TRAVELING WAVES IN FERMI-PASTA-ULAM CHAINS WITH NONLOCAL INTERACTION

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Dedicated to Professor Norman Dancer on the occasion of his 70th anniversary.

Abstract. The paper is devoted to traveling waves in FPU type particle chains assuming that each particle interacts with several neighbors on both sides. Making use of variational techniques, we prove that under natural assumptions there exist monotone traveling waves with periodic velocity profile (periodic waves) as well as waves with localized velocity profile (solitary waves). In fact, we obtain periodic waves by means of a suitable version of the Mountain Pass Theorem. Then we get solitary waves in the long wave length limit.

1. Introduction. In the present paper we consider an infinite chain of identical particles on the line such that each particle interacts with $M$ neighbors on both sides. The dynamics of such a chain is governed by equations

$$
\ddot{q}_j = \sum_{m=1}^{M} [U'_m(q_{j+m} - q_j) - U'_m(q_j - q_{j-m})], \quad j \in \mathbb{Z},
$$

(1)

where $q_j(t)$ is the position of $j$th particle at time $t$. The potentials $U_m$, $m = 1, 2, \ldots, M$, represent the interaction between a particle and its neighbors so that $U_1$ corresponds to the interaction with nearest neighbors, $U_2$ with second nearest neighbors, and so on.

In the case $M = 1$ this is the famous Fermi-Pasta-Ulam (shortly, FPU) lattice introduced and studied numerically in the pioneering paper [6]. Since that time the FPU lattice constantly attracts a lot of interest in physics and mathematics communities. A representative overview of the subject can be found in the monograph [14]. Notice, however, that [14] does not contain any information about traveling waves on the FPU lattice.

A traveling wave solution of (1) is a solution of the form $q_j(t) = u(j - ct)$, where $u(s)$ and $c > 0$ are the profile function and speed of the wave, respectively. For definiteness we consider the case of positive wave speed, the case $c < 0$ being similar. The relative displacement profile is defined by

$$
r(s) = u(s + 1) - u(s) = \int_s^{s+1} u'(t) dt,
$$

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where the derivative $u'(s)$ is the velocity profile. Two types of traveling waves are of interest: solitary waves and periodic waves. A solitary wave is localized in space in the sense that its velocity profile vanishes at infinity, while the velocity profile of a periodic wave is periodic. Automatically, these waves have vanishing at infinity and periodic displacement profiles, respectively. Notice that the profile function of a periodic wave is not necessarily periodic.

The profile function of a traveling wave is a solution of the following forward-backward differential-difference equation

$$c^2 u''(s) = \sum_{m=1}^{M} \left[ U'_m(D^+_m u(s)) - U'_m(D^-_m u(s)) \right], \quad (2)$$

where

$$D^+_m u(s) = u(s + m) - u(s)$$

and

$$D^-_m u(s) = u(s) - u(s - m).$$

We exclude trivial waves with linear profile functions $u(s) = as + b$. Notice that physically meaningful are waves with monotone, either nondecreasing or nonincreasing, profile functions. Only such waves are considered in this paper.

The first result on the existence of supersonic monotone solitary FPU traveling waves is obtained in [7]. This is done by minimizing the average kinetic energy subject to the constraint that the average potential energy is given. In this approach the wave speed appears as the Lagrange multiplier. Another constrained minimization is used in [16] to study both periodic and solitary waves in FPU with convex potentials.

Completely different approach to the existence of solitary waves is suggested in [32]. With the aid of the Mountain Path Theorem, it is obtained an existence result for supersonic waves with prescribed speed. The starting point of paper [28] is to study periodic waves, still supersonic. More precisely, it is shown that periodic waves with arbitrarily large periods do exist. Then a solitary wave is obtained as the limit of a sequence of periodic waves with periods tending to infinity (periodic approximation). The approach of [28] makes use of the Mountain Path Theorem and the Nehari manifold. A detailed presentation of these and other results on the FPU system, including the existence of subsonic periodic waves, can be found in [26]. Notice that the results mentioned so far concern the case of superquadratic interaction potential. The case of asymptotically quadratic (saturable) potential is studied in [29]. Other results on FPU traveling waves obtained with the help of variational methods can be found in [16, 31]. Let us point out that periodic approximation approach is proven to be efficient in many other variational problems (see, e.g., [25, 27, 29]).

In remarkable paper [9] it is discovered that near-sonic, supersonic FPU solitary waves can be obtained as perturbations of KdV solitons. In subsequent papers [10, 11, 12] the stability of such waves is studied. Recently, with the help of this approach cnoidal FPU waves have been found [8]. In [3, 4, 13, 20] generalized KdV equations is used to study other FPU type systems. Another perturbation approach is used in [18] where different types of traveling waves on FPU lattices are obtained by means of bifurcation theory.

More general lattice systems (1) of FPU type are not well-studied. To the best of our knowledge, there are only two papers devoted to this subject. In [34] certain approximations of solitary waves on lattices with second-neighbor interaction are
suggested and numerical results are discussed. Recent paper [17] is devoted to near-sonic solitary waves for (1) of KdV type and follows the ideas developed in [9]. However, the techniques used there is different in many aspects due to the nonlocal character of the interaction. Let us point out paper [1] where the nonlinear Klein-Gordon lattice with non-local interaction is considered and solitary traveling waves are obtained as perturbations of continuum Klein-Gordon solitary waves. Notice that nonlocal discrete Klein-Gordon equations are extensively studied in physics literature (see, e.g., [30] and references therein). Also we mention other types of lattice systems with nonlocal interaction such as discrete nonlinear Schrödinger equations (see [5] and references therein).

The organization of the paper is as follows. In Section 2 we provide a variational setting of the problem and formulate the main result of the paper. Section 3 contains some technical preliminaries, while in Section 4 we discuss the Mountain Pass Geometry of the energy functionals associated with the problem considered and Nehari type characterization of mountain pass values of these functionals. In Section 5 we prove the existence of periodic traveling waves and obtain a uniform bound for their energy levels. In Section 6 we make the passage to the limit as the period of periodic waves tends to infinity and obtain the existence of solitary waves. At this point we make use of a rudimentary concentration-compactness based on inequality (13). Deep results from [22] are not needed here because the equation for traveling waves is set up on the real line. Finally, in Section 7, we show that, under additional assumptions, periodic waves converge to solitary ones in certain non-local sense and consider some examples.

2. Setting of problem and variational formulation. Let us express the interaction potentials in the form

\[ U_m(r) = \frac{a_m}{2} r^2 + V_m(r), \quad m = 1, \ldots, M, \]

where \( a_m \geq 0 \) and \( V_m \) is \( C^1 \) with \( V_m(0) = V'_m(0) = 0 \) for all \( m = 1, \ldots, M \). Throughout the paper we employ the following assumptions by default.

(A1) The speed of sound \( c_0 \) defined by

\[ c_0^2 = \sum_{m=1}^{M} a_m m^2 \]

is positive and \( c > c_0 \).

(A2) \( V'_m(r) = o(r) \) as \( r \to 0 \) for all \( m = 1, \ldots, M \).

(A3) All functions \( V_m(r), \ m = 1, 2, \ldots, M, \) are nonnegative and at least one of them, say \( V_{m_0}, \) satisfies the condition

\[ \lim_{|r| \to \infty} r^{-2} V_{m_0}(r) = \infty. \]

Considering \( T \)-periodic problem with integer \( T \) we assume, in addition, that \( m_0 = 1 \).

(A4) All functions \( |r|^{-1} V'_m(r), \) extended by 0 to \( r = 0, \) are nondecreasing and at least one of them is strictly increasing.

Notice that, in general, not all of the functions \( V_m \) are nonzero.

Remark 1. By Assumption (A4), all functions

\[ G_m(r) = \frac{1}{2} V'_m(r)r - V_m(r), \quad m = 1, 2, \ldots, M, \]
are nondecreasing (respectively, nonincreasing) for \( r > 0 \) (respectively, \( r < 0 \)). A simple proof in a more general setting can be found in [23, Lemma 2.3]. Furthermore, \( G_m(r) > 0 \) for all \( r \neq 0 \).

We are looking for solutions of equation (2) subject to one of the following conditions

\[
\frac{d}{ds}(u(s) + T) = u'(s) \quad \forall s \in \mathbb{R},
\]

where \( T > 0 \) is a given period (periodic waves), or

\[
u'(s) \to 0 \quad \text{as} \quad |s| \to \infty
\]

(solitary waves).

Given \( T > 0 \), we introduce the space

\[
X_T = \{ u \in H^1_{\text{loc}}(\mathbb{R}) : u'(s + T) = u'(s) \quad \forall s \in \mathbb{R}, \quad u(0) = 0 \}.
\]

Since all functions in \( H^1_{\text{loc}}(\mathbb{R}) \) are continuous, condition \( u(0) = 0 \) makes sense. The inner product on \( X_T \) is given by

\[
(u, v)_T = \int_{I_T} u'(s)v'(s) \, ds.
\]

Here and later on, \( I_T = [-T/2, T/2] \). Also we introduce the Hilbert space

\[
X = \{ u \in H^1_{\text{loc}}(\mathbb{R}) : u' \in L^2(\mathbb{R}) \quad u(0) = 0 \}
\]

endowed with the inner product

\[
(u, v) = \int_{\mathbb{R}} u'(s)v'(s) \, ds.
\]

We denote by \( \| \cdot \|_T \) and \( \| \cdot \| \) the norms on \( X_T \) and \( X \) induced by the inner products \((\cdot, \cdot)_T\) and \((\cdot, \cdot)\), respectively.

We denote by \( X_T^+ \) and \( X^+ \) the cones of nondecreasing functions in \( X_T \) and \( X \), respectively. The cones of nonincreasing function are \( X_T^- = -X_T^+ \) and \( X^- = -X^+ \). All these cones are closed.

**Remark 2.** It is easily seen that if \( T \geq M \) is an integer, all operator \( D_m^\pm \), \( m = 1, 2, \ldots, M \), have nontrivial kernels in the space \( X_T \) and

\[
\ker D_1^\pm \subset \ker D_m^\pm, \quad m = 2, \ldots, M.
\]

Conversely, if one of those operators has a nonzero kernel in \( X_T, T \geq M \), then \( T \) is an integer. In the space \( X \) all operators \( D_m^\pm \) have trivial kernels.

We associate to problems (2), (4) and (2), (5) the following energy functionals

\[
J_T(u) = \int_{I_T} \left( \frac{c^2}{2} u'^2 - \sum_{m=1}^{M} U_m(D_m^+ u) \right) \, ds
\]

\[
= \int_{I_T} \left( \frac{c^2}{2} u'^2 - \frac{1}{2} \sum_{m=1}^{M} a_m^2 D_m^+ u - \sum_{m=1}^{M} V_m D_m^+ u \right) \, ds
\]

on \( X_T \) and

\[
J(u) = \int_{\mathbb{R}} \left( \frac{c^2}{2} u'^2 - \sum_{m=1}^{M} U_m(D_m^+ u) \right) \, ds
\]

\[
= \int_{\mathbb{R}} \left( \frac{c^2}{2} u'^2 - \frac{1}{2} \sum_{m=1}^{M} a_m^2 D_m^+ u - \sum_{m=1}^{M} V_m D_m^+ u \right) \, ds
\]
on $X$, respectively. Later on we shall show that these are well-defined $C^1$ functionals, and their critical points are solutions of the problems we consider. The quadratic and non-quadratic parts of $J_T$ (respectively, $J$) are denoted by $\frac{1}{2}q_T(u)$ and $\Psi_T(u)$ (respectively, $\frac{1}{2}q(u)$ and $\Psi(u)$). The spaces $X_T$ and $X$, as well as the functionals $J_T$ and $J$, are invariant with respect to modified translations

$$(S_a v)(s) = v(s + a) - v(a), \quad a \in \mathbb{R}.$$ 

Let us point out that the only trivial solution of equation (2) in the space $X$ is $u = 0$, while every space $X_T$ contains trivial solutions of the form $u(s) = ks$, $k \in \mathbb{R}$.

Our main result is the following.

**Theorem 2.1.** Under Assumptions $(A_1)$–$(A_4)$, suppose that $c > c_0$.

(a) There exists $T_c \geq M$ such that for every noninteger $T \geq T_c$, problem (2), (4) has a nontrivial solution in $X^+_T$ (respectively, in $X_T$). If $m_0 = 1$ in Assumption $(A_3)$, then the previous statement holds for all integer $T \geq T_c$ as well.

(b) Let $T_n \geq T_c$ be a sequence of noninteger numbers such that $T_n \to \infty$. Then there exist a sequence of nontrivial solutions $u_n \in X^+_T$ (respectively, $u_n \in X_T$) to problem (2), (4), and a nontrivial solution $u \in X^+$ (respectively, $u \in X^-$) of problem (2), (5) such that, along a subsequence, $J_{T_n}(u_n) \to J(u)$ and $u_n \to u$ uniformly on compact intervals together with first and second derivatives. If $m_0 = 1$ in Assumption $(A_3)$, then the same holds for all integer sequences $T_n \geq T_c$.

**Remark 3.** In [17] it is not assumed that all the coefficients $a_m$ in the quadratic parts of interaction potentials are nonnegative. We do not know whether variational methods may work if some of $a_m$ are negative. Therefore, in the last case the existence of traveling waves with arbitrary supersonic speed is an open problem.

Looking for monotone profile functions, may add, without loss of generality, the following assumption

$(E)$ $V_m(-r) = V_m(r)$ for all $m = 1, 2, \ldots, M$.

The proof of main result shall be carried out in the case of nondecreasing waves, the other case being similar.

3. Preliminaries. We begin with simple estimates for difference operators $D_m^\pm$.

**Lemma 3.1.** (a) Let $T \geq M$. If $u \in X_T$, then for all $m = 1, 2, \ldots, M$

$$|D_m^\pm u(t)| \leq m^{1/2} \|u\|_T,$$

$$\|D_m^\pm u\|_{L^2(I_T)} \leq m \|u\|_T$$

and

$$\|D_m^\pm u\|_{H^1(I_T)}^2 \leq (m^2 + 4) \|u\|_T^2.$$

(b) If $u \in X$, then for all $m \in \mathbb{N}$

$$|D_m^\pm u(t)| \leq m^{1/2} \|u\|,$$

$$\|D_m^\pm u\|_{L^2(\mathbb{R})} \leq m \|u\|$$

and

$$\|D_m^\pm u\|_{H^1(\mathbb{R})}^2 \leq (m^2 + 4) \|u\|_T^2.$$

and $D_m^\pm u(t) \to 0$ as $|t| \to \infty.$
Proof. We prove part (a) in the case of \( D^+_m u \). The other one is similar. Since
\[
D^+_m u(t) = \int_t^{t+m} u'(s) \, ds
\]
and \( m \leq M \leq T \), we have that
\[
|D^+_m u(t)| \leq \int_t^{t+m} |u'(s)| \, ds \leq m^{1/2} \int_t^{t+T} u'(s)^2 \, ds \leq m^{1/2} \|u\|_T.
\]
Now
\[
\|D^+_m u\|_{L^2(I_T)}^2 \leq m \int_{I_T} \int_t^{t+m} u'(s)^2 \, ds \, dt.
\]
Interchanging the order of integration, we obtain that
\[
\|D^+_m u\|_{L^2(I_T)}^2 \leq m^2 \int_{m+I_T} u'(s)^2 \, ds = m^2 \|u\|_T^2.
\]
Since
\[
\|D^+_m u\|_{H^1(I_T)}^2 = \|D^+_m u\|_{L^2(I_T)}^2 + \int_{I_T} (u(s + m) - u(s))^2 \, ds,
\]
the inequality for the \( H^1 \) norm follows immediately. \( \square \)

From Lemma 3.1 it follows immediately that the quadratic parts of functionals \( J_T, T \geq M \), and \( J \) are bounded on \( X_T \) and \( X \), respectively. Furthermore,
\[
q_T(u) \geq (c^2 - c^2_0)\|u\|_T^2, \quad u \in X_T,
\]
and
\[
q(u) \geq (c^2 - c^2_0)\|u\|_T^2, \quad u \in X,
\]
that is, the quadratic parts are positive definite.

By assumption \((\mathcal{A}_2)\), for every \( R > 0 \) there exists a constant \( C_R > 0 \) such that \( |V'_m(r)| \leq C_R|r| \) whenever \( |r| \leq R \). Then standard arguments together with estimates from Lemma 3.1 give rise to the following result.

**Proposition 3.1.** Functionals \( J_T, T \geq M \), and \( J \) are of class \( C^1 \), and their derivatives are given by
\[
J'_T(u)h = \int_{I_T} \left( c^2 u'(s)h'(s) - \sum_{m=1}^M U'_m(D^+_m u(s))D^+_m h(s) \right) \, ds, \quad h \in X_T, \quad (6)
\]
and
\[
J'(u)h = \int_{\mathbb{R}} \left( c^2 u'(s)h'(s) - \sum_{m=1}^M U'_m(D^+_m u(s))D^+_m h(s) \right) \, ds, \quad h \in X. \quad (7)
\]

Let \( u \in X_T \) be a critical point of \( J_T \) and \( \varphi \) be a \( T \)-periodic \( C^\infty \) function. Making use of (6), equation \( J_T(u)h = 0 \) with \( h(s) = \varphi(x) - \varphi(0) \) reduces to
\[
\int_{I_T} \left( c^2 u'(s)\varphi'(s) + \sum_{m=1}^M (U'_m(D^+_m u(s)) - U'_m(D^-_m u(s)))\varphi(s) \right) \, ds = 0,
\]
that is, \( u \) is a weak solution of (2) and, hence, a classical solution because the right-hand side of (2) is a continuous function of \( s \). Taking as \( \varphi \) a finitely supported \( C^\infty \) function and making use of (7), we obtain that critical points of \( J \) are solutions of equation (2).
Remark 4. Not all solutions of (2) in \( X_T \) are critical points of \( J_T \). Indeed, taking \( h(s) = s \) as a test function in (6), we see that any critical point of \( J_T \) satisfies
\[
c^2[u(T/2) - u(-T/2)] = \int_{I_T} \sum_{m=1}^{M} mU_m'(D_m^+u(s))ds.
\] (8)
The function \( u(s) = ks \) is a solution in \( X_T \) for all \( T \). By (8), it is a critical point of \( J_T \) if and only if \( k \) is a solution of
\[
(c^2 - c_0^2)k = \sum_{m=1}^{M} mV_m'(mk).
\]
Under our assumptions, this equation has exactly two nonzero solutions \( k_+ > 0 \) and \( k_- < 0 \). It would be interesting to find examples of solutions to (2) in \( X \) which are not critical points of \( J \).

4. Mountain Pass Geometry. Let \( \Phi \) be a \( C^1 \) functional on a Banach space \( E \). We say that \( \Phi \) possesses the Mountain Pass Geometry if \( \Phi(0) = 0 \), and there exist constants \( \alpha > 0 \), \( \rho > 0 \) and a vector \( e \in E \), with \( \|e\|_E > \rho \), such that \( \Phi(u) \geq \alpha \) for all \( u \in E \), \( \|u\|_E = \rho \), and \( \Phi(e) < 0 \).

The Cerami condition for \( \Phi \) reads as follows.

(C) If \( u_n \in E_n \) is a Cerami sequence at a level \( l \), that is, \( \Phi(u_n) \rightarrow l \) and
\[
(1 + \|u_n\|_E)\|\Phi'(u_n)\|_{E'} \rightarrow 0
\]
as \( n \rightarrow \infty \), then \( u_n \) contains a convergent subsequence.

Now we remind a version of the Mountain Pass Theorem for functionals satisfying the Cerami condition (see, e.g., [15, 24]).

Theorem 4.1. Let \( \Phi \) be a \( C^1 \) functional on a Banach space \( E \) that satisfies the Cerami condition, and let \( P : E \rightarrow E \) be a continuous mapping such that \( \Phi(Pu) \leq \Phi(u) \) for all \( u \in E \). Assume that \( \Phi \) possesses the Mountain Pass geometry with extra condition \( P(e) = e \). Then \( \Phi \) has a nonzero critical point \( u \in P(E) \) such that \( \Phi(u) = d \geq \alpha \). The critical value \( d \) has the following minimax characterization
\[
d = \inf_{\gamma \in \Gamma_{\Phi}} \max_{t \in [0,1]} \Phi(\gamma(t)),
\] (9)
where
\[
\Gamma_{\Phi} = \{ \gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) \neq 0, \Phi(\gamma(1)) < 0 \}.
\]

Notice that [15, 24] contain “no \( P \)” version of this theorem. Under the Palais-Smale condition, Theorem 4.1 is obtained in [2], but the arguments used in that paper apply in the case of Cerami condition as well.

If the functional does not satisfy the Cerami condition, we shall still consider Mountain Pass values defined by (9). In general, such a value is not necessarily a critical value.

Lemma 4.1. There exist constants \( \alpha > 0 \) and \( \rho > 0 \) independent of \( T \) such that
\[
(a) J_T(v) \geq \alpha \text{ if } \|v\|_T = \rho \text{ and each nonzero critical point } u \text{ of } J_T \text{ satisfy } \|u\|_T \geq \rho \text{ whenever } T \geq M;
\]
\[
(b) J(v) \geq \alpha \text{ if } \|v\| = \rho \text{ and each nonzero critical point } u \text{ of } J \text{ satisfy } \|u\| \geq \rho.
\]
Proof. We sketch the standard argument just to highlight the independence of \( \alpha \) and \( \rho \) on \( T \). Notice that Assumption (A2) implies that all interaction potentials \( U_m \) are subquadratic at 0. Making use of Lemma 3.1(a), we see that given \( \varepsilon > 0 \), there exists \( \rho > 0 \) independent of \( T \geq M \) and such that \( \Psi_T(v) \leq \varepsilon \|v\|^2_T \) for all \( v \in X_T \) with \( \|v\| \leq \rho \). The same lemma implies that

\[
q_T(v) \geq (c^2 - c_0^2)\|v\|^2_T.
\]

Hence,

\[
J_T(v) \geq \frac{c_0^2}{2} \|v\|^2_T - \varepsilon \|v\|^2_T, \quad \|v\| \leq \rho.
\]

Similarly, changing \( \rho \), we have that

\[
J_T'(v)v \geq (c^2 - c_0^2)\|v\|^2_T - \varepsilon \|v\|^2_T, \quad \|v\| \leq \rho.
\]

Since \( J_T'(v)v = 0 \) for all critical points, we obtain the required after an appropriate choice of \( \varepsilon \).

Part (b) is similar. \( \square \)

Remark 5. The proof of Lemma 4.1 shows that \( J_T'(v)v > 0 \) if \( 0 \neq v \in X_T \) and \( \|v\|_T \leq \rho \). Similar conclusion holds for \( J'(v)v \) as well.

Now we define the mapping \( P \) by

\[
P(v)(s) = \int_0^s |v'(\tau)|d\tau.
\]

It is easily seen that \( P : X_T \rightarrow X_T \) is a continuous mapping. We have that

\[
D_m^TP(v)(t) = \int_t^{t+m} |v'('d\tau| \geq |\int_t^{t+m} v'(\tau)|d\tau = |D_m^+v(t)| \geq D_m^+v(t).
\]

Since the interaction potentials \( U_m(r) \) are supposed to be even and increasing as \( |r| \) increases, we obtain that \( J_T(P(v)) \leq J_T(v) \). Furthermore, if \( 0 \neq e \in X_T^+ \), then \( D_m^+ > 0 \) on certain interval and, by (A3), \( J_T(te) < 0 \) for large enough \( t > 0 \). Hence, \( J_T \) possesses the Mountain Pass geometry.

The operator \( P \) acts continuously in the space \( X, J(P(v)) \leq J(v) \) for all \( v \in X \), and \( J \) possesses Mountain Pass geometry as well. As we will see below, \( J_T \) satisfies the Cerami condition (C), while (C) does not hold for \( J \).

Notice that \( P(X_T) = X_T^+ \) and \( P(X) = X^+ \) are closed.

It is easy that, defining Mountain Pass values \( d_T \) and \( d \) for the functionals \( J_T \) and \( J \), we may assume that \( \Gamma_{J_T} \) and \( \Gamma_J \) consist of paths with values in \( X_T^+ \) and \( X^+ \), respectively.

Mountain Pass values \( d_T \) and \( d \) have another minimax characterization. We introduce partial Nehari manifolds

\[
\mathcal{N}_T^+ = \{ v \in X_T^+ : v \neq 0, J_T'(v)v = 0 \}
\]

and

\[
\mathcal{N}^+ = \{ v \in X^+ : v \neq 0, J'(v)v = 0 \}.
\]

Notice that, in general, these sets are not smooth manifolds, but we follow the traditional terminology.

**Proposition 4.1.** The following identities hold true:

\[
d_T = \inf_{v \in \mathcal{N}_T^+} J_T(tv) = \inf_{v \in X_T^+, v \neq 0} \max_{t > 0} J_T(tv)
\]

(11)
if $T \geq M$, and
\[ d = \inf_{u \in N^+} J(tv) = \inf_{v \in X^+, v \neq 0} \max_{t > 0} J(tv). \] (12)

Proof. Let $v \in X^+$. Since $D_m^+ v \neq 0$ for all $m = 1, 2, \ldots, M$, then, by (A3), the function $\varphi(t) = J(tv)$, $t > 0$, attains its maximum value, while Assumption (A4) implies that
\[ \varphi'(t) = t(q(v) - t^{-1} \Psi'(tv)v) = t^2 J'(tv)(tv) \]
has the only positive zero $t_0$, and $tv \in N$. This yields the second equality in (12).

Denote by $d^*$ the common value of the two right-hand sides in (12). If $v \in X^+$, then a part of the row $\{tv, t > 0\}$, after an appropriate rescaling, is a member of $\Gamma_J$. Hence, $d^* \geq d$. On the other hand, let $\gamma \in \Gamma_J$. By Remark 5, in a neighborhood of 0 there exists $t > 0$ such that $J'(\gamma(t)) \gamma(t) > 0$. Since $J(\gamma(1)) < 0$, making use of Remark 1 we obtain that
\[ J'(\gamma(1)) \gamma(1) = q(\gamma(1)) - \Psi'((1)) \gamma(1) \leq q(\gamma(1)) - 2\Psi(\gamma(1)) = 2J(\gamma(1)) < 0. \]
Therefore, $\gamma(t_0) \in N^+$ for some $t_0 \in (0, 1)$. Since
\[ d^* \leq J(\gamma(t_0)) \leq \max_{t \in [0, 1]} J(\gamma(t)), \]
and $\gamma \in \Gamma_J$ is an arbitrary path, we get $d^* \leq d$.

The proof of (12) is similar. \qed

Remark 6. The Nehari manifold approach developed in [33] does not apply in our situation because the whole of the Nehari manifold for $J_T$ is not homeomorphic to the unit sphere in $X_T$ if $T$ is an integer.

5. Periodic waves. In this section we prove part (a) of Theorem 2.1 making use of Theorem 4.1.

In the proof of the next lemma, as well as in the proof of Lemma 6.1, we employ a normalization trick that goes back to [19] and is now used in many paper (see, e.g., [21, 23, 33]).

Lemma 5.1. Let $T \geq M$ and $m_0 = 1$ if $T$ is an integer. Then the functional $J_T$ satisfies condition (C).

Proof. Let $u_n \in X_T$ be a Cerami sequence for the functional $J_T$ at a level $l$. First, we prove that the sequence $u_n$ is bounded. Assuming the contrary and passing to a subsequence, we have that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$. Then $\|v_n\| = 1$, and, passing to a subsequence again, we may assume that $v_n \to v$ weakly in $X_T$. Lemma 3.1 and the compactness of embedding
\[ H^1(I_T) \subset C(I_T), \]
imply that
\[ D_m^+ v_n \to D_m^+ v_0, \quad m = 1, 2, \ldots, M, \]
uniformly on $[-T/2, T/2]$. We consider two cases.

Case 1. Assume that $D_m^+ v_0 \neq 0$ for some $m = 1, 2, \ldots, M$, and prove that this is impossible. If $T$ is noninteger, then $D_m^+ v_0 \neq 0$ for all $m = 1, 2, \ldots, M$, otherwise $D_1^+ v_0 \neq 0$ by Remark 2. Anyway, $D_m^+ v_0 \neq 0$, where $m_0$ is introduced in Assumption (A3). Then there exists an interval $I_0 \subset I_T$ and $\varepsilon_0 > 0$ such that $\|D_m^+ v_0\| \geq \varepsilon_0$ on $I_n$ for all $n$ large enough. Hence,
\[ |D_m^+ u_n| \geq \varepsilon_0 \|u_n\| \to \infty. \]
on $I_0$. Since $J_T(u_n) \to l$, we have that
\[
\frac{q_T(u_n)}{2} - (l + o(1)) = \Psi_T(u_n).
\]
Observing that $q_T(u) \leq c^2 \|u\|^2$ and making use of Assumption $(A_3)$, we obtain
\[
\frac{c^2}{2} \frac{l + o(1)}{\|u_n\|^2} \geq \int_{\alpha}^{\beta} \frac{V_m(D_m^+u_n)}{|D_m^+u_n|} |D_m^+u_n|^2 ds \to \infty.
\]
This is a contradiction.

Case 2. Now we rule out the case $D_m^+v_0 = 0$ for all $m = 1, 2, \ldots, M$. With this aim we choose $r_n \in [0, 1]$ such that
\[
J_T(r_n u_n) = \max_{r \in [0, 1]} J_T(r u_n).
\]
Let $w_n = kv_n$, where $k > 0$. For all $n$ large enough, $0 < k/\|u_n\| < 1$ and, hence,
\[
J_T(r_n u_n) \geq J_T(w_n) = \frac{(c^2 - c_0^2)k}{2} - \Psi_T(w_n).
\]
Notice that $D_m^+v_n \to 0$, uniformly on $I_T$ for all $m = 1, 2, \ldots, M$ and, hence, $\Psi_T(w_n) \to 0$. Since $k > 0$ is an arbitrary number, we obtain that $J_T(r_n u_n) \to \infty$. Observe that $0 < r_n < 1$ for sufficiently large $n$ because $J_T(0) = 0$ and $J_T(u_n) \to l$. As consequence, $r_n$ is an interior maximum point of the function $J_T(r u_n)$ on $[0, 1]$ and, therefore,
\[
J_T'(r_n u_n)(r_n u_n) = r_n(J_T'(r u_n))(r_n) = 0.
\]
As consequence,
\[
J_T(r_n u_n) = \frac{1}{2} \Psi_T(r_n u_n) (r_n u_n) - \Psi_T(r_n u_n).
\]
By Remark 1, we obtain that
\[
\frac{1}{2} \Psi_T(u_n) u_n - \Psi_T(u_n) \geq \frac{1}{2} \Psi_T(r_n u_n) (r_n u_n) - \Psi_T(r_n u_n) = J_T(r_n u_n) \to \infty.
\]
On the other hand,
\[
\lim_{n \to \infty} \left( \frac{1}{2} \Psi_T(u_n) u_n - \Psi_T(u_n) \right) = \lim_{n \to \infty} (J_T(u_n) - \frac{1}{2} J_T'(u_n) u_n) = l,
\]
and we obtain a contradiction.

Thus, $u_n$ is bounded, and then $u_n \to u_0$ weakly in $X_T$ along a subsequence. Hence, $D_m^+u_n \to D_m^+u_0$ weakly in $H^1(I_T)$ and, by the compactness of embedding, uniformly on $I_T$ for all $m = 1, 2, \ldots, M$. A straightforward calculation shows that
\[
q_T(u_n - u_0) = (J_T'(u_n) - J_T'(u_0))(u_n - u_0)
\]
\[+ \sum_{m=1}^{M} \int_{I_T} (V_m(D^+_m u_n) - V_m(D^+_m u_0))(D^+_m u_n - D^+_m u_0) ds.
\]
The first term in the right-hand side converges to 0 because $u_n$ is a Cerami sequence and $u_n \to u_0$ weakly in $X_T$. The second term converges to 0 due to uniform convergence $D_m^+u_n \to D_m^+u_0$, $m = 1, 2, \ldots, M$. Therefore, $q_T(u_n - u_0) \to 0$. Since the quadratic form $q_T$ is positive definite, $u_n \to u_0$ strongly in $X_T$.

Part (a) of Theorem 2.1 is a consequence of the following proposition.
Proposition 5.1. There exist $T_c \geq 2M$ and $C > 0$ independent of $T$ such that for all noninteger $T \geq T_c$ the functional $J_T$ possesses a nontrivial Mountain Pass critical point $w_T \in X_T^+$ such that $J_T(w_T) \leq C$. The same holds for integer $T \geq T_c$ if $m_0 = 1$ in Assumption (A3).

Proof. Due to Lemmas 4.1 and 5.1, to apply Theorem 4.1 we need an element of $X_T$, with large enough norm, on which $J_T$ is negative. Let $T \geq 2M$. Choose a smooth function $\varphi \geq 0$ on $\mathbb{R}$ such that $\text{supp} \varphi = [-M, 0]$. Let $\varphi_T$, $T \geq 2M$, be a unique $T$-periodic function such that $\varphi_T = \varphi$ on $I_T$ and let

$$e_T(s) = \int_0^s \varphi_T(\tau)d\tau \in X_T.$$  

It is easily seen that $J_T(te_T)$, $t \geq 0$, is independent of $T \geq 2M$ and, by Assumption (A3), is negative for sufficiently large $t > 0$.

By Theorem 4.1 with $P$ defined in (10), the functional $J_T$, $T \geq 2M$, possesses a Mountain Pass critical point $w_T \in X_T^+$ with critical value $d_T = J_T(w_T)$. Obviously,

$$d_T \leq C = \max_{t>0} J_{2M}(te_{2M}).$$

By Remark 4, the only trivial non-zero critical point in $X_T$ is the function $u(s) = k_+ s$. The value of $J_T$ at this point is $\text{const} \cdot T$ and, hence, can not be the Mountain Pass critical value $d_T$ if $T$ is large. \hfill $\square$

6. Infinite wave length limit. To obtain solutions in $X$ we pass to the limit as $T \to \infty$. With this aim first we need a uniform bound for the norms $\|u_T\|_{L^p}$, where $u_T \in X_T^+$ stands for nontrivial Mountain Pass critical points.

Lemma 6.1. If $T_n \to \infty$, $u_n \in N_{T_n}^+$ and the sequence $J_{T_n}(u_n)$ is bounded, then the sequence $\|u_n\|_{L^p}$ is bounded.

Proof. Assume the contrary. Then, passing to a subsequence, we may assume that $\|u_{T_n}\|_{L^p} \to \infty$ and $J_{T_n}(u_n) \to I$. Notice that $l \geq \rho > 0$ by Lemma 4.1. Let $v_n = u_n/\|u_n\|_{T_n}$. Making use of suitable modified translations $S_n$, we may assume that $|D_{m_0}^{+}v_0|$ attains its maximum value at 0. Since $\|v_n\|_{T_n} = 1$, also we may assume that $u_n \to v_0$ weakly in $H_{loc}^1(\mathbb{R})$ and uniformly on compact intervals. It is easily seen that $v_0 \in X$ and $\|v_0\|_{X} \leq 1$.

Consider two cases.

Case 1. Suppose that $v_0 \neq 0$. Then, by Remark 2, $D_{m_0}^{+}v_0 \neq 0$ and the maximum value of $|D_{m_0}^{+}v_0|$ is attained at 0. As consequence, there is an interval $I_0$ centered at 0 on which $|D_{m_0}^{+}v_0| \geq \varepsilon_0$ for some $\varepsilon_0 > 0$. Now the same reasoning as in Case 1 of the proof of Lemma 5.1 leads to a contradiction.

Case 2. Suppose that $v_0 = 0$ and, hence, $D_{m}^{+}v_0 = 0$ for all $m = 1, 2, \ldots, M$. Straightforwardly adapting the arguments used in Case 2 of the proof of Lemma 5.1, we obtain a contradiction in this case as well. \hfill $\square$

In the proof of the next lemma we use the following elementary inequality. If $f \in L^\infty(T) \cap L^2(I)$, then $f \in L^p(I)$, $p > 2$, and

$$\|f\|_{L^p} \leq \|f\|_{L^\infty}^{1-p} \|f\|_{L^2}^p. \tag{13}$$

Here $I \subset \mathbb{R}$ is an interval (not necessarily finite).
Lemma 6.2. Let \( T_n \to \infty \). Assume that either \( T_n \) is a noninteger sequence, or it is integer and \( m_0 = 1 \). Then for every sufficiently large \( T_n \) there exist a nontrivial Mountain Pass critical point \( u_n \in X^+_T \) for \( J_{T_n} \), and a Mountain Pass critical point \( u \in X^+ \) such that, along a subsequence, \( u_n \to u \) uniformly on compact intervals together with first and second derivatives.

Proof. Proposition 5.1 yields the existence of nontrivial Mountain Pass critical points \( u_n \in X_{T_n} \) for all sufficiently large \( T_n \) and the sequence \( d_{T_n} = J_{T_n}(u_n) \) is bounded. Due to the invariance of \( J_T \) with respect to translations \( S_n \), we may assume that \( D^+_m u_n \geq 0 \) attains its (positive) maximum value at \( 0 \). By Lemmas 4.1 and 6.1, the norms \( |u_n|_{T_n} \) are bounded below and above by positive constants independent of \( n \) and, therefore, \( u_n \to u \) weakly in \( H_{loc}^1(\mathbb{R}) \) along a subsequence. Obviously, \( u \) is nondecreasing. The uniform boundedness of \( |u_n|_{T_n} \) implies that \( u \in X^+ \).

By the compactness of Sobolev embedding, \( D^+_m u_n \to D^+_m u \) for all \( m = 1, 2, \ldots, M \), uniformly on compact intervals. Due to the standard approximation argument, to show that \( u \) is a critical point of \( J \) it is enough to verify that \( J'(u)h = 0 \) for all \( h \in X \) with compactly supported \( h' \). For any such \( h \), all \( D^+_m h \) are compactly supported as well. Let

\[
A = \text{supp} h \bigcup \left( \bigcup_{m=1}^M \text{supp} D^+_m h \right).
\]

Then \( A \subset I_{T_n} \) for all sufficiently large \( k \), and there is a unique \( h_n \in X_{T_n} \) such that \( h_n = h \) on \( A \). Then

\[
0 = J_{T_n}(u_n)h_n = \int_A \left( c^2 u_n' h' - \sum_{m=1}^M U_m(D^+_m u_n) D^+_m h \right) ds.
\]

Since \( u_n' \to u' \) weakly in \( L^2_{loc}(\mathbb{R}) \) and \( D^+_m u_n \to D^+_m u \) uniformly on compact intervals, we see that \( J(u)h = 0 \).

Finally, we show that \( u \) is a nontrivial critical point of \( J \). Notice that in this case nontrivial means nonzero. Assume the contrary. Then \( D^+_m u = 0 \) and, therefore, \( D^+_m u_n \to 0 \) uniformly on \( \mathbb{R} \) for all \( m = 1, 2, \ldots, M \). Let \( p > 2 \). Since the norms \( \|D^+_m u_n\|_{L^2(I_{T_n})}, \ m = 1, 2, \ldots, M \), are bounded by a constant independent of \( n \), inequality (13) implies that

\[
\|D^+_m u_n\|_{L^p(I_{T_n})} \to 0, \quad m = 1, 2, \ldots, M.
\]

Let \( R \geq \|D^+_m u_n\|_{L^\infty(\mathbb{R})} \) for all \( n \) and \( m = 1, 2, \ldots, M \). By Assumption (A2), for every \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that

\[
|V_m'(r)| \leq \varepsilon |r| + C_\varepsilon |r|^{p-1}, \quad m = 1, 2, \ldots, M,
\]

for all \( |r| \leq R \). As consequence, renaming \( \varepsilon \) and \( C_\varepsilon \), and making use of the boundedness of \( \|D^+_m u_n\|_{L^2(I_{T_n})} \), we obtain that

\[
0 < \Psi_{T_n}(u_n) u_n \leq \varepsilon + C_\varepsilon \sum_{m=1}^M \|D^+_m u_n\|_{L^p(I_{T_n})}.
\]

Since \( \varepsilon > 0 \) is arbitrary and all \( L^p \)-norms in the right-hand side tend to 0, we have that \( \Psi_{T_n}'(u_n) u_n \to 0 \). Then equation \( J_{T_n}'(u_n) u_n = 0 \) implies that

\[
(c^2 - c_0^2) |u_n|_{T_n}^2 \leq q_{T_n}(u_n) = \Psi_{T_n}'(u_n) u_n \to 0.
\]
This contradicts the fact that the norms $\|u_n\|^2_{T_n}$ are bounded below by a positive constants. Thus, $u \neq 0$ and the proof is complete.

The next lemma finalize the proof of Theorem 2.1(b).

**Lemma 6.3.** Let $T_n \to \infty$ be a sequence as in Lemma 6.2. Assume that $u_n \in X^+_{T_n}$ is a sequence of nontrivial Mountain Pass critical points for $J_{T_n}$ such that $u_n \rightharpoonup u$ weakly in $H^1_{\text{loc}}(\mathbb{R})$ and, hence, uniformly on compact intervals, where $u \in X^+$ is a nontrivial critical point of $J$. Then $u$ is a Mountain Pass critical point and

$$d_{T_n} = J_{T_n}(u_n) \to d = J(u).$$

**Proof.** Let $v \in \mathcal{N}^+$. Since $D^+_m v \in H^1(\mathbb{R})$, it vanishes at infinity, and we can assume that its maximum value, say, $r_0 > 0$ is attained at 0. Choose $v_n \in X^+$ such that $v'_n$ is supported on $[-T_n/2 + M, T_n/2 - M]$ and $v_n \to v$ in $X$. Then $D^+_m v_n \to D^+_m v$, $m = 1, 2, \ldots, M$, uniformly on compact intervals. Let $w_n \in X^+_{T_n}$ be a unique function such that $w_n = v_n$ on $T_n$. Obviously, $J_{T_n}(w_n) = J(v_n) \to J(v)$. Let $\tau_n > 0$ be a unique number such that $\tau_n v_n \in \mathcal{N}^+_{T_n}$. In fact, we have also that $\tau_n v_n \in \mathcal{N}^+$.

Let us prove that $\tau_n \to 1$. For, it is enough to show that the sequence $\tau_n$ is bounded. Assuming the contrary, we have that $\tau_n \to \infty$ and $J(\tau_n v_n) = J_{T_n}(\tau_n w_n) > 0$. On the other hand, $D^+_m v_n \geq r_0/2$ for all $n$ large enough. Due to Assumption (A3), this implies that $\tau_n^{-2} \Psi(\tau_n v_n) \to \infty$ and

$$J(\tau_n v_n) = \tau_n^2 \left( \frac{q(v_n^2)}{2} - \tau_n^{-2} \Psi(\tau_n v_n) \right) < 0$$

for all sufficiently large $n$, a contradiction.

Thus, $\tau_n \to 1$, $\tau_n v_n \to v$ and, therefore, $J_{T_n}(\tau_n w_n) \to J(v)$. Since $d_{T_n} \leq J_{T_n}(\tau_n w_n)$ and $v \in \mathcal{N}^+$ is arbitrary, we obtain that

$$\limsup d_{T_n} \leq d.$$ 

Observe that

$$d_{T_n} = \sum_{m=1}^{M} \int_{I} G_m(D^+_m u_n) ds,$$

where the functions $G_m$ are introduced in Remark 1. Let $I \subset \mathbb{R}$ be a compact interval. Since all $G_m$ are nonnegative,

$$d_{T_n} \geq \sum_{m=1}^{M} \int_{I} G_m(D^+_m u_n) ds$$

for all $n$ large enough. Uniform convergence $D^+_m u_n \to D^+_m u$ on compact intervals yields

$$\liminf d_{T_n} \geq \sum_{m=1}^{M} \int_{I} G_m(D^+_m u) ds.$$

As $I$ is arbitrary, $u$ is a critical point of $J$ and $G_m \geq 0$, we obtain that

$$\liminf d_{T_n} \geq \sum_{m=1}^{M} \int_{\mathbb{R}} G_m(D^+_m u) ds = J(u) \geq d.$$ 

Therefore, $\lim d_{T_n} = d = J(u)$. 

$\square$
7. Additional results and examples. First we show that, under certain additional assumptions, the convergence in Theorem 2.1(b) can be made nonlocal in some sense.

**Theorem 7.1.** In addition to assumptions of Theorem 2.1, suppose that potentials $V_m$ are $C^2$ functions and there exists $p > 2$ such that

$$0 \leq pV_m(r) \leq rV_m'(r)$$

for all $m = 1, 2, \ldots, M$. Then, along a subsequence, the solutions obtained in Theorem 2.1 satisfy

$$\|u_n' - u'\|_{L^2(I_n)} \to 0.$$  (15)

**Proof.** Let $v_n \in X^+$ be a function such that $v_n \to u$ in $X$ and $v_n'$ is supported on the interval $[-T_n/2 + M, T_n/2 - M]$. Obviously, $D^+_m v_n \to D^+_m u$, $m = 1, 2, \ldots, M$, uniformly on compact intervals and in $L^2(\mathbb{R})$. If $w_n \in X^+_T$ is a unique function such that $w_n = v_n$ on $I_{T_n}$, then it is easily seen that

$$\|u' - w_n'\|_{L^2(I_{T_n})} \to 0$$

and

$$J_{T_n}(w_n) \to d = J(u).$$

It is enough to prove that

$$J_{T_n}(u_n - w_n) \to 0$$  (16)

and

$$\|J'_{T_n}(u_n - w_n)\| \to 0.$$  (17)

Indeed, if these relations hold true, then, by (14),

$$J_{T_n}(u_n - w_n) - \frac{1}{p} J'_{T_n}(u_n - w_n)(u_n - w_n) \geq \left(\frac{1}{2} - \frac{1}{p}\right) q_{T_n}(u_n - w_n).$$

Since $\|u_n - w_n\|_{L^2(I_{T_n})}$ is bounded, we obtain that $q_{T_n}(u_n - w_n) \to 0$, and the result follows.

To prove (16) we use the following elementary identity

$$J_{T_n}(u_n - w_n) = J_{T_n}(u_n) - J_{T_n}(w_n) + \int_{I_{T_n}} c^2(w_n' - u_n') w_n' ds$$

$$- \int_{I_{T_n}} \sum_{m=1}^M \left[U_m(D^+_m(u_n - w_n)) - U_m(D^+_m u_n) + U_m(D^+_m w_n)\right] ds.$$  (18)

The difference of $J_{T_n}$’s in the right-hand side is $o(1)$.

Let $I$ be any finite interval and $n$ is large enough so that $I \subset I_{T_n}$. We split the second integral in (18) into the sum of integrals over $I$ and $I_{T_n} \setminus I$. Since $u_n - w_n \to 0$ in $H^1_{\text{loc}}(\mathbb{R})$ and $\|w_n\|_{T_n}$ is bounded, while $D^+_m(u_n - w_n) \to 0$ uniformly on compact intervals and $D^+_m w_n$ is bounded in $L^\infty(\mathbb{R})$ for every $m = 1, 2, \ldots, M$, the integral over $I$ tends to 0. As the norm $\|u_n - w_n\|_{I_{T_n}}$ is bounded, $w_n = v_n$ on $I_{T_n}$ and $v_n \to u$ in $X$, the integral over $I_{T_n} \setminus I$ does not exceed a constant multiple of

$$\|u_n - w_n\|_{I_{T_n}} \|v_n'\|_{L^2(I_{T_n} \setminus I)} \leq C \|u\|_{L^2(\mathbb{R} \setminus I)}$$

which can be made arbitrarily small by choosing sufficiently large interval $I$. Here and thereafter $C$ stands for a generic positive constant.

Splitting the second integral in (18) similarly, we see that its part over $I$ tends to 0 because $u_n \to u$ and $w_n \to u$ uniformly on compact intervals. Now we estimate the
integral over $I_{T_n} \setminus I$. By the Mean Value Theorem, there exists $\theta_n = \theta_n^m(s) \in [0, 1]$, $m = 1, 2, \ldots , M$, such that

$$U_m(D_m^+(u_n - u_n)) = U_m(D_n^+u_n) - U_m(D_n^+(u_n - \theta_n^m w_n))D_n^+w_n.$$ 

As the sequences $D_n^+u_n$ and $D_n^+w_n$ are uniformly bounded on $\mathbb{R}$, there is $R > 0$ such that $D_n^+u_n, D_n^+w_n$ and $D_n^+(u_n - \theta_n^m w_n)$ belong to the ball in $L^\infty(\mathbb{R})$, with the radius $R$ and center 0, for all $m = 1, 2, \ldots , M$. If $|r| \leq R$, then $0 \leq V_m(r) \leq Cr^2$ and $|V_m'(r)| \leq C|r|$, with some $C > 0$, for all $m = 1, 2, \ldots , M$. Therefore,

$$\int_{I_{T_n} \setminus I} | |D_n^+w_n|^2 ds \leq C \sum_{m=1}^{M} |D_m^+w_n|^2 ds$$

$$+ \sum_{m=1}^{M} \int_{I_{T_n} \setminus I} \left( |D_m^+u_n||D_m^+w_n| + \theta_n^m |D_m^+w_n|^2 \right) ds$$

$$\leq C \sum_{m=1}^{M} \int_{I_{T_n} \setminus I} |D_m^+w_n|^2 ds$$

$$+ \sum_{m=1}^{M} \left( \int_{I_{T_n} \setminus I} |D_m^+u_n|^2 ds \right)^{1/2} \left( \int_{I_{T_n} \setminus I} |D_m^+w_n|^2 ds \right)^{1/2}.$$ 

By Lemma 3.1,

$$\int_{I_{T_n} \setminus I} |D_m^+u_n|^2 ds \leq m^2 \|u_n\|_{T_n}.$$ 

Since $w_n = v_n$ on $T_n$ and $v_n \to u$ in $X$, the integrals that involve $w_n$ can be made arbitrarily small by choosing $I$ large enough. This implies (16).

To prove (17) we utilize the identity

$$J_{T_n}'(u_n - w_n)h = J_{T_n}'(u_n)h - J_{T_n}'(w_n)h$$

$$- \sum_{m=1}^{M} \int_{I_{T_n}} \left[ U_m(D_m^+(u_n - w_n)) - U_m(D_n^+u_n) + U_m(D_n^+w_n) \right] h ds$$

for all $h \in X_{T_n}$. Notice that $J_{T_n}'(u_n) = 0$. As in the proof of (16), we have that $\|J_{T_n}'(w_n)\| \to 0$. Making use of the $C^2$ smoothness of the potentials and adapting the arguments used on the end of the proof of (16), we obtain that

$$|J_{T_n}'(u_n - w_n)h| \leq \varepsilon_n \|h\|_{T_n},$$

where $\varepsilon_n \to 0$.

The proof is complete.

In the case $M = 1$ the velocity profile of a solitary wave decays exponentially fast at infinity [26] (see also [9] for near-sonic waves). The proof given there is based on a careful analysis of certain special function in complex domain. It seems that such approach does not work in general. Nevertheless, the following conjecture makes sense.

**Conjecture.** If $V_m \in C^2$ and $V_m(0) = V_m'(0) = V_m''(0) = 0$ for all $m = 1, 2, \ldots , M$, then for any solution $u \in X$ of equation (2) the velocity profile $u'$ decays exponentially fast at infinity.
Following the same lines as in the proof of [26, Proposition 3.7] one can show that, under additional $C^2$ smoothness of the interaction potentials, any monotone nontrivial solution of (2) in the space $X$ is strictly monotone provided its velocity profile decays exponentially fast at infinity.

Finally, we consider two examples to illustrate the results obtained above.

**Example 1** (FPU $\alpha$-potentials). Let

$$V_m(r) = \frac{\alpha_m}{3} r^3, \quad m = 1, 2, \ldots, M,$$

where either 1) all $\alpha_m \geq 0$ and at least one of them is positive, or 2) all $\alpha_m \leq 0$ and at least one of them is negative. Introduce new potentials as follows.

*Case 1.* $\tilde{V}_m(r) = V_m(|r|)$ for all $m = 1, 2, \ldots, M$.

*Case 2.* $\tilde{V}_m(r) = V_m(-|r|)$ for all $m = 1, 2, \ldots, M$.

Then Theorems 2.1 and 7.1 apply in both cases. As consequence, in case 1) we obtain the existence of nondecreasing periodic and solitary traveling waves as well as the nonlocal “periodic-to-solitary” convergence. In case 2) the same holds for nonincreasing waves.

**Example 2** (FPU $\beta$-potentials). Let

$$V_m(r) = \frac{\beta_m}{3} r^4, \quad m = 1, 2, \ldots, M,$$

where all $\beta_m \geq 0$ and at least one is positive. Then our results provide us the existence and nonlocal “periodic-to-solitary” convergence for both types of waves — nondecreasing and increasing.

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**REFERENCES**

[1] P. W. Bates and C. Zhang, **Traveling pulses for the Klein-Gordon equation on a lattice or continuum with long-range interaction**, Discrete Contin. Dyn. Syst. A, 16 (2006), 235–252.

[2] H. Berestycki, J. Capuzzo-Dolcetta and L. Nirenberg, Variational methods for indefinite superlinear homogeneous elliptic problems, Nonlin. Differ. Equat. Appl., 2 (1995), 553–572.

[3] E. Dumas and D. Pelinovsky, Justification of the log-KdV equation in granular chains: The case of precompression, SIAM J. Math. Anal., 46 (2014), 4075–4103.

[4] T. E. Faver and J. D. Wright, Exact diatomic Fermi-Pasta-Ulam-Tsingou solitary waves with optical band ripples at infinity, SIAM J Math. Anal., 50 (2018), 182–250.

[5] M. Fečkan and V. Rothos, Traveling waves of discrete nonlinear Schrödinger equations with optical band ripples at infinity, SIAM J Math. Anal., 89 (2010), 1387–1411.

[6] E. Fermi, J. Pasta and S. Ulam, Studies of nonlinear problems, Los Alamos Sci. Lab. Rept., LA-1940 (1955).

[7] G. Friesecke and J. A. D Wattis, Existence theorem for solitary waves on lattices, Commun. Math. Phys., 161 (1994), 391–418.

[8] G. Friesecke and A. Mikikis-Leitner, Cnoidal waves on Fermi-Pasta-Ulam lattices, J. Dyn. Diff. Equat., 27 (2015), 627–652.

[9] G. Friesecke and R. L. Pego, Solitary waves on FPU lattices. I. Qualitative properties, renormalization and continuum limit, Nonlinearity, 12 (1999), 1601–1627.

[10] G. Friesecke and R. L. Pego, Solitary waves on FPU lattices. II. Linear implies nonlinear stability, Nonlinearity, 15 (2002), 1343–1359.

[11] G. Friesecke and R. L. Pego, Solitary waves on FPU lattices. III. Howland type Floquet theory, Nonlinearity, 17 (2004), 207–227.

[12] G. Friesecke and R. L. Pego, Solitary waves on FPU lattices. IV. Proof of stability at low energy, Nonlinearity, 17 (2004), 229–251.
[13] J. Gaison, S. Moskow, J. D. Wright and Q. Zhang, Approximation of polyatomic FPU lattices by KdV equations, *Multiscale Model. Simul.*, 12 (2014), 953–995.
[14] G. Galavotti, *The Fermi-Pasta-Ulam Problem, A Status Report*, Springer, Berlin, 2008.
[15] L. Gasinski and N. Papageorgiou, *Nonlinear Analysis*, Chapman & Hall/CRC, Boca Raton, 2006.
[16] M. Herrmann, Unimodal wavetrains and solitons in convex Fermi-Pasta-Ulam chains, *Proc. Roy. Soc. Edinburgh A*, 140 (2010), 753–785.
[17] M. Herrmann and A. Mikikis-Leitner, KdV waves in atomic chains with nonlocal interaction, *Discrete Contin. Dyn. Syst. A*, 36 (2016), 2047–2067.
[18] G. Iooss, Traveling waves in the Fermi-Pasta-Ulam lattice, *Nonlinearity*, 13 (2000), 849–866.
[19] L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh*, 129 (1999), 787–809.
[20] A. Khan and D. Pelinovsky, Long-time stability of small FPU solitary waves, *Discrete Contin. Dyn. Syst. A*, 37 (2017), 2065–2075.
[21] Y. Li, Z.-Q. Wang and J. Zheng, Ground states of nonlinear Schrödinger equations with potentials, *Ann. Inst. H. Poincaré, Anal. Non Lin.*., 23 (2006), 829–837.
[22] P. L. Lions, The concentration-compactness method in the calculus of variations. The locally compact case. II, *Ann. Inst. H. Poincaré, Anal. Non Lin.*., 1 (1984), 223–283.
[23] S. Liu, On superlinear problems without the Ambrosetti-Rabinowitz condition, *Nonlin. Anal.*, 73 (2010), 788–795.
[24] D. Motreanu, V. Motreanu and N. Papageorgiou, *Topological and Variational Methods with Applications to Boundary Value Problems*, Springer, New York, 2014.
[25] A. Pankov, Periodic nonlinear Schrödinger equation with applications to photonic crystals, *Milan J. Math.*, 73 (2005), 259–287.
[26] A. Pankov, *Traveling Waves and Periodic Oscillations in Fermi-Pasta-Ulam Lattices*, Imperial College Press, London, 2005.
[27] A. Pankov, Nonlinear Schrödinger equations on periodic metric graphs, *Discrete Contin. Dyn. Syst. A*, 38 (2018), 697–714.
[28] A. Pankov and K. Pflüger, Traveling waves in lattice dynamical systems, *Math. Meth. Appl. Sci.*, 23 (2000), 1223–1235.
[29] A. Pankov and V. Rothos, Traveling waves in Fermi-Pasta-Ulam lattices with saturable nonlinearities, *Discrete Contin. Dyn. Syst. A*, 30 (2011), 835–849.
[30] Z. Rapti, Multibreather stability in discrete Klein-Gordon equations: beyond nearest neighbor interaction, *Phys. Lett. A*, 377 (2013), 1543–1553.
[31] H. Schwetlick and J. Zimmer, Solitary waves for nonconvex FPU lattices, *J. Nonlin. Sci.*, 17 (2007), 1–12.
[32] D. Smets and M. Willem, Solitary waves with prescribed speed on infinite lattices, *J. Funct. Anal.*, 149 (1997), 266–275.
[33] A. Szulkin and T. Weth, The method of Nehari manifold, *Handbook of Nonconvex Analysis and Applications*, Int. Press, Somerville, MA, 2010, 597–632.
[34] J. A. D. Wattis, Approximations to solitary waves on lattices: III. The monoatomic lattice with second-neighbour interaction, *J. Phys. A: Math. Gen.*, 29 (1996), 8139–8157.

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