Bootstrap approach to 1+1 dimensional integrable quantum field theories: the case of the Sinh-Gordon model

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1+1 dimensional integrable quantum field theories correspond to a sparse subset of quantum field theories where the calculation of physically interesting observables can be brought to explicit, closed and manageable expressions thanks to the factorisability of the S matrices which govern the scattering in these models. In particular, the correlation functions are expressed in terms of explicit series of multiple integrals, this non-perturbatively for all values of the coupling. However, the question of convergence of these series, and thus the mathematical well-definiteness of these correlators, is mostly open. This paper reviews the overall setting used to formulate such models and discusses the recent progress relative to solving the convergence issues in the case of the 1+1 dimensional massive integrable Sinh-Gordon quantum field theory.

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1 Introduction

1.1 Scattering matrices for quantum integrable field theories

It was discovered in the early XX century that the description of matter at low-scales demands to wave-off some of the existing at the time paradigms governing the motion and very structure of particles in interactions. This led to the development of the theory of relativity on the one hand, and quantum mechanics on the other. In the latter setting, the state of a physical system is described by a vector, the wave function, belonging to some Hilbert space and supposed to encapsulate all the physical degrees of freedom of that system. On the classical level, the time evolution of particles’ momenta and positions is governed by a set of generically non-linear ordinary differential equations which can be written in the form of Hamilton’s equations. In its turn, the time evolution of a wave function is governed by a first order ordinary linear differential equation driven by the Hamiltonian operator. This operator is obtained through a quantisation procedure: its symbol is given by the classical Hamiltonian of the system or, said differently, it is obtained from the classical Hamiltonian upon replacing the classical momenta and positions by operators. While the success of the approach was astonishing relatively to the amount of experiments which could have been explained, soon after the early development of the theory it became clear that in order to describe physics at even smaller scales or higher energies, one needs to develop a quantum theory of fields which would bring together the quantum and relativistic features in the setting of uncountably many degrees of freedom. In loose words, such a theory would be reached by producing operator valued generalised functions, viz. formal kernels of distributions, depending on the space-time coordinates which would satisfy analogues of non-linear, relativistically invariant, evolution equations arising in classical field theory. While it was rather straightforward to construct the quantum theory of the free field (and nowadays such a construction is fully rigorous) the construction of interacting theories, the sole relevant for physics, appeared to be a tremendously hard task, this even on a formal level of rigour. The various approaches that were developed quickly met serious problems, the most prominent being the divergence of coefficients supposed to describe the formal perturbative expansions of physical observables around the free theories. Eventually, these problems could have been formally circumvented in certain cases by the use of the so-called renormalisation procedure. The latter, while being able to produce numbers which were measured with great agreement in collider experiments, eluded for very long any attempts at making it rigorous. Some progress was eventually achieved for several instances of truly interacting, viz. non-free, quantum field theories within the so-called constructive quantum field theory approach, see [35] for a review. While successful in rigorously showing the existence and certain overall properties of such theories, the approach did not lead yet to rigorous and manageable expressions for the correlation functions, which are the quantities measured in experiments and thus of prime interest to the theory.

Among the various alternatives to renormalisation, one may single out the S-matrix program which aimed at describing a quantum field theory directly in terms of the quantities that are measured in experiments. This led to a formulation of the theory in terms of matrix valued functions in \( n \) complex variables, with \( n = 0, 1, 2, \ldots \), that correspond to the entries of the S-matrix between asymptotic states. The S-matrix program was actively investigated in the 60s and 70s and numerous attempts were made to characterise the S-matrix which is the central object in this approach, see e.g. [14], [18]. However, these investigations led to rather unsatisfactory results in
spacial dimensions higher than one, mainly due to the incapacity of constructing viable, explicit, S-matrices for non-trivial models.

The interest in the S-matrix approach was revived by the pioneering work of Gryanik and Vergeles [16]. These authors set forth the first features of an integrable structure based-method for determining S-matrices for the 1+1 dimensional quantum field theories whose classical analogues exhibit an infinite set of independent local integrals of motion. Indeed, the existence of analogous conservation laws on the quantum level heavily constrains the possible form of the scattering basically by reducing it to a concatenation of two-body processes and hence making the calculations of S-matrices feasible. The work [16] focused on the case of models only exhibiting one type of asymptotic particles, the main example being given by the quantum Sinh-Gordon model. This 1+1 dimensional quantum field theory which will be taken as a guiding example from now on. It corresponds to the appropriate quantisation of the classical evolution equation of a scalar field $\varphi(x,t)$ under the partial differential equation

$$
\left(\partial^2_t - \partial^2_x\right)\varphi + \frac{m^2}{g} \sinh(g\varphi) = 0 \quad (x,t) \in \mathbb{R}^2 .
$$

(1.1)

For this model, the asymptotic "in" states of the theory are described by vectors $f = (f^{(0)}, \ldots, f^{(n)}, \ldots)$ which belong to the Fock Hilbert space

$$
b_{\text{in}} = \bigoplus_{n=0}^{+\infty} L^2(\mathbb{R}^n_{\geq}) \quad \text{with} \quad \mathbb{R}^n_{\geq} = \left\{ \beta_n = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n : \beta_1 > \cdots > \beta_n \right\} .
$$

(1.2)

This means that $f^{(n)} \in L^2(\mathbb{R}^n_{\geq})$ has the physical interpretation of an incoming $n$-particle wave-packet density in rapidity space. More precisely, on physical grounds, one interprets elements of the Hilbert space $b_{\text{in}}$ as parameterised by $n$-particles states, $n \in \mathbb{N}$, arriving, in the remote past, with well-ordered rapidities $\beta_1 > \cdots > \beta_n$ prior to any scattering which would be enforced by the interacting nature of the model.

For the 1+1 dimension quantum Sinh-Gordon model the S-matrix proposed in [16] is purely diagonal and thus fully described by one scalar function of the relative "in" rapidities of the two particles:

$$
S(\beta) = \frac{\tanh \left( \frac{i\beta - i\pi b}{2} \right)}{\tanh \left( \frac{i\beta + i\pi b}{2} \right)} \quad \text{with} \quad b = \frac{1}{2} \frac{g^2}{8\pi + g^2} .
$$

(1.3)

This S-matrix satisfies the unitarity $S(\beta)S(-\beta) = 1$, and crossing $S(\beta) = S(\pi - \beta)$ symmetries. These are, in fact, fundamental symmetry features of an S-matrix and arise in many other integrable quantum field theories. Within the physical picture, throughout the flow of time, the "in" particles approach each other, interact, scatter and finally travel again as asymptotically free outgoing, viz. "out", particles. Within such a scheme, an "out" $n$-particle state is then parameterised by $n$ well-ordered rapidities $\beta_1 < \cdots < \beta_n$ and can be seen as a component of a vector belonging to the Hilbert space

$$
b_{\text{out}} = \bigoplus_{n=0}^{+\infty} L^2(\mathbb{R}^n_{<}) \quad \text{with} \quad \mathbb{R}^n_{<} = \left\{ \beta_n = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n : \beta_1 < \cdots < \beta_n \right\} .
$$

(1.4)

The S-matrix will allow one to express the "out" state $g = (g^{(0)}, \ldots, g^{(n)}, \ldots)$ which results from the scattering of an "in" state $f = (f^{(0)}, \ldots, f^{(n)}, \ldots)$ as

$$
g^{(n)}(\beta_1, \ldots, \beta_n) = \prod_{a<\beta}^{n} S(\beta_a - \beta_b) \cdot f^{(n)}(\beta_n, \ldots, \beta_1) .
$$

(1.5)
Note that in this integrable setting, there is no particle production and that the scattering is a concatenation of two-body processes.

Over the years, it turned out to be possible to characterise thoroughly the $S$-matrices for more involved quantum field theories underlying to other integrable classical field theories in $1+1$ dimensions. Such models possess several types of asymptotic particles which can also form bound states. Then, the "in" Fock Hilbert space is more complicated and takes the form $\bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^n, \otimes^n \mathbb{C}^p)$ where the $L^2$-space refers to $\otimes^n \mathbb{C}^p$ valued functions on $\mathbb{R}^n$, with $p$ corresponding to the number of different asymptotic particles in the given theory. The most celebrated example corresponds to the Sine-Gordon quantum field theory. Building on Faddeev-Korepin’s [23] semi-classical quantisation results of the solitons in the classical Sine-Gordon model, one concludes that the underlying quantum field theory possesses two distinct types of asymptotic particles of equal mass, the soliton and the anti-soliton, as well as a certain number, which depends on the coupling constant, of bound states thereof. These all have distinct masses and are called breathers. Zamolodchikov argued the explicit form of the $S$-matrix governing the soliton-antisoliton scattering [39] upon using the factorisability of the $n$-particle $S$-matrix into two-particle processes, the independence of the order in which a three particle scattering process arises from a concatenation of two-particle processes as well as the fact that equal mass particles may solely exchange their momenta during scattering, this due to the existence of many conservation laws. This enforces that the $S$ matrix satisfies the Yang-Baxter equation, which originally appeared in rather different contexts [6, 37], and strongly restricts its form. We do stress that the Yang-Baxter equation is the actual cornerstone of quantum integrability, so that it is not astonishing to recover it also in this setting. The missing pieces of the Sine-Gordon $S$-matrix capturing the soliton-breather and breather-breather scattering were then proposed in [19]. Nowadays, $S$-matrices of many other models have been proposed, see e.g. [1, 38].

1.2 The operator content & the Bootstrap program

1.2.1 The basic operators

Having in mind the per se full construction of the quantum field theory, identifying the content in asymptotic particles, viz. the "in" particles’ Hilbert space $h_{in}$, and the $S$-matrix which describes their scattering only arises as the first step. Indeed, one should build, in a way that is compatible with the form of the scattering encapsulated in the $S$-matrix of interest, a family $O_\alpha$ of operator-valued distributions, $\alpha$ running through some set $\mathcal{S}$. More precisely, the $O_\alpha$ should be distributions acting on smooth, compactly supported functions $d(x)$ of the Minkowskian space-time coordinate

$$x = (x_0, x_1) \in \mathbb{R}^{1,1} \quad \text{with} \quad x \cdot y = x_0 y_0 - x_1 y_1 . \quad (1.6)$$

Then $O_\alpha[d]$ is some densely defined operator on $h_{in}$ whose domain could, in principle, depend on $d$. It is useful from the point of view of connecting this picture to physics to express $O_\alpha$ directly in terms of its generalised operator valued function

$$O_\alpha[d] = \int_{\mathbb{R}^2} d^2 x d(x) O_\alpha(x) . \quad (1.7)$$

In fact, in physics’ terminology, it is the $O_\alpha(x)$s which correspond to the quantum fields of the theory. Moreover, as will be apparent in the following, it turns out that in most handlings $O_\alpha(x)$ does actually make sense as a bona fide operator valued function on the Minkowski space having a well-defined dense domain. Hence, unless it is mandatory so as to make an appropriate sense out of the formula, we will make use of the generalised function notation $O_\alpha(x)$.
On top of being compatible with the scattering date, the operators $O_\alpha(x)$ should form an algebra, viz. the product $O_\alpha(x)O_{\alpha'}(y)$ should be a well-defined dense operator for almost all $x$ and $y$, and satisfy causality, viz. that for purely Bosonic theories as the Sinh-Gordon model

$$[O_\alpha(x), O_{\alpha'}(y)] \equiv O_\alpha(x)O_{\alpha'}(y) - O_{\alpha'}(y)O_\alpha(x) = 0 \quad \text{if} \quad (x - y)^2 < 0,$$

namely when $x - y$ is space-like. The family $O_\alpha(x)$ should in particular contain the per se quantised counterparts of the classical fields arising in the original evolution equation, for instance $\Phi(x)$ or $e^{i\Phi(x)}$ in the Sinh-Gordon quantum field theory case. Moreover, these operators should comply with the various other symmetries imposed on a quantum field theory, such as invariance under Lorentz boosts of space-time coordinates or translational invariance. In the quantum Sinh-Gordon field theory on which we shall focus from now on, the latter means that the model is naturally endowed with a unitary operator $U_T$, such that for any operator $O(x)$

$$U_T \cdot O(x) \cdot U_T^{-1} = O(x + y).$$

(1.9)

The operator $U_T$ acts diagonally on $b_{in}$ given in (1.2):

$$U_T \cdot f = \left( U_T^{(0)} \cdot f^{(0)}, \ldots, U_T^{(n)} \cdot f^{(n)} \right) \quad \text{with} \quad f = (f^{(0)}, \ldots, f^{(n)}, \ldots)$$

(1.10)

and where

$$U_T^{(n)} \cdot f^{(n)}(\beta_n) = \exp \left\{ i \sum_{a=1}^{n} p(\beta_a) \cdot y \right\} f^{(n)}(\beta_n)$$

(1.11)

with $p(\beta) = (m \cosh(\beta), m \sinh(\beta))$ and $\beta_n = (\beta_1, \ldots, \beta_n)$.

Should the construction of quantum fields fulfilling to the above be achieved, the ultimate goal would consist in computing in closed and explicit form the model’s vacuum-to-vacuum $n$-point correlation functions:

$$\langle O_{\alpha_1}(x_1) \cdots O_{\alpha_n}(x_n) \rangle = \text{Tr}_{b_{in}} \left[ P_0 O_{\alpha_1}(x_1) \cdots O_{\alpha_n}(x_n) P_0 \right]$$

(1.12)

with $P_0$ being the orthogonal projection on the 0-particle Fock space. We do stress that the above objects are still generalised functions and, as such, should be considered in an appropriate distributional interpretation. That will be made precise below.

### 1.2.2 The bootstrap program for the zero particle sector

By virtue of the above, in the case of the $b_{in}$ Hilbert space, one may represent an operator $O(x)$ as an integral operator acting on the $L^2$-based Fock space

$$O(x) \cdot f = \left( O^{(0)}(x) \cdot f, \ldots, O^{(n)}(x) \cdot f, \ldots \right)$$

(1.13)

with $O^{(n)}(x) : b_{in} \to L^2(\mathbb{R}^n)$. Later on, we will discuss more precisely the structure of the operators $O^{(n)}(x)$ that one needs to impose so as to end up with a consistent quantum field theory. However, first, we focus our attention on the $0^{th}$ space operators whose action may be represented, whenever it makes sense, as

$$O^{(0)}(x) \cdot f = \sum_{m \geq 0} \int_{\mathbb{R}^n} d^n \beta \Lambda_{\nu m}^{(0)}(\beta_m) \prod_{a=1}^{m} \left\{ e^{-ip(\beta_a) \cdot x} \right\} f^{(m)}(\beta_m).$$

(1.14)

The oscillatory $x$-dependence is a simple consequence of the translation relation (1.9) along with the explicit form of the action of the translation operator (1.11).
In order for $O^{(0)}(x)$ to comply with the scattering data encoded by $S$, one needs to impose a certain amount of constraints on the integral kernels $M_{\omega m}^{(0)}(\beta_m)$. First of all, general principles of quantum field theory impose that, in order for these to correspond to kernels of quantum fields, the $M_{\omega m}^{(0)}(\beta_m)$ have to correspond to a boundary value $F_{m+}(\beta_m)$ on $\mathbb{R}^+$ of a meromorphic function $F^{(0)}_m(\beta_m)$ of the variables $\beta_\alpha$ belonging to the strip

$$\mathcal{S} = \{ z \in \mathbb{C} : 0 < \Im(z) < 2\pi \}. \quad (1.15)$$

Traditionally, in the physics literature, the functions $F^{(0)}_m(\beta_m)$ are called form factors.

Further, one imposes a set of equations on the $F^{(0)}_m$'s. These constitute the so-called form factor bootstrap program. On mathematical grounds, one should understand the form factor bootstrap program as a set of axioms that one imposes as a starting point of the theory given the data $(h_{\text{hnr}}, S)$. Upon solving them, one has to check a posteriori that their solutions do provide one, through (1.14) and (1.16), with a collection of operators satisfying all of the requirements of the theory discussed earlier on.

The bootstrap program axioms take the form of a Riemann-Hilbert problem for a collection of functions in many variables. In the case of the Sinh-Gordon model, since there are no bound states, these take the below form.

**Form Factor Axioms 1.1.** Find functions $F^{(0)}_n$, $n \in \mathbb{N}$, such that, for each $k \in \{ 1 \ldots n \}$ and fixed $\beta_a \in \mathcal{S}$, $a \neq k$, the maps $\beta_k \mapsto F^{(0)}_n(\beta_k)$ are

- meromorphic on $\mathcal{S}$;
- admit $+$, resp. $-$, boundary values $F^{(0)}_{n,\pm}$ on $\mathbb{R}$, resp. $F^{(0)}_{n,-}$ on $\mathbb{R} + 2\pi i$;
- are bounded at infinity by $C \cdot \cosh (\ell \mathcal{R}(\beta_k))$ for some $n$ and $k$ independent $\ell$.

The $F^{(0)}_n$ satisfy the multi-variable system of Riemann-Hilbert problems:

i) $F^{(0)}_n(\beta_1, \ldots, \beta_a, \beta_{a+1}, \ldots, \beta_n) = S(\beta_a - \beta_{a+1}) \cdot F^{(0)}_n(\beta_1, \ldots, \beta_{a-1}, \beta_{a+1}, \ldots, \beta_n)$;

ii) For $\beta_1 \in \mathbb{R}$, and given generic $\beta'_n = (\beta_2, \ldots, \beta_n) \in \mathcal{S}^{n-1}$,

$$F^{(0)}_{n,\pm}(\beta_1 + 2\pi i, \beta'_n) = F^{(0)}_{n,\pm}(\beta'_n, \beta_1) = \prod_{a=2}^n \left[ S(\beta_a - \beta_1) \right] \cdot F^{(0)}_{n,\pm}(\beta'_n);$$

iii) The only poles of $F^{(0)}_n$ are simple, located at $i \pi$ shifted rapidities and

$$-\text{iRes}\left( F^{(0)}_{n,\pm}(\alpha + i\pi, \beta, \beta_n) : d\alpha, \alpha = \beta \right) = \left\{ 1 - \prod_{a=1}^n S(\beta - \beta_a) \right\} \cdot F^{(0)}_n(\beta_n).$$

iv) $F^{(0)}_n(\beta_n + i\mathbf{e}_n) = e^{s_0} \cdot F^{(0)}_n(\beta_n)$ for some number $s_0$ and with $\mathbf{e}_n = (1, \ldots, 1)$.

Note that the reduction occurring at the residues of $F^{(0)}_n(\beta_n)$ at the points $\beta_{ab} = i\pi$ where $\beta_{ab} = \beta_a - \beta_b$ can be readily inferred from i) and iii).

One may already comment on the origin of the axioms. The first one illustrates the scattering properties of the model on the level of the operator’s kernel. The second and third axioms may be interpreted heuristically as a consequence of the LSZ reduction [17], and locality of the operator, see e.g. [3, 34] for heuristics on that matter. Finally, the last axiom is a manifestation of the Lorentz invariance of the theory. The number $s_0$ arising in iv) is called the spin of the operator. Moreover, the number $\ell$ depends on the type of operator being considered. Finally, for more complex models, one would also need to add an additional axiom which would encapsulate the way how the presence of bound states in the model governs certain additional poles in the form factors, c.f. [34].
1.2.3 The bootstrap program for the multi-particle sector

It is convenient to represent the action of the operators $0^n(x)$ in the form

$$ (0^n(x) \cdot f)(\gamma_n) = \sum_{m \geq 0} \prod_{a=1}^n [e^{ip(\gamma_n) x}] \cdot \mathbf{K}_0^{(m)}(x | \gamma_n)[f^{(m)}]. $$

(1.16)

There $\mathbf{K}_0^{(m)}(x | \gamma_n)$ are distribution valued functions which act on appropriate spaces of sufficiently regular functions in $m$ variables. The regularity assumptions will clear out later on, once that we provide the explicit expressions (1.18) for these distributions. In fact, it is convenient, in order to avoid heavy notations, to represent their action as generalised integral operators

$$ \mathbf{K}_0^{(m)}(x | \gamma_n) = \int_{\mathbb{R}^m} d^m \beta \ M_n^{(0)}(\gamma_n; \beta_m) \prod_{a=1}^m [e^{-ip(\beta_a) x}] f^{(m)}(\beta_m), $$

(1.17)

in which one understands of the kernels $M_n^{(0)}(\gamma_n; \beta_m)$ as generalised functions.

The last axiom of the bootstrap program provides one with a way to compute these kernels. Heuristically, it can be seen as a consequence of the LSZ reduction [17].

- $M_n^{(0)}(\alpha_n; \beta_m) = M_{n-1,m+1}^{(0)}(\alpha_n'; (\alpha_1 + i\pi, \beta_m))$
  $$ + 2\pi \sum_{a=1}^m \delta_{\alpha_a, \beta_a} \prod_{k=1}^m \{S(\beta_k - \alpha_1) \cdot M_{n-1,m-1}^{(0)}(\alpha_n', (\beta_1, \ldots, \beta_a, \ldots, \beta_m)). $$

In the above expression, $\beta_a$ means that the variable $\beta_a$ should be omitted and $\delta_{x,y}$ refers to the Dirac mass distribution centred at $x$ and acting on functions of $y$. Finally, the evaluation at $\alpha_1 + i\pi$ is understood in the sense of a boundary value of the meromorphic continuation in the strip $0 \leq \Im(z) \leq \pi$ from $\mathbb{R}$ up to $\mathbb{R} + i\pi$. This axiom is to be complemented with the initialisation condition $M_0^{(0)}(\emptyset; \beta_n) = \mathcal{T}_n^{(0)}(\beta_n)$ when $\beta_n \in \mathbb{R}_0^n$. It is direct to establish that the recursion may be solved in closed form allowing one to determine the distributional kernel $M_n^{(0)}(\alpha_n; \beta_m)$ in terms of $\mathcal{T}_n^{(0)}(\beta_n)$.

$$ M_{n,m}^{(0)}(\alpha_n; \beta_m) = \sum_{p=0}^{\min(n,m)} \sum_{k_1 < \cdots < k_p} \sum_{\ell_1 < \cdots < \ell_p} \prod_{a=1}^p \left[2\pi \delta_{\alpha_a, \beta_a} \right] S(\alpha_n | \alpha_n^{(1)}) $$

$$ \quad \times S(\beta_n^{(1)} | \beta_n) \cdot \mathcal{T}_{n+m-2,p}^{(0)}(\alpha_n^{(2)} + i\pi \alpha_n - \beta_m^{(2)}) $$

(1.18)

There, we have used the shorthand notations $\alpha_n^{(1)} = (\alpha_{k_1}, \ldots, \alpha_{k_p})$ and $\alpha_n^{(2)} = (\alpha_{\ell_1}, \ldots, \alpha_{\ell_p})$ where $\{\ell_1, \ldots, \ell_n\} = \mathbb{R} \setminus \{k_m\}$, $\ell_1 < \cdots < \ell_{n-p}$, and analogously $\beta_m^{(1)} = (\beta_{j_1}, \ldots, \beta_{j_p})$ and $\beta_m^{(2)} = (\beta_{j_1}, \ldots, \beta_{j_{n-p}})$ where $\{j_1, \ldots, j_n\} = \mathbb{R} \setminus \{i_m\}$, $j_1 < \cdots < j_{n-p}$. Moreover, we have introduced

$$ S(\alpha_n | \alpha_n^{(1)}) = \prod_{a=1}^p \prod_{j=1}^{n-p} S(\alpha_{k_j} - \alpha_{\ell_j}) , \quad S(\beta_n^{(1)} | \beta_n) = \prod_{a=1}^p \prod_{b=1}^m S(\beta_{a} - \beta_{a,b}) \cdot \prod_{a>b} S(\beta_{a} - \beta_{b}) $$

Finally, we agree upon $\mathcal{T}_N = (\gamma_N, \ldots, \gamma_1)$ for any $\gamma_N = (\gamma_1, \ldots, \gamma_N)$.

It is clear on the level of the explicit expression (1.18) that this generalised function is well defined, even though it involves a multiplication of distributions.
1.2.4 The road towards the bootstrap program

The first calculation of certain of the operators’ kernels was initiated by Weisz [36] who built on the full characterisation of the $S$ matrix of the Sine-Gordon model to argue with the help of general principles of quantum field theory an expression for the kernel $M_{1;1}^{(O)}(\alpha;\beta)$ of the electromagnetic current operator only involving one dimensional variables $\alpha, \beta$. The setting up of a systematic approach allowing one to calculate all the collection of kernels characterising an operator starting from a given model’s $S$-matrix has been initiated by Karowski and Weisz [20] who proposed a set of equation satisfied by that model’s equivalent of $F^{(O)}_{n}(\beta_{n})$. These allowed them to provide closed expressions for two particle form factors in several models. However these equations were still far from forming the full bootstrap program as described above.

After long investigations [30, 31, 32] which revealed a deeper structure of the form factors of the Sine-Gordon model, Smirnov [33] formulated the equivalent of axioms i) − ii) in that model. Subsequently, Kirillov and Smirnov [21] proposed the full set of the bootstrap program axioms, exemplified in the case of the Massive Thirring model. See also [34]

2 Solving the bootstrap program

The resolution of the bootstrap program was systematised over the years and these efforts led to explicit expressions for the form factors of local operators in numerous 1+1 dimensional massive quantum field theories, see e.g. [34]. The first expressions for the form factors were rather combinatorial in nature. Later, a substantial progress was achieved in simplifying the latter, in particular by exhibiting a deeper structure at their root. Notably, one can mention the free field based approach, also called angular quantisation, to the calculation of form factors. It was introduced by Lukyanov [26] and allowed to obtain convenient representations for certain form factors solving the bootstrap program. In particular, the construction lead to closed and manageable expressions [27, 9] for the form factors of the exponential of the field operators in the Sinh-Gordon and the Bullough-Dodd models. Later, Babujian, Fring, Karowski, Zapetal [3] and Babujian, Karowski [4, 5] developed the more powerful $K$-transform approach which will be described below on the example of the Sinh-Gordon model. The construction of [4, 5] was improved in [15, 25] so as to encompass more complicated operators, the so-called descendants of the Sinh-Gordon exponential of the field operator.

2.1 The 2-particle sector solution

The constructions of solutions to the Bootstrap program starts from obtaining a specific solution to the equations i) − iv) when $n = 2$, i.e. for two variables. This was first achieved in [20].

Lemma 2.1. [20]

Let $\mathcal{F}^{(O)}_{2}(\beta_{1},\beta_{2})$ solve i) − iv) at $n = 2$. Then, there exists $k \in \{0; \ell/2\}$, $\kappa_{a} \in \mathbb{C}$, $a = 1,\ldots,k$, such that

$$
\mathcal{F}^{(O)}_{2}(\beta_{1},\beta_{2}) = N_{0} \prod_{a=1}^{k} \left\{ \sinh \left[ \beta_{12} - \kappa_{a} \right] \cdot \sinh \left[ \beta_{12} + \kappa_{a} \right] \right\} e^{\frac{2}{\tau}(\beta_{1}+\beta_{2})} F(\beta_{12})
$$

(2.1)

for some $N_{0} \in \mathbb{C}$ and where $F$ is given by the integral representation valid for $0 < \Im(\beta) < 2\pi$:

$$
F(\beta) = \exp \left\{-4 \int_{0}^{\infty} dx \frac{\sinh(x\hat{b}) \cdot \sinh(x\hat{\beta}) \cdot \sinh(\frac{x}{2})(\beta)}{x \sinh^{2}(x)} \cos \left( \frac{x}{\hat{\beta}}(i\pi - \beta) \right) \right\} \text{ with } \hat{b} = \frac{1}{2} - \hat{\beta}.
$$

(2.2)
Proof. Axiom iv) ensures that \( F_2^{(0)}(\beta_1, \beta_2) = e^{2i\pi(\beta_1 + \beta_2)} F_2(\beta_1 - \beta_2) \) for some function \( F_2(\beta) \) that is holomorphic on the strip \( 0 < \Im(\beta) < 2\pi \), bounded at infinity by \( C \cosh(\beta) \), and such that \( F_+(-2\pi) = F_+(\beta) \), \( \beta \in \mathbb{R} \). One first looks for a particular solution to this scalar Riemann-Hilbert problem, namely a holomorphic function \( F \) with \( \Im(F) \) periodic and meromorphic function of each variable taken singly; it is easy to check that the holomorphic function \( F \) satisfies

\[
F = \exp \left\{ \int_{\mathbb{R}} \frac{\mathrm{d}s}{4i\pi} \coth \left( \frac{1}{2} (s - \beta) \right) \ln S(s) \right\}. \tag{2.4}
\]

The \( s \) integral can then be taken by means of the integral representation (2.3) for \( \ln S(s) \) and leads to (2.2). Now it is easy to check that the holomorphic function \( G(\beta) = F(\beta)/F(\beta) \) on the strip \( 0 < \Im(\beta) < 2\pi \) admits \( \pm \) boundary values and satisfies \( G_+(\beta + 2i\pi) = G_+(\beta) \) for \( \beta \in \mathbb{R} \) and is bounded by \( C \cosh(\ell\Re(\beta)) \) as \( \Re(\beta) \to \infty \) in this strip. As a consequence, it admits a unique extension into a \( 2i\pi \) periodic entire function bounded by \( C \cosh(\ell\beta) \) and hence is of the form \( P_\ell(e^\beta) \), where \( P_\ell \) is a Laurent polynomial of maximal positive and negative degree \( \ell \). Since it is \( 2i\pi \) periodic and even, \( P_\ell(e^\beta) \) necessarily takes the form

\[
P_\ell(e^\beta) = \prod_{a=1}^k \sin \left[ \frac{\beta - x_a}{2} \right] \cdot \sin \left[ \frac{\beta + x_a}{2} \right] \quad \text{for some} \quad 2k \leq \ell. \tag{2.5}
\]

\[
\begin{align*}
\mathcal{F}_n^{(0)}(\beta_n) &= \sum_{a<b}^n F(\beta_{ab}) \cdot \mathcal{K}_n^{(0)}(\beta_n) \quad \text{with} \quad \beta_{ab} = \beta_a - \beta_b, \tag{2.6}
\end{align*}
\]

with \( F \) as defined through (2.2). Then \( \mathcal{F}_n^{(0)} \) solves the bootstrap axioms i) – iv) if any only if

I) \( \mathcal{K}_n^{(0)} \) is a symmetric function of \( \beta_n \);

II) \( \mathcal{K}_n^{(0)} \) is a \( 2i\pi \) periodic and meromorphic function of each variable taken singly;

III) the only poles of \( \mathcal{K}_n^{(0)} \) are simple and located at \( \beta_a - \beta_b \in i\pi(1 + 2\mathbb{Z}) \). The associated residues are given by

\[
\text{Res} \left( \mathcal{K}_n^{(0)}(\beta_n) \cdot \mathrm{d}\beta_1, \beta_1 \right) = \frac{i}{F(i\pi)} \cdot \frac{1}{\prod_{a=3}^n F(\beta_{2a})} \cdot \left[ \mathcal{K}^{(0)}_{n-2}(\beta''_n) \right] \tag{2.7}
\]

where \( \beta'' = (\beta_3, \ldots, \beta_n) \).
This first transformation simplifies the symmetry properties of the problem. However, the inductive reductions provided by the computation of the residues are still quite intricate. The idea is then to proceed to yet another change of unknown function, this time by means of a more involved transform. The latter will then lead to structurally much simpler, and thus easier to solve, equations satisfied by the new unknown function. As already mentioned, the dawn of this approach goes back to [27, 9] and it was put in the present form in [3, 4, 5]. In particular, we refer to [4] for the proof.

**Proposition 2.3.** [3]

Let $\ell_n \in [0, 1]^n$ and $p_n^{(0)}(\beta_n | \ell_n)$ be a solution to the below constraints

\begin{align*}
\text{a)} & \quad \beta_n \mapsto p_n^{(0)}(\beta_n | \ell_n) \text{ is a collection of } 2\pi \text{ periodic holomorphic functions on } \mathbb{C} \text{ that are symmetric in the two sets of variables jointly, viz. for any } \sigma \in \mathcal{S}_n \text{ it holds } p_n^{(0)}(\beta^\sigma_n | \ell_n^\sigma) = p_n^{(0)}(\beta_n | \ell_n) \text{ with } \beta^\sigma_n = (\beta_{\sigma(1)}, \ldots, \beta_{\sigma(n)}); \\
\text{b)} & \quad p_n^{(0)}(\beta_2 + i \pi, \beta'_n | \ell_n) = g(\ell_1, \ell_2) p_n^{(0)}(\beta''_n | \ell''_n) + h(\ell_1, \ell_2 | \beta'_n) \text{ where } h \text{ does not depend on the remaining set of variables } \ell''_n \text{ and } \\
& \quad g(0, 1) = g(1, 0) = \frac{-1}{\sin(2\pi b) F(i\pi)}; \\
\text{c)} & \quad p_n^{(0)}(\beta_n + \theta \mathcal{E}_n | \ell_n) = e^{\theta s_0} \cdot p_n^{(0)}(\beta_n | \ell_n).
\end{align*}

Then, its $\mathcal{K}$-transform

\begin{equation}
\mathcal{K}_n[p_n^{(0)}(\beta_n)] = \sum_{\ell_n \in [0, 1]^n} (-1)^{\ell_n} \prod_{a < b} \left\{ 1 - \frac{i \ell_{ab} \cdot \sin[2\pi b]}{\sinh(\beta_{ab})} \right\} \cdot p_n^{(0)}(\beta_n | \ell_n),
\end{equation}

in which $\overline{\ell}_n = \sum_{a=1}^n \ell_a$, solves I) – IV).

Note that arguments were given in [13] in favour of some form of bijection between certain classes of solutions to a) – c) and I) – IV). However, we do stress that, so far, the question whether there does exist a clear cut correspondence between all solutions to a) – c) and I) – IV) is still open.

### 3 Towards physical observables and the convergence problem

The resolution of the bootstrap program provides one with the expressions for the integral kernels of certain operators which are candidates for the quantum fields of the 1+1 dimensional Sinh-Gordon quantum field theory. However, for this construction to really provide one with the quantum field theory of interest, one should establish several facts. First of all, the operators so constructed should form an algebra in the sense discussed in Subsection 1.2.1. By virtue of the translational invariance (1.9), this means that, for any $n, m \in \mathbb{N}$, the series of multiple integrals arising in the operator product $U_{\tau_k}^{-1} P_n \varOmega_1(x) \varOmega_2(0) P_m$, where $P_k$ is the orthogonal projector on the $k$-particle Fock space, should converge in the weak sense. Namely, any sufficiently regular functions $\alpha_n \mapsto f^{(m)}(\alpha_n)$ and $\beta_m \mapsto g^{(m)}(\beta_m)$ belonging, resp., to a dense subset of $L^2(\mathbb{R}_n)$ and $L^2(\mathbb{R}_m)$ and for any $d \in C^\infty(\mathbb{R}^2)$
of the form factors \( \langle O_1(x)O_2(0) \rangle \) in terms of the base form factors \( \mathcal{F}_n^{(O)}(\beta) \) of the \( n \)-fold integrals issuing from the operators products, Kirillov and Smirnov showed this property in the Sine-Gordon case in [21, 22]. Their method readily applies to the Sinh-Gordon case. Hence, convergence is the only remaining problem so as to set this construction of quantum field theories on rigorous grounds.

### 3.1 The well-poised series expansion for two-point functions

First of all, by translation invariance, it is enough to focus on \( \langle O_1(x)O_2(0) \rangle \). Recall that, at least in principle, this quantity is a generalised function and should thus be understood, in the first place, as the formal integral kernel of the distribution \( \langle O_1O_2 \rangle \). So that, for \( d \in C_\infty_c(\mathbb{R}^2) \), provided convergence holds, one has

\[
\langle O_1O_2 \rangle[d] = \int_{\mathbb{R}^2} d^2 x d(x) \langle O_1(x)O_2(0) \rangle = \sum_{n \geq 0} \frac{1}{n!} \mathcal{I}^{(O_1O_2)}_n[d]
\]

with

\[
\mathcal{I}^{(O_1O_2)}_n[d] = \int_{\mathbb{R}^n} \frac{d^n \beta}{(2\pi)^n} \mathcal{F}_n^{(O_1)}(\beta) \mathcal{M}^{(O_2)}_{n,0}(\beta_n; 0) \int_{\mathbb{R}^2} d^2 x d(x) \prod_{a=1}^n \left[ e^{-i m \left( \cosh(\beta_{n,a}) - x \sinh(\beta_{n,a}) \right)} \right].
\]

It is a direct consequence of the kernel reduction axiom \( \nu \) and of Lorentz invariance one \( \eta \) that

\[
\mathcal{M}^{(O_2)}_{n,0}(\beta_n; 0) = \mathcal{F}_n^{(O_2)}(\beta_n + i \pi \epsilon_n) = e^{i \pi s_n} \mathcal{F}_n^{(O_2)}(\beta_n).
\]

This identity along with the growth bounds in each \( \beta_n \) of the form factors \( \mathcal{F}_n^{(O)}(\beta_n) \) ensures the well definiteness of the \( n \)-fold integrals since the space-time integral over \( x \) produces a decay in each \( \beta_n \) that is faster than any
exponential $e^{\pm k\beta_n}$, $\Re(\beta_n) \to \pm \infty$. By virtue of the Morera theorem, this rapid decay at infinity along with the holomorphic properties of the integrands allow one to deform, simultaneously for each integration variable $\beta_n$, $a = 1, \ldots, n$ the integration curves to $\mathbb{R} + i\xi \operatorname{sgn}(x)$ when $x$ is space-like and, when $x$ is time-like, to $\gamma(\mathbb{R})$ where $\gamma(u) = u + i\theta(u)$, where $\theta$ is smooth, $|\theta| < \pi/4$, and such that there exists $M > 0$ large enough and $0 < \epsilon < \pi/2$ so that $\theta(u) = -\operatorname{sgn}(t)\operatorname{sgn}(u)e$ when $|u| \geq M$. This operation turns the $\beta_n$ integrals into absolutely convergent ones irrespectively of the presence of $d(x)$.

In particular, for the space-like regime, one gets that

$$I^{(0)}_n = e^{\eta(x)} \int_{\mathbb{R}^d} d^2 x d(x) \int_{\mathbb{R}^n} \frac{d^n \beta}{(2\pi)^n} F_n^{(0)}(\beta_n) \overline{F}_n^{(0)}(\beta_n) \prod_{a=1}^n e^{-mr \cosh(\beta_n)},$$

in which $r = \sqrt{x^2 - t^2}$, $\tanh(\theta) = t/x$ while $\eta(x) = i\pi s_0 + (i\nu + \theta)(s_0 + s_0)\operatorname{sgn}(x)$. Hence, provided convergence holds, one has the well-defined in the usual sense of numbers representation for the two-point function

$$\langle O_1(x)O_2(0) \rangle = e^{\eta(x)} \sum_{n \geq 0} \frac{1}{n!} \int_{\mathbb{R}^n} \frac{d^n \beta}{(2\pi)^n} F_n^{(0)}(\beta_n) \overline{F}_n^{(0)}(\beta_n) \prod_{a=1}^n e^{-mr \cosh(\beta_n)}$$

(3.4)

### 3.2 Convergence of series representation for two-point functions

Thus, the well-definedness of the two-point functions boils down to providing an appropriate upper bound for the below class of $N$-fold integrals for $x > 0$

$$Z_N(x) = \int_{\mathbb{R}^N} d^n \beta \prod_{a \neq b} e^{\frac{1}{2} \omega(\beta_{ab})} \prod_{a=1}^n \left[ e^{-2x \cosh(\beta_a)} \mathcal{K}_n[p^{(0)}_N(\beta_a)] \mathcal{K}_n[p^{(0)}_N(\beta_N)] \overline{\mathcal{K}_n[p^{(0)}_N(\beta_N)]} \right].$$

(3.5)

The two-body potential $\omega$ is defined through the relation $F(\lambda)F(-\lambda) = e^{\omega(\lambda)}$.

**Theorem 3.1.** [24] Assume that there exist $C_1, C_2$ and $k \in \mathbb{N}$ such that given $s \in \{1, 2\}$

$$|p_N^{(0)}(\beta_N | \ell_N)| \leq C_1^{N} \prod_{a=1}^N e^{C_2 \beta_a} \quad \text{for any} \quad \ell_N \in \{0, 1\}^N$$

(3.6)

uniformly in $N$. Then, it holds

$$|Z_N(x)| \leq \exp \left[ -\frac{3\pi^2 \beta \cdot N^2}{4 \cdot (\ln N)^3} \left( 1 + O\left( \frac{1}{\ln N}\right) \right) \right]$$

(3.7)

The proof of this theorem was the goal of the author’s work [24]. The proof relies on Riemann–Hilbert problem techniques for inverting singular integral operators of truncated-Wiener Hopf type along with the Deift-Zhou non-linear steepest descent method [12, 13], concentration of measure and large deviation techniques which were developed for dealing with certain $\beta$-ensembles multiple integrals [10, 2, 28], and some generalisations thereof to the case of $N$-dependent integrands in $N$-dimensional integrals as it was developed in [8].

### 4 The proof of the convergence of the form factor series

In this section we shall describe the main steps of the proof. The details can be found in Proposition 3.1 of [24].
4.1 An simpler upper bound
The starting point consists in obtaining a structurally simpler upper bound on \( \mathcal{Z}_N(x) \) when \( x > 0 \).

**Proposition 4.1.** There exists \( C > 0 \) such that
\[
|\mathcal{Z}_N(x)| \leq \left( C \cdot \ln N \right)^N \cdot \max_{p \in [0:N]} \left| \mathcal{Z}_N,p(x) \right|
\]
(4.1)
where \( \mathcal{Z}_N,p(x) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^p} \mathcal{Z}_N,p(\lambda_N-p, \nu_p) \) whose integrand is expressed as
\[
\mathcal{Z}_N,p(\lambda_N-p, \nu_p) = \prod_{a=1}^{p} \left\{ e^{-V_N(\nu_a)} \right\} \cdot \prod_{a=1}^{N-p} \left\{ e^{-V_N(\lambda_a)} \right\} \times \prod_{a<b}^{p} \left\{ e^{w_N(\nu_{ab})} \right\} \cdot \prod_{a<b}^{N-p} \left\{ e^{w_N(\lambda_{ab})} \right\} \cdot \prod_{b=1}^{N-p} \left\{ e^{w_{tot,N}(\nu_{ab})} \right\} .
\]
(4.2)

Above, there appears the \( N \)-dependent functions
\[
V_N(\lambda) = x \cosh(\tau_N \lambda) , \quad w_N(\lambda) = w(\tau_N \lambda) , \quad w_{tot,N}(\lambda) = w_{tot}(\tau_N \lambda) ,
\]
(4.3)
with \( \tau_N = \ln N \) and one has
\[
w_{tot}(\lambda) = w(\lambda) + v_{res,0}^+(\lambda) \quad \text{with} \quad v_{res,0}^+(\lambda) = \ln \left( \frac{\sinh(\lambda + i \eta) \sinh(\lambda - i \eta)}{\sinh(\lambda + i \eta) \sinh(\lambda - i \eta)} \right).
\]
(4.4)

4.2 Energetic bounds

**Proposition 4.2.** The partition function \( \mathcal{Z}_N,p(x) \) admits the upper bound:
\[
\mathcal{Z}_N,p(x) \leq \exp \left\{ -N^2 \inf_{(\mu, \nu) \in M^1(\mathbb{R}) \times M^1(\mathbb{R})} \mathcal{E}_{\mathcal{H},\mathcal{F}}[\mu, \nu] \right\} + O(N \tau_N^2),
\]
(4.5)
in which the control is uniform in \( p \in [0; N] \) and where
\[
\mathcal{E}_{\mathcal{H},\mathcal{F}}[\mu, \nu] = \frac{1}{N} \left( t \int V_N(s) d\nu(s) + (1-t) \int V_N(s) d\mu(s) \right) - \frac{t^2}{2} \int w_N(s-u) d\nu(s) d\nu(u) - \frac{(1-t)^2}{2} \int w_N(s-u) d\mu(s) d\mu(u) + t(1-t) \int w_{tot,N}(s-u) d\mu(s) d\nu(u).
\]

One may obtain such an upper bound within the standard approach to establishing large deviation bounds for \( N \)-fold integrals as pioneered in [2], adjoined to the local regularisation of the empirical distribution of the integration variables proposed in [28] and some fine bounds due to the \( N \)-dependence of the integrand which were also considered in [8]. The details can be found in Lemmata 4.2-4.3 of [24].
4.3 Characterisation of the minimiser and a lower-bound minimiser

The upper bound established in Proposition 4.2 does not allow one to conclude directly on the convergence of the series. Indeed, even if one could prove that the infimum in (4.5) gives a strictly positive number, the \(N\)-dependence of the energy functional could make the infimum \(N\)-dependent and, in principle, the latter could give rise to a behaviour in \(N\) which, when multiplied by the \(N^2\) prefactor, could turn out to be subdominant in respect to the corrections \(O(N\tau^2_N)\). Hence, the longest part of the proof is devoted to obtaining some sharp and explicit lower bound for the infimum which can then be computed in closed form so that one may explicitly check that the above scenario does hold.

For that purpose, one starts by showing the

**Proposition 4.3.** For \(0 < t < 1\) \(E_{N,t}\) admits a unique minimiser \((\mu^{eq(N,t)}_t, v^{eq(N,t)}_t)\) on \(M^1(\mathbb{R}) \times M^1(\mathbb{R})\). Similarly, \(E_{N,0}\) and \(E_{N,1}\) admit unique minimisers on \(M^1(\mathbb{R})\).

This is established by showing that, for \(0 < t < 1\), \(E_{N,t}\) is lower semi-continuous and strictly convex on \(M^1(\mathbb{R}) \times M^1(\mathbb{R})\), has compact level sets, is not identically \(+\infty\) and is bounded from below. In principle, this result could be already enough to obtain sharp in \(N\) estimates for \(E_{N,t}[\mu^{eq(N,t)}_t, v^{eq(N,t)}_t]\). Indeed, by relying on the analogous to the case of \(\beta\)-ensembles variational characterisation of the minimisers and showing that these are actually Lebesgue continuous with compact connected supports, one may establish a system of two-coupled singular linear integral equations of truncated Wiener-Hopf type depending on the large-parameter \(N\). These may be analysed within the method developed by Krein’s school after generalising the work [29] and solving the \(4 \times 4\) associated Riemann–Hilbert problem in the large-\(N\) regime by the Deift-Zhou non-linear steepest descent method [12] [13]. However, these steps would definitely lead to an extremely cumbersome and long clamber, especially taken the minimal amount of information one needs, in the end, from such handlings. Therefore, it is more convenient to reduce the numbers of minimisers which ought to be thoroughly determined by providing a lower bound for \(E_{N,t}[\mu^{eq(N,t)}_t, v^{eq(N,t)}_t]\) whose estimation would demand less effort while still leading to the desired result.

A direct calculation shows that one has a simpler representation for \(E_{N,t}\) in terms of functionals only acting on one copy of a space of bounded measures:

\[
E_{N,t}[\mu, \nu] = \sum_{\nu = \pm} E_{N}^{(\nu)}[\sigma^{(\nu)}_t] \quad \text{with} \quad \sigma^{(\pm)}_t = t\nu \pm (1 - t)\mu ,
\]

in which \(E_{N}^{(\pm)}\) is a functional on \(M^1(\mathbb{R})\) while \(E_{N}^{(\pm)}\) is a functional on \(M^2_{(2t-1)}(\mathbb{R})\), the space of signed, bounded, measures on \(\mathbb{R}\) of total mass \(2t - 1\). These take the form

\[
E_{N}^{(\pm)}[\sigma] = \frac{1}{N} \int V_N(s) \, d\sigma(s) - \frac{1}{2} \int w_{N}^{(\pm)}(s - t) \cdot d\sigma(s) \, d\sigma(u) ,
\]

\[
E_{N}^{(-)}[\sigma] = -\frac{1}{2} \int w_{N}^{(-)}(s - u) \cdot d\sigma(s) \, d\sigma(u) .
\]

The two-body interactions appearing above involve \(w\) and \(v_{a,b}\) introduced in (3.5) and (3.4):

\[
w_{N}^{(\pm)}(u) = w^{(\pm)}(\tau_N u) \quad \text{with} \quad \begin{cases} w^{(+)}(u) = w(u) + \frac{1}{2} v_{2\pi b,0}^{(+)}(u) \\ w^{(-)}(u) = -\frac{1}{2} v_{2\pi b,0}^{(-)}(u) . \end{cases}
\]

By going to Fourier space, one observes that

\[
E_{N}^{(\pm)}[\sigma] = \frac{1}{2} \int d\lambda |F[\sigma](\lambda)|^2 \frac{\sinh(\pi b,\lambda) \cdot \sinh(\pi b,\lambda)}{\lambda \sinh(\pi b,\lambda)} \geq 0 ,
\]
where $\mathcal{F}[\sigma](\lambda)$ stands for the Fourier transform of the signed measure $\sigma$. Thus,

$$E_{N;\lambda}[\mu_{\text{eq}}^{(N;\lambda)}, \nu_{\text{eq}}^{(N;\lambda)}] \geq E_{N}^{(+)}[\nu_{\text{eq}}^{(N;\lambda)} + (1 - t)\mu_{\text{eq}}^{(N;\lambda)}] \geq E_{N}^{(+)}[\sigma_{\text{eq}}^{(N)}]. \tag{4.11}$$

In the last line, we have used that $E_{N}^{(+)}$ is lower-continuous, has compact level sets, is strictly convex on $\mathcal{M}^{1}(\mathbb{R})$, bounded from below and not identically $+\infty$ so as to ensure the existence of a unique minimiser thereof: $E_{N}^{(+)}[\sigma_{\text{eq}}^{(N)}] = \inf\{E_{N}^{(+)}[\sigma] : \sigma \in \mathcal{M}^{1}(\mathbb{R})\}$.

### 4.4 Singular integral equation characterisation of the minimiser $\sigma_{\text{eq}}^{(N)}$

By using the variational characterisation of the minimiser, see e.g. [1] for an exposition in the $\beta$-ensemble case, one reduces the construction of $\sigma_{\text{eq}}^{(N)}$ to finding a solution to a singular integral equation on the Sobolev space $H_{s}([a_{N}; b_{N}])$ driven by the operator

$$S_{N}[\phi](\xi) = \int_{a_{N}}^{b_{N}} (w^{(+)})(\tau_{N}(\xi - \eta)) \cdot \phi(\eta)d\eta. \tag{4.12}$$

Indeed, upon introducing the effective potential subordinate to a function $\phi \in H_{s}([a_{N}; b_{N}])$

$$V_{\text{eff}}[\phi](\xi) = \frac{1}{N}V_{N}(\xi) - \int_{a_{N}}^{b_{N}} w^{(+)}(\tau_{N}(\xi - \eta)) \cdot \phi(\eta)d\eta \tag{4.13}$$

one may formulate the

**Proposition 4.4.** Let $a_{N} < b_{N}$ and $\xi_{\text{eq}}^{(N)} \in H_{s}([a_{N}; b_{N}])$, $1/2 < s < 1$ solve

$$\frac{1}{N\tau_{N}}V_{N}'(x) = S_{N}[\xi_{\text{eq}}^{(N)}](x) \quad \text{on} \quad [a_{N}; b_{N}], \tag{4.14}$$

be subject to the conditions

$$\xi_{\text{eq}}^{(N)}(\xi) \geq 0 \quad \text{for} \quad \xi \in [a_{N}; b_{N}], \quad \int_{a_{N}}^{b_{N}} \xi_{\text{eq}}^{(N)}(\xi)d\xi = 1 \tag{4.15}$$

and

$$V_{\text{eff}}[\xi_{\text{eq}}^{(N)}](\xi) > \inf[V_{\text{eff}}[\xi_{\text{eq}}^{(N)}](\eta) : \eta \in \mathbb{R}] \quad \text{for any} \quad \xi \in \mathbb{R} \setminus [a_{N}; b_{N}]. \tag{4.16}$$

Then, the equilibrium measure $\sigma_{\text{eq}}^{(N)}$ is supported on the segment $[a_{N}; b_{N}]$ and continuous in respect to Lebesgue’s measure with density $\xi_{\text{eq}}^{(N)}$. Moreover, the density takes the form

$$\xi_{\text{eq}}^{(N)}(\xi) = \sqrt{(b_{N} - \xi)(\xi - a_{N})} \cdot h_{N}(\xi) \quad \text{with} \quad h_{N} \in C^{\alpha}([a_{N}; b_{N}]). \tag{4.17}$$

The above proposition thus provides one with the following strategy for determining the equilibrium measure. One starts by solving the singular integral equation (4.14) for any endpoints $a_{N}$ and $b_{N}$. The inversion should be carried out in an appropriate functional space which is dictated by the local structure (4.17) of the equilibrium.
measure’s density, as can be inferred from an analysis of the systems of loop equations associated with the probability measure on \( \mathbb{R}^N \) naturally subordinate to the energy functional \( E_N^{(+)} \). The fact that \( S_N \) should be inverted on \( H_s([a_N; b_N]) \), \( 0 < s < 1 \), imposes a constraint on \( a_N \) and \( b_N \). A second constraint is obtained from the fact that the equilibrium measure has mass one (4.13). This is still not enough so as to be sure that the solution constructed in this way provides one with the equilibrium measure. For that to happen, one still needs to verify that the two positivity constraints (4.15)-(4.16) are fulfilled. The realisation of such a program demands to have a thorough control on the inversion of \( S_N \). The latter may be reached within the scheme developed in [29], by solving an auxiliary 2 \( \times \) 2 Riemann–Hilbert problem.

### 4.5 The Riemann–Hilbert based inversion of the operator

In the following, we adopt the shorthand notations

\[
\begin{align*}
\overline{a}_N &= \tau_N a_N \ , \quad \overline{b}_N = \tau_N b_N \ , \quad \overline{x}_N = \tau_N (b_N - a_N) \ .
\end{align*}
\] (4.18)

Consider the Riemann–Hilbert problem for a 2 \( \times \) 2 matrix function \( \chi \in \mathcal{M}_2(O(\mathbb{C} \setminus \mathbb{R})) \):

- \( \chi \) has continuous \( \pm \)-boundary values on \( \mathbb{R} \);
- there exist constant matrices \( \chi^{(a)} \) with \( \chi^{(1)}_{12} \neq 0 \) such that when \( \lambda \to \infty \)

\[
\chi(\lambda) = \begin{cases}
\mathcal{P}_L \chi(\lambda) \cdot \begin{pmatrix}
-\xi, & e^{i\pi N} \\
1, & 0
\end{pmatrix} \frac{(-i\lambda)^{1/3}}{e^{i \pi/3}} \cdot \left( I_2 + \frac{\chi^{(1)}}{2} + O(\lambda^{-1}) \right) \cdot Q(\lambda) & \lambda \in \mathbb{H}^+ \\
\mathcal{P}_L \chi(\lambda) \cdot \begin{pmatrix}
-1, & e^{-i\pi N} \\
0, & 1
\end{pmatrix} \cdot (i\lambda)^{1_3} \cdot \left( I_2 + \frac{\chi^{(1)}}{2} + O(\lambda^{-1}) \right) \cdot Q(\lambda) & \lambda \in \mathbb{H}^-
\end{cases}
\]

in which the matrix \( Q(\lambda) \) takes the form

\[
Q(\lambda) = \begin{pmatrix}
0 & \chi^{(1)}_{12} \\
\chi^{(-1)}_{12} & q_1 + \lambda
\end{pmatrix}
\]

with \( q_1 = \left( \chi^{(1)}_{11} \chi^{(1)}_{12} - \chi^{(2)}_{12} \right) \cdot \chi^{(2)}_{12}^{-1} \);

- \( \chi^+ (\lambda) = G_\lambda \chi^- (\lambda) \) for \( \lambda \in \mathbb{R} \) where

\[
G_\lambda = \begin{pmatrix}
e^{i\pi N} & 0 \\
\frac{1}{\pi} \cdot R(\lambda) & -e^{-i\pi N}
\end{pmatrix}
\]

with \( R(\lambda) = 2 \frac{\sinh(\pi \theta \lambda) \cdot \sinh(\pi \hat{\theta} \lambda) \cdot \sinh(\pi \hat{\theta} \lambda)}{\cosh^2(\pi \lambda)} \).

Here \( s_\lambda = \text{sgn}(\Re \lambda) \), \( O(A) \) stands for the ring of holomorphic functions on \( A \), while the \( O \) remainder appearing in matrix equalities should be understood entrywise. Moreover, we point out that the matrix \( Q(\lambda) \) appearing in the asymptotic expansion for \( \chi \) is chosen such that \( \chi \) has the large-\( \lambda \) behaviour

\[
\chi(\lambda) = \chi^{(1)}_{1/\lambda} (\lambda) \cdot (\mp i \lambda)^{1_3} \quad \lambda \in \mathbb{H}^\pm ,
\] (4.19)

with \( \chi^{(1)}_{1/\lambda} (\lambda) \) bounded at \( \infty \).

The Deift-Zhou non-linear steepest descent method [12, 13] allows one to reduce the above Riemann-Hilbert problem into one that is uniquely solvable by the singular integral equation method of [7], provided that \( N \) is large enough and \( b_N - a_N > c > 0 \) uniformly in \( N \).

The solution \( \chi \) then provides one with a full description of the inverse of \( S_N \).
Proposition 4.5. Let $0 < s < 1$. The operator $S_N : H_s([a_N \; b_N]) \to H_s(\mathbb{R})$ is continuous and invertible on its image:

$$\mathcal{X}_s(\mathbb{R}) = \left\{ H \in H_s(\mathbb{R}) : \int_{\mathbb{R}^+e^r} \chi_{12}(\mu) F[H](\tau_{N\mu}) e^{-i\mu N} \cdot \frac{d\mu}{(2\pi )^2} = 0 \right\}.$$ (4.20)

More specifically, one has the left and right inverse relations

$$\mathcal{W}_N \circ S_N = \text{id} \quad \text{on} \quad H_s([a_N \; b_N]) \quad \text{and} \quad S_N \circ \mathcal{W}_N[H](\xi) = H(\xi) \quad \text{a.e. on} \quad [a_N \; b_N]$$

for any $H \in \mathcal{X}_s(\mathbb{R})$. The operator $\mathcal{W}_N : \mathcal{X}_s(\mathbb{R}) \to H_s([a_N \; b_N])$ is given, whenever it makes sense, as an encased oscillatorily convergent Riemann integral transform

$$\mathcal{W}_N[H](\xi) = \frac{\tau_N^2}{\pi} \int_{\mathbb{R}^+2ie^r} \int_{\mathbb{R}^+2ie^r} \frac{d\mu}{2i\pi} e^{-i\tau_N\lambda(\xi - \mu)} W(\lambda, \mu) e^{-i\mu N} F[H](\tau_{N\mu}),$$ (4.21)

where $\epsilon' > 0$ is small enough. The integral kernel

$$W(\lambda, \mu) = \frac{1}{\mu - \lambda} \left( \mu : \chi_{11}(\lambda) \chi_{12}(\mu) - \chi_{11}(\mu) \chi_{12}(\lambda) \right),$$ (4.22)

is expressed in terms of the entries of the matrix $\chi$.

These pieces of information along with the explicit, uniform on $\mathbb{C}$, large-$N$ expansion of the solution $\chi$ to the above Riemann–Hilbert problem and several technical estimates which allow one to check that (4.15)-(4.16) hold, allow one to formulate the

Theorem 4.6. Let $N \geq N_0$ with $N_0$ large enough. Then the unique minimiser $\sigma_{eq}^{(N)}$ of the functional $\mathcal{E}_{eq}^{(\pm)}$ introduced in (4.7) is absolutely continuous in respect to the Lebesgue measure with density $\varrho_{eq}^{(N)}$ and is supported on the segment $[a_N \; b_N]$. The endpoints are the unique solutions to the equations

$$a_N + b_N = 0 \quad \text{and} \quad \theta \cdot \frac{(\overline{b}_N)^2}{N} \cdot t(2\overline{b}_N) \cdot \left\{ 1 + O\left(\frac{1}{\overline{b}_N} \right) \right\} = 1,$$

for any $1 > \epsilon > 0$ and the remainder is smooth and differentiable in $\overline{b}_N$. Above, one has

$$\theta = \frac{2\pi}{3(2\pi)^2} \cdot \frac{\Gamma(b, b)}{b^b b^b b^b b^b b^b} ,$$

while, upon using the constants $w_k$ introduced below in (4.24),

$$t(x_N) = \frac{6}{(x_N)^7} \left( 2 + w_2 - w_1 - \frac{w_1 w_2}{w_2} \right) \sim 1 + O\left( \frac{1}{x_N} \right).$$ (4.23)

In particular, $\overline{b}_N$ is uniformly away from zero and admits the large-$N$ expansion

$$\overline{b}_N = \ln N - 2\ln \ln N - \ln \theta + O\left( \frac{\ln \ln N}{\ln N} \right).$$

Finally, the density $\varrho_{eq}^{(N)}$ of the equilibrium measure is expressed in terms of the integral transform of the potential $\varrho_{eq}^{(N)} = \mathcal{W}_N[V_N']/(N\tau_N)$.

In the statement of the theorem, we made use of the coefficients $w_k$ arising in the $\lambda \to 0$ expansion below

$$2i \frac{b^2 b^2 + b b^3 + b^3}{\lambda^3 b b e^1 \varrho_N} \Gamma \left( \frac{1}{2} + i \frac{b}{2} \right) \left( 1 - i b \lambda, 1 - i b \lambda, 1 - i \frac{b}{2} \right) = \sum_{\ell=0}^{3} \frac{(-i)^{\ell} w_{\ell}}{\lambda^{3-\ell}} + O(\lambda).$$ (4.24)
4.6 Estimation of the minimum

The closed expression for $\sigma_{eq}^{(N)}$ in terms of the solution $\chi$ to the above Riemann–Hilbert problem and the close relation between the two-body interaction in the potential and the $S_N$ operator’s kernel allow one to exploit the system of jumps for $\chi$ so as to recast $\mathcal{E}_N^{(+)}[\sigma_{eq}^{(N)}]$ only in terms of $N$, $\mathcal{B}_N$ and $\chi$ evaluated at special points:

$$
\mathcal{E}_N^{(+)}[\sigma_{eq}^{(N)}] = \frac{k}{2N} \cosh(\mathcal{B}_N) + \frac{\kappa^2 e^{2\mathcal{B}_N}}{8\pi N^2} \left( \chi_{12}'(i) + 2[\chi_{12}(i)\chi_{11}'(i) - \chi_{11}(i)\chi_{12}'(i)] \right)
- \frac{2e^{\mathcal{B}_N}}{4N} \left( 1 + e^{-\mathcal{B}_N} + \chi_{2\bar{2}1\bar{1}}(0)[2\chi_{11}(i) + i\chi_{12}(i)] - 2\chi_{2\bar{1}1\bar{2}}(0)\chi_{12}(i) \right).
$$

Once that one arrives to the above closed expression, it is a matter of direct calculations which build on the uniform on $\mathbb{C}$ large-$N$ asymptotic expansion for $\chi$ provided by the non-linear steepest descent so as to infer the large-$N$ asymptotics

**Proposition 4.7.** One has the large-$N$ asymptotic behaviour

$$
\mathcal{E}_N^{(+)}[\sigma_{eq}^{(N)}] = \frac{3\pi^4}{4(\mathcal{B}_N)^3} \bar{\hat{w}}_1 + \frac{9\pi^4}{8(\mathcal{B}_N)^4} \left( 1 - \frac{2\bar{\hat{w}}_1}{\mathcal{B}_N} \right) + O\left(e^{-2\mathcal{B}_N(1-\epsilon)}\right),
$$

(4.25)

where $t$ is as introduced in Theorem 4.6 and we have rescaled the $w_k$ variables:

$$
w_1 = 2\mathcal{B}_N\bar{\hat{w}}_1, \quad w_2 = 2(\mathcal{B}_N)^2 \bar{\hat{w}}_2, \quad \text{with} \quad \bar{\hat{w}}_k = 1 + O\left(\frac{1}{\mathcal{B}_N}\right) \quad \text{as} \quad N \to +\infty.
$$

Together with Propositions 4.2, 4.3 and the lower bound in (4.11), the above theorem yields Theorem 5.1

5 Conclusion

In this paper we reviewed the bootstrap program approach to the rigorous construction of 1+1 dimensional integrable quantum field theories arising as appropriate quantisations of integrable classical evolution equations of 1+1 dimensional field theory. This was done on the example of the Sinh-Gordon quantum field theory which is the simplest and non-trivial instance of such model. The approach starts by proposing an appropriate Hilbert space on which such a model is realised. Then, it produces the form of the $S$-matrix which governs the scattering in such a case. This $S$-matrix arises as a solution of certain symmetry constraints on the scattering in a relativistically invariant theory along with the requirement of the factorisability of scattering into a concatenation of two-particle processes. Then, the quantum fields, which are operator valued distributions on functions of the space-time variables, are constructed as integral operators whose integral kernels satisfy a set of equations, the bootstrap program axioms i) – v), which should be taken as the basic axioms of the theory. These axiomatic equations strongly depend on the form of the $S$-matrix for the given theory. It turns out that the bootstrap program equations can be solved explicitly with the help of the algebraic setting provided by the quantum integrability of the model and, in particular, the Yang-Baxter equation satisfied by the $S$-matrix. Once one ends up with the set of explicit solutions to i) – v), it remains to check the consistency of the whole construction, in particular, that the so-constructed quantum fields do form an algebra and that they commute at space-like separations. The latter requirement is crucial for guaranteeing the causality of the so-constructed theory and thus it being viable as a per se quantum field theory. To check these last steps of the construction, one must show that the series of multiple integrals resulting from the integral operator’s multiplications do converge. This was a long standing open question in this field and its solution [24], in the simplest case scenario, was discussed by the author in the last section of this paper.
There are still numerous open questions related to these topic. First of all, to implement the method of [24] for establishing the convergence of form factor expansions for time-like separated two-point functions in the model as well as for general multi-point correlation functions regardless of the space or time-like intervals between the operators. These questions do definitely seem to be manageable within a finite time. Further, to extend the methods of proving the convergence to more challenging but also more physically relevant models such as the 1+1 dimensional integrable Sine-Gordon quantum field theory. There, the multitude of asymptotic particles along with the presence of bound states and equal mass asymptotic particles will definitely be a challenging, but hopefully surmountable task.

Last but not least, one should provide a thorough description of the correlation functions in the infrared limit, viz. when the Minkowski separation between the operator approaches zero. In the case of the two-point function given in (3.4) that would correspond to extracting the \( r \to 0^+ \) limit.

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