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Multidegree for bifiltered $D$-modules and hypergeometric systems

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Abstract

In that paper, we recall the notion of the multidegree for $D$-modules, as exposed in a previous paper[2], with a slight simplification. A particular emphasis is given on hypergeometric systems, used to provide interesting and computable examples.

Introduction

This paper is an introduction to the theory of the multidegree for $D$-modules, as exposed in a previous paper[2]. The multidegree has been defined by E. Miller[10]: that is a generalization in multigraded algebra of the usual degree known in projective geometry. In our previous paper[2] we adapted it to the setting of bifiltered modules over the ring $D$ of linear partial differential operators. Here the definition of the multidegree is slightly simplified: it becomes the identical counterpart, in the category of bifiltered $D$-modules, of the definition of Miller. We give detailed examples from the theory of $A$-hypergeometric systems of I.M. Gelfand, A.V. Zelevinsky and M.M. Kapranov[3].

The paper is organized as follows. In Section 1 we recall the definition of the objects we are interested in, up to the definition of $(F,V)$-bifiltered free resolutions of $D$-modules, following T. Oaku and N. Takayama[11]. Then we define the multidegree for bifiltered $D$-modules in Section 2. Some examples of $A$-hypergeometric systems are discussed in Section 3, leading to open questions which generalize known facts about the holonomic rank of $A$-hypergeometric systems. In Section 4 we give details on the simplification of the definition of the multidegree we give in Section 2, which consists in proving that a $D$-module $M$ and its homogenization $\mathcal{R}_V(M)$ with respect to the $V$-filtration have same codimension.
1 Bifiltered free resolution of $D$-modules

Let $D = \mathbb{C}[x_1, \ldots, x_n, t_1, \ldots, t_p]((\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{t_1}, \ldots, \partial_{t_p}))$ denote the Weyl algebra in $n + p$ variables. Denoting the monomial

\[ x_1^{a_1} \cdots x_n^{a_n} t_1^{b_1} \cdots t_p^{b_p} \partial x_1^{n_1} \cdots \partial x_n^{n_2} \partial t_1^{n_1} \cdots \partial t_p^{n_2} \]

by $x^{u\mu} \partial x^{v}\partial t^{r}$, every element $P$ in $D$ is written uniquely as a finite sum with complex coefficients

\[ P = \sum a_{\alpha, \beta, \mu, \nu} x^{\mu} \partial x^{\nu} \partial t^{r}. \]  \hspace{1cm} (1)

The fundamental relations are $\partial x_i x_i = x_i \partial x_i + 1$ for $i = 1, \ldots, n$ and $\partial t_i t_i = t_i \partial t_i + 1$ for $i = 1, \ldots, p$.

We describe two important filtrations on $D$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$, let $|\lambda| = \lambda_1 + \cdots + \lambda_n$. Let $\text{ord}^F(P)$ (resp. $\text{ord}^V(P)$) be the maximum of $|\beta| + |\nu|$ (resp. $|\nu| - |\mu|$) over the monomials $x^{u\mu} \partial x^{v}\partial t^{r}$ appearing in (1). Then we define the filtrations

\[ F_d(D) = \{ P \in D, \text{ord}^F(P) \leq d \} \]

for $d \in \mathbb{Z}$ (with $F_d(D) = \{ 0 \}$ for $d < 0$) and

\[ V_k(D) = \{ P \in D, \text{ord}^V(P) \leq k \} \]

for $k \in \mathbb{Z}$. The filtration $(F_d(D))$ is the most classical one, and the filtration $(V_k(D))$ is the so-called $V$-filtration along $t_1 = \cdots = t_p = 0$ of Kashiwara-Malgrange.

The graded ring $\text{gr}^F(D) = \oplus_d F_d(D)/F_{d-1}(D)$ is a commutative polynomial ring, whereas the graded ring $\text{gr}^V(D) = \oplus_k V_k(D)/V_{k-1}(D)$ is isomorphic to $D$. Let $M$ be a left finitely generated $D$-module. We will define the notion of a good $F$-filtration of $M$. For $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$, let $D^r[\mathbf{n}]$ denote the free module $D^r$ endowed with the filtration

\[ F_d(D^r[\mathbf{n}]) = \oplus_{i=1}^r F_{d-i}(D). \]

A good $F$-filtration of $M$ is a sequence $(F_d(M))_{d \in \mathbb{Z}}$ of sub-vector spaces of $M$ such that there exists a presentation

\[ M \xrightarrow{\phi} \frac{D^r}{N}, \]

with $N$ a sub-$D$-module of $D^r$, and a vector shift $\mathbf{n} \in \mathbb{N}^r$, such that

\[ F_d(M) \xrightarrow{\phi} \frac{F_d(D^r[\mathbf{n}]) + N}{N}. \]

The module $\text{gr}^F(M) = \oplus_d F_d(M)/F_{d-1}(M)$ is a graded finitely generated module over $\text{gr}^F(D)$. It is proved that the radical of the annihilator of $\text{gr}^F(M)$ does not depend on the good filtration of $M$. Then $\text{codim} M$ is defined as the codimension of the ring $\text{gr}^F(D)/\text{Ann}(\text{gr}^F(M))$, which does not depend on the
good filtration. A fundamental fact is that \( \text{codim} M \leq n + p \) if \( M \neq 0 \). When \( \text{codim} M = n + p \), the module \( M \) is said to be holonomic.

Now we introduce the bifiltration

\[
F_{d,k}(D) = F_d(D) \cap V_k(D)
\]

for \( d, k \in \mathbb{Z} \), and we define the notion of a good \((F, V)\)-bifiltration of \( M \). For \( \mathbf{n} = (n_1, \ldots, n_r) \) and \( \mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r \), let \( D^r[\mathbf{n}][\mathbf{m}] \) denote the free \( D \)-module \( D^r \) endowed with the bifiltration

\[
F_{d,k}(D^r) = \bigoplus_{i=1}^r F_{d-i,k-i}(D).
\]

A good \((F, V)\)-bifiltration of \( M \) is a sequence \((F_{d,k}(M))_{d,k \in \mathbb{Z}}\) of sub-vector spaces of \( M \) such that there exists a presentation

\[
M \xrightarrow{\phi} D^r \xrightarrow{\Phi} N,
\]

with \( N \) a sub-\( D \)-module of \( D^r \), and vector shifts \( \mathbf{n}, \mathbf{m} \in \mathbb{N}^r \), such that

\[
F_{d,k}(M) \xrightarrow{\phi} F_{d,k}(D^r[\mathbf{n}][\mathbf{m}]) + N.
\]

We are now in position to define the notion of a bifiltered free resolution of a module \( M \) endowed with a good bifiltration \((F_{d,k}(M))_{d,k}\). Such a resolution is an exact sequence

\[
0 \to D^r[\mathbf{n}^{(d)}][\mathbf{m}^{(d)}] \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_r} D^r[\mathbf{n}^{(0)}][\mathbf{m}^{(0)}] \xrightarrow{\phi_0} M \to 0
\]

such that for any \( d, k \in \mathbb{Z} \), we have an exact sequence

\[
0 \to F_{d,k}(D^r[\mathbf{n}^{(d)}][\mathbf{m}^{(d)}]) \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_r} F_{d,k}(D^r[\mathbf{n}^{(0)}][\mathbf{m}^{(0)}]) \xrightarrow{\phi_0} F_{d,k}(M) \to 0.
\]

**Example 1.1.** Let \( D = \mathbb{C}[t_1, t_2][\partial_{t_1}, \partial_{t_2}] \). Let \( I \) be the ideal generated by \( \partial_{t_1} - \partial_{t_2} \) and \( t_1 \partial_{t_1} + t_2 \partial_{t_2} \), and \( M = D/I \) endowed with the good bifiltration \( F_{d,k}(M) = (F_{d,k}(D) + I)/I \). That is the hypergeometric system \( M_A(0, 0) \) associated with \( A = \begin{pmatrix} 1 & 1 \end{pmatrix} \), see Section 3. We have a bifiltered free resolution:

\[
0 \to D^1[2][1] \xrightarrow{\phi_2} D^2[1, 1][1, 0] \xrightarrow{\phi_1} D^1[0][0] \to M \to 0
\]

with

\[
\begin{align*}
\phi_1(1, 0) &= \partial_{t_1} - \partial_{t_2}, \\
\phi_1(0, 1) &= t_1 \partial_{t_1} + t_2 \partial_{t_2}, \\
\phi_2(1) &= (t_1 \partial_{t_1} + t_2 \partial_{t_2}, -(\partial_{t_1} - \partial_{t_2})).
\end{align*}
\]

Bifiltered free resolutions have been introduced by T. Oaku and N. Takayama[11]. The same authors[12] show that such resolutions are at the heart of the computation of important objects in \( D \)-module theory: the restriction of a \( D \)-module.
along a smooth subvariety, the algebraic local cohomology, the tensor product and localization.

However, the $F$-filtration they use is slightly different from ours: that is the filtration by the total order (i.e. $x_i$, $t_i$, $\partial x_i$, and $\partial t_i$ are given the weight 1 for all $i$), which allows them to define the notion of a minimal bifiltered free resolution. The ranks $r_i$, called Betti numbers, and the shifts $n_i$, $m_i$ do not depend on the minimal bifiltered free resolution, but do depend on the bifiltration $(F_{d,k}(M))_{d,k}$. Our choice for the filtration $F$ comes from the common use of it, together with the $V$-filtration, in the theory of slopes and irregularity for $D$-modules, see e.g. the course of Y. Laurent[7]. In our setting the notion of a minimal bifiltered free resolution no longer makes sense. However, we will derive from the Betti numbers and shifts of an arbitrary bifiltered free resolution an invariant of the $D$-module $M$: the multidegree.

2 Multidegree for bifiltered $D$-modules

The multidegree has been introduced by E. Miller[10] in a commutative multi-graded context. It is a generalization of the notion of the degree known in projective geometry. We adapt it to the setting of bifiltered $D$-modules. First, we define the $K$-polynomial of a bifiltered $D$-module. Let

$$0 \to D^0[n][m] \overset{\varphi_1}{\to} \cdots \overset{\varphi_r}{\to} D^r[n][m] \overset{\varphi_r}{\to} M \to 0$$

be a bifiltered free resolution of $M$.

**Definition 2.1.** The $K$-polynomial of $D^r[n][m]$ with respect to $(F, V)$ is defined by

$$K_{F,V}(D^r[n][m]; T_1, T_2) = \sum_{i=1}^{r} T_1^{n_i} T_2^{m_i} \in \mathbb{Z}[T_1, T_2, T_1^{-1}, T_2^{-1}].$$

The $K$-polynomial of $M$ with respect to $(F, V)$ is defined by

$$K_{F,V}(M; T_1, T_2) = \sum_{i=0}^{\delta} (-1)^i K_{F,V}(D^i[n][m]; T_1, T_2) \in \mathbb{Z}[T_1, T_2, T_1^{-1}, T_2^{-1}].$$

The definition of $K_{F,V}(M; T_1, T_2)$ does not depend on the bifiltered free resolution (Proposition 3.2 of our previous paper[2]), thus that is an invariant of the bifiltered module $(M, (F_{d,k}(M))_{d,k})$ very close to the data of the Betti numbers and shifts.

**Example 2.1** (Continuation of Example 1.1). From the bifiltered free resolution

$$0 \to D^1[2][1] \overset{\varphi_2}{\to} D^2[1,1][1,0] \overset{\varphi_1}{\to} D^1[0][0] \to M \to 0,$$

we compute the $K$-polynomial $K(M; T_1, T_2) = 1 - (T_1 T_2 + T_1) + T_1^2 T_2$. 

4
But $K_{F,V}(M; T_1, T_2)$ depends on the bifiltration chosen, as shown in the following example.

**Example 2.2.** Let $D = \mathbb{C}[t][\partial_t]$ and $M = D$ endowed with the good bifiltration $F_{d,k}(M) = F_{d,k}(D)$. Then $M$ admits the bifiltered free resolution

$$0 \to D[0][0] \to M \to 0$$

and thus $K(M; T_1, T_2) = 1$. Now let

$$M' = \frac{D^2}{D(1,1)}.$$  

We have an isomorphism $M \simeq M'$, given by $1 \mapsto (1,0)$. We endow $M'$ with the bifiltration

$$F_{d,k}(M') = F_{d,k}(D^2[1,0][0,1]) + D(1,1).$$

Then we have the following bifiltered free resolution:

$$0 \to D[1][1] \xrightarrow{\phi} D^2[1,0][0,1] \to M' \to 0,$$

with $\phi(1) = (1,1)$. Then $K(M', T_1, T_2) = T_1 + T_2 - T_1 T_2$.

We now define the multidegree.

**Definition 2.2.** $K_{F,V}(M; 1 - T_1, 1 - T_2)$ is a well-defined power series in $T_1, T_2$. We denote by $C_{F,V}(M; T_1, T_2)$ the sum of the terms whose total degree in $T_1, T_2$ equals $\text{codim} M$ in the expansion of $K_{F,V}(M; 1 - T_1, 1 - T_2)$. This is called the **multidegree** of $M$ with respect to $(F, V)$.

**Example 2.3** (Continuation of Example 2.1). We have $\text{codim} M = 2$ and

$$K(M; 1 - T_1, 1 - T_2) = 1 - ((1 - T_1)(1 - T_2) + (1 - T_1)) + ((1 - T_1)^2(1 - T_2)) = (T_1^2 + T_1 T_2) - T_1^2 T_2,$$

thus $C(M; T_1, T_2) = T_1^2 + T_1 T_2$.

The multidegree $C_{F,V}(M; T_1, T_2)$ is a coarser invariant than $K(M; T_1, T_2)$, but its advantage is that it does not depend on the good bifiltration. In Example 2.2, we have $C_{F,V}(M; T_1, T_2) = C_{F,V}(M'; T_1, T_2) = 1$.

**Theorem 2.1.** $C_{F,V}(M; T_1, T_2)$ does not depend on the good bifiltration of $M$.

**Proof.** This theorem is similar to Theorem 3.1 of our previous paper[2], proved using an argument from Y. Laurent and T. Monteiro-Fernandes[8]. But our definition of the multidegree is slightly simpler than that given in our previous paper[2]. Let $\mathbb{K}$ denote the fraction field of $\mathbb{C}[x]$ and $\mathcal{R}_V(M)$ denote the Rees module associated with $M$ considered as a $V$-filtered module. $\mathcal{R}_V(M)$ is naturally endowed with a $F$-filtration. In our previous paper[2], we have
defined the multidegree as the sum of the terms whose total degree equals \( \text{codim}(K \otimes \text{gr}^F(\mathcal{R}_V(M))) \) in the expansion of \( K_{F,V}(M; 1 - T_1, 1 - T_2) \). Here for the sake of simplicity we no longer use \( K \): we define the multidegree as the sum of the terms whose total degree equals \( \text{codim}(\text{gr}^F(\mathcal{R}_V(M))) \). The proof of the invariance also works.

The remaining problem, so as to be in accordance with Definition 2.2, is to prove that \( \text{codim(\text{gr}^F(\mathcal{R}_V(M))) = \text{codim} M} \). We postpone the proof of it to Section 4.

3 Examples from the theory of hypergeometric systems

Let us consider a class of \( D \)-modules introduced by I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky[3], generalizing the Gauss hypergeometric equations, called \( A \)-hypergeometric systems. They are constructed as follows. In this section \( D = \mathbb{C}[x_1, \ldots, x_n][\partial_1, \ldots, \partial_n] \). Let \( A \) be a \( d \times n \) integer matrix and \( \beta_1, \ldots, \beta_d \) be complex numbers. We assume that the abelian group generated by the columns \( a_1, \ldots, a_n \) of \( A \) is equal to \( \mathbb{Z}^d \). One defines first the toric ideal \( I_A \): that is the ideal of \( \mathbb{C}[\partial_1, \ldots, \partial_n] \) generated by the elements \( \partial^u - \partial^v \) with \( u, v \in \mathbb{N}^n \) such that \( A.u = A.v \). Then one defines the hypergeometric ideal \( H_A(\beta) \): that is the ideal of \( D \) generated by \( I_A \) and the elements \( \sum a_i x_i \partial_j - \beta_i \) for \( i = 1, \ldots, d \). Finally the \( A \)-hypergeometric system \( M_A(\beta) \) is defined by the quotient \( D/H_A(\beta) \). A. Adolphson[1] proved (in the general case) that these \( D \)-modules are holonomic. As do M. Schulze and U. Walther[14], we assume that the columns of \( A \) lie in a single open halfspace. The book of M. Saito, B. Sturmfels and N. Takayama[13] is a complete reference on (homogeneous) \( A \)-hypergeometric systems.

Our purpose in that section is to give some calculations of multidegree for hypergeometric systems. We will make the \( V \)-filtration of \( D \) vary, but the module \( M_A(\beta) \) will always be endowed with the bifiltration \( F_{d,k}(M_A(\beta)) = (F_{d,k}(D) + H_A(\beta))/H_A(\beta) \) once the \( V \)-filtration of \( D \) is chosen. The computations are done using the computer algebra systems Singular[5] and Macaulay2[4].

3.1 \( V \)-filtration along the origin

At first we consider the \( V \)-filtration along \( x_1 = \cdots = x_n = 0 \).

Example 3.1. Let

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3
\end{pmatrix}.
\]

Then \( I_A \) is generated by \( \partial_2 \partial_4 - \partial_3^2, \partial_1 \partial_4, \partial_2 \partial_3, \partial_3 \partial_4 - \partial_2^2 \). The ideal \( H_A(\beta) \) is generated by \( I_A \) and the elements \( x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + x_4 \partial_4 - \beta_1, x_2 \partial_2 + 2x_3 \partial_3 + 3x_4 \partial_4 - \beta_2 \). For all \( \beta \), we have

\[
C_{F,V}(M_A(\beta); T_1, T_2) = 3T_1^4 + 6T_1^3T_2 + 3T_1^2T_2^2 = 3T_1^2(T_1 + T_2)^2.
\]
Let $\text{vol}(A)$ denote the normalized volume (with $\text{vol}([0,1]^d) = d!$) of the convex hull in $\mathbb{R}^d$ of the set $\{0,a_1,\ldots,a_n\}$. Then we have in Example 3.1:

$$C_{F,V}(M_A(\beta);T_1,T_2) = \text{vol}(A).T_1^n(T_1 + T_2)^{n-d}. \tag{2}$$

Let us take a homogenizing variable $h$ and let $H(I_A) \subset \mathbb{C}[\partial,h]$ denote the homogenization of $I_A$ with respect to the total order in $\partial_1,\ldots,\partial_n$. We proved the following (Theorem 5.2 of our previous paper[2]): if $\mathbb{C}[\partial,h]/H(I_A)$ is a Cohen-Macaulay ring and the parameters $\beta_1,\ldots,\beta_d$ are generic, then the formula (2) holds.

**Question 3.1.** Does the formula (2) hold if $\beta_1,\ldots,\beta_d$ are generic, but without the Cohen-Macaulay assumption?

For a holonomic module $M$, let us write $C_{F,V}(M;T_1,T_2) = b_0(M)T_1^n + b_1(M)T_1^{n-1}T_2 + \cdots + b_n(M)T_2^n$. From (2) we have $b_0(M_A(\beta)) = \text{vol}(A)$. We claim that the latter equality holds for all $A$, if $\beta_1,\ldots,\beta_d$ are generic. Indeed if $M$ is any holonomic $D$-module we have $b_0(M) = \text{rank}(M)$ (Remark 5.1 of our previous paper[2]), where $\text{rank}(M)$, the holonomic rank of $M$, is the dimension of the vector space of local holomorphic solutions of $M$ (considered as a system of linear partial differential equations) at a generic point. Let us remark that the niceness assumption in Remark 5.1 of our previous paper[2] can be dropped because of Proposition 4.1. Now, by Adolphson[1], $\text{rank}(M_A(\beta)) = \text{vol}(A)$ for generic $\beta_1,\ldots,\beta_d$, which concludes to prove our claim.

**Example 3.2** (from Saito-Sturmfels-Takayama[13]). Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$ 

Then $I_A$ is generated by $\partial_2\partial_2^2 - \partial_2^3, \partial_1\partial_3 - \partial_3\partial_1, \partial_3\partial_2 - \partial_2^2\partial_4, \partial_1^2\partial_4 - \partial_2^3$. Here $\mathbb{C}[\partial,h]/H(I_A)$ is not Cohen-Macaulay. However for $(\beta_1,\beta_2) \neq (1,2)$, we have

$$C_{F,V}(M_A(\beta);T_1,T_2) = 4T_1^4 + 8T_1^3T_2 + 4T_1^2T_2^2 = 4T_1^2(T_1 + T_2)^2,$$

which agrees with the formula (2).

For $(\beta_1,\beta_2) = (1,2)$, we have

$$C_{F,V}(M_A(\beta);T_1,T_2) = 5T_1^4 + 12T_1^3T_2 + 10T_1^2T_2^2 + 4T_1T_2^3 + T_2^4.$$ 

Let us remark that $(T_1 + T_2)^2$ is still a factor, indeed

$$5T_1^4 + 12T_1^3T_2 + 10T_1^2T_2^2 + 4T_1T_2^3 + T_2^4 = (T_1 + T_2)^2(5T_1^2 + 2T_1T_2 + T_2^2).$$

### 3.2 V-filtration along coordinate hyperplanes

Let us reconsider Examples 3.1 and 3.2, with the $V$-filtration along $x_i = 0$ for some fixed $i$.

**Example 3.3** (continuation of Example 3.1).
• V-filtration along $x_1 = 0$: for any $\beta_1, \beta_2$,
$$C_{F,V}(MA(\beta)) = 3T_1^4 + 2T_1^3T_2.$$  

• V-filtration along $x_2 = 0$: for any $\beta_1, \beta_2$,
$$C_{F,V}(MA(\beta)) = 3T_1^4 + 3T_1^3T_2.$$  

• V-filtration along $x_3 = 0$: same as V-filtration along $x_2 = 0$.

• V-filtration along $x_4 = 0$: same as V-filtration along $x_1 = 0$.

Example 3.4 (Continuation of Example 3.2). Let us take the V-filtration along $x_4 = 0$. Then if $(\beta_1, \beta_2) \neq (1, 2)$, then
$$C_{F,V}(MA(\beta)) = 4T_1^4 + 3T_1^3T_2.$$  

For $(\beta_1, \beta_2) = (1, 2)$, we have
$$C_{F,V}(MA(\beta)) = 5T_1^4 + 4T_1^3T_2.$$  

3.3 Dependency of the multidegree on the parameters

Studying the dependency of the multidegree on the parameters $\beta_1, \ldots, \beta_d$ is a natural problem. A basic known fact is that the multidegree remains constant outside of an algebraic hypersurface of $\mathbb{C}^d$ (analogous to Proposition 1.5 of our previous paper[2]). The dependency of the holonomic rank of $MA(\beta)$ has been deeply studied by several authors, for instance in Saito-Sturmfels-Takayama[13] and L.F. Matusevich, E. Miller and U. Walther[9]. In particular it has been proved in the latter paper that the holonomic rank of $MA(\beta)$ is upper semi-continuous as a function on the parameter $(\beta_1, \ldots, \beta_d)$, which means that the holonomic rank at an exceptional parameter is greater than the holonomic rank at a generic parameter. Thus the coefficient $b_0(MA(\beta))$ is an upper semi-continuous function on $\beta_1, \ldots, \beta_d$. Nobuki Takayama pointed out the problem of that semi-continuity for the other coefficients of the multidegree, as also suggest Examples 3.2 and 3.4.

Question 3.2. Let us fix the matrix $A$, the V-filtration on $D$ and $0 \leq i \leq n$. Is the coefficient $b_i(MA(\beta))$ a upper semi-continuous function on $\beta_1, \ldots, \beta_d$ ?

3.4 Positivity

The following positivity problem is due to an observation by Michel Granger.

Question 3.3. Let us fix $A$, $\beta_1, \ldots, \beta_d$, the V-filtration on $D$ and $0 \leq i \leq n$. Do we always have $b_i(MA(\beta)) \geq 0$ ?

That is the case in all the examples we considered. Moreover, since $b_0(MA(\beta)) = \text{rank}(MA(\beta))$, the positivity is true for $b_0(MA(\beta))$. 

8
4 Proof of Theorem 2.1

Here we complete the proof of Theorem 2.1. Again \( D = \mathbb{C}[x, t]/\langle \partial_x, \partial_t \rangle \). We recall the definition of the Rees ring \( \mathcal{R}_V(D) = \oplus V_k(D)T^k \) endowed with the filtration \( F_d(\mathcal{R}_V(D)) = \oplus_k F_{d,k}(D)T^k \). We have a ring isomorphism \( \mathcal{R}_V(D) \cong D[\theta] \) given by \( x_i T^0 \mapsto x_i; t_i T^{-1} \mapsto t_i; T^1 \mapsto \theta; \partial_x T^0 \mapsto \partial_x, \) and \( \partial_t T^1 \mapsto \partial_t \).

The \( F \)-filtration on \( D[\theta] \) induced by this isomorphism is given by assigning the weights \( (0, 0, 1, 0) \) to \( (x, t, \partial_x, \partial_t, \theta) \). Suppose that, as a bifiltered \( D \)-module, \( M = D^r[m][m]/N \). We endow \( M \) with the \( V \)-filtration \( V_k(M) = V_k(D^r[m]/N) \).

Then we associate with \( M \) a Rees module \( \mathcal{R}_V(M) = \oplus V_k(M)T^k \) endowed with the \( F \)-filtration \( F_d(\mathcal{R}_V(M)) = \oplus_k F_{d,k}(M)T^k \). We have an isomorphism of graded modules

\[
\frac{D[\theta]^r[m]}{H^V(N)} \cong \mathcal{R}_V(M)
\]

given by \( \mathbf{e}_i \mapsto \mathbf{e}_iT^{m_i} \), with \( (e_i) \) the canonical base either of \( D[\theta]^r \) or of \( D^r \), and \( H^V(N) \) the homogenization of \( N \) with respect to \( V \). Furthermore it is an isomorphism of \( F \)-filtered modules

\[
\frac{D[\theta]^r[n]}{H^V(N)} \cong \mathcal{R}_V(M)
\]

We denote by \( \text{codim} \mathcal{R}_V(M) \) the codimension of the module \( \text{gr}^F(\mathcal{R}_V(M)) \). In fact it is allowed to replace the weight vector \( (0, 0, 1, 0) \) defining the \( F \)-filtration by any non-negative weight vector \( G \), giving rise to a filtration on \( D[\theta] \) also denoted by \( G \), such that \( \text{gr}^G(D[\theta]) \) is a commutative ring (see Proposition 5.1 of our previous paper[2]).

**Proposition 4.1.** \( \text{codim} \mathcal{R}_V(M) = \text{codim} M \).

**Lemma 4.1.** \( \text{codim} \mathcal{R}_V(M) \leq \text{codim} M \).

**Proof.** We make use of the characterization of the codimension by means of extension groups (see R. Hotta, K. Takeuchi and T. Tanisaki[6], Theorem D.4.3):

\[
\text{codim} M = \inf \{ i : \text{Ext}^i_D(M, D) \neq 0 \},
\]

\[
\text{codim} \mathcal{R}_V(M) = \inf \{ i : \text{Ext}^i_{\mathcal{R}_V(D)}(\mathcal{R}_V(M), \mathcal{R}_V(D)) \neq 0 \}.
\]

Let

\[
\cdots \rightarrow \mathcal{L}_1 \xrightarrow{\phi_1} \mathcal{L}_0 \xrightarrow{\phi_0} M \rightarrow 0
\]

be a \( V \)-adapted free resolution of \( M \), with \( \mathcal{L}_i = D^r[m^{(i)}] \). It induces a graded free resolution

\[
\cdots \rightarrow \mathcal{R}_V(\mathcal{L}_1) \xrightarrow{R_V(\phi_1)} \mathcal{R}_V(\mathcal{L}_0) \xrightarrow{R_V(\phi_0)} \mathcal{R}_V(M) \rightarrow 0,
\]

with \( \mathcal{R}_V(\mathcal{L}_i) \cong D[\theta]^r[m^{(i)}] \).

If \( N \) is a left \( D \)-module, let \( N^* = \text{Hom}_D(N, D) \). If \( N \) is a left \( \mathcal{R}_V(D) \)-module, let \( N^* = \text{Hom}_{\mathcal{R}_V(D)}(N, \mathcal{R}_V(D)) \).
The complex $\mathcal{L}^*$:

$$0 \to \mathcal{L}_0^* \xrightarrow{\phi_0^*} \mathcal{L}_1^* \xrightarrow{\phi_1^*} \cdots$$

gives the groups $\text{Ext}^1_D(M, D)$. The complex $\mathcal{R}_V(\mathcal{L})^*$:

$$0 \to \mathcal{R}_V(\mathcal{L}_0)^* \xrightarrow{\mathcal{R}_V(\phi_1)^*} \mathcal{R}_V(\mathcal{L}_1)^* \xrightarrow{\mathcal{R}_V(\phi_2)^*} \cdots$$

gives the groups $\text{Ext}^1_{\mathcal{R}_V}(\mathcal{R}_V(M), \mathcal{R}_V(D)).$

If $N$ is a left $D$-module endowed with a good $V$-filtration, then we endow $N^*$ by the exhaustive filtration

$$V_k(N^*) = \{u : N \to D \text{ such that } \forall j, u(V_j(N)) \subset V_{j+k}(D)\}.$$ Let us consider $\phi_i : \mathcal{L}_i \to \mathcal{L}_{i-1}$ together with $\phi_i^* : \mathcal{L}_{i-1}^* \to \mathcal{L}_i^*.$ We endow $\ker(\phi_i^*)$ with the induced $V$-filtration. We claim that

$$\mathcal{R}_V(\ker(\phi_i^*)) \simeq \ker(\mathcal{R}_V(\phi_i)^*).$$

Indeed, let $\mathcal{L} = D'[\mathfrak{m}]$ be a $V$-filtered free module. We have a bijection $\mathcal{R}_V(\mathcal{L}^*) \simeq (\mathcal{R}_V \mathcal{L})^*$, by mapping $\oplus u_k T^k$, with $u_k \in V_k(\mathcal{L})^*$, to $\sum \mathcal{R}_V(u_k) \in (\mathcal{R}_V \mathcal{L})^*$. Under that bijection, $\mathcal{R}_V(\ker(\phi_i^*)) \subset \mathcal{R}_V(\mathcal{L}_{i-1})^*$ is seen as a subset of $(\mathcal{R}_V \mathcal{L}_{i-1})^*$ and is equal to $\ker(\mathcal{R}_V(\phi_i)^*)$.

On the other hand, let us endow $\text{Im}(\phi_i^*)$ with the induced $V$-filtration. Using the identification $\mathcal{R}_V(\mathcal{L}_i^*) \simeq (\mathcal{R}_V \mathcal{L}_i)^*$, we have

$$\text{Im}(\mathcal{R}_V(\phi_i)^*) \subset \mathcal{R}_V(\text{Im}(\phi_i^*)).$$

Then if $H_i(\mathcal{L}^*) = \ker(\phi_{i+1}^*)/\text{Im}(\phi_i^*)$ is endowed with the quotient $V$-filtration, we have

$$H_i(\mathcal{R}_V(\mathcal{L}^*)) = \frac{\ker(\mathcal{R}_V(\phi_{i+1})^*)}{\text{Im}(\mathcal{R}_V(\phi_i)^*)} \to \frac{\mathcal{R}_V(\ker(\phi_{i+1}^*))}{\mathcal{R}_V(\text{Im}(\phi_i^*)} = \mathcal{R}_V(H_i(\mathcal{L}^*).$$

Thus if $H_i(\mathcal{R}_V(\mathcal{L}^*)) = 0$, then $\mathcal{R}_V(H_i(\mathcal{L}^*)) = 0$ which implies that $H_i(\mathcal{L}^*) = 0$. The lemma follows. \hfill \Box

**Lemma 4.2.** $\text{codim} M \leq \text{codim} \mathcal{R}_V(M)$.

**Proof.** Here we make use of the theory of Gelfand-Kirillov dimension, see e.g. G.G. Smith[15]. Let now $F$ denote the Bernstein filtration on $D$, i.e. each variable $x_i, \partial_x, t_i, \partial_t$ has weight 1. We endow $M$ with the good $(F, V)$-filtration, still denoted by $F_{d,k}(M)$, given by the quotient $D'[\mathfrak{m}]$.

Let $\phi : N \to \mathfrak{m}$. Let $\gamma(\phi) = \inf \{i : f(d) \leq d^i \text{ for } d \text{ large enough}\}$. By Gelfand-Kirillov theory, we have $\gamma(d \mapsto \dim \mathcal{E}_d(M)) = \dim M = 2n - \text{codim} M$.

Let us define the filtration $(G_d(D[\theta]))$ by giving the weight 1 to all the variables. Then $gr^{\mathcal{U}}(D[\theta])$ is commutative and for any $d$, $G_d(D[\theta])$ is finitely dimensional over $\mathbb{C}$. Then we endow $\mathcal{R}_V(M)$ with the $G$-filtration $G_d(\mathcal{R}_V(M)) = G_d(D[\theta]^r[0]/H^V(N))$. 


Let $d \in \mathbb{N}$ and $E_d$ be the interval $[\min_i m_i - d, \max_i m_i + d]$. We have

$$G_d(\mathcal{R}_V(M)) \subseteq \bigoplus_{k \in E_d} F_{d,k}(M)T^k \subseteq \bigoplus_{k \in E_d} F_d(M)T^k.$$ 

Then $\dim_C G_d(\mathcal{R}_V(M)) \leq (2d + c)\dim_C F_d(M)$, with $c = \max_i m_i - \min_i m_i + 1$. Thus

$$\dim \mathcal{R}_V(M) = \gamma(d \mapsto \dim_C G_d(\mathcal{R}_V(M)))$$
$$\leq \gamma(d \mapsto (2d + c)\dim_C F_d(M))$$
$$= 1 + \gamma(d \mapsto \dim_C F_d(M))$$
$$= 1 + \dim M.$$

\[\square\]

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