Differential rotation and r-modes in magnetized neutron stars

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ABSTRACT

Rezzolla et al. [ApJ 531 (2000), L139; Phys. Rev. D 64 (2001), 104013; Phys. Rev. D 64 (2001), 104014] draw attention to the second order secular drift associated with r-modes and claimed that it should lead to magnetic field enhancement and suppression of r-mode instability in magnetized neutron stars. We critically revise these results. We present a particular second order r-mode solution with vanishing secular drift, thus refuting a widely believed statement that secular drift is an unavoidable feature of r-modes. This non-drifting solution is not affected by magnetic field $B$, if $B \ll B_{\text{crit}} \approx 10^{17} (\nu/600 \text{ Hz})$ G ($\nu$ is a spin frequency) and does not lead to secular evolution of magnetic field. For general second order r-mode solution the drift does not necessarily vanish, but the solution can be presented as a superposition of two solutions: one describes evolution of differential rotation in nonoscillating star (which describes secular drift; for nonmagnetized star it is arbitrary stationary rotation stratified on cylinders; for magnetized star differential rotation evolves on the Alfvén timescale and may lead to magnetic energy enhancement), and another one is non-drifting r-mode solution mentioned above. This representation allows us to conclude that enhancement of magnetic field energy is limited by initial energy of differential rotation, which is much less (for a factor $\propto \alpha^2$, where $\alpha$ is mode amplitude) than the total energy of r-mode. Hence, magnetic field enhancement by drift cannot suppress r-mode instability. Results can be generalized for any oscillation mode in any medium, if this mode has non-drifting solution for $B = 0$.

Key words: stars: neutron – stars: oscillations – MHD – dynamo

1 INTRODUCTION

Observations of millisecond pulsars demonstrate that some neutron stars (NSs) rotate very rapidly (the fastest known pulsar PSR J1748-2446ad has spin frequency $\nu = \Omega/2\pi \approx 716$ Hz; Hessels et al. 2006). However, as was shown by Andersson (1998) and Friedman & Morsink (1998), rapidly rotating NSs are subject to gravitation-driven instability associated with enhancement of r-modes (toroidal oscillation mode of rotating star controlled by Coriolis force). It is a particular case of the Chandrasekhar-Friedman-Schutz (CFS) instability Chandrasekhar 1974; Friedman & Schutz 1978a,b. R-mode instability probably plays an important role in NS physics. Indeed, it is argued to be a mechanism which limits pulsars’ spin frequencies Bildsten 1993, Andersson, Kokkotas & Stergioulas 1998, Gusakov, Chugunov & Kantor 2014b(a). Rapidly rotating hot NSs, observed in low mass X-ray binaries (LMXBs), can be affected by r-mode instability even more profoundly: some of these stars should be unstable within the standard model of r-mode oscillations (see Ho, Andersson & Haskell 2011, Haskell, Degenaar & Ho 2012, Gusakov et al. 2014b, for example) and their existence provides one with a good opportunity to test microphysical models of NS’s core (see, e.g., Ho et al. 2011, Haskell et al. 2012, Gusakov et al. 2014b(a)). Furthermore, Chugunov, Gusakov & Kantor (2014) suggest that r-mode instability can support a new class of rapidly rotating NSs, which do not accrete matter from companion even transiently (as NSs in LMXBs), but still keep high temperature via r-mode instability. Some of these NSs might have been already observed, but erroneously classified as quiescent LMXB candidates (see Chugunov, Gusakov & Kantor 2014 for detailed discussion). Finally, as argued by Rezzolla et al. (2000), r-mode instability can be also important for generation of NS magnetic fields. The main aim of the paper is to discuss the role, which magnetic field plays in nonlinear evolution of r-modes and role of r-modes in magnetic field evolution in NSs.

Most of the papers discussing r-modes in NSs either
neglect magnetic field or simply mention that it can be important. Indeed, dipolar magnetic field of rapidly rotating NSs ($\nu \gtrsim 200 \text{ Hz}$) is not very large $\sim 10^8 \text{ G}$, which seems to be clearly negligible (see Rezzolla, Lamb & Shapiro 2004; Asai, Lee & Yoshida 2015 and Sec. 3). Recent papers (Lee 2005; Abbassi, Rieutord & Rezania 2012; Chirenti & Skákal 2013; Asai, Lee & Yoshida 2015, for instance) confirm by detailed numerical calculations that only very high magnetic field can affect r-modes strongly for example (see fig. 7 by Chirenti & Skákal 2013), magnetic field $B \gtrsim 3 \times 10^{15} \text{ G}$ modifies r-mode frequency for a less than for a one percent for $\nu \approx 220 \text{ Hz}$. However, just after r-mode instability was discovered, Rezzolla et al. (2004, 2001a,b) pointed that r-modes are able to increase seeding magnetic field through secular drift of fluid elements, which takes place within second order (in oscillation amplitude) perturbation theory. They argue that this mechanism can be responsible for generation of strong magnetic fields in NSs, but at the same time it finally suppresses r-mode perturbation theory. They argue that this mechanism can be responsible for generation of strong magnetic fields in NSs, but at the same time it finally suppresses r-mode instability. A strong support for these results was provided by subsequent papers, which claim differential rotation, i.e. secular drift of fluid elements, to be a necessary feature of r-modes (in nonmagnetized NS) on the base of analytical (Sá 2004; Sá & Tóth 2003) and numerical calculations (see Stergioulas & Font 2001; Lindblom et al. 2001, for example).

In this paper we reanalyze these results using analytical derivations within second order (in oscillation amplitude) perturbation theory. In Sec. 2 we discuss nonmagnetized ($B = 0$) NSs, following the approach suggested by Sá (2004): Sá & Tóth (2003). As they have shown, the second order r-mode solution is not unique, but determined up to any differential rotation, which can exist in nonoscillating star (i.e. arbitrary stationary differential rotation, stratified on the cylinders). However, an important point of these papers, namely that “differential rotation, producing large scale drifts of fluid elements along stellar latitudes, is an unavoidable feature of r-modes in the nonlinear theory” (see abstract in Sá & Tóth 2003) is not true: we present a parameter set for analytical second order solution obtained by Sá (2004); Sá & Tóth (2003) with vanishing secular drift (below we refer to it as to “non-drifting solution”)

We argue that existence of second order solution with vanishing drift is not coincidental feature, specific for r-modes only, but rather is a general property of oscillation modes. In Sec. 3 we reconsider the effect of magnetic field on r-modes in second order in amplitude by accurate treatment of magnetic stresses within ideal magnetohydrodynamics (MHD). Rezzolla et al. (2000, 2001a,b) argued that r-mode velocity profile is not affected by magnetic field up to $B \ll 10^{16} \text{ G}$. Assuming secular drift to be unaffected, they conclude that r-modes enhance magnetic field due to differential rotation, associated with drift. We demonstrate that non-drifting solution is indeed unaffected by magnetic field up to $B \ll B_{\text{crit}} \approx 10^{17} \text{ G}$, but it does not lead to secular evolution of magnetic field. However, in a general second order solution the second order velocity profile stays unaffected by magnetic field only for a timescale much less then the Alfvén timescale (while back-reaction of magnetic field can be neglected). On the contrary, at the Alfvén timescale the drift contribution to the second order velocity profile (corresponding to arbitrary stationary differential rotation) is modified by magnetic field back-reaction is crucial and cannot be neglected for the same reasons as why stationary differential rotation is forbidden in magnetized stars (Ferraro’s law of isorotation). To describe back-reaction properly, we demonstrate that general second order r-mode solution in a magnetized NS with $B < B_{\text{crit}}$ can be presented as a superposition of two independent solutions of MHD equations: (a) solution which describes evolution of differential rotation in nonoscillating magnetized NS (drift solution, which describes evolution of secular drift), and (b) non-drifting r-mode solution. These solutions are decoupled (at second order in oscillation amplitude) as far as Eulerian perturbations of all quantities (except magnetic field) in drift solution are second order in amplitude of non-drifting r-mode solution. Evolution of the drift solution is governed by the same equations as in non-oscillating magnetized NS, thus it evolves at Alfvén timescale (see Spruit 1994, for example). Secular evolution of the magnetic field is solely determined by the drift, thus initial energy of differential rotation provides an upper bound for the increase of magnetic field energy. Furthermore, energy of the non-drifting mode is conserved in the leading order. As a result, r-modes cannot convert their energy to magnetic field. Hence we conclude, that magnetic energy enhancement by secular drift cannot suppress r-mode instability.

2 R-MODES IN NONMAGNETIZED NS

In this section we consider r-modes oscillations in slowly rotating Newtonian NSs with barotropic equation of state (i.e. pressure depends only on density).

1 Sá (2004) and Sá & Tóth (2003) base their conclusion on the oscillation averaged second order perturbations of Eulerian velocity. Indeed, these perturbations are nonvanishing for all possible second order solutions. However, the drift of fluid elements, i.e. secular increase of the Lagrangian displacement, is not determined by the second order Eulerian velocity only, but has correction associated with Stokes drift induced by the first order motion of the fluid elements (see Longuet-Higgins 1954, Rezzolla et al 2001a or Eqs. (28) and (12) in Sá 2004 and Sá & Tóth 2004, respectively). This correction allows to have second order solution with vanishing secular drift, but nonvanishing oscillation averaged Eulerian velocity (see Sec. 2 for details).

2 As it is stressed by Alfvén & Feltham 1963 (Sec. 3.14.5), the twist of magnetic field caused by inhomogeneous rotation, propagates along magnetic lines of force with Alfvén velocity $v_A$, thus if rotation inhomogeneity is too small, the force lines will have enough time to straighten up. As a result, to produce significant field by differential rotation, relative velocity of different parts of the star should be at least comparable with Alfvén velocity. However, this point was not discussed by Rezzolla et al. (2004, 2001a,b). On the contrary, they suppose that arbitrary small secular drift can enhance magnetic field.

3 As argued by Mendell (2001), Kinney & Mendell (2003), magnetic field can affect r-mode damping at the crust-core boundary, but it requires rather strong radial field ($> 10^{11} \text{ G}$), which is much larger than typical dipolar magnetic fields of rapidly rotating NSs ($\approx 10^8 \text{ G}$). It is also unlikely that strong radial magnetic fields can be generated by drift.
In general of case, relativistic oscillation equations become much more complicated (see Ruoff & Kokkotas 2002; Lockitch, Friedman & Andersson 2003, for example). Furthermore, Lockitch, Andersson & Friedman (2000) demonstrate fully relativistic equations for r-modes in NS with barotropic and nonbarotropic equations of state to be qualitatively different. For barotropic equation of state a relativistic analogue for Newtonian r-modes was found by Lockitch et al. (2003), and they conclude that “unstable r-modes remain essentially unaltered when the problem is studied in full general relativity” [3]. It allows us to take advantage of Newtonian r-modes for the sake of simplicity.

Second order (in oscillation amplitude) r-mode solution was obtained by Sá (2004). This solution is applicable if gravitational constant. In leading order in Ω, linear r-mode gravitational radiation growth time is
\[ t_{\text{gw}} \approx \frac{\Omega R}{v_0} \left( \frac{R}{a} \right)^2 \sqrt{\frac{Gm}{\pi}}. \]

Thus, below we do not restrict our consideration to r-modes, but we apply Sa’s r-mode solution as an example.

Let’s look at the equations describing oscillations up to second order in oscillation amplitude α. The first order equations are:
\[ \partial_\alpha \delta^{(1)} v_i + \delta^{(1)} v_i \nabla_i \delta^{(1)} v_i + \gamma_0 \nabla_i \delta^{(1)} v_i = -\nabla_\alpha \delta^{(1)} U, \]
\[ \delta^{(1)} v_i + \gamma_0 \nabla_i \rho \delta^{(1)} v_i + \nabla_i [\gamma_0 (\rho \delta^{(1)} v)] = 0, \]
\[ \delta^{(1)} v = 4\pi G \delta^{(2)} \rho, \]
where \( v, \rho, \Phi, p \) are, respectively, velocity, density, gravitational potential, and pressure in the unperturbed NS. \( \delta^{(i)} f = O(\alpha^i) \) represents the \( i^{th} \) order Eulerian perturbation of quantity \( f \), and \( \delta^{(i)} U = \delta^{(1)} \rho / \rho + \delta^{(i)} \Phi \). Finally, \( G \) is gravitational constant. In leading order in \( \Omega \), linear r-mode solution is well known (see Proveost, Berthomieu & Rocca 1981, for example) and velocity perturbation can be written in form [4]
\[ \delta^{(1)} v = \alpha \frac{\Omega R}{\sqrt{l(l+1)}} \left( \frac{R}{a} \right)^l \nabla \times (rY_{lm}) e^{i\omega t}. \]

\( Y_{lm} \) is the spherical harmonic with the multipolarity \( l \), and \( Y_{l} \equiv \sqrt{2l + 1} / \Gamma(1/2) \left( 2l + 2 \right) \sin \theta \) \( e^{i\phi} \).

Here \( \omega \) is the oscillation frequency in the inertial frame, given by (also to leading order in \( \Omega \))
\[ \omega = \frac{\Omega}{l+1}. \]

The second order (in \( \alpha \)) equations are:
\[ \partial_\alpha \delta^{(2)} \nabla_i v_i + \delta^{(2)} \nabla_i v_i + \gamma_0 \nabla_i \delta^{(2)} v_i + \delta^{(2)} v_i \nabla_\alpha \delta^{(1)} v_i = -\nabla_\alpha \delta^{(2)} U + \frac{\delta^{(1)} \rho}{\rho} \nabla_\alpha \frac{\delta^{(1)} p}{p}, \]
\[ \partial_\alpha \delta^{(2)} \rho + \gamma_0 \nabla_i \delta^{(2)} \rho + \nabla_\alpha (\rho \delta^{(2)} v_i) + \nabla_i (\delta^{(1)} \rho \delta^{(1)} \nu) = 0, \]
\[ \delta^{(2)} \Phi = 4\pi G \delta^{(2)} \rho. \]
Let us stress two features of these equations:

1. For a given first order solution, general solution of second order equations is a sum of partial solution of homogeneous equations [Eqs. (1-3), which contains terms induced by the first order solution] and a general solution of homogeneous part of Eqs. (1-3) [without terms induced by the first order solution].
2. Homogeneous part of equations (1-3) have the same form as linearized equations [Eqs. (1-3)]. Thus, evolution of all quantities in arbitrary solution of homogeneous equations is the same as for (linear) perturbations of initial (nonoscillating) state.

Thus, with accuracy up to the second order in \( \alpha \), one can select an arbitrary partial solution of the oscillation equations and present a general solution as combination of selected oscillating solution and solution, which describes arbitrary initial \( O(\alpha^2) \) perturbations, evolving in a same way as in nonoscillating NS. In fact, it is obvious even without explicit equations, since second order perturbations cannot affect oscillation solution nor be affected by it (within \( \alpha^2 \) accuracy). Let’s restrict subsequent consideration to nonoscillating solutions of homogeneous Eqs. (1-3). In case of r-modes it is stationary differential rotation, stratified on cylinders.

The freedom in a homogeneous part of the solution can be also understood as excitation of r-mode in a differentially rotating star. As far as this background differential rotation is weak (second order in oscillation amplitude) it does not affect r-mode. This result is in agreement with Karino, Yoshida & Eriguchi (2001), who discuss r-mode oscillation of differentially rotating polytropic stars and obtain r-mode solution in a case, when degree of differential rotation is not too high. Recently, Chirenti, Skákal & Yoshida (2013); Chirenti, Skákal & Yoshida (2013) confirm that r-modes are rather insensitive to small differential rotation within general relativity Cowling approximation. Note, however, that in case of very strong differential rotation, r-modes can be suppressed (see Karino et al. (2001), for more details).

Partial solution, mentioned above, can have drift, i.e. the Lagrangian displacements \( \xi \) can increase with time producing large scale secular motion of fluid elements. Corresponding velocity (so-called mass transfer velocity; see Longuet-Higgins 1953, for example) \( v^{(m)} = \partial \xi / \partial t \) is second order in oscillation amplitude. Here overline means averaging over oscillations. Generally, mass transfer velocity is not equal to oscillation averaged Eulerian velocity, but has a correction associated with Stokes drift velocity \( v^{(S)} \) induced by first order pure oscillating solution \( v^{(S)} = (\xi^{(1)} \nabla) \delta^{(1)} v / (\xi^{(1)} \nabla) \delta^{(1)} v \).
For r-modes mass transfer velocity $v^{(\text{int})}$ can be easily derived from Eqs. (30-31) in Sánchez (2004), which gives the second order Lagrangian displacement. Only $\phi$ component of $v^{(\text{int})}$ is nonvanishing and it can be written in form:

$$v^{(\text{int})} = \alpha^2 \Omega \left[ \bar{A} \left( \frac{r \sin \theta}{R} \right)^{2l-2} + \sum_{N=0}^{\infty} A_N \left( \frac{r \sin \theta}{R} \right)^N \right],$$

where $A_N$ are arbitrary constants, and

$$\bar{A} = \frac{(2l+1)!}{2^{2l+1} \pi} \frac{2l-1}{l^2}.$$

For $A_{2l-2} = -\bar{A}$ and $A_N = 0$ for all other $N$ we obtain a second order r-mode solution with vanishing mass transfer velocity. For this solution Lagrangian displacements do not increase with time, thus secular drift of fluid elements vanishes. However, oscillation averaged Eulerian velocities are not zero for this solution (see Eqs. (18) and (21) in Sánchez (2004)). Such a solution corresponds to a very specific set of initial conditions and it is not surprising, that it was not obtained in numerical calculations by Stergioulas & Font (2001); Lindblom et al. (2001). Nonetheless, it simplifies discussion of r-modes in magnetized NSs (see Sec. 3).

Presence of a second order solution with vanishing drift is not an accidental specific feature of r-modes. This property can be generalized for any oscillation mode, if mass transfer velocity $v^{(\text{int})}$ for initial partial solution of second order Eulerian equations corresponds to a solution of linear perturbation equations (or, equally, homogeneous second order equations). In this case we can construct a partial solution with vanishing drift by substituting $\delta v \to \delta v - \delta^{(d)} v$, $\delta \rho \to \delta \rho - \delta^{(d)} \rho$, $\delta \Phi \to \delta \Phi - \delta^{(d)} \Phi$. Here $\delta^{(d)} \rho$ and $\delta^{(d)} \Phi$ are perturbations of density and gravitational potential in a solution of linear perturbation equations, which corresponds to velocity perturbation $\delta^{(d)} v$.

Concluding, in this section we demonstrate, that general second order r-mode solution can be presented as a sum of oscillating solution with vanishing drift and a solution, which describes differential rotation in nonoscillating NS. As far as we consider nonmagnetized star, this differential rotation can be described by arbitrary stationary profile, stratified on cylinders. If we restrict our consideration to r-modes excited by the gravitational radiation in initially uniformly rotated NSs, this profile is well defined (Friedman, Lindblom & Lockitch 2013). However, stationary differential rotation is forbidden for magnetized star (Ferraro isorotation law), thus for magnetized NS a general r-mode solution cannot be presented as sum of oscillations and free stationary differential rotation. Magnetic field modifies general r-mode solution and we discuss this modification in the next section.

### 3. R-MODES IN MAGNETIZED NS

Oscillations in nonmagnetized NS can have nonvanishing drift, leading to large (and increasing) Lagrangian displacements, but Eulerian perturbation of all quantities stays small. This is not the case for a magnetized NS. Magnetic field is frozen into fluid elements, so that large Lagrangian displacements generally lead to large Eulerian perturbations of magnetic field. As a result, one generally cannot expand Eulerian perturbation of magnetic field in powers on oscillation amplitude and apply perturbed Eulerian approach to describe r-modes with nonvanishing drift. Hence, a stricter formalism is required to describe oscillations with nonvanishing secular drift in magnetized NS. Here we describe such formalism based on the idea to present a general second order r-mode solution as a superposition of two solutions: (a) solution which describes evolution of the differential rotation in nonoscillating NS within exact MHD equations (later referred to as ‘drift solution’, because it will describe drift), and (b) non-drifting oscillating solution for $B = 0$, described in the previous section. To achieve that, we apply perturbations in two steps. First, we apply initial perturbation, which is second order in $\alpha$, and evolve it using exact MHD equations. This solution describes evolution of drift motion [i.e. it is the drift solution mentioned above]. Second, we perturb drift solution, write down perturbation equations up to second order in $\alpha$, and show that non-drifting oscillating solution of amplitude $\alpha$ is a (partial) solution of these equations. Finally, we demonstrate that r-mode with arbitrary second order differential rotation at initial moment can be presented in form of this superposition, thus the superposition can be applied to describe a general solution in magnetized case. It is worth to note that this procedure is suitable for any oscillation mode in any media, if the mode have non-drifting solution for $B = 0$.

Let $(\rho, p, v, B)$ be a stationary configuration described...
by solution of ideal MHD equations
\[ \partial_t v + (v \nabla) v = -\frac{\nabla p}{\rho} - \nabla \Phi + \frac{1}{4\pi \rho} [B \times [\nabla \times B]], \]  
(13)
\[ \partial_t \rho + \nabla (\rho v) = 0, \]  
(14)
\[ \Delta \Phi = 4\pi G \rho, \]  
(15)
\[ \frac{\partial B}{\partial t} = [\nabla \times [v \times B]]. \]  
(16)
Let \( c(a,t) \) be a fluid element trajectory, corresponding to this stationary solution. Solution of the induction equation can be written in the form (see, e.g., Rezzolla et al. 2001a)
\[ B_i'(c(a,t),t) = B_i'(a,t = 0) \frac{\partial c_i(a,t)}{\partial a^i} = 0, \]  
(17)
For the sake of simplicity we assume \( c_i'(a,t) = a^i v^t \), thus, stationary condition implies \( \rho(c(a,t), t) = \rho(a,t) \), i.e. the density \( \rho \) is constant along trajectory \( c(a,t) \).

As the first step let us consider the drift solution. We define it as a perturbation of unperturbed stationary configuration which is an exact solution of MHD equations. Let drift solution be given by the trajectory \( c^{(d)}(t) = c(a,t) + \xi^{(d)}(a,t) \) and Eulerian variables \( (\rho^{(d)}(r,t), p^{(d)}(r,t), v^{(d)}(r,t), B^{(d)}(r,t)) \). However, let us assume that Eulerian perturbations of \( \Phi, \rho, p, v \) with respect to unperturbed stationary configuration, \( \delta^{(d)} f = f^{(d)} - f \), are second order in \( \alpha \); \( \delta^{(d)} f = O(\alpha^2) \) (at least for a time of consideration) and their space and time derivatives have same order in \( \alpha \) (i.e. they are not too large: \( \nabla \delta^{(d)} f \lesssim f \delta^{(d)} f/R \); \( \partial_t \delta^{(d)} f \lesssim \Omega \delta^{(d)} f \)). Here \( f = \Phi, \rho, p, v \) and \( \alpha \ll 1 \).

We do not assume Eulerian perturbations of magnetic field to be small and thus do not expand them in series on \( \alpha \). Instead, we consider evolution of total magnetic field, which is given by the equation:
\[ \frac{(B^{(d)})^j(r,t)}{\rho^{(d)}(r,t)} = \frac{B_j'(a,t = 0)}{\rho(a,t = 0)} \frac{\partial c_j'(a,t)}{\partial a^j}. \]  
(18)

Let us now perturb drift solution by rapidly oscillating perturbation of amplitude \( \alpha \). To describe its evolution, we apply perturbed (with respect to the drift solution) MHD equations of the first and the second order in Eulerian form. In these equations one can substitute \( \rho^{(d)}, \rho^{(d)} \), \( \Phi^{(d)} \), and \( v^{(d)} \) with correspondent values of unperturbed stationary configuration because they differ only in the second order in \( \alpha \). Thus, linear perturbation equations can be written in the second order in \( \alpha \):
\[ \partial_t \delta^{(1)} v_i + v^k \nabla_k \delta^{(1)} v_i + v^k \nabla_k \delta^{(1)} v_i = -\nabla_i \delta^{(1)} U + \delta^{(1)} F^{(m)}, \]  
(19)
\[ \partial_t \delta^{(1)} \rho + \nabla_i (\rho \delta^{(1)} v^i) = 0, \]  
(20)
\[ \triangle \delta^{(1)} \Phi = 4\pi G \delta^{(1)} \rho. \]  
(21)

Magnetic stress of the first order is:
\[ \delta^{(1)} F^{(m)} = \frac{1}{4\pi \rho} \left\{ \delta^{(1)} B \times \text{rot} B^{(d)} + [B^{(d)} \times \text{rot} \delta^{(1)} B] \right\} \]  
(22)
The second order equations in \( \alpha \) are:
\[ \partial_t \delta^{(2)} v_i + v^k \nabla_k \delta^{(2)} v_i + v^k \nabla_k \delta^{(2)} v_i + \delta^{(1)} v^k \nabla_k \delta^{(1)} v_i = -\nabla_i \delta^{(2)} U + \frac{\delta^{(1)} \rho}{\rho} \nabla_i \left( \frac{\delta^{(1)} v^i}{\rho} \right) + \delta^{(2)} F^{(m)}, \]  
(23)
\[ \partial_t \delta^{(2)} \rho + \nabla_i (\rho \delta^{(2)} v^i) + \nabla_i (\rho \delta^{(2)} v^i) = 0, \]  
(24)
\[ \triangle \delta^{(2)} \Phi = 4\pi G \delta^{(2)} \rho. \]  
(25)
where second order magnetic stress is:
\[ \delta^{(2)} F^{(m)} = \frac{1}{4\pi \rho} \left\{ \delta^{(2)} B \times \text{rot} B^{(d)} + [B^{(d)} \times \text{rot} \delta^{(2)} B] \right\} \]  
(26)
Here \( \delta B = \delta^{(1)} B + \delta^{(2)} B + \ldots = B^{(d)} - B^{(d)} \) is Eulerian variation of magnetic field on course of oscillations and \( B^{(d)} \) is total magnetic field (in accurate solution of MHD equations). However, in contrast to \( B^{(d)} \), we are interested only to the first and second order perturbations of the magnetic field (with respect of drift solution), assuming Eulerian variation \( \delta B^{(d)} \) to be small. Let us present a trajectory of fluid elements for this solution in form
\[ c^{(o)}(a,t) = c^{(d)}(a,t) + \xi^{(o)}(a,t), \]  
(27)
This form allows us to use the same relations between Eulerian variation of the velocity \( \delta^{(1)} v \) and \( \delta^{(2)} v \) and \( \xi^{(o)} \) as we do in absence of drift motions [Eqs. (10) and (11)].

Magnetic field and trajectory are coupled by the induction equation, which leads to vanishing Lagrangian perturbation of \( B/\rho \) (see Glampedakis & Andersson 2007, for example). Thus, Lagrangian variation \( \delta B^{(d)} = -B^{(d)} \nabla k \xi^{(d)} k \) and Eulerian variation of magnetic field can be derived following general formalism for vector fields (Friedman & Schutz 1978a; Sah 2004):
\[ \delta^{(1)} B^{(m)} = -B^{(d)} \nabla_k \xi^{(1)} j \nabla_j B^{(d)} - B^{(d)} \nabla_k \xi^{(1)} j \nabla_j B^{(d)} + B^{(d)} \nabla_k \xi^{(1)} j \nabla_j B^{(d)} = 0, \]  
(28)
\[ \delta^{(2)} B^{(m)} = -B^{(d)} \nabla_k \xi^{(2)} j \nabla_j B^{(d)} - B^{(d)} \nabla_k \xi^{(2)} j \nabla_j B^{(d)} - B^{(d)} \nabla_k \xi^{(2)} j \nabla_j B^{(d)} = 0, \]  
(29)
Equations (11) and (24) differ from linear equations in nonmagnetized case (13) and (14) only by magnetic stresses \( \delta^{(1)} F^{(m)} \) and \( \delta^{(2)} F^{(m)} \) in Eqs. (10) and (25).

Even the last equalities in Eqs. (28) and (29) can be applied for unperturbed stationary configuration corresponding to solid state rotation as far as drift motion perturb stationary velocity only in the second order in \( \alpha \): \( \delta^{(1)} v \sim \alpha^2 v \).
respectively. Let us analyse effects of magnetic field onto nonmagnetized \((B = 0)\) solutions by comparison of magnetic and other stresses in these equations. Nonmagnetic (hydrodynamical) stresses can be estimated as \(\delta^{(1)} F^n \approx \partial_t \delta^{(1)} v = O(\alpha) R \omega^2\) for the first order Eq. \((19)\) and as \(\delta^{(2)} F^n \approx \partial_t \delta^{(2)} v = O(\alpha^2) R \omega^2\) for the second order Eq. \((23)\). Before estimating magnetic stresses let us note, that for \(B = 0\) solutions with nonvanishing drift, displacement \((\xi^{(2)})^0\) contains terms \(\alpha^\omega v_t\), which are finite i.e. \((\xi^{(2)})^0 = O(1)\) at \(\omega_t > 1/\alpha\). For such solution Eulerian variation \(\delta^{(2)} B \sim R \alpha^\omega v\) becomes large \((\delta^{(2)} B \gtrsim B \gg \delta^{(1)} B)\). Presence of large Eulerian variation makes it doubtful, that the theory based on the expanding of Eulerian variations in series in \(\alpha\) is appropriate here.

Thus, let us concentrate on the non-drifting solution at \(B = 0\) (see Sec. 2) and demonstrate that it indeed can be applied to describe perturbations of the drift solutions (i.e. to describe oscillations at nonstationary background and \(B \neq 0\)). For this solution Lagrangian displacement \(\xi^{(0)}\) stays always small \(\xi^{(0)} = O(\alpha)\) and even \(O(\alpha^2)\) terms do not increase with time, thus \(\delta^{(1)} B = O(\alpha) = \alpha B^{(0)}\) and \(\delta^{(2)} B = O(\alpha^2) \approx \alpha^2 B^{(0)}\). Now we can estimate magnetic stresses of the first and the second order [given by Eqs. \((22)\) and \((26)\)] as \(\delta^{(1)} F^n \approx O(\alpha) B^{(0)}\) and \(\delta^{(2)} F^n \approx \alpha^2 B^{(0)}\). Thus, magnetic to nonmagnetic stresses ratio in Eqs. \((19)\) and \((23)\) is \(\delta^{(1)} F^n / \delta^{(1)} F^n \approx \delta^{(2)} F^n / \delta^{(2)} F^n \approx (B^{(0)})^2 / (\alpha^2 \pi R)\) respectively. Thus, magnetic to nonmagnetic stresses in both of these equations are negligible, if \(B^{(0)} \ll B_{crit} \approx R \omega^2 \sqrt{\pi \rho}\). For a r-mode in NS and fiducial parameters \((\nu = 4 \times 10^{14} \, \text{g cm}^{-3}, \omega = 4/3 \Omega = 8 \pi/3 \nu = 5000(\nu/600\,\text{Hz})^{-1}, R = 10^6 \, \text{cm})\) we obtain \(B_{crit} \approx 2 \times 10^{17} \, \text{G}\). This, \(B = 0\) non-drifting solution can be used as the first approximation to describe non-drifting oscillation mode for magnetized NS [first approximation to the solution of Eqs. \((19)\) \((21)\) and \((22)\) \((23)\)].

Let us analyse corrections to this solution, which can be associated with magnetic field. In the first order these effects were studied accurately by Lee (2005); Abbassi et al. (2012); Chirenti & Skákal (2013); Asai et al. (2013). For simplicity, here we apply analytical estimates, which is enough for the purpose of the study. Magnetic stresses, which oscillate along with oscillations, can modify oscillation frequency slightly by providing additional stiffness to oscillations. In analogy with effect of small magnetic field on sound waves in plasma, we anticipate increasing of the oscillation frequency to the value \(\delta \omega \approx \omega (v_A/v_t)^2\), where \(v_t\) is a phase velocity in absence of magnetic field. For \(r\)-modes, taken \(\omega R\) as a phase velocity, we obtain \(\delta \omega / \omega \approx (B^{(0)}/B_{crit})^2\). This estimate is in a reasonable agreement with accurate calculation by Asai et al. (2013) and by Chirenti & Skákal (2013) (for rotation, which is not too slow). The terms in \(\delta^{(2)} F^n\) with nonvanishing oscillation average (for example, \(\delta^{(1)} B \times \omega \delta^{(1)} B\)) cannot lead to significant displacement across the magnetic field, thus significantly affecting magnetic field, for a simple reason: additional displacement of second order \(\delta \xi B = O(\alpha^2)\) across magnetic field would produce compensating stress due to deformation of the magnetic force lines. As a result, we suppose that such terms do not lead to significant modification of magnetic field and global drift motion. Concluding, for a given constant \(B^{(0)} = \text{const}\), none of magnetic terms leads to significant modification of nonmagnetized solution, which would affect evolution of \(B\). In general case \(B^{(0)}\) is not a constant, but varies on a timescale \(t^{(d)} \approx R/v^{(d)} \approx 1/(\alpha^2 \omega) \gg 1/\omega\). Hence, adiabatic invariant of the first order oscillation equation \(E/\omega\) should be conserved during \(B^{(0)}\) evolution. As far as \(B^{(0)} \ll B_{crit}\), corrections to the oscillation frequency are small and oscillation energy is also conserved in leading order.

Let us now demonstrate that a general second order solution can be presented in a form of superposition of drift solution and non-drifting \(r\)-mode. To do that, we analyse evolution of the oscillation mode with nonvanishing mass transfer velocity at the initial moment of time. It is especially important to analyze this possibility as far as Friedman et al. (2015) clearly demonstrated that \(r\)-modes excited by gravitational radiation in uniformly rotating nonmagnetized NS correspond to well defined nonvanishing mass transfer velocity. Let us assume that at this moment Eulerian perturbations of the first order in \(\alpha\) were given by an oscillation mode for initial magnetic field \(B^{(0)}\). In this case an arbitrary perturbations of the second order can be presented as a sum of second order perturbation, corresponding to the second order non-drifting solution and corrections, associated with nonvanishing mass transfer velocity. Let us use these corrections, which are second order in \(\alpha\), to define initial state and determine drift solution \(\xi^{(d)}(a, t)\) by solving exact MHD equations. As long as initial velocity perturbations are small \(\delta \xi v = \xi^{(int)} = O(\alpha^2)\), we expect that temporal evolution of this velocity field takes place on Alfvén timescale, preserving Eulerian variation of all quantities (with possible exception of the magnetic field) to be of order of \(\alpha^2\), i.e. trajectory \(\xi^{(d)}(a, t)\) satisfies requirements, imposed to the drift motion solution in the beginning of section. Thus, we present

\[\begin{align*}
\delta \xi v = \xi^{(int)} = O(\alpha^2),
\delta \omega / \omega \approx (B^{(0)}/B_{crit})^2.
\end{align*}\]
initial condition as a composition of drift solution and non-drifting mode. It allows us to apply this solution to describe evolution of initial perturbations. Thus secular evolution of the magnetic field follows the drift solution, i.e. it should be the same as in absence of non-drifting r-mode. As a result, we can estimate an upper limit for the enhancement of the magnetic field energy because it exceeds drift energy for a fast r-mode. The total energy of the r-mode cannot be converted into gravitational radiation because it exceeds drift energy for a fast r-mode. Thus, its initial energy gives an upper limit for the enhancement of the magnetic field energy by drift significantly.

If, on the contrary, initial field $B \ll 10^8 (\alpha/10^{-4})^2$ G, it can be enhanced up to $10^8 (\alpha/10^{-4})^2$ G.

This section’s conclusion can be formulated in almost the same form as for nonmagnetized NS: a general second order r-mode solution can be presented as a superposition of oscillating solution with vanishing drift and a solution, which describes differential rotation in nonoscillating NS. The main effect of magnetic field is that it modifies differential rotation in nonoscillating NSs. In case of magnetized star differential rotation cannot be stationary, but should evolve on the Alfvén timescale.

4 DISCUSSION AND CONCLUSION

We reanalyze second order effects in r-mode oscillations. For nonmagnetized NS we refute widely believed statement that (second order) drift (secular motion of fluid elements) is an essential feature of r-modes. We present the second order r-mode solution with vanishing drift (non-drifting solution; see Sec. 2). Strictly speaking, this solution is applicable in a case of slow rotation ($\Omega \ll \Omega_k$) and $\alpha \gg (\Omega/\Omega_k)^2$. However, we suppose that these restrictions are not crucial for the main results of paper (see footnote 5), but we leave a strict proof of this conjecture beyond the scope of the paper. In Sec. 4 we demonstrate that this non-drifting solution is valid for magnetized NS. On the contrary, $B = 0$ r-mode solutions with nonvanishing drift cannot be applied directly to magnetized NS because magnetic field leads to evolution of the mass transfer velocity, which describes drift, on the Alfvén timescale. However this evolution is decoupled from oscillations. This means that the general r-mode solution can be presented as superposition of two solutions: (a) the solution characterizing evolution of mass transfer velocity (drift motion), which is the same as evolution of differential rotation in nonoscillating NS (initial conditions correspond to the mass transfer velocity at $t = 0$); and (b) non-drifting r-mode solution. Secular evolution of magnetic field is entirely determined by drift motion, thus its initial energy gives an upper limit for the enhancement of magnetic field energy. The total energy of the r-mode cannot be converted into magnetic energy because it exceeds drift energy for a factor of $1/\alpha^2$. Thus we conclude that magnetic field energy enhancement does not prove to be relevant for oscillation energy decrease and therefore cannot prevent instability.

Still, in our consideration we neglect gravitational radiation-reaction force. As shown by Friedman et al. (2013) (published in arXiv, while the present manuscript has been under revision), this force excites r-mode with well defined nonvanishing mass transfer velocity (differential rotation) in nonmagnetized NSs. Strictly speaking, in the present paper we demonstrate that this mode does not dump regardless of magnetic field enhancement (the opposite to what follows from results by Rezzolla et al. (2000, 2001)), in case when gravitational radiation force stop acting. In reality, gravitational radiation force acts permanently and accurate description of second order r-modes under combined action of gravitational radiation and magnetic field is particularly interesting task. We expect that two limiting cases are possible, depending on the ratio of Alfvén timescale $\tau_A$ to r-mode growing time $\tau$. If $\tau_A < \tau$, then magnetic field has enough time to affect mass transfer rate and the theory described here remains applicable. In the opposite case, namely when $\tau_A \gg \tau$, r-modes will saturate rapidly, since they are almost unaffected by the magnetic field.

Saturated r-modes require special consideration. It is typically assumed that saturation of r-mode instability is associated with the lowest parametric instability threshold in nonlinearly coupled triplets of oscillation modes [Schenk, Arras, Flanagan, Teukolsky & Wasserman 2002; Arras, Flanagan, Morsink, Schenk, Teukolsky & Wasserman 2003; Brink, Teukolsky & Wasserman 2003; Bondarescu, Teukolsky & Wasserman 2007, 2009; Bondarescu & Wasserman 2013]. Within this model, saturated r-modes is always accompanied by excited daughter modes (at least two). If amplitudes of these latter modes are comparable with r-mode amplitude, the first order velocity field should be presented as a sum of three oscillation modes, and each of them affects second order equations. Then the total mass transfer velocity, in principle, can differ from Eq. (11), so that the very existence of nondrifting solution might be challenged as well as applicability of the theory discussed here. However, if saturation amplitude is not too large ($\alpha \ll 1$), we expect that such scenario does not occur and we can obtain nondrifting solution even for arbitrary number of eigenmodes excited in first order. However, a more detailed analysis of this question is left beyond the scope of this paper.

Strictly speaking, the present results are accurate only at second order in mode amplitude. However, the drift of the fluid elements in the next orders (the third and higher) cannot be excluded. Even if mass transfer velocity for such drift is very small $v_{\text{drift}}(n) \approx \alpha \Omega R \approx 4 \times 10^{-3} (\nu/600 \text{ Hz}) (\alpha/10^{-4})^3 \text{ cm s}^{-1}$, it can lead to large Lagrangian displacements on a timescale of years (in nonmagnetic case). The second order drift and first order oscillations are coupled in the third order. As a result the theory in third order can be much more complicated than the one in second order described here. Furthermore, for very large amplitudes $\alpha \gtrsim 3$, which can be studied only numerically, r-modes decay nonlinearly and strong differential rotation develops [Gressman et al. 2002; Kastaun 2011]. Such a strong differential rotation (angular velocities in the range $0.5 \ldots 1.2$ of the initial one according to Kastaun 2011) can strongly affect magnetic field (formally, it does not contradict with our estimates, which predict enhancement of the magnetic field up to $\approx 10^{17} \text{ G}$ for $\alpha \approx 3$). However, r-
mode saturation amplitudes calculated by Bondarescu et al. (2009); Bondarescu & Wasserman (2013) are small $\alpha \ll 1$. Thus it seems reasonable to restrict consideration to small amplitudes. In this case, the total mass transfer velocity $\dot{m}$ is small.

Thus it seems reasonable to restrict consideration to small amplitudes. In this case, the total mass transfer velocity $\dot{m}$ is small. Consequently, high order corrections can hardly result in strong magnetic field generation. However, we leave a more accurate analysis of high order effects outside the paper.

To conclude we note, that results discussed here can be easily generalized for any oscillating mode in any magnetized medium, if secular drift is a perturbation, allowed by the linear perturbation theory. This property seems to be rather general, at least it holds true for r-modes and ocean waves (Longuet-Higgins 1953; Moore 1970) and we plan to study whether it is indeed a general property of second order waves (Longuet-Higgins 1953; Moore 1970) and we plan to study whether it is indeed a general property of second order oscillations or not.

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