From the Kirsch-Kress potential method via the range test to the singular sources method

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Abstract. We review three reconstruction methods for inverse obstacle scattering problems. We will analyse the relation between the Kirsch-Kress potential method 1986, the range test of Kusiak, Potthast and Sylvester (2003) and the singular sources method of Potthast (2000). In particular, we show that the range test is a logical extension of the Kirsch-Kress method into the category of sampling methods employing the tool of domain sampling. Then we will show how a multi-wave version of the range test can be set up and we will work out its relation to the singular sources method. Numerical examples and demonstrations will be provided.

1. Introduction
We consider the scattering of time-harmonic acoustic waves by some possibly multiply connected scatterer $D$ in $\mathbb{R}^m$ for $m = 2, 3$. An incident wave $u^i$ is scattered into the scattered field $u^s$ with far field pattern $u^\infty$. The direct problem is to calculate the scattered field $u^s$ with the knowledge of the incident field $u^i$. The considered inverse problem is to reconstruct the location, shape and properties of the unknown scatterer with the knowledge of the incident field $u^i$ and the far field pattern $u^\infty$.

Over the last 20 years a large number of different reconstruction methods for the inverse problem have been proposed. Table 1 gives a survey about the different reconstruction methods. Our goal here is to work out an approach which shows the strong relation between three different schemes, the potential method of Kirsch-Kress, the range test of Kusiak, Potthast and Sylvester and the singular sources method of Potthast.

Potential method. In 1986 Kirsch and Kress [7], [8], [9] proposed the potential method to the obstacle reconstruction problem. They decomposed the reconstruction into two steps. The first step is the reconstruction of the scattered field $u^s$ from its far field pattern $u^\infty$ by fitting the far field pattern of some single-layer potential. Then, the evaluation of the single-layer potential yields an approximation to the true scattered field $u^s$. Using a known incident field $u^i$ it a second step it is possible to search for the unknown obstacle as the zero set of the total field $u^i + u^s$.

We will present this method in Section 3

Range test. The basic idea of the original range test (see [6]) is to determine the maximal set on to which the scattered field may be analytically extended. Then, the complement of this set is a subset of the unknown scatterer $D$. The one-wave range test does not deliver full reconstructions of the shape of scatterers, but reconstruct a subset which we call the scattering support of the obstacle with respect to some far field patterns scattered from one or some incident waves. The determination of the scattering support is efficiently carried out using
Table 1. Methods for reconstructing the shape of an unknown scatterer when the physical properties of the scatterer are not known.

| Method for shape reconstruction                                      | Year    |
|---------------------------------------------------------------------|---------|
| Colton-Kirsch / Linear Sampling Method                              | 1995/96 |
| see [1]                                                             |         |
| Kirsch / Factorization Method                                       | 1998    |
| see [2]                                                             |         |
| Ikehata / Probe Method                                              | 1998    |
| see [3]                                                             |         |
| Potthast / Singular Sources Method                                  | 1999/2000 |
| see [4]                                                             |         |
| Luke-Potthast / No-Response Test                                    | 2002    |
| see [5]                                                             |         |
| Potthast-Sylvester-Kusiak / Range Test                              | 2002    |
| see [6]                                                             |         |

integral equations of the first kind on the boundaries of a number of test domains $G$, where the integral equation has a solution in $L^2(\partial G)$ if and only if the scattered field can be analytically extended up to the boundary $\partial G$ of the test domain $G$. Since this tests the range of the integral operator the method is called range test.

From the knowledge of one-wave only and without the knowledge of the boundary condition we cannot hope to calculate the full shape of an unknown scatterer $D$. But if we have more data available ist is well known that the far field patterns $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in S$ uniquely determine the unknown scatterer even if the boundary condition is not known, see [10], Chapter 3. In [11] the range has been extended to the full reconstruction of the shape of $D$ in this multi-wave situation. In particular, in a first step one constructs an approximation to the scattered field of an incident point source $\Phi(\cdot, z)$ for incident point sources $\Phi(\cdot, z)$. Then, the extensiability of this far field pattern into the exterior of the test domain $G$ is tested. This extensiability is valid uniformly for all $z \in \mathbb{R}^m \setminus G$ if and only if $D \subset G$. We will present details of this approach in Section 4.

**Singular sources method.** The idea of the singular sources method is to construct the scattered field of an incident point source $\Phi(\cdot, z)$ in its source point $z$ and to use the blow-off property

$$|\Phi^s(z, z)| \to \infty, \quad z \to \partial D$$

(1.1)

to find the unknown shape $\partial D$. We will outline a potential approach to this method in Section 5, i.e. in contrast to the original approach which is presented in [4] and [10] this approach uses the potential method of Kirsch-Kress to calculate an approximation to the function $\Phi^s(z, z)$ from the far field patterns $u_\infty(\hat{x}, d)$, $\hat{x}, d \in S$.

In our last section 6 we show examples and numerical results for all three methods. In particular, we provide details about a simple two-dimensional numerical realization of the methods.

2. Obstacle scattering problems.

This section serves to briefly review the key elements of scattering by bounded objects. Let $u^i$ be an incident field that satisfies the Helmholtz equation,

$$\Delta u + \kappa^2 u = 0,$$  

(2.1)
with wave number $\kappa > 0$ on $\mathbb{R}^m$. The incident field produces a scattered field $u^s$ that solves the Helmholtz equation on the exterior of the scatterer $D$ and is radiating, i.e. it satisfies the Sommerfeld radiation condition

$$r^{\frac{m-1}{2}} \left( \frac{\partial}{\partial r} - i\kappa \right) u(x) \to 0, \quad r = |x| \to \infty$$

uniformly in all directions. For impenetrable scatterers we consider cases where the scatterer is either sound-soft (a perfect conductor) or sound-hard (a perfect reflector). Each of these types of scatterers is modeled by a total field, $u = u^i + u^s$, that satisfies either Dirichlet or Neumann boundary conditions. These boundary conditions are given respectively as

$$u|_{\partial D} = 0, \quad \frac{\partial u}{\partial \nu}|_{\partial D} = 0.$$

**Dirichlet Problem**

For the solution of the Dirichlet problem we represent the scattered field as a combined single- and double layer potential

$$u^s(x) = \int_{\partial D} \left\{ \frac{\partial \Phi(x, y)}{\partial \nu(y)} - i\Phi(x, y) \right\} \varphi(y) ds(y), \quad x \in \mathbb{R}^m \setminus \partial D.$$

For this representation of the scattered field and the boundary condition, the density $\varphi$ must satisfy the integral equation

$$\varphi + K\varphi - iS\varphi = -2u^i,$$  \hspace{1cm} (2.2)

where $S$ is the single-layer operator,

$$(S\varphi)(x) := 2\int_{\partial D} \Phi(x, y)\varphi(y) ds(y), \quad x \in \partial D$$

and $K$ is the double-layer operator,

$$(K\varphi)(x) := 2\int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)} \varphi(y) ds(y), \quad x \in \partial D.$$

The equation (2.2) has a unique solution that depends continuously on the right-hand side in $C(\partial D)$, see [12], equation (3.26).

**Neumann Problem**

For the Neumann problem we use the modified approach due to Panich [13]

$$u^s(x) = \int_{\partial D} \left\{ \Phi(x, y)\varphi(y) + i\frac{\partial \Phi(x, y)}{\partial \nu(y)} (S_0^2\varphi)(y) \right\} ds(y), \quad x \in \mathbb{R}^m \setminus \partial D,$$  \hspace{1cm} (2.3)

where $S_0$ denotes the single layer operator in the limit as $\kappa \to 0$. For this representation of the scattered field, the density $\varphi$ can be shown to satisfy the boundary integral equation

$$\varphi - K'\varphi - iTS_0^2\varphi = 2\frac{\partial u^i}{\partial \nu},$$  \hspace{1cm} (2.4)
where 
\[(K^\prime \varphi)(x) := 2 \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(x)} \varphi(y) ds(y), \ x \in \partial D,\]

and 
\[(T\varphi)(x) := 2 \frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)} \varphi(y) ds(y), \ x \in \partial D.\]

The equation (2.4) has a unique solutions that depends continuously on the incident field in $C(\partial D)$, compare [12], equation (3.28).

3. The potential method of Kirsch-Kress

The basic idea of the potential method of Kirsch-Kress is to search for the scattered field $u^s$ in the form of a single-layer potential
\[(S_{\partial G} \varphi)(x) := \int_{\partial G} \Phi(x, y) \varphi(y) ds(y), \ x \in \mathbb{R}^m \setminus G\] (3.1)

with some auxiliary domain $G \subset \mathbb{R}^m$ such that the interior Dirichlet problem for $G$ is uniquely solvable for the wave number $\kappa$. The far field pattern of the single-layer operator is given by
\[(S^\infty_{\partial G} \varphi)(\hat{x}) := \gamma_m \int_{\partial G} e^{-i\kappa \hat{x} \cdot y} \varphi(y) ds(y), \ \hat{x} \in \mathbb{S}.\] (3.2)

with the constant
\[\gamma_m := \begin{cases} \frac{e^{i\pi/4}}{\sqrt{8\pi \kappa}}, & m = 2, \\ \frac{1}{4\pi}, & m = 3. \end{cases}\] (3.3)

Given some measured far field pattern $u^\infty_{\text{meas}}$ we solve the integral equation
\[S^\infty_{\partial G} \varphi = u^\infty\] (3.4)

with some appropriate auxiliary domain $G$. This equation is an ill-posed integral equation of the first kind. With an approximate solution $\varphi_\alpha$ of (3.4) we can obtain an approximation $u^s_\alpha$ to the scattered field $u^s$ by
\[u^s_\alpha(x) := (S_{\partial G} \varphi_\alpha)(x), \ x \in \mathbb{R}^m \setminus G.\] (3.5)

In the original works of Kirsch-Kress and their collaborators the auxiliary domain $G$ has been chosen as a fixed subset of the unknown scatterer $D$. Then, the reconstructed function $u^s_\alpha$ provides an extension of the scattered field $u^s$ into the interior $D \setminus \overline{G}$ of the unknown scatterer. For convergence of the solution of the integral equation (3.4) of the method of Kirsch and Kress it is a crucial condition that this extension is possible. If it is not possible, then the equation does not have a solution and for $\alpha \to 0$ the norm of the density will tend to infinity. Thus, it is a natural idea to use this behaviour to test the extensiability of the field $u^s$ by calculation of the norm $\|\varphi_\alpha\|$ for solutions with a number of different auxiliary domains (test domains) $G$. This was employed by Kusiak, Potthast and Sylvester and leads to the range test presented in section 4. Another possibility to remedy the problem with the solvability of the equation is to combine the linear problem with the search for zeros of the total field into one large optimization functional, compare [12].

Please note that the Kirsch-Kress method (as the point-source method) needs to know the boundary condition such that the unknown shape can be reconstructed. We will see that the
range test does not need the boundary condition, but from the knowledge of the far field pattern for one incident wave it can not reconstruct the shape of \( D \) but only some subset of the convex hull of \( D \) [6]. The multiwave version will be able to generate full reconstructions.

For the modified Kirsch-Kress method (with testing of the solvability of the equation) we will demonstrate reconstructions \( u_\alpha := u^i + u_\alpha^s \) of the total field \( u \) in the last section. Also, we will show how the Kirsch-Kress method can be made efficient when the auxiliary domain \( G \) is chosen large enough to contain the unknown scatterer \( D \) in its interior and when translations and rotations of \( G \) are used to test the solvability of (3.4) and calculate the potential (3.5) in the exterior of \( G \).

4. The range test for one or multiple waves
Consider a test domain \( G \) with boundary \( \partial G \) such that the interior Dirichlet problem for \( G \) is uniquely solvable for the wave number \( \kappa \).

**One-wave range test.** We consider a single-layer operator (3.1) with far field pattern (3.2). Given the far field pattern \( u^\infty(\cdot, d) \) for a plane wave with direction \( d \in \mathbb{S} \) we can reconstruct the scattered field \( u^s(x, d) \) for \( x \in \mathbb{R}^m \setminus G \) by the potential approach (3.4) and (3.5) under the condition that \( u^s \) can be analytically extended into the set \( \mathbb{R}^m \setminus G \). For the original range test [6] the solvability of the equation

\[
S^\infty_{\partial G} \varphi = u^\infty(\cdot, d) \tag{4.1}
\]

has been used as a criterion for the analytic extensiability of \( u^s \) into \( \mathbb{R}^m \setminus G \). If the equation is solvable, then the norm \( \|\varphi_\alpha\| \) of the Tikhonov solution

\[
\varphi_\alpha := (\alpha I + S^\infty(\cdot, d) S^\infty_{\partial G})^{-1} S^\infty u^\infty \tag{4.2}
\]

will be bounded in the limit \( \alpha \to 0 \). If the equation does not admit a solution, then we have

\[
\|\varphi_\alpha\| \to \infty, \quad \alpha \to 0. \tag{4.3}
\]

This can be used as a criterion to test for analytic extensiability.

**Multiwave range test.** For the multiwave setting we assume that for the test domain \( G \) we have analytic extensiability into \( \mathbb{R}^m \setminus G \) such that the equation (4.1) is solvable for all directions \( d \in \mathbb{S} \). In this case we can calculate

\[
u^s(x, d) = S_{\partial G} ((S^\infty_{\partial G})^{-1} u^\infty(\cdot, d)) (x), \tag{4.4}\]

where for numerical purposes we will have to regularize the inverse \( (S^\infty_{\partial G})^{-1} \) of \( S^\infty_{\partial G} \).

As the next step consider the mixed reciprocity relation

\[
u^s(x, -d) = \frac{1}{\gamma_m} \Phi^\infty(d, x) \tag{4.5}\]

as worked out in [10], Chapter 2. Then with the substitution \( d \to -d \) we obtain

\[
\Phi^\infty(-d, x) = \gamma_m S_{\partial G} ((S^\infty_{\partial G})^{-1} u^\infty(\cdot, d)) (x) \tag{4.6}\]

for \( d \in -\mathbb{S} \) and \( x \in \mathbb{R}^m \setminus \overline{G} \).

Given the far field pattern \( \Phi^\infty(\cdot, x) \) on the set \( \mathbb{S} \) (keep \( x \) fixed) we can now test the extensibility of this field into the exterior of \( \mathbb{R}^m \setminus G \) in the same way as above. This is carried out by solving the equation

\[
S^\infty_{\partial G} \psi_x := \Phi^\infty(\cdot, x), \quad \forall x \in \mathbb{R}^m \setminus \overline{G}. \tag{4.7}
\]

If \( \Phi^\infty(\cdot, x) \) can be analytically extended into \( \mathbb{R}^m \setminus G \), then the equation (4.7) can be solved.

With the following theorem we study the extensiability of the field \( \Phi^\infty(\cdot, x) \) into the interior of \( G \). A proof of the following theorem has been presented in [11].
Theorem 4.1 (Characterization of the scatterer by extensiability properties.) Assume that \( D \subset G \). Then the field \( \Phi^s(\cdot, x) \) can be analytically extended up to \( \mathbb{R}^m \setminus G \) uniformly for all \( x \in \mathbb{R}^m \setminus G \), i.e. the \( L^2 \)-norms of the densities \( \psi_x \) on \( \partial G \) are uniformly bounded for \( x \in \mathbb{R}^m \setminus \overline{G} \).

If \( D \not\subset G \), then the fields \( \Phi^s(\cdot, x) \) cannot be extended up to \( \mathbb{R}^m \setminus G \) uniformly for all \( x \in \mathbb{R}^m \setminus G \), i.e. the extensions will not be uniform in the sense that the \( L^2 \)-norms of the densities \( \psi_x \) on \( \partial G \) will not be uniformly bounded for \( x \in \mathbb{R}^m \setminus \overline{G} \).

With this characterization we know that if all densities \( \psi_x \) are uniformly bounded for \( x \in \mathbb{R}^m \setminus G \), i.e.
\[
\mu(G) := \sup_{x \in \mathcal{G}} \| \psi_x \|_{L^2(\partial G)} \leq C
\]  
(4.8)
with some sampling grid \( \mathcal{G} \) and an appropriately chosen constant \( C \), we know that \( \overline{D} \subset G \) and \( \overline{D} \not\subset G \) otherwise. Now, using a set of sampling domains \( G_\tau \) in the domain of observation with the index \( \tau \in \mathcal{T} \) and index set \( \mathcal{T} \) we obtain a reconstruction of \( D \) by
\[
D_{\text{rec}} := \bigcap_{\tau \in \mathcal{T}, \text{th.} \mu(G_\tau) \leq C} G_\tau.
\]  
(4.9)
With an proper choice of this family of sampling domains we can in principle obtain full reconstructions of the unknown scatterer \( D \).

Clearly, we do not want to calculate the fields for all \( x \in \mathbb{R}^m \setminus \overline{G} \). In the numerical section we will describe how to obtain an efficient implementation of this test.

5. The singular sources method
The singular sources method reconstructs the scattered field \( \Phi^s(z, z) \) from the far field patterns \( u^\infty(\hat{x}, d) \), \( \hat{x}, d \in \mathbb{S} \). Then, the behaviour
\[
|\Phi^s(z, z)| \to \infty, \quad z \to \partial D
\]  
(5.1)
(compare (1.1)) is used to find the shape of the unknown obstacle.

The original work [4] and [10] employs the point source method, i.e. an application of Green’s formula and point source approximations to construct the particular function \( \Phi^s(x, z) \). Here, we will describe a different approach by Schulz and Potthast [14] which is not based on the point source method but on the potential method of Kirsch-Kress.

Consider some test domain \( G \subset \mathbb{R}^m \) such that \( \overline{D} \subset G \) is satisfied. First, we solve the far field equation (3.4) with right-hand side \( u^\infty(\cdot, d) \) for some fixed \( d \in \mathbb{S} \). Then by the evaluation of the single-layer potential we obtain an approximation \( u^s(x, d) \) for \( u^s(x, d) \) for all \( x \in \mathbb{R}^m \setminus \overline{G} \).

Due to the reciprocity relation (4.5) the calculation of \( u^s(x, d) \) is the same as the calculation of the function
\[
\Phi^\infty(-d, x), \quad x \in \mathbb{R}^m \setminus \overline{G}, \quad d \in \mathbb{S},
\]  
(5.2)
just in the same way as used for the multi-wave range test in (4.6). Thus, if we are given \( u^\infty(\hat{x}, d) \) for \( \hat{x}, d \in \mathbb{S} \) we obtain the full far field pattern \( \Phi^\infty(\cdot, x) \) of the function \( \Phi^s(\cdot, x) \) for all \( x \in \mathbb{R}^m \setminus \overline{G} \).

Now, given the field \( \Phi^\infty(\cdot, x) \) for some \( x \in \mathbb{R}^m \setminus \overline{G} \) we can apply the above steps once more to reconstruct the scattered field
\[
\Phi^s(y, x), \quad x, y \in \mathbb{R}^m \setminus \overline{G}.
\]  
(5.3)
This means that we solve the far field equation (3.4) and evaluate the single-layer potential to calculate \( \Phi^s \). Now, for \( y = x = z \) we obtain an approximation for the function \( \Phi^s(z, z) \), which
is known to blow-off when \( z \) tends to the unknown boundary. Thus, if there is any intersection \( D \cap \mathbb{R}^m \setminus \overline{G} \), then we can find a sequence of points \( z_n \in \mathbb{R}^m \setminus \overline{G} \) such that

\[
\lim_{n \to \infty} z_n = z^* \in \partial D.
\]  

Then

\[
\left| \Phi^s(z_n, z_n) \right| \to \infty, \quad n \to \infty.
\]

This is an independent approach to the singular sources method different from the point source approach. Here, the use of the point source method for analytic continuation is replaced by the use of the potential method for analytic continuation of the field \( u^\infty \) into \( u^s \) and \( \Phi^{\infty}(\cdot, z) \) into \( \Phi^s(z, z) \). We will discuss the numerical expense of this method and propose a way to obtain a speed comparable to that of the point source method in the last section.

6. Numerical demonstrations and examples.
Let \( Q \) be some area in \( \mathbb{R}^m \) where we want to find the unknown scatterer \( D \). We have formulated the theoretical parts for the full-aperture case where the far field pattern is known on \( \Sigma \). It can be easily transferred to the limit aperture case where \( u^\infty \) is measured on some open subset \( \Lambda \subset \Sigma \). Thus, let \( -\Lambda \subset \Sigma \) be the set of directions of given incident plane waves and let the far field pattern \( u^\infty(\cdot, d) \) be given on \( \Lambda \).

![Figure 1](image)

**Figure 1.** Illustration for the results of the direct scattering problem. Left: Dirichlet case. Right: Neumann case. The incident plane wave is coming from the left with wave number \( \kappa = 3 \).

**Basic numerics for all three methods.** First, we choose a discretization \( y_k, k = 1, \ldots, n_G \) of the boundary \( \partial G \) of the domain \( G \). In all tests we used circles as test domains \( G \). A discretization of the set \( \Lambda \) is given by the points \( \hat{x}_j \in \Lambda, j = 1, \ldots, n_S \). We assume that the far field pattern \( u^\infty(\hat{x}_j, d_l) \) is given for evaluation points

\[
\Lambda := \{\hat{x}_j, \ j = 1, \ldots, n_S\}
\]

and all directions of incidence \( -d \in -\Lambda \). Thus, a matrix

\[
u^\infty = \left( u^\infty(\hat{x}_j, -d_l) \right)_{j,l=1,\ldots,n_S}
= \left( (u^\infty(\hat{x}_j, -d_1))_{j=1,\ldots,n_S}, \ldots, (u^\infty(\hat{x}_j, -d_{n_S}))_{j=1,\ldots,n_S} \right)
\]

is given as input data. The \( l \)-th column of \( u^\infty \) contains the far field pattern for scattering of plane waves \( e^{-i\kappa y \cdot d_l}, y \in Q \) by the scatterer \( D \).
The multi-wave range test consists of the following algorithm. We will use a Nyström method for the evaluation of the single-layer boundary operator $S$ and for the calculation of its far field pattern $S^\infty_{\partial G}$. For the theoretical foundation of the Nyström method we refer to [15]. The realization of Nyström’s method is performed by straightforward integration on the boundary of the domains. The operators $K'$ and $T$ are implemented with by splitting off the singularity (see [15] p.67–70).

We employ the exponentially convergent trapezoidal rule for the numerical approximation of the operator $S^\infty_{\partial G} : L^2(\partial G) \to L^2(\Lambda)$, which leads to the matrix

$$S^\infty_{\partial G} := \gamma_m (e^{i\kappa \hat{x} \cdot y_k s_k})_{j=1, \ldots, n_G, k=1, \ldots, n_G}.$$  (6.3)

Then, we use the Tikhonov regularization in $L^2(\lambda) \to L^2(\partial G)$ for the stabilized inversion of $S^\infty_{\partial G} : L^2(\partial G) \to L^2(\Lambda)$, $\alpha > 0$

$$R_\alpha := (\alpha I + S^\infty_{\partial G} S^{\infty*}_{\partial G})^{-1} S^{\infty*}_{\partial G}.$$  (6.4)

Here $*$ denotes the complex conjugate transposed matrix and $I$ is the identity matrix and we choose the regularization parameter as $\alpha = 10^{-9}$.

Again using the trapezoidal rule for $2\pi$-periodic analytic functions, the single-layer potential operator $S_{\partial G}$ is represented by a row vector

$$S_{\partial G, z} = (\Phi(z, y_k s_k))_{k=1, \ldots, n_G}$$  (6.5)

and evaluated by

$$S_{\partial G, z} \circ \varphi = \sum_{k=1, \ldots, n_G} \Phi(z, y_k s_k) \varphi_k$$  (6.6)

for some column vector $\varphi$. Numerical results for the direct problems are shown in Figure 1.

**Figure 2.** Examples for the realization of the Kirsch-Kress potential method. Left: testdomain contains obstacle. Right: testdomain doesn’t contain obstacle. The data are generated by an incident plane wave from the left with $\kappa = 3$.

**The Kirsch-Kress method.** The Kirsch-Kress method can be carried out by a straightforward application of the above numerics. For each point $z$ the field $u^\alpha(z, d)$, $d \in \Lambda$ are approximated by the row vector

$$u^\alpha(z) := S_{\partial G, z} \circ R_\alpha \circ u^\infty.$$  (6.7)
Please note that in (6.7) we have carried out the Kirsch-Kress reconstruction step simultaneously for \( n_s \) far field patterns. A simultaneous evaluation of the functional for different \( z \in \mathbb{R}^m \setminus \overline{G} \) can be easily obtained by definition of the matrix

\[
S_{\partial G} := (S_{\partial G, z})_{l=1, \ldots, n_l}.
\]

Then, in (6.7) we need to replace \( S_{\partial G, z} \) by \( S_{\partial G} \) and calculate a matrix

\[
u^s_\alpha := S_{\partial G} \circ R_\alpha \circ \nu^\infty.
\]

which contains the approximate values of \( u^s \) in the \( n_l \) points \( z_1, \ldots, z_{n_l} \) (rows) for each \( d \in \Lambda \) (columns).

![Figure 3. Rangestest. Left: Dirichlet, Right: Neumann. one incident plane wave from the left with \( \kappa = 3 \), 20 different rotations.](image)

**Numerics for the range test.** First, we study the one-wave range test. Here, we need to calculate the density

\[
\varphi_\alpha(d) := R_\alpha \circ \nu^\infty(d)
\]

for one far field pattern \( \nu^\infty(d) \) and its norm

\[
\mu(d) = \sqrt{\varphi^T_\alpha(d) \circ \varphi_\alpha(d)}.
\]

The size of the real number \( \mu(d) \) is an indicator for the extensiability of the field \( \nu^\infty(d) \) into the exterior of the domain \( G \).

Next, we consider the numerical realization of the multiwave range test. First, we carry out (6.7). If the functions are all extensible into \( \mathbb{R}^m \setminus \overline{G} \), then we proceed as follows. The calculation of the range test functional is carried out by matrix multiplication

\[
\psi_\alpha(z) = R_\alpha \circ P \circ (u^s_\alpha(z))^T = R_\alpha \circ P \circ (S_{\partial G, z} \circ R_\alpha \circ P \circ \nu^\infty)^T = R_\alpha \circ P \circ (\nu^\infty)^T \circ R^T_\alpha \circ S^T_{\partial G, z}.
\]

where \( P \) is a permutation matrix which maps the element with direction \(-d\) onto the element with direction \( d \). Using the norm evaluation

\[
\mu(z) = \sqrt{\psi^T_\alpha(z) \circ \psi_\alpha(z)}
\]
Figure 4. Multiwaverangetest. Left: Dirichlet, Right: Neumann. 20 different incident plane waves with $\kappa = 3$, 20 different rotations.

Figure 5. Singular sources method: reconstruction of some kite shaped domain with Neumann boundary conditions. Here we chose $\kappa = 2$ and used 200 incident waves and measurement points.

from the analysis we know that

$$\mu(z) \to \infty \text{ for } z \to \partial D. \quad (6.14)$$

Thus, evaluating the functional for a number of points $z$ in the exterior $G$ the maximum over $\mu(z)$ is unbounded if $D \not\subset G$.

**Numerics for the singular source method.** The singular sources method is the next step after (6.12). An application of the potential evaluation can be used to calculate

$$\Phi_s^*(z, z) := S_{\partial G, z} \circ R_\alpha \circ P \circ (u^\infty)^T \circ R_{\eta}^T \circ S_{\partial G, z}^T \quad (6.15)$$

as an approximation to the scattered field $\Phi^*(z, z)$ for an incident point source with source point $z$. The size of $|\Phi^*(z, z)|$ can be used according to (1.1) to detect the shape of $\partial D$. 

Efficiency questions. Finally, we describe an efficient way to do the calculation of equation 6.12 for all \( z \in Q \setminus G \). First, we construct test domains \( G_x \) from a reference domain \( G_0 \) by translations

\[
G_x := G_0 + x
\]

where \( G_0 \) is chosen such that the interior Dirichlet problem is uniquely solvable for the wave number \( \kappa \). \( G_x \) being a translation of the domain \( G_0 \) provides a quick way to calculate the solution of the corresponding integral equations. With \( \tilde{\varphi}(y) := \varphi(y + x) \) we derive the following equivalence

\[
\left( S^\infty_{\partial G_x} \varphi \right)(\hat{x}) = \int_{\partial G_x} e^{-i\hat{x} \cdot y} \varphi(y) ds(y)
= e^{-i\hat{x} \cdot x} \int_{\partial G_0} e^{-i\hat{x} \cdot y} \tilde{\varphi}(y) ds(y)
= e^{-i\hat{x} \cdot x} \left( S^\infty_{\partial G_0} \tilde{\varphi} \right)(\hat{x}).
\]

We only need to set up one version of the operator \( S^\infty_{\partial G} \) and obtain the other solutions by multiplication with an exponential factor.

Further, we only choose evaluation points \( z \in \mathbb{R}^m \setminus G_x \) neatly around the boundary of the test domain \( G_x \).

We used two different methods to construct a visualisation of the unknown shape. The first and direct way is to make the intersections of the test domains \( G \) for which the indicator functional \( \mu \) is small according to equation (4.9). For the second approach we map the calculated \( \mu \) (eq. 6.13) onto a sampling grid for only one choice \( z = x \), where \( G_0 \) is chosen as

\[
G_0 = M_\alpha B_R((R + \rho, 0)), \quad R > 0, \rho > 0
\]

with some rotation matrix \( M_\alpha \), \( \alpha \in [0, 2\pi) \). The result with \( \alpha = 0 \) and \( \alpha = \pi/2 \) is demonstrated in Figure 6, we obtain reconstructions of different parts of the scatterer which have to be combined into a full reconstruction by using different rotation angles \( \alpha \).

![Figure 6](image)

**Figure 6.** Reconstruction steps using only one choice of the source point \( z \) for each sampling domain \( G_z \) as described in (6.16).

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