Solving coupled Lane-Emden equations by Green’s function and decomposition technique

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Abstract

In this paper, the Green’s function and decomposition technique is proposed for solving the coupled Lane-Emden equations. This approach depends on constructing Green’s function before establishing the recursive scheme for the series solution. Unlike, standard Adomian decomposition method, the present method avoids solving a sequence of transcendental equations for the undetermined coefficients. Convergence and error estimation is provided. Three examples of coupled Lane-Emden equations are considered to demonstrate the accuracy of the current algorithm.

Keyword: Coupled Lane-Emden equations; Green’s function; Adomian decomposition method; Convergence analysis.

1 Introduction

This paper aims to extend the application of the Adomian decomposition method with Green’s function [1, 2] for solving the following coupled Lane-Emden boundary value problems

\[
\begin{align*}
    y_1''(x) + \frac{\alpha_1}{x} y_1'(x) &= f_1(x, y_1(x), y_2(x)), & x \in (0, 1) \\
    y_2''(x) + \frac{\alpha_2}{x} y_2'(x) &= f_2(x, y_1(x), y_2(x)), \\
    y_1'(0) &= 0, & y_2'(0) = 0, \\
    a_1y_1(1) + b_1y_1'(1) &= c_1, & a_2y_2(1) + b_2y_2'(1) = c_2,
\end{align*}
\]

(1.1)

where \(a_1, a_2, b_1, b_2, c_1, c_2\) are real constants. In recent years, singular boundary value problems for ordinary differential equations have been studied extensively [1–21] and references therein. However, we find only the following results on coupled Lane-Emden equations.

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Recently, in \[22–24\] authors studied (1.1) with boundary conditions \(y_1'(0) = y_2'(0) = 0, y_1(1) = y_2(1) = 1\) and \(\alpha_1 = \alpha_2 = 2\) that relates the concentration of the carbon substrate and the concentration of oxygen. In \[25, 26\], authors considered the coupled Lane–Emden equations (1.1) with boundary conditions \(y_1'(0) = y_2'(0) = 0, y_1(1) = 1, y_2(1) = 2\) and \(\alpha_1 = \alpha_2 = 2\) occurs in catalytic diffusion reactions. In \[24, 26\], the Adomian decomposition method was applied to obtain a convergent analytic approximate solution of (1.1) with \(\alpha_1 = \alpha_2 = 2\). Later, in \[27\], the variational iteration method was applied to obtain approximations to solutions of (1.1) for shape factors \(\alpha_1, \alpha_2 = 1, 2, 3\). In \[28\], the Sinc-collocation method was used to obtain the solution of (1.1). In \[29\] authors used the reproducing kernel Hilbert space method for solving to obtain the solution of (1.1).

### 2 Adomian decomposition method

Recently, many researchers \[30, 36\] have applied the Adomian decomposition method to deal with many different scientific models. According to the Adomian decomposition method we rewrite (1.1) in a operator form as

\[
\begin{cases}
L_1y_1(x) = f_1(x, y_1(x), y_2(x)), & x \in (0, 1) \\
L_2y_2(x) = f_2(x, y_1(x), y_2(x)),
\end{cases}
\tag{2.1}
\]

where \(L_1 = x^{-\alpha_1} \frac{d}{dx} [x^{\alpha_1} \frac{d}{dx}]\) and \(L_2 = x^{-\alpha_2} \frac{d}{dx} [x^{\alpha_2} \frac{d}{dx}]\) are differential operators and their inverse integral operators are defined as

\[
\begin{cases}
L_1^{-1}[\cdot] = \int_0^x x^{-\alpha_1} \int_0^x x^{\alpha_1}[\cdot] \, dx \, dx, \\
L_2^{-1}[\cdot] = \int_0^x x^{-\alpha_2} \int_0^x x^{\alpha_2}[\cdot] \, dx \, dx.
\end{cases}
\tag{2.2}
\]

Operating \(L_1^{-1}[\cdot], L_2^{-1}[\cdot]\) on (2.1) and using \(y_1'(0) = y_2'(0) = 0\), we get

\[
\begin{cases}
y_1(x) = y_1(0) + L_1^{-1}[f_1(x, y_1(x), y_2(x))], \\
y_2(x) = y_2(0) + L_2^{-1}[f_2(x, y_1(x), y_2(x))].
\end{cases}
\tag{2.3}
\]

According to the ADM, we decompose \(y_i(x)\) and \(f_i(x, y_1(x), y_2(x))\) as

\[
\begin{align*}
y_1(x) &= \sum_{j=0}^{\infty} y_{1j}(x), & f_1(x, y_1(x), y_2(x)) &= \sum_{j=0}^{\infty} A_{1j}, \\
y_2(x) &= \sum_{j=0}^{\infty} y_{2j}(x), & f_2(x, y_1(x), y_2(x)) &= \sum_{j=0}^{\infty} A_{2j},
\end{align*}
\tag{2.4}
\]

where \(A_{ij}\) are Adomian’s polynomials \[30\] are given

\[
A_{in} = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ f_i \left( x, \sum_{j=0}^{\infty} y_{1j} \lambda^j, \sum_{j=0}^{\infty} y_{2j} \lambda^j \right) \right]_{\lambda=0}, \quad i = 1, 2.
\tag{2.5}
\]
Substituting (2.4) into (2.3), we get

\[
\begin{align*}
\sum_{j=0}^{\infty} y_{1j}(x) &= y_1(0) + L_1^{-1} \left[ \sum_{j=0}^{\infty} A_{1j} \right], \\
\sum_{j=0}^{\infty} y_{2j}(x) &= y_2(0) + L_2^{-1} \left[ \sum_{j=0}^{\infty} A_{2j} \right].
\end{align*}
\] 

(2.6)

Upon comparing both sides of (2.6), we have

\[
\begin{align*}
y_{10}(x) &= \delta_1, \quad y_{20}(x) = \delta_2, \\
y_{1j}(x, \delta_1, \delta_2) &= L_1^{-1}[A_{1,j-1}], \\
y_{2j}(x, \delta_1, \delta_2) &= L_2^{-1}[A_{2,j-1}], \quad j = 1, 2, 3, \ldots
\end{align*}
\]

(2.7)

where \(y_1(0) = \delta_1, \ y_2(0) = \delta_2\) are unknown constants to be determined. The \(n\)-term series solutions are given as

\[
\begin{align*}
\phi_{1n}(x, \delta_1, \delta_2) &= \sum_{j=0}^{n} y_{1j}(x, \delta_1, \delta_2), \\
\phi_{2n}(x, \delta_1, \delta_2) &= \sum_{j=0}^{n} y_{2j}(x, \delta_1, \delta_2).
\end{align*}
\]

(2.8)

The unknown constants may be obtained by imposing boundary condition at \(x = 1\) on \(\phi_{1n}(x, \delta_1, \delta_2)\), that leads to

\[
\begin{align*}
a_1 \phi_{1n}(1, \delta_1, \delta_2) + b_1 \phi'_{1n}(1, \delta_1, \delta_2) - c_1 &= 0, \\
a_2 \phi_{2n}(1, \delta_1, \delta_2) + b_2 \phi'_{2n}(1, \delta_1, \delta_2) - c_2 &= 0.
\end{align*}
\]

(2.9)

Solving above transcendental equations for \(\delta_i\) require additional computational work, and \(\delta_i\) may not be uniquely determined.

To avoid solving the above sequence of difficult transcendental equations, the Adomian decomposition method with Green’s function was introduced in [1, 2]. This technique relies on constructing Green’s function before establishing the recursive scheme for the solution components. Unlike the standard Adomian decomposition method, this avoids solving a sequence of transcendental equations for the undetermined coefficients.

### 3 Green’s function and decomposition technique

In this section, we extend the application of the Adomian decomposition method with Green’s function \([2, 12, 37, 38]\), where we transformed the singular boundary value problem into the integral equation before establishing the recursive scheme for the approximate
solution. To apply this technique to coupled Lane-Emden boundary value problems \((1.1)\), we first consider the equivalent integral form of coupled Lane-Emden equation \((1.1)\) as

\[
\begin{align*}
  y_1(x) &= \frac{c_1}{a_1} + \frac{1}{a_1} \int_0^1 G_1(x, s) s^{\alpha_1} f_1(s, y_1(s), y_2(s)) ds, \\
  y_2(x) &= \frac{c_2}{a_2} + \frac{1}{a_2} \int_0^1 G_2(x, s) s^{\alpha_2} f_2(s, y_1(s), y_2(s)) ds,
\end{align*}
\]

(3.1)

where \(G_i(x, s)\) are given by

\[
G_i(x, s) = \begin{cases} 
\ln s, & x \leq s, \\
\ln x, & s \leq x,
\end{cases} \quad \alpha_i = 1, \ i = 1, 2,
\]

(3.2)

and

\[
G_i(x, s) = \begin{cases} 
\frac{s^{1-\alpha_i} - 1}{1 - \alpha_i}, & x \leq s, \quad \alpha_i > 1, \ i = 1, 2, \\
\frac{s^{\alpha_i - 1} - 1}{\alpha_i}, & s \leq x.
\end{cases}
\]

(3.3)

Substituting the series \((2.4)\) into \((3.1)\), we obtain

\[
\begin{align*}
\sum_{j=0}^{\infty} y_{1j}(x) &= \frac{c_1}{a_1} + \frac{1}{a_1} \int_0^1 G_1(x, s) s^{\alpha_1} \sum_{j=0}^{\infty} A_{1j} ds, \\
\sum_{j=0}^{\infty} y_{2j}(x) &= \frac{c_2}{a_2} + \frac{1}{a_2} \int_0^1 G_2(x, s) s^{\alpha_2} \sum_{j=0}^{\infty} A_{2j} ds.
\end{align*}
\]

(3.4)

Comparing components from both sides of \((3.4)\), we have the following recursive scheme

\[
\begin{align*}
  y_{10}(x) &= \frac{c_1}{a_1}, \quad y_{20}(x) = \frac{c_2}{a_2}, \\
  y_{1j}(x) &= \frac{1}{a_1} \int_0^1 G_1(x, s) s^{\alpha_1} A_{1,j-1} ds, \\
  y_{2j}(x) &= \frac{1}{a_2} \int_0^1 G_2(x, s) s^{\alpha_2} A_{2,j-1} ds.
\end{align*}
\]

(3.5)

Then, we obtain the approximate series solutions as

\[
\begin{align*}
\psi_{1n}(x) &= \sum_{j=0}^{n} y_{1j}(x), \\
\psi_{2n}(x) &= \sum_{j=0}^{n} y_{2j}(x).
\end{align*}
\]

(3.6)

Unlike ADM or MADM, the proposed recursive schemes \((3.6)\) do not require any computation of unknown constants.
4 Convergence and error analysis

Let $E = (C[0, 1], \| y \|)$ be a Banach space with norm

$$\| y \| = \max \{ \| y_1 \|, \| y_2 \| \}, \quad y \in E;$$

(4.1)

where, $\| y_1 \| = \max_{x \in I = [0, 1]} | y_1(x) |$ and $\| y_2 \| = \max_{x \in I} | y_2(x) |$.

From (3.5) and (3.6), we have

$$\Psi_n = \sum_{j=0}^{n} y_j(x) = \frac{c}{a} + \sum_{j=1}^{n} \int_{0}^{1} G(x, s) s^{a} A_{j-1} ds = \frac{c}{a} + \int_{0}^{1} G(x, s) s^{a} \sum_{j=1}^{n} A_{j-1} ds,$$

(4.2)

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad y_j = \begin{pmatrix} y_{1j} \\ y_{2j} \end{pmatrix}, \quad G(x, s) = \begin{pmatrix} G_1(x, s) \\ G_2(x, s) \end{pmatrix}, \quad A_j = \begin{pmatrix} A_{1j} \\ A_{2j} \end{pmatrix},$$

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad s^a = \begin{pmatrix} s^{a_1} \\ s^{a_2} \end{pmatrix}, \quad \frac{c}{a} = \begin{pmatrix} \frac{c_1}{a_1} \\ \frac{c_2}{a_2} \end{pmatrix}, \quad \Psi_n = \begin{pmatrix} \psi_{1n} \\ \psi_{2n} \end{pmatrix}.$$

**Definition 1.** The function $f(x, y_1, y_2)$ satisfy Lipschitz condition as

$$| f(x, y_1, y_2) - f(x, y_1^*, y_2^*) | \leq \sum_{j=1}^{2} l_j | y_j - y_j^* |, \quad \forall (x, y_1, y_2), (x, y_1^*, y_2^*) \in D;$$

(4.3)

where $D = \{ [0, 1] \times R \times R \}$, and $l_1, l_2$ are Lipschitz constants.

**Theorem 4.1.** Suppose that the nonlinear function $f(x, y_1, y_2)$ satisfy Lipschitz condition (4.3), then the series solution $\sum_{j=0}^{\infty} y_j$ defined by (3.6) is convergent whenever $\gamma < 1$.

**Proof.** Define

$$\Psi_0 = x_0, \quad \Psi_1 = x_0 + x_1, \ldots, \Psi_n = \sum_{k=0}^{n} y_k.$$

For $n > m$ and using (4.2), we have

$$\| \Psi_n - \Psi_m \| = \max_{x \in I} \left| \int_{0}^{1} G(x, s) s^{a} \left( \sum_{j=1}^{n} A_{j-1} - \sum_{j=1}^{m} A_{j-1} \right) ds \right|.$$

Using $\sum_{j=0}^{n} A_j \leq f(s, \psi_{1n}, \psi_{2n})$ from (40), we have

$$\| \Psi_n - \Psi_m \| \leq \max_{x \in I} \left| \int_{0}^{1} G(x, s) s^{a} \left[ f(s, \psi_{1n}, \psi_{2n}) - f(s, \psi_{1m}, \psi_{2m}) \right] ds \right|.$$
Applying the Lipschitz condition, we get

\[
\|\Psi_n - \Psi_m\| \leq \max_{x \in I} \left| \int_0^1 G(x,s) s^\alpha ds \right| \sum_{i=1}^2 l_i \max_{x \in I} |\psi_{i,n-1} - \psi_{i,m-1}| \leq \max_{x \in I} \left| \int_0^1 G(x,s) s^\alpha ds \right| 2l \max \left\{ \|\psi_{1,n-1} - \psi_{1,m-1}\|, \|\psi_{2,n-1} - \phi_{2,m-1}\| \right\} \leq 2ml\|\Psi_{n-1} - \Psi_{m-1}\| = \delta \|\Psi_{n-1} - \Phi_{m-1}\|,
\]

where

\[
m = \max \left\{ \max \left| \int_0^1 G_1(x,s) s^{\alpha_1} ds \right|, \max \left| \int_0^1 G_2(x,s) s^{\alpha_2} ds \right| \right\},
\]

\[
l = \max\{l_1, l_2\}, \quad \gamma = 2 \, ml.
\]

Thus, we have

\[
\|\Psi_n - \Psi_m\| \leq \gamma \|\Psi_{n-1} - \Psi_{m-1}\|.
\]

By taking \(n = m + 1\) in (1.5), we see that

\[
\|\Psi_{m+1} - \Psi_m\| \leq \gamma \|\Psi_{m} - \Psi_{m-1}\| \leq \gamma^2 \|\Psi_{m-1} - \Psi_{m-2}\| \leq \ldots \leq \gamma^m \|\Psi_1 - \Psi_0\|.
\]

For all \(n, m \in \mathbb{N}\), with \(n > m\), consider

\[
\|\Psi_n - \Psi_m\| = \| (\Psi_n - \Psi_{n-1}) + (\Psi_{n-1} - \Psi_{n-2}) + \ldots + (\Psi_{m+1} - \Psi_{m}) \| \\
\leq \|\Psi_n - \Psi_{n-1}\| + \|\Psi_{n-1} - \Psi_{n-2}\| + \ldots + \|\Psi_{m+1} - \Psi_{m}\| \\
\leq \left[ \gamma^{n-1} + \gamma^{n-2} + \ldots + \gamma^m \right] \|\Psi_1 - \Psi_0\| \\
= \gamma^m \left[ 1 + \gamma + \gamma^2 + \ldots + \gamma^{n-m-1} \right] \|y_1\| \\
= \gamma^m \left( \frac{1 - \gamma^{n-m}}{1 - \gamma} \right) \|y_1\|.
\]

It follows that as \(\gamma < 1\),

\[
\|\Psi_n - \Psi_m\| \leq \frac{\gamma^m}{1 - \gamma} \|y_1\|. \tag{4.6}
\]

Letting \(n, m \to \infty\), we obtain \(\lim_{n,m \to \infty} \|\Psi_n - \Psi_m\| = 0\). Hence, \(\{\Psi_n\}\) is a Cauchy sequences in the Banach space \(E\). \(\square\)

**Theorem 4.2.** If the approximate solution \(\Psi_n\) converges to \(y(x)\), then the maximum absolute truncated error is estimated

\[
\|y(x) - \Psi_m\| \leq \frac{\gamma^m m}{1 - \gamma} \max_{x \in I} |f(x, y_{10}, y_{20})|. \tag{4.7}
\]
Proof. From (4.6), we have
\[ \|\Psi_n - \Psi_m\| \leq \gamma^m \frac{1}{1 - \gamma} \| y_1 \|. \]
Since \( \Psi_n \to x(t) \) as \( n \to \infty \), and the above inequality reduces to
\[ \|y(x) - \Psi_m\| \leq \gamma^m \frac{1}{1 - \gamma} \| y_1 \|. \] (4.8)
From (3.5), we have
\[ y_1 = \int_0^1 G(x, s) s^\alpha A_0 ds, \]
and we find
\[ \|y_1\| = \max_{x \in I} \left| \int_0^1 G(x, s) s^\alpha A_0 ds \right| \leq m \max_{x \in I} |f(x, y_{10}, y_{20})|. \] (4.9)
Combining (4.8) and (4.9), we obtain error estimate as
\[ \|y(t) - \Psi_m\| \leq \gamma^m \frac{m}{1 - \gamma} \max_{x \in I} |f(x, y_{10}, y_{20})|. \] (4.10)
which completes the proof. \( \square \)

5 Numerical Results

In this section, we consider three coupled Lane-Emden type boundary value problems to examine the accuracy of the present method. Since the exact solution of the problems is not known, we examine the accuracy and applicability of the present method by the absolute residual error
\[
\begin{align*}
    r_{1n}(x) &:= \psi''_{1n}(x) + \frac{\alpha_1}{x} \psi'_{1n}(x) + f_1(x, \psi_{1n}(x), \psi_{2n}(x)) , \quad x \in (0,1), \\
    r_{2n}(x) &:= \psi''_{2n}(x) + \frac{\alpha_2}{x} \psi'_{2n}(x) + f_2(x, \psi_{1n}(x), \psi_{2n}(x)) ,
\end{align*}
\] (5.1)
where \([r_{1n}, r_{2n}]^T\) is the absolute residual error and \([\psi_{1n}, \psi_{2n}]^T\) is the present approximate solution. The maximum residual errors are defined as
\[
\begin{align*}
    \max r_{1n} &:= \max_{x \in [0,1]} r_{1n}(x), \\
    \max r_{2n} &:= \max_{x \in [0,1]} r_{2n}(x).
\end{align*}
\] (5.2)
5.1 The catalytic diffusion reactions problem [25, 26]

Example 5.1. Consider the coupled Lane-Emden equation occurs in catalytic diffusion reactions [25, 26] as

\[
\begin{align*}
 y''_1(x) + \frac{2}{x} y'_1(x) &= -k_1 y_1^2(x) - k_2 y_1(x)y_2(x), \quad x \in (0, 1), \\
y''_2(x) + \frac{2}{x} y'_2(x) &= -k_3 y_1^2(x) - k_4 y_1(x)y_2(x), \quad (5.3) \\
y'_1(0) = 0, \quad y'_2(0) = 0, \quad y_1(1) = 1, \quad y_2(1) = 2,
\end{align*}
\]

where the parameters \(k_1, k_2, k_3\) and \(k_4\) are the actual chemical reactions. Here \(a_1 = a_2 = 1, b_1 = b_2 = 0, c_1 = 1\) and \(c_2 = 2\).

In Tables 1 and 3 we list the numerical results of the approximate solution and the absolute error obtained by the proposed method of Example 5.1 for \((k_1 = 1, k_2 = 2/5, k_3 = 1/2, k_4 = 1)\) and \((k_1 = k_2 = k_3 = k_4 = 1/2)\), respectively. We also compare the numerical results of the maximum residual error \(\max r_{1n}, \max r_{2n}\) obtained by the present method and the results obtained by the modified ADM [26] in Table 2. In Table 4 we list the numerical results of the maximum residual error.

5.1.1 When \(k_1 = 1, k_2 = 2/5, k_3 = 1/2, k_4 = 1\)

By applying the proposed scheme (3.5) with the initial guesses \(y_{10}(x) = 1, y_{20}(x) = 2\), we obtain the 5-terms series solutions as

\[
\begin{align*}
 \psi_{15}(x) &= 0.776218 + 0.199501x^2 + 0.018823x^4 + 0.005706x^6 - 0.0003741x^8 + 0.000125x^{10}, \\
 \psi_{25}(x) &= 1.68423 + 0.283069x^2 + 0.0258518x^4 + 0.007133x^6 - 0.0004418x^8 + 0.000152x^{10}.
\end{align*}
\]

Table 1 Numerical results of the approximate solution \([\psi_{1n}(x), \psi_{2n}(x)]\) and the absolute error \([r_{1n}(x), r_{2n}(x)]\) when \(k_1 = 1, k_2 = 2/5, k_3 = 1/2, k_4 = 1\) of Example 5.1

| \(x\)  | \(\psi_{15}(x)\) | \(\psi_{25}(x)\) | \(r_{15}(x)\) | \(r_{25}(x)\) | \(\psi_{1,10}(x)\) | \(\psi_{2,10}(x)\) | \(r_{1,10}(x)\) | \(r_{2,10}(x)\) |
|-------|-----------------|-----------------|-------------|-------------|-----------------|-----------------|-------------|-------------|
| 0.1   | 0.7782151       | 1.6870682       | 7.00E-2     | 8.79E-2     | 0.7836523       | 1.6938487       | 5.51E-3     | 6.84E-3     |
| 0.2   | 0.7842287       | 1.6955995       | 6.55E-2     | 8.23E-2     | 0.7893632       | 1.7020042       | 5.10E-3     | 6.34E-3     |
| 0.3   | 0.7943299       | 1.7099257       | 5.85E-2     | 7.36E-2     | 0.7989874       | 1.7157379       | 4.48E-3     | 5.57E-3     |
| 0.4   | 0.8086434       | 1.7302167       | 4.97E-2     | 6.26E-2     | 0.8126872       | 1.7352661       | 3.71E-3     | 4.63E-3     |
| 0.5   | 0.8273577       | 1.7567278       | 3.98E-2     | 5.02E-2     | 0.8306972       | 1.7609008       | 2.89E-3     | 3.62E-3     |
| 0.6   | 0.8507387       | 1.7898165       | 2.97E-2     | 3.76E-2     | 0.8533324       | 1.7930601       | 2.10E-3     | 2.64E-3     |
| 0.7   | 0.8791464       | 1.8299638       | 2.02E-2     | 2.56E-2     | 0.8809992       | 1.8322829       | 1.39E-3     | 1.75E-3     |
| 0.8   | 0.9130553       | 1.8778003       | 1.18E-2     | 1.51E-2     | 0.9142115       | 1.8792486       | 7.90E-4     | 1.01E-3     |
| 0.9   | 0.9530791       | 1.9341363       | 5.09E-3     | 6.53E-3     | 0.9536120       | 1.9348042       | 3.30E-4     | 4.29E-4     |
Table 2 Comparison of the numerical results of the maximum residual error \([maxr_{1n}, maxr_{2n}]\) when \(k_1 = 1, k_2 = 2/5, k_3 = 1/2, k_4 = 1\) of Example 5.1

| \(n\) | The present method | Modified ADM [26] |
|-------|------------------|------------------|
|       | \(maxr_{1n}\)    | \(maxr_{2n}\)    | \(maxr_{1n}\)    | \(maxr_{2n}\)    |
| 2     | 8.72E-2          | 1.12E-1          | 4.13E-1          | 5.67E-1          |
| 3     | 3.95E-2          | 5.05E-2          | 2.36E-1          | 3.09E-1          |
| 4     | 2.00E-2          | 2.53E-2          | 6.43E-2          | 8.52E-2          |
| 5     | 1.07E-2          | 1.35E-2          | 4.79E-2          | 6.09E-2          |
| 6     | 6.01E-3          | 7.58E-3          | 2.09E-2          | 2.51E-2          |
| 7     | 3.47E-3          | 4.37E-3          | 1.09E-2          | 1.36E-2          |
| 8     | 2.06E-3          | 2.59E-3          | 6.21E-3          | 7.32E-3          |
| 9     | 1.24E-3          | 1.56E-4          | 3.35E-3          | 3.28E-3          |
| 10    | 7.60E-4          | 9.53E-4          | 1.79E-3          | 2.09E-3          |
| 11    | 5.91E-4          | 5.91E-4          | 9.61E-4          | 1.11E-3          |

5.1.2 When \(k_1 = k_2 = k_3 = k_4 = 1/2\)

On applying the proposed scheme (3.5) with the initial guesses \(y_{10}(x) = 1, y_{20}(x) = 2\), we obtain the 5-terms series solutions as

\[
\begin{align*}
\psi_{15}(x) &= 0.80364 + 0.17704x^2 + 0.017049x^4 + 0.00223x^6 - 0.00001x^8 + 0.0000316x^{10}, \\
\psi_{25}(x) &= 1.80365 + 0.17704x^2 + 0.017049x^4 + 0.00223x^6 - 0.00001x^8 + 0.0000316x^{10}.
\end{align*}
\]

Table 3 Numerical results of the approximate solution \([\psi_{1n}(x), \psi_{2n}(x)]\) and the absolute error \([r_{1n}(x), r_{2n}(x)]\) when \(k_1 = k_2 = k_3 = k_4 = 1/2\) of Example 5.1

| \(x\)  | \(\psi_{15}(x)\) | \(\psi_{25}(x)\) | \(r_{15}(x)\) | \(r_{25}(x)\) | \(\psi_{110}(x)\) | \(\psi_{210}(x)\) | \(r_{110}(x)\) | \(r_{210}(x)\) |
|--------|-----------------|-----------------|--------------|--------------|-----------------|-----------------|--------------|--------------|
| 0.1    | 0.8054213      | 1.8054213      | 1.42E-2      | 1.42E-2      | 0.8065106       | 1.8065106      | 2.63E-4      | 2.63E-4      |
| 0.2    | 0.8107583      | 1.8107583      | 1.33E-2      | 1.33E-2      | 0.8117871       | 1.8117871      | 2.43E-4      | 2.43E-4      |
| 0.3    | 0.8197227      | 1.8197227      | 1.18E-2      | 1.18E-2      | 0.8206563       | 1.8206563      | 2.13E-4      | 2.13E-4      |
| 0.4    | 0.8324215      | 1.8324215      | 1.00E-2      | 1.00E-2      | 0.8332325       | 1.8332325      | 1.76E-4      | 1.76E-4      |
| 0.5    | 0.8490101      | 1.8490101      | 8.05E-3      | 8.05E-3      | 0.8496803       | 1.8496803      | 1.37E-4      | 1.37E-4      |
| 0.6    | 0.8696982      | 1.8696982      | 5.98E-3      | 5.98E-3      | 0.8702190       | 1.8702190      | 9.91E-5      | 9.93E-5      |
| 0.7    | 0.8947568      | 1.8947568      | 4.05E-3      | 4.05E-3      | 0.8951290       | 1.8951290      | 6.53E-5      | 6.53E-5      |
| 0.8    | 0.9245277      | 1.9245277      | 2.36E-3      | 2.36E-3      | 0.9247601       | 1.9247601      | 3.71E-5      | 3.71E-5      |
| 0.9    | 0.9594352      | 1.9594352      | 1.00E-3      | 1.01E-3      | 0.9595423       | 1.9595423      | 1.55E-5      | 1.55E-5      |
Table 4  The numerical results of the maximum residual error \([maxr_{1n}, maxr_{2n}]\) when \(k_1 = k_2 = k_3 = k_4 = 1/2\) of Example 5.1

| \(n\) | \(maxr_{1n}\) | \(maxr_{2n}\) |
|-------|----------------|----------------|
| 2     | 4.15E-2        | 4.15E-2        |
| 3     | 1.41E-2        | 1.41E-2        |
| 4     | 5.35E-3        | 5.35E-3        |
| 5     | 2.15E-3        | 2.15E-3        |
| 6     | 9.05E-4        | 9.05E-4        |
| 7     | 3.91E-4        | 3.91E-4        |
| 8     | 1.73E-4        | 1.73E-4        |
| 9     | 7.83E-5        | 7.83E-5        |
| 10    | 3.58E-5        | 3.58E-5        |
| 11    | 1.66E-5        | 1.66E-5        |

5.2  The concentration of the carbon substrate and the concentration of oxygen problem [27]

Example 5.2. Consider the coupled Lane-Emden equations, which was used to study the concentration of the carbon substrate and the concentration of oxygen, as

\[
\begin{align*}
\psi_1''(x) + \frac{\alpha_1}{x} \psi_1'(x) &= -b + \frac{a y_1(x) y_2(x)}{(l_1 + y_1)(m_1 + y_2)} + \frac{c y_1(x) y_2(x)}{(l_2 + y_1(x))(m_2 + y_2(x))}, \quad x \in (0, 1), \\
\psi_2''(x) + \frac{\alpha_2}{x} \psi_2'(x) &= \frac{d y_1(x) y_2(x)}{(l_1 + y_1(x))(m_1 + y_2(x))} + \frac{e y_1(x) y_2(x)}{(l_2 + y_1(x))(m_2 + y_2(x))}, \\
y_1(0) = 0, \quad y_1(1) = 1, \quad y_2'(0) = 0, \quad y_2(1) = 1,
\end{align*}
\]

(5.4)

where the parameters \(l_1 = l_2 = m_1 = m_2 = 1/10000, a = 5, b = 1, c = d = 1/10, e = 5/100\) are fixed as given in [24, 27]. Here, \(a_1 = a_2 = 1, b_1 = b_2 = 0, c_1 = c_2 = 1\).

In Table 5, 6 and 7, we list the numerical results of the approximate solution and the absolute error obtained by the proposed method of Example 5.2 for \((\alpha_1 = \alpha_2 = 1), \alpha_1 = \alpha_2 = 2\) and \(\alpha_1 = \alpha_2 = 3\), respectively. We also compare the numerical results of the maximum residual error \([maxr_{1n}, maxr_{2n}]\) obtained by the present method and the results obtained by the modified ADM [24] in Table 8 for \(\alpha_1 = \alpha_2 = 2\).

5.2.1  When the shape factors \(\alpha_1 = \alpha_2 = 1\)

By applying the proposed scheme (3.5) with the initial guesses \(y_{10}(x) = 1, y_{20}(x) = 1\), we obtain the 4-terms series solutions as

\[
\begin{align*}
\psi_{14}(x) &= 2.02484 - 1.02488x^2 + 0.00006968x^4 - 0.0000308x^6 + 8.57 \times 10^{-6}x^8, \\
\psi_{24}(x) &= 1.0375 - 0.0374966x^2 + 2.049 \times 10^{-6}x^4 - 9.06 \times 10^{-7}x^6 + 2.52 \times 10^{-7}x^8.
\end{align*}
\]
Table 5 Numerical results of the approximate solution \([\psi_{1n}(x), \psi_{2n}(x)]\) and the absolute error \([r_{1n}(x), r_{2n}(x)]\) when \(\alpha_1 = \alpha_2 = 1\) of Example 5.2

| \(x\) | \(\psi_{12}(x)\) | \(\psi_{22}(x)\) | \(r_{12}(x)\) | \(r_{22}(x)\) | \(\psi_{14}(x)\) | \(\psi_{24}(x)\) | \(r_{14}(x)\) | \(r_{24}(x)\) |
|------|-----------------|-----------------|---------------|---------------|-----------------|-----------------|---------------|---------------|
| 0.1  | 2.0145977       | 1.0371205       | 2.61E-4       | 7.68E-6       | 2.0145872       | 1.0371202       | 2.67E-4       | 7.87E-6       |
| 0.2  | 1.9838514       | 1.0359956       | 2.49E-4       | 7.33E-6       | 1.9838408       | 1.0359953       | 2.40E-4       | 7.07E-6       |
| 0.3  | 1.9326076       | 1.0341208       | 2.29E-4       | 6.76E-6       | 1.9325971       | 1.0341205       | 1.99E-4       | 5.86E-6       |
| 0.4  | 1.8608665       | 1.0314960       | 2.03E-4       | 5.98E-6       | 1.8608563       | 1.0314957       | 1.50E-4       | 4.42E-6       |
| 0.5  | 1.7686285       | 1.0281214       | 1.70E-4       | 5.01E-6       | 1.7686191       | 1.0281211       | 1.00E-4       | 2.95E-6       |
| 0.6  | 1.6558940       | 1.0239968       | 1.32E-4       | 3.90E-6       | 1.6558857       | 1.0239966       | 5.69E-5       | 1.67E-6       |
| 0.7  | 1.5226632       | 1.0191224       | 9.16E-5       | 2.69E-6       | 1.5226567       | 1.0191222       | 2.49E-5       | 7.33E-7       |
| 0.8  | 1.3689369       | 1.0134981       | 5.07E-5       | 1.49E-6       | 1.3689325       | 1.0134980       | 6.88E-6       | 2.02E-7       |
| 0.9  | 1.1947156       | 1.0071239       | 1.61E-5       | 4.76E-7       | 1.1947134       | 1.0071239       | 6.08E-7       | 1.78E-8       |

5.2.2 For shape factors \(\alpha_1 = \alpha_2 = 2\)

Using the proposed scheme (3.5) with the initial guesses \(y_{10}(x) = 1, y_{20}(x) = 1\), we obtain the 4-terms series solutions as

\[
\begin{align*}
\psi_{14}(x) &= 1.66653 - 0.666545x^2 + 0.0000172x^4 - 5.2743 \times 10^{-6}x^6 + 2.054 \times 10^{-6}x^8, \\
\psi_{24}(x) &= 1.025 - 0.0249964x^2 + 5.172 \times 10^{-7}x^4 - 1.582 \times 10^{-7}x^6 + 6.164 \times 10^{-8}x^8.
\end{align*}
\]

Table 6 Numerical results of the approximate solution \([\psi_{1n}(x), \psi_{2n}(x)]\) and the absolute error \([r_{1n}(x), r_{2n}(x)]\) when \(\alpha_1 = \alpha_2 = 2\) of Example 5.2

| \(x\) | \(\psi_{12}(x)\) | \(\psi_{22}(x)\) | \(r_{12}(x)\) | \(r_{22}(x)\) | \(\psi_{14}(x)\) | \(\psi_{24}(x)\) | \(r_{14}(x)\) | \(r_{24}(x)\) |
|------|-----------------|-----------------|---------------|---------------|-----------------|-----------------|---------------|---------------|
| 0.1  | 1.6598747       | 1.0247462       | 1.31E-4       | 3.94E-6       | 1.6598657       | 1.0247459       | 5.70E-5       | 1.71E-6       |
| 0.2  | 1.6398780       | 1.0239963       | 1.25E-4       | 3.75E-6       | 1.6398694       | 1.0239960       | 5.10E-5       | 1.53E-6       |
| 0.3  | 1.6065503       | 1.0227465       | 1.14E-4       | 3.44E-6       | 1.6065422       | 1.0227462       | 4.20E-5       | 1.26E-6       |
| 0.4  | 1.5598915       | 1.0209967       | 1.00E-4       | 3.01E-6       | 1.5598843       | 1.0209965       | 3.14E-5       | 9.43E-7       |
| 0.5  | 1.4999020       | 1.0187470       | 8.34E-5       | 2.50E-6       | 1.4998950       | 1.0187468       | 2.07E-5       | 6.23E-7       |
| 0.6  | 1.4265818       | 1.0159974       | 6.38E-5       | 1.91E-6       | 1.4265769       | 1.0159973       | 1.15E-5       | 3.47E-7       |
| 0.7  | 1.3399312       | 1.0127479       | 4.31E-5       | 1.29E-6       | 1.3399276       | 1.0127478       | 4.97E-6       | 1.49E-7       |
| 0.8  | 1.2395055       | 1.0089985       | 2.32E-5       | 6.97E-7       | 1.2399482       | 1.0089984       | 1.33E-6       | 4.00E-8       |
| 0.9  | 1.1266400       | 1.0047492       | 7.12E-6       | 2.13E-7       | 1.1266389       | 1.0047491       | 1.13E-7       | 3.40E-9       |

5.2.3 For shape factors \(\alpha_1 = \alpha_2 = 3\)

Making use of the proposed scheme (3.5) with the initial guesses \(y_{10}(x) = 1, y_{20}(x) = 1\), we obtain the 4-terms series solutions as

\[
\begin{align*}
\psi_{14}(x) &= 1.49989 - 0.4999x^2 + 8.1818 \times 10^{-6}x^4 - 1.295 \times 10^{-6}x^6 + 7.802 \times 10^{-7}x^8, \\
\psi_{24}(x) &= 1.01875 - 0.018747x^2 + 2.454 \times 10^{-7}x^4 - 3.886 \times 10^{-8}x^6 + 2.340 \times 10^{-8}x^8.
\end{align*}
\]
Table 7 Numerical results of the approximate solution \([\psi_{1n}(x), \psi_{2n}(x)]\) and the absolute error \([r_{1n}(x), r_{2n}(x)]\) when \(\alpha_1 = \alpha_2 = 3\) of Example 5.2

|   | \(\psi_{12}(x)\) | \(\psi_{22}(x)\) | \(r_{12}(x)\) | \(r_{22}(x)\) | \(\psi_{14}(x)\) | \(\psi_{24}(x)\) | \(r_{14}(x)\) | \(r_{24}(x)\) |
|---|-----------------|-----------------|-------|-------|-----------------|-----------------|-------|-------|
| 0.1 | 1.4948975 | 1.0185594 | 8.20E-5 | 2.46E-6 | 1.4948929 | 1.0185592 | 2.00E-5 | 6.01E-7 |
| 0.2 | 1.4799003 | 1.0179970 | 7.79E-5 | 2.33E-6 | 1.4798959 | 1.0179968 | 1.79E-5 | 5.37E-7 |
| 0.3 | 1.4549050 | 1.0170596 | 7.12E-5 | 2.13E-6 | 1.4549010 | 1.0170595 | 1.47E-5 | 4.41E-7 |
| 0.4 | 1.4199117 | 1.0157473 | 6.21E-5 | 1.86E-6 | 1.4199081 | 1.0157472 | 1.09E-5 | 3.28E-7 |
| 0.5 | 1.3749204 | 1.0140601 | 5.11E-5 | 1.53E-6 | 1.3749175 | 1.0140600 | 7.17E-6 | 2.15E-7 |
| 0.6 | 1.3199313 | 1.0119979 | 3.88E-5 | 1.16E-6 | 1.3199290 | 1.0119978 | 3.96E-6 | 1.18E-7 |
| 0.7 | 1.2549445 | 1.0095608 | 2.59E-5 | 7.77E-7 | 1.2549429 | 1.0095607 | 1.68E-6 | 5.04E-8 |
| 0.8 | 1.1799602 | 1.0067488 | 1.37E-5 | 4.12E-7 | 1.1799593 | 1.0067487 | 4.43E-7 | 1.33E-8 |
| 0.9 | 1.0949786 | 1.0035618 | 4.12E-6 | 1.23E-7 | 1.0949782 | 1.0035618 | 3.69E-8 | 1.10E-9 |

Table 8 Comparison of the numerical results of the maximum residual error \([\text{max}r_{1n}, \text{max}r_{2n}]\) when \(\alpha_1 = \alpha_2 = 2\) of Example 5.2

|   | \(\text{The present method}\) | \(\text{Modified ADM [24]}\) |
|---|-------------------------------|-------------------------------|
|   | \(\text{max}r_{1n}\) | \(\text{max}r_{2n}\) | \(\text{max}r_{1n}\) | \(\text{max}r_{2n}\) |
| 2 | 1.33E-4 | 4.01E-6 | 1.18E-3 | 3.48E-5 |
| 3 | 8.87E-5 | 2.66E-6 | 8.53E-4 | 2.51E-5 |
| 4 | 5.91E-5 | 1.77E-6 | 6.21E-4 | 1.82E-5 |
| 5 | 3.94E-5 | 1.18E-6 | 4.52E-4 | 1.33E-5 |
| 6 | 2.62E-5 | 7.87E-7 | 3.29E-4 | 9.69E-6 |
| 7 | 1.74E-5 | 5.25E-7 | 2.39E-4 | 7.06E-6 |
5.3 The steady-state concentrations of CO$_2$ and PGE

Example 5.3. We consider the system of nonlinear differential equations, which arising in the study of the steady-state concentrations of CO$_2$ and PGE as

\begin{equation}
\begin{aligned}
y''_1(x) &= \frac{\alpha_1 y_1(x) y_2(x)}{1 + \beta_1 y_1(x) + \beta_2 y_2(x)}, \quad x \in (0, 1),
y''_2(x) &= \frac{\alpha_2 y_1(x) y_2(x)}{1 + \beta_1 y_1(x) + \beta_2 y_2(x)},
y_1(0) &= 1, \quad y_1(1) = k, \quad y'_2(0) = 0, \quad y_2(1) = 1,
\end{aligned}
\end{equation}

where the constants $\alpha_1, \alpha_2, \beta_1, \beta_2, k$ are normalized parameters, $x$ is the dimensionless distance as measured from the center, and $k$ is the dimensionless concentration of CO$_2$ at the surface of the catalyst [41]. We fix the parameters $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 1, \beta_2 = 3, k = 0.5$ as in [41]. By applying the proposed scheme, we obtain the approximate solutions as

\begin{align*}
\psi_{14}(x) &= 1 - 0.58005 x + 0.09292 x^2 - 0.01397 x^3 + 0.00173 x^4 - 0.000703 x^5 + 0.000057 x^6 \\
&\quad + 0.00001590 x^7 + 8.1396 \times 10^{-7} x^8, \\
\psi_{24}(x) &= 0.83989 + 0.185843 x^2 - 0.027952 x^3 + 0.003475 x^4 - 0.0014068 x^5 + 0.0001145 x^6 \\
&\quad + 0.0000318 x^7 + 1.627 \times 10^{-6} x^8.
\end{align*}

In Table 9 we list the numerical results of the approximate solution and the absolute error obtained by the proposed method of Example 5.3. We also compare the numerical results of the maximum residual error $[\text{max} r_{1n}, \text{max} r_{2n}]$ obtained by the present method and the results obtained by the numerical method [42] in Table 10.

**Table 9** Numerical results of the approximate solution $[\psi_{1n}(x), \psi_{2n}(x)]$ and the absolute error $[r_{1n}(x), r_{2n}(x)]$ when $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 1, \beta_2 = 3, k = 0.5$ of Example 5.3

| $x$ | $\psi_{12}(x)$ | $\psi_{22}(x)$ | $r_{12}(x)$ | $r_{22}(x)$ | $\psi_{14}(x)$ | $\psi_{24}(x)$ | $r_{14}(x)$ | $r_{24}(x)$ |
|-----|----------------|----------------|-------------|-------------|----------------|----------------|-------------|-------------|
| 0.1 | 0.9428976      | 0.8415025      | 1.30E-4     | 2.60E-4     | 0.9429113      | 0.8417515      | 8.59E-7     | 1.71E-6     |
| 0.2 | 0.8875704      | 0.8468806      | 4.90E-5     | 9.80E-5     | 0.8875994      | 0.8471355      | 3.27E-7     | 6.55E-7     |
| 0.3 | 0.8339413      | 0.8556549      | 2.27E-4     | 4.55E-4     | 0.8339853      | 0.8559153      | 1.40E-6     | 2.81E-6     |
| 0.4 | 0.7819361      | 0.8676770      | 4.08E-4     | 8.17E-4     | 0.7819931      | 0.8679387      | 2.39E-6     | 4.78E-6     |
| 0.5 | 0.7314823      | 0.8828019      | 5.87E-4     | 1.17E-3     | 0.7315485      | 0.8830574      | 3.14E-6     | 6.28E-6     |
| 0.6 | 0.6825087      | 0.9008784      | 7.54E-4     | 1.50E-3     | 0.6825785      | 0.9011254      | 3.36E-6     | 6.73E-6     |
| 0.7 | 0.6349449      | 0.9217922      | 8.93E-4     | 1.78E-3     | 0.6350110      | 0.9219982      | 2.63E-6     | 5.26E-6     |
| 0.8 | 0.5887200      | 0.9453750      | 9.81E-4     | 1.96E-3     | 0.5887737      | 0.9455316      | 4.55E-7     | 9.10E-7     |
| 0.9 | 0.5437627      | 0.9714928      | 9.91E-4     | 1.98E-3     | 0.5437943      | 0.9715808      | 3.48E-6     | 6.97E-6     |
Table 10 Comparison of the numerical results of the maximum residual errors \([\max r_{1n}, \max r_{2n}]\) when \(\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 1, \beta_2 = 3, k = 0.5\) of Example 5.3

| \(n\) | \(\max r_{1n}\) | \(\max r_{2n}\) | \(\max r_{1n}\) | \(\max r_{2n}\) |
|------|----------------|----------------|----------------|----------------|
| 2    | 4.36E-3        | 8.71E-3        | 3.86E-2        | 7.73E-2        |
| 3    | 9.99E-4        | 2.00E-3        | 1.14E-3        | 2.28E-3        |
| 4    | 3.54E-5        | 7.07E-5        | 4.87E-4        | 9.75E-4        |
| 5    | 3.38E-6        | 6.77E-6        | 4.07E-5        | 8.14E-5        |
| 6    | 5.29E-7        | 1.06E-6        | 1.01E-6        | 2.03E-6        |
| 7    | 1.05E-7        | 2.10E-7        | 3.90E-7        | 7.81E-7        |
| 8    | 2.76E-8        | 5.52E-8        | 4.88E-7        | 9.76E-9        |
| 9    | 2.80E-9        | 5.61E-9        | 3.64E-9        | 7.28E-9        |
| 10   | 2.59E-10       | 5.17E-10       | 1.22E-10       | 2.43E-10       |

6 Concluding remarks

The theory of coupled Lane-Emden equation finds its vital presence in many of the natural or physical processes such as occurs in catalytic diffusion reactions \([26]\), and some coupled Lane-Emden equations that relate the concentration of the carbon substrate and the concentration of oxygen \([24]\). An analytical approach has been presented for the approximate series solution of the coupled Lane-Emden equation. Unlike the standard Adomian decomposition method, the proposed technique does not require the computation of unknown constants. Unlike the numerical methods, our approach does not require any linearization or discretization of variables. Convergence and error estimation of the method is provided.

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