COMPUTING THE MINIMAL MODEL FOR THE QUANTUM
SYMMETRIC ALGEBRA

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Abstract. In this note, we use some of the tensor categorial machinery developed by
the quantum algebra community to study algebraic objects which appear in representation
stability. In [SS16], Sam and Snowden prove that the twisted commutative algebra Sym is
Morita equivalent to the horizontal strip category. Their proof relies on a lemma proved by
Olver in [Olv87]. We give a self contained proof that replaces Olver’s lemma with information
about the associator in the underlying category of polynomial GL(∞)-representations. In
fact, we prove a quantum analogue of the theorem. The classical version follows by letting
the parameter converge to 1.

1. Introduction

Let \( \mathcal{S} \) be the category of polynomial GL(∞)-representations studied by Sam and Snowden
in [SS16]. This category contains the algebra \( \text{Sym} = \mathbb{C}[x_1, x_2, \ldots] \) which is Morita equivalent
to \( \text{FI} \), the category of finite sets with injections. A proof can be found in [SS17]. In Section 3
of [SS16], Sam and Snowden prove that \( \text{Sym} \) is Morita equivalent to \( \text{HS} \), the category whose
objects are partitions and whose morphisms are defined by

\[
\text{HS}(\lambda, \mu) = \begin{cases}
\mathbb{C}\{\mu \backslash \lambda\} & \lambda \subseteq \mu, \ \mu \backslash \lambda \in \text{HS} \\
0 & \text{otherwise}
\end{cases}
\]

Composition is defined as follows: Assume that \( \mu \backslash \lambda \) and \( \nu \backslash \mu \) are horizontal strips. If \( \nu \backslash \lambda \) is
a horizontal strip, then

\((\nu \backslash \mu)(\mu \backslash \lambda) = \nu \backslash \lambda.\)

If \( \nu \backslash \lambda \) is not a horizontal strip, then the composition is zero. Now let \( \mathcal{H} \) be the category of
polynomial type 1 representations of \( U_a(\mathfrak{gl}_\infty) \) defined in Definition 6. Inside \( \mathcal{H} \), we have the
quantum symmetric algebra \( \text{QSym} \). In this chapter, we prove the following:

**Theorem 1.** The quantum symmetric algebra \( \text{QSym} \) is Morita equivalent to \( \text{HS} \) for generic
\( a \).

Theorem 1 implies that many of the results in [SS16] which hold for \( \text{Sym} \) are also true for
\( \text{QSym} \).

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2. Preliminaries

Definition 2. Let $X$ be a semi-simple tensor category. Index the simple objects with a set $\Lambda$. Choose a basis for each $X(\mu, \lambda \otimes \nu)$ denoted by

\[ e_1, e_2, \ldots \]

and let

\[ e_1, e_2, \ldots \]

be the dual basis of $X(\lambda \otimes \nu, \mu)$. We call these diagrams trivalent vertices. It is important to notice that trivalent vertices are not canonically defined.

Definition 3. Pick a distinguished simple object $X \in X$. The fusion graph of $X$ has vertices $\Lambda$ and the edges from $\lambda$ to $\mu$ are the distinguished basis vectors in $X(\mu, \lambda \otimes X)$.

Proposition 4. Fix $\lambda \in \Lambda$. Then $X(\lambda, X^\otimes n)$ has dimension the number of paths from the tensor unit to $\lambda$ in the fusion graph for $X$ of length $n$. Moreover, an explicit basis is given by string diagrams of the form

\[ X X X X X X X \]

\[ f_1 f_2 f_3 f_4 f_n \]

\[ \lambda \]
In this diagram, each $f_i$ is a trivalent vertex of the form

```
\lambda_i \quad X \\
\downarrow \\
\lambda_{i+1}
```

we call such string diagrams trivalent basis vectors

Proof. Decompose $X^\otimes n$ using the fusion graph for $X$. 

\begin{definition}
If $X$ is a semi-simple tensor category over $\mathbb{C}$ with finite dimensional morphism spaces, the Artin-Wedderburn Theorem implies that $\text{End}(X^\otimes n)$ is a product of matrix algebras. Proposition 4 implies that in the trivalent basis, the matrix units in $\text{End}(X^\otimes n)$ look like

```
\begin{array}{c}
e_1 \\
e_2 \\
\vdots \\
e_n \\
f_n \\
f_{n-1} \\
\end{array}
```

Equivalently, the irreducible representations of $\text{End}(X^\otimes n)$ are parameterized by the simple objects in $X$ which have a length $n$ path from the tensor unit in the fusion graph for $X$. The string diagrams defined in Proposition 4 form a basis for the corresponding representation.
\end{definition}

\begin{definition}
The Iwahori-Hecke algebra, denoted by $H_m$, is the algebra generated over $\mathbb{C}(a)$ by $1, g_1, \ldots, g_{m-1}$ subject to the relations

\begin{align*}
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \\
g_i g_j &= g_j g_i \quad \text{if } |i-j| \geq 2 \\
g_i^2 &= (a - a^{-1}) g_i + 1.
\end{align*}

We define the category $H$ which has objects the natural numbers and morphisms

```
H(m, n) = \begin{cases} 
H_m & m = n \\
0 & \text{otherwise}
\end{cases}
```

The inclusion $H_m \otimes H_n \to H_{m+n}$ defined by $g_i \otimes g_j \mapsto g_i g_{m+j}$ equips $H$ with a tensor structure. We define $\mathcal{H} \subseteq [H^{\text{op}}, \text{Vec}]$ to be the idempotent completion of $H$. The monoidal structure on $H$ extends to $\mathcal{H}$ via Day convolution. Morally, the category $\mathcal{H}$ can be described as finite dimensional type 1 representations of the quantum group $U_a(\mathfrak{g}_\infty)$. The Grothendieck ring for $\mathcal{H}$ has basis given by partitions and multiplication given by the Littlewood-Richardson
rule. A special case of the Littlewood-Richardson rule is the Pieri rule:

$$\lambda \boxtimes \square = \sum_{\lambda \subseteq \mu \vdash n+1} \mu$$

This implies that the fusion graph for $\square$ is Young’s graph:

Paths in the Young graph are in bijection with standard partition fillings. It follows that the trivalent basis vectors

$$e_{m+1} e_{m+2} e_{m+3} e_{m+4} e_n$$

are in bijection (up to scaling) with standard skew tableaux of shape $\mu \setminus \lambda$. We abuse notation and identify these tree basis vectors with the corresponding standard skew tableaux. In [LR97], Ram and Leduc computed semi-normal forms for the Iwahori-Hecke algebras. More precisely, suppose that $\lambda \subseteq \mu \vdash n+2$ are partitions such that $\mu \setminus \lambda$ is not contained in a single row or column. Then there are exactly two partitions which satisfy $\lambda \subseteq \nu \subseteq \mu$. Call them $\nu$ and $\nu'$. The multiplicity space $\mathcal{H}(\mu, \lambda \boxtimes \square \otimes \square)$ is 2-dimensional with basis

$$\lambda, \nu, \lambda, \nu'$$

and $g_1$ acts via the matrix

$$m(g_1) = \begin{pmatrix}
\frac{a^d}{[d]} & \frac{[d-1][d+1]}{[d]^2} \\
1 & \frac{a^{-d}}{-d}
\end{pmatrix}$$
where
\[
[n] = \frac{a^n - a^{-n}}{a - a^{-1}}
\]
and \(d = d_1 + d_2\) is the axial distance in \(\mu \setminus \lambda\):

More formally, if \(\mu \setminus \lambda\) contains the boxes \((a_1, b_1), (a_2, b_2)\), then the axial distance is defined by
\[
d = |a_1 - a_2| + |b_1 - b_2|.
\]
These formulas are quantum analogues of the well known Young semi-normal form for the representation theory of the symmetric group [JK81]. Indeed, when \(a \to 1\), they recover the classical Young semi-normal formulas.

3. Morita Theory

In this section, we prove a very mild generalization of classical Morita theory. In classical Morita theory, we replace an object with its presentation with respect to a single projective. We are going to replace an object with its presentation with respect to a family of projectives. For the remainder of this section, \(X\) is an abelian category enriched over \(\text{Vec}_k\), closed under colimits, \(D\) is a category enriched over \(\text{Vec}_k\) and \(D : D^{\text{op}} \to X\) is a functor.

**Theorem 7.** If \(X\) has enough projectives, then \(X\) is equivalent to the category of representations of \(D\) where \(D^{\text{op}}\) is a full subcategory of \(X\) whose objects are compact, projective and generate \(X\).

We can prove this in a very clean way using coends. They can be motivated as follows: Suppose that \(A\) is a \(k\)-algebra, \(M\) is a left \(A\)-module and \(N\) is a right \(A\)-module. Then we can form the tensor product \(M \otimes_A N\) which is a vector space. It is built by taking the tensor product \(M \otimes_k N\) and quotienting by the relations
\[
am \otimes n = m \otimes na.
\]
We can generalize the second step in the following way. Suppose that \(F : D \otimes_k D^{\text{op}} \to \text{Vec}_k\) is a bifunctor. Then we can form the vector space
\[
\int_{d \in D} F = \bigoplus_{d \in D} F(d, d) / \text{nf} = vf \quad v \in F(d, d'), f : d' \to d.
\]
This vector space is called the **coend** of \(F\). We can use coends to generalize tensor products from modules to functors. Suppose that \(F : D \to \text{Vec}\) and \(G : D^{\text{op}} \to \text{Vec}\) are functors. Then we define
\[
F \otimes_D G = \int_{d \in D} F(d) \otimes G(d).
\]
A clear exposition of the theory of coends can be found in [Rie14]. Let $D$ a category enriched over $\text{Vec}$. Suppose that we have a functor $D : \mathcal{D}^{\text{op}} \to \mathcal{X}$. Then we get a functor

$$X \mapsto [D, \text{Vec}]$$

and

$$X \mapsto X(D(-), X)$$

This functor has a left adjoint given by

$$[D, \text{Vec}] \to \mathcal{X}$$

$$V \mapsto V \otimes_D D = \int^d V_d \otimes D^d$$

The following computation demonstrates why these functors are adjoint:

$$X(V \otimes_D D, X) = X\left(\int^d V_d \otimes D^d, X\right)$$

$$= \int_d X(V_d \otimes D^d, X)$$

$$= \int_d \text{hom}(V_d, X(D^d, X))$$

$$= [D, \text{Vec}](V, X(D(-), X))$$

**Definition 8.** We call $X \in \mathcal{X}$ a compact object if $X(X, -)$ commutes with filtered colimits.

**Proposition 9.** Assume that $D$ is fully faithful and each $D(d)$ is projective and compact. Then $[D, \text{Vec}] \to \mathcal{X}$ is fully faithful.

**Proof.** We need to prove that the unit

$$V \to X(D(-), V \otimes_D D)$$

is an isomorphism. It suffices to prove this pointwise, so we need to prove that the linear map

$$V(d) \to X(D(d), V \otimes_D D)$$

is an isomorphism. Since $D(d)$ is projective and compact, it follows that $X(D(d), -)$ commutes with all colimits. Therefore

$$X\left(D(d), \int^x V(x) \otimes D(x)\right) = \int^x V(x) \otimes X(D(d), D(x))$$

$$= \int^x V(x) \otimes D(x, d)$$

$$= V(d)$$

The second equality is true because $D$ is fully faithful. $\Box$

**Proposition 10.** In addition to the hypotheses of proposition [9], assume that every $X \in \mathcal{X}$ admits an epimorphism $\bigoplus_i D(d_i) \to X$ for some family $\{d_i\}$. Then $[D, \text{Vec}] \to \mathcal{X}$ is essentially surjective.

**Proof.** By assumption, it follows that for every $X \in \mathcal{X}$, the counit

$$X(D(-), X) \otimes_D D \to X$$
is an epimorphism. Then we have an exact sequence
\[ 0 \to K \to X(D(-), X) \otimes D D \to X \to 0 \]
This gives us an exact sequence
\[ X(D(-), K) \otimes D D \to X(D(-), X) \otimes D D \to X \to 0 \]
Since \(- \otimes D D\) is fully faithful, we can write the first map as \(f \otimes D D\) for some map \(f\):
\[ X(D(-), K) \to X(D(-), X) \otimes D D \to X \to 0 \]
This proves essential surjectivity. \(\square\)

**Proof of theorem 7.** Let \(D^\text{op}\) be a full subcategory of \(X\) whose objects are compact, projective and generate \(X\). Let \(D : D^\text{op} \to X\) be the embedding. By proposition, \(9\), the functor \(- \otimes D D\) is fully faithful. By Proposition \(10\) the functor is essentially surjective. \(\square\)

**Definition 11.** If \(X\) is an abelian category with enough compact projectives, define \(M(X)\) to be the opposite of the full subcategory with objects the indecomposable compact projectives. We call \(M(X)\) the **minimal model** for \(X\). By theorem \(7\) the functor category \([M(X), \text{Vec}]\) is equivalent to \(X\).

### 4. Modules over Tensor Algebras

In this section, we work inside a fixed semi-simple tensor category \(C\). We use Morita theory to study the category of modules over an algebra internal to \(C\). Choose a distinguished simple object \(X \in C\). Define
\[ T = \bigoplus_{n \geq 0} X \otimes^n \]
This is the tensor algebra generated by \(X\). Define \(\text{Rep}(T)\) to be the category of right modules over \(T\) internal to \(C\). The forgetful functor \(F : \text{Rep}(T) \to C\) has left adjoint \(L : C \to \text{Rep}(T)\) defined by \(V \mapsto V \otimes T\). Since the right adjoint \(F\) is exact, it follows that \(L\) preserves projectives. Define
\[ T^+ = \bigoplus_{n \geq 1} X \otimes^n \]

**Lemma 12.** If \(V \in C\) is irreducible, then \(V \otimes T\) is an indecomposable projective in \(\text{Rep}(T)\).

**Proof.** Since \(V \otimes T = L(V)\), the module is projective. Suppose that \(V \otimes T = A \oplus B\) as \(T\)-modules. When we tensor with \(T/T^+\), we get
\[ V = A/AT^+ \oplus B/ BT^+ \]
in \(C\). Since \(V\) is irreducible in \(C\), we can assume without loss of generality that \(A/AT^+ = 0\). Suppose that \(A \neq 0\). Choose \(0 \neq Y \subseteq A \subseteq V \otimes T\) irreducible in \(C\). This implies that
\[ Y \subseteq \bigoplus_{n=0}^N V \otimes X \otimes^n \]
for some large \(N\). Since \(A = A(T^+)^{N+1}\), it follows that
\[ Y \subseteq A(T^+)^{N+1} \subseteq \bigoplus_{n \geq N+1} V \otimes X \otimes^n. \]
This implies that \(Y = 0\), which is a contradiction. Therefore we must have \(A = 0\). \(\square\)
**Proposition 13.** Let $G$ be the fusion graph for $X$ considered as a category where the objects are vertices and the morphisms are paths. Then $\text{Rep}(T)$ is Morita equivalent to $[G, \text{Vec}]$.

**Proof.** The indecomposable compact projectives $\lambda \otimes T$, where $\lambda$ is an irreducible in $\mathcal{C}$, generate $\text{Rep}(T)$. Using the adjunction $(L, F) : \text{Rep}(T) \to \mathcal{C}$, we have

$$\text{hom}_T(\mu \otimes T, \lambda \otimes T) = \mathcal{C}(\mu, \lambda \otimes T).$$

The right hand side has a basis consisting of vectors of the form

$$\lambda \quad X \quad X \quad X$$

which is exactly a path in the fusion graph for $X$ from $\lambda$ to $\mu$. Post composing with the corresponding morphism in $\text{hom}_T(\mu \otimes T, \lambda \otimes T)$ is the map

This implies that composition of basis vectors is exactly concatenation of paths in the fusion graph for $X$. This completes the proof. □

**Example 14.** Let $\mathcal{C} = \mathcal{H}$, which was defined in Example 6 and let $X = \Box$. The fusion graph for $X$ has objects partitions and the edges $G(\lambda, \mu)$ are the standard skew tableaux of shape $\mu \setminus \lambda$. 

![Diagram](attachment:image_url)
5. Modules over the quantum symmetric algebra

In this section, we work inside the category $\mathcal{H}$ defined in Example 6. Define $T = \bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}$. Consider the submodule $I$ of $T$ spanned by all maps $\lambda$ where $\lambda$ is a partition with two or more rows. The grading on the Grothendieck ring implies that $I$ is a 2-sided ideal in $T$, so we can form the quotient algebra $S = T/I$. We have

$$S = \emptyset \oplus \cdots \oplus \cdots$$

Define $\text{Rep}(S)$ to be category of right modules over $S$ internal to the category $\mathcal{H}$. Just like the tensor algebra, every projective $S$-module is free and the indecomposable projectives are of the form $\lambda \otimes S$ where $\lambda$ is a partition. Define $F$ to be the fusion graph for $\square$ inside $\mathcal{H}$ interpreted as a category. Define $M$ to be the category whose objects are partitions and whose morphisms are defined by

$$M(\lambda, \mu) = \text{hom}_S(\mu \otimes S, \lambda \otimes S).$$

Then we have the functor $Q = - \otimes_T S : F \to M$. By definition, this functor is the identity on objects. Since all the projectives involved are free, it follows that $Q$ is full. We can describe $Q$ more concretely as follows. Each hom space in $F$ is a skew representation of some Iwahori-Hecke algebra. We have:

**Lemma 15.** On morphisms, $Q$ projects onto the Hecke algebra invariants.
Proof. Recall that given a vector $f \in F(\lambda, \mu)$, post composition by the induced map $\text{hom}_T(\mu \otimes T, \lambda \otimes T)$ is given by

More precisely, the map $f : \mu \rightarrow \lambda \otimes X^{\otimes n}$ induces a map $\mu \otimes T \rightarrow \lambda \otimes T$ defined by $g : \mu \otimes T \xrightarrow{f \otimes 1} \lambda \otimes X^{\otimes n} \otimes T \xrightarrow{1 \otimes m} \lambda \otimes T$

where $m$ is the multiplication map. The diagram depicts post composing a map $\nu \rightarrow \mu \otimes T$ with $g$. By Yoneda’s lemma, this determines $g$. If we tensor along the projection $p : T \rightarrow S$ we have

The second equality is true because $p$ is an algebra homomorphism, so it commutes with multiplication. \hfill \Box

**Proposition 16.** Suppose that $\lambda \subseteq \mu$ are partitions. Then $\mathcal{H}(\mu, \lambda \otimes X^{\otimes n})$ has Hecke algebra invariants if and only if $\mu \setminus \lambda$ is a horizontal strip. In this case, the invariants are 1-dimensional and any skew tableaux projects onto a nonzero invariant.

**Proof.** The invariants in $\mathcal{H}(\mu, \lambda \otimes X^{\otimes n})$ are the same as maps

$$\mu \rightarrow \lambda \otimes X^{\otimes n}$$
By Pieri’s rule, $\mathcal{H}(\mu, \lambda \otimes \mathbf{1}^\otimes n)$ has invariants if and only if $\mu \backslash \lambda$ is a horizontal strip. Suppose that $\mu \backslash \lambda$ is a horizontal strip and $P$ is a skew tableaux of shape $\mu \backslash \lambda$. Then from the semi-normal form, we know that $P$ generates $\mathcal{H}(\mu, \lambda \otimes \mathbf{1}^\otimes n)$. This implies that $\mathcal{H}(\mu, \lambda \otimes \mathbf{1}^\otimes n)$ has a 1-dimensional space of invariants and $P$ projects onto a nonzero invariant. □

Proof of Theorem 1. The minimal model for $S$ is $M$. From Lemma 15, $Q : F \rightarrow M$ is a full functor which is projection onto the Hecke algebra invariants. From Proposition 16 we have

$$M(\mu, \lambda) = F(\mu, \lambda)^{H_n} = \mathcal{H}(\mu, \lambda \otimes \mathbf{1}^\otimes n)^{H_n} = HS(\mu, \lambda).$$

□

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