Electromagnetic natural convection flow in a vertical microchannel with Joule heating: exact solution

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ABSTRACT
This article explores the combined role of Joule heating and transversely applied magnetic field on natural convection flow in a vertical microchannel with asymmetric wall heating. The governing momentum and energy equations are obtained and transformed to their corresponding dimensionless form using suitable dimensionless parameters. Exact solutions are obtained for the coupled momentum and energy equations and presented graphical to understand the role of governing dimensionless parameters. During the course of numerical and graphical simulations, the results indicate that the presence of Joule heating and electromagnetic field lead to enhancement of fluid velocity, temperature, skin-friction and Nusselt number in the microchannel. In addition, the heat transfer can be improved when heat is been transported from the walls of the microchannel to the fluid and for purely asymmetric wall heating compared to the case when heat is transfer from the fluid to the wall.

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Nomenclature

\[ B_0 \] magnetic field strength
\[ Br \] Brinkman number
\[ C_p, C_v \] specific heats at constant pressure and constant volume, respectively
\[ E \] electric field
\[ g \] acceleration due to gravity
\[ H \] width of the channel
\[ \mathbf{B} \] magnetic field induction vector
\[ K \] dimensionless Joule heating parameter
\[ \frac{-E_z}{(u_0 B_0)} \]
\[ Kn \] Knudsen number
\[ M \] Hartmann number
\[ Nu \] Nusselt number
\[ Q \] dimensionless volume flow rate
\[ T \] dimensional temperature
\[ T_0 \] reference temperature
\[ u \] dimensional axial velocity
\[ U \] dimensionless axial velocity
\[ u_0 \] mean velocity
\[ x, y \] axial and transverse coordinate respectively
\[ X \] dimensionless axial coordinate
\[ Y \] dimensionless transverse coordinate
\[ \beta \] thermal expansion coefficient
\[ \beta_t, \beta_v \] dimensionless variables
\[ \xi \] wall-ambient temperature difference ratio
\[ f_t, f_v \] thermal and tangential momentum accommodation coefficients, respectively
\[ \gamma_s \] ratio of specific heats \( (C_p/C_v) \)
\[ ln \] fluid–wall interaction parameter, \( \beta_t/\beta_v \)
\[ \mu \] dynamic viscosity

\[ \nu \] kinematic viscosity
\[ \rho_0 \] density
\[ \theta \] dimensionless temperature
\[ \lambda \] mean free path
\[ \sigma \] electric conductivity
\[ \tau \] skin-friction

Subscripts

0 value at \( y = 0 \)
1 value at \( y = H \)
m mean

1. Introduction
With continuous increase in customers’ demand for micro-electro-mechanical system (MEMS) and nano-electro-mechanical systems due to their applications in cooling or heating in micro-reactor devices, many studies have been devoted to satisfy their demands. Recent applications of such devices include microchannel heat sink, microjet impingement cooling and micro-heat pipe. Therefore, a thorough understanding of flow behaviours in microchannel is becoming increasingly important for accurate prediction of performance during the design process. Although there are a remarkably growing number of realized scientific and engineering applications found for MEMS devices, understanding the fluid dynamics and heat transfer processes in such MEMS devices is still far from being thorough. On the other hand, the performance of MEMS often defies predictions made using scaling laws developed for large...
systems. So, heat can be easily built up in a densely packed MEMS protective housing, which may cause undesirable or even destructive deformation. Therefore, there is a pressing need of reliable computational capabilities for accurate predictions of these devices.

The Knudsen number is the major parameter that decides the division of rarefied gas flow and is defined as the ratio of the molecular mean free path (\(\lambda\)) to characteristic length scale. For a given fluid flow problem, when \(Kn < 10^{-3}\), the flow domain can be treated as a continuum, in which the Navier--Stokes equation in conjunction with the no-slip wall boundary conditions becomes applicable. For \(Kn > 10\), the flow becomes free molecular in nature, because of negligible molecular collisions. For \(10^{-3} < Kn < 10^{-1}\), the slip flow regime is considered and the no-slip boundary condition becomes invalid, though continuum conservation equations are used to characterize the bulk flow. For \(10^{-1} < Kn < 10^{-1}\), the transition flow regime occurs and the continuum hypothesis progressively ceases to work totally [1,2]. In this study, the slip flow regime \((10^{-3} < Kn < 10^{-1})\) is considered in order to study the hydrodynamic and thermal behaviour of fully developed electromagnetohydrodynamics natural convection flow in a vertical channel. A lot of articles have been devoted to solve flow formation problems in microchannel. Chen and Weng [3] studied analytically the fully developed natural convection in an open-ended vertical parallel plate microchannel with asymmetric wall temperature. They concluded that the effects of rarefaction and fluid wall interaction enhanced the volume flow and reduced the heat transfer rate. This result was further extended by taking into account suction/injection on the microchannel walls by Jha et al. [4]. They resolved that skin-friction as well as rate of the heat transfer strongly depends on the suction/injection parameter. Other related work on flow formation in microchannel can be found in [5–8].

Joule heating effect on flow of electrically conducting fluid in the presence of transversely applied magnetic field on the other hand has yielded a promising result in the area of power generation, nuclear energy production, electric oven, transformers, soldering iron, electric hotplate, electric radioactive space heater, incandescent bulbs, electronic cigarettes and magnetohydrodynamics (MHD) pump. Some of these notable achievements can be found in the work of Cramer and Pai [9], Chawla [10] and Soundalgekar and Takhar [11]. Recently, Jha et al. [12] investigated the MHD natural convection flow in a vertical parallel plate microchannel and found that the magnitude of skin-friction is higher in the case of symmetric heating in comparison with asymmetric wall heating of microchannel plates. The study of the effect of electrically conducting walls with MHD is significant in the design of MHD micro-pumps [13]. Soundalgekar [14,15] investigated the role of electrically conducting walls on MHD channel flow of an incompressible, viscous rarefied gas as affected by wall conductance. It was concluded that velocity is a decreasing function of magnetic field. Later, Soundalgekar [16] derived an expression for the Nusselt number by incorporating slip/jump boundary conditions at the walls for parallel plate microchannel of an electrically conducting fluid. In the other related article, Shojaeian and Shojaeian [17] presented an analytical solution of mixed electromagnetic/pressure-driven gaseous flows in microchannels. They examined the Nusselt number and Poiseuille number and concluded that the role of electromagnetic field is to increase the Nusselt number and Poiseuille number in regions with less rarefaction. Other researchers who have contributed to the study of electromagnetic field effect on flow formation are the authors of [18–25].

Despite all these contributions, to the best of authors’ findings, no analytical work has been carried out to investigate the electromagnetic flow in a vertical microchannel when the flow is driven by buoyancy (natural convection).

In view of this, the aim of this article is to examine the role of Joule heating and electromagnetic field on natural convection flow formation and heat transfer in a microchannel. Exact solutions are obtained for fluid velocity, temperature, skin-friction and Nusselt number. The role of governing parameters is graphical represented, discussed and significant conclusions are drawn. The novelty of this current article is to present an exact solution for combined magnetic field and Joule heating of natural convection flow in a vertical microchannel.

### 1.1. Problem description

Consider a fully developed natural convection flow of electrically conducting fluid in a vertical microchannel with a width \(H\) in the presence of electromagnetic field, buoyancy and Joule heating. The \(x\)-axis is parallel to the gravitational acceleration \(g\) but in the opposite direction, while the \(y\)-axis is orthogonal to the vertical parallel plates. A magnetic field of uniform strength \((0, B_0, 0)\) is assumed to be applied in the direction perpendicular to the direction of flow. It is assumed that the magnetic Reynolds number is very small, which corresponds to negligibly induced magnetic field compared to the externally applied one. The plates are heated asymmetrically with one plate \((y = 0)\) maintained at a temperature \(T_2\) greater than the fluid temperature \(T_0\) while the other plate \((y = H)\) at a temperature \(T_1\) where \(T_1 > T_2\). Due to this temperature difference at the channel walls which would result to density difference, hence natural convection flow is set up in the microchannel. The geometry of the system under consideration in this present study is shown in Figure 1. The flow is assumed to be steady and incompressible with constant properties. Also, the viscous dissipation term in the energy
equation is neglected and the external electric field exists in the z-direction only such that \( E_x = E_y = 0 \). Following Cai and Liu [26] and Shojaeian and Shojaee [27], the electrical field can be combined with the magnetic field with \( E_z = -Ku_Bn \), where \( 0 < K < 1 \).

1.2. Mathematical formulation

Considering all the above assumptions, the mathematical models representing the magnetohydrodynamics flows are obtained by incorporating the Lorentz force and Joule heating term in the momentum and energy equations respectively are as follows:

\[
\begin{align*}
\rho \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} & = -\nabla p + \rho \mathbf{V} \times \mathbf{E} + \mathbf{B} \times \mathbf{V} \\
\rho C_p \left( \frac{\partial \mathbf{T}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{T} \right) & = k \nabla^2 \mathbf{T} + \mathbf{Q}
\end{align*}
\]

where \( \mathbf{Q} \) is the Joule heating term and \( \nabla = (\partial/\partial x) \mathbf{i} + (\partial/\partial y) \mathbf{j} + (\partial/\partial z) \mathbf{k} \).

Considering a steady, incompressible, electrically conducting, fully developed flow \( (\mathbf{V} = u(r, t) \mathbf{i}) \) in the absence of pressure gradient, the governing equations respectively reduce to

\[
\begin{align*}
\frac{\partial^2 u}{\partial y^2} + g \beta(T - T_0) \rho - \sigma E_z B_0 u + \frac{M^2}{k} & = 0 \quad (3) \\
\frac{d^2 T}{dY^2} + \frac{\sigma E_z (E_z + B_0 u)}{k} & = 0 \quad (4)
\end{align*}
\]

Using the following dimensionless parameters, Equations (3)–(4) become [12,27]

\[
\begin{align*}
Y = \frac{y}{H}, \quad \theta = \frac{T - T_0}{T_1 - T_0}, \quad U = \frac{u}{u_0}, \quad Br = \frac{M^2}{(T_1 - T_0)}, \\
\beta = \frac{\sigma B_0^2 H^2}{\mu}, \quad u_0 = \frac{g \beta \rho H^2 (T_1 - T_0)}{(T_1 - T_0)}, \\
E_z = -Ku_B, \quad n = \frac{\lambda}{H}, \quad \ln = \frac{\beta_1}{\beta}, \\
\beta_v = \frac{2 - f_v}{f_v}, \quad \beta_2 = \frac{2 - f_2}{f_2}, \quad F_r = \frac{1}{1 + \gamma_f}, \\
\beta_1 = \frac{2 - f_1}{f_1}, \quad \beta_2 = \frac{2 - f_2}{f_2}, \quad \beta_3 = \frac{2 - f_3}{f_3}
\end{align*}
\]

1.2.1. Analysis

Differentiating Equation (6) twice and substituting Equation (7), the coupled equations can be decoupled as follows:

\[
\begin{align*}
\frac{d^4 U}{dY^4} - M^2 \frac{d^2 U}{dY^2} + M^2 KBr U & = M^2 K^2 Br \quad (9)
\end{align*}
\]

The exact solution of Equation (9) with boundary conditions (8) for the momentum and energy equations can be obtained respectively as follows:

\[
\begin{align*}
U(Y) & = c_1 \exp(\delta_1 Y) + c_2 \exp(-\delta_1 Y) + c_3 \exp(\delta_2 Y) + c_4 \exp(-\delta_2 Y) + K \quad (10) \\
\theta(Y) & = [M^2 - \delta_1^2] c_1 \exp(\delta_1 Y) + c_2 \exp(-\delta_1 Y) \\
& + \frac{1}{\sqrt{2}} [M^2 - \delta_2^2] c_3 \exp(\delta_2 Y) + c_4 \exp(-\delta_2 Y) + K \quad (11)
\end{align*}
\]

where \( c_1, c_2, c_3, c_4, \delta_1 \) and \( \delta_2 \) are constant defined as follows:

\[
\begin{align*}
& c_1 = \Delta_1 \Delta, \quad c_2 = \Delta_2 \Delta, \quad c_3 = \Delta_3 \Delta, \quad c_4 = \Delta_4 \Delta, \\
& \delta_1 = \frac{1}{\sqrt{2}} \left[ M^2 + \sqrt{M^4 - 4M^2 K Br} \right]^{\frac{1}{2}}, \\
& \delta_2 = \frac{1}{\sqrt{2}} \left[ M^2 - \sqrt{M^4 - 4M^2 K Br} \right]^{\frac{1}{2}} \quad (12)
\end{align*}
\]

where \( \Delta, \Delta_1, \Delta_2, \Delta_3, \Delta_4 \) are constants defined in the Appendix.

The drag effect at the surfaces of the microchannel is given by

\[
\begin{align*}
\left. \frac{dU}{dY} \right|_{Y=0} & = \delta_1 [c_1 - c_2] + \delta_2 [c_3 - c_4] \quad (13) \\
\left. \frac{dU}{dY} \right|_{Y=1} & = \delta_1 [c_1 \exp(\delta_1) - c_2 \exp(-\delta_1)] + \delta_2 [c_3 \exp(\delta_2) - c_4 \exp(-\delta_2)] \
& + \frac{1}{\sqrt{2}} [M^2 - \delta_2^2] c_3 \exp(\delta_2) + c_4 \exp(-\delta_2) \quad (14)
\end{align*}
\]

Another important quantity of interest is the volume flow rate \( Q \) which is defined in the dimensionless form

\[
\begin{align*}
Q & = \frac{1}{H^2} \left( \frac{dU}{dY} \right|_{Y=0} \right) \quad (15)
\end{align*}
\]
as follows:

$$Q = \int_{0}^{1} U(Y) dY$$

$$= \frac{[C_1(e^{\xi_1} - 1) - C_2(e^{\xi_1} - 1)]}{\delta_1}$$

$$+ \frac{[C_3(e^{\xi_2} - 1) - C_4(e^{-\xi_2} - 1)]}{\delta_2} + K$$

(15)

Also, the rate of heat transfer represented by the Nusselt number ($Nu$) in the dimensionless form for fully developed laminar flow in a microchannel is given as follows:

$$Nu_0 = \frac{\frac{dn}{dY}|_{Y=0}}{\theta_m - \xi}$$

and

$$Nu_1 = \frac{\frac{dn}{dY}|_{Y=1}}{\theta_m - 1}$$

(16)

where $\theta_m$ is the bulk temperature defined as follows:

$$\theta_m = \int_{0}^{1} U(Y) \theta(Y) dY$$

$$= \frac{\sum_{i=1}^{19} P_i}{X_{18}}$$

(17)

where $P_i$’s are constants defined in the Appendix.

$$Nu_0 = \frac{[C_1 - C_2][M^2 \delta_1 - \xi_1^2] + [C_3 - C_4][M^2 \delta_2 - \xi_2^2]}{\theta_m - \xi}$$

$$Nu_1 = \frac{[C_3e^{\xi_1} - C_2e^{-\xi_1}][M^2 \delta_1 - \xi_1^2] + [C_4e^{\xi_2} - C_3e^{-\xi_2}][M^2 \delta_2 - \xi_2^2]}{\theta_m - 1}$$

(18)

(19)

2. Results and discussion

The coupled momentum and energy equations are solved exactly and the solutions obtained are used to obtain the expressions of interest, such as bulk temperature, skin-friction and Nusselt number. The graphs are depicted to reveal the role of pertinent parameters such as Hartmann number, Brinkman number, Knudsen number and the electric field strength. Throughout this article, the Hartmann number has been selected over the range of $0 \leq M \leq 5$ in order to avoid induced magnetic field, $0 < K < 1$, $0 \leq \xi \leq 1$ to capture the cases of purely asymmetric and symmetric wall heating, $-1 < \zeta < 1$ to illustrate the cases when heat is supplied to the fluid from the walls and vice versa.

2.1. Velocity field

The exact solution obtained for the velocity profile in Equation (10) is graphical represented to show the effect of various governing parameters. Figure 2 shows the velocity profile for different flow regimes ($Kn$) and electric field strength at a fixed value of Hartmann number ($M$), $Br$ and $\xi$. It is found from this figure that the velocity as well as the velocity slip increases with decrease in $Kn$ and $K$. Also, the maximum velocity is obtained for the slip regime. This is due to the fact that for slip regime, the ratio of molecular mean free path to the channel width is higher and thereby breaking the continuum assumption, hence giving the fluid free movement, thereby enhancing fluid motion in the vertical microchannel.

On the other hand, Figure 3 illustrates the combined role of Hartmann number and Joule heating parameter for the fixed value of $Br$, $Kn$ and $\xi$. It is evident from this graph that the role of magnetic field is to increase the fluid velocity in the presence of Joule heating. This trend could be attributed to the presence of Joule heating which is converse to the well-known result that “Hartmann number reduces fluid velocity due to the presence of Lorentz force which opposes fluid velocity”. As an accuracy check, it is observed that for small value of $K$ (in the absence of Joule heating), velocity profile decreases with increase in Hartmann number ($M$). This accuracy check corresponds to the results of Jha et al. [12] (Table 1).
Table 1. Numerical comparison of the present dimensionless velocity for different flow regimes with those of Jha et al. [12] at $M = 0$.

| Y   | $Kn$ | Jha et al. [12] | Present work ($K \to 0$) |
|-----|------|-----------------|---------------------------|
| 0.0 | 0.0  | 0.0000          | 0.0000                    |
| 0.05| 0.0  | 0.0048          | 0.0048                    |
| 0.1 | 0.0  | 0.0083          | 0.0082                    |
| 0.1 | 0.0  | 0.0127          | 0.0127                    |
| 0.2 | 0.0  | 0.0151          | 0.0150                    |
| 0.1 | 0.0  | 0.0182          | 0.0181                    |
| 0.1 | 0.0  | 0.0197          | 0.0196                    |
| 0.3 | 0.0  | 0.0204          | 0.0204                    |
| 0.05| 0.0  | 0.0219          | 0.0219                    |
| 0.1 | 0.0  | 0.0232          | 0.0230                    |
| 0.4 | 0.0  | 0.0233          | 0.0233                    |
| 0.05| 0.0  | 0.0245          | 0.0245                    |
| 0.1 | 0.0  | 0.0255          | 0.0253                    |
| 0.5 | 0.0  | 0.0251          | 0.0251                    |
| 0.05| 0.0  | 0.0261          | 0.0261                    |
| 0.1 | 0.0  | 0.0270          | 0.0269                    |
| 0.6 | 0.0  | 0.0256          | 0.0256                    |
| 0.05| 0.0  | 0.0268          | 0.0267                    |
| 0.1 | 0.0  | 0.0277          | 0.0277                    |
| 0.7 | 0.0  | 0.0245          | 0.0245                    |
| 0.05| 0.0  | 0.0261          | 0.0261                    |
| 0.1 | 0.0  | 0.0273          | 0.0272                    |
| 0.8 | 0.0  | 0.0210          | 0.0210                    |
| 0.05| 0.0  | 0.0235          | 0.0234                    |
| 0.1 | 0.0  | 0.0252          | 0.0251                    |
| 0.9 | 0.0  | 0.0136          | 0.0136                    |
| 0.05| 0.0  | 0.0178          | 0.0177                    |
| 0.1 | 0.0  | 0.0206          | 0.0205                    |
| 1.0 | 0.0  | 0.0000          | 0.0000                    |
| 0.05| 0.0  | 0.0071          | 0.0070                    |
| 0.1 | 0.0  | 0.0117          | 0.0117                    |

2.2. Temperature distribution

This section is concerned with the discussion of role of transversely applied magnetic field and Joule heating on the temperature distribution in the microchannel for different values of governing parameters. It is good to state that for small value of Joule heating parameter, the temperature distribution corresponds exactly with that of Jha et al. [12]. Figure 4 presents the combined role of Knudsen number ($Kn$) and wall-ambient temperature ratio ($\xi$) on the temperature distribution in the microchannel for the fixed value of Joule heating parameter ($K = 0.5$) and Hartmann number ($M = 2.0$) for the case when heat is supplied from the wall to the fluid ($Br = 1.0$). It is found that temperature jump increases with increase in $Kn$ and $\xi$ at the wall with ambient temperature ratio, while the reverse result is noticed at the wall with symmetric heating. This can be attributed to the fact that as $\xi \to 1$, the temperature also increases gradually until symmetric wall heating is achieved and thereby enhancing the temperature distribution in the microchannel. On the other hand, temperature distribution is lower for the slip flow regime ($Kn = 0.1$) at the wall with symmetric heating. This accounts to the fact that for the slip regime, the continuum assumption is no more valid and hence the mean molecular free path is higher, therefore leading to drop in the temperature distribution in the microchannel.

For variation of temperature distribution for different values of Joule heating parameter ($K$) and Hartmann number ($M$). Figure 5 depicts the role of these parameters on the dimensionless fluid temperature near the region close to the slip regime ($Kn = 0.05$). It is established that the combined role of Joule heating and magnetic field (electromagnetic) is to increase the fluid temperature. As expected, in the absence of Joule heating ($K \to 0$), the temperature distribution is independent of the electromagnetic field. This corresponds to the findings of Jha et al. [12], where their energy equation is independent of the electromagnetic field. This finding can help enhancement of heat transfer appliances, such as heaters, soldering iron, pressing iron and other electrical appliances.

2.3. Skin-friction

The solution to the momentum equation obtained is utilized to compute the drag force (skin-friction) between the fluid and the surfaces of the vertical...
microchannel. This parameter is of great interest in determining and understanding the capacity of dam and hydropower generation. Figure 6 exhibits the role of Joule heating and transversely applied magnetic field on skin-friction at the wall with ambient temperature ratio at different flow regimes for the fixed value $Br = 1.0$. It is obvious from this figure that in the absence of magnetic field, skin-friction is independent of $K$ or $Kn$. On the other hand, the role of $M$ is to increase the skin-friction at this surface as Joule heating parameter ($K$) increases, while the reverse is the result in the absence of Joule heating parameter ($K$). In order to get further understanding on the present research skin-friction, Figure 7 gives the combined role of Brinkman number ($Br$) and wall-ambient temperature ratio ($\xi$) on drag effect at the microchannel wall for the fixed value of $Kn = 0.05$ and $K = 0.5$. It is obvious that skin-friction is higher when heat is supply from the wall to the fluid ($Br = 1.0$) than for the case when heat is supplied from the fluid to the wall ($Br = -1.0$). In addition, as the wall-ambient temperature ratio approaches symmetric heating ($\xi = 1.0$), the maximum skin-friction is achieved regardless of the source of heating. In general, the reverse results are found for skin-friction at the wall with symmetric wall heating and for brevity, it has not been reported in this article.

2.4. Heat transfer

One of the most important analyses in fluid mechanics is the heat transfer. This is due to the need to satisfy customers’ demands and make life easier both in living apartment and industrial applications which include thermostat mechanism, drying machines, air conditioners and ovens. This section is concerned with the role of governing parameter on the overall heat transfer (represented by the Nusselt number) between the microchannel walls and the fluid. Figure 8 displays the role of Joule heating parameter ($K$) and Hartmann ($M$) on the heat transfer at the wall with ambient wall temperature ratio for the fixed value of $Br = 1.0$ and $\xi = 0.5$. It is interesting to find that the Nusselt number is enhanced by increase in $K$ and $M$ regardless of its flow regime. Also, as expected, the heat transfer is higher for continuum regime ($Kn = 0.0$) than in slip flow regime ($Kn = 0.1$). On the other hand, in the absence of Joule heating, the heat transfer is independent of $M$ and it assumes the value of the wall-ambient temperature ratio ($\xi$).

Figure 9 illustrates the Nusselt number as a function of $M$, $\xi$ and $Br$ for the fixed value of $K$ at slip flow regime. One can infer that the heat transfer increases with $M$ when heat is supplied to the fluid from the heated wall and decreases otherwise. Obviously, for purely asymmetric heating ($\xi = 0.0$), more heat is transferred from the heated wall to the fluid, while the lowest heat is transferred as $\xi \rightarrow 1.0$. This is because for symmetric heating, the temperature change between the fluid and the heated wall is very infinitesimal, hence low heat is transferred as it almost attains saturation state. It is good to note that in the absence of magnetic field ($M \rightarrow 0$), the heat transfer in the microchannel is independent of $Br$. 

![Figure 6](image1.png)  
Figure 6. Skin-friction for different values of $K$ and $Kn$ at $\zeta = 0.5$, $Br = 1$ ($Y = 0$).

![Figure 7](image2.png)  
Figure 7. Skin-friction for different values of $\zeta$ and $Br$ at $Kn = 0.05$, $K = 0.5$ ($Y = 1$).

![Figure 8](image3.png)  
Figure 8. Nusselt number for different values of $Kn$ and $K$ at $\zeta = 0.5$, $Br = 1.0$. 

![Figure 9](image4.png)  
Figure 9. Nusselt number as a function of $M$, $\xi$ and $Br$ for the fixed value of $K$ at slip flow regime.
3. Conclusion

This article is devoted to investigate the combined role of Joule heating and transversely applied magnetic field on buoyancy-driven flow of an electrically conducting fluid in a vertical microchannel with asymmetric wall heating. An exact solution is obtained for the momentum and energy equations and depicted graphically to see the role of governing parameters. Based on the solutions and graphical representations obtained, the following conclusions can be drawn:

(1) Velocity, temperature, skin-friction and Nusselt number increase with Hartmann number in the presence of Joule heating.
(2) Skin-friction and Nusselt number are higher in the continuum regime than the slip regime.
(3) The maximum heat transfer is achieved when heat is transferred from the walls to the fluid and for purely asymmetric wall heating.

Disclosure statement

No potential conflict of interest was reported by the authors.

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\[ X_1 = 1 - \delta_1 \beta_1 K_n, \quad X_2 = 1 + \delta_1 \beta_1 K_n, \quad X_3 = 1 - \delta_2 \beta_1 K_n, \]
\[ X_4 = 1 + \delta_2 \beta_1 K_n, \quad X_5 = (1 + \delta_1 \beta_1 K_n) e^{\delta_1}, \]
\[ X_6 = (1 - \delta_1 \beta_1 K_n) e^{-\delta_1}, \quad X_7 = (1 + \delta_2 \beta_1 K_n) e^{\delta_2}, \]
\[ X_8 = (1 - \delta_2 \beta_1 K_n) e^{-\delta_2}, \]
\[ X_9 = M^2 - \delta_2^2 - \delta_1 K_n (M^2 \delta_1 - \delta_1^2), \]
\[ X_{10} = M^2 - \delta_2^2 + \delta_1 K_n (M^2 \delta_1 - \delta_1^2), \]
\[ X_{11} = M^2 - \delta_2^2 + \delta_1 K_n (M^2 \delta_2 - \delta_2^2), \]
\[ X_{12} = M^2 - \delta_2^2 + \delta_1 K_n (M^2 \delta_2 - \delta_2^2), \]
\[ X_{13} = \xi, \]
\[ X_{14} = e^{\delta_1} (M^2 - \delta_2^2 + \delta_1 K_n (M^2 \delta_2 - \delta_2^2)), \]
\[ X_{15} = e^{-\delta_1} (M^2 - \delta_1^2 - \delta_1 K_n (M^2 \delta_1 - \delta_1^2)), \]
\[ X_{16} = e^{\delta_2} (M^2 - \delta_2^2 + \delta_1 K_n (M^2 \delta_2 - \delta_2^2)), \]
\[ X_{17} = e^{-\delta_2} (M^2 - \delta_2^2 + \delta_1 K_n (M^2 \delta_2 - \delta_2^2)), \]
\[ \Delta = \text{det} \left( \begin{array}{cccc} X_1 C_1 & X_2 C_2 & X_3 C_3 & X_4 C_4 \\ X_5 C_1 & X_6 C_2 & X_7 C_3 & X_8 C_4 \\ X_9 C_1 & X_{10} C_2 & X_{11} C_3 & X_{12} C_4 \\ X_{14} C_1 & X_{15} C_2 & X_{16} C_3 & X_{17} C_4 \end{array} \right), \]
\[ \Delta_1 = \text{det} \left( \begin{array}{cccc} X_1 C_1 & X_2 C_2 & X_3 C_3 & X_4 C_4 \\ -K & X_2 C_2 & X_3 C_3 & X_4 C_4 \\ X_9 C_1 & X_{10} C_2 & X_{11} C_3 & X_{12} C_4 \\ 1 & X_{15} C_2 & X_{16} C_3 & X_{17} C_4 \end{array} \right), \]
\[ \Delta_2 = \text{det} \left( \begin{array}{cccc} X_1 C_1 & -K & X_3 C_3 & X_4 C_4 \\ X_2 C_1 & -K & X_3 C_3 & X_4 C_4 \\ X_9 C_1 & X_{13} & X_{11} C_3 & X_{12} C_4 \\ X_{14} C_1 & 1 & X_{16} C_3 & X_{17} C_4 \end{array} \right), \]
\[ \Delta_3 = \text{det} \left( \begin{array}{cccc} X_1 C_1 & X_2 C_2 & -K & X_4 C_4 \\ X_2 C_1 & -K & X_3 C_3 & X_4 C_4 \\ X_9 C_1 & X_{10} C_2 & X_{13} & X_{12} C_4 \\ X_{14} C_1 & X_{15} C_2 & 1 & X_{17} C_4 \end{array} \right), \]
\[ \Delta_4 = \text{det} \left( \begin{array}{cccc} X_1 C_1 & X_2 C_2 & X_3 C_3 & -K \\ X_2 C_1 & -K & X_3 C_3 & X_4 C_4 \\ X_9 C_1 & X_{10} C_2 & X_{11} C_3 & X_{13} \\ X_{14} C_1 & X_{15} C_2 & X_{16} C_3 & 1 \end{array} \right), \]
\[ X_{18} = \frac{C_1}{\delta_1} (e^{\delta_1} - 1) + \frac{C_2}{\delta_2} (1 - e^{-\delta_2}) + \frac{C_1}{\delta_2} (e^{\delta_2} - 1) + \frac{C_4}{\delta_2} (1 - e^{-\delta_2}), \]
\[ P_1 = \frac{C_1 (M^2 - \delta_1^2) (e^{-2\delta_1} - 1)}{2\delta_1}, \quad P_2 = 2C_1 C_2 (M^2 - \delta_1^2), \]
\[ P_3 = \frac{C_1 C_3 (1 - \delta_2^2) (e^{\delta_1 + \delta_2} - 1)}{(\delta_1 + \delta_2)}, \]
\[ P_4 = \frac{C_1 C_4 (1 - \delta_2^2) (e^{\delta_1 - \delta_2} - 1)}{(\delta_1 - \delta_2)}, \]
\[ P_5 = \frac{C_2 (M^2 - \delta_2^2) (1 - e^{-2\delta_2})}{2\delta_2}, \]
\[ P_6 = \frac{C_1 C_3 (1 - \delta_2^2) (e^{\delta_1 + \delta_2} - 1)}{(\delta_1 + \delta_2)}, \]
\[ P_7 = \frac{C_1 C_4 (1 - \delta_2^2) (e^{\delta_1 - \delta_2} - 1)}{(\delta_1 - \delta_2)}, \]
\[ P_8 = \frac{C_2 (M^2 - \delta_2^2) (1 - e^{-2\delta_2})}{2\delta_2}, \]
\[ P_9 = \frac{C_1 C_3 (1 - \delta_2^2) (e^{\delta_1 + \delta_2} - 1)}{(\delta_1 + \delta_2)}, \]
\[ P_{10} = \frac{C_2 (1 - \delta_2^2) (e^{\delta_2} - 1)}{2\delta_2}, \]
\[ P_{11} = 2C_3 C_4 (1 - \delta_2^2), \]
\[ P_{12} = \frac{C_1 C_3 (1 - \delta_2^2) (e^{\delta_1 + \delta_2} - 1)}{(\delta_1 + \delta_2)}, \]
\[ P_{13} = \frac{C_2 (M^2 - \delta_2^2) (1 - e^{-2\delta_2})}{2\delta_2}, \]
\[ P_{14} = \frac{C_2 (1 - \delta_2^2) (1 - e^{-2\delta_2})}{2\delta_2}, \]
\[ P_{15} = \frac{K C_1 (M^2 - \delta_1^2) (e^{\delta_1} - 1)}{\delta_1}, \]
\[ P_{16} = \frac{K C_2 (M^2 - \delta_2^2) (e^{-\delta_1} - 1)}{\delta_1}, \]
\[ P_{17} = \frac{K C_3 (1 - \delta_2^2) (e^{\delta_1} - 1)}{\delta_2}, \]
\[ P_{18} = \frac{K C_4 (1 - \delta_2^2) (e^{-\delta_1} - 1)}{\delta_2}. \]