Rigorous results on the bipartite mean-field model

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Abstract
We consider a bipartite mean-field model in which interaction coefficients and magnetic fields depend only on the groups the particles belong to. We rigorously compute the value of the limiting pressure per particle using tail estimation techniques. We study the phase space of the model in the symmetric regime without an external field and when the interaction coefficients within the two groups are identical. Magnetic field perturbations are considered.

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(Some figures may appear in colour only in the online journal)

1. Introduction
The bipartite mean-field model was introduced in the 1950s to reproduce the phase transition of some materials called metamagnets. In particular, in [1–4] a bipartite mean-field model is used to approximate a two-sublattice with nearest-neighbor and next-nearest-neighbor exchanging interactions. The same model has also been used to study the loss of Gibbsianess for a system that evolves according to a Glauber dynamics [5]. Interesting closely related questions can be found in [6].

In recent times different versions of these mean-field models have received renewed attention both at a mathematical level [7] and in the attempt to describe the large-scale behavior of some socio-economic systems [8, 9], assuming that individuals’ decisions depend upon the other decisions.

The investigation of the model introduced in [8] has been pursued from the mathematical point of view in [10]. The existence of the thermodynamic limit of the pressure exploiting a monotonicity condition on the Gibbs state of the Hamiltonian (see [11]) has been shown. The factorization of the correlation functions has been proved for almost every choice of parameters, and the exact solution of the thermodynamic limit is computed whenever the Hamiltonian is a convex function of the magnetizations.
In this paper, we carry out the analysis of the mathematical properties of the bipartite mean-field model. Firstly, we compute the exact solution of the thermodynamic limit. We exploit a tail estimation on the number of configurations that share the same value of the vector of magnetization. This technique is the same used by Talagrand to compute the thermodynamic limit for the Curie–Weiss model [12]. Then we analyze the critical points of the pressure functional associated with a symmetric bipartite mean-field model in the case in which the external field is absent or small. This analysis highlights for which values of parameters the model undergoes a phase transition.

This work is organized as follows. Section 1 introduces the notations and states the main results. Section 2 contains the proofs. The appendices contain the proofs of the lemmas that make the paper self-contained.

2. Definition and statement

We consider a system of $N$ Ising spin variables $\sigma_i$ that can be divided into two subsets $P_1$ and $P_2$ with $P_1 \cap P_2 = \emptyset$, and sizes $|P_i| = N_i$, where $N_1 + N_2 = N$. Spins interact with each other and with an external field according to the mean-field Hamiltonian:

$$H_N(\sigma) = -\frac{1}{2N} \sum_{i,j=1}^{N} J_{ij} \sigma_i \sigma_j - \sum_{i=1}^{N} h_i \sigma_i,$$

where $J_{ij}$ that tunes the mutual interaction between the spins $i$ and $j$ takes values according to the following symmetric matrix:

$$
\begin{pmatrix}
N_1 & \tilde{J}_{12} \\
\tilde{J}^T_{12} & N_2
\end{pmatrix}
$$

where each block $J_{ls}$ has constant elements $J_{ls}$. For $l = s$, $J_{ls}$ is a square matrix, whereas the matrix $J_{ls}$ is rectangular and all entries can be either positive or negative allowing both ferromagnetic and anti-ferromagnetic interactions. The vector field also takes different values depending on the subset the particles belong to as specified by

$$
\begin{pmatrix}
N_1 \\
N_2
\end{pmatrix}
\begin{pmatrix}
h_1 \\
h_2
\end{pmatrix},
$$

where each $h_l$ is a vector of constant elements $h_l$. The joint distribution of a spin configuration $\sigma = (\sigma_1, \ldots, \sigma_N)$ is given by the Boltzmann–Gibbs measure:

$$P_{N, J, h}(\sigma) = \frac{\exp(-\beta H_N(\sigma))}{Z_N(J, h)},$$

where $Z_N(J, h)$ is the partition function

$$Z_N(J, h) = \sum_{\sigma} \exp(-\beta H_N(\sigma)).$$

By introducing the magnetization of a set of spins $A$ as

$$m_A(\sigma) = \frac{1}{|A|} \sum_{i \in A} \sigma_i$$
and indicating by \( m_i(\sigma) \) the magnetization of the set \( i \), and by \( \alpha_i = N_i/N \) the relative size of the set \( i \), we may easily express Hamiltonian (1) as
\[
H_N(\sigma) = -Ng(m_1(\sigma), m_2(\sigma))
\]
where the function \( g \),
\[
g(m_1(\sigma), m_2(\sigma)) = \frac{1}{2} (\alpha_1^2 J_{11} m_1(\sigma)^2 + 2\alpha_1 \alpha_2 J_{12} m_1(\sigma) m_2(\sigma) + \alpha_2^2 J_{22} m_2(\sigma)^2) + \alpha_1 h_1 m_1(\sigma) + \alpha_2 h_2 m_2(\sigma),
\]
depends on the reduced interaction matrix \( J \) and the external field vector \( h \):
\[
J = \begin{pmatrix} J_{11} & J_{12} \\ J_{12} & J_{22} \end{pmatrix}, \quad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.
\]
The existence of the thermodynamic limit of the pressure,
\[
p_N(J, h) = \frac{1}{N} \ln Z_N(J, h),
\]
associated with the model defined by Hamiltonian (1) and distribution (2) is shown in the paper [10] exploiting an existence theorem provided for the mean-field model in [11]. In this paper, we deal with the exact calculation of this limit for all the values of the parameters. **Proposition 1.** Consider the Hamiltonian defined in (1) and the function
\[
f(\mu_1, \mu_2) = \beta g(\mu_1, \mu_2) - \alpha_1 \mathcal{J}(\mu_1) - \alpha_2 \mathcal{J}(\mu_2),
\]
where \( \beta \) is the inverse temperature, \( g \) is given by (4) and
\[
\mathcal{J}(x) = \frac{1}{2} ((1 + x) \ln(1 + x) + (1 - x) \ln(1 - x)).
\]
Given parameters \( J_{11}, J_{12}, J_{22}, h_1, h_2 \) and \( \alpha \), the limit for large \( N \) of the pressure \( p_N(J, h) \) associated with Hamiltonian (1) and defined in (5) is the following:
\[
\lim_{N \to \infty} p_N(J, h) = \ln 2 + \max_{(\mu_1, \mu_2)} f(\mu_1, \mu_2).
\]
Therefore, to compute the exact value of the thermodynamic limit of the pressure \( p_N(J, h) \), we have to maximize the function \( f \) with respect to the variables \( \mu_1 \) and \( \mu_2 \). Differentiating \( f \) we obtain
\[
\frac{\partial f}{\partial \mu_1}(\mu_1, \mu_2) = \beta (\alpha_1^2 J_{11} \mu_1 + \alpha_1 \alpha_2 J_{12} \mu_2 + \alpha_1 h_1) - \alpha_1 \tanh^{-1}(\mu_1)
\]
\[
\frac{\partial f}{\partial \mu_2}(\mu_1, \mu_2) = \beta (\alpha_1 \alpha_2 J_{12} \mu_1 + \alpha_2^2 J_{22} \mu_2 + \alpha_2 h_2) - \alpha_2 \tanh^{-1}(\mu_2).
\]
Thus, the mean-field equations of the model are
\[
\begin{cases} 
\mu_1 = \tanh(\beta (\alpha_1 J_{11} \mu_1 + \alpha_2 J_{12} \mu_2 + h_1)) \\
\mu_2 = \tanh(\beta (\alpha_1 J_{12} \mu_1 + \alpha_2 J_{22} \mu_2 + h_2)).
\end{cases}
\]
We observe that considering the symmetric case of the model, in which \( \alpha = 1/2 \) and \( J_{11} = J_{22} \), and changing the sign of both the parameter of mutual interaction \( J_{12} \) and the first component of the external field \( h_1 \), we obtain a system of mean-field equations symmetric to the previous one with respect to the vertical axis. Therefore, to analyze the critical points of the pressure functional \( f \) in the symmetric case, it suffices to consider only one sign for the parameter \( J_{12} \). In particular, we will study the cases in which \( J_{11} \) and \( J_{12} \) have opposite signs in order to keep the graphical approach introduced in [2] useful to understand the analytical results.
Proposition 2. Consider the Hamiltonian defined in (1) with \( \alpha = 1/2 \), \( J_{11} = J_{22} \) and \( h_1 = h_2 = 0 \). Denoted with \( \lambda_M \) and \( \lambda_m \) respectively the bigger and the smaller eigenvalues of the reduced interaction matrix \( J \) associated with the Hamiltonian, as \( J_{11} \) and \( J_{12} \) have opposite signs, the pressure functional \( f \) given by (6) admits the following structure of critical points.

1. If \( J_{11} > 0 \) and \( J_{12} < 0 \) (\( \lambda_M = J_{11} - J_{12} ; \lambda_m = J_{11} + J_{12} \)).
   
   (a) \( 0 < \beta \leq \frac{1}{\lambda_m} \) : the unique critical point is the origin \((0, 0)\) which is a maximum point.
   
   (b) \( \beta > \frac{1}{\lambda_m} \), \( \frac{1}{\lambda_M} < \beta < \frac{1}{\lambda_m} \), \( \lambda_m > 0 \), \( \lambda_m > 0 \); there are three critical points, 
      \((0, 0)\) and \( \pm(\hat{x}, -\hat{x}) \), where \( \hat{x} \) is a solution of
      \[ \hat{x} = \tanh \left( \frac{\beta \lambda_M}{2} \hat{x} \right). \]  
      (9)

   The origin is an inflection point, while the other two are the maximum points.

   (c) \( \frac{1}{\lambda_m} < \beta < \frac{1}{\lambda_M} \) (\( \lambda_M > 0 \)): there are five critical points, \((0, 0)\), \( \pm(\hat{x}, -\hat{x}) \) and \( \pm(\hat{x}, \hat{x}) \), 
      where \( \hat{x} \) is a solution of (9) and \( \hat{x} \) is a solution of
      \[ \hat{x} = \tanh \left( \frac{\beta \lambda_m}{2} \hat{x} \right), \]  
      (10)

   and \( \hat{\beta} \) is defined as a solution of
      \[ \frac{\hat{\beta} \tanh^{-1}(\sqrt{1 - \hat{\beta}})}{\sqrt{1 - \hat{\beta}}} = \frac{\lambda_m}{\lambda_M}. \]  
      (11)

   The origin is a minimum point, \( \pm(\hat{x}, -\hat{x}) \) are the maximum points and \( \pm(\hat{x}, \hat{x}) \) are the inflection points.

   (d) \( \beta > \frac{1}{\lambda_M} \) (\( \lambda_m > 0 \)): there are nine critical points, \( \pm(\hat{x}, -\hat{x}) \), \( \pm(x_1^1, x_2^1) \), \( \pm(x_1^2, x_2^2) \), \( (0, 0) \), where \( \hat{x} \) is a solution of (9), \( \hat{x} \) is a solution of (10), while for the other four points there is no simplified description. 

   The origin is a minimum point, and \( \pm(\hat{x}, -\hat{x}) \) and \( \pm(\hat{x}, \hat{x}) \) are the maximum points. The other four cannot be the maximum points.

2. If \( J_{11} < 0 \) and \( J_{12} > |J_{11}| \) (\( \lambda_M = J_{11} + J_{12} > 0 ; \lambda_m = J_{11} - J_{12} < 0 \)).
   
   (a) \( 0 < \beta < \frac{1}{\lambda_m} \) : the unique critical point is the origin \((0, 0)\) which is a maximum point.
   
   (b) \( \beta > \frac{1}{\lambda_m} : there are three critical points, \((0, 0)\) and \( \pm(\hat{x}, \hat{x}) \), where \( \hat{x} \) is a solution of (9). The origin is an inflection point and \( \pm(\hat{x}, \hat{x}) \) are the maximum points.

3. If \( J_{11} < 0 \) and \( 0 < J_{12} < |J_{11}| \) (\( \lambda_M = J_{11} + J_{12} < 0 ; \lambda_m = J_{11} - J_{12} < 0 \)).
   
   The unique critical point is the origin \((0, 0)\) which is a maximum point.

Corollary 1. Consider the Hamiltonian defined in (1) with \( \alpha = 1/2 \), \( J_{11} = J_{22} \) and \( h_1 = h_2 = 0 \). If \( J_{11} \) and \( J_{12} \) have the same signs, the pressure functional \( f \) given by (6) admits a structure of critical points symmetric with respect to the vertical axis to those presented in proposition 1.

The corollary follows from proposition 2 and the properties of symmetry previously exposed.

Proposition 3. Consider the Hamiltonian defined in (1) with \( \alpha = 1/2 \), \( J_{11} = J_{22} > 0 \) and \( J_{12} < 0 \).

1. As \( h_1 = h_2 = 0 \), the pressure of the system is given by
   \[ \lim_{N \to \infty} \frac{p_N(J, 0)}{N} = \begin{cases} \ln 2 + f(0, 0) = \ln 2 & \text{when } t \geq 1 \\ \ln 2 + f(\hat{x}(t), -\hat{x}(t)) & \text{when } t < 1 \end{cases} \]

   where \( f \) is given by (6), \( \hat{x} \) is a solution of (9) and \( t = 2(\beta \lambda_M)^{-1} \), with \( \lambda_M \) being defined as in the previous proposition.
The application of an infinitesimal field \( \mathbf{h} = (h_1, h_2) \) with \( h_1 \neq h_2 \) selects the state such that the scalar product between the state and the field is positive; a field with equal components retains the symmetry of the system and does not allow us to select a preferred status.

For all other possible combinations of the parameters \( J_{11} \) and \( J_{12} \), similar results can be obtained simply by considering the model’s symmetry, proposition 2 and its corollary.

We observe that as the external field is away and the bigger eigenvalue of the reduced interaction matrix is positive, at the rescaled temperature \( t = 2(\beta \lambda_M)^{-1} = 1 \) the system undergoes a transition phase from zero to non-zero magnetization.

3. Proofs

Proof of proposition 1. We compute the exact solution of the thermodynamic limit exploiting a tail estimation on the number of configurations that share the same vector of the magnetization. In this way we obtain a lower and an upper bounds for the partition function that converge to the same value as \( N \to \infty \). This technique is used by Talagrand in [12] to compute the thermodynamic limit for the Curie–Weiss model.

Denoted with \( \sigma_l \) the configuration of the spins of the set \( P_l \) and:

\[
A_{\mu_l} = \text{card}(\sigma_l \in \Omega_{N_l} | m_l(\sigma) = \mu_l),
\]

we can write

\[
Z_N(J, \mathbf{h}) = \sum_{\mu} A_{\mu_1} A_{\mu_2} \exp(\beta N g(\mu_1, \mu_2)).
\]

Lemma 1. Consider the set \( \Omega_{N_l} = \{-1, 1\}^{N_l} \) of all possible configuration \( \sigma_l \). Let \( A_{\mu_l} \) be a positive number defined by (12). Then the following inequality holds:

\[
\frac{1}{C} 2^{N_l} \exp(-N_l I(\mu_1)) \leq A_{\mu_l} \leq 2^{N_l} \exp(-N_l I(\mu_1)), \tag{13}
\]

where \( C \) is a constant and \( I \) is given by (7).

See appendix A for the proof. This lemma allows us to bound the partition function in the following way:

\[
\frac{1}{C} 2^{N_l} \sum_{\mu} \exp(\beta N g(\mu) - N_l I(\mu_1) - N_2 I(\mu_2)) \leq Z_N(J, \mathbf{h}) \leq 2^{N_l} \sum_{\mu} \exp(\beta N g(\mu) - N_l I(\mu_1) - N_2 I(\mu_2)).
\]

Then, since a sum of positive elements is always greater than its addends and always less than the largest addend times the number of elements, we have

\[
\frac{1}{C} 2^{N_l} \exp\left(N \max_{\mu} f(\mu)\right) \leq Z_N(J, \mathbf{h}) \leq 2^{N_l} (N_1 + 1)(N_2 + 1) \exp\left(N \max_{\mu} f(\mu)\right),
\]

where the function \( f \) is defined in (6). Hence for the pressure (5) we have

\[
\ln 2 - \frac{1}{N} \left( \ln C + \frac{1}{2} \ln N_1 N_2 \right) + \max_{\mu} f(\mu) \leq p_N(J, \mathbf{h}) \leq \ln 2 + \frac{1}{N} \ln((N_1 + 1)(N_2 + 1)) + \max_{\mu} f(\mu).
\]

Taking the limit for \( N \to \infty \), proposition 1 is proved.
Proof of proposition 2. To study the space phase of the model defined by Hamiltonian (1) under the hypothesis that \( a = 1/2, J_{11} = J_{22}, h_1 = h_2 = 0 \) and the signs of \( J_{11} \) and \( J_{12} \) are opposite, we rescale the model’s parameter with respect to the maximal eigenvalue \( \lambda_M \) of the reduced interaction matrix \( J \) in the following way:

\[
a := \frac{J_{11}}{|\lambda_M|}; \quad b := \frac{J_{12}}{|\lambda_M|}; \quad t := \frac{2}{\beta |\lambda_M|}.
\]

(14)

In this way, the system of mean-field equations (8) becomes:

\[
\begin{aligned}
x_1 &= \tanh \left[ \frac{1}{t} (ax_1 + bx_2) \right] \\
x_2 &= \tanh \left[ \frac{1}{t} (bx_1 + ax_2) \right].
\end{aligned}
\]

By inverting the hyperbolic tangent in the two equations, we can write \( x_1 \) as a function of \( x_2 \), and vice versa \( x_2 \) as a function of \( x_1 \). Therefore, when \( b \neq 0 \) (that is the model does not degenerate toward two Curie–Weiss models), we can rewrite the equations in the following fashion:

\[
\begin{aligned}
x_2 &= \frac{1}{b} (-ax_1 + t \tanh^{-1}(x_1)) \\
x_1 &= \frac{1}{b} (-ax_2 + t \tanh^{-1}(x_2)).
\end{aligned}
\]

Such a system lends itself to a graphic resolution. In fact, the points that verify simultaneously the two equations are the intersections of curves \( \gamma_1 \) and \( \gamma_2 \), respectively, the graphs of the functions \( x_2(x_1) \) and \( x_1(x_2) \); \( \gamma_2 \) is the symmetric curve of \( \gamma_1 \) with respect to both the bisectors of the Cartesian plane. In particular, we will plot the curve \( \gamma_1 \) by studying the function \( x_2(x_1) \) and the curve \( \gamma_2 \) by symmetry. Before starting to analyze the different cases as the interaction parameters and the temperature change, we observe that \( x_2(x_1) \) is an odd function in the interval \((-1, 1)\); it diverges as \( x_1 \to \pm 1 \) depending on the sign of \( b \) and its derivatives are

\[
\begin{aligned}
x'_2(x_1) &= \frac{1}{b} \left( \frac{t}{1-x_1^2} - a \right); \quad x'_2(0) = \frac{1}{b} (t - a); \\
x''_2(x_1) &= \frac{2t}{b} \frac{x_1}{(1-x_1^2)^2}.
\end{aligned}
\]

(15)

We are now ready to analyze in detail the different cases.

1. \( J_{11} > 0 \) and \( J_{12} < 0 \).

The maximal eigenvalue of the reduced interaction matrix is \( \lambda_M = J_{11} - J_{12} \); thus, from (14) we have \( 0 \leq a < 1, -1 \leq b < 0 \) and \( a - b = 1 \). In particular, since \( b \) is negative, the function \( x_2(x_1) \) tends to \(+\infty\) as \( x_1 \to -1 \) and to \(-\infty\) as \( x_1 \to 1 \), and the sign of its second derivative is always opposite to that of the variable.

(a) \( t > 1 \) (i.e. \( 0 < \beta < \frac{2}{\lambda_M} \)). From (15), we get that the function \( x_2(x_1) \) is monotonically decreasing, and the value of its first derivatives in the origin is less than \(-1\). Thus, the curve \( \gamma_1 \) lies above the bisector of the second and fourth quadrants when \( x_1 < 0 \) and over the bisector when \( x_1 > 0 \); the symmetric curve \( \gamma_2 \) lies over the bisector when \( x_1 < 0 \) and above when \( x_1 > 0 \); therefore, the unique intersection is at the origin (figure 1).

In order to understand what type of critical point is the origin, we consider the function \( f \) defined in (6); we rescale it as follows (14):

\[
f(x_1, x_2) = \frac{1}{2} \left( \frac{t}{2} \left( x_1^2 + x_2^2 \right) + bx_1x_2 \right) - \mathcal{F}(x_1) - \mathcal{F}(x_2);
\]
we compute its second derivatives,
\[
\frac{\partial^2 f}{\partial x_1^2}(x_1, x_2) = \frac{1}{2t} \left( a - \frac{t}{1 - x_1^2} \right)
\]
\[
\frac{\partial^2 f}{\partial x_2^2}(x_1, x_2) = \frac{1}{2t} \left( a - \frac{t}{1 - x_2^2} \right)
\]
\[
\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2) = \frac{b}{2t},
\]
and the determinant of its Hessian matrix:
\[
H_f(x_1, x_2) = \frac{1}{4t^2} \left( \left( a - \frac{t}{1 - x_1^2} \right) \left( a - \frac{t}{1 - x_2^2} \right) - b^2 \right). \tag{16}
\]
In the origin, we have
\[
H_f(0, 0) = \frac{1}{4t^2} ((a - t)^2 - b^2) = \frac{1}{4t^2} ((b + 1 - t)^2 - b^2)
\]
\[
= \frac{1}{4t^2} (t - 1)(t - 1 - 2b)
\]
\[
\frac{\partial^2 f}{\partial x_1^2}(0, 0) = \frac{1}{2t} (a - t).
\]
Thus, since \( t > 1, 0 \leq a < 1 \) and \(-1 \leq b < 0\), the origin is a maximum point of \( f \).

(b) \( t = 1 \) (i.e. \( \beta = \frac{2}{\lambda^2} \)). The function \( x_2(x_1) \) is still monotonically decreasing, and the value of its first derivatives in the origin is equal to \(-1\). Therefore, the positions of the curves \( \gamma_1 \) and \( \gamma_2 \) with respect to the bisector of the second and fourth quadrants are the same as those of the previous case. And so there is only one intersection between them at the origin. The
The determinant of the Hessian matrix (16) at this point is equal to zero. To understand what type of critical point is the origin we consider the following change of variable that diagonalizes the reduced interaction matrix:

$$
\begin{align*}
X &= \frac{x_1 + x_2}{2} \\
Y &= \frac{x_2 - x_1}{2}.
\end{align*}
$$

The Taylor expansion of the function $f$ expressed in the new variables ensures that such a point is a maximum point. (For the details see appendix B).

(c) $a \leq t < 1$ (i.e. $\frac{1}{\sqrt{b^2}} < \frac{1}{\sqrt{b^2}}$). The function $x_2(x_1)$ is still decreasing but its derivative in the origin (15) is larger than $-1$; thus, the curve $\gamma_1$ intersects the bisector of the second and fourth quadrants (and therefore the curve $\gamma_2$) at $(0, 0)$ and at other two points symmetric with respect to the origin (figure 2), $(-\tilde{x}, \tilde{x})$ and $(\tilde{x}, -\tilde{x})$, where $\tilde{x}$ is the positive solution of the equation $v(x) = 0$:

$$
v(x) = t \tanh^{-1}(x) - x.
$$

Since the function $v$, given by (18), intersects the horizontal axis at the origin, reaches its minimum in $x = \sqrt{1-t}$ and goes to infinity for large $x$, it follows

$$
\tilde{x} > \sqrt{1-t}.
$$

This inequality turns to be useful to evaluate the nature of the critical points $\pm(\tilde{x}, -\tilde{x})$. On these points, the determinant of the Hessian matrix of the function $f$ (16) is

$$
H_f(-\tilde{x}, \tilde{x}) = H_f(\tilde{x}, -\tilde{x}) = \frac{1}{4t^2} \left( \left( a - \frac{t}{1 - \tilde{x}} \right)^2 - b^2 \right).
$$

Figure 2. $t = 0.75, b = -0.8$. The blue curve is $\gamma_1$, while the red is $\gamma_2$. 

-1.0 -0.8 -0.6 -0.4 -0.2 0.2 0.4 0.6 0.8 1
-1 \(x_2\) \(x_1\)
Since in the considered range of temperature the function on the rhs of (20) is monotonically increasing for positive \(x\), by inequality (19) we have

\[
H_f(-\hat{x}, \hat{x}) = H_f(\hat{x}, -\hat{x}) > \frac{1}{4t^2} \left( \left( a - \frac{t}{1 - (1 - t)} \right)^2 - b^2 \right) = \frac{1}{4t^2}(a - 1)^2 - b^2 = 0.
\]

On the other hand, the first term of the Hessian matrix in \(\pm(\hat{x}, -\hat{x})\) is a decreasing function and thus

\[
\frac{\partial^2 f}{\partial x^2}(-\hat{x}, \hat{x}) = \frac{\partial^2 f}{\partial x^2}(\hat{x}, -\hat{x}) = \frac{1}{2t} \left( a - \frac{t}{1 - \hat{x}^2} \right) < \frac{1}{2t}(a - 1) < 0.
\]

Then, \(\pm(\hat{x}, -\hat{x})\) are the maximum points. On the contrary, the origin is an inflection point because the determinant of the Hessian matrix (16) computed in such a point for the considered range of temperature is negative.

(d) \(2a - 1 < t < a\) with \(2a - 1 > 0\) or \(0 < t < a\) (i.e. \(\lambda_m > 0\) and \(\frac{\lambda_m}{\beta} < \beta < \frac{1}{2}\) or \(\lambda_m < 0\) and \(\beta > \frac{1}{2}\)). The function \(x_2(x_1)\) has a minimum point in \(x_m = -\sqrt{1 - t/a}\) and a maximum point in \(x_M = \sqrt{1 - t/a}\). Therefore, the curves \(\gamma_1\) and \(\gamma_2\) intersect each other at at least three points: at the origin and \(\pm(\hat{x}, -\hat{x})\), where \(\hat{x}\) is again the positive solution of \(v(x) = 0\), with \(v\) defined in (18). However, since at the origin the first derivative of \(x_2(x_1)\) is positive and less than 1, \(\gamma_1\) lies above the bisector of the first and third quadrants when \(x_1 < 0\) and over when \(x_1 > 0\). Instead, \(\gamma_2\) lies over the bisector when \(x_1 < 0\) and above when \(x_1 > 0\). This scenario excludes the possibility of additional intersections between the two curves (figure 3).

Again \(\pm(\hat{x}, -\hat{x})\) are the maximum points and the origin is an inflection point.

(e) \(t = 2a - 1\) with \(2a - 1 > 0\) (i.e. \(\lambda_m > 0\) and \(\beta = \frac{1}{2}\)). In this case, the only difference from the previous one is the value of the first derivative of the function \(x_2(x_1)\) at the origin, which now is exactly equal to 1. This does not imply any qualitative change in the relative positions of the curves \(\gamma_1\) and \(\gamma_2\) that continue to intersect at the points \((-\hat{x}, \hat{x})\), \(0, 0\) and \((\hat{x}, -\hat{x})\). The analysis of the Hessian matrix of the function \(f\) ensures that \(\pm(\hat{x}, -\hat{x})\) are still the maximum points but does not give information about the nature of the origin. Considering again the change of variable (17) we can conclude that the origin is an inflection point (see appendix B).

(f) \(\tilde{t} < t < 2a - 1\) where \(\tilde{t}\) is a solution of \(\tilde{t}\tan^{-1}(\sqrt{1 - \tilde{t}}) - \sqrt{1 - \tilde{t}} = 2b\sqrt{1 - \tilde{t}}\) (i.e. \(\lambda_m > 0\) and \(\frac{\lambda_m}{\beta} < \beta < \frac{\lambda_m}{\beta_{\lambda_M}}\) where \(\beta\) is a solution of (11). The function \(x_2(x_1)\) has the same intervals of monotony of the previous case, but the value of its first derivatives in the origin is larger than 1. This implies that it intersects the bisector of the first and third quadrants at two other points beyond the origin (figure 4), \((-\hat{x}, -\hat{x})\) and \((\hat{x}, \hat{x})\), where \(\hat{x}\) is the positive solution of the equation \(v(x) = 2bx\), where \(v\) is the function defined in (18). Thus, the curves \(\gamma_1\) and \(\gamma_2\) intersect each other at least at five points: \((0, 0)\), \(\pm(\hat{x}, -\hat{x})\) and \(\pm(\hat{x}, \hat{x})\). The graphical representation presented in [2] excludes the possibility of further solutions that will appear as \(t < \tilde{t}\).

The computations made on the Hessian matrix ensure that the origin is a minimum point and \(\pm(\hat{x}, -\hat{x})\) are the maximum points. The new solutions \(\pm(\hat{x}, \hat{x})\) are the inflection points because at these points the determinant of the Hessian matrix of the function \(f\) can be expressed
in the following way:

\[ H_f(\hat{x}, \hat{x}) = \frac{1}{4t^2} \left( \left( \frac{a - t}{1 - \hat{x}^2} \right)^2 - b^2 \right) = \frac{b^2}{4t^2} (\hat{x}'(\hat{x}))^2 - 1, \]

where \(-1 \leq (\hat{x}'(\hat{x})) < 1\) in the considered range of temperature. To prove this statement we consider the inequality

\[ \hat{x} > \sqrt{1 - \frac{t}{1 + 2b}} \]

true for the solutions of the equation \(v(x) = 2bx\) for the same reason for which the inequality (19) is true for the solution of \(v(x) = 0\). Since

\[ x_1^2 \left( \sqrt{1 - \frac{t}{1 + 2b}} \right) = 1 \]

and the function \(x_1^2(x_1)\) is monotonically decreasing for positive values of the abscissa, we have that \(x_1^2(\hat{x}) < 1\). In particular, the value of this derivative, initially positive, decreases as the temperature \(t\) decreases till the value \(-1\), that is, it reaches for \(t = \hat{t}\). At this temperature, the determinant of the Hessian matrix computed in \(\pm(\hat{x}, \hat{x})\) is equal to zero. Nevertheless, without need to resort to the diagonalization of the reduced interaction matrix, we can say that these points, equal to \(\pm(\sqrt{1 - \hat{t}}, \sqrt{1 - \hat{t}})\), are the inflection points. In fact,

\[ \frac{\partial^3 f}{\partial x_1^3}(\pm \sqrt{1 - \hat{t}}, \pm \sqrt{1 - \hat{t}}) = \frac{\partial^3 f}{\partial x_2^3}(\pm \sqrt{1 - \hat{t}}, \pm \sqrt{1 - \hat{t}}) = -\frac{\sqrt{1 - \hat{t}}}{\hat{t}^2} \neq 0 \]
and the other partial derivatives of the third order of the function $f$ are equal to zero.

(g) $0 < t < \bar{t}$ (i.e. $\lambda_m > 0$ and $\beta > \frac{1}{\beta_0}$). The graphical representation presented in [2] shows that as the temperature decreases and reaches below the value $\bar{t}$ a further solution $(x_1^*, x_2^*)$ of the mean-field equations appears, with $x_2^* > x_1^* > 0$; the other three solutions are generated by the symmetry with respect to both the bisectors (figure 5). On these points it is not always possible to detect the sign of the determinant of the Hessian matrix. In any case, since they are linked to a maximum point by a monotonic piece of one of the two curves $\gamma_1$ and $\gamma_2$, we can exclude that they are in turn the maximum points. (For further details see [10].)

When $t$ is small enough, it can be shown analytically that they are inflection points.

As in the previous case, the origin is a minimum point and $\pm (\hat{x}, -\hat{x})$ are the maximum points; in contrast, $\pm (\bar{x}, \bar{x})$ becomes the maximum points. In fact, since $x_2' (\hat{x}) < -1$ as $t < \bar{t}$, the determinant of the Hessian matrix of $f$ on these points is positive and

$$\frac{\partial^2 f}{\partial x_1^2} (\hat{x}, \hat{x}) = \frac{\partial^2 f}{\partial x_1^2} (-\hat{x}, -\hat{x}) = -\frac{b}{2t} x_2'(\hat{x}) < 0.$$

2. $J_{11} < 0$ and $J_{12} > |J_{11}|$

The maximal eigenvalue of the reduced interaction matrix is $\lambda_M = J_{11} + J_{12}$; thus, from (14) we have $a < 0$, $b > 1$ and $a + b = 1$. In particular, since $b$ is positive, the function $x_2(x_1)$ tends to $-\infty$ as $x_1 \to -1$ and to $+\infty$ as $x_1 \to 1$ and the sign of the second derivative of $x_2(x_1)$ is always the same as that of the variable. Moreover, the signs of $a$ and $b$ ensure that the function $x_2(x_1)$ is always monotonically increasing. Therefore, we can distinguish only two scenarios as follows.

(a) $t \geq 1$ (i.e. $0 < \beta \leq \frac{1}{\beta_0}$). The value of the first derivative of $x_2(x_1)$ computed in the origin is larger or equal to 1. Thus, $\gamma_1$ lies over the bisector of the first and third quadrants when $x_1 < 0$ and above when $x_1 > 0$. Therefore, since $\gamma_2$ is symmetric to $\gamma_1$ with respect to
this bisector, the curves intersect each other once at the origin. Analyzing the Hessian matrix of the function \( f \) on this point for \( t > 1 \), we have

\[
H_f(0, 0) = \frac{1}{4t^2} (t - 1)(t - 1 + 2b) > 0
\]

\[
\frac{\partial^2 f}{\partial x_1^2}(0, 0) = \frac{1}{2t} (a - t) < 0.
\]

Thus \((0, 0)\) is a maximum point.

As \( t = 1 \) at the origin, the determinant of the Hessian matrix of the function \( f \) is equal to zero; however, considering the change of variable (17) and the Taylor expansion of the function \( f \) we get that such a point is a maximum point (see appendix B).

(b) \( 0 < t < 1 \) (i.e., \( \beta > \frac{\lambda}{M} \)). The value of the first derivative of \( x_2(x_1) \) at the origin is less than 1, and the curves \( \gamma_1 \) and \( \gamma_2 \) intersect each other at three points: at \((0, 0)\) and at other two points symmetric with respect to the origin \((-\tilde{x}, -\tilde{x})\), \((\tilde{x}, \tilde{x})\), where \( \tilde{x} \) is the positive solution of the equation \( v(x) = 0 \).

By inequalities (19) we have

\[
H_f(-\tilde{x}, -\tilde{x}) = H_f(\tilde{x}, \tilde{x}) = \frac{1}{4t^2} \left( \left( \frac{t}{1 - (1 - t) - a} \right)^2 - b^2 \right)
\]

\[
= \frac{1}{4t^2} \left( (1 - a)^2 - b^2 \right) = 0 \quad \text{and}
\]

\[
\frac{\partial^2 f}{\partial x^2} (-\tilde{x}, -\tilde{x}) = \frac{\partial^2 f}{\partial x^2} (\tilde{x}, \tilde{x}) = \frac{1}{2t} \left( a - \frac{t}{1 - \tilde{x}^2} \right)
\]

\[
= \frac{1}{2t} (a - 1) < 0.
\]
Therefore, ±(̅x, ̅t) are the maximum points. On the other hand, since
\[ H_f(0, 0) = \frac{1}{4t^2}(t - 1)(t - 1 + 2b) < 0, \]
the origin is an inflection point.

Therefore, by (14) we have \( a < -1, b > 0 \) and \( a + b = -1 \). Since the signs of parameters \( a \) and \( b \) are the same as those of the previous case, the comments made about the limit behavior, the monotony and the concavity of the function \( x_2(x_1) \) are still valid. Furthermore, since \( a > -1 \), the value of the first derivative of \( x_2(x_1) \) computed in the origin is larger than 1 for all \( t > 0 \).

As we have seen before, this implies that the curves \( γ_1 \) and \( γ_2 \) intersect each other only at the origin. Since
\[ H_f(0, 0) = \frac{1}{4t^2}(t + 1)(t + 2b + 1) > 0 \]
\[ \frac{\partial^2 f}{\partial x_1^2}(0, 0) = \frac{1}{2t}(a - t) < 0, \]
such a point is a maximum point. \( \Box \)

**Proof of proposition 3.**

(1) If \( 2a - 1 < 0 \), then the statement follows directly from proposition 2. If \( 2a - 1 > 0 \), then the same is true only for \( t > \hat{t} \). For \( 0 < t < \hat{t} \), the function \( f \) admits four maximum points: \( ±(\hat{x}, -\hat{t}) \) and \( ±(\tilde{x}, \tilde{t}) \); therefore, to find what is the absolute maximum between them we have to compare the values of the function on these points. For this purpose, it is more convenient to consider the function \( P(x_1, x_2) = tf(x_1, x_2) \):
\[ P(x_1, x_2) = \frac{1}{4}a(x_1^2 + x_2^2) + \frac{1}{2}bx_1x_2 - \frac{t}{2}(\mathcal{F}(x_1) + \mathcal{F}(x_2)). \]

Since the function \( P \) is pair, we choose to compute the value it assumes at the points with positive abscissa, \((\tilde{x}, -\tilde{t})\) and \((\hat{x}, \hat{t})\), where \( \tilde{x} \) and \( \hat{x} \) are positive:
\[ P(\tilde{x}, -\tilde{t}) = \frac{1}{2}a\tilde{x}^2 - \frac{1}{2}b\tilde{t}^2 - t\mathcal{F}(\tilde{x}) = \frac{1}{2}\tilde{x}^2 - t\mathcal{F}(\tilde{x}) \]
\[ P(\hat{x}, \hat{t}) = \frac{1}{2}a\hat{x}^2 + \frac{1}{2}b\hat{t}^2 - t\mathcal{F}(\hat{x}) = \frac{1}{2}(a + b)\hat{x}^2 - t\mathcal{F}(\hat{x}). \]

We observe that on the points \((\tilde{x}, -\tilde{t})\) and \((\hat{x}, \hat{t})\), the first and the second derivatives with respect to the temperature of the function \( P \) can be written as
\[ \frac{dP(M(t))}{dt} = -\mathcal{F}(x_M(t)); \]
\[ \frac{d^2 P(M(t))}{dt^2} = -\mathcal{F}(x_M(t))x_M'(t), \]
where \( M(t) \) denotes in a generic way one of the two maximum points and \( x_M(t) \) is its abscissa. Since the function \( \mathcal{F} \) is pair, convex and monotonically increasing for positive abscissa, and \( x_M(t) \) is decreasing, the function \( P \) is decreasing and concave along the maximum points. As the temperature tends to zero, both \( \tilde{x} \) and \( \hat{x} \) tend to 1; therefore
\[ \lim_{t \to 0} P(\tilde{x}, -\tilde{t}) = \frac{1}{2}; \]
\[ \lim_{t \to 0} P(\hat{x}, \hat{t}) = \frac{2a - 1}{2}. \]

Since \( 0 < a < 1 \), we obtain that as \( t \to 0 \) the pressure reaches the higher value in \((\tilde{x}, -\tilde{t})\).

Also, as \( t = \hat{t} \), it is easy to show that
\[ P(\hat{x}, \hat{t}) < P(\tilde{x}, -\tilde{t}). \]

Thus, since \( \tilde{x}(t) > \hat{x}(t) \) as \( 0 < t < \hat{t} \), we conclude that
\[ P(\tilde{x}, -\tilde{t}, t) > P(\hat{x}, \hat{t}, t) \quad \text{for} \quad 0 < t < \hat{t}. \]
(2) If we apply an infinitesimal field \( \mathbf{h} = (h_1, h_2) \) to the considered system, then the maximum points of the function \( f \) remain around the points that were maximum points when the field was away. We are interested in showing the field’s ability to select a preferred state; thus, we omit the case \( t \gg 1 \) in which there is only one possible state. In contrast, for other temperature values the function \( f \) admits at least two maximum points. The gap between the values that the function \( f \) assumes on the points \( \pm (\tilde{x}, -\tilde{x}) \) and \( \pm (\tilde{x}, \tilde{x}) \) in the absence of the field ensures that when the field is infinitesimal, the maximum points close to \( \pm (\tilde{x}, \tilde{x}) \) cannot be the effective state of the system. In order to understand which of the two remaining points is the preferred one, we distinguish three cases.

If the field components are equal, \( \mathbf{h} = (h_1, h_1) \), then the mean-field equations and their solutions retain the symmetry with respect to the bisector of the first and third quadrants that they presented as the field was away. Therefore, for each temperature value the pressure computed on the two global maximum points is the same and the field \( \mathbf{h} \) is not able to select the preferred state between them.

If the field components are opposite, \( \mathbf{h} = (h_1, h_2) \) with \( |h_1| \neq |h_2| \). Decomposing the field along the eigenvectors of the reduced interaction matrix \( \mathbf{J} \):

\[
(h_1, h_2) = (h', -h') + (h'', h'''),
\]

we apply the field components in two steps. Firstly, considering only \( (h', -h') \), as we have seen above, we obtain two solutions \( s^{(1)} = (\tilde{x}^{(1)}, -\tilde{x}^{(1)}) \) and \( s^{(2)} \) respectively close to \( (\tilde{x}, -\tilde{x}) \) and \( (-\tilde{x}, \tilde{x}) \). The subsequent application of \( (h'', h''') \) causes an infinitesimal shift of \( s^{(1)} \) and \( s^{(2)} \) in this direction. We look for the coordinates of \( s^{(1)} \) in the form

\[
\begin{align*}
x_1^{(1)}(t) &= \tilde{x}^{(1)}(t) + x_1'(t) \\
x_2^{(1)}(t) &= -\tilde{x}^{(1)}(t) + x_2'(t),
\end{align*}
\]

where \( x_1' \) is the infinitesimal shift in the diagonal direction. From the mean-field equation, we obtain

\[
\begin{align*}
x_1'' &= \tanh \left[ \frac{1}{t} \left( \tilde{x}^{(1)} + (a + b)x_1'' + h'' \right) \right] \\
-\tilde{x}^{(1)} + x_2'' &= \tanh \left[ \frac{1}{t} \left( -\tilde{x}^{(1)} + (a + b)x_2'' + h''' \right) \right].
\end{align*}
\]

Developing the hyperbolic tangent and neglecting the infinitesimal terms of second order of the type \( x''h'' \), we obtain two equations both equal to the following:

\[
x_1'' = (1 - (\tilde{x}^{(1)})^2) \tanh \left[ \frac{a + b}{t} x_1'' + \frac{h''}{t} \right]
\]

whose principal solution is those that has the same sign of \( h'' \). Since the same argument applies to the second solution, we conclude that the field \( \mathbf{h} = (h_1, h_2) \) with \( |h_1| \neq |h_2| \) moves \( \pm (\tilde{x}, -\tilde{x}) \) in two solutions which remain on opposite half-plane with respect to
the bisector of the first and third quadrants. Furthermore, the field ability to break the symmetry of the function $f$ ensures that the absolute maximum point is the one whose scalar product with the field is positive.

4. Comments

In this paper, we have thoroughly investigated the behavior of the symmetric bipartite mean-field model as the external field is away or infinitesimal. In particular, by studying the nature of the critical points of the pressure functional we have highlighted for which values of parameters the model undergoes a phase transition. In addition to being interesting in themselves, the obtained results can provide good tools to describe more complex models, like the bipartite non-symmetric mean-field model, already used in the statistical approach to socio-economical sciences [8], or the multipartite symmetric mean-field model, that seems to have a very interesting behavior with non-trivial phase transitions. These models will be the subject of further investigation by the authors. By analyzing them it is expected to relate the critical and non-critical phases to different limiting behaviors of the sums of the spins of such models [13].

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Appendix A. Proof of lemma 1

As $m_l(\sigma) = \mu_l$, the configuration $\sigma_l$ contains $N_l(1 + \mu_l)/2$ times the value 1 and $N_l(1 - \mu_l)/2$ times the value $-1$; thus we have

$$A_{\mu_l} = \left( \frac{N_l}{N_l(1+\mu_l)/2} \right).$$

Using Stirling’s formula, $n! \sim n^n e^{-n} \sqrt{2\pi n}$, we obtain

$$A_{\mu_l} \geq \frac{\sqrt{\pi}}{\sqrt{N_l(1 - \mu_l)^2}} \left( \frac{N_l(1+\mu_l)}{2} \right)^{N_l} \left( \frac{N_l(1-\mu_l)}{2} \right)^{N_l}.$$

$$\geq \frac{1}{C} \frac{2^{N_l} \sqrt{N_l(1 + \mu_l)^{N_l(1+\mu_l)/2}(1 - \mu_l)^{N_l(1-\mu_l)/2}}}{\sqrt{N_l}} \exp(-N_l f(\mu_l)).$$

In this way we obtain a lower bound of $A_{\mu_l}$.

To obtain an upper bound for $A_{\mu_l}$ we suppose that the spins $\sigma_i$ are independent Bernoulli random variables. In this case, $P(\sigma_i = 1) = P(\sigma_i = -1) = 1/2$, and all configurations $\sigma_l$ have the same probability; hence,

$$A_{\mu_l} = 2^{N_l} P[ m_l(\sigma) = \mu_l ] \leq 2^{N_l} P[ m_l(\sigma) \geq \mu_l ],$$

where by the definition of the magnetization

$$P[ m_l(\sigma) \geq \mu_l ] = P \left\{ \sum_{i \in P_l} \sigma_i \geq \mu_l N_l \right\}.$$
Taking $\lambda > 0$ by Chebyshev’s inequality, we can bound the above probability:

$$
P \left\{ \sum_{i \in F} \sigma_i \geqslant \mu_1 N_1 \right\} \leqslant e^{-\lambda \mu_1 N_1} \prod_{i=1}^{N_1} E[\exp(\lambda \sigma_i)] = \exp(N_1(-\lambda \mu_1 + \ln \cosh \lambda)) \leqslant \min_{\lambda} \{\exp(N_1(-\lambda \mu_1 + \ln \cosh \lambda))\},
$$

where $E$ denotes the expectation value.

If $|\mu_1| < 1$, then the previous exponent is minimized for

$$
\lambda = \tanh^{-1}(\mu_1) = \frac{1}{2} \ln \left( \frac{1 + \mu_1}{1 - \mu_1} \right).
$$

Since $1/(\cosh^2 y) = 1 - \tanh^2 y$, the following equality holds:

$$
\ln \cosh \lambda = -\frac{1}{2} \ln(1 - \mu_1^2).
$$

Thus, by (A.1) and (A.2),

$$
\min_{\lambda} \{\exp(N_1(-\lambda \mu_1 + \ln \cosh \lambda))\} = \exp(-N_1 \mathcal{J}(\mu_1)).
$$

Hence, the upper bound for $A_{\mu_1}$ is

$$
A_{\mu_1} \leqslant 2^{N_1} \exp(-N_1 \mathcal{J}(\mu_1)).
$$

### Appendix B. Diagonalization

In the proof of proposition 1, we have seen that in three cases the determinant of the Hessian matrix of the function $f$, given by (6), is equal to zero when computed on the critical point $(0, 0)$. In particular, this happen as $J_{11} > 0$ and $J_{12} < 0$ for $t = 1$ and $t = 2b + 1$ and as $J_{11} < 0$ and $J_{12} > |J_{11}|$ for $t = 1$. Thus, in order to determinate the nature of the origin we consider the change of variable that diagonalizes the reduced interaction matrix $\mathbf{J}$:

$$
\begin{align*}
X &= \frac{x_1 + x_2}{2} \\
Y &= \frac{x_2 - x_1}{2}
\end{align*}
\implies
\begin{align*}
x_1 &= X - Y \\
x_2 &= X + Y
\end{align*}
$$

We apply it to the function $f$:

$$
f(X, Y) = \frac{1}{2} \left( \frac{1}{t}((a + b)X^2 + (a - b)Y^2) - \mathcal{J}(X - Y) - \mathcal{J}(X + Y) \right),
$$

and we compute the partial derivatives of $f$ with respect to the new variables. Obviously, the derivatives of the first order are equal to zero in the origin. The derivatives of the second order are

$$
\begin{align*}
\frac{\partial^2 f}{\partial X^2}(X, Y) &= \frac{a + b}{t} - \frac{1}{2} \left( \frac{1}{1 - (X + Y)^2} + \frac{1}{1 - (X - Y)^2} \right) \\
\frac{\partial^2 f}{\partial Y^2}(X, Y) &= \frac{a - b}{t} - \frac{1}{2} \left( \frac{1}{1 - (X + Y)^2} + \frac{1}{1 - (Y - X)^2} \right) \\
\frac{\partial^2 f}{\partial X \partial Y}(X, Y) &= -\frac{1}{2} \left( \frac{1}{1 - (X + Y)^2} - \frac{1}{1 - (X - Y)^2} \right).
\end{align*}
$$
In particular, in $(0,0)$ we have
\[
\frac{\partial^2 f}{\partial x^2}(0,0) = \frac{a + b}{t} - 1
\]
\[
\frac{\partial^2 f}{\partial y^2}(0,0) = \frac{a - b}{t} - 1
\]
\[
\frac{\partial^2 f}{\partial x \partial y}(0,0) = 0.
\]
Thus, since $a - b = 1$ as $J_{11} > 0$ and $J_{12} < 0$ and $a + b = 1$ as $J_{11} < 0$ and $J_{12} > |J_{11}|$, in all the three cases the determinant of the Hessian matrix is equal to zero. This is the reason why we need to compute partial derivatives of order higher than the second. Those of the third order are
\[
\frac{\partial^3 f}{\partial x^3}(X, Y) = \frac{\partial^3 f}{\partial x \partial y^2}(X, Y) = \frac{1}{2} \left( \frac{2(X + Y)}{(1 - (X + Y)^2)^2} + \frac{2(X - Y)}{(1 - (X - Y)^2)^2} \right)
\]
\[
\frac{\partial^3 f}{\partial y^3}(X, Y) = \frac{\partial^3 f}{\partial x^2 \partial y}(X, Y) = \frac{1}{2} \left( \frac{2(X + Y)}{(1 - (X + Y)^2)^2} + \frac{2(Y - X)}{(1 - (X - Y)^2)^2} \right).
\]
It is easy to check that these derivatives computed in $(0,0)$ are equal to zero. At least the partial derivative of the fourth order are
\[
\frac{\partial^4 f}{\partial x^4}(X, Y) = \frac{\partial^4 f}{\partial y^4}(X, Y) = \frac{4(X + Y)^2}{(1 - (X + Y)^2)^2} + \frac{4(X - Y)^2}{(1 - (X - Y)^2)^2}
\]
\[
\frac{\partial^4 f}{\partial x^3 \partial y}(X, Y) = \frac{1}{(1 - (X + Y)^2)^2} - \frac{4(X + Y)^2}{(1 - (X + Y)^2)^3} - \frac{4(X - Y)^2}{(1 - (X - Y)^2)^3}.
\]
In particular, in $(0,0)$ these derivatives become
\[
\frac{\partial^4 f}{\partial x^4}(0,0) = \frac{\partial^4 f}{\partial y^4}(0,0) = \frac{\partial^4 f}{\partial x^3 \partial y}(0,0) = -2
\]
\[
\frac{\partial^4 f}{\partial x^2 \partial y^2}(0,0) = 0.
\]
Now, we can approximate the function $f$ in a neighborhood of the origin with its Taylor expansion till the fourth order. In particular, as $J_{11} > 0$ and $J_{12} < 0$ and $t = 1$, we have
\[
f(X, Y) = 2bX^2 - \frac{1}{12}(X + Y)^2 + o(X^2 + Y^2).
\]
Since for these values of the parameters $b$ is negative, this polynomial is negative. Therefore, the origin is a maximum point. As $J_{11} > 0$ and $J_{12} < 0$ and $t = 2b + 1$, the Taylor approximation of $f$ is
\[
f(X, Y) = -2bY^2 - \frac{1}{12}(X + Y)^2 + o(X^2 + Y^2).
\]
It is easy to check that this polynomial is negative when computed in the point of the horizontal axis and positive in the point of the vertical axis. Thus, the origin is an inflection point. At least as $J_{11} < 0$ and $J_{12} > |J_{11}|$ and $t = 1$, we have
\[
f(X, Y) = -2bX^2 - \frac{1}{12}(X + Y)^2 + o(X^2 + Y^2).
\]
In this case, $b$ is positive; thus, this polynomial is negative and the origin is a maximum point.
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