Greybody Factors for Black Holes in Four Dimensions:
Particles with Spin

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Abstract

We compute the emission spectrum of minimally coupled particles with spin that are Hawking radiated from four dimensional black holes in string theory. For a range of the black hole parameters the result has a product structure that may be interpreted in terms of the respective right- and left-moving thermal correlation functions of an effective string model. For spin-one and spin-two particles a novel cancellation between contributions to the wave function is needed to ensure this outcome. The form of the spectra suggests that the four-dimensional effective string description is “heterotic”: particles with spin are emitted from the right-moving sector, only.
1 Introduction

Recently a precise correspondence has been established between black holes and collective states in string theory (for review see, e.g., [1]). One of the results of these developments is that, in some cases, a classical black hole behaves as an effective string. The relevant low energy excitations of this effective description are the right- and left-moving modes of the string. A characteristic feature of this interpretation is that two independent inverse temperatures $\beta_{R,L}$ can be introduced. Their physical significance becomes apparent when Hawking radiation is interpreted microscopically as the result of colliding right- and left-moving excitations. In the simplest case of minimally coupled scalar fields this yields an emission spectrum with the characteristic factorized form [2, 3]:

$$\Gamma_{\text{em}}(\omega) = \frac{P_R(\frac{\omega}{2})}{e^{\beta_R \omega/2} - 1} \frac{P_L(\frac{\omega}{2})}{e^{\beta_L \omega/2} - 1} \frac{d^3k}{(2\pi)^3},$$

where $P_R$ and $P_L$ are polynomials in the frequency, with coefficients that depend on the inverse temperatures $\beta_R$ and $\beta_L$, respectively.

The identification of the black hole with an effective string faces a stringent test in the comparison of the emission spectrum with the semi-classical result given by Hawking [4]:

$$\Gamma_{\text{em}}(\omega) = \sigma_{\text{abs}}(\omega) \frac{1}{e^{\beta_H \omega} - 1} \frac{d^3k}{(2\pi)^3},$$

where $\sigma_{\text{abs}}(\omega)$ is the classical absorption cross-section for scalar particles impinging on the black hole and $\beta_H = \frac{1}{2}(\beta_R + \beta_L)$ is the inverse Hawking temperature. The agreement between the two descriptions requires:

$$\sigma_{\text{abs}}(\omega) = \frac{P_R(\frac{\omega}{2})}{e^{\beta_R \omega/2} - 1} \frac{P_L(\frac{\omega}{2})}{e^{\beta_L \omega/2} - 1} (e^{\beta_H \omega} - 1).$$

It is a striking confirmation of the effective string description that the absorption cross-section indeed takes this form for a large range of the black holes backgrounds and energy ranges of the scattered particles.

The functions $P_{R,L}(\frac{\omega}{2})$ have been calculated in the effective string theory for minimally coupled scalars in the S-wave and in this case a complete agreement with has been established that includes the numerical coefficient [2, 3]. Scalar particles with orbital angular momentum or non-minimal couplings are presently understood with less precision, because in these cases it is not known how to calculate fully the functions $P_{R,L}(\frac{\omega}{2})$ in the effective string theory. However, qualitative arguments indicate that multiparticle interactions could account for these processes [3, 4, 6, 7, 8, 9]. It is therefore reasonable to take an absorption cross-section of the form eq. 3 as evidence for some underlying effective string description, even though the specifics of this theory remain unknown.

The purpose of the present paper is to calculate greybody factors for the minimally coupled massless particles with spin-1/2, spin-1 and spin-2 in the background of static four-dimensional black holes that are parametrized by 4 $U(1)$ charges (a prototype
black hole solution of toroidally compactified string theory [11, 12] . The work is a
contribution to a broad program to address the dynamical properties of black holes
in string theory and thus further shed light on the microscopic structure of such black
holes, in particular to elucidate the effective string interpretation.

The work also serves as a step toward a complete decoupling of the full set of
perturbation equations for particle with different spins. We therefore report on partial
results that are more general than are strictly needed here. Other recent work on
greybody factors for particles with spin can be found in [13, 14, 15, 16].

Let us summarize the main results. In the simplest case of massless (minimally
coupled) Weyl fermions the absorption cross-section takes the following form:

$$\sigma_{\text{abs}}(\omega) = \frac{P_R(\omega)}{e^{\beta_R\omega/2} + 1} \frac{P_L(\omega)}{e^{\beta_L\omega/2} - 1} (e^{\beta_H\omega} + 1),$$  \hspace{1cm} (4)

where again $P_R$ and $P_L$ are polynomials in the frequency that depend on $\beta_R$ and $\beta_L$, respectively. A striking feature of this expression is the Fermi-Dirac factor
$e^{\beta_H\omega} + 1$ in the numerator, which precisely cancels the Fermi-Dirac factor in Hawking’s expression
for the emission rate:

$$\Gamma_{\text{em}}(\omega) = \sigma_{\text{abs}}(\omega) \frac{1}{e^{\beta_H\omega} + 1} \frac{d^3k}{(2\pi)^3},$$  \hspace{1cm} (5)

and thus yields an emission rate that is compatible with an interpretation as a correlation function in a weakly coupled effective string theory. In this description of emission the statistical factors in the denominator of eq. 4 have their origin as phase space factors of the right- and left-moving string states. The fact that the right-moving factor is of the Fermi-Dirac type while the left factor is of the Bose-Einstein type indicates that the effective string theory for the four dimensional black hole is “heterotic”, i.e. the right-moving sector has both fermionic and bosonic degrees of freedom, while the left-moving sector has only bosonic degrees of freedom. It has previously been observed that the entropy formula for rotating black holes in four dimensions suggests that the angular momentum is carried only by the right-moving modes [17, 18]. This is in harmony with our result because it is exactly the worldsheet fermions that carry the spacetime angular momentum [18].

For minimally coupled spin-1 and spin-2 fields we again find absorption cross-
sections of the factorized form eq. 4. The factorization turns out to be nontrivial
for particles with spin $s \geq 1$: in intermediate steps of the calculation there is a polynomial dependence on $\beta_H = \frac{1}{2}(\beta_R + \beta_L)$. However, this dependence cancels in the final expression for emission rates and so our results provide a nontrivial test of the effective string model.

The emission rate for particles with arbitrary spin and total angular momen-
tum can be written in a concise form that contains fermionic and bosonic degrees of freedom on equal footing. This result suggests that the microscopic theory is supersymmetric even though we consider black hole backgrounds that are not necessarily extremal. The regularity that we see may be a manifestation of the supersymmetry that is present in the theory, but broken by the presence of the black hole. Our re-
sult indicates that for black holes in theories with supersymmetry there is a relation
between the absorption cross-sections for particles with different spins. This would seem to suggest that the effective string is a superstring with supersymmetry broken only by the effects of thermodynamics.

The paper is organized as follows: in sec. 2 we summarize the classical geometry of the black holes backgrounds we consider, and in section 3 we describe these black holes in the Newman-Penrose formalism. This sets the stage for the derivation of field equations for perturbations, in sec. 4. For spin-1/2 and spin-1 we consider “spectator” particles that respond to the gravitational field but not to the background gauge fields, and are thus minimally coupled. In the case of gravitons we are not able to decouple the general perturbation equations but we can do so in an approximation that is sufficient to establish absorption cross-sections of the effective string form. In sec. 5 we comment on the features of the field equations, and find the general relation between their solutions and the absorption cross-section. This is exploited in sec. 6, where approximate wave functions are calculated. In sec. 7 we discuss the final results and consider some special cases and in sec. 8 we discuss their relation to the effective string theory model. Finally, in sec. 9, we conclude with comments on the generality of the type of greybody factors considered here.

2 The Black Hole Geometry

We consider the black holes in string theory that are characterized by their mass $M$ and four $U(1)$ charges $Q_i$. These quantum numbers are parametrized as:

\begin{align*}
M &= \frac{1}{2} \mu \sum_{i=1}^{4} \cosh 2 \delta_i , \\
Q_i &= \frac{1}{2} \mu \sinh 2 \delta_i ; \quad i = 1, 2, 3, 4.
\end{align*}

The gravitational coupling constant in four dimensions is $G_4 = \frac{1}{\kappa}$. In string theory this corresponds to the relation $g^2 (2\pi)^6 (\alpha')^4 / V_6 = 1$ between the gauge coupling $g$, the string tension $\alpha'$ and the volume of the six-dimensional compactified space $V_6$. The black hole metric can be written:

\begin{equation}
{ds}^2 = -\frac{\Delta}{\Sigma} dt^2 + \Sigma \left( \frac{1}{\Delta} dr^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right),
\end{equation}

where:

\begin{align*}
\Delta &= r^2 - \frac{1}{16} \mu^2 , \\
\Sigma &= \prod_i (r + \frac{1}{4} \mu \cosh 2 \delta_i)^{\frac{1}{4}} .
\end{align*}

Note that in these coordinates the two horizons are located at:

\begin{equation}
{r}_\pm = \pm \frac{1}{4} \mu .
\end{equation}

\footnote{The $m$ of \cite{17} is $m = \frac{1}{4} \mu$ and the $r_0$ of \cite{12} is $r_0 = \frac{1}{2} \mu$.}
The black hole can be interpreted as a generalization of the standard Reissner-Nordström solution. This special case is obtained by identifying the four charges through:

\[2Q_{RN} = Q_1 = Q_2 = Q_3 = Q_4 = \frac{1}{2}\mu \sinh 2\delta,\]
\[M_{RN} = 2\mu \cosh 2\delta,\]

and transforming our radial coordinate to the conventional one:
\[r_{RN} = r + \frac{1}{4}\mu \cosh \delta.\]

Then eqs. 9-10 become:

\[\Sigma_{RN} = r_{RN}^2,\]
\[\Delta_{RN} = r_{RN}^2 - 2\frac{1}{8}M_{RN} + Q_{RN}^2,\]

and the metric eq. 8 indeed reproduces the Reissner-Nordström line element with \(G_N = \frac{1}{8}\).

The entropy and the inverse temperature of the general black hole are:

\[S \equiv \frac{A}{4G_N} = 2\pi \mu^2 \prod_i \cosh \delta_i,\]
\[\beta_H \equiv \frac{2\pi}{\kappa_+} = 2\pi \mu \prod_i \cosh \delta_i,\]

where \(A\) and \(\kappa_+\) denote the area and the surface acceleration, both measured at the outer horizon. We will also need the right (R) and left (L) temperatures that combine the respective surface accelerations \(\kappa_{\pm}\) at the outer and inner horizons [19]:

\[\beta_R = \frac{2\pi}{\kappa_+} + \frac{2\pi}{\kappa_-} = 2\pi \mu \left( \prod_i \cosh \delta_i + \prod_i \sinh \delta_i \right),\]
\[\beta_L = \frac{2\pi}{\kappa_+} - \frac{2\pi}{\kappa_-} = 2\pi \mu \left( \prod_i \cosh \delta_i - \prod_i \sinh \delta_i \right).\]

The metric is a solution to the equations of motion low energy string theory in the presence of matter that can be represented as a collection of \(U(1)\) gauge fields and scalars. For the study greybody factors we will not need the explicit form of these matter fields. This state of affairs points to a robustness of this kind of calculation: these black holes can be considered solutions to different theories. It is perhaps natural to consider the low energy limits of toroidally compactified string theories, namely the \(N = 4\) or \(N = 8\) supergravity theories. However, the full family of solutions is still allowed in the bosonic part of the S-T-U–model of \(N = 2\) supergravity and they are therefore also solutions to \(N = 2\) supergravity. Thus, the greybody factors of the type found in this paper are generic for black holes that can be embedded in \(N = 4, 8\) supergravity theory, even when this possibility is not realized.
3 The Newman-Penrose Formalism

The Newman–Penrose formalism greatly simplifies the consideration of particles with spin in curved spacetime. In the following we summarize the basic features and define the notation. Here and throughout the paper we rely heavily on the book by Chandrasekhar [20]. For other introductions to the Newman–Penrose formalism we refer to [21, 22]. Some relevant discussion and applications were given in [15].

The starting point is the choice of a complex null-tetrad. We take:

\[
\begin{align*}
l &= -dt + \frac{\Sigma}{\Delta} dr, \\
n &= -\frac{\Delta}{2\Sigma} dt - \frac{1}{2} dr, \\
m &= \sqrt{\Sigma} \left( d\theta + i \sin \theta d\phi \right).
\end{align*}
\]

The dual basis is of the form:

\[
\begin{align*}
l^\mu \partial_\mu &= \partial_r + \frac{\Sigma}{\Delta} \partial_t; \\
n^\mu \partial_\mu &= -\frac{\Delta}{2\Sigma} \partial_r + \frac{1}{2} \partial_t; \\
m^\mu \partial_\mu &= \frac{1}{\sqrt{2\Sigma}} \left( \partial_\theta - \frac{i}{\sin \theta} \partial_\phi \right).
\end{align*}
\]

These directional partial derivatives are denoted \((D, \Delta, \delta)\) in the Newman-Penrose literature but we will not do so, to avoid confusion with other meanings of those symbols. The tetrad is normalized so that \(l \cdot n = -1\) and \(m \cdot \bar{m} = 1\). With our choice of basis the vectors \(n^\mu\) and \(l^\mu\) parametrize infalling and outgoing null-geodesics, with the condition that the latter geodesic is affinely parametrized.

In the local frame a vector index is equivalent to two spinor indices. The Clebsch-Gordon coefficients \(\sigma^\mu_{AA'}\) needed for this translation are conventionally suppressed. Thus the covariant directional derivatives can be written:

\[
\nabla_i = \nabla_{00'}; \quad \nabla_{\hat{n}} = \nabla_{11'}; \quad \nabla_{\hat{m}} = \nabla_{01'}; \quad \nabla_{\bar{m}} = \nabla_{10'}.
\]

Tensors are always symmetric in primed and unprimed spinor indices independently, but there is no special symmetry relating the two kinds of indices. Spinor indices are raised and lowered using the convention \(\chi^0 = \epsilon^{01} \chi_1\) where \(\epsilon^{01} = 1 = -\epsilon^{10}\).

In the Newman–Penrose formalism the components of the spin-connection, needed for the evaluation of covariant derivatives, are referred to as the spin–coefficients. The 12 complex spin-coefficients have conventional names given as:

\[
\begin{align*}
\gamma_{00'00} &= \kappa, & \gamma_{00'10} &= \epsilon, & \gamma_{00'11} &= \pi, \\
\gamma_{10'00} &= \rho, & \gamma_{10'10} &= \alpha, & \gamma_{10'11} &= \lambda, \\
\gamma_{01'00} &= \sigma, & \gamma_{01'10} &= \beta, & \gamma_{01'11} &= \mu, \\
\gamma_{11'00} &= \tau, & \gamma_{11'10} &= \gamma, & \gamma_{11'11} &= \nu.
\end{align*}
\]

The most efficient way to calculate the spin-coefficients is to use Cartan’s structure equations. By explicit calculation we find:

\[
dl = 0,
\]
\[ d\mathbf{n} = \partial_r \frac{\Delta}{2\Sigma} \imath \wedge \mathbf{n}, \quad (28) \]
\[ d\mathbf{m} = \frac{\Delta \partial_r \Sigma}{4\Sigma^2} \imath \wedge \mathbf{m} - \frac{\partial_r \Sigma}{2\Sigma} \mathbf{n} \wedge \mathbf{m} + \cot \theta \frac{\mathbf{m} \wedge \mathbf{m}}{\sqrt{2\Sigma}}. \quad (29) \]

Comparison with the standard form of the structure equations gives the nonvanishing spin-coefficients:

\[ \beta = \frac{\cot \theta}{2\sqrt{2\Sigma}} = -\alpha, \quad (30) \]
\[ \gamma = \partial_r \frac{\Delta}{4\Sigma}, \quad (31) \]
\[ \mu = \frac{\Delta \partial_r \Sigma}{4\Sigma^2}, \quad (32) \]
\[ \rho = \frac{-\partial_r \Sigma}{2\Sigma}. \quad (33) \]

The spin-coefficients provide the information that is needed to evaluate covariant derivatives or, in physical terms, to translate polarization vectors along geodesics.

The Weyl tensor is the irreducible part of the Riemann curvature tensor after the Ricci tensor has been projected out. In four dimensions it has 10 independent components that are represented as 5 complex numbers \( \Psi_i \) in the Newman-Penrose formalism. For the specific metric we consider most components vanish:

\[ \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0. \quad (34) \]

This property is the definition of a spacetime that is of type D in the Petrov classification. The last component of the Weyl tensor is non-trivial:

\[ \Psi_2 = -\partial_r \Delta \frac{\partial \Sigma}{4\Sigma^2} - \frac{\Delta}{3} \frac{\partial^2 \Sigma}{2\Sigma^2} - \frac{(\partial \Sigma)^2}{\Sigma^3}. \quad (35) \]

In its simplest form the Goldberg-Sachs theorem states that the type D property is equivalent to:

\[ \kappa = \sigma = \lambda = \nu = 0, \quad (36) \]

in vacuum spacetimes. This version of the theorem is not applicable in our case, because matter is present. However, the spacetime we consider nevertheless has both simplifying features, eqs. [34] and [36]. This is important for the consideration of perturbations.

The 10 components of the Ricci tensor are represented in the Newman-Penrose formalism as a spinorial tensor \( \Phi_{A\dot{B}A'\dot{B}'} \) with 9 independent components and the scalar \( \Lambda \) that is proportional to the Ricci scalar. In a spherically symmetric spacetime it is only the real components that do not vanish:

\[ \Phi_{00'0'} = -\frac{\partial_r^2 \Sigma}{2\Sigma} + \frac{(\partial_r \Sigma)^2}{4\Sigma^2}, \quad (37) \]
\[ \Phi_{01'1'} = \frac{1}{2\Sigma} - \partial_r \frac{\partial_r \Sigma}{4\Sigma^2} - \frac{\Delta}{4} \frac{\partial^2 \Sigma}{2\Sigma^2} - \frac{3(\partial_r \Sigma)^2}{4\Sigma^3}. \quad (38) \]
\[ \Phi_{111'} = -\frac{\Delta^2}{4\Sigma^2}(\frac{\partial^2 \Sigma}{2\Sigma} - \frac{(\partial_r \Sigma)^2}{4\Sigma^2}) \],
\[ \Lambda = -\frac{\Delta}{12} \frac{\partial^2 \Sigma}{2\Sigma^2} - \frac{(\partial_r \Sigma)^2}{4\Sigma^2} \].

In deriving these formula the special form of \( \Delta \) has been exploited to write \( \partial_r \Delta = 2 \), but \( \Sigma \) has been kept arbitrary.

### 4 Field Equations for Perturbations

In the present article we take the field equation:
\[ \nabla_{AA'}\psi^{AB_1 \cdots B_{2s-1}} = 0 \]
as the starting point for the discussion. This is the natural covariantization of the field equation for spin \( s \) in flat space. For spin \( s = 2 \) the equation is inadequate as it stands and additional non-linear terms must be added for consistency. We will discuss the required modification for \( s = 2 \) later in this section.

For \( s = \frac{1}{2} \) and \( s = 1 \) the field equation is consistent but the theory may not contain fields that couple to the background in this way. A case in point is the \( N = 8 \) supergravity. In this case the linearized field equations for the fermions with spin \( s = \frac{1}{2} \) are (see, e.g., [23]):
\[ \nabla_{AA'}\psi^A_{ij} + \epsilon_{ijklmnop}F^{lm}_{A'B'}\psi^{lnop} = 0 , \]
where the \( SO(8) \) indices have been denoted by small roman letters. (We have also suppressed terms proportional to the gravitino perturbations.) The four background \( U(1) \) fields can be represented after diagonalization as \( F^{12}, F^{34}, F^{56}, \) and \( F^{78} \). It is therefore apparent that in this case the fermions of \( N = 8 \) supergravity generically couple to the background gauge fields.

A similar situation is found for the vector particles in \( N = 8 \) supergravity. They satisfy the linearized field equations of the type:
\[ \nabla_{AA'}(\delta_{ij}^{kl} + iM_{ijkl}^{kl})F^{A'B'}_{kl} = 0 \]
where the \( M_{ijkl}^{kl} \) parametrize the scalar moduli fields. The vector particles in \( N = 8 \) supergravity therefore generically couple to the scalar particles.

In the present work we only take into account the gravitational couplings of the fermions and vectors. The philosophy is similar to that of considering \textit{minimally coupled} scalars in general relativity: it is the simplest couplings that display the effects in the scattering phenomena. Naturally it would be interesting to investigate the effect of background \( U(1) \) charges and scalars in the context of, e.g., \( N = 8 \) supergravity.

Let us now return to the field equations. In component form they are:
\[ -k\sigma\psi^{0 \cdots 1} + m^\mu \partial_\mu + 2(s-k)\beta - (k+1)\tau] \psi^{0 \cdots 1} + (44) \]
\[ + [n^\mu \partial_\mu - 2(s - k - 1)\gamma + (2s - k)\mu]\psi^{0\cdots 01\cdots 1} + (2s - k - 1)\nu\psi^{0\cdots 01\cdots 1} = 0 \]

\[ - k\kappa\psi^{0\cdots 01\cdots 1} + [n^\mu \partial_\mu + 2(s - k)\epsilon - (k + 1)\rho]\psi^{0\cdots 01\cdots 1} + \]

\[ + [\bar{m}^\mu \partial_\mu - 2(s - k - 1)\alpha + (2s - k)\pi]\psi^{0\cdots 01\cdots 1} + (2s - k - 1)\lambda\psi^{0\cdots 01\cdots 1} = 0 \]

for \( k = 0, \cdots, 2s - 1 \). Each equation is a relation between only two components of the wave function, when \( \kappa = \sigma = \lambda = \nu \). In this case each pair of equations in fact determines the two components of the wave function. In this way we recognize the central role of the type-D property for the decoupling of perturbation equations.

At this point we use the explicit expressions for the spin-coefficients and find:

\[ \mathcal{L}_{s-k}\psi^{0\cdots 01\cdots 1} - \Delta^{1/2}(D_{k+s}^\dagger - \frac{2k + 1 - 2s}{2} \psi^{0\cdots 01\cdots 1}) = 0 \]

\[ \mathcal{L}_{k+1-s}^\dagger\psi^{0\cdots 01\cdots 1} - \Delta^{1/2}(D_{k/2} + \frac{2k + 1 - 2s}{2} \psi^{0\cdots 01\cdots 1}) = 0 \]

where:

\[ D_n = \partial_r - \frac{i\omega}{\Delta} + n\frac{\partial_r \Delta}{\Delta} , \]

\[ D_n^\dagger = \partial_r + \frac{i\omega}{\Delta} + n\frac{\partial_r \Delta}{\Delta} , \]

\[ \mathcal{L}_n = \partial_\theta + \frac{m}{\sin \theta} + n\cot \theta , \]

\[ \mathcal{L}_n^\dagger = \partial_\theta - \frac{m}{\sin \theta} + n\cot \theta . \]

We also introduced a rescaled wave function \( \tilde{\psi} \) through:

\[ \psi^{0\cdots 01\cdots 1} = \Delta^{-k/2}(2\Sigma)^{k/2-s} \tilde{\psi}^{0\cdots 01\cdots 1} . \]

The equations take their most symmetric form when written in terms of \( \tilde{\psi} \). We can separate variables by writing:

\[ \psi^{0\cdots 01\cdots 1} = \Delta^{-k/2}(2\Sigma)^{k/2-s} P_{k-s}(r)S_{k-s}(\Omega) . \]

The angular functions \( S_{k-s} \) satisfy:

\[ \mathcal{L}_{k+1-s}^\dagger\mathcal{L}_{s-k}S_{k-s} = -\Lambda_{k-s}^{(-)}S_{k-s} , \]

\[ \mathcal{L}_{s-k}^\dagger\mathcal{L}_{k+1-s}S_{k+1-s} = -\Lambda_{k+1-s}^{(+)}S_{k+1-s} , \]
where the $\Lambda_k^{(\pm)}$ are separation constants. The radial function $P_{k-s}$ similarly satisfy:

$$\Delta^{1/2}[D_{k+1-s}^{\pm}] - (2k + 1 - 2s) \frac{\partial \Sigma}{\partial \Omega} \Delta^{1/2}[D_{k+1-s}^{\pm}] = \Lambda_k^{(\pm)} P_{k-s}$$

$$\Delta^{1/2}[D_{k-1}^{\pm}] + (2k + 1 - 2s) \frac{\partial \Sigma}{\partial \Omega} \Delta^{1/2}[D_{k+1-s}^{\pm}] = \Lambda_k^{(\pm)} P_{k+1-s}$$

There is one equation for each of the upper and lower components of the wave function $P_s$ and $P_{-s}$. For the remaining components $P_{\lambda}; \lambda = -s + 1, \cdots, s - 1$ there are two equations that are in general distinct. The functions are therefore overdetermined and it is only because of the underlying algebraic structure that there are solutions at all. Similar comments apply to the angular equations.

The angular functions embody the properties of the rotation group. We denote the total angular momentum of the particle $j$, its projection on a fixed axis $m$, and the helicity $\pm s$. The angular momentum satisfies $j \geq s$. The separation constants $\Lambda_k^{(\pm)}$ can be determined by algebraic or analytical methods, with the result:

$$\Lambda_k^{(\pm)} = j(j+1) - (k-s)(k-s \mp 1).$$

We will only consider the highest and lowest component of the wave function. Each is associated with a single separation constant, and their respective values coincide:

$$\Lambda \equiv \Lambda_{-s}^{(-)} = \Lambda_{+s}^{(+)} = j(j+1) - s(s-1).$$

The explicit angular wave functions are known. They are closely related to the Jacobi-Polynomials and they are proportional to the rotation matrices:

$$S_k(\Omega) = e^{im\phi} d_{km}^{(j)}(\theta),$$

where $k = -s, \cdots, s$.

Our primary interest is the radial equation for the upper and lower components of the wave function. The upper one can be written:

$$\left\{ \Delta^s \partial_\Omega \Delta^{1-s} \partial_\Omega + \frac{\Sigma}{2\omega} \Delta^2 - \frac{i s \omega \Sigma}{\Delta} \partial_\Omega \Delta + 2 i s \omega \Delta \partial_\Omega + \right.$$  

$$+ \Delta(s - \frac{1}{2}) \left[ \partial_\Omega \left( \frac{\partial_\Omega}{\Sigma} \right) + (s - \frac{1}{2}) \left( \frac{\partial_\Omega}{\Sigma} \right)^2 + (1 - s) \frac{\partial_\Omega}{\Delta} \left. \frac{\partial_\Omega}{\Sigma} \right] \right\} P_s = \Lambda P_s$$

and the lower one is the complex conjugate. An equation equivalent to this one was found by Gubser [15].

We now consider gravitons in more detail. In this case the full perturbation equations involve all the components of the curvature tensor, a formidable problem. However, in type D spacetimes the radiative parts of the field decouple and form a manageable subset. Namely, we consider perturbations of the $(\Psi_0, \Psi_1, \Psi_3, \Psi_4)$ components.

\footnote{The total angular momentum is often denoted $l$. Our notation is intended to avoid confusion with the orbital angular angular momentum that plays no role in the present computation.}
of the Weyl tensor and identify them with the fields \((\psi_{1111}, -\psi_{0111}, -\psi_{0001}, \psi_{0000})\) in the general calculation. The Weyl tensor is related to the covariant derivatives of the spin–coefficients:

\[
\Psi_0 = (l^\mu \partial_\mu - \rho - \rho^* - 3\epsilon + \epsilon)\sigma - (m^\mu \partial_\mu - \tau + \pi^* - \alpha^* - 3\beta)\kappa
\]

\[
\Psi_4 = (n^\mu \partial_\mu + \mu + \mu^* + 3\gamma - \gamma^*)\lambda - (\bar{m}^\mu \partial_\mu + 3\alpha + \beta^* + \pi - \tau^*)\nu
\]

(62) \hspace{1cm} (63)

Perturbations in the Weyl tensor are therefore necessarily accompanied by perturbations in the spin–coefficients \((\kappa, \sigma, \nu, \lambda)\). In the spin-2 case such terms should therefore be kept in the derivation above. They always appear multiplied by the non-vanishing component of the Weyl-tensor, namely \(\Psi_2\).

There is an additional complication that must be taken into account: in the spin-2 case the rationale for the field equation eq. 41 is its relation to the Bianchi identity. In vacuum it is indeed exactly the Bianchi identity but in the presence of matter there are additional couplings between \((\kappa, \sigma, \nu, \lambda)\) and the Ricci tensor given in eqs. 37-40.

The components \(\Phi_{0000'}\) and \(\Phi_{1111'}\) vanish in the Reissner-Nordström case but in the general case they lead to couplings between \((\kappa, \sigma, \nu, \lambda)\) and the complex conjugate fields \((\kappa^*, \sigma^*, \nu^*, \lambda^*)\). This is reminiscent of the Pauli couplings in eqs. 42-43.

The net effect of all this is that we can justify eq. 61 for gravitons, with the amendment that certain source terms that are proportional to \((\kappa, \sigma, \nu, \lambda)\) and their complex conjugates must be added. In the Reissner-Nordström limit the combinations of spin–coefficients that appear in these source terms are exactly such that they can be eliminated, using the Ricci-identities eqs. 62-63. Moreover, the resulting term proportional to \(\Psi_4\) precisely cancel the term in the square bracket of eq. 61. For general \(U(1)\) charges we are not able to decouple the field equations in a similar way and we are left with the source terms. However, a calculation shows that, in the approximation that we will use in sec. 6, most source terms are negligible: they are multiplied by an explicit factor of \(\Delta\) that make them vanish close to the horizon. Moreover, the cancellations noted in the Reissner-Nordström case are still approximately valid and ensure that source terms fall off at large distances at a rate faster than the angular momentum term \(\Lambda\). In the following we will use use eq. 61 with the term in the brackets omitted as an approximate decoupled field equation for gravitons. This procedure is exact in the Reissner-Nordstöm case and it should give a reliable indication in general.

The case of gravitini, i.e. \(s = \frac{3}{2}\), is similar to that of gravitons: eq. 61 should give some guidance but the full equations may contain additional terms. It is possible that the correct equations can be found by omitting the term in square bracket, but we have made no effort to substantiate this speculation for \(s = \frac{3}{2}\). However, the square bracket vanishes for \(s = \frac{1}{2}\), and for \(s = 1\) it is simple to verify that it will not contribute within our approximation scheme. We will therefore omit this term in general and take the approximate equation:

\[
[\Delta^s \partial_t \Delta^{-s} \partial_r + \frac{\Sigma^2 \omega^2 - is \omega \Sigma \partial_r \Delta}{\Delta} + 2i s \omega \partial_r \Sigma] P_s = \Lambda P_s \ ,
\]

(64)
as a basis for further exploration. This decoupled radial equation applies also for spin-0, as can be verified by inspection. An equation of this type was first found
by Teukolsky, in the case of the neutral Kerr black hole background, and so it is sometimes referred to as the Teukolsky equation \cite{24}.

5 The Flux Factors

The Teukolsky equation eq. \eqref{64} is satisfied by the upper component of the wave function. The lower component satisfies a different differential equation, namely the complex conjugate one. In order to solve the scattering problem we must find solutions to each of the two equations and, importantly, we must find a relation between the two components of the wave function that have thus been identified. The solution to this problem is straightforward in the spin-1/2 case: the original first order differential equation essentially gives one component as the derivative of the other one. Similar results are valid in the higher spin cases where the needed relations are known as the Teukolsky-Starobinsky identities. In general it requires a comprehensive analysis of the complete system of first order equations to find this result, but for our purpose the following approximate procedure is sufficient: at the horizon the two linearly independent solutions to the equation for the upper component of the wave function are an infalling wave and a solution that vanishes. The corresponding solutions for the lower component of the wave function are the complex conjugate ones. In the absorption geometry we consider the solution with ingoing flux at the horizon. Using the differential equation for the upper component we find the associated amplitude at infinity. This cannot be the whole story because the flux at the horizon and at infinity are not the same and so there must also be an outgoing flux at infinity, corresponding to the reflected wave. This flux is carried by the lower component of the wave function and the magnitude can be determined from flux conservation. In the cases we consider the transmission is in fact small and so the reflected flux is identical to the incoming one, up to quantities of subleading order. It is therefore sufficient to consider only the upper component of the wave function.

The discussion in the previous paragraph applies to a particle in a specific helicity state. The corresponding treatment of particles with the opposite helicity simply involves complex conjugation and the interchange of upper and lower components of the wave function. For this reason the two helicities lead to the same absorption cross-section, as expected for scattering off a parity invariant target. We consider the particles with left-handed chirality, for definiteness.

We now turn to the main task of this section, to determine the flux factors that are needed to convert the upper component of a wave function to an absorption cross-section. The general result:

\[
\sigma_{\text{abs}}(\omega) = \frac{\pi}{\omega}(2j+1)|T|^2 ,
\]

reduces the problem to one of transmission in one spatial dimension. We normalize the wave functions at infinity as:

\[
P_{s}^{(\infty)} \sim A_{s}^{(\infty)}(2r\omega)^{2s-1} e^{-i\omega r} .
\]
Similarly, close to the outer horizon of the black hole:

\[ P_s^{(0)} \sim A_s^{(0)} (r - r_+) \frac{e^{i\mu \omega}}{4\pi} \quad . \]  

(67)

After this general statement of the problem we consider each case independently.

**spin 1/2**: The conserved current is:

\[ \frac{1}{\sqrt{2}} J^\mu = \sigma^\mu_{AB'} \psi^A \bar{\psi}^{B'} \quad , \]  

(68)

where:

\[ \sigma^\mu_{AB'} = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} l^\mu & m^\mu \\ \bar{m}^\mu & \bar{n}^\mu \end{array} \right) \quad . \]  

(69)

Recalling the definition of the radial wave functions eq. 53 the radial current becomes:

\[ J^r = -\frac{1}{2\Sigma} (|P_{1/2} S_{1/2}|^2 - |P_{-1/2} S_{-1/2}|^2) \quad . \]  

(70)

We normalize the angular wave functions:

\[ \frac{1}{4\pi} \int |S_{\pm s}|^2 d\Omega = 1 \quad , \]  

(71)

and so the final result for the infalling flux becomes:

\[ \frac{1}{2\pi} \frac{dN}{dt} = -\frac{1}{2\pi} \int J^r \Sigma \sin \theta d\theta d\phi = |P_{1/2}|^2 - |P_{-1/2}|^2 \quad . \]  

(72)

This formula can be used both at the horizon and at infinity. The effective two-dimensional transmission coefficient is therefore simply:

\[ |T_{s=1/2}|^2 = \frac{|A_{1/2}^{(0)}|^2}{|A_{1/2}^{(\infty)}|^2} \quad . \]  

(73)

**spin 1**: For spin \( s > \frac{1}{2} \) there are no conserved currents. However, the flux factors can be inferred from the flow of energy. For a spin-1 field the energy momentum tensor is:

\[ T^\mu\nu = 2\sigma^\mu_{AA'} \sigma^\nu_{BB'} \psi^A \bar{\psi}^{A'} \psi^{B'} \quad . \]  

(74)

This works out to:

\[ T^{\nu\ell} = -\frac{1}{4\Sigma \Delta} (|P_1 S_1|^2 - |P_{-1} S_{-1}|^2) \quad . \]  

(75)

The local energy of photons is corrected for redshift according to:

\[ E = \frac{\Sigma}{\Delta} E_\infty = \frac{\Sigma \omega}{\Delta} \quad , \]  

(76)
and so the inflowing flux becomes:

\[
\frac{1}{2\pi} \frac{dN}{dt} = - \frac{1}{2\pi} \int \frac{1}{E} T^{rt} \Sigma d\Omega = - \frac{1}{2\pi \omega} \int \Delta T^{rt} d\Omega = \frac{1}{2\pi \omega} (|P_1|^2 - |P_{-1}|^2) .
\]  

(77)

In the vicinity of the horizon \( \Sigma \simeq \frac{1}{8\pi} \mu \beta_H \) while at infinity \( \Sigma \sim r^2 \). The effective transmission coefficient becomes:

\[
|T_{s=1}|^2 = \frac{2\pi}{\mu \beta_H} \left| \frac{A^{(0)}_1}{A^{(\infty)}_1} \right|^2 .
\]  

(78)

**spin 2:** In the spin two case there is in general no conserved energy momentum tensor. For this and related reasons it is somewhat ambiguous to refer to a spin-2 field propagating in a general curved spacetime. We proceed as follows. First consider the Bel-Robinson tensor:

\[
T^{\mu\nu\rho\sigma} = 4 \sigma^{\mu A}_{AA'} \sigma^{\nu B}_{BB'} \sigma^{\rho C}_{CC'} \sigma^{\sigma D}_{DD'} \psi^{ABCD} \bar{\psi}^{A'B'C'D'} .
\]  

(79)

In spherically symmetric spacetimes a special role is played by the specific component:

\[
T^{rrtt} = - \frac{1}{16\Sigma \Delta^4} (|P_2 S_2|^2 - |P_{-2} S_{-2}|^2)
\]  

(80)

The energy momentum tensor for weak gravitational field can be formed using the timelike Killing vector \( \partial_t \):

\[
\frac{1}{4\pi} \frac{dE}{dt} = - \int T^{rrtt} g_{tt} \left( \frac{\Delta}{\Sigma \omega} \right)^2 \frac{\Sigma d\Omega}{4\pi} = \frac{1}{16\Sigma \Delta^2 \omega^2} (|P_2|^2 - |P_{-2}|^2)
\]  

(81)

This expression is applicable only in the weak field case, i.e. in the asymptotic regime. If it were rewritten in terms of the metric tensor it would be related to the second time derivative \( \ddot{h}^2 \). This should contrasted with, e.g., electromagnetic waves where only a single time derivative appears. The extra derivative was compensated for by dividing out with the square of the frequency, thus arriving at an expression with correct dimensions.

The energy momentum inferred from the Bel-Robinson tensor is conserved in all static black hole spacetimes. However, in order to arrive at a proper measure of graviton number close to the horizon we must integrate with respect to proper distance along the world line, rather than simply dividing by the redshifted frequency. The resulting energy flux close to the horizon is given by eq. (81) with the replacement \( \omega^2 \rightarrow \omega^2 + \left( \frac{2\pi}{\beta_H} \right)^2 \). The flux factor now becomes:

\[
|T_{s=2}|^2 = \frac{1}{64\Sigma_{\text{hor}}^3 \omega^4} \frac{1}{\omega^2 + \left( \frac{2\pi}{\beta_H} \right)^2} \left| \frac{A^{(0)}_2}{A^{(\infty)}_2} \right|^2 ,
\]  

(82)

where \( \Sigma_{\text{hor}} = \frac{1}{8\pi} \mu \beta_H \). The peculiar polynomial dependence in \( \beta_H \) is novel, and destined for a special role in the argument.
In the book by Chandrasekhar [20] a different procedure is employed to find the flux factor for a spin-2 field. The starting point is the first law of thermodynamics that relates changes of the black hole mass to changes of the horizon area. Subsequently variations in the area are related to variations in the spin–coefficients, using a focusing theorem (essentially eq. [22]). Finally the variations in spin-coefficients are identified with the perturbation in the Weyl tensor, using the Ricci identities. We have adapted this procedure to our case and recovered the result eq. [82]. This gives confidence that the flux factor for the graviton has been correctly identified.

6 The Absorption Cross-section

The general field equations cannot be solved exactly. However, we are interested in various limits where the absorption is weak. These cases have the dual advantage that analytical approximations are available, and that the results have the same form as correlation functions in string theory. The limits are:

- The dilute gas regime: exactly three of the four boost parameters are large, say $\delta_i \gg 1$; $i = 1, 2, 3$ with $\delta_4 \sim 1$. In this case the black hole parameters are such that the inverse temperatures $\beta_R \sim \beta_L \sim \beta_H$ are much larger than the size of the black hole.

- Very low energy perturbations, $\omega \to 0$. The black hole parameters can be arbitrary.

- Large partial wave number $j \gg \sqrt{\mu \omega}$. The black hole parameters can be arbitrary.

Either of these limits is sufficient to allow the approximation scheme that has recently been exploited in similar calculations (references include [23, 20, 19, 27]) : first we solve the equations in the vicinity of the black hole, then in the region far from the black hole, and finally the solutions are matched in the intermediate region. A particular point is that some results apply to arbitrary black holes and so suggest that the effective string model applies even far from extremality [14, 27].

The horizon region: To find an equation that is accurate in the region close to the horizons we expand the function $\Sigma$ defined in eq. (11):

$$\Sigma^2 \simeq \frac{1}{8} \mu^3 (r + 1) \prod_i \cosh^2 \delta_i - \frac{1}{8} \mu^3 (r - 1) \mu \prod_i \sinh^2 \delta_i + O(r^2 - \frac{1}{16} \mu^2)$$

$$\Sigma \simeq \frac{1}{2} \mu (r + 1) \mu \prod_i \cosh \delta_i - \frac{1}{2} \mu (r - 1) \mu \prod_i \sinh \delta_i + O(r^2 - \frac{1}{16} \mu^2) .$$

In the first equation the omitted term is a polynomial with an explicit factor of $r^2 - \frac{1}{16} \mu^2$ while in the second it is an irrational function that vanishes at both the horizons. Introducing the surface accelerations at the outer and inner horizon:

$$\frac{1}{\kappa_+} = \mu \prod_i \cosh \delta_i ; \quad \frac{1}{\kappa_-} = \mu \prod_i \sinh \delta_i ,$$

$$\mu \prod_i \cosh \delta_i ; \quad \mu \prod_i \sinh \delta_i ,$$

15
and the radial variable \(x\), defined through \(r = \frac{1}{2} \mu x\), we find the field equation that applies in the horizon region:

\[
\{(x^2 - \frac{1}{4})^s \frac{\partial}{\partial x} (x^2 - \frac{1}{4})^{1-s} \frac{\partial}{\partial x} - (j + s)(j + 1 - s) + \frac{1}{x - \frac{1}{2}} \left[\left(\frac{\omega}{2\kappa_+}\right)^2 - is\omega \right] - \frac{1}{x + \frac{1}{2}} \left[\left(\frac{\omega}{2\kappa_-}\right)^2 + is\omega \right]\} P_s^{(0)} = 0 .
\]

We consider the equation for the upper component of the wave function, with the one for the lower component found by complex conjugation. This field equation is a second order differential equation and so has two linearly independent solutions. A basis can be chosen so that they satisfy ingoing and outgoing boundary conditions at the horizon, respectively. In the absorption geometry it is the ingoing solution that is relevant:

\[
P_s^{(0)}(x) = A_s^{(0)}(x - \frac{x}{2})^{-i\frac{\beta H \omega}{4\pi}} (x + \frac{1}{2})^{s-1-j} × \times F(1 + j - i\frac{\beta R \omega}{4\pi}, 1 + j - s - i\frac{\beta L \omega}{4\pi}, 1 - s - i\frac{\beta H \omega}{2\pi}, x - \frac{1}{2}),
\]

where \(F\) is the hypergeometric function. At large \(x \gg 1\) the last argument of the hypergeometric function approaches the radius of convergence \(|z| = 1\) so this representation of the wave function is inappropriate for asymptotic expansion at large \(x\). This situation can be rectified by a modular transformation of the hypergeometric function\(^3\). In the new representation:

\[
P_s^{(0)} \sim A_s^{(0)} x^{s+j} \frac{\Gamma(2j + 1)\Gamma(1 - s - i\frac{\beta \mu \omega}{2\pi})}{\Gamma(1 + j - i\frac{\beta \mu \omega}{4\pi})\Gamma(1 + j - s - i\frac{\beta \mu \omega}{4\pi})}[1 + \mathcal{O}\left(\frac{1}{x}\right) + \mathcal{O}\left(\frac{\log x}{x^2}\right)] .
\]

The leading power corrections and the leading logarithmic corrections have been indicated independently.

**The asymptotic region:** The next step is to expand for large \(r\). Terms of order \(r^2\) and \(r\) are retained, as is the angular momentum eigenvalue. After introducing the rescaled variable \(x = 2r/\mu\) that was also used in the horizon region the equation can be written:

\[
[x^2 \partial_x x^{2(1-s)} \partial_x + \frac{1}{4} \mu^2 \omega^2 x^2 + \frac{1}{4} \mu M \omega^2 x + is \mu \omega x - (j + s)(j + 1 - s)] P_s^{(\infty)} = 0 \quad (89)
\]

The regular solution is proportional to:

\[
P_s^{(\infty)} = A_s^{(\infty)} \frac{\Gamma(1 + j + s + \frac{1}{4} M \omega)}{\Gamma(2j + 2)} (\mu \omega x)^{j+s} e^{-\frac{1}{4} \mu \omega x + \frac{i}{4} \pi M \omega} × \times M_K(1 + j - s - \frac{i}{4} M \omega, 2j + 2, i\mu \omega x) . \quad (90)
\]

\(^3\)Note that in the hypergeometric function of eq. \([87]\) the sum of the first two arguments minus the third equals an integer. This is a degenerate case that needs special considerations, as given in, e.g., \([28]\). The appearance of logarithmic terms is related to this special circumstance.
With the overall normalization that has been indicated the asymptotic behavior at large \(x\) is:

\[
P_{s}^{(\infty)} \simeq A_{s}^{(\infty)} (\mu \omega x)^{2s-1} e^{-\frac{i}{2} \mu \omega x + \frac{i}{2} (1 + j - s)} \left[ 1 + \mathcal{O}\left(\frac{1}{x}\right) \right],
\]

(91)

while at small \(x\):

\[
P_{s}^{(\infty)} \simeq A_{s}^{(\infty)} \frac{e^{\frac{2}{3} M \omega} \Gamma(1 + j + s + \frac{i}{2} M \omega)}{\Gamma(2j + 2)} (\mu \omega x)^{j + s} \left[ 1 + \mathcal{O}(x) \right].
\]

(92)

**The matching:** At this point we combine the approximate solutions that apply in separate regions and form a single wave function that can be used throughout. To justify this procedure in the case of particles with spin we must extend the arguments previously given for scalar particles. The wave equation includes a term of the schematic form:

\[
\frac{\Sigma^{2} \omega^{2}}{\Delta} \sim x^{2} \mu^{2} \omega^{2} + x \mu M \omega^{2} - \mu^{2} \omega^{2} e^{4 \delta} + \frac{1}{x} \beta_{R} \beta_{L} \omega^{2}.
\]

(93)

In the dilute gas regime the parameters of the black holes are \(M \sim \mu e^{2\delta}\) and \(\beta_{R} \sim \mu e^{3\delta}\); and frequencies satisfy \(\mu \omega \ll e^{-2\delta}\). The restriction on frequency is mild enough that the interesting region \(\beta_{R} \omega \sim \beta_{L} \omega \sim 1\) is covered. Now, for \(x \sim e^{2\delta}\) we find by inspection of eq. 93 that *all terms are small.* Similarly, in the same range of \(x\):

\[
\frac{\omega \Sigma \partial_{r} \Delta}{\Delta} = \frac{\omega \Sigma}{\sqrt{\Delta}} \frac{\partial_{r} \Delta}{\sqrt{\Delta}} \sim \frac{\omega \Sigma}{\sqrt{\Delta}} \ll 1
\]

(94)

and, after a short calculation:

\[
\omega s \partial_{r} \Sigma \sim \mu \omega e^{2\delta} \ll 1.
\]

(95)

Thus the radial dependence of the wave function is determined by the kinetic term for \(x\) in this range. At smaller \(x\) the horizon approximation gives the dominant terms, at larger \(x\) the asymptotic equation is accurate, and in the matching region both equations apply and are dominated by the kinetic operator. It is therefore justified, in the dilute gas regime, to identify the large \(x\) approximation of the horizon wave function (eq. 88) with the small \(x\) limit of the asymptotic wave function (eq. 92).

When the frequency is *very small* \(\mu \omega \rightarrow 0\) the matching procedure is similarly justified: the matching region can be chosen at \(x \sim 1\), while avoiding the outer horizon at \(x = \frac{1}{2}\). For large partial wave number \(j \gg (\mu \omega)^{1/2}\) the argument is slightly different [27, 10]: in this case there exist a matching region \(x \sim 1\) where the angular momentum term \(\sim j^{2}\) *dominates all other terms.* In this region the wave function is determined by the kinetic term and the angular momentum term while, at \(x\) smaller or larger than the matching region, the horizon and the asymptotic equation applies, respectively.

Applying the matching procedure we find:

\[
\left| \frac{A_{s}^{(0)}}{A_{s}^{(\infty)}} \right|^{2} = \left( \mu \omega \right)^{2j + 2s} \frac{e^{\frac{2}{3} M \omega} \Gamma(1 + j + s + \frac{i}{2} M \omega)}{\Gamma(2j + 2)^{2} \Gamma(2j + 1)^{2}} \times
\]

\[
\times \frac{\Gamma(1 + j + i \frac{\beta_{R} \omega}{4 \pi}) \Gamma(1 + j - s + i \frac{\beta_{L} \omega}{4 \pi})}{\Gamma(1 - s + i \frac{\beta_{H} \omega}{2 \pi})} \right|^2.
\]

(96)
Taking into account the flux factors, considered in the previous section, this translates into the absorption cross-section:

\[ \sigma_{abs}(\omega) = \omega^{2j-1} \mu^{2j+1} \frac{e^{\frac{\pi M \omega}{4}} |\Gamma(1+j+s + \frac{i}{4} M \omega)|^2}{\Gamma(2j+2)^2 \Gamma(2j+1)^2} (2j+1) \times \]

\[ \times |\Gamma(1+j+i \frac{\beta R \omega}{4\pi})\Gamma(1+j-s + i \frac{\beta L \omega}{4\pi})|^2 \times \left\{ \frac{|\Gamma(\frac{1}{2} + \frac{i\beta_H \omega}{4\pi})|^2}{2|\Gamma(1+\frac{i\beta_H \omega}{4\pi})|^2} \right\} (97) \]

where the upper line is for half-integer spin \( s = \frac{1}{2} \) and the lower line is for integer spin \( s = 0, 1, 2 \). At this point we recall the identities:

\[ e^{-\frac{\pi}{2}} |\Gamma(\frac{1}{2} + i \frac{\alpha}{2\pi})|^2 = \frac{2\pi}{e^{\alpha}+1} ; \quad e^{-\frac{\pi}{2}} |\Gamma(1+i \frac{\alpha}{2\pi})|^2 = \frac{\alpha}{e^{\alpha}-1} . \] (98)

The last terms in the absorption cross-section therefore give rise precisely to the thermal factors in the numerator that are needed to cancel the thermal factors in the denominator of Hawking’s expressions eq. 2 and eq. 3 for the emission rates of bosons and fermions, respectively. Thus the emission rate becomes:

\[ \Gamma_{em}(\omega) = \omega^{2j-1} \mu^{2j+1} \frac{e^{\frac{\pi M \omega}{4}} |\Gamma(1+j+s + \frac{i}{4} M \omega)|^2}{2\Gamma(2j+2)^2 \Gamma(2j+1)^2} (2j+1) \times \]

\[ \times e^{-\frac{\beta R \omega}{4}} |\Gamma(1+j+i \frac{\beta R \omega}{4\pi})|^2 e^{-\frac{\beta L \omega}{4}} |\Gamma(1+j-s + i \frac{\beta L \omega}{4\pi})|^2 \frac{d^3 k}{(2\pi)^3} \] (99)

This expression is our main result. It exhibits interesting features: the dependence on Hawking temperature disappeared and so the factorized form expected from an effective string description is manifest. Moreover, the spin dependence takes a surprisingly simple form.

In eq. 96 for the ratio of the wave function normalizations at the horizon and at infinity a factor of \( |\Gamma(1-s + i \frac{\beta_H \omega}{2\pi})|^2 \) appears in the denominator. This is a potential source of problems for \( s > \frac{1}{2} \) because, after expansion of the Gamma-function, there will be a polynomial dependence on \( \beta_H \) in addition to the thermal factors. It would be very difficult to reconcile a dependence on \( \beta_H = \frac{1}{2}(\beta_R + \beta_L) \) with the factorized correlation functions expected from an effective string description. It is therefore important that the flux factors derived in the previous section precisely cancel these potentially dangerous terms. The cancellation that appears here is a novel one that does not show up when considering spin-0 particles.

The non-extremality parameter \( \mu \) parameterizes the coordinate distance between the two horizons \( \mu = 2(r_+ - r_-) \). This quantity therefore does not appear to have fundamental significance. In macroscopic applications it is natural to trade it for the entropy \( S \) and \( \beta_H \):

\[ S = 2A = \mu \beta_H . \] (100)

It is less clear what the appropriate microscopic variable is, but a natural representation is to eliminate \( \mu \) in favor of the effective string length \( L \) given by \[ \frac{\beta R \beta L}{2\pi} = 2\pi \mu^3 \left( \prod_{i=1}^{3} \cosh^2 \delta_i - \prod_{i=1}^{3} \sinh^2 \delta_i \right) . \] (101)
7 Results

In this section we write the results of the previous section in a more digested form and discuss their structure.

Large angular momentum: In this case \((j >> \sqrt{\mu \omega})\) the expression for the emission rate (eq. [99]) is valid for arbitrary black hole background. The result can be rewritten explicitly in terms of the thermal factors as follows:

\[
\Gamma_{\text{em}}^{\text{boson}}(\omega) = A_j C_{j+s} \frac{P_R(\omega)}{e^{\beta_R \omega/2}} - 1 \frac{P_L(\omega)}{e^{\beta_L \omega/2} - 1} \frac{d^3k}{(2\pi)^3},
\]

\[
\Gamma_{\text{em}}^{\text{fermions}}(\omega) = A_j C_{j+s} \frac{P_R(\omega)}{e^{\beta_R \omega/2} + 1} \frac{P_L(\omega)}{e^{\beta_L \omega/2} - 1} \frac{d^3k}{(2\pi)^3}.
\]

The \(A_j\) are the prefactors:

\[
A_j = \frac{\omega^{2j-1} \mu^{2j+1}(2j + 1)}{2((2j + 1)!(2j)!)^2},
\]

that are independent of the mass \(M\) and the spin \(s\). This dependence is carried by the Coulomb factors \(C_{j+s}\) given by:

\[
C_{j+s} = \frac{P_C(\omega)}{1 - e^{-\pi M \omega/2}},
\]

where:

\[
P_C(\omega) = \pi \omega \left( \frac{M}{2} \right)^{2j+2s+1} \prod_{k=1}^{j+s} \left( \frac{\omega}{2} \right)^2 + \left( \frac{2k}{M} \right)^2.
\]

Note that \(C_{j+s} \to (j + s)!^2\) in the limit \(M \omega \to 0\). The exponential factors with parameter \(\frac{1}{4} \pi M = 2\pi G_N M\) reflect the long range interaction caused by gravity in four dimensions. These terms are not specific to the black hole geometry, but it is nevertheless interesting that their form is reminiscent of the thermal factors. This result suggests that Coulomb factors may also have a statistical origin in the microscopic theory.

The polynomials \(P_R(\omega)\) are:

\[
P_R^{\text{bosons}}(\omega) = \pi \omega \left( \frac{\beta_R}{2\pi} \right)^{2j+1} \prod_{k=1}^{j} \left( \frac{\omega}{\beta_R} \right)^2 + \left( \frac{2\pi k}{\beta_R} \right)^2,
\]

\[
P_R^{\text{fermions}}(\omega) = 2\pi \left( \frac{\beta_R}{2\pi} \right)^{2|j|+2} \prod_{k=1}^{|j|+1} \left( \frac{\omega}{\beta_R} \right)^2 + \left( \frac{\pi(2k - 1)}{\beta_R} \right)^2,
\]

for bosons and fermions, respectively, and the \(P_L(\omega)\) are:

\[
P_L(\omega) = \pi \omega \left( \frac{\beta_L}{2\pi} \right)^{2j-2s+1} \prod_{k=1}^{j-s} \left( \frac{\omega}{\beta_L} \right)^2 + \left( \frac{2\pi k}{\beta_L} \right)^2.
\]
for both fermions and bosons. The emission rates eqs. 102-103 has precisely the factorized form that was advertised in the introduction (eq 2 and eq. 5). For scalars the result $s = 0$ agrees with that of [6, 10]. Interestingly, the thermal factor of the left-moving sector is always of the Bose-Einstein type, but the thermal factors of the right-moving sector correlate with the spin-statistics of the emitted particles: they are of the Bose-Einstein type for bosons and of the Fermi-Dirac type for fermions. This result is in agreement with the structure of the thermal quantities of rotating four-dimensional black holes: only the right-moving temperature and entropy depend on the angular momentum of the black hole.

**Minimal total angular momentum:** In this case $j = s$ and the emission rates eqs. 102-103 are valid for the black hole parameters of the dilute gas regime. Then the rates take a relatively simple form:

$$
\Gamma_{em}^{s=0}(\omega) = \frac{1}{2\omega^2} \frac{\beta_R \omega/2}{e^{\beta_R \omega/2} - 1} \frac{\beta_L \omega/2}{e^{\beta_L \omega/2} - 1} \frac{d^3 k}{(2\pi)^3},
$$

$$
\Gamma_{em}^{s=\frac{1}{2}}(\omega) = \frac{1}{4\mu^2} \frac{\beta^2}{\omega^2} \left[ \left( \frac{\beta_R}{\beta} \right)^2 + \left( \frac{\beta_L}{\beta} \right)^2 \right] \frac{\beta_L \omega/2}{e^{\beta_L \omega/2} - 1} \frac{d^3 k}{(2\pi)^3},
$$

$$
\Gamma_{em}^{s=1}(\omega) = \frac{1}{4\pi^2} \pi \omega^2 \rho \left[ \left( \frac{\beta_R}{\beta} \right)^2 + \left( \frac{2\pi k}{\beta} \right)^2 \right] \frac{\beta_L \omega/2}{e^{\beta_L \omega/2} - 1} \frac{d^3 k}{(2\pi)^3},
$$

$$
\Gamma_{em}^{s=2}(\omega) = \frac{\omega^3 \mu^5}{10(4!^2)} \frac{\pi \omega^2 \beta_R}{e^{\beta_R \omega/2} - 1} \frac{(2\pi k)^2}{e^{\beta_L \omega/2} - 1} \frac{d^3 k}{(2\pi)^3}.
$$

To arrive at these formulae we used the low energy form of the Coulomb factor $C_{2s} = [(2s)!]^2$. In each case the left-moving sector reduces to a single term thus indicating a “minimal” (bosonic) excitation in this sector, while the right-moving sector indicates progressively more involved structure of excitations as the spin of the particle increases. The spin-dependence of the emission rates is “stored” in the right-moving sector, only.

**The low-energy absorption cross-section:** At very low energy the absorption is dominated by the leading partial wave $j = s$ and the expression eq. 97 is valid in any black hole background. We find:

$$
\sigma_{abs}^{s=0}(\omega \rightarrow 0) = \frac{1}{2} \mu \beta_R A,
$$

$$
\sigma_{abs}^{s=\frac{1}{2}}(\omega \rightarrow 0) = \frac{\pi \mu^2}{8},
$$

$$
\sigma_{abs}^{s=1}(\omega \rightarrow 0) = \frac{\beta_H \omega^2 \mu^3}{24},
$$

$$
\sigma_{abs}^{s=2}(\omega \rightarrow 0) = \frac{\beta_H \omega^4 \mu^5}{1440}.
$$
The result for the minimally coupled fermions agrees with the expression found in [13]. As noted in that reference the absorption cross-section of minimally coupled fermions vanish in the extremal limit $\mu \to 0$. The result is also in agreement with the low energy absorption cross-section for the Schwarzschild black hole backgrounds with minimally coupled scalars and Dirac fermions [30].

8 Microscopic Interpretation

The factorized form of the emission rates eq. (99) indicates an interpretation in terms of multi-body annihilation rates in an effective string model. We can write the result for the emission rate as:

$$\Gamma_{em}(\omega) = A_j C_{L+j-s} \Gamma(2h_R)\Gamma(2h_L)(\beta R)^{2h_R-1}(\beta L)^{2h_L-1}G_{\beta R}^{h_R}(\omega)\cdot G_{\beta L}^{h_L}(\omega), \quad (118)$$

where:

$$G_{\beta}^{h}(\omega) \equiv \left(\frac{2\pi}{\beta}\right)^{2h-1}e^{-\frac{\omega}{4\beta}}\left|\frac{\Gamma(h+\frac{i\omega}{4\beta})}{\Gamma(2h)}\right|^2. \quad (119)$$

The prefactor $A_j$ and the Coulomb factor $C_{L+j-s}$ are defined in eq. (104) and eq. (105), respectively. In the effective string description the $G_{\beta R,L}^{h,R,L}(\omega)$ are the Fourier space representations of the canonically normalized thermal Green’s functions for a primary field with the respective right- and left-moving conformal dimensions $h_{R,L}$:

$$G_{\beta R}^{h_R}(z) = \left(\frac{\pi}{\beta R \sinh(\beta R)}\right)^{2h_R} , \quad G_{\beta L}^{h_L}(\bar{z}) = \left(\frac{\pi}{\beta L \sinh(\beta L)}\right)^{2h_L}. \quad (120)$$

We now arrive at a qualitative interpretation of the emission in terms of a $(4,0)$ super-conformal field theory. The fields responsible for the emission have conformal dimensions:

$$(h_R, h_L) = (j+1, j-s+1). \quad (121)$$

The left-moving sector involves only fields with the integer conformal dimensions (bosonic string excitations), while the right-moving sector involves conformal fields with both integer and half-integer conformal dimensions (both bosonic and fermionic string excitations). The effective string description is therefore “heterotic”. Note that we are describing the emission in terms of irrelevant operators, i.e. $(h_R, h_L) \geq 1$. This result embodies the correct physics: the amplitudes vanish for low energies with the power of frequency related to the conformal weight in a simple way.

A few remarks are in order:

- Particles with the minimal total angular momentum $j = s$ correspond to the conformal dimensions:

  $$(h_R, h_L) = (1+s, 1). \quad (122)$$

$^4$The $g_H$ of [13] is $g_H = \frac{8\beta^2}{2\pi A} = 16 \prod_i \cosh \delta_i$. 
We may interpret the result as the emission of 1 boson and 2s fermions in the right-moving sector, colliding with 1 boson in the left-moving sector. The pattern of conformal weights also suggests that the emission rates of the nearby integer and half-integer spin particles are correlated in the microscopic picture with the emission of the nearby components of the world-sheet superfields of the right-moving sector. It is possible that the super conformal symmetry of the right-moving sector would enable one to derive a precise relationship between emission rates for particles with different spins.

- The frequency dependence of the emission rates can be understood as follows. First of all the many-body kinematics of the initial state is taken into account by the Green’s function. The prefactor scales as $\omega^{2j-1}$ and is interpreted in the same way as the higher partial waves of minimally coupled scalar fields: the outgoing wave has a normalization $\omega^{-1}$ and each unit of angular momentum requires one derivative in the interaction term. This gives $\omega^j$ in the amplitude, and $\omega^{2j}$ in the rate. The same argument also applies to particles with spin, except that for fermions the wave function normalization is independent of $\omega$, and only the integer part of the angular momentum gives rise to a derivative term.

- The dependence of the remaining dimensionful quantities is not understood very well. A particular feature of the present problem is that the powers of $\beta_R$ and $\beta_L$ that appear in eq. 118 differ by a factor $2(h_R - h_L) = 2s$. This asymmetry in the couplings is likely to be related to the asymmetry between the two sectors. It is an important problem to construct a concrete model that exhibits this feature. This asymmetry also prevents us from combining the $\beta_{R,L}$ and the $\mu$ parameter into a single parameter $L$ — the string length (eq. 101). In general, there are therefore different relevant scales.

- A microscopic calculation of the overall numerical coefficient in eq. 118 is not feasible, yet. However, the microscopic picture that is needed should be robust and so one expects that such a factor could be derived from group theoretic factors that could follow directly from the algebra of the microscopic theory.

The quantitative microscopic description of the effective string is thus incomplete. However the qualitative picture in terms of correlation functions of primary fields of a (4,0) conformal field theory is already apparent at the current level of our understanding. In the recent microscopic derivation of the entropy of extremal black holes in four dimensions an auxiliary (4,0) conformal field theory appears prominently [31, 32]. This is very suggestive, but the precise connection is not clear.

## 9 Concluding remarks

We would like to conclude with a number of outstanding questions.

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Footnote: Conformal dimensions of the components of the world-sheet superfield differ by 1/2.
• It is an important task to determine the generality of the kind of greybody factors discussed in this paper. For example, it is not yet clear if the special properties of black holes that can be interpreted as solutions of the effective toroidally compactified string theory, i.e. solutions to specific $N = 8$ or $N = 4$ supergravity are needed.

• We did not complete the decoupling of the gravitational perturbation equations. It is actually not clear that this is possible; indeed it was a surprise when it was first accomplished for the Reissner-Nordström case. It would be desirable if the underlying supersymmetry supplies the structure needed to ensure such a decoupling.

• The Newman-Penrose formalism is peculiarly well-adapted to the problem of black hole perturbations but it seems specific to four dimensions. It would be interesting to consider higher spins also in the five-dimensional case. For the specific example of spin-1/2 fermions some results were obtained in [16].

• Another major challenge is to allow for rotating black hole backgrounds. In this case it is not obvious that the variables in the field equations can be separated thereby reducing the problem to effective one dimensional scattering. It is indeed surprising that, for minimally coupled scalar fields in rotating backgrounds, the equations do exhibit the desired separability both for four-dimensional [10] and five-dimensional [27] general rotating black hole backgrounds. If a similar simplification takes place for fields with spin the resulting greybody factors would necessarily be of the form considered in this paper and the main qualitative change that would be expected is the replacement:

\[ \beta_R \omega \rightarrow \beta_R \omega - m \beta_H \Omega, \]  

where $\Omega$ is the rotational velocity of the black hole and $m$ is the projection of the total angular momentum onto the axis of rotation. The thermodynamic quantities $\beta_{R,S}$ and $S_{R,L}$ would also acquire dependence on the rotational parameter of the background, as it was made explicit in [10]. The reason that it is possible to anticipate the answer in the rotating case on general grounds is that the computations leading to greybody factors of particular interest in string theory are independent of many details: they only depend on the structure of the black hole in the vicinity of the two horizons and at asymptotically large distances; and these features can be inferred from general principles (global spacetime structure of the such black holes), rather than explicit calculations. The robustness that manifests itself in this way suggests universal properties of the underlying microscopic theory. Specifically it would seem that the structure involving independent potentials of left and right moving modes should be generic.

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