Large eddy simulation of stably stratified turbulence

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Abstract Stably stratified turbulence is a common phenomenon in atmosphere and ocean. In this paper the large eddy simulation is utilized for investigating homogeneous stably stratified turbulence numerically at Reynolds number $Re=\frac{uL}{\nu}=10^2\sim10^3$ and Froude number $Fr=\frac{u}{NL}=10^{-2}\sim10^{0}$ in which $u$ is root mean square of velocity fluctuations, $L$ is integral scale and $N$ is Brunt-Väisälä frequency. Three sets of computation cases are designed with different initial conditions, namely isotropic turbulence, Taylor Green vortex and internal waves, to investigate the statistical properties from different origins. The computed horizontal and vertical energy spectra are consistent with observation in atmosphere and ocean when the composite parameter $ReFr^2$ is greater than $O(1)$. It has also been found in this paper that the stratification turbulence can be developed under different initial velocity conditions and the internal wave energy is dominated in the developed stably stratified turbulence.

1. Introduction

Stable stratification turbulence is anisotropic with high Reynolds numbers and low Froude numbers in atmosphere and ocean that theoretical, experimental and numerical studies become difficult. Since the late 20th century considerable efforts have been made into the study of stratification turbulence by both geophysical and fluid dynamical scientists, among whom are Godeferd & Cambon (1994), Riley & Lelong (2000), Brethouwer et al. (2007) and so on. Although considerable progress has been made in understanding this particular flow phenomenon, the numerical simulation is not fully adequate. For instance, field measurements have shown that a $k^{-3}$ law was observed for vertical energy spectrum by Garrett and Munk (1979) whereas horizontal energy spectrum showed a $k^{-5/3}$ law (1984). However the present laboratory measurements and numerical computation do not coincide with field measurements. Generally the researchers interpret the inconsistence as the lower Reynolds number and higher Froude number in numerical and laboratory experimental studies. Currently Brethouwer et al. (2007) proposed a new scaling, $ReFr^2>1$, for experimental and numerical simulation of stratification turbulence in addition to $Re>>1$ and $Fr<<1$. The new scaling was verified by previous numerical and laboratory experimental results as well as Brethowers’ own numerical computation. The idea of Brethowers is useful in that the numerical computation and laboratory experiment can simulate the real stratification turbulence as long as the composite parameter satisfies $ReFr^2>1$ together with large Reynolds number and small Froude number which are not necessarily the same as those in the real
environment.

There are two numerical methods for simulating turbulence, namely, direct numerical simulation (DNS) and large eddy simulation (LES). Direct numerical simulation requires very high spatial and temporal resolution that it is beyond simulation by the computer available at present time. Therefore the large eddy simulation is employed in this paper with dynamic Smagorinsky subgrid stress model (Sagaut, 2002).

For concentrating on the study of buoyancy effect the mean shear and rotation are ignored and the turbulence is considered homogeneous in stratified fluid in this paper. Three sets of testing cases have been performed on the different initial flow field. The simple internal wave is posed as the initial condition in the first set; simple potential vortex, i.e. Taylor-Green vortex is used as the initial condition in the second set and an isotropic turbulence is prescribed as the initial condition in the third set.

2. Governing equation and computational method

The homogeneous stably stratified turbulence is investigated in this paper and its governing equations are the filtered Navier-Stokes equation and heat balance equation with Boussinesq hypothesis as follows

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p/\rho_0 + \boldsymbol{\nabla} k + \left(v + v'_r\right) \Delta \mathbf{u} \\
\nabla \cdot \mathbf{u} &= 0 \\
\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta + w G_r &= \left(\kappa + \kappa_r\right) \Delta \theta
\end{align*}
\]

In equations (1) ~ (3) \( \mathbf{u}, p \) and \( \theta \) are the filtered fluctuations of velocity, pressure and temperature respectively; \( T_0, \rho_0 \) are the ambient temperature and density respectively, \( g \) is the gravity acceleration, \( G_r \) is the mean temperature gradient in vertical direction \( z \) and \( w \) is the vertical velocity fluctuation. In this paper the constant eddy viscosity \( \nu_r \) and diffusivity \( \kappa_r \) are accepted in large eddy simulation. The dynamic Smagorinsky model is used for the eddy viscosity, and constant turbulence Prandtl number \( Pr_r=0.75 \) is assumed for eddy diffusivity that \( \kappa_r=\nu_r/Pr_r \). In homogeneous stably stratified turbulence equations (1) ~ (3) can be transferred to spectral space as follows

\[
\begin{align*}
\hat{\mathbf{u}}(k, t) &= \sum_k \hat{u}_k(k, t) \exp(ik \cdot x), \quad p(x, t) = \sum_k \hat{p}_k(k, t) \exp(ik \cdot x), \\
\hat{\theta}(k, t) &= \sum_k \hat{\theta}_k(k, t) \exp(ik \cdot x)
\end{align*}
\]

\[
\begin{align*}
\hat{\mathbf{u}}(k, t) / \hat{\theta}(k, t) &= \left(\hat{\mathbf{u}} \cdot \hat{\boldsymbol{\Omega}}\right)_k - ik \hat{\nu} + \hat{\nabla} \cdot \hat{\mathbf{u}}(k, t) g \delta_z / T_0 - (v + v'_r) k^2 \hat{u}_k(k, t) \\
-ik \hat{u}_k(k, t) &= 0 \\
\hat{\nabla} / \hat{\theta}(k, t) &= -\mathbf{u} \cdot \hat{\nabla} \hat{\theta} - \hat{w} G_r - (\kappa + \kappa_r) k^2 \hat{\theta}
\end{align*}
\]

In equation (5) \( \hat{\nu} = \hat{\nu}^2 + u_r^2/2 - \hat{\nu}^2/3 \) and \( \hat{\boldsymbol{\Omega}} = \hat{\nabla} \times \mathbf{u} \). The spectral equations (4) ~ (7) are solved numerically by the pseudo spectral method, which can be found in Canuto et al. (1987), with three sets of initial conditions that in set A an initial internal wave is posed as follows

\[
\begin{align*}
u = u_0 \cos(x + z), \quad \hat{w} = -u_0 \cos(x + z), \quad \hat{\theta} = u_0 T_0 N \cos(x + z) / g
\end{align*}
\]

In computation \( u_0=0.0502 \) m/s, \( N=0.062 \) s\(^{-1}\) and girds points equal 128\(^3\).

In set B initial Taylor-Green vortex is prescribed as

\[
\begin{align*}
u = u_0 \cos(z) \cos(x) \sin(y), \quad \hat{w} = -u_0 \cos(z) \cos(y) \sin(x), \quad \hat{\theta} = 0
\end{align*}
\]

In computation \( u_0=0.0502 \) m/s, \( N=0.062 \) s\(^{-1}\) and girds points equal 128\(^3\).
In set C the stably stratified turbulence is developed from a pre-computed isotropic turbulence. The total initial kinetic energy equals to Case B with $N=0.062s^{-1}$ and grid points $144^3$.

3. The results and analyses

3.1 The evolution of flow parameters

The flow is decaying in the stably stratified fluid under prescribed initial conditions and is approaching to turbulence regime at dimensionless time (scaled by initial characteristic time) over 100. The flow parameters are shown in Figure 1 for three cases. The Reynolds number and Froude Number are defined as follows

$$Re = \frac{uL}{\nu}$$  \hspace{1cm} (10)

$$Fr = \frac{u}{NL}$$  \hspace{1cm} (11)

In above equations $u$ is the root mean square of velocity fluctuation, $L$ is integral length defined as $L=k^{3/2}/\varepsilon$ in which $k$ and $\varepsilon$ are turbulent kinetic energy and its dissipation rate respectively.

The flow parameters shown in Figure 1 indicate that all computational cases satisfy the condition: $Re>>1.0$, $Fr<1.0$ and $ReFr^2>1.0$ at the later stage when the fluid flow reaches the typical stratification turbulence regime confirmed by the kinetic energy spectra below.

![Figure 1. Flow parameters for case A, B and C. Solid: Re, dashed: Fr; Dot dashed: ReFr$^2$](image)

3.2 Evolution of turbulent kinetic energy

It is well known that both potential vortex and internal wave are involved in the stratification turbulence; hence it is interesting to see how they are developed in the stratification turbulence. To decompose the total flow energy into internal wave and potential vortex energy the Craya-Herring frame $(e^1, e^2, e^3)$ is used that

$$e^1 = k \times n / |k \times n|$$  \hspace{1cm} (12)

$$e^2 = k \times e^1 / |k \times e^1|$$  \hspace{1cm} (13)

$$e^3 = e^1 \times e^2$$  \hspace{1cm} (14)

In above equations $n$ is the in the gravity direction. It can be proved that the velocity of potential vortex is perpendicular to gravity direction and the velocity of internal wave is perpendicular to the wave number vector in spectral space (Godeferd and Cambon, 1994). Hence the final decomposition formula can be written as

$$\hat{u}^p_i(k) = [\hat{u}(k) \cdot e^i(k)] e^i(k)$$  \hspace{1cm} (15)
in which the superscript ‘pv’ and ‘iw’ stand for the potential vortex and inter wave respectively.

The corresponding energy are then calculated by

\[ E_{\text{total}} = \frac{1}{2} \int_0^{k_{\text{max}}} \int_0^{k_{\text{max}}} \int_0^{k_{\text{max}}} \hat{u}^2 (k) dk_1 dk_2 dk_3 \]  \hspace{1cm} (17)

\[ E_{\text{pv}} = \frac{1}{2} \int_0^{k_{\text{max}}} \int_0^{k_{\text{max}}} \int_0^{k_{\text{max}}} \hat{u}^2^{\text{pv}} (k) \hat{u}^2^{\text{pv}} (k) dk_1 dk_2 dk_3 \]  \hspace{1cm} (18)

\[ E_{\text{iw}} = \frac{1}{2} \int_0^{k_{\text{max}}} \int_0^{k_{\text{max}}} \int_0^{k_{\text{max}}} \hat{u}^2^{\text{iw}} (k) \hat{u}^2^{\text{iw}} (k) dk_1 dk_2 dk_3 \]  \hspace{1cm} (19)

Figure 2 shows the time evolution of kinetic energy including total, potential vortex and internal wave energy form three different initial conditions. It is interesting that the internal wave energy is dominant in the developing stably stratified turbulence no matter what the initial states are.

![Figure 2](image1.png)

(a) Case A  (b) Case B  (c) Case C

**Figure 2.** Evolution of kinetic energy. Solid line: total energy; Dashed: energy of potential vortex; Dot dashed: energy of internal wave

### 3.3 The horizontal and vertical kinetic energy spectra

In atmosphere and ocean the horizontal kinetic energy has a \( k^{-5/3} \) spectrum while the vertical kinetic energy spectrum has a \( k^{-3} \) law at small wave number [3]. It is the critical examination of the computed horizontal and vertical kinetic energy spectra if they are in agreement with observed power laws. Figure 3 (a) presents the horizontal and vertical spectra for Case A at \( t=386 \) when the Reynolds number is of \( O(10^3) \), Froude number of \( O(10^{-1}) \) and composite parameter \( \text{ReFr}^2 \) of \( O(10^5) \), see Figure 1 (a). At this time the vertical kinetic energy spectrum has a \( k^{-3} \) law at small wave number and horizontal energy spectrum shows \( k^{-5/3} \) law. This is consistent with observation. At later decay time \( t=849 \) the composite parameter drops to \( O(10^1) \) and the vertical spectrum, shown in Figure 3 (b), does not have the \( k^{-3} \) law although the Reynolds number is still higher enough and Froude number as low as \( O(10^{-2}) \). The results of Figure 3 (a) and (b) indicate that the composite parameter \( \text{ReFr}^2 \) is an important parameter in numerical simulation as shown by Brethouwer and Lindborg (2007) before.

Similar horizontal and vertical spectra are presented in Figure 4 for the Case B and Case C. For case B, shown in Figure 4 (a), from initial Taylor-Green vortex there is a \( k^{-5/3} \) law in horizontal spectrum and a \( k^{-3} \) law in vertical spectrum at small wave number \( k<2 \) when the characteristic turbulence parameters satisfy \( \text{Re}>>1, \text{Fr}<1.0 \) and \( \text{ReFr}^2>1 \) at dimensionless time 258. Similar spectra exist in Case C, shown in Figure 4(b), from initial isotropic turbulence at time 118.
Discussions and concluding remarks

The above numerical results are simulated in decaying turbulence. It is interesting to see if same property exists in the stationary stratification turbulence. The homogeneous stationary turbulence can be performed by external forcing in numerical simulation. Case C$^+$ is computed by external forcing method in which the energy at first two wave numbers keeps constant and this is equivalent to input energy in two largest scales turbulence. The initial condition of Case C$^+$ is same as that of Case C. The stationary state is approaching through a decaying stage from initial turbulence. Figure 5 (a) and (b) show the evolution of turbulent kinetic energy and characteristic parameters of case C$^+$ respectively. In Figure 5(a) it is shown that the kinetic energy distribution of case C$^+$ is similar to that of case C at dimensionless time 50 that the internal wave energy is greater than potential vortex. Figure 5 (b) shows that the Reynolds number reaches to $10^4$, Froude number is $O(1)$ and composite parameter $ReFr^2$ is $O(10^4)$ at time 50. Although $Fr\sim O(1)$, which does not exactly meet the condition for buoyancy turbulence, the predicted spectra still show $k^{-3}$ law in vertical spectrum and the $k^{-5/3}$ law in horizontal spectrum approximately. The horizontal spectrum has a peak which is caused by the forcing in low wave numbers. This indicates that the stationary stratification turbulence can be simulated numerically, however the initial parameters and forcing method need to be further investigated to have perfect agreement with observed data.

In summary large eddy simulation is capable of predicting buoyancy turbulence with less computation cost. In order to obtained the results in good agreement with measured data in atmosphere or ocean the turbulence Reynolds number must be great enough, say greater than $10^5$. 

(a) t=386, ReFr$^2$$\sim$1

(b) t=849, ReFr$^2$<1

Figure 3. Horizontal and vertical energy spectra for case A. Solid line: Vertical spectrum; Dashed: horizontal spectrum; Dotted: $E(k)\propto k^{5/3}$; Long dashed: $E(k)\propto k^3$.

(a) t=258

(b) t=118

Figure 4. Horizontal and vertical energy spectra for Case B and Case C. Solid line: vertical spectra; Dashed: horizontal spectra; Dot dashed: $E(k)\propto k^{5/3}$; Dotted: $E(k)\propto k^3$.

4. Discussions and concluding remarks

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(a) Case B, t=258

(b) Case C, t=118

Figure 4. Horizontal and vertical energy spectra for Case B and Case C. Solid line: vertical spectra; Dashed: horizontal spectra; Dot dashed: $E(k)\propto k^{5/3}$; Dotted: $E(k)\propto k^3$.
and Froude number must be less than $O(1)$. In addition a composite parameter $\text{ReFr}^2$ is important and must be greater than $O(1)$ at least. The internal wave energy is dominant in buoyancy turbulence whatever the initial velocity field is internal wave or potential vortex, or isotropic turbulence. This phenomenon has not been noted in previous investigation in numerical simulation of buoyancy turbulence and it should be further studied in considerable details. To understand this phenomenon the key issue is the energy transfer between internal wave and potential vortex in buoyancy turbulence and the authors are pursuing the research on this issue.

![Figure 6](image)

(a) Turbulence kinetic energy (b) Flow parameters (c) Horizontal and vertical spectra

**Figure 6.** The characteristics of forcing stratification turbulence. (a) Solid line: total energy; Dashed line: potential vortex energy; Dot dashed line: internal wave energy. (b) Solid line: Re; Dashed line: Fr; Dot dashed line: ReFr$^2$. (c) Solid line: vertical energy spectrum; Dashed line: horizontal energy spectrum; Dotted line: $E(k) \propto k^{-5/3}$; Long dashed line: $E(k) \propto k^{-3}$ at time 45

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