Double-Recurrence Fibonacci Numbers and Generalizations

Ana Paula Chaves
Instituto de Matemática e Estatística
Universidade Federal de Goiás
apchaves@ufg.br

Carlos Alirio Rico Acevedo
Departamento de Matemática
Universidade de Brasília
alirio@mat.unb.br

Brazil

Abstract

Let \((F_n)_{n \geq 0}\) be the Fibonacci sequence given by the recurrence \(F_{n+2} = F_{n+1} + F_n\), for \(n \geq 0\), where \(F_0 = 0\) and \(F_1 = 1\). There are several generalizations of this sequence and also several interesting identities. In this paper, we investigate a homogeneous recurrence relation that, in a way, extends the linear recurrence of the Fibonacci sequence for two variables, called double-recurrence Fibonacci numbers, given by \(F(m, n) = F(m-1, n-1) + F(m-2, n-2)\), for \(n, m \geq 2\), where \(F(m, 0) = F_m\), \(F(m, 1) = F_{m+1}\), \(F(0, n) = F_n\) and \(F(1, n) = F_{n+1}\). We exhibit a formula to calculate the values of this double recurrence, only in terms of Fibonacci numbers, such as certain identities for their sums are outlined. Finally, a general case is studied.

1 Introduction

Fibonacci numbers are known for their amazing properties, association with geometric figures, among others [7, 4]. Using the usual notation for such numbers, \((F_n)_{n \geq 0}\), they are given by the following linear recurrence of order two: \(F_{n+2} = F_{n+1} + F_n\), for \(n \geq 0\), where \(F_0 = 0\) and \(F_1 = 1\). The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, and others by studying high order recurrences with similar initial conditions [6, 3].

Our interest relies in a generalization that uses a recurrence for two indices (called a double-recurrence), such as the one studied by Hosoya [2], who defined a set of integers \(\{f_{m,n}\}\) satisfying:

\[f_{m,n} = f_{m-1,n} + f_{m-2,n}\]
\[ f_{m,n} = f_{m-1,n-1} + f_{m-2,n-2}, \]
for all \( m \geq 2, m \geq n \geq 0, \) where
\[ f_{0,0} = f_{1,0} = f_{1,1} = f_{2,1} = 1. \]

Those numbers, when arranged triangularly, provide the famous \textit{Fibonacci Triangle} (also known as \textit{Hosoya’s Triangle}). One of our goals is to construct an analogue of the Fibonacci Triangle, studying a similar double-recurrence. The set of numbers \( \{F(m,n)\} \), will be required to satisfy the following,
\[ F(m, n) = F(m - 1, n - 1) + F(m - 2, n - 2), \text{ for } m, n \geq 2, \] (1)
with initial values
\[ F(m, 0) = F_m, \quad F(1, n) = F_{n+1}, \]
\[ F(m, 1) = F_{m+1}, \quad F(0, n) = F_n. \]

The initial conditions above, along with (1), are sufficient to calculate the value of \( F(m, n) \) at each \((m, n) \in \mathbb{N}^2\). We call the values of the set \( \{F(m, n)\} \), \textit{double-recurrence Fibonacci numbers}. Note that \( F(m, n) \) is a symmetric function, since the initial conditions above and below the main diagonal are the same, and that \( F(k, i) = F(k, k - i) \) for all \( 0 \leq i \leq \lfloor k/2 \rfloor \).

Figure 1, displays a few values for \( F(m, n) \), considering the bottom left corner as the origin \((0,0)\), and the \((m, n)\) coordinate having the value for \( F(m, n) \).

Consider the value of the coordinate \((7, 4)\), given by \( F(7, 4) = 19 \), and then draw a parallel to the antidiagonal from this point towards the axis, where the interactions begin with initial values \( F(3, 0) = F_3 \) and \( F(4, 1) = F_5 \). This means that, in order to determine \( F(7, 4) \), we only needed the pair \( F_3 \) and \( F_5 \), in other words, only Fibonacci numbers. The following proposition, asserts that this property is true for all \( F(m, n) \), meaning that these values can be obtained using only Fibonacci numbers.
Proposition 1. Let \( m, n \in \mathbb{N} \), and \( F(m, n) \) be a double-recurrence Fibonacci number, with \( k := \min\{m, n\} \). Then,

\[
F(m, n) = F_k F_{|m-n|+2} + F_{k-1} F_{|m-n|}.
\] (2)

Proof. We proceed by the induction principle for two variables. It is straightforward that \( F(0, 0) = F_0 = F_0 F_2 + F_{-1} F_0 \). So, supposing that (2) holds for all \( i \leq m \) and \( j \leq n \), we have

\[
F(m + 1, n) = F(m, n - 1) + F(m - 1, n - 2)
\]

\[
= F_{k'} F_{|m-n|+2} + F_{k'-1} F_{|m-n+1|} + F_{k'-1} F_{|m-n+1|+2} + F_{k'-2} F_{|m-n+1|},
\]

where \( k' = \min\{m, n - 1\} \Rightarrow k' - 1 = \min\{m - 1, n - 2\} \). Therefore,

\[
F(m + 1, n) = F_{k'+1} F_{|(m+1)-n|+2} + F_{k'} F_{|(m+1)-n|},
\]

and since \( k' + 1 = \min\{m + 1, n\} \), the identity holds in this case. Analogously, following the same steps, the identity also holds for \( F(m, n + 1) \), which completes the proof.

In the homogeneous double-recurrence (1), one could replace the initial conditions by a general linear recurrence sequence of order two, or even arithmetic functions. In other words, we have the following:

Definition 2. Let \( m, n \in \mathbb{N} \). The function \( H(m, n) \) satisfying

\[
H(m, n) = H(m - 1, n - 1) + H(m - 2, n - 2)
\] (3)

for all \( m, n \geq 2 \), where the following initial conditions are given
with $H_1$, $H_2$, $H_1^2$ and $H_2^1$ arithmetic functions, is called a double-recurrence function. If $H_1$, $H_2$, $H_1^2$ and $H_2^1$ are linear recurrence sequences of order two, the function satisfying (3) is called a spin Function.

In this way, double-recurrence Fibonacci numbers are values of a spin Function, such as every Fibonacci and Lucas numbers. Now, let $H(m, n)$ be a spin Function, where

$$
H(m, 0) = H_1(m), \quad H(0, n) = H_2(n),
$$

$$
H(m, 1) = H_1^2(m), \quad H(1, n) = H_2^1(n),
$$

and if $m = n$, we have a linear recurrence sequence of order two, given by:

$$
H(m, m) = H_1^1(m), \quad \text{with } H_1^1(0) = a \quad \text{and} \quad H_1^1(1) = c.
$$

The motivation for the term spin function, relies on the way that we can reach, from the initial terms, all pairs of $(m, n) \in \mathbb{N}^2$, where the function is evaluated, using every secondary diagonal on it, that we refer as strings. A graphical representation of it, can be seen next.

---

**Figure 2:** A spin Function and its strings

**Figure 3:** Double-recurrence Fibonacci function

---

2 Properties and Identities

Among several generalizations for Fibonacci numbers, we now consider the ones that satisfies the Fibonacci recurrence relation, but with arbitrary initial conditions.
**Definition 3.** Let \((G_n)_n\) a linear recurrence sequence of order two, where \(G_1 = a\), \(G_2 = b\) and \(G_{n+2} = G_{n+1} + G_n\), \(n \geq 1\). The ensuing sequence is called a generalized Fibonacci sequence (GFS).

The following, is a classical result, that can be easily proved by induction, which states that every term on a GFS, can be written only in terms of Fibonacci numbers and their initial conditions.

**Theorem 4.** Let \(G_n\) denote the \(n\)th term of the GFS. Then \(G_{n+2} = bF_{n+1} + aF_n\), \(n \geq 1\).

**Proof.** See [5, Th. 7.1].

Note that, Proposition 1 can be seen as a generalization of Theorem 4 for double-recurrence Fibonacci numbers. Our immediate purpose is to show that an analogous result also holds for spin functions. In order to do so, we introduce a double-recurrence function that will play the same role as Fibonacci numbers on Theorem 4. Let \(m, n, a, b \in \mathbb{N}\). Then, define

\[
F_a^b(m, n) := bF_nF_{|m-n|+2} + aF_{n-1}F_{|m-n|}.
\]  
(6)

It is easy to see that \(F_a^b(m, n)\) is a double-recurrence function, but not necessarily a spin function, i.e.,

\[
F_a^b(m + 2, n + 2) = F_a^b(m + 1, n + 1) + F_a^b(m, n),
\]
but the functions on the initial conditions are not necessarily linear recurrence sequences of order two. For that, we have the following result.

**Proposition 5.** Let \(m, n \in \mathbb{N}\) and the spin function \(H(m, n)\), such as on Definition 2. Then,

i. If \(n \leq m - 1\), then \(H(m, n) = F_a^c(m - 1, n) + F_b^d(m - 2, n)\).

ii. If \(m - 1 < n\), then \(H(m, n) = F_a^c(n - 1, m) + F_b^d(n - 2, m)\).

**Proof.** Let \(H(m, n)\) be a spin function for \(n \leq m - 1\), with functions \(H_1^2\) and \(H_1\) given by the initial conditions described previously. Similarly to the Proposition 1, we have

\[
H(m, n) = F_nH_2^2(m - n + 1) + F_{n-1}H_1(m - n),
\]
and since \(H_1^2\) and \(H_1\) are linear recurrence sequences, using Theorem 4, we get

\[
H(m, n) = F_n(cF_{m-n+1} + dF_{m-n}) + F_{n-1}(bF_{m-n} + aF_{m-n-1})
= cF_nF_{m-n+1} + aF_{n-1}F_{m-n-1} + dF_nF_{m-n} + bF_{n-1}F_{m-n}.
\]

Using that \(bF_{n-1}F_{m-n} = b \cdot (F_{n-1}F_{m-n-2} + F_{n-1}F_{m-n-1})\), we obtain

\[
H(m, n) = cF_nF_{m-n+1} + (a + b) \cdot F_{n-1}F_{m-n-1} + dF_nF_{m-n} + bF_{n-1}F_{m-n-2}
= F_a^c(m - 1, n) + F_b^d(m - 2, n).
\]

Analogously, for \(m - 1 < n\), considering \(H_2^1\) and \(H_2\), we get

\[
H(m, n) = F_a^c(n - 1, m) + F_b^d(n - 2, m),
\]
which completes the proof. \(\Box\)
Figure 4: Graphical representation of Proposition 5

Figure 5: Combination of Proposition 5 and Definition 2

Now, we return our attention to sums of double-recurrence Fibonacci numbers. But first, we recall an interesting identity for Generalized Fibonacci Numbers \([9]\), giving an alternative proof for it.

**Proposition 6.** Let \((G_n)_n\) be a GFS, where \(G_n = G_{n-1} + G_{n-2}\) with initial conditions \(G_0 = g_0\) and \(G_1 = g_1\). Then

\[
\sum_{i=1}^{n} iG_i = nG_{n+2} - G_{n+3} + G_3
\] (7)

**Proof.** Straightforward from Theorem 4, we have \(G_n = g_0F_{n-1} + g_1F_n\). Thus,

\[
\sum_{i=1}^{n} iG_i = g_0\sum_{i=1}^{n} iF_{i-1} + g_1\sum_{i=1}^{n} iF_i
\]

\[
= g_0\sum_{i=0}^{n-1} (i + 1)F_i + g_1\sum_{i=0}^{n} iF_i \quad (8)
\]

\[
= g_0((n - 1)F_{n+1} - F_{n+2} + 2 + F_{n+1} - 1) + g_1(nF_{n+2} - F_{n+3} + 2)
\]

\[
= n(g_0F_{n+1} + g_1F_{n+2}) - (g_0F_{n+2} + g_1F_{n+3}) + 2g_0 + g_1
\]

\[
= nG_{n+2} - G_{n+3} + G_3
\] (9)

Where, from (8) to (9), the identity \(\sum_{i=1}^{n} iF_i = nF_{n+2} - F_{n+3} + 2\), [8, p.16, Ex.10], is used. □

The following proposition, consists of a closed form to calculate the sums of Double-Fibonacci numbers, where the indices are in \(\{1, \ldots, m\}^2\).

**Proposition 7.** Let \(F(i, j)\) be Double-Fibonacci Numbers, where \(i, j \in \{0, 1, \ldots, m\}\). Then,
i. The sum of all Double-Fibonacci Numbers with indices below the main diagonal, including it, is given by

$$\sum_{i,j=0}^{m} F(i,j) = \frac{2}{5} (mL_{m+3} - L_{m+4} + 2F_{m+2}) + 2. \quad (10)$$

ii. The sum of all Double-Fibonacci Numbers, with indices on the square $m \times m$, is

$$\sum_{i,j=0}^{m} F(i,j) = \frac{4}{5} (mL_{m+3} - L_{m+4} + 2F_{m+2}) - F_{m+2} + 5.$$

Proof. First, we proceed to prove (i), and use it to prove (ii). Rewriting (10), and using the closed form on Proposition 1, we have

$$\sum_{i,j=0}^{m} F(i,j) = \sum_{i=0}^{m} \sum_{j=0}^{i} F_j F_{i-j+2} + F_{j-1} F_{i-j}$$

$$= \sum_{i=0}^{m} \sum_{j=0}^{i} F_j F_{i-j+1} + F_{j-1} F_{i-j} + F_j F_{i-j},$$

and since $F_i = F_{j} F_{i-j+1} + F_{j-1} F_{i-j}$, it follows,

$$= \sum_{i=0}^{m} \sum_{j=0}^{i} F_i + F_j F_{i-j}$$

$$= \sum_{i=0}^{m} \left( (i+1) F_i + \sum_{j=0}^{i} F_{i-j} F_j \right).$$

Now, we observe that the sum $\sum_{j=0}^{i} F_{i-j} F_j$, is referenced as sequence A001629 on [1], where is established that it is equal to $((i - 1)F_i + 2iF_{i-1})/5 = (iL_i - F_i)/5$, $(L_n)_{n \geq 0}$ being the Lucas Sequence, and the last equality follows from [5, Eq. 32.13, p. 375]. Thus,

$$\sum_{i,j=0}^{m} F(i,j) = \sum_{i=0}^{m} (i+1) F_i + \sum_{i=0}^{m} \frac{iL_i - F_i}{5}.$$

From Proposition 6 and $\sum_{i=1}^{n} F_i = F_{n+2} - 1$, we have,

$$= m \left( \frac{L_{m+2}}{5} + F_{m+2} \right) - \left( \frac{L_{m+3}}{5} + F_{m+3} \right) + \frac{4}{5} F_{m+2} + 2,$$
then, finally by $L_{n-1} + L_{n+1} = 5F_n$ (see [5, Cor. 5.5, p. 80]), it follows that

$$
\frac{m(L_{m+2} + L_{m+1} + L_{m+3})}{5} - \frac{(L_{m+4} + L_{m+2} + L_{m+3})}{5} + \frac{4}{5}F_{m+2} + 2
$$

$$
\therefore \sum_{i,j=0 \atop i\geq j}^m F(i, j) = \frac{2}{5}(mL_{m+3} - L_{m+4} + 2F_{m+2}) + 2,
$$

completing the proof for (i). For (ii), we use the symmetry satisfied by double-recurrence Fibonacci Numbers, $F(m, n) = F(n, m)$, giving us that the sum on (ii) is two times the sum on (i), minus the sum for indices on the main diagonal:

$$
\sum_{i,j=0 \atop i\geq j}^m F(i, j) = 2\sum_{i,j=0 \atop i\geq j}^m F(i, j) - \sum_{i=0}^m F(i, i)
$$

$$
= \frac{4}{5}(mL_{m+3} - L_{m+4} + 2F_{m+2}) + 4 - \sum_{i=0}^m F_i
$$

$$
= \frac{4}{5}(mL_{m+3} - L_{m+4} + 2F_{m+2}) - F_{m+2} + 5.
$$

Out of curiosity, equation (10) happens to be the same formula for the path length of the Fibonacci tree of order $n$. (A178523 of [1])

3 Acknowledgements

During the preparation of this paper, Ana Paula Chaves was supported in part by CNPq Universal 01/2016 - 427722/2016-0 grant, and Carlos Alirio Rico Acevedo was fully supported by a Masters Scholarship from CNPq.

References

[1] The On-Line Encyclopedia of Integer Sequences, https://oeis.org/.
[2] H. Hosoya, Fibonacci triangle, Fibonacci Quart. 14 (1976), no. 2, 173–179.
[3] E. P. Miles Jr., Generalized Fibonacci numbers and associated matrices, Amer. Math. Monthly 67 (1960), 745–752. MR 0123521
[4] D. Kalman and R. Mena, *The Fibonacci numbers—exposed*, Math. Mag. **76** (2003), no. 3, 167–181.

[5] T. Koshy, *Fibonacci and Lucas numbers with applications*, Pure and Applied Mathematics (New York), Wiley-Interscience, New York, 2001.

[6] M. D. Miller, *Mathematical Notes: On Generalized Fibonacci Numbers*, Amer. Math. Monthly **78** (1971), no. 10, 1108–1109. MR 1536552

[7] A. S. Posamentier and I. Lehmann, *The (fabulous) Fibonacci numbers*, Prometheus Books, Amherst, NY, 2007, With an afterword by Herbert A. Hauptman.

[8] N. N. Vorobiev, *Fibonacci numbers*, Birkhäuser Verlag, Basel, 2002, Translated from the 6th (1992) Russian edition by Mircea Martin.

[9] C. R. Wall, *Problem b-40*, Fibonacci Quart. **2. (4)** (1964), 327–328.

---

2010 Mathematics Subject Classification: Primary 11B39; Secondary 11J86.

Keywords: Fibonacci numbers, double-recurrence sequence, closed form.

(Concerned with sequences A001629, A002940, A006478, A010049, A014286, A122491, A178523, A190062.)

---

**Appendix**

The following table explicit some interesting sequences founded on [1], that can be obtained from the sum of the terms of $H(i,j)$, with initial conditions $a, b, c$ and $d$, considering $0 \leq j < i \leq n$, $0 \leq i \leq j \leq n$, and all $i,j \in \{0,1,\ldots,n\}^2$. 

---

9
| Initial Condition $[a, b, c, d]$ | $\sum_{i=1}^{n} \sum_{j=0}^{i-1} H(i, j)$ | $\sum_{j=0}^{n} \sum_{i=0}^{j} H(i, j)$ | $\sum_{i,j=0}^{m} H(i, j)$ |
|----------------------------------|---------------------------------|---------------------------------|------------------|
| $[0, 0, 0, 1]$                  | A006478($n$)                    | A001629($n + 1$)                | A006478($n + 1$) |
| $[0, 0, 1, 0]$                  | A002940($n - 2$)                | A006478($n + 1$)                | -                |
| $[0, 0, 1, 1]$                  | -                               | A122491($n + 2$)                | -                |
| $[0, 1, 0, 0]$                  | A001629($n + 1$)                | A006478($n$)                    | A006478($n + 1$) |
| $[0, 1, 0, 1]$                  | A006478($n + 1$)                | A006478($n + 1$)                | A178523($n + 1$) |
| $[0, 1, 1, 0]$                  | A014286($n$)                    | A002940($n - 1$)                | -                |
| $[0, 1, 1, 1]$                  | -                               | A178523($n + 1$)                | -                |
| $[1, 0, 0, 0]$                  | A001629($n$)                    | A010049($n + 1$)                | -                |
| $[1, 0, 0, 1]$                  | A122491($n + 1$)                | A001629($n + 2$)                | -                |
| $[1, 0, 1, 0]$                  | A178523($n$)                    | -                               | -                |
| $[1, 0, 1, 1]$                  | -                               | A006478($n + 2$)                | -                |
| $[1, 1, 0, 0]$                  | A006478($n + 1$)                | A190062($n + 1$)                | -                |
| $[1, 1, 1, 0]$                  | A002940($n - 1$)                | -                               | -                |
| $[1, 1, 1, 1]$                  | -                               | A014286($n + 11$)               | -                |

Table 1: Related sequences.