NAMBU STRUCTURES ON FOUR-DIMENSIONAL REAL LIE GROUPS AND RELATED SUPERINTEGRABLE SYSTEMS

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We determine all third- and fourth-order Nambu tensors (Nambu structures) on four-dimensional real Lie groups. In addition, we obtain superintegrable systems using the fourth-order Nambu structures on some of these Lie groups as phase spaces with the symmetry groups A_{4,8} and A_{4,10}.

Keywords: Nambu structure, superintegrable system, Lie group

1. Introduction

In 1973 [1], Nambu studied a dynamical system described as a Hamiltonian system with respect to a generalization of the Poisson bracket (Poisson-like bracket) defined by a Jacobian determinant. After some years (about two decades), Takhtajan [2] introduced the concept of a Nambu–Poisson (or simply Nambu) structure using an axiomatic formulation for the n-bracket and gave the basic properties of this operation and also geometric formulations of Nambu manifolds. This new approach motivated a series of papers about some new concepts.

A Nambu manifold is a C∞ manifold endowed with a Nambu tensor, a skew-symmetric contravariant tensor field on a manifold such that the induced bracket operation satisfies the fundamental identity, which is a generalization of the usual Jacobi identity [3]–[7] (there is another generalization, the so-called generalized Poisson bracket [8], [9]; the two concepts were compared in [10], [11]). The concept of a Nambu Lie group was presented in [12] and [13]. In [13], Vaisman extended Nambu brackets to 1-forms and by generalizing the Poisson–Lie case defined Nambu–Lie groups as the Lie groups endowed with a multiplicative Nambu structure. Nakanishi proved the decomposability of Nambu structures for Lie groups and also the correspondence between the set of left-invariant Nambu tensors of order n on m-dimensional Lie groups G with the set of n-dimensional Lie subalgebras of G (the Lie algebra of G) in [14]. He also determined the multiplicative Nambu structures on three-dimensional real Lie groups [15]. Here, in the same way, we try to determine the multiplicative Nambu structure of orders four and three on four-dimensional real Lie groups.

The outline of the paper is as follows. In Sec. 2, for self-sufficiency of the paper, we review some definitions and theorems. In Sec. 3, using the method in [15], we determine the multiplicative Nambu structures of orders four and three on the four-dimensional real Lie groups. Finally, in Sec. 4, using the fourth-order Nambu structures on some Lie groups (i.e., four-dimensional real Lie groups with symplectic

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structures [16]), we obtain superintegrable systems with these Lie groups as phase spaces and $A_{4,8}$ and $A_{4,10}$ as symmetry Lie groups.

2. Basic definitions and theorems

We recall some basic definitions and theorems about the Nambu structure [12], [15]. Let $G$ be an $m$-dimensional Lie group with the Lie algebra $G$. Let $\Gamma(\Lambda^n TG)$ be the set of antisymmetric $n$-vector fields (contravariant tensors) on $G$. Then for each $\eta \in \Gamma(\Lambda^n TG)$, an $n$-bracket of functions on $G$ can be defined as

$$\{f_1, \ldots, f_n\} = \eta(df_1, \ldots, df_n), \quad f_i \in \mathcal{F}(G), \quad i = 1, \ldots, n,$$

where $\mathcal{F}(G)$ is the algebra of $C^\infty$ functions on $G$. Moreover, because the bracket satisfies the Leibniz rule, a vector field $X_{f_1, \ldots, f_{n-1}}$ can be defined by

$$X_{f_1, \ldots, f_{n-1}}(g) = \{f_1, \ldots, f_{n-1}, g\}, \quad g \in \mathcal{F}(G).$$

This vector field is said to be Hamiltonian, and the space of Hamiltonian vector fields is denoted by $\mathcal{H}$.

We present the following concepts [7], [14], [15].

**Definition 1.** An element $\eta \in \Gamma(\Lambda^n TG)$ for $n \geq 3$ is called a Nambu tensor of order $n$ if it satisfies $\mathcal{L}_{X_{f_1, \ldots, f_{n-1}}} \eta = 0$ for all $X_{f_1, \ldots, f_{n-1}} \in \mathcal{H}$ with $f_i \in \mathcal{F}(G)$; here $\mathcal{L}$ denotes the Lie derivative.

**Definition 2.** An element $\eta \in \Gamma(\Lambda^n TG)$ is called a multiplicative tensor if for all $g_1, g_2 \in G$, we have

$$\eta_{g_1 g_2} = L_{g_1*} \eta_{g_2} + R_{g_2*} \eta_{g_1},$$

where $R_{g_2}$ and $L_{g_1}$ are respectively right and left translations in $G$. A Lie group $G$ endowed with a multiplicative Nambu tensor $\eta$ is called a Nambu–Lie group [13].

The following theorem and its corollary were proved in [13].

**Theorem 1.** Let $G$ be an $m$-dimensional Lie group and $\mathfrak{h}$ be an $n$-dimensional Lie subalgebra of $\mathfrak{g}$ with $n \geq 3$. For a basis $X_1, \ldots, X_n$ of $\mathfrak{h}$, we set $\eta = X_1 \wedge \ldots \wedge X_n$. Then $\eta$ is a left-invariant Nambu tensor of order $n$ on $G$. Conversely, if $\eta = X_1 \wedge \ldots \wedge X_n \in \Lambda^n \mathfrak{g}$ is a left-invariant Nambu tensor on $G$, then $\mathfrak{h} = \{X_1, \ldots, X_n\}$ is a Lie subalgebra of $\mathfrak{g}$.

**Corollary 1.** There is a one-to-one correspondence up to a coefficient (function) between the set of left-invariant Nambu tensors of order $n$ on $G$ and the set of $n$-dimensional Lie subalgebras of $\mathfrak{g}$.

We note that for a Nambu tensor $\eta$ of order $n \geq 3$, if $f$ is a smooth function, then $f \eta$ is again a Nambu tensor [7].

**Theorem 2** [6]. Let $(G, \eta)$ be an $m$-dimensional compact or semisimple Nambu–Lie group and $\eta$ be of top order. Then $\eta = 0$.

The following theorem gives one of the characterizations of Nambu–Lie groups, proved by Vaisman [13].

**Theorem 3.** If $G$ is a connected Lie group endowed with a Nambu tensor $\eta$ that vanishes on the unit $e$ of $G$, then $(G, \eta)$ is a Nambu–Lie group if and only if the $n$-bracket of any $n$ left- or right-invariant 1-forms of $G$ is respectively a left- or right-invariant 1-form.
Using this theorem, we can characterize a multiplicative tensor \( \eta \) of top order. Let \( \mathcal{G} \) be a Lie algebra of \( G \) with basis \( X_1, \ldots, X_m \). It is clear that left-invariant vector fields can be considered a basis of \( \mathcal{G} \). We let the same letters \( X_i \) denote these left-invariant vector fields. Because \( \eta \) is of top order, \( \eta \) can be represented as \( \eta = f X_1 \wedge \ldots \wedge X_m \) with some function \( f \in \mathcal{F}(G) \). Using this notation, we give a theorem proved in [15].

**Theorem 4.** Let \( \eta = f X_1 \wedge \ldots \wedge X_m \), where \( f \in \mathcal{F}(G) \), be a tensor of top order on \( G \) (such a tensor is always a Nambu tensor). Then \( \eta \) is multiplicative if and only if \( f(e) = 0 \) and

\[
X_i f + \left( \sum_{k=1}^{m} C_{ik}^k \right) f = q_i, \quad i = 1, \ldots, m,
\]

where \( C_{ij}^k \) are the structure constants of \( \mathcal{G} \) with respect to the basis \( X_1, \ldots, X_m \) and \( q_i \) are some constants.

This theorem was used in [15] to obtain the Nambu structures of order three for the three-dimensional real Lie groups. Here, in the same way, we determine the Nambu structure of order four (top order) for four-dimensional real Lie groups and also use Theorem 1 to calculate the Nambu structure of order three for these Lie groups.

### 3. Nambu structures on four-dimensional real Lie groups

Using Theorems 1 and 4 in this section, we calculate the Nambu structures of orders four and three on four-dimensional real Lie groups. We note that we use the Patera–Winternitz classification [17] for four-dimensional Lie algebras and their subalgebras.

We let \( \mathcal{G} \) denote the four-dimensional real Lie algebra corresponding to a simply connected Lie group \( G \) and also let \( X_1, X_2, X_3, \) and \( X_4 \) denote the left-invariant vector fields. To calculate these vector fields, we must determine the left-invariant 1-forms; these calculations were already done in [18]. We use those results here to obtain the left-invariant vector fields. In general, for a Lie group \( G \) with the Lie algebra \( \mathcal{G} \) with an abstract basis \( \{ T_i \} \), the left-invariant 1-forms can be determined as

\[
g^{-1} dg = e^i_{\mu} T_i dx^\mu, \quad g \in G,
\]

and for the left-invariant vector fields, we hence have

\[
X_i = V_i^\mu \partial_\mu,
\]

where \( V_i^\mu = (e_i^\mu)^t \), \( e_i^\mu \) is the inverse of \( e_i^\mu \), and \( t \) denotes the transpose of the matrix previously obtained in [18]. Hence, we can calculate the left-invariant vector fields. We present the results in Table 1 in the appendix.

Now, to calculate the Nambu structures, we can write \( \eta \in \Gamma(\Lambda^4 TG) \) as \( \eta = f X_1 \wedge X_2 \wedge X_3 \wedge X_4 \) and \( \eta \in \Gamma(\Lambda^3 TG) \) as \( \eta = f X_1 \wedge X_2 \wedge X_3 \), where \( f \in \mathcal{F}(G) \). We then use Theorem 4 to calculate the Nambu structures of order four on four-dimensional real Lie groups and use Theorems 1 and 4 to find the Nambu structures of order three on four-dimensional real Lie groups.

We demonstrate how our method works in the example of the Lie algebra \( A_{4,8} \). This is a Lie algebra isomorphic to the Heisenberg algebra with the commutation relations [17]

\[
[X_2, X_4] = X_2, \quad [X_3, X_4] = -X_3, \quad [X_2, X_3] = X_1.
\]

The left-invariant vector fields for this Lie algebra have the forms

\[
X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = (x^3 e^{-x^4}) \frac{\partial}{\partial x^1} + e^{-x^4} \frac{\partial}{\partial x^2}, \quad X_3 = e^{x^4} \frac{\partial}{\partial x^3}, \quad X_4 = \frac{\partial}{\partial x^4},
\]
and they satisfy commutation relations (1). By Theorem 4, a function \( f(x^1, x^2, x^3, x^4) \) must satisfy \( f(0, 0, 0, 0) = 0 \) and

\[
X_i f + \left( \sum_{k=1}^{m} C_{ik}^f \right) f = q_i \quad i = 1, \ldots, 4, 
\]

where \( q_i \) are some constants. We can thus obtain a solution of (2) as \( f = q_4 x^4 \), and consequently

\[
\eta = q_4 x^4 \frac{\partial}{\partial x^4} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^1},
\]

which gives a Nambu–Lie structure of order four on the corresponding Lie group \( A_{4,8} \).

In the same way, for three-dimensional Lie subalgebras of \( A_{4,8} \) [17], we obtain the left-invariant vector fields:

the set of left-invariant vector fields \( \{X_2, X_3; X_1\} \) given by

\[
X_2 = \frac{\partial}{\partial x^2} - x_3 \frac{\partial}{\partial x^1}, \quad X_3 = \frac{\partial}{\partial x^3}, \quad X_1 = \frac{\partial}{\partial x^1}
\]

for \( A_{3,1} \),

the set of left-invariant vector fields \( \{X_4, X_1; X_2\} \) given by

\[
X_4 = \frac{\partial}{\partial x^4} + x_2 \frac{\partial}{\partial x^2}, \quad X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^2}
\]

for \( A_2 \oplus A_1 \), and

the set of left-invariant vector fields \( \{X_4, X_1; X_3\} \) given by

\[
X_4 = \frac{\partial}{\partial x^4} - x_3 \frac{\partial}{\partial x^3}, \quad X_1 = \frac{\partial}{\partial x^1}, \quad X_3 = \frac{\partial}{\partial x^3}
\]

for \( A_2 \oplus A_1 \).

This gives the corresponding Nambu structures of order three on \( A_{4,8} \):

\[
\eta_1 = (q_1 x^2 + q_2 x^3) \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3},
\]

\[
\eta_2 = (q_3 x^3 + q_1 (e^{x^4} - 1)) \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^4},
\]

\[
\eta_3 = (q_3 x^3 + q_1 (e^{-x^4} - 1)) \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^4}.
\]

We have thus determined all Nambu structures of order four and three on four-dimensional real Lie groups.

We list the results in Tables 1 and 2 in the appendix.

4. Superintegrable systems with Nambu structures

A Hamiltonian system with \( n \) degrees of freedom is integrable in the Liouville sense if it has \( n \) invariants in involution [19], and it is superintegrable if it has at least \( 2n - 1 \) additional independent invariants [20]. In this section, we construct superintegrable dynamical systems using Nambu structures of order four on four-dimensional real Lie groups. In fact, we consider some systems related to the Nambu structure on some Lie groups as a phase space (i.e., Lie groups with a symplectic structure [16]). We do this in two stages. In the first stage, we consider superintegrable systems with \( A_{4,8} \) as the symmetry group. In the second stage, we consider systems with \( A_{4,10} \) as the symmetry group. We note that \( A_{4,8} \) and \( A_{4,10} \) are the only Lie groups with invertible ad-invariant metrics. In what follows, \( (x^1, \ldots, x^4) \) are the coordinates on Lie groups.
4.1. Superintegrable systems with $A_{4,8}$ as a symmetry group. We consider several cases.

- Lie group $A_{4,9}^0$ as the phase space. The symplectic structure on $A_{4,9}^0$ has the form [16]

$$\{x^1, x^4\} = \alpha, \quad \{x^2, x^3\} = -\alpha,$$

where $\alpha$ is an arbitrary nonzero real constant. We introduce the Darboux coordinates

$$y^1 = -\frac{1}{\alpha} x^2, \quad y^2 = x^1, \quad y^3 = x^3, \quad y^4 = \frac{1}{\alpha} x^4, \quad (3)$$

which satisfy the standard Poisson brackets $^1$

$$\{y^1, y^3\} = 1, \quad \{y^2, y^4\} = 1. \quad (4)$$

Using the method in [21], we can now construct a dynamical system taking the Nambu structure on the Lie group $A_{4,9}^0$ as the phase space. For this, we regard the dynamical quantities $Q_a$ as functions of $x^i$ satisfying the relation

$$\{Q_a, Q_b\} = f_{ac} \{Q_c, Q\}, \quad (5)$$

where $f_{ab}$ are the structure constants of the symmetry Lie algebra $A_{4,8}$. After some calculations, we can rewrite the Nambu four-brackets (weighted by the structure constants) in terms of the Poisson bracket as [21]

$$g^{ac} g^{bd} f_{cd} \{A, Q_a, Q_b, Q_c\} = 3 g^{ac} g^{bd} f_{cd} f_{be} \eta\{A, Q_a\} Q_f, \quad (6)$$

where $g^{ac}$ is the inverse of the ad-invariant nondegenerate metric on the Lie algebra $A_{4,8}$ and $\eta$ is the Nambu structure on the Lie group $A_{4,9}^0$ in terms of $y^i, i = 1, 2, 3, 4$. After some calculations, we find that for the Lie algebra $A_{4,8}$, the ad-invariant metric has the form

$$g_{ab} = \begin{pmatrix} 0 & 0 & 0 & s \\ 0 & 0 & -s & 0 \\ 0 & -s & 0 & 0 \\ s & 0 & 0 & 0 \end{pmatrix}, \quad (6)$$

where $s$ is an arbitrary nonzero real constant.

On the other hand, after some calculations, we obtain

$$g^{ac} g^{bd} f_{cd} f_{be} Q_a Q_f = -\frac{2}{s^2} Q_1^2, \quad (7)$$

where $Q_1$ is the Casimir function of $A_{4,8}$ [22], while for the dynamical system with the symmetry Lie algebra $A_{4,8}$, this expression is proportional to the system Hamiltonian. In connection with this, we have

$$g^{ac} g^{bd} f_{cd} f_{be} \{A, Q_a, Q_b, Q_c\} = -\frac{3\eta}{s^2} \{A, H\} = -\frac{3\eta}{s^2} \frac{\partial A}{\partial t}, \quad (8)$$

where $\eta$ is the value of the Nambu structure of the Lie group $A_{4,9}^0$ and $H = Q_1^2$. Therefore, the evolution of the dynamical system can be described in terms of the Nambu structure as

$$\frac{\partial A}{\partial t} = -\frac{s^2}{3\eta} g^{ac} g^{bd} f_{cd} \{A, Q_a, Q_b, Q_c\}. \quad (8)$$

\footnote{We note that in the subsequent examples, the Darboux coordinates satisfy standard Poisson brackets (4).}
Using the realization of the Lie algebra $A_{4,8}$ [23] in $\mathbb{R}^4$

\[
X_1 = \frac{\partial}{\partial y_1}, \quad X_2 = \frac{\partial}{\partial y_2}, \quad X_3 = y_2^2 \frac{\partial}{\partial y_1}, \quad X_4 = y_2^2 \frac{\partial}{\partial y_2},
\]

we now obtain the forms\(^2\) for the $Q_i$

\[
Q_1 = -p_1 = -y^3, \quad Q_2 = -p_2 = -y^4, \quad Q_3 = -y^2p_1 = -y^2y^3, \quad Q_4 = -y^2p_2 = -y^2y^4.
\]

They satisfy (5) by virtue of relations (4). We can write the corresponding Hamiltonian as

\[
H = p_1^2 = (y^3)^2.
\]  

(9)

We can thus describe the dynamics of the superintegrable system (with involutive functions,\(^3\) e.g., $(H, Q_1, Q_2)$) in terms of the Nambu structure as\(^4\)

\[
\frac{\partial A}{\partial t} = -\frac{s^2}{3q_4(e^{-2\alpha y^4} - 1)}g^{ac}g^{bd}f_{cd}\{A, Q_a, Q_b, Q_c\},
\]

where we use $\eta = q_4(e^{-2\alpha y^4} - 1)$ (see Table 1) and Darboux coordinates (3).

- **Lie group $A_{4,1}^{-1}$ as the phase space.** For the Lie group $A_{4,2}^{-1}$, the symplectic structure has the form [16]

\[
\{x^1, x^2\} = 2\alpha, \quad \{x^1, x^3\} = -\alpha, \quad \{x^2, x^4\} = \beta e^{-x^4},
\]

where $\alpha$ and $\beta$ are arbitrary nonzero real numbers. The Darboux coordinates for $A_{4,2}^{-1}$ are [24]

\[
y^1 = \frac{e^{x^4}}{\beta} + x^3, \quad y^2 = -\frac{2\alpha e^{x^4} - \beta x^1 + \alpha \beta x^2}{\alpha \beta^2}, \quad y^3 = \frac{2e^{x^4}}{\beta} + \frac{x^1}{\alpha}, \quad y^4 = e^{x^4},
\]

and the Nambu structure on the Lie group $A_{4,2}^{-1}$ (see Table 1) in terms of $y^i$, $i = 1, 2, 3, 4$, is $\eta = q_4(1/y^4 - 1)$. Therefore, we can describe the dynamics of the superintegrable system using (8) as

\[
\frac{\partial A}{\partial t} = -\frac{s^2}{3q_4(1/y^4 - 1)}g^{ac}g^{bd}f_{cd}\{A, Q_a, Q_b, Q_c\}.
\]

In this case, the Hamiltonian has the form

\[
H = (y^3)^2 = \left(\frac{2e^{x^4}}{\beta} + \frac{x^1}{\alpha}\right)^2.
\]

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\(^2\)We note that we use the quantum mechanical realization of the operators $p_i = -\partial/\partial y_i$, where $p_1 = y^3$ and $p_2 = y^4$ for standard Poisson bracket (4).

\(^3\)In the subsequent examples, the functions that are involutive can be assumed to be one of the sets $(H, Q_1, Q_2)$, $(H, Q_1, Q_3)$, or $(H, Q_1, Q_4)$.

\(^4\)We note that all of the systems obtained in what follows have the same dynamical evolution as (8) and the same Hamiltonian (9) (in the Darboux coordinates) with differences in the value $\eta$ (the coefficient function of the Nambu structure).
● Lie group $A_{4,3}$ as the phase space. The nondegenerate Poisson structure on $A_{4,3}$ is [16]

$$\{x^1, x^2\} = \alpha x^4 e^{-x^4}, \quad \{x^1, x^3\} = \beta e^{-x^4}, \quad \{x^1, x^4\} = \gamma e^{-x^4}, \quad \{x^2, x^3\} = \lambda,$$

where $\alpha$, $\beta$, $\gamma$, and $\lambda$ are arbitrary nonzero real numbers. The Darboux coordinates for $A_{4,3}$ are written as [24]

$$y^1 = \frac{\beta x^2}{\lambda} + \frac{\alpha \gamma (x^3)^2}{2\beta \lambda} - \frac{\alpha x^3 x^4}{\lambda}, \quad y^2 = \frac{x^1}{\gamma} - \frac{\beta e^{-x^4} x^2}{2\gamma} - \frac{\alpha e^{-x^4} (x^3)^2}{2\beta \lambda} + \frac{\alpha e^{-x^4} x^3 x^4}{\gamma \lambda},$$

$$y^3 = \frac{x^3}{\beta}, \quad y^4 = e^{-x^4},$$

and the Nambu structure on the Lie group $A_{4,3}$ (see Table 1) is expressed in terms of $y^i$, $i = 1, 2, 3, 4$, as

$$\eta = q_1 \gamma y^2 + q_1 y^1 + \frac{\alpha q_1 y^3 \log y^4}{2\gamma} - \frac{q_1 \alpha \beta y^3 \log y^4}{\lambda y^4} + \frac{q_4}{y^4} - q_4.$$

The dynamics of the superintegrable system can be described using (8) and the Nambu bracket as

$$\frac{\partial A}{\partial t} = -\frac{s^2}{3(q_1 \gamma y^2 + q_1 y^1 / y^4 + (\alpha q_1 y^3 \log y^4)/(2\gamma y^4) - (q_1 \alpha \beta y^3 \log y^4)/(\lambda y^4) + q_4 / y^4 - q_4)} \times g^{ac} g^{bd} f^{ec}_{cd} \{A, Q_a, Q_b, Q_c\}$$

with the Hamiltonian

$$H = (y^3)^2 = \left(\frac{x^3}{\beta}\right)^2.$$

● Lie group $A_{4,6}^{a,0}$ as the phase space. For the Lie group $A_{4,6}^{a,0}$, the nondegenerate Poisson structure has the form [16]

$$\{x^1, x^4\} = \gamma e^{-\alpha x^4}, \quad \{x^2, x^3\} = \beta,$$

where $\alpha$, $\beta$, and $\gamma$ are arbitrary nonzero real numbers. The Darboux coordinates for $A_{4,6}^{a,0}$ are [24]

$$y^1 = x^3, \quad y^2 = \frac{e^{2\alpha x^4} x^1}{\alpha \gamma}, \quad y^3 = \frac{x^2}{\beta}, \quad y^4 = e^{-\alpha x^4},$$

and the Nambu structure on $A_{4,6}^{a,0}$ (see Table 1) is therefore expressed in terms of $y^i$, $i = 1, 2, 3, 4$, as

$$\eta = q_1 \alpha \gamma (y^4)^2 y^2 + q_4 y^4 - q_4,$$

and the dynamics of the superintegrable system are described using (8) as

$$\frac{\partial A}{\partial t} = -\frac{s^2}{3(q_1 \alpha \gamma (y^4)^2 y^2 + q_4 y^4 - q_4)} g^{ac} g^{bd} f^{ec}_{cd} \{A, Q_a, Q_b, Q_c\}$$

with the Hamiltonian

$$H = (y^3)^2 = \left(\frac{x^3}{\beta}\right)^2.$$
• **Lie group $A_{4,7}$ as the phase space.** The nondegenerate Poisson structure on $A_{4,7}$ is [16]

$$
\{x^1, x^3\} = -2\alpha x^3 e^{-2x^4}, \quad \{x^1, x^4\} = \alpha e^{-2x^4}, \quad \{x^2, x^3\} = 2\alpha e^{-2x^4},
$$

where $\alpha$ is an arbitrary nonzero real number. The Darboux coordinates for $A_{4,7}$ are [24]

$$
y^1 = \frac{e^{2x^4} x^2}{2\alpha}, \quad y^2 = \frac{-1 - e^{2x^4} + e^{4x^4} x^1 + e^{4x^4} x^2 x^3}{2\alpha}, \quad y^3 = x^3, \quad y^4 = e^{-2x^4},
$$

and the Nambu structure on $A_{4,7}$ (see Table 1) can consequently be expressed in terms of $y^i$, $i = 1, 2, 3, 4$, as

$$
\eta = q_4((y^4)^2 - 1).
$$

The dynamical equation for the superintegrable system can then be obtained from (8), and we have

$$
\frac{\partial A}{\partial t} = -\frac{s^2}{3q_4((y^4)^2 - 1)} g^{ac} g^{bd} f_{cd} \{A, Q_a, Q_b, Q_c\}
$$

with the Hamiltonian

$$
H = (y^3)^2 = (x^3)^2.
$$

• **Lie group $A_{4,9}$ as the phase space.** The symplectic structure on $A_{4,9}^1$ is [16]

$$
\{x^1, x^3\} = 2\alpha x^3 e^{-2x^4}, \quad \{x^1, x^4\} = -\alpha e^{-2x^4}, \quad \{x^2, x^3\} = -2\alpha e^{-2x^4},
$$

where $\alpha$ is an arbitrary nonzero real number. The Darboux coordinates for $A_{4,9}^1$ are [24]

$$
y^1 = -\frac{e^{2x^4} x^2}{2\alpha}, \quad y^2 = \frac{-1 - e^{2x^4} + e^{4x^4} x^1 + e^{4x^4} x^2 x^3}{2\alpha}, \quad y^3 = x^3, \quad y^4 = e^{-2x^4},
$$

and the Nambu structure on $A_{4,9}^1$ (see Table 1) can be written in terms of $y^i$, $i = 1, 2, 3, 4$, as above using (10). In this case, the dynamical equation of the superintegrable system takes form (11) with Hamiltonian (12).

• **Lie group $A_{4,12}$ as phase space.** The nondegenerate Poisson structure on $A_{4,12}$ is [16]

$$
\{x^1, x^3\} = -\gamma e^{-x^3}(\alpha \cos x^4 + \beta \sin x^4), \quad \{x^1, x^4\} = \gamma e^{-x^3}(-\beta \cos x^4 + \alpha \sin x^4),
$$

$$
\{x^2, x^3\} = \gamma e^{-x^3}(-\beta \cos x^4 - \alpha \sin x^4), \quad \{x^2, x^4\} = -\gamma e^{-x^3}(\alpha \cos x^4 + \beta \sin x^4),
$$

where $\alpha$ and $\beta$ are arbitrary nonzero real numbers and $\gamma = 1/(\alpha^2 + \beta^2)$. The Darboux coordinates for $A_{4,12}$ are [24]

$$
y^1 = e^{2x^3}(\alpha x^1 \cos x^4 - \beta x^2 \cos x^4 + \beta x^1 \sin x^4 + \alpha x^2 \sin x^4),
$$

$$
y^2 = e^{-x^3}(\beta x^1 \cos x^4 + \alpha x^2 \cos x^4 - \alpha x^1 \sin x^4 + \beta x^2 \sin x^4),
$$

$$
y^3 = e^{x^3}, \quad y^4 = x^4,
$$

and the Nambu structure on $A_{4,12}$ (see Table 1) is expressed in terms of $y^i$, $i = 1, 2, 3, 4$, as

$$
\eta = q_4\left(\frac{1}{(y^3)^2} - 1\right).
$$

In this case, the dynamical equation of the superintegrable system is obtained from (8) in the form

$$
\frac{\partial A}{\partial t} = -\frac{s^2}{3q_4(1/(y^3)^2 - 1)} g^{ac} g^{bd} f_{cd} \{A, Q_a, Q_b, Q_c\}
$$

with the Hamiltonian

$$
H = (y^3)^2 = e^{2x^3}.
$$
4.2. Superintegrable systems with $A_{4,10}$ as a symmetry group. As in the preceding subsection, we construct some dynamical systems using the Nambu structure on some Lie groups as a phase space and with the symmetry group $A_{4,10}$. We find the ad-invariant nondegenerate metric on $A_{4,10}$:

$$
g_{ab} = \begin{pmatrix}
0 & 0 & 0 & m \\
0 & m & 0 & 0 \\
0 & 0 & m & 0 \\
n & 0 & 0 & 0
\end{pmatrix},
$$

(13)

where $m$ and $n$ are arbitrary nonzero real numbers. In this case, instead of (7) and (8), we have the relations

$$g^{ac}g^{bd}f_{ce}f^e_{cd}Q_aQ_f = \frac{2}{mn}Q_1^2$$

and

$$\frac{\partial A}{\partial t} = \frac{mn}{3\eta}g^{ac}g^{bd}f_{cd}\{A, Q_a, Q_b, Q_c\}.$$

Hence, using the realization [23] in $\mathbb{R}^4$

$$X_1 = \frac{\partial}{\partial y_1}, \quad X_2 = \frac{\partial}{\partial y_2}, \quad X_3 = y^1 \frac{\partial}{\partial y_1} + y^2 \frac{\partial}{\partial y_2}, \quad X_4 = y^2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2},$$

we obtain

$$Q_1 = -y^3, \quad Q_2 = -y^4, \quad Q_3 = -y^1 y^3 - y^2 y^4, \quad Q_4 = -y^2 y^3 + y^1 y^4.$$ 

In this case, the Hamiltonian of the superintegrable systems with the symmetry group $A_{4,10}$ are the same as in the preceding section with the differences in only dynamical equations (8) (i.e., in the inverse metrics of (6) and (13) in (8) and also structure constants $f^a_{bc}$ of $A_{4,8}$ and $A_{4,10}$).

We thus write the dynamical equations for the four-dimensional real Lie groups taken as the phase space:

- With $A_{4,9}^0$ as the phase space, we have
  $$\frac{\partial A}{\partial t} = \frac{mn}{3q_4(e^{-2\alpha y^4} - 1)}g^{ac}g^{bd}f_{cd}\{A, Q_a, Q_b, Q_c\}.$$

- With $A_{4,2}^{-1}$ as the phase space, we have
  $$\frac{\partial A}{\partial t} = \frac{mn}{3q_4(1/y^4 - 1)}g^{ac}g^{bd}f_{cd}\{A, Q_a, Q_b, Q_c\}.$$

- With $A_{4,3}$ as the phase space, we have
  $$\frac{\partial A}{\partial t} = \frac{mn}{3(q_1 \gamma y^2 + q_1 y^4/ y^4 + (\alpha q_1 y^3 \log y^4)/(2\lambda y^4) - (q_1 \alpha \beta y^3 \log y^4)/(\lambda y^4) + q_4/y^4 - q_4)} \times$$
  $$\times g^{ac}g^{bd}f_{cd}\{A, Q_a, Q_b, Q_c\}.$$
• With $A^a_{4,6}$ as the phase space, we have

$$\frac{\partial A}{\partial t} = \frac{mn}{3(q_1\alpha\gamma(y^4)^2y^2 + q_4y^4 - q_4)} g^{ac} g^{bd} f_{cd} \{A, Q_a, Q_b, Q_c\}.$$ 

• With $A_{4,7}$ as the phase space, we have

$$\frac{\partial A}{\partial t} = \frac{mn}{3q_4((y^4)^2 - 1)} g^{ac} g^{bd} f_{cd} \{A, Q_a, Q_b, Q_c\}.$$ 

• With $A^1_{4,9}$ as the phase space, we have

$$\frac{\partial A}{\partial t} = \frac{mn}{3q_4((y^4)^2 - 1)} g^{ac} g^{bd} f_{cd} \{A, Q_a, Q_b, Q_c\}.$$ 

• With $A_{4,12}$ as the phase space, we have

$$\frac{\partial A}{\partial t} = \frac{mn}{3q_4(1/(y^2)^2 - 1)} g^{ac} g^{bd} f_{cd} \{A, Q_a, Q_b, Q_c\}.$$ 

5. Conclusions

We have obtained all Nambu structures of order four and three on four-dimensional real Lie groups. We also obtained new superintegrable systems with some four-dimensional real Lie groups as the phase space and the symmetry groups $A_{4,8}$ and $A_{4,10}$ such that their dynamical evolution equations are described in terms of the related Nambu structures of order four.

Appendix

In Table 1, we present the Nambu structures of order four on four-dimensional real Lie groups and the corresponding left-invariant vector fields. In Table 2, we present the corresponding Lie algebras with left-invariant vector fields and the Nambu structure of order three on the Lie groups. We use the notation $\partial_i = \partial/\partial x_i$, $i = 1, 2, 3, 4$, in the tables.
| Lie algebra  | Structure constants | Left-invariant vector fields | $\eta^{1234}$ |
|-------------|---------------------|-----------------------------|---------------|
| 4A₁        | $f^k_{ij} = 0$      | $X_1 = \partial_1$, $X_2 = \partial_2$, $X_3 = \partial_3$, $X_4 = \partial_4$ | $q_1x^1 + q_2x^2 + q_3x^3 + q_4x^4$ |
| $A_2 \oplus 2A_1$ | $f^2_{12} = 1$ | $X_1 = \partial_1 - x^2\partial_2$, $X_2 = \partial_2$, $X_3 = \partial_3$, $X_4 = \partial_4$ | $q_2x^2 + q_1(e^{-x^1} - 1)$ |
| 2A₂      | $f^2_{12} = 1$, $f^1_{34} = 1$ | $X_1 = \partial_1 - x^2\partial_2$, $X_2 = \partial_2$, $X_3 = \partial_3 - x^4\partial_4$, $X_4 = \partial_4$ | $q_3(e^{-(x^1+x^3)} - 1)$ |
| $A_{3,1} \oplus A_1$ | $f^1_{23} = 1$ | $X_1 = \partial_1$, $X_2 = -x^3\partial_1 + \partial_2$, $X_3 = \partial_3$, $X_4 = \partial_4$ | $q_2x^2 + q_3x^3 + q_4x^4$ |
| $A_{3,2} \oplus A_1$ | $f^1_{13} = 1$, $f^1_{23} = 1$, $f^2_{23} = 1$ | $X_1 = e^{-x^2}\partial_1$, $X_2 = -x^3e^{-x^2}\partial_1 + e^{-x^3}\partial_2$, $X_3 = \partial_3$, $X_4 = \partial_4$ | $q_3(e^{-2x^3} - 1)$ |
| $A_{3,3} \oplus A_1$ | $f^1_{13} = 1$, $f^2_{23} = 1$ | $X_1 = e^{-x^3}\partial_1$, $X_2 = e^{-x^2}\partial_2$, $X_3 = \partial_3$, $X_4 = \partial_4$ | $q_3(e^{-2x^3} - 1)$ |
| $A_{3,4} \oplus A_1$ | $f^1_{13} = 1$, $f^2_{23} = -1$ | $X_1 = e^{-x^2}\partial_1$, $X_2 = e^{-x^3}\partial_2$, $X_3 = \partial_3$, $X_4 = \partial_4$ | $q_3x^3 + q_4x^4$ |
| $A_{3,5} \oplus A_1$ | $F^1_{13} = 1$, $f^2_{23} = a$ | $X_1 = e^{-x^2}\partial_1$, $X_2 = e^{-x^3}\partial_2$, $X_3 = \partial_3$, $X_4 = \partial_4$ | $q_3(e^{-(a+1)x^3} - 1)$ |
| $A_{3,6} \oplus A_1$ | $f^2_{13} = -1$, $f^1_{23} = 1$ | $X_1 = \cos x^3\partial_1 + \sin x^3\partial_2$, $X_2 = -\sin x^3\partial_1 + \cos x^3\partial_2$, $X_3 = \partial_3$, $X_4 = \partial_4$ | $q_3x^3 + q_4x^4$ |
| $A_{3,7} \oplus A_1$ | $f^1_{13} = a$, $f^2_{13} = -1$, $f^1_{23} = 1$, $f^2_{23} = a$ | $X_1 = \partial_1$, $X_2 = \partial_2$, $X_3 = \partial_3$, $X_4 = \partial_4$ | $q_3(e^{2ax^3} - 1)$ |
| $A_{3,8} \oplus A_1$ | $f^2_{13} = -2$, $f^1_{12} = 1$, $f^2_{23} = 1$ | $X_1 = e^{-x^2}\partial_1 + (2x^2)\partial_2 - (x^3)^2\partial_3$, $X_2 = \partial_2 - x^3\partial_3$, $X_3 = \partial_3$, $X_4 = \partial_4$ | $q_4e^{-x^2}x^4$ |

continued
| Lie algebra | Structure constants | Left-invariant vector fields | \eta^{1234} |
|-------------|---------------------|----------------------------|------------|
| \( A_{3,9} \oplus A_1 \) | \( f_{12}^1 = 1, f_{13}^2 = -1, f_{12}^3 = 1 \) | \( X_1 = \frac{\cos x^2}{\cos x^2} \partial_1 + \sin x^3 \partial_2 - \frac{\cos x^3 \sin x^2}{\cos x^2} \partial_3, \) | \( q_4 x^4 \) |
| \( A_{4,1} \) | \( f_{14}^1 = 1, f_{24}^2 = 1 \) | \( X_1 = \partial_1, \ X_2 = -x^4 \partial_1 + \partial_2, \) | \( q_3 x^3 + q_4 x^4 \) |
| \( A_{4,2}^a \) | \( f_{14}^1 = a, f_{24}^2 = 1, f_{34}^3 = 1 \) | \( X_1 = e^{-ax} \partial_1, \ X_2 = e^{-a x} \partial_2, \) | \( q_4(e^{-(a+2)x^4} - 1) \) |
| \( A_{4,2}^b \) | \( f_{14}^1 = 1, f_{24}^2 = 1, f_{34}^3 = 1 \) | \( X_1 = e^{-x} \partial_1, \ X_2 = x^4 \partial_1 + \partial_2, X_3 = -x^4 e^{-x} \partial_1 + e^{-x} \partial_2, \) | \( q_4(e^{-3x^4} - 1) \) |
| \( A_{4,3} \) | \( f_{14}^1 = 1, f_{24}^2 = 1 \) | \( X_1 = e^{-x} \partial_1, \ X_2 = x^2 \partial_1 + \partial_2, \) | \( q_1 x^1 + q_4(e^{-x^4} - 1) \) |
| \( A_{4,4} \) | \( f_{14}^1 = 1, f_{24}^2 = 1, f_{34}^3 = 1 \) | \( X_1 = e^{-x} \partial_1, \ X_2 = x^4 \partial_1 + \partial_2, X_3 = -x^4 e^{-x} \partial_1 + e^{-x} \partial_2, \) | \( q_4(e^{-3x^4} - 1) \) |
| \( A_{4,5}^a \) | \( f_{14}^1 = 1, f_{24}^2 = a, f_{34}^3 = b \) | \( X_1 = e^{-x} \partial_1, \ X_2 = e^{-ax} \partial_2, \) | \( q_1 x^1 + q_3 x^3 e^{(a+b+1)x^4} + q_4(e^{-(a+b+1)x^4} - 1) \) |
| \( A_{4,5}^b \) | \( f_{14}^1 = 1, f_{24}^2 = a, f_{34}^3 = a \) | \( X_1 = e^{-x} \partial_1, X_2 = e^{-ax} \partial_2, X_3 = -e^{-ax} \partial_3, X_4 = \partial_4 \) | \( q_4(1 - e^{-(2a+1)x^4}) \) |
| \( A_{4,5}^a^1 \) | \( f_{34}^1 = 1, f_{14}^2 = 1, f_{24}^3 = a \) | \( X_1 = e^{-x} \partial_1, X_2 = e^{-ax} \partial_2, \) | \( q_1 x^1 + q_3 x^3 + q_4(e^{-a+2)x^4} - 1) \) |
| \( A_{4,5}^b^1 \) | \( f_{14}^1 = 1, f_{24}^2 = 1, f_{34}^3 = 1 \) | \( X_1 = e^{-x} \partial_1, \ X_2 = e^{-ax} \partial_2, \) | \( q_4(e^{-3x^4} - 1) \) |
| Lie algebra | Structure constants | Left-invariant vector fields | $\eta^{1234}$               |
|-------------|----------------------|-----------------------------|-----------------------------|
| $A_{4,6}^{a,b}$ | $f_{14}^1 = a, f_{34}^1 = 1, f_{34}^2 = b, f_{23}^2 = b$ | $X_1 = e^{-ax^4} \partial_1$, $X_2 = e^{2x^4} \cos x^4 \partial_2 + \sin x^4 \partial_3$, $X_3 = -e^{-bx^4} \sin x^4 \partial_2 + \cos x^4 \partial_3$, $X_4 = \partial_4$ | $q_1 x^1 e^{-2b x^4} + q_4 (e^{-(a+2b) x^4} - 1)$ |
| $A_{4,7}$ | $f_{14}^1 = 2, f_{24}^1 = 1, f_{23}^1 = 1, f_{34}^2 = 1$ | $X_1 = e^{-2x^4} \partial_1$, $X_2 = x^3 e^{-x^4} \partial_1 + e^{-x^4} \partial_2$, $X_3 = -x^3 x^4 e^{-x^4} \partial_1 - x^4 e^{-x^4} \partial_2 + e^{-x^4} \partial_3$, $X_4 = \partial_4$ | $q_4 (e^{-4x^4} - 1)$ |
| $A_{4,8}$ | $f_{23}^1 = 1, f_{24}^3 = 1, f_{34}^1 = -1$ | $X_1 = \partial_1$, $X_2 = x^3 e^{-x^4} \partial_1 + e^{-x^4} \partial_1$, $X_3 = x^4 e^{-x^4} \partial_3$, $X_4 = \partial_4$ | $q_4 x^4$ |
| $A_{4,9}^b$ | $f_{14}^1 = 1 + b, f_{24}^1 = 1, f_{23}^1 = b$ | $X_1 = e^{-(b+1)x^4} \partial_1$, $X_2 = -x^3 e^{-x^4} \partial_1 + e^{-x^4} \partial_2$, $X_3 = e^{-bx^4} \partial_3$, $X_4 = \partial_4$ | $q_2 x^2 + q_4 (e^{-2(b+1)x^4} - 1)$ |
| $A_{4,9}^1$ | $f_{23}^1 = 1, f_{24}^1 = 1, f_{14}^2 = 2, f_{34}^3 = 1$ | $X_1 = e^{-2x^4} \partial_1$, $X_2 = -x^3 e^{-x^4} \partial_1 + e^{-x^4} \partial_2$, $X_3 = e^{-x^4} \partial_3$, $X_4 = \partial_4$ | $q_4 (e^{-4x^4} - 1)$ |
| $A_{4,9}^0$ | $f_{14}^1 = 1, f_{23}^1 = 1, f_{24}^3 = 1$ | $X_1 = e^{-x^4} \partial_1$, $X_2 = -x^3 e^{-x^4} \partial_1 + e^{-x^4} \partial_2$, $X_3 = \partial_3$, $X_4 = \partial_4$ | $q_4 (e^{-2x^4} - 1)$ |
| $A_{4,10}$ | $f_{23}^1 = 1, f_{24}^3 = -1, f_{34}^3 = 1$ | $X_1 = \partial_1$, $X_2 = -x^3 \cos x^4 \partial_1 + \cos x^4 \partial_2 + \sin x^4 \partial_3$, $X_3 = x^3 \sin x^4 \partial_1 - \sin x^4 \partial_2 + \cos x^4 \partial_3$, $X_4 = \partial_4$ | $q_4 x^4$ |

continued
### Table 1 (cont.)

| Lie algebra | Structure constants | Left-invariant vector fields | $\eta^{1234}$ |
|-------------|---------------------|-----------------------------|---------------|
| $A^q_{3,11}$ | $f^1_{23} = 1, f^2_{23} = a,$<br>$f^1_{14} = 2a,$<br>$f^3_{24} = -1,$<br>$f^2_{34} = 1,$<br>$f^3_{34} = a$ | $X_1 = e^{-2ax^i} \partial_1,$<br>$X_2 = -x^3 e^{-ax^i} \cos x^4 \partial_1 + e^{-ax^i} \cos x^4 \partial_2 +$<br>$+ e^{-ax^i} \sin x^4 \partial_3,$<br>$X_3 = e^{-ax^i} x^3 \sin x^4 \partial_1 - e^{-ax^i} \sin x^4 \partial_2 +$<br>$+ e^{-ax^i} \cos x^4 \partial_3,$<br>$X_4 = \partial_4$ | $q_4(e^{-4ax^4} - 1)$ |
| $A_{4,12}$  | $f^1_{13} = 1, f^2_{23} = 1,$<br>$f^1_{14} = -1,$<br>$f^2_{24} = 1$ | $X_1 = e^{-x^3} \cos x^4 \partial_1 + e^{-x^3} \sin x^4 \partial_2,$<br>$X_2 = -e^{-x^3} \sin x^4 \partial_1 + e^{-x^3} \cos x^4 \partial_2,$<br>$X_3 = \partial_3,$<br>$X_4 = \partial_4$ | $q_3(e^{-2x^3} - 1)$ |

### Table 2

| Lie subalgebra | Basis Lie subalgebra | Left-invariant vector fields | $\eta^{ijk}$ |
|----------------|----------------------|-----------------------------|---------------|
| $3A_1 \subset 4A_1$ | $X_1 + aX_4,$<br>$X_2 + bX_4,$<br>$X_3 + cX_4$ | $X_1 + aX_4 = \partial_1 + a\partial_4,$<br>$X_2 + bX_4 = \partial_2 + b\partial_4,$<br>$X_3 + cX_4 = \partial_3 + c\partial_4$ | $\eta^{123} = q_1 x^1 + q_2 x^2 + q_3 x^3,$<br>$\eta^{124} = c(q_1 x^1 + q_2 x^2 + q_3 x^3),$<br>$\eta^{143} = b(q_1 x^1 + q_2 x^2 + q_3 x^3),$<br>$\eta^{234} = a(q_1 x^1 + q_2 x^2 + q_3 x^3)$ |
| $3A_1 \subset A_2 + 2A_1$ | $X_1 + aX_3,$<br>$X_2 + bX_3,$<br>$X_4$ | $X_1 + aX_3 = \partial_1 + a\partial_3,$<br>$X_2 + bX_3 = \partial_2 + b\partial_3,$<br>$X_4 = \partial_4$ | $\eta^{123} = q_1 x^1 + q_2 x^2 + q_3 x^4,$<br>$\eta^{124} = a(q_1 x^1 + q_2 x^2 + q_3 x^4),$<br>$\eta^{134} = a(q_1 x^1 + q_2 x^2 + q_3 x^4),$<br>$\eta^{234} = a(q_1 x^1 + q_2 x^2 + q_3 x^4)$ |
| $3A_1 \subset 4A_1$ | $X_1, X_3, X_4$ | $X_1 = \partial_1,$<br>$X_3 = \partial_3,$<br>$X_4 = \partial_4$ | $\eta^{123} = q_1 x^1 + q_2 x^3 + q_3 x^4,$<br>$\eta^{234} = q_1 x^1 + q_2 x^2 + q_3 x^4,$<br>$\eta^{134} = q_1 x^1 + q_2 x^3 + q_3 x^4,$<br>$\eta^{124} = a(q_1 x^1 + q_2 x^2 + q_3 x^4)$ |
| $3A_1 \subset A_2 + 2A_1$ | $X_2, X_3, X_4$ | $X_2 = \partial_2,$<br>$X_3 = \partial_3,$<br>$X_4 = \partial_4$ | $\eta^{123} = q_1 x^1 + q_2 x^2 + q_3 x^4,$<br>$\eta^{124} = a(q_1 x^1 + q_2 x^2 + q_3 x^4),$<br>$\eta^{134} = a(q_1 x^1 + q_2 x^2 + q_3 x^4),$<br>$\eta^{234} = q_1 x^1 + q_2 x^2 + q_3 x^4$ |

continued
| Lie subalgebra | Basis Lie subalgebra | Left-invariant vector fields | $\eta^{ij,k}$ |
|----------------|----------------------|-----------------------------|-------------|
| $A_2 \oplus A_1 \subset A_2 \oplus 2A_1$ | $X_1 + a(X_3 \cos \phi + X_4 \sin \phi)$, $X_3 \sin \phi - X_4 \cos \phi$, $X_2$ | $X_1 + a(X_3 \cos \phi + X_4 \sin \phi) = \partial_1 + a \cos \phi \partial_3 + a \sin \phi \partial_4$, $X_3 \sin \phi - X_4 \cos \phi = \sin \phi \partial_3 - \cos \phi \partial_4$, $X_2 = \partial_2$ | $\eta^{132} = (q_3 x^2 + q_1 (e^{-x_1} - 1)) \sin \phi$, $\eta^{142} = (q_3 x^2 + q_1 (e^{-x_1} - 1)) \cos \phi$, $\eta^{432} = a(q_3 x^2 + q_1 (e^{-x_1} - 1))$ |
| $A_1 \oplus A_2 \subset 2A_2$ | $X_1, X_3, X_2$, $X_1, X_4, X_2$, $X_1, X_3, X_4$, $X_2, X_3, X_4$ | $X_1 = \partial_1 - x_2 \partial_2$, $X_3 = \partial_3$, $X_2 = \partial_2$, $X_1 = \partial_1 - x_2 \partial_2$, $X_4 = \partial_4$, $X_2 = \partial_2$, $X_3 = \partial_3$, $X_4 = \partial_4$ | $\eta^{132} = q_3 x^2 + q_1 (e^{-x_1} - 1)$, $\eta^{142} = q_3 x^2 + q_1 (e^{-x_1} - 1)$, $\eta^{324} = q_3 x^2 + q_2 (e^{-x_3} - 1)$ |
| $A_{3,3} \subset 2A_2$ | $X_1 + X_3, X_2, X_4$ | $X_1 + X_3 = \partial_1 - x_2 \partial_2 - x_4 \partial_4$, $X_2 = \partial_2$, $X_4 = \partial_4$ | $\eta^{124} = q_1 e^{-2x_1}$ |
| $A_{3,4} \subset 2A_2$ | $X_1 - X_3, X_2, X_4$ | $X_1 - X_3 = \partial_1 - \partial_3 - x_2 \partial_2 + x_4 \partial_4$, $X_2 = \partial_2$, $X_4 = \partial_4$ | $\eta^{124} = q_1 (e^{-i\theta} - 1)/(2x_1^{2})$, $\eta^{324} = -q_1 (e^{-i\theta} + 1)$ |
| $A_{3,5}^{a} \subset 2A_2$, $a = \begin{cases} \hat{a}, & 0 < |\hat{a}| < 1, \\ 1/\hat{a}, & 1 < |\hat{a}| < \infty \end{cases}$ | $X_1 + \hat{a}X_3, X_2, X_4$ | $X_1 + \hat{a}X_3 = \partial_1 + \hat{a} \partial_3 - x_2 \partial_2 - \hat{a} x_4 \partial_4$, $X_2 = \partial_2$, $X_4 = \partial_4$ | $\eta^{124} = q_1 (e^{-i\hat{a} - 1}) - 1$, $\eta^{324} = -\hat{a} q_1 (e^{-i\hat{a} + 1} - 1)$ |
| $3A_1 \subset A_{3,1} \oplus A_1$ | $X_1$, $X_2 \cos \phi + X_3 \sin \phi$, $X_4$ | $X_1 = \partial_1$, $X_2 \cos \phi + X_3 \sin \phi = \cos \phi \partial_2 + \sin \phi \partial_3$, $X_4 = \partial_4$ | $\eta^{124} = (q_1 x^1 + q_2 ((x^3)^2 + (x_2)^2)^{1/2} + q_3 x^4) \cos \phi$, $\eta^{134} = q_1 x^1 + q_2 (x^3)^2 + (x_2)^2 \cos \phi$, $\eta^{324} = (q_1 x^2 + q_2 x^3)^2 + (x_2)^2 \cos \phi$, $\eta^{341} = b(q_1 x^2 + q_2 x^3)$, $\eta^{341} = b(q_1 x^2 + q_2 x^3)$ |
| $A_{3,1} \subset A_{3,1} \oplus A_1$ | $X_2 + aX_4$, $X_3 + bX_4, X_1$ | $X_2 + aX_4 = \partial_2 + a \partial_4 - x^2 \partial_1$, $X_3 + bX_4 = \partial_3 + b \partial_4$, $X_1 = \partial_1$ | $\eta^{231} = q_1 x^2 + q_2 x^4$, $\eta^{321} = b(q_1 x^2 + q_2 x^3)$, $\eta^{341} = b(q_1 x^2 + q_2 x^3)$ |
| $3A_1 \subset A_{3,2} \oplus A_1$ | $X_1, X_2, X_4$ | $X_1 = \partial_1$, $X_2 = \partial_2$, $X_4 = \partial_4$ | $\eta^{124} = q_1 x^1 + q_2 x_2^2 + q_3 x^4$ |
| $A_2 \oplus A_1 \subset A_{3,2} \oplus A_1$ | $X_3, X_4, X_1$ | $X_3 = \partial_3 + x^1 \partial_1$, $X_4 = \partial_4$, $X_1 = \partial_1$ | $\eta^{134} = q_3 x^1 + q_1 (e^{-x_1} - 1)$ |

continued
| Lie subalgebra | Basis Lie subalgebra | Left-invariant vector fields | $\eta^{j,k}$ |
|---------------|----------------------|----------------------------|------------|
| $A_{3,2} \subset A_{3,2} \oplus A_1$ | $X_3 + aX_4, X_1, X_2$ | $X_3 + aX_4 = \partial_3 + x^4\partial_1 + (x^4 + x^2)\partial_1 + x^2\partial_1$, $X_1 = \partial_1$, $X_2 = \partial_2$ | $\eta^{123} = q_1(e^{x^3} + x^4/a - 1)$, $\eta^{124} = aq_1(e^{x^3} + x^4/a - 1)$ |
| $3A_1 \subset A_{3,3} \oplus A_1$ | $X_1, X_2, X_4$ | $X_1 = \partial_1$, $X_2 = \partial_2$, $X_4 = \partial_4$ | $\eta^{124} = q_1x^1 + q_2x^2 + q_3x^4$ |
| $A_2 \oplus A_1 \subset A_{3,3} \oplus A_1$ | $X_3, X_4$, $X_1 \cos \phi + X_2 \sin \phi$ | $X_3 = \partial_3 + ((x^1)^2 + (x^2)^2)^{1/2} \times \cos \phi \partial_1 + \sin \phi \partial_2$, $X_4 = \partial_4$, $X_1 = \partial_1$, $X_2 = \partial_2$ | $\eta^{123} = q_3((x^1)^2 + (x^2)^2)^{1/2} + q_1(e^{x^3} - 1)\cos \phi$, $\eta^{124} = q_3((x^1)^2 + (x^2)^2)^{1/2} + q_1(e^{x^3} - 1)\sin \phi$ |
| $A_{3,3} \subset A_{3,3} \oplus A_1$ | $X_3 + aX_4, X_1, X_2$ | $X_3 + aX_4 = \partial_3 + a\partial_4 + x^1\partial_1 + x^2\partial_2$, $X_1 = \partial_1$, $X_2 = \partial_2$ | $\eta^{123} = q_3(e^{x^3} + x^4/a - 1)$, $\eta^{124} = q_3(e^{x^3} + x^4/a - 1)$ |
| $3A_1 \subset A_{3,4} \oplus A_1$ | $X_1, X_2, X_4$ | $X_1 = \partial_1$, $X_2 = \partial_2$, $X_4 = \partial_4$ | $\eta^{124} = q_1x^1 + q_2x^2 + q_3x^4$ |
| $A_2 \oplus A_1 \subset A_{3,4} \oplus A_1$ | $X_3, X_4, X_1$ | $X_3 = \partial_3 + x^4\partial_1$, $X_4 = \partial_4$, $X_1 = \partial_1$ | $\eta^{123} = q_3x^1 + q_1(e^{x^3} - 1)$ |
| $3A_1 \subset \mathfrak{A}_{3,5} \oplus A_1$ | $X_1, X_2, X_4$ | $X_1 = \partial_1$, $X_2 = \partial_2$, $X_4 = \partial_4$ | $\eta^{123} = q_3x^1 + q_1(e^{x^3} - 1)$ |
| $A_2 \oplus A_1 \subset \mathfrak{A}_{3,5} \oplus A_1$ | $X_3, X_4, X_1$ | $X_3 = \partial_3 + x^4\partial_1$, $X_4 = \partial_4$, $X_1 = \partial_1$ | $\eta^{123} = q_3x^1 + q_1(e^{x^3} - 1)$ |
| $\mathfrak{A}_{3,5} \subset \mathfrak{A}_{3,5} \oplus A_1$ | $X_3 + aX_4, X_1, X_2$ | $X_3 + aX_4 = \partial_3 + a\partial_4 + x^1\partial_1 + x^2\partial_2$, $X_1 = \partial_1$, $X_2 = \partial_2$ | $\eta^{123} = q_3(e^{a+1})((x^1)^2 + (x^2)^2/a)^{1/2} - 1)$, $\eta^{124} = aq_3(e^{a+1})((x^1)^2 + (x^2)^2/a)^{1/2} - 1))$ |
| $3A_1 \subset \mathfrak{A}_{3,6} \oplus A_1$ | $X_1, X_2, X_4$ | $X_1 = \partial_1$, $X_2 = \partial_2$, $X_4 = \partial_4$ | $\eta^{123} = q_3x^1 + q_2x^2 + q_3x^4$ |
| $A_{3,6} \subset A_{3,6} \oplus A_1$ | $X_3 + aX_4, X_1, X_2$ | $X_3 + aX_4 = \partial_3 + a\partial_4 + x^2\partial_1 - x^1\partial_2$, $X_1 = \partial_1$, $X_2 = \partial_2$ | $\eta^{123} = q_3x^1 + q_2x^2 + q_3x^4$ |
| $3A_1 \subset \mathfrak{A}_{3,7} \oplus A_1$ | $X_1, X_2, X_4$ | $X_1 = \partial_1$, $X_2 = \partial_2$, $X_4 = \partial_4$ | $\eta^{123} = q_3x^1 + q_2x^2 + q_3x^4$ |
| $\mathfrak{A}_{3,7} \subset \mathfrak{A}_{3,7} \oplus A_1$ | $X_3 + aX_4, X_1, X_2$ | $X_3 + aX_4 = \partial_3 + a\partial_4 + (x^2 + ax^1)\partial_1 + (ax^2 - x^1)\partial_2$, $X_1 = \partial_1$, $X_2 = \partial_2$ | $\eta^{123} = q_3(e^{x^3} + x^4 - 1)$, $\eta^{124} = q_3(e^{x^3} + x^4 - 1)$ |
| Lie subalgebra  | Basis Lie subalgebra | Left-invariant vector fields | $\eta^{ijk}$                       |
|-----------------|----------------------|-------------------------------|-----------------------------------|
| $A_2 \oplus A_1 \subset A_{3,8} \oplus A_1$ | $X_2, X_4, X_1$ | $X_2 = \partial_2 + x^1 \partial_1$, $X_4 = \partial_4$, $X_1 = \partial_1$ | $\eta^{124} = q_1 x^1 + q_1(e^{x^2} - 1)$ |
| $A_{3,8} \subset A_{3,8} \oplus A_1$ | $X_3 + aX_4, X_1, X_2$ | $X_1 = e^{-x^2} \partial_1 + 2x^3 \partial_2 - (x^3)^2 \partial_3$, $X_2 = \partial_2 - x^3 \partial_3$, $X_3 = \partial_3$ | $\eta^{123} = 0$ |
| $A_{3,8} \subset A_{3,9} \oplus A_1$ | $X_1, X_2, X_3$ | $X_1 = \frac{\cos x^3}{\cos x^2} \partial_1 + \sin x^3 \partial_2 - \frac{\cos x^3 \sin x^2}{\cos x^2} \partial_3$, $X_2 = -\sin x^3 \partial_1 + \cos x^3 \partial_2 + \frac{\sin x^3 \sin x^2}{\cos x^2} \partial_3$, $X_3 = \partial_3$ | $\eta^{123} = 0$ |
| $3A_1 \subset A_{4,1}$ | $X_1, X_2, X_3$ | $X_1 = \partial_1$, $X_2 = \partial_2$, $X_3 = \partial_3$ | $\eta^{123} = q_1 x^1 + q_2 x^2 + q_3 x^3$ |
| $A_{3,1} \subset A_{4,1}$ | $X_4 + aX_3, X_2, X_1$ | $X_4 + aX_3 = \partial_4 + a\partial_3 + x^3 \partial_1$, $X_2 = \partial_2$, $X_1 = \partial_1$ | $\eta^{121} = q_1(x^3/a + x^4) + q_2 x^3$, $\eta^{121} = a(q_1(x^3/a + x^4) + q_2 x^3)$ |
| $3A_1 \subset A_4^{\nu}$ | $X_1, X_2, X_3$ | $X_1 = \partial_1$, $X_2 = \partial_2$, $X_3 = \partial_3$ | $\eta^{123} = q_1 x^1 + q_2 x^2 + q_3 x^3$ |
| $A_{3,2} \subset A_4$ | $X_4, X_2, X_3$ | $X_4 = \partial_4 + (x^3 + x^2) \partial_2 + x^3 (\partial_3 + a\partial_1)$, $X_2 = \partial_2$, $X_3 = \partial_3 + a\partial_1$ | $\eta^{234} = q_1(e^{x^4} - 1)$ |
| $A_{3,4} \subset A_4$ | $X_4, X_1, X_2$ | $X_4 = \partial_4 + ax^1 \partial_1 + x^2 \partial_2$, $X_1 = \partial_1$, $X_2 = \partial_2$ | $\eta^{124} = q_1 x^4$ |
| $A_{5,5}^{\nu} \subset A_4$, $\nu = \begin{cases} a, & |a| < 1, \\ 1/a, & |a| > 1 \end{cases}$ | $X_4, X_1, X_2$ | $X_4 = \partial_4 - x^1 \partial_1 + x^2 \partial_2$, $X_1 = \partial_1$, $X_2 = \partial_2$ | $\eta^{124} = q_1(e^{(a+1)x^4} - 1)$ |
| $3A_1 \subset A_4$ | $X_1, X_2, X_3$ | $X_1 = \partial_1$, $X_2 = \partial_2$, $X_3 = \partial_3$ | $\eta^{123} = q_1 x^1 + q_2 x^2 + q_3 x^3$ |
| $A_{3,2} \subset A_4$ | $X_4, X_2, X_3 + aX_1$ | $X_4 = \partial_4 + (x^2 + x^3) \partial_2 + x^3 \partial_3$, $X_2 = \partial_2$, $X_3 + aX_1 = \partial_3 + a\partial_1$ | $\eta^{234} = q_1(e^{x^4} - 1)$, $\eta^{124} = -aq_1(e^{x^4} - 1)$ |
| $A_{3,3} \subset A_4$ | $X_4, X_1, X_2$ | $X_4 = \partial_4 + x^1 \partial_1 + x^2 \partial_2$, $X_1 = \partial_1$, $X_2 = \partial_2$ | $\eta^{124} = q_1(e^{x^4} - 1)$ |

**Table 2 (cont.)**
| Lie subalgebra | Basis Lie subalgebra | Left-invariant vector fields | \( \eta^{jk} \) |
|----------------|----------------------|----------------------------|---------------------|
| \( 3A_1 \subset A_{4,3} \) | \( X_1, X_2, X_3 \) | \( X_1 = \partial_1, \ X_2 = \partial_2, \ X_3 = \partial_3 \) | \( \eta^{123} = q_1 x^1 + q_2 x^2 + q_3 x^3 \) |
| \( A_2 \oplus A_1 \subset A_{4,3} \) | \( X_4 + aX_3, X_2, X_1 \) | \( X_4 + aX_3 = \partial_1 + a\partial_3 + x^1\partial_1, \ X_2 = \partial_2, \ X_1 = \partial_1 \) | \( \eta^{123} = a(q_3 x^1 + q_1 (e^{(x^2+x^3/a)/2} - 1)), \eta^{124} = q_3 x^1 + q_1 (e^{(x^2+x^3/a)/2} - 1) \) |
| \( A_{3,1} \subset A_{4,3} \) | \( X_3, X_4, X_2 \) | \( X_3 = \partial_3 - x^2 \partial_2, \ X_4 = \partial_1, \ X_2 = \partial_2 \) | \( \eta^{123} = q_1 x^1 + q_2 x^2 + q_3 x^3 \) |
| \( 3A_1 \subset A_{4,4} \) | \( X_1, X_2, X_3 \) | \( X_1 = \partial_1, \ X_2 = \partial_2, \ X_3 = \partial_3 \) | \( \eta^{124} = q_1 x^4 - 1 \) |
| \( A_{3,2} \subset A_{4,4} \) | \( X_4, X_1, X_2 \) | \( X_4 = \partial_4 + (x^2 x^3)\partial_2, \ X_1 = \partial_1, \ X_2 = \partial_2 \) | \( \eta^{124} = q_1 (e^{(a+1)x^4} - 1) \) |
| \( A_{3,3} \subset A_{4,5} \) | \( X_4, X_2, X_3 \) | \( X_4 = \partial_4 + (x^2 x^3)\partial_2 + (x^2 x^3)\partial_3, \ X_2 = \partial_2, \ X_3 = \partial_3 \) | \( \eta^{234} = q_1 (e^{(a+b)x^4} - 1) \) |
| \( A_{3,4} \subset A_{4,5} \) | \( X_4, X_1, X_2 \cos \phi + X_3 \sin \phi \) | \( X_4 = \partial_4 + x^1 \partial_1 + a((x^2 x^3)^2 + (x^2 x^3) x^2 + (x^2 x^3)^2 \cos \phi \partial_2 + \sin \phi \partial_3), \ X_1 = \partial_1, \ X_2 = \partial_2 \cos \phi + X_3 \sin \phi \cos \phi \partial_2 + \sin \phi \partial_3 \) | \( \eta^{124} = q_1 (e^{(a+1)x^4} - 1) \cos \phi, \eta^{124} = q_1 (e^{(a+1)x^4} - 1) \cos \phi \) |
| \( 3A_1 \subset A_{4,6} \) | \( X_1, X_2, X_3 \) | \( X_1 = \partial_1, \ X_2 = \partial_2, \ X_3 = \partial_3 \) | \( \eta^{123} = q_1 x^1 + q_2 x^2 + q_3 x^3 \) |
| \( A_{3,5} \subset A_{4,6} \) | \( X_4, X_1, X_3 \cos \phi + X_3 \sin \phi, X_2 \) | \( X_4 = \partial_4 + x^1 \partial_1 + x^2 \partial_3, \ X_1 = \partial_1, \ X_3 = \partial_3 \) | \( \eta^{234} = q_1 (e^{(a+1)x^4} - 1) \cos \phi, \eta^{234} = q_1 (e^{(a+1)x^4} - 1) \sin \phi \) |
| \( 3A_1 \subset A_{4,5} \) | \( X_1, X_2, X_3 \) | \( X_1 = \partial_1, \ X_2 = \partial_2, \ X_3 = \partial_3 \) | \( \eta^{123} = q_1 x^1 + q_2 x^2 + q_3 x^3 \) |

\( \nu = \begin{cases} a/b, & |a/b| < 1, \\ b/a, & |a/b| > 1 \end{cases} \)
| Lie subalgebra | Basis Lie subalgebra | Left-invariant vector fields | \( \eta^{j,k} \) |
|----------------|----------------------|----------------------------|------------------|
| \( A_{3,3} \subset A_{4,5}^{1/2} \) | \( X_4, X_1 + aX_2, X_3 \) | \( \begin{align*}
X_1 &= \partial_1 + x^1(\partial_1 + a\partial_2) + x^2(\partial_2 + b\partial_3), \\
X_2 + bX_3 &= \partial_2 + b\partial_3 \\
X_4 &= \partial_1 + x^2(\partial_1 + a\partial_2) + x^3\partial_3, \\
X_1 + aX_2 &= \partial_1 + a\partial_2, \quad X_3 = \partial_3
\end{align*} \) | \( \eta^{124} = q_1(\varepsilon^{2x^4} - 1), \) \( \eta^{134} = aq_1(\varepsilon^{2x^4} - 1) \) |
| \( 3A_1 \subset A_{4,6}^{a,b} \) | \( X_1, X_2, X_3 \) | \( \begin{align*}
X_1 &= \partial_1, \quad X_2 = \partial_2, \quad X_3 = \partial_3
\end{align*} \) | \( \eta^{123} = q_1x^1 + q_2x^2 + q_3x^3 \) |
| \( A_{3,7} \subset A_{4,6}^{a,b} \) | \( X_4, X_2, X_3 \) | \( \begin{align*}
X_4 &= \partial_1 + (bx^2 + x^3)\partial_2 + (bx^3 - x^2)\partial_3, \\
X_2 &= \partial_2, \quad X_3 = \partial_3
\end{align*} \) | \( \eta^{123} = q_1(\varepsilon^{2bx^4} - 1) \) |
| \( A_{3,1} \subset A_{4,7} \) | \( X_2, X_3, X_1 \) | \( \begin{align*}
X_2 &= \partial_1 - x^1\partial_1, \quad X_3 = \partial_3, \quad X_1 = \partial_1
\end{align*} \) | \( \eta^{124} = q_1x^2 + q_2x^3 \) |
| \( A_{3,5}^{1/2} \subset A_{4,7} \) | \( X_4, X_1, X_2 \) | \( \begin{align*}
X_4 &= \partial_1 + 2x^1\partial_1 + x^2\partial_2, \\
X_1 &= \partial_1, \quad X_2 = \partial_2
\end{align*} \) | \( \eta^{124} = q_1(\varepsilon^{2x^4} - 1) \) |
| \( A_{3,1} \subset A_{4,8} \) | \( X_2, X_3, X_1 \) | \( \begin{align*}
X_2 &= \partial_2 - x^2\partial_1, \quad X_3 = \partial_3, \quad X_1 = \partial_1
\end{align*} \) | \( \eta^{123} = q_1x^2 + q_2x^3 \) |
| \( A_{2} \oplus A_{1} \subset A_{4,8} \) | \( X_4, X_1, X_2 \) | \( \begin{align*}
X_4 &= \partial_1 - x^1\partial_2, \quad X_1 = \partial_1, \quad X_2 = \partial_2
\end{align*} \) | \( \eta^{134} = q_1x^2 + q_1(\varepsilon^{x^4} - 1) \) |
| \( A_{2} \oplus A_{1} \subset A_{4,9} \) | \( X_2, X_3, X_1 \) | \( \begin{align*}
X_2 &= \partial_2 - x^1\partial_1, \quad X_3 = \partial_3, \quad X_1 = \partial_1
\end{align*} \) | \( \eta^{123} = q_1x^2 + q_2x^3 \) |

Continued
| Lie subalgebra   | Basis Lie subalgebra   | Left-invariant vector fields                                      | $\eta^{ijk}$                      |
|------------------|------------------------|------------------------------------------------------------------|-----------------------------------|
| $A_{3,5}^{1/2} \subset A_{4,9}^{1}$ | $X_4, X_1, X_2 \cos \phi + X_3 \sin \phi$ | $X_4 = \partial_4 + 2x^1 \partial_1 + (x^2 \cos \phi + \sin \phi)(\cos \phi \partial_2 + \sin \phi \partial_3), X_1 = \partial_1, X_2 \cos \phi + X_3 \sin \phi = \cos \phi \partial_2 + \sin \phi \partial_3$ | $\eta^{124} = q_1(e^{3x^3} - 1) \cos \phi, \eta^{134} = q_1(e^{3x^4} - 1) \sin \phi$ |
| $A_{3,1} \subset A_{4,9}^{0}$ | $X_2, X_3, X_1$ | $X_2 = \partial_2 - x^3 \partial_1, X_3 = \partial_3, X_1 = \partial_1$ | $\eta^{123} = q_1 x^2 + q_2 x^3$ |
| $A_2 \oplus A_1 \subset A_{3,9}^{1}$ | $X_3, X_4, X_1$ | $X_3 = \partial_3, X_4 = \partial_1 + x^1 \partial_1, X_1 = \partial_1$ | $\eta^{134} = q_3 x^1 + q_2(e^{-x^3} - 1)$ |
| $A_{3,3} \subset A_{4,9}^{0}$ | $X_4, X_1, X_2$ | $X_4 = \partial_4 + x^1 \partial_1 + x^2 \partial_2, X_1 = \partial_1, X_2 = \partial_2$ | $\eta^{124} = q_1(e^{2x^4} - 1)$ |
| $A_{3,2} \subset A_{4,9}^{0}$ | $X_4 + aX_3, X_1, X_2$ | $X_4 + aX_3 = \partial_4 + a\partial_3 + (ax^2 + x^1)\partial_1 + x^2 \partial_2, X_1 = \partial_1, X_2 = \partial_2$ | $\eta^{124} = q_1(e^{x^4 + x^7/a} - 1), \eta^{123} = a\eta_1(e^{x^4 + x^7/a} - 1)$ |
| $A_{3,1} \subset A_{4,10}$ | $X_2, X_3, X_1$ | $X_2 = \partial_2 - x^3 \partial_1, X_3 = \partial_3, X_1 = \partial_1$ | $\eta_1 = q_1 x^2 + q_2 x^3$ |
| $A_{3,1} \subset A_{4,11}$ | $X_2, X_3, X_1$ | $X_2 = \partial_2 - x^3 \partial_1, X_3 = \partial_3, X_1 = \partial_1$ | $\eta_1 = q_1 x^2 + q_2 x^3$ |
| $A_{3,3} \subset A_{4,12}$ | $X_3, X_1, X_2$ | $X_3 = \partial_3 + x^1 \partial_1 + x^2 \partial_2, X_1 = \partial_1, X_2 = \partial_2$ | $\eta^{123} = q_1(e^{2x^3} - 1)$ |
| $A_{3,6} \subset A_{4,12}$ | $X_4, X_1, X_2$ | $X_1 = \partial_4 + x^2 \partial_1 - x^1 \partial_2, X_1 = \partial_1, X_2 = \partial_2$ | $\eta^{124} = q_1 x^4$ |
| $A_{3,7}^{[a]} \subset A_{4,12}, a \neq 0$ | $X_4 + aX_3, X_1, X_2$ | $X_4 + aX_3 = \partial_4 + a\partial_3 + (ax^1 + x^2)\partial_1 + (ax^2 - x^1)\partial_2, X_1 = \partial_1, X_2 = \partial_2$ | $\eta^{124} = q_1(e^{x^3 + ax^4} - 1), \eta^{123} = a\eta_1(e^{x^3 + ax^4} - 1)$ |
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