Multi-Task Variational Information Bottleneck

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Abstract

In this paper we propose a multi-task deep learning model called multi-task variational information bottleneck (in short MTVIB). The structure of the variational information bottleneck (VIB) is used to obtain the latent representation of the input data; the task-dependent uncertainties are used to learn the relative weights of task loss functions; and the multi-task learning can be formulated as a constrained multi-objective optimization problem. Our model can enhance the latent representations and consider the trade-offs among different learning tasks. It is examined with publicly available datasets under different adversarial attacks. The overall classification performance of our model is promising. It can achieve comparable classification accuracies as the benchmarked models, and has shown a better robustness against adversarial attacks compared with other multi-task deep learning models.

1 Introduction

Multi-task learning is a popular sub-field in machine learning. In the conventional setting, different tasks share the same representation and are learned simultaneously. Each task has its contribution to the total learning loss and a good balance among tasks can result in good predictive capability. Multi-task learning models are usually believed to be computationally efficient and can obtain better generalized representations compared to learning a single task at a time. Generally speaking, there are two groups of approaches which are widely used to improve the multi-task learning performance. The first group is focused on improving the shared latent representation and the other group aims to find the optimal relative weights for sub-learning tasks [23, 30]. The proposed model in our research will work on both issues.

Many multi-task learning models use the deterministic encoding techniques to obtain latent representations. While in this study we use the variational encoding method to improve the latent representation. Specifically, we adopt the information bottleneck method [25] to obtain the latent codes. The information bottleneck method can be implemented via variational inference [5, 29], called the variational information bottleneck (VIB) [2]. Variational inference has widely been used in deep learning such as variational autoencoders (VAEs) and their variants [15, 21, 11, 7, 19], Bayesian neural networks (BNNs) [6], variational dropout [15, 20] and deep variational prior [3]. In addition, more recently, variational inference has been used for representation learning, for instance, mutual information estimation and maximization [26].

In terms of optimal task weights, the grid search (GS) is a simple way to find the optimal trade-off among tasks, which uses a set of different combinations of constant weights to find the appropriate relative weights of different losses. However, this method is not computationally efficient [18]. Recently, the gradient normalization is used to balance the losses in multi-task deep neural net-
works [8]. Alternatively, from the perspective of optimization, if different learning tasks are regarded as different objectives, then multi-task learning can be formulated as a multi-objective optimization problem [24, 18]. One can also make use of the homoscedastic uncertainties of the losses to compute the optimal weights for each learning goal [13]. Since our approach is probabilistic, we naturally adopt this uncertainty-based method by using likelihoods of the predictions to weigh the different losses and we also find that this method helps stabilize the training process in practice. In order to extend the VIB for multi-task learning, we leverage the predictive likelihoods of the VIB decoder to calculate the task-related weights for different losses. Based on this, we propose the multi-task variational information bottleneck (MTVIB).

Our research has two methodological contributions. First, the MTVIB adopts the VIB structure to obtain the latent representations of the input data. Compared to deterministic latent representations, the variational latent representations are regularized and thereby expected to be more robust to noises, for instance, under adversarial attacks. Second, the MTVIB uses the task-related uncertainties to assign the relative weights for each task loss. This not only helps us to find a good trade-off among different tasks but also ensures the multi-task training process steady.

2 Model Setup

Our model assumptions are based on a typical information Markov chain, where the information associated with the input data can be represented by a latent distribution. Let $X$ denote the input data, $Z$ denote the latent representation and $Y$ denote the output. The information Markov chain is then $X \rightarrow Z \rightarrow Y$, which can be presented in an encoder-decoder structure. A similar idea was proposed by [1], where the information is stored in the neural network weights.

**Assumption 1.** There exists a statistic of the input $X$ which is solely sufficient enough to learn the posterior probability of $Z$, i.e., $P(Z|X,Y) = P(Z|X)$.

This assumption was proposed by [2]. It suggests that the input data contains the needed information in order to compute the latent distribution. The latent distribution in our model will be learnt by an encoder network. Different to the deterministic codes computed by vanilla auto-encoders [12], we adopt the VIB method and accordingly the learned latent codes can be seen as a disentangled representation. In addition, in an ideal situation, the latent representation can not only be sufficient but also be minimal. Consequently, only the task-related information will be retained.

**Assumption 2.** The learned representation $Z$ is solely sufficient enough to learn the likelihood of $Y$, i.e., $P(Y|X,Z) = P(Y|Z)$.

This assumption was made by [1]. It indicates that the sufficiency of the latent representation is ensured by the decoder network. This can be easily evaluated through the loss function, e.g., the cross-entropy for the classification problem. In short, we expect that our model can compress the data maximally while also express the output as much as possible under Assumptions 1 and 2.

**Assumption 3.** If there are multiple learning tasks (e.g., $Y = [Y_1, Y_2, \ldots, Y_K]$), they are conditionally independent and share the same representation, i.e., $P(Y|Z) = \prod_{k=1}^{K} P(Y_k|Z)$.

This assumption has also been used in [13]. It allows us to divide the output into different sub-outputs which can be represented by different decoders and then we can make use of the factorial probabilities to weigh the losses.

3 Latent Representation

We seek a sufficient minimal representation for the input features. For supervised learning, according to the information bottleneck theory [25], the following optimization problem can be formulated

\[
\begin{align*}
\max & \quad I(Z;Y), \\
\text{s.t.} & \quad I(Z;X) \leq I_C,
\end{align*}
\]

where $I$ denotes the mutual information between two variables, and $I_C$ is the information constraint.

To solve this optimization problem, the Karush-Kuhn-Tucker (KKT) conditions can be applied and the corresponding Lagrangian yields

\[
I(Z;Y) - \beta (I(Z;X) - I_C),
\]

2
where $\beta$ is a non-negative Lagrangian multiplier and $I_C$ can be ignored as it is a constant.

Direct computing the mutual information in Eq. 3 is intractable, instead, we can use other techniques to approximate it. Based on Assumptions 1-2, the following variational lower bound of the information bottleneck can be obtained:

$$I(Z; Y) - \beta I(Z; X) \geq \mathbb{E}_{p(z,y)} \left[ \log \{ p(y|z) \} \right] - \beta \mathbb{E}_x \left[ D_{KL}(p(z|x)||q(z)) \right], \quad (4)$$

where $q(z)$ is an uninformative prior distribution. The detailed derivation of Eq. (18) is provided in the supplementary material.

Since $p(z, y) = \int p(y|z)p(z|x)p(x)dx$, we can approximate $p(z, y)$ by drawing $x$ from the dataset $\mathcal{D}$ first, then drawing $z$ from $p(z|x)$ and $y$ from the dataset simultaneously. Then Eq. (18) becomes

$$I(Z; Y) - \beta I(Z; X) \geq \mathbb{E}_{\mathcal{D}} \left[ \mathbb{E}_{z \sim p(z|x)} \left[ \log \{ p(y|z) \} \right] - \beta D_{KL}(p(z|x)||q(z)) \right], \quad (5)$$

Similar to $\beta$-VAEs and VIB, we adopt an encoder-decoder structure. The encoder $p(z|x)$ is parameterized by $\phi$ and the latent variable $z$ can be sampled via $p_\phi(z|x)$. The decoder $p_\theta(y|z)$ is parameterized by $\theta$. Maximizing the lower bound in Eq. (5) is equivalent to minimizing the following loss function:

$$\mathcal{L}(\mathcal{D}, \theta, \phi) = \mathbb{E}_{\mathcal{D}} \left[ \mathbb{E}_{z \sim p_\phi(z|x)} \left[ - \log \{ p_\theta(y|z) \} \right] + \beta D_{KL}(p_\phi(z|x)||q(z)) \right]. \quad (6)$$

Eq. (6) is the VIB structure to acquire the stochastic latent representation. Note that the trade-off between the encoding term and the decoding term is governed by $\beta$. A larger $\beta$ means that the VIB is more compressed for the input and less expressed for the output, and vice versa. When it comes to supervised learning, a relatively small value of $\beta$ is preferred to ensure the prediction performance.

4 Multi-Task Variational Information Bottleneck

We now extend the VIB structure for multi-task learning. The uncertainty weighted losses method [13] is used to balance the weights among different tasks. The learning tasks in Eq. (6) are to compute the task-related likelihoods. With Assumption 3, different tasks are conditionally independent from each other. Therefore, the likelihood of the output for $K$ tasks is

$$p(y_1, \ldots, y_K | z) = \prod_{k=1}^K p(y_k | z) = \prod_{k=1}^K \prod_{i=1}^I \left[ \mathbb{P}(y_k = i) \right]^{m_i}, \quad (7)$$

where $m_i = 1$ if $y_k = i$ otherwise $m_i = 0$.

We take the homoscedastic uncertainties of the losses into account, and we can use the softmax function to construct a Boltzmann distribution (or Gibbs distribution) for classification problem. Let $f_{\theta, k}(z)$ be the output vector of the decoder network for task $k$ and it is parameterized by weight vector $\theta$. We follow [13] and the classification likelihood of class $i$ for task $k$ is adapted with a scaling squash through the softmax function:

$$\mathbb{P}(y_k = i) = \text{softmax} \left\{ f_{\theta, k}^i(z), \sigma_k \right\}$$

$$= \frac{\exp \left\{ \frac{1}{\sigma_k} f_{\theta, k}^i(z) \right\}}{\sum_{j=1}^I \exp \left\{ \frac{1}{\sigma_k} f_{\theta, k}^j(z) \right\}}$$

$$\approx \frac{1}{\sigma_k} \left[ \text{softmax} \left\{ f_{\theta, k}^i(z) \right\} \right]^{1/\sigma_k}, \quad (8)$$

where $\sigma_k$ is a positive scalar indicating the homoscedastic uncertainty of task $k$ and $f_{\theta, k}^i(z)$ is the $i$th element of the decoder output, and $\text{softmax} \left\{ f_{\theta, k}^i(z) \right\}$ is the likelihood of class $i$ for task $k$ without squash scaling.
Thus, the loss function based on the negative log-likelihood for $K$ tasks can be obtained:

$$ - \log\{p(y_1, \ldots, y_K | z)\} = - \sum_{k=1}^{K} \sum_{i=1}^{I} m_i \log \{P(y_k = i)\} $$

$$ = \sum_{k=1}^{K} \left[ \frac{1}{\sigma_k} \delta_k + \log\{\sigma_k\} \right], $$

where $\delta_k = - \sum_{i=1}^{I} m_i \log \left\{ \text{softmax}\{f_{\phi,k}(z)\} \right\}$ is the cross-entropy for task $k$.

Our proposed model combines the VIB structure with the uncertainty weighted losses. Based on Eq. (5), the multi-task learning can be formulated as follows

$$ \arg\max_{\theta, \phi} \mathbb{E}_{\theta} \left[ \mathbb{E}_{p_{\theta}(z|\mathbf{x})} \left[ \log \{ p_{\theta}(y_1, \ldots, y_K | z) \} \right] \right], $$

$$ \text{s.t. } D_{KL}(p_{\theta}(z|\mathbf{x}), q(z)) \leq \epsilon, $$

where $\phi$ is the encoder parameter, $\theta$ is the decoder parameter, and $\epsilon$ is a constant. Applying the KKT condition to Eq. (10) then gives the Lagrangian form

$$ \mathcal{L} = \mathbb{E}_{\theta} \left[ \mathbb{E}_{z \sim p_{\theta}(z|\mathbf{x})} \left[ - \log \{ p_{\theta}(\theta_1, \ldots, \theta_K | z) \} + \beta D_{KL}(p_{\theta}(z|\mathbf{x})||q(z)) \right] \right] $$

$$ = \mathbb{E}_{\theta} \left[ \mathbb{E}_{z \sim p_{\theta}(z|\mathbf{x})} \left[ - \sum_{k=1}^{K} \log \{ p_{\theta_k}(y_k | z) \} + \beta D_{KL}(p_{\theta}(z|\mathbf{x})||q(z)) \right] \right]. $$

To solve the KL divergence term in Eq. (12), we resort to the re-parameterization method [16]:

$$ f_{\phi}(x, \epsilon_z) = \mu_z(x) + \sigma_z(x) \odot \epsilon_z $$

where $f_{\phi}(x, \epsilon_z)$ is the deterministic function used in encoder $p_{\phi}(z|x)$, $\mu_z$ and $\sigma_z$ are the deterministic functions to calculate the mean and variance of the latent Gaussian distribution, $\odot$ is the Hadamard product, and $\epsilon_z$ is a random noise sampled from a standard diagonal Gaussian distribution $\mathcal{N}(0, I)$.

Eq. (12) can be solved with Eq. (13) and Monte Carlo sampling. The final loss of the MTVIB is

$$ \mathcal{L} = \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}_{\epsilon_z \sim p(\epsilon_z)} \left[ \sum_{k=1}^{K} - \log \{ p_{\theta_k}(y_n | f_{\phi}(x_n, \epsilon_z)) \} + \beta D_{KL}(p_{\phi}(z|x_n)||q(z)) \right] $$

$$ = \sum_{k=1}^{K} \left[ \frac{1}{\sigma_k} \delta_k + \log\{\sigma_k\} \right] + \beta D_{KL}(p_{\phi}(z|x)||q(z)) $$

where $N$ is the size of mini batch for Monte Carlo sampling.
Similar to the VIB, in model testing, we sample the latent variable $z$ using encoder $p_\phi(z|x)$, then $z$ will be used as the input for decoders $p_w$, and the output of decoders are our model’s output. The architecture of the MTVIB is exhibited in Fig. 1. The encoder-decoder architecture with a latent distributions has been used in VAEs and $\beta$-VAEs. It should be noted that $\beta$-VAEs also adopt the information bottleneck method to acquire better disentanglement. However, they are designed to sample new imagines for unsupervised learning while our model is for supervised multi-task learning. Additionally, the proposed model is in line with the minimum description length (MDL) principle and provides a general solution to the overfitting problem [22], [10]. It can be said that the KL divergence $D_{KL}(p(z|x)||q(z))$ is the expected number of additional bits for the $q(z)$-optimal encoding of outcomes generated by $p(z|x)$ [10]. Alternatively, for the mutual information term $I(X; Z)$, it can be estimated by using the method proposed in [4] instead of using the re-parameterization trick. More detailed discussions are provided in the supplementary material.
5 Experiments

Our proposed model is evaluated with three publicly available datasets.

**Two-task classification:** We use the MultiMNIST dataset and the MultiFashionMNIST dataset [18]. The former is created from the MNIST dataset [17] while the latter is created based on the Fashion-MNIST dataset [27]. Specifically, each image of the MultiMNIST or MultiFashionMNIST dataset is overlapped by two images from the MNIST or Fashion-MNIST dataset, respectively. The samples of the MultiMNIST and Fashion-MNIST datasets are provided in the supplementary material. The input
dimension of data is $36 \times 36$ and the each input is related to a 2-label target. For each dataset, the training instance number is 120000 and the test instance number is 10000.

**Four-task classification:** We use the multi-task facial landmark (MTFL) dataset [31]. This dataset has four classification tasks, whose target category numbers are 2, 2, 2 and 5, respectively. The dimension of the images is resized to $150 \times 150$. The training image number is 9000 and the test image number is 1000.

Different adversarial attacks are taken into account. In the experiments, we use the fast gradient sign method (FGSM) [9] to produce the adversarial samples:

$$x_{adv} = x + \eta \cdot \text{sign}(\nabla_x J(w; x; y))$$

(15)

where $\eta$ is a positive scalar (we use different values, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3 for the experiments). $J$ is the loss function and $w$ is the model parameter. The original samples and their adversarial for the used datasets are presented in Figs. 2-4.

The benchmarked models include the grid search (GS), the uncertainty weighted losses (UWL) [13], the single-task learning (STL), and the single-task variational information bottleneck (STVIB). The GS and the UWL are multi-task learning models while the STL and the STVIB are single task learning models. We follow [28] and use the Small AlexNet as the base structure for STL, GS and UWL. In STVIB and MTVIB, the latent dimensions are set as 256 and $\beta$ is set to be $1e^{-3}$. The learning rate is 0.0001; the optimizer is Adam [14]; the training epoch is 200; and the minibatch size is 200. The implementation details of models are presented in Tables 1-2.

Our model has shown promise in multi-task classification. As presented in Figs. 5-6, the model is on a par with the benchmarked single-task and multi-task learning models in terms of the overall classification accuracy of each task. Without adversarial attacks, the average classification accuracy is 0.9435 for the MultiMNIST dataset, 0.8537 for the MultiFashionMNIST dataset, and 0.7418 for the MTFL dataset. When the noise level increases, our model is more robust and significantly outperforms the benchmarked multi-task learning models (i.e., the UWL and the GS).

### 6 Conclusion

We propose the MTVIB in this paper. Our model is based on the VIB structure, which can obtain the latent representation of the input data. The task-dependent uncertainties is used to learn the relative weights of task loss functions and the multi-task learning can be formulated as a constrained multi-objective optimization problem. The MTVIB can enhance the latent representation and consider the trade-offs among different learning tasks. It is examined with publicly available datasets under different adversarial attacks. It has achieved comparable classification accuracy as the benchmarked models, and has shown a better robustness against adversarial attacks compared with other multi-task learning models.

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As the mutual information can be defined as the Kullback-Leibler (KL) divergence between the joint density and the product of the marginal densities, if Assumptions 1-2 are met, then

\[
I(Z; Y) = D_{KL}(p(z, y) || p(z)p(y))
\]

\[
= \mathbb{E}_{p(z, y)} \left[ \log \left( \frac{p(z, y)}{p(z)p(y)} \right) \right]
\]

\[
= \mathbb{E}_{p(z, y)} \left[ \log \left( \frac{p(z, y)}{p(z)p(y)} \right) - \log \{p(y)\} \right]
\]

\[
= \mathbb{E}_{p(z, y)} \left[ \log \left( \frac{p(y|z)p(z)}{p(z)} \right) - \log \{p(y)\} \right]
\]

\[
\geq \mathbb{E}_{p(z, y)} \left[ \log \{p(y|z)\} \right].
\]

(16)

On the other hand, according to Assumption 1, the posterior \(p(Z|X)\) can be calculated, and then

\[
I(Z; X) = D_{KL}(p(z, x) || p(z)p(x))
\]

\[
= \mathbb{E}_{p(z, x)} \left[ \log \left( \frac{p(z, x)}{p(z)p(x)} \right) \right]
\]

\[
= \sum_z \sum_x p(z, x) \log \left( \frac{p(z, x)}{p(z)p(x)} \right)
\]

\[
= \sum_z \sum_x p(z, x) \log \left( \frac{p(z|x)p(x)}{p(z)q(z)} \right)
\]

\[
= \sum z \sum x p(z, x) \left[ \log \left( \frac{p(z|x)q(z)}{p(z)} \right) - \log \left( \frac{p(z)}{q(z)} \right) \right]
\]

\[
= \sum z \sum x p(z|x)p(x) \log \left( \frac{p(z|x)}{q(z)} \right) - \sum_z \sum x p(z|x)p(x) \log \left( \frac{p(z)}{q(z)} \right)
\]

\[
= \mathbb{E}_x \left[ D_{KL}(p(z|x) || q(z)) \right] - D_{KL}(p(z) || q(z))
\]

\[
\leq \mathbb{E}_x \left[ D_{KL}(p(z|x) || q(z)) \right],
\]

(17)

where \(q(z)\) is an uninformative prior distribution.

Substituting Eqs. (16)-(17) into the corresponding Lagrangian then give a variational lower bound of the information bottleneck:

\[
I(Z; Y) - \beta I(Z; X) \geq \mathbb{E}_{p(z, y)} \left[ \log \{p(y|z)\} \right] - \beta \mathbb{E}_x \left[ D_{KL}(p(z|x) || q(z)) \right].
\]

(18)
6.2 Derivation of $\mathbb{P}(y_k = i)$

Let $f_{\theta,k}(z)$ be the output vector of the decoder network for task $k$ and it is parameterized by weight vector $\theta$. The likelihood of the $i$th element in vector $f_{\theta,k}(z)$ with dimension $I$ is

$$
\text{softmax}\{f_{\theta,k}^i(z), \sigma\} = \frac{\exp\left\{\frac{1}{\sigma^2}f_{\theta,k}^i(z)\right\}}{\sum_{j=1}^I \exp\left\{\frac{1}{\sigma^2}f_{\theta,k}^j(z)\right\}}
= \left[\exp\left\{f_{\theta,k}^i(z)\right\}\right]^\frac{1}{\sigma^2}
\sum_{j=1}^I \exp\left\{\frac{1}{\sigma^2}f_{\theta,k}^j(z)\right\}. \tag{19}
$$

If without squash scaling, the likelihood of the $i$th element in vector $f_{\theta}(z)$ is

$$
\text{softmax}\{f_{\theta,k}^i(z)\} = \frac{\exp\left\{f_{\theta,k}^i(z)\right\}}{\sum_{j=1}^I \exp\left\{f_{\theta,k}^j(z)\right\}}. \tag{20}
$$

Therefore, we have

$$
\text{softmax}\{f_{\theta,k}^i(z), \sigma\} = \frac{\left[\text{softmax}\{f_{\theta,k}^i(z)\}\right]^\frac{1}{\sigma^2} \left[\sum_{j=1}^I \exp\left\{f_{\theta,k}^j(z)\right\}\right]^\frac{1}{\sigma^2}}{\sum_{j=1}^I \exp\left\{\frac{1}{\sigma^2}f_{\theta,k}^j(z)\right\}}
\approx \frac{1}{\sigma_k} \left[\text{softmax}\{f_{\theta,k}^i(z)\}\right]^\frac{1}{\sigma_k}. \tag{21}
$$
Table 1: Implementation details of models for the two-task datasets.

| Model        | Input Size | Convolutional Layers | Fully Connected Layers | Output |
|--------------|------------|----------------------|------------------------|--------|
| (a) Small AlexNet model | 36 × 36 | 1. Conv 64 (Kernel: 3 × 3; Stride: 1) + BN + ReLU | FC 3136 × 384 + BN + ReLU | FC 192 × 10 |
|              |            | 2. MaxPool 2 × 2 | FC 384 × 192 + BN + ReLU | softmax |
| (b) Small AlexNet-adapted model | 36 × 36 | 1. Conv 64 (Kernel: 3 × 3; Stride: 1) + BN + ReLU | FC 3136 × 384 + BN + ReLU | FC 192 × 10 |
|              |            | 2. MaxPool 2 × 2 | FC 384 × 192 + BN + ReLU | softmax |
| (c) STVIB model | 36 × 36 | 1. Conv 64 (Kernel: 3 × 3; Stride: 1) + BN + ReLU | FC 3136 × 384 + BN + ReLU | FC 192 × 10 |
|              |            | 2. MaxPool 2 × 2 | FC 384 × 192 + BN + ReLU | softmax |
| (d) MTVIVB model | 36 × 36 | 1. Conv 64 (Kernel: 3 × 3; Stride: 1) + BN + ReLU | FC 3136 × 384 + BN + ReLU | FC 192 × 10 |
|              |            | 2. MaxPool 2 × 2 | FC 384 × 192 + BN + ReLU | softmax |

Table 2: Implementation details of models for the four-task dataset.

| Model        | Input Size | Convolutional Layers | Fully Connected Layers | Output |
|--------------|------------|----------------------|------------------------|--------|
| (a) Small AlexNet model | 150 × 150 | 1. Conv 64 (Kernel: 5 × 5; Stride: 2) + BN + ReLU | FC 4096 × 384 + BN + ReLU | FC 192 × 10 |
|              |            | 2. MaxPool 2 × 2 | FC 384 × 192 + BN + ReLU | softmax |
| (b) Small AlexNet-adapted model | 150 × 150 | 1. Conv 64 (Kernel: 5 × 5; Stride: 2) + BN + ReLU | FC 4096 × 384 + BN + ReLU | FC 192 × 10 |
|              |            | 2. MaxPool 2 × 2 | FC 384 × 192 + BN + ReLU | softmax |
| (c) STVIB model | 150 × 150 | 1. Conv 64 (Kernel: 5 × 5; Stride: 2) + BN + ReLU | FC 4096 × 384 + BN + ReLU | FC 192 × 10 |
|              |            | 2. MaxPool 2 × 2 | FC 384 × 192 + BN + ReLU | softmax |
| (d) MTVIVB model | 150 × 150 | 1. Conv 64 (Kernel: 5 × 5; Stride: 2) + BN + ReLU | FC 4096 × 384 + BN + ReLU | FC 192 × 10 |
|              |            | 2. MaxPool 2 × 2 | FC 384 × 192 + BN + ReLU | softmax |