The Blessings of Multiple Causes: A Reply to Ogburn et al. (2019)

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Abstract

Ogburn et al. (2019) discuss “The Blessings of Multiple Causes” (Wang and Blei, 2018). Many of their remarks are interesting. However, they also claim that the paper has “foundational errors” and that its “premise is...incorrect.” These claims are not substantiated. There are no foundational errors; the premise is correct.

Wang and Blei (2018) provide assumptions, theory, and algorithms for multiple causal inference. The deconfounder method involves modeling the causes, using the model to infer a substitute confounder, and then using the substitute confounder in a downstream causal inference. The deconfounder is not a black-box solution to causal inference. Rather, it’s a way to use careful domain-specific modeling in the service of causal inference.

In Ogburn et al. (2019), Ogburn, Shpitser, and Tchetgen Tchetgen (OSTT) provide a technical meditation on some of the theoretical aspects of Wang and Blei (2018). Many of their remarks are interesting, and we are glad to participate in such a vigorous intellectual discussion. Other commentary includes Imai and Jiang (2019) and D’Amour (2019a).

In their discussion, OSTT claim that there are “foundational errors” with the paper and that “the premise of the deconfounder is...incorrect.” These claims are not substantiated. There are no foundational errors; the premise is correct.\textsuperscript{1}

The identification results in Theorems 6–8 capitalize on two requirements: (1) the distribution of the causes $p(\mathbf{a})$ can be described by a factor model and (2) the factor model pinpoints the substitute confounder $Z$, i.e. $Z \overset{d}{=} f_\theta(\mathbf{A})$ for some $f_\theta$. The first requirement relies on the successful execution of the deconfounder, i.e., finding a factor model that captures $p(\mathbf{a})$. The conditional independence structure of factor models guarantees that the substitute confounder $Z$ pick up all multi-cause confounders and no multi-cause mediators or colliders. The second requirement is the “consistency of the substitute confounder.” It is satisfied when the number of causes goes to infinity and $Z$ remains finite-dimensional. From Lemma 4, it guarantees that $Z$ cannot pick up single-cause mediators or colliders.

\textsuperscript{1}All references refer to Wang and Blei (2018), version 3 as of Apr 15, 2019. This reply was revised in response to version 3 of Ogburn et al. (2019), which was revised in response to the earlier version of our reply.
Single-cause variables, colliders, and Lemma 4

OSTT’s main concern revolves around Lemma 4, which states the substitute confounder cannot pick up information about multi-cause mediators, single-cause mediators (OSTT Figures 1a and 1b), or single-cause colliders (OSTT Figure 1c). Further consequences of Lemma 4 are that the substitute cannot pick up single-cause M-bias (OSTT Figure 1d). Lemma 4 is correct, as is the proof in the paper.

OSTT invoke a series of arguments, but none is valid. (a) The identification theorems in the paper all require a pinpointed substitute confounder (Definition 4). In OSTT Figure 1a, R cannot be pinpointed. When causes are causally dependent, there may not be a valid substitute confounder (see Section 2.6.6). (b) $Z \perp \perp Y \mid A$ does not imply “no confounding.” Any $Z$ that is a deterministic function of $A$ will satisfy this statement. Consider the second equation in Theorem 6; it satisfies the independence statement in question, and there is confounding. (c) In Figures 1b and 1c, OSTT worry about single-cause mediators and colliders. The substitute confounder is constructed to render the causes conditionally independent. It cannot be both single-cause and consistent, which precludes it from picking up single-cause mediators and colliders. (See the discussion of Lemma 4 below.) (d) OSTT Figure 1d concerns single-cause M-bias. They worry that substitute confounder picks up multi-cause confounders and single-cause M-colliders, but misses single-cause confounders. For the same reason as in (c), the substitute confounder cannot pick up single-cause M-colliders.

Points (c) and (d) rest on the correctness of Lemma 4. Lemma 4 and its consequences says the following. Suppose the distribution of causes can be represented by a factor model and the substitute confounder can be pinpointed. Then the substitute cannot contain information about post-treatment variables, whether single-cause or multi-cause, or single-cause variables, including single-cause confounders and single-cause colliders. This fact might seem surprising. Here we expand on its statement and provide an alternative proof.

**Restatement of Lemma 4.** No single-cause pre-treatment variable, single-cause post-treatment variable, or multi-cause post-treatment variable can be measurable with respect to a consistent substitute confounder.

**Proof.** First, the substitute cannot pick up any multi-cause post-treatment variables. Otherwise, the substitute can not render all the causes conditionally independent.

The substitute also cannot pick up any single-cause variables. These variables include pre-treatment variables, such as single-cause confounders, and single-cause post-treatment variables, such as single-cause mediators or colliders.

The key idea behind the proof is the following. We assume the causes pinpoint the substitute confounder $Z \overset{\text{def}}{=} f_0(A)$, as is the case where there are many causes. The deconfounder further requires that the converse is not true, i.e., that the substitute does not pinpoint the causes. This fact holds in a probabilistic model of the causes, such as when the dimension
of the substitute stays fixed as the number of causes increases. Further, the deconfounder requires that the factor model can not have one component of the substitute \emph{a priori} be a deterministic function of another component; this fact also holds in probabilistic factor models. The proof then follows by contradiction: if the substitute picks up single-cause variables then the factor model must be “degenerate,” i.e., non-probabilistic.

Here are the details. Suppose the substitute \( Z \) does pick up a single-cause variable. Then separate \( Z \) into a single-cause component and a multi-cause one, \( Z = (Z_s, Z_m) \). Without loss of generality, assume the single-cause component only depends on the first cause. The assumption of a consistent substitute confounder says

\[
p(z | a, \theta) = p(z_s, z_m | a, \theta) = \delta(f_s(a; \theta), f_m(a; \theta)),
\]

where \( a = (a_1, \ldots, a_m) \) are the \( m \) causes and \( f_s(\cdot), f_m(\cdot) \) are the deterministic functions that map causes to substitute confounders.

Now calculate the conditional distribution of the single-cause component given the causes,

\[
p(z_s | a) = p(z_s | a, z_m = f_m(a; \theta))
= p(z_s | a, z_m = f_m(a; \theta))
= p(z_s | a_1, z_m = f_m(a; \theta))
= \frac{p(z_s | z_m = f_m(a; \theta)) \cdot p(a_1 | z_s, z_m = f_m(a; \theta))}{p(a_1 | z_m = f_m(a; \theta))}.
\]

Equation 2 is due to the consistency of substitute confounder. Equation 3 is due to \( Z_s \perp A_2, \ldots, A_m | A_1, Z_m \). Equation 4 is due to the definition of conditional probability.

Equation 4 and Equation 1 imply that at least one of \( p(z_s | z_m = f_m(a; \theta)) \) and \( p(a_1 | z_s, z_m = f_m(a; \theta)) \) is a point mass. But this is a contradiction: either term being a point mass implies that the factor model is degenerate. The former is a point mass when one component \( Z_s \) of the substitute is a deterministic function of another component \( Z_m \). The latter is a point mass when the first cause is a deterministic function of the latent \( Z \).

Note the same argument would not reach a contradiction for multi-cause variables \( Z_m \). The reason is that

\[
p(z_m | a)
= p(z_m | a, z_s = f_s(a; \theta))
= \frac{p(a_1, z_m | z_s = f_s(a; \theta)) \cdot \prod_{j=2}^m p(a_j | z_m)}{p(a)}
\]

where \( \prod_{j=2}^m p(a_j | z_m) \) can converge to a point mass with non-degenerate factor models and \( m \to \infty \). \( \square \)
Other remarks of OSTT

OSTT question the random variable on which we used the Kallenberg construction in Lemmas 1 and 2. Definition 3 is the Kallenberg construction we intended, and it involves potential outcomes; see Eq. 39 in the paper.

OSTT further argue that potential outcomes cannot only be included in only one of Eq. 38 and Eq. 39. This claim is not true. Eq. 38 and Eq. 39 jointly define the Kallenberg construction: Eq. 39 describes a requirement that the random variables \( U_{ij} \)’s in Eq. 38 must satisfy in the Kallenberg construction.

OSTT claim a counterexample where a random variable separates a multi-cause confounder into single-cause confounders. However, a consistent substitute cannot separate a multi-cause confounder into single-cause confounders. Returning now to Lemmas 1 and 2, it is these lemmas that link factor models of the causes to their Kallenberg construction and unconfoundedness, thanks to the consistency of the substitute confounder.

OSTT discuss a “missing assumption” in Theorem 6 that \( f_1 \) is less smooth than \( f_2 \). This fact is sufficient for the theorem, but it is not necessary. Identification requires the differentiability of \( f_1, f_2 \) and the non-differentiability of \( f \) (Assumption 1 of Theorem 6).

OSTT further provide a counterexample where the \( f_1 \) is equal to \( f_2(f(\cdot)) \). Under the assumptions of Theorem 6, this cannot be true. Theorem 6 requires that \( f \) be piecewise constant and \( f_1, f_2 \) be continuously differentiable; thus the function \( f_2(f(\cdot)) \) can only be piecewise constant or constant. When \( f_2(f(\cdot)) \) is piecewise constant, it is non-differentiable and hence cannot be equal to the continuously differentiable \( f_1 \). When \( f_2(f(\cdot)) \) is a constant function, there is no confounding; Theorem 6 (Eq. 41) is correct.

OSTT claim that the paper leaves open that Theorem 7 is “vacuous” because the overlap condition may be impossible to satisfy. D’Amour’s discussion of the paper (D’Amour, 2019a) shows how Theorem 7 can be useful.

OSTT remark that requiring a pinpointed substitute implies that the unobserved multi-cause confounding is effectively observed. Their intuition is correct—the multiplicity of the causes and the consistent estimability of factor models enable us to effectively observe such multi-cause confounding. It is these two features that form the basis of the deconfounder.
Discussion

In their discussion, OSTT remind us that all causal inference requires assumptions, and we agree. Causal inference with the deconfounder involves a number of assumptions and trade-offs. Among them are the following. (1) There can be no unobserved single-cause confounders. (2) When we apply the deconfounder, we trade an increase in variance for a reduction in confounding bias; there is no free lunch. (3) We do not recommend using the deconfounder with causally dependent causes, such as a time series.

There are many directions for research. We need a more complete picture of identification; D'Amour (2019a,b) and Imai and Jiang (2019) make good progress. We need to understand the finite-sample properties of the deconfounder (or, how much is lunch?). We need rigorous methods of model criticism for assessing the validity of the substitute confounder. But these are directions for research; the foundations of Wang and Blei (2018) are intact.

References

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