QUANTUM COHOMOLOGY AND CONFORMAL BLOCKS ON $\overline{M}_{0,n}$

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Abstract. We give several necessary and sufficient conditions for conformal blocks divisors on $\overline{M}_{0,n}$ to be nonzero. We show that conformal block divisors in type A satisfy new symmetries when levels and ranks are interchanged in non-standard ways. Higher level conformal block divisors are made accessible using techniques from quantum cohomology and representation theory. Several examples are given, and relations to Hassett spaces are obtained.

1. Introduction

Conformal blocks bundles give a link between the geometry of $\overline{M}_{0,n}$ and representation theory. The first Chern classes of these bundles, computed by Fakhruddin [Fak12], called conformal blocks divisors, represent fundamentally new non-negative objects in representation theory, evoking memories of other important non-negative quantities such as intersection numbers of Schubert varieties, and tensor product multiplicities.

We develop the theory of conformal blocks divisors in higher levels (where explicit formulas are hard to use), producing new and unexpected symmetries, vanishing and nonvanishing results, using a variety of different methods such as the quantum cohomology of Grassmannians, and the geometric theory of conformal blocks (as generalized theta functions). Most fundamental is the problem of when a given conformal blocks divisor is nontrivial (Thms 1.11 and 7.1). We use work on conformal blocks divisors in relation to the birational geometry of $\overline{M}_{0,n}$ [Fak12], and [AGS10, AGSS11, Gia11, GG12, GJMS12], the relation between quantum cohomology and conformal blocks in type A [Wit95], a Lie algebraic study of conformal blocks from [FSV95, Bea96], work on strange duality [Bel04, Bel08b, MO07, Oud11] and the Horn conjecture (cf. [Ful00, Bel10]).

To state our results, we briefly recall a minimal amount of notation. For a finite dimensional simple Lie algebra $\mathfrak{g}$, and a positive integer $\ell$ (called the level), let $P_\ell(\mathfrak{g})$ denote the set of dominant integral weights $\lambda$ with $(\lambda, \theta) \leq \ell$. To a triple $(\mathfrak{g}, \vec{\lambda}, \ell)$, such that $\vec{\lambda} \in P_\ell(\mathfrak{g})^n$, there corresponds a vector bundle $V_{\mathfrak{g}, \vec{\lambda}, \ell}$ of conformal blocks on the stack $\overline{M}_{g,n}$ [TUY89, Fak12].

In case $g = 0$, as we consider in this work, these bundles are globally generated, and give rise to morphisms on $\overline{M}_{0,n}$, as follows. Given a dominant integral weight $\lambda$ for $\mathfrak{g}$, $V_\lambda$ denotes the corresponding finite dimensional highest weight irreducible representation of $\mathfrak{g}$. For an $n$-tuple $\vec{\lambda}$, we let $A = A_{\mathfrak{g}, \vec{\lambda}}$ denote the constant vector bundle, which at a point $X = (C, (p_1, \ldots, p_n)) \in \overline{M}_{0,n}$, is the vector space of co-invariants $A = (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})_{\mathfrak{g}}$.

\footnote{Here $\theta$ is the highest root, and $(\ ,\ )$ is the Killing form, normalized so that $(\theta, \theta) = 2$.}
Fakhruddin proved, in [Fak12, Lemma 2.5], that there is a surjection
\[ A_{\mathbf{g},\lambda}|_x = A \to \mathbb{V}_{\mathbf{g},\lambda,\ell}|_x. \]
In particular, one obtains a morphism to a Grassmannian variety of quotients of \( A \):
\[ \overline{M}_{0,n} \xrightarrow{f_V} \text{Grass}^{\text{quo}}(\text{rk} V, A) \xrightarrow{p} \mathbb{P}^{(\text{rk} V)} - 1, \]
and this morphism \( p \circ f_V \) is given by the conformal blocks divisor \( c_1(\mathbb{V}_{\mathbf{g},\lambda,\ell}) = \mathbb{D}_{\mathbf{g},\lambda,\ell} \).

In this work we discuss the following aspects of these bundles, divisors, and morphisms:

1. We give two different levels: the critical level (Def. 1.1, valid for \( \mathbf{g} = \mathfrak{sl}_{r+1} \)), and the theta level (Def. 1.7, valid for any \( \mathbf{g} \)) such that for \( \ell \) greater than either, the bundles \( \mathbb{V}_{\mathbf{g},\lambda,\ell} \) coincide with \( A_{\mathbf{g},\lambda} \), and are hence trivial. The connection between the critical level and the quantum cohomology of Grassmannians plays a key role (cf. Rmk. 4.1, Sec. 4.1).

2. Our numerical criterion for the critical level is symmetric in \( r \) and \( \ell \): If \( \ell \) is the critical level for the pair \( (\mathfrak{sl}_{r+1}, \lambda) \), then \( r \) is the critical level for the pair \( (\mathfrak{sl}_\ell, \lambda^T) \). We show there are critical level symmetries \( \mathbb{D}_{\mathfrak{sl}_{r+1},\lambda,\ell} = \mathbb{D}_{\mathfrak{sl}_\ell,\lambda^T,r} \) (Cor. 1.5), and these critical level partner divisors give the same map to the Grassmannians, up to “Grassmann duality” (see Sect. 2.3). This identity, different from the level-rank dualities of [NT92], is a consequence of a new (and non-standard) symmetry of conformal blocks, which is new even at the level of ranks (see Thm. 1.4, Equality (1.3) and Sect. 18).

3. Given a triple \( (\mathbf{g}, \lambda, \ell) \), with \( \lambda \in P_\ell(\mathbf{g})^n \), we consider the question: when is \( \mathbb{D}_{\mathbf{g},\lambda,\ell} \neq 0? \) We answer this question completely in the simplest non-trivial case when the critical level and theta level coincide and are equal to \( \ell \) (Thm. 1.11, Prop. 16.5), give general nonvanishing criteria (Thm. 7.1), and also a complete answer for \( \mathbf{g} = \mathfrak{sl}_2 \) (Cor. 8.2(2)).

4. In Props 5.3 and 5.4, we give simple criteria which allow us to detect certain curves that are contracted by particular divisors \( \mathbb{D}_{\mathbf{g},\lambda,\ell} \). In Thms 6.2 and 6.3, we show that particular associated morphisms factor through maps to Hassett spaces.

5. We find other geometric identities between conformal blocks divisors. In Prop. 17.1, we show that if certain rank conditions are satisfied, classes are subject to an additive structure. Applications include identifying images of morphisms given by many higher level divisors (cf. Cor. 17.3, and Ex. 7.4.1). In Prop. 17.6 we give a relation between divisors that comes from an involution of the Weyl chamber.

6. In Prop. 7.3 and Cor. 7.5 the family of critical level divisors \( \mathbb{D}_{\mathfrak{sl}_{r+1},\omega,\lambda,\ell} \) is studied extensively. Many other examples are given.

1.1. Results. Finite dimensional irreducible polynomial representations for \( \text{GL}_{r+1} \) are parameterized by Young diagrams
\[ \lambda = (\lambda^{(1)} \geq \lambda^{(2)} \geq \cdots \geq \lambda^{(r)} \geq \lambda^{(r+1)} \geq 0). \]
Note that Young diagrams \( \lambda \) and \( \mu \) give the same representation of \( \text{SL}_{r+1} \) (equivalently \( \mathfrak{sl}_{r+1} \)) if \( \lambda^{(a)} - \mu^{(a)} \) is a constant independent of \( a \). We use the notation \( |\lambda| = \sum_{i=1}^r \lambda^{(i)} \), and
$\lambda \in P_t(sl_{r+1})$ if and only if $\lambda^{(1)} - \lambda^{(r+1)} \leq \ell$. We will say that $\lambda$ is normalized if $\lambda^{(r+1)} = 0$. The normalization of $\lambda$ is the Young diagram $\lambda = \lambda^{(r+1)} \cdot (1, 1, \ldots, 1)$.

**Definition 1.1.** Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be an $n$-tuple of normalized integral weights for $sl_{r+1}$. Assume that $r + 1$ divides $\sum_{i=1}^n |\lambda_i|$. We define the critical level to be

$$\ell(sl_{r+1}, \vec{\lambda}) = -1 + \frac{1}{r+1} \sum_{i=1}^n |\lambda_i|.$$  

We can define $\ell(sl_{r+1}, \vec{\lambda})$ in general, by replacing each $\lambda_i$ by its normalization.

A vector bundle $V_{sl_{r+1}, \vec{\lambda}, \ell}$ is said to be a critical level bundle if $\ell = \ell(sl_{r+1}, \vec{\lambda})$ and $\vec{\lambda} \in P_t(sl_{r+1})^n$. In this case we say that $\ell$ is the critical level for the pair $(sl_{r+1}, \vec{\lambda})$.

If $\ell$ is the critical level for $(sl_{r+1}, \vec{\lambda})$, then $r$ is the critical level for $(sl_{\ell+1}, \vec{\lambda}^T)$.

**Proposition 1.2.** Suppose $\vec{\lambda} \in P_t(sl_{r+1})^n$, and $\ell > \ell(sl_{r+1}, \vec{\lambda})$. Then $A_{sl_{r+1}, \vec{\lambda}} = V_{sl_{r+1}, \vec{\lambda}, \ell}$.

The proof of Proposition 1.2, given in Section 4, follows from enumerative interpretations of conformal blocks [Wit95, Agn95] and classical invariants for $sl_{r+1}$. The main point is that quantum cohomology structure coefficients include the classical structure constants. Def. 1.1 and Proposition 1.2 appear in Fakhruddin’s work for $r = 1$ (cf. [Fak12, Sec. 4.3]).

**Remark 1.3.** One may look for levels at which parabolic semistable bundles are necessarily trivial (as bundles). The resulting bounds for vanishing are weaker than the critical level bounds (by one), and are unrelated to the theta level bounds (cf. Sec. 4.1 and Question 4.2).

**Theorem 1.4.** Suppose that $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is an $n$-tuple of elements in $P_t(sl_{r+1})$ where $\ell = \ell(sl_{r+1}, \vec{\lambda})$. Then, there is a natural isomorphism over $M_{0,n}$:

$$A^*_{sl_{r+1}, \vec{\lambda}} / V_{sl_{r+1}, \vec{\lambda}, \ell} \cong V_{sl_{r+1}, \vec{\lambda}, \ell}.$$  

An overall sketch of the proof of Theorem 1.4 is given in Section 9.

**Corollary 1.5.** Suppose that $\vec{\lambda} \in P_t(sl_{r+1})^n$, where $\ell = \ell(sl_{r+1}, \vec{\lambda})$. Then, (a) one has that:

$$rk V_{sl_{r+1}, \vec{\lambda}, \ell} + rk V_{sl_{r+1}, \vec{\lambda}, \ell} = rk A_{sl_{r+1}, \vec{\lambda}} = rk A_{sl_{r+1}, \vec{\lambda}};$$

(b) critical level partner divisors are the same:

$c_1(V_{sl_{r+1}, \vec{\lambda}, \ell}) = c_1(V_{sl_{r+1}, \vec{\lambda}, \ell})$;

(c) and Chern classes above the critical level vanish:

$c_j(V_{sl_{r+1}, \vec{\lambda}, \ell}) = 0$, for $j > 0$, and $c > 0$.

Theorem 1.4 and Corollary 1.5 part (a) hold for all $n \geq 3$. Structure constants in the cohomology of a Grassmannian are decomposed into a sum of two quantum cohomology structure constants for different Grassmannians, yielding (1.3) (see Section 12.8). Corollary 1.5 part (a) is proved in Section 12. It follows from a (new?) degeneration of the classical
Grassmannian into a union of smooth quot-schemes on $\mathbb{P}^1$ meeting transversally (see Section 12.1).

**Remark 1.6.** Our critical level symmetries are different from, but related to, the strange dualities considered in the literature (e.g., [Bel08b, MO07, NT92, Oud11]). See Section 14.3.

### 1.2. The theta level.

**Definition 1.7.** Given a pair $(g, \vec{\lambda})$, we refer to
\[
\theta(g, \vec{\lambda}) = -1 + \frac{1}{2} \sum \lambda_i (H_{\theta}) \in \frac{1}{2} \mathbb{Z}
\]
as the theta level. Here $H_{\theta}$ is the co-root corresponding to the highest root $\theta$.

**Lemma 1.8.** (1) $\text{rk} V_{g, \vec{\lambda}, \ell} \leq \text{rk} V_{g, \vec{\lambda}, \ell+1} \leq \text{rk} A_{g, \vec{\lambda}}$;

(2) If $\text{rk} V_{g, \vec{\lambda}, \ell} = \text{rk} A_{g, \vec{\lambda}}$, then $V_{g, \vec{\lambda}, \ell}$ is trivial as a vector bundle on $M_{0,n}$.

(3) Suppose that $\sum \lambda_i (H_{\theta}) < 2(\ell + 1)$, then $V_{g, \vec{\lambda}, \ell} = A_{g, \vec{\lambda}}$ and $V_{g, \vec{\lambda}, \ell}$ is trivial as a vector bundle. Here $H_{\theta}$ is the co-root corresponding to the highest root $\theta$.

**Remark 1.9.** The non-zeroness of $\text{rk} A_{g, \vec{\lambda}}$ (similarly $\text{rk} V_{g, \vec{\lambda}, \ell}$) for $g = \mathfrak{sl}_{r+1}$ is controlled by a non-trivial system of inequalities [Kly98, KT99, Ful00, Bel08, Bel10].

The theta and critical levels coincide for $\mathfrak{sl}_2$, but are generally incomparable (see Sect. 8.2).

### 1.3. Non-vanishing results.

The non-zeroness question for conformal block divisors is subtle, and is comparable to other questions in representation theory such as the existence of invariants in tensor products, or of non-zeroness in Schubert calculus [Ful00]. Explicit formulas for Chern classes, or of ranks of invariants, or of Schubert structure constants, do not give a complete geometric understanding of these problems. Similarly, while there is a formula for the classes of the conformal blocks divisors [Fak12], one can not determine from this formula whether the divisors are zero, even at high levels where they are zero.

**Remark 1.10.** For $\mathfrak{sl}_2$, where the critical and theta level are the same, at levels $\tilde{\ell}$ not greater than the critical level, the divisor is non-zero as long as $\text{rk} V_{\mathfrak{sl}_2, \vec{\mu}, \ell} \neq 0$ (cf. Corollary 8.2).

However, this is not true more generally. There are instances where $\text{rk} V_{\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell} \neq 0$ and $\text{rk} V_{\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell} \neq 0$, and yet $c_1(V_{\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell}) = c_1(V_{\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell}) = 0$. Such occurrences were first found by Fakhruddin using [Swi10], (cf. lines (3), (6), and (7) of the table in Section 18).

**Theorem 1.11.** Suppose $V_{\mathfrak{sl}_2, \vec{\mu}, \ell}$ is conformal blocks bundle on $\overline{M}_{0,n}$, for $n \geq 4$, such that

(1) $\ell = \ell(\mathfrak{sl}_{r+1}, \vec{\lambda}) = \theta(\mathfrak{sl}_{r+1}, \vec{\lambda})$;

(2) $\lambda_i \neq 0$ for all $i$, normalized;

(3) $\text{rk} V_{\mathfrak{sl}_2, \vec{\mu}, \ell} > 0$, and if $r \geq 3$, $\text{rk} V_{\mathfrak{sl}_{r-1}, \vec{\nu}, \ell} > 0$.

where $V_{\mathfrak{sl}_2, \vec{\mu}, \ell}$, and $V_{\mathfrak{sl}_{r-1}, \vec{\nu}, \ell}$ are auxiliary bundles constructed below.

Then $\mathbb{D}_{\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell} \neq 0$. 

Auxiliary bundles for Theorem 1.11. Given a bundle $\mathcal{V}_{sl_{r+1}, \bar{\lambda}, \ell}$ such that $r \geq 2$, and with $\lambda_i \neq 0$ for all $i$, and normalized (so the last rows of $\lambda_i$ are zero). One forms auxiliary bundles $\mathcal{V}_{sl_2, \bar{\mu}, \ell}$ and $\mathcal{V}_{sl_{r-1}, \bar{\nu}, \ell}$, where $\mu_i$ correspond to the $2 \times \ell$ Young diagrams formed by the first and last rows of $\lambda_i$, and $\nu_i$ are given by the $(r - 1) \times \ell$ diagram obtained by removing the first and last rows of $\lambda_i$, $i = 1, \ldots, n$. Note that $\nu_i$ may not be normalized, and can be zero.

The main idea in the proof is to degenerate a general rank $(r+1)$-parabolic bundle (with weights $\bar{\lambda}$) into a direct sum of semi-stable parabolic bundles of ranks 2 and $r-1$, with weight data $\bar{\mu}$ and $\bar{\nu}$ respectively (and taking care of boundary issues).

Remark 1.12. In Theorem 1.11, a converse statement (see Proposition 16.5) holds as well: Assume conditions (1) and (2). If $r > 2$ and $\text{rk} \mathcal{V}_{sl_{r-1}, \bar{\nu}, \ell} = 0$, then $\mathcal{D}_{sl_{r+1}, \bar{\lambda}, \ell} = 0$. We do not know of a converse statement for Theorem 7.1.

1.3.1. Extremality in the Nef cone. If a conformal blocks divisor $\mathcal{D} = \mathcal{D}_{g, \bar{\lambda}, \ell}$ intersects a curve in degree zero, then it is not ample, but rather extremal in the nef cone. The results in Theorem 1.4 and Lemma 1.8 give rise to the simple criteria of Propositions 5.3 and 5.4 that in degree zero, then it is not ample, but rather extremal in the nef cone. The results in Theorem 1.4 and Lemma 1.8 give rise to the simple criteria of Propositions 5.3 and 5.4 that in degree zero, then it is not ample, but rather extremal in the nef cone.

Remark 1.13. This gives four conditions that are each sufficient for a critical level conformal blocks divisor $\mathcal{D}_{sl_{r+1}, \bar{\lambda}, \ell}$ to contract a given F-curve: Proposition 5.3 for $\mathcal{D}_{sl_{r+1}, \bar{\lambda}, \ell}$ and $\mathcal{D}_{sl_{r+1}, \bar{\lambda}^*, \ell}$ (see Proposition 17.6); and Proposition 5.4 for each of $\mathcal{D}_{sl_{r+1}, \bar{\lambda}, \ell}$ and $\mathcal{D}_{sl_{r+1}, \bar{\lambda}^*, \ell}$.

In Theorem 6.2 we prove that morphisms given by $\mathcal{D}_{sl_{r+1}, \bar{\lambda}, \ell}$ where $r$, $\ell$, and $\bar{\lambda}$ satisfy certain conditions, factor through maps to Hassett spaces $M_{0,A}$, where the weight data $A$ is determined by $r$, $\ell$, and $\bar{\lambda}$, and in Theorem 6.3 we prove a similar result for morphisms $\phi_{\mathcal{D}}$ associated to conformal blocks divisors $\mathcal{D}$ of any type.

In Corollary 7.5 we give a complete description of all F-curves contracted by the critical level divisors $\mathcal{D}_{sl_{r+1}, \omega^m_{\bar{\lambda}, \ell}}$ on $M_{0,n}$, where $n = (r+1)(\ell+1)$, for all $r$ and $\ell$. In Section 7.4 we give many examples showing that the divisor maps to a space that lies beyond the Hassett space (that the Hassett space maps to but isn’t isomorphic to).

1.3.2. Relations. In Section 17, we study relations of the form $\mathcal{D}_{g, \bar{\mu} + \bar{\nu}, \ell+m} = \text{rk} \mathcal{V}_{sl_{r+1}, \bar{\lambda}, \ell} + \mathcal{D}_{g, \bar{\mu}, \ell} + \mathcal{D}_{g, \bar{\nu}, m}$, which hold under certain conditions (see Proposition 17.1). Together with the quantum generalization of a conjecture of Fulton [Bel07, BKn13, Remark 8.5] in invariant theory, we show in Corollary 17.3 that if $\text{rk} \mathcal{V}_{sl_{r+1}, \bar{\lambda}, \ell} = 1$, then $\mathcal{D}_{sl_{r+1}, \bar{\lambda}, \ell} = N \cdot \mathcal{D}_{sl_{r+1}, \bar{\lambda}, \ell}$ for all positive integers $N$. As an application, in Corollary 17.5 we identify images of the maps $\phi_{\mathcal{D}}$ for $\mathcal{D} = \mathcal{D}_{sl_{r+1}, \bar{\lambda}, \ell}$, $\ell = \mathcal{D}_{sl_{r+1}, \bar{\lambda}, 1}$, as the generalized Veronese quotients of $Gia11$ and $GJM11$.

1.4. A note on our methods. The main results of this paper are proved by using relations of conformal blocks to generalized theta functions, and by arguments used in the geometric strange duality theory of vector bundles [Bel08b, MO07, Oud11], and in the study of quantum cohomology of Grassmannians ([Wit95], also cf. [Bel10]). The applications use the geometry of $M_{0,n}$ and the factorization formulas of [TUY89]. We recommend the Bourbaki article of Sorger [Sor96] for some of the background on conformal blocks.
2. Schubert varieties and Witten’s Dictionary

In this section we recall Theorem 2.2, a special case of [Bel08, Proposition 3.4], which gives a dictionary that rephrases the ranks of the bundles $V(\mathfrak{sl}_r, \ell, \lambda)$, in terms of an enumerative problem, counting certain sub-bundles of evenly split bundles on $\mathbb{P}^1$.

2.1. Notation. We write $[m] = \{1, \ldots, m\}$ for all positive integers $m$. For an $m$-dimensional vector space $V$, denote by $\text{Fl}(V)$ the space of complete flags of vector subspaces of $V$: $F_\bullet : 0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_m = V$.

The determinant line $\Lambda^m V$ is denoted by $\text{det} V$. We fix a collection of $n$ distinct and ordered points $S = \{p_1, \ldots, p_n\} \subseteq \mathbb{P}^1$, and for a vector bundle $\mathcal{W}$ on $\mathbb{P}^1$, define $\text{Fl}_S(\mathcal{W}) = \prod_{p \in S} \text{Fl}(\mathcal{W}_p)$. If $\mathcal{E} \in \text{Fl}_S(\mathcal{W})$, we will assume that it is written in the form $\mathcal{E} = (E_p^\bullet \mid p \in S)$.

2.2. Schubert varieties, and their tangent spaces. Let $E_\bullet \in \text{Fl}(W)$ be a complete flag in an $N = \tilde{r} + k$-dimensional vector space $W$. Suppose $\lambda$ is a Young diagram that fits into a $\tilde{r} \times k$ box. It is useful to associate to $\lambda$ a $\tilde{r}$-element subset $I$ of $[N]$ by

$$I = \{i_1 < \cdots < i_{\tilde{r}}\}, \quad i_a = k + a - \lambda^{(a)}, \quad i_0 = 0, \quad i_{\tilde{r}+1} = N.$$ 

The open Schubert cell $\Omega^\alpha_\lambda(E_\bullet) = \Omega^\alpha_\lambda(E_\bullet)$ is defined to be the smooth subvariety of the Grassmannian $\text{Gr}(\tilde{r}, W)$ of $\tilde{r}$-dimensional vector subspaces of $W$ given by

$$\Omega^\alpha_\lambda(E_\bullet) = \{V \in \text{Gr}(\tilde{r}, W) \mid \text{rk}(V \cap E_i) = a \text{ for } i_a \leq j < i_{a+1}, a \in [\tilde{r}]\}.$$ 

The closure $\overline{\Omega^\alpha_\lambda(E_\bullet)}$ is denoted by $\Omega_\lambda(E_\bullet)$. This is the set of $V \in \text{Gr}(\tilde{r}, W)$ such that $\text{rk}(V \cap E_i) \geq a$, $a \in [\tilde{r}]$. The codimension of $\Omega_\lambda(E_\bullet)$ in $\text{Gr}(\tilde{r}, W)$ is $|\lambda|$.

Suppose $V \in \Omega^\alpha_\lambda(E_\bullet)$. Let $Q = W/V$. It is easy to see that $V$ and $Q$ each receive induced flags from $E_\bullet$; denote these by $F_\bullet$ and $G_\bullet$. Then $T \Omega^\alpha_\lambda(E_\bullet)_V \subseteq T \text{Gr}(\tilde{r}, W)_V = \text{Hom}(V, Q)$ is

$$T \Omega^\alpha_\lambda(E_\bullet)_V = \{\phi \in \text{Hom}(V, Q) \mid \phi(F_a) \subseteq G_{k-a(\lambda)} \text{, } a \in [\tilde{r}]\}. \tag{2.1}$$ 

2.3. Grassmann duality. Let $W$ be an $M = \tilde{r} + k$ dimensional vector space. A natural identification $\text{Gr}(\tilde{r}, W) = \text{Gr}(k, W^*)$ takes a subspace $V \subseteq W$ to the kernel of the surjective map $W^* \rightarrow V^*$. There is similarly an identification of the complete flag varieties $\text{Fl}(W)$ and $\text{Fl}(W^*)$. A Schubert variety $\Omega_\lambda(E_\bullet) \subseteq \text{Gr}(\tilde{r}, W)$ can be identified with $\Omega_{\lambda^T}(E'_\bullet) \subseteq \text{Gr}(k, W^*)$, where $E_\bullet$ and $E'_\bullet$ correspond under the identification of $\text{Fl}(W)$ and $\text{Fl}(W^*)$. A natural group isomorphism $\text{GL}(W) \rightarrow \text{GL}(W^*)$ acts equivariantly on the above identifications.

Let $\lambda$ be an Young diagram which fits into a $(\tilde{r} \times k)$ box. Let $\lambda^c$ be the complement of the Young diagram $\lambda$. We consider the line bundles $\mathcal{L}_{\lambda^c}$ and $\mathcal{L}_\lambda$ on $\text{Fl}(W)$ and $\text{Fl}(W^*)$ as defined in Section 10. The line bundle $\mathcal{L}_\lambda$ pulls back to $\mathcal{L}_{\lambda^c}$ under the natural map from $\text{Fl}(W)$ to $\text{Fl}(W^*)$, equivariantly under the action of groups $\text{SL}(W) \rightarrow \text{SL}(W^*)$.

2.4. Witten’s Dictionary.

Definition 2.1. A vector bundle $\mathcal{W}$ on $\mathbb{P}^1$ is said to be evenly split if $\mathcal{W} = \bigoplus_{j=1}^N \mathcal{O}_{\mathbb{P}^1}(a_j)$ with $|a_i - a_j| \leq 1$ for $0 < i < j \leq N$. It is easy to see that there are evenly split bundles of every degree and (non-zero) rank. These bundles are generic bundles of the given degree and rank.
2.4.1. **Notation for Witten’s Dictionary.** Suppose $\mu_1, \ldots, \mu_n$ are Young diagrams that fit into a $\tilde{r} \times k$ box (not necessarily normalized). Assume further that $\sum_{i=1}^n |\mu_i| = \tilde{r}x$ for some positive integer $x$. We set up the enumerative problem in the following way: First write

$$\sum_{i=1}^n |\mu_i| = \tilde{r}x = \tilde{r}k - D\tilde{r}, \ D = k - x \in \mathbb{Z}.$$  

Let $\mathcal{W}$ be an evenly split vector bundle of rank $\tilde{r} + k$ and degree $-D\tilde{r}$, and $E \in \text{Fl}(\mathcal{W})$ a general point (cf. Sections 2.1 and 15).

**Theorem 2.2 (Witten’s Dictionary).** Following the notation of Section 2.2, one has that the rank of $V_{\mathfrak{sl}_{\tilde{r}}, \vec{\mu}, k}$ is equal to the number of elements in the set

$$\left\{ \subb(\mathcal{V}) \subset \mathcal{W} : \begin{array}{l} \deg(\mathcal{V}) = 0 \\
\text{rk}(\mathcal{V}) = \tilde{r} \\
\mathcal{V}_{\mu_i} \in \Omega_{\mu_i}(E_{\mu_i}) \subseteq \text{Gr}(\tilde{r}, \mathcal{W}_{\mu_i}), \ i \in [n] \end{array} \right\},$$

which is finite, by Kleiman-transversality.

See [Bel08, Theorem 3.6 and Remark 3.8] for a proof of Witten’s Dictionary, modeled on the proof of its classical counterpart described just after Remark 2.3 below (cf. [Ful00, Section 6.2]). A cohomological version of Witten’s dictionary is given in Section 3.4.

**Remark 2.3.** In the original form of Witten’s Dictionary one has quantum cohomology structure coefficients on one side (at any degree $d$), and the ranks of Verlinde bundles at degree $-d$ (the underlying bundle of parabolic bundles have degree $-d$ which may not be zero). We choose to (cyclically) twist both sides so that the plain degree of the parabolic bundles is now zero and give conformal blocks as in our paper ([Bel08] explains how to twist the enumerative side so that instead of counting subbundles of the trivial bundle (as in quantum cohomology), one counts subbundles of arbitrary evenly split bundles).

The classical counterpart of Witten’s Dictionary, going back at least to L. Lesieur [Les47] (also [Ful00, Section 6.2]) is the following: Suppose $\lambda_1, \ldots, \lambda_n$ are Young diagrams that each fit into a $\tilde{r} \times k$ box (not necessarily normalized). Assume further that $\sum_{i=1}^n |\lambda_i| = \tilde{r}k$. Then, the rank of the classical co-invariants $(V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n})_{\mathfrak{sl}_\tilde{r}}$ equals the number of points in

$$\bigcap_{i=1}^n \Omega_{\lambda_i}(E_{\mu_i}^*) \subseteq \text{Gr}(\tilde{r}, W),$$

where $W$ is a vector space of dimension $\tilde{r} + k$, and $(E_{\lambda_1}^*, \ldots, E_{\lambda_n}^*)$ is a general element of $\text{Fl}(W)^n$.

The rank of classical co-invariants above also equals the multiplicity of the $\mathfrak{sl}_r$ (or $\text{GL}_r$) representation $V_{\lambda_n}^\vee$ in $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_{n-1}}$. Here $\lambda_n^\vee$ is the complement of the Young diagram $\lambda_n$ in a $\tilde{r} \times k$ box (and flipped over) and corresponds to the dual of $V_{\lambda_n}$ as a $\mathfrak{sl}_r$ representation.

### 3. Ranks of Conformal Blocks and Classical Co-invariants

It is useful to have the cohomological version of Witten’s dictionary 2.2 and its classical counterpart, especially for numerical computations.
3.1. Classical cohomology of Grassmannians. Recall that the cohomology \( H^*(X, \mathbb{Z}) \) of a Grassmannian \( X = \text{Gr}(r+1, M) = \text{Gr}(r+1, \mathbb{C}^M) \), \( M = r+1+k \) is a commutative and associative ring, with an additive basis of cycle classes \( \sigma_\mu \) of Schubert varieties \( \Omega_\mu(E_\bullet) \). Here \( \mu \) runs through all Young diagrams which fit into a \((r+1) \times k\) box \( (\mu \text{ need not be normalized}) \). Note that \( \sigma_\mu \in H^{2|\mu|}(X, \mathbb{Z}) \). The class of a point \([pt]\) is \( \sigma_{(k,k,\ldots,k)} \).

The coefficient of \( \sigma_{\lambda_n} \) \((\lambda_n^c \text{ is the complement of } \lambda_n \text{ in a } (r+1) \times k \text{ box})\) in the product
\[
\sigma_{\lambda_1} \cdot \sigma_{\lambda_2} \cdots \sigma_{\lambda_{n-1}} \in H^*(X) = H^*(X, \mathbb{Z})
\]
has the following enumerative interpretation: Pick a general element \((E_1^\bullet, \ldots, E_n^\bullet)\) of \( \text{Fl}(W)^n \) where \( W = \mathbb{C}^M \), and count number of points in \((\text{count as zero if this number is infinite})\)
\[
\cap_{i=1}^n \Omega_{\lambda_i}(E_i^\bullet).
\]

3.2. Rank of classical co-invariants. Given a pair \((\mathfrak{s}l_{r+1}, \tilde{\lambda})\), we find the rank of the classical co-invariants \( A_{\mathfrak{s}l_{r+1}, \tilde{\lambda}} \) as follows: Write \( \sum |\lambda_i| = (r+1)k \) with \( k \geq \lambda_i^{(1)} \) for all \( i \in [n] \), by possibly adding some \((r+1) \times s\) boxes if necessary, or padding on columns of length \((r+1)\) to the \( \lambda_i \)'s. Now compute the multiplicity of the class of a point \([pt]\) = \( \sigma_{(k,k,\ldots,k)} \) in the product
\[
\sigma_{\lambda_1} \cdot \sigma_{\lambda_2} \cdots \sigma_{\lambda_n} \in H^*(X), \ X = \text{Gr}(r+1, r+1+k).
\]

We may also write this rank as the multiplicity of the class of \( \sigma_{\lambda_n} \) \((\lambda_n^c \text{ is the complement of } \lambda_n \text{ in a } (r+1) \times k \text{ box})\) in
\[
\sigma_{\lambda_1} \cdot \sigma_{\lambda_2} \cdots \sigma_{\lambda_{n-1}} \in H^*(X).
\]
The classical Pieri rule gives the decomposition of \( \sigma_{(a,0,\ldots,0)} \cdot \sigma_\lambda \): It decomposes as a direct sum of \( \sigma_\mu \) such that \( \mu \) is obtained from \( \lambda \) by adding \( a \) boxes, no two in the same column.

3.3. Quantum cohomology. We refer the reader to [Ber97] for the basic notions of quantum cohomology. As an abelian group, the quantum cohomology group of a Grassmannian \( X = \text{Gr}(r+1, M) = \text{Gr}(r+1, \mathbb{C}^M) \), \( M = (r+1) + k \) is
\[
QH^*(X) = QH^*(X, \mathbb{Z}) = H^*(X, \mathbb{Z}) \otimes \mathbb{Z}[q].
\]
The multiplication in this graded commutative and associative ring is \( q \)-linear (\( \sigma_\nu \) has degree \(|\nu| \) and \( q \) has degree \( M \)). The coefficient of \( q^d \sigma_{\lambda_n} \) \((\lambda_n^c \text{ is the complement of } \lambda_n \text{ in a } (r+1) \times k \text{ box})\) in the quantum product
\[
\sigma_{\lambda_1} \ast \sigma_{\lambda_2} \cdots \ast \sigma_{\lambda_{n-1}} \in QH^*(X)
\]
has the following enumerative interpretation: Fix distinct points \( p_1, \ldots, p_n \in \mathbb{P}^1 \), and a general element \((E_1^\bullet, \ldots, E_n^\bullet)\) of \( \text{Fl}(W)^n \) where \( W = \mathbb{C}^M \), and count number of maps \((\text{count as zero if this number is infinite}) f : \mathbb{P}^1 \to X \) of degree \( d \) so that
\[
f(p_i) \in \Omega_{\lambda_i}(E_i^\bullet) \subseteq X, \forall i \in [n].
\]
By setting \( q = 0 \) in the product \( \sigma_\mu \ast \sigma_\nu \), we recover the classical product \( \sigma_\mu \cdot \sigma_\nu \in H^*(X) \).

Recall that maps \( f : \mathbb{P}^1 \to X \) of degree \( d \) are in one-one correspondence with subbundles \( S \subseteq \mathcal{O}^\oplus M \) of rank \( r+1 \) and degree \(-d\) (the correspondence is by pulling back the tautological
subbundle on \( X \) by \( f \). Therefore the multiplicity of the previous paragraph can be interpreted as follows: Let \( \mathcal{W} = \mathcal{O}^{\oplus M} \) and pick a general point \( \mathcal{E} \in \text{Fl}_S(\mathcal{W}) \) (see Section 2.1). The enumerative problem is to count (count as zero if this number is infinite) subbundles \( \mathcal{V} \subseteq \mathcal{W} \) of degree \(-d\) and rank \( r + 1 \) such that

\[
\mathcal{V}_{p_i} \in \Omega_{\mu_i}(E_{p_i}^n) \subseteq \text{Gr}(r + 1, \mathcal{W}_{p_i}), \ i \in [n].
\]

The product \( \sigma_\gamma \star \sigma_{(a,0,\ldots,0)} \in QH^*(X) \) is described by the quantum Pieri rule \cite{Ber97}.

**Example 3.1.** The quantum product \( \sigma_{(k,0,\ldots,0)} \star \sigma_\mu \in QH^*(X) \) is always a single term of the form \( q^d \sigma_\nu \), where

1. If \( \mu^{(r+1)} > 0 \), then \( d = 1 \), and \( \nu \) is obtained by deleting one box from each row of \( \mu \).
2. If \( \mu^{(r+1)} = 0 \), then \( d = 0 \) and \( \nu = (k, \mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r)}) \).

### 3.4. Cohomological form of Witten’s dictionary

To compute the rank of a conformal block \( \mathbb{V}_{\text{sl}_{r+1},\tilde{\lambda},\ell} \) in terms of quantum cohomology of Grassmannians we proceed as follows. Write \( \sum |\lambda_i| = (r + 1)\ell + s(r + 1) \). If \( s \leq \binom{\ell}{2} \) the block coincides with co-invariants and we use the classical rules from Section 3.2. Assume \( s > 0 \). Let \( \lambda = (\ell, 0, \ldots, 0) \). Then the rank of \( \mathbb{V}_{\text{sl}_{r+1},\tilde{\lambda},\ell} \) is the coefficient of \( q^s [\text{pt}] = q^s \sigma_{(\ell,\ell,\ldots,\ell)} \) in

\[
\sigma_{\lambda_1} \star \cdots \star \sigma_{\lambda_n} \star \sigma^s_\lambda \in QH^*(Y), \ Y = \text{Gr}(r + 1, r + 1 + \ell).
\]

This relation to quantum cohomology follows from \cite{Wit95} and the twisting procedure of \cite{Bel08}, see Eq (3.10) from \cite{Bel08}. Note that \( \sigma^s_\lambda \) is the \( s \)-fold quantum \( \star \) product of \( \sigma_\lambda \).

We can write the above multiplicity also as the coefficient of \( q^s \sigma_{\lambda^c_n} \) (\( \lambda^c_n \) is the complement of \( \lambda_n \) in a \((r + 1) \times \ell \) box) in

\[
\sigma_{\lambda_1} \star \sigma_{\lambda_2} \star \cdots \star \sigma_{\lambda_{n-1}} \star \sigma^s_\lambda \in QH^*(Y).
\]

### 4. The proof of Proposition 1.2

Let \( \tilde{\ell} = \ell(\text{sl}_{r+1},\tilde{\lambda}) + 1 \). We divide the proof into two cases. The first case is if \( \lambda_1, \ldots, \lambda_n \) are in \( P_{\ell}(\text{sl}_{r+1}) \), and the second is when they are not.

In case (1), it suffices to prove (using Lemma 1.8) that \( \text{rk} \mathbb{V}_{\text{sl}_{r+1},\tilde{\lambda},\tilde{\ell}} = \text{rk} A_{\text{sl}_{r+1},\tilde{\lambda}} \). The enumerative translation of \( \text{rk} \mathbb{V}_{\text{sl}_{r+1},\tilde{\lambda},\tilde{\ell}} \) is the following (see Section 2.4): Let \( \mathcal{W} \) be an evenly split bundle of rank \( N = r + \tilde{\ell} + 1 \) and degree \(-D = 0 \) (using (2.2)), i.e., \( \mathcal{W} = \mathcal{O}^{\oplus N} = \mathcal{W} \otimes \mathcal{O} \) for an \( N \)-dimensional vector space \( W \). Pick a general point \( \mathcal{E} \in \text{Fl}_S(\mathcal{W}) \) (see Section 2.1). The enumerative problem is to count (a finite list by Kleiman-transversality) subbundles \( \mathcal{V} \) of \( \mathcal{W} \) of degree 0 and rank \( r + 1 \) such that

\[
\mathcal{V}_{p_i} \in \Omega_{\lambda_i}(E_{p_i}^n) \subseteq \text{Gr}(r + 1, \mathcal{W}_{p_i}), \ i \in [n].
\]

\footnote{See the case \( D > 0 \) in \cite{Bel08} Section 3.5.}
Any such subbundle is clearly trivial, that is, of the form $V \otimes O$ for some $r + 1$-dimensional subspace $V \subseteq W$, and our count therefore equals the number of points in

\[ \bigcap_{i=1}^{n} \Omega_{\lambda_i}(E_i^*) \subseteq \text{Gr}(r + 1, W), \]

where $\text{rk} W = N$, and $(E^1, \ldots, E^n)$ is a general element of $\text{Fl}(W)^n$. By Section 2.4, this classical count is $\text{rk} A_{\text{sl}_{r+1}, \vec{\lambda}}$ and we are done.

**Remark 4.1.** We could have argued cohomologically as follows: The rank of $V_{(\text{sl}_{r+1}, \vec{\ell}, \vec{\lambda})}$ is, by Section 3.4, the coefficient of $[\text{pt}]$ in the quantum product $\sigma_{\lambda_1} \star \sigma_{\lambda_2} \star \cdots \star \sigma_{\lambda_n} \in QH^*(Y)$, where $\ell = \text{Gr}(r + 1, r + 1 + \ell)$, since $s = 0$ in our case. This is the classical part of quantum cohomology, i.e., the above coefficient is the same if we replace $QH^*(Y)$ by $H^*(Y)$. Therefore, by Section 3.2, our coefficient equals rank of $A_{\text{sl}_{r+1}, \vec{\lambda}}$, as desired.

In case (2), we claim that $A_{\text{sl}_{r+1}, \vec{\lambda}}$ is trivial, and hence all $\text{sl}_{r+1}$ conformal blocks bundles and divisors for $\vec{\lambda}$ are zero. Assume that $k = \lambda_1^{(1)} > \tilde{\ell}$ is the maximum of the $\lambda_i^{(1)}$ and write

\[ \sum_{i=1}^{n} |\lambda_i| = (r + 1)k - p(r + 1) \]

so that $\lambda_i$ fit into boxes of size $(r + 1) \times k$ and $p > 0$. Therefore if we set $\mu_1 = \cdots = \mu_p = (1, 1, \ldots, 1)$ then

\[ \sum_{i=1}^{n} |\lambda_i| + \sum_{j=1}^{p} |\mu_j| = (r + 1)k. \]

The representation $V_{(1, \ldots, 1)}$ of $\text{GL}(r + 1)$ is trivial as a representation of $\text{sl}_{r+1}$. Therefore applying the classical theorem relating intersection numbers for $X = \text{Gr}(r + 1, r + 1 + k)$ and invariants for $\text{sl}_{r+1}$, we find that the rank of $A_{\text{sl}_{r+1}, \vec{\lambda}}$ equals the multiplicity of the class of a point in the cup product

\[ \sigma_{\lambda_1} \cdots \sigma_{\lambda_n} \cdot \sigma^p_{(1,1,\ldots,1)} \in H^*(X). \]

But it is easy to see that $\sigma_{\lambda_1} \cdot \sigma_{(1,1,\ldots,1)} = 0 \in H^*(X)$ (for example, Grassmann dualize and apply the Pieri rule from Section 3.2). The desired vanishing therefore holds.

**4.1. Parabolic semistability.** The duals of spaces of conformal blocks $\nabla_{\text{sl}_{r+1}, \vec{\lambda}, \vec{\ell}}$ over points of $\mathcal{M}_{0,n}$ can be identified with global sections of certain line bundles on suitable moduli spaces of semi-stable parabolic bundles on $\mathbb{P}^1$. We note that it is not necessary (but perhaps sufficient) for all parabolic semistable bundles to have trivial underlying bundles for $\nabla_{\text{sl}_{r+1}, \vec{\lambda}, \vec{\ell}}$ to be equal to the trivial bundle $A_{\text{sl}_{r+1}, \vec{\lambda}, \ell}$ (see Remark 14.5 for an example). As has been pointed out to us, it is instructive to determine conditions on $\ell$ so that any semistable parabolic bundle $\nabla$ with weights $|\lambda_i|/\ell$ has the trivial splitting type. If the underlying bundle $\nabla$ has
\( \mathcal{O}(a) \) (with \( a > 0 \)) as a direct summand, and \( \mathcal{O}(a) \) meets the flags of \( \mathcal{V} \) generically, and for it to contradict semistability (by an easy calculation),

\[
1 + \frac{1}{1} \cdot 0 > 0 + \frac{1}{(r+1)} \cdot \sum |\lambda|, \\
\text{or that}
\]

\[
\ell > \frac{1}{r+1} \sum |\lambda| = \ell(\mathfrak{sl}_{r+1}, \vec{\lambda}) + 1.
\]

This parabolic method misses \( \ell = \ell(\mathfrak{sl}_{r+1}, \vec{\lambda}) + 1 \): Proposition 1.2 says that \( \mathcal{V}_{\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell} \) coincides with coinvariants for \( \ell = \ell(\mathfrak{sl}_{r+1}, \vec{\lambda}) + 1 \) whereas this argument is inconclusive. Therefore, the critical level bound is strictly better than the bound obtained via parabolic semistability by one. In fact, for \( \ell = \ell(\mathfrak{sl}_{r+1}, \vec{\lambda}) + 1 \) there are semistable parabolic bundles with non-trivial splitting type (cf. e.g., Example (2) in Table 18 and Remark 14.5). Theorems 1.4 and 1.11 (and many others) require the critical level as defined, and the connection of the critical level to quantum cohomology is decisive. However, we believe such a parabolic semistability argument should give bounds for all Lie algebras \( \mathfrak{g} \), and should be further developed.

We ask the following natural question:

**Question 4.2.** Given a triple \((g, \vec{\lambda}, \ell)\), with \( \vec{\lambda} \in P_{\ell}(g)^n \), what are necessary and sufficient conditions so that \( \text{rk} \mathcal{V}_{g, \vec{\lambda}, \ell} = \text{rk} A_{g, \vec{\lambda}} \)?

This answer must be subtle. For example, Corollary 1.5 says that it is sufficient to take \( \ell \) greater than the critical level, but that if \( \ell \) is equal to the critical level to have this rank identity it is necessary that the critical level partner bundle has rank zero. A look at the first four lines of Table 18 shows that sometimes this happens, while other times it does not.

Another sufficient condition is given by the theta level (Lemma 1.8).

5. Extremality of the \( \mathcal{D} = \mathcal{D}_{g, \vec{\lambda}, \ell} \) in Nef Cone

In this section we give criteria (in Propositions 5.3 and 5.4) which come from Corollary 1.5 and Lemma 1.8 to detect certain so-called F-curves (cf. Def. 5.1) that get contracted by the associated conformal blocks maps \( \phi_D \). This enables us to show in Section 6 that the morphisms \( \phi_D \) factor through certain Hassett contractions.

5.1. Notation.

**Definition 5.1.** Fix a partition of \([n] = \{1, \ldots, n\}\) into four nonempty sets \( N_1, N_2, N_3, N_4 = [n] \setminus N_1 \cup N_2 \cup N_3 \), and consider the morphism

\[
\overline{M}_{0,4} \longrightarrow \overline{M}_{0,n}, \quad (C, (a_1, a_2, a_3, a_4)) \mapsto (X, (p_1, \ldots, p_n))
\]

where \( X \) is the nodal curve obtained as follows. If \( |N_i| \geq 2 \), then one glues a copy of \( \mathbb{P}^1 \) to the spine \((C, (a_1, a_2, a_3, a_4))\) by attaching a point

\[
(\mathbb{P}^1, \{p_j : j \in N_i\} \cup \{\alpha_i\}) \in M_{0,|N_i|+1}
\]
to $a_i$ at $\alpha_i$. If $|N_i| = 1$, one does not glue any curve at the point $a_i$, but instead labels $a_i$ by $p_i$. We refer to any element of the numerical equivalence class of the image of this morphism the F-Curve $F(N_1, N_2, N_3)$ or by $F(N_1, N_2, N_3, N_4)$, depending on the context.

**Conjecture 5.2** (The F-Conjecture). A divisor $D$ on $\overline{\mathcal{M}}_{0,n}$ is nef if and only if it non-negatively intersects all F-curves.

5.2. Contraction results.

**Proposition 5.3.** Suppose that $r \geq 1$, $\ell \geq 1$, and let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be an $n$-tuple in $P_{t}(\mathfrak{g})$. Let $N_1, N_2, N_3, N_4$ be any partition of $[n]$ into four nonempty subsets, ordered so that, without loss of generality, if $\lambda(N_j) = \sum_{i \in N_j} |\lambda_i|$, then $\lambda(N_1) \leq \cdots \leq \lambda(N_4)$. If

$$\sum_{j \in \{1, 2, 3\}} \lambda(N_j) \leq \ell + 1,$$

then the (possibly constant) morphism $\phi_D$ given by the divisor $D = D_{\mathfrak{sl}_{r+1}, \vec{x}, \ell}$ contracts $F(N_1, N_2, N_3)$ and in particular, is not ample.

**Proof.** The F-curve $F(N_1, N_2, N_3)$ is contained in the boundary divisor

$$\Delta_{N_1 \cup N_2 \cup N_3} \cong \overline{\mathcal{M}}_{0,|N_1 \cup N_2 \cup N_3|+1} \times \overline{\mathcal{M}}_{0,|N_4|+1},$$

and is actually isomorphic to a curve contained in $\overline{\mathcal{M}}_{0,|N_1 \cup N_2 \cup N_3|+1}$ under the attaching map, which attaches the leg with $N_4$ marked points to the extra attaching point. Therefore, to show that $\phi_D$ contracts $F(N_1, N_2, N_3)$, it suffices to show that $D$ is trivial on $\overline{\mathcal{M}}_{0,|N_1 \cup N_2 \cup N_3|+1} \times x$ for a fixed $x \in \overline{\mathcal{M}}_{0,|N_4|+1}$. Let $I = N_1 \cup N_2 \cup N_3$.

When pulled to $M = \overline{\mathcal{M}}_{0,|N_1 \cup N_2 \cup N_3|+1} \times \overline{\mathcal{M}}_{0,|N_4|+1}$, our bundle $V_{\mathfrak{sl}_{r+1}, \vec{\lambda}}$ breaks up (by factorization) into a direct sum

$$\bigoplus_{\mu \in P_{t}} V_{\mathfrak{sl}_{r+1}, \{\lambda_i : i \in I\} \cup \{\mu\}, \ell} \otimes V_{\mathfrak{sl}_{r+1}, \{\lambda_i : i \in N_4\} \cup \{\mu\}}$$

of tensor products of vector bundles pulled back from the two projections of $M$. It therefore suffices to show that $c_1(V_{\mathfrak{sl}_{r+1}, \{\lambda_i : i \in I\} \cup \{\mu\}, \ell}) = 0$ for any $\mu \in P_{t}$. But $|\mu| \leq \ell r$ and $\sum_{i \in I} |\lambda_i| + |\mu| \leq \ell + r + \ell r < (\ell + 1)(r + 1)$ and hence $\ell$ is greater than the critical level for $(\mathfrak{sl}_{r+1}, \{\lambda_i : i \in I\} \cup \{\mu\})$. We may therefore apply Proposition 1.2 to conclude that the desired $c_1$ vanishes. \qed

Using Lemma 1.8 in place of Corollary 1.5 gives the following general result.

**Proposition 5.4.** Suppose that $r \geq 1$, $\ell \geq 1$, and let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be an $n$-tuple in $P_{t}(\mathfrak{g})$. Let $N_1, N_2, N_3, N_4$ be any partition of $[n]$ into four nonempty subsets, ordered so that, without loss of generality, if $L(N_j) = \sum_{i \in N_j} (\lambda_i, \theta)$, then $L(N_1) \leq L(N_2) \leq L(N_3) \leq L(N_4)$. If

$$\sum_{j \in \{1, 2, 3\}} L(N_j) \leq \ell + 1,$$

then the (possibly constant) morphism $\phi_D$ afforded by the divisor $D = D_{0, \vec{\lambda}, \ell}$ contracts the F-Curve $F(N_1, N_2, N_3)$, and in particular, is not ample.
6. Conformal blocks morphisms and Hassett contractions

As a step towards understanding the images of the morphisms \( \phi_\mathcal{D} \), Theorems 6.2 and 6.3 show that under certain conditions, they factor through maps to Hassett’s moduli spaces \( \overline{M}_{0,A} \) of stable weighted pointed rational curves (cf. Section 6.1), where the weight data \( a_i \), are explicitly determined by the \( g \), \( \ell \) and \( \lambda_i \) and satisfy certain properties.

6.1. Background on Hassett spaces. Consider an \( n \)-tuple \( \mathcal{A} = \{a_1, \ldots, a_n\} \), with \( a_i \in \mathbb{Q} \), \( 0 < a_i \leq 1 \), such that \( \sum_i a_i > 2 \). In [Has03], Hassett introduced moduli spaces \( \overline{M}_{0,A} \), parameterizing families of stable weighted pointed rational curves \( (C,(p_1, \ldots, p_n)) \) such that

1. \( C \) is nodal away from its marked points \( p_i \);
2. \( \sum_{j \in J} a_i \leq 1 \), if the marked points \( \{p_j : j \in J\} \) coincide; and
3. If \( C' \) is an irreducible component of \( C \) then
   \[ \sum_{p_i \in C'} a_i + \text{number of nodes on } C' > 2. \]

These Hassett spaces \( \overline{M}_{0,A} \) receive birational morphisms \( \rho_\mathcal{A} \) from \( \overline{M}_{0,n} \) that are characterized entirely by which F-Curves (cf. Def. 5.1) they contract.

Definition/Lemma 6.1. For any Hassett space \( \overline{M}_{0,A} \), with \( \mathcal{A} = \{a_1, \ldots, a_n\} \), there are birational morphisms \( \rho_\mathcal{A} : \overline{M}_{0,n} \rightarrow \overline{M}_{0,A} \), which contract all F-curves \( F(N_1, N_2, N_3, N_4) \) satisfying:

\[ \sum_{i \in N_1 \cup N_2 \cup N_3} a_i \leq 1, \]

and no others, where without loss of generality, the leg \( N_4 \) carries the most weight.

6.2. Results on Hassett spaces. The following theorems generalize [Fak12, Proposition 4.7], where \( g = sl_2 \) was considered.

Theorem 6.2. Let \( \mathcal{D} = \mathbb{D}_{\mathfrak{sl}_{r+1},\chi,\ell} \) be a (possibly trivial) conformal blocks divisor such that:

1. \( 0 < |\lambda_i| \leq \ell + r \) for all \( i \in \{1, \ldots, n\} \);
2. \( \sum_{i=1}^n |\lambda_i| > 2(r + \ell) \).

Then the morphism \( \phi_\mathcal{D} \) given by \( \mathcal{D} \) factors through the contraction map

\[ \rho_\mathcal{A} : \overline{M}_{0,n} \rightarrow \overline{M}_{0,A}, \text{ where } \mathcal{A} = \{a_1, \ldots, a_n\}, \text{ a}_i = \frac{|\lambda_i|}{r+\ell}. \]

Proof. For \( \mathcal{A} = \{a_1, \ldots, a_n\} \), \( a_i = \frac{|\lambda_i|}{r+\ell} \), as in the hypothesis, the condition \( |\lambda_i| \leq \ell + r \), guarantees that \( a_i \leq 1 \) for all \( i \), and \( \sum_{i=1}^n |\lambda_i| > 2(r + \ell) \) guarantees that \( \sum_{i=1}^n a_i > 2 \).

By [Fak12, Lemma 4.6], we need to show that any F-curve \( F(N_1, N_2, N_3, N_4) \) contracted by \( \rho_\mathcal{A} \) is also contracted by \( \phi_\mathcal{D} \). Suppose that \( \rho_\mathcal{A} \) contracts the F-curve \( F(N_1, N_2, N_3, N_4) \), so that in particular, by Definition/Lemma 6.1 \( \sum_{i \in N_1 \cup N_2 \cup N_3} a_i \leq 1 \). Then

\[ \sum_{j \in \{1,2,3\}} \lambda(N_j) = (r + \ell) \sum_{j \in \{1,2,3\}} a_j \leq r + \ell, \]

and hence by Proposition 5.3, \( \phi_\mathcal{D} \) contracts F-curve \( F(N_1, N_2, N_3, N_4) \). ∎
Using Proposition 5.4 in place of Proposition 5.3, we get

**Theorem 6.3.** Let $D = D_{g, \lambda, \ell}$ be a (possibly trivial) conformal blocks divisor such that $\lambda_i \neq 0$ for all $i$ and

$$\sum_{i=1}^{n} \lambda_i (H_\theta) > 2(\ell + 1).$$

Then the morphism $\phi_D$ given by $D$ factors through the contraction map

$$\rho_A : \overline{M}_{0, n} \to \overline{M}_{0, A}, \text{ where } A = \{a_1, \ldots, a_n\}, \quad a_i = \frac{\lambda_i (H_\theta)}{\ell + 1}.$$  

Note that if $D = D_{g, \lambda, \ell}$ is non trivial then by Lemma 1.8, $\sum_{i=1}^{n} |\lambda_i(H_\theta)| \geq 2(\ell + 1)$.

7. The divisor $D = D_{\mathfrak{sl}_{r+1}, \omega, \ell}, n = (r+1)(\ell + 1)$

We next consider the critical level divisor $D = D_{\mathfrak{sl}_{r+1}, \omega, \ell}, n = (r+1)(\ell + 1)$. To determine the F-curves that it contracts, we use a more general form of Theorem 1.11 which we describe below (and prove in Section 16).

7.1. A more general formulation of Theorem 1.11. We next give a generalization of Theorem 1.11 in that the auxiliary bundles can be chosen from any two rows of the $\lambda_i$.

7.1.1. Auxiliary bundles. Given $\tilde{\lambda} \in P(\mathfrak{sl}_{r+1})^n$, such that for each $i \in [n]$, $\lambda_i$ is normalized. For each $i \in [n]$, choose a two element subset $A_i = \{\alpha_i < \beta_i\} \subseteq [r+1]$. Consider associated conformal blocks bundles $V_{\mathfrak{sl}_2, \mu, \ell}$ and $V_{\mathfrak{sl}_{r-1}, \nu, \ell}$ where $\mu_i$ is the $2 \times \ell$ Young diagram formed by the $\alpha_i$th and $\beta_i$th rows of $\lambda_i$, and $\nu_i$ is the $(r-1) \times \ell$ Young diagram formed by removing the $\alpha_i$th and $\beta_i$th rows of $\lambda_i$, $i = 1, \ldots, n$. Note that $\mu_i$ and $\nu_i$ may not be normalized.

7.1.2. Statement of Theorem. It is easy to see that Theorem 1.11 is a special case of the following result, which is proved in Section 16.

**Theorem 7.1.** Given $\tilde{\lambda} \in P(\mathfrak{sl}_{r+1})^n$, such that for each $i$, $\lambda_i$ is normalized. Suppose that:

(a)

$$\frac{1}{2} \sum_{i=1}^{n} |\mu_i| = \frac{1}{r-1} \sum_{i=1}^{n} |\nu_i| = \frac{1}{r+1} \sum_{i=1}^{n} |\lambda_i| = \delta \in \mathbb{Z}$$

(b) Assume that $\ell$ is not greater than the critical level for $\mu$ (one needs to normalize $\mu$ to find the critical level), and $\text{rk} V_{\mathfrak{sl}_2, \mu, \ell} \neq 0$.

(c) If $r > 2$, then $\text{rk} V_{\mathfrak{sl}_{r-1}, \nu, \ell} \neq 0$ (so condition (c) is vacuous for $r = 2$).

Then $D_{\mathfrak{sl}_{r+1}, \tilde{\lambda}, \ell} \neq 0$.

The following addition statement is useful in calculations:
Proposition 7.2. Suppose $\vec{\nu}$ is an $n$-tuple of dominant integral weights in $P((\mathfrak{sl}_r))$ ($\vec{\nu}$ may not be normalized). Suppose $\tilde{r} = a + b$ with $a$ and $b$ positive integers. Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be $n$-tuples of subsets of $[\tilde{r}]$ such that $|A_i| = a$ and $|B_i| = b$, $[\tilde{r}] = A_i \cup B_i$, $i = 1, \ldots, n$. Let $\vec{\nu}_A = (\nu_{1,A_1}, \ldots, \nu_{n,A_n})$ be the $n$-tuple of a $\times \ell$ Young diagrams formed by taking the $A_i$-rows of $\nu_i$ for each $i$ (similarly $\vec{\nu}_B$). Suppose

\begin{equation}
\frac{1}{\tilde{r}} \sum_{i=1}^{n} |\nu_i| = \frac{1}{a} \sum_{i=1}^{n} |\nu_i,A_i| = \frac{1}{b} \sum_{i=1}^{n} |\nu_i,B_i| = \delta \in \mathbb{Z}.
\end{equation}

1. If $a > 1$, then $\text{rk} \mathcal{V}(\mathfrak{sl}_a, \ell, \vec{\nu}_A) \neq 0$.
2. If $b > 1$, then $\text{rk} \mathcal{V}(\mathfrak{sl}_b, \ell, \vec{\nu}_B) \neq 0$.

Then $\text{rk} \mathcal{V}(\mathfrak{sl}_r, \ell, \vec{\nu}) \neq 0$.

Proposition 7.2 will be proved in Section 15.3.

7.2. F-curves contracted by $\mathbb{D}_{\mathfrak{sl}_{r+1}, \omega^a_{\ell}}$, $n = (r + 1)(\ell + 1)$. Propositions 5.3 and 7.3 together give a complete description of the F-curves contracted by the divisors $\mathbb{D}_{\mathfrak{sl}_{r+1}, \omega^a_{\ell}}$.

Proposition 7.3. Suppose that $r \geq 1$ and $\ell \geq 1$. For $n = (r + 1)(\ell + 1)$, the divisor $\mathbb{D} = \mathbb{D}_{\mathfrak{sl}_{r+1}, \omega^a_{\ell}}$ positively intersects all F-curves $F(N_1, N_2, N_3, N_4)$ with $i \in \{1, 2, 3, 4\}$, $|N_i| = n_i$, where $n_1 \leq n_2 \leq n_3 \leq n_4 = (r + 1)(\ell + 1) - \sum_{1 \leq i \leq 3} n_i$, and $\sum_{i=1}^{3} n_i \geq r + \ell + 1$.

This proposition is known for $r = 1$ [Fak12], or $\ell = 1$ [AGSS11, Gia11, GG12]. Our proof carries a larger induction hypothesis, and we prove a stronger statement for these cases.

7.3. Proof of Proposition 7.3. We want to show that any F-curve $F(N_1, N_2, N_3, N_4)$, $|N_i| = n_i$ with $n_1 \leq (r + 1)(\ell + 1) - (r + \ell + 1) = r\ell$, $i = 1, \ldots, 4$ is not contracted by $\mathbb{D}$ (so we drop the hypothesis that $n_1 \leq n_2 \leq n_3 \leq n_4$). By [Fak12],

\[\mathbb{D} \cdot F(N_1, N_2, N_3, N_4) = \sum_{\vec{\lambda} = (\lambda_1, \ldots, \lambda_4) \in P^4_{\ell}} \text{deg} \mathcal{V}_{\vec{\lambda}} \Pi_{i=1}^{4} \text{rk} \mathcal{V}_{\mathfrak{sl}_{r+1}, (\omega_{\ell}^{n_i}, \lambda_i^{0}), \ell},\]

where $\mathcal{V}_{\vec{\lambda}} = \mathcal{V}_{\mathfrak{sl}_{r+1}, \vec{\lambda}}$. This is a sum of nonnegative numbers. Therefore, to show that the sum is nonzero, it is enough to show that there is at least one element $\vec{\lambda} = (\lambda_1, \ldots, \lambda_4) \in P^4_{\ell}$ for which

\[\text{deg} \mathcal{V}_{\vec{\lambda}} \Pi_{i=1}^{4} \text{rk} \mathcal{V}_{\mathfrak{sl}_{r+1}, (\omega_{\ell}^{n_i}, \lambda_i^{0}), \ell} > 0.\]

We note that if $\lambda_i$ are normalized dominant integral weights for $\mathfrak{sl}_{r+1}$ in $P_{\ell}(\mathfrak{sl}_{r+1})$ (so they fit into boxes of size $r \times \ell$) with $|\lambda_i| = n_i$, then $\text{rk} \mathcal{V}_{\mathfrak{sl}_{r+1}, (\omega_{\ell}^{n_i}, \lambda_i^{0}), \ell} > 0$, since this classical, and we may use the Pieri rule. Therefore it suffices to establish the following claim:

Claim 7.4. Suppose $(n_1, n_2, n_3, n_4) \in [r\ell]^{4}$. Then there are Young diagrams $\lambda_i$, $i = 1, \ldots, 4$ fitting into boxes of size $r \times \ell$, so that $|\lambda_i| = n_i$, and

\[\text{deg} \mathcal{V}_{\mathfrak{sl}_{r+1}, (\lambda_1, \lambda_2, \lambda_3, \lambda_4), \ell} > 0.\]
Proof. We will do this by induction on $r$. The weights $\lambda_j$’s will be such Theorem 7.1 is applicable. So in addition to $\tilde{\lambda}$ we will have subsets $A_i = \{\alpha_i < \beta_i\} \subseteq [r + 1], i = 1, \ldots, 4$ and associated conformal blocks bundles $V_{s_{1i}, \tilde{\lambda}}$ and $V_{s_{2r-1}, \tilde{\lambda}}$. This data will be such that conditions (a), (b) and (c) of Theorem 7.1 hold, with $\delta = \ell + 1$. For $r = 1$, $\sum n_i = 2(\ell + 1)$ and $0 < n_i \leq \ell$, so any choice of $\lambda_i$ will work (use Fakhruddin’s result that critical level $\mathfrak{s}_2$ conformal blocks divisors are non-zero).

Assume that the claim holds for $r$ and prove it for $r+1 \geq 2$ as follows. Let $m_1, m_2, m_3, m_4 \in [(r + 1)\ell]$ be positive integers which sum to $(r + 2)(\ell + 1)$.

We get $(n_1, \ldots, n_4) \in [r\ell]^4$ and $(q_1, \ldots, q_4) \in [\ell]^4$ from $(m_1, \ldots, m_4)$ by applying Lemma 7.6 below. Apply the claim (with the stronger burden of induction) for $r$ with data $n_1, \ldots, n_4$, and obtain the data $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}$ etc. Now add on a row of size $q_i$ to $\lambda_i$ and get a new Young diagram $\lambda_i'$ (and permute rows so that one gets a legitimate Young diagram). The old $\mu_i$ corresponds to rows $(\alpha_i' < \beta_i')$ of $\lambda_i'$. The new $\lambda_i'$ satisfies our requirement by using Theorem 7.1 and Proposition 7.2. Note that Proposition 7.2 is applied to the $n$-tuple of weights of $\mathfrak{s}_r$ obtained by adding rows of sizes $q_i$ to $\nu_i$. (so, $a = r - 1$ and $b = 1$).

Corollary 7.5. Suppose that $r > 1$ and $\ell > 1$. Put $n = (r + 1)(\ell + 1)$, $D = D_{\mathfrak{d}_{r+1}, \omega_{\ell}, \ell}$, and $\mathcal{A} = (a_1, \ldots, a_n)$, with $a_i = \frac{1}{r+\ell}$. Then the maps $\phi_D$ and $\rho_A$ contract the same $F$-curves.

Proof. This follows from Theorem 6.2 and Proposition 7.3.

Lemma 7.6. It is possible to write $m_i = n_i + q_i, i = 1, \ldots, 4, m_i, q_i \in \mathbb{Z}$ with $0 < n_i \leq r\ell$ and $0 \leq q_i \leq \ell$ and $\sum n_i = (r + 1)(\ell + 1)$ (so $\sum q_i = \ell + 1$).

Proof. First note that not more than two of the $m_i$ can be one since $(r+1)\ell+3 < (r+2)(\ell+1) = r+2\ell+r\ell+2$.

- If $m_i$ are all $\leq r\ell$: Write $m_i = 1 + \delta_i$. Then $\sum \delta_i = (r + 2)(\ell + 1) - 4 \geq \ell + 1$, since $r + \ell + r\ell \geq 3$. There are at least two of the $m_i$ from which we may subtract, so we may restrict $q_i$ to be $\leq \ell$.
- If $m_1 = r\ell + k, k > 0, 1 \leq m_2, m_3, m_4 \leq r\ell$, suppose $m_2 > 1$. Then subtract $\ell$ from $m_1$ and $1$ from $m_2$. So $q_1 = \ell, q_2 = 1, q_3 = 0, q_4 = 0$.
- If $m_1 = r\ell + k_1, m_2 = r\ell + k_2$ are $> r\ell$, and $1 < m_3, m_4 \leq r\ell$ (up to reordering). Then $k_1 + k_2 \leq (r + 2)(\ell + 1) - 2r\ell = r + 2\ell - r\ell \leq \ell + 1$ since $r + \ell - r\ell - 1 = -(r - 1)(\ell - 1) \leq 0$. So we can subtract $k_1$ from $m_1$ and $(\ell + 1 - k_1)$ from $m_2$. That is $q_1 = k_1, q_2 = (\ell + 1) - k_1$ and $q_3 = q_4 = 0$.
- If $m_1 = r\ell + k_1, m_2 = r\ell + k_2$ and $m_3 = r\ell + k_3$ are $> r\ell$ and $1 < m_4 \leq r\ell$. Then $k_1 + k_2 + k_3 \leq (r + 2)(\ell + 1) - 3r\ell - 1 = [(r + 2)(\ell + 1) - 2r\ell - 2] + [1 - r\ell] \leq \ell + 1 + 0 = \ell + 1$. So we may set $q_1 = k_1, q_2 = k_2, q_3 = (\ell + 1) - (k_1 + k_2)$ and $q_4 = 0$.
- If $m_i > r\ell$ for all $i$, then $4r\ell < (r + 2)(\ell + 1) = r\ell + 2\ell + r + 2$. Hence, $3r\ell - 2\ell - r - 2 < 0$ hence that $(r - 1)(\ell - 2) - 4 + 2r\ell < 0$. If $\ell \geq 2$, then this cannot happen. If $\ell = 1$, then we get $3r - r - 2 < 0$ or that $r < 2$ and hence $r = 1$. Writing $m_i = 1 + \delta_i$, we see that $\sum \delta_i = 6 - 4 = 2$ but $\delta_i > 0$ by assumption, so this case cannot happen.

□
7.4. Examples: GIT constructions and images of conformal blocks maps. While we have shown that many conformal blocks divisors give rise to maps that factor through Hassett spaces, their images are not in general isomorphic to Hassett spaces. As the following examples show, it does seem a likely possibility, that the type A conformal blocks divisors give maps to spaces that have modular interpretations and have constructions as GIT quotients. These particular examples have images that are birational to $\overline{M}_{0,n}$, but in Section 7.4.1, we exhibit a divisor for which all the weights are nonzero and whose corresponding morphism has positive dimensional fibers.

7.4.1. If $D = D_{s_{l+1},i}$ is a nontrivial level one divisor, so that necessarily $\sum |\lambda_i| = (r + 1)(d + 1)$, for some $d \geq 1$, then the image of $\phi_D$ is isomorphic to the generalized Veronese quotient $U_{d,n} / (\alpha_{s_{l+1}}, \omega_i^2, \lambda_i)$ for $a_i = 1/(1 + r)$ \cite{Gia11, GG12}. These spaces, which receive morphisms from $\overline{M}_{0,A}$, are birational to $\overline{M}_{0,n}$ and have modular interpretations \cite{Gia11, GJMT11}. In Corollary 7.5, we prove that for all $\ell \geq 1$, the images of maps given by divisors $D_{s_{l+1},i,\ell}$ are also isomorphic to $U_{d,n} / (\alpha_{s_{l+1}}, \omega_i^2, \lambda_i)$.

7.4.2. For all $\ell \geq 1$, and $r \geq 1$, the non-trivial (by Proposition 7.3) divisor $D = D_{s_{l+1},i,\ell}$ is $S_n$-invariant, and therefore by \cite{KeelMcKernan, Gib09}, it is big, and the corresponding morphism $\phi_D$ is birational. By Corollary 7.5 for $A = (\frac{1}{\ell + r}, \ldots, \frac{1}{\ell + r})$, $\ell > 1$, and $r > 1$, the maps $\rho_A$ and $\phi_D$ contract the same $F$-curves. According to the F-conjecture (Conjecture 5.2), the divisors $D$ and $\rho_A^\ell(A)$, where $A$ is any ample divisor on $\overline{M}_{0,A}$ conjecturally lie on the same face of the nef cone of $\overline{M}_{0,n}$. In particular, the (normalization of the) image of the morphism $\phi_D$ should be isomorphic to $\overline{M}_{0,A}$. Han-Bom Moon has shown that $\overline{M}_{0,A}$ can be constructed as a GIT quotient of $\overline{M}_{0,A}(\mathbb{P}^1, 1)$ by $\text{SL}(2)$. The case $\ell = 1$, the image of $\phi_D$ was shown in \cite{Fak12} to be isomorphic to $(\mathbb{P}^1)^n / (\alpha_{s_{l+1}}, \omega_i^2, \lambda_i)$ for $a_i = 1/(1 + r)$.

7.4.3. Consider the critical level conformal blocks divisor $D = D_{s_{l+1},i,\ell}$ where $\omega_i^2 = 2\omega_1 = (2, 2, \ldots, 0)$. Apply Theorem 7.4 with $\alpha_i = 1, \beta_i = 4$ for $i = 1, \ldots, 14$ and $\alpha_i = 3, \beta_i = 4$ for $i = 15, 16$ to see that $D$ is non-zero. Theorem 6.2 gives us a morphism $\rho_A$, where $a_i = \frac{2}{10}$. However, $F$-curves such that $N_1 \cup N_2 \cup N_3$ corresponds to eight copies of $\omega_2$ are contracted by $\phi_D$ (applying Proposition 5.4), and not by $\rho_A$.

7.4.4. In \cite{Fak12}, Fakhruddin showed that if $D_{s_{l+1},i,\ell}$ is a critical level divisor, so necessarily $\sum |\lambda_i| = 2(\ell + 1)$, then the image of $\phi_D$ is isomorphic to $\mathbb{P}^1 / (\alpha_{s_{l+1}}, \omega_i^2, \lambda_i)$ for $a_i = |\lambda_i| / (\ell + 1)$. We wonder if a similar statement is true for divisors of the form $D_{s_{2n},i,\ell}$, where $n = 2(\ell + 1)$. We think in this case that $D_{s_{2n},i,\ell}$ is proportional to $D_{s_{l+1},i,\ell}$, and if so then the images will be isomorphic to $(\mathbb{P}^1)^n / (\alpha_{s_{l+1}}, \omega_i^2, \lambda_i)$, where $a_i = 1/(\ell + 1)$, by \cite{Fak12}. Such a GIT scaling identity
holds in general for level one divisors, and using \cite{Swi10} we have checked it holds for higher level for low \( r \). For example, \( r \geq 1 \),

1. \( \mathcal{D}_{\mathfrak{sl}_2, \omega_1^{2(d+1)}, 1} = \frac{1}{k} \mathcal{D}_{\mathfrak{sl}_2, \omega_1^{2(d+1)}, 1} \); \cite{GG12} Prop. 1.3;
2. \( \mathcal{D}_{\mathfrak{sl}_2, \omega_2^2, 2} = \frac{1}{3} \mathcal{D}_{\mathfrak{sl}_4, \omega_2^2, 2} = (\frac{1}{2})^4 \mathcal{D}_{\mathfrak{sl}_6, \omega_2^4, 2} = (\frac{1}{7})^3 \mathcal{D}_{\mathfrak{sl}_8, \omega_2^8, 2} \);
3. \( \mathcal{D}_{\mathfrak{sl}_2, \omega_3^3, 3} = (\frac{1}{2}) \mathcal{D}_{\mathfrak{sl}_4, \omega_3^3, 3} = (\frac{1}{7})^5 \mathcal{D}_{\mathfrak{sl}_6, \omega_3^6, 3} \);
4. \( \mathcal{D}_{\mathfrak{sl}_2, \omega_3^4, 4} = (\frac{1}{2}) (\frac{1}{7}) (\frac{1}{19}) \mathcal{D}_{\mathfrak{sl}_6, \omega_3^6, 4} \).

In particular, while the corresponding maps factor through the Hassett spaces, their images are not Hassett spaces, but are GIT quotients of \((\mathbb{P}^1)^n\) by \( \text{SL}(2) \).

8. Analogous vanishing above the theta level

8.1. Proof of Lemma 1.8. The first and second statements in Lemma 1.8 are standard (the first one also follows from the reasoning below).

For (3) let \( x = (z_1, \ldots, z_n) \) be an \( n \)-tuple of distinct points in \( A^1 \subseteq \mathbb{P}^1 \). Set \( A = A_{g, \lambda} \) and denote by \( C_x \) the image of the map \( T_x^{r+1} : W \to W \), where \( W = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \) and \( T_x = \sum_{i=1}^n z_i e^{(i)}_\theta \) with \( e^{(i)}_\theta \) acting on the \( i \)th coordinate.

Then by \cite[Proposition 4.1]{Bea96} and \cite[Section 1.1]{FSV95}, the fiber of \( V_{g, \lambda, \ell} \) at \( x \) is the cokernel of the natural map \( C_x \to A \). Now assume \( \sum \lambda_i(H_\theta) < 2(\ell + 1) \). Writing elements of \( V_{\lambda_i} \) as obtained from lowest weight vectors by application of the operators \( e_\alpha \), we see that any element of \( C_x \) is a sum of eigenvectors for \( H_\theta \) with strictly positive eigenvalues, and hence maps to zero under \( C_x \to A \). Therefore \( C_x \to A \) is the zero map. This proves (3).

8.2. Comparison between critical and theta levels. It is easy to see that:

\[ \theta(\mathfrak{sl}_{r+1}, \lambda) < \ell(\mathfrak{sl}_{r+1}, \lambda) \iff \frac{1}{r} \leq \frac{\sum_{i=1}^n \lambda^{(1)}_i}{\sum_{i=1}^n |\lambda_i|} < \frac{2}{r+1} \].

If for example, \( \lambda_i = \lambda_i^{(1)} \omega_1 \) for all \( i \), then the theta level is never the better one:

\[ (\ell(\mathfrak{sl}_{r+1}, \lambda) + 1) = \frac{2}{r+1} \theta(\mathfrak{sl}_{r+1}, \lambda) + 1) \].

As \( r \) grows, the critical level becomes increasingly smaller than the theta level. The opposite is true for \( \omega_r \).

8.3. The fixed part of conformal blocks. In the setting of Lemma 1.8, consider the “fixed part”:

\[ \mathcal{F}(\mathfrak{g}, \ell, \lambda) = \cap_{x \in \mathcal{M}_{0,n}} (\mathcal{V}_{\mathfrak{g}, \lambda, \ell})^+_x \subseteq A^* = A^*_{g, \lambda} \]

(note that the intersection is the same if we intersect over all points \( x \in \overline{\mathcal{M}_{0,n}} \)). Note that \( c_1 \mathcal{V}_{\mathfrak{sl}_{r+1}, \lambda, \ell} \neq 0 \) if and only if the rank of \( \mathcal{F}(\mathfrak{g}, \ell, \lambda) \) is smaller than that of \( \mathcal{V}_{\mathfrak{sl}_{r+1}, \lambda, \ell} \).
Let $C' \subseteq V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$ be the $C$-linear span of elements of the form

$$e^{a_1}_\theta v_1 \otimes \ldots \otimes e^{a_n}_\theta v_n, \; v_i \in V_{\lambda_i}, \; 0 \leq a_i \leq \ell + 1, \; \sum_{i=1}^{n} a_i = \ell + 1.$$

Lemma 8.1. (1) $\mathbb{F}(g, \ell, \tilde{\lambda})^*$ is the cokernel of the natural map $C' \to A$.

(2) $c_1 \nabla_{g, \tilde{\lambda}, \ell} = 0$ if and only if $C'$ and $C_x$ have the same image in $A$. $C_x$ was defined in Lemma 1.8, $C_x \subseteq C'$.

Proof. An element $\alpha \in A^*$ is in the fixed part, if and only if $\alpha(T^{\ell+1}_x(v_1 \otimes \ldots \otimes v_n)) = 0$ as a polynomial in $z_1, \ldots, z_n$, where $v_i$ are arbitrary elements of $V_{\lambda_i}$. This polynomial is zero if all its coefficients are zero. So $\alpha \in \mathbb{F}(g, \ell, \tilde{\lambda})$ if and only if $\alpha(C') = 0$ as desired. This gives (1). It is easy to see that (2) follows from (1). □

Part (2) of the corollary below says that sub-critical level conformal blocks divisors for $\mathfrak{sl}_2$ are non-zero as long as their ranks are not equal to zero (this is not surprising given the results in [Fak12] for $\mathfrak{sl}_2$, and should be compared with B. Alexeev’s formula [Swi11, (3.5)]). Part (1) of the corollary will be used in the proof of non-vanishing criteria for conformal blocks divisors (Theorems 1.11 and 7.1).

Corollary 8.2. Suppose $g = \mathfrak{sl}_2$, and $\ell$ the critical level for $\tilde{\lambda}$. Suppose $\tilde{\ell} \leq \ell$ and $\tilde{\lambda}$ is in $P_{\tilde{\ell}}(\mathfrak{sl}_{r+1})$.

1. $\mathbb{F}(g, \tilde{\ell}, \tilde{\lambda}) = 0$.

2. If $\text{rk} \nabla_{g, \tilde{\ell}, \tilde{\lambda}} \neq 0$, then $c_1 \nabla_{g, \tilde{\ell}, \tilde{\lambda}} \neq 0$.

Proof. For (1) it suffices to consider the case $\tilde{\ell} = \ell$. Let $C''$ be the set of $\mathfrak{h}$-invariants of $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$. It is easy to see that $C''$ surjects on to $A$. We will show $C'' \subseteq C'$ and hence prove (1). A tensor $\gamma$ in $C''$ can be written as sum of vectors of the form $e^{a_1}_\theta v_1 \otimes \ldots \otimes e^{a_n}_\theta v_n$ with $v_i$ lowest weight vectors. Since $\gamma$ is $\mathfrak{h}$-invariant, we should have (in each term) $2 \sum a_i - \sum \lambda_i = 0$, so $\sum a_i = \ell + 1$. Therefore $\gamma \in C'$. This gives (1). Now (2) follows from the lemma above and (1). □

Question 8.3. Compute $\text{rk} \mathbb{F}(g, \ell, \tilde{\lambda})$.

9. The proof of Theorem 1.4: The main reductions

For Theorem 1.4 in addition to [Wit95, Agn95], we use the geometric procedure of creating invariants from Schubert calculus [Bel04]. Together with a standard formalism, this leads to a duality isomorphism of classical invariant theory (reviewed in Section 10):

$$A^*_g(\mathfrak{sl}_{r+1}, \lambda) \xrightarrow{\sim} A_g(\mathfrak{sl}_{r+1}, \tilde{\lambda}).$$

The proof of Theorem 1.4 then breaks up into two parts:

1. We prove the equality (1.3) by working over $M_{0,n}$ and using the connection between quantum cohomology and conformal blocks, and a degeneration argument for Gromov-Witten invariants (but working over a fixed point of $M_{0,n}$).
(2) We show that the natural morphism \( \mathcal{V}^{*}\mathfrak{sl}_{r+1,\vec{\lambda},\ell} \to \mathcal{V}^{*}\mathfrak{sl}_{r+1,\vec{\lambda}^r,\ell} \) arising from the isomorphism (9.1) is the zero map. To do this, we invoke the interpretation \([\text{Pan}96]\) of \( \mathcal{V}^{*}\mathfrak{sl}_{r+1,\vec{\lambda},\ell} \) as global sections of line bundles on moduli spaces of parabolic bundles (which is available over \( M_{0,n} \), and not over \( \overline{M}_{0,n} \)). Again, it suffices to work over \( M_{0,n} \).

These two steps lead to an isomorphism (1.2): Let \( \mathcal{K}_{r+1,\vec{\lambda},\ell} \) be the kernel of \( \mathcal{A}_{\mathfrak{sl}_{r+1,\vec{\lambda}}} \to \mathcal{V}_{\mathfrak{sl}_{r+1,\vec{\lambda}},\ell} \):

Consider the exact sequences

\[
0 \to \mathcal{V}^{*}\mathfrak{sl}_{r+1,\vec{\lambda},\ell} \to \mathcal{A}^{*}\mathfrak{sl}_{r+1,\vec{\lambda}} \to \mathcal{K}^{*}_{r+1,\vec{\lambda},\ell} \to 0
\]

and

\[
0 \to \mathcal{K}^{*}_{r+1,\vec{\lambda}} \to \mathcal{A}^{*}_{\mathfrak{sl}_{r+1,\vec{\lambda}}} \to \mathcal{V}_{\mathfrak{sl}_{r+1,\vec{\lambda}},\ell} \to 0.
\]

Step (2) gives rise to vertical arrows from the first exact sequence above to the second. We therefore find a map (1.2). By the snake lemma, (1.2) is a surjective morphism of vector bundles of the same rank, hence an isomorphism.

10. Classical strange duality and the map (9.1)

10.1. We recall some results of [Bel04]: How Schubert calculus of Grassmannians produces natural bases in the spaces of invariants and hence produces the duality isomorphism (9.1) by the (standard) strange duality formalism. The following construction is inspired by the tangent space of a Schubert cell \( \Omega_{\vec{\lambda}}(\mathcal{E}^\bullet) \) in a Grassmannian (see Section 2.1).

Let \( V \) and \( Q \) be vector spaces of dimensions \( r+1 \) and \( \ell+1 \) respectively. Let \( N = r+\ell+2 \) (as before). Let \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \) be an \( n \)-tuple of Young diagrams that each fit into a \((r+1) \times (\ell+1)\) box. Assume

\[
\sum_{i=1}^{n} |\lambda_i| = (r+1)(\ell+1).
\]

We will define a divisor in \( D \subseteq \text{Fl}(V)^n \times \text{Fl}(Q)^n \): A point \( (F^1_\bullet, \ldots, F^n_\bullet, G^1_\bullet, \ldots, G^n_\bullet) \in D \) iff there is a non-zero map \( \phi : V \to Q \) so that for \( a \in [r+1] \) and \( i \in [n] \),

\[
\phi(F^i_a) \subseteq G^i_{\ell+1-\lambda^{(a)}_i}.
\]

We will make this precise by defining \( D \) as a determinantal scheme (cf. Section 14) and also identify \( \mathcal{O}(D) \). We choose to write the answers in a symmetric manner. We set \( T = Q^* \). Note that \( \text{Fl}(Q) \) is canonically identified with \( \text{Fl}(T) \). For \( a \in [1, r+1] \), let \( \mathcal{L}_a \in \text{Pic}(\text{Fl}(V)) \) be the pull back of the ample generator (top exterior power of the dual of the universal subbundle) of \( \text{Gr}(a, V) \) by the tautological map

\[
\text{Fl}(V) \to \text{Gr}(a, V), \ F_\bullet \mapsto F_a.
\]

For non-negative integers \( \nu_1, \ldots, \nu_{r+1} \), define a Young diagram \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r+1)}) \) by

\[
\lambda^{(a)} = \nu_a + \nu_{a+1} + \cdots + \nu_{r+1}
\]

and a line bundle \( \mathcal{L}_\lambda = \mathcal{L}_1^{\nu_1} \otimes \cdots \otimes \mathcal{L}_{r+1}^{\nu_{r+1}} \), whose fiber over \( F_\bullet \) is denoted by \( \mathcal{L}_\lambda(V, F_\bullet) = \mathcal{L}_\lambda(F_\bullet) \).
Proposition 10.1. (Borel-Weil) The following are isomorphic as representations of $GL(V)$.

$$H^0(\text{Fl}(V), L_\lambda) = V_{\lambda}^*$$

For a Young diagram $\lambda$, define a bundle $\mathcal{P}_\lambda$ on $\text{Fl}(V) \times \text{Fl}(Q)$ with fiber over $(F_\bullet, G_\bullet)$ given by the formula,

$$\{ \phi \in \text{Hom}(V, Q) | \phi(F_t) \subseteq G_{t+1-\lambda(t)}, t \in [r+1] \}.$$  

We can view $D$ as the zero locus of the determinant of the morphism on $\text{Fl}(V)^n \times \text{Fl}(Q)^n$ which reads at a point $(F_1^\bullet, \ldots, F_n^\bullet, G_1^\bullet, \ldots, G^n_\bullet)$ as

$$\text{Hom}(V, Q) \to \bigoplus_{i=1}^n \text{Hom}(V, Q)_{\lambda_i}(F_i^\bullet, G_i^\bullet).$$

Recall that if $\mathcal{V} \to \mathcal{W}$ is a map between vector bundles of the same rank on a space $S$, then one gets a canonical element $s \in H^0(S, \det \mathcal{W} \otimes (\det \mathcal{V})^*)$. In this way we get a line bundle on $\text{Fl}(V)^n \times \text{Fl}(Q)^n$ with a canonical section whose zero locus is $D$. This line bundle has fiber

$$\bigotimes_{i=1}^n (\mathcal{L}_{\lambda_i}(F_i^\bullet) \otimes \mathcal{L}_{\mu_i}(G_i^\bullet)) \otimes (\det V)^{\ell+1} \otimes (\det Q)^{r+1},$$

where $\mu_i$ is the partition conjugate to the partition $(\ell + 1 - \lambda_i^{(r+1)}, \ldots, \ell + 1 - \lambda_i^{(1)})$. See [Bel04] (and [BK10], formula 16, where $\mu_i$ is called the $(\ell + 1)$-flip of $\lambda_i$).

More symmetrically, one can view $s$ as a canonical section of a line bundle of $\text{Fl}(V)^n \times \text{Fl}(T)^n$, $T = Q^*$ whose fiber over $(F_1^\bullet, \ldots, F_n^\bullet, G_1^\bullet, \ldots, G^n_\bullet)$ is

$$\bigotimes_{i=1}^n (\mathcal{L}_{\lambda_i}(F_i^\bullet) \otimes \mathcal{L}_{\lambda_i^T}(G_i^\bullet)) \otimes (\det V)^{\ell+1} \otimes (\det T)^{r+1}.$$

Here $G_i^\bullet \in \text{Fl}(T)$ correspond to $G_i^\bullet \in \text{Fl}(Q)$ under the identification $\text{Fl}(T) = \text{Fl}(Q)$. It is easy to see that $s$ is $GL(V) \times GL(T)$ invariant.

We find therefore an element

$$s \in H^0(\text{Fl}(V)^n, \bigotimes_{i=1}^n \mathcal{L}_{\lambda_i})_{\text{SL}(V)} \otimes H^0(\text{Fl}(T)^n, \bigotimes_{i=1}^n \mathcal{L}_{\lambda_i^T})_{\text{SL}(T)} = A_{s_{\ell+1}}^* \otimes A_{s_{\ell+1}}^* \overline{\lambda}^r$$

and a duality map

$$A_{s_{\ell+1}} \overline{\lambda}^r \to A_{s_{\ell+1}}^* \overline{\lambda}^r.$$  

10.2. Relations to Schubert calculus. Let $W$ be an $N$-dimensional vector space equipped with $n$ flags $H_1^W, \ldots, H^n_\bullet$, in general position. Let $V$ and $Q$ be as above. By Kleiman transversality, the intersection $\cap_{i=1}^n \Omega_{\lambda_i}(H_i^W)$ is finite, and transverse. Let $V_1, \ldots, V_m$ be the points in this intersection, and let $Q_1, \ldots, Q_m$ the corresponding quotients, $Q_a = W/V_a$. Each of the vector spaces $V_1, \ldots, V_m, Q_1, \ldots, Q_m$ receive $n$ canonical induced flags (from $W$). and hence give $GL(V)$ and $GL(Q)$-orbits in $\text{Fl}(V)^n$ and $\text{Fl}(Q)^n$ respectively (by choosing isomorphisms $V_a \to V$, and $Q_a \to Q$). We can dualize the $GL(Q)$ orbits to obtain $GL(T)$ orbits in $\text{Fl}(T)^n$ corresponding to each of $Q_1, \ldots, Q_m$ (see Section 2.3).

Choose orbit representatives $x_1, \ldots, x_m \in \text{Fl}(V)^n$ and $y_1, \ldots, y_m \in \text{Fl}(T)^n$. From the geometry of the situation it is known that $m = \text{rk}(\bigotimes_{i=1}^n V_{\lambda_i}^*)_{\text{SL}(V)} = \text{rk}(\bigotimes_{i=1}^n V_{\lambda_i}^*)_{\text{SL}(T)}$, and that $s(x_i, y_j) \neq 0$ if and only if $a = b$. Therefore the sections $s(\cdot, y_a)$ (well defined up to scalars) form a basis for $(\bigotimes_{i=1}^n V_{\lambda_i}^*)_{\text{SL}(V)}$ (this was the main result of [Bel04]). Hence by
the standard strange duality formalism, the duality map (10.2) is an isomorphism, also see Lemma 11.1 with quotient stacks \( M = \text{Fl}(V)^n / \text{SL}(V) \) and \( N = \text{Fl}(T)^n / \text{SL}(T) \) (and \( \mathcal{A} \) and \( \mathcal{B} \) are descents of \( \otimes_{i=1}^n \mathcal{L}_{\lambda_i} \) and \( \otimes_{i=1}^n \mathcal{L}_{\lambda_i^T} \)).

Remark 10.2. We recall the reason for the vanishing \( s(x_a, y_b) = 0 \) for \( a \neq b \): The natural non-zero map \( \phi : V_a \to Q_b \) (inclusion in \( W \) followed by projection to \( Q_b \)) satisfies the conditions of (10.1). It is also easy to see that \( s(x_a, y_a) \neq 0 \): If \( (x_a, y_a) \in D \) then any map \( \phi \) in the definition of \( D \) gives us an element in \( \cap_{i=1}^n T\Omega_{\lambda_i}(F_i^*)_{V_i} = 0 \) (by transversality).

The map (1.2) is defined to be the inverse of (10.2). By Lemma 11.1 we have explicit control of (1.2) when we lay out a suitable enumerative problem.

11. A UNIVERSAL SITUATION

We analyze the “strange duality” setting of spaces \( M, N \) equipped with line bundles \( \mathcal{A} \) and \( \mathcal{B} \) and a section \( s \) of \( \mathcal{A} \otimes \mathcal{B} \) in some detail (an instance of this has appeared in Section 10.2). We will use this setting again in Section 14.

Fix the following data:

1. Let \( M \) and \( N \) be spaces (or quotient stacks \( X/G \)) with line bundles \( \mathcal{A} \) and \( \mathcal{B} \) respectively. Assume that \( H^0(M, \mathcal{A}) \) and \( H^0(N, \mathcal{B}) \) are both finite dimensional of the same dimension \( m \).

2. Suppose we are given a section \( s \) of \( \mathcal{A} \otimes \mathcal{B} \) on \( M \times N \). This gives rise to a (possibly degenerate) “duality” \( s \in H^0(M, \mathcal{A}) \otimes H^0(N, \mathcal{B}) \), or a map \( D : H^0(M, \mathcal{A})^* \to H^0(N, \mathcal{B}) \).

Now suppose we are able to manufacture \( x_1, \ldots, x_m \in M \) and points \( y_1, \ldots, y_m \in N \) so that \( s(x_a, y_b) = 0 \) if \( a \neq b \) and non-zero (as an element of \( \mathcal{A}_{x_a} \otimes \mathcal{B}_{y_a} \)) if \( a = b \). Set \( \alpha_a = s(x, y_a) \in H^0(M, \mathcal{A}) \otimes \mathcal{B}_{y_a} \) and \( \beta_a = s(x_a, y) \in \mathcal{A}_{x_a} \otimes H^0(N, \mathcal{B}) \).

Lemma 11.1. \((1) \)

\begin{equation}
(11.1) \quad s = \sum_{a=1}^m s(x_a, y_a)^{-1} \alpha_a \beta_a.
\end{equation}

\((2) \) The duality map \( D : H^0(M, \mathcal{A})^* \to H^0(N, \mathcal{B}) \) is an isomorphism. It carries the element (well defined up-to scalars) “evaluation at \( x_a \)” in \( H^0(M, \mathcal{A})^* \) to the section \( \beta_a \) (up-to scalars).

Proof. From the given data, it is clear that \( \{ \alpha_a : a \in [m] \} \) (resp. \( \{ \beta_a : a \in [m] \} \)) are linearly independent (we can ignore the twists by constant lines) and hence form a basis of \( H^0(M, \mathcal{A}) \) (resp. \( H^0(N, \mathcal{B}) \)) respectively. Therefore there is an expression of the form

\[ s = \sum_{a=1}^m \sum_{b=1}^m \gamma_{a,b} \alpha_a \beta_b \]

with \( \gamma_{a,b} \in \mathcal{A}_{x_a}^{-1} \otimes \mathcal{B}_{y_b}^{-1} \). Evaluating this equation at points of the form \( (x_a, y_b) \), we obtain (11.1). It is easy to see that (2) follows from (11.1). \( \square \)
12. Proof of Corollary 1.5 Part a

We set $N = r + \ell + 2$, and introduce the notation $I^p = \{i_1^p < \cdots < i_{r+1}^p\}$ for $p \in S = \{p_1, \ldots, p_n\}$, by the formula $\lambda_i^{(p)} = (\ell + 1) + a - i_a^p$ for $a \in [r + 1], i \in [n]$. Since $\vec{\lambda}$ is an $n$-tuple of normalized weights, $\tilde{i}^{p+1}_r = N$ and $i^p \neq 1$. We are next going to establish three enumerative interpretations: for $\text{rk} A_{sl_{r+1},\vec{\lambda}}$, $\text{rk} V_{sl_{r+1},\vec{\lambda},\ell}$ and $\text{rk} V_{sl_{r+1},\vec{\lambda}^T,r}$.

12.0.1. The enumerative problem corresponding to $\text{rk} A_{sl_{r+1},\vec{\lambda}}$. We first note that $A_{sl_{r+1},\vec{\lambda}}$ and $A_{sl_{r+1},\vec{\lambda}^T}$ have the same rank (by their enumerative interpretation and Grassmann duality, see Section 2.3) and that these ranks equal the following enumerative number: Let $\mathcal{W} = \mathcal{O}^{\oplus N}$. Choose a general point $E \in \text{Fl}_S(\mathcal{W})$. The enumerative problem is: Count subbundles $\mathcal{V}$ of degree 0 and rank $r + 1$ of $\mathcal{W}$ such that $\mathcal{V}_{p_i} \subset \Omega_\lambda(E_{p_i}^p) \subset \text{Gr}(r + 1,\mathcal{W}_{p_i}), i = 1, \ldots, n$.

Note that this is the same problem as counting classical intersection of generic translates of Schubert varieties (corresponding to $\lambda_i$) in a Grassmannian $\text{Gr}(r + 1, N)$.

12.0.2. The enumerative problem corresponding to $\text{rk} V_{sl_{r+1},\vec{\lambda},\ell}$. Let $\mathcal{W'} = \mathcal{O}(1) \oplus \mathcal{O}^{\oplus N-2}$. Choose a general point $E' \in \text{Fl}_S(\mathcal{W'})$. Count subbundles $\mathcal{V}$ of degree 0 and rank $r + 1$ of $\mathcal{W'}$ such that $\Omega_\lambda(E_{p_i}^p) \subset \text{Gr}(r + 1,\mathcal{W}_{p_i}), i = 1, \ldots, n$.

12.0.3. The enumerative problem corresponding to $\text{rk} V_{sl_{r+1},\vec{\lambda}^T,r}$. The enumerative problem corresponding to $V_{sl_{r+1},\vec{\lambda}^T,r}$ is of counting subbundles of $\mathcal{W}'$ of degree 0 and rank $\ell + 1$ of $\mathcal{W}' = \mathcal{O}(1) \oplus \mathcal{O}^{\oplus N-2}$ subject to incidence conditions at the points $p \in S$. We “Grassmann dualize” this problem by considering the dual of quotients $\mathcal{V} = (\mathcal{W}'/\mathcal{T})^* \subseteq (\mathcal{W}')^*$. The association $\mathcal{T}$ with $\mathcal{V}$ is $1 - 1$ and we may write down equivalent conditions on $\mathcal{V}$.

The resulting enumerative problem is the following: Let $\overline{\mathcal{W}} = \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus N-2}$. Choose a general point $E' \in \text{Fl}_S(\overline{\mathcal{W}})$. Count subbundles $\mathcal{V}$ of degree $-1$ and rank $r$ of $\overline{\mathcal{W}}$ such that $\mathcal{V}_{p_i} \subset \Omega_\lambda(E_{p_i}^p) \subset \text{Gr}(r + 1,\overline{\mathcal{W}}_{p_i}), i = 1, \ldots, n$.

12.1. Outline of proof of (1.3). The main idea is to degenerate the enumerative problem corresponding to $A_{sl_{r+1},\vec{\lambda}}$ by replacing $\mathcal{O}^{\oplus N}$ by its simplest degeneration $\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus(N-2)}$. The quotient scheme corresponding to $\mathcal{O}^{\oplus N}$ (actually a Grassmannian) degenerates (flatly) into a union of two smooth quot schemes intersecting transversally.

For every bundle $\mathcal{W}$ of rank $N$ and degree 0 we can pose an enumerative problem (to ensure conservation of numbers, we will work with compactified versions of the enumerative problems, see Section 1.5): Fix a general $E \in \text{Fl}_S(\mathcal{W})$ and “count” coherent subsheaves $\mathcal{V} \subset \mathcal{W}$ of degree 0 and rank $r + 1$ such that $\mathcal{V}_{p} \rightarrow \mathcal{W}_{p}/E_{p}^{p}$ has kernel of dimension at least $a$ for all $a \in [r + 1]$, and $p \in S$. Note that for some $\mathcal{W}$ the above problem is not enumerative, i.e., the solution scheme is not of the expected dimension (or non-reduced). But we will show that for $\mathcal{O}^{\oplus N}$ and $\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus(N-2)}$, this problem is enumerative.

When $\mathcal{W} = \mathcal{O}^{\oplus N}$, this is the counting problem of Section 12.0.1. Now replace $\mathcal{O}^{\oplus N}$ by $\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus(N-2)}$. The degenerate enumerative number breaks up into two parts and we identify these parts as ranks of conformal blocks. We show that the enumerative counts are conserved in Section 12.7 and hence obtain (1.3).
12.2. Quot schemes and degenerations. Let $\text{Quot}$ be the quot scheme of degree $0$ and rank $r+1$-subsheaves of $\mathcal{W} = \mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus N-2}$. Note that the surjection $\mathcal{W} \to \mathcal{O}(-1)$ and the inclusion $\mathcal{O}(1) \to \mathcal{W}$ are canonical (up-to scalars). Hence the corresponding kernel and quotients are canonical.

Consider the evenly split sheaves $\mathcal{W}' = \mathcal{O}(1) \oplus \mathcal{O}^{\oplus N-2} \subseteq \mathcal{W}$ and $\mathcal{W} = \mathcal{W}/\mathcal{O}(1)$. Let $\text{Quot}_1$ be (smooth) quot scheme of degree $0$ rank $r+1$ subsheaves of $\mathcal{W}'$ and $\text{Quot}_2$ the (smooth) quot scheme of degree $-1$ and rank $r$ subsheaves of $\mathcal{W}$.

**Lemma 12.1.**

1. $\text{Quot} = \text{Quot}_1 \cup \text{Quot}_2$ where $\text{Quot}_1$ and $\text{Quot}_2$ are smooth subschemes and $C = \text{Quot}_1 \cap \text{Quot}_2$ is a subscheme of smaller dimension.

2. $\text{Quot}$ is smooth in the complement of $\text{Quot}_1 \cap \text{Quot}_2$.

3. Points of $\text{Quot} \setminus \text{Quot}_1 \cap \text{Quot}_2$ correspond to subbundles of $\mathcal{W}$.

4. The dimensions of $\text{Quot}_1$ and $\text{Quot}_2$ equal $(r+1)(\ell+1)$.

12.3. Proof of Lemma 12.1. Consider a $\mathcal{V} \subseteq \mathcal{W}$ such that $\mathcal{Q} = \mathcal{W}/\mathcal{V}$ is a point of $\text{Quot}$. Clearly $\mathcal{V}$ cannot have factors of the type $\mathcal{O}(a)$, $a \geq 2$ as these do not admit non-zero maps to $\mathcal{W}$. Any map $\mathcal{O}(1) \to \mathcal{W}$ has image inside the standard copy $\mathcal{O}(1) \subseteq \mathcal{W}$. Therefore there cannot be more than one $\mathcal{O}(1)$ in such a $\mathcal{V}$.

The maps $\text{Quot}_1 \to \text{Quot}$, $j = 1,2$ are clear. Let $\mathcal{V} \subseteq \mathcal{W}$ correspond to a point on $\text{Quot}$. If the composite $\mathcal{V} \to \mathcal{W} \to \mathcal{O}(-1)$ is the zero map then the point is in $\text{Quot}_1$. Suppose that the map is not zero. Then $\mathcal{V}$ has a factor of $\mathcal{O}(1)$ which maps isomorphically to the canonical copy of $\mathcal{O}(1)$ in $\mathcal{W}$. So $\mathcal{V}$ is determined by the sub-sheaf $\mathcal{V}/\mathcal{O}(1) \subseteq \mathcal{W}$ i.e., a point of $\text{Quot}_2$.

Points in $C$ correspond to subsheaves of $\mathcal{W}'/\mathcal{O}(1)$ of degree $-1$ and rank $r$. By an easy computation, $\dim \text{Quot}_1 = \dim \text{Quot}_2 = \dim C + 1$. Let $A = \text{Quot}_1 \setminus C \subseteq \text{Quot}$ and $B = \text{Quot}_2 \setminus C$.

At points of $A$, $\mathcal{V}$ is of the form $\mathcal{O}^{\oplus r+1}$. In this case clearly $H^1(\mathbb{P}^1, \mathcal{V}^* \otimes \mathcal{W}) = 0$. Long exact sequences in cohomology imply that $H^1(\mathbb{P}^1, \mathcal{V}^* \otimes \mathcal{W}) = 0$ surjects on to $H^1(\mathbb{P}^1, \mathcal{V}^* \otimes \mathcal{W}/\mathcal{V})$, and therefore the last group is zero. Therefore $\text{Quot}$ is smooth at such points.

At points of $B$, $\mathcal{V}$ is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus r-1}$ and $\mathcal{S} = \mathcal{O}(1) \oplus \mathcal{O}^{\oplus N-2}$ surjects onto $\mathcal{W}/\mathcal{V}$. It is easy to see that $H^1(\mathbb{P}^1, \mathcal{V}^* \otimes \mathcal{S}) = 0$ and hence the required vanishing $H^1(\mathbb{P}^1, \mathcal{V}^* \otimes \mathcal{W}/\mathcal{V}) = 0$ follows. Therefore $\text{Quot}$ is smooth at such points.

12.4. Degeneration of the Gromov-Witten numbers. With $\mathcal{W}$ as above, consider a generic point $\mathcal{E} \in \text{Fl}(\mathcal{W})$. For $p \in S$, the fiber $\mathcal{W}_p$ has a canonical quotient $\tau_p : \mathcal{W}_p \to L_p$ (corresponding to $\tau : \mathcal{W} \to \mathcal{O}(-1)$) and a canonical line $M_p \subseteq \mathcal{W}_p$ (corresponding to $\mathcal{O}(1) \subseteq \mathcal{W}$). In addition, $\tau_p(M_p) = 0$. Genericity of the flags implies that $E^p_1$ surjects onto $L_p$ via $\tau_p$ and $E^p_1 \cap M_p = \{0\}$. The induced flags $\mathcal{E}' \in \text{Fl}_S(\mathcal{W}')$ (note $\ker \tau = \mathcal{W}'$) are therefore suitably general. Note that $E^p_1$ is the flag

$$0 \subseteq E^p_2 \cap \mathcal{W}_p' \subseteq E^p_3 \cap \mathcal{W}_p' \subseteq \cdots \subseteq E^p_N \cap \mathcal{W}_p' = \mathcal{W}_p'.$$

Similarly the induced flags $\overline{\mathcal{E}} \in \text{Fl}_S(\mathcal{W})$ (note $\overline{\mathcal{W}} = \mathcal{W}/\mathcal{O}(1)$) are suitably generic. Note that $\overline{E^p_1}$ is the flag (here $\gamma : \mathcal{W} \to \mathcal{W}/\mathcal{O}(1)$)

$$(12.1) \quad 0 \subseteq \gamma(E^p_1) \subseteq \gamma(E^p_2) \subseteq \cdots \subseteq \gamma(E^p_N) = \overline{\mathcal{W}}_p.$$
We now analyze the (degenerate) enumerative problem in Sections 12.0.2 and 12.0.3.

12.5. **Part of the intersection in Quot**. Take a coherent subsheaf \( V \subset \ker \tau = \mathcal{W}'\) with the corresponding induced (generic) flags. This enumerative problem is the same as counting such subsheaves of \( \mathcal{W}'\) such that \( V_p \to \mathcal{W}_p/E^p_i\) has kernel of dimension at least \( a \) for all \( a \in [r+1], \) and \( p \in S.\) So we need \( \mathcal{V}_p \to \mathcal{W}'_p/E^p_i\) to have kernel of dimension at least \( a.\) But \( E^p_i \cap \mathcal{W}_p\) has rank \( i^p_a - 1.\) Let \( J^p = \{ i^p_1 = 1 < \cdots < i^p_N - 1 \} \) and \( E^p_i\) the induced flag on \( \mathcal{W}_p.\) The conditions on \( V\) are therefore that \( V_p \to \mathcal{W}'_p/E^p_i\) has kernel of rank at least \( a.\) This is just the compactified enumerative problem of subsheaves of the evenly split bundle \( \mathcal{W}'\) of degree 0 and rank \( r + 1 + a - i^p_a = (N - 1) - (r + 1) + a - (i^p_a - 1) = \lambda^{(a)}_i, p_i = p.\)

This is the enumerative problem 12.0.2. Using Section 15, we conclude that the intersection in \( \text{Quot}_{1}\) has \( \text{rk} V_{sl(r+1),\lambda,\ell} \) number of points and that the intersection lies entirely in the open part \( A\) (see Remark 15.2).

12.6. **Part of the intersection in Quot**. Take a coherent subsheaf \( \overline{V} \subset \overline{W}\) with the corresponding induced (generic) flags. This enumerative problem is the same as counting such subsheaves (of the evenly-split bundle \( \overline{W}\)) so that, setting \( V\) to be the inverse image of \( \overline{V}\) under the surjection \( \mathcal{W} \to \overline{W}, \) \( V_p \to \mathcal{W}_p/E^p_i\) has kernel of dimension at least \( a.\)

Let \( E^p_i\) be the induced flag on \( \mathcal{W}_p\) (and \( E^p_{N-1} = \mathcal{W}_p.\) By an elementary calculation, the kernel of \( \overline{V} \to \mathcal{W}_p/E^p_i\) is isomorphic to the kernel of \( \mathcal{V}_p \to \mathcal{W}_p/E^p_i\) for \( \ell \leq r.\) For this, write exact sequences

\[
0 \to \mathcal{O}(1)_p \to \mathcal{V}_p \to \overline{V}_p \to 0
\]

\[
0 \to \mathcal{O}(1)_p \to \mathcal{W}_p/E^p_i \to \mathcal{W}_p/E^p_i \to 0.
\]

Also note that \( E^p_i\) has rank \( i^p_a.\)

So we want to count subsheaves of \( \overline{W}\) of rank \( r\) and degree \(-1\) with Schubert conditions given by the data \( \lambda_i\) at \( p = p_i;\) note

\[
(N - 1) - r + \ell - i^p_a = N - (r + 1) + \ell - i^p_a = \lambda^{(a)}_i, a \leq r
\]

\[
N - (r + 1) + (r + 1) - i^p_{r+1} = 0.
\]

Therefore the part of the enumerative problem in Quot 2 is the enumerative problem 12.0.3 and has \( \text{rk} V_{sl(r+1),\lambda,\ell} \) points, and these points lie entirely in \( B\) (again using Remark 15.2).

12.7. **Proof of 12.3**. Consider a family of vector bundles \( \mathcal{W}\) over \( \mathbb{P}^1 \times T\) where \( T\) is a smooth curve such that \( \mathcal{W}_t\) is isomorphic to \( \mathcal{O}^{\oplus m}\) for \( t \neq t_0\) and isomorphic to \( \mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus (n-2)}\) for \( t = t_0.\) Let \( E_{t_0} \in \text{Fl}_S(\mathcal{W}_{t_0})\) be generic and consider a family of \( E_t \in \text{Fl}_S(\mathcal{W}_t)\) specializing to \( E_{t_0}\) and specializing at a fixed point \( t_1\) to a general element of \( \text{Fl}_S(\mathcal{W}_{t_1})\) with \( t_1 \neq t_0.\)

We have a relative quot scheme (of quotients of degree 0 and rank \( \ell + 1\) of \( \mathcal{W}_t\)) \( \pi : \text{Quot} \to T\) and the family of solutions to the enumerative problem \( i : \mathcal{C} \hookrightarrow \text{Quot}.\) We note the following:

1. \( \pi\) is proper and \( i\) is closed.
2. \( \pi\) is smooth over \( T - \{ t_0 \}.\)
(3) Each irreducible component of $\mathcal{C}$ has dimension at least one. This is because the map $\mathcal{C} \to T$ is the pull back of an universal $\mathcal{C}' \to T'$ by a map $T \to T'$. It turns out that $\mathcal{C}'$ and $T'$ have the same dimension and hence the assertion follows from standard properties of dimensions of fibers.

(4) $\pi$ is smooth at $A \cup B \subseteq \pi^{-1}(t_0)$. $\pi$ is not smooth at $C \subseteq \pi^{-1}(t_0)$.

(5) $(\pi \circ i)^{-1}(t_0) \subseteq A \cup B$ and $(\pi \circ i)^{-1}(t_0)$ is a reduced scheme consisting of finitely many points.

By shrinking $T$ if necessary around $t_0$ we may assume that each component of $\mathcal{C}$ surjects onto $T$. Since $(\pi \circ i)^{-1}(t_0)$ is a reduced scheme consisting of finitely many points and each irreducible component of $\mathcal{C}$ has dimension at least one, we see that for $c \in \mathcal{C}$ over $t_0$, $\mathcal{O}_{C,c}$ is a discrete valuation ring (the Zariski tangent space has dimension at most one). By shrinking $T$ we can assume $\mathcal{C}$ to be smooth and equidimensional of dimension 1. It is now easy to see that $C \to T$ is finite and étale over a neighborhood of $t_0$. The generic fiber of $C \to T$ has $\text{rk} A_{\mathfrak{sl}_{r+1},\vec{\lambda}}$ number of points, and the special fiber over $t_0$ has $\text{rk} V_{\mathfrak{sl}_{r+1},\vec{\lambda}} r + \text{rk} V_{\mathfrak{sl}_{r+1},\vec{\lambda},\ell}$ number of points. Therefore (1.3) holds.

12.8. The equality (1.3) stated in terms of quantum cohomology. We return to the setting of Section 12. Assume that $\ell$ is the critical level for $\vec{\lambda}$. The rank $\text{rk} A_{\mathfrak{sl}_{r+1},\vec{\lambda}}$ is the coefficient of the class of a point $[\text{pt}]$ in the classical product

$$\sigma_{\lambda_1} \cdot \sigma_{\lambda_2} \cdots \sigma_{\lambda_n} \in H^*(\text{Gr}(r+1, r+\ell+2)).$$

Let $\lambda$ be the $(r+1) \times \ell$ Young diagram $(\ell,0,\ldots,0)$. Then $\text{rk} V_{\mathfrak{sl}_{r+1},\vec{\lambda},\ell}$ (there is a similar expression for $\text{rk} V_{\mathfrak{sl}_{r+1},\vec{\lambda}}$) is the coefficient of $q[\text{pt}]$ in the small quantum product

$$\sigma_{\lambda_1} \ast \cdots \ast \sigma_{\lambda_n} \ast \sigma_{\lambda} \in QH^*(\text{Gr}(r+1, r+\ell+1)).$$

Note that the Grassmannian appearing in (12.3) is different from the one in (12.2). In terms of Gromov-Witten numbers, the sum of

$$\langle \sigma_{\lambda_1}, \ldots, \sigma_{\lambda_n}, \sigma_{(\ell,0,\ldots,0)} \rangle_1 \quad \text{and} \quad \langle \sigma_{\lambda_1^T}, \ldots, \sigma_{\lambda_n^T}, \sigma_{(r,0,\ldots,0)} \rangle_1$$

computed for $\text{Gr}(r+1, r+\ell+1)$ and for $\text{Gr}(\ell+1, r+\ell+1)$ respectively equals the classical coefficient $\langle \sigma_{\lambda_1}, \ldots, \sigma_{\lambda_n} \rangle_0$ computed for $\text{Gr}(r+1, r+\ell+2)$.

Remark 12.2. One may ask whether structure constants in the classical cohomology of a $G/P$ (the above is a special case where $G/P = \text{Gr}(r+1, r+\ell+2)$) similarly decompose as sums of quantum cohomology structure constants (for possibly different groups). Any classical cohomology structure constant can be interpreted as the enumerative problem of counting suitable reductions of the structure group of the trivial principal $G$-bundle on $\mathbb{P}^1$ (subject to incidence conditions). We may replace the trivial principal $G$ bundle by its simplest degeneration and then look at the corresponding enumerative problem.
13. Conformal blocks as generalized theta functions

13.1. Notation.

(1) For a finite dimensional simple Lie algebra \( \mathfrak{g} \), let \( \hat{\mathfrak{g}} \) denote the corresponding affine Lie algebra. For a dominant integral weight \( \lambda \) in \( \mathfrak{p}(\mathfrak{g}) \), let \( V_\lambda \) denote the corresponding finite dimensional representation of \( \mathfrak{g} \) with highest weight \( \lambda \). Let \( \mathcal{H}_{\lambda, \ell} \) denote the corresponding irreducible representation of \( \hat{\mathfrak{g}} \). Note that \( V_\lambda \subseteq \mathcal{H}_{\lambda, \ell} \) (we simply write \( \mathcal{H}_\lambda \) when the level \( \ell \) is clear from the context).

(2) Let \( \mathcal{G}_{r+1} \) be the affine Grassmannian of rank \( r + 1 \)-vector bundles with trivialized determinants on \( \mathbb{P}^1 \), and trivialized outside of \( p \). \( \mathcal{G}_{r+1} \) is identified with the (ind-variety) \( \text{SL}_{r+1}(\mathbb{C}[[z]])/\text{SL}_{r+1}(\mathbb{C}[[z]]) \) where \( z \) is a local coordinate at \( p \). It carries a natural line bundle \( \mathcal{L} \) such that (a theorem of Kumar and Mathieu) \( H^0(\mathcal{G}_{r+1}, \mathcal{L}^\ell) = \mathcal{H}_{0, \ell}^r \).

(3) A quasi-parabolic \( \text{SL}_{r+1} \) bundle on \( \mathbb{P}^1 \) is a triple \( (\mathcal{V}, \mathcal{F}, \gamma) \) where \( \mathcal{V} \) is a vector bundle on \( \mathbb{P}^1 \) of rank \( r + 1 \) and degree 0 with a given trivialization \( \gamma : \text{det} \mathcal{V} \rightarrow \mathcal{O} \), and \( \mathcal{F} = (F_{\ell, p_1}, \ldots, F_{\ell, p_n}) \in \text{Fl}_s(\mathcal{V}) \) is a collection of complete flag on fibers over \( p_1, \ldots, p_n \). Let \( \mathcal{P}_{\text{ar}, r+1} \) be the moduli stack parameterizing quasi-parabolic \( \text{SL}_{r+1} \) vector bundles on \( \mathbb{P}^1 \).

13.2. Generalized theta functions. It is a well known result, that conformal blocks for \( \mathfrak{sl}_{r+1} \) on a smooth projective curve can be identified with the space of sections, called “generalized theta functions”, of a suitable line bundle on the moduli space of vector bundles of rank \( r + 1 \) with trivial determinant on that curve (see the survey [Sor90]). A parabolic generalization for \( \mathfrak{sl}_{r+1} \) was proved in [Pan96] which we recall now (but only for \( \mathbb{P}^1 \)).

Associated to the data \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathfrak{p}(\mathfrak{sl}_{r+1}) \), we can form a line bundle \( \mathcal{P}(\mathfrak{sl}_{r+1}, \vec{\ell}, \vec{\lambda}) \) on \( \mathcal{P}_{\text{ar}, r+1} \). The fiber over a point \( (\mathcal{V}, \mathcal{F}, \gamma) \) is a tensor product

\[
D(\mathcal{V})^\vec{\ell} \otimes \bigotimes_{i=1}^n \mathcal{L}_{\lambda_i}(\mathcal{V}_{p_i}, F_{\ell, p_i}^{\lambda_i}),
\]

where \( D(\mathcal{V}) \) is the determinant of cohomology of \( \mathcal{V} \) i.e., the line \( \text{det} H^1(\mathbb{P}^1, \mathcal{V}) \otimes \text{det} H^0(\mathbb{P}^1, \mathcal{V})^* \) and the lines \( \mathcal{L}(\mathcal{V}_{p_i}, F_{\ell, p_i}^{\lambda_i}) \) are as defined in Section 13.

It is known that the space of generalized theta functions is canonically identified (up-to scalars) with the dual of the space of conformal blocks [Pan96]: Let \( x = (p_1, \ldots, p_n) \in \mathcal{M}_{0,n} \).

\[
H^0(\mathcal{P}_{\text{ar}, r+1}, \mathcal{P}(\mathfrak{sl}_{r+1}, \vec{\ell}, \vec{\lambda})) \sim (\mathcal{V}_{\mathfrak{sl}_{r+1}, \vec{\lambda}})^*.
\]

The determinant of cohomology \( D(\mathcal{V}) \) carries a canonical section \( \theta \) (note that the determinant of \( \mathcal{V} \) is trivialized) which we normalize by requiring it to be the canonical one when \( \mathcal{V} = \mathcal{O}^{\oplus r+1} \). To construct \( \theta \), let \( P \in \mathbb{P}^1 \) and consider

\[
0 \rightarrow \mathcal{V}(-P) \rightarrow \mathcal{V} \rightarrow \mathcal{V}_P \rightarrow 0
\]

which sets up an isomorphism \( D(\mathcal{V}) = D(\mathcal{V}(-P)) \). But \( \chi(\mathbb{P}^1, \mathcal{V}(-P)) = 0 \) and hence \( D(\mathcal{V}(-P)) \) carries a canonical theta section. So one gets an element \( \theta \) of \( D(\mathcal{V}) \) which does not depend on \( P \) (otherwise we will get non-trivial functions in \( P \in \mathbb{P}^1 \)). We may also apply
the isomorphism (13.1) at level 1 with vacuum representations at \( p_i \) (i.e., \( \lambda_i = 0 \)) to construct \( \theta \). It is easy to see that \( \theta \) vanishes at \( \mathcal{V} \) iff \( \mathcal{V} \) is non-trivial as a vector bundle.

Multiplication by \( \theta \in D(\mathcal{V}) \) sets up an injective map

\[
H^0(\mathcal{P}ar_{r+1}, \mathcal{P}(\mathfrak{sl}_{r+1}, \ell, \lambda)) \to H^0(\mathcal{P}ar_{r+1}, \mathcal{P}(\mathfrak{sl}_{r+1}, \ell + 1, \lambda)).
\]

**Remark 13.1.** The image of (13.2) necessarily vanishes on points \((\mathcal{V}, \mathcal{F}, \gamma) \in \mathcal{P}ar_{r+1} \) with \( \mathcal{V} \) not isomorphic to \( \mathcal{O}\overline{\sum}^{r+1} \).

Note that \( \mathcal{P}ar_{r+1} \) has a classical part, \( \mathcal{P}ar^c_{r+1} \), the open substack where the underlying vector bundle is trivial. It is easy to see that

\[
H^0(\mathcal{P}ar^c_{r+1}, \mathcal{P}(\mathfrak{sl}_{r+1}, \ell, \lambda)) = A^*_{r+1, \lambda}
\]

via the map \( \pi : \text{Fl}(V)^n \to \mathcal{P}ar^c_{r+1} \), where \( V \) is a vector space of dimension \( r + 1 \) with trivialized determinant, which sets up \( \mathcal{P}ar^c_{r+1} \) as a stack quotient \( \text{Fl}(V)^n / \text{SL}(V) \) (\( \pi \) pulls back \( \mathcal{P}(\mathfrak{sl}_{r+1}, \ell, \lambda) \)) to \( \otimes_{i=1}^n \mathcal{L}_\lambda \). We therefore obtain injective maps

\[
H^0(\mathcal{P}ar_{r+1}, \mathcal{P}(\mathfrak{sl}_{r+1}, \ell, \lambda)) \to A^*_{r+1, \lambda}.
\]

which are compatible with (13.2) (because the canonical section of \( D(\mathcal{V}) \) is 1 on \( \text{Fl}(V) \) with our normalization). Note that if the map (13.3) is an isomorphism for \( \ell \) then it is also an isomorphism for \( \ell + 1 \).

As a final compatibility (given (13.1)), we note that (13.3) is dual to the canonical surjection (up to scalars)

\[
A^*_{r+1, \lambda} \to \left( \mathcal{V}_{\mathfrak{sl}_{r+1}, \lambda, \ell} \right)_x.
\]

**14. Proof of Theorem 1.4**

To prove that the composite

\[
(14.1) \quad \mathcal{V}^*_{\mathfrak{sl}_{r+1}, \lambda, \ell} \to A^*_r \mathfrak{A}_{r+1, \lambda} \to A^*_r \mathfrak{A}_{r+1, \lambda, r}^* \to \mathcal{V}^*_{\mathfrak{sl}_{r+1}, \lambda, r}^*,
\]

is the zero map (see Section 9 Step 2), we use the enumerative interpretations of \( A^*_{\mathfrak{sl}_{r+1}, \lambda} \) and \( A^*_{\mathfrak{sl}_{r+1}, \lambda, r} \). In the setting of Sections 11 and 10.2, the duality maps are explicit for sections that are defined by the enumerative problem of classical Schubert calculus. To get explicit representatives for the images of \( \mathcal{V}^*_{\mathfrak{sl}_{r+1}, \lambda, \ell} \) in \( A^*_{\mathfrak{sl}_{r+1}, \lambda} \), we use the (degenerate) enumerative problem of Section 12.4. We will show that there are natural isomorphisms (assuming \( \ell \) is the critical level for \( \lambda \))

\[
H^0(\mathcal{P}ar_{r+1}, \mathcal{P}(\mathfrak{sl}_{r+1}, \ell + 1, \lambda)) \cong A^*_{r+1, \lambda}
\]

\[
H^0(\mathcal{P}ar_{r+1}, \mathcal{P}(\mathfrak{sl}_{r+1}, \ell + 1, \lambda^T)) \cong A^*_{r+1, \lambda^T}.
\]

By Section 2.3 we already know the equality of ranks in the above isomorphisms. The resulting duality map

\[
H^0(\mathcal{P}ar_{r+1}, \mathcal{P}(\mathfrak{sl}_{r+1}, \ell + 1, \lambda)) \cong H^0(\mathcal{P}ar_{r+1}, \mathcal{P}(\mathfrak{sl}_{r+1}, \ell + 1, \lambda^T))^*
\]
can be controlled in terms of a basis of sections coming from the enumerative problem from Section 12.4. This setup and Remark 13.1 allow us to successfully chase elements through the composition (14.1).

**Remark 14.1.** The isomorphism (14.2) is an example of a “strange duality” isomorphism which should be considered classical. In terms of conformal blocks it reads as (recall that \(\sum |\lambda_i| = (r + 1)(\ell + 1)\))

\[
\nabla_{sl_{\ell+1},x,\ell+1}|_x^* \sim \nabla_{sl_{\ell+1},x,\ell+1}\big|_x.
\]

The above isomorphism is identified with the isomorphism (9.1). It is important for our considerations to know that (9.1) is independent of any choices (of \(x = (p_1, \ldots, p_n)\)).

### 14.1. Enlargement of the duality divisor \(D\).

Let \(\mathcal{M} = \mathcal{P}ar_{r+1}, \mathcal{N} = \mathcal{P}ar_{\ell+1}, \mathcal{M}^c = \mathcal{P}ar_{r+1}^c\) and \(\mathcal{N}^c = \mathcal{P}ar_{\ell+1}^c\). There are natural maps \(\mathcal{M}^c \to \mathcal{M}\) and \(\mathcal{N}^c \to \mathcal{N}\).

We will define a divisor in \(D \subseteq \mathcal{M} \times \mathcal{N}\) extending the divisor on \(\mathcal{M}^c \times \mathcal{N}^c\) of Section 10.

A point \((\mathcal{V}, \mathcal{F}, \mathcal{T}, \tilde{\mathcal{G}})\) is in \(D\) iff there is a non-zero map \(\phi: \mathcal{V} \to \mathcal{Q}\) so that for \(a = 1, \ldots, r + 1\) and \(i = 1, \ldots, n\),

\[
\phi(Fa^p_i) \subseteq C_{\ell+1, a_i}^{p_i}.
\]

where \(\mathcal{Q} = T^*\) and \(\mathcal{G} \in \text{Fl}(\mathcal{Q})\) the flags induced from \(\tilde{\mathcal{G}} \in \text{Fl}(\mathcal{T})\).

We can recast this into a “determinantal scheme” as follows: define a locally free sheaf \(\mathcal{K}\) on \(\mathcal{M} \times \mathcal{N} \times \mathbb{P}^1\) as follows. The fiber of \(\mathcal{K}\) at a point \(b = (\mathcal{V}, \mathcal{F}, \mathcal{T}, \tilde{\mathcal{G}})\) is given by (as a vector bundle on \(\mathbb{P}^1\))

\[
0 \to \mathcal{K}_b \to \text{Hom}(\mathcal{V}, \mathcal{Q}) \to \bigoplus_{i=1}^n \text{Hom}(\mathcal{V}_{p_i}, \mathcal{Q}_{p_i}) \to 0.
\]

With the conditions that we have assumed, \(\chi(\mathbb{P}^1, \mathcal{K}_b) = 0\), so there is a canonical element \(s \in D(\mathcal{K}_b)\) which vanishes if and only if \(x \in D\). The determinant of cohomology \(D(\mathcal{K}_b)\) is given by

\[
\bigotimes_{i=1}^n (\mathcal{L}_\lambda(F_{p_i}^p) \otimes \mathcal{L}_{\lambda_i}^T(\tilde{G}_{p_i}^p)) \otimes D(\mathcal{V}^* \otimes T^*).
\]

We can rewrite the above as the following:

\[
(\bigotimes_{i=1}^n (\mathcal{L}_\lambda(F_{p_i}^p)) \otimes D(\mathcal{V}^k) \otimes (\bigotimes_{i=1}^n \mathcal{L}_{\lambda_i}(\tilde{G}_{p_i}^p)) \otimes D(T^*)^{r+1}) = A_{\mathcal{V}, \mathcal{F}} \otimes B_{T, \tilde{\mathcal{G}}}
\]

where we have introduced the notation \(A = \mathcal{P}(\mathfrak{sl}_{r+1}, \ell + 1, \tilde{\lambda})\) and \(B = \mathcal{P}(\mathfrak{sl}_{\ell+1}, r + 1, \tilde{\lambda}^T)\), used the canonical isomorphisms \(D(\mathcal{V}) = D(\mathcal{V}^*)\) and \(D(\mathcal{T}, T^*)\) (the determinants of \(\mathcal{V}\) and \(\mathcal{T}\) are trivialized), and the following (see e.g., [Fal93], Theorem I.1):

**Lemma 14.2.** Let \(\mathcal{V}, \mathcal{W}\) be vector bundles on \(\mathbb{P}^1\). Suppose \(\deg \mathcal{V} = \deg \mathcal{W} = 0\). There is a natural isomorphism \(D(\mathcal{V} \otimes \mathcal{W}) \to D(\mathcal{V})^{rk \mathcal{V}} \otimes D(\mathcal{W})^{rk \mathcal{W}}\) which specializes to the obvious one when \(\mathcal{V} = \mathcal{O}^{rk \mathcal{V}}\) (note that \(D(\mathcal{O})\) is canonically trivial).

Let \(A^c\) (resp. \(B^c\)) be the pull back of \(A\) (resp. \(B\)) under the map \(\mathcal{M}^c \to \mathcal{M}\) (resp. \(\mathcal{N}^c \to \mathcal{N}\)). The duality divisor \(D\) of \(\mathcal{M} \times \mathcal{N}\) pulls back to the duality divisor of Section 10 in \(\mathcal{M}^c \times \mathcal{N}^c\).

We introduce the following notation:
\[ V \in S \text{ trivializations of their determinants.} \]

To set the stage for the final verification, note that (comparing ranks and the enumerative interpretation of Section 12.0.1),

\[(14.4) \quad H^0(\mathcal{M}, \mathcal{A}) \sim H^0(\mathcal{M}^c, \mathcal{A}^c) \sim A_{r+1, \lambda}^* \]

are isomorphisms. There are also natural maps (see (13.2)) \( H^0(\mathcal{M}, \mathcal{A}_t) \to H^0(\mathcal{M}, \mathcal{A}) \). These maps are compatible with (14.4) (see (13.3)). Let us also note that each \( y \in \mathcal{N} \) gives us an element “evaluation at \( y \)” \( \text{ev}_y \in H^0(\mathcal{N}, \mathcal{B}^*) \) which is well defined up-to scalars.

**Remark 14.3.** By Remark 13.1, it follows that the image of \( \text{ev}_y \) vanishes in \( H^0(\mathcal{N}, \mathcal{B}_r)^* \) if the underlying bundle of \( y \) is non-trivial.

14.2. **Geometric data.** Recall the enumerative problem of Section 12.3 corresponding to a general point \( E \in \text{Fl}_5(\mathcal{W}) \) with \( \mathcal{W} = O(1) \oplus O(-1) \oplus O^{\oplus N-2} \). Let \( \alpha = \text{rk} \mathbb{V}_{sl_{r+1}, \lambda, \ell} \), \( \beta = \text{rk} \mathbb{V}_{sl_{r+1}, \lambda, r, \ell} \), so \( \alpha + \beta = \text{rk} A_{sl_{r+1}, \lambda} \). The solutions to the enumerative problem in \( A \subseteq \text{Quot} \) produce vector bundles on \( \mathbb{P}^1 \) equipped with complete (induced) flags \((\mathcal{V}_1, \mathcal{F}_1), \ldots, (\mathcal{V}_a, \mathcal{F}_a)\), where \( \mathcal{F}_i \in \text{Fl}_5(\mathcal{V}_i) \). We also obtain quotients equipped with complete (induced) flags \((\mathcal{Q}_1, \mathcal{G}_1), \ldots, (\mathcal{Q}_a, \mathcal{G}_a)\) where \( \mathcal{Q}_i = \mathcal{W}/\mathcal{V}_i \) and \( \mathcal{G}_i \in \text{Fl}_5(\mathcal{Q}_i) \). Note that the vector bundles \( \mathcal{Q}_i \) are non-trivial since they have \( O(-1) \) as a direct summand.

The solutions to the enumerative problem in \( B \subseteq \text{Quot} \) similarly produce vector bundles and quotients (each equipped with flags at points of \( S \)): \((\mathcal{V}_1', \mathcal{F}_1'), \ldots, (\mathcal{V}_a', \mathcal{F}_a')\) and \((\mathcal{Q}_1', \mathcal{G}_1'), \ldots, (\mathcal{Q}_a', \mathcal{G}_a')\) (again \( \mathcal{V}_a \subseteq \mathcal{W} \) and \( \mathcal{Q}_a = \mathcal{W}/\mathcal{V}_a \)). Note that the vector bundles \( \mathcal{V}_a \) are non-trivial since they have \( O(-1) \) as a subsheaf.

Let \((\mathcal{T}_i, \mathcal{G}_i)\) (resp. \((\mathcal{T}_a', \mathcal{G}_a')\)) be the duals of \((\mathcal{Q}_i, \mathcal{G}_i)\) (resp. \((\mathcal{Q}_a, \mathcal{G}_a)\)), \( i \in [\alpha] \) (resp. \( a \in [\beta] \)). The bundles \( \mathcal{V}_i, \mathcal{T}_i, \mathcal{V}_a, \mathcal{T}_a, i \in [\alpha], a \in [\beta] \) are of degree zero. We choose and fix trivializations of their determinants.

We therefore obtain

1. Points \( x_1, \ldots, x_\alpha; \ x'_1, \ldots, x'_\beta \in \mathcal{M} \). Here, \( x_i \) (resp. \( x'_i \)) are the points \((\mathcal{V}_i, \mathcal{F}_i)\) (resp. \((\mathcal{V}_i', \mathcal{F}_i')\)).
2. Points \( y_1, \ldots, y_\alpha; \ y'_1, \ldots, y'_\beta \in \mathcal{N} \). Here, \( y_i \) (resp. \( y'_i \)) are the points \((\mathcal{T}_i, \mathcal{G}_i)\) (resp. \((\mathcal{T}_a', \mathcal{G}_a')\)).
3. \( s(x_i, y'_a) = 0 \) and \( s(x'_a, y_i) = 0 \) for all \( i \in [\alpha] \) and \( a \in [\beta] \). This is because there are natural non-zero maps \( \mathcal{V}_i \to \mathcal{Q}_a \) “inclusion in \( \mathcal{W} \) followed by projection” (resp. \( \mathcal{V}_a \to \mathcal{Q}_j \)) which satisfy the conditions of (14.3) (with the corresponding flags \( \mathcal{F}_i \) and \( \mathcal{G}_a \) (resp. \( \mathcal{F}_a \) and \( \mathcal{G}_i \))).
4. \( s(x'_a, y'_a) \neq 0 \) iff \( a = b \), and \( s(x_i, y_j) \neq 0 \) iff \( i = j, i, j \in [\alpha] \) and \( a, b \in [\beta] \). This is a consequence of transversality in the enumerative problem (as in Remarks 10.2 and Remark 15.3).
5. By Lemma 11.1, the sections \( \delta_i = s(., y_i) \) and \( \gamma_a = s(., y'_a) \), \( i \in [\alpha], a \in [\beta] \), together form a basis for \( H^0(\mathcal{M}, \mathcal{A}) \).

The desired vanishing of \( \mathbb{V}_{sl_{r+1}, \lambda, \ell}^* \to \mathbb{V}_{sl_{r+1}, \lambda, r, \ell}^* \) follows from
**Proposition 14.4.** The composition of the following maps is zero:

\[ H^0(\mathcal{M}, A_\ell) \to H^0(\mathcal{M}, A) \to H^0(\mathcal{N}, B)^* \to H^0(\mathcal{N}', B_r)^*. \]

*Proof.* The image of a section \( \tau \in H^0(\mathcal{M}, A_\ell) \) in \( H^0(\mathcal{M}, A) \) can be expressed as a linear combination of \( \delta_i \)'s and \( \gamma_a \)'s. Now evaluate such an expression at the points \( x'_a \). By Remark 13.1, \( \tau(x'_a) = 0 \) because the vector bundle \( \mathcal{V}_a \) underlying \( x'_a \) is non-trivial. Therefore, \( \tau \) is a linear combination of the sections \( \delta_i \). By Lemma 11.1, the duality map \( H^0(\mathcal{M}, A) \to H^0(\mathcal{N}, B)^* \) sends \( \delta_i \) to \( ev_{y_i} \). Applying Remark 14.3, we see that the image of \( ev_{y_i} \) in \( H^0(\mathcal{N}', B_r)^* \) is zero (because the vector bundle \( \mathcal{Q}_i^\prime \) underlying \( y_i \) is non-trivial). Hence the image of \( \tau \) in \( H^0(\mathcal{N}', B_r)^* \) is zero as desired. \( \square \)

**Remark 14.5.** The points \( x'_a \) in \( \mathcal{M} \) have non-trivial underlying bundles, and are yet parabolic-semistable for the linearization \( \mathcal{P}(\mathfrak{sl}_{\ell+1}, \ell + 1, \lambda) \). This is because of the existence of parabolic bundles \( y'_a \) and the non-existence of non-zero maps of parabolic bundles from \( x'_a \) to \( y'_a \) (using (4) above and a parabolic generalization of a method of Faltings, see [Bel08b, A.1]).

### 14.3. Strange duality and critical level symmetries.

Several rank-level (or “strange”) dualities have been proposed by many authors (inspired by work in mathematical physics e.g., [NT92, Bel08b, MO07, NT92, Oud11], over smooth pointed curves. In genus 0 these take the form of isomorphisms (well defined up to scalars) over \( M_{0,n} \), the moduli space of \( n \)-pointed genus 0 curves. In genus 0 these take the form of isomorphisms (well defined up to scalars) of the form \( \mathcal{V}_{(\lambda_1, \lambda_2), (\vec{\mu}_1, \vec{\mu}_2)} \to \mathcal{V}_{(\lambda_2, \lambda_1), (\vec{\mu}_2, \vec{\mu}_1)} \).

While our critical level symmetries are reminiscent of this “strange duality”, we emphasize that the vector bundles \( \mathcal{V}_{\mathfrak{sl}_{\ell+1}, \lambda, \ell} \) and \( \mathcal{V}_{\mathfrak{sl}_{\ell+1}, \lambda', \ell} \) on \( \overline{M}_{0,n} \) are not dual. In fact, their ranks may not be the same (see the examples in Section 14.8). Because they are both globally generated, their first Chern classes are base point free, and hence effective. For two effective divisors to be dual, would mean they are trivial.

Moreover, the classical duality (9.1) can (eventually) be viewed as a special case of the general strange duality (14.6) given above (see Remark 14.1). We could say that the critical level identities of this paper are orthogonality relations “via” the strange duality (14.6). So whenever both sides of (14.3) are classical, i.e., coincide with co-invariants, there is perhaps a corresponding symmetry of conformal blocks divisors (at levels \( \ell_1 - 1, \ell_2 - 1 \) for \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \)).

However, in [Muk13], the third author studies identities that do come from standard level-rank dualities associated to conformal embeddings of affine Lie algebras. Very roughly speaking, a conformal embedding of affine Lie algebras \( \mathfrak{g}_1 \oplus \mathfrak{g}_2 \subseteq \mathfrak{g} \) producing a level rank duality, gives rise to a corresponding relation of conformal blocks divisors which takes the shape:

\[ D_{\mathfrak{g}_1, \lambda, \ell_1} + D_{\mathfrak{g}_2, \vec{\mu}, \ell_2} = c \cdot D_{\mathfrak{g}, \vec{\nu}, 1} + E, \]
where $E$ is a sum of boundary divisors and $\psi$ classes and $c \neq 0$, determined by the embedding.

15. Compactifications and Gromov-Witten numbers

Let $\mathcal{W}$ be a vector bundle on $\mathbb{P}^1$ of rank $N$ and degree $-D$. Let $d$ and $\tilde{r}$ be integers with $N > \tilde{r} > 0$ (and $d$, $D$ possibly negative). Define $\text{Gr}(d, \tilde{r}, \mathcal{W})$ to be the moduli space of sub bundles of $\mathcal{W}$ of degree $-d$ and rank $\tilde{r}$. This is an open subset of the quot scheme $\text{Quot}(d, \tilde{r}, \mathcal{W})$ of quotients of $\mathcal{W}$ of degree $d - D$ and rank $N - \tilde{r}$.

**Proposition 15.1.** Suppose $\mathcal{W}$ is evenly-split

1. $\text{Quot}(d, \tilde{r}, \mathcal{W})$ is a smooth projective variety (possibly empty) of dimension $\tilde{r}(N - \tilde{r}) + dN - D\tilde{r}$.
2. $\text{Gr}(d, \tilde{r}, \mathcal{W}) \subseteq \text{Quot}(d, \tilde{r}, \mathcal{W})$ is dense and connected.
3. The subset of $\text{Gr}(d, \tilde{r}, \mathcal{W})$ consisting of evenly split sub bundles $\mathcal{V} \subseteq \mathcal{W}$ such that $\mathcal{W}/\mathcal{V}$ is also evenly split, is open and dense in $\text{Gr}(d, \tilde{r}, \mathcal{W})$.

See Section 12.1 of [Bel08] for a proof of the above (certainly well known).

Consider a 5-tuple $(d, \tilde{r}, D, N, I)$ where $d, D, \tilde{r}, N$ are as above and $I$ assigns to each $p \in S$ an element $I_p \in \left(\begin{array}{c} N \\ \tilde{r} \end{array}\right)$. We will use the notation $I_p = \{i_1^p < \cdots < i_{\tilde{r}}^p\}$ for $p \in S$, also introduce Young diagrams $\lambda_p$ by the formula $\lambda_p(a) = N - \tilde{r} + a - i_a^p$ for $a = 1, \ldots, \tilde{r}$.

Let $\mathcal{W}$ be an evenly split bundle of degree $-D$ and rank $N$. Let $E \in \text{Fl}_S(\mathcal{W})$. Also assume that the expected number of subbundles of $\mathcal{W}$ of degree $-d$ and rank $\tilde{r}$ whose fibers at points $p \in S$ lie in the Schubert cell $\Omega^{(E_p)}_{I_p} \subseteq \text{Gr}(\tilde{r}, \mathcal{W}_p)$ is finite:

\[
(15.1) \quad \tilde{r}(N - \tilde{r}) - D\tilde{r} + dN = \sum_{p \in S} |\lambda_p|.
\]

It is important to work with compact parameter spaces while degenerating enumerative problems. Consider the following modified enumerative problem: Let $E \in \text{Fl}_S(\mathcal{W})$ be general. Count subsheaves $\mathcal{V} \subseteq \mathcal{W}$ such that for all $p \in S, a \in [\tilde{r}]$, the map $\mathcal{V}_p \to \mathcal{W}_p/E_{i_a^p}$ has kernel of dimension at least $a$. The set of such $\mathcal{V}$ is parameterized by $\text{Quot}(d, \tilde{r}, \mathcal{W})$. It is a result of Bertram [Ber97] that any such $\mathcal{V}$ necessarily lies in $\text{Gr}(d, \tilde{r}, \mathcal{W})$ when $D = 0$. For completeness, we include a proof here for arbitrary $D$. To count subbundles $\mathcal{V} \subseteq \mathcal{W}$ with the above properties, we form the natural morphism

\[
\pi : \text{Gr}(d, \tilde{r}, \mathcal{W}) \to \prod_{i=1}^n \text{Gr}(\tilde{r}, \mathcal{W}_{p_i})
\]

and count the number of points in the intersection

\[
\Omega = \pi^{-1}\left(\prod_{i=1}^n \Omega_{I_{p_i}}(E_{i_a^p})\right).
\]

Since $\prod_{i=1}^n \text{Gr}(\tilde{r}, \mathcal{W}_{p_i})$ is a homogeneous space and $\text{Gr}(d, \tilde{r}, \mathcal{W})$ a smooth variety (and the flags general), $\Omega$ is a transverse intersection consisting of finitely many points (by Kleiman’s transversality theorem).
Remark 15.2. In fact $\Omega$ may be assumed to be a subset of any predetermined non-empty open subset of $\text{Gr}(d, \tilde{r}, \mathcal{W})$: For example those subbundles $\mathcal{V}$ such that $\mathcal{V}$ and $\mathcal{W}/\mathcal{V}$ are again evenly split. The points of $\Omega$ also map to the intersection of open Schubert cells under $\pi$.

Remark 15.3. Drop the assumption that $\mathcal{W}$ is general. But assume that the flags $\mathcal{E}$ are in general position and pick a point $\mathcal{V} \in \Omega$ such that $\text{Gr}(d, \tilde{r}, \mathcal{W})$ is smooth, and of the expected dimension at $\mathcal{V}$. Then by Kleiman’s transversality theorem, $\Omega$ is a transverse and isolated intersection at $\mathcal{V}$. Let $\mathcal{Q} = \mathcal{W}/\mathcal{V}$, and $\mathcal{F} \in \text{Fl}_s(\mathcal{V})$, $\mathcal{G} \in \text{Fl}_s(\mathcal{Q})$ the induced flags. Transversality and zero dimensionality of the intersection $\Omega$ at $\mathcal{V}$ imply that there are no nonzero maps $\phi : \mathcal{V} \to \mathcal{Q}$ so that for $a = 1, \ldots, r + 1$ and $i = 1, \ldots, n$, the inclusion (14.3) holds. This is because the tangent space to $\text{Gr}(d, \tilde{r}, \mathcal{W})$ at $\mathcal{V}$ is $\text{Hom}(\mathcal{V}, \mathcal{Q})$, and the tangent spaces to Schubert cells are as described in Section 2.1.

15.1. We give a proof of Bertram’s result from above, adapted to our notation. Assume $\mathcal{W}$ is evenly-split and stratify $\text{Quot}(d, \tilde{r}, \mathcal{W})$: Given a function $\epsilon : S \to \mathbb{Z}$ such that $0 \leq \epsilon(p) \leq \tilde{r}$, let $\text{Gr}(\epsilon)$ be the scheme of subsheaves $\mathcal{V} \subseteq \mathcal{W}$ such that the map $\mathcal{V}_p \to \mathcal{W}_p$ has kernel $K_p$ of rank $\epsilon(p)$, $p \in S$. Let $\mathcal{V}$ be a generic point in $\text{Gr}(\epsilon)$ which satisfies the enumerative problem. Suppose the subset $\mathcal{V}_p/K_p$ is a subspace of $\mathcal{W}_p$ in the Schubert cell corresponding to $J^p = \{j_1^p < \cdots < j_{\tilde{r} - \epsilon(p)}^p\}$. We clearly need

$$j_a^p + \epsilon(p) \leq i_{a+\epsilon(p)}, \quad a \in [\tilde{r} - \epsilon(p)].$$

One therefore gets that the codimension of the Schubert cell corresponding to $J^p$ is at least the codimension of the Schubert cell corresponding to $I^p$ minus

$$\sum_{a=1}^{\epsilon(p)} (N - \tilde{r} + a - i_a^p).$$

The dimension of each irreducible component of the solutions to the enumerative problem in $\text{Gr}(\epsilon)$ is therefore (by a simple calculation, using (15.1) and Kleiman transversality) at most

$$\dim \text{Gr}(\epsilon) - \dim \text{Gr}(d, \tilde{r}, \mathcal{W}) + \sum_{p \in S} \sum_{a=1}^{\epsilon(p)} (N - \tilde{r} + a - i_a^p).$$

(15.2)

Lemma 15.4. Suppose $\text{Gr}(\epsilon) \neq \emptyset$. Then, $\text{Gr}(\epsilon)$ is smooth and connected, and

$$\dim \text{Gr}(\epsilon) = \dim \text{Gr}(d, \tilde{r}, \mathcal{W}) - n \sum_{p \in S} \epsilon(p) + \sum_{p \in S} \epsilon(p)(\tilde{r} - \epsilon(p)).$$

Given the lemma, we see that the quantity (15.2), is less than or equal to

$$\sum_{p \in S} \sum_{a=1}^{\epsilon(p)} (a - i_a^p - \epsilon(p)) < 0$$

unless $\epsilon(p) = 0$ for each $p$. Therefore the enumerative intersection takes place over the stratum corresponding to $\epsilon(p) = 0$. This stratum has an open dense subset given by $\text{Gr}(d, \tilde{r}, \mathcal{W})$. By Kleiman’s transversality, the intersection occurs entirely on this subset.
15.1.1. Proof of Lemma 15.4. Consider the variety $\mathcal{A}$ of tuples $(K_{p_1}, \ldots, K_{p_s}, V)$ where $V$ is a coherent subsheaf of $W$ of degree $-d$ and rank $r$ and $K_p \subset V_p$ are $\epsilon(p)$ dimensional subspaces such that the composites $K_p \to V_p \to W_p$ are zero for each $p$. Our $\text{Gr}(\epsilon)$ is an open subset of $\mathcal{A}$ and so it suffices to find the dimension of $\mathcal{A}$.

Given a datum such as above we can shift $V$ along the spaces $K_p$ to find a new coherent subsheaf $\tilde{V}$ of $W$ so that $V \subset \tilde{V}$ and the kernel $V_p \to \tilde{V}_p$ is $K_p$ and image $Q_p$ of $V_p \to \tilde{V}_p$ has rank $\tilde{r} - \epsilon(p)$. More precisely, $\tilde{V}$ coincides with $V$ outside of $S$, and in a neighborhood $U_p$ of $p \in S$ sections of $\tilde{V}$ are meromorphic sections of $V$ such that upon multiplication by a uniformizing parameter are in $V_p$ and have fiber in $K_p$. Note also that the point in $\mathcal{A}$ can be recovered from $\tilde{V}$ and the data $Q_p$ (this is a one-one correspondence, i.e., isomorphism of schemes).

Note that $\tilde{V}$ has rank $r$ and degree $-d + \sum_p \epsilon(p)$. The dimension of the space of such $\tilde{V}$ is $\dim \text{Gr}(d, r, W) - n \sum_{p \in S} \epsilon(p)$. Adding the dimension of the Grassmann bundle of $Q_p$, we get the formula given in the lemma.

15.2. Non zero sections of conformal blocks bundles from enumerative geometry. Consider $x = (\mathbb{P}^1, p_1, \ldots, p_n) \in M_{0,n}$. The space $\mathcal{V}_{\mathfrak{s}l_{r+1}, \lambda, \ell}^*$ is identified with $H^0(\mathcal{P}ar_{r+1}, \mathcal{P}_{r+1})$ where $\mathcal{P}_{r+1}$ is the line bundle $\mathcal{P}(\mathfrak{s}l_{r+1}, \ell, \lambda)$. Write $\sum_{i=1}^n |\lambda_i| = (r + 1)\ell - D(r + 1)$. Let $\mathcal{W}$ be an evenly split bundle of rank $r + \ell + 1$ and degree $-D$. Let $E \in \text{Fl}_S(\mathcal{W})$ be a general element. Recall from Theorem 2.2 that the rank of $\mathcal{V}_{\mathfrak{s}l_{r+1}, \lambda, \ell}^*$ equals the number of subbundles $\mathcal{V} \subset \mathcal{W}$ of degree 0 and rank $r + 1$ so that $V_p \in \Omega_{\lambda_p}(\mathfrak{F}_D^*) \subseteq \text{Gr}(r + 1, \mathcal{W}_p)$ for $p \in S$.

Lemma 15.5. The following are equivalent (see [Bel08])

1. $\text{rk} \mathcal{V}_{\mathfrak{s}l_{r+1}, \lambda, \ell}^* \neq 0$.
2. There exist vector bundles $\mathcal{V}$ and $\mathcal{Q}$ of degrees 0 and $-D$ respectively, and ranks $r + 1$ and rank $\ell$ respectively, and $\mathcal{F} \in \text{Fl}_S(\mathcal{V})$ and $\mathcal{G} \in \text{Fl}_S(\mathcal{Q})$ so the vector space

$$\{ \phi_p \in \text{Hom}(\mathcal{V}, \mathcal{Q}) \mid \phi(F_p^\rho) \subseteq G^{\rho}_{\ell - \lambda(a), p = p_i, a \in [r + 1]} \}$$

is zero.

This is a special case of Proposition 5.5 from [Bel08]. For a fixed $y = (\mathcal{Q}, \mathcal{G})$ we may think of (15.3) as defining a section $s_y \in H^0(\mathcal{P}ar_{r+1}, \mathcal{P}_{r+1})$ following Section 14.1. The section $s_y$ does not vanish at $(\mathcal{V}, \mathcal{F}, \gamma) \in \mathcal{P}ar_{r+1}$ if and only if (15.3) is zero. Similarly as in the classical situation [Bel04], the sections $s_y$ over all $y$ span $H^0(\mathcal{P}ar_{r+1}, \mathcal{P}_{r+1})$ [Oud11].

15.3. Proof of Proposition 7.2. Proposition 7.2 can be proved with the same methods as those used in Lemma 15.5. It is however easiest to deduce it from (one of the forms of) the quantum generalization of the Horn conjecture [Bel08 Proposition 3.4].

For every dominant integral weight $\lambda$ of $\mathfrak{s}l_{r+1}$ define a diagonal matrix $\alpha(\lambda, \ell) = \alpha_{ij}$ in the special unitary group $\text{SU}(r + 1)$ with diagonal entries

$$\alpha_{aa} = c^{-1} \exp\left( \frac{2\pi i \lambda(a)}{\ell} \right), \quad c = \exp\left( \frac{2\pi i |\lambda|}{\ell(r + 1)} \right).$$
Consider a conformal blocks bundle $\mathcal{V}_{\mathfrak{sl}_{r+1}, \lambda, \ell}$ such that $(r + 1)$ divides $\sum |\lambda_i|$.

**Proposition 15.6.** [Bel08, Proposition 3.4] The following are equivalent:

1. $\text{rk} \mathcal{V}_{\mathfrak{sl}_{r+1}, \lambda, \ell} \neq 0$.
2. There exist matrices $A_i \in \text{SU}(r + 1)$ with $A_1 A_2 \ldots A_n = \text{Id}$ where each $A_i$ is conjugate to $\alpha(\lambda_i, \ell)$.
3. There exist matrices $A_i$ in the unitary group $U(r + 1)$ with $A_1 A_2 \ldots A_n = \gamma \text{Id}$ where each $A_i$ is conjugate to a diagonal matrix with entries $\exp\left(\frac{2 \pi i \lambda_i(n)}{\ell}\right)$, and $\gamma = \exp\left(\frac{2 \pi i \sum |\lambda_i|}{2(r+1)}\right)$.

Proposition 15.2 follows immediately. This is because if we have matrices $A_i$ in $U(a)$ and $B_i \in U(b)$, with $A_1 A_2 \ldots A_n = \gamma \text{Id}$ and $B_1 B_2 \ldots B_n = \gamma \text{Id}$, then we can form matrices $C_i \in U(a + b)$ with $C_1 C_2 \ldots C_n = \gamma \text{Id}$ by direct sum. If $b = 1$, we let $B_i = (b_i)$ be the $1 \times 1$ matrices with $b_i = \exp\left(\frac{2 \pi i \mu_i(n)}{\ell}\right)$.

### 16. Non-vanishing criteria: Proof of Theorem 7.1

#### 16.1. Weyl group translates of highest weight vectors. Let $S = \mathbb{C}^{r+1}$ with basis vectors $\epsilon_1, \ldots, \epsilon_{r+1}$ and dual basis $L_1, \ldots, L_{r+1}$. Let $U = \mathbb{C} \epsilon_1 \oplus \mathbb{C} \epsilon_2$ and $W = \mathbb{C} \epsilon_3 \oplus \cdots \oplus \mathbb{C} \epsilon_{r+1}$ so that one has an internal direct sum $S = U + W$. There is a natural map $GL(U) \times GL(W) \to GL(S)$. Identify $GL(U) = GL(2), GL(W) = GL(r - 1)$ and $GL(S) = GL(r + 1)$ in the evident way. Let $\mathfrak{h}_U, \mathfrak{h}_W$ and $\mathfrak{h}_S$ be the Cartan algebras of $\mathfrak{sl}_2, \mathfrak{sl}_{r-1}$ and $\mathfrak{sl}_{r+1}$ respectively. The Weyl group $S_{r+1}$ of $\mathfrak{sl}_{r+1}$ can be considered to be a subgroup of $GL(S)$ (as permutation matrices), and acts on $\mathfrak{h}_S$ and $\mathfrak{h}_S^*$: $\pi \in S_{r+1}$ acts as $\pi \epsilon_i = \epsilon_{\pi(i)}$ and $\pi \cdot L_i = L_{\pi(i)}$.

- If $\lambda \in \mathfrak{h}_S^*$ then $(\pi \cdot \lambda)(\epsilon_i) = \lambda(\epsilon_{\pi^{-1}(i)})$. Therefore $\pi L_i(\epsilon_j) = L_i(\epsilon_{\pi^{-1}(j)}) = \delta_{i,\pi^{-1}(j)}$ and hence $\pi L_i = L_{\pi(i)}$.

Now let $V_\lambda$ be an irreducible representation of $GL_{r+1}$ with highest weight vector $v$, and highest weight $\lambda$. Let $\pi \in S_{r+1}$.

**Lemma 16.1.**

1. The vector $\pi v$ is a weight vector of weight $\pi \lambda$.
2. $\pi v$ is a highest weight vector of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_{r-1}$ if and only if

\begin{equation}
\pi^{-1}(1) < \pi^{-1}(2), \quad \pi^{-1}(3) < \cdots < \pi^{-1}(r + 1)
\end{equation}

**Proof.** If $h \in \mathfrak{h}_S, h(\pi v) = \pi(h^{-1} h \pi)v = \lambda(h^{-1} \cdot h) \pi v = (\pi \cdot \lambda)(h) \pi v$. Therefore $\pi v$ is a weight vector, of weight $\pi \lambda$.

Let $e_{ij} \in \mathfrak{sl}_{r+1}$ take $\epsilon_j$ to $\epsilon_i$ and all others to zero. Then $\pi^{-1} e_{ij} \pi = e_{\pi^{-1}(i), \pi^{-1}(j)}$ and so $\pi^{-1} e_{ij} \pi = e_{\pi^{-1}(i), \pi^{-1}(j)}$. Moreover, $e_{ij} \pi v = (\pi^{-1} e_{ij} \pi)v = \pi e_{\pi^{-1}(i), \pi^{-1}(j)}v$ which is zero if $\pi^{-1}(i) < \pi^{-1}(j)$. \hfill $\square$

Assuming (16.1), denote the corresponding irreducible representation of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_{r-1}$ by $V_\mu \otimes V_\nu$. The representation of $\mathfrak{sl}_2$ corresponds to $\pi^{-1}(1)$ and $\pi^{-1}(2)$ rows of $\lambda$ (which gives $\mu$), and the representation corresponding to $\mathfrak{sl}_{r-1}$ corresponds to the remaining rows of $\lambda$ (which gives $\nu$): This is because $\pi \lambda(\epsilon_i) = \lambda(\epsilon_{\pi^{-1}(i)})$.

It is easy to see that $\mathcal{H}_{\mu, \ell} \otimes \mathcal{H}_{\nu, \ell} \to \mathcal{H}_{\lambda, \ell}$ compatibly with the above $V_\mu \otimes V_\nu \subseteq V_\lambda$ (and equivariant with an appropriate map of Kac-Moody algebras).
16.2. Maps of conformal blocks. Returning to the proof of Theorem 7.1 assume that \( r - 1 \geq 2 \) (we will indicate modifications for \( r = 2 \)) note that one finds a diagram

\[
\begin{array}{c}
\text{Vol} (\mathfrak{sl}_2, \bar{\mu}, \ell) \otimes \text{Vol} (\mathfrak{sl}_{r-1}, \bar{\nu}, \ell) \\
\phi \\
\text{Vol} (\mathfrak{sl}_{r+1}, \bar{\lambda})
\end{array}
\]

\[ (\text{16.2}) \]

Here, at the point \( i \) we choose the permutation \( \pi_i \) so that \( \pi_i^{-1}(1) = \alpha_i \) and \( \pi_i^{-1}(2) = \beta_i \) (and satisfying \[ (16.1) \]) and apply the previous considerations. Suppose we manage to prove that \( (1) \) \( \text{rk} \text{Vol} (\mathfrak{sl}_{r+1}, \bar{\lambda}) \neq 0 \).

(2) \( \phi \) is a generically non-zero map over \( \mathcal{M}_{0,n} \).

Suppose \( c_1 \text{Vol} (\mathfrak{sl}_{r+1}, \bar{\lambda}) = 0 \). Then \( \text{Vol} (\mathfrak{sl}_{r+1}, \bar{\lambda}) \) is trivial as a vector bundle. The image of \( \text{Vol}^* (\mathfrak{sl}_{r+1}, \bar{\lambda}) \) in \( A^* \) with \( A = \mathbb{A} (\mathfrak{sl}_2, \bar{\mu}, \ell) \otimes \mathbb{A} (\mathfrak{sl}_{r-1}, \bar{\nu}, \ell) \) is a constant non-zero subspace which lies inside the image of the dual of \( \text{Vol} (\mathfrak{sl}_2, \bar{\mu}, \ell) \otimes \text{Vol} (\mathfrak{sl}_{r-1}, \bar{\nu}, \ell) \) inside \( A^* \). But, by Corollary 8.2,

\[ \bigcap_{x \in \mathcal{M}_{0,n}} \text{Vol} (\mathfrak{sl}_2, \bar{\mu}, \ell)^x = 0 \subset \mathbb{A} (\mathfrak{sl}_2, \bar{\mu}). \]

We can therefore reach a contradiction using Lemma 8.2 and Corollary 16.2 below. Therefore \( c_1 \text{Vol} (\mathfrak{sl}_{r+1}, \bar{\lambda}) \neq 0 \) as desired.

**Lemma 16.2.** Suppose \( V_1, V_2 \) are subspaces of a finite dimensional vector space \( V \), and \( W_1 \) and \( W_2 \) are subspaces of a finite dimensional vector space \( W \). Then \( V_1 \otimes W_1 \) and \( V_2 \otimes W_2 \) are naturally subspaces of \( V \otimes W \) whose intersection is \( (V_1 \cap V_2) \otimes (W_1 \cap W_2) \subseteq V \otimes W \).

16.3. Geometrization of branching. Let \( X_a = \text{SL}(a)/B_a \), where \( B_a \) is a chosen Borel subgroup. There is a natural map \( \iota : X_a \times X_{a-1} \to X_{a+1} \) given by \( (g, h) \mapsto gh \pi \). There is a natural map \( X_{a+1} \to \mathbb{P}(V) \). The pull backs of \( O(1) \) to \( X_a \times X_{a-1} \) and \( X_{a+1} \) are \( L_{\mu} \otimes L_{\nu} \) and \( L_{\lambda} \) respectively. Then (compatibly) \( H^0(X_a \times X_{a-1}, L_{\mu} \otimes L_{\nu}) = (V_\mu \otimes V_\nu)^* \) and \( H^0(X_{a+1}, L_{\lambda}) = V_{\lambda}^* \).

Note further that \( \iota \) is the map \( \text{Fl}(U) \times \text{Fl}(W) \to \text{Fl}(S) \) given by \( (F_\bullet, G_\bullet) \mapsto H_\bullet \), where \( H_\bullet \) is computed as follows:

\[ H_a = F_m \oplus G_k, \quad m = \pi^{-1} \{1, 2\} \cap [i], \quad k = a - m. \]

16.4. The final step. We will produce an element \( \delta \in \text{Vol}^* (\mathfrak{sl}_{r+1}, \bar{\lambda}) \) whose image via \( \phi^* \) in \( (\text{Vol} (\mathfrak{sl}_2, \bar{\mu}, \ell) \otimes \text{Vol} (\mathfrak{sl}_{r-1}, \bar{\nu}, \ell))^* \) is non-zero. We do so by working over \( x = (\mathbb{P}^1, p_1, \ldots, p_n) \in \mathcal{M}_{0,n} \).

Consider the maps of moduli stacks

\[ \beta : \mathcal{P}ar_2 \times \mathcal{P}ar_{r-1} \to \mathcal{P}ar_{r+1} \]

\( (\mathcal{P}ar_2 \) are the moduli stacks from Section 13 with \( n \)-marked points \( p_1, \ldots, p_n \)). Here \( \beta \) is the map that sends \( (\mathcal{V}, \mathcal{F}, \gamma) \times (\mathcal{W}, \mathcal{G}, \gamma') \mapsto (\mathcal{V} \oplus \mathcal{W}, \mathcal{H}, \gamma \wedge \gamma') \) where \( H_p = F_m^p \oplus G_k^p \) where \( m \) is the number of elements in \( \pi_i^{-1} \{1, 2\} \) that are less than or equal to \( a \), \( k = a - m \).

Consider line bundles \( \mathcal{P}_2 = \mathcal{P}(\mathfrak{sl}_2, \ell, \bar{\mu}) \) on \( \mathcal{P}ar_2 \), \( \mathcal{P}_{r-1} = \mathcal{P}(\mathfrak{sl}_{r-1}, \ell, \bar{\nu}) \) on \( \mathcal{P}ar_{r-1} \) and \( \mathcal{P}_{r+1} = \mathcal{P}(\mathfrak{sl}_{r+1}, \ell, \bar{\lambda}) \) on \( \mathcal{P}ar_{r+1} \).
Lemma 16.3. The map $\beta$ pulls back $\mathcal{P}_{r+1}$ to $\mathcal{P}_2 \boxtimes \mathcal{P}_{r-1}$ and induces the dual of the map $\phi$ at the level of global sections.

Proof. Introduce a new point $p \in \mathbb{P}^1$. The map $\beta$ is dominated by a map

$$(Gr_2 \times \text{Fl}(U)^n) \times (Gr_{r-1} \times \text{Fl}(W)^n) \to (Gr_{r+1} \times \text{Fl}(S)^n).$$

Therefore our final task can be restated in geometric terms as: The map

$$(16.3) \quad H^0(Par_{r+1}, \mathcal{P}_{r+1}) \to H^0(Par_2, \mathcal{P}_2) \otimes H^0(Par_{r-1}, \mathcal{P}_{r-1})$$

is non-zero.

Write $\ell - D = \frac{1}{r+1} \sum |\lambda_i|$. Consider an evenly split bundle $Q$ of degree $-D$ and rank $\ell$. Let $G \in \text{Fl}_S(Q)$ be generic. Using the results of Section 15.2, we will show that $s_{(Q,G)} \in H^0(Par_{r+1}, \mathcal{P}_{r+1})$ maps to a non-zero element under the map (16.3). So we need to show that $s_{(Q,G)}$ is non-zero on images of generic elements of the form $(\mathcal{V}, \mathcal{F}, \gamma) \times (\mathcal{W}, G, \gamma')$ via $\beta$.

Suppose not, then we will find maps $\psi_1 : \mathcal{V} \to Q$ and $\psi_2 : \mathcal{W} \to Q$, such that the resulting map $\mathcal{V} \oplus \mathcal{W} \to Q$ is non-zero and

$$(\psi_1)_p(F_a^p) \subseteq G_{\ell-\mu_i(a)}, (\psi_1)_p(F_b^p) \subseteq G_{\ell-\mu_i(b)}, a \in [2], b \in [r-1], p = p_i \in S.$$ 

But there are no such non-zero maps by Lemma 15.5 applied to the non-zero vector spaces from conditions (b) and (c) of the theorem. Here we note that if we fix a $(\mathcal{V}, \mathcal{F})$ in (15.3), the zeroness holds for generic $(Q, G)$.

Remark 16.4. At the point $p = p_i$, $F_a^p$ maps to $H^p_{\ell-\mu_i(a)}$ but $\lambda_i^{(a)} = \mu_i^{(a)}$; similarly for $\nu$.

16.5. Case $r = 2$. We just omit the $\mathfrak{sl}_{r-1}$ factor. The transversality statement boils down to the following: Let $L$ be a one dimensional complex vector space. Then there are no non-zero maps $\psi$ such that for all $p = p_i, i = 1, \ldots, n$,

$$\psi : L \otimes \mathcal{O} \to Q, \quad \psi_p(L_p) \subseteq G_{\ell-\mu_i(1)}.$$ 

One can prove this by converting the above transversality assertion into the non-zeroness of a generalized Gromov-Witten number (using an argument of the type used in Proposition 15.3), the fact that the small quantum cohomology ring of a projective space is simply governed by degree constraints, and the shifting operations from [Bel08]. Here we sketch a more direct argument: If $\psi_p$ are all non-zero, then the above follows from Kleiman’s transversality. If some $\psi_p$ are zero, say for $p_1, \ldots, p_m$ then $\psi$ gives rise to a map $L(\sum_{i \leq m} p_i) \to Q$, we may apply Kleiman’s transversality and find the expected dimension to be negative.
16.6. The converse to Theorem 1.11. Consider a conformal blocks bundle \( \mathbb{V}_{\mathfrak{sl}_{r+1}, \lambda, \ell} \) with \( \lambda_i \neq 0 \) and \( r > 2 \) normalized (so the last rows of \( \lambda_i \) are zero), and the associated conformal blocks bundles \( \mathbb{V}_{\mathfrak{sl}_{2}, \mu, \ell} \) and \( \mathbb{V}_{\mathfrak{sl}_{r-1}, \nu, \ell} \), where \( \mu_i \) is the \( 2 \times \ell \) Young diagram formed by the first and last rows of \( \lambda_i \) and \( \nu_i \) is the \( (r-1) \times \ell \) Young diagram formed by removing the first and last rows of \( \lambda_i, i = 1, \ldots, n \). Assume further that the critical levels for \( \ell(\mathfrak{sl}_{r+1}, \lambda) = \ell(\mathfrak{sl}_{2}, \mu) = \ell \).

**Proposition 16.5.** If \( \text{rk} \mathbb{V}_{\mathfrak{sl}_{r-1}, \nu, \ell} = 0 \), then \( \mathbb{D}_{\mathfrak{sl}_{r+1}, \lambda, \ell} = 0 \).

**Proof.** Note that \( \pi_i \) are the same permutation \( \pi \) here. Let \( \mathfrak{g}' = \mathfrak{sl}_g \oplus \mathfrak{sl}_{r-1} \) be the \( \pi^{-1} \) conjugate embedding of \( \mathfrak{sl}_2 \oplus \mathfrak{sl}_{r-1} \). Let \( v_i \in V_{\lambda_i} \) be the highest weight vectors. Break each \( V_{\lambda_i} \) into a direct sum \( M_i \oplus \bigotimes_{j \in I} W_j^i \) of irreducible modules for \( \mathfrak{g}' \) where \( M_i \) is the irreducible module with highest weight vector \( v_i \). It is easy to see there are no eigenvectors for \( h_\theta \) with weight \( \lambda_i(1) \) in any of the \( W_j^i \) (because they will involve at least one application of \( f_{\alpha_i} \) or \( f_{\alpha_r} \), which lower the \( h_\theta \) weight).

Therefore under the quotient \( T_{x+1}^\ell: V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n} \rightarrow A_{\mathfrak{sl}_{r+1}, \lambda} \), the only term that survives is the image of \( T_{x+1}^\ell M \) with \( M = M_1 \otimes \ldots \otimes M_n \). But the image of the coinvariants \( M_\ell' \) in \( \mathbb{V}_{\mathfrak{sl}_{r+1}, \lambda, \ell} \) is zero since it factors through \( \mathbb{V}_{\mathfrak{sl}_2, \mu, \ell} \otimes \mathbb{V}_{\mathfrak{sl}_{r-1}, \nu, \ell} = 0 \), from (16.2). This implies that the image of \( T_{x+1}^\ell M \) in \( A_{\mathfrak{sl}_{r+1}, \lambda} \) is equal to the image of \( M_\ell' \), which is constant.

Therefore, \( \mathbb{V}_{\mathfrak{sl}_{r+1}, \lambda, \ell} \) is a constant quotient of \( A_{\mathfrak{sl}_{r+1}, \lambda} \), and has zero first Chern class.

\[ \square \]

17. Behavior under tensor products

Let \( G \) be a simple, simply connected algebraic group with Borel subgroup \( B \) and Lie algebra \( \mathfrak{g} \). Note that if \( \lambda \) and \( \nu \) are dominant integral weights in \( P(\mathfrak{g}) \) and \( P_m(\mathfrak{g}) \), then one has a canonical inclusion (mapping highest weight vectors to tensor products of highest weight vectors, see [Man09]) \( \mathcal{H}_{\mu + \nu, \ell + m} \subseteq \mathcal{H}_{\mu, \ell} \otimes \mathcal{H}_{\nu, m} \) which restricts to a natural inclusion \( V_{\mu + \nu} \subseteq V_\mu \otimes V_\nu \).

Suppose \( \mu = (\mu_1, \ldots, \mu_n) \) and \( \nu = (\nu_1, \ldots, \nu_n) \) are \( n \)-tuples of dominant integral weights in \( P(\mathfrak{g}) \) and \( P_m(\mathfrak{g}) \). There is a natural diagram of vector bundles on \( \overline{M}_{0, n} \), with surjective vertical arrows (cf. [Man09]).

\[
\begin{align*}
\mathcal{A}_{\mathfrak{g}, \mu + \nu} & \longrightarrow \mathcal{A}_{\mathfrak{g}, \mu} \otimes \mathcal{A}_{\mathfrak{g}, \nu} \\
\mathbb{V}_{\mathfrak{g}, \mu + \nu, \ell + m} & \longrightarrow \mathbb{V}_{\mathfrak{g}, \mu, \ell} \otimes \mathbb{V}_{\mathfrak{g}, \nu, m}
\end{align*}
\]

Now suppose

\[
\text{rk} \mathbb{V}_{\mathfrak{g}, \mu, \ell} = 1.
\]

Then we claim

**Proposition 17.1.** The map \( \phi \) is a surjection. If

\[
\text{rk} \mathbb{V}_{\mathfrak{g}, \mu + \nu, \ell + m} = \text{rk} \mathbb{V}_{\mathfrak{g}, \nu, m} = \delta,
\]
then the map $\phi$ is an isomorphism, and hence
\[
\mathbb{D}_{g,\vec{\mu}+\vec{\nu},\ell+m} = \delta \cdot \mathbb{D}_{g,\vec{\mu},\ell} + \mathbb{D}_{g,\vec{\nu},m}.
\]

Proof. We will assume (17.3) and show that the dual map of $\phi$ is an isomorphism (below we show that the dual map is always injective fiber wise under the assumption (17.2)).

Let $y$ be an arbitrary closed point of $M_{0,n}$. Let $u$ and $v$ be non-zero elements of $V_{g,\vec{\mu},\ell}|_y$ and $V_{g,\vec{\nu},m}|_y$ respectively. Let $\bar{u}$ and $\bar{v}$ denote their (non-zero) images in $A_{g,\vec{\mu}}|_y$ and $A_{g,\vec{\nu}}|_y$ respectively. We want to show that the image of $\bar{u} \otimes \bar{v}$ in $A_{g,\vec{\mu}+\vec{\nu}}|_y$ is non-zero. Note that all elements of the dual of $V_{g,\vec{\mu},\ell}|_y \otimes V_{g,\vec{\nu},m}|_y$ are pure tensors.

It therefore suffices to prove the following classical non-vanishing theorem: If $\alpha$ and $\beta$ are non-zero elements in $A_{g,\vec{\mu}}$ and $A_{g,\vec{\nu}}$ respectively, then the image of $\alpha \otimes \beta$ under $A_{g,\vec{\mu}} \otimes A_{g,\vec{\nu}} \to A_{g,\vec{\mu}+\vec{\nu}}$ is non-zero. This is a well known consequence of the Borel-Weil theorem. To see this, write commutative diagrams for each $i$

\[
\begin{align*}
G/B \xrightarrow{\Delta} G/B \times G/B & \xrightarrow{\text{P}(V_{\mu_i} \otimes V_{\nu_i})} \text{P}(V_{\mu_i}+V_{\nu_i})
\end{align*}
\]

For every dominant integral weight $\lambda$ there is a line bundle $L_\lambda$ on $G/B$ whose global sections equal $V_\lambda^*$ ($L_\lambda$ is the pull back of $O(1)$ via the map $G/B \to \text{P}(V_\lambda)$). Introduce the product of flag varieties $X = (G/B)^n$ which carries a diagonal action of $G$. Define the following line bundle for every $\vec{\lambda}$:
\[
L_{\vec{\lambda}} = \bigotimes_{i=1}^n L_{\lambda_i}
\]

Note that $A_{g,\vec{\mu}} = H^0(X, L_{\vec{\mu}})^G$. Under the multiplication map (induced by $n$ fold product of the diagram (17.4)),
\[
H^0(X, L_{\vec{\mu}})^G \times H^0(X, L_{\vec{\nu}})^G \to H^0(X, L_{\vec{\mu}+\vec{\nu}})^G
\]

the image of $\alpha \times \beta$ is non-zero (because locally we are forming the product of non-zero functions on $X$). This implies the the desired non-vanishing. □

Proposition 17.1 explains the vanishing of the first Chern class in the third example from Section 18. With $\ell = 1$, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$, and $m = 2$, $\nu_1 = 0$, $\nu_2 = \nu_3 = \nu_4 = \omega_1 + \omega_3$.

17.1. Non-trivial conformal blocks divisors that do not give birational morphisms. Morphisms associated to divisors $\mathbb{D} = \mathbb{D}_{g,\lambda,\ell}$ with (some) zero weights give rise to maps with images that are not birational to $\overline{M}_{0,n}$. In all of the examples discussed in Section 7.4 where the weights are nonzero, the images are birational to $\overline{M}_{0,n}$. However, this is not always so:

Proposition 17.2. The divisor $\mathbb{D}_{g,\lambda_1,\omega_2,\omega_2,2\omega_1,2\omega_1,\omega_1,2}$ gives the projection map from $\overline{M}_{0,5}$ to $\overline{M}_{0,4}$, dropping the 5th marked point.
Proof. The bundles $\mathbb{V}_{\mathfrak{sl}_3,(\omega_2\omega_2\omega_2,\omega_2\omega_1,\omega_1),2}$, $\mathbb{V}_{\mathfrak{sl}_3,(0,0,\omega_1,\omega_1,\omega_1),1}$, $\mathbb{V}_{\mathfrak{sl}_3,(\omega_2\omega_2\omega_2,\omega_2\omega_1,\omega_1,0),1}$ all have rank one. This can be check by using \cite{Swi10}. One can write:

$$\mathbb{D}_{\mathfrak{sl}_3,(0,0,\omega_1,\omega_1,\omega_1),1} = \pi_{1,2}^* \mathbb{D}_{\mathfrak{sl}_3,(\omega_1,\omega_1,\omega_1),1},$$

and

$$\mathbb{D}_{\mathfrak{sl}_3,(\omega_2,\omega_2,\omega_2,\omega_2,\omega_1,\omega_1),1} = \pi_5^* \mathbb{D}_{\mathfrak{sl}_3,(\omega_2,\omega_2,\omega_1,\omega_1,\omega_1),1},$$

where $\pi_{1,2}$ is the composition onto $\overline{\mathcal{M}}_{0,3}$ of the projection maps dropping the first two points and $\pi_5$ is the projection onto $\overline{\mathcal{M}}_{0,4}$ given by dropping the fifth marked point. Since $\overline{\mathcal{M}}_{0,3}$ is a point, the divisor $\mathbb{D}_{\mathfrak{sl}_3,(\omega_1,\omega_1,\omega_1),1}$ is trivial on it, and so it’s pullback is trivial as well. Using Theorem \ref{thm:corollary} one may check that $\mathbb{D}_{\mathfrak{sl}_3,(\omega_2,\omega_2,\omega_1,\omega_1,\omega_1),1}$ is nontrivial on $\overline{\mathcal{M}}_{0,4}$. Therefore, by Proposition \ref{prop:involution} $\mathbb{D}_{\mathfrak{sl}_3,(\omega_2,\omega_2,\omega_2,\omega_2,\omega_1,\omega_1),2}$ is a sum

$$\mathbb{D}_{\mathfrak{sl}_3,(\omega_2,\omega_2,\omega_2,\omega_2,\omega_1,\omega_1),2} = \mathbb{D}_{\mathfrak{sl}_3,(0,0,\omega_1,\omega_1,\omega_1),1} + \mathbb{D}_{\mathfrak{sl}_3,(\omega_2,\omega_2,\omega_1,\omega_1,\omega_1),1},$$

In particular, as the first summand is trivial, $\mathbb{D}_{\mathfrak{sl}_3,(\omega_2,\omega_2,\omega_2,\omega_2,\omega_1,\omega_1),2}$ gives $\pi_5$. \hfill \Box

17.2. Fulton’s conjecture. Fulton conjectured that if $\text{rk} \mathbb{A}_{\mathfrak{sl}_{r+1},\tilde{\lambda}} = 1$ then $\text{rk} \mathbb{A}_{\mathfrak{sl}_{r+1},N\tilde{\lambda}} = 1$ for all $N \in \mathbb{Z}_{>0}$. This was proved by Knutson, Tao and Woodward \cite{KTW04}. Using Witten’s dictionary, the quantum generalization of Fulton’s conjecture \cite{Bel07, BK13} is the following: Suppose $\text{rk} \mathbb{V}_{\mathfrak{sl}_{r+1},\tilde{\lambda},\ell} = 1$ ($\ell$ is not necessarily the critical level) then $\text{rk} \mathbb{V}_{\mathfrak{sl}_{r+1},N\tilde{\lambda},N\ell} = 1$ for all positive integers $N$. Using this generalization and Proposition \ref{prop:involution} we obtain (by induction):

**Corollary 17.3.** If $\text{rk} \mathbb{V}_{\mathfrak{sl}_{r+1},\tilde{\lambda},\ell} = 1$, then $\mathbb{D}_{\mathfrak{sl}_{r+1},N\tilde{\lambda},N\ell} = N \cdot \mathbb{D}_{\mathfrak{sl}_{r+1},\tilde{\lambda},\ell}$, $\forall N \in \mathbb{Z}_{>0}$.

**Remark 17.4.** Corollary \ref{cor:17.3} appears in case $r = 1$ and $\tilde{\lambda} = (\omega_1, \ldots, \omega_1)$ in \cite{GJMS12}.

**Corollary 17.5.** Let $\mathbb{D} = \mathbb{D}_{\mathfrak{sl}_{r+1},\ell\tilde{\lambda},\ell}$ be a nontrivial conformal blocks divisor, so that $\sum_i |\lambda_i| = (r+1)(d+1)$. The image of the morphism $\phi_{\mathbb{D}}$ is isomorphic to the Veronese quotient $U_{d,n}/\langle (a_i) \rangle$, where $a_i = |\lambda_i|/(r+1)$.

**Proof.** By assumption, $\lambda_1, \ldots, \lambda_n \in P_{\ell}(\mathfrak{sl}_{r+1})$ and $\text{rk} \mathbb{V}_{\mathfrak{sl}_{r+1},\ell\tilde{\lambda},\ell} \neq 0$. Therefore, by Proposition \ref{prop:factorization} $\text{rk} \mathbb{V}_{\mathfrak{sl}_{r+1},\ell\tilde{\lambda},1} \neq 0$. Using factorization, $\text{rk} \mathbb{V}_{\mathfrak{sl}_{r+1},\tilde{\lambda},1} \neq 0$ implies that $\text{rk} \mathbb{V}_{\mathfrak{sl}_{r+1},\tilde{\lambda},1} = 1$ \cite{Fak12}. So by Corollary \ref{cor:17.3} $\mathbb{D}_{\mathfrak{sl}_{r+1},\ell\tilde{\lambda},1} = \ell \cdot \mathbb{D}_{\mathfrak{sl}_{r+1},\tilde{\lambda},1}$ for all positive integers $\ell$. Applying \cite[Theorem 1.2]{GG12} gives the assertion. \hfill \Box

17.3. Involution Identities. Here, in Proposition \ref{prop:factorization} (compare with \cite[Proposition 2.8]{Bea96}), we give relations between conformal blocks divisors for any finite dimensional simple Lie algebra $\mathfrak{g}$ induced by a particular involution of the Weyl chamber.

There is an involution on $P_{\ell}(\mathfrak{g})$ which sends a weight $\lambda$ to its “dual” $\lambda^* = -w_0(\lambda)$, where $w_0$ is the longest element of the Weyl group of $\mathfrak{g}$. For example if $\mathfrak{g} = \mathfrak{sl}_{r+1}$ is of type $A$, then $\lambda^*$ is given by $\lambda^c$ for any weight $\lambda$ represented by a Young diagram of size $(r+1) \times \ell$.

**Proposition 17.6.** On $\overline{\mathcal{M}}_{0,n+1}$, $\mathbb{D}_{\mathfrak{g},\tilde{\lambda}} = \mathbb{D}_{\mathfrak{g},\tilde{\lambda}^*,\ell}$, where $\tilde{\lambda} \in P_{\ell}(\mathfrak{g})^n$.

By results in \cite{Fak12}, this reduces to the case $n = 4$:

**Lemma 17.7.** On $\overline{\mathcal{M}}_{0,4}$, $\deg \mathbb{V}_{\mathfrak{g},\tilde{\mu},\ell} = \deg \mathbb{V}_{\mathfrak{g},\tilde{\mu}^*,\ell}$. 
Proof. Let $\mu$ be any dominant integral weight of $\mathfrak{g}$ and $\mu^* = -w_0(\mu)$. Since the Cartan killing form is invariant under the action of the Weyl group of $\mathfrak{g}$, we get the following

$$(\mu, \mu + 2\rho) = (\mu^*, \mu^* + 2\rho),$$

where $\rho$ is the half sum of positive roots. We conclude the proof using Corollary 3.5 in [Fak12], and the above equality. \qed

18. Table

In these examples, computed using [Swi10], $\ell$ is a critical level for the pair $(\mathfrak{sl}_{r+1}, \tilde{\lambda})$, and $\text{Deg}$ (when $n = 4$), denotes the degree of $V_{\mathfrak{sl}_{r+1}, \tilde{\lambda}}$ (and of $V_{\mathfrak{sl}_{r+1}, \tilde{\lambda}^r, r}$) over $\mathbb{M}_{0,4} = \mathbb{P}^1$. In particular, $\text{Deg} = 0$ means $D_{\mathfrak{sl}_{r+1}, \tilde{\lambda}} \subset \mathbb{P}^{\mathfrak{sl}_{r+1}, \tilde{\lambda}^r, r} = 0$, even when the rank of $V_{\mathfrak{sl}_{r+1}, \tilde{\lambda}}$ (and of $V_{\mathfrak{sl}_{r+1}, \tilde{\lambda}^r, r}$) is nonzero. A * in the $\text{Deg}$ column indicates more than 4 marked points.

| $\text{Deg}$ | $(r + 1, \ell + 1)$ | $n$ | $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_n)$ | $\text{rk} A_{\mathfrak{sl}_{r+1}, \tilde{\lambda}}$ | $\text{rk} kV_{\mathfrak{sl}_{r+1}, \tilde{\lambda}, \ell}$ | $\text{rk} kV_{\mathfrak{sl}_{r+1}, \tilde{\lambda}^r, r}$ |
|-------------|------------------|-----|---------------------------------|-----------------|-------------------------------|-----------------|
| *           | (3, 2)           | 6   | $(\omega_1, \ldots, \omega_1)$ | 5               | 1                             | 4               |
| 1           | (3, 2)           | 4   | $(\omega_1, \omega_1, \omega_2, \omega_2)$ | 2               | 1                             | 1               |
| 0           | (4, 4)           | 4   | $(\omega_1, (2\omega_1 + 3\omega_2)^4)$ | 2               | 1                             | 1               |
| *           | (3, 6)           | 5   | $(2\omega_1 + 2\omega_2, 2\omega_2, 2\omega_2, 2\omega_1, 2\omega_2, \omega_1 + \omega_2)$ | 7               | 7                             | 0               |
| *           | (3, 5)           | 5   | $(2\omega_1 + 2\omega_2, \omega_1, 2\omega_1, 2\omega_1, 2\omega_2, \omega_1 + \omega_2)$ | 9               | 8                             | 1               |
| 0           | (4, 4)           | 4   | $(\omega_2 + \omega_3, \omega_1, \omega_1, 2\omega_2, 2\omega_2, 2\omega_1 + \omega_3)$ | 2               | 1                             | 1               |
| 0           | (4, 5)           | 4   | $(\omega_1, 2\omega_1 + \omega_2 + \omega_3, (3\omega_1 + \omega_3)^2)$ | 2               | 1                             | 1               |
| 4           | (4, 6)           | 4   | $(\omega_1 + \omega_2 + \omega_3, \omega_1, \omega_1 + \omega_1 + \omega_2 + \omega_2, \omega_1 + \omega_2 + \omega_3, 2\omega_1 + \omega_2)$ | 18              | 14                            | 4               |
| 6           | (4, 6)           | 4   | $(\omega_1 + \omega_2 + \omega_3, 2\omega_2, 2\omega_1 + 2\omega_2 + \omega_3, 2\omega_1 + \omega_2)$ | 19              | 13                            | 6               |
| 2           | (4, 5)           | 4   | $(2\omega_1 + \omega_3, 2\omega_1 + 2\omega_2, 2\omega_1 + 2\omega_2, 3\omega_1)$ | 7               | 2                             | 5               |
| 1           | (4, 5)           | 4   | $(\omega_1 + \omega_3, 2\omega_1 + 2\omega_2, 2\omega_1 + 2\omega_2, 4\omega_1)$ | 4               | 1                             | 3               |
| *           | (3, 6)           | 8   | $((2\omega_1)^6, \omega_2, 2\omega_2)$ | 150             | 136                           | 14              |

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