On 1-skeleton of the polytope of pyramidal tours with step-backs

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Abstract. Pyramidal tours with step-backs are Hamiltonian tours of a special kind: the salesperson starts in city 1, then visits some cities in ascending order, reaches city \( n \), and returns to city 1 visiting the remaining cities in descending order. However, in the ascending and descending direction, the order of neighboring cities can be inverted (a step-back). It is known that on pyramidal tours with step-backs the traveling salesperson problem can be solved by dynamic programming in polynomial time.

We define the polytope of pyramidal tours with step-backs \( PSB(n) \) as the convex hull of the characteristic vectors of all possible pyramidal tours with step-backs in a complete directed graph. The 1-skeleton of \( PSB(n) \) is the graph whose vertex set is the vertex set of the polytope, and the edge set is the set of geometric edges or one-dimensional faces of the polytope. We present a linear-time algorithm to verify vertex adjacencies in 1-skeleton of the polytope \( PSB(n) \) and estimate the diameter and the clique number of 1-skeleton: the diameter is bounded above by 4 and the clique number grows quadratically in the parameter \( n \).

Keywords: Pyramidal tour with step-backs · 1-skeleton · Vertex adjacency · Graph diameter · Clique number · Pyramidal encoding.

1 Introduction

The 1-skeleton of a polytope \( P \) is the graph whose vertex set is the vertex set of \( P \) and edge set is the set of geometric edges or one-dimensional faces of \( P \). In this paper, we consider 3 characteristics of 1-skeleton: vertex adjacency, graph diameter, and clique number.

Two vertices of a graph \( G \) are called adjacent iff they share a common edge. Vertex adjacency in 1-skeleton is of interest as it can be directly applied to develop simplex-like combinatorial optimization algorithms that move from one feasible solution to another along the edges of the 1-skeleton. This class includes, for example, the blossom algorithm by Edmonds for constructing maximum matchings [15], set partitioning algorithm by Balas and Padberg [4], Balinski’s algorithm for the assignment problem [5], Ikura and Nemhauser’s algorithm for the set packing problem [20], etc.

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The diameter of a graph $G$ is the maximum edge distance between any pair of vertices. The study of 1-skeleton’s diameter is motivated by its relationship to the simplex-method and similar edge-following algorithms since the diameter serves as a lower bound for the number of iterations of such algorithms (see [14,19]), as well as the famous Hirsch conjecture [14,25].

The clique number of a graph $G$, denoted by $\omega(G)$, is the number of vertices in a maximum clique of $G$. It is known that the clique number of 1-skeleton is a lower bound for computational complexity in a class of direct-type algorithms based on linear comparisons [7,8]. Besides, this characteristic is polynomial for known polynomially solvable problems and is superpolynomial for intractable problems (see, for example, [6,10,27]).

### 2 Traveling salesperson polytope

We consider an asymmetric traveling salesperson problem: given a complete weighted digraph $K_n = (V, E)$ (whose vertices are called cities), it is required to find a Hamiltonian tour of minimum weight [17]. With each Hamiltonian tour $x$ in $K_n$ we associate a characteristic vector $v(x) \in \mathbb{R}^E$ by the following rule:

$$v(x)_e = \begin{cases} 1, & \text{if an edge } e \in E \text{ is contained in the tour } x, \\ 0, & \text{otherwise.} \end{cases}$$

An example of constructing a characteristic vector $v(x)$ for a Hamiltonian tour $x$ is shown in Fig. 1.

The polytope

$$\text{ATSP}(n) = \text{conv}\{v(x) \mid x \text{ is a Hamiltonian tour in } K_n\}$$

is called the asymmetric traveling salesperson polytope.
The symmetric traveling salesperson polytope $TSP(n)$ is defined similarly as the convex hull of characteristic vectors of all possible Hamiltonian cycles in the complete undirected graph $K_n$.

The traveling salesperson polytope was introduced by Dantzig, Fulkerson, and Johnson in their classic work on solving the traveling salesperson problem for 49 US cities by integer linear programming [13]. State-of-the-art exact algorithms for the traveling salesperson problem are based on a partial description of the facets of the traveling salesperson polytope and the branch and cut method for integer linear programming [2].

The 1-skeleton of the traveling salesperson polytope has long been the object of close attention in the field of polyhedral combinatorics. The classic result by Papadimitriou [23] states that the question of whether two vertices of the ATSP($n$) (or TSP($n$)) are not adjacent is NP-complete. It is known that the graph diameter of 1-skeleton equals 2 for ATSP($n$) [22] and is at most 4 for TSP($n$) [24]. An open conjecture by Grötzsch and Padberg states that the diameter is 2 for both polytopes [19]. As for the clique number of ATSP($n$) (and TSP($n$)), Bondarenko proved that it is superpolynomial in the parameter $n$ [7].

Note that, historically, the traveling salesperson polytope was the first combinatorial polytope for which both the NP-completeness of verifying the vertex non-adjacency and the superpolynomial clique number of the 1-skeleton were established.

Since vertex adjacency is a hard problem for the traveling salesperson polytope, various special cases are of interest. In particular, Sierksma et al. [26] studied the faces of diameter 2, Arthanari [3] considered the pedigree polytope which provided a sufficient condition for non-adjacency in the traveling salesperson polytope, and Bondarenko et al. [9,11] studied the polytope of pyramidal tours.

In this paper, we consider the polytope associated with Hamiltonian tours of a special kind: pyramidal tours with step-backs.

### 3 Pyramidal tours

We suppose that the cities are labeled from 1 to $n$. Let $\tau$ be a Hamiltonian tour. We denote the successor of $i$-th city as $\tau(i)$, and the predecessor as $\tau^{-1}(i)$. For any natural $k$, we denote the $k$-th successor of $i$ as $\tau^k(i)$, the $k$-th predecessor of $i$ as $\tau^{-k}(i)$. The city $i$ satisfying $\tau^{-1}(i) < i$ and $\tau(i) < i$ is called a peak. A pyramidal tour, introduced by Aizenshtat and Kravchuk [4], is a Hamiltonian tour with only one peak $n$. In other words, the salesperson starts in city 1, then visits some cities in ascending order, reaches city $n$ and returns to city 1, visiting the remaining cities in descending order.

Enomoto, Oda, and Ota introduced a more general class of pyramidal tours with step-backs [16]. A step-back peak (see Fig. 2) is the city $i$, such that

$$\tau^{-1} < i, \quad \tau(i) = i - 1, \quad \tau^2(i) > i, \quad \text{or} \quad \tau^{-2} > i, \quad \tau^{-1}(i) = i - 1, \quad \tau(i) < i.$$ 

A proper peak is a peak $i$ which is not a step-back peak. A pyramidal tour with step-backs is a Hamiltonian tour with exactly one proper peak $n$. 

Pyramidal tours and pyramidal tours with step-backs are of interest, since, on the one hand, the minimum cost pyramidal tour (with step-backs) can be found in \( O(n^2) \) time by dynamic programming, and, on the other hand, there are known restrictions on the distance matrix that guarantee the existence of an optimal tour that is pyramidal (with step-backs). See [12,18] for pyramidal tours, and [16] for pyramidal tours with step-backs.

We consider a complete digraph \( K_n = (V, E) \). With each pyramidal tour (with step-backs) \( x \) in \( K_n \) we associate a characteristic vector \( v(x) \in \mathbb{R}^E \):

\[
v(x)_e = \begin{cases} 
1, & \text{if an edge } e \in E \text{ is contained in the tour } x, \\
0, & \text{otherwise}.
\end{cases}
\]

The polytope

\[
\text{PYR}(n) = \text{conv}\{v(x) \mid x \text{ is a pyramidal tour in } K_n\}
\]

is called the polytope of pyramidal tours.

The polytope

\[
\text{PSB}(n) = \text{conv}\{v(x) \mid x \text{ is a pyramidal tour with step-backs in } K_n\}
\]

is called the polytope of pyramidal tours with step-backs.

The polytope of pyramidal tours \( \text{PYR}(n) \) was introduced in [11] and later considered in [9] by Bondarenko et al. It was established that vertex adjacency in 1-skeleton of the \( \text{PYR}(n) \) polytope can be verified in linear time \( O(n) \), the diameter of 1-skeleton equals 2, and the asymptotically exact estimate of clique number is \( \Theta(n^2) \).

The polytope of pyramidal tours with step-backs was introduced in [21], where a necessary and sufficient condition for vertex adjacency in the 1-skeleton of the polytope is given. Based on this condition, we develop a linear-time algorithm to verify vertex adjacencies in the polytope \( \text{PSB}(n) \) and study the diameter and the clique number of the 1-skeleton.
4 Pyramidal encoding

Following [21], we introduce a special pyramidal encoding to represent the pyramidal tours with step-backs. With each pyramidal tour with step-backs $x$ in $K_n$ we associate a vector $x^{0,1,\text{sb}}$ of length $n - 2$, each coordinate corresponds to a city from 2 to $n - 1$, by the following rule:

$$x^{0,1,\text{sb}}_i = \begin{cases} 
1, & \text{if } i \text{ is visited by } x \text{ in ascending order}, \\
\overline{1}, & \text{if } i \text{ is a step-back peak in ascending order}, \\
0, & \text{if } i \text{ is visited by } x \text{ in descending order}, \\
\overline{0}, & \text{if } i \text{ is a step-back peak in descending order}.
\end{cases}$$

Note that a step-back on $i$ also involves the previous coordinate $i - 1$. An example of a pyramidal tour with step-backs and the corresponding encoding vector $x^{0,1,\text{sb}}$ is shown in Fig. 3.

We denote by $x^{0,1,\text{sb}}_{[i,j]}$ a fragment of encoding on coordinates from $i$ to $j$. The superscript indicates what we consider in the encoding: descending order ($0$), ascending order ($1$), or step-backs ($\text{sb}$). For example, $x^{1,\text{sb}}_{[i,j]}$ means a fragment of the encoding only in ascending order taking into account step-backs; $x^{0,1}_{[i,j]}$ – a fragment of the encoding disregarding step-backs, etc.

5 Vertex adjacency

We consider 12 blocks of the following form (a wavy line means that the corresponding coordinate can either contain a step-back or not):

$$
\begin{align*}
U_{11} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & U_{00} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & U_{1111} &= \begin{pmatrix} \overline{1} \\ 1 \\ 1 \end{pmatrix}, & U_{0000} &= \begin{pmatrix} \overline{0} \\ \overline{0} \\ \overline{0} \end{pmatrix}, \\
L_{1110} &= \begin{pmatrix} \overline{1} \\ \overline{1} \\ 0 \end{pmatrix}, & L_{1011} &= \begin{pmatrix} 1 \\ \overline{0} \\ 1 \end{pmatrix}, & L_{0001} &= \begin{pmatrix} \overline{0} \\ \overline{0} \\ 1 \end{pmatrix}, & L_{0100} &= \begin{pmatrix} \overline{0} \\ 0 \\ 0 \end{pmatrix}, \\
R_{1101} &= \begin{pmatrix} \overline{1} \\ \overline{1} \\ \overline{0} \end{pmatrix}, & R_{0111} &= \begin{pmatrix} \overline{0} \\ \overline{1} \\ 1 \end{pmatrix}, & R_{0010} &= \begin{pmatrix} \overline{0} \\ \overline{0} \\ \overline{1} \end{pmatrix}, & R_{1000} &= \begin{pmatrix} \overline{1} \\ 0 \\ 0 \end{pmatrix}.
\end{align*}
$$
Theorem 1 (Nikolaev [21]). Vertices \( v(x) \) and \( v(y) \) of the polytope PSB\((n)\) are not adjacent if and only if the following conditions are satisfied.

- There exists a city \( i \) (called a left block) such that the tours \( x \) and \( y \) on the coordinate \( i \) (coordinates \( i \) and \( i+1 \) for double blocks) in the pyramidal encoding have the form of \( U, L \), or \( i = 1 \).
- There exists a city \( j \) (called a right block) such that the tours \( x \) and \( y \) on the coordinate \( j \) (coordinates \( j-1 \) and \( j \) for double blocks) in the pyramidal encoding have the form of \( U, R \), or \( j = n \).

We denote by \( i_0 \) the first city after the left block: \( i_0 = i + 1 \) for single blocks and \( i_0 = i + 2 \) for double blocks. We denote by \( j_b \) the last city before the right block: \( j_b = i - 1 \) for single blocks and \( j_b = j - 2 \) for double blocks.

Two blocks cut the encoding of the tours into three parts: the left (less than \( i_0 \)), the central (from \( i_0 \) to \( j_b \)), and the right (larger than \( j_b \)).

- In the central part, the coordinates of \( x^{0,1} \) and \( y^{0,1} \) completely coincide: \( x^{0,1}_{[i_0,j_b]} = y^{0,1}_{[i_0,j_b]} \).

We say that two tours

- differ in the left part if \( x^{0,1}_{[1,i_0-1]} \neq y^{0,1}_{[1,i_0-1]} \);
- differ in the right part if \( x^{0,1}_{[j_b+1,n]} \neq y^{0,1}_{[j_b+1,n]} \);
- differ in the central part in ascending order if \( x^{1,0}_{[i_0,j_b]} \neq y^{1,0}_{[i_0,j_b]} \);
- differ in the central part in descending order if \( x^{0,1}_{[i_0,j_b]} \neq y^{0,1}_{[i_0,j_b]} \).

The remaining conditions are divided into four cases depending on the values of \( x^{0,1}_i \) and \( x^{0,1}_j \).

1. If \( x^{0,1}_i = x^{0,1}_j = 1 \), then the tours differ
   - in the central part in ascending order;
   - in the left part, or in the central part in descending order, or in the right part.
2. If \( x^{0,1}_i = x^{0,1}_j = 0 \), then the tours differ
   - in the central part in descending order;
   - in the left part, or in the central part in ascending order, or in the right part.
3. If \( x^{0,1}_i = 1, x^{0,1}_j = 0 \), then the tours differ
   - in the central part in ascending order or in the right part;
   - in the central part in descending order or in the left part.
4. If \( x^{0,1}_i = 0, x^{0,1}_j = 1 \), then the tours differ
   - in the central part in descending order or in the right part;
   - in the central part in ascending order or in the left part.

Cities \( 1 \) and \( n \) can be considered in the encoding as visited in ascending or descending order, if required.

The idea of sufficient conditions is that if from the edges of the tours \( x \) and \( y \) we can assemble two complementary pyramidal tours with step-backs \( z \) and \( t \), then the segment \([v(x), v(y)]\) intersects with the segment \([v(z), v(t)]\), and the corresponding vertices of the polytope PSB\((n)\) are not adjacent (see [21]).
The examples of the first and third sufficient conditions for non-adjacency are shown in Fig. 4 (edges of $x$ are solid, edges of $y$ are dashed, left and right blocks in pyramidal encodings are highlighted with dashed boxes).

In [21] it was proved that the necessary and sufficient condition for non-adjacency of Theorem 1 can be verified by exhaustive search in $O(n^3)$ time. We improve this estimate by introducing a linear-time algorithm.

**Theorem 2.** The question of whether two vertices of the polytope PSB($n$) are adjacent can be verified in linear time $O(n)$.

**Proof.** We consider two pyramidal tours with step-backs $x$ and $y$, and the corresponding vertices $v(x)$ and $v(y)$ of the polytope PSB($n$). Each of the four sufficient non-adjacency conditions of Theorem 1 can be verified in a single pass through the pyramidal encodings $x^{0,1,\text{sb}}$ and $y^{0,1,\text{sb}}$, when we sequentially find the left block, the right block, and check additional conditions. The pseudo-code to construct a tour $z$ and verify the first and second sufficient conditions is given in the Algorithm 1. The other two sufficient conditions are verified similarly.

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**6 Graph diameter and clique number**

Based on Theorem 1, we estimate the diameter of 1-skeleton of PSB($n$).

**Theorem 3.** The diameter of 1-skeleton of PSB($n$) is bounded above by 4.
Algorithm 1 Verifying 1st and 2nd sufficient conditions for non-adjacency

procedure NonAdjacencyTest(x, y, n)
    LBlock $\leftarrow$ TRUE \textsuperscript{\small $\triangleright$ Consider the city 1 as a left block}
    RBlock, zNotx, zNoty $\leftarrow$ FALSE
    for $i \leftarrow 2$ to $n - 1$
        if LBlock = TRUE and RBlock = FALSE then \textsuperscript{\small $\triangleright$ Central part}
            if $z_i$, $y_i$ is different from $x_i$, then
                $z$Notx $\leftarrow$ TRUE \textsuperscript{\small $\triangleright$ $z$ visits $i$ by the edges of $y$}
            end if
            if $z$Notx = TRUE and we found $U$ or $R$ block then
                RBlock $\leftarrow$ TRUE \textsuperscript{\small $\triangleright$ Starting the right part}
            end if
            if the conditions of the central part are violated then
                LBlock, zNotx $\leftarrow$ FALSE \textsuperscript{\small $\triangleright$ Return to the left part}
            end if
        end if
        if RBlock = TRUE then \textsuperscript{\small $\triangleright$ Left part}
            if $x_i$, $y_i$ is different from $y_i$, then \textsuperscript{\small $\triangleright$ $z$ visits $i$ by the edges of $x$}
                zNoty $\leftarrow$ TRUE
            end if
            if we found $U$ or $L$ block then
                LBlock $\leftarrow$ TRUE \textsuperscript{\small $\triangleright$ Starting the central part}
            end if
        end if
        if LBlock = TRUE then \textsuperscript{\small $\triangleright$ Right part}
            if $x_i$, $y_i$ is different from $y_i$, then \textsuperscript{\small $\triangleright$ $z$ visits $i$ by the edges of $y$}
                zNoty $\leftarrow$ TRUE
            end if
        end if
    end for
    RBlock $\leftarrow$ TRUE \textsuperscript{\small $\triangleright$ Consider the city $n$ as a right block}
    if LBlock, RBlock, zNotx, zNoty = TRUE then
        return 1st/2nd sufficient condition for non-adjacency is satisfied
    else
        return 1st/2nd sufficient condition for non-adjacency is not satisfied
    end if
end procedure

Proof. The idea is as follows. For an arbitrary pyramidal tour with step-backs $x$ we construct a pyramidal tour $\hat{x}$ where

$$\hat{x}_{0,1} = \begin{cases} 0, & \text{if } i \text{ is a part of step-back in } x \\ 1, & \text{otherwise.} \end{cases}$$

For instance:

$$x : \langle 1 \overline{1} 1 1 0 0 1 \overline{0} 0 1 \rangle, \quad \hat{x} : \langle 1 0 0 1 \overline{1} 1 1 1 0 0 1 \rangle.$$

By construction, the encodings of the tours $x$ and $\hat{x}$ can only contain blocks $U_{11}$, which restricts us to the first sufficient condition of Theorem [1]. However,
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Fig. 5. Path of length 4 between an arbitrary pair of PSB\((n)\) vertices \(v(x)\) and \(v(y)\) by (1) the tours \(x\) and \(\hat{x}\) cannot differ in the central part in the ascending order. Hence, by Theorem 1, the vertices \(v(x)\) and \(v(\hat{x})\) are adjacent. And for any pyramidal tour \(\hat{x}\), the vertex \(v(\hat{x})\) is adjacent to the vertices corresponding to the tours \(\langle 1, 1, \ldots, 1 \rangle\) and \(\langle 0, 0, \ldots, 0 \rangle\) (see [9]). Thus, between any pair of vertices of the polytope PSB\((n)\) we can construct a path of no more than 4 edges. The corresponding scheme is shown in Fig. 5.

Now we apply the necessary and sufficient condition of Theorem 1 to estimate the clique number of the 1-skeleton of the polytope PSB\((n)\).

**Theorem 4.** The clique number of 1-skeleton of the polytope PSB\((n)\) is quadratic in the parameter \(n\):

\[
\omega(\text{PSB}(n)) = \Theta(n^2).
\]

**Proof.** Upper bound. Let \(Y^v\) be a set of pairwise adjacent vertices of PSB\((n)\), and \(Y\) be the set of corresponding pyramidal tours with step-backs. Let us estimate the cardinality of \(Y\).

**Step 1.** Consider the pyramidal encodings of the tours. We call a tour \(x \in Y\) unique with respect to a pair of neighboring coordinates \(k, k + 1\) if

\[
\forall y \in Y \setminus \{x\} : x^0,1_{[k,k+1]} \neq y^0,1_{[k,k+1]}.
\]

A pair of neighboring coordinates in the pyramidal encoding can take 18 different values. Hence the number of tours in \(Y\) that are unique in a pair of neighboring coordinates does not exceed \(18(n - 3)\). We construct the set \(W\) by excluding from \(Y\) all tours that are unique in a pair of neighboring coordinates.

**Step 2.** Let a pyramidal tour with step-backs in a pair of neighboring cities \(k\) and \(k + 1\) visit one of the cities in ascending order, and the other in descending order. Since the city may or may not be a part of step-back, we get 8 possible types of pyramidal encodings at coordinates \(k\) and \(k + 1\):

\[
\langle 1 \ 0 \rangle, \langle 1 \ 0 \ 0 \rangle, \langle 1 \ 0 \ 0 \ 0 \rangle, \langle 0 \ 0 \ 1 \rangle, \langle 0 \ 0 \ 1 \ 1 \rangle.
\]

We call such sections of encoding 0/1-segments.

We consider a pyramidal tour with step-backs \(x \in W\) with 0/1-segment at the coordinates \(k, k + 1\). By construction, we have excluded from \(W\) all tours that are unique in a pair of neighboring coordinates, so there exists a tour \(y \in W \setminus \{x\}\) with the same 0/1-segment at coordinates \(k, k + 1\), i.e. \(x^0,1_{[k,k+1]} = y^0,1_{[k,k+1]}\).
Thus, pyramidal encodings of tours $x$ and $y$ on coordinates $k, k + 1$ have the form of a pair of blocks $U$, where the coordinate $k$ is in the left block and $k + 1$ is in the right block. Since the corresponding vertices $v(x)$ and $v(y)$ of the polytope $PSB(n)$ are adjacent, by Theorem 1, the encodings of the tours $x$ and $y$ coincide either in the left part ($x^{0,1}_{[1,k]} = y^{0,1}_{[1,k]}$), or in the right part ($x^{0,1}_{[k+1,n]} = y^{0,1}_{[k+1,n]}$). For instance:

$$x : \langle \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \rangle_{k,k+1},
\quad y : \langle \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \rangle_{k,k+1}. $$

Note that for any subset of tours in $W$ with a common 0/1-segment, the coinciding parts of the encoding are on the same side of the segment. Indeed, suppose that three tours $x, y, z \in W$ have the same 0/1-segment on the cities $k, k + 1$, but the coinciding parts of the encodings are on different sides of the segment. Without loss of generality, let

$$x^{0,1}_{[1,k]} = y^{0,1}_{[1,k]} = x^{0,1}_{[k+1,n]} = y^{0,1}_{[k+1,n]},
\quad x : \langle \begin{array}{c} L \\ 0 \\ 1 \end{array} \rangle_{k,k+1},
\quad y : \langle \begin{array}{c} L \\ 0 \\ 1 \end{array} \rangle_{k,k+1},
\quad z : \langle \begin{array}{c} 0 \\ 1 \end{array} \rangle_{k,k+1}. $$

However, since the corresponding vertices $v(y)$ and $v(z)$ of the polytope $PSB(n)$ are adjacent, by Theorem 1 the pyramidal encodings of the tours $y$ and $z$ also coincide either in the left part: $y^{0,1}_{[1,k]} = y^{0,1}_{[k,1]}$ (in this case $x = z$), or in the right part: $y^{0,1}_{[k+1,n]} = y^{0,1}_{[k+1,n]}$ (in this case $x = y$). We got a contradiction.

Thus, for any pyramidal tour with step-backs from the set $W$, all 0/1-segments can be divided into left segments (for which tours with a common segment coincide in the left part) and right segments (for which tours coincide in the right part).

Let us show that the left and right 0/1-segments in tours from $W$ are ordered, i.e. if some tour $x \in W$ contains a left 0/1-segment on cities $k, k + 1$ and a right 0/1-segment on cities $s, s + 1$, then $k < s$.

Assume that $k > s$. Consider a tour $y \in W$ that shares a left 0/1-segment on $k, k + 1$ with $x$, then $x^{0,1}_{[1,k]} = y^{0,1}_{[1,k]}$. Hence the tours $x$ and $y$ coincide on the cities $s, s + 1$ since $s < k$, and have common right 0/1-segments:

$$x : \langle \begin{array}{c} * \\ * \\ 0 \\ 1 \end{array} \rangle_{L},
\quad y : \langle \begin{array}{c} * \\ * \\ 0 \\ 1 \end{array} \rangle_{R}. $$

It remains to note that

$$x^{0,1}_{[1,k]} = y^{0,1}_{[1,k]} = y^{0,1}_{[s,n]},
\quad x^{0,1}_{[k+1,n]} = y^{0,1}_{[k+1,n]}.$$
hence, \( x = y \), a contradiction.

**Step 3.** Consider a pyramidal tour with step-backs \( x \in W \). We choose the left 0/1-segment \( L_{\text{max}} \) at the largest coordinates \( i, i+1 \). If there is no such segment, then we set \( i = 1 \). We choose the right 0/1-segment \( R_{\text{min}} \) at the smallest coordinates \( j-1, j \). If there is no such segment, then we set \( j = n \).

By construction, \( i \leq j \) and cities from \( i+1 \) to \( j-1 \) are visited in the same direction. Let’s call the part of pyramidal encoding \([i+1, j-1]\) a 0-sequence if the cities from \( i+1 \) to \( j-1 \) are visited in the descending order, and 1-sequence if the cities are visited in ascending order.

Since the tours in \( W \) coincide to the left of the common left 0/1-segment and to the right of the common right 0/1-segment, each pyramidal tour with step-backs \( x \in W \) corresponds to a unique 0-sequence or 1-sequence. For example:

\[
\langle 1 0 0 0 0 1 1 0 1 0 0 0 1 0 1 0 0 0 0 1 1 \rangle.
\]

**Step 4.** We consider in \( W \) the subset of all pyramidal tours with step-backs that have a 0-sequence starting at position \( i \). Let’s denote this subset as \( W_i^0 \).

We consider in the set \( W_i^0 \) all tours containing at least one unique 0-coordinate inside the 0-sequence in the pyramidal encoding. There are \( n-i \) possible positions of the unique 0-coordinate, which can take 3 different values: start of a step-back, end of a step-back, not a step-back. Thus, the total number of such unique tours in \( W_i^0 \) does not exceed \( 3(n-i) = O(n) \). Let us construct the set \( W_i^0 \) by excluding from \( W_i^0 \) all tours with unique coordinates in the 0-sequence.

Consider some tour \( x \in W_i^0 \). By construction, for any coordinate \( s \) within the 0-sequence, there exists a second tour \( y \in W_i^0 \) such that \( x_{s}^{0, sb} = y_{s}^{0, sb} \) and \( s \) belongs to the 0-sequence of \( y \). Then the tours \( x \) and \( y \) on the coordinates \( i-1 \) (segment \( L_{\text{max}} \)) form a block \( U_{11} \) (or \( U_{1111} \)), and on the coordinate \( s \) – one of the blocks \( U_{00} \) or \( U_{0000} \). Therefore, by Theorem 1 the pyramidal encodings of the tours \( x \) and \( y \) coincide either in the central part between \( i-1 \) and \( s \) in descending order \( (x_{[i, s]}^{0, sb} = y_{[i, s]}^{0, sb}) \) or to the right of \( s \) \( (x_{[s, n]}^{0, sb} = y_{[s, n]}^{0, sb}) \). For instance:

\[
x: \langle 0 1 0 1 0 0 0 1 0 1 0 1 \rangle,
\]

\[
y: \langle 0 1 0 1 0 0 0 0 0 0 0 1 1 \rangle.
\]

Otherwise, the corresponding vertices \( v(x) \) and \( v(y) \) of the polytope \( \text{PSB}(n) \) are not adjacent.

Further reasoning completely repeats similar ones for common 0/1-segments. For any subset of tours in \( W_i^0 \) with a common 0-coordinate, the coinciding parts of the encoding must be on the same side of the coordinate. This allows us to divide all 0-sequence coordinates into left 0-coordinates (tours with a common coordinate coincide on the left side) and right 0-coordinates (for which tours coincide to the right of the common coordinate). Note also that all coordinates
are ordered, i.e. if \( s \) is the left 0-coordinate and \( t \) is the right 0-coordinate, then \( s < t \):

\[
\begin{array}{ccc}
0 & 1 & 1 \\
0 & 0 & 0 \\
R & L & 0 \\
\end{array}
\]

We denote by \( \bar{W}_{i,s}^0 \) the subset of all tours from \( \bar{W}_{i,s}^0 \) for which \( s \) is the number of the largest left 0-coordinate, and \( s + 1 \) is the number of the smallest right 0-coordinate.

The key idea is that the coordinates \( s \) and \( s + 1 \) of any tour from \( \bar{W}_{i,s}^0 \) can take only 2 values: \( (0) \) and \( (0 0) \). Moreover, the values of this pair of coordinates uniquely determine the tour since tours with the same left coordinate coincide to the left and with the same right coordinate – to the right. Thus, the subset \( \bar{W}_{i,s}^0 \) contains at most 4 tours.

And now we rise back to the original set of pairwise adjacent tours. Firstly,

\[
\bar{W}_{i,s}^0 = \bigcup_{s=i+1}^{n-1} \bar{W}_{i,s}^0,
\]

whence, with the excluded tours with unique 0-coordinates, \( |W^0_i| = O(n) \). Similarly, we get that \( |W_1^0| = O(n) \).

Secondly,

\[
W = \left( \bigcup_{i=2}^{n-1} W_{i}^0 \right) \cup \left( \bigcup_{i=2}^{n-1} W_{i}^1 \right),
\]

whence we get that \( |W| = O(n^2) \).

Finally, returning the excluded tours with unique pairs of neighboring coordinates, we arrive at the upper bound \( |Y| = |W| + O(n) = O(n^2) \). Thus,

\[
\omega(\text{PSB}(n)) = O(n^2).
\]

**Lower bound.** We construct an example of the set \( Z^v \) of pairwise adjacent vertices of the polytope \( \text{PSB}(n) \) such that \( |Z^v| = \Omega(n^2) \). Let \( n \) be even. With each pair of integers \( k, s \) such that \( 0 \leq k, s \leq \frac{n}{2} \) we associate a pyramidal tour with step-backs \( x(k, s) \in Z \) such that

- \( \forall i \ (2 \leq i \leq k + 1): x_{i}^{0,1,1} = 1 \);
- \( \forall j \ (n - s \leq j \leq n - 1): x_{j}^{0,1,1} = 1 \);
- all other coordinates of \( x_{i}^{0,1,1} \) are equal to 0.

The tours from the set \( Z \) do not contain step-backs, therefore, they cannot differ in the central part between the left and right blocks. Moreover, if for two tours \( x, y \in Z \) along some coordinate \( i \): \( x_{i}^{0,1,1} = y_{i}^{0,1,1} = 1 \), then

- if \( i \leq \frac{n}{2} \), then the encodings coincide in the left part (i.e. \( x_{[1,i]}^{0,1,1} = y_{[1,i]}^{0,1,1} \));
- if \( i \geq \frac{n}{2} + 1 \), then the pyramidal encodings coincide in the right part (i.e. \( x_{[i,n]}^{0,1,1} = y_{[i,n]}^{0,1,1} \)).
Hence, by Theorem 1, the corresponding vertices \( v(x) \) and \( v(y) \) of the polytope \( \text{PSB}(n) \) are adjacent.

The case of odd \( n \) is reduced to an even case, it suffices to fix one of the cities in all tours from \( Z \) in ascending or descending order. Thus,

\[
\omega(\text{PSB}(n)) \geq |Z| = \left\lfloor \frac{n}{2} \right\rfloor^2 = \Omega(n^2).
\]

An example of a set \( Z \) for \( n = 8 \) is given in Table 1.

Combining the upper and lower bounds, we obtain the desired asymptotically exact quadratic estimate (2).

### 7 Conclusion

The results of the research are summarized in Table 2.

|                      | Hamiltonian cycles ATSP(n) and TSP(n) | Pyramidal tours PYR(n) | Pyramidal tours with step-backs PSB(n) |
|----------------------|--------------------------------------|------------------------|---------------------------------------|
| Complexity of TSP problem | NP-hard                              | \( O(n^2) \)            | \( O(n^2) \)                          |
| Vertex adjacency in 1-skeleton | co-NP-complete                      | \( O(n) \)             | \( O(n) \)                            |
| Diameter of 1-skeleton | 2                                    | 2                      | \( \Theta(n^2) \)                     |
| Clique number of 1-skeleton | \( \Omega \left( 2^{\sqrt{n} - 1} \right) \) | \( \Theta(n^2) \)     | \( \Theta(n^2) \)                     |

We have considered several versions of the traveling salesperson problem and the associated combinatorial polytopes.

The general traveling salesperson problem is NP-hard \([17]\). The question of whether two vertices of the traveling salesperson polytope are not adjacent is NP-complete \([23]\). The clique number of 1-skeleton of the polytope ATSP(n) is superpolynomial in \( n \) \([7]\).

On the other hand, the traveling salesperson problem on pyramidal tours and pyramidal tours with step-backs is solvable in polynomial time by dynamic programming \([16,18]\). The vertex adjacency for the polytopes PYR(n) and PSB(n) can be verified in linear time \( O(n) \). The clique numbers of 1-skeletons are quadratic in \( n \).
Thus, the properties of 1-skeletons of the polytopes associated with the traveling salesman problem directly correlate with the complexity of the problem itself.

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