Abstract. In this work, a proof of the orbital stability of the black soliton solution of the quintic Gross-Pitaevskii equation in one spatial dimension is obtained. We first build and show explicitly black and dark soliton solutions and we prove that the corresponding Ginzburg-Landau energy is coercive around them by using some orthogonality conditions related to perturbations of the black and dark solitons. The existence of suitable perturbations around black and dark solitons satisfying the required orthogonality conditions is deduced from an Implicit Function Theorem. In fact, these perturbations involve dark solitons with sufficiently small speeds and some proportionality factors arising from the explicit expression of their spatial derivative.

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1. Introduction

In this work we consider the one-dimensional quintic Gross-Pitaevskii equation (quintic GP)

\[ \begin{align*}
  iu_t + u_{xx} &= (|u|^4 - 1)u, \quad (t, x) \in \mathbb{R}^2, \\
  u(0, x) &= u_0(x),
\end{align*} \tag{1.1} \]

where \( u \) is a complex-value function and the initial data \( u_0 \) satisfies the boundary condition

\[ \lim_{|x| \to +\infty} |u_0(x)|^2 = 1. \tag{1.2} \]

From the physical point of view it is interesting to look for solutions \( u(t, x) \) of (1.1) satisfying the boundary condition (1.2) for all \( t \geq 0 \). This is a defocusing nonlinear Schrödinger equation modeling for example ultra-cold dilute Bose gases in highly elongated traps. More specifically, it describes dynamics of weak density modulations of one dimensional bosonic clouds (Tonks-Girardeau gases) when the tight transverse confinement potential is turned off. In fact (1.1), in the case of one dimensional atomic strings, allows to explain many fermionic properties arising in one dimensional chains of bosons, phenomena usually named as bosonic fermionization. See \[7, 17, 18, 20\] and references therein for a complete background on the physical phenomena accounted for by this quintic defocusing model.

The quintic GP equation is phase (also called \( U(1) \) invariance) and translation invariant, meaning that if \( u \) is a solution of (1.1), then

\[ e^{i\theta} u(t, x + a), \quad a \in \mathbb{R}, \quad \theta \in \mathbb{R}, \]

is also a solution of (1.1). The quintic GP (1.1) also bears Galilean invariance, namely

\[ e^{i(\xi x - \xi^2 t)} u(t, x - ct), \quad c \in \mathbb{R}, \]

but this will not be used in our approach. Note moreover that in (1.2), the asymptotic value 1 can be changed to any number \( \zeta > 0 \) without loss of generality by rescaling the values of \( u \) through \( v = \zeta u(\zeta^4 t, \zeta^2 x) \). Under this change (1.1) recasts as

\[ iv_t + v_{xx} = (|v|^4 - \zeta^4) v, \quad (t, x) \in \mathbb{R}^2. \]

Furthermore, and as far as we know, the quintic GP (1.1)-(1.2) is a non-integrable hamiltonian model (see \[24, 6\]), with well known low order conservation laws for regular solutions, such as the mass

\[ M[u](t) := \int_\mathbb{R} (1 - |u|^2) dx = M[u](0), \]

and the classical energy

\[ E_1[u](t) := \int_\mathbb{R} \left( |u_x|^2 - \frac{1}{3}(1 - |u|^6) \right) dx = E_1[u](0). \]

In this work will be important the so called quintic Ginzburg-Landau energy given by

\[ E_2[u] = E_1[u] + M[u], \]

or explicitly

\[ E_2[u](t) := \int_\mathbb{R} \left( |u_x|^2 + \frac{1}{3}(1 - |u|^2)^2(2 + |u|^2) \right) dx, \tag{1.3} \]

which is also preserved along the flow. Another conserved quantity of (1.1) is the momentum, which in the context of solutions verifying (1.2) can be suited in different forms. For example, for nonvanishing solutions (see \[10\]), is written as

\[ P_1[u](t) := \int_\mathbb{R} \langle iu, u_x \rangle_C \left( 1 - \frac{1}{|u|^2} \right) dx. \tag{1.4} \]

Moreover, considering vanishing solutions, in \[2\] it was introduced a renormalized version of the momentum (1.4), namely, \( \langle u(t, x) = A(t, x)e^{i\varphi(t,x)} \)
Here by regular solutions we will understand those solutions that belong to the energy space associated to (1.1):

\[ \Sigma = \left\{ u \in H^1_{loc}(\mathbb{R}) : u_x \in L^2(\mathbb{R}) \text{ and } 1 - |u|^4 \in L^2(\mathbb{R}) \right\}. \]  

(1.6)

Notice that if \( u \in \Sigma \), then \( 1 - |u|^2 \in L^2(\mathbb{R}) \). Hence,

\[(1 - |u|^2)^2(2 + |u|^2) = (1 - |u|^2)^2 + (1 - |u|^2)(1 - |u|^4) \in L^1(\mathbb{R}), \]  

(1.7)

and \( E_2[u] \) is well-defined.

Some previous results on the Cauchy problem of (1.1) are well known in the literature. For example, local well-posedness in the context of a Zhidkov space \( \{ u \in L^\infty(\mathbb{R}), \partial_x u \in L^2(\mathbb{R}) \} \) was shown in [22] and global well-posedness of (1.1)-(1.2) was established in [9], where it was considered the general model

\[ iu_t + u_{xx} + f(|u|^2)u = 0, \]  

(1.8)

with regular nonlinearity \( f : \mathbb{R}^+ \to \mathbb{R} \) satisfying \( f(1) = 0 \) and \( f'(1) < 0 \). (1.8) includes, as particular cases, other important equations such as

- Pure powers: \( f(r) = 1 - r^p, \, p \in \mathbb{Z}^+ \).
- Cubic case \( (p = 1) \): \( f(r) = 1 - r \), the cubic Gross-Pitaevskii (cubic GP) equation.
- Cubic-Quintic case: \( f(r) = (r - 1)(2a + 1 - 3r) \) with \( 0 < a < 1 \).
- Quintic case \( (p = 2) \): \( f(r) = 1 - r^2 \), the quintic GP equation (1.1).

More precisely, in [9] Theorem 1.1] was proved that the Cauchy problem for the quintic GP equation (1.1)-(1.2) is globally well-posed in the space

\[ \phi + H^1(\mathbb{R}), \]  

for any \( \phi \) verifying

\[ \phi \in C_0^2(\mathbb{R}), \quad \phi' \in H^2(\mathbb{R}), \quad |\phi|^2 - 1 \in L^2(\mathbb{R}). \]

See [3][10] and [11] for further reading on these generalized Schrödinger models.

Concerning solutions, complex constants with modulus one are the simplest solutions contained in (1.1). Moreover, and with respect to particular soliton solutions, and specifically to the stability of solitonic waves for (1.8), the situation is well understood in the case of the cubic GP equation, profiting its integrable character ([25]). Indeed, it is well known that the black soliton of the cubic GP is

\[ \nu_0(x) = \tan\left(\frac{x}{\sqrt{2}}\right), \]  

which is a stationary, i.e. time independent, wave solution. Furthermore, the study of orbital and asymptotic stability for \( \nu_0(x) \) was considered in several works [2][8][12][13]. Beside that, for some cases of the cubic-quintic model \( (f(r) = (r - 1)(2a + 1 - 3r)) \), the stability of traveling solitonic bubbles was shown in [19]. See [1][2][21] for more details on these models. Finally, [4] dealt with stability (and instability) problems for stationary and subsonic traveling waves giving an explicit condition on a general \( C^2 \) nonlinearity \( f \) in the NLS model. Once in this work we have obtained exact traveling wave solitons (also named as dark solitons), [4] Theorem 24] can be applied to study the orbital stability of stationary solutions (i.e. black solutions) but in another metric, well adapted to the \( \Sigma \) space (1.6), different from the metric used in the current work.

In comparison with the cubic GP, the non-integrability of the quintic GP equation makes the search of solutions even harder as well as the rigorous study of the analytical properties related to them. Actually, and as far as we know, the black soliton solution for the quintic GP was discovered in [17] eqn.(12)]. Beside that, we present in this work the explicit expression of this solution as well as its formal derivation (see Section 2). Namely, the black soliton of (1.1) is given by
\[ \phi_0(x) = \sqrt{2} \frac{\tanh(x)}{\sqrt{3 - \tanh^2(x)}}, \] (1.9)

which is a solution of
\[ \phi'' + (1 - \phi^4)\phi = 0, \] (1.10)

the corresponding differential equation describing stationary real waves of (1.1) with \( u(0,x) = \phi(x) \) (see Section 2 for further details). Therefore, it is natural to question whether, in the case of the quintic GP, the stability of \( \phi_0 \) is preserved in some sense. In fact, the main result of this work is the following (see Section 5 for a more detailed version and proof of this result):

**Theorem 1.1.** The black soliton solution \( \phi_0 \) (1.9) of the quintic GP equation (1.1) is orbitally stable in a subspace of the energy space \( \Sigma \) (1.6).

The black soliton (1.9), stationary by nature, belongs to a greater family of traveling waves. As far as we know, an explicit and correct expression of a traveling wave family of solutions for the quintic GP (1.1) was missed. In fact, we show in this work that the quintic GP (1.1)-(1.2) also bears explicit traveling-wave solutions. These waves, with the form
\[ u(t,x) = \Phi_c(x-ct), \]

are currently known as dark solitons, a reminiscent terminology coming from nonlinear optics (see [15]). The function \( \Phi_c \) satisfies the complex nonlinear ordinary differential equation
\[ \Phi''_c - ic\Phi'_c + (1 - |\Phi_c|^4)\Phi_c = 0. \] (1.11)

Indeed, for \(|c| < 2\) we are able to obtain the following explicit family of dark solitons:
\[ \Phi_c(\xi) = \frac{i\mu_1(c) + \mu_2(c)\tanh(\kappa(c)\xi)}{\sqrt{2}\sqrt{1 + \mu(c)\tanh^2(\kappa(c)\xi)}}, \] (1.12)

with \( \xi = x - ct \) and where
\[ \kappa \equiv \kappa(c) = \frac{\sqrt{4 - c^2}}{2}, \] (1.13)
\[ \mu_1 \equiv \mu_1(c) = \frac{3c^2 - 4 + 2\sqrt{3c^2 + 4}}{\sqrt{18c^2 - 8 + (3c^2 + 4)^{3/2}}}, \] (1.14)
\[ \mu_2 \equiv \mu_2(c) = \frac{3c\sqrt{4 - c^2}}{\sqrt{18c^2 - 8 + (3c^2 + 4)^{3/2}}}, \]

and \( \mu \equiv \mu(c) \) verifying the constraint relation
\[ \frac{\mu_1^2 + \mu_2^2}{2 + 2\mu} = 1, \] (1.15)

for all \(|c| < 2\) and which comes from (1.2). Therefore, \( \mu \) is explicitly
\[ \mu \equiv \mu(c) = \frac{3c^2 + 20 - 8\sqrt{4 + 3c^2}}{3(-4 + c^2)}. \] (1.16)

Note that
\[ \lim_{c \to 0^+} \mu_1 = 0, \quad \lim_{c \to 0^+} \mu_2 = \pm \frac{2}{\sqrt{3}} \quad \text{and, from (1.15),} \quad \lim_{c \to 0} \mu = -\frac{1}{3} \quad \text{with} \quad -\frac{1}{3} \leq \mu \leq 0. \]

Also notice that, as a consequence of the above limits, we get
\[ \lim_{c \to 0^\pm} \Phi_c(x) = \pm \Phi_0(x) = \pm \sqrt{2} \frac{\tanh(x)}{\sqrt{3 - \tanh^2(x)}}, \] (1.17)
Finally hereafter, since \( \pm \Phi_c \) are both solutions of \((1.11)\), it is better to consider a \( c \)–smooth continuation of \((1.9)\) in the following way:

\[
\phi_c = \begin{cases} 
\Phi_c, & c \geq 0, \\
-\Phi_c, & c < 0.
\end{cases}
\] (1.18)

Hence,

\[
\lim_{c \to 0^\pm} \phi_c = \phi_0.
\] (1.19)

**Remark 1.2.** The main ingredient in the orbital stability proof is the use of the associated family of complex dark profiles \( \phi_c \) whose real parts are odd functions with respect to the speed \( c \) and are laterally approximated to the stationary black solution when \( c \to 0 \), as shown in \((1.17)\).

Our motivation to deal with the orbital stability of black solitons with this particular quintic nonlinearity (note that the cubic GP case was approached in [14]) comes firstly from the physical relevance that this model has in quantum gases as we said above. We were also motivated to prove this orbital stability result getting rid of a hydrodynamical formulation because, this approach only describes non-vanishing solutions, and therefore excluding the black soliton solution. From a specific mathematical point of view, we avoid technical issues coming from the non integrable character of the model and therefore not being allowed to use classical integrability methods.

More specifically, our proof establishes the coercivity of the functional \( Q_c [\phi] \) \((3.45)\), when the function \( \phi \) satisfies suitable orthogonality conditions well adapted to this specific quintic nonlinearity. These orthogonality conditions are guaranteed by the introduction of modulation parameters (see Proposition \[4.2\]) and needed to perturb a stationary object as the black soliton of the quintic GP. This approach has the advantage to show a better control on the perturbation with respect to the black soliton.

Besides that, we are able to explicitly obtain (despite the nonintegrable nature of the model) dark solitons (traveling wave kinks) of the quintic GP, being one of the special cases where it is still possible to get these solutions (the other one is the cubic GP). The knowledge about dark solitons of quintic GP allowed us to perform precise perturbations on the black soliton.

Summarizing, we highlight here the main results involved in the proof of the orbital stability of the black soliton of the quintic GP \((1.1)\). Our proof introduces new theoretical and technical tools, with respect to the integrable cubic GP equation \([14]\) or more recently with respect to
systems of cubic GP equations [5]. These tools are specially suited to deal with the associated nonlinear solutions of (1.1), namely the black soliton $\phi_0$ (1.9) and the dark soliton $\phi_c$ profile (2.4). Specifically we introduced

- a new family of traveling wave solutions $\phi_c$ (1.18), close to the stationary black soliton $\phi_0$, for the quintic GP equation (1.1). Obtaining non-constant solutions of this non-integrable equation is not a simple task, and even more in the case when one has to solve a coupled nonlinear ODE system (1.11). Only by proposing a suitable ansatz and a careful tuning of the free parameters allowed us to obtain them. Just, compare these solutions of the quintic GP equation with the corresponding ones of the cubic GP equation, where a simple complex constant translation gives the traveling family. See Section 2.2 for further reading.

- a modified metric $d_c$. This is a weighted metric with nonlinear weight $\phi^3_c$ as it is dictated from the coercivity estimates that we need to prove on the quintic Ginzburg-Landau energy on black and dark solitons. See (2.14) for a precise definition of $d_c$ and also Propositions 3.2 - 3.4.

- new functional spaces, in order to correctly measure the distance between black and dark solitons and their perturbations $z$. See Section 2.3 for details.

- new orthogonality conditions associated to perturbations of the black and dark solitons (3.22) and (3.43) and specially adapted to the spectral properties of the quintic GP equation.

In this work, we were able to overcome several technical issues coming from the nonlinear functional structure of the quintic GP and its black and dark solitons, by working in a small speed region $|c| < c$. Moreover, the apparent structural difference between black solitons in the cubic GP and the quintic GP, is reflected in many identities and related functions around these black (and dark) solitons, e.g. the quintic Ginzburg-Landau energy $E_2$ (1.3) or the spatial derivative $\phi_0'$.

The strategy we used for the proof of the orbital stability result for the black soliton $\phi_0$ of (1.1) was focused to first show that the quintic Ginzburg-Landau energy $E_2$ is coercive around the black and dark solitons. This was done by using some orthogonality relations based on perturbations $z$ of the black and dark solitons and arising from the particular spectral problem related to (1.1), suitable nonlinear identities and some proper Gagliardo-Nirenberg estimates on functions of the black and dark solitons of (1.1).

We notice that the orthogonality conditions arising from the coercivity result (Proposition 3.2) on the black soliton $\phi_0$, do not include a linear term appearing after expansion of $E_2$ (1.3) around the black and dark solitons, and therefore we must deal with this remaining linear term along the proof, estimating it in a suitable way to obtain the expected bounds, in contrast with previous approaches ([14]) where their natural orthogonality conditions imposed its cancellation.

After that main step, we continued with Proposition 4.2 proving, through a modulation of parameters, the existence of suitable perturbations $z$ of the dark soliton which satisfy the orthogonality conditions defined in (3.43).

Finally note that, related with the orbital stability, is the concept of asymptotic stability which essentially states the convergence of perturbations of the black soliton to a special element in the tubular neighborhood generated by its symmetries, e.g. phase and translation invariances. A detailed study on the asymptotic stability of the black soliton (1.9) of the quintic GP (1.1) is currently being made and it will appear elsewhere.

1.1. Final remarks.

- Our work does not get an orbital stability result for dark solitons (1.18) in $d_c$ metric (2.14) for speeds close to 0. However, this kind of stability for dark profiles can be obtained in an alternative metric as the used in [19, Theorem 1.1]. In fact, computing directly
\[ P_{1}[\phi_c] = \frac{\mu_1 \mu_2}{\sqrt{\mu}} \arctan(\sqrt{\mu}) - 2 \arctan \left( \frac{\mu_2}{\mu_1} \right) , \]

we get

\[ \frac{dP_{1}}{d\phi_c} < 0, \]

as it can also be seen in Figure 2

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Momentum $P_{1}$ \eqref{eq:momentum_p1} at $\phi_c$ \eqref{eq:phi_c}.}
\end{figure}

- Remember that \eqref{eq:phi_c} is phase invariant, and therefore since $e^{i\tau_0 \phi_c} \xrightarrow{c \to 0} e^{i\tau_0 \phi_0}$, $\tau_0 \in (0, 2\pi)$, we also have orbital stability for this phase transformed family of black solitons.
- The quintic NLS:

\[ iv_t + v_{xx} - |v|^4 v = 0. \]

The application to this model is rather direct, because it only involves the introduction of a rotation in time transformation $u = e^{it} v$ to connect \eqref{eq:phi_c} with the quintic NLS.

- Note that some recent works (see \cite{22, 23}) have approached another NLS model with modified dispersion terms, and dealing with orbital stability of black solitons using dark solitons with small speed, close to 0, but without an explicit expression of them and resorting to symmetries to simplify the coercivity analysis.

1.2. Structure of the paper. In Section 1 we introduce the problem and the main result. In Section 2 we obtain the black and dark solitons of \eqref{eq:phi_c} and describe some properties and nonlinear identities and norms based on them. In Section 3 we present the coercivity properties of the quintic Ginzburg-Landau energy $E_2$ around black and dark solitons. In Section 4 we study the existence and time growth of some modulation parameters associated to black and dark solitons. Finally in Section 5 we prove the main Theorem on the orbital stability of the black soliton of \eqref{eq:phi_c}, gathering the results obtained in the previous sections.

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2. DERIVATION OF BLACK AND DARK SOLITONS FOR THE QUINTIC GP

In this section we explain the derivation of the black and dark solutions given in (1.9) and (1.18). The following basic result will be useful for obtaining the black family (1.9).

Lemma 2.1. Let \(b > 0\). Then,
\[
\int_0^y \frac{ds}{(b - s^2) \sqrt{s^2 + 2b}} = \frac{1}{2b \sqrt{3}} \ln \left( \frac{\sqrt{2b + y^2 + \sqrt{3}y}}{\sqrt{2b + y^2 - \sqrt{3}y}} \right),
\]
for all \(|y| < \sqrt{b}\).

See Appendix B for a proof of this identity.

2.1. Derivation of black solitons. Using (1.10), we get after multiplication by \(\phi'(x)\)
\[
\phi(x)\phi'(x) + \phi''(x)\phi'(x) = \phi^5(x)\phi'(x),
\]
and then
\[
\frac{d}{dx} \left[ \phi(x)^2 + (\phi'(x))^2 - \frac{\phi^6(x)}{3} \right] = 0,
\]
which yields
\[
\phi(x)^2 + (\phi'(x))^2 - \frac{\phi^6(x)}{3} = K_0.
\]

From the boundary conditions at infinity in (1.2) we conclude that \(K_0 = \frac{2}{3}\) and we obtain the following first order ODE
\[
(\phi')^2 = \frac{1}{3} \phi^6 - \phi^2 + \frac{2}{3} = \frac{1}{3}(1 - \phi^2)^2(2 + \phi^2).
\]

Assuming that \(\phi' > 0\) and integrating, we get
\[
\int_{x_0}^{x} \frac{\phi'(\hat{x}) d\hat{x}}{(1 - \phi(\hat{x})^2)^2(\phi(\hat{x})^2 + 2)} = \frac{x - x_0}{\sqrt{3}},
\]
where we consider \(x_0 = \phi^{-1}(0)\). Then, making the change \(s = \phi(\hat{x})\) we have
\[
\int_0^{\phi(x)} \frac{ds}{\sqrt{(1 - s^2)^2(s^2 + 2)}} = \frac{x - x_0}{\sqrt{3}}.
\]

Without loss of generality we can assume \(x_0 = 0\). Since \(|\phi(x)| < 1\), using Lemma 2.1 it follows that
\[
\frac{1}{2\sqrt{3}} \ln \left( \frac{\sqrt{2 + \phi^2(x)} + \sqrt{3}\phi(x)}{\sqrt{2 + \phi^2(x)} - \sqrt{3}\phi(x)} \right) = \frac{x}{\sqrt{3}},
\]
which yields
\[
\phi(x) = \sqrt{2 + \phi^2(x)} = \frac{e^{2x} - 1}{\sqrt{3} e^{2x} + 1} = \frac{1}{\sqrt{3}} \tanh(x),
\]
and consequently
\[
\phi(x) = \sqrt{\frac{2}{3}} \frac{\tanh(x)}{\sqrt{3} - \tanh^2(x)},
\]
which, in fact is the unique (up to symmetries of the equation) non-trivial stationary solution of the quintic GP (1.1)-(1.2) and named as black soliton.

An important observation is that the black soliton \(\phi_0\) (1.9) has a definite variational structure. More precisely, considering the quintic Ginzburg-Landau energy \(E_2[u]\) defined in (1.3) as the corresponding Lyapunov functional, and considering a small perturbation \(\zeta\) of the black soliton \(\phi_0\), namely a \(\zeta \in H_0(\mathbb{R})\) with \(H_0(\mathbb{R}) \subset H^1_{loc}(\mathbb{R})\) to be defined in (2.9), we get, after a power expansion in \(\zeta\) of \(E_2\) (1.3),
Because the first variation of $E_2$ vanishes for (1.10), the black soliton is characterized as critical point of the functional $E_2$ associated to the quintic GP (1.1). In fact, it is easy to see that

$$E_2[\phi_0] := 2\sqrt{3} \arctanh \left( \frac{1}{\sqrt{3}} \right).$$  

Moreover, it is possible to state the following minimality’s characterization on the black soliton solution

**Proposition 2.2** ([2, Lemma 2.6]). Let $E_2 (1.3)$ and let $\phi_0$ be the black soliton solution (1.9). Then we have

$$E_2[\phi_0] = \inf \left\{ E_2[\phi] : \phi \in H^1_{loc}(\mathbb{R}), \inf_{x \in \mathbb{R}} |\phi(x)| = 0 \right\}.$$  

Moreover, if $E_2[\phi] < E_2[\phi_0]$, then $\inf_{x \in \mathbb{R}} |\phi(x)| > 0$.

**Proof.** This result is essentially contained in [2, Lemma 2.6], where the black soliton case for the cubic GP was considered. The extension to the quintic GP case, once we work with the energy $E_2$ is direct and does not require additional steps. We therefore skip the details. □

### 2.2. Derivation of dark solitons.

Once obtained the black soliton (1.9), the detailed construction of the dark soliton solution (1.18) to (1.11) is presented in the Appendix A. A sketch of the derivation is the following: bearing in mind that (1.11) reduces to (1.10) at $c = 0$, we proposed a suitable ansatz like

$$\Phi_c(x) = \frac{ia_1 + a_2 \tanh(kx)}{\sqrt{1 + a_3 \tanh^2(kx)}},$$  

with $a_1$, $a_2$, $a_3$ and $k$ as free parameters to be determined in order that (2.3) is actually a solution of (1.11), and verifying the asymptotic behavior

$$\lim_{x \to \pm \infty} |\Phi_c(x)|^2 = 1.$$  

Hence, substituting (2.3) into (1.11) and after lengthy manipulations, we got (1.18), with $\mu_1 = \sqrt{2}a_1$, $\mu_2 = \sqrt{2}a_2$ and $a_3 = \mu$, satisfying the relation (1.15) and $k = \kappa$ as in (1.13).

Finally, we introduce the notion of dark profile.

**Definition 2.3** (Dark profile). Let $c \in (-2, 2)$, and $x_0 \in \mathbb{R}$ be fixed parameters. We define the complex-valued dark profile $\phi_c$ with speed $c \neq 0$ as follows

$$\phi_c(x) := \phi_c(x; c, x_0) = \text{sgn}(c) \frac{i\mu_1(c) + \mu_2(c) \tanh(\kappa(c)(x + x_0))}{\sqrt{2} \sqrt{1 + \mu(c) \tanh^2(\kappa(c)(x + x_0))}}.$$  

**Remark 2.4.** Note that the profile $\phi_c$ is the standard profile associated to the dark soliton solution (1.18). Note moreover, that although $\phi_c$ is not an exact solution of (1.1), it can be interpreted as follows: for each $(t, x) \in \mathbb{R}^2$,

$$(t, x) \mapsto \phi_c(x; c, x_0 - ct),$$  

is an exact dark soliton solution of (1.1) moving with speed $c$. 

\[E_2[\phi_0 + \delta] = E_2[\phi_0] - 2\text{Re} \left[ \int_\mathbb{R} \delta(\phi_0'' + (1 - |\phi_0|^4)\phi_0) \right] + O(\delta^2).\]
2.3. Preliminaries. First of all, we introduce the following notation for the nonlinear weights

\[ \eta_0(x) = 1 - \phi_0^4(x) \quad \text{and} \quad \eta_c(x) = 1 - |\phi_c(x)|^4, \] (2.5)

and for the real and imaginary parts of the dark soliton

\[ R_c(x) = \text{Re} \, \phi_c(x) = \frac{\mu_2 \tanh(\kappa x)}{\sqrt{2} \sqrt{1 + \mu \tanh^2(\kappa x)}}, \] (2.6)

\[ I_c(x) = \text{Im} \, \phi_c(x) = \frac{\mu_1}{\sqrt{2} \sqrt{1 + \mu \tanh^2(\kappa x)}}, \] (2.7)

To simplify the notation, we shall also denote

\[ \langle f, g \rangle_c = \text{Re}(f \bar{g}). \] (2.8)

Moreover, we define the following functional spaces: given \( c \in (-2, 2) \) we consider the weighted Sobolev space

\[ H_c(\mathbb{R}) := \{ f \in C^0(\mathbb{R}, \mathbb{C}) : f' \in L^2(\mathbb{R}) \text{ and } \eta_c^{1/2} f \in L^2(\mathbb{R}) \}, \] (2.9)

with the norm

\[ \| f \|_{H_c} := \left( \int_{\mathbb{R}} |f'|^2 + \eta_c |f|^2 \right)^{1/2}. \] (2.10)

We will also use \( H_c^{\text{real}}(\mathbb{R}) \) to denote the set of real-valued functions in \( H_c(\mathbb{R}) \), that is,

\[ H_c^{\text{real}}(\mathbb{R}) = \{ f \in C^0(\mathbb{R}, \mathbb{R}) : f' \in L^2(\mathbb{R}) \text{ and } \eta_c^{1/2} f \in L^2(\mathbb{R}) \}. \] (2.11)

Using the exponential decay of \( \eta_c \) we can check that the space \( H_c \) does not depend on the velocity \( c \) when \( |c| \leq \bar{c} \), for some \( \bar{c} \) small enough. Even more, the norms \( \| \cdot \|_{H_c} \) are equivalent with \( \| \cdot \|_{H_0} \). For further details see Lemma 2.6. Therefore, hereafter we simplify the notation using the identification

\[ \mathcal{H} := H_c \quad \text{and} \quad \mathcal{H}^{\text{real}} := H_c^{\text{real}}, \] (2.12)

for all \( |c| < \bar{c} \). Beside that, we define a proper subset of \( Z(\mathbb{R}) \subseteq \mathcal{H}(\mathbb{R}) \), namely

\[ Z(\mathbb{R}) := \{ u \in \mathcal{H}(\mathbb{R}) : 1 - |u|^4 \in L^2(\mathbb{R}) \}, \] (2.13)

which has metric structure with the distance

\[ d_c(u_1, u_2) := \left( \| u_1 - u_2 \|_{H_c}^2 + \| \phi_c^4(|u_1|^2 - |u_2|^2) \|_{L^2}^2 \right)^{1/2}, \] (2.14)

for all \( |c| < \bar{c} \). Also notice that if \( u \in Z(\mathbb{R}) \), from the computations in (1.7) we see that, the energy \( E_2 \) is well defined for elements in \( Z(\mathbb{R}) \).

**Remark 2.5.** Similarly to the theory developed in [14] in the context of the cubic GP model, here we also have that the unique global solution \( u \) of (1.1) with initial data \( u_0 \in Z(\mathbb{R}) \) remains continuous from \( \mathbb{R} \) to \( Z(\mathbb{R}) \) endowed with the metric structure induced by \( d_c \).
2.4. **Nonlinear identities and estimates for black and dark solitons.** Now we present some nonlinear identities related to the black soliton and dark soliton profile (1.9) and (2.4), which shall be useful along the work. Firstly we note that from (2.1) we get

\[
\phi_0'(x) = \frac{1}{\sqrt{3}} (1 - \phi_0^2(x)) \sqrt{2 + \phi_0^2(x)}. \tag{2.15}
\]

In comparison, the dark soliton satisfies the following identity:

\[
\phi_d'(x) = \frac{1}{\sqrt{2}} \frac{\kappa \text{sech}^2(kx)}{(1 + \mu \tanh^2(kx))^{3/2}} (\mu_2 - i \mu_1 \tanh(kx)),
\]

and which shows the localized character of \( \phi_d' \).

Notice that from (1.10), (2.1) and (2.2), the black soliton solution (1.9) satisfies the identity

\[
||\phi_0||_{H^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \left( \eta_0 \phi_0^2 + (\phi_0')^2 \right) dx = \int_{\mathbb{R}} \left( -\phi_0 \phi_0'' + \frac{1}{2} (1 - |\phi_0|^2)^2 (2 + |\phi_0|^2) \right) dx = E_2[\phi_0] = 2\sqrt{3} \arctanh \left( \frac{1}{\sqrt{3}} \right),
\]

and by direct calculation we have also \( ||\eta_0||_{L^2(\mathbb{R})}^2 = 2\sqrt{3} \arctanh \left( \frac{1}{\sqrt{3}} \right) \). Hence,

\[
||\phi_0||_{H^2(\mathbb{R})}^2 = ||\eta_0||_{L^2(\mathbb{R})}^2 = E_2[\phi_0] = 2\sqrt{3} \arctanh \left( \frac{1}{\sqrt{3}} \right). \tag{2.16}
\]

Coming back to (1.18) and using (1.10) and (1.15), we get that the explicit quintic Ginzburg-Landau energy (1.3) of the dark soliton (1.18) is

\[
E_2[\phi_c] := \frac{s_1 + s_2 \arctanh(\sqrt{|\mu|})}{32 \kappa \mu^2}, \tag{2.17}
\]

with

\[
s_1 := (2\mu(\mu + 3) - \mu_2^2(\mu - 1)) \left( \mu_2^4 - 4 \mu_2^2 \mu + 4 \mu (\mu + \kappa^2) \right),
\]

\[
s_2 := 12 \mu \kappa^2 \left( \mu_1^2(\mu - 3) \mu + \mu_2^2(3\mu - 1) \right) - \left( \mu_2^4 - 2 \mu_2^2(3\mu^2 - 2\mu + 3) - 2 \mu (3\mu^2 + 10\mu - 9) \right). \tag{2.18}
\]

On the other hand, we get the right convergence of (2.17) to (2.16) when the speed \( c \) goes to 0, namely (see Figure 3)

\[
\lim_{c \to 0} E_2[\phi_c] = E_2[\phi_0] = 2\sqrt{3} \arctanh \left( \frac{1}{\sqrt{3}} \right). \tag{2.19}
\]

In fact, the expansion of (2.17) around \( \phi_0 \) up to \( c^2 \) order is

\[
E_2[\phi_c] = E_2[\phi_0] - \frac{1}{4} \left( 3 + E_2[\phi_0] \right) c^2 + \mathcal{O}(c^3), \tag{2.20}
\]

and therefore, for \( c \) small enough one gets

\[
E_2[\phi_c] - E_2[\phi_0] \geq -2\sqrt{3} c^2. \tag{2.21}
\]

For the sake of completeness, we also show here the following related amounts:

\[
d_0^2(\phi_0, \phi_c) := ||\phi_0 - \phi_c||_{H^2}^2 + ||\phi_0^2(\phi_0^2 - \phi_0^2)||_{L^2}^2. \tag{2.22}
\]

We now have that if \( |c| < \epsilon \), with some \( \epsilon \ll 1 \),

\[
||\phi_0 - \phi_c||_{H^2}^2 = \mathcal{O}(c^2) \tag{2.23}
\]
and
\[ \| \phi_0^3 (|\phi_c|^2 - \phi_0^2) \|_{L^2}^2 = O(c^4). \]
Therefore, we get that
\[ d_0^2 (\phi_0, \phi_c) = O(c^2). \] (2.24)
In the following lines, we show some interesting computations, which are justified in Appendix D. Firstly we have that
\[ \| \phi_c' \|_{L^2}^2 \leq \frac{\pi}{3 \sqrt{3}} + \frac{2}{\sqrt{3}} \arctanh \left( \sqrt{3} \right), \] (2.25)
for all \(|c| < 2\). On the other hand, because \(-\frac{1}{3} \leq \mu < 0\) for all \(|c| < 2\), we have
\[ \| I_c \|_{L^\infty} = \frac{\mu_1}{\sqrt{2 + 2 \mu}} = O(c), \] (2.26)
with \( I_c \) defined in (2.7). We also have the next useful estimates:
\[ \| \phi_c - \phi_0 \|_{L^2} = O(c^2), \] (2.27)
\[ \| \eta_0 \phi_0 - R_c \eta_c \|_{L^2} = O(c^2), \] (2.28)
\[ \| \eta_c |\phi_c|^2 - \eta_0 \phi_0^2 \|_{L^2} = O(c^2), \] (2.29)
and
\[ \| \eta_c |\phi_c|^2 R_c - \eta_0 \phi_0^4 \|_{L^2} = O(c^2), \] (2.30)
\[ \| \phi_c - \phi_0^2 \|_{L^\infty}^2 = O(c^2) \] (2.31)
for all \(|c| \leq \epsilon\) with some \( \epsilon \ll 1\). Also we have the uniform pointwise estimate.
\[ |\phi_0(x)| \lesssim |\phi_c(x)|, \quad \forall x \in \mathbb{R}, |c| \leq c, \ c \ll 1. \quad (2.32) \]

For more details on these \( L^2 \) and \( L^\infty \)-norms, see subsections C.1 and C.2 in Appendix D.1 and D.2 respectively.

The next estimate will be useful in subsequent technical results on perturbations of the black soliton \( \phi_0 \), and therefore we present a brief proof of it.

**Lemma 2.6 (Equivalent norms).** Let \( \phi_0 \) and \( \phi_c \) be the black soliton and dark soliton profile \( (1.9) \) and \( (2.4) \) respectively. Then, there exist positive constants \( \sigma_1 \) and \( \sigma_2 \) such that

\[ \sigma_1 \|z\|_{\mathcal{H}_c}^2 \leq \|z\|_{\mathcal{H}_c}^2 \leq \sigma_2 \|z\|_{\mathcal{H}_0}. \quad (2.34) \]

**Proof.** From \( (2.31) \) and using the identity \( z(x) = z(0) + \int_0^x z'(\tilde{x})d\tilde{x} \), which implies

\[ |z(x)| \leq |z(0)| + |x|^{1/2}\|z\|_{L^2}, \]

we have that

\[ \int_\mathbb{R} |\phi_c|^4 - \phi_0^4|dz|^2dx \leq 2 \int_\mathbb{R} |\phi_c|^2 - \phi_0^2|dz|^2dx \]

\[ \leq c^2 \int_\mathbb{R} (1 + x^2)\eta_c(x)|z|^2dx \]

\[ \leq c^2 \left( |z(0)|^2 + \int_\mathbb{R} (1 + x^2)\eta_c(x)dx + \|z'\|_{L^2}^2 \int_\mathbb{R} (1 + x^2)\eta_c(x)|x|dx \right). \quad (2.35) \]

Now we show that the last two integrals are uniformly bounded in \( c \), with \( |c| \leq 1 \). Firstly, to do this we observe that

\[ \max \{1 + x^2, (1 + x^2)|x|\} \lesssim \cosh(\kappa(c)x), \]

for all \( x \in \mathbb{R} \) and \( |c| \leq 1 \). Furthermore, due to the exponential decay of \( \eta_c(x) \) one gets

\[ (I) + (II) \lesssim \int_\mathbb{R} \cosh(\kappa(c)x)\eta_c(x)dx = \int_\mathbb{R} \cosh(\kappa(c)x) \left( 1 - \frac{(\mu_1^2 + \mu_2^2 \tanh^2(\kappa(c)x))^2}{4(1 + \mu \tanh(\kappa(c)x))^2} \right)dx \]

\[ = \frac{\pi}{4\sqrt{2}} \frac{12 - 4\mu_1^2 - \mu_2^2}{\kappa(c) \sqrt{\mu_1^2 + \mu_2^2}}. \]

So, using this control, from \( (2.35) \) we conclude that

\[ \int_\mathbb{R} |\phi_c|^4 - \phi_0^4|dz|^2dx \lesssim c^2 (|z(0)|^2 + \|z\|_{\mathcal{H}_c}^2). \quad (2.36) \]

To estimate \( |z(0)|^2 \) we consider a cut-off function \( \chi \in C^\infty(\mathbb{R}, [0, 1]) \) such that

\[ \chi = 1 \text{ on } [-1, 1] \quad \text{and} \quad \chi = 0 \text{ on } \mathbb{R} \setminus [-2, 2]. \]

Then, using that the functions

\[ \chi(x)/\sqrt{\eta_c(x)} \quad \text{and} \quad \chi'(x)/\sqrt{\eta_c(x)} \]

are bounded on \( \mathbb{R} \) (uniformly for \( |c^*| < c \)), combined with the Sobolev embedding, we have

\[ |z|^2 \leq \|\chi z\|_{L^\infty} \leq \|\chi z\|_{L^2} \left( ||\chi z'||_{L^2} + ||\chi z'\|_{L^2} \right) \lesssim \|z\|_{\mathcal{H}_c}^2. \quad (2.37) \]
Thus, (2.33) follows by substituting (2.37) into (2.36). Finally, in view of (2.33), and using the relation
\[ \|\Delta\|_{\mathcal{H}_c}^2 - \|\Delta\|_{\mathcal{H}_0}^2 = \int_{\mathbb{R}} (\phi_0^4 - |\phi_0|^4)|\Delta|^2 \, dx, \]
we check that $\mathcal{H}_c \equiv \mathcal{H}_0$ for all $|c| < c$ and further we have (2.34).

3. Coercivity of the quintic Ginzburg-Landau energy

In this section we establish that the quintic Ginzburg-Landau energy $E_2$ (1.3) is coercive around $\phi_0$ and $\phi_c$ solitons respectively. First of all, we establish some preliminary notation and results.

We first expand the energy $E_2$ in (1.3) around $\phi_0$ given in (1.9). Let, $\vec{z} := \vec{z}_1 + i\vec{z}_2$, with $\vec{z}_1, \vec{z}_2 \in \mathbb{R}$, and define
\[ \rho_0(\vec{z}) := |\phi_0 + \vec{z}|^2 - |\phi_0|^2 = 2\Re(\phi_0\vec{z}) + |\vec{z}|^2 = 2\phi_0\vec{z}_1 + |\vec{z}|^2. \]  

Then,
\[ E_2[\phi_0 + \vec{z}] = \int_{\mathbb{R}} \left[ |\phi_0 + \vec{z}|^2 + \frac{1}{4}(1 - |\phi_0 + \vec{z}|^2)^2(2 + |\phi_0 + \vec{z}|^2) \right] \, dx \]
\[ = \int_{\mathbb{R}} \left[ (\phi_0^2 + 2\Re(\phi_0\vec{z}_1) + |\vec{z}_1|^2 + \frac{1}{3}(1 - \phi_0^2 - \rho_0)^2(2 + \phi_0^2 + \rho_0) \right] \, dx \]
\[ = E_2[\phi_0] - 2\Re \int_{\mathbb{R}} \vec{z}(\phi_0^2 + \eta_0\phi_0) \, dx \]
\[ + \int_{\mathbb{R}} (|\vec{z}_1|^2 - \eta_0 |\vec{z}|^2) \, dx + \int_{\mathbb{R}} (\phi_0^3 + \frac{1}{3}\rho_0^3) \, dx, \]
thus, using (1.10), we have
\[ E_2[\phi_0 + \vec{z}] - E_2[\phi_0] = 2Q_0[\vec{z}] + N_0[\vec{z}], \]  
where $Q_0[\vec{z}]$ is the quadratic form
\[ Q_0[\vec{z}] := \frac{1}{2} \int_{\mathbb{R}} (|\vec{z}_1|^2 - \eta_0 |\vec{z}|^2) \, dx, \]  
and $N_0[\vec{z}]$ is the nonlinear term
\[ N_0[\vec{z}] := \int_{\mathbb{R}} (|\phi_0|^2 \rho_0^2 + \frac{1}{3}\rho_0^3) \, dx. \]  
In the case of $\vec{z} = f$ is a real-valued function belonging to the space $\mathcal{H}^{\text{real}}(\mathbb{R})$ (see (2.12)),
\[ Q_0[f] := \frac{1}{2} \int_{\mathbb{R}} [(f')^2 - \eta_0 f^2] \, dx. \]  
Then, considering now $\mathcal{H}^{\text{real}}(\mathbb{R})$ endowed with the inner product
\[ \langle f, g \rangle_0 := \int_{\mathbb{R}} (f'g' + \eta_0 fg) \, dx, \]  
we have that $(\mathcal{H}^{\text{real}}(\mathbb{R}), \langle \cdot, \cdot \rangle_0)$ is a Hilbert space with the induced norm
\[ \|f\|_{\mathcal{H}_0}^2 = \int_{\mathbb{R}} [(f')^2 + \eta_0 f^2] \, dx. \]  
For a fixed $f \in \mathcal{H}^{\text{real}}(\mathbb{R})$ we have $g \mapsto \int_{\mathbb{R}} \eta_0 fg \in [\mathcal{H}^{\text{real}}(\mathbb{R})]'$. Indeed,
\[ \left| \int_{\mathbb{R}} \eta_0 fg \, dx \right| \leq \|\eta_0^{1/2} f\|_{L^2} \|\eta_0^{1/2} g\|_{L^2} \leq \|\eta_0^{1/2} f\|_{L^2} \|g\|_{\mathcal{H}_0}. \]  
Therefore, by the Riesz Theorem, there exists a bounded and self-adjoint operator $T_0$ such that
\[ \langle T_0 f, g \rangle_0 = \int_{\mathbb{R}} \eta_0 f g \, dx, \quad \forall g \in H^{real}(\mathbb{R}), \]  
\begin{equation}
(3.9)
\end{equation}
and also

\[ \|T_0 f\|_{H_0} \leq \|\eta_0^{1/2} f\|_{L^2}. \]

Moreover, the quadratic form \( Q_0 \) satisfies

\[ Q_0[f] = \frac{1}{2} \int_{\mathbb{R}} \left( f' \right)^2 + \eta_0 f^2 \, dx - \int_{\mathbb{R}} \eta_0 f^2 \, dx = \langle (\frac{1}{2} I - T_0) f, f \rangle_0, \]  
\begin{equation}
(3.10)
\end{equation}
for all \( f \in H^{real}(\mathbb{R}) \).

**Lemma 3.1** (Compactness of \( T_0 \)). The operator \( T_0 : H^{real}(\mathbb{R}) \to H^{real}(\mathbb{R}) \) is compact.

**Proof.** Throughout the proof we will use \( M_j, j = 1, 2, \ldots, 6 \), to denote some universal constants.

Consider now a sequence \( f_n \in H^{real}(\mathbb{R}) \) such that

\[ \|f_n\|_{H_0}^2 = \|f_n'\|_{L^2}^2 + \|\eta_0^{1/2} f_n\|_{L^2}^2 \leq M_1, \quad \forall n \in \mathbb{N}. \]  
\begin{equation}
(3.11)
\end{equation}
Then, we can assume that

\[ f_n \rightharpoonup f^* \in H^{real}(\mathbb{R}), \quad \text{when} \quad n \to \infty. \]

**Claim 1:** It holds that

\[ \|\eta_0^{1/2} f_n\|_{H^1}^2 \leq M_2, \quad \text{for all} \quad n \in \mathbb{N}. \]  
\begin{equation}
(3.12)
\end{equation}
To obtain this estimate we note that

\[ \|\eta_0^{1/2} f_n\|_{H^1}^2 \leq \|\eta_0^{1/2} f_n\|_{H^2}^2 + \|\eta_0^{1/2} f_n'\|_{L^2}^2 + \| - \frac{2\phi_0^3 \phi_0'}{\eta_0^{1/2}} f_n\|_{L^2}^2. \]  
\begin{equation}
(3.13)
\end{equation}
Now, using \ref{2.15} and that \( |\phi_0| \leq 1 \), we get

\[ \frac{2|\phi_0|^3 |\phi_0'|}{\eta_0^{1/2}} = \frac{2}{\sqrt{3}} \frac{|\phi_0|^3 (2 + \phi_0^2)^{1/2}}{(1 + \phi_0^2)^{1/2}} (1 - \phi_0^2)^{1/2} \leq 2\eta_0^{1/2}, \]  
\begin{equation}
(3.14)
\end{equation}
so we have

\[ \| - \frac{2\phi_0^3 \phi_0'}{\eta_0^{1/2}} f_n\|_{L^2} \leq 2\|\eta_0^{1/2} f_n\|_{L^2}. \]  
\begin{equation}
(3.15)
\end{equation}
Then, using that \( \eta_0(x) \leq 3 \text{sech}^2(x) \), combined with \ref{3.11}, \ref{3.13} and \ref{3.15} we obtain the statement in \ref{3.12} and Claim 1 \ref{3.12} is proved.

In particular, from \ref{3.12} we conclude that

\[ |f_n(0)| \leq \|\eta_0^{1/2} f_n\|_{L^\infty} \leq M_3, \quad \forall n \in \mathbb{N}, \]  
\begin{equation}
(3.16)
\end{equation}
and also we can assume that

\[ \eta_0^{1/2} f_n \rightharpoonup \eta_0^{1/2} f^* \in C_0^{1, \infty}(\mathbb{R}), \]  
\begin{equation}
(3.17)
\end{equation}
i.e. we get uniform convergence on compact subsets of \( \mathbb{R} \).

**Claim 2:** It holds that

\[ \|\eta_0^{1/4} f_n\|_{L^2} \leq M_4, \quad \text{for all} \quad n \in \mathbb{N}. \]  
\begin{equation}
(3.18)
\end{equation}
To prove this estimate we first observe that

\[ f_n(x) = f_n(0) + \int_0^x f_n'(s) \, ds, \quad \forall n \in \mathbb{N}, \]
which implies that
\[ \eta_0^{1/4} |f_n(x)| \leq \eta_0^{1/4} |f_n(0)| + |x|^{1/2} \eta_0^{1/4} \|f_n'\|_{L^2}. \]
Then, from (3.11), (3.16) and using the exponential decay of \( \eta_0 \) we get the estimate (3.18) in Claim 2.

Now given \( \epsilon > 0 \), due to the exponential decay of \( \eta_0 \), we can take \( a_\epsilon > 0 \) such that
\[ \eta_0^{1/2} < \epsilon, \quad \forall |x| > a_\epsilon. \]
Then, using (3.18), we have
\[ \int_{|x| > a_\epsilon} \eta_0 (f_n - f^*)^2 dx \leq \epsilon \int_{|x| > a_\epsilon} \eta_0^{1/2} (f_n - f^*)^2 dx \leq \epsilon \eta_0^{1/4} (f_n - f^*) \|_{L^2} \leq M_5 \epsilon. \] (3.19)
Then, from (3.17) one gets
\[ \|\eta_0^{1/2} (f_n - f^*)\|_{L^\infty(|x| \leq a_\epsilon)} < \epsilon, \quad \forall n > n_\epsilon, \text{ with some } n_\epsilon \gg 1. \]
Therefore, by using (3.18), we get
\[ \int_{|x| \leq a_\epsilon} \eta_0 (f_n - f^*)^2 dx \leq \epsilon \int_{|x| \leq a_\epsilon} \eta_0^{1/2} |f_n - f^*| dx \leq \epsilon \|\eta_0^{1/4}\|_{L^2} \|\eta_0^{1/4} (f_n - f^*)\|_{L^2} \leq M_6 \epsilon. \] (3.20)
Now, from (3.19) and (3.20) notice that
\[ \int_{-\infty}^{+\infty} \eta_0 (f_n - f^*)^2 dx \lesssim \epsilon, \]
for all \( n \gg n_\epsilon \), so \( \lim_{n \to \infty} \|\eta_0^{1/2} (f_n - f^*)\|_{L^2} = 0 \). Finally, from (3.9) we have that
\[ \|T_0 (f_n - f^*)\|_{H_0} \leq \|\eta_0^{1/2} (f_n - f^*)\|_{L^2}, \]
which implies that
\[ T_0 f_n \xrightarrow{n \to \infty} T_0 f^* \quad \text{in} \quad H^{real}(\mathbb{R}), \]
and the proof is finished.

**Proposition 3.2 (Coercivity of \( E_2 \) around the black soliton).** Let \( \tilde{\eta} \in H(\mathbb{R}) \) be such that the perturbation \( \phi_0 + \tilde{\eta} \in Z(\mathbb{R}) \) and set \( \rho_0 = 2 \text{Re}(\phi_0 \tilde{\eta}) + |\tilde{\eta}|^2 \) as in (3.1). Then there exists a universal positive constant \( \Lambda_0 > 0 \) such that
\[ E_2[\phi_0 + \tilde{\eta}] - E_2[\phi_0] \geq \Lambda_0 (\|\tilde{\eta}\|_{H_0}^2 + \|\phi_0 \tilde{\eta}\|_{L^2}^2 + \|\tilde{\eta} \rho_0\|_{L^2}^2) - \frac{1}{\Lambda_0} \|\tilde{\eta}\|_{H_0}^3 \] (3.21)
as soon as
\[ \int_{\mathbb{R}} \langle \eta_0, \tilde{\eta} \rangle_{C} = 0, \quad \int_{\mathbb{R}} \langle i \eta_0, \tilde{\eta} \rangle_{C} = 0, \quad \text{and} \quad \int_{\mathbb{R}} \langle i \phi_0 \eta_0, \tilde{\eta} \rangle_{C} = 0. \] (3.22)

**Proof.** The proof will be divided into 3 steps.

**Step 1:** There exists a constant \( \Lambda_1 > 0 \) such that
\[ Q_0[f] \geq \Lambda_1 (f, f)_0 = \Lambda_1 \int_{\mathbb{R}} [(f')^2 + \eta_0 f^2] dx, \]
for any function \( f \in H^{real}(\mathbb{R}) \) such that
\[ (1a) \int_{\mathbb{R}} f \eta_0 dx = 0 \quad \text{and} \quad \int_{\mathbb{R}} f \phi_0 \eta_0 dx = 0. \]
Furthermore,
(1b) \( Q_0[f] \geq 0 \) if only the first orthogonality condition in (1a) is satisfied.

**Proof of Step 1.** Recall that from (3.10) we have

\[
Q_0[f] = \langle (\frac{1}{2} \mathbb{I} - T_0)f, f \rangle_0 = \langle \tilde{Q}_0 f, f \rangle_0,
\]

where

\[
\tilde{Q}_0 := \frac{1}{2} \mathbb{I} - T_0.
\]  

(3.23)

Then, using the Spectral Theorem, there exists a sequence \( \{\lambda_n\} \) of eigenvalues for \( \tilde{Q}_0 \) with \( \lim_{n \to +\infty} \lambda_n = \frac{1}{2} \) and a Hilbert basis \( \{e_n\} \) of \( H_{\text{real}}(\mathbb{R}) \) such that

\[
\tilde{Q}_0 e_n = \lambda_n e_n, \quad n \in \mathbb{N}.
\]

Notice that

\[
Q_0[f] \leq \frac{1}{2} \langle f, f \rangle_0 \quad \forall f \in H_{\text{real}}(\mathbb{R}),
\]

consequently,

\[
\tilde{Q}_0 \leq \frac{1}{2} \mathbb{I}, \quad (\lambda_n)_{n \in \mathbb{N}} \subset (-\infty, \frac{1}{2}] \quad \text{and} \quad \lambda_n \nearrow \frac{1}{2}.
\]

Now, let \( \lambda \in (-\infty, \frac{1}{2}] \) be an eigenvalue with \( f \) as the corresponding eigenfunction. Then for all \( g \in H_{\text{real}}(\mathbb{R}) \) we have

\[
\langle \tilde{Q}_0 f, g \rangle_0 = \lambda \langle f, g \rangle_0.
\]

So, from (3.9), it holds that

\[
\frac{1}{2} \int_{\mathbb{R}} f'g' dx - \frac{1}{2} \int_{\mathbb{R}} \eta_0 fg dx = \lambda \left[ \int_{\mathbb{R}} f'g' dx + \int_{\mathbb{R}} \eta_0 fg dx \right]
\]

which yields

\[
\int_{\mathbb{R}} \left[ (1 - 2\lambda)f'' + (2\lambda + 1)\eta_0 f \right] g dx = 0,
\]

for all \( g \in H_{\text{real}}(\mathbb{R}) \). Thus,

\[
(1 - 2\lambda)f'' + (2\lambda + 1)\eta_0 f = 0 \implies -f'' - \eta_0 f = \frac{4\lambda}{1 - 2\lambda} \eta_0 f,
\]

therefore

- \( f = 1 =: e_0 \) is a solution for \( \lambda = -\frac{1}{2} \),
- \( f = \phi_0 =: e_1 \) is a solution for \( \lambda = 0 \).

Note that, since \( \phi_0 \) has exactly one zero, the Sturm-Liouville theory guarantees that \( \lambda = -\frac{1}{2} \) is the only negative eigenvalue of \( Q_0 \) with kernel given by the \( \text{span}(\phi_0) \), more precisely

\[
\lambda_0 = -\frac{1}{2} < 0 = \lambda_1 < \lambda_2 < \cdots < \frac{1}{2},
\]

and

\[
\text{Ker}(\tilde{Q}_0 + \frac{1}{2} \mathbb{I}) = \mathbb{R}, \quad \text{Ker}(\tilde{Q}_0) = \mathbb{R} \cdot \phi_0.
\]

Thus, expanding \( f \in H_{\text{real}} \) on the normalized basis of eigenfunctions

\[
f = \sum_{n=0}^{+\infty} \langle f, \tilde{e}_n \rangle_0 \tilde{e}_n, \quad \tilde{e}_n = \frac{e_n}{\|e_n\|_{H_0}},
\]

if \( \langle f, 1 \rangle_0 = \langle f, \phi_0 \rangle_0 = 0 \) we get

\[
f = \sum_{n=2}^{+\infty} \langle f, \tilde{e}_n \rangle_0 \tilde{e}_n, \quad \text{and then}
\]

\[
Q_0[f] = \langle \tilde{Q}_0[f], f \rangle_0 = \sum_{n=2}^{+\infty} \lambda_n \langle f, \tilde{e}_n \rangle_0^2 \geq \lambda_2 \sum_{n=2}^{+\infty} \langle f, \tilde{e}_n \rangle_0^2 = \lambda_2 \langle f, f \rangle_0.
\]  

(3.24)

Hence, under hypothesis:
inequality, we obtain the estimate
\[ \langle f, 1 \rangle_0 = \int_\mathbb{R} \eta_0 f \, dx = 0 \quad \text{and} \quad \langle f, \phi_0 \rangle_0 = 2 \int_\mathbb{R} f \phi_0 \eta_0 = 0, \]
and taking \( \Lambda_1 := \lambda_2 \) we finish the proof of Step 1.

**Step 2:** Let \( z \in \mathcal{H}(\mathbb{R}) \) fulfilling the orthogonality conditions (3.22). Then, it follows that
\[ E_2[\phi_0 + z] - E_2[\phi_0] \geq 2Q_0[\bar{z}] + \frac{1}{\bar{z}}\|\phi_0 \rho_0 \|^2_{L^2} + \frac{1}{\bar{z}}\|\bar{z} \rho_0 \|^2_{L^2} + 2\Lambda_1|\bar{z}|^2_{\mathcal{H}_0}, \]
with \( \Lambda_1 \) as in Step 1.

**Proof of Step 2.** Recall that from the expansion of \( E_2 \) in (3.2) we have
\[ E_2[\phi_0 + z] - E_2[\phi_0] = 2Q_0[\bar{z}] + N_0[\bar{z}], \]
where \( \bar{z} = \bar{z}_1 + i\bar{z}_2, \rho_0 = 2\phi_0 \bar{z}_1 + |\bar{z}|^2 \) and \( Q_0 \) satisfying \( Q_0[\bar{z}] = Q_0[\bar{z}_1] + Q_0[\bar{z}_2] \). Applying Young’s inequality, we obtain the estimate
\[ N_0[\bar{z}] = \int_\mathbb{R} (\phi_0^2 \rho_0 + \frac{1}{\bar{z}}\rho_0^3) \, dx \leq \int_\mathbb{R} \phi_0^2 \rho_0 \, dx + \frac{1}{\bar{z}} \int_\mathbb{R} (2\phi_0 \bar{z}_1 + |\bar{z}|^2) \rho_0^2 \, dx \geq \int_\mathbb{R} \phi_0^2 \rho_0 \, dx + \frac{1}{\bar{z}} \int_\mathbb{R} |\bar{z}|^2 \rho_0^2 \, dx - \int_\mathbb{R} \phi_0 \bar{z}_1 \rho_0^2 \, dx \]
(3.25)
\[ \geq \frac{2}{\bar{z}} \int_\mathbb{R} \phi_0^2 \rho_0 \, dx + \frac{1}{\bar{z}} \int_\mathbb{R} (|\bar{z}|^2 - \bar{z}_1^2) \rho_0^2 \, dx \]
\[ = \frac{2}{\bar{z}} \int_\mathbb{R} \phi_0^2 \rho_0 \, dx + \frac{1}{\bar{z}} \int_\mathbb{R} \bar{z}_2 \rho_0^2 \, dx. \]
On the other hand, from Step 1, the first two orthogonality conditions in (3.22) imply that
\[ Q_0[\bar{z}_1] \geq 0 \quad \text{and} \quad Q_0[\bar{z}_2] \geq 0, \]
(3.26)
while, in addition, the last orthogonality condition in (3.22) ensures that
\[ Q_0[\bar{z}_2] \geq \Lambda_1|\bar{z}_2|^2_{\mathcal{H}_0}, \]
(3.27)
where \( \Lambda_1 := \lambda_2 \) is the first positive eigenvalue obtained in Step 1. Then, putting the bounds given in (3.25), (3.26) and (3.27) into the expansion of \( E_2 \), we obtain the claimed estimate in Step 2.

Since \( Q_0[\bar{z}_1] \geq 0 \), in order to complete the proof of Proposition 3.2 we remark that, bearing in mind the estimate in the Step 2, we only need to show the coercivity property for the operator \( Q_0 \) on the full variable \( \bar{z} \). We will explain this in the next step.

**Step 3:** Now we proceed with the proof of (3.21).

**Proof of Step 3.** We begin by estimating the term \( \frac{2}{\bar{z}}\|\phi_0 \rho_0 \|^2_{L^2} \) which appears in the lower estimate of the Step 2
\[ \frac{2}{\bar{z}}\|\phi_0 \rho_0 \|^2_{L^2} = \frac{2}{\bar{z}}\|\phi_0 \rho_0 \|^2_{L^2} + I, \]
(3.28)
where
\[ I := \frac{2}{\bar{z}} \int_\mathbb{R} \eta_0 \phi_0^2 \rho_0^2 \, dx \]
\[ = \frac{2}{\bar{z}} \int_\mathbb{R} \eta_0 \phi_0^2 \bar{z}_1^4 \, dx + \frac{8}{3} \int_\mathbb{R} \eta_0 \phi_0^4 \bar{z}_1^2 \, dx + \frac{8}{3} \int_\mathbb{R} \eta_0 \phi_0^2 |\bar{z}_1|^2 \, dx \]
(3.29)
\[ := I_1 + I_2 + I_3 \geq I_2 - |I_3|. \]
Now, bearing in mind (1.10) and integrating by parts, we simplify \( I_3 \) as follows:
\[ I_3 = -\frac{8}{3} \int_\mathbb{R} \phi_0' \phi_0^2 \bar{z}_1^2 \, dx = \frac{8}{3} \int_\mathbb{R} 2(\phi_0')^2 \phi_0 \bar{z}_1 |\bar{z}_1|^2 \, dx + \frac{8}{3} \int_\mathbb{R} \phi_0 |\phi_0'(\bar{z}_1 |\bar{z}_1|^2)' \, dx := I_{3,1} + I_{3,2}. \]
(3.30)
Using (2.15), the inequality \(0 < 1 - \phi_0^2 \leq 1 - \phi_0^4\) and a Gagliardo-Nirenberg inequality we obtain

\[
|I_{3,1}| \leq \frac{16}{3} \int_R (1 - \phi_0^2)^2 (2 + \phi_0^2) |\phi_0| |\phi_0'| |\phi_0|^2 dx
\]

\[
\leq \frac{16}{3} \| (1 - \phi_0^2)^{2/3} \|_{L^2}^3
\]

\[
\leq \| (1 - \phi_0^2)^{1/2} \|_{L^2}^3
\]

\[
\leq \| (1 - \phi_0^2)^{1/2} \|_{L^2}^{1/2} \| (1 - \phi_0^2)^{1/2} \|_{L^2}^{1/2}
\]

\[
\leq \| |\phi_0| |\phi_0'| (1 - \phi_0^2)^{1/2} \|_{L^2}^{1/2} + \| |(1 - \phi_0^2) \|_{L^2}^{1/2}
\]

\[
\leq \| \| \phi_0 \|_{\mathcal{H}_0} \| (1 - \phi_0^2)^{1/2} \|_{L^2}^3
\]

\[
\leq \| |\phi_0| \|_{\mathcal{H}_0}^3
\]

and in a similar way we deduce

\[
|I_{3,2}| \leq \frac{8}{3} \int_R \phi_0^2 (1 - \phi_0^2) (2 + \phi_0^2)^{1/2} |\phi_0'| (3\phi_1^2 + \phi_2^2) + 2\phi_3 \phi_4 \phi_5^2 dx
\]

\[
\leq \frac{8}{3} \int_R (1 - \phi_0^2) |\phi_0'| (3\phi_1^2 + \phi_2^2) + 2\phi_3 \phi_4 \phi_5^2 dx
\]

\[
\leq \int_R (1 - \phi_0^2) (|\phi_0'| + |\phi_0'|) |\phi_0'| dx
\]

\[
\leq \| |\phi_0| \|_{\mathcal{H}_0} \| |\phi_0| \|_{L^2}^3
\]

\[
\leq \| |\phi_0| \|_{\mathcal{H}_0} \| (1 - \phi_0^2)^{1/2} \|_{L^2}^3
\]

\[
\leq \| |\phi_0| \|_{\mathcal{H}_0} \| (1 - \phi_0^2)^{1/2} \|_{L^2}^3
\]

\[
\leq \| |\phi_0| \|_{\mathcal{H}_0}^3
\]

Therefore, combining (3.28), (3.29), (3.30), (3.31) and (3.32) we get, for some positive number \(\gamma\)

\[
\frac{2}{3} \| |\phi_0| \|_{L^2}^3 \geq \frac{2}{3} \| |\phi_0| \|_{L^2}^3 + \frac{8}{3} \int_R \eta_0 |\phi_0|^2 dx - \gamma \| |\phi_0| \|_{\mathcal{H}_0}^3.
\]

Now, we consider the real function \(\tilde{\phi}_1 := \tilde{\phi}_1 - (\tilde{\phi}_1, e_1)e_1\), where \(e_1 = \phi_0/|\phi_0|_{\mathcal{H}_0}\). Then, using the first orthogonality condition in (3.22) we have \((\tilde{\phi}_1, e_0) = (\tilde{\phi}_1, e_1) = 0\). Thus, the expansion of \(\tilde{\phi}_1\) is given by

\[
\tilde{\phi}_1 = \sum_{n=2}^{+\infty} (\tilde{\phi}_1, e_n) e_n
\]

and \(Q_0[\tilde{\phi}_1] = Q_0[\tilde{\phi}_1]\). Hence, from Step 1, it follows that

\[
Q_0[\tilde{\phi}_1] \geq \lambda_2 \| \tilde{\phi}_1 - (\tilde{\phi}_1, e_1)e_1 \|_{\mathcal{H}_0}^2 = \lambda_2 \| \tilde{\phi}_1 \|_{\mathcal{H}_0}^2 - \lambda_2 (\tilde{\phi}_1, e_1)^2.
\]

Now, for any number \(0 < \nu < 1\) which will be chosen later, using the identities (2.16) and

\[
\| |\phi_0| \|_{\mathcal{H}_0}^2 = 2\| |\phi_0| \|_{L^2}^2 \quad \text{and} \quad \int_R \eta_0 dx = 3,
\]

and combined with the Cauchy-Schwarz inequality we have
Remark 3.3. We remind that in the case of cubic GP treated in [14], the corresponding black soliton (denoted as $U_0$) satisfies the relation $U_0 = \frac{1}{\sqrt{2}}(1 - U_0^3)$, and hence, the orthogonality condition

$$\langle f, 1 \rangle_0 = \int_{\mathbb{R}} (1 - U_0^3) f dx = \sqrt{2} \int_{\mathbb{R}} U_0 f dx = 0,$$

but in the case of the quintic GP (1.1) this relation is not satisfied anymore.
Now we perturb the black soliton $\phi_0$ of (1.1) with a function $u \in \mathcal{H}(\mathbb{R})$ belonging to the orbit generated by the symmetries of (1.1), namely

$$\mathcal{U}_0(\alpha) := \left\{ w \in \mathcal{H}(\mathbb{R}) : \inf_{(b, c) \in \mathbb{R}^2} ||e^{-it}w(\cdot + b) - \phi_0||_{\mathcal{H}_0(\mathbb{R})} < \alpha \right\},$$

(3.39)

for some $\alpha > 0$ and then, given a function $u \in \mathcal{U}_0(\alpha)$ we can choose $(c, \iota, b) \in (-2, 2) \times \mathbb{R}^2$ in such a way that

$$e^{-it}u(\cdot + b) = \phi_c + \zeta,$$

with $\zeta$ satisfying the orthogonality conditions (3.43) around the dark soliton.

Finally note that we can define the following tubular subset of $\mathcal{U}_0(\alpha)$,

$$\mathcal{V}_0(\alpha) := \left\{ v \in \mathcal{Z}(\mathbb{R}) : \inf_{(b, c) \in \mathbb{R}^2} d_0(e^{-it}v(\cdot + b), \phi_0) < \alpha \right\} \subset \mathcal{U}_0(\alpha).$$

(3.40)

Coming back to the main question on the orbital stability of the black soliton, we use the coercivity of $E_2$ around the black soliton $\phi_0$ to small perturbations around the dark soliton $\phi_c$ (see Proposition 3.4).

In fact, the idea to introduce a dark soliton family in this argument is to give an extra degree of freedom which allows us to satisfy the third constraint in (3.1) rewritten as (3.12). In that case, the situation is different with respect to the cubic GP equation, because we can not assume the cancellation of the linear term $\langle i\phi'_c, \zeta \rangle_\mathcal{C}$ in our approach, given the orthogonality conditions arising naturally, from the particular structure of the associated spectral problem as we already saw in Proposition 3.2 e.g. (3.24). In fact, this extra technical difficulty introduced by the linear term $\langle i\phi'_c, \zeta \rangle_\mathcal{C}$ is overcome in Proposition 3.4 by using a previously computed $L^2$ norm (2.25).

Before establishing the next result, and with (2.8), we fix the following notation:

$$\rho_c(\zeta) := |\phi_c + \zeta|^2 - |\phi_c|^2 = 2(\phi_c, \zeta)_\mathcal{C} + |\zeta|^2 = 2 \text{Re}(\phi_c, \zeta) + |\zeta|^2.$$

(3.41)

**Proposition 3.4** (Coercivity of $E_2$ around the dark soliton). There exists $\epsilon \in (0, 2)$ small enough such that the following holds. For all $|c| \leq \epsilon$ and for any $\zeta \in \mathcal{H}(\mathbb{R})$ satisfying

$$\phi_c + \zeta \in \mathcal{Z}(\mathbb{R}), \text{ with } \|\rho_c(\zeta)\|_{L^2} < C,$$

(3.42)

for some constant $C$ and the generalized orthogonality conditions

$$\int_{\mathbb{R}} \langle \eta_c, \zeta \rangle_\mathcal{C} = 0, \quad \int_{\mathbb{R}} \langle i\eta_c, \zeta \rangle_\mathcal{C} = 0 \quad \text{and} \quad \int_{\mathbb{R}} \langle iR_c\eta_c, \zeta \rangle_\mathcal{C} = 0,$$

(3.43)

there exists $\hat{\Gamma} > 0$, not depending on $c$, such that

$$E_2[\phi_c + \zeta] - E_2[\phi_0] \geq \hat{\Gamma}(||\zeta||^2_{\mathcal{H}_0} + ||\phi_c^3\rho_c||^2_{L^2}) - \frac{1}{\hat{\Gamma}}(c^2 + ||\phi_c||^2_{\mathcal{H}_0}).$$

(3.44)

**Remark 3.5.** Note that the quadratic term $||\phi_c^2\rho_c||^2_{L^2}$ is not appearing in (3.44) because the lower bound is already guaranteed only with the current terms. Hereafter and for the sake of simplicity we will not include such a quadratic term but note that keeping it, we would recover (3.21) in the limit $c \to 0$.

**Proof.** First of all, we remind that $\zeta = \zeta_1 + i\zeta_2 \in \mathcal{H}(\mathbb{R})$ and that defining the quadratic form in $\zeta$

$$Q_c[\zeta] := \frac{1}{2} \int_{\mathbb{R}} \left( |\zeta|^2 - \eta_c |\zeta|^2 \right) dx,$$

(3.45)

and the nonlinear term

$$N_c[\zeta] := \int_{\mathbb{R}} \left( |\phi_c|^2 \rho_c^2 + \frac{1}{2} \rho_c^3 \right) dx,$$

(3.46)

1Note that $\epsilon$ is chosen as the minimum of the values that guarantee that some precise estimates in the proof hold for, e.g. (2.33) (3.48).
we have
\[ E_2[\phi_c + \beta] - E_2[\phi_c] = -c \int_\mathbb{R} 2 \text{Re}(i\phi_c')dx + 2Q_c[3] + N_c[3]. \]  
(3.47)

Also (see (2.19), (2.20) and (2.21)) we already know that for small \( c \)
\[ E_2[\phi_c] - E_2[\phi_0] = -\frac{1}{4} \left( 3 + E_2[\phi_0] \right) c^2 + O(c^3) \geq -2\sqrt{3}c^2. \]  
(3.48)

We recall (see (3.2)) that
\[ E_2[\phi_0 + \beta] - E_2[\phi_0] = 2Q_0[31] + 2Q_0[32] + N_0[3]. \]  
(3.49)

which implies
\[ E_2[\phi_c + \beta] - E_2[\phi_c] = (E_2[\phi_c + \beta] - E_2[\phi_0 + \beta]) + 2Q_0[31] + 2Q_0[32] + N_0[3]. \]  
(3.50)

We begin by computing the first term on the r.h.s. of (3.50). Note that subtracting (3.47) and (3.49), we have
\[ E_2[\phi_c + \beta] - E_2[\phi_c] = E_2[\phi_c] - E_2[\phi_0] \]
\[ - c \int_\mathbb{R} 2 \text{Re}(i\phi_c')dx + \int_\mathbb{R} (|\phi_c|^4 - |\phi_0|^4)|3|^2dx + \Delta N[3], \]  
(3.51)

where
\[ \Delta N[3] := N_c[3] - N_0[3] = \int_\mathbb{R} (|\phi_c|^2 \rho_c^2 - \phi_0^2 \rho_0^2)dx + \int_\mathbb{R} \frac{1}{3} (\rho_c^3 - \rho_0^3)dx. \]  
(3.52)

Substituting (3.51) in the r.h.s. of (3.50) we get
\[ E_2[\phi_c + \beta] - E_2[\phi_0] = 2Q_0[31] + 2Q_0[32] + (E_2[\phi_c] - E_2[\phi_0]) \]
\[ + N_c[3] - c \int_\mathbb{R} 2 \text{Re}(i\phi_c')dx + \int_\mathbb{R} (|\phi_c|^4 - |\phi_0|^4)|3|^2dx. \]  
(3.53)

Hereinafter, unless otherwise noted, we shall consider the constant \( c \) as defined in (2.34). Now we proceed to estimate the last three terms in (3.53) and we begin by the last integral. Using (2.33) we have
\[ \left| \int_\mathbb{R} (|\phi_c|^4 - |\phi_0|^4)|3|^2dx \right| \leq \int_\mathbb{R} ||\phi_c|^4 - |\phi_0|^4||3|^2dx \]  
(3.54)

for some positive constant \( \beta \) and all \( |c| \leq \beta \). Now we continue estimating the linear term
\[ - c \int_\mathbb{R} 2 \text{Re}(i\phi_c')dx. \]  
In fact, we use (2.34) and (2.25) to get
\[ \left| - c \int_\mathbb{R} 2 \text{Re}(i\phi_c')dx \right| \leq 2|c| \left\| \frac{\phi_c'}{\sqrt{\eta_c}} \right\|_{L^2} \left\| \sqrt{\eta_c} \cdot 3 \right\|_{L^2} \]  
(3.55)

with a larger constant \( \beta \) if necessary, and all \( |c| \leq \beta \). Now we estimate the nonlinear term \( N_c[3] \). Using that \( |\text{Re}(\phi_c')| \leq \frac{|\phi_c|^2 + |3|^2}{2} \) we get
\[ N_c[3] = \int_\mathbb{R} |\phi_c|^2 \rho_c^2dx + \frac{1}{3} \int_\mathbb{R} (2 \text{Re}(\phi_c' + |3|^2) \rho_c^2dx \]
\[ \geq \int_\mathbb{R} |\phi_c|^2 \rho_c^2dx + \frac{1}{3} \int_\mathbb{R} |3|^2 \rho_c^2dx - \frac{2}{3} \int_\mathbb{R} \text{Re}(\phi_c') \rho_c^2dx \]  
(3.56)
Combining (3.48), (3.53), (3.54), (3.55) and (3.56) and Young’s inequality we obtain

\[
E_2[\phi_c + \zeta] - E_2[\phi_0] \geq 2Q_0[\zeta_1] + 2Q_0[\zeta_2]
\]

\[
- 2\sqrt{3}c^2 + \frac{2}{3}\|\phi_c \rho_c\|_{L^2}^2 - \beta|c|\|\zeta_1\|_{H^0} - \beta c^2\|\zeta_2\|_{H^0}
\]

\[
\geq 2Q_0[\zeta_1] + 2Q_0[\zeta_2]
\]

\[
- 2\sqrt{3}c^2 + \frac{2}{3}\|\phi_c \rho_c\|_{L^2}^2 - \beta_1 c^2 - \beta_2\|\zeta_1\|_{H^0} - \beta c^2\|\zeta_2\|_{H^0},
\]

(3.57)

for all $|c| \leq \epsilon$ and $\beta_2$ to be fixed later. Now we split the components of the perturbation $\zeta = \zeta_1 + i\zeta_2$ in the following way

\[
\zeta_1 = \zeta_1^* + \omega_1(c)\eta_0
\]

\[
\zeta_2 = \zeta_2^* + \omega_2(c)\eta_0 + \omega_3(c)\phi_0\eta_0,
\]

(3.58)

with $\omega_1$, $\omega_2$ and $\omega_3$ real-valued functions chosen so that $\zeta^* := \zeta_1^* + i\zeta_2^*$ satisfies the orthogonality conditions in (3.22). Thus, using (3.43), the functions $\omega_i$ satisfy the relations

\[
\int_{\mathbb{R}} \langle \eta_0 - \eta_c, \bar{\zeta} \rangle_c dx = \int_{\mathbb{R}} \langle \eta_0, \bar{\zeta} \rangle_c dx = \omega_1(c) \int_{\mathbb{R}} \eta_0^2 dx,
\]

\[
\int_{\mathbb{R}} \langle i\eta_0 - i\eta_c, \bar{\zeta} \rangle_c dx = \int_{\mathbb{R}} \langle i\eta_0, \bar{\zeta} \rangle_c dx = \omega_2(c) \int_{\mathbb{R}} \eta_0^2 dx + \omega_3(c) \int_{\mathbb{R}} \phi_0\eta_0^2 dx,
\]

\[
\int_{\mathbb{R}} \langle i\phi_0\eta_0 - iRc\eta_c, \bar{\zeta} \rangle_c dx = \int_{\mathbb{R}} \langle i\phi_0\eta_0, \bar{\zeta} \rangle_c dx = \omega_2(c) \int_{\mathbb{R}} \phi_0\eta_0^2 dx + \omega_3(c) \int_{\mathbb{R}} \phi_0^2\eta_0^2 dx,
\]

(3.59)

and the system has a solution, because it has a nonvanishing determinant $\epsilon$. \footnote{Note that for parity reasons $\int_{\mathbb{R}} \phi_0\eta_0^2 dx = 0$.}

Now, using (2.27) and (2.28), the integrals in the left hand of (3.59) can be estimated as follows

\[
|\int_{\mathbb{R}} \langle \eta_0 - \eta_c, \bar{\zeta} \rangle_c dx| + |\int_{\mathbb{R}} \langle i\eta_0 - i\eta_c, \bar{\zeta} \rangle_c dx| + |\int_{\mathbb{R}} \langle i\phi_0\eta_0 - iRc\eta_c, \bar{\zeta} \rangle_c dx| \lesssim c^2 \|\sqrt{\eta_0}\|_{L^2} \lesssim c^2\|\eta_0\|_{H^0},
\]

and therefore there exists a positive constant $\beta_3$ such that

\[
|\omega_1(c)| + |\omega_2(c)| + |\omega_3(c)| \leq \beta_3 c^2\|\eta_0\|_{H^0}.
\]

(3.60)

Notice that we can obtain the following estimates, by using Step 1 of Proposition 3.2 applied to $\zeta_1$ and $\zeta_2$, and combined with (3.60),

\[
Q_0[\zeta_1] = Q_0[\zeta_1^*] - \omega_1^2 Q_0[\eta_0] + \omega_1 \left( \int_{\mathbb{R}} \zeta_1^* \eta_0^2 dx - \int_{\mathbb{R}} \zeta_1 \eta_0^2 dx \right)
\]

\[
\geq Q_0[\zeta_1^*] - \omega_1^2 |Q_0[\eta_0]| - |\omega_1| \left( \|\zeta_1^*\|_{L^2} \|\eta_0\|_{L^2} + \|\zeta_1 \sqrt{\eta_0}\|_{L^2} \|\eta_0\|_{H^0} \right) + \|\zeta_1 \sqrt{\eta_0}\|_{L^2} \|\eta_0\|_{H^0} \right)
\]

\[
\geq Q_0[\zeta_1^*] - \beta_4 c^2\|\zeta_1\|_{H^0}^2,
\]

(3.61)

where $Q_0[\zeta_1^*] \geq 0$, $\beta_4$ is a positive constant and $|c| < \epsilon < 1$. Analogously, since $\zeta_2$ verifies the inequality $Q_0[\zeta_2] \geq \Lambda_1\|\zeta_2\|_{H^0}^2$, we deduce that

\[
Q_0[\zeta_2] \geq \Lambda_1\|\zeta_2\|_{H^0}^2 - \beta_4 c^2\|\zeta_2\|_{H^0}^2.
\]

(3.62)

for a larger $\beta_4$ if necessary and for all $|c| < \epsilon < 1$.

On the other hand, since $\zeta_1^*$ is orthogonal to $\eta_0$ in $L^2$, the same arguments used to obtain (3.55) in Step 3 of Proposition 3.2 allow us to conclude the existence of positive $\Lambda_\nu$ such that
Note that we shall proceed in a similar way as in (3.28) - (3.29), estimating the following \( \beta < \nu < 1 \) such that \( \frac{3(1+\nu)^2}{2\|\phi_0\|_{H_0}^2} < 1 \).

Now, combining (3.61), (3.62) and (3.63) we have

\[
Q_0[\beta] + Q_0[\beta'] \geq \Lambda \nu \|\beta\|_{H_0}^2 + \Lambda_1 \|\beta'\|_{H_0}^2 - (2\beta_4 + \beta_5) c^2 \|\beta\|_{H_0}^2 - J[\beta] \tag{3.64}
\]

and substituting (3.64) in (3.57) it follows that

\[
E_2[\phi_c + \zeta] - E_2[\phi_0] \geq 2 \Lambda \nu \|\beta\|_{H_0}^2 + 2 \Lambda_1 \|\beta'\|_{H_0}^2 + \frac{2}{3} \|\phi_c \rho_c\|_{L^2}^2 - 2 J[\beta] - \beta_2 \|\beta\|_{H_0}^2 - (\beta_1 + 2\sqrt{3}) c^2 - \beta_6 c^2 \|\beta\|_{H_0}^2, \tag{3.65}
\]

where \( \beta_6 = \beta + 4\beta_4 + 2\beta_5 \). At this point, to control the effect of \( J[\beta] \) on the lower bound of (3.65), we shall proceed in a similar way as in (3.28) - (3.29), estimating the following \( L^2 \) norm:

\[
\frac{2}{3} \|\phi_c \rho_c\|_{L^2}^2 \geq \frac{2}{3} \|\phi_c \rho_c\|_{L^2}^2 + \frac{8}{3} \int_R \eta_0 \phi_0^4 \zeta dx + \frac{8}{3} \int_R \left( \eta_c |\phi_c|^2 R_c^2 - \eta_0 \phi_0^2 \right) \zeta dx \tag{3.66}
\]

\[
- \frac{8}{3} \int_R \eta_c |\phi_c|^2 R_c \beta_1 + I_c \beta_2 \|\beta\|_{H_0}^2 dx - \frac{16}{3} \int_R \eta_c |\phi_c|^2 R_c I_c \beta_1 \beta_2 dx,
\]

where \( R_c, I_c \) are defined in (2.2) and (2.7). Now, using that \( |\phi_c| \leq 1 \), (2.26) and (2.34) one gets

\[
|J_c[\beta]| \lesssim c \|\beta\|_{H_0}^2. \tag{3.67}
\]

On the other hand, since \( |\phi_c| \leq 1 \) we have

\[
J_{b}[\beta] \leq \int_R \eta_c |\phi_c|^3 \|\beta\|^3 dx \leq \int_R \left( \eta_c |\phi_c|^2 - \eta_0 \phi_0^2 \right) \|\beta\|^3 dx + \int_R \eta_0 \phi_0^2 \|\beta\|^3 dx \tag{3.68}
\]

\[
:= J_{b1}[\beta] + J_{b2}[\beta].
\]

Note that

\[
J_{b2}[\beta] = - \int_R \phi_0^2 |\phi_0|^3 dx;
\]

so in a similar way as for the integral \( I_3 \) in (3.30) we have

\[
|J_{b2}[\beta]| \lesssim \|\beta\|_{H_0}^2. \tag{3.69}
\]

Remembering that (see Appendix \[C\] for details)

\[
\|\beta\|_{L^\infty} \lesssim (1 + \|\rho_c\|_{L^2})(1 + \|\beta\|_{H_0}), \tag{3.70}
\]

then using (3.42), we obtain

\[
\|\beta\|_{L^\infty} \lesssim 1 + \|\beta\|_{H_0}, \tag{3.71}
\]
and finally by (2.29), we obtain
\[ |J_{b1}[\xi]| \lesssim (1 + \|\zeta\|_{H_0})^2 \int_R \eta_c |\phi_c|^2 - \frac{\eta_0 |\phi_0|^2}{\sqrt{\tau_0}} |\zeta| dx \]
\[ \lesssim c^2 (\|\zeta\|_{H_0} + \|\zeta\|^3_{H_0}). \]  
(3.72)

Following the same procedure as in estimate \(J_{b1}[\xi]\) and using (2.30) we get
\[ |J_{a1}[\xi]| \lesssim c^2 (\|\zeta\|_{H_0} + \|\zeta\|^3_{H_0}). \]
(3.73)

Therefore, collecting estimates \(J_a, J_b, J_c\), we conclude that
\[ \frac{2}{3} \|\phi_c \rho_c\|_{L^2}^2 \geq \frac{2}{3} \|\phi_c^3 \rho_c\|_{L^2}^2 + \frac{8}{3} \int R |\phi_c|^2 R^2 |\zeta|^2 dx - \gamma_1 \|\zeta\|^3_{H_0} - \gamma_2 (c^2 + \|\zeta\|^2_{H_0}) - \beta_7 c^2, \]
(3.74)

for some positive numbers \(\gamma_1, \gamma_2, \beta_7\). Finally, we fix \(\beta_7\) such that \(\beta_7 < \min(\Lambda_1, \Lambda_2)\) and 0 < \(\nu < 1\), smaller if necessary, such that
\[ 6(1 + \nu)^2 \|\phi_0\|_{H_0}^2 < \frac{8}{3}. \]

Then, substituting (3.74) into (3.65), the second term on the right hand side of (3.74) allows to control the integral \(J[\xi]\) and consequently we can take a positive constant \(\Gamma_\varepsilon\) such that
\[ E_2[\phi_c + \xi] - E_2[\phi_0] \geq \Gamma_\varepsilon (\|\zeta\|^3_{H_0} + \|\phi_c^3 \rho_c\|^2_{L^2}) - \frac{1}{\Gamma_\varepsilon} (c^2 + \|\zeta\|^3_{H_0}), \]
for all \(|c| < \varepsilon\).

Notice that in the process of obtaining the constant \(\Gamma_\varepsilon\) we see that this coercivity constant is lower bounded by a constant \(\tilde{\Gamma}\) when \(\varepsilon \to 0\). In other words, \(\Gamma_\varepsilon \geq \tilde{\Gamma}\) and \(-\frac{1}{\varepsilon} \geq -\frac{1}{\tilde{\Gamma}}\) for all \(\varepsilon\) in a small interval \((0, \tilde{\Gamma})\). Hence, we get
\[ E_2[\phi_c + \xi] - E_2[\phi_0] \geq \tilde{\Gamma} (\|\zeta\|^3_{H_0} + \|\phi_c^3 \rho_c\|^2_{L^2}) - \frac{1}{\tilde{\Gamma}} (c^2 + \|\zeta\|^3_{H_0}) \]
as claimed in (3.44). \(\square\)

### 4. Modulation of Parameters

In order to apply the coercivity property of the quintic Ginzburg-Landau energy \(E_2\) shown in Section 3 we have to ensure that the orthogonality relations hold.

In this section, we prove that there exist small perturbations \(\xi \in \mathcal{H}(\mathbb{R})\) such that the orthogonality conditions (3.22) for the black soliton are satisfied. In fact, we will prove a more general result, valid for \(c \neq 0\), and dealing with generalized orthogonality conditions (3.43) for perturbations \(\xi \in \mathcal{H}(\mathbb{R})\) around the dark soliton and therefore obtaining the desired orthogonality conditions related with the black soliton (3.22) in the limit \(c = 0\).

Firstly, and for the sake of completeness, we will present a preliminary result on the continuous dependence for the shift and phase parameters \(b, \theta\) on the corresponding dark soliton profile.

**Lemma 4.1.** Let \((c, a, \theta) \in (-\varepsilon, \varepsilon) \times \mathbb{R}^2\) and set \(\phi_{c,a,\theta} := e^{i\theta} \phi_c(\cdot - a)\). Given a positive number \(\delta\), there exists a positive number \(\tilde{\delta}\) such that if
\[ \|
\phi_{c,b_1,\theta_1} - \phi_{c,b_2,\theta_2}\|_{H_c} < \tilde{\delta}, \]
then we have \(|b_2 - b_1| + |e^{i\theta_2} - e^{i\theta_1}| < \delta\).

**Proof.** The proof runs exactly as [13 Lemma 2.1]. \(\square\)
Proposition 4.2 (Modulation). Let \((c, a, \theta) \in (-c, c) \times \mathbb{R}^2\). There exist two positive numbers \(\tilde{r}_c\) and \(\tilde{s}_c\), depending continuously on \(c\), for which there exist a map \((\tilde{c}, \tilde{a}, \tilde{\theta}) : B_{\mathcal{H}_0}(\phi_{c,a,\theta}; \tilde{r}_c) \to (-c, c) \times \mathbb{R}^2\) with \((\tilde{c}, \tilde{a}, \tilde{\theta})(\phi_{c,a,\theta}) = (c, a, \theta)\) and such that for any \(w \in B_{\mathcal{H}_0}(\phi_{c,a,\theta}; \tilde{r}_c)\) the perturbation of the dark soliton profile

\[
3 := e^{-i\hat{\theta}(w)}w(\cdot + \hat{a}(w)) - \phi_{\tilde{c}(w)},
\]

satisfies the generalized orthogonality conditions \((3.43)\). Moreover, \(\tilde{c}, \tilde{a}, \tilde{\theta}\) are \(C^1\)-functions in \(B_{\mathcal{H}_0}(\phi_{c,a,\theta}; \tilde{r}_c)\) and given any \(w \in B_{\mathcal{H}_0}(\phi_{c,a,\theta}; \tilde{r}_c)\), the vector \((\tilde{c}, \tilde{a}, \tilde{\theta})(w)\) is the unique element in the ball \(B((c, a, \theta); \tilde{s}_c) \subset \mathbb{R}^3\) verifying \((3.43)\).

**Proof of Proposition 4.2.** The proof of this result is a classical application of the Implicit Function Theorem. We begin by considering the functional

\[
F : \mathcal{H} \times (-c, c) \times \mathbb{R}^2 \to \mathbb{R}^3,
\]

given by

\[
F(w, \sigma, \sigma, \sigma) := \left( \int_{\mathbb{R}} \langle \eta_{\sigma}, 3 \rangle, \int_{\mathbb{R}} \langle i\eta_{\sigma}, 3 \rangle, \int_{\mathbb{R}} \langle iR_{\sigma} \eta_{\sigma}, 3 \rangle \right),
\]

where \(3 := e^{-i\hat{\theta}(w)}(\cdot + \hat{a}(w)) - \phi_{\tilde{c}(w)}\).

Notice that, similarly to the context of the cubic GP \([14]\), the functional \(F\) has \(C^1\)-regularity. Recall now the notation introduced in Lemma 4.1

\[
\phi_{c,a,\theta} := e^{i\theta} \phi_c(\cdot - a).
\]

Then

\[
F(\phi_{c,a,\theta}, c, a, \theta) = 0, \quad \text{for all} \quad (c, a, \theta) \in (-c, c) \times \mathbb{R}^2,
\]

where \(0 := (0, 0, 0)\). On the other hand, we have that\(^3\)

\[
\partial_x F(\phi_{c,a,\theta}, c, a, \theta) = 0, \int_{\mathbb{R}} \langle i\eta_{\sigma}, -\partial_x \phi_{\sigma} \rangle, \{0\},
\]

\[
\partial_y F(\phi_{c,a,\theta}, c, a, \theta) = \left( \int_{\mathbb{R}} \langle \eta_{\sigma}, \partial_y \phi_{\sigma} \rangle, \{0\}, \int_{\mathbb{R}} \langle iR_{\sigma} \eta_{\sigma}, \partial_y \phi_{\sigma} \rangle \right),
\]

\[
\partial_z F(\phi_{c,a,\theta}, c, a, \theta) = \left( \int_{\mathbb{R}} \langle \eta_{\sigma}, -i\phi_{\sigma} \rangle, \{0\}, \int_{\mathbb{R}} \langle iR_{\sigma} \eta_{\sigma}, -i\phi_{\sigma} \rangle \right).
\]

Let \(F(c)\) be the \(3 \times 3\) matrix \(F(c) := (\partial_x F, \partial_y F, \partial_z F)(\phi_{c,a,\theta}, c, a, \theta)\), which is a continuously differentiable function on the interval \(c \in (-c, c)\).

From \((4.4)\), we have that for all \(c \in (-c, c)\) (see Appendix \(E\) for a detailed computation of \(\text{det} F(c)\))

\[
\text{det} F(c) = -\int_{\mathbb{R}} \langle i\eta_{\sigma}, \partial_x \phi_{\sigma} \rangle \times \mathcal{D}(c) \neq 0,
\]

where

\[
\mathcal{D}(c) = \left( \int_{\mathbb{R}} \langle \eta_{\sigma}, \partial_y \phi_{\sigma} \rangle \times \int_{\mathbb{R}} \langle iR_{\sigma} \eta_{\sigma}, -i\phi_{\sigma} \rangle - \int_{\mathbb{R}} \langle \eta_{\sigma}, -i\phi_{\sigma} \rangle \times \int_{\mathbb{R}} \langle iR_{\sigma} \eta_{\sigma}, \partial_y \phi_{\sigma} \rangle \right).
\]

Therefore, by the Implicit Function Theorem, there exists a neighborhood

\[
B_{\mathcal{H}_0}(\phi_{c,a,\theta}; \tilde{r}_c) \times B((c, a, \theta); \tilde{s}_c) \subset \mathcal{H} \times (-c, c) \times \mathbb{R}^2
\]

and a unique \(C^1\) map \((\tilde{c}, \tilde{a}, \tilde{\theta}) : B_{\mathcal{H}_0}(\phi_{c,a,\theta}; \tilde{r}_c) \to B((c, a, \theta); \tilde{s}_c)\) such that

\[
F(w, \tilde{c}(w), \tilde{a}(w), \tilde{\theta}(w)) = 0,
\]

for any \(w \in B_{\mathcal{H}_0}(\phi_{c,a,\theta}; \tilde{r}_c)\), and consequently we get \((3.43)\). \(\square\)

\(^3\)By parity arguments, some of the terms vanish in \((4.4)\).
Before establishing the next result, we remember the neighborhood (4.39) of the orbit of \( \phi_0 \),

\[
\mathcal{U}_0(\alpha) := \left\{ w \in \mathcal{H}(\mathbb{R}) : \inf_{(b, i)} \| e^{-iu}w(\cdot + b) - \phi_0 \|_{\mathcal{H}_0(\mathbb{R})} < \alpha \right\},
\]

where we split \( e^{-iu}w(\cdot + b) = \phi_c + z \). By taking \( \alpha \) smaller, if necessary, we can apply a well known standard theory of modulation for the solution \( u(\cdot) \in \mathcal{U}_0(\alpha) \) of the Cauchy problem (1.1).

**Corollary 4.3.** Let \( \tilde{r}_0 \) and \( \tilde{s}_0 \) be the constants established in Proposition 4.2 for the case \( c = 0 \), chosen in such a way that \( \tilde{r}_0 \tilde{s}_0 < 1 \). There exists \( \alpha > 0 \) such that for a given \( w \in \mathcal{U}_0(\alpha) \) there exist numbers \( a, \theta \) such that

\[
w \in B_{\mathcal{H}_0}(\phi_{0, a, \theta}; \tilde{r}_0/2)
\]

and the map \((\tilde{c}, \tilde{a}, \tilde{\theta})\) established in Proposition 4.2 in each ball \( B_{\mathcal{H}_0}(\phi_{0, a, \theta}; \tilde{r}_0/2) \) is well defined from the neighborhood \( \mathcal{U}_0(\alpha) \) with values in \( \mathbb{R}^2 \times \mathbb{R}/2\pi \). More precisely, the functions \( \tilde{c}(w), \tilde{a}(w) \) and \( \tilde{\theta}(w) \) (modulo \( 2\pi \)) do not depend on which \( (a, \theta) \) parameters are chosen.

**Proof.** Taking \( \alpha \leq \alpha_0 := \min \{\tilde{r}_0/2, \tilde{\delta}/4\} \) (with \( \tilde{\delta} \) provided in Lemma 4.1 in the case \( c = 0 \)), the proof follows in a similar way as it was done in the first part of the Step 2 in the proof of [13] Proposition 2.

**Corollary 4.4.** Consider \( \alpha \) as in Corollary 4.3 and let \( u(t, \cdot) \) be the solution of (1.1) - (1.2) with initial data \( u_0 \) satisfying \( d_0(u_0, \phi_0) < \alpha \). Then, there exist \( T > 0 \) and mappings

\[
[-T, T] \ni t \mapsto (c(t), a(t), \theta(t)),
\]

such that \( F(u(t, \cdot), c(t), a(t), \theta(t)) = 0 \).

**Proof.** As a direct consequence of the continuity of the quintic GP flow in \( \mathcal{Z}(\mathbb{R}) \), we can find \( T > 0 \) such that

\[
\|u(t, \cdot) - \phi_0\|_{\mathcal{H}_0} < d_0(u_0, \phi_0) < \alpha, \quad \forall t \in [-T, T],
\]

and consequently \( u(t, \cdot) \in B_{\mathcal{H}_0}(\phi_0; \alpha) \subset \mathcal{U}_0(\alpha) \) for all \( t \in [-T, T] \). So, from Corollary 4.3 we can define the mappings

\[
t \mapsto c(t), \quad t \mapsto a(t), \quad t \mapsto \theta(t),
\]

on \([-T, T]\) by setting \( c(t) := \tilde{c}(u(t, \cdot)), a(t) := \tilde{a}(u(t, \cdot)), \theta(t) := \tilde{\theta}(u(t, \cdot)) \). Moreover, the perturbation \( \tilde{z}(t) = e^{-i\theta(t)}u(\cdot + a(t)) - \phi_{c(t)} \) satisfies

\[
F(u(t, \cdot), c(t), a(t), \theta(t)) = \left( \int_{\mathbb{R}} \langle \eta_{c(t), \tilde{z}(t)}(t) \rangle, \int_{\mathbb{R}} \langle i\eta_{c(t), \tilde{z}(t)}(t) \rangle, \int_{\mathbb{R}} \langle iC_{c(t)}(t) \eta_{c(t), \tilde{z}(t)}(t) \rangle \right) = 0,
\]

for all \( t \in [-T, T] \).

Furthermore, using the definition in (4.3), we also have an estimate on the size of the modulation parameters involved in the perturbation

\[
\tilde{z}(t, \cdot) = e^{-i\theta(t)}u(t, \cdot) - \phi_{c(t)}(\cdot, a(t)),
\]

namely the following result:

**Corollary 4.5.** Let \( \alpha \) be given in Corollary 4.3 and \( u(t, \cdot), c(t), \theta(t), a(t) \) as in Corollary 4.4. There exist positive constants \( K_0 \) and \( A_0 \) such that if for some \( (a, \theta) \in \mathbb{R}^2 \) and \( 0 < \epsilon \leq \min\{1, \alpha\} \) is satisfied

\[
\|u(t, \cdot) - \phi_{0, a, \theta}\|_{\mathcal{H}_0} = \|u(t, \cdot) - e^{i\theta} \phi_0(\cdot - a)\|_{\mathcal{H}_0} \leq \epsilon, \quad t \in [-T, T],
\]

then it follows that

\[
|c(t)| + |a(t) - a| + |e^{i\theta(t)} - e^{i\theta}| \leq K_0 \epsilon \quad \text{and} \quad \|z(t, \cdot)\|_{\mathcal{H}_0} \leq A_0 \sqrt{\epsilon}.
\]
Proof. First of all, note that all components in the mapping
\[ w \in B_{\mathcal{H}_0}(\phi_0, a, \alpha, \theta) \mapsto (\vec{c}(w), \vec{a}(w), \vec{\theta}(w)) \in B((0, a, \alpha), \vec{s}_0), \]
are $C^1$-functions, and therefore, Lipschitz continuous with Lipschitz constant $K_0$. So, from (4.10), we have that
\[
|c(t)| + |a(t) - a| + |\theta(t) - \theta| = |c(u(t, \cdot))| + |\vec{a}(u(t, \cdot)) - a| + |\vec{\theta}(u(t, \cdot)) - \theta| \leq K_0\|u(t, \cdot) - \phi_{0, a, \theta}\|_{\mathcal{H}_0} \leq K_0\epsilon, \tag{4.12}
\]
for all $t \in [-T, T]$. This implies the first estimate in (4.11).

On the other hand, using (2.23) we have that
\[
\|\phi_{c(t), a(t), \theta(t)} - \phi_{0, a, \theta}\|_{\mathcal{H}_0}^2 = \|e^{it\theta(t)}\phi_{c(t)}(\cdot - a(t)) - e^{it\theta(t)}\phi_0(\cdot - a)\|_{\mathcal{H}_0}^2 \\
\lesssim \|e^{it\theta(t)}\phi_{c(t)}(\cdot - a(t)) - e^{it\theta(t)}\phi_0(\cdot - a)\|_{\mathcal{H}_0}^2 \\
+ \|e^{it\theta(t)}\phi_0(\cdot - a(t)) - e^{it\theta(t)}\phi_0(\cdot - a)\|_{\mathcal{H}_0}^2 \tag{4.13}
\]
\[
\lesssim c^2(t) + \|\phi_0(\cdot - a(t)) - \phi_0(\cdot - a)\|_{\mathcal{H}_0}^2 + \|e^{it\theta(t)}\phi_0(\cdot - a)\|_{\mathcal{H}_0}^2 \\
\lesssim c^2(t) + \|\phi_0(\cdot - a(t)) - \phi_0(\cdot - a)\|_{\mathcal{H}_0}^2 + |\theta(t) - \theta|^2 \|\phi_0\|_{\mathcal{H}_0}^2.
\]

Now, using the Mean Value Theorem there exist $\nu_i = \nu_i(t, x) \in (0, 1)$, $i = 1, 2$, such that
\[
\|\phi_0(\cdot - a(t)) - \phi_0(\cdot - a)\|_{\mathcal{H}_0}^2 = \int_{\mathbb{R}} \eta_0 |\phi_0'(\cdot - a(t)) - \phi_0'(\cdot - a)|^2 \\
+ \int_{\mathbb{R}} |\phi_0'(\cdot - a(t)) - \phi_0'(\cdot - a)|^2 \\
= |a(t) - a|^2 \int_{\mathbb{R}} \eta_0 |\phi_0'(\cdot - \nu_1 a + (1 - \nu_1)a(t))|^2 \\
+ |a(t) - a| \int_{\mathbb{R}} |\phi_0'(\cdot - a(t)) - \phi_0'(\cdot - a)| |\phi_0'(\cdot - \nu_2 a + (1 - \nu_2)a(t))| \\
\lesssim |a(t) - a| \|\eta_0\|_{L^1} |a(t) - a| + 2 \|\phi_0'\|_{L^1} \\
\lesssim |a(t) - a|^2 + |a(t) - a|^2.
\] Combining (4.13) and (4.14) we have
\[
\|\phi_{c(t), a(t), \theta(t)} - \phi_{0, a, \theta}\|_{\mathcal{H}_0}^2 \leq K(c(t)^2 + |a(t) - a|^2 + |a(t) - a| + |\theta(t) - \theta|^2), \tag{4.15}
\]
for some universal constant $K$ for all $|c| < c$.

Now, from (4.12) and (4.15), and using that $\epsilon < 1$, one gets
\[
\|\tilde{\mathbf{s}}(t)\|_{\mathcal{H}_0} = \|u(t, \cdot) - \phi_{c(t), a(t), \theta(t)}\|_{\mathcal{H}_0} \leq \|u(t, \cdot) - \phi_{0, a, \theta}\|_{\mathcal{H}_0} \\
+ \|\phi_{c(t), a(t), \theta(t)} - \phi_{0, a, \theta}\|_{\mathcal{H}_0} \leq (1 + \sqrt{K}(K_0 + \sqrt{K_0}))\sqrt{\epsilon}, \tag{4.16}
\]
which yields the second estimate in (4.11) with $A_0 = 1 + \sqrt{K}(K_0 + \sqrt{K_0})$. \(\square\)

Now, we will determine the growth in time of the modulation parameters $c(t), a(t)$ and $\theta(t)$ for any $t \in [-T, T]$. We will first show the evolution equation satisfied by the perturbation
\[
\tilde{\mathbf{s}}(t) \equiv \tilde{\mathbf{s}}(t) = e^{-it\theta(t)}u(t, \cdot + a(t)) - \phi_{c(t)}(\cdot).
\]
Lemma 4.6 (Evolution equation for $\tilde{\mathbf{s}}$). Let $\tilde{\mathbf{s}}(t) = e^{-it\theta(t)}u(t, \cdot + a(t)) - \phi_{c(t)}(\cdot)$ the perturbation of the dark soliton profile $\phi_{c(t)}(\cdot)$. Then we have that
\[
\partial_t \tilde{\mathbf{s}}(t) := -c'(t)\partial_c \phi_{c(t)} - i\theta'(t)(\phi_{c(t)} + \tilde{\mathbf{s}}(t)) + a'(t)(\partial_x \phi_{c(t)} + \partial_x \tilde{\mathbf{s}}(t)) + iZ(t), \tag{4.17}
\]
with
\[
Z(t) := \partial_{xx} \tilde{\mathbf{s}}(t) + ic(t)\partial_x \phi_{c(t)} + \eta_c(t)\tilde{\mathbf{s}}(t) - (\rho_{c(t)}^2 + 2|\phi_{c(t)}|^2\rho_{c(t)})(\phi_{c(t)} + \tilde{\mathbf{s}}(t)). \tag{4.18}
\]
and $\rho_{c(t)} = \rho_{c(t)}(\tilde{\mathbf{s}}(t))$. 

Proof. First consider the explicit time derivative of $\delta(t, \cdot)$:

$$\partial_t \delta(t) = -c'(t)\partial_x \phi_{c(t)} - i\theta'(t)(\phi_{c(t)} + \delta(t)) + a'(t)(\partial_x \phi_{c(t)} + \partial_x \delta(t)) + e^{-i\theta(t)}\partial_t u(t, \cdot + a(t)).$$

Now computing the last term $\partial_t u(t, \cdot + a(t))$, bearing in mind that $u$ fulfills (1.1), and also (2.5), (3.41) and that

$$|u|^4 = |\phi_{c(t)}|^4 + \rho_{c(t)}^2 + 2\rho_{c(t)}|\phi_{c(t)}|^2,$$

a direct calculation gives us (4.17). □

We now look for an expression for the growth in time of the modulation parameters $c(t)$, $a(t)$ and $\theta(t)$. In order to do that, we resort to the continuity of the quintic Gross-Pitaevskii flow in $Z(\mathbb{R}) \subset H(\mathbb{R})$. Specifically, if an initial data $u_0$ is chosen such that $d_0(u_0, \phi_0) < \alpha$, then we get a time $T$ such that the corresponding solution $u(t, \cdot)$ along the quintic Gross-Pitaevskii flow belongs to $V_0(\alpha)$, for any $t \in [-T, T]$.

We will see in Section 5, that we can fix the smallness parameter $\alpha$ in such a way that the solution $u(t, \cdot)$ of the Cauchy problem (1.1) still belongs to $V_0(\alpha)$ for all $t \in \mathbb{R}$.

Proposition 4.7 (Estimates on the growth of the modulation parameters). There exist numbers $\alpha_1 > 0$ and $A_1(\alpha_1) > 0$ such that if the solution $u(t, \cdot)$ lies in $V_0(\alpha_1)$ for any $t \in [-T, T]$, then the functions $c$, $a$ and $\theta$ are $C^1([-T, T]; \mathbb{R})$ and satisfy

$$|c'(t)| + |a'(t)| + |\theta'(t)| \leq A_1^2(\alpha_1)\|\delta(t, \cdot)\|_{H_0},$$

for all $t \in [-T, T]$.

Proof. We differentiate with respect to time the three generalized orthogonality conditions (3.43) for perturbations around the dark soliton profile $\phi_{c(t)}$.

Since we need to compute the derivatives in time of the orthogonality conditions, we initially consider regular enough initial data, for example $\partial_x u_0 \in H^2(\mathbb{R})$. In fact with this regularity we can justify (4.20), (4.22) and (4.23) below. We consider initially $\alpha$ and $u_0$ as in Corollary 4.4 so that the solution $u(t, \cdot) \in U_0(\alpha)$ for all $t \in [-T, T]$ and then we can set the modulation parameters $(c(t), a(t), \theta(t)) \in (-\epsilon, \epsilon) \times \mathbb{R}^2$ for any $t \in [-T, T]$.

Note that $c$, $a$ and $\theta$ belong to $C^1([-T, T], \mathbb{R})$ by the Chain Rule Theorem and moreover note that $\delta(t) \in C^1([-T, T], H(\mathbb{R}))$, and therefore we can get (4.17). Therefore, derivating the first orthogonality condition in (3.43), and with the notation $m_{ij}$, $i, j = 1, 2, 3$, for integrals independent of $\delta(t)$ and $n_k$, $k = 1, \ldots, 9$ for integrals with terms depending on $\delta(t)$, we get

$$\begin{align*}
\partial_t \int_{\mathbb{R}} \langle \eta_{c(t)}, \delta(t) \rangle_C & = \int_{\mathbb{R}} \left( \langle c'(t) \partial_x \eta_{c(t)}, \delta(t) \rangle_C + \langle \eta_{c(t)}, c'(t) \partial_x \delta(t) \rangle_C + \langle \eta_{c(t)}, \partial_x \delta(t) \rangle_C \right) \\
& = \int_{\mathbb{R}} \left( \langle c'(t) \partial_x \eta_{c(t)}, \delta(t) \rangle_C + \langle \eta_{c(t)}, c'(t) \partial_x \delta(t) \rangle_C \right) \\
& + \langle \eta_{c(t)}, (c'(t) \partial_x \phi_{c(t)} - i\theta'(t)(\phi_{c(t)} + \delta(t)) + a'(t)(\partial_x \phi_{c(t)} + \partial_x \delta(t)) + iZ(t)) \rangle_C.
\end{align*}$$
Thus,

\[ \partial_t \int_R \langle \eta_c(t), \mathfrak{z}(t) \rangle \]

\[ = a'(t) \left( \int_R \langle \eta_c(t), \partial_x \phi_c(t) \rangle + \int_R \langle \eta_c(t), \partial_x \mathfrak{z}(t) \rangle \right) \]

\[ + c'(t) \left( - \int_R \langle \eta_c(t), \partial_x \phi_c(t) \rangle + \int_R \langle \partial_x \eta_c(t), \mathfrak{z}(t) \rangle + \int_R \langle \eta_c(t), \partial_x \mathfrak{z}(t) \rangle \right) \]

\[ + \theta'(t) \left( \int_R \langle \eta_c(t), -i \phi_c(t) \rangle + \int_R \langle \eta_c(t), -i \mathfrak{z}(t) \rangle \right) + \int_R \langle \eta_c(t), i R(t) \rangle \]

\[ = a'(t)(m_{11} + n_1) + c'(t)(n_2 - m_{12}) + \theta'(t)(m_{13} + n_3) + \int_R \langle \eta_c(t), i R(t) \rangle = 0. \tag{4.21} \]

Now, we differentiate the second orthogonality condition in (3.43), and we obtain

\[ \partial_t \int_R \langle i \mathcal{R}_c(t) \eta_c(t), \mathfrak{z}(t) \rangle \]

\[ = a'(t) \left( \int_R \langle i \mathcal{R}_c(t) \eta_c(t), \partial_x \phi_c(t) \rangle + \int_R \langle i \mathcal{R}_c(t) \eta_c(t), \partial_x \mathfrak{z}(t) \rangle \right) \]

\[ + c'(t) \left( - \int_R \langle i \mathcal{R}_c(t) \eta_c(t), \partial_x \phi_c(t) \rangle + \int_R \langle i \partial_x \eta_c(t), \mathfrak{z}(t) \rangle + \int_R \langle i \mathcal{R}_c(t) \eta_c(t), \partial_x \mathfrak{z}(t) \rangle \right) \]

\[ + \theta'(t) \left( \int_R \langle i \mathcal{R}_c(t) \eta_c(t), -i \phi_c(t) \rangle + \int_R \langle i \mathcal{R}_c(t) \eta_c(t), -i \mathfrak{z}(t) \rangle \right) + \int_R \langle i \mathcal{R}_c(t) \eta_c(t), i R(t) \rangle \]

\[ = a'(t)(m_{21} + n_4) + c'(t)(n_5 - m_{22}) + \theta'(t)(m_{23} + n_6) + \int_R \langle i \mathcal{R}_c(t) \eta_c(t), i R(t) \rangle = 0. \tag{4.22} \]

Finally, we differentiate the third orthogonality condition in (3.43), and we get

\[ \partial_t \int_R \langle i \mathcal{R}_c(t) \eta_c(t), \mathfrak{z}(t) \rangle \]

\[ = a'(t) \left( \int_R \langle i \mathcal{R}_c(t) \eta_c(t), \partial_x \phi_c(t) \rangle + \int_R \langle i \mathcal{R}_c(t) \eta_c(t), \partial_x \mathfrak{z}(t) \rangle \right) \]

\[ + c'(t) \left( - \int_R \langle i \mathcal{R}_c(t) \eta_c(t), \partial_x \phi_c(t) \rangle + \int_R \langle i \partial_x \eta_c(t), \mathfrak{z}(t) \rangle + \int_R \langle i \mathcal{R}_c(t) \eta_c(t), \partial_x \mathfrak{z}(t) \rangle \right) \]

\[ + \theta'(t) \left( \int_R \langle i \mathcal{R}_c(t) \eta_c(t), -i \phi_c(t) \rangle + \int_R \langle i \mathcal{R}_c(t) \eta_c(t), -i \mathfrak{z}(t) \rangle \right) + \int_R \langle i \mathcal{R}_c(t) \eta_c(t), i R(t) \rangle \]

\[ = a'(t)(m_{31} + n_7) + c'(t)(n_8 - m_{32}) + \theta'(t)(m_{33} + n_9) + \int_R \langle i \mathcal{R}_c(t) \eta_c(t), i R(t) \rangle = 0. \tag{4.23} \]

Gathering all three previous derivatives, we obtain the following linear system

\[ \mathcal{M}(c, \mathfrak{z}) \begin{pmatrix} \alpha'(t) \\ c'(t) \\ \theta'(t) \end{pmatrix} = \mathcal{B}(c, \mathfrak{z}), \tag{4.24} \]
Namely, choosing parameter \( \alpha \) and therefore \( M \) a positive number \( A \) such that from (4.11) in Proposition 4.2, it holds verifying invertible. In fact, having in mind the Neumann Series Theorem, it is enough to consider \( (c, z) \) with the matrix \( M \) for a suitable choice of the constant and the matrix \( B \) modulation parameters (4.24) to a general solution. Therefore we get the continuous differentiability property of the \( \alpha \) operator norm of its inverse depending on (1.1) is continuous with respect to initial data in \( \mathcal{Z} \) (4.24).

Note that, in the case of null perturbation in (4.25), and considering the limit case of \( c = 0 \), it turns out that \( \mathcal{M}(0, 0) \) has a nonvanishing determinant, namely

\[
\det \mathcal{M}(0, 0) = \frac{8}{3} E_2[\phi_0],
\]

and therefore \( \mathcal{M}(0, 0) \) is invertible. By using a continuity argument, we can select a small enough parameter \( \alpha_1 < \alpha \) such that for small speeds and perturbations \( (c, z) \), the matrix \( \mathcal{M}(c, z) \) is still invertible. In fact, having in mind the Neumann Series Theorem, it is enough to consider \( (c, z) \) verifying

\[
\| \mathcal{M}(c, z) - \mathcal{M}(0, 0) \|_{\mathcal{M}_{2\times2}(\mathbb{C})} \leq \alpha_1 < \| \mathcal{M}^{-1}(0, 0) \|_{\mathcal{M}_{2\times2}(\mathbb{C})}^{-1}.
\]

Namely, choosing \( \alpha_1 < \alpha \) small enough such that \( u(t, \cdot) \in \mathcal{U}_0(\alpha_1) \), for all \( t \in [-T, T] \), and therefore that from (4.11) in Proposition 4.2 it holds

\[
\| \delta(t, \cdot) \|_{\mathcal{H}_0} + |c(t)| \leq A_0 \alpha_1,
\]

with \( \det \mathcal{M}(c, z) \neq 0 \) and, consequently, the operator norm of its inverse is bounded by some positive number \( A_1(\alpha_1) \). In the same way, the r.h.s. of (4.24) is bounded as follows:

\[
\| \mathcal{B}(c, z) \|_{\mathbb{R}^3} \leq A_1(\alpha_1) \| \delta(t, \cdot) \|_{\mathcal{H}_0},
\]

for a suitable choice of the constant \( A_1(\alpha_1) \). Therefore, from (4.24), we finally get that

\[
|a'(t)| + |c'(t)| + |\theta'(t)| \leq \| \mathcal{M}(c, z)^{-1} \cdot \mathcal{B}(c, z) \| \leq A_1^2(\alpha_1) \| \delta(t, \cdot) \|_{\mathcal{H}_0},
\]

for all \( t \in [-T, T] \).

Finally, we extend the above estimate (4.25) for general initial data \( u_0 \in \mathcal{Z}(\mathbb{R}) \). In fact, the flow of (1.1) is continuous with respect to initial data in \( \mathcal{Z}(\mathbb{R}) \) (see [9]). Moreover, from the continuity of the modulation parameters \( c(t), a(t) \) and \( \theta(t) \), we have that the matrices \( \mathcal{M}(c, z) \) and \( \mathcal{B}(c, z) \) depend continuously on \( u \in \mathcal{H}(\mathbb{R}) \). Therefore, since the matrix \( \mathcal{M}(c, z) \) is invertible with an operator norm of its inverse depending on \( \alpha_1 \), we can use a standard density argument to extend (4.24) to a general solution. Therefore we get the continuous differentiability property of the modulation parameters \( c(t), a(t) \) and \( \theta(t) \), and we obtain the corresponding estimates (4.19) from (4.24). □
5. Proof of the Main Theorem

In this section we prove a detailed version of Theorem 5.1.

**Theorem 5.1 (Orbital stability of the black soliton).** Let \( \phi_0 \) be the black soliton \(^{(1.9)}\) of the quintic GP equation \(^{(1.1)}\). Given \( \epsilon > 0 \) there exists \( \delta(\epsilon) > 0 \) and a positive constant \( A_\epsilon \) such that if the initial data \( u_0 \) verifies

\[
u_0 \in Z(\mathbb{R}) \quad \text{and} \quad d_0(u_0, \phi_0) < \delta(\epsilon),
\]

then there exist functions \( a, \theta \in C^1(\mathbb{R}, \mathbb{R}) \) such that the solution \( u \) of the Cauchy problem for the quintic GP equation \(^{(1.1)}\), with initial data \( u_0 \), satisfies

\[
d_0(e^{-i\theta(t)}u(t, \cdot + a(t)), \phi_0) < \epsilon
\]

and

\[
|a'(t)| + |\theta'(t)| < A_\epsilon \epsilon\tag{5.2}
\]

for any \( t \in \mathbb{R} \).

**Remark 5.2.** With respect to the cubic case \(^{(14)}\), a difference appears in the proof of the orbital stability Theorem for black solitons of \(^{(1.1)}\), that is, we could not achieve a lipschitzian control of the metric, i.e.

\[
d_0(e^{-i\theta(t)}u(t, \cdot + a(t)), \phi_{c(t)}) \lesssim d_0(u_0, \phi_0), \quad \text{for all} \quad c \in (0, c)\tag{5.3}
\]

The main reason is that if the momentum \(^{(14)}\) p.313, (1.27)] is used in our problem, a linear term

\[
\int_\mathbb{R} \langle i\phi_{c(t)}', \phi_{c(t)} \rangle \, \text{d}c,
\]

appears when expanding it around dark solitons \( \phi_{c(t)} \). Unfortunately this term does not match with any orthogonality relation \(^{(3.12)}\) and it can not be bounded from above nor controlled in the right and proper way. This is a big difference with respect to the cubic GP case (see p.314,l.7 in \(^{(14)}\)) which allows them to get an upper bound on the speed \( c(t) \) as shown in p.314,l.7-3 in \(^{(14)}\).

In our case, instead \(^{(5.3)}\) we get

\[
d_0(e^{-i\theta(t)}u(t, \cdot + a(t)), \phi_{c(t)}) \lesssim d_0(u_0, \phi_0) + c\tag{5.5}
\]

Moreover, and again in view of this technical issue with the linear term \(^{(5.4)}\), we decided to change this uniform control on \( c(t) \) on a fixed speed interval \((-c, c)\) by using the following strategy: for each fixed \( \epsilon > 0 \) we choose a suitable interval \((0, c(\epsilon))\) which allows us to select initial data in an appropriate ball with center \( \phi_0 \) in the \( d_0 \) metric and such that the solution remains in the \( \epsilon \)-neighborhood for all time by a bootstrap argument (note that \(^{(5.1)}\) is not as strong as the corresponding statement in \(^{(14)}\) p.308, (1.9))]. Obviously with this approach, once we reduce \( \epsilon, \) the speed interval \((-c, c)\) can also be reduced. As a consequence of our approach, we lose any possibility to say something about the orbital stability of the dark soliton solution.

**Proof of theorem 5.1.** In order to simplify the explanation we show the proof for \( t \geq 0 \).

Let \( \alpha, \ A_0 \) and \( \alpha_1 \) as in Corollary \(^{(1.3)}\), Corollary \(^{(4.5)}\) and Proposition \(^{(4.7)}\) respectively. Consider now \( \epsilon > 0 \) such that

\[
0 < \epsilon \leq \min\{1, \alpha\}, \quad 0 < \epsilon < \alpha_1 \quad \text{and} \quad A_0 \epsilon < 1\tag{5.6}
\]

Firstly, we take \( u_0 \in \mathbb{Z}(\mathbb{R}) \) \(^{(2.13)}\) such that \( d_0(u_0, \phi_0) < \epsilon /2 \) and \( u \in C(\mathbb{R}, \mathbb{Z}) \) the corresponding solution to \(^{(1.1)}\).

Now we define

\[
T^* := \sup \left\{ T > 0 : \forall t \in [0, T], \ \inf_{(t,b) \in \mathbb{R}^2} d_0(e^{-it}u(t, \cdot + b), \phi_0) < \epsilon \right\}\tag{5.7}
\]

and the idea is to use a contradiction argument under the assumption \( T^* < \infty \) when \( d_0(u_0, \phi_0) \) is small enough.
Note that since \( d_0(u_0, \phi_0) < \epsilon \), then as a direct consequence of the continuity of the quintic GP flow in \( Z(\mathbb{R}) \) with respect to the metric \( d_0 \), we can find \( T_0 > 0 \) such that
\[
d_0(u(t, \cdot), \phi_0) < \epsilon, \quad \text{for all } t \in [0, T_0],
\]
which, in particular, implies that \( T^* \) is well-defined in (5.7). Furthermore,
\[
u(t, \cdot) \in V_0(\epsilon) \subset U_0(\epsilon) \subset U_0(\alpha),
\]
for all \( t \in [0, T^*) \).

Consider now the functions \( c(t), a(t), \theta(t) \) given in Corollary 4.4 and notice that, in view of 5.8, we can consider these functions defined on the whole interval \( [0, T^*) \).

Now suppose that \( T^* < +\infty \) and consider
\[
\mathfrak{z}(t, \cdot) = e^{-i\theta(t)} u(t, \cdot + a(t)) - \phi_{c(t)}(\cdot), \quad t \in [0, T^*),
\]
where \( (c(t), a(t), \theta(t)) \in (-c, c) \times \mathbb{R}^2 \).

Then, having in mind the global theory in [9] which guarantees that \( \|\rho_c(\mathfrak{z})\|_{L^2} \) verifies (3.42), using the coercivity of \( E_2(\mathfrak{z}) \) around the dark soliton (3.44) in Proposition 3.4 and Corollary 4.3 with \( (\alpha, \theta) = (0, 0) \), we obtain
\[
\|\mathfrak{z}(t, \cdot)\|_{H_0}^2 + \|\phi^3_{c(t)}(\mathfrak{z}(t, \cdot))\|_{L^2}^2 \leq \frac{1}{16} \left( \mathfrak{T}(E_2[\phi_{c(t)} + \mathfrak{z}] - E_2[\phi_0]) + c^2(t) + \|\mathfrak{z}(t, \cdot)\|_{H_0}^2 \right) \tag{5.9}
\]

Selecting \( \epsilon \) such that \( A_0 \sqrt{\epsilon} < \frac{1}{2} \sqrt{2} \) and using the conservation of the \( E_2 \) energy (1.3) one gets
\[
\|\mathfrak{z}(t, \cdot)\|_{H_0}^2 + \|\phi^3_{c(t)}(\mathfrak{z}(t, \cdot))\|_{L^2}^2 \leq \frac{2}{\sqrt{2}} \left( \mathfrak{T}(E_2[u_0] - E_2[\phi_0]) + c(t)^2 \right), \tag{5.10}
\]
for all \( t \in [0, T^*) \).

Now, from the expansion (3.2) with \( \mathfrak{z} = u_0 - \phi_0 \) there exists a positive constant \( \tilde{k}_1 \) such that
\[
E_2[u_0] - E_2[\phi_0] \leq \tilde{k}_1 d_0^2(u_0, \phi_0), \tag{5.11}
\]
with \( \tilde{k}_1 \) independent of \( u_0 \). Then, putting this estimate into (5.10) and using (2.32) we have that there exists a universal positive constant \( \tilde{K} \), non depending on \( \epsilon \), such that
\[
\|\mathfrak{z}(t, \cdot)\|_{H_0}^2 + \|\phi^3_{c(t)}(\mathfrak{z}(t, \cdot))\|_{L^2}^2 \leq \tilde{K} \left( \|\mathfrak{z}(t, \cdot)\|_{H_0}^2 + \|\phi^3_{c(t)}(\mathfrak{z}(t, \cdot))\|_{L^2}^2 \right) \tag{5.12}
\]

So, we have
\[
d_0(e^{-i\theta(t)} u(t, \cdot + a(t)), \phi_{c(t)}) = \left( \|\mathfrak{z}(t, \cdot)\|_{H_0}^2 + \|\phi^3_{c(t)}(\mathfrak{z}(t, \cdot))\|_{L^2}^2 \right)^{\frac{1}{2}} \leq \left( \frac{2\tilde{K}\tilde{k}_1}{\Gamma} d_0^2(u_0, \phi_0) + \frac{2\tilde{K}}{\Gamma^2} \epsilon^2 \right)^{\frac{1}{2}}
\]
and hence from (2.24) we have
\[
d_0(e^{-i\theta(t)} u(t, \cdot + a(t)), \phi_{c(t)}) \leq \left( \frac{2\tilde{K}\tilde{k}_1}{\Gamma} d_0^2(u_0, \phi_0) + \frac{2\tilde{K}}{\Gamma^2} \epsilon^2 \right)^{\frac{1}{2}} + d_0(\phi_{c(t)}, \phi_0) \tag{5.13}
\]
for some positive constant \( \tilde{k}_2 \). Now we reduce \( \epsilon \), if necessary, to hold
\[
\left( \frac{\sqrt{2\tilde{K}}}{\Gamma} + \tilde{k}_2 \right) \epsilon < \frac{\epsilon}{4}, \tag{5.14}
\]
and we also consider $u_0$ satisfying

$$d(u_0, \phi_0) < \min \left\{ \frac{\epsilon}{2}, \frac{\epsilon}{4} \left( \frac{\Gamma}{2Kk_1} \right)^{\frac{1}{2}} \right\}. \quad (5.15)$$

Then, combining (5.13), (5.14) and (5.15), we get

$$d_0(e^{-it(t)}u(t, \cdot + a(t)), \phi_0) < \frac{\epsilon}{2}, \quad \text{for all } t \in [0, T^*), \quad (5.16)$$

which contradicts the definition of $T^* < \infty$ in (5.7), due to the continuity of the flow of the solution $u(t, \cdot)$ with respect to the metric $d_0$. Then, $T^* = \infty$ and the proof of (5.1) is finished.

Finally, from Proposition 4.7 one gets

$$\text{from Proposition 4.7 one gets }$$

$$\sup_{t \in \mathbb{R}} |a'(t)| + |\theta'(t)| < A_* \epsilon,$$

for some positive constant $A_*$. This completes the proof of Theorem 5.1. \hfill \square

APPENDIX A. PROOF OF (1.18)

In order to prove that (1.18) is a solution of (1.11), we propose a suitable ansatz (2.3):

$$\Phi_\epsilon(\xi) = \frac{ia_1 + a_2 \tanh(k\xi)}{\sqrt{1 + a_3 \tanh^2(k\xi)}}, \quad \xi = x - ct. \quad (A.1)$$

This ansatz must reduce to the black solution (1.9) when $c = 0$, therefore this implies that $a_1$ has to be dependent on $c$ in some way. We make the following selection for

$$a_1 = c\tilde{a}_1 a_2,$$

with $\tilde{a}_1$ to be determined. Hence, we recast (A.1) as follows:

$$\Phi_\epsilon(\xi) = \frac{i \epsilon \tilde{a}_1 a_2 + a_2 \tanh(k\xi)}{\sqrt{1 + a_3 \tanh^2(k\xi)}}, \quad (A.2)$$

where $\tilde{a}_1, k, a_2, a_3$ are parameters to be determined imposing that (A.2) is a solution of (1.11). Therefore, substituting (A.2) into (1.11) and simplifying (here $X = \tanh(k\xi)$, $D = 1 + a_3 \tanh^2(k\xi)$), we get

$$\Phi_\epsilon'' - i c \Phi_\epsilon' + (1 - |\Phi_\epsilon|^4) \Phi_\epsilon = \frac{a_2}{D^{5/2}} \sum_{i=0}^{5} r_i X^i, \quad (A.3)$$

where $r_i, \ i = 0, \ldots, 5$, are the following complex coefficients

$$r_0 = -ic(k + \tilde{a}_1 k^2 a_3 + \tilde{a}_1^5 c^4 a_2^4 - \tilde{a}_1),$$
$$r_1 = (\tilde{a}_1 c^2 k a_3 + k^2(-3a_3 - 2) - \tilde{a}_1 c^4 a_3^2 + 1),$$
$$r_2 = -ic(2\tilde{a}_1 k^2 (-a_3 - 2)a_3 - k(1 - a_3) + 2\tilde{a}_1^3 c^2 a_2^4 - 2\tilde{a}_1 a_3),$$
$$r_3 = -(\tilde{a}_1 c^2 k a_3(1 - a_3) + k^2(-4a_3 - 2) + 2\tilde{a}_1^2 c^2 a_3^4 - 2a_3),$$
$$r_4 = -ic(\tilde{a}_1 a_4^2 - \tilde{a}_1 a_3^2 - a_3 k + \tilde{a}_1 a_3(3 + 2a_3)k^2),$$
$$r_5 = -(a_2^4 - a_3(k^2 - \tilde{a}_1 c^2 k a_3 - a_3)). \quad (A.4)$$

Now, we impose that

$$r_i = 0, \ \forall i = 0, \ldots, 5, \quad (A.5)$$

and look for non trivial solutions (i.e. $\phi \neq 0$). Starting with the last equation $r_5 = 0$, we get

$$a_2^4 = a_3(k^2 + \tilde{a}_1 c^2 k a_3 + a_3). \quad (A.6)$$

Substituting the above value for $a_2^4$ into system (A.5), we get that the equation $r_4 = 0$ is solved for
\[
a_3 = -\frac{-1 + 2k\tilde{a}_1}{\tilde{a}_1(2k + c^2\tilde{a}_1)}, \tag{A.7}
\]

Therefore, with these values for \(a_2^4, a_3\), the group of (A.4) is recasted as follows (\(H = \frac{1 + c^2\tilde{a}_1^2}{(2k + c^2\tilde{a}_1)^2}\))

\[
\begin{align*}
  r_0 &= -ickHM_0 = 0, \\
  r_1 &= \frac{kH}{\tilde{a}_1} M_0 = 0, \\
  r_2 &= -ickHM_1 = 0, \\
  r_3 &= -\frac{kH}{\tilde{a}_1} M_1 = 0, \\
  r_4 &= 0, \\
  r_5 &= 0,
\end{align*} \tag{A.8}
\]

with

\[
\begin{align*}
  M_0 &= 6k^2 + 6\tilde{a}_1^4k^2 + \tilde{a}_1 k(5c^2 - 4(k^2 + 1)) + \tilde{a}_1^3c^2k(-5c^2 + 4(k^2 + 1)) \\
       &\quad + \tilde{a}_1^2(c^4 - 4c^2(2k^2 + 1)), \\
  M_1 &= -8k^2 + 8\tilde{a}_1 k^3 + c^2(1 - 10\tilde{a}_1 k + 12\tilde{a}_1^2 k^2) - 4 + 8\tilde{a}_1 k.
\end{align*} \tag{A.9}
\]

Solving \(M_1 = 0\),

for \(\tilde{a}_1\), we get (selecting e.g. the + root)

\[
\tilde{a}_1 = \frac{(5c^2 - 4) - 4k^2 + \sqrt{13c^4 + 8c^2(7k^2 + 1) + 16(k^2 + 1)^2}}{12c^2k}. \tag{A.10}
\]

Now, rewriting \(M_0\) \(\text{(A.9)}\) with \(\tilde{a}_1\) as in \(\text{(A.10)}\), we get

\[
M_0 = \frac{(4k^2 + c^2 - 4)}{144c^2k^2} m_0(c, k), \tag{A.11}
\]

with

\[
m_0(c, k) = \left[ 16 - 16c^2 + 19c^4 + 32k^2 + 80c^2k^2 + 16k^4 \\
+ (5c^2 - 4k^2 - 4) \sqrt{13c^4 + 8c^2(7k^2 + 1) + 16(k^2 + 1)^2} \right]. \tag{A.12}
\]

Finally, selecting

\[
k = \frac{1}{2}\sqrt{4 - c^2}, \tag{A.13}
\]

we get

\[
M_0 = 0,
\]

and we have solved system \(\text{(A.5)}\), and therefore \(\text{(A.2)}\) is a solution. Note that for these values of \(\tilde{a}_1\) and \(k\), the factor \(H\) is well defined; in fact, \(H = \frac{6c^2 + (3c^2 - 4)\sqrt{3c^2 + 4 + 8}}{c^2(\sqrt{3c^2 + 4 + 8})^2}\). In order to compare this solution with \(\text{(1.18)}\), we rewrite it as follows: firstly note that with this value of \(k\), \(\text{(A.10)}\) and \(\text{(A.7)}\) reduce to

\[
\tilde{a}_1 = \frac{3c^2 - 4 + 2\sqrt{4 + 3c^2}}{3c^2\sqrt{4 - c^2}}, \tag{A.14}
\]

and

\[
a_3 = -\frac{3(4 - c^2)(\sqrt{3c^2 + 4} - 2)}{(4 + \sqrt{3c^2 + 4})(3c^2 - 4 + 2\sqrt{3c^2 + 4})}. \tag{A.15}
\]
and hence, from (A.6), with the above values of \( \tilde{a}_1, k, a_3 \), and simplifying, we get (taking for instance a real + root)

\[
a_2 = \frac{3c \left( c^2 - 4 \right)}{\sqrt{2} \sqrt{-18c^4 + \left( 3\sqrt{4 - c^2}\sqrt{-3c^4 + 8c^2 + 16 + 80 \right) c^2 + 4 \left( \sqrt{4 - c^2}\sqrt{-3c^4 + 8c^2 + 16 - 8 \right)}}} = \frac{3c \sqrt{4 - c^2}}{\sqrt{2} \sqrt{3 \left( \sqrt{3c^2 + 4} + 6 \right) c^2 + 4 \left( \sqrt{3c^2 + 4 - 2 \right)}}} = \frac{3c \sqrt{4 - c^2}}{\sqrt{2} \sqrt{18c^2 - 8 + (3c^2 + 4)^{3/2}}}
\]  

Therefore

\[
\sqrt{2} a_2 = \mu_2.
\]  

Now, from ansatz (A.2), and values (A.14) and (A.16), we get that

\[
c \tilde{a}_1 a_2 = c \times \left( \frac{3c^2 + 2 \sqrt{3c^2 + 4} - 4}{3c^2 \sqrt{4 - c^2}} \right) \times \left( \frac{3c \sqrt{4 - c^2}}{\sqrt{2} \sqrt{18c^2 - 8 + (3c^2 + 4)^{3/2}}} \right)
\]

and hence

\[
\sqrt{2} c \tilde{a}_1 a_2 = \mu_1.
\]

Finally note that

\[
k = \kappa,
\]

and with (A.17) and (A.16), we get

\[
\frac{\mu_2^2 + \mu_2^2}{2 + 2a_3} - 1 = 0,
\]

and then

\[
a_3 = \mu.
\]

**Appendix B. Proof of Lemma 2.1**

The proof of this identity is made by quadratures. Making the change \( s = \sqrt{2b} \tan \theta \) we get the following equalities for the indefinite integrals

\[
\int \frac{ds}{(b - s^2) \sqrt{s^2 + 2b}} = \int \frac{\sec \theta \ d\theta}{b(1 - 2 \tan^2 \theta)} = \int \frac{\cos \theta \ d\theta}{b(1 - 3 \sin^2 \theta)}.
\]  

Now, by using the change \( \rho = \sqrt{3} \sin \theta \) we obtain

\[
\int \frac{\cos \theta \ d\theta}{b(1 - 3 \sin^2 \theta)} = \int \frac{d\rho}{\sqrt{3b(1 - \rho^2)}} = \frac{1}{2b^2} \ln \left( \frac{1 + \rho}{1 - \rho} \right).
\]

Combining (B.1) and (B.2) the result follows from the Fundamental Theorem of Calculus.

**Appendix C. Proof of (3.70)**

Having in mind that \( \rho_c = |\phi_c + j|^2 - |\phi_c|^2 = 2 \Re(\phi_c j) + |j|^2 \), it turns out that

\[
|\phi_c + j|^2 = |\phi_c|^2 + \rho_c,
\]

and therefore, we have

\[
||j||_{L^\infty} \lesssim (1 + ||\rho_c||_{L^\infty}^{1/2}) \lesssim (1 + ||\rho_c||_{L^\infty}).
\]

(C.1)
On the other hand,
\[ \|\rho_c\|_{L^\infty} \lesssim \|\rho_c\|_{L^2}^{1/2} \|\rho'_c\|_{L^2}^{1/2} \lesssim \|\rho_c\|_{L^2} \sqrt{\|\delta\|_{L^\infty} + \|\delta\|_{L^2} + \|\delta\|_{L^\infty}} \left(\|\delta\|_{L^2}\right)^{1/2}. \]

Hence, using Lemma 2.6 we get
\[ \|\rho_c\|_{L^\infty} \lesssim \|\rho_c\|_{L^2}^{1/2} \|\delta\|_{H_0}^{1/2} + \|\delta\|_{H_0}^{1/2} + \|\delta\|_{H_0}^{1/2}. \tag{C.2} \]

Now, substituting (C.2) into (C.1) and using Young’s inequality, we obtain
\[ \|\delta\|_{L^\infty} \lesssim \left(1 + \|\rho_c\|_{L^2} + \|\rho_c\|_{L^2}^{1/2} \|\delta\|_{H_0}^{1/2} + \|\rho_c\|_{L^2} \|\delta\|_{H_0} \right) \lesssim \left(1 + \|\rho_c\|_{L^2}\right)\left(1 + \|\delta\|_{H_0}\right). \tag{C.3} \]

**Appendix D. Computation of some $L^2$ and $L^\infty$ norms**

We collect some $L^2$ and $L^\infty$ norms needed along this work, in the following sections. Hereafter, we will consider by $K$ the smallest of the constants that allow us to get the corresponding upper bound.

**D.1. $L^2$ norms.** We first compute the associated $H_0$ norm in the distance $d_0$ (2.22). By definition,
\[ \|\phi_0 - \phi_c\|_{H_0}^2 = \|\phi_0' - \phi'_c\|_{L^2}^2 + \|\sqrt{\eta_0}(\phi_0 - \phi_c)\|_{L^2}^2, \]

therefore we split the computation in two steps: first we consider (with $R_c$ in (2.6))
\[ \|\phi_0' - \phi'_c\|_{L^2}^2 = \int_{\mathbb{R}} (\phi_0' - \phi'_c)(\phi_0' - \tilde{\phi}_c') = \int_{\mathbb{R}} ((\phi_0')^2 + |\phi_c'|^2 - 2R_c\phi_0^2). \]

Then, expanding in $c$ the last integrand, we note that this $L^2$ norm is bounded above, at small speeds $|c| \leq c$, with $c \ll 1$, by
\[ \|\phi_0' - \phi'_c\|_{L^2}^2 \leq K \int_{\mathbb{R}} \left(-\frac{9}{8} \left(\frac{\tanh^2(x) \sech^4(x)}{(\tanh^2(x) - 3)^3}\right)\right) dx = \frac{K}{32} \left(12 + 5\sqrt{3}\log(2 + \sqrt{3})\right) c^2. \tag{D.1} \]

On the other hand, we consider
\[ \|\sqrt{\eta_0}(\phi_0 - \phi_c)\|_{L^2}^2 = \int_{\mathbb{R}} \eta_0(\phi_0 - \phi_c)(\phi_0 - \tilde{\phi}_c) = \int_{\mathbb{R}} \eta_0((\phi_0)^2 + |\phi_c|^2 - 2R_c\phi_0), \]

which again behaves (proceeding as above), at small speeds $|c| \leq c$, with $c \ll 1$, as
\[ \leq K \int_{\mathbb{R}} \left(\frac{27}{8} \frac{\tanh^4(x) + 2\tanh^2(x) - 3}{(\tanh^2(x) - 3)^3}\right) dx = \frac{K}{32} \left(12 - \sqrt{3}\log(2 - \sqrt{3})\right) c^2. \tag{D.2} \]

Finally summing (D.1) and (D.2) and simplifying, we get the $H_0$ norm in (2.22)
\[ \|\phi_0 - \phi_c\|_{H_0}^2 \leq \frac{Kc^2}{16} \left(24 + \sqrt{3}\log(2 + \sqrt{3})\right). \]

With respect to (2.25), we first compute $|\phi'_c/\sqrt{\eta_c}|^2$ as
\[ |\phi'_c/\sqrt{\eta_c}|^2 = \frac{\partial_x \phi_c \partial_x \tilde{\phi}_c}{(\sqrt{\eta_c})^2} = -2\kappa^2 \sech^4(\kappa x) \left(\mu_1^2 \mu_2^2 \tanh^2(\kappa x) + \mu_2^2\right) \left(1 + \mu \tanh^2(\kappa x)\right) \left(\mu_1^2 + 2(\mu_1^2 \mu_2^2 - 4\mu) \tanh^2(\kappa x) + (\mu_2^2 - 4\mu^2) \tanh^4(\kappa x) - 4\right). \]
Therefore, integrating and having in mind the constraint relation \((1.15)\) we have that

\[
\left\| \frac{\phi'_c}{\sqrt{\mu L}} \right\|_{L^2}^2 := \int_\mathbb{R} -2\kappa^2 \text{sech}^4(\kappa x) \left( \mu_1^2 \mu^2 \tanh^2(\kappa x) + \mu_2^2 \right) dx
\]

\[
= \frac{-4\kappa}{(\mu_1^2 - 2)} \left( \sqrt{\mu} \arctanh(\sqrt{\mu}) + \left( \mu_2^2 + 2\mu + \mu_1^2 \right) \arctan \left( \frac{2\mu + \mu_2^2}{2 + \mu_1^2} \right) \right)
\]

\[
\leq \frac{\pi}{\sqrt{3}} + 2 \frac{1}{\sqrt{3}} \arctanh \left( \sqrt{3} \right).
\]

With respect to the \(L^2\)-norms in \((2.27)\) and \((2.28)\), we get after an expansion in \(c\), \(|c| < \epsilon\), with \(\epsilon \ll 1\), in the integrand of \((2.27)\), that this \(L^2\)-norm is bounded above by

\[
\left\| \frac{\phi_c}{\sqrt{\eta_0}} - \frac{\phi_0}{\sqrt{\eta_0}} \right\|_{L^2}^2 \leq K \int_\mathbb{R} \frac{3 \text{sech}^2(x) (\tanh^2(x) + 9)^2}{64 (\tanh^2(x) - 3)^2 (\tanh^2(x) + 3)} c^4 dx
\]

\[
\leq \frac{K c^4}{192} \left( 36 + \sqrt{3} \pi + 12 \sqrt{3} \log(2 + \sqrt{3}) \right)
\]

\[
\leq \frac{K}{4} c^4,
\]

and therefore we obtain \((2.27)\). Now in \((2.28)\), expanding again in \(|c| < \epsilon\), with \(\epsilon \ll 1\), we get that

\[
\left\| \frac{\phi_c \eta_0 - R_c \eta_0}{\sqrt{\eta_0}} \right\|_{L^2}^2 \leq K \int_\mathbb{R} \frac{3}{512} c^4 \left( -1 + \tanh^4(x) \right) \tanh^2(x) (3 + \tanh^2(x)) dx
\]

\[
\leq \frac{K}{4} c^4.
\]

We now compute the \(L^2\) norm in \((2.29)\). Firstly we write explicitly the integrand

\[
\frac{(\eta_c | \phi_c |^2 - \eta_0 \phi_0^2)^2}{\eta_0} = \frac{\eta_c \mu_2^2 + \mu_3^2 \mu^2 \tanh^2(\kappa x) - \eta_0 \mu_2^2 \tanh^2(x) - 2 \tanh^4(x) + 1}{(3 - \tanh^2(x))^2 (3 + \tanh^2(x))}.
\]

In fact, in the same small speed region \(|c| < \epsilon\), with \(\epsilon \ll 1\), we get, after an expansion of the above expression, that

\[
\left\| \frac{\eta_c | \phi_c |^2 - \eta_0 \phi_0^2}{\sqrt{\eta_0}} \right\|_{L^2}^2 \leq K \int_\mathbb{R} \frac{81 \text{sech}^4(x) x^4}{8 (3 - \tanh^2(x))^2} c^4 dx = K \frac{9 c^4}{32} \sqrt{3} \log \left( \frac{1}{2 - \sqrt{3}} \right)
\]

\[
\leq \frac{K}{\sqrt{2}} c^4.
\]

We now compute the \(L^2\) norm in \((2.30)\) proceeding in the same way. In fact after an expansion of the integrand in the small speed region \(|c| < \epsilon\), with \(\epsilon \ll 1\), we get

\[
\left\| \frac{\eta_c | \phi_c |^2 R_c^2 - \eta_0 \phi_0^2}{\sqrt{\eta_0}} \right\|_{L^2}^2 \leq K \int_\mathbb{R} \frac{27 \text{sech}(x) \tanh^4(x)}{8 (3 - \tanh^2(x))^3} c^4 dx \leq \frac{K}{2} c^4.
\]
D.2. \(L^\infty\) norms. We now compute the \(L^\infty\) norm in (2.31). Expanding it in the small speed region \(|c| < c\), with \(c \ll 1\), we get

\[
\frac{\phi_c^2 | - \phi_0^2}{(1 + x^2)\eta_c} \leq \frac{(9 - 4x \tanh(x) + \tanh^2(x))}{8(1 + x^2)(3 + \tanh^2(x))^3} c^2,
\]

uniformly in \(x \in \mathbb{R}\), and whose maximum value \((\frac{3}{8})\) is attained at \(x = 0\). Therefore we get that

\[
\| \frac{\phi_c^2 | - \phi_0^2}{(1 + x^2)\eta_c} \|_{L^\infty} \leq \frac{3}{8} c^2. \tag{D.5}
\]

Now, we justify the uniform pointwise estimate in (2.32). First we note that for any given \(x \in \mathbb{R}\) we have

\[
\sqrt{\frac{3}{2}} |x| = \kappa(1)|x| \leq \kappa(c)|x|, \quad \forall \ |c| \leq 1,
\]

with \(\kappa\) defined in (1.13). Now observe that

\[
\lim_{x \to 0^=} \frac{\phi_0(x)^2}{\phi_0(\sqrt{3}x/2)^2} = \frac{4}{3}\quad \text{and} \quad \lim_{x \to \pm \infty} \frac{\phi_0(x)^2}{\phi_0(\sqrt{3}x/2)^2} = 1,
\]

from which we can conclude that

\[
\phi_0(x)^2 \lesssim \frac{\tanh^2(\sqrt{3}|x|/2)}{3 - \tanh^2(\sqrt{3}|x|/2)}, \quad \forall \ x \in \mathbb{R}. \tag{D.8}
\]

Now, selecting \(s := \tanh(\sqrt{3}|x|/2)\) and using that the function \(s \mapsto \frac{s^2}{3 - s^2}\) is increasing on the interval \([0, 1]\) we have, combining (D.6) and (D.8), that

\[
\phi_0(x)^2 \lesssim \frac{\tanh^2(\kappa(c)|x|)}{3 - \tanh^2(\kappa(c)|x|)}, \quad \forall \ (x, c) \in \mathbb{R} \times [-1, 1]. \tag{D.9}
\]

Finally, using that

\[
\lim_{c \to 0^=} \mu_2(c) = \pm \frac{2}{\sqrt{3}} \quad \text{and} \quad -\frac{1}{3} \leq \mu(c) \leq 0
\]

we conclude, from (D.9), that there exists \(c \ll 1\) such that

\[
\phi_0(x)^2 \lesssim \frac{\mu_1^2(c) + \mu_2(c) \tanh^2(\kappa(c)|x|)}{2 + 2\mu(c) \tanh^2(\kappa(c)|x|)} \quad \sim \phi_0(x)^2 \tag{D.10}
\]

for all \((x, c) \in \mathbb{R} \times (-c, c)\).

Appendix E. Computation of det\(\mathcal{F}(c)\) and Matrix Elements of \(\mathcal{M}(c, z)\)

First of all, we remember the expression of det\(\mathcal{F}(c)\) (4.5):

\[
det\mathcal{F}(c) = \int_{\mathbb{R}} (i\eta_c, -\partial_c \phi_c) \mathcal{C}
\]

\[\times \left\{ \int_{\mathbb{R}} (\eta_c, -i\phi_c) \mathcal{C} \int_{\mathbb{R}} (iR_c \eta_c, \partial_x \phi_c) \mathcal{C} - \int_{\mathbb{R}} (\eta_c, \partial_x \phi_c) \mathcal{C} \int_{\mathbb{R}} (iR_c \eta_c, -i\phi_c) \mathcal{C} \right\}, \tag{E.1}
\]

for all \(c \in (-2, 2)\). Now, we compute the five different elements in (E.1) at \(c = 0\). We start with the first factor in (E.1)

\[
\int_{\mathbb{R}} (i\eta_c, -\partial_c \phi_c) \mathcal{C} = -\int_{\mathbb{R}} \text{Re} \left( i\eta_c \partial_c \phi_c \right), \tag{E.2}
\]

and at \(c = 0\) we get
- \Re \left( i \eta_c \partial_c \bar{\phi}_c \right) \bigg|_{c=0} = - \frac{9}{2 \sqrt{2}} \frac{\left( \tanh^4(x) + 2 \tanh^2(x) - 3 \right)}{\sqrt{3 - \tanh^2(x) \left( \tanh^2(x) - 3 \right)^2}}.

Now, integrating the above expression, we obtain
\[ \int_R \langle i \eta_c, -\partial_c \phi_c \rangle_C \bigg|_{c=0} = -2. \] (E.3)

The second factor is
\[ \int_R \langle i R_c \eta_c, \partial_x \phi_c \rangle_C = \int_R \Re \left( i R_c \eta_c \partial_x \bar{\phi}_c \right), \] (E.4)
and at \( c = 0 \) we have that
\[ \Re \left( i R_c \eta_c \partial_x \bar{\phi}_c \right) \bigg|_{c=0} = 0, \]
and we get that
\[ \int_R \langle i R_c \eta_c, \partial_x \phi_c \rangle_C \bigg|_{c=0} = 0. \] (E.5)

The corresponding third factor is
\[ \int_R \langle \eta_c, -i \phi_c \rangle_C = \int_R \Re \left( i \eta_c \bar{\phi}_c \right), \] (E.6)
with
\[ \Re \left( i \eta_c \bar{\phi}_c \right) \bigg|_{c=0} = 0. \]
Therefore, as above we have,
\[ \int_R \langle \eta_c, -i \phi_c \rangle_C \bigg|_{c=0} = 0. \] (E.7)

The fourth factor is
\[ \int_R \langle \eta_c, \partial_x \phi_c \rangle_C = \int_R \Re \left( \eta_c \partial_x \bar{\phi}_c \right), \] (E.8)
with
\[ \Re \left( \eta_c \partial_x \bar{\phi}_c \right) \bigg|_{c=0} = \frac{9 \sqrt{2}}{\sqrt{3 - \tanh^2(x) \left( \tanh^2(x) - 3 \right)^3}} \frac{\left( \tanh^4(x) + 2 \tanh^2(x) - 3 \right)}{\sqrt{3 - \tanh^2(x) \left( \tanh^2(x) - 3 \right)^2}} \]
Therefore integrating, we get
\[ \int_R \langle \eta_c, \partial_x \phi_c \rangle_C \bigg|_{c=0} = \frac{8}{5}. \] (E.9)

Finally the last factor is
\[ \int_R \langle i R_c \eta_c, -i \phi_c \rangle_C = \int_R \Re \left( - R_c \eta_c \bar{\phi}_c \right), \] (E.10)
with
\[ \Re \left( - R_c \eta_c \bar{\phi}_c \right) \bigg|_{c=0} = - \frac{6 \left( \tanh^2(x) \left( \tanh^4(x) + 2 \tanh^2(x) - 3 \right) \right)}{\left( \tanh^2(x) - 3 \right)^2} \]
Integrating, we get
\[ \int_R \langle i R_c \eta_c, -i \phi_c \rangle_C \bigg|_{c=0} = \frac{1}{2} \sqrt{3} \log \left( \sqrt{3} + 2 \right). \] (E.11)
Finally, gathering the five terms above, we have that
\[
det F(0) = -\frac{8\sqrt{3}}{5} \log \left( \sqrt{3} + 2 \right) = -\frac{8}{5} E_2[\phi_0]. \quad \text{(E.12)}
\]

Now, using a classical continuity argument, we get that, for \( c \in [0, \varepsilon) \), with smaller \( \varepsilon \ll 1 \), if necessary,

\[
\det F(c) \neq 0. \quad \text{(E.13)}
\]

We now list here the computed matrix elements of \( M(c, z) \) in \( (4.25) \). By parity reasons some terms vanish. Namely

\[
m_{11} = \int \langle \eta_{c(t)}, \partial_x \phi_{c(t)} \rangle_C = [\text{E.8}], \quad \text{(E.14)}
\]

\[
m_{12} = \int \langle \eta_{c(t)}, \partial_c \phi_{c(t)} \rangle_C = 0, \quad \text{and} \quad m_{13} = \int \langle \eta_{c(t)}, -i\phi_{c(t)} \rangle_C = [\text{E.6}]. \quad \text{(E.15)}
\]

Now, we list the products coming from the second orthogonality condition in \( (3.43) \):

\[
m_{21} = \int \langle i\eta_{c(t)}, \partial_x \phi_{c(t)} \rangle_C = 0, \quad \text{(E.16)}
\]

\[
m_{22} = \int \langle i\eta_{c(t)}, \partial_c \phi_{c(t)} \rangle_C = -[\text{E.2}], \quad \text{and} \quad m_{23} = \int \langle i\eta_{c(t)}, -i\phi_{c(t)} \rangle_C = 0, \quad \text{(E.17)}
\]

and finally, the products coming from the third orthogonality relation of \( (3.43) \):

\[
m_{31} = \int \langle iR_{c(t)} \eta_{c(t)}, \partial_x \phi_{c(t)} \rangle_C = [\text{E.4}], \quad \text{(E.18)}
\]

\[
m_{32} = \int \langle iR_{c(t)} \eta_{c(t)}, \partial_c \phi_{c(t)} \rangle_C = 0, \quad \text{and} \quad m_{33} = \int \langle iR_{c(t)} \eta_{c(t)}, -i\phi_{c(t)} \rangle_C = [\text{E.10}]. \quad \text{(E.19)}
\]

In the limit case when \( (c = 0, z = 0) \), the matrix \( (4.25) \) has the following simple expression

\[
M(0, 0) := \begin{pmatrix}
\frac{2}{5} & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -\frac{1}{2} E_2[\phi_0]
\end{pmatrix}, \quad \text{(E.20)}
\]

with

\[
\det M(0, 0) = -\frac{8}{5} E_2[\phi_0].
\]

\section*{References}

[1] J. Angulo and C.A. Melo, \textit{On stability properties of the Cubic-Quintic Schrödinger equation with δ-point interaction}. Comm. Pure Appl. Anal. 18 (4), (2019), 2093-2116.

[2] F. Béthuel, P. Gravejat, J-C. Saut and D. Smets, \textit{Orbital Stability of the Black Soliton for the Gross-Pitaevskii Equation}. Indiana University Mathematics Journal 57, No. 6, (2008), 2611-2642.

[3] F. Béthuel, P. Gravejat and J-C. Saut, \textit{Existence and properties of travelling waves for the Gross-Pitaevskii equation}, Stationary and Time Dependent Gross-Pitaevskii Equations (A. Farina and J.-C. Saut, eds.), Contemporary Mathematics, 473, Amer. Math. Soc., Providence, RI, 2008, pp. 55-103.

[4] D. Chiron, \textit{Stability and instability for subsonic Traveling waves of the non linear Schrödinger equation in dimension one}, Analysis & PDE, 6, No. 6, 2013, pp. 1327-1420.

[5] A. Contreras, D.E. Pelinovsky and M. Plum, \textit{Orbital Stability of Domain Walls in Coupled Gross–Pitaevskii Systems}, SIAM J. Math. Anal. 50 (2018), 810-833.

[6] P. Deift and X. Zhou, \textit{Perturbation theory for infinite-dimensional integrable systems on the line. A case study}. Acta Mathematica 188, 163-262, (2002).

[7] D.J. Frantzeskakis, N.P. Proukakis and P.G. Kevrekidis, \textit{Dynamics of shallow dark solitons in a trapped gas of impenetrable bosons}. Phys. Rev. A. 70, 015601, (2004).
[8] T. Gallay and D. Pelinovsky, *Orbital stability in the cubic defocusing NLS equation: II. The black soliton*, J. Diff. Eq., 258, No. 10, (2015), 3639-3660.

[9] C. Gallo, *The Cauchy Problem for Defocusing Nonlinear Schrödinger Equations with Non-Vanishing Initial Data at Infinity*, Commun. in Partial Diff. Eq. 33 (2008), 729-771.

[10] P. Gerard, *The Gross-Pitaevskii equation in the energy space*, Stationary and time dependent Gross-Pitaevskii equations, (A. Farina and J.-C. Saut, eds.), Contemporary Mathematics, 473 Amer. Math. Soc., Providence, RI, 2008, pp. 129-148.

[11] P. Gerard, *The Cauchy problem for the Gross-Pitaevskii equation*, Ann. Inst. Henri Poincare, Analyse Non Lineaire 23 (2006), 765-779.

[12] P. Gerard and Z. Zhang, *Orbital stability of traveling waves for the one-dimensional Gross-Pitaevskii equation*, J. Math. Pures Appl. 91 (2009) 178-210

[13] P. Gravejat and D. Smets, *Asymptotic Stability of the Black Soliton for the Gross-Pitaevskii Equation*, Proceedings of the London Mathematical Society 111, No. 2, (2015), 305-353.

[14] P. Gravejat and D. Smets, *Asymptotic Stability of the Black Soliton for the Gross-Pitaevskii Equation*, Proceedings of the London Mathematical Society 111, No. 2, (2015), 305-353.

[15] Y.S. Kivshar and B.L. Davies, *Dark optical solitons: physics and applications*. Physics Reports 298 (1998) 81-197.

[16] Y.S. Kivshar and X. Yang, *Perturbation-induced dynamics of dark solitons*, Phys. Rev. E.3/ 49:2 (1994), 1657-1670.

[17] E.B. Kolomeisky, T.J. Newman, J.P. Straley and X. Qi, *Low-Dimensional Bose Liquids: Beyond the Gross-Pitaevskii Approximation*, Phys. Rev. Lett. 85, 1146, (2000).

[18] E. Lieb, R. Seiringer and Y. Yngvasson, *One-Dimensional Bosons in Three-Dimensional Traps*, Phys. Rev. Lett. 91 15 (2003).

[19] Z. Lin, *Stability and Instability of Traveling Solitonic Bubbles*. Adv. in Diff. Equations. 47, n. 8 (2002), 897-918.

[20] A. Minguzzi, P. Vignolo, M.L. Chiofalo and M.P. Tosi, *Hydrodynamic excitations in a spin-polarized Fermi gas under harmonic confinement in one dimension*. Phys. Rev. A. 64, 033605, (2001).

[21] D.E. Pelinovsky, Y.S. Kivshar and V.V. Afanasjev, *Internal modes of envelope solitons*. Physica D 116 (1998) 121-142.

[22] D.E. Pelinovsky and M. Plum, *Stability of black solitons in optical systems with intensity-dependent dispersion*, arXiv:2205.10177

[23] D.E. Pelinovsky and M. Plum, *Dynamics of the black soliton in a regularized nonlinear Schrödinger equation*, arXiv:2304.04823

[24] J. Yang, *Nonlinear Waves in Integrable and Non-integrable Systems*. SIAM Mathematical Modeling and Computation (2010).

[25] V.E. Zakharov and A.B. Shabat, *Interaction between solitons in a stable medium*. Sov.Phys. JETP 37 (1973), 823-828.

[26] P.E. Zhidkov, *Korteweg-de Vries and Nonlinear Schrödinger Equations: Qualitative Theory*. Lecture Notes in Mathematics 1756, Springer (2001)

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