Quantum gravity and quantum probability

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We argue that in quantum gravity there is no Born rule. The quantum-gravity regime, described by a non-normalisable Wheeler-DeWitt wave functional $\Psi$, must be in quantum nonequilibrium with a probability distribution $P \neq |\Psi|^2$ (initially and always). A Born rule can emerge only in the semiclassical regime of quantum systems on a classical spacetime background, with normalisable Schrödinger wave functions $\psi$. Conditioning on the underlying quantum-gravitational ensemble yields a nonequilibrium distribution $\rho \neq |\psi|^2$ at the beginning of the semiclassical regime, with quantum relaxation $\rho \to |\psi|^2$ taking place only afterwards. Quantum gravity naturally creates an early nonequilibrium universe. We also show how small corrections to the Schrödinger equation yield an intermediate regime in which the Born rule is unstable: an initial distribution $\rho = |\psi|^2$ can evolve to a final distribution $\rho \neq |\psi|^2$. These results arise naturally in the de Broglie-Bohm pilot-wave formulation of quantum gravity. We show that quantum instability during inflation generates a large-scale deficit $\sim 1/k^3$ in the primordial power spectrum at wavenumber $k$, though the effect is too small to observe. Similarly we find an unobservably large timescale for quantum instability in a radiation-dominated universe. Quantum instability may be important in black-hole evaporation, with a final burst of Hawking radiation that violates the Born rule. Deviations from the Born rule can also be generated for atomic systems in the gravitational field of the earth, though the effects are unlikely to be observable. The most promising scenario for the detection of Born-rule violations appears to be in radiation from exploding primordial black holes.

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1 Introduction

An unfinished task of theoretical physics concerns how to combine the two great theories of the twentieth century – general relativity and quantum mechanics – into a coherent theory of quantum gravity. One approach has been to apply the methods of canonical quantisation to the gravitational field. Despite some initial confusion, by the late 1960s a formal theory was arrived at in which the quantum state $\Psi[g_{ij}, \phi]$ is a functional of the 3-metric $g_{ij}$ (together with other quantum fields $\phi$) and obeys a time-independent Wheeler-DeWitt equation of the form $\hat{H}\Psi = 0$ (for an appropriate operator $\hat{H}$) [1, 2]. A consensus was reached on the formal equations defining the theory, but to this day there remains widespread controversy over how to interpret them – a controversy which is often summarised under the general heading of the ‘problem of time’ [2–8].

The underlying theory appears to be timeless, and yet in some semiclassical approximation it must yield the standard time-dependent quantum mechanics of systems on a classical spacetime background. There are many approaches to solving this problem, and authors differ as to whether or not the problem has in fact been solved. Controversy over the physical interpretation of the theory lingers despite the fundamental equations having been written down more than half a century ago. Much progress has been made in recent decades rewriting the equations in terms of alternative sets of variables, resulting in a version of the theory widely known as loop quantum gravity, which has certain technical advantages while retaining similar interpretational problems [9–12]. The problems of interpretation seem unrelated to questions concerning ultraviolet divergences and perturbative non-renormalisability, arising as they do even in finite-dimensional minisuperspace models. Instead the problems appear to signal a basic conceptual difficulty that afflicts the formalism. It has long been suspected that progress in understanding quantum gravity might require some change in our understanding of quantum mechanics. Some authors have proposed, for example, that we must reformulate quantum mechanics itself as a timeless theory [13–15].

In this paper we suggest that some of the conceptual difficulties in understanding quantum gravity arise from the unwarranted assumption that the Born rule is still relevant in the deep quantum-gravity regime. Our everyday laboratory experience tells us that quantum systems have time-dependent wave functions $\psi$ and the Born rule states that $|\psi|^2$ is a probability density. It has long been assumed that some analogue of the Born rule must hold even at the Planck scale. Some early workers assumed that the Wheeler-DeWitt wave functional $\Psi$ yields a probability density $|\Psi|^2$, but this ‘naive Schrödinger interpretation’ encountered difficulties and has been widely abandoned. Among other problems the density $|\Psi|^2$ is non-normalisable – and not for merely technical reasons but for the deep physical reason that the Wheeler-DeWitt equation has the structure of a Klein-Gordon-like wave equation on configuration space. Even so the many approaches to solving the problem of time generally assume that some form of Born rule must be applied. We argue here that this is a mistake, that in the deep quantum-gravity regime there is no such thing as the Born rule,
and that the Born rule emerges only in the semiclassical regime for quantum systems on a classical spacetime background.

To make sense of this proposal requires a formulation of quantum mechanics in which the Born rule is not an axiom or law but is merely emergent. Such a formulation is provided by the pilot-wave theory of de Broglie and Bohm [16–21] – at least when correctly interpreted [22–30]. In pilot-wave theory a quantum system has an evolving configuration \( q(t) \) whose motion is determined by the configuration-space wave function \( \psi(q,t) \). For standard systems the components of the velocity \( dq/dt \) are proportional to the gradient of the phase \( S = \text{Im} \ln \psi \). This nonclassical theory of motion – originally due to de Broglie [2] – provides a deterministic theory of trajectories for quantum systems. But the equations determine a trajectory \( q(t) \) only given the initial configuration \( q(t_i) \) (and the initial wave function \( \psi(q,t_i) \)) at some initial time \( t_i \). Over an ensemble of systems with the same \( \psi(q,t_i) \) we will have some initial distribution \( \rho(q,t_i) \) of configurations. If we assume that \( \rho(q,t_i) = |\psi(q,t_i)|^2 \) it follows from the equations that this initial Born-rule density is preserved in time in the sense that we obtain \( \rho(q,t) = |\psi(q,t)|^2 \) at later times \( t \). The Born rule describes a state of ‘quantum equilibrium’. With this assumption about the initial conditions the empirical predictions of textbook quantum mechanics are recovered (as first shown in full detail by Bohm) [18–21]. However, the status of the Born rule as an initial condition in pilot-wave theory has been disputed. Most authors take it as an extra postulate or law of the theory (along with the Schrödinger equation for \( \psi \) and the de Broglie ‘guidance equation’ for the velocity \( dq/dt \)) [19, 31–33]. This author has long argued that this is a mistake [20–30, 34]. There is a basic conceptual difference between initial conditions on the one hand and laws of motion on the other. For an ensemble of systems with the same \( \psi(q,t_i) \), the actual initial distribution \( \rho(q,t_i) \) of configurations is in principle arbitrary (just as, in classical mechanics, for an ensemble of systems with the same Hamiltonian the actual initial distribution \( \rho(p,q,t_i) \) on phase space is in principle arbitrary).

In particular \( \rho(q,t_i) \) may or may not be equal to \( |\psi(q,t_i)|^2 \). If we take pilot-wave theory seriously as a physical theory, it tells us that more general non-Born-rule distributions \( \rho(q,t_i) \neq |\psi(q,t_i)|^2 \) (corresponding to ‘quantum nonequilibrium’) are possible at least in principle, resulting in a wider physics beyond that allowed by the usual quantum formalism [20–30, 34–36]. In other words, quantum theory is merely a special case of a much wider physics [37].

It has been argued that in pilot-wave theory the Born rule emerges by a dynamical process of ‘quantum relaxation’, broadly analogous to thermal relaxation in classical physics, a view which has been supported by extensive numerical simulations [22, 24, 26, 30, 38–44]. On this view relaxation \( \rho(q,t) \rightarrow |\psi(q,t)|^2 \) to the Born rule took place (on a coarse-grained level) some time in the remote past, probably in the very early universe [23–26, 30, 34, 37], and may have left traces in the cosmic microwave background [34, 36, 45–50] or in relic cosmo-

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4Pilot-wave theory was first presented by de Broglie, for a many-body system, at the 1927 Solvay conference [16]. For a complete translation of the conference proceedings, and a detailed historical analysis, see ref. [17].
logical particles \[26, 36, 45, 51, 52\]. Simulations of quantum relaxation have been carried out for a variety of systems on a classical spacetime background, including quantum scalar fields in a cosmological setting. It has been shown that quantum relaxation tends to be suppressed for long-wavelength (super-Hubble) field modes on expanding space \[36, 45–47, 49\], but in most respects numerical studies have confirmed the general picture of the Born rule emerging by dynamical relaxation.

In non-gravitational physics quantum equilibrium is stable in the sense that it is preserved in time. Under standard operations and interactions an initial equilibrium state \(\rho = |\psi|^2\) will evolve into a final equilibrium state \(\rho = |\psi|^2\). Thus, for example, the Born rule continues to hold in high-energy collisions (as probed by scattering cross-sections). In pilot-wave theory this stability is a simple consequence of the dynamics, which evolves an initial Born-rule state to a final Born-rule state. We might then ask what happens to quantum equilibrium in the presence of gravitational processes. It has been suggested that the Born rule could become unstable during the formation and complete evaporation of a black hole, so as to compensate for the apparent information loss that would otherwise occur \[36, 53, 54\]. On this scenario an evaporating black hole would emit Hawking radiation in a state of quantum nonequilibrium \(\rho \neq |\psi|^2\). However, the arguments given are semiclassical, for quantum fields on a classical spacetime background. The suggested mechanism, which involves entanglement between ingoing and outgoing field modes, depends on the assumption that quantum nonequilibrium is somehow generated behind the horizon (close to the singularity). This lacks a sound theoretical basis. To make further progress requires the application of pilot-wave theory to the gravitational field itself.

A pilot-wave theory of quantum gravity, in which the Wheeler-DeWitt wave functional \(\Psi[g_{ij}, \phi]\) is supplemented by a de Broglie-Bohm trajectory \((g_{ij}(t), \phi(t))\), has been considered by a number of authors. Beginning with the early papers of Vink \[55\], Horiguchi \[56\] and Shtanov \[57\], the theory has been extensively applied to quantum cosmology in particular by Pinto-Neto and collaborators \[58–61\]. An alternative approach to pilot-wave quantum gravity considers a Schrödinger-like equation for a time-dependent wave functional \(\Psi[g_{ij}, \phi, t]\) with a preferred time parameter \(t\) \[24, 25, 62\]. In such a theory the Born-rule distribution \(P[g_{ij}, \phi, t] = |\Psi[g_{ij}, \phi, t]|^2\) should be stable by construction. However the consistency and completeness of the latter approach remains in doubt \[63\].

Here we restrict ourselves to pilot-wave theory as applied to the timeless Wheeler-DeWitt equation with a time-independent wave functional \(\Psi[g_{ij}, \phi]\). In previous work the focus has been on properties of the trajectories for the evolving 3-metric (for example whether or not the trajectories are singularity-avoiding in cosmological scenarios \[64\]), while little has been done in considering probabilities and the Born rule. As we shall see, the usual problems emerge when trying to interpret the non-normalisable (and static) Wheeler-DeWitt density \(|\Psi|^2\) as a probability density. Perhaps for this reason most authors in the field have confined themselves to considering properties of the trajectories alone, without attempting to define a theory of ensembles or of probabilities.
In this paper we propose a new approach to understanding the Born rule in quantum gravity. Adopting a pilot-wave theory of gravity with the Wheeler-DeWitt equation, we argue that at the fundamental level a probability density $P$ must be, and always remains, unequal to the apparent ‘Born-rule’ density $|\Psi|^2$, since $P$ is normalisable (by construction) while $|\Psi|^2$ is not. Quantum relaxation cannot take place in the usual way because there is no well-defined equilibrium state. In effect the deep quantum-gravity regime is in a perpetual state of quantum nonequilibrium $P \neq |\Psi|^2$. Even so, the Born rule can be recovered in the semiclassical regime, with the Schrödinger approximation for an effective time-dependent (and normalisable) wave function $\psi$. In that regime quantum relaxation $\rho \to |\psi|^2$ can proceed in the usual way and over time we recover the Born rule $\rho = |\psi|^2$ as an equilibrium state. At the beginning of the semiclassical regime, however, the system is expected to be in a state of nonequilibrium $\rho \neq |\psi|^2$, with $\rho$ arising as a conditional probability from the underlying quantum-gravitational nonequilibrium state $P \neq |\Psi|^2$. Relaxation $\rho \to |\psi|^2$ takes place only afterwards. In this way the hypothesis of primordial quantum nonequilibrium [22–27, 30, 34, 36, 37] is derived as a consequence of quantum gravity.

We also study an intermediate regime, with quantum-gravitational corrections to the Schrödinger approximation, following methods developed by Kiefer and collaborators [68–71]. A long-standing difficulty in the field has been the appearance of small, quantum-gravitationally induced, non-Hermitian corrections in the effective Hamiltonian (for a field system on a classical background). In standard quantum theory such terms violate the conservation of probability. For this reason, in such calculations usually only the Hermitian corrections are kept and the non-Hermitian terms are discarded. We show that, when reformulated in terms of pilot-wave theory, probabilities are fully conserved even in the presence of non-Hermitian terms, whose effect is instead to generate a small instability of the Born rule: an initial equilibrium distribution $\rho = |\psi|^2$ can in principle evolve to a nonequilibrium distribution $\rho \neq |\psi|^2$ (at least in circumstances where relaxation is relatively negligible). In this paper we study various systems where such ‘quantum instability’ can occur: a scalar field on de Sitter space, a field on a radiation-dominated background, a field in the spacetime of an evaporating black hole, and an atomic system with a rapidly-changing Hamiltonian in a curved background. In general the effects are found to be extremely small, with the possible exception of the final stages of black-hole evaporation where the effects may well be significant.

An outline of this paper now follows. We employ units with $\hbar = c = 16\pi G = 1$. In Section 2 we review the status of the Born rule in pilot-wave theory on a classical spacetime background. In Section 3 we review the difficulties with understanding and applying the Born rule in canonical quantum gravity in both its conventional and pilot-wave versions. Our new approach to the Born rule is explained and outlined in general terms in Section 4. In Section 5 we con-

\[ ^{3} \text{Related and preliminary versions of these proposals were given in refs.} \ 65, 66. \text{ For a concise overview of the present work see ref.} \ 67. \]
sider quantum-gravitational corrections to the Schrödinger approximation and in Section 6 we show how the non-Hermitian part of such corrections can be understood in pilot-wave theory as generating a small instability of the Born rule. In Section 7 we apply these proposals to quantum cosmology and implement a simplified model to enable tractable calculations. This model is applied in Section 8 to study the gravitational creation of quantum nonequilibrium for a scalar field on de Sitter space. In Section 9 these results are employed to derive an approximate correction to the cosmological primordial power spectrum, in the form of a small power deficit scaling with wave number $k$ as $1/k^3$, whose magnitude is however estimated to be too small to be observable in the cosmic microwave background. In Section 10 we study similar effects for a scalar field in a radiation-dominated expanding universe. In Section 11 we show how quantum nonequilibrium is expected to be created by quantum-gravitational effects in the spacetime of an evaporating black hole, where the effects are estimated to be significant only in the final stages of evaporation when the mass $M$ of the hole approaches the Planck mass $m_P$. Finally, in Section 12 we consider how comparable effects can occur for atomic systems with rapidly-changing Hamiltonians in the gravitational field of the earth, however the effects are so tiny as to be seemingly of theoretical interest only. Our conclusions are drawn in Section 13.

2 Pilot-wave theory and the Born rule

In this section we outline the status of the Born rule in pilot-wave theory.

2.1 Pilot-wave theory and quantum equilibrium

In pilot-wave theory a general system has an evolving configuration $q(t)$ with a velocity law [16–21]

$$\frac{dq}{dt} = v(q, t)$$

(1)

where $v(q, t)$ is determined by the wave function $\psi(q, t)$. The time parameter $t$ is associated with a foliation of spacetime by spacelike hypersurfaces. For a system evolving on a classical spacetime background, $q$ is the configuration of fields and particles on a curved 3-space (as defined by the hypersurfaces). The de Broglie velocity field $v(q, t)$ is defined as follows. The usual Schrödinger equation on configuration space

$$i\frac{\partial \psi}{\partial t} = \hat{H}\psi,$$ 

(2)

implies a continuity equation

$$\frac{\partial |\psi|^2}{\partial t} + \partial_q \cdot j = 0$$

(3)

4Systems with spin have multi-component wave functions. It will not be necessary to consider such systems here.
for a density $|\psi|^2$ and a current $j = j[\psi] = j(q, t)$, where $\partial_q$ is a generalised gradient and $j$ satisfies

$$\partial_q \cdot j = 2 \text{Re} \left( i\psi^* \hat{H} \psi \right). \quad (4)$$

The explicit expression for $j$ in terms of $\psi$ is determined by the form of the Hamiltonian $\hat{H}$, and such a current exists whenever $\hat{H}$ is given by a differential operator $[72]$. Given an expression for $j$, the de Broglie velocity field is defined by

$$v(q, t) = \frac{j(q, t)}{|\psi(q, t)|^2}. \quad (5)$$

The equation of motion (1) then determines a trajectory $q(t)$ in configuration space, given an initial configuration $q(0)$ and an initial wave function $\psi(q, 0)$. Physical Hamiltonians are often quadratic in the canonical momenta, in which case the components $v_a$ of $v$ are proportional to the components of the phase gradient:

$$v_a \propto \partial_q v = \text{Im} \left( \frac{\partial_q \psi}{\psi} \right). \quad (6)$$

(where $\psi = |\psi| e^{iS}$). It is important to note that the ‘pilot wave’ $\psi$ is a field on configuration space that guides the deterministic motion of an individual system. Fundamentally $\psi$ has nothing to do with probability.

We can consider an ensemble of systems with the same wave function $\psi(q, t)$. The ensemble will have an evolving distribution $\rho(q, t)$ of configurations $q(t)$. The initial distribution $\rho(q, t_i)$ at time $t_i$ need not be equal to the squared-amplitude $|\psi(q, t_i)|^2$ of the initial pilot wave $\psi(q, t_i)$. Because the individual configurations evolve by the equation of motion (1), the ensemble distribution $\rho(q, t)$ will necessarily evolve by the continuity equation

$$\frac{\partial \rho}{\partial t} + \partial_q \cdot (\rho v) = 0. \quad (7)$$

This is the same as the continuity equation (3) for the time evolution of $|\psi|^2$. We then have a simple ‘quantum equilibrium theorem’: if $\rho$ and $|\psi|^2$ happen to be equal at an initial time $t_i$ then they will remain equal at later times $t$. Thus an initial distribution $\rho(q, t_i) = |\psi(q, t_i)|^2$ evolves into a final distribution

$$\rho(q, t) = |\psi(q, t)|^2. \quad (8)$$

This is the state of quantum equilibrium, in which the ensemble obeys the Born rule.

It is sometimes useful to consider how the ratio

$$f = \frac{\rho}{|\psi|^2} \quad (9)$$

evolves along a trajectory. From (11) and (8) it is easy to show that

$$\frac{df}{dt} = 0, \quad (10)$$
where \( \frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla_q \) is the time derivative along a trajectory with local velocity \( v \).

As was first shown in detail by Bohm [18], in the state (8) of quantum equilibrium the statistical predictions for the outcomes of general quantum measurements agree with the usual predictions of textbook quantum theory. In contrast, for a ‘quantum nonequilibrium’ ensemble with a non-Born-rule distribution \( \rho(q, t) \neq |\psi(q, t)|^2 \), the statistical predictions generally differ from those of quantum theory [20–26, 28–30, 34, 36, 37]. Such ensembles entail new physics outside the domain of conventional quantum physics. If pilot-wave theory is taken seriously, we must conclude that quantum physics is merely an effective theory of an equilibrium state – and that, at least in principle, there is a much wider nonequilibrium physics beyond the physics that is currently known.

For all systems that are currently accessible to us, experiments have confirmed the Born rule \( \rho = |\psi|^2 \). This state can be understood as having arisen from a dynamical process of quantum relaxation (analogous to thermal relaxation). Because \( \rho \) and \( |\psi|^2 \) obey the same continuity equation, the fine-grained \( H \)-function

\[
H(t) = \int dq \, \rho \ln(\rho/|\psi|^2)
\]  

(minus the relative entropy of \( \rho \) with respect to \( |\psi|^2 \)) is constant in time, \( dH/dt = 0 \). But if we assume that \( \rho \) and \( |\psi|^2 \) have no fine-grained structure at some initial time \( t_i \), the coarse-grained \( H \)-function

\[
\bar{H}(t) = \int dq \, \bar{\rho} \ln(\bar{\rho}/|\psi|^2)
\]  

obeys an \( H \)-theorem

\[
\bar{H}(t) \leq \bar{H}(t_i),
\]

where \( \bar{H}(t) \) is bounded below by zero and is equal to zero if and only if \( \bar{\rho} = |\psi|^2 \) everywhere [22, 24, 26]. Coarse-graining is needed because of the fine-grained conservation (10) of \( f \), which is analogous to the classical Liouville theorem on phase space. As in classical thermal relaxation for an isolated system, we must assume that the initial state has no fine-grained structure.

Wide-ranging numerical simulations show that, when \( \psi \) is a superposition of energy eigenstates, there is rapid coarse-grained relaxation \( \bar{\rho} \rightarrow |\psi|^2 \) [22, 24, 26, 30, 38–44], with \( \bar{H}(t) \) decaying approximately exponentially towards zero [38, 40, 42],

\[
\bar{H}(t) \approx \bar{H}(0) \exp\left(-t/\tau_{\text{relax}}\right),
\]

where the quantum relaxation timescale \( \tau_{\text{relax}} \) depends on \( \psi \) (as well as on the coarse-graining length \( \varepsilon \)) [40]. Numerical results for two-dimensional systems have yielded values \( \tau_{\text{relax}} \) roughly comparable to the quantum timescale

\[5\] See refs. [19, 20] for detailed accounts of textbook quantum mechanics in terms of pilot-wave theory.

\[6\] For a detailed discussion see ref. [30].
over which $\psi$ evolves, however there is no general relation between the two timescales. As expected relaxation tends to be faster for larger numbers $M$ of superposed modes (as well as for larger $\varepsilon$). For particles in a two-dimensional box, for example, there is strong numerical evidence for an approximate inverse scaling $\tau_{\text{relax}} \propto 1/M$ (at fixed $\varepsilon$) \cite{10}. Similar results are found in pilot-wave field theory, where a single scalar field mode is mathematically equivalent to a two-dimensional oscillator. For fields on expanding space the results are somewhat modified: quantum relaxation takes place efficiently at short (sub-Hubble) wavelengths but is suppressed at long (super-Hubble) wavelengths \cite{30, 45, 46}.

It has been suggested that quantum relaxation took place in the very early universe \cite{23–26, 30, 34, 37}. Ordinary laboratory systems have a long and violent astrophysical history, and are expected to have reached quantum equilibrium a long time ago. For such systems we can expect to see the Born rule today to exceedingly high accuracy. However, quantum nonequilibrium in the early universe can leave an observable imprint in the cosmic microwave background (CMB) \cite{34, 36, 45, 46, 48}, with a specific signature \cite{47, 49} (caused by super-Hubble suppression of relaxation) which has been searched for in recent CMB data \cite{50}. Furthermore, early nonequilibrium might survive to the present day in relic cosmological particles if they decoupled at sufficiently early times \cite{26, 30, 45, 51}. This could potentially be observed in the form of anomalous spectra for decaying or annihilating dark matter \cite{52}. Thus the wider physics of quantum nonequilibrium may have existed in the very early universe, before quantum relaxation took place, possibly leaving traces in the CMB and in relic particles today.

Finally, we mention an alternative approach to understanding the Born rule in pilot-wave theory, which will be relevant later. Beginning with ref. \cite{31} there has developed a distinctive school of de Broglie-Bohm theory called ‘Bohmian mechanics’ \cite{32}. This school has been particularly influential among philosophers \cite{33, 74}. The Bohmian mechanics school claims that a fundamental role is played by the initial Born-rule measure $|\Psi(q,0)|^2$ for the universe, where $\Psi(q,0)$ is the initial universal wave function at $t = 0$. It is claimed that $|\Psi(q,0)|^2$ provides a fundamental measure of ‘typicality’ (or equivalently, of probability) for the initial universal configuration $q(0)$, implying Born-rule probabilities for subsystems. In effect, the Born rule is asserted to have a law-like status at the beginning of the universe, and the Born rule we see at later times is simply a consequence of the Born rule at $t = 0$. However, a different choice of initial measure, such as $|\Psi(q,0)|^4$, yields non-Born rule distributions for subsystems at $t = 0$ \cite{25, 26, 30}. To obtain the Born rule at later times we would then have to appeal to some form of dynamical relaxation. But the Bohmian mechanics school claims that $|\Psi(q,0)|^2$ is the natural choice at $t = 0$, and that this suffices as an explanation, rendering dynamical relaxation superfluous. The argument

\footnote{This terminology should properly be applied to Bohm’s 1952 reformulation \cite{15} of de Broglie’s original 1927 dynamics \cite{16}. Bohm’s version of the theory (based on an equation for acceleration with a pseudo-Newtonian ‘quantum potential’) has been shown to be unstable \cite{73}.}
is, however, circular and unjustified. The Born rule is assumed to apply to the whole universe, in order to obtain the Born rule for subsystems. As we will see in Section 4, even leaving the circularity aside, a careful analysis of the role of probability in pilot-wave quantum gravity definitively resolves the dispute in favour of dynamical relaxation.

2.2 Quantum equilibrium on a globally-hyperbolic spacetime

We have seen that the existence of a quantum equilibrium state is a trivial consequence of the structure of pilot-wave dynamics for any system that obeys a Schrödinger equation with an associated conserved current \( j \). Given such a current, \( \rho \) and \( |\psi|^2 \) will by construction obey the same continuity equation (with the same velocity field \( v = j/|\psi|^2 \)) and the quantum equilibrium theorem immediately follows. It is instructive to illustrate this for field theory on a classical curved spacetime background – assuming that the spacetime is globally hyperbolic.

A globally hyperbolic spacetime can always be foliated (usually nonuniquely) by spacelike hypersurfaces \( \Sigma(t) \) labelled by a global time function \( t \). The line element \( d\tau^2 = (4)g_{\mu\nu}dx^\mu dx^\nu \) (with 4-metric \( (4)g_{\mu\nu} \)) can then be written in the form

\[
d\tau^2 = (N^2 - N_i N^i)dt^2 - 2N_i dx^i dt - g_{ij} dx^i dx^j,
\]

where \( N \) is the lapse function, \( N_i \) is the shift vector, and \( g_{ij} \) is the 3-metric on \( \Sigma(t) \). We have a proper time element \( d\tau = N dt \) normal to the slices \( \Sigma(t) \), where the normal deviates from lines of constant \( x^i \) by \( dx^i = -N_i(x^j, t)dt \). For simplicity we can take \( N^i = 0 \) so that lines \( x^i = \text{const.} \) are normal to the slices (this can always be done provided such lines do not meet singularities).

Consider for example a massless, minimally-coupled real scalar field \( \phi \) with Lagrangian density

\[
L = \frac{1}{2} \sqrt{-g} (\frac{1}{4}g_{\mu\nu}\partial_\mu \phi \partial_\nu \phi)
\]

(with \( g = \det g_{\mu\nu} \)). This implies a canonical momentum density

\[
\pi = \frac{\partial L}{\partial \dot{\phi}} = \frac{\sqrt{g}}{N} \pi
\]

(with \( g = \det g_{ij} \) and taking \( N^i = 0 \)) and a Hamiltonian

\[
H = \int d^3x \left( \frac{1}{2} N \sqrt{g} \left( \frac{1}{g} \pi^2 + g^{ij} \partial_i \phi \partial_j \phi \right) \right).
\]

The system may then be quantised in the usual way, with field operators \( \hat{\phi}(x) \) and \( \hat{\pi}(x) \) on \( \Sigma(t) \). In the functional Schrödinger picture, with the realisations

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8See ref. [30] for a detailed critique of this approach – as well as of the Bohmian mechanics school generally.
\( \hat{\phi}(x) \rightarrow \phi(x) \) and \( \hat{\pi}(x) \rightarrow -i\delta/\delta\phi(x) \), the wave functional \( \Psi[\phi, t] \) obeys the Schrödinger equation 6

\[
\frac{\partial \Psi}{\partial t} = \int d^3x \left[ \frac{1}{2} N \sqrt{g} \left( -\frac{1}{g} \frac{\delta^2}{\delta\phi^2} + g^{ij} \partial_i \phi \partial_j \phi \right) \right] \Psi .
\] (19)

So far we have simply written down a standard quantum field theory on a curved background. To convert this into a pilot-wave theory we note that (19) implies a continuity equation

\[
\frac{\partial |\Psi|^2}{\partial t} + \int d^3x \frac{\delta}{\delta\phi} \left( |\Psi|^2 \frac{N \delta S}{\sqrt{g} \delta\phi} \right) = 0
\] (20)

(where \( \Psi = |\Psi| e^{iS} \)) with a current

\[
j = |\Psi|^2 \frac{N \delta S}{\sqrt{g} \delta\phi} \] (21)

From (5) we then have a de Broglie velocity

\[
\frac{\partial \phi}{\partial t} = \frac{N \delta S}{\sqrt{g} \delta\phi} .
\] (22)

In addition to the evolving wave \( \Psi[\phi, t] \) on configuration space we also have an evolving field \( \phi(x, t) \) on 3-space.

Equations (19) and (22) define the dynamics for an individual field on a curved background. We can also consider a theoretical ensemble of fields guided by the same wave functional \( \Psi \) (on the same curved background). The ensemble will have some arbitrary initial distribution \( P[\phi, t_i] \), which need not be equal to \( |\Psi[\phi, t_i]|^2 \). Because each field has the velocity (22), the distribution \( P[\phi, t] \) will necessarily evolve by the continuity equation

\[
\frac{\partial P}{\partial t} + \int d^3x \frac{\delta}{\delta\phi} \left( P \frac{N \delta S}{\sqrt{g} \delta\phi} \right) = 0 \] (23)

Again, by construction, this is the same continuity equation (20) that is satisfied by \( |\Psi|^2 \) and so the quantum equilibrium theorem immediately follows: if \( P = |\Psi|^2 \) holds at some initial time then \( P = |\Psi|^2 \) will hold at future times. In this way we can easily establish the existence of a quantum equilibrium state for a field on a classical globally-hyperbolic spacetime.

Pilot-wave theory with a stable Born rule covers a wide range of physics, including high-energy field theory on a curved spacetime background – provided the background is globally hyperbolic. 10 It is, however, unclear if a similar construction can be given when the background spacetime is not globally

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9 In this context it is usual to implicitly assume some form of regularisation – for example, dimensional regularisation. 10

10 For completeness we note that, in pilot-wave theory, fermions can be described in terms of a Dirac sea with particle trajectories generated by a many-body Dirac wave function 77, 78. A less well-developed approach describes fermions in terms of anti-commuting Grassmann fields 24, 25, 63.
hyperbolic. It has been argued that such a spacetime is generated by the formation and complete evaporation of a black hole [79]. Such arguments remain controversial, but if they are correct we may be forced to rethink the idea of a stable Born rule in a gravitational context. It has been suggested that quantum equilibrium could be unstable for fields and particles propagating on a background non-globally-hyperbolic spacetime [36, 53, 54]. Hawking radiation from an evaporating black hole could then be in a state of quantum nonequilibrium, even if the original collapsing matter was initially in quantum equilibrium. Because nonequilibrium radiation can contain more information than is possible for conventional radiation, this opens up a new approach to the (still controversial) puzzle of information loss in black holes. The results reported in this paper support the suggestion that evaporating black holes can create quantum nonequilibrium (see Section 11).

3 Quantum gravity and the Born rule

In this section we review canonical quantum gravity, in both its conventional and pilot-wave versions, focussing on the difficulties with understanding and applying the Born rule.

3.1 Canonical quantum gravity

We begin by outlining the standard canonical quantisation of general relativity [1, 2], whose starting point is the classical Hamiltonian formulation of Einstein’s field equations [80, 81]. We first review the formalism for pure gravitation, with the classical field equations $(4) G_{\mu\nu} = 0$ in free space, before generalising to gravitation with a matter field $\phi$.

Classically, second time-derivatives of the metric appear only in the space-space components $(4) G_{ij} = 0$, which are the dynamical part of the field equations. The other components, $(4) G_{0\mu} = 0$, contain only first-order time derivatives and are constraints on the initial data (defined on an initial spacelike slice $\Sigma(0)$). To rewrite this system in Hamiltonian form, the 4-metric is first split into the 3+1 form (15). After dropping surface terms the Einstein-Hilbert action

$$I = -\int d^4x \left( -4 g \right)^{1/2} (4) R$$

becomes

$$I = \int dt \int d^3x \; Ng^{1/2} (K_{ij} K^{ij} - K^2 + R),$$

where

$$K_{ij} = \frac{1}{2N} \left( \frac{\partial g_{ij}}{\partial t} + D_i N_j + D_j N_i \right)$$

(with $D_i$ the 3-covariant derivative with respect to $x^i$) is the extrinsic curvature tensor, $K = K_i^i$ and $R$ is the 3-scalar curvature (the intrinsic curvature of $\Sigma$).
From the Lagrangian density
\[ \mathcal{L} = N g^{1/2}(K_{ij}K^{ij} - K^2 + R) \] (27)
we obtain the canonical momentum density
\[ p^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{g}_{ij}} = -g^{1/2}(K^{ij} - g^{ij} K) . \] (28)
This relation can be inverted:
\[ K^{ij} = -g^{-1/2}(g^{ij} - \frac{1}{2} g^{ij} p) . \] (29)

From (26) and (29) we then have
\[ \frac{\partial g_{ij}}{\partial t} = 2 N G_{ijkl} p^{kl} + D_i N_j + D_j N_i , \] (30)
where
\[ G_{ijkl} = \frac{1}{2} g^{-1/2}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij} g_{kl}) . \] (31)

The functions \( N, N^i \) are not dynamical variables and their canonical momenta vanish. The gravitational Hamiltonian is then
\[ H = \int d^3 x \ (p^{ij} \dot{g}_{ij} - \mathcal{L}) = \int d^3 x \ (N \mathcal{H} + N_i \mathcal{H}^i) \] (32)
where
\[ \mathcal{H} = G_{ijkl} p^{ij} p^{kl} - g^{1/2} R \] (33)
and
\[ \mathcal{H}^i = -2 D_j p^{ij} . \] (34)
Treating \( N, N^i \) as Lagrange multipliers, the conditions \( \delta H/\delta N = 0, \delta H/\delta N_i = 0 \) imply the respective Hamiltonian and momentum constraints
\[ \mathcal{H} = 0 , \quad \mathcal{H}^i = 0 . \] (35)
These are respectively equivalent to the initial-value constraints \( ^{(4)}G_{00} = 0 \) and \( ^{(4)}G_{0i} = 0 \).

We can now write down Hamilton’s first-order dynamical equations
\[ \dot{g}_{ij} = \frac{\delta H}{\delta p^{ij}}, \quad \dot{p}^{ij} = -\frac{\delta H}{\delta g_{ij}} . \] (36)
The first is equivalent to (30). This can be used to eliminate \( p^{ij} \) from the second, yielding the second-order result \( ^{(4)}G_{ij} = 0 \).

To quantise this Hamiltonian system, the canonical variables \( g_{ij}, p^{ij} \) are promoted to operators \( \hat{g}_{ij}, \hat{p}^{ij} \) satisfying the usual commutation relations on
the hypersurface $\Sigma$. We employ the functional Schrödinger picture with the configuration-space operator realisations

$$\hat{g}_{ij}(x) \rightarrow g_{ij}(x), \quad \hat{p}^{ij}(x) \rightarrow -i\frac{\delta}{\delta g_{ij}(x)}$$

where $x$ labels a spatial point on $\Sigma$. We might have expected the formal wave functional $\Psi[g_{ij}, t] = \langle g_{ij}|\Psi(t)\rangle$ to satisfy a Schrödinger equation $i\partial\Psi/\partial t = \hat{H}\Psi$, where $t$ is the time function labelling the hypersurfaces $\Sigma$. But, following the method of Dirac, the classical constraints (35) are promoted to operator constraints on $\Psi$:

$$\hat{H}\Psi = 0, \quad \hat{H}^i\Psi = 0.$$ (38)

A formal Schrödinger equation then reads $i\partial\Psi/\partial t = \hat{H}\Psi = 0$. The functional $\Psi = \Psi[g_{ij}]$ depends on the 3-metric only and not on $t$ – a first hint that the quantum-gravitational state is a different kind of thing from a conventional quantum state. Note further that, according to (38), $\Psi$ is restricted not merely to eigenstates of zero energy but to eigenstates of zero energy density.

In configuration space the constraints (38) take the form

$$\left(-G_{ijkl}\frac{\delta^2}{\delta g_{ij}\delta g_{kl}} - g^{1/2}R\right)\Psi = 0,$$ (39)

$$D_j\left(\frac{\delta\Psi}{\delta g_{ij}}\right) = 0.$$ (40)

The first is the Wheeler-DeWitt equation (or Hamiltonian constraint). We have written it with a specific operator ordering in the kinetic term, but in fact the ordering is ambiguous and this should be understood. Different orderings will be considered later. The second is the momentum constraint, which ensures that $\Psi$ is really a function on the space of coordinate-independent 3-geometries (that is, on superspace).

It is straightforward to write down the generalisation to quantum gravity in the presence of a scalar matter field $\phi$ with potential $V(\phi) [56, 58]$. The wave functional $\Psi[g_{ij}, \phi]$ obeys an extended Wheeler-DeWitt equation

$$(\hat{H}_g + \hat{H}_\phi)\Psi = 0,$$ (41)

where

$$\hat{H}_g = -G_{ijkl}\frac{\delta^2}{\delta g_{ij}\delta g_{kl}} - \sqrt{g}R$$

and

$$\hat{H}_\phi = \frac{1}{2}\sqrt{g}\left(-\frac{1}{g}\frac{\delta^2}{\delta \phi^2} + g^{ij}\partial_i\phi\partial_j\phi\right) + \sqrt{g}V(\phi)$$

are respectively the gravitational and matter-field Hamiltonian constraint operators. The momentum constraint now reads

$$-2D_j\frac{\delta\Psi}{\delta g_{ij}} + \partial^i\phi\frac{\delta\Psi}{\delta \phi} = 0$$

(44)
(where classically $H^i = -2D_j p^{ij} + \pi^i \partial^i \phi$).

In our general considerations below we often work for simplicity with the purely gravitational wave functional $\Psi[g_{ij}]$, where it is understood that the extension to a system including a matter field is straightforward.

### 3.2 The problem of time and of probability

The time independence of the Wheeler-DeWitt wave functional $\Psi[g_{ij}]$ sets quantum gravity apart from other quantum theories. For a non-gravitational system with configuration $q$ we usually have a time-dependent Schrödinger equation for a wave function(al) $\psi(q, t)$. For a general quantum observable $\hat{\omega} = f(\hat{q}, \hat{p})$ (where symbolically $\hat{p} = -i\partial_q$ is the canonical momentum) we have a time-dependent expectation value

$$\langle \hat{\omega} \rangle = \int dq \, \psi^*(q, t) f(q, -i\partial_q) \psi(q, t).$$  (45)

In quantum gravity, in contrast, we have a seemingly static theory. Formally, an arbitrary observable $\hat{\omega} = f(\hat{g}_{ij}, \hat{p}^{ij})$ appears to have a time-independent expectation value

$$\langle \hat{\omega} \rangle = \int Dg \, \Psi^*[g_{ij}] f[g_{ij}, -i\delta/\delta g_{ij}] \Psi[g_{ij}]$$  (46)

(where $\int Dg$ is an appropriate functional integral over the 3-metric).

Despite the apparent lack of time evolution, it is usually assumed that conventional quantum mechanics with the Born rule still applies in some (perhaps modified) form. However, even taking into account the impressive technical advances of recent decades, it is fair to say that the physical interpretation of canonical quantum gravity remains controversial. The difficulties are usually discussed under the general heading of the ‘problem of time’. This is essentially the problem of recovering an approximate time-dependent Schrödinger-like evolution and time-dependent probabilities – for example in the limit of quantum fields, perhaps including metric perturbations, on a classical spacetime background – from an underlying theory with no time. There have been many attempts to solve this problem since the pioneering work of DeWitt and Wheeler in the late 1960s [1, 82]. Numerous approaches have been tried, with varying degrees of success. Particularly insightful reviews of the problem, and of various potential solutions, were given by Unruh and Wald [3], Isham [4, 6] and Kuchař [5, 7]. An exhaustive and up-to-date review has recently been given by Anderson [8].

Some authors seek to avoid the problem from the outset by identifying a preferred time parameter prior to quantisation. This approach has often been called the ‘internal Schrödinger interpretation’. The aim is to obtain, after quantisation, a time-dependent Schrödinger-like equation with an appropriately-defined Hamiltonian. This approach has a long history with its own problems. Two pilot-wave theory models along these lines have been proposed. The first
adopts a slicing with a uniform lapse function $N = 1$ [24], while the second assumes that the preferred slices are foliations of constant York time $\gamma$ [25] [62]. The resulting Schrödinger-like equations should ensure that the Born rule has the usual status as an equilibrium state and there is no instability. In fact these models resemble some formulations of Hořava-Lifshitz gravity [83], for which Lorentz invariance is broken at very high energies and there is a preferred foliation of spacetime. But the consistency and completeness of such models remains a matter for future research [63].

Many authors in the field take the view that time is fundamentally undefined in the deep quantum-gravity regime and that an effective time evolution emerges only in certain conditions and in certain approximations [1, 13–15, 84, 85]. In these approaches it is common to suppose that an effective physical ‘time’ is hidden in the 3-metric $g_{ij}$ [1] [82]. On this view some function $\mathcal{I}$ of $g_{ij}$ must be extracted to play the role of time (where now the extraction of a time function is attempted after quantisation). The functional $\Psi[g_{ij}]$ then takes the schematic form $\Psi[\sigma, \mathcal{I}]$, where $\sigma$ represents the remaining metric variables. In appropriate conditions we might recover an effective and well-defined time evolution.

Such approaches, with no fundamental notion of time, often run into what some authors regard as natural limitations and other authors regard as conceptual difficulties. For example, in quantum cosmology a common choice for $\mathcal{I}$ is the cosmological scale factor $a$. In a minisuperspace model we might have a wave function of the form $\psi(\phi, \sigma, a)$, where $\phi$ is a homogeneous matter field, $\sigma$ are reduced metric degrees of freedom (perhaps representing perturbations around a homogeneous and isotropic 3-metric), and $a$ is regarded as an effective time. In a closed universe that expands and recontracts, the ‘time’ $a$ can have pathological properties: distinct states can be associated with the same value of ‘time’ and certain ‘times’ might never be reached at all. It is not obvious that a parameter measuring spatial volume can be consistently reinterpreted as a parameter that measures time [11] [3]. We might be able to derive a conventional time evolution in some local region of configuration space (corresponding, for example, to an expanding cosmological phase), but globally we are likely to find pathologies. Some authors are unconcerned by the pathologies, concluding that our conventional ideas about time have limited validity and emerge only in certain restricted circumstances. Other authors are troubled by the pathologies and argue that the formalism suffers from a deep conceptual problem.

Issues also arise concerning the application of basic rules of quantum mechanics in a fundamentally timeless theory [9] [14] [84] [86]. Again, authors differ on the significance of these questions.

It is not our purpose here to review or critique the numerous approaches to the problem of time developed over more than half a century [8]. Proposals that are still being actively pursued include, for example, evolving constants of motion and conditional probability interpretations (to name just two among many). It is noteworthy that this area remains active and controversial with as

\[\text{11 It has been argued that such problems will be generic for any realistic degree of freedom } \mathcal{I} \text{ hidden in } g_{ij} \text{.}\]
yet no definitive resolution.

There are, however, three well-known approaches that are particularly relevant for our purposes. The first, known as the ‘Klein-Gordon interpretation’, exploits the mathematical parallel between the Wheeler-DeWitt equation (on superspace) and the Klein-Gordon equation for a single particle (on a curved space with an arbitrary potential), with the hope of obtaining well-defined time-dependent probabilities at least in some regime. This approach was studied in particular by Kuchař and was shown to have seemingly unresolvable difficulties. The second approach, which came to be known as the ‘naive Schrödinger interpretation’, was championed in particular by Hawking and collaborators in the 1980s but was also shown to have serious difficulties and has been widely abandoned. The third or ‘WKB’ approach dates from the 1960s and is still widely used (in particular in quantum cosmology). It will be helpful to review these three approaches before we proceed.

### 3.2.1 Klein-Gordon interpretation

As is well known the DeWitt metric \( G_{ijkl} \) defines a manifold with hyperbolic signature \(-++\ldots++\) (for a 6-dimensional space of ‘points’ \( g_{ij} \) at each spatial point \( x \)) [1]. For this reason the Wheeler-DeWitt equation is formally analogous to an infinite-dimensional Klein-Gordon equation (with a ‘mass-squared’ term \( g^{1/2}R \)). By extracting an appropriate time functional \( T[g_{ij}, x] \) (a functional of \( g_{ij} \) at each \( x \)), the Wheeler-DeWitt equation \( (39) \) for \( \Psi[g_{ij}] \) can be written as a Klein-Gordon-like equation [6]

\[
\left( -\frac{\delta^2}{\delta T^2} + F^{ab}_{\sigma^a \sigma^b} - g^{1/2}R \right) \Psi = 0 \tag{47}
\]

for \( \Psi[\sigma, T] \), where \( \sigma^a(g, x) (a = 1, \ldots, 5) \) are the remaining metric variables and \( F^{ab} = F^{ab}[T, \sigma; x] \).

The Klein-Gordon interpretation attempts to understand the physics of the Wheeler-DeWitt equation by exploiting the formal analogy with the single-particle Klein-Gordon equation

\[
\left( \frac{\partial^2}{\partial t^2} + \delta^{ij} \frac{\partial^2}{\partial x^i \partial x^j} - m^2 \right) \psi = 0 \tag{48}
\]

(written for simplicity on Minkowski spacetime). This implies a continuity equation \( \partial_t \rho_{\mathrm{KG}} + \partial_i j_{\mathrm{KG}}^i = 0 \) with a density

\[
\rho_{\mathrm{KG}} = i(\psi^* \dot{\psi} - \psi \dot{\psi}^*) = -2 |\psi|^2 \dot{S} \tag{49}
\]

and a spatial current

\[
j_{\mathrm{KG}}^i = -i(\psi^* \partial_i \psi - \psi \partial_i \psi^*) = 2 |\psi|^2 \partial_i S \tag{50}
\]

(with \( \psi = |\psi| e^{iS} \)). The global quantity \( \int d^3 x \rho_{\mathrm{KG}} \) is preserved in time. However \( \rho_{\mathrm{KG}} \) is not positive definite and so cannot be interpreted as a probability density.
If we multiply $\rho_{KG}$ and $j_{KG}$ by $1/2m$, the current appears to take a standard form $|\psi|^2 (\partial_i S/m)$ and we might attempt to take $|\psi|^2$ as the true probability density. But in that case we find that $\int d^3x |\psi|^2$ is not preserved in time. For these reasons it is unclear how to associate probabilities with the single-particle Klein-Gordon equation.

Similar problems arise for the Wheeler-DeWitt equation written in the Klein-Gordon form (47). The infinite-dimensional analogue of the Klein-Gordon density is again not positive definite [1]. In developing the Klein-Gordon interpretation it was hoped that the density would turn out to be positive on some appropriately-defined subspace of solutions $\Psi$, but unfortunately this program was beset with difficulties and has been widely abandoned [5, 6]. Even so, as we shall see, important mathematical and physical insights can be gained by considering the Wheeler-DeWitt equation in the Klein-Gordon form (47).

### 3.2.2 Naive Schrödinger interpretation

According to the naive Schrödinger interpretation we can treat $|\Psi[g_{ij}]|^2$ directly as the probability density for the 3-metric $g_{ij}$ [2]. More precisely, $|\Psi[g_{ij}]|^2$ is taken to be the probability density (on superspace) for finding a spacelike hypersurface with 3-geometry represented by $g_{ij}$. In a minisuperspace model with a wave function $\psi(\phi, \sigma, a)$, the quantity $|\psi(\phi, \sigma, a)|^2$ is taken to be the probability density for finding a matter field $\phi$, metric perturbations $\sigma$, and a scale factor $a$. This is analogous to taking $|\psi(x,t)|^2$ for a single particle to be the probability density for finding a time $t$ with the particle at $x$. This kind of reasoning was applied by Hawking and his school to argue that certain universes are more likely than others (concluding, for example, that the universe is likely to be flat and large) [87].

One difficulty with this approach is that it seems unable to answer what we might call dynamical questions, specifically, how to calculate the probability of finding a measured value given the outcome of a preceding measurement. Attempts have been made to refine the interpretation using conditional probabilities, but this ‘conditional probability interpretation’ raises difficult questions about which variables can be appropriately chosen as conditional statements for the remaining variables [5, 6].

The naive Schrödinger interpretation, and attempts to refine it, in any case founder on one insuperable problem: solutions $\Psi[g_{ij}]$ to the Wheeler-DeWitt equation (39) are generally not normalisable (that is, not square-integrable). To see why, consider the Wheeler-DeWitt equation written in the Klein-Gordon form (48). A solution $\Psi[g_{ij}]$ of (39) corresponds to a solution $\Psi[\sigma, T]$ of (48), with a change of variables from $g_{ij}$ to $\sigma, T$. Therefore we can write

$$\int Dg |\Psi[g_{ij}]|^2 \sim \int D\sigma \int DT |\Psi[\sigma, T]|^2 = \infty .$$

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12 Unruh and Wald [3] were the first to call this ‘the “naive interpretation” of canonical quantum gravity’. 

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18
The right-hand side of (51) diverges because it is just a higher-dimensional analogue of the integral

\[ \int d^3 x \int_{-\infty}^{+\infty} dt \ |\psi(x, t)|^2 = \infty \]  

(52)

for a solution \(\psi(x, t)\) of the single-particle Klein-Gordon equation (48), in which we integrate not only over \(x\) but also over \(t\). Thus the Klein-Gordon-like character of the Wheeler-DeWitt equation ensures that solutions \(\Psi[g_{ij}]\) are not square-integrable.

It might be thought that the diverging integral (51) could be rendered finite by some appropriate regularisation (for example replacing continuous 3-space with a discrete lattice). But for as long as we integrate over the whole ‘time axis’ the integral (51) will remain divergent just like its lower-dimensional counterpart (52). The divergence reflects a basic fact about wave propagation: a solution of the wave equation can be confined (or decay rapidly) with respect to \(x\) but not with respect to \(t\).

Because \(|\Psi[g_{ij}]|^2\) is not square-integrable it cannot be employed as a probability density. Similar problems afflict the conditional probability interpretation. For this reason the naive Schrödinger interpretation fails. Some authors have tried to view the divergence of (51) as a technical issue to be resolved by more sophisticated mathematics. In this paper we argue instead that the divergence of (51) points to an important physical fact: that there is no fundamental Born rule in quantum gravity.

To be clear, we should remark that there are technical issues with normalisation and the definition of integration measures that afflict any continuous field theory. Even in classical physics it would be mathematically delicate to define a probability density on the space of continuous electromagnetic fields. For a classical scalar field, for example, we might consider a formal probability density \(P[\phi(x)]\) on the space of field configurations \(\phi(x)\) and try to define the functional integral \(\int D\phi\) (with \(\int D\phi \ P[\phi] = 1\)) as a limiting integral \(\int \ldots \int \ d\phi_1 d\phi_2 \ldots d\phi_n \ldots\) over field values \(\phi_i\) on a discrete lattice of spatial points \(x_i\). But, as is well known, the Lebesgue measure \(d\phi_1 d\phi_2 \ldots d\phi_n \ldots\) is not well-defined in the continuum limit. But in practice such problems are routinely evaded, for example, by introducing periodic boundary conditions and moving to Fourier space, and perhaps adding a high-frequency cutoff, so as to make the system effectively discrete and finite. Similar techniques are routinely applied in quantum field theory. Despite such technical issues with measures we can, for example, calculate the probability distribution for vacuum field fluctuations in inflationary cosmology and compare successfully with observation. But the normalisability problem for the Wheeler-DeWitt equation is much deeper: the very structure of the equation ensures what is in effect a wavelike propagation in configuration space, so that attempting to normalise a Wheeler-DeWitt functional \(\Psi[g_{ij}]\) in superspace is as misguided as attempting to normalise a Klein-Gordon function \(\psi(x, t)\) in spacetime. We might say that \(\Psi[g_{ij}]\) is not merely ‘technically’ non-normalisable but ‘intrinsically’ so.
To conclude, the mathematical structure of the Wheeler-DeWitt equation ensures a wave-like propagation in configuration space, just as if some of the degrees of freedom played the role of time in analogy with the Klein-Gordon equation on spacetime. For this reason solutions of the Wheeler-DeWitt equation are intrinsically non-normalisable and the naive Schrödinger interpretation fails. While there is a general consensus in the literature that the naive Schrödinger interpretation is unworkable, in this paper we suggest that the physical reason for its failure has not been properly identified. It is often thought, for example, that it fails because some of the metric degrees of freedom really play the role of time. As we shall see, from the point of view of pilot-wave theory, the naive Schrödinger interpretation fails because there is no such thing as quantum equilibrium or the Born rule in the deep quantum-gravity regime. This observation will form the starting point for our new approach to the Born rule in quantum gravity (Section 4).

3.2.3 WKB approach and Schrödinger approximation

One of the most successful and still widely-used approaches to the problem of time employs the semiclassical WKB method, in which trajectories for an evolving 3-geometry are associated with a WKB wave functional $\Psi$.

If we write $\Psi = |\Psi|e^{iS}$ and assume that $|\Psi|$ varies slowly compared with $S$, the Wheeler-DeWitt equation (39) implies an approximate time-independent Hamilton-Jacobi equation

$$G_{ijkl} \frac{\delta S}{\delta g_{ij}} \frac{\delta S}{\delta g_{kl}} - g^{1/2} R = 0$$

(53)

for $S$. In the WKB approach it is assumed that a solution $S$ generates classical trajectories with a canonical momentum

$$p^{ij} = \frac{\delta S}{\delta g_{ij}}.$$  

(54)

From (30) we then have a first-order equation of motion

$$\frac{\partial g_{ij}}{\partial t} = 2NG_{ijkl} \frac{\delta S}{\delta g_{kl}} + D_i N_j + D_j N_i$$

(55)

for the trajectories. If we differentiate (55) with respect to time, then from (53) (together with the momentum constraint (10), which implies two similar constraints on $|\Psi|$ and $S$) it is possible to recover the classical Einstein equations [1].

Thus, given a WKB wave functional $\Psi[g_{ij}]$, we can define approximately classical trajectories for the 3-metric and recover an approximately classical spacetime background. If we include a scalar matter field $\phi$, we can also recover an approximate time-dependent Schrödinger equation

$$i\frac{\partial \psi[\phi, t]}{\partial t} = \hat{H}_{\text{eff}} \psi[\phi, t]$$

(56)
for the propagation of $\phi$ on the classical spacetime background (where $\hat{H}_{\text{eff}}$ is an effective Hamiltonian). Details are given in Section 3.4. To summarise the method, we begin with the extended Wheeler-DeWitt equation (41) for $\Psi[g_{ij}, \phi]$. We take an approximate solution of the form

$$
\Psi[g_{ij}, \phi] \approx \Psi_{\text{WKB}}[g_{ij}] \psi[\phi, g_{ij}],
$$

where $\Psi_{\text{WKB}}$ is a WKB solution for the 3-metric alone whose phase $S_{\text{WKB}} = \text{Im} \ln \Psi_{\text{WKB}}$ satisfies the classical Hamilton-Jacobi equation (53). We can then use $S_{\text{WKB}}$ to generate trajectories for the classical background via (55). The wave function $\psi[\phi, g_{ij}]$ of the matter field is evaluated along a trajectory $g_{ij} = g_{ij}(t)$ for the background and so we can define an effective time-dependent function

$$
\psi_{\text{eff}}[\phi, t] = \psi[\phi, g_{ij}(t)].
$$

In a small time $\delta t$ the quantity $\psi_{\text{eff}}$ will change by

$$
\delta \psi_{\text{eff}} = \int d^3x \frac{\delta \psi_{\text{eff}}}{\delta g_{ij}} \delta g_{ij} = \int d^3x \frac{\delta \psi_{\text{eff}}}{\delta g_{ij}} \dot{g}_{ij} \delta t.
$$

We then have a time derivative

$$
\frac{\partial}{\partial t} = \int d^3x \ \dot{g}_{ij} \frac{\delta}{\delta g_{ij}},
$$

where $\dot{g}_{ij}$ is given by (55).

Similarly, in a minisuperspace model with wave function $\psi(\phi, \sigma, a)$, we have a scale factor $a = a(t)$ for the classical background and an effective time-dependent wave function

$$
\psi_{\text{eff}}(\phi, \sigma, t) = \psi(\phi, \sigma, a(t))
$$

with time derivative

$$
\frac{\partial}{\partial t} \equiv \dot{a} \frac{\partial}{\partial a},
$$

where $\dot{a}$ is found from the minisuperspace analogue of the relation (55).

Note the crucial role played by the WKB trajectories for the classical background. They allow us to define an effective time parameter $t$ (often called ‘WKB time’ [58]) and an effective time-dependent wave functional for a matter field on the background. The WKB trajectories are in fact de Broglie-Bohm trajectories evaluated in the WKB approximation, and so the above construction is entirely natural in pilot-wave theory.

### 3.3 Pilot-wave theory and quantum gravity

We now consider canonical quantum gravity with the Wheeler-DeWitt equation supplemented by a de Broglie-Bohm trajectory for the 3-geometry [55–58].
3.3.1 Pilot-wave geometrodynamics

Collecting together the basic general equations, the pilot wave \( \Psi[g_{ij}] \) obeys the Wheeler-DeWitt equation \( \dot{\Psi} = 0 \) or

\[
- G_{ijkl} \frac{\delta^2 \Psi}{\delta g_{ij} \delta g_{kl}} - g^{1/2} R \Psi = 0
\]

(subject to the constraint (60) or \( D_j (\delta \Psi/\delta g_{ij}) = 0 \)), while the trajectories \( g_{ij}(t) \) for the evolving 3-geometry are determined by the de Broglie guidance equation

\[
\frac{\partial g_{ij}}{\partial t} = 2 N G_{ijkl} \frac{\delta S}{\delta g_{kl}} + D_i N_j + D_j N_i
\]

(where \( S = \text{Im} \ln \Psi \)). As in the WKB approach (61) can be motivated from the classical canonical relations (54) and (30). However note that here (61) is assumed to be valid for any Wheeler-DeWitt wave functional \( \Psi \) – even outside the WKB limit. Note also the (implicitly understood) ambiguity in the ordering of the kinetic term in (60).

Equations (60) and (61) (with (40)) are the fundamental laws of pilot-wave geometrodynamics for pure gravitation. Together they define a dynamics for an individual system, with no mention of ensembles or probabilities. As in non-gravitational pilot-wave theory, \( \Psi \) is a physical object in configuration space that guides the motion of an individual system – it has no intrinsic connection with probability.

If we substitute \( \Psi = |\Psi| e^{iS} \) into (60) the real part yields a modified Hamilton-Jacobi equation

\[
G_{ijkl} \frac{\delta S}{\delta g_{ij}} \frac{\delta S}{\delta g_{kl}} - g^{1/2} R + q = 0 ,
\]

where

\[
q = -\frac{1}{|\Psi|} G_{ijkl} \frac{\delta^2 |\Psi|}{\delta g_{ij} \delta g_{kl}}
\]

is the quantum potential density, while the imaginary part yields the equation

\[
G_{ijkl} \frac{\delta}{\delta g_{ij}} \left( |\Psi|^2 \frac{\delta S}{\delta g_{kl}} \right) = 0 .
\]

The results (63) and (64) follow if we adopt the explicit ordering \( G_{ijkl}(\delta/\delta g_{ij})(\delta/\delta g_{kl}) \) in (60) but are modified for other orderings. For example with the alternative ordering \( (\delta/\delta g_{ij})G_{ijkl}(\delta/\delta g_{kl}) \) the result (64) instead reads

\[
\frac{\delta}{\delta g_{ij}} \left( |\Psi|^2 G_{ijkl} \frac{\delta S}{\delta g_{kl}} \right) = 0 .
\]

DeWitt [1] advocated a rule whereby \( \delta/\delta g_{ij} \) is understood to give zero when acting on \( g_{kl} \) at the same spatial point, in which case the two forms (64) and (65) are the same. Horiguchi [56] employs the general form \( g^{-p}(\delta/\delta g_{ij})g^p G_{ijkl}(\delta/\delta g_{kl}) \)
for a fixed parameter $p$ (where $p = -1/2$ corresponds to a Laplace-Beltrami operator ordering).

Instead of motivating the guidance equation (61) from the classical relations (54) and (30), alternatively it can be motivated as the natural velocity field appearing in (65), which can be rewritten as

$$\frac{\delta}{\delta g^i j} \left( |\Psi|^2 \frac{\partial g^i j}{\partial t} \right) = 0 \quad (66)$$

(for simplicity setting $N_i = 0$). However we must be careful not to over-interpret (66). As we shall discuss in Section 3.3.2, some workers integrate (66) over $x$ and (mistakenly) regard the result as a physical continuity equation for a probability density $|\Psi|^2$. But the static functional $|\Psi|^2$ has pathological properties: as we saw in Section 3.2.2 it is not integrable and cannot be a physical probability density.

An important issue concerns the status of the spacetime foliation in the above dynamics. It seems fair to say that at present this aspect of the theory is still not fully understood. The arbitrary choice of lapse and shift functions $N$ and $N^i$, which appear in the guidance equation (61), should not affect the overall 4-geometry that is traced out by the time evolution of the 3-metric (for given initial conditions on a spacelike slice). Otherwise the initial-value problem would not be well-posed. Shtanov [57] presented an example where the 4-geometry seemed to depend on $N$ and suggested on this basis that the dynamics breaks foliation invariance. One possible way to proceed would then be to impose a specific choice of $N$ as part of the theory. But later work by Pinto-Neto and Santini [90] suggests that the above pilot-wave dynamics of geometry is well-posed for arbitrary $N$. The question is addressed by writing the dynamics as a classical Hamiltonian system with (32) replaced by

$$H_q = \int d^3 x \left( N H_q + N^i H^i \right), \quad (67)$$

where $H_q = H + q$ and $q$ is the quantum potential density (63). If we include $p^i j = \delta S/\delta g^i j$ as an initial condition on the momenta, Hamilton’s equations then generate the same trajectories as pilot-wave dynamics with the guidance equation $p^i j = \delta S/\delta g^i j$ applied at all times. The question is whether the time evolution of an initial 3-geometry will yield the same 4-geometry for any choice of $N$, $N^i$. Classically this question is answered by the following theorem: Hamilton’s equations, with a Hamiltonian $\hat{H} = \int d^3 x \left( N \hat{H} + N^i \hat{H}^i \right)$, generate a unique locally-Lorentzian 4-geometry if and only if $\hat{H}$ and $\hat{H}^i$ satisfy the Dirac-Teitelboim algebra [91]. The classical quantities $H$ and $H^i$ (appearing in (32)) satisfy this algebra and so the classical Hamiltonian dynamics indeed generates a 4-geometry that is independent of $N$ and $N^i$. In contrast, for a system with Hamiltonian (67), the algebra satisfied by $H_q$ and $\hat{H}$ is found to be closed for all

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13 Alternatively, as we have noted, we might abandon foliation invariance already at the classical level and obtain a time-dependent Schrödinger equation for $\Psi[g^i j, t]$ with a preferred time parameter $t$ [24, 25, 32].
Ψ (when evaluated on de Broglie-Bohm trajectories) but modified when \( q \neq 0 \). Pinto-Neto and Santini conclude that the 4-geometry is again independent of \( N \) and \( N' \) but now forms a non-Lorentzian spacetime. On this interpretation the nonlocality associated with \( q \neq 0 \) breaks local Lorentz invariance for individual trajectories. A locally-Lorentzian spacetime is obtained only in the classical limit \( q \to 0 \) where the nonlocality vanishes. The physical implication appears to be that, for a given solution \( \Psi \) of the Wheeler-DeWitt equation and for a given initial 3-geometry, there is in effect a preferred foliation of the resulting spacetime.

Intuitively this is consistent with the ‘Aristotelian’ structure of pilot-wave dynamics, where de Broglie’s law of motion determines velocity rather than acceleration [92]. For standard systems of fields and particles the pilot wave \( \Psi \) determines the canonical momentum for a given configuration. In the case of gravity the canonical momentum density \( p^{ij} \), expressed by (28), is essentially equal to the extrinsic curvature tensor \( K^{ij} \). Thus for a given 3-metric \( \Psi \) determines \( K^{ij} \), which describes how the 3-geometry is embedded in spacetime. In effect the slicing is determined by the 3-geometry and wave functional – just as the momentum of a particle is determined by its position and wave function. A preferred foliation for a quantum-gravitational system is also consistent with a preferred foliation for the limiting case of a field on a classical spacetime background [10, 24, 25, 93]. On this issue we note, finally, that for quantum nonequilibrium ensembles of entangled systems, nonlocality can manifest as statistical nonlocal signalling [23, 27, 28], which arguably requires a physical preferred foliation of spacetime [94].

In this paper we focus on the question of probability in pilot-wave quantum gravity. Equations (60) and (61) are assumed to define a pilot-wave dynamics of 3-geometry for an individual system with wave functional \( \Psi \). We can then consider a theoretical ensemble of such systems, with the same wave functional \( \Psi \), and with an arbitrary initial probability distribution \( P[g_{ij}, t_i] \) at some initial time \( t_i \). Because the 3-metric evolves in time, with a velocity \( \partial g_{ij}/\partial t \) given by (61), by construction the distribution \( P[g_{ij}, t] \) will evolve according to the continuity equation

\[
\frac{\partial P}{\partial t} + \int d^3x \frac{\delta}{\delta g_{ij}} \left( P \frac{\partial g_{ij}}{\partial t} \right) = 0 .
\]

(68)

The above equations are readily generalised to include matter fields. We simply take the extended Wheeler-DeWitt equation (11) for \( \Psi[g_{ij}, \phi] \), the same guidance equation (61) for \( g_{ij}(t) \), and add the guidance equation (22) for the matter field trajectory \( \phi(x, t) \). For a theoretical ensemble with the same wave functional \( \Psi \), a general distribution \( P[g_{ij}, \phi, t] \) will then evolve by the extended continuity equation

\[
\frac{\partial P}{\partial t} + \int d^3x \frac{\delta}{\delta g_{ij}} \left( P \frac{\partial g_{ij}}{\partial t} \right) + \int d^3x \frac{\delta}{\delta \phi} \left( P \frac{\partial \phi}{\partial t} \right) = 0 .
\]

(69)

A crucial question remains: how do we construct the theory of a quantum equilibrium ensemble? As we saw in Section 2, for a non-gravitational system
this is usually straightforward: the continuity equation (7) for a general distribution $\rho$ coincides with the continuity equation (3) for $|\psi|^2$ (derived from the time-dependent Schrödinger equation), enabling us to deduce that $\rho = |\psi|^2$ is an equilibrium state. As we shall see, for a gravitational system obeying the Wheeler-DeWitt equation, this reasoning breaks down.

3.3.2 Naive Schrödinger interpretation in pilot-wave theory

Some workers try to interpret $|\Psi|^2$ as the equilibrium probability density for pilot-wave theory with a Wheeler-DeWitt wave functional $\Psi$. Even though $|\Psi|^2$ is static, individual trajectories depend on time and we might hope to recover time-dependent probabilities for subsystems. This approach was originally suggested by Vink [55] in the context of a minisuperspace model, and it has recently been advocated in general terms by Dürr and Struyve [89]. However, this amounts to applying the naive Schrödinger interpretation to pilot-wave quantum gravity. While having time-dependent trajectories adds a novel element, the essential difficulty of the naive Schrödinger interpretation remains.

To see the difficulty let us include a matter field $\phi$, so that $\Psi[g_{ij}, \phi]$ obeys the extended Wheeler-DeWitt equation (41), and consider how Dürr and Struyve attempt to motivate $|\Psi[g_{ij}, \phi]|^2$ as the equilibrium density [89]. Writing $\Psi = |\Psi| e^{iS}$ we find from (41) (with an appropriate operator ordering) that $|\Psi[g_{ij}, \phi]|^2$ satisfies

$$\frac{\delta}{\delta g_{ij}} \left( |\Psi|^2 \frac{\partial g_{ij}}{\partial t} \right) + \frac{\delta}{\delta \phi} \left( |\Psi|^2 \frac{\partial \phi}{\partial t} \right) = 0$$

at each spatial point $x$. This is not yet a continuity equation. Comparing with the standard continuity equation for a single particle, which for a static density ($\partial \rho / \partial t = 0$) takes the form $\nabla \cdot j = 0$, equation (70) is analogous to the set of equations $\partial_x j_x = \partial_y j_y = \partial_z j_z = 0$ (one for each degree of freedom). A proper continuity equation instead takes the form (69) satisfied by a general (time-dependent) distribution $P[g_{ij}, \phi, t]$. However, following Dürr and Struyve, such an equation for $|\Psi|^2$ can be obtained by integrating (70) over $x$. Noting that $|\Psi|^2$ has no explicit time dependence, $\partial |\Psi|^2 / \partial t = 0$, we can then write what we shall refer to as a ‘pseudo-continuity equation’

$$\frac{\partial |\Psi|^2}{\partial t} + \int d^3x \frac{\delta}{\delta g_{ij}} \left( |\Psi|^2 \frac{\partial g_{ij}}{\partial t} \right) + \int d^3x \frac{\delta}{\delta \phi} \left( |\Psi|^2 \frac{\partial \phi}{\partial t} \right) = 0 .$$

This is now formally the same as the physical continuity equation (69) satisfied by $P$. It might then be thought that we can deduce a quantum equilibrium state $P = |\Psi|^2$ in the usual way: because $P$ and $|\Psi|^2$ obey the same evolution equation, if they are equal initially they will be equal later, and so $P = |\Psi|^2$ is an equilibrium state – just as usual for non-gravitational systems. Indeed Dürr and Struyve write down the equation (71) (without the vanishing first term) and

\footnote{For completeness we note that Horiguchi [56] tries to implement a form of the Klein-Gordon interpretation.}
assert on this basis that $|\Psi|^2$ is a quantum equilibrium measure for the system [89]. It is claimed further that $|\Psi|^2$ can be employed as a measure of ‘typicality’ for initial configurations of the universe, from which the Born rule for subsystems can be derived [15]. But this is another version of the naive Schrödinger interpretation, albeit in the context of pilot-wave theory. It fails for the same reason as before: $|\Psi|^2$ is not normalisable and so cannot be used to define an equilibrium measure. As we saw in Section 3.2.1, and as has been known since the 1960s, the Wheeler-DeWitt equation has the character of a Klein-Gordon equation. For this reason trying to normalise a solution $\Psi[g_{ij},\phi]$ with respect to the gravitational degrees of freedom $g_{ij}$ is mathematically analogous to trying to normalise a single-particle Klein-Gordon wave function $\psi(x,t)$ in spacetime.

Dürr and Struyve suggest that the non-normalisability of $|\Psi|^2$ for the Wheeler-DeWitt equation is analogous to the non-normalisability of $|\Psi|^2$ encountered in relational theories of dynamics [95] and that it can be handled in the same way. But the two cases are not analogous. In relational theories $\Psi$ is non-normalisable because of symmetries associated with unobservable and unphysical (absolute) structure. As discussed in ref. [95] this can be handled essentially by factoring out the unobservable structure. A correct analogy might be drawn between relational theories and quantum gauge theories, where for the latter we need to factor out an infinite unobservable gauge volume when normalising the relevant wave functional (though there the gauge volume can be rendered finite by the simple device of putting the field theory on a lattice [96]). But the case of the Wheeler-DeWitt wave functional is fundamentally different: the non-normalisability is not caused by unobservable structure but by the wave-like propagation of $\Psi$ in configuration space (Section 3.2.2).

While the pseudo-continuity equation (71) is strictly speaking correct, it disguises the fact that the Wheeler-DeWitt equation implies not just the single equation (71) but the infinity of equations (70) (one per space point $x$). Summing the equations (70) to produce the one equation (71), in order to give the appearance that $|\Psi|^2$ obeys a standard continuity equation, is analogous to summing the equations $\partial_x j_x = \partial_y j_y = \partial_z j_z = 0$ for a single particle to produce $\nabla \cdot \mathbf{j} = 0$. The result is mathematically correct but physically misleading. If $|\Psi|^2$ really did behave like a conventional density, as suggested by equation (71), we would expect it to be normalisable. But in fact the Klein-Gordon-like structure of the Wheeler-DeWitt equation shows that $|\Psi|^2$ evidently cannot be normalised and is not a physical density. Thus we should beware of artificial attempts to make $|\Psi|^2$ take on the appearance of a conventional density, when in fact it is a different kind of object. As numerous authors have pointed out, the naive Schrödinger interpretation is physically misguided, and this remains true in pilot-wave theory.

It will be argued below that there is no concept of quantum equilibrium in the deep quantum-gravity regime, and that this is the physical significance of the non-normalisability of the Wheeler-DeWitt wave functional $\Psi$. To interpret

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15This mimics the general approach to the Born rule advocated by the Bohmian mechanics school of de Broglie-Bohm theory (noted at the end of Section 2.1).
$|\Psi|^2$ as a Born-rule probability measure (or typicality measure) is a category mistake. At the fundamental level there is no Born rule. The Born rule can emerge only in the Schrödinger approximation (Section 4.3).

### 3.4 Schrödinger approximation for a matter field

It will be useful to consider carefully how the Schrödinger approximation, with a time-dependent effective wave function, arises from the underlying quantum-gravitational formalism.

#### 3.4.1 Semiclassical expansion of the Wheeler-DeWitt equation

To convey the general method we first outline the original treatment given by Kiefer and Singh [68]. We consider quantum gravity in the presence of a scalar field $\phi$. The extended Wheeler-DeWitt equation (41) for $\Psi[g_{ij},\phi]$ is solved by successive approximation, by means of a semiclassical expansion in powers of a parameter $\mu = c^2/32\pi G$ (dimensionally a mass per length),

$$\Psi = \exp i \left( \mu S_0 + S_1 + \mu^{-1} S_2 + \ldots \right),$$

where the terms in the round brackets are generally complex. The expansion (72) is inserted into the left-hand side of (41), terms of the same order in $\mu$ are collected and the sum is set equal to zero.

The highest order that appears is $\mu^2$, for which it is found that

$$\frac{\left( \delta S_0 \right)^2}{\delta \phi} = 0.$$  \hfill (73)

Thus $S_0 = S_0[g_{ij}]$ depends only on $g_{ij}$. The function $S_0$ is associated with a classical spacetime background which does not depend on the matter field (or matter-field perturbations) propagating on it.

At order $\mu$ we obtain a classical Hamilton-Jacobi equation

$$G_{ijkl} \frac{\delta S_0}{\delta g_{ij}} \frac{\delta S_0}{\delta g_{kl}} - g^{1/2} R = 0.$$  \hfill (74)

As we have noted this is equivalent to the vacuum Einstein equations. A realistic model would include a matter term in (74), but for present purposes (74) describes a classical spacetime background on which quantum perturbations propagate.

At order $\mu^0$ we obtain an equation for $S_1$:

$$G_{ijkl} \frac{\delta S_0}{\delta g_{ij}} \frac{\delta S_1}{\delta g_{kl}} - i 2 G_{ijkl} \frac{\delta^2 S_0}{\delta g_{ij} \delta g_{kl}} + \frac{1}{2 \sqrt{g}} \left( \frac{\delta S_1}{\delta \phi} \right)^2 - i \frac{\delta^2 S_1}{2 \sqrt{g} \delta \phi^2} + u = 0,$$  \hfill (75)

where

$$u = \frac{1}{2} \sqrt{g} g^{ij} \partial_i \phi \partial_j \phi + \sqrt{g} V(\phi).$$  \hfill (76)
This can be written as an effective time-dependent Schrödinger equation for a matter wave functional \( \psi^{(0)}[\phi, g_{ij}, t] \), which we may write more simply as \( \psi^{(0)}[\phi, t] \) and which describes the quantum evolution of \( \phi \) on a classical background with metric \( g_{ij} \) (where the background is determined by (74)).

This is accomplished by using the trajectories of the classical background to define an effective time parameter \( t \) by means of the relation
\[
\frac{\partial}{\partial t} = \int d^3x \ 2NG_{ijkl} \frac{\delta S_0}{\delta g_{kl}} \frac{\delta}{\delta g_{ij}}.
\]

The factor \( 2NG_{ijkl}\delta S_0/\delta g_{kl} \) coincides with the WKB (or de Broglie) velocity \( \dot{g}_{ij} \) generated by \( S_0 \) (with \( N_i = 0 \)). Thus this definition is the same as the WKB time defined above in equation (59). Here \( t \) parameterises integral curves of the de Broglie velocity field associated with the classical background. If we then define
\[
\psi^{(0)} = D[g_{ij}] \exp(iS_1),
\]
where \( D \) is chosen so as to satisfy the condition
\[
G_{ijkl} \frac{\delta S_0}{\delta g_{ij}} \frac{\delta D}{\delta g_{kl}} - \frac{1}{2}G_{ijkl} \frac{\delta^2 S_0}{\delta g_{ij} \delta g_{kl}} D = 0,
\]
it is found that
\[
\frac{i\partial \psi^{(0)}}{\partial t} = \int d^3x \ \hat{H}_\phi \psi^{(0)}
\]
(of the general form advertised in (56)). This equation describes a quantum field \( \phi \) evolving on a classical spacetime background, where \( \psi^{(0)}[\phi, t] \) is the zeroth-order or uncorrected wave functional.

Note that, to this order, \( \Psi \) does indeed take the WKB form (57) with
\[
\Psi_{\text{WKB}}[g_{ij}] = \frac{1}{D[g_{ij}]} \exp(iMS_0)
\]
and \( \psi[\phi, g_{ij}] = \psi^{(0)}[\phi, t] \).

### 3.4.2 Semiclassical expansion of the de Broglie velocity

We have just reviewed the semiclassical reduction of the extended Wheeler-DeWitt equation (41) for \( \Psi[g_{ij}, \phi] \) to an effective time-dependent Schrödinger equation (80) for \( \psi^{(0)}[\phi, t] \). In pilot-wave theory we must also consider the de Broglie guidance equations. If the metric \( g_{ij} \) is coupled to a scalar matter field \( \phi \), then as well as the guidance equation (61) for \( g_{ij} \) we also have the guidance equation (22) for \( \phi \). We must consider what happens to (22) under the semiclassical reduction, and in particular how the field velocity \( \phi \) is related to the effective wave functional \( \psi^{(0)}[\phi, t] \).

Note that the presence of a de Broglie trajectory \( q(t) = (g_{ij}(t), \phi(t)) \) for the combined metric-plus-field system does not affect the equations for the guiding wave functional \( \Psi[g_{ij}, \phi] \). Thus the semiclassical reduction of the Wheeler-DeWitt equation proceeds exactly as before (though arguably our understanding
of WKB time is improved by the explicit presence of a de Broglie-Bohm trajectory). The question is only what form the guidance equation (22) for \( \phi \) takes after performing the semiclassical expansion (72) of \( \Psi[g_{ij}, \phi] \).

Inserting the expansion (72) into the guidance equation (22) we find

\[
\frac{\partial \phi}{\partial t} = N \sqrt{g} \frac{\delta}{\delta \phi} (\mu \text{Re} S_0 + \text{Re} S_1 + \mu^{-1} \text{Re} S_2 + \ldots) \tag{82}
\]

(where \( S = \text{Im} \ln \Psi \) and \( \text{Im}(iz) = \text{Re} z \)). From (73) we know that \( S_0 \) does not depend on \( \phi \) and so the first term in (82) vanishes. Thus we have

\[
\frac{\partial \phi}{\partial t} = N \sqrt{g} \frac{\delta}{\delta \phi} \left( \text{Re} S_1 + \mu^{-1} \text{Re} S_2 + \ldots \right) . \tag{83}
\]

From equation (75) it is clear that \( S_1 \) is generally complex. Furthermore, in (78) we have defined a zeroth-order wave function \( \psi^{(0)} = D \exp(i S_1) \), where \( D \) satisfies the condition (79) and can be chosen to be real. Thus \( \text{Re} S_1 \) is equal to the phase of \( \psi^{(0)} \) – that is, \( \text{Re} S_1 = \text{Im} \ln \psi^{(0)} \). If we keep only the first term in (83), to lowest order we then have the usual de Broglie velocity,

\[
\left( \frac{\partial \phi}{\partial t} \right)^{(0)} = N \frac{\delta}{\sqrt{g} \delta \phi} \text{Re} S_1 = N \frac{\delta}{\sqrt{g} \delta \phi} \left( \text{Im} \ln \psi^{(0)} \right) , \tag{84}
\]

generated in the usual way by \( \psi^{(0)} \). To lowest order the de Broglie velocity for the matter field is unchanged.

### 3.4.3 The problem of the Born rule

Given the effective Schrödinger equation (80) for the wave function \( \psi^{(0)} \) of the matter field \( \phi \), writing \( \psi^{(0)} = \left| \psi^{(0)} \right| e^{i S^{(0)}} \) we can readily show that \( \left| \psi^{(0)} \right|^2 \) satisfies a continuity equation

\[
\frac{\partial \left| \psi^{(0)} \right|^2}{\partial t} + \int d^3x \frac{\delta}{\delta \phi} \left( \left| \psi^{(0)} \right|^2 N \frac{\delta S^{(0)}}{\sqrt{g} \delta \phi} \right) = 0 . \tag{85}
\]

This implies that the squared-norm \( \int D\phi \left| \psi^{(0)} \right|^2 \) is conserved in time. In standard quantum mechanics we might then simply assume that \( \left| \psi^{(0)} \right|^2 \) represents a physical probability density for the field \( \phi \) propagating on the classical background. But it would be more satisfactory if we could derive this interpretation from first principles starting from the underlying quantum-gravitational description.

If we begin with the Wheeler-DeWitt wave functional \( \Psi[g_{ij}, \phi] \), it may seem natural to interpret \( \left| \psi^{(0)} \right|^2 \) as a conditional probability density derived from an underlying joint density \( \left| \Psi[g_{ij}, \phi] \right|^2 \) – where the probability for \( \phi \) is conditional on the metric coinciding with the classical background. Recalling the standard
conditional probability formula $p(A \mid B) = p(A \cap B) / p(B)$ for events $A$ and $B$, we might consider the conditional probability (at a given time $t$)

$$\rho(\phi, t) D\phi = p(\phi \in D\phi \mid g_{ij} \in Dg_{ij}) . \quad (87)$$

If we write down the formal expressions

$$p(\phi \in D\phi \cap g_{ij} \in Dg_{ij}) = \left| \Psi[g_{ij}, \phi] \right|^2 D\phi Dg_{ij} \quad (88)$$

and

$$p(g_{ij} \in Dg_{ij}) = \left( \int \left| \Psi[g_{ij}, \phi] \right|^2 D\phi \right) Dg_{ij} \quad (89)$$

we find

$$\rho(\phi, t) = \frac{\left| \Psi[g_{ij}, \phi] \right|^2}{\left( \int \left| \Psi[g_{ij}, \phi] \right|^2 D\phi \right)} . \quad (90)$$

If we take the WKB approximation $\Psi[g_{ij}, \phi] \approx \Psi_{\text{WKB}}[g_{ij}] \rho(0)[\phi, t]$ we then have

$$\rho(0)[\phi, t] \approx \left| \rho(0)[\phi, t] \right|^2 \quad (91)$$

(assuming that $\rho(0)$ is normalised).

It may then appear that we have succeeded in deriving the Born rule $\rho(0) = \left| \psi(0) \right|^2$ for the effective Schrödinger regime from an underlying Born-rule density $\left| \Psi[g_{ij}, \phi] \right|^2$ for the joint Wheeler-DeWitt system. Unfortunately, however, the derivation rests on treating $\left| \Psi[g_{ij}, \phi] \right|^2$ as if it were a well-defined physical probability density, which as we saw in Section 3.2.2 is impossible, reflecting the well-known and long-standing difficulties with the naive Schrödinger interpretation. The Wheeler-DeWitt wave functional $\Psi[g_{ij}, \phi]$ is intrinsically non-normalisable and the quantity $\left| \Psi[g_{ij}, \phi] \right|^2$ cannot be interpreted as a physical probability density. Therefore the above derivation of the effective Born rule $\rho(0) = \left| \psi(0) \right|^2$ is in fact without foundation. Even if the resulting conditional probability $\rho$ is normalisable, it makes no physical sense to derive it from a parent density $\left| \Psi \right|^2$ which is not a physical probability.

In pilot-wave theory we can go a little further. The velocity field appearing in the continuity equation (85) for $\left| \psi(0) \right|^2$ coincides with the de Broglie velocity for $\phi$ obtained from the Wheeler-DeWitt equation. We can then readily show that $\left| \psi(0) \right|^2$ is a state of quantum equilibrium. A general distribution $\rho(0)$ will by construction satisfy

$$\frac{\partial \rho(0)}{\partial t} + \int d^3x \frac{\delta}{\delta \phi} \left( \rho(0) \frac{N}{\sqrt{g}} \delta S(0) \right) = 0 . \quad (92)$$
This takes the same form as (85). If $\rho^{(0)}$ and $|\psi^{(0)}|^2$ happen to be equal at some initial time then they will necessarily remain equal at later times. Thus we have an equilibrium state $\rho^{(0)} = |\psi^{(0)}|^2$. But a basic question still remains. We have a well-defined Born rule and equilibrium state in the effective Schrödinger regime, but this can only be an emergent approximation. What happens to the Born rule and to the concept of quantum equilibrium at the underlying level of the Wheeler-DeWitt equation? The quantity $|\Psi[g_{ij}, \phi]|^2$ cannot be a physical density and there are no other obvious candidates for a general Born rule in the quantum-gravity regime. There is then a logical gap: we appear to have a well-defined Born rule at the emergent level but not at the fundamental level.

To make sense of this, we must reconsider the nature of probability in quantum gravity from first principles. We shall see that, if we are careful to follow the internal logic of pilot-wave theory, the problem resolves itself naturally and simply.

4 New approach to the Born rule in quantum gravity

In this paper we propose a new approach to the Born rule in quantum gravity. We begin by accepting that the time-independent and non-normalisable Wheeler-DeWitt wave functional $\Psi$ is a different kind of thing from the time-dependent and normalisable Schrödinger wave functions $\psi$ that we are used to from non-gravitational physics. In particular we take the non-normalisability of $\Psi$ as an indication that $|\Psi|^2$ can never be equal to a physical probability density $P$. To make sense of this, we will consider the pilot-wave theory of the Wheeler-DeWitt equation on its own terms, without making any undue assumptions taken from standard quantum theory. We find that there is no physical equilibrium or Born-rule state in the deep quantum-gravity regime. Furthermore this implies the presence of quantum nonequilibrium at the beginning of the emergent semiclassical regime. Quantum gravity naturally creates an early nonequilibrium universe. Once the Schrödinger approximation is established, quantum relaxation to the Born rule can take place in the usual way. Fundamentally, however, the physics of the deep quantum-gravity regime undermines the Born rule as we know it, rendering it unstable.

4.1 Absence of an equilibrium state in the deep quantum-gravity regime

The starting point for our new approach is to recognise that the non-integrable quantity $|\Psi|^2$ is not and cannot be a physical Born-rule probability density $P$. As we saw in Section 3.2.2, $\Psi$ is intrinsically non-normalisable and for a deep reason: the Wheeler-DeWitt equation has the character of a Klein-Gordon equation. This problem cannot be eliminated by a standard device such as discretising the system. Instead the problem suggests that something is deeply
wrong with our interpretation of the formalism. In our view the failure of the naive Schrödinger interpretation shows that $\Psi$ is not an ordinary wave function. If we accept this starting point we nevertheless need to discuss probabilities for ensembles – and somehow recover an effective Born rule for systems propagating on a classical spacetime background. By carefully following the internal logical of pilot-wave theory, this turns out to be straightforward.

We begin with the deterministic pilot-wave dynamics of an individual gravitational system with wave functional $\Psi[g_{ij}, \phi]$, where $\Psi[g_{ij}, \phi]$ satisfies the extended Wheeler-DeWitt equation (41). Each gravitational system has a trajectory determined by the de Broglie guidance equations (61) and (22) for $g_{ij}$ and $\phi$ respectively. The wave functional $\Psi[g_{ij}, \phi]$ acts as a ‘pilot wave’ on configuration space guiding the motion of the system. The question is how to relate this dynamics to ensembles and how to recover an effective Born rule in some limit.

Let us consider a theoretical ensemble of similar systems each guided by the same wave functional $\Psi[g_{ij}, \phi]$. At time $t$ each element of the ensemble has an evolving configuration $q(t) = (g_{ij}(x, t), \phi(x, t))$. Over the ensemble we will then have an evolving distribution $P[g_{ij}, \phi, t]$ of configurations. As usual in pilot-wave theory, at an initial time $t_i$, the initial distribution $P[g_{ij}, \phi, t_i]$ is in principle arbitrary. By definition, of course, $P$ is a physical probability density and must be normalisable,

$$\int \int DgD\phi \ P[g_{ij}, \phi, t] = 1 , \quad (93)$$

at all times $t$. There are the usual technical issues concerning the rigorous definition of the measure $DgD\phi$. These might be avoided by some form of discretisation; in any case we leave them aside and focus on the conceptual questions. Each configuration $q(t) = (g_{ij}(x, t), \phi(x, t))$ evolves with the velocities (61) and (22). It follows that $P[g_{ij}, \phi, t]$ evolves in time according to the continuity equation (69). In this way pilot-wave dynamics defines the time evolution of an arbitrary ensemble of gravitational systems.

As we saw in Section 3.3.2, $|\Psi[g_{ij}, \phi]|^2$ formally satisfies the same continuity equation (71). However $|\Psi[g_{ij}, \phi]|^2$ is non-integrable and cannot count as a physical density.

Now we have seen that, in non-gravitational pilot-wave theory, the initial density $\rho(q, t_i)$ need not be equal to the initial equilibrium density $|\psi(q, t_i)|^2$. The Born rule $\rho(q, t) = |\psi(q, t)|^2$ can nevertheless emerge at later times $t$ by a process of quantum relaxation (Section 2.1). Here we argue that, for gravitational systems in the deep quantum-gravity regime, we must go a step further: the density $P[g_{ij}, \phi, t]$ can never be equal to $|\Psi[g_{ij}, \phi]|^2$, neither at the initial time nor subsequently. In other words, in quantum gravity there is no such thing as ‘quantum equilibrium’. By definition $P[g_{ij}, \phi, t]$ is a physical probability density which (subject to some mathematical caveats) can be normalised. In contrast, the failure of the naive Schrödinger interpretation shows that $|\Psi[g_{ij}, \phi]|^2$ cannot be a physical probability density. Therefore we must have

$$P[g_{ij}, \phi, t] \neq |\Psi[g_{ij}, \phi]|^2 \quad (94)$$
at all times. The two quantities \( P[g_{ij}, \phi, t] \) and \(|\Psi[g_{ij}, \phi]|^2\) are different kinds of physical things. The left-hand side of (94) can and must be normalised, the right-hand side can never be. Hence they cannot be equal, neither initially nor subsequently.

At this point it might be asked what determines the initial \( P \). This question has already been addressed at length in non-gravitational pilot-wave theory and the same principles apply here. The initial \( P \) is not fixed by any law; it is instead to be determined empirically, or at least constrained as far as possible by observation, as is generally the case for initial conditions in physics. In short, \( P \) is in principle arbitrary and in practice empirically constrained \(^{30}\).

In non-gravitational pilot-wave theory we have the analogue (10) of Liouville’s theorem together with the analogue (13) of the coarse-graining \( H \)-theorem \(^{22}\). If we write

\[ P[g_{ij}, \phi, t] = |\Psi[g_{ij}, \phi]|^2 f[g_{ij}, \phi, t] \tag{95} \]

the result (10) still applies because \( P \) and \(|\Psi|^2\) obey the same continuity equations (69) and (71). If we define a fine-grained \( H \)-function

\[ H(t) = \int \int DgD\phi \ln\left(\frac{P}{|\Psi|^2}\right) \tag{96} \]

we again have \( dH/dt = 0 \). Furthermore, if we assume that \( P \) and \(|\Psi|^2\) have no initial fine-grained structure, then even though \(|\Psi|^2\) is non-normalisable we can still show that the coarse-grained \( H \)-function

\[ \bar{H}(t) = \int \int DgD\phi \bar{P} \ln\left(\frac{\bar{P}}{|\Psi|^2}\right) \tag{97} \]

obeys the \( H \)-theorem (13) (since none of the steps in ref. \(^{22}\) depend on the normalisability of \(|\Psi|^2\)). It might then be thought that quantum relaxation, \( \bar{P} \to |\Psi|^2 \), could still take place on a coarse-grained level. However, when \( \Psi \) is non-normalisable, the \( H \)-function has no lower bound and any physical distribution \( \bar{P} \) will always be infinitely far away from ‘equilibrium’ (as defined by the minimum value of \( H \)).

To see this consider a general system with \( H \)-function (11) for which

\[ \int dq \ |\psi|^2 = N \tag{98} \]

is finite but arbitrarily large (while of course \( \int dq \rho = 1 \)). Normally we have \( N = 1 \), in which case \( H \) is bounded below by zero. This follows from the general inequality \( x \ln(x/y) \geq x - y \) (with equality if and only if \( x = y \)). Putting \( x = \rho \) and \( y = |\psi|^2 \) we have \( H \geq \int dq (\rho - |\psi|^2) = 0 \) (with \( H = 0 \) if and only if \( \rho = |\psi|^2 \)). Thus \( H = 0 \) normally corresponds to equilibrium. However, for
general $N$, the $H$-function is bounded below by $-\ln N$. To see this write

$$H = \int dq \left( \rho \ln(\rho/|\psi|^2) - \rho + |\psi|^2/N \right)$$

(99)

$$= -\ln N + \int dq \left( \rho \ln \left( \frac{\rho}{(|\psi|^2/N)} \right) - \rho + |\psi|^2/N \right).$$

(100)

Using $x \ln(x/y) - x + y \geq 0$ with $x = \rho$ and $y = |\psi|^2/N$ we have

$$H \geq -\ln N,$$

(101)

with $H = -\ln N$ if and only if $\rho = \frac{1}{N} |\psi|^2$. Clearly, as $N \to \infty$ there is no lower bound on $H$ and so there is no physical equilibrium state. The coarse-grained function $\bar{H}(t)$ could continue to decrease indefinitely, without ever reaching a minimum and hence without ever reaching equilibrium. Note that if the initial nonequilibrium distribution $\rho$ is localised in some finite region $R$ of configuration space, the initial $H$-function $H = \int_R dq \rho \ln(\rho/|\psi|^2)$ will have a finite value. As $\bar{\rho}$ evolves the coarse-grained function $\bar{H}(t)$ will remain finite and in this sense the system forever remains infinitely far away from the ‘equilibrium’ state $\bar{H} = -\infty$.

It would be instructive to explore quantum relaxation numerically for such systems. This could be done for a minisuperspace model of quantum cosmology, for which the Wheeler-DeWitt equation has the usual Klein-Gordon-like structure and the wave function $\psi$ is intrinsically non-normalisable. (For an example of such a model see Section 7.4.) Given the analogy with a Klein-Gordon wave function on spacetime, we may expect that $\rho$ will continue to spread out indefinitely over its unbounded domain. In principle we might obtain some degree of relaxation in a limited region of configuration space, but even then the system will necessarily remain infinitely far from equilibrium overall. We leave such numerical studies for future work [97]. In any case the conclusion is clear: there can be no proper quantum relaxation for a system with a non-normalisable wave function.

We conclude that, in all circumstances, there is no physical Born rule at the fundamental level of quantum gravity: $P$ can never be equal to $|\Psi|^2$. We might say that the deep quantum-gravity regime is necessarily and perpetually in a state of quantum nonequilibrium. It would be more precise, however, to say that in this regime there is no physical state of quantum equilibrium.

If we accept this reasoning, it might appear that we now have no hope of ever recovering the Born rule even in the limit appropriate to ordinary quantum physics on a classical spacetime background. However, if we follow the internal logic of pilot-wave theory, we find that in fact the relevant physics can be recovered. But first we need to consider how probabilities emerge for systems on a classical spacetime background.
4.2 Early quantum nonequilibrium as a consequence of quantum gravity

Let us consider what we can expect to find as the early universe emerges from the deep quantum-gravity regime and enters the approximate Schrödinger regime. In particular, we would like to know the probability distribution for a quantum matter field $\phi$ evolving on a classical spacetime background at the beginning of the Schrödinger regime.

We already have the tools required to address this question. At the fundamental level we have the timeless Wheeler-DeWitt equation (41) for the wave functional $\Psi[g_{ij}, \phi]$. As we saw in Section 3.4.1, in an appropriate limit we recover the approximate time-dependent Schrödinger equation (80) with an effective wave functional $\psi(0)[\phi, t]$ for $\phi$ on the classical background. In addition, the fundamental de Broglie equation of motion (22) for the trajectory $\phi(x, t)$ reduces to the effective equation (84) with the field velocity $\partial \phi / \partial t$ determined by $\psi(0)$ in the usual way. The effective Schrödinger equation (80) is just the usual functional Schrödinger equation for a free scalar field on a curved spacetime background. If we allow ourselves to employ the usual regularisation methods — for example some form of discretisation, or dimensional regularisation [76] — then the emergent wave function $\psi(0)$ is normalisable as well as time-dependent. In standard quantum mechanics this might seem sufficient to obtain the Born rule. But in pilot-wave theory so far all we have done is recover an approximate pilot-wave dynamics for an individual field system. To speak of probabilities, we must consider ensembles.

Consider, then, a theoretical ensemble of matter fields $\phi$ with the same effective wave functional $\psi(0)[\phi, t]$. The ensemble will have some distribution $\rho(0)[\phi, t]$. Before considering how the distribution evolves in time, let us first focus on its ‘initial’ value $\rho(0)[\phi, t_i]$ defined at some time $t_i$ when the system has just emerged from the deep quantum-gravity regime and entered the approximate Schrödinger regime. What do we expect to find? In Section 3.4.3 we noted the difficulty of trying to derive $\rho(0)[\phi, t_i]$ as a conditional probability from an underlying non-normalisable distribution $|\Psi[g_{ij}, \phi]|^2$. According to our new approach there is no such physical distribution. But we can still consider a theoretical ensemble of systems with the same wave functional $\Psi[g_{ij}, \phi]$. As we have argued, such an ensemble will have an arbitrary initial distribution $P[g_{ij}, \phi, t_i]$ which must be constrained empirically. The effective distribution $\rho(0)[\phi, t_i]$ for the matter field can then be derived, as a conditional probability, not from $|\Psi[g_{ij}, \phi]|^2$ but from $P[g_{ij}, \phi, t_i]$ (with the probability for $\phi$ being conditional on the metric coinciding with the classical background). Following steps similar to those in Section 3.4.3, but with $|\Psi[g_{ij}, \phi]|^2$ replaced by $P[g_{ij}, \phi, t_i]$, we find

$$\rho(0)[\phi, t_i] = \frac{P[g_{ij}, \phi, t_i]}{\int P[g_{ij}, \phi, t_i] D\phi}.$$  
(102)

Note that on the right-hand side it is understood that we have inserted the known value of $g_{ij}$. The resulting conditional distribution $\rho(0)[\phi, t_i]$ is of course
normalised by construction.

Now, as we have noted, \( P[g_{ij}, \phi, t_i] \) is arbitrary and can only be constrained empirically. It follows from (102) that \( \rho^{(0)}[\phi, t_i] \) is also arbitrary and can only be constrained empirically. Furthermore, because of the fundamental nonequilibrium condition (94) of the deep quantum-gravity regime, in general we will have

\[
\rho^{(0)}[\phi, t_i] \neq |\psi^{(0)}[\phi, t_i]|^2
\]

(except for the special case where \( P[g_{ij}, \phi, t_i] = \Pi[g_{ij}] |\psi^{(0)}[\phi, t_i]|^2 \) for some \( \Pi[g_{ij}] \)). Thus, as we enter the Schrödinger regime, the matter-field system can be expected to be in a state of quantum nonequilibrium.

In effect we have derived the general hypothesis of primordial quantum nonequilibrium (103) – for matter fields in the early Schrödinger regime – as a consequence of quantum gravity. This might seem like a failure, as we have yet to recover the Born rule. However, in reasonable circumstances, after the initial time \( t_i \), we can expect the effective density \( \rho^{(0)}[\phi, t] \) to relax towards \( |\psi^{(0)}[\phi, t]|^2 \) (on a coarse-grained level), simply as a consequence of the dynamics. Thus we do not derive the Born rule at early times (at the beginning of the Schrödinger regime). On the contrary, we derive nonequilibrium at early times. The Born rule emerges only later, via the by-now much-studied process of quantum relaxation.

4.3 Quantum relaxation in the Schrödinger approximation

To see how the Born rule emerges by quantum relaxation, let us look again at the effective equations (80) and (84) for the pilot-wave dynamics of the matter field \( \phi \) on a classical background spacetime. For an ensemble of similar systems, with the same effective wave function \( \psi^{(0)} \), at an early ‘initial’ time \( t_i \), we will have (as argued above) an arbitrary nonequilibrium distribution \( \rho^{(0)}[\phi, t_i] \). Subsequently the distribution \( \rho^{(0)}[\phi, t] \) will, by construction, evolve in time according to the continuity equation

\[
\frac{\partial \rho^{(0)}}{\partial t} + \int d^3x \frac{\delta}{\delta \phi} \left( \rho^{(0)} \frac{\partial \phi}{\partial t} \right) = 0 ,
\]

where the velocity \( \partial \phi/\partial t \) is given by (84). This is the same continuity equation that is satisfied by \( |\psi^{(0)}[\phi, t]|^2 \) (as derived from the Schrödinger equation (80)). We can then define an \( H \)-function

\[
H(t) = \int D\phi \rho^{(0)} \ln \left( \frac{\rho^{(0)}}{|\psi^{(0)}|^2} \right) .
\]

This will have the usual properties. Because \( \rho^{(0)} \) and \( |\psi^{(0)}|^2 \) satisfy the same continuity equation the exact or fine-grained value of \( H(t) \) will be conserved in time. If we assume that \( \rho^{(0)} \) and \( |\psi^{(0)}|^2 \) have no fine-grained structure at
the initial time $t_i$, we obtain the coarse-graining $H$-theorem \[13\]. As usual, $\dot{H}(t)$ is bounded below by zero and is equal to zero if and only if $\rho(0) = |\psi(0)|^2$ everywhere.

For field theory it is more convenient to discuss quantum relaxation in terms of modes in Fourier space \[36, 45, 46\]. For a scalar field on an expanding homogeneous background with scale factor $a(t)$, each field mode is mathematically equivalent to a two-dimensional oscillator with mass $m = a^3$ and angular frequency $\omega = k/a$. Extensive numerical simulations have shown that relaxation occurs efficiently at sub-Hubble wavelengths ($a\lambda < H^{-1}$) and is slowed or suppressed at super-Hubble wavelengths ($a\lambda > H^{-1}$) \[46, 47, 49\]. In this way, at least at the short wavelengths relevant to laboratory physics, we recover the Born rule as a consequence of quantum relaxation.

We emphasise that, according to this reasoning, efficient quantum relaxation emerges only in the Schrödinger regime. It is only in this regime that a physical state of quantum equilibrium exists at all.

It is worth noting that this new approach definitively resolves a controversy about the origin of the Born rule in pilot-wave theory, which we mentioned at the end of Section 2.1. The argument for a dynamical origin remains valid, and is in fact strengthened by the observation that we will generally find nonequilibrium as we emerge from the deep quantum-gravity regime, with quantum relaxation to the Born rule taking place only afterwards. In contrast, the argument from typicality given by the Bohmian mechanics school – even in its usual circular form – can no longer even be formulated: there is no physical typicality measure $|\Psi|^2$ for a universe governed by quantum gravity.

### 4.4 Gravitational instability of the Born rule

We have argued that, at the fundamental level of quantum gravity, there is no Born rule and no such thing as a state of quantum equilibrium. Furthermore, as the very early universe emerges from the deep quantum-gravity regime, it will be in a state of quantum nonequilibrium as defined in the emergent Schrödinger approximation. And finally, once the Schrödinger approximation is established, quantum relaxation to the Born rule can take place in the usual way. We are then able to explain why we observe the Born rule today, even though (according to our interpretation) there is no such rule in the fundamental quantum-gravitational theory. A natural question then arises: once quantum equilibrium is reached, can subsequent gravitational effects drive a system away from quantum equilibrium? In other words, might quantum gravity render the Born rule unstable?

Consider, for example, a bouncing cosmology \[98\]. On some scenarios there is a pre-big-bang phase during which the universe is macroscopic, contracting, and behaving essentially classically. As the universe contracts it eventually approaches a phase in which quantum effects become important, resulting in a ‘bounce’ with conventional big-bang cosmology emerging on the other side as the universe expands. During the bounce quantum-gravitational effects could
be important, in which case the following scenario could arise [65]. During the contracting phase we have an effective Schrödinger approximation with matter fields obeying equations of the form (80) and (84). If the contracting phase has lasted a long time, we can reasonably expect the matter fields to have reached quantum equilibrium. As we approach the bounce, then, the Born rule should hold. However, once we are sufficiently close to the bounce the Schrödinger approximation may break down and we could enter the deep quantum-gravity regime. During that phase there will be no Born rule and no state of quantum equilibrium. We may then reasonably expect that, as we emerge from the bounce and back into a Schrödinger regime, the matter fields will be in a state of quantum nonequilibrium. Thus quantum-gravitational effects during a cosmological bounce might render the Born rule unstable.

We can envisage similar effects involving black holes. During the contracting phase of gravitational collapse there is a fairly well-understood regime described by quantum field theory on a classical spacetime background. In this regime the Born rule is usually assumed. But according to classical general relativity, the formation of a black hole entails the formation of a classical spacetime singularity, where it is expected that quantum-gravitational effects will be important (possibly preventing the formation of a true singularity). Close to the singularity we can then expect to find a breakdown of the Born rule and of the very idea of quantum equilibrium. The semiclassical mechanism discussed in refs. [36, 53, 54] could then come into play. This mechanism depends on the well-known entanglement between ingoing and outgoing field modes in the natural vacuum state (for quantised fields propagating on the background spacetime) [99]. In pilot-wave theory entanglement can provide a channel whereby information can propagate nonlocally [23]. Such effects are erased in equilibrium but could be relevant if nonequilibrium already exists for internal degrees of freedom behind the horizon. If the ingoing field modes interact (locally) with internal nonequilibrium degrees of freedom, then according to pilot-wave theory the outgoing field modes will evolve away from quantum equilibrium. In effect, entanglement enables the nonlocal propagation of quantum nonequilibrium from the interior to the exterior [53], as has been confirmed analytically and numerically for a simple field-theoretical model [54]. We then have a mechanism whereby information can escape from behind the classical event horizon – provided the black hole contains nonequilibrium degrees of freedom, presumably because of quantum-gravitational effects near the singularity. We may also mention the possible phenomenon of quantum tunnelling from a black hole to a white hole, which has been argued to be induced by quantum-gravitational effects [100]. As in a bouncing cosmology, we might expect some degree of quantum nonequilibrium to be generated during the black-to-white transition or bounce.

Finally, we may consider a more immediately tractable scenario, which will be developed in detail in the rest of this paper. We have argued that in the deep quantum-gravity regime there is no state of quantum equilibrium and that such a state emerges only in the Schrödinger approximation. This raises the question of what might happen in an intermediate regime with quantum-gravitational
corrections to the Schrödinger approximation. It seems reasonable to expect that such corrections could generate a small instability of the Born rule, whereby quantum nonequilibrium can be generated from a prior equilibrium state. We shall now show that this last expectation is in fact justified.

5 Gravitational corrections to the Schrödinger approximation

In this section we discuss quantum-gravitational corrections to the Schrödinger approximation. These take the form of small corrections to the effective Hamiltonian. The corrections arise from higher-order terms in the semiclassical expansion (72) of the Wheeler-DeWitt wave functional. Such corrections were first derived by Kiefer and Singh [68]. Similar results have been derived for quantum-cosmological models and have been applied extensively to inflationary cosmology [69–71]. Remarkably, the quantum-gravitational corrections to the effective Hamiltonian consist of both Hermitian and non-Hermitian terms. In standard quantum mechanics it is difficult to interpret the non-Hermitian terms as they appear to violate the conservation of probability. For this reason the non-Hermitian terms are usually ignored or dropped by fiat. However, as we shall see in Section 6, such terms have a clear interpretation in pilot-wave theory: in their presence probability is still conserved but the Born rule becomes unstable.

5.1 Corrections to the effective Schrödinger equation

Following ref. [68] we continue the analysis of Section 3.4.1 and consider higher orders in the semiclassical expansion (72) of the Wheeler-DeWitt equation for $\Psi[g_{ij}, \phi]$. At order $\mu^{-1}$ Kiefer and Singh obtain an equation for $S_2$:

$$G_{ijkl} \frac{\delta S_0}{\delta g_{ij}} \frac{\delta S_2}{\delta g_{kl}} + \frac{1}{2} G_{ijkl} \frac{\delta S_1}{\delta g_{ij}} \frac{\delta S_1}{\delta g_{kl}} - i \frac{1}{2} G_{ijkl} \frac{\delta^2 S_1}{\delta g_{ij} \delta g_{kl}} + \frac{1}{\sqrt{g}} \frac{\delta S_1}{\delta \phi} \frac{\delta S_2}{\delta \phi} - \frac{i}{2 \sqrt{g}} \frac{\delta^2 S_2}{\delta \phi^2} = 0$$

(106)

Using (78) to write $S_1$ in terms of $\psi^{(0)}$, and writing

$$S_2 = \sigma_2 [g_{ij}] + \eta [\phi, g_{ij}]$$

(107)

where $\sigma_2$ is chosen to satisfy the condition

$$G_{ijkl} \frac{\delta S_0}{\delta g_{ij}} \frac{\delta \sigma_2}{\delta g_{kl}} - \frac{1}{D^2} G_{ijkl} \frac{\delta D}{\delta g_{ij}} \frac{\delta D}{\delta g_{kl}} + \frac{1}{2D} G_{ijkl} \frac{\delta^2 D}{\delta g_{ij} \delta g_{kl}} = 0 ,$$

(108)

the corrected matter wave functional

$$\psi^{(1)} = \psi^{(0)} \exp(i\eta/\mu)$$

(109)

is found to obey a corrected Schrödinger equation
\[ i \frac{\partial \psi^{(1)}}{\partial t} = \int d^3 x \left[ \hat{\mathcal{H}}_\phi + \frac{1}{8 \mu} \frac{1}{\sqrt{gR}} \hat{\mathcal{H}}_\phi^2 + \frac{1}{8 \mu} \delta \left( \frac{\hat{\mathcal{H}}_\phi}{\sqrt{gR}} \right) \right] \psi^{(1)}, \quad (110) \]

where the convenient shorthand
\[ \frac{\delta}{\delta \tau} = 2G_{ijkl} \frac{\delta S_0}{\delta g_{ij}} \frac{\delta}{\delta g_{kl}} \quad (111) \]
denotes a many-fingered time derivative. Equation (110) is the main result of ref. [68].

Thus we have
\[ i \frac{\partial \psi^{(1)}}{\partial t} = \int d^3 x \left( \hat{\mathcal{H}}_\phi + \hat{\mathcal{H}}_a + i \hat{\mathcal{H}}_b \right) \psi^{(1)}, \quad (112) \]

where
\[ \hat{\mathcal{H}}_a = \frac{1}{8 \mu} \frac{1}{\sqrt{gR}} \hat{\mathcal{H}}_\phi^2 \quad (113) \]

and
\[ \hat{\mathcal{H}}_b = \frac{1}{8 \mu} \frac{\delta}{\delta \tau} \left( \frac{\hat{\mathcal{H}}_\phi}{\sqrt{gR}} \right) \quad (114) \]

are both Hermitian operators.

At this order the total effective Hamiltonian takes the form
\[ \hat{H} = \hat{H}_\phi + \hat{H}_a + i \hat{H}_b, \quad (115) \]

where
\[ \hat{H}_\phi = \int d^3 x \hat{\mathcal{H}}_\phi, \quad \hat{H}_a = \int d^3 x \hat{\mathcal{H}}_a, \quad \hat{H}_b = \int d^3 x \hat{\mathcal{H}}_b \quad (116) \]

are Hermitian operators. The Hamiltonian has a Hermitian correction \( \hat{H}_a \) together with a non-Hermitian correction \( i \hat{H}_b \).

Similar corrections have been derived and discussed for minisuperspace models of quantum cosmology [69–71]. As in the above discussion, it is found that the effective wave function of the matter field \( \phi \) obeys a Schrödinger-like equation with a corrected Hamiltonian of the form (115), where again \( \hat{H}_\phi \) is the field (or ‘matter’) Hamiltonian, the first gravitational correction \( \hat{H}_a \) is Hermitian, while the second gravitational correction \( i \hat{H}_b \) is non-Hermitian. Several studies have included the term \( \hat{H}_a \) in models of inflationary cosmology and have calculated its effect on the primordial power spectrum [69–71]. It is found that \( \hat{H}_a \) generates a correction to the spectrum at large scales, which is however too small to be observable. In these studies the unitarity-violating term \( i \hat{H}_b \) is neglected (see Section 9.3).

\[ ^{16}\text{As noted by Kiefer and Singh the correction terms in (110) turn out to be independent of the factor ordering chosen in the gravitational part of the Wheeler-DeWitt equation.} \]
The approximate ratio of the non-Hermitian correction $i\hat{H}_b$ to the Hermitian correction $\hat{H}_a$ can be easily estimated for an expanding cosmological background with scale factor $a$. The ratio is, schematically, of order

$$\frac{1}{H_\phi^2} \frac{d\hat{H}_\phi}{dt} \sim \frac{1}{H_\phi^2} \frac{d\hat{H}_\phi}{da} \sim H \frac{\dot{a}}{E},$$

where $H = \dot{a}/a$ is the Hubble parameter and $E$ is a typical energy scale for the matter field [68]. This ratio is exceedingly small during the late cosmological era, but it might be large in the very early universe where $H$ can be large for small $t$ (for example $H = 1/2t$ for a radiation-dominated expansion with $a \propto t^{1/2}$).

As we shall see in Section 11, the non-Hermitian term can be large in the final stages of black-hole evaporation.

5.2 Origin and status of non-Hermitian corrections

The origin of the non-Hermitian or unitarity-violating correction $i\hat{H}_b$ may be understood as follows. The full Wheeler-DeWitt equation implies the conservation of a Klein-Gordon-like current and not of a Schrödinger-like current (as discussed in Section 3.2.1) [1]. Expanding the Klein-Gordon-like continuity equation implies a Schrödinger-like continuity equation that is modified by small gravitational corrections in such a way that the usual Schrödinger norm is no longer conserved ($\frac{d}{dt} \int D\phi \psi^* \psi \neq 0$) [101]. Thus mathematically the appearance of the non-Hermitian correction $i\hat{H}_b$ is readily understandable. Even so, its physical status has been disputed, because in its presence the standard probabilistic interpretation of quantum mechanics breaks down. However, as we will see in Section 6, in pilot-wave theory a non-Hermitian Hamiltonian does not generate any inconsistency: probability is still fully conserved but the Born rule is no longer stable.

We have seen that, in pilot-wave quantum gravity, there is no physical Born-rule state in the deep quantum-gravity regime, and the Born rule emerges only in a semiclassical Schrödinger approximation. It then seems unsurprising to find that quantum-gravitational corrections to the Schrödinger equation can yield an intermediate regime in which the Born rule suffers a small instability. In fact it would be reasonable to expect such a regime on general physical grounds. Thus in pilot-wave theory there seems to be no reason to be alarmed by the appearance of small non-Hermitian corrections in the effective Hamiltonian.

From the viewpoint of standard quantum mechanics, however, the non-Hermitian terms appear unphysical and might be regarded as artifacts. Similar terms are found if one carries out an analogous approximation scheme in scalar QED: one may derive an approximate Schrödinger equation for a charged scalar field in a background electromagnetic field, and corrections to that equation contain similar non-Hermitian terms [68]. But the analogy with scalar QED is misleading. In the latter theory there is a conserved Schrödinger-like current in the full configuration space, and so at the fundamental level there is a well-defined Born rule from which one may derive a marginal or conditional
distribution for the matter field. In contrast, in the gravitational case there is no underlying conserved Schrödinger current. On the contrary, the Wheeler-DeWitt equation possesses a conserved Klein-Gordon-like current whose density is not positive-definite. There appears to be no well-defined Born-rule distribution at the fundamental level, and therefore it is no longer obvious that a small violation of unitarity in the emergent Schrödinger regime is necessarily a mathematical artifact.

Some authors working with standard quantum mechanics have proposed to remove the non-Hermitian terms by an appropriate redefinition of the effective wave function. To see the general method, consider a Schrödinger equation

\[ i\frac{\partial \psi}{\partial t} = \hat{H}\psi \]

with Hamiltonian \( \hat{H} = \hat{H}_0 + \Delta \hat{H} \), where the correction term \( \Delta \hat{H} \) may or may not be Hermitian. The term \( \Delta \hat{H} \) can be eliminated by redefining the original wave function \( \psi(q,t) \) as

\[ \psi'(q,t) = \exp \left( i \int_0^t d\tau \Delta \hat{H}(\tau) \right) \psi(q,t) \]

(118)

(where we assume for simplicity that \( \Delta \hat{H}(t) \) commutes with \( \hat{H}_0 \) and with itself at different times). The new wave function \( \psi' \) then satisfies a modified Schrödinger equation \( i\frac{\partial \psi'}{\partial t} = \hat{H}_0\psi' \) with Hamiltonian \( \hat{H} = \hat{H}_0 \) only. Normally this procedure would be regarded as illegitimate: the new wave function would not be the correct physical wave function, and in particular \( |\psi'|^2 \) would not be the correct measured probability density for \( q \). One cannot simply remove terms from the Hamiltonian by redefining the wave function.

Even so a similar procedure has been considered by some authors in order to remove the unwanted non-Hermitian terms. In a quantum-cosmological model Bini et al. [70] show how the term \( i\hat{H}_b \) can be eliminated by redefining the wave function in the general form (118). However, as noted by the authors, it is not clear which of the two wave functions (the original or the new) should be regarded as physically correct in the sense associated with the Born rule. More recently, a general scheme for eliminating the non-Hermitian terms has been developed by Kiefer and Wichmann [102]. Their starting point is the observation that the WKB ansatz \( \Psi[g_{ij}, \phi] \approx \Psi_{\text{WKB}}[g_{ij}] \psi[\phi, g_{ij}] \) (equation (57)) is invariant under a rescaling

\[ \Psi_{\text{WKB}} \to e^A \Psi_{\text{WKB}} , \quad \psi \to e^{-A} \psi , \]

(119)

where \( A[g_{ij}] \) is an arbitrary complex functional analogous to a choice of ‘gauge’. Different choices of \( A \) will affect the time evolution of \( \psi \). Kiefer and Wichmann obtain a general effective Schrödinger equation for \( \psi \) whose Hamiltonian \( \hat{H} \) is defined only implicitly. As before non-Hermitian terms make an appearance. These cannot be eliminated by a simple redefinition of the form (118), as the Hamiltonian is not known explicitly. However, solving the equations iteratively, an appropriate redefinition of the form (118) can be imposed order by order in such a way that at each order the redefined wave functions \( \psi'_0, \psi'_1, \psi'_2, \ldots \) evolve unitarily (with respect to the usual Schrödinger norm). The authors draw an analogy with the Born-Oppenheimer ansatz for the Schrödinger equation as
employed in molecular physics.\textsuperscript{17} But again, as the authors themselves note, the analogy is limited because the Wheeler-DeWitt equation is fundamentally different from the Schrödinger equation (possessing in particular a quite different form of conserved current).

At this point we must address a difficult question. Does the method of ref.\textsuperscript{102} correctly identify the true physical wave functions $\psi'$ (with purely Hermitian Hamiltonians), or are the wave function redefinitions an elaborate means of disguising the unwanted but physical non-Hermitian terms? The correct answer is not known for certain. Our experience with time-dependent wave functions on a classical spacetime background tells us that the Hamiltonian should be Hermitian, but that is in conditions where quantum-gravitational effects can be neglected. Perhaps the non-Hermitian terms are a natural consequence of quantum gravity (where as we shall see those terms can be easily understood in pilot-wave theory) and the modified expansions with wave function redefinitions are really an artificial means of disguising genuine physical effects. At issue here is exactly how to identify the familiar Schrödinger-regime wave functions as emergent features of the semiclassical regime. It may be that the question can be answered only by experiment. Without some empirical input it is difficult to know exactly how the wave functions we observe in the laboratory are to be identified from the quantum-gravitational formalism.

Of course in standard quantum mechanics there is good reason to redefine the effective wave functions so as to eliminate the non-Hermitian terms. But as we shall see that motivation is lost in pilot-wave theory where such terms are perfectly consistent with the conservation of probability. Furthermore, again, given our argument that there is no fundamental Born rule in quantum gravity, it appears quite plausible on general physical grounds that there will be an intermediate regime with a small instability of the Born rule. Ultimately, however, the only way to resolve the dispute will probably be through experiment. In the rest of this paper we will explore the consequences of the non-Hermitian terms in pilot-wave theory, with a view to possible experimental tests.

5.3 Absence of corrections to the de Broglie velocity

We now continue the analysis of Section 3.4.2 and consider the next order in the semiclassical expansion \textsuperscript{72} of the de Broglie guidance equation \textsuperscript{22} for the matter field. Noting that the term $\sigma_2$ in \textsuperscript{107} satisfies $\delta\sigma_2/\delta\phi = 0$, the de Broglie velocity \textsuperscript{83} takes the form

$$\frac{\partial \phi}{\partial t} = \frac{N}{\sqrt{g}} \frac{\delta}{\delta\phi} \left( \text{Re} \, S_1 + \mu^{-1} \text{Re} \, \eta + ... \right). \quad (120)$$

Equation \textsuperscript{109} defines a corrected wave function $\psi^{(1)} = \psi^{(0)} \exp(i\eta/\mu)$, satisfying a corrected Schrödinger equation \textsuperscript{112}. The total phase of the wave function now includes a correction equal to $\mu^{-1} \text{Re} \, \eta$ (in addition to the uncorrected phase $\text{Re} \, S_1$ of $\psi^{(0)}$). Thus the corrected wave function $\psi^{(1)}$ has a total

\textsuperscript{17}In this context see also refs. \textsuperscript{103} \textsuperscript{104}. 43
phase

\[ \text{Im} \ln \psi^{(1)} = \text{Re} S_1 + \mu^{-1} \text{Re} \eta . \]  

(121)

The corrected canonical de Broglie velocity (120) can then be written as

\[ \left( \frac{\partial \phi}{\partial t} \right)^{(1)} = \frac{N}{\sqrt{g}} \delta S^{(1)} \delta \phi , \]  

(122)

where \( S^{(1)} = \text{Im} \ln \psi^{(1)} \) is the phase of \( \psi^{(1)} \).

We arrive at an important conclusion. Despite the presence of the non-Hermitian term in the Schrödinger equation (112), the de Broglie velocity of the matter field is still given by the standard guidance equation (22) with the phase \( S \) equal to the corrected phase \( S^{(1)} = \text{Im} \ln \psi^{(1)} \) of the effective wave function \( \psi^{(1)} \). This is just the usual guidance equation that we would normally associate with the Hermitian part of the Hamiltonian. In other words: while the non-Hermitian term in (112) affects the time evolution of \( \psi^{(1)} \) (and hence indirectly affects the trajectories), the presence of the non-Hermitian term does not change the form of the guidance equation. As we shall now see, this implies that the Born rule for the matter field is unstable.

Thus, if the semiclassical expansion is to be trusted, the fundamental equations of pilot-wave quantum gravity imply the instability of the Born rule – as described by quantum-gravitational corrections to the Schrödinger approximation.

6 Pilot-wave theory with an unstable Born rule

By performing a semiclassical expansion of the Wheeler-DeWitt equation together with the associated de Broglie guidance equation, we have derived an effective pilot-wave theory of a matter field on a classical background. The effective Schrödinger equation (112) contains a non-Hermitian contribution to the Hamiltonian, while the effective de Broglie guidance equation (122) continues to take the standard form (associated with only the Hermitian part). Before proceeding it will be helpful to understand the implications of these results in general terms.

6.1 Pilot-wave dynamics with a non-Hermitian Hamiltonian

Let us consider how pilot-wave theory can be applied to a general system with a Hamiltonian

\[ \hat{H} = \hat{H}_1 + i\hat{H}_2 , \]  

(123)
where $\hat{H}_1$ and $\hat{H}_2$ are both Hermitian\footnote{Any operator $\hat{H}$ can of course be decomposed into a sum of a Hermitian part $\frac{1}{2}(\hat{H} + \hat{H}^\dagger)$ and an anti-Hermitian part $\frac{1}{2}(\hat{H} - \hat{H}^\dagger)$.}. The wave function $\psi(q,t)$ will evolve according to the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = (\hat{H}_1 + i\hat{H}_2)\psi \ .$$

(124)

This implies a continuity equation

$$\frac{\partial |\psi|^2}{\partial t} + \partial_q \cdot j_1 = s \ ,$$

(125)

where $j_1$ is the standard current associated with $\hat{H}_1$ (satisfying $\partial_q j_1 = 2 \text{Re} \left( i\psi^* \hat{H}_1 \psi \right)$) and

$$s = 2 \text{Re} \left( \psi^* \hat{H}_2 \psi \right)$$

(126)

is an effective ‘source’ term.

Integrating (125) over configuration space we find

$$\frac{d}{dt} \int dq \ |\psi|^2 = \int dq \ s = 2 \langle \hat{H}_2 \rangle \ ,$$

(127)

where the standard quantum expectation value $\langle \hat{H}_2 \rangle \equiv \int dq \ (\psi^* \hat{H}_2 \psi)$ is real for Hermitian $\hat{H}_2$ and we assume as usual that $j_1$ vanishes at infinity.

We can then define a de Broglie velocity field

$$v = \frac{j_1}{|\psi|^2} \ .$$

(128)

This amounts to assuming that the velocity is generated by the Hermitian part of the Hamiltonian only. Note, however, that $j_1$ is a functional of $\psi$ whose time evolution is determined via (124) by the total Hamiltonian $\hat{H}$ (including the non-Hermitian part).

Equations (124) and (128) define a pilot-wave dynamics for an individual system whose Hamiltonian has a non-Hermitian part. Remarkably, as we have seen, a dynamics of precisely this type emerges naturally in an appropriate limit of quantum gravity.

### 6.2 Instability of the Born rule

For an ensemble of systems with the same wave function $\psi$, the probability density $\rho(q,t)$ will satisfy

$$\frac{\partial \rho}{\partial t} + \partial_q \cdot (\rho v) = 0$$

(129)

(since each system follows the velocity field $v$). Equations (124), (125) and (129) then define a pilot-wave theory of an ensemble with an unstable Born rule.
To see this, rewrite (125) as
\[ \frac{\partial |\psi|^2}{\partial t} + \partial_q \cdot (|\psi|^2 v) = s . \] (130)

From (129) and (130) it follows that the ratio
\[ f \equiv \rho / |\psi|^2 \] (131)
is not conserved along trajectories. Instead we find
\[ \frac{df}{dt} = - \frac{s}{|\psi|^2} f \] (132)
(20)

(20)

where again \( d/dt = \partial / \partial t + v \cdot \partial_q \) is the time derivative along a trajectory. An initial distribution with \( f = 1 \) everywhere will generally evolve into a final distribution with \( f \neq 1 \). In other words, an initial Born-rule distribution \( \rho = |\psi|^2 \) at \( t = t_i \) generally evolves into a non-Born-rule distribution \( \rho \neq |\psi|^2 \) at \( t > t_i \).

Thus (124), (128) and (129) indeed define a pilot-wave dynamics of ensembles in which the Born-rule distribution \( \rho = |\psi|^2 \) is unstable.

6.3 Timescale for quantum instability. Condition for the growth of quantum nonequilibrium

In such a theory quantum nonequilibrium is created on a timescale \( \tau_{\text{noneq}} \) which can be estimated from the rate of change of the \( H \)-function (11). Taking the time derivative of (11) and using (129) and (130) it is easy to show that
\[ \frac{dH}{dt} = - \int dq \, \frac{\rho}{|\psi|^2} s . \] (133)

If we are close to equilibrium (\( \rho \approx |\psi|^2 \)) we have
\[ \frac{dH}{dt} \approx - \int dq \, s = -2 \left\langle \hat{H}_2 \right\rangle . \] (134)

We can then define \( \tau_{\text{noneq}} \) as the time required for \( H \) to change by a factor of order unity,
\[ \tau_{\text{noneq}} \left| \frac{dH}{dt} \right| \approx 1 , \] (135)
yielding the estimate
\[ \tau_{\text{noneq}} \approx \frac{1}{2 \left\langle \hat{H}_2 \right\rangle} . \] (136)

19 If \( \rho \) and \( |\psi|^2 \) obey the same continuity equation the exact \( H \) is constant in time while the coarse-grained value tends to decrease (if there is no initial fine-grained structure). But here the continuity equation (130) for \( |\psi|^2 \) contains a source term \( s \) and so the exact \( H \) is no longer constant.
We might instead estimate $\tau_{\text{noneq}}$ from the rate of change of the squared-norm $\int dq \ |\psi\|^2$ by defining

$$\frac{1}{\tau_{\text{noneq}}} = \frac{1}{\int dq \ |\psi|^2} \left| \frac{d}{dt} \int dq \ |\psi|^2 \right| = \left| \frac{d}{dt} \ln \left( \int dq \ |\psi|^2 \right) \right| \ .$$

(137)

Close to equilibrium we have $\int dq \ |\psi|^2 \approx 1$ and so from (127) we find the same estimate (136).

We saw in Section 4.1 that, for general unnormalised wave functions,

$$H \geq -\ln \left( \int dq \ |\psi|^2 \right) .$$

It is then not surprising that the definitions (135) and (137) yield essentially the same timescales.

It must be noted that, for realistic systems, there will usually be two competing effects: the creation of quantum nonequilibrium on a timescale $\tau_{\text{noneq}}$ and the relaxation of quantum nonequilibrium on a timescale $\tau_{\text{relax}}$. The quantum-gravitationally generated violations of the Born rule can build up over time only if the usual quantum relaxation process is relatively negligible, that is, only if

$$\tau_{\text{relax}} > \tau_{\text{noneq}} .$$

(138)

In other words, quantum nonequilibrium must be generated faster than relaxation can remove it. As we will see the condition (138) can be realistically satisfied for a scalar field during inflation (Section 8). Whether or not it can be satisfied in other circumstances (such as black-hole evaporation, Section 11) will depend on the results of more detailed modelling including quantum relaxation.

7 Quantum cosmology and the Born rule

Brizuela, Kiefer and Krämer [71] have developed a semiclassical expansion of the Wheeler-DeWitt equation for a minisuperspace quantum-cosmological model and have used it to derive quantum-gravitational corrections to the Schrödinger regime for perturbations on a classical background spacetime. The method is similar to that presented in Sections 3.4.1 and 5.1 for the full superspace, though there are some differences. We begin by summarising their model and results, adapted to pilot-wave theory, where again both Hermitian and non-Hermitian corrections appear in the effective Schrödinger equation. We then study the de Broglie velocity and show that it remains uncorrected at the relevant order. It is then instructive to consider again why the theory has no fundamental equilibrium state, and to reconsider the emergence and instability of the Born rule in this quantum-cosmological context. Finally, we write down a simplified model for the slow-roll limit, with a view to enabling more tractable calculations.
7.1 Pilot-wave quantum cosmology

In the model of ref. [71] both scalar and tensor perturbations propagate on a background flat Friedmann-Lemaître universe, where the background is also treated quantum-mechanically. For our purposes we can ignore the tensor contributions. At the classical level, in terms of conformal time $\eta$ the background line element is

$$d\tau^2 = a^2(d\eta^2 - dx^2),$$  \hspace{1cm} (139)

where $\eta$ is related to the usual cosmic time $t$ by $d\eta/dt = a^{-1}$. We employ the common convention whereby $\lambda$ denotes a comoving wavelength at some reference time $\eta_0$ when $a_0 = 1$. The physical wavelength of a field mode at time $\eta$ is then given by $\lambda_{\text{phys}}(\eta) = a(\eta)\lambda$.

The background contains a homogeneous scalar field $\phi$ with a potential $V(\phi)$ that drives the expansion (here $\phi$ is the homogenous part of a massive and minimally-coupled inflaton field). Using primes to denote derivatives with respect to $\eta$, the classical action for the background is taken to be

$$S_b = \frac{1}{2} \int d\eta \mathcal{L}^3 \left[ -\frac{3}{4\pi G} \left( a'' - \frac{\phi'}{a} \phi' + a^2 \frac{dV}{d\phi} \right) \right],$$  \hspace{1cm} (140)

where $\mathcal{L}$ is an arbitrary length scale associated with the spatial integration (to be regarded as an infrared cutoff) [105]. The action (140) implies the standard equations of motion

$$\frac{3}{4\pi G} \frac{a''}{a} + (\phi')^2 - 4a^2 V = 0 ,$$

$$\phi'' + 2 a \frac{\phi'}{a} \frac{dV}{d\phi} = 0 ,$$

which in terms of $t$ take the more familiar forms

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left( \rho + 3p \right) ,$$

$$\frac{\ddot{\phi}}{a} + 3 \frac{\dot{a}}{a} \frac{\dot{\phi}}{a} + \frac{dV}{d\phi} = 0 ,$$

where

$$\rho = \frac{1}{2} \dot{\phi}^2 + V$$  \hspace{1cm} (141)

is the energy density and $p = \rho - 2V$ is the pressure.

Scalar perturbations of the background metric are usually described by scalar functions $A, B, \psi$ and $E$:

$$d\tau^2 = a^2 \left[ (1 - 2A)d\eta^2 - 2(\partial_i B)dx^i d\eta - ((1 - 2\psi)\delta_{ij} + 2\partial_i \partial_j E) dx^i dx^j \right].$$

It is convenient to combine the inflaton perturbation $\varphi \equiv \delta \phi$ with $A, B, \psi, E$ to form the Mukhanov-Sasaki variable [106]

$$v = a \left( \tilde{\varphi} + \varphi \frac{\Phi_B}{H} \right) ,$$  \hspace{1cm} (142)
where

\[ \tilde{\varphi} \equiv \varphi + \phi'(B - E') , \]  
\[ \Phi_B \equiv A + \frac{1}{a} [a(B - E')]' \]  

and

\[ \mathcal{H} \equiv a'/a = Ha \]  

(where \( H = \dot{a}/a \) is the usual Hubble parameter).

The action \( S_p \) of the perturbations is found by expanding the total (including the Einstein-Hilbert) action around the background. In terms of \( \upsilon \),

\[ S_p = \frac{1}{2} \int d\eta d^3x \left[ (\upsilon')^2 - \delta^{ij} \partial_i \upsilon \partial_j \upsilon + \frac{z''}{z} \upsilon^2 \right] , \]  

where

\[ z = a \sqrt{\epsilon} \]  

and

\[ \epsilon = - \frac{\dot{H}}{H^2} = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} \]  

(the first slow-roll parameter). Brizuela et al. work with a Fourier transform

\[ \upsilon(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3/2} \upsilon_k(\eta) e^{i\mathbf{k} \cdot \mathbf{x}} . \]  

The wave vector \( \mathbf{k} \) is then defined so that the magnitude \( k \) is equal to the inverse of a wavelength (without the usual factor \( 2\pi \)). Because \( \upsilon \) is real, \( \upsilon_k^* = \upsilon_{-\mathbf{k}} \). In terms of the \( \upsilon_k \)'s we then have

\[ S_p = \frac{1}{2} \int d\eta \int d^3k \left[ \upsilon_k' \upsilon_k'^* + \upsilon_k \upsilon_k^* \left( \frac{z''}{z} - k^2 \right) \right] . \]  

At this point the modes are discretised by the replacement

\[ \int d^3k \rightarrow \frac{1}{2\pi^3} \sum_k . \]  

In addition, for simplicity, the \( \upsilon_k \)'s are treated as real (a proper procedure defines a new set of real variables but gives the same results). The action of the perturbations is then

\[ S_p = \frac{1}{2} \int d\eta \frac{1}{2\pi^3} \sum_k \left[ (\upsilon_k')^2 + \upsilon_k^2 \left( \frac{z''}{z} - k^2 \right) \right] . \]  

Following ref. [105], Brizuela et al. eliminate the appearance of \( \mathcal{L} \) in the equations by appropriate rescalings:

\[ a_{\text{new}} = a_{\text{old}} \mathcal{L} , \quad \eta_{\text{new}} = \eta_{\text{old}} / \mathcal{L} , \quad \upsilon_{\text{new}} = \upsilon_{\text{old}} / \mathcal{L}^2 , \quad k_{\text{new}} = k_{\text{old}} \mathcal{L} . \]  

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In terms of these rescaled variables, the background has a Lagrangian

$$ L_b = \frac{1}{2} \left[ -\frac{3}{4\pi G} (a')^2 + a^2 (\phi')^2 - 2a^4 \mathcal{V} \right] $$

(where $S_b = \int d\eta \, L_b$) and canonical momenta

$$ \pi_a = -\frac{3}{4\pi G} a' \, , \, \pi_\phi = a^2 \phi' \, , \quad (153) $$

while the perturbations have a Lagrangian

$$ L_p = \frac{1}{2} \sum_k \left[ (\nu'_k)^2 + (\nu_k)^2 \left( \frac{z''}{z} - k^2 \right) \right] \quad (154) $$

(with $S_p = \int d\eta \, L_p$) and canonical momenta

$$ \pi_k = \nu'_k \, . \quad (155) $$

The Hamiltonian of the background is then

$$ H_b = -\frac{2\pi G}{3} \pi_a^2 + \frac{1}{2a^2} \pi_\phi^2 + a^4 \mathcal{V} \, , \quad (156) $$

(noting the crucial sign difference between the terms in $\pi_a^2$ and $\pi_\phi^2$), while the Hamiltonian of the perturbations is

$$ H_p = \frac{1}{2} \sum_k \left[ \pi_k^2 + \nu_k^2 \left( k^2 - \frac{z''}{z} \right) \right] \, . \quad (157) $$

The system is then quantised. We have a Wheeler-DeWitt equation

$$ \left( \hat{H}_b + \hat{H}_p \right) \Psi = 0 $$

for a wave functional $\Psi = \Psi(a, \phi, \{\nu_k\})$. The sign difference between the terms in $\pi_a^2$ and $\pi_\phi^2$ in the background Hamiltonian $\hat{H}_b$ signals the Klein-Gordon-like character of the wave equation.

Taking a product ansatz \[69\]

$$ \Psi = \Psi_0(a, \phi) \prod_k \Psi_k(a, \phi, \nu_k) \, , \quad (158) $$

where $\hat{H}_b \Psi_0 = 0$ defines a wave functional $\Psi_0$ for the background, Brizuela et al. define a wave function

$$ \Psi_k = \Psi_k(a, \phi, \nu_k) = \Psi_0(a, \phi) \tilde{\Psi}_k(a, \phi, \nu_k) \quad (159) $$

for each mode $k$, where $\Psi_k$ obeys a Wheeler-DeWitt equation

$$ \frac{1}{2m_p^2} \frac{1}{a} \frac{\partial}{\partial a} \left( a \frac{\partial \Psi_k}{\partial a} \right) - \frac{1}{2a^2} \frac{\partial^2 \Psi_k}{\partial \phi^2} + \frac{1}{2} V \Psi_k - \frac{1}{2} \omega_k^2 \nu_k \Psi_k = 0 \quad (160) $$

50
(with appropriate factor ordering), where

\[ m_P^2 = \frac{3}{4\pi G} \]  \hspace{1cm} (161)

is a rescaled Planck mass,

\[ V = \frac{2}{m_P^2} a^4 \nu \]  \hspace{1cm} (162)

is an auxiliary potential, and

\[ \omega_k^2 = k^2 - \frac{z''}{z} \]  \hspace{1cm} (163)

is a time-dependent frequency.

It is now straightforward to define a corresponding pilot-wave theory. We can identify de Broglie velocities for the evolving variables \( a(\eta), \phi(\eta) \) and \( \nu_k(\eta) \) by setting

\[ \pi_a = \frac{\partial S_k}{\partial a}, \quad \pi_\phi = \frac{\partial S_k}{\partial \phi}, \quad \pi_k = \frac{\partial S_k}{\partial \nu_k}, \]  \hspace{1cm} (164)

where \( S_k = \text{Im} \ln \Psi_k \) is the phase of \( \Psi_k \). From the classical expressions (153) and (155) we then have the de Broglie guidance equations

\[ \frac{da}{d\eta} = -\frac{1}{m_P^2} \frac{\partial S_k}{\partial a}, \quad \frac{d\phi}{d\eta} = \frac{1}{a^2} \frac{\partial S_k}{\partial \phi} \]  \hspace{1cm} (165)

for the background and

\[ \frac{d\nu_k}{d\eta} = \frac{\partial S_k}{\partial \nu_k} \]  \hspace{1cm} (166)

for the perturbations. Together with the wave equation (160) these define a pilot-wave dynamics for the variables \( a, \phi \) and \( \nu_k \).

It is instructive to check that the expressions (165) and (166) agree with the velocities obtained from the polar decomposition \( \Psi_k = |\Psi_k|e^{iS_k} \) of the wave equation (160). This is done by inserting \( \Psi_k = |\Psi_k|e^{iS_k} \) into (160) and taking real and imaginary parts. The real part can be written as a modified Hamilton-Jacobi equation

\[ -\frac{1}{2m_P^2} \left( \frac{\partial S_k}{\partial a} \right)^2 + \frac{1}{2a^2} \left( \frac{\partial S_k}{\partial \phi} \right)^2 + \frac{1}{2} m_P^2 V + \frac{1}{2} \left( \frac{\partial S_k}{\partial \nu_k} \right)^2 + \frac{1}{2} \omega_k^2 \nu_k^2 + Q = 0 \]

where

\[ Q = \frac{1}{2m_P^2} \frac{1}{|\Psi_k|} \frac{1}{a} \frac{\partial}{\partial a} \left( a \frac{|\Psi_k|}{|\Psi_k|} \right) - \frac{1}{2a^2} \frac{1}{|\Psi_k|} \frac{\partial^2 |\Psi_k|}{\partial \phi^2} - \frac{1}{2} \frac{1}{|\Psi_k|} \frac{\partial^2 |\Psi_k|}{\partial \nu_k^2} \]

is the quantum potential (terms depending on \( |\Psi_k| \)), while the imaginary part can be written in the form of what we call a pseudo-continuity equation

\[ \frac{\partial}{\partial a} \left( a |\Psi_k|^2 a' \right) + \frac{\partial}{\partial \phi} \left( a |\Psi_k|^2 \phi' \right) + \frac{\partial}{\partial \nu_k} \left( a |\Psi_k|^2 \nu_k' \right) = 0 \]  \hspace{1cm} (167)
for a density \( a |\Psi_k|^2 \) with velocities \( a', \phi', \upsilon'_k \) given by the expressions \( (165) \) and \( (166) \) obtained from the canonical momenta. The measure

\[
d\mu = |\Psi_k|^2 a da d\phi d\upsilon_k
\]  

is preserved by the de Broglie velocity field defined by \( (165) \) and \( (166) \).

In pilot-wave theory we can consider the time evolution of a general ensemble, with probability density \( P_k(a, \phi, \upsilon_k, \eta) \) (defined with respect to \( da d\phi d\upsilon_k \)), and with each system guided by the same wave function \( \Psi_k \). By definition this will satisfy the continuity equation

\[
\frac{\partial P_k}{\partial \eta} + \frac{\partial}{\partial a} (P_k a') + \frac{\partial}{\partial \phi} (P_k \phi') + \frac{\partial}{\partial \upsilon_k} (P_k \upsilon'_k) = 0
\]  

(169)

with velocities \( a', \phi', \upsilon'_k \) again given by \( (165) \) and \( (166) \). This is the same as equation \( (167) \) satisfied by the density \( a |\Psi_k|^2 \) (which has no explicit \( \eta \) dependence). However, as we shall discuss in Section 7.4, there is no equilibrium or Born-rule state \( P_k = a |\Psi_k|^2 \) since \( a |\Psi_k|^2 \) is non-normalisable.

For completeness we note that the model may of course be expressed in terms of standard cosmological time \( t \) (where \( dt = ad\eta \)). Since \( a' = \dot{a}, \phi' = a\dot{\phi}, \) and \( \upsilon'_k = a\dot{\upsilon}_k \), \( (167) \) can be rewritten as a pseudo-continuity equation

\[
\frac{\partial}{\partial a} \left( a^2 |\Psi_k|^2 \dot{a} \right) + \frac{\partial}{\partial \phi} \left( a^2 |\Psi_k|^2 \dot{\phi} \right) + \frac{\partial}{\partial \upsilon_k} \left( a^2 |\Psi_k|^2 \dot{\upsilon}_k \right) = 0
\]  

(170)

for a density \( a^2 |\Psi_k|^2 \), with a preserved measure

\[
d\mu = |\Psi_k|^2 a^2 da d\phi d\upsilon_k
\]  

(171)

and with velocities

\[
\dot{a} = -\frac{1}{m_p} \frac{1}{a} \frac{\partial S_k}{\partial a}, \quad \dot{\phi} = \frac{1}{a^2} \frac{\partial S_k}{\partial \phi}, \quad \dot{\upsilon}_k = \frac{1}{a} \frac{\partial S_k}{\partial \upsilon_k}
\]  

(172)

(where \( \dot{a} \) and \( \dot{\phi} \) are as originally given by Vink \[55\]). A general ensemble with probability density \( P_k(a, \phi, \upsilon_k, t) \) (defined with respect to \( da d\phi d\upsilon_k \)) will satisfy the continuity equation

\[
\frac{\partial P_k}{\partial t} + \frac{\partial}{\partial a} (P_k \dot{a}) + \frac{\partial}{\partial \phi} (P_k \dot{\phi}) + \frac{\partial}{\partial \upsilon_k} (P_k \dot{\upsilon}_k) = 0
\]  

with \( \dot{a}, \dot{\phi}, \dot{\upsilon}_k \) given by \( (172) \). Again, this is the same as equation \( (170) \) satisfied by \( a^2 |\Psi_k|^2 \), but there is no equilibrium state \( P_k = a^2 |\Psi_k|^2 \) since \( a^2 |\Psi_k|^2 \) is non-normalisable.

In standard quantum theory equation \( (167) \) (or \( (170) \)) would be taken to imply the Born-rule measure \( (168) \) (or \( (171) \)) as a probability measure on the minisuperspace with variables \((a, \phi, \upsilon_k)\). We would then encounter the difficulty that the putative ‘probability’ measure is in general non-normalisable, owing to the Klein-Gordon-like character of the wave equation \( (160) \). As we will discuss in Section 7.4, in pilot-wave theory it is clear that there is in fact no such physical Born-rule probability measure.
7.2 Semiclassical expansion

It is sometimes convenient to write
\[ \alpha = \ln a , \]  
\[ (173) \]
in terms of which the wave equation \[ (160) \] reads
\[ \frac{1}{2m_p^2} e^{-2\alpha} \frac{\partial^2 \Psi_k}{\partial \alpha^2} - \frac{1}{2} e^{-2\alpha} \frac{\partial^2 \Psi_k}{\partial \phi^2} + \frac{1}{2} m_p^2 V \Psi_k - \frac{1}{2} \frac{\partial^2 \Psi_k}{\partial v_k^2} + \frac{1}{2} \omega_k^2 v_k^2 \Psi_k = 0 . \]  
\[ (174) \]
Brizuela et al. \[ 71 \] rewrite this in the compact form
\[ - \frac{1}{2m_p^2} G_{AB} \frac{\partial^2 \Psi_k}{\partial q_A \partial q_B} + \frac{1}{2} m_p^2 V \Psi_k - \frac{1}{2} \frac{\partial^2 \Psi_k}{\partial v_k^2} + \frac{1}{2} \omega_k^2 v_k^2 \Psi_k = 0 , \]  
\[ (175) \]
where \( \Psi_k = \Psi_k(q_A, v_k) \) and the indices \( A, B \) take values 0, 1, with
\[ G_{AB} = \text{diag}(-e^{-2\alpha}, e^{-2\alpha}) \]  
\[ (176) \]
and
\[ (q_0, q_1) = (\alpha, m_p^{-1}\phi) . \]  
\[ (177) \]
The Klein-Gordon-like character of the wave equation is now more explicit.

Taking \[ (175) \] as a starting point, Brizuela et al. proceed along the lines pioneered by Kiefer and Singh \[ 68 \]. We summarise their method, since some of the details will be required in what follows.

The method begins by solving \[ (175) \] with a WKB-type or semiclassical expansion
\[ \Psi_k = \exp \left[ i \left( m_p^2 S_0 + m_p^0 S_1 + m_p^{-2} S_2 + \ldots \right) \right] \]  
\[ (178) \]
in powers of \( m_p^2 \). This is inserted into \[ (175) \], terms with the same power of \( m_p \) are collected and in each case the sum is set equal to zero.

The highest order, \( m_p^4 \), yields
\[ \partial S_0 / \partial v_k = 0 , \]  
\[ (179) \]
which tells us that the background part of the wave function does not depend on the perturbations.

The next order, \( m_p^3 \), yields a Hamilton-Jacobi equation for the background:
\[ G_{AB} \frac{\partial S_0}{\partial q_A} \frac{\partial S_0}{\partial q_B} + V(q_A) = 0 . \]  
\[ (180) \]

At order \( m_p^0 \), Brizuela et al. find
\[ 2G_{AB} \frac{\partial S_0}{\partial q_A} \frac{\partial S_1}{\partial q_B} - iG_{AB} \frac{\partial^2 S_0}{\partial q_A \partial q_B} + \left( \frac{\partial S_1}{\partial v_k} \right)^2 - i \frac{\partial^2 S_1}{\partial v_k^2} + \omega_k^2 v_k^2 = 0 , \]  
\[ (181) \]
and define an uncorrected wave function
\[ \psi_k^{(0)} = \psi_k^{(0)}(q_A, v_k) = \gamma(q_A) e^{i S_1(q_A, v_k)} , \]  
\[ (182) \]
where $\gamma$ satisfies
\[ G_{AB} \frac{\partial}{\partial q_A} \left( \frac{1}{2\gamma^2} \frac{\partial S_0}{\partial q_B} \right) = 0. \] (183)

Introducing a conformal WKB time,
\[ \frac{\partial}{\partial \eta} = G_{AB} \frac{\partial S_0}{\partial q_A} \frac{\partial}{\partial q_B}, \] (184)
the equation (181) for $S_1$ can be rewritten as a Schrödinger equation for $\psi_k^{(0)}$:
\[ i \frac{\partial \psi_k^{(0)}}{\partial \eta} = \hat{H}_k \psi_k^{(0)}, \] (185)
where
\[ \hat{H}_k = -\frac{1}{2} \frac{\partial^2}{\partial \nu^2_k} + \frac{1}{2} \omega^2_k \nu^2_k \] (186)
is the effective uncorrected Hamiltonian for the perturbations.

As before, to this order $\Psi_k$ takes the WKB form
\[ \Psi_k(q_A, \nu_k) \approx \Psi_{kWKB}(q_A) \psi_k(\nu_k, q_A), \] (187)
with
\[ \Psi_{kWKB}(q_A) = \frac{1}{\gamma(q_A)} \exp(i m^2 P S_0) \] (188)
and $\psi_k(\nu_k, q_A) = \psi_k^{(0)}(\nu_k, q_A)$.

Note also that, as in Section 3.4.1, the definition (184) of WKB time is natural in pilot-wave theory. In terms of the variables $q_A$ the de Broglie velocities (165) for the background can be written as
\[ \frac{dq_A}{d\eta} = \frac{1}{m_P} G_{AB} \frac{\partial S_k}{\partial q_B}. \] (189)
In the Hamilton-Jacobi limit we have $S_k \approx m_P^2 S_0$. The definition (184) then reads
\[ \frac{\partial}{\partial \eta} = \frac{dq_A}{d\eta} \frac{\partial}{\partial q_A}. \] (190)
This relation simply reinterpret a dependence on the changing coordinate $q_A(\eta)$ as a dependence on $\eta$, so that in effect we have
\[ \psi_k^{(0)} = \psi_k^{(0)}(\nu_k, \eta). \] (191)

The next order, $m_P^{-2}$, yields the result
\[ G_{AB} \frac{\partial S_0}{\partial q_A} \frac{\partial S_2}{\partial q_B} + \frac{1}{2} G_{AB} \frac{\partial S_1}{\partial q_A} \frac{\partial S_1}{\partial q_B} - \frac{i}{2} G_{AB} \frac{\partial^2 S_1}{\partial q_A \partial q_B} - \frac{\partial S_1}{\partial q_A} \frac{\partial S_2}{\partial q_B} - \frac{i}{2} \frac{\partial^2 S_2}{\partial \nu_k^2} = 0. \] (192)
Writing
\[ S_2 = S_2(q_A, \nu_k) = \zeta(q_A) + \chi(q_A, \nu_k), \] (193)
\( \chi \) is found to satisfy
\[
\frac{\partial \chi}{\partial \eta} = \frac{1}{\psi_k^{(0)}} \left( -\frac{1}{\gamma} G_{AB} \frac{\partial \psi_k^{(0)}}{\partial q_A} \frac{\partial \gamma}{\partial q_B} + \frac{1}{2} G_{AB} \frac{\partial^2 \psi_k^{(0)}}{\partial q_A \partial q_B} + i \frac{\partial \psi_k^{(0)}}{\partial v_k} \frac{\partial \chi}{\partial v_k} + \frac{i}{2} \frac{\psi_k^{(0)}}{\partial v_k} \frac{\partial^2 \chi}{\partial v_k^2} \right). \tag{194}
\]

Brizuela et al. then define a corrected wave function
\[
\psi_k^{(1)}(q_A, v_k) = \psi_k^{(0)}(q_A, v_k) e^{im_p^2 \chi(q_A, v_k)}, \tag{195}
\]
which is shown to obey a quantum-gravitationally corrected Schrödinger equation
\[
i \frac{\partial \psi_k^{(1)}}{\partial \eta} = \hat{H}_k \psi_k^{(1)} - \frac{1}{2m_p^2} \left( \frac{1}{V} (\hat{H}_k)^2 \psi_k^{(0)} + i \frac{\partial}{\partial \eta} \left( \frac{1}{V} \hat{H}_k \right) \psi_k^{(0)} \right) \psi_k^{(1)}, \tag{196}
\]
where \( V \) is the auxiliary potential (162) and in effect \( \psi_k^{(1)} = \psi_k^{(1)}(v_k, \eta) \).

As we saw in the general case (Section 5.1) the corrections have a non-Hermitian part. In standard quantum mechanics this violates unitarity and leads to inconsistencies: probability will not be conserved. But as we described in Section 6, such corrections can arise naturally in pilot-wave theory while maintaining a fully conserved probability, with the Born rule rendered unstable.

As noted in Section 5.1 the approximate ratio (117) of the non-Hermitian correction to the Hermitian correction is of order \( \sim H/E \), where \( H = \dot{a}/a \) is the Hubble parameter and \( E \) is a typical energy for the field. This ratio is extremely small at late times but might be large in the very early universe.

### 7.3 The de Broglie velocity

As in the general case we can ask what happens to the guidance equation (166) for the perturbations \( v_k \) in the quantum-gravitationally corrected Schrödinger approximation. We again find that the de Broglie velocity reduces to that generated by the Hermitian part of the effective Hamiltonian.

To show this, let us examine what happens to the expression (166) under the semiclassical expansion (178) of the wave function \( \Psi_k \). Noting that \( S_k = \text{Im} \ln \Psi_k \) we have
\[
\frac{dv_k}{d\eta} = \frac{\partial}{\partial v_k} \text{Im} \ln \Psi_k = \frac{\partial}{\partial v_k} \left( m_p^2 \text{Re} S_0 + m_p^0 \text{Re} S_1 + m_p^{-2} \text{Re} S_2 + ... \right). \tag{179}
\]
From (179) we then have
\[
\frac{dv_k}{d\eta} = \frac{\partial}{\partial v_k} \left( \text{Re} S_1 + m_p^{-2} \text{Re} S_2 + ... \right). \tag{198}
\]

By similar reasoning to that given for general case, we can show that the expression
\[
\text{Re} S_1 + m_p^{-2} \text{Re} S_2 + ...
\]

55
is equal to the overall phase $s^{(1)}_k = \text{Im} \ln \psi^{(1)}_k$ of the corrected wave function $\psi^{(1)}_k$. Note first that, from equation (181), $S_1$ is generally complex. The definition (182) of the uncorrected wave function $\psi_0^{(0)} = e^{iS_1}$ contains a prefactor $\gamma$, and from equation (183) we see that $\gamma$ can be chosen to be real. Thus $\text{Re} S_1$ is equal to the overall phase of the uncorrected wave function $\psi_0^{(0)}$. To lowest order, then, we recover the usual de Broglie velocity. Now in (193) we have $S_2 = \zeta + \chi$ where from (194) we see that $\chi$ is complex. From the definition (195) of the corrected wave function, $\psi^{(1)}_k = \psi_0^{(0)} e^{im^2/2 \chi}$, we see that the correction to the overall phase is equal to $m^2/2 \text{Re} \chi$. Thus (199) is indeed equal to the overall phase of the corrected wave function $\psi^{(1)}_k$.

To this order, then, the expansion (198) for the de Broglie velocity can be written as

$$\frac{d\nu_k}{d\eta} = \frac{\partial s^{(1)}_k}{\partial \nu_k},$$

(200)

where $s^{(1)}_k = \text{Im} \ln \psi^{(1)}_k$ is the phase of $\psi^{(1)}_k$. The result (200) is just the standard guidance equation associated with the Hermitian part of the Hamiltonian appearing in the corrected Schrödinger equation (196).

As in the general case (Section 5.3), we find that in the Schrödinger approximation the de Broglie velocity for the perturbations $\nu_k$ is driven by the Hermitian part of the effective Hamiltonian. Because the quantum-gravitationally corrected Hamiltonian contains a non-Hermitian part as well, it follows from the fundamental equations of quantum gravity that the emergent Born rule is unstable.

### 7.4 Absence of a fundamental equilibrium state

In Section 4 we presented a new approach to the Born rule in quantum gravity. The same reasoning applies to quantum cosmology. The minisuperspace model considered here provides a convenient testing ground for these proposals.

Let us for a moment ignore the perturbations $\nu_k$ and consider only the variables $a$ and $\phi$ in the deep quantum-gravity regime. Dropping the terms in $\nu_k$ from (160) and recalling the definition (162) of $V$, the Wheeler-DeWitt equation for $\Psi(a, \phi)$ (dropping the subscript $k$) can be written as

$$\frac{1}{m^2} a \frac{\partial}{\partial a} \left( a \frac{\partial \Psi}{\partial a} \right) - \frac{1}{a^2} \frac{\partial^2 \Psi}{\partial \phi^2} + 2a^4 V(\phi) \Psi = 0.$$  

(201)

The guidance equations for $a$ and $\phi$ are given by (165). From (168) (without $\nu_k$) we know that the dynamics preserves the measure

$$d\mu = |\Psi|^2 adad\phi.$$  

(202)

This simple model can be used to illustrate our new approach to the Born rule. Equations (201) and (165) define a deterministic pilot-wave dynamics for an individual system. For a theoretical ensemble of systems guided by the same
wave function $\Psi$, the evolving distribution $P(a, \phi, \eta)$ necessarily satisfies the continuity equation

$$\frac{\partial P}{\partial \eta} + \frac{\partial}{\partial a}(Pa') + \frac{\partial}{\partial \phi}(P\phi') = 0$$

(203)

(where the probability density $P$ is defined with respect to $dad\phi$). This is the same as the equation

$$\frac{\partial}{\partial a} \left(a |\Psi|^2 a'\right) + \frac{\partial}{\partial \phi} \left(a |\Psi|^2 \phi'\right) = 0$$

(204)

satisfied by the density $a |\Psi|^2$ (cf. equation (167)). However, as we have noted, there can be no equilibrium or Born-rule state $P = a |\Psi|^2$ because $a |\Psi|^2$ is non-normalisable.

To see this it is convenient to use the variable $\alpha = \ln a$ in terms of which (201) reads

$$\frac{1}{m_P^2} \frac{\partial^2 \Psi}{\partial \alpha^2} - \frac{\partial^2 \Psi}{\partial \phi^2} + 2e^{6\alpha}V(\phi)\Psi = 0$$

(205)

This is a two-dimensional Klein-Gordon equation with a potential term. The free part (without the potential) has the general solution

$$\Psi_{\text{free}} = f(\phi - m_P\alpha) + g(\phi + m_P\alpha)$$

(206)

where $f$ and $g$ are general packets travelling with the ‘wave speed’ $c = m_P$ in the minisuperspace $(\alpha, \phi)$. There is then no question of $\Psi$ being normalisable. In terms of $\alpha$ the measure (202) reads

$$d\mu = |\Psi|^2 e^{2\alpha}d\alpha d\phi$$

(207)

For the free part (206) we necessarily have

$$\int \int e^{2\alpha}d\alpha d\phi ~ |\Psi_{\text{free}}|^2 = \infty$$

(208)

Indeed even without the factor $e^{2\alpha}$ the integral still diverges. In terms of $a$ and $\phi$, and for a general solution $\Psi$, we will inevitably have

$$\int \int ad\alpha d\phi ~ |\Psi(a, \phi)|^2 = \infty$$

(209)

We emphasise once again that the non-normalisability of the wave function $\Psi$ is not caused by symmetries or unobservable degrees of freedom but is simply a consequence of the Klein-Gordon-like character of the Wheeler-DeWitt equation – which implies a wave-like propagation on the minisuperspace.

By definition $P(a, \phi, \eta)$ is a physical probability density satisfying

$$\int \int d\alpha d\phi ~ P(a, \phi, \eta) = 1$$

(210)
at all times \( \eta \). Apart from this condition, the initial distribution \( P(a, \phi, \eta_i) \) (at some initial conformal time \( \eta_i \)) is in principle arbitrary, and subsequently evolves via (203). In contrast, \( a |\Psi(a, \phi)|^2 \) is non-integrable and cannot be a physical density. As in the general quantum-gravitational case, we must have

\[
P(a, \phi, \eta) \neq a |\Psi(a, \phi)|^2 \tag{211}
\]

at all times. The quantities \( P \) and \( a |\Psi|^2 \) can never be equal, neither initially nor subsequently.

As in the general quantum-gravitational case, the initial \( P(a, \phi, \eta_i) \) is not fixed by any law but is instead empirical. It must be constrained by observation, just like any other initial condition in physics.

We can again consider a coarse-grained \( H \)-function

\[
\bar{H}(t) = \int \int da d\phi \ln( \bar{P}/a |\Psi|^2 ) , \tag{212}
\]

which measures (minus) the relative entropy of \( \bar{P} \) with respect to \( a |\Psi|^2 \) and which again obeys the \( H \)-theorem (13). As we saw in the general case, because \( \Psi \) is non-normalisable (212) has no lower bound and \( \bar{P} \) will always be infinitely far away from the pseudo-equilibrium state \( a |\Psi|^2 \). In this sense the system will be perpetually in a state of quantum nonequilibrium.

Further understanding of the time dependence of the \( H \)-function (212) will require numerical simulations for specific solutions \( \Psi \). This will be explored elsewhere [97].

### 7.5 Emergence and instability of the Born rule

In our approach, at the fundamental level, there is no state of quantum equilibrium and no physical Born rule. Even so, as we discussed in the general case, the Born rule can be recovered in the effective Schrödinger regime for quantum fields propagating on a classical spacetime background. Let us summarise how this comes about in terms of the above quantum-cosmological model.

We may reinstate the perturbations \( v_k \) and consider the variables \( a \) and \( \phi \) in a semiclassical regime where \( a \) and \( \phi \) define an approximately classical background. We have seen that the system can then be described by a Schrödinger approximation with a time-dependent wave function \( \psi_k(v_k, \eta) \). To zeroth-order \( \psi_k \) satisfies the time-dependent Schrödinger equation (185) so that \( \psi_k = \psi_k^{(0)} \). The de Broglie velocity is then given by the usual formula

\[
\frac{dv_k}{d\eta} = \frac{\partial s_k^{(0)}}{\partial v_k} , \tag{213}
\]

where now \( s_k^{(0)} = \text{Im} \ln \psi_k^{(0)} \) is the phase of \( \psi_k^{(0)} \). For a theoretical ensemble with the same wave function \( \psi_k^{(0)} \) we can consider an arbitrary probability distribution \( \rho_k^{(0)}(v_k, \eta) \) for \( v_k \) on a given background \( (a, \phi) \), which necessarily evolves
according to the continuity equation

\[ \frac{\partial \rho_k^{(0)}}{\partial \eta} + \frac{\partial}{\partial \nu_k} \left( \rho_k^{(0)} \nu_k' \right) = 0. \] (214)

As in our general discussion we first need to consider \( \rho_k^{(0)}(\nu_k, \eta_i) \) at some initial time \( \eta_i \) soon after the system enters the Schrödinger regime. This arises as a conditional probability from a theoretical ensemble of universes with Wheeler-DeWitt wave function \( \Psi_k(a, \phi, \nu_k) \) and arbitrary distribution \( P_k(a, \phi, \nu_k, \eta_i) \). Conditioning on the background \((a, \phi)\) we then have

\[ \rho_k^{(0)}(\nu_k, \eta_i) = \frac{P_k(a, \phi, \nu_k, \eta_i)}{\int P_k(a, \phi, \nu_k, \eta_i) d\nu_k}. \] (215)

Like \( P_k(a, \phi, \nu_k, \eta_i) \) itself, \( \rho_k^{(0)}(\nu_k, \eta_i) \) is in principle arbitrary and can only be constrained empirically. Because of the fundamental nonequilibrium condition

\[ P_k(a, \phi, \nu_k, \eta) \neq a |\Psi_k(a, \phi, \nu_k)|^2 \] (216)

(except in the special case \( P_k(a, \phi, \nu_k, \eta_i) = \Pi(a, \phi)|\psi_k^{(0)}(\nu_k, \eta_i)|^2 \) for some \( \Pi(a, \phi) \)). Thus we can expect the perturbations \( \nu_k \) to be in a state of quantum nonequilibrium at the beginning of the Schrödinger regime.

At later times the Born rule

\[ \rho_k^{(0)}(\nu_k, \eta) = |\psi_k^{(0)}(\nu_k, \eta)|^2 \] (217)

can emerge by quantum relaxation in the usual way (on a coarse-grained level). The zeroth-order Schrödinger equation \((185)\) implies that \( |\psi_k^{(0)}(\nu_k, \eta)|^2 \) obeys the same continuity equation

\[ \frac{\partial}{\partial \eta} \left| \psi_k^{(0)} \right|^2 + \frac{\partial}{\partial \nu_k} \left( \left| \psi_k^{(0)} \right|^2 \nu_k' \right) = 0 \] (218)

as is satisfied by \( \rho_k^{(0)} \) (equation \((214)\)). We can then define a coarse-grained \( H \)-function

\[ \bar{H}(t) = \int d\nu_k \rho_k^{(0)} \ln \left( \frac{\rho_k^{(0)}}{|\psi_k^{(0)}|^2} \right), \] (219)

which will satisfy the \( H \)-theorem \((13)\). Quantum relaxation will in fact be rather limited for a one-dimensional system with a single degree of freedom \( \nu_k \) \cite{24, 26}. But in a realistic cosmology different modes will be entangled at early times.
Extensive numerical simulations show that even just two dimensions suffice for efficient relaxation to occur [38, 40–42]. More precisely, for quantum fields on expanding space, efficient quantum relaxation is found for field modes with sub-Hubble wavelengths, while relaxation is suppressed at super-Hubble wavelengths [46, 47, 49]. Thus the Born rule is certainly recovered at the wavelengths observed in the laboratory.

Even so, if we include first-order gravitational corrections to the Schrödinger equation, we find small non-Hermitian terms in the Hamiltonian that can in principle drive the system away from equilibrium. The corrected Schrödinger equation (196) contains such a term, implying that the corrected wave function \( \psi^{(1)}_k \) obeys a continuity equation

\[
\frac{\partial}{\partial \eta} \left| \psi^{(1)}_k \right|^2 + \frac{\partial}{\partial \upsilon_k} \left( \left| \psi^{(1)}_k \right|^2 \upsilon'_k \right) = s
\]  

with a source term \( s \) (as discussed in Section 6). However, the de Broglie velocity still takes the standard form (200), and so the actual distribution \( \rho^{(1)}_k(\upsilon_k, \eta) \) satisfies the usual continuity equation

\[
\frac{\partial \rho^{(1)}_k}{\partial \eta} + \frac{\partial}{\partial \upsilon_k} \left( \rho^{(1)}_k \upsilon'_k \right) = 0
\]  

(where in both (220) and (221) we have \( \upsilon'_k = \partial s^{(1)}_k / \partial \upsilon_k \)). As we saw in Section 6.2, the mismatch between the continuity equations (220) and (221) implies that an initial equilibrium distribution \( \rho^{(1)}_k = \left| \psi^{(1)}_k \right|^2 \) can evolve away from equilibrium \( (\rho^{(1)}_k \neq \left| \psi^{(1)}_k \right|^2 \) at later times). As we have noted, in practice this can occur only if nonequilibrium is generated on a timescale \( \tau_{\text{noneq}} \) that is shorter than the usual timescale \( \tau_{\text{relax}} \) for quantum relaxation.

### 7.6 Simplified model for the slow-roll limit

We have seen how a semiclassical expansion of the Wheeler-DeWitt equation together with the de Broglie guidance equation leads to a pilot-wave model of quantum cosmology in which the Born rule can be unstable. We have a quantum-gravitationally corrected Schrödinger equation (196) for the wave function \( \psi^{(1)}_k(\upsilon_k, \eta) \) together with the standard de Broglie guidance equation (200) for the perturbations \( \upsilon_k \). We will now write these equations in a far slow-roll limit. The resulting simplified model can then be used to perform tractable calculations of the quantum-gravitational production of quantum nonequilibrium in cosmology.

Note first that the result (196) is written in terms of rescaled variables \( a_{\text{new}}, \eta_{\text{new}}, \upsilon_{\text{new}}, k_{\text{new}} \) (defined by (152)), where the subscript ‘new’ has been dropped. We will find it convenient to undo these rescalings and write the equations in terms of the original variables \( a_{\text{old}}, \eta_{\text{old}}, \upsilon_{\text{old}}, k_{\text{old}} \) (now dropping
the subscript ‘old’). The lengthscale $\mathcal{L}$ then reappears. Making this change, from now on it is understood that we use the symbols $a$, $\eta$, $\nu_k$, $k$ to denote the original (unrescaled) variables. We also revert to standard cosmic time $t$ via the relation $d\eta = a^{-1}dt$. In (146) we then need to make the replacements

$$\frac{\partial}{\partial \eta} \rightarrow \mathcal{L} \frac{\partial}{\partial \eta} = \mathcal{L} a \frac{\partial}{\partial t}, \quad V \rightarrow \mathcal{L}^4 V = \frac{2}{m_P^2} \mathcal{L}^4 a^4 V.$$ (222)

Our simplified limit is taken as follows. First, we ignore the factor $z''/z$ and write

$$\omega^2_k = k^2 - \frac{z''}{z} \approx k^2.$$ (223)

Second, in the expression (142) for the Mukhanov-Sasaki variable $\nu$ we neglect terms containing $\phi'$, so that $\nu_k \approx a\phi$ and

$$\nu_k \approx a\phi_k.$$ (224)

It is also convenient to rescale the variable $\phi_k$ and to write

$$\phi_k = \mathcal{L}^{3/2} \phi_k,$$ (225)

remembering that here both $\phi_k$ and $q_k$ are treated as real.

With these changes and simplifications, the uncorrected Hamiltonian $\hat{H}_k$ (given by (186)) appearing in (196) becomes

$$\hat{H}_k = \mathcal{L} a \hat{H}_k,$$ (226)

where

$$\hat{H}_k = -\frac{1}{2a^3} \frac{\partial^2}{\partial q_k^2} + \frac{1}{2} a k^2 q_k^2$$ (227)

is the Hamiltonian for the (real) degree of freedom $q_k$.

Taking into account (222) and (226), and defining

$$\bar{k} = \frac{1}{\mathcal{L}},$$ (228)

the quantum-gravitationally corrected Schrödinger equation (196) becomes

$$\frac{i}{\partial t} \psi^{(1)}_k = \hat{H}_k \psi^{(1)}_k - \frac{\bar{k}^4}{2m_P^2} \psi^{(0)}_k \left[ -\frac{1}{a^3 (2V/m_P^2)} (\hat{H}_k)^2 \psi^{(0)}_k + i \frac{\partial}{\partial t} \left( \frac{1}{a^3 (2V/m_P^2)} \hat{H}_k \right) \psi^{(0)}_k \right] \psi^{(1)}_k.$$ (229)

To complete our simplified model, we also need to rewrite the de Broglie guidance equation (200). We undo the rescalings (152), restore $t$, and use the approximation (224) with the rescaling (225). We find

$$\frac{dq_k}{dt} = \frac{1}{a^3} \frac{\partial s^{(1)}_k}{\partial q_k}.$$ (230)
For a general theoretical ensemble with wave function $\psi^{(1)}_k(q_k, t)$, the probability density $\rho^{(1)}_k(q_k, t)$ will evolve by the continuity equation

$$\frac{\partial \rho^{(1)}_k}{\partial t} + \frac{\partial}{\partial q_k} \left( \rho^{(1)}_k(q_k) \dot{q}_k \right) = 0 ,$$

(231)

with velocity field $\dot{q}_k = dq_k/dt$ given by (230).

Equations (229), (230) and (231) define a simplified pilot-wave quantum-cosmological model in which the Born rule can be unstable. This model will be used as a starting point for some of the calculations that follow.

8 Quantum instability for a scalar field on de Sitter space

In the slow-roll limit the energy density (141) of the background scalar field may be written as $\rho \approx V$ and is approximately constant in time. The Friedmann–Lemaître equation $(\dot{a}/a)^2 = (8\pi G/3)\rho$ then implies an approximate de Sitter expansion, $a \propto e^{Ht}$, with an approximately constant Hubble parameter

$$H = \sqrt{(8\pi G/3)V} .$$

(232)

Using $m_P^2 = 3/4\pi G$ we can then write

$$2V/m_P^2 = H^2 .$$

(233)

Inserting this into the corrected Schrödinger equation (229) we then have

$$i\frac{\partial \psi^{(1)}_k}{\partial t} = \hat{H}_k \psi^{(1)}_k - \frac{k^3}{2m_P^2 H^2} \frac{1}{a^3} \left( \frac{1}{a^3} \hat{H}_k^2 \psi^{(0)}_k + i \frac{\partial}{\partial t} \left( \frac{1}{a^3} \hat{H}_k \right) \psi^{(0)}_k \right) \psi^{(1)}_k .$$

(234)

The de Broglie guidance equation remains, of course, as given by (230).

We will now apply this model – defined by (234), (230) and (231) – to the gravitational production of quantum nonequilibrium on de Sitter space. Our results in effect provide a mechanism for the gravitational production of quantum nonequilibrium during inflation that was suggested in refs. [34, 65].

8.1 Approximation for the Bunch-Davies vacuum

First we estimate the timescale (136) for quantum instability on de Sitter space, where the term $\hat{H}_2$ comes from quantum-gravitational corrections to the Schrödinger equation. To a first approximation we may take

$$\langle \hat{H}_2 \rangle \approx \langle \hat{H}_2 \rangle_{B-D} ,$$

(235)

where $\langle \ldots \rangle_{B-D}$ denotes a quantum expectation value calculated with the zeroth-order Bunch-Davies vacuum wave function $\psi^{(0)}_k$ (without gravitational corrections). The gravitationally-corrected wave function $\psi^{(1)}_k$ obeys the equation
where $H$ is the quasi-static Hubble parameter, $\hat{H}_k$ is the single-mode Hamiltonian for a massless scalar field, and $\bar{k} = 1/\Sigma$ is an arbitrary scale. Note that the corrections in (234) are multiplicative, one real and one imaginary.

In our notation, with $\hat{H} = \hat{H}_1 + i\hat{H}_2$, we have

$$\hat{H}_1 = \hat{H}_k - \frac{\bar{k}^3}{2m_p^2H^2}\frac{1}{a^3}(\hat{H}_k)^2\psi_k^{(0)}$$

(236)

(where to lowest order $\hat{H}_1 \simeq \hat{H}_k$) and

$$\hat{H}_2 = -\frac{\bar{k}^3}{2m_p^2H^2}\frac{1}{a^3}\frac{\partial}{\partial t}\left(\frac{1}{a^3}\hat{H}_k\right)\psi_k^{(0)},$$

(237)

where the Bunch-Davies wave function $\psi_k^{(0)}$ satisfies

$$i\frac{\partial \psi_k^{(0)}}{\partial t} = \hat{H}_k\psi_k^{(0)}$$

(238)

(with the Minkowski boundary condition for the limit $H \to 0$).

Using $\dot{a} = Ha$ we may write

$$\hat{H}_2 = -\frac{\bar{k}^3}{2m_p^2H^2}\frac{1}{a^3}\frac{1}{\psi_k^{(0)}}\hat{H}'_k\psi_k^{(0)},$$

(239)

where we have defined a ‘distorted’ single-mode Hamiltonian

$$\hat{H}'_k = -\frac{3}{a^3}\frac{\partial^2}{\partial q_k^2} + ak^2q_k^2$$

(240)

(of the same form as (227) but with different numerical coefficients).

Since (239) is purely multiplicative we have

$$\langle \hat{H}_2 \rangle_{B-D} = \frac{\bar{k}^3}{2m_p^2H}a^3\langle \hat{H}'_k \rangle_{B-D},$$

(241)

where

$$\langle \hat{H}'_k \rangle_{B-D} = \frac{3}{a^3}\langle -\frac{\partial^2}{\partial q_k^2} \rangle_{B-D} + ak^2\langle q_k^2 \rangle_{B-D}.$$  

(242)

This can be calculated from the Bunch-Davies wave function $\psi_k^{(0)} = |\psi_k^{(0)}|e^{i\delta_k^{(0)}}$, which has amplitude and phase

$$|\psi_k^{(0)}| = \frac{1}{(2\pi\Delta_k^2)^{1/4}}e^{-\frac{q_k^2}{4\Delta_k^2}},$$

(243)

$$\delta_k^{(0)} = -\frac{ak^2q_k^2}{2H(1+k^2/H^2a^2)} + \frac{1}{2}H - \frac{1}{2}\tan^{-1}\left(\frac{k}{Ha}\right),$$

(244)
and width
\[ \langle q_k^2 \rangle_{B-D} = \Delta_k^2 = \frac{H^2}{2k^3} \left( 1 + \frac{k^2}{H^2 a^2} \right). \]  
(245)

We obtain
\[ \langle \dot{q}_k^2 \rangle_{B-D} = \frac{k}{2a} \left( 1 + \frac{H^2 a^2}{k^2} \right). \]  
(246)

From this we may find our estimate (136) for \( \tau_{\text{noneq}} \).

### 8.2 Timescale for quantum instability

From (136), (241) and (246) we have
\[ \tau_{\text{noneq}} \sim \frac{2m_p^2}{H^3} \left( \frac{k}{\bar{k}} \right)^3 \frac{x^4}{4 + x^2}, \]  
(247)

where it is convenient to define
\[ x = \frac{Ha}{k} = \frac{1}{2\pi} \frac{\lambda_{\text{phys}}}{H^{-1}}. \]  
(248)

The result (247) gives an estimated timescale for the gravitational production of quantum nonequilibrium during an approximate de Sitter phase.

We may consider a given mode with fixed \( k \). According to (247) the effect is significant – that is, \( \tau_{\text{noneq}} \) is small – only when \( x \) is small (when the mode is inside the Hubble radius). As the mode evolves in time, both \( x \) and \( \tau_{\text{noneq}} \) grow unboundedly large and the gravitational effect is frozen. Thus there is no gravitational production of quantum nonequilibrium for \( \lambda_{\text{phys}} >> H^{-1} \).

To ensure the validity and relevance of our approximate calculations, we can restrict ourselves to a range of physical wavelengths
\[ 10^{3} \lesssim \lambda_{\text{phys}} \lesssim 10H^{-1}. \]  
(249)

The lower bound is justified by the need to avoid the deep quantum-gravity regime where our calculations are invalid, while the upper bound may be justified by the standard inflationary result that the mode becomes effectively classical soon after exiting the Hubble radius (this rough approach is needed because we are using our simplified model for the slow-roll limit). In terms of \( x \) the range (249) corresponds (approximately) to \( 10^{-1} \lesssim x \lesssim 1 \).

To get a sense of orders of magnitude, let us consider the result (247) for \( k \sim \bar{k} \) and \( x \sim 1 \), so that roughly
\[ \tau_{\text{noneq}} \sim \frac{m_p^2}{H^3}. \]  
(250)

As an illustrative example, if we take the inflationary phase to have an energy scale of order \( H \sim 10^{16} \text{ GeV} \sim 10^{-3}m_p \) (which is likely to be an upper bound), we find \( \tau_{\text{noneq}} \sim 10^9t_P \) (where \( t_P \) is the Planck time). By comparison the time \( H^{-1} \sim 10^4t_P \) the mode spends in the regime \( x \sim 1 \) is only a fraction \( \sim 10^{-6} \) of
Thus the timescale over which our estimate applies is only about $10^{-6}$ of the timescale $\tau_{\text{noneq}}$ for the creation of nonequilibrium. These crude estimates suggest that the effect will indeed be small (as expected).

As noted in Section 6.3 the created quantum nonequilibrium can build up over time only if the condition \[133\] is satisfied. In the ideal limit of a Bunch-Davies vacuum there is in fact no quantum relaxation at all (essentially because the trajectories for the field modes are too simple to generate relaxation) \[34\]. In effect $\tau_{\text{relax}} = \infty$ and the condition \[133\] is indeed satisfied. Any nonequilibrium created during a de Sitter expansion will not be erased by relaxation. Small perturbations to the Bunch-Davies vacuum are unlikely to change this conclusion, since numerical studies for the oscillator (analogous to a field mode) indicate that small perturbations do not cause relaxation \[107\]. Thus de Sitter space provides a particularly simple and effective example of the gravitational creation of quantum nonequilibrium building up over time.

9 Corrections to the cosmic microwave background

The temperature anisotropy $\Delta T(\hat{n}) \equiv T(\hat{n}) - \bar{T}$ of the cosmic microwave background (CMB) can be expressed in spherical harmonics

$$\frac{\Delta T(\hat{n})}{T} = \sum_{l=2}^{\infty} \sum_{m=-l}^{+l} a_{lm} Y_{lm}(\hat{n}) ,$$

(251)

where $\hat{n}$ labels points on the sky and $\bar{T}$ is the mean temperature. The coefficients $a_{lm}$ are generated by the primordial curvature perturbation $\mathcal{R}_k$ \[108\],

$$a_{lm} = \frac{i^l}{2\pi^2} \int d^3 k \, T(k,l) \mathcal{R}_k Y_{lm}(\hat{k}) ,$$

(252)

where $T(k,l)$ is the transfer function. It is usually assumed that $\Delta T(\hat{n})$ is drawn from a theoretical ensemble with an isotropic probability distribution. This implies that the angular power spectrum $C_l \equiv \langle |a_{lm}|^2 \rangle$ is independent of $m$ (where $\langle ... \rangle$ denotes an ensemble average). Thus statistical isotropy implies that for given $l$ the $2l+1$ quantities $|a_{lm}|^2$ all have the same ensemble mean $C_l$. This allows us to probe the underlying theoretical ensemble from measurements made on our one observed sky. The measured mean statistic

$$C_{l,\text{sky}} \equiv \frac{1}{2l+1} \sum_{m=-l}^{+l} |a_{lm}|^2$$

(253)

satisfies $\langle C_{l,\text{sky}} \rangle = C_l$ and therefore provides an unbiased estimate of $C_l$ for the ensemble. For a Gaussian distribution $C_{l,\text{sky}}$ has a cosmic variance

$$\frac{\Delta C_{l,\text{sky}}}{C_l} = \sqrt{\frac{2}{2l+1}} .$$

(254)
We can probe $C_l$ accurately at large $l$, while for small $l$ the accuracy is limited. It is also usually assumed that the theoretical ensemble for $R$ is statistically homogeneous, which implies $\langle R_k^* R_k \rangle = \delta_{kk} \langle |R_k|^2 \rangle$. From (252) we then have the formula

$$C_l = \frac{1}{2\pi^2} \int_0^{\infty} \frac{dk}{k} T^2(k, l) P_R(k),$$

where

$$P_R(k) \equiv \frac{4\pi k^3}{V} \langle |R_k|^2 \rangle$$

is the primordial power spectrum for $R_k$. It is sometimes claimed that probabilities are meaningless for a single universe. But in fact, by assuming statistical isotropy and statistical homogeneity, measurements made on a single sky can constrain the primordial spectrum $P_R(k)$ – even though $P_R(k)$ is a property of a theoretical ensemble.

According to inflationary cosmology the primordial spectrum originates from quantum vacuum fluctuations in a scalar inflaton field \[109–111\]. An inflaton perturbation $\varphi_k$ generates a curvature perturbation $R_k \propto \varphi_k$ (once the physical wavelength exits the Hubble radius). The measured power spectrum $P_R(k)$ is then proportional to the variance $\langle |\varphi_k|^2 \rangle$ of $\varphi_k$ – which is usually calculated in quantum field theory by applying the Born rule. But in pilot-wave theory we can consider nonequilibrium probabilities for $\varphi_k$ and hence a general variance

$$\langle |\phi_k|^2 \rangle = \langle |\phi_k|^2 \rangle_{QT} \xi(k),$$

where $\langle |\phi_k|^2 \rangle_{QT}$ is the standard quantum-theoretical variance and the factor $\xi(k)$ quantifies the departure from equilibrium at wavenumber $k$. For a quantum nonequilibrium ensemble of inflaton perturbations we have a primordial power spectrum

$$P_R(k) = P_R^{QT}(k) \xi(k),$$

where $P_R^{QT}(k)$ is the quantum-theoretical spectrum. From (254) we then obtain a corrected angular power spectrum $C_l$.

Thus measurements of the CMB can probe primordial corrections to the Born rule as quantified by $\xi(k)$ \[34, 50\]. We will now show how to calculate (approximately) a new contribution to $\xi(k)$ from the quantum-gravitational creation of quantum nonequilibrium during inflation.

### 9.1 Approximate calculation of the nonequilibrium function $f$

In inflationary cosmology observations of the CMB can probe the width (or variance) of the distribution $\rho_k$ of inflaton perturbations $\varphi_k \propto q_k$ as a function...
of wave number $k$. In quantum nonequilibrium we have $\rho_k = |\psi_k|^2 f_k$, with $f_k \neq 1$, so that in general $\rho_k$ is different from $|\psi_k|^2$. To compare with observation we need to know how the gravitational corrections affect $\rho_k$. As well as knowing how the corrections affect $\psi_k$ we also need to know how they affect the nonequilibrium function $f_k$. An approximate result for $f_k$ will now be derived.

We use the following method. If we assume that the corrected wave function $\psi_k$ is already known, we may calculate $\rho_k$ by calculating $f_k = \rho_k / |\psi_k|^2$. As we saw in Section 6.2, in the presence of non-Hermitian gravitational corrections, $f_k$ is no longer conserved along trajectories but instead changes in accordance with the differential equation (132). This has the solution

$$f_k(q_k(t_f), t_f) = f_k(q_k(t_i), t_i) \exp \left( -\int_{t_i}^{t_f} dt \, u_k(q_k(t), t) \right),$$

(259)

where

$$u_k = s_k / |\psi_k|^2$$

(260)

and

$$s_k = 2 \text{Re} \left( \psi_k^* \hat{H}_2 \psi_k \right),$$

(261)

with $\hat{H}_2$ given by (239) and (240). The trajectories are of course determined by the exact $\psi_k$. However, as a first approximation, in the expressions for $u_k$ and $s_k$ we may insert the lowest-order Bunch-Davies wave function $\psi_k^{(0)}$ and perform the integration in (259) along the uncorrected trajectories generated by $\psi_k^{(0)}$.

We first find an expression for $u_k$ (with $\psi_k = \psi_k^{(0)}$). Since $\hat{H}_2$ is a purely multiplicative operator we have

$$u_k = 2 \text{Re} \left( \hat{H}_2 \right).$$

(262)

Employing the Schrödinger equation (238) and writing $\psi_k^{(0)} = |\psi_k^{(0)}| e^{i\phi_k^{(0)}}$ we have

$$u_k = - \frac{k^3 \hbar}{m^2 c^4} \frac{1}{a^3} \left( 6 \frac{\partial \psi_k^{(0)}}{\partial t} + 2a k^2 q_k^2 \right),$$

(263)

where from (214)

$$6 \frac{\partial \psi_k^{(0)}}{\partial t} + 2a k^2 q_k^2 = a k^2 q_k^2 \left( \frac{2 - 5x^2 - x^4}{(1 + x^2)^2} \right) - \frac{3k}{a} \frac{1}{1 + x^2}.$$  

(264)

(again defining $x = Ha/k$). Thus, with $a = kx/H$, we may write

$$u_k = - \frac{k^3 \hbar}{m^2 c^4} \frac{H^2}{k^3 x^3} \left( \frac{1}{H} k^3 \frac{2 - 5x^2 - x^4}{(1 + x^2)^2} q_k^2 - 3H x \frac{1}{1 + x^2} \right).$$

(265)

We will be integrating $u_k$ along trajectories $q_k(t)$, from arbitrary initial points $q_k(t_i)$ to arbitrary final points $q_k(t_f)$. It will be convenient to write
\( q_k^2 \) in terms of \( x \) and \( q_k(x_f) \). We may then integrate with respect to \( x \) from \( x_i = (H/k)a_i \) to \( x_f = (H/k)a_f \). The trajectories are generated by the lowest-order Bunch-Davies wave function \( \psi_k^{(0)} \) whose phase \( s_k^{(0)} \) is given by (244). The relevant de Broglie equation of motion

\[
\frac{dq_k}{dt} = \frac{1}{a^3} \frac{\partial s_k^{(0)}}{\partial q_k},
\]

(266) with \( s_k^{(1)} \) replaced by \( s_k^{(0)} \) can be solved exactly to yield the trajectories \( q_k(t) \) [34]. Reverting for a moment to conformal time \( \eta \) (where on de Sitter space \( \eta = -1/Ha \)), the solution \( q_k(\eta) = q_k(0) \sqrt{1 + k^2 \eta^2} \) can be written as

\[
q_k(\eta) = q_k(\eta_f) \sqrt{1 + k^2 \eta^2},
\]

(267) where \( \eta_f \) is a final time. Writing in terms of \( x = Ha/k = -1/k\eta \) we have

\[
q_k^2(x) = q_k^2(x_f) \frac{(x_f)^2}{1 + (x_f)^2} \frac{1 + x^2}{1 + (x_f)^2}.
\]

(268)

Our expression for \( u_k \) as a function of \( x \) then reads

\[
u_k = -\frac{\bar{k}^3}{m_P^2} \frac{H^2}{k^3 x^3} \left( \frac{1}{H} \frac{k^3}{x} \frac{(2 - 5x^2 - x^4)x}{(1 + x^2)^2} q_k^2(x_f) (x_f)^2 \frac{1 + x^2}{1 + (x_f)^2} - 3H \frac{1}{x} - 1 \right),
\]

(269)

where we need to evaluate the integral

\[
I_k = \int_{t_i}^{t_f} dt \ u_k(q_k(t), t)
\]

(270)

along trajectories, in order to find the function

\[
f_k(q_k(t_f), t_f) = f_k(q_k(t_i), t_i) \exp(- I_k)
\]

(271)

(at arbitrary final points \( q_k(t_f) \)). Since \( dt = (1/H)(1/x)dx \) we have

\[
I_k = \int_{t_i}^{t_f} u_k(q_k(t), t)dt = \frac{1}{H} \int_{x_i}^{x_f} \frac{1}{x} u_k(q_k(x), x)dx.
\]

(272)

Inserting (269) we then have

\[
I_k = -\eta_k^2(x_f) \frac{(x_f)^2}{1 + (x_f)^2} \frac{\bar{k}^3}{m_P^2} I_1 + 3H \frac{\bar{k}^3}{m_P^2} I_2
\]

(273)

where

\[
I_1 = \int_{x_i}^{x_f} \frac{(2 - 5x^2 - x^4)}{(1 + x^2)^2} \frac{1}{x^3} dx,
\]

(274)

\[
I_2 = \int_{x_i}^{x_f} \frac{1}{(1 + x^2)^2} \frac{1}{x^3} dx.
\]

(275)
For given $x_i$ and $x_f$ these integrals can easily be evaluated numerically.

To specify $x_i$ and $x_f$ let us take a range (249) of physical wavelengths, which corresponds approximately to

$$x_i = 10^{-1}, \quad x_f = 1 .$$

(276)

For these values the integrals (274) and (275) are found to be

$$I_1 \simeq 4.7 \times 10^3, \quad I_2 \simeq 2.5 \times 10^3 .$$

(277)

We then have

$$I_k \simeq -2.4 \times 10^3 \frac{k^3}{m_P^2} q_k^2(x_f) + 0.8 \times 10^4 \frac{k^3}{m_P^2} H^2. \quad (278)$$

Note that for $k \sim \bar{k}$ the magnitude of the second term is $\sim 10^4 H^2/m_P^2$, which for an illustrative (and probable upper bound) value $H \sim 10^{-3} m_P$ is of order $\sim 10^{-2}$, and so $I_k$ is indeed small.

Having evaluated $I_k$ we can write down the nonequilibrium function (271). Recalling that $f_k = \rho_k / |\psi_k|^2$, and reverting to standard time $t$, we can then find the corrected density $\rho_k(q_k(t_f), t_f)$ from the formula

$$\rho_k(q_k(t_f), t_f) = |\psi_k(q_k(t_f), t_f)|^2 \exp(-I_k) . \quad (279)$$

9.2 Nonequilibrium correction to the primordial power spectrum

We wish to calculate the correction to the primordial power spectrum as a function of wave number $k$. Focussing on a single mode with wave vector $k$, let us assume for simplicity that our mode is initially in quantum equilibrium:

$$\rho_k(q_k(t_i), t_i) = |\psi_k(q_k(t_i), t_i)|^2 .$$

(280)

From (279) we then have a final nonequilibrium density

$$\rho_k(q_k(t_f), t_f) = |\psi_k(q_k(t_f), t_f)|^2 \exp(-I_k) . \quad (281)$$

In (279) we have an expression for $I_k$ as a function of arbitrary final points $q_k(t_f)$ (or $q_k(x_f)$). In (281) we have an expression for the density $\rho_k$ at the final time $t_f$ and at arbitrary final points $q_k(t_f)$. We can then simply write $\rho_k$ at time $t_f$ as a function of the general variable $q_k$:

$$\rho_k(q_k, t_f) = |\psi_k(q_k, t_f)|^2 \exp(-I_k(q_k)) , \quad (282)$$

where

$$I_k(q_k) \simeq -2.4 \times 10^3 \frac{k^3}{m_P^2} q_k^2 + 0.8 \times 10^4 \frac{k^3}{m_P^2} H^2 . \quad (283)$$
Note that \( \psi_k(q_k, t_f) \) will be gravitationally corrected. A complete calculation of \( \rho_k(q_k, t_f) \) then requires an expression for \( |\psi_k(q_k, t_f)|^2 \) including the gravitational corrections. But the primordial power spectrum depends only on the width of \( \rho_k(q_k, t_f) \), so it suffices to consider only how \( |\psi_k(q_k, t_f)|^2 \) contributes to the width. Writing the corrected wave function as

\[
\psi_k \simeq \psi_k^{(1)} = \psi_k^{(0)} + \delta \psi_k^{(1)}
\]

(for simplicity omitting the arguments \( q_k, t_f \)), and using the fact that \( I_k \) is small, we may write

\[
\rho_k = |\psi_k|^2 \exp (-I_k) \simeq |\psi_k^{(0)}|^2 (1 - I_k)
\]

where \( \delta \psi_k^{(1)} \sim O(1/m_P^2) \) and \( I_k \sim O(1/m_P^2) \). This may be written as

\[
\rho_k = |\psi_k^{(0)}|^2 + |\psi_k^{(0)}|^2 \delta \psi_k^{(1)} + \psi_k^{(0)} \delta \psi_k^{(1)*} - I_k |\psi_k^{(0)}|^2 + O(1/m_P^4)
\]

(again omitting the arguments \( q_k, t_f \)). The second and third terms are an \( O(1/m_P^2) \) correction coming from the gravitational correction to \( \psi_k \), while the fourth term is an \( O(1/m_P^2) \) correction coming from the gravitational production of quantum nonequilibrium.

We may now consider how the separate effects change the width of \( \rho_k \). The mean-square \( \langle q_k^2 \rangle_f = \int dq_k q_k^2 \rho(q_k, t_f) \) (evaluated at the final time \( t_f \)) takes the form

\[
\langle q_k^2 \rangle_f = \langle q_k^2 \rangle_f^{(0)} + \left( \delta \langle q_k^2 \rangle_f \right)^{(1)} + \left( \delta \langle q_k^2 \rangle_f \right)^{\text{noneq}} + O(1/m_P^4),
\]

where

\[
\langle q_k^2 \rangle_f^{(0)} = \int dq_k q_k^2 |\psi_k^{(0)}(q_k, t_f)|^2
\]

is the uncorrected Bunch-Davies width (at time \( t_f \)),

\[
\left( \delta \langle q_k^2 \rangle_f \right)^{(1)} = \int dq_k q_k^2 \left( \psi_k^{(0)*}(q_k, t_f) \delta \psi_k^{(1)}(q_k, t_f) + \psi_k^{(1)*}(q_k, t_f) \delta \psi_k^{(0)}(q_k, t_f) \right)
\]

is the correction to the width from the gravitational correction to \( \psi_k \) itself, and

\[
\left( \delta \langle q_k^2 \rangle_f \right)^{\text{noneq}} = - \int dq_k q_k^2 I_k(q_k) |\psi_k^{(0)}(q_k, t_f)|^2
\]

is the correction to the width from the gravitational production of quantum nonequilibrium. We now calculate the term \[289\] and obtain an estimate of its effect on the primordial power spectrum.

We are really interested in the primordial deficit function \( \xi(k) \), defined by

\[
\xi(k) = \frac{\langle q_k^2 \rangle_f}{\langle q_k^2 \rangle_f^{(0)}},
\]

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where \( \langle q^2_k \rangle_f \) is given by (286). We may write

\[
\xi(k) = 1 + \delta \xi^{(1)}(k) + \delta \xi^{\text{noneq}}(k) + \ldots ,
\]

(291)

where

\[
\delta \xi^{(1)}(k) = \frac{\left( \delta \langle q^2_k \rangle_f \right)^{(1)}}{\langle q^2_k \rangle_f} , \quad \delta \xi^{\text{noneq}}(k) = \frac{\left( \delta \langle q^2_k \rangle_f \right)^{\text{noneq}}}{\langle q^2_k \rangle_f} .
\]

(292)

Note that, as defined by (290), the deficit function \( \xi(k) \) contains contributions from both effects (gravitational corrections to \( \psi_k \) and the gravitational production of nonequilibrium).

We are particularly interested in the term \( \delta \xi^{\text{noneq}} \), which from (289) and (283) takes the form

\[
\delta \xi^{\text{noneq}}(k) \simeq 10^{-4} \bar{k}^3 \left( \frac{1}{4} \frac{\langle q^4_k \rangle_f}{\langle q^2_k \rangle_f} - \frac{4}{5} \frac{H^2}{k^3} \right) ,
\]

(293)

where \( \langle q^2_k \rangle_f^{(0)} \) and \( \langle q^4_k \rangle_f^{(0)} \) are the respective (quantum-theoretical) means of \( q^2_k \) and \( q^4_k \) in the Bunch-Davies vacuum at the final time \( t_f \). From (245) the mean-square width \( \langle q^2_k \rangle_f^{(0)} = \Delta^2_k(t_f) \) can be written as

\[
\langle q^2_k \rangle_f^{(0)} = \frac{H^2}{2k^3} \left( 1 + \frac{1}{x_f^2} \right) .
\]

(294)

Taking \( x_f = 1 \) we have \( \langle q^2_k \rangle_f^{(0)} = H^2/k^3 \). For the Gaussian wave-function amplitude (243) we also have \( \langle q^4_k \rangle_f^{(0)} = 3 \left( \langle q^2_k \rangle_f^{(0)} \right)^2 \) yielding a ratio

\[
\frac{\langle q^4_k \rangle_f^{(0)}}{\langle q^2_k \rangle_f^{(0)}} = \frac{3H^2}{k^3} .
\]

(295)

Thus we find, finally,

\[
\delta \xi^{\text{noneq}}(k) \simeq -5 \times 10^{-2} \frac{H^2}{m_P^2} \left( \frac{\bar{k}}{k} \right)^3 .
\]

(296)

According to this approximate calculation, the nonequilibrium correction to \( \xi \) amounts to a power deficit scaling as \( \sim 1/k^3 \).

To get a sense of orders of magnitude, again taking our illustrative (and probable upper bound) value \( H \sim 10^{16} \) GeV \( \sim 10^{-3} m_P \), in the region \( k \sim \bar{k} \) the result (296) yields

\[
\delta \xi^{\text{noneq}} \sim -5 \times 10^{-4} .
\]

(297)

According to our estimate, the nonequilibrium correction to \( \xi \) is small and negative.
9.3 Comparison with gravitational corrections to the wave function

Our quantum-gravitational correction to the Born rule yields a large-scale power deficit (296). In contrast, the quantum-gravitational correction to the wave function $\psi_k$, as derived by Brizuela et al. [71], yields a large-scale power excess

$$\mathcal{P}^{(1)}(k) = \mathcal{P}^{(0)}(k) \left( 1 + 0.988 \frac{H^2}{m_p^2} \left( \frac{k}{\bar{k}} \right)^3 + O \left( \frac{H^4}{m_p^4} \right) \right)$$

or

$$\delta \xi^{(1)}(k) \simeq \frac{H^2}{m_p^2} \left( \frac{k}{\bar{k}} \right)^3.$$ (299)

The opposing signs, $\delta \xi^{\text{noneq}} < 0$ and $\delta \xi^{(1)} > 0$, mean that the overall sign of the total correction $\delta \xi^{\text{noneq}} + \delta \xi^{(1)}$ cannot be known without more accurate calculations. In particular, in our derivation of $\delta \xi^{\text{noneq}}$, the magnitude of the coefficient is not determined accurately. Thus at present we cannot say if the overall effect will be a power deficit or a power excess.

The magnitudes of both effects depend on the arbitrary scale $\bar{k}$ which appears in the correction (237) to the Hamiltonian. However the ratio

$$\left| \frac{\delta \xi^{\text{noneq}}(k)}{\delta \xi^{(1)}(k)} \right| \simeq 5 \times 10^2$$

is independent of $\bar{k}$ (and indeed independent of $k$). The difference in magnitude, by about two orders, is however not very significant given our crude estimate for $\delta \xi^{\text{noneq}}$.

It is noteworthy that such different physical effects – gravitational corrections to the wave function on the one hand, and gravitational corrections to the Born rule on the other – both yield essentially the same results except for an overall sign difference. In both cases we find a scaling $\propto \left( \bar{k}/k \right)^3$.

Finally we make a few comments on the derivation of (298) by Brizuela et al. [71]. The calculation keeps only the Hermitian correction appearing in the Schrödinger equation (196) and drops the non-Hermitian term. As we have seen, the ratio (117) of the latter correction to the former is generally of order $\sim H/E$ where $E$ is a typical energy for the field, so it is not entirely clear if the non-Hermitian term can be neglected. However, Brizuela et al. motivate dropping the non-Hermitian term for other reasons. To solve (196) for the corrected wave function $\psi_k^{(1)}$ a Gaussian ansatz is assumed. If the non-Hermitian term is included, two perceived problems arise. First, the normalisation of $\psi_k^{(1)}$ is not conserved in time and the standard probability interpretation becomes problematic: it is not possible to unambiguously take expectation values and compute the power spectrum. Second, solving the Gaussian ansatz numerically leads to large oscillations in the amplitude of $\psi_k^{(1)}$ in the distant past ($\eta \to -\infty$), which does not happen if the non-Hermitian term is dropped. From the point of view of
standard quantum mechanics these issues motivate dropping the non-Hermitian term (even if the semiclassical expansion of the Wheeler-DeWitt equation has naturally produced it). But in pilot-wave theory the two perceived problems are only apparent. First, as we discussed in Section 6 for a general system with a non-Hermitian Hamiltonian, even though the normalisation of the wave function \( \psi \) is not conserved, the actual probability density \( \rho \) remains normalised (by construction) at all times, and the effect of the non-Hermitian term is to generate a deviation of \( \rho \) from \( |\psi|^2 \). We can then use \( \rho \) to unambiguously take expectation values and compute the power spectrum. Second, because observed quantities such as power spectra are now calculated from the actual density \( \rho \) (which in general will not equal \( |\psi|^2 \)), the fact that the wave amplitude \( |\psi| \) becomes large in the remote past need not by itself signal a difficulty.

9.4 Effect on the angular power spectrum

We may ask if there is a realistic prospect of measuring the correction \( \delta \xi_{\text{noneq}} \) to the primordial power spectrum. In addition to having to consider the other contribution \( \delta \xi^{(1)} \), an immediate question is whether or not the effect on the \( C_l \)'s will be swamped by the cosmic variance (254), in which case it will be unobservable in principle. This will depend on the region of \( k \)-space that is affected – that is, on the value of \( k \).

From (255), (258) and (291) we can write

\[
C_l = C_l^{\text{QT}} + \delta C_l^{(1)} + \delta C_l^{\text{noneq}} + \ldots
\]

where

\[
\begin{align*}
C_l^{\text{QT}} &= \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} \, T^2(k,l) P^{\text{QT}}_R(k), \\
\delta C_l^{(1)} &= \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} \, T^2(k,l) P^{\text{QT}}_R(k) \delta \xi^{(1)}(k), \\
\delta C_l^{\text{noneq}} &= \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} \, T^2(k,l) P^{\text{QT}}_R(k) \delta \xi_{\text{noneq}}(k).
\end{align*}
\]

We then have a fractional correction

\[
\frac{\delta C_l^{\text{noneq}}}{C_l^{\text{QT}}} = \frac{\int_0^\infty \frac{dk}{k} \, T^2(k,l) P^{\text{QT}}_R(k) \delta \xi_{\text{noneq}}(k)}{\int_0^\infty \frac{dk}{k} \, T^2(k,l) P^{\text{QT}}_R(k)}.
\]

Let us first consider (303) at low \( l \) (say \( l \lesssim 20 \)), where the angular power spectrum is dominated by the Sachs-Wolfe effect and the (square of the) transfer function takes the approximate analytical form [109]

\[
T^2(k,l) = \pi H_0^4 j_l^2(2k/H_0),
\]

where \( j_l \) is the \( l \)th order spherical Bessel function and \( H_0 \) is the Hubble parameter today. Taking (roughly) \( P^{\text{QT}}_R(k) \simeq \text{const.} \), (303) becomes

\[
\frac{\delta C_l^{\text{noneq}}}{C_l^{\text{QT}}} \simeq 2l(l+1) \int_0^\infty \frac{dk}{k} \, j_l^2(2k/H_0) \delta \xi_{\text{noneq}}(k),
\]
where we have used $\int_0^\infty \frac{dx}{x^2} J_2^2(x) = \frac{1}{\pi x^{1/2}}$. The integral (305) is dominated by the scale $k \approx lH_0/2$, so a significant effect requires $\delta \xi_{\text{noneq}}$ to be significant for $k$ in this region—that is, for wavelengths $\lambda \approx (4\pi/l)H_0^{-1}$, which for low $l$ is comparable to the Hubble radius today ($H_0^{-1} \approx 3000 \text{ Mpc}$). If the scale $\bar{k}$ is far from this region, the effect will be negligible. And even if $\bar{k} \approx lH_0/2$, the effect will still be negligible if the coefficient $H^2/m_P^2$ in (296) is very small. For example, even for $H$ as large as $H \sim 10^{16} \text{ GeV} \sim 10^{-3} m_P$, we have $H^2/m_P^2 \sim 10^{-6}$.

If the correction $\delta \xi_{\text{noneq}}$ is indeed present in the region $k \approx lH_0/2$ then from (305) we may roughly expect to find a fractional deficit

$$\left| \frac{\delta C_l^{\text{noneq}}}{C_l^{\text{QT}}} \right| \sim |\delta \xi_{\text{noneq}}|$$

in the angular power spectrum at the $l$th multipole. Similarly, at large values of $l$, a multipole of order $l$ probes a comoving wavenumber $k \approx H_0l/2$ [109], so that if the correction $\delta \xi_{\text{noneq}}$ is present in the region $k \approx lH_0/2$ we again roughly expect to find a fractional deficit of order (306). However such a deficit can be observed in principle only if it is larger than the cosmic variance [254].

Brizuela et al. [71] suggest that $\bar{k}$ can be interpreted as an infrared cutoff. For the purpose of estimating the magnitude of $\delta \xi^{(1)}$, however, $\bar{k}$ is taken to coincide with the pivot scale $k_* = 0.05 \text{ Mpc}^{-1}$ chosen by the Planck team. Taking an upper bound $H \lesssim 10^{-5} m_P$ (motivated by limits on the tensor-to-scalar ratio) yields an estimated upper bound $|\delta \xi^{(1)}| \lesssim 2 \times 10^{-10}$ at $k \approx k_*$, which is far too small to be observable [71]. Taking the same bound $H \lesssim 10^{-5} m_P$ our result (296) implies an upper bound

$$|\delta \xi_{\text{noneq}}| \lesssim 5 \times 10^{-8}$$

at $k \approx k_*$, which is again far too small to be observable.

## 10 Quantum instability in a radiation-dominated universe

We have seen that quantum-gravitational effects can create quantum nonequilibrium for a scalar field on exponentially expanding de Sitter space. We expect there will be similar effects for fields on an expanding background generally. For example, we might consider the radiation-dominated expansion that took place during the early phase of the hot big bang. Various quantum fields were propagating on this background: the electromagnetic field, fermionic fields, and others. Quantum-gravitational corrections to the Schrödinger evolution for these fields will create quantum nonequilibrium, in particular at very early times when the expansion was extremely rapid. The particle-like excitations of those fields later correspond to relic cosmological particles, which include the photons of the CMB, the neutrinos of the expected cosmic neutrino background, as well as other more exotic particles such as gravitinos whose existence is predicted by
some theories of particle physics (and which might be a significant component of dark matter). It is then conceivable that such relic particles could violate the Born rule even today. This possibility has already been studied in some detail in a scenario where it is assumed that the universe begins in a state of quantum nonequilibrium: quantum relaxation will be important in the early universe but can be suppressed for some special systems that decouple very early \( [26, 30, 45, 51] \). We are now concerned with a novel possibility: that even particles initially in equilibrium could develop nonequilibrium over time as a result of quantum-gravitational effects. In a realistic scenario, we would have to compare the timescale \( \tau_{\text{noneq}} \) over which nonequilibrium is created with the timescale \( \tau_{\text{relax}} \) for quantum relaxation. As noted in Section 6.3, deviations from the Born rule can build up over time only if \( \tau_{\text{relax}} > \tau_{\text{noneq}} \). Estimates for \( \tau_{\text{relax}} \) for relic cosmological particles have been discussed elsewhere \( [26, 30, 45, 51] \).

Here we focus on developing an estimate for \( \tau_{\text{noneq}} \) for such particles. As we will see, the result is found to be so large as to make this phenomenon seemingly of theoretical interest only.

In Section 7 we studied the instability of the Born rule for a simple model of quantum cosmology with a background expansion driven by a scalar field \( \phi \) with a potential \( V(\phi) \). As shown in ref. \( [71] \), scalar perturbations on the background satisfy the gravitationally-corrected Schrödinger equation \( (190) \), which we have written in the simplified form \( (229) \) by taking a slow-roll limit. The gravitational corrections depend on \( V(\phi) \). A glance at the uncorrected part of \( (229) \) shows that it is just the usual Schrödinger equation for a scalar-field Fourier mode \( q_k \) propagating on a background expanding space with scale factor \( a = a(t) \). If we were to consider a scalar field propagating on a radiation-dominated background (with \( a \propto t^{1/2} \)), then the uncorrected Schrödinger equation would take precisely this form. The question is: what will the gravitational corrections look like for this case?

To answer this question rigorously and from first principles, we would need to develop a quantum-cosmological model with a background radiation-dominated expansion. This might be done along the lines of the above model for an appropriate choice of \( V(\phi) \). It is well known that, for a potential \( V(\phi) = \frac{1}{4} \lambda \phi^4 \), averaging over oscillations in \( \phi \) yields an effective equation of state \( p = \rho/3 \) corresponding to a radiation-dominated universe \( [112] \). However, instead, here we will use a shortcut to obtain a simple estimate of the correction terms.

We assume that some appropriate potential \( V(\phi) \) can be used to model the radiation-dominated expansion of the background, and that the semiclassical expansion of the Wheeler-DeWitt equation leads to an effective Schrödinger equation of the form \( (229) \) for the Fourier mode \( q_k \) of some field propagating on the background, where the relevant potential \( V(\phi) \) appears in the correction terms. We treat the propagating field as a scalar, but we expect similar equations to apply for example to modes of the electromagnetic field (only with more components).

The correction terms in the Schrödinger equation \( (229) \) depend on \( V \) which depends on the time-evolving background field \( \phi \). We can crudely estimate the magnitude of \( V \) as follows. Classically we know that the energy density \( [141] \)
is a sum of kinetic and potential terms. The Friedmann–Lemaître equation \((\dot{a}/a)^2 = (8\pi G/3)\rho\) implies that \(H^2 = (8\pi G/3)\rho\), where of course now \(H\) is time dependent. As a rough order-of-magnitude estimate we might ignore the kinetic term in \(\rho\) and write \(\rho \approx V(\phi)\). Using \(m_p^2 = 3/4\pi G\) we then have, roughly,

\[
2V(\phi)/m_p^2 \approx H^2 .
\] (308)

Inserting this into (229) we have an approximate Schrödinger equation

\[
i\frac{\partial \psi_k^{(1)}}{\partial t} \approx \hat{H}_k \psi_k^{(1)} - \frac{\bar{k}^3}{3m_P^2} \frac{1}{\psi_k^{(0)}} \left[ \frac{1}{a^3} \left( \frac{\dot{H}_k}{H^2} \right)^2 \psi_k^{(0)} + i \frac{\partial}{\partial t} \left( \frac{1}{a^3} \frac{\dot{H}_k}{H^2} \right) \psi_k^{(0)} \right] \psi_k^{(1)}
\]

for a field mode on an expanding background with time-dependent Hubble parameter \(H = H(t)\). This should suffice for the purpose of order-of-magnitude estimates. In our notation (123) the correction \(\hat{H}_2\) is then estimated to be

\[
\hat{H}_2 \approx -\frac{\bar{k}^3}{3m_P^2} \frac{1}{\psi_k^{(0)}} \frac{\partial}{\partial t} \left( \frac{1}{a^3} \frac{\dot{H}_k}{H^2} \right) \psi_k^{(0)} .
\] (309)

This is the term that causes the instability of the Born rule over time.

For a radiation-dominated expansion \(a \propto t^{1/2}\) and \(H = \dot{a}/a = 1/2t\). Following common convention we take the scale factor today, at time \(t_0\), to be \(a_0 = 1\) (so that comoving wavelengths \(\lambda\) correspond to physical wavelengths today). Writing \(a = (t/t_0)^{1/2}\) we have \(a^3 H^2 = H_0^2/a\), where \(H_0 = 1/2t_0\) is the Hubble parameter today. Using \(\dot{a} = Ha\) and the expression (227) for \(\hat{H}_k\) we can then write

\[
\hat{H}_2 \approx \frac{\bar{k}^3}{m_P^2 H_0 a} \frac{1}{\psi_k^{(0)}} \dot{H}_k' \psi_k^{(0)} ,
\] (310)

where now we have defined a distorted Hamiltonian

\[
\hat{H}_k' = -\frac{1}{2a^3} \frac{\partial^2}{\partial q_k^2} - \frac{1}{2a^2} k^2 q_k^2 .
\] (311)

To estimate the timescale \(\tau_{\text{noneq}}\) for the gravitational production of quantum nonequilibrium, as given by (136), we need to estimate the magnitude of the quantum expectation value \(\langle \hat{H}_2 \rangle\). Following the same reasoning as in Section 8.1 for de Sitter space, to a first approximation we take

\[
\langle \hat{H}_2 \rangle \approx \langle \hat{H}_2 \rangle^{(0)} ,
\] (312)

where \(\langle ... \rangle^{(0)}\) denotes a quantum expectation value calculated with the uncorrected wave function \(\psi_k^{(0)}\). As before (311) is purely multiplicative and so

\[
\langle \hat{H}_2 \rangle^{(0)} \approx \frac{\bar{k}^3}{m_P^2 H_0 a} \langle \hat{H}_k' \rangle^{(0)} ,
\] (313)
where our sought-for timescale is given approximately by

\[ \tau_{\text{noneq}} \approx \frac{1}{2 \left| \langle \hat{H}_2 \rangle^{(0)} \right|}. \] (314)

Let us write, crudely, \( \langle \hat{H}_k \rangle^{(0)} \sim E \) where \( E = E_0/a \) is the energy of a single particle propagating on the expanding background (where \( E_0 \) is the energy today). We also reintroduce the lengthscale \( \mathcal{L} \) via the definition \( k = 1/\mathcal{L} \). We then have

\[ \langle \hat{H}_2 \rangle^{(0)} \sim \mathcal{L}^{-1} \frac{E_0}{m_p^2 a} \cdot \] (315)

Using \( m_p \sim 1/l_p \) and inserting \( c \), the timescale (314) then takes the suggestive form

\[ \tau_{\text{noneq}} \sim \frac{\mathcal{L}}{c} \left( \frac{a \mathcal{L}}{l_p} \right) \left( \frac{E}{E_0} \right) \left( \frac{m_p}{E} \right) \cdot \] (316)

where \( a \mathcal{L} \) is the physical lengthscale at time \( t \) corresponding to the comoving lengthscale \( \mathcal{L} \).

Writing \( a = (t/t_0)^{1/2} \) and employing the standard temperature clock \( t \sim (1 \text{ s}) (1 \text{ MeV}/k_B T)^2 \) for a radiation-dominated phase with ambient temperature \( T \), the result (316) takes the form

\[ \tau_{\text{noneq}} \sim \frac{\mathcal{L}}{c} \sqrt{\frac{1 \text{ s}}{t_0}} \left( \frac{1 \text{ MeV}}{k_B T} \right) \left( \frac{\mathcal{L}}{l_p} \right) \left( \frac{E}{E_0} \right) \left( \frac{m_p}{E} \right) \cdot \] (317)

Writing \( E \sim k_B T \) and \( m_p \sim k_B T \), we then have our estimated timescale

\[ \tau_{\text{noneq}} \sim \frac{\mathcal{L}}{c} \sqrt{\frac{1 \text{ s}}{t_0}} \left( \frac{\mathcal{L}}{l_p} \right) \left( \frac{1 \text{ MeV}}{k_B T} \right) \left( \frac{1}{k_B T} \right) \left( \frac{k_B T}{k_B T} \right) \cdot \] (318)

To get an idea of the magnitude of \( \tau_{\text{noneq}} \), let us apply our result to the deep Planck era where \( k_B T \sim k_B T_p \sim 10^{19} \text{ GeV} \). From (313) we find

\[ \tau_{\text{noneq}} \sim 10^{-22} \frac{\mathcal{L}}{c} \sqrt{\frac{1 \text{ s}}{t_0}} \left( \frac{\mathcal{L}}{l_p} \right) \left( \frac{\mathcal{L}}{H_0^{-1}} \right) \cdot \] (319)

At this point we must choose a value for the lengthscale \( \mathcal{L} \). It seems reasonable to set

\[ \mathcal{L} \sim H_0^{-1} \sim 10^{28} \text{ cm} \cdot \] (320)

With the current age \( t_0 \sim 10^{17} \text{ s} \) of the universe we then obtain the huge result

\[ \tau_{\text{noneq}} \sim 10^{48} \text{ s} \cdot \] (321)

This is overwhelmingly large even compared to \( t_0 \). At lower temperatures \( \tau_{\text{noneq}} \) is even larger.
Our result for $\tau_{\text{noneq}}$ is so huge because of the factor $(\Sigma/l_P)$. Without some changed understanding of the infrared cutoff lengthscale $\Sigma$ there seems to be no way around this. We therefore tentatively conclude that there is no hope of a significant effect building up over realistic cosmological timescales – even in a situation where the usual quantum relaxation can be neglected. It is, however, still possible that relic cosmological particles today could show residual violations of the Born rule as a result of the universe beginning in a state of primordial quantum nonequilibrium [26, 30, 43, 51]. As we saw in Sections 4.2 and 7.5, such primordial nonequilibrium is to be expected as the universe emerges from the deep quantum-gravity regime.

11 Quantum instability in the spacetime of an evaporating black hole

Kiefer, Müller and Singh [113] considered how the procedure of ref. [68] – which we summarised in Sections 3.4.1 and 5.1 – can be generalised to the asymptotically-flat spacetime of an evaporating black hole. This requires the inclusion of a boundary term $Mc^2$ in the (integrated) Wheeler-DeWitt equation, where the mass $M$ of the black hole is asymptotically defined by the usual ADM energy. Kiefer et al. argue that the resulting quantum-gravitational corrections to the Schrödinger equation, for a matter field propagating in the spacetime of the black hole, will be of the same form as in equation (110) but with the replacement

$$\sqrt{g}R \rightarrow -16\pi GM/c^2,$$  \hspace{1cm} (322)

where the Schwarzschild radius $r_s = 2GM/c^2$ provides a natural lengthscale. It is argued that, for an evaporating black hole, the Hermitian correction to the Hamiltonian will be negligible compared to the dominant term $\hat{H}_\phi = \int d^3x \hat{H}_\phi$ (where $\hat{H}_\phi$ is the matter Hamiltonian density), since the ratio will be of order the energy of the field divided by $Mc^2$. The general non-Hermitian correction in (110) takes the suggestive form

$$\Delta \hat{H}_\phi = i \frac{4\pi \hbar G}{c^4} \int d^3x \frac{\delta}{\delta \tau} \left( \frac{\hat{H}_\phi}{\sqrt{g}R} \right),$$  \hspace{1cm} (323)

where $\delta/\delta \tau$ is the many-fingered time derivative (111) and we have inserted the expansion parameter $\mu = c^2/32G$ as well as $\hbar$ and $c$. With the replacement (322), for an evaporating black hole, (323) then takes the approximate form

$$\Delta \hat{H}_\phi \simeq -i \frac{4\pi \hbar G}{c^4} \frac{d}{dt} \left( \frac{c^2}{16\pi GM} \right) \int d^3x \hat{H}_\phi = i \frac{\hbar}{4c^2 M^2} \frac{dM}{dt} \hat{H}_\phi$$  \hspace{1cm} (324)

(neglecting the rate of change of $\hat{H}_\phi$ compared with the rate of change of the background geometry). Kiefer et al. suggest that this term might play a role in alleviating the problem of black-hole information loss.
Assuming that the time-dependent mass $M(t)$ takes the phenomenological form \[ M(t) \simeq M_0 \left(1 - \kappa \frac{m_p^3}{M_0^2} \left(\frac{t}{t_p}\right)^3\right)^{1/3}, \] where $M_0$ is the initial mass, $\kappa$ is a numerical factor, and here $m_p = \sqrt{\hbar c/G}$ denotes the standard Planck mass, the non-Hermitian correction becomes important (compared to the dominant term $\hat{H}_\phi$) if $M$ approaches the Planck mass $m_P$, which happens after a time \[ t_* \simeq (M_0/m_P)^3 t_p. \] In the same regime we can then expect the gravitational production of quantum nonequilibrium to become important.

We can now estimate the rate of production of quantum nonequilibrium during the evaporation of a black hole. Assuming the time-dependent mass we have \[ \frac{dM}{dt} \simeq -\frac{1}{3} \kappa \frac{m_p}{t_p} \left(\frac{m_p}{M}\right)^2, \] and so from we have \[ \Delta \hat{H}_\phi \simeq -\frac{1}{12} \kappa \left(\frac{m_p}{M}\right)^4 \hat{H}_\phi, \] where $M = M(t)$ is the mass of the black hole and $\hat{H}_\phi$ is the (uncorrected) Hamiltonian of a matter field $\phi$ propagating on the background spacetime. In particular, $\hat{H}_\phi$ could be the Hamiltonian of a field in the exterior region, in the vicinity of the event horizon. According to our framework, the effective Schrödinger equation for the field will contain a small non-Hermitian term of the form \[ \Delta \hat{H}_2 \simeq -\frac{1}{12} \kappa \left(\frac{m_p}{M}\right)^4 \hat{H}_k, \] where $\hat{H}_k$ is the (uncorrected) Hamiltonian for the field mode.

The timescale $\tau_{\text{noneq}}$ for the gravitational production of quantum nonequilibrium, caused by the dynamical spacetime of the evaporating black hole, is given by \[ \tau_{\text{noneq}} \simeq 6 \kappa \left(\frac{M}{m_p}\right)^4 \left(\frac{1}{E_k}\right)\left(\frac{M}{m_p}\right)^5. \] For the purposes of our estimate we take $E_k \sim k_B T$. From and we then have \[ \tau_{\text{noneq}} \simeq 6 \kappa \left(\frac{M}{m_p}\right)^4 \frac{1}{E_k} \sim 48 \kappa \left(\frac{1}{E_k}\right)\left(\frac{M}{m_p}\right)^5. \]
The creation of quantum nonequilibrium will be significant if the timescale \( \tau_{\text{noneq}} \) is not too large compared to the timescale \( t_{\text{evap}} \) over which the black hole evaporates, where
\[
\frac{1}{t_{\text{evap}}} = \left| \frac{dM}{dt} \right| .
\] (332)
From (327) we have
\[
t_{\text{evap}} \approx \frac{3}{\kappa} \kappa_{\text{P}} \left( \frac{M}{m_{\text{P}}} \right)^3 .
\] (333)
The ratio of the timescales is then
\[
\frac{\tau_{\text{noneq}}}{t_{\text{evap}}} \approx \frac{48\pi}{3} \left( \frac{M}{m_{\text{P}}} \right)^2 .
\] (334)
(where conveniently the numerical factor \( \kappa \) cancels). Thus for \( M >> m_{\text{P}} \) we find \( \tau_{\text{noneq}} >> t_{\text{evap}} \) and it seems safe to conclude that in that regime the creation of quantum nonequilibrium will be negligible. On the other hand, in the final stages of evaporation, as \( M \) approaches \( m_{\text{P}} \), according to our estimate quantum nonequilibrium will be created on a timescale that exceeds the evaporation timescale by only one or two orders of magnitude, suggesting that a significant degree of nonequilibrium will be created. In other words, in the very final stage of black-hole evaporation, outgoing field modes could show significant departures from the Born rule.

The question remains as to whether or not such departures from the Born rule can survive quantum relaxation. In the usual calculation of Hawking radiation it is assumed that the field is in a relatively simple quantum state corresponding to an appropriately defined vacuum [99]. Back reaction on the background spacetime is neglected. This suffices to obtain the rate of radiation. But in the final stages of evaporation the background spacetime changes rapidly and the quantum state is likely to be more complicated than is usually assumed. We may then expect the field to be subject to quantum relaxation on some timescale \( \tau_{\text{relax}} \). As noted in Section 6.3 quantum nonequilibrium can grow or build up over time only if the condition (138), or \( \tau_{\text{relax}} > \tau_{\text{noneq}} \), is satisfied. Whether or not nonequilibrium survives in the outgoing radiation then depends on which effect (instability or relaxation) dominates as \( M \) approaches \( m_{\text{P}} \). To answer this we need to know how \( \tau_{\text{relax}} \) scales with \( M \) in the final stages (where the above estimate has \( \tau_{\text{noneq}} \propto (M/m_{\text{P}})^5 \)). That will require a more detailed model. If it turns out that \( \tau_{\text{relax}} < \tau_{\text{noneq}} \) as \( M \) approaches \( m_{\text{P}} \), then any nonequilibrium that is created will be dissipated before it can build up: the outgoing nonequilibrium field modes will rapidly relax to equilibrium and so the resulting Hawking radiation (observed in the asymptotically flat region) will still obey the Born rule. If instead \( \tau_{\text{relax}} > \tau_{\text{noneq}} \) as \( M \) approaches \( m_{\text{P}} \), then we can expect the outgoing field modes to maintain some degree of nonequilibrium, and the resulting Hawking radiation could show deviations from the Born rule. It is perhaps optimistic to expect to be able to observe effects from the final Planckian phase of Hawking evaporation. But in principle the effects are predicted to exist. More precise predictions require a more realistic model, including quantum
relaxation, with a reliable estimate of $\tau_{\text{relax}}$. This is a matter for future work. It seems likely from the above estimate that significant nonequilibrium will be created in the very final stage of black-hole evaporation, but whether or not it quickly relaxes again remains to be seen.

Realistically, Hawking radiation might be observed from evaporating primordial black holes. It is expected that black holes with a range of masses will have formed in the early universe [115]. The evaporation timescale (333) is of order the current age $t_0 \sim 10^{17}$ s of the universe for microscopic black holes with $M \sim 10^{15}$ g. Thus today it might be possible to detect radiation from primordial black holes with masses $\lesssim 10^{15}$ g (if they are sufficiently abundant, where on some scenarios they may form a significant component of dark matter [116]). The calculated radiated power $\sim (-t)^{-2/3}$ formally diverges at $t = 0$. It is widely believed that the hole will disappear in an explosion, whose products will depend on the high-temperature behaviour of matter. For $M \sim 10^{15}$ g it is estimated that a significant fraction of the luminosity will be in the $\gamma$-ray region (peaked at $\sim 100$ MeV) [117]. It is therefore plausible that we might be able to detect $\gamma$-rays from the evaporation of primordial black holes. If so we could probe their quantum properties, and in particular search for signs of deviations from the Born rule. Such deviations could manifest as a blurring of the usual single-photon two-slit interference pattern or as anomalous polarisation probabilities (deviations from the standard $\cos^2 \theta$ modulation with angle $\theta$ [35]).

Finally, as noted in Section 4.4, it has been argued that primordial black holes can decay via quantum-gravitationally-induced tunnelling (from black holes to white holes) [100], with potentially observable signatures in radio and gamma-ray astronomy [118]. In future work it would be of interest to consider how quantum nonequilibrium might be created during such a process.

12 Quantum instability for atomic systems

It is of theoretical interest to ask if, at least in principle, the Born rule could be unstable for a laboratory atomic system in the gravitational field of the earth.

Consider again the general non-Hermitian correction (323). We saw that in the spacetime of a Schwarzschild black hole the quantity $\sqrt{gR}$, which has the dimensions of length, is replaced by $-8\pi r_S$ (equation (322)) where the Schwarzschild radius $r_S = 2GM/c^2$ is the natural lengthscale of the background spacetime. For a system in the gravitational field of the earth we may then expect to implement a replacement of the form

$$\sqrt{gR} \rightarrow -8\pi r_c,$$

(335)

where $r_c$ is the local radius of curvature ($r_c \approx 10^{13}$ cm at the surface of the earth).

A related suggestion was made on dimensional grounds by Kiefer and Singh [68], who considered the effect of the dominant Hermitian term $\sim \hat{H}_a^2$ on atomic energy levels (where $\hat{H}_a$ is the uncorrected atomic Hamiltonian), with $r_c$ taken to
be the curvature lengthscale associated with the gravitational field of the atomic nucleus. A correction term of the form \( (\hbar^4/c^4 m^2 r_c) \nabla^2 \nabla^2 \) yields tiny shifts in the energy levels, which are of course far too small to be observable.

Our interest is in the non-Hermitian term \( (323) \) for a laboratory atomic system in the gravitational field of the earth. Implementing the replacement \( (335) \), and writing \( \hat{H}_\phi \) as \( \hat{H}_a \), the atomic Hamiltonian \( \hat{H}_a = \int d^3 x \hat{H}_a \) suffers a non-Hermitian correction

\[
\Delta \hat{H}_a \sim -i \hbar G \frac{1}{r_c} \int d^3 x \frac{\delta}{\delta \tau} \left( \hat{H}_a \right) = -i \frac{l_P}{r_c} \frac{\partial \hat{H}_a}{\partial t} t_P ,
\]

which is non-zero only if the atom has a time-dependent (uncorrected) Hamiltonian \( \hat{H}_a \). The correction \( (336) \) is roughly equal to the change in \( \hat{H}_a \) over a Planck time and is suppressed by the tiny ratio \( l_P/r_c \) (and with a factor of \(-i\)).

We can now estimate the timescale \( \tau_{\text{noneq}} \) for the gravitational production of quantum nonequilibrium for an atomic system, caused by a dynamical Hamiltonian in a background curved space, where \( \tau_{\text{noneq}} \) is given by \( (136) \). In our notation \( (123) \) the correction \( \hat{H}_2 \) is

\[
\hat{H}_2 \sim -\frac{l_P}{r_c} \frac{\partial \hat{H}_a}{\partial t} t_P .
\]

If the atomic Hamiltonian \( \hat{H}_a \) changes rapidly over a small timescale \( t_a \), we can apply the sudden approximation where the atomic wave function \( \psi_a \) hardly changes over a time \( t_a \). We can then write \( \left( \frac{\partial \hat{H}_a}{\partial t} \right) \approx d \left( \hat{H}_a \right)/dt \) and define

\[
\frac{1}{t_a} = \left| \frac{1}{\left\langle \hat{H}_a \right\rangle} \frac{d}{dt} \left\langle \hat{H}_a \right\rangle \right| .
\]

We then have

\[
\left\langle \hat{H}_2 \right\rangle \sim \frac{l_P}{r_c} \frac{t_P}{t_a} \left\langle \hat{H}_a \right\rangle
\]

and a timescale (inserting \( \hbar \))

\[
\tau_{\text{noneq}} \sim \frac{r_c}{l_P} \frac{t_a}{t_P} \frac{\hbar}{\left\langle \hat{H}_a \right\rangle} .
\]

In realistic conditions we will of course have \( r_c/l_P \gg 1 \) and \( t_a/t_P \gg 1 \), and so \( \tau_{\text{noneq}} \) will be huge compared with the natural quantum timescale \( \hbar/\left\langle \hat{H}_a \right\rangle \) for the atomic system. We can also compare \( \tau_{\text{noneq}} \) with \( t_a \). Writing \( E_P = \hbar/t_P \) we have

\[
\frac{\tau_{\text{noneq}}}{t_a} \sim \frac{r_c}{l_P} \frac{E_P}{\left\langle \hat{H}_a \right\rangle} .
\]
For $r_c/l_P >>> 1$ and $E_P/\langle \hat{H}_a \rangle >>> 1$ the ratio $\tau_{\text{noneq}}/t_a$ will again be prohibitively large.

Note that, because of the rapidly-changing Hamiltonian, the atomic wave function will necessarily be a superposition of energy eigenstates. This will ensure quantum relaxation over timescales $\tau_{\text{relax}} << \tau_{\text{noneq}}$. Thus, even if we were able to probe an atomic ensemble over times of order $\tau_{\text{noneq}}$ (which realistically will far exceed the age of the universe $t_0 \sim 10^{17}$ s), any nonequilibrium that was gravitationally-generated would have quickly dissipated. It would appear that, as a point of principle, the gravitational creation of quantum nonequilibrium at the atomic scale is of theoretical interest only.

13 Conclusion

We have argued that there is no well-defined Born-rule state at the fundamental level of quantum gravity. A theoretical ensemble with a (non-normalisable) Wheeler-DeWitt wave functional $\Psi$ is necessarily in a state of quantum nonequilibrium $P \neq |\Psi|^2$ (initially and always). An equilibrium Born-rule state can exist only after we enter the semiclassical regime of quantum systems on a classical spacetime background, with time-dependent and normalisable wave functions $\psi$ satisfying a Schrödinger equation. At the beginning of the semiclassical regime, however, ‘initial’ quantum nonequilibrium $\rho \neq |\psi|^2$ is inherited from the deep quantum-gravity regime as a conditional probability. Thus quantum gravity naturally creates an early nonequilibrium universe. Quantum relaxation $\rho \rightarrow |\psi|^2$ takes place only afterwards, finally yielding the Born rule as an emergent equilibrium state. We have also shown how small quantum-gravitational corrections to the Schrödinger equation yield an intermediate regime in which the Born rule is unstable: an initial ensemble $\rho = |\psi|^2$ can evolve to $\rho \neq |\psi|^2$. We have seen that the latter effects are generally very small, though perhaps significant in the final stages of black-hole evaporation. These results have been obtained by applying the de Broglie-Bohm pilot-wave formulation of quantum mechanics to canonical quantum gravity. The results emerge naturally by following the internal logic of this particular approach to quantum physics.

A key starting point for our argument has been the Klein-Gordon-like structure of the Wheeler-DeWitt equation. We have argued that this is not a peculiarity to work around but a sign that the timeless Wheeler-DeWitt wave functional $\Psi[g_{ij}, \phi]$ is not a wave functional as we usually understand the term. The quantity $|\Psi[g_{ij}, \phi]|^2$ is non-normalisable not for some technical reason to be fixed but because, while it may superficially resemble the familiar probability densities $|\psi(q, t)|^2$ encountered in non-gravitational physics, in reality $|\Psi[g_{ij}, \phi]|^2$ is a quite different kind of physical thing. We are able to make sense of this in de Broglie-Bohm pilot-wave theory, because in this formulation of quantum mechanics there is no law-like relationship between the probability density $P$ and the wave function $\Psi$. Instead, in pilot-wave theory, the Born rule emerges as an equilibrium state by dynamical relaxation. Because
$|\Psi[g_{ij}, \phi]|^2$ is non-normalisable, however, there is no equilibrium state in the deep quantum-gravity regime, where inevitably $P[g_{ij}, \phi, t] \neq |\Psi[g_{ij}, \phi]|^2$ always. An equilibrium Born-rule state can emerge only in the semiclassical regime of time-dependent and normalisable wave functions $\psi(q, t)$, by means of (coarse-grained) relaxation $\rho(q, t) \rightarrow |\psi(q, t)|^2$. The small non-Hermitian gravitational corrections to the Schrödinger equation for $\psi$, first derived by Kiefer and Singh, then make it possible for initial equilibrium $\rho = |\psi|^2$ to evolve to nonequilibrium $\rho \neq |\psi|^2$, though the latter effects are generally very small.

It should be emphasised that our intermediate regime, with a small instability of the Born rule, has been derived from the fundamental equations of pilot-wave quantum gravity. Performing a semiclassical expansion of the Wheeler-DeWitt wave functional results in small non-Hermitian terms in the effective Hamiltonian of the Schrödinger regime, while the de Broglie velocities remain those associated with only the Hermitian part of the Hamiltonian. As a consequence, there is a mismatch between the continuity equations satisfied by the actual density $\rho$ and by the Born-rule density $|\psi|^2$, resulting in a small gravitationally-induced instability of the Born rule.

The derivation of non-Hermitian corrections to the Schrödinger equation depends, however, on a semiclassical expansion whose validity might be questioned. Previous authors have generally ignored the non-Hermitian terms because there is no consistent way to interpret them in standard quantum mechanics. It is then tempting to view such terms as an artifact to be eliminated from the effective Hamiltonian by appropriately redefining the effective wave function (as discussed in Section 5.2). But in pilot-wave theory the non-Hermitian terms are fully consistent, thereby removing the ground for viewing them as artificial or for seeking to eliminate them. It may however be that the question of whether or not the non-Hermitian terms really exist can only be settled by experiment, in the sense that some empirical input may be needed in order to know exactly how our familiar time-dependent wave functions are to be identified as they emerge from the underlying quantum-gravitational formalism.

It is also worth remarking that, even if the non-Hermitian terms do turn out to be a mathematical artifact, this will only invalidate the intermediate regime studied in this paper. It will still be the case that there is no well-defined Born rule in the deep quantum-gravity regime, which remains in a perpetual state of nonequilibrium. It will also still be the case that as the universe enters the semiclassical regime it will be in a state of nonequilibrium, with relaxation to the Born rule taking place only afterwards. The Born rule will still be unstable, in the sense that when an equilibrium system enters the deep quantum-gravity regime it can be expected to emerge in a state of nonequilibrium.

Pilot-wave theory solves the notorious quantum measurement problem \[119\], but for as long as we are confined to quantum equilibrium it remains indistinguishable from other formulations or interpretations of quantum physics. In this paper we have shown that pilot-wave theory offers a natural understanding of certain peculiarities of canonical quantum gravity. The non-normalisability of the Wheeler-DeWitt wave functional and the failure of the naive Schrödinger
interpretation are explained by the absence of a physical Born-rule equilibrium state. The emergence of the Born rule in the semiclassical regime is explained by quantum relaxation. The small non-Hermitian terms in the gravitationally-corrected Schrödinger equation no longer present an inconsistency but instead generate a small instability of the Born rule. Finally, the presence of actual de Broglie-Bohm trajectories for all wave functionals arguably gives a clearer account of WKB time and of the emergence of time-dependent wave functions in the semiclassical regime. Thus in several respects the de Broglie-Bohm formulation of quantum mechanics appears advantageous in understanding gravitation.

We have seen that a system emerging from the deep quantum-gravity regime can be expected to violate the Born rule. In future work it would therefore be of interest to study the potential creation of quantum nonequilibrium in bouncing cosmologies and in black-hole to white-hole transitions. We may also expect quantum-gravitational effects to generate nonequilibrium close to classical spacetime singularities, thereby potentially enabling the resolution of black-hole information loss by the mechanism studied in refs. [53, 54] (where ingoing Hawking field modes interact with nonequilibrium degrees of freedom behind the horizon and thereby transmit nonequilibrium to the exterior region via entanglement with the outgoing modes). Finally, it would also be of interest to reconsider the effects discussed in this paper in terms of loop quantum gravity.

The ultimate test of any physical theory is of course by experiment. We have provided approximate calculations of the effects of quantum instability for various systems. Experimentally the most promising candidates for a test appear to be exploding primordial black holes, which may form a significant component of decaying dark matter. As we have seen, radiation from the final stages of black-hole evaporation can potentially show deviations from the Born rule, for example in the form of anomalous photon polarisation probabilities. Should such effects ever be observed, a new window would be opened on the relationship between quantum theory and gravitation.

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