Abstract. State-based models of concurrent systems are traditionally considered under a variety of notions of process equivalence. In the particular case of labelled transition systems, these equivalences range from trace equivalence to (strong) bisimilarity, and are organized in what is known as the linear time - branching time spectrum. A combination of universal coalgebra and graded monads provides a generic framework in which the semantics of concurrency can be parametrized both over the branching type of the underlying transition systems and over the granularity of process equivalence. In particular, it yields a generic notion of trace logic, which is maybe surprisingly based on using graded monad algebras as formulas. In the present paper, we focus on substantiating the genericity over process equivalences by elaborating concrete graded monads for a range of equivalences from the linear time - branching time spectrum. Moreover, we complete the theory of trace logics by adding an explicit propositional layer and providing a generic expressiveness criterion that generalizes the classical Hennessy-Milner theorem to coarser notions of process equivalence. We extract trace logics for our leading examples, and give exemplaric proofs of their trace invariance and expressiveness based on our generic criterion.

1 Introduction

State-based models of concurrent systems are standardly considered under a wide range of system equivalences, typically located between two extremes respectively representing linear time and branching time views of system evolution. Over labelled transition systems, one specifically has the well-known linear time – branching time spectrum of system equivalences between trace equivalence and bisimilarity [36]. Similarly, e.g. probabilistic automata have been equipped with various semantics including strong bisimilarity [23], probabilistic (convex) bisimilarity [32], and distribution bisimilarity (e.g. [11,13]). New equivalences keep appearing in the literature, such as the recently introduced convex language semantics of nondeterministic probabilistic automata [37].

This motivates the search for unifying principles that allow for a generic treatment of process equivalences of varying degrees of granularity and for systems of different branching types (non-deterministic, probabilistic etc.). As regards the variation of the branching type, universal coalgebra [28] has emerged as a widely-used uniform framework for state-based systems covering a broad
range of branching types including besides non-deterministic and probabilistic, or more generally weighted, branching also, e.g., alternating, neighbourhood-based, or game-based systems. It is based on modelling the system type as an endofunctor on some base category, often the category of sets.

Unified treatments of system equivalences, on the other hand, are so far less well-established, and their applicability is often more restricted. Existing approaches include coalgebraic trace semantics in Kleisli [15] and Eilenberg-Moore categories [6, 19, 21, 33], respectively. Both semantics are based on decomposing the coalgebraic type functor into a monad, the branching type, and a functor, the transition type (in different orders), and require suitable distributive laws between these parts; correspondingly, they apply only to functors that admit the respective form of decomposition. In the present work, we build on a more general approach introduced by Pattinson and two of us, based on mapping the coalgebraic type functor into a graded monad [25]. Graded monads correspond to algebraic theories where operations come with an explicit notion of depth, and allow for a stepwise evaluation of process semantics. Maybe most notably, graded monads systemically support a reasonable notion of graded trace logic whose formulas are graded algebras for the given graded monad; for depth-1 graded monads, one has a compositional syntax for specifying such algebras. This approach applies to all cases covered in the mentioned previous frameworks, and additional cases that do not fit any of the earlier setups.

Our goal in the present work is to illustrate the level of generality achievable by means of graded monads in the dimension of system equivalences. We thus pick a fixed coalgebraic type, that of labelled transition systems, and elaborate how a number of equivalences from the linear time – branching time spectrum are cast as graded monads. In the process, we demonstrate how to extract logical characterizations of the respective equivalences from the given graded monads. For the time being, none of the logics we find are sensationally new, and in fact van Glabbeek already provides logical characterizations in his exposition of the linear time–branching time spectrum [36]; the emphasis in the examples is mainly on showing how the respective logics are developed uniformly from general principles.

Using these examples as a backdrop, we develop the theory of graded monads and trace logics further. In particular,

– we give a more economical characterization of depth-1 graded monads involving only two functors (rather than an infinite sequence of functors);
– we extend the logical framework by a treatment of propositional operators – previously regarded as integrated into the modalities – as first class citizens;
– we prove, as our main technical result, a generic expressiveness criterion for trace logics guaranteeing that inequivalent states are separated by a trace formula.

Our expressiveness criterion subsumes, at the branching-time end of the spectrum, the classical Hennessy-Milner theorem and its coalgebraic generalization [26, 30] as well as expressiveness of probabilistic modal logic with only conjunction [12]; we show that it also covers expressiveness of the respective trace logics for more
coarse-grained equivalences along the linear time – branching time spectrum including, e.g., trace equivalence (diamond and, optionally, disjunction) and simulation (diamond, conjunction and, optionally, disjunction).

Related Work We have already mentioned previous work on coalgebraic trace semantics in Kleisli and Eilenberg-Moore categories [6,15,19,21,33]. A common feature of these approaches is that, more precisely speaking, they model language semantics rather than trace semantics – i.e. they work in settings with explicit successful termination, and consider only successfully terminating traces. When we say that graded monads apply to all scenarios covered by these approaches, we mean more specifically that the respective language semantics are obtained by a further canonical quotienting of our trace semantics [25]. Having said that graded monads are strictly more general than Kleisli and Eilenberg-Moore style trace semantics, we hasten to add that the more specific setups have their own specific benefits including final coalgebra characterizations and, in the Eilenberg-Moore setting, generic determinization procedures. A further important piece of related work is Klin and Rot’s method of defining trace semantics via the choice of a particular flavour of trace logic [22]. In a sense, this approach is opposite to ours: A trace logic is posited, and then two states are declared equivalent if they satisfy the same trace formulas. In our approach via graded monads, we instead pursue the ambition of first fixing a semantic notion of equivalence, and then designing a logic that characterizes this equivalence. Like Klin and Rot, we view trace equivalence as an inductive notion, and in particular limit attention to finite traces; coalgebraic approaches to infinite traces exist, and mostly work within the Kleisli-style setup [7–10,17,20,35]. For finitely branching systems, the infinite-time and finite-time variants of the various equivalences typically coincide; e.g. this is the case for trace equivalence and bisimilarity. Jacobs, Levy and Rot [18] use corecursive algebras to provide a unifying categorical view on the above-mentioned approaches to traces via Kleisli- and Eilenberg-Moore categories and trace logics, respectively. This framework does not appear to subsume the approach via graded monads, and like for the previous approaches we are not aware that it covers semantics from the linear time – branching time spectrum other than the end points (bisimilarity and trace equivalence).

2 Graded Monads and Graded Theories

We proceed to recall background on trace semantics via graded monads introduced in our previous work [25]. We assume that readers are familiar with standard notions of basic category theory [24,27]. Graded monads were originally introduced by Smirnov [34] (with grades in a commutative monoid, which we instantiate to the natural numbers):

Definition 2.1 (Graded Monads). A graded monad \( \mathcal{M} \) on a category \( \mathcal{C} \) consists of a family of functors \( (M_n : \mathcal{C} \to \mathcal{C})_{n \in \mathbb{N}} \), a natural transformation \( \eta : \text{Id} \to M_0 \) (the unit) and a family of natural transformations
\[
\mu^{nk} : M_n M_k \to M_{n+k} \quad (n, k \in \mathbb{N})
\]
(the multiplication), satisfying the unit laws, \( \mu^n \cdot \eta M_n = \text{id}_{M_n} = \mu^n \cdot M_n \eta \), for all \( n \in \mathbb{N} \), and the associative law

\[
M_n M_k M_m \xrightarrow{\mu^{n+k} m} M_n M_{k+m} \xrightarrow{\mu^{n, k+m}} M_{n+k+m} \quad \text{for all } k, n, m \in \mathbb{N}.
\]

Note that it follows that \((M_0, \eta, \mu^0)\) is a (plain) monad. For \( C = \text{Set} \), the standard equivalent presentation of monads as algebraic theories carries over to graded monads. Whereas for a monad \( M \), the set \( MX \) consists of terms over \( X \) modulo equations of the corresponding algebraic theory, for a graded monad \((M_n)_{n \in \mathbb{N}}\), \( M_n X \) consists of terms of uniform depth \( n \) modulo equations:

**Definition 2.2 (Graded Theories [25]).** A graded theory \((\Sigma, E, d)\) consists of an algebraic theory, i.e. a (possibly class-sized) algebraic signature \( \Sigma \) and a class \( E \) of equations, and an assignment \( d \) of a depth \( d(f) \in \mathbb{N} \) to every operation \( f \in \Sigma \). This induces a notion of a term having uniform depth \( n \): all variables have uniform depth 0, and \( f(t_1, \ldots, t_n) \) with \( d(f) = k \) has uniform depth \( n + k \) if all \( t_i \) have uniform depth \( n \). (In particular, a constant \( c \) has uniform depth \( n \) for all \( n \geq d(c) \)). We require that all equations \( t = s \in E \) have uniform depth, i.e. that both \( t \) and \( s \) have uniform depth \( n \) for some \( n \). Moreover, we require that for every set \( X \) and every \( k \in \mathbb{N} \), the class of terms of uniform depth \( k \) over variables from \( X \) modulo provable equality is small (i.e. in bijection with a set).

Graded theories and graded monads on \( \text{Set} \) are essentially equivalent concepts; for the finitary case, this is implicit in [34]. In particular, a graded theory \((\Sigma, E, d)\) induces a graded monad \( M \) by taking \( M_n X \) to be the set of \( \Sigma \)-terms over \( X \) of uniform depth \( n \), modulo equality derivable under \( E \).

**Example 2.3.** We recall some examples of graded monads and theories [25].

1. For every endofunctor \( F \) on \( \text{Set} \), the \( n \)-fold composition \( M_n = F^n \) yields a graded monad with unit \( \eta = \text{id}_M \) and \( \mu^{nk} = \text{id}_{F^n \cdot F^k} \).

2. As indicated in the introduction, distributive laws yield graded monads: Suppose that we are given a monad \((M, \eta, \mu)\), an endofunctor \( F \) on \( \text{Set} \) and a distributive law of \( F \) over \( M \) (a so-called Kleisli law), i.e. a natural transformation \( \lambda: FM \rightarrow MF \) such that \( \lambda \cdot F\eta = \eta F \) and \( \lambda \cdot F\mu = \mu F \cdot M\lambda \cdot \lambda M \). Define natural transformations \( \lambda^n: F^n M \rightarrow MF^n \) inductively by \( \lambda^0 = \text{id}_M \) and \( \lambda^{n+1} = \lambda^n F \cdot F^n \lambda \). Then we obtain a graded monad with \( M_n = MF^n \), unit \( \eta \), and multiplication \( \mu^{nk} = \mu_{F^{n+1}} \cdot M\lambda^n F^k \). The situation is similar for distributive laws of \( M \) over \( F \) (so-called Eilenberg-Moore laws).

3. As a special case, for every monad \((M, \eta, \mu)\) on \( \text{Set} \) and every set \( A \), we obtain a graded monad with \( M_n X = M(A^n \times X) \). Of particular interest to us will be the case where \( M = \mathcal{P} \) is the powerset monad, which is generated by the infinitary algebraic theory of complete join-semilattices. The arising graded monad \( M_n = \mathcal{P}(A^n \times X) \), which the reader will guess is associated with trace
equivalence, is generated by the graded theory consisting, at depth 0, of the operations and equations of complete join-semilattices, and additionally a unary operation of depth 1 for each \( \sigma \in A \), subject to (depth-1) equations expressing that these unary operations distribute over joins.

**Depth-1 Graded Monads and Theories** For purposes of building algebras of graded monads, a particularly convenient case is the one where operations and equations have depth at most 1; in the following, we elaborate on this condition.

**Definition 2.4 (Depth-1 Graded Theory \([25]\))**. A graded theory is called depth-1 if all its operations and equations have depth at most 1. A graded monad on \( \text{Set} \) is depth-1 if it can be generated by a depth-1 graded theory.

**Proposition 2.5 (Depth-1 Graded Monads \([25]\))**. A graded monad \((M_n, \eta, (\mu^n))\) on \( \text{Set} \) is depth-1 if the diagram below is objectwise a coequalizer diagram in \( \text{Set}^{M_0} \) for every \( n \):

\[
\begin{array}{c}
M_1 M_0 M_n \xrightarrow{M_1 \mu^n} M_1 M_n \xrightarrow{\mu^{1+n}} M_{1+n}.
\end{array}
\]  

(2.1)

**Example 2.6.** All graded monads in Example 2.3 are depth 1: for (1) this is easy to see, for (3) it follows from the presentation as a graded theory, and for (2), see Section A.2 in the appendix.

One may use the equivalent property of Proposition 2.5 to define depth-1 one graded monads over arbitrary base categories \([25]\). We show next that depth-1 graded monads may be specified by \( M_0, M_1, \mu^{nk} \) for \( n+k \leq 1 \).

**Theorem 2.7.** Depth-1 graded monads are in bijective correspondence with 6-tuples \((M_0, M_1, \eta, (\mu^{00}, \mu^{10}, \mu^{01}))\) such that given data satisfy all applicable instances of the graded monad laws.

**Semantics via Graded Monads** We next recall the definition of generic (trace) semantics in terms of graded monads:

**Definition 2.8 (Graded semantics \([25]\))**. Given a set functor \( G \), a graded semantics for \( G \)-coalgebras consists of a graded monad \(((M_n), \eta, (\mu^{nk}))\) and a natural transformation \( \alpha : G \to M_1 \). The \( \alpha \)-pretrace sequence \((\gamma^{(n)} : X \to M_n X)_{n \in \mathbb{N}}\) for a \( G \)-coalgebra \( \gamma : X \to GX \) is defined by

\[
\gamma^{(0)} = \eta_X \quad \text{and} \quad \gamma^{(n+1)} = (X \xrightarrow{\alpha X \gamma} M_1 X \xrightarrow{M_1 \gamma^{(n)}} M_1 M_n X \xrightarrow{\mu^{1n}} M_{n+1} X).
\]

The \( \alpha \)-trace sequence \( T^n_\gamma \) is the sequence \((M_n! \cdot \gamma^{(n)} : X \to M_n)_n \in \mathbb{N}\).

In \( \text{Set} \), two states \( x \in X, y \in Y \) of coalgebras \( \gamma : X \to GX \) and \( \delta : Y \to GY \) are \( \alpha \)-trace (or graded) equivalent if \( M_n! \cdot \gamma^{(n)}(x) = M_n! \cdot \delta^{(n)}(y) \) for all \( n \in \mathbb{N} \).
Intuitively, $M_nX$ consists of all length-$n$ pretraces, i.e. traces ending with a poststate from the given $G$-coalgebra, and $M_1^n$ consists of all length-$n$ traces, obtained by erasing the poststate.

**Example 2.9.** For $G = \mathcal{P}(A \times -)$ on Set, where $A$ is a finite set (of actions), we recall two graded semantics that model the extreme ends of the linear time - branching time spectrum [25].

First note that coalgebras for $G$ may be identified with labelled transition systems (LTS). In fact, a coalgebra structure $\gamma: X \to \mathcal{P}(A \times X)$ can be viewed as graph with $A$-labelled edges on the set $X$ of nodes. Given a coalgebra $(X, \gamma)$, we write $I(x) = P \pi_1 \cdot \gamma(x) = \pi_1[\gamma(x)]$ for the set of actions available at $x$, the ready set of $x$. For $x, y \in X$ and $w \in A^*$, we write $x \overset{w}{\rightarrow} y$ if $y$ can be reached from $x$ on a path whose labels yield the word $w$, and $T(x) = \{ w \in A^* \mid \exists y \in X. x \overset{w}{\rightarrow} y \}$ denotes the set of traces of $x \in X$.

(1) **Trace equivalence.** To capture trace semantics of labelled transition systems we consider the graded monad with $M_nX = \mathcal{P}(A^n \times X)$ with unit given by $\eta_X(x) = \{(\varepsilon, x)\}$, and multiplication $\mu^{nk}(S) = \{(uv, x) \mid \exists (u, V) \in S. (v, x) \in V\}$ (see Example 2.3(3)). The natural transformation $\alpha$ is the identity. For a $G$-coalgebra $(X, \gamma)$ and $x \in X$ we have that $\gamma^{(n)}(x)$ is the set of pairs $(w, y)$ with $w \in A^n$ and $x \overset{w}{\rightarrow} y$, i.e. pairs of length-$n$ traces and their corresponding poststate. Consequently, the $n$-th component $M_n! \cdot \gamma^{(n)}$ of the $\alpha$-trace sequence maps $x$ to the set of its length-$n$ traces. Thus, $\alpha$-trace equivalence is the standard trace equivalence [36].

Note that the equations presenting the graded monad $M_n$ in Example 2.3(3) bear a striking resemblance to the ones given by van Glabbeek to axiomatize trace equivalence of processes, with the difference that in his axiomatization actions do not distribute over the empty join. In fact, $a.0 = 0$ is clearly not valid for processes under trace equivalence. In the graded setting, this equation just expresses the fact that a trace which ends in a deadlock after $n$ steps cannot be extended to a trace of length $n+1$.

(2) **Finite bisimilarity.** Recall that two states in labelled transition systems are finitely bisimilar [14] if Duplicator wins all finite-length bisimulation games on them. Here we take the graded monad with $M_nX = G^nX$ (see Example 2.3(1)), and we let $\alpha$ be the identity again. Then two states in $G$-coalgebras are $\alpha$-trace equivalent iff they are finitely bisimilar [25]. It follows that on finitely branching LTS, $\alpha$-trace equivalence is bisimilarity.

The second example above shows that graded equivalence may be much finer than anything one would call a trace equivalence. We will see in the next section that graded monads capture many equivalences on the spectrum between the two above examples.

### 3 A Spectrum of Graded Monads

We present graded monads for a range of equivalences on the linear time – branching time spectrum, giving in each case a graded theory and a description
of the arising graded monads obtained via normal forms. Some of our equations bear some similarity to van Glabbeek’s axioms for equality of process terms under the respective semantics. There are also important differences, however. In particular, some of van Glabbeek’s axioms are implications, while ours are purely equational; moreover, van Glabbeek’s axioms sometimes nest actions, while we take care to employ only depth-1 equations (which precludes nesting of actions) in order to enable the extraction of trace logics later. All graded theories we introduce contain the (infinitary) theory of complete join-semilattices, whose join operations, denoted as sums, are assigned depth 0; we mention only the additional operations needed. We use terminology introduced in Example 2.9.

Completed Trace Semantics refines trace semantics by distinguishing whether traces can end in a deadlock. We define a depth-1 graded theory with a unary depth-1 operation $\sigma$ for every action $\sigma \in A$ and a constant depth-1 operation $\ast$ denoting deadlock. We impose the equation $\sigma(\sum_{j \in J} x_j) = \sum_{j \in J} \sigma(x_j)$. The induced graded monad has

$$M_0 X = \mathcal{P}(X), \quad M_1 = \mathcal{P}(A \times X + 1)$$

(and $M_n X = \mathcal{P}(A^n \times X + A^{<n})$ where $A^{<n}$ denotes the set of words over $A$ of length less than $n$). The natural transformation $\alpha_X : \mathcal{P}(A \times X) \rightarrow M_1 X$ is given by $\alpha(\emptyset) = \{\ast\}$, with $\ast$ denoting the unique element of 1, and $\alpha(S) = S \subseteq A \times X + 1$ for $\emptyset \neq S \subseteq A \times X$.

Ready Trace and Failure Trace Semantics A ready trace of a state $x$ is a sequence $A_0 a_1 A_1 \ldots a_n A_n \in (\mathcal{P}A \times A)^* \times \mathcal{P}A$ such that there exist transitions $x = x_0 \xrightarrow{a_1} x_1 \ldots \xrightarrow{a_n} x_n$ where each $x_i$ has ready set $I(x_i) = A_i$. A failure trace has the same shape but we require that $A_i$ is a failure set of $x_i$, i.e. $I(x_i) \cap A_i = \emptyset$. States of labelled transition systems are ready (failure) trace equivalent if they have the same ready (failure) traces.

For ready traces, we define a depth-1 graded theory with depth-1 operations $\langle A, \sigma \rangle$ for $\sigma \in A$, $A \subseteq A$ and a depth-1 constant $\ast$, denoting deadlock, and equations $\langle A, \sigma \rangle(\sum_{j \in J} x_j) = \sum_{j \in J} \langle A, \sigma \rangle(x_j)$. The resulting graded monad is simply the graded monad capturing completed trace semantics for labelled transition systems where the set of actions is changed from $A$ to $\mathcal{P}A \times A$. For failure traces, we additionally impose the equation $\langle A, \sigma \rangle(x) + (\langle A \cup B, \sigma \rangle(x) = \langle A \cup B, \sigma \rangle(x)$, which in the set-based description of the graded monad corresponds to downward closure of failure sets.

For ready trace semantics we define the natural transformation $\alpha_X : \mathcal{P}(A \times X) \rightarrow M_1 X$ by $\alpha_X(\emptyset) = \{\ast\}$ and $\alpha_X(S) = \{((\pi_1[S], \sigma), x) \mid (\sigma, x) \in S\}$ for $S \neq \emptyset$. For failure traces we define $\alpha_X : \mathcal{P}(A \times X) \rightarrow M_1 X \subseteq \mathcal{P}((\mathcal{P}A \times A) \times X + 1)$ by $\alpha_X(\emptyset) = \{\ast\}$ and $\alpha(S) = \{((\langle A, \sigma \rangle, x) \mid (\sigma, x) \in S, A \cap \pi_1[S] \neq \emptyset\}$ for $S \neq \emptyset$; note that in the latter case, $\alpha(S)$ is closed under decreasing failure sets.
Readiness A ready pair of a state $x$ is a pair $(w,A) \in A^* \times \cal P(A)$ such that there exists $z$ such that $x \xrightarrow{w} z$ and $I(z) = A$. Two states are readiness equivalent if they have the same ready pairs. We define a depth-1 graded theory with depth-1 operations $\sigma$ for $\sigma \in \cal A$, unary depth-1 operations $(A,\sigma)$ for pairs of (ready) sets $A \subseteq A$ and actions $\sigma \in A$, a unary depth-0 operation $\circ$ and a depth-1 constant $\star$, with equations stating that all unary operations distribute over sums. The purpose of the operation $\circ$ is to mark whether its argument is of depth $\geq 1$ so we can ensure with the equation $(A,\sigma)(\circ(x)) = \sigma(\circ(x))$ that operations $(A,\sigma)$ appear at most at bottom-level; the equations $t(x) = \circ(t(x)), \star = \circ(\star)$ where $t$ ranges over all unary operations ensure that every depth-1 term can be marked as such with $\circ$ and unnecessary nesting of $\circ$ can be eliminated. An additional equation $\sigma(\circ(x)) = \sigma(x)$ ensures that spurious occurrences of $\circ$ at the bottom level of terms can be eliminated. The arising depth-1 graded monad has $M_0 X = \cal P(X + \{\circ\} \times X), M_1 X = \cal P(\cal P(A \times A + A) \times X + 1))$. The natural transformation $\alpha_X : \cal P(AX \times X) \rightarrow M_1 X$ is defined by $\alpha_X(\emptyset) = \{\emptyset\}$ and for $S \neq \emptyset$ by $\alpha_X(S) = \{(\sigma, x) | (\sigma, x) \in S\} \cup \{((\pi_1[S], \sigma), x) | (\sigma, x) \in S\}$.

Simulation Equivalence declares two states to be equivalent if they simulate each other in the standard sense. We define a depth-1 graded theory with the same signature as for trace equivalence but instead of join preservation require only that each $\sigma$ is monotone, i.e. $\sigma(x + y) + \sigma(x) = \sigma(x + y)$. The arising graded monad $M_n$ is equivalently described as follows. We define the sets $M_n X$ inductively, along with an ordering $\leq$ on $M_n X$. We take $M_0 X = \cal P(X)$, ordered by set inclusion. We then order the elements of $A \times M_n X$ by the product ordering of the discrete order on $A$ and the given ordering on $M_n X$. and take $M_{n+1} X$ to be the set of downclosed subsets of $A \times M_n X$, denoted $\cal P^1(A \times M_n X)$, ordered by set inclusion. The natural transformation $\alpha$ is defined like for trace equivalence.

Ready Simulation Equivalence A ready simulation is defined like a simulation but requiring additionally that related states have the same ready set. States $x$ and $y$ are ready similar if they are related by some ready simulation, and ready simulation equivalent if there are mutually ready similar. The depth-1 graded theory combines the signature for ready traces with the equations for simulation, i.e. only requires the operations $(A,\sigma)$ to be monotone.

4 Graded Trace Logics

Our next goal is to develop a principled notion of generic graded trace logic for a given graded semantics. The minimal requirement on a graded trace logic is that it is invariant under the given notion of graded equivalence, a property we briefly refer to as graded trace invariance. Ideally, we moreover want the trace logic to provide a logical characterization of the given system equivalence in the sense that two states are semantically equivalent iff they satisfy the same formulas of the graded trace logic, a property that we call expressiveness. The
logics we introduce are trace invariant by construction [25]; the criterion for expressiveness is new, and will be given in Section 5. In comparison to the original presentation [25], we add propositional operators as first class citizens: Trace logics typically do not include the full set of Boolean operators as this would violate trace invariance; on the other hand, expressiveness will depend on the presence of enough propositional operators, and having an explicit notion of propositional operators is a prerequisite for formulating the expressiveness theorem.

For this section and the next, we fix a functor $G$, a graded monad $\mathbb{M} = \langle (M_n)_{n<\omega}, \eta, (\mu^{nk})_{n,k<\omega} \rangle$ and a graded semantics $\alpha: G \to M_1$. Maybe unexpectedly, we let algebras for $M$, more precisely graded algebras in a sense that we will recall presently, play the role of formulas in the logic (to be quite clear, we do not mean that formulas form a graded algebra, but really that every formula is, or more precisely represents, a graded algebra). The evaluation of formulas will then be based on the universal property of the codomain $(M_n1)$ of the trace sequence as a free graded algebra over 1.

In general, the notion of graded algebra is defined as follows.

**Definition 4.1 (Graded algebras).** Let $n < \omega$. A (graded) $M_n$-algebra $A = \langle (A_k)_{k \leq n}, (a^{mk})_{m+k \leq n} \rangle$ consists of carrier sets $A_k$ and structure maps $a^{mk}: M_mA_k \to A_{m+k}$ satisfying the laws

$$
A_k \xrightarrow{\eta A_k} M_0 A_k \\
\xrightarrow{\mu^{nk}} A_k \\
M_m M_r A_k \xrightarrow{\mu^{m,r} a^{rk}} M_mA_{r+k} \\
\xrightarrow{\mu^{m,r} a^{rk}} M_mA_{r+k} \\
\xrightarrow{M_m a^{rk}} A_{m+k} \\
\xrightarrow{a^{m+r,k}} A_{m+k}
$$

for all $k \leq n$ (left) and all $m, r, k$ such that $m + r + k \leq n$ (right), respectively. A morphism $f$ from $A$ to an $M_n$-algebra $B = \langle (B_k)_{k \leq n}, (b^{mk})_{m+k \leq n} \rangle$ consists of maps $f_k: A_k \to B_k, k \leq n$, such that $f_{m+k} a^{mk} = b^{mk} M_m f_k$ for all $m, k$ such that $m + k \leq n$.

We view the carrier $A_k$ of an $M_n$-algebra as the set of algebra elements that have already absorbed operations up to depth $n$. As in the case of plain monads, we can equivalently describe graded algebras in terms of graded theories: If $\mathbb{M}$ is generated by a graded Theory $\mathcal{T} = (\Sigma, E, d)$, then an $M_n$-algebra is a structure that interprets the operations in $\Sigma$ up to accumulated depth $n$, satisfying the equations in $E$. More precisely, an operation $f \in \Sigma$ of arity $r$ and depth $d(f) = m$ is interpreted by maps $f^K_k: A^K_k \to A_{m+k}$ for all $k$ such that $m + k \leq n$; this gives rise to an inductively defined interpretation of terms of uniform depth at most $n$, and hence to the expected notion of satisfaction of equations (recall that equations in graded theories have uniform depth), where equations are only relevant to $A$ if their depth is at most $n$. We note that products of $M_n$-algebras
are formed in the expected way [25, Proposition 6.3], in particular have carriers formed by taking Cartesian products at every depth.

As expected, we have that \((M_k X)_{k \leq n}\) with multiplication as the algebra structure is the free \(M_n\)-algebra over \(X\). As indicated above, we will use this property as the underlying principle of formula evaluation: We fix a set \(\Omega\) of truth values, and take an \(\alpha\)-trace property (or graded trace property, avoiding mention of \(\alpha\)) to consist of an \(M_n\)-algebra \(A\) with carriers of the form \(A_k = \Omega^{k\text{th}}\), where \(k_n = 1\), together with a base \(\tau: 1 \rightarrow \Omega^{k_0}\). Formulas will denote \(\alpha\)-trace properties. The evaluation of \(A\) on a \(G\)-coalgebra \((C, \gamma)\) is then the morphism

\[
C \xrightarrow{\gamma(n)} M_n C \xrightarrow{M_n!} M_n 1 \xrightarrow{\tau#} A_n
\]

where \(\tau#\) is the unique homomorphism from the free \(M_n\)-algebra on \(1\), \((M_k 1)_{k \leq n}\), to \(A\) such that \(\tau_0# \cdot \eta = \tau\). Thus, the evaluation of \(\alpha\)-trace properties is \(\alpha\)-trace invariant by construction.

While in the general case, graded algebras are monolithic objects, for depth-1 graded monads we can construct them in a modular fashion [25, Theorem 7.8]: Assume from now on that \(M\) is depth 1. Then an \(M_n\)-algebra with carriers \((A_k)_{k \leq n}\) is equivalently determined by

- an \(M_0\)-algebra structure \(a^{00}: M_0 A_k \rightarrow A_k\) for each \(k \leq n\); and
- maps \(a^{1k}: M_1 A_k \rightarrow A_{k+1}\) for \(0 \leq k \leq n - 1\)

such that for each \(k \leq n - 1\), the data \(A_k, A_{k+1}, a^{1k}, a^{0,k+1}, a^{0,k}\) determine an \(M_1\)-algebra; as a slogan: An \(M_n\)-algebra is a chain of \(M_1\)-algebras with mutually compatible \(M_0\)-parts. This principle, which we refer to as the modular description of graded algebras, can be exploited to give a compositional semantics for a generic trace logic, defined next.

**Syntax** We parametrize the syntax of trace logics over

- a set \(\Theta\) of truth constants,
- a set \(O\) of propositional operators with assigned finite arities, and
- a set \(\Lambda\) of modalities with assigned arities.

For readability, we will restrict the technical exposition to unary modalities; the treatment of higher arities requires no more than additional indexing (and we will use 0-ary modalities in the examples). Formulas of the logic are restricted to have uniform depth, where truth constants and propositional operators have depth 0 and modalities have depth 1; a somewhat particular feature is that truth constants can have top-level occurrences only in depth-0 formulas. That is, the formulas \(\phi\) of depth 0 are given by the grammar

\[
\phi ::= p(\phi_1, \ldots, \phi_k) \mid c \quad (p \in O \text{ k-ary}, c \in \Theta),
\]

and formulas \(\phi\) of depth \(n + 1\) by

\[
\phi ::= p(\phi_1, \ldots, \phi_k) \mid L\psi \quad (p \in O \text{ k-ary}, L \in \Lambda)
\]

where \(\phi_1, \ldots, \phi_n\) range over formulas of depth \(n + 1\) and \(\psi\) over formulas of depth \(n\).
Semantics As indicated above, we fix a set $\Omega$ of truth values, which we moreover equip with a fixed $M_0$-algebra structure $o : M_0\Omega \rightarrow \Omega$. The $n$-th power $\Omega^n$ of $\Omega$ is again an $M_0$-algebra, whose structure map we denote as $o^{(n)}$. We interpret truth constants $c \in \Theta$ as elements of $\Omega$, understood as maps $[c] : 1 \rightarrow \Omega$. A modality $L \in A$ is interpreted as an $M_1$-algebra $A = [L]$ with carriers $A_0 = A_1 = \Omega$ and $a^{01} = a^{00} = o$. Such an $M_1$-algebra is thus specified by the map $a^{10} : M_1\Omega \rightarrow \Omega$ alone, so we often write $[L]$ for just this map, which is said to determine $[L]$. We apply this terminology more generally to $M_1$-algebras $A$ whose carriers are powers of the form $\Omega^n$, i.e. $A_0 = \Omega^k$, $A_1 = \Omega^m$, $a^{00} = o^{(k)}$, $a^{01} = o^{(m)}$. Expanding definitions, this means that a map $l : M_1\Omega^k \rightarrow \Omega^m$ determines an $M_1$-algebra if $l\mu^{00}_{0k} = l\mu_1^{00}o^{(k)}$ and $l\rho^{01}_{0k} = o^{(m)}l_0$. It remains to interpret the propositional operators, newly added here as a feature of the logic. The semantics of a $k$-ary propositional operator will be a truth function $\Omega^k \rightarrow \Omega$. Not all such truth functions can be included, as they may destroy the requisite algebra properties. Formally, we define the requisite property for truth functions returning tuples of truth values:

**Definition 4.2.** A truth function $f : \Omega^k \rightarrow \Omega^m$ is admissible for $M$ if $M_1$-algebras are stable under composition with $f$ on both sides; that is:

1. If $l : M_1\Omega^m \rightarrow \Omega^n$ determines an $M_1$-algebra, then so does $l \cdot M_1f : M_1\Omega^k \rightarrow \Omega^n$; and

2. if $l : M_1\Omega^m \rightarrow \Omega^k$ determines an $M_1$-algebra, then so does $f \cdot l : M_1\Omega^k \rightarrow \Omega^m$.

An important class of admissible truth functions is the following:

**Lemma 4.3.** Every $M_0$-homomorphism $\Omega^k \rightarrow \Omega^m$ is admissible.

This implies in particular that all morphisms $\Omega^k \rightarrow \Omega^m$ that are tuples of product projections, i.e. just copy, permute, or drop variables, are admissible. In our examples on the linear time–branching time spectrum, $M_0$ is either the identity or, most of the time, the powerset monad. In the former case, all truth functions are $M_0$-morphisms. In the latter case, the $M_0$-morphisms $\Omega^k \rightarrow \Omega^m$ are the join-continuous functions; in the standard case where $\Omega = 2$ is the set of Boolean truth values, such functions are built from projections, tupling, and disjunction. In earlier work, we have elaborated cases where $M_0$ is the distribution monad; in such cases, $M_0$-morphisms are affine maps [25]. We shall see, however, that admissible truth functions need not be $M_0$-morphisms, and in fact we will encounter cases where conjunction is admissible.

We thus complete the semantic parametrization of our logic by interpreting each $k$-ary propositional operator $p \in \mathcal{O}$ by an admissible truth function

$$[p] : \Omega^k \rightarrow \Omega.$$ 

The semantics of formulas is then defined by converting depth-$n$ formulas $\phi$ into depth-$n$ graded trace properties $[\phi]$, inductively over $n$. For a depth-0 formula, i.e. a truth constant $c$, we take $[c]$ (overloading notation) to be the graded trace property consisting of the $M_0$-algebra $\Omega$ and the base $[c] : 1 \rightarrow \Omega$. 

Graded Monads for the Linear Time – Branching Time Spectrum 11
We now proceed to develop our main result, an expressiveness criterion for finite-depth behavioural equivalence:

**Example 4.4 (Graded trace logics).** We recall the two most basic examples, fixing $\Omega = 2$ in both cases, and $\top$ as the only truth constant:

1. **Finite-depth behavioural equivalence:** Recall that the graded monad $M_n X = F^n X$ captures finite-depth behavioural equivalence on $F$-coalgebras. As mentioned above, every truth function $2^k \to 2$ is admissible since $M_0$ is the identity monad, so we can use all Boolean operators as propositional operators. An $M_1$-algebra is just a map $F 2 \to 2$, equivalently a predicate lifting [30]; summing up, we just obtain standard coalgebraic logic [26] as the graded ‘trace’ logic in this case. In our running example $F = \mathcal{P}(A \times (-))$, we can take modalities $\Diamond_\sigma$ indexed over actions $\sigma \in A$ and define $[\Diamond_\sigma] : \mathcal{P}(A \times 2) \to 2$ by $[\Diamond_\sigma](S) = \top$ iff $(\sigma, \top) \in S$, obtaining precisely classical Hennessy-Milner logic [16].

2. **Trace equivalence:** Recall that the trace semantics of labelled transition systems with actions in $A$ is modelled by the graded monad $M_n X = \mathcal{P}(A^n \times X)$. As mentioned above, in this case we can use disjunction as a propositional operator since $M_0 = \mathcal{P}$. Since the graded theory for $M_n$ specifies for each $\sigma \in A$ a unary depth-1 operation that distributes over joins, we find that the maps $[\Diamond_\sigma]$ from the previous example determine $M_1$-algebras also in this case, so we obtain a graded trace logic featuring precisely diamonds and disjunction, as expected.

We defer the discussion of further examples to the next section, where we will simultaneously illustrate the generic expressiveness result (Example 5.8). In previous work [25], we have considered also examples where $\Omega = [0,1]$, obtaining, e.g., graded trace logics for probabilistic transition systems featuring an expectation modality, and affine combinations as propositional operators.

## 5 Expressiveness

We now proceed to develop our main result, an expressiveness criterion for graded trace logics. This will require the presence of both enough modalities and enough propositional operators. Our criterion will be formulated using a class of $M_1$-algebras that we introduce next.
Graded Monads for the Linear Time – Branching Time Spectrum

**Definition 5.1.** The 0-part of an $M_1$-algebra $A$ is the $M_0$-algebra $(A_0, a^{00})$. Taking 0-parts defines a functor $U_0$ from $M_1$-algebras to $M_0$-algebras. An $M_1$-algebra is canonical if it is free, w.r.t. $U_0$, over its 0-part. For $A$ canonical and a modality $L \in \Lambda$, we denote the unique morphism $A_1 \to \Omega$ extending an $M_0$-morphism $f : A_0 \to \Omega$ to an $M_1$-morphism $A \to [L]$ by $[L](f)$, i.e. $[L](f)$ is the unique $M_0$-morphism $A_1 \to \Omega$ such that the following diagram commutes:

\[
\begin{array}{ccc}
M_1 A_0 & \xrightarrow{M_1 f} & M_1 \Omega \\
\downarrow a^{10} & & \downarrow [L] \\
A_1 & \xrightarrow{[L](f)} & \Omega
\end{array}
\] (5.1)

The main purpose of this notion is to abstract the $M_1$-algebra components of the free graded algebras:

**Lemma 5.2.** An $M_1$-algebra $A$ is canonical iff the following is a (reflexive) coequalizer diagram in the category of $M_0$-algebras:

\[
\begin{array}{ccc}
M_1 M_0 A_0 & \xrightarrow{\mu^{10}} & M_1 A_0 \\
\downarrow M_1 a^{00} & & \downarrow a^{10} \\
M_1 A_0 & \xrightarrow{a^{10}} & A_1
\end{array}
\]

**Corollary 5.3.** If $M$ is a depth-1 graded monad, then for every $n$ and every set $X$, the $M_1$-algebra with carriers $M_n X, M_{n+1} X$ and with multiplication as the algebra structure is canonical.

The expressiveness criterion depends on the truth constants, the propositional operators, and the modalities featured in a trace logic:

**Definition 5.4.** We say that a graded trace logic with set $\Omega$ of truth values and sets $\Theta$, $\mathcal{O}$, $\Lambda$ of truth constants, propositional operators, and modalities, respectively, is

1. depth-0 separating if the collection of all Kleisli liftings $\mu^{00}_{M_0, c} : M_01 \to \Omega$, for truth constants $c \in \Theta$, is jointly injective; and
2. depth-1 separating if, whenever $A$ is a canonical $M_1$-algebra and $\mathfrak{A}$ is a jointly injective collection of $M_0$-homomorphisms $A_0 \to \Omega$ that is closed under the propositional operators in $\mathcal{O}$ (in the sense that if $f_1, \ldots, f_k \in \mathfrak{A}$ and $p \in \mathcal{O}$ is $k$-ary, then $[p] \cdot (f_1, \ldots, f_k) \in \mathfrak{A}$), then the set

\[
\{[[L](f)] : A_1 \to \Omega \mid L \in A, f \in \mathfrak{A}\}
\]

of maps is jointly injective.

**Theorem 5.5 (Expressiveness).** If a graded trace logic is both depth-0 separating and depth-1 separating, then it is expressive.

**Proof (Sketch).** Induction on the depth, with depth-0 separation as the inductive base and depth-1 separation as the inductive step. $\square$
Remark 5.6. The inclusion of depth-0 separation is mostly for convenience of proofs; in reality there is nothing to separate at depth 0, since all states are equivalent at depth 0. Alternatively, one can use depth 1 as the induction base and show joint injectivity of modalities applied to truth constants at $M_1$; we encounter one case where this is in fact needed.

Example 5.7 (Logics for bisimilarity). We first consider two applications to logics for bisimilarity. We have already seen (Example 4.4) that standard coalgebraic modal logic, with the full set of Boolean connectives, is an instance of a graded trace logic, for the monad $M_nX = F^nX$ where $F$ is the coalgebraic type functor. In this case, depth-0 separation is vacuous, and for finitary $F$, depth-1 separation is equivalent to the usual notion of separation [26]. We thus recover the known coalgebraic Hennessy-Milner theorem [26], of which the standard Hennessy-Milner theorem [16] is a special case, as an instance of Theorem 5.5.

A well-known particular case is probabilistic bisimilarity on Markov chains, for which an expressive logic needs only probabilistic modalities $\lozenge_p$ with probability at least $p'$ and conjunction [12]. This result is also easily recovered as an instance of Theorem 5.5, using the same standard lemma from measure theory as in op. cit., which states that measures are uniquely determined by their values on a generating set of the underlying $\sigma$-algebra that is closed under finite intersections (corresponding to the set $\mathfrak{A}$ from Definition 5.4 being closed under conjunction).

Example 5.8 (Expressive trace logics on the linear time – branching time spectrum). We next extract graded trace logics from the graded monads for the linear time – branching time spectrum introduced in Section 3, and show how in each case, expressiveness is an instance of Theorem 5.5. Bisimilarity is already covered by the previous example. Depth-0 separation is almost always trivial and not mentioned further. Unless mentioned otherwise, all logics have disjunction, enabled by $M_0$ being powerset as discussed in the previous section. Most of the time, the logics are essentially already given by van Glabbeek (with the exception that we show that one can add disjunction) [36]; the emphasis is entirely on uniformization.

(1) Trace equivalence: As seen in Example 4.4, the graded trace logic for trace equivalence features diamond modalities $\lozenge_{\sigma}$ indexed over actions $\sigma \in A$. The ensuing proof of depth-1 separation uses canonicity of a given $M_1$-algebra $A$ only to obtain that the structure map $a^{10}$ is surjective. The other key point is that a jointly injective collection $\mathfrak{A}$ of $M_0$-homomorphisms $A_0 \to 2$, i.e. join preserving maps, has the stronger separation property that whenever $x \not\leq y$ then there exists $f \in \mathfrak{A}$ such that $f(x) = \top$ and $f(y) = \bot$.

(2) Simulation: In this case, the graded theory specifies that an $M_1$-algebra $A$ with carriers $A_0 = 2^n$, $A_1 = 2$ should provide for each action $\sigma \in A$ a monotone map $\sigma^A : 2^n \to 2$. Such maps are clearly closed under composition with conjunction on both sides, i.e. conjunction is admissible. We thus include (disjunction and) conjunction as propositional operators, and moreover use the same diamond modality as for trace equivalence. We obtain positive modal logic with only diamonds as a graded trace logic for simulation. It is well-known that expressive-
ness holds already for the fragment without disjunction (which is related to the lightweight description logic $\mathcal{EL}$ [1,2]). The proof of depth-1 separation is similar to the case of trace equivalence.

(3) Graded trace logics for the remaining cases are developed from the above by adding constants or additionally indexing modalities over sets of actions, with only little change to the proofs of depth-1 separation. For completed trace equivalence, we just add a 0-ary modality $\star$ indicating deadlock. For ready trace equivalence and ready simulation, we index the diamond modalities $\Diamond_\sigma$ with sets $I \subseteq \mathcal{A}$; formulas $\Diamond_\sigma I \varphi$ are then read ‘the current ready set is $I$, and there is an $a$-successor satisfying $\varphi$’. For failure trace equivalence we proceed in the same way but read the index $I$ as ‘$I$ is a failure set at the current state’. For readiness equivalence, we keep the modalities $\Diamond_\sigma$ unchanged and instead introduce 0-ary modalities $r_I$ indicating that $I$ is the ready set at the current state, thus ensuring that formulas do not continue after postulating a ready set. In this last case, the expressiveness proof needs to proceed according to Remark 5.6, since depth-0 separation in fact fails due to left-over markers in $M_0$.

6 Conclusion and Future Work

We have provided graded monads modelling a range of process equivalences on the linear time – branching time spectrum, presented in terms of carefully designed graded algebraic theories. From these graded monads, we have extracted characteristic modal logics for the respective equivalences systematically, following a paradigm of graded trace logics that grows out of a natural notion of graded algebra. Our main technical results concern the further development of the general framework for graded trace logics; in particular, we have introduced a first-class notion of propositional operator, and we have established a criterion for expressiveness of graded trace logics that simultaneously takes into account the expressive power of the modalities and that of the propositional base. Instances of this result include, for instance, the coalgebraic Hennessy-Milner theorem [26,30], Desharnais et al.’s expressiveness result for probabilistic modal logic with only conjunction [12], and expressiveness of positive modal logic with only diamonds for simulation equivalence. The emphasis in the examples has been on well-researched equivalences and logics for the basic case of labelled transition systems, aimed at demonstrating the versatility of graded monads and graded trace logics along the axis of granularity of system equivalence. The framework as a whole is however parametric also over the branching type of systems and in fact over the base category determining the structure of state spaces; an important direction for future research is therefore to capture equivalences and extract expressive trace logics on other system types such as probabilistic systems (see [5] for a comparison of some equivalences on probabilistic automata, which combine probabilities and non-determinism) and nominal systems, e.g. nominal automata [3,31]. Moreover, we plan to extend the framework of graded trace logics to cover also temporal logics, using graded algebras of unbounded depth.
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A Omitted Proofs and Details

A.1 Proof of Theorem 2.7

The mentioned applicable instances of the graded monad laws are the following: 

\((M_0, \eta, \mu^{00})\) is a monad, \((M_1, \mu^{10})\) a (right) \(M_0\)-module, i.e. the following diagrams commute:

\[
\begin{array}{ccc}
M_1 & \xrightarrow{\eta} & M_1M_0 \\
\downarrow & & \downarrow \mu^{10} \\
M_1 & & M_1M_0
\end{array}
\]

\[
\begin{array}{ccc}
M_1M_0M_0 & \xrightarrow{\mu^{10}M_0} & M_1M_0 \\
\downarrow & & \downarrow \mu^{10} \\
M_1M_0 & & M_1M_0
\end{array}
\]

\((M_1, \mu^{01})\) is a (left) \(M_0\)-module,

\[
\begin{array}{ccc}
M_1 & \xrightarrow{\eta} & M_0M_1 \\
\downarrow & & \downarrow \mu^{01} \\
M_1 & & M_1
\end{array}
\]

\[
\begin{array}{ccc}
M_0M_0M_1 & \xrightarrow{M_0\mu^{01}} & M_0M_1 \\
\downarrow & & \downarrow \mu^{01} \\
M_0M_1 & & M_0M_1
\end{array}
\]

and \(\mu^{10}\) is (componentwise) an \(M_0\)-algebra homomorphism, i.e. \(\mu^{10} \cdot \mu^{01} M_0 = \mu^{01} M \mu^{10}\):

\[
\begin{array}{ccc}
M_0M_1M_0 & \xrightarrow{M_0\mu^{10}} & M_0M_1 \\
\downarrow & & \downarrow \mu^{01} \\
M_1M_0 & & M_1M_0
\end{array}
\]

\[
\begin{array}{ccc}
M_1M_0 & \xrightarrow{\mu^{10}} & M_1M_0 \\
\downarrow & & \downarrow \mu^{01} \\
M_1 & & M_1
\end{array}
\]

Proof. Note first that \(\mu^{10}\) is a coequalizer in \(\text{Set}^{M_0}\), in fact, a split coequalizer

\[
\begin{array}{ccc}
M_1M_0M_0 & \xrightarrow{\mu^{10}M_0} & M_1M_0 \\
\downarrow & & \downarrow M_1\eta \\
M_1M_0 & & M_1M_0
\end{array}
\]

For every \(k \geq 1\), we now define \(M_k, \mu^{0,k+1} : M_0M_{k+1} \to M_{k+1}\) and \(\mu^{1k} : M_1M_k \to M_{k+1}\) inductively by taking the coequalizer

\[
\begin{array}{ccc}
M_1M_0M_k & \xrightarrow{M_1\mu^{0k}} & M_1M_k \\
\downarrow & & \downarrow M_1\eta \\
M_1M_k & & M_1M_k
\end{array}
\]

\[
\begin{array}{ccc}
M_1M_k & \xrightarrow{\mu^{1k}} & M_{k+1} \\
\downarrow & & \downarrow \mu^{10} \\
M_1M_k & & M_1M_k
\end{array}
\]

(objectwise) in \(\text{Set}^{M_0}\); here we use that \(M_1\) preserves \(M_0\)-algebras, in particular \(\mu^{0k}\), and \(\mu^{0,k+1}\) is given by the \(M_0\)-algebra structures of \(M_{k+1}X\) for any \(X\). Thus, we have

\[
\begin{array}{ccc}
M_{k+1} & \xrightarrow{\eta M_{k+1}} & M_0M_{k+1} \\
\downarrow & & \downarrow \mu^{0,k+1} \\
M_{k+1} & & M_{k+1}
\end{array}
\]

\[
\begin{array}{ccc}
M_0M_0M_{k+1} & \xrightarrow{M_0\mu^{0,k+1}} & M_0M_{k+1} \\
\downarrow & & \downarrow \mu^{0,k+1} \\
M_0M_{k+1} & & M_0M_{k+1}
\end{array}
\]

\[
\begin{array}{ccc}
M_0M_{k+1} & \xrightarrow{\mu^{0,k+1}} & M_{k+1} \\
\downarrow & & \downarrow \mu^{0,k+1} \\
M_0M_{k+1} & & M_0M_{k+1}
\end{array}
\]
Note that the left-hand triangle above shows the unit laws for the $\mu^{0k}$. The remaining $\mu^{nk}$ for $n, k \in \omega$ are again defined inductively, this time over $n$. In the induction step one uses the universal property of the coequalizer $\mu^{1n}M_k$ and that $\mu^{nk}$ is an $M_0$-algebra homomorphism:

$$
\begin{align*}
M_1M_0M_nM_k & \xrightarrow{M_1M_0\mu^{0n}M_k} M_1M_nM_k \xrightarrow{\mu^{1n}M_k} M_{n+1}M_k \\
M_1M_0 M_{n+k} & \xrightarrow{M_1M_0 \mu^{0,n+k}} M_1M_{n+k} \xrightarrow{\mu^{1,n+k}} M_{n+k+1}
\end{align*}
$$

(A.1)

Indeed, since the two left-hand squares commute, we obtain a unique morphism on the right-hand edge making the right-hand square commutative. We verify the necessary properties of $\mu^{n+1,k}$

(1) For $k = 0$ we have $\mu^{n+1,0} \cdot M_{n+1} \eta = \text{id}$, using that $\mu^{1n}$ is an epimorphism:

$$
\begin{align*}
M_1M_n & \xrightarrow{\mu^{1n}} M_{n+1} \\
M_1M_nM_0 & \xrightarrow{\mu^{1n}M_0} M_{n+1}M_0 \\
M_1M_n & \xrightarrow{\mu^{1n}} M_{n+1}
\end{align*}
$$

Indeed, the lower square commutes by the definition of $\mu^{n+1,0}$ and the upper one by naturality. Since the left-hand edge is the identity by induction hypothesis, and $\mu^{1n}$ is an epimorphism, the right-hand edge is the identity.

(2) The remaining associativity laws for $\mu^{nk}$ are proved as follows.

(a) For $\mu^{1k}$, $k \geq 2$, we need to show for every $m, l$ with $m + l = k$ that the following diagram commutes:

$$
\begin{align*}
M_0M_1M_m & \xrightarrow{M_0\mu^m} M_0M_k \\
M_1M_m & \xrightarrow{\mu^m} M_k
\end{align*}
$$

(A.2)

This holds since $\mu^m$ is obtained by the universal property of a coequalizer in $M_0$-algebras, hence it is an $M_0$-algebra morphism.

(b) Now we show for every $k$ and $n \geq 1$ that for every $l, m$ with $l + m = n$ the following diagram commutes:

$$
\begin{align*}
M_lM_mM_k & \xrightarrow{M_l\mu^mM_k} M_nM_k \\
M_lM_mM_{n+k} & \xrightarrow{M_l\mu^{l,m+k}} M_{n+k}
\end{align*}
$$
For \( l = 0 \) this holds since \( M = n \) and \( \mu^{nk} \) is an \( M_0 \)-algebra homomorphism. For \( l \geq 1 \), consider the following diagram:

\[
\begin{array}{c}
M_1 M_{l-1} M_m M_k & \rightarrow & M_1 M_{n-1} M_k \\
\downarrow & \mu^{1,l-1} M_m M_k & \downarrow \mu^{1,n-1} M_k \\
M_1 M_{m+k} & \rightarrow & M_1 M_{n+k}
\end{array}
\]

Its left-hand part commutes by naturality of \( \mu^{1,l-1} \), the upper, right-hand and lower parts commute by definition of \( \mu^{n+1,k} \) (Diagram A.1) or by Diagram A.1 in case \( l, n \) and \( l \) respectively, are equal to 1. Now proceed by induction on \( n \): For \( n = 1 \) we have \( l = 1 \) and \( m = 0 \); thus the outside commutes by the same argument as above, Diagram A.2. In the induction step, the outside commutes by induction hypothesis. Thus, the desired inside commutes when precomposed by the epimorphism \( \mu^{1,l-1} M_m M_k \), thus it commutes.

(c) It remains to prove for every \( k \) and \( n \geq 1 \) that for \( l,m \) with \( l + m = k \) the following diagram commutes:

\[
\begin{array}{c}
M_n M_l M_m \rightarrow M_n M_k \\
\downarrow \mu^{n,m} M_m & \downarrow \mu^{n,k} \\
M_{n+1} M_m & \rightarrow M_{n+k}
\end{array}
\]

This is done analogously, i.e. by induction on \( n \):

\[
\begin{array}{c}
M_1 M_{n-1} M_l M_m \rightarrow M_1 M_{n-1} M_k \\
\downarrow & \mu^{1,n-1} M_l M_m & \downarrow \mu^{1,n-1} M_k \\
M_1 M_{n+1} M_m & \rightarrow M_1 M_{n+k}
\end{array}
\]

Note that the upper part commutes for every \( n \) by naturality of \( \mu^{1,n-1} \). For \( n = 1 \), the left-hand square and right-hand square commutes by Diagram A.1,
and the lower square by A.1. The outside commutes because $\mu^{1n}$ is an $M_0$-algebra homomorphism, thus the desired inner square commutes (when precomposed by the epimorphism $\mu^{1n-1}M_lM_m$). For the induction step, the left-hand, right-hand and lower squares commute by the defining square (A.1) and the outside by induction hypothesis. Again, the desired inner square commutes since $\mu^{1n-1}M_lM_m$ is an epimorphism.

\[\square\]

A.2 Details for Example 2.6

**Proposition A.1.** Let $(M, \eta, \mu)$ be a monad and $F$ an endofunctor on $C$. Then the graded monad $(M_n)_{n \in \mathbb{N}}$ of Example 2.3(2) is depth-1.

**Proof.** Recall first that a *split coequalizer* in some category is a diagram of the form

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{s} \\
\downarrow{t} & & C \\
\end{array}
\]

satisfying the equations

\[
\begin{align*}
\begin{aligned}
c \cdot f &= c \cdot g \\
c \cdot s &= \text{id} \\
f \cdot t &= \text{id} \\
g \cdot t &= s \cdot c
\end{aligned}
\]  
\tag{A.1}
\]

It then follows that $c$ is indeed an (absolute) coequalizer of $f$ and $g$.

We will prove that (2.1) is componentwise a split coequalizer in the category of $M_0$-algebras. Note first that here we have

\[
\begin{align*}
\mu^{1n} &= (MFMF^n \xrightarrow{M\lambda F^n} MFMF^{n+1} \xrightarrow{\mu F^{n+1}} MF^{n+1}) \\
M_1\mu^{0n} &= (MFMMF^n \xrightarrow{MF\mu F^n} MFMF^n) \\
\mu^{10}M_n &= (MFMMF^n \xrightarrow{M\lambda MF^n} MMFMF^n \xrightarrow{\mu F MF^n} MFMF^n)
\end{align*}
\]

Note that all of these are componentwise $M_0$-algebra homomorphisms. Furthermore, so are the desired splittings $s$ and $t$, which are given by

\[
\begin{align*}
MF\eta F^n \colon MF^{n+1} &= MFF^n \xrightarrow{MF\eta F^n} MFMF^n \\
MFM\eta F^n \colon MFMF^n &= MFMMF^n
\end{align*}
\]

We readily verify the four desired equations above. The first one holds by the laws of the graded monad $M_n$. For the second one use the unit laws of the distributive law and the monad $M$:

\[
\begin{array}{ccc}
MFMF^n & \xrightarrow{M\lambda F^n} & MFMF^{n+1} \\
\downarrow{\mu^{1n}} & & \downarrow{\mu^{1n+1}} \\
MFMF^n & \xrightarrow{M\eta F^n} & MF^n \\
\end{array}
\]
The third equation again follows from one of the unit laws of the monad \( M \): just apply \( MF \) from the left and \( F^n \) from the right on both sides of \( \mu \cdot M\eta = \text{id}_M \).

Finally, the last desired equation follows from naturality

\[
\begin{align*}
MFMMF^n &\xrightarrow{\mu F^n} MFMF^n \\
MFMMF^n &\xrightarrow{\mu F^n} MFMF^n \\
MFMMF^n &\xrightarrow{\mu^n} MFMF^n
\end{align*}
\]

This completes the proof. \( \square \)

A related result concerns distributive laws of a monad over an endofunctor (so-called EM-laws). Let \((M, \eta, \mu)\) be a monad and \( F \) an endofunctor on \( \mathbf{C} \) equipped with a distributive law \( \lambda : MF \to FM \), i.e. we have

\[
\lambda \cdot \eta F = F \eta \quad \text{and} \quad \lambda \cdot \mu F = F \mu \cdot \lambda M \cdot M \lambda.
\]

Then, as shown in [25], we obtain a graded monad as follows: define \( \lambda^n : MF^n \to F^nM \) inductively by \( \lambda^0 = \text{id}_M \) and

\[
\lambda^{n+1} = (MF^{n+1} = MF^nF \xrightarrow{\lambda^n F} F^nMF \xrightarrow{F^n \lambda} F^n FM = F^{n+1} M);
\]

then we obtain a graded monad with \( M_n = F^nM \), unit \( \eta \) and multiplication given by

\[
\mu^{nk} = (F^n MF^k M \xrightarrow{F^n \lambda^k M} F^{n+k} MM \xrightarrow{F^{n+1} \mu} F^{n+k} M).
\]

**Proposition A.2.** The above graded monad is depth \( 1 \).

**Proof.** The proof is similar to the one of Proposition A.1. This time we show that (2.1) is componentwise the coequalizer of a \( U \)-split pair, where \( U : \mathbf{Set}^{M_0} \to \mathbf{Set} \) is the forgetful functor, i.e. the two splittings \( s \) and \( t \) are not required to be \( M_0 \)-algebra homomorphisms. By Beck’s theorem (see e.g. [24]), \( U \) creates coequalizers of \( U \)-split pairs. Thus, it suffices to verify the equations in (A.1) in \( \mathbf{Set} \). The calculations are completely analogous to what we have seen in the proof of Proposition A.1 using (the components of)

\[
F\eta F^n M : F^{n+1} M = FF^n M \to FFM^n M,
\]
\[
F\eta M F^n M : FMF^n M \to FMMF^n M
\]
as \( s \) and \( t \), respectively. We leave this as an easy exercise for the reader. \( \square \)
A.3 Details for Section 3

To prove that a given description \((M_n)_{n<\omega}\) of the graded monad generated by a given graded theory \(T\) is correct, we generally proceed as follows.

- We identify a notion of normal form of terms of a given depth \(n\), and show that every depth-\(n\) term can be brought into this form, by a fixed normalization procedure.

- It will then typically be easy to see that depth-\(n\) normal forms of terms over variables from a set \(X\) are in bijection with the claimed description \(M_nX\); the interpretation of the operations of \(T\) over normal forms (by term formation and subsequent normalization) then transfers to \(M_nX\) along this bijection. (This step also determines the multiplication, whose explicit description we otherwise mostly elide.)

- We finally show that under this interpretation of the operations of \(T\), the \(M_nX\) form a graded algebra for \(T\), i.e. satisfy its equations. This proves that \((M_n)_{n<\omega}\) is indeed the graded monad generated by \(T\).

The description \((M_n)_{n<\omega}\) of the graded monad then usually makes it immediate that the associated graded semantics captures the process equivalence at hand, in that the graded monad contains precisely the data in the original semantics. In some cases we need to introduce modifications of the original semantics that we prove to induce the same notion of process equivalence, notably for ready and failure traces and for readiness semantics.

Completed Trace Semantics Two states \(x, y\) in labelled transition systems are completed trace equivalent if \(\mathcal{CT}(x) = \mathcal{CT}(y)\) where \(\mathcal{CT}(x)\) is defined as follows for a state \(x\) in an LTS with carrier \(X\) [36]:

\[
\mathcal{CT}(x) = \left\{ w \in \mathcal{A}^* \mid \exists z \in X. x \xrightarrow{w} z \right\} \\
\cup \left\{ w^* \in \mathcal{A}^* \times \{\star\} \mid \exists z \in X. x \xrightarrow{w} z \land I(z) = \emptyset \right\}.
\]

An element of \(\mathcal{CT}(x)\) is thus either a trace or a completed trace, the latter being recognizable by the marker \(\star\).

Normal forms in the graded theory for completed traces as defined in Section 3 are described as follows: A depth-0 term is normalized by just collapsing nested joins into a set, identifying the term with the set of elements that occur in it (in the sequel, we will mostly keep normalization of joins into sets implicit). At depth \(n+1\), actions are distributed over the joins of depth-\(n\) terms (normalized by induction to contain joins only at the top level). We end up with normal forms of depth-\(n+1\) terms being joins of terms consisting either of \(n+1\) unary operations of the form \(\sigma\) for \(\sigma \in \mathcal{A}\), applied to a variable, or of at most \(n\) operations of the form \(\sigma\), applied to the constant \(\star\); terms of the latter kind represent completed traces, those of the former kind represent standard traces.

Clearly, these normal forms are in bijection with our claimed description \(M_nX = \mathcal{P}(\mathcal{A}^n \times X + \mathcal{A}^{<n} \times \{\star\}) \cong \mathcal{P}(\mathcal{A}^n \times X + \mathcal{A}^{<n})\). The arising interpretation
of the operations of the theory is as follows: Joins are interpreted by set union; operations $\sigma$ for $\sigma \in \mathcal{A}$ by prefixing all words in a set with $\sigma$; and the constant $\star$ by the set $\{\star\}$. In our standard programme outlined above, it remains to prove that this graded algebraic structure on the $\mathcal{M}_n X$ satisfies the equations of the graded theory. The join equations are inherited from powerset, and distributivity of the unary operations $\sigma$ over joins is clear by the description of the interpretation of $\sigma$ just given. This proves that the graded monad generated by the graded theory of completed traces is indeed $\mathcal{M}_n X = \mathcal{P}(\mathcal{A}^n \times X + \mathcal{A}^\leq n \times \{\star\})$, and thus represents completed traces. Formally, it remains to be shown that the graded semantics induced by $\alpha$ as described in Section 3 does indeed compute the completed trace semantics of a state (and not some other set of traces and completed traces). We carry this proof out explicitly for the case at hand and elide it in the sequel.

**Proposition A.3.** The $n$-th stage $\mathcal{M}_n ! \gamma^{(n)}(x)$ of the $\alpha$-trace semantics of a state $x$ in an LTS $(X, \gamma)$ contains exactly the length-$n$ traces of $x$ and the completed traces of $x$ of length less than $n$.

(Consequently, $\alpha$-trace equivalence coincides with completed trace equivalence.)

**Proof.** We show the stronger statement that $\gamma^{(n)}(x)$ contains exactly the length-$n$ pretraces of $x$ and the completed traces of $x$ of length less than $n$. We proceed by induction over $n$. For $n = 0$, $x$ has exactly one length-$n$ pretrace, namely $(\epsilon, x)$ (and, of course, no completed trace of length less than $n$). On the other hand, we also have $\gamma^{(0)}(x) = \eta_X(x) = \{(\epsilon, x)\}$. In the inductive step from $n$ to $n + 1$, if $\gamma(x) = \emptyset$, then $x$ has no traces of depth $n + 1$ and exactly one completed trace of depth $< n$, written $\star$. On the other hand, $\gamma^{n+1}(x) = \mu^{1n} \cdot M_1 \cdot \gamma^{(n)} \cdot \alpha_X \cdot \gamma(x)$ arises, in the algebraic view, by substituting into the term $\star$ that represents $\alpha \cdot \gamma(x) = \alpha(\emptyset) = \{\star\}$; since $\star$ is a constant, it is left unchanged by substitutions, so that $\gamma^{(n+1)}(x) = \{\star\}$ as required. If $\gamma(x) \neq \emptyset$, then $\alpha(\gamma(x)) = \mathcal{P} \text{inl}(\gamma(x))$ is represented in algebraic notation as

$$\alpha_X(\gamma(x)) = \sum_{(\sigma, x) \in \gamma(x)} \sigma(x).$$

By the description of the interpretation of the algebraic operations in $\mathcal{M}$, we thus have

$$\gamma^{(n+1)}(x) = \mu^{1n} \cdot M_1 \gamma^{(n)} \cdot \alpha_X \cdot \gamma(x) = \{(\sigma w, z) \mid (\sigma, y) \in \gamma(x), (w, z) \in \gamma^{(n)}(y)\},$$

whence the claim is immediate by the inductive hypothesis. □

**Ready and failure traces** The set $\mathcal{R}T(x)$ of ready traces in normal form of a state $x$ of an LTS with carrier $X$ is defined as

$$\mathcal{R}T(x) = \{A_0 \sigma_1 A_1 \ldots \sigma_n A_n \in (\mathcal{P} \mathcal{A} \times \mathcal{A})^* \times \mathcal{P} \mathcal{A} \mid \exists x_0, \ldots, x_n \in X. x = x_0 \xrightarrow{\sigma_1} x_1 \ldots \xrightarrow{\sigma_n} x_n \land \forall i \leq n. A_i = I(x_i)\}.$$
States \( x, y \) are ready trace equivalent if \( \mathcal{RT}(x) = \mathcal{RT}(y) \).

As indicated above, we work with an equivalent variant of this semantics:

**Lemma A.4.** Ready trace equivalence coincides with the equivalence defined by assigning to a state \( x \) of an LTS with carrier \( X \) the set

\[
\mathcal{RT}'(x) = \{ A_1 \sigma_1 \ldots A_n \sigma_n \in (\mathcal{PA} \times A)^* \mid \exists x_0, \ldots, x_n \in X. \\
x = x_0 \xrightarrow{\sigma_1} x_1 \ldots x_{n-1} \xrightarrow{\sigma_n} x_n \land \forall i < n. A_i = I(x_i) \} \cup \\
\{ A_1 \sigma_1 \ldots A_n \sigma_n \ast \in (\mathcal{PA} \times A)^* \times 1 \mid \exists x_0, \ldots, x_n \in X. \\
x = x_0 \xrightarrow{\sigma_1} x_1 \ldots x_{n-1} \xrightarrow{\sigma_n} x_n \land \forall i < n. I(x_i) = A_i \land I(x_n) = \emptyset \}.
\]

**Proof.** All the data in \( \mathcal{RT}'(x) \) can clearly be calculated from \( \mathcal{RT}(x) \). Conversely, given \( \mathcal{RT}'(x) \), a ready trace of \( x \) has the form \( wA \) where \( w \in \mathcal{RT}'(x) \). If \( A = \emptyset \), then \( w \ast \in \mathcal{RT}'(x) \), so the information about \( wA \) is contained in \( \mathcal{RT}'(x) \). Otherwise pick \( a \in A \); then \( wAa \in \mathcal{RT}'(x) \), so again the information about \( wA \) is in \( \mathcal{RT}'(x) \). This proves the claim. \( \Box \)

The set \( \mathcal{FT}(x) \) of failure traces in normal form of a state \( x \) of an LTS with carrier \( X \) is defined as

\[
\mathcal{FT}(x) = \{ A_0 \sigma_1 A_1 \ldots \sigma_n A_n \in (\mathcal{PA} \times A)^* \times \mathcal{PA} \mid \\
\exists x_0, \ldots, x_n \in X. x = x_0 \xrightarrow{\sigma_1} x_1 \ldots \xrightarrow{\sigma_n} x_n \land \forall i \leq n. A_i \cap I(x_i) = \emptyset \}
\]

Two states \( x, y \) are failure trace equivalent if \( \mathcal{FT}(x) = \mathcal{FT}(y) \).

**Lemma A.5.** Failure trace equivalence coincides with the equivalence defined by assigning to a state \( x \) of an LTS with carrier \( X \) the set

\[
\mathcal{FT}'(x) = \{ A_1 \sigma_1 \ldots A_n \sigma_n \in (\mathcal{PA} \times A)^* \mid \exists x_0, \ldots, x_n \in X. \\
x = x_0 \xrightarrow{\sigma_1} x_1 \ldots x_{n-1} \xrightarrow{\sigma_n} x_n \land \forall i < n. A_i \land I(x_i) = \emptyset \} \cup \\
\{ A_1 \sigma_1 \ldots A_n \sigma_n \ast \in (\mathcal{PA} \times A)^* \times 1 \mid \exists x_0, \ldots, x_n \in X. \\
x = x_0 \xrightarrow{\sigma_1} x_1 \ldots x_{n-1} \xrightarrow{\sigma_n} x_n \land \\
\forall i < n. I(x_i) \cap A_i = \emptyset \land I(x_n) = \emptyset \}
\]

**Proof.** All data in \( \mathcal{FT}'(x) \) can clearly be calculated from \( \mathcal{FT}(x) \) (noting that a state \( z \) is a deadlock iff \( A \cap I(z) = \emptyset \)). Conversely, given \( \mathcal{FT}'(x) \) a failure trace in \( \mathcal{FT}(x) \) has the form \( wA \) where \( w \in \mathcal{FT}'(x) \), and then there exists \( z \) such that \( x \xrightarrow{w} z \) and \( I(z) \cap A = \emptyset \) (where \( x \xrightarrow{B} x \) whenever \( I(x) \cap B = \emptyset \) for \( B \subseteq A \)). If \( I(z) = \emptyset \), then \( w \ast \in \mathcal{FT}'(x) \), and from this knowledge \( wA \in \mathcal{FT}(x) \) is deducible. Otherwise, pick \( a \in I(z) \). Then \( wAa \in \mathcal{FT}'(x) \), from which knowledge \( wA \in \mathcal{FT}(x) \) is deducible. This proves the claim. \( \Box \)

**Proposition A.6.** States in labelled transition systems are ready (failure) trace equivalent iff they have the same graded trace sequence for the given graded monads and graded trace semantics.
Proof. For ready trace semantics, the claim is, by Lemma A.4, shown in exactly the same way as for completed traces.

For failure traces, one similarly uses the alternative failure trace semantics \( \mathcal{FT}' \) of Lemma A.5. We note that the graded monad induced by the graded theory defined in Section 3 is described as follows. We order \( A \) by equality and \( \mathcal{P}A \) by set inclusion, and equip \( (\mathcal{P}A \times A)^n \) with the product ordering; on \( (\mathcal{P}A \times A)^{< \infty} = \sum_{i<n} (\mathcal{P}A \times A)^i \), we use the coproduct ordering (in particular, words are comparable only if they have the same length). Then, \( M_nX \) consists of the downclosed subsets of \( (\mathcal{P}A \times A)^n \times X + (\mathcal{P}A \times A)^{< \infty} \). One shows by induction over \( n \) that \( A_1 \sigma_1 \ldots A_n \sigma_n \in \mathcal{FT}'(x) \) iff \( (A_1 \sigma_1 \ldots A_n \sigma_n, \bullet) \in M_n! \cdot \gamma^{(n)}(x) \) where now \( \bullet \) denotes the unique element of 1 (in this case indicating absence of information rather than deadlock), and, for \( i < n \), \( A_1 \sigma_1 \ldots A_i \sigma_i \in \mathcal{FT}'(x) \) iff \( A_1 \sigma_1 \ldots A_i \sigma_i \in M_n! \cdot \gamma^{(n)}(x) \); again, details are like for completed traces. This proves the claim. \( \square \)

Readiness The set of ready pairs of a state \( x \) of an LTS with carrier \( X \) is defined as

\[
\mathcal{R}(x) = \{(w, A) \in A^* \times \mathcal{P}A \mid \exists z \in X . x \xrightarrow{w} z \land A = I(z)\}
\]

Again, we introduce an equivalent variant:

**Lemma A.7.** Readiness equivalence coincides with the equivalence defined by assigning to a state \( x \) of an LTS with carrier \( X \) the set

\[
\mathcal{R}'(x) = \mathcal{T}(x) \cup \{(w, A, a) \in A^* \times \mathcal{P}A \times (A + 1) \mid \exists z \in X . x \xrightarrow{w} z

\land A = I(z) \land (a \in I(z) \lor (I(z) = \emptyset \land a \in 1))\}
\]

Proof. Since every ready trace \( (w, A) \in \mathcal{R}(x) \) extends to \( (w, A, a) \in \mathcal{R}'(x) \) either by picking some \( a \in A \) or, in case \( A = \emptyset \), by taking \( a \in 1 \), we obtain \( \mathcal{R}(x) \) from \( \mathcal{R}(x') \) by just dropping the last component from the triples in \( \mathcal{R}(x') \).

Conversely, for \( (w, A, a) \in A^* \times \mathcal{P}A \times (A + 1) \) we have \( (w, A, a) \in \mathcal{R}'(x) \) iff \( (w, A) \in \mathcal{R}(x) \) and either \( a \in A \) or \( A = \emptyset \) and \( a \in 1 \); that is, we can also compute \( \mathcal{R}'(x) \) from \( \mathcal{R}(x) \), which proves the claim. \( \square \)

We refer to \( \mathcal{R}' \) as the alternative readiness semantics, and to elements \( (w, A, a) \in \mathcal{R}'(x) \) as ready triples of \( x \), with length \( \|w\| + 1 \).

We claim that the graded monad generated by the graded theory of readiness has the form

\[
M_nX = \mathcal{P}(X + \{\emptyset\} \times X)
\]

\[
M_{n+1}(X) = \mathcal{P}(\mathcal{A}^n(A + (\mathcal{P}A \times A)) \times X + A^\leq n \times \{\star\}).
\]

As per our standard programme, we will prove this by introducing a notion of normal form. In view of Lemma A.7, this description shows that the graded equivalence induced by our graded theory is exactly readiness equivalence: It shows that \( M_{n+1} \) contains sets consisting of traces and ready triples of length \( n + 1 \) and of complete traces of length at most \( n \); for a given state in an LTS,
these data are built inductively in the correct way by construction of the graded semantics α.

Normal forms of terms over variables from X in the graded theory of readiness are described as follows. Depth-0 terms are normalized by distributing the marker operation o over joins and eliminating any nesting of joins and of the marker o. Normal forms at depth 0 are thus just joins consisting of variables x ∈ X and marked variables o(x), and thus are in bijection with M₀X as defined above. Depth-n + 1 terms are normalized by first distributing operations ⟨A, σ⟩, σ and o over the top-level joins of (by induction, normalized) depth-n terms. We end up with a join of terms containing only unary operations and either a variable or the constant *. Such a term is then further normalized by replacing any occurrence of an operation ⟨A, σ⟩ with σ unless ⟨A, σ⟩ is applied directly to a depth-0 term – this is supported by the theory because any argument t of ⟨A, σ⟩ with t of positive depth will satisfy t = o(t). Moreover, we subsequently remove any occurrences of o using the equations o(op(x)) = op(x) (for unary operations op) and σ(o(x)) = σ(x), where the latter equation takes care of occurrences of o at depth 0.

Summing up, a normal form of positive depth n + 1 is a join of sequences consisting either of n σ-operations applied to a depth-1 term that is either a σ-operation applied to a variable or an ⟨A, σ⟩-operation applied to a variable, or of at most n σ-operations applied to the constant *. Such normal forms are in bijection with Mₙ₊₁X as defined above. We are done once we show that the sets MₙX form a graded algebra for the graded theory of readiness, where according to the above description of normalization we interpret join as union; σ ∈ A as prefixing all words in a set with σ, removing possible occurrences of o (found only in front of elements x ∈ X, i.e. σ(o(x)) = σx); ⟨A, σ⟩ as prefixing variables (i.e. elements of X) with ⟨A, σ⟩, and all other words in a set with σ, removing possible occurrences of o (in particular ⟨A, σ⟩(o(x)) = σx); the marker o as prefixing variables with o and leaving all other words unchanged; and * as {∗}. All equations of the graded theory are readily checked, the only case of mild interest being the equation ⟨A, σ⟩(o(y)) = σ(o(y)); but both ⟨A, σ⟩(o(−)) and σ(o(−)) act on a set by prefixing all words in the set with σ.

Simulation The description of the graded monad is seen as follows. We define a normal form for depth-n terms inductively. The description of M₀ is standard, and arises by collapsing nested joins into sets. A depth-(n + 1) term normalizes, in the same way, to a set of elements of A × MₙX (with depth-n terms normalized to elements of MₙX by induction). Monotonicity of the σ ∈ A implies that we can normalize such sets to be downclosed under the product ordering on A × MₙX; indeed, given a term ∨ᵢ∈I σᵢ(xᵢ) in Mₙ₊₁X (corresponding to the set {⟨σᵢ, xᵢ⟩ | i ∈ I}, we first expand the join by adding for every i ∈ I a copy of the summand σᵢ(xᵢ) for every y ≤ xᵢ, and then use monotonicity and the (infinitary) congruence rule in the middle step below:

\[ \bigvee_{i \in I} \sigma_i(x_i) = \bigvee_{i \in I, y \leq x_i} \sigma(x_i) = \bigvee_{i \in I, y \leq x_i} \sigma_i(y) = \bigvee_{i \in I, y \leq x_i} \sigma_i(y), \]
which corresponds to a downward closed subset of $A \times M_n X$.

We arrive at normal forms that are in bijection with the claimed description of $M_{n+1} X$. The induced interpretations of the operations of the graded theory are as follows: Unary operations $\sigma$ for $\sigma \in A$ map $t \in M_n X$ to the downclosure of $\{(\sigma, t)\}$; and joins act as set unions. We are done once we show that under this interpretation, the $M_n X$ form a graded algebra, i.e. satisfy the equations of the graded theory. This is clear for the complete join semilattices; the monotonicity equation is ensured by the formation of downclosures in the interpretation of $\sigma \in A$.

For characterization of simulation by the graded monad, we prove the stronger claim that given states $x$ and $y$ in $F$-coalgebras $(X, \gamma)$ and $(Y, \delta)$, respectively, $y$ simulates $x$ up to depth $n$ iff $M_i! \cdot \gamma(i)(x) \leq M_i! \cdot \delta(i)(y)$ for all $i \leq n$ (a condition that is automatic for $i = 0$). We proceed by induction over $n$, with trivial base case $n = 0$. For the inductive step, we note that $y$ simulates $x$ up to depth $n + 1$ iff for each $(\sigma, x') \in \gamma(x)$ there exists $(\sigma, y') \in \delta(y)$ such that $y'$ simulates $x'$ up to depth $n$, and by induction, the latter is equivalent to $M_i! \cdot \gamma(i)(x') \leq M_i! \cdot \delta(i)(y')$ for all $i \leq n$. Thus, $y$ simulates $x$ up to depth $n + 1$ iff for each $i \leq n$, the set $\{(\sigma, M_i! \cdot \gamma(i)(x')) | (\sigma, x') \in \gamma(x)\}$ is contained in the downset of $\{(\sigma, M_i! \cdot \delta(i)(y')) | (\sigma, y') \in \delta(y)\}$ (w.r.t. the product ordering on $A \times M_1$). Since for $0 < j \leq n + 1$, we can express $M_j! \cdot \gamma(j)(x)$ in terms of the graded theory as

$$\bigvee_{(\sigma, x') \in \gamma(x)} \sigma(M_{j-1}! \cdot \gamma(j-1)(x')),$$

correspondingly for $y$, this condition is, by the definition of the ordering on $M_1$, equivalent to $M_j! \cdot \gamma(j)(x) \leq M_j! \cdot \delta(j)(y)$ for $j \leq n + 1$, as desired.

A.4 Proof of Lemma 4.3

Proof. The lemma is immediate from the following more general statements. Let $A$ be an $M_1$-algebra; then the following hold:

1. For each $M_0$-algebra $B$ and each $M_0$-homomorphism $h : A_1 \rightarrow B$, taking $\tilde{a}^{0,0} = a^{0,0}$, $\tilde{a}^{0,1} = b^{0,0}$, $\tilde{a}^{1,0} = h \cdot a^{1,0}$ defines an $M_1$-algebra $\tilde{A}$.

2. For each $M_0$-algebra $B$ and each $M_0$-homomorphism $h : B \rightarrow A_0$, taking $\tilde{a}^{0,1} = b^{0,1}$, $\tilde{a}^{0,1} = a^{0,1}$, $\tilde{a}^{1,0} = a^{1,0}$, $M_1 h$ defines an $M_1$-algebra $\tilde{A}$.

These statements are proved by straightforward diagram chasing. In each case, the two conditions to check are coequalization, i.e. $\tilde{a}^{1,0}$ coequalizes $\mu^{0,1}$ and $M_1 \tilde{a}^{0,1}$, and $0$-homomorphy, i.e. $\tilde{a}^{1,0}$ is an $M_0$-homomorphism from $\mu^{0,1}$ to $\tilde{a}^{0,1}$:

$$\begin{array}{ccc}
M_1 M_0 A_0 & \xrightarrow{M_1 \tilde{a}^{0,0}} & M_1 A_0 \\
\mu^{1,0} & \downarrow \quad & M_0 M_1 A_0 \\
M_1 A_0 & \rightarrow & A_1 \\
\tilde{a}^{1,0} \quad & & M_0 \tilde{a}^{1,0} \quad & \downarrow \quad \mu^{0,1} \quad & \tilde{a}^{1,0} \rightarrow A_1
\end{array}$$

(1) It is clear that postcomposition with $M_0$-homomorphisms preserves $0$-homomorphy, and coequalization is even preserved by postcomposition with all maps.
(2) For any \( h \), \( M_1h \) is an \( M_0 \)-homomorphism by naturality of \( \mu^{0,1} \), so precomposition with \( M_1h \) preserves \( M_0 \)-homomorphy. It remains to show coequalization, i.e. commutation of the outer hexagon in

![Diagram](image)

(recall that \( \tilde{A}_0 = B_0 \) and \( \tilde{a}^{0,0} = b^{0,0} \)). The lower left square in the diagram commutes because \( h \) is an \( M_0 \)-homomorphism; the lower right square commutes because \( A \) is an \( M_1 \)-algebra; and the upper square commutes because \( \mu^{1,0} \) is natural; so the whole diagram commutes, including the outer hexagon. \( \square \)

### A.5 Proof of Lemma 5.2

Regarding the reflexivity comment, note that \( \mu^{10} \cdot M_1\eta_{A_0} = \text{id}_{M_1A_0} = M_1a^{00} \cdot M_1\eta_{A_0} \) (independently of canonicity).

‘If’: Let \( B \) be an \( M_1 \)-algebra, and let \( f : A_0 \to B_0 \) be a morphism of \( M_0 \)-algebras. We have to show that \( f \) extends uniquely to an \( M_1 \)-algebra morphism \( A \to B \). We have \( M_0 \)-algebra morphisms

\[
M_1A_0 \xrightarrow{M_1f} M_1B_0 \xrightarrow{b^{10}} B_1
\]

whose composite \( b^{10} \cdot M_1f \) coequalizes \( \mu^{10} \) and \( M_1a^{00} \):

\[
\begin{align*}
\mu^{10} \cdot M_1f \mu^{10} &= b^{10} \cdot \mu^{10} \cdot M_1M_0f \\
&= b^{10} \cdot M_0b^{00} \cdot M_1M_0f \\
&= b^{10} \cdot M_1f \cdot M_1a^{00}
\end{align*}
\]

By the coequalizer property, we thus obtain an \( M_0 \)-morphism \( f^\sharp : A_1 \to B_1 \) such that

\[
\begin{array}{ccc}
M_1A_0 & \xrightarrow{M_1f} & M_1B_0 \\
\downarrow \mu^{10} & & \downarrow b^{10} \\
A_1 & \xrightarrow{f^\sharp} & B_1
\end{array}
\]
commutes; since moreover both $f$ and $f^\sharp$ are $M_0$-algebra morphisms, this implies that $(f,f^\sharp)$ is an $M_1$-algebra morphism $A \to B$, and clearly the unique such morphism extending $f$.

‘Only if’: Let $B$ be an $M_0$-algebra, and let $f : M_1 A_0 \to B$ be an $M_0$-algebra morphism such that $f \cdot \mu^{01} = f \cdot M_1 a^{00}$. It is then immediate from the assumptions that $B = (a^{00}, b^{00}, f)$ is an $M_1$-algebra (with carriers $B_0 = A_0$, $B_1 = B_0$). By canonicity of $A$, there is a unique $M_0$-algebra morphism $g : A_1 \to B_0$ such that the pair $(id_{A_0}, g)$ forms an $M_1$-algebra morphism $A \to B$:

$$
\begin{align*}
M_1 A_0 &\xrightarrow{M_1 id_{A_0}} M_1 A_0 \\
A_1 &\xrightarrow{a^{10}} B_0
\end{align*}
$$

This shows that $f$ factorizes uniquely through $a^{10}$, proving the desired coequalizer property of $a^{10}$.

A.6 Full Proof of Theorem 5.5

For readability, we restrict to unary modalities. We have to show that for each $n$, the set of evaluation functions $J \phi : M_n 1 \to \Omega$ of depth-$n$ trace formulas $\phi$ is jointly injective. We proceed by induction on $n$. The base case $n = 0$ is immediate by depth-0 separation. For the induction step from $n$ to $n + 1$, let $\mathfrak{A}$ denote the set of evaluation maps $M_n 1 \to \Omega$ of depth-$n$ trace formulas. By the inductive hypothesis, $\mathfrak{A}$ is jointly injective. Moreover, by the construction of the logic, $\mathfrak{A}$ is closed under all propositional operators in $\mathcal{O}$. By depth-1 separation, it follows that the set

$$
\{ L([\phi]) \mid L \in \mathcal{A}, \phi \text{ a depth-}n \text{ formula} \}
$$

is jointly injective. These maps are the interpretations of depth-$(n + 1)$ trace formulas of the form $L \phi$, which proves the inductive claim.

A.7 Details for Example 5.7

Let $F$ be a finitary set functor. Then the graded monad $M_n X = F^n X$ captures behavioural equivalence in $F$-coalgebras. In this setting, $M_0$-algebras are just sets, and modalities in the arising trace logic (again, unary to save on notation) are maps $L : F^2 \to 2$, which are equivalent to predicate liftings, i.e. natural transformations $Q \to Q \circ F^{op}$ [29]. Combining a set $\Lambda$ of such modalities with full Boolean propositional logic leads to coalgebraic modal logic [26, 29]. It is well-known that the coalgebraic modal logic determined by $\Lambda$ is expressive, i.e. distinguishes behaviourally inequivalent states in $F$-coalgebras, if the set $\Lambda$ of modalities is separating, i.e. each element $t \in FX$ is uniquely determined by the set $\{ (L,f) \in \Lambda \times 2^X \mid L(F f(t)) = \top \}$.

This fact becomes an instance of Theorem 5.5 as follows. An $M_1$-algebra is just a map of the form $FA_0 \to A_1$, and canonical algebras are coequalizers of
two identical maps into \( FA_0 \), hence isomorphisms, and then w.l.o.g. identities \( \text{id} : FA_0 \to FA_0 = A_1 \); for a map \( f : A_0 \to 2 \) (since \( M_0 \)-algebras are just sets, all maps are \( M_0 \)-homomorphisms), we then have \([L](f) = L \cdot Ff : FA_0 \to 2\). To see that separation in the sense recalled above implies depth-1 separation, let \( \mathfrak{A} \) be a jointly injective set of maps \( A_0 \to 2 \), closed under all (finitary) Boolean operations. We have to show that the set of maps \( \{[L](f) \mid L \in \mathfrak{A}, f \in \mathfrak{A}\} \) is again jointly injective. Let \( t, s \) be distinct elements of \( FA_0 \). Since \( F \) is finitary, there exists a finite \( X \subseteq A_0 \) and (distinct) \( s', t' \in FX \) such that \( s = F(s') \), \( t = F(t') \) where \( i \) is the injection \( X \hookrightarrow A_0 \). Since \( A \) is separating, we have \( L \in A \) and \( f : X \to 2 \) such that \([L](f)(s') \neq [L](f)(t')\). Since \( \mathfrak{A} \) is jointly injective and closed under Boolean operations, and \( X \) is finite, there exists \( g \in \mathfrak{A} \) such that \( f = g \cdot i \). Then \( L(g) \) separates \( s \) and \( t \).

A.8 Details for Example 5.8

**Trace equivalence** Let \( A \) be a canonical \( M_1 \)-algebra; then the structure map \( \mathcal{P}(A \times A) \to A_1 \) is surjective, i.e. every element of \( A_1 \) has the form \( \bigvee_{i \in I} \sigma_i(x_i) \), for \( \sigma_i \in A \), \( x_i \in A_0 \). Since the operations \( \sigma \) are complete join semilattice morphisms, we can in fact write every element of \( A_1 \) in the form \( \bigvee_{\sigma \in A} \sigma(x_\sigma) \).

Now, to show depth-1 separation, suppose we have two distinct elements of \( A_1 \); by the above, these have the form \( x = \bigvee_{\sigma \in A} \sigma(x_\sigma) \) and \( y = \bigvee_{\sigma \in A} \sigma(y_\sigma) \), respectively, and thus there must exist \( \sigma \in A \) such that \( x_\sigma \neq y_\sigma \); w.l.o.g. \( x_\sigma \not\leq y_\sigma \). Since the \( f \in \mathfrak{A} \) preserve joins, joint injectivity of \( \mathfrak{A} \) thus implies that there exists \( f \in \mathfrak{A} \) such that \( f(x_\sigma) = \top \) and \( f(y_\sigma) = \bot \). (To see this, note that \( x_\sigma \not\leq y_\sigma \) implies that \( x_\sigma \lor y_\sigma \neq y_\sigma \), so there exists \( f \in \mathfrak{A} \) such that \( f(x_\sigma \lor y_\sigma) \neq f(y_\sigma) \), and by monotonicity of \( f \), we must have \( f(x_\sigma \lor y_\sigma) = \top \) and \( f(y_\sigma) = \bot \). But then \( f(x_\sigma) = f(x_\sigma) \lor \bot = f(x_\sigma) \lor f(y_\sigma) = f(x_\sigma \lor y_\sigma) = \top \).) Now recall that the modal operator \( \Diamond_\sigma : \mathcal{P}(A \times 2) \to 2 \) is defined by \( \Diamond_\sigma(S) = \top \) if \( (\sigma, \top) \in S \), and \( \Diamond_\sigma(S) = \bot \) otherwise. The commutativity of

\[
P(A \times A_0) \xrightarrow{\mathcal{P}(A \times f)} \mathcal{P}(A \times \Omega) \xrightarrow{\Diamond_\sigma} A_0
\]

(an instance of (5.1)), yields that, for \( \bigvee_{\sigma \in A} \sigma(z_\sigma) \) in \( A_1 \), we have

\[
\Diamond_\sigma(f)(\bigvee_{\sigma \in A} \sigma(z_\sigma)) = \Diamond_\sigma(\{(\sigma, f(z_\sigma)) \mid \sigma \in A\}) = \begin{cases} \top & \text{if } f(z_\sigma) = \top \\ \bot & \text{otherwise} \end{cases}
\]

Thus, \( \Diamond_\sigma(f) \) separates \( x \) and \( y \).

**Simulation equivalence** Let \( A \) be a canonical \( M_1 \)-algebra, and let \( \mathfrak{A} \) be a jointly injective family of \( M_0 \)-morphisms \( A_0 \to 2 \). Again, \( M_0 \)-morphisms are join-continuous maps, and the structure map \( a^{10} \) is surjective by canonicity. Now
suppose that $x,y$ are distinct elements of $A_1$; w.l.o.g. $x \not\leq y$. By surjectivity of $a^{10}$, $x$ and $y$ have the form $x = \sum_i \sigma_i x_i$ and $y = \sum_j \tau_j y_j$, respectively. Since $x \not\leq y$ and the unary operations $\sigma \in A$ are monotone, there exists $i$ such that $x_i \not\leq y_j$ for all $j$ such that $\sigma_i = \tau_j$. Since the elements of $A$ are join-continuous, we obtain as in the above case of trace equivalence that for each $j$ there exists $f_j \in A$ such that $f_j(x_i) = \top$ and $f_j(y_j) = \bot$. Take $f = \bigwedge f_j$, so that $f(x_i) = \top$ and $f(y_j) = \bot$ for all $j$ such that $\sigma_i = \tau_j$. Then $[\Diamond_{\sigma_i}](x) = \top$ and $[\Diamond_{\sigma_i}](y) = \bot$, showing that the set $\{ [\Diamond_{\sigma_i}](g) \mid \sigma \in A, g \in A \}$ is jointly injective as required.