Vibration of functionally graded Mindlin plate based on a modified strain gradient elasticity theory

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Abstract. Vibrational behavior of functionally graded (FG) microplates is investigated by a new modified strain gradient Mindlin plate (MSGMP) model. With the help of Hamilton’s principle, the dynamic equation is easily obtained. Furthermore, the general forms of boundary conditions are gotten by using coordinate transformation. The MSGMP model can be degenerated to a couple stress elastic Mindlin plate model or the classical Mindlin plate (CMP) model. Analytical solutions of vibrational problem of a rectangular microplate with four simply supported edges are gotten. Numerical results reveal significant effects of the dimensionless nonlocal parameters, the power law index and vibration mode on the free vibration behavior of FG plate.

1. Introduction
Functionally graded materials (FGMs) have gained widespread applications in many engineering fields [1-3]. In these application, the scale effects of microscale structures are often observed [4-6]. Because the classical continuum theories do not capture such scale effects, various higher-order continuum theories have been proposed to successfully predict the scale effect [4,7-12]. A modified strain gradient elasticity theory (MSGET) which contains three material length scale parameters respectively, was proposed by Lam et al. [4]. If two or all material length scale parameters are set to be zero respectively, the MSGET can be degenerated into the modified couple stress elasticity theory (MCSET) [11] or classical continuum theories. Thus, the MSGET can be deemed to a more general form of the MCSET. Several MSGET beam and plate models have been successfully used to capture scale effects of microstructures made of homogeneous materials [12-20]. For example, Wang [14] developed Timoshenko beam model to study the static bending and vibrational behaviors of microbeams. Ashoori [18] presented a microscale model of Kirchhoff plate for static bending to capture the scale effect. Recently, MSGET is extended to analyze FG microbeams and microplates [21-30].

This paper focuses on investigating the vibration behavior of FG MSGET Mindlin plate (MSGMP). The developed mode can capture the scale effects. The dynamic equation and general forms of boundary conditions are obtained simultaneously with the help of the Hamilton’s principle. Then analytical solutions for the vibration of a rectangle microplate with four simply supported edges are gotten. Numerical results indicate that dimensionless material length scale parameter, the power law index and vibration mode have important influences on the free vibration behavior of FGM plate.

2. The MSGMP Model
The strain energy \( U \) of MSGMP model is given as [4]
\[ U = \frac{1}{2} \int v \left( \sigma_{ij} \varepsilon_{ij} + \tau_{kln}^{(1)} \eta_{kln}^{(1)} + m_{kl} \chi_{kl} + p_k \gamma_k \right) dv, \]

where \( \varepsilon_{ij} \), \( \gamma_k \), \( \eta_{kln}^{(1)} \) and \( \chi_{kl} \) are the strain tensor, the dilatation gradient tensor, the deviatoric stretch gradient tensor and the symmetric rotation gradient tensor respectively. They are expressed as

\[ \varepsilon_{kl} = \frac{1}{2} (u_k, + u_i, - k) , \]
\[ \gamma_k = \varepsilon_{ss,k}, \]
\[ \eta_{kln}^{(1)} = \eta_{sl} - \frac{1}{5} (\delta_{j,s} \eta_{jkm} + \delta_{km} \eta_{jkl} + \delta_{sk} \eta_{kln}) , \]
\[ \eta_{kln}^{(1)} = \frac{1}{3} (u_{k,ln} + u_{l,mk} + u_{m,kl}) , \]
\[ \chi_{kl} = \frac{1}{4} (e_{lm} u_{n,m} + e_{ln} u_{n,lm}) , \]

where a comma indicates partial derivative and \( u_k \) is displacement field, \( e_{klm} \) and \( \delta_{kl} \) are the permutation symbol and the Kronecker symbol respectively.

The stress tensors including the classic stress tensor \( \sigma_{ij} \), and the high order stress \( \tau_{kln}^{(1)} \) and \( m_{kl} \) can be given by [4]

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{xz}
\end{bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix}
1 & \mu & 0 & 0 & 0 \\
\mu & 1 & 0 & 0 & 0 \\
0 & 0 & 1-\mu & 0 & 0 \\
0 & 0 & 0 & 1-\mu & 0 \\
0 & 0 & 0 & 0 & 1-\mu
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_{xy} \\
\varepsilon_{yz} \\
\varepsilon_{xz}
\end{bmatrix},
\]

\[ p_k = 2G I^2_0 \gamma_k , \]
\[ \tau_{kln}^{(1)} = 2G I^2_1 \eta_{kln}^{(1)} , \]
\[ m_{kl} = 2G I^2_2 \chi_{kl} , \]

where the shear modules \( G \) is expressed as [31]

\[ G = \frac{E}{2(1+\mu)}, \]

\( E \) is Young’s modulus, \( \mu \) is Poisson’s ratio, and \( I_0, I_1 \) and \( I_2 \) are the additional independent material length scale parameters.

As depicted in Figure 1, FGMs microplate of uniform thickness \( h \) that is made of two materials through thickness is considered. The material density \( \rho \) and Young’s modulus \( E \) are defined as

\[ \rho(z) = \rho_2 + (\rho_1 - \rho_2) \left( \frac{1}{2} + \frac{z}{h} \right)^p , \]
\[ E(z) = E_2 + (E_1 - E_2) \left( \frac{1}{2} + \frac{z}{h} \right)^p , \]
where $p$ is power law index, Poisson’s ratio $\mu$ is constant [28]. The displacement of the Mindlin plate model can be given as

$$u_i = -(z - z_0)\phi_i(x, y, t), \quad u_2 = -(z - z_0)\phi_i(x, y, t), \quad u_3 = w(x, y, t),$$

where $\phi_i$ and $\phi_i'$ denote the rotation angles of a transverse normal about the $y$- and $x$-axes respectively, and the physical neutral surface ($z = z_0$) can be gotten by [32]

$$z_0 = \int_{-h/2}^{h/2} zE(z)dz \int_{-h/2}^{h/2} E(z)dz. \tag{15}$$

**Figure 1.** Schematic of a FGM plate

Substituting Equation (14) into Equation (2) gives

$$\varepsilon_{xx} = -(z - z_0)\phi_{xx} + \phi_{yy}, \quad \varepsilon_{yy} = -(z - z_0)\phi_{yy} + \phi_{xy} - \frac{1}{2}(z - z_0)(\phi_{yy} + \phi_{xx}),$$

$$\varepsilon_{xy} = \frac{1}{2}(w_{yy} - \phi_y), \quad \varepsilon_{xz} = \frac{1}{2}(w_{xx} - \phi_x). \tag{16}$$

Substituting Equation (14) into Equation (6) gives

$$X_{xx} = \frac{1}{2}(w_{xy} + \phi_x), \quad X_{yy} = -\frac{1}{2}(w_{yy} + \phi_y), \quad X_{zz} = \frac{1}{2}(\phi_x + \phi_y) + 2\phi_x + \phi_y),$$

$$X_{xy} = \frac{1}{4}(w_{xx} + \phi_x - \phi_y), \quad X_{xz} = \frac{1}{4}(z - z_0)(\phi_{xx} - \phi_{yy}) + \frac{1}{4}(z - z_0)(\phi_{xy} - \phi_{yx}),$$

$$X_{yz} = \frac{1}{4}(z - z_0)(\phi_{xz} - \phi_{yz}). \tag{17}$$

By means of Equations (2), (4) and (14), one can obtain

$$\eta^{(1)}_{xxx} = -\frac{1}{5}(z - z_0)(2\phi_{xx} - 2\phi_{xy} - \phi_{xx}),$$

$$\eta^{(1)}_{yyy} = \frac{1}{5}(z - z_0)(\phi_{yy} + 2\phi_{xy} - 2\phi_{yy}),$$

$$\eta^{(1)}_{zzz} = \frac{1}{5}(2\phi_{xx} + 2\phi_{xy} - w_{xx} - w_{yy}),$$

$$\eta^{(1)}_{xyy} = \eta^{(1)}_{yxx} = \eta^{(1)}_{yyx} = -\frac{1}{15}(z - z_0)(8\phi_{y} + 4\phi_{xx} - 3\phi_{yy}),$$

$$\eta^{(1)}_{xzz} = \eta^{(1)}_{zxx} = \eta^{(1)}_{xxz} = -\frac{1}{15}(8\phi_{xz} - 2\phi_{y} - 4w_{xx} + w_{sy}),$$

$$\eta^{(1)}_{yzy} = \eta^{(1)}_{yzy} = \eta^{(1)}_{zzx} = \frac{1}{15}(z - z_0)(3\phi_{zz} - 8\phi_{yz} - 4\phi_{xzz}),$$

$$\eta^{(1)}_{zzy} = \eta^{(1)}_{zzy} = \eta^{(1)}_{zzy} = \frac{1}{15}(2\phi_{xz} - 8\phi_{yz} - w_{xx} + 4w_{yy}).$$
\[
\eta_{zz}^{(1)} = \eta_{zz}^{(1)} = \eta_{zz}^{(1)} = \frac{1}{15} (z - z_0)(3\phi_{x,xx} + 2\phi_{x,xy} + \phi_{x,yy}), \\
\eta_{yy}^{(1)} = \eta_{yy}^{(1)} = \eta_{yy}^{(1)} = \frac{1}{15} (z - z_0)(2\phi_{x,xy} + 3\phi_{y,yy} + \phi_{y,xx}), \\
\eta_{xy}^{(1)} = \eta_{xy}^{(1)} = \eta_{xy}^{(1)} = \eta_{xy}^{(1)} = \eta_{xy}^{(1)} = \frac{1}{3} (\phi_{x,y} + \phi_{y,x} - w_{xy}).
\]

Using Equations (2), (3) and (14), the dilatation gradient tensors are given as

\[
\gamma_x = -(z - z_0)(\phi_{x,xx} + \phi_{y,xy}), \quad \gamma_y = -(z - z_0)(\phi_{x,xy} + \phi_{y,yy}), \quad \gamma_z = -(\phi_{x,xy} + \phi_{y,xx}).
\]

Considering the transverse distributed load \( q = q(x, y, t) \), the virtual work can be given as

\[
\delta \int_0^T W dt = \int_0^T \delta q \delta \omega dVdt.
\]

The kinetic energy of Mindlin plate is given as

\[
K = \frac{1}{2} \int \int \left[ J_0 \left( \frac{\partial \mathbf{w}}{\partial t} \right)^2 + J_1 \left( \frac{\partial \phi}{\partial t} \right)^2 + J_1 \left( \frac{\partial \phi}{\partial t} \right)^2 \right] dxdy,
\]

where \( (J_0, J_1) \) are defined as

\[
(J_0, J_1) = \int_{-h/2}^{h/2} \rho(z)(1, (z - z_0)^2) dz.
\]

By using of Equation (21), one can obtain as follows

\[
\delta \int_0^T K dt = - \int_0^T \left( J_0 \frac{\partial^2 \mathbf{w}}{\partial t^2} + J_1 \frac{\partial^2 \phi}{\partial t^2} + J_1 \frac{\partial^2 \phi}{\partial t^2} \right) dxdydt.
\]

From Equation (1), one obtains the following

\[
\delta U = \int \sum \sigma \frac{\partial \epsilon}{\partial x} + 2\sigma_{x,y} \frac{\partial \epsilon_{x,y}}{\partial y} + 2\sigma_{y,x} \frac{\partial \epsilon_{y,x}}{\partial x} + 2\sigma_{x,x} \frac{\partial \epsilon_{x,x}}{\partial x} + p_x \delta \gamma_x + p_y \delta \gamma_y + m_{x,x} \frac{\partial \epsilon_{x,x}}{\partial x} + m_{y,y} \frac{\partial \epsilon_{y,y}}{\partial y} + 2m_{x,y} \frac{\partial \epsilon_{x,y}}{\partial y} + 2m_{y,x} \frac{\partial \epsilon_{y,x}}{\partial x} + 3\tau_{x,x} \frac{\partial \gamma_{x,x}}{\partial x} + 3\tau_{y,y} \frac{\partial \gamma_{y,y}}{\partial y} + 3\tau_{x,y} \frac{\partial \gamma_{x,y}}{\partial y} + 3\tau_{y,x} \frac{\partial \gamma_{y,x}}{\partial x} + 6\tau_{x,y} \frac{\partial \gamma_{x,y}}{\partial y} dV.
\]

Thus, the principle of Hamilton for this higher-order plate theory has the form

\[
0 = \delta \int_0^T [K - U + W] dt
\]

\[
= \int_0^T \int \left[ H_{x,x} \frac{\partial \omega}{\partial x} + H_{y,y} \frac{\partial \omega}{\partial y} + H_{x,y} \frac{\partial \omega}{\partial y} + Y_{x,x} \frac{\partial \phi}{\partial x} + Y_{y,y} \frac{\partial \phi}{\partial y} + Y_{x,y} \frac{\partial \phi}{\partial y} + Y_{y,x} \frac{\partial \phi}{\partial x} + J_0 \frac{\partial \phi}{\partial x} + J_1 \frac{\partial \phi}{\partial t} \right] dxdydt
\]

\[
= \int_0^T \int \left[ -H_{x,x} n_x + H_{y,y} n_y - H_{x,y} n_y + J_1 \frac{\partial \phi}{\partial t} \right] dxdydt
\]

\[
+ \left( \frac{1}{2} H_{x,x,xx} n_x + H_{y,y,yy} n_y \right) \frac{\partial \phi}{\partial x} + J_0 \frac{\partial \phi}{\partial x} + J_1 \frac{\partial \phi}{\partial t} \right] dxdydt
\]

\[
+ \left( \frac{1}{2} H_{x,x,xx} n_x + H_{y,y,yy} n_y \right) \frac{\partial \phi}{\partial x} + J_0 \frac{\partial \phi}{\partial x} + J_1 \frac{\partial \phi}{\partial t} \right] dxdydt
\]
in which new parameters are shown in Appendix A. By Equation (25), the motion equation of Mindlin plate can be obtained as follows

\[
\begin{align*}
H_{xx,xx} + H_{yy,yy} + H_{xy,xy} - Y_{xz,x} - Y_{yz,y} + J_\delta \ddot{w} - q &= 0, \\
Y_{xx,xx} + Y_{yy,yy} + Y_{xy,xy} - Y_{xz,x} - Y_{yz,y} + J_\phi \ddot{\phi} &= 0, \\
Y_{xx,xx} + Y_{yy,yy} + Y_{xy,xy} - Y_{xz,x} - Y_{yz,y} + J_\phi \ddot{\phi} &= 0, \\
\end{align*}
\]

and

\[
\begin{align*}
\int_0^T \int_\partial \Omega \left[ \left( -Y_{xx,xx} n_x - Y_{yy,yy} n_y - \frac{1}{2} H_{xy,xy} n_x + Y_{xz,x} + Y_{yz,y} \right) \delta w \\ + (H_{xx,xx} n_x + \frac{1}{2} H_{xx,xx} n_y) \delta w_x + (H_{yy,yy} n_y + \frac{1}{2} H_{yy,yy} n_x) \delta w_y \\ + (-Y_{xx,xx} n_x - \frac{1}{2} Y_{xx,xx} n_y - \frac{1}{2} Y_{xx,xx} n_x - Y_{xy,xy} n_y + Y_{xz,x} + Y_{yz,y}) \delta \phi_x \\ + (-Y_{xx,xx} n_x - \frac{1}{2} Y_{xx,xx} n_y - \frac{1}{2} Y_{xx,xx} n_x - Y_{xy,xy} n_y + Y_{xz,x} + Y_{yz,y}) \delta \phi_y \\ + (Y_{xx,xx} n_x + \frac{1}{2} Y_{xx,xx} n_y) \delta \phi_{x,y} + (Y_{xx,xx} n_y + \frac{1}{2} Y_{xx,xx} n_x) \delta \phi_{y,x} \\ + (Y_{xx,xx} n_x + \frac{1}{2} Y_{xx,xx} n_y) \delta \phi_{x,y} + (\frac{1}{2} Y_{xx,xx} n_x + Y_{yy,yy} n_y) \delta \phi_{y,x} \right] ds dt = 0.
\end{align*}
\]

Equation (29) is based on the coordinate system \((x, y, z)\). This can be inconvenient for a general shape microplate. Therefore the same Cartesian coordinate system \((n, s, z)\) is introduced, where the unit normal vector \(e_n = n_x e_1 + n_y e_2\) and the unit tangent vector \(e_s = -n_x e_1 + n_y e_2\) on the plate boundary \(\partial \Omega\) are shown in Figure 2 [33]. The vectors \(\{e_1, e_2, e_3\}\) and \(\{e_n, e_s, e_t\}\) are the unit base vectors \((x, y, z)\) and \((n, s, z)\) respectively. The relations of the above-mentioned two Cartesian coordinate systems are given in Appendix B.
Using Equations (73)-(80) in Appendix B, Equation (29) is simplified as follows for the plate with the closed boundary

\[ \int_0^T \int_{\partial\Omega} K_i \delta w + K_2 \delta w, + K_3 \delta \phi_n + K_4 \delta \phi_n, + K_5 \delta \phi_{n,n} + K_6 \delta \phi_{n,n,} \, ds dt = 0, \quad (30) \]

in which the parameters \( K_i (i = 1,2,\cdots,6) \) are given in Appendix C.

From Equation (30), the boundary condition of Mindlin microplate based on the MSGET for arbitrary closed boundaries are obtained as follows

\[ \begin{align*}
K_1 &= 0 \text{ or } w = \tilde{w}, \\
K_2 &= 0 \text{ or } w, = \tilde{w}, \\
K_3 &= 0 \text{ or } \phi_n = \tilde{\phi}_n, \\
K_4 &= 0 \text{ or } \phi_n = \tilde{\phi}_n, \\
K_5 &= 0 \text{ or } \phi_{n,n} = \tilde{\phi}_{n,n}, \\
K_6 &= 0 \text{ or } \phi_{n,n,} = \tilde{\phi}_{n,n,}.
\end{align*} \quad (31-36) \]

Using Equations (58)-(72) in Appendix A and Equations (26)-(28), the equations of the motion are given as

\[ \frac{1}{60} A_2 \left[ (32 l_1^2 + 15 l_2^2) (w,xxxx + 2w,xxyy + w,yyyy) - (64 l_4^2 - 15 l_2^2) (\phi,xxxx + \phi,xxyy + \phi,yyyy + \phi,yyyy) \right] + k_s A_2 (\phi,sx - \phi,sy - w,xx - w,yy) + J_0 \ddot{w} - q = 0, \quad (37) \]

\[ \begin{align*}
A_{222} \left[ \frac{2}{5} (5 l_2^2 + 2 l_1^2) \phi ,xxxx + \frac{1}{12} (24 l_6^2 + 16 l_1^2 - 3 l_2^2) \phi ,xxyy + \frac{1}{60} (32 l_1^2 + 15 l_2^2) \phi ,yyyy \right. \\
&\left. + \frac{1}{60} (120 l_6^2 + 16 l_1^2 - 15 l_2^2) (\phi ,yyyy + \phi ,yyyy) \right] - (2 A_2 l_2^2 + \frac{32}{15} A_2 l_2^2 + \frac{4}{5} A_2 l_2^2 + A_{11}) \phi ,sx \\
&\left. - (\frac{4}{3} A_2 l_2^2 + A_2 l_2^2 + A_{222}) \phi ,sy + (2 A_2 l_2^2 + \frac{4}{5} A_2 l_2^2 - \frac{3}{4} A_2 l_2^2 + A_{11}) v + A_{222} \phi ,sy \right]
\end{align*} \]

\[ \begin{align*}
+ &\frac{1}{60} A_2 (64 l_1^2 - 15 l_2^2) (w,xxxx + w,xxyy) + k_s A_2 (\phi_s - w_s) + J_1 \ddot{\phi}_s = 0, \quad (38) \]

\[ \begin{align*}
A_{222} \left[ \frac{1}{60} (32 l_1^2 + 15 l_2^2) \phi ,xxxx + \frac{1}{12} (24 l_6^2 + 16 l_1^2 + 3 l_2^2) \phi ,xxyy + \frac{2}{5} (5 l_1^2 + 2 l_1^2) \phi ,yyyy \right. \\
&\left. + \frac{1}{60} (120 l_6^2 + 16 l_1^2 - 15 l_2^2) (\phi ,yyyy + \phi ,yyyy) \right] - (2 A_2 l_2^2 + \frac{32}{15} A_2 l_2^2 + \frac{4}{5} A_2 l_2^2 + A_{11}) \phi ,sy \\
&\left. - (\frac{4}{3} A_2 l_2^2 + A_2 l_2^2 + A_{222}) \phi ,sy + (2 A_2 l_2^2 + \frac{4}{5} A_2 l_2^2 - \frac{3}{4} A_2 l_2^2 + A_{11}) v + A_{222} \phi ,sy \right]
\end{align*} \]

\[ \begin{align*}
+ &\frac{1}{60} A_2 (64 l_1^2 - 15 l_2^2) (w,yyyy + w,xxyy) + k_s A_2 (\phi_s - w_s) + J_1 \ddot{\phi}_y = 0. \quad (39) \]

In particular, if \( l_0 = 0 \) and \( l_1 = 0 \), the motion equations and boundary conditions of MSGMP model can be degenerated into the those of the MCSET. In this case, if two parameters \( E \) and \( \rho \) are constant, Equations (37)-(39) were reduced to Equations (32)-(34) in [33]. Furthermore, if \( l_0 = l_1 = l_2 = 0 \), the MSGMP model can be degenerated to classical continuum case.
3. Analytical solution of FG MSGMP

In the case of simply supported rectangular MSGMP model, the boundary conditions can be given as

\[ w = 0, \quad \phi_x = 0, \quad K_2 = 0, \quad K_3 = 0, \quad K_4 = 0, \quad K_6 = 0, \]

(40)

for four boundaries. For the edges \( x = 0 \) and \( x = a \), \( n_y = 0 \) and \( n_y = -1 \) (on \( x = 0 \)) or \( n_y = 1 \) (on \( x = a \)), using Equations (B.1)-(B.8) in Appendix B and (C.1)-(C.6) in Appendix C, Equation (40) gives

\[ w(0, y) = w(a, y) = 0, \]

(41)

\[ \phi_y(0, y) = \phi_y(a, y) = 0, \]

(42)

\[(32l_1^2 + 15l_2^2)w_{xy} - (8l_1^2 + 15l_2^2)w_{yy} - (64l_1^2 - 15l_2^2)\phi_x, + (16l_1^2 - 15l_2^2)\phi_{yy} = 0, \]

(43)

\[ A_2(6l_1^2 - 15l_2^2)w_{xy} - A_2(64l_1^2 - 15l_2^2)w_{yy} - 24A_{222}(5l_0^2 + 2l_1^2)\phi_{x,xy} - A_{222}(120l_0^2 - 48l_1^2)\phi_{yy} \]

+ (120A_{20}^2 + 128A_{11}^2 + 15A_{22}^2 + 60A_{11}^2)\phi_{x,x} + (120A_{20}^2 - 32A_{11}^2 - 15A_{22}^2 + 60A_{11}^2)\phi_{y,y} - A_{222}(120l_0^2 + 16l_1^2 - 15l_2^2)\phi_{x,xy} = 0, \]

(44)

\[(5l_0^2 + 2l_1^2)\phi_{x,xy} - (5l_0^2 - 2l_1^2)\phi_{y,xy} = 0, \]

(45)

\[(64l_1^2 - 15l_2^2)\phi_{y,xy} + (32l_1^2 + 15l_2^2)\phi_{x,xx} - 24l_1^2\phi_{x,yy} = 0, \]

(46)

at edges \( x = 0 \) and \( x = a \).

In the same way, for the edges \( y = 0 \) and \( y = b \), \( n_x = 0 \) and \( n_x = -1 \) (on \( y = 0 \)) or \( n_x = 1 \) (on \( y = b \)), Equation (40) gives

\[ w(x, 0) = w(x, b) = 0, \]

(47)

\[ \phi_x(x, 0) = \phi_x(x, b) = 0, \]

(48)

\[(32l_1^2 + 15l_2^2)w_{xy} - (8l_1^2 + 15l_2^2)w_{xx} + (16l_1^2 - 15l_2^2)\phi_x, - (64l_1^2 - 15l_2^2)\phi_{xx} = 0, \]

(49)

\[ A_2(6l_1^2 - 15l_2^2)w_{xy} - A_2(64l_1^2 - 15l_2^2)w_{xx} - A_{222}(120l_0^2 - 48l_1^2)\phi_{x,xx} - 24A_{222}(5l_0^2 + 2l_1^2)\phi_{y,yy} \]

+ (120A_{20}^2 - 32A_{11}^2 - 15A_{22}^2 + 60A_{11}^2)\phi_{x,xx} + (120A_{20}^2 + 128A_{11}^2 + 15A_{22}^2 + 60A_{11}^2)\phi_{y,yy} - A_{222}(120l_0^2 + 16l_1^2 - 15l_2^2)\phi_{x,xx} = 0, \]

(50)

\[(5l_0^2 + 2l_1^2)\phi_{x,xx} - (5l_0^2 - 2l_1^2)\phi_{y,xx} = 0, \]

(51)

\[(64l_1^2 - 15l_2^2)\phi_{y,xx} + (32l_1^2 + 15l_2^2)\phi_{x,xx} - 24l_1^2\phi_{xx,yy} = 0, \]

(52)

at \( y = 0 \) and \( y = b \).

The Fourier series solutions can be given as

\[ w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin(Ax) \sin(By) e^{i\omega t}, \]

(53)

\[ \phi_x(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} X_{mn} \cos(Ax) \sin(By) e^{i\omega t}, \]

(54)

\[ \phi_y(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn} \sin(Ax) \cos(By) e^{i\omega t}, \]

(55)

where \( i = \sqrt{-1} \), \( A = m\pi/a \), \( B = n\pi/b \), \( \omega \) is the frequency. Obviously, these displacements \((\phi_x, \phi_y, w)\) can satisfy the boundary conditions in Equations (41)-(52) of the simply supported rectangular Mindlin plate. Substituting Equations (53)-(55) into Equations (37)-(39) yields
\[
\begin{bmatrix}
    d_{11} & d_{12} & d_{13} \\
    d_{21} & d_{22} & d_{23} \\
    d_{31} & d_{32} & d_{33}
\end{bmatrix} - \omega^2 \begin{bmatrix}
    J_0 & 0 & 0 \\
    0 & J_1 & 0 \\
    0 & 0 & J_1
\end{bmatrix} \begin{bmatrix}
    W_{mn} \\
    X_{mn} \\
    Y_{mn}
\end{bmatrix} = \begin{bmatrix}
    0 \\
    0 \\
    0
\end{bmatrix},
\]

in which the values of \(d_{ij}\) \((i, j = 1, 2, 3)\) are shown in Appendix D. Natural frequencies of the Mindlin plate model can be gotten from Equation (56).

4. Numerical Results and Discussions

The FGM Mindlin micro-plate is made of alumina (material 1) and aluminum (material 2). The Young’s modulus \(E_1=380\) GPa and \(E_2=70\) GPa, the material density \(\rho_1 = 3800\) kg/m\(^3\) and \(\rho_2 = 2702\) kg/m\(^3\) are used for alumina and aluminum respectively. The Poisson’s ratio \(v = 0.3\) is used for both materials. The thickness \(h = 17.6\) \(\mu\)m and the correction factor \(k_\ell = 5/6\) are taken for the plate [28]. The dimensionless frequency is expressed as \(\tilde{\omega} = \omega a^2 / \sqrt{\rho_1 E_1} h\). For simplicity, we take \(l_1 = l_0 = l_2\). In order to verify results, non-dimensional frequencies of the simply supported FGM plate \(p=0\) are shown in Table 1. In this case, the FGM plate model is degenerated to homogeneous plate model. Compared with the analytical solution in reference [16], the frequency of simply supported Kirchhoff MSGT plate (KMSGT) is expressed

\[
\omega_{nm}^2 = \frac{C_1 P_1 + C_2 P_2}{\rho h},
\]

where, \(P_1 = \frac{Eh}{2(1+v)} \left(\frac{l_0^2}{6} + \frac{l_2^2}{15}\right)\), \(P_2 = \frac{Eh}{12(1-v^2)} + \frac{8Eh}{2(1+v)} (2l_0^2 + \frac{8}{15} l_1^2 + l_2^2)\), \(C_1 = \left((\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2\right)^2\) and \(C_2 = \left((\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2\right)\). Two results have good agreement when the value of \(a/h\) is large (\(a \geq 20h\)). However, the discrepancies are obvious when the thickness of plate increases. The dimensionless frequencies of the FGM plate based on the MSGET are shown in Table 3. Obviously, the dimensionless frequencies rise with the increase of \(l/h\) and the decrease of the power law index \(p\) monotonically.

Table 1. Dimensionless frequencies of simply supported homogeneous Kirchhoff and Mindlin plates based on the MSGT

| \(l/h\) | \(a/h=5\) | \(a/h=20\) | \(a/h=50\) |
|-------|----------|----------|----------|
|       | KMSGT    | MSGMP    | KMSGT    | MSGMP    | KMSGT    | MSGMP    |
| 0     | 5.9734   | 5.2802   | 5.9734   | 5.9199   | 5.9734   | 5.9647   |
| 0.2   | 7.6135   | 6.4173   | 7.5452   | 7.4467   | 7.5414   | 7.5253   |
| 0.4   | 11.1723  | 8.4062   | 10.9854  | 10.7110  | 10.9748  | 10.9292  |
| 0.6   | 15.3702  | 10.2978  | 15.0640  | 14.4064  | 15.0464  | 14.9340  |
| 0.8   | 19.8049  | 11.9730  | 19.3822  | 18.0722  | 19.3582  | 19.1242  |
| 1     | 24.3474  | 13.5900  | 23.8099  | 21.5433  | 23.7795  | 23.3542  |

Figure 3 presents the impact of the non-dimensional material length scale parameter \(l/h\) on the non-dimensional frequencies of the FGM plate with various power law index \(p\). The non-dimensional frequencies of the MSGMP models increase with the rise of \(l/h\) monotonically. The dimensionless frequencies increase more obviously for the lower values of index \(p\) and the greater \(l/h\). The dimensionless frequencies of the MSGMP model are greater than those of CMP at all time. The
dimensionless frequencies of the MSGMP and CMP models have obviously large differences as the value of $l/h$ is large ($h < 8l$), but it can be ignored as the value of $l/h$ is small ($h > 14l$). This demonstrates that the scale effect must be considered as the thickness of microplate is near to the material length scale parameter.

**Table 2.** Non-dimensional frequencies of simply supported FGM Reddy and Mindlin plate based on the MCSET

| $a/h$ | $l/h$ | $p=0$      | $p=5$      | $p=10$     |
|-------|-------|------------|------------|------------|
|       |       | MSGMP [27] | MSGMP [27] | MSGMP      |
| 5     | 0     | 5.2813     | 3.3938     | 3.2514     |
|       | 0.2   | 5.7699     | 4.3862     | 4.3200     |
|       | 0.4   | 7.0330     | 5.7137     | 5.3710     |
|       | 0.6   | 8.7389     | 6.9796     | 6.8795     |
|       | 0.8   | 10.6766    | 8.3338     | 8.6795     |
|       | 1     | 12.7408    | 9.7409     | 9.7620     |
| 10    | 0     | 5.7694     | 3.7682     | 3.6368     |
|       | 0.2   | 6.2537     | 4.0876     | 3.9162     |
|       | 0.4   | 7.5210     | 4.9169     | 4.6464     |
|       | 0.6   | 9.2543     | 6.0447     | 5.6487     |
|       | 0.8   | 11.2396    | 7.3338     | 6.8030     |
|       | 1     | 13.3651    | 8.7135     | 8.0448     |
| 20    | 0     | 5.9199     | 3.8884     | 3.7622     |
|       | 0.2   | 6.4027     | 4.2005     | 4.0323     |
|       | 0.4   | 7.6708     | 5.0199     | 4.7488     |
|       | 0.6   | 9.4116     | 6.1457     | 5.7453     |
|       | 0.8   | 11.4108    | 7.4397     | 6.9013     |
|       | 1     | 13.5545    | 8.8286     | 8.1494     |
| 100   | 0     | 5.9712     | 3.9299     | 3.8058     |
|       | 0.2   | 6.4535     | 4.2394     | 4.0725     |
|       | 0.4   | 7.7217     | 5.0552     | 4.7840     |
|       | 0.6   | 9.4651     | 6.1800     | 5.7782     |
|       | 0.8   | 11.4689    | 7.4755     | 6.9345     |
|       | 1     | 13.6186    | 8.8673     | 8.1846     |

The influence of the power law index $p$ on the non-dimensional frequencies of the FGM plate with different $l/h$ is shown in Figure 4. The non-dimensional frequencies of the FGM plate decrease with the power law index $p$ monotonically. The non-dimensional frequencies decrease more obvious for the smaller power law index $p$. However, the power law index $p$ has little effect on dimensionless frequencies after $p=10$. The differences between the dimensionless frequencies of MSGMP and CMP models are obvious for the greater $l/h$. 
Figure 3. The relation of the non-dimensional material length scale parameter $l/h$ and the non-dimensional frequencies of the FGM plate with different power law index $p$ ($a=b=10h$).

Table 3. Non-dimensional frequencies of a simply supported FGM MSGMP model

| $a/h$ | $l/h$ | power law index $p$ | $p=0$ | $p=5$ | $p=10$ | $p=15$ |
|-------|-------|---------------------|-------|-------|---------|--------|
| 5     | 0     | 5.2802              | 3.4500| 3.3099| 3.2101  |
|       | 0.2   | 6.4173              | 4.1770| 3.9269| 3.7935  |
|       | 0.4   | 8.4362              | 5.4782| 5.0601| 4.8701  |
|       | 0.6   | 10.2978             | 6.6845| 6.1328| 5.8935  |
|       | 0.8   | 11.9770             | 7.7748| 7.1140| 6.8321  |
|       | 1     | 13.5900             | 8.8224| 8.0628| 7.7410  |
| 10    | 0     | 5.7693              | 3.7882| 3.6580| 3.5509  |
|       | 0.2   | 7.1896              | 4.6980| 4.4416| 4.2942  |
|       | 0.4   | 10.0505             | 6.5406| 6.0632| 5.8390  |
|       | 0.6   | 12.9861             | 8.4384| 7.7574| 7.4574  |
|       | 0.8   | 15.6099             | 10.1383| 9.2860| 8.9196  |
|       | 1     | 17.8996             | 11.6234| 10.6271| 10.2036 |
| 20    | 0     | 5.9199              | 3.8939| 3.7681| 3.6588  |
|       | 0.2   | 7.4467              | 4.8740| 4.6172| 4.4651  |
|       | 0.4   | 10.7110             | 6.9796| 6.4805| 6.2421  |
|       | 0.6   | 14.4064             | 9.3704| 8.6248| 8.2926  |
|       | 0.8   | 18.0722             | 11.7454| 10.7680| 10.3445 |
|       | 1     | 21.5453             | 13.9973| 12.8062| 12.2971 |

Figure 5 shows the effects of vibrational mode on the non-dimensional frequencies of the FGM plate with different $l/h$ when the power law index $p$ is 5. The dimensionless frequencies of the MSGMP model are greater than those of CMP model all the time. Their differences increase with the order of mode monotonically. More obviously differences can be seen, when the $l/h$ increases. Thus the dimensionless frequencies are sensitive to the material length scale parameter effects in the higher modes.
5. Conclusions

The MSGMP model, containing three material length scale parameters, has the ability to describe the scale effect. The dynamic equations and corresponding general boundary conditions are obtained with the help of Hamilton’s principle. Analytical solution for the vibration of a simply supported Mindlin plate is obtained. The numerical results show that dimensionless frequencies of the FGM MSGMP model are greater than those of the FGM CMP model. Furthermore, it is found that the material length scale parameter, the power law index and vibration mode have remarkable effects on the dimensionless frequencies.

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Appendix A.

The new parameters in Equation (25) are expressed as

\[ H_{xx} = \int_{\frac{h}{2}}^{\frac{h}{2}} \left( \frac{4}{5} \tau_{xxz} - \frac{1}{5} \tau_{zzz} - \frac{1}{5} \tau_{yyz} - \frac{1}{2} m_{xy} \right) dz \]

\[ = \frac{1}{60} A_1[(16l_i^2 - 15l_i^2)\phi_{x,y} - (64l_i^2 - 15l_i^2)\phi_{x,z} + (32l_i^2 + 15l_i^2)w_{xx} - (8l_i^2 + 15l_i^2)w_{yy}] \]

\[ H_{yy} = \int_{\frac{h}{2}}^{\frac{h}{2}} \left( \frac{4}{5} \tau_{yyz} - \frac{1}{5} \tau_{zzz} - \frac{1}{5} \tau_{xxz} + \frac{1}{2} m_{xy} \right) dz \]

\[ = \frac{1}{60} A_2[(16l_i^2 - 15l_i^2)\phi_{x,x} - (64l_i^2 - 15l_i^2)\phi_{x,y} + (32l_i^2 + 15l_i^2)w_{yy} - (8l_i^2 + 15l_i^2)w_{xx}] \]

\[ H_{xy} = \int_{\frac{h}{2}}^{\frac{h}{2}} \left( 2\tau_{xy} + \frac{1}{2} m_{xx} - \frac{1}{2} m_{yy} \right) dz \]

\[ = \frac{1}{6} A_2[(8l_i^2 + 6l_i^2)w_{xy} - (8l_i^2 - 3l_i^2)\phi_{x,y} - (8l_i^2 - 3l_i^2)\phi_{y,x}] \]

\[ Y_{xxx} = \int_{\frac{h}{2}}^{\frac{h}{2}} \left( \frac{3}{5} \tau_{xxx} + \frac{3}{5} \tau_{xxz} - \frac{2}{5} \tau_{xxx} - p_x \right) zdz \]

\[ = \frac{5}{2} A_{222}[(5l_i^2 + 2l_i^2)\phi_{x,xx} + (5l_i^2 - 2l_i^2)\phi_{x,xy} - l_i^2\phi_{x,yy}] \]

\[ Y_{xyy} = \int_{\frac{h}{2}}^{\frac{h}{2}} \left( \frac{2}{5} \tau_{yy} + \frac{8}{5} \tau_{xy} + \frac{2}{5} \tau_{yyz} + \frac{1}{2} m_{xy} - p_y \right) zdz \]

\[ = \frac{1}{60} A_{222}[(120l_i^2 + 128l_i^2 + 15l_i^2)\phi_{x,yy} + 24(5l_i^2 - 2l_i^2)\phi_{y,yy} + (64l_i^2 - 15l_i^2)\phi_{y,xx}] \]

\[ Y_{xxy} = \int_{\frac{h}{2}}^{\frac{h}{2}} \left( \frac{1}{5} \tau_{xxz} - \frac{4}{5} \tau_{xy} + \frac{1}{5} \tau_{xxz} + \frac{1}{2} m_{xy} \right) zdz \]

\[ = \frac{1}{60} A_{222}[(32l_i^2 + 15l_i^2)\phi_{x,yy} + (64l_i^2 - 15l_i^2)\phi_{y,xy} - 24l_i^2\phi_{y,yy}] \]

\[ Y_{yyx} = \int_{\frac{h}{2}}^{\frac{h}{2}} \left( \frac{1}{5} \tau_{yyz} - \frac{4}{5} \tau_{xy} + \frac{1}{5} \tau_{yyz} - \frac{1}{2} m_{xy} \right) zdz \]

\[ = \frac{1}{60} A_{222}[(64l_i^2 - 15l_i^2)\phi_{x,xy} + (32l_i^2 + 15l_i^2)\phi_{x,yx} - 24l_i^2\phi_{y,yy}] \]
\[ Y_{yz} = \int_0^h \left(\frac{2}{5} \tau_{zzz} - \frac{8}{5} \tau_{zxy} + \frac{2}{5} \tau_{xyy} - \frac{1}{2} m_{yz} - p_z\right) dz \]

\[ = \frac{1}{60} A_{222} \left[ 24(5l_0^2 - 2l_1^2)\phi_{x,xx} + (64l_1^2 - 15l_2^2)\phi_{x,yy} + (120l_0^2 + 128l_1^2 + 15l_2^2)\phi_{y,yy}\right], \quad (A.8) \]

\[ Y_{yy} = \int_0^h \left(\frac{3}{5} \tau_{xy} + \frac{3}{5} \tau_{zxx} - \frac{2}{5} \tau_{yy} - p_y\right) dz \]

\[ = \frac{2}{5} A_{222} \left[ (5l_0^2 - 2l_1^2)\phi_{x,xy} + (5l_0^2 + 2l_1^2)\phi_{y,xy} - l_1^2 \phi_{y,xx}\right], \quad (A.9) \]

\[ Y_{xx} = \int_0^h \left(\frac{2}{5} \tau_{xzz} - \frac{8}{5} \tau_{xx} + \frac{2}{5} \tau_{yy} - \frac{1}{2} m_{xy} - \sigma_{xx} z - p_x\right) dz \]

\[ = \frac{1}{60} \left[ (60A_{111} + 120A_{220}^2 + 128A_{211}^2 + 15A_2^2)\phi_{x,xx} + (60A_{111}^2 + 120A_{220}^2 - 32A_2^2 - 15A_2^2)\phi_{y,yy}\right] \]

\[- A_2 (64l_1^2 - 15l_2^2)w_{xx} + A_2 (16l_1^2 - 15l_2^2)w_{yy}\], \quad (A.10) \]

\[ Y_{yx} = \int_0^h \left(\frac{1}{2} m_x - \frac{1}{2} m_y - 2 \tau_{xyz} - \sigma_{xy} z\right) dz \]

\[ = \frac{1}{3} \left( 4A_1 l_1^2 + 3A_1 l_2^2 + 3A_{222} \phi_{x,xy} \right) + \frac{1}{6} \left( 8A_1 l_1^2 - 3A_1 l_2^2 + 6A_{222} \phi_{y,xy} \right) - \frac{1}{6} A_2 (8l_0^2 - 3l_2^2)w_{xy}, \quad (A.11) \]

\[ Y_{yx} = \int_0^h \left(\frac{1}{2} m_{xx} - \frac{1}{2} m_{yy} - 2 \tau_{xyy} - \sigma_{xy} z\right) dz \]

\[ = \frac{1}{3} \left( 4A_1 l_1^2 + 3A_1 l_2^2 + 3A_{222} \phi_{x,xy} \right) + \frac{1}{6} \left( 8A_1 l_1^2 - 3A_1 l_2^2 + 6A_{222} \phi_{y,xy} \right) - \frac{1}{6} A_2 (8l_0^2 - 3l_2^2)w_{xy}, \quad (A.12) \]

\[ Y_{yy} = \int_0^h \left(\frac{2}{5} \tau_{zxx} - \frac{8}{5} \tau_{xx} + \frac{1}{2} m_{xy} - \sigma_{yy} z - p_x\right) dz \]

\[ = \frac{1}{60} \left[ (60A_{111}^2 + 120A_{220}^2 - 32A_2^2 - 15A_2^2)\phi_{x,xx} + (60A_{111}^2 + 120A_{220}^2 + 128A_{211}^2 + 15A_2^2)\phi_{y,yy}\right] \]

\[ + A_2 (16l_1^2 - 15l_2^2)w_{xx} - A_2 (64l_1^2 - 15l_2^2)w_{yy}\], \quad (A.13) \]

\[ Y_{zz} = \int_0^h \left(\frac{3}{2} \sigma_{zz} dz = -k_s A_2 [\phi_z - w_z]\right), \quad (A.14) \]

\[ Y_{zz} = \int_0^h \left(\frac{3}{2} \sigma_{zz} dz = -k_s A_2 [\phi_y - w_y]\right), \quad (A.15) \]

where \( A_{111} = \int_{-h/2}^{h/2} \frac{E(z)(z-z_0)^2}{1-v^2} dz, \quad (A_2, A_{222}) = \int_{-h/2}^{h/2} \frac{E(z)}{2(1+v)}(1,(z-z_0)^2) dz \) and \( k_s \) is correction factor.

**Appendix B.**

The transformation relations between two Cartesian coordinate systems are shown as

\[ C = \begin{pmatrix} n_x & -n_y \\ n_y & n_x \end{pmatrix}, \quad (B.1) \]
\[(w'_x, w'_y)^T = C(w_n, w'_n)^T, \] (B.2)

\[(\phi'_x, \phi'_y)^T = C(\phi_n, \phi'_n)^T, \] (B.3)

\[
\begin{pmatrix}
  w_{xx} & w_{xy} \\
  w_{yx} & w_{yy}
\end{pmatrix}
= C
\begin{pmatrix}
  w_{mn} & w_{ms} \\
  w_{ns} & w_{ss}
\end{pmatrix} C^T,
\] (B.4)

\[
\begin{pmatrix}
  \phi_{x,xx} \\
  \phi_{y,xy} \\
  \phi_{x,yx} \\
  \phi_{y,yy}
\end{pmatrix}
= C
\begin{pmatrix}
  \phi_{n,nn} & \phi_{n,ns} \\
  \phi_{n,ss} & \phi_{s,ss}
\end{pmatrix} C^T,
\] (B.5)

\[
\begin{pmatrix}
  n_x^3 - 3n_x^2n_y & n_y^2 - n_x^2 & 3n_y^2 - n_x^2 & n_x^3 - 3n_x^2n_y & -n_y^3 \\
  2n_x^2n_y^2 & n_y^3 - 3n_xn_y^2 & 3n_x^2n_y^2 & 2n_x^2n_y^2 & -n_x^3 \\
  n_x^3 - 3n_xn_y^2 & 3n_y^2 - n_x^2 & n_x^3 - 3n_xn_y^2 & 3n_y^2 - n_x^2 & n_x^3 - 3n_xn_y^2
\end{pmatrix}
\] (B.6)

\[
\begin{pmatrix}
  n_x^3 & 2n_x^2n_y & n_y^2 & n_x^3 & 2n_x^2n_y \\
  n_x^3 & 2n_x^2n_y & n_y^2 & n_x^3 & 2n_x^2n_y \\
  n_x^3 & 2n_x^2n_y & n_y^2 & n_x^3 & 2n_x^2n_y
\end{pmatrix}
\] (B.7)

\[
\begin{pmatrix}
  \phi_{x,xx} \\
  \phi_{y,xy} \\
  \phi_{x,yx} \\
  \phi_{y,yy}
\end{pmatrix}
= C
\begin{pmatrix}
  \phi_{n,nn} & \phi_{n,ns} \\
  \phi_{n,ss} & \phi_{s,ss}
\end{pmatrix} C^T.
\] (B.8)

**Appendix C.**

The parameters \(K_i (i=1,2,\cdots,6)\) in Equation (30) are given as follows.

\[
K_1 = k_s A_2 w_n - \frac{1}{60} A_2 (32l_i^2 + 15l_i^2) w_{mn} - \frac{1}{60} A_2 (72l_i^2 + 45l_i^2) w_{ns} - k_s A_2 \phi_n
\]

\[
+ \frac{1}{60} A_2 (64l_i^2 - 15l_i^2) \phi_{n,nn} + \frac{1}{6} A_2 (8l_i^2 - 3l_i^2) \phi_{n,ss} + \frac{1}{60} A_2 (64l_i^2 - 15l_i^2) \phi_{s,ss}.
\] (C.1)

\[
K_2 = \frac{1}{60} A_2 [(32l_i^2 + 15l_i^2) w_{mn} - (72l_i^2 + 15l_i^2) w_{ns} + (64l_i^2 - 15l_i^2) \phi_{n,nn} + (16l_i^2 - 15l_i^2) \phi_{s,ss}].
\] (C.2)
\[
K_1 = \frac{1}{60} A_2 (16l_i^2 - 15l_1^2) w_{mm} - \frac{1}{60} A_2 (64l_i^3 - 15l_1^3) w_{mm} + \left[ 2 A_2 l_1^3 + \frac{32}{15} A_1 l_1^3 + \frac{1}{4} A_2 l_2^3 + A_1 l_1^3 + (n_x^2 + 2n_y^2 + n_z^2) \right] \\
+ A_{222} n_x^2 n_z^2 \phi_{n,m} + \frac{n_x n_y}{v A_{11}} (2 n_x^2 - n_y^2) (v A_{11} - A_{11}) + 2 A_{222} (\phi_{n,m} + \phi_{s,m}) - \frac{2}{5} A_{222} (5l_0^2 + 2l_1^2) \phi_{n,mm}, \\
\]

\[
K_2 = -\frac{1}{6} A_2 (8l_i^3 - 3l_1^3) w_{mm} + n_x n_y (n_x^2 - n_y^2) (v A_{11} - A_{11}) + 2 A_{222} \phi_{n,n} \\
+ \frac{A_2 l_1^3}{2} + A_{222} (n_x^2 - n_y^2) + 2 A_{11} n_x^2 n_y^2 (1 - v) \phi_{s,m}, \\
+ \frac{4}{3} A_2 l_1^3 + A_{222} (n_x^2 - n_y^2) + 2 A_{11} n_x^2 n_y^2 (1 - v) \phi_{s,m} - n_x n_y (n_x^2 - n_y^2) (v A_{11} - A_{11}) + 2 A_{222} \phi_{s,s} \\
- \frac{4}{6} A_{222} (120l_0^3 + 16l_i^3 - 15l_1^3) \phi_{n,mm} - \frac{1}{60} A_{222} (64l_i^2 - 15l_1^2) \phi_{n,ss}, \\
- \frac{1}{60} A_{222} (32l_i^2 + 15l_1^2) \phi_{n,nn} - \frac{1}{60} A_{222} (120l_0^2 + 104l_i^2 + 15l_1^2) \phi_{n,ss}, \\
K_5 = \frac{2}{5} A_{222} (5l_0^2 + 2l_1^2) \phi_{n,n} - \frac{2}{5} A_{222} l_2^2 \phi_{n,ss} + \frac{2}{5} A_{222} (5l_0^2 - 2l_1^2) \phi_{n,m}, \\
K_6 = \frac{1}{60} A_{222} (64l_i^2 - 15l_1^2) \phi_{n,n} + \frac{1}{60} A_{222} (32l_i^2 + 15l_1^2) \phi_{n,ss} - \frac{2}{5} A_{222} l_2^2 \phi_{s,s}. \\
\]

**Appendix D.**

The parameters \(d_{ij}(i,j = 1,2,3)\) in Equation (56) are given by:

\[
d_{11} = A_2 (A^2 + B^2) \left[ \frac{1}{60} (A^2 + B^2)(32l_i^2 + 15l_1^2) + \kappa_1 \right], \\
d_{12} = -A_2 A \left[ \frac{1}{60} (A^2 + B^2)(64l_i^2 - 15l_1^2) + \kappa_1 \right], \\
d_{13} = -A_2 B \left[ \frac{1}{60} (A^2 + B^2)(64l_i^2 - 15l_1^2) + \kappa_1 \right], \\
d_{22} = \frac{1}{15} (8A_{222} B^4 + 20A_{222} A^3 B^2 + 12A_{222} A^4 + 20A_2 B^2 + 32A_2 A^2) l_i^2 + A_{11} A^2 + A_{222} B^2 + \kappa_1 A_2 \\
+ 2(A_{222} A^2 B^2 + A_{222} A^4 + A_2 A^2) l_i^2 + \frac{1}{4} (A_{222} B^4 + A_{222} A^2 B^2 + 4A_2 B^2 + A_2 A)^2 l_i^2, \\
d_{23} = \frac{1}{15} AB [4A_{222} (A^2 + B^2) + 12A_2] l_i^2 - \frac{1}{4} AB [A_{222} (A^2 + B^2) + 3A_2] l_i^2 \\
+ 2AB [A_{222} (A^2 + B^2) + A_2] l_i^2 + A_{11} vAB + A_{222} AB, \\
d_{33} = A_{11} B^2 + A_{222} A^2 + \kappa_1 A_2 + \left[ 12A_{222} B^4 + 20A_{222} A^3 B^2 + 8A_{222} A^4 + 32A_2 B^2 + 20A_2 A^2 \right] l_i^2 \\
+ \frac{1}{4} (A_{222} A^2 B^2 + A_{222} A^4 + A_2 B^2 + 4A_2 A^2) l_i^2 + 2(A_{222} B^4 + A_{222} A^2 B^2 + A_2 B^2) l_i^2, \\
\]
\[ d_{21} = d_{12}, \quad d_{31} = d_{13}, \quad d_{32} = d_{23}. \]  

(D.7)