Entropic uncertainty relations in a class of generalized probabilistic theories

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Abstract
Entropic uncertainty relations play an important role in both fundamentals and applications of quantum theory. Although they have been well-investigated in quantum theory, little is known about entropic uncertainty in generalized probabilistic theories (GPTs). The current study explores two types of entropic uncertainty relations, preparation and measurement uncertainty relations, in a class of GPTs which can be considered generalizations of quantum theory. Not only a method for obtaining entropic preparation uncertainty relations but also an entropic measurement uncertainty relation similar to the quantum one by Buscemi et al (2014 Phys. Rev. Lett. 112 050401) are proved in those theories. These results manifest that the entropic structure in the uncertainty relations is not restricted to quantum theory and therefore is universal one. Concrete calculations of our relations in GPTs called the regular polygon theories are also demonstrated.

Keywords: generalized probabilistic theories, uncertainty relations, measurement incompatibility, foundation of quantum physics

(Some figures may appear in colour only in the online journal)

1. Introduction

The concept of uncertainty, advocated initially by Heisenberg [1], is one of the most peculiar features in quantum theory. Much study has been devoted to proper understandings of uncertainty to demonstrate that it has two aspects: preparation uncertainty and measurement uncertainty [2]. Loosely speaking, for a pair of observables which do not commute, the former describes that there is no state on which their individual measurements output simultaneously definite values, and the latter expresses the impossibility of performing their joint measurement

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While there have been several mathematical representations of them, preparation uncertainty relations (PURs) and measurement uncertainty relations (MURs) respectively [5–9], entropic uncertainty relations [10–16] have the advantages of their compatibility with information theory and independence from the structure of the sample spaces. They have been applied to the field of quantum information in various ways [17]. On the other hand, two kinds of uncertainty have been investigated also in physical theories broader than quantum theory called generalized probabilistic theories (GPTs) [18–23]. For example, there have been researches on both types of uncertainty [24] or joint measurability of observables [25–29], which are related with MURs, in GPTs. In [30], several formulations of two types of uncertainty were generalized to GPTs, and it was revealed quantitatively that there are close relations between them not only in quantum theory [31] but also in a class of GPTs. However, although the notion of entropy has been introduced in GPTs [32–37], insights of entropic uncertainty relations in GPTs are still missing.

In the present paper, entropic uncertainty relations are studied in a class of GPTs investigated in the previous work [30]: GPTs satisfying transitivity and self-duality with respect to a certain inner product. They include finite dimensional classical and quantum theories, and thus can be regarded as generalizations of them. In those theories, we obtain an entropic inequality related with PURs in a simple way via the Landau–Pollak-type relation [38–40]. We also prove an entropic relation similar to the quantum MUR by Buscemi et al [16] with their formulations generalized to those GPTs. Moreover, they can be considered as an entropic counterpart of [30]: if there exists an entropic PUR giving certain bounds of uncertainty, then an entropic MUR also exists and can be formulated in terms of the same bounds as the PUR. We also present, as an illustration, concrete expressions of our entropic relations in a specific class of GPTs called the regular polygon theories [41].

This paper is organized as follows. In section 2, we give a short survey of GPTs including the introduction of the regular polygon theories. Section 3 is the main part of this paper, and there are shown entropic uncertainty relations in a certain class of GPTs. We conclude the present work and give brief discussions in section 4.

2. GPTs

GPTs are the most general physical theories reflecting intuitively the notion of physical experiments: to prepare a state, to conduct a measurement, and to observe a probability distribution. In this section, a brief survey of GPTs is shown according mainly to [23, 30, 34, 35, 42].

2.1. Fundamentals

Any GPT is associated with the notion of states and effects. In this paper, a compact convex set Ω in \( V \equiv \mathbb{R}^{N+1} \) with \( \dim \text{aff}(\Omega) = N \) describes the set of all states in a GPT, which we call the state space of the theory. We assume in this paper that GPTs are finite dimensional (\( N < \infty \)), and \( \text{aff}(\Omega) \) does not include the origin \( O \in V (O \notin \text{aff}(\Omega)) \). We note that the notion of probability mixture of states is reflected by the convex structure of \( \Omega \). The extreme elements of \( \Omega \) are called pure states, and we denote the set of all pure states by \( \Omega^{\text{ext}} = \{\omega^{\text{ext}}_\lambda\}_\lambda \in \Lambda \). The other elements of \( \Omega \) are called mixed states. For a GPT with its state space \( \Omega \), we define the effect space of the theory as \( \mathcal{E}(\Omega) = \{e \in V' | e(\omega) \in [0, 1] \text{ for all } \omega \in \Omega\} \), where \( V' \) is the dual space of \( V \), and call its elements effects. Remark that we follow the no-restriction hypothesis [43] in this paper, and we sometimes denote \( \mathcal{E}(\Omega) \) simply by \( \mathcal{E} \). With introducing the unit effect \( u \) as \( u \in \mathcal{E}(\Omega) \) satisfying \( u(\omega) = 1 \) for all \( \omega \in \Omega \), a measurement or observable on some sample space \( X \) is defined by a set of effects \( \{e_x\}_{x \in X} \) such that \( \sum_{x \in X} e_x = u \). In this paper, we assume
that every measurement is with finite outcomes (i.e. the sample space \( X \) is finite) and does not include the zero effect, and the trivial measurement \( \{ u \} \) is not considered. Two measurements \( A = \{ a_x \}_{x \in X} \) and \( B = \{ b_y \}_{y \in Y} \) are called \textit{jointly measurable or compatible} if there exists a joint measurement \( C = \{ c_{xy} \}_{(x,y) \in X \times Y} \) such that its marginals satisfy \( \sum_{y \in Y} c_{xy} = a_x \) and \( \sum_{x \in X} c_{xy} = b_y \) for all \( x \in X, y \in Y \). If \( A \) and \( B \) are not jointly measurable, then they are called \textit{incompatible}.

We say that two GPTs are equivalent if their state spaces \( \Omega_1 \) and \( \Omega_2 \) satisfy \( \psi(\Omega_1) = \Omega_2 \) for a linear bijection \( \psi \) on \( V \). In that case, because \( \mathcal{E}(\Omega_2) = \mathcal{E}(\Omega_1) \circ \psi^{-1} \) holds, we can see the covariance (equivalence) of physical predictions.

For a state space \( \Omega \), the \textit{positive cone} \( V_+(\Omega) \) (or simply \( V_+ \)) generated by \( \Omega \) is defined as the set of all unnormalized states, that is, \( V_+ := \{ v \in V | v = k\omega, \omega \in \Omega, k \geq 0 \} \). We can also define the \textit{dual cone} \( V_+^*(\Omega) \) (or simply \( V_+^* \)) as the set of all unnormalized effects:

\[
\forall f \in V^+, \exists v \in V_+, \text{such that } f(v) = 0.
\]

A half-line \( E \subset V_+ \) is called an \textit{extremal ray} of \( V_+ \) (respectively \( V_+^* \)) if \( l = m + n \) with \( l \in E \) and \( m, n \in V_+ \) (respectively \( m, n \in V_+^* \)) implies \( m, n \in E \). We call effects on extremal rays of \( V_+ \) \textit{indecomposable}, while it is easy to see that the half-lines \( \{ x \in V | x = k\omega_0, k \geq 0 \} \) generated by the pure states \( \omega_0 \in \Omega \) are the extremal rays of \( V_+ \). It is known that there exist pure and indecomposable effects [34], and we denote by \( \mathcal{E}^\text{ext}(\Omega) \) (or simply \( \mathcal{E}^\text{ext} \)) the set of all pure and indecomposable effects. They are thought to be a generalization of rank-1 projections in finite dimensional quantum theories (see [30]).

### 2.2. Additional notions

Let \( \Omega \subset V \) be a state space. A linear bijection \( T : V \to V \) is called a \textit{state automorphism} on \( \Omega \) if it satisfies \( T(\Omega) = \Omega \), and we denote by \( G(\Omega) \) (or simply \( G \)) the set of all state automorphisms on \( \Omega \). States \( \omega_1, \omega_2 \in \Omega \) are called \textit{physically equivalent} if there exists \( T \in G \) satisfying \( T\omega_1 = \omega_2 \). We say that \( \Omega \) is \textit{transitive} if all pure states are physically equivalent, i.e., for an arbitrary pair of pure states \( \omega_{e1}, \omega_{e2} \in \Omega^\text{ext} \) there exists \( T \in G \) such that \( T\omega_{e1} = \omega_{e2} \). When \( \Omega \) is transitive, we can define the \textit{maximally mixed state} \( \omega_M \in \Omega \) as a unique state satisfying \( T\omega_M = \omega_M \) for all \( T \in G \) [44].

There exists a useful inner product \( \langle \cdot, \cdot \rangle_G \) on \( V \), with respect to which all elements of \( G \) are orthogonal transformations on \( V \). That is,

\[
\langle Tx, Ty \rangle_G = \langle x, y \rangle_G \quad \forall x, y \in V
\]

holds for all \( T \in G \). In fact, \( \langle \cdot, \cdot \rangle_G \) is constructed in the way

\[
\langle x, y \rangle_G = \int_G \langle x, y \rangle d\mu \quad (x, y \in V)
\]

by means of the two-sided invariant Haar measure \( \mu \) on \( G \) and a reference inner product \( \langle \cdot, \cdot \rangle \) on \( V \) (such as the standard Euclidean inner product on \( V \)). This inner product allows us to introduce a convenient vector representation [30, 45, 46]. To see this, we refer a proposition in [30] which holds especially when \( \Omega \) is transitive (note that there were considered more general cases in [30]):

**Proposition (Proposition II.2 in [30]).** For a transitive state space \( \Omega \), there exists a basis \( \{ v_i \}_{i=1}^{N+1} \) of \( V \) orthonormal with respect to the inner product \( \langle \cdot, \cdot \rangle \) such that \( v_{N+1} = \omega_M \) and

\[
x \in \text{aff}(\Omega) \iff x = \sum_{i=1}^{N} a_i v_i + v_{N+1} = \sum_{i=1}^{N} a_i v_i + \omega_M (a_1, \ldots, a_N \in \mathbb{R}).
\]
It follows from this proposition that
\[ u = \omega_M = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]
which we shall see is important for deriving our main results. When \( \Omega \) is transitive, we can also prove that all pure states are of equal norm with respect to \( \langle \cdot, \cdot \rangle_G \):
\[ \| \omega^\text{ext} \|_G = \sqrt{\alpha} \quad \forall \omega^\text{ext} \in \Omega^\text{ext}, \]
where \( \| \cdot \|_G := \langle \cdot, \cdot \rangle_G^{1/2} \) and \( \alpha \) is a positive number.

For the positive cone \( V^+ \) generated by \( \Omega \) and an inner product \( \langle \cdot, \cdot \rangle \) on \( V \), the internal dual cone \( V^\text{int}(\cdot, \cdot) \) relative to \( \langle \cdot, \cdot \rangle \) is defined as \( V^\text{int}(\cdot, \cdot) := \{ w \in V | (w, v) \geq 0 \quad \forall v \in V^+ \} \), and the cone \( V^\alpha \) is called self-dual if \( V^+ = V^\text{int}(\cdot, \cdot) \) for some inner product \( \langle \cdot, \cdot \rangle \) on \( V \). Note that by virtue of the Riesz representation theorem \( V^\text{int}(\cdot, \cdot) \) can be regarded as the dual cone \( V^\ast \), i.e., the set of all unnormalized effects. Thus, the self-duality of \( V^+ \) means that (unnormalized) states can be identified with (unnormalized) effects. Let us assume that \( \Omega \) is transitive and \( V^+ \) is self-dual with the self-dualizing inner product being \( \langle \cdot, \cdot \rangle_G \). In this case, \( \Omega^\text{ext} \) holds, and we can prove that \( \alpha \omega^\text{ext} = \Omega^\text{ext} \), that is,
\[ e^\text{ext} = \frac{\omega^\text{ext}}{\alpha} \]
gives a pure and indecomposable effect for any \( \omega^\text{ext} \in \Omega^\text{ext} \) because the extreme rays of \( V^+ = V^\text{int}(\cdot, \cdot) \) are generated by \( \Omega^\text{ext} \) and \( \omega^\text{ext} \) is a (unique) state satisfying \( \langle \omega^\text{ext}, \omega^\text{ext} \rangle = 1 \) [34]. In [30], it was demonstrated that the state spaces of finite dimensional classical and quantum theories satisfy both transitivity and self-duality with respect to the inner product \( \langle \cdot, \cdot \rangle_G \). There was also shown that assuming self-duality with respect to an arbitrary inner product in stead of \( \langle \cdot, \cdot \rangle_G \) is sufficient for the observations above when \( \Omega \) satisfies transitivity and \( \| \Omega^\text{ext} \| < \infty \) (see proposition II.3 in [30] for details). In the next subsection, we will introduce other examples of GPTs with transitivity and self-duality.

2.3. Examples of GPTs: regular polygon theories

The regular polygon theories are GPTs whose state spaces are regular polygons in \( V \equiv \mathbb{R}^3 \), and if a state space is the regular polygon with \( n \) sides (\( n \geq 3 \)), then we denote it by \( \Omega_n \). In [41], we can find that \( \Omega_n \) is given by the convex hull of its pure states (its vertices)
\[ \Omega^\text{ext}_n = \{ \omega^\text{ext}_n(i) \}_{i=0}^{n-1}, \]
where
\[ \omega^\text{ext}_n(i) = \begin{pmatrix} r_n \cos \left( \frac{2\pi i}{n} \right) \\ r_n \sin \left( \frac{2\pi i}{n} \right) \\ 1 \end{pmatrix} \quad \text{with} \quad r_n = \sqrt{\frac{1}{\cos \left( \frac{\pi}{n} \right)}}, \]
\[ \text{with} \quad r_n = \sqrt{\frac{1}{\cos \left( \frac{\pi}{n} \right)}}, \]
The corresponding effect space \( E(\Omega_n) \) is given by \( V^\text{int}(\cdot, \cdot) \cap \{ u - V^\text{int}(\cdot, \cdot) \} \) in terms of the dual cone \( V^\ast(\Omega_n) = V^\text{int}(\cdot, \cdot) \) represented by the standard Euclidean inner product \( \langle \cdot, \cdot \rangle_E \) of \( V \). \( V^\ast(\Omega_n) \) is generated by the pure and indecomposable effects
\[ E^\text{ext}(\Omega_n) = \{ e^\text{ext}_n(i) \}_{i=0}^{n-1}, \]
where

\[
e_{\text{ext}}^{(i)}(n) = \begin{cases} 
\frac{1}{2} \left( \begin{array}{c} r_n \cos \left( \frac{(2i - 1)\pi}{n} \right) \\
 \frac{1}{2} \sin \left( \frac{(2i - 1)\pi}{n} \right) \end{array} \right) & (n \text{ even}); \\
\frac{1}{1 + r_n^2} \left( \begin{array}{c} r_n \cos \left( \frac{2i\pi}{n} \right) \\
 \frac{1}{2} \sin \left( \frac{2i\pi}{n} \right) \end{array} \right) & (n \text{ odd}).
\end{cases}
\]

(7)

We can also consider the case when \( n = \infty \) in (5)–(7). The state space \( \Omega_{\infty} \) is a disc with its pure states and pure and indecomposable effects being

\[
\Omega_{\infty}^{\text{ext}} = \{ \omega_{\infty}^{\text{ext}}(\theta) \}_{\theta \in [0, 2\pi)}
\]

(8)

and

\[
\mathcal{E}_{\infty}^{\text{ext}} = \{ e_{\infty}^{\text{ext}}(\theta) \}_{\theta \in [0, 2\pi)},
\]

(9)

where

\[
\omega_{\infty}^{\text{ext}}(\theta) = \left( \begin{array}{c} \cos \theta \\
 \sin \theta \\
 1 \end{array} \right) \quad \text{and} \quad e_{\infty}^{\text{ext}}(\theta) = \frac{1}{2} \left( \begin{array}{c} \cos \theta \\
 \sin \theta \\
 1 \end{array} \right),
\]

(10)

respectively.

For \( n = 3, 4, \ldots, \infty \), it can be shown that \( \Omega_n \) is transitive with respect to \( G(\Omega_n) \), and the standard Euclidean inner product \( \langle \cdot, \cdot \rangle_E \) is indeed the inner product \( \langle \cdot, \cdot \rangle_{G(\Omega_n)} \) invariant with any \( T \in G(\Omega_n) \). In particular, (1) holds in these cases: \( u = \omega_{M} = (0, 0, 1) \). Moreover, we can see from (4)–(10) that \( V_+(\Omega_n) \) is self-dual with respect to \( \langle \cdot, \cdot \rangle_E \) when \( n \) is odd or \( \infty \), whereas \( V_+(\Omega_n) \) is no more than isomorphic to \( V^\text{int}_+(\cdot, \cdot_G(\Omega_n)) \) when \( n \) is even (in this case, \( V_+(\Omega_n) \) is called weakly self-dual [47, 48]). We note that the cases when \( n = 3 \) and \( n = \infty \) correspond to a classical trit system and a qubit system restricted to real coefficients (qubit system with only \( \sigma_x \) and \( \sigma_z \)) respectively.

3. Entropic uncertainty relations in a class of GPTs

In this section, we present our main results on two types of entropic uncertainty in a certain class of GPTs. While our results reproduce entropic uncertainty relations obtained in finite dimensional quantum theories, they indicate that similar relations hold also in a broader class of physical theories. We also demonstrate entropic uncertainty relations in the regular polygon theories as an illustration of our results.

3.1. Entropic PURs

In quantum theory, it is known that we cannot prepare a state on which individual measurements of position and momentum observables, for example, take simultaneously definite values [49].
This type of uncertainty and its quantifications are called \textit{preparation uncertainty} and \textit{PURs} respectively.

In order to give general descriptions of uncertainty in GPTs, the notion of ideal measurements has to be introduced. Considering that projection-valued measures (PVMs), whose effects are sums of rank-1 projections, give ideal measurements in finite dimensional quantum theories [2], we call a measurement \( \{ e_x \}_{x \in X} \) in some GPT ideal [30] if for any \( x \in X \) there exists a finite set of pure and indecomposable effects \( \{ e_{i_x}^{\text{ext}} \}_{i_x} \) such that

\[
e_x = \sum_{i_x} e_{i_x}^{\text{ext}} \quad \text{or} \quad e_x = u - \sum_{i_x} e_{i_x}^{\text{ext}}.
\]  

(11)

It is easy to check that measurements satisfying (11) are reduced to PVMs in finite dimensional quantum theories.

Let us consider a GPT with its state space \( \Omega \), and two ideal measurements \( A = \{ a_x \}_{x \in X} \) and \( B = \{ b_y \}_{y \in Y} \) on \( \Omega \). For the probability distribution \( \{ a_x(\omega) \}_{x} \) obtained in the measurement of \( A \) on a state \( \omega \in \Omega \) (and similarly for \( \{ b_y(\omega) \}_{y} \)), its Shannon entropy is defined as

\[
H(\{ a_x(\omega) \}_{x}) = -\sum_{x \in X} a_x(\omega) \log a_x(\omega).
\]  

(12)

Note that \( H(\{ a_x(\omega) \}_{x}) \geq 0 \) and \( H(\{ a_x(\omega) \}_{x}) = 0 \) if and only if \( \{ a_x(\omega) \}_{x} \) is definite, i.e. \( a_{x'}(\omega) = 1 \) for some \( x' \) and \( a_x(\omega) = 0 \) for \( x \neq x' \). If there exists a relation such as

\[
H(\{ a_x(\omega) \}_{x}) + H(\{ b_y(\omega) \}_{y}) \geq \Gamma_{AB} \quad \forall \omega \in \Omega
\]  

with a constant \( \Gamma_{AB} > 0 \), then it is called an entropic PUR because it demonstrates that we cannot prepare a state which makes simultaneously \( H(\{ a_x(\omega) \}_{x}) \) and \( H(\{ b_y(\omega) \}_{y}) \) vanish, or \( \{ a_x(\omega) \}_{x} \) and \( \{ b_y(\omega) \}_{y} \) definite. One way to obtain an entropic PUR is to consider the Landau–Pollak-type relation [38–40]:

\[
\max_{x \in X} a_x(\omega) + \max_{y \in Y} b_y(\omega) \leq \gamma_{AB} \quad \forall \omega \in \Omega
\]  

(13)

with a constant \( \gamma_{AB} \in (0, 2] \). Remark that relations of the form (13) always can be found for any pair of measurements. It is known [14, 50] that \( \max_{x \in X} a_x(\omega) \) is related with \( H(\{ a_x(\omega) \}_{x}) \) by

\[
\exp[-H(\{ a_x(\omega) \}_{x})] \leq \max_{x \in X} a_x(\omega),
\]

and thus we can observe from (13)

\[
\exp[-H(\{ a_x(\omega) \}_{x})] + \exp[-H(\{ b_y(\omega) \}_{y})] \leq \gamma_{AB}.
\]

Considering that

\[
\exp[-H(\{ a_x(\omega) \}_{x})] + \exp[-H(\{ b_y(\omega) \}_{y})] \geq 2 \exp\left[-\frac{H(\{ a_x(\omega) \}_{x}) - H(\{ b_y(\omega) \}_{y})}{2}\right]
\]

holds, we can finally obtain an entropic relation

\[
H(\{ a_x(\omega) \}_{x}) + H(\{ b_y(\omega) \}_{y}) \geq -2 \log \frac{\gamma_{AB}}{2} \quad \forall \omega \in \Omega.
\]  

(14)
If $\gamma_{AB} < 2$, then (14) gives an entropic PUR because it indicates that it is impossible to prepare a state which makes both $H\{\{a_i(\omega)\}\}$ and $H\{\{b_j(\omega)\}\}$ zero, that is, there is no state preparation on which $A$ and $B$ take simultaneously definite values (note that (13) also gives a PUR if $\gamma_{AB} < 2$). In a finite dimensional quantum theory with its state space $\Omega_{QT}$, it can be shown that

$$\max_i a_i(\omega) + \max_j b_j(\omega) \leq 1 + \max_{x,y}(|a_i|b_j)| \quad \forall \omega \in \Omega_{QT}, \quad (15)$$

where $A = \{\{a_i\}\{a_i\}\}$ and $B = \{\{b_j\}\{b_j\}\}$ are rank-1 PVMs. In that case, (14) can be rewritten as

$$H(\{a_i(\omega)\}) + H(\{b_j(\omega)\}) \geq 2 \log \frac{2}{1 + \max_{x,y}|a_i|b_j)| \quad \forall \omega \in \Omega_{QT}, \quad (16)$$

which is the entropic PUR proved by Deutsch [13]. There have been studies to find a better bound [14] or generalization [15] of (16).

**Remark 3.1.1.** Entropic PURs in quantum theory can be derived also by means of majorization [51–56]. This method of majorization can be also applied to GPTs. To see this, let us introduce probability vectors $a(\omega)$ and $b(\omega)$ defined simply through $\{\{a_i\}\{a_i\}\}$ and $\{\{b_j\}\{b_j\}\}$, shown above respectively. By adding outcomes to either $X$ or $Y$, we can assume without loss of generality that their cardinalities are equal: $|X| = |Y| = d$, and $a(\omega)$ and $b(\omega)$ are $d$-dimensional vectors. If $d$-dimensional probability vectors $p = (p_i)$ and $q = (q_i)$ satisfy

$$\sum_{j=1}^{k} p_j^i \leq \sum_{j=1}^{k} q_j^i \quad \forall k = 1, 2, \ldots, d,$$

where $p_j^i$'s are obtained thorough ordering the components of $p$ in decreasing order: $\{p_j^i\} = \{p_i\}$, and $p_j^i \geq p_k^i \geq p_l^i \geq \cdots$ (similarly for $q_j^i$'s), then $p$ is called majorized by $q$ and we write $p \prec q$. For $a(\omega)$ and $b(\omega)$, a relation of the form

$$a(\omega) \otimes b(\omega) \prec r \quad \forall \omega \in \Omega, \quad (17)$$

where $r = (r_i)$, is a $d^2$-dimensional probability vector defined below, was proved in [52]. The vector $r$ was given by

$$r = (R_1, R_2 - R_1, \ldots, R_d - R_{d-1}, 0, 0, \ldots, 0)$$

with

$$\begin{align*}
R_k &= \max_{\omega \in \Omega} \max_{(x,y) \in T_k} a(x,y) b(y(\omega)) \\
T_k &= \{(x_1, y_1), \ldots, (x_k, y_k) \mid (x_i, y_i) \in X \times Y, \ (x_i, y_i) \neq (x_j, y_j) \ \text{for} \ i \neq j\}
\end{align*}$$

(thus we can see $R_k = 1$ for $d \leq k \leq d^2$ because $A$ and $B$ are ideal). From (17), we can derive [51]

$$H(\{a_i(\omega)\}) + H(\{b_j(\omega)\}) \geq H(\{r_i\}) \quad \forall \omega \in \Omega, \quad (18)$$
which gives a similar entropic relation to (14). Note that when $A$ and $B$ are binary, the vector $r$ is completely determined by

$$R_1 = \max_{(x,y)} a_x(\omega) b_y(\omega).$$

In [52], $R_1$ was evaluated as

$$R_1 = \max_{(x,y)} a_x(\omega) b_y(\omega) \leq \frac{\gamma^2}{4}$$

with

$$\gamma = \max_{(x,y)} (a_x + b_y)(\omega),$$

and it was shown that in quantum theory the equality holds:

$$R_1 = \max_{(x,y)} a_x(\omega) b_y(\omega) = \frac{\gamma^2}{4}.$$

We will consider in subsection 3.3 similar cases when $R_1 = \frac{\gamma^2}{4}$ holds, and give concrete value of $\gamma$.

### 3.2. Entropic MURs

When considering two measurements, they are not always jointly measurable [28]. Their incompatibility is represented quantitatively by MURs in terms of measurement error, which describes the difference between the ideal, original measurement and their approximate joint measurement [26, 27].

Let $\Omega$ be a state space which is transitive and satisfies $V_+ = V_+^{\text{int}} + \langle \cdot, \cdot \rangle_G$, and we hereafter denote the inner product $\langle \cdot, \cdot \rangle_G$ simply by $\langle \cdot, \cdot \rangle$. Then, because of the self-duality of $V_+$, we can in the following identify effects with elements of $V_+$. There can be defined measurement error in terms of entropy in the identical way with the quantum one by Buscemi et al [16]. Let in the GPT $E = \{e_x\}_{x \in X}$ be an ideal measurement defined in (11) and $M = \{m_x\}_{x \in \hat{X}}$ be a measurement. Since it was demonstrated in [30] that

$$\langle e_x', e_x \rangle = \delta_{x'}$$

holds for all $x, x' \in X$, and

$$\omega_M = u = \sum_x e_x$$

$$= \sum_x \langle u, e_x \rangle \frac{e_x}{\langle u, e_x \rangle}$$

(see (1)) holds, the joint probability distribution

$$\{p(x, \hat{x})\}_{x, \hat{x}} = \{(e_x, m_{\hat{x}})\}_{x, \hat{x}} = \left\{ \frac{e_x}{\langle u, e_x \rangle}, m_{\hat{x}} \right\}_{x, \hat{x}}$$

(21)
is considered to be obtained in the measurement of $\mathcal{M}$ on the 'eigenstates' $\{e_x/\langle u, e_x \rangle\}_x$ of $E$ (see (19)) with the initial distribution
\[
\{p(x)\}_x = \{\langle u, e_x \rangle\}_x.
\] (22)

According to [16], the conditional entropy
\[
N(\mathcal{M}; E) := H(E | \mathcal{M}) = \sum_{\hat{x}} p(\hat{x}) H \left( \left\{ \frac{m_{\hat{x}}}{\langle u, m_{\hat{x}} \rangle} \right\}_x \right)
\]
\[= \sum_{\hat{x}} \langle u, m_{\hat{x}} \rangle H \left( \left\{ \frac{m_{\hat{x}}}{\langle u, m_{\hat{x}} \rangle} \right\}_x \right)
\] (23)
calculated via (21) describes how inaccurately the actual measurement $\mathcal{M}$ can estimate the input eigenstates of the ideal measurement $E$. In fact, if we consider measuring $\mathcal{M}$ on $e_x/\langle u, e_x \rangle$ and estimating the input state from the output probability distribution
\[
\{p(\hat{x}|x)\}_\hat{x} = \left\{ \frac{m_{\hat{x}}}{\langle u, m_{\hat{x}} \rangle} \right\}_\hat{x}
\]
by means of a guessing function $f : \hat{X} \rightarrow X$, then the error probability $p_{\text{error}}^f(x)$ is given by
\[
p_{\text{error}}^f(x) = 1 - \sum_{\hat{x} : f(\hat{x}) = x} p(\hat{x}|x) = \sum_{\hat{x} : f(\hat{x}) \neq x} p(\hat{x}|x).
\]
When similar procedures are conducted for all $x \in X$ with the probability distribution $\{p(x)\}_x$ in (22), the total error probability $p_{\text{error}}^f$ is
\[
p_{\text{error}}^f = \sum_x p(x) p_{\text{error}}^f(x) = \sum_{x \in X} \sum_{\hat{x} : f(\hat{x}) \neq x} p(x, \hat{x}),
\] (24)
and it was shown in [16] that
\[
\min_f p_{\text{error}}^f \rightarrow 0 \iff N(\mathcal{M}; E) = H(E | \mathcal{M}) \rightarrow 0.
\]

We can conclude from the consideration above that the entropic quantity (23) represents the difference between $E$ to be measured ideally and $\mathcal{M}$ measured actually, and thus we can define their entropic measurement error as (23).

We are now in the position to derive a similar entropic relation to [16] with the generalized entropic measurement error (23). We continue focusing on a GPT with its state space $\Omega$ being transitive and $V_+$ being self-dual with respect to the inner product $\langle \cdot, \cdot \rangle_G \equiv \langle \cdot, \cdot \rangle$, that is, $V_+ = V_+^{\text{int}}$. Let $A = \{a_x\}_{x \in X}$ and $B = \{b_y\}_{y \in Y}$ be a pair of ideal measurements defined in (11), and consider their approximate joint measurement $\mathcal{M} = \{m_{\hat{i}\hat{j}}\}_{\hat{i}, \hat{j} \in X \times Y}$ and its marginals
\[
\mathcal{M}^A = \{m_{\hat{i}}\}_{\hat{i} \in X} \quad \text{with} \quad m_{\hat{i}} = \sum_{y \in Y} m_{\hat{i}y}
\]
\[
\mathcal{M}^B = \{m_{\hat{y}}\}_{\hat{y} \in Y} \quad \text{with} \quad m_{\hat{y}} = \sum_{x \in X} m_{x\hat{y}}.
\]

We can prove the following theorem.

**Theorem 3.2.1.** Suppose that $\Omega$ is a transitive state space with its positive cone $V_+$ being self-dual with respect to $\langle \cdot, \cdot \rangle_G \equiv \langle \cdot, \cdot \rangle$, $A = \{a_x\}_x$ and $B = \{b_y\}_y$ are ideal measurements on
\(\Omega,\) and \(\mathcal{M}\) is an arbitrary approximate joint measurement of \((A, B)\) with its marginals \(\mathcal{M}^A\) and \(\mathcal{M}^B\). If there exists a relation
\[
H\left(\{a_\omega(\omega)\}_{\omega}\right) + H\left(\{b_\omega(\omega)\}_{\omega}\right) \geq \Gamma_{A,B} \quad \forall \omega \in \Omega
\]
with a constant \(\Gamma_{A,B}\), it also holds that
\[
N(\mathcal{M}^A; A) + N(\mathcal{M}^B; B) \geq \Gamma_{A,B}.
\]

**Proof.** Since for every \(\hat{x} \in X\) and \(\hat{y} \in Y\)
\(\omega_{xy} := m_{xy}/(u, m_{xy})\) is a state due to the self-duality, it holds that
\[
H\left(\{a_\omega(\omega)\}_{\omega}\right) + H\left(\{b_\omega(\omega)\}_{\omega}\right) \geq \Gamma_{A,B}
\]
for all \(\hat{x} \in X\) and \(\hat{y} \in Y\). Therefore, taking into consideration that \(\langle u, m_{xy} \rangle \geq 0\) for all \(\hat{x}, \hat{y}\) and \(\sum_{xy} (u, m_{xy}) = \langle u, u \rangle = \langle u, \omega_M \rangle = 1\), we have
\[
\sum_{i \in I} \sum_{j \in J} (u, m_{ij}) \left[H\left(\{a_i(\omega_{ij})\}_{\omega}\right) + H\left(\{b_j(\omega_{ij})\}_{\omega}\right)\right] \geq \Gamma_{A,B},
\]
or equivalently (see (23))
\[
H(A|M) + H(B|M) \geq \Gamma_{A,B}.
\] (25)

Note that the conditional entropy \(H(A|M)\) is obtained through a joint probability distribution \(\{p(x, \hat{x}, \hat{y})\}_{x, \hat{x}, \hat{y}} := \{a_i, m_{ij}\}\), and we can also obtain \(H(A|\mathcal{M}^A)\) from its marginal distribution \(\{p(x, \hat{x})\}_{x, \hat{x}} = \{a_i, m_{ij}\}\). The quantity
\[
H(A|\mathcal{M}^A) - H(A|M)
\]
defined from those two conditional entropies is called the (classical) conditional mutual information, and it is known [57] to be nonnegative:
\[
H(A|\mathcal{M}^A) - H(A|M) \geq 0.
\]

A similar relation holds also for \(H(B|M)\) and \(H(B|\mathcal{M}^B)\), and thus, together with (25), we can conclude that
\[
H(A|\mathcal{M}^A) + H(B|\mathcal{M}^B) \geq \Gamma_{A,B}
\]
holds, which proves the theorem.

Theorem 3.2.1 is a generalization of the quantum result [16] to a class of GPTs. In fact, when we consider a finite dimensional quantum theory and a pair of rank-1 PVMs \(A = \{|a_i\rangle\langle a_i|\}_{i} \) and \(B = \{|b_j\rangle\langle b_j|\}_{j} \), our theorem results in the one in [16] with the quantum bound \(\Gamma_{A,B} = -2 \log \max_{i,j} |\langle a_i|b_j \rangle|\) by Maassen and Uffink [14]. Theorem 3.2.1 demonstrates that if there is an entropic PUR, i.e. \(\Gamma_{A,B} > 0\), then there is also an entropic MUR which shows that we cannot make both \(N(\mathcal{M}^A; A)\) and \(N(\mathcal{M}^B; B)\) vanish.

**Remark 3.2.2.** There is another type of entropic uncertainty relation on successive measurements in quantum theory [58–61]. With a suitable introduction of transformations associated with ideal measurements, we can derive similar entropic relations also in GPTs considered
above. For an ideal measurement \( E = \{e_i\}_{i \in X} \), we define a (Schrödinger) channel \([62]\) \( \Phi_E \) giving the post-measurement states as

\[
\Phi_E : \Omega \to \Omega : \omega \mapsto \sum_x \langle e_x, \omega \rangle e_x / \langle \omega, e_x \rangle
\]

(26)

in analogy with the channel associated with a rank-1 projective measurement (Lüders measurement \([2]\) for a rank-1 PVM) in quantum theory (remember \([19]\)). Note that this channel (called the ‘measure-and-prepare channel’) is shown to be completely positive for any physically reasonable tensor product \([62]\). In the Heisenberg picture, it becomes

\[
\Phi_E^* : \mathcal{E} \to \mathcal{E} : e \mapsto \sum_x \langle e_x, ax \rangle e_x.
\]

(27)

Let \( A = \{a_i\}_i \) and \( B = \{b_i\}_i \) be ideal measurements associated with the channel defined in \((26)\) (or \((27)\)). It is easy to see that

\[
H(\{a_i(\omega)\}_i) + H(\{b_i(\omega)\}_i) \geq \Gamma_A^B \quad \forall \omega \in \Omega
\]

with

\[
\Gamma_A^B := \inf_{\Phi_A} [H(\{a_i(\omega)\}_i) + H(\{b_i(\Phi_A(\omega))\}_i)]
\]

holds. We consider measuring successively \( A \) and \( B \) on a state \( \omega \): measuring \( A \) first, and then \( B \). The observed statistics are \( \{a_i(\omega)\}_i \) and \( \{b_i(\Phi_A(\omega))\}_i \), and we can derive

\[
H(\{a_i(\omega)\}_i) + H(\{b_i(\Phi_A(\omega))\}_i) \geq \Gamma_A^B
\]

(28)

with

\[
\Gamma_A^B := \inf_{\Phi_A} [H(\{a_i(\omega)\}_i) + H(\{b_i(\Phi_A(\omega))\}_i)]
\]

(29)

\[
= \inf_{\Phi_A} [H(\{a_i(\Phi_A(\omega))\}_i) + H(\{b_i(\Phi_A(\omega))\}_i)]
\]

(30)

because \( a_i(\omega) = a_i(\Phi_A(\omega)) \). We can see that \( \Gamma_A^B \geq \Gamma_A^B \) holds, and thus there is more uncertainty in the successive measurement than the individual measurements of \( A \) and \( B \). The entropic relation \((28)\) together with \((29)\) can be considered as a generalization of the quantum result \([58]\). Note that similarly to \([58]\) we can present another bound for \((28)\) in terms of the joint entropy. In fact, considering that \( \{a_i\}_i \) and \( \{\Phi_A(\omega)\}_i \) are jointly measurable \( \left\{ \left\langle b_s, \frac{\omega}{\langle \omega, a_i \rangle} / a_i \right\rangle \right\}_{i,i} \) is the joint measurement), that is, the probability distributions \( \{a_i(\omega)\}_i \) and \( \{b_i(\Phi_A(\omega))\}_i \) are obtained from the joint distribution \( \left\{ \left\langle b_s, \frac{\omega}{\langle \omega, a_i \rangle} / a_i \right\rangle \right\}_{i,i} \), it can be shown \([57]\) that

\[
H(\{a_i(\omega)\}_i) + H(\{b_i(\Phi_A(\omega))\}_i) \geq H(\left\{ \left\langle b_s, \frac{\omega}{\langle \omega, a_i \rangle} / a_i \right\rangle \right\}_{i,i})
\]

It is easy to see that the right-hand side is also greater than or equal to \( \Gamma_A^B \).
3.3. Examples: entropic uncertainty in regular polygon theories

In this part, we restrict ourselves to the regular polygon theories. Although the self-duality with respect to $\langle \cdot, \cdot \rangle_G$ holds only for regular polygons with odd sides, we can introduce the entropic measurement error (23) in the same way and prove the same theorem also for even-sided regular polygon theories. To see this, we modify our parametrization for an even-sided polygon state space in (4) and (5) as

$$\Omega_n^{\text{ext}} = \{\omega_n^{\text{ext}}(i)\}_{i=0}^{n-1} \quad \text{with} \quad \omega_n^{\text{ext}}(i) = \left( \frac{r_n^2 \cos \frac{2\pi i}{n}}{1}, \frac{r_n^2 \sin \frac{2\pi i}{n}}{1} \right).$$

(31)

It follows that the indecomposable and pure effects in (6) are modified as

$$e_n^{\text{ext}}(i) = \frac{1}{2} \left( \cos \left( \frac{(2i-1)\pi}{n} \right), \sin \left( \frac{(2i-1)\pi}{n} \right) \right).$$

(32)

Note that this modification does not change the physical predictions of the theory because it is a bijective linear transformation. Under this parametrization, we can observe easily that $V_+^{\text{int}}$ generated by (31) includes $V_+^{\text{int}, \omega}$ generated by (32); $V_+^{\text{int}, \omega} \subset V_+$. Thus, any effect can be regarded as an element of $V_+$. Moreover, we can show that any ideal measurement is of the form $\{e_n^{\text{ext}}(i), e_n^{\text{ext}}(i + \frac{n}{2})\}$ for some $i$, and satisfies a similar relation to (19), that is,

$$\langle e_n^{\text{ext}}(i), e_n^{\text{ext}}(i) \rangle_{(u, e_n^{\text{ext}}(i))} = 1,$$

and

$$\langle e_n^{\text{ext}}(i), e_n^{\text{ext}}(i + \frac{n}{2}) \rangle_{(u, e_n^{\text{ext}}(i + \frac{n}{2}))} = 0,$$

(see theorem III.3 in [30] for details). Therefore, entropic measurement error can be defined also in even-sided polygon theories, and we can obtain a similar result to theorem 3.2.1 by repeating the same proof. We restate this fact as another theorem.

**Theorem 3.3.1.** Theorem 3.2.1 also holds for the regular polygon theories. That is, for a regular polygon theory with its state space $\Omega_n$ ($n = 3, 4, \ldots, \infty$), ideal measurements $A = \{a_i\}_x$ and $B = \{b_i\}_y$ on $\Omega_n$, and an arbitrary approximate joint measurement $M$ of $(A, B)$ with its marginals $M^A$ and $M^B$, if there exists a relation

$$H(\{a_i(\omega)\}_x) + H(\{b_i(\omega)\}_y) \geq \Gamma_{A,B}(n) \quad \forall \omega \in \Omega_n,$$

then

$$N(M^A; A) + N(M^B; B) \geq \Gamma_{A,B}(n).$$

In the following, we shall give a concrete value of $\Gamma_{A,B}(n)$ in theorem 3.3.1 in the way introduced in subsection 3.1.
Let us focus on the state space $\Omega_n$. Any nontrivial ideal measurement is of the form 
$\{e_n^\text{ext}(i), u - e_n^\text{ext}(i)\}$ (see (7) and (10)). Note that although $\{e_n^\text{ext}(i)\}_{i=0,1,2}$ is also an ideal measurement when $n = 3$ (a classical trit system), we focus only on ideal measurements with two outcomes in this subsection. Thus, if we consider a pair of ideal measurements $A$ and $B$, then we can suppose that they are binary: $A = A' \equiv \{a'_0, a'_1\}$ and $B = B' \equiv \{b'_0, b'_1\}$ with $a'_0 = e_n^\text{ext}(i)$ and $b'_0 = e_n^\text{ext}(j)$ for $i, j \in \{0, 1, \ldots, n - 1\}$ (or $i, j \in [0, 2\pi)$ when $n = \infty$). On the other hand, it holds that
\[
\max_{x=0,1} a'_i(\omega) + \max_{y=0,1} b'_j(\omega) \leq \sup_{\omega \in (\Omega_n, \langle x, y \rangle) \subset [0,1]^2} [(a'_i + b'_j)(\omega)] = \max_{\omega \in (\Omega_n^\text{ext}, \langle x, y \rangle) \subset [0,1]^2} [(a'_i + b'_j)(\omega)]
\]
(33)
because $\Omega_n$ is a compact set and any state can be represented as a convex combination of pure states. Therefore, if we let $\omega_n^\text{ext}(k)$ be a pure state (5) and (10)), then the value
\[
\gamma_n^{n,A,B} := \max_{k} \max_{\langle x, y \rangle \subset [0,1]^2} [(a'_i + b'_j)(\omega_n^\text{ext}(k))]
\]
gives a Landau–Pollak-type relation
\[
\max_{x=0,1} a'_i(\omega) + \max_{y=0,1} b'_j(\omega) \leq \gamma_n^{n,A,B} \quad \forall \omega \in \Omega_n,
\]
(35)
to derive entropic relations
\[
H(\{a_i(\omega)\}_x) + H(\{b_i(\omega)\}_y) \geq -2 \log \gamma_n^{n,A,B} \quad \forall \omega \in \Omega_n,
\]
(36)
and
\[
N(M^A; A) + N(M^B; B) \geq -2 \log \gamma_n^{n,A,B}.\]
(37)

Tables 1–3 show the value of $(a'_i + b'_j)(\omega_n^\text{ext}(k))$ in terms of the angles $\theta_i, \theta_j$, and $\phi_k$ between the $x$-axis and the effects $a'_0 = e_n^\text{ext}(i)$, $b'_0 = e_n^\text{ext}(j)$, and the state $\omega_n^\text{ext}(k)$ respectively when viewed from the $z$-axis (see (4)–(10)). Maximizing the values in those tables over all pure states, we can obtain the optimal bound $\gamma_n^{n,A,B}$ in (34) for each regular polygon theory. Note that focusing only on the case $j = 0$ and $0 < i < \frac{\pi}{n}$ when $n = \infty$ is sufficient for the universal description of $\gamma_n^{n,A,B}$, due to the geometric symmetry of the regular polygon theories. $\gamma_n^{n,A,0}$ for the regular polygon theory with $n < \infty$ sides is exhibited in tables 4 and 5, and $\gamma_n^{\infty,A,0}$ for the disc theory (the regular polygon theory with $n = \infty$ sides) can be calculated from table 3 as
\[
\gamma_n^{\infty,A,0} = \max \left\{ 1 + \cos \frac{\theta'_i}{2}, 1 + \sin \frac{\theta'_i}{2} \right\},
\]
(38)
where $\theta'_i = \theta_i - \theta_0 = \theta_i$ similarly to tables 4 and 5. (38) can be regarded as giving the quantum bound in (15) for a qubit system in terms of the usual Bloch representation. Note that when $n$ is even or odd, due to the geometric symmetry, $(a'_i + b'_j)(\omega_n^\text{ext}(k))$ takes its maximum when $\omega_n^\text{ext}(k)$ lies just ‘halfway’ between the effects $a'_0$ and $b'_0$, that is, $a'_i(\omega) = b'_j(\omega)$ and thus $a'_i(\omega)b'_j(\omega) = \frac{1}{2}(a'_i(\omega) + b'_j(\omega))^2$ holds (see remark 3.1.1), while this does not hold generally when $n$ is odd. From tables 4, 5 and (38), we can obtain the corresponding entropic inequalities (36) (also (18)) and (37) for an arbitrary regular polygon theory. The value $\gamma_n^{n,A,0}$ can be used also to evaluate the nonlocality of the theory via its degree of incompatibility [30].
Remark 3.3.2. With the angle \( \theta'_i \) fixed, we can see from tables 4, 5 and (38) that \( \gamma^n_{A_i, \theta'} \geq \gamma^n_{A_i, \theta'} \) holds for all \( n \). In fact, if we assume, for example, \( n \) is odd and \( i \) is even, then

\[
\gamma^n_{A_i, \theta'} = \max \left\{ \frac{2\pi^2}{1 + r_n^2} + \frac{2}{1 + r_n^2} \cos \frac{\theta'_i}{2}, 1 + \frac{1}{\cos \frac{\theta'}{2}} \sin \frac{\theta'_i}{2} \right\}
\]
Table 5. The value $\gamma_{n}^{\alpha, B_{0}}$ when $n$ is odd.

- $i$: even $\max\left\{ \frac{2d_{i}}{1+r_{n}^{2}}, \frac{2}{1+r_{n}^{2}} \cos \frac{\theta_{i}^{'}}{2}, 1 + \cos \frac{\theta_{i}^{'}}{2} \sin \frac{\theta_{i}^{'}}{2}\right\}$
- $i$: odd $\max\left\{ \frac{2d_{i}}{1+r_{n}^{2}}, \frac{2r_{n}^{2}}{1+r_{n}^{2}} \cos \frac{\theta_{i}^{'}}{2}, 1 + \cos \frac{\theta_{i}^{'}}{2} \sin \frac{\theta_{i}^{'}}{2}\right\}$

\[\theta_{i}^{' \pi} = \frac{2i}{n} = \theta_{i} - \theta_{0} = \theta_{i}\]

Figure 1. The optimal bound $\gamma_{2\pi/3}^{3m}$ for the Landau–Pollak-type inequality on a pair of observable $(A_{m}^{\alpha}, B_{0})$ in the regular polygon theory with $n = 3m$.

(see table 5), and it can be easily shown that

\[\frac{2r_{n}^{2}}{1+r_{n}^{2}} + \frac{2}{1+r_{n}^{2}} \cos \frac{\theta_{i}^{'}}{2} \geq 1 + \cos \frac{\theta_{i}^{'}}{2},\]

\[1 + \frac{1}{\cos \frac{\theta_{i}^{'}}{2}} \geq 1 + \sin \frac{\theta_{i}^{'}}{2}\]

hold for $0 < i < \frac{n}{2}$ (or $0 < \theta_{i}^{' \pi} < \frac{\pi}{2}$). Thus, we can conclude $\gamma_{n}^{A_{m}, B_{0}} \geq \gamma_{n}^{\infty, B_{0}}$. To see this in a more explicit way, let us consider, as an illustration, regular polygon theories with $n = 3m$ ($m = 1, 2, \ldots$), and let the angle $\theta_{i}^{' \pi}$ be $\theta_{i}^{' \pi} = \frac{2i}{3} \pi$ (i.e. $i = m$). We can calculate the corresponding optimal bound $\gamma_{2\pi/3}^{3m} = \gamma_{2\pi/3}^{3m}$ for any $m$ from tables 4, 5 and (38), and describe its behavior as a function of $m$ in figure 1. There can be observed that theories with $m = 1, 2$ ($n = 3, 6$) admit $\gamma_{2\pi/3}^{3m} = 2$, that is, there is a state on which both $A_{1} = A_{m}^{\alpha}$ and $B_{0}$ take simultaneously exact values when $m = 1, 2$. It exhibits that when $m \geq 3$, there exists preparation uncertainty for this $(A_{m}^{\alpha}, B_{0})$. Hence, it follows from our theorems that there also exists measurement uncertainty for $(A_{1}^{\alpha}, B_{0})$, and their entropic representations (entropic PUR and MUR) are given by similar inequalities with the same bound. Also, it can be observed that $\gamma_{2\pi/3}^{3m} \geq \gamma_{2\pi/3}^{\infty} = 1 + \frac{\sqrt{3}}{2}$ holds for all $m$, which has been shown in the argument above. Note that we can derive easily an observable-independent relation

\[\min_{i} \gamma_{n}^{A_{m}, B_{0}} \geq \min_{i} \gamma_{n}^{\infty, B_{0}}.\]
In other words, the disc theory shows the ‘maximum uncertainty’ in terms of the Landau–Pollak-type formulation.

4. Conclusion and discussion

Overall, we examined entropic PURs and MURs in GPTs with transitivity and self-duality with respect to a specific inner product and in the regular polygon theories. We proved similar entropic relations to PURs and MURs in quantum theory also in the GPTs with the Landau–Pollak-type relation and the entropic measurement error generalized respectively. It manifests that the entropic behaviors of two kinds of uncertainty in quantum theory are also observed in a broader class of physical theories, and thus they are more universal phenomena. It is easy to obtain similar results if more than two measurements are considered. We also gave concrete calculations of our results in the regular polygon theories.

The resulting theorems (theorems 3.2.1 and 3.3.1) can be considered as entropic expressions of the ones in [30]. Our theorems demonstrate in an entropic way that MURs are indicated by PURs and both of them can be evaluated by the same bound. We note similarly to [30] that while the quantum results in [16] were based on the ‘ricochet’ property of maximally entangled states, our theorems were obtained without considering entanglement or even composite systems. It may be indicated that some of the characteristics of quantum theory can be obtained without entanglement.

Although there are researches suggesting that our assumptions on theories are satisfied in the presence of several ‘physical’ requirements [36, 45, 63], future study will need to investigate whether our theorems still hold in GPTs with weakened assumptions. On the other hand, we have proved that under the Landau–Pollak-type formulation the disc theory shows the ‘maximum uncertainty’ among the regular polygon theories (see remark 3.3.2). It would be interesting to investigate whether similar results can be proved under the entropic or other formulations, and find a characterization of quantum theory in terms of ‘the amount of uncertainty’. To give better bounds to our entropic inequalities, which itself is another future work, may contribute to this problem. Concerning the regular polygon theories, it would be possible to obtain optimal entropic bounds similarly to [64]. To find information-theoretic applications of our results is also left for future work.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary information files).

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