Self-dual Quantum Electrodynamics on the boundary of 4d Bosonic Symmetry Protected Topological States

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We study 3d (or (3 + 1)d) Quantum Electrodynamics (QED) realized on the boundary of 4d (or (4 + 1)d) bosonic symmetry protected topological (BSPT) states, using a systematic nonlinear sigma model (NLSM) field theory description of BSPT states developed in Ref. 1. We demonstrate that many of these QED states have an exact electric-magnetic duality due to the symmetry of the BSPT states in the 4d bulk. The gauge charge and Dirac monopole both carry projective representations of the bulk symmetry, and the emergent gapless photons of the QED phase also transform nontrivially under the bulk symmetry. Some of these QED boundary states can be further driven into a 3d $\mathbb{Z}_2$ topological order, and the statistics and symmetry transformation of its point particle and vison loop excitations guarantee that this topological order cannot be driven into a trivial confined or Higgs phase. With a finite fourth dimension, the entire system becomes a 3d lattice, the self-dual QED and the $\mathbb{Z}_2$ topological order can coexist on two opposite boundaries respectively, which together constitute an exotic 3d self-dual “topological photon phase”.

1. INTRODUCTION

Symmetry protected topological (SPT) phases, a new type of quantum disordered phase pioneered in Ref. 2, are intrinsically different from trivial direct product state, when and only when the system has certain symmetry $G$. In terms of its phenomena, a SPT phase on a $d$-dimensional lattice should satisfy at least the following three criteria:

(i) On a $d$-dimensional lattice without boundary, this phase is fully gapped, and nondegenerate;

(ii) On a $d$-dimensional lattice with a $(d - 1)$-dimensional boundary, if the Hamiltonian of the entire system (including both bulk and boundary Hamiltonian) preserves symmetry $G$, then this phase is either gapless or gapped but degenerate.

(iii) The boundary state of this $d$-dimensional system cannot be realized as a $(d - 1)$-dimensional lattice system with the same symmetry $G$.

Notice that the second criterion (ii) implies the following two possibilities: On a lattice with a boundary, the system is either gapless, or gapped but degenerate. For a 1d SPT state, its 0d boundary must be degenerate, which forms a projective representation of the symmetry group; for a 2d SPT state, its 1d boundary can be either gapless, or degenerate due to spontaneous discrete symmetry breaking; for a 3d SPT state, the degeneracy of its 2d boundary can correspond to either spontaneous breaking of $G$, or correspond to certain topological degeneracy at the boundary. Which case occurs in the system will depend on the detailed Hamiltonian at the boundary of the system. For example, with strong interaction, the boundary of a 3d topological band insulator can be driven into a nontrivial topological phase. 4 5. And one natural candidate 2d boundary state of most 3d bosonic SPT states is a 2d $\mathbb{Z}_2$ topological order with $e$ and $m$ excitations both carrying fractional degrees of freedom. 4 5.

In this paper we will investigate the 3d boundary states of 4d bosonic SPT states. All the possible boundary states discussed above can occur in our case, but there is one new possibility which does not occur in lower dimensions: the 3d boundary can be a deconfined gapless photon state, which does not exist in lower dimensions with gapped matter fields due to the well-known fact that the $(2+1)d$ compact QED with gapped matter fields is always confined due to the proliferation of Dirac monopole in the space-time. However in $(3 + 1)d$, a compact QED can have a deconfined photon phase with gapless photon excitations, and deconfined electric charge (denoted as $e$) and Dirac monopole (denoted as $m$) excitations. This boundary gapless photon must be very unusual, because based on the definition of an SPT state, this boundary state cannot be realized in 3d without the bulk or the opposite boundary.

Please note that this photon phase only exists at the 3d boundary; namely the bulk is still fully gapped and nondegenerate. Just like the 2d topological order at the boundary of a 3d SPT state, the $e$ and $m$ excitations of the photon phase must carry a nontrivial representation (or projective representation) of the symmetry groups, which implies that the system cannot be driven into a trivial confined or Higgs phase with a gapped and nondegenerate ground state by condensing $e$ or $m$ excitations. The quantum number of $e$ and $m$ excitations can be computed systematically using the NLSM field theory developed in Ref. 1. Besides the quantum numbers carried by $e$ and $m$, we are going to show that in many cases the boundary photon phase has an exact “self-dual” symmetry. In this work we are going to describe two examples in detail. In the first example, the self-dual symmetry is the physical time-reversal symmetry, and in the second example it is a $\mathbb{Z}_4$ symmetry.

A 4d system with infinite size is unrealistic. However, our study of the 3d boundary of a 4d SPT state can lead to exotic phases in 3d as well. We can imagine making a thin slab of the 4d SPT state, namely a 4d system
with a finite fourth dimension. Then the entire system becomes three dimensional, but one can realize two different 3d states on the two opposite boundaries. For the first example state in which time-reversal plays the role of a self-dual symmetry, we can realize the 3d self-dual photon phase on the top surface, but realize a fully gapped topological order on the bottom surface. Then at low energy only the photon phase on the top surface becomes visible, which by definition is a phase that cannot be realized in 3d at all. Only at higher energy will the topological order on the bottom surface be exposed. By contrast, for the example where the discrete Z_4 symmetry plays the role as self-duality, it seems this photon phase cannot be driven into any gapped topological order without breaking the Z_3 symmetry.

We note that in Ref. 10, 11, a QED state on the 3d boundary of a 4d bosonic short range entangled (BSRE) state was also studied, and this QED state can have a maximum SL(2, Z) duality symmetry. But in that case the 4d BSRE state does not need any symmetry to be nontrivial because the 3d boundary QED state is an “all fermion state”; namely its e and m excitations are both fermions 12, which is a fact robust against any symmetry breaking. This bulk BSRE state is also an “invertible topological state” discussed in Ref. 13–17. However, in our case, the 4d system is a nontrivial SPT state when and only when the system has certain symmetry. When the symmetry is broken, the system becomes a trivial Mott insulator.

2. SELF-DUAL PHOTON PHASE WITH \( Z_2 \times T \) SYMMETRY

Let us start with a 4d SPT state with \( Z_2 \) and time-reversal symmetry (\( T \)) symmetry only. This state can be described by the following NLSM field theory with a six component unit vector \( \mathbf{n} \):

\[
S = \int d^4x \, d\tau \, \frac{1}{g} (\partial_\mu \mathbf{n})^2 \\
+ \frac{i2\pi}{\Omega_5} \epsilon_{abcde} f^a \partial_\tau n^b \partial_x n^c \partial_x n^d \partial_x n^e n^f.
\]

(1)

Here \( |\mathbf{n}| = 1 \), and \( \Omega_5 \) is the volume of the five dimensional unit sphere. In Eq. [1] when the coupling constant \( g \) is larger than some critical value, the system is in a quantum disordered phase with a fully gapped and nondegenerate bulk spectrum, and according to Ref. 14 different SPT states correspond to different symmetry transformations on \( \mathbf{n} \) that keep the entire action, including the topological \( \Theta \)–term invariant. In this work we primarily consider the state that corresponds to the following transformation as an example:

\[
\mathcal{T} : n_1 \equiv n_4, \quad n_2 \equiv -n_5, \quad n_3 \equiv n_6.
\]

(2)

In fact, even without the \( \mathcal{T} \) symmetry this state is already a nontrivial SPT state, and this state is just a 4d generalization of the 2d SPT with \( Z_2 \) symmetry 18, which is described by a \((2 + 1)\)d NLSM with a four component unit vector \( \mathbf{n} \). Also, this state can be viewed as a 4d bosonic integer quantum Hall state with U(1) symmetry broken down to \( Z_2 \) (see more details in appendix B).

In this work we will focus on the \( Z_2 \times \mathcal{T} \) symmetry. But, Eq. [1] actually can also describe SPT states with much larger symmetries. Let us parameterize the six-component vector \( \mathbf{n} = (\cos \alpha \mathbf{N}, \sin \alpha \mathbf{M}) \), where \( \mathbf{N} \sim (n_1, n_2, n_3) \) and \( \mathbf{M} \sim (n_4, n_5, n_6) \) are both three-component unit vectors. We also tentatively introduce two more SO(3) symmetries to the system with \( \mathbf{N} \) and \( \mathbf{M} \) transforming as vectors under the two SO(3) symmetries respectively, although these exact SO(3) symmetries are unimportant to the main physics we are going to discuss. (Introducing extra symmetries and eventually breaking them has proved to be a very helpful trick for field theory analysis, as shown in Ref. [36].) At the \((3 + 1)\)d boundary, Eq. [1] will reduce to a \((3 + 1)\)d NLSM with a Wess-Zumino-Witten (WZW) term [1] at level-1:

\[
S = \int d^3x \, d\tau \, \frac{1}{g} (\partial_\mu \mathbf{n})^2 \\
+ \int_0^1 du \frac{i2\pi}{\Omega_5} \epsilon_{abcde} f^a \partial_\tau n^b \partial_x n^c \partial_x n^d \partial_x n^e n^f.
\]

(3)

Just like all WZW terms, the last term in Eq. [8] is equal to the volume of the target space \( S^5 \) enclosed by the trajectory of \( \mathbf{n} \) under a periodic evolution. \( u \in [0, 1] \) is an extra parameter introduced and \( \mathbf{n}(x, \tau, u = 0) = (0, 0, 0, 0, 0, 1) \) while \( \mathbf{n}(x, \tau, u = 1) = \mathbf{n}(x, \tau) \).

The physical meaning of this WZW term becomes explicit when we reduce this WZW term on a hedgehog monopole of \( \mathbf{N} \sim (n_1, n_2, n_3) \), which is a point defect \( \mathbf{N} \) is normalized to be a three-component unit vector). Since the hedgehog monopole is a singularity of \( \mathbf{N} \), then at the hedgehog monopole the six component unit vector \( \mathbf{n} \) will reduce to another three-component unit vector \( \mathbf{M} \sim (n_4, n_5, n_6) \), and this WZW term reduces to a \((0 + 1)\)d WZW model for \( \mathbf{M} \) (for more details please see appendix C):

\[
S_{hm} = \int d\tau \, \frac{1}{g'} (\partial_\mu \mathbf{M})^2 + \int_0^1 du \frac{i2\pi}{\Omega_2} \epsilon_{abc} M^a \partial_\tau M^b \partial_x M^c.
\]

(4)

With the extra SO(3) symmetries, the ground state of this \((0 + 1)\)d field theory Eq. [4] is two fold degenerate:

\[
|m\rangle = \left( \cos(\theta/2)e^{i\phi/2}, \sin(\theta/2)e^{-i\phi/2} \right)^t.
\]

(5)

Here we have used the standard parametrization of the vector \( \mathbf{M} \): \( \mathbf{M} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \). Thus we need to introduce a two component bosonic fields \( z^m = \)
changes. But the nature of this photon phase does not
under the duality $Z$ which is also a fractionalized CP
1

Now let us look at vector the $N \sim (n_1, n_2, n_3)$ and introduce the standard CP
1
field parametrization of $N$:

$$N = \frac{1}{2} z^e \sigma z^e.$$  \hspace{1cm} (6)

As usual, a U(1) gauge field $a_\mu$ is introduced in this
parametrization, and when the CP
1
field $z^e$ is gapped and disordered, the U(1) gauge field is in its deconfined
photon phase. In the deconfined photon phase, in
addition to gapless photon excitations, there are also two
types of basic gapped point particles. The first type is
the electric charge $e$, which is the CP
1
field $z^e$; the second kind of gapped particle is the Dirac monopole $m$, which is
nothing but the hedgehog monopole of $N$. This is
because in the standard CP
1
formalism, the U(1) gauge flux quantum $\int d^2x \frac{1}{2} \nabla \times a$ through any closed surface
is just the Skyrmion number $\int d^2x \frac{1}{2} N \cdot (\partial_x N \times \partial_y N)$. Therefore
the Dirac monopole, which is the source of the
gauge flux, is identified with the hedgehog monopole,
which is the source of the Skyrmion number. Thus the
Dirac monopole $m$ of the photon phase can also be represented by $z^m$.

Now we can also turn on the symmetry $T$. Based on
the transformation in Eq. \ref{T] \:} $T$ interchanges the electric
dirac monopole. Under the $Z_2$ and $T$ symmetries, the
electric charge $z^e$ and magnetic monopole $z^m$ transform as

$$Z_2 \ : \ z^{e,m} \rightarrow i \sigma^y (z^{e,m})^*,$$
$$T \ : \ z^e \rightarrow z^m, \ \ z^m \rightarrow z^e.$$  \hspace{1cm} (7)

The electric and magnetic field will transform as

$$Z_2 \ : \ E, \ B \rightarrow -E, \ -B;$$
$$T \ : \ E \rightarrow B, \ \ B \rightarrow E.$$  \hspace{1cm} (8)

The commutation relation $[E_i(x), B_j(x')] =
\epsilon_{ijl} \partial_{x'_l} \delta^3(x - x')$, and hence the Maxwell equation,
are invariant under the $Z_2$ and $T$ symmetries. Thus in
this photon phase the $T$ symmetry acts as a
electric-magnetic duality. This photon phase cannot be
driven into a trivial confined or Higgs phase because the
condensate of either $z^e$ or $z^m$ will inevitably generate
an order of a certain component of $n$, which therefore
breaks the $Z_2$ and $T$ symmetry. Since $T$ plays the role
as the duality symmetry, our photon phase is different from
all the 3d time-reversal symmetry enriched photon
phases classified in Ref. \ref{20}.

The two extra SO(3) symmetries make the physical
meaning of the $e$ and $m$ particles transparent: the $e$
($m$) particle is the hedgehog monopole of $-M$ ($N$ \cite{33}),
which is also a fractionalized CP
1
field of $N$ ($M$). Under the duality $Z_2^T$ symmetry, the role of $e$ and $m$
interchanges. But the nature of this photon phase does not
depend on the extra SO(3) symmetries. Thus after we
establish this photon phase, the SO(3) symmetries can be
explicitly broken without changing the physics of the
photon phase.

Another point excitation of this photon phase is the
dyon. A dyon is a bound state between $e$ and $m$ which forms a fermion (denoted as $f$). $e$ and $m$ view each other
as a $2\pi-$flux source. Therefore the effective theory that
describes the internal degree of freedom of a dyon is precisely
the $(0+1)d$ O(3) NLSM with a WZW term at level
1, i.e. Eq. \ref{1} except with $M$ replaced by $D$ where $D$
the vector that connects electric and magnetic charges.
The ground state of this model is again a spin-1/2 doublet.
Since now $e$ and $m$ interchange under $T$, this means that
in this model $T$ takes $D$ to $-D$, which implies that under $T$, the dyon is a Kramers doublet fermion: $T : f \rightarrow i \sigma^y f,$
and $T^2 = -1$. This is due to the fact that $T^2$ is effectively
a $2\pi-$rotation in space, and a spin-1/2 object acquires
a minus sign under $2\pi-$rotation. Thus the dyon transforms exactly the same under time-reversal as a physical
electron.

In the bulk of the 4d BSPT, the hedgehog monopole of
$N$, or equivalently the Dirac monopole of the gauge field
$a_\mu$ in the CP
1
formalism, is a 1d loop defect. And Eq. \ref{1} reduces to a $(1 + 1)d$ O(3) NLSM of $M$
with a topological $\Theta-$term with $\Theta = 2\pi$, which is an effective theory
for a spin-1 antiferromagnet chain \cite{21, 22}. Thus a closed
loop of hedgehog (Dirac) monopole in the 4d bulk is fully
gapped and nondegenerate, and can therefore proliferate
and drive the bulk into a gapped disordered SPT and
gauge confined phase. But when a monopole line terminates
at the boundary, its end cannot condense without
breaking symmetry. Thus it is possible for a deconfined
photon phase to exists at the 3d boundary. The physical
meaning of this boundary photon phase, including its
emergent photon excitations, can also be understood in
a different way, presented in the next section.

3. $Z_2$ TOPOLOGICAL ORDER WITH A KRAMERS DOUBLET FERMION

A 3d U(1) photon phase can usually be driven into a
fully gapped $Z_2$ topological order by condensing either
a pair of $e$ or $m$ particles. In the condensate of $2e$, $m$
and $f$ are confined because they have nontrivial mutual
statistics with $2e$, and the only deconfined point particle
is $e$. Besides the point particle, the system also has a
gapped loop excitation (usually called the “vison” loop
excitation), which has a mutual semionic statistics with
$e$; i.e. when the closed trajectory of $e$ links with the vi-
son loop by odd numbers, the system wave function will
acquire a minus sign. However, the vison loop has trivial
statistics with $2e$, and is therefore not confined in the $2e$
condensate.

In our system, the condensate of either $2e$ or $2m$ will
break the $\mathcal{T}$ symmetry. In order to construct a topological order that preserves $\mathcal{T}$, we need to condense a Cooper pair of dyons (each dyon is a fermion). Since the dyon is a Kramers doublet, we will condense the singlet Cooper pair of the dyons, just to preserve all the symmetries. In this condensate, both $e$ and $m$ excitations of the original photon phase will be confined because they have nontrivial mutual statistics with the dyon Cooper pair (this is also called oblique confinement [23, 24]), and the only deconfined but gapped point excitation is the fermionic dyon $f$.

The condensate of a pair of dyons also has a loop excitation which has mutual semionic statistics with the dyon. To understand this loop defect, let us first turn on two extra SO(2) symmetries where $(n_1, n_2)$ and $(n_4, n_5)$ transform as vectors under the first and second SO(2) group respectively:

\[ \text{SO}(2)_1 : (n_1 + in_2) \rightarrow e^{i\theta}(n_1 + in_2), \quad z^e \rightarrow e^{i2\sigma^1}z^e, \]
\[ \text{SO}(2)_2 : (n_4 + in_5) \rightarrow e^{i\theta}(n_4 + in_5), \quad z^m \rightarrow e^{i2\sigma^3}z^m \]

This means $e$ carries half charge of the first SO(2) symmetry and therefore must have a mutual semionic statistics with the vortex loop of $(n_1, n_2)$; likewise, $m$ must have a mutual semionic statistics with the vortex loop of $(n_3, n_4)$. The dyon, which is a bound state of $e$ and $m$, must have mutual semionic statistics with both types of vortex loops.

Besides their mutual statistics with point excitations, these vortex loops also have a nontrivial spectrum. The $(3 + 1)d$ WZW term in Eq. [3] will decorate each vortex line of $(n_1, n_2)$ with a $(1+1)d$ O(4) WZW-term at level-1:

\[
S_v = \int dx d\tau \frac{1}{g_2} \mathcal{L}^\mu U^\dagger \partial_\mu U + \int_0^1 \frac{i2\pi}{24\pi^2} e^{i\mu_\nu x^\lambda} \left(U^\dagger \partial_\mu U U^\dagger \partial_\nu U U^\dagger \partial_\lambda U \right) \tag{10}
\]

where $U \sim n_3e^{i\theta} + in_3e^{i\sigma^1} + in_3e^{i\sigma^2} + i\sigma^3$ is a SU(2) matrix. The condensate of $(n_1, n_2)$ will break part of the symmetries, but it preserves another new $Z'_2$ symmetry which is the combination of a $\pi$-rotation of SO(2)$_1$ and the $Z_2$ symmetry. Then this $Z'_2$ symmetry guarantees that the vortex loop of $(n_1, n_2)$ must be either gapless or degenerate, due to the 1d WZW term in Eq. [11]. The nature of the vortex loop can also be understood as the following: In the $4d$ bulk, a vortex of $(n_1, n_2)$ is a 2d membrane, and according to the bulk action Eq. [11] this 2d membrane is decorated with a 2d SPT phase with $Z'_2$ symmetry, which implies that when the vortex membrane terminates at the 3d boundary, it becomes a vortex loop, and it is also the boundary of a 2d SPT state; thus it must be either gapless or degenerate. The degeneracy of the vortex line corresponds to spontaneous symmetry breaking of the $Z'_2$ symmetry; for instance $\langle n_3 \rangle > 0$ or $\langle n_3 \rangle < 0$ along the vortex loop. Then there are two flavors of vortex loops, and the domain wall between the two flavors is precisely the hedgehog monopole of $N \sim (n_1, n_2, n_3)$.

Although a single vortex loop of $(n_1, n_2)$ must have nontrivial spectrum, a double strength vortex loop (a vortex loop of $4\pi$ vorticity) can be gapped and nondegenerate. One way to see this is that, because the $2d$ SPT phase with $Z'_2$ symmetry has a $Z_2$ classification [2, 3, 18], two copies of such a state becomes trivial; i.e. their boundary can be rendered gapped and nondegenerate.

Many interesting phases can be obtained by manipulating the dynamics of the vortex loops. For example, consider a superfluid phase with spontaneous SO(2)$_1$ symmetry breaking, i.e. a superfluid phase with condensation of complex boson $b_1 \sim n_1 + in_2$ (this phase also spontaneously breaks $\mathcal{T}$). Then according to Ref. [25] if the vortex loop has two flavors and both flavors proliferate, then this phase is precisely the photon phase described in section 2. The $Z_2$ topological order with $2f$ condensate discussed at the beginning of this section can be realized when the strength-2 (fully gapped and nondegenerate) vortex loop of $(n_1, n_2)$ proliferates.

Now let us start with a superfluid phase where the bound state of bosons $b_1 \sim n_1 + in_2$ and $b_2^* \sim n_4 - in_5$ condense. Under $\mathcal{T}$, $b_1 \rightarrow b_2$ and $b_2 \rightarrow b_1$; thus the condensate of bound state $(b_1b_2^*)$ does not break $\mathcal{T}$ (i.e. the real and imaginary parts of $(b_1b_2^*)$ are both invariant under $\mathcal{T}$). In this phase there are two types of vortex loops: vortex loops of $(n_1, n_2)$ and $(n_4, n_5)$. Now we argue that the $Z_2$ topological order with $2f$ condensate can be constructed by proliferating the bound state of these two types of vortex loops. First of all, since $e$ and $m$ particles both have mutual semionic statistics with this “bound vortex loop”, they will both be confined in this condensate; the only deconfined point particle is the dyon $f$, which views this “bound vortex loop” as a 2$\pi$–flux instead of a $\pi$–flux loop. Since in the $Z_2$ topological order the bound state of the two vortex loops already proliferate, there is only one type of well-defined loop excitation, and it can be viewed as the remnant of either the $(n_1, n_2)$ or $(n_4, n_5)$ vortex loop, either of which has the correct semionic statistics with the dyon. Thus this vortex loop can be identified as the vison loop excitation of the desired 3d $Z_2$ topological order.

More systematically, the condensate of bound state $b_1b_2^*$ can be described by the following the effective action in the Euclidean space-time:

\[
S = \sum_x \sum_{\epsilon = 1,2} -K \cos(d\theta_{\epsilon} - a), \tag{11}
\]

where $b_{1,2} \sim e^{i\phi_{1,2}}$, and $a$ is a 1-form gauge field. In this equation, when both $b_1$ and $b_2$ condense, the only gauge invariant order parameter is $b_1b_2^*$.

We can take the standard Villain form of the action by expanding the cosine function at its minimum and
introducing the 1-form fields \(l_c \in \mathbb{Z}\) and \(k_c \in \mathbb{R}\) (\(c = 1, 2\)):

\[
Z = \text{Tr} \exp \left[ \sum_{x} \sum_{c=1,2} -\frac{K}{2} (d\theta_c - a - 2\pi l_c)^2 \right]
\]
\[
\sim \text{Tr} \exp \left[ \sum_{x} \sum_{c=1,2} -\frac{1}{2K} k_c^2 + ik_c \cdot (d\theta_c - a - 2\pi l_c) \right]
\]
\[
\sim \text{Tr} \exp \left[ \sum_{x} \sum_{c=1,2} -\frac{1}{2K} k_c^2 - ik_c \cdot (a + 2\pi l_c) \delta[\partial k_c] \right]
\]
\[
\sim \text{Tr} \exp \left[ \sum_{x} \sum_{c=1,2} -\frac{1}{2K} (d\Omega_c)^2 + i(a + 2\pi l_c) \wedge d\Omega_c \right].
\]

(12)

In the last line, we introduce the 2-form fields \(\Omega_c\) (\(c = 1, 2\)) on the dual space-time manifold, such that \(k_c = *d\Omega_c\) resolves the constraint \(\partial k_c = 0\). Summing over \(l_c\) will require \(\Omega_c\) to take only integer values, which could be imposed by adding a \(\cos(2\pi \Omega_c)\) term, and the theory now becomes

\[
Z \sim \text{Tr} \exp \left[ \sum_{x} \sum_{c=1,2} -\frac{1}{2K} (d\Omega_c)^2 + ia \wedge d\Omega_c + t \cos(2\pi \Omega_c) \right]
\]

(13)

Integrating out the gauge field \(a\) will impose the constraint \(d(\Omega_1 + \Omega_2) = 0\), which can be resolved by \(\Omega_1 = \Omega - dv_1/(2\pi)\), \(\Omega_2 = -\Omega + dv_2/(2\pi)\). Therefore the final action takes the form of

\[
S \sim \sum_{x} \sum_{c=1,2} -t \cos(dv_c - 2\pi \Omega) + \frac{1}{K} (d\Omega)^2.
\]

(14)

\(v_1\) and \(v_2\) are both 1-form vector fields. \(\Psi_{1,\mu} \sim \exp(i\nu_{1,\mu})\) creates a segment of vortex line of \(b_1\) along the \(\mu\) direction, while \(\exp(idv_1)\) creates a unit vortex loop. If we are going to proliferate the bound state of the two types of vortex loops, say \(\Psi_{1,\mu} \Psi_{2,\mu} \sim \exp(i\nu_{\mu}) = \exp(i\nu_{1,\mu} + i\nu_{2,\mu})\), then the effective action for \(\nu_{\mu}\) reads

\[
S \sim \sum_{x} -t \cos(dv - 4\pi \Omega) + \frac{1}{K} (d\Omega)^2.
\]

(15)

When \(\nu_{\mu}\) proliferates, \(\Omega\) can take two inequivalent minima: \(\Omega = 0, 1/2\); therefore this state is a \(\mathbb{Z}_2\) topological order.

Once we establish the existence of this \(\mathbb{Z}_2\) topological order, the extra \(SO(2)\) symmetries can be broken, which will not affect the statistics between vison loops and the dyon \(f\). In this \(\mathbb{Z}_2\) topological order, since there is no spontaneous symmetry breaking at all, the original \(\mathbb{Z}_2\) symmetry already guarantees that the vison loop must be either gapless or degenerate because the vison loop is effectively the boundary of a 2d SPT state with \(Z_2\) symmetry. However, after we break the two \(SO(2)\) symmetries, the bound state between the \((n_1, n_2)\) and \((n_3, n_5)\) vortex loops become gapped and nondegenerate, allowing it to safely proliferate. Because the deconfined point particle excitation of this \(\mathbb{Z}_2\) topological order is a fermion, it cannot condense and drive the system into a trivial Higgs phase; similarly, because the vison loop is gapless or degenerate, it also cannot proliferate and drive the system into a gapped and nondegenerate confined phase.

While the \(\mathbb{Z}_2\) topological order itself cannot be driven into either a trivial Higgs or confined phase, two copies of this \(\mathbb{Z}_2\) topological order can indeed be trivialized. The reason is that, for two copies of the \(\mathbb{Z}_2\) topological order (labeled \(A\) and \(B\)), one can first condense the bound state of dyons from both copies (i.e. condensate of \(f_A f_B\)) to break the two copies of \(\mathbb{Z}_2\) topological order down to one \(\mathbb{Z}_2\) topological order. Then in this residual \(\mathbb{Z}_2\) topological order the only well-defined point particle is the dyon \(f_A\) (or equivalently \(f_B\) because of the background pair condensate). The vison loop is the bound state of vison loops from both copies: \(\Psi_{A,\mu} \Psi_{B,\mu}\); this is because \(\Psi_{A,\mu}\) and \(\Psi_{B,\mu}\) individually have semionic statistics with \(f_A f_B\), and hence must be confined in the condensate. Since \(\Psi_{A,\mu}\) and \(\Psi_{B,\mu}\) both carry a \((1 + 1)\)d WZW term at level-1, their bound state is fully gapped, and hence can further proliferate and drive the entire system into a trivial confined phase without any symmetry breaking. This implies that two copies of the BSPT states Eq. 11 with \(\mathbb{Z}_2 \times T\) symmetry is a trivial state, which is consistent with the classification based on the NLSM itself given in appendix A.

Having understood the self-dual photon phase and the \(\mathbb{Z}_2\) topological order, we can realize an exotic \((3 + 1)\)d self-dual topological photon phase by making a thin slab of 4d BSPT state with \(\mathbb{Z}_2 \times T\) symmetry, and realizing the self-dual photon phase on the top boundary and the \(\mathbb{Z}_2\) topological order on the bottom boundary. At low energy, only the photon phase at the top boundary will be detectable while only at higher energy will the bottom boundary be exposed.

4. SELF-DUAL PHOTON PHASE WITH \(Z_4\) SYMMETRY

Another BSPT phase that leads to a self-dual photon phase at its boundary is a state with \(Z_4\) symmetry, which is still described by Eq. 11 but now the the vector \(n\) transforms as

\[
Z_4 : (n_1 + in_2) \rightarrow e^{i\theta}(n_1 + in_2),
\]
\[
(n_3 + in_4) \rightarrow e^{i\theta}(n_3 + in_4),
\]
\[
(n_5 + in_6) \rightarrow e^{i\theta}(n_5 + in_6),
\]

(16)

where \(\theta = \frac{2\pi k}{4}\), with \(k = 0, 1, 2, 3\). Using the formalisms introduced in section 2, we can demonstrate that the 3d boundary of this \(4d\) BSPT is a self-dual photon phase.
with the following transformation of its $e$ and $m$ excitations:

$$Z_4 : z^e \rightarrow z^m, \quad z^m \rightarrow i\sigma^y(z^e)^*.$$  \hfill (17)

Note here $e$ and $m$ are hedgehog monopoles of three component vectors $\mathbf{N} \sim (n_2, n_4, n_6)$ and $\mathbf{M} \sim (n_1, n_3, n_5)$ respectively. The emergent electric and magnetic fields transform under the $Z_4$ symmetry as

$$E \rightarrow B, \quad B \rightarrow -E.$$  \hfill (18)

Again the Maxwell equation and the commutation relations between $E$ and $B$ fields are invariant under this $Z_4$ transformation.

Unlike the previous case, it is not obvious whether we can drive this self-dual photon phase into a gapped topological order with full $Z_4$ symmetry. As we discuss in appendix B, this $Z_4$ BSPT state can be constructed by breaking the $U(1)$ symmetry of the 4$d$ bosonic integer quantum Hall state down to $Z_4$. In Ref. [26], we argued that a BSPT state whose boundary has perturbative gauge anomaly after “gauging” the symmetry cannot be driven into a fully gapped topological order because the system must respond to infinitesimal external gauge field. The boundary of the 4$d$ bosonic integer quantum Hall (BIQH) state we discuss in appendix B has a perturbative gauge anomaly after the $U(1)$ symmetry is gauged. Thus the boundary of the 4$d$ BIQH state cannot be driven into a symmetric topological order. It is possible that the $Z_4$ BSPT state inherits this property from its BIQH parent state. More rigorous study will be given in the future.

5. SUMMARY

In this work we studied two examples of different self-dual photon phases that can be realized on the boundary of 4$d$ bosonic SPT states. Both states need certain symmetries to protect their boundaries, which is an important difference from the self-dual photon phase studied in Ref. [11]. Understanding these symmetry protected self-dual photon phases at the boundary of 4$d$ systems can lead to exotic photon phases on a 3$d$ system as well, as was discussed in the end of section 3. In even higher dimensions, topological orders and photon states can still exist on the boundary of BSPT states; but more exotic boundary states can also be realized, such as deconfined spin liquid phases with nonabelian gauge fields. This is due to the fact that a nonabelian gauge fields usually lead to confinement in dimensions lower than $3 + 1$, while in higher dimensions a stable deconfined phase exists.

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[34] This opposite sign is due to the fact that in the WZW term of Eq. 3 the monopole of $\mathbf{N}$ is a $(0 + 1)/2$ WZW model of $\mathbf{M}$ with level $k = 1$, while the monopole of $\mathbf{M}$ is a WZW model of $\mathbf{N}$ with level $k = -1$. 
SUPPLEMENTAL MATERIAL

A. NLSM field theory description of BSTP states

Bosonic SPT phases in all dimensions with various continuous and discrete symmetries can be systematically described and classified by semiclassical nonlinear Sigma model (NLSM) field theories with a topological Θ-term. In 1+1d, it is well known that a spin-1 chain can be described by an O(3) NLSM with topological a Θ-term at Θ = 2π where its disordered phase corresponds to a BSPT state protected by O(3) or time reversal symmetry. In general, a BSPT phase in (d + 1)-dimensional space-time can always be formulated by an O(d + 2) NLSM with a topological Θ-term, assuming the symmetry group G of the BSPT state is a subgroup of O(d + 2) and other discrete symmetries such as time reversal:

$$S^\Theta_{d+1} = \int d^{d+1}x d\tau \frac{1}{g} (\partial \mu n)^2 (19)$$

The boundary theory of the d + 1-dimensional theories with Θ = 2π are described by (d − 1) + 1-dimensional O(d + 2) NLSMs with a Wess-Zumino-Witten (WZW) term at level-1:

$$S^{WZW}_{d+1} = \int d^{d+1}x d\tau \frac{1}{g} (\partial \mu n)^2 (20)$$

To define the WZW-term we need to extend the order parameter field \( n(x_1, x_2, ..., x_{d-1}, \tau) \) to \( n(x_1, x_2, ..., x_{d-1}, \tau, u) \) with the following condition:

$$n(x_1, x_2, ..., x_{d-1}, \tau, 0) = (0, 0, ..., 1) \quad (21)$$

$$n(x_1, x_2, ..., x_{d-1}, \tau, 1) = n(x_1, x_2, ..., x_{d-1}, \tau) \quad (22)$$

The spectrum of the boundary theory above is in general non-trivial: either degenerate or gapless. For example, when \( d = 1 \), the boundary is a 0+1d O(3) NLSM with a WZW term at level-1. As the theory has full O(3) symmetry or time reversal symmetry, the ground state is a doublet with protected two-fold degeneracy. When \( d = 2 \), the boundary is a 1+1d O(4) WZW-term at level-1, which is conformal assuming the full O(4) symmetry is preserved. The spectrum can also be degenerate if we only have discrete symmetry and the degeneracy is precisely due to the symmetry breaking.

Notice that all components of \( n \) in Eq. 20 must have a nontrivial transformation under the symmetry group \( G \). Otherwise one can turn on a linear “Zeeman” term that polarizes some component of \( n \) which will trivially gap out the edge states. In this case, the Θ-term has no effect, and the bulk state is trivial.

Eq. 1 with \( Z_2 \times T \) symmetry is a nontrivial BSTP state when \( \Theta = 2\pi \). However, two copies (layers) of Eq. 1 can be trivialized after turning on symmetry allowed interlayer couplings. For instance, starting with two copies of Eq. 1 (labeled A and B), the following coupling is allowed by the symmetry:

$$H_{AB} = \int d^4x - w(n_{A,1}n_{B,4} + n_{A,2}n_{B,5} + n_{A,3}n_{B,6} + n_{A,4}n_{B,1} + n_{A,5}n_{B,2} + n_{A,6}n_{B,3}). \quad (23)$$

When \( w \) is positive and large,

$$(n_{A,1}, n_{A,2}, n_{A,3}) \sim (n_{B,4}, n_{B,5}, n_{B,6})$$

$$(n_{A,4}, n_{A,5}, n_{A,6}) \sim (n_{B,1}, n_{B,2}, n_{B,3}) \quad (24)$$

As a result, the two Θ-terms of copies A and B will cancel out, and effectively the coupled system has \( \Theta = 0 \), and is therefore a trivial state. This conclusion is consistent with the analysis based on the boundary topological orders in section 3.

B. 4d bosonic integer quantum Hall state as parent state

In this section we discuss 4d bosonic integer quantum Hall (BIQH) states and their relation with the two states discussed in this work. The 4d BIQH state is a straightforward generalization of the 2d BIQH state discussed in Ref. 33. It is described by a (4 + 1)d O(6) NLSM with \( \Theta = 2\pi \) (Eq. 1), where the six component vector \( n \) transforms under the U(1) symmetry as

$$U(1) : (n_1 + in_2) \rightarrow e^{i\theta} (n_1 + in_2), (25)$$

$$n_3 + in_4 \rightarrow e^{i\theta} (n_3 + in_4),$$

$$n_5 + in_6 \rightarrow e^{i\theta} (n_5 + in_6).$$

If we couple the U(1) charge to an external U(1) gauge field \( A_\mu \), then after integrating out the matter field \( n \), a Chern-Simons term is generated for \( A_\mu \):

$$S_{cs} = \int d^4x d\tau \frac{6i}{3! (2\pi)^2} \epsilon_{\mu\nu\rho\beta} A_\mu \partial_\nu A_\rho \partial_\alpha A_\beta \quad (26)$$

which is a CS theory at level 6.

Directly integrating out the bosonic field is technically difficult. But alternatively we can start with 8 copies of 4d fermionic integer quantum Hall model:

$$H = \sum_{k} \sum_{a=1}^{8} \psi_{a,k}^\dagger \left( \sum_{i=1}^{4} \Gamma_i \sin k_i + m (e - \sum_i \cos k_i) \Gamma_5 \right) \psi_{a,k} \quad (27)$$
\( \Gamma_1 = \sigma^{13}, \Gamma_2 = \sigma^{43}, \Gamma_3 = \sigma^{33}, \Gamma_4 = \sigma^{01}, \Gamma_5 = \sigma^{02} \) where \( \sigma^{ab} = \sigma^a \otimes \sigma^b \otimes \cdots \). We focus on the phase with \( 3 < c < 4 \), where each fermion copy (labeled by \( a \)) gives rise to a 3d chiral fermion at the boundary. Therefore there are in total 8 chiral fermions at the boundary of Eq. 27:

\[
H_{3d} = \int d^3x \sum_{n=1}^{8} \psi_n^\dagger (i\sigma \cdot \partial) \psi_n. \tag{28}
\]

Now we can couple the boundary chiral fermions to a six component vector \( \mathbf{n} \):

\[
H_{int} = \int d^3x \left( u(n_1 \text{Re}[\psi_n^\dagger \sigma^{122} \psi] + n_2 i\text{Im}[\psi_n^\dagger \sigma^{212} \psi]) + u(n_3 \text{Re}[\psi_n^\dagger \sigma^{202} \psi] + n_4 i\text{Im}[\psi_n^\dagger \sigma^{022} \psi]) + u(n_5 \text{Re}[\psi_n^\dagger \sigma^{322} \psi] + n_6 i\text{Im}[\psi_n^\dagger \sigma^{232} \psi]) \right). \tag{29}
\]

The same WZW term as Eq. 3 will be generated after integrating out the fermions.

This fermion model (Eq. 27) has at most a U(8) symmetry, which contains three U(1) symmetries as a subgroup. The three U(1) symmetries are generated by \( \sigma^{330}, \sigma^{220}, \) and \( \sigma^{110} \) so that \( (n_1, n_2), (n_3, n_4), \) and \( (n_5, n_6) \) transform as two-component vectors under these three U(1) symmetries, respectively. Now let us couple the fermion model Eq. 27 to three U(1) gauge fields: \( \frac{1}{2} A^{(1)}_\mu \sigma^{330}, \frac{1}{2} A^{(2)}_\mu \sigma^{220}, \frac{1}{2} A^{(3)}_\mu \sigma^{110} \). We give the fermions charge\(-1/2\) because we want the bosons to carry charge\(-1\) under these gauge fields. Then after integrating out the fermions, the following Chern-Simons field theory is generated:

\[
S = \frac{1}{8} \text{tr}[\sigma^{330}\sigma^{220}\sigma^{110}] \int d^4x d\tau \times \frac{i}{3!(2\pi)^2} \epsilon^{\mu\nu\rho\sigma} A^{(1)}_\mu (\partial_\nu A^{(2)}_\rho (\partial_\sigma A^{(3)}_\sigma)) + \text{ permutation of } 1, 2, 3; \tag{30}
\]

After breaking these three U(1) gauge symmetries down to a single U(1) gauge symmetry, Eq. 28 is generated.

The two BSPT states we discussed in this paper can be obtained by breaking the U(1) symmetry down to either \( Z_2 \) or \( Z_4 \) symmetry. Notice that the BIQH state with U(1) symmetry has a \( Z \) classification with U(1) symmetry and \( \Theta = 2\pi k \) where each integer \( k \) corresponds to a different BIQH state. Notice that the coupling Eq. 28 explicitly breaks the U(1) symmetry.

\[ C. \text{ Dimensional Reduction of Topological Terms} \]

In this Appendix, we are going to derive the effective field theory of a monopole core of an O(6) NLSM, namely Eq. 4. In \( 3 + 1d \) a monopole configuration of an O(3) order parameter, for instance \( (n_1, n_2, n_3) \), can be understood as an intersection point of the domain walls of the three order parameter fields respectively. So we can derive the theory on a monopole core by three domain wall projections, which is described below.

To derive the theory on the domain wall of one of the order parameter fields, e.g. \( n_1 \), we first construct a domain wall configuration of \( n_1 \). Consider the following configuration of the vector \( \mathbf{n} \):

\[
\mathbf{n} = (\cos \theta, \sin \theta N_2, \sin \theta N_3, \sin \theta N_4, \sin \theta N_5, \sin \theta N_6), \]

where \( \mathbf{N} \) is an O(5) vector with unit length and \( \theta \) is a function of coordinate \( z \) only with

\[
\theta(z = +\infty) = \pi, \quad \theta(z = -\infty) = 0. \tag{31}
\]

By inserting this parametrization of \( \mathbf{n} \) into Eq. 3 and integrating along the \( z \) direction, the O(6) WZW-term reduces to an O(5) WZW-term with the same level. More explicitly, the theory on the domain wall is:

\[
S_{dw} = \int d^2xd\tau \frac{1}{g} (\partial_\mu \mathbf{N})^2 + \int_0^1 du \frac{2\pi}{\Omega_4} \epsilon_{abcd} N^a \partial_x N^b \partial_y N^c \partial_z N^d \partial_u N^e. \tag{32}
\]

A domain wall projection reduces both the spatial dimension and the dimension of the order parameter field by one, and the effective field theory on the domain wall inherits the topological term from the original theory.

We can repeat this domain wall projection procedure once more. On the \( n_1 \) domain wall we just made, consider a domain wall of \( n_2 \) along the \( y \)-direction. We can integrate over the \( y \)-direction, and the resulting theory is an O(4) WZW-term with level-1:

\[
S_c = \int d^2xd\tau \frac{1}{g} (\partial_\mu \mathbf{N})^2 + \int_0^1 du \frac{2\pi}{\Omega_3} \epsilon_{abcde} N^a \partial_x N^b \partial_y N^c \partial_z N^d \partial_u N^e. \tag{33}
\]

This field theory can be thought of as the effective field theory on a \( 2\pi \)-vortex of \( (n_1, n_2) \) components. Notice that this field theory is equivalent to an \( 1 + 1d \) SU(2) principle chiral model by introducing SU(2) matrix field \( U = n_3 \sigma^0 + in_4 \sigma^1 + in_5 \sigma^2 + in_6 \sigma^3 \). The SU(2) principle chiral model is precisely written as in Eq. 10.

Based on the configuration we already have, if we further make a domain wall of \( n_3 \) on the vortex core, then the whole configuration of the order parameter field corresponds to a monopole configuration of the O(3) order parameter. And right on the core of the monopole, the effective field theory is precisely an O(3) WZW-term at level-1 as in Eq. 4.