\mathfrak{h} = X \cap (G \times V^\mathfrak{h}) \). By assumption \( X \) is transverse to the leaf \( H \times \{0\} \) of \( \mathcal{H} \). It follows that \( X \) is transverse to \( G \times V^\mathfrak{h} \), so \( X^\mathfrak{h} \) is a submanifold of \( X \). This shows that \( M^\mathfrak{h} \) is a submanifold of \( M \).

QED

On each orbit type stratum \( M_{(b)} \) the anchor map \( t : \mathfrak{g} \ltimes \mathcal{F} \to TM \) has constant rank equal to \( \dim(\mathfrak{g}) - \dim(\mathfrak{h}) + \dim(\mathcal{F}) \) and therefore defines a regular foliation. We finish by showing that the orbit-type stratification is locally finite. We partially order the set of conjugacy classes of Lie subalgebras of \( \mathfrak{g} \) by declaring
\[
(4.9) \quad (\mathfrak{h}_1) \preceq (\mathfrak{h}_2)
\]
if \( \mathfrak{h}_2 \) is conjugate to a subalgebra of \( \mathfrak{h}_1 \).

**4.10. Theorem.** Every \( x \in M \) has a \( \mathfrak{g} \ltimes \mathcal{F} \)-invariant open neighbourhood \( U \) with the property that for all \( y \in U \) the orbit type of \( y \) is greater than or equal to the orbit type of \( x \), and that \( U \cap M^\mathfrak{g} \) is empty for all but finitely many conjugacy classes \( \mathfrak{h} \). In particular, if \( M / \mathcal{F} \) is compact, the collection of \( \mathfrak{h} \) for which \( M^\mathfrak{g} \) is nonempty is finite.
Proof. Let \( x \in M \) and \( h = \text{stab}(x, g \ltimes \mathcal{F}) \). The projection \( M \to M/\mathcal{F} \) is open, so it is enough to show that \( x \) has an open neighbourhood \( U \) such that (a) \( \{ \text{stab}(y, g \ltimes \mathcal{F}) \} \geq \{ \text{stab}(x, g \ltimes \mathcal{F}) \} \) for all \( y \in U \), and (b) \( U \) intersects only finitely many orbit type strata \( M_{(\mathfrak{t})} \). It follows from (4.8) that the stabilizer of every point in the model manifold \( X \) is conjugate to a subalgebra of \( h \), so assertion (a) follows from Proposition 4.6. Assertion (b) is proved by induction on the codimension \( q \) of the foliation \( \mathcal{F} \). The case \( q = 0 \) is trivial. Suppose the assertion is proved for all Lie algebras \( \mathfrak{a} \) and all Riemannian foliated manifolds \( (Y, \mathcal{F}_Y) \) equipped with isometric transverse \( a \)-actions, where \( \mathcal{F}_Y \) is of codimension less than \( q \). Let us deduce that the assertion is true for \( M \). Again computing in the model manifold \( X \) at \( x \), for each subalgebra \( \mathfrak{t} \) of \( \mathfrak{g} \) we have \( X_{(\mathfrak{t})} = \emptyset \) if \( (\mathfrak{t}) \not\geq (\mathfrak{h}) \), and

\[
X_{(\mathfrak{t})} = G \times V^h \times W_{(\mathfrak{t})}
\]

if \( (\mathfrak{t}) \geq (\mathfrak{h}) \), where \( W \) is the orthogonal complement of \( V^h \) in \( V \). The orbit type stratum \( W_{(\mathfrak{t})} \) of the orthogonal \( \mathfrak{h} \)-module \( W \) is preserved by multiplication by nonzero scalars, and \( W_{(\mathfrak{h})} = W^h = \{ \emptyset \} \). Hence \( W_{(\mathfrak{t})} \) for \( \mathfrak{t} \neq \mathfrak{h} \) intersects the unit sphere \( S(W) \), which is an \( \mathfrak{h} \)-invariant submanifold of \( W \). In other words, the set \( \{ (\mathfrak{t}) > (\mathfrak{h}) \mid X_{(\mathfrak{t})} \neq \emptyset \} \) is in bijective correspondence with the set \( \{ (\mathfrak{t}) > (\mathfrak{h}) \mid S(W_{(\mathfrak{t})}) \neq \emptyset \} \). Since \( S(W) \) is compact and has dimension \( < q \), the induction hypothesis tells us that \( S(W) \), and hence \( X \), contains only finitely many orbit types, which completes the induction step. QED

We mention without proof the following “regular orbit type” theorem. The proof is similar to that of Theorem 4.10.

4.11. Theorem. If \( M \) is connected, the set of orbit types of elements of \( M \) has a greatest element \( (\mathfrak{b}_0) \) with respect to the partial order (4.9). The corresponding stratum \( M_{(\mathfrak{b}_0)} \) is connected, open, and dense.

5. Borel-Atiyah-Segal localization

In this section we derive from the stratification theorems of Section 4 a “contravariant” localization theorem in equivariant basic cohomology, Theorem 5.1, akin to the Borel-Atiyah-Segal theorem of ordinary equivariant cohomology theory. This answers a question posed in [14, Remark 3.22]. The important case where the Lie algebra is Molino’s structural Lie algebra of a Killing foliation was handled earlier in [15, § 5].

Notation and conventions. Throughout this section \( \mathfrak{g} \) denotes a finite-dimensional abelian Lie algebra and \( M \) denotes a connected manifold equipped with a complete Riemannian foliation \( (\mathcal{F}, g) \) and an isometric transverse action \( a: \mathfrak{g} \to \mathfrak{X}(M, \mathcal{F}, g) \). For \( \xi \in \mathfrak{g} \) we denote the transverse vector field \( a(\xi) \) by \( \xi_M \).

Contravariant localization. The following theorem is a foliated version of results due to Borel, Atiyah and Segal, which can be found in [2, § 3], [19, § III.2], or [12, § III.3]. Let \( i_\mathfrak{g}: M^0 \to M \) be the inclusion of the fixed-leaf set, which by Theorem 4.7(i) is a \( \mathfrak{g} \ltimes \mathcal{F} \)-invariant closed submanifold of \( M \). Recall that, \( \mathfrak{g} \) being abelian, the equivariant basic cohomology groups \( H_\mathfrak{g}(M, \mathcal{F}) \) and \( H_\mathfrak{g}(M^0, \mathcal{F}) \) (defined in Section 2.5) are modules over the symmetric algebra \( S_\mathfrak{g}^* \).

5.1. Theorem. Let \( \mathfrak{g} \) be abelian and \( M/\mathcal{F} \) compact. The kernel and the cokernel of the restriction homomorphism \( i_\mathfrak{g}^*: H_\mathfrak{g}(M, \mathcal{F}) \to H_\mathfrak{g}(M^0, \mathcal{F}) \) have support in the cone \( \Gamma_M = \bigcup_{x \in M \setminus M^0} \text{stab}(\mathfrak{g} \ltimes \mathcal{F}, x) \).
Proof. Since $M/\mathcal{F}$ is compact, the collection of subalgebras $\mathfrak{h}$ for which $M_{\mathfrak{h}}$ is nonempty is finite by Theorem 4.10. Hence the cone $\Gamma_M$ is the union of a nonempty finite collection of proper linear subspaces $h_j$. For each $j$ choose a nonzero $f_j \in g^*$ which vanishes on $h_j$ and let $f = \prod_{j} f_j \in Sg^*$. It is enough to show that $i^*_g$ becomes an isomorphism after inverting $f$, i.e. after localizing $Sg^*$ at the multiplicative set $S = \{ f^n \mid n \in \mathbb{N} \}$. Let $U$ be a $g \ltimes \mathcal{F}$-invariant tubular neighbourhood of $M^0$ (the existence of which is guaranteed by Proposition 3.3.4) and let $V = M \setminus M^0$. Choose $\xi \in g$ satisfying $f(\xi) \neq 0$. Then $\xi \in h \setminus \Gamma_M$, so $\xi_{M,x} \neq 0$ for all $x \in V$, which is to say that the transverse action of the Lie subalgebra $\mathfrak{t} = RH\mathfrak{g}$ of $g$ on $V$ is free. Let $\mathfrak{h}$ be a hyperplane contained in the zero locus of $f$; then $g = \mathfrak{t} \oplus \mathfrak{h}$. Let $\mathbf{A}$ be the commutative $g$-differential graded algebra $\Omega(V, \mathcal{F}|_V)$. The transverse $\mathfrak{t}$-action on $V$ is free, so by [15, Lemma 3.18] $\mathbf{A}$ possesses an $\mathfrak{h}$-invariant $\mathfrak{t}$-connection. Applying Theorem 2.4.6 to $\mathbf{A}$ we obtain a homotopy equivalence $A_\mathfrak{h} \simeq (A_{\mathfrak{h}_{\mathfrak{bas}}})_\mathfrak{h}$, i.e. $A_\mathfrak{h} \simeq B_{\mathfrak{h}_{\mathfrak{bas}}}$, where

$$B = W\mathfrak{h} \otimes A_{\mathfrak{h}_{\mathfrak{bas}}}.$$ 

Since $f$ vanishes on $\mathfrak{h}$, multiplication by $f$ kills the subalgebra $W\mathfrak{h}$ of $B$. The subalgebra $A_{\mathfrak{h}_{\mathfrak{bas}}}$ of $B$ is a subcomplex of the de Rham complex of the finite-dimensional manifold $V$, and $f$ has positive degree, so a sufficiently high power of $f$ annihilates $A_{\mathfrak{h}_{\mathfrak{bas}}}$. Hence $S^{-1}B = 0$ and $S^{-1}B_{\mathfrak{h}_{\mathfrak{bas}}} = 0$. It follows that the localization of $H_\mathfrak{h}(V, \mathcal{F}|_V) = H_\mathfrak{h}(A) \cong H(B_{\mathfrak{h}_{\mathfrak{bas}}})$ at $S$ is equal to $0$. By the same argument, the localization of $H_\mathfrak{h}(U \cap V, \mathcal{F}|_{U \cap V})$ at $S$ is equal to $0$. The equivariant basic Mayer-Vietoris sequence

$$\cdots \to H_\mathfrak{h}(M, \mathcal{F}) \to H_\mathfrak{h}(U, \mathcal{F}|_U) \oplus H_\mathfrak{h}^*(V, \mathcal{F}|_V) \to H_\mathfrak{h}(U \cap V, \mathcal{F}|_{U \cap V}) \to \cdots$$

is exact (Proposition 3.3.7), so after localizing at $S$ we get an isomorphism

$$i^*_g: S^{-1}H_\mathfrak{h}(M, \mathcal{F}) \xrightarrow{\cong} S^{-1}H_\mathfrak{h}(U, \mathcal{F}|_U) \xrightarrow{\cong} S^{-1}H_\mathfrak{h}(M^0, \mathcal{F}|_{M^0}),$$

where the second isomorphism follows from Lemma 2.5.1.

5.2. Corollary. Let $g$ be abelian and $M/\mathcal{F}$ compact. Then

$$\begin{align*}
\text{rank}(H^\text{even}_\mathfrak{h}(M, \mathcal{F})) &= \dim(H^\text{even}(M^0, \mathcal{F})), \\
\text{rank}(H^\text{odd}_\mathfrak{h}(M, \mathcal{F})) &= \dim(H^\text{odd}(M^0, \mathcal{F})).
\end{align*}$$

Proof. Let $S$ be the multiplicative set $S = Sg^* \setminus \{0\}$. Then $F = S^{-1}(Sg^*)$ is the fraction field of $Sg^*$. By definition the rank of an $Sg^*$-module $M$ is equal to the dimension of the $F$-vector space $S^{-1}M$. It follows from Theorem 5.1 that $S^{-1}H_\mathfrak{h}(M, \mathcal{F}) \cong S^{-1}H_\mathfrak{h}(M^0, \mathcal{F})$, so the $Sg^*$-modules $H_\mathfrak{h}(M, \mathcal{F})$ and $H_\mathfrak{h}(M^0, \mathcal{F})$ have the same rank. Because the $g$-action on $M^0$ is trivial, we have $H_\mathfrak{h}(M^0, \mathcal{F}) \cong Sg^* \otimes H(M^0, \mathcal{F})$, so the module $H_\mathfrak{h}(M^0, \mathcal{F})$ is free of rank equal to the dimension of the $R$-vector space $H(M^0, \mathcal{F})$. Therefore $\text{rank}(H_\mathfrak{h}(M, \mathcal{F})) = \dim(H(M^0, \mathcal{F}))$. The same argument applies to the even and odd parts, because localization of a $Z$-graded $Sg^*$-module at an ideal preserves the $Z/2Z$-grading.

6. Atiyah-Bott-Berline-Vergne Localization

In this section we derive from the Borel-Atiyah-Segal localization theorem, Theorem 5.1, a “covariant” localization formula in equivariant basic cohomology, Theorem 6.4.1. This uses the equivariant basic Thom isomorphism theorem of our paper [24]. Pushing forward to a point then yields an integration formula, Theorem 6.4.2, akin to the formulas of Atiyah and Bott [2] and Berline and Vergne [4]. An immediate consequence is a Duistermaat-Heckman formula for transversely symplectic foliations, Theorem 6.4.4.
6.1. **Notation and conventions.** In §6 we denote by $\mathfrak{g}$ a finite-dimensional Lie algebra and by $M$ a manifold equipped with a complete Riemannian foliation $(\mathcal{F}, g)$ of codimension $q$ and an isometric transverse action $a: \mathfrak{g} \to \mathfrak{X}(M, \mathcal{F}, g)$. For $\xi \in \mathfrak{g}$ we denote the transverse vector field $a(\xi)$ by $\xi_M$.

If $(C, d)$ is a cochain complex we write $(C[k], d[k])$ for the $k$-shifted complex, which is defined by $C[k]' = C' + k$ and $d[k] = (-1)^k d$. See also the notation index at the end.

6.2. **Pushforward homomorphisms and transverse integration.** We review the definition of the pushforward homomorphism (also called wrong-way or Thom-Gysin homomorphism) in equivariant basic de Rham theory given in [14, Appendix A], and we state a few of its properties. In particular we show how the fundamental class in equivariant basic Thom form of its normal bundle. These facts make use of the notion of a transverse integral of a basic differential form, which is an integral “across” the leaves of a Riemannian foliation. Unlike the integral of a differential form over a manifold, the transverse integral is not canonically defined, but depends on choices. However, it does satisfy Stokes’ formula and Poincaré duality.

Let $X$ be a co-oriented $\mathfrak{g} \ltimes \mathcal{F}$-invariant closed submanifold of $M$ of codimension $r$. Let $i = i_X: X \to M$ be the inclusion map, let $\pi = \pi_X: NX \to X$ be the normal bundle, and let $\zeta = \zeta_X: X \to NX$ be the zero section. The normal bundle is a foliated bundle equipped with a foliation $\mathcal{F}_X$, and we denote by $\Omega_{\alpha, cv}(NX, \mathcal{F}_X)$ the complex of equivariant basic forms on $NX$ which are vertically compactly supported. Integration over the fibres of $NX$ gives a morphism of complexes

$$\pi_*: \Omega_{\alpha, cv}(E, \mathcal{F}_E)[r] \longrightarrow \Omega_{\alpha}(M, \mathcal{F}).$$

An equivariant basic Thom form of $NX$ is an $r$-form $\tau_X \in \Omega^r_{\alpha, cv}(NX, \mathcal{F}_NX)$ which satisfies $\pi_*\tau_X = 1$ and $d_q \tau_X = 0$. Such forms exist by [24, Proposition 5.2.1]. The equivariant basic Thom isomorphism theorem, [24, Theorem 4.4.1], states that $\pi_*$ is a homotopy equivalence with homotopy inverse equal to the map

$$\zeta_*: \Omega_{\alpha}(M, \mathcal{F}) \longrightarrow \Omega_{\alpha, cv}(E, \mathcal{F}_E)[r]$$

defined by $\zeta_*(\alpha) = \tau_X \wedge \pi^*\alpha$. Let $f = f_X: NX \to M$ be an equivariant foliate tubular neighbourhood embedding as in Proposition 3.3.4. The pushforward homomorphism

$$(6.2.1) \quad i_* = i_{X,*}: \Omega(X, \mathcal{F}_X) \longrightarrow \Omega(M, \mathcal{F})[r]$$

is defined by $i_* = f_* \circ \zeta_*$, i.e. $i_*\beta = f_*(\zeta_X \wedge \pi_X^*\beta)$. Here for each $\gamma \in \Omega_{\alpha}(NX, \mathcal{F}_NX)$ the form $f_*(\gamma) \in \Omega(M)$ is defined by extending the form $(f^{-1})^*\gamma \in \Omega(f(NX), \mathcal{F})$, which is supported near $X$, by zero. The properties of the pushforward map $i_*$ include an equivariant basic long exact Thom-Gysin sequence (see [24, Proposition 5.3.3]), as well as the next lemma, which says that the map in cohomology induced by $i_*$ is $H_q(M, \mathcal{F})$-linear and is independent of the choice of the Thom form and the tubular neighbourhood embedding.

6.2.2. **Lemma.** Let $(M, \mathcal{F}, g)$ be a complete Riemannian foliated manifold equipped with an isometric transverse Lie algebra action $\mathfrak{g} \to \mathfrak{X}(M, \mathcal{F}, g)$. Let $X$ be a co-oriented $\mathfrak{g} \ltimes \mathcal{F}$-invariant closed submanifold of $M$ of codimension $r$.

(i) The pushforward homomorphism $i_*$ is a degree $r$ morphism of $\mathfrak{g}$-differential graded modules. Reversing the co-orientation of $X$ changes the sign of $i_*$. 

(ii) A different choice of equivariant basic Thom form $\tau_X$ or of tubular neighbourhood embedding $f$ affects $i_*$ by a homotopy of $\mathfrak{g}$-differential graded modules. It follows

...
that the maps in cohomology $i_* : H(X, \mathcal{F}_X) \to H(M, \mathcal{F})[r]$ and $i^* : H_q(X, \mathcal{F}_X) \to H_q(M, \mathcal{F})[r]$ induced by $i_*$ are independent of the choice of $\tau_X$ and $f$.

(iii) The degree $r$ morphism of $\mathfrak{g}$-differential graded modules $\Omega(X, \mathcal{F}_X) \otimes \Omega(M, \mathcal{F}) \to \Omega(M, \mathcal{F})[r]$ defined by

$$\beta \otimes \alpha \mapsto i_* \beta \wedge i^* \alpha$$

is null-homotopic. It follows that $i_* : H(X, \mathcal{F}_X) \to H(M, \mathcal{F})[r]$ is $\mathfrak{g}$-equivariant and that $i^* : H_q(X, \mathcal{F}_X) \to H_q(M, \mathcal{F})[r]$ is $H_q(M, \mathcal{F})$-linear.

**Proof.** (i) That $i_* = f \circ \zeta_*$ is morphism of $\mathfrak{g}$-dgm follows from the fact that $\tau_X$ is $\mathfrak{g}$-equivariant. If we equip $N X$ with the opposite orientation, the Thom form changes to $-\tau_X$, so the pushforward map changes to $-i_*$. 

(ii) Let $\tau'_X$ be another equivariant basic Thom form for $N X$ and let $f' : N X \to M$ be another equivariant foliate tubular neighbourhood embedding. Then we have a second Thom map $\zeta'_X \beta = \tau'_X \wedge \pi'_X \beta$ and a second pushforward morphism $i'_* = f' \circ \zeta'_*$, and we must show that $i'_*$ is homotopic to $i_*$. It suffices to show that $\zeta'_*$ is homotopic to $\zeta_*$ and that $f'_*$ is homotopic to $f_*$. It follows from the equivariant basic Thom isomorphism theorem, [24, Theorem 4.4.1], that $\tau'_X - \tau_X = d \nu$ for some form $\nu \in \Omega^+_{X, \mathfrak{g}}(N X, \mathcal{F}_{N X})$, so $l(\tau'_X) - l(\tau_X) = [d, l(\nu)]$, where “l” means left multiplication. This implies

$$\zeta'_* - \zeta_* = [d, l(\nu) \circ \pi^*],$$

which shows that $\zeta'_*$ is homotopic to $\zeta_*$. It follows from Proposition 3.3.4 that there is an equivariant foliate isotopy $F : [0, 1] \times N X \to M$ from $f$ to $f'$. The track of this isotopy is an embedding $\bar{F} : [0, 1] \times N X \to [0, 1] \times M$, so we have an extension by zero map

$$\bar{F}_* : \Omega([0, 1] \times N X, \star \times \mathcal{F}_{N X}) \to \Omega([0, 1] \times M, \star \times \mathcal{F}).$$

Let $\text{pr}_{N X} : [0, 1] \times N X \to N X$ and $\text{pr}_M : [0, 1] \times M \to M$ be the cartesian projections and define $\kappa$ to be the composition of the maps

$$\Omega(N X, \mathcal{F}_{N X}) \xrightarrow{\text{pr}_{N X}^*} \Omega([0, 1] \times N X, \star \times \mathcal{F}_{N X}) \xrightarrow{\bar{F}_*} \Omega([0, 1] \times M, \star \times \mathcal{F}) \xrightarrow{\text{pr}_M*} \Omega(M, \mathcal{F})[-1].$$

Define $j_* : M \to [0, 1] \times M$ by $j_*(x) = (t, x)$. Then $[d, \text{pr}_{M,*}] = j^* - j'_0$ by Stokes’ formula; see e.g. [24, Corollary B.4]. It follows that

$$[d, \kappa] = [d, \text{pr}_{M,*}] \circ \bar{F}_* \circ \text{pr}_{N X}^* = (j^* - j'_0) \circ \bar{F}_* \circ \text{pr}_{N X}^* = f'_* - f_*,$$

which shows that $f'_*$ is homotopic to $f_*$. 

(iii) For $\alpha \in \Omega(M, \mathcal{F})$ and $\beta \in \Omega(X, \mathcal{F}_X)$ we have

$$i_* \beta \wedge \alpha - i_*(\beta \wedge i^* \alpha) = f_*(\zeta_\mathfrak{g} \beta \wedge (\text{id}_{N X} - \pi^* \zeta_\mathfrak{g}) f^* \alpha).$$

The map $\zeta \circ : N X \to N X$ is homotopic to the identity via a $\mathfrak{g}$-equivariant foliate homotopy. Let $\kappa : \Omega(N X, \mathcal{F}_{N X}) \to \Omega(N X, \mathcal{F}_{N X})[-1]$ be a homotopy from $\text{id}_{N X}^*$ to $\pi^* \zeta_*$ as in Lemma 2.5.1. Define

$$\tilde{\kappa} : \Omega(N X, \mathcal{F}_{N X}) \to \Omega(N X, \mathcal{F}_{N X})[r - 1]$$
by \( \tilde{k}(\beta \otimes \alpha) = (-1)^{r+|\beta|} f_*(\zeta, \beta \wedge \kappa f^*\alpha) \). A calculation shows that \([\iota(\tilde{\xi}), \tilde{k}] = [L(\xi), \tilde{k}] = 0\) for all \( \xi \in \mathfrak{g} \) and that

\[
\begin{align*}
    i_* \beta \wedge \alpha - i_*(\beta \wedge i^*\alpha) &= f_*(\zeta, \beta \wedge (d\kappa + \kappa d)f^*\alpha) \\
    &= d\tilde{k}(\beta \otimes \alpha) + (-1)^r \tilde{k}(d\beta \otimes \alpha + (-1)^{|\beta|} \beta \otimes d\alpha) \\
    &= [d, \tilde{k}](\beta \otimes \alpha).
\end{align*}
\]

Therefore \( \tilde{k} \) is a homotopy of \( \mathfrak{g} \)-dgm of degree \( r - 1 \). QED

Let us call a subset of \( M \) transversely compact if it is closed and its image in the leaf closure space \( M/\mathcal{F} \) is compact. A subset \( C \) of \( M \) is transversely compact if and only if the corresponding subset \( \varrho(\pi^{-1}(C)) \) of the Molino manifold \( W \) is compact. We denote the family of all transversely compact subsets by “\( \text{ct} \)”. The basic differential forms with transversely compact supports constitute a subcomplex \( \Omega_{\text{ct}}(M, \mathcal{F}) \) of the basic de Rham complex \( \Omega(M, \mathcal{F}) \).

For simplicity we will from now on assume that the foliation \( \mathcal{F} \) is transversely oriented and that the top exterior power \( \det(\cdot) \) of the Molino centralizer bundle is a trivial flat line bundle. This assumption implies that the Molino manifold \( W \) is orientable. Without this assumption the following definition is not quite correct, but needs to be amended by using forms with coefficients in the homological orientation sheaf of \( [29] \).

**6.2.3. Definition.** A transverse integral \( \mathbf{f} \) is a collection of \( \mathbb{R} \)-linear functionals

\[
\mathbf{f}_X : \Omega_{\text{ct}}(X, \mathcal{F}_X) \to \mathbb{R},
\]

one for each co-oriented \( \mathcal{F} \)-invariant closed submanifold \( X \) of \( M \), which satisfy the following conditions:

(a) \( \mathbf{f}_M \alpha = \mathbf{f}_M \alpha_q \) for all \( \alpha \in \Omega_{\text{ct}}(M, \mathcal{F}) \), where \( \alpha_q \) is the component of \( \alpha \) of degree equal to \( q = \text{dim}(M) - \text{dim}(\mathcal{F}) \);

(b) for \( \alpha \in \Omega(M, \mathcal{F}) \) we have \( \alpha = 0 \) if and only if \( \mathbf{f}_M \alpha \wedge \beta = 0 \) for all \( \beta \in \Omega_{\text{ct}}(M, \mathcal{F}) \);

(c) Stokes’ formula holds: \( \mathbf{f}_M d\alpha = 0 \) for all \( \alpha \in \Omega_{\text{ct}}(M, \mathcal{F}) \);

(d) Poincaré duality holds: the pairing

\[
H(M, \mathcal{F}) \times H_{\text{ct}}(M, \mathcal{F}) \to \mathbb{R}
\]

defined by \( ([\alpha], [\beta]) \mapsto \mathbf{f}_M \alpha \wedge \beta \) induces an isomorphism \( H(M, \mathcal{F})[q] \to H_{\text{ct}}(M, \mathcal{F})^q \); and

(e) the form \( i_X \star 1 = f_X^* \tau_X \in \Omega(M, \mathcal{F}) \) represents the basic fundamental class of \( X \) in the sense that \( \mathbf{f}_X i_X^* \alpha = \mathbf{f}_M i_X \star 1 \wedge \alpha \) for all \( \alpha \in \Omega_{\text{ct}}(M, \mathcal{F}) \), where \( i_X \) denotes the pushforward map \((6.2.1)\).

**6.2.4. Remark.** It follows from Lemma 6.2.2(ii) and Stokes’ formula (c) that the integral \( \mathbf{f}_M i_X \star 1 \wedge \alpha \) is independent of the choice of the Thom form \( \tau_X \) and the embedding \( f_X \). However, reversing the co-orientation of \( X \) changes the sign of \( i_* \) (Lemma 6.2.2(i)) and therefore, by condition (e), changes the sign of the transverse integral \( \mathbf{f}_X \).

Transverse integrals have the following elementary properties.

**6.2.5. Lemma.** Let \( (M, \mathcal{F}, g) \) be a complete Riemannian foliated manifold equipped with an isometric transverse Lie algebra action \( \mathfrak{g} \to \mathfrak{X}(M, \mathcal{F}, g) \). Let \( \mathbf{f} \) be a transverse integral and let \( i = i_X : X \to M \) be a co-oriented \( \mathcal{F} \)-invariant closed submanifold of \( M \).
(i) The homomorphisms \( i^* \) and \( i \) are adjoint in the sense that
\[
\int_X \beta \wedge i^* \alpha = \int_M i_\beta \wedge \alpha
\]
for all \( \alpha \in \Omega^d(M, \mathcal{F}) \) and \( \beta \in \Omega(X, \mathcal{F}_X) \).

(ii) Let \( \int_X' \) be another transverse integral. There exists a nonzero constant \( k \), independent of \( X \), such that
\( \int_X' \beta = k \int_X \beta \) for all closed \( \beta \in \Omega_{cl}(X, \mathcal{F}_X) \).

(iii) Let \( \beta \in \Omega_{cl}(X, \mathcal{F}_X) \). Then
\( \int_X L(v)\beta = \int_X i(v)\beta = 0 \) for all transverse Killing vector fields \( v \in \mathfrak{X}(X, \mathcal{F}_X, g) \).

(iv) Suppose \( X \) is \( g \ltimes \mathcal{F} \)-invariant. The \( \mathcal{W}_\Omega \)-linear functional
\[
\int_X = \text{id}_{\mathcal{W}_\Omega} \otimes \int_X : \mathcal{W}_\Omega \otimes \Omega_{cl}(X, \mathcal{F}_X) \to \mathcal{W}_\Omega
\]
is a morphism of \( g \)-differential graded modules of degree \(-\text{codim}(\mathcal{F}_X)\). In particular \( \int_X \beta \in (S^q \Omega^* \mathcal{F}^*)^0 \) and \( \int_X d_\beta = 0 \) for all \( \beta \) in the Weil complex \( \Omega_{cl}(X, \mathcal{F}_X) \).

\textbf{Proof.}\ (i) This follows from Definition 6.2.3(e) and the fact that \( i_* \beta = i \wedge f_* \pi^*_X \beta \).

(ii) By Poincaré duality (Definition 6.2.3(d)), each of the transverse integrals \( \int_M \) and \( \int_X' \) gives an isomorphism \( H^d_{cl}(M, \mathcal{F})^* \cong H^d(M, \mathcal{F}) \cong \mathbb{R} \). Hence there exists a nonzero constant \( k \) such that \( \int_X' \beta = k \int_M \alpha \) for all \( \alpha \in \Omega^d_{cl}(M, \mathcal{F}) \). It follows from (i) that
\[
\int_X \beta = \int_M i_* \beta \quad \text{and} \quad \int_X \beta = \int_M i_\beta \quad \text{for all } \beta \in \Omega_{cl}(X, \mathcal{F}_X).
\]
Therefore \( \int_X' \beta = c \int_X \beta \) for all \( \beta \in \Omega_{cl}(X) \).

(iii) Using \( \int_X \beta = \int_M i_* \beta \) and Definition 6.2.3(a) we get \( \int_X \beta = \int_X \beta_{q_X} \) for all \( \beta \in \Omega_{cl}(X, \mathcal{F}_X) \), where \( \beta_{q_X} \) is the component of \( \beta \) of degree equal to \( q_X = \dim(X) - \text{dim}(\mathcal{F}) \). This implies the identity \( \int_X i(v)\beta = 0 \), because a basic form \( \beta \) on \( X \) of degree \( q_X + 1 \) is equal to 0. We also have \( \int_X d\beta = \int_M i_* d\beta = \int_M d_i \beta = 0 \) because of Stokes’ formula (Definition 6.2.3(c)). Hence \( \int_X L(v)\beta = \int_X (d_i(v)\beta + i(v)d\beta) = 0 \).

(iv) That \( \int_M \) is a morphism of \( g \)-differential modules follows from (iii). By definition \( \alpha \in \Omega_{cl}(M, \mathcal{F}) \) is a \( g \)-basic element of \( \mathcal{W}_\Omega \otimes \Omega_{cl}(M, \mathcal{F}) \). Hence \( \int_M \alpha \in \mathcal{W}_\Omega \text{bas} = (S^q \Omega^* \mathcal{F}^*)^0 \). The differential of the Weil algebra \( d \mathcal{W}_\Omega \) vanishes on basic elements (see e.g. [24, Proposition A.4.2(ii)]), so \( \int_M d_\alpha = d \mathcal{W}_\Omega \int_M \alpha = 0 \).

QED

We are aware of two approaches to transverse integration. The first, which works if \( M \) is compact, is that of Kamber and Tondeur (see [31, Ch. 7] or [14, § 3]) and the second is that of Sergiescu [29]. We review Sergiescu’s approach and show that it defines a transverse integral in our sense.

Let \( \gamma \) be the volume element on the Lie algebra \( \mathfrak{g} = \mathfrak{o}(q) \) defined by the normalized invariant inner product \( \mathfrak{g}_1 \). Let \( \theta \), be the transverse Levi-Civita connection of the Molino bibundle \( P \) (Section 3.2). Then \( \gamma = \theta_\mathcal{F}^\mathcal{F}(\gamma) \) is an \( \mathcal{F}_P \)-basic form on \( P \) that restricts to a normalized invariant volume form on each fibre of the bundle \( \pi : P \to M \). The projection formula gives \( \pi_*(\nu \wedge \pi^* \alpha) = \pi_* \nu \wedge \alpha = \alpha \) for all forms \( \alpha \) on \( M \), which means that the map
\[
\Omega(M, \mathcal{F}) \to \Omega(P, \mathcal{F}_P) : \quad \alpha \mapsto \nu \wedge \pi^* \alpha
\]
is a right inverse of the fibre integration map \( \pi_* \). Choose a nonzero global flat section \( s \) of the real line bundle \( \det(c) \), where \( c \) is the centralizer bundle of \( (M, \mathcal{F}) \) defined in Theorem 3.2.6. For every \( \gamma \in \Omega(P, \mathcal{F}_P) \) the contraction \( \iota_{\pi'(s)} \gamma \) with the multivector field \( \pi'(s) \) is well-defined. Every \( \mathcal{F}_P \)-basic form is \( \mathcal{F}_P \)-invariant, so \( \iota_{\pi'(s)} \gamma \) is \( \mathcal{F}_P \)-basic and therefore equal to \( q^* \mu \) for a unique form \( \mu \) on \( W \), which we will denote by \( \mu = q^*(\pi'(s)) \gamma \).

Let \( X \) be a co-oriented \( g \ltimes \mathcal{F} \)-invariant closed submanifold of \( M \) of codimension \( r \). Then
the corresponding $G \times K$-invariant submanifold $X_W = \varrho(\pi(X))$ of the Molino manifold $W$ is oriented. For simplicity let us write $\pi$ for the restriction of $\pi$ to $X_P = \pi^{-1}(X)$, and similarly for $\varrho$, $\nu$, and $s$. Then for $\beta \in \Omega^*(X, \mathcal{F})$ we define

$$\int_X \beta = \int_{X_W} \varrho_! t_{\pi|s}(\pi^* \beta \wedge \nu).$$

Here we adopt the usual convention that $\int_{X_W}$ $\mu$ is the integral of the component of $\mu$ of degree $\dim(X_W)$. If $\beta$ is homogeneous of degree $j$, then the degree of $\varrho_! t_{\pi|s}(\pi^* \beta \wedge \nu)$ is $j + \frac{1}{2} q(q - 1) - l$. Using $\dim(X_W) = q_X + \frac{1}{2} q(q - 1) - l$, where $q_X = \dim(X) - \dim(\mathcal{F})$ and $l$ is the rank of the centralizer bundle $\epsilon$ (see table of dimensions at the end of the notation index), we see that $\int_X \beta$ is the transverse integral of the component of $\beta$ of degree $q_X = \dim(X) - \dim(\mathcal{F})$. In particular the transverse integral $\int_X 1 \in \Omega^0(M, \mathcal{F})$ is defined if $X$ is a closed leaf of $\mathcal{F}$.

6.2.7. **Proposition.** Let $(M, \mathcal{F}, g)$ be a complete Riemannian foliated manifold equipped with an isometric transverse Lie algebra action $g : X(M, \mathcal{F}, g)$. Suppose that the foliation $\mathcal{F}$ is transversely oriented and that the line bundle $\det(\epsilon)$ is trivial. Then the formula (6.2.6) defines a transverse integral in the sense of Definition 6.2.3.

**Proof.** Conditions (a)–(d) of Definition 6.2.3 are verified in [29, §2]. Condition (e) is checked as follows. By definition

$$\int_M i_* 1 \wedge \alpha = \int_M f_\ast \tau_X \wedge \alpha = \int_W \varrho_! t_{\pi|s}(\pi^* f_\ast \tau_X \wedge \alpha \wedge \nu).$$

In this formula we may take $\tau_X$ to be any equivariant basic Thom form for the normal bundle $NX$. Let us take $\tau_X$ to be the universal equivariant basic Thom form defined in [24, §4.5]. Then $\tau_X$ is $\mathcal{F}NX$-basic by [24, Lemma 4.5.2(iv)], so $t_{\pi|s}(\pi^* f_\ast \tau_X) = \pi^* f_\ast t_{\pi|s}(\tau_X) = 0$. Under the Molino correspondence $M \leftrightarrow W$ the submanifold $X$ corresponds to the submanifold $X_W = \varrho(\pi^{-1}(X))$ and we have an isomorphism of metric vector bundles with connection $\pi^* NX \cong \varrho^* XW$. (See Remark 3.3.5.) The universality property of the Thom form ([24, Lemma 4.5.2(ii)]) gives $\pi^* (\tau_X) = \varrho^* (\tau_{X_W})$, where $\tau_{X_W}$ is the universal Thom form of the bundle $NX_W$. Hence

$$\int_M i_* 1 \wedge \alpha = \int_W \varrho_! t_{\pi|s}(\pi^* f_\ast \tau_X \wedge \alpha \wedge \nu) = \int_W f_{\ast W} \tau_{X_W} \wedge \varrho_! t_{\pi|s}(\pi^* \alpha \wedge \nu) = \int_{X_W} i_{\ast W} \varrho_! t_{\pi|s}(\pi^* \alpha \wedge \nu) = \int_X i^* \alpha,$$

where we used the fact that the Thom form $\tau_{X_W}$ represents the Poincaré dual of the submanifold $X_W$; see e.g. [5, Proposition 6.24].

We close this section by reiterating a point made by Töben. Suppose the submanifold $X$ has the property that all leaves of $\mathcal{F}_X$ are closed. Then the leaf space $X/\mathcal{F}_X = X/\mathcal{F}_X = X_W/K$ is an orbifold (Proposition 3.3.10), and a basic differential form $\beta$ on $X$ represents a differential form on the orbifold. The transverse integral $\int_X \beta$ relates to the orbifold integral $\int_{X/\mathcal{F}_X} \beta$ as follows. The principal stratum of $X$ is the open dense subset $X^{\text{prin}}$ consisting of all $x \in X$ for which the holonomy of $\mathcal{F}_X$ vanishes at $x$. The corresponding submanifold
\( \varrho(\pi(X^{\text{prin}})) \) of the Molino manifold \( W \) is equal to the principal orbit type stratum of \( X_W \) considered as a \( K \)-manifold.

6.2.8. Proposition ([30, Lemma 5.4]). Under the hypotheses of Proposition 6.2.7, let \( X \) be a co-oriented \( \mathcal{F} \)-invariant closed submanifold of \( M \). Suppose that the leaves of \( \mathcal{F}_X \) are closed. Then the transverse integral \( c = \int_{\mathcal{F}(X)} 1 \) is independent of \( x \in X^{\text{prin}} \) and for all \( \beta \in \Omega_c(X, \mathcal{F}_X) \) we have

\[
\int_X \beta = (-1)^{\frac{1}{2}h(q+1)} \int_{X/\mathcal{F}_X} \beta.
\]

6.3. Euler forms. Let \((M, \mathcal{F}, g)\) be a complete Riemannian foliated manifold equipped with an isometric transverse action of a Lie algebra \( g \). By Theorem 4.7(i) the fixed-leaf manifold \( M^g \) is a \( g \times \mathcal{F} \)-invariant closed submanifold of \( M \). We now show that if \( g \) is abelian every connected component \( X \) of \( M^g \) is co-oriented and of even codimension \( 2l \).

Thus each \( X \) has an equivariant basic Thom form \( \tau_X \in \Omega^{2l}_{c,v_0}(NX, \mathcal{F}_NX) \). Its restriction to the zero section \( \eta_X = \zeta_X^* \tau_X \in \Omega^{2l}(X, \mathcal{F}_X) \) is known as the associated equivariant basic Euler form. The next result, which is well known for torus actions on manifolds (see e.g. [2, § 3] or [7, § 1]), gives a simple formula for the restriction of the Euler form to a point.

6.3.1. Proposition. Let \((M, \mathcal{F}, g)\) be a complete Riemannian foliated manifold equipped with an isometric transverse action of an abelian Lie algebra \( g \) and let \( X \) be a connected component of the fixed-leaf manifold \( M^g \). Let \( r \) be the codimension of \( X \) in \( M \) and let \( E = NX \) be the normal bundle of \( X \). Let \( j_x : \{x\} \to X \) be the inclusion of \( x \in X \), and let \( a_x : g \to \mathfrak{o}(E_x, g_{E,x}) \) be the \( g \)-action on the fibre \( E_x \).

(i) The bundle \( E \) has a \( g \)-invariant almost complex structure \( J \). The weights \( \lambda_1, \lambda_2, \ldots, \lambda_l \in \mathfrak{g}^* \) of the action \( a_x \) with respect to \( J_x \) are nonzero and are independent of \( x \in X \). We have a weight space decomposition \( E = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_l} \) into \( g \)-equivariant foliated subbundles. In particular \( E \) is orientable and the rank \( r = 2l \) of \( E \) is even.

(ii) Let \( \eta_X = \zeta_X^* \tau_X \in \Omega^r_0(X, \mathcal{F}_X) \) be an equivariant basic Euler form with respect to the orientation given by \( \eta \) and let \( \eta_0 \) be the component of \( \eta_X \) in \( \mathfrak{g}^* \otimes \Omega^r(X, \mathcal{F}_X) \). Then \( \eta_0 = \lambda_1 \lambda_2 \cdots \lambda_l \in \mathfrak{g}^* \). Reversing the co-orientation of \( X \) has the effect of changing the sign of the \( \eta_X \) and \( \eta_0 \).

(iii) The Euler form \( \eta_X \) becomes invertible in the algebra \( \Omega^r_0(X, \mathcal{F}_X) \) after inverting the weights \( \lambda_1, \lambda_2, \ldots, \lambda_l \).

Proof. (i) The transverse \( g \)-action on \( X \) being trivial, the transverse \( g \)-action on \( E \) is tangent to the fibres of \( E \). The foliation induced by \( \mathcal{F}_E \) on each fibre \( E_x \) is the trivial 0-dimensional foliation, so on \( E_x \) we have a linear \( g \)-action in the usual sense. The fibre metric of \( E \) is by assumption \( g \)-invariant, so for each \( x \in X \) the transverse \( g \)-action on \( E_x \) is given by an infinitesimal orthogonal representation \( a_x : g \to \mathfrak{o}(E_x, g_{E,x}) \). Our vector bundle comes equipped with an invariant metric connection, so a parallel transport argument shows that the representation \( a_x : g \to \mathfrak{o}(E_y, g_{E,y}) \) is equivalent to \( a_x \) for all \( y \in X \). Since \( a_x \) leaves only the origin of \( E_x \) fixed, we have a real isotypical decomposition

\[
E_x \cong \bigoplus_{\lambda \in P} \mathbb{R}^2(\lambda)^{\oplus \mu(\lambda)}.
\]

Here \( \mathbb{R}^2(\lambda) \) is a copy of \( \mathbb{R}^2 \) on which \( g \) acts through the infinitesimal rotations

\[
\xi \mapsto 2\pi \lambda(\xi) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
$m(\lambda)$ is the multiplicity of $R^j(\lambda)$, and $P$ is a finite subset of $g^* \setminus \{0\}$. Since $R^j(\lambda) \equiv R^j(-\lambda)$, the weights $\lambda$ are defined only up to sign, so $P$ is not unique. To remove the ambiguity we choose an element $\xi_0 \in g$ which fixes only the origin of $E_x$ and require all $\lambda \in P$ to satisfy $\lambda(\xi_0) > 0$. We give $E_x$ the almost complex structure $J_x$ which is equal to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on $R^j(\lambda)$. This almost complex structure depends on the choice of $\xi_0$, but not on the isomorphism (6.3.2) (see [17, § 8.5.3]), and gives us the desired almost complex structure $J$ and the corresponding isotypical decomposition of $E$.

(ii) Since $g$ acts trivially on $X$, the Weil complex of $X$ is $\Omega(\pi, \mathcal{F}_X) = S\eta^* \otimes \Omega(X, \mathcal{F}_X)$ with differential $d_\eta = \text{id} \otimes d$. Because the Euler form $\eta_X$ is closed, this shows that its component $\eta_0 \in S\eta^* \otimes \Omega^0(X, \mathcal{F}_X)$ is closed. In other words, for any $x \in X$, $\eta_0 \in S\eta^*$ is equal to $j_x^*(\eta_X)$, which is a $g$-equivariant Euler form for the bundle $E_x$ over the one-point space $\{x\}$. To calculate this, we may take $\tau_X$ to be the universal equivariant basic Thom form of [24, § 4.5] with respect to a $g$-invariant basic metric connection $\theta$ on $E$. Then the Euler form is

$$\eta_X = (-2\pi)^{-\frac{1}{2}}c_{\theta, \theta}(\text{Pf}),$$

where $c_{\theta, \theta} : (S\theta^*)^j \rightarrow \Omega(\pi, \mathcal{F}_X)$ is the $g$-equivariant characteristic, or Cartan-Chern-Weil, homomorphism (2.4.5), with $\theta = \theta(2\pi)$, and $\text{Pf} \in S\theta^*$ is the Pfaffian. The $g$-equivariant characteristic homomorphism $(S\theta^*)^j \rightarrow \Omega(g(\pi)) \equiv S\eta^*$ for the bundle $E_x \rightarrow \{x\}$ is just the restriction map $\alpha_x : (S\theta^*)^j \rightarrow S\eta^*$ induced by the action $\alpha_x : g \rightarrow \theta$, so the naturality property of the universal Euler form (see [24, Proposition 4.6.1]) with respect to the map $j_x$ yields $j_x^*(\eta_X) = (-2\pi)^{-\frac{1}{2}}\alpha_x^*(\text{Pf})$. The polynomial $\alpha_x^*(\text{Pf})$ is the product of the Pfaffians of the matrices (6.3.3), i.e. $\alpha_x(\text{Pf}) = (-2\pi)^{\frac{1}{2}}l_1 l_2 \cdots l_\ell$, and hence $j_x^*(\eta_X) = l_1 l_2 \cdots l_\ell$. The last assertion follows from the fact that if we reverse the orientation of $E$ the Thom form changes sign.

(iii) The form $\eta_X - \eta_0 \in S\eta^* \otimes \Omega^0(X, \mathcal{F}_X)$ is nilpotent. It follows from (ii) that $\eta_0 = l_1 l_2 \cdots l_\ell$ is a nonzero element of $S\eta^*$. Therefore $\eta_X$ is invertible in the localization of the $S\eta^*$-algebra $\Omega(\pi, \mathcal{F}_X)$ at the weights $l_\ell$.

The utility of (iii) lies in the fact that the localization map

$$\Omega(\pi, \mathcal{F}_X) \rightarrow \Omega(\pi, \mathcal{F}_X)[(l_1 \cdots l_\ell)^{-1}]$$

is injective, because the weights in $g^* \setminus \{0\}$ are not zero divisors in $S\eta^*$. Identities involving equivariant basic forms can therefore be verified in the localized algebra, which is easier to handle.

6.4. Covariant localization. Let $(M, \mathcal{F}, g)$ be a complete Riemannian foliated manifold equipped with an isometric transverse action of an abelian Lie algebra $g$. By Proposition 6.3.1 connected components $X$ of the fixed-leaf manifold $M^0$ are co-orientable and of even codimension $2l_X$. Fixing a co-orientation for each $X$ we have pushforward homomorphisms $i_{X, \ast}$, which we can assemble to a single map

$$i_{\ast, \ast} = \bigoplus_X i_{X, \ast} : \bigoplus_X \Omega(\pi, \mathcal{F}_X)[-2l_X] \rightarrow \Omega(\pi, \mathcal{F}_X),$$

where $X$ ranges over the connected components of $M^0$. For each $X$ the normal bundle $NX$ has an equivariant basic Euler form $\eta_X \in \Omega^0(\pi, \mathcal{F}_X)$, and a well-defined weight product $l_X \in S\eta^*$. By Proposition 6.3.1(iii) the forms $\eta_X$ become invertible after extending the ring of scalars $S\eta^*$ by inverting the $l_X$, so the following statement makes sense.

6.4.1. Theorem (covariant localization). Let $(M, \mathcal{F}, g)$ be a transversely compact complete Riemannian foliated manifold equipped with an isometric transverse action of an abelian
Lie algebra \( g \). The \( S^g \)-linear homomorphism \( i_{n,*} = \bigoplus_X i_{X,*} \) induces an isomorphism on cohomology after inverting the weight products \( \lambda_X \in S^g \) and any \( f \in S^g \) that vanishes on the cone \( \Gamma_M = \bigcup_{x \in M \setminus M^0} \text{stab}(g \times \mathcal{F}, x) \). The inverse of this isomorphism is induced by the map \( i_n^0 \) defined by

\[
i_n^0(\alpha) = \sum_X \eta_X^{-1} \wedge i_X^* \alpha,
\]

where the sum is over the connected components \( X \) of the fixed-leaf manifold \( M^0 \).

**Proof.** For all \( \beta \in \Omega_g(M^0, \mathcal{F}|M^0) \) and for all components \( X \) of \( M^0 \) we have

\[
i_X^* i_X i_* \beta = \xi_X^* \xi_X \beta = \xi_X^* (\tau_N(NX, \mathcal{F}|NX) \wedge \pi^*_X \beta) = \eta_X \wedge i_X^* \beta,
\]

and therefore

\[
i_n^0(i_{n_*} \beta) = \sum_X \eta_X^{-1} \wedge i_X^* i_X i_* \beta = \sum_X \eta_X^{-1} \wedge \eta_X \wedge i_X^* \beta = \sum_X i_X^* \beta = \beta,
\]

which shows that \( i_n^0 \circ i_{n_*} = \text{id} \). Let \( i_n^0 = \sum_X i_X^0 : \Omega_g(M, \mathcal{F}) \to \Omega_g(M^0, \mathcal{F}|M^0) \) be the restriction map. For all \( \alpha \in \Omega_g(M, \mathcal{F}) \) we have

\[
i_n^0 i_{n_*} i_n^0(\alpha) = \sum_X i_X^0 i_X i_* (\eta_X^{-1} \wedge i_X^* \alpha) = \sum_X \eta_X \wedge \eta_X^{-1} \wedge i_X^* \alpha = \sum_X i_X^* \alpha = i_n^0 \alpha,
\]

which shows that \( i_n^0 \circ i_{n_*} \circ i_n^0 = i_n^0 \). By the contravariant localization theorem, Theorem 5.1, \( i_n^0 \) induces an isomorphism on cohomology, so we conclude that \( i_{n_*} \circ i_n^0 \) induces the identity on cohomology. \( \text{QED} \)

Pushing forward to a point now gives an integration formula, which is a foliated version of the formulas of Atiyah and Bott [2] and Berline and Vergne [4].

**6.4.2. Theorem.** Let \( (M, \mathcal{F}, g) \) be a transversely compact complete Riemannian foliated manifold equipped with an isometric transverse action of an abelian Lie algebra \( g \). Suppose that the foliation \( \mathcal{F} \) is transversely oriented and that the bundle \( \det(\cdot) \) is trivial. Then

\[
\int_M \alpha = \sum_X \int_X \eta_X^{-1} \wedge i_X^* \alpha
\]

for all \( d_2 \)-closed \( \alpha \in \Omega_g(M, \mathcal{F}) \), where the sum is over the connected components \( X \) of the fixed-leaf manifold \( M^0 \).

**Proof.** It follows from Theorem 6.4.1 and from Stokes’ formula that

\[
\int_M \alpha = \int_M i_{n_*} i_n^0(\alpha) = \sum_X \int_M i_X i_* (\eta_X^{-1} \wedge i_X^* \alpha).
\]

Lemma 6.2.5(i) now yields the desired result. \( \text{QED} \)

**6.4.3. Remark.** Reversing the co-orientation of \( X \) changes the sign of the integral \( \int_X \) (Remark 6.2.4) and of the Euler form \( \eta_X \) (Proposition 6.3.1(ii)), so the integral \( \int_X \eta_X^{-1} \wedge i_X^* \alpha \) is independent of the choice of co-orientation.

This immediately gives a foliated version of the Duistermaat-Heckman formula [13], where we consider the case of a transverse symplectic form, i.e. a closed basic 2-form \( \omega \in \Omega^2(M, \mathcal{F}) \) which is nondegenerate on the normal bundle \( N\mathcal{F} \). Then the codimension \( q = 2n \) of \( \mathcal{F} \) is even. Suppose that the transverse \( g \)-action is Hamiltonian with moment map \( \Phi : M \to \mathfrak{g}^* \). Then the equivariant transverse symplectic form \( \omega_q = \omega + \Phi \) is \( g \)-equivariant and \( \mathcal{F} \)-basic. The fixed-leaf components \( X \) are transversely symplectic of dimension \( 2n_X \),...
and the moment map has a constant value $\Phi_X$ on $X$. Applying the integration formula Theorem 6.4.2 to the form $\alpha = \exp(\omega_\partial)$ gives the following result.

6.4.4. **Theorem.** Let $(M, \mathcal{F}, g)$ be a transversely compact complete Riemannian foliated manifold equipped with an isometric transverse action of an abelian Lie algebra $\mathfrak{g}$. Suppose that $(M, \mathcal{F})$ is transversely symplectic, that the line bundle $\det(\cdot)$ is trivial, and that the transverse $\mathfrak{g}$-action is Hamiltonian with moment map $\Phi: M \to \mathfrak{g}^\ast$. Then

$$
\int_M e^{\Phi \omega_\partial} = \sum_{\chi} e^{\Phi_X} \int_X \eta_X^{-1} \wedge \omega_X^\ast.
$$

See [18, § 10] for a different take on the foliated Duistermaat-Heckman theorem.

**Notation Index**

| Notation | Description |
|----------|-------------|
| $\mathcal{F}$ | foliation, 3 |
| $\mathcal{F}(x)$ | leaf of $x$, 3 |
| $\Omega(M, \mathcal{F})$ | basic de Rham complex, 7 |
| $\Omega(M)$ | equivariant basic de Rham complex, 9 |
| $\mathcal{F}$ | foliated manifold, 3 |
| $\mathcal{F}(M, \mathcal{F}, g)$ | Riemannian foliated manifold, 9 |
| $\mathfrak{g}$ | Lie algebra, 4 |
| $\mathfrak{g}^\ast$ | dual Lie algebra, 6 |
| $\mathfrak{g}^\dagger$ | natural descent homomorphism, 10 |
| $\mathfrak{g}^\dagger_0$ | natural lifting homomorphism, 10 |
| $\mathfrak{g}_K$ | normalized bi-invariant Riemannian metric on $K = \Omega(q)$, 10 |
| $\mathfrak{g}_L$ | inner product on $\mathfrak{g}$ induced by $g_K$, 10 |
| $H_{\mathfrak{g}, \mathfrak{b}}(M)$ | $\mathfrak{g}$-basic cohomology of $M$, 7 |
| $H_{\mathfrak{g}}(M)$ | equivariant cohomology of $M$, 8 |
| $\iota$ | contraction, 7 |
| $L$ | Lie derivative, 7 |
| $(\mathcal{F}, \mathcal{F})$ | $\mathcal{F}$-invariant submanifold of $M$ with induced foliation, 15 |
| $X_W$ | submanifold of $W$ corresponding to $X$, 15 |
| $\xi_M$ | transverse vector field induced by $\xi \in \mathfrak{g}$, 4 |
| $\mathfrak{X}(\mathcal{F})$ | vector fields tangent to $\mathcal{F}$, 3 |
| $\mathfrak{X}(M)$ | vector fields on $M$, 3 |
| $\mathfrak{X}(\mathcal{F})$ | transverse vector fields on $(M, \mathcal{F})$, 3 |
| $\mathfrak{X}(\mathcal{F}, g)$ | transverse Killing vector fields on $(M, \mathcal{F}, g)$, 10 |
| $\mathfrak{X}(W, g_W)$ | Killing vector fields on Molino manifold $(W, g_W)$, 14 |
Dimensions and ranks

| Symbol | Equation |
|--------|----------|
| \(\dim(M)\) | \(m\) |
| \(\text{codim}_M(\mathcal{F})\) | \(q\) |
| \(\text{rank}(\mathcal{c})\) | \(l\) |
| \(\dim(K)\) | \(q(q - 1)/2\) |
| \(\dim(P)\) | \(m + q(q - 1)/2\) |
| \(\dim(W)\) | \(q + q(q + 1)/2 - l\) |
| \(\text{codim}_X(\mathcal{F}_X)\) | \(q_X\) |
| \(\dim(X_W)\) | \(q_X + q(q + 1)/2 - l\) |

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