ON REALIZATION OF TANGENT CONES OF HOMOLOGICALLY AREA-MINIMIZING COMPACT SINGULAR SUBMANIFOLDS

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ABSTRACT. We show that every area-minimizing hypercone and every oriented Lawlor cone in \cite{Law91} can be realized as a tangent cone at a point of some homologically area-minimizing singular compact submanifold. In particular this generalizes the result of N. Smale \cite{Sma99}.

1. Introduction

Let $C$ be a $k$-dimensional cone over link $L \subset S^{n-1}(1)$ in an Euclidean space $(\mathbb{R}^n, g_E)$. We call $C$ area-minimizing (mass-minimizing) if $C_1 = C \cap B^n(1)$ has least mass among all integral (normal) currents (see \cite{FF60}) with boundary $L$. We say that a $d$-closed compactly supported integral current in a Riemannian manifold is homologically area-minimizing (mass-minimizing) if it has least mass in its homology class of integral (normal) currents.

A well-known result of Federer (Theorem 5.4.3 in \cite{Fed69}, also see Theorem 35.1 and Remark 34.6 (2) in Simon \cite{Sim83}) asserts that a tangent cone at a point of an area-minimizing rectifiable current is itself area-minimizing. This paper studies its converse realization question by compact submanifolds ($\star$):

Can any area-minimizing cone be realized as a tangent cone at a point of some homologically area-minimizing compact singular submanifold?

Through techniques of geometric analysis and Allard’s regularity theorem, N. Smale found realizations for all strictly minimizing, strictly stable hypercones (see \cite{HS85}) in \cite{Sma99}. They are first examples of codimension one homological area-minimizers with singularities.

Very recently, different realizations of many area-minimizing cones, including all homogeneous minimizing hypercones (classified by Lawlor \cite{Law91}, also see \cite{Law72} and \cite{Zhab}) and special Lagrangian cones, by extending local calibration pairs were discovered in \cite{Zha}.

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However in general the answer to (⋆) is still far to be known. In this paper, we focus on two important classes of mass-minimizing cones – minimizing hypercones\(^1\) and oriented Lawlor cones.

For hypercones, two long-term standing conjectures (or equivalent versions) raised by Simon, Hardt and Simon respectively are the followings.

**Conjecture 1.1.** Except trivial examples in low dimensions, all minimizing hypercones are strictly area-minimizing?

**Conjecture 1.2.** Any non-trivial strictly area-minimizing hypecone is always strictly stable?

Up to now it is unclear how far it is for a minimizing hypercone to be strictly stable and strictly area-minimizing. An important characterization of minimizing hypercones in [HS85] is that each of them possesses a canonical singular “calibration”.

By Lawlor cones we mean area-minimizing cones shown in [Law91]. He studied when certain preferred bundle structure (somewhat analogous to that in [HS85] for hypercones, nevertheless involving curvatures more heavily without the limitation to codimension one) of some angular neighborhood of a minimal cone exists, and successfully added quite a few interesting new oriented area-minimizing cones (and non-orientable area-minimizing cones in the sense of modulo 2 as well). In the oriented case, such bundle structure naturally induces a “calibration” of the cone that is singular in a set of codimension one and possibly also along the cone.

By virtue of these peculiar calibrations of minimizing hypercones and oriented Lawlor cones, we obtain realizations for them.

**Theorem 1.3.** Every minimizing hypercone can be realized to (⋆).

**Remark 1.4.** Our construction removes the requirements of a minimizing hypercone’s being strictly stable and being strictly minimizing in [Sma99]. Hence the case of codimension one is completely settled.

**Theorem 1.5.** Every oriented Lawlor cone can be realized to (⋆).

**Remark 1.6.** This answers affirmatively to (⋆) for lots of area-minimizing cones of higher codimensions, for instance, a minimal cone \(C\) over a product of two or more spheres satisfying (1) \(\dim(C) > 7\), or (2) \(\dim(C) = 7\) with none of the spheres being a circle (cf. Theorem 5.1.1 in [Law91]). These cones do not split. Namely, they cannot be written as products of two or more area-minimizing cones of lower dimensions (vs. N. Smale [Sma00]).

\(^1\) By [Fed74] or [Mor86], the area-minimality of a hypercone is equivalent to its mass-minimality. So we say minimizing for short.
The paper is organized as follows. In §2 our preferred model $S$ of construction is introduced. By a monotonicity result of Allard, we get Lemma 3.1 which helps us transform the global realization question to a local problem around $S$ in §4. Thus, we only need to construct a smooth metric $\bar{g}$ on some neighborhood $\bar{U}$ of $S$ such that $S$ is homologically area-minimizing in $\bar{U}$.

We discuss the case of codimension one in §5. There are two steps. First suitably extend the canonical (local, singular and non-coflat) calibrations around $p_1$ and $p_2$ (see §2) to a $C^1$ closed form $\Phi$ in a neighborhood $\bar{U}$ of $S$. Then a smooth metric $\bar{g}$ can be created to make $\Phi$ a $C^1$ calibration of $S$. Hence we gain the homological area-minimality of $S$ in $\bar{U}$.

In §6 realizations of oriented Lawlor cones are constructed. The idea is roughly the same. However the calibration is discontinuous in a set of codimension one. So we consider its regularization through convolution for the desired local homological area-minimality of $S$. Although the approximating closed forms may have comass greater than one somewhere, by the mildness of calibrations in [Law91] and Lebesgue’s bounded convergence theorem, the needed area-minimality can be achieved.

2. Model of Construction

Given a $k$-dimensional cone $C \subset \mathbb{R}^N$. As in [Sma99], consider $\Sigma_C \triangleq (C \times \mathbb{R}) \cap S^N(1)$ in $\mathbb{R}^{N+1}$. Let $M$ be an embedded oriented connected compact $k$-dimensional submanifold in some $N$-dimensional oriented compact manifold $T$ with $[M] \neq [0] \in H_k(T; \mathbb{Z})$. Within smooth balls round a point of $M$ and a regular point of $\Sigma_C$ respectively one can connect $T$ and $S^N(1)$, $M$ and $\Sigma_C$ simultaneously through one connected sum. Denote by $X$ and $S$ the resulting manifold and submanifold (singular at two points $p_1$ and $p_2$). Apparently $[S] \neq [0] \in H_k(X; \mathbb{Z})$.

3. Positive Lower Bound of Mass

The lemma below will play a key role in §4

**Lemma 3.1.** Let $g$ be a metric on a compact manifold $X$, $W \subseteq X$ an open domain where $\overline{W}$ forms a manifold with nonempty boundary $\partial \overline{W}$, and $\alpha$ a positive number. Then there exists $\beta = \beta_{\alpha, g|_W} > 0$ such that every rectifiable current $K$ in $W$ with no boundary, vanishing generalized mean curvature vector field $\delta K$ and at least one point in its support $\alpha$ away from $\partial \overline{W}$ has mass greater than $\beta$.

**Proof.** By Nash’s embedding theorem [Nas56], $(\overline{W}, g|_{\overline{W}})$ can be isometrically embedded through a map $f$ into some Euclidean space $(\mathbb{R}^s, g_{E})$. Then $f_\#K$ is a rectifiable current of $f(\overline{W})$. Denote the induced varifold by $V_{f_\#K}$. Since $K$ has no boundary in $W$ and $\delta K$ vanishes, the norm of $\delta V_{f_\#K}$ in $\mathbb{R}^s$ is bounded from above a.e. by a constant $A$ depending upon $f$ only.
Let $\overline{W}_\alpha = \{ x \in W : \text{dist}_g(x, \partial W) \geq \alpha \}$. Define $2\mu = \text{dist}_{\text{gs}}(f(\overline{W}_\alpha), f(\partial W))$. Note that the density of $V_{fK}$ is a.e. at least one on the support $\text{spt}(f_k K)$ of $f_k K$. Therefore there exists some point $p \in \text{spt}(f_k K) \cap f(W)$ with $\lambda = \text{dist}_{\text{gs}}(p, f(\partial W)) > \mu$ and density at least one.

By applying the following monotonicity result of Allard to $A$, $p$, $\mu$ and $U$ the open $\lambda$-ball centered at $p$, we obtain our statement.

**Theorem 3.2** ([All72]). Suppose $0 \leq A < \infty$, $p \in \text{support } \|V\|$, $V \in \mathcal{V}_m(U)$, where $U$ is an open region of $\mathbb{R}^s$. If $0 < \mu < \text{dist}_{\text{gs}}(p, \mathbb{R}^s - U)$ and

$$\|\delta V\|B(p, r) \leq A\|V\|B(p, r)$$

whenever $0 < r \leq \mu$, then $r^{-m}\|V\|B(p, r) \exp Ar$ is nondecreasing in $r$ for $0 < r \leq \mu$.

\[\square\]

4. **Reduction of (⋆) from Global to Local**

The following theorem indicates that the essential difficulty of (⋆) comes from local. Hence in §5 and §6 we make constructions on some neighborhood of $S$ only.

**Theorem 4.1.** Suppose $S$ is homologically area-minimizing in $(U, \bar{g})$ where $U$ is an open neighborhood of $S$ and $\bar{g}$ is a smooth metric on $U$. Then there exists a smooth metric $\hat{g}$ on the compact manifold $X$ such that $S$ is homologically area-minimizing in $(X, \hat{g})$.

**Proof.** Take open neighborhoods $W$, $W'$ and $W''$ of $S$ so that $W'' \Subset W' \Subset W \Subset U$ and the closer of $W$ (W' and W'' respectively) is a manifold with nonempty boundary. Extend $\bar{g}$ to a metric $\tilde{g}$ on $X$ with

$$\tilde{g}|_W = \bar{g}|_W.$$ 

Set $\alpha = \text{dist}_{\tilde{g}}(\partial W', \partial W)$. Let $\beta$ be the lower bound in Lemma 3.1 for $\alpha$, domain $\overline{W'}$ and $\tilde{g}|_{\overline{W'}}$. Choose $\gamma = (t\beta^{-1}\text{Vol}_g(S))^{-\frac{1}{2}} < 1$ for some large constant $t > 1$. Then construct $\hat{g}$ as follows.

\[
\hat{g} = \begin{cases} 
\gamma \tilde{g} & \text{on } W'' \\
h \tilde{g} & \text{on } W' \sim W \\
\tilde{g} & \text{on } X \sim W'
\end{cases}
\]

where $h$ is a smooth function on $\overline{W} \sim W''$, no less than $\gamma$ and equal to one near $\partial \overline{W}'$.

Now we show that $S$ is homologically area-minimizing in $(X, \hat{g})$.

By the celebrated compactness result in Federer and Fleming ([FF60]) there exists an area-minimizing current $T$ in $[S]$ with nonempty $\text{spt}T$. 

Case One: $\text{spt}T$ is not contained in $W$. According to our construction, $M(S) = \beta < \beta < M(T)$ by Lemma 3.1. Contradiction.

Case Two: $\text{spt}T \subset W$. By assumption and (4.1) $S$ is homologically area-minimizing in $(W, \hat{g}|_W)$. As a result, $S$ and $T$ share the same mass. Hence $S$ is homologically area-minimizing in $(X, \hat{g})$. □

**Remark 4.2.** $[S] \neq [0] \in H_k(X; \mathbb{Z})$ is crucial in our proof.

## 5. Realization of Minimizing Hypercones

Choose a metric $g$ for our model in §2 such that
(i). balls $B^g_{p_i}(1)$ of radius one centered at $p_i$ are disjoint, and
(ii). local model $S \cap B^g_{p_i}(1)$ in $(B^g_{p_i}(1), g|_{B^g_{p_i}(1)})$ is exactly $C_1$ in $(B^V(1), g_E|_{B^V(1)})$.

Now take $U$ to be an open neighborhood of $S$ shown in the picture.

Let us recall a beautiful result due to Hardt and Simon.

**Theorem 5.1** (Theorem 2.1 in [HS85]). Assume $C$ is an area-minimizing hypercone in $\mathbb{R}^N$. If $E$ is either one of the components $E_+, E_-$ of $\mathbb{R}^N \sim C$, then there is a unique oriented connected embedded real analytic minimizing hypersurface $H \subset E$ with $H = \partial[F], \overline{F} \subset \overline{E}, F$ open, the singular set of $H$ empty and the distance of $H$ and the origin equal to one. Moreover, $H$ has the property that for any $\xi \in E$ the ray $\{t\xi : t > 0\}$ intersects $H$ in a single point.
Hence \( E_+ \) is foliated by \( \Gamma_+ = \{ tH_+ : t > 0 \} \). Let \( X_+ \) be the oriented unit normal vector of \( \Gamma_+ \) with limit \( v_\omega \) (pointing into \( E_+ \)) along \( C \sim 0 \), and \( \phi_+ \) the oriented volume form of \( \Gamma_+ \). On \( \mathbb{R}^{N+1} \sim 0 \), define

\[
\phi = \begin{cases} 
\phi_+ & \text{in } E_+ \\
\lim_{t \to 0} \phi_+(= \lim_{t \to 0} \phi_-) & \text{in } C \sim 0 \\
\phi_- & \text{in } E_-
\end{cases}
\]

According to [HS85], outside some large ball, each \( H_\pm \) is a graph of some \( C^2 \) function on \( C \), so \( \phi \) is \( C^1 \) along \( C \sim 0 \) and smooth elsewhere.

Our strategy is the following.

**Step 1**: glue such forms around \( p_1 \) and \( p_2 \) to a form \( \Phi \) in some neighborhood of \( S \).

**Step 2**: construct a smooth metric on the neighborhood so that \( \Phi \) is a singular calibration of \( S \).

In this way a realization of a minimizing hypercone can be produced based upon §4.

Assume, for some \( 0 < 3R < 1 \), \( B_{p_1}(3R) \subset U \). Let \( r \) be the distance to the origin along \( C \) and \( \Theta \) a small angular neighborhood over \( C \cap \{ 1.4R < r < 2R \} \) shown in the figure.

Set \( \omega \) to be the unit volume form of the link \( L \) of \( C \) and \( \psi = r\omega \). Then \( d\psi \) is the oriented unit \( N \)-dimensional form of \( C \sim 0 \). Since \( \text{div } X_+ = 0 \), one has (shrink \( \Theta \) if necessary)

\[
\phi|_\Theta = [\pi^*d\psi]|_\Theta = [d(\pi^*\psi)]|_\Theta,
\]

where \( \pi \) is the projection along \( X_+ \). On \( \Theta \), let \( \pi_\omega \) be the projection to the nearest point on \( C \) and \( r = r(\pi_\omega) \). Define

\[
\Phi = d[\tau(r)(\pi^*\psi) + (1 - \tau(r))(\pi_\omega^*\psi)],
\]

where \( \tau \) is a decreasing smooth function from value one to zero on \( [1.4R, 2R] \) with the support of \( d\tau \) contained in \( [1.6R, 1.7R] \). Note that \( \Phi \) is the unit volume form of the cone in \( \{ 1.4R < r < 2R \} \cap C \).
For **Step 2**, we do some estimate on \( \Phi \). Let \( V \) be the parallel extension of \( \nu_C \) along fibers of \( \sigma, V^\perp \) the oriented unit \( N \)-vector perpendicular to \( V \). Then on \( E, \Gamma \Theta \)

\[
L_V \Phi = L_V \tau(\pi^* \psi) + (1 - \tau(\pi^* \psi)]
\]

(5.1)

\[
= d[L_V(\pi^* \psi) + (1 - \tau(\pi^* \psi))] = d[(1 - \tau(\pi^* \psi))] \]

By the foliation structure, it follows from (5.1) that (5.2)

\[
\sigma(\Gamma V^\perp) = [1 + O(d_{ge}^2)] V^\perp |_C
\]

for the minimal cone \( C \), where \( d_{ge}(\cdot) \) is the Euclidean distance to \( C \). Consequently, (5.3)

\[
(L_V V^\perp) |_C = 0.
\]

Therefore by (5.1) and (5.3)

\[
(L_V [\Phi(V^\perp)]) |_C = (L_V \Phi) |_C(V^\perp |_C).
\]

By the foliation structure, it follows from (5.1) that

\[
(L_V \Phi) |_C = \tau(\pi^* \psi) |_C.
\]

Since \( \phi \) is a calibration, we obtain

(5.4)

\[
(L_V [(\Phi(V^\perp)]) |_C = \tau(\pi^* \psi) |_C \leq 0.
\]

The same argument on \( E, \Gamma \Theta \) produces

(5.5)

\[
(L_{-V} [(\Phi(V^\perp)]) |_C = \tau(\pi^* \psi) |_C \leq 0.
\]

Hence, (5.4), (5.5) and the compactness of \([1.4R, 2R]\) imply that there exists a positive constant \( K \) such that in a sufficiently small neighborhood \( \Xi \) of \( C \cap \Theta \) in \( \Theta \)

(5.6)

\[
\Phi(V^\perp) \leq 1 + K d_{ge}^2.
\]

Now consider the smooth metric on \( \Xi \)

(5.7)

\[
\tilde{g} = (1 + K \phi(r)d_{ge}^2)^\frac{1}{2} g_E,
\]

where \( \phi \) is a smooth increasing function with value zero on \([1.4R, 1.5R]\) and value one on \([1.6R, 2R]\). Set

(5.8)

\[
\tilde{g} = \rho(r)\tilde{g} + (1 - \rho(r))(\pi^* d\psi)^{\frac{1}{2}} g_E,
\]

where \( \rho \) is one on \([1.4R, 1.8R]\), decreases to zero on \([1.8R, 1.9R]\) and keeps value zero on \([1.9R, 2R]\). On \([1.7R, 2R]\), since \( \Phi(V^\perp) = \|\pi^* d\psi\|_E^* \) (5.6) guarantees

\[
\tilde{g} \geq (\|\pi^* d\psi\|_E^*)^\frac{1}{2} g_E.
\]
Therefore, on $1.4R \leq r \leq 2R$,

$$\Phi(V^\perp_{\tilde{g}}) \leq 1,$$

where $V^\perp_{\tilde{g}}$ is the oriented unit $N$-vector perpendicular to $V$ under $\tilde{g}$.

By Lemmas 2.12 and 2.14 in Harvey and Lawson [HL82b] there exists a continuously varying 1-dimensional plane field $\mathcal{W}$ transverse to $V^\perp_{\tilde{g}}$ for $1.4R \leq r \leq 2R$ such that under the orthogonal combination $\tilde{g} = g|_{V^\perp} \oplus \tilde{\alpha}g|_{\mathcal{W}}$ for some sufficiently large constant $\tilde{\alpha}$

$$\|\Phi\|_{\tilde{g}}^* = \Phi(V^\perp_{\tilde{g}}) \leq 1.$$

However a vital flaw is that $\tilde{g}$ may be NOT smooth. To conquer this, note that the angle between $V$ and $W$ can be assumed strictly less than $\frac{\pi}{4}$ (the angle of $V$ and $W$ being 0 along $C \cap \Xi$) in $\Xi$ on $1.4R \leq r \leq 2R$. We define a smooth metric

$$\bar{g} = g|_{V^\perp} \oplus [1 + \phi(r)\rho(r + 0.1R) \sqrt{2\tilde{\alpha}}]g|_{V}$$

on $\Xi$. (The shift term 0.1R is in fact not necessary.) Since

- on $[1.4R, 1.6R)$, $\|\Phi\|_{\bar{g}}^* \leq \|\Phi\|_{\tilde{g}}^* = \|\phi\|_{\tilde{g}}^* \leq \|\phi\|_{g_E}^* = 1$;
- on $[1.6R, 1.7R)$, $\|\Phi\|_{\bar{g}}^* \leq \|\Phi\|_{\tilde{g}}^* \leq 1$; and
- on $[1.7R, 2R)$, $\|\Phi\|_{\bar{g}}^* \leq \|\Phi\|_{\tilde{g}}^* = \Phi(V^\perp_{\tilde{g}}) \leq 1$,

we have

$$\|\Phi\|_{\bar{g}}^* \leq 1.$$

On $[1.4R, 1.5R]$, $\Phi = \phi$ and $\bar{g} = g_E$. Meanwhile, on $[1.9R, 2R]$, $\Phi = \varpi^*(d\psi)$ and $\bar{g} = \tilde{\varpi} = (\|\varpi^*d\psi\|_{g_E})\tilde{g}$. Meanwhile, on $[1.9R, 2R]$, $\Phi = \varpi^*(d\psi)$ and $\bar{g} = \tilde{\varpi} = (\|\varpi^*d\psi\|_{g_E})\tilde{g}$. Meanwhile, on $[1.9R, 2R]$, $\Phi = \varpi^*(d\psi)$ and $\bar{g} = \tilde{\varpi} = (\|\varpi^*d\psi\|_{g_E})\tilde{g}$.

It is apparent that this calibration pair of the $C^1$-calibration $\Phi$ and the smooth metric $\bar{g}$ can naturally extend on some neighborhood $\tilde{U}$ of $S$ in our model in $\S 2$. According to Theorem 6.2 in [Fed74] $S$ is homologically area-minimizing in $\tilde{U}$. 
6. Realization of Oriented Lawlor Cones

Lawlor found many mass-minimizing cones in [Law91] by constructing particular calibrations discontinuous along boundary \( \mathcal{B} \) of some open angular neighborhood \( \mathcal{N} \) for each of them. They are of form \( \phi = d(f \psi) \) where \( \psi \) is a smooth \((k - 1)\)-form on \( \bar{\mathcal{N}} \) and where \( f \) is at least \( C^2 \) along the cone and smooth elsewhere on \( \mathcal{N} \), Lipschitzian along \( \mathcal{B} \) with value zero on \( \bar{\mathcal{B}} \). Although \( \phi \) is not continuous, through mollifications all oriented cones with such calibrations can be shown mass-minimizing. We will use the same idea.

First, one can similarly follow Step 1 and Step 2 in §5 with certain modifications. Here most notations are taken directly from §5.

Recall \( \psi = \mathbf{r} \omega \) where \( \omega \) is the unit volume form of the link \( L \) of an oriented Lawlor cone \( C \). Then \( d\psi \) is the oriented unit \( k \)-dimensional form of \( C \sim 0 \), and

\[
\phi = d(f \cdot \sigma^* \psi)
\]

where \( f(q) = \tilde{f}(\tan(\theta(q))) \) and \( \theta(q) \) is the angle between \( \mathbf{O}q \) and \( \mathbf{O}(\sigma(q)) \). Set \( t = \tan(\theta(q)) = \frac{d\psi}{\tau(q)} \). According to [Law91] \( \tilde{f}(t) = 1 - at^2 - bt^3 + \cdots \) near \( t = 0 \).

Define

\[
\Phi = d[\tau(\mathbf{r})(f \cdot \sigma^* \psi) + (1 - \tau(\mathbf{r}))((\sigma^* \psi))].
\]

For \( q \in \mathcal{N} \sim C \), define \( V_q = \frac{\sigma(q)q^*}{|\sigma(q)q^*|} \). Then we get a unit vector field \( V \) on \( \mathcal{N} \sim C \) whose limits on \( C \sim 0 \) give normal directions of \( C \sim 0 \). For \( q \in \mathcal{N} \), denote by \( F_q^\perp \) the oriented unit \( k \)-vector perpendicular to the fiber through \( q \) and it gives a \( k \)-vector field \( F^\perp \) in \( \mathcal{N} \). Since \( L_V(\sigma^* \psi) = 0 \),

\[
L_V \Phi = L_V d[\tau(\mathbf{r})(f \sigma^* \psi) + (1 - \tau(\mathbf{r}))((\sigma^* \psi))]
\]

\[
= d[L_V(\tau(\mathbf{r})f \sigma^* \psi) + (1 - \tau(\mathbf{r}))((\sigma^* \psi))] + d[L_V(\tau(\mathbf{r})f \sigma^* \psi)] + d[\tau(\mathbf{r})L_V((\sigma^* \psi))] + d[1 - \tau(\mathbf{r})L_V((\sigma^* \psi))]
\]

\[
= d[L_V f \tau(\mathbf{r}) \sigma^* \psi]
\]

Let \( \gamma(s) = \exp_{p,s} \nu \) for \( 0 \leq s < \epsilon \) where \( \nu \) is a normal direction at a point \( p \) of \( C \sim 0 \) and \( \epsilon \) is small enough. So \( \gamma'(s) = V_{\gamma(s)} \) for \( 0 < s < \epsilon \) with \( \lim_{s \to 0} V_{\gamma(s)} = \nu \). By Lemma 2.3.2 in [Law91],

\[
\lim_{s \to 0} (L_V F^\perp)_{\gamma(s)} = \left( \frac{d}{ds} \right)_{s=0} \det[I - sh^s_{ij}]^{-1} F^\perp_p = 0,
\]

where \( h^s_{ij} \) is the second fundamental form at \( p \) in normal direction \( \nu \). Note that by (6.1)

\[
\lim_{s \to 0} (L_V \Phi)_{\gamma(s)}
\]

involves a normal direction. Therefore

\[
\lim_{s \to 0} (L_V(\Phi(F^\perp)))_{\gamma(s)} = 0.
\]
Hence there exists a positive constant $K$ such that in a sufficiently small neighborhood $\Xi$ of $C \cap \Theta$ in $\Theta$

\begin{equation}
\Phi(F^{-1}) \leq 1 + Kd_{g_\Sigma}^2.
\end{equation}

Then following the procedures in §5 one can obtain a pair of $\Phi$ and $\bar{g}$ on some neighborhood $\bar{U}$ of $S$, such that

1. $\bar{g}$ is a smooth metric,
2. the comass of $\Phi$ is no larger than 1 where it is defined, and
3. $\Phi$ is the oriented volume form of the cone along $C \sim 0$.

Take a smaller neighborhood $Y$ of $S$ where $Y \subset \bar{U}$ and $(\bar{Y}, \bar{g}|_{\bar{Y}})$ forms a manifold with boundary. Isometrically embed $\bar{Y}$ into some Euclidean space $(\mathbb{R}^\xi, g_\xi)$ thru $F$. By the compactness of $F(\bar{Y})$ there is $\tau > 0$ such that the exponential map restricted to the $\tau$-disk normal bundle $\mathcal{N}$ over $F(Y)$ is a diffeomorphism. Denote by $\mathcal{N}$ the image of $\mathcal{N}$ and by $\nu$ the induced projection. Choose an open neighborhood $W \subset Y$ of $S$. Let $\lambda = \text{dist}_g(\partial F(Y), \partial F(W))$. Then mollify $\pi^*((F^{-1})^*(\Phi))$ with averaging radius $\epsilon < \epsilon_0 = \frac{1}{2} \min\{\lambda, \tau\}$ in the region $\{x \in \mathcal{N} : \text{dist}_g(x, \partial \mathcal{N}) \geq \epsilon_0\}$ of $\mathbb{R}^\xi$. Denote the generated smooth forms by $\Phi_\epsilon$ and set $\Phi_\epsilon = F^*(\Phi_\epsilon|_{(F(W)})$. By the commutativity of the exterior differentiation and mollification in $\mathbb{R}^\xi$, it follows $d\Phi_\epsilon = 0$.

Now we show that $S$ is homologically area-minimizing in $(\bar{W}, \bar{g}|_{\bar{W}})$. By [FF60] there exists a disjoint union of countably many $C^1$ submanifolds (see [Fed69]) and denote the bad set $(\text{spt}\Phi \sim \mathcal{S}) \cap B \sim 0$ by $\mathcal{B}$. Then $\mathcal{B} = \mathcal{C} \cup \mathcal{O}$ where $\mathcal{C} = \{x \in \mathcal{B} : \text{dist}_g(x, \partial \mathcal{B}) \geq \epsilon_0\}$ and $\mathcal{O} = \mathcal{B} \sim \mathcal{C}$. The decomposition is unique up to a $\||T||$-measure 0 set. Obviously $\mathcal{O}$ is of $\||T||$-measure 0. Although $\Phi$ is not well defined along $\mathcal{B}$, $\Phi_\epsilon(\bar{T}, \cdot)$ makes sense on $\text{spt}\Phi$ (i.e., almost $\||T||$-everywhere). Applying Lebesgue’s bounded convergence theorem we have

\[ M(S) = \int_S \Phi = \lim_{\epsilon \downarrow 0} \int_S \Phi_\epsilon = \lim_{\epsilon \downarrow 0} \int \Phi_\epsilon(\bar{T})d||T|| = \int \Phi(\bar{T})d||T|| \leq M(T). \]

**Remark 6.1.** $\Phi_\epsilon$ for $0 < \epsilon < \epsilon_0$ may have comass greater than one under $\bar{g}$.

**Remark 6.2.** Similar argument shows that all Cheng’s examples of homogeneous area-minimizing cones of codimension 2 in [Che88] (e.g. minimal cones over $U(7)/U(1) \times SU(2)^3$ in $\mathbb{R}^{42}$, $Sp(n) \times Sp(3)/Sp(p(1)^3 \times Sp(n - 3)$ in $\mathbb{R}^{12n}$ for $n \geq 4$, and $Sp(4)/Sp(1)^4$ in $\mathbb{R}^{27}$) can be realized as well.
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References

[All72] William K. Allard, On the first variation of a varifold, Ann. Math. 95 (1972), 417–491.
[Che88] Benny N. Cheng, Area-minimizing cone-type surfaces and coflat calibrations, Indiana Univ. Math. J. 37 (1988), 505–535.
[FF60] Herbert Federer and Wendell H. Fleming, Normal and integral currents, Ann. Math. 72 (1960), 458–520.
[Fed69] Herbert Federer, Geometric Measure Theory, Springer-Verlag, New York, 1969.
[Fed74] ———, Real flat chains, cochains and variational problems, Indiana Univ. J. Math. 24 (1974), 351–407.
[HS85] Robert Hardt and Leon Simon, Area minimizing hypersurfaces with isolated singularities, J. Reine. Angew. Math. 362 (1985), 102–129.
[HL82a] F. Reese Harvey and H. Blaine Lawson, Jr., Calibrated geometries, Acta Math. 148 (1982), 47–157.
[HL82b] ———, Calibrated foliations, Amer. J. Math. 104 (1982), 607–633.
[Law91] Gary R. Lawlor, A Sufficient Criterion for a Cone to be Area-Minimizing, Vol. 91, Mem. of the Amer. Math. Soc., 1991.
[Law72] H. Blaine Lawson, Jr., The equivariant Plateau problem and interior regularity, Trans. Amer. Math. Soc. 173 (1972), 231-249.
[Mor86] Frank Morgan, On finiteness of the number of stable minimal hypersurfaces with a fixed boundary, Indiana Univ. Math. J. 35 (1986), 779-833.
[Nas56] John Nash, The embedding problem for Riemannian manifolds, Ann. of Math. 63 (1956), 20–63.
[Sim83] Leon Simon, Lectures on Geometric Measure Theory, Vol. 3, Proc. Centre Math. Anal. Austral. Nat. Univ., 1983.
[Sma99] Nathan Smale, Singular homologically area minimizing surfaces of codimension one in Riemannian manifolds, Invent. Math. 135 (1999), 145-183.
[Sma00] ———, A construction of homologically area minimizing hypersurfaces with higher dimensional singular sets, Trans. Amer. Math. Soc. 352 (2000), 2319-2330.
[Zhaa] Yongsheng Zhang, On extending calibration pairs. Available at [arXiv:1511.03953]
[Zhab] ———, On Lawson’s area-minimizing hypercones. Available at [arXiv:1501.04681]

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