On Lorentzian two-symmetric manifolds of dimension-four

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Abstract. We study curvature properties of four-dimensional Lorentzian manifolds with two-
symmetry property. We then consider Einstein-like metrics, Ricci solitons and homogeneity over
these spaces.

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1. Introduction

Symmetries of the mathematical models have a lot of applications in applied sciences. For
example, molecular symmetries studied in [20] and [6], obtaining the group of symmetries of the
molecules. K-symmetry spaces are a natural generalization of symmetric manifolds. A (pseudo-)
Riemannian space \((M,g)\) is called \(k\)-symmetric if the following condition is valid:

\[
\nabla^k R = 0, \quad \nabla^{k-1} R \neq 0,
\]

where \(k \geq 1\) and \(R\) is the curvature tensor of \((M,g)\). In the Riemannian setting, contrary to the
pseudo-Riemannian case, a \(k\)-symmetric space is necessarily locally symmetric, i.e., \(\nabla R = 0\) [29].
Examples of pseudo-Riemannian \(k\)-symmetric spaces with \(k \geq 2\) can be found in [28] [8] [24]. Many
interesting results about Lorentzian two-symmetric spaces were presented in [28], in particular the
author proved that any two-symmetric Lorentzian manifold admits a parallel null vector field. A
classification of four-dimensional two-symmetric Lorentzian spaces is obtained in [8], based on the
Petrov classification of the Weyl tensors, and it is shown that such spaces are some special pp-waves.
For wide applications in physics, many authors studied pp-wave manifolds which are spacial kind
of pr-wave spaces. In [7], local symmetry, conformal flatness, Einstein-like metrics and existence of
non-trivial Ricci solitons studied on the conformally flat pr-wave manifolds. Homogeneous plane
wave manifolds, other special kind of pr-waves, investigated in [9] and one geodesically complete
family of the spaces under consideration were found. The generalization of the results of [5] is the
subject of [3], where it is proven that a locally indecomposable Lorentzian manifold of dimension $n + 2$ is two-symmetric if and only if there exist local coordinates $(v, x^1, \ldots, x^n, u)$ such that

\begin{equation}
\label{eq:1.1}
g = 2dvdu + \sum_{i=1}^{n} (dx^i)^2 + (H_{ij}u + F_{ij})x^i x^j (du)^2,
\end{equation}

where $H_{ij}$ is a nonzero diagonal matrix with diagonal elements $\lambda_1 \leq \cdots \leq \lambda_n$, and $F_{ij}$ is a symmetric real matrix. According to this general form of Lorentzian two-symmetric manifolds, in the four-dimensional case, there exist local coordinates $(x^1, \ldots, x^4)$ such that the metric $g$ of a Lorentzian two-symmetric space is

\begin{equation}
\label{eq:1.2}
g = 2dx^1 dx^4 + (dx^2)^2 + (dx^3)^2 + (x^4(a(x^2)^2 + b(x^3)^2) + p(x^2)^2 + 2qx^2 x^3 + s(x^3)^2) (dx^4)^2,
\end{equation}

where $a, b, p, q, s$ are real constants and $a^2 + b^2 \neq 0$. Our main goal is to study some geometric properties of four-dimensional Lorentzian two-symmetric spaces.

This paper is organized in the following way. Curvature properties of Lorentzian two-symmetric four-spaces will be studied in the section two and Einstein-like metric of the spaces under consideration is the subject of section three. Ricci solitons and homogeneous four-dimensional Lorentzian two-symmetric spaces will be studied in section four and five respectively.

2. TWO-SYMMETRIC LORENTZIAN FOUR-MANIFOLDS

The first step for study the geometry of (pseudo-)Riemannian manifolds is to determine the Lievi-Civita connection. By using the Koszul identity $2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y])$, and applying the metric (1.2), one can determine the components of the Levi-Civita connection. We use $\partial_i = \frac{\partial}{\partial x^i}$ as a local basis for the tangent space and have:

\begin{equation}
\label{eq:2.1}
\begin{align*}
\nabla_{\partial_x} \partial_4 &= (ax^2 x^4 + px^2 + qx^3)\partial_1, \\
\nabla_{\partial_y} \partial_4 &= (bx^3 x^4 + sx^3 + qx^2)\partial_1, \\
\nabla_{\partial_z} \partial_4 &= \frac{a(x^2)^2 + b(x^3)^2}{2}\partial_1 - (ax^2 x^4 + px^2 + qx^3)\partial_2 - (bx^3 x^4 + qx^2 + sx^3)\partial_3.
\end{align*}
\end{equation}

Applying the relation $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ we immediately determine the curvature tensor. If we set $R(\partial_k, \partial_l)\partial_j = R^i_{jkl}\partial_i$, then by contraction on the first and third indices of the curvature tensor, the Ricci tensor $\rho$ will be deduced. The scalar curvature tensor $\tau$ is also obtained by full contraction of coefficients of the curvature tensor.
Theorem 2.2. A four-dimensional two-symmetric Lorentzian space admits zero scalar curvature. Also, the non-zero components of curvature tensor and Ricci tensor are

\[
\begin{align*}
R(\partial_2, \partial_4) &= (ax^4 + p)\partial_1 dx^2 + q\partial_1 dx^3 - (ax^4 + p)\partial_2 dx^4 - q\partial_3 dx^4, \\
R(\partial_3, \partial_4) &= q\partial_1 dx^2 + (bx^4 + s)\partial_1 dx^3 - q\partial_2 dx^4 - (bx^4 + s)\partial_3 dx^4,
\end{align*}
\]

(2.2)

\[
\varrho(\partial_4, \partial_4) = -(a + b)x^4 - (s + p).
\]

A (pseudo-)Riemannian manifold \((M, g)\) is called Einstein if \(\varrho = cg\), for a real constant \(c\). Being Ricci flat means that the Ricci tensor vanishes identically. Also, conformal flatness translates into the following system of algebraic equations:

\[
W_{ijkh} = R_{ijkh} - \frac{1}{2}(g_{ik}\varrho_{jh} + g_{jh}\varrho_{ik} - g_{jk}\varrho_{ih} - g_{jh}\varrho_{ik}) + \frac{1}{6}(g_{ih}g_{jh} - g_{ih}g_{jh}) = 0
\]

(2.3)

for all indices \(i, j, k, h = 1, \ldots, 4\),

where \(W\) denotes the Weyl tensor and \(\tau\) is the scalar curvature. Although two-symmetric spaces clearly aren’t flat, we can check Ricci flatness.

Theorem 2.3. Let \((M, g)\) be a two-symmetric four-dimensional Lorentzian manifold such that the metric \(g\) is described by the Equation (1.2) in local coordinates \((x^1, x^2, x^3, x^4)\). The following statements satisfy:

a) \((M, g)\) is Einstein if and only if be Ricci flat if and only if \(b = -a, s = -p\).

b) \((M, g)\) is conformally flat if and only if \(b = a, s = p, q = 0\).

Proof. Let \((M, g)\) be an Einstein manifold. Using the Equation (2.2) we set \(\varrho = cg\). The following relations must be established.

\[
c = (a + b)x^4 + s + p = 0.
\]

So the Einstein property is equivalent to satisfying \(b = -a, s = -p\) and \(c = 0\) which is clearly equivalent to Ricci flatness. Using the Equation (2.3), the non-zero components of the Weyl tensor would be

\[
W_{2424} = -W_{3434} = \frac{(b-a)x^4+s-p}{2}, \quad W_{2434} = -q,
\]

it is obvious that the Weyl tensor vanishes if and only if \(b = a, s = p\) and \(q = 0\). □

3. Einstein-like Lorentzian two-symmetric spaces

Two new classes of Riemannian manifolds which are defined through conditions on the Ricci tensor, introduced by A. Gray in [23]. These types of manifolds which are famous as \(\mathcal{A}\) and \(\mathcal{B}\) classes, would be extended at once to the pseudo-Riemannian geometry. \(\mathcal{A}\) and \(\mathcal{B}\) classes are defined in the following way:

**Class \(\mathcal{A}\):** A pseudo-Riemannian manifold \((M, g)\) belongs to class \(\mathcal{A}\) if and only if its Ricci tensor \(\varrho\) is cyclic-parallel, that is,

\[
(\nabla_X\varrho)(Y, Z) + (\nabla_Y\varrho)(Z, X) + (\nabla_Z\varrho)(X, Y) = 0,
\]

(3.1)
for all vector fields $X, Y$ and $Z$ on $M$. The Equation (3.1) is equivalent to requiring that $\rho$ is a Killing tensor, that is,

\[(\nabla_X \rho)(X, X) = 0.\]

(3.2)

To note that Equation (3.2), also known as the first odd Ledger condition, is a necessary condition for a (pseudo-)Riemannian manifold to be a D'Atri space. Hence, identifying two-symmetric manifolds of a given dimension satisfying (3.2), is the first step to understand D'Atri spaces.

**Class B**: a pseudo-Riemannian manifold $(M, g)$ belongs to class $B$ if and only if its Ricci tensor be a Codazzi tensor, that is,

\[(\nabla_X \rho)(Y, Z) = (\nabla_Y \rho)(X, Z).\]

(3.3)

A pseudo-Riemannian manifold which belongs to one of the above classes is called an Einstein-like manifold. We denote the class of Ricci parallel, Einstein and manifolds with constant scalar curvature by $\mathcal{P}, \mathcal{E}$ and $\mathcal{C}$ respectively. One can easily see that the intersection of two Einstein-like classes consists of Ricci parallel manifolds. This situation can be summarized in the following diagram:

\[\mathcal{E} \subset \mathcal{P} = \mathcal{A} \cap \mathcal{B} \subset \mathcal{A} \cup \mathcal{B} \subset \mathcal{C}.\]

Einstein-like metrics are deeply investigated through the different kinds of homogeneous spaces in both Riemannian and pseudo-Riemannian signatures. Three-dimensional Riemannian homogeneous spaces studied in [11]. In [13], authors study three- and four-dimensional Einstein-like Riemannian manifolds which are Ricci-curvature homogeneous. They could completely classify three-dimensional case of the mentioned spaces, while in the four-dimensional case, they partially classified the special case where the manifold is locally homogeneous. They also presented explicit examples of four-dimensional locally homogeneous Riemannian manifolds whose Ricci tensor is cyclic-parallel and has distinct eigenvalues. These examples invalidated the expectation stated by F. Podestá and A. Spiro in [27]. Three-dimensional ball-homogeneous spaces, semi-symmetric spaces, Sasakian spaces and three-dimensional contact metric manifolds are other Riemannian classes which were the subject of research for the Einstein-like properties [15, 10, 2, 16].

**Theorem 3.1.** Every four-dimensional two-symmetric Lorentzian manifold $(M, g)$ belongs to class $\mathcal{A}$, if and only if $b = -a$.

*Proof.* Let $v = v^1 \partial_1 + v^2 \partial_2 + v^3 \partial_3 + v^4 \partial_4$ be an arbitrary vector space on $(M, g)$, where $v^1, \ldots, v^4$ are smooth functions on $M$. As mentioned before, $(M, g)$ belongs to class $\mathcal{A}$ of Einstein-like manifolds if and only if the Equation (3.2) satisfies. By straightforward calculations it is implied that

\[(\nabla_v \rho)(v, v) = -(v^4)^3(a + b).\]

Thus, $(M, g)$ belongs to class $\mathcal{A}$ if and only if $b = -a$. \qed
Theorem 3.2. Every four-dimensional two-symmetric Lorentzian manifold \((M, g)\) belongs to class \(\mathcal{B}\) of the Einstein-like manifolds.

Proof. Let \(v = \sum_{i=1}^{4} v^i \partial_i, u = \sum_{i=1}^{4} u^i \partial_i\) and \(w = \sum_{i=1}^{4} w^i \partial_i\) be three arbitrary smooth vector fields on \((M, g)\). Every two-symmetric space \((M, g)\) belongs to class \(\mathcal{B}\) of Einstein-like manifolds if and only if the Equation (3.3) satisfies. Direct calculations yield that 
\[
(\nabla u \varphi)(v, w) = (\nabla v \varphi)(u, w) = -u^i v^i w^i (a + b),
\]
and so, the Equation (3.3) always establishes and proves the claim. ☐

4. Two-symmetric Lorentzian Ricci solitons

We now report some basic information on Ricci solitons, referring to [19] for a survey and further references. A Ricci soliton is a pseudo-Riemannian manifold \((M, g)\) admitting a smooth vector field \(V\), such that
\[
\mathcal{L}_V g + g = \lambda g,
\]
where \(\mathcal{L}\) denotes the Lie derivative and \(\lambda\) a real constant. A Ricci soliton is said to be shrinking, steady or expanding depending on whether \(\lambda > 0\), \(\lambda = 0\) or \(\lambda < 0\), respectively. Ricci solitons are self-similar solutions of the Ricci flow.

Originally introduced in the Riemannian case, Ricci solitons have been intensively studied in pseudo-Riemannian settings in recent years. The Ricci soliton equation is also a special case of Einstein field equations.

Theorem 4.1. Every four-dimensional two-symmetric Lorentzian manifold \((M, g)\) is shrinking, expanding and steady Ricci soliton.

Proof. Let \((M, g)\) be a four-dimensional two-symmetric Lorentzian manifold, where \(g\) is described by the Equation (1.2). Suppose that \(v = \sum_{i=1}^{4} v^i \partial_i\) is a smooth vector field on \((M, g)\) such that the Equation (4.1) satisfies for a real constant \(\lambda\). The Lie derivative of \(g\) in direction \(v\) is
\[
\mathcal{L}_v g = 2\partial_1 v^1 (dx^1)^2 + 2(\partial_2 v^1 + \partial_1 v^2)dx^1 dx^2 + 2(\partial_3 v^1 + \partial_1 v^3)dx^1 dx^3 + \frac{2}{3}v^1 (dx^1)^3 + 2\partial_2 v^2 (dx^2)^2 + 2(\partial_3 v^2 + \partial_2 v^3)dx^2 dx^3 + 2(\partial_4 v^2 + \partial_2 v^4)dx^2 dx^4 + 2\partial_3 v^3 (dx^3)^2 + 2\partial_3 v^4 (dx^3)^2 + 2\partial_4 v^4 (dx^4)^2 + 2\partial_4 v^4 (dx^4)^2 + 2\partial_4 v^4 (dx^4)^2 + 2\partial_4 v^4 (dx^4)^2 + 2\partial_4 v^4 (dx^4)^2 + 2\partial_4 v^4 (dx^4)^2.
\]

Applying Equations (1.2) and (2.2) in the Ricci soliton Equation (4.1), we have a system of PDEs which admits the following solution
\[
\begin{align*}
\lambda &= 2c, \\
v^1 &= \frac{\lambda + s}{4}(a + b) + \frac{s}{2}(s + p) + 2cx^1, \\
v^2 &= cx^2, \\
v^3 &= cx^3, \\
v^4 &= 0,
\end{align*}
\]
for a real constant $c$. Since $c$ is arbitrary, $(M, g)$ can be an expanding, shrinking or steady Ricci soliton. \hfill \Box

A Ricci soliton $(M, g)$ is called gradient if the Equation (4.1) holds for a vector field $X = \text{grad} f$, for some potential function $f$. In this case, (4.1) can be rewritten as

$$2 \text{Hess}_f + \varrho = \lambda g,$$

where $\text{Hess}_f$ denotes the Hessian of $f$. Studying locally conformally flat Lorentzian gradient Ricci solitons, as well as quasi-Einstein spaces, in [11] and [12] proved that such spaces are locally isometric to a plane-wave, if the gradient of the potential function is null.

**Theorem 4.2.** Every four-dimensional two-symmetric Lorentzian space $(M, g)$ is a gradient Ricci soliton if and only if be a steady Ricci soliton. In this case, the potential function is $f = \frac{a+b}{4}(x^4)^3 + \frac{c}{4}(x^4)^2 + c_1 x^4 + c_2$, for arbitrary real constants $c_1, c_2$.

**Proof.** Let $f = f(x^1, x^2, x^3, x^4)$ be a smooth function on $(M, g)$ and $v = \sum_{i=1}^4 v^i \partial_i$ be a gradient Ricci soliton with the potential function $f$. So the coefficient $v^i$ must be $v^i = \sum_{j=1}^4 g^{ij} \partial_j(f)$. By applying $v$ to the Equation (4.1) the following equations must establish

$$\begin{cases} f_{11} = f_{12} = f_{13} = f_{23} = 0, \\
\lambda = 2f_{14} = 2f_{22} = 2f_{33}, \\
2f_{24} - 2af_1 x^2 x^4 - 2pf_1 x^2 - 2qf_1 x^3 = 0, \\
2f_{34} - 2bf_1 x^3 x^4 - 2qf_1 x^2 - 2sf_1 x^3 = 0, \\
\lambda(x^2)^2(a x^4 + p) + 2\lambda q x^2 x^3 + \lambda(x^3)^2(b x^4 + s) + (a+b)x^4 + s + p - 2f_{44} \\
+f_1(a x^2)^2 + b(x^3)^2 - 2f_2(ax^2 x^4 + px^2 + qx^3) - 2f_3(qx^2 + bx^3 x^4 + s x^3) = 0, \end{cases}$$

where $f_i := \partial_i f$. After solving the above system of PDEs we get that $\lambda$ must be vanished and $f$ must be the same function of the statement, this matter ends the proof. \hfill \Box

To note that, from the above Theorem 4.2 we get $\nabla f = \frac{a+b}{4}(x^4)^3 + \frac{c}{2} x^4 + c_1$, which is a null vector field. This result is compatible with main Theorem in [11].

5. **Homogeneous two-symmetric four-dimensional spaces**

Study of homogeneous spaces is one of the most interesting topics in differential geometry, where a deep connection between geometry and algebra appears. A (pseudo-)Riemannian manifold $M$ is called homogeneous, if for any points $p, q \in M$, there is an isometry $\phi$ of $M$ such that $\phi(p) = q$. In short, $I(M)$, the group of isometries of $M$, acts transitively on $M$. If $M$ is homogeneous, then evidently any geometrical properties at one point of $M$ holds at every point. For a detailed introduction to homogeneous spaces see e.g., [5, 26, 31].

Homogeneous Riemannian structures introduced by Ambrose and Singer in [4] and deeply studied in [30]. The notation is generalized to homogeneous pseudo-Riemannian structures by Gadea and Oubiña in [22], in order to obtain a characterization of reductive homogeneous pseudo-Riemannian manifolds. A pretty application of homogeneous structures on three dimensional Lorentzian manifold is shown in [13].
Let $(M,g)$ be a connected pseudo-Riemannian manifold, the following definition introduced by Gadea and Oubiña:

**Definition 5.1.** [22] A homogeneous pseudo-Riemannian structure on $(M,g)$ is a tensor field $T$ of type $(1,2)$ on $M$, such that the connection $\tilde{\nabla} = \nabla - T$ satisfies

\[(5.1) \quad \tilde{\nabla} g = 0, \quad \tilde{\nabla} R = 0, \quad \tilde{\nabla} T = 0.\]

The above conditions are equivalent to the following system of equations which are famous as Ambrose-Singer equations.

\[(5.2) \quad g(T_XY, Z) + g(Y, T_XZ) = 0,\]
\[(5.3) \quad (\nabla_X R)YZ = [T_X, RYZ] - RT_XYZ - RYT_XZ,\]
\[(5.4) \quad (\nabla_X T)Y = [T_X, TY] - TT_XY,\]

for all vector fields $X,Y,Z$.

Existence of homogeneous pseudo-Riemannian structures shows the homogeneity of the space. This fact is the subject of the following Lemma

**Lemma 5.2.** [22] Let $(M,g)$ be a simply connected and complete pseudo-Riemannian manifold, then $(M,g)$ admits a pseudo-Riemannian homogeneous structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.

**Case 1: Reductive cases:**

By applying the above lemma, we consider reductive homogeneous four-dimensional two-symmetric Lorentzian spaces. The result is the following theorem,

**Theorem 5.3.** Every Lorentzian four-dimensional two-symmetric manifold is not reductive homogeneous.

**Proof.** Let $(M,g)$ be a four-dimensional two-symmetric manifold. There exist local coordinates $(x^1, \ldots, x^4)$ such that the metric $g$ is defined using the Equations (1.2). According to the Lemma 5.2, $(M,g)$ is (locally) reductive homogeneous if and only if the tensor field $T$ of type $(1,2)$ exists, such that the Ambrose-Singer equations satisfy. Let $T_{ij} = T_{ij}^k \partial_k, 1 \leq i,j,k \leq 4$ be a homogeneous structure on $(M,g)$ where $\partial_1 = \partial_v, \partial_2 = \partial_{x_1}, \partial_3 = \partial_{x_2}, \partial_4 = \partial_u$ and $T_{ij}^k$ are smooth functions on $M$.

From the Equations (5.2) and (5.3), besides the relations between the components $T_{ij}^k$, one of the following relations for the constants $a, b, p, q, s$ must be valid:

1. $a = p = q = 0$,
2. $a \neq 0, q = 0, s = \frac{bp}{a}$,
3. $b = a, s = p, q = 0$,
4. $b = -a, s = -p$,
but each of these solutions makes a contradiction with Equation (5.4) (or equivalently with $\tilde{\nabla} T = 0$). For example, in the case 1, for the components $T_{ij}^k$ have

$$
T_{ij}^k = 0, \quad 1 \leq j, k \leq 4, \quad (j,k) \neq (4,1), \quad T_{14}^1 = -T_{44}^1 = \frac{b}{2(6x^4 + y^4)},
$$

$$
T_{ij}^k = 0, \quad 1 \leq j, k \leq 4, \quad k \neq 1,

T_{ij}^k = 0, \quad 1 \leq j \leq 4, \quad k \neq 1,

T_{ij}^k = 0, \quad 1 \leq j \leq 3, \quad T_{44}^1 = \frac{b(x^3)^2}{2},

T_{ij}^k = -T_{kj}^i, \quad 1 \leq j \leq 4, \quad 2 \leq k \leq 3,

T_{ij}^k = 0, \quad 1 \leq j \leq 3.
$$

On the other hand, we have

$$
(\tilde{\nabla}_{\partial_t} T)_4^4 = \partial_t T_{44}^4 + T_{42}^4 (T_{44}^2 + x^2(p + ax^4) + qx^3) + T_{43}^4 (T_{44}^3 + x^3(s + bx^4) + qx^2)
$$

$$
+ T_{42}^4 (x^2(p + ax^4) + qx^3) + T_{43}^4 (x^3(s + bx^4) + qx^2)
$$

$$
+ T_{41}^4 (T_{44}^4 - \frac{a(x^2)^2 + b(x^4)^2}{2} - T_{44}^4 a(x^2)^2 + b(x^4)^2) + (T_{44}^4)^2.
$$

If we substitute the previous solutions in the above relation we get $(\tilde{\nabla}_{\partial_t} T)_4^4 = \frac{3b^2}{4(6x^4 + y^4)}$, and so the Equation (5.4) satisfies if $b = 0$ which is a contradiction, since in this case the matrix $H$ in (1.1) vanishes.

**Case 2: Non-reductive cases:**

Consider a homogeneous manifold $M = G/H$ (with $H$ connected), the Lie algebra $\mathfrak{g}$ of $G$, the isotropy subalgebra $\mathfrak{h}$, and $m = \mathfrak{g}/\mathfrak{h}$ the factor space, which identifies with a subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}$. The pair $(\mathfrak{g}, \mathfrak{h})$ uniquely defines the isotropy representation

$$
\psi : \mathfrak{g} \to \mathfrak{gl}(m), \quad \psi(x)(y) = [x,y]_m \quad \text{for all} \quad x \in \mathfrak{g}, y \in \mathfrak{m}.
$$

Given a basis $\{h_1, ..., h_r, u_1, ..., u_n\}$ of $\mathfrak{g}$, where $\{h_j\}$ and $\{u_i\}$ are bases of $\mathfrak{h}$ and $\mathfrak{m}$, respectively, a bilinear form on $\mathfrak{m}$ is determined by the matrix $g$ of its components with respect to the basis $\{u_i\}$ and is invariant if and only if $\psi(x) \circ g + g \circ \psi(x) = 0$ for all $x \in \mathfrak{h}$. Invariant pseudo-Riemannian metrics $g$ on the homogeneous space $M = G/H$ are in a one-to-one correspondence with nondegenerate invariant symmetric bilinear forms $g$ on $m$ [25].

Then, $g$ uniquely defines its invariant linear Levi-Civita connection, described in terms of the corresponding homomorphism of $\mathfrak{h}$-modules $\Lambda : \mathfrak{g} \to \mathfrak{gl}(m)$, such that $\Lambda(x)(y)_m = [x,y]_m$ for all $x \in \mathfrak{h}, y \in \mathfrak{g}$. Explicitly, one has

$$
\Lambda(x)(y)_m = \frac{1}{2}[x,y]_m + v(x,y), \quad \text{for all} \quad x,y \in \mathfrak{g},
$$

where $v : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{m}$ is the $\mathfrak{h}$-invariant symmetric mapping uniquely determined by

$$
2g(v(x,y), z)_m = g(x_m, [z,y]_m) + g(y_m, [z,x]_m), \quad \text{for all} \quad x,y,z \in \mathfrak{g}.
$$

The curvature tensor is then determined by

$$
R : m \times m \to \mathfrak{gl}(m)
$$

$$(x,y) \mapsto [\Lambda(x), \Lambda(y)] - \Lambda([x,y]).$$
Finally, the Ricci tensor $\varrho$ of $g$, described in terms of its components with respect to \( \{ u_i \} \), is given by

\[
(5.6) \quad \varrho(u_i, u_j) = \sum_{r=1}^{4} R_{ri}(u_r, u_j), \quad i, j = 1, \ldots, 4
\]

and the scalar curvature $\tau$ is the trace of $\varrho$.

Non-reductive homogeneous manifolds of dimension 4 were classified in [21], in terms of the corresponding non-reductive Lie algebras. The corresponding invariant pseudo-Riemannian metrics, together with their connection and curvature, were explicitly described in [17, 18]. These spaces categorized in eight classes, $A_1, \ldots, A_5, B_1, B_2, B_3$. The invariant metrics of types $A_1, A_2, A_3$ can be both of Lorentzian or neutral signature while the cases $A_4, A_5$ are always Lorentzian and cases $B_1, B_2, B_3$ admit the neutral signature $(2,2)$.

**Theorem 5.4.** Every Lorentzian four-dimensional non-reductive homogeneous manifold is locally symmetric if and only if $\nabla^2 R = 0$.

**Proof.** Let $(M, g)$ be a Lorentzian non-reductive homogeneous four-dimensional manifold, then $(M, g)$ is isometric to a homogeneous space $G/H$ equipped with a Lorentzian invariant metric $g$, where the corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are described in the cases $A_1, \ldots, A_5$ of [17].

We bring the details of the case $A_1$. The other cases can be treated in the similar way. In this case, $\mathfrak{g} = \mathfrak{A}_1$ is the decomposable 5-dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{s}(2)$, where $\mathfrak{s}(2)$ is the 2-dimensional solvable algebra. There exists a basis $\{ e_1, \ldots, e_5 \}$ of $\mathfrak{A}_1$, such that the non-zero products are:

\[
[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_4, e_5] = e_4
\]

and the isotropy subalgebra is $\mathfrak{h} = \text{Span}\{ h_1 = e_3 + e_4 \}$. So, we can take

\[
m = \text{Span}\{ u_1 = e_1, u_2 = e_2, u_3 = e_3, u_4 = e_3 - e_4 \}.
\]

With respect to $\{ u_i \}$, we have the following isotropy representation $H_1$ for $h_1$ and consequently the following description of invariant metric $g$:

\[
(5.7) \quad H_1 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & -1/2 & 0
\end{pmatrix}, \quad g = \begin{pmatrix}
a & 0 & -\frac{a}{2} & 0 \\
0 & b & c & a \\
-\frac{a}{2} & c & d & 0 \\
0 & a & 0 & 0
\end{pmatrix},
\]
which is nondegenerate whenever \( a(a - 4d) \neq 0 \) and is Lorentzian if and only if \( a(a - 4d) < 0 \). Putting \( \Lambda[i] := \Lambda(u_i) \) for all indices \( i = 1, \ldots, 4 \), we find:

\[
\begin{align*}
\Lambda[1] &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{b}{a} & -\frac{c}{a} & -1
\end{pmatrix}, & \Lambda[2] &= \begin{pmatrix}
0 & -\frac{b(2d+a)}{a(-4d+a)} & -\frac{c}{a} & -1 \\
-1 & 0 & -\frac{b}{2a} & 0 \\
0 & 0 & 0 & 0 \\
-\frac{b}{a} & -\frac{b}{2a} & 0 & \frac{1}{2}
\end{pmatrix}, \\
\Lambda[3] &= \begin{pmatrix}
0 & 0 & \frac{2}{3} & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{c}{a} & -\frac{b}{2a} & 0 & \frac{1}{2}
\end{pmatrix}, & \Lambda[4] &= 0.
\end{align*}
\]

Moreover, applying (5.5) and (5.6), by setting \( R_{ij} = R(u_i, u_j) \), some standard calculations give that with respect to \( \{u_i\} \), the non-zero curvature components are determined as follows:

\[
\begin{align*}
R_{12} &= \begin{pmatrix}
0 & \frac{b(20d+a)}{a(-4d+a)} & -\frac{c}{a} & -1 \\
1 & 0 & -\frac{b}{2a} & 0 \\
0 & \frac{12b}{a(-4d+a)} & 0 & 0 \\
\frac{4b}{a} & -\frac{12b}{a(-4d+a)} & \frac{b}{a} & 0
\end{pmatrix}, & R_{13} &= \begin{pmatrix}
0 & -\frac{b(4d+a)}{2a(-4d+a)} & -\frac{c}{a} & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{c}{a} & 0 & -\frac{c}{2a} & 0
\end{pmatrix}, \\
R_{14} &= \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & -\frac{1}{2} & 0
\end{pmatrix}, & R_{23} &= \begin{pmatrix}
0 & -\frac{b(4d+a)}{2a(-4d+a)} & -\frac{c}{a} & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{c}{a} & 0 & -\frac{c}{2a} & 0
\end{pmatrix}, \\
R_{24} &= \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{b}{a} & \frac{4}{a} & \frac{1}{2}
\end{pmatrix}, & R_{34} &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & -\frac{2c}{a} & -\frac{1}{2} & 0
\end{pmatrix}.
\end{align*}
\]

and the Ricci tensor \( \rho \) is determined by

\[
\rho = \begin{pmatrix}
-2 & 0 & 1 & 0 \\
0 & \frac{2b(a+12d)}{a(a-4d)} & -\frac{2c}{a} & -2 \\
1 & -\frac{2c}{a} & -\frac{1}{2} & 0 \\
0 & -2 & 0 & 0
\end{pmatrix}.
\]

By using description of the curvature tensor and Levi-Civita connection, we set \( \nabla R = 0 \) and have, the homogeneous spaces \( G/H \) is locally symmetric if and only if \( b = 0 \). Also, \( \nabla^2 R = 0 \) if and only if \( b = 0 \), so we conclude that the homogeneous spaces of type \( A_1 \) are locally symmetric if and only if \( \nabla^2 R = 0 \). Similar arguments will be applied for the other Lorentzian cases and this finishes the proof.

The following remark is the direct conclusion of the Theorems 5.3 and 5.4

**Remark 5.5.** Let \((M, g)\) be a homogeneous Lorentzian four-dimensional manifold, then \((M, g)\) is not a two-symmetric space.
Classification of four-dimensional pseudo-Riemannian homogeneous spaces with non-trivial isotropy has been studied in [25] in order to find the local classification of four-dimensional Einstein-Maxwell homogeneous spaces with an invariant pseudo-Riemannian metric of arbitrary signature. The presented list is a good reference to study pseudo-Riemannian homogeneous four-manifolds. We apply the mentioned classification to find an example of a pseudo-Riemannian two-symmetric homogeneous four-manifold, equipped with an invariant metric of neutral signature.

**Example 5.6.** Let $M = G/H$ be a homogeneous four-dimensional manifold of type $1.A^1 : 9$ of the Komrakov’s list [25]. In this case, the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are described as following:

$$\mathfrak{g} = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & \lambda x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & -\mu x \end{pmatrix} | x \in \mathbb{R}, \lambda, \mu \in \mathbb{R} \right\} \ltimes (\mathfrak{n}_3 \times \mathbb{R}), \quad \mathfrak{h} = \langle p \rangle,$$

where $\mathfrak{n}_3 = \langle h, p, q \rangle$ with the only non-zero bracket $[p, q] = h$. If $\mathfrak{g} = \text{span}\{u_1, u_2, u_3, u_4\}$ and $\mathfrak{h} = \text{span}\{h_1\}$, the table of Lie brackets is:

| $\ [ \ , \ ]$ | $h_1$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
| --- | --- | --- | --- | --- |
| $h_1$ | 0 | 0 | $u_1$ | $u_2$ | 0 |
| $u_1$ | 0 | 0 | 0 | $u_1$ | 0 |
| $u_2$ | $-u_1$ | 0 | 0 | $\lambda h_1 + u_2 + u_4$ | 0 |
| $u_3$ | $-u_2$ | $-u_1$ | $-\lambda h_1 - u_2 - u_4$ | 0 | $\mu u_4$ |
| $u_4$ | 0 | 0 | 0 | $-\mu u_4$ | 0 |

The invariant metric will be calculated as following

$$g = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix},$$

for arbitrary real constants $a \neq 0, b, c, d$. This metric admits both Lorentzian and neutral signatures while for $d = -9a$, $g$ is of neutral signature. We also set $\mu = -\frac{5}{2}$ and $\lambda = \frac{3}{2}$. Keeping in mind $\Lambda[i] = \Lambda(u_i)$ for all indices $i = 1, \ldots, 4$, the components of the Levi-Civita connection are:

$$\Lambda[1] = 0, \quad \Lambda[2] = \begin{pmatrix} 0 & 1 & \frac{c}{2a} & -\frac{9}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

$$\Lambda[3] = \begin{pmatrix} -1 & \frac{c}{2a} & \frac{6b^2 - c^2}{6a^2} & \frac{5c}{2a} \\ 0 & 0 & \frac{c}{a} & -\frac{9}{2} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{c}{6a} & 0 \end{pmatrix}, \quad \Lambda[4] = \begin{pmatrix} 0 & -\frac{9}{2} & \frac{5c}{2a} & -\frac{45}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{2} & 0 \end{pmatrix}.$$
also, by using Equation (5.5), if set $R_{ij} = R(u_i, u_j)$, the non-zero components of the curvature tensor are

$$R_{23} = \begin{pmatrix} 0 & 6 & -\frac{2c}{a} & 18 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \quad R_{34} = \begin{pmatrix} 0 & -18 & \frac{6c}{a} & -54 \\ 0 & 0 & -18 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{pmatrix}.$$ 

The space is Ricci flat and the only non-zero covariant derivative of the curvature tensor is in the direction of $u_3$. We set $(\Lambda[k]R)_{ij} = (\Lambda(u_k)R)(u_i, u_j)$, the non-zero components are

$$(\Lambda[3]R)_{23} = \begin{pmatrix} 0 & 6 & -\frac{2c}{a} & 18 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \quad (\Lambda[3]R)_{34} = \begin{pmatrix} 0 & -18 & \frac{6c}{a} & -54 \\ 0 & 0 & -18 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{pmatrix}.$$ 

Clearly, $(M = G/H, g)$ is never locally symmetric but by straightforward calculations we get $\nabla^2 R = 0$ which shows that the spaces is two-symmetric.

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