Spray dynamics as a multi-scale process

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Abstract. The analysis of the processes in sprays, taking into account the contribution of all spatial and temporal scales, is not feasible in most cases due to its complexity. The approach used in most applications is based on separate analysis of the processes at various scales, and the analysis of the link between these processes. This approach is demonstrated for the analysis of spray break-up and penetration in Diesel engine-like conditions, and vortex ring-like structures in gasoline engine-like conditions. The conventional WAVE, TAB, stochastic and modified WAVE (taking into account transient effects) models are reviewed. It is pointed out that the latter model leads to the prediction of spray penetration in Diesel engine-like conditions closest to the one observed experimentally. In gasoline engine-like conditions, spray penetration is often accompanied by the formation of vortex ring-like structures, the spatial scale of which is comparable with the scale of spray penetration. The general expression of the velocity of the vortex ring centroid can be simplified for short and long times, the latter simplification being particularly simple and useful for engineering applications. The thickness of the vortex ring is expressed as $\ell = a\theta^b$, where $a$ is an arbitrary constant and $1/4 \leq b \leq 1/2$. The cases when $b = 1/2$ and $b = 1/4$ refer to laminar and turbulent vortex rings respectively. The model is compatible with the observation of vortex ring-like structures in gasoline engine-like conditions.

1. Introduction
Various types of sprays are observed in the environment (e.g. rain, waterfalls and volcanic eruptions) and are widely used in medical and engineering applications, including those in direct injection internal combustion engines. Some recent developments in sprays modelling have been summarised in a number of monographs and review papers, including [1]-[4]. In contrast to general reviews such as [1]-[3], review [4] was specifically focused on the relevance and importance of multiple scales in spray modelling. In the present paper a review rather similar to [4] is presented, but specific topics to be covered will be different from those discussed in [4]. These topics will include the models of droplet break-up (Sections 2 and 3) and the vortex ring models (Section 4). These models will be focused on specific applications to sprays in internal combustion (Diesel and gasoline) engines, although their generalisation to other applications is expected to be straightforward in most cases.

This review will be addressed primarily to mathematicians not familiar with the engineering and physics background of the problems. Hence, relevant elementary introductions to specific topics will be included in the format of this paper.
Typical photographs of sprays observed in Diesel and gasoline engines are shown in Figs. 1 and 2 respectively.

**Figure 1.** A typical image of a spray in Diesel engine-like conditions.

**Figure 2.** A typical image of a spray in gasoline engine-like conditions.

The main difference between Diesel and gasoline sprays lies in the injection and ambient gas pressures. Diesel sprays are typically injected at pressures 100-200 MPa into a gas at pressure exceeding 3 MPa, while gasoline sprays are typically injected at pressures 0.35-10 MPa into a gas at pressures close to atmospheric pressure. As can be seen from Fig. 1, the Diesel spray has an essentially conical shape with some oscillations superimposed on it. The shape of a gasoline spray, shown in Fig. 2, is rather different from the Diesel spray, especially in the region close to its tip, where vortex ring-like structures can be clearly traced. In both cases, however, the underlying physics of spray formation is essentially the same, especially at the initial stage of spray development. Initially a jet of liquid fuel leaves the nozzle. The shape of this jet can be perturbed by the cavitation processes developing inside the nozzle [5]-[8]. Whether this happens or not, outside the nozzle the jet is disintegrated into liquid sheets and liquid ligaments due to the development of various hydrodynamic instabilities [9]-[11]. Then these ligaments are disintegrated into droplets, which, in their turn, are disintegrated into smaller droplets due to the development of Rayleigh-Taylor instability (caused by normal stresses at the droplet surfaces) or Kelvin-Helmholtz instability (caused by tangential stresses at the droplet surfaces). These processes can be accompanied by the interaction between droplets, sometime leading to
their coalescence [12]-[13], heating and evaporation of droplets and eventually the autoignition of fuel vapour/air mixture [3]. It is very difficult, and sometimes impossible, to experimentally observe these processes. As a result, their models are mainly validated based on the integral characteristics such as spray penetration, dynamics of vortex rings and the autoignition timing. This makes it necessary to link the processes which take place at different scales: including typical scales associated with individual droplets ($10^{-5} - 10^{-4}$ m) and typical scales of spray penetration or the sizes of the vortex ring-like structures ($10^{-2} - 10^{-1}$ m).

The current review will be focused on the analysis of jet and spray break-up models of the processes described above and their links with the integral characteristics of sprays on the one hand, and the analysis of the models capable of explaining some features of vortex ring-like structures on the other hand. The analysis of the first part will be mainly focused on applications to Diesel engine sprays shown in Fig. 1, while the analysis of the second part will be mainly focused on applications to gasoline sprays shown in Fig. 2.

2. Spray break-up models

A brief review of spray break-up models, most relevant to engineering applications, and the most recent developments in this field, are presented in [14]. In what follows, an updated version of the results presented in this paper, will be given.

The nature of the breakup process depends on a spray region. Primary breakup takes place near the nozzle exit. Here disintegration of the liquid jet in liquid sheets and liquid sheets into ligaments takes place. In the far-field spray, where the liquid phase is dispersed in the gas, the secondary breakup of large droplets into smaller ones takes place [15]. Although the analysis of each of the above mentioned individual processes has been reported in the literature (e.g. [9]-[11]), no quantitative models, that take into account all of the details of spray break-up, suitable for engineering applications, have been reported to the best of our knowledge. This is mainly related to the fact that in most practically important engineering applications, including internal combustion engines, the spray break-up process needs to be modelled alongside a number of other processes, including the effects of gas turbulence and chemical kinetics, often in a complex geometry [16]. Perhaps the most successful approach to perform spray break-up modelling when taking into account the complexity of the engineering environment, has been based on the application of research and commercial computational fluid dynamics (CFD) codes (e.g. KIVA, PHOENICS, FLUENT). In these codes the primary requirement to spray break-up modelling is not just their accuracy, but a compromise between accuracy and computer (CPU) efficiency. In most cases in these codes, unified models have been used both for primary and secondary breakup. In these models, the jet is approximated by a chain of droplets, with initial diameters equal to the diameter of the nozzle, or slightly less than this diameter if the effects of cavitation are taken into account. For implementation into CFD codes, designed for computation of three-dimensional sprays, the so called TAB (Taylor Analogy Breakup), WAVE models of breakup, stochastic break-up model and their modifications are commonly applied [17, 18]. One of the important modifications of the WAVE model, that takes into account the transient processes during spray injection, is described in [14]. In what follows, all these models will be briefly described.

2.1. Conventional WAVE model

The WAVE model, originally developed by Reitz[19], is based on the analysis of the Kelvin-Helmholtz instability of a liquid jet. This instability leads to stripping of child droplets from the liquid core. The core is approximated by parent droplet parcels injected from the nozzle. The radii of the droplets in these parcels ($R_d$) continuously decrease during the breakup process, as
described by the following equation:

\[
\frac{dR_d}{dt} = -\frac{R_d - R_{d(eq)}}{t_{ba}},
\]

where \( t_{ba} \) is the characteristic breakup time, \( R_{d(eq)} \) is the radius of equilibrium (stable) droplets:

\[
R_{d(eq)} = \begin{cases} 
B_0 \Lambda, & B_0 \Lambda \leq R_d \\
\mathcal{R}, & B_0 \Lambda > R_d,
\end{cases}
\]

\[
\mathcal{R} = \min \left( \left( 3\pi R_d^2 U/(2\Omega) \right)^{0.33}, \left( 3R_d^3 \Lambda/4 \right)^{0.33} \right)
\]

\( B_0 = 0.61 \) is the model constant, \( \Lambda \) and \( \Omega \) are the wavelength and the frequency of the fastest growing disturbance on the surface of a liquid jet, \( U \) is the jet velocity. The breakup time \( t_{ba} \) is estimated as:

\[
t_{ba} = 3.726 \frac{B_1 R_d}{\Lambda \Omega}.
\]

The breakup constant \( B_1 \) is usually taken equal to 10 based on the experimental results reported in[20], although the actual value of this constant can vary widely[21].

The further development of the WAVE model took into account the effect of the Rayleigh-Taylor (RT) instability of droplets [22]. The wavelength corresponding to the maximal increment of this instability is given by the expression \( \Lambda_{RT} = 2\pi \sqrt{3\sigma_s/(a_{RT} \rho_l)} \), where \( \sigma_s \) is the surface tension, \( a_{RT} = 3C_D \rho_g U^2/\rho_l R_d^2 \), \( C_D \) is the drag coefficient, \( \rho_g \) and \( \rho_l \) are densities of gas and liquid, respectively. When \( \Lambda_{RT} \) is less than the diameter of a droplet, breakup is expected to take place. In this case \( R_{d(eq)} \) is calculated as \( C_{RT} \Lambda_{RT} \), where \( C_{RT} = 2.5 \) is the model constant [22]. The breakup time is estimated as:

\[
t_{ba \ RT} = \sqrt{\frac{3}{2a_{RT}}} \sqrt{\frac{3\sigma_s}{\rho_l a_{RT}}}
\]

Kelvin-Helmholtz instability is mainly responsible for the primary breakup, while the Rayleigh-Taylor mechanism prevails at the secondary atomisation stage [22].

2.2. TAB model

The Taylor Analogy Breakup (TAB) model describes the process in terms of the critical deformation of an oscillating-distorting droplet [23, 24]. The external force is caused by the relative droplet motion, the restoring force is the surface tension force, and the damping term results from the liquid viscosity. The normalised radial deformation of the droplet \( y = 2x/R_d \) is described by the equation of the forced damped linear harmonic oscillator:

\[
\frac{d^2y}{dt^2} = \frac{2\rho_g U^2}{3\rho_l R_d^2} \frac{8\sigma_s}{\rho_l R_d^2 y} - \frac{5\mu_l}{\rho_l R_d^2} \frac{dy}{dt},
\]

where \( \rho_g \) is the density of gas and \( \mu_l \) is the dynamic viscosity of liquid.

It is assumed that breakup occurs when the droplet deformation \( x \) exceeds \( R_d/2 \). The Sauter Mean Radius (SMR) of the product droplets at the moment of breakup is found from the conservation of droplet energy during the breakup process:

\[
SMR = \frac{R_d}{\frac{7}{3} + \frac{\rho_l R_{d(eq)^2}}{4\sigma_s}}.
\]
where $R_d$ is the parent droplet radius, $v_{\text{def}}$ is the velocity of droplet deformation at the moment of breakup, $v_{\text{def}} = 2 \dot{r}$.

In contrast to the WAVE model, the product droplets’ radii $R_{d,pr}$ follow the distribution:

$$f(R_{d,pr}) = \frac{1}{R} \exp\left(-\frac{R_{d,pr}}{R}\right)$$

after breakup in the TAB model, where $R = \text{SMR}/3$ is the number averaged product droplet radius.

2.3. Stochastic model

As follows from the previous analysis, the WAVE model is essentially a deterministic model, in which the radii of product droplets are determined by Equation (1). The TAB model has a stochastic element in choosing the radii of product droplets assuming that the distribution function of these droplets is a priori given, but it still focuses on sample droplets rather than on the whole spectrum. The model suggested by Gorokhovsky and Saveliev [18] is based on a completely different approach to breakup modelling. Their approach is based on the assumption, originally suggested by Kolmogorov [25], that the breakup of parent particles into secondary particles does not depend on the instantaneous sizes of the parent particles. This assumption is obviously not valid when $R_d$ is close to $R_{d,(eq)}$. In high pressure injection sprays, characterised by large Weber numbers, the hydrodynamic mechanism of atomisation due to the mean velocity difference at the liquid-gas surface, can be complicated by the impact of turbulent fluctuations on jet breakup [18]. Under such conditions, when the specific mechanism of atomisation and the scale of the breakup length cannot be clearly defined, stochastic approaches to the modelling of breakup become more appropriate than deterministic ones.

Kolmogorov’s assumption allowed Gorokhovsky and Saveliev [18] to derive the conservation equation for the normalised distribution function of product droplets $f(R)$ in the form:

$$\frac{\partial f(R_d)}{\partial t} = \left[\dot{I}_+ - 1\right] \nu f(R_d), \quad (5)$$

where

$$\dot{I}_+ = \int_0^1 f\left(\frac{R_d}{\alpha}\right) \frac{q(\alpha)}{\alpha} d\alpha \quad (6)$$

is the break-up operator, $\alpha \in [0,1]$ is the parameter linking the radii of product ($R_d$) and parent ($R_p$) droplets ($\alpha = R_d/R_p$), $q(\alpha) d\alpha$ is the normalised probability that the radius of each product droplet is within the range $[\alpha R_d, (\alpha + d\alpha)R_d]$, $\nu = \nu_0 q_0$, $\nu_0$ is the breakup frequency of an individual droplet, $q_0$ is the average number of droplets produced after each breakup action.

There are several assumption made during the derivation of Equation (5). Firstly it was assumed that $q_0$ is finite and independent of the scale $R_d$ and the ratio $\alpha$. The authors of [26] managed to eliminate this assumption by considering a new conserved quantity, the total volume occupied by droplets, and replacing number distribution by volume distribution. Secondly it was assumed that $q$ depends only on $\alpha$. This is a rather strong hypothesis (sometimes called ‘scaling similarity’ or ‘homogeneous kernel’), which is justified for very high Weber numbers and low Ohnesorge numbers [26]. These assumptions are generally valid for sprays in Diesel engines.

Equation (5) cannot be solved due to the unknown function $q(\alpha)$, which takes into account the unknown details of the break-up process. However, using the Taylor expansion:

$$\frac{1}{\alpha} f\left(\frac{R_d}{\alpha}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \left(\frac{\partial^n R}{\partial R_d}\right) f(R_d) \ln^n \alpha$$
it was shown by [18] that in the limit of large times $t \to \infty$, Equation (5) is equivalent to the Fokker-Planck type equation for the evolution of the droplet number distribution function $F(R_d) = N f(R_d)$:

$$
\frac{\partial F(R_d)}{\partial t} = \left[ -3 \langle \ln \alpha \rangle - \frac{9}{2} \langle \ln^2 \alpha \rangle - \frac{\partial}{\partial R_d} R_d \langle \ln \alpha \rangle + \frac{1}{2} \frac{\partial}{\partial R_d} R_d \frac{\partial}{\partial R_d} R_d \langle \ln^2 \alpha \rangle \right] \nu F(R_d),
$$

(7)

where $N$ is the total number of droplets per unit volume, and

$$
\langle \ln^n \alpha \rangle = \int_0^1 \ln^n \alpha q(\alpha) \, d\alpha.
$$

Equation (7) depends on two unknown constants $\langle \ln \alpha \rangle$ and $\langle \ln^2 \alpha \rangle$. To reach an agreement between the predictions of this model and the measurements by [20], it was assumed that $\langle \ln \alpha \rangle = -1/2$ and $\langle \ln^2 \alpha \rangle = 1$. The frequency of breakup $\nu$ was obtained from the relation:

$$
\nu = \frac{1}{B_1 R_d} \sqrt{\frac{\rho_g}{\rho_l}}.
$$

(8)

The value of constant $B_1 = \sqrt{3}$ was chosen in order to match experimental data on the stripping breakup of droplets. For further analysis of stochastic models see [26].

The above described models have mostly been validated for quasi-steady-state sprays, injected at constant or slowly varying velocities. However, to the best of our knowledge, the effect of unsteady injection at the initial stage of formation of a Diesel spray has not been addressed in the literature, and none of the above models have been tested specifically for this transient period of injection, except in [14], where the generalised WAVE model was suggested. In the next subsection the main features of this model are briefly summarised.

2.4. Modified WAVE model

In order to account for the transient nature of injection on breakup in the WAVE model, it was suggested to consider some parameters of this model as functions of acceleration [14]. $\Omega$ was assumed to be inversely proportional to injection acceleration, while the wavelength of critical instability, $\Lambda$, was not affected by the transient nature of the flow. The decrease in $\Omega$ with increasing injection acceleration can be related to the observation that flow acceleration is expected to lead to relaminarisation of the flow and thickening of the boundary layer in the gas phase around the jet for a certain range of Reynolds numbers. The increase in the boundary layer thickness is, in turn, expected to stabilise the gas-liquid interface. Since $t_{bu} \sim 1/\Omega$, the effect of flow acceleration was accounted for by modifying the expression for $B_1$ in Equation (3) to:

$$
B_1 = B_{1st} + c_1 (a^+) c_2,
$$

(9)

where

$$
a^+ = 2 \sqrt{\frac{\text{Re}}{U_{inj}^2}} \frac{R_d}{U_{inj}^2} \frac{dU_{inj}}{dt}
$$

is the acceleration parameter taking into account the effect of flow acceleration; $c_1$ and $c_2$ are adjustable constants. In the steady-state limit $a^+$ is zero and $B_1 = B_{1st}$. It was assumed that $B_{1st} = 10$. The acceleration parameter $a^+$ is constructed by analogy with the local pressure gradient parameter $p^+$ suggested by [27], assuming the laminar-type dependence of the local skin friction coefficient on the Reynolds number.

In order to describe the propagation of disturbances along the liquid core, a rigid body concept was used, which is an additional assumption of the model. In order to describe the
continuous structure of the liquid core in the spray, the collision algorithm by [23] was been modified; the droplets constituting the liquid core are allowed to move together during the injection acceleration stage, even when these droplets are distributed over several computational cells. According to this algorithm, an increase in the injection velocity at the nozzle, caused by injection acceleration, is immediately transferred to all the droplets in the liquid core. This method naturally accounts for incompressibility of liquid, and eliminates the grid dependence of the results. In the region downstream of the spray breakup length, droplet collisions were described following [23].

The detailed testing of the models described in this section is not feasible at the moment, as this would require simultaneous measurements of droplet distribution by radii and velocities in the whole domain. However, these models can be tested indirectly via the comparison between the integral characteristics of sprays predicted by the models and the experimental observations. The most commonly used and practically important such integral characteristic is spray penetration. This will be discussed in the next section.

3. Spray penetration

Droplet break-up models discussed in the previous section referred to typical scales in the range between 1 \( \mu \)m and 100 \( \mu \)m. At the same time typical scales of spray penetration in internal combustion engine-like conditions lie in the range 1 cm and 10 cm. Despite this difference is scales droplet break-up processes play decisive role in determining the scale of spray penetration. This will be illustrated later in this section.

Various spray penetration models have been discussed in a number of reviews, including [4]. In what follows the focus will be on the comparison between the spray penetration predicted by the models discussed in Section 2 and available experimental data, based on our earlier publication [14]. Comparisons were performed for sprays injected through a single-hole nozzle of 0.2 mm in diameter into compressed air at temperature 572 K and pressure 4 MPa. Spray penetration data were obtained from the analysis of video recordings, combined with the mass flow rate measurements [28]. Based on the measured mass flow rate, the average velocity of injection was calculated.

Spray computations were performed using the KIVA 2 code, in which the models described in Section 2 have been implemented. Droplet parcels were injected into a cylindrical gas-filled domain using the blob injection method (the jet was approximated as a set of spherical droplets with diameters equal to the diameter of the nozzle with possible corrections for the effects of cavitation). Spray computations were performed in two dimensions, taking into account the observed axial symmetry of the flow. The flow domain was represented by a constant volume gas chamber of 2 cm in radius and 10 cm in length, neglecting the piston motion during the injection pulse. The domain was covered by a uniform 2-dimensional grid, typically with 20 cells in the radial and 48 cells in the axial directions. The calculations were performed using all four droplet breakup models described in Section 2.

The results of calculations of spray tip penetration, using the TAB, conventional WAVE (with \( B_1 = 10 \)), modified WAVE and stochastic models, and the corresponding experimental data for a highly transient spray are shown in Fig. 3. As follows from this figure, the conventional WAVE, TAB and stochastic models under-predict significantly the penetration at the initial stage of this process. The increase of the parameter \( B_1 \) from 10 to 60 leads to some improvement in the accuracy of the prediction of the conventional WAVE model. At the same time the modified version of the WAVE model, described in Section 2.4, gives much better agreement between the predictions of the model and experimental data.

At a later stage in the injection, the cluster shedding from the tip of the spray was experimentally observed. This led to fluctuation of the observed tip penetration length at around 4 cm (see Fig. 3). This phenomenon has not been addressed in the present study. In our
computations, the spray penetration length was identified as the distance from the nozzle to the leading droplet parcel. This leads to the deviation between the predictions of all models and experimental data at times of 0.6 ms or greater. We anticipate that the actual tip penetration length is larger than that shown in Fig. 3, as the spray at distances greater than about 4 cm from the nozzle is not actually detected by the available equipment as it is outside the optical window.

4. Vortex ring-like structures in Sprays

As can be seen in Fig. 2, the scales of vortex ring-like structures are comparable with the scales of spray penetration. The details of the distribution of flow velocities in these structures could be potentially obtained using Computational Fluid Dynamics (CFD) packages. Unfortunately the complexity of these packages in many cases hides the underlying physics of the phenomena. For example, in the case of the deviation between the predictions of the model and experimental data it turns out to be very difficult, sometimes impossible, to establish the exact reason for this. These CFD packages can be effectively complemented by much more simple analytical models. These models cannot be used to analyse the phenomena in all their complexity, but they can show the underlying physics of specific elements of the phenomena.

The focus of this section will be on one of these models describing vortex rings. A brief overview of the vortex ring models developed so far is presented in Section 4.1. In Section 4.2 some predictions of these models are compared with experimental data for gasoline engine-like conditions.

4.1. Vortex ring models

Since the pioneering paper by Helmholtz [29], the theory of vortex rings has been extensively developed and the results have been reported in several review papers and monographs (e.g. [30, 31]). Among more recent publications we can mention [32]–[36]. In what follows, some results referring to the axial translational velocity of the vortex rings, under various approximations, will be briefly summarised.

The general analytical expression for this velocity was found only in the case when the vortex ring Reynolds number, defined as \( \text{Re} = \zeta_0 \ell^2 / \nu \), where \( \zeta_0 \) is the characteristic initial vorticity, \( \ell \) is the characteristic vortex ring thickness, \( \nu \) is the kinematic viscosity, is small. In the case of laminar vortex rings, \( \ell \) can be defined as \( \ell = \sqrt{2\nu t} \), where \( t \) is time. Using these assumptions and ignoring changes with time of the radius of the vortex ring \( R_0 \) (distance from its axis to the point where the fluid velocity in the frame of reference moving with the vortex ring centroid, is equal to zero), the following general equation for the normalised vortex ring axial translational
velocity has been obtained [33]:

\[ U_x = \frac{V_x}{v_n} = \sqrt{\pi} \theta \left\{ 3 \exp \left( -\frac{\theta^2}{2} \right) I_1 \left( \frac{\theta^2}{2} \right) + \frac{\theta^2}{12} 2F_2 \left[ \frac{3}{2}, \frac{5}{2}; \frac{7}{2}; -\theta^2 \right] \right\} \]

where

\[ V_x = \left( \int_0^\infty \int_{-\infty}^{\infty} (\Psi - 6x r v_r) \zeta dx dr \right) \left( \int_0^\infty \int_{-\infty}^{\infty} r^2 \zeta dx dr \right) \]  

(11)

\[ 2F_2 \left[ a_1, a_2; b_1, b_2; x \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k x^k}{(b_1)_k (b_2)_k k!} \]

(12)
is the generalised hypergeometric function with the coefficients defined as

\[ (\alpha)_0 = 1; \quad (\alpha)_1 = \alpha; \quad (\alpha)_k = \alpha (\alpha + 1) \ldots (\alpha + k - 1) \quad (k \geq 2), \]

\[ \theta = \frac{R_0}{\ell}, \quad v_n = \frac{M}{4\pi^2 R_0^4} = \frac{\Gamma_0}{4\pi R_0}, \]

\[ \Gamma_0 = M/(\pi R_0^3) \] is the initial circulation of the vortex ring, and

\[ M = \pi \int_0^\infty \int_{-\infty}^{\infty} r^2 \zeta dx dr \]

is the specific momentum of the vortex ring [34].

The complexity of Equation (10) makes it difficult to apply it directly to modelling the engineering processes. It can, however, be considerably simplified in the limits of small and large \( \theta \) (long and short times). In the first case it reduces to:

\[ U_x = \frac{7}{30} \sqrt{\pi} \theta^3. \]  

(13)

In the second case it reduces to:

\[ U_x = \ln \theta + \frac{3}{2} - \frac{\gamma}{2} - \psi(3/2) + O(1/\theta), \]

(14)

where \( \gamma = 0.57721566 \) is the Euler constant, \( \psi(x) \) is the di-gamma function defined as:

\[ \psi(x) = \frac{d}{dx} \log \Gamma(x), \]

(15)

\( \Gamma(x) \) is the Gamma function.

Equations (13) and (14) are identical with the results obtained by [37] and [38] respectively.

Although Equations (10), (13) and (14) were originally derived, assuming that \( \ell = \sqrt{2\nu t} \), they remain valid in the more general case when

\[ \ell = at^b, \]

(16)

where \( a \) and \( b \) are constants. The main advantage of approximation (16) is that it incorporates the case of laminar and turbulent vortex rings for which \( b = 1/2 \) and \( b = 1/4 \) respectively (cf. [39]).
For practical applications, the long time limit of Equation (10), given in Equation (13), turned out to be the most useful. In view of (16), Equation (13) can be rewritten as:

\[ U_x = \frac{7 \sqrt{\pi}}{30} R_0^3 a^{-3} t^{-3b}. \]  

(17)

Note that the velocity \( U_x \), predicted by Equation (17), is different from the velocities of the fluid in the region of maximal vorticity, \( U_{\omega x} = V_{\omega x}/v_n \). In the case of long times, they are linked by the following equation:

\[ U_{\omega x} \equiv \frac{V_{\omega x}}{v_n} = \frac{U_x}{1 + \frac{2\pi \theta^2}{\pi}} \int_{\mu_0}^{\infty} \mu^2 \text{erfc}\left(\frac{\mu}{\sqrt{2}}\right) J_1(\theta \mu) J_0(\sigma_{\text{max}} \mu) d\mu, \]  

(18)

where \( \theta \) and \( \sigma_{\text{max}} = r_{\text{max}}/R_0 \) are determined for the point where the vorticity is maximal,

\[ \text{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt. \]

For sufficiently large \( \mu_0 \), the contribution of \( \mu > \mu_0 \) in the integral in Equation (18) can be ignored. For \( \theta \ll \mu_0^{-1} \) the Bessel function \( J_1 \) in Equation (18) can be simplified in the limit that the argument is small. As a result, remembering Equations (16) and (17), Equation (18) can be simplified to:

\[ U_{\omega x} \equiv \frac{V_{\omega x}}{v_n} = \frac{U_x}{1 + \frac{2\pi \theta^2}{\pi}} \int_{\mu_0}^{\infty} \mu^2 \text{erfc}\left(\frac{\mu}{\sqrt{2}}\right) J_0(\mu) d\mu \]  

\[ \times R_0^3 a^{-3} t^{-3b}. \]  

(19)

For sufficiently large \( t \ (\theta < 1) \), \( \sigma_{\text{max}} \approx 1 \). Hence, Equation (19) can be further simplified to:

\[ U_{\omega x} = \left[ \frac{7 \sqrt{\pi}}{30} + \pi \int_0^{\infty} \mu^2 \text{erfc}\left(\frac{\mu}{\sqrt{2}}\right) J_0(\mu) d\mu \right] R_0^3 a^{-3} t^{-3b}. \]  

(20)

The expression in the square bracket in Equation (20) does not depend on time.

The radial component of vortex ring velocity, both in the region where the velocity is close to zero, in the frame of reference moving with the ring, and in the region where the vorticity is maximal, is equal to zero.

4.2. Vortex ring-like structures in gasoline engines

Experimental investigations were performed on two modern production gasoline injectors; a low pressure, port fuel injector (PFI) and a high pressure, direct fuel injector (G-DI). The contrasting choice of fuel injection systems was adopted to highlight the difference in the spray dynamics and to assess the robustness of the vortex ring models. In what follows only data referring to the G-DI injector will be presented and analysed. These data turned out to be more consistent and showed less scatter than those obtained using the PFI injector. A typical photograph of vortex ring-like structures observed in a fuel spray in a quartz chamber with quiescent and atmospheric gas conditions is shown in Fig. 2. Iso-octane fuel at a pressure of 100 bar was injected for 2 ms. The image was recorded with an initial camera delay of 1.5 ms after the start of injection. The formation of vortex ring-like structures was observed in a large, atmospheric pressure and temperature, quiescent chamber.

Two sets of experiments were performed to determine the characteristics of vortex ring-like structures: high-speed cinéc, still photography and Phase Doppler Anemometry (PDA). In general, the fuel sprays were observed to comprise three, widely recognised phases. Initially, a developing phase was observed where a poorly atomised, unsteady, high velocity jet was formed,
combined with small dispersed droplets. This was followed by the main, quasi-steady period, where the mass flow of liquid fuel was approximately constant and the position of the leading edge of spray penetration was observed to move almost linearly with time. In the final phase, the spray momentum decayed and vortex ring-like structures were seen to be formed, translated and destroyed as the jet collapsed inwards towards the injector axis.

Reverse flow structures of differing scales were identified over the entire injection duration. The vortex ring-like structures observed in the decaying phase were used in our study as these could be identified with sufficient precision for quantitative analysis. In the first instance, the high-speed photographic results were used to identify the regions of investigation for the PDA study. Secondly, the centres of the vortex ring-like structures were located by identifying the regions corresponding to maximal vorticity magnitude computed from the interpolated PDA measurement grid. These occurred from approximately 1.75 ms to 5 ms after start of injection (SOI). In [40], the points where the fluid velocities are close to zero were manually traced between frames. Here, computer tracing of the regions of maximal vorticity, calculated from a velocity field measured by the PDA, is considered instead.

In each time interval, the mean vorticity magnitude of the observed spatial velocity distribution was calculated where clearly defined vortex centres were identified. The regions corresponding to maximal vorticity magnitude were identified. The time evolution of the axial and radial velocity components at these regions were obtained by direct measurement. In addition, the velocity components at each of these locations were computed from the displacement of the vortex centres divided by the time interval between frames. Implicit within this method was the assumption that the velocities were constant within a given time step of 0.25 ms. However, the averaged-interval method showed a greater fluctuation in the prediction of the vortex core precession than in the case of direct measurements. It was concluded that this method proved less accurate than the direct method due to the greatest experimental error being present in the determination of the vortex centre. In what follows the focus will be on direct measurements of translational velocities.

The evolution of the values of the axial and radial velocities, obtained by direct measurement in the mixture of air and fuel droplets in the region corresponding to maximal vorticity was investigated. The radial component of velocity showed the most scatter in data. In most cases, this component was close to zero, being either positive or negative. This is consistent with the model prediction as described in the previous section. The axial component of velocity was positive for all values of \( t \) and tended to decrease with time over the range identified. The experimental results were approximated as \( V_x(t) = A t^B \), where \( t \) is the time after SOI. Constants \( A \) and \( B \) were found using a linear least squares curve fitting technique by minimising the sum of the squares of the vertical distance of the points from the curve: \( A = 74.1 \, \text{ms}^{-1} \) and \( B = -1.57 \). Oscillations in both the axial and radial velocities, with a frequency of approximately 1 kHz were observed, especially far from the nozzle during the decaying phase of injection.

Although vortex ring-like structures, observed in gasoline engines look rather different from the structures of the classical vortex rings [30], we can expect that the integral characteristics of these structures, such as translational axial and radial velocities are similar. This can be supported by the fact that although the finite values of the Reynolds number lead to noticeable changes in vorticity distribution, especially far from the ring core, the effects of this number on these velocities is weak [41].

A comparison of the experimental data with the prediction of the vortex ring model, described in the previous section, identified several unknown parameters including the values of \( R_0 \) and initial time. To eliminate the effect of these values, the observed axial velocities were normalised by the times at which the vortex rings were first observed, \( t = t_{\text{init}} \). Also, the non-dimensional time, \( \tilde{t} = t/t_{\text{init}} \) was introduced. The non-dimensional axial velocity component \( \tilde{U}_{\omega x}(t) = U_{\omega x}(t)/U_{\omega x}(t_{\text{init}}) = V_{\omega x}(t)/V_{\omega x}(t_{\text{init}}) \) with \( \tilde{U}_{\omega x}(1) = 1 \) was introduced. An attempt
was made to approximate the experimental data as \( U_{\omega x}(t) = t^{-C} \), where \( C \) is the new positive constant. In a similar manner to the case described earlier, the value of \( C \) was calculated by the use of the least squares best fit method. The value \( C = 1.14 \) was found. This result is compatible with Equation (20) which predicts \( U_{\omega x}(t) \propto t^{-\alpha} \), where \( 3/4 \leq \alpha \leq 3/2 \).

5. Conclusions

The direct numerical simulation of the processes in sprays is not feasible in most cases due to their complexity. The approach used in most engineering applications is based on separate analysis of the processes at various spatial and temporal scales, and the analysis of the link between these processes. This approach is demonstrated for the analysis of spray break-up and penetration in Diesel engine-like conditions, and vortex ring-like structures in gasoline engine-like conditions. Four spray break-up models have been reviewed. These are conventional WAVE, TAB, stochastic and modified WAVE models. Only the latter model took into account the transient effects during spray injection. In Diesel engine-like conditions the spatial scales of these processes are expected to be in the range approximately between 1 \( \mu \)m and 100 \( \mu \)m. It is pointed out that the modified WAVE model leads to the prediction of spray penetration in Diesel engine-like conditions closest to the one observed experimentally. Typical scales of spray penetration in these conditions are close to approximately 10 cm. This shows that the link between processes taking place at spatial scales which differ by approximately three orders of magnitude, cannot be ignored. Also, it demonstrates that the transient effects need to be taken into account when modelling spray break-up processes.

In gasoline engine-like conditions, spray penetration is often accompanied by the formation of vortex ring-like structures, the spatial scale of which is comparable with the scale of spray penetration. Using a two-dimensional approximation, these structures can be described in terms of explicit analytical formulae. The velocity of the vortex ring centroid is expressed in terms of generalised hypergeometric functions. The general expression of this velocity can be simplified for short and long times, the latter simplification being particularly simple and useful for engineering applications. Although all these expressions were originally derived for the laminar vortex rings, when their thickness is given by the expression \( \ell = \sqrt{2\nu t} \), where \( \nu \) is the kinematic velocity of fluid, they remain valid for the more general expression for this thickness \( \ell = at^b \), where \( a \) is an arbitrary constant and \( 1/4 \leq b \leq 1/2 \). The case when \( b = 1/2 \) refers to conventional laminar vortex rings, while the case when \( b = 1/4 \) refers to a fully turbulent state of fluid. The new generalised expression for vortex ring velocity was used to derive the expression for the fluid velocity in the region of maximal vorticity. In the long time limit, this expression predicts that the velocity in this region is proportional to \( t^{-3b} \). This result was compared with the observed velocity in the region of maximal vorticity, normalised by the corresponding velocity when the vortex ring-like structures were first observed in gasoline engine-like conditions. The observed normalised velocity was approximated by the expression \((t/t_{\text{init}})^{-1.14}\), which is compatible with the prediction of the model remembering that \( 1/4 \leq b \leq 1/2 \). Hence, the long time limit is applicable for the analysis of vortex ring-like structures in gasoline engine-like conditions.

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