ON UNIQUENESS RESULTS FOR THE BENJAMIN EQUATION

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Abstract. We prove that the uniqueness results obtained in [22] for the Benjamin equation, cannot be extended for any pair of non-vanishing solutions. On the other hand, we study uniqueness results of solutions of the Benjamin equation. With this purpose, we showed that for any solutions $u$ and $v$ defined in $\mathbb{R} \times [0, T]$, if there exists an open set $I \subset \mathbb{R}$ such that $u(\cdot, 0)$ and $v(\cdot, 0)$ agree in $I$, $\partial_t u(\cdot, 0)$ and $\partial_t v(\cdot, 0)$ agree in $I$, then $u \equiv v$. To finish, a better version of this uniqueness result is also established.

1. Introduction

In this work, we study the initial-value problem (IVP) concerning the Benjamin equation
\begin{equation}
\begin{cases}
  u_t + \mathcal{H} \partial_x^2 u + \partial_x^3 u + uu_x = 0, & x, t \in \mathbb{R} \\
  u(0, x) = \phi(x),
\end{cases}
\tag{1.1}
\end{equation}
where $u = u(t, x)$ is a real-valued function and $\mathcal{H}$ stands for the Hilbert transform defined as
\begin{align*}
  \mathcal{H} f(x) &= \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \int_{|y| \geq \epsilon} \frac{f(x - y)}{y} dy \\
  &= -i (\text{sgn}(\xi) \hat{f}(\xi))^{\vee}(x).
\end{align*}

The integral-differential equation (1.1) is a mathematical model to describe a class of the intermediate waves in the stratified fluid. It was deduced by Benjamin [1], to study gravity-capillary surface waves of solitary type on deep water. He also obtained the following conservation laws
\begin{align}
  I_1(u) &= \int_{-\infty}^{\infty} u(t, x) dx, \tag{1.2} \\
  I_2(u) &= \int_{-\infty}^{\infty} u^2(t, x) dx \tag{1.3}
\end{align}
and
\begin{align*}
  I_3(u) &= \int_{-\infty}^{\infty} \left\{ \frac{1}{2} (\partial_x u(t, x))^2 - \frac{1}{2} u(t, x) \mathcal{H} \partial_x u(t, x) - \frac{1}{3} u^3(t, x) \right\} dx.
\end{align*}

The IVP (1.1) has been extensively studied with respect existence, uniqueness and regularity of solutions in the Sobolev space $H^s(\mathbb{R})$. In [19] Linares, using the Fourier restriction method together with the conservation quantity (1.3) obtained global well-posedness of the IVP (1.1) in $L^2(\mathbb{R})$. In the last years, this result was improved by many authors. Kosovo et al. [16] proved local well-posedness in $H^s(\mathbb{R})$, where $s > -3/4$. Li and Wu [18], using the I-method, showed global well-posedness of the Benjamin equation in $H^s(\mathbb{R})$, for $s > -3/4$. Finally, the best result is due

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a Chen, Guo and Xiao [5], that extended the global well-posedness to $H^s(\mathbb{R})$ for $s \geq -3/4$.

There are many works about uniqueness properties for dispersive models. We can mentioned that for IVPs associated to the $k$-generalized Korteweg–de Vries (k-gKdV) equation

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad t, x \in \mathbb{R},$$

and for the semi-linear Schrödinger (NLS) equation

$$\partial_t u = i(\nabla u + V(x,t)u), \quad \text{in } \mathbb{R}^n \times [0,T],$$

Escauriaza, Kenig, Ponce and Vega [10, 11] obtained uniqueness results for any pair of solutions $u_1$ and $u_2$ in a appropriate class functions. Also in this direction, in [2], for the Ostrovsky equation and in [3], for the Zakharov–Kuznetsov (ZK) equation, Bustamante et al., showed that if the difference of two sufficiently regular solutions $u_1$ and $u_2$ have a suitable decay in two different times, then they are equal. Recently, the results in [3] was improved in [6].

On the other hand, also there exist results on Benjamin equation in the weighted Sobolev spaces

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(x^{2r} dx) \quad \text{and} \quad \hat{Z}_{s,r} = \{ f \in Z_{s,r} : \hat{f}(0) = 0 \},$$

where $s, r \geq 0$. In such spaces, Urrea [22], studied well-posedness and unique continuation properties of solutions. Here it can be seen that the condition $s \geq 2r$ is necessary for solution $u$ of the Benjamin equation satisfies the persistence property in $Z_{s,r}$. Next, these results are put in order.

The first is about persistence property and well-posedness of solutions.

**Theorem A**

(i) Let $s \geq 1$, $r \in [0,s/2]$ and $r < 5/2$. If $\phi \in Z_{s,r}$, then the solution $u(x,t)$ of the IVP (1.1) satisfies that $u \in C([0,\infty) : Z_{s,r})$. Furthermore if $s > 3/2$, $r \in [0,s/2]$ and $r < 5/2$, then the IVP (1.1) is globally well-posed in $Z_{s,r}$.

(ii) For $r \in [5/2,7/2)$, $r \leq s/2$, the IVP (1.1) is GWP in $\hat{Z}_{s,r}$. The second is a necessary condition for the persistence of solution in $Z_{s,r}$, where $s \geq 5$ and $r \geq 5/2$.

**Theorem B** Let $u \in C([0,T] : Z_{4,2})$ be a solution of the IVP (1.1). If there exist two different times $t_1, t_2 \in [0,T]$ such that $u(\cdot,t_j) \in Z_{5,5/2}$, for $j = 1,2$, then $\phi(0) = 0$ and therefore $u(\cdot,t) \in Z_{4,2}$, for all $t \in [0,T]$.

In the following, was established the unique continuation property.

**Theorem C** Suppose that $u \in C([0,T] : \hat{Z}_{7,7/2})$ is a solution of the IVP (1.1). If there exist three different times $t_1, t_2, t_3 \in [0,T]$ such that $u(\cdot,t_j) \in \hat{Z}_{7,7/2}$, for $j = 1, 2, 3$, then $u(x,t) \equiv 0$.

**Theorem D** Suppose that $u \in C(\mathbb{R} : \hat{Z}_{7,7/2})$ is a nonzero solution of the IVP (1.1). If $u(0) = \phi \in \hat{Z}_{10,4}$ and

$$\int_{-\infty}^{\infty} x\phi(x)dx \neq 0,$$

then, there exists $t^* \neq 0 \in \mathbb{R}$ such that $u(t^*) \in Z_{8,4}$. 


As observed in [22], Theorem D shows us that the hypothesis of three different times in Theorem C cannot be reduced for two different times. Hence, we see that it’s may expect to have uniqueness results with two different times on $\dot{Z}_{s,r}$, only for decay $r > 4$. In addition, about the decay of solutions, we also can mention that the author in [12], by using conservation laws of the Benjamin-Ono equation improved the decay results (see Theorem 2) in [13].

Next, we point out the main contributions of this paper. Our first result (see Theorem 1.1 below) is inspired by the work of Fonseca, Linares and Ponce [13]. Here, the authors, for the Benjamin-Ono (BO) equation

$$u_t + \mathcal{H} \partial_x^2 u + uu_x = 0, \quad x, t \in \mathbb{R},$$

shows that the uniqueness result obtained by Fonseca and Ponce in [14] cannot be extended to any pair of non-vanishing solutions of the BO equation.

**Theorem 1.1.** Let $u$ and $v$ solutions of the IVP (1.1) with initial data $\phi$ and $\varphi$, respectively. Suppose that $\phi, \varphi \in Z_{9-4}$ satisfies $\phi \neq \varphi$,

$$\|\phi\| = \|\varphi\|,$$  \hspace{1cm} (1.4)

$$\int \phi(x)dx = \int \varphi(x)dx$$  \hspace{1cm} (1.5)

and

$$\int x\phi(x)dx = \int x\varphi(x)dx.$$  \hspace{1cm} (1.6)

Then $u \neq v$ and for all $T > 0$

$$u - v \in L^\infty([-T, T]; Z_{9-4}).$$  \hspace{1cm} (1.7)

The above theorem tell us that uniqueness results established in [22] do not extend to any solutions $u \neq 0$ and $v \neq 0$ of the Benjamin equation.

In what follows, by assuming higher-order decay in initial data, we improve the last theorem.

**Theorem 1.2.** Let $\theta \in (0, 1/2)$ and $u, v$ solutions of the IVP (1.1) with initial data $\phi$ and $\varphi$, respectively. Suppose that $\phi, \varphi \in Z_{9+2\theta, 4+\theta}$ satisfies $\phi \neq \varphi$, \hspace{1cm} (1.4)

$$\|\phi\| = \|\varphi\|,$$ \hspace{1cm} (1.5)

and \hspace{1cm} (1.6), above.

Then $u \neq v$ and for all $T > 0$

$$u - v \in L^\infty([-T, T]; Z_{9+2\theta, 4+\theta}).$$ \hspace{1cm} (1.8)

For the next two results, we show a version of the Theorems 1.1 and 1.2 for initial data with more low regularity.

**Theorem 1.3.** Let $u$ and $v$ solutions of the IVP (1.1) with initial data $\phi$ and $\varphi$, respectively. Suppose that $\phi, \varphi \in Z_{8, 4}$ satisfies $\phi \neq \varphi$, \hspace{1cm} (1.4)

$$\|\phi\| = \|\varphi\|,$$ \hspace{1cm} (1.5)

and \hspace{1cm} (1.6), above.

Then $u \neq v$ and for all $T > 0$

$$u - v \in L^\infty([-T, T]; Z_{8, 4}).$$ \hspace{1cm} (1.9)

**Theorem 1.4.** Let $\theta \in (0, 1/2)$ and $u, v$ solutions of the IVP (1.1) with initial data $\phi$ and $\varphi$, respectively. Suppose that $\phi, \varphi \in Z_{8+2\theta, 4+\theta}$ satisfies $\phi \neq \varphi$, \hspace{1cm} (1.4)

$$\|\phi\| = \|\varphi\|,$$ \hspace{1cm} (1.5)

and \hspace{1cm} (1.6), above.

Then $u \neq v$ and for all $T > 0$

$$u - v \in L^\infty([-T, T]; Z_{8+2\theta, 4+\theta}).$$ \hspace{1cm} (1.10)
The next result corresponds to an improvement of the Theorem D, in the sense that, it requires more higher decay on the initial data $\phi$.

**Theorem 1.5.** Let $\theta \in (0, 1/2)$. Suppose that $u \in C([0, T]; Z_{7/2-})$ is a solution of the IVP \[1.11\]. If $u(0) = \phi \in \mathcal{Z}_{9+2\theta,4+\theta}$ and

$$\int_{-\infty}^{\infty} x\phi(x)dx \neq 0,$$

then

$$u(t^*) \in \mathcal{Z}_{9+2\theta,4+\theta},$$

where

$$t^* = -\frac{4}{||\phi||^2} \int_{-\infty}^{\infty} x\phi(x)dx.$$

Note that the Theorem \[1.16\] implies that we only may expect to have uniqueness results on two different times in $\mathcal{Z}_{s,r}$, for a decay $r \geq 9/2$.

Our results on uniqueness are inspired in the recent work of Kenig, Ponce and Vega \[15\]. In this paper, the authors use techniques of Complex analysis to establish uniqueness results for the Benjamin Ono equation.

In the following we present our first main result.

**Theorem 1.6.** Let $u$ and $v$ be the solutions of the IVP \[1.12\] for $(x,t) \in \mathbb{R} \times [0,T]$ such that

$$u, v \in C([0, T]; H^s(\mathbb{R})) \cup C^1((0, T); H^{s-3}(\mathbb{R})), \ s > 7/2. \quad (1.11)$$

If there exist an open set $I \subset \mathbb{R}$ such that

$$u(x,0) = v(x,0) \quad \text{and} \quad \partial_t u(x,0) = \partial_t v(x,0), \ \text{for all} \ x \in I, \quad (1.12)$$

then

$$u \equiv v. \quad (1.13)$$

In particular, if $u \equiv 0$ in $I \times \{0\}$ and $\partial_t u(x,0) = 0, \ \forall x \in I$, then

$$u \equiv 0. \quad (1.14)$$

Note that the hypotheses of the above theorem are more general than from those in \[13\] Theorem 1.1. Here we require conditions only on initial data and the time derivative at the origin.

The next result is an improvement of the above Theorem.

**Theorem 1.7.** Let $u$ and $v$ be the solutions of the IVP \[1.12\] for $(x,t) \in \mathbb{R} \times [0,T]$ such that

$$u, v \in C([0, T]; H^s(\mathbb{R})) \cup C^1((0, T); H^{s-3}(\mathbb{R})), \ s > 7/2. \quad (1.15)$$

If there exists an open set $I \subset \mathbb{R}$, $0 \in I$, such that

$$u(x,0) = v(x,0), \ x \in I, \quad (1.16)$$

and for each $N \in \mathbb{Z}^+$

$$\int_{|x| \leq R} |\partial_t u(x,0) - \partial_t v(x,0)|^2 dx \leq c_N R^N, \ \text{as} \ R \downarrow 0, \quad (1.17)$$

then

$$u(x,t) = v(x,t), \ (x,t) \in \mathbb{R} \times [0,T]. \quad (1.18)$$
The rest of this paper is organized as follows. Section 2 contains some notation and preliminary estimates that will be useful in the proof of our results. In Section 3 we give the proof of Theorems 1.1–1.4. In Section 4 is proved the Theorem 1.5. Finally, in Section 5 is presented the proof of Theorems 1.6 and 1.7.

2. Notation and Preliminaries

Let us introduce the notation that is being used in this paper. We write $a \lesssim b$ if there exists a constant $c$ such that $a \leq cb$; by $a \sim b$ we mean that $a \lesssim b$ and $b \lesssim a$. The notation $a \lesssim_l b$ indicate that the constant $c$ depends on parameter $l$. For the usual $L^2(\mathbb{R})$ norm we will write only $\| \cdot \|$. The Fourier transform of $f$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.$$ 

Given any $s \in \mathbb{R}$, we denote the following operators, via their Fourier transform as follows

$$\hat{J}^s f(\xi) = (1 + \xi^2)^{s/2} \hat{f}(\xi) \quad \text{and} \quad \hat{D}^s f(\xi) = |\xi|^s \hat{f}(\xi).$$

We also define the $L^2$ based Sobolev space $H^s = H^s(\mathbb{R})$ by

$$\{ f \in S'(\mathbb{R}) : \| \langle \xi \rangle^s \hat{f} \| < \infty \},$$

where $S'(\mathbb{R})$ is the space of tempered distributions and $\langle \xi \rangle := (1 + \xi^2)^{1/2}$.

For all $r \in \mathbb{R}$, we setting $L^2_r(\mathbb{R})$ as being the set of all functions such that

$$\int (1 + x^2)^r |f(x)|^2 dx < \infty.$$

If $s, r \in \mathbb{R}$ we denote the weighted Sobolev space (as we already mentioned) by

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2_r(\mathbb{R}),$$

endowed with the norm

$$\| \cdot \|_{Z_{s,r}}^2 = \| \cdot \|_{H^s}^2 + \| \cdot \|_{L^2_r}^2.$$ 

We also write $\hat{Z}_{s,r} = \{ f \in Z_{s,r} : \hat{f}(0) = 0 \}$.

For our estimates in weighted spaces, we setting the truncated weights $\langle x \rangle_N$, $N \in \mathbb{Z}^+$, which are given by

$$\langle x \rangle_N := \begin{cases} 
\langle x \rangle & \text{if } |x| \leq N, \\
2N & \text{if } |x| \geq 3N,
\end{cases}$$

where $\langle x \rangle = (1 + x^2)^{1/2}$. Also, we assume that $\langle x \rangle_N$ is smooth and non-decreasing in $|x|$ with $\langle x \rangle_N'(x) \leq 1$, for any $x \geq 0$, and there exists a constant $c$ independent of $N$ such that $|\langle x \rangle''_N(x)| \leq c \partial_x^2 \langle x \rangle$.

Finally, putting $\mu(\xi, t) := e^{it(\xi^2 - \xi \langle \xi \rangle)}$, follows that the group generated by the linear part of Benjamin equation can be written, via their Fourier transform as

$$\hat{U}(t) \hat{\phi}(\xi, t) = \mu(\xi, t) \hat{\phi}(\xi).$$

Thus, the integral equation associated with the IVP (1.1) is given by

$$u(t) = U(t) \phi - \int_0^t U(t - \tau) \kappa(\tau) d\tau, \quad (2.1)$$

where $\kappa(\tau) := \frac{1}{2} \partial_x u^2$. 
Proposition 2.1. Let $\delta, \nu > 0$ such that $J^{\delta} f \in L^2(\mathbb{R})$ and $\langle x \rangle^\nu f \in L^2(\mathbb{R})$. Then for any $\beta \in (0, 1)$
\[ \|J^{\beta}\langle x \rangle(1-\beta)^\nu f\| \leq c\|\langle x \rangle^\nu f\|^{1-\beta}\|J^{\delta} f\|^\beta. \]

Proof. See Lemma 1 in [14]. □

Lemma 2.2. Let $-1/2 < \nu < 1/2$, then the Hilbert transform $\mathcal{H}$ is a bounded operator in $L^2(\omega^\nu dx)$, i.e.
\[ \|\omega'\mathcal{H} f\| \leq c\|\omega^\nu f\|, \]
where $\omega = |x|$ or $\omega = \langle x \rangle_N$ and $c$ is a constant independent of $N$.

Proof. See [14] and references therein. □

The following result helps us to estimate the $L^2$ norm of some terms that will appear in the proof of Theorems 1.2 and 1.5.

Proposition 2.3. For all $\theta \in (0, 1)$
\[ \|J^{2\theta}(x^\theta \partial_x^k f)\| \lesssim \|J^{2(\theta+j)+k} f\| + \|\langle x \rangle^{\theta+j+k/2} f\|, \tag{2.2} \]

and
\[ \|\langle x \rangle^\theta \partial_x^k (x^l f)\| \lesssim \|J^{2(4\theta)} f\| + \|\langle x \rangle^{2(4\theta)(j+k+1)/(4\theta+1)} f\|, \tag{2.3} \]

where $j$ and $k$ are non-negative integers such that $k < 2(4\theta)$.

Proof. First, we will deal with (2.3). We divide the proof in two cases.

Case $k \leq j$.

For all $0 \leq l \leq k$, it is possible to apply Lemma 2.1 to obtain
\[ \|J^l(\langle x \rangle^{j-k+1+l}\partial_x^k f)\| \lesssim \|J^{2(\theta+j)} f\| + \|\langle x \rangle^{2(\theta+j)} f\|. \tag{2.4} \]

Thus, by using Leibniz rule for derivatives and (2.4) it seems that
\[ \|\langle x \rangle^\theta \partial_x^k (x^l f)\| \lesssim \sum_{l=0}^k \|\langle x \rangle^{j-k+l+\theta} \partial_x^k f\| \lesssim \sum_{l=0}^k \|J^l(\langle x \rangle^{j-k+1+l+\theta} f)\| \lesssim \sum_{l=0}^k (\|J^{2(\theta+j)} f\| + \|\langle x \rangle^{2(\theta+j)} f\|) \leq \|J^{2(\theta+j)} f\| + \|\langle x \rangle^{2(\theta+j)} f\|, \tag{2.5} \]

where above we used the inequality
\[ \frac{2(4+\theta)(j-k+l+\theta)}{2(4+\theta)-l} \leq \frac{2(4+\theta)(j+\theta)}{2(4+\theta)-k}. \]

Case $k > j$.

For any $0 \leq l \leq j$, again using Lemma 2.1
\[ \|J^{k-j}(\langle x \rangle^{j-l+\theta} f)\| \lesssim \|J^{2(\theta+j)} f\| + \|\langle x \rangle^{2(\theta+j)} f\|. \tag{2.6} \]
Then, by Leibnitz rule and (2.6) follows that
\[
\|x^\theta \partial_x^k (x^j f)\| \lesssim \sum_{l=0}^{j} \|x^{j-l+\theta} \partial_x^{k-j} f\|
\]
\[
\lesssim \sum_{l=0}^{j} \|J^{k-j} (x^{j-l+\theta} f)\|
\]
\[
\leq \sum_{l=0}^{j} \left( \|J^{2(4+\theta)} f\| + \|x^{2(4+\theta)(j-l+\theta)} f\| \right)
\]
\[
\lesssim \|J^{2(4+\theta)} f\| + \|x^{2(4+\theta)(j+\theta)} f\|, \quad (2.7)
\]
where above its used the inequality
\[
\frac{2(4 + \theta)(j - l + \theta)}{2(4 + \theta) - k + l} \leq \frac{2(4 + \theta)(j + \theta)}{2(4 + \theta) - k}.
\]
With respect to inequality (2.2), the Plancherel identity implies that
\[
\|J^b \theta (x^j \partial_x^k f)\| = \|\langle \xi \rangle^{2b} \partial_x^k (\xi^j \hat{f})\|, \quad (2.8)
\]
hence, from here we deal by similar way to above.
This finishes the proof. \[\square\]

Next, we present a characterization of Sobolev spaces defined as \(L^p_s := (1 - \Delta)^{-s/2} L^p(\mathbb{R}^n).\)

**Theorem 2.4.** Let \(b \in (0, 1)\) and \(2n/(n + 2b) < p < \infty.\) Then \(f \in L^p_s(\mathbb{R}^n)\) if and only if
\[
\begin{align*}
&\text{a) } f \in L^p(\mathbb{R}^n), \\
&\text{b) } \mathcal{D}^b f(x) = \left( \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+2b}} dy \right)^{1/2} \in L^p(\mathbb{R}^n), \text{ with,}
\end{align*}
\]
\[
\|f\|_{b,p} \equiv \|(1 - \Delta)^{b/2} f\|_p = \|J^b f\|_p \simeq \|f\|_p + \|D^b f\|_p \simeq \|f\|_p + \|\mathcal{D}^b f\|_p, \quad (2.9)
\]
where, for \(s \in \mathbb{R}, D^s = (-\Delta)^{s/2}.\)

**Proof.** See Theorem 1 in [20]. \[\square\]

The above operator \(\mathcal{D}^b\) sometimes is named of Stein derivative of order \(b.\) The advantage in using it, is that is possible deduce a few useful pointwise estimates.

Moreover, from the above theorem, part b), with \(p = 2\) and \(b \in (0, 1),\) it’s possible to obtain the following product estimate
\[
\|\mathcal{D}^b (fg)\|_2 \leq \|f\| \|\mathcal{D}^b g\|_2 + \|g\| \|\mathcal{D}^b f\|_2. \quad (2.10)
\]
In what follows, the next two results, are useful to obtain the Lemma 2.7 below.

**Proposition 2.5.** Let \(b \in (0, 1).\) For any \(t > 0,\)
\[
\mathcal{D}^b (e^{-it|x|}) \leq c(t^{b/2} + t^b |x|^b). \quad (2.11)
\]

**Proof.** See Proposition 2 in [21]. See also [8]. \[\square\]
Lemma 2.6. Let $b \in (0, 1)$. There exists a constant $C_b > 0$ such that, for all $t > 0$ and $x \in \mathbb{R}$,

$$\mathcal{D}^b(e^{itx^3}) \leq C_b \left( t^{b/3} + t^{1/3+2b/9} + (t^{1/3+2b/3} + t^{2b/3})|x|^{2b} \right).$$

Proof. See Lemma 2.2 in [4].

The following result is a key ingredient to deal with the estimates in Theorems 1.2 and 1.5.

Lemma 2.7. For all $\theta \in (0, 1)$ and $t \in (0, \infty)$

$$\|\mathcal{D}_x^\theta ((\mu(\xi, t) \hat{f}) \| \lesssim \rho(t)(\|J^{2b} \hat{f}\| + \|\|x\|^\theta \hat{f}\|),$$

where $\rho$ is increasing in $t$.

Proof. Using inequality (2.10), Lemma 3.53 and Proposition 2.5 we obtain

$$\|\mathcal{D}_x^\theta ((\mu(\xi, t) \hat{f}) \| \lesssim \|\mathcal{D}_x^\theta (e^{-it|\xi|}e^{itx^3} \hat{f})\| + \|e^{-it|\xi|}\|\mathcal{D}_x^\theta (e^{itx^3} \hat{f})\|$$

$$\lesssim \| (t^{\frac{\theta}{2}} + t^{\theta}|\xi|^\theta) \hat{f}\| + \| (t^{\frac{\theta}{2}} + t^{\frac{\theta}{3}} + t^{\frac{\theta}{3}} + t^{\frac{\theta}{3}} + t^{\frac{\theta}{3}})|\xi|^{2b} \hat{f}\|$$

$$+ \|\mathcal{D}_x^\theta \hat{f}\|$$

$$\lesssim \rho(t)(\|\hat{f}\| + \|\|x\|^\theta \hat{f}\| + \|\|x\|^{|\xi|^2} \hat{f}\| + \|\mathcal{D}_x^\theta \hat{f}\|)$$

$$\lesssim \rho(t)(\|\|\xi\|^{|\xi|^2} \hat{f}\| + \|\mathcal{D}_x^\theta \hat{f}\|).$$

Hence, the Plancherel identity and inequality (2.9) give us the desired result.

The next Lemma will be useful in the proof of Theorem 1.6.

Lemma 2.8. Let $f \in H^s(\mathbb{R})$, $s > 1/2$ be a real valued function. If there exists an open set $I \subset \mathbb{R}$ such that

$$f(x) = \mathcal{H}_f(x) = 0, \quad \forall x \in I,$$  (2.13)

then $f \equiv 0$.

Proof. See Corollary 2.2 in [15].

To prove the Theorem we made use of the following result.

Lemma 2.9. Let $f \in L^2(\mathbb{R})$ be a real valued function. If there exists an open set $I \subset \mathbb{R}$, $0 \in I$, such that

$$f(x, 0) = 0, \quad x \in I,$$  (2.14)

and for each $N \in \mathbb{Z}^+$

$$\int_{|x| \leq R} |\mathcal{H}_f(x)|^2 dx \leq c_NR^N \quad \text{as} \quad R \downarrow 0,$$  (2.15)

then

$$f(x) = 0, \quad x \in \mathbb{R}.$$  (2.16)

Proof. See Lemma 4.1 in [15].
3. Proof of Theorems 1.1–1.4

This section is devoted to proof Theorems 1.1–1.4. For Theorems 1.1 and 1.2 we use estimates on the norm of weighted spaces of the group associated with the linear part of the Benjamin equation. Such technique is also present in [7] and [9].

In Theorems 1.3 and 1.4 we use the approach given by the authors in [13].

Proof of Theorem 1.1. Let \(u\) and \(v\) be the solutions of the IVP (1.1), with initial data \(\phi\) and \(\varphi\), respectively. Suppose also \(\phi \neq \varphi\), then by putting \(\sigma := \phi - \varphi\), \(w := u - v\) and \(z := \frac{1}{2}\partial_{\xi}(u^2 - v^2)\), the integral equation (2.1) gives us

\[
w(t) = U(t)\sigma - \int_0^t U(t - \tau)z(\tau)d\tau.
\] (3.1)

We observe that for all \(T > 0\), the Theorem A implies that \(u, v \in C([-T, T]; L^p_{\text{loc}})\). Choosing \(1/4 < \epsilon < 1/2\) and setting \(s = s_\epsilon = \frac{2 + \epsilon}{\epsilon}\), it follows that

\[s < 9 \quad \text{and} \quad 2 + \epsilon < 5/2.
\] (3.2)

As a consequence, the constant

\[N := \sup_{[-T,T]} \{|u(t)| + |\xi,v\|_{L^2_{\text{loc}}} + |v(t)|_{L^2_{\text{loc}}}\},
\] (3.3)

is finite.

Multiplying the last identity by \(x^4\) and taking the Fourier transform follows that

\[
\partial_x^4(w(t)) = \partial_x^4(\mu(\xi, t)\hat{\sigma}) - \int_0^t \partial_x^4(\mu(\xi, t - \tau)\hat{z}(\tau))d\tau,
\] (3.4)

we recall that \(\mu(\xi, t) = e^{it(\xi^3 - \ell|x|)}\).

Before to deal with the second right-hand of the last identity, we will need of the third derivative of the function \(\mu \tilde{\sigma}\), that, after several computations is given by

\[
\partial_x^3(\mu(\xi, t)\hat{\sigma}) = \left(4i\ell\delta + 6it - 12it^2\xi + 54t^3|\xi|\xi - 54t^2\xi^3 + 8it^3|\xi|^2 - 36it^3\xi^4 + 54it^3|\xi|^4 - 27it^3\xi^6\right)\hat{\sigma} + \left(-6it\text{sgn}(\xi) + 18it\xi - 12it^2\xi^2 + 36it^2|\xi|^2 - 27it^2\xi^4\right)\partial_x\hat{\sigma}
\]

\[
+ \left(9it\xi^2 - 6it|x|\right)\partial_x^2\hat{\sigma} + \partial_x^3\hat{\sigma}\mu.
\] (3.5)

Using the hypothesis (1.5) the term above that involves the delta Dirac function, can be computed as

\[-4it\hat{\sigma}\psi\delta = -4i\left(\int \sigma(x)dx\right)\delta = -4i\left(\int \phi(x)dx - \int \varphi(x)dx\right)\delta = 0.
\] (3.6)

Then, the delta Dirac function does not appear in identity (3.5). Using this fact, we can take the derivative with respect to \(\xi\) variable in both sides of identity (3.5), to obtain
\( \frac{\partial^2}{\partial t^2}(\mu(\xi, t) \hat{\sigma}) = \left( -12t^2 - 120i|\xi| + 180\xi^2 - 288it^2\xi^3 - 540i|\xi|^4 + 324\xi^5 + 48it^3\xi|\xi| - 216t^2\xi^4 - 216it|\xi|^6 + 81\xi^8 + 96it^3|\xi|^4 + 16t^4\xi^4 \right) \hat{\sigma} \\
- \left( -4t - 4it\delta + 6\xi - 12t^2\xi - 54i\xi^4 + 54\xi^3 - 36t^2\xi^4 - 54it|\xi|^4 + 27\xi^6 + 8it\delta |\xi|^2 \right) \partial_\xi \hat{\sigma} + 6\left( -2it \text{sgn}(\xi) + 6\xi - 4t^2\xi^2 - 12it|\xi|^2 + 9\xi^4 \right) \partial_\xi^2 \hat{\sigma} \\
+ 4\left( 3\xi^2 - 2it|\xi| \right) \partial_\xi^2 \hat{\sigma} + \partial_\xi^3 \hat{\sigma} \right) \mu. \tag{3.7} \\
\]

The next step is to employ the identity \( B_{\delta} \) to deal with the first and second terms on the right-hand side of the identity \( B_{\delta} \). First, we will compute explicitly the term in \( B_{\delta} \) which includes the delta Dirac function. To do this, it’s seen that the hypothesis \( B_{\delta} \) implies

\[
16it\partial_\xi \hat{\sigma} \psi \delta = -16 \left( \int x\sigma(x)dx \right) \delta = -16 \left( \int x\psi(x)dx - \int x\varphi(x)dx \right) \delta = 0. \tag{3.8} \\
\]

From \( (3.6) \) and \( (3.8) \), to estimate the first term on the right-hand side of \( (3.4) \) we may write

\[
\| x^4 U_1(t) \sigma \| \lesssim \| \hat{\sigma} \| + \| \xi \| \| \hat{\sigma} \| + \| \xi^2 \| \| \hat{\sigma} \| + \| \xi^3 \| \| \hat{\sigma} \| + \| \xi^4 \| \| \hat{\sigma} \| \\
+ \| \xi^5 \| \| \hat{\sigma} \| + \| \xi^6 \| \| \hat{\sigma} \| + \| \xi^7 \| \| \hat{\sigma} \| + \| \xi^8 \| \| \hat{\sigma} \| \\
+ \| \xi^9 \| \| \hat{\sigma} \| + \| \xi^{10} \| \| \hat{\sigma} \| + \| \xi^{11} \| \| \hat{\sigma} \| + \| \xi^{12} \| \| \hat{\sigma} \| \\
+ \| \xi^{13} \| \| \hat{\sigma} \| + \| \xi^{14} \| \| \hat{\sigma} \| + \| \xi^{15} \| \| \hat{\sigma} \| + \| \xi^{16} \| \| \hat{\sigma} \| \\
+ \| \xi^{17} \| \| \hat{\sigma} \| + \| \xi^{18} \| \| \hat{\sigma} \| + \| \xi^{19} \| \| \hat{\sigma} \| + \| \xi^{20} \| \| \hat{\sigma} \| \\
+ \| \xi^{21} \| \| \hat{\sigma} \| + \| \xi^{22} \| \| \hat{\sigma} \| + \| \xi^{23} \| \| \hat{\sigma} \| + \| \xi^{24} \| \| \hat{\sigma} \| \\
+ \| \xi^{25} \| \| \hat{\sigma} \| + \| \xi^{26} \| \| \hat{\sigma} \| + \| \xi^{27} \| \| \hat{\sigma} \| + \| \xi^{28} \| \| \hat{\sigma} \| \\
:= B_1 + \cdots + B_{28}. \tag{3.9} \\
\]

We will estimate only some terms in \( B_{\delta} \). Using Lemma \( 2.1 \) (with \( \delta = 4, \nu = 8 \)) and Plancherel identity it follows that

\[
\| B_{19} \| \lesssim \| \xi^6 \partial_\xi \hat{\sigma} \| \\
\lesssim \| \langle \xi \rangle^5 \hat{\sigma} \| + \| J_\xi (\langle \xi \rangle^6 \hat{\sigma}) \| \\
\lesssim \| J_\xi^5 \| \| \hat{\sigma} \| + \| J_\xi^6 \| \| \hat{\sigma} \| \| \langle \xi \rangle^8 \| \| \hat{\sigma} \| \\
\lesssim \| J_\xi^8 \| \| \langle \xi \rangle^{10} \| \| \hat{\sigma} \|. \tag{3.10} \\
\]

Also by Proposition \( 2.1 \) (with \( \delta = \nu = 4 \)) we obtain

\[
\| B_{27} \| \lesssim \| \xi \partial_\xi^2 \hat{\sigma} \| \\
\lesssim \| \hat{\sigma} \| + \| J_\xi (\langle \xi \rangle \hat{\sigma}) \| + \| J_\xi^2 (\langle \xi \rangle \hat{\sigma}) \| \\
\lesssim \| J_\xi^3 \| \| \hat{\sigma} \| + \| \langle \xi \rangle^4 \| \| \hat{\sigma} \| \| \langle \xi \rangle^8 \| \| \hat{\sigma} \|. \tag{3.11} \\
\]

To estimate the other terms we use the same argument above. Thus, proceeding in this way we conclude that

\[
\| B_j \| \lesssim \| J_\xi^j \| \| \langle \xi \rangle^4 \|, \quad j = 1, \ldots, 28. \tag{3.12} \\
\]
Then, (3.6)–(3.12) imply that

\[ \|x^4U(t)\sigma\| \lesssim \|J^8\sigma\| + \|x^4\sigma\|. \tag{3.13} \]

In the following we will estimate the second term on the right-hand side of (3.4). First, observe that \( \hat{z}(\tau, \xi) = \frac{i}{2} \xi (\hat{u}^2 - \hat{v}^2) \), then for any \( \tau \in [-T, T] \)

\[ \hat{z}(\tau, 0) = 0, \tag{3.14} \]

and

\[ \partial_\xi \hat{z}(\tau, 0) = \frac{i}{2} (u^2 - v^2)^\wedge (\tau, 0) = \frac{i}{2} (\|u(\tau)\|^2 - \|v(\tau)\|^2) = \frac{i}{2} (\|\phi\|^2 - \|\varphi\|^2) = 0, \tag{3.15} \]

where in the last identity we used the conservation law (1.3) and (1.4). Hence, from (3.6)–(3.12) implies that for all \( \tau \in [-T, T] \)

\[ \|x^4U(t - \tau)\hat{z}(\tau)\| \lesssim \|J^8\hat{z}(\tau)\| + \|x^4\hat{z}(\tau)\|. \tag{3.16} \]

Since \( z = \frac{1}{2} (\partial_x w(u + v) + w\partial_x (u + v)) \), \( z = \frac{1}{2} \) follows that for all \( \tau \in [-T, T] \)

\[ \|x^4U(t - \tau)z(\tau)\| \lesssim \|J^8z(\tau)\| + \|x^4z(\tau)\|. \tag{3.17} \]

Next, we will estimate the terms on the right-hand side of the inequality (3.18). By using (3.3), Sobolev’s embedding and Proposition 2.1 (with \( \delta = s \) and \( \nu = 2 + \epsilon \)) we obtain

\[ \|x^4\partial_x w(u + v)\| \lesssim \|x^2(u + v)\| \|x^2\partial_x w\| \]

\[ \lesssim (\|x^2 u\|_{L^\infty_x} + \|x^2 v\|_{L^\infty_x})(\|xw\| + \|J(\langle x^2 w\rangle)\|) \]

\[ \lesssim (\|J(\langle x^2 u\rangle)\| + \|J(\langle x^2 v\rangle)\|)(\|J^s\| + \|\langle x^2 + \tau w\|\|) \]

\[ \lesssim (\|J^s u\| + \|\langle x^2 + \tau u\| + \|J^s v\| + \|\langle x^2 + \tau v\|\|)^2 \]

\[ \lesssim N^2. \tag{3.19} \]

Analogously to the last inequality we see that

\[ \|x^4w\partial_x (u + v)\| \lesssim N^2. \tag{3.20} \]

To estimate the first term on the right-hand side of (3.18), since \( H^8(\mathbb{R}) \) is a Banach algebra, follow that

\[ \|J^8(\partial_x w(u + v))\| \lesssim \|J^8(\partial_x w)\| \|J^8(u + v)\| \lesssim \|J^9(u - v)\| \|J^8(u + v)\| \lesssim N^2. \tag{3.21} \]

Also, in a similar way to (3.21) it’s seen that

\[ \|J^8(w\partial_x (u + v))\| \lesssim N^2. \tag{3.22} \]

Thus, from (3.13)–(3.22) we deduce that

\[ \|x^4U(t - \tau)z(\tau)\| \lesssim N^2, \quad \text{for all } \tau \in [-T, T]. \tag{3.23} \]
Therefore, (3.11), (3.13) and (3.23) imply that

\[ \|x^4 w(t)\| \lesssim \|J^6 \sigma\| + \|x^4 \sigma\| + N^2 \int_0^t d\tau \lesssim \|J^8 \phi\| + \|x^4 \phi\| + \|J^8 \varphi\| + \|x^4 \varphi\| + |t|N^2, \]

for any \( t \in [-T, T] \).

This gives us the desired result.

\( \Box \)

Proof of Theorem 1.2. Suppose that \( u \) and \( v \) are solutions of the IVP (1.1), with \( u(0) = \phi, v(0) = \varphi, \varphi \neq \phi \) and \( \varphi, \phi \in Z_{9+2\theta, 4+\theta} \). Here we are assuming the same notation as at beginning of the proof of Theorem 1.1 that is \( \sigma := \phi - \varphi, w := u - v \) and \( z := \frac{1}{2}\partial_4 (u^2 - v^2) \). Using Theorem A, we see that for all \( T > 0, u, v \in C([-T, T]; Z_{9+2\theta, 5/2-}) \). In addition, Theorem 1.1 implies that \( w \in C([-T, T]; L^2(|x|^8dx)) \). Hence, the constant

\[ M := \sup_{[0,T]} \{ \|u(t)\|_{Z_{9+2\theta, 2+\theta}} + \|v(t)\|_{Z_{9+2\theta, 2+\theta}} + \|x^4 w(t)\| \}, \]

(3.24)
is finite.

Then, proceeding by similar way as the proof of Theorem 1.1 we obtain

\[ \partial_\xi^1(\hat{w}(t)) = \partial_\xi^1(\mu(\xi, t)\hat{\sigma}) - \int_0^t \partial_\xi^1(\mu(\xi, t - \tau)\hat{\zeta}(\tau)) d\tau. \]

From the last identity we conclude that

\[ D_\xi^6 \partial_\xi^1(\hat{w}(t)) = D_\xi^6 \partial_\xi^1(\mu(\xi, t)\hat{\sigma}) - \int_0^t D_\xi^6 \partial_\xi^1(\mu(\xi, t - \tau)\hat{\zeta}(\tau)) d\tau. \]

(3.25)

In the next, we will estimate the terms in the right-hand side of identity (3.25). To do this, we observe that from the hypothesis (1.5) and (1.6), follows that the identity (3.7) still hold, that is

\[ \partial_\xi^1(\mu(\xi, t)\hat{\sigma}) = \left( -12t^2 - 120i|\xi| + 180\xi^2 - 288t^2\xi^3 - 540i|\xi|^3 + 324\xi^5 + 48it^3|\xi|^3 - 216t^2|\xi|^5 - 216it|\xi|^6 + 81t^3|\xi|^4 + 16t^4|\xi|^4 \right) \hat{\sigma} - 4\left( 6 - 12t^2\xi - 54i|\xi| + 54t^3\xi^3 - 36t^2\xi^4 - 54it|\xi|^4 + 27\xi^6 + 8it^3|\xi|^2 \right) \delta_\xi \hat{\sigma} + 6\left( 2t\text{sgn}(\xi) + 6\xi - 4t^2\xi^2 - 12t|\xi|^2 + 9\xi^4 \right) \partial_\xi^2 \hat{\sigma} + 4\left( 3\xi^2 - 2it|\xi| \right) \partial_\xi^2 \hat{\sigma} + \partial_\xi^2 \hat{\sigma} \right) \mu, \]

where above, the delta Dirac function does not appear.
Using Plancherel identity and the last equality, the first term on the right-hand side of (3.25) can be estimated as follows

$$
|||x|t\theta U(t)| \leq ||D_0^0(\mu \sigma) + ||D_0^0(\mu \xi \sigma) + ||D_0^0(\mu \xi^2 \sigma)|| +
+ ||D_0^0(\mu \xi^3 \sigma)|| + ||D_0^0(\mu \xi^5 \sigma)|| + ||D_0^0(\mu \xi^9 \sigma)|| +
+ ||D_0^0(\mu \xi^{11} \sigma)|| + ||D_0^0(\mu \xi^{13} \sigma)|| + ||D_0^0(\mu \xi^{17} \sigma)||
+ ||D_0^0(\mu \xi^{25} \sigma)|| + ||D_0^0(\mu \xi^{29} \sigma)|| + ||D_0^0(\mu \xi^{33} \sigma)||
+ ||D_0^0(\mu \xi^{41} \sigma)|| + ||D_0^0(\mu \xi^{45} \sigma)|| + ||D_0^0(\mu \xi^{49} \sigma)||
+ ||D_0^0(\mu \xi^{53} \sigma)|| + ||D_0^0(\mu \xi^{57} \sigma)|| + ||D_0^0(\mu \xi^{61} \sigma)||
+ ||D_0^0(\mu \xi^{65} \sigma)|| + ||D_0^0(\mu \xi^{69} \sigma)|| + ||D_0^0(\mu \xi^{73} \sigma)||
+ ||D_0^0(\mu \xi^{77} \sigma)|| + ||D_0^0(\mu \xi^{81} \sigma)|| + ||D_0^0(\mu \xi^{85} \sigma)||
+ ||D_0^0(\mu \xi^{89} \sigma)|| + ||D_0^0(\mu \xi^{93} \sigma)|| + ||D_0^0(\mu \xi^{97} \sigma)||
+ ||D_0^0(\mu \xi^{101} \sigma)|| + ||D_0^0(\mu \xi^{105} \sigma)|| + ||D_0^0(\mu \xi^{109} \sigma)||
+ ||D_0^0(\mu \xi^{113} \sigma)|| + ||D_0^0(\mu \xi^{117} \sigma)|| + ||D_0^0(\mu \xi^{121} \sigma)||
+ ||D_0^0(\mu \xi^{125} \sigma)|| + ||D_0^0(\mu \xi^{129} \sigma)|| + ||D_0^0(\mu \xi^{133} \sigma)||$$

$$
:= E_1 + \cdots + E_{28}.
$$

(3.26)

Next, we will give details of the estimate of some terms in (3.26). From Lemma 2.7, Proposition 2.2 (with $\omega = |x|^\theta$) and Proposition 2.3 (with $j = 2, k = 0$) it follows that

$$
\|E_{28}\| \leq ||J_2^{2\theta}H(x^2 \sigma)|| + ||x|^{2\theta}H(x^2 \sigma)||
\leq ||J_2^{2\theta}(x^2 \sigma)|| + ||x|^{2\theta}||
\leq ||J_2^{2(\theta+2)}\sigma|| + ||x|^{2\theta+\sigma}||
\leq ||J_2^{2(\theta+1)\sigma}|| + ||x|^{2\theta+\sigma}||.
$$

Also by Lemmas 2.7 and Proposition 2.3 (with $j = 3, k = 2$) we obtain

$$
\|E_{28}\| \leq ||J_2^{2\theta}(x^3 \sigma)|| + ||x|^{2\theta}(x^3 \sigma)||
\leq ||J_2^{2\theta}(x \sigma)|| + ||x|^{2\theta}(x^3 \sigma)|| + ||x|^{2\theta}(x^3 \sigma)||
\leq ||J_2^{2(\theta+1)\sigma}|| + ||x|^{2\theta+\sigma}|| + ||x|^{2\theta+\sigma}||
\leq ||J_2^{2(\theta+3)\sigma}|| + ||x|^{2\theta+\sigma}|| + ||x|^{2\theta+\sigma}||
\leq ||J_2^{2(\theta+4)\sigma}|| + ||x|^{2\theta+\sigma}||.
$$

To deal with the other terms we can proceed in a similar way to above. Thus, we can deduce that

$$
\|E_j\| \leq ||J_2^{2(\theta+1)\sigma}|| + ||x|^{2\theta+\sigma}||, \quad j = 1, \ldots, 28.
$$

(3.27)

Then, gathering together (3.26) and (3.27), we see that

$$
|||x|t\theta U(t)| \leq ||J_2^{2(\theta+1)\sigma}|| + ||x|^{2\theta+\sigma}||.
$$

(3.28)

Now, we will estimate the second term on the right-hand side of (3.25). First we recall that $\hat{\xi}(\tau, 0) = 0$, for all $\tau \in [-T, T]$. Moreover, by analogous way to proof of Theorem 1.1 the conservation law (1.13) and the identity (1.14) imply that $\partial_\tau \hat{\xi}(\tau, 0) = 0$, for any $\tau \in [-T, T]$. As a consequence, putting $z(\tau)$ instead of $\sigma$, in (3.18), we obtain
\[
\| x^{1+\theta} U(t-\tau)z(\tau) \| \lesssim \| J^{2(1+\theta)} z(\tau) \| + \| x^{1+\theta} z(\tau) \|. 
\] (3.29)

In view of
\[
z = \frac{1}{2} (\partial_x w(u+v) + w \partial_x (u+v)),
\] by (3.16) we can write, for all \( \tau \in [-T,T] \)
\[
\| x^{1+\theta} U(t-\tau)z(\tau) \| \lesssim \| J^{2(1+\theta)} \partial_x w(u+v) \| + \| J^{2(1+\theta)} (w \partial_x (u+v)) \|
+ \| x^{1+\theta} \partial_x w(u+v) \| + \| x^{1+\theta} \partial_x w(u+v) \|
\] (3.30)

In what follows, we will deal with terms on the right-hand side of the inequality (3.18).

In view of definition (3.24), the Sobolev's embedding and Proposition 2.1 (with \( \delta = \nu = 2 + \theta \) and \( \delta = \nu = 4 \)) imply that
\[
\| x^{1+\theta} \partial_x w(u+v) \| \leq \| x^{1+\theta} (u+v) \| L_\infty \| x^3 \partial_x w \|
\lesssim \| J\langle x \rangle^{1+\theta} (u+v) \| (\| x^2 w \| + \| J\langle x \rangle^3 w \|)
\lesssim \| J^{2+\theta} (u+v) \| + \| \langle x \rangle^{2+\theta} (u+v) \| (\| J^4 w \| + \| \langle x \rangle^4 w \|)
\lesssim M^2.
\] (3.31)

Moreover, proceeding as above
\[
\| x^{1+\theta} w \partial_x (u+v) \| \leq \| x^\theta \partial_x (u+v) \| L_\infty \| x^4 w \| \lesssim M^2.
\] (3.32)

In a similar way to (3.21), the first term on the right-hand side of (3.18) can be estimated as follows
\[
\| J^{2(1+\theta)} \partial_x w(u+v) \| \lesssim M^2.
\] (3.33)

From way analogous to (3.33) we also deduce that
\[
\| J^{2(1+\theta)} (w \partial_x (u+v)) \| \lesssim M^2.
\] (3.34)

Finally, the inequalities (3.30)–(3.34) imply that, for all \( \tau \in [-T,T] \)
\[
\| x^{1+\theta} U(t-\tau)z(\tau) \| \lesssim M^2.
\] (3.35)

Therefore, by Plancherel identity, (3.24), (3.28) and (3.35) we obtain
\[
\| x^{1+\theta} w(t) \| \lesssim \| J^{2(1+\theta)} \sigma \| + \| x^{1+\theta} \sigma \| + |t| M^2
\lesssim \| J^{2(1+\theta)} \phi \| + \| J^{2(1+\theta)} \varphi \| + \| x^{1+\theta} \phi \| + \| x^{1+\theta} \varphi \| + |t| M^2.
\]

By the last inequality we conclude the proof of Theorem. \( \square \)

**Proof of Theorem 1.3.** Follows similar ideas contained in \( \ref{13} \). \( \square \)
Proof of Theorem 1.4. Assume that \( \theta \in (0, 1/2) \). Let \( u \) and \( v \) solutions of the IVP (1.1) with initial data \( \phi \) and \( \varphi \), respectively. As before, let \( \sigma = \phi - \varphi \) and \( w = u - v \). Hence, we see that \( w \) satisfies the following equation

\[
 w_t + \mathcal{H} \partial_x^2 w + \partial_x^3 w + u \partial_x w + \partial_x v w = 0, \quad x, t \in \mathbb{R}.
\] (3.36)

The global well-posedness in Sobolev spaces yield us \( u, v \in C([-T, T]; H^{8+2\theta}) \), for all \( T > 0 \). Moreover, the Theorem 1.3 implies that \( w \in L^\infty([-T, T]; Z_{8+2\theta}, 4) \). Thus, the constant

\[
 M_1 := \sup_{[-T,T]} \{ \| u(t) \|_{H^{8+2\theta}} + \| v(t) \|_{H^{8+2\theta}} + \| w(t) \|_{Z_{8+2\theta}, 4} \},
\]

is finite.

We observe that by (1.2) and (1.3), the solution \( w \) of (3.36) satisfies the following conservation laws

\[
 \int w(x,t)dx = \int \sigma(x)dx
\] (3.37)

and

\[
 \frac{d}{dt} \int xw(x,t)dx = \frac{1}{2}(\| \phi \|^2 - \| \varphi \|^2).
\] (3.38)

Then, from (3.37), (3.38) and the hypothesis (1.4)–(1.6) it follows that

\[
 \int w(x,t)dx = 0
\] (3.39)

and

\[
 \int xw(x,t)dx = 0,
\] (3.40)

for all \( t \) in which the solution there exists.

Multiplying (3.36) by \( \langle x \rangle^{2+2\theta} x^6 \) \( w \) and integrating on \( \mathbb{R} \) we obtain

\[
 \frac{1}{2}\| \langle x \rangle_N^{1+\theta} x^3 w \|^2 + \int \langle x \rangle_N^{1+\theta} x^3 \mathcal{H} \partial_x^2 w(x) \langle x \rangle_N^{1+\theta} x^3 w + \int \langle x \rangle_N^{2+2\theta} x^6 \partial_x^3 w + \int \langle x \rangle_N^{2+2\theta} x^6 (u \partial_x w + \partial_x v w) = 0. \] (3.41)

Next, we will estimate the second term on the left-hand side of (3.41). It’s seen that the identities

\[
 x \mathcal{H} \partial_x^2 w = \mathcal{H} \partial_x^2 (xw) - 2 \mathcal{H} \partial_x w,
\]

\[
 x^2 \mathcal{H} \partial_x^2 w = \mathcal{H} \partial_x^2 (x^2 w) - 4 \mathcal{H} \partial_x (xw) + 2 \mathcal{H} w
\]

and (3.39) imply that

\[
 x^3 \mathcal{H} \partial_x^2 w = \mathcal{H} \partial_x^2 (x^3 w) - 6 \mathcal{H} \partial_x (x^2 w) - 2 \mathcal{H} (xw).
\]

Thus,

\[
 \langle x \rangle_N^{1+\theta} x^3 \mathcal{H} \partial_x^2 w = \langle x \rangle_N^{1+\theta} \mathcal{H} \partial_x^2 (x^3 w) - 6 \langle x \rangle_N^{1+\theta} \mathcal{H} \partial_x (x^2 w) - 2 \langle x \rangle_N^{1+\theta} \mathcal{H} (xw)
\]

\[
 := \mathcal{A} + \mathcal{B} + \mathcal{C}.
\] (3.42)
Next, we will deal with each one of the terms above.

We can write

\[ A = \langle x \rangle_{N}^{1+\theta} \mathcal{H} [\partial_{x}^{2} (x^{3}w) + \mathcal{H} \langle x \rangle_{N}^{1+\theta} \partial_{x}^{2} (x^{3}w) ] \]

\[ = A_{1} + \mathcal{H} \partial_{x}^{2} \langle x \rangle_{N}^{1+\theta} x^{3}w - 2 \mathcal{H} \langle x \rangle_{N}^{1+\theta} \partial_{x} \langle x \rangle_{N}^{1+\theta} x^{3}w \]  \hspace{1cm} (3.43)

\[ = A_{1} + \cdots + A_{4}. \]

Thus, by the Calderón commutator estimate (see Theorem 6 in [14] and references therein) we get

\[ A_{1} \lesssim \| \partial_{x}^{2} \langle x \rangle_{N}^{1+\theta} \|_{L^{\infty}} \| x^{3}w \| \lesssim M_{1}. \]  \hspace{1cm} (3.44)

Also, the fact that \( \mathcal{H} \) is a bounded operator in \( L^{2} \) implies that

\[ A_{4} \lesssim \| \partial_{x}^{2} \langle x \rangle_{N}^{1+\theta} x^{3}w \| \lesssim \| \partial_{x}^{2} \langle x \rangle_{N}^{1+\theta} \|_{L^{\infty}} \| x^{3}w \| \lesssim M_{1}. \]  \hspace{1cm} (3.45)

By returning the term \( A_{2} \) in \ref{comp2} and using Plancherel identity we see that

\[ \int \mathcal{H} \partial_{x}^{2} \langle x \rangle_{N}^{1+\theta} x^{3}w \langle x \rangle_{N}^{1+\theta} x^{3}w = 0. \]  \hspace{1cm} (3.46)

To estimate \( A_{3} \), the inequality \( |x \partial_{x} \langle x \rangle_{N}| \lesssim \langle x \rangle_{N} \) yields us

\[ \| A_{3} \| \lesssim \| \langle x \rangle_{N}^{\theta} \partial_{x} \langle x \rangle_{N} x^{2}w \| + \| \langle x \rangle_{N}^{\theta} \partial_{x} \langle x \rangle_{N} x^{3} \partial_{x}w \| \]

\[ \lesssim \| \langle x \rangle_{N}^{1+\theta} xw \| + \| \langle x \rangle_{N}^{1+\theta} x^{2} \partial_{x}w \| \]  \hspace{1cm} (3.47)

\[ \lesssim M_{1} + P. \]

Using inequality \( \langle x \rangle_{N} \lesssim 1 + |x| \), Proposition \ref{comp2} (with \( \omega = \langle x \rangle_{N} \)) and identity \( \partial_{x}(x^{2}w)(0,t) = 0 \), the term \( B \) can be estimated as follows

\[ B \lesssim \| \langle x \rangle_{N}^{\theta} \mathcal{H} \partial_{x} (x^{2}w) \| + \| \langle x \rangle_{N}^{\theta} \mathcal{H} \partial_{x} (x^{2}w) \| \]

\[ \lesssim \| \langle x \rangle_{N}^{\theta} \partial_{x} (x^{2}w) \| + \| \langle x \rangle_{N}^{\theta} \mathcal{H} (x^{3}w) \| \]

\[ \lesssim \| \langle x \rangle_{N}^{\theta} xw \| + \| \langle x \rangle_{N}^{\theta} x^{2} \partial_{x}w \| + \| \langle x \rangle_{N}^{\theta} x^{3} \partial_{x}w \| \]

\[ \lesssim M_{1} + \| \langle x \rangle_{N}^{1+\theta} x^{3}w \| + \| \langle x \rangle_{N}^{\theta} x^{3} \partial_{x}w \|. \]  \hspace{1cm} (3.48)

The third term on the right-hand side of the last inequality can be estimated as

\[ F \lesssim \| J \langle x \rangle_{N}^{\theta} \langle x \rangle_{N}^{3}w \| + \| x^{3}w \| + \| \langle x \rangle_{N}^{\theta} \langle x \rangle_{N}^{3}w \| \]

\[ \lesssim \| J^{1+\theta} (x) \| + \| \langle x \rangle_{N}^{1+\theta} \langle x \rangle_{N}^{3}w \| \]

\[ \lesssim \| J^{1+\theta} \| + \| \langle x \rangle_{N}^{1+\theta} \langle x \rangle_{N}^{3}w \| \]  \hspace{1cm} (3.49)

\[ \lesssim M_{1} + \| \langle x \rangle_{N}^{1+\theta} x^{3}w \|, \]

where above its used Lemma \ref{comp2} (with \( \gamma = \nu = 1 + \theta \) and \( \gamma = 4(1 + \theta), \nu = 4 \)).

Also by the Proposition \ref{comp2} and equality \ref{comp4} we get
and Lemma 2.1 it follows that
\[ C \lesssim \| \langle x \rangle_N^\theta \mathcal{H}(xw) \| + \| \langle x \rangle_N^\theta x \mathcal{H}(xw) \| \]
\[ \lesssim \| \langle x \rangle_N^\theta xw \| + \| \langle x \rangle_N^\theta x \mathcal{H}(x^2w) \| \]
\[ \lesssim \| \langle x \rangle_N^\theta xw \| + \| \langle x \rangle_N^\theta x^2w \| \]
(3.50)

where above, we used \( \| x^{-2} \mathcal{H}(x) \| \lesssim 1 \).

In what follows we deal with the other terms in (3.41).

An application of integration by parts gives us
\[ \int \langle x \rangle_N^{2+2\theta} x^6 w \partial_x^3 w = -\frac{1}{2} \int \partial_x^2 \langle x \rangle_N^{2+2\theta} w \partial_x w - \frac{3}{2} \int \partial_x \langle x \rangle_N^{2+2\theta} w \partial_x^2 w \]
\[ \lesssim \| x^3 \langle x \rangle_N^{1+\theta} \| \| x^4 \langle x \rangle_N^{1+\theta} \| + \| x^2 \langle x \rangle_N^{1+\theta} \| \| x^3 \langle x \rangle_N^{1+\theta} \| \]
\[ \lesssim \| x^3 \langle x \rangle_N^{1+\theta} \| (F + L), \]
(3.51)

where above we also used the inequalities
\[ |\partial_x (\langle x \rangle_N^{2+2\theta})| \lesssim |x|^5 \langle x \rangle_N^{2+2\theta} \]
\[ \text{and } |\partial_x^2 (\langle x \rangle_N^{2+2\theta})| \lesssim x^4 \langle x \rangle_N^{2+2\theta}. \]

To estimate \( L \), we observe that by the equality
\[ \partial_x^2 (\langle x \rangle_N^{2+2\theta}) = \partial_x (\langle x \rangle_N^{2+2\theta}) \partial_x w + \langle x \rangle_N^{2+2\theta} \partial_x^2 w \]
and Lemma 2.1, it follows that
\[ L \lesssim \| J^2 (\langle x \rangle_N^{1+\theta} w) \| + M_1 \]
\[ \lesssim \| J^6 (\langle x \rangle_N^{1+\theta} w) \| + \| x^3 \langle x \rangle_N^{1+\theta} w \| + M_1 \]
\[ \lesssim \| J^{8+2\theta} w \| + \| x^3 (\langle x \rangle_N^{1+\theta} w \| + \| x^3 \langle x \rangle_N^{1+\theta} w \| + M_1 \]
(3.52)

where above, we used Lemma 2.1 moreover the inequality \( \langle x \rangle_N^3 \lesssim (1 + x^3). \)

In a similar way to the term \( L \), we conclude that the term \( P \) (in (3.41)) satisfies
\[ P \lesssim \| x^3 (\langle x \rangle_N^{1+\theta} w \| + M_1. \]
(3.53)

Hence, by (3.43)–(3.47)
\[ \mathcal{A} \lesssim \| x^3 \langle x \rangle_N^{1+\theta} w \| + M_1. \]
(3.54)

About the fourth term in (3.41), integration by parts, Sobolev’s embedding and the inequality \( |\partial_x (\langle x \rangle_N^{2+2\theta})| \lesssim (1 + x^6) \langle x \rangle_N^{2+2\theta} \) imply that
\[
\int (x)^{2+2\theta} x^6 w(u\partial_x w + \partial_x vw) = -\frac{1}{2} \int \left( \partial_x (x^6 (x)^{2+2\theta}) u + x^6 (x)^{2+2\theta} \partial_x u \right) w^2 + \int x^6 (x)^{2+2\theta} w^2 \partial_x v \\
\lesssim (\|u\|_{L^\infty} + \|\partial_x v\|_{L^\infty}) \|x^3 (x)^{1+\theta} w\|^2 + M_1^2 \\
\lesssim M_1 (M_1 + \|x^3 (x)^{1+\theta} w\|^2).
\]

Gathering together the above inequalities we obtain
\[
\frac{d}{dt} \|\langle x\rangle^{1+\theta} x^3 w\|^2 \lesssim (1 + M_1) \|\langle x\rangle^{1+\theta} x^3 w\|^2 + M_1^2.
\]
Using Gronwall’s Lemma we conclude that
\[
\sup_{[-T,T]} \|\langle x\rangle^{1+\theta} x^3 w\| \leq c(T).
\]
Therefore, from the last inequality we conclude the proof.

\section{Proof of Theorem 1.5}

\textbf{Proof of Theorem 1.5.} First of all, as we already mentioned the IVP is GWP in \(Z_{s,r}\), where \(5/2 < r < 7/2\) and \(s > 2r\). Hence, from the hypothesis \(\phi \in Z_{9+2\theta, \frac{9+2\theta}{2}}\) it follows that \(u \in C(\mathbb{R}; Z_{9+2\theta, \frac{9+2\theta}{2}})\).

We recall that the integral equation associated with IVP (1.1) is given by
\[
u(t) = U(t)\phi - \frac{1}{2} \int_0^t U(t - \tau)(\partial_x u^2)(\tau) d\tau.
\]

Then, (4.1) and Plancherel’s identity imply that
\[
\partial_x^j \left( u(t) \right) = \partial_x^j (\mu(\xi, t) \phi) - \int_0^t \partial_x^j (\mu(\xi, t - \tau) \kappa(\tau)) d\tau,
\]
where \(\kappa := \frac{1}{2} \partial_x u^2\).

Since \(\phi\) has zero mean value, we can use the identity (3.7) to write
\[
\partial_x^j (\mu(\xi, t) \phi) = \left( -12t^2 - 120i|\xi| + 180\xi^2 - 288t^2\xi^3 - 540i|\xi|^3 + 324\xi^5 \\
+ 48i t^3 |\xi| - 216t^2\xi^6 - 216it|\xi|\xi^6 + 81\xi^8 + 96i t^3 |\xi|^2 + 16t^4 |\xi|^4 \right) \phi \\
- 4( -4it\delta + 6 - 12t^2 \xi - 54i\xi|\xi| + 54\xi^3 - 36t^2\xi^4 - 54it|\xi|^4 + 27\xi^6 + \\
+ 8it^3 |\xi|^2 \partial_x \phi + 6(-2itsgn(\xi) + 6\xi - 4t^2\xi^2 - 12it|\xi|^2 + 9\xi |\phi|) \partial_x^2 \phi \\
+ 4(3\xi^2 - 2it|\xi|)\partial_x^3 \phi + \partial_x^4 \phi \right) \mu \\
= 16it\mu \partial_x \phi \delta + A_1 + \cdots + A_{28},
\]
where the terms \(A_j\) are such that \(A_j = A_j(t, \xi, \phi)\).
Using (4.2) and (4.3) we obtain

\[
\partial_4 \hat{u}(t, \xi) = 16i \left( -t \mu(t, \xi) \partial_\xi \hat{\phi} + \int_0^t (t - \tau) \mu(t - \tau, \xi) \partial_\xi \hat{\kappa} d\tau \right) \delta + \sum_{1 \leq j \leq 28} A_j(t, \xi, \hat{\phi})
\]

\[
- \sum_{1 \leq j \leq 28} \int_0^t A_j(t - \tau, \xi, \hat{\kappa}(\tau)) d\tau
\]

\[
= G(t, 0) \delta + \sum_{1 \leq j \leq 28} A_j(t, \xi, \hat{\phi}) - \sum_{1 \leq j \leq 28} \int_0^t A_j(t - \tau, \xi, \hat{\kappa}(\tau)) d\tau.
\]

(4.4)

The conservation law (1.3) implies that

\[
\partial_\xi \hat{\kappa}(\tau, 0) = i 2 \hat{u}^2(\tau, 0) = i \|\phi\|^2.
\]

(4.5)

Since \(\mu(t, 0) = \mu(t - \tau, 0) = 1\), by the last identity we can write

\[
G(t, 0) = 16i \left( -t \partial_\xi \hat{\phi}(0) + \int_0^t (t - \tau) \partial_\xi \hat{\kappa}(\tau, 0) d\tau \right)
\]

\[
= -16 \left( \int x\phi(x) dx + \frac{1}{2} \int_0^t (t - \tau) \|\phi\|^2 d\tau \right)
\]

\[
= -16t \left( \int x\phi(x) dx + \frac{t}{4} \|\phi\|^2 \right)
\]

\[
= 0,
\]

if

\[
t = t^* = \frac{-4}{\|\phi\|^2} \int x\phi(x) dx.
\]

Therefore, putting \(t = t^*\) in (4.4) we obtain

\[
\partial_4^2 (\hat{u}(t^*)) = \sum_{1 \leq j \leq 28} A_j(t^*, \xi, \hat{\phi}) - \sum_{1 \leq j \leq 28} \int_0^{t^*} A_j(t^* - \tau, \xi, \hat{\kappa}(\tau)) d\tau,
\]

(4.7)

where the delta Dirac function is not contained in the last identity. In the follow argument we are always assuming \(t = t^*\). Thus, taking \(D_\theta^\delta\) in both sides of (4.4) and using Plancherel identity it’s seen that

\[
(|x|^{4+\theta} u(t))^w(\xi) = \sum_{1 \leq j \leq 28} D_\theta^\delta A_j(t, \xi, \hat{\phi}) - \sum_{1 \leq j \leq 28} \int_0^t D_\theta^\delta A_j(t - \tau, \xi, \hat{\kappa}(\tau)) d\tau.
\]

(4.8)

The next step is to show that all terms on the right-hand side of (4.8) belongs to \(L^2(\mathbb{R})\). We will estimate some only terms.
The Lemma 2.7 and Proposition 2.3 (with \( j = 2, k = 4 \)) imply that
\[
\|D^\theta_\xi A_{25}(t, \cdot, \hat{\phi}(\cdot))\| \lesssim \|J^{2\theta} \partial_x^4(x^2 \phi)\| + \|\|x\|^{\theta} \partial_x^4(x^2 \phi)\|
\lesssim \|J^{2\theta} \partial_x^2 \phi\| + \|J^{2\theta} (x^2 \partial_x^3 \phi)\| + \|\|x\|^{\theta} \partial_x^4(x^3 \phi)\|
\lesssim \|J^{2(\theta+1)+3}\phi\| + \|\langle x \rangle^{\theta+1+\frac{1}{2}} \phi\| + \|J^{2(\theta+4)} \phi\| + \|\langle x \rangle^{4+\theta} \phi\|
+ \|\langle x \rangle^{\frac{2(\theta+1)(2+\theta)}{2\theta+\theta+1+\theta}} \phi\|
\lesssim \|J^{2(\theta+\theta)} \phi\| + \|\|x\|^{4+\theta} \phi\|.
\]

(4.9)

In addition, an application of Lemma 2.7, Lemma 2.2 and Proposition 2.3 (with \( j = 3, k = 1 \)) gives us
\[
\|D^\theta_\xi A_{28}(t, \cdot, \hat{\phi}(\cdot))\| \lesssim \|J^{2\theta} \mathcal{H} \partial_x(x^3 \phi)\| + \|\|x\|^{\theta} \mathcal{H} \partial_x(x^3 \phi)\|
\lesssim \|J^{2\theta} (x^3 \phi)\| + \|\|x\|^{\theta} \partial_x(x^3 \phi)\|
\lesssim \|J^{2(\theta+3)+1}\phi\| + \|\langle x \rangle^{\theta+3+\frac{1}{2}} \phi\| + \|J^{2(\theta+4)} \phi\| + \|\langle x \rangle^{\frac{2(\theta+3)(3+\theta)}{2(\theta+4)-\theta}} \phi\|
\lesssim \|J^{2(\theta+\theta)} \phi\| + \|\|x\|^{4+\theta} \phi\|.
\]

(4.10)

Also, by using Proposition 2.2 (with \( \omega = |x|^\theta \)) we get
\[
\|D^\theta_\xi A_{21}(t, \cdot, \hat{\phi}(\cdot))\| \lesssim \|J^{2\theta} \mathcal{H}(x^2 \phi)\| + \|\|x\|^{\theta} \mathcal{H}(x^2 \phi)\|
\lesssim \|J^{2\theta} (x^2 \phi)\| + \|\|x\|^{\theta} x^2 \phi\|
\lesssim \|J^{2(\theta+2)} \phi\| + \|\|x\|^{2+\theta} \phi\|.
\]

(4.11)

To deal with the other terms \( A_j(t, \xi, \hat{\phi}) \) we can proceed in a similar way, to obtain, for all \( j = 1, \ldots, 28 \)
\[
\|D^\theta_\xi A_j(t, \cdot, \hat{\phi}(\cdot))\| \lesssim \|J^{2(\theta+\theta)} \phi\| + \|\|x\|^{4+\theta} \phi\|.
\]

(4.12)

About the integral terms in (4.13), putting \( k = \frac{1}{2} \partial_x u^2 \) instead of \( \phi \), in the above estimates, follows that, for any \( j = 1, \ldots, 28 \) and \( \tau \in [0, t] \)
\[
\|D^\theta_\xi A_j(t, \cdot, \hat{\kappa}(\tau))\| \lesssim \|J^{2(\theta+\theta)} \kappa(\tau)\| + \|\|x\|^{4+\theta} \kappa(\tau)\|.
\]

(4.13)

Next, as in (3.21) follows that
\[
\|J^{2(\theta+\theta)} \kappa(\tau)\| \lesssim \|u(\tau)\|^2_{\mathcal{H}^{\theta+2\theta}}.
\]

(4.14)

In addition, from Sobolev’s embedding and Proposition 2.4 with \( \beta = \frac{1}{\theta+2\theta} \) and \( \nu = \frac{(2+\theta)(8+2\theta)}{7+2\theta} \), follow that
\[
\|\|x\|^{4+\theta} \kappa(\tau)\| \lesssim \|\|x\|^{4+\theta} u \|_{L^\infty} \|\|x\|^{2+\theta} u \|_{L^\infty}
\lesssim \|\langle x \rangle^{2+\theta} u \|_{L^\infty} (\|\partial_x (|x|^{2+\theta} u)\| + \|\|x\|^{1+\theta} u\|)
\lesssim \|J (|x|^{2+\theta} u)\| (\|J (|x|^{2+\theta} u)\| + \|\|x\|^{1+\theta} u\|)
\lesssim \|J^{2(\theta+\theta)} u\|^2 + \|\langle x \rangle^{\nu} u\|^2.
\]
We observe that the terms above are finite, in view of \( \nu < \frac{7}{2} \) and \( u \in C(\mathbb{R}; Z_{9+2\theta}) \).

From (4.12), (4.13) and (4.14)

\[
\int_0^t \|D_\theta^j A_j(t, \cdot, \hat{\kappa}(\tau))\|d\tau \lesssim \int_0^t \left( \|J^{2(4+\theta)}u\|^2 + \|\langle x \rangle^\nu u\|^2 \right)d\tau
\]

\[
\lesssim \sup_{[0,T]} \|u(t)\|^2_{Z_{s,\nu}}, \quad j = 1, \ldots, 28.
\]

(4.15)

Therefore, using (4.7), (4.11) and (4.15) we conclude that

\[ u(t^*) \in Z_{9+2\theta}, 4+\theta \]

This finishes the proof.

\[ \square \]

5. Uniqueness Results

**Proof of Theorem 1.6.** Let \( w := u - v \), then the IVP (1.1) implies that

\[ \partial_t w + \mathcal{H}\partial_x^2 w + \partial_x^3 w + u\partial_x w + w\partial_x v = 0, \quad (x, t) \in \mathbb{R} \times [0, T]. \]

(5.1)

The hypothesis (1.12) gives us, for all \( x \in I \)

\[ w(x, 0) = \partial_x w(x, 0) = \partial_t w(x, 0) = \partial_x^3 w(x, 0) = 0. \]

(5.2)

By (5.1) and (5.2), we obtain

\[ \mathcal{H}\partial_x^2 w(x, 0) = 0, \quad \text{for any } x \in I. \]

(5.3)

Using (5.2) and (5.3) it follows that

\[ \partial_x^2 w(x, 0) = \mathcal{H}\partial_x^2 w(x, 0) = 0, \quad \forall x \in I, \]

(5.4)

where \( \partial_x^2 w(\cdot, 0) \in H^s(\mathbb{R}), s > 1/2. \)

Thus, from Lemma 2.8 we obtain \( \partial_x^2 w(\cdot, 0) \equiv 0. \)

Therefore \( w(x, 0) = 0 \), for all \( x \in \mathbb{R} \), and by the uniqueness in \( H^s(\mathbb{R}) \) it follows that

\[ u(x, t) = v(x, t), \quad (x, t) \in \mathbb{R} \times [0, T]. \]

This finishes the proof.

\[ \square \]

**Proof of Theorem 1.7.** Let \( w(x, t) = (u - v)(x, t) \), then by the IVP (1.1)

\[ \partial_t w + \mathcal{H}\partial_x^2 w + \partial_x^3 w + u\partial_x w + w\partial_x v = 0, \quad (x, t) \in \mathbb{R} \times [0, T]. \]

(5.5)

From (1.10) \( w(x, 0) = 0 \), for any \( x \in I \). Then \( \partial_x^2 w(x, 0) = 0 \), for any \( x \in I \), where \( j = 0, \ldots, 3. \) As a consequence, by (5.5)

\[ \mathcal{H}\partial_x^2 w(x, 0) = -\partial_t w(x, 0), \quad \text{for all } x \in I. \]

(5.6)

Using (1.17) and (5.6) we conclude that

\[
\int_{|x| \leq R} |\partial_t u(x, 0) - \partial_t v(x, 0)|^2 dx = \int_{|x| \leq R} |\partial_t w(x, 0)|^2 dx
\]

\[
= \int_{|x| \leq R} |\mathcal{H}\partial_x^2 w(x, 0)|^2 dx \leq c_N R^N, \quad \text{as } R \downarrow 0.
\]

(5.7)
Therefore, by the Lemma (2.9)
\[
\partial_x^2 w(x, 0) = 0, \quad \text{for all } x \in \mathbb{R}.
\] (5.8)
The rest run as in the proof of Theorem 1.6.
This ends the proof. □

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