SHARP INTERFACE LIMIT OF A DIFFUSE INTERFACE MODEL FOR TUMOR-GROWTH

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Abstract. We consider the asymptotic limit of a diffuse interface model for tumor-growth when a parameter \( \varepsilon \) proportional to the thickness of the diffuse interface goes to zero. An approximate solution which shows explicitly the behavior of the true solution for small \( \varepsilon \) will be constructed by using matched expansion method. Based on the energy method, a spectral condition in particular, we establish a smallness estimate of the difference between the approximate solution and the true solution.

1. Introduction

In this paper we consider the singular limit, as \( \varepsilon \to 0 \), of the solutions of the following system for \((u^\varepsilon, \sigma^\varepsilon)\):

\[
\begin{cases}
    u^\varepsilon_t - \Delta u^\varepsilon = 2\sigma^\varepsilon + u^\varepsilon - \mu^\varepsilon, & \text{in } \Omega \times (0, T), \\
    \sigma^\varepsilon_t - \Delta \sigma^\varepsilon = -(2\sigma^\varepsilon + u^\varepsilon - \mu^\varepsilon), & \text{in } \Omega \times (0, T), \\
    \varepsilon \mu^\varepsilon = -\varepsilon^2 \Delta u^\varepsilon + f'(u^\varepsilon), & \text{in } \Omega \times (0, T), \\
    u^\varepsilon(x, 0) = u_0^\varepsilon(x), \quad \sigma^\varepsilon(x, 0) = \sigma_0^\varepsilon(x), & \text{on } \Omega \times \{0\}, \\
    \frac{\partial u^\varepsilon}{\partial n} = \frac{\partial \mu^\varepsilon}{\partial n} = \frac{\partial \sigma^\varepsilon}{\partial n} = 0, & \text{on } \partial \Omega \times (0, T).
\end{cases}
\]

Here \( \Omega \subset \mathbb{R}^N \) is a bounded smooth domain, \( n \) is the unit outer normal to \( \partial \Omega \), \( \varepsilon^2 \) is the diffusivity corresponding to the surface energy, \( u^\varepsilon \) is the tumor cell concentration, \( \mu^\varepsilon \) is the chemical potential, \( \sigma^\varepsilon \) is the nutrient concentration, and \( f \) is a double equal-well potential taking its global minimum value 0 at \( u = \pm 1 \). Without loss of generality we take \( f(u) = (u^2 - 1)^2 \).

The morphological evolution of tumor progression has been an area of intense research interest recently (see, for instance [5, 9, 15, 16, 20, 22, 25, 26] and the references therein). System (1.1) is introduced to study the evolution of a growing solid tumor which coexists with the host tissue. The dynamics can be divided into two stages. During the first stage, two species are segregated according to the initial data and interface appears around the common boundary of two species. After a very fast time the dynamics enters the second stage in which the interface begins to involve. The second stage takes a much longer time than the first stage.

Generation of the interface will be left in the forthcoming paper. In this paper we are interested in the later stage and assume that the interface has been formed initially. There are two well-known approaches to describe the motion of the interface so far. The classical modeling approach is the so-called sharp interface approach which treats the interface between two phases as a \( N-1 \) dimensional sufficiently smooth surface with zero width. The second modeling approach (the so-called diffuse interface approach) treats the tumor/host tissue interface as a transition layer with finite (small) width. Comparing with the sharp interface model, the diffuse interface model has many advantages in numerical simulations of the interfacial motion (see, for instance, [11] and the
(1.1) is a diffuse interface model related to the dynamics of tumor growth which consists of advection-reaction-diffusion equation coupled with the Cahn-Hilliard equation.

One of important and natural problems is to investigate whether the diffuse interface model can be related to the corresponding sharp interface model by the asymptotic limit (i.e., the sharp interface limit) when the interfacial width tends to zero. Some formal asymptotic analyses regarding the sharp interface limits of some different tumor-growth models can be found in [17, 18, 21] for instance, but, up to our knowledge, only a few rigorous results are set up for such coupled systems. The authors in [28] rewrote (1.1) as a gradient flow and used the techniques related to gradient flow to prove that (1.1) converges to the corresponding sharp interface model in the sense of \( \Gamma \)-convergence as \( \varepsilon \to 0 \). One can see [12] for some rigorous sharp interface limit for a model which is introduced in [8] and coupled with the velocity field in a simplified case. More recently, the authors in [27] consider Cahn-Hilliard-Darcy system (first neglecting the nutrient \( \sigma \)) that models tumor growth and prove that weak solutions tend to varifold solutions of a corresponding sharp interface model when the interface thickness goes to zero. One can see [1, 10, 23, 29] for instance for more works on the convergence in the sense of \( \Gamma \)-convergence or varifold solutions, and [2] for the sharp interface limit of the Stokes-Allen-Cahn system.

This paper focuses on the rigorous analysis on the sharp interface limit of local classical solutions to (1.1). In [3] the authors proved the classical solutions of the Cahn-Hilliard equation tend to solutions of the Mullins-Sekerka problem (also called the Hele-Shaw problem) assuming the classical solutions of the latter exists. By employing the method used in [3] we will show more explicitly the asymptotic behaviors of local classical solutions \((u^\varepsilon, \sigma ^\varepsilon, \mu ^\varepsilon)\) in the sense of pointwise when \( \varepsilon \) goes to zero and establish a stronger convergence than the convergences in the sense of \( \Gamma \)-convergence and varifold solutions in some sense. In particular we can characterize the evolution laws of \((u^\varepsilon, \sigma ^\varepsilon, \mu ^\varepsilon)\) in the transition region, a small neighbourhood of \( \Gamma \), in which the behaviors are different from the ones in the two phase spaces.

The sharp interface model of (1.1) is the following two-phase flow (Theorem 5.8 and Theorem 5.9 in [28]):

\[
\begin{aligned}
-\Delta \mu + \mu &= 2\sigma \pm 1, & \text{in } \Omega_\pm, \\
\partial_t \sigma - \Delta \sigma + 2\sigma &= \mu \pm 1, & \text{in } \Omega_\pm, \\
[\mu] &= [\sigma] = 0, & \text{on } \Gamma, \\
[f] &= 0, & \text{on } \Gamma, \\
\frac{\partial \mu}{\partial n} &= -2V, & \text{on } \Gamma, \\
\frac{\partial \sigma}{\partial n} &= 0, & \text{on } \Gamma, \\
\mu &= \kappa \int_{-1}^{1} \sqrt{2f(u)} du, & \text{on } \Gamma, \\
\frac{\partial \mu}{\partial n} = \frac{\partial \sigma}{\partial n} &= 0, & \text{on } \partial \Omega,
\end{aligned}
\]  

(1.2)

where \( \Gamma \) is a closed sharp interface, \( \Omega_- \) and \( \Omega_+ \) are the interior and exterior of \( \Gamma \) in \( \Omega \) respectively, \( \mathbf{n} \) is the unit outer normal to \( \Gamma \) from \( \Omega_- \) to \( \Omega_+ \) or to \( \partial \Omega \), \( V \) is the normal velocity of the sharp interface \( \Gamma \), \( \kappa \) is the mean curvature of \( \Gamma \), \( f \) is the jump condition of \( f \) from \( \Omega_+ \) to \( \Omega_- \) defined by \([f] = f|_{\Omega_+} - f|_{\Omega_-}\).

Specifically speaking, firstly, we use Hilbert expansion method to construct an explicit approximate solution to (1.1) around the local classical solution of (1.2) by assuming the latter exists and this process also can recover the sharp interface model (1.2). This method has been used in [3, 4, 6, 13, 19, 30] and the references therein. In this paper Hilbert expansions will be performed in two phase space \( \Omega_\pm \), transition layer and near boundary \( \partial \Omega \). A kind of inner-outer matching condition will be imposed to ensure the outer expansions in \( \Omega_\pm \) and inner expansions in transition layer should match in the overlapped region. The same to the case that outer expansions in \( \Omega_\pm \) and boundary layer expansion near boundary \( \partial \Omega \). Such a method is also called matched asymptotic expansion method.
Secondly, based on the energy method, we derive the smallness of the error between the approximate solution and the true solution. Consequently we can prove rigorously that (1.1) converges to (1.2) as \( \varepsilon \to 0 \). To estimate the error we mainly need to prove the following inequality

\[
\int_{\Omega} \left( |\nabla v|^2 + \frac{1}{\varepsilon^2} f''(u^\varepsilon)v^2 \right) dx \geq -C \int_{\Omega} v^2 dx.
\]

holds for any small \( \varepsilon \) and \( v \in H^1(\Omega) \). Here \( u^A \) is an approximate solution of \( u^\varepsilon \) to be constructed and satisfies (1.3). This inequality has been proved in [7] and used to prove the convergence of the Cahn-Hilliard equation to the Hele-Shaw model in [3]. Here we write the details in a concise way which shows clearly how to vanish the singularity \( \frac{1}{\varepsilon^2} \). A sketch of the proof is as follows.

Noting that \( f''(\pm 1) > 0 \) and

\[
u^A \sim \pm 1, \text{ in } \Omega_{\pm},
\]

then \( \frac{1}{\varepsilon^2} f''(u^\varepsilon)v^2 \) is non-negative in \( \Omega_{\pm} \) and thus we only need to control \( \frac{1}{\varepsilon^2} f''(u^\varepsilon)v^2 \) in the transition layer \( \Gamma(\delta) \) which is defined in (2.3). According to the construction of the approximate solution,

\[
u^A \sim \theta(z) + \varepsilon \bar{u}^{(1)}(x,t,z) + O(\varepsilon^2), \quad z \sim \frac{d^{(0)}(x,t)}{\varepsilon} + d^{(1)}(x,t), \text{ in } \Gamma(\delta), \quad (1.3)
\]

where \( \theta(z) \) is the solution to

\[
\theta''(z) = f'(\theta(z)), \quad \theta(\pm \infty) = \pm 1, \quad \theta(0) = 0, \quad (1.4)
\]

and \( d^{(0)}, d^{(1)} \) are defined in (2.1), and \( \bar{u}^{(1)}(x,t,z) \) which is defined in (2.10) satisfies the important property Lemma 3.5. Therefore one has

\[
\frac{1}{\varepsilon^2} f''(u^\varepsilon)v^2 \sim \frac{1}{\varepsilon^2} f''(\theta(z))v^2 + \frac{1}{\varepsilon} f''(\theta(z))\bar{u}^{(1)}(x,t,z)v^2 + O(1)v^2, \text{ in } \Gamma(\delta).
\]

To deal with the most singular term \( \frac{1}{\varepsilon^2} f''(\theta(z))v^2 \) we will draw support from the diffusion term and use the estimates of the first eigenvalue, the corresponding eigenfunction and the second eigenvalue of the following Neumann eigenvalue problem

\[
\mathcal{L}f = -\frac{d^2 f}{dz^2} + f''(\theta)q = \lambda q, \quad z \in I_\varepsilon = (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}); \quad q'(\pm \frac{1}{\varepsilon}) = 0, \quad (1.5)
\]

which has been proved in [7] (proved by a new method in [14]). And the next singular term

\[
\frac{1}{\varepsilon} \int_{\Omega} f''(\theta(z))\bar{u}^{(1)}(x,t,z)v^2 dx \text{ will vanish in some sense due to the property Lemma 3.5 of } \bar{u}^{(1)}.
\]

To deal with the difficulty coming from the coupling term \( 2\sigma^\varepsilon + u^\varepsilon - \mu^\varepsilon \) in (1.1), we consider the auxiliary variable \( \varphi^\varepsilon = u^\varepsilon + \sigma^\varepsilon \) and construct an approximate solution \( \varphi^A, \sigma^A \) to approximate the system which \( (\varphi^\varepsilon, \sigma^\varepsilon) \) satisfies. The system for \( (\varphi^\varepsilon, \sigma^\varepsilon) \) is a gradient-flow system (28). Then we apply the energy method to estimate the errors \( \varphi^\varepsilon - \varphi^A \) and \( \sigma^\varepsilon - \sigma^A \).

Our main conclusion is

**Theorem 1.1.** Given a classical solution \( (\mu, \sigma, \Gamma) \) of (1.2) in \( \Omega \times (0,T) \) which satisfies

\[
dist(\Gamma, \partial \Omega) > 0, \quad x \in \Omega, \quad t \in [0,T]. \quad (1.6)
\]

Then there exist \( \varepsilon_0 > 0 \) and \( C > 0 \) such that the following hold: for any \( 0 < \varepsilon < \varepsilon_0 \), there exist \( u^A_0(x), \sigma^A_0(x) \) such that if \( (u^\varepsilon, \sigma^\varepsilon) \) is the solution to (1.7) in \( \Omega \times (0,T) \), we have

\[
\|u^\varepsilon - u^A\|_{C^{1,1}(\Omega \times (0,T))} + \|\sigma^\varepsilon - \sigma^A\|_{C^{2,1}(\Omega \times (0,T))} + \|\mu^\varepsilon - \mu^A\|_{C^{2,1}(\Omega \times (0,T))} \leq C\varepsilon,
\]

where \( (u^A, \sigma^A, \mu^A) \) which will be constructed in Section 5 is an approximate solution of \( (u^\varepsilon, \sigma^\varepsilon, \mu^\varepsilon) \) with the order of asymptotic expansion \( k \) large enough.

As a corollary we have
Corollary 1.2. Given a classical solution $(\mu, \sigma, \Gamma)$ of (1.2) as in Theorem 1.1. Then there exist $u_0^{(\varepsilon)}(x), \sigma_0^{(\varepsilon)}(x)$ such that if $(u^{(\varepsilon)}, \sigma^{(\varepsilon)})$ is the solution to (1.1) in $\Omega \times (0, T)$, then as $\varepsilon \to 0$, there hold

$$
\|u^{(\varepsilon)} - (\pm 1)\|_{C(\Omega \pm \Gamma(\delta))} \to 0, \quad \|u^{(\varepsilon)} - \theta\left(\frac{d^{(0)}}{\varepsilon} + d^{(1)}\right)\|_{C(\Gamma(\delta))} \to 0, \tag{1.7}
$$

$$
\|\sigma^{(\varepsilon)} - \sigma\|_{C(\Omega \times [0, T])} + \|\mu^{(\varepsilon)} - \mu\|_{C(\Omega \times [0, T])} \to 0, \tag{1.8}
$$

here and in what follows $\delta$ is a small positive constant satisfying $\delta < \frac{1}{2}\text{dist}(\Gamma, \partial\Omega)$ for $(x, t) \in \Omega \times [0, T]$.

Remark 1.3. $u_0^{(\varepsilon)}(x)$ and $\sigma_0^{(\varepsilon)}(x)$ are defined in (3.10). The initial data like this form is often called sharp interface initial data since we assume the interface has been formed initially. One can refer to [3, 24] for this kind of data.

We organize this paper as follows. In Section 2, we do $\varepsilon^k (k = 0, 1)$-order asymptotic expansion and get all the zeroth order terms and some 1th order terms. In Section 3 we establish a spectral condition and give a smallness estimation of the error between the approximate solution and the true solution. Then we complete the proof of Theorem 1.1 and Corollary 1.2. In Section 4, we do $\varepsilon^k (k \geq 2)$-order asymptotic expansion and get all the remained terms. In Section 5, an approximate solution is constructed.

Through this paper $C$ denotes a generic positive constant independent of small $\varepsilon$.

2. Matched asymptotic expansion

Let $\Gamma^{\varepsilon}$ be a smooth surface centered in the transition layer. For any $t \in [0, T]$ for fixed $T > 0$, let $d^{\varepsilon}(x, t)$ be the signed distance from $x$ to $\Gamma^{\varepsilon}$. Then $d^{\varepsilon}$ is smooth and $|\nabla d^{\varepsilon}| = 1$ in a neighborhood of $\Gamma^{\varepsilon}$. We assume

$$
d^{\varepsilon}(x, t) = d^{(0)}(x, t) + \varepsilon d^{(1)}(x, t) + \varepsilon^2 d^{(2)}(x, t) + \cdots, \tag{2.1}
$$

where $d^{(0)}$ is a signed distance to $\Gamma$ and $d^{(i)} (i \geq 1)$ is to be determined later.

Since

$$
1 = |\nabla d^{\varepsilon}|^2 = |\nabla d^{(0)}|^2 + 2\varepsilon \nabla d^{(0)} \cdot \nabla d^{(1)} + \sum_{k=2}^{+\infty} \varepsilon^k \left( \sum_{i=0}^{k} \nabla d^{(i)} \cdot \nabla d^{(k-i)} \right) = 1 + 2\varepsilon \nabla d^{(0)} \cdot \nabla d^{(1)} + \sum_{k=2}^{+\infty} \varepsilon^k \left( \sum_{i=0}^{k} \nabla d^{(i)} \cdot \nabla d^{(k-i)} \right),
$$

then

$$
\nabla d^{(0)} \cdot \nabla d^{(k)} = D_{k-1}(k \geq 1) \triangleq \begin{cases} 
0, & k = 1, \\
-\frac{1}{2} \sum_{i=1}^{k-1} \nabla d^{(i)} \cdot \nabla d^{(k-i)}, & k \geq 2.
\end{cases} \tag{2.2}
$$

Furthermore, we have $\Gamma = \{(x, t) : d^{(0)}(x, t) = 0\}$, $\Omega_{\pm} = \{(x, t) : d^{(0)}(x, t) \geq 0\}$, $V = -d^{(0)}$, and $\kappa = -\Delta d^{(0)}$. Defining

$$
\Gamma(\delta) = \{(x, t) \in \Omega \times (0, T) : |d^{(0)}(x, t)| < \delta\}. \tag{2.3}
$$

Now we do outer expansion in $\Omega_{\pm}$, inner expansion in $\Gamma(\delta)$ and boundary layer expansion in $\partial \Omega(\delta) = \{(x, t) : \text{dist}(x, \partial\Omega) < \delta, x \in \Omega, t \in [0, T]\}$. For clarity we only match zero-order and $\varepsilon$-order, and solve all the zero-order terms and some 1-order terms in this section. Matching $\varepsilon^k (k \geq 2)$-order and all the remained terms will be presented in Section 4.
2.1. **Outer expansion in $\Omega_\pm$.**

In $\Omega_\pm$, we set

\[
\begin{align*}
    u^\varepsilon &= u^{(0)}_\pm + \varepsilon u^{(1)}_\pm + \varepsilon^2 u^{(2)}_\pm + \cdots, \\
    \mu^\varepsilon &= \mu^{(0)}_\pm + \varepsilon \mu^{(1)}_\pm + \varepsilon^2 \mu^{(2)}_\pm + \cdots, \\
    \sigma^\varepsilon &= \sigma^{(0)}_\pm + \varepsilon \sigma^{(1)}_\pm + \varepsilon^2 \sigma^{(2)}_\pm + \cdots.
\end{align*}
\]

Moreover, by using Taylor expansion we write in $\Omega_\pm$

\[
f'(u^\varepsilon) = f'(u^{(0)}_\pm) + \varepsilon f''(u^{(0)}_\pm)u^{(1)}_\pm + \cdots + \varepsilon^k \left( f''(u^{(0)}_\pm)u^{(k)}_\pm + g(u^{(0)}_\pm, \ldots, u^{(k-1)}_\pm) \right) + \cdots,
\]

here $g(u^{(0)}_\pm, \ldots, u^{(k-1)}_\pm)$ depends on $u^{(0)}_\pm, \ldots, u^{(k-1)}_\pm$.

Substituting (2.4)-(2.6) into (1.1) and collecting all terms of zero order we have

\[
\begin{align*}
    \partial_t u^{(0)}_\pm - \Delta \mu^{(0)}_\pm &= 2\sigma^{(0)}_\pm + u^{(0)}_\pm - \mu^{(0)}_\pm, \\
    \partial_t \sigma^{(0)}_\pm - \Delta \sigma^{(0)}_\pm &= -(2\sigma^{(0)}_\pm + u^{(0)}_\pm - \mu^{(0)}_\pm), \\
    f'(u^{(0)}_\pm) &= 0.
\end{align*}
\]

We take

\[
u^{(0)}_\pm = \pm 1
\]

and then

\[
\begin{align*}
    -\Delta \mu^{(0)}_\pm + \mu^{(0)}_\pm &= 2\sigma^{(0)}_\pm \pm 1, \\
    \partial_t \sigma^{(0)}_\pm - \Delta \sigma^{(0)}_\pm + 2\sigma^{(0)}_\pm &= \mu^{(0)}_\pm \mp 1.
\end{align*}
\]

Substituting (2.4)-(2.6) into (1.1) and collecting all terms of $\varepsilon$-order we have

\[
\begin{align*}
    u^{(1)}_\pm &= \frac{\mu^{(0)}_\pm}{f''(u^{(0)}_\pm)}, \\
    -\Delta \mu^{(1)}_\pm + \mu^{(1)}_\pm &= 2\sigma^{(1)}_\pm + u^{(1)}_\pm - \partial_t u^{(1)}_\pm, \\
    \partial_t \sigma^{(1)}_\pm - \Delta \sigma^{(1)}_\pm + 2\sigma^{(1)}_\pm &= \mu^{(1)}_\pm - u^{(1)}_\pm.
\end{align*}
\]

2.2. **Inner expansion in $\Gamma(\delta)$.**

Let $z = \frac{x}{\varepsilon} \in (-\infty, +\infty)$. In $\Gamma(\delta)$, we set

\[
\begin{align*}
    u^\varepsilon(x, t) = \bar{u}^\varepsilon(x, t, z) &= \bar{u}^{(0)}(x, t, z) + \varepsilon \bar{u}^{(1)}(x, t, z) + \varepsilon^2 \bar{u}^{(2)}(x, t, z) + \cdots, \\
    \mu^\varepsilon(x, t) = \bar{\mu}^\varepsilon(x, t, z) &= \bar{\mu}^{(0)}(x, t, z) + \varepsilon \bar{\mu}^{(1)}(x, t, z) + \varepsilon^2 \bar{\mu}^{(2)}(x, t, z) + \cdots, \\
    \sigma^\varepsilon(x, t) = \bar{\sigma}^\varepsilon(x, t, z) &= \bar{\sigma}^{(0)}(x, t, z) + \varepsilon \bar{\sigma}^{(1)}(x, t, z) + \varepsilon^2 \bar{\sigma}^{(2)}(x, t, z) + \cdots,
\end{align*}
\]

and the following inner-outer matching conditions: there exists a fixed positive constant $\nu$ such that as $z \to \pm \infty$ there hold

\[
\begin{align*}
    D^\alpha_x D^\beta_x D^\gamma_z \left( \bar{u}^{(i)}(x, t, z) - u^{(i)}_\pm(x, t) \right) &= O(e^{-\nu |z|}), \\
    D^\alpha_x D^\beta_x D^\gamma_z \left( \bar{\mu}^{(i)}(x, t, z) - \mu^{(i)}_\pm(x, t) \right) &= O(e^{-\nu |z|}), \\
    D^\alpha_x D^\beta_x D^\gamma_z \left( \bar{\sigma}^{(i)}(x, t, z) - \sigma^{(i)}_\pm(x, t) \right) &= O(e^{-\nu |z|})
\end{align*}
\]

for $(x, t) \in \Gamma(\delta)$ and $0 \leq \alpha, \beta, \gamma \leq N$ with $N$ depending on actual expansion order.
Using Taylor expansion and (2.10) one has
\[ f'(\tilde{u}^\varepsilon) = f'(\tilde{u}^{(0)}) + \varepsilon f''(\tilde{u}^{(0)})\tilde{u}^{(1)} + \cdots + \varepsilon^k \left( f''(\tilde{u}^{(0)})\tilde{u}^{(k)} + \bar{g}(\tilde{u}^{(0)}, \cdots, \tilde{u}^{(k-1)}) \right) + \cdots, \tag{2.16} \]

here \( \bar{g}(\tilde{u}^{(0)}, \cdots, \tilde{u}^{(k-1)}) \) depends on \( \tilde{u}^{(0)}, \cdots, \tilde{u}^{(k-1)} \).

Noting that
\[
\tilde{u}_t^\varepsilon(x, t, z) = \partial_t \tilde{u}^\varepsilon + \varepsilon^{-1} \partial_z \tilde{u}^\varepsilon \partial_t d^\varepsilon, \\
\Delta_x \tilde{\mu}^\varepsilon(x, t, z) = -\varepsilon^2 \partial_{zz} \tilde{\mu}^\varepsilon + 2\varepsilon^{-1} \nabla_x \partial_z \tilde{\mu}^\varepsilon \cdot \nabla_x d^\varepsilon + \varepsilon^{-1} \partial_x \tilde{\mu}^\varepsilon \Delta_x d^\varepsilon + \Delta_x \tilde{\mu}^\varepsilon,
\]

then in the new coordinate \((x, t, z)\) the first equation in (1.1) becomes
\[
-\partial_{zz} \tilde{\mu}^\varepsilon + \varepsilon \left( \partial_z \tilde{\mu}^\varepsilon \partial_t d^\varepsilon - 2\nabla_x \partial_z \tilde{\mu}^\varepsilon \cdot \nabla_x d^\varepsilon - \partial_z \tilde{\mu}^\varepsilon \Delta_x d^\varepsilon \right) + \varepsilon^2 \left( \partial_t \tilde{u}^\varepsilon - \Delta_x \tilde{\mu}^\varepsilon - (2\tilde{\sigma}^\varepsilon + \tilde{u}^\varepsilon - \tilde{\mu}^\varepsilon) \right) = 0.
\tag{2.17}
\]

Similarly, the second equation and the third equation in (1.1) become respectively
\[
-\partial_{zz} \tilde{\sigma}^\varepsilon + \varepsilon \left( \partial_z \tilde{\sigma}^\varepsilon \partial_t d^\varepsilon - 2\nabla_x \partial_z \tilde{\sigma}^\varepsilon \cdot \nabla_x d^\varepsilon - \partial_z \tilde{\sigma}^\varepsilon \Delta_x d^\varepsilon \right) + \varepsilon^2 \left( \partial_t \tilde{u}^\varepsilon - \Delta_x \tilde{\sigma}^\varepsilon + 2\tilde{\sigma}^\varepsilon + \tilde{u}^\varepsilon - \tilde{\mu}^\varepsilon \right) = 0, \tag{2.18}
\]
\[
-\partial_{zz} \tilde{u}^\varepsilon + f'(\tilde{u}^\varepsilon) - \varepsilon \left( 2\nabla_x \partial_z \tilde{u}^\varepsilon \cdot \nabla_x d^\varepsilon + \partial_z \tilde{u}^\varepsilon \Delta_x d^\varepsilon + \tilde{\mu}^\varepsilon \right) - \varepsilon^2 \Delta_x \tilde{u}^\varepsilon = 0. \tag{2.19}
\]

Generally the inner-outer exponentially decaying matching conditions (2.13)-(2.15) may not necessarily hold. To ensure this conditions, we will modify (2.17)-(2.19) as follows motivated by [3]. For that we choose a smooth non-decreasing function \( \eta \) such that \( \eta(z) = 0 \) for \( z \leq -1 \), \( \eta(z) = 1 \) for \( z \geq 1 \) and define
\[
\eta^\pm(z) = \eta(-M \pm z), \quad z \in \mathbb{R},
\]

here the constant \( M = ||d^{(1)}||_{C^0(\Gamma(\delta))} + 2 \).

Now we modify (2.17)-(2.19) as follows
\[
-\partial_{zz} \tilde{\mu}^\varepsilon + \varepsilon \left( \partial_z \tilde{\mu}^\varepsilon \partial_t d^\varepsilon - 2\nabla_x \partial_z \tilde{\mu}^\varepsilon \cdot \nabla_x d^\varepsilon - \partial_z \tilde{\mu}^\varepsilon \Delta_x d^\varepsilon \right) + \varepsilon^2 \left( \partial_t \tilde{u}^\varepsilon - \Delta_x \tilde{\mu}^\varepsilon - (2\tilde{\sigma}^\varepsilon + \tilde{u}^\varepsilon - \tilde{\mu}^\varepsilon) \right) + \eta'' p(d^\varepsilon - \varepsilon z) + \eta' g(d^\varepsilon - \varepsilon z) - \varepsilon^2 (s^\varepsilon_+ \eta^+ + s^\varepsilon_- \eta^-) = 0, \tag{2.20}
\]
\[
-\partial_{zz} \tilde{\sigma}^\varepsilon + \varepsilon \left( \partial_z \tilde{\sigma}^\varepsilon \partial_t d^\varepsilon - 2\nabla_x \partial_z \tilde{\sigma}^\varepsilon \cdot \nabla_x d^\varepsilon - \partial_z \tilde{\sigma}^\varepsilon \Delta_x d^\varepsilon \right) + \varepsilon^2 \left( \partial_t \tilde{u}^\varepsilon - \Delta_x \tilde{\sigma}^\varepsilon + 2\tilde{\sigma}^\varepsilon + \tilde{u}^\varepsilon - \tilde{\mu}^\varepsilon \right) + \eta'' q(d^\varepsilon - \varepsilon z) + \eta' h(d^\varepsilon - \varepsilon z) - \varepsilon^2 (r^\varepsilon_+ \eta^+ + r^\varepsilon_- \eta^-) = 0, \tag{2.21}
\]
\[
-\partial_{zz} \tilde{u}^\varepsilon + f'(\tilde{u}^\varepsilon) - \varepsilon \left( 2\nabla_x \partial_z \tilde{u}^\varepsilon \cdot \nabla_x d^\varepsilon + \partial_z \tilde{u}^\varepsilon \Delta_x d^\varepsilon + \tilde{\mu}^\varepsilon \right) - \varepsilon^2 \Delta_x \tilde{u}^\varepsilon + \varepsilon' \eta^l(d^\varepsilon - \varepsilon z) = 0, \tag{2.22}
\]
Remark 2.1. Due to where can find have (2.13)-(2.15) hold. the modifications, we have changed the equation of every order such that the matching conditions the details. Therefore (2.17)-(2.19) are the same to (2.20)-(2.22) respectively. However, through

\[ \lim_{x \to \pm \infty} z = 0 \]

for some \( a \) and then \( \alpha, \beta, \gamma \)

Substituting (2.1) and (2.10)-(2.12) into (2.20)-(2.22) and collecting all terms of zero order we have

\[ \partial_{zz} \tilde{\mu}(0) = \eta'' \mu(0) d(0), \]
\[ \partial_{zz} \tilde{\sigma}(0) = \eta'' q(0) d(0), \]
\[ \partial_{zz} \tilde{u}(0) = f'(\tilde{u}(0)). \]

Now we argue in this subsection according to the following result:

\[ (\tilde{\mu}(0), \tilde{\sigma}(0)) \to (p(0), q(0), [\mu(0)], [\sigma(0)]) \to \tilde{\mu}(0). \]

\( (\tilde{\mu}(0), \tilde{\sigma}(0)) \)

From (2.23) we can write

\[ \tilde{\mu}(0)(z, x, t) = \eta(z)p(0)(x, t)d(0)(x, t) + a(x, t)z + b(x, t) \]

for some \( a(x, t) \) and \( b(x, t) \). Since the inner-outer matching condition \( \tilde{\mu}(0)(z, x, t) \to \mu_\pm(0)(x, t) \) as \( z \to \pm \infty \) needs to be satisfied, we must have

\[ a(x, t) = 0, \quad b(x, t) = \mu_\pm(0)(x, t) \]

and

\[ p(0)(x, t)d(0)(x, t) = \mu_\pm(0)(x, t) - \mu_\pm(0)(x, t). \]

Thus we have

\[ \tilde{\mu}(0)(x, t, z) = \eta(z)\mu_\pm(0)(x, t) + (1 - \eta(z))\mu_\pm(0)(x, t) \]

and then

\[ D_\alpha D_\beta D_\gamma \left( \tilde{\mu}(0)(x, t, z) - \mu_\pm(0)(x, t) \right) = O(e^{-\nu|z|}) \]

for any \( \alpha, \beta, \gamma \in \mathbb{N} \) and \( \nu > 0 \).
Similarly, we get

\[ q^{(0)}(x, t) d^{(0)}(x, t) = \sigma^{(0)}_{\pm}(x, t) - \sigma^{(0)}(x, t) \] (2.28)

and

\[ \tilde{\sigma}^{(0)}(x, t, z) = \eta(z) \sigma_{\pm}^{(0)}(x, t) + (1 - \eta(z)) \sigma^{(0)}_{\pm}(x, t) \] (2.29)

which implies

\[ D_\alpha^x D_\beta^t D_\gamma^z \left( \tilde{\sigma}^{(0)}(x, t, z) - \sigma^{(0)}_{\pm}(x, t) \right) = O(e^{-\nu|z|}) \]

for any \( \alpha, \beta, \gamma \in \mathbb{N} \) and \( \nu > 0 \).

- \((p^{(0)}, q^{(0)}, [\mu^{(0)}], [\sigma^{(0)}])\)

Moreover, according to (2.26) and (2.28) there hold on \( \Gamma \)

\[ [\mu^{(0)}] \triangleq \mu^{(0)}_{\pm} - \mu^{(0)} = 0, \quad [\sigma^{(0)}] \triangleq \sigma^{(0)}_{\pm} - \sigma^{(0)} = 0. \] (2.30)

And we can define smooth functions \( p^{(0)} \) and \( q^{(0)} \) in \( \Gamma(\delta) \) as follows

\[
p^{(0)} = \begin{cases} 
\mu^{(0)}_{\pm} - \mu^{(0)} - d^{(0)}(x, t), & \text{in } \Gamma(\delta) \setminus \Gamma, \\
\nabla_x d^{(0)} \cdot \nabla_x (\mu^{(0)}_{\pm} - \mu^{(0)}), & \text{on } \Gamma,
\end{cases}
\] (2.31)

and

\[
q^{(0)} = \begin{cases} 
\sigma^{(0)}_{\pm} - \sigma^{(0)} - d^{(0)}(x, t), & \text{in } \Gamma(\delta) \setminus \Gamma, \\
\nabla_x d^{(0)} \cdot \nabla_x (\sigma^{(0)}_{\pm} - \sigma^{(0)}), & \text{on } \Gamma.
\end{cases}
\] (2.32)

- \( \tilde{u}^{(0)} \)

By (2.25) and the inner-outer matching condition, \( \tilde{u}^{(0)} \) satisfies

\[ \partial_{zz} \tilde{u}^{(0)} = f'(\tilde{u}^{(0)}), \quad \tilde{u}^{(0)}(\pm \infty) = u^{(0)}_{\pm} = \pm 1, \quad \tilde{u}^{(0)}(0) = 0, \]

here the condition \( \tilde{u}^{(0)}(0) = 0 \) is imposed to ensure \( \tilde{u}^{(0)} \) is unique. Therefore \( \tilde{u}^{(0)} \) is independent of \((x, t)\) and then \( \tilde{u}^{(0)}(x, t, z) = \theta(z) \) which is defined in (1.4). In fact, we can get

\[ \tilde{u}^{(0)}(x, t, z) = \theta(z) = \tanh(\sqrt{2}z) \] (2.33)

and for \( k \in \mathbb{N} \cup \{0\} \)

\[ \frac{d^k}{dz^k}(\theta(z) + 1) = O(e^{-\sqrt{2}|z|}), \quad \text{as } z \to -\infty; \quad \frac{d^k}{dz^k}(\theta(z) - 1) = O(e^{-\sqrt{2}|z|}), \quad \text{as } z \to +\infty, \]

which implies

\[ D_\alpha^x D_\beta^t D_\gamma^z \left( \tilde{u}^{(0)}(x, t, z) - u^{(0)}_{\pm}(x, t) \right) = O(e^{-\sqrt{2}|z|}) \]

for any \( \alpha, \beta, \gamma \in \mathbb{N} \).
2.2.2. Matching 1st order.

Substituting (2.1) and (2.10)-(2.12) into (2.20)-(2.22) and collecting all terms of \( \varepsilon \)-order we have

\[
- \partial_{zz} \tilde{\mu}^{(1)} + \left( \partial_{zz} \tilde{\mu}^{(0)} \partial_t d^{(0)} - 2 \nabla_x \partial_z \tilde{\mu}^{(0)} \cdot \nabla_x d^{(0)} - \partial_z \tilde{\mu}^{(0)} \Delta_x d^{(0)} \right) \\
+ \eta'' \left( \mu^{(1)} d^{(0)} + p^{(0)} d^{(1)} \right) - \eta'' z p^{(0)} + \eta' g^{(0)} d^{(0)} = 0,
\]

(2.34)

\[
- \partial_{zz} \tilde{\sigma}^{(1)} + \left( \partial_{zz} \sigma^{(0)} \partial_t d^{(0)} - 2 \nabla_x \partial_z \sigma^{(0)} \cdot \nabla_x d^{(0)} - \partial_z \tilde{\sigma}^{(0)} \Delta_x d^{(0)} \right) \\
+ \eta'' \left( q^{(1)} d^{(0)} + q^{(0)} d^{(1)} \right) - \eta'' z q^{(0)} + \eta' h^{(0)} d^{(0)} = 0,
\]

(2.35)

\[
- \partial_{zz} \tilde{u}^{(1)} + f''(\tilde{u}^{(0)}) \tilde{u}^{(1)} - \left( 2 \nabla_x \partial_z \tilde{u}^{(0)} \cdot \nabla_x d^{(0)} + \partial_z \tilde{u}^{(0)} \Delta_x d^{(0)} + \tilde{\mu}^{(0)} \right) \\
+ \eta' l^{(0)} d^{(0)} = 0.
\]

(2.36)

Next we argue according to the following order:

\[
\left( \tilde{\mu}^{(1)}, g^{(0)}, \left[ \frac{\partial \mu^{(0)}}{\partial n} \right] \right) \rightarrow \left( \tilde{\sigma}^{(1)}, h^{(0)}, \left[ \frac{\partial \sigma^{(0)}}{\partial n} \right] \right) \rightarrow \left( \tilde{u}^{(1)}, l^{(0)}, \mu^{(0)}_{\pm} |_{\Gamma} \right).
\]

\( \bullet \) \( \left( \tilde{\mu}^{(1)}, g^{(0)}, \left[ \frac{\partial \mu^{(0)}}{\partial n} \right] \right) \)

For \( (x, t) \in \Gamma(\tilde{\delta}) \), we write (2.34) as

\[
- \left( \tilde{\mu}^{(1)} - \eta \left( p^{(1)} d^{(0)} + p^{(0)} d^{(1)} \right) \right)_{zz} = \eta'' z p^{(0)} - \eta' g^{(0)} d^{(0)} - \partial_z \tilde{u}^{(0)} \partial_t d^{(0)} \\
+ 2 \nabla_x \partial_z \tilde{\mu}^{(0)} \cdot \nabla_x d^{(0)} + \partial_z \tilde{\mu}^{(0)} \Delta_x d^{(0)} \\
\triangleq \Theta_{0,1}.
\]

(2.37)

It follows from Lemma 4.3 in [3] and direct computations that if

\[
\int_{-\infty}^{+\infty} \Theta_{0,1}(x, t, z) dz = 0,
\]

(2.38)

then (2.37) has a bounded solution

\[
\tilde{\mu}^{(1)}(x, t, z) = \eta(z) \left( p^{(1)} d^{(0)} + p^{(0)} d^{(1)} \right)(x, t) \\
+ \int_{-\infty}^{z} \int_{z'}^{+\infty} \Theta_{0,1}(z'', x, t) dz'' dz' + \mu^{(1)}_{\pm}(x, t)
\]

which and \( \tilde{\mu}^{(1)}(+\infty, x, t) = \mu^{(1)}_{\pm}(x, t) \) imply

\[
(p^{(1)} d^{(0)} + p^{(0)} d^{(1)})(x, t) = \mu^{(1)}_{\pm}(x, t) - \int_{-\infty}^{+\infty} \int_{z'}^{+\infty} \Theta_{0,1}(z'', x, t) dz'' dz' - \mu^{(1)}_{\pm}(x, t).
\]

Hence we obtain

\[
\tilde{\mu}^{(1)}(x, t, z) = \eta(z) \mu^{(1)}_{\pm}(x, t) + (1 - \eta(z)) \mu^{(1)}_{\pm}(x, t) \\
- \eta(z) \int_{-\infty}^{+\infty} \int_{z'}^{+\infty} \Theta_{0,1}(z'', x, t) dz'' dz' \\
+ \int_{-\infty}^{z} \int_{z'}^{+\infty} \Theta_{0,1}(z'', x, t) dz'' dz'
\]

and

\[
\left[ \mu^{(1)} \right] = \mu^{(1)}_{\pm} - \mu^{(1)}_{\pm} = p^{(0)} d^{(1)} + \int_{-\infty}^{+\infty} \int_{z'}^{+\infty} \Theta_{0,1}(z'', x, t) dz'' dz', \text{ on } \Gamma.
\]
According to the results obtained in subsection 2.2.1 one has for any $\alpha, \beta, \gamma \in \mathbb{N}$,
\[
D_x^\alpha D_t^\beta D_z^\gamma \Theta_{0,1} = O(e^{-\nu|z|}), \quad \text{for some } \nu > 0,
\]
and thus
\[
D_x^\alpha D_t^\beta D_z^\gamma \left( \bar{\mu}^{(1)}(x, t, z) - \mu^{(1)}_\pm(x, t) \right) = O(e^{-\nu|z|}), \quad \text{for some } \nu > 0.
\]
Moreover, by (2.38), (2.30), (2.31) and direct computations we can get
\[
\left[ \frac{\partial \mu^{(0)}_+}{\partial n} \right] = \nabla_x \left( \mu^{(0)}_+ - \mu^{(0)}_- \right) \cdot \nabla_x d^{(0)} = 2d_t^{(0)} \triangleq -2V, \quad \text{on } \Gamma,
\]
and define a smooth function $g^{(0)}$ in $\Gamma(\delta)$
\[
g^{(0)} = \left\{ \begin{aligned}
&\frac{(\mu^{(0)}_+ - \mu^{(0)}_-) \Delta_x d^{(0)} + 2\nabla_x (\mu^{(0)}_+ - \mu^{(0)}_-) \cdot \nabla_x d^{(0)} - \sigma^0 - 2\partial_t d^{(0)}}{d^{(0)}}, \\
&\nabla_x d^{(0)} \cdot \nabla_x \left( \mu^{(0)}_+ - \mu^{(0)}_- \right) \Delta_x d^{(0)} + 2\nabla_x (\mu^{(0)}_+ - \mu^{(0)}_-) \cdot \nabla_x d^{(0)}
\end{aligned} \right. 
\]
on $\Gamma(\delta) \setminus \Gamma$,
\[
g^{(0)} = \left\{ \begin{aligned}
&\nabla_x d^{(0)} \cdot \nabla_x \left( \mu^{(0)}_+ - \mu^{(0)}_- \right) \Delta_x d^{(0)} + 2\nabla_x (\mu^{(0)}_+ - \mu^{(0)}_-) \cdot \nabla_x d^{(0)}
\end{aligned} \right. 
\]
on $\Gamma$.

• $(\tilde{\sigma}^{(1)}, h^{(0)}, \left[ \frac{\partial \sigma^{(0)}_+}{\partial n} \right])$

Applying the above similar arguments to (2.35), we obtain that if
\[
\int_{-\infty}^{+\infty} \Theta_{0,2}(x, t, z) dz = 0,
\]
then (2.35) has a bounded solution
\[
\tilde{\sigma}^{(1)}(x, t, z) = \eta(z) \sigma^{(1)}_+(x, t) + \left( 1 - \eta(z) \right) \sigma^{(1)}_-(x, t)
\]
\[
- \eta(z) \int_{-\infty}^{+\infty} \int_{z'}^{+\infty} \Theta_{0,2}(x, t, z'' \text{d}z'' \text{d}z'') + \int_{-\infty}^{+\infty} \int_{z'}^{+\infty} \Theta_{0,2}(x, t, z'' \text{d}z'' \text{d}z''),
\]
and for any $\alpha, \beta, \gamma \in \mathbb{N}$,
\[
D_x^\alpha D_t^\beta D_z^\gamma \left( \tilde{\sigma}^{(1)}(x, t, z) - \sigma^{(1)}_\pm(x, t) \right) = O(e^{-\nu|z|}), \quad \text{for some } \nu > 0,
\]
and
\[
[\sigma^{(1)}] \triangleq \sigma^{(1)}_+ - \sigma^{(1)}_- = q^{(0)} d^{(1)} + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Theta_{0,2}(x, t, z'' \text{d}z'' \text{d}z'), \quad \text{on } \Gamma,
\]
where
\[
\Theta_{0,2} \triangleq \eta' z q^{(0)} - \eta' h^{(0)} d^{(0)} - \partial_t \tilde{\sigma}^{(0)} - 2\nabla_x \partial_t \tilde{\sigma}^{(0)} \cdot \nabla_x d^{(0)} + \partial_z \tilde{\sigma}^{(0)} \Delta_x d^{(0)}.
\]
Moreover, it follows from (2.41), (2.30), (2.32) and direct computations that
\[
\left[ \frac{\partial \sigma^{(0)}_+}{\partial n} \right] \triangleq \nabla \left( \sigma^{(0)}_+ - \sigma^{(0)}_- \right) \cdot \nabla d^{(0)} = 0, \quad \text{on } \Gamma,
\]
and
\[
h^{(0)} = \left\{ \begin{aligned}
&\frac{(\sigma^{(0)}_+ - \sigma^{(0)}_-) \Delta_x d^{(0)} - \partial_t d^{(0)} + 2\nabla_x (\sigma^{(0)}_+ - \sigma^{(0)}_-) \cdot \nabla_x d^{(0)} - q^{0}}{d^{(0)}}, \\
&\nabla_x d^{(0)} \cdot \nabla_x \left( \sigma^{(0)}_+ - \sigma^{(0)}_- \right) \Delta_x d^{(0)} - \partial_t d^{(0)}
\end{aligned} \right. 
\]
on $\Gamma(\delta) \setminus \Gamma$,
\[
h^{(0)} = \left\{ \begin{aligned}
&\nabla_x d^{(0)} \cdot \nabla_x \left( \sigma^{(0)}_+ - \sigma^{(0)}_- \right) \Delta_x d^{(0)} - \partial_t d^{(0)}
\end{aligned} \right. 
\]
on $\Gamma$. 

• $(\bar{u}^{(1)}, I^{(0)}, \mu^{(0)}_\pm |\Gamma)$
Based on the method of variation of constants for ODE and direct computations (or Lemma 4.3 in [3]) we find that if
\[
\int_{-\infty}^{+\infty} \Theta_{0,3}(x, t, z) \theta'(z) dz = 0,  \tag{2.44}
\]
then the solution to (2.36) with \( \tilde{u}^{(1)}(0, x, t) = 1 \) is
\[
\tilde{u}^{(1)}(x, t, z) = \frac{\theta'(z)}{\theta'(0)} + \theta'(z) \int_{0}^{z} (\theta'(\varsigma))^{-2} \int_{\varsigma}^{+\infty} \Theta_{0,3}(x, t, \tau) \theta'(\tau) d\tau d\varsigma,
\]
which satisfies for any \( \alpha, \beta, \gamma \in \mathbb{N} \),
\[
D_x^\alpha D_t^\beta D_z^\gamma \left( \tilde{u}^{(1)}(x, t, z) - u_{\pm}^{(1)}(x, t) \right) = O(e^{-\nu|z|}), \quad \text{for some } \nu > 0,
\]
where
\[
\Theta_{0,3} \triangleq 2\nabla_x \partial_x \tilde{u}^{(0)} \cdot \nabla_x d^{(0)} + \partial_x \tilde{u}^{(0)} \Delta_x d^{(0)} + \tilde{\mu}^{(0)} - \eta' \theta'^{0} d^{(0)}
\]
\[= \theta' \Delta_x d^{(0)} + \tilde{\mu}^{(0)} - \eta' \theta'^{0} d^{(0)}.
\]
Due to (2.44) there holds
\[
\mu_{\pm}^{(0)}(x, t) = -\Delta_x d^{(0)} \int_{-\infty}^{+\infty} (\theta'(z))^{2} dz = \kappa \int_{-\infty}^{+\infty} (\theta'(z))^{2} dz
\]
\[= 2\kappa \int_{-\infty}^{+\infty} f(\theta(z)) dz = \kappa \int_{-1}^{1} \sqrt{2f(u)} du, \quad \text{on } \Gamma, \tag{2.45}
\]
here we have used \( \theta'(z) = \sqrt{2f(\theta(z))} \).

Furthermore, by (2.44) a smooth function \( l^{(0)} \) is defined as follows:
\[
l^{(0)} = \begin{cases} 
\frac{1}{d^{(0)} f^{(0)} - \eta' \theta' dz} \left( \Delta_x d^{(0)} \int_{-\infty}^{+\infty} (\theta'(z))^{2} dz + \int_{-\infty}^{+\infty} \tilde{\mu}^{(0)} \theta'(z) dz \right), & \text{in } \Gamma(\delta) \setminus \Gamma, \\
\frac{1}{d^{(0)} f^{(0)} - \eta' \theta' dz} \nabla_x d^{(0)} \cdot \nabla_x \left( \Delta_x d^{(0)} \int_{-\infty}^{+\infty} (\theta'(z))^{2} dz + \int_{-\infty}^{+\infty} \tilde{\mu}^{(0)} \theta'(z) dz \right), & \text{on } \Gamma.
\end{cases} \tag{2.46}
\]

2.3. Boundary layer expansion in \( \partial \Omega(\delta) \).

Let \( d_B(x) < 0 \) be the signed distance from \( x \in \partial \Omega \) and \( z = \frac{d_B(x)}{\epsilon} \in (-\infty, 0] \). In \( \partial \Omega(\delta) = \{ x \in \overline{\Omega} : -\delta < d_B(x) \leq 0 \} \), we set
\[
u^\varepsilon(x, t) = u_B^{(0)}(x, t, z) = u_B^{(0)}(x, t, z) + \varepsilon u_B^{(1)}(x, t, z) + \varepsilon^2 u_B^{(2)}(x, t, z) + \cdots, \tag{2.47}
\]
\[
\mu^\varepsilon(x, t) = \mu_B^{(0)}(x, t, z) = \mu_B^{(0)}(x, t, z) + \varepsilon \mu_B^{(1)}(x, t, z) + \varepsilon^2 \mu_B^{(2)}(x, t, z) + \cdots, \tag{2.48}
\]
\[
\sigma^\varepsilon(x, t) = \sigma_B^{(0)}(x, t, z) = \sigma_B^{(0)}(x, t, z) + \varepsilon \sigma_B^{(1)}(x, t, z) + \varepsilon^2 \sigma_B^{(2)}(x, t, z) + \cdots, \tag{2.49}
\]
and the following boundary-outter matching conditions: there exists a fixed positive constant \( \nu \) such that as \( \varepsilon \to -\infty \) there hold
\[
D_x^\alpha D_t^\beta D_z^\gamma \left( u_B^{(i)}(x, t, z) - u_+^{(i)}(x, t) \right) = O(e^{-\nu|z|}), \tag{2.50}
\]
\[
D_x^\alpha D_t^\beta D_z^\gamma \left( \mu_B^{(i)}(x, t, z) - \mu_+^{(i)}(x, t) \right) = O(e^{-\nu|z|}), \tag{2.51}
\]
\[
D_x^\alpha D_t^\beta D_z^\gamma \left( \sigma_B^{(i)}(x, t, z) - \sigma_+^{(i)}(x, t) \right) = O(e^{-\nu|z|}), \tag{2.52}
\]
for \((x, t) \in \partial \Omega(\delta) \times [0, T] \) and \( 0 \leq \alpha, \beta, \gamma \leq N \) with \( N \) depending on actual expansion order.

Using Taylor expansion and (2.47) one has
\[
f'(u_B^{(0)}) = f'(u_B^{(0)}) + \varepsilon f''(u_B^{(0)}) u_B^{(1)} + \cdots + \varepsilon^k \left( f'''(u_B^{(0)}) u_B^{(k)} + g_B(u_B^{(0)}, \ldots, u_B^{(k-1)}) \right) + \cdots,
\]
here the function \( g_B(u_B^{(k)} , \cdots , u_B^{(k-1)}) \) depends on \( u_B^{(0)}, \cdots , u_B^{(k-1)} \).

We write (1.1) in \( \partial \Omega(\delta) \) in the new coordinate \((x, t, z)\) as follows

\[
- \partial_{zz} \mu_B^\varepsilon - \varepsilon \left( 2 \nabla_x \partial_z \mu_B^\varepsilon \cdot \nabla_x d_B + \partial_z \mu_B^\varepsilon \Delta_x d_B \right) + \varepsilon^2 \left( \partial_t \mu_B^\varepsilon \right) - \Delta_x \mu_B^\varepsilon - \left( 2 \sigma_B^\varepsilon + u_B^\varepsilon - \mu_B^\varepsilon \right) = 0, \tag{2.53}
\]

\[
- \partial_{zz} \sigma_B^\varepsilon - \varepsilon \left( 2 \nabla_x \partial_z \sigma_B^\varepsilon \cdot \nabla_x d_B + \partial_z \sigma_B^\varepsilon \Delta_x d_B \right) + \varepsilon^2 \left( \partial_t \sigma_B^\varepsilon \right) - \Delta_x \sigma_B^\varepsilon + 2 \sigma_B^\varepsilon + u_B^\varepsilon - \mu_B^\varepsilon = 0, \tag{2.54}
\]

\[
- \partial_{zz} u_B^\varepsilon + f'(u_B^\varepsilon) - \varepsilon \left( 2 \nabla_x \partial_z u_B^\varepsilon \cdot \nabla_x d_B + \partial_z u_B^\varepsilon \Delta_x d_B + \mu_B^\varepsilon \right) - \varepsilon^2 \Delta_x u_B^\varepsilon = 0. \tag{2.55}
\]

Moreover, homogeneous Neumann boundary conditions in (1.1) imply on \( \partial \Omega \times [0, T] \)

\[
\partial_z \mu_B^\varepsilon(x, t, 0) + \varepsilon \nabla_x \mu_B^\varepsilon(x, t, 0) \cdot \nabla_x d_B(x, t) = 0, \tag{2.56}
\]

\[
\partial_z \sigma_B^\varepsilon(x, t, 0) + \varepsilon \nabla_x \sigma_B^\varepsilon(x, t, 0) \cdot \nabla_x d_B(x, t) = 0, \tag{2.57}
\]

\[
\partial_z u_B^\varepsilon(x, t, 0) + \varepsilon \nabla_x u_B^\varepsilon(x, t, 0) \cdot \nabla_x d_B(x, t) = 0. \tag{2.58}
\]

Firstly substituting (2.47)–(2.49) into (2.53)–(2.55) and (2.56)–(2.58) and collecting all terms of zero order we have

\[
- \partial_{zz} \mu_B^{(0)} = 0,
\]

\[
- \partial_{zz} \sigma_B^{(0)} = 0,
\]

\[
- \partial_{zz} u_B^{(0)} + f'(u_B^{(0)}) = 0,
\]

and on \( \partial \Omega \times [0, T] \)

\[
\partial_z \mu_B^{(0)}(x, t, 0) = 0,
\]

\[
\partial_z \sigma_B^{(0)}(x, t, 0) = 0,
\]

\[
\partial_z u_B^{(0)}(x, t, 0) = 0.
\]

Therefore we can take

\[
\mu_B^{(0)} = \mu_+^{(0)}, \quad \sigma_B^{(0)} = \sigma_+^{(0)}, \quad u_B^{(0)} = u_+^{(0)} = 1, \tag{2.59}
\]

which satisfy (2.50)–(2.52) in the case of \( i = 0 \).

Next substituting (2.47)–(2.49) into (2.53)–(2.55) and (2.56)–(2.58) and collecting all terms of \( \varepsilon \)-order we have

\[
- \partial_{zz} \mu_B^{(1)} = 0,
\]

\[
- \partial_{zz} \sigma_B^{(1)} = 0,
\]

\[
- \partial_{zz} u_B^{(1)} + f''(1)u_B^{(1)} = \mu_B^{(0)},
\]

and on \( \partial \Omega \times [0, T] \)

\[
\partial_z \mu_B^{(1)}(x, t, 0) = - \frac{\partial \mu_B^{(0)}}{\partial n},
\]

\[
\partial_z \sigma_B^{(1)}(x, t, 0) = - \frac{\partial \sigma_B^{(0)}}{\partial n},
\]

\[
\partial_z u_B^{(1)}(x, t, 0) = 0,
\]

here \( n \) is the unit outer normal to \( \partial \Omega \).
Therefore we can take
\[ \mu_B^{(1)} = \mu_+^{(1)}, \quad \sigma_B^{(1)} = \sigma_+^{(1)}, \quad u_B^{(1)} = u_+, \]
which satisfy (2.50)-(2.52) in the case of \( i = 1 \) and imply on \( \partial \Omega \times [0, T] \)
\[ \frac{\partial \mu_+^{(0)}}{\partial n} = \frac{\partial \sigma_+^{(0)}}{\partial n} = 0. \tag{2.60} \]

2.4. Solving the leading order terms.
\( u_+^{(0)} \) and \( \bar{u}^{(0)} \) are determined by (2.7) and (2.33) respectively. Collecting (2.28), (2.31), (2.39), (2.42), (2.45) and (2.60) one has
\[
\begin{cases}
-\Delta \mu_+^{(0)} + \mu_+^{(0)} = 2\sigma_+^{(0)} \pm 1, & \text{in } \Omega_+,

\partial_t \sigma_+^{(0)} - \Delta \sigma_+^{(0)} + 2\sigma_+^{(0)} = \mu_+^{(0)} \mp 1, & \text{in } \Omega_+,

[\mu_+^{(0)}] = [\sigma_+^{(0)}] = 0, & \text{on } \Gamma,

\frac{\partial \mu_+^{(0)}}{\partial n} = -2V, & \text{on } \Gamma,

\frac{\partial \sigma_+^{(0)}}{\partial n} = 0, & \text{on } \Gamma,

\mu_+^{(0)} = \kappa \int_{-1}^{1} \sqrt{2f(w)} dw, & \text{on } \Gamma,

\frac{\partial \mu_+^{(0)}}{\partial n} = \frac{\partial \sigma_+^{(0)}}{\partial n} = 0, & \text{on } \partial \Omega,
\end{cases}
\]
which is just the sharp interface model (1.2) of (1.1). Therefore we recover the sharp interface model by the above matched asymptotic expansion method.

Let (1.2) has a local smooth solution \((\mu, \sigma, \Gamma)\) which satisfies (1.6). Let \( \mu_+^{(0)} = \mu|_{\Omega_+}, \sigma_+^{(0)} = \sigma|_{\Omega_+} \) and \( d^{(0)} \) be the signed distance to \( \Gamma \). Then \( \Gamma = \{(x, t) \in \Omega \times (0, T) : d^{(0)}(x, t) = 0\} \) and \( \Omega_+ = \{(x, t) \in \Omega \times (0, T) : d^{(0)}(x, t) \geq 0\} \). \( p^{(0)}, q^{(0)}, g^{(0)}, h^{(0)} \) and \( l^{(0)} \) are determined by (2.31), (2.39), (2.40), (2.43) and (2.46) respectively. \( \bar{\mu}^{(0)} \) and \( \bar{\sigma}^{(0)} \) are determined by (2.27) and (2.28) respectively. And \( \mu_B^{(0)}, \sigma_B^{(0)}, u_B^{(0)} \) are determined by (2.59). Moreover, the inner-outer matching conditions (2.13)-(2.15) and the boundary-outer matching conditions (2.50)-(2.52) hold for \( i = 0 \).

**Remark 2.2.** We can extend \((u_+^{(0)}, \mu_+^{(0)}, \sigma_+^{(0)})\) smoothly from \( \Omega_+ \) to \( \Omega \) as in Remark 4.1 in [3].

3. Spectral condition and error estimates

For clarity we leave the higher order expansions and the construction of the approximate solution \((u^A, \mu^A, \sigma^A, \varphi^A)\) which is defined in (5.1) to the last two sections. In this section we firstly establish a spectral condition and give a smallness estimation of the error between the approximate solution and the true solution. Then Theorem 1.1 and Corollary 1.2 will be proved.

3.1. Spectral condition.

**Theorem 3.1.** (Spectral condition) There exist two positive constants \( \varepsilon_0 \) and \( C \) such that for any \( 0 < \varepsilon < \varepsilon_0, v \in H^1(\Omega) \) and \( w \in H^2(\Omega) \) with \( \Delta w = v \) there holds
\[
\int_{\Omega} \left( \varepsilon |\nabla v|^2 + \frac{1}{\varepsilon} f''(u^A) v^2 \right) dx \geq -C \int_{\Omega} |\nabla w|^2 dx, \tag{3.1}
\]
where \( u^A \) is an approximate solution of \( u^c \) which satisfies (1.3) and will be constructed in Section 5.
Thanks to Theorem 3.1 in [7] we only need to prove the following lemma.

**Lemma 3.2.** There exist two positive constants $\varepsilon_0$ and $C$ such that for any $0 < \varepsilon < \varepsilon_0$ and $v \in H^1(\Omega)$ there holds

$$\int_{\Omega} \left( |\nabla v|^2 + \frac{1}{\varepsilon^2} f''(u^A)v^2 \right) dx \geq -C \int_{\Omega} v^2 dx. \quad (3.2)$$

In fact, if

$$\int_{\Omega} \left( |\nabla v|^2 + \frac{1}{\varepsilon^2} f''(u^A)v^2 \right) dx \geq 0,$$

then (3.1) holds obviously. If

$$\int_{\Omega} \left( |\nabla v|^2 + \frac{1}{\varepsilon^2} f''(u^A)v^2 \right) dx \leq 0,$$

and $w \in H^2(\Omega)$ with $\Delta w = v$, then the assumptions of Theorem 3.1 in [7] hold, hence we have

$$-\varepsilon \int_{\Omega} v^2 dx \geq -C \int_{\Omega} |\nabla w|^2 dx$$

which and (3.2) immediately imply (3.1).

Theorem 3.1 and Lemma 3.2 have been used to prove the convergence of the Cahn-Hilliard equation to the Hele-Shaw model in [3]. However the proof of Lemma 3.2 were established in another paper [7]. Here we write the details in a concise way which will show clearly how to vanish the singularity.

In order to prove Lemma 3.2 we show the following proposition on the spectral analyses of Neumann eigenvalue problem (1.5). The proof has been given in [7] and proved by a new method in [14].

**Proposition 3.3.** (7) (1)(Estimate of the first eigenvalue of $L_f$)

$$\lambda_1^f \equiv \inf_{\|q\| = 1} \int_{I_{\varepsilon}} \left( (q')^2 + f''(\theta)q^2 \right) dz = O(e^{-\frac{C}{\varepsilon}}), \quad (3.3)$$

here $C$ is a positive constant independent of small $\varepsilon$ and $\|q\| = \left( \int_{I_{\varepsilon}} q^2(z)dz \right)^{\frac{1}{2}}$.

(2)(Estimate of the second eigenvalue of $L_f$)

$$\lambda_2^f \equiv \inf_{\|q\| = 1} \int_{I_{\varepsilon}} \left( (q')^2 + f''(\theta)q^2 \right) dz \geq c_f > 0, \quad (3.4)$$

here $q \perp q_1^f \iff \int_{I_{\varepsilon}} qq_1^f dz = 0$, $q_1^f$ is the normalized eigenfunction corresponding to $\lambda_1^f$ and $c_f$ is a positive constant independent of small $\varepsilon$.

(3)(Characterization of the first normalized eigenfunction of $L_f$)

$$\|q_1^f - \alpha \theta\|^2 = O(e^{-\frac{C}{\varepsilon}}), \quad (3.5)$$

here $\alpha = \frac{1}{\|\theta\|}$.

Let

$$d^{[k]}(x,t) = d^{(0)}(x,t) + \varepsilon d^{(1)}(x,t) + \varepsilon^2 d^{(2)}(x,t) + \cdots + \varepsilon^k d^{(k)}(x,t),$$

then $d^{[k]}(x,t)(k \geq 1)$ is a $k$-th approximate of the signed distance from $x$ to the interface $\Gamma^\varepsilon_k \equiv \{(x,t) : d^{[k]}(x,t) = 0\}$ in the following sense.

**Lemma 3.4.** For every fixed $t \in [0,T]$, let $r_t(x)$ be the signed distance from $x$ to $\Gamma^\varepsilon_k$. Then for small $\varepsilon$

$$\|r_t(x) - d^{[k]}(x,t)\|_{C^1(\Gamma^\varepsilon_t(\delta))} = O(\varepsilon^{k+1}). \quad (3.6)$$
Proof. Noting that \(|\nabla d|^{k}|^2 = 1 + O(\varepsilon^{k+1})\), then for small \(\varepsilon\) one gets

\[
|\nabla d|^{k}| - 1 = \frac{|\nabla d|^{k}|^2 - 1}{|\nabla d|^{k}| + 1} = O(\varepsilon^{k+1}).
\]

Since \(r_t(x)\) is the signed distance, then \(|\nabla r_t| = 1\) and \(\nabla r_t\) is parallel to \(\nabla d|^{k}|\). Consequently, we obtain

\[
|\nabla r_t(x) - \nabla d|^{k}|(x,t)|^2 = |\nabla r_t(x)|^2 - 2\nabla d|^{k}|(x,t) \cdot \nabla r_t(x) + |\nabla d|^{k}|(x,t)|^2 = 1 - 2|\nabla d|^{k}|(x,t)| + |\nabla d|^{k}|(x,t)|^2 = (1 - |\nabla d|^{k}|(x,t)|)^2 = O(\varepsilon^{2(k+1)}).
\]

Choosing \(x_0 \in \Gamma^\varepsilon_k\), i.e., \(r_t(x_0) = d|^{k}|(x_0, t) = 0\), then

\[
|\nabla r_t(x) - d|^{k}|(x,t)| = |\nabla r_t(x) - d|^{k}|(x,t) - r_t(x_0) + d|^{k}|(x_0, t)| = \left|\int_0^1 (\nabla r_t(t')x + (1 - t')x_0) - \nabla d|^{k}|((t'x + (1 - t')x_0), t) \right| \cdot (x - x_0)dt' \leq C\varepsilon^{k+1}.
\]

Hence we complete the proof of this lemma. \(\Box\)

Let \(s_t(x)\) be the projection of \(x\) on \(\Gamma^\varepsilon_k\) along the normal of \(\Gamma^\varepsilon_k\). Then the transformation \(x \mapsto (r_t(x), s_t(x))\) is a diffeomorphism in \(\Gamma^\varepsilon_k(\delta)\) for small \(\delta\). Let \(J(r_t, s_t) = \det \frac{\partial x^{-1}(r_t, s_t)}{\partial (r_t, s_t)}\) be the Jacobian of the transformation, then \(J|r^\varepsilon_k| = 1\) and \(\frac{\partial J}{\partial r_t}|r^\varepsilon_k| = 0\). Thus

\[
0 < C_1 \leq J(r_t, s_t) \leq C_2, \quad |J_{r_t}(r_t, s_t) | = \frac{\partial J}{\partial r_t}(r_t, s_t) \leq C |r_t|.
\] (3.7)

In view of (2.36) and the similar arguments in P199 in [3] we obtain

Lemma 3.5. In \(\Gamma(\delta)\), \(\overline{u}^{(1)}\) can be expressed as

\[
\overline{u}^{(1)}(x, t, z) \bigg|_{z = \frac{\theta_1(x)}{\varepsilon}} = \overline{p}(s_t(x))\theta_1\left(\frac{r_t(x)}{\varepsilon}\right) + \overline{q}(x) = \overline{p}(s_t(x))\theta_1(z) + \overline{q}(x),
\]

where \(\theta_1 \in L^\infty(\mathbb{R})\), \(\overline{p} \in L^\infty(\Gamma(\delta))\) and

\[
\int_{-\infty}^{+\infty} f''(\theta(z))\theta_1(z)(\theta'(z))^2 dz = 0, \quad |\overline{q}(x)| \leq C(\varepsilon + |r_t(x)|) \leq C\varepsilon(1 + |z|).
\]

Now we focus on the proof of Lemma 3.2. \(\Box\)

Proof of Lemma 3.2. For clarity we divide into three steps to proceed.
Step 1. Noting that \( f''(\pm 1) > 0 \), then for small \( \varepsilon \) there holds

\[
\int_{\Omega} \left( |\nabla v|^2 + \frac{1}{\varepsilon^2} f''(uA) v^2 \right) dx \\
\geq \int_{\Gamma_\varepsilon^\nu(\delta)} \left( |\nabla v|^2 + \frac{1}{\varepsilon^2} f''(uA) v^2 \right) dx \\
= \varepsilon^{-2} \int_{\Gamma_\varepsilon^\nu} \int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \left( |\partial_z v|^2 + f''(uA) v^2 \right) J(r_\varepsilon(x), s_\varepsilon(x)) d\varepsilon dz \\
\geq \varepsilon^{-2} \int_{\Gamma_\varepsilon^\nu} \int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \left( |\partial_z v|^2 + f''(\theta(z)) v^2 \right) J(r_\varepsilon(x), s_\varepsilon(x)) d\varepsilon dz \\
+ \varepsilon^{-1} \int_{\Gamma_\varepsilon^\nu} \int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} f'''(\theta(z)) \bar{u}^{(1)}(x, t, z) v^2 J(r_\varepsilon(x), s_\varepsilon(x)) d\varepsilon dz - C \int_{\Omega} v^2 dx,
\]

where we have used the fact that in \( \Gamma_\varepsilon^\nu(\delta) \)

\[
f''(uA) = f'' \left( \theta(z) + \varepsilon \bar{u}^{(1)}(x, t, z) + O(\varepsilon^2) \right) \big|_{z = \frac{d|k|}{\varepsilon}} \\
= f''(\theta(z)) + \varepsilon f'''(\theta(z)) \bar{u}^{(1)}(x, t, z) + O(\varepsilon^2) \big|_{z = \frac{d|k|}{\varepsilon}} \\
= f''(\theta(z)) + \varepsilon f'''(\theta(z)) \bar{u}^{(1)}(x, t, z) + O(\varepsilon^2) \big|_{z = \frac{r_\varepsilon(x)}{\varepsilon}}.
\]

Set \( \hat{v} = v J_\varepsilon^\frac{1}{2} \), from Lemma 5.8 in [14] (we show the proof in Appendix for completeness of this paper) one gets

\[
\int_{\Gamma_\varepsilon^\nu} \left( |\partial_z \hat{v}|^2 + f''(\theta(z)) \hat{v}^2 \right) J(r_\varepsilon(x), s_\varepsilon(x)) d\varepsilon dz \\
\geq \frac{3}{4} \int_{\Gamma_\varepsilon^\nu} \int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \left( |\partial_z \hat{v}|^2 + f''(\theta(z)) \hat{v}^2 \right) d\varepsilon dz - C\varepsilon^2 \int_{\Omega} v^2 dx. \tag{3.8}
\]

Consequently

\[
\int_{\Omega} \left( |\nabla v|^2 + \frac{1}{\varepsilon^2} f''(uA) v^2 \right) dx \geq \varepsilon^{-2} \int_{\Gamma_\varepsilon^\nu} IdS + \varepsilon^{-1} \int_{\Gamma_\varepsilon^\nu} II dS - C \int_{\Omega} v^2 dx, \tag{3.9}
\]

where

\[
I = \frac{3}{4} \int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \left( |\partial_z \hat{v}|^2 + f''(\theta(z)) \hat{v}^2 \right) dz, \quad II = \int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} f'''(\theta(z)) \bar{u}^{(1)}(x, t, z) \hat{v}^2 dz.
\]
Step 2. To deal with term $I$, we decompose $\dot{v} = \gamma q_1^f + p_1$, here $p_1 \perp q_1^f$ and then $\|\dot{v}\|^2 = \gamma^2 + \|p_1\|^2$. Then

$$I = \frac{3}{4} \int_{-\delta}^{\delta} \left( |\partial_z \dot{v}|^2 + f''(\theta(z)) \dot{v}^2 \right) dz$$

$$= \frac{3}{4} \gamma^2 \int_{-\delta}^{\delta} \left( (q_1^f)' \right)^2 + f''(\theta(z)) (q_1^f)^2 \right) dz + \frac{3}{4} \int_{-\delta}^{\delta} \left( (\partial_z p_1) \right)^2 + f''(\theta(z)) p_1^2 \right) dz$$

$$\geq \frac{3}{4} \gamma^2 \lambda_1^f + \frac{3}{4} \lambda_2^f \|p_1\|^2 \geq -C \varepsilon^2 \gamma^2 + \frac{3}{4} \lambda_2^f \|p_1\|^2$$

$$\geq -C \varepsilon^2 \int_{-\delta}^{\delta} v^2 J dz + \frac{3}{4} \lambda_2^f \|p_1\|^2,$$

where we have used (3.3) and (3.4).

Step 3. To estimate $II$, we use Lemma 3.5 and then

$$\int_{-\delta}^{\delta} f'''(\theta(z)) \tilde{u}^{(1)}(x, t, z)(\theta'(z))^2 dz$$

$$= \tilde{p}(s_t(x)) \int_{-\delta}^{\delta} f'''(\theta(z)) \theta_1(z)(\theta'(z))^2 dz + \int_{-\delta}^{\delta} f'''(\theta(z)) \tilde{q}(x)(\theta'(z))^2 dz$$

$$= -\tilde{p}(s_t(x)) \int_{-\infty}^{-\delta} f'''(\theta(z)) \theta_1(z)(\theta'(z))^2 dz - \tilde{p}(s_t(x)) \int_{\delta}^{+\infty} f'''(\theta(z)) \theta_1(z)(\theta'(z))^2 dz$$

$$+ \int_{-\delta}^{\delta} f'''(\theta(z)) \tilde{q}(x)(\theta'(z))^2 dz$$

$$= O(\varepsilon).$$

Hence we find

$$II = \int_{-\delta}^{\delta} f'''(\theta(z)) \tilde{u}^{(1)}(x, t, z) \dot{v}^2 dz dS$$

$$= \gamma^2 \int_{-\delta}^{\delta} f'''(\theta(z)) (q_1^f)'^2 dz + 2\gamma \int_{-\delta}^{\delta} f'''(\theta(z)) \tilde{u}^{(1)}(x, t, z) q_1^f p_1 dz$$

$$+ \int_{-\delta}^{\delta} f'''(\theta(z)) \tilde{u}^{(1)}(x, t, z) p_1^2 dz$$

$$= \gamma^2 \alpha^2 \int_{-\delta}^{\delta} f'''(\theta(z)) \tilde{u}^{(1)}(\theta')^2 dz + O(\varepsilon^2) \gamma^2 + O(1) \|p_1\| \gamma + O(1) \|p_1\|^2$$

$$= O(\varepsilon) \gamma^2 + O(\varepsilon^2) \gamma^2 + O(1) \|p_1\| \gamma + O(1) \|p_1\|^2$$

$$\geq (O(\varepsilon^2) + O(\varepsilon)) \gamma^2 - \frac{1}{8\varepsilon} \lambda_2^f \|p_1\|^2$$

$$\geq -C \varepsilon \gamma^2 - \frac{1}{8\varepsilon} \lambda_2^f \|p_1\|^2$$

$$\geq -C \varepsilon \int_{-\delta}^{\delta} v^2 J dz - \frac{1}{8\varepsilon} \lambda_2^f \|p_1\|^2,$$
Therefore we have
\[ \varepsilon^{-2} \int_{\Gamma_k} \text{Id}S + \varepsilon^{-1} \int_{\Gamma_k} IIdS \geq -C \int_{\Omega} v^2 dx \]

which and (3.9) imply the desired conclusion. Thus we complete the proof of the lemma.

### 3.2. Error estimates

Let \( u^{err} = u^\varepsilon - u^A, \mu^{err} = \mu^\varepsilon - \mu^A, \sigma^{err} = \sigma^\varepsilon - \sigma^A, \varphi^{err} = \varphi^\varepsilon - \varphi^A \) with \( \varphi^\varepsilon = u^\varepsilon + \sigma^\varepsilon \) and impose
\[ u_0^\varepsilon(x) = \varphi^A(x, 0) - \sigma_0^A(x), \quad \sigma_0^A(x) = \sigma^A(x, 0). \quad (3.10) \]

Here \((u^A, \mu^A, \sigma^A, \varphi^A)\) is defined in (5.1).

By (1.1) and (5.2) there hold
\[
\begin{aligned}
\partial_t \varphi^{err} - \Delta \mu^{err} - \Delta \sigma^{err} &= 0, & \Omega \times (0, T), \\
\partial_t \sigma^{err} - \Delta \sigma^{err} &= -(2\sigma^{err} + u^{err} - \mu^{err}), & \Omega \times (0, T), \\
\mu^{err} &= -\varepsilon \Delta u^{err} + \frac{1}{\varepsilon} f''(u^A)u^{err} + \frac{1}{\varepsilon} \varphi - \omega^A, & \Omega \times (0, T), \\
u^{err} &= \varphi^{err} - \sigma^{err} + \omega^A, & \Omega \times (0, T), \\
\varphi^{err}(x, 0) &= \sigma^{err}(x, 0) = 0, & \Omega \times \{0\}, \\
\frac{\partial \varphi^{err}}{\partial n} &= \frac{\partial u^{err}}{\partial n} = \frac{\partial \sigma^{err}}{\partial n} = 0, & \partial \Omega \times (0, T),
\end{aligned}
\]

where
\[ \mathcal{F} = f'(u^{err} + u^A) - f'(u^A) - f''(u^A)u^{err} = 4(u^{err})^3 + 8u^A(u^{err})^2 \]

and
\[ \omega^A = -\frac{1}{|\Omega|} \int_{\Omega}^{t} \int_{\Omega} (\omega_1^A + \omega_2^A)(x, t')dxdt' + \omega_2^A + \tilde{\mu}^A = O(\varepsilon^{k-1}). \]

**Theorem 3.6.** For small \( \varepsilon \) and large \( k \), there exist \( \gamma = \gamma(k) \in (1, k) \) which is an increasing function of \( k \) and a positive constant \( C \) such that
\[
\| u^{err} \|_{L^p(\Omega \times (0, T))} + \| \varphi^{err} \|_{L^p(\Omega \times (0, T))} + \| \sigma^{err} \|_{L^p(\Omega \times (0, T))} \leq C \varepsilon^\gamma.
\]

**Proof.** For the sake of clarity we omit the superscript “err” in the proof and only in this proof. Noting that
\[ \int_{\Omega} \varphi(t, x)dx = \int_{0}^{t} \int_{\Omega} \partial_t \varphi(t, x)dx = \int_{0}^{t} \int_{\Omega} (\Delta \mu + \Delta \sigma)(t, x)dx = 0, \]
then there exists a function \( \psi(t, \cdot)(t \in (0, T)) \) which satisfies
\[
\begin{aligned}
-\Delta \psi &= \varphi, & \Omega, \\
\frac{\partial \psi}{\partial n} &= 0, & \partial \Omega, \\
\int_{\Omega} \psi(t, x)dx &= 0, & t \in (0, T),
\end{aligned}
\]

And let \( \varphi(t, \cdot)(t \in (0, T)) \) be the solution of the following equation
\[
\begin{aligned}
-\Delta \varphi &= \sigma, & \Omega, \\
\varphi &= 0, & \partial \Omega.
\end{aligned}
\]

Theorem 3.6 is proved.

**Remark.** The error estimates (3.12) and (3.13) are based on the following fact:
\[ \int_{\Omega} \varphi(t, x)dx = \int_{0}^{t} \int_{\Omega} \partial_t \varphi(t, x)dx = \int_{0}^{t} \int_{\Omega} (\Delta \mu + \Delta \sigma)(t, x)dx = 0, \]
which is valid for any large \( k \) and \( \varepsilon \) small enough.
Multiplying the first equation in (3.11) by $\psi$ and integrating by parts we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \psi|^2 \, dx + \int_{\Omega} \left( \epsilon |\nabla \varphi|^2 + \frac{1}{\epsilon} f''(u^A) \varphi^2 \right) \, dx
- \int_{\Omega} \left( \epsilon \nabla \varphi \nabla \sigma + \frac{1}{\epsilon} f''(u^A) \varphi \sigma \right) \, dx + \frac{1}{\epsilon} \int_{\Omega} F \varphi \, dx + \int_{\Omega} \nabla \psi \nabla \sigma \, dx
= \int_{\Omega} \omega_6^A \varphi \, dx,
\]
where
\[
\omega_6^A = \omega_4^A + \epsilon \Delta \omega_5^A - \epsilon^{-1} f''(u^A) \omega_5^A = O(\epsilon^{k-2}).
\]

Multiplying the second equation in (3.11) by $\sigma$ and integrating by parts we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \sigma^2 \, dx + \int_{\Omega} |\nabla \sigma|^2 \, dx + \int_{\Omega} F \sigma \, dx + \int_{\Omega} \varphi \sigma \, dx + \frac{1}{\epsilon} \int_{\Omega} F \sigma \, dx + \int_{\Omega} \nabla \psi \nabla \sigma \, dx
= \int_{\Omega} \omega_7^A \sigma \, dx,
\]
where
\[
\omega_7^A = -\omega_4^A - \omega_5^A - \epsilon \Delta \omega_5^A + \epsilon^{-1} f''(u^A) \omega_5^A = O(\epsilon^{k-2}).
\]

Combining (3.14) and (3.15) we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \psi|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \sigma^2 \, dx + \int_{\Omega} |\nabla \sigma|^2 \, dx + \int_{\Omega} \varphi^2 \, dx + \int_{\Omega} \sigma^2 \, dx
+ \int_{\Omega} \left( \epsilon |\nabla (\varphi - \sigma)|^2 + \frac{1}{\epsilon} f''(u^A) (\varphi - \sigma)^2 \right) \, dx
+ 2 \int_{\Omega} \nabla \psi \nabla \sigma \, dx + \int_{\Omega} \nabla \varphi \nabla \sigma \, dx + \int_{\Omega} (\varphi - \sigma) F \, dx
= \int_{\Omega} \omega_6^A \varphi \, dx + \int_{\Omega} \omega_7^A \sigma \, dx.
\]

Moreover, we can easily find
\[
(\varphi - \sigma) F = 4 (\varphi - \sigma)^4 + (8 u^A + 24 \omega_5^A) (\varphi - \sigma)^3 + (16 u^A \omega_5^A + 12 (\omega_5^A)^2) (\varphi - \sigma)^2
+ (8 u^A (\omega_5^A)^2 + 4 (\omega_5^A)^3) (\varphi - \sigma)
\geq -\tilde{C}_p |\varphi - \sigma|^4 + \omega_8^A |\varphi - \sigma|,
\]
\[
\geq -C_p |\varphi|^p - C_p |\sigma|^p + \omega_8^A |\varphi - \sigma|, \quad \forall p \in (1, 3],
\]
where the positive constants $\tilde{C}_p, C_p$ depend on $p$ and $\omega_8^A = O(\epsilon^{k-1})$. 

\[3.17\]
Plugging (3.17) into (3.16), using the Young’s inequality and the Sobolev inequality we arrive at

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \psi|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \sigma^2 dx + \frac{1}{2} \int_{\Omega} |\nabla \sigma|^2 dx + \int_{\Omega} \sigma^2 dx \\
+ \int_{\Omega} \left( \varepsilon |\nabla (\varphi - \sigma)|^2 + \frac{1}{\varepsilon} f''(u^A)(\varphi - \sigma)^2 \right) dx \\
\leq C \int_{\Omega} |\nabla \psi|^2 dx + C \int_{\Omega} \sigma^2 dx + \frac{C_p}{\varepsilon} \int_{\Omega} |\varphi|^p dx + \frac{C_p}{\varepsilon} \int_{\Omega} |\sigma|^p dx \\
+ \left( \int_{\Omega} |\varphi|^p dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |\omega_0^A|^q dx \right)^{\frac{1}{q}} \\
+ \left( \int_{\Omega} |\sigma|^p dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |\omega_0^{10}|^q dx \right)^{\frac{1}{q}},
\]  

(3.18)

where \( \omega_0^A = O(\varepsilon^{k-2}) \), \( \omega_0^{10} = O(\varepsilon^{k-2}) \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

According to the spectral condition Theorem 3.1 one has for small \( \varepsilon \)

\[
\int_{\Omega} \left( \varepsilon |\nabla (\varphi - \sigma)|^2 + \frac{1}{\varepsilon} f''(u^A)(\varphi - \sigma)^2 \right) dx \geq -C \int_{\Omega} |\nabla (\psi - \varphi)|^2 dx,
\]  

(3.19)

where \( C \) is a positive constant independent of \( t \).

By the Poincaré inequality and (3.13) one get

\[
\int_{\Omega} |\nabla \varphi|^2 dx \leq C \int_{\Omega} \sigma^2 dx.
\]  

(3.20)

Plugging (3.19)-(3.20) into (3.18) we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \psi|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \sigma^2 dx + \frac{1}{2} \int_{\Omega} |\nabla \sigma|^2 dx + \int_{\Omega} \sigma^2 dx \\
\leq C \int_{\Omega} |\nabla \psi|^2 dx + C \int_{\Omega} \sigma^2 dx + \frac{C_p}{\varepsilon} \int_{\Omega} |\varphi|^p dx + \frac{C_p}{\varepsilon} \int_{\Omega} |\sigma|^p dx \\
+ \left( \int_{\Omega} |\varphi|^p dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |\omega_0^A|^q dx \right)^{\frac{1}{q}} \\
+ \left( \int_{\Omega} |\sigma|^p dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |\omega_0^{10}|^q dx \right)^{\frac{1}{q}}.
\]

It follows from the Gronwall inequality that for any \( t \in (0, T] \)

\[
\sup_{0 \leq t' \leq t} \left( \|\nabla \psi\|^2_{L^2(\Omega)} + \|\sigma\|^2_{L^2(\Omega)} + \|\nabla \sigma\|^2_{L^2(\Omega \times (0, t))} + \|\sigma\|^2_{L^2(\Omega \times (0, t))} \right) \\
\leq C \left( \varepsilon^{-1} \|\varphi\|^p_{L^p(\Omega \times (0, t))} + \varepsilon^{-1} \|\sigma\|^p_{L^p(\Omega \times (0, t))} + \|\varphi\|_{L^p(\Omega \times (0, t))} \|\omega_0^A\|_{L^q(\Omega \times (0, t))} \right. \\
\left. + \|\sigma\|_{L^p(\Omega \times (0, t))} \|\omega_0^{10}\|_{L^q(\Omega \times (0, t))} \right).
\]  

(3.21)
Furthermore, by (3.18) and (3.21) one gets

\[ \| \nabla (\varphi - \sigma) \|_{L^2(\Omega \times (0,t))}^2 \leq \varepsilon^{-2} \left( - \int_0^t \int_\Omega f''(u^A)(\varphi - \sigma)^2 dx dt' \right) + C \varepsilon^{-2} \left( \| \varphi \|_{L^p(\Omega \times (0,t))}^p + \| \sigma \|_{L^p(\Omega \times (0,t))}^p \right) \]

\[ + C \varepsilon^{-1} \left( \| \varphi \|_{L^p(\Omega \times (0,t))} \| \omega_A^0 \|_{L^q(\Omega \times (0,t))} + \| \sigma \|_{L^p(\Omega \times (0,t))} \| \omega_A^1 \|_{L^q(\Omega \times (0,t))} \right) \]

\[ \leq \varepsilon^{-2} \left( \| \varphi \|_{L^p(\Omega \times (0,t))}^2 \| \nabla \sigma \|_{L^2(\Omega \times (0,t))} + \| \sigma \|_{L^p(\Omega \times (0,t))} \right) \]

\[ + C \varepsilon^{-2} \left( \| \varphi \|_{L^p(\Omega \times (0,t))}^2 \| \nabla \sigma \|_{L^2(\Omega \times (0,t))} \right) \]

\[ \leq \varepsilon^{-2} \left( \| \varphi \|_{L^p(\Omega \times (0,t))}^2 \| \nabla \sigma \|_{L^2(\Omega \times (0,t))} \right) \]

\[ + C \varepsilon^{-1} \left( \| \varphi \|_{L^p(\Omega \times (0,t))} \| \omega_A^0 \|_{L^q(\Omega \times (0,t))} + \| \sigma \|_{L^p(\Omega \times (0,t))} \| \omega_A^1 \|_{L^q(\Omega \times (0,t))} \right) \].

Thus

\[ \| \nabla \varphi \|_{L^2(\Omega \times (0,t))}^2 \leq \| \nabla (\varphi - \sigma) \|_{L^2(\Omega \times (0,t))}^2 + \| \nabla \sigma \|_{L^2(\Omega \times (0,t))}^2 \]

\[ \leq C \varepsilon^{-1-2/p} \left( \| \varphi \|_{L^p(\Omega \times (0,t))}^2 + \| \sigma \|_{L^p(\Omega \times (0,t))}^2 \right) \]

\[ + C \varepsilon^{-2} \left( \| \varphi \|_{L^p(\Omega \times (0,t))} + \| \sigma \|_{L^p(\Omega \times (0,t))} \right) \]

\[ + C \varepsilon^{-1} \left( \| \varphi \|_{L^p(\Omega \times (0,t))} \| \omega_A^0 \|_{L^q(\Omega \times (0,t))} + \| \sigma \|_{L^p(\Omega \times (0,t))} \| \omega_A^1 \|_{L^q(\Omega \times (0,t))} \right) \].

Applying the Sobolev imbedding theorem and the Hölder inequality we get

\[ \| \varphi \|_{L^p(\Omega \times (0,t))} + \| \sigma \|_{L^p(\Omega \times (0,t))} \]

\[ = \int_0^t \left( \| \varphi \|_{L^p(\Omega)} + \| \sigma \|_{L^p(\Omega)} \right) (t') dt' \]

\[ \leq C \int_0^t \| \varphi \|_{L^p(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)} (t') dt' + C \int_0^t \| \sigma \|_{L^p(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)} (t') dt' \]

\[ + C \int_0^t \| \sigma \|_{L^p(\Omega)} (t') dt' \]

\[ \leq C \int_0^t \| \nabla \psi \|_{L^2(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)} (t') dt' + C \int_0^t \| \sigma \|_{L^p(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)} (t') dt' \]

\[ + C \int_0^t \| \sigma \|_{L^p(\Omega)} (t') dt' \]

\[ \leq C \left( \int_0^t \| \nabla \psi \|_{L^2(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)} (t') dt' \right)^{1/2} \| \nabla \varphi \|_{L^2(\Omega \times (0,t))} + C \left( \int_0^t \| \sigma \|_{L^p(\Omega)} (t') dt' \right)^{1/2} \| \nabla \varphi \|_{L^2(\Omega \times (0,t))} \]

\[ + C \int_0^t \| \sigma \|_{L^p(\Omega)} (t') dt' \]

\[ \leq C \left( \sup_{0 \leq t' \leq t} \| \nabla \psi \|_{L^2(\Omega)} \right)^{1/2} \| \nabla \varphi \|_{L^2(\Omega \times (0,t))} + C \left( \sup_{0 \leq t' \leq t} \| \sigma \|_{L^p(\Omega)} \right)^{1/2} \| \nabla \varphi \|_{L^2(\Omega \times (0,t))} \]

\[ + C T \left( \sup_{0 \leq t' \leq t} \| \sigma \|_{L^p(\Omega)} \right)^p \]

\[ , \quad (3.22) \]
where \( \frac{1}{p} = \frac{\theta}{2} + \frac{(N-2)(1-\theta)}{2N} \), \( \frac{1}{\alpha} + \frac{(1-\theta)p}{2} = 1 \) and \( \frac{1}{\beta} + \frac{(1-\theta)p}{2} = 1 \).

Setting \( \Theta(t) = ||\varphi||_{L^p(\Omega \times (0,t))} + ||\sigma||_{L^p(\Omega \times (0,t))} \) \( (t \in (0,T)) \), then with the help of \((3.21) - (3.22)\), we derive a recursive inequality for \( \Theta(t) \):

\[
\begin{align*}
\Theta^p(t) & \leq \left( \varepsilon^{-1}\Theta^p(t) + \Theta(t)||\omega^A_{11}||_{L^q(\Omega \times (0,t))} \right)^{\frac{\theta p}{4}} \left( \varepsilon^{-1-\frac{2}{q}}\Theta^2(t) \right) \\
& \quad + \varepsilon^{-2}\Theta^p(t) + \varepsilon^{-1}\Theta(t)||\omega^A_{11}||_{L^q(\Omega \times (0,t))} \left( \frac{1}{2} - \frac{\theta}{4} \right) p \\
& \quad + C \left( \varepsilon^{-1}\Theta^p(t) + \Theta(t)||\omega^A_{11}||_{L^q(\Omega \times (0,t))} \right)^{\frac{p}{2}} \\
& \quad \quad (3.23)
\end{align*}
\]

where \( \omega^A_{11} = O(\varepsilon^{k-2}) \).

Noting that if we fix \( p \in (2, 3) \), then

\[
\frac{\theta p}{4} + 2\left( \frac{1}{2} - \frac{\theta}{4} \right) = 1 + \frac{\theta(p-2)}{4} > 1, \quad \frac{p}{2} > 1.
\]

According to the continuous argument, for small \( \varepsilon \) and large \( k \) there exits \( \gamma = \gamma(k) \in (1,k) \) which tends to \(+\infty\) as \( k \to +\infty \) such that

\[ \Theta(T) \leq \varepsilon^\gamma. \]

Hence

\[ ||\varphi||_{L^p(\Omega \times (0,T))} + ||\sigma||_{L^p(\Omega \times (0,T))} \leq \varepsilon^\gamma \]

and then

\[ ||u||_{L^p(\Omega \times (0,T))} = ||\varphi - \sigma + \omega^A_5||_{L^p(\Omega \times (0,T))} \]

\[ \leq ||\varphi||_{L^p(\Omega \times (0,T))} + ||\sigma||_{L^p(\Omega \times (0,T))} + ||\omega^A_5||_{L^p(\Omega \times (0,T))} \]

\[ \leq C \varepsilon^\gamma. \]

Therefore the proof of the theorem is completed. \( \square \)

In order to establish higher order regularity estimates of \( u^{err}, \varphi^{err} \) and \( \sigma^{err} \), we consider the equations for \( (u^{err}, \varphi^{err}) \)

\[
\begin{align*}
\partial_t u^{err} - \Delta \mu^{err} + \mu^{err} &= 2\sigma^{err} + u^{err} + \partial_t \omega^A_5, \quad \text{in } \Omega \times (0,T), \\
\partial_t \sigma^{err} - \Delta \sigma^{err} + 2\sigma^{err} &= \mu^{err} - u^{err}, \quad \text{in } \Omega \times (0,T),
\end{align*}
\]

which is a Cahn-Hilliard equation coupled linearly with a heat equation. Using the similar arguments in Theorem 2.3 in [3] and the boot-strap method we can give the desired conclusions. Here we omit the detailed argument. In particular we have

**Theorem 3.7.** For small \( \varepsilon \) and large \( k \),

\[ ||u^{err}||_{C^{4,1}(\overline{\Omega} \times [0,T])} + ||\varphi^{err}||_{C^{2,1}(\overline{\Omega} \times [0,T])} + ||\sigma^{err}||_{C^{1,1}(\overline{\Omega} \times [0,T])} + ||\mu^{err}||_{C^{1,1}(\overline{\Omega} \times [0,T])} \leq C \varepsilon. \]

Accordingly Theorem 1.1 can be obtained with the aid of Theorem 3.7 if we take 3.10. Next we prove Corollary 1.2

**Proof.** Recalling the definition of \( \omega^A \) in Section 5, we easily obtain (1.7). Now we prove (1.8). More concretely we only prove

\[ ||\sigma^\varepsilon - \sigma||_{C(\overline{\Omega} \times [0,T])} \to 0, \quad \text{as } \varepsilon \to 0. \]

The other one in (1.8) is similar.
The definition of \( \sigma^A \) in Section 5 yields

\[
\sigma^A = \begin{cases} 
\sigma^A_+ \bigg|_{\Omega_+ \setminus \Gamma(\delta)}, \\
\sigma^A_+ \big( \frac{d^0}{\delta} \big) + \sigma^A_+ \big( 1 - \zeta \big( \frac{d^0}{\delta} \big) \big), & \text{in } (\Gamma(\delta) \setminus \Gamma(\delta/2)) \cap \Omega_+,
\end{cases}
\]

\[
\text{the leading order of } \sigma^A = \begin{cases} 
\sigma^0_+ \bigg|_{\Gamma(\delta/2)}, \\
\sigma^0_+ \big( \frac{d^0}{\delta} \big) + \sigma^0_+ \big( 1 - \zeta \big( \frac{d^0}{\delta} \big) \big), & \text{in } \Omega_+ \setminus \Gamma(\delta),
\end{cases}
\]

\[
\text{Moreover, } \eta \cdot \sigma^0_+ \big( x, t \big) + \big( 1 - \eta \big) \sigma^0_- \big( x, t \big) \text{ in } \Omega_+ \setminus \Gamma(\delta),
\]

\[
\text{in } \Gamma(\delta),
\]

\[
\text{in } \Omega_+ \setminus \Gamma(\delta),
\]

\[
\text{in } \Omega_+ \setminus \Gamma(\delta).
\]

Based on the inner-outer matching condition we find

\[
\left\| \left( \tilde{\sigma}^0 \big( \frac{d^0}{\delta} \big) + \sigma^0_\pm \big( 1 - \zeta \big( \frac{d^0}{\delta} \big) \big) \right) - \sigma^0_\pm \right\|_{C( (\Gamma(\delta) \setminus \Gamma(\delta/2)) \cap \Omega_\pm )} \leq C \varepsilon \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.
\]

Consequently we only need to prove

\[
\left\| \tilde{\sigma}^0 - \sigma \right\|_{C(\Gamma(\delta/2))} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.
\]

Note that

\[
\tilde{\sigma}^0(x, t, z) = \begin{cases} 
\sigma^0_+(x, t), & \text{in } \{(x, t) : d^k(x, t) \geq \varepsilon\},
\eta(z) \sigma^0_+(x, t) + \big( 1 - \eta(z) \big) \sigma^0_-(x, t), & \text{in } \{(x, t) : -\varepsilon < d^k(x, t) < \varepsilon\},
\sigma^0_-(x, t), & \text{in } \{(x, t) : d^k(x, t) \leq -\varepsilon\},
\end{cases}
\]

and

\[
\sigma(x, t) = \begin{cases} 
\sigma^0_+(x, t), & \text{in } \Omega_+,
\kappa \int_{-1}^{1} \sqrt{2f(u)} \, du, & \text{on } \Gamma,
\sigma^0_-(x, t), & \text{in } \Omega_-.
\end{cases}
\]

By mean value theorem we derive in \( \Gamma(\delta/2) \)

\[
\left\| \sigma^0_+(x, t) - \kappa \int_{-1}^{1} \sqrt{2f(u)} \, du \right\|, \left\| \sigma^0_-(x, t) - \kappa \int_{-1}^{1} \sqrt{2f(u)} \, du \right\| \leq C \left| d^0(x, t) \right|.
\]

Moreover,

\[
\{(x, t) : d^k(x, t) \geq \varepsilon\} \cap \Omega_- \subseteq \{(x, t) : |d^0(x, t)| \leq C \varepsilon\},
\]

\[
\{(x, t) : -\varepsilon < d^k(x, t) < \varepsilon\} \subseteq \{(x, t) : |d^0(x, t)| \leq C \varepsilon\},
\]

\[
\{(x, t) : d^k(x, t) \leq -\varepsilon\} \cap \Omega_+ \subseteq \{(x, t) : |d^0(x, t)| \leq C \varepsilon\}.
\]

Then there holds

\[
\left\| \tilde{\sigma}^0 - \sigma \right\|_{C(\Gamma(\delta/2))} \leq C \varepsilon,
\]

which implies the desired result.

The proof of Corollary 1.2 is completed. \( \square \)

4. Matching \( \varepsilon^k (k \geq 2) \)-order expansions

4.1. Matching \( k \)th order \( (k \geq 2) \) outer expansion.
Substituting (2.4)-(2.6) into (1.1) and collecting all terms of $\varepsilon^k$-order ($k \geq 2$) we have

$$u^{(k)}_\pm = \mu^{(k-1)}_\pm + \Delta u^{(k-2)}_\pm - g(u^{(0)}_\pm, \ldots, u^{(k-1)}_\pm), \quad (4.1)$$

$$-\Delta \mu^{(k)}_\pm + \mu^{(k)}_\pm = 2\sigma^{(k)}_\pm + u^{(k)}_\pm - \partial_t u^{(k)}_\pm, \quad (4.2)$$

$$\partial_t \sigma^{(k)}_\pm - \Delta \sigma^{(k)}_\pm + 2\sigma^{(k)}_\pm = \mu^{(k)}_\pm - u^{(k)}_\pm. \quad (4.3)$$

### 4.2 Matching $k$th order ($k \geq 2$) inner expansion.

Substituting (2.1) and (2.10)-(2.12) into (2.20)-(2.22) and collecting all terms of $\varepsilon^k$-order we have

$$-\partial_{zz} \left( \tilde{\mu}^{(k)} - \eta(p^{(k)}d^{(0)} + p^{(0)}d^{(k)}) \right)$$

$$= -\sum_{i=0}^{k-1} \left( \partial_z \tilde{\mu}^{(i)} \partial_t d^{(k-1-i)} - 2\nabla_x \partial_z \tilde{\mu}^{(i)} \cdot \nabla_x d^{(k-1-i)} - \partial_z \tilde{\mu}^{(i)} \Delta_x d^{(k-1-i)} \right)$$

$$- \left( \partial_z \tilde{\mu}^{(k-2)} - \Delta_x \tilde{\mu}^{(k-2)} - (2\tilde{\sigma}^{(k-2)} + \tilde{u}^{(k-2)} - \tilde{\mu}^{(k-2)}) \right)$$

$$- \eta'' \sum_{i=1}^{k-1} p^{(i)}d^{(k-i)} + \eta'' zp^{(k-1)} - \eta'' \sum_{i=0}^{k-1} g^{(i)}d^{(k-1-i)} + z\eta' g^{(k-2)}$$

$$+ \left( s^{(k-2)} \eta^+ + s^{(k-2)} \eta^- \right) \equiv \Theta_{k-1,1}, \quad (4.4)$$

$$-\partial_{zz} \left( \tilde{\sigma}^{(k)} - \eta(q^{(k)}d^{(0)} + q^{(0)}d^{(k)}) \right)$$

$$= -\sum_{i=0}^{k-1} \left( \partial_z \tilde{\sigma}^{(i)} \partial_t d^{(k-1-i)} - 2\nabla_x \partial_z \tilde{\sigma}^{(i)} \cdot \nabla_x d^{(k-1-i)} - \partial_z \tilde{\sigma}^{(i)} \Delta_x d^{(k-1-i)} \right)$$

$$- \left( \partial_z \tilde{\sigma}^{(k-2)} - \Delta_x \tilde{\sigma}^{(k-2)} + 2\tilde{\sigma}^{(k-2)} + \tilde{u}^{(k-2)} - \tilde{\mu}^{(k-2)} \right)$$

$$- \eta'' \sum_{i=1}^{k-1} q^{(i)}d^{(k-i)} + \eta'' zq^{(k-1)} - \eta'' \sum_{i=0}^{k-1} h^{(i)}d^{(k-1-i)} + z\eta' h^{(k-2)}$$

$$+ \left( r^{(k-2)} \eta^+ + r^{(k-2)} \eta^- \right) \equiv \Theta_{k-1,2}, \quad (4.5)$$

$$-\partial_{zz} \tilde{u}^{(k)} + f''(\tilde{u}^{(0)})\tilde{u}^{(k)} = -\tilde{g}(\tilde{u}^{(0)}, \ldots, \tilde{u}^{(k-1)}) + 2\sum_{i=0}^{k-1} \nabla_x \partial_z \tilde{u}^{(i)} \cdot \nabla_x d^{(k-1-i)} + \sum_{i=0}^{k-1} \partial_z \tilde{u}^{(i)} \Delta_x d^{(k-1-i)} + \tilde{\mu}^{(k-1)} + \Delta_x \tilde{u}^{(k-2)}$$

$$- \eta \sum_{i=0}^{k-1} l^{(i)}d^{(k-1-i)} + z\eta' l^{(k-2)} \equiv \Theta_{k-1,3}. \quad (4.6)$$

**Step 1.** By induction we assume that the inner-outer matching conditions (2.13)-(2.15) hold for order up to $k-1$. It follows from Lemma 4.3 in [3] and direct computations that if
\[
\int_{-\infty}^{+\infty} \Theta_{k-1,1}(x, t, z) \, dz = 0, \tag{4.7}
\]

then (4.8) has a bounded solution
\[
\tilde{\mu}^{(k)}(x, t, z) = \eta(z) \mu_+^{(k)}(x, t) + (1 - \eta(z)) \mu_-^{(k)}(x, t)
- \eta(z) \int_{-\infty}^{+\infty} \int_{z'}^{+\infty} \Theta_{k-1,1}(x, t, z'') \, dz'' \, dz' \\
+ \int_{-\infty}^{z} \int_{z'}^{+\infty} \Theta_{k-1,1}(x, t, z'') \, dz'' \, dz' \tag{4.8}
\]

which satisfies
\[
(p^{(k)} d^{(0)} + p^{(0)} d^{(k)}) (x, t) = \mu_+^{(k)}(x, t) - \int_{-\infty}^{+\infty} \int_{z'}^{+\infty} \Theta_{k-1,1}(x, t, z'') \, dz'' \, dz' - \mu_-^{(k)}(x, t), \tag{4.9}
\]

and
\[
[\mu^{(k)}] \triangleq \mu_+^{(k)} - \mu_-^{(k)} = p^{(0)} d^{(k)} + \int_{-\infty}^{+\infty} \int_{z'}^{+\infty} \Theta_{k-1,1}(x, t, z'') \, dz'' \, dz', \quad \text{on } \Gamma, \tag{4.10}
\]

and for any \(\alpha, \beta, \gamma \in \mathbb{N},\)
\[
D_x^\alpha D_t^\beta D_z^\gamma \left( \tilde{\mu}^{(k)}(x, t, z) - \mu_\pm^{(k)}(x, t) \right) = O(e^{-\nu|z|}), \quad \text{for } |z| \gg 1 \text{ and some } \nu > 0.
\]

From (4.9) we can take
\[
p^{(k)} = \begin{cases} 
\frac{1}{d^{(0)}} \left( \mu_+^{(k)}(x, t) - \mu_-^{(k)}(x, t) - \int_{-\infty}^{+\infty} \int_{z'}^{+\infty} \Theta_{k-1,1}(x, t, z'') \, dz'' \, dz' \\
-p^{(0)} d^{(k)} \right), & \text{in } \Gamma(\delta) \setminus \Gamma, \\
\nabla_x d^{(0)} \cdot \nabla_x \left( \mu_+^{(k)}(x, t) - \mu_-^{(k)}(x, t) - \int_{-\infty}^{+\infty} \int_{z'}^{+\infty} \Theta_{k-1,1}(x, t, z'') \, dz'' \, dz' \\
-p^{(0)} d^{(k)} \right), & \text{on } \Gamma. \tag{4.11}
\end{cases}
\]

Moreover, by (4.7) we arrive at
\[
2 \partial_t d^{(k-1)} = -g^{(0)} d^{(k-1)} - g^{(k-1)} d^{(0)} + 2 \nabla_x \left( \mu_+^{(0)} - \mu_-^{(0)} \right) \cdot \nabla_x d^{(k-1)} - \left( u_-^{(k-1)} - u_-^{(k-1)} \right) \partial_t d^{(0)} \\
+ 2 \nabla_x \left( \mu_+^{(k-1)} - \mu_-^{(k-1)} \right) \cdot \nabla_x d^{(0)} + \left( \mu_+^{(k-1)} - \mu_-^{(k-1)} \right) \Delta_x d^{(0)} \\
- \sum_{i=1}^{k-2} \left( \partial_z \tilde{u}^{(i)} \partial_t d^{(k-1-i)} - 2 \nabla_x \partial_z \tilde{u}^{(i)} \cdot \nabla_x d^{(k-1-i)} - \partial_z \tilde{\mu}^{(i)} \Delta_x d^{(k-1-i)} \right) \\
- p^{(k-1)} - \sum_{i=1}^{k-2} g^{(i)} d^{(k-1-i)} + g^{(k-2)} \int_{-\infty}^{+\infty} z \eta' \, dz + \int_{-\infty}^{+\infty} \left( s_+^{(k-2)} \eta^+ + s_-^{(k-2)} \eta^- \right) \, dz \\
- \int_{-\infty}^{+\infty} \left( \partial_t \tilde{u}^{(k-2)} - \Delta_x \tilde{u}^{(k-2)} - (2 \sigma^{(k-2)} + \tilde{\mu}^{(k-2)}) \right) \, dz. \tag{4.12}
\]
In particular, for \((x, t) \in \Gamma\) there holds
\[
2 \partial_t d^{(k-1)} = -g^{(0)} d^{(k-1)} + 2 [\nabla_x \mu^{(0)}] \cdot \nabla_x d^{(k-1)} - [u^{(k-1)}] \partial_t d^{(0)} + 2 \left[ \frac{\partial \mu^{(k-1)}}{\partial n} \right] + [\mu^{(k-1)}] \Delta_x d^{(0)}
- \sum_{i=1}^{k-2} \left( \partial_x \tilde{\eta}^{(i)} \partial_t d^{(k-1-i)} - 2 \nabla_x \partial_x \tilde{\mu}^{(i)} \cdot \nabla_x d^{(k-1-i)} - \partial_x \tilde{\mu}^{(i)} \Delta_x d^{(k-1-i)} \right)
- p^{(k-1)} - \sum_{i=1}^{k-2} g^{(i)} d^{(k-1-i)} + g^{(k-2)} \int_{-\infty}^{+\infty} z \eta' dz
+ \int_{-\infty}^{+\infty} \left( s_+^{(k-2)} \eta_+ + s_-^{(k-2)} \eta_- \right) dz
- \int_{-\infty}^{+\infty} \left( \partial_t \tilde{u}^{(k-2)} - \Delta_x \tilde{\mu}^{(k-2)} - (2 \tilde{\sigma}^{(k-2)} + \tilde{u}^{(k-2)} - \tilde{\mu}^{(k-2)}) \right) dz.
\] (4.13)

Due to (2.50) there holds \([\nabla_x \mu^{(0)}] \cdot \nabla_x d^{(k-1)} = \begin{cases} 0, & k = 2, \\ -\frac{p^{(0)}}{2} \sum_{i=1}^{k-2} \nabla_x d^{(i)} \cdot \nabla_x d^{(k-1-i)}, & k \geq 3. \end{cases} \] (4.14)

Combining (4.14), (4.11) \((k \to k - 1)\), (4.10) \((k \to k - 1)\), (4.11) \((k \to k - 1)\) with (4.13) we obtain
\[
\partial_t d^{(k-1)} = \frac{1}{2} \left( p^{(0)} \Delta_x d^{(0)} - \nabla_x d^{(0)} \cdot \nabla_x p^{(0)} - g^{(0)} \right) d^{(k-1)} + \frac{1}{2} \left[ \frac{\partial \mu^{(k-1)}}{\partial n} \right] + \Lambda_{k-2,1}, \quad \text{on } \Gamma, \] (4.15)

where the function \(\Lambda_{k-2,1}\) depends only on terms up to order \(k - 2\).

And by (4.12) one has
\[
g^{(k-1)} = \begin{cases} \frac{1}{d^{(0)}} \left(-2 \partial_t d^{(k-1)} - g^{(0)} d^{(k-1)} + 2 \nabla_x \mu^{(0)}_+ - \mu^{(0)}_- \right) \cdot \nabla_x d^{(k-1)} \\
- \left( u^{(k-1)}_+ - u^{(k-1)}_- \right) \partial_t d^{(0)} + 2 \nabla_x \mu^{(k-1)}_+ - \mu^{(k-1)}_- \right) \cdot \nabla_x d^{(0)} \\
+ \left( \mu^{(k-1)}_+ - \mu^{(k-1)}_- \right) \Delta_x d^{(0)} - p^{(k-1)} + \Lambda_{k-2,2}, \end{cases} \quad \text{in } \Gamma(\delta) \setminus \Gamma, \] (4.16)

\[
\partial_t d^{(0)} \cdot \nabla \left(-2 \partial_t d^{(k-1)} - g^{(0)} d^{(k-1)} + 2 \nabla_x \mu^{(0)}_+ - \mu^{(0)}_- \right) \cdot \nabla_x d^{(k-1)} \\
- \left( u^{(k-1)}_+ - u^{(k-1)}_- \right) \partial_t d^{(0)} + 2 \nabla_x \mu^{(k-1)}_+ - \mu^{(k-1)}_- \right) \cdot \nabla_x d^{(0)} \\
+ \left( \mu^{(k-1)}_+ - \mu^{(k-1)}_- \right) \Delta_x d^{(0)} - p^{(k-1)} + \Lambda_{k-2,2}, \end{cases} \quad \text{on } \Gamma, \] (4.17)

where the function \(\Lambda_{k-2,2}\) depends only on terms up to order \(k - 2\).

Similarly, if
\[
\int_{-\infty}^{+\infty} \Theta_{k-1,2}(x, t, z) dz = 0, \] (4.17)
then (4.5) has a bounded solution

\[
\bar{\sigma}^{(k)}(x, t, z) = \eta(z)\sigma_+^{(k)}(x, t) + (1 - \eta(z))\sigma_-^{(k)}(x, t)
\]

\[
- \eta(z) \int_{-\infty}^{+\infty} \int_{z'}^{+\infty} \Theta_{k-1,2}(z'', x, t) dz'' dz' + \int_{-\infty}^{z} \int_{z'}^{+\infty} \Theta_{k-1,2}(x, t, z'') dz'' dz',
\]

which satisfies

\[
\left[ \sigma^{(k)} \right] = q^{(0)} d^{(k)} + \int_{-\infty}^{+\infty} \int_{z'}^{+\infty} \Theta_{k-1,2}(x, t, z'') dz'' dz', \quad \text{on } \Gamma,
\]

and for any \(\alpha, \beta, \gamma \in \mathbb{N},\)

\[
D_x^\alpha D_t^\beta D_z^\gamma \left( \bar{\sigma}^{(k)}(x, t, z) - \sigma_\pm^{(k)}(x, t) \right) = O(e^{-\nu|z|}), \quad \text{for } |z| \gg 1 \text{ and some } \nu > 0.
\]

Furthermore we can obtain

\[
q^{(k)} = \begin{cases}
\frac{1}{d^{(0)}} \left( \sigma_+^{(k)}(x, t) - \sigma_-^{(k)}(x, t) - \int_{-\infty}^{+\infty} \int_{z'}^{+\infty} \Theta_{k-1,2}(x, t, z'') dz'' dz' \right), & \text{in } \Gamma(\delta) \setminus \Gamma, \\
\nabla_x q^{(0)} \cdot \nabla \left( \sigma_+^{(k)}(x, t) - \sigma_-^{(k)}(x, t) - \int_{-\infty}^{+\infty} \int_{z'}^{+\infty} \Theta_{k-1,2}(x, t, z'') dz'' dz' \right) - q^{(0)} d^{(k)}, & \text{on } \Gamma, \tag{4.18}
\end{cases}
\]

and

\[
\left[ \frac{\partial \sigma^{(k-1)}}{\partial n} \right] = \left( \nabla_x q^{(0)} + h^{(0)} - q^{(0)} \Delta_x d^{(0)} \right) d^{(k-1)} + \Lambda_{k-2,3}, \quad \text{on } \Gamma, \tag{4.19}
\]

and

\[
h^{(k-1)} = \begin{cases}
\frac{1}{d^{(0)}} \left( - h^{(0)} d^{(k-1)} - (\sigma_+^{(k-1)} - \sigma_-^{(k-1)}) \partial_t d^{(0)} + 2 \nabla_x \left( \sigma_+^{(k-1)} - \sigma_-^{(k-1)} \right) \cdot \nabla_x d^{(0)} \right) + (\sigma_+^{(k-1)} - \sigma_-^{(k-1)}) \Delta_x d^{(0)} - q^{(k-1)} + \Lambda_{k-2,4}, & \text{in } \Gamma(\delta) \setminus \Gamma, \\
\nabla_x d^{(0)} \cdot \nabla \left( - h^{(0)} d^{(k-1)} - (\sigma_+^{(k-1)} - \sigma_-^{(k-1)}) \partial_t d^{(0)} \right) + 2 \nabla_x \left( \sigma_+^{(k-1)} - \sigma_-^{(k-1)} \right) \cdot \nabla_x d^{(0)} + (\sigma_+^{(k-1)} - \sigma_-^{(k-1)}) \Delta_x d^{(0)} - q^{(k-1)} + \Lambda_{k-2,4}, & \text{on } \Gamma, \tag{4.20}
\end{cases}
\]

where the functions \(\Lambda_{k-2,3}\) and \(\Lambda_{k-2,4}\) depend only on terms up to order \(k - 2\).

Step 2. Based on the method of variation of constants and direct computations (or Lemma 4.3 in [3]) we find that if

\[
\int_{-\infty}^{+\infty} \Theta_{k-1,3}(x, t, z) \theta'(z) dz = 0, \tag{4.21}
\]

then (4.6) has a bounded solution \(\bar{u}^{(k)}(x, t, z)\) satisfying \(\bar{u}^{(k)}(0, x, t) = 0\) and for any \(\alpha, \beta, \gamma \in \mathbb{N},\)

\[
D_x^\alpha D_t^\beta D_z^\gamma \left( \bar{u}^{(k)}(x, t, z) - u_\pm^{(k)}(x, t) \right) = O(e^{-\nu|z|}), \quad \text{for } |z| \gg 1 \text{ and some } \nu > 0.
\]
According to (4.8) and (4.21) we get
\[
\mu^{(k-1)}_+ \int_{-\infty}^{+\infty} \eta(z)\theta'(z)dz + \mu^{(k-1)}_- \int_{-\infty}^{+\infty} (1 - \eta(z))\theta'(z)dz
\]
\[
= -\Delta_x d^{(k-1)} \int_{-\infty}^{+\infty} \theta'(z)^2 dz + l^{(0)} d^{(k-1)} \int_{-\infty}^{+\infty} \eta'(z)\theta'(z)dz
\]
\[
+ l^{(k-1)} d^{(0)} \int_{-\infty}^{+\infty} \eta'(z)\theta'(z)dz + \Lambda_{k-2,5},
\]
where the function \(\Lambda_{k-2,5}\) depends only on terms up to order \(k - 2\). Here we have used the fact \(\bar{u}^{(k-1)}\) actually depends only on terms up to order \(k - 2\). In particular, for \((x, t) \in \Gamma\) there holds
\[
\mu^{(k-1)}_+ \int_{-\infty}^{+\infty} \eta(z)\theta'(z)dz + \mu^{(k-1)}_- \int_{-\infty}^{+\infty} (1 - \eta(z))\theta'(z)dz
\]
\[
= -\Delta_x d^{(k-1)} \int_{-\infty}^{+\infty} \theta'(z)^2 dz + l^{(0)} d^{(k-1)} \int_{-\infty}^{+\infty} \eta'(z)\theta'(z)dz + \Lambda_{k-2,5}, \text{ on } \Gamma.
\] (4.23)

It follows from (4.10) \((k \to k - 1)\) and (4.23) that
\[
\mu^{(k-1)}_+ = -\frac{1}{2} \Delta_x d^{(k-1)} \int_{-\infty}^{+\infty} \theta'(z)^2 dz + \frac{1}{2} l^{(0)} d^{(k-1)} \int_{-\infty}^{+\infty} \eta'(z)\theta'(z)dz
\]
\[
+ \frac{1}{2} l^{(0)} d^{(k-1)} \int_{-\infty}^{+\infty} \left(\frac{1}{2} - \eta(z) \pm \frac{1}{2}\right)\theta'(z)dz + \Lambda_{k-2,6}, \text{ on } \Gamma,
\]
where the functions \(\Lambda^{\pm}_{k-2,6}\) depend only on terms up to order \(k - 2\).

And by (4.22) one has
\[
l^{(k-1)} = \begin{cases} 
\frac{1}{d^{(0)} \int_{-\infty}^{+\infty} \eta'(z)\theta'(z)dz} \left( \mu^{(k-1)}_+ \int_{-\infty}^{+\infty} \eta(z)\theta'(z)dz + \mu^{(k-1)}_- \int_{-\infty}^{+\infty} (1 - \eta(z))\theta'(z)dz 
+ \Delta_x d^{(k-1)} \int_{-\infty}^{+\infty} \theta'(z)^2 dz - l^{(0)} d^{(k-1)} \int_{-\infty}^{+\infty} \eta'(z)\theta'(z)dz 
+ \Lambda_{k-2,7} \right) 
, \text{ in } \Gamma(\delta) \backslash \Gamma, \\
\frac{1}{d^{(0)} \int_{-\infty}^{+\infty} \eta'(z)\theta'(z)dz} \nabla_x d^{(0)} \cdot \nabla_x \left( \mu^{(k-1)}_+ \int_{-\infty}^{+\infty} \eta(z)\theta'(z)dz + \mu^{(k-1)}_- \int_{-\infty}^{+\infty} (1 - \eta(z))\theta'(z)dz 
+ \Delta_x d^{(k-1)} \int_{-\infty}^{+\infty} \theta'(z)^2 dz - l^{(0)} d^{(k-1)} \int_{-\infty}^{+\infty} \eta'(z)\theta'(z)dz 
+ \Lambda_{k-2,7} \right) , \text{ on } \Gamma,
\end{cases}
\] (4.25)

where the function \(\Lambda_{k-2,7}\) depends only on terms up to order \(k - 2\).

4.3. Matching kth order \((k \geq 2)\) boundary layer expansion.

Substituting (4.17 - 4.19) into (4.30), (4.31) and (4.36), (4.38) and collecting all terms of \(\varepsilon^k\)-order \((k \geq 2)\) we have
\[
- \partial_{zzz} u_B^{(k)} = \Xi_{k-1,1},
\] (4.26)
\[
- \partial_{zz} u_B^{(k)} = \Xi_{k-1,2},
\] (4.27)
\[
- \partial_{zz} u_B^{(k)} + f''(1) u_B^{(k)} = \Xi_{k-1,3},
\] (4.28)
and on $\partial \Omega \times [0,T]$

$$
\partial_x \mu_B^{(0)}(x,t,0) = 0, \quad \partial_x \mu_B^{(k)}(x,t,0) = -\nabla_x \mu_B^{(k-1)}(x,t,0) \cdot \nabla_x d_B(x,t),
$$

(4.29)

$$
\partial_x \sigma_B^{(0)}(x,t,0) = 0, \quad \partial_x \sigma_B^{(k)}(x,t,0) = -\nabla_x \sigma_B^{(k-1)}(x,t,0) \cdot \nabla_x d_B(x,t),
$$

(4.30)

$$
\partial_x u_B^{(0)}(x,t,0) = 0, \quad \partial_x u_B^{(k)}(x,t,0) = -\nabla_x u_B^{(k-1)}(x,t,0) \cdot \nabla_x d_B(x,t),
$$

(4.31)

where the functions $\Xi_{k-1,1}, \Xi_{k-1,2}$ and $\Xi_{k-1,3}$ depend only on the terms up to order $k - 1$. More concretely,

$$
\Xi_{k-1,1} = 2\nabla_x \partial_x \mu_B^{(k-1)} \cdot \nabla_x d_B + \partial_x \mu_B^{(k-1)} \Delta_x d_B - \partial_t u_B^{(k-2)} + \Delta_x \mu_B^{(k-2)}
$$

$$
+ 2\sigma_B^{(k-2)} + u_B^{(k-2)} - \mu_B^{(k-2)},
$$

$$
\Xi_{k-1,2} = 2\nabla_x \partial_x \sigma_B^{(k-1)} \cdot \nabla_x d_B + \partial_x \sigma_B^{(k-1)} \Delta_x d_B - \partial_t \sigma_B^{(k-2)} + \Delta_x \sigma_B^{(k-2)}
$$

$$
- (2\sigma_B^{(k-2)} + u_B^{(k-2)} - \mu_B^{(k-2)}),
$$

$$
\Xi_{k-1,3} = -g_B(u_B^{(0)}, \ldots, u_B^{(k-1)}) + 2\nabla_x \partial_x u_B^{(k-1)} \cdot \nabla_x d_B + \partial_x u_B^{(k-1)} \Delta_x d_B
$$

$$
+ \mu_B^{(k-1)} + \Delta_x u_B^{(k-2)}.
$$

By induction we assume that the boundary-outer matching conditions (2.51)-(2.52) hold for order up to $k - 1$. We then by (4.26)-(4.27) get

$$
\mu_B^{(k)}(x,t,z) = -\int_{-\infty}^{z} \int_{-\infty}^{z'} \Xi_{k-1,1}(x,t,z'',z') dz''dz' + \mu_+^{(k)}(x,t),
$$

(4.32)

$$
\sigma_B^{(k)}(x,t,z) = -\int_{-\infty}^{z} \int_{-\infty}^{z'} \Xi_{k-1,2}(x,t,z'',z') dz''dz' + \sigma_+^{(k)}(x,t),
$$

(4.33)

and (2.51)-(2.52) for order $k$ are satisfied by induction arguments.

In order that $\mu_B^{(k)}$ defined by (4.32) satisfies (4.29)($k \geq 2$) and $\sigma_B^{(k)}$ defined by (4.33) satisfies (4.30)($k \geq 2$), we only need to assume on $\partial \Omega \times [0,T]$ that

$$
\nabla_x d_B(x,t) \cdot \nabla_x \mu_B^{(k-1)}(x,t)
$$

$$
= -\left(2\nabla_x d_B(x,t) \cdot \nabla_x + \Delta_x d_B(x,t)\right) \int_{-\infty}^{0} \int_{-\infty}^{z'} \Xi_{k-2,1}(x,t,z'',z') dz''dz'
$$

$$
- \int_{-\infty}^{0} \left(\partial_t u_B^{(k-2)} - \Delta_x u_B^{(k-2)} - 2\sigma_B^{(k-2)} - u_B^{(k-2)} + \mu_B^{(k-2)}\right)(x,t,z) dz
$$

$$
\triangleq \Pi_{k-2,1}
$$

(4.34)

and

$$
\nabla_x d_B(x,t) \cdot \nabla_x \sigma_B^{(k-1)}(x,t)
$$

$$
= -\left(2\nabla_x d_B(x,t) \cdot \nabla_x + \Delta_x d_B(x,t)\right) \int_{-\infty}^{0} \int_{-\infty}^{z'} \Xi_{k-2,2}(x,t,z'',z') dz''dz'
$$

$$
- \int_{-\infty}^{0} \left(\partial_t \sigma_B^{(k-2)} - \Delta_x \sigma_B^{(k-2)} - 2\sigma_B^{(k-2)} - u_B^{(k-2)} + \mu_B^{(k-2)}\right)(x,t,z) dz
$$

$$
\triangleq \Pi_{k-2,2}.
$$

(4.35)
In fact, for \((x, t) \in \partial \Omega \times [0, T]\) one has

\[
-\partial_z \mu_B^{(k)}(x, t, 0) = \int_{-\infty}^0 \Xi_{k-1,1}(x, t, z)dz \\
= \left(2\nabla_x dB(x, t) \cdot \nabla_x + \Delta_x dB(x, t)\right)\mu_B^{(k-1)}(x, t, 0) \\
- \left(2\nabla_x dB(x, t) \cdot \nabla + \Delta_x dB(x, t)\right)\mu^+_B^{(k-1)}(x, t) \\
- \int_{-\infty}^0 \left(\partial_t u_B^{(k-2)} - \Delta_x u_B^{(k-2)} - 2\sigma_B^{(k-2)} - u_B^{(k-2)} + \mu_B^{(k-2)}\right)(x, t, z)dz \\
= -\left(2\nabla_x dB(x, t) \cdot \nabla_x + \Delta_x dB(x, t)\right)\int_{-\infty}^0 \int_{-\infty}^{z'} \Xi_{k-2,1}(x, t, z')dz''dz' \\
- \int_{-\infty}^0 \left(\partial_t u_B^{(k-2)} - \Delta_x u_B^{(k-2)} - 2\sigma_B^{(k-2)} - u_B^{(k-2)} + \mu_B^{(k-2)}\right)(x, t, z)dz
\]

and

\[
\nabla_x \mu_B^{(k-1)}(x, t, 0) \cdot \nabla_x dB(x, t) = -\nabla_x dB(x, t) \cdot \nabla_x \left(\int_{-\infty}^0 \int_{-\infty}^{z'} \Xi_{k-2,1}(x, t, z'')dz''dz'\right) \\
+ \nabla_x dB(x, t) \cdot \nabla_x \mu^+_B^{(k-1)}(x, t).
\]

Then we easily get (4.29) \((k \geq 2)\) with the help of (4.34) and the above equalities. The other case is complete similar.

Finally, we equip (4.28) \((k \geq 2)\) with the following boundary condition at \(z = 0\)

\[
\partial_z u_B^{(k)}(x, t, 0) = -\nabla_x u_B^{(k-1)}(x, t, 0) \cdot \nabla_x dB(x, t). \tag{4.36}
\]

for \((x, t) \in \partial \Omega(\delta) \times [0, T]\). Obviously (4.36) implies (4.34).

Since (4.28) \((k \geq 1)\) is a linear second-order ordinary differential equation with constant coefficients, we can solve it explicitly and conclude that there exists a unique solution \(u_B^{(k)}\) which satisfies (4.36) and (2.50).

### 4.4. Solving expansions of \(k\)th order \((k \geq 1)\).

Assuming \(u_B^{(k-1)}, \mu^{(k)}, \sigma^{(k-1)}; \sigma_B^{(k-1)}, q^{(k-1)}, p^{(k-1)}, q^{(k-1)}, h^{(k-1)}, l^{(k-1)}, \tilde{u}^{(k-1)}, \tilde{\mu}^{(k-1)}, \tilde{\sigma}^{(k-1)}, \mu_B^{(k-1)}, \sigma_B^{(k-1)}, u_B^{(k-1)}\) are known and the inner-outer matching conditions (2.13)-(2.15), the boundary-outer matching conditions (2.50)-(2.52) hold for order up to \(k - 1\). Then \(u^{(k)}\) are defined by (4.1). Combining (4.2), (4.3), (2.12), (4.21) \((k - 1 \rightarrow k)\), (4.19) \((k - 1 \rightarrow k)\), (4.15) \((k - 1 \rightarrow k)\), (4.35) \((k - 1 \rightarrow k)\), we have

\[
\begin{cases}
-\Delta \mu^{(k)} + \mu^{(k)} = 2\sigma^{(k)} + u^{(k)} - \partial_t u^{(k)}, & \text{in } \Omega, \\
\partial_\sigma^{(k)} + \Delta \sigma^{(k)} + 2\sigma^{(k)} = \mu^{(k)} + u^{(k)}, & \text{in } \Omega, \\
\nabla d^{(k)} \cdot \nabla d^{(k)} = D_{k-1}, & \text{in } \Gamma(\delta), \\
\mu^{(k)} = -a_0 Dd^{(k)} + a_1 d^{(k)} + \Lambda^{(k)}_{k-1,6}, & \text{on } \Gamma, \\
\left[\frac{\partial \sigma^{(k)}}{\partial n}\right] = a_2 d^{(k)} + \Lambda_{k-1,3}, & \text{on } \Gamma, \\
\partial_t q^{(k)} = a_3 d^{(k)} + \frac{1}{2} \left[\frac{\partial u^{(k)}}{\partial n}\right] + \Lambda_{k-1,1}, & \text{on } \Gamma, \\
\frac{\partial \mu^{(k)}}{\partial n} = \Pi_{k-1,1}, & \text{on } \partial \Omega, \\
\frac{\partial \sigma^{(k)}}{\partial n} = \Pi_{k-1,2}, & \text{on } \partial \Omega, \\
d^{(k)}(x, 0) = 0, & \text{on } \Gamma_0,
\end{cases} \tag{4.37}
\]
where \( a_0 \) is a positive constant, the functions \( a_1, a_2 \) and \( a_3 \) depend on \( p^{(k)}, q^{(0)}, g^{(0)}, h^{(0)}, l^{(0)} \) and \( d^{(0)} \), and \( \Gamma_0 = \Gamma_{|t=0} \). Given initial data \( \sigma^{(k)}_\pm (x,0) \) and solving (4.37) leads to \( \mu^{(k)}_\pm, \sigma^{(k)}_\pm \) and \( \overline{d}^{(k)} \). (4.37) is a “linearized” Hele-Shaw problem (P193 in [3]) coupled linearly with a heat equation satisfied by \( \sigma^{(k)}_\pm \). The first and key strategy is to get the value of \( \overline{d}^{(k)} \) on \( \Gamma \). Here we don’t aim to show the lengthy details and one can refer to the similar arguments in Section 6 of [3].

Then \( p^{(k)}, q^{(k)}, g^{(k)}, h^{(k)}, l^{(k)} \) are determined by (4.11), (4.18), (4.16), (4.20), (4.25) respectively. Moreover \( \overline{u}^{(k)}, \overline{\mu}^{(k)}, \overline{\sigma}^{(k)} \) are determined in section 4.2, \( \mu^{(k)}_B, \sigma^{(k)}_B, \overline{u}^{(k)}_B \) are determined in section 4.3 and the inner-outer matching conditions (2.13)-(2.15) and the boundary-outer matching conditions (2.50)-(2.52) hold for order \( k \).

**Remark 4.1.** We can extend \( (u^{(k)}_\pm, \mu^{(k)}_\pm, \sigma^{(k)}_\pm) \) smoothly from \( \Omega_\pm \) to \( \Omega \) as in Remark 4.1 in [3].

## 5. Construction of an approximate solution

In this section we divide into two steps to construct an approximate solution and determine the system which is satisfied by the approximate solution.

**Step 1.** In \( \Omega_+ \cup \Omega_- \) we define

\[
\overline{u}^A(x,t) = \left( \sum_{i=0}^{k} \epsilon^i \overline{u}^{(i)}_+(x,t) \right) \chi_{\Omega_+}(x,t) + \left( \sum_{i=0}^{k} \epsilon^i \overline{u}^{(i)}_-(x,t) \right) \chi_{\Omega_-}(x,t),
\]

\[
\overline{\mu}^A(x,t) = \left( \sum_{i=0}^{k} \epsilon^i \overline{\mu}^{(i)}_+(x,t) \right) \chi_{\Omega_+}(x,t) + \left( \sum_{i=0}^{k} \epsilon^i \overline{\mu}^{(i)}_-(x,t) \right) \chi_{\Omega_-}(x,t),
\]

\[
\overline{\sigma}^A(x,t) = \left( \sum_{i=0}^{k} \epsilon^i \overline{\sigma}^{(i)}_+(x,t) \right) \chi_{\Omega_+}(x,t) + \left( \sum_{i=0}^{k} \epsilon^i \overline{\sigma}^{(i)}_-(x,t) \right) \chi_{\Omega_-}(x,t),
\]

where \( \chi_{\Omega_\pm} \) is the characteristic function of \( \Omega_\pm \).

Thanks to outer matching expansion procedure, we obtain in \( \Omega_+ \cup \Omega_- \)

\[
\partial_t \overline{u}^A = \Delta \overline{\mu}^A + 2 \overline{\sigma}^A + \overline{u}^A - \overline{\mu}^A,
\]

\[
\partial_t \overline{\sigma}^A = \Delta \overline{\sigma}^A = -(2 \overline{\sigma}^A + \overline{u}^A - \overline{\mu}^A),
\]

\[
\overline{\mu}^A = -\epsilon \Delta \overline{u}^A + \epsilon^{-1} f'(\overline{u}^A) + O(\epsilon^k).
\]

In \( \Gamma(\delta) \) we define

\[
\overline{u}^A(x,t) = \sum_{i=0}^{k} \epsilon^i \overline{u}^{(i)}(x,t,z) \bigg|_{z = \frac{d^{(k)}(x,t)}{\epsilon}},
\]

\[
\overline{\mu}^A(x,t) = \sum_{i=0}^{k} \epsilon^i \overline{\mu}^{(i)}(x,t,z) \bigg|_{z = \frac{d^{(k)}(x,t)}{\epsilon}},
\]

\[
\overline{\sigma}^A(x,t) = \sum_{i=0}^{k} \epsilon^i \overline{\sigma}^{(i)}(x,t,z) \bigg|_{z = \frac{d^{(k)}(x,t)}{\epsilon}}.
\]
Thanks to inner matching expansion procedure, we obtain in $\Gamma(\delta)$

$$\begin{align*}
\partial_t u^A_t - \Delta \mu^A_t &= 2\sigma^A_t + u^A_t - \mu^A_t + O(\varepsilon^{k-1}), \\
\partial_t \sigma^A_t - \Delta \sigma^A_t &= -(2\sigma^A_t + u^A_t - \mu^A_t) + O(\varepsilon^{k-1}), \\
\mu^A_t &= -\varepsilon \Delta u^A_t + \varepsilon^{-1} f'(u^A_t) + O(\varepsilon^k).
\end{align*}$$

In $\partial \Omega(\delta)$ we define

$$\begin{align*}
u^A_B(x, t) &= \sum_{i=0}^k \varepsilon^i u^{(i)}_B(x, t, z) \bigg|_{z=d_B(x) \varepsilon} - \varepsilon^k u^{(k)}_B(0, x, t), \\
\mu^A_B(x, t) &= \sum_{i=0}^k \varepsilon^i \mu^{(i)}_B(x, t, z) \bigg|_{z=d_B(x) \varepsilon} - \varepsilon^k \mu^{(k)}_B(0, x, t), \\
\sigma^A_B(x, t) &= \sum_{i=0}^k \varepsilon^i \sigma^{(i)}_B(x, t, z) \bigg|_{z=d_B(x) \varepsilon} - \varepsilon^k \sigma^{(k)}_B(0, x, t).
\end{align*}$$

Thanks to boundary matching expansion procedure, we obtain in $\partial \Omega(\delta)$

$$\begin{align*}
\partial_t u^A_B - \Delta \mu^A_B &= 2\sigma^A_B + u^A_B - \mu^A_B + O(\varepsilon^{k-1}), \\
\partial_t \sigma^A_B - \Delta \sigma^A_B &= -(2\sigma^A_B + u^A_B - \mu^A_B) + O(\varepsilon^{k-1}), \\
\mu^A_B &= -\varepsilon \Delta u^A_B + \varepsilon^{-1} f'(u^A_B) + O(\varepsilon^{k-1})
\end{align*}$$

and

$$\frac{\partial u^A_B}{\partial n} = \frac{\partial \mu^A_B}{\partial n} = \frac{\partial \sigma^A_B}{\partial n} = 0, \text{ on } \partial \Omega \times (0, T).$$

**Step 2.** We define $(\overline{u^A}, \overline{\mu^A}, \overline{\sigma^A})$ as follows:

$$\overline{u^A} = \begin{cases} 
\overline{u^A}_B, & \text{in } \partial \Omega(\frac{\delta}{\varepsilon}), \\
\overline{u^A}_B \zeta \left( \frac{d_B}{\delta} \right) + \overline{u^A}_O \left( 1 - \zeta \left( \frac{d_B}{\delta} \right) \right), & \text{in } \partial \Omega(\delta) \setminus \partial \Omega(\frac{\delta}{\varepsilon}), \\
\overline{u^A}_O, & \text{in } \Omega \setminus (\partial \Omega(\delta) \cup \Gamma(\delta)), \\
\overline{u^A}_O \zeta \left( \frac{d_O}{\delta} \right) + \overline{u^A}_O \left( 1 - \zeta \left( \frac{d_O}{\delta} \right) \right), & \text{in } \Gamma(\delta) \setminus \Gamma(\frac{\delta}{\varepsilon}), \\
\overline{u^A}_O, & \text{in } \Gamma(\frac{\delta}{\varepsilon}),
\end{cases}$$

and $\overline{\mu^A}, \overline{\sigma^A}$ are defined similarly, where

$$\zeta \in C^\infty_c(R), \quad \zeta = 1 \quad \text{for } |\zeta| \leq \frac{1}{2}, \quad \zeta = 0 \quad \text{for } |\zeta| \geq 1.$$  

Based on the boundary-outer matching conditions (2.50)-(2.52) and inner-outer matching conditions (2.13)-(2.15), one has for small $\varepsilon$

$$\|u^A - u^A_O\|_{C^2(\partial \Omega(\delta) \setminus \partial \Omega(\frac{\delta}{\varepsilon}))} = \|u^A_B - u^A_O\|_{C^2(\partial \Omega(\delta) \setminus \partial \Omega(\frac{\delta}{\varepsilon}))} = O(\varepsilon^2 e^{-\frac{|\delta|}{\varepsilon}}) + O(\varepsilon^k)$$

and

$$\|u^A - u^A_O\|_{C^2(\Gamma(\delta) \setminus \Gamma(\frac{\delta}{\varepsilon}))} = \|u^A_B - u^A_O\|_{C^2(\Gamma(\delta) \setminus \Gamma(\frac{\delta}{\varepsilon}))} = O(\varepsilon^2 e^{-\frac{|\delta|}{\varepsilon}}).$$

And we can obtain the similar results for $\overline{\mu^A} - \overline{\mu^A}_O$ and $\overline{\sigma^A} - \overline{\sigma^A}_O$.  

Consequently \((u^A, \mu^A, \sigma^A)\) satisfies in \(\Omega \times (0, T)\)
\[
\begin{align*}
\partial_t u^A - \Delta u^A &= 2\sigma^A + u^A - \mu^A + \omega_1^A, \\
\partial_t \sigma^A - \Delta \sigma^A &= -(2\sigma^A + u^A - \mu^A) + \omega_2^A, \\
\mu^A &= -\varepsilon \Delta u^A + \varepsilon^{-1} f'(u^A) + \omega_3^A
\end{align*}
\]
and
\[
\begin{align*}
\frac{\partial u^A}{\partial n} = \frac{\partial \mu^A}{\partial n} = \frac{\partial \sigma^A}{\partial n} = 0, \quad \text{on } \partial \Omega \times (0, T),
\end{align*}
\]
where \(\omega_i^A = O(\varepsilon^{k-1})(i = 1, 2, 3)\) which depend on \((u^A, \mu^A, \sigma^A)\).

Let \(\varphi^A = u^A + \sigma^A\), then \(\varphi^A\) satisfies
\[
\begin{align*}
\begin{cases}
\partial_t \varphi^A - \Delta \varphi^A - \Delta \sigma^A = \omega_1^A + \omega_2^A, & \text{in } \Omega \times (0, T), \\
\frac{\partial \varphi^A}{\partial n} = 0, & \text{on } \partial \Omega \times (0, T).
\end{cases}
\end{align*}
\]
Define the approximate solution \((\varphi^A, \mu^A, \sigma^A)\) as follows:
\[
\begin{align*}
\begin{cases}
\varphi^A(x, t) &= \varphi^A(x, t) - \frac{1}{|\Omega|} \int_0^t \omega_1^A(x, t') dx dt', \\
\mu^A(x, t) &= \mu^A(x, t) - \tilde{\mu}^A(x, t), \\
\sigma^A(x, t) &= \sigma(x, t), \\
u^A(x, t) &= u^A(x, t) - \omega_2^A(x, t) - \tilde{\mu}^A(x, t),
\end{cases}
\end{align*}
\]
where \(\tilde{\mu}^A(x, t)\) satisfies
\[
\begin{align*}
\begin{cases}
\Delta \tilde{\mu}^A &= \omega_1^A + \omega_2^A - \frac{1}{|\Omega|} \int_\Omega (\omega_1^A + \omega_2^A)(x, t) dx, & \text{in } \Omega \times (0, T), \\
\frac{\partial \tilde{\mu}^A}{\partial n} &= 0, & \text{on } \partial \Omega \times (0, T), \\
\int_\Omega \tilde{\mu}^A(x, t) dx &= 0, & t \in (0, T).
\end{cases}
\end{align*}
\]
Then the approximate solution \((\varphi^A, \mu^A, \sigma^A)\) satisfies
\[
\begin{align*}
\begin{cases}
\partial_t \varphi^A - \Delta \varphi^A - \Delta \sigma^A &= 0, & \text{in } \Omega \times (0, T), \\
\partial_t \sigma^A - \Delta \sigma^A &= -(2\sigma^A + \nu^A - \mu^A), & \text{in } \Omega \times (0, T), \\
\mu^A &= -\varepsilon \Delta \nu^A + \varepsilon^{-1} f'(u^A) + \omega_4^A, & \text{in } \Omega \times (0, T), \\
\frac{\partial \varphi^A}{\partial n} = \frac{\partial \mu^A}{\partial n} = \frac{\partial \sigma^A}{\partial n} = 0, & \text{on } \partial \Omega \times (0, T),
\end{cases}
\end{align*}
\]
where \(\omega_4^A = O(\varepsilon^{k-2})\).

6. Appendix

In this Appendix we give the proof of (3.8) which has been proved in [14]. Here we repeat the proof out of the completeness of this paper.

Proof. Firstly we can observe that
\[
\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} ((\partial_z v)^2 + f''(\theta(z)) v^2) dz \geq \int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \left( (\partial_z \tilde{v})^2 + f''(\theta(z)) \tilde{v}^2 \right) dz - \varepsilon \int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \tilde{v} \partial_z \tilde{v} J_{\varepsilon} J_{\varepsilon}^{-1} dz - C\varepsilon^2 \int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} v^2 dz.
\]
Let \( \hat{v} = \gamma q_1 f + p_1 \), then

\[
-\varepsilon \int_{-\frac{T}{2}}^{\frac{T}{2}} \hat{v} \partial_z v J_r J^{-1} dz = -\varepsilon \gamma \int_{-\frac{T}{2}}^{\frac{T}{2}} \hat{v} (q_1 f)' J_r J^{-1} dz - \varepsilon \int_{-\frac{T}{2}}^{\frac{T}{2}} \hat{v} \partial_z p_1 J_r J^{-1} dz
\]

\[
= -\varepsilon \gamma \int_{-\frac{T}{2}}^{\frac{T}{2}} \hat{v} (q_1 f - \alpha \theta') J_r J^{-1} dz + \varepsilon \alpha \gamma \int_{-\frac{T}{2}}^{\frac{T}{2}} \hat{v} \theta'' J_r J^{-1} dz
\]

\[
- \varepsilon \int_{-\frac{T}{2}}^{\frac{T}{2}} \hat{v} \partial_z p_1 J_r J^{-1} dz. \tag{6.2}
\]

We easily find

\[
-(q_1 f - \alpha \theta')'' + f''(\theta) (q_1 f - \alpha \theta') = \lambda_1 f q_1 f, \quad (q_1 f - \alpha \theta') \bigg|_{-\frac{T}{2}}^{\frac{T}{2}} = -\alpha \theta' \bigg|_{-\frac{T}{2}}^{\frac{T}{2}}.
\]

Multiplying the above equation by \( q_1 f - \alpha \theta' \), integrating by parts and using (3.5), we have

\[
\int_{-\frac{T}{2}}^{\frac{T}{2}} \left( q_1 f - \alpha \theta' \right)'' \left( q_1 f - \alpha \theta' \right) dz = \int_{-\frac{T}{2}}^{\frac{T}{2}} f''(\theta) (q_1 f - \alpha \theta')^2 dz + \frac{\lambda_1}{\delta} \int_{-\frac{T}{2}}^{\frac{T}{2}} q_1 f (q_1 f - \alpha \theta') dz
\]

\[
- \alpha^2 \theta'' \frac{\delta}{\varepsilon} \theta' (-\frac{\delta}{\varepsilon}) + \alpha^2 \theta'' \frac{\delta}{\varepsilon} \theta' (-\frac{\delta}{\varepsilon}) = O(e^{-\frac{C}{\varepsilon}}). \tag{6.3}
\]

It follows from (6.3) that

\[
\varepsilon \gamma \int_{-\frac{T}{2}}^{\frac{T}{2}} \hat{v} (q_1 f - \alpha \theta')' J_r J^{-1} dz \leq C \varepsilon \gamma \left( \int_{-\frac{T}{2}}^{\frac{T}{2}} |\hat{v}|^2 dz \right)^{\frac{1}{2}} \left( \int_{-\frac{T}{2}}^{\frac{T}{2}} (q_1 f - \alpha \theta')' |q_1 f - \alpha \theta'|^2 dz \right)^{\frac{1}{2}}
\]

\[
\leq O(e^{-\frac{C}{\varepsilon}}) \left( \int_{-\frac{T}{2}}^{\frac{T}{2}} |\hat{v}|^2 dz \right) \leq O(e^{-\frac{C}{\varepsilon}}) \int_{-\frac{T}{2}}^{\frac{T}{2}} v^2 dz. \tag{6.4}
\]

And from (6.7) one has

\[
\varepsilon \alpha \gamma \left| \int_{-\frac{T}{2}}^{\frac{T}{2}} \hat{v} \theta'' J_r J^{-1} dz \right| \leq C \varepsilon^2 \alpha \gamma \left( \int_{-\frac{T}{2}}^{\frac{T}{2}} |\hat{v}|^2 dz \right)^{\frac{1}{2}} \left( \int_{-\frac{T}{2}}^{\frac{T}{2}} |\theta'' z|^2 dz \right)^{\frac{1}{2}}
\]

\[
\leq C \varepsilon^2 \left( \int_{-\frac{T}{2}}^{\frac{T}{2}} v^2 dz \right). \tag{6.5}
\]
Using (3.4) we can arrive at
\[
\int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} |\partial_z p_1|^2 dz = \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} \left( |\partial_z p_1|^2 + f''(\theta)(p_1)^2 \right) dz - \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} f''(\theta)(p_1)^2 dz
\]
\[
\leq C \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} (|\partial_z p_1|^2 + f''(\theta)(p_1)^2) dz
\]
\[
= C \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} (\partial_z \hat{v})^2 + f''(\theta)\hat{v}^2 dz - C\lambda_1^2 \gamma^2
\]
\[
\leq C \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} (|\partial_z \hat{v}|^2 + f''(\theta)\hat{v}^2) dz + C e^{-\frac{C}{\epsilon}} \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} \hat{v}^2 dz. \tag{6.6}
\]

By (6.6) and the Young’s inequality, one has
\[
\left| -\frac{\epsilon}{\delta} \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} \hat{v} \partial_z p_1 J_{1, J^{-1}} dz \right| \leq C \epsilon \left( \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} \hat{v}^2 dz \right)^{\frac{1}{2}} \left( \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} (\partial_z p_1)^2 dz \right)^{\frac{1}{2}}
\]
\[
\leq C \epsilon^2 \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} \hat{v}^2 dz + \frac{1}{4} \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} ((\partial_z \hat{v})^2 + f''(\theta)\hat{v}^2) dz.
\]

Thus
\[
-\frac{\epsilon}{\delta} \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} \hat{v} \partial_z p_1 J_{1, J^{-1}} dz \geq -C \epsilon^2 \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} \hat{v}^2 dz - \frac{1}{4} \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} ((\partial_z \hat{v})^2 + f''(\theta)\hat{v}^2) dz. \tag{6.7}
\]

Substituting (6.4), (6.5) and (6.7) into (6.2) we have
\[
-\frac{\epsilon}{\delta} \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} \hat{v} \partial_z \hat{v} J_{1, J^{-1}} dz \geq -C \epsilon^2 \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} \hat{v}^2 dz - \frac{1}{4} \int_{-\frac{\delta}{\epsilon}}^{\frac{\delta}{\epsilon}} ((\partial_z \hat{v})^2 + f''(\theta)\hat{v}^2) dz
\]
which together with (6.4) leads to (3.8).

Hence the proof of (3.8) is finished. \qed

Acknowledgments

The authors would like to thank Professor Z. Zhang in Peking University for helpful discussions. M. Fei is partly supported by NSF of China under Grant 11301005 and AHNSF grant 1608085MA13. W. Wang is partly supported by NSF of China under Grant 11501502 and “the Fundamental Research Funds for the Central Universities” No. 2016QNA3004.

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