The classification of the finite groups whose abelian subgroups of equal prime power order are conjugate

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Abstract

Let $G$ be a finite group and assume $p$ is a prime dividing the order of $G$. Suppose for any such $p$, that every two abelian $p$-subgroups of $G$ of equal order are conjugate. The structure of such a group $G$ has been settled in this article.

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1 Introduction

Let us consider the following classes of finite groups.

(1) $\mathcal{B} = \{G| \text{each pair of subgroups of } G \text{ of equal order consists of conjugate subgroups}\}$

(2) $\mathcal{H} = \{G| \text{each pair of supersolvable subgroups of } G \text{ of equal order consists of conjugate subgroups}\}$.

In the formulation of (2), the word *supersolvable* can successively be replaced by *nilpotent, abelian, cyclic*. In doing so, one gets the corresponding classes of finite groups $\mathcal{N}, \mathcal{A}, \mathcal{C}$. Next, let us change the phrase of *equal order* from (1) and (2) into *of equal prime power order*; as such, one gets the classes (1)$' : \mathcal{B}_\pi$ and (2)$' : \mathcal{H}_\pi$ respectively. In addition, one gets the classes of finite groups $\mathcal{N}_\pi, \mathcal{A}_\pi, \mathcal{C}_\pi$ by replacing in (2)$'$ the word *supersolvable* again successively by *nilpotent, abelian, cyclic*. 
The classification of the groups in the classes $\mathcal{B}, \mathcal{N}, \mathcal{H}$ and $\mathcal{A}$ has been subject of study in the papers [3], [8], [9], [10] and [11]; that of $\mathcal{C}$ can be read off from [2]. Notice that the following hierarchies of classes of groups are apparent: $\mathcal{B} = \mathcal{H} \subseteq \mathcal{N} \subseteq \mathcal{A} \subseteq \mathcal{C}$ (the former $\subseteq$-sign is in fact Theorem 2.2 in [11]), and $\mathcal{B}_\pi \subseteq \mathcal{H}_\pi \subseteq \mathcal{N}_\pi \subseteq \mathcal{A}_\pi \subseteq \mathcal{C}_\pi$. Since any group of prime power order is nilpotent, any abelian group is nilpotent and any nilpotent group is supersolvable, this forces immediately $\mathcal{B}_\pi = \mathcal{H}_\pi = \mathcal{N}_\pi$. As it will be obtained below, it holds that the class of groups $\mathcal{A}_\pi$ is properly contained in $\mathcal{C}_\pi$, and likewise $\mathcal{N}_\pi \subseteq \mathcal{A}_\pi$. The simple first Janko group $J_1$ is a member of the class $\mathcal{B}_\pi$, but not of $\mathcal{B}$; see [11]. An example of a solvable group in the class $\mathcal{B}_\pi$, but not appearing in the class $\mathcal{B}$, has been found by F. Gross; see pages 337 and 338 in [4]. Hence $\mathcal{B} \subseteq \mathcal{B}_\pi$ anyway.

In this paper all groups are finite. Notations, symbols etc., will be standard; see for instance [5], but also [3].

In [7] the structure of the solvable groups from the class $\mathcal{C}_\pi$ has essentially been determined, while in [12] the structure of the non-solvable groups belonging to $\mathcal{C}_\pi$ has been established, next to a plethora of other results.

We will exhibit the structure of the groups belonging to the classes $\mathcal{B}_\pi$ and $\mathcal{A}_\pi$. Salient results from [7] and [12] will be properly quoted when needed.

## 2 Non-solvable $\mathcal{A}_\pi$-groups

We start with the classification of the non-solvable $\mathcal{A}_\pi$-groups, i.e. non-solvable $G \in \mathcal{A}_\pi$. Let $X \in \mathcal{C}_\pi$, then we know from (2.3) Theorem, [12] that any chief factor of $X$ is either cyclic of prime order, or non-cyclic elementary abelian, or, it is isomorphic to a specific non-abelian simple group $N$. In the last situation, $N$ happens to be isomorphic to one of the simple groups from the following list $\mathcal{L}$, consisting of the groups labeled $a$) up to and including $h)$, see §5 in [12].

- a) the Mathieu simple groups $M_{11}$ and $M_{23}$;
- b) the simple first Janko group $J_1$;
- c) the simple groups $PSL(2, q)$, $q = p^n$, $p$ an odd prime, $n \geq 1$, $n$ odd, $q \geq 7$;
- d) the groups $SL(2, q)$, $q = p^n$, $p$ an odd prime, $n \geq 1$, $n$ odd, $q \geq 7$;
- e) the simple Suzuki group $Sz(2^{2m+1}), m \geq 1$;
- f) the simple group $PSL(2, 2^a), a \geq 3$;
- g) $SL(2, 5)$;
- h) the simple alternating group $A_5$.

It has been shown in §5 of [12] that any group from the list $\mathcal{L}$ belongs to $\mathcal{C}_\pi$. It will be implicitly used that no two simple groups in the list $\mathcal{L}$ are isomorphic to each other, see [1]. In addition, it has been shown in (6.1) Theorem, [12], that in any $X \in \mathcal{C}_\pi$ admitting a non-solvable chief factor $L/K (K \leq X, K \leq L)$, it is true that any odd prime dividing $|L/K|$ does not divide the product $|X/L| \cdot |K|$; either one of the groups $X/L$ and $K$ happens to be solvable, see (2.3) Theorem, [12].

### 2.1 Case a)

In applying $\mathcal{A}_\pi \subseteq \mathcal{C}_\pi$, one observes that none of the groups $M_{11}$ or $M_{23}$ can be isomorphic to a chief factor of some alleged $G \in \mathcal{A}_\pi$, in its role of $X \in \mathcal{C}_\pi$ mentioned above. [Indeed, suppose it were, then (4.2) Theorem, [12] guarantees the existence of an $R \leq G$ with $G \cong M \times R$, $M \in \{M_{11}, M_{23}\}$ and $(|M|, |R|) = 1$. Thus $G \in \mathcal{A}_\pi$ would imply $M \in \mathcal{A}_\pi$. Now, as each Sylow 2-subgroup of $M_{11}$ and of $M_{23}$ is neither cyclic, nor elementary abelian, it follows on the contrary from the $\mathcal{A}_\pi$-property for $G$ when applied to the abelian 2-subgroups of $G$ of equal order being conjugate, that the alleged statement "$M \in \mathcal{A}_\pi$" in untenable. All this forces that the earlier appropriate group $N$ is not a chief factor of $G \in \mathcal{A}_\pi$ isomorphic to each of $M_{11}$ and $M_{23}$. Hence such a group $G$ is not an $\mathcal{A}_\pi$-group.]
2.2 Case e)

Now assume one deals with a group \( X \in \mathcal{C}_\pi \) in which a chief factor might be isomorphic to some \( S(2^{2m+1}), m \geq 1 \). Then (5.5) and (5.6) Theorems, [12] reveal that a normal subgroup \( S \) of \( X \) exists with \( S \cong S(2^{2m+1}) \). Any Sylow 2-subgroup of \( S \) does contain cyclic subgroups of order 4, as well as subgroups isomorphic to \( C_2 \times C_2 \). Therefore, the group \( X \in \mathcal{C}_\pi \) is not an \( \mathcal{A}_\pi \)-group.

Next, we are going to deal successively with the classes of groups \( b), c), d), f) followed by g) and h).

2.3 Case b)

Due to (4.2) Theorem, [12], any \( X \in \mathcal{C}_\pi \) admitting a chief factor isomorphic to the first Janko group \( J_1 \), does satisfy the structure \( X \cong N \times R \), in which \( N \cong J_1 \) and where \( ([N], |R|) = 1 \), with \( R \leq X, N \leq X, [N, R] = 1 \). It holds that \( J_1 \in \mathcal{N} \), see Theorem 2.1, [11]. As surely now \( J_1 \in \mathcal{A}_\pi \) holds, the specialized assumption \( X \in \mathcal{A}_\pi \) implies though that also \( R \in \mathcal{A}_\pi \); notice that \( R \) is solvable as \( 2 | |R| \). The contents of Theorem C in [7] imply that the Sylow subgroups of \( R \) are cyclic or non-cyclic elementary abelian. Hence as \( ([J_1], |R|) = 1 \) and \( X \in \mathcal{A}_\pi \) (whence \( X \in \mathcal{C}_\pi \)) we conclude that in particular \( X \in \mathcal{B}_\pi \). Indeed, \( J_1 \in \mathcal{N}_\pi = \mathcal{B}_\pi \) and \( R \in \mathcal{A}_\pi \) implies here \( R \in \mathcal{B}_\pi \).

2.4 Case c)

Let \( X \in \mathcal{C}_\pi \) admit an isomorphic copy of a non-abelian simple group occurring in \( c ) \), as a chief factor \( L/K \) and assume the order of \( K \) is odd. Then it is known from (5.9) Theorem, [12], that there exists an \( N \leq X \), with \( N \cong L/K \) and \( ([X/N], |N|) = 1 \). Any Sylow subgroup of \( X/N \) is cyclic or elementary abelian, see Theorem C in [7]. The Sylow 2-subgroups of the groups from the list \( \mathcal{L} \) occurring in \( c ) \) are dihedral of order \( 2^h, h \geq 3 \) (say) or elementary abelian of order 4. The Sylow \( t \)-subgroups, \( t \neq 2 \) of \( N \) are cyclic or non-cyclic elementary abelian. Next, we specialize these conditions to \( X \in \mathcal{A}_\pi \). Hence, by the order formula

\[
(p^n - 1) \cdot p^n \cdot (p^n + 1) = |PSL(2, p^n)| = |N|,
\]

where \( N \cong PSL(2, p^n), N \leq X, X \in \mathcal{A}_\pi, N \in \mathcal{L}, \) one deduces that the Sylow 2-subgroups of \( X \) and hence of \( N \) must be isomorphic to \( C_2 \times C_2 \). It follows that \( X \) actually belongs to \( \mathcal{B}_\pi \)! In addition, the aforementioned order formula provides the number theoretic condition \( p \equiv \pm 3 \mod 8 \).

2.5 Case d)

It has been shown in (2.2) Theorem in [7] that any factor group of a group \( X \in \mathcal{C}_\pi \) belongs to \( \mathcal{C}_\pi \). Assume that \( K \leq X \) exists in which \( L \leq X \) with \( L > K \) satisfies that \( L/K \) is isomorphic to a simple group from \( c ) \) in the list \( \mathcal{L} \). Assume \( K \) is of even order. Then (5.9) Theorem, [12] yields the existence of a normal subgroup \( S \) of \( X \) with \( S \cong SL(2, p^n) \), with this \( SL(2, p^n) \) being a group in \( d ) \) from the list \( \mathcal{L} \). Notice that here \( ([X/S], |S|) = 1 \) by that same Theorem (5.9), that \( X/S \in \mathcal{C}_\pi \) and that also \( L/K \) is isomorphic to \( SL(2, p^n)/Z(SL(2, p^n)) \), i.e. isomorphic to \( S/Z(S) \). Remember \( X/L, K \) and \( X/S \) are all solvable by the Jordan-Hölder-Zassenhaus Theorem. By the way, the logically equivalent statement " \( SL(2, t) \in \mathcal{C}_\pi \iff PSL(2, t) \in \mathcal{C}_\pi \) " , where \( t \) is some specific power of an odd prime, is certainly non-trivial to figure out: one has to use the facts that the Sylow 2-subgroups of \( SL(2, t) \) are generalized quaternion, whereas the corresponding Sylow 2-subgroups of \( PSL(2, t) \) are dihedral or isomorphic to \( C_2 \times C_2 \). Now, if one specializes to \( X \in \mathcal{A}_\pi \) in this Case \( d ) \), then certainly \( X \cong SL(2, q) \times R \) for some specific \( q \) being an odd power of an odd prime, \( ([R], |SL(2, q)|) = 1 \), the Sylow subgroups of \( R \) being cyclic or non-cyclic elementary abelian. All the non-cyclic elementary abelian Sylow subgroups of \( R \) do centralize \( SL(2, q) \), following (5.8) Theorem, [12]. Hence by Sylow, the implication \( X \in \mathcal{A}_\pi \Rightarrow R \in \mathcal{B}_\pi \) holds true. It is possible under the assumption \( X \in \mathcal{A}_\pi \) to pin down more specific information on the group \( SL(2, q) \). At first, \( SL(2, q) \in \mathcal{C}_\pi \) by (5.9) Theorem, [12], implying \( PSL(2, q) \in \mathcal{C}_\pi \). Furthermore, all abelian 2-subgroups of \( SL(2, q) \) are in fact cyclic, since a Sylow 2-subgroup of \( SL(2, q) \) is generalized quaternion. It is true, but a little bit tricky to verify, that any two cyclic subgroups of equal order of \( SL(2, q) \) are conjugate in \( SL(2, q) \), see for instance [11]. Now let \( q = p^n, p \) an odd prime, \( n \geq 1 \) odd. Notice \( X/\operatorname{C}_X(S) \rightarrow \operatorname{Aut}(SL(2, p^n)) \) and that \( \operatorname{Out}(SL(2, p^n)) \) is isomorphic to a cyclic group of order \( 2n \), just as \( n \) is odd. Moreover, due to (5.4) Theorem, [12], it holds for \( X \in \mathcal{A}_\pi \subseteq \mathcal{C}_\pi \), that \( X \)
isomorphic to \((SL(2,p^n) \times U)(\varphi)\), with \(|\varphi|\) dividing \(n\), that \((|U(\varphi)|, |SL(2,p^n)|) = 1\) and that \(U \leq G_{S\Delta U}(\varphi)(S)\). The Sylow \(l\)-subgroups of \(X\) with \(l\) any prime dividing \(|U(\varphi)/U|\) are cyclic, whereas \(U(\varphi)\) has precisely cyclic or non-cyclic elementary abelian Sylow subgroups, due to (5.4) Theorem, [12]. Hence, since \(X \in \mathcal{A}_n\), it follows that \(U(\varphi) \in \mathcal{A}_n\).

We did show that the abelian 2-subgroups of our \(X \in \mathcal{A}_n\) are contained in \(S \subseteq X\) and that the Sylow \(u\)-subgroups coincide with the Sylow \(u\)-subgroups of \(X\) for any prime \(u\) dividing \(|S|\). We are going to show that the integer \(n\) occurring in \(SL(2,p^n) \cong S\) turns out to be equal to 1 or to 3. This runs as follows.

Assume \(n \geq 3\). The Sylow \(p\)-subgroups of \(X\), whence of \(X \in \mathcal{A}_n\), are non-cyclic elementary abelian. The Frattini-argument (I.7.8 Satz, [5]) provides \(X = SN_X(S_p)\), where \(S_p \in Syl_p(S) = Syl_p(X)\). Therefore, \(X/S\), being isomorphic to \(N_X(S_p)/N_S(S_p)\), is solvable, as \(2 \mid |S|\) and \(|X/S|, |S|) = 1\). Notice that also \(N_{S/Z(S)}((S_p/Z(S))/Z(S))\) is isomorphic to \(N_S(S_p)/Z(S)/Z(S)\), whence to \(N_S(S_p)/Z(S)\). The group \(N_{S/Z(S)}((S_p/Z(S))/Z(S))\) is solvable, as \(S_p/Z(S)/Z(S)\) is elementary abelian of order \(p^n\) and as \(N_{S/Z(S)}((S_p/Z(S))/Z(S)) = (S_p/Z(S))/Z(S)\) is cyclic; see II.8.21 Hilfsatz [5].

Therefore, each of the groups \(N_X(S_p)/NS(S_p), N_S(S_p)/Z(S)\) is solvable, whence the group \(N_X(S_p)\) itself is solvable. Now, remember, \(S_p\) is elementary abelian, \(S_p \in Syl_p(X)\) and \(X \in \mathcal{A}_n\).

Thus \(X\) permutes all of the \(p\)-subgroups of equal order under conjugation action. In combination with \(N_X(S_p)\) being solvable, one observes that Lemma 3, [4], due to Gross, is usable, yielding \(n = 3\).

The proof of Lemma 3, [4] is a quite a little bit involved, among others, it needs Huppert’s Transitivity Theorem for doubly-transitive solvable groups; see XII.7.3 Theorem, [5b]. Originally in this rubric Case d), the integer \(n\) in the group \(SL(2,p^n)\) satisfies \(n \equiv 1 \mod (2)\). Hence indeed the integer \(n\) equals 1 or 3. Thus \(S\) is isomorphic to \(SL(2,p)\) or to \(SL(2,p^2)\). Observe, such \(S\) belongs a fortiori to \(\mathcal{A}_n\); see Main Theorem, [8].

A little warning. Only when for an odd prime \(p\), \(p \equiv \pm 3 \mod (8)\) happens to be the case, it is true that \(SL(2,p)\) and \(SL(2,p^2)\) are groups living in \(\mathcal{B}_\pi\). Indeed, otherwise 16 would divide the order of a Sylow 2-subgroup \(Q\) of \(SL(2,p)\) and likewise of \(SL(2,p^2)\), yielding that \(Q\) would contain one cyclic subgroup of order \(\bar{Q}/2\) and simultaneously two subgroups of that order \(\bar{Q}/2\) being (generalized) quaternion.

### 2.6 Case f)

We infer from (5.5) Theorem, [12], that there exists an \(N \leq Y\) with \((|N|, |Y/N|) = 1\) and \(N \cong PSL(2,2^n)\), whenever it is assumed that \(Y \in \mathcal{A}_n\) and that \(Y\) admits a chief factor \(L/K\) being isomorphic to the group \(PSL(2,2^n)\) in case \(a \geq 3\). The Sylow 2-subgroups of \(N\) are non-cyclic elementary abelian. The other Sylow subgroups of \(N\) are cyclic. Now, as the odd primes dividing \(|N|\) are distinct from those dividing \(|Y/N|\), it appears that any Sylow \(l\)-subgroup of \(Y/N\) happens to be cyclic or non-cyclic elementary abelian; indeed, \(l \geq 3\) holds, so that (5.8) Theorem, [12] yields the conclusion. If we specialize to \(Y \in \mathcal{A}_n\) here, one deduces though that \(Y\) is henceforth included in \(\mathcal{B}_\pi\).

In summary, we did show in the cases \(b), c)\) and \(f)\), that the assumption \(G \in \mathcal{A}_n\) may lead to \(G \in \mathcal{B}_\pi\). Precisely said, in each of these three cases any non-abelian simple normal subgroup \(N\) of \(G\) originating from the cases \(b), c)\) and \(f)\) with \(G/N\) solvable and satisfying \((|N|, |G/N|) = 1\) with \(G \in \mathcal{A}_n\), does yield \(G \in \mathcal{B}_\pi\), up to the situation in which \(p \equiv \pm 1 \mod (8)\) holds with respect to the group \(N \cong PSL(2,p^n)\).

There is more to say about those cases \(b), c)\) and \(f)\). In [4] it has been brought to light that any factor group \(T/E\) with \(E \leq T\) and \(T \in \mathcal{B}_\pi\), belongs to \(\mathcal{B}_\pi\). Furthermore, due to Theorem 1, [4], it holds that, given a non-solvable group \(T \in \mathcal{B}_\pi\), the factor group \(T/O_{2^\delta}(T)\) is isomorphic to one of the groups from the list \(V\), defined by \(V = (J_1, PSL(2,p), PSL(2,p^2), SL(2,8), SL(2,p), SL(2,p^3), PGL(2,32))\), where \(p\) is any prime congruent to \(\pm 3 \mod (8)\) and where \(PGL(2,32)\) stands for the group \(PSL(2,32) \times (\vartheta)\); here \(\vartheta \in Aut(PSL(2,32))\) with \(|\vartheta| = 5\). The automorphism \(\vartheta\) operates like a Frobenius automorphism on the elements inside any matrix of the group \(SL(2,32)\); notice \(SL(2,32) \cong PSL(2,32)\). As Gross did observe, each group from \(V\) belongs to \(\mathcal{B}_\pi\). Of course, the solvable group \(O_{2^\delta}(T)\) stands for the normal subgroup of \(T\) subject to being maximal of odd order.
The results obtained so far give rise to the following theorems.

**Theorem 1** Let $G \in \mathcal{L}_2$. Assume there exists a non-abelian chief factor of $G$, isomorphic to $PSL(2, 2^a)$ for some $a \geq 3$. Then either $a = 5$, or else $G \cong PSL(2, 8) \times R$, where $(|R|, |PSL(2, 8)|) = 1$, and $R \in \mathcal{L}_2$. It holds that $PSL(2, 8) \in \mathcal{B}_\pi$; in fact, $PSL(2, 8) \in \mathcal{B}_\pi$.

**Proof** The assumptions of the theorem have been considered above. Those did lead to the implication $G \in \mathcal{L}_2 \Rightarrow G \in \mathcal{B}_\pi$. Thus in consulting the list $\mathcal{L}$ and applying the Jordan-Hölder-Zassenhaus Theorem, we see that either $a = 3$ or $a = 5$ is forced. One observes that the direct product property holds for $a = 3$; use Case $f$ and Theorem 1, [4], with factors $N \cong PSL(2, 8)$ and $R \leq G$, generating $G$ and satisfying $(|N|, |R|) = 1$. Groups in $\mathcal{B}_\pi$ are factor-group-closed in taking homomorphisms. Thus $R \in \mathcal{B}_\pi$ as $R \cong G/N$. The fact that $PSL(2, 8)$ belongs to $\mathcal{B}_\pi$ can be found in [3]. □

Similar to the proof of Theorem 1, one is able to show the truth of the statements in Theorem 2.

**Theorem 2** Let $G \in \mathcal{L}_2$. Assume $L/K$ is a chief factor of $G$ being non-abelian. Suppose $L/K$ is neither isomorphic to $A_3$ or isomorphic to $PSL(2, 32)$. In addition, suppose $|K|$ is odd. Then $G$ is a direct product of a group $N$ and a group $R$, where $N$ stems from the list $\mathcal{L}$ as to the cases $b$ and $c$ of the list $\mathcal{L}_2$, such that $(|N|, |R|) = 1$ holds. Notice $G \in \mathcal{B}_\pi$ and $R \in \mathcal{B}_\pi$. □

More involved is the proof of the next Theorem 3, in which one deals with a group $G \in \mathcal{L}_2$ admitting a chief section isomorphic to $PSL(2, 32)$.

**Theorem 3** Let $G \in \mathcal{L}_2$ and assume $G$ admits a chief factor containing a subgroup isomorphic to $PSL(2, 32)$. Then $G = (N \times B)(\alpha)$, satisfying $N \unlhd G$, $N \cong PSL(2, 32)$, $B \leq G$, $(|N|, |B(\alpha)|) = 1$, $\alpha \in G \setminus (N \times B)$ some 5-element of $G$ with $\alpha^5 \in B$ and with $\langle \alpha \rangle \in Syl_5(G)$. In addition, $G \in \mathcal{B}_\pi$ is fulfilled and $B(\alpha) \in \mathcal{B}_\pi$ too. Furthermore $N(\alpha) \in \mathcal{B}_\pi$.

**Proof** By (5.3), (5.5), (6.1) Theorems, [12], one gets that there exists $N \unlhd G$ satisfying $(|N|, |G/N|) = 1$ and $N \cong PSL(2, 32)$. The group $N$ is simple. One has $G/C_G(N) \to Aut(N)$, where $Aut(N) \cong N \rtimes \langle \theta \rangle$ with $|\theta| = 5$. Look at $NC_G(N)/C_G(N)$. That group is isomorphic to $N/(NC_G(N))$, whence to $N$. Since $G \in \mathcal{L}_2$, it cannot be that $G$ equals $NC_G(N)$, which is $N \times C_G(N)$. [Indeed, suppose the contrary. Thus one has $(|N|, |C_G(N)|) = 1$ where also $N \in \mathcal{L}_2$ would happen. However, it would follow that $N \in \mathcal{B}_\pi$, because the Sylow 2-subgroups of $N$ are elementary abelian and the other Sylow subgroups of $N$ cyclic. It is known, see page 400, [3], that $PSL(2, 32)$ is not a member of $\mathcal{B}_\pi$. So we did produce a contradiction]. Therefore, due to Theorem 1, [4] in combination with the Jordan-Hölder-Zassenhaus Theorem, one gets $G/C_G(N) \cong (PSL(2, 32) \rtimes \langle \theta \rangle)$. Thus $C_G(N)$ is the wanted group $B \unlhd G$, showing there exists a 5-element $\alpha \in G \setminus (N \times B)$ satisfying $G = (N \times B)(\alpha)$. The Sylow $t$-subgroups of $B$ are elementary abelian or cyclic for $t$ any prime with $t \geq 7$ and $t$ dividing $|B|$. Subgroups of $B$ of equal order of some power of the prime $t$ are conjugate by means of products of elements of $B$ and powers of $\alpha$, whereas the Sylow 5-subgroups of $B(\alpha)$ are cyclic. [Indeed, any Sylow 5-subgroup of $G$ is conjugate to a Sylow 5-subgroup of $B(\alpha)$; it holds that $G \in \mathcal{L}_2$ (given), so that $\alpha^5 \in B$. In case $\alpha^5 \neq 1$ this forces that Sylow 5-subgroups of $B$ cannot be non-cyclic elementary abelian; if not, any Sylow 5-subgroup of $G$ would be non-cyclic elementary abelian and contained in $B$ itself, which is not the case.] The conclusion is that $B(\alpha) \in \mathcal{B}_\pi$. As to $N(\alpha) \in \mathcal{B}_\pi$, see Theorem 9.1, [3]. Notice $G \in \mathcal{B}_\pi$ follows. □

**Theorem 4** Let $G \in \mathcal{L}_2$ and assume $N \unlhd G$ exists with $N \cong SL(2, 5)$ or $N \cong A_5$. Then $G = N \times R$ with $R \leq G$ satisfying $(|N|, |R|) = 1$. Furthermore, $SL(2, 5) \in \mathcal{B}_\pi$, $A_5 \in \mathcal{B}_\pi$, and $R \in \mathcal{B}_\pi$, implying $G \in \mathcal{B}_\pi$.

**Proof** Do combine the outcome of (5.4) Theorem, [12] with that of Theorem 1, [4]. The result is the direct product property, yielding $R \in \mathcal{B}_\pi$. The statements $SL(2, 5) \in \mathcal{B}_\pi$ and $A_5 \in \mathcal{B}_\pi$ are to be found in Lemma 6 and Theorem 11, [3]. □
In accordance with the results proved so far, one observes that details of the structure of a non-solvable group $G \in \mathcal{A}_\pi$, are related to those groups $U \in \mathcal{A}_\pi$ satisfying $(6, |U|) = 1$. Those groups $U$ are solvable by the Feit-Thompson Theorem. So let us assume $N$ is a non-cyclic minimal normal subgroup of $U$. Hence $N$ is elementary abelian of $p$-power order for some prime $p \geq 5$. Since $U \in \mathcal{A}_\pi$, $Syl_p(U)$ is a singleton, i.e. $\{N\} = Syl_p(U)$, due to (4.4) Theorem [7]. Look at the group $O_{p'}(U)$, being the normal subgroup of $U$ of highest possible order relatively prime to $p$. Now, any non-cyclic Sylow subgroup of our $U \in \mathcal{A}_\pi$, with $(6, |U|) = 1$, is elementary abelian and normal in $U$; see the proof of Theorem 12 later on. Therefore $O_{p'}(U)N/O_{p'}(U)$ is not only contained in the Fitting subgroup $F(U/O_{p'}(U))$ of $U/O_{p'}(U)$, but even better, as $Syl_p(O_{p'}(U)N/O_{p'}(U)) = O_{p'}(U)N/O_{p'}(U) \cong \{N\}$ and as $F(U/O_{p'}(U))$ is nilpotent, satisfying $Syl_p(U/O_{p'}(U)) = Syl_p(O_{p'}(U)N/O_{p'}(U))$, it holds that $F(U/O_{p'}(U)) = O_{p'}(U)N/O_{p'}(U)$. Above it was implicitly observed that all non-cyclic Sylow $t$-subgroups of $U$ (if any) are contained in $O_{p'}(U)$, whenever such prime $t$ exists unequal to $p$. Hence all Sylow subgroups of $U/O_{p'}(U)$ are cyclic, whence $U/O_{p'}(U)$ is meta-cyclic; see Theorem 5 and (6.2) Theorem [7]. Hence the group $U/(O_{p'}(U)N)$, being isomorphic to $(U/O_{p'}(U))/F(U/O_{p'}(U))$, does permute all subgroups of equal order of the elementary abelian group $O_{p'}(U)N/O_{p'}(U)$ transitively under conjugation action, as $U \in \mathcal{A}_\pi$. So we have been ended up into the contents of (7.2) Theorem [7], whence it follows that $U/O_{p'}(U) \in \mathcal{B}$ holds. Therefore, due to Theorem 7.2, [10], the order of $O_{p'}(U)N/O_{p'}(U)$ equals $p^3$; here $(6, |U|) = 1$ is used again. Also, via Theorem 5.3 of [10], one gets that $U/(O_{p'}(U)N) \cong (U/O_{p'}(U))/F(U/O_{p'}(U))$ implies, using Theorem 7.2, [10], that $U/O_{p'}(U)N$ is cyclic. Thus we have got that for any prime $p_i$ dividing $|U|$ for which $S_i \in Syl_p(U)$ is not cyclic, it is true that $S_i$ is a normal subgroup of $U$ and that $S_i$ is elementary abelian of order $p_i^3$; whence, if there are $d \geq 2$ such primes $p_i$, one has $\prod_{j=1}^d |S_j| = \prod_{j=1}^d p_i^3$ and also $U/S_i \in \mathcal{B}$, where $S_i$ stands for the product $S_1 \cdot S_2 \cdots S_i \cdots S_d$, meaning that $S_i$ is not included in taking the product of the $S_i$’s. Finally, when it happens that it occurs that $(p_i^2 + p_i + 1, p_i^2 + p_i + 1) = 1$ whenever $1 \leq r < s \leq d$, then $U \in \mathcal{B}$ holds; see Corollary 5.5 of Theorem 5.3, [10]. Conversely, if $U \in \mathcal{A}_\pi$ is given with $(6, |U|) = 1$, then $U \in \mathcal{B}$, provided that relatively-prime-condition regarding the $p_i$’s $(j = 1, \ldots, d)$ from the last sentence is due to hold; see again Theorem 5.3, [10]. Furthermore, if the previous integer $d$ happens to be equal to 1, then $U$ appears to be a member of $\mathcal{B}$; see Theorem 3, [4]. Notice that any $U \in \mathcal{A}_\pi$ with $(6, |U|) = 1$ is a member of the class $\mathcal{A}_\pi$.

2.7 Cases g) and h)

In order to handle the cases g) and h) from the list $\mathcal{L}$, we will prove the following Theorem inter alia, irrespective of the structure of any arbitrary $G \in \mathcal{A}_\pi$.

**Theorem 5** Let $G \in \mathcal{A}_\pi$. Assume $N \trianglelefteq G$ exists with $2 \nmid |N|$. Then $G/N \in \mathcal{A}_\pi$.

**Proof** Suppose $G$ is a minimal counterexample to the Theorem. Then there exists a normal subgroup $T$ of $G$ for which $G/T \notin \mathcal{A}_\pi$ and $|T|$ odd. Take $T$ a fortiori as small as possible with respect to that conclusion. Then $T$ is a minimal normal subgroup of $G$. Indeed, otherwise there exists a $V \trianglelefteq G$ with $T \geq V \geq \{1\}$, here any such $V$ satisfies $G/V \in \mathcal{A}_\pi$. The group $V$ is of odd order. Also $G/T \cong (G/V)/(T/V)$, whence, as $V \neq \{1\}$, $G/T \in \mathcal{A}_\pi$ does follow, contrary to the choice of $T$ in $G$. Now, due to $|T|$ being odd, the group $T$ is solvable, whence $T$ is a $p$-group being cyclic or elementary abelian for some specific odd prime $p$. Let $X/T$ and $Y/T$ be abelian subgroups of $G/T$ of equal prime order, $G \geq X$ and $G \geq Y$. Assume firstly that $p$ does not divide $|X/T|$. Hence by the Schur-Zassenhaus Theorem, there exists $X_1 \leq G$ and $Y_1 \leq G$ of equal order satisfying $X = TX_1$, $Y = TY_1$ and $X_1 \cap T = \{1\} = Y_1 \cap T$. Moreover, as we know, the abelian groups $X_1$ and $Y_1$ are of equal prime power order. Since $G \in \mathcal{A}_\pi$, the groups $X_1$ and $Y_1$ are conjugate in $G$. Hence $X/T$ and $Y/T$ are conjugate in $G/T$ and of equal prime power order. Thus we may also assume that $X/T$ and $Y/T$ are $p$-groups. Suppose that $T$ were non-cyclic elementary abelian. Then, as $G \in \mathcal{A}_\pi$, no Sylow $p$-subgroup $P$ of $G$ does contain a cyclic subgroup of order $p^2$. Hence, any Sylow $p$-subgroup of $X$ (and of $Y$) would be contained in $T$, due to the $\mathcal{A}_\pi$-property of $G$. This, however, contradicts $p$ dividing $|X/T|$ with $|X/T| = |Y/T|$. Therefore, because $T$ is a minimal normal subgroup of $G$, it follows that $T$ is cyclic of order $p$. Thus, as $p$ is odd, Huppert’s Theorem III.8.3 Satz, [5], forces the Sylow $p$-subgroups of $G$ to be cyclic, since $G \in \mathcal{A}_\pi$ yields $T$ being the only existing subgroup of order.
p in $G$. As $X/T$ and $Y/T$ are $p$-subgroups of $G/T$ of equal order, one observes that $X$ and $Y$ are $p$-subgroups of $G$ of equal order (remember $X/T$ and $Y/T$ were chosen to be of equal prime power order). Now $X$ is contained in some cyclic Sylow $p$-subgroup $S$ of $G$ and $Y$ is likewise contained in some cyclic Sylow $p$-subgroup $\overline{S}$ of $G$. It holds by Sylow Theory, that $S$ and $\overline{S}$ are conjugate in $G$.

Therefore in particular $X$ and $Y$ are conjugate in $G$. Then, however, the cyclic groups $X/T$ and $Y/T$, being of equal prime power order, are conjugate in $G/T$. All in all now, one gets that, irrespective of the particular prime power order of $X/T$ (and of $Y/T$), it always holds that $X/T$ and $Y/T$ are conjugate in $G/T$. This, however, contradicts the choice of $T$ appearing in the second sentence of this proof with respect to $G/T \notin \mathcal{A}_\pi$. Therefore, no such $T$ exists. The Theorem has been proved. □

**Remark** The condition $2 \mid |N|$ in Theorem 5 cannot be omitted. The group $SL(2,7)$ belongs to $\mathcal{A}$, whence to $\mathcal{A}_\pi$, but $PSL(2,7)$ being a factor group of $SL(2,7)$, does contain cyclic subgroups of order 4 as well as non-cyclic abelian subgroups of order 4. Therefore $PSL(2,7)$ is not contained in $\mathcal{A}_\pi$.

**Theorem 6** Let $G \in \mathcal{A}_\pi$ and assume there exists a chief factor $L/K$ of $G$ containing a subgroup isomorphic to $A_5$. Then $L/K \cong A_5$. Suppose $2 \mid |K|$. Then there exists an $N \leq G$ with $N \cong A_5$. It holds that $G = N \times R$, a direct product of the group $N$ and some group $R \in \mathcal{B}_\pi$ with $(|N|,|R|) = 1$.

**Proof** Look at (5.8) Theorem, [12] and the fact that $R \in \mathcal{A}_\pi$ with $6, |R| = 1$ implies that $R \in \mathcal{B}_\pi$, as we saw above. □

In the next theorem we quote (7.1) Theorem, [12], in connection to Theorem 4 in shorthand shape.

**Theorem 7** Suppose $L/K \cong A_5$ is a chief factor of $G \in \mathcal{A}_\pi$. Assume $2 \mid |K|$. Then
1) either $G \cong SL(2,5) \times U$, with $U \in \mathcal{A}_\pi$ and $(30, |U|) = 1$;
2) or else $G = (V \times T) \rtimes W, V \leq G, T \leq G, ([V]|, |T|) = 1 = ([V \times T], |W|); W$ is a meta-cyclic $\mathcal{C}_\pi$-group, or $W = \{1\}$, or $W$ is cyclic; $T = \{1\}$ or $T = \prod_{i=1}^{4}(\text{non-cyclic elementary abelian Sylow subgroups } S_1 \leq TW)$ with all $S \leq TW$; $TW$ is a solvable $\mathcal{C}_\pi$-group,

$$[[W, W], F] = \{1\}, F \neq G, V = F \rtimes S, [V, V] = V, S \cong SL(2,5); F = \prod_{i=1}^{4}(C_{p_i} \times C_{p_i})^{\delta_i},$$

with $p_1 = 11, p_2 = 19, p_3 = 29, p_4 = 59$, all $\delta_i \in \{0, 1\}$ except $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$. □

Next, in Theorem 8, we do specialize the group $G$ occurring in Theorem 7, in being a group from $\mathcal{A}_\pi$.

**Theorem 8** Let $G$ be a group from the announcement in Theorem 7 and assume $G \in \mathcal{A}_\pi$. Then, using the notations and notions from Theorem 7,
1) either $G \cong SL(2, 5) \times U$, with $U \in \mathcal{A}_\pi$ and $(30, |U|) = 1$;
2) or else $G/F \in \mathcal{A}_\pi, G/F \times Z(S) \in \mathcal{B}_\pi, [S, TW] = \{1\}$;

$TW$ is a solvable $\mathcal{B}_\pi$-group all of whose non-cyclic Sylow $t$-subgroups are of order $t^3$ and also elementary abelian, whenever $t$ is a prime dividing $|T|;

W \in \mathcal{B}_\pi$, and if $W \neq \{1\}$, then $W$ is (meta-)cyclic satisfying $([W/[W, W]], [[W, W]]) = 1, VW \in \mathcal{A}_\pi$ and $[[W, W], F] = \{1\}$.

**Proof** Re 1) Here $G \in \mathcal{A}_\pi$ has certainly the structure of part 1) in the announcement of Theorem 7. In addition, $U \in \mathcal{B}_\pi$ holds due to the results discussed earlier between Theorem 4 and Theorem 5.

Re 2) Because of Theorem 6, $G/F \in \mathcal{A}_\pi$ holds. Hence $G/F$ satisfies a structure as found in part 1) of Theorem 7 and also that from Re 1) just proven regarding Theorem 8. Therefore also $G/F \times Z(S)$ is an $\mathcal{A}_\pi$-group; notice $PSL(2,5) \in \mathcal{B}_\pi$ and $V \in \mathcal{B}_\pi$, together with $G/F \times Z(S) \cong A_5 \times TW$. The rest of the Theorem has been discussed earlier. The property $W \in \mathcal{B}_\pi$ has been shown in Theorem 3, [4] and its immediate Corollary; notice that it is implicitly used that each of the Sylow subgroups of $W$ is equal to some Sylow subgroup of $G$ and that those subgroups are cyclic. Furthermore $VW \cong G/T \in \mathcal{A}_\pi$ holds; the groups $[W, W]$ and $F$ do centralize each other, as shown in (7.1) Theorem, [12]. □

**Remark** It is surely possible that in the Theorems 7 and 8 the group $W$ does not act trivially on $F$ by conjugation. It might only happen when $p_3 = 29$ with $\delta_3 = 1$, where $W$ contains a cyclic Sylow
Theorem 10

Let \( \langle c^7, F \rangle = \{1\} \) and when \( p_1 = 59 \) with \( d_1 = 1 \) occurs and \( W \) possesses a cyclic Sylow 29-subgroup \( (d) \) such that \([d^{29}], F] = \{1\} \). Notice, however, inside the corresponding group \( G \in \mathcal{A}_\pi \), it is not possible that both the described situations happen to occur simultaneously. The easiest paradigm examples of Theorem 8, part 2), are \( G \cong (C_{29} \times C_{29}) \triangleright (SL(2, 5) \times C_7) \) and \( G \cong (C_{59} \times C_{59}) \triangleright (SL(2, 5) \times C_{20}) \); both these groups belong to \( \mathcal{B} \), whence to \( \mathcal{B}_\pi \). See Theorem 11, [3] and §7, [12].

2.8 Sylow subgroups of non-solvable \( \mathcal{A}_\pi \)-groups

We are going to find out, what structure a Sylow subgroup of a given \( Y \in \mathcal{A}_\pi \) can afford. Notice that a non-cyclic \( t \)-subgroup of an non-solvable \( Y \in \mathcal{A}_\pi \), \( t \) an odd prime, is indeed elementary abelian, see the next Theorem 9. As to that statement regarding solvable \( \mathcal{A}_\pi \)-groups, see Section 3 below.

Theorem 9

Let \( Y \in \mathcal{A}_\pi \) be non-solvable and assume \( S_p \in \text{Syl}_p(Y) \), \( p \) an odd prime. Then \( S_p \) is either cyclic or else it is non-cyclic elementary abelian.

Proof Suppose firstly \( p \geq 5 \) and assume \( S_p \) is not cyclic. Then, by (5.8) Corollary, [12], \( S_p \) is a normal subgroup of \( Y \) or some \( PSL(2, p^f)(f \geq 2) \) is isomorphic to a chief section of \( Y \). If \( S_p \triangleleft Y \in \mathcal{A}_\pi \) happens to be true, then \( S_p \) turns out to be elementary abelian. Thus suppose that some \( PSL(2, p^f)(f \geq 2) \) is isomorphic to a chief factor of \( Y \), whence \( f \) is odd. Hence \( S_p \) is elementary abelian because of \( S_p \) being isomorphic to a Sylow \( p \)-subgroup of that \( PSL(2, p^f) \); see II.8.10 Satz, [5] and (6.1) Theorem, [12]. What about \( p = 3 \)? Assume \( S_3 \) is not cyclic and that \( S_3 \) and the Fitting subgroup \( F(Y) \) of \( Y \) do intersect each other non-trivially. Then it follows that \( S_3 \) is in fact a normal subgroup of \( Y \), due to \( Y \in \mathcal{A}_\pi \) in combination with III.7.5 Hilfssatz, [5], so that \( S_3 \) is elementary abelian too, by the fact that here \( Y \in \mathcal{A}_\pi \) is 3-solvable implying, by a result of Shult VIII.7.11 Remarks, [5b], that \( S_3 \) happens to be abelian at first sight; see also (4.1) Lemma and Theorem C, [7]. Thus assume \( S_3 \cap F(Y) = \{1\} \). It follows then from the classification of the non-solvable \( \mathcal{C}_\pi \)-groups as carried out in (5.4) , (6.1) and (7.1) Theorems, [12], that either \( S_3 \) is isomorphic to a subgroup of a simple chief factor of \( \mathcal{A}_\pi \), in which indeed our non-cyclic group \( S_3 \) turns out to be elementary abelian, or else that \( S_2(2^{2m+1}) \leq Y \in \mathcal{A}_\pi \) should be fulfilled. The last eventuality however, is not possible, as the groups \( S_2(2^{2m+1}) \) do possess cyclic as well as non-cyclic subgroups all being of order 4. Therefore, for any \( S_3 \in Syl_3(Y) \) with \( Y \in \mathcal{A}_\pi \), the assumption on \( S_3 \) being non-cyclic does lead to the fact that \( S_3 \) turns out to be elementary abelian. \( \Box \)

Let us introduce some notation. \( E_{pm} \) – an elementary abelian group of order \( p^m \) \((p \) any prime, \( n \) a natural number); \( Q_\nu \) – a generalized quaternion group of order \( 2^\nu \) with \( \nu \geq 4 \); \( Q \) – the quaternion group of order 8; \( C_u \) – a cyclic group of order \( u \) with \( u \geq 1 \).

Theorem 10

Let \( X \in \mathcal{A}_\pi \) be non-solvable. Then \( S_2 \in Syl_2(X) \) is isomorphic to one of the following groups: \( Q, E_{4}, E_{8}, E_{32} \).

Proof If \( S_2 \) would be cyclic, then \( X \) would be solvable, as it is known from Burnside’s theorem that \( X/O_{2^\nu}(X) \cong S_2 \) (see IV.2.8 Satz, [5]); remember, the odd order group \( O_{2^\nu}(X) \) is solvable due to the Feit-Thompson Theorem. If \( S_2 \) is not cyclic, then the statement regarding the structure of \( S_2 \) is to be found in the results sofar. \( \Box \)

3 Solvable \( \mathcal{A}_\pi \)-groups

In this Section we do focus our attention on the structure of the Sylow subgroups of solvable \( \mathcal{A}_\pi \)-groups. To start with, let us look at the following Theorem.

Theorem 11

Let \( X \in \mathcal{C}_\pi \) be solvable and assume \( p \) is a prime dividing \( |X| \). Then \( S_p \in Syl_p(X) \) is isomorphic to a group belonging to the following classes of groups: \( \{C_{pm}\}_{a \geq 1}, \{E_{pm}\}_{b \geq 2}, \{Q\}, \{\text{Suzuki 2-groups} \neq Q\} \).

Proof The statement of the Theorem is the outcome of §3, [7]. \( \Box \)
Let us specialize Theorem 11 to the case in which $X \in \mathcal{A}_\pi$ is solvable. Notice, that if a group $Y \in \mathcal{C}_\pi$ admits Suzuki 2-groups $\not\cong Q$ as Sylow 2-subgroups, it cannot be that $Y$ belongs to $\mathcal{A}_\pi$; as such, those Suzuki 2-groups do contain cyclic subgroups of order 4 as well as non-cyclic abelian subgroups of order 4.

**Theorem 12** Let $X \in \mathcal{A}_\pi$ be solvable and assume $p$ is a prime dividing $|X|$. Then $S_p = Syl_p(X)$ is isomorphic to a group from the following classes of groups: $\{C_p, a \geq 1\}$, $\{E_p: p \not\equiv 2\}$, $\{E_3\}$, $\{E_8\}$, $\{Q\}$. Furthermore, $X/O_2^2(X)$ is isomorphic to one of the following groups: $C_2^a(a \geq 0)$, $E_4 \times C_3$, $E_8 \times (C_7 \times C_3)$, $E_8 \times C_7$, $E_{32} \times (C_{31} \times C_3)$, $Q \times C_3$.

**Proof** By using the property that for any given prime $t$, Sylow $t$-subgroups of any group are conjugate to each other in that group, it follows from Theorem 11, that given $X \in \mathcal{A}_\pi$ being solvable and observing that $Q$ has only as subgroups of order 4 the cyclic ones and that it contains only one subgroup of order 2, it holds that $X \in \mathcal{B}_\pi$. [This result will be mentioned as Theorem 15, in order to give it a prominent place.]. Hence by Theorem 1, [4]) one observes that any Sylow 2-subgroup of $X$ is either isomorphic to some cyclic 2-group, or to $E_4$, or to $E_8$, or to $E_{32}$, or to $Q$. Namely, Gross in his Theorem 1, [4] did show the truth of the statement as formulated in the sentence of our Theorem beginning with "Furthermore ...". We proceed in providing self-contained elaborated reasoning around the structure of $S_p = Syl_p(X)$, when $p$ is odd. Suppose $S_p$ is not cyclic. Then there do not exist cyclic subgroups of order $p^i$ in $S_p$ for any $i \geq 2$. [Indeed, assume it were. Then, as $S_p \leq X \in \mathcal{A}_\pi$, $S_p$ contains exactly one subgroup of order $p$. [Otherwise, as $Z(S) \geq C_p$ for some specific $C_p$, there would exist $C_p \times D$ with $D \not\cong C_p$, $D \leq S_p$, $|D| = p$; a contradiction to $X \in \mathcal{A}_\pi$.]

Hence III.7.5 Hilfssatz, [5] yields $S_p$ being cyclic, a contradiction to the assumed non-cyclcity of $S_p$.]

Thus $S_p$ does contain only as non-trivial cyclic subgroups, its cyclic subgroups of order $p$. Now $X \in \mathcal{A}_\pi$ permutes all its cyclic subgroups of prime power order transitively under conjugation action. Therefore, by VIII.5.8.b Theorem and VIII.7.11 Remarks, [5a] in which a lemma of Shult is involved, one concludes that $S_p$ is abelian. Thus, as $S_p$ does not contain elements of order $p^i$, $S_p$ is elementary abelian. Hence Gross’ Lemma 3, [4] is valid for our non-cyclic group $S_p$ with $p \not\equiv 2$ too, i.e. $S_p \cong E_{p^2}$ or $S_p \cong E_{p^3}$. □

**Corollary 13** Assume $S_p \in Syl_p(X)$ happens to be non-cyclic for some prime $p$, where $X \in \mathcal{A}_\pi$ is understood to be solvable. Then $S_p \leq X$ or $S_p \cong Q$.

**Proof** Apply Theorem 12. Hence one gets, by using (4.4) Theorem, [7], that $S_p \leq X$ unless $S_p \cong Q$. [Indeed, (4.4) Theorem, [7] says that elementary abelian non-cyclic Sylow subgroups of a solvable group $Y \in \mathcal{C}_\pi$, happen to be entirely contained in the Fitting subgroup of $Y$, whence implying these subgroups to be normal in $Y$.] □

**Corollary 14** Assume $Q$ to be isomorphic to some Sylow 2-subgroup $S_2$ of some $X \in \mathcal{A}_\pi$. Then not always $S_2 \leq X$ does hold.

**Proof** Take $X \cong E_{32} \rtimes SL(2,3)$, where the action by conjugation of $SL(2,3)$ on $E_{32}$ is Frobenius and faithful. Then $X \in \mathcal{B} \leq \mathcal{A}_\pi$, but $S_2 = Syl_2(X)$ being isomorphic to $Q$, is not a normal subgroup of $X$. Notice $X$ is a solvable group. □

To be complete, when $S_2$ is a cyclic Sylow 2-subgroup of some group $H$, it holds that $H/O_2^2(H) \cong S_2$; see IV.2.8 Satz, [5].

**Theorem 15** Let $G \in \mathcal{A}_\pi$ be solvable, then $G \in \mathcal{B}_\pi$.

**Proof** The truth of the statement of this theorem follows directly from combining Theorem 12 and Corollary 13. □

Due to Theorem 15, one observes that solvable $\mathcal{A}_\pi$-groups are closed under taking homomorphisms, as follows.
**Theorem 16** Let $N$ be a normal subgroup of a solvable group $G \in \mathcal{A}_\pi$. Then $G/N \in \mathcal{A}_\pi$.

**Proof** It follows from Theorem 15 that $G \in \mathcal{B}_\pi$ holds. Groups in $\mathcal{B}_\pi$ are closed under taking homomorphisms, as observed on page 333, [4]. Thus $G/N \in \mathcal{B}_\pi$. As $\mathcal{B}_\pi \subseteq \mathcal{A}_\pi$, we are done. □

There exists a theorem, a converse to Theorem 16 and valid for any $G$, solvable or not, as follows.

**Theorem 17** Let $G$ be a group and assume $N_1 \leq G$ and $N_2 \leq G$ do exist, satisfying $(|N_1|, |N_2|) = 1$. Suppose $G/N_1 \in \mathcal{A}_\pi$ and $G/N_2 \in \mathcal{A}_\pi$. Then $G \in \mathcal{A}_\pi$.

**Proof** Let $X$ and $Y$ be abelian subgroups of $G$ of equal prime-power order $p^m (m \geq 1)$. Then at least one of the orders $|N_1|$ or $|N_2|$, say $|N_1|$, is not divisible by $p$. Hence $XN_1/N_1$, being isomorphic to $X/(X \cap N_1)$, and $YN_1/N_1$, being isomorphic to $Y/(Y \cap N_1)$, are abelian groups each of order $p^m$, just by $X \cap N_1 = \{1\} = Y \cap N_1$. Therefore, as $G/N_1 \in \mathcal{A}_\pi$, there exists $g \in G$, satisfying $(gN_1)^{-1}(gN_1/N_1)(gN_1) = YN_1/N_1$. Thus $(g^{-1}Xg)N_1/N_1 = YN_1/N_1$. The groups $g^{-1}Xg$ and $Y$ are Sylow $p$-subgroups of $YN_1$. Hence, by Sylow’s theorem, there exists a $t \in YN_1$, such that $Y = t^{-1}(g^{-1}Xg)t = (gt)^{-1}X(gt)$. In other words, $X$ and $Y$ are conjugate in $G$. So indeed, $G \in \mathcal{A}_\pi$ does follow. □

Summarizing, according to Theorem 15, any solvable $\mathcal{A}_\pi$-group has turned out to be a $\mathcal{B}_\pi$-group, while it holds that any non-cyclic Sylow $p$-subgroup $S \neq Q$ of a solvable $\mathcal{A}_\pi$-group $G$ is normal in $G$; namely those groups $S$ are elementary abelian, making (4.4) Theorem, [7] in vogue. Due to these facts, the analysis of the structure of a solvable $\mathcal{A}_\pi$-group is rather easy to accomplish. Let us see what happens.

**Theorem 18** Let $G \in \mathcal{A}_\pi$ be solvable and assume there exists $E_{p^2} < G$ for some particular prime $p$. Then
1) either $G/O_{p'}(G)$ does contain a normal Sylow $p$-subgroup being non-cyclic elementary abelian, whereas the other Sylow subgroups of $G/O_{p'}(G)$ (if any) are cyclic;
2) or else $p = 5$ and $G/O_{p'}(G) \cong E_{5^2} \rtimes SL(2, 3)$, or $p = 11$ and $G/O_{11'}(G) \cong E_{11^2} \rtimes SL(2, 3)$, or $G/O_{11'}(G) \cong E_{11^2} \rtimes (SL(2, 3) \times C_5)$.

**Proof** Notice $G/O_{p'}(G)$ is solvable and $G/O_{p'}(G) \in \mathcal{A}_\pi$. We want to apply the contents of (7.2) Theorem, [7] with respect to the group $G/O_{p'}(G)$. In order to accomplish that goal, one has to do the following.
Assume $G/O_{p'}(G)$ does contain a non-cyclic non-elementary Sylow subgroup; whence it has to be non-abelian due to Theorem 12, for the group $G/O_{p'}(G) \in \mathcal{A}_\pi$. It holds that $E_{p^2} \leq S_p \in Sy_{p,p'}(G)$, so that $S_p$ is elementary abelian. By (4.4) Theorem, [7] it follows that $S_p < G$. Thus $S_pO_{p'}(G)/O_{p'}(G)$, being a Sylow $p$-subgroup of $G/O_{p'}(G)$, happens to be normal in $G/O_{p'}(G)$. So $p$ does not divide $|G/O_{p'}(G)|$. By definition of $O_{p'}(G)$, the fact that $S_pO_{p'}(G)/O_{p'}(G)$ is normal in the Fitting subgroup $F(G/O_{p'}(G))$, the solvability of $G/O_{p'}(G)$ and the fact that $F(G/O_{p'}(G))$ is nilpotent, all together imply that $F(G/O_{p'}(G)) = S_pO_{p'}(G)/O_{p'}(G)$. Notice now, as $G/O_{p'}(G) \in \mathcal{A}_\pi$, that $S_pO_{p'}(G)/O_{p'}(G)$ is a chief factor of $G/O_{p'}(G)$ and that there exists no other minimal normal subgroup in $G/O_{p'}(G)$ than $S_pO_{p'}(G)/O_{p'}(G)$, due to the solvability of $G/O_{p'}(G)$ and $F(G/O_{p'}(G))$ being the unique normal Sylow $p$-subgroup of $G/O_{p'}(G)$. Hence $G/O_{p'}(G)$ is a so-called (P)-subdirectly irreducible solvable $\mathcal{A}_\pi$-group in the terminology of [7], containing a non-abelian Sylow subgroup by our assumption in the beginning of the proof of the Theorem. Hence we have reached the point where (7.2) Theorem, [7] can be applied on $G/O_{p'}(G)$, namely that it enforces that $p$ divides the integer 55; as such one gets the structure of the three particular groups appearing in the statement of the Theorem. The working-out of those facts has been done in the proof of (7.2) Theorem, [7].

**Next** suppose that all Sylow subgroups of $G/O_{p'}(G) \in \mathcal{A}_\pi$ are abelian. One of them is the non-cyclic elementary abelian and normal $p$-subgroup $S_pO_{p'}(G)/O_{p'}(G)$. Any other alleged proper Sylow subgroup $S$ of $G/O_{p'}(G) \in \mathcal{A}_\pi$ is cyclic or otherwise non-cyclic elementary abelian because of $G/O_{p'}(G) \in \mathcal{A}_\pi$. If $S$ is not cyclic, it thus would be normal in $G/O_{p'}(G)$ by Corollary 13. However,
this is not possible, for then $S \subseteq F(G/O_p(G)) = S_pO_p(G)/O_p(G)$ produces the contradiction $p \mid |S| \mid |S_pO_p(G)/O_p(G)| = |S_p|$. Hence all Sylow subgroups of $G/O_p(G)$ other than $S_pO_p(G)/O_p(G)$ are cyclic. The Theorem has been proved. □

**Corollary 19** Let $G \in \mathcal{A}_\pi$ be solvable. Assume there exists a prime $p$ dividing $|G|$ such that $E_p < G$. Then $G/O_p(G) \in \mathcal{B}$.

**Proof** Look at the statements in Theorem 18. If $G/O_p(G)$ is a group occurring in the phrase just after the word "either", then $G/O_p(G) \in \mathcal{B}$ holds as it satisfies the contents of Theorem 11, [9]. As to the three exceptional groups remaining in Theorem 18, a detailed study by means of the computer language GAP has been carried out in Sektion 3.2, [6]. □

All the facts about solvable $\mathcal{A}_\pi$-groups obtained so far, can be collected in the following Portmanteau Theorem.

**Theorem 20** (Portmanteau Theorem)
Let $G$ be a solvable $\mathcal{A}_\pi$-group. Then one of the following eight statements is true and all do occur in practice.

1) Suppose all Sylow subgroups of $G$ are cyclic. Then $G \in \mathcal{B}$, $G$ is meta-cyclic or cyclic, $([|G|, G]|, |G/[G, G]|) = 1$.

2) Suppose $E_B \triangleleft G$. Then $G = (E_B \times R(a, b)$ or $G = (E_B \times T(c)$ with $R \triangleleft G$, $(a, b) \in \mathcal{B}_\pi$, $R(a, b)/R \cong C_7 \times C_3$, $\langle a \rangle \in Syl_7(G)$, $\langle b \rangle \in Syl_3(G)$, $a^7 \in R$, $b^3 \in R$, $E_B \times (a, b) \in \mathcal{B}$, $(2, |R|) = (2, |T|)$, $T \triangleleft G$, $T \langle c \rangle \in \mathcal{B}_\pi$, $T \langle c \rangle / T \cong C_7$, $\langle c \rangle \in Syl_7(G)$, $c^7 \in T$, $E_B \times \langle c \rangle \in \mathcal{B}$.

3) Suppose $E_3 \triangleleft G$. Then $G = (E_3 \times U)\langle f \rangle$ with $U \triangleleft G$, $U \langle f \rangle / U \cong C_3$, $\langle f \rangle \in Syl_3(G)$, $f^3 \in U$, $U \langle f \rangle \in \mathcal{B}_\pi$, $(2, |U|) = 1$, $E_3 \langle f \rangle \in \mathcal{B}$.

4) Suppose $E_3 < G$. Then $E_3 < G$ and $G = (E_3 \times R(a, b)$ with $R(a, b)$ as in 2), but with the integer 7 replaced everywhere by 31 and the integer 3 replaced everywhere by 5, $E_3 \times (a, b) \in \mathcal{B}$.

5) Suppose $E_3 < G$. Then either $Q \triangleleft G$ or else $QF(G)/F(G) \triangleleft G/F(G)$.

In the first case one has $G/C_3(Q) \cong A_4 \cong (C_2 \times C_2) \times C_3$ and $G/O_2'(G) \cong Q \times C_3 \cong SL(2, 3)$, with $G = (Q \times R)\langle a \rangle$, $\langle a \rangle \in Syl_3(G)$, $a^3 \in R$, $R \triangleleft G$, $(a) \in \mathcal{B}_\pi$, $(Q\langle a \rangle \in \mathcal{B}$, $(2, |R|) = 1$.

In the second case one has for $Q \not\triangleleft G$ that $(G/F(G))/(O_2(G/F(G))) \cong Q \times C_3 \cong SL(2, 3)$ and that the Sylow $p$-subgroups of $G$ with $p \neq 2$ and $p$ dividing $|G/F(G)|$ are cyclic. Moreover, one has here that $G/F(G) \in \mathcal{B}$ and that either $G = (((C_5 \times C_5)^{\delta_1} \times (C_1 \times C_1)^{\delta_2}) \times Q) \times T)\langle b \rangle$, $T \triangleleft G$, $[Q, \langle b \rangle] \neq 1$, $Q \subsetneq Q\langle b \rangle$, $T\langle b \rangle \in \mathcal{B}_\pi$, $T\langle b \rangle$ being a Hall-subgroup of $G$, $\langle b \rangle \in Syl_3(G)$, $b^3 \in T$, $\delta_j \in \{0, 1\}$ for $j \in \{1, 2\}$ except for $\delta_1 = \delta_2 = 0$, or that $G = (((C_1 \times C_1) \times Q) \times T)\langle b, d \rangle$, $T \triangleleft G$, $T\langle b, d \rangle$ being a Hall-subgroup of $G$, $\langle b \rangle \in Syl_3(G)$, $b^3 \in T$, $\langle d \rangle \in Syl_3(G)$, $d^3 \in T$, $\langle Q, \langle b \rangle \rangle \neq 1$, $Q \triangleleft Q\langle b \rangle$, $\{Q, \langle d \rangle \} = 1$, $\{\langle b \rangle, \langle d \rangle \} = 1$, $[C_1 \times C_1, \langle b \rangle] \neq 1$, $[C_5 \times C_1, \langle d \rangle] \neq 1$.

6) Suppose $C_3 < G$. Then either $G = V \langle c \rangle$, $V \subseteq G$, $(2, |V|) = 1$, $\langle c \rangle \in Syl_2(G)$, $G/O_2'(G) \cong \langle c \rangle$ or else $Q \in Syl_2(G)$ satisfying $Q \triangleleft G$.

7) Suppose $E_3^p < G$, $p$ an odd prime. Then $E_3^p < G$ and $p \nmid |G/E_3^p|$. Moreover, as $F(G) = E_3 \times V$ with $V \triangleleft G$ and $p \nmid |V|$, either it holds that $G/F(G) \in \mathcal{B}$ with all its Sylow subgroups cyclic, or else $G/F(G) \in \mathcal{B}$, where $G/F(G)$ contains a normal Sylow 2-subgroup isomorphic to $Q$ and all of its other Sylow subgroups are cyclic. Notice $G/E_3^p \in \mathcal{B}_\pi$ and $G/V \in \mathcal{B}_\pi$.

8) Suppose $E_3^p < G$, $p$ an odd prime with $p \nmid |G : E_3^p|$. Then $E_3^p < G$. Moreover, one has $F(G) = E_3 \times V$ with $V \triangleleft G$ and $p \nmid |V|$. Either it holds that $G/F(G) \in \mathcal{B}$ with all its Sylow subgroups cyclic of odd order, or else $p$ divides 55, $QF(G)/F(G)$ being a normal subgroup of $G/F(G)$ and also satisfying $G/F(G) \in \mathcal{B}$.
The rest of the statements in the Theorem under 1) is now nothing else but a recasting of the existing subgroup of order 4 in \( G \). Hence either any Sylow 2-subgroup of \( G \) is cyclic or generalized quaternion; compare with 6) in Theorem 20.

Remark In Theorem 20, 5) it was investigated what happens when \( Q \not\trianglelefteq G \) for a solvable \( \mathcal{A}_3 \)-group \( G \). Here we inspect carefully where \( Q \not\trianglelefteq G \) leads to for such groups \( G \). Notice \( Q \in \text{syl}_2(G) \). It holds.
that $Q$ intersected with the Fitting subgroup $F(G)$ of $G$ provides $Q \cap F(G) = \{1\}$.

Indeed, if $Q \cap F(G) \neq \{1\}$, then $Z(Q) \leq F(G)$ and because of $Q \not\leq G$, $F(G)$ does contain a normal Sylow 2-subgroup $S$ (say) of order at most 4 satisfying $S \geq Z(Q)$; remember that $F(G)$ is nilpotent, that each Sylow 2-subgroup of $G$ is isomorphic to $Q$, that $Q$ contains as subgroups of order 4 only cyclic ones, and that $Z(Q)$ is the only existing subgroup of order 2 inside $Q$. So, if $|S| = 4$ would hold, then, by $G \in \mathcal{A}_2$, any cyclic subgroup of $G$ being of order 4 would be contained in $F(G)$, whence in particular it happens that $Q \leq F(G)$, as $Q$ is generated by its cyclic subgroups of order 4. By assumption though it is therefore not possible that $|S| = 4$.

Thus $Z(Q) \in Syl_2(F(G))$ satisfying $Z(Q) \leq F(G)$. Then, however, $Z(Q)$ is characteristic in $F(G)$, yielding $Z(Q) \leq G$. Thus $G/Z(Q) \in \mathcal{A}_2$, where $Q/Z(Q)$ is isomorphic to $E_4$ and where now $Q/Z(Q) \in Syl_2(G/Z(Q))$. Hence $Q/Z(Q) \leq G/Z(Q)$ by (4.4) Theorem, [7]. Hence $Q$ would be a normal subgroup of $G$ yielding $Q \leq F(G)$, which is not allowed.

Thus $QF(G)/F(G)$ is a Sylow 2-subgroup of $G/F(G)$ being isomorphic to $Q$. Remember, because of $G \in \mathcal{A}_2$, that all non-cyclic elementary abelian subgroups of $G$ are contained in $F(G)$. Any other Sylow subgroup of $F(G)$ is thus cyclic as there are no Sylow subgroups of $F(G)$ existent being non-cyclic and simultaneously not elementary abelian. Hence each Sylow subgroup of $G/F(G)$ is cyclic in case it is not isomorphic to $Q$. Recall that $G/F(G) \in \mathcal{A}_2$. Then, just as it has been argued above, either $QF(G)/F(G)$ is normal in $F(G/F(G))$ or otherwise

$$QF(G)/F(G) \cap F(G/F(G)) = \{1\}$$

satisfying $QF(G)/F(G) \cong Q$.

Suppose the last equality ($\star$) is in vogue. Let us introduce the notations $\overline{G} = G/F(G)$, $\overline{Q} = QF(G)/F(G)$ and $\overline{F} = F(G/F(G))$. It holds that all the Sylow subgroups of $\overline{F}$ are cyclic. Since $\overline{F}$ is therefore abelian, it holds that not only $2 \nmid |\overline{F}|$ is fulfilled, but also that $\overline{F}$ is the direct product of cyclic subgroups of odd order being pairwise relatively prime. Since $\overline{G}$ is solvable, it is a fact that $G/\overline{F}$ is abelian; notice, that group is a subgroup of $Aut(\overline{F})$ and that here $Aut(\overline{F})$ is abelian. We have thus obtained a contradiction to ($\star$).

In summary, if $Q \leq G$, $G \in \mathcal{A}_2$ being solvable, then either $Q \leq G$ or otherwise $\overline{Q}$ is normal in $\overline{G}$ with $\overline{Q} \cong Q$; in the last situation $G$ is a $\mathcal{B}$-group as it satisfies the assumptions of Theorem 11, [9], whose structure has been fully described in Theorem 4.1, [10]. Knowing that the group $\overline{G}$ with $\overline{Q} \leq \overline{G}$ contains only as other Sylow subgroups cyclic ones, one therefore gets, that there exists a 3-nilpotent meta-cyclic $\mathcal{B}$-subgroup $L(a)$ of $\overline{G}$ such that $\overline{G} = (\overline{Q} \times L)(a)$ in which $L$ is a meta-cyclic $\mathcal{B}$-group satisfying $(\overline{a}) \in Syl_3(\overline{G})$, $([a], \overline{Q}) \neq \{1\}$, $a^3 \in L$, $L \lhd L(a)$, $\overline{Q}(a)/\overline{a}^3 \cong SL(2,3)$. As such these facts do sharpen the structure of the groups $T(b)$ and $T(b, d)$ mentioned in Theorem 20, 5).

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