A Fast Iterative Method for Chandrasekhar’s $H$-functions for General Laws of Scattering

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Abstract This work shows that notable acceleration of the speed of calculating Chandrasekhar’s $H$-functions for general laws of scattering with an iterative method can be realized by supplying a starting approximation produced by the following procedure: (i) in the cases of azimuth-angle independent Fourier components, values of the isotropic scattering $H$-function given by an accurate yet simple-to-apply formula, in particular, the one by Kawabata & Limaye (2011), and (ii) for azimuth-angle dependent Fourier components, an already obtained solution of the next lower order term.

Keywords radiative transfer: general — multiple scattering, $H$-function, numerical solution

1 Introduction

The intensities of sunlight reflected by a plane-parallel planetary atmosphere can be expanded in a Fourier series of azimuth-angle difference $\Delta \phi$ of the incident and emergent directions of light. Furthermore, if the atmosphere in question is semi-infinite in optical depth, each Fourier coefficient of the series can be expressed analytically in terms of the Chandrasekhar’s $H$-function (Chandrasekhar 1960; Sobolev 1972; van de Hulst 1980) associated with a characteristic function determined by the scattering phase function of the atmosphere.

Although it is rather straightforward nowadays to carry out multiple light scattering calculations, it is nevertheless very useful to have analytical representations of the emergent intensities of reflected or transmitted sunlight at hand for the purpose of numerical accuracy check of, say, a newly developed computer program for multiple scattering, or for gaining some physical or mathematical insights into actual problems involving light scattering. It may also be important to keep in mind that theoretical works in some disciplinary areas of physics such as electron transport in condensed matter require numerical values of the $H$-function with relatively high accuracy (Jablonsk 2012).

Each Fourier component $H^{(m)}(\varpi_0, \mu)$ of $H$-function satisfies the following integral equation:

$$H^{(m)}(\varpi_0, \mu) = 1 + \mu H^{(m)}(\varpi_0, \mu) \times \int_0^1 \frac{\psi^{(m)}(\mu')}{\mu + \mu'} H^{(m)}(\varpi_0, \mu') d\mu'$$  \hspace{1cm} (1)$$

where $m$ designates the order of the Fourier component, $\psi^{(m)}(\mu)$ is the characteristic function specified by the scattering law of our interest, $\varpi_0$ the single scattering albedo, and $\mu$ the cosine of a zenith angle of the direction of a ray of light incident on or emerging from the top surface of a semi-infinite atmosphere. The characteristic function $\psi^{(m)}(\mu)$ depends also on $\varpi_0$, but will hereafter be omitted for simplicity.

A great deal of efforts have been devoted by various authors to the study on mathematical nature of this equation, and to development of efficient numerical methods to solve Eq. (1) (see, e.g., Bosma & de Rooij 1983; Kawabata, et al. 1991; Davidovic’ et al. 2008; Kawabata & Limaye 2011 and the references cited therein). For isotropic scattering, for which $\psi^{(0)}(\mu) = \varpi_0/2$, a closed form integral representation of the solution is known, so that we can directly calculate its numerical values. Furthermore, several accurate approximation formulas are now available (Hapke 1993; Davidovic’ et al. 2008; Kawabata & Limaye 2011).

As for more general types of anisotropic scattering, the $H$-functions can be expressed in terms of the characteristic roots of an equation involving $\psi(\mu)$...
but the root-finding process is too time-consuming (see, e.g., Kawabata et al. 1991). Some sort of successive numerical iterations are essential to obtain accurate values for their $H$-functions. In fact, the iterative method (ii) proposed by Bosma & de Rooij (1983) is highly instrumental in generating numerical solutions correct to 11 significant figures or more. In view of the important role played by Chandrasekhar’s $H$-functions, it may be of some significance to further improve the efficiency of the iterative method of Bosma & de Rooij (1983). Inspired by the work of Hiroi (1994) on the $H$-function for isotropic scattering, we would like to investigate, in particular, the effect of initial approximation on the required number of iterations.

2 Formalism

As in Bosma & de Rooij (1983), we employ an alternative form of Eq. (2) to calculate the values of $H$-functions using an iterative procedure:

$$H^{(m)}(\omega_0, \mu) = (1 - 2\psi_0^{(m)})^{1/2} + \int_0^1 \frac{\mu' \psi_0^{(m)}(\mu') H^{(m)}(\omega_0, \mu')}{\mu + \mu'} d\mu', \quad (2)$$

where we have put

$$\psi_0^{(m)} = \int_0^1 \psi_0^{(m)}(\mu') d\mu'. \quad (3)$$

For numerical calculations, we replace the integral with respect to $\mu'$ over the interval $[0, 1]$ with a quadrature of degree $N$. In the present work, the $N_G$-point Gauss-Legendre quadrature is chosen for this purpose, so that Eq. (2) takes the following form:

$$H^{(m)}(\omega_0, \mu_k) = (1 - 2\psi_0^{(m)})^{1/2} + \sum_{j=1}^{N_G} \frac{\mu_j \psi_0^{(m)}(\mu_j) H^{(m)}(\omega_0, \mu_j)}{\mu_k + \mu_j} w_j \quad (k = 1, \ldots, N_G) \quad (4)$$

with $\mu_j$ and $w_j$ being the $j$-th quadrature point and the corresponding weight, and $\mu_k$ the $k$-th quadrature point at which the numerical solution is sought for.

In order to study the effect of anisotropy of scattering on each method employed in this work, a four-term phase function of the form

$$P(\cos \Theta) = \omega_0 \left\{ 1 + \sum_{j=1}^{J} x_j P_j(\cos \Theta) \right\} \quad (J \leq 3) \quad (5)$$

is employed, where $P_j(\cos \Theta)$ is the Legendre polynomial function of the $j$-th degree, $x_j$’s are the expansion coefficients, and $\Theta$ is the scattering angle (van de Hulst 1980). $J$ is the maximum degree of the Legendre functions to be taken into account, and coincides with the highest degree $M$ of the Fourier terms required to represent the azimuth-angle dependence of emergent intensities of reflected light, which is in turn the highest degree of the Fourier components $H^{(m)}(\omega_0, \mu)$: for isotropic scattering, where $J = 0$, only the azimuth-angle independent Fourier component $H^{(0)}(\omega, \mu)$ needs to be considered, which we shall refer to as $\omega_0 H(\omega_0, \mu)$. This four-term phase function is convenient for the fact that it covers, as special cases,

(i) isotropic scattering phase function given by $x_j = 0 \ (j = 1, \ldots, 3)$,

(ii) Rayleigh scattering phase function with $x_1 = 0$, $x_2 = 1/2$, and $x_3 = 0$,

(iii) linearly anisotropic scattering phase functions with $x_1 \neq 0$, and $x_2 = x_3 = 0$,

(iv) three-term phase functions with $x_3 = 0$,

(v) four-term phase functions with $x_3 \neq 0$.

The $m$-th order Fourier components $\psi^{(m)}(\mu) \ (m = 0, 1, 2, 3)$ corresponding to the four-term phase function Eq. (5) are as follows:

$$m = 0 : \psi^{(0)}(\mu) = \frac{1}{2} \omega_0 \left\{ 1 + \frac{1}{4} x_2 + \left[ h_0 x_1 - \frac{3}{4} x_2 \right] \mu^2 + \frac{3}{4} h_0 h_1 x_2 + h_0 x_3 + \frac{1}{4} h_2 x_3 \right\} \quad (6a)$$

$$m = 1 : \psi^{(1)}(\mu) = \frac{1}{2} \omega_0 \left\{ 1 - \mu^2 \right\} \left\{ \frac{1}{2} x_1 + \frac{3}{16} x_3 \right\} \mu^2 + \frac{1}{4} h_1 x_2 + \frac{1}{16} (h_1 h_2 + 15) x_3 \right\} \mu^2 + \frac{5}{16} h_1 h_2 x_3 \mu^2 \quad (6b)$$

$$m = 2 : \psi^{(2)}(\mu) = \frac{3}{16} \omega_0 (1 - \mu^2)^2 \times (x_2 + x_3 h_2 \mu^2) \quad (6c)$$

$$m = 3 : \psi^{(3)}(\mu) = \frac{5}{32} \omega_0 x_3 (1 - \mu^2)^3 \quad (6d)$$

where $h_k \ (k = 0, 1, 2, 3)$ is defined as

$$h_k = 2k + 1 - \omega_0 x_k, \quad (7)$$

following the Display 6.2 of van de Hulst (1980).
As was initially recognized by Chandrasekhar (1960) and later demonstrated mathematically by Bosma & de Rooij (1983), Eq. (2) and hence Eq. (3) is more suited for iterative methods, and in fact, tabulation of the values of the \( H \)-functions for certain laws of anisotropic scattering have been made by e.g., Chandrasekhar (1960), Kolesov & Smoktii (1972), and Bosma & de Rooij (1983) solving this alternative form of the basic equation. A recourse is correspondingly made to Eq. (4) also in the present work.

For a given value of \( \varpi \), a convergence to the solution of Eq. (4) is assumed to have been realized after the \( n \)-th iteration, if the following condition is satisfied:

\[
\max \left| H^{(m)}(\varpi, \mu_j)_n - H^{(m)}(\varpi, \mu_j)_{n-1} \right| \leq \varepsilon, \\
(j = 1, 2, 3, \ldots, N_G) \tag{8}
\]

where we adopt absolute-error tolerance \( \varepsilon = 10^{-12} \) as in Bosma & de Rooij (1983) to ensure 11 significant figure accuracy. If, on the other hand, this condition is not fulfilled, we have two alternatives for creating a next approximation \( \overline{H}(\varpi, \mu) \) for \( H^{(m)}(\varpi, \mu) \) on the right side of Eq. (4) before proceeding to the \((n+1)\)-th iteration:

\[
(1) \quad \overline{H}(\varpi, \mu) = H^{(m)}(\varpi, \mu_j)/H^{(m)}(\varpi, 0)_n. \tag{9}
\]

The values for \( H^{(m)}(\varpi, 0)_n \) required above are to be calculated at the end of the \( n \)-th iteration using Eq. (4) as

\[
\frac{1}{H^{(m)}(\varpi, 0)_n} = (1 - 2 \psi^{(m)}_0)^{1/2} + \\
+ \sum_{j=1}^{N_G} \psi^{(m)}(\mu_j)H^{(m)}(\varpi, \mu_j)_n w_j \tag{10}
\]

or

\[
(2) \quad \overline{H}(\varpi, \mu)_n = \lambda_0 H^{(m)}(\varpi, \mu_j)_n + (1 - \lambda_0)H^{(m)}(\varpi, \mu_j)_{(n-1)} \tag{11}
\]

Bosma & de Rooij (1983) suggested, in their method (i), the use of

\[
\lambda_0 = \frac{1}{2} \left[ 1 + (1 - 2 \psi^{(m)}_0)^{1/2} \right], \tag{12}
\]

Eq. (12) yields \( \lambda_0 = 0.5 \) only for the conservative case where \( \varpi = 1 \). Hiroi (1994), on the other hand, employed \( \lambda_0 = 0.5 \) exclusively to solve for the isotropic scattering \( H \)-function for arbitrary value of \( \varpi \).

3 Numerical Calculations

Table 1 shows the data set used for the present work, compiling 37 cases of phase functions that can be represented by the the four-term phase function given by Eq. (5). The last column of the table indicates an abbreviated name of the scattering law characterizing each entry: the row No.1 designated by 'ISO' is for isotropic scattering, the rows No.2 through No.7 designated by 'LIN' are linearly anisotropic scattering phase functions, each of which yields an azimuth-angle independent component \( \psi^{(0)}(\mu) \) and a first order azimuth-angle dependent Fourier component \( \psi^{(1)}(\mu) \). The row No.8 designated by 'RAY' is for Rayleigh scattering, a special case of a three-term phase function, and gives rise to three Fourier component characteristic functions \( \psi^{(m)}(\mu) \) \((m = 0, 1, 2, 3)\), the rows No. 9 through No.23 are for more general cases of the three-term phase function \((J = 2)\), each having three Fourier components as in No.8, and the rows No.24 through No.37 are for four-term phase functions \((J = 3)\), each bearing four Fourier component characteristic functions \( \psi^{(m)}(\mu) \) \((m = 0, 1, 2, 3)\). As the result, altogether 117 cases of characteristic functions arise, for each of which Eq. (4) is solved iteratively for \( H^{(m)}(\varpi, \mu) \). Asterisked entries indicate that their phase functions show negative values at some scattering angles due to insufficient degree of approximation obtained with the first four Legendre polynomials.

Following Bosma & de Rooij (1983), we also employ \( N = N_G = 128 \) for the Gauss-Legendre quadrature, and \( \varepsilon = 10^{-12} \) (see Eq. (4)). For each of the 117 cases of the Fourier components \( H^{(m)}(\varpi, \mu) \) given by the 37 phase functions enumerated in Table 1, calculations of \( H^{(m)}(\varpi, \mu) \) \((m \leq 3)\) are carried out solving Eq. (4) iteratively at mesh points \((\varpi, \mu)\) specified by 14 values of \( \varpi \), viz., \( 10^{-3}, 0.05, 0.1 \) through 0.9 (with a step of 0.1), 0.99, 0.999, and 1, and 128 quadrature points \( \mu_k \) \((k = 1, 2, 3, \ldots, 128)\). Once a set of numerical solution \( H^{(m)}(\varpi, \mu) \) has been found at 128 quadrature points for a given value of \( \varpi \), we can freely calculate the value of \( H^{(m)}(\varpi, \mu) \) at more regularly spaced non-quadrature points \( \mu \) by replacing \( \mu_k \) with \( \mu \) in Eq. (4).

For convenience of comparison with the results given in other literature, we employ 21 values for \( \mu \), viz., 0 through 1 with a step of 0.05, for tabulation.

In order to access the efficiency of each scheme we examine, we have proceeded in the following four steps successively:

Method A:
To produce a control case, the numerical solutions for \( H^{(m)}(\varpi, \mu) \) are calculated for the 117 cases of
characteristic functions by applying the method (ii) of Bosma & de Rooij (1983): the starting approximation is simply $T^{(m)}(\varpi_0, \mu) = 1$ irrespective of the characteristic function involved. Furthermore, the successive iteration ($n \geq 1$) is carried out with Eq. (9).

To verify our computer code, our numerical values for the cases No.1, No.2, and No.8 given in Table 1 were compared with those in Tables 1a through 1f of Bosma & de Rooij (1983), to find that both results are in perfect agreement to each other to the last decimal figures. For No.1, the number of iterations $N_i$ required to get the numerical solution for $H^{(0)}(\varpi_0, \mu)$ were also in fair agreement with the corresponding result given in their Table 3.

Method B:

Inspired by the work of Hiroi (1994) for isotropic scattering, we conceived of a possibility that supplying a more improved approximation formula obtained by Kawabata & Limaye (2011) simply to warn the readers that erroneously states that the Kawabata-Limaye method yielded 9077901976, while the more correct value is 9078105291, something else was desired to speed up the calculations of $H^{(m)}(\varpi_0, \mu)$.

Method D:
The rational approximation formula of Kawabata & Limaye (2011) (see also Kawabata & Limaye 2013, for erratum) for $iso^2 H(\varpi_0, \mu)$ rather than that of Davidovic’ et al. (2008) is employed to produce the starting approximation to solve Eq. (4) in the $m = 0$ cases. The formula we are going to use is of the following form:

$$H(\varpi_0, \mu) = H_{app}(1, \mu)/[1 + \sum_{k=0}^{8} C_k(\varpi_0)x^k]$$ (13)

where $x = \mu^{1/4}$, while $H_{app}(1, \mu)$ and $C(\varpi_0)$ are eighth order polynomials of $x$ and $\sqrt{1 - \varpi_0}$, respectively. As has already been mentioned, this formula is based on extensive numerical data of $iso^2 H(\varpi_0, \mu)$ produced with 11 digit accuracy by Kawabata & Limaye (2011) using a closed form solution.

For the $m(\geq 1)$-th order Fourier components, however, we substitute $H^{(m-1)}(\varpi_0, \mu)$, the solution for the $(m-1)$-th Fourier component, as the initial approximation for iterations. This procedure, though simple as it is, was found fairly promising during some preliminary experiments. The approximation $T^{(m)}(\varpi_0, \mu)$, used for the $(n+1)$-th iteration is calculated by Eq. (11).

4 Results

The left half of Table 2 compares in the case of isotropic scattering the numbers of iterations $N_i$ required for four methods A, B, C, and D to achieve a convergence within $\varepsilon = 10^{-12}$ to the solution of Eq. (4) for 14 values of $\varpi_0$. Also shown in each row of the right half of the table are the values for $N_i - \min N_i$ for the same four methods, where $\min N_i$ designates the minimum of the four $N_i$ values given in the same row on the
left. The bottom row shows sum of these quantities given in each column. We notice that Eq. (13) derived by Kawabata & Limaye (2011) enables us to cut down the number of iterations $N_{it}$ by a factor of more than two in the cases of isotropic scattering.

Of course, the true capability of each method must be assessed from the standpoint of anisotropic scattering calculations. The topmost row of Table 12 shows the values of $N_i - \min N_{it}$ required to calculate 116 sets of solutions for $H^{(m)}(\pi_0, \mu)$ ($m \geq 0$) arising from the phase functions No.2 through No.37 of Table 1. The second and third rows are a breakdown of the statistics given in the first row into two groups: the figures shown in the second row are for 37 azimuth-angle independent components $H^{(0)}(\pi_0, \mu)$, and those shown in the third row are for 79 cases of azimuth-angle dependent Fourier components $H^{(m)}(\pi_0, \mu)$ ($m \geq 1$), respectively. The parenthesized figures in each row indicate fractional contributions from the four methods tested entries where discrepancies mostly by one unit in the third decimal figures are found, the largest of which occurs for the reflection function for isotropic scattering at $\mu = 0.5$, where we have $1.0128195942$ in contrast to $1.023$ obtained by Kolesov & Smoktii (1972). It may be worth noting that our value is in agreement with 1.01282 shown in Table 12 of van de Hulst (1980).

5 Conclusion

On the basis of the numerical experiments carried out in this work in accordance with the four methods A, B, C, and D, we have reached the following conclusions:

(i) The iterative method to solve Eq. (2) for general laws of scattering works quite efficiently, if the normalization procedure of Bosma & de Rooij (1983), Eq. (14), is applied to create an approximate solution for $H$-function prior to each successive iteration. The efficiency of the iterative solution can further be improved by supplying adequate starting numerical approximations to initiate the iteration.

(ii) For azimuth-angle independent components, the number of iterations required to obtain solutions for $H^{(0)}(\pi_0, \mu)$ can be significantly reduced by using, as a starting approximation, the $\text{iso}H(\pi_0, \mu)$ generated with the approximation formula obtained by Kawabata & Limaye (2011) whose maximum relative error is $2.1 \times 10^{-4}$ %.

(iii) In the cases of higher order Fourier components $H^{(m)}(\pi_0, \mu)$ ($m \geq 1$), the substitution of the $(m-1)$-th order solution $H^{(m-1)}(\pi_0, \mu)$ as the initial approximation greatly reduce the number of iterations necessary.}

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\footnote{A Fortran77 source program to calculate $H$-functions applying the method D is available from the author on request.}
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vан де Хулст, H.C. 1980, Multiple Light Scattering: Tables, Formulas, and Applications, Vols. 1 and 2, New York: Academic Press.
Table 1  Phase functions employed for numerical experiments

| No. | J | $x_1$  | $x_2$  | $x_3$  | ref† |
|-----|---|--------|--------|--------|------|
| 1   | 0 | 0      | 0      | 0      | ISO  |
| 2   | 1 | 1      | 0      | 0      | LIN  |
| 3   | 1 | 0.9    | 0      | 0      | LIN  |
| 4   | 1 | 0.5    | 0      | 0      | LIN  |
| 5   | 1 | -0.5   | 0      | 0      | LIN  |
| 6   | 1 | -0.9   | 0      | 0      | LIN  |
| 7   | 1 | -1     | 0      | 0      | LIN  |
| 8   | 2 | 0      | 0.5    | 0      | RAY  |
| 9   | 2 | 1      | 1      | 0      | SOB  |
| 10  | 2 | 1.5    | 1      | 0      | SOB  |
| 11  | 2 | 1.076  | 0.795  | 0      | KS3  |
| 12  | 2 | 0.240  | 0.498  | 0      | KS3  |
| 13  | 2 | 0.092  | 0.497  | 0      | KS3  |
| 14  | 2 | 2.670  | 2.470  | 0      | KS3  |
| 15  | 2 | 1.269  | 0.909  | 0      | KS3  |
| 16  | 2 | 0.566  | 0.566  | 0      | KS3  |
| 17  | 2 | 2.879  | 2.740  | 0      | KS3  |
| 18  | 2 | 1.198  | 0.869  | 0      | KS3  |
| 19  | 2 | 0.540  | 0.568  | 0      | KS3  |
| 20  | 2 | 2.560  | 2.285  | 0      | KS3  |
| 21  | 2 | 1.789  | 1.265  | 0      | KS3  |
| 22  | 2 | 2.698  | 2.459  | 0      | KS3  |
| 23  | 2 | 1.759  | 1.283  | 0      | KS3  |
| 24  | 3 | 1.006  | 0.795  | 0.215  | KS4  |
| 25  | 3 | 0.208  | 0.498  | 0.098  | KS4  |
| 26  | 3 | 0.083  | 0.497  | 0.028  | KS4  |
| 27  | 3 | 1.972  | 2.470  | 1.635  | KS4  |
| 28  | 3 | 1.180  | 0.909  | 0.269  | KS4  |
| 29  | 3 | 0.529  | 0.566  | 0.113  | KS4  |
| 30  | 3 | 2.079  | 2.740  | 1.875  | KS4  |
| 31  | 3 | 1.110  | 0.869  | 0.266  | KS4  |
| 32  | 3 | 0.510  | 0.568  | 0.092  | KS4  |
| 33  | 3 | 1.948  | 2.285  | 1.432  | KS4  |
| 34  | 3 | 1.615  | 1.266  | 0.432  | KS4  |
| 35  | 3 | 2.028  | 2.450  | 1.569  | KS4  |
| 36  | 3 | 1.560  | 1.283  | 0.494  | KS4  |
| 37  | 3 | 0      | 1      | 1      | HUL  |

†ISO: isotropic scattering; LIN: linearly anisotropic scattering; RAY: Rayleigh scattering; SOB: Sobolev (1975), Table 7.1; KS3, KS4: Kolesov & Smoktii (1972), Table 1; HUL: van de Hulst (1980), Table 29

*Phase functions exhibit negative values depending on the values of scattering angle.
### Table 2  Number of iterations $N_{it}$ required to calculate $H$-function for isotropic scattering

| $\omega_0$ | $N_{it}$ | $N_{it} - \min N_{it}$ |
|------------|----------|------------------------|
|            | A    | B    | C    | D    | A    | B    | C    | D    |
| 1.000      | 12   | 31   | 10   | 7    | 5    | 24   | 3    | 0    |
| 0.999      | 12   | 29   | 9    | 6    | 6    | 23   | 3    | 0    |
| 0.990      | 12   | 27   | 10   | 6    | 6    | 21   | 4    | 0    |
| 0.900      | 13   | 20   | 10   | 6    | 7    | 14   | 4    | 0    |
| 0.800      | 14   | 17   | 10   | 6    | 8    | 11   | 4    | 0    |
| 0.700      | 15   | 15   | 11   | 5    | 10   | 10   | 6    | 0    |
| 0.600      | 14   | 13   | 11   | 6    | 8    | 7    | 5    | 0    |
| 0.500      | 13   | 11   | 10   | 5    | 8    | 6    | 5    | 0    |
| 0.400      | 12   | 10   | 9    | 5    | 7    | 5    | 4    | 0    |
| 0.300      | 11   | 9    | 8    | 4    | 7    | 5    | 4    | 0    |
| 0.200      | 10   | 7    | 7    | 4    | 6    | 3    | 3    | 0    |
| 0.100      | 8    | 6    | 6    | 4    | 4    | 2    | 2    | 0    |
| 0.050      | 7    | 5    | 5    | 3    | 4    | 2    | 2    | 0    |
| 0.001      | 4    | 2    | 2    | 2    | 2    | 0    | 0    | 0    |
| **Total**  | 157  | 202  | 118  | 69   | 88   | 133  | 49   | 0    |
Table 3  $N_{it} - \min N_{it}$ for the 116 cases involving anisotropic phase functions

| Cases | #  | A   | B   | C   | D   | Total |
|-------|----|-----|-----|-----|-----|-------|
| $m \geq 0$ | 116 | 981 | 5125 | 908 | 80  | 7094  |
|       |     | (.138) | (.722) | (.128) | (.011) |       |
| $m = 0$   | 37  | 349 | 4180 | 19  | 0   | 4548  |
|       |     | (.319) | (.077) | (.919) | (.004) | (.0)  |
| $m \geq 1$ | 79  | 632 | 945  | 889 | 80  | 2546  |
|       |     | (.681) | (.248) | (.371) | (.349) | (.031) |
Table 4  Values of $H$-functions and azimuth-angle averaged reflection function for conservative scattering with the phase function No. 34 of Table 1 using Method D

| $\mu$  | $H^{(0)}(1, \mu)$ | $H^{(1)}(1, \mu)$ | $H^{(2)}(1, \mu)$ | $H^{(3)}(1, \mu)$ | $R^{(0)}(\mu, \mu)$ |
|-------|------------------|------------------|------------------|------------------|------------------|
| 0.00  | 1.0000000000     | 1.0000000000     | 1.0000000000     | 1.0000000000     | $\infty$        |
| 0.05  | 1.1659440619     | 1.0771633075     | 1.0320505999     | 1.0076297119     | 4.2285847359    |
| 0.10  | 1.2989965575     | 1.1265567212     | 1.0516671536     | 1.0113354601     | 2.4817486757    |
| 0.15  | 1.4229520561     | 1.1661176772     | 1.0652788635     | 1.0138828020     | 1.879139139     |
| 0.20  | 1.5420072951     | 1.1995291407     | 1.0760596942     | 1.0158004425     | 1.5711341592    |
| 0.25  | 1.6579405618     | 1.2285300089     | 1.0849344306     | 1.0173173607     | 1.3827840472    |
| 0.30  | 1.7717010913     | 1.2541429670     | 1.0924264204     | 1.018568495      | 1.2569336797    |
| 0.35  | 1.8838624879     | 1.2770429808     | 1.098869831      | 1.019593779      | 1.1687277855    |
| 0.40  | 1.9947999590     | 1.2977085807     | 1.1044812796     | 1.0204763205     | 1.1057803494    |
| 0.45  | 2.1047729686     | 1.3164959702     | 1.1094300709     | 1.0212386882     | 1.0612366041    |
| 0.50  | 2.2139685305     | 1.3336798109     | 1.1138324177     | 1.0219047912     | 1.030969193     |
| 0.55  | 2.3225258489     | 1.3494776133     | 1.1177790365     | 1.0224924770     | 1.0122907758    |
| 0.60  | 2.4305125272     | 1.3640652645     | 1.1213406392     | 1.0230152976     | 1.0032916519    |
| 0.65  | 2.538127033      | 1.3775874048     | 1.1245733862     | 1.0234837620     | 1.0024844377    |
| 0.70  | 2.6453210934     | 1.3901646382     | 1.1275225888     | 1.0239061654     | 1.0085874221    |
| 0.75  | 2.7521845597     | 1.4018987024     | 1.1302252991     | 1.0242891558     | 1.0203910630    |
| 0.80  | 2.8587615184     | 1.4128762757     | 1.1327121707     | 1.0246381324     | 1.036687265     |
| 0.85  | 2.965087522      | 1.4231718428     | 1.1350088237     | 1.0249575309     | 1.0561139435    |
| 0.90  | 3.071135192      | 1.4328498923     | 1.1371368562     | 1.025210332      | 1.077293407     |
| 0.95  | 3.1771037571     | 1.4419666308     | 1.1391146657     | 1.0255217236     | 1.0986134940    |
| 1.00  | 3.2828399994     | 1.4505713372     | 1.1409579575     | 1.0257722074     | 1.1182855176    |

\#iters.\textsuperscript{b}  12  14  11  7

\textsuperscript{a}A four-term phase function with $x_1 = 1.615$, $x_2 = 1.266$, and $x_3 = 0.432$

\textsuperscript{b}The numerical figures to the right indicate the number of iterations needed to get the values of $H^{(m)}(1, \mu)$ ($m = 0, 1, 2, 3$) with $\varepsilon = 10^{-12}$. 
