A note on geometric 3-hypergraphs

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Abstract

In this note, we prove several Turán-type results on geometric hypergraphs. The two main theorems are 1) Every \( n \)-vertex geometric 3-hypergraph in the plane with no three strongly crossing edges has at most \( O(n^2) \) edges, 2) Every \( n \)-vertex geometric 3-hypergraph in 3-space with no two disjoint edges has at most \( O(n^2) \) edges. These results support two conjectures that were raised by Dey and Pach, and by Akiyama and Alon.

1 Introduction

A geometric \( r \)-hypergraph \( H \) in \( d \)-space is a pair \((V, E)\), where \( V \) is a set of points in general position in Euclidean \( d \)-space, and \( E \) is a set of closed \((r-1)\)-dimensional simplices (edges) induced by some \( r \)-tuple of \( V \). The sets \( V \) and \( E \) are called the vertex set and edge set of \( H \), respectively. Two edges in \( H \) are crossing if they are vertex disjoint and have a point in common. Notice that if \( k \) edges are pairwise crossing, it does not imply that they all have a point in common. Hence we say that \( H \) contains \( k \) strongly crossing edges if \( H \) contains \( k \) vertex disjoint edges that all share a point in common. See Figure 1.

\( \text{(a) Three strongly crossing edges.} \)

\( \text{(b) Three pairwise crossing edges with an empty intersection.} \)

\( \text{(c) Three edges not strongly crossing since two share a vertex.} \)

Figure 1: Three edges of a geometric 3-hypergraph in the plane.

A direct application of the colored Tverberg theorem (see [3], [20]) gives

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Theorem 1.1. Let $\text{ex}_d(\text{SC}^{d+1}_k, n)$ denote the maximum number of edges an $n$-vertex geometric $(d+1)$-hypergraph in $d$-space has with no $k$ strongly crossing edges. Then

$$\text{ex}_d(\text{SC}^{d+1}_k, n) = O \left( n^{d+1 - \frac{1}{(2k-1)^d}} \right).$$

Dey and Pach [5] showed that $\text{ex}_d(\text{SC}^{d+1}_2, n) = \Theta(n^d)$, and conjectured $\text{ex}_d(\text{SC}^{d+1}_k, n) = \Theta(n^d)$ for every fixed $d$ and $k$. The lower bound can easily be seen by taking all edges with a vertex in common. The main motivation for their conjecture is for deriving upper bounds on the maximum number of $k$-sets of an $n$-point set in $\mathbb{R}^d$. See [12] for more details. In this note, we settle the Dey-Pach conjecture for geometric 3-hypergraphs in the plane with no three strongly crossing edges, and improve the upper bound of $\text{ex}_2(\text{SC}^3_k, n)$.

Theorem 1.2. $\text{ex}_2(\text{SC}^3_3, n) = \Theta(n^2)$.

Theorem 1.3. For fixed $k \geq 4$, $\text{ex}_2(\text{SC}^3_k, n) \leq O(n^{3 - \frac{1}{k}})$.

As a related result, Akiyama and Alon [2] used the Borsuk-Ulam Theorem [4] to show the following.

Theorem 1.4. Let $\text{ex}_d(D^d_k, n)$ denote the maximum edges that an $n$-vertex geometric $d$-hypergraph in $d$-space has with no $k$ pairwise disjoint edges. Then

$$\text{ex}_d(D^d_k, n) \leq n^{d - (1/k)^{d-1}}.$$ 

They conjecture that for every fixed $d$ and $k$, $\text{ex}_d(D^d_k, n) = \Theta(n^{d-1})$. Again the lower bound can easily be seen by taking all edges with a vertex in common. Pach and Töröcsik [15] showed that $\text{ex}_2(D^2_k, n) = O(k^4n)$, which was later improved to $O(k^2n)$ by Tóth [17]. Here we settle the Akiyama-Alon conjecture for geometric 3-hypergraphs in 3-space with no two disjoint edges.

Theorem 1.5. $\text{ex}_3(D^3_2, n) = \Theta(n^2)$.

For clarity of the proofs, we do not make any attempts to optimize the constants.

2 Strongly crossing edges in the plane

In this section we will prove Theorems 1.2 and 1.3. Recall that a geometric graph is a graph drawn in the plane with vertices represented by points and edges by straight line segments connecting the corresponding pairs. Recently Ackerman [1] showed the following.

Lemma 2.1. Let $G = (V, E)$ be an $n$-vertex geometric graph in the plane with no four pairwise crossing edges. Then $|E(G)| \leq O(n)$.

We note that Lemma 2.1 holds for topological graphs. Before we give the proofs, we will introduce some terminology. Consider a family $\mathcal{S} = \{s_1, ..., s_k\}$ of pairwise crossing segments in the plane, and let $\mathcal{L} = \{l_1, ..., l_k\}$ be a family of lines such that $l_i$ is the line supported by segment $s_i$. Recall that the level of a point $x \in \bigcup \mathcal{L}$ is defined as the number of lines of $\mathcal{L}$ lying strictly below $x$. We
define the top level of $L$ as the closure of the set of points in $\cup L$ with level $k-1$. We define the top level of $S$ to be the top level of $L$. See Figure 2. Notice that $L$ is a (not strictly) convex function.

For each edge $t$ in a geometric 3-hypergraph in the plane, we define its base as the side with the longest $x$-projection. We define the other two sides of $t$ as its left and right side. See Figure 2. Notice that every edge in a geometric 3-hypergraph is incident to a vertex that lies strictly above or below its base. We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $H = (V, E)$ be an $n$-vertex geometric 3-hypergraph in the plane with no three strongly crossing edges. We can assume that $|E(H)| \geq 20n^2$ (since otherwise we would be done) and at most $|E(H)|/2$ edges in $H$ are incident to a vertex that lies strictly below its base. We will discard all such edges, leaving us with at least $|E(H)|/2$ edges left. Let $E_{uv}$ be the set of edges in $H$ with base $uv$. We discard all sets $E_{uv}$ for which $|E_{uv}| \leq |E(H)|/(2n^2)$. Since we have thrown away at most $|E(H)|/4$ edges in this process, we have at least $|E(H)|/4$ edges left. Therefore $|E_{uv}| = 0$ or $|E_{uv}| \geq |E(H)|/(2n^2) \geq 10$.

Now let $G_v = (V, E)$ denote the geometric graph with $V(G_v) = V(H)$ and $xy \in E(G_v)$ if $conv(x \cup y \cup v) \in E(H)$ with base $xy$.

Observation 2.2. $G_v$ does not contain four pairwise crossing edges (bases).

Proof. For sake of contradiction, suppose $G_v$ contains four pairwise crossing edges $b_1, b_2, b_3, b_4 \in E(G_v)$. Then $v$ lies above $b_i$ for all $i$. Let $L$ denote the top level of the arrangement $S = \{b_1, b_2, b_3, b_4\}$. Now the proof falls into three cases.

Case 1. Suppose $L$ intersects exactly two members of $S$, say bases $b_1$ and $b_2$ (in order from left to right along $L$). Let $p$ be the intersection point of $b_1$ and $b_2$. Then the vertical line through $p$ must intersect $b_3$ below $p$. Moreover, since segments $b_1$ and $b_3$ cross, $v$ and the right-endpoint of $b_3$ must lie on the same half-plane generated by the line supported by $b_1$. Likewise, $v$ and the left-endpoint of $b_3$ must lie on the same half-plane generated by the line supported by $b_2$. Therefore
$p \in \text{conv}(v \cup b_3)$. See Figure 4(a). Since $|E_{b_1}|, |E_{b_2}| \geq 10$, there exists vertices $x, y \in V(H)$ such that $\text{conv}(v \cup b_3), \text{conv}(x \cup b_1), \text{conv}(y \cup b_2)$ are three (vertex disjoint) strongly crossing edges in $H$ and we have a contradiction.

**Case 2.** Suppose $L$ intersects exactly three members of $S$, say $b_1, b_2, b_3$ (in order from left to right along $L$). Now $b_4$ must intersect $b_2$ either to the left or right of $b_2 \cap L$. Without loss of generality, we can assume that $b_4$ intersects $b_2$ to the right of $b_2 \cap L$. Let $p$ be the intersection point of segments $b_2$ and $b_3$. By the same argument as above, $p \in \text{conv}(v \cup b_4)$. See Figure 4(b). Since $|E_{b_1}|, |E_{b_2}| \geq 10$, there exists vertices $x, y \in V(H)$ such that $\text{conv}(v \cup b_4), \text{conv}(x \cup b_1), \text{conv}(y \cup b_2)$ are three strongly crossing edges in $H$ and we have a contradiction.

**Case 3.** Suppose $L$ intersects $b_1, b_2, b_3, b_4$ in order from left to right along $L$. Let $p$ be the intersection point of segments $b_2$ and $b_3$, and let $l$ be the vertical line through $v$. Since the right endpoint of $b_1$ lies to the right of $l$, and the left endpoint of $b_1$ lies to the left of $l$, we have $p \in \text{conv}(v \cup b_1) \cup \text{conv}(v \cup b_4)$. Therefore, either $\text{conv}(v \cup b_1)$ or $\text{conv}(v \cup b_4)$ (say $\text{conv}(v \cup b_1)$) contains $p$. See Figure 4(c). Since $|E_{b_2}|, |E_{b_3}| \geq 10$, there exists vertices $x, y \in V(H)$ such that $\text{conv}(v \cup b_1), \text{conv}(x \cup b_2), \text{conv}(y \cup b_3)$ are three strongly crossing edges in $H$ and we have a contradiction.

![Figure 4: Three cases.](image)

Therefore by Lemma 2.3, $|E(G_v)| \leq O(n)$ for every vertex $v \in V(H)$. Hence

$$\frac{|E(H)|}{4} \leq \sum_{v \in V(H)} |E(G_v)| = O(n^2),$$

which implies $|E(H)| = O(n^2)$.

Before we prove Theorem 1.3, we will need the following lemma due to Valtr [18].

**Lemma 2.3.** Let $G = (V, E)$ be an $n$-vertex geometric graph in the plane such that all of the edges in $G$ intersect the $y$-axis. If $G$ does not contain $k$ pairwise crossing edges, then $|E(G)| \leq c_k n$ where $c_k$ depends only on $k$.

**Proof of Theorem 1.3** Let $H$ be an $n$-vertex geometric 3-hypergraph in the plane with no $k$ strongly crossing edges for $k \geq 4$. Just as before, we can assume at most $|E(H)|/2$ of the edges in
$H$ are incident to a vertex that lies strictly below its base. We discard all such edges, leaving us with at least $|E(H)|/2$ edges left in $H$. Now we make the following observation.

**Observation 2.4.** Suppose $b_1, \ldots, b_k$ are $k$ pairwise crossing bases and $v_1, \ldots, v_k \in V(H)$ such that $\text{conv}(v_i \cup b_j) \in E(H)$ with base $b_j$ for all $i, j$. Then $H$ contains $k$ strongly crossing edges.

**Proof.** Let $L$ denote the top level of the segment arrangement $S = \{b_1, \ldots, b_k\}$ and assume that $b_1, \ldots, b_k$ are ordered by increasing slopes. See Figure 5.

![Figure 5: Arrangement of $b_1, b_2, b_3, b_4$.](image)

Now we define edges $t_1, t_2, \ldots, t_k \in E(H)$ as follows. Among the $k$ edges $\text{conv}(b_1 \cup v_1), \text{conv}(b_1 \cup v_2), \ldots, \text{conv}(b_1 \cup v_k) \in E(H)$, (with slight abuse of notation) let $t_1 = \text{conv}(b_1 \cup v_1)$ be the edge whose right side has the rightmost intersection with $L$. Then among the $k - 1$ edges $\text{conv}(b_2 \cup v_2), \text{conv}(b_2 \cup v_3), \ldots, \text{conv}(b_2 \cup v_k)$, (again with slight abuse of notation) let $t_2 = \text{conv}(b_2 \cup v_2)$ be the edge whose right side has the rightmost intersection with $L$. We continue this procedure until we have $k$ edges $t_1, t_2, \ldots, t_k$. Clearly these $k$ edges are vertex disjoint.

Now notice that $(t_i \cap L) \cap (t_j \cap L) \neq \emptyset$ for all pairs $i, j$. Indeed for sake of contradiction, suppose there exists two edges $t_i$ and $t_j$ for $i < j$ such that either $t_i \cap L$ lies completely to the left of $t_j \cap L$ or vice versa. See Figure 6.

![Figure 6: Assume $(t_i \cap L) \cap (t_j \cap L) = \emptyset$.](image)

**Case 1.** Suppose $t_i \cap L$ lies completely to the left of $t_j \cap L$. Then the vertical line through $v_j$ intersects the right side of $t_i$ below $v_j$. Therefore the right side of $\text{conv}(b_i \cup v_j)$ intersects $L$ more to the right than the right side of $t_i = \text{conv}(b_i \cup v_i)$ does. This contradicts the definition of $t_i$ and $t_j$. 

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Case 2. Suppose \( t_i \cap L \) lies completely to the right of \( t_j \cap L \). Then there exists a base \( b_s \) that has a point \( p \) on \( L \) between \( t_i \cap L \) and \( t_j \cap L \). Base \( b_s \) must

1. lie below \( v_i \) and \( v_j \),
2. cross \( b_i \) and \( b_j \), and
3. contain point \( p \).

However this impossible by the following argument. Let \( l \) be the vertical line through \( p \). Clearly \( l \) intersects \( b_i \) and \( b_j \). Since \( b_s \) lies below \( v_i \) and \( v_j \), \( b_s \) must intersect \( b_j \) to the left of \( l \), and intersect \( b_i \) to the right of \( l \). Since \( b_s \) intersects \( b_j \) to the left of \( l \), the slope of \( b_s \) must be greater than the slope of \( b_j \). However since the slope of \( b_i \) is less than the slope of \( b_j \), this implies that \( b_s \) cannot intersect \( b_i \) to the right of \( l \). Hence we have a contradiction.

Since \( (t_i \cap L) \cap (t_j \cap L) \neq \emptyset \) for every \( i, j \in \{1, 2, ..., k\} \), by Helly’s Theorem \([6]\) \( t_1, ..., t_k \) has a nonempty intersection on \( L \).

Notice that no \( k \) points in \( V(H) \) have \( c_k n \) bases in common. Indeed, otherwise the vertical line through any of these \( k \) points would intersect all \( c_k n \) bases, and by Lemma \([2,3]\) there would be \( k \) pairwise crossing bases. By Observation \([2,4]\) we would have \( k \) strongly crossing edges.

Now let \( G = (A \cup B, E) \) be a bipartite graph where \( A = V(H) \) and \( B = V^2(H) \), such that \((v, xy) \in E(G) \) if \( \text{conv}(x \cup y \cup v) \in E(H) \) with base \( xy \). Since \( G \) does not contain \( K_{k, c_k n} \) as a subgraph, we can use the following well known result of Kővári, Sós, Turán \([10]\).

**Theorem 2.5.** If \( G = (A \cup B, E) \) is a bipartite graph with \( |A| = n \) and \( |B| = m \) containing no subgraph \( K_{r,s} \) with the \( r \) vertices in \( A \) and the \( s \) vertices in \( B \), then

\[
|E(G)| \leq (s - 1)^{1/r} nm^{1-1/r} + (r - 1)m.
\]
By plugging in the values $m = n^2, r = k, s = c_kn$ into Theorem 2.5, we obtain

$$\frac{|E(H)|}{2} \leq |E(G)| \leq O\left(n^{3-\frac{1}{k}}\right).$$

Hence

$$|E(H)| \leq O\left(n^{3-\frac{1}{k}}\right).$$

\[\square\]

2.1 Convex geometric 3-hypergraphs

In the case when the vertices are in convex position in the plane, extremal problems on geometric 3-hypergraphs become easier due to the linear ordering of its vertices. The proof of Observation 2.4 can be copied almost verbatim to conclude the following.

**Observation 2.6.** Let $H = (V, E)$ be a geometric 3-hypergraph in the plane with vertices in convex position. Suppose $H$ contains $k$ edges of the form $t_i = \text{conv}(x_i \cup y_i \cup z_i)$, such that the vertices $(x_1, x_k, y_1, y_k, z_1, z_k)$ appear in clockwise order along the boundary of their convex hull. Then $t_1, \ldots, t_k$ are $k$ strongly crossing edges.

\[\square\]

Marcus and Klazar [9] extended the Marcus-Tardos theorem [13] by showing that the number of 1-entries in a $r$-dimensional $(0,1)$-matrix with side length $n$ which avoids an $r$-dimensional permutation matrix is $O(n^{r-1})$. As pointed out by Marcus and Klazar, it is not difficult to modify their proof to obtain an $O(n^{r-1})$ bound on the number of edges in an ordered $n$-vertex $r$-uniform hypergraph that does not contain a fixed ordered matching. Hence by Observation 2.6, we can conclude the following.

**Theorem 2.7.** Let $H = (V, E)$ be a geometric 3-hypergraph in the plane with vertices in convex position. If $H$ does not contain $k$ strongly crossing edges, then $|E(H)| \leq c_k n^2$ where $c_k$ is a constant that depends only on $k$.

\[\square\]

3 Disjoint edges in 3-space

In this section, we will prove Theorem 1.5. Recall that two edges in a geometric graph are parallel if they are the opposite edges of a convex quadrilateral. Katchalski and Last [7] and Pinchasi [16] showed that all $n$-vertex geometric graphs with more than $2n - 2$ edges contain two parallel edges. By following Pinchasi’s argument almost verbatim, one can prove the following.

**Lemma 3.1.** Let $G$ be a graph drawn on the unit sphere $S$ with vertices represented as points such that no three lie on a great circle, and edges $uv \in E(G)$ are drawn as arcs along the great circle containing points $u$ and $v$ of length less than $\pi$ (the shorter arc). We say that edges $e_1, e_2 \in E(G)$ are avoiding if the great circle supported by $e_1$ is disjoint to $e_2$, and the great circle supported by $e_2$ is disjoint from $e_1$. If $|E(G)| > 2n - 2$, then $G$ contains two avoiding edges.
Proof of Theorem 1.5. Let $H = (V, E)$ be an $n$-vertex geometric 3-hypergraph in 3-space with no two disjoint edges. Fix a pair of vertices $u, v \in V(H)$, and just consider the edges $E_{uv} = \{ t \in E(H) : u, v \text{ are vertices of } t \}$. We color $t \in E_{uv}$ red if all of the members of $E_{uv}$ lie in one of the closed half-spaces generated by the plane supported by $t$. Notice that there are at most two red edges in $E_{uv}$. Repeat this procedure for each pair of vertices, which will leave us with at most $n^2$ red edges in the end. Color the remaining edges blue, and let $d_b(v)$ denote the number of blue edges incident to $v$. Then we have

$$\sum_{v \in V(H)} d_b(v) \geq 3|E(H)| - 3n^2.$$ 

Therefore, there exists a vertex $v$ incident to at least $(3|E(H)| - 3n^2)/n$ blue edges. Now consider a small 2-dimensional sphere $S^2$ centered at $v$. Then the intersection of $S^2$ and the blue edges incident to $v$ forms a graph $G$ with at most $n$ vertices and at least $(3|E(H)| - 3n^2)/n$ edges.

If $(3|E(H)| - 3n^2)/n > 2n - 2$, then by Lemma 3.1 we know that $G$ contains two avoiding edges $xy$ and $wz$. Let $h$ be the plane supported by the blue edge $conv(w \cup z \cup v) \in E(H)$. Then the blue edge $conv(x \cup y \cup v)$ must lie in one of the closed half-spaces generated by the plane $h$. Since $conv(w \cup z \cup v)$ is blue, there must be a red edge $conv(w \cup z \cup p)$ such that $h$ separates it from $conv(x \cup y \cup v)$. Hence $conv(x \cup y \cup v)$ and $conv(w \cup z \cup p)$ are disjoint and we have a contradiction. See Figure 8. Therefore $(3|E(H)| - 3n^2)/n \leq 2n - 2$, which implies $|E(H)| \leq O(n^2)$.

![Figure 8: Disjoint edges $conv(w \cup z \cup p)$ and $conv(x \cup y \cup v)$.](image)

4 Remarks

By applying the Abstract Crossing Lemma (see [19]) to Theorem 1.2, every $n$-vertex geometric 3-hypergraph $H$ in the plane has either $O(n^2)$ edges or $\Omega(|E(H)|^{7/12})$ triples that have a point in common. In the latter case, by the fractional Helly theorem [8] this implies one can always find a point inside at least $\Omega(|E(H)|^{5/12})$ edges of $H$. However, this is not as strong as the

$$\Omega\left(\frac{|E(H)|^{3/2}}{n^6 \log^2 n}\right)$$

bound obtained by Nivasch and Sharir [14].
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