CYCLIC DESCENTS, MATCHINGS AND SCHUR-POSITIVITY

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Abstract. A new descent set statistic on involutions, defined geometrically via their interpretation as matchings, is introduced in this paper, and shown to be equi-distributed with the standard one. This concept is then applied to construct an explicit cyclic descent extensions on involutions, standard Young tableaux and Motzkin paths. Schur-positivity of associated quasi-symmetric functions follows.

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1. Introduction

The notion of descent set, for permutations as well as for standard Young tableaux, is well established. Klyachko [15] and Cellini [7] introduced a natural notion of cyclic descents for permutations. This notion was generalized to standard Young tableaux of rectangular shapes by Rhoades [19] and to other shapes and combinatorial sets in [3].

Recall the bijection sh : 2^{[n]} → 2^{[n]} induced by the cyclic shift i → i + 1 (mod n), for all i ∈ [n].

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Definition 1.1. Let $T$ be a finite set, equipped with a map $\text{Des} : T \to 2^{[n-1]}$. A cyclic extension of $\text{Des}$ is a pair $(c\text{Des}, p)$, where $c\text{Des} : T \to 2^n$ is a map and $p : T \to T$ is a bijection, satisfying the following axioms: for all $T$ in $T$,

(extension) $c\text{Des}(T) \cap [n-1] = \text{Des}(T)$,

(equivariance) $c\text{Des}(p(T)) = \text{sh}(c\text{Des}(T))$,

(non-Escher) $\emptyset \subsetneq c\text{Des}(T) \subsetneq [n]$.

A pair $(c\text{Des}, p)$, which satisfies the first two axioms but does not satisfy the third axiom is called an Escherian cyclic descent extension.

Example 1.2. Consider the symmetric group $S_n$ on $n$ letters and the standard descent set of a permutation $\pi = [\pi_1, \ldots, \pi_n]$, defined as

$$\text{Des}(\pi) := \{1 \leq i \leq n - 1 : \pi_i > \pi_{i+1}\} \subseteq [n-1],$$

where $[m] := \{1, 2, \ldots, m\}$. Its cyclic descent set was defined by Cellini [7] as

$$C\text{DES}(\pi) := \{1 \leq i \leq n : \pi_i > \pi_{i+1}\} \subseteq [n],$$

with the convention $\pi_{n+1} := \pi_1$. The map $C\text{DES}$ together with the rotation map $p : S_n \to S_n$ defined by $p([\pi_1, \ldots, \pi_n]) := [\pi_n, \pi_1, \ldots, \pi_{n-1}]$ is a cyclic descent extension for $S_n$.

Cyclic descent extensions were introduced in the study of Lie algebras [15] and descent algebras [7]. Surprising connections of cyclic descent extensions to a variety of mathematical areas were found later. For connections of cyclic descents to Kazhdan-Lusztig theory see [19]; for topological aspects and connections to the Steinberg torus see [8]; for twisted Schützenberger promotion see [19, 13]; for cyclic quasi-symmetric functions and Schur-positivity see [1, 2, 5]; for higher Lie characters see [2]; for Postnikov’s toric Schur functions and Gromov-Witten invariants see [3].

The question addressed in [2] was: which conjugacy classes in $S_n$ carry a cyclic descent extension? Cellini’s cyclic descent map does not provide a cyclic descent extension on most conjugacy classes. However, it turns out that most conjugacy classes carry a cyclic descent extension.

Example 1.3. Consider the conjugacy class of transpositions in $S_4$

$$\{2134, 3214, 4231, 1324, 1432, 1243\}.$$  

Cellini cyclic descent sets are

$$\{1, 4\}, \{1, 2, 4\}, \{1, 3\}, \{2, 4\}, \{2, 3, 4\}, \{3, 4\}$$

respectively; thus, not closed under cyclic rotation. On the other hand, letting the cyclic descent sets be

$c\text{Des}(2134) = \{1, 4\}, \ c\text{Des}(3214) = \{1, 2\}, \ c\text{Des}(4231) = \{1, 3\},$

$c\text{Des}(1324) = \{2, 4\}, \ c\text{Des}(1432) = \{2, 3\}, \ c\text{Des}(1243) = \{3, 4\}$

and the map $p$ be defined by

$$3214 \to 1432 \to 1243 \to 2134 \to 3214$$
and
\[ 4231 \rightarrow 1324 \rightarrow 4231, \]
the pair \((\text{cDes}, p)\) determines a cyclic descent extension for this conjugacy class.

A full characterization of conjugacy classes in \(S_n\) which carry a cyclic descent extension was given.

**Theorem 1.4.** [2, Theorem 1.4] A conjugacy class of permutations of cycle type \(\lambda\) carries a cyclic descent extension if and only if \(\lambda\) is not equal to \((r^s)\) for any square free integer \(r\).

The proof of Theorem 1.4, presented in [2], is algebraic and not constructive.

**Problem 1.5.** [2, Problem 7.11] Find an explicit combinatorial description of the cyclic descent extension for conjugacy classes, whenever it exists.

In this paper we present a solution for conjugacy classes of involutions. For \(n \geq k \geq 0\) with \(n - k\) even, let \(I_{n,k}\) be the conjugacy class of involutions in the symmetric group \(S_n\) with \(k\) fixed points. We present a purely combinatorial constructive proof of the following result.

**Theorem 1.6.** For every \(n > k > 0\) with \(n - k\) even, \(I_{n,k}\) carries a cyclic descent extension.

In other words, every conjugacy class of involutions with fixed points carries a regular (non-Escherian) cyclic descent extension.

In order to construct an explicit cyclic descent extension for conjugacy classes of involutions with fixed points, we have to consider first the conjugacy class of fixed-point-free involutions. It will be shown that a certain geometric set function on perfect matchings is equidistributed with the standard descent set on fixed-point-free involutions, leading to an exceptional (Escherian) cyclic descent extension for this conjugacy class of involutions and a regular (non-Escherian) extension for the rest.

For \(n \geq k \geq 0\) with \(n - k\) even, let \(M_{n,k}\) be the set of partial matchings on \(n\) points, labeled by \([n] := \{1, \ldots, n\}\), with exactly \(k\) unmatched points. The involutions in \(I_{n,k}\) may be interpreted as matchings in \(M_{n,k}\), where an involution \((i_1, i_2) \cdots (i_{r-1}, i_r) \in I_{n,k}\) is identified with the matching \(m \in M_{n,k}\) with matched pairs \(\{i_1, i_2\}, \ldots, \{i_{r-1}, i_r\}\).

The standard descent set of a matching \(m \in M_{n,k}\), denoted \(\text{Des}(m)\), is defined via the one-row notation of the corresponding involution in \(I_{n,k}\).

**Definition 1.7.** The geometric descent set of a matching \(m \in M_{n,k}\), denoted \(\text{GDes}(m)\), consists of the geometric descents of \(m\), defined as follows:

Draw the \(n\) points on the horizontal axis of the real plane and label them by \(1, \ldots, n\) from left to right. Indicate a matched pair \(\{i, j\}\), with \(i < j\), by drawing a half-circle in the upper half plane with diameter extending between the points labeled by \(i\) and \(j\). \(i \in [n-1]\) is a geometric descent of the matching \(m \in M_{n,k}\) if one of the following conditions holds:

1. \(\{i, i + 1\}\) is a matched pair in \(m\).
2. The half-circle containing \(i\) intersects the half circle containing \(i + 1\).
3. \(i\) is unmatched and \(i + 1\) is matched.
Figure 1. $m = (1, 6)(3, 4)(5, 7) \in M_{8,2}$, $GDes(m) = \{2, 3, 5, 6\}$, and $Des(m) = Des([6, 2, 4, 3, 7, 1, 5, 8]) = \{1, 3, 5\}$.

For a finite set of positive integers $J$ let $x^J := \prod_{j \in J} x_j$.

Lemma 1.8. For every $n \geq 0$

$$\sum_{m \in M_{n,0}} x^{Des(m)} y^{GDes(m)} = \sum_{m \in M_{n,0}} x^{GDes(m)} y^{Des(m)}.$$  

For a matching $m \in M_{n,k}$ let $cr(m)$ and $ne(m)$ be the crossing number and nesting number of $m$, respectively; see Definition 2.13 below. Using Lemma 1.8 we will prove the following.

Theorem 1.9. For every $n \geq k \geq 0$

$$\sum_{m \in M_{n,k}} q^{cr(m)} x^{GDes(m)} = \sum_{m \in M_{n,k}} q^{ne(m)} x^{Des(m)}.$$  

Bijective proofs of Lemma 1.8 and Theorem 1.9 will be described in Section 3.

Let $um(m)$ be the number of unmatched points in $m$. For a partition $\lambda$ let $ht(\lambda)$ be the number of parts in $\lambda$, let $oc(\lambda)$ be the number of odd parts in the conjugate partition, and let $s_{\lambda}$ be the corresponding Schur function. For definitions and more details see Subsection 4.1.

The following Schur-positivity phenomenon follows from the proof of Theorem 1.9.

Theorem 1.10. For every $n \geq 0$

$$\sum_{m \in M_n} q^{um(m)} t^{cr(m)} F_{n, GDes(m)} = \sum_{\lambda \vdash n} q^{oc(\lambda)} t^{\lfloor ht(\lambda)/2 \rfloor} s_{\lambda}.$$  

The existence of cyclic descent extensions, on conjugacy classes of involutions with fixed points and other combinatorial sets, follows.

To verify that, observe, first, that a cyclic extension for GDes on $M_{n,k}$ is most natural.

Definition 1.11. Draw $n$ points on a circle and label them by $1, \ldots, n$ counterclockwise. Indicate a matched pair by drawing a chord between the corresponding points. A point $i \in [n]$ is a cyclic geometric descent of a matching $m \in M_{n,k}$ if one of the following conditions holds (where addition is modulo $n$):
(1) \( \{i, i + 1\} \) is a short chord in \( m \).
(2) The chord containing \( i \) intersects the chord containing \( i + 1 \).
(3) \( i \) is unmatched and \( i + 1 \) is matched.

The cyclic geometric descent set of \( m \) is denoted by \( \text{cGDes}(m) \).

\[
\begin{align*}
\text{m} = (1,6)(3,4)(5,7) \in \mathbb{M}_{8,2} & \text{ has } \text{GDes}(m) = \{2,3,5,6\} \\
\text{cGDes}(m) = \{2,3,5,6\} & \text{.}
\end{align*}
\]

Figure 2. \( m = (1,6)(3,4)(5,7) \in \mathbb{M}_{8,2} \) has \( \text{GDes}(m) = \{2,3,5,6\} \) and \( \text{cGDes}(m) = \{2,3,5,6\} \). Rotating \( m \) by \( 2\pi/8 \) yields \( r(m) = (2,7)(4,5)(6,8) \) with \( \text{GDes}(r(m)) = \text{cGDes}(r(m)) = \{3,4,6,7,1\} \).

The proof of Theorem 1.9 applies an explicit bijection \( \hat{i} : \mathcal{I}_{n,k} \rightarrow \mathbb{M}_{n,k} \) for any \( n \geq k \geq 0 \), which satisfies
\[
\text{GDes}(\hat{i}(\pi)) = \text{Des}(\pi) \quad (\forall \pi \in \mathcal{I}_{n,k}).
\]

Define \( \text{cDes} : \mathcal{I}_{n,k} \rightarrow [n] \) by
\[
\text{cDes}(\pi) := \text{cGDes}(\hat{i}(\pi)) \quad (\forall \pi \in \mathcal{I}_{n,k}).
\]

Let \( r : \mathbb{M}_{n,k} \rightarrow \mathbb{M}_{n,k} \) correspond to counterclockwise rotation by \( 2\pi/n \).

Corollary 1.12. The pair \( (\text{cDes}, \hat{i}^{-1} \circ r \circ \hat{i}) \) is a cyclic descent extension for \( \mathcal{I}_{n,k} \). In case that \( k \neq 0, n \), it is a non-Escherian extension.

The cyclic descent extension from Corollary 1.12 is further refined to certain subsets of \( \mathcal{I}_{n,k} \), yielding a combinatorial cyclic descent extension for sets of standard Young tableaux of bounded height and given number of odd columns. Letting the height be \( \leq 2 \) and all columns even, or height \( \leq 3 \) with no further constraints, give explicit cyclic descent extensions for the sets of Dyck paths and Motzkin paths of fixed length, respectively. These cyclic extensions coincide with those determined by Dennis White [17] and Bin Han [12].

2. Preliminaries

2.1. Permutations and tableaux. For \( 1 \leq k \leq n \) denote \([n] := \{1,2,\ldots,n\}\) and \([k,n] := \{k,k+1,\ldots,n\}\). A partition of a positive integer \( n \) is a series \( \lambda = (\lambda_1,\ldots,\lambda_t) \) of weakly decreasing positive integers whose sum is \( n \). Denote \( \lambda \vdash n \).

Let \( S_n \) denote the symmetric group consisting of all permutations of \([n]\). A permutation \( \pi \in S_n \) will be represented by the one-row notation \( \pi = [\pi_1,\ldots,\pi_n] \in S_n \), where \( \pi_i := \pi(i) \).
\(i \in [n]\); denote also \(\text{Fix}(\pi) := \{i \in [n] : \pi(i) = i\}\), the set of fixed points of \(\pi\). Recall that the \textit{descent set} of a permutation \(\pi \in S_n\) is
\[
\text{Des}(\pi) := \{i : 1 \leq i \leq n - 1, \pi(i) > \pi(i + 1)\}.
\]

Another important family of combinatorial objects for which there is a well-studied notion of descent set is the set of standard Young tableaux (SYT). Let \(\text{SYT}(\lambda)\) denote the set of standard Young tableaux of shape \(\lambda\), where \(\lambda\) is a partition of \(n\). We draw tableaux in English notation, as in Figure 3. The \textit{descent set} of \(T \in \text{SYT}(\lambda)\) is
\[
\text{Des}(T) := \{i \in [n - 1] : i + 1 \text{ is in a lower row than } i \text{ in } T\}.
\]

For example, the descent set of the SYT in Figure 3 is \(\{1, 3, 5, 6\}\).

\[1 \quad 3 \quad 5 \quad 9\]
\[2 \quad 4 \quad 6\]
\[7 \quad 8\]

\textbf{Figure 3.} A SYT of shape \((4, 3, 2)\).

The Robinson-Schensted (RS) correspondence is a bijection \(\pi \mapsto (P_\pi, Q_\pi)\) from permutations in \(S_n\) to pairs of standard Young tableaux (SYT) of the same shape and size \(n\). The common shape \(\lambda\) of the insertion tableau \(P_\pi\) and the recording tableau \(Q_\pi\) is called the \textit{shape} of the permutation \(\pi\). We recall basic properties of the RS correspondence that will be used in the paper.

The \textit{height} \(\text{ht}(\lambda)\) of a shape \(\lambda\) is the number of rows in \(\lambda\).

**Proposition 2.1.** \([20]\) For every permutation \(\pi \in S_n\), the length of the maximal decreasing subsequence of the one-row notation of \(\pi\) is equal to the height of the shape of \(\pi\).

**Proposition 2.2.** \([4\text{ Propositions 14.4.12 and 14.10.6}]\]
\begin{enumerate}
\item \(P_\pi = Q_{\pi^{-1}}\), thus \(Q_\pi = P_{\pi^{-1}}\) and \(P_\pi = Q_\pi\) if and only if \(\pi \in S_n\) is an involution.
\item \(\text{Des}(Q_\pi) = \text{Des}(\pi)\) for all \(\pi \in S_n\).
\end{enumerate}

**Proposition 2.3.** \([21]\) The number of columns of odd length in the shape of an involution with \(k\) fixed points is equal to \(k\).

Let \(\text{SYT}_n\) denote the set of SYT of size \(n\), and let \(\text{SYT}_{n,k}\) denote the set of standard Young tableaux of size \(n\) having \(k\) columns of odd length. Consider the map \(q : S_n \to \text{SYT}_n\) defined by mapping \(\pi \in S_n\) to the corresponding RS recording tableau \(Q_\pi\).

**Corollary 2.4.** The map \(q\) determines a descent-set-preserving bijection from the set \(\mathcal{I}_{n,k}\) of involutions in \(S_n\) with \(k\) fixed points to the set \(\text{SYT}_{n,k}\) of standard Young tableaux of size \(n\) with \(k\) odd columns.

**Proof.** By Proposition 2.2.1, the map \(q\) determines a bijection from the set of involutions \(\mathcal{I}_n\) to the set of all SYT of size \(n\). By Proposition 2.2.2, this map is descent-set-preserving and, by Proposition 2.3, the pre-image of \(\text{SYT}_{n,k}\) is \(\mathcal{I}_{n,k}\). \(\square\)
Let $U$ and $V$ be disjoint finite totally-ordered sets of letters and let $\sigma$ and $\tau$ be two permutations of $U$ and $V$ respectively. The shuffle of $\sigma$ and $\tau$, denoted by $\sigma \shuffle \tau$, is the set of all permutations of the set $U \cup V$ in which the letters of $U$ appear in same order as in $\sigma$ and the letters of $V$ appear in same order as in $\tau$. For sets $A$ and $B$ of permutations on disjoint finite totally-ordered sets of letters $U$ and $V$, respectively, denote by $A \shuffle B$ the set of all shuffles of a permutation in $A$ and a permutation in $B$. For example, if $A = \{12, 21\}$ and $B = \{43\}$, then $A \shuffle B = \{1243, 1423, 1432, 4123, 4132, 2143, 2413, 2431, 4213, 4231, 4321\}$.

**Observation 2.5.** By definition of the RS correspondence, the first $k$ letters in $P_\pi$ form a sub-tableau which depends on their relative positions in $\pi$ only.

In particular, letting $\sigma$ be a permutation on $[k]$ and $\tau$ a permutation on $[n] \setminus [k]$, all $\pi \in \sigma \shuffle \tau$ have a common sub-tableau of $P_\pi$ consisting of first $k$ letters.

**Proposition 2.6.** For every $\sigma \in I_{n-k,0}$ and $\pi \in \sigma \shuffle [n-k+1, \ldots, n]$, the number of odd columns in the RS shape of $\pi$ is equal to $k$.

**Proof.** First, $\sigma \in I_{n-k,0}$, thus by Proposition 2.2 all columns of its shape are of even length, and by Observation 2.5 the shape of the sub-tableau consisting of first $n-k$ letters in $P_\pi$ is of even columns only as well. On the other hand, for every shuffle $\pi \in \sigma \shuffle [n-k+1, \ldots, n]$, $i \notin \text{Des}(\pi^{-1})$ for all $n-k < i < n$. Together with Proposition 2.2 this implies that the last $k$ letters form a horizontal strip in $P_\pi$. Finally, addition of a horizontal strip of size $k$ to a shape changes the parity of the length of $k$ columns. The result follows. \hfill \Box

The proof of Proposition 2.6 implies the following.

**Corollary 2.7.** For every $\sigma \in I_{n-k,0}$ and $\pi \in \sigma \shuffle [n-k+1, \ldots, n]$, the last $k$ letters in $P_\pi$ appear at the bottom cells of the $k$ odd columns of $P_\pi$.

### 2.2. Involutions and oscillating tableaux.
Consider the Young lattice whose elements are all partitions, ordered by inclusion of the corresponding Young diagrams. A standard Young tableau of shape $\lambda$ may be viewed as a maximal chain, in the Young lattice, from the empty partition to $\lambda$; see, e.g., [4, §14.2.5.1]. A variation of this description yields oscillating tableaux, which correspond to general paths in the Hasse diagram of the Young lattice, from the empty diagram to a diagram of shape $\lambda$. The size of the oscillating tableau is the length of the path, and its shape is $\lambda$. We focus on closed paths of length $2n$ from the empty diagram to itself; in other words, on oscillating tableaux of size $2n$ with an empty shape. The set of all such oscillating tableaux will be denoted by $O_{2n}$. A key tool in this paper is Sundaram’s bijection $s : I_{2n,0} \to O_{2n}$, from the set $I_{2n,0}$ of fixed-point-free involutions in $S_{2n}$ to the set $O_{2n}$ of oscillating tableaux of size $2n$ and empty shape; see [25]. We hereby describe this bijection.

**Definition 2.8.** (Sundaram’s bijection [25])

Let $\pi \in I_{2n,0}$. We start with $\lambda^0 = \emptyset$. For $1 \leq d \leq 2n$, define a standard Young tableau of shape $\lambda^d$, with letters forming a subset of $[2n]$, from a presumably-defined standard Young tableau of shape $\lambda^{d-1}$ as follows. Let $t_d = (i,j)$, $i < j$, be the unique transposition which affects $d$ in the factorization of $\pi$ into a product of $n$ disjoint transpositions. If $d = i$, insert
into the tableau of shape $\lambda^{d-1}$ using Robinson-Schensted insertion and get a tableau of shape $\lambda^d$. If $d = j$, delete $j$ from the tableau of shape $\lambda^{d-1}$ and apply jeu-de-taquin to get a tableau of shape $\lambda^d$. We get a sequence of $2n + 1$ tableaux of shapes $\lambda^d$, $0 \leq d \leq 2n$. Ignoring the letters in the tableaux yields a sequence of shapes, which is the oscillating tableau corresponding to $\pi$.

Example 2.9. Let $\pi = (1, 5)(2, 4)(3, 8)(6, 7) \in \mathcal{I}_{8, 0}$. The corresponding sequence of tableaux is

$$\emptyset, \begin{array}{c} \boxed{5} \\ \boxed{4} \end{array}, \begin{array}{c} \boxed{4} \\ \boxed{8} \end{array}, \begin{array}{c} \boxed{5} \\ \boxed{8} \end{array}, 8, \begin{array}{c} \boxed{7} \\ \boxed{8} \end{array}, \emptyset$$

Thus, the oscillating tableau corresponding to $\pi$ is

$$s(\pi) = (\emptyset, \begin{array}{c} \boxed{5} \\ \boxed{4} \end{array}, \begin{array}{c} \boxed{4} \\ \boxed{8} \end{array}, \begin{array}{c} \boxed{5} \\ \boxed{8} \end{array}, 8, \begin{array}{c} \boxed{7} \\ \boxed{8} \end{array}, \emptyset) \in O_8.$$

A special case of [25, Theorem 5.3] is the following.

Theorem 2.10. The map $s : \mathcal{I}_{2n, 0} \rightarrow O_{2n}$ defined above is a bijection.

A characterization of the descents of $\pi$ in the language of oscillating tableaux follows.

Observation 2.11. [Kim, Proof of Theorem 3.4] For every $\pi \in \mathcal{I}_{2n, 0}$, $i \in \text{Des}(\pi)$ if and only if what we do in the $i$th and $(i + 1)$st steps of the corresponding oscillating tableau $s(\pi)$ is either

1. add a box in the $i$th step and then delete in the next step; or
2. delete a box in the $i$th step and then delete another box in a strictly upper row in the next step; or
3. add a box in the $i$th step and then add another box in a strictly lower row in the next step.

In all other cases, $i \not\in \text{Des}(\pi)$.

Definition 2.12. For an oscillating tableau $O = (\emptyset = D_0, D_1, \ldots, D_{2n}) \in O_{2n}$, let $\text{tr} O := (D'_0, D'_1, \ldots, D'_{2n})$, where for all $0 \leq i \leq 2n$, the diagram $D'_i$ is the transpose of the diagram $D_i$.

This operation will be applied in the following sections.

2.3. Matchings. Chen et al. [6] generalized Sundaram’s bijection, presented in Subsection 2.2, and applied it to the enumeration of crossings and nestings in perfect matchings and partitions.

Definition 2.13. 1. The crossing number $\text{cr}(m)$ of a matching $m \in \mathcal{M}_n$ is the maximal $r$ such that there exist $1 \leq i_1 < \cdots < i_r < j_1 < \cdots < j_r \leq n$ where $\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_r, j_r\}$ are matched pairs in $m$.
2. The nesting number $\text{ne}(m)$ of a matching $m \in \mathcal{M}_n$ is the maximal $r$ such that there exist $1 \leq i_1 < \cdots < i_r < j_1 < \cdots < j_r \leq n$ where $\{i_1, j_r\}, \{i_2, j_{r-1}\}, \ldots, \{i_r, j_1\}$ are matched pairs in $m$. 
Example 2.14. For \( m \in M_{8,2} \) as in Figure 1, \( 1 < 5 < 6 < 7 \) is a maximal crossing, thus \( cr(m) = 2 \). Also, \( 1 < 3 < 4 < 6 \) is a maximal nestings, thus \( ne(m) = 2 \).

Chen et al. introduced the involution \( \iota : M_{2n,0} \rightarrow M_{2n,0} \), defined by
\[
\iota := s^{-1} \circ tr \circ s
\]
Here \( s \) is Sundaram’s bijection, \( s : I_{2n,0} \rightarrow O_{2n} \), described in Definition 2.8 (where we view perfect matchings in \( M_{2n,0} \) as involutions in \( I_{2n,0} \)), and \( tr \) is the conjugation operation on oscillating tableaux, as in Definition 2.12.

Example 2.15. Let \( \pi = (1, 5)(2, 4)(3, 8)(6, 7) \in I_{8,0} \) as in Example 2.9. Then
\[
tr \circ s(\pi) = (\emptyset, \begin{array}{c}
\bigcirc \\
\bigsmalltriangleup
\end{array}, \begin{array}{c}
\Box \\
\Box
\end{array}, \begin{array}{c}
\bigtriangleup \\
\bigsmalltriangleup
\end{array}, \begin{array}{c}
\Box \\
\Box
\end{array}, \begin{array}{c}
\bigcirc \\
\bigsmalltriangleup
\end{array}, \begin{array}{c}
\Box \\
\Box
\end{array}, \emptyset)
\]
and \( \iota(\pi) = s^{-1} \circ tr \circ s(\pi) = (1, 4)(2, 7)(3, 5)(6, 8) \in I_{8,0} \).

Theorem 2.16. For every \( m \in M_{2n,0} \),
\[
kr(m) = ne(\iota(m)),
\]
thus
\[
\sum_{m \in M_{2n,0}} q^{kr(m)} t^{ne(m)} = \sum_{m \in M_{2n,0}} q^{ne(m)} t^{kr(m)}.
\]

3. GEOMETRIC VERSUS STANDARD DESCENTS - EQUIDISTRIBUTION

Chen et al.’s bijection, presented in Subsection 2.3, is applied in Subsection 3.1 to prove Lemma 1.8. This bijection serves as a component in the proof of the following result in Subsection 3.2.

Theorem 3.1. For every \( n \geq k \geq 0 \) there exists an explicit bijection \( \hat{i} : I_{n,k} \rightarrow M_{n,k} \), to be described in Definition 3.6, which satisfies
\[
GDes(\hat{i}(\pi)) = Des(\pi) \quad \text{and} \quad ne(\hat{i}(\pi)) = kr(\pi) \quad (\forall \pi \in I_{n,k}).
\]

Theorem 1.9 follows. The bijection \( i : I_{n,k} \rightarrow M_{n,k} \) is used to prove Theorem 1.10 in Subsection 4.2 and to determine cyclic descents on involutions in Section 5.

3.1. PROOF OF LEMMA 1.8. Recall the involution \( \iota : I_{n,0} \rightarrow I_{n,0} \) introduced by Chen et al. [6], described in Subsection 2.3.

Proposition 3.2. The involution \( \iota : I_{n,0} \rightarrow I_{n,0} \) satisfies
\[
Des(\iota(\pi)) = GDes(\pi) \quad \forall \pi \in I_{n,0}.
\]

Proof. Recall from Subsection 2.3 that for an involution \( \pi \in I_{n,0} \), there exists a unique involution \( \hat{\pi} := \iota(\pi) \in I_{n,0} \), whose oscillating tableau, \( s(\hat{\pi}) \), is the conjugate of \( s(\pi) \). \( tr \circ s(\pi) \). Here \( s : I_{n,0} \rightarrow O_{2n} \) is Sundaram’s bijection, described in Definition 2.8 and \( tr \) is the conjugation operation on oscillating tableaux, see Definition 2.12.

We will show that \( Des(\hat{\pi}) = GDes(\pi) \).

For every \( 1 \leq i < n \) there are seven possible cases.
(1) \((i, i + 1)\) is a short chord in \(\pi\).

(2) there exists \(a < i\) and \(b > i + 1\), such that \((a, i)\) and \((i + 1, b)\) are chords in \(\pi\).

(3) there exists \(a < i\) and \(b > i + 1\), such that \((i, b)\) and \((a, i + 1)\) are chords in \(\pi\).

(4) there exists \(a < b < i\) such that \((a, i)\) and \((b, i + 1)\) are chords in \(\pi\).

(5) there exists \(a < b < i\) such that \((a, i + 1)\) and \((b, i)\) are chords in \(\pi\).

(6) there exists \(i < a < b\) such that \((a, i)\) and \((b, i + 1)\) are chords in \(\pi\).

(7) there exists \(i < a < b\) such that \((a, i + 1)\) and \((b, i)\) are chords in \(\pi\).

By definition of \(\text{GDes}\), \(i \in \text{GDes}(\pi)\) in cases (1), (3), (4) and (6) and \(i \notin \text{GDes}(\pi)\) in all other cases.

By definition of the oscillating tableau series, what we do in the corresponding cases in the \(i^{th}\) and \((i + 1)^{st}\) steps is

1. add a box and then delete a box.
2. delete a box and then add a box.
3. add a box and then delete a box.
4. delete a box and then delete another box in a weakly lower row.
5. delete a box and then delete another box in a strictly upper row.
6. add a box and then add another box in a weakly upper row.
7. add a box and then add another box in a strictly lower row.

Hence what we do in the conjugated sequence is the same in cases (1)-(3) and is switched in cases (4) and (5), and cases (6) and (7). By Observation 2.11, this translates to \(i \in \text{Des}(\hat{\pi})\) in cases (1), (3), (4) and (6), and to ascents in all other cases, completing the proof. \(\square\)

**Remark 3.3.** Arguments, similar to those used in the proof of Lemma 1.8, were used by Kim [14] to prove the symmetry of the Eulerian and Mahonian distributions on \(\mathcal{I}_{n,0}\).

The following refinement of Lemma 1.8 follows.

**Corollary 3.4.** For every \(n \geq 0\)

\[
\sum_{\pi \in \mathcal{I}_{n,0}} x^{\text{GDes}(\pi)} y^{\text{Des}(\pi)} q^{\text{cr}(\pi)} t^{\text{ne}(\pi)} = \sum_{\pi \in \mathcal{I}_{n,0}} x^{\text{Des}(\pi)} y^{\text{GDes}(\pi)} q^{\text{ne}(\pi)} t^{\text{cr}(\pi)},
\]

**Proof.** By Proposition 3.2, the involution \(\iota : \mathcal{I}_{n,0} \to \mathcal{I}_{n,0}\) satisfies \(\text{Des}(\iota(\pi)) = \text{GDes}(\pi)\) for all \(\pi \in \mathcal{I}_{n,0}\). By Theorem 2.16, \(\text{ne}(\iota(\pi)) = \text{cr}(\pi)\). Since \(\iota\) is an involution, \(\text{Des}(\pi) = \text{GDes}(\iota(\pi))\) and \(\text{ne}(\pi) = \text{cr}(\iota(\pi))\). Thus

\[
\sum_{\pi \in \mathcal{I}_{n,0}} x^{\text{GDes}(\pi)} y^{\text{Des}(\pi)} q^{\text{cr}(\pi)} t^{\text{ne}(\pi)} = \sum_{\sigma \in \mathcal{I}_{n,0}} x^{\text{Des}(\sigma)} y^{\text{GDes}(\sigma)} q^{\text{ne}(\sigma)} t^{\text{cr}(\sigma)},
\]

where \(\sigma := \iota(\pi)\). Since \(\iota\) is a bijection, proof is completed. \(\square\)

### 3.2. Proof of Theorem 1.9

In this section we describe a map

\[\hat{i} : \mathcal{I}_{n,k} \to \mathcal{I}_{n,k},\]

for any \(0 \leq k \leq n\), which generalizes the bijection \(i : \mathcal{I}_{n,0} \to \mathcal{I}_{n,0}\) used in previous section. It will be shown that \(\hat{i}\) is a bijection which maps the descent set to the geometric descent set and the crossing number to the nesting number, implying Theorem 1.9.
Remark 3.5. The bijection of Chen et al is defined for involutions with fixed points as well. It is an involution which maps the crossing number to the nesting number and preserves the fixed point set. Unfortunately, for involutions with fixed points it does not map GDes to Des and vice versa. For example, Chen et al’s involution maps \( \sigma = (1, 5)(2, 4)(3) \) to \( \pi = \{2, 3\} \neq \text{GDes}(\sigma) = \{3\} \) and \( \text{Des}(\sigma) = \{1, 2, 3, 4\} \neq \text{GDes}(\pi) = \{1, 3, 4\} \).

Definition 3.6. 1. For every \( \pi \in \mathcal{I}_{n,k} \) let \( \text{res}(\pi) \) be the direct product of the the set of fixed points in \( \pi \), \( \text{Fix}(\pi) \), with the fixed-point-free involution in \( S_{n-k} \) with same relative order as on \( [n] \setminus \text{Fix}(\pi) \) in \( \pi \).

2. For \((J, \sigma) \in \binom{[n]}{k} \times \mathcal{I}_{n,0}\) let \( \text{emb}(\sigma, J) \) be the permutation in the set of all shuffles \( \mathcal{I}_{n-k,0} \sqcup \{n-k+1, n-k+2, \ldots n\} \), for which the letters in \([n-k]\) are ordered as in \( \sigma \), and set of positions of the increasing subsequence \([n-k+1, \ldots n]\) is equal to \( J \).

3. Let \( \varphi : \mathcal{I}_{n,k} \rightarrow \mathcal{I}_{n-k,0} \sqcup \{n-k+1, \ldots n\} \) be defined as

\[
\varphi : \mathcal{I}_{n,k} \xrightarrow{\text{res}} \binom{[n]}{k} \times \mathcal{I}_{n-k,0} \xleftarrow{\iota} \binom{[n]}{k} \times \mathcal{I}_{n-k,0} \xrightarrow{\text{emb}} \mathcal{I}_{n-k,0} \sqcup \{n-k+1, \ldots n\},
\]

where \( \iota(J, \sigma) := (J, \iota(\sigma)) \).

4. For \( \pi \in \mathcal{I}_{n-k} \sqcup \{n-k+1, \ldots n\} \) let \( q(\pi) \) be the RS preimage of \((Q_{\pi}, Q_{\pi})\), where \( Q_{\pi} \) is the RS recording tableau of \( \pi \). Note that by Proposition [2.2.1], \( q(\pi) \) is an involution.

5. Let \( \hat{i} := q \circ \varphi \). Namely,

\[
\hat{i} : \mathcal{I}_{n,k} \xrightarrow{\varphi} \mathcal{I}_{n-k,0} \sqcup \{n-k+1, \ldots n\} \xrightarrow{Q_{\pi}} \text{SYT}_n \xrightarrow{\text{diagonal}} \text{SYT}_n \times \text{SYT}_n \xrightarrow{\text{RS}^{-1}} \mathcal{I}_n,
\]

where \( \text{SYT}_n \) denotes the set of standard Young tableaux of size \( n \) and \( \mathcal{I}_n \) is the set of involutions in \( S_n \).

The following proposition implies Theorem [3.1]

Proposition 3.7. The map \( \hat{i} : \mathcal{I}_{n,k} \rightarrow \mathcal{I}_n \) is a bijection from \( \mathcal{I}_{n,k} \) to itself, which satisfies

\[
\text{GDes}(\pi) = \text{Des}(\hat{i}(\pi)) \quad \text{and} \quad \text{cr}(\pi) = \text{ne}(\hat{i}(\pi)) \quad (\forall \pi \in \mathcal{I}_{n,k}).
\]

Example 3.8. Let \( \pi = [4, 2, 6, 1, 5, 3] = (1, 4)(3, 6)(2)(5) \in \mathcal{I}_{6,2} \). Then \( \varphi(\pi) \) is

\[
\varphi : [4, 2, 6, 1, 5, 3] \xrightarrow{\text{res}} (1, 3)(2, 4) \times \{2, 5\} \xrightarrow{\iota} (1, 4)(2, 3) \times \{2, 5\} \xrightarrow{\text{emb}} [4, 5, 3, 2, 6, 1],
\]

thus \( \hat{i}(\pi) \) is

\[
\hat{i} : [4, 2, 6, 1, 5, 3] \xrightarrow{\varphi} [4, 5, 3, 2, 6, 1] \xrightarrow{Q_{\pi}} \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix} \xrightarrow{\text{RS}^{-1}} [1, 6, 4, 3, 5, 2].
\]
Namely, $i(\pi) = (2,6)(3,4)(1)(5) \in \mathcal{I}_{6,2}$. Indeed, $\text{GDes}(\pi) = \text{Des}(i(\pi)) = \{2,3,5\}$ and $\text{cr}(\pi) = \text{ne}(i(\pi)) = 1$.

To prove Proposition 3.7, we generalize the concepts of crossing and nesting numbers to shuffles of fixed-point-free involutions with an increasing sequence.

**Definition 3.9.** For every $\pi \in \mathcal{I}_{n-k,0} \sqcup [n-k+1, \ldots]$, define $\text{ne}(\pi) := \text{ne}(\sigma)$ and $\text{cr}(\pi) := \text{cr}(\sigma)$, where $\sigma \in \mathcal{I}_{n-k,0}$ is obtained by deleting the letters $n-k+1, \ldots, n$ from $\pi$.

**Example 3.10.** Let $\pi = [3,4,5,1,6,2] \in \mathcal{I}_{4,0} \sqcup [5,6]$. Then $\pi$ is not an involution; however, deleting the letters $5,6$ from $\pi$ gives $\sigma = [3,4,1,2] = (1,3)(2,4)\mathcal{I}_{4,0}$. Then $\text{cr}(\pi) = \text{cr}(\sigma) = 2$ and $\text{ne}(\pi) = \text{ne}(\sigma) = 1$.

**Lemma 3.11.**

1. The map $\varphi : \mathcal{I}_{n,k} \rightarrow \mathcal{I}_{n-k,0} \sqcup [n-k+1, n-k+2, \ldots n]$ is a bijection, which satisfies

$$\text{GDes}(\pi) = \text{Des}(\varphi(\pi)) \quad (\forall \pi \in \mathcal{I}_{n,k}).$$

2. For every $n \geq k \geq 0$

$$\sum_{m \in \mathcal{I}_{n,k}} q^{\text{ne}(m)} x^{\text{GDes}(m)} = \sum_{m \in \mathcal{I}_{n-k,0} \sqcup [n-k+1, \ldots, n]} q^{\text{ne}(m)} x^{\text{Des}(m)}.$$

**Proof.**

1. By definition, $\varphi$ is a bijection. To show that $\varphi$ maps GDes to Des, we will consider the following cases: for every $1 \leq i < n$,

**Case 1.** If $i, i+1 \in \text{Fix}(\pi)$ then, by definition of GDes, $i \not\in \text{GDes}(\pi)$; on the other hand, by definition of $\varphi$, $\varphi(\pi)(i) < \varphi(\pi)(i+1)$.

**Case 2.** If $i \not\in \text{Fix}(\pi)$ and $i+1 \in \text{Fix}(\pi)$ then $i \not\in \text{GDes}(\pi)$ and $\varphi(\pi)(i) < \varphi(\pi)(i+1)$.

**Case 3.** If $i \in \text{Fix}(\pi)$ and $i+1 \not\in \text{Fix}(\pi)$ then $i \in \text{GDes}(\pi)$ and $\varphi(\pi)(i) > \varphi(\pi)(i+1)$.

**Case 4.** Finally, if $i, i+1 \not\in \text{Fix}(\pi)$ then, by definition of $\varphi$ we can ignore the fixed points and apply Proposition 3.11 which shows that $i \in \text{GDes}(\pi) \iff i \in \text{Des}(\varphi(\pi))$.

2. By definition of the map $\varphi$ and Theorem 2.16

$$\text{ne}(\pi) = \text{cr}(\varphi(\pi)) \quad \text{and} \quad \text{cr}(\pi) = \text{ne}(\varphi(\pi)) \quad (\forall \pi \in \mathcal{I}_{n,k}).$$

Combining this with the first part, the second part follows. $\square$

To prove Proposition 3.7 we further need the following lemmas.

**Lemma 3.12.** For every involution $\pi \in S_n$

$$\text{ne}(\pi) = \left\lfloor \frac{\text{ht}(Q_\pi)}{2} \right\rfloor,$$

where $Q_\pi$ is the RS recording tableau of $\pi$.

**Proof.** There is a nest of length $k$ in $\pi$ if and only if there exists a series $i_1 < \cdots < i_k < i_{k+1} < \cdots < i_{2k}$ such that for every $1 \leq j \leq 2k$, $\pi(i_j) = i_{2k+1-j}$. Then $i_{2k}, \ldots, i_1$ is a decreasing subsequence in the one row notation of $\pi$. By Proposition 2.7

$$2\text{ne}(\pi) \leq \text{ht}(Q_\pi).$$
On the other hand, the fixed points of $\pi$ form an increasing subsequence in $\pi$. Hence, letting $\pi(i_1), \ldots, \pi(i_t)$ be a decreasing subsequence of maximal length in $\pi$, it contains at most one fixed point. Assume, first, that there is no fixed point in this series. Let $k := \max\{j : \pi(i_j) > i_j\}$. If $k > t/2$ then $\pi(i_1), \ldots, \pi(i_k), i_k, \ldots, i_1$ is a decreasing subsequence of length $2k > t$ in $\pi$, in contradiction to the maximality of $\pi(i_1), \ldots, \pi(i_t)$. Similarly, if $k < t/2$ then $i_t, i_{t-1}, \ldots, i_{t-k+1}, \pi(i_{t-k+1}), \ldots, \pi(i_t)$ is a decreasing subsequence of length $2(t-k) > t$ in $\pi$, in contradiction to the maximality assumption. We deduce that $k = t/2$, and thus $\pi(i_1), \ldots, \pi(i_k), i_k, \ldots, i_1$ is a maximal decreasing sequence, which is realized by a nest.

Finally, assume that the maximal decreasing subsequence contains a fixed point, $\pi(j) = j$. Then for all $1 \leq d \leq t$, $\pi(i_d) > i_d$ if and only if $i_d < j$. By the above argument, $[t/2] < j < [t/2]$, otherwise it is possible to increase the length of the decreasing subsequence, in contradiction to its maximality. Then $\pi(i_1), \ldots, \pi(i_{t/2}), j, [t/2], \ldots, i_1$ is a maximal decreasing sequence; all the letters in this decreasing maximal sequence, excluding $j$, form a nest. By Proposition 2.1

$$2 \text{ne}(\pi) + 1 \geq \text{ht}(Q_\pi).$$

Proof is completed. \qed

Recall the map $Q : S_n \mapsto \text{SYT}_{n,k}$ defined by mapping $\pi \in S_n$ to its corresponding RS recording tableau $Q_\pi$.

**Lemma 3.13.** The map $Q$ determines a descent set preserving bijection from the set of shuffles $\mathcal{I}_{n,n-k} \sqcup [n-k+1, \ldots, n]$ to $\text{SYT}_{n,k}$.

**Proof.** First, by Proposition 2.2, $\text{Des}(Q_\pi) = \text{Des}(\pi)$, so $Q$ is decent set preserving.

Second, by Proposition 2.6, for every $\pi \in \mathcal{I}_{n,n-k} \sqcup [n-k+1, \ldots, n]$, $Q_\pi$ has $k$ odd columns, so $Q$ maps $\mathcal{I}_{n,n-k} \sqcup [n-k+1, \ldots, n]$ into $\text{SYT}_{n,k}$.

To prove that $Q$ is a bijection, we will show that the map $Q$ is invertible: assuming that $\pi \in \mathcal{I}_{n-k,0} \sqcup [n-k+1, \ldots, n]$, it will be shown that $\pi$ can be recovered from its recording tableau $Q_\pi$. Let $\pi \in \sigma \sqcup [n-k+1, \ldots, n]$, where $\sigma \in \mathcal{I}_{n-k,0}$. By Corollary 2.7, the last $k$ letters in $P_\pi$ appear at the bottom cells of the $k$ odd columns of $P_\pi$. To recover the positions of these letters in $\pi$, apply the inverse RS procedure:

- Let $T := Q_\pi$. Assume that the bottom cell of the first odd column from the right is at the $r^{th}$ row. Let $i_r$ be the entry in this cell.
- Let $i_{r-1}$ be the maximal letter in the $r - 1^{st}$ row, which is smaller than $i_r$. Delete $i_r$ from $r^{th}$ row, and replace $i_{r-1}$ by $i_r$.
- So on, till $i_1$, the maximal letter in first row smaller than $i_2$ is replaced by $i_2$.
- $i_1$ is the position of $n$ in $\pi$.

Next apply the same procedure on the resulting tableau $T$ to find the position of $n-1$, and so on.

After finding the positions of the increasing subsequence of last $k$ letters, the letters in the resulting tableau $T$ are replaced by $1, \ldots, n-k$ with same relative order. The resulting tableau is $Q_\sigma$. 

Since \( \sigma \) is an involution, Proposition 2.2.1 implies that \( P_\sigma = Q_\sigma \), so the RS preimage of \((Q_\sigma, Q_\sigma)\) is \( \sigma \).

\[\square\]

**Example 3.14.** Let \( Q_\pi = \begin{array}{cccc}
2 & 4 & 6 \\
3 & 5 & 8 \\
7 & & & 
\end{array} \). Then \( n = 8 \) and \( k = 2 \) - number of columns of odd length. So, \( \pi \in \sigma \sqcup \{7, 8\} \) for some \( \sigma \in \mathcal{I}_{6,0} \).

The bottom cell of the first odd column from the right appears in the first row, thus \( r = 1 \) and the entry \( i_r = i_1 = 6 \); so \( \pi(6) = 8 \). The resulting tableau after deleting \( i_r = 6 \) is

\[ T = \begin{array}{cccc}
1 & 2 & 4 \\
3 & 5 & 8 \\
7 & & & 
\end{array} \]

Now bottom cell of the first odd column from the right appears in the row \( r = 3 \), \( i_3 = 7 \), \( i_2 = 5 \) and \( i_1 = 4 \). Thus \( \pi(4) = 7 \). The resulting tableau is

\[ T = \begin{array}{cccc}
1 & 2 & 5 \\
3 & 7 & 8 \\
& & & 
\end{array} \]

So, \( \sigma = \text{RS}^{-1}(Q_\sigma, Q_\sigma) = [3, 5, 1, 6, 2, 4] \) and \( \pi = [3, 5, 1, 7, 6, 8, 2, 4] \).

Recall the map \( q \) from Definition 3.6.4: for \( \pi \in \mathcal{I}_{n-k,0} \sqcup \{n - k + 1, \ldots, n\} \) let \( q(\pi) \) be the RS preimage of \((Q_\pi, Q_\pi)\), where \( Q_\pi \) is the RS recording tableau of \( \pi \).

**Corollary 3.15.** The map \( q \) is a descent set and nesting number preserving bijection from the set of shuffles \( \mathcal{I}_{n-k,0} \sqcup \{n - k + 1, \ldots, n\} \) to the set of involutions \( \mathcal{I}_{n,k} \).

**Proof.** First, by Proposition 2.2.1 \( Q_\pi \) has \( k \) odd columns. Combining this with Propositions 2.2.1 and 2.3 the RS preimage of \((Q_\pi, Q_\pi)\) is an involution with \( k \) fixed points, namely \( q(\pi) \in \mathcal{I}_{n,k} \). Moreover, by Lemma 3.13 together with Corollary 2.4, \( q \) is a descent set preserving bijection. Finally, by definition \( Q_\pi = Q_{q(\pi)} \). Thus, by Lemma 3.12

\[ \text{ne}(\pi) = \left\lfloor \frac{\text{ht}(Q_\pi)}{2} \right\rfloor = \left\lfloor \frac{\text{ht}(Q_{q(\pi)})}{2} \right\rfloor = \text{ne}(q(\pi)) \quad (\forall \pi \in \mathcal{I}_{n-k,0} \sqcup \{n - k + 1, \ldots, n\}) \].

\[\square\]

**Proof of Proposition 3.7.** By Proposition 3.11.1 together with Corollary 3.15

\[ i : \mathcal{I}_{n,k} \overset{\varphi}{\to} \mathcal{I}_{n-k,0} \sqcup \{n - k + 1, \ldots, n\} \overset{q}{\to} \mathcal{I}_{n,k} \]

is a bijection. Also, Proposition 3.11.1 together with Corollary 3.15 imply

\[ \text{GDes}(\pi) = \text{Des}(\varphi(\pi)) = \text{Des}(q \circ \varphi(\pi)) = \text{Des}(i(\pi)) \].

Finally, by Equation (1) together with Lemma 3.12

\[ \text{cr}(\pi) = \text{ne}(\varphi(\pi)) = \left\lfloor \frac{\text{ht}(Q_{\varphi(\pi)})}{2} \right\rfloor = \text{ne}(q \circ \varphi(\pi)) = \text{ne}(i(\pi)) \].

\[\square\]
4. Schur-positivity

4.1. Background. Schur functions indexed by partitions of $n$ form a distinguished basis for $\Lambda_n$, the vector space of homogeneous symmetric functions of degree $n$; see, e.g., [23] Corollary 7.10.6. A symmetric function in $\Lambda_n$ is Schur-positive if all the coefficients in its expansion in the basis $\{s_\lambda : \lambda \vdash n\}$ of Schur functions are nonnegative.

For each $D \subseteq [n-1] = \{1, 2, \ldots, n-1\}$, define the fundamental quasisymmetric function

$$F_{n,D}(x) := \sum_{i_1 \leq i_2 \leq \cdots \leq i_n \atop i_j < i_{j+1} \text{ if } j \in D} x_{i_1}x_{i_2}\cdots x_{i_n}.$$ 

Let $A$ be a set of combinatorial objects, equipped with a set function $D : A \to 2^{[n-1]}$. We say that $A$ is symmetric (Schur-positive) with respect to $D$ if

$$Q_{A,D} := \sum_{\pi \in A} F_{n,D(\pi)},$$

is a symmetric (Schur-positive) function. Determining whether a given (quasi)symmetric function is Schur-positive is a major problem in contemporary algebraic combinatorics [24] §3.

The following theorem is due to Gessel.

**Theorem 4.1.** [23] Theorem 7.19.7} For every partition $\lambda \vdash n$

$$Q_{\text{SYT}(\lambda),\text{Des}} = s_\lambda,$$

thus $\text{SYT}(\lambda)$ is Schur-positive with respect to the standard descent set.

We say that a statistic $f : A \to \mathbb{N} \cup \{0\}$ is Schur-positive on $A$ with respect to the set function $D$ if

$$\sum_{\pi \in A} q^{f(\pi)} F_{n,D(\pi)},$$

is a Schur-positive symmetric function. Examples of Schur-positive statistics with respect to the standard descent set on permutations include

- Statistics on $S_n$ which are invariant under conjugation; e.g., cycle number and the number of fixed points. This follows from [11] Thm. 2.1.
- Statistics on $S_n$ which are invariant under Knuth relations; e.g., the length of the longest increasing subsequence, inverse descent number, and inverse major index. This follows from Theorem 4.1 together with Proposition 2.2.
- Inversion number on $S_n$ (reduced to inverse major index by Foata’s bijection). For a far reaching generalization see [22] Thm. 6.3.

Theorem 1.10 states that the pair $(\text{cr},\text{um})$ of crossing and unmatching numbers on the set of matchings on $n$ points $M_n$, is Schur-positive with respect to the geometric descent set $G\text{Des}$. 
4.2. Proof of Theorem 1.10. Let \( um(m) \) be the number of unmatched points in \( m \). For a partition \( \lambda \) let \( ht(\lambda) \) be the number of parts in \( \lambda \) and \( oc(\lambda) \) be the number of odd parts in the conjugate partition.

The following proposition follows from Theorem 1.9.

Proposition 4.2. For every \( n \geq 0 \)

\[
\sum_{m \in M_n} q^{um(m)} t^{cr(m)} x^{GDes(m)} = \sum_{\lambda \vdash n} q^{oc(\lambda)} t^{\lfloor ht(\lambda)/2 \rfloor} \sum_{T \in SYT(\lambda)} x^{Des(T)}.
\] (2)

Proof. We have

\[
\sum_{m \in M_n} q^{um(m)} t^{cr(m)} x^{GDes(m)} = \sum_{k=0}^{n} q^{k} \sum_{m \in M_{n,k}} t^{cr(m)} x^{GDes(m)}
\]

\[
= \sum_{k=0}^{n} q^{k} \sum_{m \in M_{n,k}} t^{ne(m)} x^{Des(m)}
\]

\[
= \sum_{m \in M_n} q^{um(m)} t^{ne(m)} x^{Des(m)}
\]

\[
= \sum_{\lambda \vdash n} q^{oc(\lambda)} t^{\lfloor ht(\lambda)/2 \rfloor} \sum_{T \in SYT(\lambda)} x^{Des(T)}.
\]

The second equality follows from Theorem 1.9. The last equality is obtained from application of the Robinson-Schensted bijection from \( M_n \), interpreted as the set of involutions in \( S_n \), to the set of all SYT of size \( n \), recalling Proposition 2.3, Lemma 3.12 and Proposition 2.2.2. □

Proof of Theorem 1.10. Consider Equation (2). Applying the vector space isomorphism from the ring of multilinear polynomials to the ring of quasisymmetric functions, defined by \( x^J \mapsto F_{n,J} \) for every subset \( J \subseteq [n-1] \), one obtains

\[
\sum_{m \in M_n} q^{um(m)} t^{cr(m)} F_{n,GDes(m)} = \sum_{\lambda \vdash n} q^{oc(\lambda)} t^{\lfloor ht(\lambda)/2 \rfloor} \sum_{T \in SYT(\lambda)} F_{n,Des(T)}
\]

\[
= \sum_{\lambda \vdash n} q^{oc(\lambda)} t^{\lfloor ht(\lambda)/2 \rfloor} s_{\lambda}.
\]

The last equality follows from Theorem 4.1. □

5. Cyclic descent extensions

The above setting is applied in this section to construct a cyclic descent extension for conjugacy classes of involutions and their refinements, that is, for involutions with fixed cycle structure and nesting number. The construction applies to other sets of combinatorial objects, in particular, to standard Young tableaux of given number of odd columns and height.
Let $M_n$ be the set of matching on $n$ points on the circle, labelled by $1, \ldots, n$ counterclockwise. Let $r: M_n \rightarrow M_n$ be the counterclockwise rotation by $\frac{2\pi}{n}$. Recall the definition of the geometric cyclic descent set map of a matching, $cGDes: M_n \mapsto 2^n$, from Definition 1.11.

**Observation 5.1.** For every $m \in M_n$, 
\[cGDes(m) \cap [n-1] = GDes(m)\]
and
\[cGDes(r(m)) = 1 + cGDes(m),\]
with addition modulo $n$.

Denote 
\[I_{n,k,j} := \{\pi \in I_{n,k}, \text{ ne}(\pi) = j\},\]
and recall the map $\hat{i}: I_{n,k} \mapsto I_{n,k}$ from Definition 3.6.5.

**Proposition 5.2.**
1. For every $0 \leq k \leq n$ and $0 < j \leq n/2$, the pair
\[(cGDes \circ \hat{i} - 1, \hat{i} \circ r \circ \hat{i} - 1)\]

is a (possibly Escherian) cyclic descent extension on $I_{n,k,j}$.

2. If $0 < k < n$ or $k = 0$ and $j \neq n/2$ this cyclic descent extension is non-Escherian.

Corollary 1.12 follows.

**Proof.** 1. First, the number of unmatched points is invariant under rotation, hence for every $\pi \in I_{n,k}$, $\hat{i} \circ r \circ \hat{i} - 1(\pi) \in I_{n,k}$. Furthermore,
\[\hat{i} \circ r \circ \hat{i} - 1(\pi) \in I_{n,k,j} \quad (\forall \pi \in I_{n,k,j}),\]

since
\[\text{ne}(\hat{i} \circ r \circ \hat{i} - 1(\pi)) = \text{cr}(r \circ \hat{i} - 1(\pi)) = \text{cr}(\hat{i} - 1(\pi)) = \text{ne}(\hat{i} \circ \hat{i} - 1(\pi)) = \text{ne}(\pi).\]

Here we applied Proposition 3.7 together with the fact that the crossing number is invariant under rotation.

Letting 
\[cDes(\pi) := cGDes(\hat{i} - 1(\pi)) \quad (\forall \pi \in I_{n}),\]

by Proposition 3.7 together with Observation 5.1 we have
\[cDes(\pi) \cap [n-1] = cGDes(\hat{i} - 1(\pi)) \cap [n-1] = GDes(\hat{i} - 1(\pi)) = \text{Des}(\pi) \quad (\forall \pi \in I_{n}),\]

and
\[cDes(\hat{i} \circ r \circ \hat{i} - 1(\pi)) = cGDes(r \circ \hat{i} - 1(\pi)) = 1 + cGDes(\hat{i} - 1(\pi)) = 1 + cDes(\pi) \quad (\forall \pi \in I_{n}).\]

2. For every $m \in M_{n,k}$, $k \neq 0, n$, $cGDes(m) \neq \emptyset, [n]$, hence
\[cDes(\pi) = cGDes(\hat{i} - 1(\pi)) \neq \emptyset, [n].\]

Similarly, for every $m \in M_{n,0,j}$, $j \neq n/2$, $cGDes(m) \neq \emptyset, [n]$, hence $cDes(\pi) \neq \emptyset, [n].$  \qed
Recall the map $i : I_{n,k} \to I_{n,k}$ from Definition 3.6 and the map $Q : S_n \to SYT_n$ defined by mapping $\pi \in S_n$ to its corresponding RS recording tableau $Q_\pi$. Let the map $h : M_{n,k} \mapsto SYT_{n,k}$ be defined by $h := Q \circ i$.

A cyclic descent extension on the set

$$SYT_{n,k,j} := \{ T \in SYT_{n,k} \mid 2j \leq ht(T) \leq 2j + 1 \}.$$

is described in the following statement.

**Proposition 5.3.** The pair $(cGDes \circ h^{-1}, h \circ r \circ h^{-1})$ is a cyclic descent extension on $SYT_{n,k,j}$.

Proof is almost identical to the proof of Proposition 5.2 and is omitted.

**Remark 5.4.** Cyclic rotation of geometric configurations was used before for construction of cyclic descent extensions on standard Young tableaux of rectangular shapes of height $\leq 3$ [17] and flag shapes [16]. These results motivated our work, and some of them are obtained as special cases:

- Letting $j = 1$ and $k = 0$ in Proposition 5.3 determines a cyclic descent extension on standard Young tableaux of shape $(n, n)$, since $SYT_{2n,0,1} = SYT(n,n)$. One can verify that this cyclic extension coincides with the one determined by Dennis White [17, Theorem 1].

- Recalling that the number of Motzkin paths of length $n$ is equal to the number of standard Young tableaux of size $n$ and at most three rows [18, 9, 4], let $j = 1$ and $k$ vary in Proposition 5.3. This determines a cyclic descent extension on Motzkin paths via Bin Han’s bijection [12], which coincides with Han’s cyclic descent extension on Motzkin paths.

6. **Equi-distribution - revisited**

In an early version of this paper the following conjecture was posed.

**Conjecture 6.1.** Let $\mu \vdash m$ and $\nu \vdash n$ be integer partitions with no common part. Let $\pi$ and $\sigma$ be permutations of cycle types $\mu$ and $\nu$, respectively, with disjoint supports. Let $A_{\pi,\sigma}$ be the subset of the conjugacy class of cycle type $\mu \sqcup \nu \vdash m + n$, for which the relative order of the letters in the union of all cycles of $\mu$ is as on $\pi$ and of the letters in the union of all cycles of $\nu$ is as on $\sigma$. Then

$$\sum_{w \in A_{\pi,\sigma}} x^{Des(w)} = \sum_{\tau \in \pi \sqcup \nu} x^{Des(\tau)}.$$

**Example 6.2.** Let $\pi = (1, 3, 2)$ and $\sigma = (4)$. Then $A_{\pi,\sigma}$ is the following subset of the conjugacy class of cycle type $(3, 1)$ in $S_4$$\ A_{\pi,\sigma} = \{(1, 3, 2)(4), (1, 4, 2)(3), (1, 4, 3)(2), (2, 4, 3)(1)\} = \{[3124], [4132], [4213], [1423]\}$$

and $\pi \sqcup \sigma = [312] \sqcup [4] = \{[3124], [3142], [3412], [4312]\}$

with same distribution of the descent set.
Conjecture 6.1 was proved by Gessel.

**Proposition 6.3.** [10] Conjecture 6.1 holds.

The proof is partly algebraic and not bijective.

**Proposition 6.3** implies the following.

**Corollary 6.4.** There exists an implicit descent set, nesting number and crossing number preserving bijection

\[ \phi : \mathcal{I}_{n-k,0} \cup [n-k+1, \ldots, n] \mapsto \mathcal{I}_{n,k}. \]

**Remark 6.5.** The explicit bijection \( q : \mathcal{I}_{n-k,0} \cup [n-k+1, n-k+2, \ldots n] \rightarrow \mathcal{I}_{n,k} \) from Lemma 3.15 preserves the descent set and nesting number, but does not preserve the crossing number. The bijection \( \phi : \mathcal{I}_{n-k,0} \cup [n-k+1, n-k+2, \ldots n] \rightarrow \mathcal{I}_{n,k} \) is implicit but preserves the crossing number as well.

**Proof of Corollary 6.4.** By Proposition 6.3 there exists an implicit bijection

\[ \phi : \mathcal{I}_{n-k,0} \cup [n-k+1, \ldots, n] \mapsto \mathcal{I}_{n,k}, \]

which preserves the descent set and satisfies the following property: for every \( \sigma \in \mathcal{I}_{n-k,0} \) and all \( \pi \in \sigma \cup [n-k+1, \ldots, n] \), the relative order of the letters in the \( \frac{n-k}{2} \)-cycles in \( \phi(\pi) \) is equal to the relative order of the letters in \( \sigma \). Since the nesting and crossing numbers of \( \pi \) are determined by the relative order of the letters in \( \sigma \) (Definition 3.9), \( \phi \) preserves both nesting and crossing numbers, as well. \( \square \)

The following refinement of Theorem 1.9 follows.

**Theorem 6.6.** For every \( n \geq k \geq 0 \)

\[
\sum_{m \in \mathcal{M}_{n,k}} q^{\text{cr}(m)} t^{\text{ne}(m)} x^{\text{GDes}(m)} = \sum_{m \in \mathcal{M}_{n,k}} q^{\text{ne}(m)} t^{\text{cr}(m)} x^{\text{Des}(m)}. 
\]

**Proof.** Replace \( \hat{\iota} := q \circ \phi \) by \( \hat{\iota} := \phi \circ \varphi \) in the proof of Theorem 1.9, and apply Corollary 6.4. \( \square \)

**Problem 6.7.** Find an explicit bijective proof of Theorem 6.6.

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