PARAMETRIC GALOIS EXTENSIONS

FRANÇOIS LEGRAND

Abstract. Given a field $k$ and a finite group $H$, an $H$-parametric extension over $k$ is a finite Galois extension of $k(T)$ of Galois group containing $H$ which is regular over $k$ and has all the Galois extensions of $k$ of group $H$ among its specializations. We are mainly interested in producing non $H$-parametric extensions, which relates to classical questions in inverse Galois theory like the Beckmann-Black problem and the existence of one parameter generic polynomials. We develop a general approach started in a preceding paper and provide new non parametricity criteria and new examples.

1. Presentation

The Inverse Galois Problem asks whether, for a given finite group $H$, there exists at least one Galois extension of $\mathbb{Q}$ of group $H$. A classical way to obtain such an extension consists in producing a Galois extension $E/\mathbb{Q}(T)$ with the same group which is regular over $\mathbb{Q}$: from the Hilbert irreducibility theorem, $E/\mathbb{Q}(T)$ has at least one specialization of group $H$ (in fact infinitely many if $H$ is not trivial).

In this paper we are interested in “parametric Galois extensions”, i.e. in finite Galois extensions $E/\mathbb{Q}(T)$ which are regular over $\mathbb{Q}$ - from now on, say for short that $E/\mathbb{Q}(T)$ is a “$\mathbb{Q}$-regular Galois extension” - and which have all the Galois extensions of $\mathbb{Q}$ of group $H$ among their specializations. More precisely, given a field $k$ and a finite group $H$, we say that a $k$-regular finite Galois extension $E/k(T)$ of group $G$ containing $H$ (with possibly $H \neq G$) is $H$-parametric over $k$ if any Galois extension of $k$ of group $H$ occurs as a specialization of $E/k(T)$ (definition 2.2). The special case $H = G$ is of particular interest.

This was introduced in our previous paper [Leg13b] in the number field case. Given a field $k$ and a finite group $G$, the question of whether there is a $G$-parametric extension over $k$ of group $G$ or not is intermediate between these classical two questions in inverse Galois theory: - if there is such an extension, then it obviously solves the Beckmann-Black problem for $G$ over $k$, which asks whether any Galois extension

\[ E \cap \mathbb{Q} = \mathbb{Q}. \] See §2.1 for basic terminology.
$F/k$ of group $G$ occurs as a specialization of some $k$-regular Galois extension $E_F/k(T)$ with the same group,
- if there are no such extension, then there obviously cannot exist a one parameter generic polynomial over $k$ of group $G$, i.e. a polynomial $P(T,Y) \in k(T)[Y]$ of group $G$ such that the splitting extension over $L(T)$ is $G$-parametric over $L$ for any field extension $L/k$.

We refer to §2.2 for more details.

If studying parametric extensions indeed seems a natural first step to these important topics, it is itself already quite challenging, especially over number fields. The question of deciding whether a given $k$-regular Galois extension of $k(T)$ of given group $G$ is $G$-parametric over a given base field $k$ or not indeed seems to be difficult, even for small groups $G$: for example, in the case $G = \mathbb{Z}/3\mathbb{Z}$ and $k = \mathbb{Q}$, the answer seems to be known for only one such extension (this extension is $\mathbb{Z}/3\mathbb{Z}$-parametric over $\mathbb{Q}$; see §1.1 below). Of course there are some obvious examples like the extensions $k(\sqrt[n]{T})/k(T)$ ($n \in \mathbb{N} \setminus \{0\}$) and $k(T)(\sqrt{T^2 + 1})/k(T)$: if $k$ contains the $n$-th roots of unity, the former is $\mathbb{Z}/n\mathbb{Z}$-parametric over $k$ (this follows from the Kummer theory) whereas, if $k \subset \mathbb{R}$, the latter is not $\mathbb{Z}/2\mathbb{Z}$-parametric over $k$ (since none of its specializations is imaginary). But they seem to be quite sparse.

1.1. Parametric extensions over various fields. In §2.3, we give some first conclusions on parametric extensions (based on previous works) over various base fields $k$ with good arithmetic properties such as PAC fields, finite fields or the field $\mathbb{Q}$ and its completions.

For example, in the case $k$ is PAC (§2.3.1), the situation is quite clear: any finite $k$-regular Galois extension of $k(T)$ is parametric over $k$ with respect to any subgroup of its Galois group. In contrast, in the case $k = \mathbb{Q}$ (§2.3.4), not much is known although it may be expected that only a few extensions are parametric. On the one hand, it is known that there is a $G$-parametric extension over $\mathbb{Q}$ of group $G$ for each of the four groups $\{1\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ and $S_3$. For any other one, it is unknown whether there exists such an extension or not. On the other hand, only a few non parametric extensions over $\mathbb{Q}$ are known.

1.2. First examples over $\mathbb{Q}$. In §3, we use ad hoc arguments to obtain some new examples of non $H$-parametric extensions over $\mathbb{Q}$ with small Galois groups $G$ and small branch point numbers (propositions 3.1, 3.3 and 3.5):

Theorem 1. (1) A given $\mathbb{Q}$-regular quadratic extension of $\mathbb{Q}(T)$ with two branch points is $\mathbb{Z}/2\mathbb{Z}$-parametric over $\mathbb{Q}$ if and only if each of them is $\mathbb{Q}$-rational.
(2) No $\mathbb{Q}$-regular Galois extension of $\mathbb{Q}(T)$ of group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with three branch points is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-parametric over $\mathbb{Q}$.

(3) The splitting extension over $\mathbb{Q}(T)$ of $Y^3 + T^2 Y + T^2$ is a four branch point $\mathbb{Q}$-regular Galois extension of group $S_3$ which is $H$-parametric over $\mathbb{Q}$ for no subgroup $H \subset S_3$.

The proof rests on the non-existence of solutions to some diophantine equations (for the first two parts) and on the non totally real behavior of the specializations (for the third part).

1.3. A systematic approach. In §4, we offer a systematic approach, already started in [Leg13b], to give more examples of non $H$-parametric extensions over $k$ of group $G$ containing $H$. Given a $k$-regular Galois extension $E_1/k(T)$ of group $H$ and a $k$-regular Galois extension $E_2/k(T)$ of group $G$, we provide two sufficient conditions which each guarantees that there exist some specializations of $E_1/k(T)$ of group $H$ which cannot be specializations of $E_2/k(T)$ (and so $E_2/k(T)$ is not $H$-parametric over $k$). The first one (Branch Point Hypothesis) involves the branch point arithmetic while the second one (Inertia Hypothesis) is a more geometric condition on the inertia of the two extensions $E_1/k(T)$ and $E_2/k(T)$. Theorem 4.2 is our precise result.

We work over base fields $k$ which are quotient fields of any Dedekind domain of characteristic zero with infinitely many distinct primes, additionally assumed to be hilbertian. Number fields or finite extensions of rational function fields $\kappa(X)$, with $\kappa$ an arbitrary field of characteristic zero (and $X$ an indeterminate), are typical examples.

1.4. Applications. In §5-7, we use our criteria to give new examples of non parametric extensions over various base fields.

1.4.1. A general result over suitable number fields. In §5, we obtain the following result (corollary 5.2) which leads to non $G$-parametric extensions of group $G$ over suitable number fields for many groups $G$.

**Theorem 2.** Let $G$ be a finite group. Assume that there exists some set $\{C_1, \ldots, C_r, C\}$ of non trivial conjugacy classes of $G$ satisfying the following two conditions:

1. the elements of $C_1, \ldots, C_r$ generate $G$,
2. the conjugacy class $C$ is a power of $C_i$ for no index $i \in \{1, \ldots, r\}$.

Then there exist some number field $k$ and some $k$-regular Galois extension of $k(T)$ of group $G$ which is not $G$-parametric over $k$.

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$^2$The case where there are only finitely many primes can also be considered. We refer to [Leg13a, chapter 3] where this situation was studied.
Many finite groups admit a conjugacy class set as above: abelian groups which are not cyclic of prime power order, symmetric groups $S_n$ ($n \geq 3$), alternating groups $A_n$ ($n \geq 4$), dihedral groups $D_n$ of order $n \geq 2$, non abelian simple groups, etc. See §5.1.1 for more details and references. Moreover the conclusion also holds if $k$ is any finite extension of the rational function field $\mathbb{C}(X)$ (§5.2) and, under some conjecture of Fried, one can even take $k = \mathbb{Q}$ (corollary 5.3).

1.4.2. Examples over given base fields. In §6 and §7, we give new examples of non $H$-parametric extensions of group $G$ containing $H$ over various given base fields $k$ (in particular over $k = \mathbb{Q}$).

To do so, we need to start from two $k$-regular Galois extensions of $k(T)$ with groups $H$ and $G$ respectively. This first step depends on the state-of-the-art in inverse Galois theory, especially in the case $k = \mathbb{Q}$, and the involved finite groups then are the classical ones in this context: abelian groups, symmetric groups, alternating groups, some other simple groups... We present our examples below in connection with those already given in [Leg13b].

(a) Examples from the Branch Point Criterion (§6). Let $k$ be a number field and $G$ a finite group. A first example is an improved version of a result of [Leg13b] for $k$-regular Galois extensions of $k(T)$ of group $G$ with four branch points. Here we drop the branch point number assumption and give pure branch point arithmetical conditions for a given $k$-regular Galois extension of $k(T)$ of group $G$ not to be $H$-parametric over $k$ for any given non trivial subgroup $H \subset G$ (corollary 6.1).

We give some concrete examples in the situation $G = \mathbb{Z}/2\mathbb{Z}$ (and so $H = \mathbb{Z}/2\mathbb{Z}$ too) where the existence of at least one $k$-regular Galois extension of $k(T)$ of group $G$ satisfying our conditions is guaranteed and which is already of some interest (corollary 6.4). Some further examples with $G = \mathbb{Z}/n\mathbb{Z}$ ($n \geq 2$) are given (corollaries 6.5 and 6.6).

(b) Examples from the Inertia Criterion (§7).

(i) Symmetric and alternating groups. A first example is an improved version of a result of [Leg13b] giving practical sufficient conditions for a given $k$-regular Galois extension of $k(T)$ of group $G = S_n$ ($n \geq 3$) not to be $G = S_n$-parametric over $k$ (§7.1.3); here $k$ is any of our allowed base fields (and even more general ones) while it was a number field in [Leg13b]. We also have an analog with $G = A_n$ (§7.2.3). Theorem 3 below is a consequence of our results:

**Theorem 3.** Let $r$ be an integer $\geq 3$ and $k$ a number field or a finite extension of the rational function field $\mathbb{C}(X)$. Then, for any integer
n \geq 8r^2$, no \( k \)-regular Galois extension of \( k(T) \) of group \( G = A_n \) with \( r \) branch points is \( G = A_n \)-parametric over \( k \).

The same conclusion holds with \( G = S_n \) over more general base fields.

Moreover our results show that several classical \( k \)-regular Galois extensions of \( k(T) \) of group \( S_n \) (resp. of group \( A_n \)) are not \( S_n \)-parametric (resp. \( A_n \)-parametric) over any of our allowed base fields \( k \). Corollaries 7.1 and 7.3 give our main examples.

(ii) Non abelian simple groups. We also show that some regular realizations of some simple groups \( G \) provided by the rigidity method are not \( G \)-parametric. For instance, using the Atlas \([C+85]\) notation for conjugacy classes of finite groups, we have (corollary 7.4):

Let \( p \) be a prime \( \geq 5 \) and \( k \) one of our allowed base fields such that \((-1)^{(p-1)/2}/p \) is a square in \( k \). Then no \( k \)-regular Galois extension of \( k(T) \) of group \( \text{PSL}_2(\mathbb{F}_p) \) provided by either one of the rigid triples \((2A, pA, pB) \) (if \((\frac{2}{p}) = -1 \)) and \((3A, pA, pB) \) (if \((\frac{3}{p}) = -1 \)) of conjugacy classes of \( \text{PSL}_2(\mathbb{F}_p) \) is \( \text{PSL}_2(\mathbb{F}_p) \)-parametric over \( k \).

We also have a similar result with the Monster group (corollary 7.5).

(iii) Examples with \( H \neq G \). We also have various examples which are specifically devoted to the case \( H \neq G \). For instance (corollary 7.6):

Let \( k \) be one of our allowed base fields. Then, with \( \text{Th} \) the Thompson group, no \( k \)-regular Galois extension of \( k(T) \) of group the Baby-Monster group \( B \) provided by the rigid triple \((2C, 3A, 5A) \) of conjugacy classes of \( B \) is \( \text{Th} \)-parametric over \( k \).

Further similar examples with various groups such as symmetric groups, other sporadic groups or \( p \)-groups are given (corollaries 7.7 and 7.8).

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2. Basics on parametric extensions

We first set up in §2.1 the terminology for the basic notions we will use in this paper. We then point out in §2.2 some connections between parametric extensions and some classical notions in inverse Galois theory and investigate in §2.3 such extensions over various fields.

2.1. Basic definitions. Let \( k \) be a field and \( \overline{k} \) an algebraic closure of \( k \). Denote the separable closure of \( k \) in \( \overline{k} \) by \( k^{\text{sep}} \) and its absolute Galois group by \( G_k \). Let \( E/k(T) \) be a finite Galois extension which is
regular over \( k \) (i.e. \( E \cap \overline{k} = k \)) and \( G \) its Galois group. To make the exposition simpler, say for short that \( E/k(T) \) is a “\( k \)-regular Galois extension of group \( G \)”.

For more on §2.1.1-3, we refer for example to [Déb09, chapter 3].

2.1.1. Branch points. Denote the integral closure of \( k[T] \) (resp. of \( k[1/T] \)) in \( Ek \) by \( \overline{B} \) (resp. by \( \overline{B^*} \)). A point \( t_0 \in \overline{k} \) (resp. \( \infty \)) is said to be a branch point of \( E/k(T) \) if the prime \( (T - t_0)[k[T] \) (resp. \( (1/T)[k[1/T]) \) ramifies in \( \overline{B} \) (resp. in \( \overline{B^*} \)). Classically \( E/k(T) \) has only finitely many branch points, denoted by \( t_1, \ldots, t_r \).

2.1.2. Inertia canonical invariant. Assume that \( k \) has characteristic zero. Fix a coherent system \( \{\zeta_n\}_{n=1}^{\infty} \) of roots of unity, i.e. \( \zeta_n \) is a primitive \( n \)-th root of unity and \( \zeta_{nm}^n = \zeta_m \) for any integers \( n \) and \( m \).

To each \( t_i \) can be associated a conjugacy class \( C_i \) of \( G \), called the inertia canonical conjugacy class (associated with \( t_i \)), in the following way. The inertia groups of \( E\overline{k}/\overline{k}(T) \) at \( t_i \) are cyclic conjugate groups of order equal to the ramification index \( e_i \). Furthermore each of them has a distinguished generator corresponding to the automorphism \( (T - t_i)^{1/e_i} \mapsto \zeta_{e_i}((T - t_i)^{1/e_i}) \) (replace \( T - t_i \) by \( 1/T \) if \( t_i = \infty \)). Then \( C_i \) is the conjugacy class of all the distinguished generators of the inertia groups at \( t_i \). The unordered \( r \)-tuple \( (C_1, \ldots, C_r) \) is called the inertia canonical invariant of \( E/k(T) \).

2.1.3. Specializations. If \( t_0 \in \mathbb{P}^1(k) \) is not a branch point, the residue field of some prime above \( t_0 \) in \( E/k(T) \) is denoted by \( E_{t_0} \) and we call the extension \( E_{t_0}/k \) the specialization of \( E/k(T) \) at \( t_0 \) (this does not depend on the choice of the prime above \( t_0 \) since the extension \( E/k(T) \) is Galois). It is a Galois extension of \( k \) of Galois group a subgroup of \( G \), namely the decomposition group of the extension \( E/k(T) \) at \( t_0 \).

This classical lemma, which is proved in [Leg13a, §B.1.4.1], is useful:

**Lemma 2.1.** Let \( P(T,Y) \in k[T][Y] \) be a monic (with respect to \( Y \)) separable polynomial of splitting field \( E \) over \( k(T) \). Then, for any \( t_0 \in k \) such that the specialized polynomial \( P(t_0,Y) \) is separable over \( k \), \( t_0 \) is not a branch point of \( E/k(T) \) and the specialization \( E_{t_0}/k \) of \( E/k(T) \) at \( t_0 \) is the splitting extension over \( k \) of \( P(t_0,Y) \).

2.1.4. Parametric extensions.

**Definition 2.2.** Let \( E/k(T) \) be a \( k \)-regular finite Galois extension of branch point set \( \{t_1, \ldots, t_r\} \).

(1) Let \( H \) be a subgroup of \( \text{Gal}(E/k(T)) \). We say that \( E/k(T) \) is \( H \)-parametric over \( k \) if, for every Galois extension \( F/k \) of group \( H \), there
exists some point $t_0 \in \mathbb{P}^1(k) \setminus \{t_1, \ldots, t_r\}$ such that $F/k$ occurs as the specialization $E_{t_0}/k$ of $E/k(T)$ at $t_0$.

(2) We say that $E/k(T)$ is \textit{parametric over $k$} if this extension is $H$-parametric over $k$ for each subgroup $H \subset \text{Gal}(E/k(T))$.

\section*{2.2. Connections with some classical notions.}

For more on below and §2.3, we refer to \cite[§2.1-2]{Leg13a}.

Let $k$ be a field and $H \subset G$ two finite groups. The notion of $H$-parametric extensions $E/k(T)$ over $k$ of group $\text{Gal}(E/k(T)) = G$ is intermediate between these classical notions in inverse Galois theory.

\subsection*{2.2.1. One parameter generic polynomials.}

Recall that a monic (with respect to $Y$) separable polynomial $P(T,Y) \in k[T][Y]$ of group $G$ is called \textit{generic over $k$} if, for any field extension $L/k$, any Galois extension of $L$ of group $G$ occurs as the splitting extension of some specialized separable polynomial $P(t_0,Y)$ with $t_0 \in L$. See \cite{JLY02} for more on generic polynomials.

Given a generic polynomial over $k$ of group $G$, its splitting extension $E/k(T)$ is $G$-parametric over $k$ (lemma 2.1). It is in fact $G$-\textit{generic over $k$}, i.e. the extension $E L/L(T)$ is $G$-parametric over $L$ for any field extension $L/k$ (which is linearly disjoint from $E$ over $k$). Of course any $G$-generic extension over $k$ of group $G$ is $G$-parametric over $k$. Remark 7.2 shows however that the converse does not hold in general\footnote{Other counter-examples are given in \cite[remark 2.1.7 and example 2.2.1]{Leg13a}.}

Moreover, if $k$ is infinite, the splitting extension $E/k(T)$ is even \textit{generic over $k$}, i.e. the extension $E L/L(T)$ is parametric over $L$ for any field extension $L/k$ (as explained in \cite[proposition 2.1.8]{Leg13a}, it is essentially \cite{Kem01}). In particular, $E/k(T)$ is parametric over $k$.

\subsection*{2.2.2. Lifting extensions.}

Given a Galois extension $F/k$ of group $H$, recall that a \textit{lifting extension of group $G$ for $F/k$} is a $k$-regular Galois extension $E_F/k(T)$ of group $G$ which has the extension $F/k$ among its specializations.

Then any $H$-parametric extension over $k$ of group $G$ obviously is a lifting extension of group $G$ for any Galois extension of $k$ of group $H$. Moreover, if there exists at least one $G$-parametric extension over $k$ of group $G$, then it obviously solves the \textit{Beckmann-Black problem for $G$ over $k$}, which asks whether any Galois extension of $k$ of group $G$ has a lifting extension with the same group.

\section*{2.3. Parametric extensions over various fields.}

Let $H \subset G$ be two finite groups. We investigate below $H$-parametric extensions of group $G$ over various base fields $k$. 
2.3.1. $k$ is a PAC field. Recall that a field $k$ is said to be PAC if every non-empty geometrically irreducible $k$-variety has a Zariski-dense set of $k$-rational points. Classical results show that in some sense PAC fields are “abundant” [FJ05, theorem 18.6.1] and a concrete example (due to Pop) is the field $\mathbb{Q}^{tr}(\sqrt{-1})$; here $\mathbb{Q}^{tr}$ denotes the field of totally real numbers (algebraic numbers such that all conjugates are real). See [FJ05] for more on PAC fields.

In the case $k$ is a PAC field, the situation is quite clear: [Dèb99, theorem 3.2] shows that any $k$-regular Galois extension of $k(T)$ of group $G$ (such an extension exists [FV91] [Pop96]) is parametric over $k$.

2.3.2. $k$ is a finite field. Since there are no (resp. only one) Galois extension of $k$ of group $H$ if $H$ is not cyclic (resp. if $H$ is cyclic), we trivially have that any $k$-regular Galois extension of $k(T)$ of group $G$ is $H$-parametric over $k$ if $H$ is not cyclic and $H'$-parametric over $k$ for at least one cyclic subgroup $H' \subset G$.

Moreover any $k$-regular Galois extension of $k(T)$ of group $G$ is known to be parametric over $k$ provided that $k$ is large enough (depending on $G$ and the branch point number) [Fri74] [Jar82] [Eke90] [DG11]. As in addition the group $G$ occurs as the Galois group of a $k$-regular Galois extension of $k(T)$ provided that $k$ is large enough (depending on $G$) [FV91] [Pop96] (see [DD97, remark 3.9(a)] for more details), conclude that there exists at least one parametric extension over $k$ of group $G$ for large enough finite fields $k$.

2.3.3. $k$ is a completion of $\mathbb{Q}$.

(a) $k = \mathbb{Q}_p$. Since any finite Galois extension of $\mathbb{Q}_p$ is solvable, we vacuously have that any $\mathbb{Q}_p$-regular Galois extension of $\mathbb{Q}_p(T)$ of group $G$ (such an extension exists [Har87]) is $H$-parametric if $H$ is not solvable.

If $H$ is solvable, it does not hold in general. Indeed, given a $\mathbb{Q}$-regular Galois extension $E/\mathbb{Q}(T)$ of group $\mathbb{Z}/8\mathbb{Z}$, the extension $E\mathbb{Q}_2/\mathbb{Q}_2(T)$ is not $\mathbb{Z}/8\mathbb{Z}$-parametric over $\mathbb{Q}_2$. Otherwise there exists some specialization point $t_0 \in \mathbb{P}^1(\mathbb{Q}_2)$ such that $(E\mathbb{Q}_2)_{t_0}/\mathbb{Q}_2$ is the unramified extension of $\mathbb{Q}_2$ of degree 8. From Krasner’s lemma, one may assume that $t_0 \in \mathbb{P}^1(\mathbb{Q})$ and one then obtains a contradiction from [Wan48].

(b) $k = \mathbb{R}$. Since the only finite extensions of $\mathbb{R}$ are the trivial one $\mathbb{R}/\mathbb{R}$ and the quadratic one $\mathbb{C}/\mathbb{R}$, we trivially have that any $\mathbb{R}$-regular Galois extension of $\mathbb{R}(T)$ of group $G$ (such an extension is known to be from a classical work of Hurwitz) is $H$-parametric over $\mathbb{R}$ if neither $H = \{1\}$ nor $H = \mathbb{Z}/2\mathbb{Z}$, and is $\{1\}$-parametric or $\mathbb{Z}/2\mathbb{Z}$-parametric over $\mathbb{R}$.

In particular, any $\mathbb{R}$-regular Galois extension of $\mathbb{R}(T)$ of group $G$ is parametric over $\mathbb{R}$ if $G$ has odd order.
If $G$ has even order, there is at least one $\mathbb{Z}/2\mathbb{Z}$-parametric extension over $\mathbb{R}$ with group $G$. Indeed, from [Hur91] [KN71], it suffices to find an even integer $r \geq 2$ and an $(r+1)$-tuple $(g_0, g_1, \ldots, g_r)$ of non-trivial elements of $G$ such that:
- $g_1 \cdots g_r = 1$ and $\langle g_1, \ldots, g_r \rangle = G$,
- $g_0$ has order 2,
- $g_{r+1-i} = g_0 g_i^{-1} g_0$ for each index $i \in \{1, \ldots, r/2\}$.

To do this, start from an element $g_1 \in G$ that has order 2. Next pick non-trivial elements $g_2, \ldots, g_r$ of $G$ such that $g_1, \ldots, g_r$ generate $G$. Denote the elements $g_r^{-1}, \ldots, g_2^{-1}$ by $g_{r+1}, \ldots, g_{2r-1}$ and set $g_0 = g_1$.

Then the following $(4r - 1)$-tuple satisfies the desired conditions:

$$ (g_0, g_0 g_{2r-1} g_0, \ldots, g_0 g_{r}^{-1} g_0, g_1, \ldots, g_{2r-1}) $$

### 2.3.4. $k = \mathbb{Q}$

The situation in the case $k = \mathbb{Q}$ is more unclear.

(a) If $G = \{1\}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$ or $S_3$, there is a parametric extension over $\mathbb{Q}$ of group $G$. This comes from the fact that these four groups (are the only ones to) have a one parameter generic polynomial over $\mathbb{Q}$ [JLY02, page 194]. Below are some examples of parametric extensions over $\mathbb{Q}$ which are in fact provided by one parameter generic polynomials:

(i) the trivial extension $\mathbb{Q}(T)/\mathbb{Q}(T)$,
(ii) the $\mathbb{Q}$-regular quadratic extension $\mathbb{Q}(\sqrt{T})/\mathbb{Q}(T)$,
(iii) the $\mathbb{Q}$-regular cyclic extension of $\mathbb{Q}(T)$ of degree 3 defined by the polynomial $Y^3 - TY^2 + (T - 3)Y + 1$ (e.g. [JLY02, §2.1]),
(iv) the $\mathbb{Q}$-regular Galois extension of $\mathbb{Q}(T)$ of group $S_3$ defined by the trinomial $Y^3 + TY + T$ (e.g. [JLY02, §2.1]).

(b) If $G$ is none of these four groups, it is unknown whether there is a $G$-parametric extension over $\mathbb{Q}$ of group $G$ or not. In the case $H \neq G$, [Dèb09, proposition 3.2.4] provides an $H$-parametric extension over $\mathbb{Q}$ of group $G$ in the case $H = \{1\}$ and $G$ abelian.

(c) In addition to the example with $G = \mathbb{Z}/2\mathbb{Z}$ from the presentation, only a few negative examples are known.

(i) No $\mathbb{Q}$-regular Galois extension of $\mathbb{Q}(T)$ of group $S_7$ and branch point set $\{0, 1, \infty\}$ is $S_7$-parametric over $\mathbb{Q}$: [Bec94, example 1.1] shows indeed that the Galois extension of $\mathbb{Q}$ of group $S_7$ defined by the polynomial $P(Y) = Y^7 + 42482Y^6 + 5643Y^5 - 21164Y^4 + 2431Y^3 + 46189Y^2 + 46189Y + 46189$ cannot be a specialization of such an extension.

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4The same claim also holds with $\mathbb{Z}/2\mathbb{Z}$ replaced by $\{1\}$ (see e.g. the proof of [Dèb09, theorem 4.3.2]).

5and the desired regular realization of $G$ over $\mathbb{R}$ even has no real branch point.
(ii) For any finite group $G \neq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $S_3$, $D_4$, $D_6$ which occurs as the Galois group of a totally real Galois extension of $\mathbb{Q}$, no $\mathbb{Q}$-regular Galois extension of $\mathbb{Q}(T)$ of group $G$ with three branch points is $G$-parametric over $\mathbb{Q}$: [DF90, proposition 1.2] shows indeed that no specialization of such an extension is totally real.

3. First examples

This section is devoted to theorem 1 from the presentation. We use ad hoc arguments to give new examples of non $H$-parametric extensions over $\mathbb{Q}$ of group $G$. Our examples have $r \in \{2, 3, 4\}$ branch points (§3.1-3). We also discuss the case $r \geq 5$ in §3.4.

3.1. An example with $r = 2$. Proposition 3.1 below unifies the two examples $\mathbb{Q}(T)(\sqrt{T^2 + 1})/\mathbb{Q}(T)$ and $\mathbb{Q}(\sqrt{T})/\mathbb{Q}(T)$:

**Proposition 3.1.** Let $a, b$ and $c$ three rational numbers such that $b^2 - 4ac \neq 0$ and $E = \mathbb{Q}(T)(\sqrt{aT^2 + bT + c})$. Then the following three conditions are equivalent:

1. $E/\mathbb{Q}(T)$ is parametric over $\mathbb{Q}$,
2. $E/\mathbb{Q}(T)$ is $\mathbb{Z}/2\mathbb{Z}$-parametric over $\mathbb{Q}$,
3. $b^2 - 4ac$ is a square in $\mathbb{Q}$.

Note that condition (3) is equivalent to the following:

4. the two branch points of $E/\mathbb{Q}(T)$ each is $\mathbb{Q}$-rational.

Indeed this easily follows from lemma 3.2 below which will be used on several occasions in this paper. We omit the proof which involves very classical tools and which is detailed in [Leg13a, §2.3.2.1].

**Lemma 3.2.** Let $k$ be a field of characteristic zero, $P(T) \in k[T]$ a separable polynomial over $k$, $n$ its degree and $\{t_1, \ldots, t_n\}$ its root set. Then the quadratic extension $k(T)(\sqrt{P(T)})/k(T)$ is $k$-regular and its branch point set $t$ is

1. $t = \{t_1, \ldots, t_n\}$ if $n$ is even,
2. $t = \{t_1, \ldots, t_n\} \cup \{\infty\}$ if $n$ is odd.

**Proof of proposition 3.1.** We successively prove implications (3) $\Rightarrow$ (1), (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). Furthermore the proof will show the following:

(a) if condition (3) holds, then any quadratic or trivial extension of $\mathbb{Q}$ is the splitting extension over $\mathbb{Q}$ of some specialized polynomial $Y^2 - (at_0^2 + bt_0 + c)$ with $t_0 \in \mathbb{Q}$.

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6More on $\mathbb{Q}$-regular finite Galois extensions of $\mathbb{Q}(T)$ with $r = 2$ branch points can be found in [Leg13a, §2.3.2].
(b) if condition (3) does not hold, then there are infinitely many distinct quadratic extensions of \( \mathbb{Q} \) which each is not a specialization of \( E/\mathbb{Q}(T) \).

(3) \( \Rightarrow \) (1). Assume that condition (3) holds. Let \( t_1 \in \mathbb{Q} \) be a root of \( aT^2 + bT + c \) and \( F/\mathbb{Q} \) a quadratic or trivial extension. Set \( F = \mathbb{Q}(\sqrt{d}) \) with \( d \) a non-zero integer.

The curve defined by the equation \( dY^2 = aT^2 + bT + c \) has a (non singular) \( \mathbb{Q} \)-rational point (for example \((0, t_1)\)). Being of genus 0, it is then birational to \( \mathbb{P}^1 \) over \( \mathbb{Q} \). Then there exist two rational numbers \( y \) and \( t_0 \) such that \( y \neq 0 \) and \( dy^2 = at_0^2 + bt_0 + c \). Hence one has \( F = \mathbb{Q}(\sqrt{at_0^2 + bt_0 + c}) \), i.e. \( F/\mathbb{Q} \) is the splitting extension over \( \mathbb{Q} \) of the specialized polynomial \( Y^2 - (at_0^2 + bt_0 + c) \) (and so statement (a) holds). Since this polynomial is separable over \( \mathbb{Q} \), one may apply lemma 2.1 and conclude that \( F/\mathbb{Q} \) is the specialization \( E_{t_0}/\mathbb{Q} \).

(1) \( \Rightarrow \) (2). This is a consequence of definition 2.2.

(2) \( \Rightarrow \) (3). Assume that condition (2) holds. There are three steps to show that \( b^2 - 4ac \) is a square in \( \mathbb{Q} \).

- Step 1: \( a \in \mathbb{Z} \setminus \{0\}, b = 0 \) and \( c \in \mathbb{Z} \setminus \{0\} \). First remark that \( \infty \) is not a branch point since \( a \neq 0 \) (lemma 3.2).

  Let \( p \) be a prime such that neither \( a \) nor \( c \) is a multiple of \( p \) and that does not ramify in \( E_{\infty}/\mathbb{Q} \). From condition (2), there exists some \( t_0 \in \mathbb{Q} \) such that \( E_{t_0} = \mathbb{Q}(\sqrt{p}) \), i.e. \( \mathbb{Q}(\sqrt{at_0^2 + c}) = \mathbb{Q}(\sqrt{p}) \) (lemmas 2.1 and 3.2). Hence there exists some non-zero rational number \( \lambda \) such that \( \lambda p^2 = at_0^2 + c \). Then there exist three non-zero integers \( x, y \) and \( z \) such that \( px^2 = ay^2 + cz^2 \) and one may assume that \( z \) is not a multiple of \( p \) (otherwise \( x \) and \( y \) are also multiples of \( p \) and, with \( n \) the \( p \)-adic valuation of \( z \), one may then replace \((x, y, z)\) by \((x/p^n, y/p^n, z/p^n)\)). By reducing modulo \( p \), \(-ac\) is a square modulo \( p \).

  Hence \( Y^2 + 4ac \) has a root modulo \( p \) for all but finitely many primes \( p \) (note that this also holds if we only assume that all but finitely many quadratic extensions of \( \mathbb{Q} \) are specializations of \( E/\mathbb{Q}(T) \), so proving statement (b)). From e.g. [Hei67, theorem 9]\(^7\), \(-4ac\) is a square in \( \mathbb{Q} \).

- Step 2: \( (a, b, c) \in \mathbb{Z}^3 \). Condition (3) trivially holds if \( a = 0 \) or \( c = 0 \). So assume that \( a \neq 0 \) and \( c \neq 0 \). Set \( \Delta = b^2 - 4ac \).

  Let \( p \) be a prime such that neither \( a \) nor \( \Delta \) is a multiple of \( p \) and that does not ramify in \( E_{\infty}/\mathbb{Q} \). From condition (2), there is some \( t_0 \in \mathbb{Q} \) such that \( \mathbb{Q}(\sqrt{p})/\mathbb{Q} = E_{t_0}/\mathbb{Q} \), i.e. \( \mathbb{Q}(\sqrt{at_0^2 + bt_0 + c}) = \mathbb{Q}(\sqrt{p}) \). Set \( t_0 = 2at_0 + b \). As \( at_0^2 - \Delta = 4a^2(at_0^2 + bt_0 + c) \), one has \( \mathbb{Q}(\sqrt{at_0^2 + bt_0 + c}) = \mathbb{Q}(\sqrt{at_0^2 - \Delta}) \). From step 1, \( 4a^2 \Delta \) is a square in \( \mathbb{Q} \) and so is \( \Delta \) too.

\(^7\)It seems that more elementary proofs exist in the quadratic case.
- Step 3: \((a, b, c) \in \mathbb{Q}^3\). Set \(a = a_1/a_2\), \(b = b_1/b_2\) and \(c = c_1/c_2\) with integers \(a_1, a_2, b_1, b_2, c_1, c_2\) such that \((a_1, a_2) = (b_1, b_2) = (c_1, c_2) = 1\).

As \(a_2^2b_2^2c_2^2(aT^2 + bT + c) = a_1a_2b_2^2c_2^2T^2 + b_1b_2a_2^2c_2^2T + c_1c_2a_2^2b_2^2\), one has

\[
E = \mathbb{Q}(T)(\sqrt{a_1a_2b_2^2c_2^2T^2 + b_1b_2a_2^2c_2^2T + c_1c_2a_2^2b_2^2})
\]

From step 2, the discriminant \(b_1^2b_2^2a_2^4c_2^4 - 4a_1c_1a_2^2c_2^3b_2^4 = (a_2b_2c_2)^4(b^2 - 4ac)\) is a square in \(\mathbb{Q}\). Hence condition (3) holds. □

3.2. An example with \(r = 3\). As for any abelian finite group, the Beckmann-Black problem for \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) over \(\mathbb{Q}\) has a positive answer: any Galois extension \(F/\mathbb{Q}\) of group \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) has a lifting extension \(E_F/\mathbb{Q}(T)\) with the same group. Moreover [Bec94, corollary 2.4] shows that \(E_F/\mathbb{Q}(T)\) may be chosen with three branch points.

The following shows however that none of these lifting extensions \(E_F/\mathbb{Q}(T)\) with three branch points is \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\)-parametric over \(\mathbb{Q}\).

**Proposition 3.3.** Let \(E/\mathbb{Q}(T)\) be a \(\mathbb{Q}\)-regular Galois extension of group \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) with three branch points. Then there exist infinitely many distinct Galois extensions of \(\mathbb{Q}\) of group \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) which each is not a specialization of \(E/\mathbb{Q}(T)\). In particular, this extension is not \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\)-parametric over \(\mathbb{Q}\).

**Proof.** Let \(P_1(T)\) and \(P_2(T)\) be two distinct separable polynomials over \(\mathbb{Q}\) such that \(E = \mathbb{Q}(T)(\sqrt[4]{P_1(T)}, \sqrt[4]{P_2(T)})\).

Given \(i \in \{1, 2\}\), it follows from the extension \(E/\mathbb{Q}(T)\) having three branch points and the \(\mathbb{Q}\)-regular quadratic one \(\mathbb{Q}(T)(\sqrt[4]{P_i(T)})/\mathbb{Q}(T)\) having an even branch point number (lemma 3.2) that the latter has two branch points. Consequently each branch point of \(E/\mathbb{Q}(T)\) is \(\mathbb{Q}\)-rational. Hence we may assume that these branch points are 0, 1 and \(\infty\). In particular, there exist two non-zero squarefree integers \(a\) and \(b\) such that \(E = \mathbb{Q}(T)(\sqrt[4]{aT}, \sqrt[4]{bT - b})\) (lemma 3.2).

Now fix two distinct squarefree integers \(d_1, d_2\) and assume that \(\mathbb{Q}(\sqrt[4]{d_1}, \sqrt[4]{d_2}) = \mathbb{Q}(\sqrt[4]{d_0}, \sqrt[4]{d_0 - b})\) for some \(t_0 \in \mathbb{Q} \setminus \{0, 1\}\). Then the quadratic subextensions coincide and one of these conditions holds:

(i) \(a d_1 t_0 \in \mathbb{Q}^2\) and \(a d_2 (b t_0 - b) \in \mathbb{Q}^2\),

(ii) \(a d_1 t_0 \in \mathbb{Q}^2\) and \(d_1 d_2 (b t_0 - b) \in \mathbb{Q}^2\),

(iii) \(a d_2 t_0 \in \mathbb{Q}^2\) and \(d_1 (b t_0 - b) \in \mathbb{Q}^2\),

(iv) \(a d_2 t_0 \in \mathbb{Q}^2\) and \(d_1 d_2 (b t_0 - b) \in \mathbb{Q}^2\),

(v) \(a d_1 d_2 t_0 \in \mathbb{Q}^2\) and \(d_1 (b t_0 - b) \in \mathbb{Q}^2\),

(vi) \(a d_1 d_2 t_0 \in \mathbb{Q}^2\) and \(d_2 (b t_0 - b) \in \mathbb{Q}^2\).

Consequently one of the following six equations has a non trivial solution, i.e. a solution \((x, y, z) \in \mathbb{Z}^3\) such that \(xyz \neq 0\):

(i) \(a d_1 X^2 - b d_2 Y^2 - Z^2 = 0\),
(ii) \( aX^2 - bY^2 - d_1 Z^2 = 0 \),
(iii) \( a d_2 X^2 - b d_1 Y^2 - Z^2 = 0 \),
(iv) \( aX^2 - b d_1 Y^2 - d_2 Z^2 = 0 \),
(v) \( a d_2 X^2 - b Y^2 - d_1 Z^2 = 0 \),
(vi) \( a d_1 X^2 - b Y^2 - d_2 Z^2 = 0 \).

We show below that there are infinitely many distinct couples \((d_1, d_2)\)

of distinct squarefree integers such that none of these six equations has

a non trivial solution. In particular, the conclusion holds (lemma 2.1).

One may assume that \( a > 0 \) or \( b < 0 \) (otherwise take \( d_1 > 0 \) and \( d_2 > 0 \) to conclude). Assume for example that \( a > 0 \) and \( b > 0 \) (the

other two cases for which \( a > 0 \) and \( b < 0 \), \( a < 0 \) and \( b < 0 \) are similar).

First assume that the squarefree integer \( b \) satisfies \( b \neq 1 \). Fix a

squarefree integer \( d_2 > 0 \) such that neither \( a b d_2 \) nor \( a d_2 \) is a square

in \( \mathbb{Q} \). As \( b \neq 1 \), the quadratic fields \( \mathbb{Q}(\sqrt{a b d_2}) \) and \( \mathbb{Q}(\sqrt{a d_2}) \) are
distinct. Hence there are infinitely many distinct primes \( p \) such that

neither \( a b d_2 \) nor \( a d_2 \) is a square modulo \( p \) (e.g. [Nag69, theorem 7]).

Then, for such a prime \( p \), none of the previous equations with \( d_1 = -p \)

has a non trivial solution, i.e. none of the following equations does:

(i) \(-a p X^2 - b d_2 Y^2 - Z^2 = 0\),
(ii) \( a X^2 - b d_2 Y^2 + p Z^2 = 0\),
(iii) \( a d_2 X^2 + p b Y^2 - Z^2 = 0\),
(iv) \( a X^2 + p b Y^2 - d_2 Z^2 = 0\),
(v) \( a d_2 X^2 - b Y^2 + p Z^2 = 0\),
(vi) \(-a p X^2 - b Y^2 - d_2 Z^2 = 0\).

Indeed, first note that neither equation (i) nor equation (vi) has such

a solution (as all coefficients are negative). If one of equations (ii)-(v)

has such a solution \((x, y, z)\), one may assume that \( x \) is not a multiple

of \( p \) (otherwise \( y \) and \( z \) are also multiples of \( p \) and, with \( n \) the \( p \)-adic

valuation of \( x \), one may then replace \((x, y, z)\) by \((x/p^n, y/p^n, z/p^n)\)). By

reducing modulo \( p \), \( a d_2 \) or \( a b d_2 \) is a square modulo \( p \); a contradiction.

Now assume that \( b = 1 \). Fix a squarefree integer \( d_2 > 0 \) such that

\( a d_2 \) is not a square in \( \mathbb{Q} \). Hence there exist infinitely many distinct

primes \( p \) such that \( a d_2 \) is not a square modulo \( p \) (e.g. [Hei67, theorem 9]). Then, for such a prime \( p \), a similar argument as that in the case

\( b \neq 1 \) shows that none of the previous six equations with \( d_1 = -p \) has

a non trivial solution, thus ending the proof. \(\square\)

Remark 3.4. The proof and proposition 3.1 show in particular that any

quadratic subextension of \( E/\mathbb{Q}(T) \) is parametric over \( \mathbb{Q} \). However their

compositum is not \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \)-parametric over \( \mathbb{Q} \).

3.3. An example with \( r = 4 \). Denote the splitting extension over

\( \mathbb{Q}(T) \) of \( Y^3 + T^2Y + T^2 \) by \( E/\mathbb{Q}(T) \). As this trinomial is absolutely
irreducible and its discriminant $\Delta(T) = -4T^6 - 27T^4$ is not a square in $\mathbb{Q}(T)$, $E/\mathbb{Q}(T)$ is $\mathbb{Q}$-regular and one has $\text{Gal}(E/\mathbb{Q}(T)) = S_3$.

**Proposition 3.5.** The extension $E/\mathbb{Q}(T)$ is $H$-parametric over $\mathbb{Q}$ for no subgroup $H \subset S_3$. More precisely, for any non trivial subgroup $H \subset S_3$, there exist infinitely many distinct Galois extensions of $Q$ of group $H$ which each is not a specialization of $E/\mathbb{Q}(T)$.

**Proof.** One easily shows that the branch point set $t$ of $E/\mathbb{Q}(T)$ is contained in $\{0, 3i\sqrt{3}/2, -3i\sqrt{3}/2, \infty\}$. Hence $t$ contains some $\mathbb{Q}$-rational point and the two complex conjugate points $3i\sqrt{3}/2, -3i\sqrt{3}/2$.

Assume that $E/\mathbb{Q}(T)$ has three branch points. The Riemann-Hurwitz formula then shows that it has genus 0. Then there exists some transcendental element $U$ over $\mathbb{Q}$ such that $E\overline{U} = \overline{Q}(U)$. Since $S_3$ is isomorphic to the finite group $\mathcal{D}$ generated by $\sigma$ and $\tau$ such that $\tau(U) = 1/U$ and $\sigma(U) = e^{2i\pi/3}U$, one has $\overline{Q}(T) = \overline{Q}(U)^{\mathcal{D}} = \overline{Q}(U^3 + U^{-3})$ (since $U$ is a root of the trinomial $Y^6 - (U^3 + U^{-3})Y^3 + 1$). Moreover the branch point set of $\overline{Q}(U)/\overline{Q}(U^3 + U^{-3})$ is contained in $\{-2, 2, \infty\}$. In particular, any branch point of $E\overline{Q}(T)/\overline{Q}(T)$ should be $\mathbb{Q}$-rational; a contradiction. Hence the extension $E/\mathbb{Q}(T)$ has four branch points.

Given a non-zero rational number $t_0$, the specialized polynomial $Y^3 + t_0^2Y + t_0^3$ is separable over $\mathbb{Q}$ and, from lemma 2.1, the specialization $E_{t_0}/\mathbb{Q}$ is its splitting extension over $\mathbb{Q}$. Since this polynomial has only one real root, the specialization $E_{t_0}/\mathbb{Q}$ is not totally real. Hence the conclusion obviously holds for $H = \{1\}$ and $H = \mathbb{Z}/2\mathbb{Z}$. Moreover, since any finite Galois extension of $\mathbb{Q}$ of odd degree is totally real, the conclusion also holds for $H = \mathbb{Z}/3\mathbb{Z}$. Finally, since it is known that there exist infinitely many distinct totally real Galois extensions of $\mathbb{Q}$ of group $S_3$ (e.g. [KM01, proposition 2]), the conclusion is also true for $H = S_3$, thus ending the proof. $\square$

3.4. **The case** $r \geq 5$. In this case, it seems difficult to give similar examples. However one has the following general conclusion:

Let $G$ be a finite group, $H$ a subgroup of $G$ and $E/\mathbb{Q}(T)$ a $\mathbb{Q}$-regular Galois extension of group $G$ with $r \geq 5$ branch points. Then, given a Galois extension $F/\mathbb{Q}$ of group $H$, there exist only finitely many distinct points $t_0 \in \mathbb{P}^1(\mathbb{Q})$ (possibly none) such that the extension $F/\mathbb{Q}$ occurs as the specialization $E_{t_0}/\mathbb{Q}$ of $E/\mathbb{Q}(T)$ at $t_0$.

Indeed denote the genus of $E/\mathbb{Q}(T)$ by $g$. The Riemann-Hurwitz formula yields $2g \geq 2 + [E : \mathbb{Q}(T)]((r/2) - 2)$. Hence $g \geq 2$ and the conclusion follows from the Faltings theorem as explained in [Dèb99, §3.3.5].

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8In fact no Galois extension of $\mathbb{Q}$ of group $\mathbb{Z}/3\mathbb{Z}$ is a specialization of $E/\mathbb{Q}(T)$. 
4. CRITERIA FOR NON PARAMETRICITY

This section is devoted to theorem 4.2 below which gives our most general criteria for a given \( k \)-regular Galois extension of \( k(T) \) not to be parametric over \( k \); it is the aim of §4.1. We also give in §4.2 several more practical forms of this statement which each will be used in the next three sections to obtain new examples of such extensions over various base fields \( k \).

Let \( A \) be a Dedekind domain of characteristic zero with infinitely many distinct primes and \( k \) its quotient field. First recall the following:

Definition 4.1. Let \( P(T) \in k[T] \) be a non constant polynomial and \( P \) a (non-zero) prime of \( A \). We say that \( P \) is a prime divisor of \( P(T) \) if there exists some \( t_0 \in k \) such that \( P(t_0) \) is in the maximal ideal \( PA_P \) of the localization \( A_P \) of \( A \) at \( P \).

4.1. General result.

4.1.1. Notation. Let \( H \) be a non trivial finite group and \( E_1/k(T) \) a \( k \)-regular Galois extension of group \( H \). Denote its branch point set by \( \{ t_{1,1}, \ldots, t_{r_1,1} \} \) and its inertia canonical invariant by \( (C_{1,1}, \ldots, C_{r_1,1}) \).

Recall some important notation from [Leg13b]. For each index \( i \in \{1, \ldots, r_1\} \), denote the irreducible polynomial of \( t_{i,1} \) (resp. of \( 1/t_{i,1} \)) over \( k \) by \( m_{i,1}(T) \) (resp. by \( m_{i,1}^*(T) \)). Set \( m_{i,1}(T) = 1 \) if \( t_{i,1} = \infty \) and \( m_{i,1}^*(T) = 1 \) if \( t_{i,1} = 0 \). Finally set \( m_{E_1}(T) = \prod_{i=1}^{r_1} m_{i,1}(T) \) and \( m_{E_1}^*(T) = \prod_{i=1}^{r_1} m_{i,1}^*(T) \).

Let \( G \) be a finite group containing \( H \) and \( E_2/k(T) \) a \( k \)-regular Galois extension of group \( G \). Define the same notation for \( E_2 \).

Moreover, given a conjugacy class \( C \) of \( H \), denote the conjugacy class in \( G \) of elements of \( C \) by \( C^G \).

4.1.2. Statement of the result. Consider the following two conditions:

(Branch Point Hypothesis) there exist infinitely many distinct primes of \( A \) which each is a prime divisor of \( m_{E_1}(T) \cdot m_{E_1}^*(T) \) but not of \( m_{E_2}(T) \cdot m_{E_2}^*(T) \),

(Inertia Hypothesis) there exists some index \( i \in \{1, \ldots, r_1\} \) satisfying the following two conditions:

(a) \( m_{i,1}(T) \cdot m_{i,1}^*(T) \) has infinitely many distinct prime divisors,

(b) the set \( \{C_{1,2}^a, \ldots, C_{r_2,2}^a / a \in \mathbb{N}\} \) does not contain \( C_{i,1}^G \).

Theorem 4.2. Under either one of these two conditions, the following non parametricity condition holds:

\(^9\)Set \( 1/t_{i,1} = 0 \) if \( t_{i,1} = \infty \) and \( 1/t_{i,1} = \infty \) if \( t_{i,1} = 0 \).
(non parametricity) there exist infinitely many distinct finite Galois extensions of \( k \) which each is not a specialization of \( E_2/k(T) \) \(^{10}\).

Furthermore these Galois extensions of \( k \) may be obtained by specializing \( E_1/k(T) \).

Recall that a set \( S \) of conjugacy classes of \( H \) is called \( g \)-complete (a terminology due to Fried \([Fri95]\)) if no proper subgroup of \( H \) intersects each conjugacy class in \( S \). For instance, the set of all conjugacy classes of \( H \) is \( g \)-complete \([Jor72]\).

**Addendum 4.2.** Under either one of the following two extra conditions:
1. \( k \) is hilbertian,
2. there exists some subset \( I \subset \{1, \ldots, r_1\} \) satisfying the following two conditions:
   a. \( m_{i,1}(T) \cdot m_{i,1}^*(T) \) has infinitely many distinct prime divisors for each index \( i \in I \),
   b. the set \( \{C_{i,1} / i \in I\} \) is \( g \)-complete,

the following more precise non \( H \)-parametricity condition holds:

(\( \text{non } H \)-parametricity) there exist infinitely many distinct Galois extensions of \( k \) of group \( H \) which each is not a specialization of \( E_2/k(T) \).

Moreover these Galois extensions of \( k \) of group \( H \) may be obtained by specializing \( E_1/k(T) \) and, in the case the base field \( k \) is assumed to be hilbertian, they may be further required to be linearly disjoint.

Theorem 4.2 is proved in §4.3.

### 4.2. Practical forms of theorem 4.2.

We now give four more practical forms of theorem 4.2. The first one rests on a sharp variant of the Branch Point Hypothesis and the other ones each uses the Inertia Hypothesis.

#### 4.2.1. Branch Point Criterion.

If \( E_1/k(T) \) has at least one \( k \)-rational branch point \( t_{i,1} \), then all but finitely many primes of \( A \) obviously are prime divisors of \( m_{i,1}(T) \cdot m_{i,1}^*(T) \), and so are of \( m_{E_1}(T) \cdot m_{E_1}^*(T) \) too.

Hence one obtains the following statement:

**Branch Point Criterion.** The (non \( H \)-parametricity) condition\(^{11}\) holds if the following three conditions are satisfied:

- (BPC-1) \( k \) is a number field,
- (BPC-2) the extension \( E_1/k(T) \) has at least one \( k \)-rational branch point,
- (BPC-3) \( \text{there exist infinitely many distinct Galois extensions of } k \text{ which each is not a specialization of } E_2/k(T) \).

\(^{10}\)In particular, the extension \( E_2/k(T) \) is not parametric over \( k \).

\(^{11}\)Here and in the next criteria, one can add as in theorem 4.2 that the Galois extensions of group \( H \) whose existence is claimed may be obtained by specialization.
(BPC-3) there exist infinitely many distinct primes of $A$ which each is not a prime divisor of $m_{E_2}(T) \cdot m_{E_2}^*(T)$.

An obvious necessary condition for condition (BPC-3) to hold is that $E_2/k(T)$ has no $k$-rational branch point. Moreover condition (BPC-1) may be replaced by either one of the two conditions of addendum 4.2.

4.2.2. Inertia Criteria. Since part (b) of the Inertia Hypothesis does not depend on the base field $k$, one obtains the following three criteria in which the (non $H$-parametricity) condition remains true after any finite scalar extension, i.e. in which the following holds:

(geometric non $H$-parametricity) for any finite extension $k'/k$, there exist infinitely many distinct Galois extensions of $k'$ of group $H$ which each is not a specialization of $E_2/k(T)$.

Moreover, given a finite extension $k'/k$, these Galois extensions of $k'$ of group $H$ may be obtained by specializing $E_1/k(T)$ and, in the case the base field $k$ is assumed to be hilbertian, they may be further required to be linearly disjoint.

**Inertia Criterion 1.** The (geometric non $H$-parametricity) condition holds if the following three conditions are satisfied:

(IC1-1) each branch point of $E_1/k(T)$ is $k$-rational,

(IC1-2) there exists some index $i \in \{1, \ldots, r_H\}$ such that $C_{i,1}^G$ is not contained in the set $\{C_{i,2}^{a_1}, \ldots, C_{r_2,2}^{a_2} / a \in \mathbb{N}\}$,

(IC1-3) the set $\{C_{1,1}, \ldots, C_{r_1,1}\}$ is $g$-complete.

Indeed, given a finite extension $k'/k$, apply theorem 4.2 to the extensions $E_1/k'(T)$ and $E_2/k'(T)$. Fix an index $i \in \{1, \ldots, r_H\}$ such that the set $\{C_{i,2}^{a_1}, \ldots, C_{r_2,2}^{a_2} / a \in \mathbb{N}\}$ does not contain $C_{i,1}^G$ (condition (IC1-2)). Then part (b) of the Inertia Hypothesis holds for this index $i$. From condition (IC1-1), $t_{i,1}$ is $k'$-rational and then part (a) of the Inertia Hypothesis also holds for this $i$ (as noted at the beginning of §4.2.1). As condition (2) of addendum 4.2 holds (with $I = \{1, \ldots, r_1\}$) from conditions (IC1-1) and (IC1-3), the conclusion follows.

**Inertia Criterion 2.** The (geometric non $H$-parametricity) condition holds if the following two conditions are satisfied:

(IC2-1) there is a $k$-rational branch point $t_{i,1}$ of $E_1/k(T)$ such that the set $\{C_{i,2}^{a_1}, \ldots, C_{r_2,2}^{a_2} / a \in \mathbb{N}\}$ does not contain $C_{i,1}^G$,

(IC2-2) $k$ is hilbertian.

Indeed, given a finite extension $k'/k$, apply theorem 4.2 to the extensions $E_1/k'(T)$ and $E_2/k'(T)$. From condition (IC2-1), the Inertia
Hypothesis is satisfied. As $k'$ is hilbertian from condition (IC2-2), i.e. condition (1) of addendum 4.2 is satisfied, the conclusion follows.

**Inertia Criterion 3.** The (geometric non $H$-parametricity) condition holds if the following two conditions are satisfied:

- (IC3-1) there exists some index $i \in \{1, \ldots, r_1\}$ such that $C^G_{i,1}$ is not contained in $\{C^a_{1,2}, \ldots, C^a_{r_2,2} / a \in \mathbb{N}\}$,
- (IC3-2) $k$ is either a number field or a finite extension of a rational function field $\kappa(X)$ with $\kappa$ an arbitrary algebraically closed field of characteristic zero (and $X$ an indeterminate).

Indeed, given a finite extension $k'/k$, apply theorem 4.2 to the extensions $E_1k'/k(T)$ and $E_2k'/k(T)$. Fix an index $i \in \{1, \ldots, r_1\}$ such that the set $\{C^a_{1,2}, \ldots, C^a_{r_2,2} / a \in \mathbb{N}\}$ does not contain $C^G_{i,1}$ (condition (IC3-1)). Then part (b) of the Inertia Hypothesis holds for this index $i$. We show below that part (a) of the Inertia Hypothesis also holds for this $i$. As condition (1) of addendum 4.2 is satisfied from condition (IC3-2), the conclusion follows.

Indeed it follows from condition (IC3-2) that any non constant polynomial $P(T) \in k'[T]$ has infinitely many distinct prime divisors: this classically follows from the Tchebotarev density theorem (e.g. [Neu99, chapter 7, theorem (13.4)]) in the case $k$ is a number field (and so $k'$ is too) and this is left to the reader as an easy exercise in the function field case.

**Remark 4.3.** Part (b) of the Inertia Hypothesis (and similar other of our conditions) has a stronger but more practical variant in terms of ramification indices instead of inertia canonical conjugacy classes.

Indeed, given an index $i \in \{1, \ldots, r_1\}$, if the ramification index of $t_{j,2}$ in $E_2k'/k(T)$ is a multiple of that of $t_{i,1}$ in $E_1k'/k(T)$ for no index $j \in \{1, \ldots, r_2\}$, then $\{C^a_{1,2}, \ldots, C^a_{r_2,2} / a \in \mathbb{N}\}$ does not contain $C^G_{i,1}$.

### 4.3. Proof of theorem 4.2.

The proof rests on the main result of [Leg13b] (theorem 3.1 there) and a classical result on the ramification in the specializations of a $k$-regular finite Galois extension of $k(T)$ (re-called as the “Specialization Inertia Theorem” in [Leg13b, §2.2.3]) and is similar to that of theorem 4.2 of [Leg13b] (which is theorem 4.2 here in the special case $H = G$ and $k$ is a number field). We reproduce this proof below with the necessary adjustments for the bigger generality.

First assume that the Branch Point Hypothesis holds. Then there exists some index $i \in \{1, \ldots, r_1\}$ such that the polynomial $m_{i,1}(T) \cdot m^*_{i,1}(T)$ has infinitely many distinct prime divisors $\mathcal{P}$ which each is not a prime divisor of $m_{E_2}(T) \cdot m^*_{E_2}(T)$. Furthermore, up to excluding
finitely many of these primes, one may also assume that such a prime $P$ satisfies the following two conditions:

(i) $P$ is a good prime for $E_1/k(T)$ in the sense of [Leg13b, definition 2.6] and $m_{i,1}(T), m_{i,1}^*(T)$ each has its coefficients in $A_P$,

(ii) $P$ is a good prime for $E_2/k(T)$ and the polynomials $m_{j,2}(T)$ and $m_{j,2}^*(T)$ have their coefficients in $A_P$ for each index $j \in \{1, \ldots, r_2\}$.

For such a prime $P$, apply [Leg13b, theorem 3.1] to construct a specialization $F_P/k$ of $E_1/k(T)$ which ramifies at $P$. From [Leg13b, corollary 2.12], $F_P/k$ does not occur as a specialization of $E_2/k(T)$ and the conclusion follows.

Now assume that the Inertia Hypothesis holds. From its part (a), there exist infinitely many distinct prime divisors $P$ of $m_{i,1}(T) \cdot m_{i,1}^*(T)$ which each may be assumed as before to further satisfy conditions (i) and (ii) above. For such a prime $P$, apply [Leg13b, theorem 3.1] to construct a specialization $F_P/k$ of $E_1/k(T)$ whose inertia group at $P$ is generated by some element of $C_{i,1}$. If $F_P/k$ is a specialization of $E_2/k(T)$, then, from the Specialization Inertia Theorem, there exist some index $j \in \{1, \ldots, r_2\}$ and some positive integer $a$ such that the inertia group of $F_P/k$ at $P$ is generated by some element of $C_{a,j}^2$. This contradicts part (b) of the Inertia Hypothesis. Hence $F_P/k$ is not a specialization of $E_2/k(T)$ and the conclusion follows.

Furthermore assume that condition (2) of addendum 4.2 holds. Instead of [Leg13b, theorem 3.1], use [Leg13b, corollary 3.4 and remark 3.5] in the previous two paragraphs. In each case, the extension $F_P/k$ may be required to have Galois group $H$ and the (non $H$-parametricity) condition follows. In the case condition (1) holds, [Leg13b, corollary 3.3] should be used (instead of [Leg13b, corollary 3.4 and remark 3.5]) to obtain the (non $H$-parametricity) condition and the extra linearly disjointness condition.

5. A GENERAL CONSEQUENCE OVER NUMBER FIELDS

Our method to obtain examples of non $G$-parametric extensions over a given base field $k$ with prescribed Galois group $G$ starts with the knowledge of two $k$-regular Galois extensions of $k(T)$ of group $G$ with some somehow incompatible ramification data. Over number fields, the state-of-the-art in inverse Galois theory does not always provide such extensions in general. Proposition 5.1, our conditional result, provides an inverse Galois theory assumption which makes the method work. This statement leads in particular to corollary 5.2 which is theorem 2 from the presentation. Corollary 5.3, our conjectural result, is the corresponding result under a conjecture of Fried.
5.1. **The number field case.** Let $k$ be a number field and $G$ a finite group. Denote the set of all conjugacy classes of $G$ by $cc(G)$.

5.1.1. **Conditional result.** To make the rest of this section simpler, we will use the following condition:

(H1/$k$) each non trivial conjugacy class of $G$ occurs as the inertia canonical conjugacy class associated with some branch point of some $k$-regular Galois extension of $k(T)$ of group $G$.

It is unknown in general if any finite group satisfies the inverse Galois theory condition (H1/$k$) for a given number field $k$. However, as recalled below, every finite group satisfies condition (H1/$k$) for suitable number fields $k$. Indeed the Riemann existence theorem classically provides the following statement (e.g. [Déb01, §12]):

(*) Any set $\{C_1, \ldots, C_r\}$ of non trivial conjugacy classes of $G$ whose all elements generate $G$ occurs as the inertia canonical conjugacy class set of some Galois extension of $\mathbb{Q}(T)$ of group $G$.

In particular, there exists some Galois extension $E/\mathbb{Q}(T)$ of group $G$ whose inertia canonical conjugacy class set is the set of all non trivial conjugacy classes of $G$. Hence condition (H1/$k$) holds over any number field $k$ that is a field of definition of $E/\mathbb{Q}(T)$.

**Proposition 5.1.** Let $E/k(T)$ be a $k$-regular Galois extension of group $G$ and inertia canonical invariant $(C_1, \ldots, C_r)$. Assume that the following condition holds:

(H2) $\{C_1^a, \ldots, C_r^a / a \in \mathbb{N}\} \neq cc(G)$.

Then, under condition (H1/$k$), the extension $E/k(T)$ satisfies the (geometric non $G$-parametricity) condition.

In particular, under the sole condition (H2), there exists some number field $k'$ containing $k$ such that the extension $Ek'/k'(T)$ satisfies the (geometric non $G$-parametricity) condition.

**Proof.** Let $C$ be a conjugacy class of $G$ which is not in $\{C_1^a, \ldots, C_r^a / a \in \mathbb{N}\}$ (condition (H2)) and $E'/k(T)$ a $k$-regular Galois extension of group $G$ such that $C$ occurs as the inertia canonical conjugacy class associated with one of its branch points (condition (H1/$k$)). Then the two extensions $E'/k(T)$ and $E/k(T)$ satisfy condition (IC3-1) of Inertia Criterion 3. As condition (IC3-2) also holds, the conclusion follows. □

Now assume that the following group theoretical condition holds:

There exists some set $\{C_1, \ldots, C_r\}$ of non trivial conjugacy classes of $G$ satisfying the following two conditions:
(1) the elements of $C_1, \ldots, C_r$ generate $G$,
(2) $\{C_1^a, \ldots, C_r^a / a \in \mathbb{N}\} \neq \text{cc}(G)$.

Then such a set $\{C_1, \ldots, C_r\}$ occurs as the inertia canonical conjugacy class set of some $k'$-regular Galois extension $E'/k'(T)$ of group $G$ for some number field $k'$ satisfying condition (H1/$k'$) (condition (1) and statement (*)). Moreover the extension $E'/k'(T)$ satisfies condition (H2) of proposition 5.1 (condition (2)). One then obtains the following:

**Corollary 5.2.** There exist some number field $k'$ and some $k'$-regular Galois extension of $k'(T)$ of group $G$ satisfying the (geometric non $G$-parametricity) condition.

Many finite groups satisfy the above group theoretical condition (and then the conclusion of corollary 5.2). Here are some of them.

(a) Given two non trivial finite groups $G_1$ and $G_2$, the product $G_1 \times G_2$ does (in particular, any abelian finite group which is not cyclic of prime power order does\(^{12}\)). Indeed the elements, and a fortiori their conjugacy classes, $(g_1, 1)$ $(g_1 \in G_1)$ and $(1, g_2)$ $(g_2 \in G_2)$ obviously generate the product $G_1 \times G_2$. And no couple of non trivial elements $(g_1, g_2) \in G_1 \times G_2$ is conjugate to a power of one of these couples.

(b) Symmetric groups $S_n$ ($n \geq 3$), alternating groups $A_n$ ($n \geq 4$) and dihedral groups $D_n$ ($n \geq 2$) obviously do.

(c) Non abelian simple groups do. Indeed, as shown in [Wag78] and [MSW94], such a group may be generated by involutions. Then, for any odd prime divisor $p$ of the order of the group, no element of order $p$ is conjugate to a power of an involution and the conclusion follows.

5.1.2. **Conjectural result.** By taking $\{C_1, \ldots, C_r\}$ to be the set of all non trivial conjugacy classes of $G$ in the following conjecture of Fried, the inverse Galois theory condition (H1/$\mathbb{Q}$) from §5.1.1 holds:

**Conjecture (Fried).** Let $\{C_1, \ldots, C_r\}$ be a set of non trivial conjugacy classes of $G$ satisfying the following two conditions:

(1) the elements of $C_1, \ldots, C_r$ generate $G$,
(2) $\{C_1, \ldots, C_r\}$ is a rational\(^{13}\) set of conjugacy classes.

Then $\{C_1, \ldots, C_r\}$ occurs as the inertia canonical conjugacy class set of some $\mathbb{Q}$-regular Galois extension of $\mathbb{Q}(T)$ of group $G$.

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\(^{12}\)Note that this does not hold if $G$ is cyclic of prime power order.

\(^{13}\)i.e. $g^m \in \cup_{i=1}^r C_i$ for each element $g \in \cup_{i=1}^r C_i$ and each positive integer $m$ relatively prime to the least common multiple of the orders of the elements of $C_1, \ldots, C_r$.\/
Under Fried’s conjecture, one then obtains corollary 5.3 below:

**Corollary 5.3.** Assume that there exists some set \( \{C_1, \ldots, C_r\} \) of non-trivial conjugacy classes of \( G \) satisfying the following three conditions:

1. the elements of \( C_1, \ldots, C_r \) generate \( G \),
2. \( \{C_1, \ldots, C_r\} \) is a rational set of conjugacy classes,
3. \( \{C_1^a, \ldots, C_r^a / a \in \mathbb{N}\} \neq \text{cc}(G) \).

Then there exists some \( \mathbb{Q} \)-regular Galois extension of \( \mathbb{Q}(T) \) of group \( G \) satisfying the (geometric non-\( G \)-parametricity) condition.

Indeed, under Fried’s conjecture, conditions (1) and (2) provide a \( \mathbb{Q} \)-regular Galois extension \( E/\mathbb{Q}(T) \) of group \( G \) whose inertia canonical conjugacy class set is \( \{C_1, \ldots, C_r\} \). Moreover \( E/\mathbb{Q}(T) \) satisfies condition (H2) of proposition 5.1 (condition (3)) and, as already noted, condition (H1/\( \mathbb{Q} \)) also holds under Fried’s conjecture.

5.2. Other base fields. Assume in this subsection that the base field \( k \) is a finite extension of a rational function field \( \kappa(X) \) with \( \kappa \) an arbitrary algebraically closed field of characteristic zero (and \( X \) an indeterminate).

In this case, condition (H1/\( k \)) is satisfied (statement (*)). Conjoining this and the proof of proposition 5.1 shows that the conclusion of this statement holds under the sole condition (H2). Moreover, under the group theoretical condition from §5.1.1\(^{14}\), corollary 5.2 holds with the suitable number field \( k' \) replaced by our given base field \( k \).

6. Applications of the Branch Point Criterion

Given a number field \( k \) and a finite group \( H \), we use below the Branch Point Criterion to show that some known \( k \)-regular finite Galois extensions of \( k(T) \) of group \( G \) containing \( H \) satisfy the (non-\( H \)-parametricity) condition.

6.1. A general result. The aim of this subsection is corollary 6.1 below. Given a field \( k \) and a finite group \( H \), we will use the following condition which has already appeared in [Leg13b]:

(H3/\( k \)) the group \( H \) occurs as the Galois group of a \( k \)-regular Galois extension of \( k(T) \) with at least one \( k \)-rational branch point.

Not all finite groups satisfy condition (H3/\( k \)) for a given number field \( k \). For example, if \( k \subset \mathbb{R} \), such a group should be of even order [DF90, corollary 1.3]. However, as recalled in [Leg13b, §4.3.1], any finite group satisfies condition (H3/\( k \)) for suitable number fields \( k \).

\(^{14}\)introduced before corollary 5.2.
6.1.1. **Statement of the result.** Let $k$ be a number field, $G$ a finite group and $E/k(T)$ a $k$-regular Galois extension of group $G$. Denote the orbits of its branch points under the action of $G_k$ by $O_1, \ldots, O_s$ and, for each $i \in \{1, \ldots, s\}$, the field generated over $k$ by all points in $O_i$ by $F_i$.

**Corollary 6.1.** Assume that either one of the following two conditions is satisfied:

1. $|O_i| \geq 2$ and the fields $F_1, \ldots, F_s$ are linearly disjoint over $k$;\footnote{i.e. $F_i$ and $F_1 \ldots F_{i-1}F_{i+1} \ldots F_s$ are linearly disjoint over $k$ ($i = 1, \ldots, s$).} 
2. $s = 2$ and $|O_1| = |O_2| = 2$.

Then the extension $E/k(T)$ satisfies the (non $H$-parametricity) condition for any subgroup $H \subset G$ satisfying condition (H3/$k$).

**Remark 6.2.** Assume that $G$ satisfies condition (H3/$k$) and $E/k(T)$ has $r \leq 4$ branch points. From corollary 6.1, we obtain that

- if (i) no branch point is $k$-rational,
- then (ii) $E/k(T)$ satisfies the (non $G$-parametricity) condition.

In particular, we reobtain [Leg13b, corollary 4.4] which is implication (i) $\Rightarrow$ (ii) in the special case $r = 4$.

Proposition 3.3 shows however that the converse (ii) $\Rightarrow$ (i) does not hold in general if $r = 3$: the extension $E/\mathbb{Q}(T)$ there has at least one $\mathbb{Q}$-rational branch point (as noted in the proof) but condition (ii) holds. Proposition 3.5 provides a similar counter-example in the case $r = 4$.

However, for $r = 2$ and number fields $k \subset \mathbb{R}$, this converse (ii) $\Rightarrow$ (i) is true. Indeed fix such a number field $k$ and assume that $E/k(T)$ has two branch points with at least one $k$-rational. Then the other one is also $k$-rational. From [DF94, theorem 1.1], $\text{Gal}(E/k(T))$ is generated by involutions and, since it is cyclic, one has $\text{Gal}(E/k(T)) = \mathbb{Z}/2\mathbb{Z}$. Next repeat the same argument as in the proof of (3) $\Rightarrow$ (1) in proposition 3.1 to conclude that $E/k(T)$ is parametric over $k$.

6.1.2. **Proof of corollary 6.1.** We show below that, under either one of conditions (1) and (2), there are infinitely many distinct primes of the integral closure $A$ of $\mathbb{Z}$ in $k$ which each is not a prime divisor of $m_{E}(T) \cdot m_{E}^*(T)$. Given a subgroup $H \subset G$ satisfying condition (H3/$k$) and a $k$-regular Galois extension $E_H/k(T)$ of group $H$ with at least one $k$-rational branch point, the conclusion then follows from the Branch Point Criterion applied to the extensions $E_H/k(T)$ and $E/k(T)$.

For each index $i \in \{1, \ldots, s\}$, pick $t_i \in O_i$ and let $m_i(T)$ be the irreducible polynomial of $t_i$ over $k$ and $d_i$ the degree of $m_i(T)$. Denote the action of $\text{Gal}(F_i/k)$ on the roots of $m_i(T)$ by $\sigma_i : \text{Gal}(F_i/k) \to S_{d_i}$ ($i = 1, \ldots, s$) and the splitting field of $\prod_{i=1}^{s} m_i(T)$ over $k$ by $F$.\footnote{i.e. $F_i$ and $F_1 \ldots F_{i-1}F_{i+1} \ldots F_s$ are linearly disjoint over $k$ ($i = 1, \ldots, s$).}
First assume that condition (1) holds. From the second part of our hypothesis, \( \text{Gal}(F/k) \) is isomorphic to \( \text{Gal}(F_1/k) \times \cdots \times \text{Gal}(F_s/k) \) and \( \sigma_1 \times \cdots \times \sigma_s : \text{Gal}(F_1/k) \times \cdots \times \text{Gal}(F_s/k) \to S_{d_1 + \cdots + d_s} \) corresponds to its action on the roots of \( \prod_{i=1}^s m_i(T) \). Given an index \( i \in \{1, \ldots, s\} \), the assumption \( |O_i| \geq 2 \) and an easy group theoretical lemma\(^\text{16}\) show that there exists some \( g_i \in \text{Gal}(F_i/k) \) such that \( \sigma_i(g_i) \) has no fixed points.

From the Tchebotarev density theorem, there exist infinitely many distinct primes of \( A \) such that the associated Frobenius is conjugate in \( \text{Gal}(F/k) \) to \( (g_1, \ldots, g_s) \). In particular, there are infinitely many distinct primes of \( A \) which are not prime divisors of \( \prod_{i=1}^s m_i(T) \), and so not of \( m_E(T) \) either. Since \( \infty \) is not a branch point of \( E/k(T) \), the same conclusion holds for \( m_E(T) \cdot m_Z(T) \) [Leg13b, remark 3.11].

Now assume that condition (2) holds. From the last two paragraphs, we may assume that \( F_1 = F_2 \). Then \( m_1(T) \) and \( m_2(T) \) have the same prime divisors up to finitely many. As \( m_1(T) \) is irreducible over \( k \) and has degree \( \geq 2 \), there exist infinitely many distinct primes of \( A \) which each is not a prime divisor of \( m_1(T) \) (e.g. [Hei67, theorem 9]), and so not of \( m_1(T) \cdot m_2(T) \) either, thus ending the proof.

Remark 6.3. (1) As pointed out by the referee, the proof shows that the conclusion of corollary 6.1 still holds under the sole assumption that there exists some \( g \in \text{Gal}(F/k) \) fixing no element of \( O_1 \cup \cdots \cup O_s \).

(2) If \( s = 2 \) and \( |O_1| \geq 3 \) or \( |O_2| \geq 3 \), the proof does not work in general. Indeed \( P(T) = (T^3 - 2)(T^2 + T + 1) \) has zero mod \( p \) for all primes \( p \) [Nag69, §7].

6.2. Examples. As already said in the presentation, \( k \)-regular Galois extensions of \( k(T) \) with given Galois group \( G \) (and \textit{a fortiori} satisfying the assumptions of corollary 6.1) are not always known yet. Of course such extensions always exist in the case \( G = \mathbb{Z}/2\mathbb{Z} \) (and then \( H = \mathbb{Z}/2\mathbb{Z} \) too). We focus in §6.2.1 on this particular situation. We next give in §6.2.2 another examples with \( H = G = \mathbb{Z}/2\mathbb{Z} \) and conclude in §6.2.3 by some examples with larger cyclic groups.

6.2.1. Application of corollary 6.1. Let \( k \) be a number field and \( P(T) \in k[T] \) a separable polynomial over \( k \) of even degree.

Lemma 3.2 shows that the branch points of the \( k \)-regular quadratic extension \( k(T)(\sqrt{P(T)})/k(T) \) are the roots of \( P(T) \). Hence the orbits \( O_1, \ldots, O_s \) of corollary 6.1 exactly correspond to the root sets of the irreducible factors \( P_1(T), \ldots, P_s(T) \) over \( k \) of \( P(T) \). Thus part (1)

\(^{16}\)Namely, given a finite group \( \Gamma \) acting transitively on a finite set \( S \) with at least two elements, there exists some \( \gamma \in \Gamma \) such that \( \gamma \cdot s = s \) for no \( s \in S \). This can be easily obtained from the Burnside lemma [Bur55].
Proof. Set \( \text{denote the } n \) The extension \( \text{Corollary 6.5.} \)

Corollary 6.4. Denote the splitting fields over \( k \) of the polynomials \( P_1(T), \ldots, P_s(T) \) by \( F_1, \ldots, F_s \) respectively. Assume that \( \deg(P_i(T)) \geq 2 \) for each index \( i \in \{1, \ldots, s\} \) and the fields \( F_1, \ldots, F_s \) are linearly disjoint over \( k \). Then the extension \( k(T)(\sqrt{P(T)})/k(T) \) satisfies the (non \( \mathbb{Z}/2\mathbb{Z} \)-parametricity) condition.

In particular, we reobtain implication \( (2) \Rightarrow (3) \) in proposition 3.1.

6.2.2. Cyclotomic polynomials. Fix a positive integer \( s \) and an \( s \)-tuple \( (n_1, \ldots, n_s) \) of distinct integers \( \geq 3 \). For each index \( i \in \{1, \ldots, s\} \), denote the \( n_i \)-th cyclotomic polynomial by \( \phi_{n_i}(T) \).

Corollary 6.5. The extension \( \mathbb{Q}(T)(\sqrt{\phi_{n_1}(T) \cdots \phi_{n_s}(T)})/\mathbb{Q}(T) \) satisfies the (non \( \mathbb{Z}/2\mathbb{Z} \)-parametricity) condition.

Proof. Set \( E = \mathbb{Q}(T)(\sqrt{\phi_{n_1}(T) \cdots \phi_{n_s}(T)}) \). We show below that there are infinitely many distinct primes which each is not a prime divisor of \( m_E(T) \cdot m_E^*(T) \). The conclusion follows from the Branch Point Criterion applied to the extensions \( \mathbb{Q}(\sqrt{T})/\mathbb{Q}(T) \) (for example) and \( E/\mathbb{Q}(T) \).

Since \( \phi_{n_1}(T) \cdots \phi_{n_s}(T) \) has even degree, \( \infty \) is not a branch point of \( E/\mathbb{Q}(T) \) (lemma 3.2). Hence, from [Leg13b, remark 3.11], the two polynomials \( m_E(T) \cdot m_E^*(T) \) and \( m_E(T) \) have the same prime divisors (up to finitely many). Moreover the branch points of \( E/\mathbb{Q}(T) \) are the roots of \( \phi_{n_1}(T) \cdots \phi_{n_s}(T) \) and, with \( \varphi \) the Euler function, one then has \( m_E(T) = \phi_{n_1}(T)^{\varphi(n_1)} \cdots \phi_{n_s}(T)^{\varphi(n_s)} \). Since, for each index \( i \in \{1, \ldots, s\} \), the prime divisors of \( \phi_{n_i}(T) \) are known to be exactly all primes \( p \) such that \( p \equiv 1 \mod n_i \) (up to finitely many), any prime divisor \( p \) of \( m_E(T) \cdot m_E^*(T) \) satisfies \( p \equiv 1 \mod n_i \) for some index \( i_p \in \{1, \ldots, s\} \) (up to finitely many). From the Dirichlet theorem, there exist infinitely many distinct primes \( p \) which each satisfies \( p \equiv 1 \mod n_i \) for no index \( i \in \{1, \ldots, s\} \), thus ending the proof. \( \square \)

6.2.3. Larger cyclic groups. Let \( n \) be a positive integer \( \geq 3 \). As shown in [Des95], there exists at least one \( \mathbb{Q} \)-regular Galois extension of \( \mathbb{Q}(T) \) of group \( \mathbb{Z}/n\mathbb{Z} \) and branch point set \( \{e^{2ik\pi/n} / (k, n) = 1\} \). Let \( E_n/\mathbb{Q}(T) \) be such an extension.

Corollary 6.6. The extension \( E_n/\mathbb{Q}(T) \) satisfies the (non \( \mathbb{Z}/m\mathbb{Z} \)-parametricity) condition for any positive divisor \( m \) of \( n \) satisfying either one of the following two conditions:

(1) \( m \) is even,

(2) \( m \not\in \{1, n\} \) and, if \( n \equiv 2 \mod 4 \), \( m \neq n/2 \).
Proof. First assume that condition (1) holds. Then the group $\mathbb{Z}/m\mathbb{Z}$ satisfies condition (H3/\Q) \textsuperscript{17} (e.g. [Leg13a, lemma 3.3.5]). Applying part (1) of corollary 6.1 provides the conclusion.

Now assume that condition (2) holds. First remark that, since $\infty$ is not a branch point of $E_n/\Q(T)$, the polynomials $m_{E_n}(T) \cdot m_{E_n}^*(T)$ and $m_{E_n}(T)$ have the same prime divisors (up to finitely many) [Leg13b, remark 3.11] and, since $m_{E_n}(T)$ is a power of the $n$-th cyclotomic polynomial, these prime divisors are exactly all primes $p$ such that $p \equiv 1 \text{ mod } n$ (up to finitely many).

One may assume from above that $m \geq 3$. From the Dirichlet theorem, there exist infinitely many distinct primes $p$ which each satisfies $p \equiv 1 \text{ mod } m$ and $p \not\equiv 1 \text{ mod } n$. Hence, with $E_m/\Q(T)$ any $\Q$-regular Galois extension obtained as $E_n/\Q(T)$ but for the integer $m$ (i.e. $E_m/\Q(T)$ has Galois group $\mathbb{Z}/m\mathbb{Z}$ and branch point set \{e$^{2ik\pi/m}$ / $(k,m) = 1$\}, the original Branch Point Hypothesis of theorem 4.2 applied to the extensions $E_m/\Q(T)$ and $E_n/\Q(T)$ holds. As condition (1) of addendum 4.2 holds, the conclusion follows. \qed

7. Applications of the Inertia Criteria

For this section, let $A$ be a Dedekind domain of characteristic zero with infinitely many distinct primes and $k$ its quotient field.

Given a finite group $H$, we use below Inertia Criteria 1-3 to show that some known $k$-regular finite Galois extensions of $k(T)$ of group $G$ containing $H$ satisfy the (geometric non $H$-parametricity) condition. We first consider the case $H = S_n$ (§7.1) and then the case $H = A_n$ (§7.2). §7.3 is devoted to some other cases $H$ is a non abelian simple group and we conclude in §7.4 with the case $H$ is a $p$-group.

7.1. The case $H = S_n$. Let $n$ be an integer $\geq 3$. The aim of this subsection is corollary 7.1 below which gives our main examples in the situation $H = G = S_n$. The involved regular realizations of $S_n$ are recalled in §7.1.1. Corollary 7.1 is stated in §7.1.2 and proved in §7.1.3.

7.1.1. Some classical regular realizations of symmetric groups. Recall that a permutation $\sigma \in S_n$ is said to have type $1^{i_1} \ldots n^{i_n}$ if, for each index $i \in \{1, \ldots, n\}$, there are $l_i$ disjoint cycles of length $i$ in the cycle decomposition of $\sigma$ (for example, an $n$-cycle is of type $n^1$). Denote the conjugacy class in $S_n$ of elements of type $1^{i_1} \ldots n^{i_n}$ by $[1^{i_1} \ldots n^{i_n}]$.

(a) Morse polynomials. Recall that a degree $n$ monic polynomial $M(Y) \in k[Y]$ is a Morse polynomial if the zeroes $\beta_1, \ldots, \beta_{n-1}$ of the derivative

\textsuperscript{17}introduced at the beginning of §6.1.
$M'(Y)$ are simple and $M(\beta_i) \neq M(\beta_j)$ for $i \neq j$. For example, $M(Y) = Y^n \pm Y$ is a Morse polynomial.

Given a degree $n$ Morse polynomial $M(Y)$, the polynomial $P(T, Y) = M(Y) - T$ provides a $k$-regular Galois extension $E_1/k(T)$ of group $S_n$, branch point set $\{\infty, M(\beta_1), \ldots, M(\beta_{n-1})\}$ and inertia canonical invariant $([n^1], [1^{n-2}2^1], \ldots, [1^{n-2}2^1])$. See [Ser92, §4.4].

(b) Trinomial realizations. Let $m$, $q$, $s$ be positive integers satisfying $1 \leq m \leq n$, $(m, n) = 1$ and $s(n - m) - qn = 1$. The trinomial $Y^n - T^q Y^m + T^s$ provides a $k$-regular Galois extension $E_2/k(T)$ of group $S_n$, branch point set $\{0, \infty, m^n(n-m)^{n-m}n^{-n}\}$ and inertia canonical invariant $([m^1(n-m)^1], [n^1], [1^{n-2}2^1])$. We note for later use that the set of these three conjugacy classes of $S_n$ is g-complete [Sch00, §2.4].

(c) A realization with four branch points. Assume that $n$ is even and $n \geq 6$. From [HRD03], there exists some $k$-regular Galois extension $E_3/k(T)$ of group $S_n$ with four $\mathbb{Q}$-rational branch points and inertia canonical invariant $([1^2(n-2)^1], [1^{n-3}3^1], [2^{(n/2)}], [1^{2(n/2)}])$.

7.1.2. Examples with $G = S_n$. Assume that $n \geq 4$.

**Corollary 7.1.** (1) The three extensions $E_1/k(T)$ (for arbitrary $n \geq 4$), $E_2/k(T)$ (if $n \notin \{4, 6\}$) and $E_3/k(T)$ (if $n \geq 6$ is even) each satisfies the (geometric non $S_n$-parametricity) condition.

(2) Assume that $n = 6$ and $k$ is hilbertian. Then the extension $E_2/k(T)$ satisfies the (geometric non $S_n$-parametricity) condition.

**Remark 7.2.** Fix a PAC field $\kappa$ of characteristic zero and a $\kappa$-regular Galois extension $E/\kappa(T)$ of group $S_n$ (with $n \geq 4$) provided by some degree $n$ Morse polynomial with coefficients in $\kappa$. As noted in §2.3.1, the extension $E/\kappa(T)$ is $S_n$-parametric over $\kappa$. But, with $X$ an indeterminate, the extension $E(X)/\kappa(X)(T)$ is not (corollary 7.1). Hence $E/\kappa(T)$ is not $S_n$-generic over $\kappa$. To our knowledge, no such example was known before.

7.1.3. Proof of corollary 7.1. The proof has two main parts. The first one consists in showing the following general result:

Let $E/k(T)$ be a $k$-regular Galois extension of group $S_n$ and inertia canonical invariant $(C_1, \ldots, C_r)$. Denote the set of all integers $m$ such that $1 \leq m \leq n$ and $(m, n) = 1$ by $I_n$. Then the extension $E/k(T)$ satisfies the (geometric non $S_n$-parametricity) condition provided that one of the following three conditions holds:

\[18\text{As explained in [Leg13a, remark 2.2.4(b)], corollary 7.1 does not hold in the case } n = 3.\]
(1) \([n^1]\) is not in \(\{C_1, \ldots, C_r\}\).
(2) \([m^1(n-m)^1]\) is not in \(\{C_1, \ldots, C_r\}\) for some \(m \in I_n\),
(3) \(k\) is hilbertian, \(n \geq 6\) is even and \([1^2(n-2)^1]\) is not in \(\{C_1, \ldots, C_r\}\).

In particular, the extension \(E/k(T)\) satisfies the (geometric non \(S_n\)-parametricity) condition if \(r \leq \varphi(n)/2\).

This statement is a generalization of [Leg13b, corollary 4.7] and provides theorem 3 from the presentation in the case \(G = S_n\) (as \(\varphi\) satisfies the classical inequality \(\varphi(n) \geq \sqrt{n/2} (n \geq 2)\)).

The second part of the proof consists in checking that each extension \(E_i/k(T)\) \((i = 1, 2, 3)\) satisfies one of the conditions above.

Part 1. The proof consists in each case in applying Inertia Criterion 1 (if there are no assumption on the base field \(k\)) or Inertia Criterion 2 (if \(k\) is assumed to be hilbertian) to some suitable extension \(E_j/k(T)\) (§7.1.1) and the given one \(E/k(T)\).

First assume that \([n^1]\) is not in \(\{C_1, \ldots, C_r\}\). Then \([n^1]\) is not in \(\{C_1^n, \ldots, C_r^n / a \in \mathbb{N}\}\) either, i.e. condition (IC1-2) of Inertia Criterion 1 applied to the extensions \(E_2/k(T)\) and \(E/k(T)\) holds. As conditions (IC1-1) and (IC1-3) hold (as noted in §7.1.1(b)), the conclusion follows.

If \([m^1(n-m)^1]\) is not in \(\{C_1, \ldots, C_r\}\) for some integer \(m \in I_n\), repeat the same argument with \([n^1]\) replaced by \([m^1(n-m)^1]\).

Now assume that \(k\) is hilbertian, \(n \geq 6\) is even and \([1^2(n-2)^1]\) is not in \(\{C_1, \ldots, C_r\}\). Then \([1^2(n-2)^1]\) is not in \(\{C_1^n, \ldots, C_r^n / a \in \mathbb{N}\}\) either, i.e. condition (IC2-1) of Inertia Criterion 2 applied to \(E_3/k(T)\) and \(E/k(T)\) holds. As condition (IC2-2) holds, the conclusion follows.

Part 2. Let \(i \in \{1, 2, 3\}\).
(a) If \(i = 1\), condition (2) holds (with \(m = 1\)).
(b) Assume that \(i = 2\). If \(n \not\in \{4, 6\}\), one has \(\varphi(n) \geq 4\) and condition (2) holds. In the case \(n = 6\), condition (3) holds.
(c) If \(i = 3\), condition (1) holds.

7.2. The case \(H = A_n\). Let \(n\) be an integer \(\geq 4\). The aim of this subsection is corollary 7.3 below which gives our main examples in the situation \(H = G = A_n\). The involved regular realizations of \(A_n\) are recalled in §7.2.1. Corollary 7.3 is stated in §7.2.2 and proved in §7.2.3.

7.2.1. Some classical regular realizations of alternating groups. Recall that, if the conjugacy class \([1^{i_1} \ldots n^{i_n}]\) in \(S_n\) of permutations of type \(1^{i_1} \ldots n^{i_n}\) is contained in \(A_n\), then \([1^{i_1} \ldots n^{i_n}]\) is a conjugacy class of \(A_n\) if and only if there exists some index \(p \in \{1, \ldots, n\}\) such that \(l_p \geq 2\) or

\[\varphi(n) \geq \sqrt{n/2} (n \geq 2)\]

\[\varphi(n)/2\]

Here and in §7.2.3, \(\varphi\) denotes the Euler function.
l_2p \geq 1. Otherwise [1^1 \ldots n^{l_n}] splits into two distinct conjugacy classes of A_n which we denote below by [1^1 \ldots n^{l_n}]_1 and [1^1 \ldots n^{l_n}]_2.

(a) Mestre’s realizations. Assume that n is odd. In [Mes90], Mestre produces some k-regular Galois extensions E'_1/k(T) of group A_n with n − 1 branch points and inertia canonical invariant ([1^{n−3}3^1], \ldots, [1^{n−3}3^1]).

(b) From the trinomial realizations. Fix a positive integer m ≤ n such that \( (m,n) = 1 \). Apply the “double group trick” [Ser92, lemma 4.5.1] to the extension E_2/k(T) from §7.1.1(b) to obtain a three branch point k-regular Galois extension E_2/k(T) of group A_n and, from the branch cycle lemma [Fri77] [Völ96, lemma 2.8], with inertia canonical invariant -\([n^1, n^1, m^1(n−m)/2^2]\) if n is even,
-\([n^1, n^1, m^1(n−m)/2^2]\) if n and m are odd,
-\([n^1, n^1, m^2(n−m)/2^2]\) if n is odd and m is even.

Note that the branch cycle lemma shows that the branch point corresponding to the following conjugacy class (in each case) is \( \mathbb{Q} \)-rational:
-\([n/2^2]\) if n is even,
-\([m^1(n−m)/2^2]\) if n is odd and m is odd,
-\([m^2(n−m)/2^2]\) if n is odd and m is even.

(c) From the realization with four branch points. Assume that n is even and \( n \geq 6 \). As explained in [HRD03, §3.3], the extension E_3/k(T) from §7.1.1(c) induces a k-regular Galois extension E_3/k(T) of group A_n with five branch points and inertia canonical invariant -\([1^2(n−2)/2^2], [1^{n−3}3^1], [1^{n−3}3^1], [1^{2}2^{n}/2], [1^{2}2^{n}/2]\) if n/2 is even,
-\([1^2(n−2)/2^2], [1^{n−3}3^1], [1^{n−3}3^1], [1^{2}2^{n−2}/2], [1^{2}2^{n−2}/2]\) otherwise.

Note that the branch point of E_3/k(T) associated with \([1^2((n−2)/2)^2]\) (in each case) is \( \mathbb{Q} \)-rational from the branch cycle lemma (if \( n \geq 8 \))^20.

7.2.2. Examples with \( G = A_n \). Assume that \( n \geq 5 \).

Corollary 7.3. (1) Assume that \( k \) is hilbertian. Then the three extensions \( E'_n/k(T) \) (if \( n \) is odd), \( E'_2/k(T) \) (for any \( n \neq 6 \)) and \( E'_3/k(T) \) (if \( n \) is even) each satisfies the geometric non \( A_n \)-parametricity condition.

(2) Assume that \( n = 6 \) and \( k \) is either a number field or a finite extension of a rational function field \( \kappa(X) \) with \( \kappa \) an arbitrary algebraically closed field of characteristic zero (and \( X \) an indeterminate). Then \( E'_n/k(T) \) satisfies the geometric non \( A_n \)-parametricity condition.

7.2.3. Proof of corollary 7.3. As in the case \( H = S_n \), the proof has two parts. The first one consists in showing the following general result:

^20In the case \( n = 6 \), the involved branch point might not be \( \mathbb{Q} \)-rational as the two conjugacy classes \([1^2((n−2)/2)^2]\) and \([1^22^{(n−2)/2}\]) coincide.
Let $E'/k(T)$ be a $k$-regular Galois extension of group $A_n$ and inertia canonical invariant $(C_1, \ldots, C_r)$. Denote the set of all integers $m$ such that $1 \leq m \leq n$ and $(m, n) = 1$ by $I_n$. Then the extension $E'/k(T)$ satisfies the (geometric non $A_n$-parametricity) condition provided that either one of the following two conditions holds:

1. $k$ is hilbertian and one of the following four conditions holds:
   
   a. $n$ is odd and $[m^1((n - m)/2)^2]$ is not in $\{C_1, \ldots, C_r\}$ for some odd $m \in I_n$,
   
   b. $n$ is odd and $[(m/2)^2(n - m)]$ is not in $\{C_1, \ldots, C_r\}$ for some even $m \in I_n$,
   
   c. $n$ is even and $[(n/2)^2]$ is not in $\{C_1, \ldots, C_r\}$,
   
   d. $n \geq 8$ is even and neither $[2^1(n - 2)^1]$ nor $[1^2((n - 2)/2)^2]$ is in the set $\{C_1, \ldots, C_r\}$.

2. $k$ is either a number field or a finite extension of a rational function field $\kappa(X)$ with $\kappa$ an arbitrary algebraically closed field of characteristic zero (and $X$ an indeterminate) and either one of the following two conditions holds:

   a. $n$ is odd and neither $[n^1]_1$ nor $[n^1]_2$ is in $\{C_1, \ldots, C_r\}$,
   
   b. $n$ is even and neither $[m^1(n - m)]_1$ nor $[m^1(n - m)]_2$ is in the set $\{C_1, \ldots, C_r\}$ for some $m \in I_n$,
   
   c. $n = 6$ and neither $[2^14^1]$ nor $[1^22^2]$ is in $\{C_1, \ldots, C_r\}$.

In particular, the extension $E'/k(T)$ satisfies the (geometric non $A_n$-parametricity) condition if $r \leq \varphi(n)/2$ and $k$ is either a number field or a finite extension of a rational function field $\kappa(X)$ with $\kappa$ an arbitrary algebraically closed field of characteristic zero.

The second part of the proof consists in checking that each extension $E'_i/k(T)$ ($i = 1, 2, 3$) satisfies one of the conditions above.

**Part 1.** The proof is very similar to that in the case $H = G = S_n$. The only differences are that

- the extensions of §7.2.1 should be used (instead of those of §7.1.1),
- Inertia Criterion 3 should be used if $k$ is assumed to be either a number field or a finite extension of a rational function field $\kappa(X)$ with $\kappa$ an arbitrary algebraically closed field of characteristic zero.

We only indicate in each case which conjugacy class of $A_n$ is not in $\{C^a_1, \ldots, C^a_n / a \in \mathbb{N}\}$ and which extension $E'_j/k(T)$ from §7.2.1 should be used to conclude. A unified proof is given in [Leg13a, §3.4.2.2].

Assume that $k$ is hilbertian. In case (1)-(a), the conjugacy class $[m^1((n - m)/2)^2]$ is not in $\{C^a_1, \ldots, C^a_n / a \in \mathbb{N}\}$. Then use the extension $E'_2/k(T)$. Replace $[m^1((n - m)/2)^2]$ by $[(m/2)^2(n - m)]$ in case
(1)-(b) and by \( [(n/2)^2] \) in case (1)-(c). In case (1)-(d), \( [1^2((n - 2)/2)^2] \) is not in \( \{ C_1^a, \ldots, C_r^a / a \in \mathbb{N} \} \). Then use the extension \( E'_3/k(T) \).

Now assume that \( k \) is either a number field or a finite extension of \( \kappa(\mathbf{X}) \). In case (2)-(a), either one of the classes \([n_1]\) and \([n_2]\) is not in \( \{ C_1^a, \ldots, C_r^a / a \in \mathbb{N} \} \). Then use the extension \( E'_2/k(T) \). Replace \( [n_1]\) and \([n_2]\) by \([m_1(n - m)]_1\) and \([m_1(n - m)]_2\) in case (2)-(b). In case (2)-(c), \([1^22^2] \) is not in \( \{ C_1^a, \ldots, C_r^a / a \in \mathbb{N} \} \). Then use \( E'_3/k(T) \).

**Part 2.** Let \( i \in \{ 1, 2, 3 \} \).

(a) If \( i = 1 \), condition (1)-(a) holds (with \( m = 1 \)).

(b) Assume that \( i = 2 \). If \( n \geq 8 \) is even (resp. \( n = 6 \)), condition (1)-(d) (resp. (2)-(c)) holds. If \( n \) is odd and \( m \in \{ 1, n - 1 \} \) (resp. \( m \notin \{ 1, n - 1 \} \)), condition (1)-(b) (resp. (1)-(a)) holds.

(c) If \( i = 3 \), condition (1)-(c) holds.

Proceeding similarly, one may also show that the three \( k \)-regular Galois extensions \( E_1/k(T) \), \( E_2/k(T) \) and \( E_3/k(T) \) of group \( S_n \) from §7.1.1 each satisfies the (geometric non \( A_n \)-parametricity) condition if \( k \) is hilbertian (for suitable integers \( n \)). This is done in [Leg13a, §3.4.3].

**7.3. Some other cases \( H \) is a non abelian simple group.** We now give some examples involving some \( k \)-regular Galois extensions of \( k(T) \) provided by the rigidity method. We use below standard Atlas [C+85] notation for conjugacy classes of finite groups.

**7.3.1. Examples with \( \text{PSL}_2(\mathbb{F}_p) \).** Let \( p \geq 5 \) be a prime such that \( \left( \frac{2}{p} \right) = -1 \) (resp. \( \left( \frac{2}{p} \right) = -1 \)) and \( E_1/k(T) \) (resp. \( E_2/k(T) \)) a \( k \)-regular Galois extension of group \( \text{PSL}_2(\mathbb{F}_p) \) and inertia canonical invariant \( (2A, pA, pB) \) (resp. \( (3A, pA, pB) \)) [Ser92, propositions 7.4.3-4 and theorem 8.2.2].

**Corollary 7.4.** Assume that \( k \) is hilbertian and \( (-1)^{(p-1)/2}p \) is a square in \( k \). Then the extensions \( E_1/k(T) \) (if \( \left( \frac{2}{p} \right) = -1 \)) and \( E_2/k(T) \) (if \( \left( \frac{2}{p} \right) = -1 \)) satisfy the (geometric non \( \text{PSL}_2(\mathbb{F}_p) \)-parametricity) condition.

**Proof.** Let \( E/k(T) \) be a \( k \)-regular Galois extension of group \( \text{PSL}_2(\mathbb{F}_p) \) with three \( k \)-rational branch points and inertia canonical invariant \( (2A, 3A, pA) \) [Ser92, proposition 7.4.2 and theorem 8.2.1]. As \( 3 \) does not divide \( 2p \) (resp. \( 2 \) does not divide \( 3p \)), condition (IC2-1) of Inertia Criterion 2 applied to \( E/k(T) \) and \( E_1/k(T) \) (resp. and \( E_2/k(T) \)) holds (remark 4.3). As condition (IC2-2) holds, the conclusion follows. \( \square \)

**7.3.2. Examples with the Monster group.** Let \( E_1/k(T) \) be a \( k \)-regular Galois extension of group the Monster group \( M \) with three \( k \)-rational branch points and inertia canonical invariant \( (2A, 3B, 29A) \) [Ser92,
proposition 7.4.8 and theorem 8.2.1]. Let $E_2/k(T)$ be a $k$-regular Galois extension of group $M$ with three $k$-rational branch points and corresponding ramification indices $2, 3, 71$ [Tho84] (if $-71$ is a square in $k$). Applying twice Inertia Criterion 2 (and remark 4.3) to these extensions leads to corollary 7.5 below:

**Corollary 7.5.** Assume that $k$ is hilbertian and $-71$ is a square in $k$. Then the two extensions $E_1/k(T)$ and $E_2/k(T)$ each satisfies the (geometric non $M$-parametricity) condition.

7.3.3. **Examples with $H \neq G$.** Let $E/k(T)$ be a $k$-regular Galois extension of group the Baby-Monster $B$ and inertia canonical invariant $(2C, 3A, 55A)$ [MM99, chapter II, proposition 9.6 and chapter I, theorem 4.8].

**Corollary 7.6.** Assume that $k$ is hilbertian. Then, with $Th$ the Thompson group, the extension $E/k(T)$ satisfies the (geometric non $Th$-parametricity) condition.

**Proof.** It suffices to apply Inertia Criterion 2 (and remark 4.3) to the extensions $E'/k(T)$ and $E/k(T)$, where $E'/k(T)$ denotes any $k$-regular Galois extension of group $Th$ with three $k$-rational branch points and inertia canonical invariant $(2A, 3A, 19A)$ [MM99, chapter II, proposition 9.5 and chapter I, theorem 4.8].

Any finite group $H$ is a subgroup of $G = S_n$ provided that $n \geq |H|$. This allows us to give some examples of non $H$-parametric extensions of group $S_n$ for large enough integers $n$. For instance, the extension $E_1/k(T)$ from §7.1.1(a) satisfies the following:

**Corollary 7.7.** Let $n$ be a positive integer $\geq 604800$. Assume that either one of the following two conditions holds:

1. $7 \nmid n$ and $k$ is hilbertian,
2. $5 \nmid n$ and $k$ is either a number field or a finite extension of a rational function field $\kappa(X)$ with $\kappa$ an arbitrary algebraically closed field of characteristic zero (and $X$ an indeterminate).

Then, with $J_2$ the Hall-Janko group, the extension $E_1/k(T)$ satisfies the (geometric non $J_2$-parametricity) condition.

**Proof.** It suffices to apply Inertia Criterion 2 if condition (1) holds or Inertia Criterion 3 if condition (2) holds (and remark 4.3 in both situations) to the extensions $E/k(T)$ and $E_1/k(T)$, where $E/k(T)$ denotes any $k$-regular Galois extension of group $J_2$, inertia canonical invariant $(5A, 5B, 7A)$ and such that the branch point corresponding to $7A$ is $k$-rational [Ser92, proposition 7.4.7 and theorem 8.2.2].
7.4. **The case $H$ is a $p$-group.** Let $G$ be a finite group, $p$ a prime divisor of $|G|$ and $E/k(T)$ a $k$-regular Galois extension of group $G$.

**Corollary 7.8.** Assume that the following two conditions hold:

1. $p$ divides none of the ramification indices of the branch points,
2. $k$ is a number field or a finite extension of a rational function field $k(X)$ with $k$ an arbitrary algebraically closed field of characteristic zero.

Then the extension $E/k(T)$ satisfies the (geometric non $H$-parametricity) condition for any $p$-subgroup $H \subset G$ which occurs as the Galois group of a $k$-regular Galois extension of $k(T)$. Furthermore condition (2) can be removed in the case $p = 2$ and $H = \mathbb{Z}/2\mathbb{Z}$.

**Remark 7.9.** Assume that $k$ is a number field. Under condition (1), one has the following two conclusions.

(a) The extension $E/k(T)$ satisfies the (geometric non $\mathbb{Z}/p\mathbb{Z}$-parametricity) condition.

(b) There is a finite extension $k'/k$ such that $Ek'/k(T)$ satisfies the (geometric non $H$-parametricity) condition for any $p$-subgroup $H \subset G$.

In the case $k$ is a finite extension of a rational function field $k(X)$ with $k$ an arbitrary algebraically closed field of characteristic zero, then, under condition (1), statement (b) holds with $k' = k$.

Corollary 7.8 may be applied to various $k$-regular Galois extensions of $k(T)$. For example, consider those of group the Conway group $Co_1$ and inertia canonical invariant $(3A, 5C, 13A)$ [MM99, chapter II, proposition 9.3 and chapter I, theorem 4.8]. The set of prime divisors of $|Co_1|$ is $\{2, 3, 5, 7, 11, 13, 23\}$ and condition (1) holds for any prime $p$ in $\{2, 7, 11, 23\}$. Moreover many $k$-regular Galois extensions of $k(T)$ recalled in this paper also satisfy condition (1) (for suitable primes $p$).

**Proof.** Given a $p$-subgroup $H \subset G$ as in corollary 7.8, the conclusion follows from Inertia Criterion 3 (and remark 4.3) applied to any $k$-regular Galois extension $E_H/k(T)$ of group $H$ and $E/k(T)$.

In the special case $p = 2$ and $H = \mathbb{Z}/2\mathbb{Z}$, take $E_H = k(\sqrt{T})$ and use Inertia Criterion 1 (instead of Inertia Criterion 3) to conclude. □

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E-mail address: Francois.Legrand@math.univ-lille1.fr

Laboratoire Paul Painlevé, Mathématiques, Université Lille 1, 59655 Villeneuve d’Ascq Cedex, France