Intersection Matrices Revisited

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Abstract: Several intersection matrices of \(s\)-subsets versus \(k\)-subsets of a \(v\)-set are introduced in the literature. We study these matrices systematically through counting arguments and generating function techniques. A number of new or known identities appear as natural consequences of this viewpoint; especially, use of the derivative operator \(d/dz\) and some related operators reveals some connections between intersection matrices and the “combinatorics of creation-annihilation.” As application, the eigenvalues of several intersection matrices including some generalizations of the adjacency matrices of the Johnson scheme are derived; two new bases for the Bose–Mesner algebra of the Johnson scheme are introduced and the associated intersection numbers are obtained as well. Finally, we determine the rank of some intersection matrices. © 2012 Wiley Periodicals, Inc. J. Combin. Designs 20: 383–397, 2012

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1. INTRODUCTION

Let \(s\), \(k\), and \(v\) be integers satisfying \(0 \leq s, k \leq v\). We fix a \(v\)-set \(V\) throughout this paper. The inclusion matrix \(W_{s,k}\) is a \((0, 1)\)-matrix whose rows and columns are indexed...
by $s$-subsets and $k$-subsets of $V$, respectively, and $W_{s,k}(S, K) = 1$ if and only if $S \subseteq K$. This matrix has interesting properties and arises in many combinatorial problems, especially in design theory and extremal set theory (see [3, 8, 9, 19, 20, 22]). It satisfies several nice identities among which is

$$W_{i,s}W_{s,k} = \binom{k - i}{s - i}W_{i,k}, \quad (1)$$

which holds for $i \leq s \leq k \leq v$ (see [15, 16, 19, 20, 22]). Another $(0, 1)$-matrix that is closely related to $W_{s,k}$ is the exclusion matrix $\overline{W}_{s,k}$ with the same row and column indices as $W_{s,k}$, where the $(S, K)$th entry is 1 if and only if $S \cap K = \emptyset$.

Both of the inclusion and exclusion matrices may be regarded as intersection matrices in the sense that the $(S, K)$th entry only depends on $|S \cap K|$. Some other intersection matrices are also studied in the literature in different contexts in combinatorics, including design theory, association scheme, and extremal set theory, the most significant properties of which are the combinatorial identities they satisfy. The goal of the present paper is to introduce and investigate a more general framework in which several intersection matrices arise as special cases and the identities involving them are derived more naturally. This uniform framework demonstrates the relation between intersection matrices and some operators of the form $\phi(z)\frac{d}{dz}$. These operators were studied previously in [18] and more recently in [5] and the references therein.

The paper is organized as follows. Section 2 contains basic properties of the derivative operator and some binomial identities that we will use later. In Section 3, we introduce several intersection matrices and study relations between them. Particularly, we show that all these matrices can be extracted from one, namely $F_{s,k,t}(z)$, a matrix with polynomial entries in the variable $z$. We also show that studying identities containing this matrix, produces identities containing the other matrices. In Section 4, we calculate the matrix product $W_{i,s}F_{i,k,t}$ as a linear combination of derivatives of $F_{s,k,t}$. This reveals a close connection between this matrix product and the operator $\frac{d}{dz}$. Section 5 is the application section in which we introduce two new bases for the Bose–Mesner algebra of the Johnson scheme using the intersection matrices above. The eigenvalues of some generalizations of the adjacency matrices of the Johnson scheme are also derived. All the eigenvalues of these matrices can be expressed in terms of the polynomials $\mu_j(z) = \sum_{i=j}^{t} \binom{k - j}{i-j} \binom{v - j - i}{k - i} z^i$ for $j = 0, 1, \ldots, t$.

2. OPERATORS AND BASIC NOTATION

Let $D$ denote the derivative operator $\frac{d}{dz}$, and $(zD)_n$ denote the falling factorial $(zD)(zD - 1) \cdots (zD - (n - 1))$, where $1$ is the identity operator. For convenience we replace $\alpha 1$ by $\alpha$ to write instead $(zD)(zD - 1) \cdots (zD - (n - 1))$. Here are some of the identities containing the derivative operator:

1. $(zD)^n = \sum_{k=1}^{n} S(n, k) z^k D^k$ for $S(n, k)$ the Stirling numbers of the second kind;
2. $(zD)_n = z^n D^n;$
3. $(zD - k)_n = n! \sum_{r=0}^{n} \binom{n + k - r - 1}{n - r} (-1)^{n-r} z^r D^r;$
4. $D^t (z^r D^r) = \sum_{j=0}^{t} \binom{t}{j} (r)_{t-j} z^{r-t+j} D^{r+j}.$
We frequently make use of the following binomial identities (see Chapter 5 of [10]):

\[ \sum_k (-1)^k \binom{\ell}{m+k} \binom{s+k}{n} = (-1)^{\ell+m} \binom{s-m}{n-\ell}, \quad \ell \geq 0, \quad \text{and} \]

\[ \sum_{k \leq \ell} (-1)^k \binom{\ell-k}{m} \binom{s}{k-n} = (-1)^{\ell+m} \binom{s-m-1}{\ell-n}, \quad \ell, m, n \geq 0. \] (3)

The coefficient of \( z^i \) in a polynomial (or a generating function) \( p(z) \) is denoted as \([z^i]p(z)\). For two matrices \( A \) and \( B \), we write \( A \equiv B \) if \( A \) can be obtained from \( B \) by a permutation of the rows and a permutation of the columns. For instance \( W_{sk} \equiv W_{s,v-k} \); this is because for \( s \)-subset \( S \) and \( k \)-subset \( K \) of \( V \), \(|S \cap K| = \emptyset \) if and only if \( S \subseteq V \setminus K \).

3. INTERSECTION MATRICES

Let \( s, k, \) and \( v \) be integers satisfying \( 0 \leq s, k \leq v \). Let \( \binom{V}{s} \) and \( \binom{V}{k} \) denote the sets of \( s \)-subsets and \( k \)-subsets of \( V \), canonically ordered somehow. Then, an intersection matrix (relative to this setup) is a \((v_s \times v_k)\) matrix with \((S,K)\) entry as a function of \(|S \cap K|\) (but not otherwise dependent on the specific subsets \( S \) and \( K \)). The inclusion matrix \( W \) is one such since

\[ \binom{|S \cap K|}{s} = \begin{cases} 1 & \text{if } S \subseteq K, \\ 0 & \text{otherwise}. \end{cases} \]

The exclusion matrix \( W \) is another such since

\[ \binom{s - |S \cap K|}{s} = \begin{cases} 1 & \text{if } S \cap K = \emptyset, \\ 0 & \text{otherwise}. \end{cases} \]

Definitions. The intersection matrices considered herein are:

1. \( A_{s,k,t} \) with \((S,K)\) entry \( \binom{|S \cap K|}{t} \), a generalization of \( W_{s,k} = A_{s,k,s} \);
2. \( F_{s,k,t}(z) \) with \((S,K)\) entry \( \sum_{i=0}^t (-1)^{i} \binom{|S \cap K|}{i} z^i \);
3. \( U_{s,k,t,\ell} \) with \((S,K)\) entry \( \sum_{i=0}^t (-1)^{i+\ell} \binom{|S \cap K|}{i} \binom{|S \cap K|-\ell}{t-i-\ell} \);
4. \( N_{s,k,t} \) with \((S,K)\) entry \( \sum_{i=0}^t (-1)^i \binom{|S \cap K|}{i} \binom{|S \cap K|-1}{t-i} \);
5. \( F_{s,k}(z) = F_{s,k,s}(z) \);
6. \( U_{s,k,\ell} = U_{s,k,s,\ell} \), which is a \((0,1)\)-matrix whose \((S,K)\) entry is 1 if and only if \(|S \cap K| = \ell \).

The obvious relations among these include

1. \( F_{s,k,t}(z) = \sum_{j=0}^t A_{s,k,t,j} z^j \);
2. \( F_{s,k,t}(z) = \sum_{j=0}^t U_{s,k,t,\ell}(z+1)^j \);
3. \( A_{s,k,t} = \frac{D^t}{\ell!} F_{s,k,t}(z) \big|_{z=0} \);
4. \( U_{s,k,t,\ell} = \frac{D^t}{\ell!} F_{s,k,t}(z) \big|_{z=-1} \);
5. \( F_{s,k}(z)(S,K) = (z+1)^{|S \cap K|} \).

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(6) \( U_{s,k,t,\ell} = \sum_{i=0}^{t}(-1)^{t-i} \binom{t}{i} A_{s,k,i}; \)

(7) \( A_{s,k,i} = \sum_{l=i}^{t} \binom{t}{l} U_{s,k,t,\ell}; \)

(8) \( N_{s,k,t} = (-1)^{t} U_{s,k,t,0}; \)

(9) \( N_{s,k,t} = \sum_{i=0}^{t}(-1)^{t-i} A_{s,k,i}; \)

(10) \( A_{s,k,t} = N_{s,k,t} + N_{s,k,t-1}. \)

The following properties of the intersection matrices are straightforward.

**Proposition 1.**

(i) \( A_{t,v-k,t} = \overline{W}_{t,k} \) and \( A_{v-t,k,k} = \overline{W}_{t,k}. \)

(ii) \( U_{t,k,t} = W_{t,k} \) and \( U_{k,t,t} = W_{t,k}. \)

(iii) If \( t \geq \min(s,k), \) then \( F_{s,k,t} = F_{s,k} \) and \( U_{s,k,t,\ell} = U_{s,k,t}. \)

(iv) \( U_{t,k,0} = \overline{W}_{t,k}. \)

(v) \( A_{s,k,t} = W_{t,s}^{T} W_{t,k}. \)

(vi) \( F_{s,k,t} = F_{s,k,t}. \) Hence, \( F_{s,k,t} \) is a symmetric matrix.

(vii) \( U_{s,k,t,\ell}(S,K) \neq 0 \) only if \( |S \cap K| \in B, \) where \( B = \{ \ell \} \cup \{ t + 1, t + 2, \ldots, \min(s,k) \}. \)

(viii) If \( \ell \leq t \leq \min(s,k), \) then the number of nonzero elements in each row of \( U_{s,k,t,\ell} \) is \( \sum_{i \in B} \binom{t}{i} \binom{s-1}{t-i}. \)

(ix) \( U_{s,k,t,\ell} = \sum_{i \in B} \binom{t}{i} \binom{s-1}{t-i} U_{s,i,k,i}. \)

(x) There are exactly \( \binom{v-1}{k} \) \( + \sum_{i=t+1}^{\min(s,k)} \binom{i}{t} \binom{v-s}{k-i} \) nonzero elements in each row of \( N_{s,k,t}. \)

(xi) There are exactly \( \binom{v-1}{k} \) \( + \binom{s-1}{k-1} \) nonzero elements in each row of \( N_{s,k,s-1}. \)

(xii) There are exactly \( \binom{v-1}{k} \) \( + \binom{s-1}{k-1} \) nonzero elements in each row of \( N_{s,k,s-1}. \)

**Remark 1.** The matrix \( N_{s,k,t} \) was introduced in [13] and discussed subsequently in [17] as an auxiliary tool to speed up an algorithmic search for finding \( t \)-designs. More precisely, the matrices \( N_{7,7,6} \) with \( v = 14 \) and \( N_{6,6,5} \) with \( v = 13 \) had important roles in finding \( 6-(14, 7, 4) \) designs (see [13]). It was observed that the small number of nonzero elements in each row of \( N_{s,k,t} \) (which is obtained by choosing proper values of \( s \) as it can be seen from Proposition 1) is a useful property for this. The matrix \( N_{7,7,6} \) with \( v = 14 \) has only two nonzero elements in each of its rows and the matrix \( N_{6,6,5} \) with \( v = 13 \) has eight nonzero elements in each of its rows.

2. **Proposition.** The followings hold:

(i) \( U_{s,k,\ell} \equiv U_{s,v-k,s-\ell} \equiv U_{v-s,k,k-\ell} \equiv U_{v-s,v-k,v-s-k+\ell}; \)

(ii) \( F_{v-s,k}(z) \equiv (z+1)^{k} F_{s,k}(\frac{z}{z+1}); \)

(iii) \( F_{s,v-k}(z) \equiv (z+1)^{s-k} F_{s,k}(\frac{z}{z+1}); \)

(iv) \( F_{v-s,v-k}(z) \equiv (z+1)^{v-s-k} F_{s,k}(z). \)

**Proof.** Note that \( |S \cap K| = \ell \) if and only if \( |S \cap (V \setminus K)| = s - \ell. \) Thus, we have \( U_{s,k,\ell} \equiv U_{s,v-k,s-\ell}. \) The rest of part (i) is proved similarly. Since the proofs of the remaining parts are similar, we only prove (ii):

\[
F_{v-s,k}(z)(V \setminus S, K) = (z+1)^{|(V \setminus S) \cap K|} = (z+1)^{v-|S \cap K|}
\]

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\[= (z + 1)^k \left( 1 - \frac{z}{z + 1} \right)^{|S \cap K|} \]
\[= (z + 1)^k F_{i,k} \left( \frac{-z}{z + 1} \right) (S, K). \]

4. MATRIX PRODUCTS

The following theorem gives some useful identities, among which (4) and (6) are new. Moreover, two new proofs are given for the known identity (5) that appeared first in [19].

**Theorem 3.** The following hold:

\[U_{a,b,i}U_{b,c,j} = \sum_{\ell=0}^{\min(a,c)} \sum_{n=0}^{\ell} \binom{\ell}{n} \binom{c-\ell}{j-n} \binom{a-\ell}{i-n} \binom{v-a-c+\ell}{b-i-j+n} U_{a,c,\ell},\]  

(4)

\[W_{a,k}W_{b,k}^T = \sum_{n=0}^{\min(a,b)} \binom{v-b-a}{v-k-n} A_{a,b,n},\]  

(5)

\[A_{a,b,i}A_{b,c,j} = \sum_{n=0}^{\min(i,j)} \binom{a-n}{i-n} \binom{c-n}{j-n} \binom{v-i-j}{b+n-i-j} A_{a,c,n}.\]  

(6)

**Proof.** For any given \(a\)-subset \(A\) and \(c\)-subset \(C\), using simple counting arguments, the entry \((A, C)\) of the matrix product \(U_{a,b,i}U_{b,c,j}\) is calculated as

\[ (U_{a,b,i}U_{b,c,j})(A, C) = |\{B \subseteq \{1, \ldots, v\} : |B| = b, |B \cap A| = i, |B \cap C| = j\}|. \]

To count the number of \(b\)-sets \(B\) with the above constraints, let \(\ell = |A \cap C|\) and \(n = |A \cap B \cap C|\). To construct \(B\), one should select \(n\) points from \(A \cap C\), \(i - n\) points from \(A \setminus C\), \(j - n\) points from \(C \setminus A\), and \(b - i - j + n\) points from \(A' \cap C'\). Hence,

\[ (U_{a,b,i}U_{b,c,j})(A, C) = \sum_{\ell=0}^{\min(a,c)} \delta_{|A \cap C|, \ell} \sum_{n=0}^{\ell} \binom{\ell}{n} \binom{c-\ell}{j-n} \binom{a-\ell}{i-n} \binom{v-a-c+\ell}{b-i-j+n}, \]

which proves (4).

To prove (5), note that \(W_{a,k} \equiv W_{v-k,v-a}^T\ because A \subseteq K\ if and only if V \setminus K \subseteq V \setminus A.\ If we apply simultaneously the same permutation on the columns of \(W_{a,k}\ and on the rows of \(W_{b,k}^T,\ then we see that \(W_{a,k}W_{b,k}^T \equiv W_{v-k,v-a}^T W_{v-k,v-b}.\ Hence,

\[W_{a,k}W_{b,k}^T = A_{v-a,v-b,v-k} = [z^{v-k}] F_{v-a,v-b}\]
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\[\equiv [z^{v-k}](z + 1)^{v-a-b} F_{a,b} \]

\[= \sum_{n=0}^{\min(a,b)} \binom{v-a-b}{v-k-n} [z^n] F_{a,b} \]

\[= \sum_{n=0}^{\min(a,b)} \binom{v-a-b}{v-k-n} A_{a,b,n},\]

concluding (5). (We notice that the same ordering is used for the rows and the columns of both matrices.) An alternative way to prove (5), is as follows:

\[W_{a,k} W_{b,k}^\top = U_{a,k,a} U_{k,b,b} \]

\[= \sum_\ell \binom{v-a-b+\ell}{k-a-b+\ell} U_{a,b,\ell} \quad \text{(by 4)} \]

\[= \sum_\ell \binom{v-a-b+\ell}{v-k} \sum_n (-1)^{n-\ell} \binom{n}{\ell} A_{a,b,n} \]

\[= \sum_n (-1)^n A_{a,b,n} \sum_\ell (-1)^\ell \binom{n}{\ell} \binom{v-a-b+\ell}{v-k} \]

\[= \sum_n \binom{v-a-b}{v-k-n} A_{a,b,n} \quad \text{(by 2)}.\]

Now, replacing \(a\) by \(i\) and \(k\) by \(j\) in (5) and multiplying the identity from left and right by \(W_{j,c}^\top\) and \(W_{i,a}\), respectively, and using (1), we conclude (6). \[\square\]

We now turn to calculating the matrix product \(W_{s,j}^\top F_{j,k,t} \). This calculation reveals the relationship between this matrix product and the derivative operator. The general form of these identities contains expressions of the form \((\phi(z)D + \rho(z))^n\); such expressions are studied firstly by Scherk in [18] and extensively by some authors in recent years (see [4] and the references therein). Recently, many interesting properties associated with the algebra of two operators, \(A\) and \(B\), satisfying \(AB - BA = 1\) in quantum physics. This relation is called the creation–annihilation axiom. A simple representation of this algebra is obtained by taking \(A = D\) and \(B = z\) (see Section 2.4 of [4]). Also in [5], systematic evaluation of expressions of the form \((\phi(z)D + \rho(z))^n\) using several combinatorial models involving set partitions, permutations, increasing trees, and weighted lattice paths is studied. This is discussed more in Remark 3 at the end of this section.

**Theorem 4.** The following identities hold:

(i) \(W_{s-1,s}^\top F_{s-1,k,t}(z) = s F_{s,k,t}(z) - z D F_{s,k,t}(z)\),

(ii) \(W_{s-1,s}^\top U_{s-1,k,t,\ell} = (s - \ell) U_{s,k,t,\ell} + (\ell + 1) U_{s,k,t,\ell+1}\).

**Proof.** By Proposition 1(i) and (1), we have \(W_{s-1,s}^\top A_{s-1,k,i} = (s - i) A_{s,k,i}\). Therefore,

\[W_{s-1,s}^\top F_{s-1,k,i}(z) = \sum_{i=0}^s (s - i) A_{s,k,i} z^i\]
which proves (i). By applying the operator $D^\ell$ to (i) we get,

$$W_{s-1,s}^T D^\ell F_{s-1,k,i}(z) = (s - \ell)D^\ell F_{s,k,i}(z) - zD^{\ell+1} F_{s,k,i}(z).$$

Setting $z = -1$, we obtain (ii). □

**Remark 2.** From Theorem 4, some new identities can be derived. By setting $t = s$ in (i) and using Proposition 1(ii),

$$W_{s-1,s}^T F_{s-1,k,i}(z) = sF_{s,k}(z) - zDF_{s,k}(z).$$

Transposing the two sides of this equation and exchanging $k$ and $s$, we have

$$F_{s,k-1}(z)W_{k-1,k} = kF_{s,k,t}(z) - zDF_{s,k,t}(z),$$

and

$$F_{s,k-1}(z)W_{k-1,k} = kF_{s,k}(z) - zDF_{s,k}(z).$$

In Theorem 4(ii), the expression $W_{i,s}^T U_{i,k,t,\ell}$ is calculated for $i = s - 1$, but how can we calculate this expression in general? In the special case of $t = k$, the answer is simply obtained by using (4) as follows:

$$W_{i,s}^T U_{i,k,t,\ell} = \sum_{h=\ell}^s \binom{h}{\ell} \binom{s-h}{i-\ell} U_{s,k,t,h}.$$

The following theorem gives the answer in general.

**Theorem 5.** Let $L_{s,j}(z) = \sum_{r=\ell}^{s-1}(-1)^r \binom{s-r}{i}^{\ell} r! D^r$. Then the following identities hold:

(i) $W_{i,s}^T F_{i,k,t}(z) = L_{s,i}(z)F_{s,k,j}(z)$,

(ii) $W_{i,s}^T U_{i,k,t,\ell} = \sum_{h=\ell}^{\ell+i} \binom{h}{\ell} \binom{s-h}{i-\ell} U_{s,k,t,h}$.

**Proof.** (i) By Theorem 4(i),

$$W_{s-1,s}^T F_{s-1,k,i}(z) = (s - zD)F_{s,k,t}(z).$$

On the other hand, from (1) it follows that

$$W_{i,i+1}W_{i+1,i+2} \cdots W_{s-1,s} = (s - i)!W_{i,s}.$$

Now, by iterative use of Theorem 4(i), we have

$$W_{i,s}^T F_{i,k,t} = \frac{1}{(s - i)!} W_{s-1,s}^T W_{s-2,s-1}^T \cdots W_{i,i+1}^T F_{i,k,t}$$

$$= \frac{1}{(s - i)!} (s - zD)(s - 1 - zD) \cdots (i + 1 - zD)F_{s,k,t}.$$
By the property 3 of Section 2, the operator \( \frac{1}{(s-j)!}(s-zD)(s-1-zD)\cdots(i+1-zD) \) in the last expression can be simplified as \( L_{s,i}(z) = \sum_{r=0}^{s-i}(-1)^r\binom{s-r}{i}D^r. \)

(ii) Applying the operator \( D^\ell/\ell! \) on (i), we have

\[
W_{i,s}^T \frac{D^\ell}{\ell!} F_{i,k,t} = \frac{1}{\ell!} \sum_{r=0}^{s-i} \frac{(-1)^r}{r!} \binom{s-r}{i} D^\ell(z^r\ell)F_{s,k,t}
\]

\[
= \frac{1}{\ell!} \sum_{r=0}^{s-i} \frac{(-1)^r}{r!} \binom{s-r}{i} \sum_{j=0}^{\ell} \binom{\ell}{j} (r)_{\ell-j} z^{-\ell+j} D^\ell+j F_{s,k,t}(z).
\]

Letting \( z = -1 \), we get

\[
W_{i,s}^T U_{i,k,t,\ell} = \sum_{r=0}^{s-i} \sum_{j=0}^{\ell} \binom{s-r}{i} \binom{r+j}{j} \binom{r}{\ell-j} (-1)^{\ell-j} U_{s,k,t,r+j}
\]

\[
= \sum_{h=0}^{\ell+s-i} a_h U_{s,k,t,h},
\]

where \( a_h = \sum_{j=0}^{h} (-1)^{\ell-j} \binom{h}{j} \binom{h-j}{s-h+j} \). From (3), it follows that \( a_h = \binom{h}{s-h} \). \( \square \)

**Remark 3.** The calculations in this section show some connections between intersection matrices and operators of the form \( (\phi(z)D)^n \) (or more generally \( \phi(z)D + \rho(z) \)). These operators are studied in [4] and in more details in [5, Section 6.3]. There are more such connections. For instance, we can prove that

\[
W_{s,j} F_{j,k}(z) = (z+1)^{-v+j+k} \sum_{r=0}^{j-s}(-1)^r\binom{v-s-r}{v-j} \frac{z^{r}D^r}{r!}((z+1)^{v-s-k}F_{s,k}(z)).
\]

(7)

On the other hand, by (5), \( W_{j-1,k} F_{j,k}(z) = (\phi(z)D + \rho_j(z))F_{j-1,k}(z) \), where \( \phi(z) = -z^2 - z \) and \( \rho_j(z) = kz + v - j + 1 \). Using techniques similar to the ones used in the proof of Theorem 5, it follows that

\[
W_{s,j} F_{j,k}(z) = \frac{1}{(j-s)!}(\phi(z) + \rho_{s+1})(\phi(z) + \rho_{s+1} - 1)\cdots(\phi(z) + \rho_{s+1} - j + s + 1)F_{s,k}(z).
\]

We believe that deeper relations of this form help one in studying more useful properties of intersection matrices. Such operators were studied firstly in [18] and more recently in [5].

### 5. SOME APPLICATIONS

This section contains some applications of the results obtained so far. One important application is deriving the eigenvalues of the matrices \( F_{k,k,t}(z), U_{k,k,t,\ell} \), and \( N_{k,k,t} \) based
on Wilson’s method for computing the eigenvalues of $A_{k,k,i}$ [19] (cf. [20, 21]). To the best of our knowledge, the rest of the results of this section are new. Among these are introducing two new bases for the Bose–Mesner algebra of the Johnson scheme and obtaining the associated intersection numbers. We also determine the rank of some intersection matrices.

5.1 Johnson Scheme

An association scheme with $d$ classes is a set of $d+1$ square $(0, 1)$-matrices $X_0, X_1, \ldots, X_d$, which satisfy

(i) $\sum_{i=0}^{d} X_i = J$,
(ii) $X_0 = I$,
(iii) $X_i = X_i^\top$, for $i = 0, 1, \ldots, d$,
(iv) $X_i X_j = \sum_{\ell=0}^{d} a_{ij}^{\ell} X_{\ell}$, for $i, j \in \{0, 1, \ldots, d\}$.

The numbers $a_{ij}^{\ell}$ are called the intersection numbers of the association scheme. From (i), we see that the matrices $X_i$ are linearly independent, and by use of (ii)–(iv), we see that they generate a commutative $(d+1)$-dimensional algebra of symmetric matrices with constant diagonal. This algebra is called the Bose–Mesner algebra of the association scheme.

A Bose–Mesner algebra has a basis $\{E_0 = \frac{1}{n} J, E_1, \ldots, E_d\}$ of idempotents, that is, $E_i E_j = \delta_{i,j} E_i$, where $\delta_{i,j}$ is the Kronecker symbol. The change-of-coordinates matrix $P = [p_{ij}]$ defined by $X_j = \sum_i p_{ij} E_i$ has the property that $p_{ij}$ is an eigenvalue of $X_j$ whose eigenspace is the column space of $E_i$. The matrix $P$ of eigenvalues contains many properties of the scheme from which many parameters of the scheme (such as $a_{ij}^{\ell}$, etc.) can be obtained (see [7]). In this regard, the eigenvalues of different bases of an association scheme are important subjects worth studying.

The Johnson scheme $J(v, k)$ is a $k$-class association scheme in which the rows and the columns of each $X_i$ are indexed by all $k$-subsets of a $v$-set and $X_i(K_1, K_2) = 1$ if and only if $|K_1 \cap K_2| = k - i$, for $i = 0, 1, \ldots, k$. In other words, $X_i = U_{k,k,k-i}$. In this section, we introduce two new bases for the Bose–Mesner algebra of $J(v, k)$ and obtain the associated intersection numbers.

The first new basis for the Bose–Mesner algebra of $J(v, k)$ is $\{A_{k,k,i} : i = 0, \ldots, k\}$; this follows from the identities $U_{s,k,t,\ell} = \sum_{i=0}^{t} (-1)^{i-\ell} \binom{t}{i} A_{s,k,i}$ and $A_{s,k,i} = \sum_{\ell=0}^{t} \binom{t}{\ell} U_{s,k,t,\ell}$.

To introduce the second basis, we define the matrix $B_{s,k,\ell}$ as

$$B_{s,k,\ell}(S, K) = \begin{cases} 1 & \text{if } |S \cap K| \geq \ell, \\ 0 & \text{otherwise}. \end{cases}$$

Thus, we have $B_{s,k,\ell} = \sum_{t=\ell}^{k} U_{s,k,t,\ell}$ and $U_{s,k,\ell} = B_{s,k,\ell} - B_{s,k,\ell+1}$. This shows that the matrices $\{B_{s,k,\ell} : \ell = 0, \ldots, k\}$ form a basis for the Bose–Mesner algebra of $J(v, k)$. The relation between the two new bases is demonstrated below.
Proposition 6. If $\ell > 0$, then
\[
B_{s,k,\ell} = \sum_{i=\ell}^{s} (-1)^{i-\ell} {i - 1 \choose \ell - 1} A_{s,k,i}.
\] (8)

Proof. Let $G_{s,k}(z) = F_{s,k}(z - 1)$, $H_{s,k}(z) = \frac{1}{z-1}(G_{s,k}(z) - G_{s,k}(1))$, and $G^+_{s,k}(z) = \sum_{\ell} B_{s,k,\ell} z^\ell$. Then, $G_{s,k}(z) = \sum_{i=0}^{s} U_{s,k,\ell} z^\ell$ and $H_{s,k}(z) = \sum_{i=1}^{s} A_{s,k,i} (z - 1)^{i-1}$. Moreover, by $U_{s,k,\ell} = B_{s,k,\ell} - B_{s,k,\ell+1}$, we have
\[
(z - 1)G^+_{s,k}(z) = zG_{s,k}(z) - G_{s,k}(1)
= z\left(G_{s,k}(z) - G_{s,k}(1)\right) + (z - 1)G_{s,k}(1)
= z(z - 1)H_{s,k}(z) + (z - 1)G_{s,k}(1).
\]

Hence, $G^+_{s,k}(z) = zH_{s,k}(z) + G_{s,k}(1)$ and for $\ell > 0$, we get
\[
B_{s,k,\ell} = [z^\ell]G^+_{s,k}(z)
= [z^{\ell-1}]H_{s,k}(z)
= D^{\ell-1} \left( \sum_{i=1}^{s} A_{s,k,i} (z - 1)^{i-1} \right) \bigg|_{z=0}
= \sum_{i=\ell}^{s} (-1)^{i-\ell} {i - 1 \choose \ell - 1} A_{s,k,i}.
\]

Define the intersection numbers $r_{ij}^\ell$ and $p_{ij}^\ell$ by
\[
A_{k,k,i} A_{k,k,j} = \sum_{\ell=0}^{k} r_{ij}^\ell A_{k,k,\ell}, \text{ and } U_{k,k,i} U_{k,k,j} = \sum_{\ell=0}^{k} p_{ij}^\ell U_{k,k,\ell}.
\]
From (6) and (4), it follows that:

Proposition 7. The values of intersection numbers $r_{ij}^\ell$ and $p_{ij}^\ell$ are as follows:
\[
r_{ij}^\ell = \binom{v - i - j}{k - i - j + \ell} \binom{k - \ell}{i - \ell} \binom{k - \ell}{j - \ell}, \quad \text{and}
\]
\[
p_{ij}^\ell = \sum_{e=0}^{\ell} \binom{\ell}{e} \binom{k - \ell}{i - e} \binom{k - \ell}{j - e} \times \binom{v - 2k + \ell}{k - i - j + e}.
\]

5.2 Eigenvalues and Rank of Intersection Matrices
The eigenvalues of $U_{k,k,k-\ell}$ (for $\ell = 0, 1, \ldots, k$), the adjacency matrices of the Johnson scheme $J(v, k)$, can be expressed in terms of “Eberlein polynomials” (see [1, 7]), which
are

\[ E_j = \sum_{i=0}^{\ell} (-1)^{\ell-i} \binom{k-i}{\ell-i} \binom{k-j}{i} \binom{v-k+i-j}{i}, \]

with multiplicity \( \binom{v}{j} - \binom{v}{j-1} \), for \( j = 0, 1, \ldots, k \). In this section, we obtain the eigenvalues of \( F_{k,k,t}(z) \) and \( U_{k,k,k-\ell} \) as well as \( B_{k,k,k-\ell} \). The eigenvalues of \( W_{k,k}^T F_{k,k,j}(z) \) and \( W_{k,k}^T U_{k,k,k-\ell} \) are also determined. Moreover, we give a closed form for the eigenvalues and the rank of \( N_{k,k,k-1} \). The rank of \( U_{t,k,\ell} \) is also investigated. It is interesting that all the eigenvalues of the matrices above can be expressed in terms of the polynomials \( \mu_j(z) = \sum_{i=0}^{s} \binom{v-j-i}{i} \binom{v-j-i}{s} k^i \) for \( j = 0, 1, \ldots, t \).

The following lemma that gives the eigenvalues and the corresponding eigenvectors of \( A_{k,k,i} \) was proved by Wilson [19] with a proof based on Equation (6). The following decomposition of \( \mathbb{R}(\mathcal{G}) \) is used in [19]: fix \( k \) and let \( R_j \) denote the row-space of \( W_{j,k} \) over the field \( \mathbb{R} \). From (1), it follows that \( R_0 \subseteq R_1 \subseteq \cdots \subseteq R_k = \mathbb{R}(\mathcal{G}) \). Now, let \( V_0 = R_0 \), and \( V_j := R_j \cap R_{j-1}^\perp \) for \( j = 1, \ldots, k \). Then, \( \mathbb{R}(\mathcal{G}) = V_0 \oplus V_1 \oplus \cdots \oplus V_k \), and \( V_j \) has dimension \( \binom{v}{j} - \binom{v}{j-1} \). We note that, as it is well known, if \( s \leq k \leq v-s \), then \( \text{rank } W_{sk} = \binom{v}{s} \) (see [6, 11, 12, 22]); moreover, an explicit right inverse for \( W_{sk} \) in this case is given in [2, 11, 14].

**Lemma 8.** ([19]) With the above definitions, for any \( x \in V_j \), \( A_{k,k,t} x^\top = \lambda_j x^\top \), where

\[
\lambda_j = \begin{cases} 
\binom{v-j-i}{k-i} & \text{if } i \geq j, \\
0 & \text{otherwise.}
\end{cases}
\]

In other words, the vectors of \( R_j^\perp \) are eigenvectors corresponding the eigenvalue 0 and the vectors in \( V_j \), for \( j = 0, \ldots, t \) are eigenvectors corresponding the eigenvalue \( \binom{k-j}{j} \binom{v-j-i}{k-i} \).

The following theorem determines the eigenvalues of \( F_{k,k,t}(z) \). Before that, we need further definitions: fix \( k \) and let \( R_j(z) \) denote the row-space of \( W_{j,k} \) over the field of rational functions \( \mathbb{R}(z) \) and let \( V_0(z) := R_0(z) \), \( V_j(z) := R_j(z) \cap R_{j-1}(z)^\perp \) for \( j = 1, \ldots, k \). Note that a basis of \( R_j \) (resp. \( V_j \)) over the ground field \( \mathbb{R} \) is also a basis for \( R_j(z) \) (resp. \( V_j(z) \)) over the field \( \mathbb{R}(z) \).

**Theorem 9.** Let \( 0 \leq t \leq k \leq v/2 \). Consider \( F_{k,k,t}(z) \) as a matrix with entries in the field of rational functions \( \mathbb{R}(z) \). Then, the eigenvalues of \( F_{k,k,t}(z) \) are

\[
\mu_0(z)^{(\mathcal{G})}, \mu_1(z)^{(\mathcal{G})}-\mathcal{G}_{(\mathcal{G})}, \ldots, \mu_t(z)^{(\mathcal{G})}-\mathcal{G}_{(\mathcal{G})}, 0^{(\mathcal{G})}-\mathcal{G}_{(\mathcal{G})},
\]

where the exponents indicate the multiplicity and

\[
\mu_j(z) = \sum_{i=j}^{t} \binom{k-j}{i-j} \binom{v-j-i}{k-i} z^i,
\]

\( j = 0, 1, \ldots, t \).
for \( j = 0, 1, \ldots, t \). Furthermore, with the above notation, the vectors in \( V_j(z) \) are eigenvectors corresponding to \( \mu_j \), for \( j = 0, \ldots, t \). The vectors of \( R_t(z)^\perp \) are eigenvectors corresponding to the eigenvalue 0.

Proof. Considering \( F_{k,k,t}(z) = \sum_{i=0}^{t} A_{k,k,i} z^i \), the proof follows from Lemma 8. \( \square \)

Now, it is easily seen that Eberlin polynomials, defined at the beginning of this section, are obtained from the polynomials \( \mu_j(z) \), in the case \( t = k \) as follows:

\[
E_j = \frac{D^{k-\ell}}{(k-\ell)!} \mu_j(z) \bigg|_{z=\frac{-1}{\ell}}.
\]

**Corollary 10.** Let \( 0 \leq t \leq k \leq v/2 \). The eigenvalues of \( U_{k,k,t,\ell} \) are

\[
\lambda_j = \frac{D^\ell}{\ell!} \mu_j(z) \bigg|_{z=-1} = \sum_{i=\ell}^{t} (-1)^{i+1} \binom{i}{\ell} \binom{k-j}{i-j} \binom{v-j-i}{k-i},
\]

for \( j = 0, 1, \ldots, t \).

In the previous subsection, we saw that \( \{B_{k,k,\ell} : \ell = 0, \ldots, k \} \) gives a new basis for the Johnson scheme, so it is important to calculate their eigenvalues that are given in the following corollary.

**Corollary 11.** Let \( \ell \geq 0 \) and \( k \leq v/2 \) and let \( \mu_j(z) \) be as in (9) with the additional condition \( t = k \). Moreover, let

\[
v_j(z) = \frac{z \mu_j(z-1) - \mu_j(0)}{z-1}, \quad j = 0, 1, \ldots, k.
\]

The eigenvalues of \( B_{k,k,\ell} \) are \( \frac{D^\ell}{\ell!} v_j(z) \big|_{z=0} \) with multiplicities \( \binom{v}{j} - \binom{v-1}{j-1} \), for \( j = 0, 1, \ldots, k \). Furthermore, if \( \ell > 0 \), then

\[
\frac{D^\ell}{\ell!} v_j(z) \bigg|_{z=0} = \sum_{i=\ell}^{k} (-1)^{i+1} \binom{i-1}{\ell-1} \binom{k-j}{i-j} \binom{v-j-i}{k-i}.
\]

Proof. Considering the notation used in the proof of Proposition 6, we have

\[
G_{k,k}^+(z) = \frac{z}{z-1}(F_{k,k}(z-1) - F_{k,k}(0)) + F_{k,k}(0).
\]

Now the result follows from Theorem 9. \( \square \)

**Corollary 12.** Let \( k \leq v/2 \). Then
(i) the eigenvalues of $N_{k,k,t}$ are
\[ \lambda_j = (-1)^{k-t} \left( \frac{2k - v - 1}{k - j} \right) - \sum_{i=t+1}^{k} (-1)^{i-t} \left( \frac{k - j}{i - j} \right) \left( \frac{v - j - i}{k - i} \right); \]

(ii) the eigenvalues of the matrix $N_{k,k,k-1}$ are
\[ \lambda_0^{(t)}, \lambda_1^{(t)} - (v), \ldots, \lambda_{k-1}^{(t)} - \binom{v}{k-1}, 0^{(t)} - (v), \]
where
\[ \lambda_j = 1 - \left( \frac{2k - v - 1}{k - j} \right), \text{ for } j = 0, 1, \ldots, k - 1; \]

(iii) with $v = 2k$, rank $N_{k,k,k-1} = \frac{1}{2} \binom{2k}{k}$;
(iv) rank $N_{k,k,k-1} = \binom{v}{k-1}$ provided that $k < v/2$.

Proof.

(i) By Corollary 10 and (3), the eigenvalues of $N_{k,k,t} = (-1)^t U_{k,k,t,0}$ are
\[ \lambda_j = (-1)^{k-t} \left( \frac{2k - v - 1}{k - j} \right) - \sum_{i=t+1}^{k} (-1)^{i-t} \left( \frac{k - j}{i - j} \right) \left( \frac{v - j - i}{k - i} \right). \]

(ii) This is an immediate consequence of part (i).

(iii) Setting $v = 2k$ in part (ii) yields $\lambda_j = 1 + \binom{-1}{k-j} = 1 + (-1)^{k-j+1}$ for $j = 0, 1, \ldots, k - 1$. Hence, $\lambda_j \neq 0$ if and only if $j + k - 1$ is even. Therefore,
\[ \text{rank } N_{k,k,k-1} = \sum_{0 \leq j \leq k - 1 \atop 2 \mid j + k - 1} \left( \binom{2k}{j} - \binom{2k}{j-1} \right). \]

The result now follows from the identities $\sum_j \binom{2k}{j} = 2^{2k-1}$ and $2 \sum_j \binom{2k}{j} = 2^{2k} - \binom{2k}{2}$, where $j$ runs over the same set as in the above sum. (We remark that a direct proof is obtained simply by considering the entries of $N_{k,k,k-1}$.)

(iv) It is easily seen that in this case, $\lambda_j \neq 0$ for all $j = 0, 1, \ldots, k - 1$, thus
\[ \text{rank } N_{k,k,k-1} = \sum_{j=0}^{k-1} \left( \binom{v}{j} - \binom{v}{j-1} \right) = \binom{v}{k-1}. \]

\[ \square \]

Example 1. Considering Remark 1, we give eigenvalues of the matrices $N_{7,7,6}$ with $v = 14$ and $N_{6,6,5}$ with $v = 13$. For the first matrix, $\lambda_j = 1 - \binom{-1}{7-j} = 1 + (-1)^j$, for $j = 0, 1, \ldots, 6$. Thus, the set of eigenvalues is $\{2^{1716}, 0^{1716}\}$ and the rank of this matrix is 1716. For the second matrix, $\lambda_j = 1 - \binom{-2}{6-j} = 1 + (-1)^{j+1}(7 - j)$, for $j = 0, 1, \ldots, 6$. Thus, the set of eigenvalues is
\[ \{(-6)^1, 7^{12}, (-4)^6^5, 5^{208}, (-2)^{429}, 3^{572}, 0^{429}\} \]
and the rank of this matrix is 1287.

**Theorem 13.** Let \( 0 \leq t \leq s \leq k \leq v/2 \). Consider \( F_{s,k,t}(z) \) as a matrix with entries in the field of rational functions \( \mathbb{R}(z) \). Then, the eigenvalues of the matrix \( W_{s,k}^T F_{s,k,t}(z) \) are

\[
\alpha_0(z)^{\binom{v}{0}}, \alpha_1(z)^{\binom{v}{1}} - v_0, \ldots, \alpha_t(z)^{\binom{v}{t}} - v_{t-1}, 0^{\binom{v}{s}} - v_k,
\]

where

\[
\alpha_j(z) = L_{k,s} \mu_j(z) = (-1)^{k+s} \sum_{i=j}^t \binom{k-j}{i-j} \binom{v-j-i}{k-i} \binom{i-s-1}{k-s} z^i.
\]

**Proof.** Again, we remark that in Theorem 9, the eigenspace of a given eigenvalue of \( F_{k,k,t}(z) \) has a basis independent of \( z \). From this and the equation \( W_{s,k}^T F_{s,k,t} = L_{k,s} F_{k,k,t} \) (see the proof of Theorem 5), it turns out that the eigenvalues of \( W_{s,k}^T F_{s,k,t} \) are of the following form:

\[
\left( L_{k,s} \mu_0(z) \right)^{\binom{v}{0}}, \left( L_{k,s} \mu_1(z) \right)^{\binom{v}{1}} - v_0, \ldots, \left( L_{k,s} \mu_t(z) \right)^{\binom{v}{t}} - v_{t-1}, 0^{\binom{v}{s}} - v_k,
\]

which yields the result. \( \square \)

**Corollary 14.** Let \( 0 \leq s \leq k \leq v/2 \) and let \( \alpha'_j(z) \) be the polynomial obtained from \( \alpha_j(z) \) in the previous theorem by setting \( t = k \). Then, the eigenvalues of the matrix \( W_{s,k}^T U_{s,k,\ell} \) are

\[
\tau_0^{\binom{v}{0}}, \tau_1^{\binom{v}{1}} - v_0, \ldots, \tau_s^{\binom{v}{s}} - v_{s-1}, 0^{\binom{v}{s}} - v_k,
\]

where

\[
\tau_j = \left. \frac{D^\ell}{\ell!} \alpha'_j(z) \right|_{z=-1} = (-1)^{k+s+\ell} \sum_{i=\min(j,\ell)}^k (-1)^i \binom{i}{\ell} \binom{k-j}{i-j} \binom{v-j-i}{k-i} \binom{i-s-1}{k-s},
\]

for \( j = 0, 1, \ldots, s \). Hence,

\[
\text{rank } U_{s,k,\ell} = \sum_{0 \leq j \leq s} \left( \binom{v}{j} - \binom{v}{j-1} \right).
\]

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REFERENCES

[1] E. Bannai and T. Ito, Algebraic Combinatorics. I. Association Schemes, The Benjamin/Cummings Publishing Co., Inc., Menlo Park, CA, 1984.
[2] R. B. Bapat, Moore–Penrose inverse of set inclusion matrices, Linear Algebra Appl, 318 (2000), 35–44.
[3] T. Beth, D. Jungnickel, and H. Lenz, Design Theory, Cambridge University Press, Cambridge, MA, 1993.
[4] P. Blasiak, Combinatorics of boson normal ordering and some applications, Ph. D. Thesis, available at http://arxiv.org/pdf/quant-ph/0507206v2.
[5] P. Blasiak and Ph. Flajolet, Combinatorial models of creation-annihilation, preprint, available online at http://arxiv.org/abs/1010.0354.
[6] D. de Caen, A note on the ranks of set-inclusion matrices, Electron J Combin, 8 (2001), N5, 2 pp.
[7] P. Delsarte, An algebraic approach to the association schemes of coding theory, Philips Res Rep Suppl, 10 (1973), vi+97 pp.
[8] P. Frankl, Intersection theorems and mod p-rank of inclusion matrices, J Combin Theory Ser A, 54 (1990), 85–94.
[9] E. Ghorbani, G. B. Khosrovshahi, Ch. Maysoori, and M. Mohammad-Noori, Inclusion matrices and chains, J Combin Theory Ser A, 115 (2008), 878–887.
[10] R. L. Graham, D. E. Knuth, and O. Patashnik, Concrete Mathematics: A Foundation for Computer Science, 2nd ed., Addison Wesley Publishing Company, New York, 1994.
[11] R. L. Graham, S.-Y.R. Li, and W. C. W. Li, On the structure of t-designs, SIAM J Alg Discr Meth, 1 (1980), 8–14.
[12] J. E. Graver and W. B. Jurkat, The module structure of integral designs, J Combin Theory Ser A, 15 (1973), 75–90.
[13] G. B. Khosrovshahi, M. Mohammad-Noori, and B. Tayfeh-Rezaie, Classification of 6-(14, 7, 4) designs with nontrivial automorphism groups, J Combin Des, 10 (2002), 180–194.
[14] H. Kramer, Inversion of incidence mappings, Sém Lothar Combin, 39 (1997), Art. B39f, 20 pp.
[15] E. S. Kramer and D. M. Mesner, t-Designs on hypergraphs, Discrete Math., 15 (1976), 263–296.
[16] D. L. Kreher, An incidence algebra for t-designs with automorphisms, J Combin Theory Ser A, 42 (1986), 239–251.
[17] M. Mohammad-Noori; Some computational aspects of t-designs; Dejean’s conjecture and sturmian words, Ph. D. Thesis, Université Paris XI, 2005, available at https://sites.google.com/site/mortezamohammadnoori/publications/ph-d-thesis.
[18] H. F. Scherk, De Evolvenda Functione, Dissertationes Nonnullae Analyticae, Ph. D. Thesis, Berlin, 1823, Publicly available from Göttinger Digitalisierungszentrum (GDZ).
[19] R.M. Wilson, Incidence matrices of t-designs, Linear Algebra Appl, 46 (1982), 73–82.
[20] R.M. Wilson, The exact bound in the Erdös–Ko–Rado theorem, Combinatorica, 984, 247–257.
[21] R.M. Wilson, On the Theory of t-Designs, D. M. Jackson, S. A. Vanstone (Editors), Enumeration and Design (Proceeding of Waterloo Silver Jubilee Conference), Academic Press, Toronto, ON, 1984, pp. 19–51.
[22] R. M. Wilson, A diagonal form for the incidence matrices of t-subsets vs. k-subsets, European J Combin., 11 (1990), 609–615.