Short-Circuit Logic

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Abstract

Short-circuit evaluation denotes the semantics of propositional connectives in which the second argument is evaluated only if the first argument does not suffice to determine the value of the expression. In programming, short-circuit evaluation is widely used.

A short-circuit logic is a variant of propositional logic (PL) that can be defined by short-circuit evaluation and implies the set of consequences defined by a module SCL. The module SCL is defined using Hoare’s conditional, a ternary connective comparable to if-then-else, and implies all identities that follow from four basic axioms for the conditional and can be expressed in PL (e.g., axioms for associativity of conjunction and double negation shift). In the absence of side effects, short-circuit evaluation characterizes PL. However, short-circuit logic admits the possibility to model side effects. We use sequential conjunction as a primitive connective because it immediately relates to short-circuit evaluation. Sequential conjunction gives rise to many different short-circuit logics. The first extreme case is FSCL (free short-circuit logic), which characterizes the setting in which evaluation of each atom (propositional variable) can yield a side effect. The other extreme case is MSCL (memorizing short-circuit logic), the most identifying variant we distinguish below PL. In MSCL, only static side effects can be modelled, while sequential conjunction is non-commutative. We provide sets of equations for FSCL and MSCL, and for MSCL we have a completeness result.

Extending MSCL with one simple axiom yields SSSCL (static short-circuit logic, or sequential PL), for which we also provide a completeness result. We briefly discuss two variants in between FSCL and MSCL, among which a logic that admits the contraction of atoms and of their negations.

Key Words and Phrases: Non-commutative conjunction, conditional composition, reactive valuation, sequential connective, short-circuit evaluation, side effect.

1 Introduction

Propositional statements are built up from propositional variables and connectives, and possibly constants for true and/or false, for which we use the notation $T$ and $F$, respectively. For brevity, a propositional variable is further called an atom.

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Our starting point concerns short-circuit evaluation and side effects, and can be captured by the following question: “Given some programming language, what is the logic that implies the equivalence of conditions, notably in if-then-else and while-do constructs and the like?” In the (hypothetical) case that a programming language prohibits the use of conditions that yield side effects this underlying logic is just propositional logic (PL), but in most cases it certainly is not PL. In this paper we consider a few sequential variants of PL in which conjunction is not commutative and we shall use a fresh notation for sequential conjunction. We specify these logics in an equational style (cf. an equational basis of PL comprising $x \lor \neg x = T$ as an axiom) because otherwise we would have to mix sequential connectives with classical connectives such as $\leftrightarrow$, which would obscure the presentation.

Consider the following example, taken from the MatLab documentation about short-circuit evaluation of conjunction that explains a certain form of sequential conjunction.

Example 1. Logical Operators: Short-circuit $\&\&$ $||$

Logical operations, with short-circuiting capability

**Syntax**

- expr1 $\&\&$ expr2
- expr1 $||$ expr2

**Description**

expr1 $\&\&$ expr2 represents a logical AND operation that employs short-circuiting behavior. With short-circuiting, the second operand expr2 is evaluated only when the result is not fully determined by the first operand expr1. For example, if $A = 0$, then the following statement evaluates to false, regardless of the value of $B$, so the MATLAB software does not evaluate $B$:

$$A \&\& B$$

These two expressions must each be a valid MATLAB statement that evaluates to a scalar logical result.

expr1 $||$ expr2 represents a logical OR operation that employs short-circuiting behavior.

**Note** Always use the $\&\&$ and $||$ operators when short-circuiting is required. Using the elementwise operators ($\&$ and $|$) for short-circuiting can yield unexpected results.

**Examples**

In the following statement, it doesn’t make sense to evaluate the relation on the right if the divisor, $b$, is zero. The test on the left is put in to avoid generating a warning under these circumstances:

$$x = (b \neq 0) \&\& (a/b > 18.5)$$

By definition, if any operands of an AND expression are false, the entire expression must be false. So, if $(b \neq 0)$ evaluates to false, MATLAB assumes the entire expression to be false and terminates its evaluation of the expression early. This avoids the warning that would be generated if MATLAB were to evaluate the operand on the right.

**End example.**

Clearly, conjunction as discussed in this example is not commutative: the results of a short-circuit evaluation of $A \&\& B$ and of $B \&\& A$ can be different (observe that negation is evaluated in the expected way). This raises the following question: “Which (equational)
laws are valid in a setting with short-circuit evaluation, in particular if we restrict to true and false as the only possible truth values?" Here is a list of more specific questions, where we write $\neg A$ for the negation of $A$:

- Is the double negation shift ($\neg\neg A = A$) valid?
- Are $\&\&$ and $||$ associative?
- Are De Morgan’s laws ($\neg(\neg A \&\& B) = \neg A || \neg B$ and its dual) valid?
- Is the law of the excluded middle valid, that is, does $A || \neg A = T$ hold?
- Does $A \&\& F = F$ hold?
- Are $\&\&$ and $||$ idempotent?
- Are the sequential versions of laws for distributivity of $\&\&$ and $||$ valid?
- Do $\&\&$ and $||$ satisfy sequential versions of absorption, e.g., $A || B \&\& A = A$?

The answer to the last question with respect to the mentioned version of absorption is NO if we allow erroneous statements: assume $A = ((b \sim= 0) \&\& (a/b > 18.5))$

as in Example[1] and $B = (a/b > 18.5)$, and suppose $(b \sim= 0)$ does not hold, thus evaluation of $A$ yields false. Then $B$ will be evaluated in $(A || B) \&\& A$, which will lead to an error, and thus $(A || B) \&\& A \neq A$. But also if each statement evaluates to either true or false, the answer to the last question should be NO because the evaluation of $B$ can yield a side effect that changes the second evaluation of $A$. For the same reason, right-distributivity of $\&\&$, thus

$$(A || B) \&\& C = (A \&\& C) || (B \&\& C)$$

is not valid, and neither are the law of the excluded middle and the identity $A \&\& F = F$.

A side effect takes place in the evaluation of a propositional statement $A$ if the evaluation of one of its atoms changes the evaluation value of one or more atoms to be subsequently evaluated in $A$. For example, a valuation (interpretation function) $f$ can be such that $f(A) = f(B) = T$, while $f(A \&\& B) = F$ because the side effect of $f(A)$ is that $f(B)$ has become $F$. This immediately implies that side effects and symmetric connectives do not go together. Side effects can be modeled by valuations that are defined on strings of atoms instead of only on a set of atoms. We call a side effect static if within the evaluation of a propositional statement the value of each atom remains fixed after its first evaluation. So, a valuation that admits static side effects is determined by its definition on the set of strings of atoms in which each atom occurs at most once. In the setting of static side effects, sequential conjunction is not commutative if at least two atoms are involved.

In order to analyze short-circuit evaluation in a systematic way, we use so-called left-sequential conjunction

$$x \circ \& \& y$$

where the particular asymmetric notation is taken from [2]: the small circle indicates which argument must be evaluated first. As a consequence, other connectives such as left-sequential disjunction $\lor^\circ$ and right-sequential conjunction $\&^\circ$ have a straightforward notation. Thus, conjunction that is subject to short-circuit evaluation is further called left-sequential conjunction, and in $x \circ \& \& y$ it is required that first $x$ is evaluated and if this yields false then the
conjunction yields false; if the evaluation of \( x \) yields true, then \( y \) is evaluated and determines the overall evaluation result. So \( x \land y \) satisfies the “equation”

\[
x \land y = \text{if } x \text{ then } y \text{ else } F,
\]

which characterizes the short-circuit evaluation of left-sequential conjunction. We use Hoare’s notation \( x \triangleleft y \triangleright z \) for if \( y \) then \( x \) else \( z \) and four of his equational laws that express basic identities between conditional expressions, and we define an (indirect) axiomatization SCL (abbreviating short-circuit logic) that implies all consequences that follow from these four axioms and that can be expressed using only \( T, \neg \) and \( \land \). As an example, axioms expressing the associativity of conjunction and the double negation shift are consequences of SCL. We formally define SCL with help of module algebra \([3]\).

We define short-circuit logic in a generic way: a short-circuit logic is any logic that implies all consequences of SCL. We note that in each short-circuit logic the constant \( F \) and left-sequential disjunction \( \lor \) are definable in the expected way, and SCL implies De Morgans’s laws for left-sequential connectives. So, the answers to the first three questions listed above are YES. We use the name free short-circuit logic, or briefly FSCL, for the most basic one among these logics: FSCL implies no more consequences than SCL (thus identifies “as least as possible” propositional statements) and is a reasonable short-circuit logic in the sense that realistic examples and counter-examples can be found. We do not know whether the logic FSCL has a straightforward and simple equational basis, but we come up with a list of axioms that is sound and that provides the answers to all remaining questions: all these answers are NO.

Next we propose Memorizing short-circuit logic, or briefly MSCL, as the variant of SCL that identifies “as much as possible” below PL. We claim that MSCL is a reasonable candidate for modeling short-circuit evaluation in a context where only static side effects occur. MSCL has a straightforward and relatively simple equational basis. We will show that in MSCL the answers to the “remaining questions” mentioned are NO (\( x \lor \neg z = T \)), NO (\( x \land F = F \)), YES (idempotency of \( \land \) and \( \lor \)), SOME (distributivity laws), SOME (absorption laws).

The further contents of the paper can be summarized as follows: Hoare’s conditionals were the starting point for our work on proposition algebra, and both these works are briefly discussed in Section 2 because they are used to define SCL. In Section 3 we provide a generic definition of a short-circuit logic and we define FSCL as the least identifying short-circuit logic we distinguish. In Section 4 we define MSCL, the most identifying short-circuit logic below PL that we distinguish. In Section 5 we consider some other variants of short-circuit logic. We end the paper with some conclusions in Section 6. The three appendices A, B and C contain an example program in Perl, some detailed proofs, and a simple axiomatization of PL, respectively.

## 2 Short-circuit evaluation and proposition algebra

In this section we briefly discuss proposition algebra \([6]\), which has sort-circuit evaluation as its natural semantics and provides the means to define short-circuit logic.
2.1 Hoare’s conditional and proposition algebra

In 1985, Hoare introduced in the paper \[10\] the ternary connective
\[
x \triangleleft y \triangleright z,
\]
and called this connective the conditional.\(^3\) A more common expression for the conditional \(x \triangleleft y \triangleright z\) is
\[
\text{if } y \text{ then } x \text{ else } z
\]
with \(x, y\) and \(z\) ranging over propositional statements. However, in order to reason systematically with conditionals, a notation such as \(x \triangleleft y \triangleright z\) seems indispensable. In \[10\], Hoare proves that PL can be equationally characterized over the signature \(\Sigma_{CP} = \{ T, F, \triangleleft, \triangleright \}\) and provides a set of elegant axioms to this end, including those in Table 1.

In our paper \[6\] we introduce proposition algebra and we define varieties of so-called valuation algebra\’s. These varieties serve the interpretation of a logic over \(\Sigma_{CP}\) by means of short-circuit evaluation: in the evaluation of \(t_1 \triangleleft t_2 \triangleright t_3\), first \(t_2\) is evaluated, and the result of this evaluation determines further evaluation; upon evaluation result \(true\), \(t_1\) is evaluated and determines the final evaluation result (\(t_3\) is not evaluated); upon evaluation result \(false\), \(t_3\) is evaluated and determines the final evaluation result (\(t_1\) is not evaluated).

All varieties discussed in \[6\] satisfy the axioms in Table 1 while the most distinguishing variety is axiomatized by exactly these four axioms. We write \(CP\) for this set of axioms (where \(CP\) abbreviates conditional propositions) and \(\equiv_{fr}\) (free valuation congruence) for the associated valuation congruence. Thus for each pair of closed terms \(t, t\) over \(\Sigma_{CP}\), i.e., terms that do not contain variables, but that of course may contain atoms (propositional variables),
\[
\text{CP} \vdash t = t' \iff t =_{fr} t'.
\]

In \[12\] it is shown that the axioms of \(CP\) are independent, and also that they are \(\omega\)-complete if the set of atoms involved contains at least two elements.

With the conditional as a primitive connective, negation can be defined by
\[
\neg x = F \triangleleft x \triangleright T,
\]

\(^3\)Not to be confused with Hoare’s conditional introduced in his 1985 book on CSP \[9\] and in his well-known 1987 paper Laws of Programming \[8\] for expressions \(P \triangleleft b \triangleright Q\) with \(P\) and \(Q\) programs and \(b\) a Boolean expression; these sources do not refer to \[10\] that appeared in 1985.
and the following consequences are easily derived from the extension of CP with negation:

\[
F = \neg T, \\
\neg\neg x = x, \\
\neg(x \triangleleft y \triangleright z) = \neg x \triangleleft y \triangleright \neg z, \\
x \triangleleft \neg y \triangleright z = z \triangleleft y \triangleright x.
\]

Furthermore, left-sequential conjunction \(x \triangleleft y\) can be defined in CP by

\[
x \triangleleft y = y \triangleleft x \triangleright F,
\]

and left-sequential disjunction \(\triangledown\) can be defined by

either \(x \triangledown y = \neg (\neg x \land y)\) or \(x \triangledown y = T \triangleleft x \triangleright y\).

A proof of the latter inter-definability is as follows:

\[
\neg (\neg x \land y) = F \triangleleft (F \triangleleft y \triangleright T) \triangleleft (F \triangleleft x \triangleright T) \triangleright T \\
= F \triangleleft (F \triangleleft y \triangleright T) \triangleright T \\
= T \triangleleft x \triangleright (F \triangleleft y \triangleright F) \\
= T \triangleleft x \triangleright y.
\]

The connectives \(\triangleleft\) and \(\triangledown\) are associative and the dual of each other. For \(\triangleleft\) a proof of this is as follows:

\[
(x \triangleleft y) \triangleleft z = z \triangleleft (y \triangleleft x \triangleright F) \triangleright F \\
= (z \triangleleft y \triangleright F) \triangleleft (z \triangleleft F \triangleright F) \\
= z \triangleleft y \triangleright F \triangleleft x \triangleright F \\
= x \triangleleft (y \triangleleft z),
\]

and duality (a sequential version of De Morgan’s laws) immediately follows from the first definition of \(\triangledown\). Finally, observe that from CP extended with defining equations for \(\triangleleft\) and \(\triangledown\) the following equations (and their duals) are derivable:

\[
T \triangleleft x = x, \quad x \triangleleft T = x, \quad T \triangledown x = T,
\]

in contrast to \(x \triangledown T = T\) (and \(x \triangleleft F = F\)).

A typical inequality is \(a \triangleleft a \neq_T a\) where \(a\) is an atom: an evaluation can be such that \(a \triangleleft a\) yields \text{false}, while \(a\) yields \text{true}.

### 2.2 Memorizing CP

CP can be strengthened in various ways, among which its extension to CP\(_{\text{mem}}\) defined by adding this axiom to CP:

\[
x \triangleleft y \triangleright (z \triangleleft u \triangleright (v \triangleleft y \triangleright w)) = x \triangleleft y \triangleright (z \triangleleft u \triangleright w). \quad \text{(CPmem)}
\]
The axiom \( \text{CPmem} \) expresses that the first evaluation value of \( y \) is memorized. With \( u = F \) we find the contraction law

\[
x \odot y \odot (v \odot y \odot w) = x \odot y \odot w,
\]
and replacing \( y \) by \( \neg y \) yields with \( x \leftarrow \neg y \rightarrow z = z \odot y \rightarrow x \) the symmetric contraction law

\[
(w \odot y \odot v) \odot y \rightarrow x = w \odot y \rightarrow x.
\]

A simple consequence of contraction is the idempotence of \( \land \):

\[
x \land x = x \odot x \odot F
= (T \odot x \odot F) \odot x \odot F
= T \odot x \odot F
= x.
\]

Furthermore, we will use the fact that replacing in axiom \( \text{CPmem} \) the variables \( y \) and/or \( u \) by their negation yields various equivalent versions of this axiom, in particular,

\[
(x \odot y \odot (z \odot u \odot v)) \odot u \odot w = (x \odot y \odot z) \odot u \odot w,
\]
\( \text{(CPmem')} \)

\[
x \odot y \odot ((z \odot u \odot v) \odot y \odot w) = x \odot y \odot (v \odot u \odot w).
\]
\( \text{(CPmem'')} \)

We use the name “memorizing CP” for the signature and axioms of \( \text{CPmem} \). In memorizing CP extended with defining equations for \( \neg \) and \( \land \), the conditional is definable: \( \forall \) is expressible, and

\[
(y \land x) \forall (\neg y \land z) = T \odot (x \odot y \odot F) \odot (z \odot (F \odot y \odot T) \odot F)
= T \odot (x \odot y \odot F) \odot (F \odot y \odot z)
= (T \odot x \odot F) \odot y \odot (F \odot y \odot z)
= T \odot x \odot F \odot y \odot z
= x \odot y \odot z.
\]
\( \text{(CP3)} \)

Another (equivalent) definition is \( x \odot y \rightarrow z = (y \forall z) \land (\neg y \forall x) \). In Section 4 we provide axioms over the signature \( \{T, \neg, \land\} \) that define \( F \) and \( \forall \), and that constitute an equational basis for \( \text{CPmem} \).

We write \( =_{\text{mem}} \) (memorizing valuation congruence) for the valuation congruence axiomatized by \( \text{CPmem} \): in [6] we prove that for each pair of closed terms \( t, t' \) over \( \Sigma_{\text{CP}} \),

\[
\text{CP}_{\text{mem}} \vdash t = t' \iff t =_{\text{mem}} t'.
\]

From the above-mentioned consequence \( x \land x = x \) it follows that \( =_{\text{mem}} \) identifies more than \( =_{fr} \), and a typical inequality is \( a \land b \neq_{\text{mem}} a \land a \) when \( a \) and \( b \) are different atoms: an evaluation can be such that \( a \land b \) yields \text{true} and \( b \land a \) yields \text{false}. (In the deprecated case that propositional statements may contain at most one atom, \( \land \) is commutative in \( \text{CPmem} \), see [6,12] for more details.)
2.3 Static CP

The most identifying extension of CP that we distinguished in [6] is defined by adding to CP both the axiom

\[(x \triangleleft y \triangleright z) \triangleleft u \triangleright v = (x \triangleleft u \triangleright v) \triangleleft y \triangleright (z \triangleleft u \triangleright v)\] (CPstat)

and the contraction law (2):

\[x \triangleleft y \triangleright (v \triangleleft y \triangleright w) = x \triangleleft y \triangleright w.\]

We write CPstat for this extension and we use the name “static CP” for the signature and axioms of CPstat. The axiom (CPstat) expresses how the order of evaluation of u and y can be swapped. In [6] we prove that CPstat and Hoare’s axiomatization in [10] are inter-derivable, which implies that all tautologies of PL can be proved in CPstat according to the following procedure: first, replace all symmetric connectives by their left-sequential counterparts, then use the translations to conditional expressions discussed in Section 2.1. We return to this point in Section 5.1. The valuation congruence that is axiomatized by CPstat is called static valuation congruence.

A simple consequence in CPstat is

\[v = v \triangleleft y \triangleright v\] (3)

(take \(u = F\) in axiom (CPstat)), and with this equation one easily derives \(x \triangleleft y = y \triangleleft x\):

\[x \triangleleft y = y \triangleleft x \triangleright F\]
\[= (T \triangleleft y \triangleright F) \triangleleft x \triangleright F\]
\[= (T \triangleleft x \triangleright F) \triangleleft y \triangleright (F \triangleleft x \triangleright F)\] by (CP3)
\[= x \triangleleft y \triangleright F\]
\[= y \triangleleft x\] by (CP3) and (3)

We stated in Section 1 that the presence of side effects refutes the commutativity of \(\triangleleft\). This implies that in static CP and PL, it is not possible to express propositions with side effects, which perhaps explains best why static valuation congruence seems the most interesting and well-known (two-valued) valuation congruence. Nevertheless, we found that short-circuit evaluation and sequential connectives brought an interesting perspective on this valuation congruence. In fact, short-circuit logic turned out to be a crucial tool in finding an axiomatization of static valuation congruence that is more simple and elegant than CPstat, as we will show in Section 5.1.

3 Free short-circuit logic: FSCL

In this section we provide a generic definition of a short-circuit logic and a definition of FSCL, the least identifying short-circuit logic we consider. In Section 5.2 we define a set of equations that is sound in FSCL and raise the question of its completeness.
3.1 A generic definition of short-circuit logics

We define short-circuit logics using Module algebra [3]. Intuitively, a short-circuit logic is a logic that implies all consequences that can be expressed in the signature \( \{ T, \neg, \land \} \) of some CP-axiomatization. The definition below uses the export-operator \( \square \) of module algebra to define this in a precise manner, where it is assumed that CP satisfies the format of a module specification. In module algebra, \( S \square X \) is the operation that exports the signature \( S \) from module \( X \) while declaring other signature elements hidden. In this case it declares conditional composition to be an auxiliary operator.

Definition 1. A short-circuit logic is a logic that implies the consequences of the module expression

\[
SCL = \{ T, \neg, \land \} \square (CP
+ \langle \neg x = F \triangleright x \bowtie T \rangle
+ \langle x \land y = y \triangleright x \bowtie F \rangle).
\]

As an example, \( SCL \vdash \neg \neg x = x \) can be proved as follows:

\[
\neg \neg x = F \triangleleft (F \triangleright x \bowtie T) \triangleright T
= (F \triangleright F \bowtie T) \triangleleft x \bowtie (F \triangleleft T \bowtie T)
= T \triangleleft x \bowtie F
= x.
\]

Following Definition 1, the most basic (least identifying) short-circuit logic we distinguish is the following (but see item 1 in the paragraph on Future work in Section 6 about leaving out \( T \) in the exported signature).

Definition 2. FSCL (free short-circuit logic) is the short-circuit logic that implies no other consequences than those of the module expression \( SCL \).

3.2 Equations for FSCL

Although the constant \( F \) does not occur in the exported signature of \( SCL \), we discuss FSCL using this constant to enhance readability. This is not problematic because

\[
CP + \langle \neg x = F \triangleleft x \triangleright T \rangle \vdash F = \neg T,
\]

so \( F \) can be used as a shorthand for \( \neg T \) in FSCL.

In Table 2 we provide equations for FSCL: equation \( \text{SCL1} \) defines \( F \), and equations \( \text{SCL2} \rightarrow \text{SCL7} \) are derivable in FSCL (cf. the similar equations in Section 2.1). Also, equations \( \text{SCL8} \rightarrow \text{SCL10} \) are derivable in FSCL (see Proposition 1). Some comments:

- Equation \( \text{SCL8} \) defines a property of the mix of negation and the sequential connectives: its left-hand side states that if evaluation of \( x \) yields false, then \( y \land (z \land F) \) is subsequently evaluated and this is also the case for its right-hand side; if evaluation of \( x \) yields true, then \( z \land F \) is evaluated next (while \( y \) is not) and leads to false, which also is the case for its right-hand side (the second conjunct is not evaluated).

\footnote{Or, if one prefers the semantical point of view, “satisfies”}
evaluation of $t \land x = x$ (SCL4)
$x \land T = x$ (SCL5)
$F \land x = F$ (SCL6)

- Equation \( (x \land y) \land z = x \land (y \land z) \) defines a restricted form of right-distributivity of \( \land \), and so does equation \( (x \land F) \land z = x \land F \) (observe that \( (x \land F) \land z = x \land F \) is derivable).

We use the name EqFSCL for this set of equations and we note that equations \( (SCL2) \) and \( (SCL3) \) imply sequential versions of De Morgan’s laws, which allows us to use the duality principle.

Observe that EqFSCL does not contain the equation

$$x \land F = F.$$  

This illustrates a typical property of a logic that models side effects: although it is the case that for each closed term \( t \), evaluation of \( t \land F \) yields false, the evaluation of \( t \) might also yield a side effect. However, the same side effect and evaluation result are obtained upon evaluation of \( \neg t \land F \), and this explains a simple consequence of equation \( (SCL8) \) (take \( y = z = F \)):

$$x \land F = \neg x \land F.$$  

(SCL8*)

This consequence implies another useful consequence that can be derived as follows:

$$x \land y = ((x \land F) \land T) \land y = (x \land F) \land (T \land y)$$

With \( y = T \) we find \( (x \land T) \land F = \neg x \land F = x \land F \), and with the dual \( (x \land F) \land T = x \land T \) we derive

$$x \land y = (x \land F) \land (T \land y) = (x \land F) \land y.$$  

\( \text{(SCL10)} \)

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Table 2: EqFSCL, a set of equations for FSCL

| Equation | Description |
|----------|-------------|
| \( (x \land y) \land z = x \land (y \land z) \) | Restricted right-distributivity of \( \land \) |
| \( (x \land F) \land z = x \land F \) | Right-distributivity of \( \land \) for \( F \) |
| \( x \land T = x \) | Side effect yields \( x \) |
| \( F \land x = F \) | \( F \) has no side effects |
| \( T \land x = x \) | \( T \) has no side effects |

\( \text{These are } \neg(x \land y) = \neg x \lor \neg y \) and \( \neg(x \lor y) = \neg x \land \neg y \).
Proposition 1 (Soundness). The equations in EqFSCL (see Table 2) are derivable in FSCL.

Proof. Trivial. As an example we prove equation \( \text{SCL10} \), where we use that \( \vee \) can be defined in FSCL in exactly the same way as is done in Table 2 and that \( x \vee y = \neg (\neg x \land \neg y) = T \land x \lor y \), where the latter identity follows in CP extended with defining equations for \( \neg \) and \( \land \) (cf. Section 2.1):

\[
\begin{align*}
((x \land F) \lor y) \land z &= z \land (T \land (F \land x \lor F) \lor y) \lor y \\
&= z \land (y \land x \lor y) \lor F \\
&= (z \land y \lor F) \lor x \lor (z \land y \lor F) \\
&= T \land (F \land x \lor F) \lor (z \land y \lor F) \\
&= (x \land F) \lor (y \land z).
\end{align*}
\]

The programming language Perl can be used to illustrate our claim that FSCL defines a reasonable logic because Perl’s language definition is rather liberal with respect to conditionals and satisfies all consequences of FSCL. In Perl, the simple assignment operator is written \( = \) and there is also an equality operator \( == \) that tests equality and returns either \( T \) or \( F \). An assignment is comparable to a procedure that is evaluated for the side effect of modifying a variable and regardless of which kind of assignment operator is used, the final value of the variable on the left is returned as the value of the assignment as a whole. This implies that in Perl assignments can occur in if-then-else statements and then the final value of the variable on the left is interpreted as a Boolean. In particular, any number is evaluated \( true \) except for \( 0 \). For this reason, Perl can be used to demonstrate that certain laws that perhaps seem reasonable, should \( not \) be added to FSCL, as for example these:

\[
x \land x = x,
\]
\[
x \land \neg x = \neg x \land x.
\]

It is not hard to write Perl programs that demonstrate the non-validity of these identities. As an example, consider the run perl Not.pl of the Perl program Not.pl depicted in Figure 1 that shows that

\[ A \land not(A) = not(A) \land A \]

does not hold in Perl, not even if \( A \) is an “atom”, as in the code of Not.pl (see Appendix A for this code).
\[ F = \neg T \]  
\[ x \lor y = \neg (\neg x \land \neg y) \]  
\[ \neg \neg x = x \]  
\[ T \land x = x \]  
\[ x \land T = x \]  
\[ F \land x = F \]  
\[ (x \land y) \land z = x \land (y \land z) \]  
\[ x \land F = \neg x \land F \]  
\[ x \land (x \lor y) = x \]  
\[ x \land (y \lor z) = (x \land y) \lor (x \land z) \]  
\[ (x \lor y) \land (\neg x \lor z) = (\neg x \lor z) \land (x \lor y) \]  
\[ ((x \land y) \lor (\neg x \land z)) \land u = (x \lor (z \land u)) \land (\neg x \lor (y \land u)) \]  

Table 3: EqMSCL, a set of axioms for MSCL.

While not having found any equations that are derivable in FSCL and not from EqFSCL, we failed to prove completeness of EqFSCL in the following sense:

For all SCL-terms \( t \) and \( t' \), \( \text{EqFSCL} \vdash t = t' \iff \text{FSCL} \vdash t = t' \).

(Of course, \( \implies \) follows from Proposition 1.)

4 Memorizing short-circuit logic: MSCL

In this section we define a “most liberal” short-circuit logic in which (only static) side effects can occur and in which \( \land \) is not commutative.

Definition 3. MSCL (memorizing short-circuit logic) is the short-circuit logic that implies no other consequences than those of the module expression

\[ \{ T, \neg, \land \} \square (\text{CP}_{\text{mem}} \lor \langle \text{CP}_{\text{mem}} \rangle) \lor \langle \neg x = F \land x \land T \rangle \lor \langle x \land y = y \land x \land F \rangle. \]

According to Definition 1, MSCL is a short-circuit logic because CP_{mem} is an axiomatic extension of CP (\( \text{CP}_{\text{mem}} = \text{CP} + (\text{CP}_{\text{mem}}) \)), see Section 2.2. In Section 4.1 we provide axioms for MSCL and in Section 4.2 we prove their completeness.

4.1 Axioms for MSCL

In Table 3 we present a set of axioms for MSCL and we call this set EqMSCL. Axioms (SCL1) – (SCL7) occur in EqFSCL (see Table 2) and thus need no further comment, and
We use that

We end this section with a result on the soundness of MSCL.

Proposition 2 (Soundness). The axioms of EqMSCL (see Table 3) are derivable in MSCL.

Proof. We use that \( \forall \) can be defined in exactly the same way in MSCL as in EqMSCL (cf. the proof of Proposition 1). With respect to the soundness of EqMSCL, axiom (MSCL4), i.e.,

\[
((x \land y) \lor (\neg x \land z)) \land u = (x \lor (z \land u)) \land (\neg x \land (y \land u)),
\]
is the only non-trivial case. Write $L = R$ for axiom \textbf{MSCL4}, then

\[
L = u \triangleleft (T \triangleleft (y \triangleleft x \triangleright F) \triangleright F)
\]

\[
= u \triangleleft (y \triangleleft x \triangleright F) \triangleright (u \triangleleft (F \triangleleft x \triangleright z) \triangleright F)
\]  
by \textbf{CP4} and \textbf{CP1}

\[
= u \triangleleft (y \triangleleft x \triangleright F) \triangleright F
\]  
by \textbf{CP4} and \textbf{CP2}

\[
= [u \triangleleft y \triangleright (F \triangleleft x \triangleright (u \triangleleft z \triangleright F)) \triangleleft x \triangleright F \triangleleft x \triangleright (u \triangleleft z \triangleright F)]
\]  
by \textbf{CP4}

\[
= (u \triangleleft y \triangleright F) \triangleleft x \triangleright (u \triangleleft z \triangleright F)
\]  
by \textbf{CPmem'} and \textbf{1}

\[
= [((u \triangleleft y \triangleright F) \triangleleft x \triangleright T) \triangleleft (u \triangleleft z \triangleright F) \triangleright F]
\]  
by \textbf{2} and \textbf{CPmem''}

\[
= [((u \triangleleft y \triangleright F) \triangleleft x \triangleright T) \triangleleft (T \triangleleft x \triangleright (u \triangleleft z \triangleright F)) \triangleright F]
\]  
by \textbf{CP4} and \textbf{CP1}

\[
= R.
\]

\[
\square
\]

\[
4.2 \text{ A complete axiomatization of MSCL}
\]

In this section we prove that EqMSCL is a complete axiomatization of MSCL. We first derive a number of useful consequences that follow from EqMSCL, notably:

\[
x \triangleleft \neg x = x \triangleleft F,
\]

\[
x \triangleleft y = x \triangleleft (\neg x \triangleright y),
\]

\[
x \triangleleft (y \triangleright (x \triangleleft z) \triangleright (\neg x \triangleleft u)) = x \triangleleft (y \triangleright z),
\]

\[
x \triangleleft (y \triangleright x) = x \triangleleft y,
\]

\[
(x \triangleleft y) \triangleright z = (x \triangleleft (y \triangleright z)) \triangleright (\neg x \triangleleft z).
\]

Equations (10) and (11) will be used in the next section and in Appendix B respectively.

\bullet \text{ Equation (7) can be derived as follows:}

\[
x \triangleleft \neg x = (x \triangleright F) \triangleleft (\neg x \triangleright F)
\]

\[
= (x \triangleright F) \triangleright (\neg x \triangleleft F)
\]

\[
= (x \triangleright F) \triangleright (x \triangleleft F)
\]

\[
= x \triangleleft F,
\]

\[
\text{by \textbf{SCL8}}
\]

and hence, $\neg x \triangleleft x = \neg x \triangleleft \neg \neg x = \neg x \triangleleft F = x \triangleleft \neg x$. Note that the dual of (7), thus $x \triangleright \neg x = x \triangleright T$, can be seen as a weak version of the law of the excluded middle.

\bullet \text{ Equation (8) can be derived as follows:}

\[
x \triangleleft y = x \triangleleft (F \triangleright y)
\]

\[
= (x \triangleleft F) \triangleright (x \triangleleft y)
\]

\[
= (x \triangleleft \neg x) \triangleright (x \triangleleft y)
\]

\[
= x \triangleleft (\neg x \triangleright y).
\]

\[
\text{by \textbf{MSCL2}}
\]
Two immediate consequences of this identity are

\[ x = x \land (\neg x \lor T) \quad \text{and its dual} \quad x = x \lor (\neg x \land F). \quad (12) \]

• Equation (9) can be derived as follows:

\[
x \land (y \lor (x \land z) \lor (\neg x \land u)) = (x \land y) \lor ((x \land z) \lor (x \land (\neg x \land u))) \quad \text{by MSCL2}
\]

\[
x \land (x \land y) \lor ((x \land z) \lor (\neg x \land (x \land u))) = (x \land y) \lor ((x \land z) \lor (\neg x \land (x \land u))) \quad \text{by MSCL1}
\]

• Equation (10), i.e., \( x \land (y \land x) = x \land y \), expresses a property of MSCL that one may call memory (the first evaluation result of \( x \) is memorized). We employ (9) to prove equation (10):

\[
x \land (y \land x) = x \land (y \land (\neg y \lor x)) \quad \text{by (8)}
\]

\[
= x \land (\neg y \lor x) \land y \quad \text{by MSCL3}
\]

\[
= (x \land (\neg y \lor x)) \land y \quad \text{by (12)}
\]

\[
= (x \land (\neg y \lor (x \land (\neg x \land F)))) \land y \quad \text{by (12)}
\]

\[
= (x \land (\neg y \lor T)) \land y \quad \text{by (9)}
\]

\[
= x \land (\neg y \lor T) \land y \quad \text{by MSCL3}
\]

\[
= x \land (y \land (\neg y \lor T)) \quad \text{by MSCL3}
\]

\[
= x \land y. \quad \text{by (12)}
\]

Note that with \( y = T \) we find \( x \land x = x \).

• Equation (11) can be derived as follows:

\[
(x \land y) \lor z = (x \land (\neg x \lor y)) \lor z \quad \text{by (8)}
\]

\[
= ((x \lor F) \land (\neg x \lor y)) \lor z \quad \text{by (8)}
\]

\[
= (x \land (y \lor z)) \lor (\neg x \land (F \lor z)) \quad \text{by (8)}
\]

\[
= (x \land (y \lor z)) \lor (\neg x \land z). \quad \text{by (8)}
\]

**Theorem 1 (Completeness).** For all SCL-terms \( t \) and \( t' \),

\[ \text{EqMSCL} \vdash t = t' \iff \text{MSCL} \vdash t = t'. \]

**Proof.** According to Proposition 2 it suffices to prove that the axioms of \( \text{CP}_{mem} \) are derivable from EqMSCL. In this proof we use \( F \) and \( \lor \) in the familiar way and also the definability of the conditional by \( x \land y \triangleright z = (y \land x) \lor (\neg y \land z) \) (see Section 2.2), and equation (4), i.e.,

\[
(y \land x) \lor (\neg y \land z) = (y \lor z) \land (\neg y \lor x).
\]
With (CP3):

\[ x \triangleleft T \triangleright y = (T \wedge x) \vee (F \triangleleft y) = x \vee F = x. \]

With (CP4):

\[ x \triangleleft F \triangleright y = (F \wedge x) \vee (T \triangleleft y) = F \vee y = y. \]

To derive (CP4), i.e.,

\[ L = (X \triangleright y) \wedge (\neg x \triangleleft F) = x \wedge (\neg x \triangleleft F) = x \] by (12).

To derive (CP4), i.e.,

\[ x \triangleleft (y \vee z) \triangleright (x \triangleleft u \triangleright v) = (x \triangleleft y \triangleright v) \triangleleft z \triangleright (x \triangleleft u \triangleright v), \]

to which we refer to by \( L = R \), we use the identity

\[ (x \vee (y \triangleleft z)) \triangleleft (\neg x \vee (u \triangleleft z)) = ((x \vee y) \triangleleft (\neg x \vee u)) \triangleleft z, \]

which can be easily derived from equations (9) and (MSCL3). Then

\[ L = (X \triangleleft x) \vee (\neg X \triangleleft v), \]

with \( X = (z \triangleleft y) \vee (\neg z \triangleleft u) \). Hence, \( X = (z \vee u) \triangleleft (\neg z \vee y) \) is derivable from EqMSCL, and so is \( \neg X = (z \vee u) \triangleleft (\neg z \vee y) \). We derive

\[ L = ([z \vee (u \triangleleft x)] \triangleleft [\neg z \vee (y \triangleleft x)]) \vee \]

\[ ([z \vee (\neg u \triangleleft v)] \triangleleft [\neg z \vee (\neg y \triangleleft v)]) \]

by (13)

\[ = (z \triangleleft (y \triangleleft x)) \vee (\neg z \triangleleft (u \triangleleft x)) \vee \]

\[ (z \triangleleft (\neg y \triangleleft v)) \vee (\neg z \triangleleft (\neg u \triangleleft v)) \]

by (1) and (SCL7)

\[ = (z \triangleleft (y \triangleleft x)) \vee (\neg z \triangleleft (\neg y \triangleleft v)) \vee \]

\[ (\neg z \triangleleft (u \triangleleft x)) \vee (\neg z \triangleleft (\neg u \triangleleft v)) \]

by (MSCL3)

\[ = (z \triangleleft ((y \triangleleft x) \vee (\neg y \triangleleft v))) \vee (\neg z \triangleleft ((u \triangleleft x) \vee (\neg u \triangleleft v))) \]

by (MSCL2)

\[ = R. \]

As argued in Section 2.2 it is sufficient to derive axiom (CPmem), i.e.,

\[ (w \triangleleft y \triangleright (z \triangleleft x \triangleright u)) \triangleleft x \triangleright v = (w \triangleleft y \triangleright z) \triangleleft x \triangleright v, \]

say \( L = R \). We will use equation (9), which we repeat here:

\[ x \triangleleft (y \vee [(x \triangleleft z) \vee (\neg x \triangleleft u)]) = x \triangleleft (y \vee z). \]

We derive

\[ L = (x \triangleleft (w \triangleleft y \triangleright (z \triangleleft x \triangleright u))) \vee (\neg x \triangleleft v) \]

\[ = (x \triangleleft [(y \vee (z \triangleleft x \triangleright u)) \triangleleft (\neg y \vee w)]) \vee (\neg x \triangleleft v) \]

\[ = (x \triangleleft [(y \vee [(x \triangleleft z) \vee (\neg x \triangleleft u)]) \triangleleft (\neg y \vee w)]) \vee (\neg x \triangleleft v) \]

\[ = (x \triangleleft [(y \vee [(x \triangleleft z) \vee (\neg x \triangleleft u)])] \triangleleft (\neg y \vee w)) \vee (\neg x \triangleleft v) \]

\[ = (x \triangleleft (y \vee z)) \triangleleft (\neg y \vee w) \vee (\neg x \triangleleft v) \]

by (SCL7)

\[ = (x \triangleleft (y \vee z) \triangleleft (\neg y \vee w)) \vee (\neg x \triangleleft v) \]

by (9)

\[ = (x \triangleleft ((y \vee z) \vee (\neg y \vee w))) \vee (\neg x \triangleleft v) \]

by (SCL7)

\[ = R. \]

\[ \square \]

5 Other short-circuit logics

In this section we consider some other variants of short-circuit logic. First we discuss a sequential variant of PL, and then two more variants that stem from axiomatizations of proposition algebra that are in between CP and CPmem.
5.1 Static short-circuit logic: SSCL

In this section we prove that the equation \( x \land F = F \) marks the distinguishing feature between MSCL and PL: adding this axiom to EqMSCL yields an equational characterization of PL (be it in sequential notation and defined with short-circuit evaluation).

**Definition 4. SSCL (static short-circuit logic)** is the short-circuit logic that implies no other consequences than those of the module expression

\[
\{ T, \neg, \land \} \sqcap (\text{CP}_{\text{mem}} + \langle F \land x \triangleright F = F \\
\quad + \langle \neg x = F \land x \triangleright T \rangle + \langle x \land y = y \land x \triangleright F \rangle).
\]

Furthermore, we define EqSSCL as the extension of EqMSCL with the axiom \( x \land F = F \).

Our first result is a very simple corollary of Theorem 1.

**Theorem 2 (Completeness).** For all SCL-terms \( t \) and \( t' \),

\[ \text{EqSSCL} \vdash t = t' \iff \text{SSCL} \vdash t = t'. \]

**Proof.** Soundness, i.e., \( \iff \) follows trivially from Proposition 2 and the fact that \( (x \land F = F) \in \text{EqSSCL} \). In order to show \( \implies \) it is by Theorem 1 sufficient to show that the axiom \( F \land x \triangleright F = F \) is derivable in EqSSCL: \( F \land x \triangleright F = (x \land F) \lor (\neg x \land F) = F \lor F = F \).

Combining identity (7) (i.e., \( x \land \neg x = x \land F \)) and \( x \land F = F \) yields

\[ x \land \neg x = F \]

and thus also \( x \lor \neg x = T \).

We prove that EqSSCL \( \vdash x \land (y \land \neg x) = F \). The derivation is similar to the one we used to prove the memory property (10):

\[
\begin{align*}
x \land (y \land \neg x) &= x \land (y \land (\neg y \lor \neg x)) \\
&= x \land ((\neg y \lor \neg x) \land y) \\
&= (x \land (\neg y \lor \neg x)) \land y \\
&= (x \land (\neg y \lor (\neg x \lor (x \land F)))) \land y \\
&= (x \land (\neg y \lor ((\neg x \land T) \lor (x \land F)))) \land y \\
&= (x \land (\neg y \lor (x \land F) \lor (\neg x \land T))) \land y \\
&= (x \land (\neg y \lor F)) \land y \\
&= x \land (\neg y \land y) \\
&= F.
\end{align*}
\]

From this equation and the memory property, commutativity of \( \land \) can be easily derived:

\[
\begin{align*}
x \land y &= (y \lor \neg y) \land (x \land y) \\
&= (y \land (x \land y)) \lor (\neg y \land (x \land y)) \\
&= y \lor x. \quad \text{(by (10) and the above equation) (14)}
\end{align*}
\]

As a consequence, all distributivity and absorption axioms follow from EqSSCL, and it is not difficult to see that EqSSCL defines the mentioned variant of “sequential PL”: this
follows for example immediately from [13] in which equational bases for Boolean algebra are provided, and each of these bases can be easily derived from EqSSCL (below we return to this point).

Using equation [14] (i.e., the commutativity of $\land$) we can establish in a simple way an equivalent characterization of SSCL. Recall that CP$_{stat}$ is defined in Section 2.3 as the extension of CP with the contraction law [1] (i.e., $x \land y \downarrow (v \land y \downarrow w) = x \land y \downarrow w$) and the axiom [CP$_{stat}$], i.e.,

$$(x \land y \downarrow z) \land u \downarrow v = (x \land u \downarrow v) \land y \downarrow (z \land u \downarrow v).$$

We show that CP$_{stat}$ and (CP$_{mem} + (F \land x \uparrow F = F)$) are equally strong:

$$\text{CP}_{stat} \vdash x \land y \downarrow (z \land u \downarrow (v \land y \downarrow w))$$

$$= x \land y \downarrow ((v \land y \downarrow w) \land (F \land u \uparrow T) \downarrow z)$$

$$= x \land y \downarrow ((v \land (F \land u \uparrow T) \downarrow z) \land y \downarrow (w \land (F \land u \uparrow T) \downarrow z))$$

$$= x \land y \downarrow (w \land (F \land u \uparrow T) \downarrow z)$$

$$= x \land y \downarrow (z \land u \downarrow w),$$

and hence CP$_{stat} \vdash$ (CP$_{mem}$). Furthermore, we showed in Section 2.3 that CP$_{stat} \vdash v = v \land y \downarrow v$ and thus CP$_{stat} \vdash F \land x \uparrow F = F$, which proves the first part of the mentioned characterization. Now, recall that the contraction law [1] is derivable in CP$_{mem}$. Using equation [13] in conditional notation we derive

$$(\text{CP}_{mem} + (F \land x \uparrow F = F)) \vdash (x \land y \downarrow z) \land u \downarrow v$$

$$= (x \land y \downarrow (z \land u \downarrow v)) \land u \downarrow v$$

$$= (x \land y \downarrow (z \land u \downarrow v)) \land u \downarrow v$$

$$= x \land (y \land u \downarrow F) \uparrow (z \land u \downarrow v)$$

$$= x \land (u \land y \downarrow F) \uparrow (z \land u \downarrow v)$$

$$= (x \land u \downarrow (z \land u \downarrow v)) \land y \downarrow (z \land u \downarrow v)$$

$$= (x \land u \downarrow (z \land u \downarrow v)) \land y \downarrow (z \land u \downarrow v).$$

Hence (CP$_{mem} + (F \land x \uparrow F = F)) \vdash$ (CP$_{stat}$) and we have the following corollary.

**Corollary 1.** SSCL equals

$$\{T, \neg, \land\} \sqcap (\text{CP}_{stat} + (\neg x = F \land x \uparrow T) + (x \land y = y \land x \uparrow F)).$$

Hoare proved in [10] that each tautology in PL can be (expressed and) proved with his axioms for the conditional. According to [6], this also holds for CP$_{stat}$, and thus also for EqSSCL if we identify the symmetric connectives with their left-sequential counterparts.

### 5.2 Contractive and Repetition-Proof short-circuit logic

We briefly discuss two other variants of short-circuit logics which both involve explicit reference to a set $A$ of atoms (propositional variables). Both these variants are located in between SCL and MSCL.
In [6] we introduced CP<sub>cr</sub> (contractive CP) which is defined by the extension of CP with these axiom schemes ($a \in A$):

\[
\begin{align*}
(x \triangleleft a \triangleright y) \triangleleft a \triangleright z &= x \triangleleft a \triangleright z, \\
(x \triangleleft a \triangleright (y \triangleleft a \triangleright z)) &= x \triangleleft a \triangleright z.
\end{align*}
\]

(CP<sub>cr1</sub>)

(CP<sub>cr2</sub>)

These schemes contract for each atom $a$ respectively the true-case and the false-case. We write Eq<sub>cr</sub>($A$) to denote the set of these axioms schemes in the format of module algebra [3].

**Definition 5.** **CSCL** (**contractive short-circuit logic**) is the short-circuit logic that implies no other consequences than those of the module expression

\[
\{T, \neg, \triangleleft, a \mid a \in A\} \Box (\text{CP} + \text{Eq}_{cr}(A) + (\neg x = F \triangleleft x \triangleright T) + (x \triangleleft y = y \triangleleft x \triangleright F)).
\]

The equations defined by CSCL include those that are derivable from EqFSCL (see Table 2) and the following immediate counterparts of the axiom schemes (CP<sub>cr1</sub>) and (CP<sub>cr2</sub>) (for $a \in A$):

\[
\begin{align*}
a \triangleleft (a \triangleright x) &= a, \\
(a \triangleright a) &= a.
\end{align*}
\]

(15)

(16)

Observe that from EqFSCL and equation schemes (15) and (16) the following equations can be derived ($a \in A$):

\[
\begin{align*}
a \triangleleft a &= a, \\
\neg a \triangleleft (\neg a \triangleright x) &= \neg a, \\
\neg a \triangleleft \neg a &= \neg a, \\
\neg a \triangleright a &= \neg a, \\
\neg a \triangleright (\neg a \triangleright x) &= \neg a, \\
\neg a \triangleright \neg a &= \neg a.
\end{align*}
\]

Furthermore, it is not hard to prove that the following two equation schemes ($a \in A$) are also valid in CSCL:

\[
\begin{align*}
a \triangleright a &= a \triangleright T, \\
a \triangleright \neg a &= a \triangleright F.
\end{align*}
\]

(17)

(18)

The question whether the extension of EqFSCL with the equation schemes (15) − (18) provides for closed terms an axiomatization of CSCL is left open. An example that illustrates the use of CSCL concerns atoms that define manipulation of Boolean registers:

- Consider atoms set:i:j and eq:i:j with $i \in \{1, ..., n\}$ (the number of registers) and $j \in \{T, F\}$ (the value of registers).
- An atom set:i:j can have a side effect (it sets register $i$ to value $j$) and yields upon evaluation always true.
- An atom eq:i:j has no side effect but yields upon evaluation only true if register $i$ has value $j$.

Clearly, the consequences mentioned above are derivable in CSCL, but $x \triangleleft x = x$ is not: assume register 1 has value $F$ and let $t = \text{eq}:1:F \triangleleft \text{set}:1:T$. Then $t$ yields true upon evaluation in this state, while $t \triangleleft t$ yields false.
It is easily seen that the axiom schemes (CPrp1) and (CPrp2) are derivable in CP or, which implies no other consequences than those of the module expression.

Definition 6. RPSCL

These axiom schemes (CPrp1) and (CPrp2) follow from EqFSCL extended with the equation schemes (15) − (20).

It is easily shown that all equation schemes in Table 4 whether the extension of EqFSCL with the equation schemes in Table 4 provides for closed terms an axiomatization of RPSCL. It is easily shown that all equation schemes in Table 4 follow from EqFSCL extended with the equation schemes (15) − (20) for CSCL.

An example that illustrates the use of RPSCL is a combination of the examples on FSCL (Fig. 1) and CSCL. Consider simple arithmetic expressions over the natural numbers (or the

Table 4: Equation schemes (a ∈ A) for RPSCL

In [6] we also introduced CPcr (repetition-proof CP) for the axioms in CP extended with these axiom schemes (a ∈ A):

\begin{align*}
(x \lor y) &\lor (a \lor \neg a) = (\neg x \lor (a \lor \neg a)) \lor (y \lor (a \lor \neg a)) \\
(x \lor y) &\lor (\neg a \land a) = (\neg x \lor (\neg a \land a)) \lor (y \lor (\neg a \land a)) \\
(x \lor y) &\lor (a \lor \neg a) = (x \lor (a \lor \neg a)) \lor (y \lor (a \lor \neg a)) \\
(x \lor y) &\lor (\neg a \lor a) = (x \lor (\neg a \lor a)) \lor (y \lor (\neg a \lor a)) \\
((x \lor y) \lor (\neg a \lor a)) &\lor (z \lor (a \lor \neg a)) \\
((\neg a \lor a) \lor (y \lor (a \lor \neg a))) &\lor (x \lor (a \lor \neg a)) \\
((\neg a \lor a) \lor (y \lor (a \lor \neg a))) &\lor (x \lor (a \lor \neg a)) \\
((\neg a \lor a) \lor (y \lor (a \lor \neg a))) &\lor (x \lor (a \lor \neg a)) \\
((\neg a \lor a) \lor (y \lor (a \lor \neg a))) &\lor (x \lor (a \lor \neg a))
\end{align*}

\begin{align}
(x \lor y) &\lor (a \lor \neg a) = (\neg x \lor (a \lor \neg a)) \lor (y \lor (a \lor \neg a)) \\
(x \lor y) &\lor (\neg a \land a) = (\neg x \lor (\neg a \land a)) \lor (y \lor (\neg a \land a)) \\
(x \lor y) &\lor (a \lor \neg a) = (x \lor (a \lor \neg a)) \lor (y \lor (a \lor \neg a)) \\
(x \lor y) &\lor (\neg a \lor a) = (x \lor (\neg a \lor a)) \lor (y \lor (\neg a \lor a)) \\
((x \lor y) \lor (\neg a \lor a)) &\lor (z \lor (a \lor \neg a)) \\
((\neg a \lor a) \lor (y \lor (a \lor \neg a))) &\lor (x \lor (a \lor \neg a)) \\
((\neg a \lor a) \lor (y \lor (a \lor \neg a))) &\lor (x \lor (a \lor \neg a)) \\
((\neg a \lor a) \lor (y \lor (a \lor \neg a))) &\lor (x \lor (a \lor \neg a))
\end{align}

It is easily seen that the axiom schemes (CPrp1) and (CPrp2) are derivable in CPcr (so CPcr is also an extension of CPcr). We write Eqcr(A) to denote the set of these axioms schemes in the format of module algebra [3].

Definition 6. RPSCL (repetition-proof short-circuit logic) is the short-circuit logic that implies no other consequences than those of the module expression

\[ \{T, \neg, \land, a \mid a \in A\} \boxtimes (CP + Eq_{cr}(A)) + (\neg x = F < a \lor T) + (x \land y = y < a \lor F) \].

The equations defined by RPSCL include those that are derivable from EqFSCL (see Table 2 and Table 4). In Table 4, equation schemes (19) and (20) are the immediate counterparts of the axiom schemes (CPrp1) and (CPrp2), and equations schemes (21) and (22) are the counterparts of the identity \( T \land x = F \lor x \). Equation schemes (23) − (30) are the counterparts of the remaining EqFSCL-equations that involve \( T \) or \( F \). We do not know whether the extension of EqFSCL with the equation schemes in Table 4 provides for closed terms an axiomatization of RPSCL. It is easily shown that all equation schemes in Table 4 are derivable in the module algebra.

An example that illustrates the use of RPSCL is a combination of the examples on FSCL (Fig. 1) and CSCL. Consider simple arithmetic expressions over the natural numbers (or the
integers) and a programming notation for imperative programs or algorithms in which each atom is either a test or an assignment. Assume that assignments when used as conditions always evaluate to true (next to having their intended effect). Then, these atoms satisfy the equations in Tables 2 and 3. However, the atom \((n=n+1)\) clearly does not satisfy the contraction law \(a \land r \triangleright a = a\) because \(((n=n+1) \land (n=n+1)) \land (n==2)\) and \((n=n+1) \land (n==2)\) yield different evaluation results. Hence we have a clear example of the repetition-proof characteristic of RPSCL.

6 Conclusions and future work

In our paper [6] we introduced proposition algebra using Hoare’s conditional \(x \preceq y \triangleright z\) and the constants \(T\) and \(F\). We distinguished various valuation congruences that are defined by means of short-circuit evaluation, and provided axiomatizations of these congruences: CP (four axioms) characterizes the least identifying valuation congruence we consider, and the extension CP\(_{mem}\) (one extra axiom) characterizes the most identifying valuation congruence below propositional logic. Various other valuation congruences in between these two and axiomatizations thereof are also described in [6]. In our paper [7] we provide an alternative valuation semantics for proposition algebra in the form of Hoare McCarthy algebras (HMAs) that is more elegant than the semantical framework introduced in [6]: HMA-based semantics has the advantage that one can define a valuation congruence without first defining the valuation equivalence it is contained in.

This paper arose by an attempt to answer the question whether the extension of CP\(_{mem}\) with \(\neg\) and \(\land\) characterizes a reasonable logic if one restricts to axioms defined over the signature \(\{T, \neg, \land\}\) (and with \(F\) and \(\lor\) being definable). After having found an axiomatization of MSCL (memorizing short-circuit logic), we distinguished FSCL (free short-circuit logic) as the most basic (least identifying) short-circuit logic, where we took CP as a point of departure. We used the module expression

\[
SCL = \{T, \neg, \land\} \Box (CP + (\neg x = F \preceq x \triangleright T) + (x \land y = y \preceq x \triangleright F))
\]

in our generic definition of short-circuit logics (Definition [1]) and presented the set EqFSCL of equations that are derivable in FSCL. However, we did not prove independence and completeness of EqFSCL, although we have not found FSCL-identities that are not derivable from EqFSCL.

We used a Perl program to illustrate that FSCL is a reasonable logic. A next question is to provide a natural example that supports MSCL. A first idea is to consider only programs (say, in a Perl-like language) that allow in conditions only (comparative) tests on scalar variables (no side effects), but also special conditions that test whether a program-variable has been evaluated (initialized) before, say eval\(x\) for scalar variable \(x\). This combines well with the consequences of MSCL, but refutes identities that are typically not in MSCL, such as for example \((x==x) \land (\text{eval}\ x) = (\text{eval}\ x) \land (x==x)\) where the left-hand side always evaluates to true, while the right-hand side can yield false.

The extension of CP\(_{mem}\) with the axiom \(F \preceq x \triangleright F = F\) that defines SSCL (static short-circuit logic) yields an axiomatization of static CP (a sequential variant of PL) that is more elegant than the one provided in Section 2.3 and in [6]: it is a simple exercise to derive the axiom \(\text{CPstat}\) using the expressibility of conditional composition and commutativity of \(\land\) and \(\lor\) (and hence full distributivity). We say a little more about this matter in Appendix C.
In [6], more variants in between CP and CP\textsubscript{mem} are distinguished, and thus more short-circuit logics can be defined. In this paper we defined CSCL (contractive short-circuit logic) and RPSCL (repetition-proof short-circuit logic), and listed some obvious equations for the associated extensions of EqFSCL, but leave their completeness open. Furthermore, we provided examples on the applicability of CSCL and RPSCL.

The focus on left-sequential conjunction that is typical for this paper leads to the following considerations. First, the valuation congruences that we consider can be axiomatized in a purely incremental way: the axiom systems CP\textsubscript{rp} up to and including CP\textsubscript{stat} as defined in [6] all share the axioms of CP and each next system can be defined by the addition of either one or two axioms, in most cases making previously added axiom(s) redundant (the last inter-derivability $\dashv \vdash$ is explained in Section 5.1):

- \( CP\textsubscript{rp} = CP + (\text{CPrp1}) + (\text{CPrp2}) \),
- \( CP\textsubscript{cr} \vdash CP\textsubscript{rp} + (\text{CPcr1}) + (\text{CPcr2}) \),
- \( CP\textsubscript{mem} \vdash CP\textsubscript{cr} + (\text{CPmem}) \),
- \( CP\textsubscript{stat} \vdash CP\textsubscript{mem} + \langle F \triangleleft x \triangleright F = F \rangle \).

(In [6] we also define weakly memorizing valuation congruence that has an axiomatization with the same property in between CP\textsubscript{cr} and CP\textsubscript{mem}.) Secondly, the focus on left-sequential conjunction led us to a new, elegant axiomatization of CP\textsubscript{stat} (see Appendix C) consisting of five axioms that are simpler than those of CP\textsubscript{stat} (which stem from the eleven axioms in Hoare’s paper [10]). Of course, “simple and few in number” is not an easy qualification in this case: with the axioms \( x \triangleleft T \triangleright y = x \) and \( x \triangleleft F \triangleright y = y \), each other pair of axioms \( L_1 = R_1 \) and \( L_2 = R_2 \) can be combined with a fresh variable, say \( u \), to a single axiom \( L_1 \triangleleft u \triangleright L_2 = R_1 \triangleleft u \triangleright R_2 \). So, each extension of CP can be defined by adding a single (and ugly) axiom to CP (and CP itself can be axiomatized with three axioms).

**Future work**

1. A perhaps interesting variant of SCL is obtained by leaving out the constant \( T \) in the exported signature (and thus also leaving out \( F \) as a definable constant). This weakened variant of SCL can be motivated by the fact that these constants are usually absent in conditions in imperative programming (although they may be always used; also \( T \) can be mimicked by a void equality test such as \((1 == 1)\) in Perl). On the other hand, it might be judged inappropriate to define a logic about truth and falsity in which one cannot express true as a value (up to and including MSCL). Observe that in “SCL without \( T \)” only the EqFSCL-equations expressing duality, double negation shift, and associativity (equations \( \text{SCL2}, \text{SCL3} \) and \( \text{SCL7} \), respectively) remain. Moreover, these axioms also yield a complete axiomatization of SCL-congruence (this is work to appear). A more pragmatic point of view on “SCL without \( T \)” is to define the set of conditions over set \( A \) of atoms by

\[ \{T, F\} \cup \{t \mid t \text{ a closed term over } \{\land, \lor, \neg, a \mid a \in A\}\} \]

(so \( T \) and \( F \) do not occur in compound conditions). In this case, the EqFSCL-equations \( \text{SCL1, SCL2}, \text{SCL3} \) and \( \text{SCL7} \) yield a complete axiomatization of SCL-congruence on conditions.

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2. In [1] a connection is proposed between short-circuit logic and instruction sequences as studied in program algebra [4, 5]. In particular, the relation between non-atomic test instructions (i.e., test instructions involving sequential connectives) and their decomposition in atomic tests and jump instructions is considered, evolving into a discussion about the length of instruction sequences and their minimization. Also, the paper [1] contains a discussion about a classification of side effects, derived from a classification of atoms (thus partitioning \( A \)), and a consideration about “real time” atoms, as for example (\( \text{height} > 3000 \text{ meter} \)). We expect that these matters will lead to future work.

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A  The Perl program Not.pl

my $x = 0;
print "\n \$x=$x (assignment)\n";

if ( (($x=$x+1) && not($x=$x+1)) || $x==2 )
    {print " "((($x=$x+1) && not($x=$x+1)) || \$x==2)" is true \n\n";}
else
    {print " \" is false \n\n";}

$x = 0;
print " \$x=$x (assignment)\n";

if ( (not($x=$x+1) && ($x=$x+1)) || $x==2 )
    {print " \" is true \n\n";}
else
    {print " \"((not($x=$x+1) && ($x=$x+1)) || \$x==2)" is false \n\n";}

B  Derivability of axioms SCL8 - SCL10 from EqMSCL

We prove that the EqFSCL-equations \( - \) can be derived from the axiomatization EqMSCL (see Table 3 of MSCL. (Of course, derivability of all closed instances of these equations follows from Theorem 1 and the fact that MSCL identifies more than FSCL.)

In the derivations below we use equation 11 and its dual 11’, that is,

\[
(x \lor y) \land z = (x \lor (y \land z)) \land (\neg x \lor z).
\]

EqMSCL ⊢ SCL8. With the identity \( (z \land F) = (z \land F) \land \neg u \) we derive

\[
(x \lor y) \land (z \land F) = (x \lor (y \land (z \land F))) \land (\neg x \lor (z \land F)) \land (\neg x \lor (z \land F)).
\]

by 11’
EqMSCL ⊢ (SCL9). With the identity \((z ∨ T) = (z ∨ T) ∨ u\) we derive

\[(x ∨ y) \land (z ∨ T)\]

\[= (x ∨ (y \land (z ∨ T))) \land (\neg x ∨ (z ∨ T))\]  
by \((11)\)

\[= (x \land (z ∨ T)) ∨ (\neg x ∨ (y ∨ (z ∨ T)))\]  
by \((\text{\ref{r}})\)

\[= (x \land ((z ∨ T) ∨ (y ∨ (z ∨ T)))) ∨
\[\neg x ∨ (y ∨ (z ∨ T)))\]  
by \((\text{\ref{r}})\)

\[= (x \land (z ∨ T)) ∨ (y ∨ (z ∨ T)).\]  
by \((11)\)

EqMSCL ⊢ (SCL10). This equation can be derived as follows:

\[(x \land (z ∨ y)) \land y = \neg x \land (y ∨ (z ∨ T)) \land z\]  
by \((\text{\ref{r}})\)

\[= ((x ∨ (z ∨ y)) \land y) \land z\]  
by \((\text{\ref{r}})\)

\[= ((x ∨ y) ∨ (\neg x ∨ (z ∨ T))) \land z\]  
by \((\text{\ref{r}})\)

\[= (x ∨ y) \land (y ∨ (z ∨ T))\]  
by \((\text{\ref{scl4}})\)

\[= (x ∨ y) \land (y ∨ (z ∨ T)).\]  
by \((\text{\ref{r}})\)
A very simple axiomatization of static CP (see Section 2.3) is provided in Table 5 and we write CP$^*_{stat}$ for this set of axioms.

Compared to CP, axiom (CP$^3_\text{stat}$) is a strengthening of axiom (CP$^3$) (i.e., $T \triangleleft x \triangleright F = x$) that also implies $x \lor y = y \lor x$ and thus $x \lor T = T \lor x = T$. Axiom (CP$^5$) implies the absorption law $x \land (x \lor y) = x$, and also $x \land \neg x = F$. Observe that the duality principle is derivable in CP$^*_{stat}$ (because it derivable in CP, cf. Section 2.1), so $x \land y = y \land x$, and thus $x \land \neg x = x \land F = F \land x = F$. Also, distributivity is derivable in CP$^*_{stat}$:

\[
\begin{align*}
  x \lor (y \land z) &= T \triangleleft (z \triangleleft y \triangleright F) \\
  &= T \triangleleft (z \triangleleft (y \triangleright F)) \triangleleft x \\
  &= (T \triangleleft x \triangleright z) \triangleleft y \triangleright (T \triangleleft x \triangleright F) \quad \text{by (CP$^3_\text{stat}$)} \\
  &= (T \triangleleft x \triangleright z) \triangleleft y \triangleright (T \triangleleft (x \triangleright z) \triangleleft x \triangleright F) \quad \text{by (CP$^5$)} \\
  &= (T \triangleleft x \triangleright z) \triangleleft (T \triangleleft y \triangleright x) \triangleright F \quad \text{by (CP$^4$) and (CP$^1$)} \\
  &= (T \triangleleft x \triangleright y) \triangleright (T \triangleleft x \triangleright y) \triangleright F \quad \text{by (CP$^*_{stat}$)} \\
  &= (x \lor y) \land (x \lor z).
\end{align*}
\]

With the translation $x \triangleleft y \triangleright z = (y \land x) \lor (\neg y \land z)$ it is a simple exercise to derive the CP$^*_{stat}$-axioms (CP$\text{stat}$ and 1) in CP$^*_{stat}$:

\[
\begin{align*}
  (x \triangleleft u \triangleright v) \triangleleft y \triangleright (z \triangleleft u \triangleright v) \\
  &= (y \land [(u \land x) \lor (\neg u \land v)]) \lor (\neg y \land [(u \land z) \lor (\neg u \land v)]) \\
  &= (y \land u \land x) \lor (y \land \neg u \land v) \lor (\neg y \land u \land z) \lor (\neg y \land \neg u \land v) \\
  &= (y \land u \land x) \lor (\neg y \land (\neg u \land v)) \lor (\neg y \land u \land z) \\
  &= (u \land [(y \land x) \lor (\neg y \land z)]) \lor (\neg u \land v) \\
  &= (x \triangleleft y \triangleright z) \triangleleft u \triangleright v,
\end{align*}
\]
and
\[
(x \triangleleft y \triangleright z) \triangleleft y \triangleright u = (y \land [(y \land x) \lor (\neg y \land z)]) \lor (\neg y \land u)
\]
\[
= (y \land y \land x) \lor (y \land \neg y \land z) \lor (\neg y \land u)
\]
\[
= (y \land x) \lor (\neg y \land u)
\]
\[
= x \triangleleft y \triangleright u.
\]

Hence, CP* stat is a complete axiomatization of static CP. In [12] it is proved that CP stat is \(\omega\)-complete, so CP* stat is \(\omega\)-complete as well.

Next, we show that the axioms of CP* stat are independent. Inspired by [12] we provide the following independence-models, where \(P, Q\) and \(R\) range over closed terms, and where \(a\) and \(b\) are two atoms and \(\phi\) is the interpretation function. The domain of the first two independence-models is \(\{T, F\}\) and the interpretation refers to classical propositional logic.

1. The model defined by \(\phi(T) = F, \phi(F) = \phi(a) = \phi(b) = T\) and \(\phi(P \triangleleft Q \triangleright R) = \phi(Q) \land \phi(R)\) satisfies all axioms but (CP1).

2. The model defined by \(\phi(T) = \phi(a) = \phi(b) = T, \phi(F) = F\) and \(\phi(P \triangleleft Q \triangleright R) = \phi(P)\) satisfies all axioms but (CP2).

3. Each model for CP mem that does not satisfy \(a \lor \phi(b) = b \lor \phi(a)\), satisfies all axioms but (CP3*). Such models exist and are discussed in [6, 7].

4. The model with the natural numbers as its domain, and the interpretation defined by
   \[
   \phi(T) = 0, \phi(F) = 1, \phi(a) = 2, \phi(b) = 3,
   \phi(P \triangleleft Q \triangleright R) = \begin{cases} 
   \phi(P) & \text{if } \phi(Q) = 0, \\
   \phi(R) & \text{if } \phi(Q) = 1, \\
   \phi(Q) \cdot \phi(R) & \text{otherwise},
   \end{cases}
   \]
satisfies all axioms but (CP3): \(\phi(F \triangleleft a \triangleright T) = \phi(a) \cdot \phi(T) = 0\), so
   \[
   \phi(F \triangleleft (F \triangleleft a \triangleright T) \triangleright T) = \phi(F) = 1,
   \]
while \(\phi((F \triangleleft F \triangleright T) \triangleleft a \triangleright (F \triangleleft T \triangleright T)) = \phi(a) \cdot \phi(F) = 2\).

5. The model with the integers numbers as its domain, and the interpretation defined by
   \[
   \phi(T) = 0, \phi(F) = 1, \phi(a) = 2, \phi(b) = 3,
   \phi(P \triangleleft Q \triangleright R) = (1 - \phi(Q)) \cdot \phi(P) + \phi(Q) \cdot \phi(R),
   \]
satisfies all axioms but (CP3*): \(\phi(T \triangleleft a \triangleright F) = 2\), while
   \[
   \phi((T \triangleleft a \triangleright F) \triangleleft a \triangleright F) = (1 - 2) \cdot 2 + 2 \cdot 1 = 0.
   \]

A proof of the validity of axioms (CP3*) and (CP4) is a simple arithmetical exercise.

Finally, we note that CP* stat has an elegant, symmetric nature: exchanging (CP3*) with \(x \triangleleft y \triangleright F = y \triangleleft x \triangleright F\) and (CP5) with \(T \triangleleft x \triangleright (y \triangleleft x \triangleright z) = T \triangleleft x \triangleright z\) yields an equally strong axiomatization. Moreover, the resulting axioms are also independent (modifying the independence proofs given above is nothing more than a simple exercise).