A general formula for the type-I seesaw mechanism

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By performing an approximate spectral decomposition to the inverse mass matrix of the right-handed neutrinos $M_{R}^{-1}$, we obtain a formula for the type-I seesaw mechanism in a general basis. The formula indicates conditions for naturalness, the existence of a $CP$ symmetry, and constraints for the flavor structure of the Yukawa matrix of neutrinos $Y_{\nu}$.

The type-I seesaw mechanism [1–3], which has the potential to explain neutrino oscillation [4] and baryon asymmetry of the universe [5], is one of the most studied subjects in particle phenomenology. However, the general nature of the mechanism is difficult to analyze, because it involves eighteen terms in each matrix element of light neutrinos $m_{\nu}$. E. Witten has argued that “the considerations have always been qualitative, and, despite some interesting attempts, there has never been a convincing quantitative model of the neutrino masses” [6, 7]. In this letter, we derive a concise formula of the type-I seesaw mechanism in a general basis, by using a spectral decomposition in the first-order perturbation to the inverse mass matrix of the right-handed neutrinos $M_{R}^{-1}$. We will see that the formula places significant constraints on the naturalness of the seesaw mechanism, the existence of a $CP$ symmetry, and flavor structures.

First of all, the mass matrix $M$ of the right-handed neutrinos $\nu_{Ri}$ is defined as

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{pmatrix}.$$  \hspace{1cm} (1)
These matrix elements $M_{ij}$ are in general complex. The inverse matrix of $M$ is

$$M^{-1} = \frac{1}{|M|} \begin{pmatrix}
M_{22}M_{33} - M_{23}^2 & M_{13}M_{23} - M_{12}M_{33} & M_{12}M_{23} - M_{13}M_{22} \\
M_{13}M_{23} - M_{12}M_{33} & M_{11}M_{33} - M_{13}^2 & M_{12}M_{13} - M_{11}M_{23} \\
M_{12}M_{23} - M_{13}M_{22} & M_{12}M_{13} - M_{11}M_{23} & M_{11}M_{22} - M_{12}^2
\end{pmatrix},$$

(2)

$$M^{-1} = \frac{1}{|M|} \begin{pmatrix}(M_2 \times M_3)^T \\
(M_3 \times M_1)^T \\
(M_1 \times M_2)^T
\end{pmatrix},$$

(3)

where $|M| = \det M$ and $(M_i)_j \equiv (M_{ij})$ are three-dimensional vectors composed from the rows of the mass matrix. If the Yukawa matrix of neutrinos $Y$ is hierarchical like the other fermions, $M$ should have a strong hierarchy about the order of $Y Y^T$. Then we assume

$$M_{33} \gg M_{23}, M_{22}, \gg M_{13}, M_{12}, M_{11}.$$

(4)

The mass eigenvalues $M_i$ of $M$ (1) is approximately calculated as

$$(M_3, M_2, M_1) \simeq (M_{33}, M_{22} - \frac{M_{23}^2}{M_{33}}, \frac{|M|}{M_2M_3}).$$

(5)

Here, we will consider an approximate spectral decomposition of Eq. (2). A matrix $M^{(1)}$ whose rank is one is defined as;

$$M^{(1)} \equiv \begin{pmatrix}
(M^{-1})_{11} & (M^{-1})_{12} & (M^{-1})_{13} \\
(M^{-1})_{12} & (M^{-1})_{12}^2 & (M^{-1})_{12}(M^{-1})_{13} \\
(M^{-1})_{13} & (M^{-1})_{12}(M^{-1})_{13} & (M^{-1})_{11}
\end{pmatrix},$$

(6)

where $(M^{-1})_{ij}$ is an element of Eq. (2). Explicitly, matrix elements of $M^{(1)}$ are found to be

$$\frac{1}{|M|} \begin{pmatrix}
M_{22}M_{33} - M_{23}^2 & M_{13}M_{23} - M_{12}M_{33} & M_{12}M_{23} - M_{13}M_{22} \\
M_{13}M_{23} - M_{12}M_{33} & (M_{13}M_{23} - M_{12}M_{33})^2 & (M_{12}M_{23} - M_{13}M_{22})(M_{13}M_{23} - M_{12}M_{33}) \\
M_{12}M_{23} - M_{13}M_{22} & (M_{12}M_{23} - M_{13}M_{22})(M_{13}M_{23} - M_{12}M_{33}) & (M_{13}M_{23} - M_{12}M_{22})^2
\end{pmatrix}.$$

(7)

By using Eq. (3), $M^{(1)}$ can also be written as follows

$$M^{(1)} = \frac{1}{(M_{22}M_{33} - M_{23}^2)|M|} (M_2 \times M_3) \otimes (M_2 \times M_3)^T.$$

(8)
Since $M^{(1)}_{11} = (M^{-1})_{11} = (M_{22}M_{33} - M_{23}^2)/|M| \simeq 1/M$ holds, this $M^{(1)}$ can be regarded as a matrix composed of an eigenvector $V_1$ belonging to $M_1$. Note that this is not an exact form, but an approximation of a first-order perturbation for the off-diagonal component $M_{ij}$ with $i \neq j$.

Similarly, we can consider the spectral decomposition for $M_2$ and $M_3$. By subtracting $M^{(1)}$ from $M^{-1}$,

$$\tilde{M} \equiv M^{-1} - M^{(1)} = \begin{pmatrix}
0 & 0 & 0 \\
0 & (M^{-1})_{22} - \frac{(M^{-1})_{12}^2}{(M^{-1})_{11}} & (M^{-1})_{23} - \frac{(M^{-1})_{12}(M^{-1})_{13}}{(M^{-1})_{11}} \\
0 & (M^{-1})_{23} - \frac{(M^{-1})_{12}(M^{-1})_{13}}{(M^{-1})_{11}} & (M^{-1})_{33} - \frac{(M^{-1})_{13}^2}{(M^{-1})_{11}}
\end{pmatrix}. \quad (9)$$

A matrix $M^{(2)}$ whose rank is one is extracted from Eq. (9) as

$$M^{(2)} \equiv \begin{pmatrix}
0 & 0 & 0 \\
0 & \tilde{M}_{22} & \tilde{M}_{23} \\
0 & \tilde{M}_{23} & \frac{\tilde{M}_{23}^2}{\tilde{M}_{22}}
\end{pmatrix} = \frac{1}{M_{22}M_{33} - M_{23}^2} \begin{pmatrix}
0 & 0 & 0 \\
0 & M_{33} & -M_{23} \\
0 & -M_{23} & \frac{M_{23}^2}{M_{33}}
\end{pmatrix}. \quad (10)$$

Since the non-zero eigenvalue of this matrix is

$$\frac{M_{23}^2 + M_{33}^2}{M_{33}(M_{22}M_{33} - M_{23}^2)} \simeq \frac{1}{(M_{22} - \frac{M_{23}^2}{M_{33}})} \simeq \frac{1}{M_2}, \quad (11)$$

this can be regarded as a matrix composed of the eigenvector $V_2$ belonging to $M_2$. The last remaining element is

$$M^{(3)} \equiv M^{-1} - M^{(1)} - M^{(2)} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1/M_{33}
\end{pmatrix}. \quad (12)$$

Then this is almost a matrix from an eigenvector $V_3$ that belongs $M_3$. Note that this spectral decomposition is an approximate one of the first-order perturbation, and the matrix $M^{(i)}$ does not strictly satisfy the condition $M^{(i)}M^{(j)} \propto \delta^{ij}M^{(j)}$. This decomposition can be regarded as a reconstruction of $M$ by a perturbative rotation from the diagonalized basis. In this construction, the three eigenvectors are not strictly orthogonal;

$$V'_1 \propto \left((M^{-1})_{11} \ (M^{-1})_{12} \ (M^{-1})_{13}\right), \quad V'_2 \propto \left(0 \ M_{33} \ -M_{23}\right), \quad V'_3 = \left(0 \ 0 \ 1\right). \quad (13)$$

However, the relation $M^{(1)} + M^{(2)} + M^{(3)} = M^{-1}$ is exact and no approximation is used.
The Yukawa matrix of neutrinos $Y$ is defined by

$$Y = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} \equiv \begin{pmatrix} Y_1^T \\ Y_2^T \\ Y_3^T \end{pmatrix}, \tag{14}$$

where $(Y_i)_j \equiv Y_{ij} = (A_i, B_i, C_i)$ are 3-dimensional vectors and $A_i, B_i, C_i (i = 1, 2, 3)$ are complex parameters. By neglecting the vacuum expectation value of the Higgs field $v$, the mass dimension of $Y$ becomes one.

By considering the seesaw mechanism with this decomposition (Eqs. (8), (11) and (12)), the mass matrix $m$ for the light neutrinos $\nu_i$ is found to be

$$m = Y(M^{(1)} + M^{(2)} + M^{(3)})Y^T \equiv m^{(1)} + m^{(2)} + m^{(3)} \tag{15}$$

$$= \frac{|M^{(23)}|}{\det M} \begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{pmatrix} + \frac{M_{33}}{|M^{(23)}|} \begin{pmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_3 & b_2b_3 & b_3^2 \end{pmatrix} + \frac{1}{M_{33}} \begin{pmatrix} C_1^2 & C_1C_2 & C_1C_3 \\ C_1C_2 & C_2^2 & C_2C_3 \\ C_1C_3 & C_2C_3 & C_3^2 \end{pmatrix}, \tag{16}$$

where $|M^{(23)}| \equiv M_{22}M_{33} - M_{23}^2$ and

$$a_i \equiv \frac{\det(Y_i, M_2, M_3)}{|M^{(23)}|}, \quad b_i \equiv \frac{(Y_i \times M_3)_1}{M_{33}} = Y_{i2} - Y_{i3} \frac{M_{23}}{M_{33}}. \tag{17}$$

This is an exact expression in a general basis without approximations. Since $a_i$ is proportional to the scalar triple product, it represents a component of $Y_i$ in a direction orthogonal to $M_2$ and $M_3$ (or parallel to $M_2 \times M_3$). From the hierarchy (11) or (5), $a_i, b_i$ are approximately equal to the first and second elements of $Y_i$. Substituting $(a_i, b_i) \sim (A_i, B_i)$, we obtain

$$m_{ij} \simeq \frac{1}{M_1} A_i A_j + \frac{1}{M_2} B_i B_j + \frac{1}{M_3} C_i C_j. \tag{18}$$

The equality holds for diagonal $M$. Each matrix roughly represents contributions from $M_{1,2,3}$.

The general formula (16) has implications for several phenomenologies.

1. **Flavor structure.** Since a relation $C_3 \gg C_2 \gg C_1$ that holds for many Yukawa matrices and the hierarchy of $M$ (11), at most only the 33 element contribute to $m$ in the third term $m^{(3)}$ of Eq. (16). Therefore, the first and second terms $m^{(1,2)}$ make important contributions in the type-I seesaw mechanism. On the other hand, for the MNS matrices, the trimaximal mixing [8–15] is observed with good accuracy. Thus, if there is no nontrivial relation between $a_i$ and $b_i$, the
trimaximal mixing requires \( a_1 \simeq a_2 \simeq a_3 \) or \( a_1 + a_2 + a_3 \simeq 0 \) (\( b_i \) must satisfy another condition of \( a_i \) to avoid \( m_2 \simeq 0 \) or \( m_{1,3} \simeq 0 \)). In particular, if we set \( A_1 \simeq A_2 \simeq A_3 \simeq B_1 \simeq \delta, B_2 \simeq -B_3 \simeq \lambda \) with \( \lambda \gtrsim \delta \), the matrices \( Y \) and \( m \) become

\[
Y \simeq \begin{pmatrix}
\delta & \delta & C_1 \\
\delta & \lambda & C_2 \\
\delta & -\lambda & C_3
\end{pmatrix}, \quad m \simeq \frac{1}{M_2} \begin{pmatrix}
\delta^2 + \delta^2 & \delta \lambda + \delta^2 & -\delta \lambda + \delta^2 \\
\delta \lambda + \delta^2 & \lambda^2 + \delta^2 & -\lambda^2 + \delta^2 \\
-\delta \lambda + \delta^2 & -\lambda^2 + \delta^2 & \lambda^2 + \delta^2
\end{pmatrix},
\]

where \( \delta^2 \simeq M_2 \delta^2/M_1 \). The form of \( Y \) recalls the cascade hierarchy \( C_1 \simeq \delta, C_2 \simeq -\lambda \) suggested by the CKM matrix and is rather desirable from a viewpoint of unified models. Then, it is very plausible to speculate that the matrices \( Y \) and \( m \) have these forms \([16]\). Similar results have been obtained by (constrained) sequential dominance \([18–23]\) and previous studies discussing fine-tuning in the seesaw mechanism \([24–26]\).

2. naturalness. If one of \( a_i, b_i \) becomes too large, it will be necessary to cancel out the other terms. Then, the naturalness \([27]\) requires parameters \( a_i \) and \( b_i \) to stay within certain ranges. Since \( M_1 \) is the lightest eigenvalue, such a restriction is strongest for \( a_i \). In other words, the component of \( Y_i \) in the direction of \( M_2 \times M_3 \), which is approximately equal to \( Y_{i1} = A_i \), must be the same order of magnitude for any \( i \). By using \( a \sim \sqrt{m_{i1} M_1} \) as a standard parameter, these constraints are rewritten as

\[
|b_i| \lesssim \left| \frac{\lambda}{\delta} a \sqrt{\frac{M_2}{M_1}} \right|, \quad |C_i| \lesssim \left| \frac{\lambda}{\delta} a \sqrt{\frac{M_3}{M_1}} \right|,
\]

where \( \lambda/\delta \) accounts for the mass hierarchy of \( m \). For example, if alignments \( Y_i \propto M_{2,3} \) holds approximately in Eq. \([17]\), such naturalness conditions are easily achieved \([28]\). A more detailed analysis will be done in future work.

3. CP symmetry. In the case of the normal hierarchy, either the first or second term of Eq. \([16]\) is a dominant matrix with rank one that produces \( m_3 \). Since phases of \( a_i \) or \( b_i \) can be eliminated approximately by field redefinitions, this case indicates the existence of an approximate generalized CP symmetry \([29–32]\) for neutrino mass \( m \) in some basis. One such example is diagonal reflection symmetries that imposed on \( X = m, Y, M \) and the mass matrix of electrons \( Y_e \) \([33–36]\):

\[
R X^* R = X, \quad Y_e^* = Y_e,
\]

(21)
where

\[
R = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\quad
X_{ij} = \begin{pmatrix}
X_{11} & iX_{12} & iX_{13} \\
iX_{21} & X_{22} & X_{23} \\
iX_{31} & X_{32} & X_{33}
\end{pmatrix},
\]

(22)

with real parameters \(X_{ij}\) and \((Y_e)_{ij}\). Combined with the four-zero texture \((M_f)_{11} = (M_f)_{13,31} = 0\) for \(f = u, d, \nu, e\) [37], these symmetries reproduce the bestfit of the CKM and MNS matrices with an accuracy of \(10^{-3}\).

To conclude, by performing an approximate spectral decomposition to the inverse mass matrix of the right-handed neutrinos \(M^{-1}\), we obtain a concise formula for the type-I seesaw mechanism in a general basis. The general formula indicates conditions for naturalness, the existence of a \(CP\) symmetry. Moreover, the bimaximal and trimaximal conditions restrict the flavor structure of Yukawa matrix of neutrinos \(Y\). Similar expressions are expected for other analogs of the seesaw mechanism.

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