ON THE GENERALIZED VOLUME CONJECTURE AND REGULATOR

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Abstract. In this paper, by using the regulator map of Beilinson-Deligne on a curve, we show that the quantization condition posed by Gukov is true for the $SL_2(C)$ character variety of the hyperbolic knot in $S^3$. Furthermore, we prove that the corresponding $C^*$-valued closed 1-form is a secondary characteristic class (Chern-Simons) arising from the vanishing first Chern class of the flat line bundle over the smooth part of the character variety, where the flat line bundle is the pullback of the universal Heisenberg line bundle over $C^* \times C^*$. Based on this result, we give a reformulation of Gukov’s generalized volume conjecture from a motivic perspective.

1. Introduction

It is a very important question in knot theory to find the geometric and topological interpretation of the Jones polynomial of a knot. Observed by Kashaev [Kas], H. Murakami and J. Murakami [MM], the asymptotic rate of (N-colored) Jones polynomial is related to the volume of the hyperbolic knot complement. It is known as the Volume conjecture. Following Witten’s $SU(2)$ topological quantum field theory, Gukov [Guk] proposed a complex version of Chern-Simons theory and generalized the volume conjecture to a $C^*$-parametrized version with parameter lying on the zero locus of the $A$-polynomial of the knot in $S^3$.

In this paper, we prove that the quantization condition posed by Gukov [Guk, Page 597] is true for hyperbolic knots in $S^3$. The key ingredient of the proof is the construction of the regulator map of an algebraic curve studied by Beilinson, Bloch, Deligne and many others ([Bei, Bl, De, Ram]). Let $K$ be a hyperbolic knot in $S^3$. For each irreducible component $Y$ of the zero locus of the $A$-polynomial $A(l, m)$ of $K$, we show that the symbol $\{l, m\} \in K_2(C(Y))$ is a torsion. Associated to $\{l, m\}$, there is a cohomology class $r(l, m)$ in $H^1(Y_h, C^*)$, where $Y_h$ is some open Riemann surface. The detail is given in Section 3. As Deligne noted, $H^1(Y_h, C^*)$ is the group of isomorphism classes of flat line bundles over $Y_h$. Thus, our class $r(l, m)$ corresponds to a flat line bundle. Moreover, the line bundle $r(l, m)$ can be constructed explicitly as the pullback of the universal Heisenberg line bundle over $C^* \times C^*$, see [Bl, Ram]. We then derive a closed 1-form from it and show that this 1-form is the Chern-Simons class of the first Chern class $C_1$ of $r(l, m)$. Note that our Chern-Simons class is not the usual Chern-Simons class as a closed 3-form of the second Chern class for a 3-dimensional manifold. We also reformulate the generalized volume conjecture via this closed 1-form Chern-Simons class. Our proof is motivic in nature.

The paper is organized as follows. In section 2, we introduce the notations used in the paper. In section 3, we discuss the generalized volume conjecture and the regulator of a curve, then we prove the theorem about the quantization condition and give a reformulation of the generalized volume conjecture. In the end, we remark the motivic aspect of the proof.
Acknowledgements. We would like to thank Professor Alexander Goncharov and Professor Xiao-Song Lin for their helpful comments and suggestions on an earlier version of the paper.

2. Terminology and Notation

2.1. Let $K$ be a knot in $S^3$ and $M_K$ its complement. That is, $M_K = S^3 - N_K$ where $N_K$ is the open tubular neighborhood of $K$ in $S^3$. $M_K$ is a compact 3-manifold with boundary $\partial M_K = T^2$ a torus. Denote by $R(M_K) = \text{Hom}(\pi_1(M_K), SL_2(\mathbb{C}))$ and $R(\partial M_K) = \text{Hom}(\pi_1(\partial M_K), SL_2(\mathbb{C}))$. It is known that they are affine algebraic sets over the complex numbers $\mathbb{C}$ and so are the corresponding character varieties $X(M_K)$ and $X(\partial M_K)$ (See [CS]). We also have the canonical surjective morphisms $t: R(M_K) \to X(M_K)$ and $t: R(\partial M_K) \to X(\partial M_K)$ which map a representation to its character. The natural homomorphism $i: \pi_1(\partial M_K) \to \pi_1(M_K)$ induces the restriction maps $r: X(M_K) \to X(\partial M_K)$ and $r: R(M_K) \to R(\partial M_K)$.

2.2. Since $\pi_1(\partial M_K) = \mathbb{Z} \oplus \mathbb{Z}$, we shall fix two oriented simple curves $\mu$ and $\lambda$ as its generators. They are called the meridian and longitude respectively. Let $R_D$ be the subvariety of $R(\partial M_K)$ consisting of the diagonal representations. Then $R_D$ is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. Indeed, for $\rho \in R_D$, we obtain

$$\rho(\lambda) = \begin{bmatrix} l & 0 \\ 0 & l^{-1} \end{bmatrix} \text{ and } \rho(\mu) = \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix},$$

then we assign the pair $(l, m)$ to $\rho$. Clearly this is an isomorphism. We shall denote by $t_D$ the restriction of the morphism $t: R(\partial M_K) \to X(\partial M_K)$ on $R_D$.

2.3. Next we recall the definition of the $A$-polynomial of $K$ which was introduced in [CCGLS]. Denote by $X'(M_K)$ the union of the irreducible components $Y'$ of $X(M_K)$ such that the closure $r(Y')$ in $X(\partial M_K)$ is 1-dimensional. For each component $Z'$ of $X'(M_K)$, denote by $Z$ the curve $t_D^{-1}(r(Y')) \subset R_D$. We define $D_K$ to be the union of the curves $Z$ as $Z'$ varies over all components of $X'(M_K)$. Via the above identification of $R_D$ with $\mathbb{C}^* \times \mathbb{C}^*$, $D_K$ is a curve in $\mathbb{C}^* \times \mathbb{C}^*$. Now by definition the $A$-polynomial $A(l, m)$ of $K$ is the defining polynomial of the closure of $D_K$ in $\mathbb{C} \times \mathbb{C}$.

From now on, we shall assume that $K$ is a hyperbolic knot. Denote by $\rho_0: \pi_1(M_K) \to PSL_2(\mathbb{C})$ the discrete, faithful representation corresponding to the hyperbolic structure on $M_K$. Note that $\rho_0$ can be lifted to a $SL_2(\mathbb{C})$ representation. Moreover, there are exactly $|H^1(M_K; \mathbb{Z}_2)| = 2$ such lifts.

3. $A$-Polynomial, Regulator and $K_2$ of a Curve

In this section, we briefly recall Gukov’s formulation of the generalized volume conjecture. Using the regulator map of a curve, we show that the form $r(l, m) = \xi(l, m) + in(l, m)$ over the 1-dimensional character variety $Y_h$ has exact imaginary part and rational real part. This provides an affirmative answer to Gukov’s quantization over $Y_h$. Moreover, $dr(l, m) = \frac{dl}{l} \wedge \frac{dm}{m} = 2\pi iC_1(L) = 0$ justifies that $r(l, m)$ is the Chern-Simons class from the first Chern class $C_1$. 
3.1. Let \( \overline{D_K} \) be the zero locus of the \( A \)-polynomial \( A(l, m) \) in \( \mathbb{C}^2 \). Let \( y_0 \in \overline{D_K} \) correspond to the character of the representation of the hyperbolic structure on \( M_K \) and \( m(y_0) = 1 \). For a path \( c \) in \( \overline{D_K} \) with the initial point \( y_0 \) and endpoint \( (l, m) \), the following quantities are defined in [Guk, (5.2)]:

\[
\text{(3.1)} \quad V ol(l, m) = V ol(K) + 2 \int_c \left[ -\log |l| \, d(\arg m) + \log |m| \, d(\arg l) \right],
\]

\[
\text{(3.2)} \quad CS(l, m) = CS(K) - \frac{1}{\pi} \int_c \left[ \log |m| \, d \log |l| + (\arg l) \, d(\arg m) \right],
\]

where \( V ol(K) \) and \( CS(K) \) are the volume and the Chern-Simons invariant of the complete hyperbolic metric on \( M_K \).

In [Guk, (5.12)], Gukov proposes his **Generalized Volume Conjecture**: for a fixed number \( a \) and \( m = -\exp(i\pi a) \),

\[
\text{(3.3)} \quad \lim_{N,k \to \infty; \frac{N}{k} = a} \frac{\log J_N(K, e^{2\pi i/k})}{k} = \frac{1}{2\pi} (V ol(l, m) + i2\pi^2 CS(l, m)),
\]

where \( J_N(K, q) \) is the \( N \)-colored Jones polynomial of \( K \), \( V ol(l, m) \) and \( CS(l, m) \) as in (3.1) and (3.2), are the functions on the zero locus of the \( A \)-polynomial of the hyperbolic knot \( K \). Note that for \( m = 1 \) and \( a = 1 \), we get the usual **Volume Conjecture**:

\[
\text{(3.4)} \quad \lim_{N \to \infty} \frac{\log |J_N(K, e^{2\pi i/N})|}{N} = \frac{1}{2\pi} V ol(K).
\]

Gukov’s generalized volume conjecture links the Jones invariants of knots with the topological and geometric invariants arising from the character variety of the knot complement. It has received lot of attention. But other than the verification of a few examples, there is no essential mathematical evidence to support this interesting conjecture. And there appeared some confusions in the study of this conjecture (see [Mu], [Gu-Mu]). Understanding those terms in Gukov’s generalized volume conjecture (3.3) would be the first step.

**Remark 1.** In [Mu], Murakami reformulated a parametrized volume conjecture where \( 2\pi i \) is replaced by \( 2\pi i + u \). He conjectured that

\[
H(K, u) = (2\pi i + u) \lim_{N \to \infty} \frac{\log(J_N(K, e^{2\pi i + u}/N))}{N}
\]

is analytic on some open subset of \( \mathbb{C} \) and the volume function \( V(K, u) \) is given by

\[
V(K, u) = \text{Im}(H(K, u)) - \text{Re}(u) \cdot \text{Im}( \frac{dH(K, u)}{du} ).
\]

This parametrization is different from Gukov’s original one \((m = \exp(u), l = \exp(2\frac{dH}{du} - 2\pi i))\). It comes from a different choice of polarization when \( \text{Re}(u) \neq 0 \). Throughout this paper, we shall keep the same polarization as Gukov’s.

For \( V ol(l, m) \) in (3.1), it is understood that it measures the change of volumes of the representations on the path \( c \) in \( \overline{D_K} \), the zero locus of \( A \)-polynomial of the hyperbolic knot. See [CCGLS, Sect. 4.5] and [Dun, Sect. 2] for more detail.
For $CS(l, m)$ in (3.2), to our knowledge, it has not been understood mathematically. It was derived from the point of view of physics, see [Guk, Sect. 3]. On the other hand, since $M_K$ has boundary a torus $T$, by [RSW] and [KK], its 3-form Chern-Simons functional is only well-defined as a section of a circle bundle over the gauge equivalence classes of $T$. By [KK, Theorem 3.2, 2.7], if $\chi_t, t \in [0, 1]$ is a path of characters of $SL_2(\mathbb{C})$ representations of $M_K$ and $z(t)$ is the Chern-Simons invariant of $\chi_t$, then:

\begin{equation}
z(1)z(0)^{-1} = \exp(2\pi i \int_0^1 \alpha \frac{d\beta}{dt} - \beta \frac{d\alpha}{dt}) = \exp\left(\frac{1}{2\pi i} \int_0^1 \left(\log m \, d\log l - \log l \, d\log m\right)\right)
\end{equation}

where $(\alpha(t), \beta(t))$ is a lift of $\chi_t$ to $\mathbb{C}^2$, under the $(l, m)$ coordinates, $\alpha = \frac{1}{2\pi i} \log m$ and $\beta = \frac{1}{2\pi i} \log l$ for a fixed branch of logarithm.

It is clear that (3.5) and (3.2) are not the same. Moreover, the Chern-Simons 3-form in [KK] is the secondary class from the second Chern class (a closed 4-form). The term in (3.2) defined in [Guk, (5.6)] is a 1-form. It may be the secondary class of the first Chern class (a closed 2-form) of some line bundle over $D_K$.

In the following subsections, we show that $dCS(l, m)$ indeed arises from the first Chern class of a (universal) line bundle over the Heisenberg group. Furthermore, we relate both $dVol$ and $dCS$ to the imaginary and real parts of the secondary Chern-Simons class respectively. We also give a mathematical proof of Gukov’s quantization statement of the Bohr-Sommerfeld condition by using some torsion element of $K_2$ and the regulator map.

### 3.2. The regulator map of $K_2$

Let $X$ be a smooth projective curve over $\mathbb{C}$ or a compact Riemann surface. Let $f, g$ be two meromorphic functions on $X$. Denote by $S(f)$ (resp. $S(g)$) the set of zeros and poles of $f$ (resp. $g$). Notice that $S(f) \cup S(g)$ is a finite set. Put $X' = X \setminus (S(f) \cup S(g))$.

Following Beilinson [Bei], see also [De], we define an element $r(f, g) \in H^1(X'; \mathbb{C}^*)$, equivalently, as an element of $\text{Hom}(\pi_1(X'), \mathbb{C}^*)$: for a loop $\gamma$ in $X'$ with a distinguished base point $t_0 \in X'$,

\begin{equation}
r(f, g)(\gamma) = \exp \left(\frac{1}{2\pi i} \int_{\gamma} \log f \, \frac{dg}{g} - \log g(t_0) \int_{\gamma} \frac{df}{f}\right),
\end{equation}

where the integrals are taken over $\gamma$ beginning at $t_0$.

It is well-known that this definition is independent of the choices of the base point $t_0$ and the branches of $\log f$ and $\log g$. From now on, we shall take $\log z : \mathbb{C}^* \to \mathbb{C}$ with $0 \leq \arg z < 2\pi$. Then it is well-defined, but discontinuous on the positive real line $[0, +\infty)$ and it is holomorphic on the cut plane $\mathbb{C} \setminus [0, +\infty)$.

In [De], Deligne noticed that $H^1(X'; \mathbb{C}^*)$ is the group of isomorphism classes of the line bundles over $X'$ with flat connections. Hence $r(f, g)$ corresponds to such a line bundle with a flat connection.

**Proposition 3.1.** (1) The curvature of the line bundle associated to the class $r(f, g)$ is $\frac{df}{f} \wedge \frac{dg}{g}$. 


Let \( \{l, m\} \in K_2(\mathbb{C}(Y)) \) be a torsion element.

**Proof.** By \([\text{CCGLS}], \text{Proposition 2.2, 4.1}\), there is a finite field extension \( F \) of \( \mathbb{C}(Y) \) such that \( \{l, m\} \in K_2(F) \) is of order at most 2. We have a homomorphism \( i : K_2(\mathbb{C}(Y)) \to K_2(F) \) induced by the inclusion of \( \mathbb{C}(Y) \) into \( F \). We also have the transfer map \( t : K_2(F) \to K_2(\mathbb{C}(Y)) \). It is well-known that the composition \( t \circ i \):

\[
K_2(\mathbb{C}(Y)) \to K_2(F) \to K_2(\mathbb{C}(Y))
\]

is given by the multiplication of \( n = [F : \mathbb{C}(Y)] \), the degree of the finite extension. Hence \( t(i(\{l, m\}) = t(\{l, m\}) = n\{l, m\} \). This implies that \( \{l, m\} \in K_2(\mathbb{C}(Y)) \) is a torsion and its order divides \( 2n \).

Suppose the component \( Y \) contains \( y_0 \in \tilde{D}_K \) which corresponds to the discrete faithful character \( \chi_0 \) of the hyperbolic structure and \( m(y_0) = 1 \). Let \( S(l, m) \) be the finite set of poles and zeros of \( l \) and \( m \). Put \( Y_h = \tilde{Y} \setminus S(l, m) \) as the \( X' \) in \S 3.2. We choose the distinguished point \( t_0 \) as follows. If \( y_0 \) is a smooth point, we take \( t_0 = y_0 \); if \( y_0 \) is a singular point, we fix a
point in the pre-images of \( y_0 \) in \( \tilde{Y} \) and take \( t_0 \) as this fixed point. This is equivalent to fixing a branch at the singular point \( y_0 \).

**Theorem 3.3.** (i) The closed real 1-form \( \eta(l, m) = \log |l| \, d \arg m - \log |m| \, d \arg l \) is exact on \( Y_h \);

(ii) For any loop \( \gamma \) with initial point \( t_0 = \chi_0 \) in \( Y_h \)

\[
\frac{1}{4\pi^2} \int_\gamma \left( \log |m| \, d \log |l| + \arg l \, d \arg m \right) = \frac{p}{q},
\]

where \( p \) is some integer and \( q \) is the order of the symbol \( \{l, m\} \) in \( K_2(\mathbb{C}(Y)) \).

**Proof.** By (3.4), we have an element \( r(l, m) \in H^1(Y_h, \mathbb{C}^*) \). By Proposition 3.2, it is a torsion of order \( q \). By the definition of \( r(l, m) \) in (3.6), we conclude that for any loop \( \gamma \) in \( Y_h \),

\[
\{\exp \left( \frac{1}{2\pi i} \left( \int_\gamma \log l \, \frac{dm}{m} - \log m(t_0) \int_\gamma \frac{dl}{l} \right) \right)\}^q = 1
\]

Write \( \int_\gamma \log l \, \frac{dm}{m} - \log m(t_0) \int_\gamma \frac{dl}{l} = \text{Re} + i\text{Im} \), where \( \text{Re} \) and \( \text{Im} \) are the real and imaginary parts respectively. (3.8) means that \( \exp \left( \frac{q \cdot \text{Im}}{2\pi i} + \frac{q \cdot \text{Re}}{2\pi i} \right) = 1 \). Therefore, \( \text{Im} = 0 \) and \( \frac{q \cdot \text{Re}}{2\pi i} = 2\pi ip \), for some integer \( p \). Our result follows from the following lemma. \(\square\)

**Lemma 3.4.** Denote \( \int_\gamma \log l \, \frac{dm}{m} - \log m(t_0) \int_\gamma \frac{dl}{l} = \text{Re} + i\text{Im} \) as above. Then

\[
\text{Im} = \int_\gamma \left( \log |l| \, d \arg m - \log |m| \, d \arg l \right) = \int_\gamma \eta(l, m),
\]

and

\[
\text{Re} = -\int_\gamma \left( \log |m| \, d \log |l| + \arg l \, d \arg m \right) = \int_\gamma \xi(l, m),
\]

where \( \xi(l, m) \) depends on the branches of \( \arg \) function and \( \text{Re} \) is well-defined up to \( (2\pi)^2 \mathbb{Z} \).

**Proof.** Let \( F \) be a smooth non-zero complex-valued function, and \( F = \text{Re}(F) + i\text{Im}(F) \), where \( \text{Re}(F) \) denotes its real part and \( \text{Im}(F) \) its imaginary part. Then we have

\[
d \log F := \frac{dF}{F} = \frac{d|F|}{|F|} + i\frac{\text{Re}(F)d\text{Im}(F) - \text{Im}(F)d\text{Re}(F)}{|F|^2}.
\]

So the real part of \( d \log F \) is \( d \log |F| \) which is exact and the imaginary part is denoted by \( d \arg F \).

By a straightforward calculation, we have

\[
\text{Im} = \int_\gamma \left( \log |l| \, d \arg m + \arg l \, d \log |m| \right) - \log |m(t_0)| \int_\gamma d \arg l.
\]

Integration by parts, we obtain:

\[
\int_\gamma \arg l \cdot d \log |m| = \log |m(t_0)| \int_\gamma d \arg l - \int_\gamma \log |m| \cdot d \arg l.
\]

Therefore,

\[
\text{Im} = \int_\gamma \left( \log |l| \, d \arg m - \log |m| \, d \arg l \right).
\]
For the real part $Re$, it is equal to
\[ \int_{\gamma} (\log |l| \, d \log |m| - \arg l \, d \arg m) + \arg m(t_0) \int_{\gamma} d \arg l. \]

Integration by parts, we get
\[ \int_{\gamma} \log |l| \, d \log |m| = - \int_{\gamma} \log |m| \, d \log |l|. \]

By the choice of $t_0 = \chi_{\rho_0}$, $\arg m(t_0) = 0$. Hence the result follows. \[
\]

Remark 2. (i) The first part of the theorem was also proved in [CCGLS, Sect. 4.2]. Our proof gives the real and imaginary part simultaneously.

(ii) The result of the second part is stronger than the one in [Guk, 3.29] where he derived that the value of the integral is in $\mathbb{Q}$ from the quantizable Bohr-Sommerfield condition.

(iii) The class $r(l, m) \in H^1(Y_h; \mathbb{C}^*)$ corresponds to a flat line bundle $L$ over $Y_h$ which is the pullback of the universal Heisenberg line bundle on $\mathbb{C}^* \times \mathbb{C}^*$, see [Bl, Ram]. Formally,
\[ d(\xi(l, m) + i\eta(l, m)) = dl \wedge dm = 0. \]

Hence, \( \frac{1}{2\pi i}(\xi(l, m) + i\eta(l, m)) \) is the 1-form Chern-Simons. Denote it by $CS_1(l, m)$. Then $dCS_1(l, m) = C_1(L) = \frac{1}{2\pi i} \frac{dl}{l} \wedge \frac{dm}{m} = 0$.

By Theorem 3.3 and Lemma 3.4, we would like to propose the corresponding generalized volume conjecture as the following:

For a path $c : [0, 1] \to Y_h$ with $c(0) = t_0$ and $c(1) = (l, m)$, write $c(t) = (l(t), m(t))$. Recall that $q$ is the order of the symbol \( \{l, m\} \) in $K_2(\mathbb{C}(Y))$. We denote
\[ U(l, m) = -q \cdot \int_c [\log |m(t)| \, d \log |l(t)| + \arg l(t) \, d \arg m(t)]. \]

and call it the special Chern-Simons invariant of $K$.

For a fixed number $a$ and $m = -\exp(i\pi a)$, we reformulate the generalized volume conjecture as the following:

**Conjecture:** (The reformulated generalized volume conjecture)

\[
\lim_{N,k \to \infty} \frac{\log J_N(K, e^{2\pi i/k})}{k} = \frac{1}{2\pi} (Vol(l, m) + i \frac{1}{2\pi} U(l, m)).
\]

Remark 3. By Theorem 3.3 (ii), \( \frac{1}{(2\pi i)^2} U(l, m) \) is well-defined in $\mathbb{R}/\mathbb{Z}$. The classical Chern-Simons invariant is also well-defined in $\mathbb{R}/\mathbb{Z}$.

Our reformulated generalization gives a $\mathbb{C}^*$-parametrized version of the volume conjecture. Using Fuglede-Kadison determinant, W. Zhang and the first author [LZ] defined an $L^2$-version twisted Alexander polynomial which can be identified with an $L^2$-Reidemeister torsion. By Luck and Schick’s result, this $L^2$-Alexander polynomial provides a $\mathbb{C}^*$-parametrization of the hyperbolic volumes. For other discussions on the volume conjecture, see [Oh, Sections 1.3, 7.3] and the related references within.
Remark 4. In \[\text{GuMu}\], Gukov and Murakami showed that the difference of their conjectures (in \[\text{Guk}\] and \[\text{Mu}\] respectively) comes from the different choice of polarization. One choice leads to Gukov’s, the other gives Murakami’s. Our generalization \[3.4\] only focuses on Gukov’s original one in \[\text{Guk}\]. Moreover, it indicates that the special Chern-Simons term \[U(l, m)\] comes from the regulator over the character variety. It would be interesting to extend our reformulated generalized conjecture \[3.4\] to other polarizations.

3.4. A Motivic Perspective. In \[\text{Gon}\], A. Goncharov proved that the volume of a hyperbolic 3-manifold is the period of a mixed Tate motive, and he gave the explicit construction of this motive there. The Hopf algebra \[\mathcal{H}\] of framed mixed Tate motives was also defined there, for the detail, see \[\text{Gon}\] and the references therein.

For our case, the equation \[3.1\] can be thought of as the variation formula of the volume on the deformation space \(X_0\) of hyperbolic structures on \(M_K\), see \[\text{Th1, Hod, CCGLS}\]. On the motivic level, when we deform the hyperbolic structures on \(M_K\), we can think of it as the variation of the corresponding mixed motives. Professor A. Goncharov pointed out to the second-name author that in this picture, the equation \[3.1\] is related to the coproduct of the Hopf algebra \(\mathcal{H}\). This indicates an interesting link for a future study.

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