Constrained convex bodies with extremal affine surface areas

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Abstract

Given a convex body $K \subseteq \mathbb{R}^n$ and $p \in \mathbb{R}$, we introduce and study the extremal inner and outer affine surface areas

\[ IS_p(K) = \sup_{K' \subseteq K} \left( \text{as}_p(K') \right) \quad \text{and} \quad os_p(K) = \inf_{K' \supseteq K} \left( \text{as}_p(K') \right), \]

where $\text{as}_p(K')$ denotes the $L_p$-affine surface area of $K'$, and the supremum is taken over all convex subsets of $K$ and the infimum over all convex compact subsets containing $K$.

The convex body that realizes $IS_1(K)$ in dimension 2 was determined in [3] where it was also shown that this body is the limit shape of lattice polytopes in $K$. In higher dimensions no results are known about the extremal bodies. We use a thin shell estimate of [22] and the Löwner ellipsoid to give asymptotic estimates on the size of $IS_p(K)$ and $os_p(K)$. Surprisingly, it turns out that both quantities are proportional to a power of volume.

1 Introduction

F. John proved in [31] that among all ellipsoids contained in a convex body $K \in \mathbb{R}^n$, there is a unique ellipsoid of maximal volume, now called the John ellipsoid of $K$. Dual to the John ellipsoid is the Löwner ellipsoid, the ellipsoid of minimal volume containing $K$. These ellipsoids play fundamental roles in asymptotic convex geometry. They are related to the isotropic position, to the study of volume concentration, volume ratio, reverse isoperimetric inequalities, Banach-Mazur distance of normed spaces, and many more, including the hyperplane conjecture, one of the major open problems in asymptotic geometric analysis. We refer to e.g., the books [1, 11] for the details and more information.

In this paper, we introduce the analogue to John’s theorem, when volume is replaced by affine surface area. In parallel to John’s maximal volume ellipsoid, respectively the minimal volume Löwner ellipsoid, we investigate these convex bodies contained in $K$, respectively containing $K$, that have the largest, respectively smallest, $L_p$-affine surface areas,

\[ IS_p(K) = \sup_{K' \subseteq K} \left( \text{as}_p(K') \right) \quad \text{and} \quad os_p(K) = \inf_{K' \supseteq K} \left( \text{as}_p(K') \right). \]
By compactness and continuity, the supremum and infimum are in fact a maximum and minimum, i.e., \( IS_p(K) = a s_p(K_0) \) for some convex body \( K_0 \subset K \) and \( os_p(K) = a s_p(K_1) \) for some convex body \( K_1 \supset K \).

For \( p > 1 \), the \( L_p \)-affine surface area was introduced by E. Lutwak in his ground breaking paper [36] in the context of the \( L_p \)-Brunn-Minkowski theory and in [51] for all other \( p \), (see also [28] [41]). \( L_1 \)-affine surface area is classical and goes back to W. Blaschke [7].

The definition of \( L_p \)-affine surface area is given below in (2.1), where we also list some of its properties. Due to its remarkable properties, this notion is important in many areas of mathematics and applications. We only quote characterizations of \( L_p \)-affine surface areas by M. Ludwig and M. Reitzner [34], the \( L_p \)-affine isoperimetric inequalities, proved by E. Lutwak [36] for \( p > 1 \) and for all other \( p \) in [58]. The classical case \( p = 1 \) goes back to W. Blaschke [7]. These inequalities are related to various other inequalities, see e.g., E. Lutwak, D. Yang and G. Zhang [37] [39]. In particular, the affine isoperimetric inequality implies the Blaschke-Santalo inequality and it proved to be the key ingredient in the solution of many problems, see e.g. the books by R. Gardner [16] and R. Schneider [46] and also [30] [33] [35] [52] [53] [54] [58]. Recent developments include extensions to an Orlicz theory, e.g., [17] [27] [33] [59], to a functional setting [12] [13] and to the spherical and hyperbolic setting [43] [60].

Applications of affine surface areas have been manifold. For instance, affine surface area appears in best and random approximation of convex bodies by polytopes, see, e.g., K. Böröczky [8] [9], P. Gruber [20] [21], M. Ludwig [32], M. Reitzner [44] [45] and also [18] [19], [16] [17], appearing in best and random approximation of convex bodies by polytopes, see, e.g., K. Böröczky [8] [9], P. Gruber [20] [21], M. Ludwig [32], M. Reitzner [44] [45] and also [18] [19], [16] [17].

In dimension 2 and for \( p = 1 \), \( IS_1(K) \) was determined exactly by I. Bárány [3]. Moreover, he showed in [3] that the extremal body \( K_0 \) of (1.1) is unique and that \( K_0 \) is the limit shape of lattice polygons contained in \( K \).

In higher dimensions and for \( p \neq 1 \), there are no results available on \( IS_p(K) \), \( os_p(K) \) and related notions \( OS_p(K) \) and \( is_p(K) \), defined in (2.2) and (2.3) below. We observe that only certain \( p \)-ranges are meaningful for the various notions.

We use a thin shell estimate by O. Guédon and E. Milman [22], see also G. Paouris [22], on concentration of volume to show in our main theorem that \( IS_p(K) \) is proportional to a power of the volume \( |K| \) of \( K \). It involves the Euclidean unit ball \( B^2_2 \) centered at 0, and the isotropic constant \( L^2_K \) of \( K \), defined by

\[
nL^2_K = \min \left\{ \frac{1}{|TK|^{1+\frac{2}{n}}} \int_{a+TK} ||x||^2 dx : a \in \mathbb{R}^n, T \in GL(n) \right\}.
\]

**Theorem 3.4.** There is a constant \( C > 0 \) such that for all \( n \in \mathbb{N} \), all \( 0 \leq p \leq n \) and all convex bodies \( K \subset \mathbb{R}^n \),

\[
\frac{1}{n^{3/2}} \left( \frac{C}{|L_K|} \right)^{2np} \frac{IS_p(B^2_2)}{|B^2_2|^{\frac{2np}{n+2p}}} \leq \frac{IS_p(K)}{|K|^{\frac{2np}{n+2p}}} \leq \frac{IS_p(B^2_2)}{|B^2_2|^{\frac{2np}{n+2p}}}.
\]

Equality holds trivially in the right inequality if \( p = 0, n \). If \( p \neq 0, n \), equality holds in the right inequality iff \( K \) is a centered ellipsoid.

Since \( \frac{IS_p(B^2_2)}{|B^2_2|^{\frac{2np}{n+2p}}} = n |B^2_2|^{\frac{2p}{n+2p}} \), which is asymptotically equivalent to \( \frac{c}{n^{\frac{2p}{n+2p}}} \) with an absolute constant \( c \), the theorem shows that \( IS_p(K) \) is proportional to a power of \( |K| \).
We use the Löwner ellipsoid of $K$ (e.g., [11] or the survey [25]), to give asymptotic estimates on the size of $os_p(K)$ and $OS_p(K)$, also in terms of powers of $|K|$, in Theorem 3.5. For instance, we show that for $-n < p \leq 0$,

$$
\frac{os_p(B^n_2)}{|B^n_2|^\frac{n+p}{n}} \leq \frac{os_p(K)}{|K|^\frac{n+p}{n}} \leq n^n \frac{n-p}{n+p} \frac{os_p(B^n_2)}{|B^n_2|^\frac{n+p}{n}}.
$$

Equality holds trivially in the left inequality if $p = 0$. If $p \neq 0$, equality holds in the left inequality iff $K$ is a centered ellipsoid.

If $K$ is centrally symmetric, $n^{n-p} |K|$ can be replaced by $n^{n-p} |B^n_2|$.

We refer to Theorem 3.5 for the details.

2 Background and definitions

Throughout the paper, $c, C$ etc., denote absolute constants that may change from line to line. We will always assume throughout the paper that $0$ is the center of gravity of $K$,

$$
\int_K x \, dx = 0.
$$

For real $p \neq -n$, the $L_p$-affine surface areas are defined as [36, 41, 51]

$$
as_p(K) = \int_{\partial K} \frac{\kappa(x)^{\frac{1}{n+p}}}{'(x, N(x))^{\frac{n-p}{n+p}}} \, d\mu(x), \quad (2.1)
$$

where $N(x)$ is the outer unit normal vector at $x$ to $\partial K$, the boundary of $K$, and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$ which induces the Euclidian norm $\| \cdot \|$. The case $p = 1$ is the classical affine surface area whose definition goes back to Blaschke [7].

We denote by $\mathcal{K}_K$ the collection of all compact convex subsets of $K$ that have center of gravity at $0$ and by $\mathcal{K}^K$ the collection of all compact convex sets containing $K$ that have center of gravity at $0$.

For $-\infty \leq p \leq \infty$, $p \neq -n$, we then define the inner and outer maximal affine surface areas by

$$
IS_p(K) = \sup_{C \in \mathcal{K}_K} (as_p(C)), \quad OS_p(K) = \sup_{C \in \mathcal{K}^K} (as_p(C)), \quad (2.2)
$$

and the inner and outer minimal affine surface areas by

$$
is_p(K) = \inf_{C \in \mathcal{K}_K} (as_p(C)), \quad os_p(K) = \inf_{C \in \mathcal{K}^K} (as_p(C)). \quad (2.3)
$$

We show in section 3.1 that $is_p$ is identically equal to $0$ for all $p$ and all $K$ and that the only meaningful $p$-range for $IS_p$ is $[0, n]$ for $OS_p$ it is $[n, \infty]$ and for $os_p$ it is $(-n, 0]$.

By Blaschke’s selection theorem, $\mathcal{K}_K$ is compact with respect to the Hausdorff metric. For a fixed convex body $K \subset \mathbb{R}^n$ there is $R > 0$ such that the Euclidean ball $B^n_2(0, R)$ centered at $0$ with radius $R$ contains $K$. Then, again by Blaschke’s selection theorem, also $\mathcal{K}^K$ is compact with respect to the Hausdorff metric. Proposition 3.2 below, proved in [36], shows that the functional $K \mapsto as_p(K)$ is upper semicontinuous with respect to the Hausdorff metric, if $-\infty \leq p < -n$ or $0 \leq p \leq \infty$, respectively lower semicontinuous.
It was shown in [49] that for any convex body $K$ independently by Bárány and Larman [4] and Schütt and Werner [49],

$$IS_p(K) = \alpha_p(K_0) \quad \text{and} \quad OS_p(K) = \alpha_p(K_1)$$

for some convex body $K_0 \subset K$, respectively $K \subset K_1$, and that the second infimum in (2.3) is in fact a minimum for $-n < p \leq 0$,

$$os_p(K) = \alpha_p(K_2),$$

for some $K_2$ in $K^K$.

It was shown [36, 51] that for all $p \neq -n$ and for all invertible linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$\alpha_p(T(K)) = |\det(T)|^\frac{n-p}{n} \alpha_p(K).$$

It then follows immediately from the definitions (2.2) and (2.3) that for all invertible linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$IS_p(T(K)) = |\det(T)|^\frac{n-p}{n} IS_p(K), \quad OS_p(T(K)) = |\det(T)|^\frac{n-p}{n} OS_p(K) \quad (2.4)$$

and

$$is_p(T(K)) = |\det(T)|^\frac{n-p}{n} is_p(K), \quad os_p(T(K)) = |\det(T)|^\frac{n-p}{n} os_p(K). \quad (2.5)$$

For a general convex body $K$ in $\mathbb{R}^n$, a particularly useful way to define $\alpha_1(K)$ is the following. For $u \in \mathbb{R}^n$ and $t \geq 0$, define the half-spaces

$$H^+(t, u) = \{ x \in \mathbb{R}^n \mid \langle x, u \rangle \geq t \}, \quad H^-(t, u) = \{ x \in \mathbb{R}^n \mid \langle x, u \rangle \leq t \}.$$

For a convex body $K \subset \mathbb{R}^n$ and $\delta > 0$, the (convex) floating body $K_\delta$ was introduced independently by Bárány and Larman [4] and Schütt and Werner [49],

$$K_\delta = \bigcap_{|H^+(t, u) \cap K| \leq \delta |K|} H^-(t, u). \quad (2.6)$$

It was shown in [49] that for any convex body $K$ in $\mathbb{R}^n$,

$$\alpha_1(K) = 2 \left( \frac{B_n^{n-1}}{n+1} \right)^\frac{n}{n+1} \lim_{\delta \to 0} \frac{|K| - |K_\delta|}{\delta |K|} \quad (2.7)$$

Here, and in what follows, $B_n^2$ denotes the unit Euclidean ball in $\mathbb{R}^n$.

It was also shown in [49] that for an invertible affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, we have $(T(K))_\delta = T(K_\delta)$. Then, in particular,

$$\alpha_1(T(K)) = |\det(T)|^\frac{n}{n+1} \alpha_1(K). \quad (2.8)$$

By (2.8) and the definitions it follows that for $p = 1$ the expressions in (2.2) and (2.3) are also affine invariant: If $T : \mathbb{R}^n \to \mathbb{R}^n$ is affine and invertible, then

$$IS_1(TK) = |\det(T)|^{\frac{n-1}{n}} IS_1(K), \quad is_1(TK) = |\det(T)|^{\frac{n-1}{n}} is_1(K),$$

$$OS_1(TK) = |\det(T)|^{\frac{n-1}{n}} OS_1(K), \quad os_1(K) = |\det(T)|^{\frac{n-1}{n}} os_1(K). \quad (2.9)$$

Geometric descriptions in the sense of (2.6) and (2.7) of $L_p$-affine surface area also exist. We refer to e.g., [28, 50, 51, 50, 58].
3 Main results

Our main results give quantitative estimates for the inner and outer extremal affine surface areas. We observe first that for some $p$, the values for the extremal affine surface areas can be given explicitly and the $p$-ranges can be restricted accordingly in the quantitative estimates of Theorems 3.4 and 3.5 below.

3.1 The relevant $p$-ranges

(i) The case $IS_p(K)$

If $p = 0$, then for all $K$,

$$IS_0(K) = \sup_{K' \in K_K} (as_0(K')) = n \sup_{K' \in K_K} |K'| = n|K|.$$ 

If $p = n$, then for all $K$,

$$IS_n(K) = n|B_2^n|.$$ 

Indeed, on the one hand, we have by (2.4),

$$IS_n(K) \geq \sup_{\rho B_2^n \in K_K} (as_n(\rho B_2^n)) = \sup_{\rho B_2^n \in K_K} (as_n(B_2^n)) = n|B_2^n|.$$ 

The equi-affine isoperimetric inequality [36] says that $as_n(K) \leq as_n(B_2^n)$. Therefore,

$$IS_n(K) = \sup_{K' \in K_K} (as_n(K')) \leq \sup_{K' \in K_K} (as_n(B_2^n)) = n|B_2^n|.$$ 

If $n < p \leq \infty$, then $IS_p(K) = \infty$. This holds as by (2),

$$IS_p(K) \geq \sup_{\varepsilon B_2^n \in K_K} (as_p(\varepsilon B_2^n)) = \sup_{\varepsilon} \varepsilon^{n \frac{n-p}{n+p}} n|B_2^n| = \infty,$$

since $\frac{n-p}{n+p} < 0$.

If $-n < p < 0$, then for all $K$, $IS_p(K) = \infty$. Indeed, we have for all polytopes $P$

$$IS_p(K) \geq \sup_{P \in K_K} (as_p(P)) = \sup_{P \in K_K} \int_{\partial P} \frac{\kappa(x)^{\frac{n}{n+p}}}{\nu(x)^{\frac{n-p}{n+p}}} d\mu(x) = \infty,$$

since $\kappa(x) = 0$ almost everywhere.

If $-\infty \leq p < -n$, then for all $K$, $IS_p(K) = \infty$. Indeed, as above,

$$IS_p(K) \geq \sup_{\varepsilon B_2^n \in K_K} (as_p(\varepsilon B_2^n)) = \sup_{\varepsilon} \varepsilon^{n \frac{n-p}{n+p}} n|B_2^n| = \infty,$$

since $\frac{n-p}{n+p} < 0$.

Conclusion. The relevant $p$-range for $IS_p$ is $p \in [0, n]$.

We note also that for $p \in [0, n]$,

$$IS_p(B_2^n) = n|B_2^n| = as_p(B_2^n).$$ (3.1)
(ii) The case $OS_p(K)$.

If $p = n$, then for all $K$, $OS_n(K) = n|B^n_2|$. Similarly, to (i) above,\[OS_n(K) \geq \sup_{RB^2_n \in K^K} (as_n(RB^2_n)) = \sup_{RB^2_n \in K^K} (as_n(B^n_2)) = n|B^n_2|\]
and again by the equi-affine isoperimetric inequality,

$$OS_n(K) = \sup_{K' \in K^K} (as_n(K')) \leq \sup_{K' \in K^K} (as_n(B^n_2)) = n|B^n_2|.$$\[\text{If } 0 \leq p < n, \text{ then, } OS_p(K) = \infty. \text{ This holds as we can again take polytopes } P \text{ that contain } K.\]

If $-n < p < 0$, then $OS_p(K) = \infty$. This holds as we can again take polytopes $P$ that contain $K$.

If $-\infty \leq p < -n$, then for all $K$, $OS_p(K) = \infty$. Let $C_\varepsilon$ be a rounded cube centered at 0 containing $K$ and such that each vertex is rounded by replacing it by a Euclidean ball with radius $\varepsilon$. More specifically, $C_\varepsilon$ is the convex hull of the $2^n$ Euclidean balls

$$B^n_2(t \cdot \delta, \varepsilon) \quad \delta = (\delta_1, \ldots, \delta_n)$$

where $\delta_i = \pm 1$ for all $i = 1, \ldots, n$ and $t$ is sufficiently big so that the convex hull contains $K$. The boundary of $C_\varepsilon$ contains all the $2^n$-tants of the boundary of $B^n_2$. Therefore, in order to estimate $as_p(C_\varepsilon)$ from below it suffices to restrict the integration over the boundary of $C_\varepsilon$ to those $2^n$-tants of the boundary of $B^n_2$. The curvature there equals $\varepsilon^{-n+1}$, while

$$\langle x, N(x) \rangle \leq 2t \cdot \sqrt{n}.$$\[\text{Then,} \quad OS_p(K) \geq as_p(C_\varepsilon) \geq \frac{\varepsilon^{n(n-1)_{n-p}}(2t \sqrt{n})^{n-p}}{n^{n-p}}|B^n_2|,\]

which can be made arbitrarily large for $\varepsilon$ arbitrarily small.

**Conclusion.** The relevant $p$-range for $OS_p$ is $p \in [n, \infty]$. We note also that for $p \in [n, \infty]$,

$$OS_p(B^n_2) = n|B^n_2| = as_p(B^n_2). \quad (3.2)$$

(iii) The case $os_p(K)$.

If $p = 0$, then for all $K$,

$$os_0(K) = \inf_{K' \in K^K} (as_0(K')) = n \inf_{K' \in K^K} |K'| = n|K|.$$\[\text{If } 0 < p \leq \infty \text{ or if } -\infty < p < -n, \text{ then for all } K, os_p(K) = 0. \text{ Indeed, for polytopes } P \in \mathcal{K}^K, \text{ we have for those } p\text{-ranges}\]

$$isp_p(K) \leq \inf_{P \in \mathcal{K}^K} as_p(P) = 0.$$
Conclusion. The relevant $p$-range for $os_p$ is $p \in (-n, 0]$.

We note also that for $p \in (-n, 0]$,

\[ os_p(B_2^n) = n|B_2^n| = as_p(B_2^n). \]  (3.3)

(iv) The case $is_p(K)$.

We have that $is_p(K) = 0$ for all $p$ and for all $K$.

If $0 < p \leq \infty$ or if $-\infty \leq p < -n$ we get for polytopes $P \in K_K$,

\[ is_p(K) \leq \inf_{P \in K_K} as_p(P) = 0. \]

If $-n < p \leq 0$, then for all $K$,

\[ is_p(K) \leq \inf_{\epsilon B_2^n \in K_K} (as_p(\epsilon B_2^n)) = n|B_2^n| \inf_{\epsilon} \epsilon^n \frac{n}{p+n} = 0. \]

Conclusion. There is no interesting $p$-range for the inner minimal affine surface area $is_p$.

3.2 Continuity, monotonicity and isoperimetricity

It was proved by Lutwak [36] that for $p \geq 1$, $L_p$-affine surface area is an upper semicontinuous functional with respect to the Hausdorff metric. In fact, it follows from Lutwak’s proof that the same holds for all $0 \leq p < 1$ (aside from the case $p = 0$, which is just volume and hence continuous) and for all $-\infty \leq p < -n$. For $-n < p \leq 0$, the functional is lower semicontinuous.

**Proposition 3.1.** [36] Let $-\infty \leq p < -n$ or $0 \leq p \leq \infty$. Then the functional $K \mapsto as_p(K)$ is upper semicontinuous with respect to the Hausdorff metric on $\mathbb{R}^n$. For $-n < p \leq 0$, the functional is lower semicontinuous.

It is natural to ask about the continuity properties of inner and outer maximal, respectively minimal, affine surface areas in the $p$-ranges that are not already settled by the above considerations.

**Proposition 3.2.** Let the set of convex bodies in $\mathbb{R}^n$ be endowed with the Hausdorff metric.

For $0 \leq p \leq n$, the functional $K \mapsto IS_p(K)$ is continuous.

For $n \leq p \leq \infty$, the functional $K \mapsto OS_p(K)$ is continuous.

For $-n < p \leq 0$, the functional $K \mapsto os_p(K)$ is continuous.

The next proposition lists affine isoperimetric inequalities and monotonicity properties for the the functionals $IS_p$, $OS_p$ and $os_p$. 

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Proposition 3.3. Let $K$ be a convex body in $\mathbb{R}^n$.

(i) Let $0 \leq p \leq n$. Then $\frac{\text{IS}_p(K)}{|K|^{\frac{n}{n+p}}}$ holds trivially if $p = 0$ or $p = n$. Equality holds trivially if $p = 0$.

Let $n \leq p \leq \infty$. Then $\frac{\text{OS}_p(K)}{|K|^{\frac{n}{n+p}}}$ holds trivially if $p = n$.

(ii) $\frac{\text{OS}_p(K)}{|K|^{\frac{n}{n+p}}} - \frac{\text{IS}_p(K)}{|K|^{\frac{n}{n+p}}}$ is strictly increasing in $p \in (0,n]$.

$p \rightarrow \frac{\text{IS}_p(K)}{|n|^{\frac{n}{n+p}}}$ is strictly decreasing in $p \in [n, \infty)$.

$p \rightarrow \frac{\text{OS}_p(K)}{|n|^{\frac{n}{n+p}}}$ is strictly decreasing in $p \in (-n,0)$.

3.3 Asymptotic estimates

The next theorems provide estimates for the inner and outer extremal affine surface areas in the $p$-ranges that are not already settled above. There, $L_K$ is the isotropic constant of $K$.

Theorem 3.4. There is a constant $C > 0$ such that for all $n \in \mathbb{N}$, all $0 \leq p \leq n$ and all convex bodies $K \subseteq \mathbb{R}^n$,

$$\frac{1}{n^{5/6}} \left( \frac{C}{L_K} \right)^{\frac{2p}{n+p}} \frac{\text{IS}_p(B_2^n)}{|B_2^n|^{\frac{n}{n+p}}} \leq \frac{\text{IS}_p(K)}{|K|^{\frac{n}{n+p}}} \leq \frac{\text{IS}_p(B_2^n)}{|B_2^n|^{\frac{n}{n+p}}}. \quad (3.4)$$

Equality holds trivially in the right inequality if $p = 0, n$. If $p \neq 0, n$, equality holds in the right inequality iff $K$ is a centered ellipsoid.

By (3.1), $\frac{\text{IS}_p(B_2^n)}{|B_2^n|^{\frac{n}{n+p}}} = n|B_2^n|^{\frac{2p}{n+p}}$. Therefore, Theorem 3.4 states that

$$\frac{1}{n^{5/6}} \left( \frac{C}{L_K} \right)^{\frac{2p}{n+p}} \leq \frac{\text{IS}_p(K)}{n |B_2^n|^{\frac{2p}{n+p}} |K|^{\frac{n}{n+p}}} \leq 1.$$ 

Stirling’s formula yields that with absolute constants, $c_1, c_2$,

$$\frac{c_2^{\frac{2p}{n+p}}}{n^{\frac{n(p-1)-p}{n+p}}} \leq \frac{\text{IS}_p(B_2^n)}{|B_2^n|^{\frac{n}{n+p}}} = n|B_2^n|^{\frac{2p}{n+p}} \leq \frac{c_1^{\frac{2p}{n+p}}}{n^{\frac{n(p-1)-p}{n+p}}}.$$ 

Thus Theorem 3.4 shows that $\text{IS}_p(K)$ is proportional to a power of $|K|$. 
As noted, the upper bound is sharp when e.g., $K$ is $B_2^n$. However, in general we have $\text{IS}_p(K) > \text{as}_p(K)$. For example, for the $n$-dimensional cube $B_2^n$ centered at 0 with sidelength 2, $\text{as}_p(B_2^n) = 0$, but $B_2^n \subseteq B_\infty^n$ and so $\text{IS}_p(B_\infty^n) \geq \text{as}(B_2^n) > 0$.

**Theorem 3.5.** Let $K \subseteq \mathbb{R}^n$ be a convex body.

(i) Let $n \leq p \leq \infty$. Then

$$n^{n \frac{p-n}{p+1}} \frac{\text{OS}_p(B^n_2)}{|B^n_2|^\frac{n}{p+1}} \leq \frac{\text{OS}_p(K)}{|K|^\frac{n}{p+1}} \leq \frac{\text{OS}_p(B^n_2)}{|B^n_2|^\frac{n}{p+1}}. \quad (3.5)$$

Equality holds trivially in the right inequality if $p = n$. If $p \neq n$, equality holds in the right inequality iff $K$ is a centered ellipsoid.

(ii) Let $-n < p \leq 0$. Then

$$\frac{\text{os}_p(B^n_2)}{|B^n_2|^\frac{n-p}{p+1}} \leq \frac{\text{os}_p(K)}{|K|^\frac{n-p}{p+1}} \leq n^{n \frac{p-n}{p+1}} \frac{\text{os}_p(B^n_2)}{|B^n_2|^\frac{n-p}{p+1}}. \quad (3.6)$$

Equality holds trivially in the left inequality if $p = 0$. If $p \neq 0$, equality holds in the left inequality iff $K$ is a centered ellipsoid.

If $K$ is centrally symmetric, $n^{n \frac{p-n}{p+1}}$ can be replaced by $n^{n \frac{p-n}{2(p+1)}}$.

### 3.4 Relation to quermassintegrals

Finally, we turn to the relation of the extremal affine surface areas to quermassintegrals. While some of the (trivial) extremal affine surface areas are quermassintegrals, we will see that in general this is not the case.

Given a convex body $K \subseteq \mathbb{R}^n$ and $t \geq 0$, the Steiner formula (see, for example [46]) says that there exist non-negative numbers $W_0(K), \ldots, W_n(K)$, such that

$$|K + t B^n_2| = W_0(K) + \binom{n}{1} W_1(K) t + \binom{n}{2} W_2(K) t^2 + \cdots + W_n(K) t^n.$$  

The numbers $W_0(K), \ldots, W_n(K)$ are called the quermassintegrals. In particular, $W_0(K) = |K|$ and $W_n(K) = |B^n_2|$. Therefore, by section [5.1] $\text{IS}_0(K) = \text{os}_0(K) = n|K| = nW_0(K)$ and $\text{IS}_n(K) = \text{OS}_n(K) = n|B^n_2| = nW_n(K)$ are (multiples of) quermassintegrals. However, as shown in the next proposition, in general the extremal affine surface areas are not (multiples, or powers of) quermassintegrals.

We only treat the cases $\text{IS}_1$, $\text{os}_{-1}$ and $\text{OS}_{n^2}$. The other relevant $p$-cases are treated similarly.

**Proposition 3.6.** (i) If $\beta > 0$, then $\text{IS}_1^\beta$ and $\text{os}_{-1}$ are not equal to $W_i$, for any $0 \leq i \leq n$, and if $\beta < 0$, then $\text{OS}_{n^2}^\beta$ is not equal to $W_i$, for any $0 \leq i \leq n$.

(ii) The quantities $\text{IS}_1$, $\text{os}_{-1}$ and $\text{OS}_{n^2}$ are not a linear combination of quermassintegrals. In particular, those quantities are not valuations.
Remark 3.1. From [48] it is known that affine surface area is a valuation, that is, for every \( K, L \subseteq \mathbb{R}^n \) convex,

\[
as_1(K \cap L) + as_1(K \cup L) = as_1(K) + as_1(L).
\]

It is also known by Hadwiger’s characterization theorem [24], that every continuous rigid motion invariant valuation on the set of convex bodies is a linear combination of quermassintegrals. Thus, Proposition 3.6 (ii) shows in particular that \( IS_1, os_{-1} \) and \( OS_{n,2} \) are not valuations.

4 Proofs

Proof of Proposition 3.2

By section (3.1) (i), \( IS_0(K) = n|K| \) is just volume, which is continuous and \( IS_0(K) = n|B_n^2| \), which is constant and hence continuous. Thus for \( IS_p(K) \) we only need to consider \( p \in (0, n) \). Recall that we assume always that 0 is the center of gravity of \( K \), that is,

\[
\int_K x \, dx = 0.
\]

Hence, there exists \( \rho > 0 \) such that \( \rho B_n^2 \subseteq K \). Let \( \{K_l\}_{l=1}^\infty \) be a sequence of convex bodies, all having center of gravity at the origin, that converges to \( K \) in the Hausdorff metric. That is, for every \( \varepsilon > 0 \), there exists \( l_0 \in \mathbb{N} \) such that for all \( l \geq l_0 \),

\[
K_l \subseteq K + \varepsilon B_n^2 \quad \text{and} \quad K \subseteq K_l + \varepsilon B_n^2.
\]

If \( \varepsilon > 0 \) is sufficiently small, then we can assume that for all \( l \geq l_0 \), \( \varepsilon B_n^2 \subseteq K_l \). Thus, for all \( l \geq l_0 \),

\[
K_l \subseteq K + \varepsilon B_n^2 \subseteq K + \frac{\varepsilon}{\rho} K = \left(1 + \frac{\varepsilon}{\rho}\right) K,
\]
and

\[
K \subseteq K_l + \varepsilon B_n^2 \subseteq K_l + \frac{10\varepsilon}{\rho} K_l = \left(1 + \frac{10\varepsilon}{\rho}\right) K_l.
\]

Hence,

\[
\left(1 + \frac{\varepsilon}{\rho}\right)^n IS_p(K) \overset{\text{def}}{=} IS_p\left(\left(1 + \frac{\varepsilon}{\rho}\right) K\right) \overset{\text{4.1}}{\geq} IS_p(K_l),
\]
and

\[
\left(1 + \frac{10\varepsilon}{\rho}\right)^n IS_p(K_l) \overset{\text{def}}{=} IS_p\left(\left(1 + \frac{10\varepsilon}{\rho}\right) K_l\right) \overset{\text{4.2}}{\geq} IS_p(K).
\]

Altogether, for all \( l \geq l_0 \),

\[
\left(1 + \frac{\varepsilon}{\rho}\right)^{-n} IS_p(K_l) \leq IS_p(K) \leq \left(1 + \frac{10\varepsilon}{\rho}\right)^{-n} IS_p(K_l).
\]

Since \( \varepsilon > 0 \) is arbitrary, the result follows.

Continuity for outer maximal affine surface area \( OS_p \) and outer minimal surface area \( OS_p \) is treated similarly.
For the proof of Proposition 3.3 and Theorem 3.4 we use the $L^p$-affine isoperimetric inequalities which were proved by Lutwak [36] for $p > 1$ and for all other $p$ by Werner and Ye [58]. The case $p = 1$ is the classical case.

For $p > 0$,

$$\frac{a_p(K)}{a_p(B^n_2)} \leq \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n-p}{n+p}}, \quad (4.3)$$

and for $-n < p < 0$,

$$\frac{a_p(K)}{a_p(B^n_2)} \geq \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n+p}{n-p}}, \quad (4.4)$$

Equality holds in both inequalities if $K$ is an ellipsoid. Equality holds trivially in both inequalities if $p = 0$.

**Proof of Proposition 3.3** (i) When $0 < p \leq n$ and $K' \subseteq K$, we use (4.3) and (3.1),

$$\text{IS}_p(K) = \sup_{K' \in K} (a_p(K')) \leq \sup_{K' \in K} a_p(B^n_2) \left( \frac{|K'|}{|B^n_2|} \right)^{\frac{n-p}{n+p}} \leq a_p(B^n_2) \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n-p}{n+p}},$$

and for $-n < p < 0$,

$$\text{OS}_p(K) = \sup_{K' \in K} (a_p(K')) \geq \inf_{K' \in K} a_p(B^n_2) \left( \frac{|K'|}{|B^n_2|} \right)^{\frac{n-p}{n+p}} \geq a_p(B^n_2) \left( \frac{|K|}{|B^n_2|} \right)^{\frac{n-p}{n+p}}.$$
increasing in $p \in (0, \infty)$. Therefore we get for $0 < p < q \leq n,$
\[
\left( \frac{\text{IS}_p(K)}{|n|K^n} \right)^{\frac{n+p}{p}} = \sup_{K' \in K_n} \left( \frac{\text{as}_p(K')}{|n|K^n} \right)^{\frac{n+p}{p}} \leq \frac{(n|K|)^{\frac{p}{n+p}}}{(n|K|)^{\frac{n}{p}}} \left( \sup_{K' \in K_n} \text{as}_q(K') \right)^{\frac{n+p}{q}} = \left( \frac{\text{IS}_q(K)}{|n|K^n} \right)^{\frac{n+p}{q}}.
\]
It was also shown in [58] (see also [43]) that the function $p \to \left( \frac{\text{as}_p(K)}{|n|K^n} \right)^{n+p}$ is strictly decreasing in $p \in (0, \infty)$ Therefore we get for $n < p < q < \infty,$
\[
\left( \frac{\text{OS}_p(K)}{|n|K^n} \right)^{n+p} = \sup_{K' \in K_n} \left( \frac{\text{as}_p(K')}{|n|K^n} \right)^{n+p} \geq \left( \frac{\text{OS}_q(K)}{|n|K^n} \right)^{n+p}
\]
and for $-n \leq p < q \leq 0,$
\[
\left( \frac{\text{os}_p(K)}{|n|K^n} \right)^{n+p} = \inf_{K' \in K_n} \left( \frac{\text{as}_p(K')}{|n|K^n} \right)^{n+p} \geq \left( \frac{\text{os}_q(K)}{|n|K^n} \right)^{n+p}
\]

In part of the proof below it is most convenient to work with a body which is in isotropic position. A body $K \subseteq \mathbb{R}^n$ is said to be in isotropic position if $|K| = 1$ and there exists $L_K > 0$ such that for all $\theta \in S^{n-1},$
\[
\int_K \langle x, \theta \rangle dx = 0, \quad \int_K \langle x, \theta \rangle^2 dx = L_K^2.
\]
Here and in what follows, $S^{n-1}$ denotes the unit Euclidean sphere in $\mathbb{R}^n$. It is known that for every convex body $K \subseteq \mathbb{R}^n$, there exists $T: \mathbb{R}^n \to \mathbb{R}^n$ affine and invertible such that $TK$ is isotropic. See for example [11] for this and other facts on isotropic position used here.

**Proof of Theorem** [22] The upper bound, together with the equality characterizations, follows immediately from Proposition [33] (i).

Now we turn to the lower bound in the case (i). As noted above, $IS_p(TK) = \det(T)^{\frac{n}{n+p}} IS_p(K)$ for any invertible linear map $T$. Therefore, to prove the lower bound for $0 < p < \infty$, it is sufficient to consider $K$ in isotropic position. Let $L_K$ be the isotropic constant of $K$. By the thin shell estimate of O. Guéron and E. Milman [22] (see also [15, 42]), we have with universal constants $c$ and $C$, that for all $t \geq 0,$
\[
|K \cap \{ x \in \mathbb{R}^n : ||x|| - L_K \sqrt{n} < tL_K \sqrt{n} \}| > 1 - C \exp(-cn^{1/2}\min(v^3, t)).
\]
Taking \( t = O(n^{-1/6}) \), there is a new universal constant \( c > 0 \) such that for all \( n \in \mathbb{N} \),
\[
\left| K \cap \left\{ x \in \mathbb{R}^n : \|x\| - L_K \sqrt{n} < c L_K n^{1/3} \right\} \right| \geq \frac{1}{2} \tag{4.5}
\]
This set consists of all \( \|x\| \in K \) for which
\[
L_K \left( n^{1/2} - cn^{1/3} \right) < \|x\| < L_K \left( n^{1/2} + cn^{1/3} \right).
\]
We consider those \( n \in \mathbb{N} \) for which \( n^{1/6} > c \).
We will truncate the above set. For \( i = 0, 1, 2, \ldots, k_n = \left[ n \log_2 \frac{n^{1/2} + cn^{1/3}}{n^{1/2} - cn^{1/3}} \right] \), consider the sets
\[
L_i := K \cap \left\{ x \in \mathbb{R}^n : 2^{i/n} \left( L_K (n^{1/2} - cn^{1/3}) \right) < \|x\| \leq 2^{(i+1)/n} \left( L_K (n^{1/2} - cn^{1/3}) \right) \right\}.
\]
Then
\[
2^{i/n} \leq 2^{\log_2 \frac{n^{1/2} + cn^{1/3}}{n^{1/2} - cn^{1/3}}} = \frac{n^{1/2} + cn^{1/3}}{n^{1/2} - cn^{1/3}}
\]
and thus
\[
K \cap \left\{ x \in \mathbb{R}^n : \|x\| - L_K \sqrt{n} < c L_K n^{1/3} \right\} \subset \bigcup_{i=0}^{k_n} L_i. \tag{4.6}
\]
Moreover, with a new absolute constant \( C_0 \),
\[
k_n \leq n \log_2 \frac{n^{1/2} + cn^{1/3}}{n^{1/2} - cn^{1/3}} = n \log_2 \frac{1 + cn^{-1/6}}{1 - cn^{-1/6}} \leq C_0 n^{5/6}.
\]
By (4.5) and (4.6), there exists \( i_0 \in \{1, 2, \ldots, \lfloor C_0 n^{5/6} \rfloor \} \) such that
\[
|L_{i_0}| \geq \frac{1}{2 \lfloor C_0 n^{5/6} \rfloor}. \tag{4.7}
\]
We set \( R = 2^{i_0/n} \left( L_K (n^{1/2} - cn^{1/3}) \right) \). In particular, we have
\[
L_{i_0} = K \cap \left\{ x \in \mathbb{R}^n : R < \|x\| \leq 2^{1/n} R \right\}.
\]
Let
\[
O = \left\{ \theta \in S^{n-1} : \rho_K (\theta) > R \right\}, \text{ and } S_O = \{ r \theta : \theta \in O \text{ and } r \in [0, 1] \} \subset K,
\]
where \( \rho_K (\theta) = \max \{ r \geq 0 : r \theta \in K \} \) is the radial function of \( K \).
Now we claim that
\[
L_{i_0} \subset 2^{1/n} S_O. \tag{4.8}
\]
Indeed, let \( y \in L_{i_0} \). We express \( y = r \theta \) in polar coordinates. By definition, we have \( R < r < 2^{1/n} R \) and \( r \theta \in K \). Thus, \( \rho_K (\theta) \geq r > R \) and hence \( \theta \in O \). Therefore, \( r \theta \in 2^{1/n} S_O \) because \( r \in [0, 2^{1/n} R] \). By (4.7) and (4.8) we conclude that
\[
|S_O| \geq 2^{-1/n} \frac{1}{n} |L_{i_0}| \geq \frac{1}{4 \lfloor C_0 n^{5/6} \rfloor}. \tag{4.9}
\]
Now, we consider $a_s (K \cap RB_2^n)$. For $\theta \in O$, $R\theta$ is a boundary point of $K \cap RB_2^n$. Thus,

$$a_s (K \cap RB_2^n) \geq \int_{RO} \frac{K \cap \beta}{(n-1) \beta x n} d\mu (x) = \int_{RO} \frac{R^{-\frac{1}{n^2}}}{\beta x} d\mu (x)$$

$$= \mu (RO) \left( \frac{1}{R} \right)^{\frac{n}{n^2} - 1} = \mu (RO) \left( \frac{1}{R} \right)^{\frac{n}{n^2} + 1},$$

where $\mu$ is the surface area measure of $RS^{n-1}$. We can compare surface area and volume,

$$\frac{\mu (RO) \cdot R}{n} = |SO|.$$

Hence,

$$a_s (K \cap RB_2^n) \geq \left( \frac{1}{R} \right)^{\frac{n}{n^2} - 1} n |SO| = \left( \frac{1}{R} \right)^{\frac{n}{n^2} + 1} n |SO|$$

$$\geq \left( \frac{1}{R} \right)^{\frac{n}{n^2} + 1} \frac{n}{4 |C_0 n^{\frac{3}{4}}|}.$$

Since $R \leq 2\sqrt{n} L K$, this finishes the proof for the lower bound.

**Proof of Theorem 3.3** The upper bound of (i) and the lower bound of (ii), together with the equality characterizations, follow immediately from Proposition 3.3 (i).

For the other estimates, we will assume without loss of generality that $K$ is in L"owner position, i.e., L"owner ellipsoid $L (K)$, which is the ellipsoid of minimal volume containing $K$, is the Euclidean ball $\frac{|L (K)|}{|B_2^n|} B_2^n$. We also have that

$$K \subset L (K) \subset n K,$$

and that for a 0-symmetric convex body $K$,

$$K \subset L (K) \subset \sqrt{n} K.$$

(i) We get with (2.4), (3.2) and (4.10),

$$OS_p (K) \geq a_s (L (K)) = \left( \frac{|L (K)|}{|B_2^n|} \right)^{\frac{n}{n^2} + 1} n |B_2^n| \geq n^{\frac{n}{n^2} + 1} \left( \frac{|K|}{|B_2^n|} \right)^{\frac{n}{n^2} + 1} OS_p (B_2^n),$$

which finishes the lower estimate of (i).

(ii) Similarly, now using (2.5), (3.2) and (4.11),

$$os_p (K) \leq a_s (L (K)) = \left( \frac{|L (K)|}{|B_2^n|} \right)^{\frac{n}{n^2} + 1} n |B_2^n| \leq n^{\frac{n}{n^2} + 1} \left( \frac{|K|}{|B_2^n|} \right)^{\frac{n}{n^2} + 1} os_p (B_2^n).$$

In the 0 -symmetric case we use (4.11) to get the estimate with $n^{\frac{n}{n^2} + 1}$ instead of $n^{\frac{n}{n^2} + 1}$.

\qed
Proof of Proposition 3.6. We only give the proofs for \( IS_1 \). The proofs for \( os_{-1} \) and \( OS_{n^2} \) are the same with the obvious modifications.

(i) To prove the first assertion, note that by (2.4), \( IS_1^\beta \) is homogeneous of degree \( \frac{\beta n(n-1)}{n+1} \). Also, it is known that \( W_i \) is homogeneous of degree \( n - i \). Hence, if \( IS_1^\beta = W_i \) for some \( i \), then \( \frac{\beta n(n-1)}{n+1} \in \mathbb{N} \) and in particular \( \beta \in \mathbb{Q} \). On the other hand, it is known that \( W_i(B_2) = |B_2^n| \). Thus, we must also have

\[
|B_2^n| = IS_1^\beta(B_2^n) \overset{(\ast)}{=} n^\beta |B_2^n|,
\]

where in (\ast) we used (3.1). Therefore, we have \( |B_2^n|^{1 - \beta} \in \mathbb{N} \). Now, it is known that

\[
|B_2^n| = \frac{\pi^\frac{n}{2}}{\Gamma \left( \frac{n}{2} + 1 \right)} \begin{cases} 
\frac{\pi^\frac{n}{2}}{(n/2)!} \frac{n^{-1}}{2^{(n-1)/2}(4\pi)^{n/2}} & 2 \mid n, \\
\frac{\pi^\frac{n}{2}}{(n/2)!} \frac{n^{-1}}{2^{(n-1)/2}(4\pi)^{n/2}} & 2 \nmid n.
\end{cases}
\]

In other words, for every \( n \in \mathbb{N} \), we have \( |B_2^n| = Q_n \pi^\frac{n}{2} \) or \( |B_2^n| = Q_n \pi^\frac{n}{2}^{-1} \), where \( Q_n \in \mathbb{Q} \). Therefore, if \( |B_2^n|^{1 - \beta} \in \mathbb{N} \) with \( \beta \in \mathbb{Q} \), that would imply that \( \pi \) is an algebraic number, which is not the case. This proves the first assertion.

(ii) Suppose that \( IS_1 \) is a linear combination of quermassintegrals. Then, for \( K \) given, there exist \( \lambda_i \), \( 0 \leq i \leq n \), not all of them equal to 0, such that \( IS_1(K) = \sum_{i=0}^n \lambda_i W_i(K) \). The respective homogeneity properties then imply that for all \( \alpha \in \mathbb{R} \),

\[
\alpha \frac{n-1}{n+1} IS_1(K) = \sum_{i=0}^n \lambda_i \alpha^{n-i} W_i(K),
\]

and in particular, for \( K = B_2^n \), that for all \( \alpha \in \mathbb{R} \),

\[
n \alpha \frac{n-1}{n+1} = \sum_{i=0}^n \lambda_i \alpha^{n-i} = \lambda_0 \alpha^n + \lambda_1 \alpha^{n-1} + \cdots + \lambda_n. \tag{4.12}
\]

Letting \( \alpha = 0 \) in (4.12) shows that \( \lambda_n = 0 \). This means that for all \( \alpha \in \mathbb{R} \),

\[
n \alpha \frac{n-1}{n+1} = \sum_{i=0}^{n-1} \lambda_i \alpha^{n-i} = \lambda_0 \alpha^n + \lambda_1 \alpha^{n-1} + \cdots + \lambda_{n-1} \alpha.
\]

Differentiation gives

\[
n \left( n \frac{n-1}{n+1} \right) \alpha^{\frac{n-1}{n+1}-1} = n \lambda_0 \alpha^{n-1} + (n-1) \lambda_1 \alpha^{n-2} + \cdots + \lambda_{n-1}. \tag{4.13}
\]

Letting \( \alpha = 0 \) in (4.13) shows that \( \lambda_{n-1} = 0 \). We continue differentiating till the largest \( k \in \mathbb{N} \) for which the exponent \( \frac{n-1}{n+1} - k \) of \( \alpha \) on the left hand side of the equality is strictly larger than 0. We can take \( k = n - 2 \) and get that \( \lambda_n = \lambda_{n-1} = \cdots = \lambda_2 = 0 \). Thus equality (4.12) reduces to the following: there exist \( \lambda_0 \) and \( \lambda_1 \) such that for all \( \alpha \in \mathbb{R} \),

\[
\frac{n}{\alpha^{\frac{1}{n+1}}} = \lambda_0 \alpha + \lambda_1,
\]

which is not possible. The proof is therefore complete. \( \square \)
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