ON SIGNED $p$-KOSTKA MATRICES

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ABSTRACT. We show that the signed $p$-Kostka numbers depend just on $p$-Kostka numbers and the multiplicities of projective indecomposable modules in certain signed Young permutation modules. We then examine the signed $p$-Kostka number $k_{(\alpha|\beta),(\lambda|\mu)}$ in the case when $|\beta| = p|\mu|$. This allows us to explicitly describe the multiplicities of direct summands of a signed Young permutation module lying in the principal block of $FS_{mp}$ in terms of the $p$-Kostka numbers.

1. INTRODUCTION

In 2001, Donkin introduced signed Young modules as the indecomposable summands of signed Young permutation modules [7]. In the same article, the author also defines listing modules as the indecomposable direct summands of the mixed tensor product of symmetric and exterior powers of the natural module for the Schur algebras over fields of positive characteristics $p$. These objects generalise important classical objects such as Young modules and tilting modules respectively. The study of the decomposition of Young permutation modules into indecomposable summands (Young modules) is central in the modular representation theory of symmetric groups. The Krull-Schmidt multiplicities of Young modules in Young permutation modules are called $p$-Kostka numbers. The complete determination of these multiplicities is well-known to be equivalent to the complete determination of the decomposition matrix for Schur algebras.

In this article we focus on signed $p$-Kostka numbers. These numbers can be defined as the Krull-Schmidt multiplicities of signed Young modules into signed Young permutation modules, but appear in several other contexts. For instance, signed $p$-Kostka numbers can also be regarded as multiplicities of listing modules in mixed tensor product of symmetric and exterior powers of the natural module [7, 3.1.3(3)], or as the dimensions of weight spaces of irreducible modules for Schur superalgebras [7, 2.3(7)].

Our first main result is Theorem 3.3 below. Avoiding for the moment the introduction of the necessary technical notation we summarise it as follows.

Main Result A. We give a closed formula for signed $p$-Kostka numbers. This reduces the computation of signed $p$-Kostka numbers to the knowledge of $p$-Kostka numbers and of projective signed $p$-Kostka numbers.

The formula announced above and given in Theorem 3.3 generalizes and extends to signed $p$-Kostka numbers the well known Klyachko’s multiplicity formula [21]. Moreover, using our Main Result A we are able to deduce an analogue Steinberg Tensor Product Theorem for the irreducible modules for Schur superalgebras [20, II.3.17]. This is done in Corollary 3.5. We are aware that Corollary 3.5 can be directly obtained from Steinberg Tensor Product Theorem by counting the

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weight spaces. Nevertheless, we find interesting to highlight that our proof uses only techniques from the representation theory of symmetric groups.

In Section 4 we analyze the signed $p$-Kostka matrix and we obtain the following result.

**Main Result B.** We show that the signed $p$-Kostka matrix is lower unitriangular and has diagonal blocks given by the Kronecker products of some suitable $p$-Kostka matrices.

We refer the reader to Corollary 4.4 for full details on our second main result. We remark that Corollary 4.4 is obtained as a consequence of Theorem 4.3. Similarly, using Theorem 4.3 we are also able to explicitly describe the multiplicities of direct summands of signed Young permutation modules lying in the principal block of $FG_n$, when $n$ is a multiple of $p$. This is done in Corollary 4.5.

Signed Young permutation $FG_n$-modules are naturally labelled by pairs of compositions $(\lambda|\mu)$ such that $|\lambda| + |\mu| = n$. We devote the last part of the article to address the following problem.

**Main Result C.** Given two pairs of compositions of $n$, we determine exactly when the corresponding signed Young permutation modules are isomorphic.

Our third main result is stated and proved in all details in Theorem 6.1. The proof uses the signed Young Rule, recently discovered by Tan and the second author [24].

Our paper is organised as follows. In Section 2 we set up the notation and give some basic background that we will repeatedly use throughout the paper. In Section 3, we prove our Main Result A. In the first part of Section 4 we generalized the row-removal formula for $p$-Kostka numbers obtained in [2] to the case of signed $p$-Kostka numbers. The second part of Section 4 is devoted to prove Theorem 4.3, which in turn implies our Main Result B. In Section 5, we study Theorem 4.3 in a more general setting by dropping the assumption $|\beta| = p|\mu|$. In Section 6, we prove Main Result C by addressing the problem of deciding when two signed Young permutation modules are isomorphic.

2. **Generalities**

Throughout this paper $F$ denotes a field of characteristic $p > 0$. Whenever $A$ is a finite dimensional algebra over $F$, an $A$-module is a finite-dimensional left $A$-module. All groups considered in this paper are finite groups.

Suppose that $V_1$ and $V_2$ are $A$-modules such that $V_1$ is isomorphic to a direct summand of $V_2$. We write $V_1 \mid V_2$. If $V_1$ is indecomposable then we denote by $[V_2 : V_1]$ the Krull–Schmidt multiplicity of $V_1$ in $V_2$, that is, in every indecomposable direct sum decomposition of $V_2$, there are precisely $[V_2 : V_1]$ direct summands that are isomorphic to $V_1$.

Let $H$ be a subgroup of a group $G$ and, $V$ and $W$ be $FG$- and $FH$-modules respectively. We denote by $\text{Res}^G_H(V)$ the restriction of $V$ to $H$, and we denote by $\text{Ind}^G_H(W)$ the induction of $W$ to $G$. Suppose further that $V$ is indecomposable. A Green vertex $Q$ of $V$ is a minimal (with respect to inclusion) subgroup $Q$ of $G$ such that $V \mid \text{Ind}^G_Q\text{Res}^G_Q(V)$. In this case, a Green source with respect to $Q$ is an indecomposable $FQ$-module $S$ such that $V \mid \text{Ind}^G_Q S$.

If $H$ is normal in $G$ and $U$ is an $F[G/H]$-module then we denote by $\text{Inf}^G_{G/H}(U)$ the inflation of $U$ to $G$. Similarly, if $H$ is a subgroup of $G$ acting trivially on an $FG$-module $V$, we denote by $\text{Def}^G_{G/H} V$ the $F[G/H]$-module which is the same vector space $V$ with the action $(gH) \cdot v = gv$, i.e., the
deflation of \( V \) to \( G/H \). Clearly, in this case, if \( K \subseteq H \), then \( \text{Def}_{(G/K)/(H/K)}^{G/K} V \cong \text{Def}_{G/H}^{G/K} V \) via the identification \( (G/K)/(H/K) \cong G/H \). Let \( V \) and \( W \) be \( FG \)- and \( FH \)-modules for some groups \( G \) and \( H \) respectively. The exterior tensor product \( V \otimes_F W \) is an \( F[G \times H] \)-module in the obvious way. We shall denote this module by \( V \boxtimes W \).

We have the following easy lemma.

**Lemma 2.1** (see [9, Lemma 2.1]). Let \( V, W \) be \( FG \)-modules such that \( W \) is indecomposable and suppose that \( H \) is a normal subgroup of \( G \) acting trivially on both the modules \( V, W \). Then \([V : W] = [\text{Def}_{G/K}^G V : \text{Def}_{G/H}^G W] \).

### 2.1. Symmetric groups and their modules.

Let \( \mathbb{N}_0, \mathbb{N} \) be the sets of nonnegative integers and positive integers respectively. From now on and thereafter, all the infinite sums and infinite products encountered in this paper are indeed finite sums and finite products respectively. For instance, \( \sum_{i=1}^{\infty} \delta_i \) means there exists \( r \in \mathbb{N} \) such that \( \delta_i = 0 \) for all \( i > r \) and hence \( \sum_{i=1}^{\infty} \delta_i = \sum_{i=1}^{r} \delta_i \).

Let \( n \in \mathbb{N}_0 \). We denote \( \text{Seq}(n) \) for the set consisting of infinite sequences of nonnegative integers \( \delta = (\delta_i)_{i=1}^{\infty} \) such that \( n = \sum_{i=1}^{\infty} \delta_i \). In this case, we write \(|\delta| = n\) and denote the unique sequence \( \delta \) such that \(|\delta| = 0\) by \( \emptyset \). Let \( \text{Seq} = \bigcup_{n \in \mathbb{N}_0} \text{Seq}(n) \). Let \( \delta, \gamma \in \text{Seq} \) and \( q, r \in \mathbb{N}_0 \). We define

\[
\delta + \gamma = \gamma + \delta = (\delta_i + \gamma_i)_{i=1}^{\infty},
q \cdot \delta = (q \delta_i)_{i=1}^{\infty},
\delta \preceq_r = (\delta_i)_{i=1}^{r},
\delta \succeq_r = (\delta_i)_{i=1}^{\infty}.
\]

Let \( n \in \mathbb{N}_0 \). A composition of \( n \) is a sequence of nonnegative integers \( \alpha = (\alpha_1, \ldots, \alpha_r) \) for some \( r \in \mathbb{N} \) such that \( n = \sum_{i=1}^{r} \alpha_i \). In this case, we call \( \ell(\alpha) = r \) the length of \( \alpha \) and, by abuse of notation, \(|\alpha| = n \). By convention, we assume that \( \alpha_i = 0 \) if \( i > r \). The set of compositions of \( n \) is denoted as \( \mathcal{C}(n) \). The Young diagram \([\alpha] \) of \( \alpha \) is the set

\[
[\alpha] = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq j \leq \alpha_i \}.
\]

A composition is called a partition if the terms are non-increasing and nonzero. The set of all partitions of \( n \) is denoted as \( \mathcal{P}(n) \). By abuse of notation, we also denote the unique partition of \( 0 \) as \( \emptyset \). The set \( \mathcal{P}(n) \) is equipped with the usual dominance order \( \succeq \).

There is an obvious inclusion of \( \mathcal{P}(n) \) into \( \text{Seq}(n) \) by adding zeroes to the tail. Let \( \lambda \in \mathcal{P}(n) \), \( \delta \in \text{Seq}(n) \) and \( \alpha, \beta \in \mathcal{C}(n) \). We may write \( \lambda = \delta \) if \( \lambda_i = \delta_i \) for all \( i \in \mathbb{N} \). If the length of a composition is out of the question, we may also write \( \lambda = \alpha \) if \( \lambda_i = \alpha_i \) for all \( i \in \mathbb{N} \). The notions \( q \cdot \alpha \) and \( \alpha + \beta \) are defined similarly as before. The partition obtained from \( \delta \) (respectively, \( \alpha \)) by rearranging its parts and deleting zero entries is denoted as \( \varphi(\delta) \) (respectively, \( \varphi(\alpha) \)). Suppose further that \( r = \ell(\alpha) \) and \( s = \ell(\beta) \). We define

\[
\alpha \# \beta = (\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s),
\alpha \# \delta = (\alpha_1, \ldots, \alpha_r, \delta_1, \delta_2, \ldots),
\alpha \cup \beta = \varphi(\alpha \# \beta).
\]

Let \( r \in \mathbb{N}_0 \), \( \lambda \in \mathcal{P}(n) \) and \( k = \ell(\lambda) \). We define the partitions

\[
\lambda^{Tr} = (\lambda_1, \ldots, \lambda_r),
\]

...
\(\lambda^{Br} = (\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_k)\),

if \(0 \leq r \leq k - 1\) and, \(\lambda^T = \lambda\) and \(\lambda^B = \emptyset\) if \(r \geq k\). Therefore, \(\lambda = \lambda^T \neq \lambda^B\). A pair of partitions \((\alpha, \lambda)\), not necessarily of the same size, admits a horizontal \(r\)-row cut if \(|\alpha^T| = |\lambda^T|\) for some \(r \in \mathbb{N}_0\).

Let \(p\) be a prime number. A partition \(\lambda\) is \(p\)-restricted if \(0 \leq \lambda_{i+1} - \lambda_i \leq p - 1\) for all \(i \in \mathbb{N}\). A partition \(\lambda\) is \(p\)-regular if and only if its conjugate \(\lambda'\) is \(p\)-restricted. The set of \(p\)-restricted partitions of \(n\) is denoted by \(\mathcal{R}(\mathcal{P}(n))\). The sum

\[\lambda = \sum_{i=0}^{\infty} p^i \cdot \lambda(i)\]

is the \(p\)-adic expansion of \(\lambda\) if, for each \(i \in \mathbb{N}_0\), \(\lambda(i)\) is a \(p\)-restricted partition. For technical computation, we assume that \(\lambda(-1) = \emptyset\).

Let \(n \in \mathbb{N}_0\). We denote the symmetric group of degree \(n\) by \(\mathcal{S}_n\) and its alternating subgroup by \(\mathcal{A}_n\). Let \(\alpha \in \mathcal{C}(n)\). The standard Young subgroup of \(\mathcal{S}_n\) labelled by \(\alpha\) is denoted by \(\mathcal{S}_\alpha = \prod_{j=1}^{\infty} \mathcal{S}_{\alpha_j}\), here the \(j\)th factor \(\mathcal{S}_{\alpha_j}\) acts on the set

\[\left\{ 1 + \sum_{i=1}^{j-1} \alpha_i, 2 + \sum_{i=1}^{j-1} \alpha_i, \ldots, \sum_{i=1}^{j} \alpha_i \right\}.

By our convention, if \(k = \ell(\alpha)\), then \(\mathcal{S}_\alpha = \mathcal{S}_{\alpha_1} \times \cdots \times \mathcal{S}_{\alpha_k}\). Let \(G\) be a finite group. We identify the wreath product \(G \wr \mathcal{S}_\alpha\) with the direct product \(\prod_{i=1}^{\infty} G \wr \mathcal{S}_{\alpha_i}\). Furthermore, if \(G \leq \mathcal{S}_m\) for some \(m \in \mathbb{N}_0\), we view \(G \wr \mathcal{S}_\alpha\) as a subgroup of \(\mathcal{S}_{mn}\) in the obvious way.

We recall, for instance from [18, 4.1.22, 4.1.24], the structure of the Sylow \(p\)-subgroups of \(\mathcal{S}_n\). If \(n = \sum_{i=0}^{\infty} a_i p^i\) is the \(p\)-adic expansion of \(n\) where \(0 \leq a_i \leq p - 1\) then let \(P_n\) be a fixed Sylow \(p\)-subgroup of the Young subgroup \(\prod_{i=0}^{\infty} (\mathcal{S}_{p^i})^{a_i}\) of \(\mathcal{S}_n\). Thus \(P_n = \prod_{i=0}^{\infty} (P_{p^i})^{a_i}\) where \(P_{p^i}\) is a Sylow \(p\)-subgroup of \(\mathcal{S}_{p^i}\). Moreover, for each \(i \in \mathbb{N}_0\), the Sylow \(p\)-subgroup \(P_{p^i}\) of \(\mathcal{S}_{p^i}\) may be identified with the \(i\)-fold iterated wreath product \(C_p \wr \cdots \wr C_p\). Furthermore, we write \(N_n = N_{\mathcal{S}_n}(P_n)\). Whenever \(\alpha \in \mathcal{C}(n)\), we denote by \(P_\alpha\) the fixed Sylow \(p\)-subgroup of \(\mathcal{S}_\alpha\) given by \(P_\alpha = \prod_{i=0}^{\infty} P_{a_i}\).

We use the notation \(\rho = ((p^i)^{n_i})_{i=0}^{\infty}\) to denote an element in \(\text{Seq}\) where the first \(n_0\) terms of \(\rho\) are \(1\), the next \(n_1\) terms are \(p\), and so on, so that \(\rho \in \text{Seq}(n)\) if \(\sum_{i=0}^{\infty} n_i p^i = n\). By convention, if \(n = 0\) then \(\rho = \emptyset\). Therefore we get \(P_\rho = \prod_{i=0}^{\infty} (P_{p^i})^{n_i}\). In particular, for \(i \in \mathbb{N}_0\), if \(|\rho| = n\), the group \(P_\rho\) has precisely \(n_i\) orbits of sizes \(p^i\) acting on the set \(\{1, \ldots, n\}\). We write \(N_\rho = N_{\mathcal{S}_n}(P_\rho)\) and further identify \(N_\rho\) with the following subgroup of \(\mathcal{S}_n\):

\[N_\rho = \prod_{i=0}^{\infty} (N_{p^i} \wr \mathcal{S}_{n_i}) \subseteq \prod_{i=0}^{\infty} \mathcal{S}_{n_i p^i}.

We now describe some standard modules for the symmetric group algebras. The trivial and sign representations of \(F\mathcal{S}_n\) are denoted by \(F(n)\) and \(\text{sgn}(n)\) respectively (or simply \(F\) and \(\text{sgn}\) if it is clear in the context). For each \(\lambda \in \mathcal{P}(n)\), we denote by \(S^\lambda\) the Specht \(F\mathcal{S}_n\)-module labelled by \(\lambda\) as defined by James in [15, Section 4]. As \(\lambda\) varies over the set of \(p\)-regular partitions of \(n\), the quotient modules \(D^\lambda := S^\lambda / \text{rad}(S^\lambda)\) vary over a set of representatives of the isomorphism classes of simple \(F\mathcal{S}_n\)-modules. Analogously, as \(\mu\) varies over the set of \(p\)-restricted partitions of \(n\) the submodules \(D_\mu := \text{soc}(S^\mu)\) vary over a set of representatives of the isomorphism classes of simple
$F\Sigma_n$-modules as well. If $\mu \in \mathcal{R}(n)$ then one has

$$D_{\mathcal{M}(\mu)} = D_{\mu} \otimes \text{sgn} \cong D_{\mu^{'}}$$

where $\mu^{'}$ is the conjugate partition of $\mu$ and $\mathcal{M}$ is the Mullineux map on the $p$-restricted partitions. We refer the reader to [4] for the explicit map $\mathcal{M}$.

Let $H$ be a subgroup of $\Sigma_n$. We denote the trivial and sign representations of $F\Sigma_n$ restricted to the subgroup $H$ by $F(H)$ and $\text{sgn}(H)$ respectively. In the case when $H = \Sigma_n$ for some $\alpha \in \mathcal{C}(n)$, for simplicity, we write $F(\alpha)$ and $\text{sgn}(\alpha)$ for $F(\Sigma_n)$ and $\text{sgn}(\Sigma_n)$ respectively. Let $G$ be a finite group and suppose further that $n \geq 0$. Let $V$ be an $FG$-module. The $n$-fold tensor product of $V$ is an $F(G \wr \Sigma_n)$-module with the action given by

$$(g_1, \ldots, g_n; \sigma) \cdot (v_1 \otimes \cdots \otimes v_n) = \text{sgn}(\sigma)g_1v_{\sigma^{-1}(1)} \otimes \cdots \otimes g_nv_{\sigma^{-1}(n)}$$

for any $(g_1, \ldots, g_n; \sigma) \in G \wr \Sigma_n$ and $v_1, \ldots, v_n \in V$. We denote this module by $\hat{V}^{\otimes n}$. By convention, $\hat{V}^{\otimes n}$ is the trivial module for the trivial group. Furthermore, we write $\hat{V}^{\otimes n} = \bigotimes_{i=1}^{n} \hat{V}^{\otimes n}$ for the $F(G \wr \Sigma_n)$-module.

For further background on the representation theory of the symmetric group we refer the reader to [15, 18].

2.2. Signed Young permutation modules, listing modules and signed $p$-Kostka numbers. Let $\alpha, \beta$ be compositions such that $|\alpha| + |\beta| = n$. We call $(\alpha|\beta)$ a pair of compositions of $n$. Let $\mathcal{C}^2(n)$ be the set of all pairs of compositions of $n$. Moreover, let $\mathcal{P}^2(n)$ be the set consisting of those pairs $(\alpha|\beta) \in \mathcal{C}^2(n)$ such that both $\alpha$ and $\beta$ are partitions. The set $\mathcal{P}^2(n)$ is partially ordered by $\trianglerighteq$ where $(\alpha|\beta) \trianglerighteq (\lambda|\mu)$ if, for all $k \in \mathbb{N}$, we have both

(a) $\sum_{i=1}^{k} \alpha_i \geq \sum_{i=1}^{k} \lambda_i$, and
(b) $|\alpha| + \sum_{i=1}^{k} \beta_i \geq |\lambda| + \sum_{i=1}^{k} \mu_i$.

In the case $(\alpha|\beta) \trianglerighteq (\lambda|\mu)$ but $(\alpha|\beta) \nprec (\lambda|\mu)$, we write $(\alpha|\beta) \triangleright (\lambda|\mu)$.

Let $p$ be an odd prime. We denote by $\mathcal{P}_p^2(n)$ the subset of $\mathcal{P}^2(n)$ consisting of all pairs of partitions of $n$ of the form $(\lambda|\mu p)$. We view $\mathcal{C}(n)$ and $\mathcal{P}(n)$ as subsets of $\mathcal{C}^2(n)$ and $\mathcal{P}^2(n)$, respectively, by identifying $\alpha$ with $(\alpha|\emptyset)$. The identification restricts the dominance order of $\mathcal{P}^2(n)$ to the usual dominance order of $\mathcal{P}(n)$ we introduced earlier.

Let $S_F(m, n)$ be the Schur algebra over $F$ as defined in [12]. The category of finite dimensional $S_F(m, n)$-modules $S_F(m, n)$-$\text{mod}$ is Morita equivalent to the category of polynomial $GL_m(F)$-modules of degree $n$. Furthermore, in the case when $m \geq n$, we have the Schur functor

$$f : S_F(m, n)$-$\text{mod} \rightarrow F\Sigma_n$-$\text{mod}$$

defined by $f(V) = eV$ for certain idempotent $e$ of $S_F(m, n)$. Furthermore, the functor $f$ is exact. For simplicity, we assume $m \geq n$ throughout for the Schur algebra $S_F(m, n)$.

Let $E$ be the natural $GL_m(F)$-module, and let $S^\alpha E$ and $\bigwedge^\beta E$ be the symmetric and exterior powers of $E$ with respect to the compositions $\alpha, \beta$, respectively; namely,

$$S^\alpha E = S^{\alpha_1}E \otimes \cdots \otimes S^{\alpha_q}E,$$

$$\bigwedge^\beta E = \bigwedge^{\beta_1}E \otimes \cdots \otimes \bigwedge^{\beta_q}E.$$

Let $K(\alpha|\beta) = S^\alpha E \otimes \bigwedge^\beta E$ for each $(\alpha|\beta) \in \mathcal{C}^2(n)$. Then $K(\alpha|\beta)$ is a polynomial $GL_m(F)$-module of degree $n$. By [7, §3], the indecomposable summands of the mixed powers $K(\alpha|\beta)$ are called the
listing modules and their isomorphism classes are labelled by the set \( \mathcal{P}_p^2(n) \). The listing module labelled by \((\lambda|\mu) \in \mathcal{P}_p^2(n)\) is denoted by List\((\lambda|\mu)\). Furthermore,

\[
K(\lambda|\mu) \cong \text{List}(\lambda|\mu) \oplus C(\lambda|\mu)
\]

where \(C(\lambda|\mu)\) is a direct sum of listing modules labelled by pairs \((\xi|\eta) \in \mathcal{P}_p^2(n)\) such that \((\xi|\eta) \trianglerighteq (\lambda|\mu)\).

For \((\alpha|\beta) \in \mathcal{C}_2(n)\), one defines the signed Young permutation \(F\mathfrak{S}_n\)-module

\[
M(\alpha|\beta) := \text{Ind}_{\mathfrak{S}_\alpha \times \mathfrak{S}_\beta}^{\mathfrak{S}_n} (F(\alpha) \boxtimes \text{sgn}(\beta)).
\]

In the case when \(\beta = \emptyset\) this yields the usual Young permutation \(F\mathfrak{S}_n\)-module \(M^\alpha = M(\alpha|\emptyset)\). Notice that \(M(\alpha|\beta) \otimes \text{sgn} \cong M(\beta|\alpha)\) and \(M(\alpha|\beta) \cong M(\varphi(\alpha)|\varphi(\beta))\). Therefore, without loss of generality, up to isomorphism, we may only consider \(M(\alpha|\beta)\) when \((\alpha|\beta) \in \mathcal{P}_p^2(n)\). For \(\gamma, \delta \in \text{Seq}\), suppose that \(\alpha = \gamma \) and \(\beta = \delta\), we write \(M(\gamma|\delta)\) for \(M(\alpha|\beta)\) (and hence \(M^\gamma\) for \(M^\alpha\)).

For each \((\lambda|\mu) \in \mathcal{P}_p^2(n)\), let \(Y(\lambda|\mu) = f(\text{List}(\lambda|\mu))\) where \(f\) is the Schur functor. By convention, \(Y(\emptyset|\emptyset)\) is the trivial \(F\mathfrak{S}_n\)-module. Following [7, §3], we have \(f(K(\alpha|\beta)) \cong M(\alpha|\beta)\).

Since \(f\) is exact, we obtain that \(\{Y(\lambda|\mu) : (\lambda|\mu) \in \mathcal{P}_p^2(n)\}\) is a set consisting of the isomorphism classes of the indecomposable summands of the signed Young permutation modules. We call \(Y(\lambda|\mu)\) a signed Young permutation module. As such, when \((\lambda|\mu)\) varies over the set \(\mathcal{P}_p^2(n)\), both subsets consisting of the isomorphism classes containing \(M(\lambda|\mu)\) and \(Y(\lambda|\mu)\), respectively, are bases for the Green ring for \(F\mathfrak{S}_n\) generated by all signed Young permutation modules.

In the case when \(\mu = \emptyset\), one obtains \(Y(\lambda|\emptyset) \cong Y^\lambda\), the usual indecomposable Young \(F\mathfrak{S}_n\)-module labelled by the partition \(\lambda\) of \(n\). The indecomposable Young \(F\mathfrak{S}_n\)-module \(Y^\lambda\) is projective if and only if \(\lambda\) is a \(p\)-restricted partition. In this case, \(Y^\lambda\) is a projective cover of the simple \(F\mathfrak{S}_n\)-module \(D_\lambda\).

For the remainder of this paper, we fix a total order on \(\mathcal{P}_p^2(n)\) as follows. For each \(n \in \mathbb{N}_0\), we fix a total order \(\succeq_n\) on the set \(\mathcal{P}(n)\) refining the dominance order \(\trianglerighteq\) on \(\mathcal{P}(n)\). Let \(\succeq\) be a total order on the set \(\mathcal{P}_p^2(n)\) refining the dominance order on \(\mathcal{P}_p^2(n)\) such that \((\alpha|p\beta) \succeq (\delta|p\beta)\) if and only if \(\alpha \succeq_m \delta\) where \(|\alpha| = m = |\delta|\).

We remark that, since \(\succeq\) refines \(\trianglerighteq\), if \((\alpha|p\beta) \trianglerighteq (\lambda|\mu) \trianglerighteq (\delta|p\beta)\), then \((\alpha|p\beta) \trianglerighteq (\lambda|\mu) \trianglerighteq (\delta|p\beta)\) and hence \(\beta \trianglerighteq \mu \trianglerighteq \beta\) which implies that \(\beta = \mu\). Also, the restriction of \(\succeq\) on \(\mathcal{P}_p^2(n)\) to the subset \(\mathcal{P}(n)\) is precisely \(\succeq_n\).

We now define the signed \(p\)-Kostka numbers and signed \(p\)-Kostka matrices.

**Definition 2.2.**

(i) Let \((\alpha|\beta), (\lambda|\mu) \in \mathcal{P}_2(n)\). The Krull–Schmidt multiplicity

\[
k_{(\alpha|\beta),(\lambda|\mu)} = [M(\alpha|\beta) : Y(\lambda|\mu)] = [K(\alpha|\beta) : \text{List}(\lambda|\mu)]
\]

is called a signed \(p\)-Kostka number. If \(\beta = \emptyset = \mu\) then

\[
k_{\alpha,\lambda} = [M^\alpha : Y^\lambda] = [S^\alpha E : P(\lambda)]
\]

is called a \(p\)-Kostka number where \(P(\lambda)\) is the projective cover of the simple \(S_F(m, n)\)-module \(L(\lambda)\) of highest weight \(\lambda\) and \(m \geq n\). By convention, \(k_{(\varnothing|\varnothing),(\varnothing|\varnothing)} = 1 = k_{\varnothing,\varnothing}\). Furthermore, if \(\delta, \gamma \in \text{Seq}\) such that \(\varphi(\delta) = \alpha\) and \(\varphi(\gamma) = \beta\), then we write

\[
k_{(\delta|\gamma),(\lambda|\mu)} = k_{(\alpha|\beta),(\lambda|\mu)}.
\]
(ii) The signed $p$-Kostka matrix $K_n^\pm$ is the square matrix whose entries are signed $p$-Kostka numbers of the form $k_{(\alpha|\beta),(\lambda|\mu)}$ where $(\alpha|\beta),(\lambda|\mu) \in \mathcal{P}_p^2(n)$ with respect to the fixed total order $\succ$ on $\mathcal{P}_p^2(n)$. The $p$-Kostka matrix $K_n$ is the square matrix whose entries are $p$-Kostka numbers $k_{\alpha,\lambda}$ where $\alpha, \lambda \in \mathcal{P}(n)$ with respect to the total order $\succ_n$ on $\mathcal{P}(n)$.

**Remark 2.3.**

(i) By [7, §2.3], the number $k_{(\alpha|\beta),(\lambda|\mu)}$ is also equal to the dimension of the $(\alpha|\beta)$-weight space $L(\lambda|\mu)^{(\alpha|\beta)}$ of the irreducible module $L(\lambda|\mu)$ of highest weight $(\lambda|\mu)$ of the Schur superalgebra $S_F(a|b,n)$ where $a, b \geq n$.

(ii) By definition, it is clear that $K_n$ is the top left $(\ell \times \ell)$-submatrix of $K_n^\pm$ where $\ell = |\mathcal{P}(n)|$.

(iii) In [23], for each $(\alpha|\beta) \in \mathcal{P}^2(n)$, the second author gave an explicit formula to write $M(\alpha|\beta)$ as a linear combination of $M(\zeta|\eta)$’s for $(\zeta|\eta) \in \mathcal{P}_p^2(n)$ in the Green ring of the group algebra $F \mathfrak{S}_n$. Therefore, theoretically, all signed $p$-Kostka numbers are determined whenever the signed $p$-Kostka numbers of the form $k_{(\zeta|\eta),(\lambda|\mu)}$ are known. Therefore $K_n^\pm$ is only labelled by $\mathcal{P}_p^2(n) \times \mathcal{P}_p^2(n)$.

We end this subsection with the following general row removal formula was proved by Bowman and the first author. We remark that the case when $r = 1$ was first proved by Fang-Henke-Knoenig (see [8, Corollary 9.1]).

**Theorem 2.4** ([2, Corollary 1.1]). Let $\alpha, \lambda \in \mathcal{P}(n)$ such that the pair $(\alpha, \lambda)$ admits a horizontal $r$-row cut. Then

$$k_{\alpha,\lambda} = k_{\alpha_{\ell \cap r},\lambda_{\ell \cap r}}k_{\alpha_{\ell \setminus r},\lambda_{\ell \setminus r}}.$$  

2.3. **Brauer correspondence for $p$-permutation modules.** We recall now the definition and the basic properties of the Brauer quotients for $FG$-modules. Let $G$ be a finite group. Given an $FG$-module $V$ and $Q$ a $p$-subgroup of $G$, the set of fixed points $V$ of $Q$ is denoted by

$$V^Q = \{ v \in V : gv = v \text{ for all } g \in Q \}.$$  

It is easy to see that $V^Q$ is an $FN_G(Q)$-module on which $Q$ acts trivially. For $P$ a proper subgroup of $Q$, the relative trace map $\text{Tr}_P^Q : V^P \rightarrow V^Q$ is the linear map defined by

$$\text{Tr}_P^Q(v) = \sum_{g \in Q/P} gv,$$

where $Q/P$ is a left transversal of $P$ in $Q$. The map is independent of the choice of the left transversal. We observe that

$$\text{Tr}^Q(V) = \sum_{P \prec Q} \text{Tr}_P^Q(V^P)$$

is an $FN_G(Q)$-module on which $Q$ acts trivially. Therefore we can define the $F[N_G(Q)/Q]$-module called the Brauer quotient of $V$ with respect to $Q$ by

$$V(Q) = V^Q/\text{Tr}^Q(V).$$

An $FG$-module $V$ is called a $p$-permutation module if for every Sylow $p$-subgroup $P$ of $G$ there exists an $F$-linear basis $\mathcal{B}_P$ of $V$ that is permuted by $P$, or equivalently, $V$ has trivial Green source. It follows from the definition that the class of $p$-permutation modules is closed under taking direct
sum, direct summand, tensor product, restriction and induction. If \( V \) is an indecomposable \( FG \)-module and \( Q \) is a \( p \)-subgroup of \( G \), then \( V(Q) \neq 0 \) implies that \( Q \) is contained in a Green vertex of \( V \). Broué proved in [3] that the converse holds in the case of \( p \)-permutation modules.

**Theorem 2.5** ([3, Theorem 3.2]). Let \( V \) be an indecomposable \( p \)-permutation module and \( Q \) be a Green vertex of \( V \). Let \( P \) be a \( p \)-subgroup of \( G \), then \( V(P) \neq 0 \) if and only if \( P \leq \langle Q \rangle \) for some \( g \in G \).

Another important tool we will be using to study the signed \( p \)-Kostka numbers is the Brauer correspondence of \( p \)-permutation modules developed by Broué as follows.

**Theorem 2.6** ([3, Theorem 3.2 and 3.4]). An indecomposable \( p \)-permutation module \( V \) has Green vertex \( Q \) if and only if \( V(Q) \) is a projective \( F[N_G(Q)/Q] \)-module. Furthermore, we have the following statements.

(i) The Brauer map sending \( V \) to \( V(Q) \) is a bijection between the isomorphism classes of the indecomposable \( p \)-permutation \( FG \)-modules with Green vertex \( Q \) and the isomorphism classes of the indecomposable projective \( F[N_G(Q)/Q] \)-modules. Furthermore, \( \text{Inf}^{N_G(Q)}/Q}_{N_G(Q)} V(Q) \) is the Green correspondent of \( V \) with respect to the subgroup \( N_G(Q) \).

(ii) Let \( V \) be a \( p \)-permutation \( FG \)-module and let \( U \) be an indecomposable \( p \)-permutation module with Green vertex \( Q \). Then \( U \) is a direct summand of \( V \) if and only if \( U(Q) \) is a direct summand of \( V(Q) \). Moreover,

\[ [V : U] = [V(Q) : U(Q)]. \]

2.4. Brauer correspondences for signed Young and signed Young permutation modules.

In this subsection, we describe the Brauer correspondents of signed Young and signed Young permutation modules. Recall that, for each \( k \in \mathbb{N}_0 \), \( P_k \) is the fixed Sylow \( p \)-subgroup of \( \mathfrak{S}_k \) and \( N_k = N_{\mathfrak{S}_k}(P_k) \).

**Definition 2.7.** Let \( p \) be an odd prime, \( s, n \in \mathbb{N}_0 \) such that \( s \leq n \), \( U \) and \( V \) be \( F\mathfrak{S}_s \)- and \( F\mathfrak{S}_{n-s} \)-modules respectively, and let \( X \) be an \( FG \)-module. We define the \( F[G \wr \mathfrak{S}_n] \)-module

\[ G_X(U, V) = \text{Inf}^{G\mathfrak{S}_n}_{G\mathfrak{S}_s \times \mathfrak{S}_{n-s}} \left( (\text{Inf}^{G\mathfrak{S}_n}_{\mathfrak{S}_s} U) \boxtimes (\text{Inf}^{G\mathfrak{S}_{n-s}}_{\mathfrak{S}_{n-s}} V) \otimes \hat{X}^{\otimes n-s} \right). \]

For each \( k \in \mathbb{N}_0 \), we write

\[ G_k(U, V) = G_{\text{Def}^{N_k/P_k}_{N_k/P_k} \text{sgn}(N_k)}(U, V). \]

Notice that, if \( H \) acts trivially on \( X \), then subgroup \( H^n \) of the base group of \( G\mathfrak{S}_n \) acts trivially on \( G_X(U, V) \) and we have \( G_{\text{Def}^{G\mathfrak{S}_n}_{G\mathfrak{S}_{n}} X}(U, V) \cong \text{Def}^{G\mathfrak{S}_s}_{G\mathfrak{S}_s} G_X(U, V) \). Let \( \mathfrak{A}_k \) be the alternating subgroup of \( \mathfrak{S}_k \). For \( k \geq 2 \), since \( N_{\mathfrak{A}_k}(P_k) \) acts trivially on \( \text{sgn}(N_k) \) and via the identification \((N_k/P_k)/(N_{\mathfrak{A}_k}(P_k)/P_k)) \cong C_2 \), we therefore have

\[ \text{Def}^{(N_k/P_k)\mathfrak{S}_s}_{((N_k/P_k)/(N_{\mathfrak{A}_k}(P_k)/P_k))\mathfrak{S}_s} G_k(U, V) \cong G_{\text{Def}^{N_k}_{N_k/P_k} \text{sgn}(N_k)}(U, V) \cong G_{\text{sgn}(C_2)}(U, V) \cong G_2(U, V). \]

**Definition 2.8.** Let \((\alpha|\beta) \in \mathcal{C}^2(n) \) and \( \rho = ((\rho^i)^{n_i})_{i=0}^{\infty} \in \text{Seq}(n) \).

(i) We denote by \( \Lambda((\alpha|\beta), \rho) \) the set of tuples \((\gamma|\delta) = ((\gamma^i)^{n_i}|(\delta^i)^{n_i})_{i=0}^{\infty} \) such that

(a) for each \( i \in \mathbb{N}_0 \), \((\gamma^i), (\delta^i) \in \text{Seq}, \)

(b) \( \alpha = \sum_{i=0}^{\infty} p^i \cdot \gamma^{(i)}, \beta = \sum_{i=0}^{\infty} p^i \cdot \delta^{(i)}, \) and
(c) for each \( i \in \mathbb{N}_0, \) \( |\gamma^{(i)}| + |\delta^{(i)}| = n_i. \)

By our convention, \( \Lambda((\emptyset|\emptyset), \emptyset) = \{(\emptyset|\emptyset)\} \) where \( \emptyset = (\emptyset)_{i=1}^{\infty}. \)

(ii) For each \( (\gamma|\emptyset) \in \Lambda((\alpha|\beta), \emptyset), \) we define the \( F[N_{\rho}/P_{\rho}] \)-module

\[
W(\gamma|\emptyset) = \bigotimes_{i=0}^{\infty} G_{p^i}(M^{\gamma^{(i)}}, M^{\delta^{(i)}}).
\]

Again, we observe that this is indeed a finite outer tensor product because, for \( p^i > n, \) we have \( \gamma^{(i)} = \emptyset = \delta^{(i)} \).

Given \( (\alpha|\beta) \) and \( \rho \) as in the definition above, the following describes the structure of the Brauer quotient \( M(\alpha|\beta)(P_\rho) \).

**Proposition 2.9** ([9, Proposition 3.12]). Let \( \rho = ((p^i)^{n_i})_{i=0}^{\infty} \in \text{Seq}(n) \) and let \( (\alpha|\beta) \in \mathcal{S}^2(n) \). Then we have an isomorphism of \( F[N_{\rho}/P_{\rho}] \)-modules

\[
M(\alpha|\beta)(P_\rho) \cong \bigoplus_{(\gamma|\delta) \in \Lambda((\alpha|\beta), (\rho))} W(\gamma|\delta).
\]

To describe the Brauer correspondents of signed Young modules, we introduce the following notation.

**Definition 2.10.** For each \( (\lambda|\mu_p) \in \mathcal{S}^2_p(n), \) let

\[
\lambda = \sum_{i=0}^{\infty} p^i \cdot \lambda(i) \quad \text{and} \quad \mu = \sum_{i=0}^{\infty} p^i \cdot \mu(i),
\]

be the \( p \)-adic expansions of \( \lambda \) and \( \mu \) respectively and let \( n_i = |\lambda(i)| + |\mu(i-1)| \) for all \( i \in \mathbb{N}_0 \) (recall that, by convention, \( \mu(-1) = \emptyset \)).

(i) Let \( \rho : \mathcal{S}^2_p(n) \rightarrow \text{Seq}(n) \) be the function defined by

\[
\rho((\lambda|\mu_p)) = ((p^i)^{n_i})_{i=0}^{\infty} \in \text{Seq}(n).
\]

(ii) Let \( \rho = \rho((\lambda|\mu_p)) \). We define the \( F[N_{\rho}/P_{\rho}] \)-module

\[
Q(\lambda|\mu_p) := \bigotimes_{i=0}^{\infty} G_{p^i}(Y^{\lambda^{(i)}}, Y^{\mu^{(i-1)}}).
\]

By [7, 5.1(3) and 5.2(2)] and [5, Theorem 3.8(b) and Corollary 3.18], we obtain the following result.

**Theorem 2.11.** Let \( (\lambda|\mu_p) \in \mathcal{S}^2_p(n) \) and \( \rho = \rho((\lambda|\mu_p)) \). Then

(i) \( Y(\lambda|\mu_p) \) has Green vertex \( P_\rho, \) and

(ii) \( Y(\lambda|\mu_p)(P_\rho) \cong Q(\lambda|\mu_p) \) as \( F[N_{\rho}/P_{\rho}] \)-modules.

3. Klyachko’s multiplicity formula for signed \( p \)-Kostka numbers

Let \( p \) be an odd prime. We are now ready to exploit the Brauer correspondences for signed Young permutation and signed Young modules obtained in Proposition 2.9 and Theorem 2.11 to deduce the main result of this section (see Theorem 3.3) which is the signed version of the Klyachko’s multiplicity formula. We then use the result to deduce Steinberg Tensor Product Theorem for the irreducible modules for the Schur superalgebras.

We begin with a lemma.
Lemma 3.1. Let $\gamma, \delta$ be partitions and $\lambda, \mu$ be $p$-restricted partitions such that $n = |\lambda| + |\mu| = |\gamma| + |\delta|$. Then

$$[G_2(M^\gamma, M^\delta) : G_2(Y^\lambda, Y^\mu)] = \begin{cases} k_{\gamma, \lambda}k_{\delta, \mu} & \text{if } |\gamma| = |\lambda| \text{ and } |\delta| = |\mu|, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $M^\gamma \cong \bigoplus_{\alpha \supseteq \gamma} k_{\gamma, \alpha}Y^\alpha$ and $M^\delta \cong \bigoplus_{\beta \supseteq \delta} k_{\delta, \beta}Y^\beta$. Then, by Definition 2.7,

$$G_2(M^\gamma, M^\delta) \cong \bigoplus_{\alpha \supseteq \gamma, \beta \supseteq \delta} k_{\gamma, \alpha}k_{\delta, \beta} \cdot G_2(Y^\alpha, Y^\beta).$$

Since $\lambda, \mu$ are $p$-restricted partitions, by [9, Lemma 4.4 and Proposition 4.5], we see that $G_2(Y^\lambda, Y^\mu) \cong G_2(Y^\alpha, Y^\beta)$ if and only if $\lambda = \alpha$ and $\mu = \beta$. Therefore, we obtain our desired equality. Alternatively, using [22, Propositions 1.2 and 5.1], we see that each $F[C_2 \wr S_n]$-module $G_2(Y^\alpha, Y^\beta)$ is indecomposable. The complexity of $G_2(Y^\alpha, Y^\beta)$ is the sum of the complexities of $Y^\alpha$ and $Y^\beta$ and therefore is a projective module (i.e., has complexity zero) if and only if both $\alpha$ and $\beta$ are $p$-restricted. Now use [5, A.1].

In view of Lemma 3.1, we consider the following subset of $\Lambda((\alpha|\beta), \rho(\lambda|\mu))$.

**Definition 3.2.** Suppose that $(\alpha|\beta) \in \mathcal{C}^2(n)$ and $(\lambda|\mu) \in \mathcal{P}^2(n)$. Let $\Lambda^\circ((\alpha|\beta), (\lambda|\mu))$ be the subset of $\Lambda((\alpha|\beta), \rho(\lambda|\mu))$ consisting of $(\gamma|\delta)$ such that, for all positive integers $i$,

(a) $|\gamma(i)| = |\lambda(i)|$ and $\lambda(i) \trianglerighteq \rho(\gamma(i))$,

(b) $|\delta(i)| = |\mu(i-1)|$ and $\mu(i-1) \trianglerighteq \rho(\delta(i))$, and

(c) $(\lambda(0)|\emptyset) \trianglerighteq (\rho(\gamma(0))|\rho(\delta(0)))$.

The main result in this section is the following Klyachko’s multiplicity formula for signed $p$-Kostka numbers. The formula basically says that the computation of signed $p$-Kostka numbers can be reduced to the computation of $p$-Kostka numbers and signed $p$-Kostka numbers of the form $k_{(\alpha|\beta), (\lambda|\emptyset)}$ where $\lambda$ is $p$-restricted, i.e., the multiplicity of each projective indecomposable module as a summand of $M(\alpha|\beta)$.

**Theorem 3.3.** Let $(\alpha|\beta), (\lambda|\mu) \in \mathcal{P}^2(n)$. We have

$$k_{(\alpha|\beta), (\lambda|\mu)} = \sum_{(\gamma|\delta) \in \Lambda^\circ((\alpha|\beta), (\lambda|\mu))} k_{(\gamma(0)|\delta(0)), (\lambda(0)|\emptyset)} \prod_{i=1}^{\infty} k_{\gamma(i), \lambda(i)}k_{\delta(i), \mu(i-1)}.$$

Proof. Let $\rho = \rho(\lambda|\mu)$. By Theorem 2.6(ii), since $P_\rho$ is a Green vertex of $Y(\lambda|\mu)$ by Theorem 2.11(i), we have $[M(\alpha|\beta) : Y(\lambda|\mu)] = [M(\alpha|\beta)(P_\rho) : Y(\lambda|\mu)(P_\rho)]$ and therefore by Proposition 2.9 and Theorem 2.11(ii), we have

$$[M(\alpha|\beta) : Y(\lambda|\mu)] = \sum_{(\gamma|\delta) \in \Lambda((\alpha|\beta), \rho)} \prod_{i=0}^{\infty} (G_1(M^{\gamma(i)}, M^{\delta(i)}) : G_1(Y^{\lambda(i)}, Y^{\mu(i-1)})).$$

Fix $(\gamma|\delta) \in \Lambda((\alpha|\beta), \rho)$. When $i = 0$, for

$$[G_1(M^{\gamma(0)}, M^{\delta(0)}) : G_1(Y^{\lambda(0)}, Y^{\emptyset})] = [M(\gamma(0)|\delta(0)) : Y^{\lambda(0)}] = k_{(\gamma(0)|\delta(0)), (\lambda(0)|\emptyset)} \neq 0,$$
it is necessary that $(\lambda(0)|\emptyset) \supseteq (\varphi(\gamma(0))|\varphi(\delta(0)))$. When $i \geq 1$, by Equation 2.1, Lemmas 2.1 and 3.1, we have

$$\left[\mathcal{G}_\rho(M^{\gamma(i)}, M^{\delta(i)}) : \mathcal{G}_\rho(Y^{\lambda(i)}, Y^{\mu(i-1)})\right]$$

$$= \left[\mathcal{G}_2(M^{\gamma(i)}, M^{\delta(i)}) : \mathcal{G}_2(Y^{\lambda(i)}, Y^{\mu(i-1)})\right]$$

$$= \begin{cases} k_{\gamma(i), \lambda(i)} k_{\delta(i), \mu(i-1)} & \text{if } |\gamma(i)| = |\lambda(i)| \text{ and } |\delta(i)| = |\mu(i-1)|, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $k_{\gamma(i), \lambda(i)} k_{\delta(i), \mu(i-1)} = 0$ unless $\lambda(i) \supseteq \varphi(\gamma(i))$ and $\mu(i-1) \supseteq \varphi(\delta(i))$. This shows that we only need to consider the subset $\Lambda^0((\alpha|\beta), (\lambda|\mu))$ in Equation 3.1. The proof is now complete. □

We end this subsection with the following remark about Klyachko’s multiplicity formula [21, Corollary 9.2] and Steinberg Tensor Product Theorem.

**Corollary 3.4** (Klyachko’s multiplicity formula). Let $\alpha, \lambda \in \mathcal{P}(n)$. We have

$$k_{\alpha, \lambda} = \sum_{i=0}^{\infty} \prod_{j=0}^{\infty} k_{\gamma(i), \lambda(i)}$$

where the sum is taken over all $\gamma(0), \gamma(1), \ldots \in \text{Seq}$ such that $\alpha = \sum_{i=0}^{\infty} p^i \cdot \gamma(i)$ and $\lambda(i) \supseteq \varphi(\gamma(i))$ for all $i \in \mathbb{N}_0$.

**Proof.** Take $\beta = \emptyset = \mu$ in Theorem 3.3, the condition Definition 3.2(b) is trivially satisfied and we have only $\lambda(i) \supseteq \varphi(\gamma(i))$ for all $i \in \mathbb{N}_0$. □

**Corollary 3.5** (Steinberg Tensor Product Theorem). Let $(\lambda|\mu) \in \mathcal{P}_p^2(n)$ and $r$ be the maximal such that either $\lambda(r) \not\supseteq \emptyset$ or $\mu(r-1) \not\supseteq \emptyset$. Then

$$L(\lambda|\mu) \cong L(\lambda(0)|\emptyset) \otimes L(p\lambda(1)|\mu(0)) \otimes \cdots \otimes L(p^r\lambda(r)|p^r\mu(r-1)).$$

**Proof.** Let $N = L(\lambda(0)|\emptyset) \otimes L(p\lambda(1)|\mu(0)) \otimes \cdots \otimes L(p^r\lambda(r)|p^r\mu(r-1))$ and $\rho = \rho(\lambda|\mu)$. Let $\text{GL}(m|n)(F)$ denote the super general linear group as defined in [4, Section 2]. The Frobenius map induces a map $\mathcal{F} : \text{GL}(m|n)(F) \rightarrow \text{GL}(m)(F) \times \text{GL}(n)(F)$. By [4, Remark 4.6(iii)], $\mathcal{F}^*(L(\sigma) \boxtimes L(\tau)) \cong L(p\sigma|p\tau)$ is the irreducible $S_F(m|n)$-module obtained by inflating through $\mathcal{F}$, where $L(\sigma), L(\tau)$ are the irreducible $S_F(m, s)$- and $S_F(n, s)$-modules of highest weights $\sigma, \tau$ respectively. Since $L(\sigma) \boxtimes L(\tau)$ has weights of the form $(\gamma|\delta)$, $\mathcal{F}^*(L(\sigma) \boxtimes L(\tau))$ has weights of the form $(p^i\gamma(i)|p^i\delta(i))$ of which the highest weight is $(p^i\lambda(i)|p^i\mu(i-1))$. Furthermore,

$$\dim_F L(p^i\lambda(i)|p^i\mu(i-1))(p^i\gamma(i)|p^i\delta(i)) = \dim_F L(p^i\lambda(i)|p^i\gamma(i)) \dim_F L(p^i\mu(i-1)|p^i\delta(i))$$

$$= k_{\gamma(i), \lambda(i)} k_{\delta(i), \mu(i-1)}.$$ 

When $i = 0$, $\dim_F L(\lambda(0)|\emptyset)(\gamma(0)|\delta(0)) = k_{(\alpha|\delta),(\lambda(0)|\emptyset)}$. Therefore, for each $(\alpha|\beta) \in \mathcal{P}^2(n)$, by Theorem 3.3, we have

$$\dim_F N^{\alpha|\beta} = \sum_{(\gamma|\delta) \in \Lambda((\alpha|\beta), \rho)} \prod_{i=0}^{r} \dim_F L(p^i\lambda(i)|p^i\mu(i-1))(p^i\gamma(i)|p^i\delta(i))$$
result is Theorem 4.3 and we use it to deduce Corollaries 4.4 and 4.5. The proof of Theorem 4.3 follows easily by noting that the parts (i) and (ii) follow from Definition 3.2. The fact that \( |\gamma| = p \) implies \( |\delta| = p \) and the map \( \phi : \Lambda^\circ((\alpha|\beta), (\lambda|\mu)) \to \Lambda^\circ((\alpha|\emptyset), (\lambda|\emptyset)) \times \Lambda^\circ((\beta|\emptyset), (\mu|\emptyset)) \)

given by \( \phi((\gamma|\delta)) = ((\gamma|\emptyset), (\delta|\emptyset)) \) is bijective. Furthermore, if \( \delta^{(0)} = \emptyset \) for all \( (\gamma|\delta) \in \Lambda^\circ \) and the map \( \phi : \Lambda^\circ((\alpha|\beta), (\lambda|\mu)) \to \Lambda^\circ((\alpha|\emptyset), (\lambda|\emptyset)) \times \Lambda^\circ((\beta|\emptyset), (\mu|\emptyset)) \)

given by \( \phi((\gamma|\delta)) = ((\gamma|\emptyset), (\delta|\emptyset)) \) is bijective. Furthermore, if \( \delta^{(0)} = \emptyset \) for all \( (\gamma|\delta) \in \Lambda^\circ \) and \( \Lambda^\circ \neq \emptyset \) then \( |\beta| = p|\mu| \).

Proof. Most parts of parts (i) and (ii) follow from Definition 3.2. The fact that \( \phi \) is a bijection follows easily by noting that \( \lambda(i) \trianglerighteq \varphi(\gamma^{(i)}) \) and \( \mu(i) = \mu(i-1) \trianglerighteq \varphi(\delta^{(i)}) \) for all \( i \in \mathbb{N}_0 \).

We have the following immediate proposition. This serves as a motivation why we are interested in the case when \( |\beta| = p|\mu| \).

**Proposition 4.2.** Let \((\alpha|\beta), (\lambda|\mu) \in \mathcal{P}^2(\mathbb{N}_0)\) for some nonnegative integer \( m \). If \( \lambda(0) = \emptyset \), i.e., \( Y(\lambda|\mu) \) lies in the principal block of \( \mathcal{S}_{mp} \), then \( k_{(\alpha|\beta), (\lambda|\mu)} = 0 \) unless \( |\beta| = p|\mu| \).

Proof. By [14, Corollary 5.2.9], the signed Young module \( Y(\lambda|\mu) \) belongs to the block labelled by the \( p \)-core of \( \lambda \). Therefore \( \lambda(0) = \emptyset \) is equivalent to \( Y(\lambda|\mu) \) lies in the principal block of \( F\mathcal{S}_{mp} \). By Lemma 4.1(i) and (ii), we see that \( \Lambda^\circ((\alpha|\beta), (\lambda|\mu)) = \emptyset \) unless \( |\beta| = p|\mu| \). This translates to our result using Theorem 3.3.

We are now ready to state the main of this section. It generalizes Theorem 2.4 to the signed \( p \)-Kostka numbers.

**Theorem 4.3.** Let \((\alpha|\beta), (\lambda|\mu) \in \mathcal{P}^2(\mathbb{N}_0)\) such that \( |\beta| = p|\mu| \) and the pairs \((\alpha, \lambda)\) and \((\beta, \mu)\) admit horizontal \( r \)- and \( s \)-row cuts respectively. Then

\[
k_{(\alpha|\beta), (\lambda|\mu)} = k_{\alpha r, \lambda r} k_{\beta s, \mu s} k_{\alpha b r, \lambda b s} k_{\beta b s, \mu b s}.
\]
Proof. By Lemma 4.1(ii), Theorems 2.4 and 3.3, we have

\[ k_{(\alpha|\beta), (\lambda|\mu)} = \sum_{(\gamma|\delta) \in \Lambda^2((\alpha|\beta), (\lambda|\mu))} k_{\gamma(i), \lambda(i)} \prod_{i=1}^{\infty} k_{\gamma(i), \lambda(i)} \prod_{i=1}^{\infty} k_{\delta(i), \mu(i-1)} \]

\[ = \left( \sum_{(\gamma|\delta) \in \Lambda^2((\alpha|\beta), (\lambda|\mu))} \prod_{i=0}^{\infty} k_{\gamma(i), \lambda(i)} \right) \left( \sum_{(\delta|\zeta) \in \Lambda^2((\beta|\delta), (\mu|\nu))} \prod_{i=0}^{\infty} k_{\delta(i), \mu(i-1)} \right) \]

\[ = k_{\alpha, \lambda} k_{\beta, \mu} \]

\[ = k_{\alpha(\lambda-\alpha), \lambda} k_{\beta, \mu} k_{\lambda-\alpha, \beta} k_{\mu, \lambda-\alpha} \].

Recall that the $p$-Kostka and signed $p$-Kostka matrices are denoted by $K_n$ and $K_n^\pm$ with respect to fixed total orders $\succ_n$ and $\succ$ on $\mathcal{P}(n)$ and $\mathcal{P}_p^2(n)$ respectively such that the restriction of $\succ$ to $\mathcal{P}(n)$ is $\succ_n$ and $\succ_n$ refines the dominant order $\triangleright$. Also, we denote the Kronecker product of two matrices $A, B$ as

\[ A \otimes B = \begin{pmatrix} A_{11} B & \cdots & A_{1n} B \\ \vdots & \ddots & \vdots \\ A_{m1} B & \cdots & A_{mn} B \end{pmatrix} \]

if $A$ is an $(m \times n)$-matrix. Theorem 4.3 implies the following corollaries. The first one gives a proof of our Main Result B from the introduction.

Corollary 4.4.

(i) Let $(\alpha|\beta), (\lambda|\mu) \in \mathcal{P}_p^2(n)$ such that $|\beta| = p|\mu|$. We have

\[ k_{(\alpha|\beta), (\lambda|\mu)} = k_{\alpha, \lambda} \cdot k_{\beta, \mu}. \]

In this case, the signed $p$-Kostka number $k_{(\alpha|\beta), (\lambda|\mu)}$ is nonzero if and only if there exists expansions (not necessarily $p$-adic expansions) $\alpha = \sum_{i=0}^{\infty} p^i \gamma(i)$ and $\beta = \sum_{i=1}^{\infty} p^i \delta(i)$ such that, for all $i \in \mathbb{N}_0$, $|\lambda(i)| = |\gamma(i)|$, $\lambda(i) \triangleright \varphi(\gamma(i))$, $|\mu(i-1)| = |\delta(i)|$ and $\mu(i-1) \triangleright \varphi(\delta(i))$.

(ii) Let $(\alpha|\beta), (\lambda|\mu) \in \mathcal{P}_p^2(n)$ such that $|\alpha| = |\lambda| - |\lambda(0)|$. We have

\[ k_{(\alpha|\beta), (\lambda|\mu)} = k_{\alpha, \lambda - \lambda(0)} \cdot k_{\beta, \varphi(\lambda(0)) + \mu}. \]

(iii) The signed $p$-Kostka matrix $K_n^\pm$ is lower unitriangular of the form

\[ \begin{pmatrix} K_n & * & K_{n-p} & * & K_2 \otimes K_{n-2p} \\ * & * & * & * & K_3 \otimes K_{n-3p} \\ * & * & * & * & \ddots \\ * & * & * & * & \vdots \\ * & * & * & * & \cdots K_d \otimes K_{n-dp} \end{pmatrix} \]

where $0 \leq n - dp \leq p - 1$ and, for each $0 \leq s \leq d$, $K_s \otimes K_{n-sp}$ is the Kronecker product of the $p$-Kostka matrices $K_s$ and $K_{n-sp}$.
Proof. For part (i), the equality is obtained by taking $r = 0 = s$ in Theorem 4.3. Therefore, $k_{(\alpha|\beta),(\lambda|\mu)} \neq 0$ if and only if $k_{\alpha,\lambda} \neq 0 \neq k_{\beta,\mu}$ and hence the second assertion follows from [6, Remark, page 55] noting that $(p_\mu)(i) = \mu(i-1)$.

For part (ii), by [5, Theorem 3.18] and part (i),

$$k_{(\alpha|\beta),(\lambda|\mu)} = k_{\beta|\alpha}(\lambda(0)+\mu(\lambda-\lambda(0))) = k_{\beta,\mu}(\lambda(0)+\mu)|k_{\alpha,\lambda}(\lambda(0)) = k_{\alpha,\lambda}(\lambda(0)) - K_{\alpha,\lambda}(\lambda(0)).$$

For part (iii), since the total order $\succ$ on $P^2_p(n)$ refines the dominance order, $K^n_\alpha$ is lower unitriangular. Suppose that $(\zeta|p\eta), (\lambda|p\mu) \in P^2_p(n)$ and $s = |\mu| = |\eta|$. By parts (i) and (ii), we have

$$(K_{\alpha})_{(\zeta|p\eta), (\lambda|p\mu)} = k_{\zeta|p\eta}(\lambda(0)) = k_{\zeta,\lambda}(\lambda(0)) = (K_{\alpha})(\lambda(0)).$$

We have chosen the total order $\succ$ so that the signed $p$-Kostka matrix has the form as in the statement. \qed

Corollary 4.5. Let $B_0$ be the principal block of $\frak{S}_{mp}$ and $(\alpha|\beta) \in P^2(mp)$. Then

$$M(\alpha|\beta)_{B_0} \cong \bigoplus_{(\lambda|p\mu) \in P^2_p(mp), \lambda(0) = \emptyset, |\beta| = |\mu|} Y(\lambda|p\mu) \oplus^{k_{\alpha,\lambda}(\lambda|p\mu)}.$$ 

Proof. Let $Y(\lambda|p\mu)$ belong to $B_0$, i.e., $\lambda(0) = \emptyset$, and suppose that $k_{(\alpha|\beta),(\lambda|p\mu)} > 0$. Our result now follows from Proposition 4.2 and Corollary 4.4(i). \qed

5. THE SIGNED $p$-KOSTKA NUMBERS $k_{(\alpha|\beta),(\lambda|p\mu)}$ IN THE GENERAL CASE

In this section, we examine the validity of Theorem 4.3 without the condition $|\beta| = p|\mu|$. This general case is slightly more complicated and we need a separate treatment. More precisely, we need to examine the Specht series for skew representations introduced by James-Peel in [19].

We now describe the minimal ingredient from [19] that we shall require in the proof of Lemma 5.1. A diagram is a finite subset of $Z \times Z$. Let $\alpha$ and $\beta$ be partitions such that $\alpha_i \geq \beta_i$ for all $i \in \mathbb{N}$. We have the skew diagram

$$D = [\alpha \setminus \beta] = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq \ell(\alpha), \beta_i < j \leq \alpha_i\}.$$ 

Let $-D$ (or $-\alpha \setminus \beta$) be the diagram $\{(i, -j) : (i, j) \in D\}$ and let $n = |\alpha| - |\beta|$. There is a $F\frak{S}_n$-module $S^{\alpha \setminus \beta}$ (or $S^D$) labelled by the diagram $D$ called a skew representation. In the case when $\beta = (\lambda^s_{s+1})$ for some $s \in \mathbb{N}_0$, we have that $S^{\alpha \setminus \beta}$ is isomorphic to $\text{Ind}^\frak{S}_n \times \frak{S}_{n-s} \rightarrow S^{\alpha \setminus \beta} \otimes S^{\alpha_{s+1}^s}$ where $d = |\alpha_T| - |\beta|$ (see [19, Page 345]). There is a unique node $(r, s) \in D$ such that, if $(i, j) \in D$, then $i \geq r$ and $j \leq s$. A node $(i, j) \in D$ is deposited in $D$ if $(i', j') \in D$ whenever $r \leq i' \leq i$ and $j \leq j' \leq s$. All the operations in the next paragraph fix the node $(r, s)$.

Let $D$ be a skew diagram and $(r, s) \in D$ be the unique node as in the previous paragraph. The operation $X$ is defined so that $D_X$ is a diagram obtained from $D$ by permuting the rows or columns of $D$. In this case, $S^D \cong S^{D_X}$ (see [19, 2.1]). Let $(i_1, j_1), (i_2, j_2) \in D$ but $(i_1, j_1) \notin D$ and $(i_2, j_2) \notin D$. The diagram $D^I$ (respectively, $D^K$), with respect to the nodes $(i_1, j_1)$ and $(i_2, j_2)$, is defined as the diagram obtained from $D$ by moving the nodes $(i_1, j_1) \in D$ such that $(i_2, j_2) \notin D$ to $(i_2, j)$ (respectively, $(i, j_1) \in D$ such that $(i, j_2) \notin D$ to $(i, j_2)$) and keeping the remaining nodes; namely, $D^I$ and $D^K$ are obtained from $D$ by moving all the nodes in the $i_1$th row (respectively, $j_1$th column) to the respective vacant positions in $i_2$th row (respectively, $j_2$th column). Suppose
there exists \( i^* > r \) such that \((i^*, j) \notin D\) for any \( j \in \mathbb{Z} \) but there exists a node \((i_1, j_1) \in D\) for some \( i_1 > i^* \) or there exists \( j^* < s \) such that \((i, j^*) \notin D\) for any \( i \in \mathbb{Z} \) but there exists a node \((i_1, j_1) \in D\) for some \( j_1 < j^* \), i.e., we have the condition [19, Theorem 4.7 (ii)–(iii)]. Let \( Y \) be the operation so that \( D^Y \) is the diagram obtained from \( D \) by exchanging the \((i + 1)\)th row (respectively, \((j + 1)\)th column) of with the \(i\)th row (respectively, \(j\)th column) successively in the order of and for each \( i = i^*, i^* + 1, \ldots \) (respectively, \( j = j^*, j^* + 1, \ldots \)). Literally, \( D^Y \) is obtained from \( D \) by deleting the \(i^*\)th row (respectively, \(j^*\)th column). In this case, \( \mathcal{S}^D \cong \mathcal{S}^{D^Y} \) (see [19, Lemma 4.8]).

**Lemma 5.1.** Let \( \lambda \in \mathcal{P}(n) \), \( \lambda = \xi \# \zeta \) for some partitions \( \xi, \zeta \) and \( d = |\xi| \). Then

(i) \( \mathcal{S}^\lambda \) is isomorphic to a submodule of the induced module \( \text{Ind}_{S_d \times S_{n-d}}^{S_n} (S^\xi \boxtimes S^\zeta) \), and

(ii) \( \left[ \text{Ind}_{S_d \times S_{n-d}}^{S_n} (Y^\xi \boxtimes Y^\zeta) : Y^\lambda \right] = 1. \)

**Proof.** We use the notation as in the paragraphs prior to the statement. Let \( k = \ell(\xi), \beta = (\zeta_1^k), \alpha = (\xi + \beta) \# \zeta \) and let \( D \) be the diagram of \( \alpha \backslash \beta \). Then \( S^\alpha \backslash \beta \) is a skew representation isomorphic to \( \text{Ind}_{S_d \times S_{n-d}}^{S_n} (S^\xi \boxtimes S^\zeta) \). By [19, Theorems 4.15 and 4.16], it suffices to show that we get the diagram \(-[\lambda]\) by successive application of the operations either \( X \), \( Y \) or \( K \) (and without \( I \)). We refer to Figure A for the following explanation. The red dot denotes the unique node \((r, s)\) in each diagram. The pair of blue dots in each diagram denotes the pair satisfying the operation \( K \). We first apply the operation \( X \) where \(-\xi \) and \(-\zeta \) denote the mirror images of \( \xi \) and \( \zeta \) respectively in the obvious way. All the nodes in \(-\xi \) are deposited. Applying the process \( K \), the most right column of \(-\zeta \) moves to the bottom of the most right column of \(-\xi \). We now apply the operation \( Y \) to rearrange the columns, i.e., deleting the empty column. Now the pair of blue nodes satisfies the operation \( K \). We can repeat the arguments for \( \zeta_1 \) times, since, by assumption \( \xi_k \geq \zeta_1 \), and get the diagram \(-\lambda \). Since we applied the operations \( X \), \( Y \) or \( K \) each time, the Specht module \( \mathcal{S}^\lambda \) appears a submodule in some Specht series of \( \text{Ind}_{S_d \times S_{n-d}}^{S_n} (S^\xi \boxtimes S^\zeta) \). The proof of part (i) is now complete.

![Figure A](image-url)
Therefore, we conclude that $S^\lambda \subseteq \text{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_{n-d}}(S^\xi \boxtimes S^\zeta) \subseteq \text{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_{n-d}}(Y^\xi \boxtimes Y^\zeta) | \text{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_{n-d}}(M^\xi \boxtimes M^\zeta) = M^\lambda$.

Therefore, we conclude that

\[ 1 \leq \left[ \text{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_{n-d}}(Y^\xi \boxtimes Y^\zeta) : Y^\lambda \right] \leq \left[ M^\lambda : Y^\lambda \right] = 1. \]

\[ \square \]

**Corollary 5.2.** Let $(\gamma|\delta), (\lambda|\varnothing) \in \mathcal{P}^2(n)$ and $\gamma', \gamma'', \delta', \delta''$ be partitions such that $\gamma = \gamma' \cup \gamma''$, $\delta = \delta' \cup \delta''$, $|\lambda^T_r| = |\gamma'| + |\delta'|$ and $|\lambda^{B_r}| = |\gamma''| + |\delta''|$ for some nonnegative integer $r$. Then

\[ k_{(\gamma|\delta),(\lambda|\varnothing)} \geq k_{(\gamma'|\delta'),(\lambda^T_r|\varnothing)}k_{(\gamma''|\delta''),(\lambda^{B_r}|\varnothing)}. \]

**Proof.** Let $a = k_{(\gamma'|\delta'),(\lambda^T_r|\varnothing)}$ and $b = k_{(\gamma''|\delta''),(\lambda^{B_r}|\varnothing)}$. Since $M(\gamma|\delta) \cong \text{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_{n-d}}(M(\gamma'|\delta') \boxtimes M(\gamma''|\delta''))$ where $d = |\lambda^T_r|$, we have

\[ \left( \text{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_{n-d}}(Y^{\lambda^T_r} \boxtimes Y^{\lambda^{B_r}}) \right)^{\oplus ab} | M(\gamma|\delta). \]

By Lemma 5.1(ii), we have

\[ [M(\gamma|\delta) : Y^\lambda] \geq ab \cdot \left[ \text{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_{n-d}}(Y^{\lambda^T_r} \boxtimes Y^{\lambda^{B_r}}) : Y^\lambda \right] = ab. \]

\[ \square \]

Corollary 5.2 is a special case of our main result Theorem 5.6. To prove the general version Theorem 5.6, we require the following key lemmas.

**Lemma 5.3.** Let $\lambda$ be a partition, $r \in \mathbb{N}_0$, $b_i = \lambda(i)_{r+1}$ for each nonnegative integer $i$ and $b = \sum_{i=0}^{\infty} p^i b_i = \lambda_{r+1}$. Then, for any nonnegative integer $i$, we have both

\[ \lambda^{B_r}(i) = \lambda(i)^{B_r}, \]

\[ (\lambda^T_r - (b^r))(i) = \lambda(i)^T_r - (b^r). \]

**Proof.** Since $\lambda(i)$ is a $p$-restricted partition, it is clear that $\lambda(i)^{B_r}$ is also a $p$-restricted partition. Observe that $\lambda^{B_r} = \sum_{i=0}^{\infty} p^i \lambda(i)^{B_r}$. This shows the first equality. For the second equality, first observe that

\[ \sum_{i=0}^{\infty} p^i \left( \lambda(i)^T_r - (b^r) \right) = \sum_{i=0}^{\infty} (p^i \lambda(i)^T_r - ((p^i b_i)^r)) = \lambda^T_r - (b^r). \]

It remains to show that, for each $i$, $\gamma := \lambda(i)^T_r - ((b_i)^r)$ is a $p$-restricted partition. Notice that $\gamma$ is a $p$-restricted partition because $\gamma_r = \lambda(i)^T_r - b_i = \lambda(i)^T_r - \lambda(i)_{r+1}^T$ and, for each $1 \leq j \leq r - 1$, we have

\[ \gamma_j - \gamma_{j+1} = (\lambda(i)^T_j - b_i) - (\lambda(i)^T_{j+1} - b_i) = \lambda(i)^T_j - \lambda(i)_{j+1}^T. \]

\[ \square \]

**Lemma 5.4.** Let $\lambda \in \mathcal{P}(n), k \in \mathbb{N}_0$ and $\gamma \in \mathcal{C}(n)$ such that $\ell(\gamma) \leq k$. If $\lambda \supseteq \varphi(\gamma)$ then $\gamma_j \geq \lambda_k$ for all $1 \leq j \leq k$. 

Proof. By assumption, \( s := \ell(\varphi(\gamma)) \leq k \). Let \( \varphi(\gamma) = (a_1, \ldots, a_k) \) (assume that \( a_i = 0 \) if \( i > s \)) and suppose on the contrary that \( a_i < \lambda_k \) for some \( 1 \leq i \leq k \). Then

\[
n - (k - i + 1)\lambda_k < n - \sum_{j=i}^{k} a_j = \sum_{j=1}^{i-1} a_j \leq \sum_{j=1}^{i-1} \lambda_j \leq n - \sum_{j=i}^{k} \lambda_j \leq n - (k - i + 1)\lambda_k.
\]

Alternatively, one observes that the Young diagram of \( \varphi(\gamma) \) is obtained from \( \lambda \) by successively moving a node from a higher row to a lower row such that, at each step, it remains as a partition. Therefore, since we cannot move a node lower than row \( k \), the block \(((\lambda_k)^k)\) in \( \lambda \) will remain in each step and therefore in \( \varphi(\gamma) \).

\( \square \)

Lemma 5.5. Let \((\alpha|\beta), (\lambda|\mu) \in \mathcal{P}^2(n)\) such that the pairs \((\alpha, \lambda)\) and \((\beta, \mu)\) admit horizontal \( r \)- and \( s \)-row cuts respectively. For each \( i \in \mathbb{N}_0 \), let \( b_i = \lambda(i)_{r+1}, c_i = \mu(i)_{s+1}, b = \sum_{i=0}^{\infty} b_i = \lambda_{r+1} \) and \( c = \sum_{i=0}^{\infty} B^{j+1} c_i = p\mu_{s+1} \).

(i) We have that both \( \alpha^{T_r} - (b^r) \) and \( \beta^{T_s} - (c^s) \) are partitions.

(ii) Let \( \Gamma_1 = \lambda \circ \alpha^{T_r} - (b^r), \lambda^{T_r} - (b^r) \), \( \Gamma_2 = \lambda \circ (\beta^{T_s} - (c^s), p\mu^{T_s} - (c^s)) \), \( \Gamma_3 = \lambda \circ (\alpha^{\beta_r}|\beta^{B_r}), (\lambda^{B_r}|p\mu^{B_r}) \), \( \Gamma_4 = \lambda \circ (\alpha|\beta), (\lambda|\mu) \).

\( \Gamma = \Gamma_1 \times \Gamma_2 \times \Gamma_3 \), and, for each \( \sigma \in \Gamma_1, \tau \in \Gamma_2 \) and \( (\gamma|\delta) \in \Gamma_3 \), let \( \eta^{(i)} = (\sigma^{(i)} \leq_r + (b_i)^r) \gamma^{(i)} \) and \( \theta^{(i)} = (\tau^{(i)} \leq_s + (c_i-1)^s) \delta^{(i)} \) (assume \( c_{-1} = 0 \) so that \( \theta^{(0)} = \emptyset \leq_s \delta^{(0)} \)). Then the map \( \iota : \Gamma \to \Gamma_4 \) defined as \( \iota(\sigma, \tau, (\gamma|\delta)) = (\eta|\theta) \) is injective.

Proof. Part (i) follows from Lemma 5.4. For part (ii), it suffices to check that \((\eta|\theta) \in \Gamma_4 \). Notice that \( \sigma^{(i)} = \sigma^{(i)} \# \emptyset \) and \( \tau^{(i)} = \tau^{(i)} \# \emptyset \). We will be using Lemmas 5.3 and 5.4 repeatedly. First of all, we have

\[
\sum_{i=0}^{\infty} b_i \eta^{(i)} = \sum_{i=0}^{\infty} b_i \sigma^{(i)} + (b_i) \sum_{i=0}^{\infty} b_i \gamma^{(i)} = (\alpha^{T_r} - (b^r) + (b^r)) \alpha^{B_r} = \alpha,
\]

\[
\sum_{i=0}^{\infty} b_i \theta^{(i)} = \sum_{i=0}^{\infty} b_i \tau^{(i)} + (c_i-1)^s \sum_{i=0}^{\infty} b_i \delta^{(i)} = (\beta^{T_s} - (c^s) + (c^s)) \beta^{B_s} = \beta,
\]

\[
|\eta^{(0)}| + |\theta^{(0)}| = |\lambda(0)^{T_r}| + |\gamma^{(0)}| + |\delta^{(0)}| = |\lambda(0)^{T_r}| + |\lambda^{B_r}(0)| = |\lambda(0)|,
\]

and, for \( i \geq 1 \),

\[
|\eta^{(i)}| = |(\sigma^{(i)} \leq_r + (b_i)^r) \gamma^{(i)}| = |(\lambda^{T_r} - (b^r)) (i)| + rb_i + |\lambda^{B_r}(i)|
\]

\[
= |\lambda(i)^{T_r}| - |(b_i)^r| + rb_i + |\lambda(i)^{B_r}| = |\lambda(i)|,
\]

\[
|\theta^{(i)}| = |(\tau^{(i)} \leq_s + (c_i-1)^s) \delta^{(i)}| = |(p\mu^{T_s} - (c^s))(i)| + sc_{i-1} + |(p\mu^{B_s})(i)|
\]

\[
= |(p\mu)(i)^{T_s}| - |(c_i-1)^s| + sc_{i-1} + |(p\mu)(i)^{B_s}| = |(p\mu)(i)| = |\mu(i-1)|.
\]

Since \( b_i = \lambda(i)_{r+1} \), we see that \( \varphi(\eta^{(i)}) = \varphi(\sigma^{(i)} \leq_r + (b_i)^r) \varphi(\gamma^{(i)}) \). For all \( 1 \leq k \leq r \) and \( \ell > r \), we have

\[
\sum_{j=1}^{k} \lambda(i) j = \sum_{j=1}^{k} \lambda(i) j = kb_i + \sum_{j=1}^{k} (\lambda^{T_r} - (b^r))(i) j \geq \sum_{j=1}^{k} \varphi(\sigma^{(i)} \leq_r + (b_i)^r) j = \sum_{j=1}^{k} \varphi(\eta^{(i)}) j,
\]

\( \square \)
\[
\sum_{j=1}^{\ell} \lambda(i)_j = |\lambda(i)^T_r| + \sum_{j=1}^{\ell-r} \lambda(i)_{B_r}^T = |(\lambda^T - (b^r))(i)| + rb_i + \sum_{j=1}^{\ell-r} \lambda(B_r(i))_j \\
= |\sigma(i)_{s_r}| + rb_i + \sum_{j=1}^{\ell-r} \lambda(B_r(i))_j \geq |\sigma(i)_{s_r}| + rb_i + \sum_{j=1}^{\ell-r} \varphi(\gamma(i))_j = \sum_{j=1}^{\ell} \varphi(\eta(i))_j.
\]

Therefore \(\lambda(i) \trianglerighteq \varphi(\eta(i))\). The proofs of \(\lambda(0) \trianglerighteq (\varphi(\eta(0)))\trianglerighteq \varphi(\theta(0))\) and, when \(i \geq 1\), \(\varphi(\theta(i)) = \varphi(\tau(i)_{s_r} + (c_{i-1} - s))\trianglerighteq \varphi(\delta(i))\) and \(\mu(i) - 1 \trianglerighteq \varphi(\theta(i))\) are similar. 

We are now ready to state and prove our main result of this section. Observe in the proof that the main obstruction for the inequality in Theorem 5.6 to be an equality lies in both Corollary 5.2 and Lemma 5.5. If the inequality in Corollary 5.2 is an equality and the map \(\iota\) in Lemma 5.5 is bijective then we would have obtained an equality in Theorem 5.6.

**Theorem 5.6.** Let \((\alpha|\beta), (\lambda|\mu) \in \mathcal{P}^2(n)\) such that the pairs \((\alpha, \lambda)\) and \((\beta, \mu)\) admit horizontal \(r\)- and \(s\)-row cuts respectively. Then

\[
k(\alpha|\beta), (\lambda|\mu) = k(\alpha^T_r, \lambda^T_r, k_{\beta^T_s, \mu^T_s}, k(\lambda_{B_r}|\beta_{B_s}), (\lambda_{B_r}|\mu_{B_s})).
\]

**Proof.** We use the notation as in Lemma 5.5 and will be using Theorem 3.3, [2, Corollary 1.1], Corollary 5.2 and Lemmas 5.3, 5.4, 5.5 repeatedly in the calculation. We first notice that \(\tau(0) = \varnothing\) for any \(\tau \in \Gamma_2\).

\[
\begin{align*}
k_{\alpha^T_r, \lambda^T_r, k_{\beta^T_s, \mu^T_s}, k(\lambda_{B_r}|\beta_{B_s}), (\lambda_{B_r}|\mu_{B_s})} \\
= \sum_{(\sigma, \tau, (\gamma|\delta)) \in \Gamma} \prod_{i=0}^{\infty} k_{\sigma(i), (\lambda^T_r - (b^r))_i} k_{T_r, (\mu^T_s - (c^s))_i} k_{(\gamma(0)|\delta(0))_i, (\lambda_{B_r}|\beta_{B_s})_i} \prod_{i=1}^{\infty} k_{\gamma(0), \lambda_{B_r}} k_{\delta(i), \mu_{B_s} (i-1)} \\
= \sum_{(\sigma, \tau, (\gamma|\delta)) \in \Gamma} \prod_{i=0}^{\infty} k_{\sigma(i), (\lambda^T_r - (b^r))_i} k_{T_r, (\mu^T_s - (c^s))_i} k_{(\gamma(0)|\delta(0))_i, (\lambda_{B_r}|\beta_{B_s})_i} \prod_{i=1}^{\infty} k_{\gamma(0), \lambda_{B_r}} k_{\delta(i), \mu_{B_s} (i-1)} \\
= \sum_{(\sigma, \tau, (\gamma|\delta)) \in \Gamma} k_{\sigma(i) + (b^r), (\lambda^T_r - (b^r))_i} k_{(\gamma(0)|\delta(0))_i, (\lambda_{B_r}|\beta_{B_s})_i} \prod_{i=1}^{\infty} k_{\sigma(i) + (b^r), (\lambda(0)|\beta(0))_i} k_{\gamma(0), (\lambda(0)|\beta(0))_i} k_{\delta(i), \mu_{B_s} (i-1)} \\
\leq \sum_{(\gamma|\delta) \in \Gamma_4} k_{(\sigma(i) + (b^r))\trianglerighteq (\gamma(0)|\delta(0))_i, (\lambda(0)|\beta(0))_i} \prod_{i=1}^{\infty} k_{\sigma(i) + (b^r), (\lambda(i)|\beta(i))_i} k_{\gamma(i), (\lambda(i)|\beta(i))_i} k_{\delta(i), \mu_{B_s} (i-1)} \\
\leq \sum_{(\gamma|\delta) \in \Gamma_4} \prod_{i=1}^{\infty} k_{\gamma(i), (\lambda(i)|\beta(i))_i} = k_{(\alpha|\beta), (\lambda|\mu)}.
\end{align*}
\]
6. Labelling of signed Young permutation modules and mixed powers

In this final section, we address the question regarding the labelling of signed Young permutation modules. It is known in the classical case that, if $\lambda, \mu \in \mathcal{P}(n)$, then $M^\lambda \cong M^\mu$ if and only if $\lambda = \mu$ because each Young permutation module $M^\lambda$ has a distinguished indecomposable summand the Young module $Y^\lambda$. In the case of signed Young permutation modules, it is known from the work of Donkin [7] that, if $(\alpha|\beta), (\sigma|\tau) \in \mathcal{P}_p^2(n)$, then $M(\alpha|\beta) \cong M(\sigma|\tau)$ if and only if $\alpha = \sigma$ and $\beta = \tau$, again because the signed Young permutation module $M(\alpha|\beta)$ has a distinguished indecomposable summand the signed Young module $Y(\alpha|\beta)$. However, it is not clear the exact condition for $M(\alpha|\beta) \cong M(\sigma|\tau)$. In this section, we deal with the general case and obtain the following result.

**Theorem 6.1.** Let $(\alpha|\beta), (\sigma|\tau) \in \mathcal{P}^2(n)$. Then $M(\alpha|\beta) \cong M(\sigma|\tau)$ if and only if $\alpha = \rho\#(1^a)$, $\sigma = \delta\#(1^b)$, $\beta = \zeta\#(1^c)$ and $\tau = \zeta\#(1^d)$ for some non-negative integers $a, b, c, d$ and partitions $\rho, \zeta, \delta, \tau$.

We shall first recall the definition of semistandard tableau and signed Young Rule.

**Definition 6.2.** Let $\lambda \in \mathcal{P}(n)$ and let $(\alpha|\beta) \in \mathcal{C}^2(n)$. A semistandard $\lambda$-tableau $t$ of type $(\alpha|\beta)$ is an assignment of the nodes of $[\lambda]$ with colours

$$c_1 < c_2 < \cdots < d_1 < d_2 < \cdots$$

such that the following two conditions are satisfied.

(i) The subtableau $s$ of $t$ occupied by the colours $c_1, c_2, \ldots$ is a semistandard $\mu$-tableau of type $\alpha$ for some partition $\mu$ where $[\mu] \subseteq [\lambda]$.

(ii) The skew $(\lambda/\mu)$-tableau $t/s$ occupied by the colours $d_1, d_2, \ldots$ is a conjugate semistandard of type $\beta$.

The total number of semistandard $\lambda$-tableaux of type $(\alpha|\beta)$ is denoted by $s^\lambda_{\alpha|\beta}$.

**Theorem 6.3** ([24, Theorem 2.2]). Let $(\alpha|\beta) \in \mathcal{C}^2(n)$. Then $M(\alpha|\beta)$ has a Specht filtration such that every Specht module $S^\alpha$ occurs as factors with multiplicity $s^\lambda_{\alpha|\beta}$.

Using the signed Young Rule, we obtain the following lemma we shall need in the proof of Theorem 6.1.

**Lemma 6.4.** Let $\alpha, \beta$ be partitions. Then we have $s^{\alpha\#(1^|\beta|)}_{\alpha|\beta} = 1$.

**Proof.** Let $k = \ell(\alpha)$. The only semistandard $(\alpha\#(1^|\beta|))$-tableau of type $(\alpha|\beta)$ is the colouring where the nodes in row $i$ are coloured by $c_i$ if $1 \leq i \leq k$ and, when $k + 1 \leq j \leq k + |\beta|$, the nodes $(j, 1)$ are coloured by $d_1, d_2, \ldots$ in the obvious way. \(\square\)

We are now ready to prove Theorem 6.1.

**Proof of Theorem 6.1.** The converse of the statement is easy to prove. We shall only prove the other implication. Assume that $M(\alpha|\beta) \cong M(\sigma|\tau)$ and suppose that $\alpha = \rho\#(1^a)$ and $\sigma = \nu\#(1^b)$ such that $\rho(\rho) \geq 2$ and $\nu(\nu) \geq 2$, i.e., both $\rho$ and $\nu$ have no parts of size 1. We have

$$M(\rho|\gamma) \cong M(\alpha|\beta) \cong M(\sigma|\tau) \cong M(\nu|\delta),$$

where $\gamma = \beta\#(1^a)$ and $\delta = \tau\#(1^b)$. Since signed Young permutation modules have trivial sources, they admit unique lift to trivial source modules (see [1, 2.6.3]). As such the corresponding ordinary characters of $M(\rho|\gamma)$ and $M(\nu|\delta)$ are identical, i.e., $\text{ch}(M(\rho|\gamma)) = \text{ch}(M(\nu|\delta))$. By signed Young
Rule, we deduce that $\sum_{\lambda \vdash n} s_{\rho|\gamma}^\lambda \text{ch}(S^\lambda) = \sum_{\lambda \vdash n} s_{\nu|\delta}^\lambda \text{ch}(S^\lambda)$. Since ordinary characters are linearly independent, this shows that $s_{\rho|\gamma}^\lambda = s_{\nu|\delta}^\lambda$ for all $\lambda \in \mathcal{P}(n)$.

Let $\lambda = \rho\#(1^{\nu_1})$ and $\mu = \nu\#(1^{\nu_2})$. By Lemma 6.4, we have $1 = s_{\rho|\gamma}^\lambda = s_{\nu|\delta}^\lambda$ and $1 = s_{\rho|\gamma}^{\mu} = s_{\nu|\delta}^{\mu}$. Since there is a semistandard $\lambda$-tableau $t$ of type $(\nu|\delta)$, we conclude that $\nu_1 \leq \lambda_1 = \rho_1$. Similarly, $\rho_1 \leq \mu_1 = \nu_1$, and hence $\rho_1 = \nu_1$. So the $\rho_1$ nodes in the first row of $t$ must be filled by the colour $c_1$. Again, since $t$ is semistandard, we conclude that $\nu_2 \leq \rho_2$. Similarly, we have $\rho_2 \leq \nu_2$, and hence $\rho_2 = \nu_2$. Continue in this fashion, we conclude that $\rho_i = \nu_i$ for all $i \in \{1, \ldots, m\}$ where $m = \min\{\ell(\rho), \ell(\nu)\}$. If $\ell(\nu) > m = \ell(\rho)$, since $t$ is semistandard again, we necessarily have $\nu_m = \lambda_m = 1$. But $\nu_{m+1} > \ell(\nu) \geq 2$. The contradiction shows that $\ell(\nu) \leq \ell(\rho)$. Similarly, we have $\ell(\nu) \geq \ell(\rho)$ and hence $\nu = \rho$. Therefore, we have $\alpha = \rho\#(1^a)$ and $\sigma = \nu\#(1^b)$.

Tensor with the sign representation, we obtain $M(\beta|\alpha) \cong M(\tau|\sigma)$. Argue exactly the same way as the above, we also conclude that $\beta = \zeta\#(1^c)$ and $\tau = \zeta\#(1^d)$ for some non-negative integers $a, b, c, d$ and partition $\zeta$. The proof is now complete. \hfill $\square$

Passing the result back to the Schur algebra case under the Schur functor, we obtain the following corollary.

**Corollary 6.5.** Let $(\alpha|\beta), (\sigma|\tau) \in \mathcal{P}^2(n)$, and let $n \geq \dim_F E$. Then $S^\alpha E \otimes \wedge^\beta E \cong S^\beta E \otimes \wedge^\tau E$ if and only if $\alpha = \rho\#(1^a)$, $\sigma = \nu\#(1^b)$, $\beta = \zeta\#(1^c)$ and $\tau = \zeta\#(1^d)$ for some non-negative integers $a, b, c, d$ and partitions $\rho, \zeta$.

**Proof.** Under the Schur functor, we have $M(\alpha|\beta) \cong f(K(\alpha|\beta)) \cong f(K(\sigma|\tau)) \cong M(\sigma|\tau)$. Now apply Theorem 6.1. \hfill $\square$

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