Topological Fulde-Ferrell superfluid in spin-orbit coupled atomic Fermi gases

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We theoretically predict a new topological matter - topological inhomogeneous Fulde-Ferrell superfluid - in one-dimensional atomic Fermi gases with equal Rashba and Dresselhaus spin-orbit coupling near s-wave Feshbach resonances. The realization of such a spin-orbit coupled Fermi system has already been demonstrated recently by using a two-photon Raman process and the extra one-dimensional confinement is easy to achieve using a tight two-dimensional optical lattice. The topological Fulde-Ferrell superfluid phase is characterized by a nonzero center-of-mass momentum and a non-trivial Berry phase. By tuning the Rabi frequency and the detuning of Raman laser beams, we show that such an exotic topological phase occupies a significant part of parameter space and therefore it could be easily observed experimentally, by using, for example, momentum-resolved and spatially resolved radio-frequency spectroscopy.

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Topological superfluids attract tremendous interests over the past few years [1]. In addition to providing a new quantum phase of matter, topological superfluids can host exotic quasiparticles at their boundary, which are known as Majorana fermions - particles that are their own antiparticles [2, 3]. Due to their non-Abelian exchange statistics, Majorana fermions are believed to be the essential quantum bits for topological quantum computation [4]. Therefore, the pursuit for topological superfluids and Majorana fermions turns out to be one of the most important challenges in fundamental science. Theoretically, a number of settings have been proposed for the realization of topological superfluids, including the fractional quantum Hall states at filling ν = 5/2 [5], vortex states of χ + i p y superconductors [6, 7], and surfaces of three-dimensional (3D) topological insulators in proximity to an s-wave superconductor [8], and one-dimensional (1D) nanowires with strong spin-orbit coupling coated also on an s-wave superconductor [9]. In the latter setting, in-direct evidences of topological superfluid and Majorana fermions have been observed experimentally [10].

Ultracold Fermi gas with spin-orbit coupling near an s-wave Feshbach resonance is a new promising candidate to create topological superfluids [11, 15]. Due to the unprecedented controllability in interatomic interaction, dimensionality and purity, there are a number of rapid experimental advances [10]. In particular, a spin-orbit coupled Fermi gas can now be routinely realized by using two counterpropagating Raman laser beams [17-19], a scheme first advanced by Ian Spielman and coworkers [20]. Through the use of Feshbach resonances [21] and optical lattices [22], a strongly interacting spin-orbit coupled Fermi gas in low dimensions could be manipulated immediately. Theoretical proposals for engineering a topological superfluid in such Fermi gas systems have been discussed in greater detail by a number of cold-atom researchers [11, 15].

In this work, we theoretically predict a new type topological superfluid, in which the superfluid order parameter varies in real space [23-24]. This prediction is motivated by the recent discovery that by imposing an in-plane Zeeman field along one of the directions of synthetic spin-orbit coupling, the phase space for inhomogeneous Fulde-Ferrell (FF) superfluidity is greatly enlarged [25]. In low dimensions, this inhomogeneous superfluid may acquire non-trivial topological feature. Our main result is summarized in Fig. 1 which shows a zero-temperature phase diagram for a 1D interacting Fermi...
gas with the experimentally realized equal Rashba and Dresselhaus spin-orbit coupling \[17\] \[18\]. We find a large window for the topological inhomogeneous FF superfluid, characterized by both a nonzero center-of-mass momentum and a non-trivial Berry phase. This exotic superfluid phase could be easily created and probed in current experiments, once the heating issue related to the Raman process is overcome. We may also anticipate the appearance of topological inhomogeneous superfluid in two dimensions, but with Rashba spin-orbit coupling, which is so far not experimentally realized yet.

Our investigation is based on the mean-field theory which is qualitatively reliable at zero temperature. It does not capture the large phase fluctuations found in 1D, although the mean-field physics is robust against these fluctuations as shown by Ref. \[20\]. More accurate description could be obtained by using other standard techniques in 1D, for example, bosonization. In that language, the ground state may be identified as a two-component Luttinger liquid \[20\]. These possibilities will be addressed in the future study.

**Model Hamiltonian and mean-field theory.** — We consider a 1D spin-orbit coupled two-component Fermi gas near a broad Feshbach resonance. Experimentally, the 1D confinement can be easily created by imposing a tight 2D optical lattice \[22\], i.e., in the \(x \times y\) plane. The synthetic spin-orbit coupling has already been engineered in \[6\] Li or \[40\] K atoms by using the Raman scheme first demonstrated at NIST for a \(^{87}\) Rb Bose-Einstein condensate (BEC) \[20\]. In this scheme, two Raman laser beams counter-propagate along the \(z\)-direction and couple the two different hyperfine states, giving rise to the term in the model Hamiltonian: \(\Omega_R/2 \int dx \psi^\dagger \psi e^{2\pi ik_z x} + \text{H.c.}\), where \(\psi^\dagger(z)\) is the creation field operator for atoms in one of the hyperfine states \(\sigma = (\uparrow, \downarrow)\), referred to as the spin state, \(\Omega_R\) is the Rabi frequency of Raman beams, and \(k_R = 2\pi/\lambda_R\) is the recoil momentum determined by the wave length \(\lambda_R\) of two beams. Thus, during the two-photon Raman process, atoms absorb a momentum of \(2\hbar k_R\) and simultaneously change their spin state from \(\downarrow\) to \(\uparrow\), creating a correlation between spin and orbital motion. This can be seen most clearly by introducing a gauge transformation, \(\Psi(z) = e^{ik_R z} \psi(z)\) and \(\Psi^\dagger(z) = e^{-ik_R z} \psi^\dagger(z)\), which leads to a term proportional to \(k_R \sigma_z\), where \(\hat{k} = -i\partial_z\) and \(\sigma_z\) is the Pauli matrix. Near a broad Feshbach resonance, the spin-orbit coupled Fermi gas system may therefore be described by a single-channel model Hamiltonian \(\mathcal{H} = \int dz [\mathcal{H}_0 + \mathcal{H}_{\text{int}}]\), where the single-particle part

\[
\mathcal{H}_0 = \begin{bmatrix}
\psi^\dagger \psi^\dagger \\
\psi^\dagger \\
\end{bmatrix}
\begin{bmatrix}
\hat{\xi}_k + \lambda \hat{k} + \Delta / 2 & \Omega_R / 2 \\
\Omega_R / 2 & \hat{\xi}_k - \lambda \hat{k} - \Delta / 2 \\
\end{bmatrix}
\begin{bmatrix}
\psi \\
\psi^\dagger \\
\end{bmatrix},
\]  

and \(\mathcal{H}_{\text{int}} = g_{1D} \psi^\dagger \Psi^\dagger \psi \psi^\dagger \psi\) is the interaction Hamiltonian that describes the contact interaction between two spin states with 1D effective interaction strength \(g_{1D} < 0\) \[31\].

In the single-particle Hamiltonian \(\mathcal{H}_0\), \(\hat{\xi}_k = -\hbar^2 \partial^2_z / (2m) - \mu\) is the kinetic energy after we drop a constant recoil energy \(E_R = \hbar^2 k_R^2 / (2m)\), and \(\Delta\) is the two-photon detuning from the Raman resonance. For convenience, we have defined a spin-orbit coupling constant \(\lambda = \hbar k_R / E_R\), where \(k_R\) is the Fermi wavenumber and \(E_R = \hbar^2 k_R^2 / (2m)\) is the Fermi energy. In the Shanxi experiment with \(40\) K atoms \[17\], the Fermi wavelength is typically about \(k_F \approx k_R\) and the Rabi frequency \(\Omega_R \approx 2E_R\). In the quasi-1D geometry formed by a tight 2D optical lattice \[22\], it is shown by Olshanii and co-workers that the effective interaction strength \(g_{1D}\) could be expressed through a 3D s-wave scattering length \(a_{3D}\) \[31\]. It is useful to characterize the interaction strength \(g_{1D}\) by using a dimensionless interaction parameter \(\gamma = -mg_{1D} / (\hbar^2 n)\), where \(n = 2k_F / \pi\) is the linear density in 1D.

In the absence of detuning \(\delta = 0\), the model Hamiltonian \(\mathcal{H}_0\) for cold-atoms has been previously solved by the present authors \[13, 14\] and Mueller and co-workers \[15\]. It is known that when the Rabi frequency \(\Omega_R\) is above a threshold \(\Omega_R / 2 \equiv \sqrt{\Delta^2 + \mu^2}\), where \(\Delta\) is the pairing gap, the system becomes a topological superfluid. The 3D counterpart of the model Hamiltonian has also been investigated \[22\]. In this case, a nonzero detuning leads to the inhomogeneous FF superfluid state in which Cooper pairs carry a single valued center-of-mass momentum \[23\]. Therefore, it is natural to anticipate that in 1D a topological inhomogeneous FF superfluid may arise at finite detuning \(\delta \neq 0\) and at a large Rabi frequency. In the following, we confirm this anticipation by detailed numerical calculations.

For this purpose, we assume a FF-like order parameter \(\Delta(z) = -g_{1D} \langle \psi^\dagger_\uparrow(z) \psi^\dagger_\downarrow(z) \rangle = \Delta e^{izq}\) and consider the mean-field decoupling of the interaction Hamiltonian, \(\mathcal{H}_{\text{int}} \approx -[\Delta(z) \psi^\dagger_\uparrow(z) \psi^\dagger_\downarrow(z) + \text{H.c.}] - \Delta^2 / g_{1D}\). By using a Nambu spinor \(\Phi(z) \equiv [\psi^\dagger_\uparrow(z), \psi^\dagger_\downarrow(z), \psi_\uparrow(z), \psi_\downarrow(z)]^T\), the total Hamiltonian can be written into a compact form, \(\mathcal{H} = (1 / 2) \int dz [\Phi^\dagger \mathcal{H}_{\text{BdG}} \Phi(z) - L \Delta^2 / U_0 + \sum_\kappa \xi_\kappa]\), where \(L\) is the length of the system and the Bogoliubov Hamiltonian takes the form

\[
\mathcal{H}_{\text{BdG}} = \begin{bmatrix}
S^+_k & 0 & 0 & -\Delta(z) \\
\Omega_R / 2 & \delta^+ \Delta(z) & 0 \delta^+ S^+_{-k} - \Omega_R / 2 \\
0 & \Delta(z) & -S^+_{-k} - \Omega_R / 2 \\
-\Delta^*(z) & 0 & -\Omega_R / 2 & -S^+_{-k} \\
\end{bmatrix}
\]  

with \(S^+_k \equiv \hat{\xi}_k + \lambda \hat{k} + \Delta / 2\). It is straightforward to diagonalize the Bogoliubov Hamiltonian \(\mathcal{H}_{\text{BdG}}\) with quasiparticle wave-function \(\Phi_{\kappa \eta}(z) = e^{ik_z / \sqrt{L}} [u_{\kappa \eta} e^{i \eta q / 2}, u_{\kappa \eta} e^{i \eta q / 2}, v_{\kappa \eta} e^{i \eta q / 2}, v_{\kappa \eta} e^{i \eta q / 2}]^T\) and quasiparticle energy \(E_{\kappa \eta}(\eta = 1, 2, 3, 4)\). The
mean-field thermodynamic potential $\Omega$ at temperature $T$ is given by,

$$
\frac{\Omega}{L} = \frac{1}{2L} \left[ \sum_k (\xi_k + q/2 + \xi_k - q/2) - \sum_{kq} E_{kq} \right]
- \frac{k_B T}{L} \sum_{kq} \ln \left( 1 + e^{-E_{kq}/k_B T} \right) - \frac{\Delta^2}{g_{1D}}.
$$

Here $\xi_k \equiv \hbar^2 k^2/(2m) - \mu$ and the summation over quasiparticle energy should be restricted to $E_{kq} \geq 0$ because of an inherent particle-hole symmetry in the Nambu spinor representation [32]. For a given set of parameters (i.e., the temperature $T$, interaction strength $\gamma$, etc.), different mean-field phases can be determined using the self-consistent stationary conditions: $\partial\Omega/\partial\Delta = 0$, $\partial\Omega/\partial q = 0$, as well as the conservation of total atom number, $N = n_L = -\partial\Omega/\partial \mu$. At finite temperatures, the ground state has the lowest free energy $F = \Omega + \mu N$.

Topological Fulde-Ferrell superfluid. — Let us focus on the phase diagram at zero temperature. The Raman Rabi frequency $\Omega_R$ and the detuning $\delta$ can be experimentally tuned [17, 18]. Thus, we shall present the phase diagram as functions of $\Omega_R$ and $\delta$. Throughout the paper, we consider a realistic dimensionless spin-orbit coupling constant. According to the typical number of atoms in experiments [17], we take $k_R = 5k_F/4$, corresponding to a dimensionless spin-orbit coupling constant $\lambda = \lambda k_F/E_F = 2.5$. We also choose a typical 3D $s$-wave scattering length near the broad Feshbach resonance and obtain the 1D effective interaction strength by using the standard relation for confinement induced resonance [31]. This leads to a dimensionless interaction parameter $\gamma = 3$ [22, 33, 34].

It is known from the previous studies that, in a 3D Fermi gas with equal Rashba and Dresselhaus spin-orbit coupling, a FF superfluid with a single-valued center-of-mass momentum (i.e., the FF momentum) is always energetically favorable at any finite detuning and nonzero Rabi frequency [27]. This superfluid corresponds to the solution with $\Delta \neq 0$ and $q \neq 0$. In Fig. 2 we check that this observation also holds in 1D, where the transverse spatial degree of freedom of the atoms is frozen by a tight 2D optical lattice. Fig. 2 compares the free energy of a FF superfluid and of a standard BCS superfluid at the Rabi frequency $\Omega_R = 3E_F$. The solution of the BCS superfluid is obtained by artificially restricting the FF momentum $q = 0$. A normal state is also considered, but is found to have higher energy than superfluid phases and is only favorable at sufficiently large detuning and/or Rabi frequency. All the three competing phases (normal, BCS or FF) are stable against phase separation (i.e., $\partial^2\Omega/\partial \Delta^2 \geq 0$). Thus, we do not include the phase-separation phase which involves two or more competing phases. It is clear from Fig. 2 that the free energy of FF superfluid is lower by an amount of $0.05N E_F$ than that of the BCS superfluid at the typical detuning $\delta = 1.5E_F$, suggesting that the FF superfluid should also be energetically favorable at finite temperatures below the superfluid phase transition which would occur at about one-tenth of the Fermi temperature [14]. The detuning dependence of the FF pairing gap and momentum is shown in the inset of Fig. 2. The FF momentum increases rapidly with increasing the detuning. At the same time, the pairing gap decreases as the detuning behaves like an effective Zeeman field. The system will become a normal Fermi gas at sufficiently large detuning beyond the so-called Chandrasekhar-Clogston limit.

Now, let us investigate in greater detail the properties of the FF superfluid. At zero detuning, where $q = 0$, the FF superfluid reduces to a BCS superfluid [27]. In this limit, it is known that there is a topological phase transition at a threshold Rabi frequency ($\Omega_R)_{c_1}$ [13, 14]. This threshold corresponds to a critical point where the energy gap close and then open. The topology of the Fermi surface of the system changes beyond the critical point, by developing non-trivial spin texture in real space charachterized by Berry phase or topological invariant. Indeed, by analyzing the analytical expression of the energy spectrum at $\delta = 0$, it is easy to obtain $(\Omega_R)_{c_2} = 1.5E_F$. By increasing the detuning of Raman beams, the BCS superfluid changes smoothly into a FF superfluid with a nonzero FF momentum, $q \neq 0$ [27]. In this case, it is reasonable to argue that the procedure of closing and opening energy gap may persist. Thus, the topology of the FF superfluid will change accordingly. As a result, we may obtain a topological inhomogeneous FF superfluid.

In Fig. 3 we present the energy gap of the FF superfluid at a nonzero detuning $\delta = 0.6E_F$ as a func-
With increasing the Rabi frequency, the energy gap of the system (solid line) close and then open. The Berry phase $\gamma_B$ is $\pi$ and 0 at the topologically trivial and non-trivial regimes (circles). The insets shows the order parameter and momentum of the FF superfluid, as a function of the Rabi frequency.

![Plot](image)

**FIG. 4:** (color online) The excitation spectrum $E_i(k)$ ($i = 1, 2, 3$ and 4, from bottom to top) at different Rabi frequencies: (a) $\Omega_R = 2.0E_F$, (b) the critical field $\Omega_R \simeq 2.45E_F$, (c) $\Omega_R = 3.0E_F$ and (d) $\Omega_R = 4.0E_F$. Here we take $\delta = 0.6E_F$. Due to the particle-hole symmetry, $E_2(k) = -E_3(-k)$ and $E_1(k) = -E_4(-k)$.

The four excitation spectra of Bogoliubov quasiparticles are shown in Fig. 4 at different Rabi frequencies. At the threshold, as shown in Fig. 4(b), the bottom (top) of the second (third) spectrum of hole (particle) excitations touches zero, giving rise to a zero energy gap. Before and after this threshold, see for example, Fig. 4(a) and 4(c) respectively, the excitation spectrum is more or less the same. Both of them are gapped. The asymmetry of the excitation spectrum with respect to the point $k = 0$ is due to the nonzero detuning and FF momentum. In the inset of Fig. 3 we also show the pairing gap and FF momentum with increasing the Rabi frequency. Across the topological transition point, the pairing gap and FF momentum decreases and increases more rapidly as the Rabi frequency increases.

To characterize in a more quantitative way the topological phase transition, we calculate the Berry phase defined by

$$\gamma_B = \int_{-\infty}^{+\infty} dk \left[ W_1^*(k) \partial_k W_1(k) + W_2^*(k) \partial_k W_2(k) \right].$$

Here $W_\eta(k)$ is the wave function of the $\eta$-th energy band: $W_\eta(k) = [u_{k\eta 1}e^{iqz/2}, u_{k\eta 1}e^{-iqz/2}, v_{k\eta 1}e^{-iqz/2}, v_{k\eta 1}e^{iqz/2}]^T$. In Fig. 3 the Berry phase is shown by circles. It jumps from $\pi$ to 0, right across the topological superfluid transition. It is somewhat counter-intuitive the $\gamma_B = 0$ sector corresponds to the topologically non-trivial superfluid state. But it agrees well with the known result in the limit of zero detuning [13], where a standard BCS superfluid translates into a topological BCS superfluid.

**Experimental detection.** — To experimentally probe the topological inhomogeneous superfluid, one should measure the nonzero FF momentum $q$ and at the same time confirm the non-trivial topological feature of the system. For the former, the nonzero FF momentum might be determined by using momentum-resolved radio-frequency spectroscopy, with a similar scenario for a 3D spin-orbit coupled atomic Fermi gas, where the value of $q/2$ can be directly read from the spectroscopy [27].

To confirm the non-trivial topological feature, we may consider using a spatially resolved radio-frequency spectroscopy to image the resulting Majorana fermions localized at the boundary of the Fermi gas system [13,14]. In reality, both the finite temperature and the existence of a harmonic trap will decrease the signal in the spectroscopy. These issues will be addressed in the later study by solving the Bogoliubov-de-Gennes equation of a harmonically trapped 1D spin-orbit coupled Fermi gas with nonzero laser detuning at finite temperatures.

**Conclusions.** — In summary, we have proposed that a new topological superfluid with inhomogeneous order parameter in real space could be potentially realized in current cold-atom settings, by using a one-dimensional atomic Fermi gas with equal Rashba and Dresselhaus spin-orbit coupling that has already been created in laboratories through two-photon Raman process [14,15]. The one dimensional confinement is straightforward to implement via a tight two-dimensional optical lattice, as demonstrated recently by Randy Hulet group at Rice University [22]. We believe such a topological inhomogeneous superfluid is within the reach in the near future once the temperature of the Fermi cloud could be cooled...
down to one-tenth of the Fermi temperature. The inhomogeneity and the non-trivial topological feature of the superfluid can be revealed by using momentum-resolved and spatially resolved radio-frequency spectroscopies, respectively. It is also of great interest to predict similar topological inhomogeneous superfluid in solid-state systems, but the unprecedented controllability in cold-atom systems would pave a new way to explore the long-sought topological and inhomogeneous superfluidity.

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Note added. — In completing this work, we are aware of related works which theoretically propose the topological inhomogeneous superfluid in a 1D or 2D spin-orbit coupled Fermi gas with additional optical lattices [35, 36], or in a 2D Rashba spin-orbit coupled Fermi gas in free space [37].

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