Photon propagator in skewon electrodynamics

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Electrodynamics with a local and linear constitutive law is used as a framework for models violating Lorentz covariance. The constitutive tensor of such a construction is irreducibly decomposed into three independent pieces. The principal part is the anisotropic generalisation of the standard electrodynamics. The two other parts, axion and skewon, represent non-classical modifications of electrodynamics. We derive the expression for the photon propagator in the Minkowski spacetime endowed with a skewon field. For a relatively small (antisymmetric) skewon field, a modified Coulom law is exhibited.

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I. INTRODUCTION. MAXWELL EQUATIONS IN LINEAR RESPONSE MEDIA

In Minkowski vacuum, the standard Maxwell electrodynamics is represented by a system of eight partial differential equations

\[ F_{[ij,k]} = 0, \quad \text{and} \quad F^{ij} \cdot J^j = J^i \quad (1.1) \]

for six independent components of the antisymmetric tensor \( F_{ij} \). Two tensors \( F^{ij} \) and \( F_{ij} \) appearing here are related by the use of the metric tensor

\[ F^{ij} = \frac{1}{2} \left( g^{im} g^{jn} - g^{in} g^{jm} \right) F_{mn} \quad (1.2) \]

The Lorentz and CPT invariance of Eqs.(1.1–1.2) is exhibited in the fact that propagation of electromagnetic waves in this system generates the perfect light cone structure that is a fundamental expression of Minkowski space-time geometry.

Modern field models, such as loop quantum gravity [1], [2], string theory [3], [4] and the very special relativity models [5], [6], often predict violation of the Lorentz and CPT invariance.

The violation of the local Lorentz symmetry would also violate the Einstein equivalence principle. In order to test the Einstein equivalence principle and to incorporate the Lorentz symmetry violation in electrodynamics, one starts, see [7], [8], [9],[10], [11], with a generic extension of the electromagnetic Lagrangian

\[ \mathcal{L} \sim \kappa^{ijkl} F_{ijkl} F_{ijkl} \quad (1.3) \]

Due to its definition, the phenomenological tensor \( \chi^{ijkl} \) is antisymmetric in two pairs of its indices and symmetric under the exchange of these pairs, \( \chi^{ijkl} = \chi^{klji} \). Altogether, it is left with 21 independent components. For most magnitudes of these parameters, the resulting field equations indeed have solutions with breaking CPT symmetry and doubled light cone structure (birefringence). A similar approach was worked out in the wider context of standard model called standard model extension (SME), [12], and in gravity, [13]. Even in a relatively simple case of electrodynamics, such construction is left with a big set of undetermined phenomenological parameters. Moreover, a physical meaning of the individual terms of the Lagrangian (1.3) and of the corresponding phenomenological dimensionless coefficients remains undefined.

In this paper, we apply an alternative approach based on a modified constitutive relation instead of a modified Lagrangian. In other words, we are dealing with a non-trivial vacuum as a type of a medium and thus apply some constructions familiar from solid state physics. In order to extend the vacuum system (1.1–1.2) to a non-trivial medium, it is modified in the following way, [14], [15]: One considers two antisymmetric tensors – the field strength \( F_{ij} \) and the excitation \( H^{ij} \). The field equations for this system are assumed to be of the standard Maxwell form:

\[ F_{[ij,k]} = 0, \quad \text{and} \quad H^{ij} \cdot J^j = J^i \quad (1.4) \]

Instead of the trivial relation (1.2), a generic linear local constitutive relation between two tensor fields,

\[ H^{ij} = \frac{1}{2} \chi^{ijmn} F_{mn} \quad (1.5) \]

is postulated. Since the metric tensor is not involved in Eqs.(1.4–1.5), this approach is often referred to as pre-metric electrodynamics. The construction presented by Eqs.(1.4–1.5) allows to describe different types of equivalence principle violation models [7], and Lorentz violation models in electrodynamics [10], [11]. It is wide enough to include in a natural way the vacuum birefringence [16], [17], the axion construction [7], [18], [19], [20], [21] and some special types of Finsler geometry [22], [23]. Recently an intensive theoretical study of wave propagation in the system (1.4–1.5) was provided, see [24], [25], [26], [27], [28], [29], [30]. An important output of this analysis is a proof that in the linear response system (1.4–1.5), the

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generalized Fresnel hypersurface (dispersion relation) is quartic in general. Thus birefringence turns out to be a generic property of these models.

In (1.5), the constitutive tensor $\chi^{ijmn}$ is not restricted by the pair interchanging symmetry, i.e., $\chi^{ijmn} \neq \chi^{mnij}$, in general. Consequently, $\chi^{ijmn}$ has 36 independent components, i.e, 15 extra components relative to the Kostelecky tensor $\kappa^{ijmn}$. This additional set of components forms a $GL(4)$-irreducible part of the constitutive tensor, called skewon, [15]. It is a main object of our current consideration.

In the pioneer work of Obukhov and Hehl [31], various types of the skewon media with the corresponding dispersion relations were outlined. In [32], we presented a new form of dispersion relation in the skewon sector of the premetric electrodynamics and derive some new features of this model, such as superluminal wave propagation and Higgs-type symmetry breaking. The analysis given in [31], [32], [33], [34], and [36] was restricted to the wave propagation problems, i.e., to the vacuum case. In the present paper, we extend this analysis to the non-empty regions and derive the explicit expression of the skewon modified photon propagator (Green’s function in the momentum representation). This tensor is important for a description of the interaction between waves and charges and for an account of the quantum effects. Usually, the propagator tensor in Lotents violation models is derived from the Lagrangian, see [37], [38], [39]. Since skewon represents the non-Lagrangian sector of the model, this method is not available in our case. Instead, we use an algebraic technique based on the notion of the second adjoint tensor, see [21], [40]. The skewon propagator is shown to have an ordinary form of the second order tensor that is singular on the doubled lightcone. Due to the anisotropic property of the skewon field, the propagator has a non zero antisymmetric part. We derive the expressions in the Feynman, Landau and Coulomb gauge. We apply the latter type of propagator for derivation of the skewon modification of the Coulomb law. The ellipsoidal equipotential surfaces in this case represent the violation of the isotropy in antisymmetric skewon model.

The organization of the paper is as follows. In the next section, we present the geometric optic approximation of the model (1.4–1.5). As a result, we arrive in a system of algebraic equations for the electromagnetic potential. The coefficients of this system are linear in the constitutive tensor components and quadratic in the wave covector. On the tangent bundle, these coefficients compose a second order tensor that we call optic tensor. In section 3 we define the photon propagator and derive its generic expression for the case of Maxwell’s system with a generic linear constitutive relation defined on a metric space. In section 4, we derive the explicit expression of the photon propagator for the case of a medium with a simplest Minkowski principal part but a most generic skewon part. Application of this formalism for specific examples of antisymmetric skewon is presented in section 5. Here we derive a specific example of the skewon modified Coulomb law. In the conclusion section, we discuss some extensions of our construction. Certain relevant algebraic facts and some explicit matrix calculations are given in Appendix.

II. CHARACTERISTIC SYSTEM AND DISPERSION RELATION

In order to discuss the solutions of the system (1.4)–(1.5), we start with the ansatz

$$F_{ij} = (1/2) (A_{i,j} - A_{j,i}) .$$ (2.1)

that solves the homogeneous equation in terms of vector potential. Substituting (1.5) and (2.1) into the second equation of (1.4), we have

$$\left(\chi^{ijmn} A_{m,n}\right)_{,j} = J^j .$$ (2.2)

Now we apply the geometric optics approximation. First we assume the tensor $\chi^{ijmn}$ to be a slowly varying function of the coordinates with respect to the fast varying function $A_m$. Consequently, Eq.(2.2) is approximated as

$$\chi^{ijmn} A_{m,nj} = J^i .$$ (2.3)

Using the Fourier transforms of the current

$$J^i(x) = \int j^i(q) \exp \left( i q_k x^k \right) d^4 q ,$$ (2.4)

and of the potential

$$A_m(x) = \int a_m(q) \exp \left( i q_k x^k \right) d^4 q ,$$ (2.5)

we obtain from Eq.(2.3) the characteristic system

$$\left(\chi^{ijmn} q_j q_n\right) a_m = j^i .$$ (2.6)

In the following, it is convenient to use the notation

$$M^{im} = \chi^{ijmn} q_j q_n .$$ (2.7)

We refer to $M^{ij}$ as the optic tensor. It is analogous to the acoustic (Christoffel) tensor of elasticity theory. Consequently, we rewrite Eq.(2.6) in the form of a general system of four linear equations for four independent variables

$$M^{im} a_m = j^i .$$ (2.8)

The basic feature of this system is that the matrix $M$ is singular and thus for given functions $M^{im}(q)$ and $j^i(q)$, the solution $a_m$ to Eq.(2.8) is not unique. Indeed, we immediately observe from Eq.(2.7) two linear relations

$$M^{ij} q_j = 0 , \quad \text{and} \quad M^{ij} q_i = 0 .$$ (2.9)

Being trivial consequences of the symmetries of $\chi^{ijmn}$, these relations nevertheless reflect the basic physical facts of electromagnetism:
(i) The first relation of Eq. (2.9) expresses the *gauge invariance* of our system: If $a_m$ is a solution of Eq. (2.8) then $a_m + C q_m$ is also a solution. This is an algebraic expression of the gauge invariance of the electromagnetic potential $A_i \rightarrow A_i + \partial_i \varphi$.

(ii) The second relation of Eq. (2.9) corresponds to the *electric charge conservation law*. Indeed, the multiplication of both sides of Eq. (2.8) by $q_i$ yields $j^i q_i = 0$. It is an algebraic expression of the charge conservation law $J^{k,i} = 0$.

Now we use the following algebraic fact (see [27] for the proof): For a tensor $M^{ij}$ satisfying the relations (2.9), the adjoint tensor $A_{ij} = \text{adj} M$ is proportional to the tensor square of the covector $q_i$, i.e.,

$$A_{ij} = \lambda(q) q_i q_j .$$  (2.10)

Here $\lambda(q)$ is a fourth order homogeneous polynomial of the wave covector $q_i$. We recall that a polynomial is called homogeneous of the order $r$ if for every constant $C$ it satisfies $\lambda(Cq) = C^r \lambda(q)$. Since the adjoint matrix is constructed from the determinants of $3 \times 3$ matrices, $\lambda(q)$ is a third-order homogeneous polynomial of the constitutive tensor $\chi$.

In source-free regions where $J^i = 0$, a non-trivial solutions to Eq. (2.8) exist only if $A_{ij} = 0$. Consequently, Eq. (2.10) yields the *dispersion relation* for the electromagnetic waves in the form

$$\lambda(q) = 0 .$$  (2.11)

In the Minkowski case, $\lambda(q) = q^2$. It yields the standard Lorentz light cone $q^2 = 0$. It is an algebraic expression of the charge conservation law $J^{k,i} = 0$.

### III. PROPAGATOR FOR LINEAR RESPONSE MEDIA

In this section, we study solutions of the system (2.8) in a region where $J^i \neq 0$. These solutions are represented by the Green functions (Green tensors). We use the momentum-space representation similar to the usually used in quantum field theory and refer to Green functions as *photon propagators*.

#### A. The propagator tensor and its gauge transformations

A formal solution of the linear system in Eq. (2.8) can be written as

$$a_i = D_{ik} j^k .$$  (3.1)

The tensor $D_{ik}$ is called the *propagator tensor*. In standard quantum electrodynamics based on the Minkowski constitutive tensor, it suffices to consider symmetric propagators, $D_{ik} = D_{ki}$. We will see, however, that in the case of a generic linear constitutive relation, the propagator tensor is asymmetric. Notice that the antisymmetric part of the photon propagator is already derived in the axion model [21].

Any second order tensor can be decomposed into symmetric and antisymmetric parts:

$$D_{ij} = \tilde{D}_{ij} + \tilde{D}_{ij} .$$  (3.2)

Here the symmetric part is denoted by

$$\tilde{D}_{ij} = D_{(ij)} = (1/2) (D_{ij} + D_{ji}) ,$$  (3.3)

while the antisymmetric part is denoted by

$$\tilde{D}_{ij} = D_{[ij]} = (1/2) (D_{ij} - D_{ji}) .$$  (3.4)

Evidently,

$$\tilde{D}_{ij} = \tilde{D}_{ji} , \quad \tilde{D}_{ij} = -\tilde{D}_{ji} .$$  (3.5)

In standard electrodynamics, propagator has only the simplest symmetric scalar part $D_{ik} \sim D g_{ik}$. Since the diagonal Minkowski metric is considered, it means the vectors $a_i$ and $j_i$ are proportional one to another. In particular, static charge creates an electric field only, and no magnetic. Additional terms in the propagator tensor must exhibit a richer class of phenomena.

Due to the electric charge conservation, $q_k j^k = 0$, the propagator $D_{ik}$ in Eq. (3.1) is defined only up to an additional term proportional to $q_k$. Also a term proportional to $q_i$ can be freely added to $D_{ik}$. It is due to the gauge invariance of the potential, $a_i \rightarrow a_i + C q_i$. Consequently, we arrive to the generic *gauge transformation of the propagator*

$$D_{ik} \rightarrow D_{ik} + q_i \phi_k + q_k \psi_i .$$  (3.6)

Here $\phi_k$ and $\psi_i$ are arbitrary functions of the wave covector $q_i$. We recall that for an ordinary symmetric propagator, the transformation (3.6) is used with only one arbitrary function, i.e., the relation $\phi_i = \psi_i$ is assumed. When $\phi_k$ and $\psi_i$ are treated as covectors, we obtain in (3.6) a Lorentz invariant gauge transformation. Alternatively, in some situations, it may be advantageous to use a non-invariant Coulomb-type gauge. In this case, some components of $\phi_k$ and $\psi_i$ are treated as fixed or even equal to zero.

We write the gauge transformation laws for the symmetric and antisymmetric parts separately

$$\tilde{D}_{ik} \rightarrow \tilde{D}_{ik} + q_i \rho_k + q_k \rho_i ,$$  (3.7)

and

$$\tilde{D}_{ik} \rightarrow \tilde{D}_{ik} + q_i \sigma_k - q_k \sigma_i .$$  (3.8)

Here,

$$\rho_k = (1/2)(\phi_k + \psi_k) , \quad \sigma_k = (1/2)(\phi_k - \psi_k) .$$  (3.9)
B. Feynman, Landau and Coulomb propagators

The transformation relations (3.6) may be used for simplification the expressions of the propagator.

1. Feynman propagator

Starting with an arbitrary propagator $D_{ik}$ and choosing certain special functions for $\psi_k$ and $\phi_i$, we can freely remove from $D_{ik}$ all the terms proportional to $q_i$ and $q_k$. In a correspondence with the usual practice, we will call such simplest propagator, without longitudinal components, as the Feynman propagator and denote it as $F_{D_{ik}}$. We denote by $\tilde{F}_{D_{ik}}$ its symmetric part, and by $\tilde{F}_{D_{ik}}$ its remaining antisymmetric part.

2. Landau propagator

The Landau propagator is defined to be purely transverse, i.e., $D_{ik}q^k = 0$. Notice that Feynman’s and Landau’s propagators are both covariant. In asymmetric case, the generalization of the Landau-gauge propagator $L_{D_{ik}}$ is naturally to define by two independent conditions

$$L_{D_{ik}}q^k = 0, \quad L_{D_{ik}}q^i = 0.\tag{3.10}$$

These equations are equivalent to two separate conditions for symmetric and antisymmetric parts respectively

$$L\tilde{D}_{ik}q^k = 0, \quad L\tilde{D}_{ik}q^i = 0.\tag{3.11}$$

The Landau propagator can be derived by the gauge transformation from the Feynman propagator. For the purposes of the derivation, it is convenient to deal with the symmetric part $L\tilde{D}_{ik}$ and antisymmetric part $L\tilde{D}_{ik}$ separately.

For the symmetric part of the Landau propagator, we write

$$L\tilde{D}_{ik} = F_{\tilde{D}_{ik}} + q_i\phi_k + q_k\phi_i.\tag{3.12}$$

We use the first gauge condition of (3.11) to get

$$F_{\tilde{D}_{ik}}q^k + q_i(q, \phi) + q^2\phi_i = 0.\tag{3.13}$$

Multiplying two sides of this equation by $q^i$, we obtain

$$(q, \phi) = -\frac{\Delta}{2q^2}, \quad \text{where} \quad \Delta = F_{D_{mn}}q^mq^n.\tag{3.14}$$

Consequently, Eq. (3.13) yields

$$\phi_i = -\frac{F_{\tilde{D}_{ik}}}{2q^2}q^i + \frac{\Delta}{2q^4}q_i.\tag{3.15}$$

From Eq.(3.12), the symmetric part of the Landau propagator is expressed now as

$$L\tilde{D}_{ik} = F_{\tilde{D}_{ik}} - (F_{\tilde{D}_{im}}q_k + F_{\tilde{D}_{km}}q_i) \frac{q^m}{q^2} + \frac{\Delta}{q^4}q_iq_k.\tag{3.16}$$

For the antisymmetric part of the propagator, we write

$$L\tilde{D}_{ik} = F_{\tilde{D}_{ik}} + q_i\psi_k - q_k\psi_i.\tag{3.17}$$

Contracting with $q^k$, we obtain

$$L\tilde{D}_{ik}q^k = F_{\tilde{D}_{ik}}q^k + q_i(\psi, q) - q^2\psi_i = 0.\tag{3.18}$$

This algebraic equation for $\psi$ yields $(\psi, q) = 0$. Consequently, we obtain now

$$\psi_i = F_{\tilde{D}_{ik}} \frac{q_k}{q^2}.\tag{3.19}$$

Hence, the relation between the antisymmetric parts of the Feynman and Landau propagators is given by

$$L\tilde{D}_{ik} = F_{\tilde{D}_{ik}} + (F_{\tilde{D}_{km}}q_i - F_{\tilde{D}_{im}}q_k) \frac{q^m}{q^2} + \frac{\Delta}{q^4}q_iq_k.\tag{3.20}$$

The sum of the right hand sides of Eqs.(3.16, 3.20) yields finally the expression of the Landau propagator in term of the Feynman one

$$L_{D_{ik}} = F_{D_{ik}} - (F_{D_{im}}q_k + F_{D_{km}}q_i) \frac{q^m}{q^2} + \frac{\Delta}{q^4}q_iq_k.\tag{3.21}$$

It is straightforward to check for this expression the identities $L_{D_{ik}}q^i = 0$ and $L_{D_{ik}}q^k = 0$.

3. Coulomb propagator

In static electromagnetic problems, it is useful to use a propagator that is covariant only under the group $SO(3)$ of 3-dimensional rotations. We denote such Coulomb-gauge propagator as $C_{D_{ij}}$. We use the $(1+3)$-decomposition of the wave vector as

$$q_i = (\omega, k_\alpha), \quad q^i = (-\omega, k^\alpha).\tag{3.22}$$

Here and in the sequel, the Greek indices take values in the range $\{1, 2, 3\}$. We use the standard Minkowski metric in the Cartesian coordinates $g^{ij} = \text{diag}(-1, 1, 1, 1)$. In the case of an asymmetric propagator, it is natural to extend the standard Coulomb gauge condition, $C_{D_{ij}k^\mu} = 0$, to a pair of two separate conditions for symmetric and skew-symmetric parts respectively,

$$C_{D_{ij}k^\mu} = 0, \quad C_{D_{ij}k^\mu} = 0.\tag{3.23}$$

or, equivalently,

$$C\tilde{D}_{ij}k^\mu = 0, \quad C\tilde{D}_{ij}k^\mu = 0.\tag{3.24}$$

Let us start with the symmetric part, $C\tilde{D}_{ij}$. Using the transformation law (3.7), we write

$$C\tilde{D}_{ik} = F\tilde{D}_{ik} + q_i\rho_k + q_k\rho_i.\tag{3.25}$$

The first equation of (3.24) yields

$$C\tilde{D}_{ij}k^\mu = F\tilde{D}_{ij}k^\mu + q_i(k, \rho) + \rho_ik^2 = 0.\tag{3.26}$$
Here we use three dimensional notations: \( k^2 = k_\alpha k^\alpha \) and \( (k, \rho) = k^\alpha \rho_\alpha \). For \( i = 0 \), we have in Eq.(3.26)
\[
F \tilde{D}_{0\mu} k^\mu + \omega (k, \rho) + \rho_0 k^2 = 0,
\]
and, for \( i = \alpha \),
\[
F \tilde{D}_{\alpha \mu} k^\mu + k_\alpha (k, \rho) + \rho_\alpha k^2 = 0.
\]
Multiplying the latter equation by \( k^\alpha \) we derive
\[
(k, \rho) = -\frac{\mathcal{D}}{2k^2}, \quad \text{where} \quad \mathcal{D} = F \tilde{D}_{\alpha \beta} k^\alpha k^\beta.
\]
Consequently, Eqs.(3.27,3.28) yield
\[
\rho_0 = \frac{\omega^2 \mathcal{D}}{2k^4} - \frac{F \tilde{D}_{0\mu} k^\mu}{k^2},
\]
and
\[
\rho_\alpha = \frac{k_\alpha \mathcal{D}}{2k^4} - \frac{F \tilde{D}_{\alpha \beta} k^\beta}{k^2}.
\]
Substituting (3.30) and (3.31) into Eq.(3.7) we obtain for the symmetric part of the Coulomb propagator
\[
C \tilde{D}_{00} = F \tilde{D}_{00} + \frac{\omega^2 \mathcal{D}}{k^4} - \frac{2F \tilde{D}_{0\mu} \omega k^\mu}{k^2},
\]
\[
C \tilde{D}_{\beta 0} = F \tilde{D}_{\beta 0} + \frac{\omega k_\beta \mathcal{D}}{k^4} - \frac{k^\mu}{k^2} \left( F \tilde{D}_{\beta \mu} \omega + F \tilde{D}_{0\mu} k_\beta \right),
\]
and
\[
C \tilde{D}_{\alpha \beta} = F \tilde{D}_{\alpha \beta} + \frac{k_\alpha k_\beta \mathcal{D}}{k^4} - \frac{k^\mu}{k^2} \left( F \tilde{D}_{\alpha \mu} k_\beta + F \tilde{D}_{\beta \mu} k_\alpha \right).
\]
We consider now the skew-symmetric Coulomb propagator. Due to the transformation law (3.7), it can be written as
\[
C \tilde{D}_{ik} = F \tilde{D}_{ik} + q_i \sigma_k - q_k \sigma_i,
\]
In (1 + 3)-notations, it reads
\[
C \tilde{D}_{0\alpha} = F \tilde{D}_{0\alpha} + \omega \sigma_\alpha - k_\alpha \sigma_0,
\]
and
\[
C \tilde{D}_{\alpha \beta} = F \tilde{D}_{\alpha \beta} + k_\alpha \sigma_\beta - k_\beta \sigma_\alpha.
\]
The solution of the Coulomb gauge condition \( \tilde{D}_{ik} k^\mu = 0 \) with the expressions (3.36) and (3.37) substituted is not unique (a term proportional to \( \sigma_0 \) can be freely added to \( \sigma_\alpha \)). But, for our purposes, we need only a partial solution. We observe that
\[
\sigma_0 = \frac{F \tilde{D}_{0\beta} k^\beta}{k^2}, \quad \sigma_\alpha = \frac{F \tilde{D}_{\alpha \beta} k^\beta}{k^2},
\]
is one of such solutions. Consequently we get the skew-symmetric part of the Coulomb propagator
\[
C \tilde{D}_{0\beta} = F \tilde{D}_{0\beta} + \frac{k^\mu}{k^2} \left( F \tilde{D}_{\beta \mu} \omega + F \tilde{D}_{0\mu} k_\beta \right),
\]
and
\[
C \tilde{D}_{\alpha \beta} = F \tilde{D}_{\alpha \beta} + \frac{k^\mu}{k^2} \left( F \tilde{D}_{\beta \mu} k_\alpha - F \tilde{D}_{\alpha \mu} k_\beta \right).
\]

C. Propagator of a generic linear response system

In order to derive the expression for \( D_{ik} \) for the singular system (2.8) we will use the second adjoint of the matrix \( M^ij \). It comes instead of the ordinary first adjoint (a factor of the inverse matrix) used for the regular systems. Please note that the adjoint tensors can be defined for any tensor and this concept does not require a metric structure. In Appendix A, we provide formal definitions and basic formulas related to the first and the second adjoint tensors. Recall that the first adjoint of a \((2,0)\)-order tensor \( M^{ij} \) is expressed by a \((0,2)\)-order tensor \( A_{ij} \). The full contraction of these two tensors is proportional to the determinant of the matrix, \( M^{ij} A_{jk} \sim \det M \). The second adjoint of a \((2,0)\)-order tensor \( M^{ij} \) is expressed by a \((0,4)\)-order tensor \( B_{ijkl} \). Its contraction with the tensor \( M^{ij} \) is proportional to the first adjoint \( A_{ij} \), see (A12).

Our goal is to derive the solution \( a_m \) of the singular system (2.8). It is instructive to recall how one deals with a similar regular system \( M^{ik} a_k = j^i \) in basic linear algebra. First one multiplies both sides of this equation by the adjoint matrix \( A_{ki} = \text{adj}(M) \). Due to the Laplace identity (A4), one remains with the equation \( \text{det}(M) a_i = A_{ik}j^k \). For the invertible matrix \( M \), the unique solution is obtained by dividing both sides of the equation by the scalar factor \( \text{det} M \), i.e. by forming the inverse matrix. In our case, the above procedure does not work because the matrix is not invertible, i.e., \( \text{det} M = 0 \).

For a singular system, we apply a similar procedure but with the use of the second adjoint \( B_{ijkl} \) instead of the first adjoint \( A_{ij} \). We multiply two sides of the equation
\[
M^{ij} a_j = j^i
\]
by \( B_{irks} \) to obtain
\[
B_{irks} M^{ij} a_j = B_{irks} j^i.
\]
Applying the second order Laplace expansion (A12), we rewrite this equation as
\[
A_{rs} a_k - A_{rk} a_s = B_{irks} j^i.
\]
Substituting here the expression for the adjoint (2.10), we obtain
\[
\lambda q_r (q_s a_k - q_k a_s) = B_{irks} j^i.
\]
The problem now is to extract the potential \( a_i \) from this equation. In [40], we provide a metric-free procedure based on the homogeneity of our algebraic equation. In the presence of the metric structure, the problem can be solved much simpler. First we multiply both sides of Eq.(3.44) by the metric tensor \( g^{rs} \) and use the Lorenz gauge condition \( a_s q^s = 0 \). Consequently, Eq.(3.44) takes the form
\[
\lambda q_r q_s a_s = -B_{ijrs} j^i.
\]
Contracting now two sides of this equation with the use of the metric tensor, we remain with

$$a_k = \frac{1}{\lambda q^2} g^{rs} B_{irks} j^i .$$  \hfill (3.46)

This expression can be made to satisfy the Lorenz gauge condition by adding a term of the form $C q_k$ with a specially chosen scalar $C$. Since we are interested in a generic propagator and remember the propagator gauge freedom (3.6), we do not need to redefine the potential. It is instructive to check our solution (3.46) by substituting it into the original equation (3.41). We calculate

$$M^{pk} a_k = \frac{1}{\lambda q^2} g^{rs} M^{pk} B_{irks} j^i .$$  \hfill (3.47)

Using the rule (A12), we obtain

$$M^{pk} a_k = \frac{1}{\lambda q^2} g^{rs} (\delta_i^p A_{rs} - \delta_r^p A_{is}) j^i .$$  \hfill (3.48)

Substituting here the expression for the adjoint (2.10), we get

$$M^{pk} a_k = \frac{1}{q^2} g^{rs} (\delta_i^p q_r q_s - \delta_r^p q_i q_s) j^i .$$  \hfill (3.49)

Using the charge conservation equation $j^k q_k = 0$, we obtain in the right hand side of Eq.(3.49) the $p$-th component of the current $j^p$. Thus (3.46) indeed represents a solution to Eq.(3.41). Even if it is only a special solution, the whole set of solutions can be readily reinstated by the gauge transformation of the potential.

From Eq.(3.46), we extract the propagator tensor

$$F_{Dij} = \frac{1}{\lambda q^2} B^{mijm} .$$  \hfill (3.50)

This expression is derived by removing the longitudinal terms, thus it must be treated as the Feynman-type propagator. Since the tensor $B^{mijm}$ is quadratic in the entries of the matrix $M_{ij}$, it is a homogeneous fourth order polynomial in the components of $q_i$. Also $\Lambda$ is a homogeneous fourth order polynomial in $q_i$. Consequently the propagator $F_{Dij} \sim q^{-2}$ as the ordinary non-modified photon propagator. From (3.21), we conclude that the Landau-type propagator has the same behavior $F_{Dij} \sim q^{-2}$.

### IV. PROPAGATOR FOR SKEWON MEDIA

In this section, we calculate the propagator (3.50) for a special case of a linear response medium – the Minkowski vacuum modified by a generic skewon field.

#### A. Optic tensors and skewon covector

The general constitutive tensor satisfies the symmetry relations

$$\chi^{ijmn} = -\chi^{jimn} = -\chi^{ijnm} ,$$  \hfill (4.1)

and thus has 36 independent components. Under the action of the general linear group $GL(4, \mathbb{R})$, this tensor is uniquely irreducibly decomposed into the sum of three independent pieces [15],

$$\chi^{ijkl} = (1)\chi_1^{ijkl} + (2)\chi_2^{ijkl} + (3)\chi_3^{ijkl} .$$  \hfill (4.2)

The principal part of 20 independent components is expressed as

$$(1)\chi_1^{ijkl} = \frac{1}{6} \left( 2\chi^{ijkl} + 2\chi^{klij} - \chi^{iklj} - \chi^{ijlk} - \chi^{ijkl} - \chi^{iklj} \right) .$$  \hfill (4.3)

It is a generalization of the standard Minkowski factor

$$(1)\chi_1^{ijkl} = g^{ik} g^{jl} - g^{il} g^{jk} .$$  \hfill (4.4)

appearing in (1.2). Two additional parts in (4.2) do not have classical analogs.

The axion part $(3)\chi_3^{ijkl}$ is represented by the complete antisymmetrization of $\chi^{ijkl}$ in all its four indices. This pseudotensor has only one independent component. Consequently, it is represented by a pseudoscalar $\alpha$,

$$(3)\chi_3^{ijkl} = \chi^{ijkl} = \alpha \epsilon^{ijkl} .$$  \hfill (4.5)

It is a rather popular object in modified electrodynamics.

The skewon part $(2)\chi_2^{ijkl}$ of 15 independent components is expressed as

$$(2)\chi_2^{ijkl} = \frac{1}{2} \left( \chi^{ijkl} - \chi^{klij} \right) .$$  \hfill (4.6)

It is a primary object of our interest in the current paper. In order to clarify the skewon contribution to electrodynamics, we will assume the principal part to be of the classical Minkowski form (4.4). Moreover, we assume the flat Minkowski metric $g_{ij} = \text{diag}(-1, +1, +1, +1)$. Note that in a more general case of electrodynamics on a curved pseudo-Riemann manifold, $(1)\chi_1^{ijkl}$ must be endowed with a factor $\sqrt{-g}$. In our case, this factor is equal to 1 and thus omitted.

With the constitutive tensor decomposed into the three independent pieces (4.2), we obtain the optic tensor decomposed into two parts

$$M^{ij} = P^{ij} + Q^{ij} .$$  \hfill (4.7)

Here the principal part of the optic tensor is given by

$$P^{ij} = (1)\chi_1^{imjn} q_m q_n ,$$  \hfill (4.8)

while the skewon part is

$$Q^{ij} = (2)\chi_2^{imjn} q_m q_n .$$  \hfill (4.9)

The axion part drops out from the optic tensor. Consequently, in the framework of geometric optics approximation, axion does not contribute to the dispersion relation and to the photon propagator.

The contributions of the principal and skewon parts to the optic tensor are of very special forms. In particular,
the principal part builds up the symmetric optic tensor while the skewon optic tensor is skew-symmetric
\begin{equation}
P^{ij} = P^{ji}, \quad Q^{ij} = -Q^{ji}, \quad (4.10)
\end{equation}
Moreover, both these matrices are singular. Indeed, we evidently have the identities
\begin{equation}
P^{ij} q_i = 0, \quad Q^{ij} q_i = 0, \quad (4.11)
\end{equation}
that mean linear relations between the rows and the columns of the corresponding matrices.
For the antisymmetric tensor $Q^{ij}$, we can derive from (4.11) a simple representation, see [32]. Indeed, the expression
\begin{equation}
Q^{ij} = \varepsilon^{ijkm} q_j Y_m, \quad (4.12)
\end{equation}
with an arbitrary covector $Y_m$ satisfies the second equation of (4.11). For a given tensor $Q_{ij}$, the skewon covector $Y_m$ is determined from (4.12) only up to an addition of a covector proportional to the wave covector, $Y_m \rightarrow Y_m + C q_m$. This is an independent gauge invariant property, that can be used for simplification of the calculations, see [33].

We consider the special Minkowski principal part (4.4). In this case, Eq. (4.8) yields
\begin{equation}
P^{ij} = g^{ij} q^2 - q^i q^j. \quad (4.13)
\end{equation}
The skewon part, however, is left to be of the generic form.

**B. First adjoint**

In this sub-section we present the first adjoint of the tensor $M^{ij}$ with the Minkowski symmetric part given in Eq.(4.13) and the skewon antisymmetric part represented in Eq.(4.12).

It is convenient to start with a more general case with arbitrary principal and skewon parts. In this case, the first adjoint tensor can be expressed as,
\begin{equation}
A_{ij} = (\lambda_0 + P^{mn} Y_m Y_n) q_i q_j, \quad (4.14)
\end{equation}
see [32] for details and proofs. Here the term $\lambda_0 q_i q_j$ is the adjoint of the principal part tensor $P^{ij}$ alone. Consequently, the $\lambda$-term for the skewon modified media is given by
\begin{equation}
\lambda = \lambda_0 + P^{mn} Y_m Y_n. \quad (4.15)
\end{equation}
For source-free regions with $j^i = 0$, a non-trivial solution (electromagnetic wave) exists if and only if the adjoint is equal to zero. This condition represents the dispersion relation that in our case takes the form
\begin{equation}
\lambda_0 + P^{mn} Y_m Y_n = 0. \quad (4.16)
\end{equation}
For the Minkowski principale part (4.4) with the tensor $P^{ij}$ given in Eq.(4.13), the $\lambda$-term (4.15) takes a simple form [32]
\begin{equation}
\lambda = q^4 - Y^2 q^2 + (Y, q)^2. \quad (4.17)
\end{equation}
The dispersion relation, in this case, is represented by a 4-th order homogeneous equation
\begin{equation}
q^4 - Y^2 q^2 + (Y, q)^2 = 0. \quad (4.18)
\end{equation}

**C. Second adjoint**

We calculate now the second adjoint of the optic tensor $M^{ij}$. Substituting the decomposition of the constitutive tensor (4.7) into the expression for the second adjoint
\begin{equation}
B_{ijkl} = \frac{1}{2!} \varepsilon_{ijkl \sigma} M^{\alpha \beta} M^\beta_{\sigma}, \quad (4.19)
\end{equation}
we obtain it decomposed into a sum of three pieces
\begin{equation}
B_{ijkl} = (1) B_{ijkl} + (2) B_{ijkl} + (3) B_{ijkl}, \quad (4.20)
\end{equation}
Here, $(1) B_{ijkl}$ denotes the $P^2$-term expressed as
\begin{equation}
(1) B_{ijkl} = \frac{1}{2!} \varepsilon_{ijmn} \varepsilon_{klrs} P^{mr} P^{ns}. \quad (4.21)
\end{equation}
Substituting here the expression (4.13) we calculate, see Appendix B,
\begin{equation}
(1) B_{ijkl} = q^2 (q_i q_k g_{jl} - q_j q_k g_{il} + q_j q_l q_{ik} - q_i q_l g_{jk}). \quad (4.22)
\end{equation}
The contraction of this expression reads
\begin{equation}
(1) B_{ijkl} g^{ik} = - (q^2 g_{il} + 2 q_i q_l) q^2. \quad (4.23)
\end{equation}
This is an expression related to the standard electrodynamics. The term $(2) B_{ijkl}$ represents the mixed $PQ$-contribution
\begin{equation}
(2) B_{ijkl} = \varepsilon_{ijmn} \varepsilon_{klrs} P^{mr} Q^{ns}. \quad (4.24)
\end{equation}
Calculations provided in Appendix B give
\begin{equation}
(2) B_{ijkl} = \varepsilon_{ijmn} \varepsilon_{klrs} P^{mr} Q^{ns}. \quad (4.25)
\end{equation}
The contraction of this expression yields
\begin{equation}
(2) B_{ijkl} g^{ik} = - \varepsilon_{ijml} P^{lm} q^2. \quad (4.26)
\end{equation}
The third term $(3) B_{ijkl}$ represents the pure skewon contribution
\begin{equation}
(3) B_{ijkl} = \frac{1}{2!} \varepsilon_{ijmn} \varepsilon_{klrs} Q^{mr} Q^{ns}. \quad (4.27)
\end{equation}
Its computation in Appendix B yields
\begin{equation}
(3) B_{ijkl} = (q_i Y_j - q_j Y_i)(q_k Y_l - q_l Y_k), \quad (4.28)
\end{equation}
The contraction of this expression reads
\begin{equation}
(3) B_{ijkl} g^{ik} = (q, Y)(q_i Y_j + q_j Y_i) - q^2 Y_l Y_i Y_j - Y^2 q_i q_l. \quad (4.29)
\end{equation}
D. Propagator

Substituting (4.23,4.26) and (4.29) into the formula (3.50) we obtain the propagator tensor

$$D_{ij} = \frac{1}{\lambda q^2} \left( (q^2 g_{ij} + 2q_i q_j) q^2 + \epsilon_{ikmj} q^k Y^m q^2 + q^2 Y_i Y_j + Y^2 q_i q_j - (q, Y)(q_i Y_j + q_j Y_i) \right).$$

We can simplify this expression by applying the gauge invariance of the propagator. In order to have the Feynman propagator, we merely remove from this expression all the terms proportional to the components $q_i$ and $q_j$. Consequently, the $q^2$ factor in the denominator cancels out and we are left with the following compact expression for the Feynman propagator

$$D_{ij} = -\frac{1}{\lambda} \left( g_{ij} q^2 + \epsilon_{ikmj} q^k Y^m + Y_i Y_j \right).$$

Explicitly, it reads

$$D_{ij} = -g_{ij} q^2 + \epsilon_{ikmj} q^k Y^m + Y_i Y_j.$$

The Landau propagator also takes a simple form

$$L_{ij} = -\frac{(g_{ij} q^2 - q_i q_j) + \epsilon_{ikmj} q^k Y^m + Y_i Y_j}{q^4 - q^2 Y^2 + (q, Y)^2}.$$

Observe some basic properties of these propagators:

1. For the vanishing skewon field, the propagators return to their standard vacuum form

$$F_{ij} = \frac{g_{ij}}{q^2}, \quad L_{ij} = -\frac{g_{ij} q^2 - q_i q_j}{q^2}.$$  

The same is true also for the non-zero but physically trivial skewon covector $Y_i \sim q_i$.

2. Skewon propagator has nontrivial symmetric and antisymmetric parts. In the case of the Feynman gauge, they are expressed as

$$F_{(ij)} = -\frac{g_{ij} q^2 + Y_i Y_j}{q^4 - q^2 Y^2 + (q, Y)^2},$$  

and

$$F_{[ij]} = -\frac{\epsilon_{ikmj} q^k Y^m}{q^4 - q^2 Y^2 + (q, Y)^2},$$

respectively.

3. The propagator is singular in at most four roots of the denominator. As a result, there are at most two light cones at every space-time point – biofringence.

4. If the skewon covector $Y_i$ is regular in $q$ then in the first order approximation for a small skewon field

$$F_{(ij)} = \frac{g_{ij}}{q^2}, \quad F_{[ij]} = -\frac{\epsilon_{ikmj} q^k Y^m}{q^4}.$$  

Thus, the small skewon field affects only the antisymmetric part of the propagator.

V. SKEWON AND MODIFIED COULOMB LAW

Since skewon modifies the propagator expression, it is naturally to expect the corresponding modification of the basic electrodynamics relationships, in particular, the Coulomb law. In order to describe such modification, we need an expression of propagator in the Coulomb gauge.

A. Skewon propagator in the Coulomb gauge

Since we are dealing with the static problem, we need only one component of the propagator in the Coulomb gauge, namely $C_{D_{00}}$. From Eq. (4.31), we extract the following symmetric components:

$$F_{D_{00}} = \frac{q^2 - Y_0^2}{\lambda}, \quad F_{D_{0 \alpha}} = -\frac{Y_0 Y_\alpha}{\lambda},$$

and

$$F_{D_{\alpha \beta}} = -\frac{q^2 g_{\alpha \beta} + Y_\alpha Y_\beta}{\lambda}.$$  

With these expressions, we calculate the temporal part (3.32) of the skewon propagator in the Coulomb gauge. It reads

$$C_{D_{00}} = \frac{q^4 k^2 - (k^2 Y_0^2 - \omega(k, Y))^2}{\lambda k^4},$$

where we use the 3-dimensional scalar product $(k, Y) = g^{\alpha \beta} k_\alpha Y_\beta$. Applying the Lorentz gauge for the skewon covector, $(Y, q) = 0$, we replace the scalar product $(k, Y)$ by $\omega Y_0$ and rewrite this expression in the form

$$C_{D_{00}} = \frac{q^4 (k^2 - Y_0^2)}{\lambda k^4}.$$  

With the expression of the $\lambda$-function listed in Eq.(4.17), we have here

$$C_{D_{00}} = \frac{q^4 (k^2 - Y_0^2)}{k^4 (q^2 - Y^2 + (q, Y)^2)}.$$  

For zero skewon, we are left here with the standard expression $C_{D_{00}} = 1/k^2$.

B. Antisymmetric skewon

We consider now some explicit model of the skewon field. As it is shown in [15], a generic skewon $^{(2)}\chi^{ijkl}$ of 15 independent components can be uniquely represented by a traceless matrix $S^{ij}$. Since we are dealing with the space endowed with a metric tensor $g_{ij}$, the mixed tensor $S^{ij}$ can be replaced by the covariant and contravariant tensors

$$S^{ij} = g_{ik} S_k^j, \quad S_{ij} = g_{jk} S^{kj}.$$  

(5.6)
These two tensors depend not only of the skewon itself, but also of the metric tensor. In this paper, we restrict ourselves to the Minkowski metric, so this fact is irrelevant. The tensors (5.6) have some advantage because they can be invariantly decomposed into symmetric and skew-symmetric parts. These two parts do not mix under coordinate transformations, those they can be studied separately.

Up to an addition of an arbitrary term proportional to the wave covector $q_i$, the skewon covector $Y_i$ can be presented in the form

$$Y_i = S_{ij} q^j.$$  

(5.7)

The case of an antisymmetric matrix $S_{ij} = -S_{ji}$ is especially simple because it provides the skewon covector of the regular polynomial form (5.7) even for the Lorentz-type gauge, $(Y, q) = 0$. We use here the wave covector parametrized as $q = (\omega, k^\mu)$. Observe that the skewon (5.9) indeed satisfies the gauge condition $(Y, q) = 0$.

We calculate the square of the skewon covector

$$Y^2 = \alpha^2 \omega^2 - (\alpha, k)^2.$$  

(5.10)

Consequently, the denominator of the propagator takes the form

$$\lambda = (\omega^2 - k^2) \left( (1 + \alpha^2) \omega^2 - (\alpha, k)^2 - k^2 \right).$$  

(5.11)

The zero set of this function has two branches – two light cones. In addition to the ordinary circular light cone

$$\omega^2 = k^2;$$  

(5.12)

there is an elliptic light cone

$$(1 + \alpha^2) \omega^2 = (\alpha, k)^2 + k^2.$$  

(5.13)

Comparing the expressions (5.12) and (5.13) we read off the following consequences, see [32] for detailed discussion:

1. For an arbitrary real non-zero vector $\alpha_\mu$, there are two separated cones. In optics, this behavior is refereed to as birefringence.

2. The elliptic cone is exterior to the circular one. It means superluminal wave propagation along it. In other words, the causality is broken down in this model for an arbitrary non-zero vector $\alpha_\mu$.

3. The cones meet one another along two straight lines – optic axes.

C. Modification of the Coulomb law

In order to study how the skewon field modifies the Coulomb law, we use the skewon propagator in the Coulomb gauge. Substituting (5.9 – 5.10) into (5.5), we obtain

$$C_{D_{00}} = \frac{(-\omega^2 + k^2)(k^2 - (\alpha, k)^2)}{k^4 \left( -\omega^2(1 + \alpha^2) + k^2 + (\alpha, k)^2 \right)}.$$  

(5.14)

In the non-relativistic limit, i.e., neglecting the radiation effects, we can neglect the term $\omega^2/c^2$ relative to $k^2$. Consequently we are left with

$$C_{D_{00}} = \frac{1}{k^2} \frac{k^2 - (\alpha, k)^2}{k^2 + (\alpha, k)^2}.$$  

(5.15)

Due to the experimental bounds provided recently by Ni, see [34],[35], the skewon effect must be very small. Thus we use the approximation $(\alpha, k)^2 << k^2$. Consequently we obtain the temporal part of the skewon modified photon propagator in the form

$$C_{D_{00}} = \frac{1}{k^2} \left( 1 - 2 \frac{(\alpha, k)^2}{k^2} \right).$$  

(5.16)

The electromagnetic potential for a point-wise charge $q$ is calculated by the inverse Fourier transform

$$\varphi(r) = q \int C_{D_{00}} e^{i(r, k)} \frac{d^3k}{(2\pi)^3}.$$  

(5.17)

The first term of (5.16) provides the standard Coulomb potential $\varphi = q/(4\pi r)$. The second term presents the skewon addition, that is given by the integral

$$\delta \varphi = -2q \int \frac{(\alpha, k)^2}{k^4} e^{i(r, k)} \frac{d^3k}{(2\pi)^3}.$$  

(5.18)

This integral expression is familiar from the Breit equation in QED, see [41]. We have the result of the integration

$$\delta \varphi = -\frac{q}{4\pi r} \left( \alpha^2 - \frac{(\alpha, r)^2}{r^2} \right).$$  

(5.19)

Consequently, the skewon addition provides anisotropic modifications of the Coulomb potential

$$\varphi = -\frac{q}{4\pi r} \left( 1 - \alpha^2 + \frac{(\alpha, r)^2}{r^2} \right).$$  

(5.20)

The corresponding equipotential surfaces are ellipsoids. They are similar to the wave-front surfaces of the skewon model [36].

VI. RESULTS AND DISCUSSION

For a generic skewon part of the constitutive tensor, we evaluated the modified photon propagator in Feynman,
Landau and Coulomb gauges. This tensor is asymmetric and singular on the Fresnel hypersurface. It is quite naturally to guess that the same features will emerge for a more generic medium with an arbitrary linear response tensor. In the lowest order approximation, we derived the anisotropic modification of the Coulomb law.

Some natural problems arise in this context:

- What physical interpretation can be given to the antisymmetric part of the propagator?
- Which new types of the modification of the Coulomb law emerge from the higher order approximations?
- How the two branches of the birefringent Fresnel hypersurface influence the modified Coulomb law?
- Which new observational effects such modification can provide for the atom spectrum? These effects must be similar to the recently calculated splitting of the energy levels originated in the Finsler-type modifications of the Coulomb law [23].

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Appendix A: Determinant and adjoints

1. Determinant of a tensor

In four-dimensional space, the determinant of a covariant (up-indexed) second order tensor $M^{ij}$ can be written as

$$\det M = \frac{1}{4!} \epsilon_{ijkl} \epsilon_{prst} M^{ip} M^{jr} M^{ks} M^{lt}. \quad (A1)$$

The Levi-Civita symbol takes the values $\epsilon_{ijkl} = \{+1, -1, 0\}$, for even, odd, and no permutation of the indices $\{0123\}$, respectively; the analogous is assumed for the up-indexed Levi-Civita’s symbol $\epsilon^{ijkl}$. It is can be checked straightforwardly that the expression on the right-hand side of Eq.(A1) includes all proper products of the elements of the matrix $M^{ij}$ with the proper signs that usually appear in the determinant. The leading factor $1/4!$ provides the correct value of the determinant, thus the formula in Eq.(A1) is proved.

2. First adjoint tensor

The cofactor matrix of a square matrix is constructed by removing one row and one column of the original matrix and calculating the determinants of the remaining matrices. The transpose of the cofactor matrix is called (classical) adjoint or adjugate, or adjunct matrix. In order to simplify the formulas above we use the adjoint without transpose.

For a covariant (up-indexed) $4 \times 4$ tensor $M^{ij}$ we define the adjoint tensor $A_{ij} = \text{adj}(M)$ to be expressed by a contravariant (down-indexed) second order tensor

$$A_{ij} = \frac{1}{3!} \epsilon_{ijabc} \epsilon_{jdef} M^{ad} M^{be} M^{cf}. \quad (A2)$$

For some applications, it is convenient to see the adjoint tensor as a derivative of the determinant relative to the tensor entries

$$A_{ij} = \frac{\partial \det M}{\partial M^{ij}}. \quad (A3)$$

The classical Laplace expansion of the determinant is written in tenor notations as

$$M^{ij} A_{kj} = (\det M) \delta_{ik}, \quad (A4)$$

or,

$$M^{ij} A_{ik} = (\det M) \delta_{lj}. \quad (A5)$$

Here the first equation represents the expansion of a determinant along its rows while the second equation means the expansion along the columns. This equation follows straightforwardly from the definitions (A1–A2). Using the transpose of $A_{kj}$ we return in Eq.(A4) to the classical expansion of the determinant with the standard ”row-times-column” rule.

In the non-singular case, $\det(M) \neq 0$, tensor $M^{ij}$ has an inverse tensor that is necessary down-indexed. It is expressed as

$$(M^{-1})_{ij} = \frac{1}{\det(M)} A_{ij}. \quad (A6)$$

3. Second adjoint

The second adjoint is defined by removing two rows and two columns from the original matrix. Formally it can be written as

$$B_{ijkl} = \frac{1}{2!} \epsilon_{ijabc} \epsilon_{kld} M^{ac} M^{bd}. \quad (A7)$$

From this expression, we read off the symmetries

$$B_{ijkl} = -B_{jikl} = -B_{ijkl}. \quad (A8)$$

We can also arrange this tensor as the derivative of the first adjoint tensor and even of the determinant itself

$$B_{ijkl} = \frac{\partial A_{ik}}{\partial M^{jl}} = \frac{\partial^2 \det M}{\partial M^{ik} \partial M^{jl}}. \quad (A9)$$

The Laplace identity for the second adjoint is readily derived from the identity given in Eq.(A4). Taking a
We multiply the last row and the last column of the second Laplace identity
\[ (A5) \]
\[ \delta^{s}_{r} \delta^{l}_{s} A_{kj} + M^{ij} B_{ijkl} = \delta^{l}_{r} A_{rs} . \]  
(A11)
Consequently, the Laplace identity reads
\[ M^{ij} B_{ijkl} = \delta^{l}_{r} A_{rs} - \delta^{l}_{r} A_{ks} . \]  
(A12)
We observe some consequences of this identity. Contracting the indices \( i \) and \( s \), we get
\[ M^{ij} B_{krj} = A_{rk} - A_{kr} . \]  
(A13)
This is in a correspondence with the symmetries (A8). Contracting of (3.42) relative to the indices \( i \) and \( k \), we have
\[ M^{ij} B_{irs} = 3 A_{rs} . \]  
(A14)
The latter identity is a consequence of the order 3 homogeneity of the adjoint tensor.
Using the same derivative procedure for the identity (A5) we derive the second Laplace identity
\[ M^{ij} B_{irs} = \delta^{l}_{r} A_{rs} - \delta^{l}_{r} A_{rk} . \]  
(A15)

**Appendix B: Calculation of the second adjoint**

In this section, we provide the details of the calculations of the second adjoint (4.21).

1. Calculation of \((1) B_{ijkl}\)

Substituting (4.13) into the first part of (4.21) we have
\[ (1) B_{ijkl} = \frac{1}{2!} \varepsilon_{ijmn} \varepsilon_{klrs} P^{mr} P^{ns} \]
\[ = \frac{1}{2} \varepsilon_{ijmn} \varepsilon_{klrs} \left( g^{mr} q^2 - q^m q^r \right) \left( g^{ns} q^2 - q^n q^s \right) \]  
(B1)
Using the standard formulas for the contraction of two \( \varepsilon \)-symbols, we obtain
\[ (1) B_{ijkl} = q^4 \left| \begin{array}{ccc} g_{ik} & g_{il} & -q^2 g_{ir} \\ g_{jk} & g_{jl} & q_i \\ g_{mk} & g_{ml} & g_{mr} \end{array} \right| \]  
(B2)
We multiply the last row and the last column of the second determinant by \( q^r \) and \( q^m \) correspondingly
\[ (1) B_{ijkl} = q^4 \left| \begin{array}{ccc} g_{ik} & g_{il} & q_i \\ g_{jk} & g_{jl} & q_i \\ q_k & q_l & q^2 \end{array} \right| \]  
(B3)
Evaluating the second determinant, we obtain
\[ (1) B_{ijkl} = q^2 \left( q_j g_{ik} g_{jl} - q_i g_{jk} g_{jl} \right) . \]  
(B4)
Consequently,
\[ (1) B_{ijkl} = q^2 \left( q_ik q_k g_{jl} - q_j g_k g_{ik} + q_j q_l g_{ik} - q_i q_l g_{jk} \right) . \]  
(B5)

2. Calculation of \((2) B_{ijkl}\)

Let us calculate the mixed term of the second adjoint
\[ (2) B_{ijkl} = \varepsilon_{ijmn} \varepsilon_{klrs} P^{mr} Q^{ns} \]
\[ = \varepsilon_{ijmn} \varepsilon_{klrs} \left( g^{mr} q^2 - q^m q^r \right) \varepsilon_{nspq} q_p Y_q \]  
(B6)
Evaluating the product of two \( \varepsilon \)-symbols in the term of a determinant, we have
\[ (2) B_{ijkl} = \varepsilon_{ijmn} \left( g^{mr} q^2 - q^m q^r \right) q_p Y_q \left| \begin{array}{ccc} \delta^m_k & \delta^p_k & \delta^m_l \\ \delta^m_l & \delta^p_l & \delta^m_r \\ \delta^m_r & \delta^p_r & \delta^m_i \end{array} \right| \]
\[ = \varepsilon_{ijmn} \left( Y^m q^2 - q^m (q, Y) \right) \left| \begin{array}{ccc} \delta^m_k & \delta^p_k & \delta^m_l \\ \delta^m_l & \delta^p_l & \delta^m_r \\ \delta^m_r & \delta^p_r & \delta^m_i \end{array} \right| \]  
(B7)
Consequently, we obtain
\[ (2) B_{ijkl} = \left( \varepsilon_{ijmk} q_l - \varepsilon_{ijml} q_k \right) \left( Y^m q^2 - q^m (q, Y) \right) \]  
(B8)

3. Calculation of \((3) B_{ijkl}\)

Let us calculate this expression in the term of the skew-won optic covector. Substituting (4.12) into (4.27), we write it as
\[ (3) B_{ijkl} = \frac{1}{2!} \varepsilon_{ik_1i_2} \varepsilon_{j_1j_2} Q^{i_1j_1} Q^{i_2j_2} \]
\[ = \frac{1}{2} \left( \varepsilon_{ijmn} \varepsilon^{mrab} q_a Y_b \right) \varepsilon_{klrs} \varepsilon^{nscd} q_c Y_d \]  
(B9)
Using the standard formulas for the product of two permutation tensors, we have
\[ \varepsilon_{ijmn} \varepsilon^{mrab} q_a Y_b = \left| \begin{array}{ccc} \delta^r_i & \delta^s_i & \delta^b_i \\ \delta^r_j & \delta^s_j & \delta^b_j \\ \delta^r_n & \delta^s_n & \delta^b_n \end{array} \right| q_a Y_b \]  
(B10)
Similarly,

\[ \varepsilon_{klrs}e_{\text{nsc}}^{abcd} q_c Y_d = \begin{vmatrix} \delta^l_i & \delta^k_j & \delta^s_k & \delta^r_l \\ \delta^r_i & \delta^d_j & \delta^s_d & \delta^l_r \\ \delta^l_i & \delta^n_j & \delta^m_k & \delta^q_r \\ \delta^q_i & \delta^n_j & \delta^m_d & \delta^r_q \end{vmatrix} q_c Y_d = \begin{vmatrix} \delta^r_i & \delta^q_k \\ \delta^q_i & \delta^r_k \end{vmatrix} q_i Y_k . \]  

(B11)

Thus

\[ (3) B_{ijkl} = \frac{1}{2} \begin{vmatrix} \delta^q_i & q_i Y_j \\ \delta^q_i & q_i Y_k \end{vmatrix} \cdot \begin{vmatrix} \delta^r_i & q_i Y_l \\ \delta^r_i & q_i Y_r \end{vmatrix} \]  

(B12)

Expanding the third-order determinants, we have

\[ (3) B_{ijkl} = \frac{1}{2} \begin{vmatrix} q_j Y_i - \delta^r_i q_j Y_i \end{vmatrix} \begin{vmatrix} q_i Y_k - \delta^r_i q_i Y_k \end{vmatrix} \]  

(B13)

Term by term multiplication yields

\[ (3) B_{ijkl} = \begin{vmatrix} q_j Y_i \\ q_k Y_k \end{vmatrix} \begin{vmatrix} q_i Y_i \end{vmatrix} - \begin{vmatrix} q_j Y_j \end{vmatrix} \begin{vmatrix} q_k Y_k \end{vmatrix} = (q_j Y_k - q_k Y_j)(q_i Y_i - q_i Y_j) - (q_j Y_i - q_i Y_j)(q_k Y_k - q_k Y_i) = (q_i Y_j - q_j Y_i)(q_k Y_k - q_k Y_i). \]  

(B14)

Expanding these expressions, we come to a compact formula

\[ (3) B_{ijkl} = (q_i Y_j - q_j Y_i)(q_k Y_k - q_k Y_i). \]  

(B15)

It can be written also in a matrix form

\[ (3) B_{ijkl} = \begin{vmatrix} q_i Y_i \\ q_j Y_j \end{vmatrix} \begin{vmatrix} q_k Y_k \end{vmatrix} . \]  

(B16)