Some Curvature Problems in Semi-Riemannian Geometry

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Abstract In this survey article we review several results on the curvature of semi-Riemannian metrics which are motivated by the positive mass theorem and have been obtained within the Priority Program “Globale Differentialgeometrie” of the Deutsche Forschungsgemeinschaft. The main themes are estimates of the Riemann tensor of an asymptotically flat manifold and the construction of Lorentzian metrics which satisfy the dominant energy condition.

In this survey article we review recent progress on several curvature problems in semi-Riemannian geometry, each of which has a certain relation to the positive mass theorem (PMT). The focus is on work in which we were involved within the Priority Program “Globale Differentialgeometrie.”

The time-symmetric version of the PMT says in particular that an asymptotically flat Riemannian manifold with zero mass is flat. Section 1 investigates whether an asymptotically flat manifold whose mass is almost zero must be almost flat in a suitable sense. The general, not necessarily time-symmetric, situation is considered as well. The main tool in this work is the spinor which occurs in Witten’s proof of the PMT.

The PMT implies that if the energy $E$ and the momentum $P$ of an asymptotically flat spacelike hypersurface $M$ of a Lorentzian manifold in which the dominant energy condition holds satisfy $E = |P|$, then the Lorentzian metric is flat along $M$. Schoen–Yau proved that in this situation, $M$ with its given second fundamental form can be isometrically embedded as a spacelike graph into Minkowski spacetime. The short Sect. 2 presents an alternative proof of this fact, based on the Lorentzian version of the fundamental theorem of hypersurface theory due to Bär–Gauduchon–Moroianu.
Section 3 deals with the question which smooth manifolds admit a Lorentzian metric that satisfies the dominant energy condition. Since every closed or asymptotically flat spacelike hypersurface of a Lorentzian manifold can potentially yield a PMT-like obstruction to the dominant energy condition, one should avoid in the construction of dominant energy metrics that such spacelike hypersurfaces exist at all. This can indeed be accomplished in many situations.

1 Analysis of Asymptotically Flat Manifolds via Witten Spinors

Asymptotically flat Lorentzian manifolds describe isolated gravitating systems (like a star or a galaxy) in the framework of general relativity. As discovered by Arnowitt, Deser and Misner [1], to an asymptotically flat Lorentzian manifold one can associate the total energy and the total momentum, defined globally via the asymptotic behavior of the metric near infinity. Moreover, the energy-momentum tensor gives a local concept of energy and momentum. These global and local quantities are linked by Einstein’s field equations, giving rise to an interesting interplay between local curvature and the global geometry of space-time. The first result which shed some light on the nature of this interplay is the positive energy theorem [29,30], which states that if the local energy density is positive (in the sense that the dominant energy condition holds), then the total energy is also positive. More recently, the proof of the Riemannian Penrose inequalities [5,19] showed that in the time-symmetric situation, the total energy is not only positive, but it is even larger than the energy of the black holes, as measured by the surface area of their horizons. Despite this remarkable progress, many important problems remain open (see for example [22]).

The aim of our research project was to get a better understanding of how total energy and momentum control the geometry of space-time. In the special case that energy and momentum vanish, the positive energy theorem yields that the space-time manifold is flat [27,30]. This suggests that if total energy and momentum are small, then the manifold should be almost flat, meaning that curvature should be small. But is this conjecture really correct? Suppose we consider a sequence of space-time metrics such that total energy and momentum tend to zero. In which sense do the metrics converge to the flat Minkowski metric?

Although our considerations could not give definitive answers to these questions, at least they led to a few inequalities giving some geometric insight, as we will outline in what follows. For simplicity, we begin in the Riemannian setting (in general dimension \(n\)), whereas the generalizations to include the second fundamental form will be explained in Sect. 1.7. All our methods use the Witten spinor as introduced in [37]. But in contrast to the spinor proof of the positive energy theorem [27], we consider second derivatives of the Witten spinor \(\psi\). Our starting point is a basic inequality involving the \(L^2\)-norm of the second derivatives of \(\psi\) (Sect. 1.2).