Anti-self-dual instantons with Lagrangian boundary conditions

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March 29, 2022

Abstract

We study a nonlocal boundary value problem for anti-self-dual instantons on 4-manifolds with a space-time splitting of the boundary. The model case is \( \mathbb{R} \times Y \), where \( Y \) is a compact oriented 3-manifold with boundary \( \Sigma \). The restriction of the instanton to each time slice \( \{t\} \times \Sigma \) is required to lie in a fixed (singular) Lagrangian submanifold of the moduli space of flat connections over \( \Sigma \). We establish the basic regularity and compactness properties (assuming \( L^p \)-bounds on the curvature for \( p > 2 \)) as well as the Fredholm theory in a compact model case. The motivation for studying this boundary value problem lies in the construction of an instanton Floer homology for 3-manifolds with boundary. The present paper is part of a program proposed by Salamon for the proof of the Atiyah-Floer conjecture for homology-3-spheres.

1 Introduction

Let \( X \) be a manifold with boundary, let \( G \) be a compact Lie group, and consider a principal \( G \)-bundle \( P \to M \). The natural boundary condition for the Yang-Mills equation \( d^*_A F_A = 0 \) on \( P \) is \( *F_A|_{\partial X} = 0 \). For this boundary value problem there are regularity and compactness results, see for example [U1, U2, W1]. Every solution is gauge equivalent to a smooth solution, and Uhlenbeck compactness holds: Every sequence of solutions with \( L^p \)-bounded curvature (where \( 2p > \text{dim} \, X \)) is gauge equivalent to a sequence that contains a \( C^\infty \)-convergent subsequence. On an oriented 4-manifold, the anti-self-dual instantons, i.e. connections satisfying \( F_A + *F_A = 0 \), are special first order solutions of the Yang-Mills equation. An important application of Uhlenbeck’s theorem is the compactification of the moduli space of anti-self-dual instantons over a manifold without boundary leading to the Donaldson invariants of smooth 4-manifolds [D1] and to the instanton Floer homology groups of 3-manifolds [Fl].

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On a 4-manifold with boundary the boundary condition $\star F_A|_{\partial X} = 0$ for anti-self-dual instantons implies that the curvature vanishes altogether at the boundary. This is an overdetermined boundary value problem comparable to Dirichlet boundary conditions for holomorphic maps. As in the latter case it is natural to consider weaker Lagrangian boundary conditions. The Cauchy-Riemann equation becomes elliptic when augmented with Lagrangian or more generally totally real boundary conditions. We consider a version of such Lagrangian boundary conditions for anti-self-dual instantons on a 4-manifold with a space-time splitting of the boundary, and prove that they suffice to obtain the analogue of the above mentioned regularity and compactness results for Yang-Mills connections.

More precisely, we consider oriented 4-manifolds $X$ such that each connected component of the boundary $\partial X$ is diffeomorphic to $\mathcal{S} \times \Sigma$, where $\mathcal{S}$ is a 1-manifold and $\Sigma$ is a closed Riemann surface. We shall study a boundary value problem associated to a gauge invariant Lagrangian submanifold $L$ of the space of flat connections on $\Sigma$: The restriction of the anti-self-dual instanton to each time-slice of the boundary is required to belong to $L$. This boundary condition arises naturally from examining the Chern-Simons functional on a 3-manifold $Y$ with boundary $\Sigma$. Namely, the Langrangian boundary condition renders the Chern-Simons 1-form on the space of connections closed, see [S]. The resulting gradient flow equation leads to the boundary value problem studied in this paper (for the case $X = \mathbb{R} \times Y$). Our main results establish the basic regularity and compactness properties as well as the Fredholm theory, the latter for the compact model case $X = S^1 \times Y$.

Boundary value problems for Yang-Mills connections were already considered by Donaldson in [D2]. He studies the Hermitian Yang-Mills equation for connections induced by Hermitian metrics on holomorphic bundles over a compact Kähler manifold $Z$ with boundary. Here the unique solubility of the Dirichlet problem (prescribing the metric over the boundary) leads to an identification between framed holomorphic bundles over $Z$ (meaning a holomorphic bundle with a fixed trivialization over $\partial Z$) and Hermitian Yang-Mills connections over $Z$. In particular, when $Z$ has complex dimension 1 and boundary $\partial Z = S^1$, this links loop groups to moduli spaces of flat connections over $Z$. This observation suggests an alternative approach to Atiyah’s [A1] correspondence between holomorphic curves in the loop group of a compact Lie group $G$ and anti-self-dual instantons on $G$-bundles over the 4-sphere. The correspondence might be established via an adiabatic limit relating holomorphic spheres in the moduli space of flat connections over the disc to anti-self-dual instantons over the product (of sphere and disc). Our motivation for studying the present boundary value problem lies more in the direction of another such correspondence between holomorphic curves in moduli spaces of flat connections and anti-self-dual instantons – the Atiyah-Floer conjecture for Heegard splittings of a homology-3-sphere.

A Heegard splitting $Y = Y_0 \cup Y_1$ of a homology 3-sphere $Y$ into two handlebodies $Y_0$ and $Y_1$ with common boundary $\Sigma$ gives rise to two Floer homologies (i.e. generalized Morse homologies) as follows: Firstly, the moduli space $M_\Sigma$ of gauge equivalence classes of flat connections on the trivial $SU(2)$-bundle over $\Sigma$
is a symplectic manifold (with singularities) and the moduli spaces $L_Y$, of flat connections over $\Sigma$ that extend to $Y_i$ are Lagrangian submanifolds of $M_\Sigma$ as explained in [W2]. The symplectic Floer homology $HF^{\text{symp}}_\ast(M_\Sigma, L_{Y_0}, L_{Y_1})$ is now generated by the intersection points of the Lagrangian submanifolds, and the generalized connecting orbits (that define the boundary operator) are pseudoholomorphic strips with boundary values in the two Lagrangian submanifolds.

It was conjectured by Atiyah [A2] and Floer that this should be isomorphic to the instanton Floer homology $HF^{\text{inst}}_\ast(Y)$. For the latter, the critical points are the flat SU(2)-connections over $Y$. These are the actual critical points of the Chern-Simons functional, and the connecting orbits are given by its generalized flow lines, i.e. anti-self-dual instantons on $\mathbb{R} \times Y$.

The program by Salamon [S] for the proof of this conjecture is to define the instanton Floer homology $HF^{\text{inst}}_\ast(Y, L)$ for 3-manifolds with boundary $\partial Y = \Sigma$ using boundary conditions associated with a Lagrangian submanifold $L \subset M_\Sigma$. Then the conjectured isomorphism might be established in two steps via the intermediate $HF^{\text{inst}}_\ast([0, 1] \times \Sigma, L_{Y_0} \times L_{Y_1})$.

Fukaya was the first to suggest the use of Lagrangian boundary conditions in order to define a Floer homology for 3-manifolds $Y$ with boundary, [Fu]. He studies a slightly different equation, involving a degeneration of the metric in the anti-self-duality equation, and uses SO(3)-bundles that are nontrivial over the boundary $\partial Y$. Now there are interesting examples, where one has to work with the trivial bundle. For example, on a handlebody $Y$ there exists no nontrivial G-bundle for connected G. So if one considers any 3-manifold $Y$ with the Lagrangian submanifold $L_Y$, the space of flat connections on $\partial Y = \partial Y'$ that extend over a handlebody $Y'$, then one also deals with the trivial bundle. Consequently, if one wants to use Floer homology on 3-manifolds with boundary to prove the Atiyah-Floer conjecture, it is crucial to extend this construction to the case of trivial SU(2)-bundles. There are two approaches that suggest themselves for such a generalization. One would be the attempt to extend Fukaya’s construction to the case of trivial bundles, and another would be to follow the alternative construction outlined in [S]. The present paper follows the second route and sets up the basic analysis for this theory. We will only consider trivial G-bundles. However, our main theorems A, B, and C below generalize directly to nontrivial bundles – just the notation becomes more cumbersome.

The main theorems are described below; they are proven in sections 2 and 3. The appendix reviews the regularity theory for the Neumann and Dirichlet problem in the weak formulation that will be needed throughout this paper. Here we moreover introduce a technical tool for extracting regularity results for single components of a 1-form from weak equations that are related to a combination of Neumann and Dirichlet problems.
The main results

Throughout this paper, $P$ is the trivial $G$-bundle over a 4-manifold $X$. So a connection on $P$ is a 1-form $A \in \Omega^1(X, g)$ with values in the Lie algebra $g$. Its curvature is given by $F_A = dA + \frac{1}{2}[A \wedge A]$. For more details on gauge theory and the notation used in this paper see [W2] or [W1].

We consider the following class of Riemannian 4-manifolds. Here and throughout all Riemann surfaces are closed 2-dimensional manifolds. Moreover, unless otherwise mentioned, all manifolds are allowed to have a smooth boundary.

Definition 1.1 A 4-manifold with a boundary space-time splitting is a pair $(X, \tau)$ with the following properties:

(i) $X$ is an oriented 4-manifold which can be exhausted by a nested sequence of compact deformation retracts.

(ii) $\tau = (\tau_1, \ldots, \tau_n)$ is an $n$-tuple of embeddings $\tau_i : S_i \times \Sigma_i \to X$ with disjoint images, where $\Sigma_i$ is a Riemann surface and $S_i$ is either an open interval in $\mathbb{R}$ or is equal to $S^1 = \mathbb{R}/\mathbb{Z}$.

(iii) The boundary $\partial X$ is the union

$$\partial X = \bigcup_{i=1}^n \tau_i(S_i \times \Sigma_i).$$

Definition 1.2 Let $(X, \tau)$ be a 4-manifold with a boundary space-time splitting. A Riemannian metric $g$ on $X$ is called compatible with $\tau$ if for each $i = 1, \ldots, n$ there exists a neighbourhood $U_i \subset S_i \times [0, \infty)$ of $S_i \times \{0\}$ and an extension of $\tau_i$ to an embedding $\bar{\tau}_i : U_i \times \Sigma_i \to X$ such that

$$\bar{\tau}_i^* g = ds^2 + dt^2 + g_{s,t}.$$ 

Here $g_{s,t}$ is a smooth family of metrics on $\Sigma_i$ and we denote by $s$ the coordinate on $S_i$ and by $t$ the coordinate on $[0, \infty)$.

We call a triple $(X, \tau, g)$ with these properties a Riemannian 4-manifold with a boundary space-time splitting.

Remark 1.3 In definition 1.2 the extended embeddings $\bar{\tau}_i$ are uniquely determined by the metric as follows. The restriction $\bar{\tau}_i|_{t=0} = \tau_i$ to the boundary is prescribed, and the paths $t \mapsto \bar{\tau}_i(s, t, z)$ are normal geodesics.

Example 1.4 Let $X := \mathbb{R} \times Y$, where $Y$ is a compact oriented 3-manifold with boundary $\partial Y = \Sigma$, and let $\tau : \mathbb{R} \times \Sigma \to X$ be the obvious inclusion. Given any two metrics $g_-$ and $g_+$ on $Y$ there exists a metric $g$ on $X$ such that $g = ds^2 + g_-$ for $s \leq -1$, $g = ds^2 + g_+$ for $s \geq 1$, and $(X, \tau, g)$ satisfies the conditions of definition 1.2. The metric $g$ cannot necessarily be chosen in the form $ds^2 + g_s$ (one has to homotop the embeddings and the metrics).
Now let \((X, \tau, g)\) be a Riemannian 4-manifold with a boundary space-time splitting and consider a trivial \(G\)-bundle over \(X\) for a compact Lie group \(G\). The Sobolev spaces of connections and gauge transformations are denoted by

\[
A^{k,p}(X) = W^{k,p}(X, T^* X \otimes \mathfrak{g}), \\
G^{k,p}(X) = W^{k,p}(X, G).
\]

Let \(p > 2\), then for each \(i = 1, \ldots, n\) the Banach space of connections \(A^{0,p}(\Sigma_i)\) carries the symplectic form \(\omega(\alpha, \beta) = \int_{\Sigma_i} \langle \alpha \wedge \beta \rangle\). Note that the Hodge \(*\) operator for any metric on \(\Sigma_i\) is an \(\omega\)-compatible complex structure on \(A^{0,p}(\Sigma_i)\) since \(\ast \ast = -\text{id}\) and \(\omega(\cdot, \ast \cdot)\) defines a positive definite inner product – the \(L^2\)-metric. We call a submanifold \(L \subset A^{0,p}(\Sigma_i)\) Lagrangian if it is isotropic, i.e. \(\omega|_L \equiv 0\), and if \(T_A L\) is maximal for all \(A \in L\) in the following sense: If \(\alpha \in A^{0,p}(\Sigma_i)\) satisfies \(\omega(\alpha, T_A L) = \{0\}\), then \(\alpha \in T_A L\).

We fix an \(n\)-tuple \(L = (L_1, \ldots, L_n)\) of Lagrangian submanifolds \(L_i \subset A^{0,p}(\Sigma_i)\) that are contained in the space of flat connections and that are gauge invariant,

\[
L_i \subset A^{0,p}_{\text{flat}}(\Sigma_i) \quad \text{and} \quad u^* L_i = L_i \quad \forall u \in G^{1,p}(\Sigma_i).
\]

Here \(A^{0,p}_{\text{flat}}(\Sigma_i)\) is the space of weakly flat \(L^p\)-connections on \(\Sigma_i\) as introduced in [W2]. It is shown in [W2, Lemma 4.2] that the assumptions on the \(L_i\) imply that they are totally real with respect to the Hodge \(*\) operator for any metric on \(\Sigma_i\), i.e. for all \(A \in L_i\) one has the topological sum

\[
A^{0,p}(\Sigma_i) = T_A L_i \oplus \ast T_A L_i.
\]

Now we consider the following boundary value problem for connections \(A \in A^{1,p}_{\text{loc}}(X)\)

\[
\begin{align*}
*F_A + F_A &= 0, \\
\tau^* A|_{\tau^{-1} \Sigma_i} &= L_i \quad \forall \sigma \in S_i, \ i = 1, \ldots, n. \tag{1}
\end{align*}
\]

Observe that the boundary condition is meaningful since for every neighbourhood \(U \times \Sigma\) of a boundary component one has the continuous embedding \(W^{1,p}(U \times \Sigma) \subset W^{1,p}(U, L^p(\Sigma)) = C^0(U, L^p(\Sigma)).\) The first nontrivial observation is that every connection in \(L_i\) is gauge equivalent to a smooth connection on \(\Sigma_i\) and hence \(L_i \cap A(\Sigma)\) is dense in \(L_i\), as shown in [W2, Theorem 3.1]. Moreover, the \(L_i\) are modelled on \(L^p\)-spaces, and every \(W^{1,p}_{\text{loc}}\)-connection on \(X\) satisfying the boundary condition in (1) can be locally approximated by smooth connections satisfying the same boundary condition, see [W2, Corollary 4.4, 4.5].

Note that the present boundary value problem is a first order equation with first order boundary conditions (flatness in each time-slice). Moreover, the boundary conditions contain some crucial nonlocal (i.e. Lagrangian) information. We moreover emphasize that while \(L_i\) is a smooth Banach submanifold of \(A^{0,p}(\Sigma_i)\), the quotient \(L_i/G^{1,p}(\Sigma_i)\) is not required to be a smooth submanifold of the moduli space \(M_{\Sigma_i} := A^{0,p}_{\text{flat}}(\Sigma_i)/G^{1,p}(\Sigma_i)\), which itself might be singular.

\[\text{1The subscript \text{loc} indicates that the regularity only holds on all compact subsets of the noncompact domain of definition.}\]
For example, \( \mathcal{L}_i \) could be the set of flat connections on \( \Sigma_i \) that extend to flat connections over a handlebody with boundary \( \Sigma_i \), as introduced in [W2, Lemma 4.6]. To overcome the difficulties arising from the singularities in the quotient, we work with the (smooth) quotient by the based gauge group.

The following two theorems are the main regularity and compactness results for the solutions of (1) generalizing the regularity theorem and the Uhlenbeck compactness for Yang-Mills connections on 4-manifolds without boundary. They will be proven in section 2.

**Theorem A (Regularity)**

Let \( p > 2 \). Then every solution \( A \in A^{1,p}_{\text{loc}}(X) \) of (1) is gauge equivalent to a smooth solution, i.e. there exists a gauge transformation \( u \in G^{2,p}_{\text{loc}}(X) \) such that \( u^*A \in \mathcal{A}(X) \) is smooth.

**Theorem B (Compactness)**

Let \( p > 2 \) and let \( g^\nu \) be a sequence of metrics compatible with \( \tau \) that uniformly converges with all derivatives on every compact set to a smooth metric. Suppose that \( A^\nu \in A^{1,p}_{\text{loc}}(X) \) is a sequence of solutions of (1) with respect to the metrics \( g^\nu \) such that for every compact subset \( K \subset X \) there is a uniform bound on the curvature \( \|F_{A^\nu}\|_{L^p(K)} \). Then there exists a subsequence (again denoted \( A^\nu \)) and a sequence of gauge transformations \( u^\nu \in G^{2,p}_{\text{loc}}(X) \) such that \( u^\nu^*A^\nu \) converges uniformly with all derivatives on every compact set to a smooth connection \( A \in \mathcal{A}(X) \).

The difficulty of these results lies in the global nature of the boundary condition. This makes it impossible to directly generalize the proof of the regularity and compactness theorems for Yang-Mills connections, where one chooses suitable local gauges, obtains the higher regularity and estimates from an elliptic boundary value problem, and then patches the gauges together. With our global Lagrangian boundary condition one cannot obtain local regularity results.

However, an approach by Salamon can be generalized to manifolds with boundary. Firstly, Uhlenbeck’s weak compactness theorem yields a weakly \( W^{1,p}_{\text{loc}} \)-convergent subsequence. Its limit serves as reference connection with respect to which a further subsequence can be put into relative Coulomb gauge globally (on large compact sets). Then one has to establish elliptic estimates and regularity results for the given boundary value problem together with the relative Coulomb gauge equations. The crucial point in this last step is to establish the higher regularity for the \( \Sigma \)-component of the connections in a neighbourhood \( \mathcal{U} \times \Sigma \) of a boundary component. The global nature of the boundary condition forces us to deal with a Cauchy-Riemann equation on \( \mathcal{U} \) with values in the Banach space \( \mathcal{A}^{0,p}(\Sigma) \) and with Lagrangian boundary conditions. At this point we will make use of the regularity results in [W2] that are established in the general framework of a Cauchy-Riemann equation for functions with values in a complex Banach space and with totally real boundary conditions. Some more general analytic tools for this approach will be taken from [W1].
The case $2 < p \leq 4$, when $W^{1,p}$-functions are not automatically continuous, poses some special difficulties in this last step. Firstly, in order to obtain regularity results from the Cauchy-Riemann equation, one has to straighten out the Lagrangian submanifold by going to suitable coordinates. This requires a $C^0$-convergence of the connections, which in case $p > 4$ is given by a standard Sobolev embedding. In case $p > 2$ one still obtains a special compact embedding $W^{1,p}(U \times \Sigma) \hookrightarrow C^0(U, L^p(\Sigma))$ that suits our purposes. Secondly, the straightening of the Lagrangian introduces a nonlinearity in the Cauchy-Riemann equation that already poses some problems in case $p > 4$. In case $p \leq 4$ this forces us to deal with the Cauchy-Riemann equation with values in an $L^2$-Hilbert space and then use some interpolation inequalities for Sobolev norms.

For the definition of the standard instanton Floer homology it suffices to prove a compactness result like theorem B for $p = \infty$. In our case however the bubbling analysis [W3] requires the compactness result for some $p < 3$. This is why we have taken some care to deal with this case.

In order to define a Floer homology for 3-manifolds with boundary as outlined in [S] one has to consider the moduli space of finite energy solutions of

\[(1)\]

writing $L$ for the $n$-tuple of Lagrangian submanifolds $L_i$, $M(L) := \{ A \in A_{1,p}^{A_p}(X) \mid A \text{satisfies (1)}, \| F_A \|_{L^2} < \infty \} / G_{1,p}^{2,p}(X)$.

Theorem A implies that for every equivalence class $[A] \in M(L)$ one can find a smooth representative $A \in A(X)$. Theorem B is one step towards a compactness result for $M(L)$: Every closed subset of $M(L)$ with a uniform $L^p$-bound for the curvature is compact. In addition, theorem B allows the metric to vary, which is relevant for the metric-independence of the Floer homology.

Our third main result is the Fredholm theory in section 3. It is a step towards proving that the moduli space $M(L)$ of solutions of (1) is a manifold whose components have finite (but possibly different) dimensions. This also exemplifies our hope that the further analytical details of Floer theory will work out along the usual lines once the right analytic setup has been found in the proof of theorems A and B.

In the context of Floer homology and in Floer-Donaldson theory it is important to consider 4-manifolds with cylindrical ends. This requires an analysis of the asymptotic behaviour which will be carried out elsewhere. Here we shall restrict the discussion of the Fredholm theory to the compact case. The crucial point is the behaviour of the linearized operator near the boundary; in the interior we are dealing with the usual anti-self-duality equation. Hence it suffices to consider the following model case. Let $Y$ be a compact oriented 3-manifold with boundary $\partial Y = \Sigma$ and suppose that $(g_s)_{s \in S^1}$ is a smooth family of metrics on $Y$ such that

\[X = S^1 \times Y, \quad \tau : S^1 \times \Sigma \to X, \quad g = ds^2 + g_s\]

satisfy the assumptions of definition 1.2. Here the space-time splitting $\tau$ of the boundary is the obvious inclusion $\tau : S^1 \times \Sigma \hookrightarrow \partial X = S^1 \times \Sigma$, where
Lagrangian submanifolds

\[ \Sigma = \bigcup_{i=1}^{n} \Sigma_i \] might be a disjoint union of Riemann surfaces \( \Sigma_i \). An \( n \)-tuple of Lagrangian submanifolds \( L_i \subset A^{0,p}(\Sigma_i) \) as above then defines a gauge invariant Lagrangian submanifold \( \mathcal{L} := L_1 \times \ldots \times L_n \) of the symplectic Banach space \( A^{0,p}(\Sigma) = A^{0,p}(\Sigma_1) \times \ldots \times A^{0,p}(\Sigma_n) \) such that \( \mathcal{L} \subset A^{0,p}_{\text{flat}}(\Sigma) \).

In order to linearize the boundary value problem (1) together with the local slice condition, fix a smooth connection \( A + \Phi \text{ds} \in A(S^1 \times Y) \) such that \( A_s := A(s)|_{\partial Y} \in \mathcal{L} \) for all \( s \in S^1 \). Here \( \Phi \in C^\infty(S^1 \times Y, g) \), and \( A \in C^\infty(S^1 \times Y, T^* Y \otimes g) \) is an \( S^1 \)-family of 1-forms on \( Y \) (not a 1-form on \( X \) as previously). Now let \( E_{\Lambda}^{1,p} \) be the space of \( S^1 \)-families of 1-forms \( \alpha \in W^{1,p}(S^1 \times Y, T^* Y \otimes g) \) that satisfy the boundary conditions

\[ \ast \alpha(s)|_{\partial Y} = 0 \quad \text{and} \quad \alpha(s)|_{\partial Y} \in T_{A_s} \mathcal{L} \quad \text{for all} \quad s \in S^1. \]

Then the linearized operator

\[ D_{(A, \Phi)} : E_{\Lambda}^{1,p} \times W^{1,p}(S^1 \times Y, g) \to L^p(S^1 \times Y, T^* Y \otimes g) \times L^p(S^1 \times Y, g) \]

is given with \( \nabla_s = \partial_s + [\Phi_s, \cdot] \) by

\[ D_{(A, \Phi)}(\alpha, \varphi) = (\nabla_s \alpha - d_A \varphi + \ast d_A \alpha, \nabla_s \varphi - d_A^* \alpha). \]

The second component of this operator is \( -d_A^* + \Phi \text{ds}(\alpha + \varphi \text{ds}) \), and the first boundary condition is \( \ast (\alpha + \varphi \text{ds})|_{\partial Y} = 0 \), corresponding to the choice of a local slice at \( A + \Phi \text{ds} \). In the first component of \( D_{(A, \Phi)} \) we have used the global space-time splitting of the metric on \( S^1 \times Y \) to identify the self-dual 2-forms \( \ast \gamma_s = \gamma_s \wedge \text{ds} \) with families \( \gamma_s \) of 1-forms on \( Y \). The vanishing of this component is equivalent to the linearization \( d_A^* + \Phi \text{ds}(\alpha + \varphi \text{ds}) = 0 \) of the anti-self-duality equation. Furthermore, the boundary condition \( \alpha(s)|_{\partial Y} \in T_{A_s} \mathcal{L} \) is the linearization of the Lagrangian boundary condition in the boundary value problem (1).

**Theorem C (Fredholm properties)**

Let \( Y \) be a compact oriented 3-manifold with boundary \( \partial Y = \Sigma \) and let \( S^1 \times Y \) be equipped with a product metric \( ds^2 + g_s \) that is compatible with \( \tau : S^1 \times \Sigma \to S^1 \times Y \). Let \( A + \Phi \text{ds} \in A(S^1 \times Y) \) such that \( A(s)|_{\partial Y} \in \mathcal{L} \) for all \( s \in S^1 \). Then the following holds for all \( p > 2 \).

(i) \( D_{(A, \Phi)} \) is Fredholm.

(ii) There is a constant \( C \) such that for all \( \alpha \in E_{\Lambda}^{1,p} \) and \( \varphi \in W^{1,p}(S^1 \times Y, g) \)

\[ \|(\alpha, \varphi)\|_{W^{1,p}} \leq C \left( \|D_{(A, \Phi)}(\alpha, \varphi)\|_{L^p} + \|(\alpha, \varphi)\|_{L^p} \right). \]

(iii) Let \( q \geq p^* \) such that \( q \neq 2 \). Suppose that \( \beta \in L^q(S^1 \times Y, T^* Y \otimes g) \), \( \zeta \in L^q(S^1 \times Y, g) \), and assume that there exists a constant \( C \) such that for all \( \alpha \in E_{\Lambda}^{1,p} \) and \( \varphi \in W^{1,p}(S^1 \times Y, g) \)

\[ \left| \int_{S^1 \times Y} \{ D_{(A, \Phi)}(\alpha, \varphi), (\beta, \zeta) \} \right| \leq C \|(\alpha, \varphi)\|_{L^{p^*}}. \]

Then in fact \( \beta \in W^{1,q}(S^1 \times Y, T^* Y \otimes g) \) and \( \zeta \in W^{1,q}(S^1 \times Y, g) \).
Here and throughout we use the notation $\frac{1}{p} + \frac{1}{p^*} = 1$ for the conjugate exponent $p^*$ of $p$. The above inner product $(\cdot, \cdot)$ is the pointwise inner product in $T^* Y \otimes g \times g$. The reason for our assumption $q \neq 2$ in theorem C (iii) is a technical problem in dealing with the singularities of $L/G_{1,p}(\Sigma)$. We resolve these singularities by dividing only by the based gauge group. This leads to coordinates of $L^p(\Sigma, T^* \Sigma \otimes g)$ in a Banach space that comprises based Sobolev spaces $W^{1,p}_z(\Sigma, g)$ of functions vanishing at a fixed-point $z \in \Sigma$. So these coordinates that straighten out $T_L$ along $A|_{S^1 \times \partial Y}$ are well-defined only for $p > 2$.

Now in order to prove the regularity claimed in theorem C (iii) we have to use such coordinates either for $\beta$ or for the test 1-forms $\alpha$, i.e. we have to assume that either $q > 2$ or $q^* > 2$. This is completely sufficient for our purposes – concluding a higher regularity of elements of the cokernel. This will be done via an iteration of theorem C (iii) that can always be chosen such as to jump across $q = 2$. However, we believe that the use of different coordinates should permit to extend this result.

**Conjecture** Theorem C (iii) continues to hold for $q = 2$.

One indication for this conjecture is that the $L^2$-estimate in theorem C (ii) is true (for $W^{1,p}$-regular $\alpha$ and $\phi$ with $p > 2$), as will be shown in section 3. This $L^2$-estimate can be proven by a much more elementary method than the general $L^p$-regularity and -estimates. In fact, it was already stated in [S] as an indication for the wellposedness of the boundary value problem (1).

**Outlook**

We give a brief sketch of Salamon’s program for the proof of the Atiyah-Floer conjecture (for more details see [S]) in order to point out the significance of this paper for the whole program.

The first step of the program is to define the instanton Floer homology $\text{HF}^{\text{inst}}(Y, L)$ for a 3-manifold with boundary $\partial Y = \Sigma$ and a Lagrangian submanifold $L = L/G_{1,p}(\Sigma) \subset M_\Sigma$ in the moduli space of flat connections. The Floer complex will be generated by the gauge equivalence classes of irreducible flat connections $A \in \mathcal{A}(Y)$ with Lagrangian boundary conditions $A|_\Sigma \in \mathcal{L}$. For any two such connections $A^+, A^-$ one then has to study the moduli space of Floer connecting orbits,

$$\mathcal{M}(A^-, A^+) = \{ \hat{A} \in \mathcal{A}(\mathbb{R} \times Y) \mid \hat{A} \text{ satisfies (1), } \lim_{s \rightarrow \pm \infty} \hat{A} = A^\pm \}/G(\mathbb{R} \times Y).$$

A connection $A \in \mathcal{A}_\text{flat}(Y)$ is called irreducible if its isotropy subgroup of $G(Y)$ (the group of gauge transformations that leave $A$ fixed) is discrete, i.e. $d_A|_{\mathbb{R}^\rho}$ is injective. There should be no reducible flat connections with Lagrangian boundary conditions other than the gauge orbit of the trivial connection. This will be guaranteed by certain conditions on $Y$ and $L$, for example this is the case when $L = L_{Y'}$ for a handlebody $Y'$ with $\partial Y' = \Sigma$ such that $Y \cap \Sigma' Y'$ is a homology-3-sphere.
Theorem A shows that the boundary value problem (1) is wellposed. In particular, the spaces of smooth connections and gauge transformations in the definition of the above moduli space can be replaced by suitable Sobolev completions. The next step in the construction of the Floer homology groups is the analysis of the asymptotic behaviour of the finite energy solutions of (1) on $\mathbb{R} \times Y$, which will be carried out elsewhere. Combining this with theorem C one obtains an appropriate Fredholm theory and proves that for a suitably generic perturbation the spaces $\mathcal{M}(A^-, A^+)$ are smooth manifolds. In the monotone case the connections in the $k$-dimensional part $\mathcal{M}^k(A^-, A^+)$ have a fixed energy.

Theorem B is a major step towards a compactification of these moduli spaces $\mathcal{M}^k(A^-, A^+)$. It proves their compactness under the assumption of an $L^p$-bound on the curvature for $p > 2$, whereas the energy is only the $L^2$-norm. So the key remaining analytic task is an analysis of the possible bubbling phenomena. This will be carried out in [W3] and draws upon the techniques developed in this paper. When this is understood, the construction of the Floer homology groups should be routine. In particular, for the metric independence note that one can interpolate between different metrics on $Y$ as in example 1.4, and theorem B allows for the variation of metrics on $X$. So this paper sets up the basic analytic framework for the Floer theory of 3-manifolds with boundary.

The further steps in the program for the proof of the Atiyah-Floer conjecture are to consider a Heegard splitting $Y = Y_0 \cup_\Sigma Y_1$ of a homology 3-sphere, and identify $\text{HF}_{\text{inst}}^\ast([0,1] \times \Sigma, L_{Y_0} \times L_{Y_1})$ with $\text{HF}_{\text{inst}}^\ast(Y)$ and $\text{HF}_{\text{sym}}^\ast(M_\Sigma, L_{Y_0}, L_{Y_1})$ respectively. In both cases, the Floer complexes can be identified by elementary arguments, so the main task is to identify the connecting orbits.

In the case of the two instanton Floer homologies, the idea is to choose an embedding $(0, 1) \times \Sigma \hookrightarrow Y$ starting from a tubular neighbourhood of $\Sigma \subset Y$ at $t = \frac{1}{2}$ and shrinking $\{t\} \times \Sigma$ to the 1-skeleton of $Y_t$ for $t = 0, 1$. Then the anti-self-dual instantons on $\mathbb{R} \times Y$ pull back to anti-self-dual instantons on $\mathbb{R} \times [0, 1] \times \Sigma$ with a degenerate metric for $t = 0$ and $t = 1$. On the other hand, one can consider anti-self-dual instantons on $\mathbb{R} \times [\varepsilon, 1 - \varepsilon] \times \Sigma$ with boundary values in $L_{Y_0}$ and $L_{Y_1}$. As $\varepsilon \to 0$, one should be able to pass from this genuine boundary value problem to solutions on the closed manifold $Y$. This is a limit process for the boundary value problem studied in this paper.

The identification of the instanton and symplectic Floer homologies requires an adaptation of the adiabatic limit argument in [DS] to boundary value problems for anti-self-dual instantons and pseudoholomorphic curves respectively. Here one again deals with the boundary value problem (1) studied in this paper. As the metric on $\Sigma$ converges to zero, the solutions, i.e. anti-self-dual instantons on $\mathbb{R} \times [0, 1] \times \Sigma$ with Lagrangian boundary conditions in $L_{Y_0}$, $L_{Y_1}$ should be in one-to-one correspondence with connections on $\mathbb{R} \times [0, 1] \times \Sigma$ that descend to pseudoholomorphic strips in $M_\Sigma$ with boundary values in $L_{Y_0}$ and $L_{Y_1}$. The basic elliptic properties of the boundary value problem (1) that are established in this paper will also play an important role in this adiabatic limit analysis.

I would like to thank Dietmar Salamon for his constant help and encouragement in pursuing this project.
2 Regularity and compactness

Let \((X, \tau)\) be a 4-manifold with boundary space-time splitting. This means that \(X\) is oriented and
\[
X = \bigcup_{k \in \mathbb{N}} X_k,
\]
where all \(X_k\) are compact submanifolds and deformation retracts of \(X\) such that \(X_k \subset \text{int} X_{k+1}\) for all \(k \in \mathbb{N}\). Here the interior of a submanifold \(X' \subset X\) is to be understood with respect to the relative topology, i.e. we define \(\text{int} X' = X \setminus \text{cl}(X \setminus X')\). Moreover,
\[
\partial X = \bigcup_{i=1}^n \tau_i(S_i \times \Sigma_i),
\]
where each \(\Sigma_i\) is a Riemann surface, each \(S_i\) is either an open interval in \(\mathbb{R}\) or is equal to \(S^1 = \mathbb{R}/\mathbb{Z}\), and the embeddings \(\tau_i : S_i \times \Sigma_i \to X\) have disjoint images. We then consider the trivial \(G\)-bundle over \(X\), where \(G\) is a compact Lie group with Lie algebra \(\mathfrak{g}\). For \(i = 1, \ldots, n\) let \(L_i \subset A_{0, p}(\Sigma_i)\) be a Lagrangian submanifold and suppose that \(L_i \subset A_{0, p}(\Sigma_i)\) and \(G_{1, p}(\Sigma_i)^* L_i \subset L_i\).

Furthermore, let \(X\) be equipped with a metric \(g\) that is compatible with the space-time splitting \(\tau\). This means that for each \(i = 1, \ldots, n\) the map \(S_i \times [0, \infty) \times \Sigma_i \to X, (s, t, z) \mapsto \gamma_{(s, z)}(t)\) given by the normal geodesics \(\gamma_{(s, z)}\) starting at \(\gamma_{(s, z)}(0) = \tau_i(s, z)\) restricts to an embedding \(\tau_i : U_i \times \Sigma_i \hookrightarrow X\) for some neighbourhood \(U_i \subset S_i \times [0, \infty)\) of \(S_i \times \{0\}\). Now consider the boundary value problem (1) for connections \(A \in A_{\text{loc}}^{1, p}(X)\), restated below.
\[
\begin{cases}
* F_A + F_A = 0, \\
\tau_i^* A_{\{s\} \times \Sigma_i} \in L_i \quad \forall s \in S_i, i = 1, \ldots, n.
\end{cases}
\]

The anti-self-duality equation is welldefined for \(A \in A_{\text{loc}}^{1, p}(X)\) with any \(p \geq 1\), but in order to be able to state the boundary condition correctly we have to assume \(p > 2\). Then the trace theorem for Sobolev spaces (e.g. [Ad, Theorem 6.2]) ensures that \(\tau_i^* A_{\{s\} \times \Sigma_i} \in A^{0, p}(\Sigma_i)\) for all \(s \in S_i\).

The aim of this section is to prove the regularity theorem A and the compactness theorem B for this boundary value problem. Both theorems are dealing with the noncompact base manifold \(X\). However, we shall use an extension argument by Donaldson and Kronheimer [DK, Lemma 4.4.5] to reduce the problem to compact base manifolds. For the following special version of this argument a detailed proof can be found in [W1, Propositions 8.6,10.8]. At this point, the assumption that the exhausting compact submanifolds \(X_k\) are deformation retracts of \(X\) comes in crucially. It ensures that every gauge transformation on \(X_k\) can be extended to \(X\), which is a central point in the argument of Donaldson and Kronheimer that proves the following proposition.
Proposition 2.1 Let the 4-manifold $\hat{M} = \bigcup_{k \in \mathbb{N}} M_k$ be exhausted by compact submanifolds $M_k \subset \text{int } M_{k+1}$ that are deformation retracts of $\hat{M}$, and let $p > 2$.

(i) Let $A \in \mathcal{A}_{\text{loc}}^{1,p}(\hat{M})$ and suppose that for each $k \in \mathbb{N}$ there exists a gauge transformation $u_k \in \mathcal{G}^{2,p}(M_k)$ such that $u_k^* A|_{M_k}$ is smooth. Then there exists a gauge transformation $u \in \mathcal{G}^{2,p}(\hat{M})$ such that $u^* A$ is smooth.

(ii) Let a sequence of connections $(A^\nu)_{\nu \in \mathbb{N}} \subset \mathcal{A}_{\text{loc}}^{1,p}(\hat{M})$ be given and suppose that the following holds:

For every $k \in \mathbb{N}$ and every subsequence of $(A^\nu)_{\nu \in \mathbb{N}}$ there exist a further subsequence $(\nu_{k,i})_{i \in \mathbb{N}}$ and gauge transformations $u_i^{k,i} \in \mathcal{G}^{2,p}(M_k)$ such that
\[
\sup_{i \in \mathbb{N}} \| u_i^{k,i} A^{\nu_{k,i}} \|_{W^{1,p}(M_k)} < \infty \quad \forall \ell \in \mathbb{N}.
\]

Then there exists a subsequence $(\nu_i)_{i \in \mathbb{N}}$ and a sequence of gauge transformations $u_i \in \mathcal{G}^{2,p}(\hat{M})$ such that
\[
\sup_{i \in \mathbb{N}} \| u_i^* A^{\nu_i} \|_{W^{1,p}(M_k)} < \infty \quad \forall k \in \mathbb{N}, \ell \in \mathbb{N}.
\]

So in order to prove theorem A it suffices to find smoothing gauge transformations on the compact submanifolds $X_k$ in view of proposition 2.1 (i). For that purpose we shall use the so-called local slice theorem. The following version is proven e.g. in [W1, Theorem 9.1]. Note that we are dealing with trivial bundles, so we will be using the product connection as reference connection in the definition of the Sobolev norms of connections.

Proposition 2.2 (Local Slice Theorem)

Let $M$ be a compact 4-manifold, let $p > 2$, and let $q > 4$ be such that $\frac{1}{q} > \frac{1}{p} - \frac{1}{4}$ (or $q = \infty$ in case $p > 4$). Fix $\hat{A} \in \mathcal{A}_{1,p}(M)$ and let a constant $c_0 > 0$ be given. Then there exist constants $\varepsilon > 0$ and $C_{CG}$ such that the following holds. For every $A \in \mathcal{A}_{1,p}(M)$ with
\[
\| A - \hat{A} \|_q \leq \varepsilon \quad \text{and} \quad \| A - \hat{A} \|_{W^{1,p}} \leq c_0
\]

there exists a gauge transformation $u \in \mathcal{G}^{2,p}(M)$ such that
\[
\begin{cases}
\frac{d}{\overline{\partial}}(u^* A - \hat{A}) = 0, \\
*(u^* A - \hat{A})|_{\overline{\partial}M} = 0
\end{cases}
\]

and
\[
\| u^* A - \hat{A} \|_q \leq C_{CG} \| A - \hat{A} \|_q, \quad \| u^* A - \hat{A} \|_{W^{1,p}} \leq C_{CG} \| A - \hat{A} \|_{W^{1,p}}.
\]

Remark 2.3

(i) If the boundary value problem in proposition 2.2 is satisfied one says that $u^* A$ is in Coulomb gauge relative to $A$. This is equivalent to $v^* \hat{A}$ being in Coulomb gauge relative to $A$ for $v = u^{-1}$, i.e. the boundary value problem can be replaced by
\[
\begin{cases}
\frac{d}{\overline{\partial}}(v^* \hat{A} - A) = 0, \\
*(v^* \hat{A} - A)|_{\overline{\partial}M} = 0.
\end{cases}
\]
(ii) The assumptions in proposition 2.2 on $p$ and $q$ guarantee that one has a compact Sobolev embedding

$$W^{1,p}(M) \hookrightarrow L^q(M).$$

(iii) One can find uniform constants for varying metrics in the following sense.

Fix a metric $g$ on $M$. Then there exist constants $\varepsilon, \delta > 0$, and $C_{CG}$ such that the assertion of proposition 2.2 holds for all metrics $g'$ with $\|g - g'\|_{C^1} \leq \delta$.

In the following we shall briefly outline the proof of theorem A. Given a solution $A \in A^{1,p}_{\text{loc}}(X)$ of (2) one fixes $k \in \mathbb{N}$ and proves the assumption of proposition 2.1 (i) as follows. One finds some sufficiently large compact submanifold $M \subset X$ with $X_k \subset M$. Then one chooses a smooth connection $A_0 \in A(M)$ sufficiently $W^{1,p}$-close to $A$ and applies the local slice theorem with the reference connection $\tilde{A} = A$ to find a gauge transformation that puts $A_0$ into relative Coulomb gauge with respect to $A$. This is equivalent to finding a gauge transformation that puts $A$ into relative Coulomb gauge with respect to $A_0$. We denote this gauge transformed connection again by $A \in A^{1,p}(M)$. It satisfies the following boundary value problem:

$$\begin{cases}
d^*_{A_0}(A - A_0) = 0, \\
*F_A + F_A = 0, \\
*(A - A_0)|_{\partial M} = 0, \\
\tau_i^*A|_{(s) \times \Sigma_i} \in L_i \quad \forall s \in S_i, \ i = 1, \ldots, n. 
\end{cases} \tag{3}$$

More precisely, the Lagrangian boundary condition only holds for those $s \in S_i$ and $i \in \{1, \ldots, n\}$ for which $\tau_i((s) \times \Sigma_i)$ is entirely contained in $\partial M$. If $M$ was chosen large enough, then the regularity theorem 2.6 below will assert the smoothness of $\tilde{A}$ on $X_k$.

The proof outline of the proof of theorem A goes along similar lines. We will use proposition 2.1 (ii) to reduce the problem to compact base manifolds. On these, we shall use the following weak Uhlenbeck compactness theorem (see [U1], [W1, Theorem 8.1]) to find a subsequence of gauge equivalent connections that converges $W^{1,p}$-weakly.

**Proposition 2.4 (Weak Uhlenbeck Compactness)**

Let $M$ be a compact 4-manifold and let $p > 2$. Suppose that the sequence of connections $A^\nu \in A^{1,p}(M)$ is such that $\|F_{A^\nu}\|_p$ is uniformly bounded. Then there exists a subsequence (again denoted $(A^\nu)_{\nu \in \mathbb{N}}$) and a sequence $u^\nu \in G^{2,p}(M)$ of gauge transformations such that $u^\nu \ast A^\nu$ weakly converges in $A^{1,p}(M)$.

The limit $A_0$ of the convergent subsequence then serves as reference connection $\tilde{A}$ in the local slice theorem, proposition 2.2, and this way one obtains a $W^{1,p}$-bounded sequence of connections $A^\nu$ that solve the boundary value problem (3). This makes crucial use of the compact Sobolev embedding $W^{1,p} \hookrightarrow L^q$.  

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on compact 4-manifolds (with \( q \) from the local slice theorem). The estimates in the subsequent theorem 2.6 then provide the higher \( W^{k,p} \)-bounds on the connections that will imply the compactness. One difficulty in the proof of this regularity theorem is that due to the global nature of the boundary conditions one has to consider the \( \Sigma \)-components of the connections near the boundary as maps into the Banach space \( A^{0,p}(\Sigma) \) that solve a Cauchy-Riemann equation with Lagrangian boundary conditions. In order to prove a regularity result for such maps one has to straighten out the Lagrangian submanifold by using coordinates in \( A^{0,p}(\Sigma) \). (This is done in [W2].) Thus on domains \( U \times \Sigma \) at the boundary a crucial assumption is that the \( \Sigma \)-components of the connections all lie in one such coordinate chart, that is one needs the connections to converge strongly in the \( L^{\infty}(U,L^{p}(\Sigma)) \)-norm. In the case \( p > 4 \) this is ensured by the compact embedding \( W^{1,p} \hookrightarrow L^{\infty}(U \times \Sigma) \). To treat the case \( 2 < p \leq 4 \) we shall make use of the following special Sobolev embedding. The proof mainly uses techniques from [Ad].

**Lemma 2.5** Let \( M, N \) be compact manifolds and let \( p > m = \dim M \) and \( p > n = \dim N \). Then the following embedding is compact,

\[
W^{1,p}(M \times N) \hookrightarrow L^{\infty}(M,L^{p}(N)).
\]

**Proof of lemma 2.5:**

Since \( M \) is compact it suffices to prove the embedding in (finitely many) coordinate charts. These can be chosen as either balls \( B_{2} \subset \mathbb{R}^{m} \) in the interior or half balls \( D_{2} = B_{2} \cap \mathbb{H}^{m} \) in the half space \( \mathbb{H}^{m} = \{ x \in \mathbb{R}^{m} \mid x_{1} \geq 0 \} \) at the boundary of \( M \). We can choose both of radius 2 but cover \( M \) by balls and half balls of radius 1. So it suffices to consider a bounded set \( K \subset W^{1,p}(B_{2} \times N) \) and prove that it restricts to a precompact set in \( L^{\infty}(B_{1},L^{p}(N)) \), and similarly with the half balls. Here we use the Euclidean metric on \( \mathbb{R}^{m} \), which is equivalent to the metric induced from \( M \).

For a bounded subset \( K \subset W^{1,p}(D_{2} \times N) \) over the half ball define the subset \( K' \subset W^{1,p}(B_{2} \times N) \) by extending all \( u \in K \) to \( B_{2} \setminus \mathbb{H}^{m} \) by \( u(x_{1},x_{2},\ldots,x_{m}) := u(-x_{1},x_{2},\ldots,x_{m}) \) for \( x_{1} \leq 0 \). The thus extended function is still \( W^{1,p} \)-regular with twice the norm of \( u \). So \( K' \) also is a bounded subset, and if this restricts to a precompact set in \( L^{\infty}(B_{1},L^{p}(N)) \), then also \( K \subset L^{\infty}(B_{1},L^{p}(N)) \) is compact. Hence it suffices to consider the interior case of the full ball.

The claimed embedding is continuous by the standard Sobolev estimates – check for example in [Ad] that the estimates generalize directly to functions with values in a Banach space. In fact, one obtains an embedding

\[
W^{1,p}(B_{2} \times N) \subset W^{1,p}(B_{2},L^{p}(N)) \hookrightarrow C^{\alpha,\lambda}(B_{2},L^{p}(N))
\]

into some H"older space with \( \lambda = 1 - \frac{m}{p} > 0 \). One can also use this Sobolev estimate for \( W^{1,p}(N) \) with \( \lambda' = 1 - \frac{n}{p} > 0 \) combined with the inclusion \( L^{p} \hookrightarrow L^{1} \) on \( B_{2} \) to obtain a continuous embedding

\[
W^{1,p}(B_{2} \times N) \subset L^{p}(B_{2},W^{1,p}(N)) \hookrightarrow L^{p}(B_{2},C^{\alpha,\lambda'}(N)) \subset L^{1}(B_{2},C^{0,\lambda'}(N)).
\]
Now consider a bounded subset $\mathcal{K} \subset W^{1,p}(B_2 \times N)$. The first embedding ensures that the functions $u \in \mathcal{K}$, $u : B_2 \to L^p(N)$ are equicontinuous. For some constant $C$

$$\|u(x) - u(y)\|_{L^p(N)} \leq C|x - y|^{\lambda} \quad \forall x, y \in B_2, u \in \mathcal{K}. \quad (4)$$

The second embedding asserts that for some constant $C'$

$$\int_{B_2} \|u\|_{C^{0,\lambda'}(N)} \leq C' \quad \forall u \in \mathcal{K}. \quad (5)$$

In order to prove that $\mathcal{K} \subset L^\infty(B_1, L^p(N))$ is precompact we now fix any $\varepsilon > 0$ and show that $\mathcal{K}$ can be covered by finitely many $\varepsilon$-balls.

Let $J \in C^\infty(\mathbb{R}^m, [0, \infty))$ be such that $\text{supp } J \subset B_1$ and $\int J = 1$. Then $J_\delta(x) := \delta^{-m} J(x/\delta)$ are mollifiers for $\delta > 0$ with $\text{supp } J_\delta \subset B_{\delta}$ and $\int J_\delta = 1$. Let $\delta \leq 1$, then $J_\delta * u|_{B_1} \in C^\infty(B_1, L^p(N))$ is welldefined. Moreover, choose $\delta > 0$ sufficiently small such that for all $u \in \mathcal{K}$

$$\|J_\delta * u - u\|_{L^\infty(B_1, L^p(N))} = \sup_{x \in B_1} \int_{B_\delta} J_\delta(y) \|u(x - y) - u(x)\|_{L^p(N)} dy \leq \sup_{x \in B_1} \int_{B_\delta} J_\delta(y) C|y|^\lambda dy \leq C\delta^\lambda \leq \frac{\varepsilon}{2}.$$

Now it suffices to prove the precompactness of $\mathcal{K}_\delta := \{J_\delta * u \mid u \in \mathcal{K}\}$, then this set can be covered by $\frac{\varepsilon}{2}$-balls around $J_\delta * u_i$ with $u_i \in \mathcal{K}$ for $i = 1, \ldots, I$ and above estimate shows that $\mathcal{K}$ is covered by the $\varepsilon$-balls around the $u_i$. Indeed, for each $u \in \mathcal{K}$ one has $\|J_\delta * u - J_\delta * u_i\|_{L^\infty(B_1, L^p(N))} \leq \frac{\varepsilon}{2}$ for some $i = 1, \ldots, I$ and thus

$$\|u - u_i\| \leq \|u - J_\delta * u\| + \|J_\delta * u - J_\delta * u_i\| + \|J_\delta * u_i - u_i\| \leq \varepsilon.$$

The precompactness of $\mathcal{K}_\delta \subset L^\infty(B_1, L^p(N))$ will follow from the Arzéla-Ascoli theorem (see e.g. [L, IX §4]). Firstly, the smoothened functions $J_\delta * u$ are still equicontinuous on $B_1$. For all $u \in \mathcal{K}$ and $x, y \in B_1$ use (4) to obtain

$$\|(J_\delta * u)(x) - (J_\delta * u)(y)\|_{L^p(N)} \leq \int_{B_\delta} J_\delta(z) \|u(x - z) - u(y - z)\|_{L^p(N)} |z|^\lambda dz \leq \int_{B_\delta} J_\delta(z) C|x - y|^{\lambda} dz = C|x - y|^{\lambda}.$$

Secondly, the $L^\infty$-norm of the smoothened functions is bounded by the $L^1$-norm of the original ones, so for fixed $\delta > 0$ one obtains a uniform bound from $(5)$:
For all \( u \in \mathcal{K} \) and \( x \in B_1 \)

\[
\|(J_\delta \ast u)(x)\|_{C^{0,\lambda'}(N)} \leq \int_{B_2} J_\delta(x - y) \|u(y)\|_{C^{0,\lambda'}(N)} \, d^m y
\]

\[
\leq C' \|J_\delta\|_\infty.
\]

Now the embedding \( C^{0,\lambda'}(N) \to L^p(N) \) is a standard compact Sobolev embedding, so this shows that the subset \( \{(J_\delta \ast u)(x) \mid u \in \mathcal{K}\} \subset L^p(N) \) is precompact for all \( x \in B_1 \). Thus the Arzela-Ascoli theorem asserts that \( \mathcal{K}_\delta \subset L^\infty(B_1, L^p(N)) \) is compact, and this finishes the proof of the lemma. \( \square \)

In the proof of theorem B, the weak Uhlenbeck compactness together with the local slice theorem and this lemma will put us in the position to apply the following crucial regularity theorem that also is the crucial point in the proof of theorem A. Here \((X, \tau)\) is a 4-manifold with a boundary space-time splitting as described in definition 1.1 and in the beginning of this section.

**Theorem 2.6** For every compact subset \( K \subset X \) there exists a compact submanifold \( M \subset X \) such that \( K \subset M \) and the following holds for all \( p > 2 \).

(i) Suppose that \( A \in A^{1,p}(M) \) solves the boundary value problem (3). Then \( A|_K \in \mathcal{A}(K) \) is smooth.

(ii) Fix a metric \( g_0 \) that is compatible with \( \tau \) and a smooth connection \( A_0 \in \mathcal{A}(M) \) such that \( \tau^*_i A_0|_{\{s\} \times \Sigma} \in \mathcal{L}_i \) for all \( s \in \Sigma \) and \( i = 1, \ldots, n \). Moreover, fix a compact neighbourhood \( \mathcal{V} = \bigcup_{i=1}^n \tau_{0,i}(\mathcal{U}_i \times \Sigma_i) \) of \( K \cap \partial X \).

(Here \( \tau_{0,i} \) denotes the extension of \( \tau_i \) given by the geodesics of \( g_0 \).) Then for every given constant \( C_1 \) there exist constants \( \delta > 0 \), \( \delta_k > 0 \), and \( C_k \) for all \( k \geq 2 \) such that the following holds:

Fix \( k \geq 2 \) and let \( g \) be a metric that is compatible with \( \tau \) and satisfies \( \|g - g_0\|_{C^{k+2}(M)} \leq \delta_k \). Suppose that \( A \in A^{1,p}(M) \) solves the boundary value problem (3) with respect to the metric \( g \) and satisfies

\[
\|A - A_0\|_{W^{1,p}(M)} \leq C_1,
\]

\[
\|\tau_{0,i}^*(A - A_0)|_{\Sigma_i}\|_{L^\infty(\mathcal{U}_i \times A^{0,p}(\Sigma_i))} \leq \delta \quad \forall i = 1, \ldots, n.
\]

Then \( A|_K \in \mathcal{A}(K) \) is smooth by (i) and

\[
\|A - A_0\|_{W^{k,p}(K)} \leq C_k.
\]

We first give some preliminary results for the proof of theorem 2.6. The interior regularity as well as the regularity of the \( \mathcal{U}_i \)-components on a neighbourhood \( \mathcal{U}_i \times \Sigma_i \) of a boundary component \( \Sigma \times \Sigma_i \) will be a consequence of the following regularity result for Yang-Mills connections. The proof is similar to that of lemma A.2 and can be found in full detail in [W1, Proposition 10.5]. Here \( M \) is a compact Riemannian manifold with boundary \( \partial M \) and outer unit normal \( \nu \). One then deals with two different spaces of test functions,

\[
C_c^\infty(M, g) := \{ \phi \in C^\infty(M, g) \mid \phi|_{\partial M} = 0 \},
\]

\[
C_c^\alpha(M, g) := \{ \phi \in C^\infty(M, g) \mid \frac{\partial \phi}{\partial \nu}|_{\partial M} = 0 \}.
\]

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Proposition 2.7 Let $(M, g)$ be a compact Riemannian 4-manifold. Fix a smooth reference connection $A_0 \in \mathcal{A}(M)$. Let $X \in \Gamma(TM)$ be a smooth vector field that is either perpendicular to the boundary, i.e. $X|_{\partial M} = h \nu$ for some $h \in C^\infty(\partial M)$, or is tangential, i.e. $X|_{\partial M} \in \Gamma(T\partial M)$. In the first case let $\mathcal{T} = C^\infty_M(M, g)$, in the latter case let $\mathcal{T} = C^\infty_\nu(M, g)$. Moreover, let $N \subset \partial M$ be an open subset such that $X$ vanishes in a neighbourhood of $\partial M \setminus N \subset M$. Let $1 < p < \infty$ and $k \in \mathbb{N}$ be such that either $kp > 4$ or $k = 1$ and $2 < p < 4$. In the first case let $q := p$, in the latter case let $q := \frac{4p}{8 - p}$. Then there exists a constant $C$ such that the following holds.

Let $A = A_0 + \alpha \in \mathcal{A}^{k,p}(M)$ be a connection. Suppose that it satisfies

\[
\begin{cases}
\delta_{A_0}^* \alpha = 0, \\
\ast \alpha|_{\partial M} = 0 \quad \text{on } N \subset \partial M,
\end{cases}
\]

and that for all 1-forms $\beta = \phi \cdot \iota_X g$ with $\phi \in \mathcal{T}$

\[
\int_M \langle F_A, d_A \beta \rangle = 0.
\]

Then $\alpha(X) \in W^{k+1,q}(M, g)$ and

\[
\|\alpha(X)\|_{W^{k+1,q}} \leq C \left(1 + \|\alpha\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}^3\right).
\]

Moreover, the constant $C$ can be chosen such that it depends continuously on the metric $g$ and the vector field $X$ with respect to the $C^{k+1}$-topology.

Remark 2.8 In the case $k = 1$ and $2 < p < 4$ the iteration of proposition 2.7 also allows to obtain $W^{2,p}$-regularity and -estimates from initial $W^{1,p}$-regularity and -estimates.

Indeed, the Sobolev embedding $W^{2,q} \hookrightarrow W^{1,p'}$ holds with $p' = \frac{4q}{4 - q}$ since $q < 4$. Now as long as $p' < 4$ one can iterate the proposition and Sobolev embedding to obtain regularity and estimates in $W^{1,p_i}$ with $p_0 = p$ and

\[p_{i+1} = \frac{4q_i}{4 - q_i} = \frac{2p_i}{4 - p_i} \geq \theta p_i > p_i.
\]

Since $\theta := \frac{2}{4 - p} > 1$ this sequence terminates after finitely many steps at some $p_N \geq 4$. In case $p_N > 4$ the proposition even yields $W^{2,p_N}$-regularity and -estimates. In case $p_N = 4$ one only uses $W^{1,p_N}$ for some smaller $p'_{N} > \frac{8}{3}$ in order to conclude $W^{2,\frac{8}{3}}$-regularity and -estimates for $p'_{N+1} > 4$.

Similarly, in case $k = 1$ and $p = 4$ one only needs two steps to reach $W^{2,p'}$ for some $p' > 4$.

The above proposition and remark can be used on all components of the connections in theorem 2.6 except for the $\Sigma$-components in small neighbourhoods $U \times \Sigma$ of boundary components $S \times \Sigma$. For the regularity of their higher derivatives in $\Sigma$-direction we shall use the following lemma. The crucial regularity of the derivatives in direction of $U$ of the $\Sigma$-components will then follow from the general regularity theory for Cauchy-Riemann equations in [W2].
Lemma 2.9 Let $k \in \mathbb{N}_0$ and $1 < p < \infty$. Let $\Omega$ be a compact manifold, let $\Sigma$ be a Riemann surface, and equip $\Omega \times \Sigma$ with a product metric $g_\Omega \oplus g$, where $g = (g_x)_{x \in \Sigma}$ is a smooth family of metrics on $\Sigma$. Then there exists a constant $C$ such that the following holds:

Suppose that $\alpha \in W^{k,p}(\Omega \times \Sigma, T^*\Sigma)$ such that both $d_\Sigma^{\alpha}g$ and $d_{\Sigma}^{\alpha}g$ are of class $W^{k,p}$ on $\Omega \times \Sigma$. Then $\nabla_\Sigma^{\alpha}$ also is of class $W^{k,p}$ and one has the following estimate on $\Omega \times \Sigma$

$$\|\nabla_\Sigma^{\alpha}\|_{W^{k,p}} \leq C(\|d_\Sigma^{\alpha}g\|_{W^{k,p}} + \|d_{\Sigma}^{\alpha}g\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}).$$

Here $\nabla_\Sigma$ denotes the family of Levi-Civita connections on $\Sigma$ that is given by the family of metrics $g$. Moreover, for every fixed family of metrics $g$ one finds a $C^k$-neighbourhood of metrics for which this estimate holds with a uniform constant $C$.

Proof of lemma 2.9:
We first prove this for $k = 0$, i.e. suppose that $\alpha \in L^p(\Omega \times \Sigma, T^*\Sigma)$ and that $d_\Sigma^{\alpha}$, $d_{\Sigma}^{\alpha}$ (defined as weak derivatives) are also of class $L^p$. We introduce the following functions

$$f := d_\Sigma^{\alpha} \in L^p(\Omega \times \Sigma), \quad g := -d_{\Sigma}^{\alpha} \in L^p(\Omega \times \Sigma),$$

and choose sequences $f^\nu, g^\nu \in C^\infty(\Omega \times \Sigma)$, and $\alpha^\nu \in C^\infty(\Omega \times \Sigma, T^*\Sigma)$ that converge to $f, g, \alpha$ respectively in the $L^p$-norm. Note that $\int_{\Omega} f = \int_{\Omega} g = 0$ in $L^p(\Sigma)$, so the $f^\nu$ and $g^\nu$ can be chosen such that their mean value over $\Omega$ also vanishes for all $z \in \Sigma$. Then fix $z \in \Sigma$ and find $\xi^\nu, \zeta^\nu \in C^\infty(\Omega \times \Sigma)$ such that

$$\begin{cases}
\Delta_\Sigma \xi^\nu = f^\nu, \\
\xi^\nu(x, z) = 0 \quad \forall x \in \Omega,
\end{cases} \quad \begin{cases}
\Delta_\Sigma \zeta^\nu = g^\nu, \\
\zeta^\nu(x, z) = 0 \quad \forall x \in \Omega.
\end{cases}$$

These solutions are uniquely determined since $\Delta_\Sigma : W^{j+2,p}_x(\Sigma) \to W^{j,p}_x(\Sigma)$ is a bounded isomorphism for every $j \in \mathbb{N}_0$ depending smoothly on the metric, i.e. on $x \in \Omega$. Here $W^{j,p}_x(\Sigma)$ denotes the space of $W^{j,p}$ functions with mean value zero and $W^{j+2,p}_x(\Sigma)$ consists of those functions that vanish at $z \in \Sigma$.

Furthermore, let $\pi : \Omega^1(\Sigma) \to h^1(\Sigma, g_x)$ be the projection of the smooth 1-forms to the harmonic part $h^1(\Sigma) = \ker \Delta_\Sigma = \ker d_\Sigma \cap \ker d_{\Sigma}$ with respect to the metric $g_x$ on $\Sigma$. Then $\pi$ is a family of bounded operators from $L^p(\Sigma, T^*\Sigma)$ to $W^{j,p}(\Sigma, T^*\Sigma)$ for any $j \in \mathbb{N}_0$, and it depends smoothly on $x \in \Omega$. So the harmonic part of $\tilde{\alpha}^\nu$ is also smooth, $\pi \circ \tilde{\alpha}^\nu \in C^\infty(\Omega \times \Sigma, T^*\Sigma)$. Now consider

$$\alpha^\nu := d_\Sigma \xi^\nu + *_\Sigma d_\Sigma \zeta^\nu + \pi \circ \tilde{\alpha}^\nu \in C^\infty(\Omega \times \Sigma, T^*\Sigma).$$

We will show that the sequence $\alpha^\nu$ of 1-forms converges to $\alpha$ in the $L^p$-norm and that moreover $\nabla_\Sigma \alpha^\nu$ is an $L^p$-Cauchy sequence. For that purpose we will use the following estimate. For all 1-forms $\beta \in W^{1,p}(\Sigma, T^*\Sigma)$ abbreviating $d_\Sigma = d$

$$\|\beta\|_{W^{1,p}(\Sigma)} \leq C(\|d_\Sigma^\ast \beta\|_{L^p(\Sigma)} + \|d_\Sigma \beta\|_{L^p(\Sigma)} + \|\pi(\beta)\|_{W^{1,p}(\Sigma)})$$

$$\leq C(\|d_\Sigma^\ast \beta\|_{L^p(\Sigma)} + \|d_\Sigma \beta\|_{L^p(\Sigma)} + \|\beta\|_{L^p(\Sigma)}). \quad (8)$$

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Here and in the following $C$ denotes any finite constant that is uniform for all metrics $g_\alpha$ on $\Sigma$ in a family of metrics that lies in a sufficiently small $C^k$-neighbourhood of a fixed family of metrics. To prove (8) we use the Hodge decomposition $\beta = d\xi + *d\zeta + \pi(\beta)$. (See e.g. [Wa, Theorem 6.8] and recall that one can identify 2-forms on $\Sigma$ with functions via the Hodge $*$-operator.) Here one chooses $\xi, \zeta \in W^{2,p}_x(\Sigma)$ such that they solve $\Delta \xi = d^*\beta$ and $\Delta \zeta = *d\beta$ respectively and concludes from proposition A.1 for some uniform constant $C$

$$
\|d\xi\|_{W^{1,p}(\Sigma)} \leq \|\xi\|_{W^{2,p}(\Sigma)} \leq C\|d^*\beta\|_{L^p(\Sigma)},
$$

$$
\|*d\zeta\|_{W^{1,p}(\Sigma)} \leq \|\zeta\|_{W^{2,p}(\Sigma)} \leq C\|d\beta\|_{L^p(\Sigma)}.
$$

The second step in (8) moreover uses the fact that the projection to the harmonic part is bounded as map $\pi: L^p(\Sigma, T^*\Sigma) \to W^{1,p}(\Sigma, T^*\Sigma)$.

Now consider $\alpha - \alpha' \in L^p(\Omega \times \Sigma)$ for almost all $x \in \Omega$. For almost all $x \in \Omega$ we have $\alpha(x, \cdot) - \alpha'(x, \cdot) \in L^p(\Sigma, T^*\Sigma)$ as well as $*d_\Sigma (\alpha(x, \cdot) - \alpha'(x, \cdot)) \in L^p(\Sigma)$ and $d_\Sigma (\alpha(x, \cdot) - \alpha'(x, \cdot)) \in L^p(\Sigma)$. Then for these $x \in \Omega$ one concludes from the Hodge decomposition that in fact $\alpha(x, \cdot) - \alpha'(x, \cdot) \in W^{1,p}(\Sigma, T^*\Sigma)$. So we can apply (8) and integrate over $x \in \Omega$ to obtain for all $\nu \in \mathbb{N}$

$$
\|\alpha - \alpha'\|_{L^p(\Omega \times \Sigma)}^p \leq \int_\Omega \|\alpha(x, \cdot) - \alpha'(x, \cdot)\|_{L^p(\Sigma, g_\alpha)}^p
$$

$$
\leq C \int_\Omega \left(\|d_\Sigma^* (\alpha - \alpha')\|_{L^p(\Sigma)}^p + \|d_\Sigma (\alpha - \alpha')\|_{L^p(\Sigma)}^p + \|\pi(\alpha - \alpha')\|_{W^{1,p}(\Sigma)}^p\right)
$$

$$
\leq C \left(\|f - f'\|_{L^p(\Omega \times \Sigma)}^p + \|g - g'\|_{L^p(\Omega \times \Sigma)}^p + \|\alpha - \alpha'\|_{L^p(\Omega \times \Sigma)}^p\right).
$$

In the last step we again used the continuity of $\pi$. This proves the convergence $\alpha' \to \alpha$ in the $L^p$-norm, and hence $\nabla_\Sigma \alpha' \to \nabla_\Sigma \alpha$ in the distributional sense.

Next, we use (8) to estimate for all $\nu \in \mathbb{N}$

$$
\|\nabla_\Sigma \alpha'\|_{L^p(\Omega \times \Sigma)}^p = \int_\Omega \|\nabla_\Sigma \alpha'(x, \cdot)\|_{L^p(\Sigma, g_\alpha)}^p
$$

$$
\leq C \int_\Omega \left(\|d_\Sigma^* \alpha'\|_{L^p(\Sigma)}^p + \|d_\Sigma \alpha'\|_{L^p(\Sigma)}^p + \|\alpha'\|_{L^p(\Sigma)}^p\right)
$$

$$
\leq C \left(\|d_\Sigma^* \alpha\|_{L^p(\Omega \times \Sigma)}^p + \|d_\Sigma \alpha\|_{L^p(\Omega \times \Sigma)}^p + \|\alpha\|_{L^p(\Omega \times \Sigma)}^p\right).
$$

Here one deals with $L^p$-convergent sequences $d_\Sigma^* \alpha' = \Delta_\Sigma \xi' = f' \to f = d_\Sigma^* \alpha$, $-d_\Sigma \alpha' = \Delta_\Sigma \zeta' = g' \to g = -d_\Sigma \alpha$, and $\alpha' \to \alpha$. So $(\nabla_\Sigma \alpha')_{\nu \in \mathbb{N}}$ is uniformly bounded in $L^p(\Omega \times \Sigma)$ and hence contains a weakly $L^p$-convergent subsequence. The limit is $\nabla_\Sigma \alpha$ since this already is the limit in the distributional sense. Thus we have proven the $L^p$-regularity of $\nabla_\Sigma \alpha$ on $\Omega \times \Sigma$, and moreover above estimate is preserved under the limit, which proves the lemma in the case $k = 0$,

$$
\|\nabla_\Sigma \alpha\|_{L^p(\Omega \times \Sigma)} \leq \liminf_{\nu \to \infty} \|\nabla_\Sigma \alpha'\|_{L^p(\Omega \times \Sigma)}
$$

$$
\leq \liminf_{\nu \to \infty} C \left(\|d_\Sigma^* \alpha'\|_{L^p(\Omega \times \Sigma)} + \|d_\Sigma \alpha'\|_{L^p(\Omega \times \Sigma)} + \|\alpha'\|_{L^p(\Omega \times \Sigma)}\right)
$$

$$
= C \left(\|d_\Sigma^* \alpha\|_{L^p(\Omega \times \Sigma)} + \|d_\Sigma \alpha\|_{L^p(\Omega \times \Sigma)} + \|\alpha\|_{L^p(\Omega \times \Sigma)}\right).
$$
In the case $k \geq 1$ one can now use the previous result to prove the lemma. Let $\alpha \in W^{k,p}(\Omega \times \Sigma, T^*\Sigma)$ and suppose that $d_\Sigma \alpha, d_\Sigma^* \alpha$ are of class $W^{k,p}$. We denote by $\nabla$ the covariant derivative on $\Omega \times \Sigma$. Then we have to show that $\nabla^k \nabla_\Sigma \alpha$ is of class $L^p$. So let $X_1, \ldots, X_k$ be smooth vector fields on $\Omega \times \Sigma$ and introduce

$$\tilde{\alpha} := \nabla_{X_1} \ldots \nabla_{X_k} \alpha \in L^p(\Omega \times \Sigma, T^*\Sigma).$$

Both $d_\Sigma \tilde{\alpha}$ and $d_\Sigma^* \tilde{\alpha}$ are of class $L^p$ since

$$d_\Sigma \tilde{\alpha} = \left[ d_\Sigma, \nabla_{X_1}, \ldots, \nabla_{X_k} \right] \alpha = \nabla_{X_1} \ldots \nabla_{X_k} d_\Sigma \alpha,$$

$$d_\Sigma^* \tilde{\alpha} = \left[ d_\Sigma^*, \nabla_{X_1}, \ldots, \nabla_{X_k} \right] \alpha = \nabla_{X_1} \ldots \nabla_{X_k} d_\Sigma^* \alpha.$$

So the result for $k = 0$ implies that $\nabla_\Sigma \tilde{\alpha}$ is of class $L^p$, hence $\nabla^k \nabla_\Sigma \alpha$ also is of class $L^p$ since for all smooth vector fields $X_i$ and cutting them off – one obtains the estimate

$$\|\nabla^k \nabla_\Sigma \alpha\|_{L^p(\Omega \times \Sigma)} \leq C \left( \|d_\Sigma d^*_\Sigma \alpha\|_{L^p(\Omega \times \Sigma)} + \|\nabla d_\Sigma \alpha\|_{L^p(\Omega \times \Sigma)} + \|\alpha\|_{W^{k,p}(\Omega \times \Sigma)} \right).$$

Now this proves the lemma,

$$\|\nabla_\Sigma \alpha\|_{W^{k,p}(\Omega \times \Sigma)} \leq \|\nabla_\Sigma \alpha\|_{W^{k-1,p}(\Omega \times \Sigma)} + \|\nabla^k \nabla_\Sigma \alpha\|_{L^p(\Omega \times \Sigma)} \leq C \left( \|d_\Sigma d^*_\Sigma \alpha\|_{W^{k,p}(\Omega \times \Sigma)} + \|d_\Sigma^* \alpha\|_{W^{k,p}(\Omega \times \Sigma)} + \|\alpha\|_{W^{k,p}(\Omega \times \Sigma)} \right).$$

\[\square\]

\textbf{Proof of theorem 2.6 :}

Recall that a neighbourhood of the boundary $\partial X$ is covered by embeddings $\tilde{\tau}_{0,i} : \mathcal{U}_i \times \Sigma_i \hookrightarrow X$ such that $\tilde{\tau}_{0,i} : \mathcal{U}_i \times \Sigma_i \hookrightarrow X$ for some $\delta > 0$ and $I_{0,i} \subset S_i$ that are either compact intervals in $\mathbb{R}$ or equal to $S^1$. Moreover, one can ensure that $K \hookrightarrow \mathcal{U}$ lies in the interior of the fixed neighbourhood of $K \cap \partial X$. Since $X$ is exhausted by the compact submanifolds $X_k$ one then finds $M := X_k \subset X$ such that both $K_{\text{bdy}}$ and $K_{\text{int}}$ are contained in the interior of $M$ and thus also $K \subset M$. Now let $A \in A^{1,p}(M)$ be a solution of the boundary value problem (3) with respect to a metric $g$ that is compatible with $\tau$. Then we will prove its regularity and the corresponding estimates in the interior case on $K_{\text{int}}$ and in the boundary case on $K_{\text{bdy}}$ separately.

\textbf{Interior case :}

Firstly, since $K_{\text{int}} \subset \text{int} M$ and $K_{\text{int}} \subset \text{int} X = X \setminus \partial X$ we find a sequence of compact submanifolds $M_k \subset \text{int} X$ such that $K_{\text{int}} \subset M_{k+1} \subset \text{int} M_k \subset M$ for all $k \in \mathbb{N}$. We will prove inductively $A_{|M_k} \in A^{k,p}(M_k)$ for all $k \in \mathbb{N}$ and thus
$A|_{K_{\text{int}}} \in A(K_{\text{int}})$ is smooth. Moreover, we inductively find constants $C_k, \delta_k > 0$ such that the additional assumptions of (ii) in the theorem imply

$$\|A - A_0\|_{W^{k,p}(M_k)} \leq C_k. \tag{9}$$

Here we use the fixed smooth metric $g_0$ to define the Sobolev norms – for a sufficiently small $C^k$-neighbourhood of metrics, the Sobolev norms are equivalent with a uniform constant independent of the metric. Moreover, recall that the reference connection $A_0$ is smooth.

To start the induction we observe that this regularity and estimate are satisfied for $k = 1$ by assumption. For the induction step assume this regularity and estimate to hold for some $k \in \mathbb{N}$. Then we will use proposition 2.7 on $A|_{M_k} \in A^{k,p}(M_k)$ to deduce the regularity and estimate on $M_{k+1}$.

Every coordinate vector field on $M_{k+1}$ can be extended to a vector field $X$ on $M_k$ that vanishes near the boundary $\partial M_k$. So it suffices to consider such vector fields, i.e. use $\alpha := A - A_0$ satisfies the assumption (6). For the weak equation (7) we calculate for all $\beta = \phi \cdot \iota_X g$ with $\phi \in \mathcal{T} = C^\infty_0(M_k, g)$

$$- \int_{M_k} \langle F_A, d\alpha \beta \rangle = \int_{M_k} \langle d_A (\phi \cdot \iota_X g) \wedge F_A \rangle = \int_{\partial M_k} \langle \phi \cdot \iota_X g \wedge F_A \rangle = 0.$$

We have used Stokes’ theorem while approximating $A$ by smooth connections $\tilde{A}$, for which the Bianchi identity $d_A F_{\tilde{A}} = 0$ holds. Now proposition 2.7 and remark 2.8 imply that $A|_{M_{k+1}} \in A^{k+1,p}(M_{k+1})$. In the case (ii) of the theorem the proposition moreover provides $\delta_{k+1} > 0$ and a uniform constant $C$ for all metrics $g$ with $\|g - g_0\|_{C^{k+1}(M_k)} \leq \delta_{k+1}$ such that the following holds: If (9) holds for some constant $C_k$, then

$$\|A - A_0\|_{W^{k+1,p}(M_{k+1})} \leq C \left( 1 + \|A - A_0\|_{W^{k,p}(M_k)} + \|A - A_0\|^3_{W^{k,p}(M_k)} \right) \leq C \left( 1 + C_k + C_k^3 \right) =: C_{k+1}.$$

Here we have used the fact that the Sobolev norm of a 1-form is equivalent to an expression in terms of the Sobolev norms of its components in the coordinate charts. In case $k = 1$ and $p \leq 4$, this uniform bound is not found directly but after finitely many iterations of proposition 2.7 that give estimates on manifolds $N_1 = M_1$ and $M_2 \subset N_{i+1} \subset \text{int} N_i$. In each step one chooses a smaller $\delta_2 > 0$ and a bigger $C_2$. This iteration uses the same Sobolev embeddings as remark 2.8. This proves the induction step on the interior part $K_{\text{int}}$.

**Boundary case:**

It remains to prove the regularity and estimates on $K_{\text{bdy}}$ near the boundary. So consider a single boundary component $K' := \partial_0 (I_0 \times [0, \delta_0] \times \Sigma)$. We identify $I_0 = S^1 \cong \mathbb{R}/\mathbb{Z}$ or shift the compact interval such that $I_0 = [-r_0, r_0]$ and hence $K' = \partial_0 (-r_0, r_0) \times [0, \delta_0] \times \Sigma$ for some $r_0 > 0$. Since $K_{\text{bdy}}$ (and thus also $K'$) lies in the interior of $M$ as well as $\mathcal{V}$, one then finds $R_0 > r_0$ and $\Delta_0 > \delta_0$ such that
\( \tau_0([\!-R_0, R_0] \times [0, \Delta_0] \times \Sigma) \subset M \cap \mathcal{V} \). Here \( \tau_0 \) is the embedding that brings the metric \( g_0 \) into the standard form \( ds^2 + dt^2 + g_{0,s,t} \). A different metric \( g \) compatible with \( \tau \) defines a different embedding \( \bar{\tau} \) such that \( \bar{\tau}^* g = ds^2 + dt^2 + g_{s,t} \). However, if \( g \) is sufficiently \( C^1 \)-close to \( g_0 \), then the geodesics are \( C^0 \)-close and hence \( \bar{\tau} \) is \( C^0 \)-close to \( \tau_0 \). (These embeddings are fixed for \( t = 0 \), and for \( t > 0 \) given by the normal geodesics.) Thus for a sufficiently small choice of \( \delta > 0 \) one finds \( R > r > 0 \) and \( \Delta > \delta > 0 \) such that for all \( \tau \)-compatible metrics \( g \) in the \( \delta \)-ball around \( g_0 \)

\[ K' \subset \bar{\tau}([\!-r, r] \times [0, \delta] \times \Sigma) \quad \text{and} \quad \bar{\tau}([\!-R, R] \times [0, \Delta] \times \Sigma) \subset M \cap \mathcal{V}. \]

(In the case (i) this holds with \( r_0, \delta_0, R_0 \), and \( \Delta_0 \) for the fixed metric \( g = g_0 \).)

We will prove the regularity and estimates for \( \bar{\tau}^* A \) on \( [-r, r] \times [0, \delta] \times \Sigma \). This suffices because for \( C^{k+2} \)-close metrics the embedding \( \bar{\tau} \) will be \( C^{k+1} \)-close to the fixed \( \tau_0 \), so that one obtains uniform constants in the estimates between the \( W^{k,2} \)-norms of \( A \) and \( \bar{\tau}^* A \). Furthermore, the families \( g_{s,t} \) of metrics on \( \Sigma \) will be \( C^{k} \)-close to \( g_{0,s,t} \) for \( (s, t) \in [-R, R] \times [0, \Delta] \) if \( \delta_0 \) is chosen sufficiently small. Now choose compact submanifolds \( \Omega_k \subset \mathbb{H} := \{(s, t) \in \mathbb{R}^2 \mid t \geq 0\} \) such that for all \( k \in \mathbb{N} \)

\[ [-r, r] \times [0, \delta] \subset \Omega_{k+1} \subset \text{int} \Omega_k \subset [-R, R] \times [0, \Delta]. \]

We will prove the theorem by establishing the regularity and estimates for \( \bar{\tau}^* A \) on the \( \Omega_k \times \Sigma \) in Sobolev spaces of increasing differentiability. We distinguish the cases \( p > 4 \) and \( 4 \geq p > 2 \). In case \( p > 4 \) one uses the following induction.

**I)** Let \( p > 2 \) and suppose that \( A \in A^{1,2p}(M) \) solves (3). Then we will prove inductively that \( \bar{\tau}^* A_{|\Omega_k \times \Sigma} \in A^{k,q}(\Omega_k \times \Sigma) \) for all \( k \in \mathbb{N} \) and with \( q = p \) or \( q = 2p \) according to whether \( k \geq 2 \) or \( k = 1 \). Moreover, we will find a constant \( \delta > 0 \) and constants \( C_k, \delta_k > 0 \) for all \( k \geq 2 \) such that the following holds:

If in addition \( \|g - g_0\|_{C^{k+2}(M)} \leq \delta_k \) and

\[ \|A - A_0\|_{W^{1,2p}(M)} \leq C_1, \]

\[ \|\bar{\tau}_0^* (A - A_0)|_{\Sigma}\|_{L^\infty(\mathcal{U}, A^{0,q}(\Sigma))} \leq \delta, \]

then for all \( k \in \mathbb{N} \)

\[ \|\bar{\tau}^* (A - A_0)\|_{W^{k,q}(\Omega_k \times \Sigma)} \leq C_k. \]

This is sufficient to conclude the theorem in case \( p > 4 \) as follows. One uses I) with \( p \) replaced by \( \frac{1}{2}p \) to obtain regularity and estimates of \( A - A_0 \) in \( A^{1,p}(\Omega_1 \times \Sigma), A^{2,\frac{1}{2}p}(\Omega_2 \times \Sigma), \) and \( A^{k,\frac{1}{2}p}(\Omega_k \times \Sigma) \) for all \( k \geq 3 \). Recall that the component \( K' \) of \( K_{\text{bdy}} \) is contained in each \( \bar{\tau}(\Omega_k \times \Sigma) \). In addition, one has the Sobolev embeddings \( W^{k+1,2} \hookrightarrow W^{k,p} \hookrightarrow C^{k-1} \) on the compact 4-manifolds \( \Omega_{k+1} \times \Sigma \), c.f. [Ad, Theorem 5.4]. So this proves the regularity and estimates on \( K_{\text{bdy}} \).
In the case $4 \geq p > 2$ a preliminary iteration is required in order to achieve the regularity and estimates that are assumed in I). In contrast to I) the iteration is in $p$ instead of $k$.

II) Let $4 \geq p > 2$ and suppose that $A \in \mathcal{A}^{1,p}(M)$ solves (3). Then we will prove inductively that $\tau^* A|_{\Omega_j \times \Sigma} \in \mathcal{A}^{1,p_j}(\Omega_j \times \Sigma)$ for a sequence $(p_j)$ with $p_1 = p$ and $p_{j+1} = \theta(p_j) \cdot p_j$, where $\theta : (2,4] \rightarrow (1,\frac{17}{16})$ is monotonely increasing and thus the sequence terminates with $p_N > 4$ for some $N \in \mathbb{N}$.

Moreover, we will find constants $\delta > 0$ and constants $C_{1,j}, \delta_{1,j} > 0$ for $j = 2, \ldots, N$ such that the following holds:

If for some $j = 1, \ldots, N$ in addition $\|g - g_0\|_{C^3(M)} \leq \delta_{1,j}$ and

$$\|A - A_0\|_{W^{1,p_j}(\Omega_j \times \Sigma)} \leq C_1,$$

$$\|\tau_0^*(A - A_0)\|_{L^\infty(\Omega, A_0^0, p_j(\Sigma))} \leq \delta,$$

then

$$\|\tau^*(A - A_0)\|_{W^{1,p_j}(\Omega_j \times \Sigma)} \leq C_{1,j}.$$ 

Assuming I) and II) we first prove the theorem for the case $4 \geq p > 2$. After finitely many steps the iteration of II) gives regularity and estimates in $\mathcal{A}^{1,p_N}(\Omega_N \times \Sigma)$ with $p_N > 4$ and under the assumption $\|g - g_0\|_{C^3(M)} \leq \delta_{1,N}$ on the metric. Now if necessary decrease $p_N$ slightly such that $2p \geq p_N > 4$, then one still has $\mathcal{A}^{1,p_N}$-regularity and estimates on all components of $K_{bdy}$ as well as on $K_{int}$ (from the previous argument on the interior). Thus the assumptions of I) are satisfied with $p$ replaced by $\frac{p}{2}p_N$ and $C_1$ replaced by a combination of $C_{1,N}$ and a constant from the interior iteration (both of which only depend on $C_1$). One just has to choose $\delta_2 \leq \delta_{1,N}$ and choose the $\delta > 0$ in I) smaller than the $\delta > 0$ from II). Then the iteration in I) gives regularity and estimates of $A - A_0$ in $\mathcal{A}^{k,p_N}(\Omega_k \times \Sigma)$ for all $k \geq 2$. This proves the theorem in case $2 < p \leq 4$ due to the Sobolev embeddings $W^{k+1,\frac{17}{16}} \hookrightarrow W^{k,p} \hookrightarrow C^{k-2}$. So it remains to establish I) and II).

**Proof of I):**

The start of the induction $k = 1$ is true by assumption (after replacing $C_1$ by a larger constant to make up for the effect of $\tau^*$). For the induction step assume that the claimed regularity and estimates hold for some $k \in \mathbb{N}$ and consider the following decomposition of the connection $A$ and its curvature:

$$\tau^* A = \Phi \, ds + \Psi \, dt + B,$$

$$\tau^* F_A = F_B + (d_B \Phi - \partial_t B) \wedge ds + (d_B \Psi - \partial_t B) \wedge dt$$

$$+ (\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi]) \wedge ds \wedge dt. \quad (10)$$

Here $\Phi, \Psi \in W^{k,q}(\Omega_k \times \Sigma, g)$, and $B \in W^{k,q}(\Omega_k \times \Sigma, T^* \Sigma \otimes g)$ is a 2-parameter family of 1-forms on $\Sigma$. Choose a further compact submanifold $\Omega \subset \text{int} \Omega$, such that $\Omega_{k+1} \subset \text{int} \Omega_k$. Now we shall use proposition 2.7 to deduce the higher
regularity of $\Phi$ and $\Psi$ on $\Omega \times \Sigma$. For this purpose one has to extend the vector fields $\partial_s$ and $\partial_t$ on $\Omega \times \Sigma$ to different vector fields on $\Omega_k \times \Sigma$, both denoted by $X$, and verify the assumptions (6) and (7) of proposition 2.7. These extensions will be chosen such that they vanish in a neighbourhood of $(\partial \Omega_k \times \partial \mathbb{H}) \times \Sigma$. Then

$$\alpha := \bar{\tau}^*(A - A_0)$$

satisfies (6) on $M = \bar{\tau}(\Omega_k \times \Sigma)$ with $N = \bar{\tau}((\partial \Omega_k \cap \partial \mathbb{H}) \times \Sigma)$.

Choose a cutoff function $h \in C^\infty(\Omega_k, [0, 1])$ that equals 1 on $\Omega$ and vanishes in a neighbourhood of $\partial \Omega_k \setminus \partial \mathbb{H}$. Then firstly, $X := h \partial_t$ is a vector field as required that is perpendicular to the boundary $\partial \Omega_k \times \Sigma$. For this type of vector field we have to check the assumption (7) for all $\beta = \phi h \cdot ds$ with $\phi \in C^\infty(\Omega_k \times \Sigma, g)$. Note that $\bar{\tau}_s \beta = (\phi \cdot h) \circ \bar{\tau}^{-1} \cdot t(\bar{\tau}, \partial_t) g$ can be trivially extended to $M$ and then vanishes when restricted to $\partial M$. So we can use partial integration as in the interior case to obtain

$$\int_{\Omega_k \times \Sigma} \langle F_{\bar{\tau}^* A}, d_{\bar{\tau}^* A} \beta \rangle = \int_M \langle F_A, d_A \bar{\tau}_s \beta \rangle = - \int_{\partial M} \langle \bar{\tau}_s \beta \wedge F_A \rangle = 0.$$

Secondly, $X := h \partial_s$ also vanishes in a neighbourhood of $(\partial \Omega_k \setminus \partial \mathbb{H}) \times \Sigma$ and is tangential to the boundary $\partial \Omega_k \times \Sigma$. So we have to verify (7) for all $\beta = \phi h \cdot ds$ with $\phi \in T = C^\infty(\Omega_k \times \Sigma, g)$. Again, $\bar{\tau}_s \beta$ extends trivially to $M$. Then the partial integration yields

$$\int_{\Omega_k \times \Sigma} \langle F_{\bar{\tau}^* A}, d_{\bar{\tau}^* A} \beta \rangle = - \int_{\bar{\tau}^{-1}(\partial M)} \langle \beta \wedge \bar{\tau}^* F_A \rangle$$

$$= - \int_{(\Omega_k \cap \partial \mathbb{H}) \times \Sigma} \langle \phi h \cdot ds \wedge F_B \rangle = 0.$$

The last step uses the fact that $B(s, 0) = \bar{\tau}^* A|_{(s) \times \Sigma} \in \mathcal{L} \subset A^{0,p}_{\text{Mat}}(\Sigma)$, and hence $F_B$ vanishes on $\partial \mathbb{H} \times \Sigma$. However, we have to approximate $A$ by smooth connections in order that Stokes’ theorem holds and $F_B$ is welldefined. So this calculation crucially uses the fact that a $W^{1,p}$-connection with boundary values in the Lagrangian submanifold $L$ can be $W^{1,p}$-approximated by smooth connections with boundary values in $\mathcal{L} \cap \mathcal{A}(\Sigma)$. This was proven in [W2, Corollary 4.5].

So we have verified the assumptions of proposition 2.7 for both $\Phi = \bar{\tau}^* A(\partial_s)$ and $\Psi = \bar{\tau}^* A(\partial_t)$ and thus can deduce $\Phi, \Psi \in W^{k+1,q}(\Omega \times \Sigma)$. Moreover, under the additional assumptions of (ii) in the theorem we have the estimates

$$\|\Phi - \Phi_0\|_{W^{k+1,q}(\Omega \times \Sigma)} \leq C_s \left(1 + C_k + C_k^3 \right) =: C_{k+1}^s,$$

$$\|\Psi - \Psi_0\|_{W^{k+1,q}(\Omega \times \Sigma)} \leq C_t \left(1 + C_k + C_k^3 \right) =: C_{k+1}^t. \quad (11)$$

The constants $C_s$ and $C_t$ are uniform for all metrics in some small $C^{k+1}$-neighbourhood of $g_{0,s,t}$, so by a possibly smaller choice of $\delta_{k+1} > 0$ they become independent of $g_{s,t}$. Note that in the above estimates we also have decomposed the reference connection in the tubular neighbourhood coordinates, $\bar{\tau}^* A_0 = \Phi_0 \, ds + \Psi_0 \, dt + B_0$.

It remains to consider the $\Sigma$-component $B$ in the tubular neighbourhood.
The boundary value problem (3) becomes in the coordinates (10)

\[
\begin{aligned}
d^*_B(B - B_0) &= \nabla_s(\Phi - \Phi_0) + \nabla_t(\Psi - \Psi_0), \\
\ast F_B &= \partial_t \Phi - \partial_s \Psi + [\Psi, \Phi], \\
\partial_s B + \ast \partial_t B &= d_B \Phi + \ast d_B \Psi, \\
\Psi(s, 0) - \Psi_0(s, 0) &= 0 \quad \forall (s, 0) \in \partial \Omega_k, \\
B(s, 0) &\in \mathcal{L} \quad \forall (s, 0) \in \partial \Omega_k.
\end{aligned}
\]

Here $d_B$ is the exterior derivative on $\Sigma$ that is associated with the connection $B$, $d^*_B$ is the codervative associated with $B_0$, $\ast$ is the Hodge operator on $\Sigma$ with respect to the metric $g_{s,t}$, and $\nabla_s \Phi := \partial_s \Phi + [\Phi_0, \Phi]$, $\nabla_t \Phi := \partial_t \Phi + [\Psi_0, \Phi]$. We rewrite the first two equations in (12) as a system of differential equations for $\alpha := B - B_0$ on $\Sigma$. For each $(s, t) \in \Omega_k$

\[
d^*_S \alpha(s, t) = \xi(s, t), \quad d_S \alpha(s, t) = *\zeta(s, t).
\]

Here we have abbreviated

\[
\begin{aligned}
\xi &= *[B_0 \wedge * (B - B_0)] + \nabla_s(\Phi - \Phi_0) + \nabla_t(\Psi - \Psi_0), \\
\zeta &= - * d_S B_0 - \frac{1}{2} [B \wedge B] + \partial_t \Phi - \partial_s \Psi + [\Psi, \Phi].
\end{aligned}
\]

These are both functions in $W^{k,q}(\Omega \times \Sigma, g)$ due to the smoothness of $A_0$ and the previously established regularity of $\Phi$ and $\Psi$. (This uses the Sobolev embedding $W^{k,a,\alpha} \hookrightarrow W^{k,q} \hookrightarrow L^\infty$. So lemma 2.9 asserts that $\nabla_S(B - B_0)$ is of class $W^{k,a}$ on $\Omega \times \Sigma$, and under the assumptions of (ii) in the theorem we obtain the estimate

\[
\begin{aligned}
&\|\nabla_S(B - B_0)\|_{W^{k,a}(\Omega \times \Sigma)} \\
&\leq C(\|\xi\|_{W^{k,a}} + \|\zeta\|_{W^{k,a}} + \|B - B_0\|_{W^{k,a}}) \\
&\leq C(1 + \|B - B_0\|_{W^{k,a}} + \|\Phi - \Phi_0\|_{W^{k+a,1}} + \|\Psi - \Psi_0\|_{W^{k+1,a}}) \\
&\quad + \|B - B_0\|^2_{W^{k,a}} + \|\Phi - \Phi_0\|^2_{W^{k,a}} + \|\Psi - \Psi_0\|^2_{W^{k,a}} \\
&\leq C(1 + C_k + C^*_{k+1} + C^t_{k+1} + C^2_k) := C^\Sigma_{k+1}.
\end{aligned}
\]

Here $C$ denotes any constant that is uniform for all metrics in a $C^{k+1}$-neighbourhood of the fixed $g_{s,t}$, so this might again require a smaller choice of $\delta_{k+1} > 0$ in order that the constant $C^\Sigma_{k+1}$ becomes independent of the metric $g_{s,t}$.

Now we have established the regularity and estimate for all derivatives of $B$ of order $k+1$ containing at least one derivative in $\Sigma$-direction. (Note that in the case $k = 1$ we even have $\mathcal{L}^q$-regularity with $q = 2p$ where only $\mathcal{L}^p$-regularity was claimed. This additional regularity will be essential for the following argument.)

It remains to consider the pure $s$- and $t$- derivatives of $B$ and establish the $\mathcal{L}^p$-regularity and -estimate for $\nabla^{k+1}_B B$ on $\Omega_{k+1} \times \Sigma$, where $\nabla_B$ is the standard covariant derivative on $\mathbb{H}$ with respect to the metric $ds^2 + dt^2$. The reason for this regularity, as we shall show, is the fact that $B \in W^{k,\beta}(\Omega, A^{0,\beta}(\Sigma))$ satisfies
a Cauchy-Riemann equation with Lagrangian boundary conditions,

\[
\begin{align*}
\partial_s B + \ast \partial_t B &= G, \\
B(s, 0) &\in \mathcal{L} \quad \forall (s, 0) \in \partial \Omega.
\end{align*}
\] (15)

The inhomogeneous term is

\[G := d_B \Phi + \ast d_B \Psi \in W^{k, q}(\Omega, A^{0, p}(\Sigma)).\]

Here one uses the fact that \(W^{k, q}(\Omega \times \Sigma, T^* \Sigma \otimes g) \subset W^{k, q}(\Omega, A^{0, p}(\Sigma))\) since the smooth 1-forms are dense in both spaces and the norm on the second space is weaker than the \(W^{k, q}\)-norm on \(\Omega \times \Sigma\), c.f. [W2, Lemma 2.2].

Now one has to apply the regularity result [W2, Theorem 1.2] for the Cauchy-Riemann equation on the complex Banach space \(A^{0, p}(\Sigma)\). As reference complex structure \(J_0\) we use the Hodge \(*\) operator on \(\Sigma\) with respect to the fixed family of metrics \(g_{\delta, s, t}\) on \(\Sigma\) (that varies smoothly with \((s, t) \in \Omega\)). The smooth family \(J\) of complex structures in the equation is given by the Hodge operators with respect to the metrics \(g_{s, t}\). The Lagrangian submanifold \(\mathcal{L} \subset A^{0, p}(\Sigma)\) is totally real with respect to any Hodge operator, and it is modelled on a closed subspace of \(L^p(\Sigma, \mathbb{R}^n)\) for some \(n \in \mathbb{N}\) (see [W2, Lemma 4.2, Corollary 4.4]). In the case (ii) of the theorem moreover a family of connections \(B_0 \in C^\infty(\Omega, A(\Sigma))\) is given such that \(B_0(s, 0) \in \mathcal{L}\) for all \((s, 0) \in \partial \Omega\) and \(B\) satisfies

\[
\|B - B_0\|_{L^\infty(\Omega, A^{0, p}(\Sigma))} = \|\tilde{\tau}^*(A - A_0)|\Sigma\|_{L^\infty(\Omega, A^{0, p}(\Sigma))} \leq C\|\tilde{\tau}_0^*(A - A_0)|\Sigma\|_{L^\infty(\Omega, A^{0, p}(\Sigma))} \leq C\delta.
\]

Here one uses the fact that \(\tilde{\tau}(\Omega \times \Sigma) \subset \tilde{\tau}_0(U \times \Sigma)\) lies in a component of the fixed neighbourhood \(U\) of \(K \cap \partial X\). The assumption of closeness to \(A_0\) in \(A^{0, p}(\Sigma)\) was formulated for \(\tilde{\tau}_0^*(A - A_0)|\Sigma\). However, for a metric \(g\) in a sufficiently small \(C^2\)-neighbourhood of the fixed metric \(g_0\) the extensions \(\tilde{\tau}\) and \(\tilde{\tau}_0\) are \(C^1\)-close and one obtains the above estimate with a constant \(C\) independent of the metric. So \(B \in W^{k, q}(\Omega, A^{0, p}(\Sigma))\) satisfies the assumptions of [W2, Theorem 1.2] if \(\delta > 0\) is chosen sufficiently small. (Note that this choice is independent of \(k \in \mathbb{N}\).)

Now [W2, Theorem 1.2] asserts \(B \in W^{k+1, p}(\Omega_{k+1}, A^{0, p}(\Sigma))\). By [W2, Lemma 2.2] this also proves \(\nabla_{k+1}^\partial B \in L^p(\Omega_{k+1}, A^{0, p}(\Sigma)) = L^p(\Omega_{k+1} \times \Sigma, T^* \Sigma \otimes g)\), and this finishes the induction step \(\tilde{\tau}^* A|\Omega_{k+1} \times \Sigma \in A^{k+1, p}(\Omega_{k+1} \times \Sigma)\) for the regularity near the boundary. The induction step for the estimate in case (ii) of the theorem now follows from the estimate from [W2, Theorem 1.2],

\[
\begin{align*}
\|\nabla_{k+1}^\partial (B - B_0)\|_{L^p(\Omega_{k+1} \times \Sigma)} &\leq \|B - B_0\|_{W^{k+1, p}(\Omega_{k+1}, A^{0, p}(\Sigma))} \\
&\leq C\left(1 + \|G\|_{W^{k, q}(\Omega, A^{0, p}(\Sigma))} + \|B - B_0\|_{W^{k, q}(\Omega, A^{0, p}(\Sigma))}\right) \\
&\leq C\left(1 + C_k + C^2_k + C^2_{k+1} + C^4_{k+1}\right) =: C_{k+1}^\partial. \tag{16}
\end{align*}
\]

Here the constant from [W2, Theorem 1.2] is uniform for a sufficiently small \(C^{k+1}\)-neighbourhood of complex structures. In this case, these are the families
of Hodge operators on $\Sigma$ that depend on the metric $g_{s,t}$. Thus for sufficiently small $\delta_{k+1} > 0$ that constant (and also the further Sobolev constants that come into the estimate) becomes independent of the metric. The final constant $C_{k+1}$ then results from all the separate estimates, see the decomposition (10) and the estimates in (11), (14), and (16),

$$\| \tilde{r}^*(A - A_0) \|_{W^{k+1,p}(\Omega_{k+1} \times \Sigma)} \leq C_k + C_{k+1}^* + C^1_{k+1} + C^2_{k+1}.$$ 

**Proof of II):**

Except for the higher differentiability of $B$ in direction of $\mathbb{H}$ this iteration works by the same decomposition and equations as in I). The start of the induction $k = 1$ is given by assumption. For the induction step assume that the claimed $W^{1,pk}$-regularity and -estimates hold for some $k \in \mathbb{N}$ with $p_k \leq 4$. Then proposition 2.7 gives $\Phi, \Psi \in W^{2,q_k}(\Omega \times \Sigma)$ with corresponding estimates and

$$q_k = \begin{cases} \frac{4p_k}{8-p_k} & \text{if } p_k < 4, \\ 3 & \text{if } p_k = 4. \end{cases}$$

(In the case $p_k = 4$ one applies the proposition only assuming $W^{1,p_k'}$-regularity for $p_k' = \frac{24}{7} < 4$, then one obtains $W^{2,q_k}$-regularity with $q_k = 3$.) Now the right hand sides in (13) lie in $W^{1,q_k}(\Omega \times \Sigma)$, so lemma 2.9 gives $W^{1,q_k}$-regularity and -estimates for $\nabla_\Sigma B$ on $\Omega \times \Sigma$. Next, $B \in W^{1,p_k}(\Omega, A^{0,p}(\Sigma))$ satisfies the Cauchy-Riemann equation (15) with the inhomogeneous term $G \in W^{1,q_k}(\Omega \times \Sigma, T^* (\Omega \times \Sigma) \otimes g)$. Now we shall use the Sobolev embedding $W^{1,q_k}(\Omega \times \Sigma) \hookrightarrow L^{r_k}(\Omega \times \Sigma)$ with

$$r_k = \frac{4q_k}{4 - q_k} = \begin{cases} \frac{2p_k}{4 - p_k} & \text{if } p_k < 4, \\ 12 & \text{if } p_k = 4. \end{cases}$$

Note that $r_k > p_k$ due to $p_k > 2$, so that we have $G \in L^{r_k}(\Omega, A^{0,p}(\Sigma))$. We cannot apply [W2, Theorem 1.2] directly because that would require the initial regularity $B \in W^{1,2p}(\Omega, A^{0,p}(\Sigma))$ for some $p > 2$. However, we still proceed as in its proof and introduce the coordinates from [W2, Lemma 4.3] that straighten out the Lagrangian submanifold,

$$\Theta_{s,t} : W_{s,t} \to A^{0,p}(\Sigma).$$

Here $W_{s,t} \subset Y \times Y$ is a neighbourhood of zero, $Y$ is a closed subspace of $L^p(\Sigma, \mathbb{R}^m)$ for some $m \in \mathbb{N}$, $\Theta$ is in $C^{k+1}$-dependence on $(s,t)$ in a neighbourhood $U \subset \Omega$ of some $(s_0,0) \in \Omega \cap \partial \mathbb{H}$ and it maps diffeomorphically to a neighbourhood of $B(s,t)$ or $B_0(s,t)$ in case (ii). Thus one can write

$$B(s,t) = \Theta_{s,t}(v(s,t)) \quad \forall (s,t) \in U$$

with $v = (v_1,v_2) \in W^{1,p_k}(U, Y \times Y)$. Moreover, we have already seen that both $B$ and $\nabla_\Sigma B$ are $W^{1,q_k}$-regular on $U \times \Sigma$, so we have the regularity $B \in$
Here a smooth cutoff function as in the proof of [W2, Theorem 1.2]. We now approximate $f$ which will be important later on.

L ([W1]). So we use the embedding $W^{1,q_k}(\Sigma) \hookrightarrow L^{s_k}(\Sigma)$, see [Ad, Theorem 5.4], for

$$s_k = \begin{cases} \frac{2q_k}{2-q_k} = \frac{4p_k}{8-3p_k} & \text{if } p_k < \frac{8}{3}, \\ \frac{4p_k-80}{8-p_k} & \text{if } p_k \geq \frac{8}{3}, \\ \frac{31}{2} & \text{if } p_k = 4. \end{cases}$$

(Here we have chosen suitable values of $s_k$ for later calculations in case $p_k \geq \frac{8}{3}$ and thus $q_k \geq 2$.) The special structure of the coordinates $\Theta$ in [W2, Lemma 4.3] (it also is a local diffeomorphism between $A^{0,s_k}(\Sigma)$ and a closed subset of $L^{s_k}(\Sigma, \mathbb{R}^{2m})$ since $s_k > p_k > 2$) implies that $v \in W^{1,q_k}(U, L^{s_k}(\Sigma, \mathbb{R}^{2m}))$, which will be important later on.

The Cauchy-Riemann equation (15) now becomes

$$\begin{aligned} \partial_s v + I \partial_t v &= f, \\ v_2(s, 0) &= 0 \quad \forall s \in \mathbb{R}. \end{aligned}$$

Here $I = (d_s \Theta)^{-1} \ast (d_s \theta) \in W^{1,p_k}(U, \text{End}(Y \times Y))$ and

$$f = (d_s \Theta)^{-1}((G - \partial_s \Theta(v) - \ast \partial_t \Theta(v)) \in L^{s_k}(U, Y \times Y).$$

We now approximate $f$ in $L^{s_k}(U, Y \times Y)$ by smooth functions that vanish on $\partial U$, then partial integration in [W2, (9)] yields for all $\phi \in C^\infty(U, Y^* \times Y^*)$ and a smooth cutoff function as in the proof of [W2, Theorem 1.2]

$$\begin{aligned} \int_U \langle hv, \Delta \phi \rangle &= \int_U \langle f, \partial_s (h \phi) - \partial_t (h \cdot I^* \phi) \rangle + \int_U \langle \tilde{F}, \phi \rangle \\ &+ \int_{\partial U \cap \partial \Sigma} \langle v_1, \partial_t (h \phi_1) + \partial_s (h \phi_2) \rangle. \end{aligned}$$

(17)

Here $\tilde{F} = (\Delta h)v + 2(\partial_s h) \partial_s v + 2(\partial_t h) \partial_t v + h(\partial_t I) \partial_s v - h(\partial_s I) \partial_t v$ contains the crucial terms $\partial_t (h \cdot I^* \partial_t \phi)$ and $\partial_s (h \cdot I^* \partial_s \phi)$ and thus lies in $L^{\frac{p_k}{2}}(U, Y \times Y)$. This is a weak Laplace equation with Dirichlet boundary conditions for $hv_2$, Neumann boundary conditions for $hv_1$, and with the inhomogeneous term in $W^{-1,r_k}(U, Y \times Y)$. The latter is the dual space of $W^{1,r_k'}(U, Y^* \times Y^*)$ with $\frac{1}{r_k} + \frac{1}{r_k'} = 1$. (The inclusion $L^{\frac{p_k}{2}}(U) \hookrightarrow W^{-1,r_k}(U)$ is continuous as can be seen via the dual embedding that is due to $\frac{1}{2} - \frac{1}{r_k} \geq -1 + \frac{1}{p_k - 2}$.) Recall that $Y \subset L^p(\Sigma, \mathbb{R}^m)$ is a closed subspace. Since $r_k > p$ the special regularity result [W2, Lemma 2.1] for the Laplace equation with values in a Banach space cannot be applied to deduce $hv \in W^{1,r_k}(U, Y \times Y)$. However, the general regularity theory for the Laplace equation extends to functions with values in a Hilbert space (c.f. [W1]). So we use the embedding $L^p(\Sigma) \hookrightarrow L^2(\Sigma)$. Then (17) is a weak Laplace equation with the inhomogeneous term in $W^{-1,r_k}(U, L^2(\Sigma, \mathbb{R}^{2m}))$ and enables us to deduce $hv \in W^{1,r_k}(U, L^2(\Sigma, \mathbb{R}^{2m}))$ and thus $v \in W^{1,r_k}(U, L^2(\Sigma, \mathbb{R}^{2m}))$. 28
with the corresponding estimates for some smaller domain $\bar{U}$ (a finite union of these still covers a neighbourhood of $\Omega \cap \partial \mathbb{H}$). Furthermore, recall that $v \in W^{1,q_k}(\bar{U}, L^{s_k}(\Sigma, \mathbb{R}^{2m}))$. Now we claim that the following inclusion with the corresponding estimates holds for some suitable $p_{k+1}$

$$W^{1,r_k}(\bar{U}, L^2(\Sigma)) \cap W^{1,q_k}(\bar{U}, L^{s_k}(\Sigma)) \subset W^{1,p_{k+1}}(\bar{U}, L^{p_{k+1}}(\Sigma)).$$

(18) To show (18) it suffices to estimate the $L^{p_{k+1}}(\bar{U} \times \Sigma)$-norm of a smooth function by its $L^{r_k}(\bar{U}, L^2(\Sigma))$- and $L^{q_k}(\bar{U}, L^{s_k}(\Sigma))$-norms. Let $\alpha > 2$ and $t \in [1, 2)$, then the Hölder inequality gives for all $f \in C^\infty(\bar{U} \times \Sigma, \mathbb{R}^{2m})$

$$\|f\|_{L^\alpha(\bar{U} \times \Sigma)}^\alpha = \int_{\bar{U}} \int_{\Sigma} |f|^\alpha = \int_{\bar{U}} \int_{\Sigma} |f|^t |f|^{\alpha - t} \leq \int_{\bar{U}} \|f\|_{L^2(\Sigma)}^t \|f\|_{L^{\alpha/(\alpha - t)}(\Sigma)}^{\alpha - t} \leq \|f\|_{L^t(\bar{U}, L^2(\Sigma))}^t \|f\|_{L^{\alpha/(\alpha - t)}(\bar{U}, L^{2/(\alpha - t)}(\Sigma))}^{\alpha - t} \leq \|f\|_{L^t(\bar{U}, L^2(\Sigma))}^t + \|f\|_{L^{\alpha/(\alpha - t)}(\bar{U}, L^{2/(\alpha - t)}(\Sigma))}^{\alpha - t}.$$ Here we abbreviated $r := r_k > p_k > 2$. Now we want

$$q_k = \frac{r_k(\alpha - t)}{r_k - t} \quad \text{and} \quad s_k = \frac{2(\alpha - t)}{2 - t}. \quad (19)$$

Indeed, in the case $p_k = 4$ our choices $q_k = 3$, $r_k = 12$, and $s_k = \frac{11}{2}$ together with $t := \frac{5}{3}$ and $\alpha := \frac{17}{4}$ solve these equations. So we obtain $p_{k+1} = \alpha = \frac{17}{16}p_k$. In case $p_k < 4$ the first equation gives

$$\alpha = \frac{4 + t}{8 - p_k}p_k. \quad (20)$$

If moreover $p_k \geq \frac{8}{3}$, then we choose $t := \frac{8}{3}$ to obtain $\alpha = \frac{17}{24 - 3p_k}p_k \geq \frac{17}{16}p_k$. This also solves (19) with our choice $s_k = \frac{44p_k - 80}{8 - p_k}$, so we obtain $p_{k+1} = \frac{17}{16}p_k$. Finally, in case $\frac{8}{3} > p_k > 2$ one obtains from (19)

$$t = \frac{p_k^2}{-p_k^2 + 7p_k - 8} \in [1, 2).$$

Inserting this in (20) yields $\alpha = \theta(p_k) \cdot p_k$ with

$$\theta(p_k) = \frac{3p_k - 4}{-p_k^2 + 7p_k - 8}.$$ One then checks that $\theta(2) = 1$ and $\theta'(p) > 0$ for $p > 2$, thus $\theta(p) > 1$ for $p > 2$. Moreover, $\theta'(\frac{8}{3}) = 0$, so $\theta(p') = \frac{17}{16}$ for some $p' \in (2, \frac{8}{3})$. Now for $p \geq p'$ we extend the function constantly to obtain a monotonely increasing function $\theta : (2, 4] \to (1, \frac{17}{16})$. With this modified function we finally choose
\( p_{k+1} = \theta(p_k) \cdot p_k \) for all \( 2 < p_k \leq 4 \). This finishes the proof of (18) and thus shows that \( v \in W^{1,p_{k+1}}(U, L^{p_{k+1}}(\Sigma)) \).

In addition, note that our choice of \( p_{k+1} \leq \alpha \) will always satisfy \( p_{k+1} \leq r_k \).

In case \( p_k = 4 \) see the actual numbers, in case \( p_k < 4 \) this is due to (20), \( t \leq 2 \), and \( p_k > 2 \),

\[
\alpha \leq \frac{6}{8 - p_k} p_k \leq \frac{2}{4 - p_k} p_k = r_k.
\]

Now we again use the special structure of the coordinates \( \Theta \) in [W2, Lemma 4.3] to deduce that \( B = \Theta \circ v \in W^{1,p_{k+1}}(U, A^{0,p_{k+1}}(\Sigma)) \) with the corresponding estimates. Above, we already established the \( W^{1,r_k} \)- and thus \( W^{1,p_{k+1}} \)-regularity and -estimates for \( \Phi \) and \( \Psi \) as well as \( B \in L^{p_{k+1}}(U, A^{1,p_{k+1}}(\Sigma)) \). (Recall the Sobolev embedding \( W^{1,q_k} \hookrightarrow L^{r_k} \), and that \( p_k \geq q_k \) and \( r_k \geq p_{k+1} \), so we have \( L^{r_k}(\tilde{U}, L^{r_k}(\Sigma)) \)-regularity of \( B \) as well as \( \nabla B \).) Putting all this together we have established the \( W^{1,p_{k+1}} \)-regularity and -estimates for \( \tilde{\tau}^* A \) over \( \tilde{U}_i \times \Sigma \), where the \( \tilde{U}_i \) cover a neighbourhood of \( \Omega_{k+1} \cap \partial \mathcal{H} \). The interior regularity again follows directly from proposition 2.7.

This iteration gives a sequence \((p_k)\) with \( p_{k+1} = \theta(p_k) \cdot p_k \geq \theta(p) \cdot p_k \). So this sequence grows at a rate greater or equal to \( \theta(p) > \theta(2) = 1 \) and hence reaches \( p_N > 4 \) after finitely many steps. This finishes the proof of II) and the theorem.

\[\square\]

**Proof of theorem A:**

Fix a solution \( A \in A^{1,p}_{\text{loc}}(X) \) of (2) with \( p > 2 \). We have to find a gauge transformation \( u \in G^{2,p}_{\text{loc}}(X) \) such that \( u^* A \in \mathcal{A}(X) \) is smooth. Recall that the manifold \( X = \bigcup_{k \in \mathbb{N}} X_k \) is exhausted by compact submanifolds \( X_k \) meeting the assumptions of proposition 2.1. So it suffices to prove for every \( k \in \mathbb{N} \) that there exists a gauge transformation \( u \in G^{2,p}(X_k) \) such that \( u^* A |_{X_k} \) is smooth.

For that purpose fix \( k \in \mathbb{N} \) and choose a compact submanifold \( M \subset X_k \) that is large enough such that theorem 2.6 applies to the compact subset \( K := X_k \subset M \). Next, choose \( A_0 \in \mathcal{A}(M) \) such that \( \| A - A_0 \|_{W^{1,p}(M)} \) and \( \| A - A_0 \|_{L^q(M)} \) are sufficiently small for the local slice theorem, proposition 2.2, to apply to \( A_0 \) with the reference connection \( \tilde{A} = A \). Here due to \( p > 2 \) one can choose \( q > 4 \) in the local slice theorem such that the Sobolev embedding \( W^{1,p}(M) \hookrightarrow L^q(M) \) holds.

Then by proposition 2.2 and remark 2.3 (ii) one obtains a gauge transformation \( u \in G^{2,p}(M) \) such that \( u^* A \) is in relative Coulomb gauge with respect to \( A_0 \). Moreover, \( u^* A \) also solves (2) since both the anti-self-duality equation and the Lagrangian submanifolds \( L_i \) are gauge invariant. The latter is due to the fact that \( u \) restricts to a gauge transformation in \( G^{1,p}(\Sigma_i) \) on each boundary slice \( \tau_i(\{s \times \Sigma_i \}) \) due to the Sobolev embedding \( G^{2,p}(\mathcal{U}_i \times \Sigma) \subset W^{1,p}(\mathcal{U}_i, G^{1,p}(\Sigma_i)) \hookrightarrow C^0(\mathcal{U}_i, G^{1,p}(\Sigma_i)) \). So \( u^* A \in A^{1,p}(M) \) is a solution of (3) and theorem 2.6 (i) asserts that \( u^* A |_{X_k} \in \mathcal{A}(X_k) \) is indeed smooth.

Such a gauge transformation \( u \in G^{2,p}(X_k) \) can be found for every \( k \in \mathbb{N} \), hence proposition 2.1 (i) asserts that there exists a gauge transformation \( u \in G^{2,p}_{\text{loc}}(X) \) on the full noncompact manifold such that \( u^* A \in \mathcal{A}(X) \) is smooth as claimed.

\[\square\]
Proof of theorem B:

Fix a smoothly convergent sequence of metrics $g^\nu \to g$ that are compatible to $\tau$ and let $A^\nu \in A_{0,c}^{1,0}(X)$ be a sequence of solutions of (2) with respect to the metrics $g^\nu$. Recall that the manifold $X = \bigcup_{k \in \mathbb{N}} X_k$ is exhausted by compact submanifolds $X_k$ meeting the assumptions of proposition 2.1. We will find a subsequence (again denoted $A^\nu$) and a sequence of gauge transformations $u^\nu \in G^\nu_{loc}(X)$ such that the sequence $u^\nu \ast A^\nu$ is bounded in the $W^{1,p}$-norm on $X_k$ for all $\ell \in \mathbb{N}$ and $k \in \mathbb{N}$. Then due to the compact Sobolev embeddings $W^{1,p}(X_k) \hookrightarrow C^{\ell-2}(X_k)$ one finds a further (diagonal) subsequence that converges uniformly with all derivatives on every compact subset of $X$.

By proposition 2.1 (ii) it suffices to construct the gauge transformations and establish the claimed uniform bounds over $X_k$ for all $k \in \mathbb{N}$ and for any subsequence of the connections (again denoted $A^\nu$). So fix $k \in \mathbb{N}$ and choose a compact submanifold $M \subset X$ such that theorem 2.6 holds with $K = X_k \subset M$. Choose a further compact submanifold $M' \subset X$ such that theorem 2.6 holds with $K = M \subset M'$. Then by assumption of the theorem $\|F_{A^\nu}\|_{L^p(M')}$ is uniformly bounded. So the weak Uhlenbeck compactness theorem, proposition 2.4, provides a subsequence (still denoted $A^\nu$), a limit connection $A_0 \in A^{1,0}(M')$, and gauge transformations $u^\nu \in G^2(M')$ such that $u^\nu \ast A^\nu \to A_0$ in the weak $W^{1,p}$-topology. The limit $A_0$ then satisfies the boundary value problem (2) with respect to the limit metric $g$. So as in the proof of theorem A one finds a gauge transformation $u_0 \in G^2(M)$ such that $u_0^\ast A_0 \in \mathcal{A}(M)$ is smooth. Now replace $A_0$ by $u_0^\ast A_0$ and $u^\nu$ by $u^\nu \ast u_0 \in G^2(M)$, then one still has a $W^{1,p}$-bound, $\|u^\nu \ast A^\nu - A_0\|_{W^{1,p}(M)} \leq c_0$ for some constant $c_0$, see [W2, Lemma A.5].

Due to $p > 2$ one can now choose $q > 4$ in the local slice theorem such that the Sobolev embedding $W^{1,p}(M) \hookrightarrow L^q(M)$ is compact. Hence for a further subsequence of the connections $u^\nu \ast A^\nu \to A_0$ in the $L^q$-norm. Let $\varepsilon > 0$ be the constant from proposition 2.2 for the reference connection $\tilde{A} = A_0$, then one finds a further subsequence such that $\|u^\nu \ast A^\nu - A_0\|_{L^q(M)} \leq \varepsilon$ for all $\nu \in \mathbb{N}$. So the local slice theorem provides further gauge transformations $\tilde{u}^\nu \in G^2(M)$ such that the $\tilde{u}^\nu \ast A^\nu$ are in relative Coulomb gauge with respect to $A_0$. The gauge transformed connections still solve (2), hence the $\tilde{u}^\nu \ast A^\nu$ are solutions of (3). Moreover, we have $\|\tilde{u}^\nu \ast A^\nu - A_0\|_q \leq C_{CC} \|\tilde{u}^\nu \ast A^\nu - A_0\|_q$, hence $\tilde{u}^\nu \ast A^\nu \to A_0$ in the $L^q$-norm, and

$$\|\tilde{u}^\nu \ast A^\nu - A_0\|_{W^{1,p}(M)} \leq C_{CC} c_0.$$ 

The higher $W^{k,p}$-bounds will now follow from theorem 2.6, so we first have to verify its assumptions. Fix the metric $g_0 := g$ and a compact neighbourhood $\mathcal{V} = \bigcup_{i=1}^{n} \mathring{\Sigma}_i(\mathcal{U}_i \times \Sigma_i)$ of $K \cap \partial X$. Then the $\mathring{\pi}_{0,i}(\tilde{u}^\nu \ast A^\nu - A_0)|_{\Sigma_i}$ are uniformly $W^{1,p}$-bounded and converge to zero in the $L^q$-norm on $\mathcal{U}_i \times \Sigma_i$ as seen above.

Now the embedding

$$W^{1,p}(\mathcal{U}_i \times \Sigma_i, T^*\Sigma_i \otimes g) \hookrightarrow L^\infty(\mathcal{U}_i, A^{0,p}(\Sigma_i))$$

is compact by lemma 2.5. Thus one finds a subsequence such that the $\mathring{\pi}_{0,i}(\tilde{u}^\nu \ast A^\nu)|_{\Sigma_i}$
converge in $L^\infty(U_i, A^0, p(\Sigma_i))$. The limit can only be $\bar{\tau}_0^* A_0|_{\Sigma_i}$ since this already is the $L^q$-limit. Now in theorem 2.6 (ii) choose the constant $C_1 = C_{C_0 C_0}$ and let $\delta > 0$ be the constant determined from $C_1$. Then one can take a subsequence such that

$$\|\bar{\tau}_0^* (\hat{\varphi}^* A^\nu - A_0)\|_{L^\infty(U_i, A^0, p(\Sigma_i))} \leq \delta \quad \forall i = 1, \ldots, n, \forall \nu.$$ 

Now theorem 2.6 (ii) provides the claimed uniform bounds as follows. Fix $\ell \in \mathbb{N}$, then

$$\|\hat{\varphi}^* A^\nu - g\|_{C^{\ell+2}(M)} \leq \delta_\ell \quad \forall \nu \geq \nu_\ell$$

with some $\nu_\ell \in \mathbb{N}$, and thus

$$\|\hat{\varphi}^* A^\nu - A_0\|_{W^{\ell,p}(X_k)} \leq C_\ell \quad \forall \nu \geq \nu_\ell.$$ 

This finally implies the uniform bound for this subsequence,

$$\sup_{\nu \in \mathbb{N}} \|\hat{\varphi}^* A^\nu\|_{W^{\ell,p}(X_k)} < \infty.$$ 

Here the gauge transformations $\hat{\varphi}^* \in \mathcal{G}^{2,p}(X_k)$ still depend on $k \in \mathbb{N}$ and are only defined on $X_k$. But now proposition 2.1 (ii) provides a subsequence of $(A^\nu)$ and gauge transformations $\hat{\varphi}^* \in \mathcal{G}^{2,p}(X)$ defined on the full noncompact manifold such that $\hat{\varphi}^* A^\nu$ is uniformly bounded in every $W^{\ell,p}$-norm on every compact submanifold $X_k$. Now one can iteratively use the compact Sobolev embeddings $W^{\ell+2,p}(X_k) \hookrightarrow C^{\ell}(X_k)$ for each $\ell \in \mathbb{N}$ to find a further subsequence of the connections that converges in $C^{\ell}(X_k)$. If in each step one fixes one further element of the sequence, then this iteration finally yields a sequence of connections that converges uniformly with all derivatives on every compact subset of $X$ to a smooth connection $A \in \mathcal{A}(X)$. \hfill \Box

### 3 Fredholm theory

This section concerns the linearization of the boundary value problem (1) in the special case of a compact 4-manifold of the form $X = S^1 \times Y$, where $Y$ is a compact orientable 3-manifold whose boundary $\partial Y = \Sigma$ is a disjoint union of connected Riemann surfaces. The aim of this section is to prove theorem C.

So we equip $S^1 \times Y$ with a product metric $\hat{g} = ds^2 + g_s$ (where $g_s$ is an $S^1$-family of metrics on $Y$) and assume that this is compatible with the natural space-time splitting of the boundary $\partial X = S^1 \times \Sigma$. This means that for some $\Delta > 0$ there exists an embedding

$$\tau : S^1 \times [0, \Delta) \times \Sigma \hookrightarrow S^1 \times Y$$

preserving the boundary, $\tau(s, 0, z) = (s, z)$ for all $s \in S^1$ and $z \in \Sigma$, such that

$$\tau^* \hat{g} = ds^2 + dt^2 + g_{s,t}.$$ 

Here $g_{s,t}$ is a smooth family of metrics on $\Sigma$. This assumption on the metric implies that the normal geodesics are independent of $s \in S^1$ in a neighbourhood of
the boundary. So in fact, the embedding is given by \( \tau(s, t, z) = (s, \gamma_z(t)) \), where \( \gamma \) is the normal geodesic starting at \( z \in \Sigma \). This seems like a very restrictive assumption, but it suffices for our application to Riemannian 4-manifolds with a boundary space-time splitting. Indeed, the neighbourhoods of the compact boundary components are isometric to \( S^1 \times Y \) with \( Y = [0, \Delta] \times \Sigma \) and a metric \( ds^2 + dt^2 + g_{s,t} \).

Now fix \( p > 2 \) and let \( \mathcal{L} \subset A^{0,p}_\text{flat}(\Sigma) \) be a gauge invariant Lagrangian submanifold of \( A^{0,p}(\Sigma) \) as in the introduction. Then for \( \tilde{A} \in A^{1,p}(S^1 \times Y) \) we consider the nonlinear boundary value problem

\[
\begin{cases}
*F_{\tilde{A}} + F_\tilde{A} = 0, \\
\tilde{A}_{|_{\{s\} \times \partial Y}} \in \mathcal{L} \quad \forall s \in S^1.
\end{cases}
\tag{21}
\]

Fix a smooth connection \( \tilde{A} \in A(S^1 \times Y) \) with Lagrangian boundary values (but not necessarily a solution of this boundary value problem). It can be decomposed as \( \tilde{A} = A + \Phi ds \) with \( \Phi \in \mathcal{C}^\infty(S^1 \times Y, g) \) and with \( A \in \mathcal{C}^\infty(S^1 \times Y, T^*Y \otimes g) \) satisfying \( A_s := A(s)|_{\partial Y} \in \mathcal{L} \) for all \( s \in S^1 \). Similarly, a tangent vector \( \tilde{\alpha} \) to \( A^{1,p}(S^1 \times Y) \) decomposes as \( \tilde{\alpha} = \alpha + \varphi ds \) with \( \varphi \in W^{1,p}(S^1 \times Y, g) \) and \( \alpha \in W^{1,p}(S^1 \times Y, T^*Y \otimes g) \). Now let \( E^{1,p}_A \subset W^{1,p}(S^1 \times Y, T^*Y \otimes g) \) be the subspace of \( S^1 \)-families of 1-forms \( \alpha \) that satisfy the boundary conditions from the linearization of (21) and the Coulomb gauge,

\[ *\alpha(s)|_{\partial Y} = 0 \quad \text{and} \quad \alpha(s)|_{\partial Y} \in T_{A_s} \mathcal{L} \quad \text{for all} \ s \in S^1. \]

Then the linearized operator for the study of the moduli space of gauge equivalence classes of solutions of (21) is as in the introduction

\[ D_{(A, \Phi)} : E^{1,p}_A \times W^{1,p}(S^1 \times Y, g) \longrightarrow L^p(S^1 \times Y, T^*Y \otimes g) \times L^p(S^1 \times Y, g) \]

given by

\[ D_{(A, \Phi)}(\alpha, \varphi) = (\nabla_s \alpha - d_A \varphi + *a_A \alpha, \nabla_s \varphi - d^*_A \alpha). \]

Here \( d_A \) denotes the exterior derivative corresponding to the connection \( A(s) \) on \( Y \) for all \( s \in S^1 \), \( * \) denotes the Hodge operator on \( Y \) with respect to the \( s \)-dependent metric \( g_s \) on \( Y \), and we use the notation \( \nabla_s \alpha := \partial_s \alpha + [\Phi, \alpha] \). Our main result, theorem C (i), is the Fredholm property of \( D_{(A, \Phi)} \). We now give an outline of its proof.

The first crucial point is the estimate, theorem C (ii), which ensures that \( D_{(A, \Phi)} \) has a closed image and a finite dimensional kernel. It can be rephrased as follows due to the identities

\[
\begin{align*}
d_A^* \tilde{\alpha} &= \frac{1}{2} * (\nabla_s \alpha - d_A \varphi + *d_A \alpha) - \frac{1}{2} \left( \nabla_s \alpha - d_A \varphi + *d_A \alpha \right) \wedge ds, \\
d_A^* \tilde{\alpha} &= -\nabla_s \varphi + d_A^* \alpha.
\end{align*}
\]

**Lemma 3.1** There is a constant \( C \) such that for all \( \tilde{\alpha} \in W^{1,p}(X, T^*X \otimes g) \) satisfying

\[ *\tilde{\alpha}|_{\partial X} = 0 \quad \text{and} \quad \tilde{\alpha}|_{\{s\} \times \partial Y} \in T_{A_s} \mathcal{L} \quad \forall s \in S^1 \]

satisfying

\[ *\tilde{\alpha}|_{\partial X} = 0 \quad \text{and} \quad \tilde{\alpha}|_{\{s\} \times \partial Y} \in T_{A_s} \mathcal{L} \quad \forall s \in S^1 \]

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one has the estimate
\[ \|\tilde{\alpha}\|_{W^{1,p}} \leq C(\|d^*_A\tilde{\alpha}\|_p + \|d^*_A\tilde{\alpha}\|_p + \|\tilde{\alpha}\|_p). \]

The second part of the Fredholm theory for \( D_{(A,\Phi)} \) is the identification of the cokernel with the kernel of a slightly modified linearized operator, which will be used to prove that the cokernel is finite dimensional. To be more precise let \( \sigma : S^1 \times Y \to S^1 \times Y \) denote the reflection given by \( \sigma(s,y) := (-s,y) \), where \( S^1 \cong \mathbb{R}/\mathbb{Z} \). Then we will establish the following duality:

\[ (\beta,\zeta) \in (\text{im } D_{(A,\Phi)})^\perp \iff (\beta \circ \sigma, \zeta \circ \sigma) \in \ker D_{\sigma^*(A,\Phi)}, \]

where \( D_{\sigma^*(A,\Phi)} \) is the linearized operator at the connection \( \sigma^* \tilde{A} = A \circ \sigma - \Phi \circ \sigma \) with respect to the metric \( \sigma^*\tilde{g} \) on \( S^1 \times Y \). Once we know that \( \text{im } D_{(A,\Phi)} \) is closed, this gives an isomorphism between \( (\ker D_{(A,\Phi)})^\perp \cong (\ker D_{\sigma^*(A,\Phi)})^\perp \) and \( \ker D_{\sigma^*(A,\Phi)} \). Here \( Z^* \) denotes the dual space of a Banach space \( Z \), and for a subspace \( Y \subset Z \) we denote by \( Y^\perp \subset Z^* \) the space of linear functionals that vanish on \( Y \). Now the estimate in theorem C (ii) will also apply to \( D_{\sigma^*(A,\Phi)} \), and this implies that its kernel – and hence the cokernel of \( D_{(A,\Phi)} \) – is of finite dimension. The main difficulty in establishing the above duality is the regularity result theorem C (iii).

This regularity as well as the estimate in theorem C (ii) or lemma 3.1 will be proven analogously to the nonlinear regularity and estimates in section 2. Again, the interior regularity and estimate is standard elliptic theory, and one has to use a splitting near the boundary. We shall show that the \( S^1 \)- and the normal component both satisfy a Laplace equation with Neumann and Dirichlet boundary conditions respectively. The \( \Sigma \)-component will again gives rise to a (weak) Cauchy-Riemann equation in a Banach space, only this time the boundary values will lie in the tangent space of the Lagrangian. In contrast to the required \( L^p \)-estimates we shall first show that the \( L^2 \)-estimate for \( L^p \)-regular 1-forms can be obtained by more elementary methods. These were already outlined in [S] as a first indication for the Fredholm property of the boundary value problem (21).

Let \( \tilde{\alpha} \in W^{1,p}(X,T^*X \otimes g) \) be as supposed for some \( p > 2 \). From the first boundary condition \( \partial \tilde{\alpha}(\partial X) = 0 \) one obtains

\[ \|\nabla\tilde{\alpha}\|^2 = \|d\tilde{\alpha}\|^2 + \|d^*\tilde{\alpha}\|^2 - \int_{\partial X} \tilde{g}(Y_{\tilde{\alpha}}, \nabla_{Y_{\tilde{\alpha}}} \nu). \]

Here one has \( \int_{\partial X} \tilde{g}(Y_{\tilde{\alpha}}, \nabla_{Y_{\tilde{\alpha}}} \nu) \geq -C\|\tilde{\alpha}\|^2_{L^2(\partial X)} \) since the vector field \( Y_{\tilde{\alpha}} \) is given by \( \nu_{Y_{\tilde{\alpha}}} \tilde{g} = \tilde{\alpha} \). In this last term one uses the following estimate for general \( 1 < p < \infty \).

Let \( \tau : [0,\Delta] \times \partial X \to X \) be a diffeomorphism to a tubular neighbourhood of \( \partial X \) in \( X \). Then for all \( \delta > 0 \) one finds a constant \( C_\delta \) such that for all
f ∈ W^{1,p}(X)

\[ \|f\|_{L^p(\partial X)}^p \]
\[ = \int_{\partial X} \int_0^1 \frac{d}{ds} \left( (s - 1)|f(\tau(s, z))|^p \right) ds \, d^3z \]
\[ \leq \int_{\partial X} \int_0^1 |f(\tau(s, z))|^p \, ds \, d^3z + \int_{\partial X} \int_0^1 |p|f(\tau(s, z))|^{p-1}|\partial_s f(\tau(s, z))| \, ds \, d^3z \]
\[ \leq C(\|f\|_{L^p(\partial X)}^p + \|f\|_{L^p(\partial X)}^{p-1} \|\nabla f\|_{L^p(\partial X)}) \]
\[ \leq (\delta \|f\|_{W^{1,p}(\partial X)} + C_\delta \|f\|_{L^p(\partial X)})^p. \] (22)

This uses the fact that for all \( x, y \geq 0 \) and \( \delta > 0 \)
\[ \delta^p y^p \leq \left\{ \begin{array}{ll} \delta^p y^p & ; x \leq \delta \frac{y}{\alpha} \\ \delta^{-1} \alpha^2 x^p & ; x \geq \delta \frac{y}{\alpha} \end{array} \right\} \leq (\delta y + \delta^{-\frac{1}{p}} x)^p. \]

So we obtain
\[ \|\tilde{\alpha}\|_{W^{1,2}} \leq C\left( \|d_{\tilde{A}}\tilde{\alpha}\|_2 + \|d_{\tilde{A}}^*\tilde{\alpha}\|_2 + \|\tilde{\alpha}\|_2 \right). \] (23)

In fact, the analogous \( W^{1,p}\)-estimates hold true for general \( p \), as is proven e.g. in [W1, Theorem 6.1]. However, in the case \( p = 2 \) one can calculate further for all \( \delta > 0 \)
\[ \|d_{\tilde{A}}\tilde{\alpha}\|^2_2 = \int_X \langle d_{\tilde{A}}\tilde{\alpha}, 2d_{\tilde{A}}^*\tilde{\alpha} \rangle - \int_X \langle d_{\tilde{A}}\tilde{\alpha} \wedge d_{\tilde{A}}\tilde{\alpha} \rangle 
\[ = 2\|d_{\tilde{A}}\tilde{\alpha}\|^2_2 - \int_X \langle \tilde{\alpha} \wedge [F_{\tilde{A}} \wedge \tilde{\alpha}] \rangle - \int_{\partial X} \langle \tilde{\alpha} \wedge d_{\tilde{A}}\tilde{\alpha} \rangle 
\[ \leq 2\|d_{\tilde{A}}\tilde{\alpha}\|^2_2 + C_\delta \|\tilde{\alpha}\|_2^2 + \delta\|\tilde{\alpha}\|_{W^{1,2}}^2. \] (24)

Here the boundary term above is estimated as follows. We use the universal covering of \( S^1 = \mathbb{R}/\mathbb{Z} \) to integrate over \([0, 1] \times \partial Y\) instead of \( \partial X \). Introduce \( A := (A_s)_{s \in S^1} \), which is a smooth path in \( \mathcal{L} \). Then using the splitting \( \tilde{\alpha}|_{\partial X} = \alpha + \varphi \) with \( \alpha : S^1 \times \Sigma \to T^*\Sigma \otimes \mathfrak{g} \) and \( \varphi : S^1 \times \Sigma \to \mathfrak{g} \) one obtains
\[ - \int_{\partial X} \langle \alpha \wedge d_{\tilde{A}}\tilde{\alpha} \rangle 
\[ = - \int_0^1 \int_{\Sigma} \langle \varphi, d_A\alpha \rangle \, d\omega \wedge ds - \int_0^1 \int_{\Sigma} \langle \alpha \wedge (d_A\varphi - \nabla_s\alpha) \rangle \wedge ds \]
\[ = \int_0^1 \int_{\Sigma} \langle \alpha \wedge \nabla_s\alpha \rangle \wedge ds \]
\[ \leq \delta\|\tilde{\alpha}\|_{W^{1,2}}^2 + C_\delta \|\tilde{\alpha}\|_{L^2(\Sigma)}. \]

Firstly, we have used the fact that \( d_A\alpha|_{\Sigma} = 0 \) since \( \alpha(s) \in T_{A_s}\mathcal{L} \cap \ker d_{\tilde{A}} \) for all \( s \in S^1 \). Secondly, we have also used that both \( \alpha \) and \( d_A\varphi \) lie in \( T_A\mathcal{L} \), hence the symplectic form \( \int_{\Sigma} \langle \alpha \wedge d_{\tilde{A}}\varphi \rangle \) vanishes for all \( s \in S^1 \). This is not strictly
true since $\tilde{\alpha}$ only restricts to an $L^p$-regular 1-form on $\partial X$. However, as 1-form on $[0,1] \times Y$ it can be approximated as follows by smooth 1-forms that meet the Lagrangian boundary condition on $[0,1] \times \Sigma$.

We use the linearization of the coordinates in [W2, Lemma 4.3] at $A_s$ for all $s \in [0,1]$. Since the path $s \mapsto A_s \in \mathcal{L} \cap \mathcal{A}(\Sigma)$ is smooth, this gives a smooth path of diffeomorphisms $\Theta_s$ for any $q > 2$,

$$\Theta_s : Z \times Z \rightarrow L^q(\Sigma, T^*\Sigma \otimes \mathfrak{g})$$

where $Z := W^{1,q}(\Sigma, \mathfrak{g}) \times \mathbb{R}^m$ and the $\gamma_i \in C^\infty([0,1] \times \Sigma, T^*\Sigma \otimes \mathfrak{g})$ satisfy $\gamma_i(s) \in T_A \mathcal{L}$ for all $s \in [0,1]$. We perform above estimate on $[0,1] \times Y$ since we can not necessarily achieve $\gamma_i(0) = \gamma_i(1)$. In these coordinates, we mollify to obtain the required smooth approximations of $\alpha$ near the boundary. Furthermore, we use these coordinates for $q = 3$ to write the smooth approximations on the boundary as $\alpha(s) = d_A \xi(s) + \sum_{i=1}^m v_i(s) \gamma_i(s)$ with $\|\xi(s)\|_{W^{1,3}(\Sigma)} + |v(s)| \leq C\|\alpha(s)\|_{L^3(\Sigma)}$. Then for all $s \in [0,1]$,

$$\int_{\Sigma} \langle \alpha(s) \wedge \nabla_s \alpha(s) \rangle = \int_{\Sigma} \langle \alpha \wedge (d_A, \partial_s \xi + \sum_{i=1}^m \partial_s v_i \cdot \gamma_i) \rangle$$

$$+ \int_{\Sigma} \langle \alpha \wedge ([\Phi, \alpha] + [\partial_s A, \xi] + \sum_{i=1}^m v_i \cdot \partial_s \gamma_i) \rangle$$

$$\leq C\|\alpha(s)\|_{L^2(\Sigma)} \|\alpha(s)\|_{L^3(\Sigma)}.$$}

Here the crucial point is that $d_A \partial_s \xi$ and $\partial_s v \cdot \gamma_i$ are tangent to the Lagrangian, hence the first term vanishes. Now one uses (22) for $p = 2$ and the Sobolev trace theorem (the restriction $W^{1,2}(X) \rightarrow L^2(\partial X)$ is continuous by e.g. [Ad, Theorem 6.2]) to obtain the estimate,

$$\int_0^1 \int_{\Sigma} \langle \alpha \wedge \nabla_s \alpha \rangle \wedge ds \leq C\|\tilde{\alpha}\|_{L^2(\partial X)} \|\tilde{\alpha}\|_{L^3(\partial X)}$$

$$\leq \delta \|\tilde{\alpha}\|_{W^{1,2}(X)}^2 + C\|\tilde{\alpha}\|_{L^2(\partial X)} \|\tilde{\alpha}\|_{W^{1,2}(X)}$$

$$\leq \delta \|\tilde{\alpha}\|_{W^{1,2}(X)}^2 + C\|\tilde{\alpha}\|_{L^2(\partial X)}^2.$$}

This proves (24). Now $\delta > 0$ can be chosen arbitrarily small, so the term $\|\tilde{\alpha}\|_{W^{1,2}}$ can be absorbed into the left hand side of (23), and thus one obtains the claimed estimate

$$\|\tilde{\alpha}\|_{W^{1,2}} \leq C(\|d^+_A \tilde{\alpha}\|_2 + \|d^+_A \tilde{\alpha}\|_2 + \|\tilde{\alpha}\|_2).$$

**Proof of theorem C (ii) or lemma 3.1:**

We will use lemma A.2 for the manifold $M := S^1 \times Y$ in several different cases to obtain the estimate for different components of $\tilde{\alpha}$. The first weak equation in lemma A.2 is the same in all cases. For all $\eta \in C^\infty(M; \mathfrak{g})$

$$\int_M \langle \tilde{\alpha}, d\eta \rangle = \int_M \langle d^*_A \tilde{\alpha}, \eta \rangle + \int_{\partial M} \langle \eta, *\tilde{\alpha} \rangle$$

$$= \int_M \langle d^*_A \tilde{\alpha} + *[\hat{A} \wedge *\tilde{\alpha}], \eta \rangle = \int_M \langle f, \eta \rangle.$$
Here one uses the fact that $\ast \hat{\alpha}|_{\partial M} = 0$. Then $f \in L^p(M, g)$ and

$$
||f||_p \leq ||d^{A}_x \hat{\alpha}||_p + 2||\hat{\Lambda}||_\infty ||\hat{\alpha}||_p.
$$

(25)

For the second weak equation lemma A.2 we obtain for all $\lambda \in \Omega^1(M; g)$

$$
\int_M \langle \hat{\alpha}, d^*d\lambda \rangle = \int_M \langle \hat{\alpha}, d^*d\lambda + d^* * d\lambda \rangle = \int_M \langle \gamma, d\lambda \rangle - \int_{S^1 \times \partial Y} \langle \hat{\alpha} \wedge * d\lambda \rangle - \int_{S^1 \times \partial Y} \langle \hat{\alpha} \wedge d\lambda \rangle,
$$

(26)

where $\gamma = d\alpha + * d\alpha = 2d^+_A \alpha - 2[\hat{\Lambda} \wedge \alpha]^+ \in L^p(M, \Lambda^2 T^* M \otimes g)$ and

$$
||\gamma||_p \leq 2||d^+_A \alpha||_p + 4||\hat{\Lambda}||_\infty ||\hat{\alpha}||_p.
$$

(27)

Now recall that there is an embedding $\tau : S^1 \times [0, \Delta) \times \Sigma \hookrightarrow S^1 \times Y$ to a tubular neighborhood of $S^1 \times \partial Y$ such that $\tau^* \hat{g} = ds^2 + dt^2 + g_{s,t}$ for a family $g_{s,t}$ of metrics on $\Sigma$. One can then cover $M = S^1 \times Y$ with $\tau(S^1 \times [0, \frac{\Delta}{2}] \times \Sigma)$ and a compact subset $V \subset M \setminus \partial M$.

For the claimed estimate of $\hat{\alpha}$ over $V$ it suffices to use lemma A.2 for vector fields $X \in \Gamma(TM)$ that equal to coordinate vector fields on $V$ and vanish on $\partial M$. So one has to consider (26) for $\lambda = \phi \cdot \iota_X \hat{g}$ with $\phi \in C^\infty_s(M, g)$. Then both boundary terms vanish and hence lemma A.2 directly asserts, with some constants $C$ and $C_V$, that

$$
||\hat{\alpha}||_{W^{1, p}(V)} \leq C \left(||f||_{L^p(M)} + ||\gamma||_{L^p(M)} + ||\hat{\alpha}||_{L^p(M)}\right)
$$

$$
\leq C_V \left(||d^+_A \alpha||_{L^p(M)} + ||d^*_A \hat{\alpha}||_{L^p(M)} + ||\hat{\alpha}||_{L^p(M)}\right).
$$

So it remains to prove the estimate for $\hat{\alpha}$ near the boundary $\partial M = S^1 \times \Sigma$.

For that purpose we can use the decomposition $\tau^* \hat{\alpha} = \varphi ds + \psi dt + \alpha$, where $\varphi, \psi \in W^{1, p}(S^1 \times [0, \Delta) \times \Sigma, g)$ and $\alpha \in W^{1, p}(S^1 \times [0, \Delta) \times \Sigma, T^* \Sigma \otimes g)$. Let $\Omega := S^1 \times [0, \frac{\Delta}{2}]$ and let $K := S^1 \times [0, \frac{\Delta}{2}]$. Then we will prove the estimate for $\varphi$ and $\psi$ on $\Omega \times \Sigma$ and for $\alpha$ on $K \times \Sigma$.

Firstly, note that $\psi = \alpha(\tau_x \partial_t) \circ \tau$, where $-\tau_x \partial_t|_{\partial M} = \nu$ is the outer unit normal to $\partial M$. So one can cut off $\tau_x \partial_t$ outside of $\tau(\Omega \times \Sigma)$ to obtain a vector field $X \in \Gamma(TM)$ that satisfies the assumption of lemma A.2, that is $X|_{\partial M} = -\nu$ is perpendicular to the boundary. Then one has to test (26) with $\lambda = \phi \cdot \iota_X \hat{g}$ for all $\phi \in C^\infty_s(M, g)$. Again both boundary terms vanish. Indeed, on $S^1 \times \partial Y$ we have $\phi \equiv 0$ and $\iota_X \hat{g} = \tau_x dt$, hence $d\lambda|_{S^1 \times \partial Y} = 0$ and $*d\lambda|_{S^1 \times \partial Y} = -\frac{\partial \phi}{\partial s} \tau_x (dt \wedge dt) = 0$. Thus lemma A.2 yields the following estimate.

$$
||\psi||_{W^{1, p}(\Omega \times \Sigma)} \leq C ||\hat{\alpha}(X)||_{W^{1, p}(M)}
$$

$$
\leq C \left(||f||_{L^p(M)} + ||\gamma||_{L^p(M)} + ||\hat{\alpha}||_{L^p(M)}\right)
$$

$$
\leq C_V \left(||d^+_A \alpha||_{L^p(M)} + ||d^*_A \hat{\alpha}||_{L^p(M)} + ||\hat{\alpha}||_{L^p(M)}\right).
$$

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Here \( C \) denotes any finite constant and the bounds on the derivatives of \( \tau \) enter into the constant \( C_1 \).

Next, for the regularity of \( \phi = \hat{\alpha}(\partial_s) \circ \tau \) one can apply lemma A.2 with the tangential vector field \( X = \partial_s \). Recall that \( \tau \) preserves the \( S^1 \)-coordinate. One has to verify the second weak equation for all \( \phi \in C_0^\infty (M, g) \), i.e. consider (26) for \( \lambda = \phi \cdot t_X \hat{g} = \phi \cdot ds \). The first boundary term vanishes since one has \( *d\lambda|_{S^1 \times \partial \Sigma} = -\frac{\partial \phi}{\partial t} \text{dvol}_{\partial \Sigma} = 0 \). For the second term one can choose any \( \delta > 0 \) and then finds a constant \( C_\delta \) such that for all \( \phi \in C_0^\infty (M, g) \)

\[
\left| \int_{S^1 \times \partial \Sigma} \langle \hat{\alpha} \wedge d\lambda \rangle \right| = \left| \int_{S^1} \int_{\Sigma} \langle \alpha(s, 0) \wedge d\Sigma(Q \circ \tau)(s, 0) \rangle \wedge ds \right|
\]

\[
= \left| \int_{S^1 \times \partial \Sigma} \langle \hat{\alpha} \wedge [\hat{A}, \phi] \rangle \wedge ds \right|
\]

\[
\leq \| \alpha \|_{L^p(\partial M)} \| \hat{A} \|_{L^\infty(\partial M)} \| \phi \|_{L^p(\partial M)}
\]

\[
\leq (\delta \| \hat{\alpha} \|_{W^{1, p}(M)} + C_\delta \| \hat{\alpha} \|_{L^p(M)}) \| \phi \|_{W^{1, p}(M)}.
\]

This uses the fact that \( \alpha(s, 0) \) and \( d_{A_s}(\phi \circ \tau) \) \( (s, 0) \times \Sigma \) both lie in the tangent space \( T_{\Sigma} L \) to the Lagrangian, on which the symplectic form vanishes, that is \( \int_{\Sigma} \langle \alpha \wedge d_{A_s}(\phi \circ \tau) \rangle = 0 \). Moreover, we have used the trace theorem for Sobolev spaces, in particular the estimate (22). Now lemma A.2 and remark A.3 yield with \( c_1 = \| f \|_{p}, c_2 = \| \gamma \|_{L^p(M)} + \delta \| \hat{\alpha} \|_{W^{1, p}(M)} + C_\delta \| \hat{\alpha} \|_{L^p(M)} \), and using (25), (27)

\[
\| \phi \|_{W^{1, p}(\Omega \times \Sigma)}
\]

\[
\leq C \left( \| f \|_{L^p(M)} + c_2 + \| \hat{\alpha} \|_{L^p(M)} \right)
\]

\[
\leq \delta \| \hat{\alpha} \|_{W^{1, p}(M)} + C_\delta(\| \hat{d}_{\tilde{A}} \|_{L^p(M)} + \| \hat{d}_\tilde{A} \|_{L^p(M)} + \| \hat{\alpha} \|_{L^p(M)}).
\]

Here again \( \delta > 0 \) can be chosen arbitrarily small and the constant \( C_\delta(\delta) \) depends on this choice.

It remains to establish the estimate for the \( \Sigma \)-component \( \alpha \) near the boundary. In the coordinates \( \tau \) on \( \Omega \times \Sigma \), the forms \( d^{\uparrow}_{\tilde{A}} \hat{\alpha} \) and \( d_{\tilde{A}} \hat{\alpha} \) become

\[
\tau^* d^{\uparrow}_{\tilde{A}} \hat{\alpha} = -\partial_\tau \phi - \partial_\psi \psi + d_{\Sigma} \alpha - \tau^*(\hat{\alpha} \wedge \hat{\alpha}_+),
\]

\[
\tau^* d_{\tilde{A}} \hat{\alpha} = \frac{1}{\delta}(-(\partial_\alpha + \psi \partial_\alpha) \wedge ds + \psi \partial_\alpha \wedge d\tau + \partial_{\Sigma} \alpha \wedge d\tau)
\]

\[
+ \frac{1}{\delta}(d_{\Sigma} \alpha + \psi d_{\Sigma} \alpha)ds \wedge dt + \tau^*(\hat{\alpha} \wedge \hat{\alpha}_+).
\]

So one obtains the following bounds: The components in the mixed direction of \( \Omega \) and \( \Sigma \) of the second equation yields for some constant \( C_1 \)

\[
\| \partial_\alpha + \psi \partial_\alpha \|_{L^p(\Omega \times \Sigma)} \leq \| \tau^* d^{\uparrow}_{\tilde{A}} \hat{\alpha} \|_{L^p(\Omega \times \Sigma)} + \| \tau^*(\hat{\alpha} \wedge \hat{\alpha}_+) \|_{L^p(\Omega \times \Sigma)}
\]

\[
\leq C_1 \left( \| d^{\uparrow}_{\tilde{A}} \hat{\alpha} \|_{L^p(M)} + \| \hat{\alpha} \|_{L^p(M)} \right).
\]

Similarly, a combination of the first equation and the \( \Sigma \)-component of the second
equation can be used for every \( \delta > 0 \) to find a constant \( C_2(\delta) \) such that
\[
\|d_\Sigma^\alpha\|_{L^p(\Omega \times \Sigma)} + \|d_\Sigma^r \alpha\|_{L^p(\Omega \times \Sigma)} \leq C(\|d_\Sigma^r \tilde{\alpha}\|_{L^p(M)} + \|d_\Sigma^r \alpha\|_{L^p(M)} + \|\phi\|_{L^p(M)} + \|\psi\|_{L^p(\Omega \times \Sigma)}) \\
\leq \delta \|\tilde{\alpha}\|_{W^{1,p}(M)} + C_2(\delta)(\|d_\Sigma^r \tilde{\alpha}\|_{L^p(M)} + \|d_\Sigma^r \alpha\|_{L^p(M)} + \|\tilde{\alpha}\|_{L^p(M)}).
\]

Now firstly, lemma 2.9 provides an \( L^p \)-estimate for the derivatives of \( \alpha \) in \( \Sigma \)-direction,
\[
\|\nabla_\Sigma^\alpha\|_{L^p(\Omega \times \Sigma)} \leq C(\|d_\Sigma^r \alpha\|_{L^p(\Omega \times \Sigma)} + \|\alpha\|_{L^p(\Omega \times \Sigma)}) \\
\leq \delta \|\tilde{\alpha}\|_{W^{1,p}(M)} + C_2(\delta)(\|d_\Sigma^r \tilde{\alpha}\|_{L^p(M)} + \|d_\Sigma^r \alpha\|_{L^p(M)} + \|\tilde{\alpha}\|_{L^p(M)}),
\]
where again \( C_2(\delta) \) depends on the choice of \( \delta > 0 \). For the derivatives in \( s \)- and \( t \)-direction, we will now apply [W2, Theorem 1.3] on the Banach space \( X = L^p(\Sigma, T^*\Sigma \otimes g) \) with the complex structure \( *_c \) determined by the metric \( g_{s,t} \) on \( \Sigma \) and hence depending smoothly on \((s,t) \in \Omega \). The Lagrangian submanifold \( \mathcal{L} \subset X \) is totally real with respect to all Hodge operators and it is modelled on a closed subspace of \( L^p(\Sigma, \mathbb{R}^n) \) as seen in [W2, Lemma 4.2, Corollary 4.4].

Now \( \alpha \in W^{1,p}(\Omega, X) \) satisfies the Lagrangian boundary condition \( \alpha(s,0) \in T_A \mathcal{L} \) for all \( s \in S^1 \), where \( s \mapsto A_s \) is a smooth loop in \( \mathcal{L} \). Thus [W2, Corollary 1.4] yields a constant \( C \) such that the following estimate holds,
\[
\|\nabla_\Omega^\alpha\|_{L^p(K \times \Sigma)} \leq \|\alpha\|_{W^{1,p}(K, X)} \\
\leq C(\|d_{\Sigma}^r \alpha\|_{L^p(\Omega, X)} + \|\alpha\|_{L^p(\Omega, X)}) \\
\leq C_2(\delta)(\|d_\Sigma^r \tilde{\alpha}\|_{L^p(M)} + \|\tilde{\alpha}\|_{L^p(M)}).
\]

Here \( C_2(\delta) \) includes the above constant \( C_1 \). Now adding up all the estimates for the different components of \( \tilde{\alpha} \) gives for all \( \delta > 0 \)
\[
\|\tilde{\alpha}\|_{W^{1,p}} \leq (C_V + C_t + C_s(\delta) + C_2(\delta) + C_K)(\|d_\Sigma^r \tilde{\alpha}\|_{L^p} + \|d_\Sigma^r \alpha\|_{L^p} + \|\tilde{\alpha}\|_{L^p}) \\
+ 2\delta \|\tilde{\alpha}\|_{W^{1,p}}.
\]

Finally, choose \( \delta = \frac{1}{4} \), then the term \( \|\tilde{\alpha}\|_{W^{1,p}} \) can be absorbed into the left hand side and this finishes the proof of the lemma. \( \square \)

**Proof of theorem C (iii):**

Let \( \beta \in L^q(S^1 \times Y, T^*Y \otimes g) \) and \( \zeta \in L^q(S^1 \times Y, g) \) be as supposed in theorem C. Then there exists a constant \( C \) such that for all \( \alpha \in E_A^1 \) and \( \varphi \in W^{1,q}(S^1 \times Y, g) \)
\[
\left| \int_{S^1} \int_Y \left( \nabla_s \alpha - d_A \varphi + *d_A \alpha, \beta \right) + \int_{S^1} \int_Y \left( \nabla_s \varphi - d_A^* \alpha, \zeta \right) \right| \leq C \|((\alpha, \varphi))\|_{q^r}.
\]
The higher regularity of $\zeta$ is most easily seen if we go back to the notation $\tilde{\alpha} = \alpha + \varphi ds$ with $D_{(A, \varphi)}(\alpha, \varphi) = (2\gamma, -d_A^* \tilde{\alpha})$ for $d_A^* \tilde{\alpha} = *\gamma - \gamma \wedge ds$. Abbreviate $M := S^1 \times Y$, then we have for all $\tilde{\alpha} \in C^\infty(M, T^* M \otimes \mathfrak{g})$ with $*\tilde{\alpha}|_{\partial M} = 0$ and $\tilde{\alpha}|_{(s) \times \partial Y} \in T_{A_s} \mathcal{L}$ for all $s \in S^1$

$$\left| \int_M \langle 2d_A^* \tilde{\alpha}, \beta \wedge ds \rangle + \int_M \langle d_A^* \tilde{\alpha}, \zeta \rangle \right| \leq C\|\tilde{\alpha}\|_{q^*}.$$

Now use the embedding $\tau : S^1 \times [0, \Delta) \times \Sigma \hookrightarrow M$ to construct a connection $\hat{A} \in \mathcal{A}(M)$ such that $\tau^* \hat{A}(s, t, z) = A_s(z)$ near the boundary (this can be cut off and then extends trivially to all of $M$). Then $\tilde{\alpha} := d_A \phi$ satisfies the above boundary conditions for all $\phi \in C^\infty_c(M, \mathfrak{g})$ since $d_A \phi(\nu) = \frac{\partial \phi}{\partial \nu} + [\hat{A}(\nu), \phi] = 0$ and $d_A \phi|_{(s) \times \partial Y} = d_{A_s} \phi \in T_{A_s} \mathcal{L}$ for all $s \in S^1$. Thus we obtain for all $\phi \in C^\infty_c(M, \mathfrak{g})$

$$\int_M \langle \Delta \phi, \zeta \rangle = \int_M \langle d_A^* \tilde{\alpha} + *\hat{A} \wedge *\tilde{\alpha} - d^* [\hat{A}, \phi], \zeta \rangle \leq C\|\tilde{\alpha}\|_{q^*} + \int_M \langle d_A^\perp \phi, \beta \wedge ds \rangle + \int_M \langle *[\hat{A} \wedge d_A \phi] - d^* [\hat{A}, \phi] \rangle \leq C\|\phi\|_{W^{1, q^*}}.$$

The regularity theory for the Neumann problem, e.g. proposition A.1, then asserts that $\zeta \in W^{1, q}(M)$.

To deduce the higher regularity of $\beta$ we will mainly use lemma A.2. The first weak equation in the lemma is given by choosing $\alpha = 0$ in (28). For all $\eta \in C^\infty(M, \mathfrak{g})$

$$\int_M \langle \beta, d\eta \rangle = \left| \int_{S^1} \int_Y \langle \beta, d_A \eta - [A, \eta] \rangle \right| \leq C\|\eta\|_{q^*} + \int_{S^1} \int_Y \langle \nabla_s \zeta, \eta \rangle \leq C\|\eta\|_{q^*}.$$

For the second weak equation let $\varphi = 0$ and $\alpha = *d\lambda - \partial_s \lambda$ for $\lambda = \phi \cdot 1_X \hat{g}$ with $\phi \in \mathcal{T}$ in the function space $C^\infty_c(M, \mathfrak{g})$ or $C^\infty_c(M, \mathfrak{g})$ corresponding to the vector field $X \in C^\infty(M, TY)$. If the boundary conditions for $\alpha \in E^{1, q}_{A^\perp}$ are satisfied,
then we obtain with \( d = dy \)
\[
\left| \int_M \langle \beta, d^*_M d_M \lambda \rangle \right| = \left| \int_{S^1} \int_Y \langle \beta, * d_\lambda - \partial_\lambda \rangle \right| \\
= \left| \int_{S^1} \int_Y \langle \beta, * d_\lambda - \partial^2_\lambda \rangle \right| \\
= \left| \int_{S^1} \int_Y \langle \beta, * d_\lambda - * \partial(\partial_\lambda) \rangle \right| \\
+ \partial_\lambda \lambda - \partial_\lambda * d_\lambda - * (\partial_\lambda) \partial_\lambda \rangle \\
\leq C\| \lambda \|_{W^{1,\alpha}} + \left| \int_{S^1} \int_Y \langle \zeta, d^*_A \lambda \rangle \right| + \left| \int_{S^1} \int_Y \langle \beta, * d_A \partial_\lambda - \partial_\lambda \rangle \right| \\
\leq C\| \phi \|_{W^{1,\alpha}}.
\]

Here we have used the identity
\[
* d_A \partial_\lambda - \partial_\lambda * d_\lambda = * [A \wedge \partial_\lambda] - [\Phi, * d_\lambda] - (\partial_\lambda) d_\lambda.
\]
Moreover, we have used partial integration with vanishing boundary term \(* \alpha |_{\partial Y} = 0\) to obtain
\[
\int_{S^1} \int_Y \langle \zeta, d^*_A \lambda \rangle = \int_{S^1} \int_Y \langle d_A \zeta, * d_\lambda - \partial_\lambda \rangle.
\]

Now let \( X \in C^\infty(M, TY) \) be perpendicular to the boundary \( \partial M = S^1 \times \partial Y \), then for all \( \phi \in C^\infty(M) \) the boundary conditions for \( \alpha = * d_\lambda - \partial_\lambda \in E^\alpha_{\lambda_\phi} \) are satisfied. Indeed, on the boundary \( \partial M = S^1 \times \partial Y \) the 1-form \( \lambda = \phi \cdot \iota_X \tilde{\gamma} \) vanishes, we have \( \iota_X \tilde{\gamma} = h \cdot \tau_\delta dt \) for some smooth function \( h \), and moreover \( d\phi = -\frac{\partial h}{\partial \tau} \cdot \tau_\delta dt \). Hence
\[
* \alpha |_{\partial Y} = d\lambda |_{\partial Y} - * \partial_\lambda |_{\partial Y} = 0, \\
\alpha |_{\partial Y} = * d\lambda |_{\partial Y} - \partial_\lambda |_{\partial Y} = -\frac{\partial h}{\partial \tau} (\tau_\delta dt \wedge \tau_\delta dt) = 0.
\]

Thus for all vector fields \( X \in C^\infty(M, TY) \) that are perpendicular to the boundary, lemma A.2 asserts that \( \beta(X) \in W^{1,q}(M, g) \). In particular, this implies \( W^{1,q}_\delta \)-regularity of \( \beta \) on all compact subsets \( K \subset \text{int } M \). So it remains to prove the regularity on the neighbourhood \( \tau(S^1 \times [0, \frac{\Delta}{2}] \times \Sigma) \) of the boundary \( \partial M \).

In these coordinates we decompose
\[
\tau^* \beta = \xi dt + \beta.
\]
Now firstly, lemma A.2 applies as described above to assert the regularity \( \xi = \beta(\tau_\delta \delta) \circ \tau \in W^{1,q}(\Omega \times \Sigma, g) \) on \( \Omega := S^1 \times [0, \frac{\Delta}{2}] \). Here a vector field \( X \) that is perpendicular to the boundary is constructed by cutting off \( \tau_\delta \delta \) outside of \( \tau(\Omega \times \Sigma) \).

So it remains to consider \( \beta \in L^q(\Omega \times \Sigma, T^* \Sigma \otimes g) \) and establish its \( W^{1,q} \) regularity. In order to derive a weak inequality for \( \beta \) on \( \Omega \times \Sigma \) from (28) we
use $\tilde{\alpha} = \tau_s(\varphi ds + \psi dt + \dot{\alpha})$ with $\varphi \in C^\infty_s(\Omega \times \Sigma, g)$, $\psi \in C^\infty_s(\Omega \times \Sigma, g)$, and $\dot{\alpha} \in W^{1,p}(\Omega \times \Sigma, T^*\Sigma \otimes g)$ such that $\dot{\alpha}(s, \frac{3}{4}\Delta, \cdot) = 0$ and $\dot{\alpha}(s, 0, \cdot) \in T_A\mathcal{L}$ for all $s \in S^1$. This $\tilde{\alpha}$ satisfies the boundary conditions for (28) and it can be extended trivially to a $W^{1,p}$-regular 1-form on all of $M$. Thus we obtain

$$\int_{\Omega \times \Sigma} \langle \nabla_s \tilde{\alpha} + *\nabla_t \tilde{\alpha} - d_A \varphi - *d_A \psi, \tilde{\beta} \rangle \leq \int_{\Omega \times \Sigma} \langle -\nabla_s \psi + \nabla_t \varphi - *d_A \tilde{\alpha}, \xi \rangle + \int_{\Omega \times \Sigma} \langle \nabla_s \varphi + \nabla_t \psi - d_A^* \tilde{\alpha}, \zeta \rangle + C\|\varphi ds + \psi dt + \tilde{\alpha}\|_{g^*}.$$  

Here we have introduced the decomposition $\tau^*\tilde{\alpha} = \Phi ds + \Psi dt + A$, where $A \in C^\infty(\Omega \times \Sigma, T^*\Sigma \otimes g)$ with $A(s, 0) = A_s \in \mathcal{L}$ for all $s \in S^1$. We have also used the notation $\nabla_t \varphi = \partial_t \varphi + [\Psi, \varphi]$, and moreover $d_A$ and $*$ denote the differential and Hodge operator on $\Sigma$. Now firstly put $\tilde{\alpha} = 0$, then we obtain for all $\varphi, \psi \in C^\infty_s(\Omega \times \Sigma, g)$ by partial integration

$$\int_{\Omega \times \Sigma} \langle d_A \varphi, \tilde{\beta} \rangle \leq (C + \|\nabla_t \varphi - \nabla_s \varphi\|_{L^q(\Omega \times \Sigma)})\|\varphi\|_{L^{q^*}(\Omega \times \Sigma)},$$  

$$\int_{\Omega \times \Sigma} \langle *d_A \psi, \tilde{\beta} \rangle \leq (C + \|\nabla_s \xi + \nabla_t \xi\|_{L^q(\Omega \times \Sigma)})\|\psi\|_{L^{q^*}(\Omega \times \Sigma)}.$$  

This shows that the weak derivatives $d_A \tilde{\alpha}$ and $*d_A \tilde{\alpha}$ are of class $L^q$ on $\Omega \times \Sigma$, so we have verified the assumptions of lemma 2.9 for $\tilde{\beta}$ and conclude that $\nabla_\Sigma \tilde{\beta}$ also is of class $L^q$. So it remains to deduce the $L^q$-regularity of $\partial_s \tilde{\beta}$ and $\partial_t \tilde{\beta}$ on $S^1 \times [0, \frac{2\pi}{q}] \times \Sigma$ from the above inequality for $\varphi = \psi = 0$, namely from

$$\int_{\Omega \times \Sigma} \langle \nabla_s \tilde{\alpha} + *\nabla_t \tilde{\alpha}, \tilde{\beta} \rangle \leq (C + \|d_A \xi + *d_A \xi\|_{L^q(\Omega \times \Sigma)})\|\tilde{\alpha}\|_{L^{q^*}(\Omega \times \Sigma)}.$$  

This holds for all $\tilde{\alpha} \in W^{1,p}(\Omega \times \Sigma, T^*\Sigma \otimes g)$ such that $\tilde{\alpha}(s, \frac{3}{4}\Delta, \cdot) = 0$ and $\tilde{\alpha}(s, 0, \cdot) \in T_A\mathcal{L}$ for all $s \in S^1$. We now have to employ different arguments according to whether $q > 2$ or $q < 2$.  

**Case $q > 2$ :**  

In this case the regularity of $\partial_s \tilde{\beta}$ and $\partial_t \tilde{\beta}$ will follow from [W2, Theorem 1.3] on the Banach space $X = L^q(\Sigma, T^*\Sigma \otimes g)$ with the complex structure given by the Hodge operator on $\Sigma$ with respect to the metric $g_{s,1}$. From (29) one obtains the following estimate for some constant $C$ and all $\tilde{\alpha}$ as above:

$$\int_{\Omega} \int_{\Sigma} \|\tilde{\beta} + \partial_s \tilde{\alpha} + \partial_t (*\tilde{\alpha})\|_{L^{q^*}(\Omega, X^*)} \leq C\|\tilde{\alpha}\|_{L^{q^*}(\Omega, X^*)}. $$  

Note that this extends to the $W^{1,q}(\Omega, L^{q^*}(\Sigma))$-closure of the admissible $\tilde{\alpha}$ from above. In particular the estimate above holds for all $\tilde{\alpha} \in W^{1,q}(\Omega, X)$ that are supported in $\Omega$ and satisfy $\tilde{\alpha}(s, 0, \cdot) \in T_A\mathcal{L}$ for all $s \in S^1$. To see that these
can be approximated by smooth $\hat{\alpha}$ with Lagrangian boundary conditions one uses the Banach submanifold coordinates for $L$ given by [W2, Lemma 4.3] as before. Here the Lagrangian submanifold $L \subset X$ is totally real with respect to all Hodge operators as before, and it is the $L^2$-restriction or -completion of the original submanifold in $A^{0,p}(\Sigma)$, hence it is modelled on $W^{1,q}(\Sigma, \mathbb{R}^n)$, a closed subspace of $L^q(\Sigma, \mathbb{R}^n)$ (see [W2, Lemma 4.2, 4.3]). However, in order to be able to apply [W2, Theorem 1.3], we need to extend this estimate to all $\hat{\alpha} \in W^{1,\infty}(\overline{\Sigma}, X^*)$ with supp $\alpha \subset \Omega$ and $\alpha(s, 0) \in (\ast T_A L)^\perp$ for all $s \in S^1$. This is possible since any such $\hat{\alpha}$ can be approximated in $W^{1,q}(\Omega, X^*)$ by $\hat{\alpha}_i \in C^\infty(\Omega, X)$ that are compactly supported in $\Omega$ and satisfy the above stronger boundary condition $\hat{\alpha}_i(s, 0) \in T_A L$ for all $s \in S^1$.

Indeed, [W2, Lemma 2.2] provides such an approximating sequence $\alpha_i$ without the Lagrangian boundary conditions. From the proof via mollifiers one sees that the approximating sequence can be chosen with compact support in $\Omega$. Now for all $s \in S^1$ one has the topological splitting $X = T_A L \oplus \ast T_A L$ and thus $X^* = (T_A L)^\perp \oplus (\ast T_A L)^\perp$. Since $q > 2$ the embedding $X \hookrightarrow X^*$ is continuous. This identification uses the $L^2$-inner product on $X$ which equals the metric $\omega(\cdot, \ast \cdot)$ given by the symplectic form $\omega$ and the complex structure $\ast$. So due to the Lagrangian condition this embedding maps $T_A L \hookrightarrow (\ast T_A L)^\perp$ and $\ast T_A L \hookrightarrow (T_A L)^\perp$. We write $\hat{\alpha} = \gamma + \delta$ and $\alpha_i = \gamma_i + \delta_i$ according to these splittings to obtain $\gamma, \delta \in C^\infty(\Omega, X^*)$ and $\gamma_i, \delta_i \in C^\infty(\Omega, X)$ such that $\ast T_A L \ni \gamma_i \rightarrow \gamma \in (T_A L)^\perp$ and $T_A L \ni \delta_i \rightarrow \delta \in (\ast T_A L)^\perp$ with convergence in $W^{1,q}(\Omega, X^*)$. The boundary condition on $\hat{\alpha}$ gives $\gamma|_{t=0} \equiv 0$. Moreover, $\partial_t \gamma$ is uniformly bounded in $X^*$, so one can find a constant $C$ such that $\|\gamma(s, t)\|_{X^*} \leq Ct$ for all $t \in [0, \delta \Delta)$ and hence for sufficiently small $\varepsilon > 0$

$$\|\gamma\|_{L^\varepsilon(S^1 \times [0, \varepsilon], X^*)} \leq \frac{C}{1+\varepsilon^2} e^{1+\frac{1}{\varepsilon^2}}.$$

Now let $\delta > 0$ be given and choose $1 > \varepsilon > 0$ such that $\|\gamma\|_{L^\varepsilon(S^1 \times [0, \varepsilon], X^*)} \leq \varepsilon \delta$, and $\|\gamma\|_{W^{1,q}(S^1 \times [0, \varepsilon], X^*)} \leq \delta$. Next, choose $i \in \mathbb{N}$ sufficiently large such that $\|\gamma_i - \gamma\|_{W^{1,q}(\Omega, X^*)} \leq \varepsilon \delta$, and let $h \in C^\infty([0, \frac{1}{4} \Delta], [0, 1])$ be a cutoff function with $h(0) = 0$, $h|_{t \geq \varepsilon} \equiv 0$, and $|h'| \leq \frac{1}{2}$. Then $\hat{\alpha}_i := h \gamma_i + \delta_i \in C^\infty(\Omega, X)$ satisfies the Lagrangian boundary condition $\hat{\alpha}_i(s, 0) \in T_A L$ and approximates $\hat{\alpha}$ in view of the following estimate,

$$\|\hat{\alpha}_i - \hat{\alpha}\|_{W^{1,q}(\Omega, X^*)} \leq \|h(\gamma_i - \gamma)\|_{W^{1,q}(\Omega, X^*)} + \|(1 - h)\gamma\|_{W^{1,q}(\Omega, X^*)} + \frac{\delta}{2} \|\gamma\|_{L^\varepsilon(S^1 \times [0, \varepsilon], X^*)} \leq 6 \delta.$$

This approximation shows that (30) holds indeed true for all $\hat{\alpha} \in C^1(\overline{\Omega}, X^*)$ with supp $\alpha \subset \Omega$ and $\alpha(s, 0) \in (\ast T_A L)^\perp$ for all $s \in S^1$. Thus [W2, Theorem 1.3] asserts that $\beta \in W^{1,q}(K, X)$ for $K := S^1 \times [0, \frac{1}{4} \Delta]$, and hence $\partial_s \beta$ and $\partial_t \beta$ are of class $L^q$ on $S^1 \times [0, \frac{1}{4} \Delta] \times \Sigma$ as claimed.
Case $q < 2$:

In this case we cover $S^1$ by two intervals, $S^1 = I_1 \cup I_2$ such that there are isometric embeddings $(0, 1) \hookrightarrow S^1$ identifying $\left[\frac{1}{2}, \frac{3}{2}\right]$ with $I_1$ and $I_2$ respectively. Abbreviate $K := \left[\frac{1}{2}, \frac{3}{2}\right] \times [0, \Delta]$ and let $\Omega' \subset (0, 1) \times [0, \frac{\Delta}{2}]$ be a compact submanifold of the half space $\mathbb{H}$ such that $K \subset \text{int}(\Omega')$. Then for each of the above identifications $S^1 \setminus \{pt\} \cong (0, 1)$ one has $L^q$-regularity of $\beta$ on $\Omega' \times \Sigma$ by assumption and of $\pm d_A^\xi + d_A^\zeta$ from above. Now the task is to establish in both cases the $L^q$-regularity of $\partial_\gamma \beta$ and $\partial_\xi \beta$ on $K \times \Sigma$ using (29). For that purpose choose a cutoff function $h \in C^\infty(\mathbb{H}, [0, 1])$ supported in $\Omega'$ such that $h|_{K} \equiv 1$. Then it suffices to find a constant $C$ such that for all $\gamma \in C^\infty_0(\Omega' \times \Sigma, T^*\Sigma \otimes g)$ (these are compactly supported in $\text{int}(\Omega') \times \Sigma$)

$$\left| \int_{\Omega' \times \Sigma} (\partial_\gamma \beta, h \beta) \right| \leq C\|\gamma\|_{q'}. $$

This gives $L^q$-regularity of the weak derivative $\partial_\gamma (h \beta)$ and hence of $\partial_\gamma \beta$ on $K \times \Sigma$. For the regularity of $\partial_\xi \beta$ one has to replace $\partial_\xi \gamma$ by $\partial_\xi \gamma$, then the argument is the same as the following argument for $\partial_\gamma \beta$.

We linearize the submanifold chart maps along $(A_s)_{s \in (0, 1)} \in \mathcal{A} \cap \mathcal{A}(\Sigma)$ given by [W2, Lemma 4.3] for the Lagrangian $\mathcal{L} \subset \mathcal{A}^{0,q}(\Sigma)$. Note that this uses the $L^q'$-completion of the actual Lagrangian in $\mathcal{A}^{0,p}(\Sigma)$. Abbreviate $Z := W^{1,q'}_2(\Sigma, g) \times \mathbb{R}^m$ and let $*_{s,t}$ denote the Hodge operator on $\Sigma$ with respect to the metric $g_{s,t}$. Then one obtains a smooth family of bounded isomorphisms

$$\Theta_{s,t} : Z \times Z \xrightarrow{\sim} L^{q'}(\Sigma, T^*\Sigma \otimes g) =: X$$

defined for all $(s, t) \in \Omega'$ by

$$\Theta_{s,t}(\xi, \nu, \zeta, \omega) = (d_A^\xi + \sum_{i=1}^m v^i_\gamma(s) + *_{s,t} d_A^\zeta + \sum_{i=1}^m w^i_{*_{s,t} \gamma}(s)).$$

Here $\gamma_i \in C^\infty((0, 1) \times \Sigma, T^*\Sigma \otimes g)$ with $\gamma_i(s) \in T_A^\Sigma \mathcal{L}$ for all $s \in (0, 1)$. Abbreviate $Z^\infty := C^\infty(\Sigma, g) \times \mathbb{R}^m \subset Z$, then $\Theta_{s,t}$ maps $Z^\infty \times Z^\infty$ into the set of smooth 1-forms $\Omega^1(\Sigma, g)$. So given any $\gamma \in C^\infty_0(\Omega' \times \Sigma, T^*\Sigma \otimes g)$ we have $f := \Theta^{-1} \circ \gamma \in C^\infty_0(\Omega', Z^\infty \times Z^\infty)$ and for some constant $C$,

$$\|f\|_{L^{q'}(\Omega', Z \times Z)} \leq C\|\gamma\|_{L^{q'}(\Omega', X)} = C\|\gamma\|_{L^{q'}(\Omega' \times \Sigma)}.$$

Write $f = (f_1, f_2)$ with $f_i \in C^\infty_0(\Omega', Z^\infty)$ and note that $f_i \partial_\psi f_1 = 0$ due to the compact support. So one can solve $\Delta_{\Omega'} \phi_1 = \partial_\psi f_1$ with $\phi_1 \in C^\infty_0(\Omega', Z^\infty)$ and $\phi_2 \in C^\infty_0(\Omega', Z^\infty)$. (For the $C^\infty_0(\Sigma, g)$-component of $Z^\infty$ one has solutions of the Laplace equation on every $\Omega' \times \{x\}$ that depend smoothly on $x \in \Sigma$.) Now let $\Phi := (\phi_1, \phi_2) \in C^\infty(\Omega', Z \times Z)$ and consider the 1-form

$$\alpha_\gamma := h \cdot \Theta(-\partial_\psi \Phi + J_\psi \partial_\Phi) \in C^\infty(\Omega', X).$$

This extends to a 1-form on $\Omega \times \Sigma$ that is admissible in (29). Indeed, $\alpha_\gamma$ vanishes for $s$ close to 0 or 1 and thus trivially extends to $s \in S^1$. The Lagrangian
boundary condition is met since for all $s \in S^1$
\[
\bar{\alpha}_\gamma(s,0) = h(s,0) \cdot \Theta_{s,0}(-\partial_s \phi_1 - \partial_t \phi_2, -\partial_s \phi_2 + \partial_t \phi_1) \in \Theta_{s,0}(Z,0) = T_{A_s}L.
\]
So (29) provides a constant $C$ such that for all $\bar{\alpha}_\gamma$ of the above form
\[
\left| \int_{\Omega^i \times \Sigma} \langle \hat{\beta}, \partial_s \bar{\alpha}_\gamma + \partial_t (*\bar{\alpha}_\gamma) \rangle \right| \leq C \| \bar{\alpha}_\gamma \|_{L^q(\Omega^i \times \Sigma)}
\]
Moreover, one has for all $\gamma \in C^\infty_0(\Omega' \times \Sigma, T^*\Sigma \otimes \mathfrak{g})$ and the associated $f$, $\Phi$ and $\bar{\alpha}_\gamma$ and denoting all constants by $C$
\[
\| \bar{\alpha}_\gamma \|_{L^q(\Omega, \Sigma)} \leq C \| \Phi \|_{W^{1,q}(\Omega', \Sigma)} \leq C \| f \|_{L^q(\Omega', \Sigma)} \leq C \| \gamma \|_{L^q(\Omega \times \Sigma)}.
\]
Here the second inequality follows from $\Delta_{\Omega'} \Phi = \partial_s f$ and [W2, Lemma 2.1] as follows. In the $\mathbb{R}^m$-component of $Z$, this is the usual elliptic estimate for the Dirichlet or Neumann problem; for the components in the infinite dimensional part $Y := W^{1,q}_z(\Sigma, \mathfrak{g})$ of $Z$ (still denoted by $\phi_i$ and $f_i$) this uses the following estimate. For all $\psi \in C^\infty_0(\Omega', Y^*)$ in the case $i = 1$ and for all $\psi \in C^\infty_0(\Omega', Y^*)$ in the case $i = 2$
\[
\left| \int_{\Omega' \times \Sigma} \langle \phi_i, \Delta_{\Omega'} \psi \rangle \right| = \left| \int_{\Omega' \times \Sigma} \langle \Delta_{\Omega'} \phi_i, \psi \rangle \right| = \left| \int_{\Omega' \times \Sigma} \langle \partial_s f_i, \psi \rangle \right| = \left| \int_{\Omega' \times \Sigma} \langle f_i, \partial_s \psi \rangle \right| \leq \| f_i \|_{L^q(\Omega', Y^*)} \| \psi \|_{W^{1,q}(\Omega', Y^*)}.
\]
Now a calculation shows that
\[
\partial_s \bar{\alpha}_\gamma + \partial_t (*\bar{\alpha}_\gamma) = h \cdot \Theta(\Delta \Phi) + \partial_s (h \cdot \Theta)(-\partial_s \Phi + J_0 \partial_t \Phi) + \partial_t (h \cdot \Theta)(\partial_t \Phi - J_0 \partial_s \Phi).
\]
We then use $\Delta \Phi = \partial_s f$ to obtain, denoting all constants by $C$,
\[
\left| \int_{\Omega' \times \Sigma} \langle h \cdot \hat{\beta}, \partial_s \gamma \rangle \right|
\leq \left| \int_{\Omega' \times \Sigma} \langle \hat{\beta}, h \cdot \Theta(\Delta \Phi) + h \cdot \partial_s \Theta(f) \rangle \right|
\leq \left| \int_{\Omega' \times \Sigma} \langle \hat{\beta}, \partial_s \bar{\alpha}_\gamma + \partial_t (*\bar{\alpha}_\gamma) \rangle \right|
\leq C \| \hat{\beta} \|_{L^q(\Omega', \Sigma)} \left( \| - \partial_s \Phi + J_0 \partial_t \Phi \|_{L^q(\Omega', Z \times Z)} + \| f \|_{L^q(\Omega', Z \times Z)} \right)
\]
This proves the $L^q$-regularity of $\partial_s \hat{\beta}$ (and analogously of $\partial_t \hat{\beta}$) on $S^1 \times [0, \frac{\alpha}{2}] \times \Sigma$ in the case $q < 2$ and thus finishes the proof of theorem C (iii).

**Proof of theorem C (i):**

Lemma 3.1 and the subsequent remark imply that for some constant $C$ and for all $(\alpha, \varphi)$ in the domain of $D_{(A, \Phi)}$
\[
\||\alpha, \varphi||_{W^{1,p}} \leq C (||D_{(A, \Phi)}(\alpha, \varphi)||_p + ||(\alpha, \varphi)||_p).
\]
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Note that the embedding $W^{1,p}(X) \hookrightarrow L^p(X)$ is compact, so this estimate asserts that $\ker D_{(A,\Phi)}$ is finite dimensional and $\im D_{(A,\Phi)}$ is closed (see e.g. [Z, 3.12]). So it remains to consider the cokernel of $D_{(A,\Phi)}$. We abbreviate $Z := L^p(S^1 \times Y; T^*Y \otimes g) \times L^p(S^1 \times Y; g)$, then $\coker D_{(A,\Phi)} = Z/\im D_{(A,\Phi)}$ is a Banach space since $\im D_{(A,\Phi)}$ is closed. So it has the same dimension as its dual space $(Z/\im D_{(A,\Phi)})^* \cong (\im D_{(A,\Phi)})^*$. Now let $\sigma : S^1 \times Y \to S^1 \times Y$ denote the reflection $\sigma(s,y) := (-s,y)$ on $S^1 \cong \mathbb{R}/\mathbb{Z}$, then we claim that there is an isomorphism

$$
(\im D_{(A,\Phi)})^* \xrightarrow{\sim} \ker D_{\sigma^*(A,\Phi)}
$$

Here $D_{\sigma^*(A,\Phi)} = D_{(A',\Phi')}$ is the linearized operator at the reflected connection $\sigma^*A = A' + \Phi' ds$ with respect to the metric $\sigma^*g$ on $X$. Note that $\ker D_{\sigma^*(A,\Phi)}$ is finite dimensional since the estimate in lemma 3.1 also holds for the operator $D_{\sigma^*(A,\Phi)}$. So this would indeed prove that $\coker D_{A}$ is of finite dimension and hence $D_{A}$ is a Fredholm operator.

To establish the above isomorphism consider any $(\beta, \zeta) \in (\im D_{(A,\Phi)})^*$, that is $\beta \in L^p(S^1 \times Y; T^*Y \otimes g)$ and $\zeta \in L^p(S^1 \times Y; g)$ such that for all $\alpha \in E^{1,p}_A$ and $\varphi \in W^{1,p}(S^1 \times Y, g)$

$$
\int_{S^1 \times Y} (D_{(A,\Phi)}(\alpha, \varphi), (\beta, \zeta)) = 0.
$$

Iteration of theorem C (iii) implies that $\beta$ and $\zeta$ are in fact $W^{1,p}$-regular: We start with $q = p^* < 2$, then the lemma asserts $W^{1,p^*}$-regularity. Next, the Sobolev embedding theorem gives $L^{q_1}$-regularity for some $q_1 \in (\frac{4}{3}, 2)$ with $q_1 > p^*$. Indeed, the Sobolev embedding holds for any $q_1 \leq \frac{4p}{4-p^*}$, and $\frac{4}{3} < \frac{4p}{4-p}$ as well as $p^* < \frac{4p}{4-p}$ holds due to $p^* > 1$. So the lemma together with the Sobolev embeddings can be iterated to give $L^{q_{i+1}}$-regularity for $q_{i+1} = \frac{4p}{4-p}$ as long as $4 > q_i > 2$ or $2 > q_i \geq p^*$. This iteration yields $q_2 \in (2, 4)$ and $q_3 > 4$. Thus another iteration of the lemma gives $W^{1,q_3}$ and thus also $L^p$-regularity of $\beta$ and $\zeta$. Finally, since $p > 2$ the lemma applies again and asserts the claimed $W^{1,p}$-regularity of $\beta$ and $\zeta$. Now by partial integration

$$
0 = \int_{S^1 \times Y} (D_{(A,\Phi)}(\alpha, \varphi), (\beta, \zeta))
$$

$$
= \int_{S^1} \int_Y \langle \nabla_s \alpha - d_A \varphi + *d_A \alpha, \beta \rangle + \int_{S^1} \int_Y \langle \nabla_s \varphi - d_A^* \alpha, \zeta \rangle
$$

$$
= \int_{S^1} \int_Y \langle \alpha, -\nabla_s \beta - d_A \zeta + *d_A \beta \rangle + \int_{S^1} \int_Y \langle \varphi, -\nabla_s \zeta - d_A^* \beta \rangle
$$

$$
- \int_{S^1} \int_{\Sigma} \langle \alpha \land \beta \rangle - \int_{S^1} \int_{\Sigma} \langle \varphi, *\beta \rangle.
$$

(32)

Testing this with all $\alpha \in C_0^\infty(S^1 \times Y; T^*Y \otimes g) \subset E^{1,p}_A$ and $\varphi \in C_0^\infty(S^1 \times Y, g)$ implies $-\nabla_s \beta - d_A \zeta + *d_A \beta = 0$ and $-\nabla_s \zeta - d_A^* \beta = 0$. Then furthermore we
deduce $\ast \beta(s)|_{\partial Y} = 0$ for all $s \in S^1$ from testing with $\varphi$ that run through all of $C^\infty(S^1 \times \Sigma, g)$ on the boundary. Finally, $\int_{S^1} \int_{\Sigma} (\alpha \wedge \beta) = 0$ remains from (32). Since both $\alpha$ and $\beta$ restricted to $S^1 \times \Sigma$ are continuous paths in $A^{0,p}(\Sigma)$, this implies that for all $s \in S^1$ and every $\alpha \in T_{A^s} \mathcal{L}$

$$0 = \int_{\Sigma} (\alpha \wedge \beta(s)) = \omega(\alpha, \beta(s)),$$

where $\omega$ is the symplectic structure on $A^{0,p}(\Sigma)$. Since $T_{A^s} \mathcal{L}$ is a Lagrangian subspace, this proves $\beta(s)|_{\partial Y} \in T_{A^s} \mathcal{L}$ for all $s \in S^1$ and thus $\beta \in E^{1,p}_{A^s}$, or equivalently $\beta \circ \sigma \in E_{A^s \sigma}$. So $(\beta \circ \sigma, \zeta \circ \sigma)$ lies in the domain of $D_{\sigma^*(A, \Phi)}$. Now note that $\sigma^* A = A \circ \sigma - (\Phi \circ \sigma) ds$, thus one obtains $(\beta \circ \sigma, \zeta \circ \sigma) \in \ker D_{\sigma^*(A, \Phi)}$ since $D_{\sigma^*(A, \Phi)}(\beta \circ \sigma, \zeta \circ \sigma) = ((-\nabla_s \beta - d_A \zeta + \ast d_A \beta) \circ \sigma, (-\nabla_s \zeta - d_A^\ast \beta) \circ \sigma) = 0$.

This proves that the map in (31) indeed maps into $\ker D_{\sigma^*(A, \Phi)}$. To see the surjectivity of this map consider any $(\beta, \zeta) \in \ker D_{\sigma^*(A, \Phi)}$. Then the same partial integration as in (32) shows that $(\beta \circ \sigma, \zeta \circ \sigma) \in (\text{im} D_{(A, \Phi)})^1$, and thus $(\beta, \zeta)$ is the image of this element under the map (31). So this establishes the isomorphism (31) and thus shows that $D_{(A, \Phi)}$ is Fredholm.

\[\square\]

A Dirichlet and Neumann problem

Throughout this paper we use various regularity results for the Laplace operator. For convenience these are summarized in this appendix.

We deal with (homogeneous) Dirichlet boundary conditions and with possibly inhomogeneous Neumann boundary conditions. Often, the equations are formulated weakly with the help of the following test function spaces:

$$C^\infty_0(M) = \{ \phi \in C^\infty(M) \mid \phi|_{\partial M} = 0 \},$$

$$C^\infty_{\Sigma}(M) = \{ \phi \in C^\infty(M) \mid \frac{\partial \phi}{\partial \nu}|_{\partial M} = 0 \}.$$

Here and throughout this appendix let $M$ be a manifold with boundary. We abbreviate $\Delta := \ast d^* d$, and denote by $\partial_M$ the Lie derivative in the direction of the outer unit normal. Moreover, we use the notation $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, \ldots\}$. The regularity theory for the Dirichlet and Neumann problem that is used in this paper can be summarized as follows. References are for example [GT] and [W1, Theorems 2.3,3.2,D.2].

**Proposition A.1** Let $k \in \mathbb{N}$, then there exists a constant $C$ such that the following holds. Let $f \in W^{k-1,p}(M)$ and $G \in W^{k,p}(M)$ and suppose that $u \in W^{k,p}(M)$ is a weak solution of the Dirichlet problem (or the Neumann problem with inhomogeneous boundary conditions), that is for all $\psi \in C^\infty(M)$ (or for all $\psi \in C^\infty_{\Sigma}(M)$)

$$\int_M u \cdot \Delta \psi = \int_M f \cdot \psi + \int_{\partial M} G \cdot \psi.$$
Then \( u \in W^{k+1,p}(M) \) and
\[
\|u\|_{W^{k+1,p}} \leq C(\|f\|_{W^{k-1,p}} + \|G\|_{W^{k,p}} + \|u\|_{W^{k,p}}).
\]

In the special case \( k = 0 \) there exists a constant \( C \) such that the following holds:
Suppose that \( u \in L^p(M) \) and that there exists a constant \( c \) such that for all \( \psi \in C_\infty(M) \) (or for all \( \psi \in C_\infty(\nu(M)) \))
\[
\int_M u \cdot \Delta \psi \leq c \|\psi\|_{W^{1,p}}.
\]

Then \( u \in W^{1,p}(M) \) and
\[
\|u\|_{W^{1,p}} \leq C(c + \|u\|_{L^p}).
\]

We also frequently encounter Laplace equations for 1-forms, where the components satisfy different boundary conditions. In these cases the following lemma allows to obtain regularity results for the components separately. The proof relies on the above standard regularity theory for the Laplace operator.

Lemma A.2 Let \((M, g)\) be a compact Riemannian manifold (possibly with boundary), let \( k \in \mathbb{N}_0 \) and \( 1 < p < \infty \). Let \( X \in \Gamma(TM) \) be a smooth vector field that is either perpendicular to the boundary, i.e. \( X|_{\partial M} = h \cdot \nu \) for some \( h \in C_\infty(\partial M) \), or tangential, i.e. \( X|_{\partial M} \in \Gamma(T\partial M) \). In the first case let \( \mathcal{T} = C_\infty^0(M) \), in the latter case let \( \mathcal{T} = C_\infty^0(\nu(M)) \). Then there exists a constant \( C \) such that the following holds:

Let \( f \in W^{k,p}(M) \), \( \gamma \in W^{k,p}(M, \Lambda^2 T^* M) \), and suppose that the 1-form \( \alpha \in W^{k,p}(M, T^* M) \) satisfies
\[
\int_M \langle \alpha, d\eta \rangle = \int_M f \cdot \eta \quad \forall \eta \in C_\infty(M),
\]
\[
\int_M \langle \alpha, d^* \omega \rangle = \int_M \langle \gamma, \omega \rangle \quad \forall \omega = d(\phi \cdot \iota_X g), \phi \in \mathcal{T}.
\]

Then \( \alpha(X) \in W^{k+1,p}(M) \) and
\[
\|\alpha(X)\|_{W^{k+1,p}} \leq C(\|f\|_{W^{k,p}} + \|\gamma\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}).
\]

Remark A.3 In the case \( k = 0 \) let \( \frac{1}{p} + \frac{1}{p^*} = 1 \), then the weak equations for \( \alpha \) can be replaced by the following: There exist constants \( c_1 \) and \( c_2 \) such that
\[
\left| \int_M \langle \alpha, d\eta \rangle \right| \leq c_1 \|\eta\|_{p}, \quad \forall \eta \in C_\infty(M),
\]
\[
\left| \int_M \langle \alpha, d^* (\phi \cdot \iota_X g) \rangle \right| \leq c_2 \|\phi\|_{W^{1,p^*}}, \quad \forall \phi \in \mathcal{T}.
\]

The estimate then becomes \( \|\alpha(X)\|_{W^{1,p}} \leq C(c_1 + c_2 + \|\alpha\|_p) \).
Proof of lemma A.2 and remark A.3:

Let \( \alpha^\nu \in C^\infty(M, T^*M) \) be an \( L^p \)-approximating sequence for \( \alpha \) such that \( \alpha^\nu \equiv 0 \) near \( \partial M \). Then one obtains for all \( \phi \in T \)

\[
\int_M \alpha(X) \cdot \Delta \phi = \lim_{\nu \to \infty} \left( \int_M \left( L_X \alpha^\nu, d\phi \right) - \int_M \left( \iota_X d\alpha^\nu, d\phi \right) \right)
\]

\[
= \lim_{\nu \to \infty} \left( - \int_M \left( \alpha^\nu, L_X d\phi \right) - \int_M \left( \alpha^\nu, \text{div} X \cdot d\phi \right) \right)
\]

\[
- \int_M \left( \alpha^\nu, \iota_{Y_{d\phi}} L_X g \right) - \int_M \left( \iota_X d\phi, \iota_X L_X g \right)
\]

\[
= \int_M \left( \alpha, d\left(-L_X \phi - \text{div} X \cdot \phi \right) \right) - \int_M \left( \alpha, d^* (\iota_X g \wedge d\phi) \right)
\]

\[
+ \int_M \left( \alpha, \phi \cdot \text{div} X \right) - \int_M \left( \alpha, \phi \cdot \iota_{Y_{d\phi}} L_X g \right)
\]

\[
= \int_M \left( f, -L_X \phi - \text{div} X \cdot \phi \right) + \int_M \left( \gamma, \phi \cdot \iota_X g \right)
\]

\[
- \int_M \left( \alpha, d^* (\phi \cdot L_X g) \right) + \int_M \left( \alpha, \phi \cdot (\text{div} X) - \iota_{Y_{d\phi}} L_X g \right).
\]

Here the vector field \( Y_{d\phi} \) is given by \( \iota_{Y_{d\phi}} g = d\phi \). In the case \( k \geq 1 \) further partial integration yields for all \( \phi \in T \)

\[
\int_M \alpha(X) \cdot \Delta \phi = \int_M F \cdot \phi + \int_{\partial M} G \cdot \phi,
\]

where \( F \in W^{k-1,p}(M), G \in W^{k,p}(M) \), and for some constant \( C \)

\[
\|F\|_{W^{k-1,p}} + \|G\|_{W^{k,p}} \leq C(\|f\|_{W^{k,p}} + \|\gamma\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}).
\]

So the regularity proposition A.1 for the weak Laplace equation with either Neumann (\( T = C^\infty_\nu(M) \)) or Dirichlet (\( T = C^\infty(M) \)) boundary conditions proves that \( \alpha(X) \in W^{k+1,p}(M) \) with the according estimate.

In the case \( k = 0 \) one works with the following inequality: Let \( \frac{1}{p} + \frac{1}{p'} = 1 \), then there is a constant \( C \) such that for all \( \phi \in T \)

\[
\left| \int_M \alpha(X) \cdot \Delta \phi \right| \leq C(\|f\|_{p} + \|\gamma\|_{p} + \|\alpha\|_{p})\|\phi\|_{W^{1,p'}}.
\]

(Under the assumptions of remark A.3, one simply replaces \( \|f\|_{p} \) and \( \|\gamma\|_{p} \) by \( c_1 \) and \( c_2 \) respectively.) The regularity and estimate for \( \alpha(X) \) then follow from proposition A.1. \( \square \)
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