A LOWER BOUND OF THE DISTORTION OF THE TORELLI GROUP IN THE MAPPING CLASS GROUP WITH BOUNDARY COMPONENTS

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Abstract. We prove that each Torelli group of an orientable surface with any number of boundary components is at least exponentially distorted in the mapping class group by using Broaddus-Farb-Putman’s techniques. Further we show that the distortion of each Torelli group in the level $d$ mapping class group is the same as that of in the mapping class group.

1. Introduction

Let $S_{g,b}$ be a compact connected orientable surface of genus $g \geq 3$ with $b \geq 0$ boundary components. The mapping class group $\mathcal{M}_{g,b}$ of $S_{g,b}$ is the group consists of the isotopy classes of orientation preserving homeomorphisms on $S_{g,b}$, fixing the boundary components pointwise. The Torelli group $\mathcal{I}_{g,b}$ of $S_{g,b}$ is the kernel of the natural homomorphism $\Phi: \mathcal{M}_{g,b} \to \text{Aut}(H_1(S_{g,b};\mathbb{Z}))$. There are several definitions of Torelli groups with $b \geq 2$ boundary components (see [6]). In case of our definition, the Torelli group is finitely generated, and hence it is equipped with a word norm. The geometry of the mapping class groups of orientable surfaces is well understood. Therefore, understanding the geometry of the inclusion homomorphism $\mathcal{I}_{g,b} \hookrightarrow \mathcal{M}_{g,b}$ may allow one to deduce geometric properties of the Torelli group from geometric properties of the mapping class group.

Let $G$ be a finitely generated group and $K$ a finitely generated subgroup of $G$. In this paper, we denote by $\| \cdot \|_G$ a word metric of finitely generated group $G$. Then, there exists $C > 0$ such that $\| h \|_G \leq C \| h \|_K$ for any $h \in K$. A natural question is what the smallest function $\delta: \mathbb{N} \to \mathbb{R}$ which satisfies $\| h \|_K \leq \delta(\| h \|_G)$ is. The distortion of $K$ in $G$ is at most $\delta$ if there exists $C, C'$ such that for each $h \in K$, it follows that $\| h \|_K \leq C\delta(\| h \|_G) + C'$. The distortion of $K$ in $G$ is at least $\delta$ if there exists a sequence $\{h_i\}$ ($h_i \in K$) such that the word norm of $h_i$ in $G$ grows linearly, while the word norm in $K$ grows $\delta$. If $K$ is at most and at least $\delta$ distorted in $G$, then we say the distortion of $K$ in $G$ is (exactly) $\delta$. If the distortion of $K$ in $G$ is linear, we call $K$ is undistorted in $G$.

The distortions of various subgroups in the mapping class groups have investigated. For example, the mapping class groups of subsurfaces are undistorted in the mapping class group by Masur-Minsky [5] and Hamenstädt [3], or the handle body group is exponentially distorted in the corresponding mapping class group by Hamenstädt-Hensel [4]. Moreover, Broaddus-Farb-Putman [1] proved that $\mathcal{I}_{g,0}$ (resp. $\mathcal{I}_{g,1}$) is at least exponentially and at most double exponentially distorted in.

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\( M_{g,0} \) (resp. \( M_{g,1} \)). From the result of Cohen \([2]\), it follows that the distortion of \( \mathcal{I}_{g,0} \) (resp. \( \mathcal{I}_{g,1} \)) in \( M_{g,0} \) (resp. \( M_{g,1} \)) is exponential. We find a lower bound of the distortion of the Torelli groups in the mapping class groups for the orientable surfaces with \( b \geq 2 \) boundary components by applying Broaddus-Farb-Putman’s arguments:

**Theorem 1.1.** For \( g \geq 3 \) and \( b \geq 2 \), \( \mathcal{I}_{g,b} \) is at least exponentially distorted in \( M_{g,b} \).

For \( d \geq 2 \), the level \( d \) mapping class group \( M_{g,b}[d] \) of \( S_{g,b} \) \((b = 0, 1)\) is the kernel of the natural homomorphism \( \Phi_d: M_{g,b} \to \text{Aut}(H_1(S_{g,b}; \mathbb{Z}/d\mathbb{Z})) \). We show that the distortion of the Torelli groups in the level \( d \) mapping class group is the same as that of in the mapping class groups, and so we obtain the following.

**Theorem 1.2.** For \( g \geq 3 \), \( b \leq 1 \) and \( d \geq 2 \), \( \mathcal{I}_{g,b} \) is exponentially distorted in \( M_{g,b}[d] \).

We note that we cannot apply Broaddus-Farb-Putman’s arguments to the upper bound of the distortion of \( \mathcal{I}_{g,b} \) in \( M_{g,b} \) for \( g \geq 3 \) and \( b \geq 2 \). The reason is as follows. For \( b \leq 1 \), the image \( \Phi(M_{g,b}) \) is isomorphic to the symplectic group \( \text{Sp}(2g; \mathbb{Z}) \), and they use a property of \( \text{Sp}(2g; \mathbb{Z}) \) for the upper bound of the distortion of \( \mathcal{I}_{g,b} \) in \( M_{g,b} \). However, for \( b \geq 2 \) we don’t know whether the image \( \Phi(M_{g,b}) \) is isomorphic to some symplectic group, and so we cannot use the same approach as that of Broaddus-Farb-Putman.

2. **Proof of Theorem 1.1**

In this section, we prove Theorem 1.1. Firstly, we give the definition of the partially hyperbolic matrices.

**Definition 2.1.** Let \( V \) be a free abelian group. We say that an element of the automorphism group \( \text{GL}(V) := \text{Aut}(V) \) is **partially hyperbolic** if the corresponding linear transformation of \( V \otimes \mathbb{C} \) has some eigenvalue \( \lambda \) with \( |\lambda| > 1 \). We call such a matrix a **partially hyperbolic matrix**.

We use the following proposition by Broaddus-Farb-Putman \([1]\) to prove Theorem 1.1.

**Proposition 2.2.** (\([1]\) Proposition 2.3) Let \( G \) be a finitely generated group and \( K \) be a finitely generated subgroup of \( G \). Suppose that \( V \) is a free abelian subgroup equipped with a \( G \)-action \( \rho: G \to \text{GL}(V) \) and that \( \psi: K \to V \) is a surjective homomorphism which is \( G \)-equivariant, where \( G \) acts on \( K \) by conjugation. If \( \rho(G) \) contains a partially hyperbolic matrix, then the distortion of \( K \) in \( G \) is at least exponential.

We assume that \( g \geq 3 \). We set \( H = H_1(S_{g,1}; \mathbb{Z}) \). Broaddus-Farb-Putman \([1]\) proved that \( G = \mathcal{M}_{g,1}, K = \mathcal{I}_{g,1}, V = \wedge^3 H \), and \( \psi: K \to V \) satisfy the assumptions in Proposition 2.2, where \( \psi \) is the Johnson homomorphism. Hence \( \mathcal{I}_{g,1} \) is at least exponentially distorted in \( \mathcal{M}_{g,1} \). We use this fact in the proof.

**Proof of Theorem 1.1** We set \( G = \mathcal{M}_{g,b} \) and \( K = \mathcal{I}_{g,b} \) \((g \geq 3, b \geq 2)\). Then \( \mathcal{M}_{g,b} \) acts on \( \mathcal{I}_{g,b} \) by conjugation. Further, we put \( V = \wedge^3 H \). We regard \( S_{g,1} \) as the surface obtained by capping \( S_{g,b} \) by \( b - 1 \) disks. Let \( \varphi: \mathcal{M}_{g,b} \to \mathcal{M}_{g,1} \) be a homomorphism defined by \( \varphi(f) = [\tilde{F}] \) \((f = [F] \in \mathcal{M}_{g,b})\), where \( \tilde{F}(x) = \ldots \)
$F(x)$ if $x \in S_{g,b}$ and $\bar{F}(x) = x$ if $x \notin S_{g,b}$. Let $\tau : \mathcal{I}_{g,1} \to \wedge^3 H$ be the Johnson homomorphism, and $\psi$ the composition of $\varphi|_K$ with $\tau$. The homomorphism $\psi$ is surjective since $\varphi$ induces the surjective homomorphism $\varphi|_K : K \to I_{g,b}$. We denote by $\rho$ a homomorphism defined by the composition of $\varphi$ with the homomorphism $\varphi' : M_{g,1} \to GL(\wedge^3 H)$. Since $\varphi$ is surjective and Broaddus-Farb-Putman \cite{1} showed that $\text{Im}(\varphi')$ contains a partially hyperbolic matrix, $\text{Im}(\rho)$ also contains a partially hyperbolic matrix.

We show that they satisfy the assumptions of Proposition 2.2. It is sufficient to prove the surjective homomorphism $\psi : \mathcal{I}_{g,b} \to \wedge^3 H$ is $M_{g,b}$-equivariant, namely, $f \cdot \psi(g) = \psi(f \cdot g)$ for any $f \in M_{g,b}$ and $g \in I_{g,b}$. Since $f \cdot \psi(g) = (\varphi(f))(\psi(g))$ and $\psi(f \cdot g) = \psi(fgf^{-1})$, we show $(\varphi(f))(\psi(g)) = \psi(fgf^{-1})$. Since $\tau$ is $M_{g,b}$-equivariant, we have $(\varphi(f))(\psi(g)) = \tau(\varphi(f) \cdot \varphi(g)) = \tau(\varphi(fgf^{-1})) = \psi(fgf^{-1})$. Hence, $\psi$ is $M_{g,b}$-equivariant, and we have finished the proof.

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2.

Proof of Theorem 1.2. We can show $M_{g,b}[d]$ has finite index in $M_{g,b}$. Then, $M_{g,b}[d]$ is quasi-isometrically embedded in $M_{g,b}$, that is, there exists $\lambda > 0$ such that $\| f \|_{M_{g,b}} \leq \lambda \| f \|_{M_{g,b}[d]} + \lambda$ for any $f \in M_{g,b}[d]$. From the fact that the distortion of $\mathcal{I}_{g,b}$ in $M_{g,b}$ is exponential, there exist $C_1, C_2 > 0$ such that $\| f \|_{\mathcal{I}_{g,b}} \leq C_1 e^{\| f \|_{M_{g,b}}} + C_2$ for any $f \in I_{g,b}$. Hence, it follows that $\| f \|_{I_{g,b}} \leq C_1 e^{\| f \|_{M_{g,b}[d]}} + \lambda + C_2$. Then we see the distortion of $\mathcal{I}_{g,b}$ in $M_{g,b}[d]$ is at most exponential. If $L \leq K \leq G$, then the distortion of $L$ in $G$ is at most the composition of the distortion of $L$ in $K$ with the distortion of $K$ in $G$. We know $\mathcal{I}_{g,b} \leq M_{g,b}[d] \leq M_{g,b}$. Now $M_{g,b}[d]$ has finite index in $M_{g,b}$. Thus $M_{g,b}[d]$ is undistorted in $M_{g,b}$. Therefore the distortion of $\mathcal{I}_{g,b}$ in $M_{g,b}[d]$ is at least exponential, and we are done.

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