Analytical results for the distribution of cover times of random walks on random regular graphs

Ido Tishby, Ofer Biham and Eytan Katzav∗

Racah Institute of Physics, The Hebrew University, Jerusalem 9190401, Israel
E-mail: ido.tishby@mail.huji.ac.il, biham@phys.huji.ac.il and eytan.katzav@mail.huji.ac.il

Received 10 August 2021, revised 25 October 2021
Accepted for publication 16 November 2021
Published 8 December 2021

Abstract
We present analytical results for the distribution of cover times of random walks (RWs) on random regular graphs consisting of $N$ nodes of degree $c$ ($c \geq 3$). Starting from a random initial node at time $t = 1$, at each time step $t \geq 2$ an RW hops into a random neighbor of its previous node. In some of the time steps the RW may visit a new, yet-unvisited node, while in other time steps it may revisit a node that has already been visited before. The cover time $T_C$ is the number of time steps required for the RW to visit every single node in the network at least once. We derive a master equation for the distribution $P_t(S = s)$ of the number of distinct nodes $s$ visited by an RW up to time $t$ and solve it analytically. Inserting $s = N$ we obtain the cumulative distribution of cover times, namely the probability $P(T_C \leq t) = P_t(S = N)$ that up to time $t$ an RW will visit all the $N$ nodes in the network. Taking the large network limit, we show that $P(T_C \leq t)$ converges to a Gumbel distribution. We calculate the distribution of partial cover (PC) times $P(T_{PC_k} = t)$, which is the probability that at time $t$ an RW will complete visiting $k$ distinct nodes. We also calculate the distribution of random cover (RC) times $P(T_{RC_k} = t)$, which is the probability that at time $t$ an RW will complete visiting all the nodes in a subgraph of $k$ randomly pre-selected nodes at least once. The analytical results for the distributions of cover times are found to be in very good agreement with the results obtained from computer simulations.

Keywords: random network, random regular graph, random walk, cover time, Gumbel distribution

(Some figures may appear in colour only in the online journal)

∗Author to whom any correspondence should be addressed.
1. Introduction

Random walk (RW) models [1, 2] were studied extensively in different geometries, including continuous space [3], regular lattices [4], fractals [5] and random networks [6–10]. These models are useful for the analysis of a large variety of stochastic processes such as diffusion [11, 12], polymer structure [13–15] and random search [16, 17]. In the context of complex networks [18–20] they provide useful insight on the spreading of rumors, opinions and infections [21, 22]. Consider an RW on a random network that starts at time \( t = 1 \) from a random initial node \( x_1 \). At each time step \( t \geq 2 \) the RW hops randomly to one of the neighbors of its previous node. The RW thus generates a trajectory of the form \( x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_t \rightarrow \ldots \), where \( x_t \) is the node visited at time \( t \). In some of the time steps the RW hops into nodes that have not been visited before, while in other time steps it hops into nodes that have already been visited at an earlier time. Since RWs on random networks may visit some of the nodes more than once, the number of distinct nodes visited up to time \( t \) is typically smaller than \( t \). The mean number \( \langle S \rangle_t \) of distinct nodes visited by an RW on a random network up to time \( t \) was recently studied [23–25]. It was found that in the infinite network limit it scales linearly with \( t \), namely \( \langle S \rangle_t \sim rt \), where the coefficient \( r < 1 \) depends on the network topology. These scaling properties resemble those obtained for RWs on high dimensional lattices [26–29], Bethe lattices and Cayley trees [30–33]. For finite networks of size \( N \), the linear relation between \( \langle S \rangle_t \) and \( t \) holds as long as \( t \ll N \). At longer times, the probability of an RW to enter a yet-unvisited node gradually decreases. As a result, \( \langle S \rangle_t \) eventually saturates, converging toward \( N \) at \( t \to \infty \).

RWs on random networks exhibit a variety of first passage processes [34], which take place over a broad range of time scales. These include the first hitting time \( T_{FH} \), which is the first time at which an RW steps into a node that has already been visited before [35–41]. The mean first hitting time satisfies \( \langle T_{FH} \rangle \sim \min\{c, \sqrt{N}\} \), where \( N \) is the network size and \( c \) is the mean degree [38, 41]. This implies that \( \langle T_{FH} \rangle \sim c \) in dilute networks and \( \langle T_{FH} \rangle \sim \sqrt{N} \) in dense networks. The first passage time \( T_{FP} \) is the first time at which an RW starting from a given initial node \( i \) visits a given target node \( j \) [42]. In a finite network that consists of a single connected component, the mean first passage time satisfies \( \langle T_{FP} \rangle \sim N \). A special case of the first passage time is the first return time \( T_{FR} \), which is the first time at which an RW returns to the initial node \( i \) [25, 43]. The mean first return time satisfies \( \langle T_{FR} \rangle \sim N \) [8, 25]. Interestingly, this result can be obtained directly from the Kac lemma, which employs general properties of discrete stochastic processes [44].

The cover time is the number of time steps required for an RW to visit every single node in a finite network of size \( N \) (consisting of a single connected component) at least once [45–62]. The cover time is relevant to a broad range of random search processes. These include search processes involving multiple targets in which all the targets need to be found. More specifically, in case that the number of targets is unknown one needs to perform an exhaustive search in which all the nodes in the network must be visited. Examples of such situations include the chase of pathogens by immune-system cells [63], foragers searching for food [64–66] and cleaning and demining processes in random environments. In the special case of \( c = N - 1 \) (complete graph) the cover-time problem is analogous to the coupon collector problem [61, 67, 68]. Another interesting relation is to the random deposition model [69], in the sense that the cover time corresponds to the time at which the last exposed substrate site is covered by a particle.

The mean cover time \( \langle T_C \rangle \) of RWs on general graphs has been studied extensively since the late 1980s. In particular, upper and lower bounds for \( \langle T_C \rangle \) for specific families of graphs were derived. Regarding the lower bound, it was shown that for any connected graph of \( N \)
nodes, the mean cover time satisfies $\langle T_C \rangle > [1 + \alpha(1)]N \ln N$ [49]. Actually, this lower bound was proven more directly, and is believed to be tighter, in the context of Erdős–Rényi (ER) networks [51]. In fact, in sparse ER networks, where $c < \ln N$ the mean cover time scales like $\langle T_C \rangle \sim N(\ln N)^2$, while in dense ER networks, where $c > \ln N$ the mean cover time scales like $\langle T_C \rangle \sim N \ln N$ [54, 70]. As for the upper bound, it was shown that for any connected graph of $N$ nodes, $\langle T_C \rangle \leq 4N^3/27 + O(N^{5/3})$ [50]. For any regular graph (a graph in which all the nodes are of the same degree) that consists of a single connected component it was shown that $\langle T_C \rangle \leq 8N^2$ [48]. Finally, for random regular graphs (RRGs) consisting of $N$ nodes of degree $c$ it was shown that in the asymptotic limit [71]

$$\langle T_C \rangle \sim \frac{c-1}{c-2} N \ln N. \quad (1)$$

However, very little is known about the distribution of cover times $P(T_C = t)$. For graphs of small size the distribution of cover times can be calculated using the method of reference [55]. This method yields an approximation scheme that can be used for larger graphs, and whose computational complexity scales like $O(2N)$. Nevertheless, no explicit analytical results for the distribution of cover times are available for random graphs.

In this paper we present analytical results for the distribution of cover times of RWs on RRGs consisting of $N$ nodes of degree $c \geq 3$. To this end, we derive a master equation for the distribution $P_i(S = s)$ of the number of distinct nodes $s$ visited by an RW up to time $t$. Using a generating function formalism, we solve the master equation and obtain a closed-form analytical expression for $P_i(S = s)$. Applying this result to the special case of $s = N$, we obtain the cumulative distribution of cover ($C$) times, which is given by $P(T_C \leq t) = P_i(S = N)$. We also calculate the mean and variance of the distribution of cover times. Taking the large network limit, we show that $P(T_C \leq t)$ follows a Gumbel distribution. We also study two interesting generalizations of the cover time: the partial cover (PC) time $T_{PC,k}$, which is the time it takes an RW to visit $k$ distinct nodes and the random cover (RC) time $T_{RC,k}$, which is the time it takes an RW to cover a set of $k$ randomly pre-selected nodes [72–74]. The analytical results for the distributions of cover times are found to be in very good agreement with the results obtained from computer simulations.

The paper is organized as follows. In section 2 we briefly describe the RRG. In section 3 we present the RW model. In section 4 we derive the master equation for $P_i(S = s)$. In section 5 we present the solution of the master equation in the infinite network limit. In section 6 we present the solution of the master equation for finite networks. In section 7 we calculate the mean and variance of $P_i(S = s)$. In section 8 we calculate the distribution of cover times. In section 9 we calculate the mean cover time. In section 10 we calculate the variance of the distribution of cover times. In section 11 we present the distribution of partial cover times and in section 12 we consider the distribution of random cover times. The relation between these two distributions is discussed in section 13. The results are discussed in section 14 and summarized in section 15. In appendix A we present the solution of the master equation for $P_i(S = s)$. In appendix B we consider the distribution $P(T = t|s)$ that an RW has pursued $t$ times steps given that it has visited $s$ distinct nodes. In appendix C we calculate the moments of $P_i(S = s)$. In appendix D we calculate the generating function of the distribution of cover times.

2. The random regular graph

A random network (or graph) consists of a set of $N$ nodes that are connected by edges in a way that is determined by some random process. For example, in a configuration model network the degree of each node is drawn independently from a given degree distribution $P(k)$ and the
connections are random and uncorrelated [75–77]. Configuration model networks belong to
the class of small-world networks in which the mean distance \( \langle L \rangle \) between pairs of random
nodes scales logarithmically with the network size, namely \( \langle L \rangle \sim \ln N \) [77]. The RRG is a
special case of a configuration model network, in which the degree distribution is a degenerate
distribution of the form \( P(k) = \delta_{k,c} \), namely all the nodes are of the same degree \( c \). Here we
focus on the case of \( c \geq 3 \), in which for a sufficiently large value of \( N \) the RRG consists of a
single connected component [78]. In the infinite network limit the RRG exhibits a tree structure
with no cycles. Thus, in this limit it coincides with a Bethe lattice whose coordination number
is equal to \( c \) [79]. In contrast, RRGs of a finite size exhibit a local tree-like structure, while at
larger scales there is a broad spectrum of cycle lengths. In that sense RRGs differ from Cayley
trees [80], which maintain their tree structure by reducing the most peripheral nodes to leaf
nodes of degree 1.

A special property of RRGs is that there is a great deal of uniformity in the local neigh-
borhood of all nodes in the network. This property makes it an ideal model for mean-field
analysis, which often provides exact results. For example, the distribution of shortest path
lengths (DSPL) of RRGs [81,82] as well as the distribution of shortest cycles [83] are known
exactly.

A convenient way to construct an RRG of size \( N \) and degree \( c \) is to prepare the \( N \) nodes such
that each node is connected to \( c \) half edges or stubs [19,84]. At each step of the construction,
one connects a random pair of stubs that belong to two different nodes \( i \) and \( j \) that are not
already connected, forming an edge between them. This procedure is repeated until all the
stubs are exhausted. The process may get stuck before completion in case that all the remaining
stubs belong to the same node or to pairs of nodes that are already connected. In such case one
needs to perform some random reconnections in order to complete the construction.

3. The random walk model

Consider an RW on an RRG of degree \( c \geq 3 \) and size \( N \). At each time step the RW hops from
its current node to one of its neighbors, such that the probability of hopping to each neighbor
is \( 1/c \). For sufficiently large \( N \) the RRG consists of a single connected component, thus an RW
starting from any initial node can reach any other node in the network. In the long time limit
\( t \gg N \) an RW on an RRG visits all the nodes with the same frequency, namely on average each
node is visited once every \( N \) steps. However, over shorter periods of time there may be large
fluctuations such that some nodes may be visited several times in a given time interval while
other nodes are not visited at all.

In some of the time steps an RW may visit nodes that have not been visited before while in
other time steps it may revisit nodes that have already been visited before. For example, at each
time step \( t \geq 3 \) the RW may backtrack into the previous node with probability of \( 1/c \). In the
infinite network limit the RRG exhibits a tree structure. Therefore, in this limit the backtracking
mechanism is the only way in which an RW may hop from a newly visited node to a node that
has already been visited before. Such backtracking step may be followed by retroceding steps
in which the RW continues to go backwards along its own path. However, in finite networks
the RW may also utilize cycles to retrace its path and revisit nodes it has already visited three or
time steps earlier. In figure 1 we present a schematic illustration of some of the events that
take place along the path of an RW on an RRG. In figure 1(a) we show a path segment in
which at each time step the RW enters a node that has not been visited before. In figure 1(b) we
show a path segment that includes a backtracking step, in which the RW moves back into the
previous node (step no. 4). In figure 1(c) we show a path segment that includes a backtracking
step (step no. 4) which is followed by a retroceding step (step no. 5). In figure 1(d) we show
Figure 1. Schematic illustrations of possible events taking place along the path of an RW on an RRG: (a) a path segment in which at each time step the RW enters a node that has not been visited before; (b) a path segment that includes a backtracking step into the previous node (step no. 4); (c) a path segment that includes a backtracking step (step no. 4), which is followed by a retroceding step (step no. 5); (d) a path that includes a retracing step (step no. 6) in which the RW hops into a node that was visited a few time steps earlier. Retracing steps are not possible in the infinite network limit and take place only in finite networks, which include cycles. Note that in this illustration the RRG is of degree $c = 4$. 
a path segment that includes a retracing step (step no. 6), in which the RW enters a node that was visited five time steps earlier.

The mean number of distinct nodes that are visited by an RW on an RRG up to time $t$ is denoted by $\langle S \rangle_t$. The mean number of nodes in the complementary set of nodes that have not been visited up to time $t$ is given by

$$\langle U \rangle_t = N - \langle S \rangle_t.$$  \hfill (2)

The probability that an RW will step into a yet-unvisited node at time $t$ is given by

$$\Delta_t = \langle S \rangle_t - \langle S \rangle_{t-1}. \hfill (3)$$

Using a generating function formulation based on the cavity method, it was shown that in the infinite network limit at sufficiently long times $\Delta_t \rightarrow \Delta$, where [24]

$$\Delta = \frac{c - 2}{c - 1}. \hfill (4)$$

A similar result was obtained for RWs on Bethe-lattices [30, 31]. More precisely, $\Delta_t$ converges toward equation (4) on a time scale of [25]

$$\tau = \frac{2}{\ln \left[ \frac{c^2}{4(c - 1)} \right]}.$$

On a finite RRG of size $N$, the probability of an RW that has already visited $s$ distinct nodes to enter a yet-unvisited node in the next time step is given by [25]

$$\Delta(s) = \frac{c - 2}{c - 1} \left( 1 - \frac{s}{N} \right). \hfill (6)$$

The complementary probability of an RW that has already visited $s$ distinct nodes to enter a previously visited node in the next time step is given by

$$1 - \Delta(s) = \frac{1}{c - 1} + \left( \frac{c - 2}{c - 1} \right) \frac{s}{N}. \hfill (7)$$

The first term on the right-hand side of equation (7) accounts for the probability of backtracking/retroceding (figures 1(b) and (c)), while the second term accounts for the probability of retracing (figure 1(d)). The saturation term in equation (6) is negligible at short times and becomes dominant once the RW covers a large fraction of the network. The analytical results for $\Delta(s)$ are presented in figure 4 of reference [25]. It is found to be in very good agreement with the results obtained from computer simulations [25]. Note that equation (6) is a slightly approximated version of the corresponding equation from reference [25]. In this approximation, we replaced $s - 2$ by $s$ and $N - 2$ by $N$. The difference is negligible in the time scales considered in this paper and helps to simplify the analysis.

4. The master equation for $P_t(S = s)$

Consider a trajectory of an RW on an RRG of size $N$. There is no limit on the length of the trajectory and thus the time $t$ may take values in the range $1 \leq t < \infty$. However, the number of distinct nodes $s$ visited by the RW is bounded from above by the network size, namely $1 \leq s \leq N$. The probability that an RW will visit $s$ distinct nodes up to time $t$ is denoted
by $P_t(S = s)$. Clearly, $P_t(S = s) = 0$ for $s \geq t + 1$. Below we derive a master equation for the probability $P_t(S = s)$. To this end we utilize equation (6), which provides the probability $\Delta(s)$ that an RW that has already visited $s$ distinct nodes will step into a yet-unvisited node in the next time step. The complementary probability that the RW will step into a node that has already been visited before, is given by $1 - \Delta(s)$. Note that the probability $P_{t+1}(S = s)$ is comprised of two contributions: (a) the probability that the RW has visited $s - 1$ distinct nodes up to time $t$, and that it subsequently entered a previously unvisited node at time $t + 1$; (b) the probability that the RW has already visited $s$ distinct nodes up to time $t$, and then entered a previously visited node at time $t + 1$. Taking into account these two contributions, we obtain

$$P_{t+1}(S = s) = \frac{c - 2}{c - 1} \left(1 - \frac{s - 1}{N}\right) P_t(S = s - 1) + \left[1 - \frac{c - 2}{c - 1} \left(1 - \frac{s}{N}\right)\right] P_t(S = s). \quad (8)$$

The time evolution of $P_t(S = s)$ can be expressed in terms of the forward difference

$$D_t P_t(S = s) = P_{t+1}(S = s) - P_t(S = s). \quad (9)$$

Subtracting $P_t(S = s)$ from both sides of equation (8), we obtain the discrete master equation [85–87]

$$D_t P_t(S = s) = \frac{c - 2}{c - 1} \left[1 - \frac{s - 1}{N}\right] P_t(S = s - 1) - \left(1 - \frac{s}{N}\right) P_t(S = s). \quad (10)$$

The master equation consists of a set of coupled difference equations for $P_t(S = s)$, $s = 2, \ldots, N$, at $t \geq 2$. The initial condition is given by $P_{t=2}(S = s) = \delta_{s,2}$. Note also that in the first time step $P_{t=1}(S = s) = \delta_{s,1}$, while at $t \geq 2$ the probability $P_t(S = 1) = 0$. A special property of equation (10) is that probability flows only upwards along the $s$ axis from $s - 1$ to $s$. This means that equation (10) does not support a steady-state solution, apart from the absorbing state solution imposed by the finite size of the network, which is given by $P(S = s) = \delta_{s,N}$. In general, an absorbing state is a state which once entered cannot be left. The state $s = N$ is the only absorbing state of the RW. Moreover, every single trajectory will eventually reach this absorbing state. Thus, the Markov chain describing the covering process of an RRG by an RW is referred to as an absorbing chain [88].

5. The solution of the master equation for $N \rightarrow \infty$

In the process of solving the master equation, it is instructive to first consider the infinite network limit. In this limit the discrete master equation (10) is reduced to

$$D_t P_t(S = s) = \left(\frac{c - 2}{c - 1}\right) \left[P_t(S = s - 1) - P_t(S = s)\right]. \quad (11)$$

Unlike the case of a finite network in which the master equation consists of $N - 1$ equations for $s = 2, 3, \ldots, N$, in the infinite network limit the master equation consists of an infinite number of equations, for $s \geq 2$. It is easy to verify that the solution of equation (11) for $t \geq 2$ is given by a binomial distribution of the form

$$P_t(S = s) = \binom{t - 2}{s - 2} \left(\frac{c - 2}{c - 1}\right)^{s-2} \left(\frac{1}{c - 1}\right)^{t-s}, \quad (12)$$

where $\binom{i}{j}$ is the binomial coefficient.
The mean number of distinct nodes visited by an RW up to time $t$ is given by

$$\langle S \rangle_t = 2 + \frac{c - 2}{c - 1}(t - 2), \quad (13)$$

while the variance of $P_t(S = s)$ is given by

$$\text{Var}_t(S) = \frac{c - 2}{(c - 1)^2}(t - 2). \quad (14)$$

In the next section we present the results for $P_t(S = s)$ in finite networks. In the limit of $N \to \infty$ these results must converge to equation (12). In order to show this, it is useful to express equation (12) in a different form. Inserting the binomial expansion

$$\left( \frac{1}{c - 1} \right)^{t-s} = \left( 1 - \frac{c - 2}{c - 1} \right)^{t-s} = \sum_{m=0}^{t-s} (-1)^m \left( \frac{c - 2}{c - 1} \right)^m \frac{t-s}{m}, \quad (15)$$

into equation (12), we obtain

$$P_t(S = s) = \left( \frac{t-2}{s-2} \right) \sum_{m=0}^{t-s} (-1)^m \left( \frac{c - 2}{c - 1} \right)^m \frac{t-s}{m} \frac{c - 2}{c - 1}^{m+s-2}. \quad (16)$$

In the next section we indeed show that equation (16) is obtained as the $N \to \infty$ limit of the distribution $P_t(S = s)$ in finite networks.

6. The solution of the master equation for finite networks

In appendix A we use a generating function approach to solve the discrete master equation (equation (10)) for the case of a finite network that consists of $N$ nodes. The solution is given by

$$P_t(S = s) = \sum_{v=s-2}^{t-2} (-1)^{v-s} \left( \frac{t-2}{v} \right) \left( \frac{c - 2}{c - 1} \right)^v \frac{1}{v} \sum_{m=s-2}^{\min(v,N-2)} m! \left\{ \frac{v}{m} \right\} \left( \frac{N-2}{m} \right) \left( \frac{m}{s-2} \right), \quad (17)$$

where

$$\left\{ \frac{v}{m} \right\} = \frac{1}{m!} \sum_{k=0}^{m} (-1)^k \left( \frac{m}{k} \right) (m-k)^v \quad (18)$$

is the Stirling number of the second kind [89]. The Stirling number of the second kind (which is also denoted by $S(v, m)$) represents the number of ways to partition a set of $v$ labeled objects into $m$ non-empty subsets. Clearly, this solution presented by equation (17) satisfies the condition that $P_t(S = s) = 0$ for $s > t + 1$.

In the limit of $N \to \infty$, the solution for $P_t(S = s)$ on a finite network, given by equation (17), is reduced to the solution on an infinite network, given by equation (16). To show this property we expand the right-hand side of equation (17) in powers of $1/N$, under the condition that $t < N$. The zero-order term of this expansion is obtained from the $m = v$ term in the second sum.
Replacing the second sum by this term alone, the resulting expression is found to be identical to equation (16). Thus, in the infinite system limit the solution of equation (10), which describes the time evolution of $P_t(S = s)$ on finite networks is reduced to the solution of equation (11) that describes the infinite system limit.

Below we derive an alternative expression for $P_t(S = s)$. Exchanging the order of the summations in equation (17) and rearranging terms, we obtain

$$P_t(S = s) = \sum_{m=2}^{N-2} (-1)^{m-s} \binom{N-2}{m} \binom{m}{s-2} \times (-1)^m t \sum_{v=0}^{t-2} \binom{t-2}{v} [ -\left( \frac{c-2}{c-1} \right) \frac{1}{N^v} ]^v. \tag{19}$$

Note the lower limit of the second sum in equation (19) is 0 rather than $m$. This is due to the fact that for $v < m$ the Stirling number satisfies $\left\{ \frac{v}{m} \right\} = 0$. Using identity (7.7) in reference [90], which is given by

$$(-1)^m t \sum_{v=0}^{t-2} \binom{t-2}{v} \binom{v}{m} x^v = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (1 + zk)^v, \tag{20}$$

we obtain

$$P_t(S = s) = \sum_{m=2}^{N-2} (-1)^{m-s} \binom{N-2}{m} \binom{m}{s-2} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left[ 1 - \left( \frac{c-2}{c-1} \right) k \frac{1}{N^{t-2}} \right]. \tag{21}$$

The right hand sides of equations (17) and (21) include double sums. It is thus useful to consider the number of terms included in these sums, in order to compare the computational effort involved in the calculation of $P_t(S = s)$ using equations (17) and (21). In equation (17), for $t < N$ the number of terms in the double sum scales like $t^2$ (assuming that $\left\{ \frac{v}{m} \right\}$ is evaluated using a lookup table). For $t > N$ the number of terms scales like $Nt$. As a result, for $t > N$ it becomes difficult to evaluate the right-hand side of equation (17). The number of terms in the double sum of equation (21) scales like $N^2$ and does not depend on the time $t$. As a result, at long times $t > N$ the evaluation of $P_t(S = s)$ using equation (21) is more efficient than equation (17). In light of these considerations, the evaluation of $P_t(S = s)$ is done using equation (17) for $t \lesssim N$ and using equation (21) for $t > N$. It is important to emphasize that regardless of the efficiency considerations discussed above, for any value of $t$ equations (17) and (21) are equivalent.

In the long time limit, where $t \gg N$, one can use the approximation

$$\left[ 1 - \left( \frac{c-2}{c-1} \right) k \frac{1}{N^{t-2}} \right] \approx \exp \left[ -\left( \frac{c-2}{c-1} \right) k \frac{1}{N^{t-2}} \right]. \tag{22}$$

Inserting the right-hand side of equation (22) into equation (21) and carrying out the summation over $k$, we obtain

$$P_t(S = s) \simeq \sum_{m=2}^{N-2} (-1)^{m-s} \binom{N-2}{m} \binom{m}{s-2} \left[ 1 - e^{-\left( \frac{c-2}{c-1} \right) \frac{1}{N^{t-2}}} \right]^m. \tag{23}$$
Writing the binomial coefficients explicitly in terms of the factorials, equation (23) becomes

\[
P_t(S = s) \simeq \frac{(N - 2)!}{(s - 2)!} \sum_{m=s-2}^{N-2} \frac{(-1)^{m-s}}{(m-s+2)!} \left[ 1 - e^{-\left(\frac{s-2}{N} \Delta_t\right)} \right]^m.
\]  

(24)

Shifting the summation index from \(m\) to \(r = m - (s - 2)\), we obtain

\[
P_t(S = s) \simeq \frac{(N - 2)!}{(s - 2)!} \sum_{r=0}^{N-s} \frac{(-1)^r}{r!} \left[ 1 - e^{-\left(\frac{s-2}{N} \Delta_t\right)} \right]^{r+s-2}.
\]  

(25)

Multiplying the numerator and the denominator by \((N - s)!\) and carrying out the summation over \(r\), we obtain

\[
P_t(S = s) \simeq \frac{(N - 2)!}{(s - 2)!} \left[ 1 - e^{-\left(\frac{s-2}{N} \Delta_t\right)} \right]^{s-2} \left[ e^{-\left(\frac{s-2}{N} \Delta_t\right)} \right]^{N-s}.
\]  

(26)

This implies that \(P_t(S = s)\) can be expressed as a binomial distribution. The term in the first square bracket represents the probability that a random node has already been visited by the RW up to time \(t\) while the term in the second square bracket represents the probability that a random node has not yet been visited up to time \(t\).

The tail distribution \(P_t(S > s)\) is given by

\[
P_t(S > s) = \sum_{s'=s+1}^{N} P_t(S = s').
\]  

(27)

In figure 2 we present the tail distribution \(P_t(S > s)\) for a network of size \(N = 1000\) with degree \(c = 3\) and nine values of \(t\) from \(t = 2000\) to \(t = 18000\). At early times the analytical results (solid lines), obtained from equation (27), deviate significantly from the results obtained from computer simulations (circles). As time proceeds the agreement between the analytical and the simulation results gradually improves and becomes very good beyond \(t = 10000\).

The discrepancy between the analytical and the simulation results in figure 2 are due to short range temporal correlations in the RW trajectories for RRGs of low degree and are most pronounced at \(c = 3\). To explain this point consider an RW on an RRG of low degree \(c\) that has visited \(s\) distinct nodes up to time \(t - 1\) and at time \(t\) it steps into a previously visited node \(x_i\) (such that the number of distinct nodes visited remains \(s\)). Under these conditions, the probability that at time \(t + 1\) the RW will also step into a previously visited node is higher than mean-field value of \(1 - \Delta(s)\), where \(\Delta(s)\) is given by equation (6). This is due to the fact that if \(x_i\) has been visited before, it must have entered \(x_i\) via one of its neighbors and must have left \(x_i\) via the same neighbor or another neighbor. In the case of a small degree \(c\), the one or two neighbors visited before and after the previous visit of \(x_i\) represent a large fraction of all the neighbors of \(x_i\). As a result, the probability that the RW will step into a previously visited node at time \(t + 1\) is higher than the mean-field result. Similarly, in case that the node \(x_i\) visited at time \(t\) has not been visited before, the probability that at time \(t + 1\) the RW will step into a previously visited node is lower than the mean-field result. Due to these correlations, in RRGs of a low degree \(c\) there may be instances of the RW trajectory which include long streaks of steps in which the RW revisits previously visited nodes. Similarly, there may be instances of the RW trajectory which include long streaks of steps in which the RW visits new, yet-unvisited nodes. These correlations thus broaden the distribution \(P_t(S = s)\)
Figure 2. Analytical results (solid lines) for the probability $P_t(S > s)$ that an RW will visit at least $s$ distinct nodes up to time $t$ on a random regular graph of size $N = 1000$ and node degree $c = 3$. The results are presented for nine values of the time $t$ from $t = 2000$ to $t = 18000$. At early times the analytical results, obtained from equation (27), deviate significantly from the results obtained from computer simulations (circles). As time proceeds the agreement between the analytical and the simulation results gradually improves and becomes very good beyond $t = 10000$.

obtained from computer simulations, compared to the analytical results. While the temporal correlations are short ranged, their effect on the number of distinct nodes visited up to time $t$ may accumulate, giving rise to a non-diminishing long term effect. From equation (1) one concludes that by the time the whole network is covered, each node has been visited on average $\sim \ln N$ times. This implies that for $c > \ln N$ the correlations discussed above are negligible and the results for $P_t(S = s)$, obtained from the master equation, are probably exact. In fact, these results remain highly accurate even for much smaller values of $c$. Moreover, noticeable discrepancies in $P_t(S = s)$ are observed only for the smallest possible values of $c$, namely $c = 3$ and $4$.

In figure 3 we present the tail distribution $P_t(S > s)$ for a network of size $N = 1000$ with degrees $c = 10$ (a) and $c = 30$ (b) at times $t = 2000$ (left), $t = 6000$ (center) and $t = 10000$ (right). The analytical results are in very good agreement with the results obtained from computer simulations. As time evolves the sigmoid curve in $P_t(S > s)$ slides to the right and becomes narrower as it approaches the boundary at $s = N$. At late times the probability $P_t(S > N - 1)$, which is the probability that the RW has completed covering the whole network up to time $t$, becomes nonzero and continues to increase as time evolves.
Figure 3. The probability $P_t(S > s)$ that an RW will visit at least $s$ distinct nodes up to time $t$ on a random regular graph of size $N = 1000$ and node degree $c = 10$ (a) and $c = 30$ (b). The results obtained for three values of the time $t$ are shown: $t = 2000$ (left), $t = 6000$ (center), $t = 10000$ (right). The analytical results (solid lines), obtained from equation (27) are in very good agreement with the results obtained from numerical simulations (circles).

For the simulations we generated a large number (typically 100) of random instances of the RRG consisting of $N$ nodes of degree $c$, using the procedure presented in section 2. For each network instance, we generated a large number (typically 100) of RW trajectories, where each trajectory starts from a random initial node $i = x_1$ at time $t = 1$. The simulation results are obtained by averaging over all these trajectories. In the simulations, at each time step $t$ the RW selects randomly one of the $c$ neighbors of the node $x_{t-1}$, where the probability of each neighbor to be selected is $1/c$. It then hops to the selected node, denoted by $x_t$. The number of distinct nodes visited up to time $t$ in a given RW trajectory is denoted by $s_t$. Each RW trajectory is terminated once it covers all the nodes in the network, namely when $s_t = N$. The cover time is thus equal to the length of the trajectory. The trajectory $x_0, x_1, \ldots, x_t$ is recorded for further analysis.

Another interesting probability is the inverse of the distribution $P_t(S = s)$, namely the conditional probability $P(T = t|s)$ that an RW has pursued $t$ time steps, given that it has visited $s$ distinct nodes. This probability can be obtained by marginalizing $P_t(S = s)$, namely

$$P(T = t|s) = \frac{P_t(S = s)}{\sum_{s=1}^{\infty} P_t(S = s)}.$$  (28)

Clearly, $P(T = 1|1) = 1$. The probability $P(T = t|s)$ is defined in the range of $2 \leq s \leq N - 1$ and $t \geq s$. In appendix B we use a generating function approach to obtain a closed-form expres-
Figure 4. The conditional tail distribution \( P(T > t | s) \) on the elapsed time given that the RW has visited \( s \) distinct nodes, for an RRG of size \( N = 1000 \), degree \( c = 10 \) and \( s = N/4 \) (left), \( s = N/2 \) (center) and \( s = 3N/4 \) (right). The analytical results (solid lines), obtained from equations (B.3) and (B.4) are in very good agreement with the results obtained from computer simulations (circles).

In figure 4 we present the tail distribution \( P(T > t | s) \) for a network of size \( N = 1000 \) with degree \( c = 10 \) and \( s = N/4 \) (left), \( s = N/2 \) (middle) and \( s = 3N/4 \) (right). The analytical results are in very good agreement with the results obtained from computer simulations. It is found that as \( s \) is increased the sigmoid-like function shifts to the right and broadens.

7. The mean and variance of \( P_t(S = s) \)

In appendix C we use a generating function formulation to calculate the moments of \( P_t(S = s) \). The first moment is given by

\[
\langle S \rangle_t = 2 + (N - 2) \left[ 1 - e^{\left( \frac{t}{c} \right) \frac{1}{c^2}} \right],
\]

which coincides with previous results obtained using other methods [25]. Thus, the mean number of nodes in the complementary set of nodes that have not been visited by the RW up to time \( t \) is given by

\[
\langle U \rangle_t = (N - 2)e^{\left( \frac{t}{c} \right) \frac{1}{c^2}}.
\]

In figure 5 we present analytical results for the mean number \( \langle S \rangle_t \) of nodes visited by an RW up to time \( t \) on an RRG of size \( N = 1000 \) and degrees \( c = 3 \) (solid line), \( c = 10 \) (dashed line) and \( c = 30 \) (dotted line). The analytical results, obtained from equation (29), are in very good agreement with the results obtained from computer simulations (circles).

The variance of \( P_t(S = s) \) is given by

\[
\text{Var}_t(S) = \langle S^2 \rangle_t - \langle S \rangle_t^2.
\]
In appendix C we obtain the second moment $\langle S^2 \rangle_t$. Inserting the results for the first and second moments into equation (31), we obtain

$$\text{Var}_t(S) = (N - 2) \left[ 1 - \left( \frac{c - 2}{c - 1} \right) \frac{1}{N} \right]^{t-2} - (N - 2)^2 \left[ 1 - \left( \frac{c - 2}{c - 1} \right) \frac{1}{N} \right]^{2(t-2)}$$

$$+ (N - 3)(N - 2) \left[ 1 - 2 \left( \frac{c - 2}{c - 1} \right) \frac{1}{N} \right]^{t-2}.$$  \hspace{1cm} (32)

This is a new result, which could not be obtained using the methods of reference [25].

In figure 6 we present analytical results for the variance $\text{Var}_t(S)$ as a function of $t$ for an RW on an RRG of size $N = 1000$ and degrees $c = 3$ (solid line, left), $c = 10$ (dashed line, center) and $c = 30$ (dotted line, right). For $c = 3$ the simulation results (triangles) exhibit a significant deviation from the analytical results (solid line), obtained from equation (32). This deviation is due to temporal correlations between the probabilities to visit previously unvisited nodes in successive time steps, which are most pronounced for small values of $c$. However, note that at the peak of $\text{Var}_t(S)$ for $c = 3$ in figure 6, obtained for $t \approx 2000$, the standard deviation obtained from the simulations is $\sigma_t(S) = \sqrt{\text{Var}_t(S)} \approx 24$, compared to $\sigma_t(S) \approx 14$ obtained from the theoretical calculations. Both values are small compared to $\langle S \rangle_t \approx 600$ at $t \approx 2000$, namely $P_t(S = s)$ exhibits a narrow peak around $\langle S \rangle_t$. For $c = 10$ the analytical results (dashed line) are in better agreement with the results obtained from computer simulations (squares), while for $c = 30$ there is a good agreement, showing that the mean-field argument becomes more accurate as $c$ is increased.

8. The distribution of cover times

Inserting $s = N$ in the distribution $P_t(S = s)$ one obtains $P_t(S = N)$, which is the probability that the RW has visited all the nodes in the network up to time $t$. In fact, this coincides with...
Figure 6. Analytical results for the variance $\text{Var}_t(S)$ of the distribution $P_t(S = s)$ of the number of distinct nodes visited up to time $t$ for an RW on an RRG of size $N = 1000$ and degree $c = 3$ (solid line, left), $c = 10$ (dashed line, center) and $c = 30$ (dotted line, right), and the corresponding simulation results (triangles, squares and circles, respectively). For $c = 3$ the results for the variance obtained from computer simulations (triangles) are much larger than those obtained from the analytical calculations. This discrepancy is due to short range temporal correlations in the RW trajectories that are most pronounced at small values of $c$. For $c = 10$ the agreement between the analytical results (dashed line) and the simulation results (squares) is significantly improved, while for $c = 30$ there is a good agreement.

the cumulative probability of the cover time, namely

\[ P(T_C \leq t) = P_t(S = N). \]  

The tail distribution of cover times is given by

\[ P(T_C > t) = 1 - P(T_C \leq t). \]  

Therefore,

\[ P(T_C > t) = 1 - P_t(S = N). \]  

Inserting $s = N$ in equation (17) and plugging in the right-hand side into equation (35), we obtain

\[ P(T_C > t) = 1 - (N - 2)! \sum_{v=N-2}^{t-2} (-1)^v v^{t-2} \binom{v}{N-2} \left( \frac{c-2}{c-1} \right)^v \left( 1 - \frac{c-2}{c-1} \right)^v. \]  

The number of terms in the sum on the right-hand side of equation (36) scales like $t - N$. It is thus efficient as long as $t - N$ is not too large. For longer times it is more efficient to extract the distribution of cover times from equation (21). Inserting $s = N$ in equation (21), we obtain

\[ P(T_C > t) \approx 1 - \sum_{k=0}^{N-2} (-1)^k \binom{N-2}{k} \left[ 1 - \left( \frac{c-2}{c-1} \right) \right]^{t-2}. \]  

In the long time limit $t \gg N$, one could use equation (26) to approximate the cover time. Inserting $s = N$ in equation (26), we obtain

\[ P(T_C > t) \approx 1 - \left[ 1 - e^{-\left( \frac{c-2}{c-1} \right) t} \right]^{N-2}. \]
Moreover, equation (38) can be approximated by
\[
P(T_C > t) \simeq 1 - \exp \left[ -(N - 2)e^{-\left(\frac{t}{c - 2}\right)^{c - 1}} \right].
\] (39)

Interestingly, equation (39) can be written in the form
\[
P(T_C > t) \simeq 1 - \exp \left[ -\langle U \rangle_t \right],
\] (40)
where \(\langle U \rangle_t\) is the mean number of yet-unvisited nodes at time \(t\).
Rearranging terms in the exponent, it is found that the distribution of cover times is a discrete Gumbel distribution, known from extreme value theory, which takes the form [91]
\[
P(T_C > t) \simeq 1 - \exp \left[ -\exp \left( -\frac{t - \mu}{\beta} \right) \right],
\] (41)
where
\[
\mu = 2 + \frac{c - 1}{c - 2}N \ln(N - 2)
\] (42)
is called the location parameter and
\[
\beta = \frac{c - 1}{c - 2}N
\] (43)
is called the scale parameter. The location parameter \(\mu\) is equal to the mode of the Gumbel distribution. The scale parameter \(\beta\) is equal to the standard deviation up to a constant factor of order 1.

The Gumbel distribution often emerges as the distribution of the maxima among sets of \(n\) independent random variables drawn from the same distribution. It is one of the three possible families of extreme value distributions specified by the extreme value theory, namely the Gumbel, Fréchet and Weibull families [91–95]. The Gumbel distribution appears in various problems that involve structural and dynamical processes on random networks. These include the distribution of diameters in an ensemble of subcritical ER networks [96, 97], the distribution of the number of neighbors of a set of nodes [98–100], the distribution of take-over times of infections [101], the distribution of extinction times of infections [102] and the distribution of flooding times [103, 104].

Using the point of view of the extreme-value theory, the distribution \(P(T_C \leqslant t)\) can be considered as the distribution of the maximum among \(N - 2\) distributions of first passage times from the initial node \(i\) to all the other nodes in the network (apart from the two nodes visited in the first two time steps). Therefore, under the assumption that the distributions of first passage times for different nodes are independent, the distribution of cover times satisfies
\[
P(T_C > t) = 1 - [P(T_{FP} \leqslant t)]^{N-2}.
\] (44)
Comparing equations (38) and (44), we conclude that the distribution of first passage times is given by
\[
P(T_{FP} \leqslant t) = 1 - e^{-\left(\frac{t}{c - 2}\right)^{c - 1}}.
\] (45)
Indeed, the distribution of first passage times exhibits an exponential tail. It thus meets the criterion for the emergence of the Gumbel distribution in the Fisher–Tippet–Gnedenko theorem.
Note that the first passagetimes of adjacent target nodes may be correlated. However, such correlations appear to have little effect on the distribution of covertimes.

The probability mass function of covertimes can be obtained by taking the difference

\[ P(T_C = t) = P(T_C > t - 1) - P(T_C > t). \] (46)

Using equation (35) it is found that

\[ P(T_C = t) = P_t(S = N) - P_{t-1}(S = N). \] (47)

Alternatively, the distribution of covertimes can be expressed in the form

\[ P(T_C = t) = P_{t-1}(S = N - 1)\Delta(N - 1), \] (48)

where, using equation (6),

\[ \Delta(N - 1) = \frac{c - 2}{(c - 1)N}. \] (49)

Equation (48) expresses the fact that in order for the RW to enter the last unvisited node at time \( t \) it has to visit \( N - 1 \) nodes up to time \( t - 1 \) and step into a previously unvisited node at time \( t \).

In the top row of figure 7 we present the tail distribution of cover times, \( P(T_C > t) \) vs \( t \), for an RRG of size \( N = 1000 \) consisting of nodes of degree \( c = 3 \) (left), 4 (middle) and 10 (right). For \( c = 4 \) and \( c = 10 \) the analytical results (solid lines), obtained from equation (36), are in very good agreement with the results obtained from computer simulations (circles). In the case of \( c = 3 \) there is a slight discrepancy, where the analytical results for the cover time are shifted to the right by a few hundreds of time steps compared to the simulation results. This discrepancy is due to subtle correlations that emerge in low-degree RRGs, which are most pronounced in the case of \( c = 3 \).

In the bottom row of figure 7 we present analytical results for the corresponding probability density functions \( P(T_C = t) \) vs \( t \). The results obtained from equation (48) (circles) are found to be in very good agreement with the results obtained from the Gumbel distribution (equation (41)) (+ symbols). It can be seen that as the mean degree is increased, the cover times becomes shorter, and the distribution more centralized around its mean. This is consistent with the general trend seen in previous sections, namely the fact that in sparser networks the discovery of new nodes is generally slower compared to denser networks. As mentioned, this can be explained by the fact that in low-degree networks there is a higher probability of backtracking to previously visited domains of the network. Note that for \( t \leq N - 1 \) the distribution of cover times satisfies \( P(T_C = t) = 0 \). The corresponding Gumbel distribution is vanishingly small in this regime, but not strictly zero.

9. The mean cover time

The mean cover time is given by

\[ \langle T_C \rangle = \sum_{t=N}^{\infty} tP(T_C = t). \] (50)

It can be expressed in the form

\[ \langle T_C \rangle = \frac{d}{d\omega} J(\omega) \bigg|_{\omega = 1}, \] (51)
Figure 7. Top row: analytical results for the tail distribution \( P(T_C > t) \) (solid lines) of the cover times of RWs on RRGs of size \( N = 1000 \) (top row) and degree \( c = 3 \) (left), 4 (middle) and 10 (right). The analytical results, obtained from equation (36) are in very good agreement with the results obtained from computer simulations (circles). Bottom row: the corresponding probability mass functions \( P(T_C = t) \), obtained from equation (48) (circles) and from the Gumbel distribution (equation (41)) (+ symbols). As the degree of the network increases, the cover times become shorter and more concentrated around the mean.

where

\[
J(\omega) = \sum_{t=0}^{\infty} \omega^t P(T_C = t) \tag{52}
\]

is the generating function of \( P(T_C = t) \). In appendix D we expand the generating function \( J(\omega) \) in powers of \( \omega - 1 \). Inserting the expansion of \( J(\omega) \) from equation (D.6) into equation (51), we obtain

\[
\langle T_C \rangle = 2 + \left( \frac{c-1}{c-2} \right) (N-2) H_{N-2}, \tag{53}
\]

where \( H_m \) is the \( m \)th harmonic number [89]. In the limit of \( m \gg 1 \) the harmonic numbers can be approximated by

\[
H_m = \ln m + \gamma + \frac{1}{2m} + O\left( \frac{1}{m^2} \right), \tag{54}
\]

where \( \gamma \approx 0.577 \) is the Euler–Mascheroni constant [26]. Using this approximation, it is found that in the large network limit the mean cover time can be expressed by

\[
\langle T_C \rangle = \left( \frac{c-1}{c-2} \right) N \ln N + \left( \frac{c-1}{c-2} \right) \gamma N + \frac{c-5}{2(c-2)} + O\left( \frac{1}{N} \right). \tag{55}
\]
Figure 8. Analytical results for the mean cover time $\langle T_C \rangle$ (solid line) for RRGs of size $N = 1000$ as a function of the degree $c$. The analytical results, obtained from equation (53) are in very good agreement with the results obtained from computer simulations (circles). The leading order asymptotic approximation, obtained from equation (55), is also shown (dashed line). The asymptotic value of the mean cover time in the limit of high degree $c$, namely $N \ln N$, is also shown (dotted line).

The leading term on the right-hand side of equation (55) coincides with the result obtained in reference [71], using other methods. Our solution also includes sub-leading terms which can be significant even for very large networks.

In figure 8 we show the mean cover time vs node degree $c$, for networks of size $N = 1000$. The analytical result obtained from equation (53) is shown in solid line, and the leading order asymptotic expression from equation (55) is shown in dashed line. For comparison, we also show the large $c$ limiting value $N \ln N$ in dotted line. The solid curve is in good agreement with numerical simulations, and the dashed asymptotic limit also gives a good estimate, but with a noticeable discrepancy for networks of this size, coming from the $O(N)$ correction to the leading $O(N \ln N)$ behavior.

10. The variance of the distribution of cover times

The second moment of the distribution of cover times can be expressed in terms of the generating function $J(\omega)$ in the form

$$\langle T_C^2 \rangle = \frac{d^2}{d\omega^2} J(\omega) \bigg|_{\omega=1} + \frac{d}{d\omega} J(\omega) \bigg|_{\omega=1}.$$  \hspace{1cm} (56)

Inserting the expansion of $J(\omega)$ from equation (D.6) into equation (56), we obtain

$$\langle T_C^2 \rangle = 4 + \frac{3c-1}{c-2}(N-2)H_{N-2} + \left(\frac{c-1}{c-2}\right)^2 (N-2)^2 \left[ (H_{N-2})^2 + H_{N-2}^{(2)} \right],$$  \hspace{1cm} (57)

where $H_n^{(2)}$ is the generalized harmonic number of the second order [89]. Combining the results for the first and second moments, we obtain the variance of the distribution of cover times,
Figure 9. Analytical results for the variance $\text{Var}(T_C)$ of the distribution of cover times as a function of the degree $c$ for an RW on an RRG of size $N = 1000$ (solid line). The analytical results, obtained from equation (59), are in very good agreement with the results obtained from computer simulations (circles), except for the case of $c = 3$ in which there is a significant deviation.

which is given by

$$\text{Var}(T_C) = \left(\frac{c-1}{c-2}\right)^2(N-2)^2H_N^{(2)} - \left(\frac{c-1}{c-2}\right)(N-2)H_{N-2}.$$

(58)

In the limit of $N \gg 1$ the variance can be approximated by

$$\text{Var}(T_C) \approx \frac{\pi^2}{6} \left(\frac{c-1}{c-2}\right)^2N^2 - \left(\frac{c-1}{c-2}\right)N \ln N - \left(\frac{c-1}{c-2}\right) (\gamma + \frac{c-1}{c-2}) N - \frac{3(c-1)}{2(c-2)^2} + O\left(\frac{1}{N}\right).$$

(59)

The leading term on the right-hand side of equation (59) scales like $N^2$, which means that the standard deviation of $P(T_C = t)$ scales like $N$, while the mean $\langle T_C \rangle$ scales like $N \ln N$.

In figure 9 we present analytical results for the variance $\text{Var}(T_C)$ of the distribution of cover times as a function of the degree $c$ for an RW on an RRG of size $N = 1000$ (solid line). The analytical results, obtained from equation (59), are in very good agreement with the results obtained from computer simulations (circles), except for the case of $c = 3$ in which there is a significant deviation.

11. The distribution of partial cover times

One can generalize the concept of cover time to the $k$th order partial cover time, which is the first time in which the RW visits $k$ distinct nodes [72–74]. We denote the probability that the RW will complete visiting $k$ distinct nodes at time $t$ by $P(T_{PC,k} = t)$. This probability can be
expressed in the form
\[ P(T_{PC,k} = t) = \left( \frac{c-2}{c-1} \right) \left( 1 - \frac{k-1}{N} \right) P_{t-1}(S = k-1), \] (60)
where \( P_{t-1}(S = k-1) \) is the probability that at time \( t-1 \) the RW has visited \( k-1 \) distinct nodes. This probability is multiplied by the probability that at time \( t \) the RW will step into a new node, which has not been visited before, namely by \( \Delta(k-1) \), given by equation (6).

Comparing equation (60) with equation (B.3) in appendix B, it is found that
\[ P(T_{PC,k} = t) = P(T = t-1|k-1). \] (61)

Below we present an identity that will be useful for the calculation of the moments of the distribution of partial cover times. Multiplying equation (61) by \( t^r \) and summing over \( t \), we obtain
\[ \langle T^r_{PC,k} \rangle = E[(T + 1)^r|S = k-1]. \] (62)
Using equation (B.9), we obtain the mean of the \( k \)th order partial cover time, which is given by
\[ \langle T_{PC,k} \rangle = 2 + \left( \frac{c-1}{c-2} \right) (N-2)(H_{N-2} - H_{N-k}). \] (63)
Approximating the harmonic numbers using equation (54), we obtain
\[ \langle T_{PC,k} \rangle = 2 + \frac{c-1}{c-2} N \ln \left( \frac{N-2}{N-k} \right) + O \left( \frac{1}{N} \right), \] (64)
where \( 2 \leq k \leq N - 1 \). In the limit of \( k \ll N \), equation (64) is reduced to
\[ \langle T_{PC,k} \rangle \simeq 2 + \frac{c-1}{c-2} (k-2). \] (65)
Note that equation (64) does not hold in the special case of \( k = N \), in which the partial cover time coincides with the cover time, namely \( \langle T_{PC,N} \rangle = \langle T_C \rangle \). Using equation (62) with \( r = 2 \) and equation (B.14), we obtain the variance of the distribution of partial cover times, which takes the form
\[ \text{Var}(T_{PC,k}) = \left( \frac{c-1}{c-2} \right)^2 (N-2)^2 \left[ H_{N-2}^{(2)} - H_{N-k}^{(2)} \right] - \left( \frac{c-1}{c-2} \right) (N-2)(H_{N-2} - H_{N-k}). \] (66)
In the limit of \( k \ll N \), equation (66) is reduced to
\[ \text{Var}(T_{PC,k}) = \frac{c-1}{(c-2)^2} (k-2). \] (67)

12. The distribution of random cover times

Another generalization of the concept of the cover time is referred to as the random cover time. This is the first time at which the RW completes visiting a specific pre-selected set of \( k \)
randomly selected target nodes \([72–74]\). The distribution of random cover times is denoted by \(P(T_{RC,k}=t)\). The tail distribution of random cover times can be expressed by

\[
P(T_{RC,k}>t) = 1 - \sum_{s=k}^{N} \frac{(N-k)}{N} P_t(S=s), \tag{68}
\]

where the ratio between the binomial coefficients provides the probability that all the \(k\) pre-selected nodes are included in the \(s\) distinct nodes visited by the RW up to time \(t\). The mean of the \(k\)th random cover time is given by the tail-sum formula

\[
\langle T_{RC,k} \rangle = \sum_{t=1}^{\infty} P(T_{RC,k}>t-1). \tag{69}
\]

Inserting \(P(T_{RC,k}>t)\) from equation (68) into equation (69), we obtain

\[
\langle T_{RC,k} \rangle = \sum_{t=1}^{\infty} \left[1 - \sum_{s=k}^{N} \frac{(N-k)}{N} P_t(S=s)\right]. \tag{70}
\]

It will be useful to introduce a generating function of the form

\[
\rho_k(\omega) = \sum_{t=1}^{\infty} \omega^t P(T_{RC,k} \geq t). \tag{71}
\]

Comparing equations (69) and (71), it is found that the mean \(\langle T_{RC,k} \rangle\) can be expressed using this generating function, in the form

\[
\langle T_{RC,k} \rangle = \lim_{\omega \to 1} \rho_k(\omega). \tag{72}
\]

Carrying out the summation in equation (71), we obtain

\[
\rho_k(\omega) = \frac{\omega}{1 - \omega} - \omega \sum_{s=k}^{N} \frac{(N-k)}{s} L_s(\omega), \tag{73}
\]

where \(L_s(\omega)\) is defined in appendix A. Being interested in the limit of \(\omega \to 1\), we would like to expand the generating function \(\rho_k(\omega)\) in powers of \(\omega - 1\). Inserting \(L_s(\omega)\) from equation (A.27) (for \(s \leq N - 1\)) and from (A.36) (for \(s = N\)), we obtain

\[
\rho_k(\omega) = 2 + \frac{c-1}{c-2} (N-2) \left[ H_{N-2} - \sum_{s=k}^{N-1} \frac{(N-k)}{s} \left( \frac{1}{N-s} \right) \right] + O(\omega - 1). \tag{74}
\]

Taking the limit of \(\omega \to 1\) and carrying out the summation, we obtain

\[
\langle T_{RC,k} \rangle = 2 + (N-2) \left( \frac{c-1}{c-2} \right) [H_{N-2} - (H_N - H_k)]. \tag{75}
\]

After some algebraic simplifications, we obtain a simple expression for the random cover time, which takes the form

\[
\langle T_{RC,k} \rangle = \left( \frac{c-1}{c-2} \right) \left[ (N-2)H_k - \frac{2}{c-1} - \frac{1}{N-1} \right]. \tag{76}
\]
Note that the mean cover time \( \langle T_C \rangle \) can be recovered by inserting \( k = N \) in equation (75). For values of \( k \) which are not too small, one can approximate the harmonic number \( H_k \) using equation (54). This leads to

\[
\langle T_{RC,k} \rangle = \left( \frac{c - 1}{c - 2} \right) N \ln k + \left( \frac{c - 1}{c - 2} \right) \gamma N + O \left( \frac{N}{k} \right). \tag{77}
\]

Below we show that for \( t \gg N \) the distribution of random cover times follows a Gumbel distribution. This will allow us to evaluate its variance \( \text{Var}(T_{RC,k}) \). Inserting \( P_t(S = s) \) from equation (26), which is valid for \( t \gg N \), into equation (68), we obtain

\[
P(T_{RC,k} > t) = 1 - \sum_{s=k}^{N} \left( \frac{N-k}{s-k} \right) \left( \frac{N-2}{s-2} \right) \left[ 1 - e^{-\left( \frac{s}{N-k} \right) \frac{t}{k}} \right]^{s-2} \left[ e^{-\left( \frac{s}{N-k} \right) \frac{t}{k}} \right]^{N-s}. \tag{78}
\]

Equation (78) can be simplified to

\[
P(T_{RC,k} > t) = 1 - \sum_{s=k}^{N} \left( \frac{N-k}{s-k} \right) s(s-1) \left[ 1 - e^{-\left( \frac{s}{N-k} \right) \frac{t}{k}} \right]^{s-2} \left[ e^{-\left( \frac{s}{N-k} \right) \frac{t}{k}} \right]^{N-s}. \tag{79}
\]

Carrying out the summation, we obtain

\[
P(T_{RC,k} > t) = 1 - \left[ 1 - e^{-\left( \frac{k}{N-k} \right) \frac{t}{k}} \right]^k \left\{ 1 - 2 \frac{k}{N} \left[ e^{-\left( \frac{k}{N-k} \right) \frac{t}{k}} \right] + \frac{k(k-1)}{N(N-1)} \left[ e^{-\left( \frac{k}{N-k} \right) \frac{t}{k}} \right]^2 \right\}. \tag{80}
\]

In the limit of \( t \gg N \) equation (80) can be approximated by

\[
P(T_{RC,k} \leq t) = 1 - P(T_{RC,k} > t) = \exp \left[ -e^{-\left( \frac{k}{N-k} \right) \frac{t}{k}} \right] \frac{N \ln t}{k}, \tag{81}
\]

which is a Gumbel distribution. This distribution can be expressed by equation (41) with

\[
\mu = 2 + \frac{c - 1}{c - 2} N \ln k \tag{82}
\]

and

\[
\beta = \frac{c - 1}{c - 2} N. \tag{83}
\]

The variance of the distribution of random cover times, up to leading orders in \( N \), is thus given by

\[
\text{Var}(T_{RC,k}) \simeq \frac{\pi^2}{6} \left( \frac{c - 1}{c - 2} \right)^2 N^2 - \left( \frac{c - 1}{c - 2} \right) N \ln k. \tag{84}
\]
Figure 10. Analytical results for the mean partial cover time $\langle T_{PC,k} \rangle$ (solid line) and the mean random cover time $\langle T_{RC,k} \rangle$ (dashed line), as a function of the number of nodes $k$, for an RRG of size $N = 1000$ and degree $c = 10$. The analytical results, obtained from equations (63) and (76) respectively, are in very good agreement with the results obtained from computer simulations (circles and squares, respectively). Note that the two curves exhibit a reflection symmetry between them, as discussed in the main text.

The mean number of yet-unvisited nodes up to time $t$ in the random subgraph of $k$ nodes is given by

$$\langle U_k \rangle_t = \frac{k}{N} \langle U \rangle_t,$$  

(85)

where $\langle U \rangle_t$ is given by equation (30). Thus, equation (81) can be rewritten in the form

$$P(T_{RC,k} \leq t) = e^{-\langle U_k \rangle t},$$  

(86)

13. The relation between $P(T_{PC,k} = t)$ and $P(T_{RC,k} = t)$

In figure 10 we present the mean partial cover time $\langle T_{PC,k} \rangle$, as a function of $k$, in a network of size $N = 1000$ and node degree $c = 10$. The analytical results (solid line) given by equation (63), are found to be in very good agreement with the results obtained from simulations (symbols). It can be seen that the mean partial cover time grows very rapidly only when $k$ approaches $N$. We also present the mean random cover time $\langle T_{RC,k} \rangle$ vs $k$. The analytical result (solid line) given by equation (76), is found to be in very good agreement with the results obtained from computer simulations (symbols). It is found that for any value of $1 \leq k < N$ the random cover time $\langle T_{RC,k} \rangle$ is much larger than the corresponding partial cover time $\langle T_{PC,k} \rangle$. This reflects the fact that it takes longer to visit $k$ pre-selected nodes than to visit a set of $k$ unspecified nodes.

Figure 10 reveals a surprising reflection symmetry between $\langle T_{RC,k} \rangle$ and $\langle T_{PC,k} \rangle$. Taking each one of these functions and applying the inversions $k \rightarrow N - k$ and $\langle T \rangle \rightarrow \langle T_c \rangle - \langle T \rangle$
one obtains the other functions up to a very small shift. The reflection symmetry can be expressed by
\[ \langle T_{RC,N-k} \rangle + \langle T_{PC,k} \rangle = \langle T_C \rangle + \frac{2}{c-2} + \left( \frac{c-1}{c-2} \right) \frac{1}{N-1}. \] (87)

Note that the two correction terms on the right-hand side do not depend on \( k \). Moreover, the first correction term is of the order of \( 1/c \) while the second term is of the order of \( 1/N \). Both are negligible compared to \( \langle T_C \rangle \), which scales like \( N \ln N \). Thus, for sufficiently large networks
\[ \langle T_{RC,N-k} \rangle + \langle T_{PC,k} \rangle = \langle T_C \rangle. \] (88)

This symmetry reflects the following property: starting the RW from some random node \( i \), the mean number of time steps it will take to visit \( k \) distinct nodes (which were not specified beforehand) is given by \( \langle T_{PC,k} \rangle \). At this stage there are \( N-k \) remaining nodes to cover in order to complete the cover time. However, the remaining \( N-k \) nodes are specific ones, because these are the nodes that have not been visited up to that time. Therefore, the time that will take the RW to cover the remaining \( N-k \) nodes follows the distribution of random cover times of \( N-k \) nodes, whose mean is \( \langle T_{RC,N-k} \rangle \).

To put the three types of cover times on a common footing, we summarize their scaling behavior: the mean of the distribution of cover times scales like \( \langle T_C \rangle \sim N \ln N \), the mean of the distribution of partial cover times scales like \( \langle T_{PC,k} \rangle \sim N \ln \left( \frac{N}{N-k} \right) \) and the mean of the distribution of random cover times scales like \( \langle T_{RC,k} \rangle \sim N \ln k \).

14. Discussion

A characteristic property of the cover time problem is that at early times the RW is highly efficient in covering new nodes. This efficiency is gradually reduced as the fraction of nodes that have already been visited increases, until at late times it takes a large number of steps to reach each one of the few yet-unvisited nodes that remain. Such situations are often described by the 80/20 law (or Pareto principle), which states that in certain systems roughly 80 percent of the outcome is a result of only 20 percent of the effort, while the remaining 20 percent or so of the outcome consumes 80 percent of the effort [105, 106]. In light of this observation, it is interesting to find the value of \( 0 < f < 1 \) for which \( f \langle T_C \rangle \) time steps of an RW would cover, on average, \( (1-f)N \) nodes of the RRG. The fraction \( f \) can be calculated by solving the equation
\[ \langle T_{PC,(1-f)N} \rangle = f \langle T_C \rangle. \] (89)

Taking the large \( N \) limit, we insert the leading term for \( \langle T_{PC,(1-f)N} \rangle \) from equation (64) and for \( \langle T_C \rangle \) from equation (55) into equation (89). We obtain
\[ (1-f) \ln N = \ln (fN) + \gamma f. \] (90)

Equation (90) can be written in the form
\[ fN \ln f = e^{-\gamma f}, \] (91)

which implies that
\[ f = \frac{W(\gamma + \ln N)}{\gamma + \ln N}, \] (92)
where $W(x)$ is the Lambert $W$ function [89]. The fraction $f$ is a monotonically decreasing function of $N$. For example, in case that $N = 1000$ we obtain $f \simeq 0.21$. This is consistent with the $80/20$ law.

The declining efficiency of the process of covering the network as the time evolves is reminiscent of the economic law of diminishing returns [107, 108]. Consider the production process of a commodity, in which a single input component is increased while all the other input components are held fixed. The law states that at some point the resulting increase in the output per unit increase in the input will become progressively smaller or diminishing. In this analogy the time steps of the RW are considered as the input resource and the number of distinct nodes visited by the RW is the output or product. As the time evolves the number of distinct nodes visited by the RW per time step diminishes. The practice of working on a task past the point of diminishing returns is often referred to as gold plating [109]. While the economic literature focuses on the negative side of gold plating, there are often great advantages and importance in bringing things to perfection or completion.

It is interesting to compare the results obtained in this paper for the cover times of RWs on RRGs with the corresponding results for RWs on regular lattices with the same coordination numbers. For example, the coordination number of a hypercubic lattice in $d$ dimensions is $2d$. Thus, in terms of the connectivity the $d$-dimensional hypercubic lattice is analogous to an RRG of degree $c = 2d$. In the case of an RW on a one-dimensional lattice of $N$ sites with periodic boundaries, it was found that the mean cover time is given by $\langle T_C \rangle = N(N - 1)/2$ [110, 111]. For an RW on a two-dimensional square lattice consisting of $N = L^2$ sites with periodic boundaries (forming a torus), it was found that $\langle T_C \rangle \propto N(\ln N)^2$ [111–113]. For dimensions $d \geq 3$ it was found that the mean cover time of an RW on a cubic lattice consisting of $N = L^d$ sites with periodic boundaries is given by $\langle T_C \rangle = A_d N \ln N$, where the coefficient $A_d$ depends on the dimension $d$ [111]. Thus, for $d \geq 3$ the leading term in the expression for $\langle T_C \rangle$ on regular lattices has the same functional form as in the case of RRGs. However, the values of the coefficient $A_d$, $d = 3, 4, \ldots$, for regular lattices are known only approximately from computer simulations. For example, it was found that $A_3 \simeq 1.63$ and $A_4 \simeq 1.23$. These values are larger than the coefficient obtained for the corresponding RRG with $c = 2d$, which is given by $(c - 1)/(c - 2)$. This implies that the cover time of an RW on a regular lattice is larger than the cover time on an RRG with the same coordination number. This is sensible in light of the fact that the RRG is a small world network on which it is less likely that the RW will remain for a long time in the same neighborhood and revisit the same nodes again and again. Interestingly, beyond any differences in the prefactor of the mean cover time, the limit distribution of cover times on RRGs and lattices with $d \geq 3$ is Gumbel in both cases [114]. Based on the experience gained in the study of other problems on RRGs [25, 115, 116] we expect that the results for $\langle T_C \rangle$ provide the asymptotic large $d$ behavior on regular lattices. More precisely, we conjecture that for hypercubic lattices of high dimension $d$ the coefficient $A_d$ is given by $A_d \simeq (2d - 1)/(2d - 2)$.

Another type of RW model is the non-backtracking random walk (NBW) [39, 117]. At each time step the NBW hops from its present node to one of its neighbors, except for the node it visited in the previous time step. It is thus similar to the RW, except for the backtracking step which is eliminated. As a result, all the subsequent retroceding steps are also eliminated. The paths of NBWs have been studied on regular lattices and random graphs [117]. It was shown that they explore the network more efficiently than RWs. The elimination of the backtracking step implies that on RRGs in the $N \to \infty$ limit the probability that an NBW will step into a yet unvisited node is $\Delta = 1$. Therefore, in the case of NBWs equation (6) is replaced by

$$\Delta(s) = 1 - \frac{s}{N},$$

(93)
As a result, the distribution of cover times of NBWs on RRGs is given by equation (36) where \((c - 2)/(c - 1)\) is replaced by 1. In the long time limit \(t \gg N\) it can be approximated by the Gumbel distribution, which is given by equation (41) with \(\mu = N \ln N\) and \(\beta = N\).

Another interesting direction in the context of exploration of networks using RWs is that of edge coverage [118]. A typical problem is to determine the number of distinct edges visited by an RW up to time \(t\). The time it takes an RW on an RRG to visit every single edge in the network is called the edge cover time. The distribution \(P(T^{c}_{e} = t)\) of edge cover times of RWs on RRGs can also be calculated using the approach developed in this paper. The number of edges in an RRG that consists of \(N\) nodes of degree \(c\) is \(N_{e} = Nc/2\). The probability that an RW that has already visited \(s_{e}\) distinct edges will step into a yet-unvisited edge in the next time step is given by

\[
\Delta_{e}(s_{e}) = \frac{c - 2}{c - 1} \left(1 - \frac{s_{e}}{N_{e}}\right).
\]  

(94)

Thus, the distribution \(P_{e}(S_{e} = s_{e})\) of distinct edges visited by an RW up to time \(t\) can be expressed by equation (17) or by equation (21) where \(N\) is replaced by \(N_{e}\). Similarly, the tail distribution of edge cover times \(P(T^{c}_{e} > t)\) is given by equation (36), where \(N\) is replaced by \(N_{e}\).

The cover time problem was studied for a broad range of random search processes [72]. These search processes correspond to generalized RW models such as Lévy walks, intermittent RWs and persistent RWs. It was shown that in all these systems the distribution of cover times follows the Gumbel distribution. It was thus concluded that the Gumbel distribution is a universal distribution of cover times. Interestingly, it was recently shown that by accelerating the search process one can modify the distribution of cover times from the Gumbel distribution to narrower distributions such as the Gaussian distribution [119].

An interesting strategy for accelerating the covering of a network is by using \(k\) independent RWs [120–122]. In particular, it was shown rigorously that on random networks, as long as the number of RWs is not too large, namely \(k < (\ln N)^{1-\epsilon}\) (where \(\epsilon\) is a small number), the acceleration is by a factor of at least \(k\). In simple terms, the meaning of this is that a low concentration of RWs overlap very mildly.

**15. Summary**

We presented analytical results for the distribution of cover times of RWs on RRGs consisting of \(N\) nodes of degree \(c \geq 3\). To this end, we derived a master equation for the distribution \(P_{t}(S = s)\) of the number of distinct nodes \(s\) visited by an RW up to time \(t\). Using a generating function formalism, we solved the master equation and obtained a closed-form analytical expression for \(P_{t}(S = s)\). Applying this result to the special case of \(s = N\), we obtained the cumulative distribution of cover times \(P(T_{c} \leq t) = P(S = N)\) and calculated its mean and variance. Taking the large network limit, we showed that the distribution of cover times follows a Gumbel distribution. We also studied two interesting generalizations of the cover time: the partial cover time \(T_{PC,k}\), which is the time it takes an RW to visit \(k\) distinct nodes and the random cover time \(T_{RC,k}\), which is the time it takes an RW to cover a set of \(k\) random pre-selected nodes. The analytical results were compared to the results obtained from computer simulations and found to be in very good agreement.
Acknowledgments

We thank C Cooper, S N Dorogovtsev, A M Frieze, N Masuda and P Sollich for useful discussions. This work was supported by the Israel Science Foundation Grant No. 1682/18.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Solution of the master equation for $P_t(S = s)$

The discrete master equation (equation (10)) consists of $N - 1$ coupled difference equations for the discrete time derivative of $P_t(S = s)$, $s = 2, \ldots , N$ at $t \geqslant 2$. The initial condition is given by $P_2(S = s) = \delta_{s,2}$. Note also that in the first time step $P_1(S = s) = \delta_{s,1}$, while at $t \geqslant 2$ the probability $P_t(S = 1) = 0$. The master equation can also be written in the form

$$\frac{D_t}{D_t}P_t(S = s) = \left( \frac{c - 2}{c - 1} \right) \left( \frac{s - 1}{N} P_t(S = s - 1) - \frac{s}{N} P_t(S = s) \right).$$

(A.1)

The generating function of $P_t(S = s)$, is given by

$$G_t(x) = \sum_{s=2}^{N} x^s P_t(S = s).$$

(A.2)

The generating function $G_t(x)$ represents a discrete Laplace transform of $P_t(S = s)$ with respect to $s$. Multiplying both sides of equation (A.1) by $x^s$ and summing up over $s$ we obtain

$$\frac{D_t}{D_t}G_t(x) = \left( \frac{c - 2}{c - 1} \right) (x - 1) \left( G_t(x) - x \frac{\partial}{\partial x} G_t(x) \right).$$

(A.3)

The initial condition (at $t = 2$), expressed in terms of the generating function $G_t(x)$, takes the form

$$G_2(x) = \sum_{s=2}^{N} x^s P_2(S = s) = x^2.$$

(A.4)

We now define a second generating function, of the form

$$G_L(x, \omega) = \sum_{t=2}^{\infty} \omega^{t} G_t(x).$$

(A.5)

The generating function $G_L(x, \omega)$ represents a discrete double Laplace transform of $P_t(S = s)$ with respect to both $s$ and $t$. Multiplying equation (A.3) by $\omega^t$ and summing up over $t \geqslant 2$, we obtain

$$(1 - \omega)G_L(x, \omega) - \omega^2 x^2 = \left( \frac{c - 1}{c - 2} \right) (x - 1) \omega \left( G_L(x, \omega) - x \frac{\partial}{\partial x} G_L(x, \omega) \right).$$

(A.6)
Inserting \( x = 1 \) in equation (A.6), the right-hand side vanishes. Solving the resulting equation, we obtain
\[
GL(1, \omega) = \frac{\omega^2}{1 - \omega}. \tag{A.7}
\]

Inserting \( x = 1 \) in equation (A.5) and comparing the result with equation (A.7), one can verify that \( G_t(1) = 1 \), as expected. Using equation (A.2), this result establishes the normalization of \( P_t(S = s) \). In order to solve equation (A.6), we consider a series expansion in powers of \( x - 1 \), which is given by
\[
GL(x, \omega) = \frac{\omega^2 x^2}{1 - \omega} \sum_{m=0}^{\infty} a_m(\omega)(x - 1)^m. \tag{A.8}
\]

Inserting \( x = 1 \) in equation (A.8) and comparing the result to equation (A.7), it is found that \( a_0(\omega) = 1 \). Inserting \( GL(x, \omega) \) from equation (A.8) into equation (A.6), we obtain recursion equations for the coefficients \( a_m(\omega) \), which take the form
\[
a_m(\omega) = \left[ N - 1 - m \over m + 1 + A(\omega) \right] a_{m-1}(\omega), \tag{A.9}
\]
where
\[
A(\omega) = \left[ \left( \frac{c - 2}{c - 1} \right) \frac{1}{N} \right] \left( \frac{\omega}{1 - \omega} \right). \tag{A.10}
\]

From equation (A.9) it is clear that the recursion equations terminate at \( m = N - 1 \), and therefore \( a_{N-1}(\omega) = 0 \). Iterating the recursion equation (A.9), starting from \( a_0(\omega) = 1 \), we obtain an explicit solution for the coefficients, which is given by
\[
a_m(\omega) = \frac{(-1)^m (2 - N)_m}{\left(1 + {1 \over A(\omega)}\right)_m}, \tag{A.11}
\]
where
\[
(b)_n = b(b + 1) \ldots (b + n - 1) \tag{A.12}
\]
is the (rising) Pochhammer symbol [89]. Expressing the negative Pochhammer symbol \((2 - N)_m\) in equation (A.11) in terms of a ratio of two factorials and rearranging terms, one can express the coefficients \( a_m(\omega) \) in the form
\[
a_m(\omega) = \frac{(N - 2)!}{(N - m - 2)!} \prod_{\ell=1}^{m} \frac{1}{\ell + {1 \over A(\omega)}}. \tag{A.13}
\]

Inserting the coefficients \( a_m(\omega) \) from equation (A.11) into equation (A.8), we obtain
\[
GL(x, \omega) = \frac{\omega^2 x^2}{1 - \omega} \binom{1, 2 - N}{1 + {1 \over A(\omega)}} 1 - x, \tag{A.14}
\]
where the function \( _2F_1 [] \) is the hypergeometric function [89], which is given by

\[
_2F_1 \left[ \begin{array}{c} a,b \\ d \end{array} \right] z = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(d)_n n!}. \tag{A.15}
\]

Expanding the right-hand side of equation (A.14) in a Taylor series around \( x = 0 \), we obtain

\[
GL(x, \omega) = \frac{\omega^2}{1 - \omega} \sum_{s=2}^{N} x^s \frac{\Gamma \left( 1 + \frac{1}{A(\omega)} \right)}{\Gamma \left( s - 1 + \frac{1}{A(\omega)} \right)} \frac{(N - 2)!}{(s - 1 + \frac{1}{A(\omega)})} \frac{s - 1, s - N}{1} _2F_1 \left[ \begin{array}{c} s - 1, s - N \\ s - 1 + \frac{1}{A(\omega)} \end{array} \right] \tag{A.16}
\]

where \( \Gamma(x) \) is the gamma function [89].

In order to obtain the distribution \( P_t(S = s) \) we perform below two inverse Laplace transforms, from \( x \) to \( s \) and from \( \omega \) to \( t \). We first carry out the Laplace transform from \( GL(x, \omega) \) to the generating function

\[
L_s(\omega) = \sum_{i=2}^{\infty} \omega^i P_t(S = s). \tag{A.17}
\]

This generating function is related to \( GL(x, \omega) \) via the relation

\[
GL(x, \omega) = \sum_{i=2}^{\infty} x^i L_s(\omega). \tag{A.18}
\]

In figure 11 we illustrate the sequence of discrete Laplace transforms that is used in this appendix for the calculation of \( P_t(S = s) \). The sequence takes the form

\[
P_t(S = s) \rightarrow G_t(x) \rightarrow GL(x, \omega) \rightarrow L_s(\omega) \rightarrow P_t(S = s). \tag{A.19}
\]

It consists of four functions of two variables each and four Laplace or inverse Laplace transformations between them. The conjugate variable to the discrete time \( t \) is the continuous variable \( \omega \), while the conjugate variable to the discrete variable \( s \) is the continuous variable \( x \). It illustrates the sequence of steps pursued in the solution of the master equation. The formal structure of these functions and the relations between them are reminiscent of the four thermodynamic potentials (namely \( U(S, V) \), \( F(T, V) \), \( G(T, P) \) and \( H(S, P) \)), related to each other by Legendre transforms [115, 116].

Comparing the coefficients of \( x^s \) in equations (A.16) and (A.18), we obtain

\[
L_s(\omega) = \frac{(N - 2)!}{(N - s)!} \frac{\Gamma \left( 1 + \frac{1}{A(\omega)} \right)}{\Gamma \left( s - 1 + \frac{1}{A(\omega)} \right)} \frac{\omega^2}{1 - \omega} \frac{s - 1, s - N}{1} _2F_1 \left[ \begin{array}{c} s - 1, s - N \\ s - 1 + \frac{1}{A(\omega)} \end{array} \right] \tag{A.20}
\]

Since one of the parameters in \( _2F_1 [] \) on the right-hand side of equation (A.20) is a negative integer, the hypergeometric function can be calculated using the identity

\[
_2F_1 \left[ \begin{array}{c} a, -m \\ d \end{array} \right] 1 = \frac{(d - a)_m}{(d)_m}, \tag{A.21}
\]
Figure 11. Illustration of the sequence of two discrete Laplace transforms followed by two inverse Laplace transforms, that are used in the solution of the discrete master equation, providing a closed form expression for $P_t(S = s)$.

where $(d)_m$ is the Pochhammer symbol, given by equation (A.12), and $m$ is a positive integer. This result is similar to the Chu–Vandermonde identity (equation 15.4.24 in reference [89]). In the following we will need a few more results concerning this hypergeometric function. First, in the limit of $a \to d$, we obtain

$$2F_1 \left[ a, -m \atop d \right] \bigg|_{a \to d} (0)_m \frac{(d)_m}{(d + m)_m} = \delta_{m,0}. \quad (A.22)$$

Also, the derivative of the hypergeometric function with respect to $d$ satisfies

$$\frac{\partial}{\partial d} 2F_1 \left[ a, -m \atop d \right] \bigg|_{a \to d} \frac{\Gamma(d) \Gamma(m)}{\Gamma(d + m)}, \quad (A.23)$$

while the second derivative satisfies

$$\frac{\partial^2}{\partial d^2} 2F_1 \left[ a, -m \atop d \right] \bigg|_{a \to d} 2 \frac{\Gamma(d) \Gamma(m)}{\Gamma(d + m)} \left( H_{m-1} + H_{d-1} - H_{m+d-1} \right), \quad (A.24)$$

where $H_m$ is the $m$th harmonic number [89]. The third derivative satisfies

$$\frac{\partial^3}{\partial d^3} 2F_1 \left[ a, -m \atop d \right] \bigg|_{a \to d} 3 \frac{\Gamma(d) \Gamma(m)}{\Gamma(d + m)} \left( H_{m-1} + H_{d-1} - H_{m+d-1} \right)^2$$

$$- H_{m-1}^{(2)} - H_{d-1}^{(2)} + H_{m+d-1}^{(2)}, \quad (A.25)$$

where $H_m^{(2)}$ is the $m$th generalized harmonic number of the second order [89], which is also expressible in terms of the Riemann $\zeta$ function [89]. Applying these results to equation (A.20), we obtain

$$L_s(\omega) = \omega \left( \frac{c - 1}{c - 2} \right) \left( 1 - \frac{s}{N} \right)^{-1} \Gamma(N - 1) \frac{\Gamma[N - s + \frac{1}{(\omega)}]}{\Gamma[N - 1 + \frac{1}{(\omega)}]}. \quad (A.26)$$
Expanding the expression on the right-hand side of equation (A.26) around \( \omega = 1 \) for \( s \leq N - 1 \), we obtain

\[
L_s(\omega) = \left( \frac{c-1}{c-2} \right) (1 - \frac{s}{N})^{-1} + \left( \frac{c-1}{c-2} \right) (1 - \frac{s}{N})^{-1} \left[ 1 + \left( \frac{c-1}{c-2} \right) N(H_{N-2} - H_{N-1}) \right] (\omega - 1) + \mathcal{O} \left[ (\omega - 1)^2 \right].
\]

Equation (A.27) provides the generating function \( L_s(\omega) \) as a series in powers of \( \omega - 1 \), which will be useful in appendix B. In order to extract the distribution \( P_t(S = s) \) we need to express \( L_s(\omega) \) as a series in powers of \( \omega \).

Consider the identity [90]

\[
(-1)^m \prod_{\ell=1}^{m} \frac{1}{\ell + \frac{1}{A(\omega)}} = \sum_{v=m}^{\infty} \left\{ \begin{array}{c} v \\ m \end{array} \right\} (-A(\omega))^v,
\]

where \( \left\{ \begin{array}{c} v \\ m \end{array} \right\} \) is the Stirling number of the second kind. Inserting equation (A.28) into equation (A.13), we obtain

\[
a_m(\omega) = (-1)^m \frac{(N-2)!}{(N-m-2)!} \sum_{v=m}^{\infty} \left\{ \begin{array}{c} v \\ m \end{array} \right\} (-A(\omega))^v.
\]

Plugging equation (A.29) into equation (A.8) and expanding in powers of \( x \), we obtain

\[
L_s(\omega) = \frac{\omega^2}{1-\omega} \sum_{r=0}^{\infty} \omega^r \sum_{v=0}^{\min\{v,N-2\}} m! \left\{ \begin{array}{c} v \\ m \end{array} \right\} \left( \begin{array}{c} N-2 \\ m \end{array} \right) \left( \begin{array}{c} m \\ s-2 \end{array} \right).
\]

In order to express \( [A(\omega)]^v \) in terms of powers of \( \omega \), we use the binomial identity (equation 26.3.4 in reference [89])

\[
\frac{1}{1-\omega} = \sum_{r=0}^{\infty} \left( \begin{array}{c} r \\ v-1 \end{array} \right) \omega^{r+v-1},
\]

and obtain

\[
[A(\omega)]^v = \left[ \frac{c-2}{c-1} \right] v \sum_{r=1}^{\infty} \left( \begin{array}{c} r \\ v-1 \end{array} \right) \omega^{r+v-1}.
\]

Inserting \( [A(\omega)]^v \) from equation (A.32) into equation (A.30), we obtain

\[
L_s(\omega) = \sum_{t=s}^{\infty} \omega^t \sum_{v=0}^{\min\{v,N-2\}} (-1)^{v-s} \left( \begin{array}{c} t-2 \\ v \end{array} \right) \left[ \frac{c-2}{c-1} \right] v \sum_{m=s-2}^{\infty} \frac{1}{N} \left\{ \begin{array}{c} v \\ m \end{array} \right\} \left( \begin{array}{c} N-2 \\ m \end{array} \right) \left( \begin{array}{c} m \\ s-2 \end{array} \right). \]

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Finally, extracting the coefficients of $\omega^t$ from equation (A.33), we obtain
\[
P_t(S = s) = \sum_{v=s-2}^{t-2} (-1)^{v-s} \binom{t-2}{v} \frac{(c-2)}{c-1} \frac{1}{N}^v 
\times \sum_{m=s-2}^\min\{v,N-2\} m! \binom{v}{m} \binom{N-2}{m} \binom{m}{s-2}.
\] (A.34)

Note that this solution satisfies $P_t(S = s) = 0$ for $t < s$, which serves as a quick sanity check.

The case of $s = N$ is of special importance in the context of the cover time, because $P(T_C \leq t) = P_t(S = N)$. Inserting $s = N$ in equation (A.20), we obtain
\[
\ln(N) = \frac{(N-2)!}{1 - \omega} (1 + \frac{c-1}{c-2} (N-2) H_{N-2} - \frac{1}{2} (c-1) \frac{c-2}{(N-2)^2} (H_{N-2}^2 + H_{N-2}^{(2)}) (\omega - 1) + O[(\omega - 1)^2].
\] (A.35)

Expanding $\ln(N)$ from equation (A.35) in powers of $\omega - 1$, we obtain
\[
\ln(N) \approx \frac{1}{1 - \omega} - 2 - \frac{c-1}{c-2} (N-2) H_{N-2} - \frac{1}{2} (c-1) \frac{c-2}{(N-2)^2} (H_{N-2}^2 + H_{N-2}^{(2)}) (\omega - 1) + O[(\omega - 1)^2].
\] (A.36)

**Appendix B. The conditional distribution $P(T = t | s)$ and its moments**

The probability that the RW has pursued $t$ steps, given that it has visited $s \geq 2$ distinct nodes, is given by
\[
P(T = t | s) = \frac{P_t(S = s)}{\sum_{t'=s}^\infty P_t(S = s)}.
\] (B.1)

Using the generating function $L_s(\omega)$, defined in equation (A.17) this probability can be written in the form
\[
P(T = t | s) = \frac{P_t(S = s)}{L_s(1)}.
\] (B.2)

Inserting $L_s(\omega)$ from equation (A.27) into equation (B.2) we obtain
\[
P(T = t | s) = \left( \frac{c-2}{c-1} \right) \left( 1 - \frac{s}{N} \right) P_t(S = s).
\] (B.3)

The tail distribution $P(T > t | s)$ is given by
\[
P(T > t | s) = \sum_{t'=t+1}^\infty P(T = t | s).
\] (B.4)
The mean of the conditional probability distribution $P(T = t | s)$, for $2 \leq s \leq N - 1$, is given by

$$\mathbb{E}[T|s] = \sum_{t=2}^{\infty} t P(T = t | s). \quad (B.5)$$

Inserting $P(T = t | s)$ from equation (B.2) into equation (B.5), we obtain

$$\mathbb{E}[T|s] = \frac{1}{L_s(1)} \sum_{t=2}^{\infty} t P_t(S = s). \quad (B.6)$$

Expressing the sum on the right-hand side in terms of $L_s(\omega)$, we obtain

$$\mathbb{E}[T|s] = \frac{1}{L_s(\omega)} \frac{\partial}{\partial \omega} L_s(\omega) \bigg|_{\omega = 1}, \quad (B.7)$$

or in a more compact form

$$\mathbb{E}[T|s] = \frac{\partial}{\partial \omega} \ln L_s(\omega) \bigg|_{\omega = 1}. \quad (B.8)$$

Inserting $L_s(\omega)$ from equation (A.27) into equation (B.8), we obtain

$$\mathbb{E}[T|s] = 1 + \left( \frac{c - 1}{c - 2} \right) (N - 2)(H_{N-2} - H_{N-s-1}). \quad (B.9)$$

In the limit of $2 \leq s \ll N$, equation (B.9) can be approximated by

$$\mathbb{E}[T|s] \simeq 1 + \left( \frac{c - 1}{c - 2} \right) (s - 1), \quad (B.10)$$

which is consistent with the results presented in reference [24]. In the opposite limit of $1 \leq N - s \ll N$, we obtain

$$\mathbb{E}[T|s] \simeq 1 + \left( \frac{c - 1}{c - 2} \right) N(\ln N + \gamma - H_{N-s-1}). \quad (B.11)$$

Note that for $s = N$ the expectation value $\mathbb{E}[T|N]$ diverges. This is due to the fact that $s = N$ is an absorbing state, namely once the system reached the state in which the RW has covered all the $N$ nodes it will remain in this state forever.

The variance of the conditional distribution $P(T = t | s)$ is given by

$$\text{Var} [T|s] = \mathbb{E}[T^2|s] - \left( \mathbb{E}[T|s] \right)^2. \quad (B.12)$$

It can be expressed in terms of the generating function $L_s(\omega)$, in the form

$$\text{Var} [T|s] = \mathbb{E}[T|s] + \left. \frac{\partial^2}{\partial \omega^2} \ln L_s(\omega) \right|_{\omega = 1}. \quad (B.13)$$

Carrying out the differentiation of $L_s(\omega)$, which is given by equation (A.20), using equations (A.22)–(A.25), we obtain

$$\text{Var} [T|s] = -N \left( \frac{c - 1}{c - 2} \right) (H_{N-2} - H_{N-s-1}) + \left( \frac{c - 1}{c - 2} \right)^2 N^2 [H_{N-2}^{(2)} - H_{N-s-1}^{(2)}]. \quad (B.14)$$
In the limit of \( 2 \leq s \ll N \) the variance can be simplified to

\[
\text{Var} [T|s] = \frac{c-1}{(c-2)^2} (s-1) + \frac{c(c-1)}{2(c-2)^2} N (s-1)(s+2) + \mathcal{O} \left( \frac{1}{N^2} \right). \tag{B.15}
\]

In the opposite limit of \( N - s \ll N \), it takes the form

\[
\text{Var} [T|s] = \left( \frac{c-1}{c-2} \right)^2 N^2 \zeta(2,N-s) - \left( \frac{c-1}{c-2} \right)^2 N \ln N
+ \left( \frac{c-1}{c-2} \right)^2 \left[ N + \frac{3}{2(c-1)} \right] + \mathcal{O} \left( \frac{1}{N} \right). \tag{B.16}
\]

where \( \zeta(m,n) \) is the Hurwitz zeta function \[89\].

Summarizing the results of this appendix, it was found that in the limit of \( 2 \leq S \ll N \) there is a linear relation between \( \mathbb{E}[T|s] \) and \( s \) and the variance of \( P(T = t|s) \) is small. Thus, in this limit the number of distinct nodes visited by an RW is a good predictor for the elapsed time. This property breaks down for large values of \( s \), where \( 1 \leq N - s \ll N \). In this limit the mean \( \mathbb{E}[T|s] \) saturates and the variance \( \mathcal{V}[T|s] \) becomes very large. As a result, knowing the number of distinct nodes visited by an RW provides little information about the elapsed time \( t \).

**Appendix C. The moments of \( P_t(S = s) \)**

The \( r \)th moment of the distribution \( P_t(S = s) \) at time \( t \geq 2 \) is given by

\[
\langle S^r \rangle_t = \sum_{s=2}^{r} s^r P_t(S = s). \tag{C.1}
\]

Below we calculate the moments of \( P_t(S = s) \), using the generating function \( G_t(x) \), defined by equation (A.2). Taking the derivative of \( G_t(x) \) with respect to \( x \) and setting \( x = 1 \), we obtain

\[
\langle S \rangle_t = \frac{\partial}{\partial x} G_t(x) \bigg|_{x=1}. \tag{C.2}
\]

Taking the second derivative and setting \( x = 1 \) yields

\[
\langle S(S - 1) \rangle_t = \frac{\partial^2}{\partial x^2} G_t(x) \bigg|_{x=1}, \tag{C.3}
\]

where \( \langle S(S - 1) \rangle_t \) is the second factorial moment of \( P_t(S = s) \). In general, taking the \( n \)th derivative and setting \( x = 1 \), we obtain

\[
\langle S(S - 1) \ldots (S - n) \rangle_t = \frac{\partial^n}{\partial x^n} G_t(x) \bigg|_{x=1}, \tag{C.4}
\]

which is the \( n \)th factorial moment. The ordinary moments can be expressed in terms of the factorial moments in the form

\[
\langle S^r \rangle_t = \sum_{n=0}^{r} \binom{r}{n} \langle S(S - 1) \ldots (S - n + 1) \rangle_t. \tag{C.5}
\]
Therefore, the \( r \)th moment is given by
\[
\langle S^r \rangle_t = \sum_{n=0}^{r} \binom{r}{n} \frac{\partial^n}{\partial x^n} G_t(x) \bigg|_{x=1}.
\] (C.6)

For the analysis below, it is useful to define the generating function of the factorial moments, which is given by
\[
K_n(\omega) = \sum_{i=2}^{\infty} \omega^i \langle S(S-1) \ldots (S-n+1) \rangle_t.
\] (C.7)

Using equation (C.4), this generating function can be expressed in the form
\[
K_n(\omega) = \sum_{i=2}^{\infty} \omega^i \frac{\partial^n}{\partial x^n} G_t(x) \bigg|_{x=1}.
\] (C.8)

We also define the moment generating function
\[
M_r(\omega) = \sum_{i=2}^{\infty} \omega^i \langle S^r \rangle_t.
\] (C.9)

Inserting \( \langle S^r \rangle_t \) from equation (C.6) into equation (C.9) and inverting the order of summations, we obtain
\[
M_r(\omega) = \sum_{n=0}^{r} \binom{r}{n} K_n(\omega).
\] (C.10)

The second sum in equation (C.10) is equal to \( K_n(\omega) \), given by equation (C.8). Therefore,
\[
M_r(\omega) = \sum_{n=0}^{r} \binom{r}{n} K_n(\omega).
\] (C.11)

The generating function \( G_t(x) \) can be written as a series expansion around \( x = 1 \), which takes the form
\[
G_t(x) = \sum_{n=2}^{N} \frac{(x-1)^n}{n!} \frac{\partial^n}{\partial x^n} G_t(x) \bigg|_{x=1}.
\] (C.12)

Inserting \( G_t(x) \) from equation (C.12) into equation (A.5), exchanging the order of the summations and using equation (C.8), we obtain
\[
GL(x, \omega) = \sum_{n=2}^{N} \frac{(x-1)^n}{n!} K_n(\omega).
\] (C.13)

Comparing this sum term by term to equation (A.8), one can express the generating function \( K_n(\omega) \) in terms of \( a_n(\omega) \) and \( A(\omega) \). In order to perform such comparison, we need to express
the factor of $x^2$ in equation (A.8) in terms of powers of $x - 1$, namely $x^2 = 1 + 2(x - 1) + (x - 1)^2$. Inserting this equality into equation (A.8) and rearranging terms we obtain

\[ GL(x, \omega) = \frac{\omega^2}{1 - \omega} \left\{ 1 + 2 + \frac{N - 2}{1 + \left( \frac{x - 1}{\omega} \right) N \frac{1}{\omega}} \right\} (x - 1) \]

\[ + \sum_{n=2}^{\infty} [a_n(\omega) + 2a_{n-1}(\omega) + a_{n-2}(\omega)](x - 1)^n \].  \hspace{1cm} (C.14)

Comparing equations (C.13) and (C.14) term by term, we obtain

\[ K_0(\omega) = \frac{\omega^2}{1 - \omega}, \hspace{1cm} (C.15) \]

\[ K_1(\omega) = \frac{\omega^2}{1 - \omega} \left[ 2 + \frac{N - 2}{1 + \left( \frac{x - 1}{\omega} \right) N \frac{1}{\omega}} \right] \]

and

\[ K_n(\omega) = \frac{\omega^2}{1 - \omega} n! [a_n(\omega) + 2a_{n-1}(\omega) + a_{n-2}(\omega)]. \hspace{1cm} (C.16) \]

Going back from the factorial moments to the ordinary moments, we obtain

\[ M_0(\omega) = \frac{\omega^2}{1 - \omega}, \hspace{1cm} (C.17) \]

\[ M_1(\omega) = \frac{\omega^2}{1 - \omega} \left[ 2 + \frac{N - 2}{1 + \left( \frac{x - 1}{\omega} \right) N \frac{1}{\omega}} \right], \]

and

\[ M_r(\omega) = \frac{\omega^2}{1 - \omega} \left[ 2 + \frac{N - 2}{1 + \left( \frac{x - 1}{\omega} \right) N \frac{1}{\omega}} \right] \]

\[ + \frac{\omega^2}{1 - \omega} \sum_{n=2}^{r} \left\{ \frac{r}{n} \right\} n! [a_n(\omega) + 2a_{n-1}(\omega) + a_{n-2}(\omega)], \hspace{1cm} (C.19) \]

for $r \geq 2$. Expanding $M_1(\omega)$ from equation (C.19) in powers of $\omega$, we obtain

\[ M_1(\omega) = \sum_{t=2}^{\infty} \left\{ 2 + (N - 2) \left( 1 - \left[ 1 - \left( \frac{\omega - 1}{\omega} \right) \frac{1}{N} \right] t^{-2} \right) \right\} \omega^t. \hspace{1cm} (C.20) \]

Comparing the coefficients of $\omega^t$ in equations (C.21) and (C.9) term by term, we obtain

\[ \langle S \rangle_t = 2 + (N - 2) \left\{ 1 - \left[ 1 - \left( \frac{\omega - 1}{\omega} \right) \frac{1}{N} \right] t^{-2} \right\}, \hspace{1cm} (C.22) \]
which coincides with previous results [25], obtained using different methods. Inserting \( r = 2 \) in equation (C.20), we obtain

\[
M_2(\omega) = \left( \frac{\omega^2}{1 - \omega} \right) \frac{4 + 2A(\omega) + A(\omega)N[5 + 2A(\omega)N]}{[1 + A(\omega)][1 + 2A(\omega)]}.
\]  

(C.23)

Inserting \( A(\omega) \) from equation (A.10) into equation (C.23), rearranging terms and expanding in powers of \( \omega \), we obtain

\[
M_2(\omega) = \sum_{t=2}^{\infty} \omega^t \left\{ \frac{N^2 - (N - 2)(2N - 1)}{1 - \left( \frac{e - 2}{e - 1} \right) \frac{1}{N}} \right\}^{t-2}
+ (N - 3)(N - 2) \left[ 1 - 2 \left( \frac{e - 2}{e - 1} \right) \frac{1}{N} \right]^{t-2}.
\]  

(C.24)

Therefore, the second moment is given by

\[
\langle S^2 \rangle_t = N^2 - (N - 2)(2N - 1) \left[ 1 - \left( \frac{e - 2}{e - 1} \right) \frac{1}{N} \right]^{t-2}
+ (N - 3)(N - 2) \left[ 1 - 2 \left( \frac{e - 2}{e - 1} \right) \frac{1}{N} \right]^{t-2}.
\]  

(C.25)

Using the results for the first and second moments, we obtain the variance, which is given by

\[
\text{Var}_t(S) = \langle S^2 \rangle_t - (N - 2)^2 \left[ 1 - \left( \frac{e - 2}{e - 1} \right) \frac{1}{N} \right]^{2(t-2)}
+ (N - 3)(N - 2) \left[ 1 - 2 \left( \frac{e - 2}{e - 1} \right) \frac{1}{N} \right]^{2(t-2)}.
\]  

(C.26)

**Appendix D. The generating function of** \( P(T_C = t) \)

Inserting \( s = N \) in equation (A.34), we obtain

\[
P_t(S = N) = (N - 2)! \sum_{v=N-2}^{t-2} (-1)^{v-N} \left( \begin{array}{c} v \cr N - 2 \end{array} \right) \left( \frac{e - 2}{e - 1} \right) \frac{1}{N} \right\}^{v}.
\]  

(D.1)

In equation (33) we identify the relation \( P(T_C \leq t) = P_t(S = s) \). Therefore, the probability mass function of the cover time is given by

\[
P(T_C = t) = P_t(S = N) - P_{t-1}(S = N).
\]  

(D.2)

The generating function of \( P(T_C = t) \) is given by equation (52). Inserting \( P(T_C = t) \) from equation (D.2) into equation (52) and rearranging terms, we obtain

\[
J(\omega) = (1 - \omega) L_N(\omega).
\]  

(D.3)
In the analysis below we are interested in the limit $\omega \to 1^+$. In this limit $A(\omega) \to -\infty$ and $1/A(\omega) \to 0^-$. Thus, to analyze the denominator of equation (A.35) we apply a small $a$ expansion of

$$
\frac{1}{(1-a)_m} = \frac{1}{\prod_{k=1}^m (k-a)} = (a)_m.
$$

Using equation (D.4), the small $a$ expansion of $1/(1-a)_m$ can be obtained from known results for $(a)_n$. It is given by

$$
\frac{1}{(1-a)_m} = \frac{1}{m!} \left[ 1 + H_m a + \frac{1}{2} \left( H_m^2 - H_m^{(2)} \right) a^2 + O(a^3) \right],
$$

where $H_m^{(2)}$ is the $m$th harmonic number of the second order [89]. Applying this expansion to the right-hand side of equation (D.3), we obtain

$$
J(\omega) = 1 + \left[ 2 + \frac{c-1}{c-2} (N-2) H_{N-2} \right] (\omega - 1) + \left[ 1 + \frac{c-1}{c-2} (N-2) H_{N-2} \right] (\omega - 1)^2
$$

$$
+ \frac{1}{2} \left[ \frac{c-1}{c-2} \right]^2 (N-2)^2 \left[ (H_{N-2})^2 + H_{N-2}^{(2)} \right] (\omega - 1)^2 + O((\omega - 1)^3).
$$

Taking derivatives of $J(\omega)$ and setting $\omega = 1$ yields the moments of the distribution of cover times.

**ORCID iDs**

Eytan Katzav https://orcid.org/0000-0001-7555-7717

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