Behavior of Solutions of a Fourth Order Difference Equation

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ABSTRACT. In this paper, we introduce an explicit formula for the solutions and discuss the global behavior of solutions of the difference equation

\[ x_{n+1} = \frac{ax_{n-3}}{b - cx_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots \]

where \(a, b, c\) are positive real numbers and the initial conditions \(x_0, x_1, x_2, x_3\) are real numbers.

1. Introduction

Difference equations have played an important role in analysis of mathematical models of biology, physics and engineering. Recently, there has been a great interest in studying properties of nonlinear and rational difference equations. One can see [3, 7, 9, 10, 11, 12, 13, 14, 16, 17] and the references therein.

In [8], E.M. Elsayed determined the solutions to some difference equations. He obtained the solution to the difference equation

\[ x_{n+1} = \frac{x_{n-3}}{1 - x_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots \]  

where the initial conditions are arbitrary nonzero positive real numbers. But he didn’t point to any constraints on the initial conditions.

In fact, if we start with initial conditions \(x_0 = 2, x_1 = 1, x_2 = 1, x_3 = 0.5\) in equation (1.1), then undefined value for \(x_3\) will be obtained. Therefore, additional information about the initial conditions must be given for any solution of equation (1.1) to be well-defined.

In [4], M. Aloqeili discussed the stability properties and semicycle behavior of the solutions of the difference equation

\[ x_{n+1} = \frac{x_{n-1}}{a - x_nx_{n-1}}, \quad n = 0, 1, \ldots \]

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with real initial conditions and positive real number \( a \).

In [1], we have discussed the oscillation, boundedness and the global behavior of all admissible solutions of the difference equation

\[
x_{n+1} = \frac{Ax_{n-1}}{B - Cx_nx_{n-2}}, \quad n = 0, 1, \ldots
\]

where \( A, B, C \) are positive real numbers.

In [2], we have also discussed the oscillation, periodicity, boundedness and the global behavior of all admissible solutions of the difference equation

\[
x_{n+1} = \frac{Ax_{n-2r-1}}{B - C \prod_{i=1}^{k} x_{n-2i}}, \quad n = 0, 1, \ldots
\]

where \( A, B, C \) are positive real numbers.

In [5], the authors investigated the asymptotic behavior of solutions of the equation

\[
x_{n+1} = \frac{ax_{n-1}}{b + cx_nx_{n-1}}, \quad n = 0, 1, \ldots
\]

with positive parameters \( a \) and \( c \), negative parameter \( b \) and nonnegative initial conditions.

In [6], they also used the explicit formula for the solutions of the equation

\[
x_{n+1} = \frac{ax_{n-1}}{b + cx_nx_{n-1}}, \quad n = 0, 1, \ldots
\]

with positive parameters and nonnegative initial conditions in investigating their behavior.

In [15], H. Sedaghat determined the global behavior of all solutions of the rational difference equations

\[
x_{n+1} = \frac{ax_{n-1}}{x_nx_{n-1} + b}, \quad x_{n+1} = \frac{ax_n x_{n-1}}{x_n + bx_{n-2}}, \quad n = 0, 1, \ldots
\]

where \( a, b > 0 \).

In this paper, we introduce an explicit formula and discuss the global behavior of solutions of the difference equation

\[
(1.2) \quad x_{n+1} = \frac{ax_{n-3}}{b - cx_{n-2}x_{n-3}}, \quad n = 0, 1, \ldots
\]

where \( a, b, c \) are positive real numbers and the initial conditions \( x_{-3}, x_{-2}, x_{-1}, x_0 \) are real numbers.

2. Solution of Equation (1.2)

We define \( \alpha_i = x_{-2+i}x_{-4+i}, \quad i = 1, 2 \).
Theorem 2.1. Let \( x_{-3}, x_{-2}, x_{-1} \) and \( x_0 \) be real numbers such that for any \( i \in \{1, 2\}, \alpha_i \neq \sum_{k=1}^b x_k \forall n \in \mathbb{N} \). If \( a \neq b \), then the solution \( \{x_n\}_{n=-3}^{\infty} \) of equation (1.2) is

\[
x_n = \begin{cases} 
  x_{-3} \prod_{j=0}^{n-3} \left( \frac{b}{a} \right)^{2j} \theta_1 - c, & n = 1, 5, 9, \ldots \\
  x_{-2} \prod_{j=0}^{n-2} \left( \frac{b}{a} \right)^{2j} \theta_2 - c, & n = 2, 6, 10, \ldots \\
  x_{-1} \prod_{j=0}^{n-1} \left( \frac{b}{a} \right)^{2j} \theta_3 - c, & n = 3, 7, 11, \ldots \\
  x_0 \prod_{j=0}^{n-4} \left( \frac{b}{a} \right)^{2j} \theta_4 - c, & n = 4, 8, 12, \ldots 
\end{cases}
\]

where \( \theta_i = \frac{a-b+c\alpha_i}{\alpha_i} \), \( \alpha_i = x_{-2+i}x_{-4+i} \), and \( i = 1, 2 \).

Proof. We can write the given solution as

\[
x_{4m+1} = x_{-3} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+1} \theta_1 - c, \quad x_{4m+2} = x_{-2} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+2} \theta_2 - c, \\
x_{4m+3} = x_{-1} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+1} \theta_3 - c, \quad x_{4m+4} = x_{0} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+2} \theta_4 - c, \quad m = 0, 1, \ldots
\]

It is easy to check the result when \( m = 0 \). Suppose that the result is true for \( m > 0 \). Then

\[
x_{4(m+1)+1} = \frac{ax_{4m+1}}{b - cx_{4m+1}x_{4m+3}} = \frac{ax_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j} \theta_1 - c}{b - cx_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+2} \theta_2 - c} = \frac{ax_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j} \theta_1 - c}{b - cx_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+2} \theta_2 - c}
\]

\[
= \frac{ax_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j} \theta_1 - c}{b - cx_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+2} \theta_2 - c} = \frac{ax_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j} \theta_1 - c}{b - cx_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+2} \theta_2 - c} = \frac{ax_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j} \theta_1 - c}{b - cx_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+2} \theta_2 - c}
\]

\[
= \frac{ax_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+2} \theta_2 - c}{b - cx_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+2} \theta_2 - c} = \frac{ax_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+2} \theta_2 - c}{b - cx_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+2} \theta_2 - c} = \frac{ax_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+2} \theta_2 - c}{b - cx_{4m} \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+2} \theta_2 - c}
\]
This completes the proof.

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Assume that

\[ \alpha = \frac{b-a}{c} \]

\[ \in \{ \text{integer values} \} \]

\[ x_{n+1} = \frac{a x_n - 3 \left( \frac{b}{a} \right)^{2m+2 \theta_1} - c}{b \left( \frac{b}{a} \right)^{2m+2 \theta_1} - c} \]

Using the explicit formula of its solution.

This completes the proof.

3. Global Behavior of Equation (1.2)

In this section, we investigate the global behavior of equation (1.2) with \( a \neq b \), using the explicit formula of its solution.

We can write the solution of equation (1.2) as

\[ x_{m+1} = x_0 \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+1} \theta_2 - c \]

Similarly we can show that

\[ x_{4(m+1)+2} = x_{4(m+1)+3} \]

\[ x_{4(m+1)+4} = x_0 \prod_{j=0}^{m} \left( \frac{b}{a} \right)^{2j+2} \theta_2 - c \]

This completes the proof.

\[ \square \]

Theorem 3.1. Let \( \{x_n\}_{n=-3}^{\infty} \) be a solution of equation (1.2) such that for any \( i \in \{1, 2\} \), \( \alpha_i \neq -\frac{b}{c} \sum_{j=0}^{\infty} \left( \frac{b}{a} \right)^j \) for all \( n \in \mathbb{N} \). If \( \alpha_i = \frac{b-a}{c} \) for all \( i \in \{1, 2\} \), then \( \{x_n\}_{n=-3}^{\infty} \) is periodic with prime period 4.

Proof. Assume that \( \alpha_i = \frac{b-a}{c} \) for all \( i \in \{1, 2\} \). Then \( \theta_i = 0 \) for all \( i \in \{1, 2\} \).

Therefore,

\[ x_{m+1} = x_{-4+2t+i} \prod_{j=0}^{m} \zeta(j, t, i) = x_{-4+2t+i}, \quad m = 0, 1, \ldots \]

This completes the proof. \( \square \)
Theorem 3.2. Let \( \{x_n\}_{n=-3}^{\infty} \) be a solution of equation (1.2) such that for any \( i \in \{1, 2\} \), \( \alpha_i \neq \sum_{k=0}^{n} \frac{1}{x_k} \) for all \( n \in \mathbb{N} \). Then the following statements are true.

1. If \( a < b \), then \( \{x_n\}_{n=-3}^{\infty} \) converges to 0.

2. If \( a > b \), then \( \{x_n\}_{n=-3}^{\infty} \) converges to a period-4 solution.

Proof.

1. If \( a < b \), then \( \zeta(j, t, i) \) converges to \( \frac{2}{3} < 1 \) as \( j \to \infty \), for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \). So, for every pair \( (t, i) \in \{0, 1\} \times \{1, 2\} \) we have for a given \( 0 < \epsilon < 1 \) that, there exists \( j_0(t, i) \in \mathbb{N} \) such that, \( |\zeta(j, t, i)| < \epsilon \) for all \( j \geq j_0(t, i) \). If we set \( j_0 = \max_{0 \leq t \leq 1, 1 \leq i \leq 2} j_0(t, i) \), then for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \) we get

\[
|x_{4m+2t+i}| = |x_{-4+2t+i}|| \prod_{j=0}^{m} \zeta(j, t, i)|
\]

\[
= |x_{-4+2t+i}|| \prod_{j=j_0}^{j_0-1} \zeta(j, t, i)|| \prod_{j=j_0}^{m} \zeta(j, t, i)|
\]

\[
< |x_{-4+2t+i}|| \prod_{j=0}^{j_0-1} \zeta(j, t, i)| \epsilon^{m-j_0+1}.
\]

As \( m \) tends to infinity, the solution \( \{x_n\}_{n=-3}^{\infty} \) converges to 0.

2. If \( a > b \), then \( \zeta(j, t, i) \to 1 \) as \( j \to \infty \), \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \). This implies that, for every pair \( (t, i) \in \{0, 1\} \times \{1, 2\} \), there exists \( j_1(t, i) \in \mathbb{N} \) such that \( \zeta(j, t, i) > 0 \) for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \). If we set \( j_1 = \max_{0 \leq t \leq 1, 1 \leq i \leq 2} j_1(t, i) \), then for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \) we get

\[
x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \zeta(j, t, i)
\]

\[
= x_{-4+2t+i} \prod_{j=0}^{j_1-1} \zeta(j, t, i) \exp(\sum_{j=j_1}^{m} \ln(\zeta(j, t, i))).
\]

We shall test the convergence of the series \( \sum_{j=j_1}^{\infty} |\ln(\zeta(j, t, i))| \).

Since for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \) we have \( \lim_{j \to \infty} |\frac{\ln(\zeta(j+1, t, i))}{\ln(\zeta(j, t, i))}| = 0 \), using L’Hospital’s rule we obtain

\[
\lim_{j \to \infty} |\frac{\ln(\zeta(j+1, t, i))}{\ln(\zeta(j, t, i))}| = \left(\frac{b}{a}\right)^2 < 1.
\]
It follows from the ratio test that the series \( \sum_{j=0}^{\infty} | \ln \zeta(j, t, i) | \) is convergent. This ensures that there are four positive real numbers \( \mu_{ti}, t \in \{0, 1\} \) and \( i \in \{1, 2\} \) such that

\[
\lim_{m \to \infty} x_{4m+2t+i} = \mu_{ti}, \quad t \in \{0, 1\} \quad \text{and} \quad i \in \{1, 2\}
\]

where

\[
\mu_{ti} = x_{-4+2t+i} \prod_{j=0}^{\infty} \left( \frac{b}{a} \right)^{2j+t+1} \theta_i - c, \quad t \in \{0, 1\} \quad \text{and} \quad i \in \{1, 2\}.
\]

This completes the proof. \( \square \)

**Example (1)** Figure 1. shows that if \( a = 2, b = 3, c = 1 \) \((a < b)\), then the solution \( \{x_n\}_{n=-3}^{\infty} \) of equation (1.2) with initial conditions \( x_{-3} = 0.2, x_{-2} = 2, x_{-1} = -2 \) and \( x_0 = 0.4 \) converges to 0.

**Example (2)** Figure 2. shows that if \( a = 3, b = 1, c = 0.8 \) \((a > b)\), then the solution \( \{x_n\}_{n=-3}^{\infty} \) of equation (1.2) with initial conditions \( x_{-3} = 0.2, x_{-2} = 2, x_{-1} = -2 \) and \( x_0 = 0.4 \) converges to a period-4 solution.

**Figure 1:** \( x_{n+1} = \frac{2x_{n-3}}{3-x_{n-1}x_{n-3}} \)

**Figure 2:** \( x_{n+1} = \frac{3x_{n-3}}{1-0.8x_{n-1}x_{n-3}} \)

4. **Case** \( a = b \)

In this section, we investigate the behavior of the solution of the difference equation

\[
x_{n+1} = \frac{ax_{n-3}}{a - cx_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots
\]
Theorem 4.1. Let $x_{-3}, x_{-2}, x_{-1}$ and $x_0$ be real numbers such that for any $i \in \{1,2\}$, $\alpha_i \neq \frac{a}{r(n+1)}$ for all $n \in \mathbb{N}$. Then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (3.1) is

\[
\begin{cases}
  x_{n} = \\
  \quad x_{-3} \prod_{j=0}^{n-1} \frac{a-(2j+1)\alpha_1}{a-(2j+1)\alpha_1} & , n = 1, 5, 9, ...
  x_{-2} \prod_{j=0}^{n-2} \frac{a-(2j)\alpha_2}{a-(2j)\alpha_2} & , n = 2, 6, 10, ...
  x_{-1} \prod_{j=0}^{n-1} \frac{a-(2j)\alpha_2}{a-(2j-1)\alpha_2} & , n = 3, 7, 11, ...
  x_0 \prod_{j=0}^{n-2} \frac{a-(2j+1)\alpha_1}{a-(2j+2)\alpha_2} & , n = 4, 8, 12, ...
\end{cases}
\]  

(3.2)

Proof. The proof is similar to that of Theorem (2.1) and will be omitted. \qed

We can write the solution of equation (3.1) as

\[
x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \gamma(j, t, i),
\]

where $\gamma(j, t, i) = \frac{a-(2j+t)\alpha_1}{a-(2j+t+1)\alpha_1}$, $t \in \{0,1\}$ and $i \in \{1,2\}$.

Theorem 4.2. Let $\{x_n\}_{n=-3}^{\infty}$ be a nontrivial solution of equation (3.1) such that for any $i \in \{1,2\}$, $\alpha_i \neq \frac{a}{r(n+1)}$ for all $n \in \mathbb{N}$. If $\alpha_1 = 0$ for all $i \in \{1,2\}$, then $\{x_n\}_{n=-3}^{\infty}$ is periodic with prime period 4.

Proof. Assume that $\alpha_1 = 0$ for all $i \in \{1,2\}$. Then $\gamma(j, t, i) = 1$ for all $t \in \{0,1\}$ and $i \in \{1,2\}$. Therefore,

\[
x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \gamma(j, t, i) = x_{-4+2t+i}, \quad m = 0, 1, ...
\]

This completes the proof. \qed

In the following Theorem, suppose that $\alpha_i \neq 0$ for all $i \in \{1,2\}$.

Theorem 4.3. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of equation (3.1) such that for any $i \in \{1,2\}$, $\alpha_i \neq \frac{a}{r(n+1)}$ for all $n \in \mathbb{N}$. Then $\{x_n\}_{n=-3}^{\infty}$ converges to 0.

Proof. It is clear that $\gamma(j, t, i) \rightarrow 1$ as $j \rightarrow \infty$, $t \in \{0,1\}$ and $i \in \{1,2\}$. This implies that, for every pair $(t, i) \in \{0,1\} \times \{1,2\}$ there exists $j_2(t, i) \in \mathbb{N}$ such that, $\gamma(j, t, i) > 0$ for all $t \in \{0,1\}$ and $i \in \{1,2\}$. If we set $j_2 = \max_{0 \leq t \leq 1, 1 \leq i \leq 2} j_2(t, i)$, then for all $t \in \{0,1\}$ and $i \in \{1,2\}$ we get

\[
x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \gamma(j, t, i)
\]

\[
= x_{-4+2t+i} \prod_{j=0}^{j_2-1} \gamma(j, t, i) \exp \left(- \sum_{j=j_2}^{m} \frac{1}{\gamma(j, t, i)} \right). 
\]
We shall show that $\sum_{j=j_2}^{\infty} \frac{1}{\gamma(j,t,i)} = \sum_{j=j_2}^{\infty} \frac{a-(2j+1)\alpha_i}{a-(2j+t)\alpha_i} = \infty$, by considering the series $\sum_{j=j_2}^{\infty} \frac{-\alpha_j}{a-(2j+t)\alpha_i}$. As

$$\lim_{j \to \infty} \frac{\ln(1/\gamma(j,t,i))}{-\alpha_j/(a-(2j+t)\alpha_i)} = \lim_{j \to \infty} \frac{\ln((a-(2j+t+1)\alpha_i)/(a-(2j+t)\alpha_i))}{-\alpha_j/(a-(2j+t)\alpha_i)} = 1,$$

using the limit comparison test, we get $\sum_{j=j_2}^{\infty} \frac{-\alpha_j}{a-(2j+t)\alpha_i} = \infty$. Then

$$x_{4m+2t+i} = x_{4+2t+i} \prod_{j=0}^{j_2-1} \frac{\gamma(j,t,i)}{\gamma(j+1,t,i)} \exp \left( -\sum_{j=j_2}^{m} \frac{1}{\gamma(j,t,i)} \right)$$

converges to 0 as $m \to \infty$. Therefore, $\{x_n\}_{n=-3}^{\infty}$ converges to 0. \hfill \Box

5. Case $a = b = c$

In this section, we investigate the behavior of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1-x_{n-1}x_{n-3}}, \quad n = 0,1,\ldots$$

**Theorem 5.1.** Let $x_{-3}, x_{-2}, x_{-1}$ and $x_0$ be real numbers such that for any $i \in \{1,2\}$, $\alpha_i \neq \frac{1}{n+1}$ for all $n \in \mathbb{N}$. Then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (3.3) is

$$x_n = \begin{cases} 
  x_{-3} \prod_{j=0}^{n-1} \frac{1-(2j)\alpha_1}{1-(2j+1)\alpha_1}, & n = 1,5,9,\ldots \\
  x_{-2} \prod_{j=0}^{n-2} \frac{1-(2j)\alpha_2}{1-(2j+1)\alpha_2}, & n = 2,6,10,\ldots \\
  x_{-1} \prod_{j=0}^{n-1} \frac{1-(2j+1)\alpha_1}{1-(2j+2)\alpha_2}, & n = 3,7,11,\ldots \\
  x_0 \prod_{j=0}^{n} \frac{1-(2j+1)\alpha_2}{1-(2j+2)\alpha_2}, & n = 4,8,12,\ldots 
\end{cases}$$

**Proof.** The proof is similar to that of Theorem (2.1) and will be omitted. \hfill \Box

**Theorem 5.2.** Let $\{x_n\}_{n=-3}^{\infty}$ be a nontrivial solution of equation (3.3) such that for any $i \in \{1,2\}$, $\alpha_i \neq \frac{1}{n+1}$ for all $n \in \mathbb{N}$. If $\alpha_i = 0$ for all $i \in \{1,2\}$, then $\{x_n\}_{n=-3}^{\infty}$ is periodic with prime period 4.

**Proof.** Assume that $\alpha_i = 0$ for all $i \in \{1,2\}$. Then

$$x_{4m+2t+i} = x_{4+2t+i}, \quad m = 0,1,\ldots$$

This completes the proof. \hfill \Box

In the following Theorem, suppose that $\alpha_i \neq 0$ for all $i \in \{1,2\}$.

**Theorem 5.3.** Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of equation (3.3) such that for any $i \in \{1,2\}$, $\alpha_i \neq \frac{1}{n+1}$ for all $n \in \mathbb{N}$. Then $\{x_n\}_{n=-3}^{\infty}$ converges to 0.
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Figure 3: $x_{n+1} = \frac{x_{n-3}}{1 - 1.5x_{n-1}x_{n-3}}$

Figure 4: $x_{n+1} = \frac{x_{n-3}}{1 - x_{n-1}x_{n-3}}$

Example (3) Figure 3 shows that if $a = b = 1, c = 1.5$, then the solution $\{x_n\}_{n=-3}^\infty$ of equation (3.1) with initial conditions $x_{-3} = 5, x_{-2} = -1, x_{-1} = 1.3$ and $x_0 = -1.1$ converges to 0.

Example (4) Figure 4 shows that if $a = b = c$, then the solution $\{x_n\}_{n=-3}^\infty$ of equation (3.3) with initial conditions $x_{-3} = 5, x_{-2} = 1, x_{-1} = 1.3$ and $x_0 = -1.1$ converges to 0.

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