Layer-adapted meshes for weak boundary layers

Hans-Görg Roos, TU Dresden

April 14, 2022

1 Second order problems and linear finite elements

We start with problems of convection-diffusion type and consider

\[-\varepsilon u'' - bu' + cu = f, \quad u(1) = 0, u'(0) = 0,\]  

(1.1)

assuming V-ellipticity of the related bilinear form, \(b > 1\) and \(0 < \varepsilon << 1\).

The boundary condition at \(x = 0\) implies the existence of a weak boundary layer, i.e., we have in a solution decomposition into a smooth part \(S\) and a layer part \(E\)

\[|E^{(k)}| \leq \varepsilon^{1-k} e^{-x/\varepsilon}.\]

It follows for the Sobolev seminorms

\[|E|_1 \leq \varepsilon^{1/2}, \quad |E|_2 \leq \varepsilon^{-1/2}.\]  

(1.2)

Define the \(\varepsilon\)-weighted \(H^1\) norm by

\[\|v\|_\varepsilon := \varepsilon^{1/2}|v|_1 + |v|_0.\]

It was already observed in [8], that now the discretization with linear elements on an equidistant mesh with the mesh size \(H\) leads to a uniform (with respect to \(\varepsilon\)) error estimate in the \(\varepsilon\)-weighted \(H^1\) norm (for the upwind finite difference scheme, see [6])

\[\|u - u_H\|_\varepsilon \leq H.\]  

(1.3)

This estimate is easy to prove. First we get for the interpolation error

\[\varepsilon^{1/2}|E - E^I|_1 \leq \varepsilon^{1/2} H|E|_2 \leq H\]

and

\[|E - E^I|_0 \leq H|E|_1 \leq \varepsilon^{1/2} H.\]  

(1.4)

The factor \(\varepsilon^{1/2}\) in (1.4) allows to estimate the convective term in the error equation.
Next we study problems of reaction-diffusion type
\[ -\varepsilon^2 u'' + cu = f, \quad u'(0) = 0, \quad u'(1) = 0 \]
with \( c > 1 \). Again we have weak layers, now at \( x = 0 \) and at \( x = 1 \). In the energy norm
\[ \|v\|_\varepsilon := \varepsilon|v|_1 + |v|_0 \]
one gets immediately on an equidistant mesh
\[ \|u - u_H\|_\varepsilon \leq \varepsilon^{1/2}H + H^2. \]  
One can also obtain an estimate in the balanced norm (see [9] for a survey)
\[ \|v\|_b := \varepsilon^{1/2}|v|_1 + |v|_0. \]
Here one uses the \( L_2 \) projection for the smooth part. On a uniform mesh the projection is stable in the sense
\[ |\pi S|_1 \leq |S|_1 \]
(see [1]). It follows
\[ |S - \pi S|_1 \leq |S - S^{I}|_1, \]
and finally
\[ \|u - u_H\|_b \leq H. \]  

2 Second order problems and higher order finite elements

For higher order elements the use of a uniform mesh does not lead to uniform convergence. Of course, it is possible to use the same meshes as for strong layers.

But we try to use coarser meshes, especially an equidistant mesh with mesh size \( h \) in \([0, \tau]\) and with mesh size \( H \) in \([\tau, 1]\). We choose for \( k \)-th order elements \( \tau = \varepsilon^{(k-1)/k} \). The fine mesh is given by
\[ x_i = ih, \quad i = 0, 1, \cdots, [1/H] \quad \text{with} \quad h = \alpha H \varepsilon^{(k-1)/k} \]
This choice works because we obtain on the fine mesh
\[ |E - E^I|_1 \leq h^k|E|_{k+1} \leq \varepsilon^{k-1}H^k \varepsilon^{-(k-1)/2} \leq \varepsilon^{-1/2}H^k \]
and
\[ |E - E^I|_0 \leq h^k|E|_k \leq \varepsilon^{1/2}H^k. \]
If the layer part \( E \) in the transition point \( \tau \) is sufficiently small we get
\[ \|u - u_h\|_\varepsilon \leq H^k. \]  
Let us assume \( k = 2 \). Then the smallness condition is satisfied if
\[ \varepsilon \leq H^3 \quad \text{or} \quad \varepsilon e^{-1/(\varepsilon^{1/2})} \leq H^3. \]
Remark 1 The second condition is not very restrictive. For instance, the condition reads
\[ 5 \times 10^{-12} \leq H^3 \quad \text{if} \quad \varepsilon \leq 0.0025. \]
If \( k \) increases, this condition becomes more and more restrictive. This means, our approach makes sense if \( \varepsilon \) is extremely small or \( k \) is only of moderate size.

For the reaction-diffusion problem, we get analogously
\[ \| u - u_h \|_\varepsilon \leq \varepsilon^{1/2} H^k + H^{k+1}. \tag{2.2} \]
Using the approach of [4] it is also possible to prove some result in the balanced norm.

Remark 2 Our mesh is not locally uniform. To get this property, instead of the uniform mesh in \([\tau, 1]\) one could use a graded mesh with
\[ x_{i+1} = (1 + H)x_i \quad \text{for} \quad i \geq \lfloor 1/H \rfloor, \]
following [2]. But then the number of mesh points used depends on \( \ln(1/\varepsilon) \).

3 Fourth order problems and cubic \( C^1 \)-splines

In many fourth order problems typically weak, but no strong layers exist. For finite element methods on layer adapted meshes, see [4][10][12]. We start with a problem of convection-diffusion type:
\[ \varepsilon u^{(4)} + bu''' + L_2 u = f, \quad u(0) = u'(0) = u''(0) = u'(1) = u''(1) = 0. \tag{3.1} \]
Here \( L_2 \) is a linear second order operator and we assume that the bilinear form associated to the full operator is \( V \)-elliptic and \( b > 1 \). Then, it is well known that we have a solution decomposition into a smooth part and a layer part with (see [3][10][11]).
\[ |E^{(k)}| \lesssim \varepsilon^{2-k} e^{-x/\varepsilon}. \]
That means related to the given boundary conditions the layer is very weak.
Now we use the norm
\[ \| v \|_\varepsilon := \varepsilon^{1/2}|v|_2 + |v|_1. \]
If we choose \( \tau = \varepsilon^{1/2} \) and \( h = \alpha \varepsilon^{1/2} H \), we get for the interpolation error \( \eta \) on the fine mesh the estimates
\[ |\eta|_2 \leq h^2 |E|_4 \leq h^2 \varepsilon^{-3/2} \leq \varepsilon^{-1/2} H^2 \]
and
\[ |\eta|_1 \leq h^2 |E|_3 \leq h^2 \varepsilon^{-1/2} \leq \varepsilon^{1/2} H^2. \]
These estimates allow us to prove
\[ \| u - u_h \|_\varepsilon \leq H^2. \tag{3.2} \]
Remark 3 If the given equation is equipped with such boundary conditions that the layer is extremely weak with
\[|E^{(k)}| \leq \varepsilon^{3-k}e^{-x/\varepsilon},\]
we have for the interpolation error on an equidistant mesh
\[|\eta|_2 \leq \varepsilon^{-1/2}H^2 \quad \text{and} \quad |\eta|_1 \leq \varepsilon^{1/2}H^2.\]
This allows to prove uniform convergence if the complete boundary conditions allow to prove V-ellipticity of the related bilinear form.

But if we next study the problem
\[\varepsilon u^{(4)} + bu''' + L_2 u = f, \quad u(0) = u'(0) = u(1) = u'(1) = 0. \tag{3.3}\]
with a weak layer, i.e., with
\[|E^{(k)}| \leq \varepsilon^{1-k}e^{-x/\varepsilon},\]
the situation becomes different. We have for the interpolation error of \(E\) on the fine mesh
\[\varepsilon^{1/2}|\eta|_2 \leq \varepsilon^{1/2}h^2|E|_4 \leq h^2\varepsilon^{-2}.\]
That means, to achieve a second order result, one should use \(h = \alpha\varepsilon H\). This leads to \(\tau = \varepsilon\). But then \(E\) is in \(\tau\) only small enough if \(\varepsilon\) is extremely small. Consequently, for that problem one should prefer a Shishkin type mesh with \(\tau = \tau_0\varepsilon \ln(1/H)\).

Consider finally problems of reaction-diffusion type
\[\varepsilon^2 u^{(4)} + L_2 u = f, \quad u(0) = u''(0) = u(1) = u''(1) = 0. \tag{3.4}\]
Assume ellipticity of \(L_2\). Now we have two layers at \(x = 0\) and at \(x = 1\), and assume, for instance for the layer \(E\) at \(x = 0\)
\[|E^{(k)}| \leq \varepsilon^{2-k}e^{-x/\varepsilon}.\]
Because now we estimate in the norm
\[\|v\|_\varepsilon := \varepsilon|v|_2 + |v|_1,\]
we get with \(\tau = \varepsilon^{1/2}\)
\[\|u - u_h\|_\varepsilon \leq \varepsilon^{1/2}H^2 + H^3. \tag{3.5}\]
Using [4], an estimate in the balanced norm should also be possible.

In the case of a weak layer of the boundary value problem
\[\varepsilon^2 u^{(4)} + L_2 u = f, \quad u(0) = u'(0) = u(1) = u'(1) = 0 \tag{3.6}\]
we have
\[\varepsilon|\eta|_2 \leq \varepsilon h^2|E|_4 \leq h^2\varepsilon^{-3/2}.\]
Consequently, we get for the choice \(\tau = \varepsilon^{3/4}\) the estimate
\[\|u - u_h\|_\varepsilon \leq H^2. \tag{3.7}\]
But we see no possibility to prove a balanced norm estimate.
4 A mixed finite element method for some fourth order problems

Similarly as in [3], we consider a mixed method for the problem

\[ \varepsilon^2 u^{(4)} - bu'' + du = f, \quad u(0) = u'(0) = u(1) = u'(1) = 0. \]  

(4.1)

We assume

\[ d - \frac{1}{2} b'' > \delta > 0, \]

moreover, the existence of a decomposition

\[ u = S + E_1 + E_2 \]

(4.2)

and the corresponding estimate for the layer \( E_2 \) at \( x = 1 \).

Introducing \( w = \varepsilon u'' \), the mixed method is based on:

Find \((u, w) \in H^1_0 \times H^1\)

\[ \varepsilon(u', \phi') + (w, \phi) = 0 \quad \forall \phi \in H^1 \]  

(4.3)

and

\[ (bu', \psi') + (du, \psi) - \varepsilon(w', \psi) = (f, \psi) \quad \forall \psi \in H^1_0. \]  

(4.4)

We have coercivity in the norm

\[ \|(u, w)\|_1^2 := |u|^2 + \|w\|^2_0. \]  

(4.5)

\( u \) has weak layers, the layers of \( w \) are strong. But because in the norm (4.5) only the \( L^2 \) norm of \( w \) appears, we expect that it is possible to use a mesh coarser than a standard Shishkin type mesh used in [3].

Consequently, we consider a mixed finite element method with \( P_k \)-elements for \( u \) and \( w \) to obtain the discrete solution \((u_H, w_H)\) on a special mesh. The mesh is equidistant and fine in \([0, \tau]\) and \([1 - \tau, 1]\), in the remaining part the mesh is equidistant with the mesh size \( H \).

To define \( \tau \), we study first the interpolation error. It is sufficient to consider the error generated by \( E_1 \) on \([0, \tau]\), because on the remaining part of the interval \( E_1 \) is sufficiently small and therefore the \( L^\infty \) stability of the interpolation operator guarantees the same for the interpolation error.

Similarly as in Section 2 we get for \( \tau = \varepsilon^{1-1/(2k)} \) and \( h = \alpha H \varepsilon^{1-1/(2k)} \) for the interpolation error

\[ \|(u - u^I, w - w^I)\|_1 \leq H^k. \]  

(4.6)

The smallness of the layers in the transition points makes our approach useful for \( k = 1, 2 \).

For the study of the discrete error \((\psi_H, \phi_H) = (\pi u - u_H, \pi w - w_H)\) we first let the choice of the interpolation operator \( \pi \) open. Of course, we assume that \( \pi \) has the same approximation error properties as the Lagrange interpolation.
and, consequently, (4.6) holds as well for \( u - \pi u, w - \pi w \). For the discrete error we obtain with \( \eta = \pi u - u \) and \( \zeta = \pi w - w \) (see [3])

\[
\|(\psi_H, \phi_H)\|^2_1 \leq \varepsilon(\eta', \phi_H') + (\zeta, \phi_H) + (b\eta', \psi_H') + (d\eta, \psi_H') - \varepsilon(\zeta', \psi_H').
\] (4.7)

For linear elements we have

\[
((u - u^I)', \phi_H') = 0,
\]

because \( \phi_H' \) is piecewise constant and \( u - u^I \) vanishes in all mesh points. This property simplifies the error estimation.

Therefore, we introduce for \( k \geq 2 \) a new interpolant \( \pi \). This interpolant satisfies on the interval \([x_{i-1}, x_i]\) first \( (\pi v)(x_{i-1}) = v(x_{i-1}) \) and \( (\pi v)(x_i) = v(x_i) \), moreover

\[
\int_{x_{i-1}}^{x_i} (x - x_{i-1})^l \pi v = \int_{x_{i-1}}^{x_i} (x - x_{i-1})^l v \quad \text{for } l = 1, \ldots, k - 1.
\]

Then, the first and the last term in (4.7) vanish, see Lemma 2.66 in [7]. Moreover, the interpolant has the standard approximation properties and is \( L_\infty \) stable. Equation (4.7) reduces to

\[
\|(\psi_H, \phi_H)\|^2_1 \leq (\zeta, \phi_H) + (b\eta', \psi_H') + (d\eta, \psi_H'),
\] (4.8)

and with

\[
|\eta|_1 \leq H^k \quad \text{and} \quad \|\zeta\|_0 \leq H^k
\]

one gets easily

\[
\|((\pi u - u_H, \pi w - w_H))\|_1 \leq H^k \quad \text{and} \quad \|(u - u_H, w - w_H))\|_1 \leq H^k.
\] (4.9)

Analogously one can handle the case of very weak layers of the problem

\[
\varepsilon^2 u^{(4)} - bu'' + du = f, \quad u(0) = u''(0) = u(1) = u''(1) = 0.
\] (4.10)

Then, for linear elements we derive on a uniform mesh

\[
\|(u - u_H, w - w_H))\|_1 \leq H.
\] (4.11)

If \( k \geq 2 \), we choose \( \tau = \varepsilon^{1-3/(2k)} \) and obtain again (4.9).

References

[1] Crouzeix, M., Thomee, V.: The stability in \( L_p \) and \( W^1_p \) of the \( L_2 \)-projection onto finite element function spaces. Math. Comp. 48, 521-532 (1987)

[2] Duran, R.G., Lombardi, A.L.: Finite element approximation of convection-diffusion problems using graded meshes. Appl. Num. Math., 56(2006), 1314-1325
[3] Franz, S., Roos, H.-G.: Robust error estimation in energy and balanced norms for singularly perturbed fourth order problems. Computers and Math. with Appl. 72, 233-247 (2016)

[4] Franz, S., Roos, H.-G.: Error estimates in balanced norms of finite element methods for higher order reaction-diffusion problems. Int. J. of Num. Anal. and Model., 17(2020)532-542

[5] Gartland, E.C.: Graded-mesh difference schemes. Math. Comp. 51(1988), 631-657

[6] Linss, T.: On a convection-diffusion problem with a weak layer. Appl. Math. Comp., 160(2005), 791-795

[7] Roos, H.-G., Stynes, M., Tobiska, L.: Robust numerical methods for singularly perturbed differential equations. Springer 2008

[8] Roos, H.-G., Reibiger, Ch.: Numerical analysis of a system of singularly perturbed convection-diffusion equations related to optimal control. NMTMA, 4(2011), 562-575

[9] Roos, H.-G.: Error estimates in balanced norms of finite element methods on layer-adapted meshes for second order reaction-diffusion problems. in: Z. Huang et al. (eds.), Boundary and Interior Layers, Lecture Notes in Computational Science and Engineering 120, Springer 2017, 1-18.

[10] Sun, G., Stynes, M.: Finite element methods for singularly perturbed high-order elliptic two-point boundary value problems I, IMA J. Num. Anal., 15(1995), 117-139

[11] Sun, G., Stynes, M.: Finite element methods for singularly perturbed high-order elliptic two-point boundary value problems II, IMA J. Num. Anal., 15(1995), 197-219

[12] Xenophontos, Ch.: A parameter robust finite element method for fourth order singularly perturbed problems. Comput. Methods Appl. Math., 17(2017, 337-350