FRACTIONAL DIFFUSION LIMIT OF A LINEAR KINETIC EQUATION IN A BOUNDED DOMAIN

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Abstract. A version of fractional diffusion on bounded domains, subject to ‘homogeneous Dirichlet boundary conditions’ is derived from a kinetic transport model with homogeneous inflow boundary conditions. For nonconvex domains, the result differs from standard formulations. It can be interpreted as the forward Kolmogorov equation of a stochastic process with jumps along straight lines, remaining inside the domain.

1. Introduction. This work is an extension to bounded domains of earlier efforts [4, 19, 20] to derive fractional diffusion equations from kinetic transport models. This raises the issue of the inclusion of boundary effects, which can, however, not be reduced to boundary conditions since fractional diffusion is a nonlocal process. Our main result is the derivation of a new way of realizing ‘homogeneous Dirichlet boundary conditions’, coinciding on convex domains with an already established model, see e.g. [14].

Let $\Omega \subset \mathbb{R}^d$ denote a bounded domain with smooth boundary. We shall study the asymptotic behavior as $\varepsilon > 0$ tends to zero of the kinetic relaxation model

$$\varepsilon^\alpha \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = Q(f_\varepsilon) := \int_{\mathbb{R}^d} M_{f'} - M' f_\varepsilon \, dv', \quad (1)$$

with $f_\varepsilon = f_\varepsilon(x,v,t), (x,v,t) \in \Omega \times \mathbb{R}^d \times [0,\infty)$ (where the superscript $'$ denotes evaluation at $v'$), subject to zero inflow boundary conditions and well prepared initial data:

$$f_\varepsilon(x,v,0) = f_0 n(x,v) := \rho n(x) M(v) \quad \text{for} \ (x,v) \in \Omega, \quad (3)$$

with $\Gamma^\pm = \{(x,v) | x \in \partial \Omega, \ \text{sign}(v \cdot \nu(x)) = \pm 1\}$, where $\nu$ denotes the unit outward normal along $\partial \Omega$. We assume a ‘fat-tailed’ equilibrium distribution $M$, satisfying

$$M(v) = 1/|v|^{d+\alpha} \quad \text{for} \ |v| \geq 1, \ \text{with} \ 0 < \alpha < 2, \quad (4)$$

$$M(v) > 0, \quad M(v) = M(-v) \quad \text{for all} \ v \in \mathbb{R}^d, \quad (5)$$

$$M \in L^\infty(\mathbb{R}^d), \quad \text{and} \quad \int_{\mathbb{R}^d} M(v) \, dv = 1. \quad (6)$$

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Note that these assumptions imply that $M$ does not have finite second order moments.

The translation of homogeneous Dirichlet boundary conditions to fractional diffusion induces a certain behaviour of solutions close to the boundary. Therefore, for the weak formulation of the fractional diffusion operator we will consider the following set of test functions

$$ D_Ω := \{ ϕ ∈ C^∞_0 (Ω × [0, ∞)) : \delta(x)^{-2} ϕ(x, t) \text{ bounded} \}, $$

where $δ(x) := \text{dist}(x, ∂Ω)$ denotes the distance of a point $x ∈ Ω$ to the boundary.

A convenient functional analytic setting for the main result of this paper is the $L^2$-space $L^2_{M−1}(Ω × ℝ^d)$ of functions of $(x, v)$ with weight $1/M(v)$.

**Theorem 1.1.** Let $ρ^n ∈ L^2(Ω)$, and let $f_n$ be the solution of (1)–(3). Then, for any $T > 0$, there exists $ρ ∈ L^∞(0, T; L^2(Ω))$ such that $f_n(x, v, t) → ρ(x, t)M(v)$ as $ε → 0$, in $L^∞(0, T; L^2_{M−1}(Ω × ℝ^d))$ weak-*, and $ρ$ satisfies

$$ \int_0^T ρ^n ϕ(t = 0)dx + \int_0^T \int Ω ρ \partial_t ϕ dx dt = \int_0^T \int Ω ρ(h_α ϕ − L_α(ϕ))dx dt, $$

for all $ϕ ∈ D_Ω$, with

$$ L_α(ϕ)(x, t) = Γ(α + 1)P.V. \int \frac{ϕ(x + w, t) − ϕ(x, t)}{|w|^{d+α}} dw, $$

and

$$ h_α(x) = \int ℝ^d \frac{1}{|w|^{d+α}} e^{-\frac{|x−x_0(x, w)|}{|w|}} dw, $$

where $[x, y], x, y ∈ ℝ^d$, denotes the straight line segment connecting $x$ and $y$, and $x_0(x, w)$ is the point closest to $x$ in the intersection of $∂Ω$ with the ray starting at $x$ in the direction $w$.

The function $h_α$ is well defined by (9) and converges to $∞$ when $x → ∂Ω$, see Proposition 1 in Section 4.

**Remark 1.** Theorem 1.1 remains true with slightly modified proofs for generalized versions of the model. For example, (4) may be replaced by the more general condition

$$ M(v) ∼ 1/|v|^{d+α} \quad \text{as } |v| → ∞. $$

An example of an equilibrium distribution coming from stochastic analysis is the probability density function of an $α$-stable process, see [6].

**Remark 2.** Another possible generalization is to permit a more general collision operator, satisfying the micro-reversibility principle:

$$ Q(f) = \int ℝ^d [σ(v, v′)M(v)f(v′) − σ(v′, v)M(v′)f(v)] dv′ $$

where the cross-section $σ$ is symmetric, i.e. $σ(v, v′) = σ(v′, v)$, $v, v′ ∈ ℝ^d$, and bounded from above and away from zero:

$$ 0 < ν_1 ≤ σ(v, v′) ≤ ν_2 < ∞. $$

The derivation of macroscopic limits from kinetic equations when the collision kernel has a Maxwellian as an equilibrium distribution is a classical problem studied in the pioneering works [25], [15], and [18]. Here the essential properties of the equilibrium distribution are vanishing mean velocity and finite second order moments. In the case where the equilibrium distribution is heavy-tailed, the problem
was first studied for relaxation type collision operators in [20], [19] and [4], from an an-
alytical point of view and in [16] with a probabilistic approach, obtaining as a macro-
scopic limit a fractional heat equation. These are results on whole space, and they have recently been extended to collision operators of fractional Fokker-Planck type [8] and to the derivation of fractional diffusion with drift [1, 2, 3]. The proofs of most of these results are based on the moment method introduced in [19], which will also be used here.

To find an appropriate definition of fractional diffusion in a bounded domain is not obvious since it describes the probability distribution of a jump process. The formulation of appropriate models as macroscopic limits of kinetic equations is the subject of this work and of the very recent contribution [7], where the problem of deriving a fractional heat equation from a kinetic fractional-Fokker-Planck equation is tackled with zero inflow and specular reflection boundary conditions, where the spatial domain is a circle. The main differences between this work and [7] are that we use a relaxation type collision operator, we only consider inflow boundary conditions, but we permit general, in particular nonconvex, position domains.

There are several equivalent definitions of the fractional Laplacian in the whole domain (see [17]), however, for bounded domains there are different definitions, depending on the details of the underlying stochastic process. For instance, if we consider the stochastic process consisting of a fractional Brownian motion with an \( \alpha/2 \)-stable subordinator and killed upon leaving the domain it has as infinitesimal generator the restricted fractional Laplacian (see [14])

\[
- (-\Delta|_{\Omega})^{\alpha/2} \varphi(x) := c_{d,\alpha} \text{ P.V.} \int_{\mathbb{R}^d} \frac{\varphi(y) 1_{\Omega}(y) - \varphi(x)}{|x-y|^{d+\alpha}} dy \, , \quad c_{d,\alpha} > 0 . \tag{11}
\]

This operator has also been derived in [7] as macroscopic limit of a kinetic equation in a circle, subject to zero inflow boundary conditions. The macroscopic operator of Theorem 1.1 can be written in the similar form,

\[
- h_\alpha \varphi + L_\alpha(\varphi) = \Gamma(\alpha + 1) \text{ P.V.} \int_{\mathbb{R}^d} \frac{\varphi(y) 1_{S_\alpha(x)}(y) - \varphi(x)}{|x-y|^{d+\alpha}} dy , \tag{12}
\]

where \( S_\alpha(x) \) denotes the biggest star-shaped subdomain of \( \Omega \) with center in \( x \). Obviously, (11) and (12) coincide for convex \( \Omega \) (the situation of [7]). The difference in the stochastic process interpretations of (11) and (12) is that in the latter jumps are only permitted along straight lines, which do not leave the domain.

For completeness we also mention the spectral fractional Laplacian defined as follows: The operator \(-\Delta\) subject to homogenous Dirichlet boundary conditions along \( \partial \Omega \) has positive eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \ldots \) with corresponding normalized eigenfunctions \( \{e_k\}_{k \geq 1} \). The spectral fractional Laplacian (subject to homogenous Dirichlet boundary conditions) is defined by

\[
(-\Delta_{\Omega})^{\alpha/2} \varphi(x) := \sum_{i=1}^{\infty} \lambda_i^{\alpha/2} e_i(x) \int_{\Omega} e_i(y) \varphi(y) dy . \tag{13}
\]

It can also be interpreted as generating a stochastic process (see [9]). A representation formula similar to (11) and (12) has been derived in [23]:

\[
(-\Delta_{\Omega})^{\alpha/2} \varphi(x) = c_{d,\alpha} \text{ P.V.} \int_{\Omega} [\varphi(x) - \varphi(y)] J(x, y) dy + c_{d,\alpha} \kappa(x) \varphi(x) , \quad \text{for } x \in \Omega
\]

where the functions \( J \) and \( \kappa \) and the constant \( c_{d,\alpha} \) satisfy (with positive constants \( C_1, C_2 \text{ and } C_3 \)
\[ C_1 \delta(x) \delta(y) \leq J(x, y) \leq C_2 \min \left( \frac{1}{|x - y|^{d+\alpha}}, \frac{\delta(x) \delta(y)}{|x - y|^{d+2\alpha}} \right), \]

and

\[ C_3^{-1} \delta^{-\alpha}(x) \leq \kappa(x) \leq C_3 \delta^{-\alpha}(x). \]

In [22] it is proven that the two operators \((-\Delta_\Omega)^{\alpha/2}\) and \((-\Delta|_\Omega)^{\alpha/2}\) are different since, for instance, the eigenfunctions of the former are smooth up to the boundary whereas the eigenfunctions of the latter are no better than Hölder continuous up to the boundary. In recent years fractional Laplace operators have been extensively used since they seem to be more suitable for the description of phenomena such as contaminants propagating in water [5], plasma physics [12], among many others (see [24] and [21]). However, there is some literature where for the fractional Laplacian on bounded domains the definitions (11) and (13) are used interchangeably, thus leading to false results.

2. Uniform estimates and modified test functions. It is a standard result of kinetic theory that the initial-boundary value problem (1)–(3) with an equilibrium distribution \(M\) satisfying (4)–(6) and an initial position density \(\rho^{in} \in L^1(\,dx)\) has a unique solution, which is nonnegative, if the same holds for \(\rho^{in}\) (see, e.g. [10], Chapter XXI). This will be assumed in the following, where we always denote by \(dx, dv,\) and \(dt\) the Lebesgue measures on \(\Omega, \mathbb{R}^d,\) and, respectively, \((0, \infty)\). We start with standard estimates:

**Lemma 2.1.** Let \(\rho^{in} \in L^2_+ (\,dx)\). Then the solution \(f_\varepsilon\) of (1)–(3) satisfies

\[ f_\varepsilon \in L^\infty(\,dt, L^2_+ (\,dx \,dv/M)) \quad \text{uniformly as } \varepsilon \to 0, \]

and, with \(\rho_\varepsilon := \rho_{f_\varepsilon},\)

\[ f_\varepsilon - \rho_\varepsilon M = O(\varepsilon^{\alpha/2}) \quad \text{in } L^2(\,dx \,dv\,dt/M), \quad \text{as } \varepsilon \to 0. \]

**Proof.** Multiplication of (1) by \(f_\varepsilon/M,\) integration with respect to \(x\) and \(v,\) the divergence theorem, and the boundary condition (2) yield

\[ \frac{\varepsilon^\alpha}{2} \frac{d}{dt} \int_\Omega \int_{\mathbb{R}^d} \frac{f_\varepsilon^2}{M} \,dv \,dx + \varepsilon \int_{\Gamma^+} v \cdot \nu \frac{f_\varepsilon^2}{2M} \,dv \,dx = \int_\Omega \int_{\mathbb{R}^d} Q(f_\varepsilon) \frac{f_\varepsilon}{M} \,dv \,dx = -\|f_\varepsilon - \rho_\varepsilon M\|^2_{L^2(dx \,dv/M)}, \]

where the second equality is a well known fact and the result of a straightforward computation (see, e.g. [11]). The nonnegativity of the second term and an integration with respect to \(t\) over \((0, T)\) give

\[ \frac{\varepsilon^\alpha}{2} \|f_\varepsilon(\cdot, T)\|^2_{L^2(dx \,dv/M)} + \int_0^T \|f_\varepsilon - \rho_\varepsilon M\|^2_{L^2(dx \,dv/M)} \,dt \leq \frac{\varepsilon^\alpha}{2} \|\rho^{in}\|^2_{L^2(dx)}, \]

completing the proof. \(\square\)

For the proof of Theorem 1.1 we employ the moment method introduced in [19], which relies on test functions solving a suitably chosen adjoint problem. For given \(\varphi \in \mathcal{D}_\Omega\) the function \(\chi_\varepsilon(x,v,t)\) is the solution of the stationary kinetic equation

\[ \chi_\varepsilon - \varepsilon v \cdot \nabla_x \chi_\varepsilon = \varphi, \]

subject to the inflow boundary condition

\[ \chi_\varepsilon = 0 \quad \text{on } \Gamma^+. \]
Note that the left hand side of (15) is an adjoint version of a part of (1), where only the loss term of the collision operator and the transport operator have been kept. We can readily solve (15), (16) via the method of characteristics, obtaining
\begin{equation}
\chi_\varepsilon(x, v, t) = \int_0^{r(x, v)/\varepsilon} e^{-s} \varphi(x + \varepsilon sv, t) ds,
\end{equation}
where \( r(x, v) = \frac{|x - x_0(x, v)|}{|v|} \),
and \( x_0(x, v) \) is the point closest to \( x \) in the intersection of \( \partial \Omega \) and the ray starting at \( x \) with direction \( v \). In the following a different representation will be convenient:
\begin{equation}
\chi_\varepsilon(x, v, t) = \varphi(x, t) \left( 1 - e^{-r(x, v)/\varepsilon} \right) + \int_0^{r(x, v)/\varepsilon} e^{-s} [\varphi(x + \varepsilon sv, t) - \varphi(x, t)] ds.
\end{equation}
This already shows the main difference to the whole space situation [19], which is the boundary layer correction in the parenthesis on the right hand side of (18).
In the following we shall need a uniform boundedness result.

**Lemma 2.2.** Let \( \varphi \in \mathcal{D}_\Omega \) and let \( \chi_\varepsilon \) be given by (17). Then
\[
\| \chi_\varepsilon \|_{L^2(M \, dx \, dv \, dt)} \leq \| \varphi \|_{L^2(dx \, dt)}, \quad \| \partial_t \chi_\varepsilon \|_{L^2(M \, dx \, dv \, dt)} \leq \| \partial_t \varphi \|_{L^2(dx \, dt)}.
\]

**Proof.** Multiplication of (15) by \( M \chi_\varepsilon \) and integration with respect to \( v \) gives
\[
\| \chi_\varepsilon \|_{L^2(M \, dv \, dt)}^2 - \frac{\varepsilon}{2} \nabla_x \cdot \int_{\mathbb{R}^d} vM \chi_\varepsilon^2 dv = \varphi \int_{\mathbb{R}^d} M \chi_\varepsilon dv \leq \| \varphi \|_{L^2(M \, dv \, dt)} \| \chi_\varepsilon \|_{L^2(M \, dv \, dt)},
\]
where the Cauchy-Schwarz inequality and the normalization of \( M \) has been used. Integration with respect to \( x \) and \( t \), the divergence theorem, and the boundary condition (16) for \( \chi_\varepsilon \) lead to
\[
\| \chi_\varepsilon \|_{L^2(M \, dx \, dv \, dt)}^2 - \frac{\varepsilon}{2} \int_0^\infty \int_{\Gamma^{-}} \nu \cdot vM \chi_\varepsilon^2 dv \, ds \, dt \leq \| \varphi \|_{L^2(dx \, dt)} \| \chi_\varepsilon \|_{L^2(M \, dx \, dv \, dt)},
\]
completing the proof of the first inequality. The proof of the second is analogous after differentiation of (15) with respect to \( t \). \( \square \)

3. **Proof of Theorem 1.1.** With \( \varphi \in \mathcal{D}_\Omega \) and \( \chi_\varepsilon \) defined by (17), multiplication of (1) by \( \chi_\varepsilon \) and integration with respect to \( x, v \) and \( t \) gives
\[
- \int_0^\infty \int_{\mathbb{R}^d} \int_{\Omega} f_\varepsilon \partial_t \chi_\varepsilon \, dx \, dv \, dt - \int_{\mathbb{R}^d} \int_{\Omega} \rho^{in} M \chi_\varepsilon(t = 0) \, dx \, dv \\
= \varepsilon^{-\alpha} \int_0^\infty \int_{\mathbb{R}^d} \int_{\Omega} (\rho_\varepsilon M \chi_\varepsilon - f_\varepsilon \chi_\varepsilon + f_\varepsilon \varepsilon v \cdot \nabla_x \chi_\varepsilon) \, dx \, dv \, dt \\
= \int_0^\infty \int_{\Omega} \rho_\varepsilon \left( \varepsilon^{-\alpha} \int_{\mathbb{R}^d} M (\chi_\varepsilon - \varphi) \, dx \right) \, dt.
\]
In the sequel we shall need the following notation: For \( x, y \in \mathbb{R}^d \) we denote by \( [x, y] \) the line segment connecting \( x \) and \( y \). Furthermore, we denote by \( S_\Omega(x) \) the largest star shaped subdomain of \( \Omega \) with center \( x \), i.e.
\[
S_\Omega(x) := \{ y \in \Omega : [x, y] \subset \Omega \}.
\]
The heart of our analysis is the asymptotics for the term in parentheses on the right hand side of (19).
Lemma 3.1. Let \( \varphi \in D_\Omega \) and let \( \chi_\varepsilon \) be given by (17). Then

\[
\lim_{\varepsilon \to 0} \varepsilon^{-\alpha} \int_{\mathbb{R}^d} M(\chi_\varepsilon - \varphi) dv = -h_\alpha \varphi + \mathcal{L}_\alpha(\varphi)
\]

locally uniformly in \( x \) and \( t \), where

\[
h_\alpha(x) = \int_{\mathbb{R}^d} \frac{1}{|v|^{d+\alpha}} e^{-|x-x_0(x,v)|/\varepsilon} dv,
\]

\[
\mathcal{L}_\alpha(\varphi)(x,t) = \Gamma(\alpha + 1) \text{P.V.} \int_{S\Omega(x)} \frac{\varphi(y,t) - \varphi(x,t)}{|y-x|^{d+\alpha}} dy.
\]

Proof. The representation (18) of \( \chi_\varepsilon \) induces the splitting

\[
\varepsilon^{-\alpha} \int_{\mathbb{R}^d} M(\chi_\varepsilon - \varphi) dv = -h_\alpha \varphi + \mathcal{L}_\alpha(\varphi),
\]

with

\[
h_\alpha^\varepsilon(x) = \varepsilon^{-\alpha} \int_{\mathbb{R}^d} M(v) e^{-r(x,v)/\varepsilon} dv,
\]

\[
\mathcal{L}_\alpha^\varepsilon(\varphi)(x,t) = \varepsilon^{-\alpha} \int_{\mathbb{R}^d} \int_0^{r(x,v)/\varepsilon} M(v) e^{-s}[\varphi(x + \varepsilon sv,t) - \varphi(x,t)] ds\ dv.
\]

We shall consider these parts separately. In both cases we shall start by proving that the small velocities do not contribute to the limit. This splits the rest of the proof into 4 steps.

Step 1. We consider the contribution to \( h_\alpha^\varepsilon \) coming from the small velocities. For \( |v| \leq 1 \) we have \( r(x,v) \geq \delta(x) = \text{dist}(x, \partial \Omega) \). Therefore

\[
\varepsilon^{-\alpha} \int_{|v| \leq 1} M(v) e^{-r(x,v)/\varepsilon} dv \leq \varepsilon^{-\alpha} e^{-\delta(x)/\varepsilon} \leq c \frac{\varepsilon^{2-\alpha}}{\delta(x)^2},
\]

since the map \( z \mapsto z^2 e^{-z}, z \geq 0, \) is bounded. Here and in the following, \( c \) denotes constants independent of \( \varepsilon \).

Step 2. The previous step implies that \( h_\alpha^\varepsilon \) is asymptotically equivalent to

\[
\varepsilon^{-\alpha} \int_{|v| > 1} |v|^{-d-\alpha} e^{-r(x,v)/\varepsilon} dv.
\]

In this integral we make the coordinate transformation \( w = \varepsilon v \). Observing that

\[
\frac{r(x,w/\varepsilon)}{\varepsilon} = \frac{|x-x_0(x,w/\varepsilon)|}{|w|} = r(x,w),
\]

since \( x_0(x,w/\varepsilon) = x_0(x,w) \), the expression in (21) is equal to

\[
\int_{|w| > 1} |w|^{-d-\alpha} e^{-r(x,w)} dw.
\]

For proving that this converges to \( h_\alpha(x) \), we need to estimate

\[
\int_{|w| \leq \varepsilon} |w|^{-d-\alpha} e^{-r(x,w)} dw \leq \int_{|w| \leq \varepsilon} |w|^{-d-\alpha} e^{-\delta(x)/|w|} dw
\]

\[
= |S^d| \gamma^{-\alpha} \int_{\delta(x)/\varepsilon}^{\infty} s^{\alpha-1} e^{-s} ds
\]

\[
\leq |S^d| \frac{\varepsilon^{2-\alpha}}{\delta(x)^2} \sup_{\gamma \geq 0} \left( \gamma^{2-\alpha} \int_{\gamma}^{\infty} s^{\alpha-1} e^{-s} ds \right),
\]
where we have used the change of variables $\delta(x)/|w| = s$. The supremum is finite since the integrand is bounded and decays exponentially as $s \to \infty$.

Combining this result with Step 1 shows that
\[
|h_\alpha^\varepsilon(x) - h_\alpha(x)| \leq c \frac{\varepsilon^{2-\alpha}}{|x|^2},
\]

implying pointwise convergence of $h_\alpha^\varepsilon$ to $h_\alpha$ in $\Omega$. Since $|\varphi(x,t)| \leq c\delta(x)^2$, the convergence of $h_\alpha^\varepsilon \varphi$ to $h_\alpha \varphi$ is uniform in $(x,t)$.

**Step 3.** We analyze the contributions from the small velocities to $\mathcal{L}_\alpha^\varepsilon(\varphi)$. For the test function difference, we apply the Taylor expansion:

\[
\varepsilon^{-\alpha} \int_{|v| \leq 1} \int_0^{r(x,v)/\varepsilon} M(v) e^{-s} \left( \varepsilon sv \cdot \nabla_x \varphi(x,t) + \varepsilon \frac{s^2}{2} v^t \nabla^2_x \varphi(\hat{x},t) v \right) ds\, dv \\
\leq \varepsilon^{-\alpha} \nabla_x \varphi(x,t) \cdot \int_{|v| \leq 1} v M(v) \int_0^{r(x,v)/\varepsilon} s e^{-s} ds\, dv \\
+ \varepsilon^{2-\alpha} c \int_{|v| \leq 1} |v|^2 M(v) dv \int_0^\infty s^2 e^{-s} ds.
\]

Note that $\hat{x} \in \Omega$ lies between $x$ and $x + \varepsilon sv$, and the superscript $tr$ denotes transposition. In the first term on the right hand side we change the order of integration:

\[
\int_0^{\delta(x)/\varepsilon} \int_{|v| \leq 1} v M(v) dv \int_0^{r(x,v)/\varepsilon} s e^{-s} ds\, dv = \int_0^{\infty} \int_{|v| \leq 1} v M(v) dv \int_0^{s e^{-s}} ds\, dv \\
= \int_0^{\delta(x)/\varepsilon} \int_{|v| \leq 1} v M(v) dv ds + \int_{\delta(x)/\varepsilon}^{\infty} \int_{|v| \leq 1} v M(v) dv ds.
\]

In the first term on the right hand side, the restriction $\varepsilon s \leq r(x,v)$ can be omitted, since it is automatically satisfied for $\varepsilon s \leq \delta(x) \leq r(x,v)$. As a consequence this term vanishes by $M$ being even. The last term can be estimated by

\[
\int_{\delta(x)/\varepsilon}^{\infty} \int_{|v| \leq 1} |v|M(v) dv \leq c \frac{\varepsilon}{\delta(x)} \sup_{\gamma \geq 0} \left( \gamma \int_{\gamma}^{\infty} s e^{-s} ds \right).
\]

Since $\varphi \in \mathcal{D}_\Omega$ implies $|\nabla_x \varphi(x,t)| \leq c\delta(x)$ (by the fact that $\varphi$ vanishes on the boundary), we have the result

\[
\varepsilon^{-\alpha} \int_{|v| \leq 1} \int_0^{r(x,v)/\varepsilon} M(v) e^{-s} [\varphi(x+\varepsilon sv,t) - \varphi(x,t)] ds\, dv = O(\varepsilon^{2-\alpha}),
\]

uniformly in $(x,t)$.

**Step 4.** It remains to consider

\[
\varepsilon^{-\alpha} \int_{|v| \geq 1} \int_0^{r(x,v)/\varepsilon} |v|^{-d-\alpha} e^{-s} [\varphi(x + \varepsilon sv, t) - \varphi(x,t)] ds\, dv \\
= \int_{|w| \geq \varepsilon} \int_0^{r(x,w)} |w|^{-d-\alpha} e^{-s} [\varphi(x + sw, t) - \varphi(x,t)] ds\, dw \\
= \int_0^\infty s^{d+\alpha} e^{-s} \int_{|w| \geq \varepsilon, s \leq r(x,w)} \frac{\varphi(x + sw, t) - \varphi(x,t)}{|sw|^{d+\alpha}} dw\, ds,
\]

(22)
where we performed the change of variables \( w = \varepsilon v \). By the coordinate transformation \( x + sw = y \) the condition \( s < r(x, w) \) becomes \( s \)-independent:

\[
|x - y| < |x - x_0(x, y - x)| \iff y \in \mathcal{S}_\Omega(x) .
\]

Therefore (22) is equal to

\[
\int_0^\infty s^\alpha e^{-s} \int_{\mathcal{S}_\Omega(x)\setminus B_{r_\varepsilon}(x)} \frac{\varphi(y, t) - \varphi(x, t)}{|y - x|^{d+\alpha}} dy ds ,
\]

where \( B_r(x) \) denotes the ball with center \( x \) and radius \( r \). By the splitting \( \mathcal{S}_\Omega(x) = (\mathcal{S}_\Omega(x) \setminus B_{r_\varepsilon}(x)) \cup (\mathcal{S}_\Omega(x) \cap B_{r_\varepsilon}(x)) \) this converges to \( \mathcal{L}_\alpha(\varphi) \) if

\[
\int_0^\infty s^\alpha e^{-s} \int_{\mathcal{S}_\Omega(x)\cap B_{r_\varepsilon}(x)} \frac{(y - x) \cdot \nabla_x \varphi(x, t) + (y - x)^r \nabla^2_x \varphi(\hat{x}, t)(y - x)/2}{|y - x|^{d+\alpha}} dy ds
\]

vanishes, and we have

\[
\int_0^\infty s^\alpha e^{-s} \int_{B_{r_\varepsilon}(x)} |y - x|^{-2d-\alpha} dy ds
= c e^{2-\alpha} \int_0^\infty s^2 e^{-s} ds .
\]

The estimation of the first term is more subtle. We split the domain of integration as follows:

\[
\{ (s, x) : 0 < s < \infty, x \in \mathcal{S}_\Omega(x) \cap B_{r_\varepsilon}(x) \}
= A_1 \cup A_2 \cup A_3
= \{ (s, x) : 0 < s < \delta(x)/\varepsilon, x \in B_{r_\varepsilon}(x) \}
\cup \{ (s, x) : s > \delta(x)/\varepsilon, x \in (\mathcal{S}_\Omega(x) \cap B_{r_\varepsilon}(x)) \setminus B_{\delta(x)}(x) \}
\cup \{ (s, x) : s > \delta(x)/\varepsilon, x \in B_{\delta(x)}(x) \}
\]

using \( B_{r_\varepsilon}(x) \subset \mathcal{S}_\Omega(x) \) for \( \varepsilon \ll \delta(x) \) in \( A_1 \). Actually, the integral with respect to \( y \) has to be understood as a principal value for \( \alpha \geq 1 \). Since

\[
\text{P.V.} \int_{B_{r_\varepsilon}(x)} \frac{y - x}{|y - x|^{d+\alpha}} dy = 0, \quad \text{for } r > 0 ,
\]

the integrals over \( A_1 \) and \( A_3 \) vanish, and we have

\[
\int_0^\infty s^\alpha e^{-s} \text{P.V.} \int_{\mathcal{S}_\Omega(x)\cap B_{r_\varepsilon}(x)} \frac{(y - x) \cdot \nabla_x \varphi(x, t)}{|y - x|^{d+\alpha}} dy ds ,
\]

which can be estimated by

\[
c \delta(x) \int_{\delta(x)/\varepsilon}^\infty s^\alpha e^{-s} \int_{B_{r_\varepsilon}(x)\setminus B_{\delta(x)}} |y - x|^{1-d-\alpha} dy ds
= c \delta(x) \int_{\delta(x)/\varepsilon}^\infty s^\alpha e^{-s} \int_{\delta(x)}^{\varepsilon s} r^{-\alpha} dr ds .
\]

With

\[
\int_{\delta(x)}^{\varepsilon s} r^{-\alpha} dr \leq \begin{cases} 
c (\varepsilon s)^{1-\alpha}, & \alpha < 1, 
\log(\varepsilon s/\delta(x)), & \alpha = 1, 
c \delta(x)^{1-\alpha}, & \alpha > 1, 
\end{cases}
\]

it is straightforward to obtain that (23) is $O(\varepsilon^2)$ for $\alpha \neq 1$ and $O(\varepsilon \log(1/\varepsilon))$ for $\alpha = 1$, uniformly in $(x, t)$. This completes the proof of the uniform convergence of $L^\varepsilon_\alpha(\varphi)$ to $L_\alpha(\varphi)$.

**Corollary 1.** Let $\varphi \in D_\Omega$ and let $\chi_\varepsilon$ be defined by (17). Then

$$
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} M(v)[\chi_\varepsilon(x, v, t) - \varphi(x, t)]dv = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} M(v)[\partial_t \chi_\varepsilon(x, v, t) - \partial_t \varphi(x, t)]dv = 0,
$$

uniformly with respect to $(x, t) \in \text{supp}(\varphi)$.

**Proof.** The first statement is an immediate consequence of Lemma 3.1. The second statement follows, since $\varphi \in D_\Omega$ implies $\partial_t \varphi \in D_\Omega$ and since the map $\partial_t \varphi \mapsto \partial_t \chi_\varepsilon$ is the same as $\varphi \mapsto \chi_\varepsilon$.

The remaining steps in the proof of Theorem 1.1 are rather standard. As a consequence of Lemma 2.1 and of the estimate

$$
|\rho_\varepsilon| \leq \|f_\varepsilon\|_{L^2(dv/M)} \quad \Rightarrow \quad \|\rho_\varepsilon\|_{L^2(dx)} \leq \|f_\varepsilon\|_{L^2(dv/M)}
$$

we obtain

$$
\rho_\varepsilon \rightharpoonup^* \rho \quad \text{in} \quad L^\infty(dt; L^2(dx)) , \quad f_\varepsilon \rightharpoonup^* \rho M \quad \text{in} \quad L^\infty(dt; L^2(dx dv/M)) ,
$$

when restricting to subsequences. Now we are ready for passing to the limit in (19). We decompose the first term by using

$$
\int_{\mathbb{R}^d} f_\varepsilon \partial_t \chi_\varepsilon dv = \int_{\mathbb{R}^d} (f_\varepsilon - \rho_\varepsilon M) \partial_t \chi_\varepsilon dv + \rho_\varepsilon \int_{\mathbb{R}^d} M \partial_t \chi_\varepsilon dv .
$$

The first term on the right hand side tends to zero by

$$
\left| \int_{\mathbb{R}^d} (f_\varepsilon - \rho_\varepsilon M) \partial_t \chi_\varepsilon dv \right| \leq \|f_\varepsilon - \rho_\varepsilon M\|_{L^2(dv/M)} \|\partial_t \chi_\varepsilon\|_{L^2(M dv)} ,
$$

Lemma 2.1, and Lemma 2.2. In the second term we may pass to the limit $\rho \partial_t \varphi$ by the weak* convergence of $\rho_\varepsilon$ and the strong convergence of $\int_{\mathbb{R}^d} M \partial_t \chi_\varepsilon dv$ (Corollary 1). The limit in the second term of (19) is a consequence of Corollary 1. Finally, passing to the limit in the right hand side of (19) is justified by the weak* convergence of $\rho_\varepsilon$ and by Lemma 3.1. This completes the proof of Theorem 1.1.

4. **Discussion.** In this section we discuss properties of the fractional diffusion operator. First we show that the function $h_\alpha$ defined in (9) is well defined and tends to infinity at the boundary of $\Omega$.

**Proposition 1.** Let $h_\alpha$ be defined by (9), then there exists $C > 0$ such that

$$
0 < h_\alpha(x) \leq C \delta(x)^{-\alpha}, \quad x \in \Omega .
$$

**Proof.** In order to prove (25) let us choose $x \in \Omega$. Next, let us introduce a spherical coordinates change of variables in the integral (9), and note the following:

$$
h_\alpha(x) = \int_{|w|=1} \int_0^\infty \frac{1}{r^{d+\alpha}} \exp{-|x-z_0(x, w)|/r} r^{d-1} dr \, d\sigma(w) .
$$
where $r$ denotes the radial variable. Now, noting that $|x - x_0(x,w)| \geq \delta(x)$ and introducing the change of variables $r = \delta(x)\eta$ we obtain

$$h_\alpha(x) \leq \int_{|w| = 1} \int_0^\infty \int_0^\infty \frac{1}{\eta^{d+\alpha}} e^{-\delta(x)/r} r^{d-1} dr d\sigma(w)$$

$$\leq \int_{|w| = 1} \frac{1}{\delta(x)\eta^{1+\alpha}} e^{-1/\eta} \delta(x) d\eta d\sigma(w)$$

$$= \frac{1}{\delta^\alpha(x)} \int_{|w| = 1} \int_0^\infty \frac{1}{\eta^{1+\alpha}} e^{-1/\eta} d\eta d\sigma(w),$$

from which (25) follows. In addition, we obtain that $h_\alpha(x)$ is finite for every $x \in \Omega$. \hfill \square

In [13] it has been shown that the fractional heat equation

$$\partial_t u(x,t) = -c_{d,\alpha} \, \text{P.V.} \int_{\mathbb{R}^d} \frac{u(x,t) - u(y,t)}{|x-y|^{d+\alpha}} dy$$

in $\Omega, t > 0$,

$$u(x,t) = 0$$

in $\mathbb{R}^d \setminus \Omega$,

$$u(x,0) = u^i_n(x)$$

in $\Omega$,

has a unique solution such that for any fixed $t_0 > 0$ the following estimate holds

$$\sup_{t \geq t_0} \left\| \frac{u(\cdot,t)}{\delta^\alpha/2(\cdot)} \right\|_{C^\alpha(\Omega)} \leq C(t_0) \|u^i_n\|_{L^2(\Omega)}.$$ 

Therefore, for any fixed time $t > 0$, $u(x,t)$ behaves like $\delta^\alpha/2(x)$ when $x \to \partial \Omega$.

In this work we neither prove the uniqueness of weak solutions nor any Hölder regularity results, however, formally using $\varphi(x,t) = \rho(x,t)1_{[0,T]}(t)$ in (8) yields

$$\frac{1}{2} \|\rho(\cdot,T)\|_{L^2(\Omega)}^2 + \int_0^T \int_\Omega h_\alpha \rho^2 dx dt$$

$$+ \frac{\Gamma(\alpha + 1)}{2} \int_0^T \int_{x,y \in \partial\Omega} \frac{(\rho(x) - \rho(y))^2}{|x-y|^{d+\alpha}} dx dy dt$$

$$= \frac{1}{2} \|\rho^i_n\|_{L^2(\Omega)}^2.$$

This implies uniqueness at least formally. Also the boundedness of the second integral together with Proposition 1 induces results on the behaviour of $\rho$ close to the boundary. In particular for $\alpha > 1$, as a consequence of Proposition 1, $h_\alpha$ is not integrable, implying some decay of $\rho(x,t)$ as $\delta(x) \to 0$.

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