Derivatives of tangent function and tangent numbers

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Abstract
In the paper, by induction, the Faà di Bruno formula, and some techniques in the theory of complex functions, the author finds explicit formulas for higher order derivatives of the tangent and cotangent functions as well as powers of the sine and cosine functions, obtains explicit formulas for two Bell polynomials of the second kind for successive derivatives of sine and cosine functions, presents curious identities for the sine function, discovers explicit formulas and recurrence relations for the tangent numbers, the Bernoulli numbers, the Genocchi numbers, special values of the Euler polynomials at zero, and special values of the Riemann zeta function at even numbers, and comments on five different forms of higher order derivatives for the tangent function and on derivative polynomials of the tangent, cotangent, secant, cosecant, hyperbolic tangent, and hyperbolic cotangent functions.

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1. Main results

It is well known that the tangent function $\tan x$ can be expanded into the Maclaurin series

$$\tan x = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k} (2^{2k} - 1) B_{2k}}{(2k)!} x^{2k-1}, \quad |x| < \frac{\pi}{2},$$

see [1, p. 75, 4.3.67] and [5, p. 259], where $T_{2k-1}$ are called the tangent numbers or zag numbers and $B_n$ for $n \geq 0$ are the Bernoulli numbers which may be defined by the power series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}, \quad |x| < 2\pi.$$

The tangent numbers $T_{2k-1}$ may also be defined combinatorially as the numbers of alternating permutations on $2k - 1 = 1, 3, 5, 7, \ldots$ symbols (where permutations that are the reverses of one another counted as equivalent). The first few tangent numbers $T_{2k-1}$ for $k = 1, 2, \ldots, 5$ are 1, 2, 16, 272, 7936.

It is clear that

$$T_{2k-1} = (-1)^{k-1} \frac{2^{2k-1} (2^{2k} - 1)}{k} B_{2k} = \lim_{x \to 0} \tan^{(2k-1)} x. \quad (1.1)$$

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Consequently, one way to compute the tangent numbers $T_{2k-1}$ and the Bernoulli numbers $B_{2k}$ is to find explicit formulas for $\tan^{(2k-1)} x$.

It is well known that the Riemann zeta function $\zeta(s)$ may be defined by

$$
\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \Re(s) > 0,
$$

that the Genocchi numbers $G_k$ are given by the generating function

$$
\frac{2z}{e^z + 1} = \sum_{k=0}^{\infty} G_k \frac{z^k}{k!}, \quad |z| < \pi,
$$

and that the Euler polynomials $E_k(x)$ are defined by

$$
\frac{2e^{xz}}{e^z + 1} = \sum_{k=0}^{\infty} E_k(x) \frac{z^k}{k!}, \quad |z| < \pi.
$$

The tangent numbers $T_{2k-1}$, the Bernoulli numbers $B_{2k}$, the Genocchi numbers $G_k$, the Euler polynomials $E_k(x)$, and the Riemann zeta function $\zeta(s)$ have close relations.

In combinatorics, the Bell polynomials of the second kind $B_{n,k}$, denoted by $B_{n,k}$ for $n \geq k \geq 0$, are defined by

$$
B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{l\geq 0, \sum_{q=1}^{n-k} l_q = n} \frac{n!}{\prod_{q=1}^{n-k} l_q !} \prod_{q=1}^{n-k} \left(\frac{x_q}{l_q}\right)^{l_q}.
$$

See [5, p. 134, Theorem A]. In combinatorial analysis, the Faà di Bruno formula plays an important role and may be described in terms of the Bell polynomials of the second kind $B_{n,k}$ by

$$
\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=1}^{n} f^{(k)}(h(t)) B_{n,k}(h''(t), h'''(t), \ldots, h^{(n-k+1)}(t)).
$$

See [5, p. 139, Theorem C].

In this paper, by induction, the Faà di Bruno formula, and some techniques in the theory of complex functions, we will establish general and explicit formulas for the $n$th derivatives of $\tan x$, $\cot x$, $\sin^k x$, and $\cos^k x$ for $k \in \mathbb{N}$, presents curious identities for the sine function, and obtain explicit formulas for two Bell polynomials of the second kind

$$
B_{m,k}(-\sin x, -\cos x, \sin x, \cos x, \ldots, -\sin\left[x + (m-k)\pi/2\right])
$$

and

$$
B_{m,k}(\cos x, -\sin x, -\cos x, \sin x, \ldots, -\cos\left[x + (m-k)\pi/2\right]).
$$

By applying these formulas, we will also derive explicit formulas and recurrence relations for the tangent numbers $T_{2n-1}$, the Bernoulli numbers $B_{2n}$, the Genocchi numbers $G_{2n}$, special values $E_{2n-1}(0)$ of the Euler polynomials at 0, and special values $\zeta(2n)$ of the Riemann zeta function $\zeta(z)$ at even numbers $2n$. Finally, we will comment on five different forms of higher order derivatives for $\tan x$ and on derivative polynomials of $\tan x$, $\cot x$, $\sec x$, $\csc x$, $\tan h x$, and $\coth x$.

Our main results may be stated as the following theorems.

**Theorem 1.1.** For $n \in \mathbb{N}$, derivatives of the tangent and cotangent functions may be computed by

$$
\tan^{(n)} x = \frac{1}{\cos^{n+1} x} \left\{ \frac{1}{2} \left[ \frac{1 + 1 + (-1)^n}{2} \right] a_{n, \frac{1+(-1)^n}{2}} \sin \left[ \frac{1 + (-1)^n}{2} x + \frac{1 - (-1)^n \pi}{2} \right] + \sum_{k=1}^{\frac{n-1}{2}} a_{n, 2k+\frac{1+(-1)^n}{2}} \sin \left[ \frac{2k + 1 + (-1)^n}{2} x + \frac{1 - (-1)^n \pi}{2} \right] \right\}
$$

and

$$
\cot^{(n)} x = \frac{(-1)^n}{\sin^{n+1} x} \left\{ \frac{1}{2} \left[ \frac{1 + 1 + (-1)^n}{2} \right] a_{n, \frac{1+(-1)^n}{2}} \cos \left[ \frac{1 + (-1)^n}{2} x \right] + \sum_{k=1}^{\frac{n-1}{2}} (-1)^k a_{n, 2k+\frac{1+(-1)^n}{2}} \cos \left[ \frac{2k + 1 + (-1)^n}{2} x \right] \right\}.
$$

(1.3)
Theorem 1.2. Let \( k, m \) be nonnegative integers such that \( (k, m) \neq (0, 0) \).

For all \( k, m \geq 0 \), the sine and cosine functions satisfy

\[
\frac{d^m \sin^k x}{dx^m} = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} \frac{(-1)^\ell}{2^\ell} \binom{k}{\ell} \frac{1}{\cos^\ell x} \sum_{q=0}^{\ell} \binom{\ell}{q} (2q - \ell + 1) m + (2q - k)x \tag{1.10}
\]

and

\[
\frac{d^m \cos^k x}{dx^m} = \frac{1}{k!} \sum_{\ell=0}^{k} \frac{(-1)^\ell}{2^\ell} \binom{k}{\ell} \frac{1}{\cos^\ell x} \sum_{q=0}^{\ell} \binom{\ell}{q} (2q - \ell + 1) m + (2q - k)x \tag{1.11}
\]

For \( n \in \mathbb{N} \), derivatives of the tangent and cotangent functions may be computed by

\[
\frac{d^n \tan x}{dx^n} = -\sum_{k=1}^{n+1} \frac{1}{k} \sum_{\ell=0}^{k} \frac{(-1)^\ell}{2^\ell} \binom{k}{\ell} \frac{1}{\cos^\ell x} \sum_{q=0}^{\ell} \binom{\ell}{q} (2q - \ell + 1) (n - \ell + 1) m + (2q - \ell)x \tag{1.12}
\]

and

\[
\frac{d^n \cot x}{dx^n} = \sum_{k=1}^{n+1} \frac{1}{k} \sum_{\ell=0}^{k} \frac{(-1)^\ell}{2^\ell} \binom{k}{\ell} \frac{1}{\sin^\ell x} \sum_{q=0}^{\ell} \binom{\ell}{q} (2q - \ell + 1) (n - \ell + 1) m + (2q - \ell)x \tag{1.13}
\]

Theorem 1.3. For \( m \in \mathbb{N} \), the tangent numbers \( T_{2m-1} \) may be computed by

\[
T_{2m-1} = \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} (m - k)^{2m-1} + 2 \sum_{k=1}^{m-1} (-1)^k \sum_{\ell=0}^{m-k-1} (-1)^\ell \binom{2m}{\ell} (m - k - \ell)^{2m-1} \tag{1.14}
\]
and satisfy a recurrence

$$T_{2m+1} = \sum_{k=1}^{m} \binom{2m}{2k-1} T_{2k-1} T_{2(m-k)+1}.$$  

Theorem 1.4. For $m \in \mathbb{N}$, the Bernoulli numbers $B_{2m}$ may be computed by

$$B_{2m} = (-1)^{m-1} \frac{m}{2^{2m-1} (2^{2m-1}-1)} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} (m-k)2^{m-1} + 2 \sum_{k=1}^{m-1} (-1)^k \sum_{\ell=0}^{m-k-1} (-1)^\ell \binom{2m}{\ell} (m-k-\ell)2^{m-1} \right)$$

(1.17)

and

$$B_{2m} = \frac{m}{2^{2m-1}(2^{2m-1}-1)} \sum_{\ell=1}^{2m} (-1)^\ell \frac{1}{2^\ell} \left( \frac{1}{\ell - \frac{1}{m+1}} \right) \left( \frac{2m+1}{\ell} \right) \sum_{q=0}^{\ell} \left( \frac{\ell}{q} \right) (2q - \ell)2^m$$

(1.18)

and the sequence $B_{2m} = \frac{2^{2m}-1}{2^{2m}} B_{2m}$ satisfies a recurrence

$$B_{2(m+1)} = -\sum_{k=1}^{m} \binom{2m}{2k-1} B_{2k} B_{2(m-k+1)}.$$  

(1.19)

For $m \in \mathbb{N}$, special values $\zeta(2m)$ of the zeta function $\zeta(z)$ at $z = 2m$ may be computed by

$$\zeta(2m) = \frac{\pi^{2m} m}{(2m)! (2^{2m-1}-1)} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} (m-k)2^{m-1} + 2 \sum_{k=1}^{m-1} (-1)^k \sum_{\ell=0}^{m-k-1} (-1)^\ell \binom{2m}{\ell} (m-k-\ell)2^{m-1} \right)$$

(1.20)

and

$$\zeta(2m) = (-1)^m \frac{\pi^{2m} m}{(2m)! (2^{2m-1}-1)} \sum_{\ell=1}^{2m} (-1)^\ell \frac{1}{2^\ell} \left( \frac{1}{\ell - \frac{1}{m+1}} \right) \left( \frac{2m+1}{\ell} \right) \sum_{q=0}^{\ell} \left( \frac{\ell}{q} \right) (2q - \ell)2^m$$

(1.21)

and the sequence $Z_{2m} = (-1)^m \frac{(2m)! (2^{2m-1}-1)}{(2^m)! 2^{2m}} \zeta(2m)$ satisfies a recurrence

$$Z_{2(m+1)} = \sum_{k=1}^{m} \binom{2m}{2k-1} Z_{2k} Z_{2(m-k+1)}.$$  

(1.22)

For $m \in \mathbb{N}$, the Genocchi numbers $G_{2m}$ may be calculated by

$$G_{2m} = (-1)^{m-1} \frac{m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} (m-k)2^{m-1} + 2 \sum_{k=1}^{m-1} (-1)^k \sum_{\ell=0}^{m-k-1} (-1)^\ell \binom{2m}{\ell} (m-k-\ell)2^{m-1} \right)$$

(1.23)

and

$$G_{2m} = \frac{2m}{2^{2m-1}} \sum_{\ell=1}^{2m} (-1)^\ell \frac{1}{2^\ell} \left( \frac{1}{\ell - \frac{1}{m+1}} \right) \left( \frac{2m+1}{\ell} \right) \sum_{q=0}^{\ell} \left( \frac{\ell}{q} \right) (2q - \ell)2^m$$

(1.24)

and the sequence $G_{2m} = \frac{1}{m} G_{2m}$ satisfies a recurrence

$$G_{2(m+1)} = \frac{1}{4} \sum_{k=1}^{m} \binom{2m}{2k-1} G_{2k} G_{2(m-k+1)}.$$  

(1.25)

For $m \in \mathbb{N}$, special values $E_{2m-1}(0)$ of the Euler polynomials $E_m(x)$ at $x = 0$ may be calculated by

$$E_{2m-1}(0) = (-1)^m \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} (m-k)2^{m-1} + 2 \sum_{k=1}^{m-1} (-1)^k \sum_{\ell=0}^{m-k-1} (-1)^\ell \binom{2m}{\ell} (m-k-\ell)2^{m-1} \right)$$

(1.26)

and

$$E_{2m-1}(0) = \frac{1}{2^{2m-1}} \sum_{\ell=1}^{2m} (-1)^\ell \frac{1}{2^\ell} \left( \frac{1}{\ell - \frac{1}{m+1}} \right) \left( \frac{2m+1}{\ell} \right) \sum_{q=0}^{\ell} \left( \frac{\ell}{q} \right) (2q - \ell)2^m$$

(1.27)

and the sequence $E_{2m-1}(0)$ satisfies a recurrence

$$E_{2m+1}(0) = \frac{1}{2} \sum_{k=1}^{m} \binom{2m}{2k-1} E_{2k-1}(0) E_{2(m-k)+1}(0).$$  

(1.28)
2. Comparisons and highlights

Before proving our main results, we compare them with some known conclusions for showing their highlights.

2.1. Applications of the Bell polynomials \((1.10)\) and \((1.11)\)

By the Faà di Bruno formula \((1.2)\) and the Bell polynomials \((1.10)\) and \((1.11)\) in Theorem 1.2, it is easy to write

\[
(\sec x)^{(n)} = \left(\frac{1}{\cos x}\right)^{(n)} = \frac{1}{\cos x} \sum_{k=1}^{n} \sum_{\ell=0}^{k} \frac{(-1)^{\ell}}{\ell!} \frac{1}{x} \sum_{q=0}^{\ell} \frac{(\ell)!}{q!} \cos \left[\frac{\pi}{2} n + (2q - \ell)x\right]
\]

and

\[
(\csc x)^{(n)} = \left(\frac{1}{\sin x}\right)^{(n)} = \frac{1}{\sin x} \sum_{k=1}^{n} \sum_{\ell=0}^{k} \frac{1}{2^\ell} \frac{k!}{\ell!} \frac{1}{x} \sum_{q=0}^{\ell} \frac{(-1)^{q}}{q!} \cos \left[(n - \ell) \frac{\pi}{2} + (2q - \ell)x\right].
\]

Consequently, we may find explicit formulas for the secant and cosecant numbers. Due to the limitation of length, we do not write down them in details.

The formulas \((1.10)\) and \((1.11)\) answer a problem in [39, Section 5]. These formulas, together with the Faà di Bruno formula \((1.2)\), may be applied to calculate the \(n\)th derivatives of functions of the forms \(f(\sin x)\) and \(f(\cos x)\), such as \((\sin x)^n\), \((\cos x)^n\), \((\sec x)^n\), \((\csc x)^n\), \(e^{\sin x}\), \(e^{\cos x}\), \(\ln(\cos x)\), \(\ln(\sin x)\), \(\ln(\sec x)\), \(\ln(\csc x)\), \(\sin(\sin x)\), \(\cos(\sin x)\), \(\sin(\cos x)\), and \(\cos(\cos x)\).

2.2. Five different forms of higher order derivatives for \(\tan x\)

On [41, pp. 28–31, Chapter II], see also its second edition [42], by virtue of the formula

\[
\frac{d^n y}{dx^n} = \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{\alpha=1}^{k} (-1)^\alpha k! \frac{d^n}{dx^n} \frac{d^\alpha y}{du^\alpha}
\]

in [41, p. 12, (83)], the formulas

\[
\frac{d^{2n} \tan x}{dx^{2n}} = (-1)^{n+1} 2^{2n} \sum_{k=1}^{2n} \frac{1}{2^k} \sec^{k+1} x \sin[(k - 1)x] \sum_{\alpha=1}^{k} (-1)^\alpha \left(\frac{k}{\alpha}\right) \alpha^{2n},
\]

\[
\frac{d^{2n+1} \tan x}{dx^{2n+1}} = (-1)^{n+1} 2^{2n+1} \sum_{k=1}^{2n+1} \frac{1}{2^k} \sec^{k+1} x \cos[(k - 1)x] \sum_{\alpha=1}^{k} (-1)^\alpha \left(\frac{k}{\alpha}\right) \alpha^{2n+1},
\]

\[
\frac{d^n \tan x}{dx^n} = (-1)^{\frac{n+1}{2}} \left[\frac{\pi - (-1)^{n+1}}{2} + (k - 1)x\right] \sum_{\alpha=1}^{k} (-1)^\alpha \left(\frac{k}{\alpha}\right) \alpha^n
\]

\[
\frac{d^{2n} \tan x}{dx^{2n}} = (-1)^{n+1} 2^{2n} \sec^2 x \sum_{k=1}^{2n} \frac{1}{2^k} \sum_{\alpha=1}^{k} (-1)^\alpha \left(\frac{k}{\alpha}\right) \alpha^{2n} N_{2\beta+1},
\]

\[
\frac{d^{2n+1} \tan x}{dx^{2n+1}} = (-1)^{n+1} 2^{2n+1} \sec^2 x \sum_{k=1}^{2n+1} \frac{1}{2^k} \sum_{\alpha=1}^{k} (-1)^\alpha \left(\frac{k}{\alpha}\right) \alpha^{2n+1} N_{2\beta},
\]

\[
\frac{d^n \tan x}{dx^n} = (-1)^{\frac{n+1}{2}} 2^n \sec^2 x \sum_{k=1}^{n} \frac{1}{2^k} \sum_{\alpha=1}^{k} (-1)^\alpha \left(\frac{k}{\alpha}\right) \alpha^n N_{2\beta+1}^{-1/2 - 1/2\alpha}
\]

were obtained for \(n \in \mathbb{N}\), where

\[N_{2\beta} = \sum_{\beta=0}^{\left[\frac{n+1}{2}\right]} (-1)^\beta \binom{k-1}{2\beta} \tan^{2\beta+1} x, \quad N_{2\beta+1} = \sum_{\beta=0}^{\left[\frac{n+1}{2}\right]} (-1)^\beta \binom{k-1}{2\beta+1} \tan^{2\beta+1} x,\]

and \([x]\), whose value is the biggest integer not more than \(x\), stands for the floor function of \(x\).

The formula \((1.3)\) may be separately written as

\[
\tan^{(2n-1)} x = \frac{1}{\cos^{2n} x} \sum_{k=0}^{n-1} d_{2n-1,2k} \cos (2kx)
\]

and
\[
\tan^{(2n)} x = \frac{1}{\cos^{2n+1} x} \sum_{k=0}^{n-1} a_{2n,2k+1} \sin[(2k+1)x]
\]
(2.8)

for \( n \in \mathbb{N} \), where
\[
a_{1,0} = 1,
\]
\[
a_{2n-1,0} = 2n \sum_{\ell=0}^{n-2} (-1)^\ell \binom{2n-1}{\ell} (n - \ell - 1)^{2n-2}
\]
for \( n > 1 \), and
\[
a_{p,q} = (-1)^q (p-1)(p-3)\cdots (p-2q+1) \binom{n}{\ell} \binom{n}{\ell-1} \cdots \binom{n}{n-1} v^n
\]
for \( 0 < q < p \) with \( p - q \) being a positive odd number. See the first version of the preprint [30]. It is clear that, to some extent, the forms of the formulas (2.7) and (2.8) are apparently simpler than those of the formulas (2.1) and (2.2). On the other hand, there are 2n and 2n + 1 terms in the formulas (2.1) and (2.2) respectively, but only \( n \) terms in both of the formulas (2.7) and (2.8). This means that the formula (1.3) in Theorem 1.1 is essentially simpler than (2.3).

Let \( u = u(x) \) and \( v = v(x) \neq 0 \) be differentiable functions. In [2, p. 40], the formula
\[
d^n u \over dx^n \left( \frac{u}{v} \right) = (-1)^n \frac{A_{(n+1)} \cdot 1}{B_{(n+1) \times n}}
\]
(2.9)

for the \( n \)th derivative of the ratio \( \frac{u(x)}{v(x)} \) was listed. For easy understanding and convenient availability, we now reformulate the formula (2.9) as
\[
d^n u \over dx^n \left( \frac{u}{v} \right) = (-1)^n \frac{A_{(n+1)} \cdot 1}{B_{(n+1) \times n}} [u \ u' \ u'' \ \cdots \ 0 \ \cdots \ \cdots \ \cdots \ 0 \ \cdots \ \cdots \ \cdots \ \cdots \ 0]
\]
(2.10)

\[
x_{\lambda} = \sin x_{\lambda} \]
(2.11)

Therefore, we can find an alternative form, the fifth form, for higher order derivatives of \( \tan x \). The first four forms for higher order derivatives of \( \tan x \) are (1.3), (1.12), (2.3), and (2.6). These five different forms come from the induction and different formulas for higher order derivatives of composite functions.

### 2.3. Derivative polynomials

Suppose \( f \) is a function whose derivative is a polynomial in \( f \), that is, \( f'(x) = P(f(x)) \) for some polynomial \( P \). Then all the higher order derivatives of \( f \) are also polynomials in \( f \), so we have a sequence of polynomials \( P_n \) defined by \( f^{(n)}(x) = P_n(f(x)) \) for \( n \geq 0 \). As usual, we call \( P_n(u) \) the derivative polynomials of \( f \).

In terms of this terminology, some results in [12,15,43] may be restated as follows: when \( \lambda > 0 \) and \( t \neq -\frac{\ln 2}{\lambda} \) or when \( \lambda < 0 \) and \( t \in \mathbb{R} \), the derivative polynomials of the function \( x^{\lambda e^{-t}} \) are
\[
(-1)^n \alpha^n \sum_{m=1}^{n+1} (m - 1)!(n + 1, m)u^m
\]
(2.12)
\[ S(q, m) = \frac{1}{m!} \sum_{\ell=1}^{m} (-1)^{m-\ell} \left( \begin{array}{c} m \\ \ell \end{array} \right) \ell^q \]

for \( 1 \leq m \leq q \) are the Stirling numbers of the second kind which may be generated by
\[
\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!}, \quad k \in \mathbb{N}.
\]

In [17, p. 25, (5)] and [17, Theorem 3.2], it was obtained that the derivative polynomials \( P_n \) and \( Q_n \) defined by
\[
\frac{d^n (\tan x)}{dx^n} = P_n(\tan x) \quad \text{and} \quad \frac{d^n (\sec x)}{dx^n} = Q_n(\tan x) \sec x
\]
for \( n \geq 0 \) are polynomials of degree \( n + 1 \) and \( n \) respectively and satisfy the recurrences
\[
P_{n+1}(u) = \sum_{k=0}^{n} \binom{n}{k} P_k(u)P_{n-k}(u) + \delta_{0n},
\]
\[
Q_{n+1}(u) = \sum_{k=0}^{n} \binom{n}{k} P_k(u)Q_{n-k}(u).
\]
\[
P_{n+1}(u) = (1 + u^2) \sum_{k=0}^{n} \binom{n}{k} Q_k(u)Q_{n-k}(u). \tag{2.13}
\]

where
\[
\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}
\]

In [18], more recurrences for the derivative polynomials \( P_n \) and \( Q_n \) were obtained and they were applied to combinatorics and number theory. We observe that the formulas
\[
\tan^{(n+1)} x = (\tan^2 x)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \tan^{(k)} x \tan^{(n-k)} x,
\]
\[
\sec^{(n+1)} x = (\sec x \tan x)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \sec^{(k)} x \tan^{(n-k)} x,
\]
\[
\tan^{(n+1)} x = (\sec^2 x)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \sec^{(k)} x \sec^{(n-k)} x \tag{2.14}
\]

may be used to straightforwardly and simply recover the formulas in (2.13) for \( n \in \mathbb{N} \).

Let
\[
D = \frac{d}{dx}, \quad y = \tan x, \quad \text{and} \quad z = \sec x.
\]

For \( n \geq 0 \), define
\[
(Dy)^0(y) = y, \quad (Dy)(y) = D(y^2), \quad (Dy)^0(z) = z, \quad (Dy)(z) = D(yz),
\]
\[
(Dy)^{n+1}(y) = (Dy)(Dy)^n(y) = D(y(Dy)^n(y)),
\]
\[
(Dy)^{n+1}(z) = (Dy)(Dy)^n(z) = D(y(Dy)^n(z)),
\]
\[
(yD)^{n+1}(y) = (yD)(yD)^n(y) = yD((yD)^n(y)),
\]
\[
(yD)^{n+1}(z) = (yD)(yD)^n(z) = yD((yD)^n(z)). \tag{2.15}
\]

In [21,22], the above quantities were computed in terms of polynomials in variables \( y \) and \( z \) and connected with the Eulerian numbers and polynomials and others. We observe the following contradiction: since
\[
(Dy)^2(y) = (Dy)(Dy)(y) = (Dy)D(y^2)
\]
and
\[
(Dy)^2(y) = D(y(Dy)(y)) = D(yD(y^2)) = (Dy)D(y^2) + yD^2(y^2),
\]
it follows that
\[
(Dy)D(y^2) = (Dy)D(y^2) + yD^2(y^2),
\]
that is,
\[
yD^2(y^2) = 0,
which means that $y^2 = ax + b$, where $a, b \in \mathbb{C}$. Therefore, the authors of the papers \cite{21,22,24} and related ones using the definitions in (2.15) and their analogues should select a better manner and choose suitable symbols to express their ideas. The first author of the papers \cite{21,22,24} told the current author on May 24, 2015 that he explained in \cite[Theorem 10]{23} the above definitions in (2.15) and their analogues should select a better manner and choose suitable symbols to express their ideas. The

Because

$$
(\frac{d}{dx})^m \coth x = C_m(\coth x),
$$

$$
(\frac{d}{dx})^m \tanh x = C_m(\tanh x),
$$

$$
(\frac{d}{dx})^m \sech x = (\sech x)S_m(\tanh x),
$$

$$
(\frac{d}{dx})^m \csch x = (\csch x)S_m(\coth x),
$$

$$
P_m(z) = i^{m+1}2^m(1-iz)\omega_m \left(-\frac{1+iz}{2}\right) = -i^{m+1}(-2)^m(iz+1)\sum_{k=0}^{m} \frac{k!}{2^k} S(m, k)(iz-1)^k,
$$

$$
Q_m(z) = i^m S_m(iz)
$$

were established for $m \geq 1$, where $C_0(z) = 1$.

$$
C_m(z) = (-2)^m(z+1)\omega_m \left(-\frac{z-1}{2}\right) = (-2)^m(z+1)\sum_{k=0}^{m} \frac{k!}{2^k} S(m, k)(z-1)^k, \quad m \geq 1.
$$

$$
S_m(z) = \sum_{k=0}^{m} \left(\frac{m}{k}\right) 2^k \omega_k \left(-\frac{z+1}{2}\right) = \sum_{j=0}^{m} \left[(-1)^j j! \sum_{k=j}^{m} \left(\frac{m}{k}\right) S(k, j)2^{k-j}\right] (z+1)^j, \quad m \geq 0,
$$

and

$$
\omega_n(x) = \sum_{k=0}^{n} \frac{S(n, k)}{k!} x^k
$$

are called the geometric polynomials. We notice that the formulas

$$
tanh^{(n+1)} x = -(\tanh^2 x)^{(n)} = -\sum_{k=0}^{n} \left(\frac{n}{k}\right) tanh^{(k)} x \tanh^{(n-k)} x,
$$

$$
tanh^{(n+1)} x = (\sech^2 x)^{(n)} = \sum_{k=0}^{n} \left(\frac{n}{k}\right) \sech^{(k)} x \sech^{(n-k)} x,
$$

and

$$
\sech^{(n+1)} x = -(\sech x \tanh x)^{(n)} = -\sum_{k=0}^{n} \sech^{(k)} x \tanh^{(n-k)} x
$$

for $n \geq 1$ imply trivially the recurrences

$$
C_{n+1}(u) = -\sum_{k=0}^{n} \left(\frac{n}{k}\right) C_k(u) C_{n-k}(u),
$$

$$
C_{n+1}(u) = (1+u^2) \sum_{k=0}^{n} \left(\frac{n}{k}\right) S_k(u) S_{n-k}(u),
$$

$$
S_{n+1}(u) = -\sum_{k=0}^{n} \left(\frac{n}{k}\right) C_k(u) S_{n-k}(u)
$$

(2.16)

for $n \in \mathbb{N}$.

Because

$$
cot^{(n+1)} x = -(\cot^2 x)^{(n)} = -\sum_{k=0}^{n} \left(\frac{n}{k}\right) \cot^{(k)} x \cot^{(n-k)} x,
$$

$$
cot^{(n+1)} x = -(\csc^2 x)^{(n)} = -\sum_{k=0}^{n} \left(\frac{n}{k}\right) \csc^{(k)} x \csc^{(n-k)} x,
$$

and

$$
\csc^{(n+1)} x = -(\cot x \csc x)^{(n)} = -\sum_{k=0}^{n} \left(\frac{n}{k}\right) \cot^{(k)} x \csc^{(n-k)} x
$$

for $n \in \mathbb{N}$, if we define polynomials $P_n$ and $Q_n$ by
\[
\frac{d^n (\cot x)}{dx^n} = \mathcal{P}_n(\cot x) \quad \text{and} \quad \frac{d^n (\csc x)}{dx^n} = \mathcal{Q}_n(\cot x) \csc x.
\]

then the polynomials \(\mathcal{P}_n\) and \(\mathcal{Q}_n\) for \(n \in \mathbb{N}\) meet the recurrences

\[
\mathcal{P}_{n+1}(u) = - \sum_{k=0}^{n} \binom{n}{k} \mathcal{P}_k(u) \mathcal{P}_{n-k}(u),
\]

\[
\mathcal{Q}_{n+1}(u) = -(1 + u^2) \sum_{k=0}^{n} \binom{n}{k} \mathcal{Q}_k(u) \mathcal{Q}_{n-k}(u).
\]

What are the relations among \(\mathcal{P}_n\), \(\mathcal{Q}_n\), and \(C_n\)? What are the relations among \(\mathcal{Q}_n\), \(\mathcal{R}_n\), and \(S_n\)? Because they have similar recurrences in (2.13), (2.16), and (2.17), we conjecture that their differences are just a minus.

Finally we just mention that the asymptotic distribution of zeros of some of the above derivative polynomials were investigated in [7].

3. Proofs of main results

We are now in a position to prove Theorems 1.1 to 1.4.

**Proof of Theorem 1.1.** We prove this theorem by mathematical induction.

It is easy to obtain that

\[
(\tan x)' = \sec^2 x = \frac{1}{\cos^2 x} \quad \text{and} \quad (\tan x)'' = 2 \tan x \sec^2 x = \frac{2 \sin x}{\cos^3 x}
\]

This means \(a_{1,0} = 1\) and \(a_{2,1} = 2\). Therefore, the formula (1.3) is valid for \(n = 1, 2\).

Assume that the formulas (1.3) and (1.5) are valid for some \(n > 1\). By this inductive hypothesis and a direct differentiation, we have

\[
\tan^{(2n+1)} x = \left[ \tan^{(2n)} x \right]' = \left\{ \frac{1}{\cos^{2n+1} x} \sum_{\ell=0}^{n-1} a_{2n,2\ell+1} \sin[(2\ell + 1)x] \right\}'
\]

\[
= \sum_{\ell=0}^{n-1} a_{2n,2\ell+1} \left\{ \sin[(2\ell + 1)x] \right\}'
\]

\[
= \frac{1}{\cos^{2n(x)} x} \sum_{\ell=0}^{n-1} a_{2n,2\ell+1} \{(n + 1 + \ell) \cos (2\ell x) + (\ell - n) \cos[2(\ell + 1)x]\}
\]

\[
= \frac{1}{\cos^{2n(x)} x} \left\{ (n + 1)a_{2n,1} + \sum_{\ell=1}^{n-1} (n + 1 + \ell)a_{2n,2\ell+1} - (n + 1 - \ell)a_{2n,2\ell-1} \right\} \cos (2\ell x) - a_{2n,2n-1} \cos (2nx)
\]

and

\[
\tan^{(2n+2)} x = \left[ \tan^{(2n+1)} x \right]' = \left\{ \frac{1}{\cos^{2n+2} x} \sum_{\ell=0}^{n} a_{2n+1,2\ell} \cos (2\ell x) \right\}'
\]

\[
= \sum_{\ell=0}^{n} a_{2n+1,2\ell} \left\{ \cos (2\ell x) \right\}'
\]

\[
= \frac{1}{\cos^{2n+3} x} \sum_{\ell=0}^{n} a_{2n+1,2\ell} \{(n + 1 - \ell) \sin[(2\ell + 1)x] - (n + 1 + \ell) \sin[(2\ell - 1)x]\}
\]

\[
= \frac{1}{\cos^{2n+3} x} \left\{ (2(n + 1)a_{2n+1,0} - (n + 2)a_{2n+1,2}) \sin x + \sum_{\ell=1}^{n} (n + 1 - \ell)a_{2n+1,2\ell}
\]

\[
- (n + 2 + \ell)a_{2n+1,2(\ell+1)} \right\} \sin[(2\ell + 1)x] + a_{2n+1,2n} \sin[(2n + 1)x] \}
\]

By straightforward computation, it is not difficult to see that

\[
(n + 1)a_{2n,1} = a_{2n+1,0},
\]

\[
(n + 1 + \ell)a_{2n,2\ell+1} - (n + 1 - \ell)a_{2n,2\ell-1} = a_{2n+1,2\ell}.
\]
\[-a_{2n, 2n-1} = a_{2n+1, 2n},
2(n + 1)a_{2n+1, 0} - (n + 2)a_{2n+1, 2} = a_{2n+2, 1},
(n + 1 - \ell)a_{2n+1, 2\ell} - (n + 2 + \ell)a_{2n+1, 2(\ell+1)} = a_{2n+2, 2\ell+1},
\]
\[a_{2n+1, 2n} = a_{2n+2, 2n+1},\]
where \(1 \leq \ell \leq n - 1\). From these recurrence relations, the formulas (1.3) and (1.5) may be derived straightforwardly.

The formula (1.4) can be proved by induction as the proof of the formulas (1.3) and (1.5). However, it is easy to be derived from the formulas (1.3) and (1.5) by considering \(\cot x = -\tan(x + \frac{\pi}{2})\) and \(\cot^{(n)} x = -\tan^{(n)}(x + \frac{\pi}{2})\) for \(n \in \mathbb{N}\). Theorem 1.1 is thus proved. \(\square\)

**Proof of Theorem 1.2.** In [5, p. 133], it was listed that
\[
\frac{1}{k!} \left( \sum_{m=1}^{\infty} \frac{x^{m}}{m!} \right)^{k} = \sum_{n=k}^{\infty} b_{n,k}(x_{1}, x_{2}, \ldots, x_{n-k+1}) \frac{x^{n}}{n!} \tag{3.1}
\]
for \(k \geq 0\). Taking \(x_{m} = \cos(x + m \frac{\pi}{2})\) in (3.1) yields
\[
\sum_{n=k}^{\infty} b_{n,k} \left( -\sin x, -\cos x, \sin x, \cos x, \ldots, -\sin x + (n - k) \frac{\pi}{2} \right) \frac{x^{n}}{n!} = \frac{1}{k!} \left[ \sum_{m=1}^{\infty} \cos(x + m \frac{\pi}{2}) \frac{x^{m}}{m!} \right]^{k} = \frac{1}{k!} \left[ -2 \sin \left( \frac{t}{2} \right) \sin \left( \frac{t}{2} + x \right) \right]^{k}
\]
\[= \frac{1}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \cos^{\ell}(t + x) \cos^{k-\ell} x, \tag{3.2}
\]
Since \(\cos x = e^{ix} + e^{-ix}\), we have
\[(\cos x)^{\ell} = \frac{1}{2^{\ell}} (e^{ix} + e^{-ix})^{\ell} = \frac{1}{2^{\ell}} \sum_{q=0}^{\ell} \binom{\ell}{q} e^{iqx} e^{-(q-\ell)ix} = \frac{1}{2^{\ell}} \sum_{q=0}^{\ell} \binom{\ell}{q} e^{(2q-\ell)ix}
\]
and
\[\frac{d^{m}(\cos x)^{\ell}}{dx^{m}} = \frac{1}{2^{\ell}} \sum_{q=0}^{\ell} \binom{\ell}{q} (2q - \ell)^{m} m! e^{(2q-\ell)ix} = \frac{1}{2^{\ell}} \sum_{q=0}^{\ell} \binom{\ell}{q} (2q - \ell)^{m} e^{\pi m/2 + (2q-\ell)x} \]
which implies the formula (1.7) and the Eq. (1.9).

Differentiating \(m \geq k\) times with respect to \(t\) on the very ends of (3.2) and employing (1.7) result in
\[
\sum_{n=m}^{\infty} b_{n,k} \left( -\sin x, -\cos x, \sin x, \cos x, \ldots, -\sin x + (n - k) \frac{\pi}{2} \right) \frac{x^{n-m}}{(n-m)!} \]
\[= \frac{(-1)^{k}}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \frac{d^{m}(\cos^{\ell}(t + x))}{dx^{m}} \cos^{k-\ell} x
\]
\[= \frac{(-1)^{k}}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \left( \frac{1}{2^{\ell}} \sum_{q=0}^{\ell} \binom{\ell}{q} (2q - \ell)^{m} \cos \left( \frac{\pi}{2} m + (2q - \ell)(t + x) \right) \right) \cos^{k-\ell} x.
\]
Further taking the limit \(t \to 0\) and rearranging reveal (1.10).

Taking \(x_{m} = \sin(x + m \frac{\pi}{2})\) in (3.1) yields
\[
\sum_{n=k}^{\infty} b_{n,k} \left( \cos x, -\sin x, -\cos x, \sin x, \cos x, \ldots, -\sin x + (n - k) \frac{\pi}{2} \right) \frac{x^{n}}{n!} = \frac{1}{k!} \left[ \sum_{m=1}^{\infty} \sin(x + m \frac{\pi}{2}) \frac{x^{m}}{m!} \right]^{k} = \frac{1}{k!} \left[ 2 \sin \left( \frac{t}{2} \right) \cos \left( \frac{t}{2} + x \right) \right]^{k}
\]
\[= \frac{2}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \sin^{\ell}(t + x) \sin^{k-\ell} x. \tag{3.3}
\]
Since \( \sin x = \frac{e^{ix} - e^{-ix}}{2i} \), we have

\[
(\sin x)^\ell = \frac{1}{(2i)^\ell} \sum_{q=0}^{\ell} (-1)^{\ell-q} \binom{\ell}{q} e^{iqx} e^{-(\ell-q)ix} = \frac{(-1)^\ell}{(2i)^\ell} \sum_{q=0}^{\ell} (-1)^{\ell} \binom{\ell}{q} e^{(2\ell-q)ix}
\]

and

\[
\frac{d^m (\sin x)^\ell}{dx^m} = \frac{(-1)^\ell}{(2i)^\ell} \sum_{q=0}^{\ell} (-1)^q \binom{\ell}{q} (2q - \ell)^m i^m e^{(2\ell-q)ix}
\]

Further letting \( t \in (1.2) \) and using the formula (1.11) generate

\[
\sum_{n=m}^{\infty} B_{n,k} \left( \cos x, -\sin x, -\cos x, \sin x, \ldots, -\cos \left[ x + (n-k) \frac{\pi}{2} \right] \right) \frac{t^{n-m}}{(n-m)!}
\]

which means the formula (1.6) and the Eq. (1.8).

Differentiating \( m \geq k \) times with respect to \( t \) on the very ends of (3.3) and utilizing (1.6) lead to

\[
\sum_{n=m}^{\infty} B_{n,k} \left( \cos x, -\sin x, -\cos x, \sin x, \ldots, -\cos \left[ x + (n-k) \frac{\pi}{2} \right] \right) \frac{t^{n-m}}{(n-m)!} = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \frac{d^m \sin^\ell (t + x)}{dx^m} \sin^{k-\ell} x
\]

Further letting \( t \to 0 \) gives the formula (1.11).

Applying \( f(u) = \ln u \) and \( u = h(t) = \cos t \) in (1.2) and using the formula (1.10) give

\[
\frac{d^n \tan t}{dt^n} = -\frac{d^{n+1} \ln (\cos t)}{dt^{n+1}} = -\sum_{k=1}^{n+1} (-1)^{k-1} \binom{k}{1} B_{n+1,k} \left( \cos t, \cos^\prime t, \ldots, \cos^{(n-k+2)} t \right)
\]

The formula (1.12) follows.

Applying \( f(u) = \ln u \) and \( u = h(t) = \sin t \) in (1.2) and using the formula (1.11) generate

\[
\frac{d^n \cot t}{dt^n} = \frac{d^{n+1} \ln (\sin t)}{dt^{n+1}} = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{k}{1} B_{n+1,k} \left( \cos t, \sin t, \ldots, \sin^{(n-k+2)} t \right)
\]

The formula (1.13) follows. The proof of Theorem 1.2 is complete. \( \square \)

**Proof of Theorem 1.3.** Taking \( n = 2m - 1 \) and letting \( x \to 0 \) in (1.3) yield

\[
T_{2m-1} = \lim_{x \to 0} \tan^{(2m-1)x} = \frac{1}{2} a_{2m-1,0} + \sum_{k=1}^{m-1} a_{2m-1,2k}.
\]
where
\[ a_{2m-1,0} = 2 \sum_{\ell=0}^{m-1} (-1)^{\ell} \binom{2m}{\ell} (m - \ell)^{2m-1} \]
and
\[ a_{2m-1,2k} = (-1)^k 2 \sum_{\ell=0}^{m-k-1} (-1)^{\ell} \binom{2m}{\ell} (m - k - \ell)^{2m-1}. \]
The formula (1.14) follows.

The formula (1.15) follows.

The recurrence (1.16) follows. The proof of Theorem 1.3 is thus proved.

The formula (1.15) follows.

Taking \( x \to 0 \) in (1.12) gives
\[ \lim_{x \to 0} \frac{d^{2m-2} \tan x}{dx^{2m-2}} = - \sum_{k=1}^{2m} \frac{1}{k} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \sum_{q=0}^{\ell} \binom{\ell}{q} (2q - \ell)^{2m} \sin \left[ \frac{\pi}{2} (2m - 1) \right] \]
\[ = (-1)^m \sum_{k=1}^{2m} \frac{1}{k} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \sum_{q=0}^{\ell} \binom{\ell}{q} (2q - \ell)^{2m} \sum_{k=1}^{2m} \frac{1}{k} \binom{k}{\ell} \]
\[ = (-1)^m \sum_{\ell=1}^{2m} (-1)^{\ell} \frac{1}{2^\ell} \sum_{q=0}^{\ell} \binom{\ell}{q} (2q - \ell)^{2m} \sum_{k=1}^{2m} \frac{1}{k} \binom{k}{\ell} \]
\[ = (-1)^m \sum_{\ell=1}^{2m} (-1)^{\ell} \frac{1}{2^\ell} \binom{2m}{2m+1-\ell} \sum_{q=0}^{\ell} \binom{\ell}{q} (2q - \ell)^{2m} \]
\[ = T_{2m-1}. \]
The proof of Theorem 1.4 is complete. □

Proof of Theorem 1.4. From (1.1), it follows that
\[ B_{2k} = \frac{(-1)^{k-1} k}{2^{2k-1} (2k - 1)} T_{2k-1}. \]
Combining this with the formulas (1.14) and (1.15) gains the formulas (1.17) and (1.18).

Substituting (1.1) into (1.16) and rearranging yield (1.19).

In [1, p. 807, 23.2.16], it is listed that for \( n \geq 1 \)
\[ \xi (2n) = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}|. \] (3.4)

See also [40, p. 332]. Substituting the formulas (1.17) and (1.18) into (3.4) results in (1.20) and (1.21).

Substituting (3.4) into (1.19) and rearranging yield (1.22).

The Genocchi numbers satisfy \( G_1 = 1, G_{2n+1} = 0. \) and
\[ G_{2n} = 2 \left( 1 - 2^{2n} \right) B_{2n} = 2nE_{2n-1} (0) \]
for \( n \in \mathbb{N}. \) By similar arguments as above, we may obtain the formulas (1.23), (1.24), (1.26), and (1.27) and the recurrence relations (1.25) and (1.28). The proof of Theorem 1.4 is complete. □

4. Remarks

Finally, we give several remarks on related results.

Remark 4.1. Let \( y = \tan x. \) The relation (2.14) may be rewritten as
\[ y^{(n)} = YAY^T \]
for \( n \geq 2, \) where \( Y^T \) stands for the transpose of the \( 1 \times n \) matrix
\[ Y = (y, y', y'', \ldots, y^{(n-2)}, y^{(n-1)}) \]
and $A$ is a $n \times n$ square matrix whose elements $a_{k,\ell}$ satisfy

$$a_{k,\ell} = \begin{cases} 
(n-1)\left(\frac{1}{\ell-1}\right), & k + \ell = n + 1, \\
0, & k + \ell \neq n + 1.
\end{cases}$$

**Remark 4.2.** By the formulas in (2.11), the tangent numbers $T_{2m-1}$ for $m \in \mathbb{N}$ may be represented as

$$T_{2m-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 1 & 0 & \cdots & 0 \\
0 & -1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(-1)^{m-1} & 0 & \cdots & \cdots & \cdots & 1 \\
\end{pmatrix},$$

where

$$B_{2k} = 1 + \sum_{m=1}^{2k-1} \frac{S(2k + 1, m + 1)S(2k, 2k - m)}{(2k_m)^2} - \frac{2k}{2k + 1} \sum_{m=1}^{2k} \frac{S(2k, m)S(2k + 1, 2k - m + 1)}{(2k_m)^2}, \quad k \in \mathbb{N}$$

for computing the Bernoulli numbers $B_{2k}$ in terms of the Stirling numbers $S(q, m)$ of the second kind was derived. For more information, please refer to [12,43] and the references cited therein.

**Remark 4.7.** In [29], based on establishment of the general and explicit formula

$$\left(\frac{1}{\ln x}\right)^{(n)} = (-1)^n \frac{n!}{x^n} \sum_{q=2}^{n+1} \alpha_{n,q} \ln^n x^q, \quad n \in \mathbb{N},$$

where

$$\alpha_{n,2} = (n - 1)!$$

and, for $n + 1 \geq q \geq 3,$

$$\alpha_{n,q} = (q - 1)! (n - 1)! \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{q-2}=1}^{\ell_{q-3}-1} \frac{1}{\ell_{q-2}} \sum_{\ell_{q-1}=1}^{\ell_{q-2}} \frac{1}{\ell_{q-2} \ell_{q-1}},$$

among other things, it was discovered that the Stirling numbers of the first kind $s(n, q)$ for $1 \leq q \leq n$ may be computed by

$$s(n, q) = (-1)^{n+q} (n - 1)! \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{q-2}=1}^{\ell_{q-3}-1} \frac{1}{\ell_{q-2}} \sum_{\ell_{q-1}=1}^{\ell_{q-2}} \frac{1}{\ell_{q-1}},$$

equivalently,

$$(-1)^{n-k} \frac{s(n, k)}{(n - 1)!} = \sum_{m=k-1}^{n-1} \frac{1}{m} \left[ (-1)^{m-(k-1)} \frac{s(m, k - 1)}{(m - 1)!} \right], \quad 2 \leq k \leq n,$$

and that the Bernoulli numbers of the second kind $b_q$ for $n \geq 2$ may be computed by
Remark 4.9. In [38], it was obtained that the Bernoulli numbers of the second kind for any given $n \geq 0$ may be generated respectively by

$$b_n = (-1)^n \frac{1}{n!} \left( \frac{1}{n+1} + \sum_{k=2}^{n} \frac{\alpha_{n,k} - n\alpha_{n-1,k}}{k!} \right),$$

where the Stirling numbers of the first kind $s(n, k)$ and the Bernoulli numbers of the second kind $b_n$ for $n \geq 0$ may be generated

and

$$\left[ \ln (1 + x)^{m} / m! \right] = \sum_{k=m}^{\infty} \frac{s(k, m)}{k!} x^k, \quad |x| < 1$$

for $1 \leq k \leq n$. By virtue of (4.3), the sequence

$$\left\{ (-1)^n s(n + k, k) \right\}_{n \geq 0}$$

for any given $k \in \mathbb{N}$ is proved to be logarithmically convex.

Remark 4.10. In [31], three integral representations for the Stirling numbers of the first kind $s(n, k)$ were created, one of them is

$$s(n, k) = \left( \frac{n}{k} \right) \lim_{x \to 0} \frac{d^{n-k}}{dx^{n-k}} \left\{ \left[ \int_{0}^{\infty} \left( \int_{1/e}^{1} e^{ux-1} du \right) e^{-u} du \right]^{k} \right\}$$

for $1 \leq k \leq n$. By virtue of (4.3), the sequence

$$\left\{ (-1)^n s(n + k, k) \right\}_{n \geq 0}$$

for any given $k \in \mathbb{N}$ is proved to be logarithmically convex.

Remark 4.11. In [38], it was obtained that the Bernoulli numbers of the second kind $b_n$ may be represented by

$$b_n = (-1)^{n+1} \int_{1}^{\infty} \frac{1}{(\ln (t-1))^2 + \pi^2} t^n dt, \quad n \in \mathbb{N}.$$  

As a result of the integral representation (4.4), the sequence $\{(-1)^n b_{n+1}\}_{n \geq 0}$ of the Bernoulli numbers of the second kind is proved to be completely monotonic. In [38], among other things, the sequence $\{(-1)^q q! b_{q+1}\}_{q \geq 0}$ is proved to be logarithmically convex.

Remark 4.12. This paper is a corrected, expanded, and revised version of the preprint [30].

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