A new estimate of the mass of the gravitational scalar field for Dark Energy

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Abstract

A new estimate of the mass of the pseudo dilaton is offered by following the fundamental nature that a massless Nambu-Goldstone boson, called a dilaton, in the Einstein frame acquires a nonzero mass through the loop effects which occur with the Higgs field in the relativistic quantum field theory as described by poles of $D$, spacetime dimensionality off the physical value $D = 4$. Naturally the technique of dimensional regularization is fully used to show this pole structure to be suppressed to be finite by what is called a Classical-Quantum-Interplay, to improve our previous attempt. Basically the same analysis is extended to derive also the coupling of a pseudo dilaton to two photons.

1 Introduction

We have developed our own version of the Scalar-Tensor theory (STT) \[1, 2\] due originally to Jordan \[3\], also to Brans and Dicke \[4\], but now with the unique feature that we are then allowed to be free from the possible fine-tuning problem in understading the small size of a cosmological constant (CC), or the dark energy (DE), fitted to the observed accelerating universe \[5\]. Today’s value of $CC \sim t_0^{-2}$ with $t_0$ the present age of the universe, is this small simply because we are this old cosmologically\[1\].

This approach is an outgrowth, in retrospect, of a conceptual attempt based on a simple $\Lambda$ cosmology for the radiation-dominated universe \[6\]. According to an attractor and asymptotic solution, the Jordan frame (JF), with a variable $G$, describes unrealistically a static universe, while the Einstein frame (EF, with subscript $*$), with $G_*$ kept constant, provides fortunately with an expanding universe, $a(t_*) \sim t_*^{1/2}$, hence accepted as the physical frame. Also the scalar field density interpreted as dark energy density falls off like $t_*^{-2}$, from which follows the scenario of a decaying cosmological constant, as emphasized above. On the other hand, however, starting off by assuming a conventional mass term in JF, we come to finding that the microscopic fundamental particles, including an electron, with their masses which fall off like $a_*(t_*)^{-1} \sim t_*^{-1/2}$ in EF. This is totally in conflict with today’s view on the astronomical measurements; cosmological size is measured in reference to the microscopic units of length provided by the inverse mass of the microscopic particles. Also we have no way of detecting any variation of units themselves, implying their constancy in the physical frame, as we once called the Own-unit-insensitivity principle(OUIP) \[7\], which ultimately derives the receding speeds of distant objects in terms of the red-shifts of the observed atomic spectra.

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\[1\] We use the reduced Planckian unit system defined by $c = \hbar = M_P = (8\pi G)^{-1/2} = 1$. The units of length, time and energy in conventional units are given by $8.10 \times 10^{-33}$cm, $2.70 \times 10^{-43}$sec, $2.44 \times 10^{18}$GeV, respectively. As an example of the converse of the last entry, we find $1$GeV $= 2.44^{-1} \times 10^{-18}$ in units of the Planck energy. In the same way, the present age of the universe $t_0 \approx 1.37 \times 10^{10}$y is $10^{60.2}$ in units of the Planck time.
From these arguments we view the theory of STT to face a serious flaw when it meets the microscopic physics. Sometime ago we came across [1, 2] that this flaw can be avoided miraculously in terms of global scale invariance broken spontaneously with the scalar field playing the role of a massless Nambu-Goldstone (NG) boson [8, 9, 10], called dilaton, which, like many other examples of NG bosons, would acquire a nonzero mass hence a pseudo dilaton. We further suggested a tentative estimate of its mass-squared; 
\[ \mu^2 \sim m_q^2 M_{\text{ss}}^2 / M_P^2 \sim (10^{-9} \text{eV})^2, \]
with \( m_q \) for the averaged quark mass, while \( M_{\text{ss}} \sim 10^3 \text{GeV} \) for the supersymmetry mass scale to be prepared for the quadratic cutoff of the self-energy of a scalar field.

Now in the current article, we are going to replace quarks by the Higgs field as an origin for the masses of fundamental particles, also with an improved technique for the mass acquisition mechanism applied uniquely to the dilaton. The new numerical estimate based on the Standard Model (SM) turns out to result in \( \mu \) somewhat heavier than our previous estimate, still basically more or less in the same range of the order of magnitude, remaining responsible for the experimental searches for the DE [12] of the accelerating universe through \( \gamma \gamma \) scattering.

On the theoretical side, as we also point out, the two different concepts, scaling behavior and infinities in the quantized field theory, are described by a single common variable, the spacetime dimensionality assumed to be continuous off the physical value 4. On the basis of dimensional regularization (DR) technique [13], the formulation is then not only simple and straightforward but also continued smoothly from the first half of the spontaneously broken scale invariance.

The same mechanism applies also to the coupling of a pseudo dilaton to two photons. These experiences prompt us to present our phenomenological results even, for the moment, only with a preliminary study of the effects of the conventional renormalization procedure.

## 2 Basic equations

We start with the basic Lagrangian
\[
L = \begin{cases} 
\sqrt{-g} \left( \frac{1}{2} \xi \phi^2 R - \frac{1}{2} \epsilon g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + L_{\text{matter}} - \Lambda \right), \\
\sqrt{-g_*} \left( \frac{1}{2} R_* - \frac{1}{2} g_*^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + L_{\text{matter}} - \Lambda \exp(-4\zeta \sigma) \right),
\end{cases}
\]
for \( \Lambda > 0 \) and \( \xi > 0 \) by avoiding antigravity, expressed both in JF and EF, respectively, where the factor \( \Omega \) for the conformal transfromation satisfies
\[
\Omega(x) = g_{\mu\nu}^{-1/2} g_{\ast \mu\nu}, \quad \phi = \dot{\zeta}^{-1/2} \Omega, \quad \text{with} \quad \Omega = \exp(\zeta \sigma),
\]
where
\[
\dot{\zeta} = \xi M_P^{-2}, \quad \dot{\zeta} = \zeta M_P^{-1}, \quad \text{with} \quad \zeta^{-2} \equiv 6 + \epsilon \xi^{-1} > 0.
\]
The symbol \( \phi \) implies the gravitational scalar field in JF with \( \epsilon = \pm 1 \), or 0. Notice that \( \phi \), even with an apparently ghost nature with \( \epsilon = -1 \), has been brought to be mixed to the spinless

2 This mass corresponds approximately to the force-range \( \sim 100 \text{m} \), related to the suggested non-Newtonian force, as was discussed in [11], for example.

3 Symbols \( \varphi, \omega \) used in the original references [3, 4] are now re-expressed by our more convenient ones; \( \varphi = (1/2) \xi \phi^2, 4 \omega = \epsilon \xi^{-1} \).

4 \( \epsilon = 0 \) implies \( \zeta^2 = 1/6 \) corresponding to the coefficient \( a = 1/3 \) of the scalar component of the combined potential [11, 14]. We choose \( \eta_{00} = \eta^{00} = -1 \).
portion of $g_{\mu\nu}$ through the nonminimal coupling term, the first term on RHS of the upper line of (2.1), to emerge as a canonical nonghost scalar field $\sigma$ in EF under the condition specified at the last of (2.3).

From (2.1) we derive the cosmological equations in each of radiation-dominated JF and EF, also in the standard spatially flat Robertson-Walker metric with the matter density approximated by a uniform distribution, finding the attractor and asymptotic solutions. This is the way we reach the static universe in JF and the expanding universe in EF, as stated in Sec. 1.

During the calculation, we derived the asymptotic relation for the matter density $\rho$ in JF;

$$\rho = -3\Lambda \frac{2\xi + \epsilon}{6\xi + \epsilon}.$$  \hspace{1cm} (2.4)

Using $\Lambda > 0$ and $\xi > 0$ as stated immediately following (2.1) before also the last condition of (2.3), we find that the obvious condition $\rho > 0$ results only if

$$\epsilon = -1, \text{ hence } \frac{1}{6} < \xi < \frac{1}{2}, \text{ and } \frac{1}{4} < \zeta^2 < \infty,$$  \hspace{1cm} (2.5)

as will be used later. We also come to find

$$\phi \to t, \text{ or } \Omega \to t, \text{ as } t \to \infty.$$  \hspace{1cm} (2.6)

Many of the important features in these approximate solutions are taken over to the more exact numerical solutions, which include such a unique complication like the occasional step-like falling-off behavior of the scalar-field density, as was discussed with the help of a supporting assumption, in SubSecs 5.4.1-2 of [1] and SubSecs 6.1-2 of [2], thus allowing the wording, a decaying cosmological constant, acceptable semantically. But more urgently, we re-emphasize briefly how OUP is observed by the spontaneously broken scale invariance.

The crucial point is that we are supposed to start with the interaction term in JF;

$$-L_I = \frac{1}{2} \sqrt{-g} h\phi^2\Phi^2,$$  \hspace{1cm} (2.7)

instead of the conventional mass term $-L_m = \sqrt{-g}(1/2)m^2\Phi^2$, applied to an example of the real scalar-field matter $\Phi$, where $h$ is a dimensionless coupling constant, hence indicating scale invariance in JF. The conformal transformation to EF yields

$$-L_I = \frac{1}{2} \sqrt{-g_0} \Omega^{-4}h\hat{\xi}^{-1}\Omega^2\Omega^2\Phi^2 = \frac{1}{2} \sqrt{-g_*}m_*^2\Phi_*^2, \text{ with } \Phi = \Omega\Phi_* , \text{ } m_*^2 = h\hat{\xi}^{-1}.$$  \hspace{1cm} (2.8)

Notice that all of the $\Omega$'s cancel each other, hence leaving a truly constant $m_*$. The last result might be combined with $\xi \sim O(1)$ as obtained from the second of (2.5) to find $h \sim O(m_*^2/M_F^2)$, which will be inherited basically to the more realistic but more complicated model for the Higgs field as will be developed soon later.

In this connection, we also notice that (2.7) fails to observe a premise that $\phi$ be decoupled from the matter Lagrangian, as emphasized by Brans and Dicke [4] who realized this to be the

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\[\text{For details see [1, 2, 15].}\]
simplest way to implement Weak Equivalence Principle (WEP) as one of the most important features in the macroscopic and classical gravity.\(^6\)

Now facing the microscopic and cosmological gravity, we might try an attempt beyond their premise. This is the reason why we come to (2.7), showing remarkably the global scale invariance taking \(\phi\) into account. In a sense, we exploit the scale invariance of the whole STT terms in JF Lagrangian except for \(\Lambda\).

From none of the mass in (2.7) we have created the mass spontaneously. In fact the mass dimension has been smuggled through \(\hat{\xi}^{-1/2}\) as a VEV of \(\phi\), following the second of (2.2). The spontaneous nature might also be better interpreted by computing the Noether current of the scale transformation;

\[
\delta g_{\mu\nu} = 2\ell g_{\mu\nu}, \quad \delta \phi = -\ell \phi, \quad \delta \Phi = -\ell \Phi, \quad (2.9)
\]

which derives,

\[
J^\mu = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\nu \left( \xi \hat{\xi}^{-2} \phi^2 + \Phi^2 \right), \quad (2.10)
\]

an exact result, as detailed in Appendix M of [1]. We then re-express RHS now in EF. In this process, we obtain

\[
\zeta^{-1} \Box_\sigma \sigma = - \left( m^2 f^2 + g^{\mu\nu} \partial_\mu f \partial_\nu f \Phi \right), \quad (2.11)
\]

where LHS comes directly from the first term on RHS of (2.10), hence reaching the massless nature of \(\sigma\), to be called a dilaton, precisely as had been shown by Nambu [8]. The occurrence of the massless dilaton is expected to survive the approximations mentioned above.

We now leave the first half of the scenario of spontaneously broken scale invariance, entering its second half in which the massless dilaton grows into a massive pseudo dilaton. For this purpose, we start with the spacetime dimensionality \(D\) off, but close to, the physical value 4. We also re-interpret the above \(\Phi\) now as the Higgs field supposed to provide with the origin of the masses of all the microscopic fields. In fact the familiar Mexican-Hat potential is shown to inherit the core of the 4-dimensional scale-invariance within the realm of STT.

We extend (2.7) to

\[
- \mathcal{L}_H = \sqrt{-g} \left( \frac{1}{2} h \phi^2 \Phi^2 + \frac{\lambda}{4!} \Phi^4 \right) = \sqrt{-g} \Omega^{D-4} \left( \frac{1}{2} \tilde{m}^2 f^2 + \frac{\lambda}{4!} \Phi^4 \right), \quad \text{with} \quad \tilde{m}^2 = h \hat{\xi}^{-1}, \quad (2.12)
\]

\(^6\)The amplitude through the nonminimal coupling term observes the same tensor coupling as in General Relativity.

\(^7\)The special form of the first in the following, correponding to \(g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}\), has been selected, because, unlike another type of the coordinate transformation \(x^\mu \rightarrow \Omega x^\mu\) or \(\delta x^\mu = \ell x^\mu\), making it straightforward to be applied to the expanding universe with the 3-space simply uniform without any particular origin. See [16], for example, on introducing \(\delta x^\mu / x^2 = \beta^\mu\).
where $\Omega$ in the second of (2.2) and (2.8) are both replaced by $\Omega^{d-1}$ with $d = D/2$. Then together with $\sqrt{-g} = \Omega^{-D} \sqrt{-g_\ast}$, we come to find an overall multiplier on RHS of (2.12):

$$
\Omega^{-D} \left(\Omega^{d-1}\right)^4 = \Omega^{-D+2D-4} = \Omega^{D-4}.
$$

(2.13)

Obviously, adding the quartic self-coupling of $\Phi^4$ to (2.7) leaves our experiences of the scaling behaviors and spontaneous symmetry breaking nearly unchanged.

Following the well-known procedure, we shift the origin by the VEV $v$;

$$
\Phi^* = v + \tilde{\Phi}.
$$

(2.14)

By requiring the absence of the term linear in $\tilde{\Phi}$, we arrive finally at

$$
- \mathcal{L}_H = \exp \left(2\hat{\zeta}(d-2)\sigma\right) \sqrt{-g_\ast} \mathcal{V}, \quad \text{with} \quad \mathcal{V} = \frac{1}{2} m^2 \Phi^2 + \frac{1}{2} m \sqrt{\frac{\lambda}{3}} \Phi^3 + \frac{\lambda}{4!} \Phi^4,
$$

(2.15)

where $m$ defined by

$$
m^2 = -2\tilde{m}^2 = \frac{\lambda}{3} v^2,
$$

(2.16)

is the observed mass of the Higgs field, $1.26 \times 10^2$ GeV [17].

In this way we define the dynamics of $\tilde{\Phi}$ and $\sigma$, where the exponential factor comes from (2.13), substituted from the third of (2.2), while $\mathcal{V}$ is the Higgs potential. We have ignored the possible vacuum terms as well as part of CC, in accordance with what we stated before between (2.10) and (2.11). As we consider, the current process of mass creation and the pseudo dilaton belongs to a local physics expected not to be affected seriously by CC, or DE.

To be noticed more explicitly, the field $\sigma$ occurs only in association with $d - 2 = (D - 4)/2$ separated from what is called the Higgs potential $\mathcal{V}$. In other words, the dilaton $\sigma$ might appear to be present only for $D \neq 4$. This by no means implies that $\sigma$ is entirely outside our realistic concern at $D = 4$, because another part $\mathcal{V}$ contains infinities at $D = 4$, as will be shown shortly. We should take the same attitude as in DR that we keep $d \neq 2$ during the computation until we come back to the physical value $d = 2$ only at the very end of the calculation.

For more details of the calculation, we apply the expansion

$$
\exp(2\hat{\zeta}(d-2)\sigma) \approx 1 - 2\hat{\zeta}(2 - d)\sigma.
$$

(2.17)

On the other hand, the potential $\mathcal{V}$, representing a collection of the field $\tilde{\Phi}$, might develop certain Feynman diagrams which may happen to induce closed loops of $\tilde{\Phi}$ then exhibiting divergences. The simplest 1-loop divergence is described by

$$
\Gamma(2 - d) \sim (2 - d)^{-1}, \quad \text{as} \quad d \to 2,
$$

(2.18)

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8 The symbol $m$ in the second equation is the mass in EF, to be better denoted by $m_*$. To avoid too much notational complications in the following equations, however, we continue to use $m$ without the subscript * for the observed mass for the Higgs mass in the whole subsequent part of the article.

9 This potential $\mathcal{V}$ is shown to agree with the relevant part of the SM. See (87.3) of [18], for example. His $V(\varphi) = (\lambda_r/4)(\varphi^2 - (v^2/2))^2$ is reproduced precisely by our (2.15) by choosing $\sigma = 0$, $\lambda_r/4 = \lambda/4!$, and $\varphi = (1/\sqrt{2})(v + \tilde{\Phi})$ only for the single component, with another component vanishing, corresponding to his (87.13). Notice also that a special relation chosen between the two terms in the parentesis on RHS of $V(\varphi)$ above has the same effect of the term linear in $\tilde{\Phi}$ removed, which we required in the sentence just prior to the foregoing footnote 8.
according to the technique of DR.

By substituting (2.17) and (2.18) into the first of (2.15), we find a product like \( \zeta (d-2) \sigma \Gamma(2-d) \), for which we apply the relation

\[
(2-d)\Gamma(2-d) \to 1, \quad \text{as} \quad d \to 2, \quad (2.19)
\]

to be called a Classical-Quantum Interplay (CQI), which connects a classical factor \( 2-d \) to \( \Gamma(2-d) \) obviously representing a quantum nature. The above result implies that \( \sigma \) happens to pick up a \( \tilde{\Phi} \)-loop, which thus plays a major role in the same way as the cutoff conjectured by Nambu and Jona-Lasinio [10] for the study of the pion-nucleon system, thus re-discovered in a somewhat different but closely related context.

Notice also that we no longer suffer from infinities, as far as the relevant processes are dominated by CQI. This might be, however, related to the Brans-Dicke premise as was discussed following (2.8). The pole structure (2.18) should apply only to such fundamental fields like \( \Phi \), or quarks and leptons, but not to composite particles like hadrons. In this sense, the WEP violation effect due to the occurrence of \( \phi \) in the matter Lagrangian tends to be smaller, but might need more detailed analyses before reaching the final comparison with the observations.

In the next Section we are going to discuss how this crucial relation, CQI, can be used naturally in calculating the mass of the pseudo dilaton.

### 3 Computing the mass of the pseudo dilaton

In order to derive the mass \( \mu \) of \( \sigma \), we first consider the simplest form of the \( \sigma \) self-energy (SE) part, as shown in the upper line of Fig. 1, where the dotted and solid lines are for \( \sigma \) and \( \tilde{\Phi} \), respectively. Each vertex is read out from the second term on RHS of (2.17) times the first term in \( V \) of (2.15), deriving the effective vertex part;

\[
g_0 = -2\zeta (2-d)m^2. \quad (3.1)
\]

The loop integral of \( \tilde{\Phi} \) gives\(^{10}\)\(^{11}\)

\[
\int d^Dk \frac{1}{(k^2 + m^2)^2} = i\pi^2 \left( m^2 \right)^{d-2} \frac{\Gamma(d)\Gamma(2-d)}{\Gamma(2)} \approx i\pi^2 \Gamma(2-d). \quad (3.2)
\]

The whole contribution is

\[
\sim g_0^2 i\pi^2 \Gamma(2-d) \sim 4m^4 \varepsilon^2 (2-d)^2 \Gamma(2-d) \sim 2-d \to 0, \quad \text{as} \quad d \to 2, \quad (3.3)
\]

since the pole \( (2-d)^{-1} \) has been over-cancelled by \( (d-2)^2 \) in accordance with CQI. As a lesson, a finite nonzero result will occur only if the \( \tilde{\Phi} \) in \( V \) must include 2-loop divergences.

\(^{10}\)Some of the details of the required integrals will be found in Appendix N of [1].

\(^{11}\)Strictly speaking, the denominators should be \( ((k+q/2)^2 + m^2)((k-q/2)^2 + m^2) \), where \( q \) is the momentum of the size of \( \sim \mu \). Since we finally find \( \mu \) negligibly smaller than \( m \) as in (4.8), we might justify the approximate computation as in (3.2). The same kind of approximation applies to almost any of the loop inetrals to be encountered in the following of the present article.
Figure 1: In the upper line we show the simplified 1-loop SE part of $\sigma$, to start with. Dotted and solid lines are for the $\sigma$ and the shifted Higgs field $\tilde{\Phi}$, respectively. In the lower line, we illustrate the 2-loop amplitudes to which CQI can be applied. They are different from each other in the ways the dotted lines couple to different parts of the three terms in the potential $V$ on RHS of (2.15).

We now try to extend the argument to construct 2-loop amplitudes for the $\sigma$ mass term, as illustrated by (a)-(e) in the lower line of Fig. 1. First in the diagram (a), we show a simply doubled 1-loop diagrams, in fact by connecting the simplest one to another diagram formed by the first term and the third term of $V$ in (2.15). By computing explicitly, we do reach two poles resulting in the nonzero result;

$$J_a = \left[-i(2\pi)^4\right]^2 \left[i(2\pi)^4\right]^{-4} \lambda m^4 \left[2\tilde{\zeta}(2-d)\sigma\right]^2 \int d^Dk \frac{1}{(k^2 + m^2)^2} \int d^Dk' \frac{1}{(k'^2 + m^2)^2}$$

$$= i(2\pi)^4 \lambda m^4 \left[\frac{1}{4} i\pi^2 \sigma^2 \int d^Dk' \frac{1}{k'^2 + m^2} \int d^Dk \frac{1}{k^2 + m^2} \right] = J_a. \quad (3.4)$$

The combination in (b) is basically the same as in (a), different only where one of the dilaton lines reaches $V$, thus causing a linear denominator. Still due to

$$\int d^Dk' \frac{1}{k'^2 + m^2} = i\pi^2 (m^2)^{d-1} \frac{\Gamma(d) \Gamma(1-d)}{\Gamma(3-d)} = -i\pi^2 m^2 \Gamma(2-d), \quad (3.5)$$

somewhat different from (3.2), though, we find the result, which happens to be the same as (3.4);

$$J_b = \left[-i(2\pi)^4\right]^2 \left[i(2\pi)^4\right]^{-3} \lambda m^4 \left[2\tilde{\zeta}^2(2-d)\sigma\right]^2 \int d^Dk' \frac{1}{k'^2 + m^2} \int d^Dk \frac{1}{k^2 + m^2}$$

$$= i(2\pi)^4 \lambda m^4 \left[\frac{1}{4} i\pi^2 \tilde{\zeta}^2 \sigma^2 - \frac{1}{4} i\pi^2 \tilde{\zeta}^2 m^2 \sigma^2\right] = J_a. \quad (3.6)$$

The same result will follow for the diagram in (c);

$$J_c = -i(2\pi)^4 \left[i(2\pi)^4\right]^{-2} \lambda \left[2\tilde{\zeta}(2-d)\sigma\right]^2 \int d^Dk' \frac{1}{k'^2 + m^2} \int d^Dk \frac{1}{k^2 + m^2}$$

$$= i(2\pi)^4 \lambda \tilde{\zeta}^2 \left[\frac{1}{4} i\pi^2 m^2 \Gamma(2-d)\right]^2 = J_a. \quad (3.7)$$
We then face another diagram in (d) due to the trilinear term on RHS of (2.15):

\[
\mathcal{J}_d = \frac{-i(2\pi)^{\frac{3}{2}}}{i(2\pi)^{\frac{3}{4}}} \lambda m^2 \left[ 2\frac{\zeta(2 - d)\sigma}{2\sqrt{3}} \right]^2 \left( \frac{3m^2}{k^2 + m^2} \right)^2 \int d^D k \int d^D k' \frac{1}{(k^2 + m^2)(k'^2 + m^2)} \frac{1}{(k - k')^2 + m^2},
\]

where the last term under the double integral is overlapping not separable into the functions of \( k \) alone, and \( k' \) alone, respectively, corresponding to the inclined line in (d) in Fig. 1.

We have two alternative ways, carrying out (i) the \( k' \) integral first, or (ii) the \( k \) integral first. This also corresponds to including the overlapping integral as part of \( k' \) integral, or \( k \) integral, or expressed more explicitly as

\[
\mathcal{J}_d^{(i)} = K_d(2 - d)^2 \int d^D k \frac{1}{k^2 + m^2} \left( \int d^D k' \frac{1}{(k'^2 + m^2)^2} \frac{1}{(k - k')^2 + m^2} \right), \quad (3.9)
\]

\[
\mathcal{J}_d^{(ii)} = K_d(2 - d)^2 \int d^D k' \frac{1}{(k'^2 + m^2)^2} \left( \int d^D k \frac{1}{k^2 + m^2} \frac{1}{(k - k')^2 + m^2} \right), \quad (3.10)
\]

where

\[
K_d = \frac{-i(2\pi)^{\frac{3}{2}}}{i(2\pi)^{\frac{3}{4}}} \frac{\lambda m^2}{4\pi^2} \zeta(2 - d) \sigma^2 \left( \frac{m^2}{2} \right)^2 = 3i\lambda m^2 \zeta(2 - d) \sigma^2. \quad (3.11)
\]

We are going to show first that \( \mathcal{J}_d^{(i)} \) vanishes due to the \( k' \) integration which does not behave divergently, by analyzing only the relevant portion of (3.9):

\[
\hat{\mathcal{J}}_d^{(i)} = (2 - d) \int d^D k' \frac{1}{(k'^2 + m^2)^2} \frac{1}{(k'^2 + m^2)^2} = -2(2 - d) \int_0^1 dx \int d^D k' \frac{1}{(k'^2 + m^2)^2} = -i\pi^2 (2 - d) \int dx (\mathcal{M}^{(i)}_2)^{d - 3} \Gamma(3 - d) \sim -i\pi^2 \frac{2 - d}{\mathcal{M}^{(i)}_2} \to 0, \quad \text{as} \quad d \to 2, \quad (3.12)
\]

in which the gamma function \( \Gamma(3 - d) \) is left convergent at \( d = 2 \), hence (3.12), with the factor \( 2 - d \), vanishes in the same way as in (3.3), despite that \( \mathcal{M}^{(i)}_2 \) defined by

\[
\mathcal{M}^{(i)}_2 = m^2 + x(1 - x)k^2, \quad \text{with} \quad \tilde{k}' = k' - k. \quad (3.13)
\]

is not a pure constant.

We then move on to \( \mathcal{J}_d^{(ii)} \). Also using (3.2), we find

\[
\mathcal{J}_d^{(ii)} = K_d(2 - d)^2 \int_0^1 dx \int d^D k' \frac{1}{(k'^2 + m^2)^2} \int d^D k \frac{1}{(k^2 + m^2)^2} \left( \frac{1}{k^2 + m^2} \right)^2
\]

\[
= K_d(2 - d)i\pi^2 \int d^D k' \frac{1}{(k'^2 + m^2)^2} (\mathcal{M})^{d - 2}(2 - d) \Gamma(2 - d) = K_d(2 - d)i\pi^2 \int d^D k' \frac{1}{(k'^2 + m^2)^2}
\]

\[
= K_d(i\pi^2)^2 (2 - d) \Gamma(2 - d) = K_d(i\pi^2)^2 = -3i\pi^4 \lambda \zeta(2 - d) m^4 \sigma^2 = 3\mathcal{I}_a, \quad (3.14)
\]

3 times as large as the last term of (3.4) for \( \mathcal{I}_a \), where

\[
\mathcal{M}^{(ii)} = m^2 + x(1 - x)k^2, \quad \text{with} \quad \tilde{k} = k - k'. \quad (3.15)
\]
by the repeated use of CQI. Notice that the fact that $M^2_{(ii)}$ is not purely constant does not affect the conclusion in the limit $d \to 2$.

Basically the same analysis can be applied to the diagram (e). From a visual inspection, we readily identify the difference in the equations; we only replace the double-pole terms of the type $(k^2 + m^2)^{-2}$ by the single-pole terms $(k^2 + m^2)^{-1}$, where $k$ might be $k'$. This removes the difference between (3.9) and (3.10), leaving us to consider only (3.11), with the double-pole term at the very end of the second line replaced by the single-pole term, implying a sign change as we notice between (3.2) and (3.5). This is offset by the difference between $[-i(2\pi)^4]^3/[i(2\pi)^4]^4$ at the top of (3.8) and the corresponding factor $[-i(2\pi)^4]^2/[i(2\pi)^4]^3$ supposed to occur in $J_e$. In this way we reach the simple result

$$J_e = J_d.$$  

$$(3.16)$$

We then take the sum

$$J = J_a + 2J_b + J_c + 2J_d + 2J_e = -i \frac{16}{4} \lambda m^4 \hat{\zeta}^2 \sigma^2 = -4i\lambda m^4 \hat{\zeta}^2 \sigma^2,$$  

$$(3.17)$$

where we have doubled the contribution from (b) and (d), to recover the right-left symmetry, while another doubling in (e) has been applied because the diagrams displayed in Fig. 1 are only half of the two possible choices of the dotted lines.

We are then allowed to compare our result with the effective mass term of $\sigma$;

$$J = -L_\mu = \frac{1}{2} \mu^2 \sigma^2,$$  

$$(3.18)$$

thus allowing us to identify (3.17) with $-i(2\pi)^4 \mu^2 \sigma^2$, hence

$$\mu^2 = \frac{4}{16\pi^4} \hat{\zeta}^2 \lambda m^4, \quad \text{or} \quad \mu = \frac{1}{2\pi^2} \sqrt{\lambda} \hat{\zeta} m^2.$$  

$$(3.19)$$

In this way we have come to conclude that a massless dilaton in the presence of quantum loops does acquire a nonzero mass, thus becoming a pseudo dilaton, almost automatically in accordance with a simple interpretation of the exponential factor in (2.15). It seems even amusing to find that the same Higgs field plays indispensable roles in creating masses both of the pseudo dilaton, part of gravity, and the rest of the fundamental particles.

In spite of this desired result in deriving the mass $\mu$, we still admit some uncertainty which might arise from the contribution without being derived from the CQI process. Consider a typical example of an additional $\tilde{\Phi}$ loop inserted between the two loops in the diagram (a) of Fig. 1, for example, also connected to the neighboring loops through the $\lambda$ term in $V$ of (2.15). In the absence of $(d - 2)\sigma$ we must appeal to the conventional procedure in DR;

$$\mathcal{R} = -i(2\pi)^4 \lambda^2 [i(2\pi)^4]^{-2} \int d^D k \frac{1}{(k^2 + m^2)^2} \approx -\frac{\lambda^2}{16\pi^4} \Gamma(-\delta) \exp (-\delta/\delta_m),$$  

$$(3.20)$$

where $\delta = d - 2$, $(m^2)^\delta = \exp(\delta \ln m^2)$, and $\lambda \sim \mathcal{O}(1)$, also

$$\delta_m \equiv \frac{1}{-\ln m^2} \approx \frac{1}{75} \approx 0.013.$$  

$$\text{(3.21)}$$

$^{12}$Basically the same type of analysis can be applied to another example of a loop attached to the side of the left loop in Fig. 1 (b), for example.
The foregoing CQI calculations would be left undisturbed if $|\mathcal{R}|$ is kept well below the order unity in the reduced Planckian units.

The relation (3.20) is evaluated at $\delta \to 0^+$. The divergence from $\Gamma(-\delta) \sim -\delta^{-1}$ requires a re-regularization, which is expected to convert a pole to a cutoff $\delta_c > 0$, for example. In the absence of any general way of fixing $\delta_c$, also without related physical observables, we might choose it to be $\delta_m$ defined by (3.21), which is not only unique to the current model with the occurrence of $m^2$, but also happens to be reasonably small. In this way we have now reached;

$$|\mathcal{R}(\delta_c)| = |\mathcal{R}(\delta_m)| \approx -\frac{\ln m^2}{16\pi^2} e^{-1} \approx \frac{0.48}{2.72} \approx 0.18 \lesssim \mathcal{O}(1),$$

which turns out to be barely a lower end of $\mathcal{O}(1)$.

This result achieved by a simple but natural approach appears to be an encouraging sign for a theoretically favored idea of the CQI-dominance, at least in certain physical situations, though more details are yet to be scrutinized, probably in the future. In the next Section, we then enter the final step of our numerical analysis on the basis of the expected CQI computation.

### 4 Estimating $\mu$

In the second equation of (3.19), we first re-express $m^2 = (1.26 \times 10^2)^2(\text{GeV})^2$ into the Planckian unit system;

$$m^2 = (1.26 \times 10^2)^2 \times (2.44^{-1} \times 10^{-18})^2.
\tag{4.1}$$

However, $\hat{\zeta}$ has the mass dimension $-1$, so that the product $\hat{\zeta}m^2$ has the mass dimension $+1$, in agreement with $\mu$ on LHS of the second of (3.19), and is going to be re-expressed in units of GeV instead of $(\text{GeV})^2$;

$$\hat{\zeta}m^2 = \hat{\zeta}(1.26 \times 10^2)^2 \times (2.44^{-1} \times 10^{-18}) = \frac{1.26^2 \times 10^4}{2.44} \times 10^{-18}\text{GeV} = 0.651\zeta \times 10^{-14}\text{GeV} = 6.51\zeta \mu\text{eV}.
\tag{4.2}$$

In order to know $\lambda$, we first use (2.16). We then go through the two well-known steps in SM;

$$m_W = \frac{-gv}{2}, \quad \text{and} \quad \frac{g^2}{m_W^2} = \frac{8}{\sqrt{2}}G_F,
\tag{4.3}$$

where $m_W = 80.2\text{GeV}$ is the mass of the W meson interpreted as a Higgs mechanism for the mass of a gauge field, while $G_F = 1.17 \times 10^{-5}\text{GeV}^{-2}$ is the Fermi constant for the weak interaction. We then use the second equation of (4.3) to derive\footnote{The effect of renormalized fields might have been ignored at this moment, given the crude approximation to be allowed in (4.8). For details see [18], for example.}

$$g^2 = \frac{8}{\sqrt{2}}m_W^2, \quad \text{or} \quad g = 0.653,
\tag{4.4}$$

\footnote{For the self-masses in QED or QCD, divergences are removed simply to fit the observed values.}
which is then substituted into the first of (4.3), finding
\[ v = -\frac{2}{g} m_W = 246 \text{GeV}, \] (4.5)
finally substituted into (2.16) hence arriving at
\[ \lambda = 3 \left( \frac{1.26}{2.46} \right)^2 = 0.787, \quad \text{or} \quad \sqrt{\lambda} = 0.887. \] (4.6)

We also need an estimate of \( \zeta \). According to (2.5), we have \( \epsilon = -1 \) and \( 1/6 < \xi < 1/2 \), hence \( 1/4 < \zeta^2 < \infty \) for the radiation-dominated universe, while \( 1/2 < \xi < 3/2 \) and \( 3/16 < \zeta^2 < 1/4 \) for the dust-dominated universe, as shown in Fig. 1 of [2, 7]. Our own fit to the accelerating universe yields \( \zeta \sim 1.58 \). as read in Fig. 5.8 of [1] and Fig. 9 of [2]. Also noted is \( \xi = 1/4 \) and \( \zeta^2 = 1/2 \), or \( \zeta = 0.714 \) from Super String Theory as indicated in (3.4.58) of [19]. In view of these findings, we suggest a tentative but still convenient bound of \( \zeta \) somewhere between 0.5 and 2.0;
\[ \zeta \approx (0.5 \sim 2.0). \] (4.7)

Summarizing them all finally in the second of (3.19), we obtain
\[ \mu \approx (0.15 \sim 0.59) \mu \text{eV} \sim (150 \sim 590) \text{neV}. \] (4.8)

The far RHS suggests nearly two orders of magnitude over the previous estimate \( \sim 10^{-9} \text{eV} \).

5 Coupling of a pseudo dilaton to two photons

As an application of the current approach, we now try to re-derive the coupling term
\[ -L_3 = A \sigma \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \] (5.1)
which plays a pivotal role in the experimental searches for DE [12], hopefully on a wider perspective in exploiting the scale invariance than in the past attempts [1, 2].

To emphasize the unique nature of the pseudo dilaton, we are going to study the photon SE part, or the vacuum polarization, represented by the diagram, the same type as shown on the upper line of Fig. 1, though the solid line, used to be for the neutral Higgs field, is now re-interpreted as a charged matter field, also with the dotted line used for the pseudo dilaton, replaced by the photon line. For simplicity for the moment, we assume a singly charged and massive Dirac field \( \psi \), a representative of quarks and leptons.

We start off with the simple electromagnetic interaction of \( \psi \) first in JF, followed by moving to EF;
\[ -\mathcal{L}_{em} = bie \left( \bar{\psi} b^\mu \gamma_i \psi \right) A_\mu \rightarrow b_s \Omega^{-D} e \Omega^{D-1} \left( \bar{\psi}_s b_s^\mu \Omega \gamma_i \psi_s \right) \Omega^{d-2} A_{s\mu} \equiv -b_s \Omega^{d-2} L_{sem}, \] (5.2)

\[ ^{15} \text{It still appears that the two estimates above are more or less close to each other from a wider point of view, probably because the two approaches share the same concept on the pseudo dilaton in some way or the other.} \]
where $e$ is the elementary charge chosen to be a pure constant, with $\alpha = e^2/(4\pi) \approx 1/137$;

$$- L_{\text{sem}} = ie\bar{\psi}_s b^\mu_s \gamma_\mu \psi_s A_{s\mu},$$

(5.3)

with $b^\mu_s$ the dimensionally extended tetrad with $b = \sqrt{-g}$, also the electromagnetic field transforming as $A_\mu = \Omega^{d-2} A_{s\mu}$. The occurrence of $\Omega^{d-2}$ on the far RHS of (5.2) indicates scale invariance in 4 dimensions, hence the same nature as generating $\sigma$ as in the previous Sections.

In EF, we approximate spacetime by locally Minkowskian to apply the ordinaly Feynman rules based on (5.3) to the same type of the diagram as on the upper line of Fig. 1 in terms of DR, obtaining the gauge-invariant form

$$\Pi_{\mu\nu}(k) = -\frac{\alpha}{3\pi} \left( k_\mu k_\nu - k^2 \eta_{\mu\nu} \right) \Gamma(2 - d),$$

(5.4)

where we have reversed the overall sign due to the antisymmetric nature of $\psi$ and $\bar{\psi}$.

Further re-installing $\Omega^{d-2}$ on the far RHS of (5.2), substituted from the third of (2.2), we reach the whole result

$$\sqrt{-g_s} \exp \left( 2\hat{\zeta}(d - 2)\sigma \right) \left( k_\mu k_\nu - k^2 \eta_{\mu\nu} \right) \Gamma(2 - d).$$

(5.5)

We then pick up the term liner in $\sigma$, also using the CQI in the form of (2.19), comparing the result with (5.4) by using the relation

$$\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\epsilon^\mu_f \left( k_\mu k_\nu - k^2 \eta_{\mu\nu} \right) \epsilon^\nu_i,$$

(5.6)

with $\epsilon^\mu_f$ and $\epsilon^\nu_i$ for the polarization vectors of the final and initial photons, respectively, then identifying the constant $A$ as

$$A = -\frac{\alpha}{3\pi} \hat{\zeta}. $$

(5.7)

In this way we come to determine $A$ basically of the size of the inverse of the Planck mass, as expected to be.

For each of the quarks and the leptons, the result (5.7) adds up with the corresponding multiplicative factors for the electric charges squared. Notice also that a cancellation always takes place between the terms of the same $m$, the mass of the loop fields, to meet the gauge-invariance of the form (5.4), thus yielding $A$ independent of $m$. This has an advantage that we derive $A$ basically $\sim \alpha \hat{\zeta}$, but, on the other hand, deprives us of a familiar procedure to suppress the contribution from much heavier and uncertain loop fields.

From this point of view, we may also consider the charged scalar fields, sharing the same charge-structure as indicated by supersymmetry, for which we develop obviously the pararell computations with $\psi$ in (5.3) replaced by $\Phi$ together with the additional 4-point term $\sim e^2 \Phi \Phi g^{\mu\nu} A_\mu A_\nu$, without the sign change due to the antisymmetry of the fermionic field, ending up with a multiplicative factor $-1/4$ to (5.4),

$$A_{\text{sc}} = \frac{\alpha}{12\pi} \hat{\zeta}. $$

(5.8)

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16See (6.174)-(6.181) of [1] for details simply for the scalar loop field. Extending to the Dirac field is tedious but straightforward.

17This term contributes a term proportional to $m^2$ due to a simple 1-loop of $<0|\Phi\Phi|0> \propto m^2$ in DR, thus cancelling the term of $\sim m^2$ in the main loop term as in the upper line of Fig. 1.
in place of (5.7).

The reduction factor $1/4$ can be interpreted by $(1/2)^2$ with 1 and 2 for the spin degrees of freedom for the scalar and the Dirac fields, respectively, while the squaring takes care of the occurrence of two lines in each of the main loop diagrams. In this sense, every 4 scalar fields offset the effect of 1 Dirac field, no matter how heavy they might be. Obviously, it appears too early to make an unambiguous prediction before we develop a more general survey on what the fundamental particles are to be included in the loop, also considering wider class of spin-statistics combinations, again left to future studies, at this time. An uncertainty of this kind still unavoidable at present might result in an adjustable parameter multiplied to $\Gamma^{1/2}$ with $\Gamma$ for the decay width of $\sigma$ into two photons in the formulation in [12].

6 Summary

We started out by assuming the scale-invariance of the Higgs potential in JF in STT in 4 dimensions, reaching, somewhat unexpectedly, a time-independent particle mass in EF in conformity with today’s view on measuring cosmological size in units of the inverse of the mass of the microscopic particles. We then extended the spacetime dimensionality $D$ off the physical value 4, allowing us to analyze the behavior of the pseudo NG boson, pseudo dilaton. We derived its mass by studying the $\sigma$ SE part. By considering the 2-loop amplitudes, we obtained the nonzero and finite mass $\mu$ of the pseudo dilaton, by maximally exploiting the CQI relation, also utilizing SM, somewhere around $\mu eV$, which turns out approximately 2 orders of magnitude heavier than our previous tentative estimate. The difference is understood naturally because we now deal with a theoretical model quite different from our previous simple-minded one. It still seems helpful if we find any aspect more tractable on the non-CQI terms possibly in the future. As an extended idea, we tried also to re-derive the coupling strength of $\sigma$ into $2\gamma$, still short of the fully unique determination of the multiplier at present.

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18 The previous result Eq. (91) in [2], for example, can even be re-interpreted as our (5.7) multiplied by an “adjustable parameter” $B/A = \zeta^{-1}(2/3)Z$. 

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