The minimum covering signless laplacian energy of graph

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Abstract.
Gutman [5] has come out with the idea of graph energy as summation of numerical value of latent roots of the adjacency matrix of the given graph Γ. In this paper, we introduce the Minimum Covering Signless Laplacian energy $L^+_C E(Γ)$ of a graph Γ and obtain bounds for it. Also we find $L^+_C E(Γ)$ of some important class of graphs.

Keywords: Minimum covering set, Minimum covering signless Laplacian matrix, Minimum covering signless Laplacian eigen values, Minimum covering signless Laplacian energy.

1. Introduction
Gutman [5] has come out with the idea of graph energy as summation of numerical values of the latent roots of the adjacency matrix of the given graph Γ. Complete π-electron energy plays a significant role for a molecule in Chemical science. Likewise one can find distance energy, maximum degree energy, color energy etc. in [9, 3, 4]. We consider graphs which are simple, directionless, no loops and no multiple edges. For standard definitions and terminology regarding graph theory, we refer [8].

2. Graph Signless Laplacian Energy
Abreu et al [1] have defined graph signless Laplacian energy as

$$L^+(Γ) = \sum_{i=1}^{p} \left| \mu_i^+ - \frac{2q}{p} \right|$$

where $\mu_i^+$ is the eigen values of signless Laplacian matrix and $\frac{2q}{p}$ is the average degree.
The basic properties and signless Laplacian energy bounds are found in [1].

3. Graph Minimum Covering Signless Laplacian Energy
In [2] Adiga et al have come out with graph minimum covering energy as

\[
E_C(\Gamma) = \sum_{i=1}^{p} |\lambda_i|
\]

Motivated by graph minimum covering energy, we describe the graph minimum covering signless Laplacian energy \(L^+_C E(\Gamma)\).

Let \(L^+_C(\Gamma) = D(\Gamma) + A_c(\Gamma)\) be the minimum covering signless Laplacian matrix of \(\Gamma\) with \(D(\Gamma)\), the matrix whose diagonal elements are the vertex degrees. Then we have

\[
L^+_C E(\Gamma) = \sum_{i=1}^{p} \left| \zeta^+_i - \frac{2q}{p} \right|
\]

where \(\zeta^+_i, i = 1, 2, \ldots, p\), are the eigen values of \(L^+_C(\Gamma)\) and \(\frac{2q}{p}\) is the average degree.

**Definition 3.1.** The graph spectrum \(L^+_C(\Gamma)\) is described as arrangement of distinct eigen values \(\zeta^+_1 > \zeta^+_2 > \cdots > \zeta^+_r\) with multiplicities \(m_1, m_2, \ldots, m_r\), written as

\[
L^+_C Spec(\Gamma) = \left( \begin{array}{cccc}
\zeta^+_1 & \zeta^+_2 & \cdots & \zeta^+_r \\
m_1 & m_2 & \cdots & m_r
\end{array} \right).
\]

4. Properties of Graph Minimum Covering Signless Laplacian Energy
**Theorem 4.1.** If \(C\) is the graph minimum covering set and \(\zeta^+_1 > \zeta^+_2 > \cdots > \zeta^+_r\), the eigen values of \(L^+_C(\Gamma)\) then

(i) \(\sum_{i=1}^{p} \zeta^+_i = 2|E| + |C|\) and

(ii) \(\sum_{i=1}^{p} \zeta^+_i^2 = 2|E| + \sum_{i=1}^{p} (d_i + c_i)^2\), where \(c_i = \left\{ \begin{array}{ll} 1 & \text{if } v_i \in C \\ 0 & \text{if } v_i \notin C \end{array} \right.\)

**Proof.** (i) Here from the matrix of \(L^+_C(\Gamma)\) the total of principal diagonal elements is equivalent to \(\sum_{i=1}^{p} d_i + |C| = twice the number of edges + |C|\).

As trace is equivalent to total of determinants of 1x1 principal submatrices of \(L^+_C(\Gamma)\).
Hence \(\sum_{i=1}^{p} \zeta^+_i = 2|E| + |C|\).

(ii) As trace of \([L^+_C(\Gamma)]^2\) is the total of squares of eigen values of \(L^+_C(\Gamma)\)
\[
\sum_{i=1}^{p} \zeta^+_i^2 = \sum_{i=1}^{p} \sum_{j=1}^{p} a_{ij} a_{ji}
\]
2 \sum_{i<j} (a_{ij})^2 + \sum_{i=1}^{p} (a_{ii})^2 = 2|E| + \sum_{i=1}^{p} (d_i + c_i)^2, \text{where } c_i = \begin{cases} 1 & \text{if } v_i \in C \\ 0 & \text{if } v_i \notin C \end{cases}

The bounds for $L_C^+E(\Gamma)$ can be found using McClelland's inequalities[11].

Theorem 4.2. If $C$ is the graph minimum covering set then we have the upper bound as

$$L_C^+E(\Gamma) \leq \sqrt{p \left[ 2q + \sum_{i=1}^{p} (d_i + c_i)^2 \right]}$$

Proof. Let the eigen values of $L_C^+(\Gamma)$ be $\zeta_1^+ \geq \zeta_2^+ \geq \cdots \geq \zeta_p^+$ then by using Cauchy-Schwarz inequality we have,

$$\left[ \sum_{i=1}^{p} r_i s_i \right]^2 \leq \left[ \sum_{i=1}^{p} r_i^2 \right] \left[ \sum_{i=1}^{p} s_i^2 \right]$$

Choose $r_i = 1, s_i = |\zeta_i^+|$ and by Theorem 4.1 we have

$$\left[ \sum_{i=1}^{p} |\zeta_i^+| \right]^2 = [L_C^+E(\Gamma)]^2, \left[ \sum_{i=1}^{p} r_i^2 \right] = p$$

Consequently

$$[L_C^+E(\Gamma)]^2 \leq p \left[ 2q + \sum_{i=1}^{p} (d_i + c_i)^2 \right]$$

Hence

$$L_C^+E(\Gamma) \leq \sqrt{p \left[ 2q + \sum_{i=1}^{p} (d_i + c_i)^2 \right]}$$

Theorem 4.3. For the graph minimum covering set $C$ and $D = |\det L_C^+(\Gamma)|$ the lower bound is given as

$$L_C^+E(\Gamma) \geq \sqrt{2q + \sum_{i=1}^{p} (d_i + c_i)^2 + (p^2 - p)D^2}$$
Proof. $L_C^+E(\Gamma) = \left(\sum_{i=1}^{p} |\zeta_i^+|^2\right) = \left(\sum_{i=1}^{p} |\zeta_i^+|\right) \left(\sum_{j=1}^{p} |\zeta_j^+|\right)$

$$= \sum_{i=1}^{p} |\zeta_i^+|^2 + \sum_{i \neq j} |\zeta_i^+||\zeta_j^+|$$

As average of a set of products is not greater than average value we possess

$$\frac{1}{(p^2-p)} \sum_{i \neq j} |\zeta_i^+||\zeta_j^+| \geq \left(\prod_{i \neq j} |\zeta_i^+||\zeta_j^+|\right) \frac{1}{(p^2-p)}$$

Therefore

$$[L_C^+E(\Gamma)]^2 \geq \sum_{i=1}^{p} |\zeta_i^+|^2 + (p^2-p) \left(\prod_{i \neq j} |\zeta_i^+||\zeta_j^+|\right) \frac{1}{(p^2-p)}$$

$$\geq \sum_{i=1}^{p} |\zeta_i^+|^2 + (p^2-p) \left(\prod_{i=1}^{p} |\zeta_i^+|^{2(p-2)}\right) \frac{1}{(p^2-p)}$$

$$= 2q + \sum_{i=1}^{p} (d_i + c_i)^2 + (p^2-p)D_{\overline{\pi}}^2$$

Hence

$$L_C^+E(\Gamma) \geq \sqrt{2q + \sum_{i=1}^{p} (d_i + c_i)^2 + (p^2-p)D_{\overline{\pi}}^2}$$

5. Minimum Covering Signless Laplacian Energy of familiar Graphs

Theorem 5.1. For the complete graph $K_p$ with $p \geq 2$ we have

$$L_C^+E(K_p) = \sqrt{(p+3)(p-1)}.$$

Proof. Let $K_p$ with $p \geq 2$ be the complete graph. Let $V = \{v_1, v_2, \cdots, v_p\}$ be the set of vertices of $K_p$ and $C = \{v_1, v_2, \cdots, v_{p-1}\}$ be the minimum covering set then

$$L_C^+(K_p) = \begin{bmatrix} p & 1 & 1 & \cdots & 1 & 1 \\ 1 & p & 1 & \cdots & 1 & 1 \\ 1 & 1 & p & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & p & 1 \\ 1 & 1 & 1 & \cdots & 1 & p-1 \end{bmatrix}.$$
The characteristic equation is

\[ \lambda^2 - (3p - 3)\lambda + (2p^2 - 5p + 3)|\lambda - (p - 1)|^{p-2} = 0. \]

\[ L^+_{C_{spec}}(K_p) = \left( \begin{array}{c|c}
\frac{(3p-3)-\sqrt{p^2-2p-3}}{2} & \frac{(3p-3)+\sqrt{p^2-2p-3}}{2} \\
\hline
1 & p - 1
\end{array} \right). \]

The minimum covering signless Laplacian energy is

\[ L^+_{C_{E}}(K_p) = \left| \frac{(3p-3)-\sqrt{p^2-2p-3}}{2} \right| (1) + \left| \frac{(3p-3)+\sqrt{p^2-2p-3}}{2} \right| (1) \]

Therefore we get

\[ L^+_{C_{E}}(K_p) = \sqrt{(p + 3)(p - 1)} \]

\[ \Box \]

Theorem 5.2. For the star graph \( K_{1,p-1} \) we have

\[ L^+_{C_{E}}(K_{1,p-1}) = \sqrt{p^2 + 2p - 3} + \frac{(-2 + p)^2}{p} \]

Proof. Let \( K_{1,p-1} \) be the star graph. Let \( V = \{v_1, v_2, \ldots, v_p\} \) be the set of vertices of \( K_{1,p-1} \) and \( C = \{v_1\} \) be the minimum covering set then

\[ L^+_{C}(K_{1,p-1}) = \begin{bmatrix}
p & 1 & 1 & \ldots & 1 & 1 \\
1 & 1 & 0 & \ldots & 0 & 0 \\
1 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \ldots & 1 & 0 \\
1 & 0 & 0 & \ldots & 0 & 1
\end{bmatrix}. \]

From the matrix

\[ \lambda^2 - (p + 1)\lambda + 1 \right| \lambda^{-2+p} = 0. \]

Therefore the spectrum is

\[ L^+_{C_{spec}}(K_{1,p-1}) = \left( \begin{array}{c|c}
\frac{(p+1)-\sqrt{p^2+2p-3}}{2} & \frac{(p+1)+\sqrt{p^2+2p-3}}{2} \\
\hline
1 & -2 + p
\end{array} \right). \]

The minimum covering signless Laplacian energy is

\[ L^+_{C_{E}}(K_{1,p-1}) = \left| \frac{(p+1)-\sqrt{p^2+2p-3}}{2} \right| (1) + \left| \frac{(p+1)+\sqrt{p^2+2p-3}}{2} \right| (1) \]

\[ + \left| 1 - \frac{(2p-2)}{p} \right| (-2 + p) \]
Therefore we get

\[ L^+_C E(K_{1,p-1}) = \sqrt{p^2 + 2p - 3} + \frac{(-2 + p)^2}{p} \]

\[ \square \]

**Theorem 5.3.** For the crown graph \( S^0_p \) we have

\[ L^+_C E(S^0_p) = \sqrt{4p^2 - 8p + 5} + \sqrt{5}(p - 1) \]

**Proof.** Let \( S^0_p \) be the crown graph. Let \( V = \{u_i, v_i\} \) and \( C = \{u_i\}, i = 1, 2, \ldots, p \) be respectively the set of vertices of \( S^0_p \) of order \( 2p \) and the minimum covering set then

\[
L^+_C(S^0_p) = \begin{bmatrix}
    p & 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 1 & 1 \\
    0 & p & 0 & \ldots & 0 & 1 & 0 & \ldots & 1 & 1 \\
    0 & 0 & p & \ldots & 0 & 1 & 1 & \ldots & 1 & 1 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \ldots & p & 1 & 1 & \ldots & 1 & 0 \\
    0 & 1 & 1 & \ldots & 1 & p-1 & 0 & \ldots & 0 & 0 \\
    1 & 0 & 1 & \ldots & 1 & 0 & p-1 & \ldots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    1 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & p-1 & 0 \\
    1 & 1 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & p-1 \\
\end{bmatrix}
\]

The characteristic equation is

\[
[\lambda^2 - (2p-1)\lambda + (p^2 - p - 1)][\lambda^2 - (2p-1)\lambda + (p - 1)]^{p-1} = 0.
\]

Therefore the spectrum is

\[
L^+_C spec(S^0_p) = \left( \begin{array}{cccc}
    \frac{(2p-1)-\sqrt{5}}{2} & \frac{(2p-1)+\sqrt{5}}{2} & \frac{(2p-1)-\sqrt{4p^2-8p+5}}{2} & \frac{(2p-1)+\sqrt{4p^2-8p+5}}{2} \\
    p-1 & p-1 & 1 & 1
\end{array} \right).
\]

The minimum covering signless Laplacian energy is

\[
L^+_C E(S^0_p) = \left| \frac{(2p-1)-\sqrt{5}}{2} - (p-1) \right| (p-1) + \left| \frac{(2p-1)+\sqrt{5}}{2} - (p-1) \right| (p-1) + \left| \frac{(2p-1)-\sqrt{4p^2-8p+5}}{2} - (p-1) \right| (1) + \left| \frac{(2p-1)+\sqrt{4p^2-8p+5}}{2} - (p-1) \right| (1)
\]

Therefore we get

\[ L^+_C E(S^0_p) = \sqrt{4p^2 - 8p + 5} + \sqrt{5}(p - 1). \]

\[ \square \]
Theorem 5.4. For the cocktail party graph $K_{p \times 2}$ we have

$$L_C^+ E(K_{p \times 2}) = \sqrt{4p^2 + 4p - 7} + (-3 + 2p).$$

Proof. Let $K_{p \times 2}$ be the cocktail party graph. Let $V = \{u_i, v_i\}, i = 1, 2, \ldots, p$ be the set of vertices of $K_{p \times 2}$ and $C = \{u_i, v_i\}, i = 1, 2, \ldots, p-1$ be the minimum covering set then

$$L_C^+ (K_{p \times 2}) = \begin{bmatrix}
2p - 1 & 1 & 1 & \ldots & 1 & 0 & 1 & 1 & \ldots & 1 \\
1 & 2p - 1 & 1 & \ldots & 1 & 1 & 0 & 1 & \ldots & 1 \\
1 & 1 & 2p - 1 & \ldots & 1 & 1 & 0 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 1 & \ldots & 1 & 2p - 1 & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 \\
1 & 1 & 1 & \ldots & 0 & 1 & 1 & 1 & \ldots & 2(p-1)
\end{bmatrix}$$

The characteristic equation is

$$[\lambda - (-3+2p)]^{-2+p}[\lambda - (-2+p)][\lambda - (-1+2p)]^{-1+p}[\lambda^2 - (6p-7)\lambda + (8p^2 - 22p + 14)] = 0$$

Therefore the spectrum is

$$L_C^+ spec(K_{p \times 2}) = \left(\begin{array}{cccc}
-3+2p & -2+2p & -1+2p & (6p-7)-\sqrt{4p^2+4p-7} \\
-2+p & 1 & -1+p & \frac{(6p-7)+\sqrt{4p^2+4p-7}}{2}
\end{array}\right).$$

Hence

$$L_C^+ E(K_{p \times 2}) = \left| (-3+2p) - (-2+2p) \right| (-2+p) + \left| (-2+2p) - (-2+2p) \right| (1)$$

$$+ \left| (-1+2p) - (-2+2p) \right| (-1+p) + \left| \left(\frac{6p-7}{2}-\sqrt{4p^2+4p-7}\right) - (-2+2p) \right| (1)$$

$$+ \left| \left(\frac{6p-7}{2}+\sqrt{4p^2+4p-7}\right) - (-2+2p) \right| (1)$$

Therefore we get

$$L_C^+ E(K_{p \times 2}) = \sqrt{4p^2 + 4p - 7} + (-3 + 2p).$$

\qed
Theorem 5.5. For the complete bipartite graph $K_{q,p}$ with $q \geq p$,

$$L_C^+E(K_{q,p}) = \frac{(p-q)(pq-p)}{p+q} + \left| \frac{p+q^2 - qp + q}{p+q} \right| (1-p) + \sqrt{(p+1+q)^2 - 4p}$$

Proof. Let $K_{q,p}$ be the complete bipartite graph. Let $V = \{u_i, v_i\}$, $C = \{v_i\}$, respectively be the set of vertices of $K_{q,p}$ and the minimum covering set where $i=1, 2, ..., p$ then

$$L_C^+(K_{q,p}) = \begin{bmatrix} p & 0 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 \\ 0 & p & 0 & 0 & \ldots & 1 & 1 & 1 & 1 \\ 0 & 0 & p & 0 & \ldots & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & p & \ldots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & q+1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \ldots & 0 & q+1 & 0 & 0 \\ 1 & 1 & 1 & 1 & \ldots & 0 & 0 & q+1 & 0 \\ 1 & 1 & 1 & 1 & \ldots & 0 & 0 & 0 & q+1 \\ \end{bmatrix}.$$ 

The characteristic equation is

$$(\lambda - p)^{-1+q}[\lambda - (q+1)^{-1+p}][\lambda^2 - (p+1+q)\lambda + p] = 0$$

Therefore the spectrum is

$$L_C^{+}\text{spec}(K_{q,p}) = \left( \begin{array}{c} p \\ -1+q \\ -1+p \\ \frac{(p+1+q)\sqrt{(p+1+q)^2-4p}}{2} \\ 1 \\ \frac{(p+1+q)\sqrt{(p+1+q)^2-4p}}{2} \\ 1 \\ \frac{(p+1+q)\sqrt{(p+1+q)^2-4p}}{2} \\ 1 \\ \frac{(p+1+q)\sqrt{(p+1+q)^2-4p}}{2} \end{array} \right).$$

The minimum covering signless Laplacian energy is

$$L_C^+E(K_{q,p}) = p - \frac{2pq}{q+p} \left| (-1+q) + (q+1) - \frac{2pq}{q+p} \right| (1-p) + \left| \frac{(p+1+q)\sqrt{(p+1+q)^2-4p}}{2} - \frac{2pq}{q+p} \right| (1)$$

Therefore for $q \geq p$ we get

$$L_C^+E(K_{q,p}) = \frac{(p-q)(pq-p)}{p+q} + \left| \frac{p+q^2 - qp + q}{p+q} \right| (1-p) + \sqrt{(p+1+q)^2 - 4p}$$

Similarly for the double star graph $S_{p,p}$,

$$L_C^+E(S_{p,p}) = \sqrt{p^2 + 6p - 3} + \frac{2p^2 - 6p + 4}{p} + \sqrt{p^2 + 2p - 3}.$$
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