TREFOIL SOLITONS, ELEMENTARY FERMIONS, AND $SU_q(2)$

Robert J. Finkelstein

Department of Physics and Astronomy
University of California, Los Angeles, CA 90095-1547

Abstract. By utilizing the gauge invariance of the $SU_q(2)$ algebra we sharpen the basis of the $q$-knot phenomenology.
1 Introduction.

In constructing a knot model of the elementary particles the primary challenge is to establish a procedure for correlating the knots with the particles. In earlier work\textsuperscript{1,2} we have attempted to establish this map semi-empirically. With the aid of the quantum group $SU_q(2)$ we would now like to describe a clear set of \textit{a priori} rules for mapping the knots onto the particles.

The quantum group $SU_q(2)$ offers a possible realistic implementation of a model of elementary particles as knotted flux tubes, since on the one hand the linearization of $SU_q(2)$ approximates in low order the symmetry group of standard electroweak,\textsuperscript{1,2} while on the other hand $SU_q(2)$ underlies the description of knots.\textsuperscript{3} In this model the simplest particles (elementary fermions) are the simplest knots (trefoils). There are four families of elementary fermions and there are indeed four trefoils. If it is possible to match the four families with the four trefoils one may then ask whether the three members of each family are the three lowest states of excitation of a vibrating trefoil. Since all members of a family have the same quantum numbers, $(t,t_3,t_0)$ representing isotopic spin, its third component, and the hypercharge respectively, while each trefoil is also characterized by three integers $(N, w, r)$ representing the number of crossings, the writhe and the rotation, respectively, it is necessary to establish a correspondence between $(N, w, r)$ and $(t, t_3, t_0)$. This correspondence may be established by introducing the $D_{\text{jm}}^j$, the irreducible representations of $SU_q(2)$ and labelling both the trefoils and the elementary fermions by the same $D_{\text{jm}}^j$. In order to utilize half-integer representations of $SU_q(2)$ and also to respect the knot constraint that requires a difference in parity between $w$ and $r$ we choose

\begin{align}
  j &= \frac{N}{2} \\
  m &= \frac{w}{2} \\
  m' &= \frac{r+1}{2}
\end{align}

Then the knots will be labelled by $D_{\frac{N}{2},\frac{w}{2}}^{\frac{N/2}{2},\frac{r+1}{2}}$.

It remains to relate the $(t, t_3, t_0)$ to the $(j, m, m')$. 

2
2 Correspondence of the Four Families with the Four Trefoils.

The \((2j + 1)\)-dimensional irreducible representations of \(SU_q(2)\) are

\[
D_{mm'}^{j}(a, \bar{a}, b, \bar{b}) = \Delta_{mm'}^{j} \sum_{s,t} \left\langle \begin{array}{c} n_s \\ s \\ t \\ t \end{array} \right\rangle_{1} q_{1}^{(n_s + 1 - s)(-1)^t \delta(s + t, n'_+)} a^{s} b^{n_s - s} \bar{b}^{n'_+} \bar{a}^{n'_+ - t} \tag{2.1}
\]

where

\[
n_{\pm} = j \pm m \\
n'_{\pm} = j \pm m'
\]

\[
\left\langle \begin{array}{c} n \\ s \\ t \end{array} \right\rangle_{1} = \frac{\langle n \rangle_{1}!}{\langle s \rangle_{1}! \langle n - s \rangle_{1}!} \tag{2.2}
\]

and

\[
\Delta_{mm'}^{j} = \left[ \frac{\langle n'_+ \rangle_{1}! \langle n'_{-} \rangle_{1}!}{\langle n'_+ \rangle_{1}! \langle n'_{-} \rangle_{1}!} \right]^{1/2} q_{1} = q^{-1} \tag{2.3}
\]

Here the arguments \((a, b, \bar{a}, \bar{b})\) obey the following algebra:

\[
a b = q b a \quad a \bar{a} + b \bar{b} = 1 \quad b \bar{b} = \bar{b} b
\]

\[
a \bar{b} = q \bar{b} a \quad \bar{a} a + q^{2} \bar{b} b = 1 \quad \bar{b} \bar{a} = \bar{a} \bar{b}
\]

In the limit \(q = 1\), where \(\langle n \rangle_{1} \to n\) and the arguments commute, the \(D_{mm'}^{j}\) become the irreducible representations of \(SU(2)\). According to (2.1) the labelling of each trefoil by \(D_{w^2+r^2+1}^{3/2} \) leads to the monomials shown in Table 1.

| \((w, r)\) | \(D_{w^2+r^2+1}^{3/2} \) | \(\bar{D}_{w^2+r^2+1}^{3/2} \) |
|----------------|----------------|----------------|
| \((-3, 2)\) | \(D^{3/2}_{-\frac{3}{2}} \sim \bar{b}^3 \) | \(\bar{D}^{3/2}_{\frac{3}{2}} \sim b^3 \) |
| \((3, 2)\) | \(D^{3/2}_{\frac{3}{2}} \sim a^3 \) | \(\bar{D}^{3/2}_{\frac{3}{2}} \sim \bar{a}^3 \) |
| \((3, -2)\) | \(D^{3/2}_{\frac{3}{2}} \sim ab^2 \) | \(\bar{D}^{3/2}_{\frac{3}{2}} \sim \bar{b}^2\bar{a} \) |
| \((-3, -2)\) | \(D^{3/2}_{-\frac{3}{2}} \sim \bar{a}^2\bar{b} \) | \(\bar{D}^{3/2}_{-\frac{3}{2}} \sim ba^2 \) |

where the numerical coefficients appearing in \(D_{mm'}^{3/2}\) have been dropped. The corresponding labelling of the particles is accomplished first by attaching the \(D_{mm'}^{N/2}(a, \bar{a}, b, \bar{b})\) to the normal modes representing momentum and spin. Then the quantum fields will lie in the \((A)\) algebra.

We shall assume that the antiparticle fields are the adjoint fields in this algebra (as well as
the Dirac conjugate fields in the usual way). Therefore one must attach $D^{N/2}_{mm'}$ to the normal modes of the usual antiparticle field.

Since the $D^j_{mm'}$ are assumed to mediate the correspondence between the knots and the particles, we next assign $(t,t_3,t_0)$ as well as $(N,w,r)$ to $D^j_{mm'}$, and since the fermions all have $t = 1/2$ and $t_3 = \pm 1/2$ while the trefoils have $N = 3$ and $w = \pm 3$, we set

$$t = \frac{N}{6}$$  \hspace{1cm} (2.5)
$$t_3 = -\frac{w}{6}$$  \hspace{1cm} (2.6)

To find the third relation for $t_0$, or for $Q = t_3 + t_0$, we must first require that the particle and antiparticle have opposite charge $Q$. Next note that every term of (2.1) contains a product of non-commuting factors that may be reduced (after dropping numerical factors) to the form

$$a^{n_a} \bar{a}^{\bar{n}_a} b^{n_b} \bar{b}^{\bar{n}_b}$$  \hspace{1cm} (2.7)

where $n_a = s$ and $n_{\bar{a}} = n_- - t$.

But the factor $\delta(s + t, n'_+)$ appearing in (2.1) implies

$$n'_+ = n_a + (n_- - n_{\bar{a}})$$

or

$$n_a - n_{\bar{a}} = m + m'$$  \hspace{1cm} (2.8)

Note that (2.8) holds for all terms and all representations.

Since particles and antiparticles have adjoint symbols as well as opposite charge ($Q$) we may set

$$Q = k(n_a - n_{\bar{a}})$$  \hspace{1cm} (2.9)

or by (2.8)

$$Q = k(m + m')$$  \hspace{1cm} (2.10)

According to (2.10) one sees that $k = -1/3$ is the only choice of $k$ that agrees with the pairs $(w,r)$ of the trefoils and the charges of the four families as shown in Table 2.
Table 2.

| $(w, r)$  | $D^{3/2}_{\frac{1}{2}r + \frac{1}{2}}$ | $Q$ | Family          |
|----------|-------------------------------------|-----|-----------------|
| $(-3, 2)$ | $D^{3/2}_{\frac{3}{2} - \frac{3}{2}}$ | 0   | $(\nu_e, \nu_\mu, \nu_\tau)$ |
| $(3, 2)$  | $D^{3/2}_{\frac{1}{2} - \frac{1}{2}}$ | $-1$| $(e, \mu, \tau)$     |
| $(3, -2)$ | $D^{3/2}_{\frac{3}{2} - \frac{1}{2}}$ | $-\frac{1}{3}$| $(dsb)$          |
| $(-3, -2)$| $D^{3/2}_{-\frac{1}{2} - \frac{1}{2}}$ | $\frac{2}{3}$| $(uct)$          |

Hence $k = -1/3$ and

$$Q = -\frac{1}{3}(n_a - n_{\bar{a}}) \quad (2.12)$$

$$Q = -\frac{1}{3}(m + m') \quad (2.13)$$

It follows also from (2.9) that $Q = 0$ if

$$n_a = n_{\bar{a}} \quad (2.14)$$

In this case $a$ and $\bar{a}$ may be eliminated from (2.7) in favor of $b$ and $\bar{b}$ since

$$a^n\bar{a}^n = \prod_{s=0}^{n-1} (1 - q^{2s}b\bar{b}) \quad (2.15)$$

by (A). Therefore neutral states (neutrinos and neutral bosons) lie entirely in the $(b, \bar{b})$ subalgebra.

In knot coordinates $(N, w, r)$ the expression (2.13) for the charge of the fermions becomes

$$Q = -\frac{1}{6}(w + r + 1) \quad (2.16)$$

by (1.1). If one compares with the standard expression for $Q$

$$Q = t_3 + t_0 \quad (2.17)$$

then by (2.6) and (2.16)

$$t_3 = -\frac{w}{6} \quad (2.18a)$$

$$t_0 = -\frac{1}{6}(r + 1) \quad (2.18b)$$

so that we have Table 3 agreeing with (2.18) and the usual assignments of $Q, t_3,$ and $t_0$
The fact that there are two charges in the standard electroweak theory corresponding
to the separate groups $SU(2)$ and $U(1)$, should be evident as well in the knot model as
implemented by $SU_q(2)$. Therefore let us consider

$$Q_b = n_b - n_b$$ \hspace{1cm} (2.20)$$

where

$$n_b = n_+ - s \quad \text{and} \quad n_b = t$$ \hspace{1cm} (2.21)$$

Then the factor $\delta(s + t, n'_+) \text{ in } (2.1)$ implies

$$(n_+ - n_b) + n_b = n'_+$$ \hspace{1cm} (2.22)$$

or

$$m - m' = n_b - n_b$$ \hspace{1cm} (2.23)$$

again holding for all terms and all representations.

We may now consider the two charges derived for $D_{m m'}^j$, namely

$$Q_a = -\frac{1}{3}(n_a - n_a) = -\frac{1}{3}(m + m')$$ \hspace{1cm} (2.24)$$

$$Q_b = -\frac{1}{3}(n_b - n_b) = -\frac{1}{3}(m - m')$$ \hspace{1cm} (2.25)$$

By (1.1)

$$Q_a = -\frac{1}{6}(w + r + 1)$$ \hspace{1cm} (2.26)$$

$$Q_b = -\frac{1}{6}(w - r - 1)$$ \hspace{1cm} (2.27)$$

By (2.26), (2.27), and (2.18)

$$Q_a = -\frac{1}{3}(m + m') = t_3 + t_0$$ \hspace{1cm} (2.28)$$

$$Q_b = -\frac{1}{3}(m - m') = t_3 - t_0$$ \hspace{1cm} (2.29)$$
By (2.28) and (2.29)

\[ m = -3t_3 \quad (2.30) \]
\[ m' = -3t_0 \quad (2.31) \]

in agreement with (2.19) and assignments of \( D_{3}^{3/2} \) in (2.11).

Ignoring their derivation, let us next test the same relations (2.30) and (2.31) on the vector bosons. Since the vectors are responsible for pair production we shall represent them by ditrefoils with \( N = 6 \). Then by (2.6) and (1.1)

\[ t = \frac{N}{6} = 1 \quad (2.32) \]
\[ j = \frac{N}{2} = 3 \quad (2.33) \]

By (2.30) and (2.31) one has Table 4.

| \( W^+ \) | 1 | 1 | 0 | \( D_{3}^{3} \) | \( \sim b^3a^3 \) |
| \( W^- \) | 1 | -1 | 0 | \( D_{3}^{3} \) | \( \sim a^3b^3 \) |
| \( W^3 \) | 1 | 0 | 0 | \( D_{0}^{3} \) | \( \sim f_3(b\bar{b}) \) |
| \( W^0 \) | 1 | -1 | 1 | \( D_{3}^{3} \) | \( \sim b^6 \) |

where \( t_3 \) and \( t_0 \) are taken from standard electroweak theory. Here

\[ f_3(b\bar{b}) = \prod_{s=0}^{2}(1 - q^{2s}b\bar{b}) - q^23^2(3)^21^2(b\bar{b}) \prod_{s=0}^{2}(1 - q^{2s}b\bar{b}) + q^23^2(3)^21^2(b\bar{b})^2 \]
\[ \times (1 - b\bar{b})^2(1 - b\bar{b}) - q^{12}(b\bar{b})^3 \quad (2.34) \]

The connection with knots is expressed in the trefoil case by \( D_{3}^{N/2} \). If the ditrefoil is comprised of two connected trefoils, one has \( D_{3}^{N/2,3} \) where \( t_3 \) and \( t_0 \) are related to the knot parameters, \( w \) and \( r \), by Table 5.

Table 5 agrees with Ref. 2 where the vector ditrefoil is composed of two classically connected trefoils. It would be preferable to examine a ditrefoil as a quantum mechanical composite of two trefoils.
Table 5.

|   | $W^+$ and $W^-$ | $W^3$ | $W^0$ |
|---|-----------------|--------|-------|
| $t$ | $N/6$           | $(N-1)/6$ | $(N-1)/6$ |
| $t_3$ | $-w/6$         | $w-1$   | $w$   |
| $Q$  | $-(1/3)r$      | $r$     | $r$   |

3 Conservation Laws.

The knot-particle interpretation may be related to standard electroweak theory by ascribing additional structure to the normal modes of the quantum fields. Instead of point particles $(t, t_3, t_0)$ we now associate solitonic knots $(N, w, r)$ with the normal modes. The solitonic knots display internal degrees of freedom and excited states that are defined by the $SU_q(2)$ algebra. The gauge transformations on this algebra (A) that correspond to the charges $Q_a$ and $Q_b$ are

$$ a' = e^{i\phi_a}a $$

$$ \bar{a}' = e^{-i\phi_a}\bar{a} $$

$$ b' = e^{i\phi_b}b $$

$$ \bar{b}' = e^{-i\phi_b}\bar{b} $$

The algebra is invariant under these transformations. The induced gauge transformations on the solitonic modes are

$$ U_a D^j_{mm'} = e^{i\phi_a(n_a-n_{a'})}D^j_{mm'} $$

$$ = e^{-i3\phi_aQ_a}D^j_{mm'} $$

and

$$ U_b D^j_{mm'} = e^{i\phi_b(n_b-n_{b'})}D^j_{mm'} $$

$$ = e^{-i3\phi_bQ_b}D^j_{mm'} $$

The standard action contains a trilinear term describing the interaction of fermions and vector bosons. Here that interaction is multiplied by an additional factor of the following form:

$$ F_1W_2F_3 \rightarrow D^{3/2}_{m_1m_1'}D^1_{m_2m_2'}D^{3/2}_{m_3m_3'} $$

(3.3)
Under the gauge transformations (3.1) the product (3.3) is by (3.2) multiplied by
\[ e^{3i\varphi_a(-Q^a_1 + Q^a_2 + Q^a_3)} \] (3.4a)
and
\[ e^{3i\varphi_b(-Q^b_1 + Q^b_2 + Q^b_3)} \] (3.4b)
The invariance of the interaction (3.3) therefore implies the conservation of \( Q^a \) and \( Q^b \), or of the electric charge and hypercharge.

By (2.23) and (2.24) \( m + m' \) and \( m - m' \) are then conserved as well as \( m \) and \( m' \) separately, i.e.
\[ m_1 = m_2 + m_3 \]
\[ m'_1 = m'_2 + m'_3 \] (3.5)

4 Spectrum of the Algebra.

The operator \( b\bar{b} \) is a self-adjoint operator with real eigenvalues and orthogonal eigenstates. It follows from the algebra that
\[ b\bar{b}|n\rangle = q^{2n}|\beta|^2|n\rangle \] (4.1)
So that \( b\bar{b} \) resembles the Hamiltonian of an oscillator but with eigenvalues arranged in geometric progression and with \( |\beta|^2 \) corresponding to \( \frac{1}{2}\hbar \).

\( \bar{a} \) and \( a \) are raising and lowering operators respectively:
\[ \bar{a}|n\rangle = \lambda_n|n + 1\rangle \] (4.2)
\[ a|n\rangle = \mu_n|n - 1\rangle \] (4.3)
If the \( n \) are normalized
\[ \langle n|m \rangle = \delta(n, m) \] (4.4)
then
\[ |\lambda_n|^2 = 1 - q^{2n}|\beta|^2 \] (4.5)
\[ |\mu_n|^2 = 1 - q^{2(n-1)}|\beta|^2 \] (4.6)
If \( q \) is real as we assume, then there are no finite representations of this algebra obtained by imposing

\[
\bar{a}\left|M\right> = 0 \quad (4.7)
\]
\[
a\left|M'\right> = 0 \quad (4.8)
\]
\[
M > M' \quad (4.9)
\]

since these relations are then inconsistent.

One may obtain a finite representation, however, by imposing

\[
\bar{a}\left|M\right> = 0
\]

to cut off the spectrum at the top level and by imposing a physical boundary condition at the bottom level by interpreting \( |0\rangle \) as the state of lowest energy. We may interpret \( |n\rangle \) as the state with \( n \) nodes. This physical boundary condition is obviously externally imposed and supplements the algebra.

5 Phenomenology.

In standard electroweak theory the masses of the vector bosons are determined by the interaction of the vector and Higgs fields, while the mass of the Higgs itself is fixed by the Higgs potential. In the same theory the masses of the fermions are provisionally associated with a term of the following form

\[
\bar{\psi}_L \varphi \psi_R + \bar{\psi}_R \bar{\varphi} \psi_L
\]

(5.1)

where \( \psi \) and \( \varphi \) are the fermionic and Higgs fields respectively and where \( \psi_L \) and \( \varphi \) are isotopic doublets while \( \psi_R \) is an isotopic singlet.

Without essentially altering the Lagrangian of the standard model one could replace (5.1) by

\[
\bar{\psi}_L \varphi V(\bar{\varphi}\varphi) \psi_R + \bar{\psi}_R V(\bar{\varphi}\varphi) \bar{\varphi} \psi_L
\]

(5.2)

where \( V(\bar{\varphi}\varphi) \) is the Higgs potential, which determines the mass of the Higgs, or alternatively where \( V(\bar{\varphi}\varphi) \) may introduce different interaction terms that determine the masses of the fermions.
We shall take $\psi_R$ to be a singlet as in the standard theory. We shall also assume that all quantum fields, including the “Higgs” are defined over the algebra, and we shall additionally assume that the potential $V(\bar{\varphi}\varphi)$, determining the masses of the fermions has minima at the four points occupied by the four trefoils. These points are labelled by the four monomials in (2.2) and will be referred to as trefoil points. At these points the scalar $\varphi$ and the spinor $\psi$ of the Lorentz group have the same representation in the internal algebra.

Then (5.2) reduces at the trefoil points to

$$2\bar{\varphi}\varphi V(\bar{\varphi}\varphi) = F(\bar{\varphi}\varphi) \quad (5.3)$$

where

$$\varphi(w, r) \sim D^{3/2}_{\frac{w}{2} + \frac{1}{2}} \quad (5.4)$$

Hence the mass operator is a functional of

$$\bar{\varphi}\varphi \sim \bar{D}^{3/2}_{\frac{w}{2} + \frac{1}{2}} D^{3/2}_{\frac{w}{2} + \frac{1}{2}} \quad (5.5)$$

The eigenstates of $\bar{\varphi}\varphi$ are then the eigenstates of $\bar{b}\bar{b}$ since every $a$ is compensated by an $\bar{a}$ in the product (5.5) and therefore $\bar{D}^{3/2}_{\frac{w}{2} + \frac{1}{2}} D^{3/2}_{\frac{w}{2} + \frac{1}{2}}$ lies in the $(b, \bar{b})$ subalgebra. It follows that the eigenvalues $m_n(w, r)$, the masses of the trefoils, are given by

$$F \left( D^{3/2}_{\frac{w}{2} + \frac{1}{2}} D^{3/2}_{\frac{w}{2} + \frac{1}{2}} \right) |n\rangle = m_n(w, r)|n\rangle \quad (5.6)$$

depending on $F$. By (5.4) and (5.6) each of the four trefoils represented by $(w, r)$ may exist in various excited states $n$. We assume that only the lowest three states are occupied. In the lepton family, for example, these states $(0, 1, 2)$ are occupied by $(e, \mu, \tau)$. With an allowed choice of $V(\bar{\varphi}\varphi)$ one may obtain a finite mass spectrum for the fermions by imposing the algebraic boundary condition $\bar{a}|M\rangle = 0$ at the top level and by also imposing a physical boundary condition at the bottom level by interpreting $|0\rangle$ as the state of lowest energy.

Eq. (5.6) permits one to calculate relative masses of the twelve fermions. Ratios of masses in the same family may be computed without ambiguity. These calculations depend on either experimental or theoretical knowledge of $V(\bar{\varphi}\varphi)$. If $V = 1$, one has the standard model and the earlier results.$^{1,2}$
It is also possible to compute reaction rates by taking (3.3) between definite particle states as follows:

\[
\langle n| \bar{D}_{m_{1}m_{1}'}^{3/2} D_{m_{2}m_{2}'}^{1} D_{m_{3}m_{3}'}^{3/2} |n'\rangle = \langle n| \bar{\varphi}(w_{1}r_{1}) W(w_{2}r_{2}) \varphi(w_{3}r_{3}) |n'\rangle
\]

where

\[
\begin{align*}
n &= 0, 1, 2 \\
n' &= 0, 1, 2
\end{align*}
\]

(5.7)

In this way one may calculate relative masses of fermions by (5.6) and relative reaction rates mediated by vector bosons between fermions by (5.7). This work has been carried out in some detail in Ref. 2 but the model may be refined as more empirical information is utilized.

References.

1. R. J. Finkelstein, Int. J. Mod. Phys. A20, 487 (2005).

2. R. J. Finkelstein and A. C. Cadavid, hep-th/0507022