IRREDUCIBLE VIRASORO MODULES FROM
IRREDUCIBLE WEYL MODULES

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Abstract. We use Block’s results to classify irreducible modules
over the differential operator algebra $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$. From modules $A$
over $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ and using the “twisting technique” we construct
a class of modules $A_b$ over the Virasoro algebra for any $b \in \mathbb{C}$. These new
Virasoro modules are generally not weight modules. The necessary and sufficient conditions for $A_b$ to be irreducible are
obtained. Then we determine the necessary and sufficient condi-
tions for two such irreducible Virasoro modules to be isomorphic.
Many interesting examples for such irreducible Virasoro modules
with different features are provided at the end of the paper. In
particular the class of irreducible Virasoro modules $\Omega(\lambda, b)$ for any
$\lambda \in \mathbb{C}^*$ and any $b \in \mathbb{C}$ are defined on the polynomial algebra $\mathbb{C}[x]$.

Keywords: the Virasoro algebra, non-weight irreducible module, Weyl
module, twisting technique

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1. Introduction

We denote by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all integers, nonneg-
avtive integers, positive integers, real numbers and complex numbers,
respectively.

Let $\mathfrak{V}$ denote the complex Virasoro algebra, that is the Lie algebra
with basis \{c, $d_i : i \in \mathbb{Z}$\} and the Lie bracket defined (for $i, j \in \mathbb{Z}$) as follows:

$$[d_i, d_j] = (j - i)d_{i+j} + \delta_{i,-j}\frac{i^3 - i}{12}c; \quad [d_i, c] = 0.$$ 

The algebra $\mathfrak{V}$ is one of the most important Lie algebras both in
mathematics and in mathematical physics, see for example [KR] [IK]
and references therein. The Virasoro algebra theory has been widely
used in many physics areas and other mathematical branches, for ex-
ample, quantum physics [GO], conformal field theory [FMS], Higher-
dimensional WZW models [IKUX, IKU], Kac-Moody algebras [K, MoP],
vertex algebras [LL], and so on.

The representation theory of the Virasoro algebra has been attracting
a lot of attention from mathematicians and physicists. There are two
classical families of simple Harish-Chandra $\mathfrak{V}$-modules: highest weight
modules (completely described in [FT]) and the so-called intermediate series modules. In [Mt] it is shown that these two families exhaust all simple weight Harish-Chandra modules. In [MZ1] it is even shown that the above modules exhaust all simple weight modules admitting a nonzero finite dimensional weight space.

Naturally, the next task is to study irreducible non-weight modules and irreducible weight modules with infinite dimensional weight spaces. Irreducible weight modules with infinite dimensional weight spaces were firstly constructed by taking the tensor product of some highest weight modules and some intermediate series modules in [Zh] in 1997, and the necessary and sufficient conditions for such tensor product to be simple were recently obtained in [CGZ]. Conley and Martin gave another class of such examples with four parameters in [CM] in 2001. Then very recently, new weight simple Virasoro modules were found in [LLZ, LZ].

Various families of irreducible non-weight Virasoro modules were studied in [OW1, LGZ, FJK, Ya, GLZ, OW2, MW]. These include various versions of Whittaker modules constructed using different tricks. In particular, all the above Whittaker modules and even more were described in a uniform way in [MZ2].

The main purpose of the present paper is to construct new irreducible (non-weight) Virasoro modules. Let us first introduce the algebras we will use.

Let \( \mathbb{C}[t] \) be the (associative) polynomial algebra. By \( \partial \) we denote the operator \( \frac{d}{dt} \) on \( \mathbb{C}[t] \). We see that \( \partial t^i = t^i(\partial + i) \). Then we have the associative algebra \( \mathcal{A} = \mathbb{C}[t, \partial] \) which is a proper subalgebra of the rank 1 Weyl algebra \( \mathbb{C}[t, \frac{d}{dt}] \). Note that \( \mathcal{A} \) is the universal enveloping algebra of the 2-dimensional solvable Lie algebra \( a_1 = \mathbb{C}d_0 \oplus \mathbb{C}d_1 \) subject to \([d_0, d_1] = d_1 \). See [Bl, MZ2]. Let \( \mathcal{K} = \mathbb{C}[t, t^{-1}, \partial] \) be the Laurent polynomial differential operator algebra.

Let \( B \) be an associative or Lie algebra over \( \mathbb{C} \) and \( C \) be a subspace of \( B \). A module \( V \) over \( B \) is called \( C \)-torsion if there exists a nonzero \( f \in C \) such that \( fv = 0 \) for some nonzero \( v \in V \); otherwise \( V \) is called \( C \)-torsion-free.

The paper is organized as follows. In Sect.2, we use Block’s results to classify irreducible modules over the differential operator algebra \( \mathbb{C}[t, t^{-1}, \frac{d}{dt}] \). In Sect.3, by twisting irreducible modules \( \mathcal{A} \) over the associative algebra \( \mathcal{K} \) we construct a class of modules \( A_b \) over the Virasoro algebra for any \( b \in \mathbb{C} \). These new Virasoro modules are generally not weight modules. The necessary and sufficient conditions for \( A_b \) to be irreducible are obtained (Theorem 9). And we determine the necessary and sufficient conditions for two such irreducible Virasoro modules to be isomorphic (Theorem 12). In Sect.4, we recover some old Virasoro modules and give concrete new examples from such irreducible...
Virasoro modules with different features. In particular a class of interesting irreducible Virasoro modules \(\Omega(\lambda, b)\) for any \(\lambda \in \mathbb{C}^*\) and any \(b \in \mathbb{C}\) (see Sect. 4.3) are defined on the polynomial algebra \(\mathbb{C}[x]\). We also prove that these irreducible modules \(A_b\) are not isomorphic to any other known irreducible Virasoro modules (Theorem 17).

We like to mention that, very recently, the irreducible Virasoro modules \(\Omega(\lambda, b)\) were used to determine the necessary and sufficient conditions for the modules constructed in [MW] to be irreducible, see [TZ1, TZ2].

2. Irreducible Modules over the Associative Algebra \(\mathcal{K}\)

In this section we will obtain a classification of irreducible modules over the associative algebra \(\mathcal{K}\).

**Lemma 1.** Let \(V\) be any \(\mathbb{C}[t]\)-torsion-free irreducible module over the associative algebra \(\mathcal{A}\). Then \(V\) can be extended into a module over the associative algebra \(\mathcal{K} = \mathbb{C}[t, t^{-1}, \frac{d}{dt}]\), i.e., the action of \(\mathcal{A}\) on \(V\) is a restriction of an irreducible \(\mathbb{C}[t, t^{-1}]\)-torsion-free \(\mathcal{K}\)-module.

*Proof.* Noting that \(t \cdot V\) is a submodule of \(V\), and the action of \(t\) is injective, we know that the action of \(t\) is bijective on \(V\). Then we have the action of \(t^{-1}\) on \(V\). Thus \(V\) becomes a \(\mathbb{C}[t, t^{-1}]\)-torsion-free \(\mathcal{K}\)-module. \(\square\)

The classification for all irreducible modules over \(\mathcal{A}\) was given in [Bl].

**Lemma 2.** Let \(\beta\) be an irreducible element in the associative algebra \(\mathbb{C}(t)[\partial]\). Then

\[
\mathcal{K}/((\mathcal{K} \cap (\mathbb{C}(t)[\partial], \beta))
\]

is a \(\mathbb{C}[t, t^{-1}]\)-torsion-free irreducible module over the associative algebra \(\mathcal{K}\). Moreover any \(\mathbb{C}[t, t^{-1}]\)-torsion-free irreducible module over the associative algebra \(\mathcal{K}\) can be obtained in this way.

*Proof.* Note that \(\mathbb{C}[t, t^{-1}]\) is \(\partial\)-simple (i.e., it has no nontrivial \(\partial\)-invariant ideal). The lemma follows from corollary 4.4.1 in [Bl]. \(\square\)

**Remark:** From the above two lemmas we know that any \(\mathbb{C}[t]\)-torsion-free irreducible module over the associative algebra \(\mathcal{A}\) is a restriction of a \(\mathbb{C}[t, t^{-1}]\)-torsion-free irreducible module over the associative algebra \(\mathcal{K}\) which we have the classification as in Lemma 2. But the converse is not true. For example \(A = \mathbb{C}[t, t^{-1}]\) is naturally a \(\mathcal{K}\) module (\(\partial\) acts as derivation) which is \(\mathbb{C}[t, t^{-1}]\)-torsion-free, while it is not an irreducible \(\mathcal{A}\) module. We also remark that it is generally very hard to judge whether an element in \(\mathbb{C}(t)[\partial]\) is irreducible. See Lemmas 14 and 16.
For any $\lambda \in \mathbb{C}^*$ we can define a $\mathcal{K}$-module structure on the space $\Omega(\lambda) = \mathbb{C}[[\partial]]$, the polynomial algebra in $\partial$, by

$$t^i \partial^k = \lambda^i(\partial - i)^k, \partial \partial^k = \partial^{k+1}$$

for all $k \in \mathbb{Z}_+, i \in \mathbb{Z}$. Since the action of $t$ on $\Omega(\lambda)$ has the only eigenvalue $\lambda$, we see that different $\lambda$ give non-isomorphic $\mathcal{K}$-modules.

It is straightforward to verify that $\Omega(\lambda)$ is a irreducible module over the associative algebra $\mathcal{K}$ for any $\lambda \in \mathbb{C}^*$.

Actually these are the only irreducible modules over the associative algebra $\mathcal{K}$ on which $\mathbb{C}[[t, t^{-1}]]$ is torsion.

**Lemma 3.** Let $V$ be an irreducible module over the associative algebra $\mathcal{K}$ on which $\mathbb{C}[[t, t^{-1}]]$ is torsion. Then $V \cong \Omega(\lambda)$ for some $\lambda \in \mathbb{C}^*$.

**Proof.** Since $tt^{-1} = 1$ we know that the action of $t$ (and $t^{-1}$) is bijective on $V$. Since $\mathbb{C}[[t, t^{-1}]]$ is torsion on $V$, there exists $f(t) \in \mathbb{C}[[t, t^{-1}]]$ such that $f(t)v = 0$ for some nonzero $v \in V$. Moreover, by multiplying an appropriate power of $t$, we may assume that $f(t) \in \mathbb{C}[t]$ with nonzero constant term. So there exists $\lambda \in \mathbb{C}^*$ and a nonzero $u \in V$ such that $tu = \lambda u$. We see that $V = \mathbb{C}[[\partial]]u$. If $\mathbb{C}[[\partial]]$ is torsion on $V$, there exists $f(\partial) \in \mathbb{C}[[\partial]]$ such that $f(\partial)u = 0$. So there exists $\mu \in \mathbb{C}$ and a nonzero $w \in V$ such that $\partial w = \mu w$. We must have $V = \mathbb{C}[[t, t^{-1}]]w$. Again since $\mathbb{C}[[t, t^{-1}]]$ is torsion on $V$, there exists $g(t) \in \mathbb{C}[t]$ with nonzero constant term such that $g(t)w = 0$ since $t$ is bijective on $V$. By repeatedly applying $\partial$ to $g(t)w = 0$ we obtain that $w = 0$, which is impossible. So we may assume that $V = \mathbb{C}[[\partial]]$, the action of $\partial$ is simply multiplication on the left and $t1 = \lambda$. Consequently $t^i \partial^k = \lambda^i(\partial - i)^k$ for all $k \in \mathbb{Z}_+$ and $i \in \mathbb{Z}$. Thus $V \cong \Omega(\lambda)$. \qed

Combining the Lemmas 2 and 3, we obtain a classification for all irreducible modules over the associative algebra $\mathcal{K}$.

### 3. Irreducible modules over the Virasoro algebra

We have the complete classification for irreducible modules over the associative algebra $\mathcal{K}$ in Sect.2. These modules can be considered as modules over the Virasoro algebra $\mathfrak{V}$ since the centerless Virasoro algebra is a subalgebra of $\mathcal{K}$. In this section we will use irreducible modules over the associative algebra $\mathcal{K}$ and “the twisting technique” to construct modules over the Virasoro algebra $\mathfrak{V}$ with trivial action of the center. We will determine the necessary and sufficient conditions for such modules to be irreducible, also determine the necessary and sufficient conditions for two such $\mathfrak{V}$ modules to be isomorphic. The twisted Heisenberg-Virasoro algebra will be used in the proof. Let us first recall the definition of the twisted Heisenberg-Virasoro algebra.
The twisted Heisenberg-Virasoro algebra $\mathcal{L}$ is the universal central extension of the Lie algebra $\{f(t)\frac{d}{dt} + g(t) | f, g \in \mathbb{C}[t, t^{-1}]\}$ of differential operators of order at most one on the Laurent polynomial algebra $\mathbb{C}[t, t^{-1}]$. More precisely, the twisted Heisenberg-Virasoro algebra $\mathcal{L}$ is a Lie algebra over $\mathbb{C}$ with the basis $\{d_n, t^n, z_1, z_2, z_3 | n \in \mathbb{Z}\}$ and subject to the Lie bracket

\begin{align}
(d_n, d_m) &= (m-n)d_{n+m} + \delta_{n,-m} \frac{n^3 - n}{12} z_1, \\
(d_n, t^m) &= mt^{m+n} + \delta_{n,-m}(n^2 + n)z_2, \\
[t^n, t^m] &= n\delta_{n,-m}z_3, \\
([\mathcal{L}, z_1] = [\mathcal{L}, z_2] = [\mathcal{L}, z_3] = 0.
\end{align}

The Lie algebra $\mathcal{L}$ has a Virasoro subalgebra $\mathfrak{V}$ with basis $\{d_i, z_1 | i \in \mathbb{Z}\}$, and a Heisenberg subalgebra $\mathfrak{H}$ with basis $\{t^i, z_3 | i \in \mathbb{Z}\}$.

Let us start with an irreducible module $A$ over the associative algebra $\mathcal{K}$. For any $b \in \mathbb{C}$, we define the action of $\mathcal{L}$ on $A$ by

\begin{align}
2.1
z_k A &= 0, \quad k = 1, 2, 3, \\
2.2
w_k &= -\frac{1}{2}d_{k-1}d_1 - \frac{1}{2}d_{k+1}d_{-1} + d_k d_0 \in U(\mathfrak{V}).
\end{align}

while the action of $t^n \in \mathcal{L}$ is the same as $t^n \in \mathcal{K}$ on $A$. We denote by $A_b$ this module over $\mathcal{L}$. Also $A_b$ is a module over $\mathfrak{V}$. Now we determine the irreducibility of the $\mathfrak{V}$-modules $A_b$.

**Remark:** We do not have $\alpha \in \mathbb{C}[t, t^{-1}]$ in (3.5) as in (2.3) in [LGZ] since the resulting modules for different $\alpha$ can be obtained by changing the module $A$.

**Lemma 4.** Let $A$ be an irreducible module over the associative algebra $\mathcal{K}$. Then $A_b$ is an irreducible module over $\mathcal{L}$ for all $b \in \mathbb{C}$.

**Proof.** The statement in this Lemma is obvious. □

**Lemma 5.** For any $k \in \mathbb{Z}$, let

\begin{equation}
w_k = -\frac{1}{2}d_{k-1}d_1 - \frac{1}{2}d_{k+1}d_{-1} + d_k d_0 \in U(\mathfrak{V}).
\end{equation}

Then for all $k \in \mathbb{Z}$ and $v \in A_b$ we have $w_k v = b(b-1)t^k v$. 

**trivial**
Corollary 6. Let $A$ be an irreducible module over the associative algebra $\mathcal{K}$. If $b \in \mathbb{C} \setminus \{0,1\}$, then $A_b$ is an irreducible $\mathcal{V}$-module.

Proof. From Lemma 5, any $\mathcal{V}$-submodule of $A_b$ is also an $\mathcal{L}$-submodule. Therefore the statement follows by Lemma 4.

The next result handles the case of $A_0$ for the irreducible module $A$ over the associative algebra $\mathcal{K}$.

Lemma 7. Let $A$ be an irreducible module over the associative algebra $\mathcal{K}$.

(a). The module $A_0$ is irreducible over $\mathcal{V}$ iff $A$ is not isomorphic to the natural module $\mathbb{C}[t, t^{-1}]$ over $\mathcal{K}$.

(b). When $A = \mathbb{C}[t, t^{-1}]$ is the natural $\mathcal{K}$ module, then $(\mathbb{C}[t, t^{-1}])/\mathbb{C}$ is an irreducible module over $\mathcal{V}$.

Proof. (a). Let $v$ be any nonzero element in $A$. Denote

$$I_k(v) = \{ f \in \mathbb{C}[t, t^{-1}] : \exists x = f \partial^k + \sum_{i > k} f_i \partial^i \text{ with } xv = 0 \}$$

for all $k \in \mathbb{Z}_+$. Since $A$ is irreducible over the associative algebra $\mathcal{K}$ and the adjoint module $\mathcal{K}$ is not irreducible, there exists some $k \in \mathbb{Z}_+$ such that $I_k(v) \neq 0$. Let $m$ be the minimal non-negative integer $k$ such that $I_k(v) \neq 0$. Note that $m$ depends on $v$. Using $\mathbb{C}[t, t^{-1}]/xv = 0$, $\partial xv = 0$ and $\partial f = f \partial + \partial (f)$ we see that $I_m(v)$ is a $\partial$-invariant ideal of $\mathbb{C}[t, t^{-1}]$, yielding $I_m(v) = \mathbb{C}[t, t^{-1}]$. Then $(1 + \sum_{i > 0} f_i \partial^i) \partial^m v = 0$ for some $f_i \in \mathbb{C}[t, t^{-1}]$.

Case 1: $\partial^m v = 0$.

There exists nonzero $v' \in A$ such that $\partial v' = 0$. Then $A = \mathbb{C}[t, t^{-1}]v'$. We deduce that $\partial (t^k v') = kt^k v'$. It is easy to see that $A$ is isomorphic to the natural module $\mathbb{C}[t, t^{-1}]$ over $\mathcal{K}$ which is not irreducible over $\mathcal{V}$ since it has a 1-dimensional $\mathcal{V}$-submodule $\mathbb{C}$.

Case 2: $\partial^m v \neq 0$.

Let $v' = \partial^m v$ and $u$ be any nonzero element in $A$. Since $A$ is an irreducible module over the associative algebra $\mathcal{K}$, then $v \in \mathcal{K} v'$, say $u = (g_0 + \sum_{i > 0} g_i \partial^i) v'$ for some $g_i \in \mathbb{C}[t, t^{-1}]$. Recall that $(1 + \sum_{i > 0} f_i \partial^i) v' = 0$. 

Proof. For any $k \in \mathbb{Z}$ and $v \in A_b$, we compute

$$(d_{k-i} v) = (t^k - 1)(t^k + 1) = (t^k + t^k) = (t^k + 1)$$

$$(d_{k-i} v) + (k-i)bt^{k-i})v = ((t^k + 1) + (k-i)bt^{k-i})(t^k + bit^k) = (t^k + 1) + (k-i)bt^{k-i}(t^k + bit^k) v.$$ 

Taking $i = -1, 0, 1$ respectively, we get

$$w_k v = b(b-1)t^k v.$$ 

The lemma follows. \qed
0. We see that $u \in g_0(1 + \sum_{i>0} f_i \partial^i) v' + U(\mathfrak{W}) v' = U(\mathfrak{W}) v \subset U(\mathfrak{W}) v$. Thus $A_0$ is an irreducible module over $\mathfrak{W}$.

(b). It is easy to see that $(\mathbb{C}[t, t^{-1}])/\mathbb{C} = \text{span}\{t^k : k \in \mathbb{Z} \setminus \{0\}\}$. The action is as follows:

$$d_k t^n = t^k \partial^n = n t^{k+n}.$$ 

Thus $(\mathbb{C}[t, t^{-1}])/\mathbb{C}$ is exactly the irreducible module $V_{0,0}'$ in [KR] which is irreducible over $\mathfrak{W}$. □

The next result handles the case of $A_1$ for irreducible module $A$ over the associative algebra $K$.

**Lemma 8.** Let $A$ be an irreducible module over the associative algebra $K$. Then $\partial A_1$ is an irreducible $\mathfrak{W}$-submodule of $A_1$, and $\partial A_1$ is isomorphic to $A_0$ as modules over $\mathfrak{W}$ if $A_0$ is irreducible, or isomorphic to $V_{0,0}'$, otherwise.

**Proof.** Since $b = 1$, (3.5) becomes

(3.8) $$d_n v = (nt^n + nt^n) v = \partial \cdot t^n \cdot v, \forall n \in \mathbb{Z}, v \in A.$$ 

We see that $\partial A_1$ is a $\mathfrak{W}$-submodule of $A_1$. For any $\partial v \in \partial A_1$, we have

$$d_n \partial v = (nt^n + nt^n) \partial v = \partial \cdot t^n \partial \cdot v.$$ 

If $A_0$ is irreducible, then from the proof of Case 1 of Lemma 7, we have $\partial v \neq 0$ for any $0 \neq v \in A$. In this case it is easy to verify that the linear map

$$\varphi : \partial A_1 \rightarrow A_0, \quad \partial v \rightarrow v, \forall v \in A$$

is a $\mathfrak{W}$-module isomorphism.

If $A_0$ is not irreducible, from Lemma 7 then $A = \mathbb{C}[t, t^{-1}]$. It is easy to verify that the linear map

$$\varphi : \partial A_1 \rightarrow \mathbb{C}[t, t^{-1}]/\mathbb{C}, \quad \partial t^n \rightarrow t^n/n, \forall n \in \mathbb{Z} \setminus \{0\}$$

is a $\mathfrak{W}$-module isomorphism. □

Now we can summarize simplicity results as follows.

**Theorem 9.** Suppose that $b \in \mathbb{C}$, and $A$ is an irreducible module over the associative algebra $K$. Then $A_b$ is an irreducible $\mathfrak{W}$-module if and only if one of the following holds

(i). $b \neq 0$ or 1;
(ii). $b = 1$ and $\partial A = A$;
(iii). $b = 0$ and $A$ is not isomorphic to the natural $K$ module $\mathbb{C}[t, t^{-1}]$.

Next we determine when two $\mathfrak{W}$-modules $A_b$ are isomorphic.

**Lemma 10.** Suppose that $b, b_1 \in \mathbb{C}$ with $b \neq 0$ or 1, and $A$ and $B$ are irreducible modules over the associative algebra $K$. Then $A_b \cong B_{b_1}$ as $\mathfrak{W}$-modules if and only if $b = b_1$ and $A \cong B$ as $K$-modules.
Lemma 11. Suppose that $A$ and $B$ are irreducible modules over the associative algebra $K$. Then $A \cong B$ as $\mathfrak{V}$-modules if and only if $A \cong B$ as $\mathcal{K}$-modules.

Proof. The sufficiency of the conditions is clear. Now suppose that $\mu : A_0 \rightarrow B_0$ is a $\mathfrak{V}$-module isomorphism. From Lemma 3, for any $v \in A$ we have

$$b_1(b_1 - 1)t^k \mu(v) = w_k \mu(v) = \mu(w_k v) = b(b - 1) \mu(t^k v), \quad \forall \, k \in \mathbb{Z}.$$ 

Taking $k = 0$ we obtain that $b(b - 1) = b_1(b_1 - 1)$. In particular, $b_1 \not\in \{0, 1\}$. Noting that

$$\mu(t^k v) = \mu(\frac{w_k}{b(b - 1)} v) = \frac{w_k}{b_1(b_1 - 1)} \mu(v) = t^k \mu(v),$$

we deduce that

$$\mu(t^k \partial v) = t^k \mu(\partial v) = t^k \mu(d_0^k v) = t^k d_0^k \mu(v) = t^k \partial^j \mu(v).$$

So $A \cong B$ as $\mathcal{K}$-modules. From (3.10), we have

$$0 = \mu(d_k v) - d_k \mu(v) = \mu(t^k \partial v + k t^k b v) - (t^k \partial + k t^k b_1) \mu(v)$$

$$= k(b - b_1) t^k \mu(v), \quad \forall \, v \in A, k \in \mathbb{Z}.$$ 

Thus $b = b_1$. This completes the proof. \hfill \Box

Now we consider the case $b, b_1 \in \{0, 1\}$.

Lemma 11. Suppose that $A$ and $B$ are irreducible modules over the associative algebra $K$. Then $A_0 \cong B_0$ as $\mathfrak{V}$-modules if and only if $A \cong B$ as $\mathcal{K}$-modules.

Proof. The sufficiency of the conditions is clear. Now suppose that $\mu : A_0 \rightarrow B_0$ is a $\mathfrak{V}$ module isomorphism. Note that $d_0 = \partial$.

If $(\partial - k)v = 0$ for some $k \in \mathbb{Z}$ and a nonzero $v \in A$, then $\partial(t^{-k}v) = 0$ where $t^{-k}v \neq 0$. From Case 1 of the proof of Lemma 7 we know that $A \cong \mathbb{C}[t, t^{-1}]$, the natural $\mathcal{K}$-module. Similarly, from $(\partial - k)\mu(v) = 0$ in $B$, we deduce that $B \cong \mathbb{C}[t, t^{-1}]$. Thus $A \cong B$ as $\mathcal{K}$-modules in this case.

Now suppose that $(\partial - k)$ is injective on both $A$ and $B$ for all $k \in \mathbb{Z}$.

Then for any $v \in A$, $k \in \mathbb{Z}$ we have $\mu(t^k \partial v) = t^k \partial \mu(v)$. We deduce that

$$(\partial - k) \mu(t^k v) = \mu((\partial - k)t^k v) = \mu(t^k \partial v) = t^k \partial \mu(v)$$

$$= ((\partial - k)t^k) \mu(v) = (\partial - k)(t^k \mu(v)),$$

yielding that $\mu(t^k v) = t^k \mu(v)$ for all $v \in A$ and $k \in \mathbb{Z}$. Therefore $A \cong B$ as $\mathcal{K}$-modules in this case also. This completes the proof. \hfill \Box

Now we can summarize isomorphism results as follows.

Theorem 12. Suppose that $b, b_1 \in \mathbb{C}$, and $A$ and $B$ are irreducible modules over the associative algebra $K$. Then $A_0 \cong B_0$, as $\mathfrak{V}$-modules if and only if one of the following holds

(i) $A \cong B$ as $\mathcal{K}$-modules, and $b = b_1$;

(ii) $A \cong B$ as $\mathcal{K}$ modules, $b = 1, b_1 = 0$ and $\partial A = A$;

(iii) $A \cong B$ as $\mathcal{K}$ modules, $b = 0, b_1 = 1$ and $\partial B = B$. 


4. Old and new irreducible Virasoro modules

4.1. Intermediate series modules. Let \( \alpha \in \mathbb{C}[t, t^{-1}] \), \( b \in \mathbb{C} \). Take \( \beta = \partial - \alpha \) in Lemma 2. Then we have the irreducible \( K \)-module

\[
A = K/(K \cap (\mathbb{C}(t)[\partial][\beta])) = K/(K/\beta)
\]

which has a basis \( \{ t^k : k \in \mathbb{Z} \} \) where we have identified \( t^k \) with \( t^k + K \) (we will continue to do this later without mentioning). The actions of \( K \) are given by

\[
\partial \cdot t^n = t^n(\alpha - n), \quad t^k \cdot t^n = t^{k+n}, \forall k, n \in \mathbb{Z}.
\]

Using (3.5) we obtain Vir-modules \( A_{\alpha,b} = \mathbb{C}[t, t^{-1}] \) with the action:

\[
d_k \cdot t^n = (\alpha + n + kb)t^{k+n}, \forall k, n \in \mathbb{Z}.
\]

These modules \( A_{\alpha,b} \) are exactly the ones introduced and studied in Section 4 of the paper [LGZ]. When \( \alpha \in \mathbb{C} \) these modules \( A_{\alpha,b} \) are the intermediate series modules \( V_{\alpha,b} \) in [KR].

4.2. Fraction modules. Let \( b \in \mathbb{C} \), \( \alpha = (\alpha_0, \alpha_1, \alpha_2, ..., \alpha_n) \), \( a = (a_0, a_1, a_2, ..., a_n) \in \mathbb{C}^{n+1} \) with \( a_0 = 0 \) and \( a_i \neq a_j \) for all \( i \neq j \). In Lemma 2 take

\[
\beta = \frac{d}{dt} - \sum_{i=0}^{n} \frac{\alpha_i}{t - a_i}.
\]

Then we have the irreducible \( K \)-module

\[
A = K/(K \cap (\mathbb{C}(t)[\partial][\beta])) \subset \mathbb{C}[t, (t - a_i)^{-1} \mid i = 0, 1, 2, ..., n].
\]

The actions of \( K \) are given by

\[
\frac{d}{dt} \cdot f(t) = \frac{d}{dt}(f(t)) + f(t) \sum_{i=0}^{n} \frac{\alpha_i}{t - a_i},
\]

\[
t^k \cdot f(t) = t^k f(t), \forall k \in \mathbb{Z}, f \in A.
\]

It is not very hard to verify that the Virasoro modules \( A_b \) is a submodule of the fraction module \( V(a, \alpha - b\epsilon_0, b) \) introduced and studied in Section 2 of the paper [GLZ], where \( \epsilon_0 = (1, 0, \ldots, 0) \in \mathbb{C}^{n+1} \). And we have \( A_b = V(a, \alpha - b\epsilon_0, b) \) if \( V(a, \alpha - b\epsilon_0, b) \) is simple.

4.3. Virasoro modules \( \Omega(\lambda, b) \). Let \( \lambda \in \mathbb{C}^* \) and \( b \in \mathbb{C} \). Then we have the irreducible \( K \)-module \( \Omega(\lambda) \) which has a basis \( \{ \partial^k : k \in \mathbb{Z}_+ \} \). The actions of \( K \) are given by

\[
t^i \cdot \partial^k = \lambda^i(\partial - i)^k, \quad \partial \cdot \partial^k = \partial^{k+1}, \forall k \in \mathbb{Z}_+, i \in \mathbb{Z}.
\]

Using (3.5) we obtain \( \mathfrak{V} \)-modules \( \Omega(\lambda, b) = \mathbb{C}[\partial] \) with the action:

\[
d_n \cdot \partial^k = \lambda^n(\partial + n(b - 1))(\partial - n)^k, \forall k \in \mathbb{Z}_+, n \in \mathbb{Z}.
\]
From Theorem 10 we know that the $\mathfrak{G}$-modules $\Omega(\lambda, b)$ are irreducible if $b \neq 1$. It is easy to see that $\Omega(\lambda, 1)$ has an irreducible submodule $\partial \mathbb{C}[\partial]$ which is isomorphic to $\Omega(\lambda, 0)$.

The Virasoro modules $\Omega(\lambda, b)$ for $\lambda \in \mathbb{C}^*$ and $b \in \mathbb{C}$ are very similar to the highest-weight-like Virasoro modules $V(\xi, \lambda)$ defined in Section 3 of [GLZ]. Actually they are isomorphic.

**Lemma 13.** Suppose that $\lambda \in \mathbb{C}^*$, $b \in \mathbb{C}$. Then the Virasoro modules $\Omega(\lambda, b)$ and $V(\lambda, b - 1)$ are isomorphic.

**Proof.** Denote by $\partial'$ the operator $\partial = \frac{d}{dt}$ in [GLZ]. We know that $V(\lambda, b - 1) = \mathbb{C}[d_{-1}]v$ where $\mathbb{C}[d_{-1}]$ is the polynomial algebra in $d_{-1}$ with the properties:

\[
\begin{align*}
(d_0 - \lambda d_{-1})v &= \lambda v, \\
(d_1 - 2\lambda d_0 + \lambda^2 d_{-1})v &= 0, \\
(d_2 - 3\lambda d_1 + 3\lambda^2 d_0 - \lambda^3 d_{-1})v &= 0, \\
(d_{k-1} - \lambda^k d_{-1} - k\lambda^{k-1}(b - 1))v &= 0, \quad \forall \ k \in \mathbb{Z},
\end{align*}
\]

and these properties characterize the Virasoro module $V(\lambda, b - 1)$. It is straightforward to verify that all the above properties are satisfied by the Virasoro module $\Omega(\lambda, b - 1)$ with $v$ replaced by $1$. □

We remark that in a recent preprint [TZ], it was proved that the tensor product of $\Omega(\lambda, b)$ ($b \neq 1$) with an irreducible highest weight module or with an irreducible module defined in [MZ2] is also an irreducible Virasoro module.

### 4.4. Degree two modules.

As we mentioned before, for a given element $f(t, \partial)$ in $\mathbb{C}(t)[\partial]$ (which is a left principal ideal domain) it is generally very hard to know whether $f(t, \partial)$ is irreducible in $\mathbb{C}(t)[\partial]$ or not (see the next three lemmas). We first construct some degree two irreducible elements in $\mathbb{C}(t)[\partial]$.

**Lemma 14.** Suppose that $f(t) \in \mathbb{C}[t, t^{-1}]$. Then $\partial^2 - f(t)$ is irreducible in $\mathbb{C}(t)[\partial]$ iff $f(t)$ is not of the form

\[
(4.2) \quad h(t)^2 - \partial(h(t)) - 2 \sum_{i=1}^{n} \frac{a_i(h(t) - h(a_i))}{t - a_i}
\]

for any $h(t) \in \mathbb{C}[t, t^{-1}]$ satisfying

\[
h(a_i) = \sum_{j \neq i} \frac{a_j}{a_i - a_j} - \frac{1}{2}, \quad \forall \ i = 1, 2, \ldots, n,
\]

where $n \in \mathbb{Z}_+$, and $a_1, a_2, \ldots, a_n \in \mathbb{C}^*$ are pairwise distinct.
Proof. Suppose $\partial^2 - f(t)$ is reducible in $\mathbb{C}(t)[\partial]$. Then there exists $g_1, g_2 \in \mathbb{C}(t)$ such that $\partial^2 - f(t) = (\partial - g_1)(\partial - g_2)$, to give $\partial^2 - f(t) = \partial^2 - (g_1 + g_2)\partial + g_1g_2 - \partial(g_2)$. We see that $g_2 = -g_1$ and $f = g_1^2 - \partial(g_1) \in \mathbb{C}[t, t^{-1}]$. Write

$$g_1 = \sum_{i=1}^{n} \sum_{j=1}^{l_i} \frac{c_{i,j}}{(t-a_i)^j} + h(t),$$

where $a_i \in \mathbb{C}^*$ are pairwise distinct, $c_{i,j} \in \mathbb{C}$ and $h(t) \in \mathbb{C}[t, t^{-1}]$. We may assume that $l_i \geq 1$ and $c_{i,i} \neq 0$ for each $i = 1, 2, \ldots, n$. By computing the coefficient of $(t-a_i)^{-2h}$ in $g_1^2 - \partial(g_1) \in \mathbb{C}[t, t^{-1}]$, we have $l_i = 1$ and $c_{i,1} = -a_i$. So

$$g_1 = \sum_{i=1}^{n} \frac{-a_i}{t-a_i} + h(t).$$

Consequently,

$$f = g_1^2 - \partial(g_1) = h(t)^2 + \sum_{i>j} \frac{2a_i a_j}{(t-a_i)(t-a_j)} + 2h(t) \sum_{i=1}^{n} \frac{-a_i}{t-a_i} - \partial(h(t)) - \sum_{i=1}^{n} \frac{a_i}{t-a_i} \in \mathbb{C}[t, t^{-1}].$$

That is

$$f - h(t)^2 + \partial(h(t)) = \sum_{i>j} \frac{2a_i a_j}{(t-a_i)(t-a_j)} + 2h(t) + \frac{1}{2} \sum_{i=1}^{n} \frac{-a_i}{t-a_i}$$

$$= \sum_{i>j} \frac{2a_i a_j}{(a_i-a_j)(t-a_i)(t-a_j)} \left( \frac{1}{t-a_i} - \frac{1}{t-a_j} \right) + 2h(t) + \frac{1}{2} \sum_{i=1}^{n} \frac{-a_i}{t-a_i}$$

$$= \sum_{i=1}^{n} \left( (-h(t) + 1/2) + \sum_{j \neq i} \frac{a_j}{a_i-a_j} \right) \frac{2a_i}{t-a_i} \in \mathbb{C}[t, t^{-1}].$$

Therefore $h(t)$ satisfies the condition

$$h(a_i) = \sum_{j \neq i} \frac{a_j}{a_i-a_j} - \frac{1}{2}, \forall i = 1, 2, \ldots, n.$$

We simplify

$$\sum_{i=1}^{n} \left( -(h(t) + 1/2) + \sum_{j \neq i} \frac{a_j}{a_i-a_j} \right) \frac{2a_i}{t-a_i} = -2 \sum_{i=1}^{n} \frac{a_i(h(t)-h(a_i))}{t-a_i}.$$ 

We see that

$$f = h(t)^2 - \partial(h(t)) - 2 \sum_{i=1}^{n} \frac{a_i(h(t)-h(a_i))}{t-a_i},$$

The converse is clear from the above arguments. □
Remark that the last term in (4.2) will disappear if \( n = 0 \).

**Example 1.** Take \( n = 1, a_1 = 1, h(t) = t - 3/2 \). Using Lemma 14 we obtain that \( f(t) = t^2 - 4t + \frac{1}{4} \), and \( \partial^2 - f(t) \) is reducible in \( \mathbb{C}(t)[\partial] \). But it is not hard to verify that \( \partial^2 - f(t) \) is irreducible in \( K \).

**Example 2.** From Lemma 14 we know that, if \( f(t) \in \mathbb{C}[t, t^{-1}] \) with odd positive highest degree (or odd negative lowest degree) then \( \partial^2 - f(t) \) is irreducible in \( \mathbb{C}(t)[\partial] \). Certainly, in this case \( \partial^2 - f(t) \) is automatically irreducible in \( K \).

Now let \( f(t) \in \mathbb{C}[t, t^{-1}] \) be such that \( \partial^2 - f(t) \) is irreducible in \( \mathbb{C}(t)[\partial] \). Take \( \beta = \partial^2 - f(t) \) in Lemma 2. Then we have the irreducible \( K \)-module
\[
A = K/(K \cap (\mathbb{C}(t)[\partial]) = K/(K\beta)
\]
which has a basis \( \{t^k, t^k\partial : k \in \mathbb{Z}\} \). (Remark that the second equality of the above equation is the reason why we have assumed that \( f \in \mathbb{C}[t, t^{-1}] \) instead of in \( \mathbb{C}(t) \) in Lemma 14). The actions of \( K \) on \( A \) are given by
\[
t^k \cdot t^n = t^{k+n}, \quad t^k \cdot (t^n \partial) = t^{k+n} \partial,
\]
\[
\partial \cdot t^n = t^n(\partial + n), \quad \partial \cdot (t^n \partial) = t^n(f(t) + n\partial), \quad \forall \ k, n \in \mathbb{Z}.
\]
Using (3.5), for any \( b \in \mathbb{C} \setminus \{1\} \) we obtain irreducible \( \mathfrak{U} \)-modules \( A_b = \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}[t, t^{-1}]\partial \) with the action:
\[
d_k \cdot t^n = t^{k+n}(n + kb + \partial),
\]
\[
d_k \cdot (t^n \partial) = t^{k+n}(f(t) + kb + n\partial), \quad \forall \ k, n \in \mathbb{Z}.
\]
As far as we know, these modules \( A_b \) just constructed are new irreducible \( \mathfrak{U} \)-modules (also see Theorem 17).

The next result combining with Lemma 14 gives more irreducible elements in \( \mathbb{C}(t)[\partial] \) and more irreducible Virasoro modules.

**Lemma 15.** Suppose that \( f_1(t), f_2(t) \in \mathbb{C}[t, t^{-1}] \). Then \( \partial^2 + 2f_1\partial + f_2(t) \) is irreducible in \( \mathbb{C}(t)[\partial] \) iff \( \partial^2 - (\partial(f_1) + f_1^2 - f_2) \) is irreducible in \( \mathbb{C}(t)[\partial] \).

**Proof.** It is easy to see that the linear map \( \tau : \mathbb{C}(t)[\partial] \to \mathbb{C}(t)[\partial] \) defined by \( (\sum g_i(t)\partial^i) = \sum g_i(t)(\partial - f_1)^i \) is an automorphism of \( \mathbb{C}(t)[\partial] \). We write \( \partial^2 + 2f_1\partial + f_2(t) = (\partial + f_1(t))^2 - (\partial(f_1) + f_1^2 - f_2) \). Hence the lemma follows from the fact that \( \tau(x) \) and \( x \) (\( x \in \mathbb{C}(t)[\partial] \)) have the same irreducibility in \( \mathbb{C}(t)[\partial] \).

**s4.4** 4.5. Degree n modules. We first construct some degree \( n \) irreducible elements in \( \mathbb{C}(t)[\partial] \).

**Lemma 16.** For any nonconstant polynomial \( f(t) \in \mathbb{C}[t] \), the element \( f(\frac{d}{dt}) - t \in \mathbb{C}[t, t^{-1}][\partial] \) is irreducible in \( \mathbb{C}(t)[\partial] \).
Proof. Let \( J \) be the left ideal generated by \( f(\frac{d}{dt}) - t \) in the associative algebra \( \mathbb{C}[t][\frac{d}{dt}] \). Then the \( \mathbb{C}[t][\frac{d}{dt}] \)-module \( \mathbb{C}[t][\frac{d}{dt}] / J \) has a basis \( \{ (\frac{d}{dt})^i | i \in \mathbb{Z}_+ \} \) with the action

\[
\frac{d}{dt} \cdot (\frac{d}{dt})^i = (\frac{d}{dt})^{i+1},
\]

\[
t \cdot (\frac{d}{dt})^i = -i(\frac{d}{dt})^{i-1} + f(\frac{d}{dt}) (\frac{d}{dt})^i.
\]

For any nonzero submodule \( V \) of \( \mathbb{C}[t][\frac{d}{dt}] / J \), it is easy to see that \( V \) is an ideal of the polynomial algebra \( \mathbb{C}[\frac{d}{dt}] \). For any \( g(\frac{d}{dt}) \in V \), from

\[
-t \cdot g(\frac{d}{dt}) \in V
\]

we see that \( g'(\frac{d}{dt}) \in V \). We deduce that \( 1 \in V \) and consequently \( V = \mathbb{C}[t][\frac{d}{dt}] / J \). Hence \( \mathbb{C}[t][\frac{d}{dt}] / J \) is a simple module over the associative algebra \( \mathbb{C}[t][\frac{d}{dt}] \). Note that \( \mathbb{C}[t][\frac{d}{dt}] / J \) is \( \mathbb{C}[t] \) torsion-free. Therefore \( \mathbb{C}(t)[\frac{d}{dt}] (f(\frac{d}{dt}) - t) \) is a maximal left ideal of \( \mathbb{C}(t)[\frac{d}{dt}] \). Thus \( f(\frac{d}{dt}) - t \) is irreducible in \( \mathbb{C}(t)[\partial] \).

For any \( n \in \mathbb{N} \), take \( \beta = (\frac{d}{dt})^n - t \) in Lemma 2. Then we have the irreducible \( \mathcal{K} \)-module

\[
A = \mathcal{K} / (\mathcal{K} \cap (\mathbb{C}(t)[\partial] \beta)) = \mathcal{K} / (\mathcal{K} \beta)
\]

which has a basis \( \{ t^k(\frac{d}{dt})^m : k \in \mathbb{Z}, m = 0, 1, ..., n - 1 \} \). The actions of \( \mathcal{K} = \mathbb{C}[t, t^{-1}][(\frac{d}{dt})^t] \) are given by

\[
t^k \cdot (t^r (\frac{d}{dt})^s) = t^{k+r} (\frac{d}{dt})^m, \forall k, r \in \mathbb{Z}, 0 \leq m \leq n - 1,
\]

\[
(\frac{d}{dt}) \cdot (t^r (\frac{d}{dt})^s) = rt^{r-1} (\frac{d}{dt})^s + t^r (\frac{d}{dt})^{s+1}, \forall r \in \mathbb{Z}, 0 \leq s \leq n - 1,
\]

\[
(\frac{d}{dt}) \cdot (t^r (\frac{d}{dt})^{n-1}) = rt^{r-1} (\frac{d}{dt})^{n-1} + t^{r+1}, \forall r \in \mathbb{Z}.
\]

Using (3.5), for any \( b \in \mathbb{C} \setminus \{ 1 \} \) we obtain irreducible \( \mathfrak{G} \)-modules \( A_b = \mathbb{C}[t, t^{-1}](\sum_{i=0}^{n-1} \mathbb{C}(\frac{d}{dt})^i) \) with the action:

\[
d_k \cdot (t^r (\frac{d}{dt})^s) = (rt^{k+r} + bkt^{k+r+1})(\frac{d}{dt})^s + t^{k+r+1} (\frac{d}{dt})^{s+1}, \forall k, r \in \mathbb{Z},
\]

\[
d_k \cdot (t^r (\frac{d}{dt})^{n-1}) = (rt^{k+r} + bkt^{k+r+1})(\frac{d}{dt})^{n-1} + t^{k+r+2}, \forall k, r \in \mathbb{Z},
\]

where \( 0 \leq s < n - 1 \).
For different $\beta$ (different $n$), when we consider the $K$-modules $A$ as $\mathbb{C}[t, t^{-1}]$-modules they are not isomorphic since they are free $\mathbb{C}[t, t^{-1}]$-modules of rank $n$. From Theorem 12 we know that we have obtained many non-isomorphic irreducible Virasoro modules in this way. As far as we know, these modules $A_b$ are new irreducible Vir modules (also see Theorem 17).

We would like to conclude this paper by comparing the Virasoro modules $A_b$ with other known irreducible Virasoro modules. It is not hard to see that Virasoro weight modules of the form $A_b$ are the intermediate series modules $V_{\alpha,\beta}$ in [KR] for which $A$ is $\mathbb{C}[\partial]$-torsion. If $A$ is a $\mathbb{C}[t, t^{-1}]$-torsion irreducible module over the associative algebra $K$, then $A_b$ are the modules $\Omega(\lambda, b)$ in Sect.4.3.

All other known non-weight Virasoro modules are from [LGZ, GLZ, MZ2, MW]. We have already compared with those in [LGZ, GLZ]. Let us recall those modules in [MZ2]. Let $\mathfrak{V} = \text{span}\{d_i \mid i \in \mathbb{Z}_+\}$. Given $N \in \mathfrak{V} \_ \text{-mod}$ and $\theta \in \mathbb{C}$, consider the corresponding induced module $\text{Ind}(N) := U(\mathfrak{V}) \otimes_{U(\mathfrak{V}_+)} N$ and denote by $\text{Ind}_\theta(N)$ the module $\text{Ind}(N)/(c - \theta)\text{Ind}(N)$.

**Theorem 17.** Suppose that $b \in \mathbb{C} \setminus \{1\}$, and $A$ is an irreducible module over the associative algebra $K$ which is $\mathbb{C}[t, t^{-1}]$-torsion-free and $\mathbb{C}[\partial]$-torsion-free. Then $A_b$ is not isomorphic to $\text{Ind}_\theta(N)$ for any irreducible $N \in \mathfrak{V} \_ \text{-mod}$, or the modules $\text{Ind}_{\theta, z}(\mathbb{C}_m)$ defined in [MW] for any $\theta, m_2, m_3, m_4 \in \mathbb{C}$ and $z \in \mathbb{C}^*$.

**Proof.** It was proved that $V = \text{Ind}_\theta(N)$ is irreducible over $\mathfrak{V}$ if $N \in \mathfrak{V} \_ \text{-mod}$ is irreducible and $d_k N = 0$ for all sufficiently large $k$. For any $v \in V$ we have $d_k v = 0$ for all sufficiently large $k$. But this property cannot be shared by $A_b$ because of (3.5) and the fact that $A$ is $\mathbb{C}[t, t^{-1}]$-torsion-free and $\mathbb{C}[\partial]$-torsion-free. The statement in the theorem follows for this case.

Now we compare our module $A_b$ with the irreducible Virasoro modules $W = \text{Ind}_{\theta, z}(\mathbb{C}_m)$ defined in [MW], where $\theta, m_2, m_3, m_4 \in \mathbb{C}$ and $z \in \mathbb{C}^*$ satisfying the conditions

$$zm_3 \neq m_4, 2zm_2 \neq m_3, 3zm_3 \neq 2m_4, z^2m_2 + m_4 \neq 2zm_3.$$  

From the definition of $W$, there exists a nonzero vector $v \in W$ such that

$$(d_2 - zd_1 - m_2)v = 0, (d_3 - z^2d_1 - m_3)v = 0, (d_4 - z^3d_1 - m_4)v = 0$$

yielding that

$$(d_3 - zd_2 + zm_2 - m_3)v = 0, (d_4 - zd_3 + zm_3 - m_4)v = 0.$$  

If $A_b \simeq W$, there exists nonzero $u \in A_b$ such that

$$(d_3 - zd_2 + zm_2 - m_3)u = 0, (d_4 - zd_3 + zm_3 - m_4)u = 0,$$
i.e.,
\[
t^2(t - z)\partial + bt^2(3t - 2z) + zm_2 - m_3)u = 0, \\
t^3(t - z)\partial + bt^3(4t - 3z) + zm_3 - m_4)u = 0.
\]
We obtain that
\[
(bt^3(t - z) - t(zm_2 - m_3) + (zm_3 - m_4))u = 0.
\]
Since \(\mathbb{C}[t]\) is torsion-free on \(A_b\), we deduce that \(zm_3 = m_4\) which is a contradiction to (4.4). Therefore \(A_b \not\cong W\). \(\square\)

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