EXTENSION OF C*-BUNDLES

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Abstract. We investigate which amalgamated products of continuous C*-bundles are continuous C*-bundles and we analyse the involved extension problems for continuous C*-bundles.

Introduction

Different (fibrewise) amalgamated products of continuous C*-bundles have been studied over the last years ([1], [8], [6], [4]), one of the main question being to know when these amalgamated products are continuous C*-bundles. In order to gather these approaches in a joint framework, we first recall a few definitions from the theory of deformations of C*-algebras and we fix several notations which will be used in the sequel. Then we characterise the continuity properties of different amalgamated products of (continuous) C(X)-algebras.

1. C(X)-algebras

Let X be a compact Hausdorff space and C(X) the C*-algebra of continuous functions on X with values in the complex field C.

Definition 1.1. A C(X)-algebra is a C*-algebra A endowed with a unital *-homomorphism from C(X) to the centre of the multiplier C*-algebra M(A) of A.

Given a closed subset Y ⊂ X, we denote by C₀(X \ Y) the closed ideal of continuous functions on X that vanish of Y. If A is a C(X)-algebra, then the subset C₀(X \ Y).A is a closed ideal in A (by Cohen factorisation Theorem) and we denote by π X Y the quotient map A → A/C₀(X \ Y).A.

If the closed subset Y is reduced to a point x and the element a belongs to the C(X)-algebra A, we usually write π x, A x and a x for π x, π x(A) and π x(a).

Note that the function

\[ x \mapsto \|\pi_x^X(a)\| = \inf\{\|\pi_x^X(a)\| : f \in C(X)\} \]

is always upper semi-continuous by construction. And the C(X)-algebra A is said to be continuous (or to be a continuous C*-bundle over X) if the function x → \|\pi_x^X(a)\| is actually continuous for all a in A.

Definition 1.2. A continuous field of states on a unital C(X)-algebra A is a unital positive C(X)-linear map ϕ : A → C(X).

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Remark 1.3. A (unital) separable $C(X)$-algebra $A$ is continuous if and only if (iff) there exists a continuous field of states $\varphi : A \to C(X)$ such that for all $x \in X$, the induced state $\varphi_x : a_x \in A_x \mapsto \varphi(a)(x)$ is faithful on $A_x$ ([2]).

2. HAHN-BANACH EXTENSION PROPERTIES

Given a $C^*$-algebra $A$, a $C^*$-subalgebra $B \subset A$ and a state $\phi : B \to \mathbb{C}$, there always exists a state $\varphi$ on $A$ such that $\varphi(b) = \phi(b)$ for all $b \in B$ by Hahn-Banach extension theorem. But there is no general $C(X)$-linear version of that property. Indeed, consider:

- the compact space $Y := \{0\} \cup \{\frac{1}{n}; n \in \mathbb{N}^*\}$,
- the unital continuous $C(Y)$-algebra $A := C(Y) \oplus C(Y)$ and
- the $C(Y)$-subalgebra $B := C(Y).1_A + \left( C_0(Y \setminus \{0\}) \oplus C_0(Y \setminus \{0\}) \right) \subset A$

And let $\phi : B \to C(Y)$ be the continuous field of states on $B$ fixed by the formulae

$$\phi( (b_1, b_2) ) (\frac{1}{n}) = \begin{cases} b_1(\frac{1}{n}) & \text{if } n \text{ is odd} \\ b_2(\frac{1}{n}) & \text{otherwise} \end{cases}$$

for $(b_1, b_2) \in C_0(Y \setminus \{0\}) \oplus C_0(Y \setminus \{0\})$

Then, there cannot be any continuous field of states $\varphi : A \to C(Y)$ such that $\varphi(b) = \phi(b)$ for all $b \in B$. Indeed, if $a = 1 \oplus 0 \in A$, this would imply that $\varphi(a)(\frac{1}{n}) = 1$ if $n$ is odd and $\varphi(a)(\frac{1}{n}) = 0$ otherwise. But then, the function $y \mapsto \varphi(a)(y)$ could not be continuous at $y = 0$.

3. TIEZE EXTENSION PROPERTIES

Let $(X, d)$ be a second countable compact metric space and $Y \subset X$ a non empty closed subspace. Given a separable continuous $C(X)$-algebra $A$, any continuous field of states $\phi : \pi_X^Y(A) \to C(Y)$ on the restriction $\pi_X^Y(A)$ can always be extended to a continuous field of states $\varphi : A \to C(X)$ by Michael continuous selection theorem ([2]), i.e. such that $\varphi(a)(x) = \phi(a)(x)$ for all $a \in A$ and $x \in Y$.

The point in this section is to study the following more general extension problem: Given a continuous $C(Y)$-algebra $A$, does there exist a continuous $C(X)$-algebra $D$ with a $C(Y)$-algebra isomorphic $\pi_Y^X(D) \cong A$?

Note first that if the unital separable continuous $C(Y)$-algebra $A$ is exact, then there exists a unital embedding of the $C(Y)$-algebra $A$ into the trivial $C(Y)$-algebra $C(Y, \oplus O_2)$, where $O_2$ is the unital Cuntz $C^*$-algebra generated by two isometries $s_1, s_2$ satisfying $1_{O_2} = s_1(s_1)^* + s_2(s_2)^*$ ([3]). Hence, $D := \{ f \in C(X, O_2) ; \pi_Y^X(f) \in A \}$ answers the question in that case.

Now, in order to study the general case, define in $X \times Y \times [0, 1]$:
- the open subspace $U = \{(x, y, t) \in X \times Y \times [0, 1]; 0 < t\}$ and
- the closed subspace $Z = \{(x, y, t) \in X \times Y \times [0, 1]; 0 \leq t.d(x, Y) \leq 2d(x, Y) - d(x, y)\}$.

And let $\bar{d}$ be the metric on $Z$ given by $\bar{d}((x, y, t), (x', y', t')) = d(x, x') + d(y, y') + |t - t'|$. 

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Proposition 3.1. (6) The coordinate map \( p_1 : (x, y, t) \mapsto x \) gives a structure of \( C(X) \)-algebra on \( C(Z) \) and the ideal \( C_0(U \cap Z) \) is a continuous \( C(X) \)-algebra such that \( C_0(U \cap Z)|_Y \cong C_0(Y \times [0, 1]) \), i.e. the map \((x, y, t) \in U \cap Z \mapsto x \in X \) is open.

**Proof.** Given a function \( f \) in \( C_0(U \cap Z) \), let us prove the continuity of the function
\[
x \in X \mapsto \| \pi_x^X(f) \| = \sup\{ |f(z)| ; z \in p_1^{-1}\{ \{x\} \} \}
\]
This map is already upper semicontinuous (u. s. c.) by construction. Hence, it only remains to show that for any point \( x_0 \in X \) and any constant \( \varepsilon > 0 \), one has
\[
\| \pi_x^X(f) \| > \| \pi_{x_0}^X(f) \| - \varepsilon
\]
for all points \( x \) in a neighbourhood of \( x_0 \) in \( X \).

The uniform continuity of the function \( f \) implies that there exists \( \delta > 0 \) such that
\[
|f(z) - f(z')| < \varepsilon \text{ for all } z, z' \text{ in } Z \text{ with } d(z, z') < \delta.
\]
Now three cases can appear:

1) If \( x_0 \in Y \) and \( x, y \in X \) satisfies \( d(x_0, x) < \delta/2 \), then \( |f(x, t) - f(x_0, t)| < \varepsilon \) for all \( t \in [0, 1] \). And so \( \| \pi_x^X(f) \| > \| \pi_{x_0}^X(f) \| - \varepsilon \).

2) If \( x_0 \in Y \) and \( x, y \in X \) satisfies \( d(x_0, x) < \delta/4 \), then for all \( y \in Y \), the relation \( d(x, y) \leq 2d(x, Y) \) implies that \( d(y, x_0) \leq d(y, x) + d(x, x_0) \leq 2d(x, Y) + d(x, x_0) \leq \frac{3}{4} \delta \) and so \( |f(x, y, t) - f(x_0, y, t)| < \varepsilon \) for all \( t \in [0, 1 - \frac{d(x, Y)}{d(x, Y)}] \). Whence the inequality
\[
\| \pi_x^X(f) \| > \| \pi_{x_0}^X(f) \| - \varepsilon.
\]

3) If \( x_0 \not\in Y \) and the triple \((x_0, y_0, t_0) \in U \cap Z \) satisfies \( |f(x_0, y_0, t_0)| = \| \pi_{x_0}^X(f) \| \neq 0 \), then \( d(x_0, y_0) < 2d(x_0, Y) \). Thus, there exists by continuity a constant \( \alpha \in (0, \delta/2] \) such that all \( x \in X \) in the ball of radius \( \alpha(x_0) \) around \( x_0 \) satisfy:

a) \( d(x, Y) > 0 \), b) \( d(x, y_0) < 2d(x, Y) \), c) \( t_0 < 2 - \frac{d(x, y_0)}{d(x, Y)} + \delta/2 \).

And so
\[
\| \pi_x^X(f) \| \geq \left| f(x, y_0, \inf\{t_0, 2 - \frac{d(x, y_0)}{d(x, Y)}\}) \right| > \| \pi_{x_0}^X(f) \| - \varepsilon.
\]

**Remark 3.2.** S. Wassermann pointed out that if \( Y = \{0, 1\} \subset X = [0, 1] \), then \( Z = \{(x, 0, t) \in [0, 1] \times [0, 1] ; t \leq \frac{2 - \sqrt{2}}{\pi} \} \cup \{(x, 1, t) \in [0, 1] \times [1] ; t \leq \frac{2 - \sqrt{2}}{x} \} \). Hence, this \( C(X) \)-algebra \( C(Z) \) is not continuous at \( x = \frac{1}{3} \) and \( x = \frac{2}{3} \).

Above Proposition 3.1 implies the following.

**Corollary 3.3.** Let \( A \) be a continuous \( C(Y) \)-algebra.

a) \( B := C(X) \otimes A \otimes C([0, 1]) \) is a continuous \( C(X \times Y \times [0, 1]) \)-algebra.

b) \( D := [C_0(U).B]|_Z = C_0(U).B/C_0(U \setminus U \cap Z).B \) is a continuous \( C(X) \)-algebra.

c) There is an isomorphism of \( C(Y) \)-algebras \( D|_Y \cong A \otimes C_0((0, 1]). \)

**Proof.** b) Let \( b \in D \). Then for all \( x \in X \), we have
\[
\| \pi_x^X(b) \| = \| b + C_0(X \setminus \{x\})B \| = \sup\{ \| \pi_{x_0}^X(b) \| ; z \in p_1^{-1}(\{x\}) \},
\]
whence the continuity of the map \( x \mapsto \| \pi_x^X(b) \| \) by a) and Proposition 3.1.

4. **Amalgamated tensor products of continuous \( C(X) \)-algebras**

Given a fixed compact Hausdorff space \( X \), we study in this section the continuity properties of the different tensor products amalgamated over \( C(X) \) of two given continuous \( C(X) \)-algebras \( A \) and \( B \).
Let $A \odot B$ denote the algebraic tensor product (over $\mathbb{C}$) of $A$ and $B$, let $\mathcal{I}_X(A,B)$ be the ideal in $A \odot B$ generated by the differences $af \otimes b - a \otimes fb$ ($a \in A$, $b \in B$, $f \in C(X)$) and let $A \odot^C B$ denote the quotient of $A \odot B$ by $\mathcal{I}_X(A,B)$.

If $C_\Delta(X \times X) \subset C(X \times X)$ is the ideal of continuous function of $X \times X$ which are zero on the diagonal and $A \otimes^m B$ (resp. $A \otimes^M B$) is the minimal (resp. maximal) tensor product over $\mathbb{C}$ of the two continuous $C(X)$-algebras $A$ and $B$, then the quotient $A \otimes^C B := A \otimes B / C_\Delta(X \times X)A \otimes B$ (resp. $A \otimes^C B := A \otimes B / C_\Delta(X \times X)A \otimes B$) is the minimal (resp. maximal) completion of the algebraic amalgamated tensor product $A \odot^C B$. Further, the $*$-algebra $A \odot^C B$ embeds in the $C(X)$-algebra $A \otimes^C B$ (III) and we have

\begin{equation}
\forall x \in X, \quad (A \otimes^C B)_x \cong A_x \otimes B_x \quad \text{and} \quad (A \otimes^C B)_x \cong A_x \otimes B_x.
\end{equation}

Let us also recall a characterisation of exactness given by Kirchberg and Wassermann.

**Proposition 4.1.** ([[1]], Theorem 4.5]) Let $Y = \mathbb{N} \cup \{\infty\}$ be the one point compactification of $\mathbb{N}$ and let $D$ be a $C^*$-algebra. Then the following assertions are equivalent.

i) The $C^*$-algebra $A$ is exact.

ii) For all continuous $C(Y)$-algebra $B$, the minimal tensor product $A \otimes^m B$ is a continuous $C(Y)$-algebra with fibres $A_x \otimes B_y$ ($y \in Y$).

Then, the following holds.

**Proposition 4.2.** ([[1]], (II)) Let $X$ be a second countable compact Hausdorff space and $A$ a separable unital continuous $C(X)$-algebra.

If the topological space $X$ is perfect (i.e. without isolated point), then the following assertions $\alpha_e$ and $\gamma_e$ (resp. $\alpha_n$ and $\gamma_n$) are equivalent.

$\alpha_e$) The $C^*$-algebra $A$ is exact.

$\gamma_e$) For all continuous $C(X)$-algebra $D$, the amalgamated tensor product $A \otimes^C D$ is a continuous $C(X)$-algebra with fibres $A_x \otimes D_x$ ($x \in X$).

$\alpha_n$) The $C^*$-algebra $A$ is nuclear.

$\gamma_n$) For all continuous $C(X)$-algebra $D$, the amalgamated tensor product $A \otimes^C D$ is a continuous $C(X)$-algebra with fibres $A_x \otimes D_x$ ($x \in X$).

**Proof.** $\alpha_e \Rightarrow \gamma_e$) If the $C^*$-algebra $A$ is exact, then $A \otimes D$ is a continuous $C(X \times X)$-algebra with fibres $A_x \otimes D_{x'}$ ($x, x' \in X$) ([[1]]). Hence, its restriction to the diagonal is as desired.

$\gamma_e \Rightarrow \alpha_e$) Suppose conversely that the $C(X)$-algebra $A$ satisfies $\gamma_e$. 


a) All the fibres $A_x$ are exact ($x \in X$). Indeed, given a point $x$ in $X$, take a sequence of points $x_n$ in $X$ converging to $x$ such that there is a topological isomorphism $Y := \{x_n; n \in \mathbb{N}\} \cup \{x\} \cong \mathbb{N} \cup \{\infty\}$. Then, for any separable continuous $C(Y)$-algebra $D$, there is a continuous $C(X)$-algebra $D$ such that $D|_Y = D \otimes C_0((0, 1])$ (Corollary 3.3).

Now, the continuity of the $C(X)$-algebra $D \otimes_{C(X)} A$ given by $\gamma_e$ implies that of its restriction $\left( D \otimes_{C(X)} A \right)|_Y \cong (C_0((0, 1]) \otimes D) \otimes_{C(Y)} A_{|Y}$, whence that of the $C(Y)$-algebra $D \otimes_{C(Y)} A_{|Y}$ since there is an isometric $C(Y)$-linear embedding $D \hookrightarrow D|_Y$. And this implies the exactness of the $C^*$-algebra $A_x$ is exact by Proposition 4.1.

b) If $B$ is a $C^*$-algebra and $B$ is the constant $C(X)$-algebra $C(X; B)$, then for all $x \in X$, we have the exact sequence

$$0 \to C_2(X)A_x \otimes B \to (A \otimes B)_x = A_x \otimes B \to A_x \otimes B \to 0.$$ 

c) If $D$ is a $C(X)$-algebra, then for all point $x \in X$, we have the sequence of epimorphisms $(A \otimes D)_x \to (A_x \otimes D)_x \to A_x \otimes D_x$

d) Now, let $B$ be a $C^*$-algebra, $K \triangleleft B$ a closed two sided ideal in $B$ and take an element $d \in \ker \{(A \otimes B) \to A \otimes B/K\}$. Then for all $x \in X$, we have

$$d_x \in \ker \{(A \otimes B)_x \to (A \otimes B/K)_x\}$$

$$= \ker \{A_x \otimes B \to A_x \otimes B/K\} \quad \text{by b)}$$

$$= A_x \otimes K \quad \text{by a)}$$

$$= (A \otimes K)_x \quad \text{by c)}$$

Thus, $d \in A \otimes K$. And so, the $C^*$-algebra $A$ is exact.

The proof of $\alpha_n \Rightarrow \gamma_n$ is similar to that of $\alpha_e \Rightarrow \gamma_e$. On the other hand, if a $C^*$-algebra $A$ satisfies $\gamma_n$, then all the fibres $A_x$ ($x \in X$) are nuclear by [8, Theorem 3.2] and so the $C^*$-algebra $A$ itself is nuclear (see e.g. [2, Proposition 3.23]).

Remark 4.3. These characterisations do not hold anymore if the compact space $X$ is not perfect. Indeed, if the space $X$ is reduced to a point, then both the amalgamated tensor products $A \otimes_{C(X)} D$ and $A \otimes_{C(X)} D$ are constant, hence continuous.

5. **Amalgamated free products of continuous $C(X)$-algebras**

We now describe the continuity properties of different free products amalgamated over $C(X)$ of two given unital continuous $C(X)$-algebras $A$ and $B$.

**Proposition 5.1.** ([4, Corollary 4.8]) Let $X$ be a second countable perfect compact Hausdorff space and $A$ a separable unital continuous $C(X)$-algebra. Then the following assertions are equivalent.

$\alpha_e$) The $C^*$-algebra $A$ is exact.
γe) For all separable unital continuous $C(X)$-algebra $D$ and all continuous fields of states $\phi : A \to C(X)$, $\psi : D \to C(X)$, the $C(X)$-algebra $(A, \phi) \rtimes_{C(X)}^r (D, \psi)$ is continuous.

Remark 5.2. The is no similar result for full amalgamated free product. Indeed, the full amalgamated free product $A \underset{C(X)}{\ast f} D$ of two unital continuous $C(X)$-algebras $A$ and $D$ is always a continuous $C(X)$-algebra with fibres $A_x \ast f D_x$ $(x \in X)$ ([4, Theorem 3.7]).

REFERENCES

[1] E. Blanchard, Tensor products of $C(X)$-algebras over $C(X)$, Astérisque 232 (1995), 81–92.
[2] E. Blanchard, Déformations de C*-algèbres de Hopf, Bull. Soc. Math. France, 24 (1996), 141–215.
[3] E. Blanchard, Subtriviality of continuous fields of nuclear C*-algebras, J. Reine Angew. Math. 489 (1997), 133–149.
[4] E. Blanchard, Amalgamated free products of C*-bundles, Proc. Edinburgh Math. Soc., to appear.
[5] E. Blanchard, E. Kirchberg, Global Glimm halving for C*-bundles, J. Op. Th. 52 (2004), 385–420.
[6] E. Blanchard, S. Wassermann, Exact C*-bundles, Houston J. Math. 33 (2007), 1147–1159.
[7] S. Catterall, S. Wassermann Continuous bundles of C*-algebras with discontinuous tensor products, Bull. London Math. Soc. 38 (2006), 647–656.
[8] E. Kirchberg, S. Wassermann, Operations on continuous bundles of C*-algebras. Math. Ann. 303 (1995), 677–697.

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