MODULI OF MCKAY QUIVER REPRESENTATIONS II: GRÖBNER BASIS TECHNIQUES

ALASTAIR CRAW, DIANE MACLAGAN, AND REKHA R. THOMAS

Abstract. In this paper we introduce several computational techniques for the study of moduli spaces of McKay quiver representations, making use of Gröbner bases and toric geometry. For a finite abelian group $G \subset \text{GL}(n,k)$, let $Y_\theta$ be the coherent component of the moduli space of $\theta$-stable representations of the McKay quiver. Our two main results are as follows: we provide a simple description of the quiver representations corresponding to the torus orbits of $Y_\theta$, and, in the case where $Y_\theta$ equals Nakamura’s $G$-Hilbert scheme, we present explicit equations for a cover by local coordinate charts. The latter theorem corrects the first result from Nakamura [23]. The techniques introduced here allow experimentation in this subject and give concrete algorithmic tools to tackle further open questions. To illustrate this point, we present an example of a nonnormal $G$-Hilbert scheme, thereby answering a question raised by Nakamura.

1. Introduction

For a finite subgroup $G \subset \text{SL}(2,\mathbb{C})$, McKay [23] observed a connection between the representation theory of $G$, as encoded in the McKay quiver, and the geometry of the minimal resolution of $\mathbb{C}^2/G$. This connection was made explicit in the work of Kronheimer [20] and Ito–Nakamura [18], who showed that the moduli spaces $\mathcal{M}_\theta$ of $\theta$-stable representations of the McKay quiver are isomorphic to the minimal resolution of $\mathbb{C}^2/G$. Bridgeland-King-Reid [3] subsequently proved that for finite subgroups $G \subset \text{SL}(3,\mathbb{C})$, each of the moduli spaces $\mathcal{M}_\theta$ is isomorphic to some projective crepant resolution of $\mathbb{C}^3/G$. While the paper [3] describes only the special case where $\mathcal{M}_\theta$ is the $G$-Hilbert scheme $G$-$\text{Hilb}$, the method extends to the moduli spaces $\mathcal{M}_\theta$ for any generic parameter $\theta$ (see Craw-Ishii [8]). For a finite subgroup $G \subset \text{SL}(n,\mathbb{C})$ with $n \geq 4$, or for a finite subgroup $G \subset \text{GL}(n,\mathbb{C})$ with $n \geq 3$, the moduli spaces $\mathcal{M}_\theta$ are no longer necessarily irreducible. In Craw-Maclagan-Thomas [9], we introduced for finite abelian subgroups $G \subset \text{GL}(n,k)$, an explicit construction of an irreducible component $Y_\theta$ of $\mathcal{M}_\theta$ that is birational to $\mathbb{A}^n_k/G$; we call this component the coherent component of the moduli space $\mathcal{M}_\theta$.

This paper introduces several computational techniques for the study of moduli spaces of McKay quiver representations, making use of Gröbner bases and toric geometry. By studying $\theta$-stable quiver representations in their equivalent guise as $G$-constellations, we are able to study the corresponding modules using Gröbner theory. Our first main result determines whether a given $\theta$-stable $G$-constellation corresponds to a point on the coherent component $Y_\theta$. In addition, when $\mathcal{M}_\theta \cong G$-$\text{Hilb}$ we provide an explicit description of local coordinate charts on the original.
irreducible version of the $G$-Hilbert scheme, $\text{Hilb}^G$, introduced by Nakamura \cite{nakamura}. The techniques introduced here give concrete algorithmic tools to tackle further open questions. To illustrate this point we answer the question raised by Nakamura as to whether $\text{Hilb}^G$ is normal by exhibiting a subgroup $G \subset \text{GL}(6, k)$ for which $\text{Hilb}^G$ is not normal. Thus $\text{Hilb}^G$ is an example of a nonnormal toric variety arising naturally in a geometric context.

Note that Sardo Infirri \cite{sardo} studied the moduli spaces $M_\theta$ for a finite abelian subgroup $G \subset \text{GL}(n, k)$, and claimed that each $M_\theta$ was a toric variety. Examples 4.12 and 5.7 provide counterexamples to this statement.

We now describe the results in more detail. Let $G \subset \text{GL}(n, k)$ be a finite abelian subgroup, let $S = k[x_1, \ldots, x_n]$, and write $A := \oplus_\rho S e_\rho$ for the $G$-equivariant $S$-module with one generator for each irreducible representation $\rho$ of $G$. The McKay module of $G$ is the $A$-module

$$M_G := \langle x_i e_\rho - e_{\rho^i} : 1 \leq i \leq n, \rho \text{ irreducible} \rangle.$$ 

Each vector $w \in (\mathbb{Q}^n_{>0})^*$ determines the slice $P^\vee_w := \{ v \in (\mathbb{Q}^n)^* : w+\nu - \nu^0 \geq 0 \}$ of a polyhedral cone $P^\vee$ that arises naturally from the geometric invariant theory construction of $Y_\theta$ (see Section 2). For a parameter $\theta$ in the GIT parameter space $\Theta$ (see Definition 2.2), the vector $w \in (\mathbb{Q}^n_{\geq 0})^*$ determines a unique distinguished point of $Y_\theta$, and hence a distinguished $\theta$-semistable $G$-constellation which we denote $A/M_{\theta,w}$. The following result is proved in Theorem 4.3.

**Theorem 1.1.** For $\theta \in \Theta$ and $w \in (\mathbb{Q}^n_{\geq 0})^*$, let $v \in P^\vee_w$ be any vector satisfying $\theta \cdot v \leq \theta \cdot v'$ for all $v' \in P^\vee_w$. Then the following $G$-constellations coincide:

1. The distinguished $\theta$-semistable $G$-constellation $A/M_{\theta,w}$;
2. The cyclic $A$-module $A/M_b$, where $M_b \subset A$ is the left $A$-ideal generated by $\{ x_i e_\rho - b_i^\rho e_{\rho^i} : 1 \leq i \leq n, \rho \in G^* \}$, and $b = (b_i^\rho)$ satisfies

$$b_i^\rho = \begin{cases} 1 & \text{if } w_i + \nu - \nu^0 = 0 \\ 0 & \text{if } w_i + \nu - \nu^0 > 0 \end{cases};$$

3. The cyclic $A$-module $A/\text{in}_{(v,w)}(M_G)$, where $\text{in}_{(v,w)}(M_G)$ is the initial module of $M_G$ with respect to $(v,w)$.

Theorem 1.1 provides a simple algorithm for computing $G$-constellations. The algorithm requires that one solves a linear program and then calculates an initial module. The first task is straightforward, and the latter is particularly simple here.

Theorem 1.1 can be simplified in the case where $M_\theta \cong G$-Hilb and $Y_\theta \cong \text{Hilb}^G$ as follows. The inclusion of $G$ into $(\mathbb{k}^*)^n$ gives a map $\deg : \mathbb{Z}^n \rightarrow \text{Hom}(G, \mathbb{k}^*)$ whose kernel $M$ is a lattice. Write $I_M := \langle x^u - x^u' : u, u' \in \mathbb{N}^n, u - u' \in M \rangle$ for the lattice ideal and $\text{in}_w(I_M)$ for the initial ideal of $M$ with respect to $w \in (\mathbb{Q}^n_{\geq 0})^*$. The following result is proved in Proposition 4.10.

**Corollary 1.2.** Let $J \subseteq S$ be a monomial ideal defining a $G$-cluster $[J] \in G$-Hilb. Then $[J]$ lies in the coherent component $\text{Hilb}^G$ if and only if $J = \text{in}_w(I_M)$ for some $w \in (\mathbb{Q}^n_{\geq 0})^*$. 
Using Corollary 1.2 we exhibit a finite subgroup $G \subset \text{GL}(3, \mathbb{k})$ and a monomial ideal $J \subseteq S$ such that $[J] \in G\text{-Hilb}$ does not lie on the coherent component Hilb$^G$. This provides a counterexample to the statements of Nakamura [25, Corollary 2.4, Theorem 2.11], where it is claimed that every monomial ideal defining a $G$-cluster gives a point of the component Hilb$^G$ (the main result of that paper, that Hilb$^G$ is a crepant resolution of $\mathbb{A}^3_{\mathbb{k}}/G$ when $G \subset \text{SL}(3, \mathbb{k})$, is nevertheless correct).

Stillman–Sturmfels–Thomas [29] established that all monomial ideals in the coherent component of the toric Hilbert scheme (see [26]) are initial ideals of an associated toric ideal. Haiman–Sturmfels [12] generalized the definition of toric Hilbert schemes in their work on multigraded Hilbert schemes, including $G$-Hilb as a special case, so Corollary 1.2 extends the result of [29] to this case. See Ito [16] for details in the $G$-Hilb context for finite abelian $G \subset \text{GL}(2, \mathbb{k})$.

Our second main result constructs a cover of Hilb$^G$ by local coordinate charts.

Theorem 1.3. The scheme Hilb$^G$ is covered by affine charts $\text{Spec} \mathbb{k}[A_J]$ indexed by monomial ideals $J = \text{in}_w(I_M)$ for $w \in (\mathbb{Q}^n_{\geq 0})^*$, where $A_J$ is a semigroup associated to $J$.

Theorem 1.3 enables us to present the universal $G$-cluster over $\text{Spec} \mathbb{k}[A_J]$ in an economical way (see Corollary 5.5). In addition, we exhibit a finite subgroup of GL(6, k) and an ideal $J = \text{in}_w(I_M)$ for which $\text{Spec} \mathbb{k}[A_J]$ is not normal (see Example 5.7 and Corollary 5.9). This answers the question raised by Nakamura [25, Remark 2.10] as to whether Hilb$^G$ is normal.

Corollary 1.4. Nakamura’s $G$-Hilbert scheme Hilb$^G$ is not normal in general.

While there is an extensive literature on nonnormal toric varieties (see [30]), the focus has been applications such as integer programming (see, for example, [14]). On the other hand, the standard definition of a toric variety in algebraic geometry assumes normality. Corollary 1.3 therefore provides an example of a nonnormal toric variety arising naturally in algebraic geometry.

We now explain the division into sections. Section 2 reviews the construction of the moduli spaces $\mathcal{M}_{\theta}$, and recalls the main result from [9]. Section 3 reviews some well-known facts from the theory of Gröbner bases, and gives our first Gröbner bases result for $G$-constellations. In Section 4 we establish Theorem 1.1 and Corollary 1.2. Finally, in Section 5 we prove Theorem 1.3 and Corollary 1.4.

Conventions. For an integer matrix $C$, let $NC$ denote the semigroup generated by the columns of $C$. Similarly, $ZC$ denotes the lattice, $\mathbb{Q}_{\geq 0}C$ the rational cone and $\mathbb{Q}C$ the rational vector space generated by columns of $C$. For $u = (u_1, \ldots, u_m), u' = (u'_1, \ldots, u'_m) \in \mathbb{N}^m$ we write $u \leq u'$ if $u_i \leq u'_i$ for $1 \leq i \leq m$. By a point of a scheme over $\mathbb{k}$ we mean a closed point. We write $\mathbb{k}^*$ for the one-dimensional algebraic torus.

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2. McKay quiver representations and G-constellations

We review the construction of the moduli spaces of McKay quiver representations for a finite abelian subgroup $G \subset \text{GL}(n, \mathbb{k})$ of order $r$, where $\mathbb{k}$ is an algebraically closed field whose characteristic does not divide $r$. See [9] for a more leisurely introduction. We also recall the equivalent module-theoretic formulation of McKay quiver representations, where they are known as $G$-constellations.

2.1. Moduli of McKay quiver representations. Since $G$ is abelian, we may assume that $G$ is contained in the subgroup $(\mathbb{k}^*)^n$ of diagonal matrices with nonzero entries in $\text{GL}(n, \mathbb{k})$. We thus get $n$ elements $\rho_1, \ldots, \rho_n$ of the dual group of characters $G^* := \hom(G, \mathbb{k}^*)$, defined by setting $\rho_i(g)$ to be the $i$th diagonal element of the matrix for $g$. The elements $\rho_1, \ldots, \rho_n$ generate the group $G^*$.

**Definition 2.1.** The *McKay quiver* of $G \subset \text{GL}(n, \mathbb{k})$ is the directed graph with a vertex for each $\rho \in G^*$, and an arrow $a^\rho_i$ from $\rho \rho_i$ to $\rho$ for each $\rho \in G^*$ and $1 \leq i \leq n$. We say the arrow $a^\rho_i$ is labeled $i$.

The McKay quiver has $r$ vertices and $nr$ arrows, and can be encoded in an $(r+n) \times nr$ matrix $C$ as follows. Let $\{e_\rho : \rho \in G^*\} \cup \{e_i : 1 \leq i \leq n\}$ be the standard basis of $\mathbb{Z}^{r+n}$, and let $\{e^\rho_i : \rho \in G^*, 1 \leq i \leq n\}$ denote the standard basis of $\mathbb{Z}^{nr}$. Order the latter basis globally into $r$ blocks, one for each $\rho \in G^*$ beginning with the trivial representation $\rho_0$. Within each block the elements are listed $e^\rho_1, \ldots, e^\rho_n$. Let $C$ be the $(r+n) \times nr$ matrix with column $e_\rho - e_{\rho \rho_i} + e_i$ corresponding to $e^\rho_i$. Note that the top $r \times (nr)$ submatrix $B$ with column $e_\rho - e_{\rho \rho_i}$, corresponding to $e^\rho_i$ is the vertex-edge incidence matrix of the McKay quiver.

A representation of the McKay quiver of dimension vector $(1, \ldots, 1) \in \mathbb{N}^r$ is the assignment of a one-dimensional $\mathbb{k}$-vector space $R_\rho$ to each vertex $\rho$, and a linear map $R_{\rho \rho_i} \to R_\rho$ to each arrow $a^\rho_i$. Fix a basis for each $R_\rho$ and write $b^\rho_i \in \mathbb{k}$ for the entry of the $1 \times 1$ matrix of the linear map $R_{\rho \rho_i} \to R_\rho$. We occasionally use $b^\rho_i$ to refer to the linear map itself. Since there are $nr$ arrows in the quiver, representations define points $(b^\rho_i) \in \mathbb{A}_\mathbb{k}^{nr}$. We write $\mathbb{k}[z^\rho_i : \rho \in G^*, 1 \leq i \leq n]$ for the coordinate ring of $\mathbb{A}_\mathbb{k}^{nr}$. We consider only points $(b^\rho_i)$ of the scheme $Z$ defined by the ideal

$$I = (z^{\rho \rho_i} - z^{\rho \rho_j} : \rho \in G^*, 1 \leq i, j \leq n).$$

Thus, we consider only representations $(b^\rho_i) \in \mathbb{A}_\mathbb{k}^{nr}$ satisfying the relations

$$b^\rho_j b^\rho_i = b^{\rho \rho_i} b^\rho_j \text{ for } \rho \in G^* \text{ and } 1 \leq i, j \leq n.$$

These relations arise naturally when quiver representations are translated into the equivalent language of $G$-constellations (see Remark 3.1).

We now summarize the Geometric Invariant Theory (GIT) construction of the moduli spaces of $\theta$-stable McKay quiver representations (see [9] §2.3.4) for more
details). The algebraic torus \( k^* \) acts on each \( R_\rho \), so \((k^*)^r \) acts diagonally on the vector space \( \oplus_{\rho \in G^*} R_\rho \) by change of basis. Hence \( t = (t_\rho) \in (k^*)^r \) acts on \( b^\rho_i \in \text{Hom}(R_{\rho_1}, R_\rho) = R^*_{\rho_1} \otimes R_\rho \) as

\[
\left(t \cdot b^\rho_i\right) = t^{-1}_\rho b^\rho_i.
\]

The diagonal scalar subgroup acts trivially, leaving a faithful action of the \((r-1)\)-dimensional algebraic torus \( T_B := \text{Hom}(ZB, k^*) \) on \( A_k^{nr} \) whose character lattice \( ZB \subseteq \mathbb{Z}^r \) is generated by the columns of the matrix \( B \). This action induces a \( ZB \)-grading of the coordinate ring of \( A_k^{nr} \) by setting \( \deg(z^\rho_j) = e_\rho - e_{\rho j} \). The ideal \( I \) defining \( Z \) is homogeneous, so \( k[Z] \) is \( ZB \)-graded and, for \( b \in ZB \), we write \( k[Z]_{jb} \) for the \( jb \)-graded piece of \( k[Z] \). Then the categorical GIT quotient of \( Z \) by the action of \( T_B \) is

\[
Z / \!/ bT_B := \text{Proj} \bigoplus_{j \geq 0} k[Z]_{jb}.
\]

More generally, the quotient linearized by an element \( \theta \in \mathbb{Q} \) in the \( \mathbb{Q} \)-vector space generated by the columns of \( B \) is defined to be the GIT quotient linearized by any multiple for which \( j\theta \in ZB \). A parameter \( \theta \in \mathbb{Q} \) is generic if every point of \( Z \) that is \( \theta \)-semistable (in the sense of GIT) is in fact \( \theta \)-stable, in which case \( Z / \!/ \theta T_B \) is a geometric quotient. The subset of generic parameters decomposes into finitely many open chambers, where \( Z / \!/ \theta T_B \) remains unchanged as \( \theta \) varies in a chamber, though its polarizing line bundle varies.

**Definition 2.2.** The **GIT parameter space** is the \( \mathbb{Q} \)-vector space

\[
\Theta := ZB \otimes \mathbb{Q} = \{ (\theta_\rho) \in \mathbb{Q}^r : \sum_{\rho \in G^*} \theta_\rho = 0 \}.
\]

For \( \theta \in \Theta \), \( \mathcal{M}_\theta := Z / \!/ \theta T_B \) is the coarse moduli space of \( \theta \)-semistable McKay quiver representations of dimension vector \((1, \ldots, 1)\) satisfying the relations \eqref{2.1}. For generic \( \theta \), \( \mathcal{M}_\theta \) is the fine moduli space of \( \theta \)-stable McKay quiver representations.

The best known example of \( \mathcal{M}_\theta \) is the \( G \)-Hilbert scheme, denoted \( G \)-Hilb. This parameterizes ideals \( J \subseteq S = k[x_1, \ldots, x_n] \) defining \( G \)-invariant subschemes \( Z(J) \subseteq \mathbb{A}^n_k \) whose coordinate rings \( S/J \) are isomorphic to the group ring \( kG \) as \( kG \)-modules. Ito–Nakajima \cite[§3]{17} observed that there is a unique chamber in \( \Theta \) containing parameters \( \theta \in \Theta \mid \theta_{\rho} > 0 \) for \( \rho \neq \rho_0 \) such that \( \mathcal{M}_\theta \cong G \)-Hilb.

To state the main result of Craw–Maclagan–Thomas \cite{9}, let \( NC \subseteq \mathbb{Q}^{r+n} \) denote the subsemigroup generated by the columns of the matrix \( C \) and let \( P \subseteq \mathbb{Q}^{r+n} \) be the cone generated by the column vectors \( \{ e_\rho - e_{\rho i} + e_i : \rho \in G^*, 1 \leq i \leq n \} \) of \( C \). Also, let \( \pi_1 : \mathbb{Q}^{r+n} \to \mathbb{Q}^r \) and \( \pi_n : \mathbb{Q}^{r+n} \to \mathbb{Q}^n \cong \ker_\mathbb{Z}(\pi) \otimes_\mathbb{Z} \mathbb{Q} \) be the projections onto the first \( r \) and last \( n \) coordinates respectively.

**Theorem 2.3** (Craw–Maclagan–Thomas \cite{9}). The not-necessarily-normal toric variety \( V = \text{Spec} k[NC] \) is a \( T_B \)-invariant irreducible component of the scheme \( Z \subseteq \mathbb{A}^{nr}_k \). In addition:

1. For \( \theta \in \Theta \), the GIT quotient \( Y_\theta := V / \!/ \theta T_B \) is a not-necessarily-normal toric variety that admits a projective birational morphism \( \tau_\theta : Y_\theta \to A^{nr}_k/G \) obtained by variation of GIT quotient.

2. For generic \( \theta \in \Theta \), the variety \( Y_\theta \) is the unique irreducible component of \( \mathcal{M}_\theta \) containing the \( T_B \)-orbit closures of the points of \( Z \cap (k^*)^{nr} \).
The toric fan of $Y_{\theta}$ is the inner normal fan of the polyhedron $P_{\theta}$ obtained as the convex hull of the set $\pi_n(P \cap \pi^{-1}(\theta)) \subset \ker\pi \otimes \mathbb{Q}$.

**Definition 2.4.** For generic $\theta \in \Theta$, $Y_{\theta}$ is called the coherent component of $\mathcal{M}_{\theta}$.

In the special case where $\mathcal{M}_{\theta} \cong G$-Hilb, we established [9, Corollary 1.2] that the coherent component $Y_{\theta}$ is isomorphic to the original version of the $G$-Hilbert scheme $\text{Hilb}^G$ introduced by Nakamura [25].

### 2.2. $G$-constellations

We recall the notion of $G$-constellation and review some well-known results from representation theory for which we could not find a suitable reference.

Let $S := \mathbb{k}[x_1, \ldots, x_n]$. The group $G$ acts on $S$ by $g \cdot x_i = \rho_i(g^{-1})x_i$. We now recall the skew group algebra $S \rtimes G$. As an $S$-module, the skew group algebra is the free $S$-module with basis $G$. The ring structure is given by setting $(sg) \cdot (s'g') = s(g \cdot s')gg'$ for $s, s' \in S$ and $g, g' \in G$. Recall that an $S$-module $M$ is $G$-equivariant if it has a $G$-action such that $g \cdot (sm) = (g \cdot s)(g \cdot m)$ for $g \in G, s \in S$ and $m \in M$. An $S$-module is $G$-equivariant if and only if it is a left $S \rtimes G$-module.

**Definition 2.5.** A $G$-constellation is a $G$-equivariant $S$-module that is isomorphic as a $\mathbb{k}G$-module to $\mathbb{k}G$.

In order to apply Gröbner basis theory we reinterpret $G$-constellations as graded modules. We give $S$ a $G^*$-grading by $\deg(x_i) = \rho_i$ for $1 \leq i \leq n$. This grading comes from the inclusion of $G$ into the $n$-dimensional torus acting on $\mathbb{A}_n^\mathbb{G}$, and gives a map $\deg: \mathbb{Z}^n \rightarrow G^*$.

**Definition 2.6.** Define a $G^*$-graded $\mathbb{k}$-algebra as follows. As a free $S$-module, $A = \oplus_{\rho \in G^*} S e_{\rho}$ is the free $S$-module of rank $r$ with basis $\{e_{\rho} : \rho \in G^*\}$. Extend the $G^*$-grading of $S$ to a $G^*$-grading on $A$ by defining $\deg(e_{\rho}) = \rho$. The multiplication on $A$ is then determined by

$$e_{\rho'} \cdot x^{u} e_{\rho} = \begin{cases} x^{u} e_{\rho} & \text{if } \deg(x^{u} e_{\rho}) = \deg(e_{\rho'}) \\ 0 & \text{otherwise} \end{cases}$$

This is well-defined with the $S$-module structure.

**Remark 2.7.** The algebra $A$ is the path algebra of the McKay quiver modulo the ideal of relations corresponding to (2.1). See [1] III.1 for the definition of the path algebra. This description requires the assumption that $G$ is abelian. It is well-known that for finite $G$ in GL($n, \mathbb{k}$) (see for example [31 Chapter 10]) the algebra $A$ is Morita equivalent to the skew group algebra. In the abelian case these algebras are actually isomorphic.

**Proposition 2.8.**

1. An $S$-module is a left $A$-module if and only if it is $G^*$-graded.
2. An $S$-module homomorphism between left $A$-modules is a left $A$-module homomorphism if and only if it preserves the $G^*$-grading.
3. The algebra $A$ is isomorphic to $S \rtimes G$. 

Proof. Let $M$ be a $G^*$-graded $S$-module. Define a left $A$-module structure on $M$ by setting $e_\rho \cdot m = m_\rho$ for $m \in M$, where $m_\rho$ is the piece of $m$ in degree $\rho$. To check that this gives an $A$-module structure, it suffices to show that $(e_\rho (x^u e_\rho)) \cdot m = e_\rho \cdot (x^u e_\rho \cdot m)$. The expression on the left-hand side is $x^u m_\rho$ if $\deg(e_\rho) = \deg(x^u e_\rho)$, and zero otherwise. Since $x^u e_\rho \cdot m = x^u m_\rho$, this equals the right-hand side.

Conversely, let $M$ be an $A$-module. Define $M_\rho = e_\rho M$. We claim that $M = \bigoplus_{\rho \in G} M_\rho$ as an abelian group, and this decomposition is compatible with multiplication by elements of $S$. Indeed, if $m \in M_\rho \cap M_\rho'$, then $m = e_\rho m_1 = e_\rho' m_2$ for some $m_1, m_2 \in M$. But then $e_\rho m = e_\rho^2 m_1 = e_\rho m_1 = m$, and $e_\rho' m = e_\rho e_\rho' m_2 = 0$, so $m = 0$. Since $e = \sum_{\rho \in G} e_\rho$ is the multiplicative identity of $A$, we have $m = e \cdot m = \sum_{\rho \in G} m_\rho$. This gives the decomposition as abelian groups. If $\deg(x^u) = \rho'$ then $e_{\rho'} \cdot x^u e_\rho m_\rho = x^u m_\rho$. This gives $x^u m_\rho \in M_{\rho'}$, so (1) holds.

An $A$-module homomorphism $\phi$ satisfies $\phi(e_\rho m) = e_\rho \phi(m)$, so is exactly a degree-zero $G^*$-graded $S$-module homomorphism. This gives (2).

The $k$-linear map $\phi : \bigoplus_{\rho \in G} k e_\rho \to \bigoplus_{g \in G} k g$ given by $\phi(e_\rho) = 1/r \sum_{g \in G} \rho(g) g$ is an isomorphism since the character table is an invertible matrix for an abelian group. This extends to an isomorphism of $S$-modules $\phi : A \to S \rtimes G$. It remains to check that $\phi(e_\rho) \phi(x^u e_\rho') = \phi(e_\rho \cdot x^u e_\rho')$. Indeed, the left-hand side is

$$
\frac{1}{r^2} \sum_{h \in G} \sum_{g, g' \in G, g g' = h} (\rho \rho'')(g) \rho'(g') h,
$$

where $\deg(x^u) = \rho''^{-1}$, so $g \cdot x^u = \rho''(g) x^u$. By the orthogonality relations of the character table, this is $x^u / r \sum_{g \in G} \rho'(g) g$ when $\rho' = \rho \rho''$, and zero otherwise. This proves the final statement. \( \square \)

**Corollary 2.9.** An $S$-module $F$ is a $G$-constellation if and only if $F$ is $G^*$-graded with Hilbert function $\dim_k F = 1$ for each degree $\rho \in G^*$.

**Proof.** A $G$-constellation is an $S \rtimes G$-module, and hence is $G^*$-graded by Proposition 2.8. It remains to show that a $G^*$-graded module is isomorphic to $kG = \bigoplus_{g \in G} k g$ if and only if it has Hilbert function one in each degree. This follows from the $k$-linear isomorphism $\phi$ from the proof of Proposition 2.8 and the fact that, if $F$ is a $G^*$-graded module with Hilbert function one in each degree, there is a surjection from $\bigoplus_{\rho} S e_\rho$ where the image of each $e_\rho$ is nonzero. \( \square \)

3. **Gröbner interpretation of $G$-constellations**

In this section we first review some well-known facts from the theory of Gröbner basis for modules. We then canonically associate a submodule of $A$ to every $G$-constellation, and establish the key Gröbner result by exhibiting a Gröbner basis for this module.

3.1. **Preliminary Gröbner facts.** We start by summarizing the relevant facts about Gröbner bases (see Cox–Little–O’Shea [5] and Eisenbud [10], Chapter 15).

Let $M$ be a submodule of the free module $S^r$ for some $r \in \mathbb{N}$. An element $f \in M$ can be written as $f = \sum c_{u,i} x^u e_i$, where the sum is over $u \in \mathbb{N}^r$ and $1 \leq i \leq r$, and all but finitely many $c_{u,i}$ are zero. If $v = (v_1, \ldots, v_r) \in Q^r$, and $w \in Q^n$, then the initial term $\text{in}_{(v, w)}(f) = \sum c_{u,i} x^u e_i$, where the sum is over pairs $(u, i)$
with \( w \cdot u + v_i \geq w \cdot u' + v_j \) for any other pair \((u', j)\) with \( c_{u,j} \neq 0 \). The initial module of \( M \) is the \( S \)-module

\[
in_{(v,w)}(M) = \langle in_{(v,w)}(f) : f \in M \rangle.
\]

If \( M \) is homogeneous in some grading of \( S^r \), then the Hilbert function of \( S^r/M \) equals that of \( S^r/in_{(v,w)}(M) \). When \( in_{(v,w)}(M) \) is generated by monomials \( x^u e_i \), then a basis for \( S^r/in_{(v,w)}(M) \) consists of those monomials not lying in \( in_{(v,w)}(M) \), which we call the standard monomials of \( in_{(v,w)}(M) \).

It is important to note that the initial module cannot usually be computed by taking the initial terms of the module generators. A Gröbner basis with respect to the weight vector \((v, w)\) is a set of generators \( \{m_1, \ldots, m_s\} \) for \( M \) such that

\[
in_{(v,w)}(M) = \langle in_{(v,w)}(m_1), \ldots, in_{(v,w)}(m_s) \rangle.
\]

Gröbner bases are usually defined by giving a term order, which is a total order on the monomials \( x^u e_i \) in \( S^r \) satisfying \( e_i < x^u e_i \) for all \( u \neq 0 \), and if \( x^u e_i < x^w e_j \) then \( x^{u+w} e_i < x^{u+w} e_j \). A weight vector \((v, w)\) gives a partial order, called a weight order, by setting \( x^u e_i < x^w e_j \) if \( w \cdot u + v_i < w \cdot u' + v_j \). This partial order can be refined to a term order by breaking ties with a fixed term order. We use only the term over position lexicographic order \([6\text{, Chapter 5, Definition } 2.4]\). We will use the following proposition (see Sturmfels \[30\text{, Corollary } 1.9\]) for a proof for ideals in a polynomial ring; the extension to modules is straightforward.

**Proposition 3.1.** Let \((v, w)\) be a weight vector and \( \prec_{(v,w)} \) be a term order that refines the weight order. If \( \{m_1, \ldots, m_s\} \) is a Gröbner basis for a module \( M \) with respect to \( \prec_{(v,w)} \), then \( \{m_1, \ldots, m_s\} \) is also a Gröbner basis for the weight order given by \((v, w)\).

A criterion for a subset \( \{m_1, \ldots, m_s\} \) of \( M \) to be a Gröbner basis is given by the conditions of Buchberger’s algorithm. The key idea in this algorithm is that of an \( S \)-pair: if \( in_\prec(m_i) = c_i x^u e_k \) and \( in_\prec(m_j) = c_j x^u e_k \) involve the same basis element \( e_k \) of \( S^r \), then \( S(m_i, m_j) := m_j - c_i m_i - m_j m_i \), where \( m_{ij} = c_j x^u / \gcd(x^u, x^u) \) and \( m_{ji} = c_i x^u / \gcd(x^u, x^u) \). The set \( \{m_1, \ldots, m_s\} \) is a Gröbner basis if every \( S \)-pair can be written as \( S(m_i, m_j) = \sum_h m_i r_h \), where \( h \in S \) and \( \min_h(r_h m_i) \preceq \min_h(S(m_i, m_j)) \). In this case we say that the \( S \)-pair reduces to zero. In general, if \( f = \sum_h r_h m_i + g \), where in \( \prec_h(r_h m_i) \preceq in_\prec(f) \) then we say that \( f \) reduces to \( g \) modulo \( \{m_1, \ldots, m_s\} \).

### 3.2. The key Gröbner basis result

We now use Gröbner basis techniques to write down an explicit map that associates to each quiver representation \((b^\rho_i) \in Z\), a \( G \)-constellation with a chosen presentation. This map arises naturally via an isomorphism of categories. First, we introduce the categories.

**Definition 3.2.** Let \( \mathcal{R} \) be the category whose objects are points \((b^\rho_i) \in Z\). A morphism \( h: (b^\rho_i) \rightarrow (b^{\rho'}_i) \) consists of a scalar \( h_\rho \in k \) for every \( \rho \in G^* \), satisfying \( b^\rho_i h_\rho = b^{\rho'}_i \rho_\rho b^\rho_i \).

**Remark 3.3.** The category \( \mathcal{R} \) is obtained from the category of McKay quiver representations of dimension vector \((1, \ldots, 1)\) satisfying the given relations (see
≤ M 0 A/M module is the number of standard monomials of the module in the given degree.

Since the given generators for M A/M homomorphism ψ reduces to zero, so the generators for M x is the first term is the leading term in our term order, and that b is zero since (b_i^ρ) leads naturally to the conditions b_i^ρ h_ρ = h_ρ b_i^ρ.

Definition 3.4. Let C be the category whose objects are cyclic A-modules of the form A/M satisfying dim_k(A/M)_ρ = 1 for all ρ ∈ G^*, with e_ρ ≠ M for our chosen S-module basis e_ρ of A. The morphisms of C are A-module homomorphisms.

Remark 3.5. Corollary 2.9 shows that the objects of C are G-constellations with a chosen presentation.

We now construct an explicit isomorphism of categories between C and R. We emphasize that it is highly unusual in Gröbner theory for the minimal generating set of a module to be a Gröbner basis, as occurs in this proposition.

Proposition 3.6. There is a contravariant functor Ψ : R → C taking the McKay quiver representation b = (b_i^ρ) ∈ Z to the G-constellation A/M_b for the left A-ideal

(3.1) M_b := ⟨x_i e_ρ − b_i^ρ e_ρ e_i : 1 ≤ i ≤ n, ρ ∈ G^*⟩

that is an isomorphism of categories. Moreover, the given generators for M_b form a Gröbner basis with respect to any monomial order with x_i e_ρ ∼ e_ρ e_i.

Proof. We begin by showing that the given generators for M_b form a Gröbner basis for any term order with x_i e_ρ ∼ e_ρ e_i. Indeed, the only relevant S-pairs are between terms of the form x_i e_ρ − b_i^ρ e_ρ e_i and x_j e_ρ − b_j^ρ e_ρ e_j. This S-pair is then

x_j b_i^ρ e_ρ e_i − x_i b_j^ρ e_ρ e_j. These terms are symmetric in i, j, so we may assume that the first term is the leading term in our term order, and that b_i^ρ ≠ 0. The polynomial now reduces using the binomial x_j e_ρ e_i − b_j x_i e_ρ e_j to b_j x_i e_ρ e_j − b_j b_i^ρ e_ρ e_i. Our assumption on the term order now implies that if b_j^ρ ≠ 0, the second term is the leading term, so the binomial reduces using x_i e_ρ e_j − b_i^ρ e_ρ e_j to (b_i^ρ) b_j x_i e_ρ e_j. This is zero since (b_i^ρ) ∈ Z. If b_j^ρ = 0, then since b_i^ρ ≠ 0 and (b_i^ρ) ∈ Z we must have b_j^ρ = 0, so the intermediate binomial was already zero. In both cases the S-pair reduces to zero, so the generators for M_b form a Gröbner basis.

We next show that A/M_b is an object of C. Since deg(x_i e_ρ) = deg(e_ρ e_i), the submodule M_b is homogeneous in the G*-grading, so A/M_b is also graded by G^*.

Since the given generators for M_b form a Gröbner basis as above, M_0 = ⟨x_i e_ρ : 1 ≤ i ≤ n, ρ ∈ G^*⟩ is an initial module of M_b, and thus the Hilbert function of A/M_0 equals that of A/M_b. The Hilbert function of the quotient by a monomial module is the number of standard monomials of the module in the given degree. Since the only standard monomials of M_0 are the units e_ρ, we conclude that the Hilbert function of A/M_b is one in every degree.

We now construct Ψ and its inverse. Define Ψ on objects as above. A morphism h : (b_i^ρ) → (b_i^ρ) is a collection of h_ρ ∈ k for ρ ∈ G^*. Define an S-module homomorphism ψ(h) : A → A/M_b by ψ(h)(e_ρ) = h_ρ e_ρ. Since ψ(h)(x_i e_ρ —
\[ b'_i e_{\rho \rho} = h_\mu(x_i e_\rho - b'_i e_{\rho \rho}) \in M_b, \psi(h) \text{ defines an } S\text{-module homomorphism } \Psi(h): A/M_{b'} \to A/M_b. \] It is now straightforward to check that \( \Psi \) is a functor from \( \mathcal{R} \) to \( \mathcal{C} \). To construct the inverse functor \( \Phi \), let \( A/M \) be an object of \( \mathcal{C} \). Since \( \dim_k(A/M)_\rho = 1 \) for all \( \rho \in G^* \), \( \deg(x_i e_\rho) = \deg(e_{\rho \rho}) \), and \( e_\rho \not\in M \), there is a unique \( b'_i \in k \) with \( x_i e_\rho - b'_i e_{\rho \rho} \in M \) for each \( i \) and \( \rho \). We then have \( x_i x_j e_\rho - b'^{(i)}_i b'^{(j)}_j e_{\rho \rho} \in M \), and hence \( (b'^{(i)}_i b'^{(j)}_j) e_{\rho \rho \rho} \in M \). Since \( e_\rho \not\in M \), we conclude that \( b'^{(i)}_i b'^{(j)}_j - b'^{(j)}_j b'^{(i)}_i = 0 \), and so \( (b'_i) \in Z \). We thus set
\[ \Phi(A/M) = (b'_i). \] Given a morphism \( h: A/M \to A/M' \), lift to \( \tilde{h}: A \to A/M' \), and write \( \tilde{h}(e_\rho) = \lambda_\rho x^u e_{\rho'} \), for some \( u, \rho' \) satisfying \( \deg(x^u e_{\rho'}) = \deg(e_\rho) \). Since \( \dim_k(A/M')_\rho = 1 \), and \( e_\rho \not\in M' \), there is a unique \( \mu \in k \) with \( x^u e_{\rho'} - \mu e_\rho \in M' \), and thus \( h(e_\rho) = \lambda_\rho \mu e_\rho \). The scalar \( \lambda_\rho \mu \) is independent of the choice of \( u \) and \( \rho' \) since \( e_\rho \not\in M' \). We define \( \Phi(h) \) to be the morphism \( \Phi(h): (b'_i) \to (\tilde{b}'_i) \) with \( \Phi(h)_\rho = \lambda_\rho \mu \). The fact \( \tilde{h}(M) = 0 \) implies that \( \Phi(h) \) is a morphism in \( \mathcal{R} \). It follows that \( \Phi \) is a functor from \( \mathcal{C} \) to \( \mathcal{R} \), and \( \Phi = \psi^{-1} \). 

\textbf{Remark 3.7.} (1) The \( A \)-submodules \( M_b \subseteq A \) may be regarded as left ideals in the skew group ring \( A \cong S \rtimes G \).

(2) The relations (2.1) correspond via \( \Psi \) to the commutativity of \( x_i \) and \( x_j \) in the \( S \)-module structure on \( A/M_b \).

(3) The translation from quiver representations to modules over an algebra is a special case of a result for representations of an arbitrary finite quiver with relations (see [1, III, Proposition 1.7]). Since \( \mathcal{C} \) and \( \mathcal{R} \) involve choices of bases, we obtain an isomorphism rather than an equivalence of categories.

(4) The isomorphism of categories \( \Psi \) implies that \( M_b \) from Definition 2.2 may be regarded as the moduli space of \( \theta \)-semistable \( G \)-constellations.

\section{Distinguished \( G \)-constellations via Gröbner bases}

This section explicitly describes the distinguished \( G \)-constellations that define distinguished points on \( Y_\theta \subseteq M_\theta \). We exploit here the geometric interpretation of Gröbner bases as allowing explicit computations of flat degenerations coming from a one-parameter torus (see [10, Chapter 15]). This Gröbner description allows us to decide whether or not a given point of \( M_\theta \) lies in \( Y_\theta \).

\subsection{Distinguished \( G \)-constellations}

In order to apply the Gröbner result from Proposition 3.6 we introduce an \( A \)-module that plays a key role in the rest of the paper.

\textbf{Definition 4.1.} The \textit{McKay module} is the submodule, or left ideal, of \( A \) given by
\[ M_G = \langle x_i e_\rho - e_{\rho \rho} : \rho \in G^*, 1 \leq i \leq n \rangle. \]
Note that \( M_G = M_b \) for the point \( b \in Z \) with \( b'_i = 1 \) for all \( \rho \in G^* \) and \( 1 \leq i \leq n \).

\textbf{Lemma 4.2.} The McKay module \( M_G \) is equal to the \( A \)-module \( \langle x_i - e : 1 \leq i \leq n \rangle \), where \( e \) is the multiplicative identity in \( A \).

\textbf{Proof.} Write \( M'_G := \langle x_i - e : 1 \leq i \leq n \rangle \) and fix \( 1 \leq i \leq n \). Since \( e = \sum_{\rho \in G^*} e_\rho \), we have \( x_i e - e = \sum_{\rho \in G^*} x_i e_\rho - \sum_{\rho' \in G^*} e_{\rho'} \). Relabel \( \rho' = \rho \rho_i \) and regard the second term as a sum over \( \rho \in G^* \) to give \( x_i e - e = \sum_{\rho \in G^*} (x_i e_\rho - e_{\rho \rho_i}) \), hence \( M'_G \subseteq M_G \).
For the opposite inclusion, note that for every \( \rho \in G^* \) and \( 1 \leq i \leq n \) we have \( e_{pp_i} \cdot x_i e = x_i e_{p^i} \). This implies that \( x_i e_{p^i} - e_{pp_i} = e_{pp_i} \cdot x_i e - e_{pp_i} = e_{pp_i} (x_i - e) \), so \( M_G \subseteq M_{\theta} \) as required.

Recall from Section 2 that \( P \subseteq \mathbb{Q}^{n+n} \) is the polyhedral cone generated by the vectors \( \{e_{p} - e_{p^i}, e_i : \rho \in G^*, 1 \leq i \leq n \} \). For \( \mathbf{w} = (w_i) \in (\mathbb{Q}_{\geq 0}^n)^* \), let \( P'^\mathbf{w} := \{ \mathbf{v} \in (\mathbb{Q})^* : w_i + v_{\rho} - v_{pp_i} \geq 0 \} \) denote the slice of the dual cone \( P^\mathbf{w} \). For \( \theta \in \Theta \), the vector \( \mathbf{w} \in (\mathbb{Q}_{\geq 0}^n)^* \) determines a unique distinguished point \( [b_{\theta, \mathbf{w}}] \in Y_{\theta} \) as described in \([9\text{ Section 7}]\), and hence a distinguished \( \theta \)-semistable \( G \)-constellation \( \Psi(b_{\theta, \mathbf{w}}) \) that we denote \( A_{/M_{\theta, \mathbf{w}}} \).

**Theorem 4.3.** For \( \theta \in \Theta \) and \( \mathbf{w} \in (\mathbb{Q}_{\geq 0}^n)^* \), let \( \mathbf{v} \in P'^\mathbf{w} \) be any vector satisfying \( \theta \cdot \mathbf{v} \leq \theta \cdot \mathbf{v}' \) for all \( \mathbf{v}' \in P'^\mathbf{w} \). Then the following \( G \)-constellations coincide:

1. The distinguished \( \theta \)-semistable \( G \)-constellation \( A_{/M_{\theta, \mathbf{w}}} \);
2. The cyclic \( A \)-module \( A_{/M_{b_{\theta, \mathbf{w}}}} \), where \( M_{b_{\theta, \mathbf{w}}} \) is the left \( A \)-ideal generated by \( \{x_i e_{\rho} - b_i^\rho e_{pp_i} : 1 \leq i \leq n, \rho \in G^* \} \), and \( b = (b_i^\rho) \) satisfies

\[
\begin{align*}
\begin{cases}
1 & \text{if } w_i + v_{\rho} - v_{pp_i} = 0 \\
0 & \text{if } w_i + v_{\rho} - v_{pp_i} > 0
\end{cases};
\end{align*}
\]

3. The cyclic \( A \)-module \( A_{/\text{in}_{\mathbf{v}, \mathbf{w}}(M_G)} \), where \( \text{in}_{\mathbf{v}, \mathbf{w}}(M_G) \) is the initial module of \( M_G \) with respect to \( \mathbf{v}, \mathbf{w} \).

**Proof.** For \( \theta \in \Theta \) and \( \mathbf{w} \in (\mathbb{Q}_{\geq 0}^n)^* \), fix \( \mathbf{v} \in P'^\mathbf{w} \) with \( \theta \cdot \mathbf{v} \leq \theta \cdot \mathbf{v}' \) for \( \mathbf{v}' \in P'^\mathbf{w} \). The coordinates of the distinguished \( \theta \)-semistable quiver representation \( b = b_{\theta, \mathbf{w}} \) satisfy the conditions from the second part of Theorem 4.3 by Craw–Maclagan–Thomas \([9\text{ Theorem 7.2}]\). Then the \( G \)-constellation \( A_{/M_{\theta, \mathbf{w}}} := \Psi(b_{\theta, \mathbf{w}}) \) coincides with that from (2) by Proposition 3.6.

To see that (2) and (3) coincide, let \( \prec_{\mathbf{v}, \mathbf{w}} \) be the term order on \( A \) refining the weight order given by \( (\mathbf{v}, \mathbf{w}) \), where ties are broken using the term over position lexicographic order. Since \( (\mathbf{v}, \mathbf{w}) \in P'^\mathbf{w} \), we have \( w_i + v_{\rho} \geq v_{pp_i} \), and hence \( \text{in}_{\prec_{\mathbf{v}, \mathbf{w}}}(x_i e_{\rho} - e_{pp_i}) = x_i e_{\rho} \). Proposition 3.6 states that the generators \( \{x_i e_{\rho} - e_{pp_i} : 1 \leq i \leq n, \rho \in G^* \} \) for \( M_G \) are a Gröbner basis for the term order \( \prec_{\mathbf{v}, \mathbf{w}} \), and by Proposition 3.4 they also form a Gröbner basis for the weight order given by \( (\mathbf{v}, \mathbf{w}) \). This means that \( \text{in}_{\prec_{\mathbf{v}, \mathbf{w}}}(M_G) = \langle x_i e_{\rho} - e_{pp_i} : w_i + v_{\rho} = v_{pp_i} \rangle + \langle x_i e_{\rho} : w_i + v_{\rho} > v_{pp_i} \rangle = M_b \) for \( b = (b_i^\rho) \) satisfying (4.1). This completes the proof.

Theorem 4.3 gives an algorithm to compute the distinguished \( \theta \)-semistable \( G \)-constellation \( A_{/M_{\theta, \mathbf{w}}} \). This is the \( G \)-constellation analogue of \([9\text{ Algorithm 7.6}]\).

**Algorithm 4.4.** To compute the distinguished \( G \)-constellation \( A_{/M_{\theta, \mathbf{w}}} \).

**Input:** \( (\theta, \mathbf{w}) \in \Theta \times (\mathbb{Q}_{\geq 0}^n)^* \) and the McKay module \( M_G \).

1. Set \( M_G = \langle x_i e_{\rho} - e_{pp_i} : 1 \leq i \leq n, \rho \in G^* \rangle \).
2. Compute an optimal solution \( \mathbf{v} \) of the linear program

\[
\text{minimize} \{ \theta \cdot \mathbf{v}' : \mathbf{v}' \in P'^\mathbf{w} \};
\]

3. Compute \( \text{in}_{\mathbf{v}, \mathbf{w}}(M_G) = \langle \text{in}_{\mathbf{v}, \mathbf{w}}(x_i e_{\rho} - e_{pp_i}) : \rho \in G^*, 1 \leq i \leq n \rangle \). Then \( A_{/M_{\theta, \mathbf{w}}} \) has \( M_{\theta, \mathbf{w}} = \text{in}_{\mathbf{v}, \mathbf{w}}(M_G) \).
Proof of Correctness. This is immediate from Theorem 4.3. □

Example 4.5. Consider the group $G \cong Z/11Z$ generated by $\text{diag}(\zeta, \zeta^2, \zeta^8)$. The given three-dimensional representation decomposes as $p_1 \oplus p_2 \oplus p_8$, so

$$M_G = \langle x_1 e_{p_0} - e_{p_1}, x_2 e_{p_0} - e_{p_2}, x_3 e_{p_0} - e_{p_3}, x_1 e_{p_1} - e_{p_3}, x_2 e_{p_1} - e_{p_4}, x_3 e_{p_1} - e_{p_5}, x_1 e_{p_2} - e_{p_3}, x_2 e_{p_2} - e_{p_4}, x_3 e_{p_2} - e_{p_5}, x_1 e_{p_3} - e_{p_6}, x_2 e_{p_3} - e_{p_5}, x_3 e_{p_3} - e_{p_7}, x_1 e_{p_4} - e_{p_5}, x_2 e_{p_4} - e_{p_6}, x_3 e_{p_4} - e_{p_7}, x_1 e_{p_5} - e_{p_7}, x_2 e_{p_5} - e_{p_8}, x_3 e_{p_5} - e_{p_9}, x_1 e_{p_6} - e_{p_8}, x_2 e_{p_6} - e_{p_9}, x_3 e_{p_6} - e_{p_9}, x_1 e_{p_7} - e_{p_9}, x_2 e_{p_7} - e_{p_8}, x_3 e_{p_7} - e_{p_9}, x_1 e_{p_8} - e_{p_9}, x_2 e_{p_8} - e_{p_10}, x_3 e_{p_8} - e_{p_3}, x_1 e_{p_9} - e_{p_10}, x_2 e_{p_9} - e_{p_3}, x_3 e_{p_9} - e_{p_4}, x_1 e_{p_{10}} - e_{p_5}, x_2 e_{p_{10}} - e_{p_3}, x_3 e_{p_{10}} - e_{p_4} \rangle.$$

We compute a distinguished $\theta$-stable quiver representation for the parameter $\theta = (1, 1, 1, 1, -7, -9, 1, 1, 8, 1)$ (compare [9, Example 7.7]). The vector $w = (10, 7, 6)$ lies in the relative interior of a three-dimensional cone in the fan of $Y_{\theta}$, and hence defines a torus-invariant $G$-constellation. The vector $v = (-8, -10, -1, 3, 6, 4, -9, 0, -2, -15, -6)$ is an optimal solution to the linear program in Step (2) from Algorithm 4.3 and Step (3) gives

$$M_{\theta, w} = \langle x_1 e_{p_0}, x_2 e_{p_0} - e_{p_2}, x_3 e_{p_0} - e_{p_4}, x_1 e_{p_1}, x_2 e_{p_1} - e_{p_3}, x_3 e_{p_1} - e_{p_5}, x_1 e_{p_2}, x_2 e_{p_2} - e_{p_4}, x_3 e_{p_2} - e_{p_5}, x_1 e_{p_3}, x_2 e_{p_3} - e_{p_4}, x_3 e_{p_3} - e_{p_5}, x_1 e_{p_4}, x_2 e_{p_4} - e_{p_3}, x_3 e_{p_4} - e_{p_5}, x_1 e_{p_5}, x_2 e_{p_5} - e_{p_3}, x_3 e_{p_5} - e_{p_5}, x_1 e_{p_6}, x_2 e_{p_6} - e_{p_4}, x_3 e_{p_6} - e_{p_5}, x_1 e_{p_7}, x_2 e_{p_7} - e_{p_3}, x_3 e_{p_7} - e_{p_5}, x_1 e_{p_8}, x_2 e_{p_8} - e_{p_3}, x_3 e_{p_8} - e_{p_5}, x_1 e_{p_9}, x_2 e_{p_9} - e_{p_3}, x_3 e_{p_9} - e_{p_5}, x_1 e_{p_{10}}, x_2 e_{p_{10}} - e_{p_3}, x_3 e_{p_{10}} - e_{p_5} \rangle.$$

This coincides with the $G$-constellation from [7, Table 5.5, Line 2].

4.2. Distinguished $G$-clusters. Before specializing from $G$-constellations to $G$-clusters by choosing $\theta \in \Theta$ so that $M_\theta \cong G$-Hilb, we prove a pair of lemmas that are valid for more general $\theta$.

Definition 4.6. The quiver $Q_b$ associated to a representation $b = (b_i^\rho) \in Z$ is the subquiver of the McKay quiver with a vertex for each $\rho \in G^\ast$ and those arrows $a_i^\rho$ for which $b_i^\rho \neq 0$. When $b = b_{\theta, w}$, we write $Q_{\theta, w}$.

In what follows we identify the support of a vector $u \in Q_{nr}$ with the quiver containing the arrows $a_i^\rho$ for which $u_i^\rho \neq 0$. Conversely, given an (undirected) path in the McKay quiver, its vector is the weighted sum of those $e_i^\rho$ for which $a_i^\rho$ appears in the path, with the weight recording the number of times the edge is crossed in the forward direction minus the number of times it is crossed in the negative direction. Recall that the lattice $M \subset Z^\ast$ is the kernel of the map $\text{deg}: Z^\ast \to G^\ast$, and that $P_\theta$ is a polyhedron whose normal fan is equal to the toric fan of $Y_\theta$ (see Theorem 2.3.3 for the construction). Note that since $Y_\theta$ is constructed by GIT, we have $Y_\theta = Y_{j_\theta}$ for any $j > 0$.

Lemma 4.7. Fix $\theta' \in \Theta \cap Z^\ast$ and $w \in (Q_{nr}^\ast)^\ast$. Let $N = (nr)!$ and set $\theta = N \theta'$. Let $F_\theta$ be the face of $P_\theta$ minimizing $w$. Then there exists $u \in N_{nr}$ and $m \in \text{relint}(F_\theta) \cap M$ such that $Cu = (\theta, m)$ and $\text{supp}(u) = Q_{\theta, w}$. In particular, if $m$ is a vertex of $P_\theta$, then there exists $u \in N_{nr}$ such that $Cu = (\theta, m)$ and the quiver $\text{supp}(u) = Q_{\theta, w}$ contains no directed cycles.
Proof. Let $F$ be the smallest face of $P$ containing the preimage under $\pi_n$ of the face $F_{\theta'}$ of $P_{\theta'}$ minimizing $\mathbf{w}$. This face corresponds to the distinguished representation $b_{\theta',\mathbf{w}} = (b^\rho_i)$ by definition, so $F$ is the positive rational span of those $C^\rho_i$ with $b^\rho_i \neq 0$ by Theorem 4.8. The fact that the matrix $B$ is unimodular implies that there are integral points on all faces of $\{\mathbf{u} \geq \mathbf{0} : B\mathbf{u} = \theta'\}$. Since $P_{\theta'} = \text{conv}\{\pi_n(C\mathbf{u}) : \mathbf{u} \in \mathbb{Q}^{nr}_{\geq 0}, B\mathbf{u} = \theta'\}$, there is a lattice point $\mathbf{m}' \in \pi_n(\text{relint}(F \cap \pi^{-1}(\theta'))) \cap M$. Since $(\theta', \mathbf{m}')$ lies in the relative interior of $F$ there is $\mathbf{u}' \in \mathbb{Q}^{nr}_{\geq 0}$ with $C\mathbf{u}' = (\theta', \mathbf{m}')$ such that the support of $\mathbf{u}'$ is contained in $Q_{\theta,\mathbf{w}}$. Again, since $B$ is unimodular we may take $\mathbf{u}' \in \mathbb{N}^{nr}$. Furthermore, for every arrow $\alpha^\rho_i$ in $Q_{\theta,\mathbf{w}}$ there is such a $\mathbf{u}'$ with $u^\rho_i > 0$. Adding these together gives a vector $\mathbf{u}'' \in \mathbb{N}^{nr}$ with support exactly $Q_{\theta,\mathbf{w}}$, and $C\mathbf{u}'' = (f\theta', f\mathbf{m}')$, where $f$ is the number of arrows in $Q_{\theta,\mathbf{w}}$. Since $f$ divides $(nr)!$ we set $\mathbf{u} := (N/f)\mathbf{u}''$, which satisfies $\mathbf{u} \in \mathbb{N}^{nr}$, $\text{supp}(\mathbf{u}) = Q_{\theta,\mathbf{w}}$, and $C\mathbf{u} = (\theta, N\mathbf{m}')$. The result now follows by setting $\mathbf{m} = N\mathbf{m}'$, and observing that since $\mathbf{m}'$ lies in the face of $P_{\theta'}$ minimizing $\mathbf{w}$, we must have $\mathbf{m}$ in the face of $P_{\theta}$ minimizing $\mathbf{w}$.

To prove the final statement, suppose that $\mathbf{m} \in P_{\theta}$ is a vertex and that the quiver $Q_{\theta,\mathbf{w}}$ contains a directed cycle consisting of $\alpha_i$ arrows labeled $i$. By adding together the collection of all equations $w_i + v_{\rho} = v_{\rho,i}$ arising via (4.1) from each arrow in the cycle, we obtain $\sum \alpha_i w_i = 0$. This means that $\mathbf{w}$ is constrained to lie in a hyperplane, but this is absurd since $\mathbf{w}$ may be any vector in the relative interior of the top-dimensional cone $N_{P_{\theta}}(\mathbf{m})$ which consists of all $\mathbf{w}' \in (\mathbb{Q}^n_{\geq 0})^*$ such that the linear functional $\mathbf{w}'$ is minimized over $P_{\theta}$ at $\mathbf{m}$. \hfill $\square$

Lemma 4.8. Fix $\theta \in \Theta$ with $\theta_{\rho_0} \leq 0$ and $\theta_\rho \geq 0$ for $\rho \neq \rho_0$. Let $\mathbf{u} \in \mathbb{N}^{nr}$ be such that $B\mathbf{u} = \theta$. Then $\mathbf{u}$ decomposes as $\mathbf{u} = \mathbf{u}_0 + \sum_k \mathbf{u}_k$ where $\mathbf{u}_0$ is the vector of a union of cycles, and, for each $k$, we have $\mathbf{u}_k \in \mathbb{N}^{nr}$, $\mathbf{u}_k \leq \mathbf{u}$ and $B\mathbf{u}_k = \mathbf{e}_\rho - \mathbf{e}_{\rho_0}$ for some $\rho \in G^*$ depending on $k$. Each $\mathbf{u}_k$ is the vector of a directed path in the McKay quiver from $\rho_0$ to $\rho$.

Proof. The proof is by induction on $\sum_{\rho \in G^*} |\theta_\rho|$. When this sum is zero, $\theta = 0$, so $\mathbf{u} \in \ker \{B\} \cap \mathbb{N}^{nr}$, and thus by Exercise 38 of Bollobás [2, II.3] there is a collection of directed cycles $\gamma_i$ in the McKay quiver with vectors $\mathbf{u}_k \in \mathbb{N}^{nr}$ with $B\mathbf{u}_k = 0$, and $\sum_k \mathbf{u}_k = \mathbf{u}$ as required. We may then assume that the lemma is true for smaller $\sum |\theta_\rho|$, and that this sum is positive, so $\theta_{\rho_0} < 0$.

Let $\mathcal{A}_u$ be the collection of arrows in the McKay quiver consisting of $u_\rho^i$ copies of the arrow $\alpha^\rho_i$ for each pair $(i, \rho)$ with $u^\rho_i > 0$. Since $\theta_{\rho_0} < 0$, there exists some $\gamma_i, 1 \leq i \leq n$, such that $u_{i\rho_0 \rho_i^{-1}} = u_{i\rho_i^{-1}} > 0$ or equivalently, there exists some $i$ such that $a_{\rho_i^{-1}} \in \mathcal{A}_u$. This is the first arrow in a path in the McKay quiver from $\rho_0$ consisting of arrows from $\mathcal{A}_u$. Continue this path until you reach a vertex $\rho$ that has no arrows in $\mathcal{A}_u$ with tail at $\rho$. This means that all the entries in the row of $B$ indexed by $\rho$ and lying in columns indexed by arrows in $\mathcal{A}_u$ are $+1$s. Since $B\mathbf{u} = \theta$, we have $\theta_\rho > 0$. This constructs a path $\gamma_\rho$ from $\rho_0$ to $\rho$ with $\mathbf{u}_i := v(\gamma_\rho)$ using arrows in $\mathcal{A}_u$. By construction, $\mathbf{u}_i \leq \mathbf{u}$ and $B\mathbf{u}_i = \mathbf{e}_\rho - \mathbf{e}_{\rho_0}$. Let $\mathbf{u}' = \mathbf{u} - \mathbf{u}_i$. Then $B\mathbf{u}' = \theta - \mathbf{e}_\rho + \mathbf{e}_{\rho_0}$, which has smaller coordinate sum, so by the induction hypothesis $\mathbf{u}'$, and thus $\mathbf{u}$, has a decomposition of the desired form. \hfill $\square$
For the rest of this paper we restrict to $G$-clusters. To do this, fix a parameter $\theta \in \Theta$ satisfying
\[
(4.2) \quad \begin{cases}
\theta_{\rho_0} < 0 \text{ and } \theta_{\rho} > 0 \text{ for all } \rho \neq \rho_0; \text{ and} \\
\theta/(nr)! \in \mathbb{Z}^r \text{ has } \oplus_{j \geq 2 j} k[V]_{\theta/(nr)} \text{ generated in degree one.}
\end{cases}
\]

The result of Ito–Nakajima [17, §2] implies that $\theta$-stable $G$-constellations are precisely $G$-equivariant $S$-modules of the form $S/J$, where $J \subseteq S$ is a $G$-invariant ideal and $S/J$ is isomorphic to $kG$ as a $kG$-module. Thus, $M_0 \cong G$-Hilb, and hence $Y_\theta \cong \text{Hilb}^G$. The second assumption in (4.2) guarantees that $\theta' := \theta/(nr)!$ lies in $\mathbb{Z}^r$, so we can apply Lemma 4.7. This second assumption is required only for the proofs and is not relevant when computing examples.

**Definition 4.9.** Let $I_M$ be the lattice ideal $\langle x^u - x^{u'} : u, u' \in \mathbb{N}^n, u - u' \in M \rangle$. The scheme $Z(I_M) \subseteq \mathbb{A}^n_\mathbb{R}$ is the $G$-orbit of the point $(1, \ldots, 1) \in \mathbb{A}^n_\mathbb{R}$.

Lattice ideals are generalizations of toric ideals, and have many applications. See, for example, the book of Miller-Sturmfels [24].

**Proposition 4.10.** Let $J \subseteq S$ be an ideal defining a point $[J] \in G$-Hilb. Then $[J]$ defines the distinguished point $[b_{\theta, w}] \in \text{Hilb}^G$ if and only if $J = \text{in}_w(I_M)$.

**Proof.** The ideal $J \subseteq S$ defines a point $[J] \in \text{Hilb}^G$ if and only if $[J] = [b_{\theta, w}]$ for some $w \in (\mathbb{Q}_{\geq 0})^*$, in which case $J$ satisfies $S/J \cong A/M_{\theta, w}$ as an $S$-module.

We claim that $J$ is the kernel of the $S$-module homomorphism $\phi : S \rightarrow A/M_{\theta, w}$ defined by setting $\phi(1) = e_{\rho_0}$. Indeed, by Lemma 4.7 there is a vector $u \in \mathbb{N}^{mr}$ with support $Q_{\theta, w}$ and $Bu = \theta$. By Lemma 4.8 we may decompose $u$ as a sum of vectors $u_k \leq u$, with $u_k$ the vector of a path from $\rho_0$ to some $\rho \in G^*$, and a vector $u_0 \leq u$ with $u_0$ the vector of a union of cycles. Since the support of $u$ is $Q_{\theta, w}$, these paths and cycles lie in $Q_{\theta, w}$. By assumption (4.2) on $\theta \in \Theta$, there is at least one such vector $u_k$ for each $\rho \in G^*$. This implies that $\phi$ is surjective, as a path from $\rho_0$ to $\rho$ in $Q_{\theta, w}$ yields a binomial of the form $x^\rho e_{\rho_0} - \lambda e_\rho$ in $M_{\theta, w}$, with $\lambda \neq 0$. This gives $S/\ker(\phi) \cong A/M_{\theta, w}$, so $J = \ker(\phi)$ as required.

It remains to show that $\ker(\phi) = \text{in}_w(I_M)$. The $G^*$-graded Hilbert function of $S/I_M$ is one in every degree by Definition 4.9, and thus the same is true of $S/\text{in}_w(I_M)$ for $w \in (\mathbb{Q}_{\geq 0})^*$. Since $S/\ker(\phi) \cong A/M_{\theta, w}$, it follows that $\ker(\phi)$ and $\text{in}_w(I_M)$ have the same Hilbert function. If $x^u$ is a minimal generator of $\text{in}_w(I_M)$, then since $S/\text{in}_w(I_M)$ has $G^*$-graded Hilbert function one in every degree, there is $x^{u'} \notin \text{in}_w(I_M)$ with $u - u' \in M$ and $w \cdot u > w \cdot u'$. Now $(x^u - x^{u'})e_{\rho_0} \in M_G$ since this binomial is homogeneous under $G^*$-grading, so $x^u e_{\rho_0} \in \text{in}_{(v, w)}(M_G)$ for any $v \in (\mathbb{Q}^r)^*$, and thus $x^u e_{\rho_0} \in M_{\theta, w}$. The proof is identical for a minimal generator $x^u - x^{u'}$ of $\text{in}_w(I_M)$, so $\text{in}_w(I_M) \subseteq \ker(\phi)$. Since $\ker(\phi)$ and $\text{in}_w(I_M)$ have the same Hilbert function, they must in fact be equal. This completes the proof. \(\square\)

If $\gamma$ is a path in the McKay quiver, its *type* is the vector $u = (u_i) \in \mathbb{Z}^n$ with $u_i$ being the number of arrows labelled $i$ in $\gamma$.

**Corollary 4.11.** Let $w \in (\mathbb{Q}_{\geq 0})^*$ lie in the relative interior of a top-dimensional cone in the fan of $Y_\theta$. Then a directed path from $\rho_0$ to $\rho$ of type $u \in \mathbb{N}^n$ is supported on the arrows in $Q_{\theta, w}$ if and only if $x^u \notin \text{in}_w(I_M)$ and $\deg(x^u) = \rho^{-1}$.
We now exhibit an example to illustrate that \( G \)-Hilb need not be irreducible, thereby proving that \( \mathcal{M}_\theta \neq Y_\theta \) in general. It is known by Bridgeland–King–Reid \cite{3} and Ishii \cite{15} that \( G \)-Hilb is irreducible for finite subgroups of \( \text{SL}(n, \mathbb{k}) \) with \( n \leq 3 \) and \( \text{GL}(2, \mathbb{k}) \) respectively. Thus, the simplest possible reducible example is determined by a finite subgroup of \( \text{GL}(3, \mathbb{k}) \).

**Example 4.12.** Consider the group \( G := \mathbb{Z}/14\mathbb{Z} \) embedded in \( \text{GL}(3, \mathbb{k}) \) with generator the diagonal matrix \( g = \text{diag}(\omega^1, \omega^0, \omega^{11}) \), where \( \omega \) is a primitive fourteenth root of unity. We claim that for \( \theta \in \Theta \) as in \([1,2]\), the moduli space \( \mathcal{M}_\theta \cong G \)-Hilb is reducible, hence \( G \)-Hilb \( \neq \) Hilb\(^G\). To see this, consider the torus-fixed point \( [J] \in G \)-Hilb defined by the monomial ideal

\[
J = \langle x_2^2 x_3, x_1 x_3^2, x_1 x_2^2, x_2 x_3, x_1 x_3, x_2^2, x_3^4, x_1^4 \rangle
\]

in \( \mathbb{k}[x_1, x_2, x_3] \). We check that \( J \) is a \( G \)-cluster by checking its \( G^* \)-graded Hilbert function. In this example this can be done by hand or using Macaulay 2 \cite{11}.

To show that \([J]\) does not lie in Hilb\(^G\) we establish that \( J \) is not an initial ideal of \( I_M = \langle x_1^{14} - 1, x_2 - x_1^9, x_3 - x_1^{11} \rangle \). Suppose otherwise, so there is a weight vector \( w \in (\mathbb{Q}_{\geq 0})^* \) with \( w \cdot u > w \cdot u' \) whenever \( x^u - x^{u'} \in I_M \) for \( x^u \in J \) and \( x^{u'} \notin J \). Consider the binomials \( x_2^2 x_3 - x_2^3, x_2 x_3^2 - x_1^3 \) and \( x_1 x_3^2 - x_3^3 \) in \( I_M \) where the underlined monomials are minimal generators of \( J \) and the trailing monomial in each binomial is the unique standard monomial of \( J \) in its degree. If \( J \) was in\(_w(I_M)\) for a weight vector \( w = (w_1, w_2, w_3) \) then these binomials would imply that

\[
2w_1 + w_3 > 3w_2, \quad w_2 + 2w_3 > 3w_1, \quad w_1 + 2w_2 > 3w_3.
\]

Adding these three inequalities leads to the new inequality \( 3w_1 + 3w_2 + 3w_3 > 3w_1 + 3w_2 + 3w_3 \), which is absurd.

**Remark 4.13.** The monomial ideal \( J \) constructed in Example \([1,12]\) defines a point \([J] \in G \)-Hilb that lies off the coherent component Hilb\(^G\). This provides a counterexample to the statements of Nakamura \cite{25} Corollary 2.4, Theorem 2.11 that every monomial \( G \)-cluster \( J \subset S \) defines a point \([J] \in \text{Hilb}^G\). For each monomial \( G \)-cluster \( J \), Nakamura \cite{25} §1 defined a cone \( \sigma(\Gamma) \) associated to the set of standard monomials \( \Gamma \) (which he calls a \( G \)-graph). If \( J = \text{in}_w(I_M) \), this is the cone in the Gröbner fan of \( I_M \) corresponding to \( J \), while this set is empty otherwise. See \cite{30} Chapter 2 or \cite{22} Chapter 2 for details on the Gröbner fan.

5. **Local equations on Nakamura’s G-Hilbert scheme**

In this section we give a new description of local coordinate charts on Hilb\(^G\). We use this description to exhibit a finite subgroup \( G \subset \text{GL}(6, \mathbb{k}) \) for which Hilb\(^G\) is nonnormal.

5.1. **Local equations on Hilb\(^G\).** In \cite{9}, we showed that when \( \theta \) satisfies the conditions of \([1,2]\), local charts on the coherent component \( Y_\theta \cong \text{Hilb}^G \) are given by Spec(\( \mathbb{k}[A_\sigma] \)), where \( \sigma = N_{P_\theta}(\mathbf{m}) := \{ \mathbf{w}' \in (\mathbb{Q}^n)^* : \mathbf{w}' \text{ is minimized over } P_\theta \text{ at } \mathbf{m} \} \) for a vertex \( \mathbf{m} \) of \( P_\theta \), and

\[
A_\sigma = \mathbb{N}(\mathbf{p} - \mathbf{m} : \mathbf{p} \in P_\theta \cap M).
\]
We now provide an alternative description of these local charts. Theorem 5.2 below corrects and refines the result of Nakamura (compare Remark 4.13).

**Definition 5.1.** Let $J = \text{in}_w(I_M)$ be a monomial initial ideal. We associate to $J$ the semigroup $A_J$ generated by \{ $u - u' \in M : u, u' \in \mathbb{N}^n, x^u \in J, x^{u'} \notin J$ \}.

Note that the initial ideal $\text{in}_w(I_M)$ is a monomial ideal when $w$ is generic. For a vector $u \in \mathbb{Z}^n$ we write $\text{pos}(u)$ for the vector with $i$th component $u_i$ if $u_i > 0$ and 0 otherwise. Similarly, we write $\text{neg}(u)$ for the vector with $i$th component $-u_i$ if $u_i < 0$ and 0 otherwise, so $u = \text{pos}(u) - \text{neg}(u)$. Note that for $u \in \mathbb{Z}^n$, we have $\deg(u) \in \mathcal{G}^*$ and $\deg(-u) = \deg(u)^{-1} \in \mathcal{G}^*$.

**Theorem 5.2.** For $\theta \in \Theta$ as in (4.2), let $m \in P_\theta \cap M$ be a vertex and choose $w$ in the relative interior of $\sigma = N_{P_\theta}(m)$. Then $A_\sigma = A_J$, where $J = \text{in}_w(I_M)$. Thus $\text{Hilb}^G$ is covered by affine charts $\text{Spec} \mathbb{k}[A_J]$ defined by the monomial ideals $J = \text{in}_w(I_M)$ as $w$ varies in $(\mathbb{Q}_{>0})^n$.

**Proof.** We first show that $A_J \subseteq A_\sigma$. Let $u - u' \in A_J$, with $x^u \in J$ and $x^{u'} \notin J$. It is enough to establish that the lattice point $p := u - u' + m \in M$ lies in $P_\theta$. To show this we construct vectors $u_u, u_u', u_m \in \mathbb{N}^n$ such that $\pi_n(C(u_u - u_u' + u_m))$ lies in $P_\theta$ by construction and is equal to $p$.

By Lemma 4.7 there is a vector $u_m \in \mathbb{N}^n$ satisfying $Cu_m = (\theta, m) \in \mathcal{NC}$ with $\text{supp}(u_m) = Q_{\theta,w}$. Next, by Corollary 4.11, since $x^{u'} \notin J$, any directed path $\gamma_{u'}$ in the McKay quiver from $\rho_0$ to $\deg(-u') \in \mathcal{G}^*$ of type $u' \in \mathbb{N}^n$ is supported on arrows in the subquiver $Q_{\theta,w}$. This path determines a vector $u_u' := v(\gamma_u') \in \mathbb{N}^n$ satisfying $Cu_u' = (e_{\deg(-u')} - e_{\rho_0}, u') \in \mathcal{NC}$ and $u_m - u_u' \in \mathbb{N}^n$. Finally, pick any path $\gamma_u$ of type $u \in \mathbb{N}^n$ beginning at $\rho_0$. The head of this path is $\deg(-u)$, so we may $u := v(\gamma_u)$ satisfies $Cu_u = (e_{\deg(-u)} - e_{\rho_0}, u)$. By adding $u_u \in \mathbb{N}^n$ to $u_m - u_u' \in \mathbb{N}^n$, we obtain a vector in $\mathbb{N}^n$ satisfying $\pi_n(C(u_u - u_u' + u_m)) = (\theta, u - u' + m)$, since $\deg(u) = \deg(u')$. Hence $p = \pi_n(C(u_u - u_u' + u_m))$ lies in $P_\theta$ as claimed.

For the opposite inclusion, consider a minimal generator $p - m \in A_\sigma$. By Lemma 4.7 there exists $u_m \in \mathbb{N}^n$ such that $Cu_m = (\theta, m)$ and the quiver $\text{supp}(u_m) = Q_{\theta,w}$ contains no directed cycles. Since $p \in P_\theta$ and $p \neq m$, there exists $u_p \in \mathbb{N}^n$ such that $Cu_p = (\theta, p)$ and $u_m \neq u_p$. Lemma 4.8 enables us to decompose $u_p$ into a sum of vectors of the form $u_p(\rho) \in \mathbb{N}^n$, where each $u_p(\rho)$ satisfies $Bu_p(\rho) = e_\rho - e_{\rho_0}$ and $u_p(\rho) \leq u_p$ by construction, and also a vector $u_p(0)$, where $u_p(0) \leq u_p$, and $u_p(0)$ is the vector of a union of cycles. The same is true for $u_m$. For each $\rho \neq \rho_0$, there are $\theta_\rho$ vectors of the form $u_m(\rho)$ and $\theta_\rho$ of the form $u_p(\rho).$ Note that $u_m(0) = 0$ since $Q_{\theta,w}$ contains no cycles.

There are now two cases. Either there exists $\rho \neq \rho_0$ and vectors $u_p(\rho), u_m(\rho)$ satisfying $\pi_n(Cu_p(\rho)) \neq \pi_n(Cu_m(\rho))$, or else $u_p - u_m = u_p(\rho)$. In the latter case, $p - m = \pi_n(C(u_p(\rho)))$ lies in $\mathbb{N}^n$ and $\deg(p - m) = \rho_0$. Since the only standard monomial of $J$ of degree $\rho_0$ is 1, we have $x^{p-m} \in J$, so $p - m \in A_J$ as required. In the former case, suppose that $\pi_n(Cu_p(\rho)) \neq \pi_n(Cu_m(\rho)).$ Let $u'_p = u_p - u_p(\rho) + u_m(\rho)$, and $u'_m = u_m - u_m(\rho) + u_p(\rho)$. Note that $u'_p, u'_m \in \mathbb{N}^n$ and $Bu'_p = Bu'_m = \theta$, so $\pi_n(Cu'_p) - m$ and $\pi_n(Cu'_m) - m$ both lie in $A_\sigma$. In addition we have $(\pi_n(Cu'_p) - m) + (\pi_n(Cu'_m) - m) = \pi_n(C(u'_p - u'_m + u'_p - u'_m)) = \pi_n(Cu_p - u_m)) = p - m.$ Since $p - m$ is a minimal generator of $A_\sigma$, and
\( \pi_n(C\langle u_p(\rho) - u_m(\rho) \rangle) \neq 0 \), the second of these terms \( \pi_n(C\langle u_p(\rho) - u_m(\rho) \rangle) - m \) must be zero. This gives \( p - m = \pi_n(C\langle u_p(\rho) - u_m(\rho) \rangle) \), where \( u_p(\rho) \) and \( u_m(\rho) \) are vectors of paths from \( \rho_0 \) to \( \rho \). Since \( u_m(\rho) \leq u_m \), the support of the path defined by \( u_m(\rho) \) lies in \( Q_{\theta, w} \), so \( x^{\pi_n(C\langle u_m(\rho) \rangle)} \notin J \) by Corollary 4.11. Also, the negative part of \( p - m \) satisfies \( \text{neg}(p - m) \leq \pi_n(C\text{neg}(u_p(\rho) - u_m(\rho))) \leq \pi_n(Cu_m(\rho)) \). This gives \( x^{\text{neg}(p - m)} \notin J \). Since \( p - m \neq 0 \), and \( J \) has only one standard monomial of each degree, we must have \( x^{\text{pos}(p - m)} \in J \), so \( p - m \in A_J \) as required.

We next give a smaller generating set for \( A_J \).

**Lemma 5.3.** The semigroup \( A_J \) is generated by elements of the form \( \{u - u' \in M : u, u' \in \mathbb{N}^n, \text{deg}(u) = \text{deg}(u'), x^u \text{ is a minimal generator of } J, x^{u'} \notin J \} \).

**Proof.** The semigroup \( A' \) generated by the given elements is a subsemigroup of \( A_J \), so we need only show that if \( u - v \in A_J \), with \( x^u \in J, x^v \notin J \), and \( \text{deg}(x^u) = \text{deg}(x^v) \), then \( u - v \) is in the semigroup generated by \( A' \). Note that \( \mathcal{G} = \{x^u - x^v : u - v \in A' \} \) is a Gröbner basis for \( I_M \). Since \( x^v \) is the unique standard monomial of \( J \) of its degree, this means that \( x^u \) reduces modulo \( \mathcal{G} \) to \( x^v \). So we can write \( x^u - x^v = \sum_{i=1}^{s} x^{w_i}(x^{u_i} - x^{v_i}) \), where \( x^{u_i} - x^{v_i} \in \mathcal{G} \). We extend \( \mathcal{G} \) to include the negative part of \( x^u \), so \( u - v \) lies in the semigroup generated by \( A' \).

**Remark 5.4.** Lemma 5.3 is the content of Nakamura [25] Lemma 1.8. We provide a self-contained proof to illustrate the Gröbner argument. Note that Example 4.12 is a counterexample to the sentence following [25] Lemma 1.8.

List the elements from the generating set of \( A_J \) presented in Lemma 5.3 as \( \{u_1 - u'_1, \ldots, u_s - u'_s\} \). Let \( I_U \) denote the kernel of the \( k \)-algebra homomorphism \( k[y_1, \ldots, y_s] \to k[A_J] \) sending \( y_i \) to \( x^{u_i}/x^{u'_i} \). This ideal defines the local chart \( U := \text{Spec } k[A_J] \) in \( \text{Hilb } \mathcal{G} \).

**Corollary 5.5.** The universal family above the chart \( \text{Spec } k[A_J] \) is given by

\[
F := \langle x^{u_i} - y_ix^{u'_i} : 1 \leq i \leq s \rangle + I_U
\]

in the ring \( k[x_1, \ldots, x_n][y_1, \ldots, y_s] \).

**Proof.** Write \( R = k[x_1, \ldots, x_n][y_1, \ldots, y_s] \). Let \( Z_U := \text{Spec } (R/F) \). We must show that the map \( Z_U \to U \) is a flat family of \( k \)-schemes with \( Z_U \) being \( G \)-invariant, and that \( H^0(O_{Z_u}) \cong k[\mathcal{G}] \) for all \( u \in U \) where \( Z_u \) is the geometric fiber over \( u \).

We first exhibit a Gröbner basis for \( F \). Let \( \bar{w} \in (Q^{n+1}_0)^* \) be a weight vector for which \( J = \text{in}_w(I_M) \). We extend \( \bar{w} \) to \( \bar{w} \in (Q^{n+1}_0)^* \) by setting \( \bar{w}_i = w_i \) for \( 1 \leq i \leq n \), and \( \bar{w}_i = 0 \) for \( i > n \). Let \( \succ \) be the term order on \( R \) given by defining the order given by \( \bar{w} \) by the lexicographic order. We claim that \( \{x^{u_i} - y_ix^{u'_i} : 1 \leq i \leq s\} \cup \mathcal{G} \) is a Gröbner basis for \( F \) with respect to \( \succ \), where \( \mathcal{G} \) is a Gröbner basis for \( I_U \) in the lexicographic order. Indeed, since \( F \) is a binomial ideal with coefficients \( \pm 1 \), Buchberger’s algorithm ensures that the reduced Gröbner basis for \( F \) with respect to \( \succ \) also consists of binomials of this form. Note also that if we set \( \text{deg}(x^{u_i}) = u_i \in \mathbb{Z}^n \), and \( \text{deg}(y_i) = u_i - u'_i \) then \( F \) is homogeneous with respect to this \( \mathbb{Z}^n \)-grading, and that this refines the \( G^* \)-grading given by setting \( \text{deg}(y_i) = 0 \).
Let \( y^\alpha x^\beta - y^\gamma x^\delta \) be a homogeneous binomial in \( F \) under the \( \mathbb{Z}^n \)-grading with \( \beta \neq \delta \). Then since \( \deg(x^\beta) = \deg(x^\delta) \) either \( x^\beta \) or \( x^\delta \) lies in \( J \) and hence this binomial is reducible by an element in the first part of our proposed Gröbner basis. Thus if the given set is not a Gröbner basis, then there exists an element of the form \((y^\alpha - y^\gamma)x^\beta\) in the true Gröbner basis where we may have \( \gamma = 0 \). Then since \( \deg(y^\alpha) = \deg(y^\gamma) \) under the \( \mathbb{Z}^n \)-grading, \( y^\alpha - y^\gamma \in I_U \). But then it can be reduced to zero using the Gröbner basis \( G \). Thus, no such binomial exists, so the given set is a Gröbner basis for \( F \).

This Gröbner basis means that \( \{x^u : x^u \not\in J\} \subset R \) is a basis for \( R/F \) as a \( k[y_1, \ldots, y_s]/I_U \)-module, so \( R/F \) is a free \( k[y_1, \ldots, y_s]/I_U = k[A_J] \)-module. This implies that the map \( \mathcal{Z}_U \to U \) is flat. Since \( F \) is homogeneous with respect to the \( G^* \)-grading, where \( \deg(y_i) = 0 \) for \( 1 \leq i \leq s \), the scheme \( \mathcal{Z}_U \) is \( G \)-invariant. Since \( \text{Spec}(k[A_J]) \) is reduced, the fiber at a point \( u = (u_1, \ldots, u_s) \in U \) is obtained by specializing the values of the \( y_i \), and thus \( \mathcal{O}_{\mathcal{Z}_u} = k[x_1, \ldots, x_n]/F_u \), where \( F_u \) is the result of specialization. The Gröbner result implies that \( J = \text{in}_w(F_u) \), so \( k[x_1, \ldots, x_n]/F_u \) has the same \( G^* \)-graded Hilbert function as \( J \). By Corollary 2.9 we conclude that \( k[x_1, \ldots, x_n]/F_u \) is a \( G \)-constellation, so \( H^0(\mathcal{O}_{\mathcal{Z}_u}) \cong k[G] \). \( \square \)

5.2. An example of a nonnormal \( G \)-Hilbert scheme. Recall that a sub-semigroup \( NE \) of \( \mathbb{Z}^n \) is normal (or saturated) if \( NE = \mathbb{Z}E \cap \mathbb{Q}_{\geq 0}E \), and that a semigroup algebra is normal as a \( k \)-algebra if and only if the corresponding semigroup is normal. Theorem 5.2 implies that if \( A_J \) is not a normal semigroup for some \( J = \text{in}_w(I_M) \) then the toric variety \( \text{Hilb}^G \) is not normal.

**Algorithm 5.6.** To check whether \( \text{Hilb}^G \) is normal for a given \( G \subseteq \text{GL}(n, k) \).

**Input:** A generating set \( L \) for the lattice \( M = \ker_Z(\deg) \subset \mathbb{Z}^n \).

1. Compute the lattice ideal \( I_M : = \langle x^u - x^v : u - v \in M, u, v \in \mathbb{N}^n \rangle \). To do this, we use the result of Hösten-Sturmfels [13] that

\[
I_M = \left\{ (x^u - x^v : u - v \in L, u, v \in \mathbb{N}^n) : (\prod x_i)^\infty \right\}.
\]

2. Compute all reduced Gröbner bases of \( I_M \). This computation can be done using the software package Gfan [19].

3. For each reduced Gröbner basis \( G = \{x^{\alpha_i} - x^{\beta_i}, i = 1, \ldots, t\} \), check whether the semigroup \( \mathbb{N}\{\alpha_i - \beta_i, i = 1, \ldots, t\} \) is normal. This can be done using the software package Normaliz [4]. If all semigroups checked above are normal, then \( \text{Hilb}^G \) is normal.

**Example 5.7.** Let \( G \subseteq \text{GL}(6, k) \) be the subgroup generated by the diagonal matrices \( \text{diag}(\omega, \omega, \omega, \omega, \omega, \omega) \), \( \text{diag}(1, \omega, 1, \omega^3, \omega^4, \omega^5) \), \( \text{diag}(\omega^3, \omega^2, \omega^4, \omega^2, \omega, \omega) \), and \( \text{diag}(\omega, 1, \omega, 1, 1, 1) \), where \( \omega \) is a primitive fifth root of unity. The group \( G \) is isomorphic to \((\mathbb{Z}/5\mathbb{Z})^4\). Indeed, all four generators have order five, and the matrix

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 3 & 4 & 3 \\
3 & 2 & 4 & 2 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]
with entries in \( \mathbb{Z}/5\mathbb{Z} \) has rank 4, so no generators are redundant. The ideal 
\[ I_M \subset k[a, b, c, d, e, f] \]
leads naturally to the following conjecture.

Corollary 5.9 implies that the distinguished irreducible component 
\( V \) of \( \text{Hilb}^G \) need not be normal. 

Remark 5.8. Example 5.7 was found by applying Algorithm 5.6. The choice of 
group is a modification of an example of a nonnormal toric Hilbert scheme in [29].
We note, however, that the most natural modification of that example, using the 
same weight vector, does not work. It is straightforward to modify Example 5.7 
to get a nonnormal \( \text{Hilb}^G \) for \( G \subseteq \text{SL}(7, k) \)

This example answers the question of Nakamura [23, Remark 2.10].

Corollary 5.9. Nakamura’s \( G \)-Hilbert scheme \( \text{Hilb}^G \) need not be normal.

Remark 5.10. Corollary 5.9 implies that the distinguished irreducible component 
\( V \) of \( \text{Hilb}^G \) is not normal in general. This shows that the assumption 
\( NC = \mathbb{Q}_{\geq 0}C \cap ZC \) made implicitly by Sardo Infirri [28, Proposition 5.3] is not valid in general.

Santos [27] proved that the toric Hilbert scheme may be disconnected. This 
leads naturally to the following conjecture.

Conjecture 5.11. There exists a finite abelian subgroup \( G \subset \text{GL}(n, k) \) such that 
\( G \)-Hilb is disconnected.
Remark 5.12. For a particular $G \subseteq \text{GL}(n, k)$ the connectedness of $G$-Hilb can be checked by enumerating all monomial ideals on $G$-Hilb, and then enumerating those in the connected component of the coherent component using a modification of the flip graph algorithm from [21]. Attempting to modify Santos’ examples from [27] in a similar fashion to the above, however, would give a subgroup of $\text{GL}(26, k)$, which is computationally prohibitive to work with. In addition, just as a naive modification of the nonnormal toric Hilbert scheme example does not give a non-normal $\text{Hilb}_G$, there is no reason to expect that this subgroup of $\text{GL}(26, k)$ would have a disconnected $G$-Hilb. The philosophy remains, however, that multigraded Hilbert schemes tend to be disconnected.

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