PHASE DIAGRAM OF AN EXACTLY SOLVABLE $t$–$J$ LADDER MODEL

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We study a system of one-dimensional $t$–$J$ models coupled to a ladder system. A special choice of the interaction between neighbouring rungs leads to an integrable model with supersymmetry, which is broken by the presence of rung interactions. We analyze the spectrum of low-lying excitations and ground state phase diagram at zero temperature.

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Systems of one-dimensional magnetic or electronic chains coupled to so-called ladders have attracted increasing attention as a scenario in which two-dimensional physics can be studied by starting from well established one-dimensional theories \([1,3]\). The properties of the undoped systems, i.e. spin ladders, have been thoroughly investigated by a variety of methods \([1,3]\) and recently integrable models with tunable interaction parameters have been introduced \([7,13]\). To understand the properties of the doped ladder systems the dynamics of the holes created under doping in the background of antiferromagnetically interacting spins has to be studied \([14]\). This question has been studied within a model of coupled \(t-J\) chains forming a ladder-like structure in a series of papers \([14,16]\) with interesting results. Within this approach, however, multi hole scattering processes or the dynamics of anti-bonding hole states could not be accessed.

Our aim here is to propose a modified \(t-J\) ladder model, which becomes integrable at certain limits and therefore allows us to investigate the properties of this system exactly and in more detail. In the thermodynamic limit, we can analyze further the nature of the ground state and predict the onset of excitations, thereby identifying the phase diagram of the system. The system we consider consists of two coupled \(t-J\) chains labeled \(a=1,2\). The electrons with spin \(\sigma\) on site \(j\) of the chains are described in terms of canonical Fermi operators \(c_{j\sigma}^{(a)}\). In terms of these the chain Hamiltonian is

\[
H_{t-J}^{(a)} = \sum_{j} \left\{ H_{t-J}^{(a)} \right\}_{jj+1} \text{with}
\]

\[
\left\{ H_{t-J}^{(a)} \right\}_{jk} = -t \mathcal{P} \left( \sum_{\sigma=\uparrow,\downarrow} c_{aj}^{(a)\dagger} c_{ak}^{(a)} + \text{h.c.} \right) \mathcal{P} + J \left( S_{j}^{(a)} S_{k}^{(a)} - \frac{1}{4} n_{j}^{(a)} n_{k}^{(a)} \right) + n_{j}^{(a)} + n_{k}^{(a)}
\]

where \(\mathcal{P}\) projects out double occupancies, \(S_{j}^{(a)}\) are spin operators on site \(j\) and \(n_{j}^{(a)} = n_{j\uparrow}^{(a)} + n_{j\downarrow}^{(a)}\) is the total number of electrons on site \(j\).

The Hamiltonian of the ladder model we consider then reads

\[
H = \sum_{a} H_{t-J}^{(a)} + H_{\text{int}} + H_{\text{rung}} - \mu \hat{N}.
\]

Here, \(\mu\) is the chemical potential coupling to the total number of electrons in the system, \(H_{\text{int}}\) is the interaction between neighbouring rungs of the ladder, given by the expression \(H_{\text{int}} = -\sum_{j} \left\{ H_{t-J}^{(1)} \right\}_{jj+1} \left\{ H_{t-J}^{(2)} \right\}_{jj+1} \), and \(H_{\text{rung}}\) contains all the interactions present on a single rung of the ladder and reads

\[
H_{\text{rung}} = \sum_{j} \left\{ -t' \mathcal{P} \left( \sum_{\sigma=\uparrow,\downarrow} c_{aj}^{(1)\dagger} c_{aj}^{(2)} + \text{h.c.} \right) \mathcal{P} + J' \left( S_{j}^{(1)} S_{j}^{(2)} - \frac{1}{4} n_{j}^{(1)} n_{j}^{(2)} \right) + V n_{j}^{(1)} n_{j}^{(2)} \right\}.
\]

This is similar to the terms in \([1]\) describing the interaction of electrons on the bonds within a rung but contains independent coupling constants \(t', J'\) and an additional Coulomb interaction.
of strength $V$. Below we shall use a basis of states on a single rung which are eigenstates to (3): at half filling (one electron per site) the electrons can form a spin singlet $|s\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ or triplet state $|t_+\rangle \equiv |\uparrow\uparrow\rangle$ etc. ($|\sigma\tau\rangle = c_\sigma^{(1)} c_\tau^{(2)} |00\rangle$). Doping of the system creates the states $|\sigma_{\pm}\rangle \equiv \frac{1}{\sqrt{2}}(|\sigma0\rangle \pm |0\sigma\rangle)$, with $\sigma = \uparrow$ or $\downarrow$, describing a single electron with spin $\sigma$ in a bonding (anti-bonding) state or the Fock vacuum (two-hole state) $|d\rangle \equiv |00\rangle$ on a rung. Below we shall be particularly interested in the strong coupling regime $J' \gg 1, V \gg \mu$ near half filling. This implies that the triplet states $|t_\alpha\rangle$ in the two-particle sector will be energetically unfavourable.

To study the phase diagram of the system at low temperatures we note that by excluding the triplet states and choosing the coupling constants on the legs of the ladder as $J = 2t = 2$ the Hamiltonian (2) can be rewritten as

$$H = -\sum_j \Pi_{j,j+1} - \sum_{i=1}^{5} A_i N_i + \text{const.} \quad (4)$$

where $N_l, l = 1, 2 \ (3, 4)$ is the number of bonding (anti-bonding) single particle rung states with spin $\uparrow, \downarrow$ and $N_5$ is the number of empty rungs (on the remaining $N_0 = L - \sum_l N_l$ rungs the two electrons form a singlet). The (graded) permutation operator $\Pi_{jk}$ interchanges the states on rungs $j$ and $k$ giving a minus sign if both rungs are fermionic, i.e. singly occupied. In the restricted Hilbert space we are considering here we can absorb $J'$ into the definition of $V$ which allows to express the potentials $A_i$ in terms of the coupling constants in (3) as

$$A_1 = A_2 \equiv \mu_+ = t' - \mu + V, \quad A_3 = A_4 \equiv \mu_- = -t' - \mu + V, \quad A_5 \equiv \tilde{V} = -2\mu + V \quad (5)$$

The relative strength of these potentials will determine the ground state and spectrum of low lying excitations in the ladder.

Note that the Hamiltonian (4) is that of a $gl(2|4)$ invariant super spin chain in the presence of external symmetry breaking fields and the grading of the states $|s\rangle$ and $|d\rangle$ is bosonic while the two doublets $|\sigma_{\pm}\rangle$ carry fermionic grading. This model is integrable by means of the Bethe Ansatz (BA) (see e.g. [17–20]). In the strong coupling regime near half filling, where the ground state shows little deviations from the dimerized state $\otimes_j |s\rangle_j$ doping will lead to the condensation of either the fermionic single hole states $|\sigma_{\pm}\rangle$ or the bosonic double hole state $|d\rangle$ into the ground state – depending on the relative strength of the chemical potentials $\mu_-, \mu_+$ and the rung interaction $\tilde{V}$. For simplicity we shall consider the case $\mu_+ = \mu_- \equiv \tilde{\mu}$ below, i.e. two degenerate single particle bands. Due to the grading different sets of BA equations.
have to be considered to describe these processes (see e.g. [17, 21, 22]): since the Bethe Ansatz
states are highest weight in the $gl(2|4)$ algebra we have to deal separately with two regions
in the low temperature phase diagram of the $t–J$ ladder (2) between half- and quarter-filling,

For the first case we have to order the rung basis as \( |s\rangle, |\sigma_+\rangle, |d\rangle \) and the eigenstates of (4)
are parameterized by solutions of the BA equations corresponding to this grading (BFFFFB)

\[
\begin{align*}
\left[e_1(\lambda_j^{(1)})\right]^L &= \prod_{k=1}^{M_2} e_1(\lambda_j^{(1)} - \lambda_k^{(2)}) , \\
\prod_{k\neq j}^{M_r} e_2(\lambda_j^{(r)} - \lambda_k^{(r)}) &= \prod_{l=1}^{M_{r-1}} e_1(\lambda_j^{(r)} - \lambda_l^{(r-1)}) \prod_{m=1}^{M_{r+1}} e_1(\lambda_j^{(r)} - \lambda_m^{(r+1)}) , \quad r = 2, \ldots, 4 , \tag{6}
\end{align*}
\]

with \( e_n(x) \equiv (x + in/2)/(x - in/2) \). Here the number of roots on the \( r \)th level is related to the
numbers \( N_l \) in (4) by \( M_r = M_r - N_r \) (we define \( M_0 \equiv L \)). The energy of the corresponding
state is

\[
E \left( \{ \lambda_j^{(r)} \} \right) = -2M_1 + \sum_{j=1}^{M_1} \frac{1}{(\lambda_j^{(1)})^2 + 1/4} - \sum_{l=1}^{5} \lambda_l^2 N_l . \tag{7}
\]

In the thermodynamic limit \( L \to \infty \) with \( M_r/L \) fixed the solutions of (6) are built on two types
of so-called strings, namely complexes

\[
\lambda_k^{(r-n)} = \xi^{(r)} + \frac{i}{2}(n - 2k) , \quad k = 0, \ldots, n , \quad 0 \leq n < r \text{ and } 2 \leq r \leq 4
\]
corresponding to bound states of holes with different spin and parity with densities \( \rho^{(r)}(\xi) \) and

\[
\lambda_k^{(r)} = \Lambda_m + \frac{i}{2}(m + 1 - 2k) , \quad k = 1, \ldots, m , \quad 2 \leq r \leq 4
\]
corresponding to bound spin states with densities \( \sigma_m^{(r)}(\Lambda) \), \( r = 2, 3, 4 \). On level 1 and 5 all
solutions are real with densities \( \sigma_1^{(r)}(\lambda^{(r)}) \), \( r = 1, 5 \).

By minimizing the free energy of the system (see e.g. [18, 20]) one finds that at most the
bound states of single particle rung states \( |\sigma_+\rangle \) with density \( \rho^{(4)} \) and empty rungs \( |d\rangle \) with
density \( \sigma_1^{(5)} \) are present in the ground state. The densities are determined through the Bethe
Ansatz integral equations

\[
\begin{align*}
\rho^{(4)}(x) &= a_4(x) - \int_s dy K(x - y) \rho^{(4)}(y) - \int_d dy a_1(x - y) \sigma_1^{(5)}(y) , \\
\sigma_1^{(5)}(x) &= \int_s dy a_1(x - y) \rho^{(4)}(y)
\end{align*} \tag{8}
\]
where \( K(x) = \sum_{n=1}^{3} a_{2n}(x) \) and \( a_{n}(x) = 2n/ (\pi (4x^2 + n^2)) \). The corresponding dispersions \( \epsilon_{a} \) and \( \epsilon_{d} \) are given by similar integral equations

\[
\epsilon_{a}(x) = 2\pi a_{4}(x) - 4(2 + \bar{\mu}) - \int_{s} dy \ K(x-y)\epsilon_{a}(y) + \int_{d} dy \ a_{1}(x-y)\epsilon_{d}(y) ,
\]

\[
\epsilon_{d}(x) = \bar{\mu} - \bar{V} - \int_{s} dy \ a_{1}(x-y)\epsilon_{a}(y) .
\]

The chemical potential and the rung interaction determine the ranges of integration

\[
\int_{\alpha} dy = \left\{ \int_{-\infty}^{-Q_{\alpha}} + \int_{Q_{\alpha}}^{\infty} \right\} dy
\]

through the condition \( \epsilon_{a}(\pm Q_{\alpha}) = 0 \). In terms of the solution of (9) the density of holes is

\[
n_h = 2 \int_{s} dy \ \rho^{(4)}(y) - \int_{d} dy \ \sigma^{(5)}_{1}(y) .
\]

For \( \bar{V} < \bar{\mu} < -2 \) both \( \epsilon_{a} \) and \( \epsilon_{d} \) are positive and hence the dimer state is the ground state of the system. For larger chemical potential single hole states \( |\sigma_{\pm}\rangle \) with dispersion \( \epsilon_{a} \) begin to condensate into this ground state. The electronic properties in this phase are those of the degenerate supersymmetric \( t-J \) model \( [18] \). For \( \bar{V} > \bar{\mu} - \int_{s} dy \ a_{1}(y)\epsilon_{a}(y) \) the system enters a phase where in addition double hole rung states \( |d\rangle \) with energy \( \epsilon_{d} \) will be present in the ground state.

For \( \bar{\mu} < \bar{V} \) we have to order the the rung states as \( |s\rangle, |d\rangle, |\sigma_{\pm}\rangle \) basis leading to a different set of Bethe Ansatz equations corresponding to BBFFFF grading \( (M_{6} \equiv 0) \).

\[
\left[ e_{1}(\lambda_{j}^{(1)}) \right]^{L} = \prod_{k \neq j}^{M_{1}} e_{2}(\lambda_{j}^{(1)} - \lambda_{k}^{(1)}) \prod_{k=1}^{M_{2}} e_{1}(\lambda_{k}^{(2)} - \lambda_{j}^{(1)}) ,
\]

\[
\prod_{l=1}^{M_{1}} e_{1}(\lambda_{l}^{(2)} - \lambda_{1}^{(1)}) = \prod_{m=1}^{M_{3}} e_{1}(\lambda_{j}^{(2)} - \lambda_{m}^{(3)}) ,
\]

\[
\prod_{k \neq j}^{M_{r}} e_{2}(\lambda_{j}^{(r)} - \lambda_{k}^{(r)}) = \prod_{l=1}^{M_{r-1}} e_{1}(\lambda_{j}^{(r)} - \lambda_{l}^{(r-1)}) \prod_{m=1}^{M_{r+1}} e_{1}(\lambda_{j}^{(r)} - \lambda_{m}^{(r+1)}) . \quad r = 3, \ldots, 5 .
\]

(\text{Note that due to the reordering of the basis the number of double hole rung states is now} \( N_{1} \).\text{)}

The energy of the corresponding Bethe Ansatz eigenstates is again given by (9) with \( A_{1} = \bar{V} \) and \( A_{l} = \bar{\mu} \) for \( 2 \leq l \leq 5 \). Similarly as for (9) the solutions of these equations can be classified in strings with densities \( \sigma_{m}^{(r)} \) for positive integers \( m \) and \( r = 1, 3, 4, 5 \) and single hole bound states with densities \( \rho^{(r)} \), \( r = 3, 4, 5 \). Only real roots with density \( \sigma_{1}^{(2)} \) are possible on the
second level of (1). The ground state configuration is completely determined by

\[
\rho^{(5)}(x) = -\int_s dy \ K(x-y)\rho^{(5)}(y) + \int_d dy \ a_4(x-y)\sigma_1^{(1)}(y),
\]

\[
\sigma_1^{(1)}(x) = a_1(x) + \int_s dy \ a_4(x-y)\rho^{(5)}(y) - \int_d dy \ a_2(x-y)\sigma_1^{(1)}(y).
\]

The dispersion of these excitations are determined through the integral equations

\[
\epsilon_s(x) = 4(\tilde{V} - \tilde{\mu}) - \int_s dy \ K(x-y)\epsilon_s(y) + \int_d dy \ a_4(x-y)\epsilon_d(y),
\]

\[
\epsilon_d(x) = 2\pi a_1(x) - (2 + \tilde{V}) + \int_s dy \ a_4(x-y)\epsilon_s(y) - \int_d dy \ a_2(x-y)\epsilon_d(y).
\]

and the density of holes is \(n_h = \int_d dy \ \sigma_1^{(1)}(y) - 2 \int_s dy \ \rho^{(5)}(y)\).

With these equations the phase diagram of the \(t\)-\(J\) ladder (4) can be completed: for \(\tilde{\mu} < \tilde{V} < -2\) the system is in the dimer phase, increasing the rung interaction \(\tilde{V}\) the system enters a phase where the ground state is a Fermi sea of double hole states with dispersion \(\epsilon_d\) moving in the background of dimer states \(|s\rangle\). Increasing \(\tilde{\mu}\) in this phase beyond \(\frac{1}{8}(7\tilde{V} - 2)\) both double and single hole rung states are present in the ground state configuration.

Numerical integration of the integral equations for the densities and dressed energies allows to determine the phase boundaries as a function of the hole density \(n_h\) and the rung interaction \(V = 2\tilde{\mu} - \tilde{V}\) in Eq. (4). In Fig. 1 the resulting phase diagram is shown. For \(V \gtrsim 2\) condensation of the single hole states \(|\sigma_{\pm}\rangle\) is energetically favourable. For sufficiently strong attractive rung interaction, however, doping of the system first introduces double hole rung states \(|d\rangle\). Interestingly, we find that for \(2 > V > 2 - \frac{1}{2}(\psi\left(\frac{7}{8}\right) - \psi\left(\frac{5}{8}\right)) \approx 1.676\) (\(\psi(x)\) is the digamma function) increasing the hole concentration can drive the system from the latter phase into the one without double hole rung states.

In summary we have studied the phase diagram of a system of coupled \(t\)-\(J\) models in the framework of an integrable system. By choosing properly the interaction between the neighbouring rungs the resulting ladder model can be brought into the form of a \(\text{gl}(2|4)\)-invariant exchange model in the presence of symmetry breaking fields, which enter the Hamiltonian as kinetic and interaction terms on a single rung. The spectrum of excitations and zero temperature phase diagram have been determined by means of the Bethe Ansatz. Extensions are possible by assigning different energies to the single particle rung states of different parity \((t' \neq 0)\) and including the triplet rung states \((J' \text{ finite})\). As has been shown for the undoped case new phases can be expected for finite \(J'\).
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Figure 1: Phase diagram of the $t$–$J$ ladder with rung interaction (3) for $t' = 0$ and $J' \gg 1$ as a function of the hole concentration $n_h$: For large repulsive $V$ the ground state is a Fermi sea of single hole states $|\sigma_\pm\rangle$ propagating in the background of rung dimers. For sufficiently strong attractive rung interaction the double hole rung states $|d\rangle$ are energetically favourable. In the intermediate region (shaded) dimers and both types of hole rung states coexist in the ground state configuration.