HOLOMORPHIC FUNCTIONS UNBOUNDED ON CURVES OF FINITE LENGTH

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Abstract  Given a pseudoconvex domain $D \subset \mathbb{C}^N$, $N \geq 2$, we prove that there is a holomorphic function $f$ on $D$ such that the lengths of paths $p: [0, 1] \to D$ along which $\Re f$ is bounded above, with $p(0)$ fixed, grow arbitrarily fast as $p(1) \to bD$. A consequence is the existence of a complete closed complex hypersurface $M \subset D$ such that the lengths of paths $p: [0, 1] \to M$, with $p(0)$ fixed, grow arbitrarily fast as $p(1) \to bD$.

1. Introduction and the main results

Denote by $\Delta$ the open unit disc in $\mathbb{C}$ and by $B_N$ the open unit ball of $\mathbb{C}^N$, $N \geq 2$. In [G] it was proved that there is a closed complex hypersurface $M$ in $B_N$ which is complete, that is, every path $p: [0, 1) \to M$ such that $|p(t)| \to 1$ as $t \to 1$, has infinite length. This was a consequence of the main result of [G], a construction of a holomorphic function on $B_N$ whose real part is unbounded on every path of finite length that ends on $bB_N$.

Recall that a domain $D \subset \mathbb{C}^N$, $N \geq 2$, is pseudoconvex if it has a continuous plurisub-harmonic exhaustion function. This happens if and only if $D$ is holomorphically convex and if and only if $D$ is a domain of holomorphy [H]. Every convex domain is pseudoconvex. In the present paper we show that given a pseudoconvex domain $D$ in $\mathbb{C}^N$, $N \geq 2$, there is a holomorphic function $f$ on $D$ such that the lengths of paths $p: [0, 1] \to D$ along which the real part of $f$ is bounded above, grow arbitrarily rapidly if $p(0)$ is fixed and $p(1)$ tends to $bD$. Our main result is the following

THEOREM 1.1 Let $D \subset \mathbb{C}^N$, $N \geq 2$, be a pseudoconvex domain and let $D_n$, $n \in \mathbb{N}$ be an exhaustion of $D$ by relatively compact open sets

$$D_1 \subset D_2 \subset \cdots \subset D, \quad \bigcup_{n=1}^{\infty} D_n = D.$$ 

Let $A_n$, $n \in \mathbb{N}$, be an increasing sequence of positive numbers converging to $+\infty$. There is a function $f$, holomorphic on $D$, with the following property:
Given $L < \infty$ there is an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and if $p: [0, 1] \to D$ is a path such that

(i) $\Re[f(p(t))] \leq L$  $(0 \leq t \leq 1)$
(ii) $p(0) \in D_1$, $p(1) \in D \setminus D_n$

then the length of $p$ exceeds $A_n$.
So, in particular, for every $L < \infty$, the boundary $bD$ is infinitely far away for a traveller travelling within a sublevel set \( \{ z \in D: \Re(f(z)) < L \} \) of the real part of $f$.

**COROLLARY 1.1** Given a pseudoconvex domain $D \subset \mathbb{C}^N$, $N \geq 2$, there is a holomorphic function on $D$ whose real part is unbounded above on every path $p: [0, 1) \to D$, $p(1) \in bD$, of finite length.

It is perhaps worth mentioning that for any holomorphic function $f$ on $\mathbb{B}_N$ there are paths $p: [0, 1] \to \mathbb{B}_N$, $p(0, 1) \subset \mathbb{B}_N$, $p(1) \in b\mathbb{B}_N$ along which $f$ is constant [GS].

Let $M \subset D$ be a closed complex hypersurface, that is, a closed complex submanifold of $D$ of complex codimension one. A path $p: [0, 1) \to M$ is called divergent if $p(t)$ leaves every compact subset of $M$ as $t \to 1$. $M$ is called complete if every divergent path $p: [0, 1) \to M$ has infinite length.

Let $f$ be as in Corollary 1.1. By Sard’s theorem one can choose $c \in \mathbb{C}$ such that

\[ M = \{ z \in D : f(z) = c \} \]

is a complex manifold. By the properties of $f$, $M$ is a complete hypersurface. So we have

**COROLLARY 1.2** Let $D \subset \mathbb{C}^N$, $N \geq 2$, be a pseudoconvex domain. Then $D$ contains a complete closed complex hypersurface.

In the special case when $D = \mathbb{B}_N$, Corollary 1.1 and Corollary 1.2 were proved in [G]. In [AL] Corollary 1.2 was proved for convex domains in $\mathbb{C}^2$. Note that Theorem 1.1 implies a stronger result - given an exhaustion $D_j$, $j \in \mathbb{N}$, of a pseudoconvex domain $D$ as in Theorem 1.1, there is a complete closed complex hypersurface $M$ in $D$ such that along paths, $M \setminus D_j$ becomes arbitrarily far away as $j \to \infty$:

**COROLLARY 1.3** Let $D \subset \mathbb{C}^N$, $N \geq 2$, be a pseudoconvex domain, and let $D_j$ and $A_j$, $j \in \mathbb{N}$, be as in Theorem 1.1. There is a complete closed complex hypersurface $M$ in $D$ meeting $D_1$ with the following property: There is some $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then the length of every path $p: [0, 1) \to M$, such that $p(0) \in D_1$ and $p(1) \in M \setminus D_n$, exceeds $A_n$.

2. The main lemma. Reduction to the case $D = \mathbb{C}^N$

We assume that $N \geq 2$ and write $\mathbb{B}$ for $\mathbb{B}_N$. We shall use spherical shells. If $J$ is an interval contained in $(0, \infty)$ we shall write $\text{Sh}(J) = \{ x \in \mathbb{C}^N : |x| \in J \}$. So, if $J = [\alpha, \beta]$ then $\text{Sh}(J) = \{ x \in \mathbb{C}^N : \alpha \leq |x| \leq \beta \} = \beta \mathbb{F} \setminus \alpha \mathbb{B}$. Here is our main lemma:

**LEMMA 2.1** Let $J = (r, R)$ where $0 < r < R < \infty$ and let $A < \infty$. There is a set $E \subset \text{Sh}(J)$ such that

(i) the length of every path $p: [0, 1) \to \text{Sh}(J) \setminus E$ such that $|p(0)| = r$, $|p(1)| = R$, exceeds $A$.

(ii) given $\varepsilon > 0$ and $L < \infty$ there is a polynomial $\Phi$ on $\mathbb{C}^N$ such that $|\Phi| < \varepsilon$ on $r \mathbb{F}$ and $\Re \Phi > L$ on $E$.

We will prove Lemma 2.1 in the following sections. To prove Theorem 1.1 we need the following consequence of Lemma 2.1.
LEMMA 2.2 Let $0 < r_1 < R_1 < r_2 < R_2 < \cdots$, $r_n \to \infty$ as $n \to \infty$, and let $B_n$ be an increasing sequence of positive numbers converging to $+\infty$. There is a holomorphic function $g$ on $\mathbb{C}^N$, $N \geq 2$, such that for any $L < \infty$ there is an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and if $p: [0, 1] \to \mathbb{C}^N$ is a path such that

(i) $\Re[g(p(t))] \leq L$ \quad ($0 \leq t \leq 1$)

(ii) $|p(0)| \leq r_n$, $|p(1)| \geq R_n$

then the length of $p$ exceeds $B_n$.

Proof. Let $0 < r_1 < R_1 < r_2 < R_2 < \cdots$, $r_n \to +\infty$ as $n \to \infty$ and let $B_n$ be an increasing sequence of positive numbers, converging to $+\infty$. By Lemma 2.1 there is, for each $n$, a set $E_n \subset \text{Sh}([r_n, R_n])$ such that

- the length of every path $p: [0, 1] \to \text{Sh}([r_n, R_n]) \setminus E_n$ such that $|p(0)| = r_n$, $|p(1)| = R_n$, exceeds $B_n$

- given $\varepsilon > 0$ and $L < \infty$ there is a polynomial $\Psi$ on $\mathbb{C}^N$ such that $|\Psi| < \varepsilon$ on $r_n \overline{B}$ and $\Re \Psi > L$ on $E_n$.

Let $L_n$ be an increasing sequence converging to $+\infty$. Suppose for a moment that we have a sequence of polynomials $\Phi_n$ such that

(a) $\Re \Phi_n > L_n + 1$ on $E_n$

(b) $|\Phi_{n+1} - \Phi_n| < 1/2^n$ on $r_n \overline{B}$.

By (b) the sequence $\Phi_n$ converges uniformly on compacta on $\mathbb{C}^N$, denote by $g$ its limit. So $g$ is holomorphic on $\mathbb{C}^N$. On $E_n$ we have $\Re g = \Re [\Phi_n + \sum_{j=n}^{\infty} (\Phi_{j+1} - \Phi_j)] \geq \Re \Phi_n - \sum_{j=n}^{\infty} 2^{-j} \geq L_n + 1 - 1 = L_n$.

Let $L < \infty$. There is an $n_0$ such that $L < L_n$ for all $n \geq n_0$. Suppose that a path $p$ satisfies (i). Then there are $\alpha$, $\beta$, $0 < \alpha < \beta < 1$ such that $p((\alpha, \beta)) \subset \text{Sh}((r_n, R_n))$ and $p(\alpha) = r_n$, $p(\beta) = R_n$. If $p$ satisfies also (i) then, since $\Re g \geq L_n$ on $E_n$ it follows that $p|[(\alpha, \beta)]$ is a map to $\text{Sh}([r_n, R_n]) \setminus E_n$ so by the preceding discussion the length of $p|[(\alpha, \beta)]$ exceeds $B_n$ and consequently the length of $p$ exceeds $B_n$.

We construct the sequence $\Phi_n$ by induction. Pick a polynomial $\Phi_1$ such that $\Re \Phi_1 > L_1 + 1$ on $E_1$. Suppose that we have constructed $\Phi_n$. There is a constant $C < \infty$ such that $\Re (\Phi_n + C) \geq L_{n+1} + 1$ on $E_{n+1}$. By the preceding discussion there is a polynomial $\Psi$ such that $|\Psi| < 1/2^n$ on $r_n \overline{B}$ and $\Re \Psi > C$. Then $\Phi_{n+1} = \Phi_n + \Psi$ satisfies (b) and (a) with $n$ replaced by $n + 1$. This completes the proof.

The same proof gives an analogous result for the ball which we will not need in the sequel.

COROLLARY 2.1 Let $0 < r_1 < R_1 < r_2 < R_2 < \cdots$, $r_n \to 1$ as $n \to \infty$, and let $A_n$ be an increasing sequence of positive numbers converging to $\infty$. There is a holomorphic function $g$ on $\mathbb{D}$, such that for any $L < \infty$ there is an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and if $p: [0, 1] \to \mathbb{D}$ is a path such that

(i) $\Re[g(p(t))] \leq L$ \quad ($0 \leq t \leq 1$)

(ii) $|p(0)| \leq r_n$, $|p(1)| \geq R_n$

then the length of $p$ exceeds $A_n$.

Let $D \subset \mathbb{C}^N$, $N \geq 2$, be a pseudoconvex domain. Then $D$ is a Stein manifold so there is a proper holomorphic embedding $F: D \to \mathbb{C}^{2N+1}$ [H, Th.5.3.9].

To prove Theorem 1.1 we first prove the following consequence of Lemma 2.2.
LEMMA 2.3 Let $0 < r_1 < R_1 < r_2 < R_2 < \cdots$, $r_n \to \infty$ as $n \to \infty$, and let $A_n$ be an increasing sequence of positive numbers converging to $\infty$. There is a holomorphic function $f$ on $D$ such that for any $L < \infty$ there is an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and if $p: [0, 1] \to D$ is a path such that

(i) $\Re[f(p(t))] \leq L \ (0 \leq t \leq 1)$

(ii) $|F(p(0))| \leq r_n$, $|F(p(1))| \geq R_n$

then the length of $p$ exceeds $A_n$.

Note that Lemma 2.3 implies a more precise version of Theorem 1.1, yet for a specific exhaustion $D_n = \{z \in D: |F(z)| < R_n\}$, $n \in \mathbb{N}$.

Proof of Lemma 2.3 Let $K_n = \{z \in D: r_n \leq |F(z)| \leq R_n\}$. Let $p: [0, 1] \to K_n$ be a path. Then $q = F \circ p: [0, 1] \to \Phi^{2N+1}$ is a path whose length equals

$$\text{length}(q) = \int_0^1 \left| (DF)(p(t)) \left(\frac{dp}{dt}(t)\right) \right| dt$$

$$\leq \max_{w \in K_n} \| (DF)(w) \| \int_0^1 \left| \frac{dp}{dt}(t) \right| dt$$

$$= \max_{w \in K_n} \| (DF)(w) \| \cdot \text{length}(p).$$

The map $F$ is holomorphic and $K_n$ is compact so

$$\max_{w \in K_n} \| (DF)(w) \| < \infty.$$ 

Let $A_n$, $n \in \mathbb{N}$, be an increasing sequence converging to $+\infty$. Choose an increasing sequence $B_n$ converging to $+\infty$ such that

$$A_n \cdot \max_{w \in K_n} \| (DF)(w) \| \leq B_n \ (n \in \mathbb{N}). \quad (2.1)$$

Let $g$ be an entire function on $\Phi^{2N+1}$ given by Lemma 2.2 and let $f = g \circ F$. Let $L < \infty$ By Lemma 2.2 there is an $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and if $s: [0, 1] \to \Phi^{2N+1}$ is a path such that

(i) $\Re[g(s(t))] \leq L \ (0 \leq t \leq 1)$

(ii) $|s(0)| \leq r_n$, $|s(1)| \geq R_n$

then the length of $s$ exceeds $B_n$.

Now, let $n \geq n_0$ and let $p: [0, 1] \to D$ be a path such that $|F(p(0))| \leq r_n$, $|F(p(1))| \geq R_n$ and $\Re[f(p(t))] \leq L \ (0 \leq t \leq 1)$, that is

$$\Re[g(s(t))] \leq L \ (0 \leq t \leq 1) \quad (2.2)$$

where $s = F \circ p$. There is a segment $[\alpha, \beta] \subset [0, 1]$ such that $p[\alpha, \beta]$ maps $[\alpha, \beta]$ to $K_n$ and $|F(p(\alpha))| = r_n$, $|F(p(\beta))| = R_n$, that is, $|s(\alpha)| = r_n$, $|s(\beta)| = R_n$. By (2.2) Lemma 2.2 implies that

$$\text{length}(s|[\alpha, \beta]) \geq B_n.$$
By (2.1) it follows that

\[
\text{length}(p[\alpha, \beta]) \geq \frac{\text{length}((F \circ p)[\alpha, \beta])}{\max_{w \in K_n} \|(DF)(w)\|} = \frac{\text{length}(s[\alpha, \beta])}{\max_{w \in K_n} \|(DF)(w)\|} \geq B_n \max_{w \in K_n} \|(DF)(w)\| \geq A_n
\]

Thus, the length of \( p \) exceeds \( A_n \). This completes the proof of Lemma 2.3 provided that Lemma 2.1 has been proved.

**Proof of Theorem 1.1.** Let \( D_j, j \in \mathbb{N} \), be as in Theorem 1.1 and let \( w \in D_1 \).

Since \( F: D \rightarrow \mathbb{C}^{2N+1} \) is a proper map there are \( m_0 \in \mathbb{N} \) and a strictly increasing sequence \( R_n \rightarrow \infty \) such that if \( \Delta_n = \{ z \in D: |F(z)| < R_n \} \), \( n \in \mathbb{N} \) then \( D_1 \subset \Delta_1 \) and \( \Delta_n \subset D_n \ (n \geq m_0) \). By Lemma 2.3 there is a holomorphic function \( f \) on \( D \) such that for any \( L < \infty \) there is an \( n_0 \in \mathbb{N} \), \( n_0 \geq m_0 \), such that if \( n \geq n_0 \) and if \( p: [0, 1] \rightarrow D \) is a path such that

\[
\Re[f(p(t))] \leq L \ (0 \leq t \leq 1),
\]

\( p(0) \in \Delta_1 \) and \( p(1) \in D \setminus \Delta_n \) then the length of \( p \) exceeds \( A_n \). Since \( D_1 \subset \Delta_1 \) and \( \Delta_n \subset D_n \ (n \geq n_0) \) the same holds for any path \( p: [0, 1] \rightarrow D \) which satisfies (2.3) and \( p(0) \in D_1, p(1) \in D \setminus D_n \). This completes the proof of Theorem 1.1.

**Remark** Note that we used only the fact that \( F: D \rightarrow \mathbb{C}^{2N+1} \) is a proper holomorphic map. We did not need the fact that it is an injective immersion.

It remains to prove Lemma 2.1.

### 3. Proof of Lemma 2.1, Part 1

If \( I_1, I_2 \) are two intervals contained in \( (0, \infty) \) then we shall write \( I_1 < I_2 \) provided that \( I_1 \cap I_2 = \emptyset \) and provided that \( I_2 \) is to the right of \( I_1 \), that is, if \( x_1 < x_2 \) for every \( x_1 \in I_1, x_2 \in I_2 \). If \( \mathcal{V} \subset b\mathbb{B} \) is an open set and \( J \subset (0, \infty) \) is an interval then we call the set

\[
K(\mathcal{V}, J) = \{ tz: z \in \mathcal{V}, t \in J \}
\]

a *spherical box*. Clearly

\[
K(\mathcal{V}, J) = \{ x \in \text{Sh}(J): \frac{x}{|x|} \in \mathcal{V} \}.
\]

A set of the form \( \{ x \in b\mathbb{B}: |x - x_0| < \eta \} \) where \( x_0 \in b\mathbb{B} \) is called a *ball in b\mathbb{B} of radius \( \eta \).

We show that Lemma 2.1 follows from

**Lemma 3.1.** There is a \( \rho > 0 \) with the following property:

For every ball \( \mathcal{V} \subset b\mathbb{B} \) of radius \( \rho \), for every \( A < \infty \) and for every open interval \( J = (\alpha, \beta) \) where \( 1/2 < \alpha < \beta < 1 \) there is a set \( E \subset K(\mathcal{V}, J) \) such that
(i) the length of every path \( p: [0, 1] \to K(V, J) \setminus E \) such that \( |p(0)| = \alpha, |p(1)| = \beta \), exceeds \( A \).

(ii) given \( \varepsilon > 0 \) and \( L < \infty \) there is a polynomial \( \Phi \) on \( C^N \) such that \( |\Phi| < \varepsilon \) on \( \alpha B \) and \( \Re \Phi > L \) on \( E \).

One can view Lemma 3.1 as a local version of Lemma 2.1. Note, however, that \( \rho \) does not depend on \( J \).

**Proof of Lemma 2.1, assuming Lemma 3.1.** Let \( J = (r, R) \) where \( 0 < r < R < \infty \) and let \( A < \infty \). It is easy to see that it is enough to prove Lemma 2.1 in the case when \( R \) is close to \( r \). Hence, with no loss of generality assume that \( 1/2 < r < R < 1 \).

Let \( \rho > 0 \) be as in Lemma 3.1 and put \( \eta = \rho/4 \). Choose points \( w_1, w_2, \ldots, w_M \in B \) such that the balls

\[
V_i = \{ w \in B : |w - w_i| < 2\eta \}, \quad 1 \leq i \leq M, 
\]

cover \( B \) and then let

\[
W_i = \{ w \in B : |w - w_i| < 4\eta \}, \quad 1 \leq i \leq M.
\]

Then every ball \( V \subset B \) of radius \( 2\eta \) is contained in at least one of \( W_j \), \( 1 \leq j \leq M \). Thus, if \( p: [0, 1] \to B \) is a path then either \( p([0, 1]) \) is contained in \( W_i \) for some \( i, \quad 1 \leq i \leq M \) or else the length of \( p \) exceeds \( 2\eta \). If \( p: [0, 1] \to Sh(J) \subset Sh((1/2, 1)) \) is a path then looking at the radial projections \( \pi_R(z) = z/|z| \) we conclude that either \( (\pi_R \circ p)([0, 1]) \) is contained in \( W_i \) for some \( i, \quad 1 \leq i \leq M \), or else the length of \( \pi_R \circ p \) exceeds \( 2\eta \) which implies that the length of \( p \) exceeds \( \eta \). Thus we have

\[
\text{Let } p: [0, 1] \to Sh(J) \text{ be a path. Then either there is some } j, \quad 1 \leq j \leq M \text{ such that } p([0, 1]) \subset K(W_j, J) \text{ or else the length of } p \text{ exceeds } \eta. \quad (3.1)
\]

Choose \( \ell \in \mathbb{N} \) so large that

\[
\ell \eta > A. \quad (3.2)
\]

Divide the interval \( J \) into \( \ell \) pieces \( J_1 = (r, r + (R - r)/\ell), \ldots, J_\nu = ((r + (\nu - 1)(R - r)/\ell, r + \nu(R - r)/\ell), \ldots, J_\ell = (r + (\ell - 1)(R - r)/\ell, R). \) Clearly

\[
(0, r) < J_1 < J_2 < \cdots < J_\ell < (R, \infty)
\]

and the length \( |J_k| \) of each \( J_k \), \( 1 \leq k \leq \ell \) equals \( (R - r)/\ell \). For each \( k, \quad 1 \leq k \leq \ell \), divide \( J_k \) into \( M \) equally long pieces \( J_{ks}, 1 \leq s \leq M \), so that \( J_{ks} \) are pairwise disjoint open intervals contained in \( J_k \) such that

\[
J_{k1} < J_{k2} < \cdots < J_{kM} \quad \text{and} \quad |J_{ks}| = |J_k|/M = (R - r)/(\ell M) \quad (1 \leq s \leq M).
\]

For each \( k, s, \quad 1 \leq k \leq \ell, 1 \leq s \leq M \), we apply Lemma 3.1 for the ball \( V = W_s \) and the interval \( J_{ks} \) to get a set \( E_{ks} \subset K(W_s, J_{ks}) \) such that

\[
\text{if } J_{ks} = (\alpha_{ks}, \beta_{ks}) \text{ then the length of every path } p: [0, 1] \to K(W_s, J_{ks}) \setminus E_{ks} \text{ such that } |p(0)| = \alpha_{ks}, \ |p(1)| = \beta_{ks}, \text{ exceeds } A,
\]

(3.3)
and such that
\[
\text{given } \varepsilon > 0 \text{ and } L < \infty \text{ there is a polynomial } \Phi \text{ on } \mathbb{C}^N \text{ such that } |\Phi| < \varepsilon \text{ on } \alpha_{ks} \mathbb{B} \text{ and } \Re \Phi > L \text{ on } E_{ks}. \]
\] (3.4)

Put
\[
E = \bigcup_{s=1}^{M} \bigcup_{k=1}^{\ell} E_{ks}.
\]

We show that \( E \) has the required properties.

Clearly \( E \subset \text{Sh}(J) \). Let \( p: [0, 1] \to \text{Sh}(J) \setminus E \) be a path such that \( |p(0)| = r, |p(1)| = R \). Let \( J_k = (\alpha_k, \beta_k) \) \((1 \leq k \leq \ell)\). Since \( |p(0)| = r, |p(1)| = R \) it follows that for each \( k, 1 \leq k \leq \ell \), there are \( \gamma_k, \gamma'_k \) such that
\[
0 \leq \gamma_1 < \gamma'_1 < \gamma_2 < \gamma'_2 < \cdots < \gamma_\ell < \gamma'_\ell \leq 1,
\]
such that for each \( k, \) \( p \) maps \([\gamma_k, \gamma'_k]\) to \( \text{Sh}(J_k) \) and \((\gamma_k, \gamma'_k)\) to \( \text{Sh}(J_k) \) and satisfies \( |p(\gamma_k)| = \alpha_k, \ |p(\gamma'_k)| = \beta_k \).

Fix \( k, 1 \leq k \leq \ell \). By (3.1) there are two possibilities. Either there is some \( s, 1 \leq s \leq M \), such that \( p([\gamma_k, \gamma'_k]) \subset K(\mathcal{W}_s, J_k) \) or else the length of \( p([\gamma_k, \gamma'_k]) \) exceeds \( \eta \). Assume that the first happens. Write \( J_{ks} = (\alpha_{ks}, \beta_{ks}) \). Since \( p([\gamma_k, \gamma'_k]) \subset K(\mathcal{W}_s, J_k) \) and \( |p(\gamma_k)| = \alpha_k, \ |p(\gamma'_k)| = \beta_k \) it follows that there are \( \gamma_{ks}, \gamma'_{ks} \) such that \( \gamma_k < \gamma_{ks} < \gamma'_{ks} < \gamma'_k \), and such that \( p(\gamma_{ks}) = \alpha_{ks}, \ p(\gamma'_{ks}) = \beta_{ks} \) and \( p([\gamma_{ks}, \gamma'_{ks}]) \subset K(\mathcal{W}_s, J_{ks}) \). Clearly \( p \) maps \([\gamma_{ks}, \gamma'_{ks}]\) into \( K(\mathcal{W}_s, J_{ks}) \setminus E = K(\mathcal{W}_s, J_{ks}) \setminus E_{ks} \) and so by (3.3) the length of \( p([\gamma_{ks}, \gamma'_{ks}]) \) exceeds \( A \) and so the length of \( p([\gamma_k, \gamma'_k]) \) exceeds \( A \). This shows that for each \( k, 1 \leq k \leq \ell \), the length of \( p([\gamma_k, \gamma'_k]) \) exceeds \( \min \{\eta, A\} \) and hence by (3.2) the length of \( p \) exceeds \( A \). This shows that \( E \) satisfies (i) in Lemma 2.1.

It remains to show (ii) in Lemma 2.1. To this end, rename the intervals \( J_{ks}, 1 \leq k \leq \ell, 1 \leq s \leq M, \) into \( I_1, I_2, \cdots, I_{\ell M} \) and the sets \( E_{ks} = 1 \leq k \leq \ell, 1 \leq s \leq M, \) into \( E_1, E_2, \cdots, E_{\ell M} \) in such a way that
\[
(0, r) < I_1 < I_2 < \cdots < I_{\ell M} < (R, \infty)
\]
and that \( E_j \subset \text{Sh}(I_j) \) \((1 \leq j \leq \ell M)\). There are \( \mu_j, 1 \leq j \leq \ell M + 1, \) such that \( \mu_1 = r, \ \mu_{\ell M + 1} = R, \) and such that \( I_j = (\mu_j, \mu_{j + 1}) \) \((1 \leq j \leq \ell M)\). Recall that by the properties of \( E_j \),
\[
\text{for each } j, 1 \leq j \leq \ell M, \text{ and for each } \varepsilon > 0 \text{ and } L < \infty \text{ there is a polynomial } \Psi \text{ such that } |\Psi| < \varepsilon \text{ on } \mu_j \mathbb{B} \text{ and } \Re \Psi > L \text{ on } E_j.
\]
(3.5)

Let \( L < \infty \) and let \( \varepsilon > 0 \). Let \( \Phi_1 \) be a polynomial such that
\[
|\Phi_1| < \frac{\varepsilon}{\ell M} \text{ on } \mu_1 \mathbb{B} = r \mathbb{B} \text{ and } \Re \Phi_1 > L + \varepsilon \text{ on } E_1.
\]
which is possible by (3.5). We construct polynomials \( \Phi_j \), \( 2 \leq j \leq \ell M \), such that for each \( j \), \( 2 \leq j \leq \ell M \),

\[
|\Phi_j - \Phi_{j-1}| < \frac{\varepsilon}{M\ell} \quad \text{on} \quad \mu_j \overline{B} \quad \text{and} \quad \Re\Phi_j > L + \varepsilon \quad \text{on} \quad E_j
\]  

(3.6)

and then put \( \Phi = \Phi_{M\ell} \). We show that \( \Phi \) has the required properties. On \( r\overline{B} = \mu_1 \overline{B} \) we have \( |\Phi| \leq |\Phi_1| + |\Phi_2 - \Phi_1| + \cdots + |\Phi_{M\ell} - \Phi_{M\ell-1}| \leq M\ell\varepsilon/(M\ell) = \varepsilon \). Fix \( j \), \( 1 \leq j \leq \ell M \). On \( E_j \) we have \( \Re\Phi = \Re[\Phi_j + (\Phi_{j+1} - \Phi_j) + \cdots + (\Phi_{M\ell} - \Phi_{M\ell-1})] \geq \Re\Phi_j - (M\ell-1)\varepsilon/\ell M \geq L + \varepsilon - \varepsilon \). Thus, on \( E = \bigcup_{j=1}^{M\ell} E_j \), we have \( \Re\Phi > L \).

To find \( \Phi_2, \ldots, \Phi_{\ell M} \) satisfying (3.6) we use (3.5): Suppose that we have constructed \( \Phi_j \) where \( 1 \leq j \leq \ell M - 1 \). There is a constant \( C < \infty \) such that \( \Re\Phi_j + C \geq L + \varepsilon \) on \( E_{j+1} \). By (3.5) there is a polynomial \( \Psi \) such that \( |\Psi| < \varepsilon /(M\ell) \) on \( \mu_{j+1} \overline{B} \) and \( \Re\Psi > C \) on \( E_{j+1} \). Then \( \Phi_{j+1} = \Phi_j + \Psi \) has all the required properties. This completes the proof.

4. Proof of Lemma 3.1, Part 1.

Write \( M = 2N \) and identify \( \Phi^N \) with \( \mathbb{R}^M \) by identifying \( (p_1 + iq_1, \ldots, p_N + iq_N) \in \Phi^N \) with \( (p_1, q_1, \ldots, p_N, q_N) \in \mathbb{R}^M \). Let \( U_0, U_1, U \) be small open balls in \( \mathbb{R}^{M-1} \) centered at the origin, such that

\[
U \subset U_1 \subset U_0.
\]

Write \( W_0 = U_0 \times (0, \infty) \). This is an open half tube in \( \mathbb{R}^M = \Phi^N \). Similarly, write \( W_1 = U_1 \times (0, \infty) \), \( W = U \times (0, \infty) \). Given an interval \( J \subset (1/2, \infty) \) we shall write

\[
W_0(J) = \text{Sh}(J) \cap W_0, \quad W_1(J) = \text{Sh}(J) \cap W_1, \quad W(J) = \text{Sh}(J) \cap W.
\]

We assume that the ball \( U_0 \) is so small that for each \( r \), \( 1/2 < r < 1 \), the surface \( W_0 \cap b(r\overline{B}) \) can be written as the graph of the function

\[
\psi_r(x_1, \ldots, x_{M-1}) = \left( r^2 - \sum_{i=1}^{M-1} x_i^2 \right)^{1/2}
\]

defined on \( U_0 \), that is,

\[
W_0 \cap b(r\overline{B}) = \{(x_1, \ldots, x_{M-1}, \psi_r(x_1, \ldots, x_{M-1})): (x_1, \ldots, x_{M-1}) \in U_0\}.
\]

We now turn to the proof of Lemma 3.1. By rotation it is enough to prove that there is one ball \( V \subset b\overline{B} \) of radius \( \rho > 0 \) with the properties in Lemma 3.1. It is easy to see that to prove this it is enough to prove that there is a ball \( U \) as above such that for every \( A < \infty \) and for every segment \( J = (\alpha, \beta] \), \( 1/2 < \alpha < \beta < 1 \), there is a set \( E \subset W(J) \) such that

\[
(i) \quad \text{the length of every path } p: [0, 1] \to W(J) \setminus E \text{ such that } |p(0)| = \alpha, \ |p(1)| = \beta, \text{ exceeds } A
\]

\[
(ii) \quad \text{given } \varepsilon > 0 \text{ and } L < \infty \text{ there is a polynomial } \Phi \text{ on } \Phi^N \text{ such that } |
\]

\[
\Phi | < \varepsilon \text{ on } \alpha \overline{B} \text{ and } \Re\Phi > L \text{ on } E.
\]

(4.1)
We now use some ideas from [GS] and [G]. Let us describe briefly how the set $E$ will look like. We will construct finitely many intervals $J_j$, $1 \leq j \leq n$, $(-\infty, 1/2) < J_1 < \cdots < J_n < [1, \infty)$. For each $j$, $1 \leq j \leq n$, we will construct a convex polyhedral surface $C_j \subset \text{Sh}(J_j)$ whose facets will be simplices which is such that $W \setminus C_j$ has two components. From each $C_j$ we shall remove a tiny neighbourhood $U_j$ of the skeleton of $C_j$ and what remains intersect with $W$ to get the set $E_j$. The set $E$ will be the union of $E_j$, $1 \leq j \leq n$.

A path $p: [0, 1] \to W([\alpha, \beta]) \setminus E$, such that $|p(0)| = \alpha$, $|p(1)| = \beta$ will have to pass through each $C_j$, and will have to meet $C_j$ in the neighbourhood $U_j$ of $\text{Skel}(C_j)$. We shall show that given $A < \infty$, $n \in \mathbb{N}$, the intervals $J_j$, the convex surfaces $C_j$ and $U_j$, $1 \leq j \leq n$ can be chosen in such a way that (i) in (4.1) will hold. The fact that $C_j$ are convex and contained in disjoint spherical shells will enable us to satisfy (ii) in (4.1).

Begin with a tessellation $T$ of $\mathbb{R}^{M-1}$ into simplices which is periodic with respect to a lattice

$$\Lambda = \{ \sum_{i=1}^{M-1} n_i e_i; \; n_i \in \mathbb{Z}, \; 1 \leq i \leq M-1 \}$$

where $\{e_1, \ldots, e_{M-1}\}$ is a basis of $\mathbb{R}^{M-1}$ that is $S + e \in T$ for every simplex $S \in T$ and for every $e \in \Lambda$. What remains of $\mathbb{R}^{M-1}$ after we remove the interiors of all simplices in $T$ we call the skeleton of $T$ and denote by $\text{Skel}(T)$. More generally, we shall use the tessellations

$$\tau(T + z) = \{ \tau(S + z); \; S \in T \}$$

where $z \in \mathbb{R}^{M-1}$ and $\tau > 0$ and define $\text{Skel}(\tau(T + z))$ in the same way.

We now show how to construct the polyhedral surfaces mentioned above.

Fix $U_0, U_1$ and $U$ as above. Fix $z \in \mathbb{R}^{M-1}$ and let $\tau > 0$ be very small. Fix $r$, $1/2 < r < 1$. To get the vertices of our polyhedral surface we shall "lift" the vertices of each simplex $S \in \tau(T + z)$ contained in $U_0$ to $b(r \mathcal{B})$ in the sense that if $v_1, \ldots, v_M \in \mathbb{R}^{M-1}$ are the vertices of $S$ then $(v_i, \psi_r(v_i))$, $1 \leq i \leq M$ are the vertices of the simplex that we denote by $\Psi_r(S)$. The union of these simplices $\Psi_r(S)$ for all $S \in \tau(T + z)$ contained in $U_0$ we denote by $\Gamma(r, \tau, z)$. This is a polyhedral surface. It is the graph of the piecewise linear function $\varphi_{r, \tau, z}$ defined on the union of all simplices $S$ as above, where, on each such simplex with vertices $v_1, \ldots, v_M$ we have

$$\varphi_{r, \tau, z}\left(\sum_{i=1}^{M} \lambda_i v_i \right) = \sum_{i=1}^{M} \lambda_i \psi_r(v_i) \quad (0 \leq \lambda_i \leq 1, \; 1 \leq i \leq M, \; \sum_{i=1}^{M} \lambda_i = 1).$$

We will show later that the tessellation $T$ can be chosen in such a way that if $U_0$ is chosen small enough then for each $r$, $1/2 < r < 1$, and each $z$, the surface $\Gamma(r, \tau, z)$ will be convex in the sense that given a simplex $\Psi_r(S)$ where $S \in \tau(T + z)$ is contained in $U_0$, the intersection of the hyperplane $H$ containing $\Psi_r(S)$ with $\Gamma(r, \tau, z)$ is precisely $\Psi_r(S)$, that is, all of $\Gamma(r, \tau, z)$ except $\Psi_r(S)$ is contained in the open halfspace bounded by $H$ which contains the origin.

Let $d$ be the length of the longest edge of the simplices in $T$. Then $\tau d$ is the length of the longest edge of the simplices in $\tau(T + z)$ for any $\tau > 0$ and any $z \in \mathbb{R}^{N-1}$. There is a constant $\nu > 0$ depending on $U_0$ such that for each $r$, $1/2 < r < 1$, the length of the
longest edge of a simplex building \( \Gamma(r, \tau, z) \) does not exceed \( \gamma = \nu \tau d \). Thus the vertices of each such simplex are contained in a spherical cap \( \{ x \in b(rB) : |x - x_0| < \gamma \} \) for some \( x_0 \in b(rB) \) and consequently the simplex is contained in the convex hull of this spherical cap. If \( 1/2 < r < 1 \) then it is easy to see that this convex hull misses \((r - 2\gamma^2)B\). It follows that there is a constant \( \omega = 2
u^2 d^2 \) such that

\[
\text{for each } r, \ 1/2 < r < 1, \ \text{each } z \in \mathbb{R}^{M-1} \ \text{and each } \tau > 0 \ \text{we have } \Gamma(r, \tau, z) \subset \operatorname{Sh}((r - \omega \tau^2, r]).
\]

(4.2)

It is a simple geometric fact that there is a \( \delta > 0 \) such that

\[
\text{for every } r, \ 1/2 < r < 1, \ \text{and for every } s, \ r - \delta < s < r, \ \text{every line that meets } W((s, r)) \ \text{and misses } W_1 \cap b(sB), \ \text{misses } sB.
\]

(4.3)

There is a \( \tau_0 > 0 \) such that \( \omega \tau_0^2 < \delta \) where \( \delta \) satisfies (4.3) and which is so small that for every \( \tau, \ 0 < \tau < \tau_0 \), and for every \( z \in \mathbb{R}^{M-1} \) the union of all simplices in \( \tau(T + z) \) contained in \( U_0 \), contains \( U_1 \). Let \( 0 < \tau < \tau_0 \) and let \( z \in \mathbb{R}^{M-1} \). Then \( \Gamma(r, \tau, z) \cap W_1 \) is a graph over \( U_1 \) which is contained in \( W_1 \cap (r - \omega \tau^2, r)] \).

We now show that each hyperplane \( H \) meeting \( \Gamma(r, \tau, z) \cap W \) and tangent to \( \Gamma(r, \tau, z) \) misses \((r - \omega \tau^2)B\). This is easy to see. By the convexity of \( \Gamma(r, \tau, z) \), all of \( \Gamma(r, \tau, z) \) except the simplex \( H \cap \Gamma(r, \tau, z) \) is contained in the open halfspace bounded by \( H \), which contains the origin. If \( H \) would meet \((r - \omega \tau^2)B\) and misses \((r - \omega \tau^2)B\), then, since \( \omega \tau^2 < \delta \), by (4.3) \( H \) would meet \( b((r - \omega \tau^2)B) \cap W_1 \) at a point not contained in the simplex \( H \cap \Gamma(r, \tau, z) \), a contradiction.

Denote by \( \pi \) the projection

\[
\pi(x_1, \ldots, x_M) = (x_1, \ldots, x_{M-1}).
\]

Out of simplices building \( \Gamma(r, \tau, z) \) choose those which meet \( W \), denote them with \( T_j = \Phi_r(S_j), \ 1 \leq j \leq n \) and let \( C \) be their union. Since \( \tau < \tau_0 \) the simplices in \( \tau(T + z) \) contained in \( U_0 \) cover \( U_1 \) so the simplices \( S_j, 1 \leq j \leq n \), cover \( U \) and consequently \( C \cap W \) is a graph over \( U \) which cuts \( W \) into two connected components. Any path in \( W \) connecting points in different components will have to intersect \( C \). What remains of \( C \) after we remove the relative interiors of all simplices \( T_j, 1 \leq j \leq n \), we call the skeleton of \( C \) and denote by \( \operatorname{Skel}(C) \). Obviously \( \pi(\operatorname{Skel}(C)) \subseteq \operatorname{Skel}(\tau(T + z)) \).

For each \( j, \ 1 \leq j \leq n \), there is a linear functional \( \ell_j \) on \( \Phi^N \) such that the hyperplane

\[
H_j = \{ z \in \Phi^N: \Re(\ell_j(z)) = 1 \}
\]

contains \( T_j \). We know that each \( H_j \) misses \((r - \omega \tau^2)B\).

Since \( \Gamma(r, \tau, z) \) is convex it is easy to see that given a sufficiently small \( \nu > 0 \) the sets

\[
\{ z \in B: 1 - \nu < \Re \ell_i(z) < 1 \}, \ 1 \leq i \leq n,
\]

intersect in a small neighbourhood \( V \subseteq C \) of \( \operatorname{Skel}(C) \), and the sets \( \{ z \in B: \Re \ell_i(z) < 1 - \nu \} \) contain \((r - \omega \tau^2)B\). Note that \( V \) will be arbitrarily small neighbourhood of \( \operatorname{Skel}(C) \) provided that \( \nu > 0 \) is small enough.

Choose \( \varepsilon, \ 0 < \varepsilon < 1 \). Given \( L < \infty \) we use a one variable Runge theorem to get a polynomial \( \varphi \) of one variable such that

\[
\Re \varphi > L + 1 \ \text{on } \{ \zeta \in 2\Delta: \Re \zeta = 1 \}, \ |\varphi| < \varepsilon/n \ \text{on } \{ \zeta \in 2\Delta: \Re \zeta < 1 - \nu \}.
\]
Then $\Phi = \sum_{i=1}^{n} \varphi \circ \ell_i$ is a polynomial on $\Phi^N$ such that
\[ \Re \Phi > L \text{ on } C \setminus V, \quad |\Phi| < \varepsilon \text{ on } (r - \omega \tau^2)^{\overline{M}}. \]
The convex surface $C \subset \text{Sh}((r - \omega \tau^2, r]), C \subset \Gamma(r, \tau, z)$ is such that $W \setminus C$ has two components. Let $V(\eta) \subset C$ be the $\eta$-neighbourhood of $\text{Skel}(C)$ where $\eta > 0$ is very small. Write $G = W \cap (C \setminus V(\eta))$. Then
\begin{align}
(i) & \quad G \subset W((r - \omega \tau^2, r]), \\
(ii) & \quad \text{a path } p: [0, 1] \to W([r - \omega \tau^2, r]) \setminus G \text{ such that } |p(0)| = r - \omega \tau^2, |p(1)| = r, \text{ necessarily meets } V(\eta) \\
(iii) & \quad \text{given } L < \infty \text{ and } \varepsilon > 0 \text{ there is a polynomial } g \text{ on } \Phi^N \text{ such that } |g| < \varepsilon \text{ on } (r - \omega \tau^2)^{\overline{M}} \text{ and } \Re g > L \text{ on } G.
\end{align}
Observe also that $\pi$ maps $V(\eta)$ to the $\eta$-neighbourhood of $\text{Skel}(\tau(T + z))$ in $\mathbb{R}^{M-1}$.

5. Proof of Lemma 3.1, Part 2. Completion of the proof of Lemma 2.1.

Using an easy transversality (or ”putting into general position”) argument we see that the fact that $T$ is periodic with respect to $\Lambda$ implies that there are $q_1, \ldots, q_{M-1} \in \mathbb{R}^{M-1}$ such that
\[ \text{Skel}(T) \cap \text{Skel}(T + q_1) \cap \cdots \cap \text{Skel}(T + q_{M-1}) = \emptyset \]
which implies that there is a $\mu > 0$ such that
\[ |x_1 - x_0| + |x_2 - x_1| + \cdots + |x_{M-1} - x_{M-2}| \geq \mu \]
whenever $x_i \in \text{Skel}(T + q_i)$ $(0 \leq i \leq M - 1)$ where $q_0 = 0$. It then follows that for every $\tau > 0$
\[ |x_1 - x_0| + |x_2 - x_1| + \cdots + |x_{M-1} - x_{M-2}| \geq \tau \mu \]
whenever $x_i \in \text{Skel}(\tau(T + q_i))$, $0 \leq i \leq M - 1$.

Suppose that $1/2 < r_0 < r_M < 1$. Divide the interval $[r_0, r_M]$ into $M$ equal pieces of length $(r_M - r_0)/M$, let $r_0 < r_1 < \cdots < r_{M-1} < r_M$ be their endpoints. Choose $\tau$, $0 < \tau < \tau_0$ so small that
\[ \tau^2 \omega < \frac{r_M - r_0}{M}. \]
Fix a very small $\eta > 0$. For each $j$, $1 \leq j \leq M$, use $\Gamma(r_j, \tau, q_j-1)$ to construct $G_j \subset \Gamma(r_j, \tau, q_{j-1})$ as in the preceding section. Then for each $j$, $1 \leq j \leq M$, we have $G_j \subset W((r_{j-1}, r_j])$ and each path $p: [0, 1] \to W([r_{j-1}, r_j])$, $|p(0)| = r_{j-1}$, $|p(1)| = r_j$ meets the $\eta$-neighbourhood of $\text{Skel}(\Gamma(r_j, \tau, q_{j-1})$. Thus, if $G = \cup_{j=1}^{M} G_j$ then for each path $p: [0, 1] \to W([r_0, r_M]) \setminus G$, such that $|p(0)| = r_0$, $|p(1)| = r_M$, there are $t_j, 1 \leq j \leq M$, $0 < t_1 < \cdots < t_M < 1$ such that $p(t_j)$ is in the $\eta$-neighbourhood of $\text{Skel}(\Gamma(r_j, \tau, q_{j-1}))$, so there is a point $z_j \in \text{Skel}(\Gamma(r_j, \tau, q_{j-1})$ such that
\[ |z_j - p(t_j)| < \eta \quad (1 \leq j \leq M). \]
By (5.1) it follows that

\[
\text{length}(p) \geq \sum_{j=1}^{M-1} |p(t_{j+1}) - p(t_j)| \geq \sum_{j=1}^{M-1} |z_{j+1} - z_j| - (M - 1)\eta \geq \tau\mu - M\eta.
\]

Using (iii) in (4.4) in an induction process again one concludes that given \( \varepsilon > 0 \) and \( L < \infty \) there is a polynomial \( g \) on \( \Phi^N \) such that \( \Re g > L \) on \( G \) and \( |g| < \varepsilon \) on \( r_0\mathbb{B} \).

Thus, if \( 1/2 < r_0 < r_M < 1 \) and if \( \tau, 0 < \tau < \tau_0 \) satisfies (5.2) then there is a set \( E \subset W((r_0, r_M]) \) such that

(i) if \( p: [0, 1] \rightarrow W([r_0, r_M]) \setminus E \) is a path such that \( |p(0)| = r_0, \ |p(1)| = r_M \) then the length of \( p \) exceeds \( \tau\mu - M\eta \)

(ii) given \( L < \infty \) and \( \varepsilon > 0 \) there is a polynomial \( g \) such that \( \Re g > L \) on \( E \) and \( |g| < \varepsilon \) on \( r_0\mathbb{B} \).

We now prove that for every \( A < \infty \) and for every segment \( J = (\alpha, \beta], \ 1/2 < \alpha < \beta < 1 \) there is a set \( E \subset W(J) \) which satisfies (4.1).

So let \( 1/2 < \alpha < \beta < 1 \). Write \((\alpha, \beta] = J_1 \cup J_2 \cup \cdots \cup J_\ell \) where \( J_1 < J_2 < \cdots < J_\ell \) and where \( |J_j| = (\beta - \alpha)/\ell \) (\( 1 \leq j \leq \ell \)). For each \( j \) we shall construct a set \( E_j \subset W(J_j) \) as above. Provided that \( 0 < \tau < \tau_0 \) and \( \tau^2\omega < (\beta - \alpha)/(M\ell) \) the set \( E = \bigcup_{j=1}^{\ell} E_j \) will then have the property that if a path \( p: [0, 1] \rightarrow W([\alpha, \beta]) \setminus E \) satisfies \( |p(0)| = \alpha, \ |p(1)| = \beta \) then

\[
\text{length}(p) \geq \ell(\tau\mu - M\eta) = \ell\tau\mu - M\ell\eta.
\]

Suppose that \( A < \infty \) is given. We show that it is possible to choose \( \ell \) and \( \tau, 0 < \tau < \tau_0 \), so that

\[
\tau^2\omega < \frac{\beta - \alpha}{M\ell} \tag{5.3}
\]

and

\[
\ell\tau\mu = A + 1.
\]

In fact, \( \tau = (A + 1)/(\ell\mu) \) implies that there is an \( \ell_0 \) such that \( 0 < \tau < \tau_0 \) for every \( \ell > \ell_0 \). For (5.3) to hold we must have

\[
\left( \frac{A + 1}{\ell\mu} \right)^2 \omega < \frac{\beta - \alpha}{M\ell}
\]

which is clearly possible provided that \( \ell > \ell_0 \) is chosen large enough.

So fix such \( \ell \) and such \( \tau \) and let \( \eta > 0 \) be so small that \( M\ell\eta < 1 \). Then

\[
\text{length}(p) \geq \ell\tau\mu - M\ell\eta \geq A + 1 - 1 = A.
\]

Given \( L < \infty \) and \( \varepsilon > 0 \), an inductive process again produces a polynomial \( \Phi \) such that \( \Re \Phi > L \) on \( E \) and \( |\Phi| < \varepsilon \) on \( \alpha\mathbb{B} \). This will complete the proof of Lemma 3.1 and thus the proof of Lemma 2.1 once we have proved that the tessellation \( T \) can be chosen in such a way that, provided that the ball \( U_0 \) is small enough, the surfaces \( \Gamma(r, \tau, z) \) are convex.

6. Convexity of the surfaces \( \Gamma(r, \tau, z) \)
We shall now show how to choose a tessellation $\mathcal{T}$ in Section 4 so that after choosing $U_0$ small enough the polyhedral surfaces $\Gamma(r, \tau, z)$ will be convex. This is the fact that we used in the proof of Lemma 3.1. We essentially follow [G].

Perturb the canonical orthonormal basis in $\mathbb{R}^{M-1}$ a little to get an $(M - 1)$-tuple of vectors $e_1, e_2, \cdots, e_{M-1}$ in general position so that the lattice

$$
\Lambda = \left\{ \sum_{i=1}^{M-1} n_i e_i : n_i \in \mathbb{Z}, \ 1 \leq i \leq M - 1 \right\} \quad (6.1)
$$

will be generic, and, in particular, no more than $M$ points of $\Lambda$ will lie on the same sphere.

For each point $x \in \Lambda$ there is the Voronei cell $V(x)$ consisting of those points of $\mathbb{R}^{M-1}$ that are at least as close to $x$ as to any other $y \in \Lambda$, so

$$
V(x) = \left\{ y \in \mathbb{R}^{M-1} : \text{dist}(y, x) \leq \text{dist}(y, z) \text{ for all } z \in \Lambda \right\}.
$$

It is known that the Voronei cells form a tessellation of $\mathbb{R}^{M-1}$ and in our case they are all congruent, of the form $V(0) + x, \ x \in \Lambda$ [CS].

There is a Delaunay cell for each point that is a vertex of a Voronei cell. It is the convex polytope that is the convex hull of the points in $\Lambda$ closest to that point - these points are all on a sphere centered at this point. In our case, when there are no more than $M$ points of $\Lambda$ on a sphere, Delaunay cells are $(M - 1)$-simplices. Delaunay cells form a tessellation of $\mathbb{R}^{M-1}$ [CS]. In our case it is a true Delaunay tessellation, that is, for each cell, the circumsphere of each cell $S$ contains no other points of $\Lambda$ than the vertices of $S$.

We shall denote by $\mathcal{T}$ the family of all simplices - cells of the Delaunay tessellation for the lattice $\Lambda$ and this is to be taken as our $\mathcal{T}$ in Section 4. Clearly $\Lambda$ is precisely the set of vertices of the simplices in $\mathcal{T}$.

The construction implies that the tessellation $\mathcal{T}$ is periodic with respect to $\Lambda$. Thus, there are finitely many simplices $S_1, \cdots, S_i$ in $\mathcal{T}$ such that every other simplex of $\mathcal{T}$ is of the form $S_i + w$ where $w \in \Lambda$ and $1 \leq i \leq \ell$. It is then clear by the periodicity that there is an $\eta > 0$ such that for every simplex $S \in \mathcal{T}$ in the $\eta$-neighbourhood of the closed ball bounded by the circumsphere of $S$ there are no other points of $\Lambda$ than the vertices of $S$.

When we pass from $\mathcal{T}$ to $\tau(\mathcal{T} + z)$ where $\tau > 0$ and $z \in \mathbb{R}^{M-1}$ everything changes proportionally. For instance, for every simplex $S \in \tau(\mathcal{T} + z)$ in the $(\tau \eta)$-neighbourhood of the closed ball bounded by the circumsphere of $S$ there will be no other vertex of $\tau(\mathcal{T} + z)$ than the vertices of $S$.

We must now show that if $U_0$ is chosen sufficiently small then for every $r, \ 1/2 < r < 1$, every $\tau > 0$ and every $z \in \mathbb{R}^{M-1}$ for every simplex $S \in \tau(\mathcal{T} + z)$ contained in $U_0$ the intersection of the hyperplane $H$ containing $\Psi_r(S)$ with $\Gamma(r, \tau, z)$ is precisely $\Psi_r(S)$.

So let $S$ be such a simplex and let $H$ be the hyperplane in $\mathbb{R}^{M}$ containing $\Psi_r(S)$. Then $H$ intersects $b(r\mathbb{B})$ in a sphere $\Gamma$ that is the circumsphere of $\Psi_r(S)$ in $H$. One component of $b(r\mathbb{B}) \setminus \Gamma$ is contained in the open halfspace bounded by $H$ which contains the origin. Denote this component of $b(r\mathbb{B}) \setminus \Gamma$ by $E$.

Recall that all vertices of $\Gamma(r, \tau, z)$ are contained in $b(r\mathbb{B})$. Thus, to prove that $H$ contains no other vertex of $\Gamma(r, \tau, z)$ than the vertices of $\Psi_r(S)$ it is enough to show that all vertices of $\Gamma(r, \tau, z)$ except the vertices of $\Psi_r(S)$ are contained in $E$. \quad (6.2)
Since \( \pi|W_0 \cap b(r \mathbb{B}) : W_0 \cap b(r \mathbb{B}) \to U_0 \) is one to one, to show (6.2) it is enough to show that

\[
\begin{align*}
\{ & \text{the vertices of all simplices } T \in \tau(T + z) \text{ contained in } U_0 \\
& \text{except the vertices of } S \text{ are contained in the complement} \\
& \text{of the bounded domain in } \mathbb{R}^{M-1} \text{ bounded by } \pi(\Gamma). \\
\} \tag{6.3}
\end{align*}
\]

Since the \((\tau \eta)\)-neighbourhood of the closed ball in \( \mathbb{R}^{M-1} \) bounded by the circumsphere of \( S \) contains no other vertex of \( \tau(T + z) \) than the vertices of \( S \) it follows that to show (6.3) it is enough to show that

\[
\text{provided that } U_0 \text{ is chosen small enough on the outset then for every } S \in \tau(T + z) \text{ contained in } U_0 \text{ the projection } \pi(\Gamma) \text{ of the circumsphere} \\
\Gamma \text{ of } \Psi_r(S) \text{ in the hyperplane } H \text{ containing } \Psi_r(S) \text{ is contained in the} \\
(\tau \eta)\text{-neighbourhood of the circumsphere of } S \text{ in } \mathbb{R}^{M-1}. \tag{6.4}
\]

Given a \((M - 1)\)-simplex \( T \subset \mathbb{R}^M \) denote by \( \Gamma(T) \) the circumsphere of \( T \) in the hyperplane containing \( T \). Given a \((M - 1)\)-simplex \( S \subset \mathbb{R}^{M-1} \) with vertices \( v_1, \ldots, v_M \) and \( \omega > 0 \) denote by \( \Omega_\omega(S) \) the set of all simplices with vertices \((v_1, q_1), \ldots, (v_M, q_M)\) where \( q_i \in \mathbb{R} \) satisfy

\[
|q_i - q_j| \leq \omega|v_i - v_j| \quad (1 \leq i, j \leq M). 
\]

**Proposition 6.1** Let \( S \subset \mathbb{R}^{M-1} \) be a \((M - 1)\)-simplex. Given \( \eta > 0 \) there is an \( \omega > 0 \) such that for every \( T \in \Omega_\omega(S) \) the projection \( \pi(\Gamma(T)) \) is contained in the \( \eta \)-neighbourhood of \( \Gamma(S) \). Moreover, for any \( \tau > 0 \) and for any \( T \in \Omega_\omega(\tau S) \) the projection \( \pi(\Gamma(T)) \) is contained in the \((\tau \eta)\)-neighbourhood of \( \Gamma(\tau S) \).

**Proof.** Let \( S \subset \mathbb{R}^{M-1} \) be a simplex with vertices \( v_1, \ldots, v_M \) and let \( T \) be a simplex with vertices \((v_1, q_1), \ldots (v_M, q_M)\). Note that \( \pi(\Gamma(T)) \) does not change if we translate \( T \) in the direction of the last axis so with no loss of generality consider the simplex with the vertices \((v_1, q_1 - q_M), \ldots, (v_{M-1}, q_{M-1} - q_M), (v_M, 0)\). We now show that if \( P \) is a simplex with vertices \( w_1 = (v_1, \beta_1), \ldots, w_{M-1} = (v_{M-1}, \beta_{M-1}), w_M = (v_M, 0) \) then

\[
\pi(\Gamma(P)) \text{ is arbitrarily close to } \Gamma(S) \text{ provided that } \beta_1, \ldots, \beta_{M-1} \tag{6.5}
\]

are sufficiently small.

This implies that if \( \eta > 0 \) then there is an \( \varepsilon > 0 \) such that if \( |q_i - q_M| < \varepsilon \) \((1 \leq i \leq M - 1)\) then \( \pi(\Gamma(T)) \) is contained in the \( \eta \)-neighbourhood of \( \Gamma(S) \). Picking now \( \omega > 0 \) so small that \( \omega|v_1 - v_M| < \varepsilon \) \((1 \leq i \leq M - 1)\) completes the proof of the first part of proposition. To prove (6.5), let \( H \) be the hyperplane containing \( P \) and for each \( j, 1 \leq j \leq M - 1 \), let \( H_j \) be the hyperplane through the midpoint of the segment with endpoints \( w_j, w_M \), perpendicular to \( w_M - w_j \). The center \( C \) of \( \Gamma(P) \) is the intersection of \( H, H_1, \ldots, H_{M-1} \). Since these hyperplanes are in general position and change continuously with \( \beta_1, \ldots, \beta_{M-1} \), the point \( C \) and consequently \( \Gamma(P) \) changes continuously with \( \beta_1, \ldots, \beta_{M-1} \). When \( \beta_1 = \cdots = \beta_{M-1} = 0 \) we have \( P = S \) so \( \Gamma(P) = \pi(\Gamma(P)) = \Gamma(S) \). This implies (6.5).
To prove the second statement of the proposition assume that \( \tau > 0 \) and that \( T \in \Omega_\omega(\tau S) \). Then the vertices of \( T \) are \( (\tau v_1, p_1), \cdots (\tau v_M, p_M) \) where \( |p_i - p_j| \leq \omega|\tau v_i - \tau v_j| \) \( (1 \leq i, j \leq M) \). Writing \( p_i = \tau q_i \), we get that
\[
|q_i - q_j| \leq \omega|v_i - v_j| \quad (1 \leq i, j \leq M) \tag{6.6}
\]
Thus, the vertices of \( T \) are \( (\tau v_1, \tau q_1), \cdots (\tau v_M, \tau q_M) \) where (6.6) holds, that is \( T = \tau \tilde{S} \) where \( \tilde{S} \in \Omega_\omega(S) \). Clearly \( \Gamma(T) = \tau \Gamma(\tilde{S}) \) and so \( \pi(\Gamma(T)) = \pi(\tau \Gamma(\tilde{S})) = \tau \pi(\Gamma(\tilde{S})) \).
Since \( \tilde{S} \in \Omega_\omega(S) \) the preceding discussion shows that \( \pi(\Gamma(\tilde{S})) \) is contained in the \( \eta \)-neighbourhood of \( \pi(\Gamma(S)) \) so it follows that \( \pi(\Gamma(T)) \) is contained in the \( (\tau \eta) \)-neighbourhood of \( \Gamma(\tau S) \). This completes the proof of Proposition 6.1.

To prove (6.4) recall first that there are finitely many simplices \( S_1, \cdots, S_\ell \) in \( T \) such that every other simplex of \( T \) is of the form \( S_i + w \) where \( w \in \Lambda \) and \( 1 \leq i \leq \ell \). Thus, there is an \( \omega > 0 \) such that the statement of Proposition 6.1 holds for every simplex \( S \in T + z \). Recall that \( \text{grad} \psi_r \) vanishes at the origin so one can choose \( U_0 \), a ball centered at the origin, so small that
\[
|(\text{grad} \psi_r)(x)| < \omega \text{ for all } x \in U_0 \text{ and all } r, \; 1/2 < r < 1. \tag{6.7}
\]
This implies that for every \( S \in \tau(T + z) \) with vertices \( v_1, v_2, \cdots, v_M \), contained in \( U_0 \), the simplex \( \Psi_r(S) \) with vertices \( (v_1, \psi_r(v_1)), \cdots, (v_M, \psi_r(v_M)) \), by (6.7), satisfies
\[
|\psi_r(v_i) - \psi_r(v_j)| \leq \omega|v_i - v_j| \quad (1 \leq i, j \leq M)
\]
so the simplex \( \Psi_r(S) \) belongs to \( \Omega_\omega(S) \) so (6.4) follows by Proposition 6.1. This completes the proof of convexity of surfaces \( \Gamma(r, \tau, z) \) and completes the proof of Lemma 3.1. The proof of Lemma 2.1 is complete. Theorem 1.1 is proved.

This work was supported by the Research Program P1-0291 from ARRS, Republic of Slovenia.
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