The gauge action, DG Lie algebra and identities for Bernoulli numbers

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March 17, 2014

Abstract

In this paper we prove a family of identities for Bernoulli numbers parameterized by triples of integers \((a, b, c)\) with \(a + b + c = n - 1\), \(n \geq 4\). These identities are deduced while translating into homotopical terms the gauge action on the Maurer Cartan Set which can be seen an abstraction of the behaviour of gauge infinitesimal transformations in classical gauge theory. We show that Euler and Miki’s identities, well known and apparently non related formulas, are linear combinations of our family and they satisfy a particular symmetry relation.

Introduction

In our setting, an Euler-type identity is defined to be a convolution equation of the form

\[ \sum_{k=0}^{n} \lambda_k B_k B_{n-k} = 0, \]

*Partially supported by the Ministerio de Economía y Competitividad grant MTM2010-15831, by the grants FQM-213, 2009-SGR-119, and by the Marie Curie COFUND programme U-mobility, co-financed by the University of Málaga, the European Commission FP7 under GA No. 246550, and Ministerio de Economía y Competitividad (COFUND2013-40259).
†Partially supported by the Ministerio de Economía y Competitividad grant MTM2010-18089.
‡Partially supported by the Ministerio de Economía y Competitividad grant MTM2010-18089 and by the Junta de Andalucía grants FQM-213 and P07-FQM-2863.

Key words and phrases: Gauge action. Bernoulli numbers. Homotopy theory of Lie algebras.
where $\lambda_k$ are rationals and $B_k$ are the Bernoulli numbers. Euler equation,

$$-(n+1)B_n = \sum_{k=2}^{n-2} \binom{n}{k} B_k B_{n-k}, \quad n \geq 4,$$  \hfill (0.0.1)  

is of this kind as it is the, now classical, Miki’s identity [10].

$$2H_n B_n = \sum_{k=2}^{n-2} \frac{n}{k(n-k)} \binom{n}{k} \left(1 - \binom{n}{k}\right) B_k B_{n-k}, \quad n \geq 4,$$  \hfill (0.0.2)  

where $H_n = \sum_{j=1}^{n} \frac{1}{j}$ is the harmonic number.

Combinatorial, arithmetical, analytical and geometrical methods have been used along past years to deduced Euler-type identities [3, 4, 5, 7, 11, 12]. Some of them, including Euler and Miki’s, are particular instances of a large family of Euler-type identities whose existence we prove in this paper. Namely,

**Theorem 0.1.** For any even integer $n \geq 4$ and any triple of integers $(a, b, c)$ such that $a + b + c = n - 1$, an Euler-type identity holds with

$$\lambda_k = \binom{n}{k} \left[ (-1)^c \binom{n-k}{c} \sum_{\ell=\max(0,k-b)}^{\min(a,k)} (-1)\ell \binom{k}{\ell} \right]$$  \hfill (0.0.3)  

$$-(-1)^a \binom{n-k}{a} \sum_{\ell=\max(0,k-b)}^{\min(c,k)} (-1)\ell \binom{k}{\ell} \right].$$

At the sight of its equivalent formulations detailed below, the interest of this result lies equally in its own rite as in the methods used to deduce it, which we now briefly describe.

A fundamental principle of deformation theory due to Deligne asserts that every deformation functor is governed by a differential graded Lie algebra via solutions of Maurer-Cartan equation modulo the gauge action. This permits us to view deformations of structures in completely different settings under a common algebraic scope. The quantization theorem of Kontsevich [8] or the deformation theory of Floer homology of lagrangian submanifolds and its relation with mirror symmetry [6] are good examples of this.

Given a differential graded Lie algebra, the gauge action on its Maurer-Cartan set, which may be understood as an algebraic abstraction of the behaviour of gauge infinitesimal transformations in classical gauge theory, can be encoded via the *Lawrence-Sullivan construction* $\mathfrak{L}$, see [1, Prop.3.1]
or [2, 4.6]. Then, we show that Theorem 0.1 above is equivalent to Theorem 1 of [9], that is, $L$ is indeed a differential graded Lie algebra. This is done by transporting $L$ to the category of differential graded algebras via the universal enveloping functor and forcing it to be also a universal object or cylinder in the corresponding homotopy category [1, Thm. 3.3].

Thus, the gauge action, the existence of a particular cylinder in the homotopy category of differential graded algebras, and the Euler-type identities in Theorem 0.1 are equivalent formulations of the same statement.

In the next section, devoted to the proof of our main result and its condensed version, we will develop in detail the above brief sketch. In Section 2, we show how Miki and Euler’s identities are particular instances of our family, and we find an unexpected symmetry relation between them.

1 Euler-type identities and the gauge action

This section, entirely devoted to the proof of Theorem 0.1, begins with a brief but explicit description of the algebraic version of the gauge action. From now on, any considered algebraic object is assumed to be $\mathbb{Z}$-graded and over a coefficient field $K$ of characteristic zero.

Recall that a differential graded Lie algebra, DGL henceforth, is a graded vectors space $L = \oplus_{n \in \mathbb{Z}} L_n$ endowed of a bilinear bracket of degree zero $[\cdot, \cdot] : L \otimes L \to L$, and a differential $\partial : L \to L$ of degree $-1$ satisfying,

\[
[x, y] = (-1)^{|x||y|+1}[y, x], \\
[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}|y|[x, [x, z]], \\
\partial[x, y] = [\partial x, y] + (-1)^{|x|}[x, \partial y],
\]

for any homogeneous elements $x, y, z \in L$.

The Maurer-Cartan set $\text{MC}(L)$ of $L$ is formed by those elements $a \in L_{-1}$ whose differential satisfy the Maurer-Cartan equation,

\[
\partial a + \frac{1}{2}[a, a] = 0.
\]

Let $L$ be a DGL which is free complete (see below) or in which $L_0$ acts locally nilpotently in $L$, that is, for any $x \in L_0$, $\text{ad}_x^n = 0$ for $n$ sufficiently large. The gauge action of $L_0$ on $\text{MC}(L)$ is defined as

\[
x * a = \sum_{i \geq 0} \frac{\text{ad}_x^i(a)}{i!} - \sum_{i \geq 0} \frac{\text{ad}_x^i(\partial x)}{(i + 1)!}, \quad x \in L_0, \quad a \in \text{MC}(L).
\]
This can be geometrically interpreted as follows \cite{9}: in the DGL given by 
\[ L[t] = L \otimes \mathbb{K}[t] \] 
consider the formal differential equation
\[
\begin{align*}
  u'(t) &= \partial x - \text{ad}_x u(t), \\
  u(0) &= \alpha.
\end{align*}
\]
Then, writing \( u(t) \) as a formal power series, one has \( x^* \alpha = u(1) \). In other words, thinking of MC(\( L \)) as points (say of a “formal manifold”), the gauge action can be thought of as the flow at time 1, generated by \( x \) via the above equation, with initial point \( \alpha \).

The minimal algebraic expression of this behavior, that is, an integral line joining two distinct points, was given in \cite{9}: Consider
\[
\mathcal{L} = \mathcal{L}(\alpha, \beta, x)
\]
the complete free Lie algebra\(^1\) generated by the Maurer-Cartan elements \( \alpha, \beta \) and in which the flow generated by \( x \) moves from \( \alpha \) to \( \beta \), i.e., \( x^* \alpha = \beta \). Then,

**Theorem 1.1.** \cite{9} Thm.1] The unique choice for \( \partial x \) which makes \( \mathcal{L} \) a DGL is
\[
\partial x = \text{ad}_x \beta + \sum_{i \geq 0} \frac{B_i}{i!} \text{ad}_x^i(\beta - \alpha).
\]

It is worth to remark that the gauge action in any differential graded Lie algebra \( L \) is “controlled” by \( \mathcal{L} \). Indeed, see \cite{1} Prop.3.1 or \cite{2} 4.6], any two elements \( a, b \in \text{MC}(L) \) are gauge equivalent if and only if there is a DGL morphism \( \mathcal{L} \to L \) sending \( \alpha \) to \( a \) and \( \beta \) to \( b \). In homotopical terms, \( \mathcal{L} \) is a universal cylinder for the gauge relation, which is in turn equivalent to the classical Quillen homotopy notion for DGL’s.

Now, recall that the universal enveloping algebra of a complete free Lie algebra generated by the vector space \( V \) is precisely the complete free tensor algebra on that set, that is \( U\mathcal{L}(V) = \mathcal{T}(V) \), in which \( \mathcal{T}(V) = \prod_{n \geq 0} T^n(V) \), where \( T^n(V) = V \otimes \ldots \otimes V \). Thus, the computation of \( U\mathcal{L} \), together with the induced differential, gives an equivalent formulation of Theorem 1.1 and thus of the gauge action, in the category of associative differential graded algebras. Write
\[
\mathcal{U} = U\mathcal{L} = \mathcal{T}(\alpha, \beta, x).
\]

\(^1\)Except from antisymmetry and Jacobi identity, no other relations among brackets of the generators \( \alpha, \beta, x \) are satisfied. The term complete indicates that brackets of infinite length among generators are allowed.
Theorem 1.2. [1 Thm.3.3] The differential in \( \mathcal{X} \) is given by
\[
D\alpha = -\alpha \otimes \alpha, \quad D\beta = -\beta \otimes \beta
\]
and
\[
Dx = x \otimes \beta - \beta \otimes x + \sum_{k \geq 0} \sum_{p+q=k} (-1)^q \frac{B_{p+q}}{p!q!} x^p \otimes x^q \otimes (\beta - \alpha) \otimes x^{\otimes q}.
\]

In this context, we will devote the rest of the section to show that the equation
\[
D^2 = 0
\]
and therefore the two results above, together with all the geometrical implications underneath, are equivalent to the Euler-type identities of Theorem 0.1.

From now on, for simplicity on the notation, we will drop the \( \otimes \) sign so that
\[
D\alpha = -\alpha^2, \quad D\beta = -\beta^2
\]
and
\[
Dx = x\beta - \beta x + \sum_{k=0}^{\infty} \sum_{p+q=k} (-1)^q \frac{B_{p+q}}{p!q!} x^p (\beta - \alpha)x^q. \tag{1.0.4}
\]

We will also write
\[
y = \beta - \alpha, \quad c_{(p,q)} = (-1)^q \frac{B_{p+q}}{p!q!}, \quad \Phi = \sum_{k=0}^{\infty} \sum_{p+q=k} c_{(p,q)} x^p y x^q.
\]

Observe that, while \( D^2\alpha = D^2\beta = 0 \) holds trivially, a short computation shows that \( D^2x = 0 \) is equivalent to
\[
D\Phi = -(\Phi \otimes \beta + \beta \otimes \Phi). \tag{1.0.5}
\]

Now, as \( D \) is a derivation, it is easy to deduce from (1.0.4) that
\[
Dx^m = x^m \beta - \beta x^m + \sum_{i+j=m-1} x^i \Phi x^j.
\]

Then, we have
\[
D(x^p y x^q) = (x^p \beta - \beta x^p + \sum_{i+j=p-1} x^i \Phi x^j) y x^q + x^p (Dy)x^q
\]
\[
- x^p y (x^q \beta - \beta x^q + \sum_{i+j=q-1} x^i \Phi x^j)
\]
\[
= (x^p \beta y + x^p y x^q \beta) + \Gamma_{(p,q)}, \quad \text{(*)}
\]
\[
\text{(*) A straightforward computation shows that } x^p(\beta y + Dy + y\beta)x^q = x^py^2x^q.
\]

5
where
\[ \Gamma_{(p,q)} = x^p y^2 x^q + \left( \sum_{i+j=p-1} x^i \Phi x^j \right) y x^q - x^p y \left( \sum_{i+j=q-1} x^i \Phi x^j \right). \] (1.0.6)

Therefore,
\[
D\Phi = \sum_{k=0}^{\infty} \sum_{p+q=k} c_{(p,q)} D(x^p y x^q)
\]
\[
= \sum_{k=0}^{\infty} \sum_{p+q=k} c_{(p,q)} \left( -\left( \beta x^p y x^q + x^p y x^q \beta \right) + \Gamma_{(p,q)} \right)
\]
\[
= -\left( \Phi \beta + \beta \Phi \right) + \sum_{k=0}^{\infty} \sum_{p+q=k} c_{(p,q)} \Gamma_{(p,q)}.
\]

Hence, at the sight of (1.0.5), we conclude that \( D^2 = 0 \) is equivalent to
\[
\sum_{k=0}^{\infty} \sum_{p+q=k} c_{(p,q)} \Gamma_{(p,q)} = 0.
\] (1.0.7)

Observe that the left hand side of the above equation can be rewritten as,
\[
\sum_{k=0}^{\infty} \sum_{p+q=k} c_{(p,q)} \Gamma_{(p,q)} = \sum_{n \geq 1} \sum_{a+b+c=n-1} \gamma_{(a,b,c)} x^a y^b y^c.
\]

Thus, \( D^2 = 0 \) if and only if, for any \( n \geq 1 \) and any triple of non negative integers \((a, b, c)\) such that \( a + b + c = n - 1\),
\[ \gamma_{(a,b,c)} = 0. \]

Hence, Theorem 0.1 will be established if we show that, for \( n \geq 4 \),
\[ \gamma_{(a,b,c)} = \sum_{k=0}^{n} \lambda_k B_k B_{n-k} \]
with \( \lambda_k \) as in (0.0.3). For it, we find all the terms of the left hand side of (1.0.7) contributing to the monomial \( x^a y^b y^c \) for a fixed \((a, b, c)\).

Observe that the summand \( \left( \sum_{i+j=p-1} x^i \Phi x^j \right) y x^q \) of \( \Gamma_{(p,q)} \) in equation (1.0.6) contributes to the monomial \( x^a y^b y^c \) only if \( q = c \) and \( 1 \leq p \leq a + b + 1 \). This contribution is:
\[ c_{(a-i,b-j)} x^a y^b y^c, \quad i + j = p - 1, \quad 0 \leq i \leq a, \quad 0 \leq j \leq b. \]
In the same way, the contribution of the summand \( x^p y(x^a y^b y^c) \) of 
\( \Gamma(p,q) \) to \( x^n y^a y^b y^c \), which requires \( p = a \) and \( 1 \leq q \leq b + c + 1 \), is:

\[
c_{(b-i,c-j)} x^n y^a y^b y^c, \quad i + j = q - 1, \ 0 \leq i \leq b, \ 0 \leq j \leq c.
\]

Thus, multiplying each of the above by the corresponding coefficient \( c_{(p,q)} \) and adding them all, we conclude that, for \( b \neq 0 \),

\[
\gamma(a,b,c) = \left( \sum_{p=1}^{a+b+1} c_{(p,c)} \sum_{i+j=p-1}^{a+b+1} c_{(a-i,b-j)} \right) - \left( \sum_{q=1}^{b+c+1} c_{(a,q)} \sum_{i+j=q-1}^{b+c+1} c_{(b-i,c-j)} \right),
\]

plus the extra term \( c_{(a,c)} = (-1)^c \frac{B_{a+c}}{a!c!} \) coming from the first summand of (1.0.6), added only when \( b = 0 \).

In other words, equation \( \gamma(a,b,c) = 0 \) translates to,

\[
\sum_{p=1}^{a+b+1} B_{p+c} B_{a+b+1-p} \frac{(-1)^c}{p!c!} \left( \sum_{i,j} \frac{(-1)^{b-j}}{(a-i)!(b-j)!} \right) - \sum_{q=1}^{b+c+1} B_{a+q} B_{b+c+1-q} \frac{(-1)^q}{a!q!} \left( \sum_{i,j} \frac{(-1)^{c-j}}{(b-i)!(c-j)!} \right) = 0,
\]

again for \( b \neq 0 \), plus the extra term \( (-1)^c \frac{B_{a+c}}{a!c!} \) on the left if \( b = 0 \).

Now, writing \( k = a + b + 1 - p \) and \( \ell = a - i \) in the first term and \( k = b + c + 1 - q \) and \( \ell = c - i \) in the second one we get,

\[
\sum_{k=0}^{n} B_k B_{n-k} \frac{(-1)^k}{c!(n-c-k)!} \left( \sum_{\ell=\max(0,k-b)}^{\min(a,k)} \frac{(-1)^{k-\ell}}{\ell!(k-\ell)!} \right) - \sum_{k=0}^{n} B_k B_{n-k} \frac{(-1)^{n-k-a}}{a!(n-k-a)!} \left( \sum_{\ell=\max(0,k-b)}^{\min(c,k)} \frac{(-1)^{k-\ell}}{\ell!(k-\ell)!} \right) = 0. \quad (1.0.8)
\]

Note that, the summations, which should be \( \sum_{k=0}^{a+b} \) and \( \sum_{k=0}^{b+c} \), have been replaced by \( \sum_{k=0}^{n} \) (recall that \( a + b + c = n - 1 \)). Indeed, for the first summand, if \( k \geq a + b + 1 \), then \( \min(a,k) = a \) and \( \max(0,k-b) > a \). For the second summand, if \( k \geq b + c + 1 \), then \( \min(c,k) = c \) and \( \max(0,k-b) > c \).

A short computation shows that the term containing \( B_1 B_{n-1} \) in the above formula vanishes if \( b \neq 0 \), and equals \( (-1)^c \frac{B_{a+c}}{a!c!} \) for \( b = 0 \). This
leaves formula (1.0.8) valid for any $b$ without adding any extra term. Moreover, since $B_k = 0$ for $k$ an odd integer different from 1, only terms with $k$ even remains and then, $(-1)^{n-k-a} = (-1)^n$.

Finally, multiplying each term by $1 = \frac{n!(n-k)!k!}{n!(n-k)!k!}$ equation (1.0.8) reduces to

$$\sum_{k=0}^{n} \lambda_k B_k B_{n-k} = 0$$

with

$$\lambda_k = \binom{n}{k} \left[ (-1)^c \binom{n-k}{c} \sum_{\ell = \max(0,k-b)}^{\min(a,k)} (-1)^{k-\ell} \binom{k}{\ell} 
- (-1)^a \binom{n-k}{a} \sum_{\ell = \max(0,k-b)}^{\min(c,k)} (-1)^{k-\ell} \binom{k}{\ell} \right],$$

and Theorem 0.1 is proved.

**Corollary 1.3.** Theorems 0.1, 1.1 and 1.2 are equivalent statements. □

We finish by giving a “condensed” version of our main result which will be used in the next section. For it, as Euler-type equations can be thought of as homogeneous equations on the Bernoulli numbers they can be simplified as follows.

**Definition 1.4.** Given an Euler-type identity $\sum_{k=0}^{n} \lambda_k B_k B_{n-k} = 0$, its condensed form is defined as

$$\sum_{k=0}^{\frac{n}{2}} \mu_k B_k B_{n-k} = 0,$$

where $\mu_k = \begin{cases} \lambda_k + \lambda_{n-k} & \text{if } k < \frac{n}{2} \\ \lambda_k & \text{if } k = \frac{n}{2}. \end{cases}$

**Remark 1.5.** Observe that, whenever $\frac{n}{2}$ is odd, the last summand in the above formula is $\mu_{\frac{n}{2}} B^2_{\frac{n}{2}} = 0$, as odd Bernoulli numbers vanish. Hence, in this case, $\mu_{\frac{n}{2}}$ can be arbitrarily chosen.

Then, the condensed form of Theorem 0.1 reads,

**Theorem 1.6.** For any even integer $n \geq 4$ and any triple of integers $(a, b, c)$ such that $a + b + c = n - 1$, there is a condensed Euler-type identity

$$\sum_{k=0}^{\frac{n}{2}} \mu_k B_k B_{n-k} = 0,$$
in which

\[
\mu_k = \binom{n}{k} \left[ (-1)^c \binom{n-k}{c} \sum_{\ell = \max(0, k-b)}^{\min(a,k)} (-1)^\ell \binom{k}{\ell} \right. \\
+ (-1)^c \binom{k}{c} \sum_{\ell = \max(0, n-k-b)}^{\min(a,n-k)} (-1)^\ell \binom{n-k}{\ell} \\
- (-1)^a \binom{n-k}{a} \sum_{\ell = \max(0, k-b)}^{\min(c,k)} (-1)^\ell \binom{k}{\ell} \\
- (-1)^a \binom{k}{a} \sum_{\ell = \max(0, n-k-b)}^{\min(c,n-k)} (-1)^\ell \binom{n-k}{\ell} \right]. \tag{1.0.9}
\]

2 Miki and Euler’s identities

For any even natural number \( n \geq 4 \), denote by \( \Pi_{n-1} \) the “natural valued” plane \( x + y + z = n - 1 \), that is, \( \Pi_{n-1} = \{(a, b, c) \in \mathbb{N}^3, \ a + b + c = n - 1\} \). Observe that Theorem 0.1 defines a map

\[ f: \Pi_{n-1} \rightarrow \mathbb{Q}^{n+1}, \quad f(a, b, c) = (\lambda_0, \ldots, \lambda_n), \]

whose image is contained in the \( \mathbb{Q} \)-vector space of solutions of the Euler-type equation

\[ \sum_{k=0}^{n} \lambda_k B_k B_{n-k} = 0. \]

It is immediate to observe that the classical Euler’s equation (0.0.1) corresponds to \( f(n - 1, 0, 0) \). Indeed, applying directly Theorem 0.1, an easy computation yields,

\[
\begin{align*}
\lambda_0 &= n + 1, \quad \lambda_n = 0, \\
\lambda_k &= \binom{n}{k}, \quad 1 \leq k < n.
\end{align*}
\]

In the same way, the condensed version in Theorem 1.6 provides a map

\[ g: \Pi_{n-1} \rightarrow \mathbb{Q}^{\frac{n}{2}+1}, \quad g(a, b, c) = (\mu_0, \ldots, \mu_{n/2}), \]

9
whose image is contained in the \(\mathbb{Q}\)-vector space of solutions of the condensed Euler-type equation
\[
\sum_{k=0}^{n} \mu_k B_k B_{n-k} = 0.
\]

We will prove that the condensed form of Miki’s identity (0.0.2), which is
\[
\sum_{k=0}^{n} M_k B_k B_{n-k} = 0,
\]
where,
\[
M_0 = -H_n, \quad M_k = \frac{n}{(n-k)k} \left(1 - \binom{n}{k}\right), \quad 1 \leq k \leq \frac{n}{2},
\]
is generated by \(\text{Im} \, g\).

Write \(M = (M_0, \ldots, M_{\frac{n}{2}})\) and for simplicity, denote \(g(0, n-j-1, j)\) by \(g(j)\). Then, we show that \(M\) is a \(\mathbb{Q}\)-linear combination of the vectors \(g(j)\), \(0 \leq j \leq \frac{n}{2}\). Explicitly,

**Theorem 2.1.** For any even integer \(n \geq 4\),
\[
M = \sum_{j=0}^{\frac{n}{2}-1} \left(\frac{1}{j+1} g(j) + \frac{1}{n-j} g(j+1)\right).
\]

**Proof.** Denote by \(g(j)_k\) the \(k\)th component of \(g(j)\).

Applying directly formula (1.0.9) in Theorem 1.6 we obtain that, for \(0 \leq j \leq n-1\),
\[
g(j)_0 = (-1)^j \binom{n}{j} - 1.
\]

Then, taking into account that \(\frac{n}{j+1} \binom{n}{j} = \frac{n}{n-j} \binom{n}{j+1}\), a short computation shows then that
\[
\sum_{j=0}^{\frac{n}{2}-1} \left(\frac{1}{j+1} g(j)_0 + \frac{1}{n-j} g(j+1)_0\right) = -(1 + \frac{1}{2} + \cdots + \frac{1}{n}) = -H_n = M_0.
\]

On the other hand, for \(0 < k \leq n-1\), it is also easy to check that equation (1.0.9) in Theorem 1.6 translates to:
This reduces

\[ g(j) = \begin{cases} (-1)^j \binom{n}{k} \left[ \sum_{j=1}^{k-1} (n-j-1) \right], & \text{if } j < k, \\
(-1)^j \binom{n}{k} \left[ \sum_{j=1}^{n-k-1} (n-j-k) \right], & \text{if } j \geq k. \end{cases} \]

Then

\[
\sum_{j=0}^{n-1} \left( \frac{1}{j+1} g(j) + \frac{1}{n-j} g(j+1) \right) = \sum_{j=1}^{n-1} \frac{\binom{n}{j+1} - \binom{n}{j}}{j+1} g(j) + \frac{2}{n+2} g(\frac{n}{2}) = \binom{n}{k} (n+2) \left[ \sum_{j=1}^{k-1} \frac{(-1)^j}{(j+1)(n-j+1)} \binom{n}{j} \right] \quad (A) \\
+ \binom{n}{k} (n+2) \left[ \sum_{j=k}^{n-1} \frac{(-1)^j}{(j+1)(n-j+1)} \binom{n}{j} \right] \quad (B) \\
+ \binom{n}{k} (n+2) \left[ \sum_{j=1}^{n-2} \frac{(-1)^j}{(j+1)(n-j+1)} \binom{n}{j} \right] \quad (C) \\
+ \binom{n}{k} \frac{2}{n+2} (-1)^n \left[ \binom{n-k-1}{\frac{n}{2}-1} + \binom{n-k-1}{\frac{n}{2}-2} \right] \quad (D)
\]

Now we modify these expressions:

\[
(A) = \binom{n}{k} (n+2) \left[ \sum_{j=1}^{k-1} (-1)^j \frac{(n-j)(n-j-1) \cdots (k+1-j)(k-1)!}{(j+1)! (n-j+1)!} \right] \\
= \binom{n}{k} (n+2) \left[ \frac{(k-1)!}{(n+2)!} \sum_{j=1}^{k-1} (-1)^j P_k(j) \binom{n+2}{j+1} \right] \\
= \binom{n}{k} \frac{(k-1)!}{(n+1)!} \left[ \sum_{j=0}^{n} (-1)^j P_k(j) \binom{n+2}{j+1} - P_k(k) \binom{n+2}{k+1} \right],
\]

where \( P_k(x) = x(n-x)(n-1-x) \cdots (k+1-x) \). We now use a well known result from the theory of finite differences which asserts that, for any polynomial \( P(x) \) of degree less than \( n \),

\[
\sum_{j=0}^{n} (-1)^j P(j) \binom{n}{j} = 0. \quad (2.0.10)
\]

This reduces \( A \) to

\[
\binom{n}{k} \frac{(k-1)!}{(n+1)!} \left[ P_k(-1) + P_k(n+1) - P_k(k) \binom{n+2}{k+1} \right].
\]
On the other hand:

\[(B) = \binom{n}{k} (n + 2) \sum_{j=1}^{\frac{n}{2} - 1} (-1)^j \frac{(n - j) j (j - 1) (j - 2) \cdots (j - k + 1)}{(n - j + 1)! (j + 1)!} \]

\[= \binom{n}{k} (n + 2) \sum_{j=1}^{\frac{n}{2} - 1} (-1)^j Q_k(n - j) \binom{n + 2}{j + 1} \]

\[= \binom{n}{k} \frac{(n - k - 1)!}{(n + 2)!} \sum_{j=n-k}^{\frac{n}{2} + 1} (-1)^j Q_k(j) \binom{n + 2}{n - j + 1}, \]

where \(Q_k(x) = x(n - x)(n - 1 - x) \cdots (n - k - 1 - x)\), and in the last step we have replace \(n - j\) by \(j\).

\[(C) = \binom{n}{k} (n + 2) \sum_{j=1}^{\frac{n}{2} - 1} (-1)^j \frac{j (n - j) (n - j - 1) \cdots (n - j - k + 1)}{(j + 1)! (n - j + 1)!} \]

\[= \binom{n}{k} (n + 2) \sum_{j=1}^{\frac{n}{2} - 1} (-1)^j Q_k(j) \binom{n + 2}{n - j + 1} \]

\[= \binom{n}{k} \frac{(n - k - 1)!}{(n + 1)!} \sum_{j=1}^{\frac{n}{2} - 1} (-1)^j Q_k(j) \binom{n + 2}{n - j + 1} + \sum_{j=n-k}^{\frac{n}{2} + 1} (-1)^j Q_k(j) \binom{n + 2}{n - j + 1}. \]

Therefore,

\[(B) + (C) = \binom{n}{k} \frac{(n - k - 1)!}{(n + 1)!} \sum_{j=0}^{n} (-1)^j Q_k(j) \binom{n + 2}{n - j + 1} + (-1)^\frac{n}{2} Q_k(\frac{n}{2}) \binom{n + 2}{\frac{n}{2} + 1}, \]

which, at the sight of formula (2.0.10), becomes

\[\binom{n}{k} \frac{(n - k - 1)!}{(n + 1)!} \left[ Q_k(-1) + Q_k(n + 1) - (-1)^\frac{n}{2} Q_k(\frac{n}{2}) \binom{n + 2}{\frac{n}{2} + 1} \right]. \]

Finally

\[(D) = \frac{4}{(n + 2)} (-1)^\frac{n}{2} \frac{(n - k - 1)!}{(\frac{n}{2} - k)! (\frac{n}{2} - 1)! k! (n - k)!}. \]
By simple evaluation we have:

\[ P_k(-1) = -\frac{(n+1)!}{(k+1)!}, \quad P_k(n+1) = (n+1)(n-k)!, \quad P_k(k) = k(n-k)!; \]

\[ Q_k(-1) = -\frac{(n+1)!}{(n-k+1)!}, \quad Q_k(n+1) = (n+1)k!, \quad Q_k\left(\frac{n}{2}\right) = \frac{n}{2}\left(\frac{n}{2}-k\right)!. \]

Then, a straightforward computation, substituting the above identities in the corresponding equations, reduces \((A) + (B) + (C) + (D)\) to

\[
\binom{n}{k} \left[ -\frac{1}{(k+1)k} + \frac{1}{n(n-1)_{k-1}} \right] = \frac{n}{(n-k)k} \binom{n}{k}. 
\]

But, on the one hand,

\[ \frac{1}{n(n-1)_{k-1}} = \frac{1}{n\binom{n}{k}}. \]

while

\[ -\frac{1}{(k+1)k} - \frac{n+2}{(k+1)(n-k+1)} - \frac{1}{(n-k+1)(n-k)} = -\frac{n}{(n-k)k}. \]

Thus, we conclude that

\[
\sum_{j=0}^{\frac{n}{2}-1} \left( \frac{1}{j+1} g(j)_k + \frac{1}{n-j} g(j+1)_k \right) = \binom{n}{k} \left[ \frac{n}{(n-k)k} \binom{n}{k} - \frac{n}{(n-k)k} \right] = \frac{n}{(n-k)k} \left( 1 - \binom{n}{k} \right) = M_k,
\]

and the theorem is proved.

We finish by presenting an unexpected symmetry relation between Euler and Miki’s identities.

**Theorem 2.2.** The vector of coefficients in the condensed form of the Euler’s identity equals

\[
-\frac{2}{n} \sum_{j=0}^{\frac{n}{2}-1} \left( (n-j)g(j) + (j+1)g(j+1) \right).
\]
Proof. On the one hand, observe that the condensed form of the Euler’s formula is
\[ \sum_{k=2}^{n-1} \binom{n}{k} B_k B_{n-k} + (n+1)B_n = 0. \]

On the other hand, Taking into account that \((n-j)(j)= (j+1)(j+1)\), a short computation yields
\[ \sum_{j=0}^{n-1} \left( (n-j)g(j) + (j+1)g(j+1) \right) = -(1+2+\cdots+n) = -\frac{n}{2}(n+1), \]
that is, \(-\frac{n}{2}\) times the first coefficient of the above condensed formula. Finally, another straightforward computation gives
\[ \sum_{j=0}^{n-1} \left( (n-j)g(j) + (j+1)g(j+1) \right) = -\frac{n}{2} \binom{n}{k}, \]
and the proof is complete. \(\square\)

Remark 2.3. Fix an even integer \(n \geq 4\) and identify a given condensed Euler-type identity with the vector in \(\mathbb{Q}^{\frac{n}{2}+1}\) of coefficients of the given identity. Then, theorems 2.1 and 2.2 tell us that both the condensed Miki and Euler’s identities belong to the vector space \(\langle \text{Im } g \rangle\) generated by the image \(g: \Pi_{n-1} \rightarrow \mathbb{Q}^{\frac{n}{2}+1}\).

Thus, one may ask whether any given condensed Euler-type identity lives in \(\langle \text{Im } g \rangle\), that is, whether the latter equals the subspace \(V\) of \(\mathbb{Q}^{\frac{n}{2}+1}\) of all condensed Euler-type identities. The answer is negative as the inclusion \(\langle \text{Im } g \rangle \subset V\) is proper. Indeed \(\dim V = \lfloor \frac{n}{4} \rfloor\), the integer part of \(\frac{n}{4}\), and therefore, for \(n = 12\), \(\dim V = 3\) while a direct computation shows that \(\langle \text{Im } g \rangle\) has dimension 2. Summarizing:
\[
\left\{ \begin{array}{l}
\text{Condensed Miki and Euler’s identities} \\
\text{Euler’s identities}
\end{array} \right\} \subset \langle \text{Im } g \rangle \subset \left\{ \begin{array}{l}
\text{Condensed Euler-type identities}
\end{array} \right\} \subset \mathbb{Q}^{\frac{n}{2}+1}.
\]

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