Generalized Obata theorem and its applications on foliations

Seoung Dal Jung, Keum Ran Lee and Ken Richardson

Abstract. We prove the generalized Obata theorem on foliations. Let \( M \) be a complete Riemannian manifold with a foliation \( \mathcal{F} \) of codimension \( q \geq 2 \) and a bundle-like metric \( g_M \). Then \((M, \mathcal{F})\) is transversally isometric to \((S^q(1/c), G)\), where \( S^q(1/c) \) is the \( q \)-sphere of radius \( 1/c \) in \((q+1)\)-dimensional Euclidean space and \( G \) is a discrete subgroup of the orthogonal group \( O(q) \), if and only if there exists a non-constant basic function \( f \) such that \( \nabla_X df = -c^2 f X^b \) for all basic normal vector fields \( X \), where \( c \) is a positive constant and \( \nabla \) is the connection on the normal bundle. By the generalized Obata theorem, we classify such manifolds which admit transversal non-isometric conformal fields.

1 Introduction

In 1962, M. Obata \cite{9} proved that a complete Riemannian manifold \((M, g_M)\) is isometric with a sphere of radius \( 1/c \) in \((n+1)\)-dimensional Euclidean space if and only if \( M \) admits a non-constant function \( f \) such that

\[
\nabla^M_X df = -c^2 f X^b
\]

2000 Mathematics Subject Classification. 53C12, 53C27, 53R30

Key words and phrases. The generalized Obata theorem, transversal Killing field, transversal conformal field

This paper was supported by KRF-2008-313-C00076 from Korea Research Foundations and R01-2008-000-20370-0 from KOSEF.
for any vector $X$, where $\nabla^M$ is a Levi-Civita connection on $M$, $c$ is positive constant and $X^b$ is the $g_M$-dual form of $X$.

In Section 2, we recall basic facts on foliations as well as some lemmas we need for the main results. In Section 3, we generalize the Obata theorem to Riemannian foliations. Namely, let $M$ be a complete Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric $g_M$. Then the foliation $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$, where $S^q(1/c)$ is the $q$-sphere of radius $1/c$ in the $(q + 1)$-Euclidean space and $G$ is a discrete subgroup of $O(q)$, if and only if there exists a non-constant basic function $f$ such that $\nabla_X df = -c^2 f X^b$ for all basic normal vector fields $X$, where $\nabla$ is the connection on the normal bundle.

A Riemannian foliation is a foliation $\mathcal{F}$ on a smooth $n$-manifold $M$ such that the quotient bundle $N\mathcal{F} \cong Q = TM/T\mathcal{F}$ is endowed with a metric $g_Q$ satisfying $\theta(X)g_Q = 0$ for any vector $X \in T\mathcal{F}$, where $T\mathcal{F}$ is the tangent bundle of $\mathcal{F}$ and $\theta(X)$ is the transverse Lie derivative ([13]). Note that we can choose a Riemannian metric $g_M$ on $M$ such that $g_{N\mathcal{F}} := g_M|_{N\mathcal{F}} = g_Q$; such a metric is called bundle-like. In the last section, as applications of the generalized Obata theorem, we study the Riemannian foliations admitting transversal non-isometric conformal fields.

2 Preliminaries

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q$ and a bundle-like metric $g_M$ with respect to $\mathcal{F}$. Then we have an exact sequence of vector bundles

$$0 \to T\mathcal{F} \to TM \xrightarrow{\pi} N\mathcal{F} \to 0,$$

(2.1)

where $T\mathcal{F}$ is the tangent bundle and $N\mathcal{F} \cong Q = TM/T\mathcal{F}$ is the normal bundle of $\mathcal{F}$. We denote by $\nabla$ the connection on the normal bundle $N\mathcal{F} \subset TM$. That is,
\[ \nabla_X Y = \pi(\nabla_X^M Y) \] for any \( X, Y \in \mathcal{N}\mathcal{F} \), where \( \nabla^M \) is the Levi-Civita connection on \( M \); this connection \( \nabla \) is guaranteed to be metric and torsion-free with respect to \( g_Q = g_{\mathcal{N}\mathcal{F}} \) \([13, 14]\). Let \( R^\nabla, K^\nabla, \rho^\nabla \) and \( \sigma^\nabla \) be the transversal curvature tensor, transversal sectional curvature, transversal Ricci operator and transversal scalar curvature with respect to \( \nabla \), respectively. A differential form \( \omega \in \Omega^r(M) \) is \emph{basic} if
\[
i(X)\omega = 0, \theta(X)\omega = 0, \quad \forall X \in T\mathcal{F}. \tag{2.2}
\]
Let \( \Omega^r_B(\mathcal{F}) \) be the set of all basic \( r \)-forms on \( M \). Then \( L^2(\Omega^r(\mathcal{M})) \) is decomposed as
\[
L^2(\Omega(\mathcal{M})) = L^2(\Omega^r_B(\mathcal{F})) \oplus L^2(\Omega^r_B(\mathcal{F}))^\perp. \tag{2.3}
\]
Let \( P : L^2(\Omega^r(\mathcal{M})) \to L^2(\Omega^r_B(\mathcal{F})) \) be the orthogonal projection onto basic forms \([11]\), which preserves smoothness in the case of Riemannian foliations. For any \( r \)-form \( \phi \), we put the basic part of \( \phi \) as \( \phi_B := P\phi \). Let \( \delta_B \) be the formal adjoint operator of \( d_B = d|_{\Omega^r_B(\mathcal{F})} \). The basic Laplacian \( \Delta_B \) is given by \( \Delta_B = d_B\delta_B + \delta_B d_B \).

For any basic function \( f \), it is well-known \([2\text{ or } 11]\) that \( \int_M \Delta_B f = 0 \).

**Lemma 2.1** Let \( (M, g_M, \mathcal{F}) \) be a closed, connected Riemannian manifold with a foliation \( \mathcal{F} \) and a bundle-like metric \( g_M \). If \( \Delta_B f = \kappa^B_B(f) \) for any basic function \( f \), then \( f \) is constant.

**Proof.** Since \( \Delta_B \) on basic functions is the restriction of the elliptic operator \( \Delta + \left( \kappa^B_B - \kappa^B \right) \) on all functions \([11]\text{ Prop. 4.1}]\), we have that \( \Delta_B - \kappa^B_B \) on basic functions is the restriction of \( \Delta - \kappa^B \) on all functions. A solution \( u : M \to \mathbb{R} \) to \((\Delta - \kappa^B)u = 0\) satisfies the maximum and minimum principles locally, so that if such a \( u \) has a local maximum or minimum, then \( u \) is constant. The result follows. \( \square \)

Let \( V(\mathcal{F}) \) be the space of all vector fields \( Y \) on \( M \) satisfying \([Y, Z] \in T\mathcal{F} \) for all \( Z \in T\mathcal{F} \). An element of \( V(\mathcal{F}) \) is called an \emph{infinitesimal automorphism} of \( \mathcal{F} \).
Let 
\[ V(F) = \{ Y := \pi(Y) \mid Y \in V(F) \}. \] (2.4)

Then we have \( \Omega^r_B(F) \subset \Gamma(\Lambda^r Q^r) \) and \( V(F) \cong \Omega^1_B(F) \). If \( Y \in V(F) \) satisfies \( \theta(Y)g_Q = 0 \), then \( Y \) is called a transversal Killing field of \( F \). If \( Y \in V(F) \) satisfies \( \theta(Y)g_Q = 2f_Yg_Q \) for a basic function \( f_Y \) depending on \( Y \), then \( Y \) is called a transversal conformal field of \( F \); in this case, we have 
\[ f_Y = \frac{1}{q} \text{div}_\nabla Y. \] (2.5)

A \( Y \) is called a transversal non-isometric conformal field of \( F \) if \( Y \) is transversal conformal and not Killing, i.e., \( f_Y \neq 0 \). Note that \( Y \) is a transversal conformal field if and only if
\[ g_Q(\nabla_X Y, Z) + g_Q(\nabla_Z Y, X) = 2f_Yg_Q(X, Z) \quad X, Z \in NF. \] (2.6)

Let \( \{ E_a \} \) be a local orthonormal basic frame of \( NF \). Then we have the following lemma.

**Lemma 2.2** (2.4) Let \((M, g_M, F)\) be a Riemannian manifold with a foliation \( F \) of codimension \( q \) and a bundle-like metric \( g_M \). If \( Y \in V(F) \) is a transversal conformal field, i.e., \( \theta(Y)g_Q = 2f_Yg_Q \), then we have
\[ (\theta(Y)\text{Ric}^\nabla)(E_a, E_b) = -(q - 2)\Delta_a f_b + \delta_a^b(\Delta_B f_Y - \kappa^-(f_Y)), \] (2.7)
where \( \nabla_a = \nabla_{E_a} \), \( f_a = \nabla_a f_Y \) and \( \text{Ric}^\nabla(X, Y) = g_Q(\rho^\nabla(X), Y) \) for any \( X, Y \in NF \).

### 3 The generalized Obata theorem

Recall the following definition similar to that in [5], which is a special case of an isometric equivalence between two pseudogroups of local isometries acting on smooth manifolds. Let \((M, g_M, F)\) be a Riemannian manifold of a foliation \( F \) and a bundle-like metric \( g_M \).
Definition 3.1 Let $G$ be a discrete group. A Riemannian foliation $(M, \mathcal{F})$ is *transversally isometric* to $(W, G)$, where $G$ acts by isometries on a Riemannian manifold $(W, g_W)$, if there exists a homeomorphism $\eta : W/G \to M/\mathcal{F}$ that is *locally covered by isometries*. That is, given any $x \in M$, there exists a local smooth transversal $V$ containing $x$ and a neighborhood $U$ in $W$ and an isometry $\phi : U \to V$ such that the following diagram commutes

$$
\begin{array}{ccc}
U & \xrightarrow{\phi} & V \\
\downarrow^{Po} & \circ & \downarrow^{Poj} \\
W/G & \xrightarrow{\eta} & M/\mathcal{F}
\end{array}
$$

where $i : U \to W$ and $j : V \to M$ are inclusions and $P : W \to W/G$ and $\tilde{P} : M \to M/\mathcal{F}$ are the projections.

Then we have the following generalized Obata theorem for foliations (recall that $\nabla$ refers to the connection on $N\mathcal{F}$).

**Theorem 3.2** Let $(M, g_M, \mathcal{F})$ be a connected, complete Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q \geq 2$ and a bundle-like metric $g_M$, and let $c$ be a positive real number. Then the following are equivalent:

1. There exists a non-constant basic function $f$ such that $\nabla_X df = -c^2 f X^b$ for all vectors $X \in N(\mathcal{F})$.

2. $(M, \mathcal{F})$ is transversally isometric to $(S^q(1/c), G)$, where the discrete subgroup $G$ of the orthogonal group $O(q)$ acts by isometries on the last $q$ coordinates of the $q$-sphere $S^q(1/c)$ of radius $1/c$ in Euclidean space $\mathbb{R}^{q+1}$.

**Proof.** It is clear that the second condition implies the first, because if $f$ is the first coordinate function in $\mathbb{R}^{q+1}$ considered as a function on the sphere $S^q(1/c)$, it satisfies the first condition. Next, assume that the first condition is satisfied for the basic function $f$. This implies that for each $x \in M$,

$$-c^2 f(x) g_{N_x} = \nabla^2 f|_{N_x \mathcal{F}},$$


where $N_x \mathcal{F}$ is the normal space to the leaf through $x \in M$, and where $g_{N_x} = g_{N_x \mid N_x \mathcal{F}}$ is the metric restricted to $N_x \mathcal{F}$. For any unit speed geodesic $\gamma : [0, \beta) \rightarrow M$ that is normal to the leaves of the foliation,

\[ -c^2 (f \circ \gamma) = -c^2 (f \circ \gamma) g_M(\gamma', \gamma') \]
\[ = g_M(\nabla_{\gamma'} \text{grad } f, \gamma') \]
\[ = g_M(\text{grad } f, \gamma')' - g_M(\text{grad } f, \nabla_{\gamma'} \gamma') \]
\[ = (f \circ \gamma)'' . \]

Note that since the metric is bundle-like, every geodesic with initial velocity in $N \mathcal{F}$ is guaranteed to be orthogonal to $\mathcal{F}$ at all points ([12]). Thus

\[ (f \circ \gamma)(t) = A \cos (ct) + B \sin (ct) \]

for some constants $A$ and $B$. Let $\gamma(0) = x_0 \in M$ be either a global maximum or global minimum of $f$ on $M$. Then

\[ f(\gamma(t)) = f(x_0) \cos (ct) \quad \text{(3.1)} \]

for any unit speed geodesic $\gamma$ orthogonal to the leaf $L_{x_0}$ through $x_0$, and the maximum and minimum values along $\gamma$ must have opposite signs. Suppose that we choose the geodesic so that it connects an absolute maximum $x_0$ with an absolute minimum $x_1$; such a normal geodesic can always be found (see [4]).

Note that the nondegeneracy of the normal Hessian implies that each maximum and minimum of $f \circ \gamma$ occurs at an isolated closed leaf of $(M, \mathcal{F})$; then the set $f^{-1}(-f(x_0))$ must be a discrete union of closed leaves. The normal exponential map is surjective ([1]), and $f^{-1}([-f(x_0), -f(x_0)]) = M$ by the reasoning above. So $f^{-1}(-f(x_0))$ is a single closed leaf, say $L_{x_1}$, so that all normal geodesics through $x_0$ meet $L_{x_1}$ at the exact distance $\pi/c$. Similarly, $f^{-1}(f(x_0)) = L_{x_0}$.

Given any leaf $L$ of $M$ that is neither $L_{x_0}$ nor $L_{x_1}$, there exists a minimal normal geodesic connecting it to $L_{x_0}$ by completeness. In fact, there exists such
a minimal normal geodesic through \( x_0 \), and its initial velocity lies in \( N_{x_0} \mathcal{F} \). By equation (3.1), the gradient of \( f \) is nonzero at each \( \gamma(t) \) for \( 0 < t < \pi/c \) and is parallel to \( \gamma'(t) \). Since geodesics are determined by velocity at a single point, it is impossible that two geodesics with initial velocities through \( x_0 \) meet at the same point unless that point has distance at least \( \pi/c \) from \( x_0 \). Thus, the normal exponential map \( \exp^\perp_{x_0} : N_{x_0} \mathcal{F} \to M \) is injective on the ball \( B_{\pi/c}(x_0) \subset N_{x_0} \mathcal{F} \). This discussion is independent of the initial point of \( L_{x_0} \) chosen, because for a bundle-like metric the distance from a point \( x_0 \) on one leaf closure to another is independent of the choice \( x_0 \in L_{x_0} \) (see [4]). We have
\[
\bigcup_{x \in L_{x_0}} \exp^\perp_x(\overline{B_{\pi/c}(x)}) = M.
\]
Let \( M_s = \{ L_y \mid \text{dist}(L_{x_0}, L_y) = s \} \) for any non-negative real number \( s \), so that \( M_0 = L_{x_0} \) and \( M_{\pi/c} = L_{x_1} \). By the preceding discussion, for \( s \in (0, \pi/c) \), \( M_s \) is diffeomorphic to the unit normal sphere bundle of \( L_{x_0} \subset M \). Note that the infinitesimal holonomy group \( G \) at \( x_0 \) acts by orthogonal transformations on \( N_{x_0} \mathcal{F} \) ([8]), and this action induces an isometric group action on \( M_s \cap \exp^\perp_{x_0}(N_{x_0} \mathcal{F}) \), with the induced metric from \( g_{N \mathcal{F}} \). Each saturated submanifold \( M_s \) for \( 0 \leq s < \pi/c \) has a leaf space that is isometric to the quotient of \( M_s \cap \exp^\perp_{x_0}(N_{x_0} \mathcal{F}) \) by \( G \). Then \( (M \setminus L_{x_1})/\mathcal{F} \) is diffeomorphic to \( B_{\pi/c}/G \). The map
\[
\eta : B_{\pi/c}/G \to (M \setminus L_{x_1})/\mathcal{F}
\]
is defined by \( \eta(O_\xi) = L_{\exp^\perp_{x_0}(\xi)} \), where \( \xi \in B_{\pi/c} \subset N_{x_0} \mathcal{F} \), \( O_\xi \) is the \( G \)-orbit of \( \xi \) in \( B_{\pi/c} \) and \( L_{\exp^\perp_{x_0}(\xi)} \) is the leaf containing \( \exp^\perp_{x_0}(\xi) \). Letting \( B_{\pi/c}^+ \) denote the one-point compactification of \( B_{\pi/c} \), \( \eta \) can be extended to a homeomorphism
\[
\overline{\eta} : B_{\pi/c}^+/G \to M/\mathcal{F}.
\]
Thus \( M/\mathcal{F} \) is homeomorphic to \( S/G \), where \( S = B_{\pi/c}^+ \) is a sphere. Next we will show that the pullback of the transverse metric of \( (M, \mathcal{F}) \) endows \( S \) with the standard metric of \( S^q(1/c) \).

Let \( v \) and \( w \) be any two nonzero orthonormal vectors in \( N_{x_0} \mathcal{F} \), and let \( W_s \) denote the \( \mathcal{N} \mathcal{F} \)-parallel translate of \( w = W_0 \) along the geodesic \( \gamma(s) \) with initial
velocity $v$; thus $W_s \in N_{\gamma(s)}F$ is a well-defined vector at each $\gamma(s)$ for $0 \leq s < \pi/c$. We see that $W_s$ is tangent to $M_s$ for $s \in (0, \pi/c)$. Let $(y_j)$ be geodesic normal coordinates for the normal ball $\exp_{x_0}^{\perp} (B_{\pi/c} (x_0))$. Suppose that these coordinates are chosen at $x_0$ such that $y_1(\gamma(s)) = s$ and each of $\frac{\partial}{\partial y_j}$ for $j > 1$ is orthogonal to $v = \gamma'(0)$ at $x_0 = 0$. We extend $s$ to be the function $s(y) = \sqrt{\sum y_j^2}$ and write $y_j = s\theta_j$, so that each $\theta_j$ is independent of $s$. Thus, $\gamma'(s)(\theta_j) = 0$ and $W_s(s) = 0$. Further, we let $\frac{\partial}{\partial s}$ denote the radial vector field, which agrees with $\gamma'(s)$ along $\gamma$. In the calculations that follow, we extend $y_j, \theta_j, \frac{\partial}{\partial s}$ to be well-defined and basic in a small neighborhood of the transversal $\exp_{x_0}^{\perp} (B_{\pi/c})$. From the calculation of $f$ above, we see that $\nabla f = -c \sin(cs)f(x_0)\frac{\partial}{\partial s}$.

Since $\nabla$ is torsion-free and $\nabla_{\gamma(s)}W_s = 0$ by construction,

$$\pi \left[ \frac{\partial}{\partial s}, W_s \right] = -\nabla_{W_s} \frac{\partial}{\partial s} = \frac{1}{c \sin(cs)f(x_0)} \nabla_{W_s} \nabla f = -\frac{c^2}{c \sin(cs)f(x_0)} f(\gamma(s))W_s = -\frac{c \cos(cs)}{\sin(cs)} W_s,$$

by assumption, since $\nabla$ is a metric connection and thus commutes with raising indices. Since $\theta_j$ is a locally defined basic function, for $0 < s < \pi/c$,

$$\frac{d}{ds} W_s(\theta_j) = \frac{\partial}{\partial s} W_s(\theta_j) = \left[ \frac{\partial}{\partial s}, W_s \right] (\theta_j) = \pi \left[ \frac{\partial}{\partial s}, W_s \right] (\theta_j) = -\frac{c \cos(cs)}{\sin(cs)} W_s(\theta_j).$$

Solving the differential equation above, we have

$$W_s(\theta_j) = \frac{1}{\sin cs} W_{\pi/2c}(\theta_j), 0 < s < \frac{\pi}{c}. \quad (3.2)$$

Since $W_s(s) = 0$ we have

$$W_s(y_j) = s W_s(\theta_j)$$
for $0 < s < \pi/c$. Then, for all $j$,

$$W_0(y_j) = \lim_{s \to 0} W_s(y_j) = \frac{1}{c} W_{s \theta} (\theta_j) = \frac{\sin(cs)}{c} W_s (\theta_j) = \frac{\sin(cs)}{cs} W_s (y_j).$$

Note that since the vectors $\frac{\partial}{\partial \theta_j}$ for $j > 1$ form a basis of the tangent space for $M_s \cap \exp_{x_0}^{1/2} (B_{\pi/c})$ at $\gamma(s)$ with $s > 0$, the equation above uniquely defines the vector $W_s$ in terms of $W_0$. Since the metric on the sphere $S^q (1/c)$ satisfies the same hypothesis, a corresponding fact is true for geodesic normal coordinates on $S^q (1/c)$.

We now show that the equation above implies that the pullback of the metric $g_{NF}$ to $B_{\pi/c}$ is the same as the standard metric $g_S$ corresponding to geodesic normal coordinates on $S^q (1/c)$. As above, let $W_s$ denote the parallel displacement of $W_0$ along $\gamma(s)$, and let $\overline{W_s}$ denote the parallel displacement of $W_0$ along the geodesic in $(B_{\pi/c}, g_S)$ with unit tangent vector $v$. Then

$$W_s (y_j) = \frac{cs}{\sin(cs)} W_0 (y_j) = \overline{W_s} (y_j).$$

Since the actions of the vectors $\overline{W_s}$ on the coordinate functions $y_j$ determine their values along the geodesic with initial velocity $v$, we conclude that $\overline{W_s} = W_s$. Thus, the metrics $g_{NF}$ and $g_S$ on $B_{\pi/c}$ yield identical parallel displacements of vectors orthogonal to $v$ along the line containing $v$, and $g_{NF}|_{x_0} = g_S|_{x_0}$. Since it follows from previous calculations that the initial vector $v \in N_{x_0} \mathcal{F}$ is arbitrary, we conclude that $g_{NF} = g_S$. We may reverse the roles of $x_0$ and $x_1$ and obtain a similar result.

Now, given any point $x \in M$, there is a minimal geodesic connecting this point to a point $x'_0$ on the leaf containing $x_0$. If $x \not\in L_{x_1}$, the above analysis
shows that the map $\exp_{x_0}^\perp$ restricted to $(B_{\pi/c}(x_0'),g_S)$ is an isometry onto its image, and that image contains $x$. Further, the map $\exp_{x_0}^\perp$ locally covers the map $\bar{\eta} : \left( B_{\pi/c}^+,G,g_S \right) \to (M/F,g_N/F)$. If $x \notin L_{x_0}$, a similar fact is true for $\exp_{x_1}^\perp$. Thus the map $\bar{\eta}$ is locally covered by isometries, and we conclude that $(M,F)$ is transversally isometric to $(S^q(1/c),G)$. \hfill \Box

4 Applications

In this section, we give some applications of the generalized Obata theorem. Let $M$ be a Riemannian manifold admitting a transversal non-isometric conformal field. For more details about transversal conformal fields, see [2,10].

**Theorem 4.1** Let $(M,g_M,F)$ be a compact Riemannian manifold with a foliation $F$ of codimension $q$ and a bundle-like metric $g_M$. Assume that the transversal scalar curvature $\sigma^\nabla$ is a positive constant. If $M$ admits a transversal non-isometric conformal field $Y$ such that $Y = \nabla h$ for some basic function $h$, then $(M,F)$ is transversally isometric to $(S^q(1/c),G)$, where $c^2 = \frac{\sigma^\nabla}{q(q-1)}$ and $G$ is a discrete subgroup of $O(q)$ acting on the $q$-sphere.

**Proof.** Let $\theta(Y)g_Q = 2f_Yg_Q(f_Y \neq 0)$ and $\bar{Y} = \nabla h$ for some basic function $h$. From [2,6], we have that, for any $X,Z \in \Gamma Q$,

$$
2f_Yg_Q(X,Z) = g_Q(\nabla_X\nabla h,Z) + g_Q(\nabla_Z\nabla h,X)
= \nabla\nabla h(X,Z) + \nabla\nabla h(Z,X)
= 2\nabla\nabla h(X,Z).
$$

Hence we have

$$
\nabla\nabla h = f_Yg_Q. \tag{4.1}
$$

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Note that the function $f_Y$ satisfies

$$
\Delta_B f_Y = \frac{\sigma}{q-1} f_Y + \kappa^*_B(f_Y).
$$

(4.2)

Since $\Delta_B h = -\sum_a \nabla_{E_a} \nabla_{E_a} h + \kappa^*_B(h)$, we have from (4.1) and (4.2)

$$
\Delta_B \left( f_Y + \frac{\sigma}{q(q-1)} h \right) = \kappa^*_B \left( f_Y + \frac{\sigma}{q(q-1)} h \right).
$$

From Lemma 2.1, we have that

$$
f_Y + \frac{\sigma}{q(q-1)} h = \text{constant},
$$

(4.3)

which yields

$$
\nabla \nabla f_Y + \frac{\sigma}{q(q-1)} \nabla \nabla h = 0.
$$

From (4.1), we have

$$
\nabla \nabla f_Y = -\frac{\sigma}{q(q-1)} f_Y g_Q.
$$

(4.4)

By the generalized Obata theorem (Theorem 3.2), $(M,F)$ is transversally isometric to $(S^q(1/c),G)$, where $c^2 = \frac{\sigma}{q(q-1)}$. □

**Theorem 4.2** Let $(M,g_M,F)$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q$ and a bundle-like metric $g_M$. Assume that the transversal scalar curvature $\sigma^\mathcal{V}$ is a positive constant. If $M$ admits a transversal non-isometric conformal field $Y$ such that $\theta(Y)Ric^\mathcal{V} = \mu g_Q$ for some basic function $\mu$, then $(M,F)$ is transversally isometric to $(S^q(1/c),G)$, where $c^2 = \frac{\sigma^\mathcal{V}}{q(q-1)}$ and $G$ is a discrete subgroup of $O(q)$ acting on the $q$-sphere.

**Proof.** Let $\theta(Y)g_Q = 2f_Y g_Q (f_Y \neq 0)$. From Lemma 2.2, we have

$$
\mu g_Q = -(q-2)\nabla \nabla f_Y + (\Delta_B f_Y - \kappa^*_B f_Y) g_Q.
$$

(4.5)
Hence we have
\[ \mu = \frac{2(q-1)}{q} (\Delta_B f_Y - \kappa_B^2 (f_Y)). \] (4.6)

From (4.5) and (4.6), we have
\[ \nabla^2 f_Y = -\frac{1}{q} (\Delta_B f_Y - \kappa_B^2 (f_Y)) g_Q. \] (4.7)

From (4.2), we have
\[ \nabla^2 f_Y + \frac{\sigma}{q(q-1)} f_Y g_Q = 0. \] (4.8)

By the generalized Obata theorem, the proof is completed. □

On the other hand, we recall the following theorem.

**Theorem 4.3** ([2]) Let \((M, g_M, \mathcal{F})\) be a compact Riemannian manifold with a foliation \(\mathcal{F}\) of codimension \(q\) and a bundle-like metric \(g_M\) with \(\delta_B \kappa_B = 0\). Assume that the transversal scalar curvature \(\sigma^\nabla\) is constant and \(\rho^\nabla(X) \geq \frac{\sigma^\nabla}{q} X\) for any \(X \in \Gamma Q\). If \(M\) admits a transversal non-isometric conformal field, then \((M, \mathcal{F})\) is transversally isometric to \((S^q, G)\), where \(G\) is a discrete subgroup of \(O(q)\) acting on the \(q\)-sphere.

**Remark.** On a compact Riemannian manifold admitting a transversal non-isometric conformal field, if the scalar curvature \(\sigma^\nabla\) is constant, then the condition \(\delta_B \kappa_B = 0\) implies that \(\sigma^\nabla\) is non-negative. Moreover, if \(\sigma^\nabla\) is positive constant and the transversal Ricci curvature \(\rho^\nabla(X) \geq \frac{\sigma^\nabla}{q} X\) for any \(X \in \Gamma Q\), then \(\kappa = 0\) by the tautness theorem ([4]).

Hence Theorem 4.3 is equivalent to the following.

**Theorem 4.4** Let \((M, g_M, \mathcal{F})\) be a compact Riemannian manifold with a foliation \(\mathcal{F}\) of codimension \(q\) and a bundle-like metric \(g_M\). Assume that the transversal scalar curvature \(\sigma^\nabla\) is a positive constant and \(\rho^\nabla(X) \geq \frac{\sigma^\nabla}{q} X\) for any \(X \in \Gamma Q\). If \(M\) admits a transversal non-isometric conformal field, then \((M, \mathcal{F})\) is transversally isometric to \((S^q, G)\), where \(G\) is a discrete subgroup of \(O(q)\) acting on the \(q\)-sphere.
We define an operator \( A_Y : \mathcal{N}\mathcal{F} \to \mathcal{N}\mathcal{F} \) for any vector field \( Y \in V(\mathcal{F}) \) by
\[
A_Y s = \theta(Y)s - \nabla_Y s.
\] (4.9)

Then it is proved [3] that, for any vector field \( Y \in V(\mathcal{F}) \),
\[
A_Y s = -\nabla_{Y_s} \bar{Y},
\] (4.10)
where \( Y_s = \sigma(s) \in \Gamma TM \). So \( A_Y \) depends only on \( \bar{Y} = \pi(Y) \) and is a linear operator. Moreover, \( A_Y \) extends in an obvious way to tensors of any type on \( \mathcal{N}\mathcal{F} \) (see [3] for details).

**Theorem 4.5** Let \((M, g_M, \mathcal{F})\) be a compact Riemannian manifold with a foliation \( \mathcal{F} \) of codimension \( q \) and a bundle-like metric \( g_M \). Assume that the transversal scalar curvature \( \sigma^\nabla \) is a positive constant. If \( M \) admits a transversal non-isometric conformal field \( \bar{Y} \), i.e., \( \theta(Y)g_Q = 2f_Y g_Q \) such that (i) \( \rho^\nabla(\nabla f_Y) = \frac{\sigma^\nabla}{q} \nabla f_Y \), (ii) \( \kappa_B(f_Y) = 0 \) and (iii) \( g_Q(A_{\kappa_B^*} \nabla f_Y, \nabla f_Y) \leq 0 \), then \((M, \mathcal{F})\) is transversally isometric to \((S^q, G)\), where \( G \) is a discrete subgroup of \( O(q) \) acting on the \( q \)-sphere.

**Proof.** First, we recall that, for any basic function \( g \) with \( \kappa_B^*(g) = 0 \), we have [2]
\[
\int_M \left\{ \frac{q-1}{q} g_Q(\Delta_B d_B g, d_B g) - g_Q(\rho^\nabla(\nabla f_Y), \nabla f_Y) + g_Q(A_{\kappa_B^*} \nabla f_Y, \nabla f_Y) - |\nabla^2 g + \frac{1}{q} \Delta_B g|^2 \right\} = 0.
\]
From (4.12) and assumption (ii), we have
\[
\Delta_B f_Y = \frac{\sigma^\nabla}{q-1} f_Y.
\]
Hence, from the assumptions (i) and (iii), we have
\[
\nabla^2 f_Y = -\frac{\sigma^\nabla}{q(q-1)} f_Y g_Q.
\]
By the generalized Obata theorem, the proof is completed. □

If the foliation \( \mathcal{F} \) is minimal, the conditions (ii) and (iii) in Theorem 4.5 are satisfied. Hence we have the following corollaries.
**Corollary 4.6** Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a minimal foliation $\mathcal{F}$ of codimension $q$ and a bundle-like metric $g_M$. Assume that the transversal scalar curvature $\sigma^\nabla$ is a positive constant. If $M$ admits a transversal conformal field $\nabla Y$, i.e., $\theta(Y)g_Q = 2f_Yg_Q$ such that $\rho^\nabla(\nabla f_Y) = \frac{\sigma^\nabla}{q}\nabla f_Y$, then $(M, \mathcal{F})$ is transversally isometric to $(S^q, G)$, where $G$ a discrete subgroup of $O(q)$ acting on the $q$-sphere.

**Corollary 4.7** Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a minimal, transversally Einstein foliation $\mathcal{F}$ of codimension $q$ and a bundle-like metric $g_M$. Assume that the transversal scalar curvature $\sigma^\nabla$ is a positive constant. If $M$ admits a transversal non-isometric conformal field $\nabla Y$, then $\mathcal{F}$ is transversally isometric to $(S^q, G)$, where $G$ is a discrete subgroup of $O(q)$ acting on the $q$-sphere.

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Department of Mathematics, Jeju National University, Jeju 690-756, Korea
E-mail address : sdjing@jejyunu.ac.kr
Department of Mathematics, Jeju National University, Jeju 690-756, Korea
E-mail address: niver486@jejunu.ac.kr
Department of Mathematics, Texas Christian University, Fort Worth, TX 76129, USA
E-mail address: k.richardson@tcu.edu