GLOBAL WEAK SOLUTIONS FOR AN NEWTONIAN FLUID INTERACTING WITH A KOITER TYPE SHELL UNDER NATURAL BOUNDARY CONDITIONS

HANNES EBERLEIN AND MICHAEL RŮŽIČKA

ABSTRACT. We consider an viscous, incompressible Newtonian fluid flowing through a thin elastic structure. The motion of the structure is described by the equations of a linearised Koiter shell, whose motion is restricted to transverse displacements. The fluid and the structure are coupled by the continuity of velocities and an equilibrium of surface forces on the interface between fluid and structure. On a fixed in- and outflow region we prescribe natural boundary conditions. We show that weak solutions exist as long as the shell does not self-intersect.

Keywords: Fluid-Structure Interaction, Koiter shell, Navier-Stokes equation, global weak solution, boundary values

2000MSC: 76D05, 74F10

1. INTRODUCTION

Fluid-structure interaction (FSI) problems are common in nature, most prominently seen in the blood flow through vessels. Blood generally exhibits a non-Newtonian behaviour (see [11]), while the vascular wall consists of many layers all having different biomechanical properties (see [33]). If medium or big blood vessels are considered one can use a simplified model. Firstly, we treat blood as a Newtonian fluid. Since only a small part of the circulatory system is considered we introduce artificial in- and outflow boundary conditions. Secondly, we will model the vascular wall as a linearly elastic Koiter shell whose motion is restricted to transverse displacements.

Due to the apparent regularity incompatibilities between the parabolic fluid phase and the hyperbolic dispersive solid phase, the interaction between an elastic structure and a fluid is exceedingly difficult, see e.g. [8] [18] [25] and the references therein. For the existence proof of weak solutions, two different strategies have been proven successful. The first one is working directly on the fluid domain and therefore preserving the structure of the equations. Since by the non-cylindrical space-time domain the usual Bochner-space theory is not applicable, the key element in this approach is to find an appropriate compactness argument. This was accomplished in [22], [24] and [23], by generalising methods from [8], [18]. There, the existence of a global-in-time weak solution to the interaction of an incompressible generalised Newtonian fluid completely surrounded with an linearised transversal Koiter shell was shown. The second approach consists of transforming the fluid equation by an ALE mapping to a reference domain and using a semi-discrete, operator splitting Lie scheme. This method was used in [29], [30] and [31] to show the existence of a global-in-time weak solution to the interaction between a Newtonian fluid and a (semi-) linear transversal Koiter shell, where the flow is driven by a dynamical pressure condition and no other external forces apply. Furthermore, in [32] the Navier slip boundary condition was used for the coupling between the fluid and an elastic structure in a two-dimensional setting.

The present paper is based on the first authors Ph.D. thesis [16] and extends the result of [24] to the case of an in- and outflow region. We will use the same method as in [24] and thus the structure and the arguments of the present paper are similar to [24]. Since the in- and outflow region admits a flow through the domain, various new difficulties have to
be solved. Moreover, since the reference domain has no $C^4$-boundary, but only a Lipschitz one, special care has to be taken to transfer the compactness argument to our situation. In comparison to \cite{30} and \cite{31}, we consider a different in- and outflow condition and a more general geometry. Furthermore, we allow external forces acting on the fluid and the shell.

This paper is organised as follows: In the next Subsections, we introduce our setting and formally derive an a priori estimate for the resulting system. In Section 2 we develop the mathematical framework for our non-Lipschitz in- and outflow domains. In particular, in Subsection 2.1 we define a generalised trace operator and show some density results for functions vanishing on a part of the boundary, and in Subsection 2.2 we construct a suitable extension operator of test functions for the shell equation. In Section 3, we can finally formulate the main result of this paper, the existence of a global-in-time weak solution. The proof of the main Theorem is then carried out by looking first at a decoupled, regularised limiting process to eliminate the regularisation, and formally derive an a priori estimate for the resulting system. In Section 2 we develop an extension operator of test functions vanishing on a part of the boundary, and in Subsection 2.4 we construct a suitable extension operator of test functions for the shell equation. In Section 3, we can finally formulate the main result of this paper, the existence of a global-in-time weak solution.

### 1.1. Koiter’s energy and statement of the problem

By $\Omega \subset \mathbb{R}^3$ we denote a reference domain with $\partial \Omega = \Gamma \cup M$, where $\Gamma$ is assumed to be the fixed in- and outflow region and $\partial \Gamma = \partial M \neq \emptyset$. Moreover, let $M$ represent the middle surface of the thin elastic shell of thickness $2 \varepsilon_0 > 0$ in its rest state. The deformation of the shell is then given by the displacement $\eta$ relatively to $M$. By $v$ we denote the unit normal on $\partial \Omega$, by $g$ and $h$ the first and second fundamental form of $M$ induced by the ambient Euclidean space, and by $dA$ the surface measure of $M$ or $\partial \Omega$, respectively. As in \cite{24} \cite{30} we restrict the deformations to transverse displacements, i.e. $\eta = \eta v$. Following \cite{24}, we assume further that the elastic shell, clamped on $\partial M$, consists of a homogeneous, isotropic material whose linear elastic behaviour is characterised by the Lamé constants $\lambda$ and $\mu$, and the elastic energy is given by Koiter’s energy for linearly elastic shells and transverse displacements $K(\eta) := K(\eta, \eta)$ with

$$K(\eta, \zeta) := \frac{1}{2} \int_M \varepsilon_0 (C, \sigma(\eta) \otimes \sigma(\zeta)) + \frac{\varepsilon_0^3}{3} (C, \xi(\eta) \otimes \xi(\zeta)) \, dA.$$  

Here $C$ denotes the elasticity tensor of the shell,  

$$C_{\alpha\beta\gamma\delta} = \frac{4\mu\lambda}{\lambda + 2\mu} g_{\alpha\beta} g_{\gamma\delta} + 2\mu (g_{\alpha\gamma} g_{\beta\delta} + g_{\alpha\delta} g_{\beta\gamma})$$

and

$$\sigma(\eta) = -h \eta, \quad \xi(\eta) = \nabla^2 \eta - k \eta$$

are the linearised strain tensors, where $k_{\alpha\beta} := h^i_{\alpha} h_{i\beta}$, $\nabla$ denotes the Levi-Civita connection of $M$, and $\Delta$ is the corresponding Laplacian. We refer the interested reader to \cite{9} \cite{10} for additional details to the Koiter model. As shown in \cite{10} Theorem 4.4-2], $K(\eta, \zeta)$ is a symmetric bilinear form (and therefore $K(\eta)$ a quadratic form) which is coercive on $H^2_0(M)$. Moreover, as has been shown by partial integration in \cite{22} Chapter 3], the $L^2$-gradient of this Koiter energy satisfies

$$2K(\eta, \zeta) = \int_M \text{grad}_L K(\eta) \xi \, dA, \quad \eta, \xi \in H^2_0(M). \quad (1.1)$$

Throughout the paper, we use standard notations for (vector valued) function spaces. We denote the fluid domain, which depends on a time $t \in I$ and is a priori not known, by $\Omega_{\eta(t)} \subset \mathbb{R}^3$. Then the motion of a homogeneous, incompressible, Newtonian fluid on the
space-time domain $\Omega^t := \bigcup_{t \in I} \{t\} \times \Omega_{\eta(t)}$ is governed by the system

$$\rho \partial_t u + \rho (u \cdot \nabla) u = \text{div} \left( 2 \sigma \delta u - \pi I d + f \right) \text{ in } \Omega^t, \quad (1.2)$$

$$\text{div} u = 0 \text{ in } \Omega_{\eta}^t, \quad (1.3)$$

where $u$ is the velocity field, $Du$ the symmetric part of the gradient of $u$, $\pi$ the pressure field, $\rho$ the density, $\sigma$ the dynamic viscosity, $f$ an external body force and $Id$ the $3 \times 3$ unit matrix. We will assume no-slip boundary conditions on the deformed boundary and natural boundary conditions on $\Gamma$, i.e.

$$u(\cdot, t) + \eta \nu = \partial_t \phi \text{ on } I \times M, \quad (1.4)$$

$$(2 \sigma \delta u - \pi I d) \nu = \frac{\rho}{2} (u \cdot \nu) u \text{ on } I \times \Gamma. \quad (1.5)$$

Using the map $\Phi_{\eta(t)} : M \to \partial \Omega_{\eta(t)} \setminus \Gamma$, $\Phi_{\eta(t)}(q) := \eta(t, q) \nu(q)$ to parametrize the deformed boundary, the force exerted by the fluid on this boundary is given by

$$F := (\partial \Phi_{\eta(t)} \nu_{\eta(t)}) + \pi \nu \circ \partial \Phi_{\eta(t)} [\text{det} \Phi_{\eta(t)}],$$

where $\nu_{\eta(t)}$ is the outer normal of $\Omega_{\eta(t)}$. The external forces acting on the shell along the outer normal are thereby composed of $F \cdot \nu$ and some given external force $g$. Using Hamilton’s principle, the displacement $\eta$ of the shell is a stationary point of the integrated difference between the kinetic and potential energy of the shell and therefore satisfies the corresponding Euler-Lagrange equation

$$2 \mu \rho \partial_t^2 \eta + \text{grad}_I^2 K(\eta) = g + F \cdot \nu \text{ in } I \times M. \quad (1.6)$$

Since the shell is clamped, we have

$$\eta = 0, \quad \nabla \eta = 0 \text{ on } I \times \partial M. \quad (1.7)$$

Finally we prescribe the initial conditions

$$u(\cdot, 0) = u_0 \text{ in } \Omega_{\eta_0} \quad \text{and} \quad \eta(\cdot, 0) = \eta_0, \partial_t \eta(\cdot, 0) = \eta_1 \text{ in } M. \quad (1.8)$$

In the following, we will analyse the system (1.2)–(1.8).

### 1.2. Formal a priori estimates.

We take (sufficiently smooth) solutions $u$ and $\eta$ of the fluid- and shell equation. Multiplying the fluid equation $(1.2)$ with $u$ and integrating over $\Omega_{\eta(t)}$ leads to

$$\rho \int_{\Omega_{\eta(t)}} (\partial_t u)(t) \cdot u(t) \, dx + \rho \int_{\Omega_{\eta(t)}} (u(t) \cdot \nabla) u(t) \cdot u(t) \, dx$$

$$= \int_{\Omega_{\eta(t)}} \text{div} (2 \sigma \delta u(t)) \cdot u(t) \, dx - \int_{\Omega_{\eta(t)}} \nabla \pi(t) \cdot u(t) \, dx + \int_{\Omega_{\eta(t)}} f(t) \cdot u(t) \, dx.$$

Using Reynolds transport theorem and our boundary conditions, i.e. the domain velocity equals the fluid velocity on the moving boundary $\partial \Omega_{\eta(t)} \setminus \Gamma$ and vanishes on $\Gamma$, the first term of the equation can be written as

$$\rho \int_{\Omega_{\eta(t)}} (\partial_t u)(t) \cdot u(t) \, dx$$

$$= \frac{\rho}{2} \frac{d}{dt} \int_{\Omega_{\eta(t)}} |u(t)|^2 \, dx - \frac{\rho}{2} \int_{\partial \Omega_{\eta(t)}} |u(t)|^2 u(t) \cdot \nu_{\eta(t)} \, dA_{\eta(t)},$$

where $\nu_{\eta(t)}$ denotes the outer normal and $dA_{\eta(t)}$ the surface measure on $\partial \Omega_{\eta(t)}$. By partial integration and taking the divergence constraint into account, the convective term vanishes except a boundary term

$$\rho \int_{\Omega_{\eta(t)}} (u(t) \cdot \nabla) u(t) \cdot u(t) \, dx = \frac{\rho}{2} \int_{\partial \Omega_{\eta(t)}} |u(t)|^2 u(t) \cdot \nu_{\eta(t)} \, dA_{\eta(t)}. \quad (1.11)$$
Likewise we get by partial integration, the divergence constraint, the symmetry of $Du$ and the the boundary condition (1.5) as well as $v_{\eta(t)} = v$ on the fixed boundary $\Gamma$.

$$
\int_{\Omega_{\eta(t)}} \text{div}(2\sigma Du(t)) \cdot u(t) \, dx - \int_{\Omega_{\eta(t)}} \nabla \pi(t) \cdot u(t) \, dx \tag{1.12}
$$

$$
= -2\sigma \int_{\Omega_{\eta(t)}} Du(t) : Du(t) \, dx + \int_{\partial\Omega_{\eta(t)}} \left(2\sigma Du(t) \cdot v_{\eta(t)}\right) \cdot u(t) \, dA_{\eta(t)}
$$

$$
- \int_{\partial\Omega_{\eta(t)}} \pi(t) u(t) \cdot v_{\eta(t)} \, dA_{\eta(t)} + \frac{p}{2} \int_{\Gamma} |u(t)|^2 u(t) \cdot v \, dA.
$$

Hence, using (1.10), (1.11) and (1.12), the equation (1.9) can be written as

$$
\frac{p}{2} \frac{d}{dt} \int_{\Omega_{\eta(t)}} |u(t)|^2 \, dx + 2\sigma \int_{\Omega_{\eta(t)}} Du(t) : Du(t) \, dx
$$

$$
= \int_{\Omega_{\eta(t)}} f(t) \cdot u(t) \, dx + \int_{\partial\Omega_{\eta(t)}} \left(2\sigma Du(t) \cdot v_{\eta(t)} - \pi(t) v_{\eta(t)}\right) \cdot u(t) \, dA_{\eta(t)}.
$$

Multiplying equation (1.6) with $\partial_\nu \eta(t)$, integrating over $M$ and using the bilinearity of the Koiter-Energy as well as $(\text{grad}_{L^2} K(\eta(t)), \partial_\nu \eta(t))_{L^2(M)} = 2K(\eta(t), \partial_\nu \eta(t))$, we get

$$
\varepsilon_\rho, \rho, d \int_M |\partial_\nu \eta(t)|^2 \, dA + d \int_M K(\partial_\nu \eta(t))
$$

$$
= \int_M g(t) \partial_\nu \eta(t) \, dA + \int_M F(t) \cdot v \, dA. \tag{1.14}
$$

Using the definition of $F$, the boundary condition (1.4) and a change of variables, adding (1.13) and (1.14) leads to the energy equality

$$
\frac{d}{dt} \left(\frac{p}{2} \int_{\Omega_{\eta(t)}} |u(t)|^2 \, dx + 2\sigma \int_0^t \int_{\Omega_{\eta(s)}} |Du(s)|^2 \, dx \, ds + \varepsilon_\rho, \rho_\rho_\rho, \int_M |\partial_\nu \eta(t)|^2 \, dA + K(\eta(t))\right)
$$

$$
= \int_{\Omega_{\eta(t)}} f(t) \cdot u(t) \, dx + \int_M g(t) \partial_\nu \eta(t) \, dA. \tag{1.15}
$$

We denote the expression in the parentheses on the left-hand side of (1.15) by $E(t)$, and set $E_0 := E(0)$, which depends by our initial conditions (1.8) only on $\eta_0, \eta_1$ and $u_0$. The coercivity of the Koiter Energy $K$ and Grönwall’s inequality (see [6], Appendix) imply the estimate

$$
\textup{esssup}_{t \in (0, T)} \sqrt{E(t)} \leq \sqrt{E_0} + \int_0^T \frac{1}{\sqrt{2p}} \|f(s, \cdot)\|_{L^2(\Omega_{\eta(s)})} + \frac{1}{2\sqrt{\varepsilon_\rho, \rho_\rho_\rho}} \|g(s, \cdot)\|_{L^2(M)} \, ds. \tag{1.16}
$$

Again by the coercivity of the Koiter Energy the quantity

$$
\|u\|_{L^2(I; L^2(\Omega_{\eta(t)}))} + \|Du\|_{L^2(I; L^2(\Omega_{\eta(t)}))} + \|\partial_\nu \eta(t)\|_{L^2(I; L^2(M))} + \|\eta(t)\|_{L^2(I; H^2(M))}^2
$$

is bounded by a constant depending only on the data. This (spatial) regularity of the displacement $\eta(t, \cdot) \in H^2(M)$ is not enough to ensure Lipschitz continuity, but only Hölder continuity $C^{\alpha,\lambda}(M)$ for any $\lambda < 1$. Hence, the boundary of the moving domain $\partial_\nu \eta(t)$ is not necessarily Lipschitz and the classical partial integration theorem, trace operators, etc. cannot be used. In the next Section we will develop the necessary framework for the moving domain, using a special reference domain. For the sake of better readability, we set the constants $\varepsilon_\rho, \rho, \rho_\rho_\rho$ and $\sigma$ equal to 1 throughout the paper, but emphasize that all the computations hold with the original constants.
2. MOVING DOMAINS

We assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with $\partial \Omega \in C^{0,1}$ and $\partial \Omega = \Gamma \cup M$, where $M, \Gamma \neq \emptyset$ are compact, oriented, embedded two-dimensional $C^4$-manifolds with smooth non-empty boundaries and $\partial M = \partial \Gamma$. Furthermore, we assume $M$ to be connected and $\Gamma$ to be the finite union of connected components $\Gamma_i$, where $\Gamma_i$ is subset of a hyperplane perpendicular to $M$, i.e. the continuous extension of the outer normal $\nu$ on $int M$ to $\partial M \cap \partial \Gamma_i$ is perpendicular to the outer normal of the hyperplane. For $\alpha > 0$ the open $\alpha$-tube $S_\alpha$ and the half-closed tube $\bar{S}_\alpha$ around $int M$ are given by

$$S_\alpha := \{ x \in \mathbb{R}^3 \mid x = q + s\nu(q), q \in int M, -\alpha < s < \alpha \},$$

$$\bar{S}_\alpha := \{ x \in \mathbb{R}^3 \mid x = q + s\nu(q), q \in M, -\alpha < s < \alpha \}.$$

We assume that there exists a $\kappa > 0$ such that the map $\Lambda : M \times (-\kappa, \kappa) \to S_\kappa$, $\Lambda(q,s) := q + s\nu(q)$ is a $C^1$-diffeomorphism, and write $\Lambda^{-1}(x) = (q(x), s(x))$ for the inverse mapping. For $0 < \alpha \leq \kappa$ we divide the boundary of $S_\alpha$ into the parts

$$\partial S_\alpha = \{ x \in \mathbb{R}^3 \mid x = q + \alpha\nu(q), q \in int M \}
\cup \{ x \in \mathbb{R}^3 \mid x = q - \alpha\nu(q), q \in int M \}
\cup \{ x \in \mathbb{R}^3 \mid x = q + s\nu(q), q \in \partial M, -\alpha \leq s \leq \alpha \}
=: M_\alpha^+ \cup M_\alpha^- \cup \Gamma_\alpha^\circ,$$

where the outer normal is again defined through $int M$. Furthermore, we can assume that the sets $M_\alpha^+, M_\alpha^-$ and $\Gamma_\alpha^\circ$ are disjoint and the representations through $x = q + \beta \nu(q)$ are unique. Hence, the orthogonality assumption implies $\partial S_\alpha \in C^{0,1}$. We will require that the domain’s deformation is taking place inside $\Omega \cup S_\kappa$. To ensure that the deformed moving boundary does not interfere with the fixed in- and outflow boundary $\Gamma$, we assume that

$$\{ x \in \mathbb{R}^3 \mid x = q + s\nu(q), q \in \Gamma \cup \Gamma_\kappa^\circ, s \in [0, \varepsilon) \} \cap (S_\kappa \cup \Omega) = \emptyset$$

for some $\varepsilon > 0$, where $\nu$ is the extension of the outer normal on $int \Gamma$. Finally, we assume that for all $0 < \alpha < \kappa$ it holds that $int \Gamma_\alpha^\circ \neq \emptyset$, where

$$\Gamma_\alpha^\circ := \Gamma \setminus \{ x \in \mathbb{R}^3 \mid x = q + s\nu(q), q \in \partial M, -\alpha < s \leq \alpha \}.$$

We call a domain $\Omega$ fulfilling these requirements an admissible in- and outflow domain. Its simplest example is the straight tubular cylinder. For the rest of the paper, we will always assume that $\Omega$ is an admissible in- and outflow domain.

---

1 Since $int M$ is an embedded $C^4$-manifold, the tubular neighbourhood theorem already ensures the existence of an $C^3$-diffeomorphism from $(-\kappa, \kappa) \times int M$ to $S_\kappa$.
Definition 2.1. Let \( \eta : M \to (-\kappa, \kappa) \) be continuous. We set the moving domain \( \Omega_\eta \) as
\[
\Omega_\eta := (\Omega \setminus S_\kappa) \cup \{ x \in S_\kappa \mid s(x) < \eta(q(x)) \},
\]
see Figure 7. Furthermore, for \( 0 < \alpha < \kappa \) we set \( B_\alpha := \Omega \cup S_\alpha \).

Remark 2.2. \( \Omega_\eta \) and \( B_\alpha \) are bounded domains, \( \partial B_\alpha = \Gamma_0^\alpha \cup \Gamma_1^\alpha \cup M_\alpha^\alpha \) and \( \partial B_\alpha \in C^{0,1} \). With \( \Gamma_\eta := \partial \Omega_\eta \setminus \{ x \in S_\kappa | s(x) = \eta(q(x)) \} \) we have \( \Gamma_\eta \subset \Gamma_0^\eta \cup \Gamma_1^\eta \) and in particular \( \Gamma_\eta = \Gamma \) for \( \eta = 0 \) on \( \partial M \).

For a given displacement \( \eta \in C^0(M) \) with \( \| \eta \|_{L_\infty(M)} < \kappa \) we choose a cut-off function \( \beta \in C^\infty(\mathbb{R}) \) with \( \beta = 0 \) in a neighbourhood of \(-1\), \( \beta = 1 \) in a neighbourhood of \(0\), and \( \| \beta \|_{L_\infty([-1,0])} < \kappa \| \eta \|_{L_\infty(M)}^{-1} \). We define the Hanzawa transform (see [19]) trough
\[
\Psi_\eta(x) := \begin{cases} 
 x + \eta(q(x)) \beta(x) \nu(q(x)) & x \in \Omega \cap S_\kappa, \\
 x & x \in \Omega \setminus S_\kappa,
\end{cases}
\]
where \( \nu \) on \( \partial M \) is the extended extension of the outer normal \( \nu \) on \( int M \) to \( \partial M \). By the choice of the cut-off function \( \beta \) and the properties of the diffeomorphism \( \Lambda \), one can show that \( \Psi_\eta : \Omega \to \Omega_\eta \), as well as \( \Phi_\eta : \partial \Omega \to \partial \Omega_\eta \) with \( \Phi_\eta := \Psi_\eta |_{\partial \Omega} \) are homeomorphisms which inherit the regularity of \( \eta \), i.e., \( \Psi_\eta \) and \( \Phi_\eta \) are \( C^k \)-diffeomorphisms if \( \eta \in C^k(M) \), \( k \in \{1,2,3\} \) and the Jacobian determinant \( \det d\Psi_\eta \) is positive. Furthermore, the components of the Jacobians of \( \Psi_\eta \), \( \Phi_\eta \) and their inverses have the form
\[
b_0 + b_1(\eta \circ q) + b \cdot (\nabla \eta \circ q)
\]
for some bounded, continuous functions \( b_0, b_1, b \), whose supports are contained in \( S_\kappa \). The details of these calculations can be found in [16]. The cut-off function \( \beta \) used in these definitions clearly can be chosen uniformly for a set of displacements fulfilling \( \| \eta \|_{L_\infty(M)} \leq \alpha < \kappa \). Furthermore, by some easy calculations we get the following result:

Lemma 2.4. Let \( \eta, \eta_0 \in C^0(M) \) with \( 0 \leq \| \eta \|_{L_\infty(M)} \leq \| \eta_0 \|_{L_\infty(M)} < \kappa \) let \( \Psi_{\eta_0}, \Psi_\eta \) be defined using the same cut-off function \( \beta \).

a) Let \( \eta_n \to \eta \) uniformly on \( M \). Then \( \Psi_{\eta_n} \) converges uniformly to \( \Psi_\eta \) on \( \Omega \).

b) Let \( \eta_n \to \eta \) uniformly on \( M \) and \( \Psi_{\eta_n}^{-1}, \Psi_\eta^{-1} \) extended by \( q \) to \( B_\kappa := S_\kappa \cup \Omega \). Then
(2.3)
\[
\Psi_{\eta_n}^{-1} \text{ converges uniformly to } \Psi_\eta^{-1} \text{ on } B_\kappa.
\]
c) Let \( \eta_n \rightharpoonup \eta \) weakly in \( H^2(M) \) and \( 1 \leq s < \infty \). Then \( \nabla \Psi_{\eta_n} \) converges to \( \nabla \Psi_\eta \) in \( L^s(\Omega) \) and the (canonically extended) functions \( (\nabla \Psi_{\eta_n}^{-1}) \chi_{\Omega_\eta} \) converge to \( (\nabla \Psi_\eta^{-1}) \chi_{\Omega_\eta} \) in \( L^s(B_\kappa) \). Also the Jacobian determinants \( \det d\Psi_{\eta_n} \) converge to \( \det d\Psi_\eta \) in \( L^1(\Omega) \) and \( \det d\Psi_{\eta_n}^{-1} \chi_{\Omega_\eta} \) converges to \( \det d\Psi_\eta^{-1} \chi_{\Omega_\eta} \) in \( L^1(B_\kappa) \).

By our formal a priori estimate, we cannot expect the deformation and therefore the Hanzawa transform to be Lipschitz. Hence, a change of variables by the Hanzawa transform does not hold in the classical sense. Since the Hanzawa transformation inherits the regularity of the displacement \( \eta \) and preserves the convergence of \( \eta_n \) in a suitable way by the preceding Lemma, an approximation argument shows that a change of variables is still possible. However, we have a small loss of regularity since the Jacobian determinant \( \det d\Psi_\eta \) is not \( L^\infty \). A careful inspection of this approximation argument, which can be found in [16], also gives a bound for the continuity constant.

Lemma 2.5. Let \( \eta \in H^2(M) \) with \( \| \eta \|_{L_\infty(M)} < \kappa \) and \( 1 < p \leq \infty \). Then for all \( 1 \leq r < p \) the linear mapping
\[
v \mapsto v \circ \Psi_\eta
\]
is continuous from \( L^p(\Omega_\eta) \) to \( L^r(\Omega) \) and from \( W^{1,q}(\Omega_\eta) \) to \( W^{1,r}(\Omega) \). The continuity constant only depends on \( \Omega, p, r, \| \eta \|_{H^2(M)} \) and the choice of \( \beta \) in the definition of the Hanzawa transform. An analogous statement holds for \( \Psi_\eta^{-1} \) instead of \( \Psi_\eta \).
Combining the last two results, we also get the following Lemma.

**Lemma 2.6.** Let $1 < p \leq \infty$, $\eta_\omega \rightharpoonup \eta$ weakly in $H^2(\Omega)$ with $\| \eta \|_{L^\infty(\Omega)} < \kappa$ and let $\Psi_{\eta_\omega}, \Psi_{\eta}$ be defined by the same cut-off function $\beta$. For $v \in W^{1,p}(B_\epsilon)$ it holds $v \circ \Psi_{\eta_\omega} \rightharpoonup v \circ \Psi_{\eta}$ in $W^{1,r}(\Omega)$ for all $1 \leq r < p$. For $v \in W^{1,p}_0(\Omega)$ it holds $v \circ \Psi_{\eta_\omega} \rightarrow v \circ \Psi_{\eta}$ in $W^{1,r}(B_\epsilon)$ for all $1 \leq r < p$, where the functions are extended by 0. Similar results hold for the convergence in the appropriate $L^p$-spaces.

To preserve divergence constraint, we introduce the Piola transform $\mathcal{T}_\eta$ of $\eta$ as push-forward of $(\det d\Psi_{\eta})^{-1}\Phi_{\eta}$ under $\Psi_{\eta}$, i.e., $\mathcal{T}_\eta = (d\Psi_{\eta})^{-1}(\Phi_{\eta}) \circ \Phi_{\eta}^{-1}$. Hence $\mathcal{T}_\eta$ with $\mathcal{T}_0 = 1$ is an isomorphism between the Lebesgue- and Sobolev spaces on $\Omega$, respectively $\Omega_\eta$, as long as the order of differentiability is not larger than one. Furthermore, $\mathcal{T}_\eta$ preserves the divergence-free property, as can be seen in [28] Theorem 7.20.

Using Lemma 2.6, we can obtain the usual Sobolev embeddings by transforming to the reference domain, but we have to take into account a loss of regularity. Also, the following trace operator, which compares the fluid velocity and the boundary velocity (which is given in Lagrange coordinates) is continuous.

**Definition 2.7.** Let $1 < p \leq \infty$ and $\eta \in H^2(\Omega)$ with $\| \eta \|_{L^\infty(\Omega)} < \kappa$. For $1 < r < p$ we define the linear, continuous trace operator $\text{tr}_\eta$ by

$$\text{tr}_\eta : W^{1,p}(\Omega_\eta) \rightarrow W^{1,\frac{p}{p-r}}(\partial \Omega), \quad v \mapsto (v \circ \Psi_{\eta})|_{\partial \Omega}.$$

Since the regularity of $\partial \Omega_\eta$ does not guarantee the existence of an outer normal, we will derive an analogous formula of Green type. We approximate $\eta \in H^2(M)$ by $(\eta_n)_{n \in \mathbb{N}} \subset C^2(M)$ in $H^2(M)$ and choose the same cut-off function $\beta$ for the definitions of $\Phi_{\eta_n}$. Since $\text{int} \, M$ is a two-dimensional $C^1$-manifold, there exist (locally) orthonormal, tangential $C^1$ vector fields $e_1, e_2$ with $e_1 \times e_2 = v$. We set the (local) vector fields $v^1_\eta := d\Phi_{\eta_n} e_1, v^2_\eta := d\Phi_{\eta_n} e_2$ and consider $d\Phi_{\eta_n}$ as a linear map from the parallelogram spanned by $e_1, e_2$ into the parallelogram spanned by $v^1_\eta, v^2_\eta$. Then we get

$$\| v^1_\eta \times v^2_\eta \| = \| \det d\Phi_{\eta_n} \| \| e_1 \times e_2 \| = \| \det d\Phi_{\eta_n} \| \| v \| = \| d\Phi_{\eta_n} \|.$$

By $\Phi_{\eta_n} \in C^2(M)$, the outer normal $\nu_{\eta_n}$ of $\Omega_{\eta_n}$ admits the representation $\nu_{\eta_n} \circ \Phi_{\eta_n} = v^1_\eta \times v^2_\eta / (v^1_\eta \times v^2_\eta)$. We set $\nu^n := v^1_\eta \times v^2_\eta$ and get $\nu^n = (\nu_{\eta_n} \circ \Phi_{\eta_n}) \det d\Phi_{\eta_n}$. Thus $\nu^n$ is independent of the choice of $e_1, e_2$ and consequently defined globally on $\text{int} \, M$. For $q \in \text{int} \, M$ and a curve $c$ on $\text{int} \, M$ with $c(0) = q$ and $\frac{d}{dt}|_{t=0} c(t) = e_1(q)$ we compute

$$v^n_q = \frac{d}{dt}|_{t=0} \Phi_{\eta_n}(c(t)) = \frac{d}{dt}|_{t=0} (c(t) + \nu_{\eta_n}(c(t)) \nu^n(c(t)))$$

$$= e_1(q) + d\nu_{\eta_n}(q) e_1(q) v^n + \nu_{\eta_n}(q) \frac{d}{dt}|_{t=0} v^n.$$  \hspace{1cm} (2.8)

In here, $h^n_1(q)$ denotes the components of the Weingarten map with respect to the orthonormal basis $e_1, e_2$. Hence, we have

$$\nu \cdot v^n = 1 - (h^n_1 + h^n_2) \eta_n + (h^n_1 h^n_2 + h^n_2 h^n_1) \eta^n_2 = 1 - 2 H \eta_n + G \eta^n_2,$$  \hspace{1cm} (2.9)

where $H$ is the mean curvature and $G$ the Gauss curvature of $\text{int} \, M$. Taking the limit $n \rightarrow \infty$ in (2.8) and (2.9) yields the convergence of $\nu^n$ towards an $\nu_{\eta}$ in $H^1(M)$, and the convergence of $\nu \cdot v^n$ towards $1 - 2 H \eta_n + G \eta^n_2$ in $H^{2-1}(M) \cap L^\infty(M)$. This brings us to the following definition.

**Definition 2.10.** Let $\eta \in H^2(\Omega)$ with $\| \eta \|_{L^\infty(\Omega)} < \kappa$. We call the above constructed vector field $v_{\eta}$ with $v_{\eta} \in L^1(M)$ for $1 \leq r < \infty$ scaled pseudonormal and define $\gamma(\eta) \in H^{2-1}(M) \cap L^\infty(M)$ by $\gamma(\eta) := 1 - 2 H \eta_n + G \eta^n_2$. 

Proposition 2.12. Let \( \eta \in H^2(M) \) with \( \|\eta\|_{L^\infty(M)} < \kappa \) and \( \eta \in H^2(M) \). Then
\[
\int_\Omega \varphi \cdot \nabla \psi \, dx + \int_{\Omega \setminus \Gamma} \text{div} \varphi \psi \, dA = \int_M \text{tr}_\eta \varphi \cdot \nu \text{tr}_\eta \psi \, dA
\]
(2.13)

Proof. Since \( \Omega_1 \) is bounded, for any \( 1 < p < \infty \) the embedding of \( W^{1,\infty}(\Omega_1) \) into \( W^{1,p}(\Omega_1) \) is continuous. Hence, it suffices to treat the case \( 1 < p, p' < \infty \).

We approximate \( \varphi \) by \( (\varphi_k)_{k \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^3) \) in \( W^{1,p}(\Omega_1) \), \( \psi \) by \( (\psi_k)_{k \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^3) \) in \( W^{1,p}(\Omega_1) \) and \( (\eta_k)_{k \in \mathbb{N}} \subset C^2(M) \) in \( H^2(M) \). Moreover, we chose the cut-off function \( \beta \) uniformly for all definitions of the Hanzawa transforms. By partial integration (notice \( \delta_{\Omega_1} \in C^{0,1} \)), we get
\[
\int_{\Omega_1} \varphi_k \cdot \nabla \psi \, dx = - \int_{\Omega_1} \text{div} \varphi_k \psi \, dx + \int_{\partial \Omega_1} \varphi_k \cdot \nu \psi \, dA_{\Omega_1}.
\]

Since \( \nu \eta_k \circ \Phi_{\eta_k} = \nu \) on \( \Gamma \), a change of variables yields
\[
\int_{\partial \Omega_1} \varphi_k \cdot \nu \psi \, dA_{\Omega_1} = \int_M \text{tr}_{\eta_k} \varphi_k \cdot (\nu \eta_k \circ \Phi_{\eta_k}) \text{tr}_{\eta_k} \psi \, |\det d\Phi_{\eta_k}| \, dA
\]
\[+ \int_{M \setminus \Gamma} \text{tr}_{\eta_k} \varphi_k \cdot \nu \text{tr}_{\eta_k} \psi \, |\det d\Phi_{\eta_k}| \, dA.
\]

Moreover, by Remark 2.11 we get
\[
\int_M \text{tr}_{\eta_k} \varphi_k \cdot (\nu \eta_k \circ \Phi_{\eta_k}) \text{tr}_{\eta_k} \psi \, |\det d\Phi_{\eta_k}| \, dA = \int_M \text{tr}_{\eta_k} \varphi_k \cdot \nu \text{tr}_{\eta_k} \psi \, |\det d\Phi_{\eta_k}| \, dA.
\]

At the construction of the scaled pseudonormal we already saw that \( \nu \eta_k \rightarrow \nu \eta \) in \( L^r(M) \) for any \( 1 \leq r < \infty \). Using Lemma 2.4, Lemma 2.6 and the definition of \( \text{tr}_\eta \), taking the limit \( n \rightarrow \infty \) yields
\[
\int_{\Omega} \varphi_k \cdot \nabla \psi \, dx = - \int_{\Omega} \text{div} \varphi_k \psi \, dx + \int_M \text{tr}_{\eta} \varphi_k \cdot \nu \text{tr}_{\eta} \psi \, dA
\]
\[+ \int_{\Gamma} \text{tr}_{\eta} \varphi_k \cdot \nu \text{tr}_{\eta} \psi \, |\det d\Phi_{\eta}| \, dA.
\]

Now passing to the limit \( k \rightarrow \infty \) and \( \ell \rightarrow \infty \) finally proofs (2.13). \( \square \)

Although domains with Hölder-continuous boundary generally do not admit a Korn’s inequality (cf. [1]), we can – allowing the typical loss of regularity – show a similar statement.

Lemma 2.14. Let \( \eta \in H^2(M) \) with \( \|\eta\|_{L^\infty(M)} < \alpha \ll \kappa \) and \( 1 < p < \infty \). Then for all \( 1 \leq r < p \) there exists a constant \( C \) such that for all \( \varphi \in C^1(\overline{\Omega_1}) \) it holds
\[
\|\nabla \varphi\|_{L^r(M)} \leq C \left( \|D\varphi\|_{L^p(\Omega_1)} + \|\varphi\|_{L^p(\Omega_1)} \right).
\]

For a fixed \( N \in \mathbb{N} \), the constant \( C \) can be chosen uniformly with respect to \( \eta \in H^2(M) \) satisfying \( \|\eta\|_{H^2(M)} \leq N \) and \( \|\eta\|_{L^\infty(M)} < \alpha \).
Proof. We proceed analogously to [23 Proposition 2.13]. Since $H^2(M) \hookrightarrow C^{0, \beta}$ for any $0 < \beta < 1$, the moving domain $\Omega_\eta$ possesses a $\beta$-Hölder boundary. Therefore by [11 Theorem 3.1] for $\varphi \in C^1(\Omega_\eta)$ the inequality

$$
\| \nabla \varphi \|_{L^p(\Omega_\eta)} \leq c \left( \| D\varphi \|_{L^p(\Omega_\eta)} + \| \varphi \|_{L^p(\Omega_\eta)} \right),
$$

holds, where $d(x)$ is the distance from $x \in \Omega_\eta$ to $\partial \Omega_\eta$. A careful inspection of the proof shows that for a fixed $N \in \mathbb{N}$ the constant $c$ can be chosen uniformly with respect to $\eta \in H^2(M)$ satisfying $\| \eta \|_{H^2(M)} \leq N$ and $\| \eta \|_{C^0(M)} < \alpha$. By Hölder’s inequality one gets

$$
\| \nabla \varphi \|_{L^p(\Omega_\eta)} \leq \| \nabla \varphi \|_{L^p(\Omega_\eta)} \| d^{(\beta-1)} \|_{L^{rp}(\Omega_\eta)},
$$

therefore we only have to estimate the second term on the right-hand side. We set $\tilde{r} := \frac{rp}{\beta - 1} \in (1, \infty)$ and deduce $\frac{rp}{\beta - 1} \in (0, \frac{1}{\beta})$, i.e. we can choose $\beta \in \left( \frac{1}{\tilde{r}}, 1 \right)$ with $1 - \frac{1}{\tilde{r}} < \beta < 1$. Moreover, for $\varepsilon > 0$ we consider the partition $\Omega_\eta = U_\varepsilon \cap V_\varepsilon \cup M_\varepsilon$ with

$$
U_\varepsilon := \{ x \in \Omega_\eta \mid \text{dist}(x, \Gamma_\eta) < \varepsilon \}, \quad V_\varepsilon := \{ x \in \Omega_\eta \mid \text{dist}(x, \partial \Omega_\eta \setminus \Gamma_\eta) < \varepsilon \}, \quad M_\varepsilon := \{ x \in \Omega_\eta \mid \text{dist}(x, \partial \Omega_\eta) \geq \varepsilon \}.
$$

Then

$$
\int_{\Omega_\eta} d^{(\beta-1)\tilde{r}} \, dx \leq \int_{U_\varepsilon} \text{dist}(x, \Gamma_\eta)^{(\beta-1)\tilde{r}} \, dx + \int_{V_\varepsilon} \text{dist}(x, \partial \Omega_\eta \setminus \Gamma_\eta)^{(\beta-1)\tilde{r}} \, dx + \int_{M_\varepsilon} d^{(\beta-1)\tilde{r}} \, dx.
$$

We take $\varepsilon$ small enough, such that $V_\varepsilon \subset S_\alpha$. By the assumptions to our in- and outflow domain, $\Gamma_\eta$ consists of the connected components $\Gamma_{\eta,i}$, $i = 1, \ldots, \ell$, each part of some hyperplane. Hence, we can take $\varepsilon > 0$ even smaller, such that $U_\varepsilon$ decomposes disjointly into the sets $U_{\varepsilon,i}$, $i = 1, \ldots, \ell$ of the form

$$
U_{\varepsilon,i} = \{ y \in \mathbb{R}^3 \mid y = q - s \nu(q) \text{ with } q \in \Gamma_{\eta,i}, \ s \in (0, \varepsilon) \}.
$$

Since $(\beta - 1)\tilde{r} \in (-1, 0)$, we get the estimate

$$
\int_{U_{\varepsilon,i}} \text{dist}(x, \Gamma_{\eta,i})^{(\beta-1)\tilde{r}} \, dx = \sum_{i=1}^{\ell} \int_{\Gamma_{\eta,i}} \int_{0}^{\varepsilon} s^{(\beta-1)\tilde{r}} \, ds \, dq \leq c(\ell, \beta, \tilde{r}, |\Gamma_{\eta,i}|) \varepsilon^{(\beta-1)\tilde{r}+1}.
$$

The embedding $H^2(M) \hookrightarrow C^{0, \frac{1}{2}}(M)$ and the properties of the square root imply for $q_1, q_2 \in M$ and $|s| < \kappa$

$$
|\eta(q_1) - s| \leq |\eta(q_1) - \eta(q_2)| + |\eta(q_2) - s| \\
\leq c(\|\eta\|_{H^2(M)}) |q_1 - q_2|^{\frac{1}{2}} + |\eta(q_2) - s| \\
\leq c(\kappa, \|\eta\|_{H^2(M)}) \left( |q_1 - q_2| + |\eta(q_2) - s| \right)^{\frac{1}{2}}.
$$

By the properties of the tubular neighbourhood we deduce for $q \in M$, $|s| < \kappa$

$$
|\eta(q) - s|^2 \leq c(\kappa, \Lambda, \|\eta\|_{H^2(M)}) \text{dist}(q + s \nu(q), \partial \Omega_\eta \setminus \Gamma_\eta).
$$
By a change of variables and \((\beta - 1)\tilde{r} \in (-\frac{1}{2}, 0)\) we get
\[
\int_{V_{\varepsilon}} \text{dist}(x, \partial \Omega_{\eta} \setminus \Gamma_{\eta})^{(\beta - 1)\tilde{r}} \, dx \leq \int_{M} \int_{-\alpha}^{0} \text{dist}(q + s v, \partial \Omega_{\eta} \setminus \Gamma_{\eta})^{(\beta - 1)\tilde{r}} |\det dA| \, ds \, d\lambda(q)
\]
\[
\leq c(\kappa, \Lambda, ||\eta||_{H^2(M)}) \int_{M} \int_{-\alpha}^{0} |\eta(q) - s^{(\beta - 1)\tilde{r}}| \, ds \, d\lambda(q)
\]
\[
\leq c(\kappa, \Lambda, ||\eta||_{H^2(M)}) \int_{M} \frac{1}{2(\beta - 1)^{\tilde{r}}+1} (\eta(q) + \alpha)^{2(\beta - 1)^{\tilde{r}}+1} \, d\lambda(q)
\]
\[
\leq c(\alpha, \tilde{r}, \beta, \kappa, \Lambda, M, ||\eta||_{H^2(M)}),
\]
where we used the embedding \(H^2(M) \hookrightarrow L^\alpha(M)\). Finally we have
\[
\int_{M_{\varepsilon}} d(\beta - 1)\tilde{r} \, dx \leq c(\beta - 1)^{\tilde{r}} |B_\varepsilon|.
\]

We set
\[
V_p(\Omega_\eta) := \{ u \in L^p(\Omega_\eta) \mid Du \in L^p(\Omega_\eta), \text{ div } u = 0 \},
\]
\[
\tilde{V}_p(\Omega_\eta) := \{ u \in L^p(\Omega_\eta) \mid Du \in L^p(\Omega_\eta) \}
\]
and equip these spaces with the norm
\[
||u||_{V_p(\Omega_\eta)} := ||u||_{L^p(\Omega_\eta)} + ||Du||_{L^p(\Omega_\eta)}.
\]

2.1. Generalised trace operator. Let \(U\) be an open subset of \(\mathbb{R}^3\) and \(1 \leq p \leq \infty\). Following \cite{36} section II.1.2, the Banach spaces
\[
E^p(U) := \{ \varphi \in L^p(U) \mid \text{div } \varphi \in L^p(U) \}, \quad L^p_\Sigma(U) := \{ \varphi \in E^p(U) \mid \text{div } \varphi = 0 \}
\]
with the norm \(||\varphi||_{L^p_\Sigma(U)} = ||\varphi||_{E^p(U)} = ||\varphi||_{L^p(U)} + ||\text{div } \varphi||_{L^p(U)}\) admit a generalised trace operator \(\text{tr}_\Omega\) for the normal component as long as \(\partial U\) is Lipschitz. In particular, this trace operator is defined by an approximation argument through Green’s formula from Proposition 2.12 as an element \(\text{tr}_\Omega u \in (H^{1-1/p', r'}(\partial U))^*\), and admits the representation
\[
\langle \text{tr}_\Omega u, \varphi \rangle = \int_{\partial U} \varphi \cdot \nu \, g \, dA \quad \text{for } \varphi \in C^0(\overline{U}), \ g \in H^{1-1/p', r'}(\partial U).
\]

It should be noted, that by mollification \(C^0_0(\overline{U})\) is dense in \(E^p(U)\) as long as \(\partial U \in C^0\) (cf. \cite{24} Prop. A.1). To extend the trace operator to our deformed domain, we will use Green’s formulae from Proposition 2.12.

Proposition 2.21. Let \(\eta \in H^2(M), ||\eta||_{L^\alpha(M)} < \alpha < \kappa\) and \(1 < p, p' < \infty\) with \(\frac{1}{p} + \frac{1}{p'} = 1\) and \(r, \tilde{r}\) with \(p' < r < \tilde{r} < \infty\). There exists a linear, continuous operator
\[
\text{tr}_\eta^\alpha : E^p(\Omega_\eta) \rightarrow (H^{1-1/r\tilde{r}}(\partial \Omega))^*
\]
satisfying
\[
\langle \text{tr}_\eta^\alpha \varphi, \text{tr}_\eta \psi \rangle = \int_{\Omega_\eta} \varphi \cdot \nabla \psi \, dx + \int_{\Omega_\eta} \text{div } \varphi \, \psi \, dx
\]
for all \(\varphi \in E^p(\Omega_\eta)\) and all \(\psi \in W^{1, \tilde{r}}(\Omega_\eta)\). For a fixed \(N \in \mathbb{N}\), the continuity constant can be chosen uniformly with respect to \(\eta \in H^2(M)\) satisfying \(||\eta||_{H^2(M)} \leq N\) and \(||\eta||_{L^\alpha(M)} < \alpha\).
Proof. We will define $\text{tr}^n_\eta \varphi$ for $\varphi \in C^1(\overline{\Omega}_\eta)$. Then, by density, the claim follows. For $b \in H^{1-1/r}((\partial \Omega))$ we set

$$
\langle \text{tr}^n_\eta \varphi, b \rangle := \int_M \text{tr}_\eta \varphi \cdot \nu_\eta \, dA + \int_{\Gamma} \text{tr}_\eta \varphi \cdot \nu \, |\det d\Phi_\eta| \, dA,
$$

i.e. $\text{tr}^n_\eta \varphi \in (H^{1-1/r}((\partial \Omega)))^*$. Since $\Omega$ is a bounded Lipschitz domain, there exists a linear, continuous extension operator $F : H^{1-1/r}((\partial \Omega)) \to W^{1,r}(\Omega)$ (see [14] Satz 6.41). By Lemma 2.5 and $p' < r$, the map $F \circ \Psi^{-1}_\eta : H^{1-1/r}(\partial \Omega) \to W^{1,p'}(\Omega)$ is linear and continuous with an appropriately bounded continuity constant. Using the definition of $\text{tr}_\eta$ and the extension property of $F$, we have

$$
\text{tr}_\eta((F \circ \Psi^{-1}_\eta)(b)) = (F b) \circ \Psi^{-1}_\eta \circ \Psi_\eta |_{\partial \Omega} = (F b) |_{\partial \Omega} = b.
$$

Thus, by definition of $\text{tr}_\eta$, and Proposition 2.12 we have

$$
\left| \langle \text{tr}^n_\eta \varphi, b \rangle \right| \leq c \| \varphi \|_{E^p(\Omega)} \| b \|_{H^{1-1/r}(\partial \Omega)}.
$$

By this inequality, we can extend $\text{tr}_\eta^n$ continuously to the space $E^p(\Omega_\eta)$, which shows the first part of our claim. For the second part, we observe that $r < \tilde{r}$ and therefore $\text{tr}_\eta \psi \in H^{1-1/r}(\partial \Omega)$ for $\psi \in W^{1,1}(\Omega_\eta)$. Thus, an approximation of $\varphi$, the definition of $\text{tr}_\eta$ for smooth functions and 2.13 shows the desired identity. \hfill \square

To obtain a trace operator defined only on a part of the boundary, we restrict $\text{tr}^n_\eta$ to the space of test functions vanishing on the rest of the boundary. For a measurable subset $\gamma \subset \partial \Omega$ and $1 < r < \infty$ we therefore set $W^{1,r}_{\partial \Omega, \gamma} := \{ u \in W^{1,r}(\Omega) \mid u |_{\partial \Omega, \gamma^c} = 0 \}$ and $H^{1-1/r}_{\partial \Omega, \gamma}$ as the image of the classical trace operator of $W^{1,r}_{\partial \Omega, \gamma}$. Hence $H^{1-1/r}_{\partial \Omega, \gamma}$ is a closed subspace of $H^{1-1/r}(\partial \Omega)$.

**Definition 2.22.** For a measurable subset $\gamma \subset \partial \Omega$ we set

$$
\text{tr}^n_\eta|_{\gamma} : E^p(\Omega_\eta) \to (H^{1-1/r}_{\partial \Omega, \gamma})^* , \quad \langle \text{tr}^n_\eta|_{\gamma} \varphi, b \rangle := \langle \text{tr}^n_\eta \varphi, b \rangle
$$

as the restriction of $\text{tr}^n_\eta$ to $H^{1-1/r}_{\partial \Omega, \gamma}$.

**Remark 2.23.** Let $\eta \in H^2(M)$, $\| \eta \|_{L^\infty(M)} < \kappa$ and $1 < p, p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $p' < r < \infty$. Then for all $\varphi \in W^{1,p}(\Omega_\eta)$ and all $b \in H^{1-1/r}(\partial \Omega)$ one has

$$
\langle \text{tr}^n_\eta \varphi, b \rangle = \int_M \text{tr}_\eta \varphi \cdot \nu_\eta \, dA + \int_{\Gamma} \text{tr}_\eta \varphi \cdot \nu \, |\det d\Phi_\eta| \, dA.
$$

For $\varphi \in W^{1,p}(\Omega_\eta)$ with $\text{tr}_\eta \varphi = \xi \nu$ for some $\xi \in L^p(\partial \Omega)$, Remark 2.17 implies

$$
\langle \text{tr}^n_\eta \varphi, b \rangle = \int_M \xi \nu \, \gamma(\eta) \, dA + \int_{\Gamma} \xi \nu \, |\det d\Phi_\eta| \, dA , \quad b \in H^{1-1/r}(\partial \Omega).
$$

Moreover, for an open subset $V \subset \mathbb{R}^3$ with $\Omega_\eta \subset V$ and $\gamma := \Phi_\eta^{-1}((\partial \Omega \setminus \partial V) \subset \partial \Omega$ measurable, a function $u \in E^p(\Omega_\eta)$ with $\text{tr}^n_\eta u = 0$ can be extended by 0 to $\overline{u} \in E^p(V)$. This can
be seen by taking \( \varphi \in C^0_0(V) \) and observing the identity

\[
\langle \text{div} \, \tilde{u}, \varphi \rangle = -\int_{\Omega_\eta} \tilde{u} \cdot \nabla \varphi \, dx
\]

\[
= -\langle \tr_\eta^\alpha \varphi, \tr_\eta \varphi \rangle + \int_{\Omega_\eta} \varphi \, \text{div} \, u \, dx = \int_{V} \varphi \, (\text{div} \, u) \chi_U \, dx.
\]

**Definition 2.24.** Let \( \eta \in H^2(M) \) with \( \|\eta\|_{L^2(M)} < \kappa \) and \( \gamma \subset \partial \Omega \) measurable. We set

\[
H_\gamma(\Omega_\eta) := \{ \varphi \in L_0^2(\Omega_\eta) \mid \tr_\eta^\alpha \varphi = 0 \}
\]

and equip this space with the \( L^2(\Omega_\eta) \)-norm. Since \( H_\gamma(\Omega_\eta) \) is a closed subspace of \( L_0^2(\Omega_\eta) \) and \( L^2(\Omega_\eta) \) respectively, it is a separable Hilbert space.

This space admits the following density result.

**Lemma 2.25.** Let \( \eta \in H^2(M) \) with \( \|\eta\|_{L^2(M)} < \alpha < \kappa \). The set

\[
\mathcal{V}_\alpha(\Omega_\eta) := \{ \varphi \in C^0_0(\Omega_\eta) \mid \text{div} \, \varphi = 0, \varphi = 0 \text{ in a neighbourhood of } \Phi_\eta(M) \}
\]

is dense in \( H_\gamma(\Omega_\eta) \).

**Proof.** By Remark 2.23 we have \( \mathcal{V}_\alpha(\Omega_\eta) \subset H_\gamma(\Omega_\eta) \). Let \( u \in H_\gamma(\Omega_\eta) \). At first, we will construct a suitable extension of \( u \) which vanishes on the whole boundary. Again by Remark 2.23 the extension \( \tilde{u} \) by zero gives \( \tilde{u} \in L_0^2(B_\alpha) \). Since \( B_\alpha \) is a bounded Lipschitz domain, \( B_\alpha \subset \subset B \) for some smooth bounded domain \( B \), see Figure 2. Moreover, by [20] (with \( \lambda_j = 0 \) since \( B_\alpha \) is connected), there exists an extension \( \tilde{u} \in L_0^2(B) \) of \( \tilde{u} \), whose support is compactly contained in \( B \). In particular, this function vanishes on the boundary, i.e., we have

\[
\int_B \tilde{u} \cdot \nabla \psi \, dx = 0 \quad \forall \psi \in W^{1,2}(B).
\] (2.26)

Next, we restrict \( \tilde{u} \) to the set \( U := B \setminus (B_\alpha \setminus \Omega_\eta) \) and define \( v := \tilde{u}|_U \). Therefore \( v \in L_0^2(U) \). Taking \( \psi \in W^{1,2}(U) \) (with \( 2 < r < \infty \) from the definition of \( \tr_\eta \)), by [14] Satz 6.10 we have the extension \( E(\psi) \in W^{1,2}(B) \). Hence, (2.26) and the extension properties imply

\[
\int_U v \cdot \nabla \psi \, dx = \int_{U \setminus B_\alpha} v \cdot \nabla \psi \, dx + \int_{U \cap B_\alpha} v \cdot \nabla \psi \, dx
\]

\[
= \int_B v \cdot \nabla E(\psi) \, dx - \int_{B_\alpha} v \cdot \nabla (E(\psi)) \, dx + \int_{\Omega_\eta} v \cdot \nabla \psi \, dx
\]

\[
= \int_{\Omega_\eta} u \cdot \nabla (\psi - E(\psi)) \, dx
\]

\[
= 0,
\]
In particular, \( G \) that every functional 
argue similarly as in [38, Theorem 1.6]. Obviously it holds 
\( \Upsilon(U) := \{ \varphi \in C_0^\infty(U) \mid \text{div} \varphi = 0 \} \)
in \( H(U) \). This implies our claim by restriction of the approximating functions to \( \Omega_\eta \). We argue similarly as in [38, Theorem 1.6]. Obviously it holds \( \Upsilon(U) \subset H(U) \). We will show that every functional \( G \in H(U)^* \) vanishing on \( \Upsilon(U) \) is the zero functional. Hahn-Banach's theorem then implies the density. Let \( G \in H(U)^* \) with \( \langle G, \varphi \rangle = 0 \) for all \( \varphi \in \Upsilon(U) \). By Riesz theorem, there exists \( g \in H(U) \) with \( \langle G, u \rangle = \int_U g \cdot u \, dx \) for all \( u \in H(U) \). In particular, \( \int_U g \cdot \varphi \, dx = 0 \) for all \( \varphi \in \Upsilon(U) \). By the theorem of De Rham (see [5, Theorem IV.2.4]) there exists \( p \in L^2_{\text{loc}}(U) \) with \( g = \nabla p \), i.e. \( \nabla p \in L^2(U) \). Approximating \( p \) through a sequence \( p_n \in C^1(U) \) with \( \nabla p_n \to \nabla p \) in \( L^2(U) \), we can deduce for all \( u \in H(U) \) (and therefore \( Fu = 0 \))
\[
\langle G, u \rangle = \int_U g \cdot u \, dx = \lim_{n \to \infty} \int_U \nabla p_n \cdot u \, dx = \lim_{n \to \infty} \langle Fu, p_n \rangle = 0,
\]
which proofs the claim. \( \square \)

**Remark 2.27.** Let \( \Omega_{\eta_\rho} = (\Omega \setminus S_{\kappa} \cup \{ x \in S_{\kappa} \mid s(x) < \eta(q(x)) - \rho \} \). Then \( \varphi \in \Upsilon_M(\Omega_\eta) \) already implies supp \( \varphi \subset \Omega_{\eta_\rho} \) for some \( 0 < \rho \) small enough.

The following result ensures that we can approximate functions of \( H_M(\Omega_\eta) \), while keeping the support uniformly away from the moving boundary. This Lemma adapts [24, Lemma A.13] to our situation and is crucial for the compactness theorem.

**Lemma 2.28.** Let \( 0 < \alpha < \kappa \) and \( \epsilon > 0 \) be given. Then there exists \( \rho > 0 \) such that for all \( \eta \in H^2(M) \) with \( \| \eta \|_{H^2(M)} \leq C \), \( \| \eta \|_{L^\infty(M)} \leq \alpha \) and for all \( \varphi \in H_M(\Omega_\eta) \) with \( \| \varphi \|_{L^2(\Omega_\eta)} \leq 1 \) there exists an function \( \Psi = \Psi(\eta, \varphi) \in H_M(\Omega_\eta) \) with supp \( \Psi \subset \Omega_{\eta_\rho} \), \( \| \Psi \|_{L^2(\Omega_\rho)} \leq 2 \) and \( \| \varphi - \Psi \|_{(H^1(\mathbb{R}^3))^*} < \epsilon \). As usual, \( \varphi - \Psi \in H^{1/4}(\mathbb{R}^3)^* \) is realised through the extension by 0 to \( \mathbb{R}^3 \) and the \( L^2 \) Riesz representation.

*Proof.* We suppose the claim is false. Hence, we find sequences \( (\rho_n)_{n \in \mathbb{N}} \searrow 0 \) with \( 0 < \rho_n < \kappa - \alpha \), \( \eta_n \in H^2(M) \) with \( \| \eta_n \|_{H^2(M)} \leq C \), \( \| \eta_n \|_{L^\infty(M)} \leq \alpha \) and \( \varphi_n \in H_M(\Omega_{\eta_n}) \) with \( \| \varphi_n \|_{L^2(\Omega_{\eta_n})} \leq 1 \) such that for all \( \Psi \in H_M(\Omega_{\eta_n}) \) satisfying supp \( \Psi \subset \Omega_{\eta_\rho} \), \( \| \varphi_n - \Psi \|_{L^2(\Omega_{\eta_n})} \leq 2 \) we have \( \| \varphi_n - \Psi \|_{(H^{1/4}(\mathbb{R}^3))^*} \geq \epsilon \). Therefore we find a (not further denoted) subsequence with
\[
\eta_n \to \eta \quad \text{weakly in } H^2(M), \quad \varphi_n \to \varphi \quad \text{weakly in } L^2(\mathbb{R}^3), \quad \eta_n \to \eta \quad \text{uniformly in } M.
\]
In particular, we have \( \| \eta \|_{H^2(M)} \leq C \), \( \| \eta \|_{L^\infty(M)} \leq \alpha \), \( \| \varphi \|_{L^2(\mathbb{R}^3)} \leq 1 \) and \( \varphi = 0 \) in \( \Omega_\eta \). Since the smooth extension by zero of \( \mu \in C_0^\infty(\Omega_\eta) \) implies \( \text{tr}_{\eta_n} \mu \in H^{-1/4,r}(M) \), we have by the weak convergence of \( \varphi_n \) and Green’s formula from Proposition 2.21
\[
(\text{div} \varphi, \mu) = -\lim_{n \to \infty} \int_{\mathbb{R}^3} \varphi_n \cdot \nabla \mu \, dx = \lim_{n \to \infty} \left( \int_{\mathbb{R}^3} \text{div} \varphi_n \mu \, dx - \langle \mu, \eta_n \varphi_n, \text{tr}_{\eta_n} \mu \rangle \right) = 0,
\]
i. e. $\varphi \in L^2_b(\Omega_\eta)$. For $\varphi \in H_{\text{loc}}(\Omega_\eta)$, by density it suffices to show $\langle \nu^b_\eta \varphi, b|_{\partial \Omega} \rangle = 0$ for all $b \in C^\infty(\overline{\Omega})$ vanishing in a neighbourhood of $\Gamma$, see [4]. For such a function $b$ we get, again by Green’s formula from Proposition [2.12]

$$\langle \nu^b_\eta \varphi, b|_{\partial \Omega} \rangle = \langle \nu^b_\eta \varphi, tr_\eta(b \circ \Psi^{-1}_\eta) \rangle = \int_{\Omega_\eta} \varphi \cdot \nabla(b \circ \Psi^{-1}_\eta) \, dx.$$  

We would like to pass to the sequence $\varphi_n$ on the right hand side, but first we have to extend $b \circ \Psi^{-1}_\eta$ appropriately. Using the operator from [14 Satz 6.10], which is locally defined as a reflection, we can extend $b \in C^\infty(\overline{\Omega}) \subset W^{1,\infty}(\Omega)$ to $\tilde{b} \in W^{1,\infty}(B_\rho)$ satisfying $\tilde{b} = 0$ on $\Gamma \cup \Gamma^\rho_r$ for some $r < r_1 < \infty$. Furthermore, by a change of variables through the inverse Hanzawa transform (which can be extended smoothly using the same construction on the set $B_\rho \setminus \overline{\Omega_\eta}$), we obtain a function $\tilde{b} \in W^{1,\infty}(B_\rho)$ satisfying $(b \circ \Psi^{-1}_\eta) = \tilde{b}$ in $\overline{\Omega_\eta}$ and $\text{tr}_{\Omega_\eta} \tilde{b} = 0$. Hence, we have

$$\langle \nu^b_\eta \varphi, b|_{\partial \Omega} \rangle = \int_{\Omega_\eta} \varphi \cdot \nabla \tilde{b} \, dx = \lim_{n \to \infty} \int_{\Omega_\eta} \varphi_n \cdot \nabla \tilde{b} \, dx = \lim_{n \to \infty} \langle \nu^b_\eta \varphi_n, \text{tr}_{\Omega_\eta} \tilde{b} \rangle = 0,$$

i. e. $\varphi \in H_{\text{loc}}(\Omega_\eta)$. By Lemma [2.25] there exists $\Psi \in \mathcal{C}(\Omega_\eta) \subset H_{\text{loc}}(\Omega_\eta)$ with $\text{supp} \Psi \subset \Omega_\eta - \rho_0$ for some $0 < \rho_0 < \kappa - \alpha$ and $\|\varphi - \Psi\|_{L^2(\Omega_\eta)} < \min\{\frac{1}{4}, \frac{\varepsilon}{2}\}$, i. e. $\|\varphi\|_{L^2(\Omega_\eta)} \leq 2$. Using the uniform convergence, it follows that $\text{supp} \Psi \subset \Omega_{\eta_0 - \rho_0}$ for $n$ big enough. By our assumption we have

$$\varepsilon \leq \|\varphi_n - \Psi\|_{H^\frac{1}{2}(\mathbb{R}^3)} \leq \|\varphi_n - \varphi\|_{H^\frac{1}{2}(\mathbb{R}^3)} + \|\varphi - \Psi\|_{H^\frac{1}{2}(\mathbb{R}^3)} < \frac{\varepsilon}{2} + \varepsilon \leq \varepsilon.$$

Choosing a ball $B$ big enough, the compact embedding $H^\frac{1}{2}(B) \hookrightarrow L^2(B)$ and the theorem of Schauder imply the compactness of $L^2(B) \hookrightarrow H^\frac{1}{2}(B)^*$. Hence, $\varphi_n \to \varphi$ in $H^\frac{1}{2}(B)^*$. Using the extensions by zero, this convergence holds also in $(H^\frac{1}{2}(\mathbb{R}^3))^*$, i. e. $\|\varphi_n - \varphi\|_{(H^\frac{1}{2}(\mathbb{R}^3))^*} < \frac{\varepsilon}{2}$ for $n$ big enough, a contradiction to (2.22).

2.2. Time-dependent function spaces. We will use an obvious substitute for the classical Bochner spaces in our moving domain. For $I := (0, T)$ with $0 < T < \infty$ and a continuous $\eta : \overline{I} \times M \to (0, \kappa, \kappa)$ we define the bounded domain $\Omega^I_\eta := \bigcup_{t \in I} \Omega_{\eta(t)}$ and set for $1 \leq p, r \leq \infty$

$$L^p(I; L^r(\Omega^I_{\eta(t)})) := \left\{ v \in L^p(I; L^r(\Omega^I_{\eta(t)})) \left| \begin{array}{l} v(t, \cdot) \in L^r(\Omega^I_{\eta(t)}) \text{ for almost all } t \in I, \\ \|v(t, \cdot)\|_{L^r(\Omega^I_{\eta(t)})} \in L^p(I) \end{array} \right. \right\},$$

$$L^p(I; W^{1,r}(\Omega^I_{\eta(t)})) := \left\{ v \in L^p(I; L^r(\Omega^I_{\eta(t)})) \left| \begin{array}{l} \nabla v \in L^p(I; L^r(\Omega^I_{\eta(t)})) \end{array} \right. \right\},$$

$$H^1(I; L^p(\Omega^I_{\eta(t)})) := \left\{ v \in L^p(I; L^p(\Omega^I_{\eta(t)})) \left| \begin{array}{l} \partial_n v \in L^2(I; L^p(\Omega^I_{\eta(t)})) \end{array} \right. \right\},$$

$$L^p(I; V_r(\Omega^I_{\eta(t)})) := \left\{ v \in L^p(I; L^r(\Omega^I_{\eta(t)})) \left| \begin{array}{l} Dv \in L^p(I; L^r(\Omega^I_{\eta(t)})) \end{array} \right. \right\},$$

$$L^p(I; V_{\ast}(\Omega^I_{\eta(t)})) := \left\{ v \in L^p(I; V_r(\Omega^I_{\eta(t)})) \left| \begin{array}{l} \text{div } v = 0 \end{array} \right. \right\}.$$

It should be noted, that $\nabla$, $\text{div}$ and $D$ are acting only with respect to the space variable and the derivatives are (partial) weak derivatives on $\Omega^I_{\eta(t)}$. These spaces, equipped with the canonical norms, are Banach spaces. Moreover, motivated by the formal a priori estimate,
we define the following spaces
\[ Y^I := W^{1,\infty}(I; L^2(M)) \cap L^\infty(I; H^2_0(M)), \]
\[ \breve{Y}^I := W^{1,\infty}(I; L^2(M)) \cap L^\infty(I; H^2(M)) \]
and for \( \eta \in \breve{Y}^I \) with \( \| \eta \|_{L^\infty(I \times M)} < \kappa \)
\[ X^I_\eta := L^\infty(I; L^2(\Omega_{\eta(t)})) \cap L^2(I; V_2(\Omega_{\eta(t)})), \]
\[ \breve{X}^I_\eta := L^\infty(I; L^2(\Omega_{\eta(t)})) \cap L^2(I; \bar{V}_2(\Omega_{\eta(t)})). \]
It means that we renounce the vanishing boundary values for \( \breve{Y}^I \) and the divergence constraint for \( X^I_\eta \).

For \( \eta \in C^0(\bar{I} \times M) \) and an appropriate cut-off function \( \beta \), applying the (stationary) Hanzawa transformation \( \Psi_{\eta(t)} \) at every time \( t \in I \) also defines a map between \( I \times \Omega \) and \( \Omega^I_\eta \). Clearly, we have the following result.

**Proposition 2.30.** Let \( \eta \in C^0(\bar{I} \times M) \) with \( \| \eta \|_{L^\infty(I \times M)} < \kappa \). Then
\[ \Psi_\eta : I \times \bar{\Omega} \to \bar{\Omega}_\eta, \quad (t, x) \mapsto (t, \Psi_{\eta(t)}(x)), \]
\[ \Phi_\eta : I \times \partial \Omega \to \bigcup_{t \in \bar{I}} \{ t \} \times \partial \Omega_{\eta(t)}, \quad (t, x) \mapsto (t, \Phi_{\eta(t)}(x)), \]
are homeomorphisms and \( C^k \)-diffeomorphisms for \( \eta \in C^k(I \times M), k \in \{1, 2, 3\} \).

As shown in [24], \( Y^I \) embeds into a space of Hölder-continuous functions.

**Lemma 2.31.** For \( \frac{1}{2} < \lambda < 1 \) the following embeddings are continuous
\[ \breve{Y}^I \hookrightarrow C^{0,1-\lambda}(\bar{I}; H^{2k}(M)) \hookrightarrow C^{0,1-\lambda}(\bar{I}; C^{2k-1}(M)) \hookrightarrow \hookrightarrow C^0(\bar{I} \times M). \]

As in the stationary case, the Hanzawa transform preserves convergence.

**Lemma 2.32.** Let \( (\eta_n)_{n \in \mathbb{N}} \subset \breve{Y}^I \) with \( \| \eta_n \|_{L^\infty(I \times M)} \leq \alpha < \kappa \) be a bounded sequence and \( \eta_n \to \eta \) uniformly in \( \bar{I} \times M \). Moreover, let the cut-off function \( \beta \) be chosen uniformly for the corresponding Hanzawa transforms. Then
\[ a) \ \Psi_{\eta_n} \to \Psi_{\eta} \text{ uniformly in } \bar{I} \times \bar{\Omega}, \]
\[ b) \ \Psi^{-1}_{\eta_n} \to \Psi^{-1}_{\eta} \text{ uniformly in } I \times \bar{\Omega}_\alpha, \]
\[ c) \ \text{Let } 1 \leq s < \infty. \text{ Then } \nabla \Psi_{\eta_n} \text{ converges towards } \nabla \Psi_{\eta} \text{ in } L^s(I \times \Omega), \text{ and } (\nabla \Psi^{-1}_{\eta_n}) \chi_{\Omega_{\eta_n}} \text{ towards } (\nabla \Psi^{-1}_{\eta}) \chi_{\Omega_{\eta}} \text{ in } L^s(I \times B_\alpha). \]

**Proof.** First we remark that using weak and weak-∗ convergent subsequences, we can deduce \( \eta \in \breve{Y}^I \) with \( \| \eta \|_{L^\infty(I \times M)} \leq \alpha \). The first two claims follow from the uniform convergence of \( \eta_n \) and the definition of the Hanzawa transform (one should note that for the inverse functions, a case differentiation has to be made, see [16 Lemma 2.13]). For \( s \geq 2 \), the embeddings \( H^2(M) \hookrightarrow W^{1,s}(M) \hookrightarrow L^2(M) \) are continuous and compact. By the definition of \( \breve{Y}^I \) and Aubin-Lions Lemma, \( \eta_n \) converges strongly towards \( \eta \) in \( L^s(I; W^{1,s}(M)) \).
Taking into account the characterisations of the (spatial) Jacobians \( (\mathbf{2.3}) \) of the transforms, the claimed convergences follow.

**Remark 2.33.** We can transfer the results of the preceding Sections to the time-dependent case by either repeating the proofs using Proposition 2.30 and Lemma 2.32 or applying them at every time \( t \in I \). In particular for \( \eta \in C^2(I \times M) \) the corresponding time-dependent Piola transform \( \mathcal{P}_\eta \) is an isomorphism between the Lebesgue and Sobolev spaces on \( I \times \Omega \) respectively \( \Omega^I_\eta \), as long as the order of differentiability is not larger than one.

The next Lemma shows that our spaces are closed under the compatibility condition
\[ \text{tr}_\eta \mathbf{u} = \partial_\eta \mathbf{v} \] on \( I \times M \) in a suitable sense.
Lemma 2.34. Let \((\delta_n)_{n \in \mathbb{N}} \subset \tilde{Y}^1\), \((\eta_n)_{n \in \mathbb{N}} \subset Y^1\) and \((u_n)_{n \in \mathbb{N}} \subset X^1_{\delta_n}\) be bounded sequences with \(\|\delta_n\|_{L^\infty(I \times M)} \leq \alpha < \kappa\), \(\partial_t \delta_n u_n = \partial \eta_n v\) on \(I \times M\) and
\[
\delta_n \to \delta \quad \text{uniformly in } I \times M, \quad u_n \to u \quad \text{weakly in } L^2(I, L^2(\mathbb{R}^3)),
\]
\[
\partial_t \eta_n \to \partial_t \eta \quad \text{weakly in } L^2(I, L^2(M))
\]
for some \(\delta \in \tilde{Y}^1\), \(\eta \in Y^1\), \(u \in X^1_\delta\), where \(u_n\), \(u\) are spatially extended by zero. Then \(\text{tr}_\delta u = \partial_t \eta \cdot v\) on \(I \times M\).

Proof. Again we choose the cut-off function \(\beta\) for the Hanzawa transforms uniformly. Let \(\varphi \in L^3(I \times \Omega)\). By Hölder’s inequality and a change of variables, we get
\[
\begin{align*}
\int_I \int_{\Omega} (u_n \circ \Psi_{\delta_n} - u \circ \Psi_{\delta}) \cdot \varphi \, dx \, dt &= \int_I \int_{\Omega} (u_n - u) \circ \Psi_{\delta_n} \cdot \varphi \, dx \, dt + \int_I \int_{\Omega} (u \circ \Psi_{\delta_n} - u \circ \Psi_{\delta}) \cdot \varphi \, dx \, dt \\
&\leq \int_{\Omega} \left( u_n - u \right) \circ \Psi_{\delta_n} \cdot \left( (\varphi \circ \Psi_{\delta_n}^{-1}) \cdot |\det d\Psi_{\delta_n}| \right) \, dx \\
&\quad \cdot \left( |\det d\Psi_{\delta_n}| \right) \, dx \\
&\quad + \int_{\Omega} \left| u \circ \Psi_{\delta_n} - u \circ \Psi_{\delta} \right| \, dx \\
&\leq \left( \int_{\Omega} \left( u_n - u \right) \circ \Psi_{\delta_n} \cdot \left( (\varphi \circ \Psi_{\delta_n}^{-1}) \cdot |\det d\Psi_{\delta_n}| \right) \, dx \right)^{\frac{1}{2}} \\
&\quad \cdot \left( \int_{\Omega} \left| u \circ \Psi_{\delta_n} - u \circ \Psi_{\delta} \right|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq \left( \int_{\Omega} \left( u_n - u \right) \circ \Psi_{\delta_n} \cdot \left( (\varphi \circ \Psi_{\delta_n}^{-1}) \cdot |\det d\Psi_{\delta_n}| \right) \, dx \right)^{\frac{1}{2}} \\
&\quad \cdot \left( \int_{\Omega} \left| u \circ \Psi_{\delta_n} - u \circ \Psi_{\delta} \right|^2 \, dx \right)^{\frac{1}{2}}.
\end{align*}
\]

By Lemma 2.33 and the time-dependent variant of Lemma 2.6 we have \(u \circ \Psi_{\delta_n} \to u \circ \Psi_{\delta}\) in \(L^{3/2}(I \times \Omega)\) and \((\varphi \circ \Psi_{\delta_n}^{-1}) \cdot |\det d\Psi_{\delta_n}| \to (\varphi \circ \Psi_{\delta}^{-1}) \cdot |\det d\Psi_{\delta}|\) in \(L^2(I \times \mathbb{R}^3)\). Together with the weak convergence of \(u_n \circ \Psi_{\delta_n}\), (2.35) implies the weak convergence of \(u_n \circ \Psi_{\delta_n}\) towards \(u \circ \Psi_{\delta}\) in \(L^{3/2}(I \times \Omega)\). Furthermore, by our Korn-type inequality and the time-dependent version of Lemma 2.5 \(u_n \circ \Psi_{\delta_n}\) is bound uniformly in \(L^2(I, W^{1,5/3}(\Omega))\), i.e. \(u_n \circ \Psi_{\delta_n}\) converges weakly towards \(u \circ \Psi_{\delta}\) in \(L^2(I, W^{1,5/3}(\Omega))\). By linearity and continuity of the trace operator and the definition of \(\text{tr}_\eta\) the sequence \(\text{tr}_\eta u_n\) converges weakly towards \(\text{tr}_\eta u\) in \(L^2(I \times M)\). Hence, the compatibility condition \(\text{tr}_\delta u_n = \partial_t \eta_n \cdot v\) on \(I \times M\) and the weak convergence of \(\partial_t \eta_n\) imply \(\text{tr}_\delta u = \partial_t \eta \cdot v\) on \(I \times M\).

Remark 2.36. In the usual Bochner spaces, control over the (generalised) time derivative implies continuity in time, i.e. for \(u \in H^1(I, L^2(\Omega))\) the pointwise evaluation in time \(u(t, \cdot) \in L^2(\Omega)\) is well defined. In our setting, i.e. \(\eta \in \tilde{Y}^1\) with \(\|\eta\|_{L^\infty(I \times M)} \leq \alpha < \kappa\), a function \(\varphi \in H^1(I, L^2(\Omega_{\eta(t)})) \cap L^2(I, W^{1,2}(\Omega_{\eta(t)}))\) satisfying \(\text{tr}_\eta \varphi = \partial_t \varphi \circ \eta\) on \(I \times M\) for some \(b \in H^1(I, L^2(M))\) can be extended by \((b \circ \eta)\circ \eta\) to obtain a function in the Bochner space \(H^1(I, L^2(Ba))\) (see [16 Lemma 2.72], [24 Remark A.14]). Hence, the evaluation in time is at least for such functions well defined.

2.3. Regularization of the displacement. To avoid the usual loss of regularity by a transformation to and from the reference domain, we construct a regularisation of the displacements. Since the initial data has to be adapted to the regularised moving domain, special care has to be taken. That means, we use a special mollification kernel and approximate from “above”. Hence, our regularisation cannot be linear and does not preserve zero boundary conditions.

Let \((\phi_b, U_b)_{b \in \mathbb{R}}\) be a finite atlas of \(M\) with subordinate partition of unity \((\psi_{b, h})_{h \in \mathbb{N}}\). We extend a given \(\delta \in C^0(\tilde{T} \times M)\) constantly by \(\delta(0, \cdot)\) and \(\delta(T, \cdot)\) to \((-\infty, 0) \times M\) and \((T, \infty) \times M\), respectively. By the generalised reflection \(E_b(\delta \circ \phi_b^{-1})\) from [14 Satz 6.10], we extend \(\delta \circ \phi_b^{-1}\) further to \(\mathbb{R}^3\) (note that without loss of regularity \(\phi_b(U_b)\) is smooth). Let \(\omega \in C^0_0(\mathbb{R}^3)\) with \(\omega \geq 0\), \(\int_{\mathbb{R}^3} \omega(t, z) \, dt \, dz = 1\) and \(\text{supp} \omega \subset \{ (t, z) \in \mathbb{R} \times \mathbb{R}^2 \mid 0 < t < 1, |z| < 1 \}\). Moreover, for \(\epsilon > 0\) we set \(\delta_\epsilon \coloneqq \epsilon^{-3} \omega(\epsilon^{-1} \cdot)\) and \(\partial_t \delta_\epsilon : \tilde{T} \times M \to \mathbb{R}\),
\[
\partial_t \delta_\epsilon(t, x) := \sum_{k=1}^N \left( w_k \ast E_b(\delta \circ \phi_b^{-1}) \right) \circ \phi_b(t, x) \psi_b(x) + \epsilon^{-1}.
\]
Hence, the summand with index $k$ is extended by $0$ to $M$. By basic properties of the mollification (see e.g. [5] Proposition II.2.25]) and the reflection we get:

**Proposition 2.37.** Let $\varepsilon > 0$. The map $\mathcal{R}_\varepsilon : C^0(I \times M) \to C^4(I \times M)$, $\delta \mapsto \mathcal{R}_\varepsilon \delta$ is continuous and satisfies for all $\delta_1, \delta_2 \in C^0(I \times M)$ the estimates

$$
\|\mathcal{R}_\varepsilon \delta_1\|_{C^4(I \times M)} \leq c_\varepsilon \|\delta_1\|_{C^0(I \times M)} + \varepsilon^{1/2}, \quad \|\mathcal{R}_\varepsilon \delta_1\|_{L^2(I \times M)} \leq \|\delta_1\|_{L^2(I \times M)} + \varepsilon^{1/2},
$$

$$
\|\mathcal{R}_\varepsilon \delta_1 - \mathcal{R}_\varepsilon \delta_2\|_{L^2(I \times M)} \leq \|\delta_1 - \delta_2\|_{L^2(I \times M)} + \varepsilon^{1/2}.
$$

**Remark 2.38.** Let $(\delta_n)_{n \in \mathbb{N}} \subset \hat{Y}$ be a bounded sequence. Then $\mathcal{R}_\varepsilon \delta_n$ is also bounded in $\hat{Y}$ independently of $0 < \varepsilon < \varepsilon_0$ and $n \in \mathbb{N}$ (see [13] Théorème 1.8.1) and [5] Proposition II.2.25]), but does not converge in this space.

Now we show the effect of the special mollification kernel and the translation by $\varepsilon^{1/2}$.

**Proposition 2.39.** Let $\eta_0 \in H^1_0(M)$ and $\delta \in C^0(I \times M)$ with $\delta(0, \cdot) = \eta_0(\cdot)$.

a) $\mathcal{R}_\varepsilon \delta(0, \cdot)$ is independent of $\varepsilon$ apart from $\delta(0, \cdot) = \eta_0$.

b) There exists $0 < \varepsilon_1 = \varepsilon_1(\eta_0)$ such that $\mathcal{R}_\varepsilon \delta(0, \cdot) > \eta_0$ for all $0 < \varepsilon < \varepsilon_1$.

**Proof.** By the extension of $\delta$ through $\delta(0, \cdot) = \eta_0$ to $(-\infty, 0) \times M$ and the properties of the kernel $w_\varepsilon$, we have for $z \in \mathcal{E}_k(U_k)$

$$
(w_\varepsilon \ast E_k(\delta \circ \varphi_k^{-1}))(0, z) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} w_\varepsilon(s, y) E_k(\delta \circ \varphi_k^{-1})(0 - s, z - y) \, dy \, ds
$$

$$
= \int_0^{\varepsilon} \int_{B_{\varepsilon}(0)} w_\varepsilon(s, y) E_k(\eta_0 \circ \varphi_k^{-1})(z - y) \, dy \, ds.
$$

Hence, $\mathcal{R}_\varepsilon \delta(0, \cdot)$ is independent of $\varepsilon$ apart from $\delta(0, \cdot) = \eta_0$. By the continuous embedding $H^1_0(M) \hookrightarrow C^0(I \times M)$ and the properties of the generalised reflection we have $E_k(\eta_0 \circ \varphi_k^{-1}) \in C^{1, \frac{1}{2}}(\mathbb{R}^2)$, i.e.

$$
|E_k(\eta_0 \circ \varphi_k^{-1})(z_1) - E_k(\eta_0 \circ \varphi_k^{-1})(z_2)| \leq c_k |z_1 - z_2|^{1/2} \quad \forall z_1, z_2 \in \mathbb{R}^2.
$$

Together with the identity above, this implies for $z \in \mathcal{E}_k(U_k)$

$$
|w_\varepsilon \ast E_k(\delta \circ \varphi_k^{-1}))(0, z) - \gamma_0 \circ \varphi_k^{-1}(z) |
$$

$$
= \left| \int_0^{\varepsilon} \int_{B_{\varepsilon}(0)} w_\varepsilon(s, y) \left( E_k(\eta_0 \circ \varphi_k^{-1})(z - y) - E_k(\eta_0 \circ \varphi_k^{-1})(z) \right) \, dy \, ds \right|
$$

$$
\leq c_k \int_0^{\varepsilon} \int_{B_{\varepsilon}(0)} w_\varepsilon(s, y) |y|^{1/2} \, dy \, ds
$$

$$
\leq c_k \varepsilon^{1/2}.
$$

Hence, $\mathcal{R}_\varepsilon \delta(0, \cdot) - \eta_0 \geq \varepsilon^{1/2} - \sum_{k=1}^{N} c_k \varepsilon^{1/2}$, i.e. there exists $0 < \varepsilon_1 = \varepsilon_1(1, \ldots, c_k)$ with $\mathcal{R}_\varepsilon \delta > \eta_0$ for $0 < \varepsilon < \varepsilon_1$. \hfill \Box

In the definition of $\mathcal{R}_\varepsilon \delta$, only $\delta$ and $w_\varepsilon$ depend (non trivially) on time. Thus, the classical properties of mollification and reflection imply the following convergences.

**Lemma 2.40.** Let $\delta \in C^4(I \times M)$ and $\varepsilon > 0$. Then

a) $\mathcal{R}_\varepsilon \delta \rightharpoonup \delta$ uniformly on $I \times M$ for $\varepsilon \to 0$.

b) If $\delta_0, \delta_2 \in L^2(I \times M)$ then $\delta_0 \mathcal{R}_\varepsilon \delta \rightharpoonup \delta_0 \delta$ in $L^2(I \times M)$ for $\varepsilon \to 0$.

c) Let $\eta_0 \in H^1_0(M)$ and $\delta \in C^0(I \times M)$ with $\delta(0, \cdot) = \eta_0$. Then $\mathcal{R}_\varepsilon \delta(0, \cdot)$ converges uniformly on $M$ towards $\eta_0$ for $\varepsilon \to 0$. This convergence is independent of the particular choice of $\delta$. \hfill \Box
2.4. Divergence-free extension operator. To extend a test function to the shell equation defined on $M$ to a test function of the fluid equation, we have to construct a divergence-free extension. As in our moving domains, we first look at the situation in the stationary case.

**Lemma 2.41.** Let $\eta \in H^2(M)$, $0 < \alpha < \kappa$ with $\|\eta\|_{L^\infty(M)} < \alpha$ and $\frac{2}{p} \leq p \leq 3$. Then there exists a linear, continuous extension operator

$$\mathcal{F}_\eta : W_0^{1,p}(M) \to W^{1,p}(B_\alpha)$$

which satisfies $\text{div} \mathcal{F}_\eta b = 0$ and $\text{tr}_\eta (\mathcal{F}_\eta b|_{\partial \Omega}) = b \nu$ on $M$. For a fixed $N \in \mathbb{N}$, the continuity constant can be chosen uniformly with respect to $\eta \in H^2(M)$ satisfying $\|\eta\|_{H^2(M)} \leq N$ and $\|\eta\|_{L^\infty(M)} < \alpha$.

**Proof.** Let $b \in W_0^{1,p}(M)$. We define for $x \in S_\alpha$

$$(\mathcal{F}_\eta b)(x) := \exp \left( -\int_{\eta(q(x))}^{s(x)} \left( \text{div}(\nu \circ q) \right) (q(x) + \tau \nu(q(x))) \, d\tau \right) \left( b \nu \right)(q(x))$$

and get with the same arguments as in [24, Proposition 2.19], i.e. using properties of the tubular neighbourhood, that $\mathcal{F}_\eta$ is a linear and continuous operator from $W_0^{1,p}(M)$ to $W^{1,p}(S_\alpha)$ satisfying $\text{div} \mathcal{F}_\eta b = 0$. In particular, the continuity constant is bounded as claimed in the Lemma and we have the identity

$$\frac{\partial_i}{\partial_\eta} \mathcal{F}_\eta b = \exp \left( -\int_{\eta(q(x))}^{s(x)} \left( \text{div}(\nu \circ q) \right) (q(x) + \tau \nu(q(x))) \, d\tau \right) \left[ \frac{\partial_i}{\partial_\eta} \left( b \nu \right)(q(x)) \right. + (b \nu) \circ q \left. - \text{div}(\nu \circ q) \right] \left( q(x) + \eta \nu(q(x)) \partial_i \eta \circ q \right)$$

and

$$= \text{div}(\nu \circ q) \left( q(x) + \eta \nu(q(x)) \partial_i \eta \circ q \right) - \text{div}(\nu \circ q) \left( q(x) + \eta \nu(q(x)) \partial_i \eta \circ q \right).$$

(2.42)

Since $\Omega$ is an admissible in- and outflow domain, the boundary of $S_\alpha$ is given by the disjoint sets $M^a$, $M^u$ and $\Gamma^0$, where the common boundary of $S_\alpha$ and $\Omega \setminus S_\alpha$ is given by $M^u$, see Figure [1]. Since $S_\alpha$ is an Lipschitz domain, an approximation argument shows $\mathcal{F}_\eta b = 0$ on $\Gamma^0$ and

$$(\mathcal{F}_\eta b \circ \Psi_{\alpha})|_{M} = b \nu \text{ on } M.$$  

(2.43)

Moreover, by the assumptions on the domain, $\text{int} \Gamma^0 \subset M^u$ and $\partial \left( \Omega \setminus S_\alpha \right)$ is not empty. Hence, there exists an open set $B \subset \subset \Gamma^0 \subset M^u$ and a function $\mu : \partial \left( \Omega \setminus S_\alpha \right) \to \mathbb{R}$ which is smooth on $B$, supp $\mu \subset \subset B$ and $\int_{\partial \left( \Omega \setminus S_\alpha \right)} \mu \, dA = 1$. We set

$$\xi : \partial \left( \Omega \setminus S_\alpha \right) \to \mathbb{R}^3, \quad \xi(s) := \left\{ \begin{array}{ll} \mathcal{F}_\eta b(s), & s \in M^a, \\
\int_{M^u} \mathcal{F}_\eta b \cdot \nu^\alpha \, dA \mu(s) \nu^\alpha \circ c(s), & s \in \Gamma^0, \end{array} \right.$$ 

where $\nu^\alpha$ is the outer normal on $S_\alpha$ and $\nu^\alpha \circ c$ the outer normal on the Lipschitz domain $\Omega \setminus S_\alpha$. Using the trace operator\footnote{Formally we extend $\mathcal{F}_\eta b \in W^{1,p}(S_\alpha)$ in an $\varepsilon$-neighbourhood “behind $\Gamma^0”$ by 0, and apply the trace operator for the extension of $\mathcal{F}_\eta b$ in $H^{1-\varepsilon,p}(\partial \Omega \setminus S_\alpha)$ to 0.} we have $\xi \in H^{1-\varepsilon,p}(\partial \left( \Omega \setminus S_\alpha \right))$ with

$$\|\xi\|_{H^{1-\varepsilon,p}(\partial \left( \Omega \setminus S_\alpha \right))} \leq c(\alpha, p, M, \mu) \|\mathcal{F}_\eta b\|_{W^{1,p}(S_\alpha)}.$$  

(2.44)

Moreover, by $\nu^\alpha \circ c = -\nu^\alpha$ on $M^u$ the identity

$$\int_{\partial \left( \Omega \setminus S_\alpha \right)} \xi \cdot \nu^\alpha = \int_{M^u} \mathcal{F}_\eta b \cdot \nu^\alpha \, dA \int_{\Gamma^0} \mu \nu^\alpha \circ c \nu^\alpha \circ c \, ds + \int_{M^u} \mathcal{F}_\eta b \cdot \nu^\alpha \circ c \, dA = 0.$$
follows. Hence, taking the unique\footnote{Unique in the set of solutions satisfying $(u)^+ \in L^2(\partial \Omega \setminus S_\Omega)$, where $(u)^+$ is the non-tangential maximal function of $u$, see \cite{17}.} weak solution of the Stokes equation for a vanishing forcing term and the boundary data $\xi$ (see \cite{7} Lemma 2.4) and \cite{34} Theorem 14), we can extend $F_\eta b \in W^{1,p}(S_\Omega)$ to $\tilde{F}_\eta b \in W^{1,p}(B_\alpha)$. By the linearity of the Stokes equation, the continuity of the solution operator and \cite{244}, the continuity constant is again bounded as claimed. It should be noted that the restriction on $p$ in the theorem comes from the existence result of Stokes equation in Lipschitz domains. □

Using the same construction as in the preceding Lemma but with the unique very weak solution from \cite{35} Theorem 0.3), we can also extend a given $L^p(M)$ function. Since this very weak solution coincides with the unique weak solution of the Stokes system if regular data is given, a simple approximation argument and Remark 2.23 show that the extended function is divergence-free and fulfills an appropriate identity for the normal trace.

**Proposition 2.45.** Let $\eta \in H^2(M)$ with $\|\eta\|_{L^\infty(M)} < \kappa$ and $2 \leq p < \infty$. Then there exists an linear, continuous extension operator

$$\tilde{F}_\eta : L^p(M) \rightarrow L^p(B_\alpha)$$

with $\triangledown |M| (\tilde{F}_\eta b)_{|\Omega_0} = b \gamma(\eta)$. If $b \in W^{1,p}_0(M)$ for some $p \in [2, 3]$, $\tilde{F}_\eta b$ is equivalent to the operator constructed in Lemma \cite{244}. For a fixed $N \in \mathbb{N}$, the continuity constant can be chosen uniformly with respect to $\eta \in H^2(M)$ satisfying $\|\eta\|_{H^2(M)} \leq N$ and $\|\eta\|_{L^\infty(M)} < \kappa$.

By the construction of our extension operators, convergence of the displacements $\eta_n$ imply also the “uniform” convergence of the extended functions.

**Lemma 2.46.** Let $2 \leq p < \infty$, $\eta_n, \eta \in H^2(M)$ with $\|\eta_n\|_{L^\infty(M)}, \|\eta\|_{L^\infty(M)} < \kappa$ and suppose that $\eta_n$ converges uniformly towards $\eta$ on $M$. Then $\tilde{F}_\eta b \rightarrow F_\eta b$ in $L^p(B_\alpha)$ uniformly with respect to $b \in L^p(M)$ and $b \in H^2_0(M)$ satisfying $\|b\|_{L^p(M)} \leq 1$ and $\|b\|_{H^2_0(M)} \leq 1$, respectively.

**Proof.** By the uniform convergence of $\eta_n$,

$$\int_{\eta_n = q} (\nabla (v \circ q)) (q + \tau v \circ q) \, d\tau \rightarrow \int_{\eta = q} (\nabla (v \circ q)) (q + \tau v \circ q) \, d\tau$$

uniformly in $S_\Omega$. Therefore, $\tilde{F}_\eta b$ converges uniformly to $F_\eta b$ in $S_\Omega$, where the convergence is also uniform with respect to $b \in L^p(M)$ satisfying $\|b\|_{L^p(M)} \leq 1$ (or $b \in H^2_0(M)$ satisfying $\|b\|_{H^2_0(M)} \leq 1$). Moreover, by the definition of the boundary data $\xi$ and the linearity and continuity of the solution operator to the Stokes system, this convergence carries over to the whole extension operator. □

Next, we look at the induced time-dependent extension operator.

**Lemma 2.47.** Let $\eta \in \tilde{Y}^I$ with $\|\eta\|_{L^\infty(I \times M)} < \kappa$ and $2 \leq p \leq 3$. The application of the operator from Lemma \cite{244} at almost all times defines a linear, continuous operator

$$\tilde{F}_\eta : H^1(I; L^2(M)) \cap L^p(I; H^1_0(M)) \rightarrow H^1(I; L^2(B_\alpha)) \cap C(T; H^1_0(B_\alpha)) \cap L^p(I; W^{1,p}(B_\alpha)),$$

which satisfies $\nabla \tilde{F}_\eta b = 0$ in $I \times B_\alpha$ and $\gamma \tilde{F}_\eta b|_{\Omega_0} = b \gamma$ on $I \times M$. Fixing $N \in \mathbb{N}$, the continuity constant can be chosen uniformly with respect to $\eta \in \tilde{Y}^I$ satisfying $\|\eta\|_{L^\infty(I \times M)} < \kappa$ and $\|\eta\|_{\tilde{Y}^I} < N$.

**Proof.** By $p \geq 2$ and the parabolic embedding (see \cite{5} Theorem II.5.14.) as well as the identification of the resulting interpolation space by \cite{37} Proposition 3.1, the embedding

$$H^1(I; L^2(M)) \cap L^p(I; H^1_0(M)) \hookrightarrow C(T; H^1_0(M))$$

is continuous. □
is continuous. Furthermore, using Lemma 2.31 and Sobolev’s embedding
\[ \tilde{Y}^l \hookrightarrow C(\tilde{T}; W^{1,2}(M)), \]
\[ H^1(I; L^2(M)) \cap L^p(I; H_0^1(M)) \hookrightarrow L^p(I; W^{1,2}_0(M)) \]
are continuous. By definition of \( \mathcal{F}_\eta b \), the characterisation of the spatial derivatives (2.42) and the extension to \( B_\alpha \) through the solution of the Stokes system, we have \( \mathcal{F}_\eta b \in C(\tilde{T}; H^1(B_\alpha)) \cap L^p(I; W^{1,p}(B_\alpha)) \) with an appropriately bounded continuous constant. Again by the definition of \( \mathcal{F}_\eta b \) in \( I \times S_\alpha \), we have for the time derivative
\[ \partial_t \mathcal{F}_\eta b = \exp \left( -\int_{\eta^{-1}}^\tau \text{div}(\nu \circ \eta) \circ (\eta(\nu \circ \eta)) \, d\tau \right) \left[ \left( (\partial_t b) \nu \right) \circ \eta \right] + (b \nu) \circ \eta \left( \text{div}(\nu \circ \eta) \circ (\eta(\nu \circ \eta)) \circ \partial_t \eta \right). \]
Taking into account \( L^p(I; H^2(M)) \hookrightarrow L^2(I; L^{\infty}(M)) \), we deduce that both \( \partial_t \mathcal{F}_\eta b|_{\eta^{-1}} \in L^2(I; L^2(S_\eta)) \) and the trace \( \partial_t \mathcal{F}_\eta b|_{\eta^{-1}} \in L^2(I; L^2(M^\alpha)) \) are appropriately bounded. Using the characterisation of the generalised time derivative by the difference quotient (see [6 Proposition A.6] and [6 Proposition A.7]) and the properties of the solution operator of the Stokes system, \( \partial_t \mathcal{F}_\eta b \in L^2(I; L^2(B_\alpha)) \) follows with a continuity constant bounded as claimed.

Analogously we can argue for the induced operator from Proposition 2.45.

**Corollary 2.49.** Let \( 0 < \alpha < \kappa, \eta \in \tilde{Y}^l \) with \( \| \eta \|_{L^{\infty}(I \times M)} < \alpha \). Moreover, let \( 1 \leq p \leq \infty \) and \( 2 \leq q < \infty \). The application of the extension operator from Proposition 2.45 at almost all times defines an linear, continuous extension operator
\[ \mathcal{F}_\eta : L^p(I; L^q(M)) \to L^p(I; L^q_\alpha(B_\alpha)), \]
which satisfies \( \mathcal{F}_\eta_{\eta^{-1}}(\mathcal{F}_\eta b)|_{\eta^{-1}} = b(\eta) \) on \( I \times M \). For higher spatial regularity, the operator coincides with the one from Lemma 2.47. Moreover, \( b \in C(\tilde{T}; L^q(M)) \) implies \( \mathcal{F}_\eta b \in C(\tilde{T}; L^q_\alpha(B_\alpha)) \). For a fixed \( N \in \mathbb{N} \), the continuity constant can be chosen uniformly with respect to \( \eta \in \tilde{Y}^l \) satisfying \( \| \eta \|_{L^{\infty}(I \times M)} < \alpha \) and \( \| \eta \|_{\tilde{Y}^l} < N \).

The next Lemma states some convergence properties of the time-dependent extension operators.

**Lemma 2.50.** Let \( 0 < \alpha < \kappa, N \in \mathbb{N} \) and \( \eta, \eta_n \in \tilde{Y}^l \) with \( \| \eta \|_{\tilde{Y}^l}, \| \eta_n \|_{\tilde{Y}^l} < N \) and \( \| \eta \|_{L^{\infty}(I \times M)}, \| \eta_n \|_{L^{\infty}(I \times M)} < \alpha \) for all \( n \in \mathbb{N} \).

a) Let \( 1 \leq p \leq \infty, 2 \leq q < \infty \) and \( b \in L^p(I; L^q(M)) \). If \( \eta_n \) converges uniformly towards \( \eta \) in \( I \times M \), then \( \mathcal{F}_\eta b \to \mathcal{F}_\eta b \) in \( L^p(I; L^q(B_\alpha)) \). Moreover, for \( b \in C(\tilde{T}; L^q(M)) \), one has \( \mathcal{F}_\eta b \to \mathcal{F}_\eta b \) in \( C(\tilde{T}; L^q(B_\alpha)) \).

b) If \( b_n \) converges weakly towards \( b \) in \( L^2(I \times M) \), \( \mathcal{F}_\eta b_n \) converges weakly towards \( \mathcal{F}_\eta b \) in \( L^2(I \times B_\alpha) \).

c) Let \( 2 \leq p \leq 3 \) and \( \eta_n \to \eta \) uniformly on \( I \times M \) and \( \partial_t \eta_n \to \partial_t \eta \) in \( L^2(I; L^2(M)) \), then \( \mathcal{F}_\eta b_n \to \mathcal{F}_\eta b \) in \( L^p(I; L^2(B_\alpha)) \) and \( \partial_t \mathcal{F}_\eta b_n \to \partial_t \mathcal{F}_\eta b \) in \( L^p(I; W^{1,2}_0(B_\alpha)) \cap C(\tilde{T}; L^q(B_\alpha)) \).

**Proof.**

a) Let \( b \in C(\tilde{T}; L^q(M)) \). By the definition of \( \mathcal{F}_\eta b \) on \( I \times S_\alpha \) and the extension to \( I \times B_\alpha \) through the solution of the Stokes system, \( \mathcal{F}_\eta b_n \) converges towards \( \mathcal{F}_\eta b \) in \( C(\tilde{T}; L^q(B_\alpha)) \). By density and the uniform bound on the continuity constants from Corollary 2.49 the claim follows.

b) By the definition of \( \mathcal{F}_\eta b \) on \( I \times S_\alpha \), \( \mathcal{F}_\eta b_n \) converges weakly towards \( \mathcal{F}_\eta b \) in \( L^2(I \times S_\alpha) \) (note that the integral term converges uniformly). Also the boundary values for the Stokes equation \( \xi_n \) converge weakly towards \( \xi \) in \( L^2(I \times M^\alpha) \). Since the solution operator of the Stokes equation is linear and continuous, the weak convergence carries over to the
whole extension operator.

c) Using the continuous embedding

\[ H^1(I; L^2(M)) \cap L^p(I; H^1_0(M)) \hookrightarrow \mathcal{C}(I; H^1_0(M)) \hookrightarrow \mathcal{C}(I; L^4(B_a)) \]

(see the proof of Lemma 2.47 and Sobolev’s embedding), a) implies \( \mathcal{F}_\eta b \rightarrow \mathcal{F}_\eta b \) in \( \mathcal{C}(I; L^4(B_a)) \). Moreover, by the parabolic embedding (see [13, Chapter I, Proposition 3.1]) it follows \( b \in L^2(I; L^{2p}(M)) \). Since \( H^1(M) \hookrightarrow W^{1,2p}(M) \hookrightarrow L^2(M) \) are continuous and compact, Aubin-Lions-Simons Lemma (see [5, Theorem II.5.16]) implies that the embeddings

\[ \{ v \in L^2(I; H^2(M)) \mid \partial_t v \in L^2(I; L^2(M)) \} \hookrightarrow L^2(I; W^{1,2p}(M)) \]

is also compact, i.e. \( \eta_n \) converges strongly towards \( \eta \) in \( L^2(I; W^{1,2p}(M)) \). Hence, by the characterisation of the spatial derivative (2.42), \( \partial_t \mathcal{F}_\eta b \) converges towards \( \partial_t \mathcal{F}_\eta b \) in \( L^p(I; L^p(S_a)) \). To treat the time derivative, we use the interpolation inequality

\[ \| \partial_t \eta_n - \partial_t \eta \|_{L^q(I; L^2(M))} \leq \| \partial_t \eta_n - \partial_t \eta \|_{L^2(I; L^2(M))} \| \partial_t \eta_n - \partial_t \eta \|_{L^2(I; L^2(M))} \]

to show the strong convergence of \( \partial_t \eta_n \) towards \( \partial_t \eta \) in \( L^5(I; L^2(M)) \). Arguing again by interpolation and Sobolev’s embedding, the embeddings

\[ H^1(I; L^2(M)) \cap L^p(I; H^1_0(M)) \hookrightarrow L^3(I; H^{1/3}(M)) \hookrightarrow L^3(I; L^\infty(M)) \]

are continuous, i.e. \( b \in L^3(I; L^\infty(M)) \). By the characterisation of the time derivative (2.48), \( \partial_t \mathcal{F}_\eta b \) converges towards \( \partial_t \mathcal{F}_\eta b \) in \( L^2(I; L^2(S_a)) \). Using the properties of the Stokes operator, these convergences carry over to the extension operator defined on \( I \times B_a \), which finishes the proof.

\[ \square \]

3. MAIN RESULT

Since our regularity is not sufficient to treat the force-coupling boundary term \( F \), we add the weak formulations of the shell and the fluid and couple the test functions in a way that the force-coupling term vanishes. This implies that the test functions depend on the solution of the shell equation. For \( \eta \in \tilde{Y}^I \) with \( \| \eta \|_{L^\infty(I \times M)} < \kappa \), we therefore define the canonically normed space \( T^f_\eta \) as the set of all couples \((b, \varphi)\), where

\[ b \in H^1(I; L^2(M)) \cap L^3(I; H^1_0(M)), \]

\[ \varphi \in W_\eta := H^1(I; L^2(\Omega_{\eta(t)})) \cap L^3(I; W^{1,3}(\Omega_{\eta(t)})) \cap L^\infty(I; L^4(\Omega_{\eta(t)})), \]

with \( b(T, \cdot) = 0 \), \( \varphi(T, \cdot) = 0 \), \( \text{div } \varphi = 0 \) (see Remark 2.36) and for which \( \varphi - \mathcal{F}_\eta b \) can be approximated by functions \( \varphi_n \in W_\eta, \varphi_n(T, \cdot) = 0 \), \( \text{div } \varphi_n = 0 \) vanishing in a neighbourhood of the moving boundary. This implies \( \text{tr}_\eta \varphi = b v \) on \( I \times M \). We call \((f, g, u_0, \eta_0, \eta_1)\) admissible data if \( f \in L^2((0, \infty) \times B_a), g \in L^2((0, \infty) \times M), \eta_0 \in H^2_0(M) \) with \( \| \eta_0 \|_{L^\infty(M)} < \kappa, \eta_1 \in L^2(M) \) and \( u_0 \in L^2(\Omega_{\eta_0}) \) with

\[ \langle \text{tr}_\eta^b, (b, \varphi) \rangle = \int_M \eta_1 \gamma(\eta_0) b dA. \]

Hence, our notion of a weak solution to (1.2)–(1.8) is the following.

**Definition 3.1** (weak solution). Let \((f, g, u_0, \eta_0, \eta_1)\) be admissible data. We call the couple \((\eta, u)\) a weak solution to (1.2)–(1.8) on the interval \( I := (0, T) \) if \( \eta \in \tilde{Y}^I \) with


\[ \|\eta\|_{L^2(I \times M)} < \kappa, \ \eta(0, \cdot) = \eta_0 \ \text{and} \ u \in X^I_\eta \ \text{with} \ \varpi_\eta u = \partial_t \eta \ \nu \ \text{on} \ I \times M \ \text{satisfies} \]

\[ - \int_{\Omega_{\eta}(t)} \mathbf{u} \cdot \partial_t \varphi \ dx \ dt - \frac{1}{2} \int_{M} (\partial_t \eta)^2 b \gamma(\eta) \ dA \ dt - 2 \int_{M} \partial_t \eta \ \partial_b b \ dA \ dt \]

\[ + 2 \int_{I} K(\eta, b) \ dt + 2 \int_{I} \mathbf{D} u : \mathbf{D} \varphi \ dx \ dt \]

\[ + \frac{1}{2} \int_{I} \int_{\Omega_{\eta}(t)} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \varphi \ dx \ dt - \frac{1}{2} \int_{I} \int_{\Omega_{\eta}(t)} (\mathbf{u} \cdot \nabla) \varphi \cdot \mathbf{u} \ dx \ dt \]

\[ = \int_{I} \int_{\Omega_{\eta}(t)} g \ b \ dA \ dt + \int_{I} \int_{\Omega_{\eta}(t)} f \ \varphi \ dx \ dt + \int_{\Omega_{\eta_0}} \mathbf{u}_0 \cdot \varphi(0, \cdot) \ dx + 2 \int_{M} \eta_1 b(0, \cdot) \ dA \]

\[ \text{for all} \ (b, \varphi) \in T^I_\eta. \]

We remark that by our regularity the appearing integrals are well-defined, as we will show exemplary for the first convective term. By the Korn-type inequality from Lemma \[2.14\] we have \[ u \in L^2(I; W^{1, \frac{3}{2}}(\Omega_{\eta}(t))). \] Moreover, Sobolev’s embedding yields \[ u \in L^2(I; L^{\frac{9}{5}}(\Omega_{\eta}(t))). \] Since \[ \frac{1}{2} + \frac{3}{5} = 1, \] Hölder’s inequality yields

\[ \left| \int_{\Omega_{\eta}(t)} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \varphi \ dx \ dt \right| \]

\[ \leq \int_{\Omega_{\eta}(t)} \left| u(t, \cdot) \right|^{\frac{8}{5}}(\Omega_{\eta}(t)) \left\| \nabla u(t, \cdot) \right\|_{L^\frac{9}{5}}(\Omega_{\eta}(t)) \left\| \varphi(t, \cdot) \right\|_{L^2(\Omega_{\eta}(t))} dt \]

\[ \leq c \left\| \mathbf{u} \right\|_{L^2(I; X^I_\eta)} \left\| \varphi \right\|_{T^I_\eta}. \]

Our main theorem is the following.

**Theorem 3.3.** Let \( \Omega \) be an admissible in- and outflow domain and \((f, g, u_0, \eta_0, \eta_1)\) admissible data. There exists a time \( 0 < T^* \leq \infty \) such that for all \( 0 < T < T^* \) a weak solution \((\eta, u)\) of (1.2)–(1.8) exists on the interval \( I := (0, T) \) and satisfies

\[ \text{ess sup}_{0 < T < T^*} \sqrt{E(t)} \leq \sqrt{E_0} + \int_{0}^{T} \frac{1}{2} \left\| \mathbf{u}(s, \cdot) \right\|_{L^2(\Omega_{\eta}(s))}^2 + \frac{1}{2} \left\| g(s, \cdot) \right\|_{L^2(M)} ds. \]  

(3.4)

If the admissible data is sufficiently small, we have \( T^* = \infty \). If \( T^* < \infty \), we find a \( T < T^* \) and a weak solution \((\eta, u)\) such that the maximal displacement \( \|\eta\|_{L^2(I \times M)} \) is arbitrary close to \( \kappa \).

**Remark 3.5.** For \( T^* < \infty \) the displacement of our solution is arbitrary close to \( \kappa \), at which point different parts of the shell could touch each other, i.e. reach a situation which is not covered by our mathematical model.

### 3.1. Compactness

By the non-linearity of the convective term, the weak convergences implied by the formal a priori estimate for some approximate solutions are insufficient to pass to the limit but a compactness argument is required. Because of our noncylindrical domain, classical arguments like Aubin-Lions \[22\] are not applicable. In particular, the representation of the dual spaces and an appropriate notion of a generalised time derivative are not clear. The proof of Aubin-Lions Lemma uses basically of the fundamental theorem of calculus, an application of Ehrling’s Lemma and an Arzela-Ascoli argument. Lengeler used in \[22\] (see also \[24\]) the weak formulation of his problem instead of the fundamental theorem of calculus (and some modified Ehrling Lemma) to prove compactness. A careful analysis of his proof shows that replacing his extension operator \( \mathcal{E}_{\eta_0}^{\eta} \) by our extension operator \( \mathcal{E}_{\eta} \) and using the framework developed above, especially the density result from Lemma \[2.28\] the result also holds in our situation. More precisely, we have the following generalisation of \[24\] Proposition 3.8:
Lemma 3.6. Let $0 < \alpha < \kappa$ and $(f, g, u_0, \eta_0, \eta_1)$ be admissible data, $(\delta_n)_{n \in \mathbb{N}} \subset \hat{Y}^I$ a bounded sequence with $\|\delta_n\|_{L^\infty(I \times M)} < \alpha$ and assume that 

$$\delta_n \to \delta \quad \text{uniformly in } I \times M$$

for some $\delta \in \hat{Y}^I$ with $\|\delta\|_{L^\infty(I \times M)} < \alpha$. Moreover, let $\{(\eta_n, u_n, v_n, \nu_n, \eta_1^n)\}_{n \in \mathbb{N}}$ be a bounded sequence in $Y^I \times X_\delta^I \times \hat{X}_\delta^I \times L^2(\Omega_{\delta_n}(0)) \times L^2(M)$ satisfying

$$- \int_I \int_{\Omega_{\delta_n}(0)} u_n \cdot \partial \varphi \, dx \, dt = \frac{1}{2} \int_M (\partial \eta_n) \cdot (\partial \delta_n) \gamma(\delta_n) \, dA \, dt$$

$$- 2 \int_I \int_M \partial \eta_n \cdot \partial b \, dA \, dt + 2 \int_I \int_{\Omega_{\delta_n}(0)} K(\eta_n, b) \, dt + 2 \int_I \int_{\Omega_{\delta_n}(0)} D\varphi : D\varphi \, dx \, dt$$

$$+ \frac{1}{2} \int_I \int_{\Omega_{\delta_n}(0)} (v_n \cdot \nabla) u_n \cdot \varphi \, dx \, dt - \frac{1}{2} \int_I \int_{\Omega_{\delta_n}(0)} (v_n \cdot \nabla) \varphi \cdot u_n \, dx \, dt$$

$$= \int_I \int_M g \cdot b \, dA + \int_I \int_{\Omega_{\delta_n}(0)} f \cdot \varphi \, dx \, dt + \int_{\Omega_{\delta_n}(0)} \nu_n \cdot \varphi(0, \cdot) \, dx + \int_M \int_{\Omega_{\delta_n}(0)} \eta^n \varphi(0, \cdot) \, dA$$

for all $(b, \varphi) \in T^I_\delta$. Furthermore, let $\partial \delta_n u_n = \partial \eta_n v$ on $I \times M$ and

$$\partial \eta_n \to \partial \eta \quad \text{weakly in } L^2(I; L^2(M)),$$

$$u_n \to u \quad \text{weakly in } L^2(I; L^2(\mathbb{R}^3))$$

for some $\eta \in Y^I$, $u \in X_\delta^I$, where $u_n, u$ are spatially extended by zero. Then $(\partial \delta_n u_n)$ converges strongly towards $(\partial \delta, \eta, u)$ in $L^2(I \times M) \times L^2(I \times \mathbb{R}^3)$.

Note that the structure of the proof of Lemma 3.6 and of the compactness result in 23, 24 is the same, but lengthy and technically demanding. The proof of Lemma 3.6 can be found in [16, Lemma 4.4]. Since already the proof in [24] is densely written we give, for the convenience of the reader, full details in the Appendix. Note that by the assumptions of the preceding Lemma, $\partial \delta_n u = \partial \eta_n v$ is implied by Lemma 2.34. This fact is missing in [22, 24].

3.2. Construction of basis functions. Since we do not transform our system to the reference domain, we have to construct an appropriate set of basis functions, at least in case of a given “smooth” deformation $\delta \in C^4(I \times M)$ with $\|\delta\|_{L^\infty(I \times M)} < \kappa$. Therefore we chose a basis $\{(\hat{Y}_k)_{k \in \mathbb{N}} \in H^2_0(M)$ and a basis $\{(\hat{X}_k)_{k \in \mathbb{N}} \in \text{the canonically normed space} X(\Omega) := \{\varphi \in W^{1,3}(\Omega) \mid \text{div } \varphi = 0, \varphi|_{M} = 0\}$.

We extend $\hat{Y}_k$ to $\mathcal{F}_0 \hat{Y}_k \in X(\Omega)$ by Lemma 2.41 and set $(\hat{W}_{2k}, \hat{W}_{2k-1}) := (0, \hat{X}_k)$, $(\hat{W}_{2k-1}, \hat{W}_{2k}) := (\hat{Y}_k, \mathcal{F}_0 \hat{Y}_k)$. Hence $\hat{W}_k|_M = \hat{W}_k v$ on $M$. Further we define the space $\hat{T}(I, \Omega)$ as the canonically normed set of all pairs

$$(b, \varphi) \in \left[H^1(I; L^2(M)) \cap L^3(I; H^2_0(M)) \right] \times \left[H^1(I; L^2(\Omega)) \cap L^3(I; H^1(\Omega)) \right]$$

satisfying $b(T, \cdot) = 0$ in $M$, $\varphi(T, \cdot) = 0$ in $\Omega$, $\text{div } \varphi = 0$ in $I \times \Omega$ and $\varphi|_{I \times M} = b v$.

Lemma 3.9. The set

$$\text{span}\left\{\varphi \hat{W}_k, \varphi \hat{W}_k \mid \varphi \in C^0((0, T), k \in \mathbb{N})\right\}$$

is dense in the space $\hat{T}(I, \Omega)$.

Proof. Obviously, the set is contained in $\hat{T}(I, \Omega)$. Let $(b, \varphi) \in \hat{T}(I, \Omega)$. We approximate $b$ by functions $b_k \in C^0((0, \infty); H^2_0(M))$ such that $b_k(T, \cdot) = 0$ and $\partial_t b_k(T, \cdot) = 0$, using
The following assertions hold.

By the inequality
\[ \left\| \bar{P}_e(t, \cdot) + \sum_{k=1}^{T} \int_{t_k}^{t} \alpha_k^{(\varepsilon)}(s) \, ds \, \bar{V}_k(\cdot) \right\|_{H^0_0(M)} \leq \int_{0}^{T} \left\| \partial_\varepsilon b_e(s, \cdot) - \sum_{k=1}^{T} \alpha_k^{(\varepsilon)}(s) \, \bar{V}_k(\cdot) \right\|_{H^0_0(M)} \, ds \]
and an appropriate coupling of \( \varepsilon \) and \( \ell \) we have \( \sum_{k=1}^{T} \int_{t_k}^{t} \alpha_k^{(\varepsilon)}(s) \, ds \, \bar{V}_k \to b \) in \( H^1(I; L^2(M)) \cap L^3(I; H^3_0(M)) \). By the linearity and continuity of the operator \( \mathcal{F}_0 \), \( \left( \sum_{k=1}^{T} \int_{t_k}^{t} \alpha_k^{(\varepsilon)}(s) \, ds \, \mathcal{F}_0 \bar{Y}_k(x) \right) \) converges to \( \mathcal{F}_0 b \) in \( H^1(I; L^2(\Omega)) \cap L^3(I; W^{1,3}(\Omega)) \).

Hence, it now suffices to approximate \((0, \varphi - \mathcal{F}_0 b)\) appropriately. This can be done analogously to the approximation of \( b \). The missing details can be found in [15, Lemma 5.3].

Using the Piola transform, we map the space \( \tilde{T}(I, \Omega) \) to the moving domain. Since we have to preserve the compatibility constraint, we have to construct an compatible diffeomorphism for the structure part. By the definition of our trace operator and the Piola transform, we have
\[ \text{tr}_\delta \mathcal{J}_\delta \varphi = (\mathcal{J}_\delta \varphi \circ \Psi_\delta)_{|_{I \times M}} = (d\Psi_\delta (\det d\Psi_\delta)^{-1} \varphi)_{|_{I \times M}}. \]

By the definition of the Hanzawa transform, \( \Psi_\delta(t, x) = x + \delta(t, \mathfrak{q}(x)) \mathfrak{q}(\mathfrak{q}(x)) \) in a neighbourhood of \( I \times M \). Hence, the differential \( d\Psi_\delta \) only scales the outer normal \( \nu \) on the boundary, i.e., there exists \( g : T \times M \to \mathbb{R} \) with
\[ d\Psi_\delta(t, x) \nu(x) = g(t, x) \nu(x), \quad (t, x) \in I \times M. \]

Since \( g(t, x) = d\Psi_\delta(t, x) \mathfrak{q}(x) \cdot \nu(x) \), we have \( g \in C^2(T \times M) \) and \( g \neq 0 \). Therefore, the map \( T_\delta(b, \varphi) = (g(\det d\Psi_\delta)_{|_{I \times M}})^{-1} b, \mathcal{J}_\delta \varphi \) is an isomorphism from \( \tilde{T}(I, \Omega) \) into the canonically normed space \( \tilde{T}(I, \Omega_{\delta}) \) of the couples
\[ (b, \varphi) \in \left( H^1(I; L^2(M)) \cap L^3(I; H^3_0(M)) \right) \times \left( H^1(I; L^2(\Omega_{\delta})) \cap L^3(I; W^{1,3}(\Omega_{\delta})) \right) \]
satisfying \( b(T, \cdot) = 0, \varphi(T, \cdot) = 0, \ \text{div} \varphi = 0 \) and \( \text{tr}_\delta \varphi = b \nu \) on \( I \times M \). By our construction, the basis functions \( \{ \tilde{W}_k, \tilde{W}_k \} \), \( k \in \mathbb{N} \), have the following properties:

**Proposition 3.11.** The following assertions hold.

a) \( \tilde{W}_k \in C(T; H^2_0(M)) \cap C^2(T; L^2(M)) \) and \( \tilde{W}_k \in L^\infty(I; W^{1,3}(\Omega_{\delta(t)}) \cap H^1(I; L^2(\Omega_{\delta(t)}))) \).

Moreover, we have \( \tilde{W}_k(t, \cdot) \in W^{1,3}(\Omega_{\delta(t)}) \) for all \( t \in T \).

b) \( \text{div} \tilde{W}_k = 0 \) and \( \text{tr}_\delta \tilde{W}_k = \tilde{W}_k \nu \) on \( I \times M \). In particular, \( \text{tr}_\delta \tilde{W}_k = 0 \) on \( I \times M \) for \( k \in \mathbb{N} \).

c) The set \( \text{span} \{ (\varphi \tilde{W}_k, \varphi \tilde{W}_k) | \varphi \in C^0(I, (0, T)) \}, k \in \mathbb{N} \) is dense in \( \tilde{T}(I, \Omega_{\delta}) \).

d) For \( t \in T \) the functions \( \{ \tilde{W}_k(t, \cdot), k \in \mathbb{N}, k \text{ even} \} \) form a basis of \( H^2_0(M) \).

e) For \( t \in \tilde{T} \) the functions \( \{ \tilde{W}_k(t, \cdot), k \in \mathbb{N}, k \text{ even} \} \) form a basis of the functions from \( W^{1,3}(\Omega_{\delta(t)}) \) vanishing on the moving boundary.

f) If for \( t \in T \) the linear combination \( \sum_{k=1, i \text{ odd}}^{n} \alpha_k \tilde{W}_k(t, \cdot) \) converges in \( H^2_0(M) \) (or \( L^2(M) \)) then \( \sum_{k=1, i \text{ odd}}^{n} \alpha_k \tilde{W}_k(t, \cdot) \) converges in \( W^{1,3}(\Omega_{\delta(t)}) \) (or \( L^2(\Omega_{\delta(t)}) \)).

### 3.3. The decoupled, regularised and linearised problem

To obtain a weak solution of 

\[ (1.2) - (1.8) \],

we first decouple the dependency of the moving domain from the solution of the shell equation, i.e. we prescribe some displacement \( \delta \) with \( \delta(0, \cdot) = \eta_0 \). We will later restore this coupling by a fixed point argument, hence we have to choose \( \delta \) in a space which \( \tilde{T} \) embeds compactly into. Therefore, we prescribe \( \delta \in C(\bar{T} \times M) \) with \( \delta = 0 \) on \( I \times \partial M \) and \( \| \delta \|_{L^\infty(\bar{I} \times M)} \leq \alpha < \kappa \). We further regularise the displacement, therefore
we also have to adapt the initial fluid velocity \( u_0 \in L^2_0(\Omega_{\eta_0}) \) and by the compatibility condition also the initial velocity of the shell equation \( \eta_1 \in L^2(M) \). To avoid the usual loss of regularity by transformation, we use the regularisation from Proposition 2.37 which approximates \( \delta \) at \( t = 0 \) from “above”. By Proposition 2.37 and Proposition 2.39 there exists \( 0 < \varepsilon_0 = \varepsilon_0(\alpha, \eta_0) \) such that

\[
\eta_0 = \delta(0, \cdot) \leq \mathcal{R}_\varepsilon \delta(0, \cdot), \quad \| \mathcal{R}_\varepsilon \delta \|_{L^2(I \times M)} \leq \frac{\alpha + \kappa}{2} \quad (3.12)
\]

holds for all \( 0 < \varepsilon < \varepsilon_0 \) and all \( \delta \in C(\bar{T} \times M) \) satisfying \( \delta(0, \cdot) = \eta_0 = \delta(0, \cdot) \leq \frac{\alpha + \kappa}{2} \), i.e. \( \Omega_{\eta_0} \subset \Omega_{\delta(0, \cdot)} \). Using \( \mathcal{R}_{\eta_0} \), \( \eta_1 \in L^2_0(B_{\alpha}) \) from Proposition 2.45 we set

\[
u^\varepsilon := \left\{ \begin{array}{ll}
\nu_0 & \text{in } \Omega_{\eta_0}, \\
\mathcal{R}_{\eta_0} \eta_1 & \text{in } \Omega_{\delta(0, \cdot)} \setminus \Omega_{\eta_0},
\end{array} \right.
\]

and have, by the compatibility condition, \( \nu_0^\varepsilon \in L^2_0(\Omega_{\delta(0, \cdot)}) \). Furthermore, defining

\[
\eta^\varepsilon := \exp \left( -\int_{I_{\varepsilon}} \text{div}(\nu)(\cdot + \tau \nu) \, d\tau \right) \eta_1,
\]

we have \( \eta^\varepsilon \in L^2_0(M) \) and, by the Definition of \( \mathcal{R}_{\eta_0} \eta_1 \) the compatibility condition, \( \mathcal{R}_{\delta(0, \cdot)} \nu_0^\varepsilon = \eta_1 \mathcal{A}(\mathcal{R}_\varepsilon \delta(0, \cdot)) \). Furthermore, using Lemma 2.40 we deduce

\[
\nu_0^\varepsilon \to \nu_0 \quad \text{in } L^2(\mathbb{R}^3), \\
\eta^\varepsilon \to \eta_1 \quad \text{in } L^2(M), (3.13)
\]

where we extended \( \nu_0 \) and \( \nu_0^\varepsilon \) by zero. Especially we have for all \( 0 < \varepsilon < \varepsilon_0 \)

\[
\| \nu_0^\varepsilon \|_{L^2(\mathbb{R}^3)} \leq 2 \| \nu_0 \|_{L^2(\mathbb{R}^3)} \quad \text{and} \quad \| \eta^\varepsilon \|_{L^2(M)} \leq 2 \| \eta_1 \|_{L^2(M)}. (3.14)
\]

Since we want to use the Galerkin method, we further linearise the convective terms by introducing a prescribed \( \nu \in L^2(I \times \mathbb{R}^3) \), where this regularity is motivated by our compactness result. Using the classical regularisation, we set \( \mathcal{A}_{\varepsilon} \nu := w_\varepsilon \ast \nu \), where \( w \in C^0_b(\mathbb{R}^d) \) is a kernel with \( w \geq 0 \), \( \text{supp } w \subset B_1(0) \) and \( \int_{\mathbb{R}^d} w \, dx = 1 \), \( w_\varepsilon(x) := \varepsilon^{-d} w(x/\varepsilon) \) and \( \nu \) is extended by zero to \( \mathbb{R}^d \). By the properties of the smoothing operator, we have \( \mathcal{A}_{\varepsilon} \nu \in C_c(\bar{T} \times \mathbb{R}^3) \) (see [5] Proposition II.2.25).

Definition 3.15. Let \( 0 < \alpha < \kappa \), \( (f, g, u_0, \eta_0, \eta_1) \) be admissible data, \( \delta = C(\bar{T} \times M) \) with \( \delta = 0 \) on \( I \times \partial M \), \( \| \delta \|_{L^2(I \times M)} \leq \alpha \), \( \delta(0, \cdot) = \eta_0 \) on \( M \), \( \nu \in L^2(I \times \mathbb{R}^3) \) and \( 0 < \varepsilon < \varepsilon_0 \).

We call the couple \( (\eta, u) \in Y^I \times X_{\delta(0, \cdot)}^\varepsilon \) a weak solution of the decoupled, regularised and linearised problem, if \( \eta(0, \cdot) = \eta_0 \) on \( M \), \( \mathcal{A}_{\varepsilon} \delta u = \partial_t \eta \nu \) on \( I \times M \) and

\[
\begin{align*}
-\int_T \int_{\Omega_{\delta(0, \cdot)}} u \cdot \nabla \phi \, dx \, dt - \frac{1}{2} \int_T \int_M (\partial_t \eta) (\partial_t \mathcal{R}_\varepsilon \delta) b \mathcal{A}(\mathcal{R}_\varepsilon \delta) \, dA \, dt \\
-2 \int_T \int_M \partial_t \eta \delta b \, dA \, dt + 2 \int_T \int_M K(\eta, b) \, dt + 2 \int_T \int_{\Omega_{\delta(0, \cdot)}} D \nu : D \phi \, dx \, dt \\
+ \frac{1}{2} \int_T \int_{\Omega_{\delta(0, \cdot)}} (\mathcal{A}_{\varepsilon} \nu) u \cdot \phi \, dx \, dt - \frac{1}{2} \int_T \int_{\Omega_{\delta(0, \cdot)}} (\mathcal{A}_{\varepsilon} \nu) \phi \cdot u \, dx \, dt \\
= \int_T \int_M g b \, dA \, dt + \int_T \int_M f \cdot \phi \, dx \, dt + \int_{\Omega_{\delta(0, \cdot)}} u_0^\varepsilon \cdot \phi(0, \cdot) \, dx + 2 \int_T \eta_1^\varepsilon b(0, \cdot) \, dA
\end{align*}
\] (3.16)

holds for all \( (b, \phi) \in T^I_{\delta(0, \cdot)} \).
In analogy to the energies defined in the formal a priori estimate, we set
\[
E(\mathcal{F}e, \eta, u, t) := \frac{1}{2} \int_{\Omega_{\mathcal{F}e,d}(0)} |u(t, \cdot)|^2 \, dx + 2 \int_0^t \int_{\Omega_{\mathcal{F}e,d(s)}} |Du(t, \cdot)|^2 \, dx \, ds \\
+ \int_M |\partial_t \eta(t, \cdot)|^2 \, dA + K(\eta(t, \cdot)),
\]
\[
E_0(\mathcal{F}e(0, \cdot), \eta_0, \eta^0_1, u^0_0) := \frac{1}{2} \int_{\Omega_{\mathcal{F}e,d}(0)} |u^0_0|^2 \, dx + \int_M |\eta^0_1|^2 \, dA + K(\eta_0).
\]

Our existence result for the decoupled, regularised and linearised problem is the following:

**Lemma 3.17.** Let \(0 < \alpha < \kappa, 0 < T < \infty, I := (0, T)\) and \((f, g, u_0, \eta_0, \eta_1)\) be admissible data. Let \(\varepsilon_0 = \varepsilon_0(\alpha, \eta_0)\) be as given above, \(\bar{\delta} \in C^0(\overline{I} \times M)\) with \(\bar{\delta} = 0\) on \(I \times \partial M\), \(\|\delta\|_{L^2(I \times M)} \leq \alpha, \delta(0, \cdot) = \eta_0\) on \(M\), \(v \in L^2(I \times \mathbb{R}^3)\) and \(0 < \epsilon < \varepsilon_0\). Then there exists a weak solution \((\eta, u) \in Y^1 \times X_{\mathcal{F}e,d}^1\) to the the decoupled, regularised and linearised problem which, for all \(0 < T_1 \leq T\), satisfies the energy inequality
\[
\text{esssup}_{t \in (0, T_1)} \sqrt{E(\mathcal{F}e, \eta, u, t)} \leq \sqrt{E_0(\mathcal{F}e(0, \cdot), \eta_0, \eta^0_1, u^0_0)} + \int_0^{T_1} \frac{1}{\sqrt{2}} \|f(s, \cdot)\|_{L^2(\Omega_{\mathcal{F}e,d(u)})} + \frac{1}{2} \|g(s, \cdot)\|_{L^2(M)} \, ds.
\]

In particular, \((\eta, u)\) is uniformly bounded in \(Y^1 \times X_{\mathcal{F}e,d}^1\) independently of \(\bar{\delta}, v\) and \(\epsilon\) given the conditions \(\delta(0, \cdot) = \eta_0\) and \(\|\delta\|_{L^2(I \times M)} \leq \alpha\).

**Proof.** We use the Galerkin method with the constructed basis functions \((W_k, \tilde{W}_k), k \in \mathbb{N}\). Therefore, for a fixed \(n \in \mathbb{N}\), we seek functions \(\alpha^j_n : [0, T] \rightarrow \mathbb{R}, 1 \leq j \leq n\), satisfying
\[
\int_{\Omega_{\mathcal{F}e,d(t)}} \partial_t u_n \cdot W_j \, dx + \frac{1}{2} \int_M (\partial_t \eta_n)(\partial_t \mathcal{F}e \bar{\delta}) W_j \mathcal{F}e \bar{\delta} \, dA \\
+ 2 \int_M \partial^2 \eta_n W_j \, dA + 2K(\eta_n, W_j) + 2 \int_{\Omega_{\mathcal{F}e,d(t)}} Du_n : DW_j \, dx \\
+ \frac{1}{2} \int_{\Omega_{\mathcal{F}e,d(t)}} (\mathcal{F}e \bar{\delta} \cdot \nabla) u_n \cdot W_j \, dx - \frac{1}{2} \int_{\Omega_{\mathcal{F}e,d(t)}} (\mathcal{F}e \bar{\delta} \cdot \nabla) W_j \cdot u_n \, dx \\
= \int_M g_n W_j \, dA + \int_{\Omega_{\mathcal{F}e,d(t)}} f_n \cdot W_j \, dx
\]
for all \(1 \leq j \leq n\) and all \(t \in [0, T]\), where
\[
u_n(t, x) := \sum_{k=1}^n \alpha^k_n(t) W_k(t, x), \quad \eta_n(t, x) := \int_0^t \sum_{k=1}^n \alpha^k_n(s) W_k(s, x) \, ds + \eta_0(x)
\]
and \(g_n, f_n\) are suitable approximations of \(g\) and \(f\), respectively, with
\[
g_n \in C^0(\overline{I} \times M), \quad g_n \rightarrow g \quad \text{in} \quad L^2(I \times M),
\]
\[
f_n \in C^0(\overline{I} \times \overline{K}), \quad f_n \rightarrow f \quad \text{in} \quad L^2(I \times B_K).
\]

Using the compatibility condition \(\text{tr}^{\nu_n, \eta_n(0, \cdot)} W_k(0, \cdot) = \eta_1 \mathcal{F}e(\delta(0, \cdot))\) and the properties of our basis functions as in \([24], [16]\), we can find \(\alpha^0_k\) with
\[
\sum_{k=1}^n (\alpha^0_k W_k(0, \cdot), \alpha^0_k W_k(0, \cdot)) \rightarrow (\eta^0_1, u^0_0)
\]
in \(L^2(M) \times L^2(\Omega_{\mathcal{F}e,d(0)})\). An easy computation using the linear independence and regularity of the basis functions (see \([15]\)) shows that equivalently we can search for a solution to the system of ordinary integro-differential equations
\[
\alpha_n(t) = \tilde{A}(t, \alpha(t)) + \int_0^t \tilde{B}(t, s, \alpha(s)) \, ds,
\]
where \(\tilde{A}, \tilde{B}\) are suitable operators.
where $\hat{A} \in C^0([0,\infty) \times \mathbb{R}^n; \mathbb{R}^n)$, $\hat{B} \in C^0([0,\infty) \times \mathbb{R}^n; \mathbb{R}^n)$ are affine linear in the last component. By (21) Theorem 1.11 and the classical extension argument, we get a solution $\alpha_\nu = (\alpha_\nu^i(\cdot),...)$ is $C^3([0,T];\mathbb{R}^n)$ of (3.23) to the initial value $\alpha_\nu(0) = (\alpha_\nu^i(0),\ldots)$, i.e. $u_i \in L^1(I,W^{1,3}(\Omega_{\alpha_\nu(\delta)}))$ and $\eta_\nu \in C^0(\bar{T},H^0_b(M)) \cap C^0(\bar{T},L^2(M))$ which satisfies (3.19) for all $1 \leq j \leq n$ and all $t \in [0,T]$. Furthermore, Proposition 3.11 implies

$$\text{tr}_{\delta \gamma \delta}(u_i) = \sum_{k=1}^n \alpha_{\nu}^k \text{tr}_{\delta \gamma \delta}(W_k) = \sum_{k=1}^n \alpha_{\nu}^k W_k \nu = \partial_t \eta_\nu \nu, \quad (3.24)$$

$$\text{div} u = \sum_{k=1}^n \alpha_{\nu}^k \text{div} W_k = 0,$$

and (3.23) to the initial value $u_i(0) = (u_i^1(0),\ldots)$. Also, by (3.20), we have $\eta_\nu(0,\cdot) = \eta_0$. In order to derive a uniform energy estimate, we multiply (3.19) with $\alpha_\nu^j(t)$, take the sum over $j = 1,\ldots,n$ and get

$$\int_{\Omega_{\alpha_\nu(\delta)}(t)} \partial_t u_i \cdot u_i \, dx + \frac{1}{2} \int_M (\partial_t \eta_\nu)^2 (\partial_t \gamma \nu) \gamma(\nu \gamma \delta) \, dA + 2 \int_M (\partial_t^2 \eta_\nu) (\partial_t \eta_\nu) \, dA$$

$$+ 2K(\eta_\nu, \eta_\nu) + 2 \int_{\Omega_{\alpha_\nu(\delta)}} \eta_\nu \, dx = \int_M g_\nu \partial_t \eta_\nu \, dA + \int_{\Omega_{\alpha_\nu(\delta)}} f_n \cdot u_i \, dx. \quad (3.25)$$

Thus, Reynolds transport theorem yields

$$\frac{d}{dt} \int_{\Omega_{\alpha_\nu(\delta)}} |u_i|^2 \, dx = \int_{\partial \Omega_{\alpha_\nu(\delta)}} |u_i|^2 \, ds + \int_{\partial \Omega_{\alpha_\nu(\delta)}} v_{\delta \gamma \delta}(t) \cdot v_{\delta \gamma \delta}(t) \, dA_{\delta \gamma \delta}(t),$$

where $v_{\delta \gamma \delta}(t)$ is the outer normal of $\Omega_{\alpha_\nu(\delta)}$ and $\nu_{\delta \gamma \delta}(t)$ the boundary velocity. By our assumption for the moving domain, the deformations happen only along the outer normal on $M$. In particular, the outer normal on $\partial M$ is perpendicular to the normal on $\Gamma_{\delta \gamma \delta}(t)$ by the orthogonality assumption for the reference domain. Hence, transforming the boundary integral to the boundary of the reference domain and taking into account the compatibility condition (3.24) and Remark 2.11, the identity

$$\int_{\partial \Omega_{\alpha_\nu(\delta)}} v_{\delta \gamma \delta}(t) \cdot v_{\delta \gamma \delta}(t) \, dA_{\delta \gamma \delta}(t) = \left( \int_M (\partial_t \eta_\nu)^2 (\partial_t \gamma \nu) \gamma(\nu \gamma \delta) \, dA \right)$$

follows. Arguing as in our formal a priori estimate, we deduce the energy inequality

$$\text{esssup}_{t \in (0,T)} \sqrt{E(\nu \gamma \delta, \eta_\nu, u_i, t)} \leq \sqrt{E_0(\gamma \nu(0,\cdot), \eta_0, \eta_0(0,\cdot), u_i(0,\cdot))} \quad (3.26)$$

$$+ \int_0^T 1 \sqrt{\frac{1}{2} \|f_n(s,\cdot)\|_{L^2(\Omega_{\alpha_\nu(\delta)})} + \frac{1}{2} \|g_\nu(s,\cdot)\|_{L^2(M)}} \, ds$$

for all $0 < T \leq T$. Hence, by the coercivity of the Koiter energy and the convergences (3.21), (3.22) and (3.14) the couples $(\eta_\nu, u_i)$ are uniformly bounded in $Y^t \times X^t_{\delta \gamma \delta}$ as claimed in the Lemma. Using the compact embedding $Y^t \hookrightarrow \hookrightarrow C(\bar{T} \times M)$, we get for a subsequence

$$\eta_\nu \xrightarrow{\ast} \eta \quad \text{weakly-ast in } L^\infty(\bar{T},H^0_b(M)) \text{ and uniformly in } \bar{T} \times M,$$

$$\partial_t \eta_\nu \xrightarrow{\ast} \partial_t \eta \quad \text{weakly-ast in } L^\infty(\bar{T},L^2(M)),$$

thus $\eta \in Y^t$. By our Korn-type inequality, the spatial extensions of $u_i$, $\nabla u_i$ and $\nabla u$ by zero are uniformly bounded in $L^1(\bar{T},L^2(\mathbb{R}^3))$, $L^2(\bar{T},L^{13/7}(\mathbb{R}^3))$ and $L^2(\bar{T};L^2(\mathbb{R}^3))$, respectively. For the convenience we use the notation $\nabla u$ and $\nabla u$ also outside $\Omega_{\delta \gamma \delta}$ but we emphasize that the usual meaning of the symbols only hold on the inside. Hence,

$$u_i \xrightarrow{\ast} u \quad \text{weakly-ast in } L^\infty(\bar{T},L^2(\mathbb{R}^3)),$$

$$\nabla u_i \xrightarrow{\ast} \nabla u \quad \text{weakly in } L^2(\bar{T},L^{13/7}(\mathbb{R}^3)),$$

$$\nabla u \xrightarrow{\ast} \nabla u \quad \text{weakly in } L^2(\bar{T};L^2(\mathbb{R}^3)),$$
Since \( \text{div} u_n = 0 \) in \( \Omega^I_{\varepsilon, \delta} \), the convergences imply \( \text{div} u = 0 \) in \( \Omega^I_{\varepsilon, \delta} \), i.e., \( u \in X^I_{\varepsilon, \delta} \). Moreover, (3.26) and the lower semi-continuity of Koiter’s energy and the norms imply (3.18) and the uniform bound on \( (\eta, u) \) as claimed by the Lemma. To show (3.16), we first take \( \varphi \in C_0^1((0, T)) \). Using again Reynolds transport theorem, the orthogonality assumption for the reference domain, the compatibility condition (3.24) and Remark 2.11, we have

\[
\int_I \int_{\Omega_{\varepsilon, \delta}(0)} \partial_i (u_n \cdot \varphi W_j) \, dx \, dt = - \int_I \int_M \partial_i \eta_n \varphi W_j (\partial_i \varepsilon) \gamma (\varepsilon) \, dA \, dt
\]

\[
- \int_{\Omega_{\varepsilon, \delta}(0)} u_n (0, \cdot) \cdot (\varphi (0) W_j (0, \cdot)) \, dx.
\]

Therefore, by multiplying (3.19) with \( \varphi \), integration over \( I \) and integration by parts with respect to time, we get

\[
- \int_I \int_{\Omega_{\varepsilon, \delta}(0)} \partial_i (\varphi W_j) \, dx \, dt - \frac{1}{2} \int_I \int_M (\partial_i \eta_n) (\partial_i \varepsilon) \varphi W_j \gamma (\varepsilon) \, dA \, dt
\]

\[
- 2 \int_I \int_M \partial_i \eta_n \partial_i (\varphi W_j) \, dA \, dt + 2 \int_I \int_K (\eta_n, \varphi W_j) \, dt + 2 \int_I \int_M D u_n : D (\varphi W_j) \, dx \, dt
\]

\[
+ \frac{1}{2} \int_I \int_{\Omega_{\varepsilon, \delta}(0)} (\varepsilon \nabla \cdot \nabla) u_n \cdot (\varphi W_j) \, dx \, dt - \frac{1}{2} \int_I \int_{\Omega_{\varepsilon, \delta}(0)} (\varepsilon \nabla \cdot \nabla) (\varphi W_j) \cdot u_n \, dx \, dt
\]

\[
= \int_I \int_M g_n \varphi W_j \, dA \, dt + \int_{\Omega_{\varepsilon, \delta}(0)} u_n (0, \cdot) \cdot (\varphi (0) W_j (0, \cdot)) \, dx
\]

\[
+ \int_I \int_{\Omega_{\varepsilon, \delta}(0)} u_n (0, \cdot) \cdot (\varphi (0) W_j (0, \cdot)) \, dA.
\]

By the linearisation of the convective term and (3.21), (3.22), (3.27) and (3.28) we can pass to the limit \( n \rightarrow \infty \). Thus, \( (\eta, u) \) satisfy (3.16) for all test functions

\[
(b, \varphi) \in \text{span} \{ (\varphi W_k, \varphi W_k) \mid \varphi \in C_0^1((0, T)), k \in \mathbb{N} \}.
\]

By Proposition 3.11, this set is dense in \( T_{\varepsilon, \delta}^I \), hence \( (\eta, u) \) fulfills (3.16) for all \( (b, \varphi) \in T_{\varepsilon, \delta}^I \). Since (3.20) implies \( \eta_n (0, \cdot) = \eta_0 \), we deduce that \( \eta (0, \cdot) = \eta_0 \), using (3.27) as well. Moreover, by (3.24) and Lemma 2.34 it follows that \( u_{\varepsilon, \delta} u = \partial_i \eta \nu \) and the proof is complete. \( \square \)

3.4. The regularised problem. To restore the coupling between the displacement and the domain as well as the convective term, we use a set-valued fixed-point theorem, i.e. the Bohnenblust-Karlin theorem [2, Theorem 17.57]. By choosing a sufficiently small time interval together with our compactness result, we satisfy the assumptions of the fixed-point theorem and get a weak solution of the regularised problem.

Definition 3.29. Let \( 0 < \alpha < \kappa, 0 < T < \infty \), \( I := (0, T) \) and let \( (f, \varphi, u_0, \eta_0, \eta_1) \) be some admissible data. Let \( \varepsilon_0 = \varepsilon_0(\alpha, \kappa) \) be as in Section 3.3 and \( 0 < \varepsilon < \varepsilon_0 \). We call the couple \( (\eta, u) \) with \( \eta \in Y^I \) and \( u \in X^I_{\varepsilon, \delta} \) a weak solution of the \( \varepsilon \)-regularised problem if

\[
\| \eta \|_{L^2(I; \mathcal{M} \varepsilon)} < \kappa, \eta (0, \cdot) = \eta_0 \text{ on } M, \text{ and } u_{\varepsilon, \delta} \eta = \partial_i \eta \nu \text{ on } I \times M \text{ and}
\]

\[
- \int_I \int_{\Omega_{\varepsilon, \delta}(0)} u \cdot \partial_i \varphi \, dx \, dt - \frac{1}{2} \int_I \int_M (\partial_i \eta) (\partial_i \varepsilon) b \gamma (\varepsilon) \, dA \, dt
\]

\[
- 2 \int_I \int_M \partial_i \eta \partial_i b \, dA \, dt + 2 \int_I \int_M K (\eta, b) \, dt + 2 \int_I \int_M D u : D \varphi \, dx \, dt
\]

\[
+ \frac{1}{2} \int_I \int_{\Omega_{\varepsilon, \delta}(0)} (\varepsilon \nabla \cdot \nabla) u \cdot \varphi \, dx \, dt - \frac{1}{2} \int_I \int_{\Omega_{\varepsilon, \delta}(0)} (\varepsilon \nabla \cdot \nabla) \varphi \cdot u \, dx \, dt
\]

\[
= \int_I \int_M g b \, dA \, dt + \int_I \int_{\Omega_{\varepsilon, \delta}(0)} f \cdot \varphi \, dx \, dt + \int_{\Omega_{\varepsilon, \delta}(0)} u_0^0 \cdot \varphi (0, \cdot) \, dx + 2 \int_M \eta_1 b (0, \cdot) \, dA
\]

(3.30)
holds for all \((b, \varphi) \in T^l_{\#c\eta}\).

Now we can formulate the existence result for the \(\varepsilon\)-regularised problem.

**Lemma 3.31.** Let \(0 < T_0 < \infty\) and let \((f, g, u_0, \eta_0, \eta_1)\) be some admissible data. Let
\[
0 < \alpha := \frac{\|\eta_0\|_{L^\infty(M)} + \kappa}{2} < \kappa.
\]
Then a time \(0 < T \leq T_0\) exists such that for all \(0 < \varepsilon < \varepsilon_0\) a weak solution of the \(\varepsilon\)-regularised problem \((\eta, u)\) exists on the interval \(I = (0, T)\) and satisfies \(\|\eta\|_{L^\infty(I \times M)} \leq \alpha\) and for all \(0 < T_1 \leq T\)
\[
\text{esssup} \sqrt{E(\mathcal{R}_\varepsilon \eta, \eta, \eta, u, t)} \leq \sqrt{E_0(\mathcal{R}_\varepsilon \eta(0, \cdot), \eta_0, \eta_1^0, u_0^0)} + \int_0^{T_1} \frac{1}{\sqrt{2}} \|f(s, \cdot)\|_{L^2(M)} + \frac{1}{2} \|g(s, \cdot)\|_{L^2(M)} \, ds.
\]
(3.32)

In particular \((\eta, u)\) is bounded independently of \(0 < \varepsilon < \varepsilon_0\) in \(Y^l \times X^l_{\#c\eta}\). The time \(T\) can be chosen independently of the data if an upper bound is given on the norms \(\|f\|_{L^2(I \times M)}\), \(\|g\|_{L^2(I \times M)}\), \(\|\eta_0\|_{H^1_0(M)}\), \(\|\eta\|_{L^2(M)}\), \(\|\eta\|_{L^2(M)}\), and \(\|u\|_{L^2(M)}\) as well as a strictly positive lower bound on \(\kappa - \|\eta_0\|_{L^\infty(M)}\).

**Proof.** We define \(I_0 := (0, T_0)\) and chose a \(\delta \in C^0(T_0 \times M)\) with \(\delta = 0\) on \(I_0 \times \partial M\), \(\|\delta\|_{L^\infty(I_0 \times M)} \leq \alpha\), \(\delta(0, \cdot) = \eta_0\) on \(M\) and \(v \in L^2(I_0 \times \mathbb{R}^3)\). The associated weak solution \((\eta, u)\) of the regularised, decoupled and linearised problem from Lemma 3.31 satisfies the energy inequality (3.32) and, by the coercivity of the Koiter energy, the estimate
\[
\|\eta\|_{Y^l} + \|u\|_{X^l_{\#c\eta}} \leq c_0.
\]
In here, the constant can be chosen independent of the admissible data, \(\delta, v\) and \(\varepsilon\), if there exists an upper bound on the norms \(\|f\|_{L^2(I_0 \times \mathbb{R}^3)}\), \(\|g\|_{L^2(I_0 \times \mathbb{R}^3)}\), \(\|\eta_0\|_{H^1_0(M)}\), \(\|\eta\|_{L^2(M)}\), \(\|\eta\|_{L^2(M)}\) and \(\|u\|_{L^2(M)}\) and as long as \(\delta\) satisfies \(\delta(0, \cdot) = \eta_0\) and \(\|\delta\|_{L^\infty(I_0 \times M)} \leq \alpha\). Extending \(u\) by zero in the spatial direction, we also get the estimate \(\|u\|_{L^2(I_0 \times \mathbb{R}^3)} \leq \sqrt{T_0} c_0\). Hence, we set \(c_1 := \max\{c_0, \sqrt{T_0} c_0\}\). By Lemma 2.31, the embedding \(Y^l_0 \hookrightarrow C^{0,1-\theta}(\overline{I_0}, C^0(M))\) is linear and continuous for some \(\frac{1}{2} < \theta < 1\) with an operator norm denoted by \(c_2\). Since \(\|\eta_0\|_{L^\infty(M)} \leq \kappa\), we can choose \(0 < T \leq T_0\) which satisfies
\[
c_1 c_2 T^{1-\theta} < \frac{\kappa - \|\eta_0\|_{L^\infty(M)}}{2},
\]
and depends only on \(c_1, c_2\), and a positive lower bound on \(\kappa - \|\eta_0\|_{L^\infty(M)}\). We set \(I := (0, T)\) and notice that this particular choice of \(T\) together with the initial condition \(\eta(0, \cdot) = \eta_0\) and the \(\theta\)-Hölder-continuity of \(\eta\), implies \(\|\eta\|_{L^\infty(I \times M)} < \alpha\). We define the space
\[
Z := C(I \times M) \times L^2(I \times \mathbb{R}^3)
\]
and the non-empty, closed, convex subset
\[
D := \left\{ (\delta, v) \in Z \mid \delta = 0 \text{ on } I \times \partial M, \delta(0, \cdot) = \eta_0, \|\delta\|_{L^\infty(I \times M)} \leq \alpha, \|v\|_{L^2(I \times \mathbb{R}^3)} \leq c_1 \right\}.
\]
We define \(F : D \subset Z \to Z^l\) as the set-valued map which assigns to \((\delta, v) \in D\) the set of all weak solutions \((\eta, u)\) of the decoupled, regularised and linearised problem on the interval \(I\) to the admissible data \((f, g, u_0, \eta_0, \eta_1)\) and the functions \(\delta\) and \(v\), which satisfy for all \(0 < T_1 \leq T\) the inequality (3.18) and the estimates
\[
\|\eta\|_{L^\infty(I \times M)} \leq \alpha, \quad \|u\|_{L^2(I \times \mathbb{R}^3)} \leq c_1,
\]
(3.34)
where \( u \) is again extended by 0 in the spatial direction. Taking into account the boundary and initial conditions of the weak solution, \( F \) maps \( D \) into its power set, \( F(D) \subset 2^D \). To show that \( F \) has a fixed point, we use the theorem of Bohnenblust-Karlin, see [2] Theorem 17.57. Therefore, we have to check prerequisites, i.e. we have to show that for all \( (\delta, v) \in D \) the set \( F(\delta, v) \) is non-empty and convex, the graph of \( F \) is closed, and the image of \( F \) is relatively compact in \( Z \).

By our choice of \( T \), Lemma 3.17 implies that for all \( (\delta, v) \in D \) the set \( F(\delta, v) \) is non-empty. By the linearity of the decoupled, regularised and linearised problem and the coercivity and bilinearity of the Koiter energy, some straightforward computations show that \( F(\delta, v) \) is convex. To show the relative compactness of \( F(D) \) in \( Z \), we take a sequence \( (\eta_n, u_n)_{n \in \mathbb{N}} \subset F(D) \). Thus, there exists a sequence \( (\delta_n, v_n)_{n \in \mathbb{N}} \subset D \) with \( (\eta_n, u_n) \in F(\delta_n, v_n) \). Since \( \varepsilon \) is fixed, by Proposition 2.37 \( (\mathcal{R}_\varepsilon \delta_n)_{n \in \mathbb{N}} \) is bounded in \( C^0(\bar{T} \times M) \). Using the Arzela-Ascoli theorem, we get, for a subsequence,

\[
\mathcal{R}_\varepsilon \delta_n \to \xi \quad \text{in } C^2(\bar{T} \times M). \tag{3.35}
\]

By (3.12), we have \( \|\mathcal{R}_\varepsilon \delta_n\|_{L^2(\bar{T} \times M)} \leq \frac{\alpha_0 + \varepsilon}{2} \). Therefore we can choose a uniform constant in the Korn-type inequality. Using the coercivity of the Koiter energy, the energy inequality (3.18) implies the uniform estimate

\[
\|\eta_n\|_{\mathcal{F} \xi} + \|u_n\|_{\mathcal{F} \eta \xi} \leq c_3. \tag{3.36}
\]

As in (3.27) and (3.28) for a subsequence we deduce

\[
\eta_n \rightharpoonup^{\ast} \eta \quad \text{weakly-}^* \text{ in } L^\infty(I; H^2_0(M)) \quad \text{and uniformly in } \bar{T} \times M,
\]

\[
\partial_t \eta_n \rightharpoonup^{\ast} \partial_t \eta \quad \text{weakly-}^* \text{ in } L^2(I; L^2(M)). \tag{3.37}
\]

for some \( \eta \in Y^t \), and

\[
u_n \rightharpoonup \nu \quad \text{weakly-}^* \text{ in } L^\infty(I; L^3(\mathbb{R}^3)),
\]

\[
\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } L^2(I; L^{13/7}(\mathbb{R}^3)),
\]

\[
Du_n \rightharpoonup Du \quad \text{weakly in } L^2(I; L^2(\mathbb{R}^3)), \tag{3.38}
\]

for some \( u \in X^t \), whereby the functions are extended spatially by zero. Hence, we emphasize again that the symbols \( \nabla u \) and \( Du \) have their usual meaning only on the set \( \Omega^t \). Since the weak solutions satisfy the identity (3.16), we can use the compactness result (Lemma 3.6) to obtain the strong convergences

\[
\partial_t \eta_n \to \partial_t \eta \quad \text{in } L^2(I \times M),
\]

\[
u_n \to \nu \quad \text{in } L^2(I \times \mathbb{R}^3), \tag{3.39}
\]

By (3.37) and (3.39), a subsequence of \( (\eta_n, u_n)_{n \in \mathbb{N}} \) converges towards \( (\eta, u) \) in \( Z \), therefore \( F(D) \) is relatively compact in \( Z \). It remains to show that \( F \) has a closed graph. To this end, we consider some sequences \( (\delta_n, v_n)_{n \in \mathbb{N}} \subset D \), \( (\eta_n, u_n)_{n \in \mathbb{N}} \subset D \) with \( (\eta_n, u_n) \in F(\delta_n, v_n) \) and

\[
\delta_n \to \delta \quad \text{in } C^0(\bar{T} \times M),
\]

\[
\eta_n \to \eta \quad \text{in } C^0(\bar{T} \times M),
\]

\[
v_n \to v \quad \text{in } L^2(I \times \mathbb{R}^3), \quad u_n \to u \quad \text{in } L^2(I \times \mathbb{R}^3) \tag{3.40}
\]

for a \( (\delta, v) \in Z \) and a \( (\eta, u) \in Z \). We will prove that \( (\delta, v), (\eta, u) \in D \) and \( (\eta, u) \in F(\delta, v) \). Since \( D \) is closed, \( (\delta, v), (\eta, u) \in D \) follows. With the same arguments as above, we find a subsequence such that (3.35), (3.37), (3.38) and (3.39) hold. By the properties of the regularisation operators, (3.35) and (3.40) imply \( \xi = \mathcal{R}_\varepsilon \delta \) and

\[
\mathcal{R}_\varepsilon v_n \to \mathcal{R}_\varepsilon v \quad \text{in } C^0(\bar{T} \times \mathbb{R}^3). \tag{3.41}
\]

Furthermore, from \( (\eta_n, u_n) \in F(\delta_n, v_n) \) and our convergences through the lower semi-continuity of the norms and the continuity of the Koiter energy we can deduce that for
all $0 < T_1 \leq T$ the couple $(\eta, u)$ satisfies the energy inequality (3.18) and the estimates

$$\|\eta\|_{L^\infty(I;L^1(M))} \leq \alpha, \quad \|u\|_{L^2(I;\mathbb{R}^3)} \leq c_1.$$  

Furthermore, $(\eta_n, u_n) \in F(\delta_n, v_n)$ implies $\eta_n(0, \cdot) = \eta_0$ on $M$ and $\text{tr}_{\delta_n} u_n = \partial_t \eta_n v$ on $I \times M$. By the uniform convergence of $\eta_n$ and Lemma 2.34 we also have $\eta(0, \cdot) = \eta_0$ and $\text{tr}_{\delta} u = \partial_t \eta v$. To finally show the identity (3.16) for $(\eta, u)$ and all test functions $(b, \varphi) \in T_{\delta, \delta}^r$, we will again use the fact that $(\eta_n, u_n) \in F(\delta_n, v_n)$. Therefore,

$$- \int_I \int_{\Omega_{\delta, \delta}(b)} \partial_t \eta_n \partial b_n \, dt \, dx + 2 \int_I \int_{\Omega_{\delta, \delta}(b)} K(\eta_n, b_n) \, dt \, dx + 2 \int_I \int_{\Omega_{\delta, \delta}(b)} \mathbf{D} \varphi_n : \mathbf{D} \varphi_n \, dt \, dx \leq \frac{1}{2} \int_I \int_{\Omega_{\delta, \delta}(b)} \mathbf{D} \varphi_n : \mathbf{D} \varphi_n \, dt \, dx$$

holds for all $(b_n, \varphi_n) \in T_{\delta, \delta}^r$. Since we cannot use $(b, \varphi) \in T_{\delta, \delta}^r$ directly as a test function in (3.42), we have to use the special structure of this space. Hence, we take $b \in H^1(I;L^2(M)) \cap L^2(I;H^1(M))$ with $b(T, \cdot) = 0$. By Lemma 2.47 $(b, \mathcal{F}_{\delta, \delta} b) \in T_{\delta, \delta}^r$. Taking into account the convergences from Lemma 2.50 and Lemma 2.46 as well as (3.35), (3.37), (3.38), (3.39) and (3.41), we can pass to the limit $n \to \infty$ in the identity (3.42) tested with $(b, \mathcal{F}_{\delta, \delta} b)$ and obtain that $(\eta, u)$ satisfies (3.16) for $(b, \mathcal{F}_{\delta, \delta} b)$. Due to our definition of $T_{\delta, \delta}^r$, it remains to show that $(\eta, u)$ satisfies the identity (3.16) for all test functions $(0, \varphi)$ with $\varphi \in W_{\delta, \delta}, \varphi = 0$ in a neighbourhood of the moving boundary, $\text{div} \varphi = 0$ and $\varphi(T, \cdot) = 0$. For such a test function $(0, \varphi)$, the uniform convergence of $\mathcal{F}_{\delta, \delta}$ towards $\mathcal{F}_{\delta}$ implies $\text{supp} \varphi \subset \Omega_{\delta, \delta}^r$ for $n$ big enough. Therefore, silently extending this function by zero in the spatial direction, we have that $(0, \varphi) \in T_{\delta, \delta}^r$ is an admissible test function for the identities (3.42) for $n$ big enough. Using again the convergences from above, we can pass to the limit $n \to \infty$ to show that $(\eta, u)$ is a weak solution of the decoupled, regularised and linearised problem, i.e. $(\eta, u) \in F(\delta, v)$. Hence, $F$ has a closed graph and we can use theorem of Bohnenblust–Karlin, ([2] Theorem 17.57), to obtain a fixed point and therefore a weak solution of the regularised problem. The energy inequality (3.32) then follows from the definition of the map $F$. 

3.5. Limiting Process. Taking weak solutions of the regularised problem to the parameter $\varepsilon$, we can now pass to the limit $\varepsilon \to 0$ to obtain a weak solution for our problem (1.2)–(1.8).

Proof. (of Theorem 3.3) We choose some $T_0 \in (0, \infty)$ and set $t_0 := (0, T_0)$. Further we set $\alpha := \frac{1}{2} (\|\partial_t \eta\|_{L^\infty(M)} + \kappa), \varepsilon_n := \frac{1}{n}$ and take $n \in \mathbb{N}$ big enough so that $\varepsilon_n < \varepsilon_0$ (with $\varepsilon_0 = \varepsilon_0(\alpha, \eta_0)$ as in Section 3.3) holds. By Lemma 5.31 we get the existence of an time interval $I = (0, T)$ with $0 < T \leq T_0$ independently of $n$ and weak solutions $(\eta_n, u_n)$ of the regularised problem to the regularization parameter $\varepsilon_n$ fulfilling $\|\eta_n\|_{L^\infty(I;L^1(M))} \leq \alpha$ and (5.32) for all $0 < T_1 \leq T$. Using the coercivity of $K$ and (5.14) we deduce the estimate

$$\|\eta_n\|_{Y^l} + \|u_n\|_{Y^l_{\delta, \delta, \delta, \delta}} \leq c$$

with a constant $c$ independent of $n$. This, together with the compact embedding $Y^l \hookrightarrow C^0(I \times M)$, implies that

$$\begin{align*}
\eta_n & \to \eta \quad \text{weakly-* in } L^\infty(I;H^2_0(M)) \text{ and uniformly in } I \times M, \\
\partial_t \eta_n & \to \partial_t \eta \quad \text{weakly-* in } L^\infty(I;L^2(M))
\end{align*}$$

(3.44)
for a subsequence and for an \( \eta \in Y^I \). Since by Remark 2.38, \( \mathcal{R}_\epsilon \eta_n \) is uniformly bounded in \( Y' \), we similarly get a subsequence with
\[
\mathcal{R}_\epsilon \eta_n \rightharpoonup^* \zeta \quad \text{weakly-* in } L^\infty(I, H^2(\Omega)) \quad \text{and uniformly in } \tilde{t} \times M,
\]
\[
\partial_t \mathcal{R}_\epsilon \eta_n \rightharpoonup \partial_t \zeta \quad \text{weakly-* in } L^\infty(I, L^2(\Omega)).
\] (3.45)
Taking Proposition 2.37 into account, we have
\[
\| \mathcal{R}_\epsilon \eta_n - \eta \|_{L^p(I \times M)} \leq \| \mathcal{R}_\epsilon \eta_n - \eta \|_{L^p(I \times M)} + \| \mathcal{R}_\epsilon \eta - \eta \|_{L^p(I \times M)} + 2(\epsilon_n)^{\frac{1}{2}}.
\]
This implies \( \zeta = \eta \), by the convergence properties of the regularization operator from Lemma 2.40. As usual, to treat the convective term in the fluid part, we need further interpolation (see [13, Proposition 3.1]) yield
\[
\text{for all } u \in Y^I = \text{sequence, the convergences}
\]
\[
\mathcal{R}_\epsilon \eta_n \rightharpoonup \mathcal{R}_\epsilon \eta \quad \text{weakly-* in } L^\infty(I, L^2(\Omega))
\]
\[
\text{implies}
\]
\[
\text{in } L^\infty(I, L^2(\Omega)), \text{ and uniformly in } \tilde{t} \times M.
\]
(3.46)
for \( q = 16/5 \). Extending \( u_n, \nabla u_n \) and \( D u_n \) by zero to \( I \times \mathbb{R}^3 \), we get, for a further subsequence, the convergences
\[
uu_n \rightarrow \nabla u \quad \text{weakly in } L^2(I; L^3(\mathbb{R}^3)),
\]
\[
D u_n \rightarrow D u \quad \text{weakly in } L^2(I; L^2(\mathbb{R}^3)),
\]
for an \( u \in X^I_1 \), which is also extended by zero in an appropriate fashion. Using the compactness result (Lemma 3.6), we deduce that
\[
\partial_t \eta_n \rightharpoonup \partial_t \eta \quad \text{in } L^2(I \times M), \quad u_n \rightarrow u \quad \text{in } L^2(I \times \mathbb{R}^3).
\] (3.47)
Again by Hölder-Interpolation, from (3.46), (3.47) we deduce the strong convergences in \( L^1(I; L^2(\mathbb{R}^3)) \) and \( L^3(I; L^3(\mathbb{R}^3)) \) for the sequence \( u_n \). With the properties of the smoothing operator (see [5, Proposition II.2.25] and [15, Théorème 1.8.1]), this implies the convergences
\[
\mathcal{R}_\epsilon u_n \rightarrow u \quad \text{in } L^2(I; L^3(\mathbb{R}^3)), \quad \mathcal{R}_\epsilon \mathcal{R}_\epsilon u_n \rightarrow u \quad \text{in } L^2(I; L^4(\mathbb{R}^3)).
\] (3.48)
Since \( (\eta_n, u_n) \) are weak solutions to the regularised problem, we have \( \eta_n(0, \cdot) = \eta_0 \) and \( \text{tr}_{\mathcal{R}_\epsilon \eta_n} u_n = \partial_\eta \eta_n \cdot v \). By the uniform convergence from (3.44) and Lemma 2.34, \( \eta(0, \cdot) = \eta_0 \) and \( \text{tr}_\eta u = \partial_\eta \eta \cdot v \) follows. Also, due to the lower semi-continuity of the norms and the Koiter energy, the energy estimates (3.14) of the regularised solutions imply the energy estimate (3.4) for all \( 0 < T_1 \leq T \). To show the integral identity (3.2) for all \( (b, \varphi) \in T^\epsilon_1 \), we have to use the special structure of the space of test functions. Let \( b \in H^1(I; L^2(\Omega)) \cap L^3(I; H^2(\Omega)) \) with \( b(T) = 0 \). Then, using the properties of our extension operator \( \mathcal{F}_{\mathcal{R}_\epsilon \eta_n} \), i.e. Lemma 2.47, we have \( (b, \mathcal{F}_{\mathcal{R}_\epsilon \eta_n} b) \in T^\epsilon_{\mathcal{R}_\epsilon \eta_n} \). Hence for all \( n \in \mathbb{N} \) the tuple \( (b, \mathcal{F}_{\mathcal{R}_\epsilon \eta_n} b) \) is an admissible test function for the corresponding integral identity of the regularised weak solution (3.30). By Lemma 2.50, \( \mathcal{F}_{\mathcal{R}_\epsilon \eta_n} b \) converges to \( \mathcal{F}_{\eta} b \) in \( L^\infty(I; L^4(B_\alpha)) \). Thanks to the convergence of \( \mathcal{R}_\epsilon u_n \rightarrow u \) in \( L^2(I; L^2(\mathbb{R}^3)) \) by (3.48) and the weak convergence of \( \nabla u_n \rightarrow \nabla u \) in \( L^2(I; L^{20/11}(\mathbb{R}^3)) \) by (3.46), we get for the first part of the convective term
\[
\int_I \int_{\Omega_{\mathcal{R}_\epsilon \eta_n(t)}} (\mathcal{R}_\epsilon \eta_n \cdot \nabla) u_n \cdot \mathcal{F}_{\mathcal{R}_\epsilon \eta_n} b \, dx \, dt \rightarrow \int_I \int_{\Omega_{\eta(t)}} (u \cdot \nabla) u \cdot \mathcal{F}_{\eta} b \, dx \, dt.
\]

In here, we also used the spatial extension by zero. Arguing similarly for the rest of the terms in (3.30) tested with \((b, \mathcal{F}_\alpha \eta_n b)\), i.e. taking into account the convergences (3.44)–(3.48) as well as the convergences from Lemma 2.50 and Lemma 2.46, the integral identity (3.2) holds for all \((b, \mathcal{F}_\alpha \eta_n b)\) with \(b \in H^1(I; L^2(M)) \cap L^2(I; H_0^1(M))\) and \(b(T, \cdot) = 0\). By the definition of our space of test functions, it only remains to show the identity (3.2) for all \((0, \varphi)\) with \(\varphi \in W_\eta\), \(\varphi = 0\) in a neighbourhood of the moving boundary, \(\text{div} \varphi = 0\) and \(\varphi(T, \cdot) = 0\). For such a test function \((0, \varphi)\), the uniform convergence of \(\mathcal{F}_\alpha \eta_n\) towards \(\eta\) implies \(\text{supp} \varphi \subset \Omega^I_{\alpha \delta_n}\) for \(n\) big enough. Hence, silently extending this function by zero in the spatial direction, we have that \((0, \varphi) \in T\mathcal{F}_{\alpha \eta_n}\) is an admissible test function for the regularised identities (3.30) for \(n\) big enough. Using again the convergences (3.44)–(3.48), we can pass to the limit \(n \to \infty\) to show that \((\eta, u)\) is a weak solution to our problem on the interval \(I = (0, T)\).

To extend this local existence to an global result, we observe that by the coercivity of the Koiter energy, the energy inequality (3.4) implies the estimate

\[
\|\eta\|_{L^2} + \|u\|_{X^h} \leq c,
\]

(3.49)

where the constant \(c\) only depends on \(\|u_0\|_{L^2(\Omega_{\eta_0})}, \|\eta_1\|_{L^2(M)}, \|f\|_{L^2((0, \infty) \times \Omega)}\) and \(\|g\|_{L^2((0, \infty) \times M)}\).

By the compatibility condition \(\text{tr}_s \eta = \partial_t \eta v\) and \(\|\eta\|_{L^\infty(\eta, M)} \leq \alpha < \kappa\), the estimate (3.49) implies that the quintuple \((\mathbb{f}(\cdot - s), g(\cdot - s), u(s), \eta(s), \partial_t \eta(s))\) is an admissible data for almost all \(s \in I\). Repeating the first part of the proof, we get for almost all \(s \in I\) a local weak solution to this data, whereby the length of the time interval \((0, \bar{T})\) can be chosen independent of \(s\) by Lemma 3.31. Choosing an \(s_0 \in I\) outside of an appropriate null set close enough to \(T\) and shifting the local weak solution by \(s_0\), we get the existence of a weak solution \((\eta^0, u^0)\) on the time interval \((s_0, T + \bar{T}/2)\).

Considering the initial conditions and the alternative integral identity (4.6), we see that the combined function (again denoted by \((\eta, u)\)) is a weak solution to (1.2)–(1.8) on the time interval \((0, T + \bar{T}/2)\). Since (3.4) holds for \((\eta^0, u^0)\) as well as for \((\eta, u)\) (for clarity we denote the initial energy for \((\eta^0, u^0)\) by \(E_0(s_0)\) and notice that \(E_0(s_0) \leq \text{esssup}_{\eta \in (0, T)} E_0(\eta)\)), we have

\[
\text{esssup}_{t \in (s_0, T + \bar{T}/2)} \sqrt{E(t)} \leq \sqrt{E_0(s_0)} + \frac{1}{\sqrt{2}} \int_0^{T + \bar{T}/2 - s_0} \|\mathbb{f}(s - s_0, \cdot)\|_{L^2(\Omega_{\eta(s_0)})} \, ds
\]

\[
+ \frac{1}{2} \int_0^{T + \bar{T}/2 - s_0} \|g(s - s_0, \cdot)\|_{L^2(M)} \, ds
\]

\[
\leq \sqrt{E_0} + \int_0^{\bar{T}/2} \frac{1}{\sqrt{2}} \|\mathbb{f}(s, \cdot)\|_{L^2(\Omega_{\eta(s)})} + \frac{1}{2} \|g(s, \cdot)\|_{L^2(M)} \, ds.
\]

Hence, the combined solution also fulfills the energy inequality (3.4) and therefore the estimate (3.49) with the same constant. Iterating this procedure, we obtain a maximal time \(0 < T^* \leq \infty\) such that for all \(0 < T < T^*\) a weak solution \((\eta, u)\) of (1.2)–(1.8) exists on the interval \(I := (0, T)\), fulfilling the energy inequality (3.4). Since by (3.49) the norms of the initial data for the extension stay uniformly bounded, Lemma 3.31 implies that \(T^* < \infty\) is only possible if the norm of the displacement converges to \(\kappa\). Since small enough data implies a uniform bound for the displacement (see (3.33)), we have \(T^* = \infty\) for this case.

**4. APPENDIX**

For the convenience of the reader, we include the proof of the compactness result. As already mentioned, we will need a modified Ehrling’s Lemma.

**Lemma 4.1.** Let \(0 < \alpha < \kappa\), \(N \in \mathbb{N}\) and, for the second inequality, \(\delta \in C^4(M)\) with \(\|\delta\|_{L^\infty(M)} < \kappa\). Moreover, for \(\varphi \in X(\Omega)\), we extend \(\partial_\delta \varphi\) by 0 to \(\mathbb{R}^3\). Then for all \(\varepsilon > 0\)
there exists a constant \( c > 0 \) such that for all \( \mathbf{v} \in V_2(\Omega_\eta) \), \( \bar{\mathbf{v}} \in V_2(\Omega_{\bar{\eta}}) \) and all \( \eta, \bar{\eta} \in H^2(M) \) with \( \|\eta\|_{H^2(M)} + \|\bar{\eta}\|_{H^2(M)} \leq N \) and \( \|\eta\|_{L^\infty(M)} \leq \alpha \), \( \|\bar{\eta}\|_{L^\infty(M)} \leq \alpha \) it holds that

\[
\sup_{\|b\|_{L^2(M)} \leq 1} \left( \int_{\Omega_\eta} \mathbf{v} \cdot \mathcal{F}_\eta b \, dx - \int_{\Omega_{\bar{\eta}}} \bar{\mathbf{v}} \cdot \mathcal{F}_{\bar{\eta}} b \, dx + \int_M (\mathbf{v} \cdot \nabla \eta \mathbf{v} - \mathbf{v} \cdot \nabla \bar{\eta} \bar{\mathbf{v}}) \, b \, dA \right)
\leq c \sup_{\|b\|_{H^1_0(M)^N} \leq 1} \left( \int_{\Omega_\eta} \mathbf{v} \cdot \mathcal{F}_\eta b \, dx - \int_{\Omega_{\bar{\eta}}} \bar{\mathbf{v}} \cdot \mathcal{F}_{\bar{\eta}} b \, dx + \int_M (\mathbf{v} \cdot \nabla \eta \mathbf{v} - \mathbf{v} \cdot \nabla \bar{\eta} \bar{\mathbf{v}}) \, b \, dA \right)
+ \varepsilon \left( \|\mathbf{v}\|_{V_2(\Omega_\eta)} + \|\bar{\mathbf{v}}\|_{V_2(\Omega_{\bar{\eta}})} \right)
\]

as well as

\[
\sup_{\|b\|_{H^1_0(M)^N} \leq 1} \left( \int_{\Omega_\eta} \mathbf{v} \cdot \mathcal{F}_\eta b \, dx - \int_{\Omega_{\bar{\eta}}} \bar{\mathbf{v}} \cdot \mathcal{F}_{\bar{\eta}} b \, dx \right)
\leq c \sup_{\|b\|_{X(\Omega)^N} \leq 1} \left( \int_{\Omega_\eta} \mathbf{v} \cdot \mathcal{F}_\eta b \, dx - \int_{\Omega_{\bar{\eta}}} \bar{\mathbf{v}} \cdot \mathcal{F}_{\bar{\eta}} b \, dx \right) + \varepsilon \left( \|\mathbf{v}\|_{V_2(\Omega_\eta)} + \|\bar{\mathbf{v}}\|_{V_2(\Omega_{\bar{\eta}})} \right).
\]

**Proof.** Assume that the first claim is wrong. Then there exist \( \varepsilon > 0 \) and bounded sequences \( (\eta_n)_{n \in \mathbb{N}}, (\bar{\eta}_n)_{n \in \mathbb{N}} \subset H^2(M), \ (\mathbf{v}_n)_{n \in \mathbb{N}} \subset V_2(\Omega_{\eta_n}) \) and \( (\bar{\mathbf{v}}_n)_{n \in \mathbb{N}} \subset V_2(\Omega_{\bar{\eta}_n}) \), which satisfy \( \|\eta_n\|_{L^\infty(M)} \leq \alpha \), \( \|\bar{\eta}_n\|_{L^\infty(M)} \leq \alpha \) and, after some scaling,

\[
\|\mathbf{v}_n\|_{V_2(\Omega_{\eta_n})} + \|\bar{\mathbf{v}}_n\|_{V_2(\Omega_{\bar{\eta}_n})} = 1
\]

as well as

\[
\sup_{\|b\|_{L^2(M)} \leq 1} \left( \int_{\Omega_{\eta_n}} \mathbf{v}_n \cdot \mathcal{F}_{\eta_n} b \, dx - \int_{\Omega_{\bar{\eta}_n}} \bar{\mathbf{v}}_n \cdot \mathcal{F}_{\bar{\eta}_n} b \, dx + \int_M (\mathbf{v}_n \cdot \nabla \eta_n \mathbf{v}_n - \mathbf{v}_n \cdot \nabla \bar{\eta}_n \bar{\mathbf{v}}_n) \, b \, dA \right)
\geq \varepsilon + \eta_n \sup_{\|b\|_{H^1_0(\Omega)^N} \leq 1} \left( \int_{\Omega_{\eta_n}} \mathbf{v}_n \cdot \mathcal{F}_{\eta_n} b \, dx - \int_{\Omega_{\bar{\eta}_n}} \bar{\mathbf{v}}_n \cdot \mathcal{F}_{\bar{\eta}_n} b \, dx \right)
\]

\[
+ \int_M (\mathbf{v}_n \cdot \nabla \eta_n \mathbf{v}_n - \mathbf{v}_n \cdot \nabla \bar{\eta}_n \bar{\mathbf{v}}_n) \, b \, dA \quad (4.2)
\]

By the Korn-type inequality and Definition 2.7, the sequences \( (\mathbf{v}_n, \bar{\mathbf{v}}_n)_{n \in \mathbb{N}}, (\mathbf{v}_n, \bar{\mathbf{v}}_n)_{n \in \mathbb{N}} \) are bounded in \( H^{1/2}(M) \). By Sobolev’s embedding theorem we get a subsequence satisfying

\[
\begin{align*}
\mathbf{v}_n \rightarrow d & \quad \text{in } L^2(M), \\
\eta_n & \quad \text{weakly in } H^2(M), \\
\bar{\mathbf{v}}_n \rightarrow \bar{d} & \quad \text{in } L^2(M), \\
\bar{\eta}_n & \quad \text{weakly in } H^2(M).
\end{align*}
\]

In particular, \( (\eta_n)_{n \in \mathbb{N}} \) and \( (\bar{\eta}_n)_{n \in \mathbb{N}} \) converge uniformly in \( M \), and therefore \( \|\eta\|_{L^\infty(M)} \leq \alpha \), \( \|\bar{\eta}\|_{L^\infty(M)} \leq \alpha \). Taking, as usual, the cut-off function \( \beta \) uniformly for all transformations, by Lemma 2.5 the sequences \( \mathbf{w}_n := \mathbf{v}_n \circ \Psi_{\eta_n} \) and \( \bar{\mathbf{w}}_n := \bar{\mathbf{v}}_n \circ \Psi_{\bar{\eta}_n} \) are bounded in \( W^{1, \frac{3}{2}}(\Omega) \).

By Sobolev’s embedding, for a subsequence we have \( \mathbf{w}_n \rightarrow \mathbf{w} \) and \( \bar{\mathbf{w}}_n \rightarrow \bar{\mathbf{w}} \) in \( L^{3/2}(\Omega) \). We extend the functions \( \mathbf{w} \circ \Psi_{\eta_n}^{-1}, \mathbf{w} \circ \Psi_{\bar{\eta}_n}^{-1} \) and \( \mathbf{v}_n \) by 0 to \( \mathbb{R}^3 \) and get, using Lemma 2.5,

\[
\|\mathbf{v}_n - \mathbf{w} \circ \Psi_{\bar{\eta}_n}^{-1}\|_{L^2(\mathbb{R}^3)} = \|\mathbf{w}_n \circ \Psi_{\eta_n}^{-1} - \mathbf{w} \circ \Psi_{\bar{\eta}_n}^{-1}\|_{L^2(\mathbb{R}^3)} \leq c \|\mathbf{w}_n - \mathbf{w}\|_{L^2(\Omega_{\eta_n})} + \|\mathbf{w} \circ \Psi_{\eta_n}^{-1} - \mathbf{w} \circ \Psi_{\bar{\eta}_n}^{-1}\|_{L^2(\mathbb{R}^3)}
\]

Taking Lemma 2.6 into account, we deduce that \( \mathbf{v}_n \rightarrow \mathbf{w} \circ \Psi_{\bar{\eta}_n}^{-1} := \mathbf{v} \) in \( L^2(\mathbb{R}^3) \). Analogously we get \( \bar{\mathbf{v}}_n \rightarrow \bar{\mathbf{w}} \circ \Psi_{\eta_n}^{-1} := \bar{\mathbf{v}} \) in \( L^2(\mathbb{R}^3) \). By Lemma 2.4, the equation

\[
\int_{\Omega_{\eta_n}} \mathbf{v}_n \cdot \mathcal{F}_{\eta_n} b \, dx - \int_{\Omega_{\bar{\eta}_n}} \bar{\mathbf{v}}_n \cdot \mathcal{F}_{\bar{\eta}_n} b \, dx + \int_M (\mathbf{v}_n \cdot \nabla \eta_n \mathbf{v}_n - \mathbf{v}_n \cdot \nabla \bar{\eta}_n \bar{\mathbf{v}}_n) \, b \, dA
\]
converges to
\[\int_{\Omega} \mathbf{v} \cdot \mathcal{T}_\eta \mathbf{b} \, dx - \int_{\Omega} \bar{\mathbf{v}} \cdot \mathcal{T}_\eta \mathbf{b} \, dx + \int_M (\mathbf{d} - \mathbf{d}) \cdot \mathbf{b} \, dA\]
uniformly with respect to \(\|b\|_{L^2(M)} \leq 1\). Therefore the left-hand side of (4.2) is bounded and
\[
\sup_{\|b\|_{L^2(M)} \leq 1} \left( \int_{\Omega_{\eta_n}} \mathbf{v}_n \cdot \mathcal{T}_\eta \mathbf{b} \, dx - \int_{\Omega_{\eta_n}} \bar{\mathbf{v}}_n \cdot \mathcal{T}_\eta \mathbf{b} \, dx + \int_M (\mathbf{tr}_{\eta_n} \mathbf{v}_n \cdot \mathbf{v} - \mathbf{tr}_{\eta_n} \bar{\mathbf{v}}_n \cdot \mathbf{v}) \, b \, dA \right)
\]
converges to 0. Since \(H^1_0(M)\) is dense in \(L^2(M)\), the left-hand side of (4.2) converges to 0 as well, a contradiction to \(\varepsilon > 0\).

Analogously, we assume that the second claim is wrong. Then there exist \(\varepsilon > 0\) and bounded sequences \((\eta_n)_{n \in \mathbb{N}}, (\tilde{\eta}_n)_{n \in \mathbb{N}} \subset H^2(M), (\mathbf{v}_n)_{n \in \mathbb{N}} \subset V_2(\Omega_{\eta_n})\) and \((\mathbf{v}_n)_{n \in \mathbb{N}} \subset V_2(\Omega_{\tilde{\eta}_n})\), which satisfy \(\|\eta_n\|_{L^\infty(M)} \leq \alpha, \|\tilde{\eta}_n\|_{L^\infty(M)} \leq \alpha\) and, after some scaling,
\[
\|\mathbf{v}_n\|_{L^2(\Omega_{\eta_n})} + \|\tilde{\mathbf{v}}_n\|_{L^2(\Omega_{\tilde{\eta}_n})} = 1
\]
as well as
\[
\sup_{\|\phi\|_{X(\Omega)} \leq 1} \left( \int_{\Omega_{\eta_n}} \mathbf{v}_n \cdot \mathcal{T}_\delta \phi \, dx - \int_{\Omega_{\eta_n}} \tilde{\mathbf{v}}_n \cdot \mathcal{T}_\delta \phi \, dx \right) > \varepsilon + \sup_{\|\phi\|_{X(\Omega)} \leq 1} \left( \int_{\Omega_{\eta_n}} \mathbf{v}_n \cdot \mathcal{T}_\delta \phi \, dx - \int_{\Omega_{\eta_n}} \tilde{\mathbf{v}}_n \cdot \mathcal{T}_\delta \phi \, dx \right).
\]
As in the first part of the proof, we find a subsequence with \(\mathbf{v}_n \to \mathbf{v}, \tilde{\mathbf{v}}_n \to \tilde{\mathbf{v}}\) in \(L^2(\mathbb{R}^3)\). By the continuity of the Piola-transform (notice \(\delta \in C^4(M)\)), for all \(\phi \in H_M(\Omega)\) with \(\|\phi\|_{H_M(\Omega)} \leq 1\) it follows
\[
\left| \int_{\Omega_{\eta_n}} \mathbf{v}_n \cdot \mathcal{T}_\delta \phi \, dx - \int_{\Omega_{\eta_n}} \tilde{\mathbf{v}}_n \cdot \mathcal{T}_\delta \phi \, dx - \int_{\mathbb{R}^3} \mathbf{v} \cdot \mathcal{T}_\delta \phi \, dx + \int_{\mathbb{R}^3} \tilde{\mathbf{v}} \cdot \mathcal{T}_\delta \phi \, dx \right|
\leq \|\mathbf{v}_n - \mathbf{v}\|_{L^2(\mathbb{R}^3)} \|\mathcal{T}_\delta \phi\|_{L^2(\mathbb{R}^3)} + \|\tilde{\mathbf{v}}_n - \tilde{\mathbf{v}}\|_{L^2(\mathbb{R}^3)} \|\mathcal{T}_\delta \phi\|_{L^2(\mathbb{R}^3)}
\leq c \left( \|\mathbf{v}_n - \mathbf{v}\|_{L^2(\mathbb{R}^3)} + \|\tilde{\mathbf{v}}_n - \tilde{\mathbf{v}}\|_{L^2(\mathbb{R}^3)} \right) \|\phi\|_{L^2(\Omega)}
\leq c \left( \|\mathbf{v}_n - \mathbf{v}\|_{L^2(\mathbb{R}^3)} + \|\tilde{\mathbf{v}}_n - \tilde{\mathbf{v}}\|_{L^2(\mathbb{R}^3)} \right).
\]
Hence, the left-hand side of (4.3) converges and is therefore bounded. This implies
\[
\sup_{\|\phi\|_{X(\Omega)} \leq 1} \left( \int_{\Omega_{\eta_n}} \mathbf{v}_n \cdot \mathcal{T}_\delta \phi \, dx - \int_{\Omega_{\eta_n}} \tilde{\mathbf{v}}_n \cdot \mathcal{T}_\delta \phi \, dx \right) \to 0.
\]
Since by Lemma 2.25 the space \(X(\Omega)\) is dense in \(H_M(\Omega)\), the left-hand side of (4.3) converges to zero as well, a contradiction to \(\varepsilon > 0\). \(\square\)

of Lemma 3.6 Using the extension operator from Section 2.4 we observe
\[
\int_{I} \int_{\mathbb{R}^3} |\mathbf{u}_n - \mathbf{u}|^2 \, dx \, dt + 2 \int_{I} \int_{M} |\partial_t \eta - \partial_t \eta|^2 \, dA \, dt
\leq \int_{I} \int_{\Omega_{\eta_n}(0)} |\mathbf{u}_n - \mathcal{T}_\delta \partial_t \eta_n|^2 \, dx \, dt + \int_{I} \int_{\Omega_{\eta_n}(0)} |\partial_t \mathbf{u}_n + \mathcal{T}_\delta \partial_t \eta_n|^2 \, dx \, dt
\]
\[
+ 2 \int_{I} \int_{M} |\partial_t \eta_n|^2 \, dA \, dt + \int_{I} \int_{\mathbb{R}^3} |\mathbf{u}_n|^2 \, dx \, dt + 2 \int_{I} \int_{M} |\partial_t \mathbf{u}_n|^2 \, dA \, dt
\]
\[
- 2 \int_{I} \int_{\mathbb{R}^3} |\mathbf{u}_n|^2 \, dx \, dt - 4 \int_{I} \int_{M} |\partial_t \eta_n|^2 \, dA \, dt.
\]
By the assumption (3.8), it therefore suffices to show
\[
\int_{I} \int_{\Omega_{\eta_n}(0)} |\mathbf{u}_n - \mathcal{T}_\delta \partial_t \eta_n|^2 \, dx \, dt \to \int_{I} \int_{\Omega_{\eta_n}(0)} |\mathbf{u} - \mathcal{T}_\delta \partial_t \eta|^2 \, dx \, dt \quad (4.4)
\]
and

\[
\begin{align*}
\int_I \int_{\Omega_{\delta(t)}^n} \mathbf{u}_n \cdot \mathcal{F}_{\delta_n} \partial_t \eta_n \, dxdt + 2 \int_I \int_M |\partial_t \eta_n|^2 \, dA dt \\
\to \int_I \int_{\Omega_{\delta(t)}^n} \mathbf{u} \cdot \mathcal{F}_{\delta} \partial_t \eta \, dx dt + 2 \int_I \int_M |\partial_t \eta|^2 \, dA dt.
\end{align*}
\]  

(4.5)

To this end, we will use equation (3.7). Since the functions which are constant in time are not admissible in (3.7), we will construct an alternative integral identity. We choose to this end, we will use equation (3.7). Since the functions which are constant in time are not admissible in (3.7), we will construct an alternative integral identity. We choose

\[
\int_0^t \frac{1}{\varepsilon} \left( \frac{t}{\varepsilon} \right) \int_{\Omega_{\delta(t)}^n} \mathbf{u}_n \cdot \varphi \, dx dt - \int_0^t \int_{\Omega_{\delta(t)}^n} \mathbf{u}_n \cdot \partial_t \varphi \, dx dt
\]

\[
\to \int_{\Omega_{\delta(t)}^n} \mathbf{u}_n (s, \cdot) \cdot \varphi (s, \cdot) \, dx - \int_0^s \int_{\Omega_{\delta(t)}^n} \mathbf{u}_n \cdot \partial_t \varphi \, dx dt.
\]

Arguing similarly for the remaining terms in (3.7), we get, for almost all \( s \in I \) and \( (b, \varphi) \in T_{\delta}^2 \), the identity

\[
\begin{align*}
\int_0^t \int_{\Omega_{\delta(t)}^n} \mathbf{u}_n \cdot \partial_t \varphi \, dx dt - \frac{1}{2} \int_0^t \int_M (\partial_t \eta_n) (\partial_t \delta_n) b \gamma (\delta_n) \, dA dt
\\
- 2 \int_0^t \int_M (\partial_t \eta_n) \partial_t b \, dA dt + 2 \int_0^t K (\eta_n, b) \, dt + 2 \int_0^t \int_{\Omega_{\delta(t)}^n} \mathbf{u}_n : \mathbf{D} \varphi \, dx dt
\\
+ \frac{1}{2} \int_0^t \int_{\Omega_{\delta(t)}^n} (\mathbf{v}_n \cdot \nabla) \mathbf{u}_n \cdot \varphi \, dx dt - \frac{1}{2} \int_0^t \int_{\Omega_{\delta(t)}^n} (\nabla \phi \cdot \mathbf{v}_n) \varphi \cdot \mathbf{u}_n \, dx dt
\\
= \int_0^t \int_M g \, b \, dA + \int_0^t \int_{\Omega_{\delta(t)}^n} \mathbf{f} \cdot \varphi \, dx dt + \int_0^t \mathbf{u}^n \cdot \mathbf{f} (0, \cdot) \, dx dt + 2 \int_0^t \int_{\Omega_{\delta(t)}^n} \eta_n b (0, \cdot) \, dA
\\
- \int_0^t \int_{\Omega_{\delta(t)}^n} \mathbf{u}_n (s, \cdot) \cdot \varphi (s, \cdot) \, dx - 2 \int_0^t \int_{\Omega_{\delta(t)}^n} \partial_t \eta_n (s, \cdot) b (s, \cdot) \, dA.
\end{align*}
\]  

(4.6)

Note that the requirements \( b (T, \cdot) = 0 \) and \( \varphi (T, \cdot) = 0 \) for the functions \((b, \varphi) \in T_{\delta}^2\) can be omitted in (4.6).

To show (4.5), we take \( b \in H^1_t (L^2 (\Omega_a)) \cap \mathcal{C} (\Omega_{\delta(t)}^n) \cap L^1 (I, H^3 (\Omega_a)) \) and \( \varphi \in H^1_t (L^2 (\Omega_a)) \cap \mathcal{C} (\Omega_{\delta(t)}^n) \cap L^1 (I, H^3 (\Omega_a)) \).

The couple \((b, \mathcal{F}_{\delta_n} b)\) is admissible in (4.6) and we have the estimate

\[
\| \mathcal{F}_{\delta_n} b \|_{H^1_t (L^2 (\Omega_a))} \leq c \| b \|_{H^1_t (L^2 (\Omega_a))} \leq c \| b \|_{H^1_t (L^2 (\Omega_a))},
\]  

(4.7)

where the constant \( c \) is independent of \( n \). Considering the integrands with respect to time in (4.6) with \( \varphi = \mathcal{F}_{\delta_n} b \), by H"older’s inequality we get

\[
\begin{align*}
\| \int_{\Omega_{\delta(t)}^n} \mathbf{u}_n \cdot \partial_t \mathcal{F}_{\delta_n} b \, dx \|_{L^{11/12} \times L^{11/12} (I)}^{12/11}
\leq \int_I \| \mathbf{u}_n (t, \cdot) \|_{L^{11/12} (\Omega_{\delta(t)}^n)} \| (\partial_t \mathcal{F}_{\delta_n} b) (t, \cdot) \|_{L^{11/12} (\Omega_{\delta(t)}^n)} \, dt
\leq \| \mathbf{u}_n \|_{L^{11/12} (L^{11/12} (\Omega_{\delta(t)}^n))} \| \partial_t \mathcal{F}_{\delta_n} b \|_{L^{11/12} (L^{11/12} (\Omega_{\delta(t)}^n))}
\leq c \| \mathbf{u}_n \|_{L^{11/12} (L^{11/12} (\Omega_{\delta(t)}^n))} \| \mathcal{F}_{\delta_n} b \|_{H^1_t (L^2 (\Omega_a))}.
\end{align*}
\]
As usual, the convective term needs a special treatment. By the Korn-type inequality, the sequences \( \mathbf{u}_n, \mathbf{v}_n \) are bounded in \( L^\infty(I; L^2(\Omega_{\Delta_n(t)})) \cap L^2(I; W^{1,r}(\Omega_{\Delta_n(t)})) \) for any \( 1 \leq r < 2 \). Using Sobolev’s embedding, \( W^{1,r}(\Omega_{\Delta_n(t)}) \hookrightarrow L^2(\Omega_{\Delta_n(t)}) \) is continuous for all \( 1 \leq r < \frac{3r}{3-r} \).

Hence, for \( 2 > r > \frac{3r}{3-r} \), the embeddings \( W^{1,r}(\Omega_{\Delta_n(t)}) \hookrightarrow L^{\frac{3r}{3-r}}(\Omega_{\Delta_n(t)}) \) and \( W^{1,r}(\Omega_{\Delta_n(t)}) \hookrightarrow L^{\frac{3r}{3-r}}(\Omega_{\Delta_n(t)}) \) are continuous. Using the Hölder interpolation (see [27, Korollar 2.10] and [13, Kapitel 1, Proposition 3.1]), we have

\[
L^\infty(I, L^2(\Omega_{\Delta_n(t)})) \cap L^2(I, L^{\frac{3r}{3-r}}(\Omega_{\Delta_n(t)})) \hookrightarrow L^{\frac{3r}{3}}(I, L^\frac{3r}{3}(\Omega_{\Delta_n(t)})),
\]

\[
L^\infty(I, L^2(\Omega_{\Delta_n(t)})) \cap L^2(I, L^{\frac{3r}{3-r}}(\Omega_{\Delta_n(t)})) \hookrightarrow L^{\frac{3r}{3-r}}(I, L^{\frac{3r}{3-r}}(\Omega_{\Delta_n(t)})).
\]

In particular, since \( \delta_n \) is bounded in \( \tilde{Y} \) and \( \|\delta_n\|_{L^\infty(I;\tilde{X})} < \alpha \), the appearing constants can be chosen independently of \( n \in \mathbb{N} \). By Hölder’s inequality (notice \( \frac{4}{7} + \frac{35}{88} + \frac{3}{4} = 1 \)) we therefore get, for the first part of the convective term,

\[
\| \int_{\Omega_{\Delta_n(t)}} (\nabla \cdot \mathbf{v}) \mathbf{u}_n \cdot \mathcal{F}_{\delta_n} b \, dx \|_{L^{12/11}(I)}^{12/11}
\leq \int I \|\nabla \mathbf{u}_n(t,\cdot)\|_{L^{17/4}(\Omega_{\Delta_n(t)})}^{12/11} \|\mathcal{F}_{\delta_n} b(t,\cdot)\|_{L^{3/4}(\Omega_{\Delta_n(t)})}^{12/11} dt
\leq \|\mathbf{u}_n\|_{L^{12/5}(I;L^{17/4}(\Omega_{\Delta_n(t)}))} \|\mathcal{F}_{\delta_n} b\|_{L^{3/4}(I;L^{12/5}(\Omega_{\Delta_n(t)}))}^{12/11}
\leq c \|\mathbf{u}_n\|_{\tilde{Y}_{\Delta_n}} \|\mathcal{F}_{\delta_n} b\|_{L^{12/11}(\tilde{X})}^{12/11}.
\]

For the second part of the convective term, we get similarly

\[
\| \int_{\Omega_{\Delta_n(t)}} (\nabla \cdot \mathbf{v}) \mathcal{F}_{\delta_n} b \cdot \mathbf{u}_n \, dx \|_{L^{12/11}(I)}^{12/11}
\leq \int I \|\nabla \mathbf{u}_n(t,\cdot)\|_{L^{17/4}(\Omega_{\Delta_n(t)})}^{12/11} \|\mathcal{F}_{\delta_n} b(t,\cdot)\|_{L^{3/4}(\Omega_{\Delta_n(t)})}^{12/11} dt
\leq \|\mathbf{u}_n\|_{L^{12/5}(I;L^{17/4}(\Omega_{\Delta_n(t)}))} \|\mathcal{F}_{\delta_n} b\|_{L^{3/4}(I;L^{12/5}(\Omega_{\Delta_n(t)}))}^{12/11}
\leq c \|\mathbf{u}_n\|_{\tilde{Y}_{\Delta_n}} \|\mathcal{F}_{\delta_n} b\|_{L^{12/11}(\tilde{X})}^{12/11}.
\]

With similar arguments for the remaining terms and taking into account (4.7), as well as \( \partial_t b = 0 \), the integrands with respect to time in (4.6), with \( \varphi = \mathcal{F}_{\delta_n} b \), are bounded in \( L^{12/11}(I) \) uniformly with respect to \( n \in \mathbb{N} \) and \( b \in B_1(0; H^5_0(M)) \), where \( B_1(0; H^5_0(M)) \) denotes the closed unit ball in \( H^5_0(M) \). By our assumptions,

\[
\int_{\Omega_{\Delta_n(t)}} \mathbf{u}_n \cdot \mathcal{F}_{\delta_n} b(0,\cdot) \, dx + 2 \int_M \eta^v_{\alpha} b(0,\cdot) \, dA
\]

is also bounded uniformly with respect to \( n \in \mathbb{N} \) and \( b \in B_1(0; H^5_0(M)) \). The identity (4.6) for \( \varphi = \mathcal{F}_{\delta_n} b \) can therefore be written as

\[
\int_0^T f_{\alpha,n}(t) dt + g_{\alpha,n} = \int_{\Omega_{\Delta_n(t)}} \mathbf{u}_n(s,\cdot) \cdot \mathcal{F}_{\delta_n} b(s,\cdot) \, dx + 2 \int_M \partial_t \eta^v_{\alpha}(s,\cdot) b(s,\cdot) \, dA, \tag{4.8}
\]

where \( f_{\alpha,n} \in L^{12/11}(I) \), \( g_{\alpha,n} \in \mathbb{R} \) are uniformly bounded with respect to \( n \in \mathbb{N} \) and \( b \in B_1(0; H^5_0(M)) \). By Hölder’s inequality, for \( s_1 \leq s_2 \in I \) we get

\[
\left| \int_{s_1}^{s_2} f_{\alpha,n}(t) dt - \int_{s_1}^{s_2} f_{\alpha,n}(t) dt \right| \leq \int_{s_1}^{s_2} |f_{\alpha,n}(t) dt | \leq \| f_{\alpha,n}(t) dt \|_{L^{12/11}(I)} |s_2 - s_1|^{\frac{11}{12}},
\]
i.e. \(\int_0^1 f_{b,n} \, dt \in C^0(\overline{T})\) is uniformly bounded with respect to \(n \in \mathbb{N}\) and \(b \in \overline{B_1(0;H^2_0(M))}\). We set

\[
\begin{align*}
 c_{b,n}(s) &:= \int_{\Omega_{\delta_0(s)}} u_n(s, \cdot) \cdot \mathcal{F}_\delta b(s, \cdot) \, dx + 2 \int_M \partial_t \eta_n(s, \cdot) b(s, \cdot) \, dA, \\
c_b(s) &:= \int_{\Omega_{\delta_0(s)}} u(s, \cdot) \cdot \mathcal{F}_\delta b(s, \cdot) \, dx + 2 \int_M \partial_t \eta(s, \cdot) b(s, \cdot) \, dA.
\end{align*}
\]

Considering (4.3), the sequence \(c_{b,n} \in C^0(\overline{T})\) is uniformly bounded with respect to \(n \in \mathbb{N}\) and \(b \in \overline{B_1(0;H^2_0(M))}\). In particular, by Arzela-Ascoli’s theorem, for any fixed \(b \in H^2_0(M)\) a subsequence of \((c_{b,n})_{n \in \mathbb{N}}\) converges uniformly on \(T\). Since Lemma 2.50 and our assumptions imply that \((c_{b,n})_{n \in \mathbb{N}}\) converges to \(c_b\) weakly in \(L^2(I)\), the sequence \(c_{b,n}\) converges to \(c_b\) uniformly on \(I\). We will show that

\[f_n(s) := \sup_{|b|_{H^2_0(M)} \leq 1} |c_{b,n}(s) - c_b(s)|\]

converges uniformly to 0 on \(I\). To this end, we recall that the sequence \((\partial_t \eta_n,u_n)_{n \in \mathbb{N}}\) is bounded in \(L^\infty(I;L^2(M)) \times L^\infty(I;L^2(\Omega_{\delta_0(I)}))\), and chose a countable dense subset \(I_0\) of \(I\) such that the functions \(u_n(s, \cdot) \in L^2(B_0)\) and \(\partial_t \eta_n(s, \cdot) \in L^2(M)\) are bounded in their respective norms for all \(n \in \mathbb{N}\) and \(s \in I_0\). Using a diagonal sequence argument, we get a (not further denoted) subsequence such that for all \(s \in I_0\) we have \(\partial_t \eta_n(s, \cdot) \rightharpoonup \eta^*_n\) weakly in \(L^2(M)\) and \(u_n(s, \cdot) \rightharpoonup u^*_n\) weakly in \(L^2(B_0)\). Since the embedding \(H^1(M) \hookrightarrow L^2(M)\) is compact, by Schauder’s theorem the embedding through the dual operator \((L^2(M))^* \hookrightarrow (H^1(M))^*\) is also compact. Therefore, \(\partial_t \eta_n(s, \cdot) \rightharpoonup \eta^*_n\) in \((H^1(M))^*\) and, analogously, \((u_n(s, \cdot))_{n \in \mathbb{N}} \rightharpoonup u^*_n\) in \((H^1(M))^*\). By the estimate

\[
\left| \int_{\Omega_{\delta_0(s)}} u_n(s, \cdot) \cdot \mathcal{F}_\delta b(s, \cdot) \, dx - \int_{\Omega_{\delta_0(s)}} u^*_n \cdot \mathcal{F}_\delta b(s, \cdot) \, dx \right| + 2 \int_M \left( \partial_t \eta_n(s, \cdot) - \eta^*_n \right) b(s, \cdot) \, dA
\]

\[
\leq \left| \int_{\Omega_{\delta_0(s)}} (u_n(s, \cdot) - u^*_n) \cdot \mathcal{F}_\delta b(s, \cdot) \, dx \right| + \left| \int_{\Omega_{\delta_0(s)}} u^*_n \cdot (\mathcal{F}_\delta b(s, \cdot) - \mathcal{F}_\delta b(s, \cdot)) \, dx \right|
\]

\[
\leq \left\| u_n(s, \cdot) - u^*_n \right\|_{(H^1(B_0))^*} \left\| \mathcal{F}_\delta b(s, \cdot) \right\|_{H^1(M)} + \left\| u^*_n \right\|_{L^2(B_0)} \left\| \mathcal{F}_\delta b(s, \cdot) - \mathcal{F}_\delta b(s, \cdot) \right\|_{L^2(B_0)}
\]

\[
+ 2 \left\| \partial_t \eta_n(s, \cdot) - \eta^*_n \right\|_{(H^1(M))^*} \left\| b(s, \cdot) \right\|_{H^1(M)}
\]

the estimate (4.7) and Lemma 2.46 imply the convergence of \((c_{b,n})_{n \in \mathbb{N}}\) on \(s \in I_0\), uniformly with respect to \(b \in \overline{B_1(0;H^2_0(M))}\). Since we already identified the weak limit of \((c_{b,n})_{n \in \mathbb{N}}\), we also have, for the original sequence, \(c_{b,n} \rightharpoonup c_b\) on \(I_0\) uniformly with respect to \(b \in \overline{B_1(0;H^2_0(M))}\). By the uniform bound on \(c_{b,n} \in C^0(\overline{T})\), for \(s, s' \in T\) we get

\[
|c_{b,n}(s) - c_{b,m}(s)| \leq |c_{b,n}(s) - c_{b,n}(s')| + |c_{b,n}(s') - c_{b,m}(s')| + |c_{b,m}(s') - c_{b,m}(s)|
\]

\[
\leq c|s - s'|^{1/2} + \left| c_{b,n}(s') - c_{b,m}(s') \right|
\]

where the constant \(c\) is independent of \(n, m \in \mathbb{N}, s, s' \in I\) and \(b \in \overline{B_1(0;H^2_0(M))}\). Let \(\varepsilon > 0\). Since \(T\) is compact and \(I_0\) dense in \(T\), we get a finite subset \(I_0^*\) of \(I_0\) such that for all \(s \in T\) there exists \(s' \in I_0^*\) with \(c|s - s'|^{1/2} < \varepsilon/2\). Since \(I_0^*\) is finite, the convergence from above implies \(|c_{b,n}(s') - c_{b,m}(s')| < \varepsilon/2\) for all \(s' \in I_0^*\) and \(n, m \geq N\) uniformly with respect to \(b \in \overline{B_1(0;H^2_0(M))}\). Therefore we have the uniform convergence of \(c_{b,n}(s)\) to \(c_{b,m}(s)\) independently of \(b \in \overline{B_1(0;H^2_0(M))}\) and therefore the uniform convergence of \(h_n\) to 0 on \(I\).
Again, let $\varepsilon > 0$. By the definitions of $c_{b,0}$, $c_{b}$ and their linearity with respect to $b$, the compatibility condition $\text{tr}_{\partial_t} \mathbf{u}_n = \partial_t \eta_n \mathbf{v}$, Lemma 2.34 and the Ehrling-type Lemma we get
\[
\left| \int_I c_{b,0} \eta_n (t) - c_{b} \eta_n (t) \right| dt 
\leq \left| \int_I \sup_{|b| \leq \varepsilon} \left( c_{b,0} (t) - c_{b} (t) \right) \right| dt 
\leq c \int_I \sup_{|b| \leq \varepsilon} \left( \int_{\Omega_{\delta(b)}} \left( \mathbf{u}_n \cdot \mathbf{F}_{\delta} b(s, \cdot) \mathbf{dx} - \int_{\Omega_{\delta(b)}} \mathbf{u}(s, \cdot) \cdot \mathbf{F}_{\delta} b(s, \cdot) \mathbf{dx} 
+ \int_M (\text{tr}_{\partial_t} \mathbf{u}_n (s, \cdot) \cdot \mathbf{v} - \text{tr}_{\partial_t} \mathbf{u}(s, \cdot) \cdot \mathbf{v}) b(s, \cdot) dA \right) ds \right) dt 
\leq c \int_I \varepsilon \mathbf{p} \left( \| \mathbf{u}_n (t) \|_{L^2 (\Omega_{\delta(b)})} + \| \mathbf{u} (t) \|_{L^2 (\Omega_{\delta(b)})} \right) dt 
+ c \int_I \sup_{|b| \leq \varepsilon} \left( c_{b,0} (t) - c_{b} (t) \right) dt 
\leq c \varepsilon \left( \| \mathbf{u}_n \|_{L^2 (I; L^2 (\Omega_{\delta(b)}))} + \| \mathbf{u} \|_{L^2 (I; L^2 (\Omega_{\delta(b)}))} \right) + c \int h_n (t) dt.
\]
By the uniform convergence of $h_n$ to 0, this implies $\left| \int_I c_{\partial_t} \eta_n - c_{\partial_t} \eta_n \right| dt \to 0$. We now consider the inequality
\[
\left| \int_{\Omega_{\delta(b)}} \mathbf{u}_n \cdot \mathbf{F}_{\delta} \partial_t \eta_n d\mathbf{x} dt - \int_{\Omega_{\delta(b)}} \mathbf{u} \cdot \mathbf{F}_{\delta} \partial_t \eta_n d\mathbf{x} dt 
+ 2 \int_{\Omega_{\delta(b)}} |\partial_t \eta_n|^2 dA dt - 2 \int_{\Omega_{\delta(b)}} |\partial_t \eta_n|^2 dA dt \right) 
\leq \left( \int_{\Omega_{\delta(b)}} \mathbf{u}_n \cdot \mathbf{F}_{\delta} \partial_t \eta_n d\mathbf{x} dt + 2 \int_{\Omega_{\delta(b)}} |\partial_t \eta_n|^2 dA dt \right) 
- \left( \int_{\Omega_{\delta(b)}} \mathbf{u} \cdot \mathbf{F}_{\delta} \partial_t \eta_n d\mathbf{x} dt + 2 \int_{\Omega_{\delta(b)}} |\partial_t \eta_n|^2 dA dt \right) 
+ \int_{\Omega_{\delta(b)}} \mathbf{u} \cdot \left( \mathbf{F}_{\delta} \partial_t \eta_n - \mathbf{F}_{\delta} \partial_t \eta_n \right) d\mathbf{x} dt + 2 \int_{\Omega_{\delta(b)}} \partial_t (\partial_t \eta_n - \partial_t \eta_n) dA dt 
\leq \left| \int_{\Omega_{\delta(b)}} c_{\partial_t} \eta_n - c_{\partial_t} \eta_n \right| dt + \int_{\Omega_{\delta(b)}} \mathbf{u} \cdot \left( \mathbf{F}_{\delta} \partial_t \eta_n - \mathbf{F}_{\delta} \partial_t \eta_n \right) d\mathbf{x} dt 
+ 2 \int_{\Omega_{\delta(b)}} \partial_t (\partial_t \eta_n - \partial_t \eta_n) dA dt.
\]
The weak convergence of $\partial_t \eta_n$, Corollary 2.49 and the convergence from above therefore imply 4.5.}

To show 4.4, we argue similarly but have to use the vanishing boundary values of $\mathbf{u}$ to get a uniformly admissible test function. Since the sequences $\{ \mathbf{u}_n \}_{n \in \mathbb{N}} \subseteq L^w (I; L^2 (\Omega_{\delta(b)}))$ and $\{ \partial_t \eta_n \}_{n \in \mathbb{N}} \subseteq L^2 (I; L^2 (M))$ are uniformly bounded and satisfy $\text{div} \mathbf{u}_n = 0$ and $\text{tr}_{\partial_t} \mathbf{u}_n = \partial_t \eta_n \mathbf{v}$, Lemma 2.34 implies that $\mathbf{u}_n (t, \cdot) - (\mathbf{F}_{\delta} \partial_t \eta_n)(t, \cdot)$ is uniformly bounded in $H^1 (\Omega_{\delta(b)})$ for all $n \in \mathbb{N}$ and almost all $t \in I$. Let $\varepsilon > 0$. By Lemma 2.28 there exist $\lambda > 0$, $c_0 > 0$ and $\Psi_{t,n} \in H^1 (\Omega_{\delta(b)})$ with $\text{supp} \Psi_{t,n} \subseteq \Omega_{\delta(b)} - \lambda \sigma$, such that for all $n \in \mathbb{N}$ and almost all $t \in I$ it holds that $\| \Psi_{t,n} \|_{H^1 (\Omega_{\delta(b)})} \leq c_0$ and
\[
\| \mathbf{u}_n (t, \cdot) - (\mathbf{F}_{\delta} \partial_t \eta_n)(t, \cdot) - \Psi_{t,n} \|_{L^2 \left( \mathbb{R}^3 \right)} < \varepsilon. \quad (4.9)
\]
Similarly to Proposition 2.39 we approximate the displacement $\delta$ “uniformly” from the inside by $\mathbf{F}_{\lambda} \delta \in C^1 (I \times M)$, i.e., $\mathbf{F}_{\lambda} \delta \leq \delta_0$ for $\lambda$ small enough and $\lambda$ big enough, see [16 Lemma 2.65] for the details. Then $\mathbf{F}_{\lambda} \delta$ and $\delta_0$ converge uniformly to $\delta$ on $I \times M$. Hence, for a (in the following fixed) $\lambda > 0$ small enough and all $n \in \mathbb{N}$ big enough, we have
\[ \delta_n - \lambda \epsilon < \sqrt{\lambda} \delta \leq \delta_n \text{ and } \| \nabla \phi \|_{L^\infty(I \times M)} < \kappa. \] Therefore, we get \( \text{supp } \Psi_{t,n} \subset \Omega_{\delta_n(t) - \lambda \epsilon} \subset \Omega_{\sqrt{\lambda} \delta(t)} \subset \Omega_{\delta_n(t)} \) for almost all \( t \in I \).

We extend a function \( \phi \in X(\Omega) \) constantly in time and transform it, using the Piola-transform, to \( \nabla \phi \). Since the Piola-transform preserves vanishing boundary values, we can extend \( \mathcal{P}^{-1}_{\mathcal{F} \phi} \phi \) further continuously by 0 to \( I \times B \epsilon \). By our uniform approximation from the inside, we have \( \text{supp } \mathcal{P}^{-1}_{\mathcal{F} \phi} \phi \subset \Omega_{\Lambda \mathcal{F} \phi} \subset \Omega_{\delta_n} \) and the estimate

\[
\| \mathcal{P}^{-1}_{\mathcal{F} \phi} \phi \|_{H^1(I;L^2(\Omega_{\delta_n(t)}))} \leq c \| \phi \|_{X(\Omega)},
\]

where the constant \( c \) depends, among others, on \( \lambda \), but not on \( n \). In particular, the couple \((0, \mathcal{P}^{-1}_{\mathcal{F} \phi} \phi)\) is admissible for the identity (4.6). We can now argue analogously to the proof of (4.5). Using (4.10) instead of (4.7), the integrands with respect to time in (4.6), tested by \((0, \mathcal{P}^{-1}_{\mathcal{F} \phi} \phi)\), are bounded in \( L^{12/11} \) uniformly with respect to \( n \in \mathbb{N} \) and \( \phi \in \overline{B}_1(0;X(\Omega)) \), where \( \overline{B}_1(0;X(\Omega)) \) denotes the closed unit ball in \( X(\Omega) \). Also the integral over the initial data \( u_0^n \) is bounded independently of \( n \in \mathbb{N} \) and \( \phi \in \overline{B}_1(0;X(\Omega)) \). The identity (4.6), tested by \((0, \mathcal{P}^{-1}_{\mathcal{F} \phi} \phi)\), can therefore be written as

\[
\int_0^t f_{\phi,\lambda,n}(t) \, dt + g_{\phi,\lambda,n} = \int_{\Omega_{\delta_n}} u_n(s, \cdot) \cdot \mathcal{P}^{-1}_{\mathcal{F} \phi} \phi(s, \cdot) \, dx,
\]

where \( f_{\phi,\lambda,n} \in L^{12/11} \) and \( g_{\phi,\lambda,n} \in \mathbb{R} \) are uniformly bounded with respect to \( n \in \mathbb{N} \) and \( \phi \in \overline{B}_1(0;X(\Omega)) \). Hence, we get \( f_{\phi,\lambda,n} f_{\phi,\lambda,n} \in C^0 \) \( \bar{T} \). We set

\[
c^\lambda_{\phi,n}(s) := \int_{\Omega_{\delta_n}} u_n(s, \cdot) \cdot \mathcal{P}^{-1}_{\mathcal{F} \phi} \phi(s, \cdot) \, dx,
\]

and deduce that \( c^\lambda_{\phi,n} \) in \( C^0 \) \( \bar{T} \) is bounded independently of \( n \in \mathbb{N} \) and \( \phi \in \overline{B}_1(0;X(\Omega)) \). For \( \phi \) fixed, by Arzela-Ascoli’s theorem and the weak convergence of \( u_n \), we get the uniform convergence of \( c^\lambda_{\phi,n} \) to \( c^\lambda_{\phi} \) on \( \bar{T} \). With the same arguments as in the proof of (4.5) we get that

\[
h^\lambda_{\phi}(s) := \sup_{\| \phi \|_{X(\Omega)} \leq 1} \left| c^\lambda_{\phi,n}(s) - c^\lambda_{\phi}(s) \right|
\]

converges to 0 uniformly in \( \bar{T} \). Since the Piola-transform \( \mathcal{P}^{-1}_{\mathcal{F} \phi} \) is an isomorphism between \( H_{\mathcal{F}}(\Omega) \) and \( H_{\mathcal{F}}(\mathcal{F}_{\mathcal{F} \phi}(\Omega)) \), the property \( \text{supp } \Psi_{t,n} \subset \Omega_{\sqrt{\lambda} \delta(t)} \) implies \( \Psi_{t,n} \in H_{\mathcal{F}}(\mathcal{F}_{\mathcal{F} \phi}(\Omega)) \) and therefore \( \mathcal{P}^{-1}_{\mathcal{F} \phi} \Psi_{t,n} \in H_{\mathcal{F}}(\Omega) \). Let \( \bar{\epsilon} > 0 \). By the definition of \( c^\lambda_{\phi,n}, c^\lambda_{\phi} \) (particularly their linearity with respect to \( \phi \)) and Lemma 4.1, we get

\[
\left| \int_{\Omega_{\delta_n}} u_n \cdot \Psi_{t,n} \, dx - \int_{\Omega_{\delta_n}} u \cdot \Psi_{t,n} \, dx \right| dt
\]

\[
= \left| \int_{\bar{T}} c^\lambda_{\phi,n}(s, \cdot) \cdot \mathcal{P}^{-1}_{\mathcal{F} \phi} \phi(s, \cdot) \, dx \right| dt
\]

\[
\leq c \int_{\bar{T}} \sup_{\| \phi \|_{X(\Omega)} \leq 1} \left( c^\lambda_{\phi,n}(s) - c^\lambda_{\phi}(s) \right) dt
\]

\[
\leq \bar{\epsilon} \int_{\bar{T}} \left( \| u_n \|_{L^2(\Omega_{\delta_n})} + \| u \|_{L^2(\Omega_{\delta_n})} \right) dt + c \int_{\bar{T}} \sup_{\| \phi \|_{X(\Omega)} \leq 1} c^\lambda_{\phi,n}(s) - c^\lambda_{\phi}(s) dt
\]

\[
\leq \bar{\epsilon} c \left( \| u_n \|_{L^2(\Omega_{\delta_n})} + \| u \|_{L^2(\Omega_{\delta_n})} \right) + c \int_{\bar{T}} h^\lambda_{\phi}(s) \, dx.
\]
Using the uniform convergence of \( h_k^j \) and the bound on \( u_n \) in \( X_{h_k}^j \), we therefore deduce

\[
\left| \int_{\Omega_{h_k}^j} u_n \cdot \Psi_{t,n} \, dx - \int_{\Omega_{h_k}^j} u \cdot \Psi_{t,n} \, dx \right| \to 0. \tag{4.12}
\]

Taking (4.9) into account, we get

\[
\left| \int_{\Omega_{h_k}^j} u_n \cdot (u_n - \mathcal{F}_{h_k} \partial_t \eta_n) \, dx \right| \leq \left| \int_{\Omega_{h_k}^j} u \cdot (u - \mathcal{F}_{h_k} \partial_t \eta) \, dx \right| + \int_{\Omega_{h_k}^j} u_n \cdot \Psi_{t,n} \, dx - \left| \int_{\Omega_{h_k}^j} u \cdot \Psi_{t,n} \, dx \right|
\]

\[
\leq \int_{\Omega_{h_k}^j} u \cdot (u_n - \mathcal{F}_{h_k} \partial_t \eta_n) \, dx - \int_{\Omega_{h_k}^j} u \cdot (u - \mathcal{F}_{h_k} \partial_t \eta) \, dx + |u_n|_{L^2(I;H^1(\mathbb{R}^3))} \left[ |u_n - \mathcal{F}_{h_k} \partial_t \eta_n - \Psi_{t,n}|_{L^\infty(I;H^\frac{1}{2}(\mathbb{R}^3))} + |u - \mathcal{F}_{h_k} \partial_t \eta - \Psi_{t,n}|_{L^\infty(I;H^\frac{1}{2}(\mathbb{R}^3))} \right]
\]

Since the extension by zero is continuous in \( H^\frac{1}{2} \) (see [16, Lemma A.3]), the convergences from (4.12) and (3.8), together with Lemma 2.50, therefore imply (4.4), which finishes the proof.

\[\square\]

ACKNOWLEDGMENT

This work has been partially supported by the DFG, namely within the project C2 of the SFB/TR "Geometric Partial Differential Equations".

REFERENCES

[1] Acosta, G., Durán, R.G., López García, F.: Korn inequality and divergence operator: counterexamples and optimality of weighted estimates. Proc. Amer. Math. Soc. 141(1), 217–232 (2013). DOI 10.1090/S0002-9939-2012-11408-X

[2] Aliprantis, C.D., Border, K.C.: Infinite dimensional analysis, third edn. Springer, Berlin (2006). A hitchhiker’s guide

[3] Beirão da Veiga, H.: On the existence of strong solutions to a coupled fluid-structure evolution problem. J. Math. Fluid Mech. 6(1), 21–52 (2004). DOI 10.1007/s00021-003-0082-5

[4] Bernard, J.M.E.: Density results in Sobolev spaces whose elements vanish on a part of the boundary. Chin. Ann. Math. Ser. B 32(6), 823–846 (2011). DOI 10.1007/s11401-011-0682-z

[5] Boyer, F., Fabrie, P.: Mathematical tools for the study of the incompressible Navier-Stokes equations and related models, Applied Mathematical Sciences, vol. 183. Springer, New York (2013). DOI 10.1007/978-1-4614-5975-0

[6] Brézis, H.: Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland Mathematics Studies, vol. 5. North-Holland Publishing Co., Amsterdam (1973)
[7] Brown, R.M., Shen, Z.: Estimates for the Stokes operator in Lipschitz domains. Indiana Univ. Math. J. 44(4), 1183–1206 (1995). DOI 10.1512/iumj.1995.44.2025

[8] Chambolle, A., Desjardins, B., Esteban, M.J., Grandmont, C.: Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate. J. Math. Fluid Mech. 7(3), 368–404 (2005). DOI 10.1007/s00021-004-0121-y

[9] Ciarlet, P.G.: Mathematical elasticity. Vol. III, Studies in Mathematics and its Applications, vol. 29. North-Holland Publishing Co., Amsterdam (2000). Theory of shells

[10] Ciarlet, P.G.: An introduction to differential geometry with applications to elasticity. Springer, Dordrecht (2005). Reprinted from J. Elasticity 78/79 no. 1-3 (2005)

[11] Cokelet, G.: The rheology and tube flow of blood. In: R. Skalak, S. Chien (eds.) Handbook of bioengineering. McGraw-Hill, New York (1987)

[12] Coutand, D., Shkoller, S.: The interaction between quasilinear elastodynamics and the Navier-Stokes equations. Arch. Ration. Mech. Anal. 179(3), 303–352 (2006). DOI 10.1007/s00205-005-0385-2

[13] DiBenedetto, E.: Degenerate parabolic equations. Universitext. Springer-Verlag, New York (1993)

[14] Dobrowolski, M.: Angewandte Funktionalanalysis: Funktionalanalysis, Sobolev-Rume und elliptische Differentialgleichungen. Springer-Lehrbuch Masterclass. Springer-Verlag, Berlin (2006)

[15] Droniou, J.: Intégration et espaces de Sobolev à valeurs vectorielles. Polycopié de l'Ecole Doctorale de Maths-Info de Marseille (2001). URL: http://www-gm3.univ-mrs.fr/polya/gm3-02/gm3-02.pdf

[16] Eberlein, H.: Globale existenz schwacher lösungen für die interaktion eines newtonschen fluids mit einer linearen, transversalen koiter-schale unter natürlichen randbedingungen. Ph.D. thesis, Albert-Ludwigs-Universität Freiburg im Breisgau (2017). DOI 10.6094/UNIFR/12847

[17] Fabes, E.B., Kenig, C.E., Verchota, G.C.: The Dirichlet problem for the Stokes system on Lipschitz domains. Duke Math. J. 57(3), 769–793 (1988). DOI 10.1215/S0012-7094-88-05734-1

[18] Grandmont, C.: Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate. SIAM J. Math. Anal. 40(2), 716–737 (2008). DOI 10.1137/070699196

[19] Hanszawa, E.i.: Classical solutions of the Stefan problem. Tôhoku Math. J. (2) 33(3), 297–335 (1981). DOI 10.2748/tmj/1178229399

[20] Kato, T., Mitrea, M., Ponce, G., Taylor, M.: Extension and representation of divergence-free vector fields on bounded domains. Math. Res. Lett. 7(5-6), 643–650 (2000). DOI 10.4310/MRL.2000.v7.n5.a10

[21] Lakshmikantham, V., Rama Mohana Rao, M.: Theory of integro-differential equations, Stability and Control: Theory, Methods and Applications, vol. 1. Gordon and Breach Science Publishers, Lausanne (1995)

[22] Lequeurre, J.: Existence of strong solutions for a system coupling the Navier-Stokes equations and a damped wave equation. J. Math. Fluid Mech. 15(2), 249–271 (2013). DOI 10.1007/s00021-012-0107-0

[23] Lions, J.L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod; Gauthier-Villars, Paris (1969)

[24] Mälek, J., Nečas, J., Rokyta, M., Růžička, M.: Weak and measure-valued solutions to evolutionary PDEs, Applied Mathematics and Mathematical Computation, vol. 13. Chapman & Hall, London (1996)

[25] Marsden, J.E., Hughes, T.J.R.: Mathematical foundations of elasticity. Dover Publications, Inc., New York (1994). Corrected reprint of the 1983 original

[26] Muha, B., Čanić, S.: Existence of a weak solution to a nonlinear fluid-structure interaction problem modeling the flow of an incompressible, viscous fluid in a cylinder with deformable walls. Arch. Ration. Mech. Anal. 207(3), 919–968 (2013). DOI 10.1007/s00205-012-0585-5

[27] Muha, B., Čanić, S.: A nonlinear, 3D fluid-structure interaction problem driven by the time-dependent dynamic pressure data: a constructive existence proof. Commun. Inf. Syst. 13(3), 357–397 (2013). DOI 10.4310/CIS.2013.v13.n3.a4

[28] Muha, B., Čanić, S.: Fluid-structure interaction between an incompressible, viscous 3D fluid and an elastic shell with nonlinear Koiter membrane energy. Interfaces Free Bound. 17(4), 465–495 (2015). DOI 10.4171/IFB/350

[29] Muha, B., Čanić, S.: Existence of a weak solution to a fluid-elastic structure interaction problem with the Navier slip boundary condition. J. Differential Equations 260(12), 8550–8589 (2016). DOI 10.1016/j.ode.2016.02.029

[30] Quarteroni, A., Tuveri, M., Veneziani, A.: Computational vascular fluid dynamics: problems, models and methods. Computing and Visualization in Science, vol. 13. Chapman & Hall, London (2010).

[31] Russo, R.: On Stokes’ problem. In: Advances in mathematical fluid mechanics, pp. 473–511. Springer, Berlin (2010). DOI 10.1007/978-3-642-04068-9
[35] Shen, Z.W.: A note on the Dirichlet problem for the Stokes system in Lipschitz domains. Proc. Amer. Math. Soc. 123(3), 801–811 (1995). DOI 10.2307/2160804

[36] Sohr, H.: The Navier-Stokes equations. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel (2001). DOI 10.1007/978-3-0348-8255-2. An elementary functional analytic approach

[37] Taylor, M.E.: Partial differential equations I. Basic theory, Applied Mathematical Sciences, vol. 115, second edition edn. Springer, New York (2011). DOI 10.1007/978-1-4419-7055-8

[38] Temam, R.: Navier-Stokes equations. Theory and numerical analysis. North-Holland Publishing Co., Amsterdam-New York-Oxford (1977). Studies in Mathematics and its Applications, Vol. 2

HANNES EBERLEIN
MATHEMATISCHES INSTITUT, ERNST-ZERMELO-STR. 1, 79104 FREIBURG, GERMANY
E-mail address: hannes.eberlein@mathematik.uni-freiburg.de

HANNES EBERLEIN
MATHEMATISCHES INSTITUT, ERNST-ZERMELO-STR. 1, 79104 FREIBURG, GERMANY
E-mail address: rose@mathematik.uni-freiburg.de