The inflationary Hubble parameter from the gravitational wave spectrum in the general slow-roll approximation

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Abstract. Improved general slow-roll formulae giving the primordial gravitational wave spectrum are derived in the present work. Also the first-and second-order general slow-roll inverse formulae giving the Hubble parameter $H$ in terms of the gravitational wave spectrum are derived. Moreover, the general slow-roll consistency condition relating the scalar and tensor spectra is obtained.

Keywords: cosmological perturbation theory, gravity waves/theory
1. Introduction

Inflation inevitably leads to scalar curvature perturbations and gravitational waves caused by the tensor perturbations of the spatial metric [1]. The relative contributions of scalar and tensor fluctuations to the cosmic microwave background (CMB) anisotropy depend upon the details of the inflationary potential. Also, the scalar and tensor fluctuations generate different patterns of polarization and contribute independently to the observed value [2] of the angular power spectrum, $C_l$. The exploration of gravitational waves might be possible on the basis of detection of $B$-mode polarization in the CMB anisotropy [3] and it is hoped to make more progress in that direction once we get the more precise data from Planck [4] and Big Bang Observer [5].

The standard slow-roll approximations used for inflationary scenarios make some strong assumptions about the properties of inflation, which have not yet been fully confirmed observationally. Hence, a more general slow-roll approximation has been put forward [6] which lifts the extra, unjustified, assumptions of the standard slow-roll approximation. The advantages of the general slow-roll approximation, compared with the standard approximation, are clearly discussed in [7].

The reconstructions of the inflationary potential from the observed scalar density perturbation spectrum [8] and also from the gravitational wave spectrum have been discussed by many authors [9,10]. The nearly scale-invariant spectrum, $P_\Psi$, of the gravitational wave traces the evolution of the Hubble parameter during inflation and it has been shown that one can reconstruct the time dependence of the very early Hubble parameter and matter energy density from the relic gravitational wave spectrum [11]. Recently, we proposed a general inverse formula [12] for extracting inflationary parameters from the scalar power spectrum. There, we inverted the single-field, general slow-roll
formula for the curvature perturbation spectrum to obtain a formula for inflationary parameters. Here, our inverse formalism is applied to the gravitational wave spectrum so as to estimate the Hubble parameter $H$.

2. First-order general slow-roll formulae

2.1. Gravitational wave spectrum

The formalism of the general slow-roll approximation and the power spectrum calculation are clearly described in [6]. With this general formalism, the first-order gravitational wave spectrum can be given by [13]

$$\ln \mathcal{P}_\psi (\ln k) = \int_0^\infty \frac{d\xi}{\xi} \left[ -k\xi W'(k\xi) \right] \ln \left[ \frac{1}{p^2} + \frac{2p'}{3p} \right],$$

(1)

where $p = 2\pi a\xi$ and $\xi = -\int dt/a = (1/aH)(1 - \dot{H}/H^2 + \cdots)$ is minus the conformal time. The window function $-x W'(x)$ is given by

$$W(x) = \frac{3\sin(2x)}{2x^3} - \frac{3\cos(2x)}{x^2} - \frac{3\sin(2x)}{2x} - 1.$$  

(2)

It has the asymptotic behaviour

$$\lim_{x\to 0} W(x) = \frac{2}{5}x^2 + \mathcal{O}(x^4)$$

(3)

and the window property

$$\int_0^\infty \frac{dx}{x} [-x W'(x)] = 1.$$  

(4)

An alternative form for the gravitational wave spectrum with a particularly simple window function is

$$\ln \mathcal{P}_\psi (\ln k) = \int_0^\infty \frac{d\xi}{\xi} \left[ -k\xi v'(k\xi) \right] \ln \left[ \frac{1}{p^2} - \frac{2p'}{p} \right],$$

(5)

where

$$v(x) = \frac{\sin(2x)}{2x} - 1.$$  

(6)

$v(x)$ has the asymptotic behaviour

$$\lim_{x\to 0} v(x) = -\frac{2}{3}x^2 + \mathcal{O}(x^4)$$

(7)

and the window property

$$\int_0^\infty \frac{dx}{x} [-x v'(x)] = 1.$$  

(8)

Figures 1 and 2 give the window functions as a function of $-\ln(k\xi)$. Large $k\xi$ corresponds to earlier times, when the mode of interest is within the horizon and oscillates rapidly. The window function starts to vanish for $k\xi < 1$ once the mode leaves the horizon because it freezes out.
The inflationary Hubble parameter

Figure 1. The window function $-k\xi v'(k\xi)$ as a function of $-\ln(k\xi)$.

Figure 2. $v(k\xi)$ as a function of $-\ln(k\xi)$.

Here, $p$ is taken as a function of $\ln \xi$ so that

$$p' \equiv \frac{dp}{d\ln \xi} = \xi \frac{dp}{d\xi} = -\frac{p}{2\pi} \frac{dp}{dt}$$

(9)

and since $d\xi/dt = -1/a$ and $H = \dot{a}/a$ we get

$$\frac{p'}{p} = 1 - \left(\frac{H}{2\pi}\right) p.$$  

(10)

Therefore,

$$\ln \left(\frac{H}{2\pi}\right)^2 = \ln \left(\frac{1}{p''}\right) - 2 \left(\frac{p'}{p}\right) - \left(\frac{p'}{p}\right)^2 - \cdots.$$  

(11)

Substituting the above form in equation (5) we get the general slow-roll formula for the gravitational wave spectrum in terms of the Hubble parameter $H$, up to the first-order correction terms, as

$$\ln P_\psi(\ln k) = \int_0^\infty \frac{d\xi}{\xi} \left[-k\xi v'(k\xi)\right] \ln \left(\frac{H}{2\pi}\right)^2.$$  

(12)

Also note that $\xi = 1/aH$ up to this order.
2.2. Inverse

Using the inverse identity
\[
\int_0^\infty \frac{dk}{k} m(k\zeta) v(k\xi) = \frac{1}{2\zeta} \left[ (\zeta - \xi) \text{sgn}(\zeta - \xi) + (\zeta + \xi) \text{sgn}(\zeta + \xi) \right] - \frac{1}{\zeta} \left[ \zeta \text{sgn}(\zeta) + \xi \text{sgn}(\xi) \right]
\] \quad (13)
and its derivative with respect to \(\xi\),
\[
\int_0^\infty \frac{dk}{k} m(k\zeta) [-k\xi v'(k\xi)] = \frac{\xi}{2\zeta} \left[ \text{sgn}(\zeta - \xi) - \text{sgn}(\zeta + \xi) \right] + \frac{\xi}{\zeta} \text{sgn}(\xi), \quad (14)
\]
where \(\text{sgn}(x) = -1\) for \(x < 0\) and \(\text{sgn}(x) = 1\) for \(x > 0\) and also where
\[
m(x) = \frac{2}{\pi} \left[ \frac{1}{x} - \frac{\cos(2x)}{x} - \sin(2x) \right]
\] \quad (15)
we get the first-order inverse expression for the gravitational wave spectrum as
\[
\ln \left( \frac{H}{2\pi} \right)^2 = \int_0^\infty \frac{dk}{k} m(k\zeta) \left[ \ln P_\psi - \frac{P'_{\psi}}{P_\psi} \right]. \quad (16)
\]
It will be interesting to note the asymptotic behaviour
\[
\lim_{x \to 0} m(x) = \frac{4}{3\pi} x^3 + \mathcal{O}(x^5) \quad (17)
\]
and the window properties
\[
\int_0^\infty \frac{dx}{x} m(x) = 1, \quad (18)
\]
\[
\int_0^\infty \frac{dx}{x} \frac{1}{x} m(x) = \frac{2}{\pi}, \quad (19)
\]
Equation (16) gives an explicit formula for \(H\) in terms of the gravitational wave spectrum \(P_\psi\).

3. Second-order general slow-roll formulae

Under the general slow-roll formalism [6], the second-order spectra for the scalar [14] and tensor perturbations [13] have been calculated. An improved second-order general slow-roll formula for the gravitational wave spectrum in terms of the Hubble parameter \(H\) is presented in this section.
\section{Gravitational wave spectrum}

The general slow-roll gravitational wave spectrum can be derived up to second-order terms as \cite{13}

\[
\ln P_\psi(\ln k) = \int_0^\infty \frac{d\xi}{\xi} \left[-k\xi W'(k\xi) \right] \left[\ln \frac{1}{p^2} + \frac{2}{3} \frac{p'}{p} \right] + \frac{\pi^2}{2} \left[\int_0^\infty \frac{d\xi}{\xi} \left(\frac{p'}{p} \right)^2 \right] 
- 2\pi \int_0^\infty \frac{d\xi}{\xi} m(k\xi) \frac{p'}{p} \int_0^\infty \frac{d\zeta}{\zeta} \frac{1}{k\xi} \frac{p'}{p}. \tag{20}\]

Using equation (10) and its derivative, we get

\[
\frac{\dot{H}}{H^2} = -\frac{\frac{p''}{p} + \frac{p'}{p} \left(1 - 2\frac{p'}{p}\right)}{\left(1 - \frac{p'}{p}\right)^2}. \tag{21}\]

Now, we can rewrite equation (20) to get the second-order general slow-roll gravitational wave spectrum in terms of $H$ as

\[
\ln P_\psi(\ln k) = \int_0^\infty \frac{d\xi}{\xi} \left[-k\xi v'(k\xi) \right] \left[\ln \left(\frac{H}{2\pi}\right) + \frac{\pi}{2} \left[\int_0^\infty \frac{d\xi}{\xi} n(k\xi) \frac{\dot{H}}{H^2}\right] \right] 
+ 2\pi \int_0^\infty \frac{d\xi}{\xi} n(k\xi) \frac{\dot{H}}{H^2} \left\{\frac{1}{k\xi} \frac{\dot{H}}{H^2} - \int_\xi^\infty \frac{d\zeta}{\zeta} \frac{1}{k\xi} \frac{\dot{H}}{H^2}\right\}, \tag{22}\]

where

\[
n(x) = \frac{1}{\pi} \left[\frac{1}{x} - \cos(2x) \right]. \tag{23}\]

$n(x)$ has the asymptotic behaviour

\[
\lim_{x \to 0} n(x) = -\frac{2}{\pi} x + O(x^3), \tag{24}\]

window property

\[
\int_0^\infty \frac{dx}{x} n(x) = 1, \tag{25}\]

and is related to $m(x)$ by

\[
m(x) = n(x) - x n'(x). \tag{26}\]

\subsection{Inverse}

Substituting the first-order inverse expression, equation (16), into equation (22) and following the same formalism as in \cite{12} we get the second-order inverse formula for the
gravitational wave spectrum as

\[
\ln \left( \frac{H}{2\pi} \right)^2 = \int_0^\infty \frac{dk}{k} m(k\xi) \left[ \ln \mathcal{P}_\psi(\ln k) - \frac{\mathcal{P}'_\psi(\ln k)}{\mathcal{P}_\psi(\ln k)} \right]
\]

\[
- \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} m(k\xi) \int_0^\infty \frac{dl}{l} \ln \left| \frac{k + l}{k - l} \right| \frac{\mathcal{P}'_\psi(\ln l)}{\mathcal{P}_\psi(\ln l)}
\]

\[
\times \int_0^\infty \frac{dq}{q} \ln \left| \frac{k + q}{k - q} \right| - \frac{2qk}{q^2 - k^2} \frac{\mathcal{P}'_\psi(\ln q)}{\mathcal{P}_\psi(\ln q)}
\]

\[
+ \int_0^\infty \frac{dl}{l} \int_0^\infty \frac{dq}{q} N(l\xi, q\xi) \frac{\mathcal{P}'_\psi(\ln l)}{\mathcal{P}_\psi(\ln l)} \frac{\mathcal{P}'_\psi(\ln q)}{\mathcal{P}_\psi(\ln q)},
\]

where \(N(x, y) = M_1(x, y) - M_2(x, y)\) with

\[
\int_0^\infty \frac{d\zeta}{\zeta} m(l\zeta) \int_0^\infty \frac{dk}{k^2} m(k\xi) m(k\zeta) \frac{\sin^2(q\zeta)}{q^2} = M_1(l\xi, q\xi)
\]

\[
\int_0^\infty \frac{d\zeta}{\zeta} m(l\zeta) \int_0^\infty \frac{dk}{k^2} \left[ (-k\xi)m'(k\xi) \right] m(k\zeta) \frac{\sin^2(q\zeta)}{q^2} = M_2(l\xi, q\xi).
\]

Carrying out the integrations in the above expression,

\[
\frac{M_j(x, y)}{j} = \frac{2}{\pi^2xy} \left[ g_j(x) + g_j(y) - \frac{1}{2} g_j(x - y) - \frac{1}{2} g_j(x + y) \right]
\]

for which the index \(j\) takes values 1 and 2.

\[
g_j(x) = x \text{Si}(2x) + \frac{j}{2} (\cos(2x) - 1),
\]

where we define

\[
\text{Si}(x) \equiv \int_0^x \frac{\sin t}{t} \, dt.
\]

\(M_j(x, y)\) has the window property

\[
\int_0^\infty \frac{dx}{x} \int_0^\infty \frac{dy}{y} \frac{M_j(x, y)}{j} = 1
\]

and the asymptotic behaviour

\[
\lim_{x,y \to 0} \frac{M_j(x, y)}{j} = \frac{4xy}{3\pi^2} \left[ 1 + \mathcal{O} \left( x^2 + y^2 \right) \right].
\]

4. General slow-roll consistency condition

The standard slow-roll consistency condition relating the scalar and tensor spectra is detailed in [9] for the single-field case and [15] discusses the same for the multi-component
scalar field inflation models. This section describes the general slow-roll generalization of the constraint on the spectra.

The inflationary Hubble parameter given by

$$\frac{1}{f^2} = \int_0^\infty \frac{dk}{k} m(k\xi) \ln \mathcal{P}_s(\ln k)$$

$$- \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} m(k\xi) \left[ \int_0^\infty \frac{dl}{l} \ln \frac{k + l - 1}{k - l} \left( \frac{\mathcal{P}_s'(\ln l)}{\mathcal{P}_s(\ln l)} \right)^2 \right]$$

$$+ \int_0^\infty \frac{dl}{l} \int_0^\infty \frac{dq}{q} M_1(l\xi, q\xi) \frac{\mathcal{P}_s'(\ln l)}{\mathcal{P}_s(\ln l)} \mathcal{P}_s'(\ln q), \quad (35)$$

where \( f = 2\pi a\dot{\phi}/H \). Also, for the tensor spectrum we have

$$\frac{1}{p^2} = \int_0^\infty \frac{dk}{k} m(k\xi) \ln \mathcal{P}_\psi(\ln k)$$

$$- \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} m(k\xi) \left[ \int_0^\infty \frac{dl}{l} \ln \frac{k + l - 1}{k - l} \left( \frac{\mathcal{P}_\psi'(\ln l)}{\mathcal{P}_\psi(\ln l)} + \frac{\mathcal{P}_s'(\ln l)}{\mathcal{P}_s(\ln l)} \right) \right]$$

$$+ \int_0^\infty \frac{dl}{l} \int_0^\infty \frac{dq}{q} M_1(l\xi, q\xi) \left\{ \frac{\mathcal{P}_\psi'(\ln l)}{\mathcal{P}_\psi(\ln l)} \mathcal{P}_\psi'(\ln q) - \frac{\mathcal{P}_s'(\ln l)}{\mathcal{P}_s(\ln l)} \mathcal{P}_s'(\ln q) \right\}. \quad (36)$$

Combining the above two inverse formulae we can write

$$\ln \left( \frac{\dot{\phi}}{H} \right)^2 \cong \int_0^\infty \frac{dk}{k} m(k\xi) \ln \left( \frac{\mathcal{P}_\psi}{\mathcal{P}_s} \right)$$

$$- \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} m(k\xi) \int_0^\infty \frac{dl}{l} \ln \left( \left( \frac{\mathcal{P}_\psi'}{\mathcal{P}_\psi} + \frac{\mathcal{P}_s'}{\mathcal{P}_s} \right) \right) \right]$$

$$\times \int_0^\infty \frac{dq}{q} \ln \left( \frac{k + q}{k - q} \right) \left( \frac{\mathcal{P}_\psi'}{\mathcal{P}_\psi} + \frac{\mathcal{P}_s'}{\mathcal{P}_s} \right)$$

$$+ \int_0^\infty \frac{dl}{l} \int_0^\infty \frac{dq}{q} \psi M_1(l\xi, q\xi) \left\{ \frac{\mathcal{P}_\psi'(\ln l)}{\mathcal{P}_\psi(\ln l)} \mathcal{P}_\psi'(\ln q) - \frac{\mathcal{P}_s'(\ln l)}{\mathcal{P}_s(\ln l)} \mathcal{P}_s'(\ln q) \right\}. \quad (37)$$

Defining

$$m_1(x) = \int \frac{dx}{x} m(x) = \frac{2}{\pi} \left[ \frac{\cos(2x)}{x} - \frac{1}{x} + \text{Si}(2x) \right] \quad (38)$$

we can rewrite our expression as

$$\left( \frac{\dot{\phi}}{H} \right)^2 \cong \left( \frac{\mathcal{P}_\psi}{\mathcal{P}_s} \right) \psi \left[ 1 - \int_0^\infty \frac{dk}{k} \left( m_1(k\xi) - \theta(k\xi - k_\xi) \right) \left( \frac{\mathcal{P}_\psi'}{\mathcal{P}_\psi} - \frac{\mathcal{P}_s'}{\mathcal{P}_s} \right) \right]$$

$$+ \frac{1}{2} \left( \int_0^\infty \frac{dk}{k} \left( m_1(k\xi) - \theta(k\xi - k_\xi) \right) \left( \frac{\mathcal{P}_\psi'}{\mathcal{P}_\psi} - \frac{\mathcal{P}_s'}{\mathcal{P}_s} \right) \right)^2$$

$$- \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} m(k\xi) \int_0^\infty \frac{dl}{l} \ln \left( \frac{k + l - 1}{k - l} \left( \frac{\mathcal{P}_\psi'(\ln l)}{\mathcal{P}_\psi(\ln l)} + \frac{\mathcal{P}_s'(\ln l)}{\mathcal{P}_s(\ln l)} \right) \right)$$

$$+ \int_0^\infty \frac{dl}{l} \int_0^\infty \frac{dq}{q} \psi M_1(l\xi, q\xi) \left\{ \frac{\mathcal{P}_\psi'(\ln l)}{\mathcal{P}_\psi(\ln l)} \mathcal{P}_\psi'(\ln q) - \frac{\mathcal{P}_s'(\ln l)}{\mathcal{P}_s(\ln l)} \mathcal{P}_s'(\ln q) \right\}. \quad (39)$$
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\[ \frac{2}{H^2} = \int_0^\infty \frac{dk}{k} m(k\xi) \left[ \frac{\mathcal{P}_\psi'(\ln k)}{\mathcal{P}_\psi'(\ln k)} - \left( \frac{\mathcal{P}_\psi'(\ln k)}{\mathcal{P}_\psi'(\ln k)} \right)' \right] \]

where \( k_\circ \) is some reference wavenumber. Now, taking the derivative of equation (27) with respect to \( \ln \xi \) we get

\[ 2 \frac{\dot{H}}{H^2} = \int_0^\infty \frac{dk}{k} m(k\xi) \left[ \frac{\mathcal{P}_\psi'(\ln k)}{\mathcal{P}_\psi'(\ln k)} - \left( \frac{\mathcal{P}_\psi'(\ln k)}{\mathcal{P}_\psi'(\ln k)} \right)' \right] \]

Using the definition of the slow-roll parameter \( \epsilon \equiv (1/2)(\dot{\phi}/H)^2 = -\dot{H}/H^2 \), one can equate the equations (39) and (40) to get the general slow-roll constraint on the spectra as

\[ \left( \frac{\mathcal{P}_\psi}{\mathcal{P}_s} \right)_\circ = - \int_0^\infty \frac{dk}{k} m(k\xi) \left[ \frac{\mathcal{P}_\psi'(\ln k)}{\mathcal{P}_\psi'(\ln k)} - \left( \frac{\mathcal{P}_\psi'(\ln k)}{\mathcal{P}_\psi'(\ln k)} \right)' \right] \]

\( 4.1. \) Single-field approximations

In the single-field approximation, \(-\dot{H}/H^2 \simeq \) constant and thus we can simplify the equation (22) to get an easy forward formula as

\[ \ln \mathcal{P}_\psi(\ln k) = \int_0^\infty \frac{d\xi}{\xi} \left[ -k\xi v'(k\xi) \right] \ln \left( \frac{H}{2\pi} \right)^2 + \frac{\pi^2}{2} \left( \frac{\dot{H}}{H^2} \right)^2. \]  

1 The general form of equation (1) will be identical if we consider the multi-scalar field inflation model also, but with a different interpretation of the quantity ‘\( \dot{v} \)’. The comparison of the single-field and multi-field general slow-roll formulas in the case of scalar perturbations is discussed in [16].
Now integrating the above equation by parts and then differentiating with respect to \( \ln k \) we get
\[
\frac{P_\psi'}{P_\psi} = \int_0^\infty \frac{d\xi}{\xi} \left[ (-k\xi)v'(k\xi) \right] \left( 2 \frac{\dot{H}}{H^2} - \frac{\dot{H}}{H^2} \right). \tag{43}
\]
Thus, we can find that the gravitational wave spectral index
\[n_\psi \equiv \frac{d \ln P_\psi}{d \ln k} = \frac{P_\psi'}{P_\psi} \simeq \text{constant} \]
for the single-field approximation.

Also from equation (27) we get the inverse for the single-field approximation as
\[
\ln \left( \frac{H}{2\pi} \right)^2 = \int_0^\infty \frac{dk}{k} m(k\xi) \left[ \ln P_\psi - \frac{P_\psi'}{P_\psi} \right] - \frac{\pi^2}{8} \left( \frac{P_\psi'}{P_\psi} \right)^2. \tag{44}
\]
Again, since \( n_\psi \simeq \text{constant} \) and neglecting the terms containing \( n_\psi'^2 \) also, we can simplify the consistency condition in equation (41) to get the single-field general slow-roll constraint on the spectra,
\[
\left( \frac{P_\psi}{P_s} \right)_o = -\int_0^\infty \frac{dk}{k} m(k\xi) \left( n_\psi - n_\psi' \right) + \left[ \frac{\pi}{2} - \frac{5}{2} + \ln(k_o) \right] n_\psi^2
+ n_\psi \int_0^\infty \frac{dk}{k} \left( m_1(k\xi) - \theta(k\xi - k_0\xi) \right)(n_s - 1). \tag{45}
\]

4.2. Multi-field consistency condition

For the multi-component scalar field inflation models, since the scalar power spectrum has contributions from both parallel and orthogonal components, \( P_S \geq P_{\text{single field}} \). Thus the tensor to scalar power spectra ratio will be smaller than that for the single-field case and so we can write the inequality up to first-order corrections as
\[
\frac{P_\psi}{P_S} < -\int_0^\infty \frac{dk}{k} m(k\xi) \left( n_\psi - n_\psi' \right). \tag{46}
\]
Note that in this case we cannot assume \( \epsilon \simeq \text{constant} \) as it is for the single field.

5. Standard slow-roll approximation

In the context of the standard slow-roll approximation, the gravitational wave spectrum has the form
\[
\ln P_\psi = \ln P_{\psi o} + n_\psi_o \ln \left( \frac{k}{k_o} \right) + \frac{1}{2} n_\psi' \ln^2 \left( \frac{k}{k_o} \right) + \cdots, \tag{47}
\]
where \( n_\psi \equiv d \ln P_\psi/d \ln k \) is the gravitational wave spectral index and \( k_o \) is some reference wavenumber. Applying our inverse formula given by equation (27), using the window properties given by equations (18) and (33), and the following results:
\[
\int_0^\infty \frac{dx}{x} m(x) \ln(x) = \alpha \tag{48}
\]
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\[
\int_0^\infty \frac{dx}{x} \frac{m(x) \ln^2(x)}{x} = \alpha^2 + \frac{\pi^2}{12} \quad (49)
\]

\[
\int_0^\infty \frac{dy}{y} \ln \left| \frac{x+y}{x-y} \right| = \frac{\pi^2}{2}, \quad \text{for } x > 0, \quad (50)
\]

where \( \alpha = 2 - \ln 2 - \gamma \simeq 0.7296 \), we get

\[
\ln \left( \frac{H}{2\pi} \right)^2 = \ln \mathcal{P}_{\psi*} + (\alpha_\phi - 1) n_{\psi*} + \frac{1}{2} \left( \alpha_\phi^2 - 2\alpha_\phi + \frac{\pi^2}{12} \right) n_{\psi*}' + \left( 1 - \frac{\pi^2}{8} \right) n_{\psi*}^2 + \cdots, \quad (51)
\]

where \( \alpha_\phi = \alpha - \ln(k_\phi \xi) \). Equation (51) reproduces the standard slow-roll inverse, which is trivially obtained from the standard slow-roll formula,

\[
\ln \mathcal{P}_\psi = \ln \frac{1}{p_*^2} - 2\alpha_* \frac{p_*'}{p_*} - \left( \alpha_*^2 - \frac{\pi^2}{12} \right) \frac{p_*''}{p_*} + \left( \alpha_*^2 - 4 + \frac{5\pi^2}{12} \right) \left( \frac{p_*'}{p_*} \right)^2 + \cdots, \quad (52)
\]

where \( \alpha_* = \alpha - \ln(k_* \xi_*) \) and \( \xi_* \) is an arbitrary evaluation point usually taken to be around horizon crossing.

Now let us deduce the consistency conditions for the standard slow-roll approximation case. Substituting the standard slow-roll gravitational wave spectrum given by equation (47) in the general constraint on the spectra given by equation (41), we get the constraint with the standard approximation as

\[
\left( \frac{\mathcal{P}_\psi}{\mathcal{P}_s} \right) = -n_{\psi*} - (\alpha_* - 1) n_{\psi*}' + (2 + \ln k_*)(n_{s*} - 1) n_{\psi*}^2 + \left( \frac{\pi}{2} - \frac{5}{2} + \ln k_* \right) n_{\psi*}^2. \quad (53)
\]

Comparison of the above expression with the general slow-roll condition shows that the second-order correction terms are different. Also we can write the standard slow-roll multi-field constraint as

\[
\left( \frac{\mathcal{P}_\psi}{\mathcal{P}_s} \right) < -n_{\psi*}. \quad (54)
\]

It is clear from the general multi-field constraint in equation (46) that the sign of the second term on the right-hand side will depend on that of \( n_\psi' \) and thus the bound could be varied depending on the tilt of \( n_\psi \).

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