INFINITY-INNER-PRODUCTS ON A-INFINITY-ALGEBRAS

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Abstract. In this paper the Hochschild-cochain-complex of an $A_\infty$-algebra $A$ with values in an $A_\infty$-bimodule $M$ over $A$ and maps between them is defined. Then, an $\infty$-inner-product on $A$ is defined to be an $A_\infty$-bimodule-map between $A$ and its dual $A^*$. There is a graph-complex associated to $A_\infty$-algebras with $\infty$-inner-product.

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1. Introduction

An $A_\infty$-algebra $(A,D)$ consists of a module $A$ (over a given ring $R$) together with a coderivation $D$ on the shifted tensor-coalgebra

$$T(sA) := \bigoplus_{i \geq 0} (sA)^{\otimes i},$$

such that $D^2 = 0$. The Hochschild-cochain-complex of $A$ is given by the space $CoDer(T(sA), T(sA))$ of coderivations on $T(sA)$, together with the differential $\delta(f) := [D,f] = D \circ f \pm f \circ D$. This will be reviewed in section 2.

In section 3, the notion of an $A_\infty$-bimodule $(M, D^M)$ over the $A_\infty$-algebra $(A,D)$ is defined, by taking a coderivation $D^M$ from the coalgebra $T(sA)$ into the bicomodule $T^sM(sA)$, which is given by

$$T^sM(sA) := R \oplus \bigoplus_{k \geq 0, l \geq 0} (sA)^{\otimes k} \otimes (sM) \otimes (sA)^{\otimes l},$$

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$\Delta^M(s_{a_1}, ..., s_{m}, ..., s_{a_{k+1}}) := \sum_{i=0}^{k} (s_{a_1}, ..., s_{a_i}) \otimes (s_{a_{i+1}}, ..., s_{m}, ..., s_{a_n}) +$

$\sum_{i=k}^{k+1} (s_{a_1}, ..., s_{m}, ..., s_{a_i}) \otimes (s_{a_{i+1}}, ..., s_{a_{k+1}}),$

such that $(D^M)^2 = 0$. Then this definition of $D^M$ implies that the differential $\delta^M$ on $\text{CoDer}(T_{sA}, T_{sM}sA)$, given by $\delta^M(f) := D^M \circ f \circ D^M$, is well-defined. So, one can call $\text{CoDer}(T_{sA}, T_{sM}sA)$ together with the differential $\delta^M$ the Hochschild-cochain-complex of $A$ with values in the bialgebra $M$.

In section 4, the concept of an $\mathbb{A}_\infty$-bimodule-map between two $\mathbb{A}_\infty$-bimodules $(M, D^M)$ and $(N, D^N)$ over $(A, D)$ is defined. This is given by a map $F : T_{sM}sA \rightarrow T_{sN}sA$ that respects the differentials: $D^N \circ F = F \circ D^M$. Again $F$ has to have good properties with respect to the coderivations $\Delta^M$ and $\Delta^N$. These properties will make the induced map $F^T : \text{CoDer}(T_{sA}, T_{sM}sA) \rightarrow \text{CoDer}(T_{sA}, T_{sN}sA)$, given by $F^T(f) := F \circ f$, being well-defined. So one sees that any $\mathbb{A}_\infty$-bimodule-map induces a map between the corresponding Hochschild-cochain-complexes.

Finally, in section 5, for a given $\mathbb{A}_\infty$-algebra $(A, D)$ one can canonically make $A$ and its dual $A^*$ into an $\mathbb{A}_\infty$-bimodule over $A$ by using the map $D$. An $\mathbb{A}_\infty$-bimodule map between $A$ and $A^*$ will be called an $\infty$-inner-product on $A$. There is a nice combinatorial interpretation for $\infty$-inner-products, which will give rise to a graph-complex associated to $\mathbb{A}_\infty$-algebras with $\infty$-inner-product.

The reason for the naming of the above definitions is that (just like for the $\mathbb{A}_\infty$-structure) one can write down lots of maps which uniquely determine the $\mathbb{A}_\infty$-bimodule-structure [etc.], and which are a direct generalization of the usual concepts of bimodule-structures [etc.].

All the spaces $(V, W, Z, A, M, N, ...,)$ in this paper are always understood to be graded modules $V = \bigoplus_{i \in \mathbb{Z}} V_i$ over a given ground ring $R$. The degree of homogeneous elements $v \in V_i$ is written as $|v| := i$, and the degree of maps $\varphi : V_i \rightarrow W_j$ is written as $|\varphi| := j - i$. All tensor-products of maps and their compositions are understood in a graded way:

$$(\varphi \otimes \psi)(v \otimes w) = (-1)^{|\psi|\cdot|v|} (\varphi(v)) \otimes (\psi(w)),$$

$$(\varphi \otimes \psi) \circ (\chi \otimes \varrho) = (-1)^{|\psi|\cdot|\chi|} (\varphi \circ \chi) \otimes (\psi \circ \varrho).$$

All objects $a_i, v_i, ...$ are assumed to be elements in $A, V, ...$ respectively, if not stated otherwise (e.g. Proposition 4.4.).

It will be necessary to look at elements of $V^{\otimes i} \otimes V^{\otimes j}$. In order to distinguish between the tensor-product in $V^{\otimes i}$ and the one between $V^{\otimes i}$ and $V^{\otimes j}$, it is convenient to write the first one as a tuple $(v_1, ..., v_i) \in V^{\otimes i}$, and then $(v_1, ..., v_i) \otimes (v'_1, ..., v''_j) \in V^{\otimes i} \otimes V^{\otimes j}$. The total degree of $(v_1, ..., v_i) \in V^{\otimes i}$ is given by $|(v_1, ..., v_i)| := \sum_{k=1}^{n} |v_k|$. Frequently there will be sums of the form $\sum_{i=0}^{n} (v_1, ..., v_i) \otimes (v_{i+1}, ..., v_n)$. Here the convention will be used that for $i = 0$, one has the term $1 \otimes (v_1, ..., v_n)$ and for $i = n$ the term in the sum is $(v_1, ..., v_n) \otimes 1$, with $1 = 1_{TV} \in TV$. Similar for terms $\sum_{i=0}^{k} (v_1, ..., v_i) \otimes (v_{i+1}, ..., v_k, w, v_{k+1}, ..., v_n)$ the expression for $i = k$ is understood to mean $(v_1, ..., v_k) \otimes (w, v_{k+1}, ..., v_n)$. 

I would like to thank Dennis Sullivan, who suggested to me that there should be a sequence of homotopies describing Poincaré-duality on the chain level, and M. Markl for many useful comments.

2. $A_\infty$-ALGEBRAS

Let us review the usual definitions about $A_\infty$-algebras that are important to the discussion of this paper, and that can be found in many sources (compare [GJ] section 1, and [S2]).

**Definition 2.1.** A coalgebra $(C, \Delta)$ over a ring $R$ consists of an $R$-module $C$ and a comultiplication $\Delta : C \rightarrow C \otimes C$ of degree 0 satisfying coassociativity:

\[
\begin{array}{ccc}
  C & \xrightarrow{\Delta} & C \otimes C \\
  \downarrow \Delta & & \downarrow \Delta \otimes id \\
  C \otimes C & \xrightarrow{id \otimes \Delta} & C \otimes C \otimes C
\end{array}
\]

Then a coderivation on $C$ is a map $f : C \rightarrow C$ such that

\[
\begin{array}{ccc}
  C & \xrightarrow{\Delta} & C \otimes C \\
  \downarrow f & & \downarrow f \otimes id + id \otimes f \\
  C & \xrightarrow{\Delta} & C \otimes C
\end{array}
\]

**Definition 2.2.** Let $V = \bigoplus_{j \in \mathbb{Z}} V_j$ be a graded module over a given ground ring $R$. The tensor-coalgebra of $V$ over the ring $R$ is given by

\[
TV := \bigoplus_{i \geq 0} V^\otimes i,
\]

\[
\Delta : TV \rightarrow TV \otimes TV, \quad \Delta(v_1, \ldots, v_n) := \sum_{i=0}^{n} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_n).
\]

Let $A = \bigoplus_{j \in \mathbb{Z}} A_j$ be a graded module over a given ground ring $R$. Define its suspension $sA$ to be the graded module $sA = \bigoplus_{j \in \mathbb{Z}} (sA)_j$ with $(sA)_j := A_{j-1}$. The suspension map $s : A \rightarrow sA$, $s : a \mapsto a$ is an isomorphism of degree +1. Now the bar complex of $A$ is given by $BA := T(sA)$.

An $A_\infty$-algebra on $A$ is given by a coderivation $D$ on $BA$ of degree $-1$ such that $D^2 = 0$.

Let’s try to understand this definition.

The tensor-coalgebra has the property to lift every module map $f : TV \rightarrow V$ to a coalgebra-map $F : TV \rightarrow TV$:

\[
\begin{array}{ccc}
  TV & \xrightarrow{F} & V \\
  \downarrow projection & & \downarrow f \\
  TV & \xrightarrow{f} & V
\end{array}
\]

A similar property for coderivations on $TV$ will make it possible to understand the definition of $A_\infty$-algebras in a different way.
Lemma 2.3. (a) Given a map \( \varphi : V^\otimes n \to V \) of degree \(|\varphi|\), which can be viewed as a map \( \varphi : TV \to V \) by letting its only non-zero component being given by the original \( \varphi \) on \( V^\otimes n \). Then \( \varphi \) lifts uniquely to a coderivation \( \tilde{\varphi} : TV \to TV \) with

\[
\begin{array}{c}
TV \\
\downarrow \text{projection} \\
TV \xrightarrow{\tilde{\varphi}} V
\end{array}
\]

by taking

\[\tilde{\varphi}(v_1, ..., v_k) := 0, \quad \text{for } k < n,\]
\[\tilde{\varphi}(v_1, ..., v_k) := \sum_{i=0}^{k-n} (-1)^{|\varphi|(|v_1|+...+|v_i|)} (v_1, ..., \varphi(v_{i+1}, ..., v_{i+n}), v_{i+k}), \quad \text{for } k \geq n.\]

Thus \( \tilde{\varphi} |_{V^\otimes k} : V^\otimes k \to V^\otimes k-n+1 \).

(b) There is a one-to-one correspondence between coderivations \( \sigma : TV \to TV \) and systems of maps \( \{\tilde{\varphi}_i : V^\otimes i \to V\}_{i\geq 0} \), given by \( \sigma = \sum_{i\geq 0} \tilde{\varphi}_i \).

**Proof.** (a) The argument here is dual to the way one lifts derivations on \( TA \). To be precise one should use induction on the output-component of \( \tilde{\varphi} \). Denote by \( \tilde{\varphi}^j \) the component of \( \tilde{\varphi} \) mapping \( TV \to V^\otimes j \). Then \( \tilde{\varphi}^1, ..., \tilde{\varphi}^{m-1} \) determine uniquely the component \( \tilde{\varphi}^m \), because of the coderivation property of \( \tilde{\varphi} \). Let's derive an equation with which this can be seen easily.

\[
\Delta(\tilde{\varphi}(v_1, ..., v_k)) = (\tilde{\varphi} \otimes id + id \otimes \tilde{\varphi})(\Delta(v_1, ..., v_k)) =
\]
\[
= (\tilde{\varphi} \otimes id + id \otimes \tilde{\varphi})(\sum_{i=0}^{k} (v_1, ..., v_i) \otimes (v_{i+1}, ..., v_k)) =
\]
\[
= \sum_{i=0}^{k} \tilde{\varphi}(v_1, ..., v_i) \otimes (v_{i+1}, ..., v_k) +
\]
\[
+(-1)^{|\varphi|(|v_1|+...+|v_i|)} (v_1, ..., v_i) \otimes \tilde{\varphi}(v_{i+1}, ..., v_k).\]

Now, projecting both sides to \( \bigoplus_{i+j=m} V^\otimes i \otimes V^\otimes j \subset TV \otimes TV \) yields

\[
\Delta(\tilde{\varphi}^m(v_1, ..., v_k)) = \sum_{i=0}^{k} \tilde{\varphi}^{m+i-k}(v_1, ..., v_i) \otimes (v_{i+1}, ..., v_k) +
\]
\[
+(-1)^{|\varphi|(|v_1|+...+|v_i|)} (v_1, ..., v_i) \otimes \tilde{\varphi}^{m-i}(v_{i+1}, ..., v_k).\]

So the righthand side depends only on \( \tilde{\varphi}^j \) with \( j < m \), except for the uninteresting terms \( \tilde{\varphi}^m(v_1, ..., v_k) \otimes 1 \) and \( 1 \otimes \tilde{\varphi}^m(v_1, ..., v_k) \), where \( 1 \in TV \).

With this, an induction argument shows that \( \tilde{\varphi}^m \) is only nonzero on \( V^\otimes k \) for \( k = m + n - 1 \), where it is

\[
\tilde{\varphi}^m(v_1, ..., v_{m+n-1}) = \sum_{i=0}^{m-1} (-1)^{|\varphi|(|v_1|+...+|v_i|)} (v_1, ..., \varphi(v_{i+1}, ..., v_{i+n}), ..., v_{m+n-1}).
\]
Proposition 2.4. Let 

\[ \alpha : \{ \varrho_i : V^{\otimes i} \to V \}_{i \geq 0} \to \text{Coder}(TV), \quad \{ \varrho_i : V^{\otimes i} \to V \}_{i \geq 0} \mapsto \sum_{i \geq 0} \varrho_i \]

be given by a system of maps \( D_i : sA^{\otimes i} \to sA \) for \( i \geq 1 \), (where \( D_0 = 0 \) is assumed,) just like in Lemma 2.3.(b), and rewrite them as

\[ m_1(m_1(a_1)) = 0, \]

\[ m_1(m_2(a_1, a_2)) - m_2(m_1(a_1), a_2) - (-1)^{|a_1|}m_2(a_1, m_1(a_2)) = 0, \]

\[ m_1(m_3(a_1, a_2, a_3)) - m_2(m_2(a_1, a_2), a_3) + m_2(a_1, m_2(a_2, a_3)) + m_3(m_1(a_1), a_2, a_3) + (-1)^{|a_1|}m_3(a_1, m_1(a_2), a_3) + (-1)^{|a_1|+|a_2|}m_3(a_1, a_2, m_1(a_3)) = 0, \]

\[ \sum_{i=1}^{k} \sum_{j=0}^{k-i+1} (-1)^{\varepsilon} \cdot m_{k-i+1}(a_1, \ldots, m_i(a_j, \ldots, a_{j+i-1}), \ldots, a_k) = 0, \]

where \( \varepsilon = i \cdot \sum_{l=1}^{i-1} |a_l| + (j - 1) \cdot (i + 1) + k - i \)

Let’s apply this to Definition 2.2.

**Proposition 2.4.** Let \( (A, D) \) be an \( A_\infty \)-algebra. Now let \( D \) be given by a system of maps \( D_i : sA^{\otimes i} \to sA \), where \( D_0 = 0 \) is assumed, just like in Lemma 2.3.(b), and rewrite them as \( m_i : A^{\otimes i} \to A \) given by \( D_i = s \circ m_i \circ (s^{-1})^{\otimes i} \). Then the condition \( D^2 = 0 \) is equivalent to the following system of equations:

\[ D_k(sa_1, \ldots, sa_k) = s \circ m_k \circ (s^{-1})^{\otimes k}(sa_1, \ldots, sa_k) = \]

\[ = (-1)^{k-1} \sum_{j=1}^{k-1} |a_j| + 1 \cdot s \circ m_k \circ ((s^{-1})^{\otimes k-1} \otimes \text{id})(sa_1, \ldots, sa_{k-1}, s^{-1}sa_k) = \]

\[ = (-1)^{k-2} \sum_{j=1}^{k-2} 2(|a_j|+1)+|a_{k-1}|+1 \cdot s \circ m_k \circ ((s^{-1})^{\otimes k-2} \otimes \text{id})^{\otimes 2} (sa_1, \ldots, sa_{k-2}, s^{-1}sa_{k-1}, s^{-1}sa_k) = \]

\[ = (-1)^{k-1} \sum_{j=1}^{k-1} (|a_j|+1) \cdot s \circ m_k(a_1, \ldots, a_k). \]

Therefore it follows from Lemma 2.3.(a) applied to the degree \(-1\) coderivation \( D \), that

\[ pr_A \circ D^2(sa_1, \ldots, sa_k) = \]

(2.1)
\[
\begin{align*}
&= \sum_{i=1}^{k} \sum_{j=0}^{k-i+1} (-1)^{\sum_{l=1}^{i-1} |a_l|} D_{k-i+1}(s_{a_1}, ..., D_i(s_{a_j}, ..., s_{a_{j+i-1}}), ..., s_{a_k}) = \\
&= \sum_{i=1}^{k} \sum_{j=0}^{k-i+1} (-1)^{\sum_{l=1}^{i-1} |a_l|+1+\sum_{l=j}^{i-1} (j+i-l-1) \cdot |a_l|+1} \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \cdot D_{k-i+1}(s_{a_1}, ..., s \circ m_i(s_{a_j}, ..., s_{a_{j+i-1}}), ..., s_{a_k}) = \\
&= \sum_{i=1}^{k} \sum_{j=0}^{k-i+1} (-1)^{m_{k-i+1}}(a_1, ..., m_i(a_j, ..., a_{j+i-1}), ..., a_k),
\end{align*}
\]

where \( \varepsilon \) can be determined by using the fact that \( m_r = \pm s^{-1} \circ D_r \circ s^@r \) is of degree \(-1 - 1 + r = r - 2 \). Instead of doing the general case it is more instructive to look at four special cases where \( i \) and \( j \) are either even or odd. Let’s take \( k = 8 \). As seen above, signs occur from applying \( D_i \), transforming \( D_i \) into \( m_i \), and transforming \( D_{k-i+1} \) into \( m_{k-i+1} \).

\[
\begin{array}{cccccccc}
& a_1 & a_2 & (a_3 & a_4 & a_5) & a_6 & a_7 & a_8 \\
D_i : & |a_1|+1 & |a_2|+1 & & & & & & \\
m_i : & & & |a_4|+1 & & & & & \\
m_{k-i+1} : & |a_1|+1 & (|a_3|+ |a_4|+ |a_5| +3-1) & |a_7|+1 & & & & \\
\end{array}
\]

\[
\begin{array}{cccccccc}
& a_1 & a_2 & a_3 & (a_4 & a_5 & a_6) & a_7 & a_8 \\
D_i : & |a_1|+1 & |a_2|+1 & |a_3|+1 & & & & & \\
m_i : & & & & |a_5|+1 & & & & \\
m_{k-i+1} : & |a_1|+1 & |a_3|+1 & |a_7|+1 & & & & \\
\end{array}
\]

So, if \( i \) is odd, then the lower two rows \( (m_i \) and \( m_{k-i+1} \) show that here the "\( a_r\)"-terms are exactly \( \sum_{l=1}^{k} (k-l) \cdot (|a_l| +1) \), and for \( j = 3 \) there is an additional \(-1\). The top \( (D_r) \) row has the terms \( \sum_{l=1}^{j-1} (|a_l|+1) = (\sum_{l=1}^{j-1} |a_l|) + j - 1 \). The additional \(-1\) can be put together with the \( j - 1 \) to give a constant depending only on \( k \). Thus the term for \( \varepsilon \) is given by

\[
\varepsilon = \sum_{i=1}^{k} (k-l) \cdot (|a_l| +1) + \sum_{l=1}^{j-1} |a_l| + k - 1
\]

\[
\begin{array}{cccccccc}
& a_1 & a_2 & (a_3 & a_4 & a_5 & a_6) & a_7 & a_8 \\
D_i : & |a_1|+1 & |a_2|+1 & & & & & & \\
m_i : & & & |a_4|+1 & |a_5|+1 & & & & \\
m_{k-i+1} : & |a_2|+1 & |a_7|+1 & & & & & & \\
\end{array}
\]

\[
\begin{array}{cccccccc}
& a_1 & a_2 & a_3 & (a_4 & a_5 & a_6) & a_7 & a_8 \\
D_i : & |a_1|+1 & |a_2|+1 & |a_3|+1 & & & & & \\
m_i : & & & |a_4|+1 & |a_5|+1 & |a_6|+1 & & & \\
m_{k-i+1} : & |a_2|+1 & (|a_4|+ |a_5|+ |a_6| +|a_7| +4 -1) & & & & & & \\
\end{array}
\]
So, if \( i \) is even then the "\( a_r \)"-terms (from all three rows) are exactly \( \sum_{i=1}^{k}(k-l)\cdot(|a_l|+1) \). The only \( j \)-dependence is the additional \(-1\) in the \( i = 4, j = 4 \) case, which will induce an alternating sign (, which starts depending on \( k \)). So, this case gives:

\[
\varepsilon = \sum_{l=1}^{k}(k-l)\cdot(|a_l|+1) + k + j - 1.
\]

Putting both cases together, and bringing the common term \( \sum_{l=1}^{k}(k-l)\cdot(|a_l|+1) \) (for both \( i \) even and \( i \) odd) to the left, we get the sign

\[
\varepsilon - \left(\sum_{l=1}^{k}(k-l)\cdot(|a_l|+1)\right) = i \cdot \left(\sum_{l=1}^{j-1}|a_l| + k - 1\right) + (i + 1) \cdot (k + j - 1) = \]

\[
= i \cdot \sum_{l=1}^{j-1}|a_l| + (j - 1) \cdot (i + 1) + k - i.
\]

Thus, dividing the equation \( D^2 = 0 \) by the sign \( (-1)^{\sum_{l=1}^{k}(k-l)\cdot(|a_l|+1)} \) yields the result.

**Example 2.5.** Any differential graded algebra \((A, \partial, \mu)\) gives an \( A_\infty \)-algebra-structure on \( A \) by taking \( m_1 := \partial, m_2 := \mu \) and \( m_k := 0 \) for \( k \geq 3 \). Then the equations from Proposition 2.4. are the defining conditions of a differential graded algebra:

\[
\begin{align*}
\partial^2(a) &= 0, \\
\partial(a \cdot b) &= \partial(a) \cdot b + (-1)^{|a|} a \cdot \partial(b), \\
(a \cdot b) \cdot c &= a \cdot (b \cdot c).
\end{align*}
\]

There are no higher equations.

**Definition 2.6.** Given an \( A_\infty \)-algebra \((A, D)\). Then the **Hochschild-cochain-complex of** \( A \) is defined to be the space \( C^*(A) := CoDer(BA, BA) \) of coderivations on \( BA \) with the differential \( \delta : C^*(A) \to C^*(A) \) given by \( \delta(f) := [D, f] = D \circ f - (-1)^{|f|} f \circ D \). It is \( \delta^2 = 0 \), because as \( D \) is of degree \(-1\) and \( D^2 = 0 \) one follows that \( \delta^2(f) = [D, D \circ f - (-1)^{|f|} f \circ D] = D \circ D \circ f - (-1)^{|f|} D \circ f \circ D - (-1)^{|f|+1} D \circ f \circ D + (-1)^{|f|+1+1} f \circ D \circ D = 0 \).

3. **\( A_\infty \)-bimodules**

Given an \( A_\infty \)-algebra \((A, D)\). The goal is now to define the concept of an \( A_\infty \)-bimodule over \( A \), which was already given in [3,3] and [3,4]. This should be a generalization of two facts. First, one can define the Hochschild-cochain-complex for any algebra with values in a bimodule, and so one should still be able to make that definition in the infinite case. Second, any algebra is a bimodule over itself by left- and right-multiplication, which should again still hold (see section 5).

The following space and map are important ingredients.

**Definition 3.1.** For modules \( V \) and \( W \) over \( R \), one writes

\[
T^W V := R \bigoplus \bigoplus_{k \geq 0, l \geq 0} V^{\otimes k} \otimes W \otimes V^{\otimes l}.
\]
Furthermore, let \( \Delta^W : T^W V \rightarrow (T V \otimes T^W V) \oplus (T^W V \otimes TV) \),
be given by
\[
\Delta^W (v_1, \ldots, v_k, w, v_{k+1}, \ldots, v_{k+l}) := \sum_{i=0}^{k} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, w, \ldots, v_n) + \\
\sum_{i=k}^{k+l} (v_1, \ldots, w, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_{k+l}).
\]

Again for modules \( A \) and \( M \) let \( B^M A \) be given by \( T^s M \triangleright A \), where \( s \) is the suspension from Definition 2.2.

Observe that \( T^W V \) is not a coalgebra, but rather a bi-comodule over \( TV \). Here is a definition for a coderivation from \( TV \) to \( T^W V \).

**Definition 3.2.** A coderivation from \( TV \) to \( T^W V \) is defined to be a map \( f : TV \rightarrow T^W V \) that makes the following diagram commute:

\[
\begin{array}{ccc}
TV & \xrightarrow{\Delta} & TV \otimes TV \\
of & \downarrow & \downarrow \text{id} \otimes f + f \otimes \text{id} \\
T^W V & \xrightarrow{\Delta^W} & (TV \otimes T^W V) \oplus (T^W V \otimes TV)
\end{array}
\]

For modules \( A \) and \( M \) let \( C^* (A, M) := \text{CoDer}(BA, B^M A) \) be the space of coderivations in the above sense. This space is called the Hochschild-cochain-complex of \( A \) with values in \( M \).

**Lemma 3.3.** (a) Given a map \( \varrho : V^\otimes n \rightarrow W \) of degree \( |\varrho| \), which can be viewed as a map \( \varrho : TV \rightarrow W \) by letting its only non-zero component being given by the original \( \varrho \) on \( V^\otimes n \). Then \( \varrho \) lifts uniquely to a coderivation \( \tilde{\varrho} : TV \rightarrow T^W V \) with

\[
\tilde{\varrho}(v_1, \ldots, v_k) := 0, \quad \text{for } k < n,
\]
\[
\tilde{\varrho}(v_1, \ldots, v_k) := \sum_{i=0}^{k-n} (-1)^{|\varrho|(|v_1|+\ldots+|v_i|)} (v_1, \ldots, \varrho(v_{i+1}, \ldots, v_{i+n}), \ldots, v_k),
\]
for \( k \geq n \).

Thus \( \tilde{\varrho} \restriction_{V^\otimes k} : V^\otimes k \rightarrow \bigoplus_{i+j=k-n} V^\otimes i \otimes W \otimes V^\otimes j \).

(b) There is a one-to-one correspondence between coderivations \( \sigma : TV \rightarrow T^W V \) and systems of maps \( \{ \varrho_i : V^\otimes i \rightarrow W \}_{i \geq 0} \), given by \( \sigma = \sum_{i \geq 0} \varrho_i \).
Proof. (a) The proof is similar to the one of Lemma 2.3(a). Now \( \tilde{\varphi}^j \) is meant to be the component of \( \tilde{\varphi} \) mapping \( TV \rightarrow \bigoplus_{r+s=j} V^{\otimes r} \otimes W \otimes V^{\otimes s} \). Let’s do again induction on \( m \) for \( \tilde{\varphi}^m \). The equation

\[
\Delta^W(\tilde{\varphi}(v_1, \ldots, v_k)) = (\tilde{\varphi} \otimes id + id \otimes \tilde{\varphi})(\Delta(v_1, \ldots, v_k)) = \\
(\tilde{\varphi} \otimes id + id \otimes \tilde{\varphi})(\sum_{i=0}^k (v_1, \ldots, v_i \otimes (v_{i+1}, \ldots, v_k)) = \\
\sum_{i=0}^k \tilde{\varphi}(v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_k) + \\
+(-1)^i|\tilde{\varphi}|(|v_1|+\ldots+|v_i|)(v_1, \ldots, v_i) \otimes \tilde{\varphi}(v_{i+1}, \ldots, v_k).
\]

is being projected to the component

\[
\bigoplus_{r+s+t=m} (V^r \otimes W \otimes V^s) \otimes V^t + V^r \otimes (V^s \otimes W \otimes V^t) \subset T^W V \otimes TV + TV \otimes T^W V.
\]

This gives

\[
\Delta^W(\tilde{\varphi}^m(v_1, \ldots, v_k)) = \sum_{i=0}^k \tilde{\varphi}^{m+i-k}(v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_k) + \\
+(-1)^i|\tilde{\varphi}|(|v_1|+\ldots+|v_i|)(v_1, \ldots, v_i) \otimes \tilde{\varphi}^{m-i}(v_{i+1}, \ldots, v_k).
\]

The righthand side depends only on \( \tilde{\varphi}^j \) with \( j < m \), except for the uninteresting terms \( \tilde{\varphi}^m(v_1, \ldots, v_k) \otimes 1 \) and \( 1 \otimes \tilde{\varphi}^m(v_1, \ldots, v_k) \). An induction shows that \( \tilde{\varphi}^m \) is only nonzero on \( V^{\otimes k} \) for \( k = m + n - 1 \), where it is

\[
\tilde{\varphi}^m(v_1, \ldots, v_{m+n-1}) = \sum_{i=0}^{m-1} (-1)^i|\tilde{\varphi}|(|v_1|+\ldots+|v_i|)(v_1, \ldots, \tilde{\varphi}(v_{i+1}, \ldots, v_{i+n}), \ldots, v_{m+n-1}).
\]

(b) Then maps

\[
\alpha : \{(\varphi_i : V^{\otimes i} \rightarrow W)_{i \geq 0}\} \rightarrow \text{Coder}(TV, T^W V), \quad \{\varphi_i : V^{\otimes i} \rightarrow W\}_{i \geq 0} \mapsto \sum_{i \geq 0} \varphi_i
\]

\[
\beta : \text{Coder}(TV, T^W V) \rightarrow \{(\varphi_i : V^{\otimes i} \rightarrow W)_{i \geq 0}\}, \quad \sigma \mapsto \{\text{pr}_W \circ \sigma|_{V^{\otimes i}}\}_{i \geq 0}
\]

are inverse to each other by (a).

Of course one wants to put a differential on \( C^*(A, M) \) just like in section 2.

**Proposition 3.4.** Given an \( A_{\infty} \)-algebra \( (A, D) \) and a module \( M \). Let \( D^M : B^M A \rightarrow B^M A \) be a map of degree \(-1\). Then the induced map \( \delta^M : \text{Coder}(BA, B^M A) \rightarrow \text{Coder}(BA, B^M A) \), given by \( \delta^M(f) := D^M \circ f - (-1)^{|f|} f \circ D \), is well-defined, (i.e. it maps coderivations to coderivations,) if and only if the following diagram commutes:

\[
\begin{array}{ccc}
B^M A & \xrightarrow{\Delta^M} & (BA \otimes B^M A) \oplus (B^M A \otimes BA) \\
\downarrow D^M & & \downarrow ((id \otimes D^M + D \otimes id) \oplus (D^M \otimes id + id \otimes D)) \\
B^M A & \xrightarrow{\Delta^M} & (BA \otimes B^M A) \oplus (B^M A \otimes BA)
\end{array}
\]
Proof. Let \( f : BA \to B^M A \) be a coderivation. One needs to investigate under which conditions \( \delta^M(f) \) is also a coderivation. This means, that

\[
(id \otimes \delta^M(f) + \delta^M(f) \otimes id) \circ \Delta = \Delta^M \circ \delta^M(f),
\]
or

\[
(id \otimes (D^M \circ f) - (-1)^{|f|}id \otimes (f \circ D) + (D^M \circ f) \otimes id - (-1)^{|f|}(f \circ D) \otimes id) \circ \Delta =
\]

\[
= \Delta^M \circ D^M \circ f - (-1)^{|f|} \Delta^M \circ f \circ D.
\]

Now, using the coderivation property for \( f \) and \( D \), one gets the following identity

\[
\Delta^M \circ f \circ D = (id \otimes f) \circ \Delta \circ D + (f \otimes id) \circ \Delta \circ D =
\]

\[
= (id \otimes f) \circ (id \otimes D) \circ \Delta + (id \otimes f) \circ (D \otimes id) \circ \Delta +
\]

\[
+ (f \otimes id) \circ (id \otimes D) \circ \Delta + (f \otimes id) \circ (D \otimes id) \circ \Delta =
\]

\[
= (id \otimes (f \circ D) + (-1)^{|f|}D \otimes f + f \otimes D + (f \circ D) \otimes id) \circ \Delta,
\]

and so the requirement for \( \delta^M(f) \) above reduces to

\[
\Delta^M \circ D^M \circ f = (id \otimes (D^M \circ f) + (D^M \circ f) \otimes id + D \otimes f + (-1)^{|f|}f \otimes D) \circ \Delta =
\]

\[
= (id \otimes D^M + D \otimes id) \circ (id \otimes f) \circ \Delta + (D^M \otimes id + id \otimes D) \circ (f \otimes id) \circ \Delta =
\]

\[
= (id \otimes D^M + D \otimes id) \circ \Delta^M \circ f + (D^M \otimes id + id \otimes D) \circ \Delta^M \circ f.
\]

The last step looks strange, because \( \Delta^M \circ f = (id \otimes f) \circ \Delta + (f \otimes id) \circ \Delta \). But as \( id \otimes D^M + D \otimes id : BA \otimes B^M A \to BA \otimes B^M A \), this map doesn’t pick up any part from \( (f \otimes id) \circ \Delta : BA \to B^M A \otimes BA \). A similar argument applies to \( D^M \otimes id + id \otimes D \).

So, \( D^M \) has to satisfy

\[
\Delta^M \circ D^M \circ f = (id \otimes D^M + D \otimes id + D^M \otimes id + id \otimes D) \circ \Delta^M \circ f,
\]

for all coderivations \( f : TA \to T^M A \). By Lemma 3.3., one sees that there are enough coderivations, to make this condition equivalent to

\[
\Delta^M \circ D^M = (id \otimes D^M + D \otimes id + D^M \otimes id + id \otimes D) \circ \Delta^M,
\]

which is the claim. \( \square \)

Again one wants to describe \( D^M \) by a system of maps.

**Lemma 3.5.** (a) Given a module \( V \) and a coderivation \( \psi \) on \( TV \) with an associated system of maps \( \{ \psi_i : V \otimes i \to V \}_{i \geq 1} \) from Lemma 2.3., where \( \psi_i \) is of degree \( |\psi_i| \). Then any map \( \varrho : T^W V \to W \) given by \( \varrho = \sum_{k \geq 0, l \geq 0} \theta_{k,l} \), with \( \theta_{k,l} : V \otimes k \otimes W \otimes V \otimes l \to W \) of degree \( |\theta_{k,l}| \), lifts uniquely to a map \( \tilde{\varrho} : T^W V \to T^W V \)
which makes the following diagram commute (compare diagram (3.1) in Propo-

\[
\begin{array}{ccc}
T^W V & \xrightarrow{\Delta^W} & (TV \otimes T^W V) \oplus (T^W V \otimes TV) \\
\hat{\vartheta} & & (id \otimes \hat{\vartheta} + \varphi \otimes id) \oplus (\hat{\vartheta} \otimes id + id \otimes \varphi) \\
T^W V & \xrightarrow{\Delta^W} & (TV \otimes T^W V) \oplus (T^W V \otimes TV)
\end{array}
\]

(3.2)

This map is given by taking

\[
\hat{\vartheta}(v_1, \ldots, v_k, w, v_{k+1}, \ldots, v_{k+l}) :=
\]

\[
\begin{aligned}
&= \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} (-1)^{|\psi_i| + \sum_{r=1}^{j-1} |v_r|} (v_1, \ldots, \psi_i(v_j, \ldots, v_{i+j-1}), \ldots, w, \ldots, v_{k+l}) + \\
&\quad + \sum_{i=0}^{k} \sum_{j=0}^{l} (-1)^{|\psi_i| + \sum_{r=1}^{k-i+1} |v_r|} (v_1, \ldots, \psi_i(v_{k-i+1}, \ldots, w, \ldots, v_{k+i}), \ldots, v_{k+l}) + \\
&\quad + \sum_{i=1}^{l} \sum_{j=1}^{l-i+1} (-1)^{|\psi_i| + \sum_{r=1}^{k+i-j-1} |v_r|} (v_1, \ldots, w, \ldots, \psi_i(v_{k+j}, \ldots, v_{k+i-j-1}), \ldots, v_{k+l}).
\end{aligned}
\]

(Notice that the condition of satisfying diagram (3.2) is not linear, i.e. if \(\chi\) and \(\varphi\) both make diagram (3.2) commute, then \(\chi + \varphi \) will not.)

(b) There is a one-to-one correspondence between maps \(\sigma : T^W V \rightarrow T^W V\) that make diagram (3.2) commute and maps \(\vartheta = \sum \vartheta_{k,i}\) like in (a), given by \(\sigma = \hat{\vartheta}\).

Proof. (a) Again one uses induction on the output-component of \(\hat{\vartheta}\). Denote by \(\hat{\vartheta}^i\) the component of \(\hat{\vartheta}\) mapping \(T^W V \rightarrow \bigoplus_{k+l=j} V^\otimes j \otimes W \otimes V^\otimes l\) and by \(\varphi^i\) the component of \(\varphi\) mapping \(TV \rightarrow V^\otimes\). Then \(\hat{\vartheta}^0, \ldots, \hat{\vartheta}^{m-1}\) will uniquely determine the component \(\hat{\vartheta}^m\).

\[
\Delta^W(\hat{\vartheta}(v_1, \ldots, v_k, w, v_{k+1}, \ldots, v_{k+l})) =
\]

\[
= (id \otimes \hat{\vartheta} + \varphi \otimes id + \hat{\vartheta} \otimes id + id \otimes \varphi)(\Delta^W(v_1, \ldots, v_k, w, v_{k+1}, \ldots, v_{k+l})) =
\]

\[
= (id \otimes \hat{\vartheta} + \varphi \otimes id + \hat{\vartheta} \otimes id + id \otimes \varphi)\left(\sum_{i=0}^{k} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, w, \ldots, v_{k+l}) + \\
\quad + \sum_{i=k}^{k+l} (v_1, \ldots, w, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_{k+l})\right) =
\]

\[
= \sum_{i=0}^{k} (-1)^{|\psi| + \sum_{r=1}^{i} |v_r|} (v_1, \ldots, v_i) \otimes \hat{\vartheta}(v_{i+1}, \ldots, w, \ldots, v_{k+l}) + \\
\quad + \sum_{i=0}^{k} \varphi(v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_{k+l}) + \\
\quad + \sum_{i=k}^{k+l} \hat{\vartheta}(v_1, \ldots, w, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_{k+l}) + \\
\quad + \sum_{i=k}^{k+l} (-1)^{|\psi| + \sum_{r=1}^{i} |v_r|} (v_1, \ldots, w, \ldots, v_i) \otimes \varphi(v_{i+1}, \ldots, v_{k+l}).
\]
Then projecting both sides to
\[ \bigoplus_{r+s+t=m} (V^r \otimes W \otimes V^{s+t}) \otimes V^t + V^r \otimes (V^s \otimes W \otimes V^{t}) \subset T^W V \otimes TV + TV \otimes T^W V \]
yields
\[ \Delta^W (\tilde{\partial}^m(v_1, ..., w, ..., v_k)) = \sum_{i=0}^{k} \pm (v_1, ..., v_i) \otimes \tilde{\partial}^{m-i}(v_{i+1}, ..., w, ..., v_{k+l}) + \]
\[ + \sum_{i=0}^{k} \psi^{m+i-k-l}(v_1, ..., v_i) \otimes (v_{i+1}, ..., w, ..., v_{k+l}) + \]
\[ + \sum_{i=k}^{k+l} \tilde{\partial}^{m+i-k-l}(v_1, ..., w, ..., v_i) \otimes (v_{i+1}, ..., v_{k+l}) + \]
\[ + \sum_{i=k}^{k+l} \pm (v_1, ..., w, ..., v_i) \otimes \psi^{m-i}(v_{i+1}, ..., v_{k+l}). \]

So the righthand side depends only on \( \psi^{j} \)'s, which are all explicitly known by Lemma 2.3., and \( \tilde{\partial}^{j} \) with \( j < m \), (except for the uninteresting terms \( \tilde{\partial}^{m}(v_1, ..., w, ..., v_{k+l}) \otimes 1 \) and \( 1 \otimes \tilde{\partial}^{m}(v_1, ..., w, ..., v_{k+l}) \)). With this, an induction argument shows that \( \tilde{\partial}^{m} \) is given by the formula of the Lemma.

(b) Let \( X := \{ \sigma : T^W V \to T^W V \mid \sigma \text{ makes diagram (3.2) commute } \} \). Then
\[ \alpha : \{ \varrho : T^W V \to W \} \to X, \quad \varrho \mapsto \tilde{\varrho}, \]
\[ \beta : X \to \{ \varrho : T^W V \to W \}, \quad \sigma \mapsto \text{pr}_W \circ \sigma \]
are inverse to each other by (a).

\[ \square \]

**Definition 3.6.** Given an \( A_{\infty} \)-algebra \((A, D)\). Then an \( A_{\infty} \)-bimodule \((M, D^M)\) consists of a module \( M \) together with a map \( D^M : B^MA \to B^MA \) of degree \(-1\), which makes the diagram (3.1) of Proposition 3.4. commute, and satisfies \((D^M)^2 = 0\).

By Proposition 3.4., one can put the differential \( \delta^M : \text{CoDer}(TA, T^MA) \to \text{CoDer}(TA, T^MA), \delta(f) := D^M \circ f - (-1)^{|f|} f \circ D \) on the Hochschild-cochain-complex. Now it satisfies \((\delta^M)^2 = 0\), because with \((D^M)^2 = 0\), one gets \((\delta^M)^2(f) = D^M \circ D^M \circ f - (-1)^{|f|} D^M \circ f \circ D - (-1)^{|f|+1} D^M \circ f \circ D + (-1)^{|f|+1} f \circ D \circ D = 0\.

The definition of an \( A_{\infty} \)-bimodule was already given in [3] section 3 and also in [M], and coincides with the one here.

**Proposition 3.7.** Let \((A, D)\) be an \( A_{\infty} \)-algebra with a system of maps \( \{ m_i : A^{\otimes i} \to A \}_{i \geq 1} \) associated to \( D \) by Proposition 2.4. (where \( m_0 = 0 \) is assumed). Let \((M, D^M)\) be an \( A_{\infty} \)-bimodule over \( A \) with a system of maps \( \{ D^M_{k,l} : sA^{\otimes k} \otimes sM \otimes sA^{\otimes l} \to sM \}_{k \geq 0, l \geq 0} \) from Lemma 3.5.(b) associated to \( D^M \). Let \( \{ b_{k,l} : A^{\otimes k} \otimes M \otimes A^{\otimes l} \to M \} \) be the induced map by \( D^M_{k,l} = s \circ b_{k,l} \circ (s^{-1})^{\otimes k+l+1} \).
There are no higher equations.

According to where the element of $M$ is located. Keeping this in mind, it is possible to redo all the steps from Proposition 2.4. \hfill \square

Then the condition $(D^M)^2 = 0$ is equivalent to the following system of equations:

\[
\begin{align*}
\sum_{i=1}^{k} \sum_{j=1}^{k-i+1} \pm b_{k-i+1,l}(a_1, \ldots, m_i(a_j, \ldots, a_{i+j-1}), \ldots, m, \ldots, a_{k+l}) + \\
+ \sum_{i=0}^{k} \sum_{j=0}^{l} \pm b_{k-i,l-j}(a_1, \ldots, b_{i,j}(a_{k-i+1}, \ldots, m, \ldots, a_{k+j}), \ldots, a_{k+l}) + \\
+ \sum_{i=1}^{k} \sum_{j=1}^{l-i+1} \pm b_{k,l-i+1}(a_1, \ldots, m, \ldots, m_i(a_{k+j}, \ldots, a_{k+i+j-1}), \ldots, a_{k+l}) = 0
\end{align*}
\]

where the signs are exactly analogous to the ones in Proposition 2.4.

Proof. The result follows immediately from Lemma 3.5., after rewriting $D^M_{k,l}$ and $D_j$ by $b_{k,l}$ and $m_j$. (Notice that this replacement only changes a sign.)

In order to get the correct sign, first notice that the lifting described in Lemma 3.5. is exactly the usual lifting as coderivations, except that one has to pick $b_{k,l}$ or $m_j$ according to where the element of $M$ is located. Keeping this in mind, it is possible to redo all the steps from Proposition 2.4.

Example 3.8. Let's pick up Example 2.5. Let $(A, \partial, \mu)$ be a differential graded algebra with the $A_\infty$-algebra-structure $m_1 := \partial$, $m_2 := \mu$ and $m_k := 0$ for $k \geq 3$. Now, let $(M, \partial', \lambda, \rho)$ be a differential graded bimodule over $A$, where $\lambda : A \otimes M \rightarrow M$ and $\rho : M \otimes A \rightarrow M$ denote the left- and right-action. It is possible to make $M$ into an $A_\infty$-bimodule over $A$ by taking $b_{0,0} := \partial'$, $b_{1,0} := \lambda$, $b_{0,1} := \rho$ and $b_{k,l} := 0$ for $k + l > 1$. Then the equations of Proposition 3.7. are the defining conditions of a differential bialgebra over $A$:

\[
\begin{align*}
(\partial')^2(m) &= 0, \\
\partial'(m.a) &= m.\partial(a) + (-1)^{|m|}\partial'(m).a, \\
\partial'(a.m) &= \partial(a).m + (-1)^{|a|}a.\partial'(m), \\
(a.m).b &= a.(m.b), \\
(m.a).b &= m.(a \cdot b), \\
a.(b.m) &= (a \cdot b).m.
\end{align*}
\]

There are no higher equations.

For later purposes it is convenient to have the following
Lemma 3.9. Given an $A_\infty$-algebra $(A,D)$ and an $A_\infty$-bimodule $(M,D^M)$, with system of maps $\{b_{k,l}:A^\otimes k \otimes M \otimes A^\otimes l \rightarrow M\}_{k \geq 0,l \geq 0}$ from Proposition 3.7.
Then the dual space $M^*:=\text{Hom}_R(M,R)$ has a canonical $A_\infty$-bimodule-structure given by maps $\{b'_{k,l}:A^\otimes k \otimes M^* \otimes A^\otimes l \rightarrow M^*\}_{k \geq 0,l \geq 0}$.

$(b'_{k,l}(a_1,...,a_k,m^*,a_{k+1},...,a_{k+l}))(m):=(-1)^{\varepsilon}m^*(b_{k,l}(a_{k+1},...,a_{k+l},m,a_1,...,a_k))$,
where $\varepsilon:=\sum l_i|a_i|$, for the middle term:

$\text{So, these terms come from terms of the } A_\infty\text{-bimodule-structure of } M.\text{ The same is true for the middle term:}$

$(b'_{k-l-1,i}(a_1,...,m_i(a_{k+j},...,a_{k+j+1}),...,a_{k+l}))(m)=$

$=\pm m^*(b_{k-l,i+1}(a_{k+j+1},...,a_{k+l},m,a_{k+1},...,m_i(a_{k+j},...,a_{k+j+1}),...,a_{k+l})).$

The only remaining question is whether the signs are correct. The proof for this is left to the reader. $
\square$

4. Morphisms of $A_\infty$-bimodules

Given two $A_\infty$-bimodules $(M,D^M)$ and $(N,D^N)$ over an $A_\infty$-algebra $(A,D)$. What is the natural notion of morphism between them?

Again a motivation is to have for any $A_\infty$-bimodule-map an induced map of their Hochschild-cochain-complexes.

Proposition 4.1. Given three modules $V$, $W$ and $Z$. Let $F:T^W V \rightarrow T^Z V$ be a map. Then the induced map $F^*:\text{CoDer}(TV,T^W V) \rightarrow \text{CoDer}(TV,T^Z V)$, given by $F^*(f):=F \circ f$, is well-defined, (i.e. it maps coderivations to coderivations,) if and only if the following diagram commutes:

$$
\begin{array}{ccc}
T^W V & \xrightarrow{\Delta^W} & (TV \otimes T^W V) \oplus (T^W V \otimes TV) \\
F & & (id \otimes F) \oplus (F \otimes id) \\
T^Z V & \xrightarrow{\Delta^Z} & (TV \otimes T^Z V) \oplus (T^Z V \otimes TV)
\end{array}
$$

(4.1)
Proof. If both $f : TV \rightarrow T^W V$ and $F \circ f : TV \rightarrow T^Z V$ are coderivations then this means that the top diagram and the overall diagram below commute.

$$
\begin{array}{ccc}
TV & \xrightarrow{\Delta} & TV \otimes TV \\
f \downarrow & & \downarrow \text{(id} \otimes f) + (f \otimes \text{id}) \\
T^W V & \xrightarrow{\Delta^W} & (TV \otimes T^W V) \oplus (T^W V \otimes TV) \\
F \downarrow & & \downarrow \text{(id} \otimes F) + (F \otimes \text{id}) \\
T^Z V & \xrightarrow{\Delta^Z} & (TV \otimes T^Z V) \oplus (T^Z V \otimes TV)
\end{array}
$$

But then the lower diagram has to commute if applied to any element in $\text{Im}(f) \subset T^W V$. By Lemma 3.3, there are enough coderivations to make this true for all $T^W V$. □

Again let’s describe $F$ by a system of maps.

Lemma 4.2. (a) Given modules $V$, $W$ and $Z$ and a map $\varrho : V^\otimes k \otimes W \otimes V^\otimes l \rightarrow Z$ of degree $|\varrho|$, which can be viewed as a map $\varrho : T^W V \rightarrow Z$ by letting its only nonzero component be the original $\varrho$ on $V^\otimes k \otimes W \otimes V^\otimes l$. Then $\varrho$ lifts uniquely to a map $\tilde{\varrho} : T^W V \rightarrow T^Z V$

$$
\begin{array}{c}
T^Z V \\
\tilde{\varrho} \\
\downarrow \text{projection} \\
T^W V \\
\text{\varrho} \\
\downarrow Z
\end{array}
$$

which makes the diagram (4.1) in Proposition 4.1. commute (put $\tilde{\varrho}$ instead of $F$). This map is given by

$$
\tilde{\varrho}(v_1, \ldots, v_r, w, v_{r+1}, \ldots, v_{r+s}) := 0, \quad \text{for } r < k \text{ or } s < l,
$$

$$
\tilde{\varrho}(v_1, \ldots, v_r, w, v_{r+1}, \ldots, v_{r+s}) := (-1)^{|\varrho|} \sum_{i=1}^{r-k} |v_i| \varrho(v_1, \ldots, \varrho(v_{r-k+1}, \ldots, w, \ldots, v_{r+l}), \ldots, v_{r+s}),
$$

for $r \geq k$ and $s \geq l$.

Thus $\tilde{\varrho} \mid_{V^\otimes r \otimes W \otimes V^\otimes s} : V^\otimes r \otimes W \otimes V^\otimes s \rightarrow V^\otimes r-k \otimes Z \otimes V^\otimes s-l$.

(b) There is a one-to-one correspondence between maps $\sigma : T^W V \rightarrow T^Z V$ making diagram (4.1) commute and systems of maps $\{\varrho_{k,l} : V^\otimes k \otimes W \otimes V^\otimes l \rightarrow Z\}_{k \geq 0, l \geq 0}$, given by $\sigma = \sum_{k \geq 0, l \geq 0} \varrho_{k,l}$.

Proof. (a) Again one uses induction on the output-component of $\tilde{\varrho}$. Denote by $\tilde{\varrho}^j$ the component of $\tilde{\varrho}$ mapping $T^W V \rightarrow \bigoplus_{r+s=j} V^\otimes r \otimes Z \otimes V^\otimes s$. Then $\tilde{\varrho}^1, \ldots, \tilde{\varrho}^{m-1}$ determine uniquely the component $\tilde{\varrho}^m$.

$$
\Delta^Z(\tilde{\varrho}(v_1, \ldots, v_r, w, v_{r+1}, \ldots, v_{r+s})) = \sum_{j=1}^{m-1} \tilde{\varrho}^j(\Delta^Z(v_1, \ldots, v_r, w, v_{r+1}, \ldots, v_{r+s})).
$$
$$= (id \otimes \tilde{\varphi} + \hat{\varphi} \otimes id) (\Delta^W (v_1, \ldots, v_r, w, v_{r+1}, \ldots, v_{r+s})) =$$
$$= (id \otimes \tilde{\varphi} + \hat{\varphi} \otimes id) (\sum_{i=0}^{r} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, w, \ldots, v_{r+s}) +$$
$$+ \sum_{i=r}^{r+s} (v_1, \ldots, w, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_{r+s}) ) =$$
$$= \sum_{i=0}^{r} (-1)^{|i|} \sum_{i=1}^{|i|} |v_i| (v_1, \ldots, v_i) \otimes \hat{\varphi}(v_{i+1}, \ldots, w, \ldots, v_{r+s}) +$$
$$+ \sum_{i=r}^{r+s} \hat{\varphi}(v_1, \ldots, w, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_{r+s}).$$

Then projecting both sides to
$$\bigoplus_{r+s+l=m} (V^\otimes r \otimes Z \otimes V^\otimes s) \otimes V^\otimes t + V^\otimes t \otimes (V^\otimes s \otimes Z \otimes V^\otimes t) \subset T Z V \otimes TV + TV \otimes T Z V$$

yields
$$\Delta^Z (\hat{\varphi}^m (v_1, \ldots, w, \ldots, v_{r+s})) = \sum_{i=0}^{r} \pm (v_1, \ldots, v_i) \otimes \hat{\varphi}^{m-i} (v_{i+1}, \ldots, w, \ldots, v_{r+s}) +$$
$$+ \sum_{i=r}^{r+s} \hat{\varphi}^{m+i-r-s} (v_1, \ldots, w, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_{r+s}).$$

So the right-hand side depends only on \( \hat{\varphi}^j \) with \( j < m \), (except for the uninteresting terms \( \hat{\varphi}^m (v_1, \ldots, w, \ldots, v_{r+s}) \otimes 1 \) and \( 1 \otimes \hat{\varphi}^m (v_1, \ldots, w, \ldots, v_{r+s}) \)). With this, an induction argument shows that \( \hat{\varphi}^m \) is only nonzero on \( V^\otimes r \otimes W \otimes V^\otimes s \) with \( r - k + s - l = m \), where it is given by
$$\hat{\varphi}^m (v_1, \ldots, v_r, w, v_{r+1}, \ldots, v_{r+s}) =$$
$$= (-1)^{|i|} \sum_{i=1}^{|i|} |v_i| (v_1, \ldots, \varphi(v_{r-k+1}, \ldots, w, \ldots, v_{r+l}), \ldots, v_{r+s}).$$

(b) Let \( X := \{ \sigma : T^W V \to T Z V \mid \sigma \text{ makes diagram (4.1) commute} \} \). Then
$$\alpha : \{ \{ \varphi_{k,l} : V^\otimes k \otimes W \otimes V^\otimes l \to Z \}_{k \geq 0, l \geq 0} \} \to X,$$
$$\beta : X \to \{ \{ \varphi_{k,l} : V^\otimes k \otimes W \otimes V^\otimes l \to Z \}_{k \geq 0, l \geq 0} \},$$
$$\sigma \mapsto \{ pr_Z \circ \sigma |_{V^\otimes k \otimes W \otimes V^\otimes l} \}_{k \geq 0, l \geq 0},$$

are inverse to each other by \( (a) \).

\( \square \)

Let’s apply this to the Hochschild-space.
Definition 4.3. Given two $A_{\infty}$-bimodules $(M, D^M)$ and $(N, D^N)$ over an $A_{\infty}$-algebra $(A, D)$. Then a map $F : B^M A \rightarrow B^N A$ of degree 0 is called an $A_{\infty}$-bimodule-map $\Leftrightarrow$ $F$ makes the diagram

\[
\begin{array}{ccc}
B^M A & \xrightarrow{\Delta^M} & (BA \otimes B^M A) \oplus (B^M A \otimes BA) \\
\downarrow F & & \downarrow (id \otimes F) \oplus (F \otimes id) \\
B^N A & \xrightarrow{\Delta^N} & (BA \otimes B^N A) \oplus (B^N A \otimes BA)
\end{array}
\]

commute, and in addition it holds that $F \circ D^M = D^N \circ F$. By Proposition 4.1., every $A_{\infty}$-bimodule-map induces (by composition $F^* : f \mapsto F \circ f$) a map between the Hochschild-spaces, which preserves the differentials, because $(F^1 \circ \Delta^M)(f) = F^D D^M \circ f + (-1)^{|f|} f \circ D) = F \circ D^M \circ f + (-1)^{|f|} F \circ f \circ D = D^N \circ f \circ f + (-1)^{|f|} F \circ f = \delta^N (F \circ f) = (\delta^N \circ F^*)(f)$.

Proposition 4.4. Let $(A, D)$ be an $A_{\infty}$-algebra with a system of maps $\{m_i : A^{\otimes i} \rightarrow A\}_{i \geq 1}$ from Proposition 2.4. associated to $D$, (where $m_0 = 0$ is assumed). Let $(M, D^M)$ and $(N, D^N)$ be $A_{\infty}$-bimodules over $A$ with systems of maps $\{b_{k,l} : A^{\otimes k} \otimes M \otimes A^\otimes l \rightarrow M\}_{k,l \geq 0}$ and $\{c_{k,l} : A^{\otimes k} \otimes N \otimes A^\otimes l \rightarrow N\}_{k,l \geq 0}$ from Proposition 3.7. associated to $D^M$ and $D^N$ respectively. Let $F : T^M A \rightarrow T^N A$ be an $A_{\infty}$-bimodule-map between $M$ and $N$, and let $\{F_{k,l} : sA^\otimes k \otimes sM \otimes sA^\otimes l \rightarrow sN\}_{k,l \geq 0}$ be a system of maps associated to $F$ by Lemma 4.2.(b). Again, rewrite the maps $F_{k,l}$ by $F_{k,l} : A^\otimes k \otimes M \otimes A^\otimes l \rightarrow N$ by using the suspension map: $F_{k,l} = s \circ f_{k,l} \circ (s^{-1})^\otimes k+l+1$

Then the condition $F \circ D^M = D^N \circ F$ is equivalent to the following system of equations:

\[
f_{0,0}(b_{0,0}(m)) = c_{0,0}(f_{0,0}(m)), \]
\[
f_{0,0}(b_{0,1}(m), a) - f_{0,1}(b_{0,0}(m), a) - (-1)^{|m|} f_{0,1}(m, m_1(a)) = c_{0,0}(f_{0,1}(m, a)) + c_{0,1}(f_{0,0}(m), a), \]
\[
f_{0,0}(b_{1,0}(a, m)) - f_{1,0}(m_1(a), m) - (-1)^{|a|} f_{1,0}(a, b_{0,0}(m)) = c_{0,0}(f_{1,0}(a, m)) + c_{1,0}(a, f_{0,0}(m)), \]
\[
\sum_{i=1}^{k} \sum_{j=1}^{k-i+1} (-1)^{c} f_{k-i+1,l}(a_1, ..., m_i(a_j, ..., a_{i+j-1}), ..., m, ..., a_{k+l+1}) + \sum_{j=1}^{k} \sum_{i=k-j+2}^{k+l-j+2} (-1)^{c} f_{j,k-l-i+j+3}(a_1, ..., b_{k-j+i+j+k-2}(a_j, ..., m, ..., a_{i+j-1}), ..., a_{k+l+1}) + \sum_{j=1}^{l} \sum_{i=k-j+2}^{k+j+2} (-1)^{c} f_{k,l-i}(a_1, ..., m, ..., m_i(a_j, ..., a_{i+j-1}), ..., a_{k+l+1}) = \sum_{j=1}^{k+l} \sum_{i=k-j+2}^{k+j+2} (-1)^{c} c_{j,k-l-i+j+3}(a_1, ..., f_{k-j+1,i+j+k-2}(a_j, ..., m, ..., a_{i+j-1}), ..., a_{k+l+1})\]

In order to simplify notation, it is assumed that in $(a_1, ..., a_{k+l+1})$ above, only the first $k$ and the last $l$ elements are elements of $A$ and $a_{k+l+1} = m \in M$. Then the
signs are given by
\[ \varepsilon = i \cdot \sum_{r=1}^{j-1} |a_r| + (j - 1) \cdot (i + 1) + (k + l + 1) - i, \]
and \[ \varepsilon' = (i + 1) \cdot (j + 1 + \sum_{r=1}^{j-1} |a_r|). \]

**Proof.** Up to signs these formulas follow immediately from the explicit lifting properties in Lemma 3.5.(a) and Lemma 4.2.(a). For the sign, the arguments of Proposition 2.4. will be applied.

Let's assume again that in \((a_1, \ldots, a_{k+l+1})\), only the first \(k\) and the last \(l\) elements are elements of \(A\) and \(a_{k+1} \in M\). Now notice that just like in Proposition 2.4. one gets
\[ F_{k,l}(sa_0, \ldots, sa_{k+l}) = (-1)^{\sum_{j=1}^{k+l+1} (k+l+1-j) \cdot (|a_j|+1)} \circ f_{k,l}(a_0, \ldots, a_{k+l}). \]

So, when writing out the term \(pr \circ F \circ DM\) \((sa_0, \ldots, sa_{k+l})\), exactly the same signs appear, that were in equation (2.1) of Proposition 2.4. for \(pr \circ D \circ D\). This is so, because \(DM\), which has to be applied in the argument of \(F\), has degree \(-1\) just like \(D\), and the application of the suspension map is the same for \(D\) or \(DM\) or \(F\). It follows that in this case the signs can simply be taken from Proposition 2.4.

Unfortunately the signs for the term \(DN \circ F\) cannot be taken directly from Proposition 2.4. like above. The difference is that \(F\), which is of degree 0 (and not \(-1\)), has to be applied in the argument of \(DN\). So, when \(F\) "jumps" over elements \(sa_i\), no signs are introduced. This means that here one gets a difference in signs compared to Proposition 2.4. given by
\[ (-1)^{\sum_{j=1}^{r-1} |a_r|+j-1} \]
(compare this with the first equality in (2.1)). Here one has to take the same interpretation for the variables \(i\) and \(j\) as in Proposition 2.4.; namely \(f_{r,s}\) takes exactly \(i\) inputs and the first variable in \(f_{r,s}\) is given by \(a_j\):
\[ (a_1, \ldots, f_{r,s}(a_j, \ldots, m, \ldots, a_{i-j-1}), \ldots, a_{k+l+1}). \]

Another difference to Proposition 2.4. is given in the last step of equation (2.1), because the interior element \(m_i(a_j, \ldots, a_{j+i-1})\) is replaced by some \(f_{r,s}(a_j, \ldots, a_{j+i-1})\), \((r + s = i - 1)\), with \(|m_i| = i - 2\) and \(|f_{r,s}| = i - 1\). So, when converting \(D_{k-i+1} \circ m_{k-i+1}\) in (2.1), the suspension map for \(a_{j+i}, \ldots, a_k\) jumps over one degree less. In the given case, this introduces a difference in signs of \((-1)^{k+l+1-i-j+1}\).

Putting this together with the sign in Proposition 2.4. gives
\[ \varepsilon - (\sum_{r=1}^{k+l+1} (k+l+1-r) \cdot (|a_r| + 1)) = \]
Proof. (a) First notice that the A system of maps for \( \lambda \) given by Lemma 5.1.

Given an A graded algebra with the A infinite-algebra structure described in Lemma 2.3.(a). Now, the equations of Proposition 3.7. become the same as the extension by coderivation described in Lemma 2.3.(a). Thus, dividing the equation \( DN \circ F = F \circ D^0 \) by the sign \((-1)^{\sum r=1^{k+l+1}} (k+l+1-r) \cdot (a_r+1)\) yields the result.

Example 4.5. Let’s pick up the examples 2.5. and 3.8. Let \((A, \partial, \mu)\) be a differential graded algebra with the \(A_\infty\)-algebra-structure \(m_1 := \partial, m_2 := \mu\) and \(m_k := 0\) for \(k \geq 3\). Now, let \((M, \partial^M, \lambda^M, \rho^M)\) and \((N, \partial^N, \lambda^N, \rho^N)\) be differential graded bimodules over \(A\), with the \(A_\infty\)-bialgebra-structures given by \(b_{0,0} := \lambda^M, b_{0,1} := \rho^M\) and \(b_{k,l} := 0\) for \(k + l > 1\), and \(c_{0,0} := \partial^N, c_{1,0} := \lambda^N, c_{0,1} := \rho^N\) and \(c_{k,l} := 0\) for \(k + l > 1\).

Given a bialgebra map \(f : M \longrightarrow N\) of degree 0. Then one makes \(f\) into a map of \(A_\infty\)-bialgebras by taking \(f_0 := f\) and \(f_{k,l} := 0\) for \(k + l > 0\). Then the equations of Proposition 4.4. are the defining conditions of a differential bialgebra map from \(M\) to \(N\):

\[
\begin{align*}
f \circ \partial^M(m) &= \partial^N \circ f(m) \\
f(m.a) &= f(m).a \\
f(a.m) &= a.f(m)
\end{align*}
\]

There are no higher equations.

5. \(\infty\)-INNER-PRODUCTS ON \(A_\infty\)-ALGEBRAS

There are canonical \(A_\infty\)-bialgebra-structures on a given \(A_\infty\)-algebra and its dual. \(A_\infty\)-bialgebra-maps between them will then be defined to be \(\infty\)-inner products.

Lemma 5.1. Given an \(A_\infty\)-algebra \((A, D)\). Let the coderivation \(D\) be given by the system of maps \(\{m_i : A^\otimes i \longrightarrow A\}_{i \geq 1}\) from Proposition 2.4.

(a) One can define an \(A_\infty\)-bimodule-structure on \(A\) by taking \(b_{k,l} : A^\otimes k \otimes A \otimes A^\otimes l \longrightarrow A\) to be given by

\[b_{k,l} := m_{k+l+1}.\]

(b) One can define an \(A_\infty\)-bimodule-structure on \(A^*\) by taking \(b_{k,l} : A^\otimes k \otimes A^* \otimes A^\otimes l \longrightarrow A^*\) to be given by

\[(b_{k,l}(a_1, \ldots, a_k, a^*, a_{k+1}, \ldots, a_{k+l}))(a) :=\]

\[\pm a^*(m_{k+l+1}(a_{k+1}, \ldots, a_{k+l}, a, a_1, \ldots, a_k)),\]

where the signs are given in Lemma 3.9.

Proof. (a) First notice that the \(A_\infty\)-bialgebra extension described in Lemma 3.5.(a) becomes in this case the same as the extension by coderivation described in Lemma 2.3.(a). Now, the equations of Proposition 3.7. become
exactly those of Proposition 2.4. and the diagram (3.1) from Proposition 3.4.
becomes the usual coderivation diagram for $D$.

(b) This follows immediately from (a) and Lemma 3.9.

Example 5.2. In the case of a differential algebra $(A, \partial, \mu)$, which by Example 2.5.
can be seen as an $A_\infty$-algebra, the above $A_\infty$-bialgebra structure on $A$ is exactly the
bialgebra structure given by left- and right-multiplication, because then $b_{1,0}(a \otimes b) = m_2(a \otimes b) = a \cdot b$ and $b_{0,1}(a \otimes b) = m_2(a \otimes b) = a \cdot b$, for $a, b \in A$.

Similar the $A_\infty$-bialgebra structure on $A^\ast$ is given by right- and left-multiplication
in the arguments: $b_{1,0}(a \otimes b^\ast)(c) = b^\ast(m_2(c \otimes a)) = b^\ast(c \cdot a)$ and $b_{0,1}(a^\ast \otimes b)(c) = a^\ast(m_2(b \otimes c)) = a^\ast(b \cdot c)$, for $a, b, c \in A$, and $a^\ast, b^\ast \in A^\ast$.

Definition 5.3. Given an $A_\infty$-algebra $(A, D)$. Then define an $\infty$-inner-product
on $A$ to be an $A_\infty$-bimodule-map from the $A_\infty$-bimodule $A$ to the $A_\infty$-bimodule $A^\ast$ given in Lemma 5.1.

Proposition 5.4. Given an $A_\infty$-algebra $(A, D)$. Then an $\infty$-inner product
on $A$ is exactly given by a system of inner-products on $A$, namely $\{< < \ldots >_{k,l} : A^{\otimes k+l+2} \to R\}_{k \geq 0, l \geq 0}$, that satisfies the following relations:

$$\sum_{i=1}^{k+l+2} (-1)^{\Sigma_{j=1}^{i-1} |a_j|} < a_1, \ldots, \partial(a_i), \ldots, a_{k+l+2} >_{k,l} = \sum_{i,j,n} \pm < a_i, \ldots, m_j(a_n, \ldots), \ldots >_{r,s},$$

where in the sum on the right side, there is exactly one multiplication $m_j$ ($j \geq 2$)
inside the inner-product $< < \ldots >_{r,s}$ and this sum is taken over all $i, j, n$ subject to
the following conditions:

(i) The cyclic order of the $(a_1, \ldots, a_{k+l+2})$ is preserved.

(ii) $a_{k+l+2}$ is always in the last slot of $< < \ldots >_{r,s}$.

(iii) It might happen that $a_{k+l+2}$ is inside $m_j$. By (ii), this is the only case, when
the inner product can start with an $a_i \neq a_1$,
(e.g. $a_{i+1}, \ldots, m_j(a_n, \ldots, a_{k+l+2}, a_1, \ldots, a_i) >_{r,s}$ for $i \geq 1$).

(iv) $a_{k+1}$ and $a_{k+l+2}$ are never inside the $m_j$ together. (This is exactly the significance of the indices $k$ and $l$.)

(v) $r$ and $s$ are given by looking at which slot the element $a_{k+1}$ ends up in the
inner-product. More exactly, $a_{k+1}$ will sit in the $(r+1)st$ spot of $< < \ldots >_{r,s}$, $s$ is then determined by saying that $< < \ldots >_{r,s}$ takes exactly $r+s+2$ arguments.

Proof. Let’s use the description given in Proposition 4.4. for $A_\infty$-bimodule-maps.
An $A_\infty$-bimodule-map from $A$ to $A^\ast$ is given by maps $f_{k,l} : A^{\otimes k} \otimes A \otimes A^{\otimes l} \to A^\ast$, for $k, l \geq 0$. These can clearly be interpreted as maps $A^{\otimes k} \otimes A \otimes A^{\otimes l} \otimes A \to R$, which are then denoted by the inner-product-symbol $< < \ldots >_{k,l}$ from above:

$$< a_1, \ldots, a_{k+l+1}, a' >_{k,l} := (-1)^{|a'|}(f_{k,l}(a_1, \ldots, a_{k+l+1}))(a').$$

Being an $A_\infty$-bimodule-map means that the general equation from Proposition 4.4. is satisfied. This equation is

$$\sum \pm f_{k,l}(..., m_i(...), \ldots, a, \ldots) + \sum \pm f_{k,l}(..., b_i(..., a, \ldots), \ldots) +$$
$$+ \sum \pm f_{k,l}(..., a, \ldots, m_i(...), \ldots) = \sum \pm c_{i,j}(\ldots, f_{k,l}(..., a, \ldots), \ldots).$$
Here $a \in A$ is the $(k+1)$st entry of an element in $A \otimes^k A \otimes A \otimes^l$, which means it comes from the $A_{\infty}$-bimodule $A$, instead of the $A_{\infty}$-algebra $A$. Now, by Lemma 5.1.(a), $b_{i,j} = m_{i+j+1}$ is just one of the multiplications, and thus the left side of the equation is just $f_{k,l}$ applied to all possible multiplications $m_i$.

As $f_{k,l}$ maps into $A^\ast$, one can apply the left side to an element $a' \in A$ and therefore use the maps $< \ldots >_{k,l}$:

\begin{equation}
\sum \pm (f_{k,l}(\ldots, m_i, \ldots))(a') = \sum \pm < \ldots, \partial a_i, \ldots, a' >_{k,l}.
\end{equation}

In order to rewrite the right side of the equation, one uses Lemma 5.1.(b) (with $f_{k,l}(\ldots, a, \ldots) \in A^\ast$) and the maps $< \ldots >_{k,l}$, when evaluating on $a' \in A$:

\begin{equation}
\sum \pm (c_{i,j}(a_1, \ldots, f_{k,l}(\ldots, a, \ldots), \ldots, a_{k+l+1}))(a') =
\end{equation}

\begin{equation}
= \sum \pm (f_{k,l}(\ldots, a, \ldots))(m_r(\ldots, a_{k+l+1}, a', a_1, \ldots)) =
\end{equation}

\begin{equation}
= \sum \pm < \ldots, a, \ldots, m_r(\ldots, a_{k+l+1}, a', a_1, \ldots) >_{k,l}.
\end{equation}

Now, with the identities (5.1) and (5.2), it is clear that the inner-products have to satisfy equations with sums over all possibilities of applying one multiplication to the arguments of the inner-product subject to the conditions (i)-(iv). This is of course just what is stated in the equation of the Proposition, when isolating the $\partial$-terms to the left. For condition (v), notice that the extensions of $D$ and $D^4$ from Lemma 2.3.(a) and Lemma 3.5.(a) record exactly the special entry $a$ in the $A_{\infty}$-bimodule $A$. Thus, the $A_{\infty}$-bimodule element $a$ determines the number $k$, and then $l$ is determined by the number of arguments of $< \ldots >_{k,l}$.

In order to see that the signs can be written as in the proposition, one has to insert the signs for the case $m_i = m_1 = \partial$. The important terms in (5.1) are

\begin{equation}
-1^e(f_{k,l}(\ldots, \partial a_j, \ldots))(a') = (-1)^{(\sum_{r<j}|a_r|)\varepsilon +1+k+l+1+s^i r} < \ldots, \partial a_j, \ldots, a' >_{k,l},
\end{equation}

where $e$ is the $e$ from Proposition 4.4. with $i = 1$. In (5.2) one only has to look at one term, namely

\begin{equation}
-1^e(c_{0,0}(f_{k,l}(a_1, \ldots, a_{k+l+1}))(a'),
\end{equation}

where Proposition 4.4. implies $e' \equiv 0 \ (mod \ 2)$, because $i = 1$. So, by Lemma 3.9., this term is given by

\begin{equation}
-1^k(f_{k,l}(\ldots, a_{k+l+1}))(\partial a') =
\end{equation}

\begin{equation}
= (-1)^{k+l+1+|a'|}\varepsilon(\sum_{r<i}|a_r|+1)(a', a_1, \ldots, a_{k+l+1}, a' >_{k,l}.
\end{equation}

Bringing this term to the left side and dividing by $(-1)^{k+l+1}$ yields the result. \qed

There is a diagrammatic way of picturing Proposition 5.4.

**Definition 5.5.** Given an $A_{\infty}$-algebra $(A, D)$ with the $\infty$-inner product $\{< \ldots, a' >_{k,l}$: $A \otimes^{k+l+2} \rightarrow R\}_{k \geq 0, l \geq 0}$ from Proposition 5.4. To the inner-product $< \ldots >_{k,l}$, one
associates the symbol

\[
\begin{array}{c}
k \ldots \ 2 \ 1 \\
k + 1 \quad k + l + 2 \\
k + 2 \ldots \ k + l + 1
\end{array}
\]

More generally, to any inner-product which has (possibly iterated) multiplications \(m_2, m_3, m_4, \ldots\) (but no differential \(\partial = m_1\)) inside, e.g.

\[<a_1, \ldots, m_j(\ldots), \ldots, m_p(\ldots, m_q(\ldots), \ldots), \ldots >_{k,l},\]

one can associate a diagram like above, by the following rules:

i) To every multiplication \(m_j\), associate a tree with \(j\) inputs and one output.

\[
\begin{array}{c}
m_j
\end{array}
\]

The symbol for the multiplication will also occur in a rotated way. It should always be clear, where the inputs and the output are located.

ii) To the inner product \(< \ldots >_{r,s}\), associate an "evaluation on an open circle":

\[
\begin{array}{c}
r \ldots \ 2 \ 1 \\
r + 1 \quad r + s + 2 \\
r + 2 \ldots \ r + s + 1
\end{array}
\]

Here there are \(r\) elements sitting on top of the circle, \(s\) elements are coming in from the bottom of the circle and the two (special) inputs \((r + 1)\) and \((r + s + 2)\) on the left and right. Thus one gets the required \(r + s + 2\) inputs.

iii) Around the diagram, one "sticks in" the elements \(a_i\) counterclockwise, (where the last element \(a_{r+s+2}\) is in the far right slot).

When multiplications \(m_j\) of the graph are performed, one uses the counterclockwise orientation of the plane to find the correct order of the arguments \(a_i\) in \(m_j\) (see examples below).
Let's refer to those diagrams as **inner-product-diagrams**.

Examples: Let $a, b, c, d, e, f, g, h, i, j, k \in A$.

$<a, b, c, d>_{2,0}, (deg = 2)$:

![Diagram 1](image1)

$<a, b, c, d, e, f, g, h, i>_{3,4}, (deg = 7)$:

![Diagram 2](image2)

$<m_2(m_2(b, c), m_2(d, e)), m_2(f, a)>_{0,0}, (deg = 0)$:

![Diagram 3](image3)
\( <a, b, m_3(c, d, m_2(e, f)), g, m_2(h, i) >_{1,2}, \ (deg = 4): \)

\[ \begin{array}{c}
  a \\
  \downarrow b \\
  \downarrow \downarrow c \\
  \downarrow \downarrow \downarrow d \\
  \downarrow \downarrow \downarrow \downarrow e \\
  \downarrow \downarrow \downarrow \downarrow \downarrow f \\
  \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow g \\
  \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow h \\
  \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow i \\
  \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow j \\
  \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow k \\
  \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow l \\
  \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow m \\
  \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow n \\
  \end{array} \]

\( <c, m_2(d, e), m_2(m_2(f, g), h), i, m_4(j, k, a, b) >_{2,1}, \ (deg = 5): \)

There is a chain-complex associated to the inner-product-diagrams:

i) Degree:

The degree of the inner-product-diagram associated to an inner-product \(< \ldots >_{k,l}\) with the multiplications \(m_1, \ldots, m_n\) inside is defined to be

\[
deg(Diagram) := k + l + \sum_{j=1}^{n} (i_j - 2).
\]

Examples are given above.

ii) Chain-complex:

For \(n \geq 0\), let \(C_n\) be the space generated by inner-product-diagrams of degree \(n\). Then let \(C := \bigoplus_{n \geq 0} C_n\).

iii) Differential:

Let’s define a differential on the inner-products to be the composition with the operator \(\tilde{\partial} := \sum_i id \otimes \ldots \otimes id \otimes \partial \otimes id \otimes \ldots \otimes id\) (where \(\partial = m_1\) is being in the \(i\)-th spot):

\[
(d(< \ldots, m(...), \ldots, >))(a_1, \ldots, a_s) :=
\]

\[
:= ( < \ldots, m(..., m(...), \ldots, >)(\sum_{i=1}^{s}(-1)^{\sum_{j=1}^{i-1}|a_j|}(a_1, \ldots, \partial(a_i), \ldots, a_s)).
\]

Why is this well-defined and what is its diagrammatic interpretation?

First let’s look at the inner-product \(< \ldots >_{k,l}\) without any multiplications
inside. Then by Proposition 5.4, this means that one puts one multiplication into the inner-product diagram in all possible places, such that the two lines on the far left and on the far right are not being multiplied (see Proposition 5.4. (iv)).

Now, if there are multiplications inside the inner-product, then one can observe from Proposition 2.3., that

\[
\sum_i m_n \circ (id \otimes \ldots \otimes \partial \otimes \ldots \otimes id) =
\]

\[
= \sum_{k=2}^{n-1} \sum_{i} m_{n+1-k} \circ (id \otimes \ldots \otimes m_k \otimes \ldots \otimes id) + \]

\[
(5.3)
\]

\[
+ \partial \circ m_n
\]

(5.4)

(The sum over \(i\) on both sides of the above equation means that one has to put \(\partial\) [or respectively \(m_k\)] in the \(i\)-th spot of the tensor-product.) Now, (5.3) "brakes" the given multiplication \(m_n\)

![Diagram of a multiplication process]

into all possible smaller parts \(m_{n+1-k}\) and \(m_k\)

![Diagram of smaller part process]

The last term (5.4) is also important. It makes an inductive argument of the above possible. One gets a term \(\partial(m_n(...))\) being inside the inner-product or possibly another multiplication, which then will have arguments applied to \(\partial\), so that the above discussion works again.

So, on the level of graphs, the differential means to take just one more multiplication in all possible places without multiplying the given far left and far right lines (c.f. Example 5.7 below).

**Theorem 5.6.** \(d : C_n \rightarrow C_{n-1}\), and \(d^2 = 0\).

**Proof.** By the formula for the degrees in Definition 5.5., a multiplication \(m_n\) with \(n\) inputs contributes by \(n - 2\). Now, taking the differential means to put in one more multiplication in all possible ways. Let's assume one wants to put \(m_n\) into the formula. Then this replaces \(n\) arguments with one argument in the higher level.
of the formula. Thus

$$\text{new degree} = (\text{old degree}) - n + 1 + (n - 2) =$$
$$= (\text{old degree}) - 1.$$  

One can prove $d^2 = 0$ in two ways:

i) Algebraically:

The definition of $d$ on the inner-products is just a composition with the operator $\partial = \sum_i id \otimes \ldots \otimes id \otimes \partial \otimes \ldots \otimes id$ ($\partial$ being in the $i$-th spot) on $TA$.

Thus $d^2$ is composition with

$$\tilde{\partial}^2 = \sum_{i,j} \pm id \otimes \ldots \otimes \partial \otimes \ldots \otimes \partial \otimes \ldots \otimes id = 0.$$  

This gives zero, because $\partial$ occurring at the $i$-th and the $j$-the spot can be obtained by first taking the one at the $i$-th and then the one at the $j$-th, or first taking the one on the $j$-th and then the one at the $i$-th spot. These two possibilities cancel each other out, because $\partial$ is of degree $-1$ and the first $\partial$ either has to "jump" over the other $\partial$, which gives a "," sign, or not.

ii) Diagrammatically (without signs):

If $d$ means to create one new multiplication inside the inner-product-diagram, then $d^2$ obviously corresponds to creating two new multiplications. For two given multiplications, there are always two ways of obtaining them.

Case 1: The multiplications are on different outputs

Clearly, one can do one first and then the other, or vice versa. In this way one always gets this term cancelling itself.

Case 2: The multiplications are on the same output

Here are the two ways of obtaining the same picture, and thus cancelling
Example 5.7. Let $a, b, c \in A$.

$k = 0, l = 0$: $d(<a, b>_{0,0}) = 0$

\[
\begin{array}{c}
\text{a} \\
\rightarrow \\
\text{b}
\end{array}
\]

$k = 1, l = 0$: $d(<a, b, c>_{1,0}) = <a \cdot b, c>_{0,0} \pm <b, c \cdot a>_{0,0}$

\[
\begin{array}{c}
\text{d(} \\
\text{b} \\
\text{a} \\
\text{c} \\
\text{)} = \\
\text{b} \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\text{c} \\
\pm \\
\text{b} \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\text{c}
\end{array}
\]

$k = 0, l = 1$: $d(<a, b, c>_{0,1}) = <a \cdot b, c>_{0,0} \pm <a, b \cdot c>_{0,0}$

\[
\begin{array}{c}
\text{d(} \\
\text{a} \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\text{c} \\
\text{)} = \\
\text{a} \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\text{c} \\
\pm \\
\text{a} \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\text{c}
\end{array}
\]
\[ k = 2, \ l = 0: \]

\[
d(\ ) = \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\]

\[ k = 0, \ l = 2: \]

\[
d(\ ) = \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\]

\[ k = 1, \ l = 1: \]

\[
d(\ ) = \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\]
In the last three pictures (with \( k + l = 2 \)) the righthand side is understood to be a sum over the five, or respectively six, little inner-product-diagrams. Then, as \( d^2 = 0 \), one can arrange these elements according to their corresponding boundaries to give a boundary-free object. In this way one gets certain polyhedra associated to the \(< \ldots >_{k,l}'s.

6. Further Remarks

Remark 6.1. The whole discussion given here for \( A_\infty \)-algebras should be transferable to different structures. It would be good to have a notion of an \( \infty \)-inner-product for any algebra over a given operad.

Remark 6.2. A natural question is to ask for morphisms of \( A_\infty \)-algebras preserving the \( \infty \)-inner-product. For this, there seem to be several approaches. A naive way of doing this is by transferring the usual diagrams for the condition of inner-product-preserving maps up to the ”infinity-world”. This seems to be too strict for the application of compact manifolds. M. Zeinalian and the author were able to show, that if one loosens the condition to ”inner-product-preserving maps up to homotopy”, then cochains on any compact manifold possess an \( \infty \)-inner-product-structure realizing Poincaré-duality, which is preserved by homotopy equivalences. One way of finding a good notion of morphisms might be given by using a general operad approach indicated in Remark 6.1.

Remark 6.3. There are certain well-known operations defined on the Hochschild-cochain-complex with values in itself or its dual. Some of these operations have a nice description for any \( A_\infty \)-algebra \( A \) (compare \([3,12]\)).

Definition 6.4. The \( \sim \)-product is a map \( \sim : C^\ast (A,A) \otimes C^\ast (A,A) \rightarrow C^\ast (A,A) \). If \( f,g \in C^\ast (A,A) \) are given by maps \( f_j : A^{\otimes j} \rightarrow A \), \( g_j : A^{\otimes j} \rightarrow A \), then \( (f \sim g)_j : A^{\otimes j} \rightarrow A \) is given by

\[
(f \sim g)_j := \sum_{k+l+m+p+q=j} \pm f_k \otimes m_q \otimes g_p \otimes id_{\otimes l}. \]

Definition 6.5. The Gerstenhaber-bracket is a map \([\ldots] : C^\ast (A,A) \otimes C^\ast (A,A) \rightarrow C^\ast (A,A) \). If \( f,g \in C^\ast (A,A) \) are given by maps \( f_j : A^{\otimes j} \rightarrow A \), \( g_j : A^{\otimes j} \rightarrow A \), then

\[
[f,g] := (f \circ g) - (-1)^{|f||g|} (g \circ f), \]

where \( f \circ g \) is given by \((f \circ g)_j : A^{\otimes j} \rightarrow A \),

\[
(f \circ g)_j := \sum_{k+l+m=j} \pm f_k \otimes g_m \otimes id_{\otimes l}. \]

Let’s assume that \( A \) has a unit \( 1 \in A \).

Definition 6.6. Connes-\( B \)-operator is a map \( B : C^\ast (A,A^\ast) \rightarrow C^\ast (A,A^\ast) \). If \( f \in C^\ast (A,A^\ast) \) is given by maps \( f_j : A^{\otimes j} \rightarrow A^\ast \), then \( B(f) \in C^\ast (A,A^\ast) \) is given by maps \( B(f)_{j-1} : A^{\otimes j-1} \rightarrow A^\ast \),

\[
((B(f)_{j-1})(a_1, \ldots, a_{j-1}))(a_j) := \sum_{\sigma \in \mathbb{Z}_j} \pm (f_j(a_{\sigma(1)}, \ldots, a_{\sigma(j)}))(1), \]

where \( \mathbb{Z}_j \) is understood as a subgroup of the \( j \)-th symmetric group.
The goal is now to show, that the induced $\sim$ and $B$-operators on Hochschild-cohomology form a BV-algebra structure whose implied Gerstenhaber structure is given by the map induced from $[,]$. The author was able to proof this for (certain) $A_\infty$-algebras with $\infty$-inner-product, by using a method of M. Chas and D. Sullivan [CS] Lemma 5.2. similar to Gerstenhaber’s proof of [G] Theorem 8.5. This will be shown in a following paper.

Ultimately, the author wants to use this to relate the Hochschild-cohomology of a cochain-model of a compact manifold to the homology of the free loop space of the given manifold. Here, the BV-structure, and therefore also the Gerstenhaber-structure, correspond to the given BV- and Gerstenhaber-structure described in [CS]. This last result was also observed in a different way by R. Cohen and J. Jones in [CJ].

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