Central charge for the Schwarzschild black hole

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Abstract

Proceeding in exactly the same way as in the derivation of the temperature of a dual CFT for the extremal black hole in the Kerr/CFT correspondence, it is found that the temperature of a chiral, dual CFT for the Schwarzschild black hole is

\[ T = \frac{1}{2\pi} \]

Comparing Cardy’s formula with the Bekenstein-Hawking entropy and using \( T \), it is found that the central charge for the Schwarzschild black hole is of the form \( c = 12J_{\text{in}} \), where \( J_{\text{in}} \) is the intrinsic angular momentum of the black hole, \( J_{\text{in}} = A/(8\pi G) \). It is shown that the central charge for any four-dimensional (4D) extremal black hole is of the same form. The possible universality of this form is briefly discussed.

1 Introduction

The microscopic origin of the Bekenstein-Hawking relation between entropy and area of a black hole

\[ S_{\text{BH}} = \frac{A}{4l_p^2} \] (1)

remains a central problem in black hole physics. The relation is universal; it holds for any type of black holes in any dimension. The universality can be a key to the problem; it implies the existence of a simple underlying structure. It is now believed that universal conformal structure of a two-dimensional conformal field theory (2D CFT) can provide a solution to the problem of the Bekenstein-Hawking entropy. According to this point of view, the Bekenstein-Hawking entropy is a representation of the Cardy formula, a universal relation in statistical mechanics, which relates the entropy of the CFT to its central charge. In the last years, considerable progress has been made in reproducing the Bekenstein-Hawking entropy using the Kerr/CFT correspondence and its extensions (see reviews [1] and [2]). Specifically, according to the Kerr/CFT correspondence quantum gravity in the near-horizon extreme Kerr (NHEK) geometry is holographically dual to a chiral, left-moving \( c_L = 12J \) 2D CFT at the temperature \( T_L = 1/2\pi \), where \( J \) is the angular momentum of an extremal Kerr black hole. Then the Bekenstein-Hawking entropy \( S_{\text{BH}} = 2\pi J \) can be reproduced by the Cardy formula

\[ S_C = \frac{\pi^2}{3} c_L T_L. \] (2)
The Kerr/CFT correspondence can be further generalized to the extremal Reissner-Nordström case regarding the $U(1)$ electric charge as an angular momentum in a higher-dimensional spacetime. Unfortunately, it is not so easy to extend the Kerr/CFT correspondence to the non-extremal black holes. The problem is that away from the extremal limit the NHEK geometry disappears and the near-horizon geometry is just Rindler space. To date, however, there are no generally accepted ways to associate a conformal field theory to the Rindler space.

This problem was circumvented by using the so-called hidden conformal symmetry. This symmetry is not derived from the conformal symmetry of spacetime geometry itself, but is probed by the perturbation fields in the near horizon region. Using the hidden symmetry and the central charges $c_L$ ($c_R = c_L$) derived at extremality, it was shown that the Bekenstein-Hawking entropy of the Kerr, Reissner-Nordström (RN) and Kerr-Newman (KN) black holes can be reproduced via the Cardy formula $S_C = \frac{\pi^2}{3} (c_L T_L + c_R T_R)$ in a 2D CFT at the temperatures $T_L, T_R$.

Recently, Bertini, Cacciatori and Klemm [3] showed that the Schwarzschild black hole also enjoys a hidden conformal symmetry and might be described by a chiral CFT. Going further in this direction, Lowe and Skanata [4] assumed that the temperature of the dual CFT is $T = T_H$ and, using the Cardy formula, found that the central charge for the Schwarzschild black hole is of the form

$$c = 12r_+^3. \quad (3)$$

Despite the considerable progress many problems remain [1], [2]. The main problem is that to date there are no calculations of $c$ for the non-extremal black holes away from extremality. Therefore, it is unclear how the result (3) agrees with others, since the derivations of $c$ for the Kerr, RN and KN black holes are done at extremality, which cannot be done in the Schwarzschild case. The importance of the central charge consists, in particular, in the fact that it is proportional to the number of degrees of freedom. If a CFT is holographically dual to a black hole, this number should be proportional to the black hole area. In the extremal case all the $c$ satisfy this requirement. However, in the non-extremal case this is not so. Therefore, it is unclear how to interpret the central charge in the non-extremal case and, in particular, the dependence of $c$ on $r_+^3$ in (3) for the Schwarzschild black hole. Moreover, it is unclear how the temperature $T = T_H$ agrees with the regime of applicability of the Cardy formula, which requires the temperature $T$ to be large compared to $c$.

In this note we propose an alternative calculation of $c$ for the Schwarzschild black hole. We do not use the concept of hidden conformal symmetry. Proceeding in exactly the same way as in the derivation of $T_L$ for the extremal Kerr black hole in the Kerr/CFT correspondence, and using the concept of the intrinsic angular momentum of a black hole $J_{in}$ introduced in [5],

$$J_{in} = \frac{A}{8\pi G}, \quad (4)$$

we get

$$T_L = \frac{1}{2\pi}. \quad (5)$$

Next, comparing Cardy’s formula with the Bekenstein-Hawking entropy $S_{BH} = 2\pi J_{in}$ and
using $T_L$ we get

$$c_L = 12J_{in} = \frac{3A}{2\pi G}. \tag{6}$$

This value is the same as obtained by Carlip [6]. Long before the Kerr/CFT correspondence Carlip in his “the horizon as a boundary” approach found that in any spacetime of dimension greater than two, the subgroup of diffeomorphisms in the $(r, t)$ plane becomes a Virasoro algebra with the central charge of the form (6). Solodukhin [7], using a similar near-horizon approach, also found that the central charge is proportional to the Bekenstein-Hawking entropy.

Universality of the Bekenstein-Hawking entropy implies a similar universality of $c_L$. In turn, universality of $c_L$ follows from that of $J_{in}$. This is a key observation of the paper. Using this universality as a guide, we show explicitly that parameters of the extremal 4D black holes, being expressed in terms of $J_{in}$, take the same forms as in the Schwarzschild case. We conjecture that the problem of universality can be resolved if there is a universal chiral CFT with $c_L$ of the form (6) for any 4D black hole.

This paper is organized as follows. In section 2 we summarize without proofs the relevant material on $J_{in}$. In section 3 it is shown that the central charge for the Schwarzschild black hole is $c_L = 12J_{in}$, our main result. Section 4 shows that this formula is valid not only for the Schwarzschild black hole, but also for the 4D extremal black holes. In section 5 we discuss our conclusions and conjectures.

2 Intrinsic angular momentum

In this section we shall show that any 4D black hole is characterized not only by its mass $M$, angular momentum $J$, and electric charge $Q$ but also by its intrinsic angular momentum $J_{in}$.

The concept of $J_{in}$ was first introduced in [5]. Later, it was generalized in the works [8]-[12]. For the convenience of the reader we repeat and summarize the relevant material from the works without proofs, thus making our exposition self-contained.

2.1 Schwarzschild black hole

The Bekenstein-Hawking entropy is a concept defined in the rest frame of an external fiducial (fixed $r$) observer. From the point of view of the observer, the event horizon is the end of space. Moreover, ordinary time stands still here and effectively disappears. So does motion. Therefore, we shall regard the Euclidean formulation as more fundamental; it is an analytic continuation of that part of the Lorentzian geometry that just lies outside or at the event horizon $r \geq 2GM$. Further, since the $(r, t)$ part of the Schwarzschild metric is well approximated, around the horizon, by flat space, where all our quantities are well-defined, we shall deal only with the Rindler section of the whole Euclidean Schwarzschild manifold. I begin with the Schwarzschild metric

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{7}$$
Analytically continuing to imaginary time $t = -it_E$, where $t_E$ is a real parameter and the subscript “E” means Euclidean, we obtain the Euclidean Schwarzschild metric

$$d s^2_E = \left( 1 - \frac{2GM}{r} \right) d t^2_E + \left( 1 - \frac{2GM}{r} \right)^{-1} d r^2 + r^2 d \Omega^2. \quad (8)$$

In the near-horizon approximation it takes the Rindler form

$$d s_E \approx \rho^2 d(k t_E)^2 + d \rho^2 + \frac{1}{4k^2} d \Omega^2, \quad (9)$$

where $\rho$ is the proper distance from the horizon,

$$\rho = \int_{r_+}^r \sqrt{g_{rr}(r')} dr', \quad (10)$$

and $k$ is the surface gravity of a Schwarzschild black hole, $k = 1/4GM$. It is the product of the metric on a two-sphere of radius $2GM$ (the last term) and the metric of ($\rho, t_E$) plane

$$d s_E = \rho^2 d(k t_E)^2 + d \rho^2. \quad (11)$$

Evidently we have invariance under rotation generated by the Killing operator $\partial/\partial t_E$ in the plane and the two-sphere $r_+ = 2GM$ is an axis of rotation. The ($\rho, t_E$) plane is smooth there if $\Theta_E = k t_E$ is an angular coordinate with period $2\pi$; $t_E$ itself has then periodicity $\beta = 2\pi/k$, the inverse of the Hawking temperature. This is the geometrical origin of the universal character of the Hawking temperature.

If there is rotation associated with $\Theta_E$, then, according to quantum mechanics, there is an angular momentum component (say, the $z$th, here denoted by $\hat{J}_{in}$) conjugate to $\Theta_E$, $\hat{J}_{in} = -i\hbar \partial/\partial \Theta_E$. $\hat{J}_{in}$ is, apart from a factor, the same as the Killing operator $\partial/\partial t_E$. Below, we shall often speak of the “angular momentum”, understanding by this the eigenvalue $J_{in}$,

$$-i\hbar \frac{\partial \psi}{\partial \Theta_E} = J_{in} \psi. \quad (12)$$

There are a number of ways to calculate $J_{in}$. One heuristic way is to derive $J_{in}$ from the action integral for a black hole. According to the Euclidean path integral approach to quantum gravity all the thermodynamical properties of a black hole are completely determined by the partition function $Z$, which in turn is determined, in the near-horizon approximation, by the Euclidean action integral for a black hole $\ln Z \approx i I = -I_E$. Gibbons and Hawking [13] found that for the Schwarzschild black hole

$$I_E = \frac{1}{2} \beta M. \quad (13)$$

On the other hand, since motion of the black hole in Euclidean sector has a character of rotation, the action integral is determined by $J_{in}$,

$$I_E = \oint J_{in} d \Theta_E = 2\pi J_{in}. \quad (14)$$
Equating these two expressions, we get

\[ J_{in} = 2GM^2 = \frac{A}{8\pi G}. \]  
\[ (15) \]

We can obtain the same result more rigorously using the commutator

\[ [J_{in}, \hat{\Theta}_E] = -i\hbar, \]
\[ (16) \]

where the operator \( \hat{\Theta}_E \) is defined as \( \hat{\Theta}_E = k\hat{t}_E \). However, as Medved \[8\] noted, this approach is rather complicated because it involves the Euclidean operator \( \hat{\Theta}_E \). Conversely, Medved pointed out that \( \Theta_E \) and \( A/8\pi G \), just as \( t_E \) and \( M \), appear as canonical conjugates in the off-shell black hole action of Bunster (Teitelboim) and Carlip \[13\]

\[ I_E = -\Theta_E \left( \frac{A}{8\pi G} \right) + I_{can} - t_E M. \]
\[ (17) \]

Here all the variables are well-defined, so that the result \( (15) \) follows from the action immediately without any technical problems. Note that similarly to the derivation of Schrödinger’s equation from the variational principle, Bunster and Carlip derived from the action \( (17) \) the Schrödinger-type equation \[14\]

\[ \hbar \frac{\partial \psi}{\partial \Theta} - \frac{A}{8\pi G} \psi = 0. \]
\[ (18) \]

It is evident that this equation gives the same result. Indeed, in the semiclassical approximation

\[ \psi = a \exp \left( \frac{-i}{\hbar} I \right), \]
\[ (19) \]

where \( I \) is the action of a black hole. Substituting this in \( (18) \) we obtain

\[ \frac{\partial I}{\partial \Theta} \psi = \frac{A}{8\pi G} \psi; \]
\[ (20) \]

the slowly varying amplitude \( a \) need not be differentiated. Under Euclidean continuation \( \Theta \to -i\Theta_E \) and \( I \to iI_E \),

\[ \frac{\partial I_E}{\partial \Theta_E} \psi = \frac{A}{8\pi G} \psi. \]
\[ (21) \]

The derivative \( \partial I_E/\partial \Theta_E \) is just a generalized momentum corresponding to the angle of rotation about one of the axes (say, the \( z \)th). Therefore \( A/8\pi G \) is what corresponds in quantum mechanics to \( J_{in} \). In fact, this conclusion can be recovered even more simply: analytically continuing \( \Theta \) and \( A/8\pi G \) to the imaginary values \( \Theta \to -i\Theta_E \) and \( (A/8\pi G) \to i(A/8\pi G)_E \) with \( (A/8\pi G)_E = (A/8\pi G) \) in \( (18) \), we immediately obtain the same result.

### 2.2 Area quantization

In classical mechanics, the \( z \)-component of the angular momentum is an adiabatic invariant. \( J_{in} \) is the \( z \)-component of the angular momentum. Since \( J_{in} \) is an adiabatic invariant, then the
black hole area is also an adiabatic invariant. So is the entropy. According to the semiclassical Bohr-Sommerfeld quantization rule

\[ I_E = \oint J_m d\Theta_E = 2\pi \hbar \cdot m, \quad m = 0, 1, 2, \ldots \]  

(22)

Therefore,

\[ J_m = m\hbar. \]  

(23)

In quantum mechanics \( m \) can also take negative values related with rotation around \( z \) axis in the negative direction; these are however associated with the negative surface gravity and can be rejected. From (23) it follows that the black hole area and entropy are quantized

\[ A = \Delta A \cdot m, \]  

(24)

\[ S_{BH} = 2\pi m, \]  

(25)

where

\[ \Delta A = 8\pi l_P^2 \]  

(26)

is the quantum of area. This approach was extended to generic theories of gravity by Medved [8] and to de Sitter space - by Jia, Mao and Ren [9]. Forty years ago, by proving that the black hole horizon area is an adiabatic invariant, Bekenstein [15], [16] showed that the area spectrum of a black hole is of the form (24). But he did not use the concept of the intrinsic angular momentum.

Quantization of the black hole area is an important concept because, as believed, the quantum number \( m \) determines the number of degrees of freedom of a black hole. Moreover, this concept explains universality of the Bekenstein-Hawking relation, i.e. proportionality of the entropy and horizon area. Indeed, if the horizon surface consists of \( m \) independent patches of area \( 8\pi l_P^2 \), \( m = A/8\pi l_P^2 \), and every patch-degree of freedom has \( k \) states available to it, then the total number of states is \( k^m \) and \( S_{BH} \propto A \).

### 2.3 Regge trajectories

The Schwarzschild black hole is unstable and decays by emitting Hawking radiation. Therefore, it can be viewed as a resonance and \( J_m \) as its angular momentum. It is interesting that \( J_m \) is proportional to the square of the mass of a black hole

\[ J_m = 2GM^2. \]  

(27)

This resembles the well-known angular momentum-mass relation for hadronic resonances. As is well known, the graph of the angular momentum \( J \) of hadronic resonances against their mass squared falls into lines \( J = \alpha' M^2 \) called Regge trajectories. Instead of terminating abruptly as in the case of nuclei, the graph continue on indefinitely, implying that quarks don’t fly apart when spun too fast. In contrast to nuclei, there is no a threshold there. This is a manifestation of quark confinement. Analogously, we can assume that the black holes lie on the Regge trajectories with the slope \( \alpha' = 2G \). Since \( J_m \) increases with the square of the black hole mass without limit, the black hole area never decreases, just as the area theorem predicts. This is a manifestation of gravitational confinement.
2.4 Kerr-Newman black hole

The intrinsic angular momentum for the Kerr, RN and KN black holes can be defined in exactly the same way as in the Schwarzschild case \([5]\). Without repeating the calculations, we give the result:

\[
J_{\text{in}} \equiv \frac{A}{8\pi G} = \begin{cases} 
2GM^2, & \text{Sch} \\
GM^2 + M\sqrt{G^2M^2 - a^2}, & \text{Kerr} \\
GM^2 - Q^2/2 + M\sqrt{G^2M^2 - GQ^2}, & \text{RN} \\
GM^2 - Q^2/2 + M\sqrt{G^2M^2 - GQ^2 - a^2}, & \text{KN}.
\end{cases}
\]

where \(Q\) is the electric charge of a black hole and \(a\) is the specific angular momentum, \(a = J/M\) (\(J\) being the angular momentum).

With \(J_{\text{in}}\), the entropy of any black hole takes the form

\[
S_{\text{BH}} = 2\pi J_{\text{in}},
\]

and all terms in the first law of black hole mechanics look more uniformly,

\[
dM = kdJ_{\text{in}} + \Omega_H dJ + \Phi_H dQ.
\]

It is believed that the laws of black hole mechanics have no independent physical significance and acquire it only after identifying with the laws of thermodynamics. However, if the concept of the intrinsic angular momentum is correct, the first term in (30) can have a direct physical interpretation: it is the change in the black hole energy due to rotation in internal space. Therefore, the first law of black-hole mechanics can have a mechanical meaning.

3 Temperature, conformal weight, and central charge

In this section, we shall calculate the central charge for the Schwarzschild black hole. Following Bertini, Cacciatori and Klemm \([3]\) and also Lowe and Skanata \([4]\) we assume that the Schwarzschild black hole is nothing but a thermal state of a chiral \(c_L (c_R = 0)\) CFT. In the canonical ensemble, the entropy of the thermal state is given by the Cardy formula

\[
S_C = \frac{\pi^2}{3} c_L T_L,
\]

where \(T_L\) is the temperature of the state. The appearance of temperature means that there is periodic evolution and the background topology is a cylinder of circle \(1/T_L\), not a complex plane. Then the entropy in the microcanonical ensemble is given by the Cardy formula

\[
S_C = 2\pi \sqrt{\frac{c_L}{6} \left( L_0 - \frac{c_L}{24} \right)},
\]

where \(L_0\) is the conformal weight, the eigenvalue of the zero-mode Virasoro operator, related to the \(T_L\) by the first law of thermodynamics.

We shall calculate \(c_L\) by comparing Cardy’s formula (31) (or (32)) with the Bekenstein-Hawking entropy. For this purpose we need \(T_L\) (or \(L_0\)).
3.1 Temperature and conformal weight

It is believed that the Schwarzschild case is more difficult than others because the derivations of the central charges of a dual CFT for the Kerr, RN or KN black holes are done at extremality, which cannot be done in the Schwarzschild case \cite{2}. However, although the Schwarzschild black hole has no extremal limit, there is an analogy between the Schwarzschild and the 4D extremal black holes. This analogy is based on the fact that for the extremal just as for the Schwarzschild black hole the mass $M$ is the only parameter that completely specifies the metric. As will be demonstrated later, the analogy becomes even more striking if we use the concept of the intrinsic angular momentum $J_{\text{in}}$. In the case of an extremal black hole the temperature of a CFT, i.e. the thermodynamic potential dual to the zero mode of the Virasoro algebra, follows from the first law of black hole thermodynamics in the near horizon region \cite{17}. To calculate $T_L$ (or $L_0$) for the Schwarzschild black hole we proceed in the same way as in the case of an extremal black hole. The Hartle-Hawking vacuum state around a Schwarzschild black hole is characterized by the Boltzmann factor at the Hawking temperature $T_H$. However, in the near horizon region we have the Rindler space and the Rindler vacuum. Like the near horizon region of an extremal black hole, the Rindler space of the Schwarzschild black hole can be treated as an isolated geometry with its own thermodynamics. By analogy with thermodynamics of the near horizon region of an extremal black hole \cite{17}, the first law of thermodynamics of the Rindler space can be written in the form

$$dS_{\text{BH}} = \frac{dJ_{\text{in}}}{T_L},$$

where $J_{\text{in}}$ and $T_L$ are the energy and temperature of the Rindler space. From this we get

$$T_L = \frac{1}{2\pi}.$$  

By analogy with the Kerr/CFT correspondence, we assume that the near horizon region of the Schwarzschild black hole, the Rindler space, is holographically dual to a chiral CFT at the left-moving temperature \cite{31} which is conjugate to the zero mode of the Virasoro algebra. Similarly, the vacuum state of quantum fields in the Rindler space, the Rindler vacuum, is characterized by the Boltzmann factor of the form

$$\exp \left( -\frac{L_0}{T_L} \right),$$

where $L_0$ is the eigenvalue of the Virasoro operator, the conformal weight,

$$L_0 = J_{\text{in}}.$$  

3.2 Central charge

Comparing now the Cardy formula \cite{31} with the Bekenstein-Hawking entropy $S_{\text{BH}} = 2\pi J_{\text{in}}$ and using the temperature \cite{31}, we find

$$c_L = 12J_{\text{in}} = \frac{3A}{2\pi G}.$$  

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Comparing the Cardy formula (32) with the Bekenstein-Hawking entropy \( S_{\text{BH}} = 2\pi J_{\text{in}} \) and using the the conformal weight (36) we get the same result. By definition, the zero mode Virasoro operators are linear combinations of dilations and rotations, so that the corresponding eigenvalues are linear combinations of the ordinary ADM mass and intrinsic angular momentum:

\[
L_0 = \frac{M r_+ + J_{\text{in}}}{2}, \quad \bar{L}_0 = \frac{M r_+ - J_{\text{in}}}{2}.
\]

Since we have a \( c_L \) (\( c_R = 0 \)) chiral CFT, the shifted values are

\[
L_0 - \frac{c_L}{24} = \frac{J_{\text{in}}}{2}, \quad \bar{L}_0 - \frac{c_R}{24} = 0.
\]

4 Central charges for the extremal black holes

It turns out that with \( J_{\text{in}} \) the parameters of a dual CFT for the extremal black hole take the same forms as those for the Schwarzschild black hole. Let us consider this fact in more detail. According to the Kerr/CFT correspondence and its extensions a 4D extremal black hole is holographically dual to a chiral, left-moving half of a \( c_L = c_R \) 2D CFT at the temperature \( T_L \) (\( T_R = 0 \)) (in the case of an extremal KN black hole there are two different CFT holographically dual to the KN black hole, so that \( c_L \) is a linear combination of two limiting cases corresponding to the Kerr/CFT and RN/CFT correspondences). The parameters of the CFTs, taken from [2], are given in Table 1. Here, \( R_\chi \) is the radius of the Kaluza-Klein circle in fifth dimension.

| Table 1: Parameters of the dual CFTs (from [2]) |
|-----------------------------------------------|
| extremal Kerr | extremal RN |
| central charge, \( c_L \) | 12\( J \) | 6\( Q^3\sqrt{G}/R_\chi \) |
| temperature, \( T_L \) | 1/2\( \pi \) | \( R_\chi/2\pi\sqrt{GQ} \) |
| entropy, \( S_C \) | 2\( \pi J \) | \( \pi Q^2 \) |

In general, \( R_\chi \) is a free parameter. However, Chen et al. [18], [19] noted that

\[
R_\chi = Q\sqrt{G}
\]

is the most reasonable choice (in particular, the near horizon geometry of the five-dimensional (5D) uplifted RN black hole is then exactly \( \text{AdS}_3 \times S^2 \)). Using this choice, let us rewrite the parameters of CFTs in terms of \( J_{\text{in}} \). Since the surface gravity \( k = 0 \), and hence \( \Theta_E = 0 \), one cannot determine the extremal \( J_{\text{in}} \) as was done in the non-extremal case. However, it is still possible to determine the extremal \( J_{\text{in}} \) as a value of \( J_{\text{in}} \) at extremality. For an extremal Kerr black hole we get

\[
J_{\text{in}} \equiv \frac{A_{\text{ext}}}{8\pi G} = GM^2
\]

and the intrinsic angular momentum becomes an ordinary spin

\[
J_{\text{in}} = J.
\]
Then
\[ c_L = 12J_{\text{in}}. \]  
(43)

For an extremal RN black hole
\[ J_{\text{in}} = \frac{A_{\text{ext}}}{8\pi G} = \frac{Q^2}{2} \]  
(44)

and the central charge takes the same form \([13]\). The temperature of a dual CFT for the extremal black hole follows from the first law of black hole thermodynamics,
\[ dS_{\text{BH}} = \frac{dJ_{\text{in}}}{T_L}, \]  
(45)

and is the same in both cases
\[ T_L = \frac{1}{2\pi}. \]  
(46)

The parameters of the Schwarzschild and extremal black holes are summarized in Table 2 for comparison.

| Parameter                  | Schwarzschild | extremal Kerr | extremal RN |
|----------------------------|---------------|---------------|-------------|
| intrinsic angular momentum, \( J_{\text{in}} \) | \( 2GM^2 \)   | \( J \)       | \( Q^2/2 \) |
| central charge, \( c_L \)  | \( 12J_{\text{in}} \) | \( 12J_{\text{in}} \) | \( 12J_{\text{in}} \) |
| temperature, \( T_L \)     | \( 1/2\pi \)  | \( 1/2\pi \)  | \( 1/2\pi \) |
| entropy, \( S_C \)         | \( 2\pi J_{\text{in}} \) | \( 2\pi J_{\text{in}} \) | \( 2\pi J_{\text{in}} \) |

As mentioned above, the extremal \( J_{\text{in}} \) cannot be defined in the same way as the non-extremal one. Therefore, unlike the non-extremal case, the extremal \( J_{\text{in}} \) is not quantized as an integer multiple of \( \hbar \). Indeed, although in the case of an extremal Kerr black hole
\[ J_{\text{in}} = \frac{A_{\text{ext}}}{8\pi G} = J = mh, \]  
(47)

where \( m \) is integer or half integer; in the case of an extremal RN black hole this is not so:
\[ J_{\text{in}} = \frac{A_{\text{ext}}}{8\pi G} = \frac{Q^2}{2} \neq mh. \]  
(48)

The last result is evident: since the fine structure constant \( \alpha \) is not an even integer, \( \alpha = e^2/\hbar c \simeq 1/137 \), the quantity \( Q^2/2 \), where the electric charge \( Q \) is quantized in terms of the electron charge \( e \), is not an integer multiple of \( \hbar \). According to the theory of the renormalization group, the value of \( \alpha \) grows logarithmically as the energy scale is increased; the usual value \( \alpha = e^2/\hbar c \simeq 1/137 \) is defined as the square of the completely screened charge, that is, the value observed at infinite distance or in the limit of zero momentum transfer. What happens if we get closer to the black hole? According to the RN/CFT correspondence the near horizon region of an extremal RN black hole is holographically described by a CFT with parameters depending on the radius of the Kaluza-Klein circle in fifth dimension \( R_{\chi} \). According to the Kaluza-Klein
theory [20], the electric charge $Q$ in four dimension is proportional to the momentum of the motion round the curled up fifth dimension $P$, 

$$Q = P\sqrt{G}.$$  \hspace{1cm} (49)

Since $P$ is quantized

$$P = \frac{m\hbar}{R_x}, \quad m \in \mathbb{Z},$$  \hspace{1cm} (50)

$Q$ is also quantized

$$Q = \frac{m\hbar\sqrt{G}}{R_x}.$$  \hspace{1cm} (51)

Using the choice (40), we get

$$Q^2 = m\hbar.$$  \hspace{1cm} (52)

Therefore, (48) becomes an equality for half-integer $m$.

In the original version of the Kaluza-Klein theory the masses of charged particles are simply the momenta along the extra dimension and are quantized in units of $1/R_x$. There is however a serious difficulty with this prescription: the masses come out to be of the order of the Planck mass. This means that these particles do not correspond to the observed charged particles.

We now turn attention to the fact that the mass of any black hole is always greater than or equal to the Planck mass. Moreover, the mass of an extremal RN black hole is of the form

$$M_m = \frac{Q^2}{r_+} = \frac{2J_{in}}{r_+} = \frac{|m|\hbar}{r_+},$$  \hspace{1cm} (53)

where, according to (10), the radius of the black hole is equal to that of the extra dimension

$$r_+ = R_x.$$  \hspace{1cm} (54)

Note also that like a Kaluza-Klein particle, an extremal RN black hole satisfies the Bogomol’nyi identity

$$M = |Q|G^{-1/2}.$$  \hspace{1cm} (55)

Therefore, we expect that the above difficulty can be overcome in part if we identify the Kaluza-Klein particles with the excitations of the near horizon region of the extremal RN black holes. It is interesting that the mass of any 4D black hole (whether or not extremal) can also be written in the Kaluza-Klein form

$$M_m = \frac{J_{in}}{r_+} = \frac{|m|\hbar}{r_+}, \quad \text{(Schwarzschild, Kerr)},$$  \hspace{1cm} (56)

or

$$M_m = \frac{J_{in} + \frac{Q^2}{2}}{r_+} = \frac{|m|\hbar}{r_+}, \quad \text{(RN, KN)}.$$  \hspace{1cm} (57)
5 Discussion

In this paper we have calculated the central charge $c_L$ for the Schwarzschild black hole. We have found that $c_L = 12J_{\text{in}}$, where $J_{\text{in}}$ is the intrinsic angular momentum of the black hole, $J_{\text{in}} = A/8\pi G$. We have also found the temperature of a dual CFT $T_L = 1/2\pi$ and the conformal weights $L_0 = (Mr_+ + J_{\text{in}})/2 = J_{\text{in}}$ and $\bar{L}_0 = (Mr_+ - J_{\text{in}})/2 = 0$.

Our result means that $c$ is proportional to the black hole area, not volume. This conclusion is important because it agrees with the meaning of $c$. Generally $c$ is proportional to the number of degrees of freedom of a CFT. According to the Kerr/CFT correspondence and its extensions the degrees of freedom of a 4D black hole reside on its horizon. If the CFT is holographically dual to a black hole, that number should be proportional to the black hole area. Therefore, $c$ should be proportional to the black hole area, not volume. This disagrees with the result of Lowe and Skanata; they found that $c$ is proportional to the black hole volume [3]. The reason for this is that they used $T = T_H$ and an indirect method to calculate $c$. We also calculated $c$ in an indirect way. However, our result agrees with that of Carlip [6] and Solodukhin [7] obtained by direct calculation.

The advantage of our approach is that it does not need the concept of hidden conformal symmetry; we have calculated $T_L$ proceeding in exactly the same way as in the derivation of the temperature of a dual CFT for the extremal black hole in the Kerr/CFT correspondence. We have found that $T_L$ is nothing but the Rindler temperature $T_{\text{Rindler}} = 1/2\pi$. This result is important because it justifies the high-temperature regime of applicability of the Cardy formula. Indeed, a local proper temperature measured by a Rindler observer at distance $\rho$ from the horizon can be obtained from the Rindler temperature by using the transformation between Rindler and proper time. The local temperature is thus given by $T_{\text{local}} = 1/2\pi \rho = T_{\text{Rindler}}/\rho$ and increases as we move toward the horizon.

Although our approach does not require a detailed knowledge of a CFT and its Virasoro algebra, we expect that, contrary to previous views, this Virasoro arises not from an enhancement of the $SL(2; R)$ symmetry appearing in the scalar wave equation in the Schwarzschild background but rather from the $U(1)$ rotation isometry generated by $J_{\text{in}}$ in the near horizon Rindler space. Since we have only one $U(1)$, we have a chiral CFT.

The importance of our result is due to the fact that, as we have demonstrated above, all the CFT parameters of the extremal black holes, including $c_L$, being written in terms of $J_{\text{in}}$, take the same forms as those for the Schwarzschild black hole. The reason is that the Schwarzschild just as the extremal black hole contains $U(1)$ rotation isometry generated by $J_{\text{in}}$ in the near-horizon geometry and the mass $M$ is the only parameter that completely specifies its metric. Therefore, the form of $c_L$ is, at least in part, universal. This can help in understanding universality of the Bekenstein-Hawking relation.

Can one extend this universality to the non-extremal Kerr, RN and KN black holes? The existence of holographically dual CFTs to non-extremal black holes is highly conjectural [2]. To date there are no calculations of $c_L$ for these black holes away from extremality; their Bekenstein-Hawking entropies are reproduced via the Cardy formula in CFTs with central charges derived at extremality. On the other hand, the metric of any non-extremal black hole can be reduced to the Rindler form with the same $U(1)$ isometry generated by $J_{\text{in}}$. Since the
entropy of any black hole (whether or not extremal) depends only on the intrinsic angular momentum $S_{BH} = 2\pi J_{in}$, we can define the chemical potential $1/T_L = dS_{BH}/dJ_{in}$. Using universality as a guide, it is natural to conjecture that any 4D black hole is dual to a chiral CFT at the temperature $T_L = 1/2\pi$. We hope that future investigation will help to support or falsify our conjecture.

**Note Added in Proof**

The central charges for the Schwarzschild and other 4D black holes can be obtained in more simple way if we assume that there exists a close relation between the central charge and the parameters of the $AdS_2$ component of the near horizon geometry. The near horizon geometry of all extreme black holes has an $AdS_2$ factor. Chen et al [21] noted that the central charges for the extremal Kerr and RN black holes satisfy the relation $c = 6\ell^2/G$, where $\ell$ is the $AdS_2$ radius of the near horizon geometry. Indeed, in the case of the extremal Kerr black hole $\ell^2 = 2GJ$ and $c = 12J = 12J_{in}$. In the case of the extremal RN black hole $\ell^2 = GQ^2$ and $c = 6Q^2 = 12J_{in}$. The above relation is valid also for the extremal KN black hole. In this case $\ell^2 = r_+^2 + a^2$ and $c = 12J_{in}$, where $J_{in} = A/8\pi G$. It turns out that the near horizon geometry of the Schwarzschild black hole also has an $AdS_2$ component. Bertini et al [3] showed that the Schwarzschild solution can be mapped into the near horizon limit $AdS_2 \times S^2$ of the extremal RN black hole by a Kinnersley transformation. In this case $\ell^2 = 4G^2M^2$. If the above relation is assumed valid for the Schwarzschild black hole, we have immediately $c = 3A/2\pi G = 12J_{in}$.

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