An $O(M \log M)$ Algorithm for Bipartite Matching with Roadmap Distances

Kyle Treleaven, Josh Bialkowski, Emilio Frazzoli

Abstract

An algorithm is presented which produces the minimum cost bipartite matching between two sets of $M$ points each, where the cost of matching two points is proportional to the minimum distance by which a particle could reach one point from the other while constrained to travel on a connected set of curves, or roads. Given any such “roadmap”, the algorithm obtains $O(M \log M)$ total runtime in terms of $M$, which is the best possible bound in the sense that any algorithm for minimal matching has runtime $\Omega(M \log M)$. The algorithm is strongly polynomial and is based on a capacity-scaling approach to the [minimum] convex cost flow problem. The result generalizes the known $\Theta(M \log M)$ complexity of computing optimal matchings between two sets of points on (i) a line segment, and (ii) a circle.

1 Introduction

Given two sets $S = (s_1, s_2, \ldots, s_M)$ and $T = (t_1, t_2, \ldots, t_M)$, of $M$ points each from a domain $\Omega$, a matching is a subset of pairs $A \subset S \times T$ such that each point appears in $A$ exactly once. Every matching represents a unique bijective mapping of $S$ onto $T$, and therefore, is uniquely determined by some permutation $\pi$ of $(1, 2, \ldots, M)$, in the sense that $A = \{(s_k, t_{\pi(k)})\}_{k=1}^M$. Given a distance metric $d : \Omega^2 \to \mathbb{R}_{\geq 0}$, we define the cost of a match $(s, t) \in A$ as $d(s, t)$ and the cost of the matching $A$ as the sum over pairs $\sum_{(s, t) \in A} d(s, t)$. The problem of finding the minimum cost matching, or minimal matching, is called the assignment problem or the bipartite matching problem.

Literature Review: Assignment problems [1] have direct applications in domains such as operations management, computer science, computational biology, and computational music. For example, they can be used to represent the problem of scheduling work to assets optimally (e.g., programs to computer processors, or jobs to machines or workers). In computational biology, the many-to-one assignment problem on a line from a domain $\Omega$, a matching is a subset of pairs $A \subset S \times T$ such that each point appears in $A$ exactly once. Every matching represents a unique bijective mapping of $S$ onto $T$, and therefore, is uniquely determined by some permutation $\pi$ of $(1, 2, \ldots, M)$, in the sense that $A = \{(s_k, t_{\pi(k)})\}_{k=1}^M$. Given a distance metric $d : \Omega^2 \to \mathbb{R}_{\geq 0}$, we define the cost of a match $(s, t) \in A$ as $d(s, t)$ and the cost of the matching $A$ as the sum over pairs $\sum_{(s, t) \in A} d(s, t)$. The problem of finding the minimum cost matching, or minimal matching, is called the assignment problem or the bipartite matching problem.

Literature Review: Assignment problems [1] have direct applications in domains such as operations management, computer science, computational biology, and computational music. For example, they can be used to represent the problem of scheduling work to assets optimally (e.g., programs to computer processors, or jobs to machines or workers). In computational biology, the many-to-one assignment problem on a line from a domain $\Omega$, a matching is a subset of pairs $A \subset S \times T$ such that each point appears in $A$ exactly once. Every matching represents a unique bijective mapping of $S$ onto $T$, and therefore, is uniquely determined by some permutation $\pi$ of $(1, 2, \ldots, M)$, in the sense that $A = \{(s_k, t_{\pi(k)})\}_{k=1}^M$. Given a distance metric $d : \Omega^2 \to \mathbb{R}_{\geq 0}$, we define the cost of a match $(s, t) \in A$ as $d(s, t)$ and the cost of the matching $A$ as the sum over pairs $\sum_{(s, t) \in A} d(s, t)$. The problem of finding the minimum cost matching, or minimal matching, is called the assignment problem or the bipartite matching problem.

Literature Review: Assignment problems [1] have direct applications in domains such as operations management, computer science, computational biology, and computational music. For example, they can be used to represent the problem of scheduling work to assets optimally (e.g., programs to computer processors, or jobs to machines or workers). In computational biology, the many-to-one assignment problem on a line from a domain $\Omega$, a matching is a subset of pairs $A \subset S \times T$ such that each point appears in $A$ exactly once. Every matching represents a unique bijective mapping of $S$ onto $T$, and therefore, is uniquely determined by some permutation $\pi$ of $(1, 2, \ldots, M)$, in the sense that $A = \{(s_k, t_{\pi(k)})\}_{k=1}^M$. Given a distance metric $d : \Omega^2 \to \mathbb{R}_{\geq 0}$, we define the cost of a match $(s, t) \in A$ as $d(s, t)$ and the cost of the matching $A$ as the sum over pairs $\sum_{(s, t) \in A} d(s, t)$. The problem of finding the minimum cost matching, or minimal matching, is called the assignment problem or the bipartite matching problem.

Literature Review: Assignment problems [1] have direct applications in domains such as operations management, computer science, computational biology, and computational music. For example, they can be used to represent the problem of scheduling work to assets optimally (e.g., programs to computer processors, or jobs to machines or workers). In computational biology, the many-to-one assignment problem on a line from a domain $\Omega$, a matching is a subset of pairs $A \subset S \times T$ such that each point appears in $A$ exactly once. Every matching represents a unique bijective mapping of $S$ onto $T$, and therefore, is uniquely determined by some permutation $\pi$ of $(1, 2, \ldots, M)$, in the sense that $A = \{(s_k, t_{\pi(k)})\}_{k=1}^M$. Given a distance metric $d : \Omega^2 \to \mathbb{R}_{\geq 0}$, we define the cost of a match $(s, t) \in A$ as $d(s, t)$ and the cost of the matching $A$ as the sum over pairs $\sum_{(s, t) \in A} d(s, t)$. The problem of finding the minimum cost matching, or minimal matching, is called the assignment problem or the bipartite matching problem.

Literature Review: Assignment problems [1] have direct applications in domains such as operations management, computer science, computational biology, and computational music. For example, they can be used to represent the problem of scheduling work to assets optimally (e.g., programs to computer processors, or jobs to machines or workers). In computational biology, the many-to-one assignment problem on a line from a domain $\Omega$, a matching is a subset of pairs $A \subset S \times T$ such that each point appears in $A$ exactly once. Every matching represents a unique bijective mapping of $S$ onto $T$, and therefore, is uniquely determined by some permutation $\pi$ of $(1, 2, \ldots, M)$, in the sense that $A = \{(s_k, t_{\pi(k)})\}_{k=1}^M$. Given a distance metric $d : \Omega^2 \to \mathbb{R}_{\geq 0}$, we define the cost of a match $(s, t) \in A$ as $d(s, t)$ and the cost of the matching $A$ as the sum over pairs $\sum_{(s, t) \in A} d(s, t)$.
transportation demands. LARGEARCS has been shown to be asymptotically optimal for large randomly generated instances (almost surely) [5], so efficient algorithms for matching are desirable for dealing with large SCP instances.

In general, the assignment problems can be solved in $O(M^3)$ time using the classical Hungarian method [6]. If the points are in $\mathbb{R}^2$ and the distance between them is Euclidean, then there is a class of algorithms [7] for the one-to-one assignment, or bipartite matching problem, which achieve $O(M^{2+\epsilon})$ time for any $\epsilon > 0$. If the points are on a line, then there is a trivial $O(M \log M)$ algorithm to solve the one-to-one assignment problem, the optimality of which is proved, e.g., in [8]. For the case that the points lie on a circle, another $O(M \log M)$ algorithm was given originally by Karp and Li [9], which has been refined somewhat by others, e.g., [10]. Colannino et. al. provided companion $O(M \log M)$ algorithms on the line for both the many-to-one [2] and many-to-many [11] versions of the assignment problem; both algorithms are fundamentally based on the original insights developed by Karp about the circle, though they provide non-trivial additional insights themselves. (The algorithms on the line are actually $O(M)$ if the points are already sorted; however, on the circle the runtime remains $O(M \log M)$.) The best known many-to-one and many-to-many algorithms on the circle remain $O(M^2)$.

**Contribution:** In this paper, we provide an algorithm which computes the minimal matching between two sets of $M$ points each, on a fixed but arbitrary roadmap with $m$ roads and $n$ vertices. The runtime of the algorithm is $O(M \log M) + \log M \times O(m(m + n \log n))$, which for a fixed roadmap is dominated asymptotically by the term $O(M \log M)$. Such is the best runtime bound achievable in terms of $M$. The design of the algorithm is influenced by the elegant treatment of bipartite matching on lines and circles in [10], which are the two roadmap structures one can construct from a single road ($m = 1$). The crucial component of the algorithm is the application of a capacity-scaling approach for the minimum convex cost flow problem, developed, e.g., in [12, Sec. 14.5].

Our algorithm can also provide a speed-up in cases where the roadmap geometry is implicit. Suppose, for example, that the minimal bipartite matching is desired between $S$ and $T$, where matches have costs equal to the lengths of shortest paths on a weighted, undirected graph $(S \cup T \cup V, E)$ with $n$ vertices and $m$ edges. A standard approach is to cast such problem as a minimum cost flow problem with unit supplies, which can be solved in $O(m(m + n \log n))$ time. If $m$ is very close to $2M$, then such approach represents a factor nearly $M$ speed-up, compared, e.g., to the Hungarian method on a metric completion graph. If additionally most of the $m$ edges have degree 2, then using the technique of this paper, an additional factor nearly $M$ speed up may be possible: Letting $M'$ denote (roughly) half the number of degree-2 vertices, one can produce an equivalent matching instance of $2M'$ points on a roadmap with $n'$ vertices and $m'$ edges, where $n'$ is the number of non-degree-2 vertices, and $m'$ is their total degree. Then using the algorithm of this paper, the instance can be solved in $O(M' \log M') + \log M' \times O(m'(m' + n' \log n'))$ time.

### 1.1 Organization

The rest of the paper is organized as follows: In Section 2 we review the existing results for bipartite matching on line segments and circles, which are the two most basic roadmaps. We state the problem of finding the minimal bipartite
matching between points on a general roadmap rigorously in Section 3. In Section 4 we generalize the analysis of bipartite matching on line and circles and introduce an optimal algorithm for bipartite matching on general roadmaps. We discuss computational complexity in Section 4.4 and demonstrate that our algorithm can be computed in $O(M \log M)$ time. We provide some discussion of the results and closing remarks in Section 5.

2 Optimal Matching on Lines and Circles

If the domain $\Omega$ is a single line segment with distance defined by $d(s, t) = |t - s|$, and both $S$ and $T$ are sorted, then it is well known (shown, e.g., in [8]) that the minimal matching is determined by the identity permutation $\pi_1$, i.e., $A = \{(s_1, t_1), (s_2, t_2), \ldots, (s_M, t_M)\}$. Therefore, if $S$ and $T$ are already sorted, the minimal matching can be transcribed in $\Theta(M)$ time. If the points are not sorted, then sorting them first takes $\Theta(M \log M)$ time in the worst case.

Such a straightforward scenario would seem to warrant little additional discussion, but [10] presents a characterization of the cost of the minimal matching, which becomes useful in various extensions: Let $[0, L)$ be the interval containing $S$ and $T$, and let us orient all matches $(s, t)$ from $s$ to $t$. If $s < t$ we say the match is forward, because the orientation of the match is in the direction of the positive coordinate axis; if $s > t$, then we say the match is backward. Let $H(y)$ denote the total number of forward matches crossing a coordinate $y$ minus the total number of backward matches crossing $y$. A matching $A$ is called unidirectional if at every such coordinate, all of the matches crossing the point have the same orientation.

**Lemma 2.1.** Every minimal matching is unidirectional.

**Proof.** A simple proof is given in Lemma 3 of [10], but we provide another proof that generalizes readily to every scenario in this paper. The proof is by contradiction: Assume that $A^*$ is a minimal match but there is some coordinate $y$ crossed by $M^+ > 0$ forward matches and $M^- > 0$ backward matches. Consider the case that there are no points at $y$. If $y$ has points, then we simply ignore all the matches that start or end there (they do not cross). With similar justification, assume that $y$ is an interior point. Then there is a neighborhood $(y - \epsilon, y + \epsilon)$ for some $\epsilon > 0$ which contains no points and is crossed by exactly the matches through $y$. Choose one of the forward matches and one of the backward matches and exchange their endpoints in $T$. The resulting matching has cost at least $4\epsilon$ less than the minimal matching $A^*$.

Note by Lemma 2.1 that the total number of matches crossing a point $y$ is $|H(y)|$ for the minimal matching $A^*$; if $H(y) > 0$ then they are forward matches; otherwise, they are backward matches. Thus, the total length of all matches, i.e., the cost of the optimal matching, can be written as

$$
\int_0^L |H(y)| \, dy. \tag{1}
$$

Let $N_S(y)$ denote the number of points $s \in S$ such that $s < y$, i.e., $|S \cap [0, y)|$, let $N_T(y)$ denote the number of points $t \in T$ such that $t < y$, i.e., $|T \cap [0, y)|$, and let $F(y) = N_S(y) - N_T(y)$. Note that since $|S| = |T|$ we have $F(0) = F(L) = 0.$
As argued in [10], \( F(y) = H(y) \) on a line segment, because whenever we cross a point \( s \in S \) in the positive direction, either a forward match is beginning, or a backward match is ending; either way, both \( H \) and \( F \) increase by 1. (Whenever we encounter a point \( t \in T \), both \( H \) and \( F \) decrease by 1.) Therefore, the cost of the minimal matching can be computed without producing one, by computing \( F \) and substituting it in (1).

Now suppose that \( \Omega \) is a circle instead of a line; for example, the circle of unit radius \([0, 2\pi]\) with distance between points \( y_1 \) and \( y_2 \) defined by \( \min\{|y_1 - y_2|, 2\pi - |y_1 - y_2|\} \). Though the circular case is quite similar to the linear one, the identity permutation is not necessarily minimal for sorted \( S \) and \( T \), because matches across the \( y = 0 \) boundary are now possible. It is easy to argue, however, that the minimal matching is among the \( M \) circular shift permutations, which leads immediately to an \( O(M^2) \) minimal matching algorithm, e.g., the one in [8]. The \( O(M^2) \) barrier was indeed broken in [6], where the authors observed that \( H(y) = F(y) + z \) for some integer \( z \). Since \( F(0) = 0 \), then \( z = H(0) \), which means \( z \) can be thought of as the (signed) number of matches crossing \( y = 0 \). Now the minimal matching has cost

\[
C(z) = \int_{0}^{2\pi} |F(y) + z| \, dy. \tag{2}
\]

Certainly, the cost of the optimal matching can be no less than the minimum value of (2) taken over integer \( z \). [10] provides an \( O(M \log M) \) algorithm to compute such minimum and an \( O(M) \) follow-up procedure to produce a matching of cost no greater than \( C(z) \).

3 Problem Statement

3.1 Notation

We use the following graph notation throughout the paper: Let \((V, A)\) denote a directed graph, or di-graph, with vertex set \( V \) and a set of directed edges \( A \). In general, \((V, A)\) might be a multi-di-graph, meaning there may be multiple distinct edges having the same endpoints. For any edge \( a \in A \), we let \( a^- \) denote the tail of \( a \) and let \( a^+ \) denote the head of \( a \). For example, if \( a = (u, v) \), then \( a^- = u \) and \( a^+ = v \).

**Definition 3.1** (Orientation). An orientation of an undirected graph \( G \) is the assignment of a direction to each edge in \( G \), resulting in a directed graph.

**Definition 3.2** (Topological ordering). A topological ordering of a directed acyclic graph \( G = (V, A) \) is a linear ordering \( \leq \) of the vertex set \( V \), so that for every \( a \in A \) it holds \( a^- \leq a^+ \).

It is known that any directed acyclic graph (DAG) has at least one topological ordering. An algorithm to compute a topological ordering of a DAG in time \( O(|V| + |A|) \) can be found in [13] Sec. 22.4.

3.2 Roadmaps

A roadmap can be described in terms of a set of lines or curves connected together into a particular pattern by their endpoints; the distance between
points on a roadmap is the minimum distance by which a particle could reach one point from the other while constrained to travel on the curves, or roads.

It is common practice, e.g., by modern postal services, to represent the topology of a roadmap using an undirected weighted graph or multi-graph \((V, R)\), possibly with loops, where the edges \(R\) correspond to roads in the roadmap and are labeled with lengths, and the vertices \(V\) describe their interconnections. Another common practice is to attach to such graph a coordinate system: Given a fixed orientation of the roadmap graph, every point on the roadmap continuum can be described unambiguously by a tuple, or address \((r, y)\), of a road \(r \in R\) and a real-valued coordinate \(y\) between 0 and the length \(L_r\) of \(r\). There is an intuitive notion of “roadmap distance” between points described by such addresses, arising from two basic assertions: (i) there is a path between any two points on the same road, of length equal to the difference between their address coordinates; (ii) there is a special point for every roadmap vertex \(u \in V\) which is at the respective endpoints of all the roads adjacent to \(u\), simultaneously. The distance between points then is the length of the shortest path between them.

### 3.3 Objective

The objective of the paper is to obtain an algorithm for the bipartite matching problem on roadmaps which for a given roadmap \(R\) has worst-case runtime bounded by \(O(M \log M)\), where \(M\) is the number of points in each of \(S\) and \(T\).

**Lemma 3.3** (Lemma 2 of [10]). \(\Omega(M \log M)\) is a lower bound for the time it takes to find the minimal cost matching in both the linear and the circular cases.

**Proof Sketch.** The lemma can be proved by a reduction of the Set Equality problem, as in [10].

\(\Omega(M \log M)\) is clearly a lower bound for the time to find a minimal cost matching on a fixed but arbitrary roadmap, because lines and circles are only two specific kinds of roadmaps.

### 4 Optimal Bipartite Matching on a Roadmap

In this section, we generalize the analysis of [10], about the cost of matchings on lines and circles, for the case of any fixed but arbitrarily complex roadmap \(R\). The cost of a match is equal to the roadmap distance between the points, i.e., the length of the shortest path between them. The result of our analysis will provide insight about the design of a novel minimal matching algorithm.

#### 4.1 Cost Characterization

Since shortest paths are minimal, they are also simple (they do not “cross themselves”). We will attribute to every match \((s, t)\) the orientation of its shortest path, in the direction from \(s\) to \(t\). Note that a match need not have the same orientation on every road; wherever its path follows a road \(r\) in the positive direction, we say it is forward; wherever it follows a road in the opposite direction, we say it is backward. Without loss of generality, we assume that shortest paths are unique, e.g., by perturbation or arbitrary tie-breaking.
We will now re-define the quantities $F$ and $H$ for roadmaps: For every road $r \in \mathbb{R}$, $0 \leq y \leq L_r$, let $N_S(y;r)$ denote the number of points $(r,y_i)$ in $S$ which are on $r$ such that $y_i < y$, and let $N_T(y;r)$ denote the number of points $(r,y_i)$ in $T$ which are on $r$ such that $y_i < y$. Let $F(y;r) := N_S(y;r) - N_T(y;r)$. Note that $F(0;r) = 0$ for all $r \in \mathbb{R}$, but now we have $F(L_r;r) = |S \cap r| - |T \cap r| =: b_r \neq 0$ in general. Let $b_r$ be called the surplus of road $r$.

Given a minimal matching $\mathcal{A}^*$, let $H(y;r)$ denote the number of forward matches crossing the coordinate $(r,y)$, minus the number of backward matches crossing it. Using the same arguments in [10], it is easy to argue that $H(y;r) = F(y;r) + z_r$ for some integer $z_r$; there is one such integer for every road $r \in \mathbb{R}$, and we can write the total cost of all the match fragments using road $r$ as

$$C(z_r; r) = \int_0^{L_r} |F(y;r) + z_r| \ dy. \quad (3)$$

Figure 1: An example road, with the points $S$ denoted by ‘×’ and the points $T$ denoted by ‘o’.

**Lemma 4.1** (Cost properties). Letting $C$ be defined as in (3) for each $r \in \mathbb{R}$ and all $z \in \mathbb{R}$ (and not just the integers), $C$ is (i) piece-wise linear, (ii) convex, and (iii) unbounded, and (iv) has constant slope in every integer interval.

**Proof.** We prove the result for a single road $r$, omitting the $r$-specific notation. Note that $F$ is piece-wise constant and takes value only on the set of integers. Let $\mathbf{F} = \{f_1, \ldots, f_m\} \subset \mathbb{Z}$ denote the set of values taken by $F$ (in increasing order), and let $I_1, \ldots, I_{m+1}$ denote the intervals $(-\infty, f_1), (f_1, f_2), \ldots, (f_{m-1}, f_m), (f_m, +\infty)$, disjointly covering $\mathbb{R}$. $C(z)$ can be written as

$$C(z) = \int_{F(y)+z>0} [F(y)+z] \ dy - \int_{F(y)+z<0} [F(y)+z] \ dy. \quad (4)$$

For each $k = 1, \ldots, m+1$ and all $z \in -I_k$ (i.e., $-z \in I_k$), we have that $\{y : F(y)+z > 0\} = \{F(y) > f_{k-1}\} =: \mathcal{Y}^+_k$ and $\{y : F(y)+z < 0\} = \{F(y) \leq f_{k-1}\} =: \mathcal{Y}^-_k$ are both constant. (Here, we let $f_0 := -\infty$.) Defining scalar constants

$$\alpha_k \doteq \int_{\mathcal{Y}^+_k} F(y) \ dy - \int_{\mathcal{Y}^-_k} F(y) \ dy,$$

and

$$\beta_k \doteq |\mathcal{Y}^+_k| - |\mathcal{Y}^-_k|, \quad (5)$$
then for all \( z \in -I_k \), (4) can be written as \( C(z) = \alpha_k + z \beta_k \).

The prequel demonstrates that \( C \) is piece-wise linear with constant slope within every integer interval. To show that \( C \) is convex, one can show that the slope of \( C \) is non-decreasing, e.g., by confirming that \( \beta_k \) is non-increasing in \( k \). Unboundedness can be proved by confirming that \( \beta_1 = L \) and \( \beta_{m+1} = -L \), where it can be assumed that the road length \( L > 0 \).

\[ \text{Lemma 4.2.} \] The minimal matching has cost bounded below by (6) of the optimal solution of Problem 4.1.

Proof. Given a matching \( A \), one can compute \( z_r := H(\cdot ; r) - F(\cdot ; r) \) for each \( r \in R \). If \( Z \in \mathbb{R}^R \) is the vector composed of such \( z_r \), then the total cost of \( A^* \) is precisely (6). For each \( u \in V \), the left hand side of (7) is equal to the number of matches entering \( u \), and the right hand side is equal to the number of matches leaving \( u \) (both signed counts). Regular conservation arguments imply that (7) must hold for any feasible matching. Therefore, the feasible set of Problem 4.1 contains all realizable \( Z \), obtaining the lemma.

The previous formulation can be extended quite easily to incorporate the notion of “one-way” roads: For example, if a road \( r \) admits only forward matches, that condition can be encoded as \( H(y; r) \geq 0 \) for all \( y \in (0, L_r) \), i.e., \( \min_y F(y; r) + z_r \geq 0 \). Note therefore that the framework can support one-way and bi-directional roads simultaneously.

\[ \text{Lemma 4.3 (Integral solutions).} \] The optimization problem Problem 4.1 has integer optimal solutions.

The significance of Lemma 4.3 is that we may hope to avoid the complexity associated with integer optimization problems by linear relaxation.

Proof. Suppose that \( Z \) is a fractional optimal solution to (4.1), and let \( r_1 \) be a road with non-integer component, i.e., \( z_{r_1} \notin \mathbb{Z} \). Note that since all the coefficients of (7) are integer, then among the roads which share endpoint \( r_1^+ \) with \( r_1 \) (including itself if \( r_1^- = r_1^+ \)), at least one must also be fractional. Using this argument, one can identify a cycle \((r_1, \ldots, r_K)\) of only roads with fractional components. For each \( k = 1, \ldots, K \), let us denote by \( I_k \subset \mathbb{R} \) the unit
interval \([z_{r_0}, z_{r_n}])\), and let \(\Pi\) denote the unit hyper-cube \(\prod_k I_k\). Note that \((z_{r_1}, \ldots, z_{r_n})\) is in the interior of \(\Pi\). In the interior of \(\Pi\), we have a gradient
\[
\nabla_{(z_{r_1}, \ldots, z_{r_n})}C(Z) = \left[ \frac{d}{dz} C(z; r_1) \bigg|_{z_{r_1}} \ldots \frac{d}{dz} C(z; r_K) \bigg|_{z_{r_K}} \right].
\]

Applying Lemma 4.1, one can argue that such gradient is constant over \(\Pi\). In particular, the gradient must be 0, since \(Z\) is optimal. Let \(g \in \mathbb{R}^K\) denote the vector which is +1 for each road traversed in the positive direction by the cycle, −1 for each road traversed in the reverse direction, and 0 for all roads not in the cycle. (It is easy to show that \(Z + \alpha g\) satisfies (7) for all \(\alpha \in \mathbb{R}\).) One can find \(\alpha \in \mathbb{R}\) such that \(Z + \alpha g =: Z^+\) lies on the boundary of \(\Pi\). \(Z^+\) is also feasible optimal, and has at least one less fractional component than \(Z\). Such procedure can be repeated until an integral optimal solution is obtained. □

4.2.1 Solving Problem 4.1

Since by Lemma 4.1 the objective functions \(C(\cdot; r)\) are all convex, Problem 4.1 is a so-called [minimum] convex cost flow problem [12 Ch. 14]. The convex cost flow problem (CCFP) is a generalization of the minimum [linear] cost flow problem, such that the edge costs needn’t be linear. Ahuja et al. provide a capacity-scaling algorithm for CCFPs like Problem 4.1 which have integral solutions [12 Sec. 14.5]. Provided that the objective functions can be evaluated in \(O(1)\) time, the algorithm obtains runtime \(O((m \log U)S(n, m, C))\), where \(n\) and \(m\) are the number of vertices and edges in the network, respectively, \(U\) and \(C\) are bounds on the total supply and cost, respectively, and \(S(n, m, C)\) is the time to solve the shortest path problem on such a network [12 Ch. 4]. Choosing, for example, the strongly polynomial \(O(m + n \log n)\) Fibonacci heap shortest-path algorithm of Fredman and Tarjan [13], and observing that \(U = M\) is an acceptable bound, one can solve Problem 4.1 in \(O(m \log M(m + n \log n))\) time. Moreover, the algorithm always provides integer solutions.

4.3 Obtaining an Optimal Roadmap Matching

In this section, we provide an algorithm for the construction of an optimal matching given the vector \(Z^+\). The algorithm generalizes, in a fairly clean way, e.g., the procedure given in [10].

Given a solution \(Z\) to Problem 4.1, the following procedure will obtain a matching with cost less than or equal to \(C(Z)\): The procedure is in two steps. During the first step, one obtains an intermediate data structure, of an integer-weighted digraph \(G(Z)\) on vertex set \(S \cup T \cup V\). In such graph, every edge corresponds to an empty interval on the roadmap (generally, one spanning the space between two adjacent points); thus, we will call \(G(Z)\) the interval graph. In the second step, to obtain the matching itself, we run a simple graph traversal algorithm (Algorithm 1 on \(G(Z)\)).

The edges of the interval graph are directed, and each is labeled with a positive integer weight, indicating the number of matches which cross the corresponding interval. For example, let \((y_1, y_2, \ldots, y_K)\) be the ordering of the coordinates of points in \(S \cup T\) which are on some road \(r\). Let \(Y_1, Y_2, \ldots, Y_{K+1}\) denote the set of intervals \((0, y_1), (y_1, y_2), \ldots, (y_{K-1}, y_K), (y_K, L_r)\), and let \(f_k\) denote the constant value taken by \(F(\cdot; r)\) per interval \(Y_k\). (Let \(l_k\) denote the
interval length.) Note that \( f_k + z_r =: h_k \) is the signed number of matches traversing \( Y_k \) under \( Z \). If \( h_k > 0 \) then \( G(Z) \) contains the edge from the earlier endpoint \((r, y_{k-1})\) to the later endpoint \((r, y_k)\); if \( h_k < 0 \), then it has the reverse edge; if \( h_k = 0 \), then there is no edge. If the edge is contained, then it has weight \(|h_k|\). Note that the endpoints \((r, 0)\) and \((r, L_r)\), of intervals \( Y_1 \) and \( Y_K \), respectively, are not points in \( S \cup T \). They are substituted in \( G(Z) \) by the roadmap vertices \( r^- \in V \) and \( r^+ \in V \), respectively, to which they correspond.

**Remark 4.4.** \( C(Z) \) is equal to the sum over all edges in \( G(Z) \) of \(|h| \times l\).

Algorithm 1 (below) produces a matching by visiting every vertex in \( G(Z) \), among which are all of \( S \cup T \). When the current vertex \( i \) is a point in \( S \), then it is “collected” for future use. When \( i \in T \), then it is matched to a point \( j \) previously collected. To ensure that there are always enough points collected for future matches, \( G(Z) \) is traversed in a topological order.

**Algorithm 1 ConstructMatching**

**Input:** an interval di-graph \( G \) (e.g., generated by inputs \( R, S, T, \) and \( Z \))

**Output:** a bipartite matching \( A \) between \( S \) and \( T \)

1: initialize: \( A \leftarrow \) an empty matching

2: initialize: Associate with each vertex \( v \in S \cup T \cup V \) an empty set \( L_v \).

3: Choose any topological ordering of \( S \cup T \cup V \) under \( G(Z) \). Enumerate \( S \cup T \cup V \) in this order:

4: for all vertices \( v \) do

5: If \( v \in S \), then add \( v \) to \( L_v \). Otherwise, if \( v \in T \) then remove some point \( u \) from \( L_v \), and insert the match \((u, v)\) into \( A \). (Because the vertices are enumerated in a topological order, if \( v \in T \) then it must be true \(|L_v| > 0\).)

6: If \( v \in V \), we do not alter \( L_v \), nor create a match.

7: For every edge \((v, w)\) leaving \( v \), move \( weight(v, w) \) elements from \( L_v \) into \( L_w \).

8: return \( A \)

**Lemma 4.5.** The cost of the matching produced by Algorithm 1 on interval graph \( G(Z) \) has cost no greater than \( C(Z) \).

**Proof.** During the execution of Algorithm 1, every point \( s \in S \) traverses a path in \( G(Z) \) to its match \( A(s) \in T \). The cost of the matching produced cannot be greater than the total length of all such paths. By design, every edge in \( G(Z) \) is traversed under Algorithm 1 by a total number of paths equal to its weight. The total length of all paths is equal therefore to the sum over the edges in \( G(Z) \) of their weight times length. That sum is equal to \( C(Z) \); recall, e.g., Remark 4.4.

Algorithm 1 has one technical caveat: It assumes that a topological ordering exists under \( G(Z) \). This is guaranteed as long as \( G(Z) \) is a DAG, since every DAG has at least one topological ordering.

**Lemma 4.6.** Let \( Z^* \) be an optimal solution to Problem 4.1. \( G(Z^*) \) is a DAG.
Proof. The proof is by contradiction. Suppose $Z^*$ is optimal, but $G(Z^*)$ has a directed cycle $\mathcal{C} = (a_1, \ldots, a_K)$. $\mathcal{C}$ corresponds to a cycle $\mathcal{C}' = (r_1, \ldots, r_K)$ of the roads in $R$. Let $g \in R^R$ denote the vector which is $+1$ for each road traversed in the positive direction by $\mathcal{C}'$, $-1$ for each road traversed in the reverse direction, and $0$ for all roads not in $\mathcal{C}'$. It is easy to argue that we can obtain $G(Z^* - g)$ from $G(Z^*)$ by subtracting $1$ from the weight on every edge in $\mathcal{C}$. Doing so strictly decreases the total weight of the edges in $G$, ultimately decreasing $C$, and thereby contradicting the optimality of $Z^*$.

In fact, $G(Z^*)$ is a special kind of DAG:

**Proposition 4.7.** $G(Z^*)$ is a multi-tree, i.e., a directed, acyclic graph in which there is at most one directed path between any two vertices.

Proof. The proof is by contradiction, and is similar to the previous one. Suppose that $G(Z^*)$ is optimal, but there are two distinct paths, $P$ and $P'$, from some vertex $u$ to a vertex $v$. (Recalling our assumption (w.l.g.) that shortest paths are unique, suppose $P$ is strictly shorter than $P'$.) Let $\mathcal{C}$ denote a cycle which traverses $P$ in the forward direction, and traverses $P'$ in the reverse direction. Let $\mathcal{C}'$ denote the corresponding cycle on roads in $R$. Defining $g$ as before, $Z^* + g$ is feasible, and we can obtain $G(Z^* + g)$ from $G(Z^*)$ by incrementing the edge weights on $P$ and decrementing the edge weights on $P'$. (We remove any edges which obtain zero weight.) This action decreases $C$, contradicting the optimality of $Z^*$.

4.4 Complexity Analysis

The analysis of the prequel suggests a minimal matching algorithm in three fundamental steps.

1. Transcribe the instance of Problem 4.1 generated by $R$, $S$, and $T$;
2. Obtain an optimal integer solution $Z^*$;
3. Use the solution vector $Z^*$ to construct a matching $A^*$ with cost $C(A^*) = C(Z^*)$, following the procedure of Section 4.3.

In this section we will demonstrate that all three steps can be completed within $O(M \log M)$ time, in terms of $M$.

4.4.1 Transcribing Problem 4.1

An instance of Problem 4.1 is specified by the vertex supplies $b$ and the cost functions $C$, which we must therefore compute. We rely on the key fact of Prop. 4.8 below.

**Proposition 4.8.** There are at most $2m + 2M$ total linear pieces among the objectives $C(\cdot;r)$.

Proof. If a road $r \in R$ has no points, then $F(\cdot;r)$ equals 0 everywhere on $r$, i.e., $F(\cdot;r)$ takes values from the set $\{0\}$, and so $C(\cdot;r)$ is linear over each of the intervals $(-\infty, 0)$ and $(0, +\infty)$. The $m = |R|$ total roads result in $2m$
As demonstrated in Section 4.2.1, Z time by a capacity-scaling algorithm, provided an O each C compute all arrays. (The dependence on 4.4.2 Solving Problem 4.1 for be accomplished by simple positional lookup in the array C, obtained in I level measure O takes at most F scan over built by the following procedure: Initialize the array to all zeros. Perform a O F Y be associated to a collection (accumulation) over S of S therefore, sorting C can be built by sorting the distinct values appearing in T into the collections {Y} for each i ∈ {1 . . . K_2} and for some j, z_i = -f_j I_r = [I_0 . . . L_{K_2}] an array of “level measures” such that I_i = \sum_j y_{j+1} - y_j where J = {j | f_j = z_i} C_r = [c_0 . . . c_{K_2}] an array of pairs c_k = (\alpha_k, \beta_k) such that C(z; r) = \alpha_k + \beta_k z, for z ∈ [z_i, z_{i+1}].

For every road r ∈ R, b_r is equal to the last entry of F_r, and C_r contains the data of the linear pieces of C(·; r) in order of appearance.

Since the road set R is known ahead of time, each point in S ∪ T can be associated to a collection Y_r in O(1) time (using, e.g., a hash function). Therefore, sorting S ∪ T into the collections {Y_r} can be done in a single sweep of S ∪ T in O(M log M) time. Each F_r can be generated by a linear scan (accumulation) over Y_r, and so the entire collection {F_r} can be obtained in O(m + M) time; the O(m) term appears if every road is processed explicitly, even if it contains no points.

Z_r can be built by sorting the distinct values appearing in F_r, and so it takes at most O(m + M log M) time to obtain the whole collection. I_r can be built by the following procedure: Initialize the array to all zeros. Perform a scan over F_r. For each k = 0, . . . , K_2, add the interval length y_{k+1} - y_k to the level measure I_j such that z_j = -f_k. There are O(M) elements among all the F_r, so the total time to build the collection {I_r} is O(m + M). Finally each C_r can be built by a scan over I_r, e.g., using [5], and the whole collection can be obtained in O(m + M) time.

Adding up the times of all the steps, we obtain O(m + M log M) time to compute all arrays. (The dependence on m can actually be removed, because all the arrays have implicit value for any road which obtains no points.)

4.4.2 Solving Problem 4.1 for Z*
As demonstrated in Section 4.2.1, Z* can be obtained in O(m(m + n log n) log M) time by a capacity-scaling algorithm, provided an O(1) procedure to compute each C(·; r). One such procedure is to find the linear data (\alpha_k, \beta_k) associated with a query point z, and then compute C(z; r) = \alpha_k + \beta_k z. The first task can be accomplished by simple positional lookup in the array C_r, since the domain of the array is an ordered list of consecutive integers (i.e., Z_r, except −∞).
4.4.3 Obtaining a Minimal Matching Given $Z^*$

To obtain the final matching one must: (i) construct the graph $G(Z^*)$, (ii) compute a topological ordering of the vertices under $G(Z^*)$, and (iii) execute a short program (lines 5-6 of Algorithm 1) once per vertex. The graph $G(Z^*)$ can be obtained in $O(m + M)$ time by enumerating the intervals in the arrays $\{F_r\}$. A topological ordering of the vertices can be obtained in $O(n + M)$ time using a standard algorithm, e.g., the one in [13, Sec. 22.4]. If the sets in the algorithm are implemented using linked-list queues, then the duties of the program per vertex are all constant time except possibly the task of splitting a list. It can be checked however, that at each of the vertices in $S \cup T$, the split is trivial and we can skip it. In the worst case we may suffer at most $n = |V|$ non-trivial splits, of at most $O(M)$ operations each. Therefore, the total time of the construction is $O(m + nM)$. Using a slightly more sophisticated data structure can reduce this time to $O(nm + M)$: Each sequence of adjacent points stored in a linked-list queue can be represented instead using a single interval range. Because $G(Z^*)$ is a multi-tree, it can be proved that such queues will never obtain more than $2m$ such ranges under Algorithm 1.

4.4.4 Total Runtime Complexity

Adding the runtimes of all three components and re-arranging terms, we find that the entire algorithm can be computed in $O(M \log M + \log M \times O(m(m + n \log n)))$ time. For $M$ sufficiently large compared to the description of the roadmap, the algorithm is dominated by the act of transcribing the matching instance as an instance of the convex cost flow problem, followed by construction of the matching given the optimal solution $Z^*$; in comparison to those steps, the search for the optimal solution $Z^*$ is relatively easy. Moreover, the space requirements of the algorithm are linear in $n, m$, and $M$, and so the algorithm is strongly polynomial.

5 Discussion

We would like to remark that the generalizations in this paper of the techniques in [10] are essentially straightforward. Instead, the crucial contribution of the present result is recognizing that Problem 4.1 can be solved efficiently and by a strongly polynomial algorithm. Such knowledge seems to be fairly weakly disseminated; for example, as far as we know, Ahuja et. al. only published the relevant min-cost flow algorithm in their text book [12]. Depending on the sophistication of one’s standard approaches, Problem 4.1 could alternatively be reduced to a network flow problem with $O(M)$ edges, or to a generic linear optimization problem in $O(M)$ constraints. In either case, most standard contemporary algorithms fail to meet the quite demanding $O(M \log M)$ runtime budget; moreover, strongly polynomial algorithms tend to be elusive. While the strongly polynomial property of our algorithm stems from the fact that the solution is integral (an assignment), it might be possible to obtain a strongly polynomial, efficient algorithm for the more general transportation problem on roadmaps. This study highlights the fact that netflow flow problems are somewhat better understood, in practice, than general linear optimization problems.
Acknowledgements

We thank Dr. Michael Otte and Kevin Spieser for helpful discussions.

References

[1] Rainer E Burkard, Mauro Dell’Amico, and Silvano Martello. Assignment problems. Siam, 2009.

[2] Justin Colannino, Mirela Damian, Ferran Hurtado, John Iacono, Henk Meijer, Suneeta Ramaswami, and Godfried Toussaint. An O((n log n)-time algorithm for the restriction scaffold assignment problem. Journal of Computational Biology, 13(4):979–989, 2006.

[3] Miguel Dıaz-Banez, Giovanna Farigu, Francisco Gómez, David Rappaport, and Godfried T Toussaint. El compás flamenco: a phylogenetic analysis. In Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science, pages 61–70, 2004.

[4] G. N. Frederickson, M. S. Hecht, and C. E. Kim. Approximation algorithms for some routing problems. Foundations of Computer Science, Annual IEEE Symposium on, 0:216–227, 1976.

[5] K. Treleaven, M. Pavone, and E. Frazzoli. Asymptotically optimal algorithms for one-to-one pickup and delivery problems with applications to transportation systems. Automatic Control, IEEE Transactions on, 58(9):2261–2276, 2013.

[6] Harold W Kuhn. The hungarian method for the assignment problem. Naval research logistics quarterly, 2(1-2):83–97, 1955.

[7] P. K. Agarwal, A. Efrat, and M. Sharir. Vertical decomposition of shallow levels in 3-dimensional arrangements and its applications. In Proceedings of the eleventh annual symposium on Computational geometry, pages 39–50, Vancouver, British Columbia, Canada, 1995. ACM.

[8] Michael Werman, Shmuel Peleg, and Azriel Rosenfeld. A distance metric for multidimensional histograms. Computer Vision, Graphics, and Image Processing, 32(3):328–336, 1985.

[9] Richard M Karp and Shuo-Yen R Li. Two special cases of the assignment problem. Discrete Mathematics, 13(2):129–142, 1975.

[10] Michael Werman, Shmuel Peleg, Robert Melter, and T. Yung Kong. Bipartite graph matching for points on a line or a circle. Journal of Algorithms, 7(2):277–284, 1986.

[11] Justin Colannino, Mirela Damian, Ferran Hurtado, Stefan Langerman, Henk Meijer, Suneeta Ramaswami, Diane Souvaine, and Godfried Toussaint. Efficient many-to-many point matching in one dimension. Graphs and combinatorics, 23(1):169–178, 2007.

[12] Ravindra K Ahuja, Thomas L Magnanti, and James B Orlin. Network flows: theory, algorithms, and applications. 1993.
A Capacity Scaling Algorithm

Algorithm 2 Capacity Scaling

Input:

Output:

1: initialize: flows $x := 0$ and potentials $\pi := 0$
2: initialize: Compute imbalances $e$
3: initialize: $\Delta := 2^{\lceil \log U \rceil}$
4: while $\Delta \geq 1$ do
5: [Re-] Compute $\Delta$-residual graph $G(x; \Delta)$ and reduced costs $c^\pi$
6: for every arc $a$ in the residual network $G(x; \Delta)$ do
7: if $c^\pi_a < 0$ then
8: Send $\Delta$ flow along arc $a$: update $x$, $e$, $G(x; \Delta)$ and $c^\pi$
9: end if
10: end for
11: $S(\Delta) := \{i : e(i) \geq \Delta\}$
12: $T(\Delta) := \{i : e(i) \leq -\Delta\}$
13: while $S(\Delta) \neq \emptyset$ and $T(\Delta) \neq \emptyset$ do
14: Choose $s \in S(\Delta)$ and $t \in T(\Delta)$
15: Determine shortest path distances $d(\cdot)$ from $s$ to all other nodes in $G(x; \Delta)$ with respect to the reduced costs $c^\pi$
16: Let $P$ be the shortest path from $s$ to $t$ in $G(x; \Delta)$
17: Augment $\Delta$ flow along the path $P$: update $x$, $e$, $G(x; \Delta)$, $S(\Delta)$, and $T(\Delta)$
18: Update $\pi := \pi - d$ and reduced costs $c^\pi$
19: end while
20: $\Delta := \Delta / 2$
21: end while
22: return $x$

[13] Charles E Leiserson, Ronald L Rivest, Clifford Stein, and Thomas H Cormen. Introduction to algorithms. The MIT press, 2001.

[14] Michael L Fredman and Robert Endre Tarjan. Fibonacci heaps and their uses in improved network optimization algorithms. Journal of the ACM (JACM), 34(3):596–615, 1987.