RINGS THAT ARE MORITA EQUIVALENT TO THEIR OPPOSITES

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Abstract. We consider the following problem suggested by D. Saltman: Under what assumptions do one or more of the following are equivalent for a ring $R$: (A) $R$ is Morita equivalent to a ring with involution, (B) $R$ is Morita equivalent to a ring with an anti-automorphism, (C) $R$ is Morita equivalent to its opposite ring. We show that (C)$\Rightarrow$(B) provided $R$ is semilocal or $Q$-finite as well as other results of similar flavor. In contrast to that, we demonstrate that (B)$\not\Rightarrow$(A), even where $R$ is a finite dimensional algebra over a field. Our methods give a new perspective on the Knus-Parimala-Srinivas proof of a theorem of D. Saltman, which state precisely when an Azumaya $C$-algebra $A$ is Brauer equivalent to an Azumaya $C$-algebra $B$ with an involution whose restriction to $C$ is a prescribed Galois automorphism $\sigma \in \text{Aut}(C)$. Our proofs use the recently introduced general bilinear forms to construct involutions and anti-automorphisms.

1. Overview

Unless specified otherwise, all rings are assumed to have a unity and ring homomorphisms are required to preserve it. Given a ring $R$, denote its set of invertible elements by $R^\times$ and its center by $\text{Cent}(R)$. The $n \times n$ matrices over $R$ are denoted by $M_n(R)$. The category of right $R$-modules is denoted by $\text{Mod-}R$ and the category of f.g. projective right $R$-modules is denoted by $\text{proj-}R$. For a subset $X \subseteq R$, we let $\text{Cent}_R(X)$ denote the centralizer of $X$ in $R$. If a module $M$ can be viewed as a module over several rings, we use $M_R$ (resp. $RM$) to denote “$M$, considered as a right (resp. left) $R$-module”. Endomorphisms of left (right) modules are applied on the right (left). Throughout, a semisimple ring means a semisimple artinian ring.

In this paper, we consider the following problem, suggested to us by David Saltman (to whom we are grateful): Let $R$ be a ring. Under what assumptions do all or some of the following conditions are equivalent:

(A) $R$ is Morita equivalent to a ring with involution,
(B) $R$ is Morita equivalent to a ring with an anti-automorphism,
(C) $R$ is Morita equivalent to $R^{\text{op}}$.

(Actually, we consider a slight refinement that takes into account the type of the involution/anti-automorphism/Morita equivalence; see section 3.) Note that obviously (A)$\Rightarrow$(B)$\Rightarrow$(C), so one is interested in showing (B)$\Rightarrow$(A) or (C)$\Rightarrow$(B).

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The motivation for the question comes from Azumaya algebras. Let $C$ be a commutative ring and let $A$ be an Azumaya $C$-algebra (we recall all the definitions in section 4). It was shown by Saltman in [14] Th. 3.1 that:

(i) $A$ is Brauer equivalent to an Azumaya algebra $B$ with an involution of the first kind $\iff A$ is Brauer equivalent to $A^{op}$.

(ii) If $C/C_0$ is a Galois extension with Galois group $G = \{1, \sigma\}$ ($\sigma \neq 1$), then $A$ is Brauer equivalent to an Azumaya algebra $B$ with an involution whose restriction to $C$ is $\sigma \iff$ the corestriction algebra $\text{Cor}_{C/C_0}(A) = (A \otimes A^\sigma)^G$ is split (i.e. trivial in the Brauer group of $C_0$).

In case $C$ is semilocal and connected, Saltman also showed that we can take $B = A$ in (i) and (ii). (The case where $C$ is a field is an earlier result of Albert, e.g. see [1] Ths. 10.19 & 10.22.) Two Azumaya algebras are Brauer equivalent if and only if they are Morita equivalent as $C$-algebras ([2] Cr. 17.2), so (i) can be understood as: $(C) \implies (A)$ for Azumaya algebras, provided the Morita equivalence is “of the first kind”.

A simpler proof of Saltman’s Theorem was later found by Knus, Parimala and Srinivas ([9] §4).

In this paper, we use general bilinear forms, introduced in [6], to give partial answer to Saltman’s problem. More precisely, we show that the conditions (A) and (B) above can be phrased in terms of existence of certain bilinear forms, and use this observation to give some positive results, Saltman’s Theorem in particular.

We show that $(C) \implies (B)$ when $R$ is semilocal or when $\dim_\mathbb{Q}(R \otimes_\mathbb{Z} \mathbb{Q})$ and $|\ker(R \to R \otimes_\mathbb{Z} \mathbb{Q})|$ are finite (i.e. when $R$ is $\mathbb{Q}$-finite). When $R$ is semi-perfect (e.g. artinian), these results can be sharpened even further: If $R$ is Morita equivalent to $R^{op}$, then $S$, the basic ring that is Morita equivalent to $R$, has an anti-automorphism. In addition, if $R$ has an involution, then so does $M_2(S)$, and provided $S$ is local with $2 \in S^\times$ or a division ring, $S$ itself has an involution. In the special case $R = M_n(D)$ with $D$ a division ring, this means that $R = M_n(D)$ has an involution if and only if $S = D$ has an involution, a result obtained by Albert (e.g. [1] Th. 10.12) when $[D : \text{Cent}(D)] < \infty$ and by Herstein (e.g. [7] Th. 1.2.2) in the general case.

We continue by describing the proof of Saltman’s Theorem by Knus, Parimala and Srinivas ([9] §4) from the perspective of our methods. Namely, we recover this proof as an application of our characterization of (A) in terms of general bilinear forms. This results in a very clean proof of Saltman’s Theorem, which suppress some of the computations of [9] §4. In particular, the proof of (i) above becomes just a few lines. In addition, we note that the proof of [9] implies that the algebra $B$ can be chosen such that $\text{rank}(B) \leq 4 \text{rank}(A)$ (in contrast to Saltman’s original proof), a fact which could be beneficial for computational aspects. Finally, we give a general explanation for why one can take $B = A$ when $C$ is connected semilocal.

We finish the paper with counterexamples, amongst are examples demonstrating that $(B) \nRightarrow (A)$ even when $R$ is a finite dimensional algebra over a field, and even when it has an anti-automorphism fixing the center. Several open questions are posed at the end.

Section 2 recalls the basics of Morita theory. In section 3, we give a refinement of “Saltman’s Problem”. Section 4 recalls some facts about scalar extensions, and section 5 recalls general bilinear forms. In section 6, we give a criterion in terms of bilinear forms to when a ring is Morita equivalent to a ring with an involution (resp. anti-automorphism). This criterion is the core of this paper and it is exploited several times later. In sections 7 and 8, we show that $(C) \implies (B)$ under certain finiteness assumptions, as well as other results of the same flavor. Section 9 recalls some facts about Azumaya algebras, and in section 10 we show how our methods...
can reproduce the proof of Saltman’s Theorem given in [11]. In section 11 we show that when the base ring is semilocal and connected, we can take \( B = A \) in (i) and (ii) above. Finally, section 12 presents counterexamples, including an example demonstrating (B) \( \not\Rightarrow \) (A), and some open questions.

2. Morita Theory

In this section, we recall some facts about Morita Theory. See [11, §18] or [13, §4.1] for proofs and further details.

Let \( R \) be a ring. A right \( R \)-module \( M \) is called a *generator* if every right \( R \)-module is an epimorphic image of \( \bigoplus_{i \in I} M \) for \( I \) sufficiently large, or equivalently, if \( R_R \) be a summand of \( M^n \) for some \( n \in \mathbb{N} \). The module \( M \) is a *progenerator* if \( M \) is a generator, finitely generated and projective. In this case, we also call \( M \) a (right) \( R \)-progenerator.

Let \( S \) be another ring. An \((S, R)\)-*progenerator* is an \((S, R)\)-bimodule \( P \) such that \( P_R \) is a progenerator and \( S = \text{End}(P_R) \) (i.e. every endomorphism of \( P_R \) is of the form \( p \mapsto sp \) for unique \( s \in S \)). In this case, \( sP \) is also a progenerator and \( R = \text{End}(sP) \).

The rings \( R, S \) are said to be *Morita equivalent*, denoted \( R \sim_{\text{Mor}} S \), if the categories \( \text{Mod-}R \) and \( \text{Mod-}S \) are equivalent. Morita’s Theorems assert that:

1. Every equivalence \[ F : \text{Mod-}S \to \text{Mod-}R \] admits an \((S, R)\)-progenerator \( P \) such that \( FM \) is *naturally* isomorphic to \( M \otimes_S P \) for all \( M \in \text{Mod-}S \).
2. Conversely, for any \((S, R)\)-progenerator \( P \) the functor \((-) \otimes_S P : \text{Mod-}S \to \text{Mod-}R \) is an equivalence of categories.
3. There is a one-to-one correspondence between equivalences of categories \( F : \text{Mod-}S \to \text{Mod-}R \) (considered up to *natural isomorphism*) and isomorphism classes of \((S, R)\)-progenerators. The correspondence maps the composition of two equivalences to the tensor product of the corresponding progenerators.

Every \((S, R)\)-progenerator \( P \) induces an isomorphism \( \sigma_P : \text{Cent}(R) \to \text{Cent}(S) \) given by \( \sigma_P(r) = s \) where \( s \) is the unique element of \( \text{Cent}(S) \) satisfying \( sp = pr \) for all \( p \in P \). As \( \sigma_P \) depends only on the isomorphism class of \( P \), it follows that any equivalence of categories \( F : \text{Mod-}S \to \text{Mod-}R \) induces an isomorphism \( \sigma_F : \text{Cent}(R) \to \text{Cent}(S) \).

Let \( C \) be a commutative ring and assume \( R \) and \( S \) are \( C \)-algebras. We say that \( R \) and \( S \) are Morita equivalent as \( C \)-algebras or *over \( C \)*, denoted \( R \sim_{\text{Mor}/C} S \), if there exists an equivalence \( F : \text{Mod-}S \to \text{Mod-}R \) such that \( \sigma_F(c \cdot 1_R) = c \cdot 1_S \) for all \( c \in C \). Equivalently, this means that there exists an \((S, R)\)-progenerator \( P \) such that \( cp = pc \) for all \( p \in P \) and \( c \in C \).

If \( S \) is an arbitrary ring that is Morita equivalent to \( R \) and \( F : \text{Mod-}S \to \text{Mod-}R \) is any equivalence, then we can make \( S \) into a \( C \)-algebra by letting \( C \) act on \( S \) via \( \sigma_F \). In this setting, we have \( S \sim_{\text{Mor}/C} R \).

3. Types

To ease phrasing of results in the next sections, we now introduce *types*.

Let \( C \) be a commutative ring and let \( R \) and \( S \) be central \( R \)-algebras. (The algebra \( R \) is central if the map \( c \mapsto c \cdot 1_R : C \to \text{Cent}(R) \) is an isomorphism.) Every equivalence of categories \( F : \text{Mod-}S \to \text{Mod-}R \) gives rise to an isomorphism

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1 According to textbooks, an equivalence between two categories \( \mathcal{A} \) and \( \mathcal{B} \) consists of a quartet \((F, G, \delta, \varepsilon)\) such that \( F : \mathcal{A} \to \mathcal{B} \) and \( G : \mathcal{B} \to \mathcal{A} \) are functors and \( \delta : \text{id}_{\mathcal{A}} \to GF \) and \( \varepsilon : \text{id}_{\mathcal{B}} \to FG \) are natural isomorphisms. We do not need this detailed description here and hence we only specify \( F \). In this case, the implicit functor \( G \) is determined up to natural isomorphism.
σ_F : Cent(R) → Cent(S). As both Cent(R) and Cent(S) are isomorphic to C, we can realize σ_F as an automorphism of C, which we call the type of F. (For example, when F is of type id_C, R is Morita equivalent to S as C-algebras.) Likewise, the type of an (S,R)-progenerator P is the type of the equivalence induced by P. Namely, it is the unique automorphism σ of C satisfying σ(c)p = pc for all p ∈ P, c ∈ C.

Let α be an anti-automorphism of R. The type of α is defined to be its restriction to C = Cent(R). For example, an involution of R is of the first kind (i.e. it fixes Cent(R)) if and only if its type is id_C.

We now make an essential sharpening of Saltman’s problem. Let R be a ring, let C = Cent(R) and let σ ∈ Aut(C). We look for sufficient conditions ensuring that some or all of the following are equivalent:

(A) R is Morita equivalent over C to a (necessarily central) C-algebra with involution of type σ.

(B) R is Morita equivalent over C to a (necessarily central) C-algebra with an anti-automorphism of type σ.

(C) R is Morita equivalent to R^op via equivalence of type σ (R^op is considered as a C-algebra in the obvious way).

Again, (A)⇒(B)⇒(C) so we want to show that (B)⇒(A) or (C)⇒(B). Saltman’s Theorem ([14, Th. 3.1]) for involutions of the first kind can now be phrased as (C)⇒(A) when R/C is Azumaya and σ = id_C.

4. Progenerators and Scalar Extension

In this section we recall several facts about the behavior of progenerators with respect to scalar extension. Throughout, C is a commutative ring and R is a C-algebra. If σ is an automorphism of C, then R^σ denotes the C-algebra obtained from R by letting C act via σ. Observe that for all M, N ∈ Mod-R, Hom_R(M,N) admits a (right) C-module structure given by (fc)m = (fm)c (f ∈ Hom_R(M,N), c ∈ C, m ∈ M).

Proposition 4.1. Let S be a C-algebra, let RS := R ⊗_C S, and set XS := X ⊗_C S for all X ∈ Mod-R. Then XS is a right RS-module via (x ⊗ s)(r ⊗ s') := xτ ⊗ ss' and for all X, Y ∈ Mod-R, there is a natural homomorphism

\[ \text{Hom}_R(X, Y) ⊗_C S \rightarrow \text{Hom}_{RS}(XS, Y_S) \]

\[ f ⊗ s \mapsto [x ⊗ s' \mapsto (fx) ⊗ (ss')] \]

(s, s' ∈ S, x ∈ X, r ∈ R). This homomorphism is an isomorphism when X_R is f.g. projective.

Proof. It is easy to see that the map in the proposition is an isomorphism in case X = R_R. Since the map is natural and additive in X (in the functorial sense), it is an isomorphism whenever X is a summand of R_R ⊗ · · · ⊗ R_R, i.e. when X is f.g. projective.

Proposition 4.2. Let S and D be C-algebras and let P be an (S,R)-progenerator of type σ ∈ Aut(Cent(R)). Put RD := R ⊗_C D, S^p = (S^p)^{op} ⊗_C D and PD = P ⊗_C D, and endow P with an (S^p,D)-bimodule structure by setting (s ⊗ d)(p ⊗ d') := (spr) ⊗ (dd'^p). Then PD is an (S^p,D)-progenerator.\(^2\)

\(^2\) To see that (B)⇒(C), let S be a central C-algebra admitting an automorphism α of type σ, and assume there is an (S,R)-progenerator P of type id_C (i.e. $S \sim_{\text{Mor}/C} R$). Let P^r be the (R^{op},S)-bimodule obtained from P by setting r^{op} · s = s^α · r. Then P^r is an (R^{op},S)-progenerator, hence P^r ⊗_S P is an (R^{op},R)-progenerator, and the latter is easily seen to have type σ.
Proof. Since $P$ has type $\sigma$, $\text{End}(P_R) \cong S^\sigma$ as $C$-algebras. By Proposition 4.1, $\text{End}_{R_D}(P_D) = \text{Hom}_{R_D}(P_D, P_D) \cong \text{Hom}_R(P, P) \otimes_C D = S^\sigma \otimes_C D = S^\sigma_D$. It is routine to verify that the action of $S^\sigma_D$ on $P_D$ via endomorphisms is the action specified in the proposition. Since $P_R$ is a progenerator, so is $(P_D)_{R_D}$, hence $P_D$ is an $(S^\sigma_D, R_D)$-progenerator.

Proposition 4.3. Let $K$ be an $(S, R)$-progenerator and let $N$ and $M$ be the prime radicals (resp. Jacobson radicals) of $R$ and $S$, respectively. View $\mathcal{P} := P/PN$ as a right $R$ := $R/N$-module. Then $PN = MP$, hence $\mathcal{P} = P/MP$ admits a left $S$-module structure. Furthermore, $\mathcal{P}$ is an $(S, R)$-progenerator.

Proof. By [11 Pr. 18.44], there is an isomorphism between the lattice of $R$-ideals and the lattice of $(S, R)$-submodules of $P$ given by $I \mapsto PI$. Similarly, the ideals of $S$ correspond to $(S, R)$-submodules of $P$ via $J \mapsto JP$, hence every ideal $I \subset R$ admits a unique ideal $J \subset S$ such that $JP = PI$. The ideal $J$ can also be described as $\text{Hom}_R(P, PI)$. This description implies that $S/J = \text{Hom}_R(P, P)/\text{Hom}(P, PI) \cong \text{End}_{R}(P/P/PI) \cong \text{End}_{R}(P/P/PI)$, thus $P/PI$ is an $(S/J, R/I)$-progenerator. Choose $I = N$. Then by [11 Cr. 18.45] (resp. [11 Cr. 18.50]) $J = M$, so we are done.

5. General Bilinear Forms

General bilinear forms were introduced in [3]. In this section, we recall their basics and record several facts to be needed later. When not specified, proofs can be found at [3] [2]. Throughout, $R$ is a (possibly non-commutative) ring.

Definition 5.1. A (right) double $R$-module is an additive group $K$ together with two operations $\odot_0, \odot_1 : K \times R \to K$ such that $K$ is a right $R$-module with respect to each of $\odot_0, \odot_1$ and

$$(k \odot_0 a) \odot_1 b = (k \odot_1 b) \odot_0 a \quad \forall k \in K, a, b \in R.$$ 

We let $K_i$ denote the $R$-module obtained by letting $R$ act on $K$ via $\odot_i$. For two double $R$-modules $K, K'$, we define $\text{Hom}(K, K') = \text{Hom}_R(K_0, K'_0) \cap \text{Hom}_R(K_1, K'_1)$. This makes the class of double $R$-modules into an abelian category (which is isomorphic to $\text{Mod}(R \otimes_R R)$ and also to the category of $(R^{op}, R)$-bimodules).

An anti-automorphism of a double $R$-module $K$ is an additive bijective map $\theta : K \to K$ satisfying

$$(k \odot_i a)^\theta = k^\theta \odot_{1-i} a \quad \forall a \in R, k \in K, \ i \in \{0, 1\}.$$ 

If additionally $\theta \circ \theta = \text{id}_K$, then $\theta$ is called an involution.

A (general) bilinear space over $R$ is a triplet $(M, b, K)$ such that $M \in \text{Mod}-R$, $K$ is a double $R$-module and $b : M \times M \to K$ is a biadditive map satisfying

$$b(xr, y) = b(x, y) \odot_0 r \quad \text{and} \quad b(xr, yr) = b(x, y) \odot_1 r$$

for all $x, y \in M$ and $r \in R$. In this case, $b$ is called a (general) bilinear form (over $R$). Let $\theta$ be an involution of $K$. The form $b$ is called $\theta$-symmetric if

$$b(x, y) = b(y, x)^\theta \quad \forall x, y \in M.$$ 

See [3] [2] for various examples of general bilinear forms.

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3 The usage of double $R$-modules, rather than $(R^{op}, R)$-modules, was more convenient in [3], so we follow the notation of that paper. In addition, the notion of double modules is more natural when considering bilinear forms as a special case of multilinear forms, where the form takes values in a (right) multi-$R$-module.
Fix a double $R$-module $K$ and let $i \in \{0, 1\}$. The $i$-$K$-dual (or just $i$-dual) of an $R$-module $M$ is defined by
\[
M^{[i]} := \text{Hom}_R(M, K_{1-i}).
\]
Note that $M^{[i]}$ is admits a right $R$-module structure given by $(fr)(m) = (fm) \circ r$ (where $f \in M^{[i]}$, $r \in R$ and $m \in M$). In fact, $M \mapsto M^{[i]}$ is a left-exact contravariant functor from $\text{Mod}-R$ to itself, which we denote by $[i]$.

Let $b : M \times M \to K$ be a (general) bilinear form. The left adjoint and right adjoint of $b$ are defined as follows:
\[
\text{Ad}_b^\ell : M \to M^{[0]}, \quad (\text{Ad}_b^\ell x)(y) = b(x, y),
\]
\[
\text{Ad}_b^r : M \to M^{[1]}, \quad (\text{Ad}_b^r x)(y) = b(y, x),
\]
where $x, y \in M$. It is straightforward to check that $\text{Ad}_b^\ell$ and $\text{Ad}_b^r$ are right $R$-linear. We say that $b$ is right (resp. left) regular if $\text{Ad}_b^\ell$ (resp. $\text{Ad}_b^r$) is bijective. If $b$ is both right and left regular, we say that $b$ is regular. Left and right regularity are not equivalent properties; see [6, Ex. 2.6].

Assume $b$ is regular. Then every $w \in \text{End}_R(M)$ admits a unique element $w^\alpha \in \text{End}_R(M)$ such that
\[
b(wx, y) = b(x, w^\alpha y) \quad \forall x, y \in M.
\]
The map $w \mapsto w^\alpha$, denoted $\alpha$, turns out to be anti-automorphism of $\text{End}_R(M)$ which is called the (right) corresponding anti-automorphism of $b$. (The left corresponding anti-automorphism of $b$ is the inverse of $\alpha$.) If $b$ is $\theta$-symmetric for some involution $\theta : K \to K$, then $\alpha$ is easily seen to be an involution.

We say that two bilinear spaces $(M, b, K)$, $(M, b', K')$ are similar if there is an isomorphism $f : K \to K'$ such that $b' = f \circ b$. It is easy to see that in this case, $b$ and $b'$ have the same corresponding anti-automorphism, provided they are regular.

**Theorem 5.2 ([B, Th. 5.7]).** Let $M$ be a right $R$-generator. Then the map sending a regular bilinear form on $M$ to its corresponding anti-automorphism gives rise to a one-to-one correspondence between the class of regular bilinear forms on $M$, considered up to similarity, and the anti-endomorphisms of $\text{End}_R(M)$. Furthermore, the correspondence maps the equivalence classes of forms which are $\theta$-symmetric w.r.t. some $\theta$ to the involutions of $\text{End}_R(M)$.

The theorem implies that every anti-automorphism $\alpha$ of $\text{End}_R(M)$, with $M$ an $R$-generator, is induced by some regular bilinear form on $M$, which is unique up to similarity. We will denote this form by $b_\alpha$ and the double $R$-module in which it takes values by $K_\alpha$. In case $\alpha$ is an involution, then $b_\alpha$ is symmetric with respect to some involution of $K_\alpha$, which we denote by $\theta_\alpha$.

The objects $K_\alpha$, $b_\alpha$ and $\theta_\alpha$ can be explicitly constructed as follows: Let $W = \text{End}_R(M)$. Then $M$ is a left $W$-module. Using $\alpha$, we may view $M$ as a right $W$-module by defining $m \cdot w = w^\alpha m$. Denote by $M^\alpha$ the right $W$-module thusly obtained. Then $K_\alpha = M_\alpha \otimes_W M$. We make $K_\alpha$ into a double $R$-module by defining $(x \otimes y) \circ_0 r = x \otimes yr$ and $(x \otimes y) \circ_1 r = xr \otimes y$. The form $b_\alpha$ is given by $b_\alpha(x, y) = y \otimes x$, and when $\alpha$ is an involution, $\theta_\alpha : K_\alpha \to K_\alpha$ is defined by $(x \otimes y)^{\theta_\alpha} = y \otimes x$.

We will also need the following proposition.

**Proposition 5.3 ([B, Lm. 7.7]).** Fix a double $R$-module $K$. For $M \in \text{Mod}-R$, define $\Phi_M : M \to M^{[0]}$ by $(\Phi_M f)(x) = f(x)$ for all $x \in M$ and $f \in M^{[1]}$. Then:

(i) $\{\Phi_M\}_{M \in \text{Mod}-R}$ is a natural transformation from $\text{id}_{\text{Mod}-R}$ to $[0][1]$ (i.e. for all $N, N' \in \text{Mod}-R$ and $f \in \text{Hom}_R(N, N')$, one has $f^{[1]} \circ \Phi_N = \Phi_{N'} \circ f$).

(ii) $\Phi$ is additive (i.e. $\Phi_{N \oplus N'} = \Phi_N \oplus \Phi_{N'}$ for all $N, N' \in \text{Mod}-R$).
(iii) $(\text{Ad}^0)^{(0, 0)} \circ \Phi_M = \text{Ad}^0_k$ for every general bilinear form $b : M \times M \to K$.

(iv) $R^{[1][0]}$ can be identified with $\text{End}_R(K_1)$. Under that identification, $(\Phi_R r) k = k \circ r$ for all $r \in R$ and $k \in K$.

To finish, we recall that the orthogonal sum of two bilinear spaces $(M, b, K)$ and $(M', b', K)$ is defined to be $(M \oplus M', b \oplus b', K)$ where $(b \oplus b')(x \oplus x', y \oplus y') = b(x, y) + b'(x', y')$. The form $b \perp b'$ is right regular $\iff$ $b$ and $b'$ are right regular.

6. Double Progenerators

The observation which forms the basis to all results of this paper is the fact that whether a ring $R$ is equivalent to a ring with an anti-automorphism (resp. involution) can be phrased in terms of existence of certain bilinear forms (resp. double $R$-modules). In this section, we state and prove this criterion (Proposition 5.3 and Theorem 6.5).

Throughout, $R$ is a ring and $C = \text{Cent}(R)$. Recall from section 4 that for all $M, N \in \text{Mod-R}$, $\text{Hom}_R(M, N)$ admits a (right) $C$-module structure. In particular, $\text{End}_R(M)$ is a $C$-algebra.

Let $K$ be a double $R$-module. Then $K$ can be viewed as a an $(R^{op}, R)$-bimodule by setting $a^{op} \cdot k \cdot b = k \circ b \circ_1 a$. If the double $R$-module obtained is an $(R^{op}, R)$-progenerator, we say that $K$ is a double $R$-progenerator. The type of a double $R$-progenerator is the type of its corresponding $(R^{op}, R)$-module. Namely, it is the automorphism $\sigma \in \text{Aut}(C)$ satisfying $k \circ_0 c = k \circ_1 \sigma(c)$ for all $c \in \text{Cent}(R)$, $k \in K$.

It turns out that Theorem 5.2 is useful for producing double progenerators.

Lemma 6.1. Let $M$ be an $R$-progenerator and let $\alpha$ be an anti-automorphism of $\text{End}_R(M)$. Then $K_{\alpha}$ is a double $R$-progenerator. Viewing $\text{End}_R(M)$ as a $C$-algebra, the type of $K_{\alpha}$ is the type of $\alpha$.

Proof. Let $W = \text{End}_R(M)$. Recall that by construction, $K_{\alpha} = M^\alpha \otimes_W M$. Since $M_R$ is an $R$-progenerator, $M$ is a $(W, R)$-progenerator. Endow $M^\alpha$ with a left $R^{op}$-module structure by putting $r^{op} m = m r$. Then $M^\alpha$ is an $(R^{op}, W)$-bimodule and, moreover, it is easily seen to be an $(R^{op}, W)$-progenerator (because $M$ is a $(W, R)$-progenerator). By Morita theory (see section 2), this means that $K_{\alpha} = M_{\alpha} \otimes_W M$ is an $(R^{op}, R)$-progenerator, as required. That the type of $K_{\alpha}$ is the type of $\alpha$ follows by straightforward computation.

Lemma 6.2. Fix a double $R$-progenerator $K$. Then:

(i) $\Phi_M$ is an isomorphism for all $M \in \text{proj-R}$ (see Proposition 5.3 for the definition of $\Phi$).

(ii) The functors $[0], [1] : \text{proj-R} \to \text{proj-R}$ are mutual inverses.

(iii) Let $(M, b, K)$ be a bilinear space with $M \in \text{proj-R}$. Then $b$ is right regular $\iff$ $b$ is left regular.

Proof. (i) Since $K$ is a double $R$-progenerator, every automorphism of $K_1$ is of the form $k \mapsto k \circ r$ for a unique $r \in R$. Hence, $\Phi_R$ is an isomorphism by Proposition 5.3(iv). By the additivity of $\Phi$ (Proposition 5.3(ii)), $\Phi_{R^n}$ is also an isomorphism for all $n$. As every $M \in \text{proj-R}$ admits an $M' \in \text{proj-R}$ with $M \oplus M' \cong R^n$, it follows that $\Phi_M$ is an isomorphism (since $\Phi_M \circ \Phi_M' = \Phi_{R^n}$).

(ii) It is easy to verify that $R^{[1]} \cong K_1 \in \text{proj-R}$. Since the functor $[i]$ is additive, it follows that $[i]$ takes proj-R into itself. By (i), $\Phi : \text{id}_{\text{proj-R}} \to [1][0]$ is a natural isomorphism. For all $M \in \text{Mod-R}$, define $\Psi_M : M \to M^{[0][1]}$ by $(\Psi_M x)f = f(x)$, where $f \in M^{[0]}$ and $x \in M$. Then a similar argument shows that $\Psi : \text{id}_{\text{proj-R}} \to [0][1]$ is a natural isomorphism. It follows that $[0]$ and $[1]$ are mutual inverses.
(iii) By Proposition 5.3(iii), \((\text{Ad}_b)^{[0]} \circ \Phi_M = \text{Ad}_b^c\), so by (i), that \(\text{Ad}_b^c\) is bijective implies \(\text{Ad}_b^c\) is bijective. The other direction follows by symmetry. (Use the identity \((\text{Ad}_b^c)^{[1]} \circ \Psi_M = \text{Ad}_b^c\) )

\[\text{Proposition 6.3.}\] The ring \(R\) is Morita equivalent over \(C\) to a (necessarily central) \(C\)-algebra with an anti-automorphism of type \(\sigma \in \text{Aut}(C)\) \(\iff\) there exists a regular bilinear space \((M, b, K)\) such that \(M\) is a \(R\)-progenerator and \(K\) is a double \(R\)-progenerator of type \(\sigma\).

\[\text{Proof.}\] If \((M, b, K)\) is a right or left regular bilinear space as a above, then it is regular by Lemma 6.2(iii). Thus, \(b\) has a corresponding anti-automorphism \(\alpha\). Since \(M\) is a \(R\)-progenerator, \(\text{End}_R(M) \sim_{\text{Mor}/C} R\) (recall that we view \(\text{End}_R(M)\) as a \(C\)-algebra). In addition, for all \(x, y \in M\) and \(c \in C\), we have \(b(x, y) \circ_1 \sigma(c) = b(x, y) \circ_0 c = b(cx, y) = b(x, c^\sigma y) = b(x, y) \circ_1 c^\alpha\), so since \(K_1\) is faithful (because it is a progenerator), the type of \(\alpha\) is \(\sigma\).

Conversely, let \(M\) be an \(R\)-progenerator and assume \(\text{End}_R(M)\) has an anti-automorphism of type \(\sigma\). Then by Theorem 6.2, \(b_\alpha : M \times M \to K_\alpha\) is a regular bilinear form and \(K_\alpha\) is a double \(R\)-progenerator of type \(\sigma\) by Lemma 6.1.

\[\text{Lemma 6.4.}\] Let \(K\) be a double \(R\)-module with involution \(\theta\). For every \(M \in \text{Mod}-R\), define \(u_{\theta,M} : M^{[0]} \to M^{[1]}\) by \(u_{\theta,M}(f) = \theta \circ f\). Then \(u_{\theta,M}\) is a natural isomorphism of right \(R\)-modules.

\[\text{Proof.}\] By computation.

The proof of following theorem demonstrates an idea that will be used several times in the paper: One can construct involutions on rings that are Morita equivalent to \(R\) by constructing symmetric general bilinear forms. All possible involutions are obtained in this manner (but this fails if we limit ourselves to standard sesquilinear forms).

\[\text{Theorem 6.5.}\] The ring \(R\) is Morita equivalent over \(C\) to a \(C\)-algebra with an anti-automorphism of type \(\sigma \in \text{Aut}(C)\) \(\iff\) there exists a double \(R\)-progenerator of type \(\sigma\) admitting an involution.

\[\text{Proof.}\] Assume \(M_R\) is an \(R\)-progenerator such that \(\text{End}_R(M)\) has an involution of type \(\sigma\). Then, as in the proof of Proposition 6.3, \(K_\alpha\) is a double \(R\)-progenerator of type \(\sigma\). Since \(\alpha\) is an involution, \(K_\alpha\) has an involution, namely \(\theta_\alpha\).

Conversely, assume \(K\) is a double \(R\)-progenerator of type \(\sigma\) with an involution \(\theta\). Let \(P\) be any right \(R\)-progenerator and let \(M = P \oplus P^{[1]}\). Define \(b : M \times M \to K\) by \(b(x \oplus f, y \oplus g) = qx + (fy)^\theta\). The form \(b\) is clearly \(\theta\)-symmetric. We claim that \(b\) is regular. By Lemma 6.2(iii) (or since \(b\) is \(\theta\)-symmetric), it is enough to show only right regularity. Indeed, a straightforward computation shows that

\[
\text{Ad}_b^c = \begin{bmatrix}
0 & \text{id}_{P^{[1]}} \\
\Phi_P & 0
\end{bmatrix} \in \text{Hom}_R(P \oplus P^{[1]}, P^{[1]} \oplus P^{[1][1]})
\]

As \(\Phi_P\) and \(u_{\theta,P^{[1]}}\) are both bijective by Lemmas 6.2(i) and 6.4, respectively, so is \(\text{Ad}_b^c\). Let \(\alpha\) be the corresponding anti-endomorphism of \(b\). Then \(\alpha\) is an involution (since \(b\) is \(\theta\)-symmetric), and, as in the proof of Proposition 6.3, \(\alpha\) is of type \(\sigma\).

7. Rings That Are Morita Equivalent to Their Opposites

Let \(R\) be a ring. In this section, we use the Proposition 6.3 to show that under certain finiteness assumptions, \(R \sim_{\text{Mor}} R^{op}\) implies that \(R\) is Morita equivalent to a ring with an anti-automorphism (i.e. (\(C\)\(\Rightarrow\)\(B\))). Whether this holds in general is still open. Henceforth, we freely consider \((R^{op}, R)\)-progenerators as double \(R\)-progenerators and vice versa. We let \(\text{Jac}(R)\) denote the Jacobson radical of \(R\).
We begin with a series of lemmas whose purpose is to show that the functors $[0]$ and $[1]$ commute with certain scalar extensions.

**Lemma 7.1.** Let $K$ be an $(R^{op}, R)$-progenerator of type $\sigma \in \text{Aut}(\text{Cent}(R))$, let $C \subseteq \text{Cent}(R)$ be a subring fixed by $\sigma$, and let $D/C$ be a commutative ring extension. For every $P \in \text{proj}-R$, let $P_D := P \otimes_C D \in \text{proj}-R_D$. Make $K_D$ into an $(R^{op}, R_D)$-module by defining $(r \otimes d)^{op}(k \otimes d') = (r^{op}k) \otimes dd'$ for all $r \in R$, $k \in K$, $d, d' \in D$. Then $K_D$ is an $(R^{op}, R_D)$-progenerator and $(P^{[1]}_D) \cong (P_D)^{[1]}$ for all $P \in \text{proj}-R$ (the functor $[1]$ is computed w.r.t. $K$ in the l.h.s. and w.r.t. $K_D$ in the r.h.s.).

**Proof.** That $K_D$ is an $(R^{op}, R_D)$-progenerator follows from Proposition 4.2 (observe that $(R^{op})^D = (R^{op})_D = (R^{op})^{[1]}$ since $\sigma$ fixes $C$). That $(P^{[1]}_D) \cong (P_D)^{[1]}$ follows from Proposition 4.1. $\square$

**Lemma 7.2.** Let $K$ be an $(R^{op}, R)$-progenerator and let $N$ be the prime radical (resp. Jacobson radical) of $R$. For all $P \in \text{proj}-R$ define $\overline{P} := P/PN \cong P \otimes_R (R/N) \in \text{proj}-R/N$. Then $\text{End}_{R^{op}}(\overline{K}) \cong R^{op}$, $\overline{K}$ is an $(R^{op}, \overline{R})$-progenerator and $\overline{P}^{[1]} \cong \overline{P}^{[1]}$ for all $P \in \text{proj}-R$ (the functor $[1]$ is computed w.r.t. $K$ in the l.h.s. and w.r.t. $\overline{K}$ in the r.h.s.).

**Proof.** By Proposition 4.3, $K_N = N^{op}K$ and $\overline{K}$ is an $(R^{op}, \overline{R})$-progenerator. Let $P \in \text{proj}-R$. We claim that $P^{[1]}_N = \text{Hom}_R(P, (K_N)_0)$ (we consider $K_N$ as a double $R$-module here). Indeed, by definition, $P^{[1]}_N = \text{Hom}_R(P, K_N)_0 \subseteq \text{Hom}_R(P, (K \otimes_1 N)_0) = \text{Hom}_R(P, (N^{op}K)_0) = \text{Hom}_R(P, (K_N)_0)$. By Lemma 7.3, $P^{[1]}_N \subseteq \text{Hom}_R(P, (K_N)_0)$. As this inclusion is additive in $P$, it is enough to verify the equality for $P = R_N$, which is routine. Now, we get

$$\overline{P}^{[1]}_N = \text{Hom}_R(P, K_0)/P^{[1]}_N = \text{Hom}_R(P, K_0)/\text{Hom}_R(P, K_0) \cong \text{Hom}_R(P, K_0) \cong \text{Hom}_R(\overline{P}, \overline{K}_0) = \overline{P}^{[1]}_N,$$

as required. $\square$

Recall that a ring $R$ is called $Q$-finite if both $\dim_Q(R \otimes \mathbb{Q})$ and the cardinality of $\ker(R \to R \otimes \mathbb{Q})$ are finite. The following lemma is based on [2 Pr. 18.2].

**Lemma 7.3.** Assume $R$ is $Q$-finite and let $N$ be the prime radical of $R$. Then $N$ is nilpotent and $R/N \cong T \times \Lambda$ where $T$ is a semisimple finite ring and $\Lambda$ is subring of a semisimple $Q$-algebra $E$ such that $NQ = E$.

**Proof.** Let $T = \ker(R \to R \otimes \mathbb{Q})$. Then $T$ is an ideal of $R$. Consider $T$ as a non-unital ring and let $J = \text{Jac}(T)$ (i.e. $J$ is the intersection of the annihilators of simple right $T$-modules $M$ with $MT = M$). Arguing as in [2 Pr. 18.2], we see that $J$ is also an $R$-ideal. Since $J$ and all its submodules are finite (as sets), $J^n = 0$ for some $n$ (because $MJ \subseteq N$ for any right $T$-module $M$ and any maximal submodule $N \subseteq M$). In particular, $J \subseteq N$.

Replacing $R$ with $R/J$, we may assume $J = 0$. Now, $T$ is semisimple and of finite length, hence it has a unit $e$. As $er, re \in T$ for all $r \in R$, we see that $er = re = re$. Thus, $e \in \text{Cent}(R)$ and $R \cong T \times (1 - e)R$. As $\Lambda := (1 - e)R$ is torsion-free, it is a subring of $E := \Lambda \otimes \mathbb{Q}$, which is a f.d. $Q$-algebra by assumption.

Let $I = \text{Jac}(E) \cap \Lambda$. Then $I$ is nilpotent, hence $I \times 0 \subseteq N$. Replacing $R$ by $R/(I \times 0)$, we may assume $E$ semisimple. We are thus finished if we show that the prime radical of $\Lambda$, denoted $N'$, is 0 (because then $N = N' \times 0 = 0$). Indeed, by [4 Th. 2.5], $\Lambda$ is noetherian (here we need $E$ to be semisimple). Thus, $N'$ is nil, hence so is $N'\mathbb{Q} \subseteq E$. But $E$ is semisimple, so we must have $N' \subseteq N'\mathbb{Q} = 0$. $\square$

**Theorem 7.4.** Assume $R$ is semilocal or $Q$-finite and let $K$ be an $(R^{op}, R)$-progenerator. Then for every $P \in \text{proj}-R$ there is $n \in \mathbb{N}$ such that $P \cong P^{[n]} = P^{[1],[1] \ldots [1]}$ ($n$ times). When $R$ is semilocal, $n$ is independent of $P$. 

RINGS THAT ARE MORITA EQUIVALENT TO THEIR OPPOSITES 9
Proof. Let \( N \) denote the Jacobson radical of \( R \) in case \( R \) is semilocal and the prime radical of \( R \) otherwise. In the latter case, \( N \) is nilpotent by Lemma 7.2, so \( N \subseteq \text{Jac}(R) \) in both cases.

Using the notation of Lemma 7.2, observe that every \( P \in \text{proj-}R \) is the projective cover of \( P/PN \). As projective covers are unique up to isomorphism, we have \( P \cong Q \iff P/P \cong Q/Q \) for all \( P, Q \in \text{proj-}R \). Therefore, using Lemma 7.2, we may assume \( N = 0 \). Thus, \( R \) is semisimple or \( R = T \times \Lambda \) with \( \Lambda \otimes \mathbb{Z} \mathbb{Q} \) semisimple and \( T \) finite and semisimple.

Assume \( R \) is semisimple and let \( V_1, \ldots, V_t \) be a complete set of simple \( R \)-modules up to isomorphism. By Lemma 7.2(iii), [1] permutes the class of isomorphism classes of f.g. projective \( R \)-modules, and since [1] is additive, it permutes the indecomposable projective modules, namely, \( V_1, \ldots, V_t \). Therefore, there is \( n \in \mathbb{N} \) (say, \( n = t! \)) such that \( V_i^{[n]} \cong V_i \) for all \( i \). As any \( P \in \text{proj-}R \) is a direct sum of simple modules, we get \( P^{[1]} \cong P \). We also record that \( \text{length}(P^{[1]}) = \text{length}(P) \).

Now assume \( R = \Lambda \times T \) as above, and set \( E := \Lambda \otimes \mathbb{Q} \mathbb{Q} \) and \( S := T \times E \). Let \( \sigma \in \text{Aut}(\text{Cent}(R)) \) be the type of \( K \) and let \( e = (1_T, 0) \in R \). Then \( e \) is the maximal torsion idempotent in \( R \), hence \( \sigma(e) = e \). Define \( C := \mathbb{Z}e + \mathbb{Z}(1 - e) \) and \( D := \mathbb{Z}e + \mathbb{Q}(1 - e) + T \times E = S \). Then \( C \) is fixed by \( \sigma \) and \( S \cong R \otimes C \). Let \( K' = K \otimes C \). Then by Lemma 7.4, \( K' \) is an \((S^\sigma, S)\)-progenerator and \((P_S)^{[1]} \cong (P^{[1]})_S \) for all \( P \in \text{proj-}R \) (where \( P_S := P \otimes_R S \)).

For every \( P \in \text{proj-}R \), define \( j(P) = \text{length}(P_S) \). Since \( S \) is semisimple, the previous paragraphs imply that \( j(P^{[1]}) = j(P) \). Therefore, we are done if we show that for all \( m \in \mathbb{N} \) there are finitely many isomorphism classes of modules \( P \) with \( j(P) = m \). Indeed, thanks to [4, Th. 2.8], up to isomorphism, there are finitely many \( E \)-modules of any given \( \mathbb{Z} \)-rank, and this is easily seen to imply that there are finitely many isomorphism classes of modules \( P \in \text{proj-}R \) with \( P_S \) of a given length, as required.

Corollary 7.5. Let \( C = \text{Cent}(R) \), \( \sigma \in \text{Aut}(C) \) and assume \( R \) is semilocal or \( \mathbb{Q} \)-finite. Then \( R \) is Morita equivalent over \( C \) to a (central) \( C \)-algebra with an anti-automorphism of type \( \sigma \) \( \iff \) \( R \) is Morita equivalent to \( R^\text{op} \) (as rings) with Morita equivalence of type \( \sigma \).

Proof. We only check the nontrivial direction. Assume \( R \) is Morita equivalent to \( R^\text{op} \) with with Morita equivalence of type \( \sigma \). Then there exists a double \( R \)-progenerator \( K \) of type \( \sigma \). By Theorem 7.4, there exists \( n \in \mathbb{N} \) such that \( R^{[n]} \cong R_R \). Let \( M = \bigoplus_{m=1}^n R^{[m]} \). Then there is an isomorphism \( f : M \to M^{[1]} \). This isomorphism gives rise to a right regular bilinear space \((M, b, K)\), namely, \( b : M \times M \to K \) is given by \( b(x, y) = (fy)x \). By Lemma 6.2(iii), \( b \) is regular, so we are done by Proposition 6.3.

The proof of Corollary 7.5 cannot be applied to arbitrary rings since there are double \( R \)-modules \( K \) for which \( M^{[1]} \neq M \) for all \( 0 \neq M \in \text{proj-}R \); see Example 12.3.

8. Semiperfect Rings

Let \( R \) be a semilocal ring that is Morita equivalent to its opposite. While Corollary 7.5 implies that \( R \) is Morita equivalent to a ring with an anti-automorphism, it does not provide any information about what this ring might be. However, when \( R \) is semiperfect, we can actually point out a specific ring which is Morita equivalent to \( R \) and has an anti-automorphism.

Recall that a ring \( R \) is semiperfect if \( R \) is semilocal and \( \text{Jac}(R) \) is idempotent lifting (e.g. if \( \text{Jac}(R) \) is nil). In this case, the map

\[
P \mapsto P/P\text{Jac}(R) : \text{proj-}R/\cong \to \text{proj-}(R/\text{Jac}(R))/\cong
\]

is an equivalence of type \( \sigma \)
is bijective (e.g. see [13 §2.9] or [3 Th. 2.1]). Thus, up to isomorphism, $R$ admits finitely many indecomposable projective right $R$-modules $P_1, \ldots, P_t$ and any $P \in \text{proj-}R$ can be written as $P \cong \bigoplus_{i=1}^{t} P_i^{m_i}$ with $m_1, \ldots, m_t$ uniquely determined. In particular, $R_R \cong \bigoplus_{i=1}^{t} P_i^{m_i}$ for some (necessarily positive) $m_1, \ldots, m_t$. The ring $R$ is called basic if $m_1 = \cdots = m_t = 1$, namely, if $R_R$ is a sum of non-isomorphic indecomposable projective modules. It is well-known that every semiperfect ring $R$ admits a basic ring that is Morita equivalent to it, and this ring is unique up to isomorphism (see [11, Prp. 18.37] and the preceding discussion). Explicitly, the basic ring $S$ that is Morita equivalent to $R$ is $\text{End}_R(M)$, where $M = P_1 \oplus \cdots \oplus P_t$. This description of $S$ allows us to consider $S$ as a $\text{Cent}(R)$-algebra. (For example, if $R = M_n(L)$ with $L$ a local ring, then $S = L$.)

Assume now that there is an $(R^{op}, R)$-progenerator $K$ of type $\sigma \in \text{Aut}(\text{Cent}(R))$. Then the functor $[1]$ must permute the isomorphism classes of $P_1, \ldots, P_t$ (because they are the only indecomposable nonzero modules in $\text{proj-}R$) and hence stabilize $M$. Therefore, as in the proof of Corollary [13, End $R(M)$, the basic ring which is Morita equivalent to $R$, has an anti-automorphism of type $\sigma$. We have thus obtained the following proposition.

**Proposition 8.1.** Let $R$ be a semiperfect ring and let $S$ be the basic ring that is Morita equivalent to $R$. Then $R$ is Morita equivalent to $R^{op}$ via equivalence of type $\sigma \in \text{Aut}(\text{Cent}(R))$ $\iff S$ has an anti-automorphism of type $\sigma$.

Proposition 8.1 has a slightly weaker version for involutions.

**Proposition 8.2.** Let $R$ be a semiperfect ring and let $S$ be the basic ring that is Morita equivalent to $R$. If $R$ has an involution of type $\sigma$, then so does $M_2(S)$.

**Proof.** Let $\alpha$ be an involution of $R$ of type $\sigma$ and let $K$ be the double $R$-module obtained from $R$ by setting $k \circ_0 r = r^\alpha k$ and $k \circ_1 r = kr$ ($k, r \in R$). Then $K$ is double $R$-progenerator of type $\sigma$ admitting an involution, namely, $\alpha$. For any $P \in \text{proj-}R$, let $b_P$ denote the bilinear form $b$ constructed in the proof of Theorem [14]. Then $(b_P, P \oplus P^{[1]}, K)$ is an $\alpha$-symmetric right regular bilinear space. Let $P_1, \ldots, P_t$ be a complete list of indecomposable projective right $R$-modules up to isomorphism. Then $b := b_{P_1} \perp \cdots \perp b_{P_t}$ is a right regular $\alpha$-symmetric bilinear form defined over $M \oplus M^{[1]}$, where $M = P_1 \oplus \cdots \oplus P_t$. Therefore, $\text{End}_R(M \oplus M^{[1]})$ has an involution of type $\sigma$, namely the corresponding anti-automorphism of $b$ (is left regular since it is $\alpha$-symmetric). However, we have seen above that $M \cong M^{[1]}$, so $\text{End}_R(M \oplus M^{[1]}) \cong M_2(\text{End}_R(M)) \cong M_2(S)$ as $\text{Cent}(R)$-algebras. $\blacksquare$

It is still open whether that $R$ has an involution implies that $S$ has an involution. However, this can be shown in special cases.

**Proposition 8.3.** Let $L$ be a local ring and let $R = M_n(L)$. Assume $2 \in L^\times$ or $L$ is a division ring. Then $R$ has an involution of type $\sigma$ $\iff L$ has an involution of type $\sigma$.

**Proof.** That $R$ has an involution when $L$ has an involution is obvious. Conversely, let $\alpha$ be an involution of $R$ and let $K$ be as in the proof of Proposition 8.2. Let $P$ be the unique indecomposable projective right $R$-module. Then necessarily $P \cong P^{[1]}$. Fix an isomorphism $f : P \to P^{[1]}$ and observe that the bilinear form $b(x, y) := (fy)x + ((fx)y)^\alpha$ (resp. $b'(x, y) := (fy)x - ((fx)y)^\alpha$) is $\alpha$-symmetric (resp. $(-\alpha)$-symmetric). In addition, $\text{Ad}_b^\alpha + \text{Ad}_{b'}^\alpha = 2f$.

Assume $2 \in L^\times$. Then $f^{-1} \circ \text{Ad}_b^\alpha + f^{-1} \circ \text{Ad}_{b'}^\alpha = 2 \text{id}_P$. Since the r.h.s. is invertible and lies in $\text{End}_R(P) \cong L$, one of $f^{-1} \circ \text{Ad}_b^\alpha, f^{-1} \circ \text{Ad}_{b'}^\alpha$ must be invertible, hence one of $b, b'$ is right regular. In any case, we get that $L \cong \text{End}_R(P)$ has an involution of type $\sigma$, namely, the corresponding anti-automorphism of $b$ or $b'$. 


When \( L \) is a division ring, \( P \) is simple, so \( b' \) is regular if \( Ad_{b'} \neq 0 \). If \( Ad_{b'} = 0 \), then \( b' = 0 \), hence \((fg)x = ((fx)y)''\) for all \( x, y \in P \). This means that the bilinear form \( b'(x, y) := (fx)y \) is \( \alpha \)-symmetric. As \( Ad_{b''} = f, b'' \) is regular.

**Remark 8.4.** In case \( L \) is a division ring, Proposition 8.3 follows from \([7, \text{Th. 1.2.2}]\). The case where \( L \) is also finite dimensional over its center was noted earlier by Albert (e.g. see \([1, \text{Th. 10.12}]\)).

### 9. Azumaya Algebras

The next of this paper concerns Saltman’s Theorem about Azumaya algebras with involution (see section 4). As preparation, we now briefly recall Azumaya algebras and several facts about them to be used later. We refer the reader to \([15]\) and \([5]\) for an extensive discussion and proofs. Throughout, \( C \) is a commutative ring and, unless specified otherwise, all tensor products are taken over \( C \).

An *Azumaya algebra* is a \( C \)-algebra \( A \) such that \( A \) is a progenerator as a \( C \)-module and the standard map \( \Psi : A \otimes A^{\text{op}} \to \text{End}_C(A) \) given by \( \Psi(a \otimes b^{\text{op}})(x) = axb \) is an isomorphism. When \( C \) is a field, being Azumaya simply means being simple and central, so Azumaya algebras are a generalization of central simple algebras.

Let \( A, B \) be Azumaya \( C \)-algebras. The following facts are well-known:

1. \( \text{Cent}(A) = C \) and \( \text{Cent}_{A \otimes B}(C \otimes B) = A \otimes C \).
2. \( A \otimes B \) and \( A^{\text{op}} \) are Azumaya \( C \)-algebras.
3. If \( \psi : C \to C' \) is a commutative ring homomorphism, then \( A \otimes C' \) is an Azumaya \( C' \)-algebra (\( C' \) is viewed as a \( C \)-algebra via \( \psi \)).
4. For every \( C \)-progenerator \( P \), \( \text{End}_C(P) \) is an Azumaya \( C \)-algebra.
5. If \( B \) is a subalgebra of \( A \), then \( B' := \text{Cent}_A(B) \) is Azumaya (over \( C \)), \( B = \text{Cent}_A(B') \) and \( B \otimes B' \cong A \) via \( b \otimes b' \mapsto bb' \).

For every \( M \in \text{Mod-}C \), we define \( \text{rank}(M) = \text{rank}_C(M) \) to be the function \( \text{Spec}(C) \to \mathbb{Z} \) sending a prime ideal \( P \) to \( \dim_k(M \otimes k_P) \) where \( k_P \) is the fraction field of \( C/P \). We write \( \text{rank}(M) = n \) to denote that \( \text{rank}(M)(P) = n \) for all \( P \in \text{Spec}(C) \). For example, when \( C \) is a field, \( \text{rank}(M) = \dim_C(M) \). In addition, if \( \psi : C \to C' \) is a commutative ring homomorphism and \( P' \in \text{Spec}(C') \), then \( \text{rank}_{C'}(M \otimes C')(P') = \text{rank}_C(M)(P) \) where \( P = \psi^{-1}(P') \). This allows us to extend scalars when computing ranks.

**Proposition 9.1.** Let \( A \) and \( B \) be Azumaya \( C \)-algebras and let \( \varphi : A \to B \) be a homomorphism of \( C \)-algebras. Then \( \varphi \) is injective. If moreover \( \text{rank}(A) = \text{rank}(B) \), then \( \varphi \) is an isomorphism.

Two Azumaya \( C \)-algebras \( A, B \) are said to be *Brauer equivalent*, denoted \( A \sim_{\text{Br}} B \), if there are \( C \)-progenerators \( P, Q \) such that

\[
A \otimes \text{End}_C(P) \cong B \otimes \text{End}_C(Q)
\]

as \( C \)-algebras. The *Brauer class* of \( A \), denoted \([A]\), is the collection of Azumaya algebras which are Brauer equivalent to \( A \). The *Brauer group* of \( C \), denoted \( \text{Br}(C) \), is the set of all Brauer classes endowed with the group operation \([A] \otimes [B] = [A \otimes B] \).

The unit element of \( \text{Br}(C) \) is the class of \( C \), namely, the class of Azumaya algebras which are isomorphic to \( \text{End}_C(P) \) for some progenerator \( P \). The inverse of \([A]\) is \([A^{\text{op}}]\).

The following theorem, which is essentially due to Bass, presents an alternative definition of the Brauer equivalence.

**Theorem 9.2** (Bass). Let \( A, B \) be two Azumaya \( C \)-algebras. Then \( A \sim_{\text{Br}} B \iff A \sim_{\text{Mor/C}} B \), i.e. there exists an \((A, B)\)-progenerator \( P \) such that \( cp = pc \) for all \( p \in P \) and \( c \in C \).
Proof. See [2 Cr. 17.2]. The assumption that Spec($C$) is noetherian in [2] can be ignored by [15 Th. 2.3]; see also [14 §1].

We will also need the following proposition.

**Proposition 9.3.** Let $A$, $B$ be Azumaya $C$-algebras and let $K$ be an $(A, B)$-bimodule satisfying $ck = kc$ for all $c \in C$. Then $K$ is an $(A, B)$-progenerator if and only if $K$ is a $(C, A^\text{op} \otimes B)$-progenerator with $A^\text{op} \otimes B$ acting via $k(a^\text{op} \otimes b) = akb$.

In particular, in this case, $\text{End}_C(CK) \cong A^\text{op} \otimes B$.

**Proof.** The "only if" part easily follows from $\text{Cent}_{A^\text{op} \otimes B}(C \otimes B) = A^\text{op} \otimes C$, so we turn to prove the "if" part. Let $D = A \otimes A^\text{op}$ and consider $A$ as a right $D$-module via $a(x \otimes y^\text{op}) = yax$. Since $CA$ is a progenerator and $D \cong \text{End}(CA)$ (because $A$ is Azumaya), $A$ is a $(C, D)$-progenerator. By Proposition 9.2 $K \otimes_C A^\text{op}$ is an $(A \otimes A^\text{op}, B \otimes A^\text{op})$-progenerator and hence $CA \otimes_D (K \otimes_C A^\text{op})$ is a $(C, B \otimes A^\text{op})$-progenerator. It is straightforward to check that $CA \otimes_D (K \otimes_C A^\text{op}) \cong K$ as $(C, B \otimes A^\text{op})$-bimodules via $x \otimes_D (k \otimes_C y^\text{op}) \mapsto yxk$ and $k \mapsto 1 \otimes_D (k \otimes_C 1)$ in the other direction, so we are done.

Let $G$ be a finite group of ring-automorphisms of $C$ and let $C_0 = C^G$ be the ring of elements fixed by $G$. For every 2-cocycle $f \in Z^2(G, C^\times)$ let $\Delta(C/C_0, G, f) = \bigoplus_{\sigma \in G} C\sigma$ where $\{\sigma\}$ are formal variables. We make $\Delta(C/C_0, G, f)$ into a ring by linearly extending the following relations

\[ \sigma \tau = f(\sigma, \tau)u_{\sigma \tau}, \quad \sigma c = \sigma(c)u_{\sigma}, \quad \forall \sigma, \tau \in G, \ c \in C. \]

This makes $\Delta(C/C_0, G, f)$ into a $C_0$-algebra whose unity is $u_1$.

We say that $C/C_0$ is a Galois extension with Galois group $G$ if $\Delta(C/C_0, G, 1)$ is an Azumaya $C_0$-algebra (1 denotes the trivial 2-cocycle). In this case:

1. $\Delta(C/C_0, G, 1) \cong \text{End}_{C_0}(C)$ via $\sum_{\sigma} c_{\sigma}u_{\sigma} \mapsto [x \mapsto \sum_{\sigma} c_{\sigma}(x)] \in \text{End}_{C_0}(C)$.
2. The $C_0$-algebra $\Delta(C/C_0, G, f)$ is Azumaya for all $f \in Z^2(G, C^\times)$.
3. $\text{rank}_{C_0}(C) = |G|$.
4. For every commutative $C_0$-algebra $D$, the extension $C \otimes_{C_0} D/C_0 \otimes_{C_0} D$ is Galois with Galois group $\{g \otimes \text{id}_D \mid g \in G\}$.

See [15 §6] for proofs. Azumaya algebras of the form $\Delta(C/C_0, G, f)$ are called crossed products.

**Proposition 9.4.** Let $C/C_0$ be a Galois extension with Galois group $G$. Assume $M$ is a $C$-module endowed with a $G$-action such that $\sigma(m\sigma) = m\sigma(c)$ for all $\sigma \in G$, $m \in M$, $c \in C$. Then $M^G \otimes_{C_0} C \cong M$ via $m \otimes c \mapsto mc$, where $M^G := \{m \in M \mid \sigma(m) = m \forall \sigma \in G\}$. In particular, $\text{rank}_{C_0}(M) = \text{rank}_{C_0}(C)\text{rank}_{C_0}(M^G) = |G|\cdot\text{rank}_{C_0}(M^G)$. Furthermore, if $M$ is an Azumaya $C$-algebra and $G$ acts via ring automorphisms on $M$, then $M^G$ is an Azumaya $C_0$-algebra.

**Proof.** See [15 Prps. 6.10 & 6.11].

Let $C/C_0$ be a Galois extension with Galois group $G$. For every Azumaya $C$-algebra $A$ and $\sigma \in G$, define $A^\sigma$ to be the $C$-algebra obtained from $A$ by viewing $A$ as a $C$-algebra via $\sigma: C \to A$. The algebra $A^\sigma$ is also Azumaya, but it may not be Brauer-equivalent to $A$. Observe that the algebra $B := \bigotimes_{\sigma \in G} A^\sigma$ admits a $G$-action given by $\tau(\bigotimes_{\sigma} a_{\sigma}) = \bigotimes_{\sigma} a_{\sigma^{-1}\tau}$ and that action satisfies $\sigma(b\sigma(c)) = \sigma(b)\sigma(c)$ for all $\sigma \in G$, $b \in B$, $c \in C$. The corestriction of $A$ (w.r.t. $C/C_0$) is defined to be

\[ \text{Cox}_{C/C_0}(A) = B^G = \left( \bigotimes_{\sigma \in G} A^\sigma \right)^G. \]

\[ 4 \] Recall that a 2-cocycle taking values in $C^\times$ is a function $f: G \times G \to C^\times$ satisfying $\sigma(f(\tau, \eta))f(\sigma, \tau\eta) = f(\sigma, \tau)f(\sigma\tau, \eta)$ and $f(1, \sigma) = f(\sigma, 1) = 1$ for all $\sigma, \tau, \eta \in G$. The 2-cocycles with the operation of point-wise multiplication form an abelian group denoted $Z^2(G, C^\times)$. 

RINGS THAT ARE MORITA EQUIVALENT TO THEIR OPPOSITES 13
By Proposition 9.4, \( \text{Cor}_{C/C_0}(A) \) is an Azumaya \( C_0 \)-algebra. Moreover, it can be shown that the corestriction induces a group homomorphism
\[
\text{Cor}_{C/C_0} : \text{Br}(C) \rightarrow \text{Br}(C_0)
\]
given by \( \text{Cor}_{C/C_0}([A]) = [\text{Cor}_{C/C_0}(A)] \); see [15, Ch. 8].
We also note that the corestriction can be defined for separable extensions \( C/C_0 \); see [15, Ch. 8] for further details.

10. Saltman’s Theorem

In this section, we show how to recover the Knus-Parimala-Srinivas proof of Saltman’s Theorem ([9, §4]) as an application of Theorem 6.5. In order to keep the exposition as self-contained and fluent as possible, we will first give a proof of Saltman’s Theorem using Theorem 6.5, and then explain how this proof relates to the exposition as self-contained and fluent as possible, we will first give a proof of

Let \( C \) be a commutative ring and let \( R \) be a \( C \)-algebra. Recall that a Goldman element of \( R/C \) is an element \( g \in R \otimes_C R \) such that \( g^2 = 1 \) and \( g(r \otimes s) = (r \otimes s)g \) for all \( r, s \in R \).

**Proposition 10.1.** All Azumaya \( C \)-algebras have a Goldman element

**Proof.** See [10, p. 112] or [15, Pr. 5.1], for instance. \( \square \)

**Theorem 10.2** (Saltman). Let \( C \) be a commutative ring and let \( A \) be an Azumaya \( C \)-algebra. Then:

(i) \( A \) is Brauer equivalent to an Azumaya algebra \( B \) with an involution of the first kind \( \iff \) \( A \sim_{\text{Br}} A^{\text{op}} \).

(ii) Let \( C/C_0 \) be a Galois extension with Galois group \( \{1, \sigma\} \) (\( \sigma \neq 1 \)). Then \( A \) is Brauer equivalent to an Azumaya algebra \( B \) with an involution whose restriction to \( C \) is \( \sigma \iff \text{Cor}_{C/C_0}(B) \sim_{\text{Br}} C_0 \).

**Proof.** (i) By Theorem 9.2 \( A \sim_{\text{Br}} A^{\text{op}} \) if and only if there exists a double \( A \)-progenerator of type \( \text{id}_C \) (i.e. an \( (A^{\text{op}}, A) \)-progenerator of type \( \text{id}_C \)). Likewise, by Theorems 6.5 and 9.2, \( A \) is Brauer equivalent to a \( C \)-algebra \( B \) with involution of the first kind if and only if there exist a double \( A \)-progenerator of type \( \text{id}_C \) with involution. Therefore, it is enough to show that every double \( A \)-progenerator \( K \) of \( \text{id}_C \) has an involution. Indeed, consider \( K \) as an \( A \otimes_C A \)-module via \( k \cdot (a \otimes a') = k \cdot a \otimes (a') \) and let \( g \) be a Goldman element of \( A \). It is easy to check that \( k \mapsto kg \) is an involution of \( K \).

(ii) Throughout, \( D = A \otimes A^{\sigma} \) and \( E = D^{\sigma} = \text{Cor}_{C/C_0}(D) \). Recall that \( \sigma \) extends to an automorphism of \( D \), also denoted \( \sigma \), given by \( \sigma(a \otimes a') = a' \otimes a \). By Theorems 6.5 and 9.2, it is enough to prove that \( \text{Cor}_{C/C_0}(A) \sim_{\text{Br}} C_0 \) if and only if then there exists a double \( A \)-progenerator of type \( \sigma \) admitting an involution.

Assume \( K \) is a double \( A \)-progenerator of type \( \sigma \) with involution \( \theta \). Then \( K \) can be considered as an \( (A^{\text{op}}, A^{\sigma}) \)-progenerator of type \( \text{id}_C \), hence by Proposition 9.3
\[
\text{End}(C \theta K) \cong A \otimes A^{\sigma} = D.
\]
Extend the left action of \( C \) on \( K \) to a left action of \( \Delta := \Delta(C/C_0, G, 1) \) on \( K \) by letting \( u_\sigma \) act as \( \theta \). Then it is easy to see that \( \text{End}(\Delta K) \cong D^{\sigma} = E \). Viewing \( D^{\sigma} \) and \( E \) as subrings of \( \text{End}(C_0 \theta K) \), this means \( E \) is the centralizer of \( D^{\sigma} \) in \( \text{End}(C_0 \theta K) \). As all these algebras are Azumaya over \( C_0 \), we have \( D^{\sigma} \otimes E \cong \text{End}(C_0 \theta K) \). In \( Br(C) \) this reads as \( [C_0] \otimes [E] = [D^{\sigma}] \otimes [E] = [C_0] \), so \( \text{Cor}_{C/C_0}(A) = E \sim_{\text{Br}} C_0 \).

Conversely, assume \( E = \text{Cor}_{C/C_0}(A) \sim_{\text{Br}} C_0 \) then there exists a \( (C, E) \)-progenerator \( Q \) of type \( \text{id}_{C_0} \). Let \( K = Q \otimes_{C_0} C \). Then by Proposition 9.3, \( K \) is a \((C_0 \otimes_C C, E \otimes_{C_0} C)\)-progenerator. Identify \( E \otimes_{C_0} C \) with \( D \) via \( (e \otimes c) \mapsto ec \). Then \( K \) is a \((C, D)\)-progenerator, hence by Proposition 9.3, \( K \) is an \((A^{\text{op}}, A^{\sigma})\)-progenerator.


via $a^\text{op} \cdot k \cdot a' := k(a \otimes a')$. We may thus view $K$ as a double $A$-progenerator of type $\sigma$. Define $\theta : K \to K$ by $(c \otimes q)^\theta = \sigma(c) \otimes q$ for all $c \in C, q \in Q$. We claim that $\theta$ is an involution of $K$. Indeed, $\theta^2 = \text{id}_K$, and for all $c, c' \in C, e \in E, q \in Q$, we have $(c \otimes q)(c'e) = (cc' \otimes q)e = \sigma(c)c'e = (\sigma(c) \otimes q)(c'e) = (c \otimes q)(c'e)$ $(e = \sigma(e)$ since $e \in E = D^2)$. This implies that $(kx)^\theta = k^3 \sigma(x)$ for all $k \in K, x \in D$. Putting $x = 1 \otimes a$ and $x = a \otimes 1$ yields that $\theta$ is an involution. □

**Remark 10.3.** (i) The proof of Theorem 10.2 also shows that a $C$-algebra $R$ with a Goldman element is Morita equivalent to its opposite over $C$ if and only if it is Morita equivalent as a $C$-algebra to an algebra with an involution of the first kind. We could not find a non-Azumaya algebra admitting a Goldman element, though.

(ii) A slightly different short proof of the “only if” part of part (ii) of Saltman’s Theorem that does not use double progenerators appears in [13, pp. 531].

In order to explain the connection of the previous proof with the Knus-Parimala-Srinivas proof of Saltman’s Theorem ([13, §4]), let us backtrack our proof to see what is the algebra $B$. According to Theorem 6.5, $B = \text{End}_A(P \oplus P^{[1]})$ with $P$ an arbitrary $A$-progenerator, and the involution of $B$ is induced by the bilinear form $b_P$ constructed in the proof of Theorem 6.5. The functor $[1]$ is computed using a double $A$-progenerator $K$ with involution which is obtained in the proof of Theorem 10.2.

Let us choose $P = A_A$. Then $B \cong \text{End}_A(A \oplus K_1)$ (since $A^{[1]} = \text{Hom}_A(A_A, K_0) \cong K_1$ via $f \mapsto f(1_A)$). This algebra with involution is essentially the algebra with involution constructed in [9, §4]. The difference between the proofs is that in [9], the involution on $B$ is constructed directly, while here we have obtained the involution from a general bilinear form on the right $A$-module $A \oplus K_1$, and hence suppressed some of the computations of [9].

Let also note that many proofs of special cases of Saltman’s Theorem in the literature, [9] and [14] in particular, involve the construction of an involutary map $\theta$ taking some $(A^{op}, A)$-bimodule or an equivalent object $K$ into itself and satisfying $(a^{op} b \theta) \theta = b^{op} k^\theta a$ $(a, b \in A, k \in K)$. For example, consider the map $\alpha$ in [14, p. 532–3, 537], the maps $\psi$ and $\sigma_\psi$ in [9, p. 71–2] and the map $u$ of [8, p. 196]. This hints that these proofs can be effectively phrased using general bilinear forms.

We finish this section by showing that the algebra $B$ in Saltman’s Theorem can be taken to have rank $4 \text{rank}(A)$ in part (i) and $4 \min\{\text{rank}(A), \text{rank}(A^\sigma)\}$ in part (ii).

**Lemma 10.4.** Let $A$ be an Azumaya $C$-algebra and let $M, N \in \text{proj}-A$. Then

$$\text{rank}(\text{Hom}_A(M, N)) = \frac{\text{rank}(M) \cdot \text{rank}(N)}{\text{rank}(A)}.$$ 

**Proof.** Let $P \in \text{Spec}(C)$ and let $F$ be an algebraic closure of $k_P$, the fraction field of $C/P$. Set $A_P = A \otimes_C F$, $M_F = M \otimes_C F$ and $N_F = N \otimes_C F$. By Proposition 1.4, we have $\text{Hom}_{A_P}(M_F, N_F) \cong \text{Hom}_A(M, N) \otimes_C F \cong (\text{Hom}_A(M, N) \otimes_C k_P) \otimes_{k_P} F$. Therefore, it is enough to verify the lemma in case $C$ is an algebraically closed field, i.e., when $C = F$, in which case $\text{rank}_C(\text{Hom}_C(\cdot, \cdot))$ and $\text{dim}_F(\text{Hom}_C(\cdot, \cdot))$ coincide. In this case, we must have $A \cong M_s(F)$ (as $F$-algebras) and $M \cong M_{s \times a}(F)$, $N \cong M_{t \times a}(F)$ (as $A$-modules) for suitable $s, t, n \in N \cup \{0\}$. But then, $\text{dim}_F \text{Hom}_A(M, N) = \text{dim}_F M_{s \times a}(F) = ts = \text{dim}_F(N) \cdot \text{dim}_F(M)/\text{dim}_F(A)$, as required. □

**Lemma 10.5.** Let $A$ be an Azumaya $C$-algebra and let $K$ be a double $A$-progenerator of type $\sigma \in \text{Aut}(C)$. Then

$$\text{rank}(K_0) = \sqrt{\text{rank}(A) \text{rank}(A^\sigma)}$$

and

$$\text{rank}(K_1) = \sqrt{\text{rank}(A) \text{rank}(A^{\sigma^{-1}})}.$$
Proof. Consider $K$ as an $(A^{\text{op}}, A)$-bimodule as explained in section 8. Since $A$ is of type $\sigma$, $\text{End}_A(K_0) = \text{End}(K_4) \cong (A^\sigma)^{\text{op}}$ as $C$-algebras, hence $\text{rank}(\text{End}_A(K_0)) = \text{rank}(A^\sigma)$. However, by Lemma 10.4 we have $\text{rank}(\text{End}_A(K_0)) = \text{rank}(K_0)^2 / \text{rank}(A)$. Comparing both expressions yields the formula for $\text{rank}(K_0)$. The formula for $\text{rank}(K_1)$ is shown in the same manner. $\square$

Proposition 10.6. In Theorem (10.2)(i) (resp. Theorem (10.2)(ii)), the algebra $B$ can be chosen such that $\text{rank}(B) = 4 \text{rank}(A)$ (resp. $\text{rank}(B) = 4 \text{min}\{\text{rank}(A), \text{rank}(A^\sigma)\}$).

Proof. Let $\sigma$ be id$_C$ in the case of Theorem (10.2)(i) and a nontrivial Galois automorphism of $C/C_0$ in the case of Theorem (10.2)(ii). In the comment after Remark 10.3 we saw that $B$ can taken to be $\text{End}_A(A \otimes K_1)$ with $K$ a double $A$-progenerator of type $\sigma$. In this case, by Lemmas 10.5 and 10.4, we have

$$\text{rank}(B) = \left( \frac{\text{rank}(A) + \sqrt{\text{rank}(A) \text{rank}(A^\sigma)}}{\text{rank}(A)} \right)^2 = \left( 1 + \frac{\text{rank}(A^\sigma)}{\text{rank}(A)} \right)^2 \text{rank}(A).$$

In the case of Theorem (10.2)(i), this means $\text{rank}(B) = 4 \text{rank}(A)$, so we are done. Regarding the case of Theorem (10.2)(ii), it is enough to show that $A$ is Brauer equivalent to an Azumaya algebra $A'$ of rank $\text{min}\{\text{rank}(A), \text{rank}(A^\sigma)\}$ (for then we can replace $A$ with $A'$ and get the desired rank).

To construct $A'$, recall that a finite collection of Galois extensions, Azumaya algebras and Brauer equivalences between them is always defined using finitely many elements. That is, there is a f.g. subring $D_0 \subseteq C_0$, a Galois extension $D/D_0$ with Galois group $\{1, \tau\}$ and an Azumaya $D$-algebra $E$ such that $C \cong D \otimes_{D_0} C_0$ (as extensions of $C_0$), $\sigma = \text{id}_D \otimes \tau$, $A \cong E \otimes D (C$-algebras) and $D \sim_{\text{Br}} (D^\sigma)^{\text{op}}$ (we can assume this since $A \sim_{\text{Br}} (A^\sigma)^{\text{op}}$). We omit the very technical proof and refer the reader to [13, §1] or [15, Th. 2.3 & Prp. 6.4] for the standard techniques of how to show this. Replacing $A, C, C_0$ with $E, D, D_0$, we may assume that $C$ is a f.g. ring, and hence noetherian. Let $\{e_1, \ldots, e_l\}$ be the set of primitive idempotents in $C$. Then $C = \bigoplus e_i C$ and $\text{Spec}(e_i C)$ is connected for all $i$. It is easy to see that every $C$-algebra $R$ factors as a product $\prod e_i R$ and $R/C$ is Azumaya if and only if $e_i R / e_i C$ is Azumaya for all $i$. Define $A' = \prod e_i A_i'$ where $A_i' = e_i A$ if $\text{rank}(e_i A) < \text{rank}(e_i A^\sigma)$ and $A_i' = e_i (A^\sigma)^{\text{op}}$ otherwise (observe that $\text{rank}(e_i A)$ is constant since $\text{Spec}(e_i C)$ is connected). Then $A' \sim_{\text{Br}} A$ (since $e_i A' \sim_{\text{Br}} e_i A$ for all $i$), and, by definition, $\text{rank}(A') = \text{min}\{\text{rank}(A), \text{rank}(A^\sigma)\}$, as required. $\square$

Remark 10.7. Assume $C$ is semilocal or $Q$-finite and let $\tau$ be any automorphism of $C$. Then Corollary 7.5 (together with Theorem 9.2) implies that $A$ is Brauer equivalent to an Azumaya algebra with an anti-automorphism of type $\tau$ if and only if $A^{\text{op}} \sim_{\text{Br}} A^\tau$. The semilocal part of this claim is somewhat standard since then $A^{\text{op}} \sim_{\text{Br}} A^\tau$ implies $A^{\text{op}} \cong A^\tau$ (see section 11). However, the $Q$-finite case seems to be unknown. When $\tau$ is a Galois automorphism of order 2 (i.e. $C/C(\tau)$ is Galois with Galois group $\{1, \tau\}$), the condition $A^{\text{op}} \sim_{\text{Br}} A^\tau$ is equivalent to $A \otimes_C A^\tau \cong \text{Res}_{C/C_0}(\text{Cor}(C/C_0)(A)) \sim C$. In this special case, it could be that the claim is true without any assumption on $C$.

11. The Semilocal Case

Let $C$ be a commutative ring and let $A$ be an Azumaya $C$-algebra such that $A \sim_{\text{Br}} A^{\text{op}}$ (resp. $\text{Cor}(C/C_0)(A) \sim_{\text{Br}} C_0$) with $C/C_0$ a Galois extension with Galois group $\{1, \sigma\}$. Then by Theorem 10.2, there exists an Azumaya $C$-algebra $B$ such that $B \sim_{\text{Br}} A$ and $B$ has an involution of the first kind (resp. involution whose restriction to $C$ is $\sigma$). In [13, Th. 4.4], Saltman shows that if $C$ is semilocal and connected, then we can take $B = A$. In this short section, we explain how to
deduce the same result using our methods. To some extent, our proof is merely a
reformulation of Saltman’s proof in our terminology.

By the proof of Theorem 10.3, there exists a double A-progenerator K of type idC (resp. σ) with an involution θ. Assume that K1 = A4 and let u ∈ A*. Then every k ∈ K = A can be written as u ⊙ i1 a for unique a ∈ A. Using this, define α : A → A by letting aα be the unique element of A satisfying u ⊙ 0 a = u ⊙ i1 aα. Then α is easily seen to be an anti-automorphism of A of type idC (resp. σ). Moreover, if uα = u, then α is an involution of A since u ⊙ 1 a = (u ⊙ 1 a)θθ = (u ⊙ 0 a)aθ = (u ⊙ 1 aθ)a = u ⊙ 0 aα = u ⊙ 1 aα. Therefore, if K1 ∼= A and θ fixes a unit of A, then A has an involution.

Now assume C is semilocal. Then A is also semilocal. We claim that for P, Q ∈ proj-A, rank(P) = rank(Q) implies P ∼= Q. Indeed, by tensoring with C/ Jac(C), we may assume C is a finite product of fields and A is a finite product of central simple algebras, in which case the claim is routine. Let τ ∈ Aut(C) be the type of K. If τ = idC or if C is connected, then rank(Aτ) = rank(A). Therefore, by Lemma 10.3, rank(Aτ) = rank(K1), hence K1 ∼= A (which in turn implies A ∼= End(AK1) ∼= EndA(K1) ∼= (Aτ)op). Identify K with A. Then we are done if we show that θ fixes a unit of A. However, this follows from the discussion following Lm. 4.5 in [13]. (The map α in [13] plays the role of θ here and the map J : A → A in [14] is given by aJ = 1A ⊙ 0 a.)

12. COUNTEREXAMPLES AND OPEN QUESTIONS

We conclude this paper with counterexamples and several questions.

The first two examples show that there are “nice” rings which are not Morita equivalent to rings with an involution (of any kind), but still admit an anti-automorphism. In the first example, that automorphism fixes the center, implying that Remark 10.3(i) does not generalize to arbitrary central algebras.

Example 12.1. Recall that a poset consists of a finite set I equipped with a transitive reflexive relation which we denote by ≤. For a field F and a poset I, the incidence algebra F(I) is defined to be the subalgebra of the I-indexed matrices over F spanned as an F-vector space by the matrix units {ei,j | i, j ∈ I, i ≤ j}. If I is not the disjoint union of two non-comparable subsets, then Cent(F(I)) = F. In this case, we say I is connected.

Let R be a ring that is Morita equivalent to a poset algebra. Then it can be shown that R is a poset algebra as well. In particular, the basic ring that is Morita equivalent to R is also a poset algebra, which we denoted by F(IR). The poset IR, which is unique up to isomorphism, can be extracted from R as follows: Let E denote the set of primitive idempotents in R. Then R× acts by conjugation on E. Define IR to be the set of equivalence classes in E, and for i, j ∈ IR, let i ≤ j if and only if eRFj = 0 for some (and hence any) e ∈ i and f ∈ j. This description of IR implies that any anti-automorphism (resp. involution) of R induces an anti-automorphism (resp. involution) on IR.

The poset algebra F(I) is basic if and only if i ≤ j and j ≤ i implies i = j for all i, j ∈ I. We say that I is basic in this case. Since any anti-automorphism of I gives rise to an F-anti-automorphism of F(I), the previous discussion implies that if I is connected, basic, has no involution, but still admits an anti-automorphism, then the ring F(I) has an anti-automorphism of type idF while not being Morita equivalent to a ring with involution. Such a poset was given in [16] by Scharlau.
(for other purposes); $I$ is the 12-element poset whose Hasse diagram is:

(Using Schafalau’s words, it is “the simplest example I could find.”) An anti-automorphism of $I$ is given by rotating the diagram ninety degrees clockwise.

**Example 12.2.** Various f.d. division algebras admitting an anti-automorphism but no involution are constructed in [12]. By Proposition 8.3, none of these algebras is Morita equivalent to a ring with involution.

The next example shows that there exists a ring $R$ with double $R$-progenerator $K$ such that $M^{[1]} ≠ M$ for all $0 ≠ M ∈ \text{proj-}R$ and $n ∈ \mathbb{N}$. In particular, Theorem 7.4 fails for arbitrary rings.

**Example 12.3.** Let $F$ be a field and let $R = \varprojlim \{M_2(F)^{⊗n}\}_{n ∈ \mathbb{N}}$. Then any f.g. projective right $R$-module is obtained by scalar extension from a f.g. projective module over $M_2(F)^{⊗n} \hookrightarrow R$. Using this, it not hard (but tedious) to show that the monoid $(\text{proj-}R/\cong, \oplus)$ is isomorphic to $(\mathbb{Z}[\frac{1}{2}] \cap [0, ∞), +)$. If $V_n$ is the unique indecomposable projective right module over $M_2(F)^{⊗n}$, then $V_n ⊗ R$ is mapped to $2^{−n}$.

Let $T$ denote the transpose involution on $M_2(F)$. Then $\hat{\tau} = \varprojlim \{T^{⊗n}\}_{n ∈ \mathbb{N}}$ is an involution of $R$. Let $K = R^2 ∈ \text{proj-}R$. Then $\text{End}_R(K) ≅ M_2(R) ≅ R$ and using $\hat{\tau}$, we can identify $\text{End}_R(K)$ with $R^{op}$, thus making $P$ into an $(R^{op}, R)$-progenerator. We claim that for $M ∈ \text{proj-}R$ and $n ∈ \mathbb{N}$, $M^{[1]} ∩ M$ implies $M = 0$. To see this, let $ϕ_1$ be the map obtained from $[1]$ by identifying $\text{proj-}R/\cong$ with $\mathbb{Z}[\frac{1}{2}] \cap [0, ∞)$. Then $ϕ_1(2) = 1$ because $(R^{op})^{[1]} ≅ K_1^1 ≅ R_R$. Therefore, $ϕ_1(x) = \frac{1}{2} x$ for all $x ∈ \mathbb{Z}[\frac{1}{2}] \cap [0, ∞)$, which means that $(ϕ_1 ∘ ⋅ ∘ ϕ_1)(x) ≠ x$ for all $0 ≠ x ∈ \mathbb{Z}[\frac{1}{2}] \cap [0, ∞)$.

In our last example, we construct a ring $R$ such that $M_2(R)$ has an involution but $R$ does not (in contrast to Proposition 5.1). In fact, the ring $R$ will also be Azumaya over its center. The example uses fractional ideals, so in order to distinguish between the $n$-th power of a fractional ideal $I$ and the direct sum of $n$ copies of that ideal, the latter will be denoted by $I^{⊗n}$.

**Example 12.4.** Let $K$ be a number field such that $\text{Gal}(K/\mathbb{Q}) = \{\text{id}_K\}$ and let $D$ be the integral closure of $\mathbb{Z}$ in $K$. Assume that there is a fractional $D$-ideal $I$ such that $I$ has order 4 in the class group of $K$ (i.e. $I^4$ is principal, but $I^2$ is not principal) and $I^2$ is not a fourth power in the class group. Let $M = D^{⊗3} ⊕ I$ and let $S = \text{End}_D(M)$. Observe that $M^{⊗4} ≅ D^{⊗16}$, hence $M_4(S) ≅ M^{16}(D)$. Thus, $M_4(S)$ has an involution. We claim that $S$ does not have an anti-automorphism. Assume by contradiction that $α$ is an anti-automorphism. Then $α$ restricts to an automorphism of $D = \text{Cent}(S)$, and hence extends to an automorphism of $K$. As $\text{Gal}(K/\mathbb{Q}) = \{\text{id}_K\}$, $α$ must fix $K$, and hence $D$. By Theorem 5.2 and Lemma 6.1 there is a regular bilinear form $b : M × M → K$ with $K$ a double $D$-progenerator of type $\text{id}_D$ (which is the type of $α$). This means that $K$ can
be understood as a fractional $D$-ideal $J$, considered double $D$-module by setting $j \odot_0 d = j \odot_1 d = jd$ for all $j \in J$ and $d \in D$. Since $b$ is regular, we have an isomorphism $D^{\oplus 3} \oplus I = M \cong M^1 = \text{Hom}_D(M, J) \cong J^{\oplus 3} \oplus JI^{-1}$. This is well-known to imply $D \cdot I \cong J^3 \cdot JI^{-1}$, i.e. $I^2 \cong J^4$, a contradiction. Therefore, we can take $R = S$ if $M_2(S)$ has an involution, and $R = M_2(S)$ otherwise. Explicit choices of $K, D, I$ are $K = \mathbb{Q}[x \mid x^3 = 43], D = \mathbb{Z}[x], I = \langle 2, x+1 \rangle$ (verified using SAGE).

We finish with several questions that we could not answer.

**Question 1.** Is there a ring $R$ such that $R \cong_{\text{Mor}} R^{op}$ but $R$ is not Morita equivalent to a ring with an anti-automorphism? (or, does $(C) \Rightarrow (B)$ in general?)

**Question 2.** Is there a semiperfect ring $R$ admitting an involution such that the basic ring that is Morita equivalent to $R$ does not have an involution?

**Question 3.** Is there a non-Azumaya algebra admitting a Goldman element?

**Question 4.** Let $C/C_0$ be a Galois extension with Galois group $\{1, \sigma\}$. Is it true that all Azumaya $C$-algebras $A$ with $A \cong_{\text{Br}} A^\sigma$ are Brauer equivalent to an Azumaya algebra with an anti-automorphism whose restriction to $C$ is $\sigma$?

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**References**

[1] A. Adrian Albert. *Structure of algebras*. Revised printing. American Mathematical Society Colloquium Publications, Vol. XXIV. American Mathematical Society, Providence, R.I., 1961.

[2] H. Bass. K-theory and stable algebra. *Inst. Hautes Études Sci. Publ. Math.*, (22):5–60, 1964.

[3] Hyman Bass. Finite-dimensional and a homological generalization of semi-primary rings. *Trans. Amer. Math. Soc.*, 95:466–488, 1960.

[4] E. Bayer-Fluckiger, C. Kearton, and S. M. J. Wilson. Decomposition of modules, forms and simple knots. *J. Reine Angew. Math.*, 375/376:167–183, 1987.

[5] Frank DeMeyer and Edward Ingraham. *Separable algebras over commutative rings*. Lecture Notes in Mathematics, Vol. 181. Springer-Verlag, Berlin, 1971.

[6] Uriya A. First. General bilinear forms. Submitted, 2013.

[7] I. N. Herstein. *Rings with involution*. The University of Chicago Press, Chicago, Ill.-London, 1976. Chicago Lectures in Mathematics.

[8] Nathan Jacobson. *Finite-dimensional division algebras over fields*. Springer-Verlag, Berlin, 1996.

[9] M.-A. Knus, R. Parimala, and V. Srinivas. Azumaya algebras with involutions. *J. Algebra*, 130(1):65–82, 1990.

[10] Max-Albert Knus and Manuel Ojanguren. *Théorie de la descente et algèbres d’Azumaya*. Lecture Notes in Mathematics, Vol. 389. Springer-Verlag, Berlin, 1974.

[11] T. Y. Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[12] P. J. Morandi, B. A. Sethuraman, and J.-P. Tignol. Division algebras with an anti-automorphism but with no involution. *Adv. Geom.*, 5(3):485–495, 2005.

[13] Louis H. Rowen. *Ring theory. Vol. I*, volume 127 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.

[14] David J. Saltman. Azumaya algebras with involution. *J. Algebra*, 52(2):526–539, 1978.

[15] David J. Saltman. *Lectures on division algebras*, volume 94 of *CBMS Regional Conference Series in Mathematics*. Published by American Mathematical Society, Providence, RI, 1999.

[16] Winfried Scharlau. Automorphisms and involutions of incidence algebras. In *Proceedings of the International Conference on Representations of Algebras (Carleton Univ., Ottawa, Ont., 1974)*, Paper No. 24, pages 11 pp. Carleton Math. Lecture Notes, No. 9, Ottawa, Ont., 1974. Carleton Univ.