Abstract

The main purpose of this work is to define planar self-intersection local time by an alternative approach which is based on an almost sure pathwise approximation of planar Brownian motion by simple, symmetric random walks. As a result, Brownian self-intersection local time is obtained as an almost sure limit of local averages of simple random walk self-intersection local times. An important tool is a discrete version of the Tanaka–Rosen–Yor formula; the continuous version of the formula is obtained as an almost sure limit of the discrete version. The author hopes that this approach to self-intersection local time is more transparent and elementary than other existing ones.

1 Introduction

Let \((W(t))_{t \geq 0}\) be planar Brownian motion (BM). Formally, its self-intersection local time at the point \(x \in \mathbb{R}^2\) up to time \(t\) is

\[
\alpha(t, x) = \int_0^t \int_0^u \delta(W(v) - W(u) - x) \, du \, dv,
\]

where \(\delta\) is the Dirac measure at zero. There exist several methods in the literature to make this definition rigorous. One natural approach, which is the topic of the present work, is to define \(\alpha(t, x)\) as an almost sure limit (when \(m \to \infty\)) of local averages of self-intersection local times \(\alpha_m(t, x)\) of a nested sequence of simple, symmetric planar random walks \((B_m(t))\). (See the next section for the...
definition of $B_m$.) For these imbedded random walks (RW’s), self-intersection local time can be defined by elementary counting:

$$\alpha_m(t, x) := 2^{-2m} \# \{(i, j) : 0 \leq i < j < t 2^{2m}, B_m(j 2^{-2m}) - B_m(i 2^{-2m}) = x\}.$$ 

Then, as it will be seen in Theorem 3, the following almost sure limit gives the Brownian self-intersection local time:

$$\alpha(t, y) = \lim_{\delta \to 0^+} \lim_{m \to \infty} \frac{1}{\pi \delta^2} \sum_{x \in 2^{-m} \mathbb{Z}^2 \cap B_\delta(y)} \alpha_m(t, x) 2^{-2m} \quad (y \neq 0),$$

where $B_\delta(y)$ denotes the disc centered at $y$ with radius $\delta$.

The author hopes that this approach to self-intersection local time is more elementary and more advantageous from a pedagogical point of view than other existing ones. This method is a special case of a strong invariance principle for self-intersection local time. It should be mentioned that using different methods, strong invariance was shown earlier by Cadre [2], and for general random walks by Bass and Rosen [1]. The method applied in this paper is based on a Tanaka-like formula, first introduced by Rosen [13] and then generalized by Yor [20]. More exactly, a discrete version of the Tanaka–Rosen–Yor formula for random walks is given below whose almost sure limit is the continuous version of the formula.

## 2 Preliminaries

A basic tool of the present paper is an elementary construction of Brownian motion. The specific construction used in the sequel, taken from [17], is based on a nested sequence of simple, symmetric random walks that uniformly converges to the Wiener process (=BM) on bounded intervals with probability 1. This will be called “twist and shrink” construction. This method is a modification of the one given by Frank Knight in 1962 [7].

We summarize the major steps of the “twist and shrink” construction here. We start with an infinite matrix of independent and identically distributed random variables $X_m(k), P\{X_m(k) = \pm 1\} = \frac{1}{2}$ ($m \geq 0, k \geq 1$), defined on the same complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. (All stochastic processes in the sequel will be defined on this probability space.) Each row of this matrix is a basis of an approximation of the Wiener process with a dyadic step size $\Delta t = 2^{-2m}$ in time and a corresponding step size $\Delta x = 2^{-m}$ in space. Thus we start with a sequence of independent simple, symmetric RW’s $S_m(0) = 0, S_m(n) = \sum_{k=1}^{n} X_m(k)$ ($n \geq 1$).

The second step of the construction is twisting. From the independent RW’s we want to create dependent ones so that after shrinking temporal and spatial step sizes, each consecutive RW becomes a refinement of the previous one. Since the spatial unit will be halved at each consecutive row, we define stopping times by $T_m(0) = 0$, and for $k \geq 0$,

$$T_m(k + 1) = \min\{n : n > T_m(k), |S_m(n) - S_m(T_m(k))| = 2\} \quad (m \geq 1)$$

These are the random time instants when a RW visits even integers, different from the previous one. After shrinking the spatial unit by half, a suitable modification of this RW will visit the same integers in the same order as the previous
Self-intersection local time based on random walks

RW. We operate here on each point \( \omega \in \Omega \) of the sample space separately, i.e. we fix a sample path of each RW. We define twisted RW’s \( \tilde{S}_m \) recursively for \( k = 1, 2, \ldots \) using \( \tilde{S}_{m-1} \), starting with \( \tilde{S}_0(n) = S_0(n) \) (\( n \geq 0 \)) and \( \tilde{S}_m(0) = 0 \) for any \( m \geq 0 \). With each fixed \( m \) we proceed for \( k = 0, 1, 2, \ldots \) successively, and for every \( n \) in the corresponding bridge, \( T_m(k) < n \leq T_m(k+1) \). Any bridge is flipped if its sign differs from the desired:

\[
\tilde{X}_m(n) = \begin{cases} 
X_m(n) & \text{if } S_m(T_m(k+1)) - S_m(T_m(k)) = 2\tilde{X}_{m-1}(k+1), \\
-X_m(n) & \text{otherwise},
\end{cases}
\]

and then \( \tilde{S}_m(n) = \tilde{S}_m(n-1) + \tilde{X}_m(n) \). Then \( \tilde{S}_m(n) \) (\( n \geq 0 \)) is still a simple symmetric RW [17, Lemma 1]. The twisted RW’s have the desired refinement property:

\[
\tilde{S}_{m+1}(T_{m+1}(k)) = 2\tilde{S}_m(k) \quad (m \geq 0, k \geq 0).
\]

The third step of the RW construction is shrinking. The sample paths of \( \tilde{S}_m(n) \) (\( n \geq 0 \)) can be extended to continuous functions by linear interpolation, this way one gets \( \tilde{S}_m(t) \) (\( t \geq 0 \)) for real \( t \). The \( m \)th “twist and shrink” RW is defined by

\[
\tilde{B}_m(t) = 2^{-m}\tilde{S}_m(t2^{2m}).
\]

Then the refinement property takes the form

\[
\tilde{B}_{m+1} \left( T_{m+1}(k)2^{-2(m+1)} \right) = \tilde{B}_m(k2^{-2m}) \quad (m \geq 0, k \geq 0). \tag{1}
\]

Note that a refinement takes the same dyadic values in the same order as the previous shrunk walk, but there is a time lag in general:

\[
T_{m+1}(k)2^{-2(m+1)} - k2^{-2m} \neq 0. \tag{2}
\]

It is clear that this construction is especially useful for local times, since a refinement approximates the local time of the previous walk, with a geometrically distributed random number of visits with half-length steps, cf. [18].

Now let me recall some important facts from [17] and [18] about the “twist and shrink” construction that will be used in the sequel.

**Theorem A.** On bounded intervals the sequence \( (\tilde{B}_m) \) almost surely uniformly converges as \( m \to \infty \) and the limit process is Brownian motion \( W \). For any \( C > 1 \), and for any \( K > 0 \) and \( m \geq 1 \) such that \( K2^{2m} > N(C) \), we have

\[
P \left\{ \sup_{0 \leq t \leq K} |W(t) - \tilde{B}_m(t)| \geq 27CK_\ast^\frac{1}{2}(\log_\ast K)^\frac{1}{2}m^{\frac{3}{2}}2^{-\frac{m}{2}} \right\} \leq \frac{6}{1 - 4^{1-C}(K2^{2m})^{1-C}},
\]

where \( K_\ast := K \vee 1 \) and \( \log_\ast K := (\log K) \vee 1 \).

\( (N(C) \) here and in the sequel denotes a large enough integer depending on \( C \), whose value can be different at each occasion.\)

Conversely, with a given Wiener process \( W \), one can define the stopping times which yield the Skorohod embedded RW’s \( B_m(k2^{-2m}) \) into \( W \). For every \( m \geq 0 \) let \( s_m(0) = 0 \) and

\[
s_m(k + 1) = \inf \{ s : s > s_m(k), |W(s) - W(s_m(k))| = 2^{-m} \} \quad (k \geq 0). \tag{3}
\]
With these stopping times the embedded dyadic walks by definition are

$$B_m(k2^{-2m}) = W(s_m(k)) \quad (m \geq 0, k \geq 0). \quad (4)$$

This definition of $B_m$ can be extended to any real $t \geq 0$ by pathwise linear interpolation.

If a Wiener process is built by the “twist and shrink” construction described above using a sequence $(\tilde{B}_m)$ of nested RW’s and then one constructs the Skorohod embedded RW’s $(B_m)$, it is natural to ask about their relationship. The next theorem shows that they are asymptotically equivalent. In general, roughly saying, $(\tilde{B}_m)$ is more useful when someone wants to generate stochastic processes from scratch, while $(B_m)$ is more advantageous when someone needs discrete approximations of given processes.

**Theorem B.** For any $C > 1$, and for any $K > 0$ and $m \geq 1$ such that $K2^{2m} \geq N(C)$ we have

$$\mathbb{P}\left\{ \sup_{0 \leq t \leq K} |W(t) - B_m(t)| \geq 27CK^\frac{3}{2}(\log K)^{\frac{9}{4}m^{\frac{3}{4}2^{-m}}}ight\} \leq \frac{8}{1 - 4^{1-C}(K2^{2m})^{1-C}}.$$ (5)

Apply the “twist and shrink” construction $d$-times independently, to obtain a $d$-dimensional Brownian motion $W = (W^1, \ldots, W^d)$ (vector components will be denoted by superscripts), the corresponding Skorohod-embedded RW’s $B_m = (B^1_m, \ldots, B^d_m)$, and stopping times $(s^1_m(n), \ldots, s^d_m)$:

$$B^j_m(k2^{-2m}) = W^j(s^j_m(k)) \quad (m \geq 0, k \geq 0, j = 1, \ldots, d).$$

Please note that in this paper $d$-dimensional random walks are defined as a vector of $d$ independent one-dimensional random walks. This means that the coordinate axes of a usual $d$-dimensional random walk are rotated and the length of a step is multiplied by $\sqrt{d}$.

Then Theorem [3] and Borel–Cantelli lemma imply

**Corollary 1.** Taking a $d$-dimensional Brownian motion $W$ and Skorohod embedded RW’s $B_m$, for any $K > 0$ and $m \geq 1$ one has

$$\sup_{0 \leq t \leq K} |W(t) - B_m(t)| = O\left( (\log n)^{\frac{3}{4}n^{-\frac{1}{4}}} \right) = O\left( m^{\frac{3}{4}2^{-m}} \right) \quad \text{a.s.,} \quad (5)$$

where $n = K2^{2m}$ denotes the number of vertices of an imbedded random walk $B_m$ over the time interval $[0, K]$.

As it was mentioned above, the above approach to Brownian motion is especially suitable to give an elementary definition of Brownian local time as an a.s. limit of RW local times, cf. [18]. In fact, this was the main motivation to find a similar definition of planar self-intersection local time as well.

The idea that random walk approximations can be applied to obtain results about Brownian local time goes back to Knight [8], who proved the celebrated Ray-Knight theory this way. Révész [11] and Csáki & Révész [4] were the first to prove strong invariance for local times.
Let \( \tilde{\ell}_m(0, x) := 0 \) and
\[
\tilde{\ell}_m(k, x) := \# \{ j : 0 \leq j < k, \tilde{S}_m(j) = x \}.
\]
(6)

Define local times of the “twist and shrink” RW \( \tilde{B}_m \) or imbedded RW’s \( B_m \) by
\[
\tilde{L}_m(t, x) := 2^{-m} \tilde{\ell}_m \left( \frac{t}{2^{2m}}, \frac{x}{2^{2m}} \right).
\]
(7)

Then [18] shows Theorem C.

On any strip \([0, K] \times \mathbb{R}\),
\[
\lim_{m \to \infty} \tilde{L}_m(t, x) = L(t, x) \quad \text{a.s.,}
\]
uniformly in \((t, x)\), where \( L(t, x) \) is the local time of BM. Hence this automatically gives a version of Brownian local time which is continuous in \((t, x)\).

Moreover, for all \( K > 0 \) and \( m \geq 1 \) one has
\[
\sup_{(t,x) \in [0,K] \times \mathbb{R}} |L(t,x) - \tilde{L}_m(t,x)| = O \left( (\log n)^{\frac{3}{4}} n^{-\frac{1}{4}} \right) \quad \text{a.s.,}
\]
where \( n = K 2^{2m} \).

Similar statements hold for \( L_m(t, x) \) computed from Skorohod embedded RW’s \( B_m \) as well. Interestingly, the rate of convergence is the same for local time as for the approximation of Brownian motion with the “twist and shrink” construction. While this rate is much weaker than the optimal Komlós–Major–Tusnády rate \((\log n)^{\frac{1}{2}} n^{-\frac{1}{2}} \) in the case of BM, it just slightly differs from the optimal \((\log n)^{\frac{3}{2}} (\log \log n)^{\frac{1}{4}} n^{-\frac{1}{4}} \) in the case of local time.

## 3 A discrete Itô’s formula

It is interesting that one can give discrete versions of Itô’s formula and of Itô–Tanaka–Meyer formula, which are purely algebraic identities, not assigning any probabilities to the terms. Despite this, the usual Itô’s formula follows fairly easily in a proper probability setting.

Discrete Itô formulas are not new. Apparently, the first such formula was given by Kudzma in 1982 [9]. The elementary algebraic approach used in the present paper is different from that; it was introduced by the author in 1989 [16].

First we need definitions of discrete line integrals and conservative vector fields on a grid. Fix an initial point \( a \in \mathbb{R}^d \) and step-size (mesh) \( h > 0 \). Consider the grid \( G(a, h) := a + h\mathbb{Z}^d \), and let \( f = (f^1, \ldots, f^d) : G(a, h) \to \mathbb{R}^d \) be a vector field on this grid. (Coordinates of a vector will always be denoted by superscripts.) Take an arbitrary broken line (a discrete path) \( \gamma \) that goes through finitely many (not necessarily distinct) oriented edges between adjoining vertices of the grid. A typical such edge is \([x, x + \mu he_j]\), where \( x \in G(a, h) \), \( e_j \) \((1 \leq j \leq d)\) is a coordinate unit vector and \( \mu = \pm 1 \). (The order of the two vertices is important!) A discrete path \( \gamma \) is a formal sum of such oriented edges (that is, a 1-chain):
\[
\gamma = \sum_{r=1}^n [x_r, x_r + \mu_r he_j_r] \quad (1 \leq j_r \leq d).
\]
By definition, the corresponding discrete path integral (or trapezoidal sum) of \( f \) over \( \gamma \) is defined as

\[
T_\gamma (f, \gamma_0') h := \frac{h}{2} \sum_{r=1}^{n} \mu_r \left( f^{+r}(x_r) + f^{-r}(x_r + h e_{j_r}) \right).
\]

Here the symbol \( \gamma_0' \) refers to the unit tangents of \( \gamma \) along edges, and \((f, \gamma_0')\) denotes dot product. When it will be convenient, \( T_\gamma (f(x), \gamma_0'(x)) h \) will be written to show the dummy variable of the summation.

The ordinary path (line) integral of a vector field \( f \) over a path \( \gamma \) will be denoted by \( \int_\gamma (f, \gamma_0') \, ds \) or by \( \int_\gamma (f(x)) \cdot dx \), where \( ds \) refers to the length element of \( \gamma \).

The above definition of discrete path integral shows that the orientation of an edge is defined by the order of its two vertices: it is positive if the edge goes increasingly in a coordinate and negative in the opposite case. If \( \gamma = \emptyset \), we define \( T_\gamma (f, \gamma_0') h = 0 \).

A vector field \( f \) is called discrete conservative on the grid \( \mathcal{G}(a, h) \) if for any \( b, c \in \mathcal{G}(a, h) \) and for any discrete path \( \gamma \) going from \( b \) to \( c \) through edges of \( \mathcal{G}(a, h) \), the discrete path integral does not depend on the path \( \gamma \), it depends only on the initial point \( b \) and endpoint \( c \). In this case the notation \( T_b^c (f, \gamma_0') h \) will be used for the trapezoidal sum. Clearly, \( f \) is discrete conservative if and only if \( T_b^c (f, \gamma_0') h = 0 \) whenever \( b = c \), that is, the path \( \gamma \) is closed.

When \( f \) is discrete conservative, one can define a discrete potential \( g : \mathcal{G}(a, h) \to \mathbb{R} \) by the formula \( g(x) := T_0^x (f, \gamma_0') h \). Then for any points \( b \) and \( c \) in the grid, and for any discrete path \( \gamma \) connecting them, one has \( T_\gamma (f, \gamma_0') h = g(c) - g(b) \).

The following discrete Itô's formula (which is a simple algebraic identity) already appeared in [16, Section 5] in the two-dimensional case. It is based on the principle that though our random walk is “diagonal”, constructed from independent one-dimensional random walks, the discrete integrals below go “parallel to the coordinate axes”, as in a standard continuous Itô’s formula. Also, the main object in our discrete formula is the “integrand” \( f \) in the stochastic sum, which corresponds to the gradient of a scalar field in a standard continuous Itô’s formula. That may explain why we suppose that \( f \) be discrete conservative. These methods make it convenient to deduce important continuous formulae from the discrete ones.

**Lemma 1.** Take \( a \in \mathbb{R}^d \), step \( h > 0 \), and a discrete conservative time-dependent vector field \( f = (f^1, \ldots, f^d) : h^2 \mathbb{Z}^+ \times \mathcal{G}(a, h) \to \mathbb{R}^d \). Consider a sequence \( X_r = (X^1_r, \ldots, X^d_r) \ (r \geq 1) \), where \( X^j_r = \pm 1 \). Define partial sums \( S_0 = a \), \( S_n = a + h(X_1 + \cdots + X_n) \ (n \geq 1) \) and discrete time instants \( t_r = rh^2 \ (0 \leq r \leq n) \). Assume that the steps of \( (S_n) \) are performed in time steps \( h^2 \). Then
the following equalities hold:

\[ T_{x=S_0}^{S_n} \left( f(t_n, x), \gamma_0'(x) \right) h \]

\[ = \sum_{r=1}^{n} T_{x=S_0}^{S_r} \left\{ \{ f(t_r, x) - f(t_{r-1}, x) \} , \gamma_0'(x) \right\} h \]

\[ + \sum_{r=1}^{n} \sum_{j=1}^{d} f^J \left( t_{r-1}, S_{r-1} + \sum_{i=1}^{j-1} hX_i^x e_i \right) \]

\[ + \frac{1}{2} \sum_{r=1}^{n} \sum_{j=1}^{d} f^J \left( t_{r-1}, S_{r-1} + \sum_{i=1}^{j-1} hX_i^x e_i \right) \frac{h^2}{2} \]

(\text{discrete Stratonovich formula}). Alternatively,

\[ T_{x=S_0}^{S_n} \left( f(t_n, x), \gamma_0'(x) \right) h \]

\[ = \sum_{r=1}^{n} T_{x=S_0}^{S_r} \left\{ \{ f(t_r, x) - f(t_{r-1}, x) \} , \gamma_0'(x) \right\} h \]

\[ + \frac{1}{2} \sum_{r=1}^{n} \sum_{j=1}^{d} f^J \left( t_{r-1}, S_{r-1} + \sum_{i=1}^{j-1} hX_i^x e_i \right) h^2 \]

(\text{discrete Itô’s formula}).

\textbf{Proof.} Algebraically,

\[ T_{x=S_0}^{S_r} \left( f(t_r, x), \gamma_0'(x) \right) h - T_{x=S_0}^{S_{r-1}} \left( f(t_{r-1}, x), \gamma_0'(x) \right) h \]

\[ = T_{x=S_0}^{S_r} \left( f(t_r, x), \gamma_0'(x) \right) h - T_{x=S_0}^{S_r} \left( f(t_{r-1}, x), \gamma_0'(x) \right) h \]

\[ + T_{x=S_0}^{S_{r-1}} \left( f(t_{r-1}, x), \gamma_0'(x) \right) h - T_{x=S_0}^{S_{r-1}} \left( f(t_{r-1}, x), \gamma_0'(x) \right) h. \]

Using the assumption that \( f \) is discrete conservative, we get that

\[ T_{x=S_0}^{S_r} \left( f(t_r, x), \gamma_0'(x) \right) h - T_{x=S_0}^{S_{r-1}} \left( f(t_{r-1}, x), \gamma_0'(x) \right) h \]

\[ = T_{x=S_0}^{S_{r-1}} \left( f(t_{r-1}, x), \gamma_0'(x) \right) h \]

and

\[ T_{x=S_{r-1}}^{S_{r-1}} \left( f(t_{r-1}, x), \gamma_0'(x) \right) h \]

\[ = \sum_{j=1}^{d} f^J \left( t_{r-1}, S_{r-1} + \sum_{i=1}^{j-1} hX_i^x e_i \right) \frac{h^2}{2} \]

\[ = \sum_{j=1}^{d} f^J \left( t_{r-1}, S_{r-1} + \sum_{i=1}^{j-1} hX_i^x e_i \right) hX_i^x \]

\[ + \frac{1}{2} \sum_{j=1}^{d} f^J \left( t_{r-1}, S_{r-1} + \sum_{i=1}^{j-1} hX_i^x e_i \right) \frac{h^2}{2}. \]
The first equality follows from the fact that if \( X^j_r = 1 \), one has a positively oriented edge, while if \( X^j_r = -1 \), one has a negatively oriented edge in the trapezoidal sum. Then the second equality follows since \( 1/X^j_r = X^j_r \). Summing up for \( r = 1, \ldots, n \), the sum on the left telescopes, and from the two equalities one obtains the two formulae, respectively.

One can introduce partial local times \( L^\mu_h(t_n, x) \) \((n \geq 0)\) of the series \((S_n)\) with spatial step \( h > 0 \), time step \( h^2 \), and \( \mu \in \{1, -1\} \): \( L^\mu_h(0, x) := 0 \) and

\[
L^\mu_h(t_n, x) := h \# \{ j : 0 \leq j < n, S_j = x, S_{j+1} = x + h\mu \},
\]

where \( n \geq 1 \) and \( x \in \mathcal{G}(a, h) \). The (total) local time is

\[
L_h(t_n, x) := \sum_{\mu \in \{1, -1\}} L^\mu_h(t_n, x) = h \# \{ j : 0 \leq j < n, S_j = x \}.
\]

Here our convention differs from the usual one: time 0 is counted, but time \( n \) is not. The reason is that this better fits the discrete formula below.

**Lemma 2.** With the same assumptions as above in Lemma 1 except that the vector field \( f \) does not depend on time, \( f : \mathcal{G}(a, h) \to \mathbb{R}^d \), one also has

\[
T^S_{x=S_0} (f(x), \gamma'_0(x)) \ h
= \sum_{r=1}^n \sum_{j=1}^d f^j \left( S_{r-1} + \sum_{i=1}^{j-1} hX^i_r e_i \right) hX^j_r
+ \frac{1}{2} \sum_{x \in a + h\mathbb{Z}^d} \sum_{\mu \in \{1, -1\}} L^\mu_h(t_n, x)
\times \sum_{j=1}^d \mu^j \left\{ f^j \left(x + \sum_{i=1}^j h\mu^i e_i \right) - f^j \left(x + \sum_{i=1}^{j-1} h\mu^i e_i \right) \right\},
\]

(discrete Itô–Tanaka–Meyer formula).

**Proof.** Continuing (10) in the proof of the previous lemma,

\[
T^S_{x=S_r} (f(x), \gamma'_0(x)) \ h
= \sum_{j=1}^d f^j \left( S_{r-1} + \sum_{i=1}^{j-1} hX^i_r e_i \right) hX^j_r
+ \frac{1}{2} \sum_{x \in a + h\mathbb{Z}^d} \sum_{\mu \in \{1, -1\}} h \mathbf{1}_{\{S_{r-1} = x, S_r = x + h\mu \}}
\times \sum_{j=1}^d \mu^j \left\{ f^j \left(x + \sum_{i=1}^j h\mu^i e_i \right) - f^j \left(x + \sum_{i=1}^{j-1} h\mu^i e_i \right) \right\}.
\]

Again, summing up for \( r = 1, \ldots, n \), the sum on the left telescopes, and on the right one obtains the asserted formula. \(\square\)
4 Constructing discrete conservative vector fields on a planar grid

For the sake of simplicity, from now on only the planar case \((d = 2)\) will be discussed, as this is the case that will be used in the sequel. The problem that we consider in this section is that given a differentiable scalar field \(g\) in the plane, its gradient \(\nabla g\) is not a discrete conservative vector field on a grid \(G(a,h) = a + h \mathbb{Z}^2\) in general. We want to modify \(\nabla g\) so that the resulting vector field \(f\) be discrete conservative on the grid, but still do not differ much from \(\nabla g\).

Let us call any \(E := [x, x + he_1] \times [x, x + he_2], \ x \in G(a,h)\) an **elementary rectangle of the grid**. It is clear by the previous definitions that a vector field \(f\) is discrete conservative on \(G(a,h)\) if and only if for the counterclockwise directed boundary \(\gamma = \partial E\) of any elementary rectangle, the **discrete curl** of \(f\):

\[
(\text{curl}_h f)(x) := \frac{1}{h^2} \left\{ f^1(x_1, x_2) + f^1(x_1 + h, x_2) + f^2(x_1 + h, x_2) \\
+ f^2(x_1 + h, x_2 + h) - f^1(x_1 + h, x_2 + h) \\
- f^1(x_1, x_2 + h) - f^2(x_1, x_2 + h) - f^2(x_1, x_2) \right\}
\]

is zero. Observe that a discrete curl is a trapezoidal sum over the edges of an elementary rectangle, divided by \(h^2\), the area of the rectangle.

Starting with a scalar field \(g \in C^3(\mathbb{R}^2)\), we introduce the following modification algorithm to obtain a discrete conservative vector field \(f\) on \(G(a,h)\). By translation, we may assume that \(a = 0\). First we set \(f^1(x_1, x_2) = (D_1 g)(x_1, x_2)\) whenever \(x_1 = 0\) or \(x_2 = 0\). \((D_j\) denotes partial differentiation with respect to \(x^j)\.)

Let us consider now elementary rectangles of the grid in the first quadrant. We proceed inductively with layers of rectangles whose lower left (SW) vertex is \((x^1, x^2),\ x^1 \wedge x^2 = rh,\ r = 0, 1, 2, \ldots\). Because of symmetry, it is enough to describe the algorithm when \(x^1 \leq x^2\). In the \(r\)th layer we proceed as \(x^1 = jh,\ j = 0, 1, 2, \ldots\). The next rectangle inherits the values of \(f\) defined on the vertices of previous rectangles, except for the upper right (NE) vertex, which is called the **“new” vertex**. For this, compute the **modified discrete curl**

\[
(\text{curl}^m_h f)(x) := \frac{1}{2h} \left\{ f^1(x_1, x_2) + f^1(x_1 + h, x_2) + f^2(x_1 + h, x_2) \\
+ (D_2 g)(x_1 + h, x_2 + h) - (D_1 g)(x_1 + h, x_2 + h) \\
- f^1(x_1, x_2 + h) - f^2(x_1, x_2 + h) - f^2(x_1, x_2) \right\},
\]

(which is not zero in general) and for \(j = 1, 2\) set

\[
f^1(x_1 + h, x_2 + h) = (D_j g)(x_1 + h, x_2 + h) + (-1)^{j-1}h (\text{curl}^m_h f)(x)
\]

at the “new” vertex. It is clear that the so defined \(f\) is discrete conservative in the first quadrant. Observe that the two modification terms \((-1)^{j-1}h (\text{curl}^m_h f)(x)\) have the same absolute value, and in \((\text{curl}_h f)(x)\) they both have minus sign.

In other quadrants the situation is analogous to the case of the first quadrant, but the “new” vertex is the upper left in the second, the lower left in the third,
and the lower right in the fourth quadrant. The signs of the modification terms $\pm h (\text{curl}_h^y f)(x)$ in (15) have to be changed accordingly as well.

It remains to see how large the difference between $f$ and $\nabla g$ is. For the sake of simplicity, consider only points in the first quadrant, points in other quadrant being analogous. We claim that these errors accumulate only diagonally.

In fact, if one considers two neighbor rectangles with a common edge, then we can see from (14) and (15) that the discrete curl of the “new” rectangle (which is right or up from the “old”) does not inherit the modification terms $\pm h (\text{curl}_h^y g h f)(x)$ of the “old” rectangle. The reason is that the modification terms of the “old” rectangle cancel in the “new” curl; out of the two pairs of edges joining at the NE vertex of the “old” rectangle, one pair of parallel edges has opposite directions, so the sign of the modification term changes, while the other pair has identical directions, so the sign of the modification term remains the same. On the other hand, in the case of two rectangles with a single common vertex (so which are in diagonal position), the curl of the “new” (NE) rectangle does inherit the modification terms of the “old” (SW) rectangle, because both pairs of parallel edges have opposite directions.

Thus (15) implies that for any $n \geq 1$ and $R > 0$,

$$\sup_{|x|,|y| = n h, |z| \leq R} \left| f^j(x) - (D_j g)(x) \right| \leq n h \sup_{|x| \leq R} |(\text{curl}_h g)(x)| \leq \frac{R}{\sqrt{2}} \sup_{|x| \leq R} |(\text{curl}_h g)(x)|.$$

Thus the error estimation reduces to an estimate between the “true” curl $\nabla g := D_{12} g - D_{21} g = 0$ and the discrete curl $\text{curl}_h g$. Or, more precisely, between the “true” path integral $\int_\gamma (\nabla g, g') ds = 0$ and the discrete path integral $T_\gamma (\nabla g, g'_0) h$ of the conservative vector field $\nabla g$ over the boundary $\gamma$ of an elementary rectangle.

Now, as it is well-known, if $\phi \in C^2(\mathbb{R})$, the error between the integral and the trapezoidal area of $\phi$ on $[x, x + h]$ is

$$\int_x^{x+h} \phi(u) du - h \phi(x) + \phi(x + h) = -\frac{h^3}{12} \phi''(x + sh),$$

where $0 \leq s \leq 1$.

When $E := [x, x + he_1] \times [x, x + he_2]$ is an elementary rectangle and $\gamma = \partial E$, it follows that for any $g \in C^3(\mathbb{R}^2)$ and for any $x \in \mathbb{R}^2$,

$$\text{curl}_h g(x) = \frac{1}{h^2} T_\gamma (\nabla g, g'_0) h = \frac{1}{h^2} \left\{ T_\gamma (\nabla g, g'_0) h - \int_\gamma (\nabla g, g'_0) ds \right\}$$

$$= \frac{h}{12} \left\{ (D_3^1 g)(x^1 + s_1 h, x^2) + (D_3^2 g)(x^1 + h, x^2 + s_2 h) - (D_3^1 g)(x^1 + s_3 h, x^2 + h) - (D_3^2 g)(x^1, x^2 + s_4 h) \right\},$$

where $0 \leq s_j \leq 1$. This implies that

$$|\text{curl}_h g(x)| \leq \frac{1}{6} h \epsilon_E,$$
where \( \epsilon_E := \sup_{x,y \in E} \{(D^1_1g)(x) - (D^1_1g)(y), (D^2_1g)(x) - (D^2_1g)(y)\} \), which goes to zero as \( h \to 0^+ \).

Combining (16) and (15), we obtain the following

**Lemma 3.** Let \( a \in \mathbb{R}^2 \), \( h > 0 \), \( R > 0 \), and \( g \in C^3(\mathbb{R}^2) \). Define \( \epsilon(h) = \epsilon_g(h,a,R) \) by

\[
\epsilon(h) := \sup \{(D^1_1g)(x) - (D^1_1g)(y) : |x-a|, |y-a| \leq R+h; |x-y| \leq h\sqrt{2}; j = 1, 2\}
\] (which goes to zero as \( h \to 0^+ \)). Let \( f \) denote the discrete conservative vector field on the grid \( G(a,h) = a + hZ^2 \), obtained from \( \nabla g \) by the modification algorithm described above. Then

\[
\sup_{|x-a| \leq R} |f(x) - (\nabla g)(x)| \leq \frac{R}{6} h \epsilon(h).
\]

This lemma expresses the fact that the error of the above modification algorithm even when divided by \( h \) can be made uniformly arbitrary small on any bounded planar set by choosing a small enough \( h \).

5 Planar Itô’s formula as an almost sure limit of the discrete formula

Let us apply now the planar \( (d = 2) \) case of the discrete Itô’s formula (16) to a random, time-dependent scalar field \( g : \Omega \times \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R} \), which is measurable in \( \omega \) for all \( (t,x) \), and is \( C^{1,3} \) in \( (t,x) \) for almost all \( \omega \).

More exactly, fixing \( a \in \mathbb{R}^2 \) and taking \( m = 0, 1, 2, \ldots \) and \( h = 2^{-m} \), construct first a sequence of discrete conservative vector fields \( f_m(\omega,t,x) \) on the grids \( G(a,2^{-m}) \) from \( (\nabla g)(\omega,t,x) \), for each \( \omega \) and \( t \) fixed, by the modification algorithm discussed in the previous section. (In this paper \( \nabla g = (D_1g, D_2g) \) denotes gradient of \( g(\omega,t,x) \) with respect to \( x \) and \( D_jg \) its derivative with respect to \( t\).) Fixing a bounded time interval \([0,K]\), by a slight generalization of Lemma 5 for any \( \omega \) fixed we get that

\[
\sup_{t \in [0,K]} \sup_{|x-a| \leq R} |f_m(\omega,t,x) - (\nabla g)(\omega,t,x)| \leq \frac{R}{6} 2^{-m} \epsilon_K^0(2^{-m}), \quad (20)
\]

where

\[
\epsilon_K^0(h) := \sup \{(D^1_1g)(\omega,t,x) - (D^1_1g)(\omega,t,y)\}
\]

and the supremum is taken for all \( |x-a|, |y-a| \leq R+h, |x-y| \leq h\sqrt{2}, t \in [0,K] \) and \( j = 1, 2 \). Then \( \epsilon_K^0(2^{-m}) \to 0 \) as \( m \to \infty \).

Moreover, it is clear from the modification algorithm (16) that at any point of the grid, \( f_m \) differs from \( \nabla g \) by a finite linear combination of \( D_jg \) values, so \( f_m(\omega,t,x) \) is continuously differentiable as a function of \( t \), like \( \nabla g \). Moreover, by taking derivative of (16) with respect to \( t \), we obtain

\[
(D_1 f_m)(\omega,t,x^1 + h, x^2 + h) = (D_1 D_1 g)(\omega,t,x^1 + h, x^2 + h) + (-1)^{j-1} h (\text{curl}_{h}^{D_1 g}(D_1 f_m))(x).
\]
By the same argument that lead to Lemma 3 for any fixed \( \omega \) it follows that

\[
\sup_{t \in [0,K]} \sup_{|x-a| \leq R} |(D_t f_m)(\omega, t, x) - (D_t \nabla g)(\omega, t, x)| \leq \frac{R}{6} h \epsilon_k(h),
\]

(21)

where

\[
\epsilon_k(h) := \sup\{|(D_t^3 P_t)g(\omega, t, x) - (D_t^3 D_t^2 g)(\omega, t, y)|\}
\]

and the supremum is taken for all \( |x-a|, |y-a| \leq R+h, |x-y| \leq h\sqrt{2}, t \in [0, K] \) and \( j = 1, 2 \). Then \( \epsilon_k(h) \to 0 \) as \( h \to 0^+ \).

Second, start with a planar Brownian motion \( (W(t))_{t \in \mathbb{R}} \) constructed as in Section 2 but shifted so that \( W(0) = a \). Then take the planar Skorohod embedded random walks \( (B_m(t))_{t \in \mathbb{R}} \), \( B_m(r2^{-2m}) = W_j(s_m(r)) \) \( (j = 1, 2) \) in (9). That is, let \( S_r := B_m(r2^{-2m}) \) and

\[
X_r = X_m(r) := 2^m \{ B_m(r2^{-2m}) - B_m((r-1)2^{-2m}) \}. \]

Then \( (X_m(r))_{r=1}^{\infty} \) is a two-dimensional, independent, \((\pm 1, \pm 1)\) symmetric coin tossing sequence. Define \emph{stochastic sums} by

\[
(f_m(\omega, u, W) \cdot W)^m := \sum_{r=1}^{n} \left\{ f_m^1(\omega, t_r-1, B_m^1(t_r-1)), B_m^2(t_r-1) \right\} 2^{-m} X_m^1(r)
\]

\[
+ \sum_{r=1}^{n} \left\{ f_m^2(\omega, t_r-1, B_m^1(t_r)), B_m^2(t_r) \right\} 2^{-m} X_m^2(r), \]

(22)

where \( t_r := r2^{-2m} \) and \( n := \lfloor 2^{2m} \rfloor \). (Of course, \( B_m, X_m, \) and \( W \) all depend on \( \omega \), but this dependence is not shown here and below, to simplify the notation.)

Now the discrete Itô’s formula (9) can be written as

\[
T_{x=\omega}^{B_m(t_n)} (f_m(\omega, t_n, x), \gamma'_{0}(x)) 2^{-m}
\]

\[
= \sum_{r=1}^{n} T_{x=\omega}^{B_m(t_r)} (\{ f_m(\omega, t_r, x) - f_m(\omega, t_r-1, x) \}, \gamma_{0}(x)) 2^{-m}
\]

\[
+ (f_m(\omega, u, W) \cdot W)^m
\]

\[
+ \frac{1}{2} \sum_{r=1}^{n} \left\{ \frac{f_m^1(\omega, t_r-1, B_m^1(t_r)), B_m^2(t_r-1)) - f_m^1(\omega, t_r-1, B_m(t_r-1))}{2^{-m} X_m^1(r)}
\]

\[
+ \frac{f_m^2(\omega, t_r-1, B_m(t_r)) - f_m^2(\omega, t_r-1, B_m^1(t_r), B_m^2(t_r))}{2^{-m} X_m^2(r)} \right\} 2^{-m}. \]

Our strategy is that we show that each term in this formula, except for the stochastic sum, almost surely uniformly converges to the corresponding term of the planar Itô’s formula, on any bounded time interval. Then it follows that the stochastic sum almost surely uniformly converges as well (to the stochastic integral), on any bounded time interval. Hence at the same time we obtain a proof of the planar Itô formula as an almost sure uniform limit of the discrete formula.

**Theorem 1.** Suppose \( g(\omega, t, x) \) is measurable in \( \omega \) for all \( (t, x) \), and is \( C^{1,3} \) in \( (t, x) \) for almost every \( \omega \). For each \( m = 0, 1, 2, \ldots \) and for all \( \omega \) and \( t \), let \( f_m(\omega, t, x) \) denote the discrete conservative modification of \( (\nabla g)(\omega, t, x) \) on the
grid $G(a, 2^{-m})$. Taking a planar Brownian motion $W$, for each $m$ define the Skorohod embedded random walk $B_m$. Then for arbitrary $K > 0$,

$$\sup_{t \in [0,K]} \left| \left( f_m(\omega, u, W) \cdot W \right)_t^m - \int_0^t (\nabla g)(\omega, u, W(u)) \cdot dW(u) \right| \to 0$$

almost surely as $m \to \infty$, and for any $t \geq 0$ we obtain the planar Itô’s formula as an almost sure limit of the discrete formula (23):

$$g(\omega, t, W(t)) - g(\omega, 0, W(0)) = \int_0^t (D_t g)(\omega, u, W(u)) \, du + \int_0^t (D_x g)(\omega, u, W(u)) \cdot dW(u) + \frac{1}{2} \int_0^t (\Delta g)(\omega, u, W(u)) \, du.$$  

**Proof.** We are going to prove (24) pathwise. For this, let $\Omega_0, P \{ \Omega_0 \} = 1$, denote a subset of the sample space $\Omega$, on which, as $m \to \infty$, $B_m$ uniformly converges to $W$ on $[0, K]$ and $g(\omega, t, x)$ is $C^{1,3}$ as a function of $(t, x)$. During the proof we fix an $\omega \in \Omega_0$. Then, obviously, $W$ has a continuous path and its range over $[0, K]$ lies in a ball $B_R(a) := \{ x : |x - a| \leq R \}$ with a finite radius $R = R(\omega)$. Also, by Corollary 1 we may assume that the range of $B_m$ over $[0, K]$ lies in the same ball for any $m \geq m_0(\omega)$.

Consider the term on the left side of (24). We want to show that it uniformly converges to $g(\omega, t, W(t)) - g(\omega, t, a)$ for $t \in [0, K]$. Define the path $\gamma_n = [a, (B_{m}(t_n), a^2)] + [(B_{m}(t_n), a^2), B_{m}(t_n)]$, where $n := [t2^{-m}]$ and $t_n = n2^{-m}$. Then we have

$$\sup_{t \in [0,K]} \left| T_{x=a}^{B_{m}(t_n)} \left( f_m(\omega, t_n, x), \gamma'_0(x) \right) 2^{-m} - \int_0^{W(t)} ((\nabla g)(\omega, t, x), \gamma'_0(x)) \, ds \right| \leq \sup_{t \in [0,K]} \left| T_{\gamma_n} (f_m(\omega, t_n, x), \gamma'_0(x)) 2^{-m} - T_{\gamma_n} ((\nabla g)(\omega, t, x), \gamma'_0(x)) 2^{-m} \right| + \sup_{t \in [0,K]} \left| \int_0^{W(t)} ((\nabla g)(\omega, t, x), \gamma'_0(x)) \, ds \right|.$$  

The first term on the right side of (25) can be bounded above by $K2^{2m} \times 2^{-m} \times (R/6)^2 \times e^2_{R}(2^{-m}) = O(\epsilon^2_{R}(2^{-m}))$. Here the first factor bounds the number of terms in the trapezoidal sum, the second factor is a multiplier in each term, and the third factor bounds the difference of the terms in the two sums by (21).

The second term on the right side of (25) can be bounded by $K2^{2m} \times (M_3/12)2^{-3m} = O(2^{-m})$. Here the first factor bounds the number of terms in the trapezoidal sum and the second factor bounds the difference of a trapezoidal term and the corresponding integral by (17), where $M_3$ is an upper bound of the magnitude of third $x$-partial derivatives of $g$ for $(t, x) \in [0, K] \times B_R(a)$.

The third term on the right side of (25) can be bounded by $M_2 \times (O(m^{\frac{1}{2}}2^{-\frac{m}{2}}) + 2^{-m}) = O(m^{\frac{3}{2}}2^{-\frac{m}{2}})$. Here $M_2$ is an upper bound of $|\nabla g|$ for $(t, x) \in [0, K] \times B_R(a)$, while the first term in the parentheses bounds $|W(t) - B_m(t)|$ by (14), and the second term bounds $|B_m(t) - B_m(t_n)|$. 

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In sum, the the term on the left side of (23) is bounded by $O\left(\epsilon_K^2(2^{-m})\right) + O(m^{\frac{3}{2}}2^{-\frac{m}{2}})$.

Let us turn now to the first term on the right side of (23). Define the paths $\gamma_r = [a, (B^1_m(t_r), a^2)] + [(B^2_m(t_r), a^2), B_m(t_r)]$ $(1 \leq r \leq n)$. Then we have

$$\sum_{r=1}^{n} T^B_{x=a}(t_r) \left\{ \{ f_m(\omega, t_r, x) - f_m(\omega, t_{r-1}, x) \}, \gamma_0(x) \right\} 2^{-m}$$

$$= \sum_{r=1}^{n} T_{\gamma_r} \left\{ \{(D_1 f^1_m(\omega, t_{r-1} + s_1^r, x), D_1 f^2_m(\omega, t_{r-1} + s_2^r, x)) \} 2^{-2m}, \gamma_0(x) \right\} 2^{-m}$$

$$= \sum_{r=1}^{n} T_{\gamma_r} \left\{ \{(D_1 D_1 g(\omega, t_{r-1} + s_1^r, x), D_1 D_2 g(\omega, t_{r-1} + s_2^r, x)) \}, \gamma_0(x) \right\} 2^{-m} 2^{-2m}$$

$$+ O(\epsilon_K^1(2^{-m}))$$

$$= \sum_{r=1}^{n} D_1 \left\{ T_{\gamma_r} (\nabla g(\omega, t_r, x), \gamma_0(x)) \right\} 2^{-2m} + O(\epsilon_K^1(2^{-m})) + O(2^{-m})$$

$$= \sum_{r=1}^{n} D_1 \left\{ T_{\gamma_r} (\nabla g(\omega, t_r, x)) \right\} ds 2^{-2m} + O(\epsilon_K^1(2^{-m})) + O(2^{-m})$$

$$= \sum_{r=1}^{n} D_1 \left\{ g(\omega, t_r, B_m(t_r)) - g(\omega, t_r, a) \right\} 2^{-2m} + O(\epsilon_K^1(2^{-m})) + O(2^{-m})$$

$$= \int^t \left( D_1 g)(\omega, u, W(u)) du - g(\omega, t, a) + g(\omega, 0, a) \right) O(\epsilon_K^1(2^{-m}))+O(m^{\frac{3}{2}}2^{-\frac{m}{2}}).$$

Above we made use of the fact that all considered functions are uniformly continuous over the bounded set $[0, K] \times B_B(a)$. The first equality used the mean value theorem in the time variable, component-wise for $f_m$, with $0 \leq s_1^r, s_2^r \leq 2^{-2m}$. The second equality applied inequality (24), combined with the largest possible number of terms in the sums and the corresponding multipliers. The third equality estimated the error, when one replaces the values of the components of $\nabla g$ at time $t_{r-1} + s_1^r$ by their values at time $t_r$. The fourth equality replaced the trapezoidal sum by the corresponding integral, using (17). In the fifth equality we evaluated the integral over the path $\gamma_r$. The last equality replaces the sum of the function $D_1 g$ in the time variable by an integral, and at the same time replaces $B_m$ by $W$, using (5).

Finally, let us consider the last term in (20). By (24), for the first term in the braces we have

$$f^1_m (\omega, t_{r-1}, B^1_m(t_r), B^2_m(t_{r-1})) - f^1_m (\omega, t_{r-1}, B^1_m(t_{r-1})))$$

$$= \frac{(D_1 g)(\omega, t_{r-1}, B^1_m(t_r), B^2_m(t_{r-1})) - (D_1 g)(\omega, t_{r-1}, B^1_m(t_{r-1}) + O(\epsilon_K^1(2^{-m}))}{2^{-m}X^1_m(r)}$$

$$= \frac{(D_1 g)(\omega, t_{r-1}, B^1_m(t_{r-1} + s_1^r, B^1_m(t_{r-1})) + O(\epsilon_K^1(2^{-m}))}{2^{-m}X^1_m(r)}$$

where $|s_1^r| \leq 2^{-m}$. For the other part in the braces of the last term of (23) involving $f^2_m$ one can obtain a similar result by the help of $D_2$. In sum, with
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with unit steps in unit time. Define self-intersection local time of the random walk by

\[ S = \text{symmetric random walk} \quad \text{(definition of discrete local time (6), (7), (11) and (12)).} \]

Take first a planar simple, the definition of discrete self-intersection local time

\[ 6 \quad \text{Discrete self-intersection local time} \]

made this assumption unnecessary.

Together with the pathwise, integration by parts stochastic integration technique

\[ \text{filtration of Brownian motion } W \]

Here, similarly as above, the sum was replaced by an integral and \( B_m \) by \( W \).

Thus we have seen that all terms, except for the stochastic sum, converge to their counterparts in (24), almost surely uniformly on \([0, K]\). (An extra term \(-g(\omega, t, u)\) has appeared too on both sides, that cancel each other.) Therefore the stochastic sum must converge to the stochastic integral in the same sense as well. This ends the proof of the theorem.

The reader may have noticed in the statement of Theorem 1 that the usual condition in Itô’s formulae that the random function \( g(\omega, t, x) \) be adapted to the filtration of Brownian motion \( W \), was not needed: the assumed smoothness of \( g \) together with the pathwise, integration by parts stochastic integration technique made this assumption unnecessary.

6 Discrete self-intersection local time

The definition of discrete self-intersection local time follows the lines of the definition of discrete local time (6), (7), (11) and (12). Take first a planar simple, symmetric random walk \((S_n)_{n=0}^{\infty} = 0, S_0 = 0, S_n = \sum_{i=1}^{n} X_i (n \geq 1)\), where \((S^1_n)\) and \((S^2_n)\) are independent one-dimensional simple, symmetric random walks with unit steps in unit time. Define self-intersection local time of the random walk by

\[ \alpha_1(n, x) := \# \{(i, j) : 0 \leq i < j < n, S_j - S_i = x \} \]

\[ = \sum_{j=i}^{n-1} \sum_{j=i}^{n-1} \mathbf{1}_{\{S_j - S_i = x\}}, \quad (27) \]

where \( n \in \mathbb{Z}_+ \) and \( x \in \mathbb{Z}^2 \). We also need partial self-intersection local times

\[ \alpha^\mu(n, x) := \# \{(i, j) : 0 \leq i < j < n, S_j - S_i = x, S_{j+1} - S_j = \mu \}, \quad (28) \]

where \( \mu \in \{-1, 1\}^2 \).

Clearly, by the strong Markovian property of random walks, each inner sum \( \ell_i(n - i, x) := \sum_{j=i}^{n-i} \mathbf{1}_{\{S_j - S_i=x\}} \) in the last term of (27) is a local time of a
random walk started from the point $S_i$, taken at time $n-i$ at the point $x$;

$$\alpha_1(n, x) = \sum_{i=0}^{n-1} \ell_i(n-i, x). \quad (29)$$

Denote the largest number of visits to a point of the random walk in the first $n$ steps by

$$\ell^*(n) := \sup_{x \in \mathbb{Z}^2} \ell(n, x).$$

Similarly, denote the largest number of visits to a point of the random walk starting from point $S_i$, in the first $n-i$ steps, by $\ell_i^*(n-i)$. Then for any $\omega \in \Omega$ one clearly has

$$\ell^*_0(n) \geq \ell^*_1(n-1) \geq \cdots \geq \ell^*_{n-1}(1). \quad (30)$$

In a classical paper [6], Erdős and Taylor showed the following inequality for the maximum number of visits of a random walk in the first $n$ steps:

$$\limsup_{n \to \infty} \frac{\ell^*(n)}{\log^2(n)} \leq \frac{1}{\pi} \quad \text{a.s.} \quad (31)$$

The next lemma is an easy consequence of this result.

**Lemma 4.**

$$\limsup_{n \to \infty} \frac{\sup_{x \in \mathbb{Z}^2} \alpha_1(n, x)}{n \log^2 n} \leq \frac{1}{\pi} \quad \text{a.s.}$$

**Proof.** By (29), (30) and (31),

$$\limsup_{n \to \infty} \frac{\sup_{x \in \mathbb{Z}^2} \alpha_1(n, x)}{n \log^2 n} \leq \limsup_{n \to \infty} \frac{n \ell^*_0(n)}{n \log^2 n} \leq \frac{1}{\pi} \quad \text{a.s.}$$

Now we apply the previous results to shrunken random walks. Let $h > 0$, $x \in h\mathbb{Z}^2 = \mathcal{G}(0, h)$ and $t \in h^2\mathbb{Z}_+$. Consider a simple, symmetric random walk $(S_n)_{n=0}^{\infty}$, $S_0 = 0$, on the grid $\mathcal{G}(0, h)$, with time steps $h^2$. (That is, the time between step $n$ and step $n+1$ of the walk is $h^2$.) Define the corresponding self-intersection local time as

$$\alpha_h(t, x) := h^2 \# \{(i, j) : 0 \leq i \leq j < t/h^2, S_j - S_i = x\}$$

$$= h^2 \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} 1_{\{S_j - S_i = x\}} = h^2 \sum_{i=0}^{n-1} \ell_i(n-i, x/h), \quad (32)$$
where \( n = t/h^2 \) and \( \xi_i(n - i, x/h) \) is defined in the same way as above. A partial self-intersection local time in the direction \( \mu \in \{1, -1\}^2 \) is

\[
\alpha_h^\mu(t, x) := h^2 \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} 1\{S_j - S_i = x; S_{j+1} - S_j = \mu h\}. \tag{33}
\]

Let us extend \( \alpha_h(t, x) \) for any \( t \in \mathbb{R}_+ \) and \( x \in \mathbb{R}^2 \) as a continuous function. First, with \( t \in h^2 \mathbb{Z}_+ \) fixed, we apply linear interpolation in \( x \). Let \( x \) be a point in a lower triangle \( \Delta \) with vertices \((a^1, a^2), (a^1 + h, a^2), \) and \((a^1, a^2 + h)\) for some \( a \in h \mathbb{Z}^2 \). Let \( A = \alpha_h(t, (a^1, a^2)), B = \alpha_h(t, (a^1 + h, a^2)), \) and \( C = \alpha_h(t, (a^1, a^2 + h)) \). Then define

\[
\alpha_h(t, x) := A + \frac{x^1 - a^1}{h}(B - A) + \frac{x^2 - a^2}{h}(C - A). \tag{34}
\]

Analogous is the case with an upper triangle.

Second, define \( \alpha_h(t, x) := \alpha_h(h[t/h], x) \) for \( t \in \mathbb{R}_+ \) and \( x \in \mathbb{R}^2 \). Similar is the extension of partial self-intersection local times as continuous functions. It will be of use later that then

\[
\int_{\Delta} \alpha_h(t, x) \, dx = \frac{h^2}{6}(A + B + C). \tag{35}
\]

Lemma \( \ref{lemma} \) clearly implies that

\[
\limsup_{h \to 0^+} \sup_{x \in \mathbb{R}^2} \frac{\sup_{t \in \mathbb{R}^2} \alpha_h(t, x)}{t \log^2(t/h^2)} \leq \frac{1}{\pi} \quad \text{a.s.}
\]

Briefly, this means that

\[
\sup_{x \in \mathbb{R}^2} \alpha_h(t, x) = O\left(\log^2(h)\right) \quad (h \to 0^+) \quad \text{a.s.} \tag{36}
\]

Take now a planar Brownian motion \((W(t))_{t \geq 0}, W(0) = 0\). Then take planar Skorohod embedded random walks \((B_m(t))_{t \geq 0}, B_m(r2^{-2m}) = W(j(s_h^m(r))) \) \((j = 1, 2)\) for \( m \in \mathbb{Z}_+ \). Clearly, \( B_m(t) \) is a shrunken random walk with \( h = 2^{-m} \). For sake of simplicity, let us denote the corresponding self-intersection local time by \( \alpha_m(t, x) \), and partial self-intersection local times by \( \alpha_m^\mu(t, x) \). It follows that

\[
\sup_{x \in \mathbb{R}^2} \alpha_m(t, x) = O(m^2) \quad (m \to \infty) \quad \text{a.s.}, \tag{37}
\]

uniformly on any bounded time interval \( t \in [0, K] \).

It will be also useful in the sequel that for a.e. \( \omega \), \( \alpha_m(t, x) = 0 \) if \( |x| > R = R(\omega) \), for any \( m \geq 0 \) and \( t \in [0, K] \), supposing \( R \) is large enough. This follows from the fact that for a.e. \( \omega \in \Omega \), the continuous function \( W(v) - W(u) \) is bounded on the compact triangle \( V_K = \{(u, v) : 0 \leq u \leq v \leq K\} \) and \( B_m(v) - B_m(u) \) almost surely uniformly converges to it on \( V_K \) as \( m \to \infty \).

### 7 A discrete Tanaka–Rosen–Yor formula

The aim of this section is to give a discrete version of the planar Tanaka–Rosen–Yor formula. Beyond its intrinsic interest, a special case of this formula will serve
where \( \epsilon \) paths

**Lemma 5.** Let \( \phi \) be a \( C^3 \) scalar field in the plane. Fix an \( h > 0 \) and let \( x \in h\mathbb{Z}^2 = \mathcal{G}(0, h) \). Consider a sequence \( X_r = (X_r^1, X_r^2) \) \( (r \geq 1) \), where \( X_r^j = \pm 1 \). Take partial sums \( S_0 = 0 \), \( S_n = h(X_1 + \cdots + X_n) \) \( (n \geq 1) \), supposing that the steps of this “walk” are performed in time units \( h^2 \). Let us take the discrete paths \( \gamma_r = [0, (S_r^1, 0)] + [(S_r^1, 0), S_r] \), \( (1 \leq r \leq n) \). Then with any \( y \in h\mathbb{Z}^2 \) fixed, one obtains the following discrete Tanaka–Rosen–Yor formula:

\[
\sum_{j=0}^{n} \{ T_{\gamma_n}(\nabla \phi(x - S_j - y), \gamma'_n(x)) \} h^2 = \sum_{r=1}^{n} \{ T_{\gamma_n}(\nabla \phi(x - S_r - y), \gamma'_n(x)) \} h^2 \\
+ \sum_{r=1}^{n} \sum_{j=0}^{r-1} \{ (D_1 \phi)(S_{r-1} - S_j - y) hX_j^1 + (D_2 \phi)(S_{r}^1, S_{r-1}^2) - S_j - y) hX_j^2 \} h^2 \\
+ \frac{1}{2} \sum_{r=1}^{n} \sum_{j=0}^{r-1} (\Delta \phi)(S_{r-1} - S_j - y) h^4 + O(h) + O(\epsilon(h)),
\]

where \( \epsilon(h) \to 0 \) as \( h \to 0 \). One has the following equality for the last term as well:

\[
L_h \phi(t_n, y) := \sum_{r=1}^{n} \sum_{j=0}^{r-1} (\Delta \phi)(S_{r-1} - S_j - y) h^4 = \sum_{x \in h\mathbb{Z}^2} \alpha_h(t_n, x) (\Delta \phi)(x - y) h^2 + O(h \log^2 h),
\]

where \( t_n = nh^2 \) and \( \alpha_h(t_n, x) \) is the self-intersection local time \( \lceil \gamma \rceil \) of the sums \( S_n \). For any \( K > 0 \) fixed, the error terms in (38) and (39) are uniform while \( t_n \in [0, K] \).

**Proof.** Define the following time dependent scalar field \( g^y : h^2\mathbb{Z}^2 \times h\mathbb{Z}^2 \to \mathbb{R} \):

\[
g^y(t, x) := \sum_{j=0}^{t/h^2} \phi(x - S_j - y) h^2,
\]

where \( y \in h\mathbb{Z}^2 \) is a parameter. Take a finite \( R > 0 \) such that the disc \( B_R(0) \) cover all the points \( (S_r)_r = 0^n \), \( n = t/h^2 \) and the point \( y \) as well. Then all points \( S_j - S_i - y \) are contained by the disc \( B_{3R}(0) \). Thus by Lemma 8 one can construct a discrete conservative vector field \( \psi \) in the plane such that

\[
\sup_{|x| \leq 3R} |\psi(x) - \nabla \phi(x)| \leq \frac{R}{2} h\epsilon(h),
\]

where \( \epsilon(h) = \epsilon_\phi(h, R) \to 0 \) as \( h \to 0 \).
Further, define
\[ f_y(t, x) := \frac{t}{h^2} \sum_{j=0}^{\psi(x - S_j - y) h^2}. \]
Then by (40) it follows that for any \( y \) and \( t \) fixed,
\[
\sup_{|x| \leq 3R} |f_y(t, x) - \nabla g_y(t, x)| = \sup_{|x| \leq 3R} \left| \frac{t}{h^2} \sum_{j=0}^{\psi(x - S_j - y) h^2} \nabla \phi(x - S_j - y) h^2 \right| \leq R t h \epsilon(h), \tag{41}
\]
and \( f_y(t, x) \) is a discrete conservative vector field in the plane.

Now apply the discrete Itô’s formula (9) to \( f_y(t, x) \). Let us denote \( t_r = rh^2 \) \((r = 0, 1, \ldots, n)\). Then we get the following terms.

The term on the left side of (9) becomes
\[
T_{S_0} \left( f_y(t_n, x), \gamma_0(x) \right) h = \sum_{j=0}^{n} \left\{ T_{\gamma_n} \left( \nabla \phi(x - S_j - y), \gamma_0(x) \right) h \right\} h^2 + O(\epsilon(h)).
\]
The error term is obtained since there are \( t/h^2 \) terms in the trapezoidal sum, there is a multiplier \( h \) in each term, and we can apply (41) to each term.

The first term on the right side of (9) becomes
\[
\sum_{r=1}^{n} T_{S_r} \left( \{f_y(t_r, x) - f_y(t_{r-1}, x)\}, \gamma_0(x) \right) h = \sum_{r=1}^{n} \left\{ T_{\gamma_r} \left( \nabla \phi(x - S_r - y), \gamma_0(x) \right) h \right\} h^2 + O(\epsilon(h)).
\]
The error term is obtained since there are \( t/h^2 \) terms in both sums, respectively; there are multipliers \( h \) and \( h^2 \) in each term, respectively; and we can apply (40) for each term.

The second term on the right side of (9) becomes
\[
\sum_{r=1}^{n} \left\{ (f_y)^1 (t_{r-1}, S_{r-1}) h X_r^1 + (f_y)^2 (t_{r-1}, (S_r^1, S_r^2)) h X_r^2 \right\}
= \sum_{r=1}^{n} \sum_{j=0}^{r-1} \left\{ (D_1 \phi) (S_{r-1} - S_j - y) h X_r^1 + (D_2 \phi) ((S_r^1, S_r^2) - S_j - y) h X_r^2 \right\} h^2 + O(\epsilon(h)).
\]
Again, the error term is obtained since there are at most \( t/h^2 \) terms in both sums, respectively; there are multipliers \( h \) and \( h^2 \) in each term, respectively; and we can apply (40) for each term.
Finally, the last term on the right side of (30) of (41) becomes

\[
\sum_{r=1}^{n} \left\{ \left( f^y \right)^1 (t_{r-1}, (S^1_{r-1}, S^2_{r-1})) - \left( f^y \right)^1 (t_{r-1}, S_{r}) \right\} \frac{hX^1_r}{h^2} + \frac{(f^y)^2 (t_{r-1}, S_r) - (f^y)^2 (t_{r-1}, (S^1_{r-1}, S^2_{r-1}))}{hX^2_r} \right\} h^2 = \sum_{r=1}^{n} \sum_{j=0}^{r-1} \left\{ \frac{(D_1 \phi)( (S^1_{r-1}, S^2_{r-1}) - S_j - y) - (D_1 \phi)(S_{r-1} - S_j - y)}{hX^1_r} \right\} \right\} h^4 + O(\epsilon(h)). \quad (42)
\]

Here the error term is obtained because there are at most \( t/h^2 \) terms in both sums, respectively; there are multipliers \( h^2 \) in both, respectively; each term is divided by \( h \); and we can apply (10) for each term.

We need to write the last term in two different ways. The first way mimics the method applied to the last term in the proof of Theorem 1. There exist \( s^1_r, s^2_r \in [-h, h] \) such that the last term equals

\[
\sum_{r=1}^{n} \sum_{j=0}^{r-1} \left\{ (D_1 \phi)( (S^1_{r-1} + s^1_r, S^2_{r-1}) - S_j - y) \right\} \right\} h^4 + O(\epsilon(h)) = \sum_{r=1}^{n} \sum_{j=0}^{r-1} (\Delta \phi)(S_{r-1} - S_j - y) h^4 + O(\epsilon(h)). \quad (43)
\]

Here the error term \( O(h) \) is obtained when one replaces the translations \( s^1_r \) by 0 in the second partial derivatives of \( \phi \), which are uniformly continuous over the bounded ball \( B_{3R}(0) \).

The second way of writing the last term (42) uses discrete self-intersection local times (39), arranging the terms in (42) according to \( x = S_{r-1} - S_r \):

\[
\sum_{x \in h Z^2} \sum_{\mu \in \{-1,1\}^2} \alpha_\mu^1(t_n, x) w^\mu(x - y) h^2 + O(\epsilon(h)) = \sum_{x \in h Z^2} \alpha_\mu(t_n, x) (\Delta \phi)(x - y) h^2 + O(h \log^2 h) + O(\epsilon(h)), \quad (44)
\]

where

\[
w^\mu(x) := \frac{(D_1 \phi)(x^1 + h\mu^1, x^2) - (D_1 \phi)(x^1, x^2)}{h\mu^1} + \frac{(D_2 \phi)(x^1 + h\mu^1, x^2 + h\mu^2) - (D_2 \phi)(x^1 + h\mu^1, x^2)}{h\mu^2} = (D_{11} \phi)(x^1 + s^1, x^2) + (D_{22} \phi)(x^1 + h\mu^1, x^2 + s^2) = (\Delta \phi)(x) + O(h), \quad (45)
\]

\( s^1, s^2 \in [-h, h] \). In (44) the error term \( O(h \log^2 h) \) is obtained when one replaces the translations \( s^1_r \) by 0 in the second partial derivatives of \( \phi \) in (45), harnessing
the upper bound \( \alpha_n(t_n, x) \) and the fact that there are at most \( t_n/h^2 \) non-zero terms (multiplied by \( h^2 \)) in the summation for \( x \).

Then, collecting the terms of the discrete Itô’s formula, by (43) and (44) we obtain the two versions claimed in the lemma.

It has to be emphasized that so far in this section all obtained formulae have been algebraic–analytic ones, independent of any randomness. Now take a planar Brownian motion \( W(t) \) and replace the sums above by a Skorohod imbedded sequence:
\[
S_n = B_m(n2^{-2m})
\]
with \( h = 2^{-m} \). Then taking limits of the discrete Tanaka–Rosen–Yor formula as \( m \to \infty \), one obtains a continuous version of the formula [20, Théorème 1].

**Theorem 2.** Suppose that \( \phi \) is a \( C^3 \) scalar field, \( W(t) \) is a Brownian motion in the plane, \( W(0) = 0 \), and \( y \in \mathbb{R}^2 \). With \( m = 0, 1, \ldots, \) and \( h = 2^{-m} \), apply the discrete Tanaka–Rosen–Yor formula (38) to the imbedded random walks \( S_n = B_m(n2^{-2m}) \). Then, as \( m \to \infty \), each term of the discrete formula almost surely tends to the corresponding term of the following continuous formula, uniformly on any bounded interval \( t \in [0, K] \):

\[
\int_0^t \phi(W(t) - W(u) - y) \, du = t\phi(y) + \int_0^t \int_0^u (\nabla \phi)(W(v) - W(u) - y) \, du \cdot dW(v) + \frac{1}{2} \int_0^t \int_0^u (\Delta \phi)(W(v) - W(u) - y) \, du \, dv. \tag{46}
\]

The last term can be written as an almost sure limit of sums involving self-intersection local times of imbedded random walks:

\[
L\phi(t, y) := \int_0^t \int_0^u (\Delta \phi)(W(v) - W(u) - y) \, du \, dv
\]

\[
= \lim_{m \to \infty} \sum_{x \in 2^{-m} \mathbb{Z}^2} \alpha_m(t, x) (\Delta \phi)(x - y) 2^{-2m}. \tag{47}
\]

**Proof.** To prove the almost sure convergence of the terms in (46), it is essentially enough to apply Theorem [11]. The only new element here is that in each term there is a Riemann sum of a continuous function, that converges to the corresponding Riemann integral for almost every path as \( m \to \infty \). The value of the parameter \( y \in \mathbb{R}^2 \) should also be approximated by a closest point \( y_m \in 2^{-m} \mathbb{Z}^2 \), for which \( |y - y_m| \leq 2^{-m} \). This does not cause any problem, since all functions in (46) are continuous in \( y \).

Thus the limit of the term on the left side of (48) is

\[
\int_0^t \{ \phi(W(t) - W(u) - y) - \phi(-W(u) - y) \} \, du.
\]

The limit of the first term on the right side is

\[
\int_0^t \{ \phi(W(u) - W(u) - y) - \phi(-W(u) - y) \} \, du.
\]
The extra \( \int_0^t - \phi (-W(u) - y) \, du \) term appears on both sides, so can be canceled. Observe that \( \int_0^t \phi (W(u) - W(u) - y) = t \phi (-y) \).

The limits of the second and the last terms of the right side of (38) are clearly the corresponding ones in (46). The equality in (47) clearly follows from the equality (39).

Since \( \phi \in C^3(\mathbb{R}^2) \) and \( W(t) \) is a.s. continuous, it follows that, almost surely, the term on the left side and the first and the last terms on the right side of (46) are continuous functions of \( (t, y) \in \mathbb{R}_+ \times \mathbb{R}^2 \). This implies the same conclusion for the second, stochastic integral term as well.

8 A definition of planar self-intersection local time

A possible definition of ordinary local time in one spatial dimension uses a special case of Tanaka’s formula applied with the function \( \phi(x) = x \vee 0 \), which is a fundamental solution of the one-dimensional Laplacian \( d^2/dx^2 \); see this kind of definition for example in [15, p. 117] and [3, Section 7.2]. The definition of self-intersection local time presented below is a suitable planar modification of it. This means that our definition uses a special case of planar Tanaka–Rosen–Yor formula with the function \( \phi(x) = \log |x| \), which is a fundamental solution of the planar Laplacian \( \Delta = D_{11} + D_{22} \), ignoring a constant multiplier.

Let \( W(t) \) be a planar Brownian motion, \( W(0) = 0 \). For each \( x \in \mathbb{R}^2 \) and \( \delta > 0 \), we define an everywhere continuously differentiable approximation of \( \log |x| \) by

\[
\phi_\delta(x) := \begin{cases} 
\frac{|x|^2 - \delta^2}{2\delta^2} + \log \delta & \text{for } |x| \leq \delta, \\
\frac{\pi}{|x|^2} & \text{for } |x| > \delta.
\end{cases}
\]

Then

\[
(\nabla \phi_\delta)(x) = \begin{cases} 
\frac{\pi}{|x|^2} & \text{for } |x| \leq \delta, \\
\frac{2}{|x|^2} & \text{for } |x| > \delta;
\end{cases}
\]

and

\[
(\Delta \phi_\delta)(x) = \begin{cases} 
\frac{2}{|x|^2} & \text{for } |x| < \delta, \\
0 & \text{for } |x| > \delta.
\end{cases}
\]

Note that \( (\Delta \phi_\delta)(x) \) is not defined for \( |x| = \delta \), but we set it to be 0 there.

Since \( \phi_\delta \) is not \( C^3 \), Theorem 2 is not directly applicable to it. However, by a standard procedure, taking a convolution with a sequence of \( C^\infty \) functions \( q_n \) with compact support shrinking to \( \{0\} \), and then taking a limit as \( n \to \infty \), solves this problem. For sake of explicitness, let \( q(z) = c \exp \left(- (1 - |z|^2)^{-1}\right) \) for \( |z| < 1 \) and 0 otherwise, where the constant \( c \) is chosen so that \( \int_{\mathbb{R}^2} q(z) \, dz = 1 \).

Put \( q_n(z) = n^2 q(nz) \) and \( \phi_\delta^n = \phi_\delta \ast q_n \) \((n \geq 1)\).

Then \( \phi_\delta^n \in C^\infty(\mathbb{R}^2) \); \( \phi_\delta \to \phi_\delta^n \), \( \nabla \phi_\delta \to \nabla \phi_\delta^n \) both uniformly in \( \mathbb{R}^2 \); while \( \Delta \phi_\delta^n \to \Delta \phi_\delta \) pointwise except for \( |x| = \delta \). Thus one can apply Theorem 2 to
where $\lambda$ denotes planar Lebesgue measure, $V_t = \{(u, v) : 0 \leq u \leq v \leq t\},$ and $y \in \mathbb{R}^2$. It also follows that each term here is an almost surely continuous function of $(t, y)$: this is clear for each term except for the second, stochastic integral term on the right side, but then it follows for this term too.

It is important that, by (47), the last term can be written as

$$
\frac{1}{2} L \phi^\delta(t, y) = \frac{1}{\delta^2} \lim_{m \to \infty} \sum_{x \in 2^{-m} \mathbb{Z}^2 \cap B_\delta(y)} \alpha_m(t, x) 2^{-2m},
$$

where $B_\delta(y)$ is the closed disc centered at $y$ with radius $\delta$.

Rosen [12] suggested the following definition of self-intersection local time of planar Brownian motion $W$. Define the following occupation measure for plane Borel sets $A$ and time $t \geq 0$:

$$
\mu_t(A) := \lambda\{\{(u, v) : 0 \leq u \leq v \leq t, W(v) - W(u) \in A\}\},
$$

where $\lambda$ is planar Lebesgue measure. Rosen [12] then proved that the self-intersection local time $\alpha(t, x) := \frac{\partial \mu_t}{\partial x}(x)$ a.s. exists when $x \neq 0$.

An alternative approach is to consider the symmetric derivative of $\mu_t$ w.r.t. $\lambda$:

$$
\alpha(t, y) := \lim_{\delta \to 0^+} \frac{\mu_t(B_\delta(y))}{\lambda(B_\delta(y))} = \lim_{\delta \to 0^+} \frac{\delta}{\pi \delta^2} \lambda\{\{(u, v) : 0 \leq u \leq v \leq t, |W(v) - W(u) - y| < \delta\}\}
$$

where $B_\delta(y)$ denotes the disc centered at $y$ with radius $\delta$. Among other things, the next theorem establishes the a.s. existence of this finite symmetric derivative for any $t \geq 0$ and $y \neq 0$. It is well-known (see e.g. Rudin [14]) that when there exists a finite symmetric derivative of $\mu_t$ w.r.t. $\lambda$ except for the point 0, then $\mu_t$ is absolutely continuous w.r.t. $\lambda$ on $\mathbb{R}^2 \setminus \{0\}$ and the symmetric derivative equals the Radon–Nikodym derivative; the support of the singular part of $\mu_t$ can only be the point 0.

**Theorem 3.** The terms of (49) almost surely converge to the corresponding terms of the following Tanaka–Rosen–Yor formula as $\delta \to 0^+$, when $y \neq 0$:

$$
\int_0^t \log |W(t) - W(u) - y| \, du = t \log |y| + \int_0^t \int_0^u \frac{W(v) - W(u) - y}{|W(v) - W(u) - y|^2} \, du \, dW(v) + \pi \alpha(t, y),
$$
Moreover, all terms, including $\alpha(t, y)$, are a.s. continuous in $(t, y)$ when $y \neq 0$.

It also follows that the self-intersection local time $\alpha(t, y)$ of planar Brownian motion is the almost sure limit of averages of self-intersection local times $\alpha_m(t, y)$ of imbedded random walks:

$$\alpha(t, y) = \lim_{\delta \to 0^+} \lim_{m \to \infty} \frac{1}{\pi \delta^2} \sum_{x \in 2^{-m} \mathbb{Z}^2 \cap B_\delta(y)} \alpha_m(t, x) 2^{-2m}$$

$$= \lim_{\delta \to 0^+} \lim_{m \to \infty} \frac{1}{\pi \delta^2} \int_{B_\delta(y)} \alpha_m(t, x) \, dx \quad (y \neq 0). \quad (53)$$

Proof. The well-known properties of planar Brownian motion imply that for any $t > 0$ and $y \neq 0$, $\inf \{ |W(t) - W(u) - y| : 0 \leq u \leq t \} > 0$, with probability 1. Hence it follows the almost sure convergence of the left side of (49) as $\delta \to 0^+$ when $y \neq 0$. Moreover, the integrand converges monotonically as $\delta \to 0^+$, so the limit and the integral can be interchanged. It also follows that the left side of (52) is a continuous function of $(t, y)$ when $y \neq 0$.

The convergence and the continuity of the first term on the right side is trivial when $y \neq 0$. By Lemma 7 in the Appendix, the second term on the right side of (49) is

$$\int_0^t \int_0^v \left( \nabla \phi \delta \right) (W(v) - W(u) - y) \, du \cdot dW(v)$$

$$= \frac{1}{\pi \delta^2} \int_0^t \int_0^v \int_{B_\delta(y)} \frac{W(v) - W(u) - z}{|W(v) - W(u) - z|^2} \, dz \, du \cdot dW(v)$$

$$= \frac{1}{\pi \delta^2} \int_{B_\delta(y)} \left\{ \int_0^t \int_0^v \frac{W(v) - W(u) - z}{|W(v) - W(u) - z|^2} \, du \cdot dW(v) \right\} \, dz. \quad (54)$$

The interchange of integrations is allowed by an extension of Fubini theorem. (Remember that by Lemma 6, the stochastic integral is an almost sure limit of discrete sums.) By Lemma 8 in the Appendix the stochastic integral has a continuous version for $y \neq 0$. Thus by the mean value theorem of integrals, (54) has an almost sure limit if $y \neq 0$ as $\delta \to 0^+$, namely the one stated in the theorem. By Lemma 8(c), the limit is a continuous function of $(t, y)$ when $y \neq 0$.

The above limits imply that the last term on the right side of (49) has an almost sure limit $\pi \alpha(t, y)$ if $y \neq 0$. (53) follows from this by (50) and Lemma 6 in the Appendix. It also follows from the above arguments that $\alpha(t, y)$ is continuous in $(t, y)$ when $y \neq 0$.

Formula (53) is the definition of planar self-intersection local time which has been the main objective of the present paper. It is an open question if the limits in (53) can be interchanged; then one would get the more impressive almost sure limit $\alpha(t, y) = \lim_{m \to \infty} \alpha_m(t, y)$.

The most interesting question is ‘What happens to the planar self-intersection local time $\alpha(t, y)$ when $y \to 0$?’ It was discovered by Varadhan in 1968 that $\alpha$ goes to $\infty$ then, and, in fact, it has a logarithmic singularity at 0. So one can
introduce renormalized self-intersection local time \( \gamma \) by the formula

\[
\gamma(t, y) = \begin{cases} 
\alpha(t, y) - \frac{1}{\pi} \log \frac{1}{|y|} & \text{when } y \neq 0, \\
\lim_{y \to 0} \alpha(t, y) - \frac{1}{\pi} \log |0| & \text{when } y = 0.
\end{cases}
\]

It was shown by Le Gall [10] that the limit above exists both almost surely and in \( L^2 \). Below we will give an alternative proof of it, together with a Tanaka–Rosen–Yor formula for \( \gamma \), based on Theorem 3. Yor [20, (2.k)] proved convergence in probability by a similar approach.

Prior to that, let us give the expectation of \( \gamma \), cf. [10].

**Corollary 2.**

\[
\mathbb{E}[\gamma(t, y)] = \begin{cases} 
\frac{1}{2} \log |y| - \frac{|y|^2 + 2t}{4t} \text{Ei} \left( -\frac{|y|^2}{2t} \right) - \frac{1}{\pi} e^{-|y|^2/2t} & \text{when } y \neq 0, \\
\frac{1}{2\pi} (\log(2t) - C - 1) & \text{when } y = 0,
\end{cases}
\]

where \( \text{Ei} \) denotes the exponential integral function and \( C \) is Euler’s constant. Thus this expectation is finite and continuous for every \((t, y) \in \mathbb{R}_+ \times \mathbb{R}^2\).

**Proof.** In formula (52) the expectation of the left side is given by Lemma 8(a), while the expectation of the stochastic integral term is 0 by Lemma 8(a), see the Appendix.

Since \( \gamma(t, y) \) has finite expectation for any \((t, y)\), quite often in the literature the renormalized self-intersection local time is defined by subtracting its expected value. Since that would complicate some formulae below, here we do not follow that practice.

**Theorem 4.** Combine the first and the last terms on the right side of (52) into a \( \gamma \) term. When \( y \to 0 \), the resulting terms in (52) converge almost surely and in \( L^2 \) to

\[
\int_0^t \log |W(t) - W(u)| \, du = \int_0^t \int_0^u \frac{W(v) - W(u)}{|W(v) - W(u)|^2} \, dW(v) + \pi \, \gamma(t, 0).
\]

Moreover, \( \gamma(t, y) \) is finite-valued and continuous in \((t, y)\) even when \( y = 0 \).

**Proof.** Let us denote the stochastic integral in (52) by \( Y(t, y) \), cf. (57) in the Appendix. Define \( Y_n(t) := Y(t, (n^{-1}, 0)), n \geq 1 \). By Lemma 8(a), for each \( n \), \( Y_n(t) \) is a continuous \( L^2 \)-martingale. By (57), for any \( k \geq 1 \),

\[
\mathbb{E}[|Y_{2^k}(t) - Y_{2^{k-1}}(t)|^3] \leq c_0(K)k^{5/2 - 3k} \quad (0 \leq t \leq K),
\]

where \( c_0(K) \) is a constant depending only on \( K \).

Thus a basic martingale inequality implies for any \( k \geq 1 \) that

\[
\mathbb{P} \left( \sup_{0 \leq t \leq K} |Y_{2^k}(t) - Y_{2^{k-1}}(t)| \geq 2^{-k/2} \right) \leq 2^{3k/2} \mathbb{E}[|Y_{2^k}(K) - Y_{2^{k-1}}(K)|^3] \leq c_0(K)k^{5/2 - 3k/2} \leq c_1(K)2^{-k},
\]

where \( c_1(K) \) is a constant depending on \( K \).
An application of the Borel–Cantelli lemma then yields that \( Y_{2k}(t) \) almost surely uniformly converges for \( t \in [0, K] \) to a continuous \( L^2 \)-martingale as \( k \to \infty \). (The convergence is also in \( L^2 \) by (57).) Using isometry, the expression

\[
\int_0^v \frac{W(v) - W(u) - (2^{-k}, 0)}{|W(v) - W(u) - (2^{-k}, 0)|^2} \, du
\]

converges in \( L^2 \) as well when \( k \to \infty \). Since for any fixed \( v \), \( W(v) - W(u) \) almost surely does not equal to 0 when \( 0 \leq u < v \), here the integrand can be dominated for any large enough \( k \geq k_0(\omega) \). So the integral and the limit can be interchanged and we get that the limit of \( Y_{2k} \) is the continuous \( L^2 \)-martingale \( Y(t, 0) \), which is the first term on the right side of (56).

Moreover, \( Y(t, y) \) converges to \( Y(t, 0) \) as well when \( y \to 0 \), a.s. uniformly for \( t \in [0, K] \). For, by an argument based on (69), similar to the above one, given any \( \epsilon > 0 \), for any \( y \), \( 2^{-k-1} \leq |y| \leq 2^{-k} \) with \( k \) large enough,

\[
|Y(t, y) - Y(t, 0)| \leq |Y(t, y) - Y_{2k-1}(t)| + |Y_{2k-1}(t) - Y(t, 0)| < \epsilon
\]
a.s. uniformly for \( t \in [0, K] \).

The convergence as \( y \to 0 \) of the term \( X(t, y) \) on the left side of (52) can be treated analogously by Lemma 9(b) of the Appendix.

The following occupation time formulae, cf. \[10\], follow from the previous results.

**Corollary 3.** Suppose that \( f : \mathbb{R}^2 \to \mathbb{R} \) is a bounded, Borel measurable function. Then

\[
\int_0^t \int_0^v f(W(v) - W(u)) \, du \, dv = \int_{\mathbb{R}^2} f(x) \alpha(t, x) \, dx
= \int_{\mathbb{R}^2} f(x) \left\{ \gamma(t, x) - \frac{t}{\pi} \log |x| \right\} \, dx. \tag{58}
\]

Alternatively,

\[
\int_0^t \int_0^v \{ f(W(v) - W(u)) - E f(W(v) - W(u)) \} \, du \, dv
= \int_{\mathbb{R}^2} f(x) \{ \gamma(t, x) - E \gamma(t, x) \} \, dx. \tag{59}
\]

**Proof.** It is enough to show \[58\] for indicator functions of discs; by standard methods, linear combinations of indicators extend to a general \( f \). So let us take a closed disc \( B_r(a) \) and show that

\[
\int_0^t \int_0^v 1_{B_r(a)}(W(v) - W(u)) \, du \, dv = \int_{B_r(a)} \alpha(t, y) \, dy
= \int_{B_r(a)} \gamma(t, y) \, dy - \frac{t}{\pi} \int_{B_r(a)} \log |y| \, dy. \tag{60}
\]
From the results above we know that the stochastic integral term on the right side of \([52]\) has a version \(Y(t, z)\) which is continuous for any \((t, y) \in [0, K] \times \mathbb{R}^2\). \([53]\) gives
\[
\int_0^t \int_0^{\theta} (\nabla \phi^z)(W(v) - W(u) - y) \, du \cdot dW(v) = \frac{1}{\pi \delta^2} \int_{B_{\delta}(y)} Y(t, z) \, dz.
\]
Substitute this into \([59]\):
\[
\int_0^t \phi^z(W(t) - W(u) - y) \, du - t \phi^z(y) - \frac{1}{\pi \delta^2} \int_{B_{\delta}(y)} Y(t, z) \, dz
\]
\[
= \frac{1}{\delta^2} \int_0^t \int_0^\infty \mathbf{1}_{B_{\delta}(y)}(W(v) - W(u)) \, du \, dv.
\]
Then integrate this equality over a closed disc \(B_r(a)\) with respect to \(y:\)
\[
\int_{B_r(a)} \left\{ \int_0^t \phi^z(W(t) - W(u) - y) \, du - t \phi^z(y) - \frac{1}{\pi \delta^2} \int_{B_{\delta}(y)} Y(t, z) \, dz \right\} dy
\]
\[
= \frac{1}{\delta^2} \int_0^t \int_0^\infty \int_{B_r(a)} \mathbf{1}_{B_{\delta}(y)}(W(v) - W(u)) \, dy \, du \, dv. \quad (61)
\]
Now, it is clear that
\[
\lim_{\delta \to 0^+} \frac{1}{\pi \delta^2} \int_{B_{\delta}(y)} \mathbf{1}_{B_r(y)}(z) \, dy = \mathbf{1}_{B_r(a)}(z) + \frac{1}{2} \mathbf{1}_{\partial B_r(a)}(z), \quad (62)
\]
where \(B_r^\circ(a) = \{x : |x - y| < r\}\). Take limit in \((61)\) as \(\delta \to 0^+\). By Theorem 3 the limit of the expression in the braces is \(\pi \alpha(t, y)\) when \(y \neq 0\). Since \(\alpha(t, y) = \gamma(t, y) - \frac{1}{\pi} \log |y|\), and \(\gamma\) is a.s. continuous everywhere while \(\int_{B_r(a)} \log |y| \, dy\) is finite even if \(0 \in B_r(a)\), this gives us the right hand sides of \((60)\).

The left hand side of \((60)\) is obtained as the limit of the right hand side of \((61)\), using \((62)\), since it has zero probability that \(W(v) - W(u) \in \partial B_r(a)\). This completes the proof of \((59)\).

Now take expectation in \((59)\): 
\[
\int_0^t \int_0^\infty \mathbb{E} f(W(v) - W(u)) \, du \, dv = \int_{\mathbb{R}^2} \int_0^t \mathbb{E\gamma}(t, x) - \frac{t}{\pi} \log |x| \, dx.
\]
Subtract this from \((59)\), and the result is \((59)\).

Finally, we add some remarks. We can combine the occupation time formula \((58)\) and formula \((56)\), with \(y = 0\), when \(\phi\) is a \(C^3\) scalar field:
\[
\int_0^t \phi^z(W(t) - W(u)) \, du - \int_0^t \int_0^{\theta} (\nabla \phi)(W(v) - W(u)) \, du \cdot dW(v)
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2} (\Delta \phi)(x) \alpha(t, x) \, dx + t \phi(0)
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2} (\Delta \phi)(x) \left\{ \gamma(t, x) - \frac{t}{\pi} \log |x| \right\} \, dx + t \phi(0). \quad (63)
\]
Green’s theorem implies that \( \phi(0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} (\Delta \phi)(x) \log |x| \, dx \) holds when \( \phi(x) - x \cdot (\nabla \phi)(x) \log |x| \) tends to 0 as \( |x| \to \infty \). In this case (63) simplifies to the following Tanaka–Rosen–Yor formula:

\[
\int_0^t \phi(W(t) - W(u)) \, du = \int_0^t \int_0^u (\nabla \phi)(W(v) - W(u)) \, du \cdot dW(v) + \frac{1}{2} \int_{\mathbb{R}^2} (\Delta \phi)(x) \gamma(t, x) \, dx.
\]

Further, comparing the first equality of (63) with (47) results the almost sure limit

\[
\lim_{m \to \infty} \sum_{x \in 2^{-m} \mathbb{Z}^2} \alpha_m(t, x) (\Delta \phi)(x) 2^{-2m} = \int_{\mathbb{R}^2} \alpha(t, x) (\Delta \phi)(x) \, dx. \tag{64}
\]

Now, almost surely, the support of the continuous \( \alpha \), and then by (63), the support of each \( \alpha_m \) for \( m \geq m_0(\omega) \), can be covered by a finite disc \( B_R(0) \) with a large enough radius \( R = R(\omega) \). Thus Lemma 6 implies that the sum here can be replaced by an integral. Also, the Poisson equation \( \Delta \phi = f \) can be solved in the plane for any continuous \( f \), \( f(x) = O(|x|^{-2-\delta}) \), \( \delta > 0 \), by the formula

\[
\phi(y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} (\Delta \phi)(x) \log |x-y| \, dx.
\]

Thus, for any such \( f \), (64) can be written as

\[
\lim_{m \to \infty} \int_{\mathbb{R}^2} \alpha_m(t, x) f(x) \, dx = \int_{\mathbb{R}^2} \alpha(t, x) f(x) \, dx \quad \text{a.s.} \tag{65}
\]

This weak convergence formula supplements the basic definition (53) of self-intersection local time.

### 9 Appendix: Some technical lemmas

This lemma says that the sum in (61) can be approximated by an integral.

**Lemma 6.** Almost surely, for any \( t \in \mathbb{R}_+ \), \( y \in \mathbb{R}^2 \) and \( \delta > 0 \) fixed,

\[
\sum_{x \in 2^{-m} \mathbb{Z}^2 \cap B_{\delta}(y)} \alpha_m(t, x) 2^{-2m} = \int_{B_{\delta}(y)} \alpha_m(t, x) \, dx + \delta \, O(m^2 2^{-m}).
\]

**Proof.** First, let us estimate the error between the sum of a discrete function and the integral of an interpolated function over a rectangular domain \( A_n = [x_0, x_n] \times [y_0, y_n] \) with vertices on a grid \( h\mathbb{Z}^2 \), in general. So let \( f : h\mathbb{Z}^2 \to \mathbb{R} \) be a discrete function and \( f(x, y), (x, y) \in \mathbb{R}^2 \), be obtained from \( f \) by linear interpolation on triangles of the grid, as it was described for \( \alpha_n \) in (54). Put

\[
S_n := h^2 \sum_{(x, y) \in A_n} f(x, y) = h^2 \sum_{i=1}^n \sum_{j=1}^n f(x_{i-1}, y_{j-1}),
\]

and

\[
T_n := \int_{A_n} f(x, y) \, dx \, dy
= \frac{h^2}{6} \sum_{i=1}^n \sum_{j=1}^n \{f(x_{i-1}, y_{j-1}) + 2f(x_i, y_{j-1}) + 2f(x_{i-1}, y_j) + f(x_i, y_j)\},
\]

\[
= \int_{A_n} f(x, y) \, dx \, dy.
\]
Then in the error \( T_n - S_n \), all contributions of inner vertices cancel and only the contribution of vertices at the boundary of \( A_n \) remain:

\[
T_n - S_n = \frac{h^2}{2} \sum_{i=1}^{n-1} \{ f(x_i, y_n) + f(x_n, y_i) - f(x_0, y_n) - f(x_i, y_0) \}
+ \frac{h^2}{6} \{ 2f(x_0, y_n) + 2f(x_n, y_0) + f(x_n, y_n) - 5f(x_0, y_0) \}.
\]

Returning to the statement of the lemma, by our definition in Section 6, \( \alpha_m(t, x) \) is obtained by linear interpolation on triangles of the grid with mesh \( h = 2^{-m} \). If one considers the difference of the sum and the integral over a disc \( B_\delta(y) \), the cancelation of inner vertices still holds, and only the contributions of vertices adjacent to the circumference remain. The number of these latter vertices is of the order of \( \delta O(2^m) \). Thus by (37) we have that

\[
\left| \sum_{x \in 2^{-m} \mathbb{Z}^2 \cap B_\delta(y)} \alpha_m(t, x) 2^{-2m} - \int_{B_\delta(y)} \alpha_m(t, x) \, dx \right| \leq \delta O(2^m) O(m^2) 2^{-2m} = \delta O(m^2 2^{-m}).
\]

This completes the proof of the lemma. \( \square \)

We need the following representation of \( \nabla \phi^\delta \).

**Lemma 7.**

\[
(\nabla \phi^\delta)(x) = \frac{1}{\pi \delta^2} \int_{B_\delta(0)} \frac{x - z}{|x - z|^2} \, dz
= -\frac{1}{\pi \delta^2} \int_{C_\delta(0)} \log |z - x| \, n_0(z) \, ds(z) \quad (x \in \mathbb{R}^2),
\]

where \( B_\delta(0) \) is the closed disc centered at the origin with radius \( \delta > 0 \), \( C_\delta(0) \) is its counterclockwise directed boundary, \( n_0 \) is the outward unit normal along the boundary, and \( ds \) denotes integration with respect to arc length.

**Proof.** The second equality follows from a standard theorem of vector analysis; note that the discontinuity of the integrand in the second term when \( |x| \leq \delta \) is not essential. To prove that the first term equals the third, because of rotational symmetry, it is enough to consider points \( x = (-a, 0) \), \( a \geq 0 \). Then the third term of (66) becomes

\[
-\frac{1}{2\pi \delta} \int_0^{2\pi} \log \left( (a + \delta \cos \theta)^2 + (\delta \sin \theta)^2 \right) \cos \theta \, d\theta = \left( -\frac{1}{\delta} \Psi \left( \frac{a}{\delta} \right), 0 \right),
\]

where

\[
\Psi(u) := \frac{1}{2\pi} \int_0^{2\pi} \log (u^2 + 1 + 2u \cos \theta) \cos \theta \, d\theta = \begin{cases} 
\frac{u}{u^2} & \text{for } 0 \leq u \leq 1, \\
\frac{1}{u^2} & \text{for } u \geq 1.
\end{cases}
\]

These prove the equality with \( \nabla \phi^\delta \) given by (45). \( \square \)
The next lemma establishes some important properties of the stochastic integral appearing in Theorem 3.

**Lemma 8.** Fix an arbitrary $K > 0$. Consider the stochastic integral

$$Y(t, y) := \int_0^t \int_0^u \frac{W(v) - W(u) - y}{|W(v) - W(u) - y|^2} \, du \cdot dW(v). \tag{67}$$

Then the following properties hold.

(a) $Y(t, y)$ is a continuous $L^2$-martingale with expectation 0 as a function of $t \in [0, K]$ for any fixed $y \in \mathbb{R}^2$.

(b) \[\mathbb{E}|Y(t, y) - Y(t, y')|^3 \leq C|y - y'|^{2+\beta}\] \tag{68}

with a finite $C = C(K, a)$ and with an arbitrary $\beta \in (0, 1)$ for any $t \in [0, K]$ and $|y|, |y'| \geq a$, where $a > 0$ is arbitrary, fixed. More exactly,

$$\mathbb{E}|Y(t, y) - Y(t, y')|^3 \leq c(K) \left(1 + \log_+ 2K a^{-2}\right) \log^4 \frac{1}{|y - y'|} |y - y'|^3, \tag{69}$$

where $c(K)$ is a finite constant depending only on $K$ and $\log_+ x := 0 \lor \log x$.

(c) $Y(t, y)$ has a version which is a.s. a continuous function of $y \neq 0$. In fact, it has a version which is a.s. a continuous function of $(t, y)$ when $y \neq 0$.

**Proof.** (a) Since for any fixed $v$, $\bar{W}(u) := W(v) - W(v - u)$ is planar Brownian motion as well that starts from 0, we have

$$\mathbb{E}|Y(t, y)|^2 = \int_0^t \mathbb{E} \left| \int_0^u \frac{W(v) - W(u) - y}{|W(v) - W(u) - y|^2} \, du \right|^2 \, dv \leq \int_0^t \mathbb{E} \left( \int_0^t \frac{1}{|W(u) - y|^2} \, du \right)^2 \, dv = t \mathbb{E} \left( \int_0^t \frac{1}{|W(u) - y|^2} \, du \right)^2.$$

Then by symmetry and by the independence of increments of $W$, we get that

$$\mathbb{E}|Y(t, y)|^2 \leq t \mathbb{E} \left( \int_0^t \, du_1 \int_0^{u_1} \frac{1}{|W(u_1) - y||W(u_2) - y|} \right) = 2t \int_{[0, t] \times \mathbb{R}^2} du_1 \, dz_1 \, e^{-\frac{|z_1|^2}{4u_1}} \int_{[u_1, t] \times \mathbb{R}^2} du_2 \, dz_2 \, e^{-\frac{|z_2-z_1|^2}{2(u_2-u_1)|z_2 - y|}}.$$

Writing $z_2 = y + r(\cos \theta, \sin \theta)$, $z_1 = y + \rho(\cos \alpha, \sin \alpha)$ and $v = u_2 - u_1$, for the inner integral here we obtain

$$\int_0^{t-u_1} \, dv \int_0^{2\pi} \, d\theta \int_0^\infty \, dr \, e^{-\frac{r^2 + \rho^2 - 2\rho \cos(\theta - \alpha)}{4v}} \leq \sqrt{2\pi} \int_0^{t-u_1} \, dv \int_0^{2\pi} \, d\theta \int_0^\infty \, dr \, e^{-\frac{(\rho - r)^2}{2\sqrt{2\pi}v}} \leq 2\sqrt{2\pi}(t - u_1).$$
Thus

\[ E|Y(t, y)|^2 \leq 2t \int_0^t \frac{du_1}{u_1} \int_0^{2\pi} \frac{d\alpha}{2\pi} \int_0^\infty d\rho e^{-\frac{\rho^2}{4u_1}} 2\sqrt{2\pi(t - u_1)} \]

\[ \leq 8\pi K^2 < \infty \quad (0 \leq t \leq K, y \in \mathbb{R}^2). \]

(b) First, by the Burkholder–Davis–Gundy inequality, for any \( m > 0 \), there exists a finite \( c_m \) such that

\[ E|Y(t, y) - Y(t, y')|^m \]

\[ \leq c_m E \left( \int_0^t \left| \int_0^t \frac{1}{|W(u) - y||W(u) - y'|} \left( W(v) - W(u) - y\right) \left( W(v) - W(u) - y'ight) \right|^2 \right)^{\frac{m}{2}}. \]

\[ = c_m E \left| y - y' \right|^m E \left( \int_0^t \frac{1}{|W(u) - y||W(u) - y'|} \right)^m, \quad (70) \]

where for any fixed \( v \), \( \tilde{W}(u) := W(v) - W(v - u) \) is planar Brownian motion as well that starts from 0.

Thus to show (88), it is enough to give a suitable upper estimate for the last expectation in (71) when the norms of \( y \) and \( y' \) are bounded below by \( a \) and \( t \in [0, K] \). Now, by symmetry and by the independence of increments of Brownian motion, with \( m = 3 \) we obtain that

\[ E \left( \int_0^t \frac{1}{|W(u) - y||W(u) - y'|} \right)^3 \]

\[ = 6 E \int_0^t du_1 \int_{u_2}^t du_2 \int_{u_3}^t du_3 \prod_{j=1}^3 \frac{1}{|W(u_j) - y||W(u_j) - y'|} \]

\[ = 6 \int_{[0,t] \times \mathbb{R}^2} du d\alpha \int_{[u_1,t] \times \mathbb{R}^2} du_2 d\beta \int_{[u_2,t] \times \mathbb{R}^2} du_3 d\gamma \]

\[ \times \frac{e^{-\frac{|d_x|^2}{2(u_2 - u_1)^2}}}{2\pi(u_2 - u_1)} \prod_{j=1}^3 \frac{1}{|z_j - y||z_j - y'|}. \]

Without loss of generality, from now on we may assume that \( 0 < |y - y'| \leq 1/2 \). \( B_r(x) \) will denote the closed disc centered at \( x \) with radius \( r \). Here and later we use the following covering:

\[ \mathbb{R}^2 = \left( B_1(y) \cup B_1(y') \right)^c \cup \bigcup_{n=-1}^N \left( C_n(y) \cup C_n(y') \right), \]
where \( C_n(y) := B_{2n}[y-y'](y) \cap B_{2n-1}[y-y'](y) \cap B_{2n-1}[y-y'](y'), \) \( 0 \leq n \leq N, \) \( N = \lceil \log(|y-y'|^{-1})/\log 2 \rceil, \) and \( C_{-1}(y) := B_{2-1}[y-y'](y). \) For \( C_n(y') \) the definitions are similar.

To show the method, let us estimate the innermost integral \( I_1 \) in (71) using the above covering of \( \mathbb{R}^2. \) First, let
\[
I_1^* := \int_{u_2}^{t} \, du_3 \int_{(B_1(y) \cup B_1(y'))^c} dz_3 \, e^{\frac{-(|z_3 - z'|^2)}{2\pi(u_3 - u_2)}} \frac{1}{|z_3 - y||z_3 - y'|} \leq \int_{0}^{t-u_2} \, dv \int_{\mathbb{R}^2} \, \frac{dz_3}{2\pi v} = t - u_2.
\]

Second, write \( z_3 = y + r(\cos \theta, \sin \theta), z_2 = y + \rho(\cos \alpha, \sin \alpha), \) and for \( n = 0, 1, \ldots, N \) obtain that
\[
I_{1,n}(y) := \int_{u_2}^{t} \, du_3 \int_{C_n} dz_3 \, e^{\frac{-(|z_3 - z'|^2)}{2\pi(u_3 - u_2)}} \frac{1}{|z_3 - y||z_3 - y'|} \leq \int_{0}^{2\pi} \frac{d\theta}{2\pi} \int_{0}^{t-u_2} \, dv \int_{2n-1}[y-y'|} \, \frac{dr \, e^{-\frac{r^2}{2\rho^2}}}{2\pi \rho^2 \rho \cos \theta} \leq \int_{0}^{2\pi} \frac{d\theta}{2\pi} \int_{0}^{t-u_2} \, dv \, e^{-\frac{r^2}{2\rho^2}}.
\]

(The value of \( \alpha \) clearly does not matter, so it was replaced by 0.) Here one can use the simple estimate
\[
\int_{0}^{t} \frac{1}{v} e^{-\frac{r^2}{2\rho^2}} \, dv \leq \frac{1}{e} + \log \frac{x}{b}.
\]

Then
\[
I_{1,n}(y) \leq \int_{0}^{2\pi} \left( \frac{1}{v} + \log \frac{2(t - u_2)}{\rho^2 \sin^2 \theta} \right) \frac{d\theta}{2\pi} \leq \frac{1}{e} + \log \frac{2(t - u_2)}{\rho^2} - \int_{0}^{2\pi} \log(\sin^2 \theta) \frac{d\theta}{2\pi} < 2 + \log \frac{2(t - u_2)}{|z_2 - y|^2}
\]

\((n = 0, 1, \ldots, N), \) since \( \rho = |z_2 - y|. \) The estimate for \( I_{1,-1}(y) \) is the same, and so is for any \( I_{1,n}(y') \) replacing \( y \) by \( y'. \)

In sum, the estimate for the innermost integral is
\[
I_1 \leq I_1^* + \sum_{n=-1}^{N} (I_{1,n}(y) + I_{1,n}(y')) \leq t - u_2 + 12 \log \frac{1}{|y - y'|} \left( 2 + \log \frac{2(t - u_2)}{|z_2 - y||z_2 - y'|} \right), \quad (72)
\]
since \( N + 2 < 6 \log(1/|y - y'|) \) when \( 0 < |y - y'| \leq 1/2 \).

The estimation of the second and third integrals in (71) can go in a similar fashion. Omitting the details, the result is

\[
E \left( \int_0^t \frac{1}{|W(u) - y||W(u) - y'|} \, du \right)^3 \leq C(K) \log^4 \left( \frac{1}{|y - y'|} \left( 1 + \log \left( \frac{2t}{|y||y'|} \right) \right) \right)
\]

for any \( y, y' \) such \( |y|, |y'| \geq a \), \( 0 < |y - y'| \leq 1/2 \), and \( t \in [0, K] \), where \( C(K) \) is a finite constant depending on \( K \).

By (b), \( Y(t, y) \) as a function of \( y \) satisfies the condition of a special case of the Kolmogorov–Chentsov theorem, so a.s. it has a continuous version as a function of \( y \) when \( y \neq 0 \).

One can similarly show that \( Y(t, y) \) has a version which is a continuous function of \( (t, y) \) when \( y \neq 0 \), e.g. considering fourth moment instead of the third.

This last lemma investigates the properties of the integral on the left side of formula (52).

**Lemma 9.** Consider the integral

\[
X(t, y) := \int_0^t \log |W(t) - W(u) - y| \, du \quad (t \geq 0, y \in \mathbb{R}^2).
\]

Then it has the following properties.

(a)

\[
E X(t, y) = t \log |y| - \frac{|y|^2 + 2t}{4} \text{Ei} \left( \frac{|y|^2}{2t} \right) - \frac{1}{2} t e^{-|y|^2/4t} \quad (y \neq 0),
\]

\[
\lim_{y \to 0} E X(t, y) = E X(t, 0) = \frac{t}{2} \left( \log(2t) - C - 1 \right),
\]

where \( \text{Ei} \) denotes the exponential integral function and \( C \) is Euler’s constant.

(b)

\[
E |X(t, y) - X(t, y')|^3 \leq C |y - y'|^{2 + \beta}
\]

with a finite \( C = C(K, a) \) and with an arbitrary \( \beta \in (0, 1) \) for any \( t \in [0, K] \) and \( |y|, |y'| \geq a \), where \( a > 0 \) is arbitrary, fixed. More exactly,

\[
E |X(t, y) - X(t, y')|^3 \leq c(K) \left( 1 + \log \left( 2Ka^{-2} \right) \right) \log^4 \left( \frac{1}{|y - y'|} |y - y'| \right),
\]

where \( c(K) \) is a finite constant depending only on \( K \).
Proof. (a) For any fixed $t$, $\tilde{W}(u) := W(t) - W(t-u)$ is planar Brownian motion as well that starts from 0. Thus

$$EX(t,y) = \int_0^t \log |\tilde{W}(u) - y| \, du = \int_0^t \int_{\mathbb{R}^2} \frac{\log |x - y|}{2\pi u} e^{-\frac{|x|^2}{2u}} \, dx \, du$$

$$= \int_0^t du \int_0^\infty dr \frac{r}{u} e^{-\frac{r^2}{2u}} \int_0^{2\pi} d\theta \frac{1}{2\pi} \log \left(r^2 + \rho^2 - 2r\rho \cos(\theta - \alpha)\right),$$

where $x = r(\cos \theta, \sin \theta)$ and $y = \rho(\cos \alpha, \sin \alpha)$. It is clear that the last integral does not depend on $\alpha$, so we can replace $\alpha$ by 0. Since

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{2} \log \left(r^2 + \rho^2 - 2r\rho \cos(\theta)\right) = \log(r \vee \rho),$$

it follows that

$$EX(t,y) = \int_0^t du \int_0^\infty dr \frac{r}{u} e^{-\frac{r^2}{2u}} \log(r \vee \rho),$$

that gives exactly the results (74) and (75).

(b)

$$E|X(t,y) - X(t,y')|^3 = E\left| \int_0^t \log \frac{|W(u) - y|}{|W(u) - y'|} \, du \right|^3$$

$$\leq E \left( \int_0^t \log \frac{|W(u) - y|}{|W(u) - y'|} \, du \right)^3.$$

Using the inequality $|\log b - \log a| \leq (a \wedge b)^{-1}|b - a| \leq (a^{-1} + b^{-1})|b - a|$ for $a, b > 0$, and then symmetry and the independent increments of $W$, we obtain that

$$E|X(t,y) - X(t,y')|^3$$

$$\leq |y - y'|^3 E \left( \int_0^t \left( |W(u) - y|^{-1} + |W(u) - y'|^{-1} \right) \, du \right)^3$$

$$\leq 6|y - y'|^3 \int_{[0,t] \times \mathbb{R}^2} du_1 dz_1 e^{-\frac{|z_1|^2}{2u_1}} \int_{[u_1,t] \times \mathbb{R}^2} du_2 dz_2 e^{-\frac{|z_2|^2}{2u_2}}$$

$$\times \int_{[u_2,t] \times \mathbb{R}^2} du_3 dz_3 e^{-\frac{|z_3|^2}{2(u_3 - u_2)}} \prod_{j=1}^3 \left( \frac{1}{|z_j - y|} + \frac{1}{|z_j - y'|} \right).$$

Since this last formula is very similar to formula (74), the remaining part of the proof is essentially the same, so omitted. The result differs from the one of Lemma 5(b) only by a constant multiplier depending on $K$. \qed
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