A remark on Fourier pairing and binomial formula for Macdonald polynomials

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Abstract

We give a concise direct proof of the orthogonality of interpolation Macdonald polynomials with respect to the Fourier pairing and briefly discuss some immediate applications of this orthogonality, such as the symmetry of the Fourier pairing and the binomial formula.

1 Introduction

Fourier pairing, introduced by Cherednik [1], is a fundamental notion in the theory of Macdonald polynomials. In its simplest instance, it pairs the algebra $\Lambda_n$ of symmetric polynomials in $n$ variables with the algebra $\mathcal{D}_n$ of Macdonald commuting difference operators acting on $\Lambda_n$ [6]. By definition,

$$\langle D, f \rangle = [D \cdot f](\hat{0}), \quad D \in \mathcal{D}_n, f \in \Lambda_n,$$

where $\hat{0}$ is a certain distinguished point. There is a natural isomorphism $\mathcal{D}_n \cong \Lambda_n$, which makes (1) a quadratic form on $\Lambda_n$. The most important and useful property of this form is its symmetry, see [1, 6, 7].

The main observation of this note is that there is a very natural orthogonal basis for the form (1). Namely, this is the basis $\{I_\mu\}$ of the interpolation Macdonald polynomials, which have been intensively studied by Knop, Olshanski, Sahi, the author, and others, see for example [3, 8, 13] and also [11], and references therein. The polynomials $I_\mu$ are defined by some very simple multivariate Newton-type interpolation conditions and have found many remarkable applications.

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The orthogonality of $I_\mu$ with respect to (1), stated as Theorem 1 below, follows rather easily from the definitions, without using any nontrivial properties of the polynomials $I_\mu$. It implies at once that (1) is symmetric.

Also, the orthogonality of $I_\mu$ gives immediately the expansion of the simultaneous eigenfunctions of $D_n$, known as the symmetric Macdonald polynomials $P_\lambda$, in the basis $\{I_\mu\}$. This expansion, reproduced in Theorem 3 below, is the binomial formula for $P_\lambda$, see 3. In fact, the orthogonality of $I_\mu$ is essentially equivalent to the binomial theorem, but it certainly appears to be a much more basic, natural, and appealing property.

The binomial theorem of [9] has been extended to a more general setting, including other classical root systems and the nonsymmetric Macdonald polynomials, see [4, 10, 14]. We are not trying to pursue the greatest possible generality in this note. Our intention, rather, is to show how the basic idea works in the simplest nontrivial example of the usual symmetric Macdonald polynomials. We even consider the (almost) trivial one-dimensional case first to give a completely elementary illustration of what is going on.

It should be pointed out that there is another source of orthogonality relations for the polynomials $I_\mu$. Namely, the polynomials $I_\mu$ can be obtained from the symmetric Macdonald polynomials of type $BC_n$ (this can be seen very explicitly by degenerating the binomial formula of [10] to the binomial formula for the polynomials $I_\mu$ [9]).

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2 Simplest example

2.1

As a warm-up, let us consider the one-dimensional case first. Let the operator $T$ act on polynomials in $x$ by

$$[Tf](x) = f(qx).$$
Obviously, the monomials \(x^n, n = 0, 1, 2, \ldots\), are the eigenfunctions of this operator with eigenvalues \(q^n\). Consider the following bilinear form
\[
\langle g, f \rangle = [g(T) \cdot f](1).
\]

In the basis \(\{x^n\}\), this form has the matrix
\[
[q^{nm}]_{n,m=0,1,\ldots} = \begin{bmatrix}
1 & 1 & 1 & 1 & \cdots \\
1 & q & q^2 & q^3 & \cdots \\
1 & q^2 & q^4 & q^6 & \cdots \\
1 & q^3 & q^6 & q^9 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]
and is clearly symmetric.

It is also obvious that
\[
x^* = T, \quad T^* = x,
\]
where \(x\) denotes the operator of multiplication by the independent variable and star denotes the adjoint operator with respect to (2). Clearly, (3) is an anti-automorphism of the \(q\)-Heisenberg algebra generated by \(T\) and \(x\) (subject to the relation \(Tx = qxT\)) and deserves to be called the Fourier transform.

2.2
Now consider the polynomial
\[
I_n = (x - 1)(x - q) \cdots (x - q^{n-1}), \quad n = 0, 1, \ldots,
\]
satisfying the following Newton interpolation conditions
\[
I_n \equiv x^n \mod \{x^m\}_{m<n}, \quad (5)
\]
\[
I_n(q^m) = 0, \quad 0 \leq m < n. \quad (6)
\]
We have the following

**Proposition 1.** The polynomials \(I_n\) are orthogonal with respect to the form (2), namely
\[
\langle I_n, I_m \rangle = \delta_{n,m} I_n(q^n). \quad (7)
\]
Proof. We will intentionally avoid using the symmetry of (2) in our argument because, in the general case, we want to obtain the analogous symmetry as a corollary.

It is clear from definition (2) that
\[ \langle x^n, f \rangle = f(q^n), \]
and since \( T \cdot x^n = q^n x^n \) we also have
\[ \langle g, x^n \rangle = g(q^n). \]
From (3) it now follows that
\[ \langle x^m, I_n \rangle = \langle I_n, x^m \rangle = 0, \quad m < n, \]
and it is also clear that
\[ \langle x^n, I_n \rangle = \langle I_n, x^n \rangle = I_n(q^n). \]
Now the property (5) concludes the proof.

The following expansion is immediate from (7) and (8)
\[ x^n = \sum_m \frac{\langle x^m, I_m \rangle}{\langle I_m, I_m \rangle} \frac{I_m(x)}{I_m(q^n)} = \sum_m \frac{I_m(q^n) I_m(x)}{I_m(q^n)}. \]
This is the Newton interpolation of \( x^n \) with nodes \( 1, q, q^2, \ldots \) and also an instance of the \( q \)-binomial theorem.

3 Symmetric Macdonald polynomials

3.1

We now turn to polynomials in \( n \) variables \( x_1, \ldots, x_n \). We denote
\[ [T_i f](x_1, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n). \]
Let \( t \) be an additional parameter and introduce, following Macdonald [4], the following operators
\[ D_k = t^{k(k-1)/2} \sum_{|S|=k} d_S(x) \prod_{i \in S} T_i, \]
where the summation is over subsets \( S \subset \{1, \ldots, n\} \) of cardinality \( k \) and
\[ d_S(x) = \prod_{i \in S, j \notin S} \frac{tx_i - x_j}{x_i - x_j}. \]
3.2

The operators $D_k$ commute and take symmetric polynomial to symmetric polynomials. They act triangularly in the basis of monomial symmetric functions, namely,

$$D_k \cdot m_\lambda \equiv e_k(\widehat{\lambda}) m_\lambda \quad \text{mod} \quad \{m_\mu\}_{\mu < \lambda},$$

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ is a partition,

$$m_\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n} + \text{permutations}$$

is the corresponding monomial symmetric function, $e_k = m_{(1^k)}$ is the $k$th elementary symmetric function, $\widehat{\lambda}$ denotes the following point

$$\widehat{\lambda} = (q^{\lambda_1}t^{n-1}, q^{\lambda_2}t^{n-2}, \ldots, q^{\lambda_n-1}t, q^{\lambda_n}),$$

and $\mu < \lambda$ stands for the dominance order on partitions

$$\mu \leq \lambda \Leftrightarrow \left(\begin{array}{c}
\mu_1 \leq \lambda_1, \\
\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2, \\
\vdots \\
\mu_1 + \cdots + \mu_n = \lambda_1 + \cdots + \lambda_n
\end{array}\right).$$

The simultaneous eigenfunctions of $D_k$

$$D_k \cdot P_\lambda = e_k(\widehat{\lambda}) P_\lambda,$$  

normalized by

$$P_\lambda \equiv m_\lambda \quad \text{mod} \quad \{m_\mu\}_{\mu < \lambda}$$

are known as the Macdonald symmetric polynomials.

3.3

Let $\Lambda_n$ denote the algebra of symmetric polynomials in $n$ variables. It is clear from (9) that the map

$$D : \Lambda_n \ni e_k \mapsto D_k$$

extends to an algebra homomorphism such that

$$D(g) \cdot P_\lambda = g(\widehat{\lambda}) P_\lambda, \quad g \in \Lambda_n.$$  

5
We now define, following Cherednik [1], the following Fourier pairing
\[ \langle g, f \rangle = \left[ D(g) \cdot f \right](\hat{0}), \quad f, g \in \Lambda_n. \] (12)
This is an analog of (2). It is clear that
\[ \langle hg, f \rangle = \langle g, D(h)f \rangle. \]
In other words, \( D(h) = h^* \), where \( h \) is considered as a multiplication operator and star denotes its Fourier transform, that is, the adjoint operator with respect to (12). It is also clear from (9) that the pairing (12) takes the normalized eigenfunction
\[ N_\lambda = \frac{P_\lambda}{P_\lambda(0)} \] (13)
to the \( \delta \)-function at \( \hat{\lambda} \), namely
\[ \langle g, N_\lambda \rangle = g(\hat{\lambda}). \] (14)

3.4

Our goal is now to produce an explicit orthogonal basis for the quadratic form (12). As in (4), this basis will consist of certain Newton interpolation polynomials.

Let \( \triangleleft \) be any total ordering of the set of partitions \( \lambda \) compatible with both the ordering of partitions by their size \( |\lambda| \) and, for partitions of the same number, the dominance ordering. Define the interpolation Macdonald polynomials \( I_\mu \) by the following generalization of (5) and (6)
\[ I_\mu \equiv m_\mu \mod \{ m_\lambda \}_{\lambda \triangleleft \mu}, \] (15)
\[ I_\mu(\hat{\lambda}) = 0, \quad \lambda \triangleleft \mu. \] (16)
The existence and uniqueness of such polynomials for generic \( q \) and \( t \) is clear from their existence and uniqueness for \( t = 1 \), which is elementary.

3.5

It can be shown, see for example [3, 8, 13] and also [11], that the polynomials \( I_\mu \) do not depend on the choice of the ordering \( \triangleleft \) and satisfy the much stronger extra vanishing property
\[ I_\mu(\hat{\lambda}) = 0, \quad \mu \nsubseteq \lambda. \] (17)
By the binomial formula \((22)\), this gives the following strengthening of \((15)\)

\[ I_\mu \equiv P_\mu \mod \{P_\lambda\}_{\lambda \subset \mu}. \tag{18} \]

The extra vanishing \((17)\) will not, however, be needed for what follows making our argument applicable in the situations where the analog of \((17)\) is not available.

### 3.6

Our main result is the following

**Theorem 1.** The polynomials \(I_\mu\) are orthogonal with respect to the Fourier pairing \((12)\).

An immediate corollary of this theorem is the following central result of the theory of Macdonald polynomials

**Corollary 1 (Koornwinder,\([6]\)).** The Fourier pairing \((12)\) is symmetric.

Koornwinder actually proved an equivalent symmetry, namely the following label-argument symmetry for the normalized polynomials \((13)\)

\[ N_\lambda(\hat{\mu}) = N_\mu(\hat{\lambda}). \]

Numerous application of this symmetry, such as, for example, Pieri-type formulas for Macdonald polynomials, can be found in \([1, 6, 7]\).

### 3.7

The proof of Theorem 1 goes in two steps. First, we claim that

\[ \langle I_\mu, I_\lambda \rangle = 0, \quad \mu \ntriangleright \lambda. \]

Indeed, by \((15)\), \((10)\), and \((14)\), this is equivalent to

\[ \langle I_\mu, N_\lambda \rangle = I_\mu(\hat{\lambda}) = 0, \quad \mu \ntriangleright \lambda, \]

which is indeed true by \((16)\).
3.8

Now we prove that
\[ \langle I_\mu, I_\lambda \rangle = 0, \quad \mu \preceq \lambda. \]
By (15) this is equivalent to proving that \( \langle m_\mu, I_\lambda \rangle = 0 \) if \( \mu \preceq \lambda \). Since
\[ e_\mu \overset{\text{def}}{=} e_{\mu_1} \cdots e_{\mu_n} \equiv m_\mu \mod \{m_\nu\}_{\nu < \mu}, \]
it suffices to prove that
\[ \langle e_\mu, I_\lambda \rangle = 0, \quad \mu \preceq \lambda. \]
By definition (12), this is equivalent to
\[ [D_\mu \cdot I_\lambda] (\hat{0}) = 0, \quad D_\mu = D_{\mu_1} \cdots D_{\mu_n}, \]
which will now be established.

3.9

It is a crucial property of the operators \( D_k \) that
\[ \left( \begin{array}{c} \lambda_i = \lambda_{i+1}, \\ i \not\in S, \ i + 1 \in S \end{array} \right) \Rightarrow d_S(\vec{\lambda}) = 0. \]
It follows that
\[ [D_k \cdot f] (\hat{\lambda}) = \sum_{\nu/\lambda=\text{vertical } k\text{-strip}} d_{S(\nu, \lambda)}(\vec{\lambda}) f(\vec{\nu}), \]
where \( S(\nu, \lambda) = \{i, \nu_i > \lambda_i\} \). It follows that
\[ [D_\mu \cdot f] (\hat{0}) = \sum_{\nu \leq \mu} c_{\mu, \nu} f(\vec{\nu}), \]
for some coefficients \( c_{\mu, \nu} \). A similar property can be established in more general context, such as e.g. for nonsymmetric Macdonald polynomials [2].

It is clear that (20) together with (16) imply (19) and this concludes the proof of Theorem 1.
3.10
Theorem 1 can be sharpened as follows

**Theorem 2.** We have

\[ \langle I_\mu, I_\nu \rangle = \delta_{\mu,\nu} I_\mu(\hat{\mu}) P_\mu(\hat{0}) = \delta_{\mu,\nu} c_{\mu,\mu} I_\mu(\hat{\mu}). \quad (21) \]

In particular, this shows that \( P_\mu(\hat{0}) = c_{\mu,\mu} \) which, after making the number \( c_{\mu,\mu} \) explicit, can be seen to be equivalent to the known formula for \( P_\mu(\hat{0}) \), see [6].

**Proof.** Arguing as in Section 3.7, we see that

\[ \langle I_\mu, I_\mu \rangle = \langle I_\mu, P_\mu \rangle = I_\mu(\hat{\mu}) P_\mu(\hat{0}). \]

On the other hand, arguing as in Section 3.8, we get

\[ \langle I_\mu, I_\mu \rangle = \left[ D_\mu \cdot I_\mu \right](\hat{0}) = c_{\mu,\mu} I_\mu(\hat{\mu}). \]

\[ \square \]

3.11
Theorem 2 implies the following Newton interpolation formula

\[ f = \sum_\mu \frac{\langle I_\mu, f \rangle}{\langle I_\mu, I_\mu \rangle} I_\mu, \quad f \in \Lambda_n. \]

In particular, applying this to \( N_\lambda \) and using (14) we obtain the following expansion (in which we explicitly kept the variables \( x \) in order to stress the label-argument symmetry).

**Theorem 3 (Binomial theorem, [9]).** We have

\[ N_\lambda(x) = \sum_\mu \frac{I_\mu(\hat{\lambda}) I_\mu(x)}{\langle I_\mu, I_\mu \rangle}. \quad (22) \]

It follows from (17) that only those \( \mu \) such that \( \mu \subset \lambda \) actually appear in this expansion.
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