Cubic interaction vertices for fermionic and bosonic arbitrary spin fields

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Abstract

Using the light-cone formulation of relativistic dynamics we study arbitrary spin fermionic and bosonic fields propagating in flat space of dimension greater than or equal to four. Generating functions of cubic interaction vertices for totally symmetric massive and massless fields are obtained. We derive restrictions on the allowed values of spins and the number of derivatives, which provide a classification of cubic interaction vertices for totally symmetric fermionic and bosonic fields. As an example of application of the light-cone formalism, we obtain simple expressions for the Yang-Mills and gravitational interactions of massive arbitrary spin fermionic fields. Interrelations between light-cone cubic vertices and gauge invariant cubic vertices are discussed.

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1 Introduction

The light-cone formalism [1]-[4] offers conceptual and technical simplifications of approaches to various problems of modern quantum field and string theories. This formalism hides some of the symmetries and makes the notation somewhat cumbersome but eventually turns out to be rather effective. A number of important problems have been solved in the framework of this formalism. For example, we mention the solution to the light-cone gauge string field theory [5]-[9] and the construction of a superfield formulation for some versions of supersymmetric theories [10]-[14]. Theories formulated within this formalism may sometimes be a good starting point for deriving a Lorentz covariant formulation [15]-[19]. Another attractive application of the light-cone formalism is the construction of interaction vertices in the theory of massless higher spin fields [20]-[23]. Some interesting applications of the light-cone formalism to study of AdS/CFT may be found in [24, 25, 26]. Discussions of super p-branes and string bit models in the light-cone gauge is given in [27, 28] and [29] respectively.

In this paper, we apply the light-cone formalism to study interaction vertices for higher spin fields. Considerable progress has been achieved in the problem of constructing the theory describing the interaction of massless higher spin fields with gravity. In Ref.[30], cubic interaction vertices for massless higher spin fields propagating in AdS$_4$ space were constructed; in Ref.[31], nonlinear equations of motion to all orders in the coupling constant for massless higher spin fields in AdS$_4$ were found. Nonlinear equations of motion for massless totally symmetric higher spin fields in AdS$_d$ space ($d \geq 4$) were found in Ref.[32] (see [33],[34] for a recent review). It now becomes apparent that constructing a self-consistent theory of massless higher spin fields interacting with gravity requires formulating the theory in AdS space. Unfortunately, despite the efforts, an action that leads to the above-mentioned nonlinear equations of motion has not yet been obtained. To quantize these theories and investigate their ultraviolet behavior, it would be important to find an appropriate action. Since the massless higher spin field theories correspond quantum mechanically to non-local point particles in a space of certain auxiliary variables, it is conjectured that such theories may be ultraviolet finite [35]. We believe that the light-cone formulation may be helpful in understanding these theories better.

In this paper, keeping these extremely important applications in mind, we apply method developed in [36] for constructing cubic interaction vertices involving both fermionic and bosonic fields. Extension of our approach to the case of fermionic fields is important because of supersymmetry, which plays important role in string theory and theory of higher-spin fields, involve fermionic fields. We believe that most of our approach to massless higher spin fields can be relatively straightforwardly generalized to the case of massless higher spin fields in AdS space. The light-cone gauge approach to dynamics of free fields in AdS space was developed in [37] (see also [38],[39]). Although the light-cone approach in AdS space is complicated compared to that in flat space, it turns out that these approaches share many properties. We therefore believe that methods developed in flat space might be helpful in analyzing dynamics of interacting massless higher spin fields in AdS space. As regards our study of massive fields, we note that our interest in light-cone gauge vertices for massive higher spin fields in flat space is motivated, among other things, by the potential of our approach for in-depth studies of the interaction vertices of the light-cone gauge (super)string field theory.

At present, a wide class of cubic interaction vertices for fields propagating in flat space is known. In particular, the self-interaction cubic vertices for the massless spin 3 field were found in [40]-[43] and the higher-derivative cubic vertex for massless spin 2 and spin 4 fields was studied in [44]. More general examples of the cubic interaction vertices for massless higher spin fields
were discovered in [20, 21, 45] and the full list of cubic interaction vertices for massless higher spin fields was given in [22]. A wide list of cubic interaction vertices for massive higher spin fields was obtained in [46] (see also [47], [48]). With the exception of Refs. [42, 43] (devoted to spin 3 field self-interactions) all the above-mentioned works were devoted to the analysis of interaction vertices for higher spin fields in 4d flat space. In view of possible applications of the higher spin field theory to string theory, it is instructive to study cubic interaction vertices for higher spin fields in space of dimension \(d \geq 4\). We do this in the present paper. We restrict our attention to the case of Dirac totally symmetric fermionic fields and consider both the massive and massless theories.

This paper is organized as follows. In Section 2, we introduce the notation and describe the standard manifestly so\((d−2)\) covariant light-cone formulation of free fermionic and bosonic fields.

In Section 3 we study restrictions imposed by kinematical and dynamical symmetries of the Poincaré algebra on cubic interaction vertices for massless and massive fields. We find equations on cubic interaction vertices.

In Section 4, we present solution to equations for cubic interaction vertices of massless fields. Section 5 is devoted to cubic interaction vertices for massless and massive fields. We apply our general results to derive the Yang-Mills and gravitational interactions of massive arbitrary spin fermionic Dirac fields. Our approach allows us to obtain simple expressions for vertices of these interactions. We obtain also simple expressions for interaction vertices of massless bosonic arbitrary spin field and two massive spin-\(\frac{3}{2}\) fermionic fields. In Section 6, we discuss cubic interaction vertices for massive fields. In Sections 4-6, we also derive restrictions on the allowed values of spins and the number of derivatives for cubic interaction vertices of the totally symmetric fermionic and bosonic fields.

2 Free light-cone gauge massive and massless fields

The method suggested in Ref. [1] reduces the problem of finding a new (light-cone gauge) dynamical system to the problem of finding a new solution of defining symmetry algebra commutators. Since in our case the defining symmetries are generated by the Poincaré algebra, we begin with a discussion of the realization of the Poincaré algebra on the space of massive and massless fields. We focus on free fields in this section.

The Poincaré algebra of \(d\)-dimensional Minkowski space is spanned by translation generators \(P^A\) and rotation generators \(J^{AB}\) (the latter span the so\((d−1, 1)\) Lorentz algebra). The Lorentz covariant form of the non-trivial Poincaré algebra commutators is

\[
[P^A, J^{BC}] = \eta^{AB} P^C - \eta^{AC} P^B,
\]

\[
[J^{AB}, J^{CD}] = \eta^{BC} J^{AD} + 3 \text{ terms},
\]

where \(\eta^{AB}\) stands for the mostly positive flat metric tensor. The generators \(P^A\) are chosen to be hermitian, and the \(J^{AB}\) to be antihermitian. To develop the light-cone formulation, in place of the Lorentz basis coordinates \(x^A\) we introduce the light-cone basis coordinates \(x^\pm, x^I\) defined by

\[
x^\pm = \frac{1}{\sqrt{2}}(x^{d-1} \pm x^0), \quad x^I, \quad I = 1, \ldots, d−2,
\]

1 Interesting recent discussion of totally symmetric fields in the context of non-abelian gauge theories may be found in [49].

\(A, B, C, D = 0, 1, \ldots, d−1\) are so\((d−1, 1)\) vector indices; ‘transverse’ indices \(I, J, K = 1, \ldots, d−2\) are so\((d−2)\) vector indices.
and treat \( x^+ \) as an evolution parameter. In this notation, the Lorentz basis vector \( X^A \) is decomposed as \( (X^+, X^-, X^I) \) and a scalar product of two vectors is then decomposed as

\[
\eta_{AB} X^A Y^B = X^+ Y^- + X^- Y^+ + X^I Y^I ,
\]

(2.3)

where the covariant and contravariant components of vectors are related as \( X^+ = X^- , X^+ = X^- , X^I = X_I \). Here and henceforth, a summation over repeated transverse indices is understood. In the light-cone formalism, the Poincaré algebra generators can be separated into two groups:

\[
P^+, \quad P^I, \quad J^{+I}, \quad J^{+}, \quad J^{IJ}, \quad \text{kinematical generators} ;
\]

(2.4)

\[
P^−, \quad J^{−I}, \quad \text{dynamical generators} .
\]

(2.5)

For \( x^+ = 0 \), the kinematical generators in the field realization are quadratic in the physical fields\(^3\), while the dynamical generators receive higher-order interaction-dependent corrections.

Commutators of the Poincaré algebra in light-cone basis can be obtained from (2.1) by using the light-cone metric having the following non vanishing elements: \( \eta^{+−} = \eta^{−+} = 1 \), \( \eta^{IJ} = \delta^{IJ} \). Hermitian conjugation rules of the Poincaré algebra generators in light-cone basis take the form

\[
P^{±†} = P^± , \quad P^{†I} = P^I , \quad J^{IJ†} = −J^{JI} , \quad J^{+−†} = −J^{−+} , \quad J^{±I†} = −J^{±I} .
\]

(2.6)

To find a realization of the Poincaré algebra on the space of massive and massless fields we use the light-cone gauge description of those fields. We discuss massive and massless fields in turn.

**Bosonic massive totally symmetric fields.** In order to obtain the light-cone gauge description of a bosonic massive totally symmetric fields in an easy–to–use form, let us introduce bosonic creation and annihilation operators \( \alpha^I, \alpha \) and \( \bar{\alpha}^I, \bar{\alpha} \) defined by the relations

\[
[\bar{\alpha}^I, \alpha^J] = \delta^{IJ} , \quad [\bar{\alpha}, \alpha] = 1 ,
\]

(2.7)

\[
\bar{\alpha}^I |0\rangle = 0 , \quad \bar{\alpha} |0\rangle = 0 .
\]

(2.8)

The bosonic oscillators \( \alpha^I, \bar{\alpha}^I \) and \( \alpha, \bar{\alpha} \) transform in the respective vector and scalar representations of the \( so(d−2) \) algebra. In \( d \)-dimensional Minkowski space, the massive arbitrary spin totally symmetric bosonic field is labelled by the mass parameter \( m \) and one spin label \( s \), where \( s \geq 0 \) is an integer number. Physical D.o.F of the massive field labeled by spin label \( s \) can be collected into a ket-vector defined by

\[
|\phi_s(p, \alpha)\rangle = \sum_{t=0}^s \alpha_{I_1} \ldots \alpha_{I_{s−t}} \alpha^t \phi_{I_1\ldots I_{s−t}}(p) |0\rangle .
\]

(2.9)

We note that the superscripts like \( I_{s−t} \) in (2.9) denote the transverse indices, while \( t \) is the degree of the oscillator \( \alpha \). In (2.9) and the subsequent expressions, \( \alpha \) occurring in the argument of ket-vectors \( |\phi(p, \alpha)\rangle \) denotes a set of the oscillators \{\( \alpha^I, \alpha \}\}, while \( p \) occurring in the argument of ket-vectors \( |\phi(p, \alpha)\rangle \) and \( \delta \)-functions denotes a set of the momenta \{\( p^I, \beta = p^+ \}\}. Also, we do not explicitly show the dependence of the ket-vectors \( |\phi(p, \alpha)\rangle \) on the evolution parameter \( x^+ \). The ket-vector (2.9) is a degree-\( s \) homogeneous polynomial in the oscillators \( \alpha^I, \alpha \):

\[
(\alpha^I \bar{\alpha}^I + \alpha \bar{\alpha} − s) |\phi_s(p, \alpha)\rangle = 0 .
\]

(2.10)

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\(^3\)Namely, for \( x^+ \neq 0 \) they have a structure \( G = G_1 + x^+ G_2 \), where \( G_1 \) is quadratic in fields, while \( G_2 \) contains higher order terms in fields.
Physical D.o.F of a massive field in $d$-dimensional Minkowski space are described by irreps of the $so(d-1)$ algebra. For the ket-vector $(2.9)$ to be a carrier of $so(d-1)$ algebra irreps, we should impose the following tracelessness constraint:

$$\left(\tilde{\alpha}^I \alpha^I + \alpha^I \tilde{\alpha}^I \right) \ket{\phi_s(p, \alpha)} = 0 . \quad (2.11)$$

To develop the light-cone gauge description of massive arbitrary spin fields on an equal footing we use a ket-vector defined by

$$\ket{\phi(p, \alpha)} \equiv \sum_{s=0}^{\infty} \ket{\phi_s(p, \alpha)} . \quad (2.12)$$

**Bosonic massless totally symmetric fields.** The light-cone gauge description of a bosonic massless totally symmetric fields can be realized by using the creation and annihilation operators $\alpha^I$ and $\tilde{\alpha}^I$. In $d$-dimensional Minkowski space, the massless totally symmetric bosonic field is labeled by one spin label $s$. Physical D.o.F of the massless bosonic field labeled by spin labels $s$ can be collected into a ket-vector defined by

$$\ket{\phi_{m=0}^s(p, \alpha)} = \alpha^{I_1} \ldots \alpha^{I_s} \phi_{I_1 \ldots I_s}(p) \ket{0}, \quad (2.13)$$

which is degree-$s$ homogeneous polynomial in the oscillators $\alpha^I$:

$$\left(\alpha^I \tilde{\alpha}^I - s \right) \ket{\phi_{m=0}^s(p, \alpha)} = 0 . \quad (2.14)$$

In $d$-dimensional Minkowski space, physical D.o.F of massless field are described by irreps of the $so(d-2)$ algebra. For the ket-vector $(2.13)$ to be a carrier of $so(d-2)$ algebra irreps we should impose a standard tracelessness constraint,

$$\tilde{\alpha}^I \alpha^I \ket{\phi_{m=0}^s(p, \alpha)} = 0 \quad (2.15)$$

By analogy with $(2.12)$, the ket-vectors of massless fields $(2.13)$ can be collected into a ket-vector $\ket{\phi_{m=0}^s(p, \alpha)}$ defined by

$$\ket{\phi_{m=0}^s(p, \alpha)} \equiv \sum_{s=0}^{\infty} \ket{\phi_{m=0}^s(p, \alpha)} . \quad (2.16)$$

We note that in $(2.13)$ and the subsequent expressions, the letter $\alpha$ occurring in the argument of the ket-vectors of massless fields $\ket{\phi_{m=0}^s(p, \alpha)}$ denotes the oscillators $\alpha^I$.

**Fermionic massive totally symmetric fields.** In order to obtain the light-cone gauge description of a fermionic massive totally symmetric fields we introduce bosonic creation and annihilation operators $\alpha^I$, $\alpha$ and $\tilde{\alpha}^I$, $\tilde{\alpha}$, and fermionic creation and annihilation operators $\theta^\alpha$, $p_{\theta \alpha}$, $\eta^\alpha$, $p_{\eta \alpha}$, where subscript and superscript $\alpha$ is used to indicate spinor indices. Bosonic operators satisfy the same commutators as in $(2.7)$, while the fermionic operators satisfy the following anticommutators

$$\{ \theta^\alpha, p_{\theta \beta} \} = \Pi^{\oplus \alpha \beta}, \quad \{ \eta^\alpha, p_{\eta \beta} \} = \Pi^{\oplus \alpha \beta}, \quad (2.17)$$

$$\theta \ket{0} = 0 , \quad p_{\eta} \ket{0} = 0 , \quad (2.18)$$

$$\theta^\dagger = p_{\theta} , \quad \eta^\dagger = p_{\eta} , \quad (2.19)$$

where $\Pi^{\oplus}$ is a projector operator (see Appendix A). In $d$-dimensional Minkowski space, the massive totally symmetric spin-$\left( s + \frac{1}{2} \right)$ fermionic field is labelled by the mass parameter $m$ and one
label \( s \), where \( s \geq 0 \) is an integer number. Physical D.o.F of the massive totally symmetric spin-(\( s + \frac{1}{2} \)) fermionic field can be collected into a ket-vector defined by

\[
|\psi_s(p, \alpha)\rangle = (p\theta \psi_s(p, \alpha) + \psi_s^\dagger(p, \alpha)\eta)|0\rangle,
\]

where we use the notation

\[
\psi_s(p, \alpha) \equiv \sum_{t=0}^{s} \alpha^{I_1} \cdots \alpha^{I_{s-t}} \psi^{I_1 \cdots I_{s-t}}(p)|0\rangle,
\]

\[
\psi_s^\dagger(p, \alpha) \equiv \sum_{t=0}^{s} \alpha^{I_1} \cdots \alpha^{I_{s-t}} \psi^{\dagger I_1 \cdots I_{s-t}}(p)|0\rangle.
\]

In (2.20)-(2.22) and the subsequent expressions, we do not show the spinor indices of the oscillators and fermionic fields explicitly. The ket-vector (2.20) is a degree-\( s \) homogeneous polynomial in the oscillators \( \alpha^I, \alpha \):

\[
(\alpha^I \bar{\alpha}^I + \alpha \bar{\alpha} - s) |\psi_s(p, \alpha)\rangle = 0.
\]

As noted above, physical D.o.F of a massive field in \( d \)-dimensional Minkowski space are described by irreps of the \( \mathfrak{so}(d-1) \) algebra. For the ket-vector (2.20) to be a carrier of \( \mathfrak{so}(d-1) \) algebra irreps, we should impose the following \( \gamma \)-tracelessness constraints:

\[
(\gamma^I \bar{\alpha}^I + \gamma_s \bar{\alpha}) \theta |\psi(p, \alpha)\rangle = 0,
\]

\[
p\eta(\gamma^I \bar{\alpha}^I - \gamma_s \bar{\alpha}) |\psi(p, \alpha)\rangle = 0.
\]

To develop the light-cone gauge description of massive arbitrary spin fermionic fields on an equal footing we use a ket-vector defined by

\[
|\psi(p, \alpha)\rangle \equiv \sum_{s=0}^{\infty} |\psi_s(p, \alpha)\rangle.
\]

**Fermionic massless totally symmetric fields.** In order to obtain the light-cone gauge description of a fermionic massless totally symmetric fields we introduce bosonic creation and annihilation operators \( \alpha^I \) and \( \bar{\alpha}^I \), and fermionic creation and annihilation operators \( \theta^\alpha, p_\theta^\alpha, \eta^\alpha, p_\eta^\alpha \), where subscript and superscript \( \alpha \) is used to indicate spinor indices. These operators satisfy the same commutators as in (2.7), (2.17). In \( d \)-dimensional Minkowski space, the massless totally symmetric spin-(\( s + \frac{1}{2} \)) fermionic field is labelled by one label \( s \), where \( s \geq 0 \) is an integer number. Physical D.o.F of the massless spin-(\( s + \frac{1}{2} \)) fermionic field can be collected into a ket-vector defined by

\[
|\psi_s^{m=0}(p, \alpha)\rangle = (p\theta \psi_s^{m=0}(p, \alpha) + \psi_s^{\dagger m=0}(p, \alpha)\eta)|0\rangle,
\]

where we use the notation

\[
\psi_s^{m=0}(p, \alpha) \equiv \alpha^{I_1} \cdots \alpha^{I_s} \psi^{I_1 \cdots I_s}(p)|0\rangle,
\]

\[
\psi_s^{\dagger m=0}(p, \alpha) \equiv \alpha^{I_1} \cdots \alpha^{I_s} \psi^{\dagger I_1 \cdots I_s}(p)|0\rangle.
\]

The ket-vector (2.27) is a degree-\( s \) homogeneous polynomial in the oscillators \( \alpha^I \):

\[
(\alpha^I \bar{\alpha}^I - s) |\psi_s^{m=0}(p, \alpha)\rangle = 0.
\]
As noted above, physical D.o.F of a massless field in \(d\)-dimensional Minkowski space are described by irreps of the \(so(d-2)\) algebra. For the ket-vectors (2.28), (2.29) to be carriers of \(so(d-2)\) algebra irreps, we should impose the following \(\gamma\)-tracelessness constraints:

\[
\gamma^I \bar{\alpha}^I \theta \psi^{m=0}(p, \alpha) = 0, \quad (2.31)
\]
\[
p_\eta \gamma^I \bar{\alpha}^I \psi^{m=0}(p, \alpha) = 0. \quad (2.32)
\]

To develop the light-cone gauge description of massless arbitrary spin fermionic fields on an equal footing we use a ket-vector defined by

\[
|\psi^{m=0}(p, \alpha)\rangle \equiv \sum_{s=0}^{\infty} |\psi^{m=0}_s(p, \alpha)\rangle. \quad (2.33)
\]

We proceed with the discussion of a realization of the Poincaré algebra on the space of massive and massless fields. A representation of the kinematical generators in terms of differential operators acting on the ket-vectors \(|\phi\rangle, |\psi\rangle\) is given by

\[
P^I = p^I, \quad P^+ = \beta, \quad (2.34)
\]
\[
J^+I = \partial_{p^I} \beta, \quad (2.35)
\]
\[
J^{+-} = \partial_{\beta} \beta + M^{+-}, \quad (2.36)
\]
\[
J^{IJ} = p^I \partial_{p^J} - p^J \partial_{p^I} + M^{IJ}, \quad (2.37)
\]

where \(M^{+-}\) is a spin operator of \(so(1, 1)\) algebra, while \(M^{IJ}\) is a spin operator of the \(so(d-2)\) algebra,

\[
[M^{IJ}, M^{KL}] = \delta^{JK} M^{IL} + 3 \text{ terms}, \quad (2.38)
\]

and we use the notation

\[
\beta \equiv p^+, \quad \partial_{\beta} \equiv \partial/\partial \beta, \quad \partial_{p^I} \equiv \partial/\partial p^I. \quad (2.39)
\]

The representation of the dynamical generators in terms of differential operators acting on the ket-vector \(|\phi\rangle\) is given by

\[
P^- = p^-, \quad p^- \equiv -\frac{p^I p^I + m^2}{2\beta}, \quad (2.40)
\]
\[
J^{-I} = -\partial_{\beta} p^I + \partial_{p^I} p^- + \frac{1}{\beta} (M^{IJ} p^J + m M^I) - \frac{p^I}{\beta} M^{+-}, \quad (2.41)
\]

where \(m\) is the mass parameter and \(M^I\) is a spin operator transforming in the vector representation of the \(so(d-2)\) algebra. This operator satisfies the commutators

\[
[M^I, M^{JK}] = \delta^{IJ} M^K - \delta^{IK} M^J, \quad [M^I, M^J] = -M^{IJ}. \quad (2.42)
\]

The spin operators \(M^{IJ}\) and \(M^I\) form commutators of the \(so(d-1)\) algebra (as it should be for the case of massive fields). The particular form of \(M^{+-}, M^{IJ},\) and \(M^I\) depends on the choice of

\[\text{Throughout this paper, without loss of generality, we analyze generators of the Poincaré algebra and their commutators for } x^+ = 0.\]
the realization of spin D.o.F of physical fields. For example, a representation of the spin operators $M^{+-}$, $M^{IJ}$ and $M^I$ for the realization of the physical fields given in (2.12), (2.20) takes the form

$$M^{+-} = \begin{cases} 0, & \text{for bosonic fields,} \\ -\frac{1}{2}p_\theta \bar{\theta} - \frac{1}{2}p_\eta \eta, & \text{for fermionic fields,} \end{cases}$$  

$$M^{IJ} = \begin{cases} \alpha^I \bar{\alpha}^J - \alpha^J \bar{\alpha}^I, & \text{for bosonic fields,} \\ \alpha^I \bar{\alpha}^J - \alpha^J \bar{\alpha}^I + \frac{1}{2}p_\theta \gamma^I J \theta + \frac{1}{2}p_\eta \gamma^I J \eta, & \text{for fermionic fields,} \end{cases}$$

$$M^I = \begin{cases} \alpha \bar{\alpha} - \alpha \bar{\alpha}, & \text{for bosonic fields,} \\ \alpha \bar{\alpha} - \alpha \bar{\alpha} - \frac{1}{2}p_\theta \gamma^I \gamma^I \bar{\gamma} \eta + \frac{1}{2}p_\eta \gamma^I \gamma^I \bar{\gamma} \eta, & \text{for fermionic fields.} \end{cases}$$

As seen from (2.41), in the limit as $m \to 0$, the Poincaré algebra generators are independent of the spin operator $M^I$, i.e. the free light-cone gauge dynamics of massive fields have a smooth limit, given by the dynamics of massless fields.

The above expressions provide a realization of the Poincaré algebra in terms of differential operators acting on the physical fields $|\phi\rangle$, $|\psi\rangle$. We now write a field theoretical realization of this algebra in terms of the physical fields $|\phi\rangle$, $|\psi\rangle$. As mentioned above the kinematical generators $G^{kin}$ are realized quadratically in $|\phi\rangle$, $|\psi\rangle$, while the dynamical generators $G^{dyn}$ are realized non-linearly. At the quadratic level, both $G^{kin}$ and $G^{dyn}$ admit the representation

$$G^{[2]} = G^{bos^{[2]}} + G^{fer^{[2]}},$$

$$G^{bos^{[2]}} = \int d^{d-1}p \langle \phi(-p)|\beta G|\phi(p)\rangle,$$

$$G^{fer^{[2]}} = \int d^{d-1}p \langle \psi(-p)|G|\psi(p)\rangle,$$

where $d^{d-1}p \equiv d\beta d^{d-2}p$ and $G$ are the differential operators given in (2.34), (2.37), (2.40), (2.41) and the notation $G^{[2]}$ is used for the field theoretical free generators. The fields $|\phi\rangle$, $|\psi\rangle$ satisfy the Poisson-Dirac commutators

$$[[\phi(p, \alpha), \phi(p', \alpha')]],_{equal, x^+} = \frac{\delta^{d-1}(p + p')}{2\beta}|\rangle\rangle',$$

$$[[\psi(p, \alpha), \psi(p', \alpha')]],_{equal, x^+} = \frac{\delta^{d-1}(p + p')}{2}|\rangle\rangle',$$

where $|\rangle\rangle'$ is defined by

$$|\rangle\rangle' = \begin{cases} \Pi|0\rangle|0'\rangle & \text{for bosonic fields,} \\ (p_\theta \eta' + \eta p_\theta) \Pi|0\rangle|0'\rangle & \text{for fermionic fields}, \end{cases}$$

and $\Pi$ is unity operator on the space of tensor fields subject to constraint (2.11). With these definitions, we have the standard commutators

$$[[\phi\rangle, G^{[2]}] = G|\phi\rangle,$$
In the framework of the Lagrangian approach the light-cone gauge action takes the standard form

\[ S = \int dx^+ d^{d-1}p \left( \langle \phi(p) | i \beta \partial^- | \phi(p) \rangle + \langle \psi(p) | i \partial^- | \psi(p) \rangle \right) + \int dx^+ P^-, \]

(2.54)

where \( P^- \) is the Hamiltonian. This representation for the light-cone action is valid for the free and for the interacting theory. The free theory Hamiltonian can be obtained from relations (2.40), (2.46).

Incorporation of the internal symmetry into the theory under consideration resembles the Chan–Paton method in string theory [50], and could be performed as in [51].

3 Equations for cubic interaction vertices

3.1 Restrictions imposed by kinematical symmetries

We begin with discussing the general structure of the Poincaré algebra dynamical generators (2.5). In theories of interacting fields, the dynamical generators receive corrections involving higher powers of physical fields, and we have the following expansion for them:

\[ G^{\text{dyn}} = \sum_{n=2}^{\infty} G^{\text{dyn}}_{[n]}, \]

(3.1)

where \( G^{\text{dyn}}_{[n]} \) stands for the \( n \)-point contribution (i.e., the functional that has \( n \) powers of physical fields) to the dynamical generator \( G^{\text{dyn}} \). The generators \( G^{\text{dyn}}_{[n]} \) of classical supersymmetric Yang-Mills theories do not receive corrections of the order higher than four in fields [10, 11, 12], while the generators \( G^{\text{kin}}_{[n]} \) of (super)gravity theories are nontrivial for all \( n \geq 2 \) [52, 53, 54].

The ‘free’ generators \( G^{\text{dyn}}_{[2]} \) (3.1), which are quadratic in the fields, were discussed in Section 2. Here we discuss the general structure of the ‘interacting’ dynamical generators \( G^{\text{dyn}}_{[3]} \). Namely, we describe those properties of the dynamical generators \( G^{\text{dyn}}_{[3]} \) that can be obtained from commutators between \( G^{\text{kin}}_{[3]} \) and \( G^{\text{dyn}}_{[3]} \). In other words, we find restrictions imposed by kinematical symmetries on the dynamical ‘interacting’ generators. We proceed in the following way.

(i) We first consider restrictions imposed by kinematical symmetries on the dynamical generator \( P^- \). As seen from (2.1), the kinematical generators \( P^I, P^+, J^+I \) have the following commutators with \( P^- \): \( [P^-, G^{\text{kin}}_{[2]}] = G^{\text{kin}}_{[2]} \). Since \( G^{\text{kin}}_{[2]} \) are quadratic in the fields, these commutators imply

\[ [P^-, G^{\text{kin}}_{[3]}] = 0. \]

(3.2)

Commutators (3.2) for \( G^{\text{kin}}_{[2]} = (P^I, P^+) \) lead to the representation for \( P^-_{[3]} \) as

\[ P^-_{[3]} = \int d\Gamma_3 \langle \Phi_{[3]} | P^- \rangle, \]

(3.3)
where we use the notation
\[
\langle \Phi_{[3]} \rangle \equiv \langle \psi(p_1, \alpha_1) \rangle \langle \psi(p_2, \alpha_2) \rangle \langle \phi(p_3, \alpha_3) \rangle, \\
|p_{[3]}^-\rangle \equiv p_{[3]}^- \prod_{a=1}^3 |0\rangle_a,
\]
(3.4)
\[
d\Gamma_3 \equiv (2\pi)^{d-1} \delta^{d-1}(\sum_{a=1}^3 p_a) \prod_{a=1}^3\frac{d^{d-1}p_a}{(2\pi)^{(d-1)/2}}.
\]
(3.5)

Here and below, the indices \(a, b = 1, 2, 3\) label three interacting fields and the \(\delta\)-functions in \(d\Gamma_3\) (3.5) respect conservation laws for the transverse momenta \(p_a^I\) and light-cone momenta \(\beta_a\). Generic densities \(p_{[3]}^-\) (3.4) depend on the momenta \(p_a^I\), \(\beta_a\), and variables related to the spin D.o.F, which we denote by \(\alpha\):
\[
p_{[3]}^- = p_{[3]}^- (p_a, \beta_a; \alpha).
\]
(3.6)

(ii) Commutators (3.2) for \(G_{[2]}^{\text{kin}} = J^+I\) tell us that the generic densities \(p_{[3]}^-\) in (3.4) depend on the momenta \(p_a^I\) through the new momentum variables \(\mathbb{P}^I\) defined by
\[
\mathbb{P}^I \equiv \frac{1}{3} \sum_{a=1}^3 \bar{\beta}_a p_a^I, \quad \bar{\beta}_a \equiv \beta_{a+1} - \beta_{a+2}, \quad \beta_a \equiv \beta_{a+3},
\]
(3.7)
i.e. the densities \(p_{[3]}^-\) turn out to be functions of \(\mathbb{P}^I\) in place of \(p_a^I\),
\[
p_{[3]}^- = p_{[3]}^- (\mathbb{P}, \beta_a; \alpha).
\]
(3.8)

(iii) Commutators between \(P^-\) and the remaining kinematical generators \(J^{IJ}, J^{+I}\) have the form \([P^-, J^{IJ}] = 0, [P^-, J^{+I}] = P^-\). Since \(J^{IJ}, J^{+I}\) are quadratic in physical fields, these commutators lead to
\[
[P_{[3]}^-, J^{IJ}] = 0, \quad [P_{[3]}^-, J^{+I}] = P_{[3]}^-.
\]
(3.9)

It is straightforward to check that commutators (3.9) lead to the respective equations for the generic densities \(p_{[3]}^- = p_{[3]}^- (p_a, \beta_a; \alpha)\) in (3.6):
\[
\sum_{a=1}^3 (p_a^I \partial_{p_a^I} - p_a^I \partial_{p_a^I} + M^{(a)IJ}) |p_{[3]}^-\rangle = 0, \quad \sum_{a=1}^3 (\beta_a \partial_{\beta_a} - M^{(a)+I}) |p_{[3]}^-\rangle = 0.
\]
(3.10), (3.11)

Using (3.7), we rewrite Eqs. (3.10), (3.11) in terms of \(p_{[3]}^- = p_{[3]}^- (\mathbb{P}, \beta_a; \alpha)\) in (3.8) as
\[
\left(\mathbb{P}^I \partial_{p_a^I} - \mathbb{P}^I \partial_{p_a^I} + \sum_{a=1}^3 M^{(a)IJ}\right) |p_{[3]}^-\rangle = 0, \\
\left(\mathbb{P}^I \partial_{p_a^I} + \sum_{a=1}^3 (\beta_a \partial_{\beta_a} - M^{(a)+I})\right) |p_{[3]}^-\rangle = 0.
\]
(3.12), (3.13)

(iv) To complete the description of the dynamical generators, we consider the dynamical generator \(J^{-I}\). Using commutators of \(J^{-I}\) with the kinematical generators, we obtain the representation for \(J_{[3]}^{-I}\) as
\[
J_{[3]}^{-I} = \int d\Gamma_3 \left( \langle \Phi_{[3]} | j_{[3]}^{-I} \rangle + \frac{1}{3} \sum_{a=1}^3 \partial p_a^I \langle \Phi_{[3]} | |p_{[3]}^-\rangle \right),
\]
(3.14)
where we introduce new densities \( j^{\lambda I} \). From the commutators of \( J^I \) with the kinematical generators, we learn that the densities \( j^{\lambda I} \) depend on the momenta \( p^I_\alpha \) through the momenta \( P^I \) in (3.7) and satisfy the equations

\[
\left( P^I \partial_{P^J} - P^J \partial_{P^I} + \sum_{a=1}^3 M(a)^{IJ} \right) |j^{\lambda K}_{[\lambda]} \rangle = 0 ,
\]

(3.15)

\[
\left( P^I \partial_{P^J} + \sum_{a=1}^3 (\beta_a \partial_{\beta_a} - M(a)^{+\dag}) \right) |j^{\lambda K}_{[\lambda]} \rangle = 0 .
\]

(3.16)

To summarize, the commutators between the kinematical and dynamical generators yield the expressions for the dynamical generators (3.3), (3.14), where the densities \( p^{\lambda I}_{[\lambda]} \) depend on \( P^I \), \( \beta_a \), and spin variables \( \alpha \) and satisfy Eqs. (3.12), (3.13), (3.15), (3.16). We note that the kinematical symmetry equations (3.12), (3.13) can be represented as:

\[
J^{IJ} |p^{\lambda I}_{[\lambda]} \rangle = 0 ,
\]

(3.17)

\[
\left( P^I \partial_{P^J} + \sum_{a=1}^3 (\beta_a \partial_{\beta_a} - M^{+\dag}) \right) |p^{\lambda I}_{[\lambda]} \rangle = 0 ,
\]

(3.18)

where we use the notation

\[
J^{IJ} \equiv L^{IJ}(P) + M^{IJ} ,
\]

(3.19)

\[
L^{IJ}(P) \equiv P^I \partial_{P^J} - P^J \partial_{P^I} ,
\]

(3.20)

\[
M^{IJ} \equiv \sum_{a=1}^3 M(a)^{IJ} , \quad M^{+\dag} \equiv \sum_{a=1}^3 M(a)^{+\dag} .
\]

(3.21)

### 3.2 Restrictions imposed by light-cone dynamical principle

To find the densities \( p^{\lambda I}_{[\lambda]} \), \( j^{\lambda I} \), we consider commutators between the respective dynamical generators; the general strategy of finding these densities consists basically of the following three steps, to be referred to as the light-cone dynamical principle:

a) Find restrictions imposed by commutators of the Poincaré algebra between the dynamical generators. Using these commutators shows that the densities \( j^{\lambda I} \) are expressible in terms of the densities \( p^{\lambda I}_{[\lambda]} \).

b) Require the densities \( p^{\lambda I}_{[\lambda]} \), \( j^{\lambda I} \) to be polynomials in the momenta \( P^I \). We refer to this requirement as the light-cone locality condition.

c) Find those densities \( p^{\lambda I}_{[\lambda]} \) that cannot be removed by field redefinitions.

We now proceed with discussing the restrictions imposed by the light-cone dynamical principle on density \( p^{\lambda I}_{[\lambda]} \) (3.8). In what follows, the density \( p^{\lambda I}_{[\lambda]} \) will be referred to as cubic interaction vertex. Following the procedure above-described, we first find the restrictions imposed by the Poincaré algebra commutators between the dynamical generators. All that is required is to consider the commutators

\[
[ P^-, J^I ] = 0 , \quad [ J^I , J^J ] = 0 ,
\]

(3.22)
which in the cubic approximation take the form

\[
[P^-_{[2]}, J^-_{[2]}] + [P^-_{[3]}, J^-_{[3]}] = 0, \quad (3.23)
\]

\[
[J^-_{[2]}, J^-_{[3]}] + [J^-_{[3]}, J^-_{[2]}] = 0. \quad (3.24)
\]

Equation (3.23) leads to the equation for \( |p^-_{[3]}(\mathbb{P}, \beta_a; \alpha)\rangle \) and \( |j^-_{[3]}(\mathbb{P}, \beta_a; \alpha)\rangle \),

\[
P^- |j^-_{[3]}\rangle = J^{-H^I} |p^-_{[3]}\rangle, \quad (3.25)
\]

where we use the notation

\[
P^- \equiv \sum_{a=1}^{3} p_a^- , \quad J^{-H^I} \equiv \sum_{a=1}^{3} J_a^{-H^I}, \quad (3.26)
\]

\[
p_a^- \equiv \frac{p^I J_a^I + m_a^2}{2 \beta_a}, \quad (3.27)
\]

\[
J_a^{-H^I} \equiv p_a^I \delta^I_{\beta_a} - p_a^- \delta^I_{\beta_a} - \frac{1}{\beta_a} (M^{(a)I^I} p_a^I + m_a M^{(a)I}) \frac{p_a^I M^{(a)+\dagger}}{\beta_a}. \quad (3.28)
\]

\( P^- \) and the differential operator \( J^{-H^I} \) in (3.26) can be expressed in terms of the momentum \( \mathbb{P}^I \):

\[
P^- \equiv \frac{\mathbb{P}^I \mathbb{P}^I}{2 \beta} - \sum_{a=1}^{3} \frac{m_a^2}{2 \beta_a}, \quad (3.29)
\]

\[
J^{-H^I} |p^-_{[3]}\rangle = -\frac{1}{3 \beta} \chi^I |p^-_{[3]}\rangle, \quad (3.30)
\]

where we use the notation

\[
\chi^I \equiv X^I \mathbb{P}^J \mathbb{P}^J + X^I + X \partial_{\mathbb{P}^I}, \quad (3.31)
\]

\[
X^I J^J \equiv \sum_{a=1}^{3} \tilde{\beta}_a ((\beta_a \partial_{\beta_a} - M^{(a)+\dagger}) \delta^I J^J - M^{(a)IJ}), \quad (3.32)
\]

\[
X^I \equiv \sum_{a=1}^{3} \tilde{\beta} m_a M^{(a)I}, \quad (3.33)
\]

\[
X \equiv -\sum_{a=1}^{3} \tilde{\beta} \beta_a m_a^2, \quad (3.34)
\]

\[
\tilde{\beta} \equiv \beta_1 \beta_2 \beta_3. \quad (3.35)
\]

Taking (3.30) into account we can rewrite Eq.(3.25) as

\[
|j^-_{[3]}\rangle = -\frac{1}{3 \beta P^-} \chi^I |p^-_{[3]}\rangle, \quad (3.36)
\]

which tells us that the density \( j^-_{[3]} \) is not an independent quantity but is expressible in terms of vertex \( p^-_{[3]} \) (3.8). By substituting \( j^-_{[3]} \) (3.36) into Eq.(3.24), we can verify that Eq.(3.24) is fulfilled.
Thus, we exhaust all commutators of the Poincaré algebra in the cubic approximation. Equations (3.17), (3.18) supplemented by relation (3.36) provide the complete list of restrictions imposed by commutators of the Poincaré algebra on the densities $p_{[3]}^{-1}$, $j_{[3]}^{-1}$. We see that the restrictions imposed by commutators of the Poincaré algebra by themselves are not sufficient to fix the vertex $p_{[3]}^{-1}$ uniquely. To choose the physically relevant densities $p_{[3]}^{-1}$, $j_{[3]}^{-1}$, i.e. to fix them uniquely, we impose the light-cone locality condition: we require the densities $p_{[3]}^{-1}$, $j_{[3]}^{-1}$ to be polynomials in $P^I$. As regards the vertex $p_{[3]}^{-1}$, we require this vertex to be local (i.e. polynomial in $P^I$) from the very beginning. However it is clear from relation (3.36) that a local $p_{[3]}^{-1}$ does not lead automatically to a local density $j_{[3]}^{-1}$. From (3.36), we see that the light-cone locality condition for $j_{[3]}^{-1}$ amounts to the equation

$$X^I|p_{[3]}^{-1}\rangle = P^-|V^I\rangle ,$$

(3.37)

where a vertex $|V^I\rangle$ is restricted to be polynomial in $P^I$. In fact, imposing the light-cone locality condition amounts to requiring that the generators of the Poincaré algebra be local functionals of the physical fields with respect to transverse directions.

The last requirement we impose on the cubic interaction vertex is related to field redefinitions. We note that by using local (i.e. polynomial in the transverse momenta) field redefinitions, we can remove the terms in the vertex $p_{[3]}^{-1}$ that are proportional to $P^{-}$ (see Appendix B in Ref.[36]). Since we are interested in the vertex that cannot be removed by field redefinitions, we impose the equation

$$|p_{[3]}^{-1}\rangle \neq P^-|V\rangle ,$$

(3.38)

where a vertex $|V\rangle$ is restricted to be polynomial in $P^I$. We note that Eqs.(3.37), (3.38) amount to the light-cone dynamical principle. If we restrict ourselves to low spin $s = 1, 2$ field theories, i.e. Yang-Mills and Einstein theories, it can then be verified that the light-cone dynamical principle and Eqs.(3.17), (3.18) allow fixing the cubic interaction vertices unambiguously (up to several coupling constants). It then seems reasonable to use the light-cone dynamical principle and Eqs.(3.17), (3.18) for studying the cubic interaction vertices of higher spin fields.

To summarize the discussion in this section, we collect equations imposed by the kinematical symmetries and the light-cone dynamical principle on vertex $p_{[3]}^{-1}$ (3.8):

$$J^{IJ}|p_{[3]}^{-1}\rangle = 0 ,$$

(3.39) \hspace{1cm} so($d-2$) invariance

$$ (P^I\partial_{P^I} + \sum_{a=1}^{3} \beta_a \partial_{\beta_a} - M^{+^{-} I})|p_{[3]}^{-1}\rangle = 0 ,$$

(3.40) \hspace{1cm} so($1, 1$) invariance

$$X^I|p_{[3]}^{-1}\rangle = P^-|V^I\rangle ,$$

(3.41) \hspace{1cm} light-cone locality condition

$$|p_{[3]}^{-1}\rangle \neq P^-|V\rangle ,$$

(3.42)

where the vertices $|V\rangle$ and $|V^I\rangle$ are restricted to be polynomials in $P^I$. Solving light-cone locality condition (3.41) leads to the representation for the density $|j_{[3]}^{-I}\rangle$ (3.36) as

$$|j_{[3]}^{-I}\rangle = -\frac{1}{3\beta}|V^I\rangle .$$

(3.43)

Equations (3.39)-(3.42) constitute a complete system of equations on vertex $p_{[3]}^{-1}$ (3.8). Equations (3.39), (3.40) reflect the invariance of the vertex $|p_{[3]}^{-1}\rangle$ under the respective so($d-2$) and so($1, 1$) rotations. Equations (3.41), (3.42) and the representation for the density $|j_{[3]}^{-I}\rangle$ (3.43) are obtainable from the light-cone dynamical principle.
Up to this point, our treatment has been applied to vertices for massive as well as massless fields. From now on, we separately consider vertices for the massless fields, vertices involving both massless and massive fields, and vertices for the massive fields.

Before proceeding, we discuss the general structure of vertices we study in this paper. First of all, we restrict our attention to the vertices for fields in space-time of even dimension and all fermionic fields are considered to be complex-valued Dirac fields.

Secondly, let $\psi \equiv \psi(x)$ be an arbitrary spin tensor-spinor fermionic field of the Lorentz algebra, while $\phi \equiv \phi(x)$ is an arbitrary spin bosonic field of the Lorentz algebra, where $x$ stands for coordinates of space-time. Let $\gamma$ be Dirac $\gamma$-matrices, while $\partial$ stands for derivatives w.r.t. space-time coordinates. In Lorentz-covariant approach, cubic vertices for fermionic and bosonic fields can schematically be represented as

$$L \sim \bar{\psi} \gamma_1 \ldots \gamma_d \partial \psi \partial \ldots \partial \phi.$$  \hfill (3.44)

In this paper, we restrict our attention to the light-cone vertices corresponding to covariant vertices that do not involve Levi-Civita antisymmetric tensor and matrix $\Gamma_\ast \sim \gamma^0 \ldots \gamma^{d-1}$.\footnote{Our light-cone vertices given below involve matrix $\gamma_\ast \sim \gamma^1 \ldots \gamma^{d-2}$. Appearance this matrix is related to the some special Fourier transformation of fields we use in this paper.}

We use the nomenclature of $K$-vertices and $F$-vertices. Our light-cone $K$-vertices correspond to covariant vertices whose low-order term in derivative expansion is supplemented with even number of $\gamma$-matrices. Accordingly, our $F$-vertices correspond to covariant vertices whose low-order term in derivative expansion is supplemented with odd number of $\gamma$-matrices. For example, light-cone counterparts of covariant vertices like $\bar{\psi} \psi \phi$ are referred to as $K$-vertices, while light-cone counterparts of the covariant vertices like $\bar{\psi} \gamma \psi \phi$ are referred to as $F$-vertices in this paper.

In general, vertices for massive fields involve contributions with different powers of derivatives. Therefore, to classify light-cone cubic vertices we need two labels at least. We use labels $k_{\text{min}}$ and $k_{\text{max}}$ which are the respective minimal and maximal numbers of transverse derivatives appearing in the light-cone cubic vertex. Labels like $k_{\text{min}}$ and $k_{\text{max}}$ can also be used to classify covariant vertices. If we denote by $k_{\text{min}}^{\text{cov}}$ and $k_{\text{max}}^{\text{cov}}$ the respective minimal and maximal numbers of derivatives appearing in the covariant cubic vertex, then one has the relations

$$k_{\text{min}}^{\text{cov}} \geq k_{\text{min}}, \quad k_{\text{max}}^{\text{cov}} \leq k_{\text{max}}.$$  \hfill (3.45)

Covariant vertices that involve only massless both fermionic and bosonic fields can be labelled $k^{\text{cov}}$ which is a number of derivatives appearing in covariant vertices. Denoting by $k$ numbers of transverse derivatives appearing in the corresponding light-cone vertices we have the relation\footnote{For the case of vertices involving only massless bosonic fields one has $k^{\text{cov}} = k$.}

$$k^{\text{cov}} = k - 1.$$  \hfill (3.46)

## 4 Cubic interaction vertices for massless fermionic and bosonic fields

We begin with discussing the cubic interaction vertex for the massless fields. This, we consider vertex involving two massless fermionic arbitrary spin fields having the respective mass parameters $m_1 = 0$ and $m_2 = 0$ and one massless arbitrary spin bosonic field having mass parameter $m_3 = 0$:

$$m_1|_F = 0, \quad m_2|_F = 0, \quad m_3|_B = 0,$$  \hfill (4.1)
where for transparency we attached subscript $F$ to mass parameter of fermionic fields and subscript $B$ to mass parameter of bosonic field. Equations for the vertex involving three massless fields are obtainable from (3.41) by letting $m_a \to 0$, $a = 1, 2, 3$. There are two types of general solutions for cubic vertex which take the form

\[ p_{[3]}^{-} = K^{(12)} V^K(B^{(a)}; Z), \quad K \text{ vertex}; \tag{4.2} \]
\[ p_{[3]}^{-} = F V^F(B^{(a)}; Z), \quad F \text{ vertex}; \tag{4.3} \]

where $V^K, V^F$ are arbitrary functions of $B^{(a)}$ and $Z$ and we use the notation

\[ K^{(12)} = \frac{1}{\beta_1 \beta_2} p_{\theta_1} \gamma^I \gamma_a \eta_2, \tag{4.4} \]
\[ F = \frac{1}{\beta_1 \beta_2} p_{\theta_1} \left( \frac{\beta_3}{\beta_2} \right) \gamma^I \eta_2 \alpha^{(3)} I, \tag{4.5} \]
\[ B^{(a)} \equiv \alpha^{(a)I} \frac{p_I}{\beta_a}, \quad a = 1, 2, 3, \tag{4.6} \]
\[ Z \equiv B^{(1)} \alpha^{(2)} + B^{(2)} \alpha^{(3)} + B^{(3)} \alpha^{(12)}, \tag{4.7} \]

and $\alpha^{(ab)}$ is defined by

\[ \alpha^{(ab)} \equiv \alpha^{(a)I} \alpha^{(b)I}. \tag{4.8} \]

We now comment on the solution obtained. The quantities $B^{(a)}$, $\alpha^{(ab)}$ and $Z$ are the respective degree 1, 2, and 3 homogeneous polynomials in oscillators. Henceforth, degree 1, 2, and 3 homogeneous polynomials in oscillators are referred to as linear, quadratic, and cubic forms respectively. We emphasize, however, that the vertices $V^K$, $V^F$ can depend on $\alpha^{(aa)}$. But the contribution of the $\alpha^{(aa)}$-terms to the Hamiltonian $H_{[3]}$ vanishes when ket-vectors (2.16), (2.33) are subjected to the tracelessness constraints. This is, the physical massless fields, being irreps of the $so(d-2)$ algebra, satisfy the tracelessness constraints

\[ \bar{\alpha}^{(a)I} \bar{\alpha}^{(a)I} \mid \psi_a^{m_a=0} \rangle = 0, \quad a = 1, 2, \quad \bar{\alpha}^{(a)I} \bar{\alpha}^{(a)I} \mid \phi_3^{m_3=0} \rangle = 0. \tag{4.9} \]

It is then clear that the $\alpha^{(aa)}$-terms do not contribute to the Hamiltonian $H_{[3]}$ (3.3). Therefore we drop $\alpha^{(aa)}$-terms in the general solution for the vertices. This, in case of massless fields belonging to irreps of the $so(d-2)$ algebra, vertices (4.2), (4.3) are governed by fermionic forms $K^{(12)}$ and $F$, by the linear forms $B^{(a)}$ and by the cubic forms $Z$. To understand the remaining important properties of the vertices we consider cubic vertices for massless fields with fixed spin values.

### 4.1 Cubic vertices for massless fields with fixed but arbitrary spin values

Vertices (4.2), (4.3) describe interaction of towers of massless bosonic and fermionic fields (2.16), (2.33). We now obtain vertex for massless spin $s^{(1)} + \frac{1}{2}$, $s^{(2)} + \frac{1}{2}$ and $s^{(3)}$ fields. This, we consider vertex involving two massless fermionic spin $s^{(1)} + \frac{1}{2}$ and $s^{(2)} + \frac{1}{2}$ fields and one massless bosonic spin-$s_3$ field:
The massless spin $s^{(1)} + \frac{1}{2}$ and $s^{(2)} + \frac{1}{2}$ fermionic fields are described by the respective ket-vectors $|\psi_{s^{(1)}}^{m_1=0}\rangle$ and $|\psi_{s^{(2)}}^{m_2=0}\rangle$, while the massless spin-$s^{(3)}$ bosonic field is described by the ket-vector $|\phi_{s^{(3)}}^{m_3=0}\rangle$. The ket-vectors of massless fermionic fields are obtainable from (2.27) by replacement $s \rightarrow s^{(a)}$, $\alpha \rightarrow \alpha^{(a)}$, $a = 1, 2$, in (2.27), while ket-vector of massless bosonic field is obtainable from (2.13) by replacement $s \rightarrow s^{(3)}$, $\alpha \rightarrow \alpha^{(3)}$ in (2.13). Taking into account that the ket-vectors $|\psi_{s^{(1)}}^{m_1=0}\rangle$, $|\psi_{s^{(2)}}^{m_2=0}\rangle$, $|\phi_{s^{(3)}}^{m_3=0}\rangle$ are the respective degree $s^{(a)}$, $a = 1, 2, 3$ homogeneous polynomials in the oscillators $\alpha^{(a)}$, it is easy to see that the vertex we are interested in must satisfy the equations

\[
(\alpha^{(a)} I \bar{\alpha}^{(a)} I - s^{(a)})|p^{(a)}_{[3]}\rangle = 0, \quad a = 1, 2, 3,
\]

which tell us that the vertex $p^{(3)}_{[3]}$ should be degree-$s^{(a)}$ homogeneous polynomial in the oscillators $\alpha^{(a)}$. Taking into account that forms $F$, $B^{(a)}$ and $Z$ (see (4.5)-(4.7)) are the respective degree 1 and 3 homogeneous polynomials in oscillators, we find the general solution of Eq. (4.11)

\[
p^{(1)}_{[3]}(s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2}, s^{(3)}; k) = K^{(12)} Z^{\frac{1}{2}(s-k+1)} Z^{\frac{1}{2}(s-k+1)} \prod_{a=1}^{3} (B^{(a)})^{s^{(a)} + \frac{1}{2}(k-s-1)}, \quad \text{for K vertex; (4.12)}
\]

\[
p^{(3)}_{[3]}(s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2}, s^{(3)}; k) = F Z^{\frac{1}{2}(s-k)} (B^{(3)})^{-1} \prod_{a=1}^{3} (B^{(a)})^{s^{(a)} + \frac{1}{2}(k-s)}, \quad \text{for F vertex; (4.13)}
\]

where we use the notation

\[
s \equiv \sum_{a=1}^{3} s^{(a)}, \quad (4.14)
\]

and integer $k$ is a freedom in our solution. The integer $k$ labels all possible cubic vertices that can be built for massless spin $s^{(1)} + \frac{1}{2}$, $s^{(2)} + \frac{1}{2}$, $s^{(3)}$ fields and has a clear physical interpretation. Taking into account that the forms $K^{(12)}$, $B^{(a)}$, $F$, and $Z$ are degree 1 homogeneous polynomials in the momentum $p^{I}$ it is easy to see that vertices (4.12), (4.13) are degree $k$ homogeneous polynomials in $p^{I}$. To summarize, the vertex $p^{(a)}_{[3]}(s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2}, s^{(3)}; k)$ describes interaction of three massless spin $s^{(1)} + \frac{1}{2}$, $s^{(2)} + \frac{1}{2}$, $s^{(3)}$ fields having $k$ powers of the momentum $p^{I}$. In Lorentz covariant approach, gauge invariant vertices corresponding to our light-cone vertices (4.12), (4.13) have $k - 1$ number of the derivatives with respect to space-time coordinates.

We now discuss the restrictions to be imposed on values of $s^{(1)}$, $s^{(2)}$, $s^{(3)}$ and the integer $k$. The powers of the forms $B^{(a)}$ and $Z$ in (4.12), (4.13) must be non–negative integers. For this to be the case, it is necessary to impose the following restrictions on the allowed spin values $s^{(1)}$, $s^{(2)}$, $s^{(3)}$ and the number of powers of the momentum $p^{I}$ (the number of the derivatives):

---

8 It is this property of the forms $K^{(12)}$, $F$, $B^{(a)}$ and $Z$ that allows us to introduce the vertex that is the homogeneous polynomial in $p^{I}$. A completely different type of a situation occurs in the case of massive fields, whose cubic interaction vertices depend on forms that are non-homogeneous polynomials in the $p^{I}$.
\[ s - 2s_{\text{min}} + 1 \leq k \leq s + 1, \quad s_{\text{min}} \equiv \min_{a=1,2,3} s^{(a)}, \]

for K-vertex \hspace{1cm} (4.15)

\[ \max(s - 2s^{(1)}, s - 2s^{(2)}, s - 2s^{(3)} + 2) \leq k \leq s, \]

for F-vertex \hspace{1cm} (4.16)

Formulas (4.12), (4.13) not only provides a surprisingly simple form for the vertices of massless higher spin fields but also gives a simple form for the vertices of the well-studied massless low spin fields. By way of example, we consider cubic vertices that describe the interaction of spin-$1/2$ fermionic fields with low-spin bosonic massless fields. This is to say that cubic interaction vertex for spin-$1/2$ fermionic field $\psi$ and massless scalar field $\phi$ takes the form

\[ p_{[3]}(1/2, 1/2, 0; 1) = K^{(12)} \sim \bar{\psi} \psi \phi. \] \hspace{1cm} (4.17)

The minimal interaction of spin-$1/2$ fermionic field with massless spin-1 field $\phi^A$ is given by

\[ p_{[3]}(1/2, 1/2, 1; 1) = F \sim \bar{\psi} \gamma^A \psi \phi^A, \] \hspace{1cm} (4.18)

while the non-minimal interaction of massless spin-$1/2$ fermionic field with massless spin-1 field takes the form

\[ p_{[3]}(1/2, 1/2, 1; 2) = K^{(12)} B^{(3)} \sim \bar{\psi} \gamma^A \psi F^{AB}, \] \hspace{1cm} (4.19)

where $F^{AB} = \partial^A \phi^B - \partial^B \phi^A$. The minimal gravitational interaction of massless spin-$1/2$ fermionic field $\psi$ is given by

\[ p_{[3]}(1/2, 1/2, 2; 1) = F B^{(3)} \sim \bar{\psi} \gamma^A D^A_{\text{Lor}} \psi, \] \hspace{1cm} (4.20)

where $D^A_{\text{Lor}}$ stands for covariant derivative w.r.t. Lorentz connection. The non-minimal interaction of massless spin-$1/2$ fermionic field $\psi$ with massless spin-2 field $\phi^{AB}$ takes the form

\[ p_{[3]}(1/2, 1/2, 2; 3) = K^{(12)} (B^{(3)})^2 \sim \bar{\psi} \partial^A \partial^B \psi \phi^{AB}. \] \hspace{1cm} (4.21)

Our approach gives simple expressions for interaction vertices of massless spin-$1/2$ fermionic field and massless arbitrary spin-$s$ bosonic field. This is to say that there are the following two interaction vertices for massless spin-$1/2$ fermionic field $\psi$ and massless arbitrary spin-$s$ bosonic field $\phi^{A_1...A_s}$:

\[ p_{[3]}(1/2, 1/2, s; s + 1) = K^{(12)} (B^{(3)})^s \sim \bar{\psi} \partial^{A_1} \ldots \partial^{A_s} \psi \phi^{A_1...A_s}, \] \hspace{1cm} (4.22)

\[ p_{[3]}(1/2, 1/2, s; s) = F (B^{(3)})^{s-1} \sim \bar{\psi} \gamma^{A_1} \partial^{A_2} \ldots \partial^{A_s} \psi \phi^{A_1...A_s}. \] \hspace{1cm} (4.23)

Higher-derivative interaction of massless spin-$3/2$ fermionic field $\psi^A$ with massless scalar field $\phi$ takes the form

\[ p_{[3]}(3/2, 3/2, 0; 3) = K^{(12)} B^{(3)} B^{(2)} \sim \bar{\psi} \partial^{A} \psi \phi^{AB}, \] \hspace{1cm} (4.24)
where

\[ \Psi^{AB} \equiv \partial^A \psi^B - \partial^B \psi^A \]  \tag{4.25} \]

is field strength of spin-$\frac{3}{2}$ field $\psi^A$. The non-minimal interaction vertices of massless spin-$\frac{3}{2}$ fermionic field with massless spin-1 field take the form

\[ p_{[3]}(\frac{3}{2}, \frac{3}{2}; 1; 2) = K^{(12)} Z \sim (\bar{\psi}^A \psi^B - \frac{1}{4} \psi^C \gamma^{AB} \psi^C) F^{AB}, \]  \tag{4.26} \]

\[ p_{[3]}(\frac{3}{2}, \frac{3}{2}; 1; 3) = F B^{(1)} B^{(2)} \sim \bar{\psi}^A \partial^A \gamma^B \psi^C F^{BC}, \]  \tag{4.27} \]

\[ p_{[3]}(\frac{3}{2}, \frac{3}{2}; 1; 4) = K^{(12)} B^{(1)} B^{(2)} B^{(3)} \sim \bar{\psi}^A \partial^A \partial^B \psi^C F^{BC}. \]  \tag{4.28} \]

The minimal gravitational interaction of massless spin-$\frac{3}{2}$ fermionic field $\psi^A$ is given by

\[ p_{[3]}(\frac{3}{2}, \frac{3}{2}; 2, 2) = F Z \sim \bar{\psi}^A \gamma^{ABC} D_{\text{Lor}} \psi^C. \]  \tag{4.29} \]

Light-cone vertices for two massless spin-$\frac{3}{2}$ fermionic fields and one massless arbitrary spin-$s$, $s \geq 2$, bosonic field take the form

\[ p_{[3]}(\frac{3}{2}, \frac{3}{2}, s; s) = F Z (B^{(3)})^{s-2}, \]  \tag{4.30} \]

\[ p_{[3]}(\frac{3}{2}, \frac{3}{2}, s; s + 1) = K^{(12)} Z (B^{(3)})^{s-1}, \]  \tag{4.31} \]

\[ p_{[3]}(\frac{3}{2}, \frac{3}{2}, s; s + 2) = F B^{(1)} B^{(2)} (B^{(3)})^{s-1}. \]  \tag{4.32} \]

\[ p_{[3]}(\frac{3}{2}, \frac{3}{2}, s; s + 3) = K^{(12)} B^{(1)} B^{(2)} (B^{(3)})^s. \]  \tag{4.33} \]

Denting by $L(\frac{3}{2}, \frac{3}{2}, s; k)$ covariant vertex that corresponds to light-cone vertex $p_{[3]}(\frac{3}{2}, \frac{3}{2}, s; k)$ in (4.30)-(4.33) we obtain the following covariant vertices:

\[ i^{s-1} L(\frac{3}{2}, \frac{3}{2}, s; s) = \bar{\psi} A_1 \gamma^{A_2 \partial A_2 \ldots \partial A_s \psi^B \partial^B \phi^{A_1 \ldots A_s}} + \bar{\psi} C \gamma^{A_1 \partial A_2 \ldots \partial A_{s-1} \psi^C A_s \phi^{A_1 \ldots A_s}}, \]  \tag{4.34} \]

\[ i^{s-1} L(\frac{3}{2}, \frac{3}{2}, s; s + 1) = \bar{\psi} A_1 \partial A_2 \ldots \partial A_s \psi^B \partial^B \phi^{A_1 \ldots A_s} + \bar{\psi} C \partial A_1 \ldots \partial A_{s-1} \psi^C A_s \phi^{A_1 \ldots A_s}, \]  \tag{4.35} \]

\[ i^{s-1} L(\frac{3}{2}, \frac{3}{2}, s; s + 2) = \bar{\psi} B \gamma^{A_1 \partial A_2 \ldots \partial A_s \psi^C \partial^C \phi^{A_1 \ldots A_s}}, \]  \tag{4.36} \]

\[ i^{s-1} L(\frac{3}{2}, \frac{3}{2}, s; s + 3) = \bar{\psi} B \partial A_1 \ldots \partial A_s \psi^C \partial^C \phi^{A_1 \ldots A_s}, \]  \tag{4.37} \]

where $\phi^{A_1 \ldots A_s}$ stands for massless spin-$s$ bosonic field. We note that covariant vertices in (4.34)-(4.37) are gauge invariant only w.r.t. linearized on-shell gauge symmetries. Also, note that, for
\( s = 2 \), the vertex in (4.34) is nothing but the vertex of the minimal gravitational interaction of massless spin-\( \frac{3}{2} \) fermionic field (4.29).

We see that the light-cone gauge approach gives a simple representation for such vertices. Another attractive property of the light-cone approach is that it allows treating the interaction vertices on an equal footing. Formulas (4.12), (4.13) provide a convenient representation for other well-known cubic interaction vertices of massless low-spin fields. These vertices and their Lorentz covariant counterparts are collected in Table I.

A few remarks are in order.

**i)** Gravitational interaction of low-spin massless fermionic fields are described by \( F \)-vertices. Therefore let us use of \( F \)-vertices to discuss gravitational vertices of massless higher-spin fermionic fields, \( s + \frac{1}{2} \geq 2 \). It is easy to see that restrictions (4.16) are precisely the restrictions that leave no place for the gravitational interaction of massless higher-spin fermionic fields. Indeed, the gravitational interaction of a massless spin-\( s \) gravitational field could be described by the vertex \( p_{\mu \nu}^-(s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2}, s^{(3)}; k) \) with \( s^{(1)} = s^{(2)} = s \geq 2, s^{(3)} = 2, k = 2 \). For these values \( s^{(a)} \) we obtain

\[
    s = 2s + 2, \quad \max(s - 2s^{(1)}, s - 2s^{(2)}, s - 2s^{(3)} + 2) = 2s, \tag{4.38}
\]

and therefore restrictions (4.16) take the form

\[
    2s \leq k \leq 2s + 2. \tag{4.39}
\]

On the one hand, these restrictions tell us that for \( s \geq 2 \), the gravitational interaction, i.e. the case \( k = 2 \), is not allowed. On the other hand, we see that all allowed cubic interactions vertices for graviton and massless higher-spin fermionic fields involve higher derivatives, \( k > 2 \).

**ii)** Restrictions (4.15), (4.16) lead to a surprisingly simple result for values of allowed \( k \) for cubic vertices \( p_{\mu \nu}^-(s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2}, s^{(3)}; k) \). Indeed, we see from (4.15) and (4.16) that for spin values \( s^{(1)}, s^{(2)}, s^{(3)} \), the integer \( k \) takes the values

\[
    k = s + 1, s - 1, \ldots, s - 2s^{(1) + 1}, \quad \text{for K vertex; (4.40)}
\]

\[
    k = s, s - 2, \ldots, \max(s - 2s^{(1)}, s - 2s^{(2)}, s - 2s^{(3)} + 2), \quad \text{for F vertex. (4.41)}
\]

Note that, for \( F \)-vertices, we should keep in mind the restriction \( s^{(3)} \geq 1 \) when \( s^{(3)} = s_{\min} \).

**iv)** Vertices (4.12), (4.13), with \( k \) in (4.40), (4.41), constitute the complete list of vertices for \( d > 4 \). For \( d = 4 \), the number of allowed vertices is decreased. This is, if \( d = 4 \), then for spin values \( s^{(1)}, s^{(2)}, s^{(3)} \), the integer \( k \) takes the values

\[
    k = s + 1, s - 2s^{(1) + 1}, \quad \text{when } s^{(3)} = s_{\min}, \quad \text{for K vertices in } d = 4. \tag{4.42}
\]

\[
    k = s + 1, \quad \text{when } s^{(3)} > s_{\min}, \quad \text{for K vertices in } d = 4. \tag{4.43}
\]

\[
    k = s - 2s^{(1)}, \quad \text{when } s^{(3)} > s_{\min}, \quad \text{for F vertices in } d = 4. \tag{4.44}
\]

This implies that, in \( 4d \), there are two K-vertices when \( s^{(3)} = s_{\min} > 0 \) and one K-vertex when \( s^{(3)} = s_{\min} = 0 \) or \( s^{(3)} > s_{\min} \). Also we note that, in \( 4d \), there is only one F-vertex when \( s^{(3)} > s_{\min} \) and there are no F-vertices when \( s^{(3)} = s_{\min} \).

\[
    \text{Footnote 9: There is simple rule which allows to choose nontrivial vertices in } 4d. \text{ This, an appearance of } K^{(12)} ZB^{(3)} \text{-factor in K-vertex leads to trivial K-vertex, while appearance of } FB^{(1)} B^{(2)} \text{-factor in F-vertex leads to trivial F-vertex. This rule can be approved by using helicity formalism in Ref. [20].}
\]

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Table I. Cubic vertices for massless fermionic and bosonic low-spin fields. In the 4th column, $\phi$, $\psi$, $\phi^A$, $\psi^A$, $\phi^{AB}$ stand for the respective spin $0$, $1/2$, $1$, $3/2$, $2$ massless fields. $F^{AB}$ and $R^{ABCE}$ stand for the respective Yang-Mills field strength and the Riemann tensor, while $\bar{\psi}^{AB} = \partial^A\psi^B - \partial^B\psi^A$ is field strength of spin $3/2$ field $\psi^A$. $D^A$ and $D^{A}_{\text{Lor}}$ stand for the respective Yang-Mills covariant derivative and Lorentz covariant derivative. $\omega^{BC}$ stands for the linearized Lorentz connection, $\omega^{AB} = -\omega^{BC}$. Most of the covariant vertices in the Table are invariant only under linearized on-shell gauge transformations.

| Spin values and number of derivatives $s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2}, s^{(3)}; k$ | Light-cone vertex $p_{[3]}^{-}(s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2}, s^{(3)}; k)$ | Covariant Lagrangian |
|---|---|---|
| $\frac{1}{2}, \frac{1}{2}, 0; 1$ | $d \geq 4$ | $K^{(12)}$ | $\bar{\psi}\psi\phi$ |
| $\frac{1}{2}, \frac{1}{2}, 1; 1$ | $d \geq 4$ | $F$ | $\bar{\psi}\gamma^A\psi\phi^A$ |
| $\frac{1}{2}, \frac{1}{2}, 1; 2$ | $d \geq 4$ | $K^{(12)}B^{[3]}$ | $\bar{\psi}\gamma^{AB}\psi F^{AB}$ |
| $\frac{1}{2}, \frac{1}{2}, 2; 2$ | $d \geq 4$ | $FB^{(3)}$ | $(\bar{\psi}\gamma^A D^A\psi)_{[3]}$ |
| $\frac{1}{2}, \frac{1}{2}, 2; 3$ | $d \geq 4$ | $K^{(12)}(B^{[3]} )^2$ | $\bar{\psi}\gamma^{AB}\partial^C\bar{\psi}\omega^{CAB}$ |
| $\frac{3}{2}, \frac{3}{2}, 0; 3$ | $d \geq 4$ | $K^{(12)}B^{(1)}B^{(2)}$ | $\bar{\psi}^{AB}\bar{\psi}^{AB}\phi$ |
| $\frac{3}{2}, \frac{3}{2}, 1; 2$ | $d \geq 4$ | $K^{(12)}Z$ | $(\bar{\psi}^{A}\psi^B - \frac{1}{3}\bar{\psi}^C\gamma^{AB}\psi^C F^{AB})$ |
| $\frac{3}{2}, \frac{3}{2}, 1; 3$ | $d > 4$ | $FB^{(1)}B^{(2)}$ | $\bar{\psi}^{A}\partial^A\gamma^B\psi^C F^{BC}$ |
| $\frac{3}{2}, \frac{3}{2}, 1; 4$ | $d \geq 4$ | $K^{(12)}B^{(1)}B^{(2)}B^{(3)}$ | $\bar{\psi}^{A}\partial^A\partial^B\psi^C F^{BC}$ |
| $\frac{3}{2}, \frac{3}{2}, 2; 2$ | $d \geq 4$ | $FZ$ | $(\bar{\psi}^{A}\gamma^{ABC} D^B_{\text{Lor}} \psi^C )_{[3]}$ |
| $\frac{3}{2}, \frac{3}{2}, 2; 3$ | $d > 4$ | $K^{(12)}ZB^{(3)}$ | $\bar{\psi}^C\partial^A\psi^B\bar{\psi}^B\partial^B\omega^{ABC}$ |
| $\frac{3}{2}, \frac{3}{2}, 2; 4$ | $d \geq 4$ | $FB^{(1)}B^{(2)}B^{(3)}$ | $\bar{\psi}^{A}\gamma^B\psi^C E^B\partial^A\omega^{BCE}$ |
| $\frac{3}{2}, \frac{3}{2}, 2; 5$ | $d \geq 4$ | $K^{(12)}B^{(1)}B^{(2)}(B^{(3)} )^2$ | $\bar{\psi}^{AB}\psi^C E^R_{ABCE}$ |

We note that $K$-vertices with $k = s + 1$ correspond to gauge theory cubic interaction vertices built entirely in terms of gauge field strengths. The vertices with $k < s + 1$ cannot be built entirely in terms of gauge field strengths. It is the vertices with $k < s + 1$ that are difficult to construct in Lorentz covariant approaches. The light-cone approach treats all vertices on an equal footing.

---

\footnote{Our result for the vertex $p_{[3]}^{-}(\frac{3}{2}, \frac{3}{2}, 2; 7)$ (see last row in Table II) implies that there is only one Lorentz covariant vertex $\bar{\psi}^{AB}\psi^C R^{EF}_{AB}$ that gives a non-trivial contribution to the 3-point scattering amplitude. Similar statement for the case of vertex $R^{AB}_{CD}R^{CD}_{EF}R^{EF}_{AB}$ was proved in Ref.\cite{59}.}

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Table II. Cubic vertices for massless fermionic spin-$\frac{5}{2}$ field and massless bosonic spin \(0,1,2\) fields. In the 4th column, \(\phi, \phi^A\) stand for the respective spin 0, 1 massless fields. \(F^{AB}\) and \(R^{ABCE}\) stand for the respective Yang-Mills field strength and the Riemann tensor, while \(\Psi^{ABCE}\) stands for field strength of spin 5/2 field. \(\omega_A^{BC}\) and \(\Omega_A^{BC}\) stand for the respective linearized Lorentz connection of spin-2 and spin-\(\frac{5}{2}\) fields, \(\omega_A^{BC} = -\omega_A^{CB}, \Omega_A^{BC} = -\Omega_A^{CB}\). Most of the covariant vertices in the Table are invariant only under linearized on-shell gauge transformations.

| Spin values and number of derivatives \(\frac{5}{2}, \frac{5}{2}, s^{(3)}; k\) | dimension of space-time | Light-cone vertex \(p^-_\nu (\frac{5}{2}, \frac{5}{2}, s^{(3)}; k)\) | Covariant Lagrangian |
|---|---|---|---|
| \(\frac{5}{2}, \frac{5}{2}, 0; 5\) | \(d \geq 4\) | \(K^{(12)} (B^{(1)} B^{(2)})^2\) | \(\bar{\Psi}^{ABCE} \Psi^{ABCE} \phi\) |
| \(\frac{5}{2}, \frac{5}{2}, 1; 4\) | \(d \geq 4\) | \(K^{(12)} Z B^{(1)} B^{(2)}\) | \(F^{AB} (\bar{\Omega}^{A,CE} B^{CE} - \bar{\Omega}^{C,EA} \Omega^{CE} B)\) |
| \(\frac{5}{2}, \frac{5}{2}, 1; 5\) | \(d \geq 4\) | \(F (B^{(1)})^2 (B^{(2)})^2\) | \(\bar{\Psi}^{ABCD} \gamma^E \Psi^{ABCD} \phi^E\) |
| \(\frac{5}{2}, \frac{5}{2}, 1; 6\) | \(d \geq 4\) | \(K^{(12)} (B^{(1)})^2 (B^{(2)})^2 B^{(3)}\) | \(\bar{\Psi}^{ACDE} \Psi^{BCDE} F^{AB}\) |
| \(\frac{5}{2}, \frac{5}{2}, 2; 3\) | \(d \geq 4\) | \(K^{(12)} Z^2\) | \(\text{See Ref.}\ [69]\) |
| \(\frac{5}{2}, \frac{5}{2}, 2; 4\) | \(d \geq 4\) | \(F Z B^{(1)} B^{(2)}\) | \(\text{See Ref.}\ [69]\) |
| \(\frac{5}{2}, \frac{5}{2}, 2; 5\) | \(d \geq 4\) | \(K^{(12)} Z B^{(1)} B^{(2)} B^{(3)}\) | \(\bar{\Psi}^{ABCD} \gamma^F \Psi^{BCDE} \omega F^{AB}\) |
| \(\frac{5}{2}, \frac{5}{2}, 2; 6\) | \(d \geq 4\) | \(F (B^{(1)})^2 (B^{(2)})^2 B^{(3)}\) | \(\bar{\Psi}^{ABCD} \gamma^F \Psi^{BCDE} \omega F^{AB}\) |
| \(\frac{5}{2}, \frac{5}{2}, 2; 7\) | \(d \geq 4\) | \(K^{(12)} (B^{(1)} B^{(2)} B^{(3)})^2\) | \(\bar{\Psi}^{CD} \Psi^{EF} R^{EF}\) |

5 Cubic interaction vertices for massless and massive fields

We now study cubic interaction vertices for massless and massive field. We consider cubic vertices for one massive field and two massless fields and cubic vertices for one massless field and two massive fields. In other words, we consider vertices for fields with the following mass values:

\[
\begin{align*}
\text{mass values} & \quad \text{mass values} \\
m_1|_F = m_2|_F = 0, & \quad m_3|_B \neq 0; \quad (5.1) \\
m_1|_B = m_2|_F = 0, & \quad m_3|_F \neq 0; \quad (5.2) \\
m_1|_F = m_2|_F \equiv m \neq 0, & \quad m_3|_B = 0; \quad (5.3) \\
m_1|_B = m_2|_F \equiv m \neq 0, & \quad m_3|_F = 0; \quad (5.4) \\
m_1|_F \neq 0, & \quad m_2|_F \neq 0, \quad m_1|_F \neq m_2|_F, \quad m_3|_B = 0. \quad (5.5) \\
m_1|_B \neq 0, & \quad m_2|_F \neq 0, \quad m_1|_B \neq m_2|_F, \quad m_3|_F = 0. \quad (5.6)
\end{align*}
\]

We study these cases in turn.
5.1 Cubic interaction vertices for two massless fermionic fields and one massive bosonic field

We start with the cubic interaction vertex for three fields with the mass values

\[ m_1|_F = m_2|_F = 0, \quad m_3|_B \neq 0, \tag{5.7} \]

i.e. the massless fermionic fields carry external line indices \( a = 1, 2 \), while the massive bosonic field corresponds to \( a = 3 \). Equations for the vertex involving two massless fields can be obtained from Eqs.(3.41) in the limit as \( m_1 \to 0, m_2 \to 0 \). The general solution for vertex is found to be

\[ p_{[3]}^- = K^{(12)} V^K(B^{(3)}; Q^{(aa+1)}) , \quad \text{K vertex} , \tag{5.8} \]
\[ p_{[3]}^- = F V^F(B^{(3)}; Q^{(aa+1)}) , \quad \text{F vertex} , \tag{5.9} \]

where we use the notation\(^\text{[1]}\)

\[ K^{(12)} = \frac{1}{\beta_1 \beta_2} p_{a1} \mathbb{P}^I \gamma^I \gamma^j \eta_2 , \tag{5.10} \]
\[ F = \frac{1}{\beta_1 \beta_2} p_{a1} \left( \left( \frac{\beta_3}{2} \mathbb{P}^I - \gamma^I \mathbb{P}^J \right) \alpha^{(3)I} + \frac{2 \beta_1 \beta_2}{\beta_3} m_3 \alpha^{(3)} \right) \eta_2 , \tag{5.11} \]
\[ B^{(a)} \equiv \frac{\alpha^{(a)I} \mathbb{P}^I}{\beta_a} , \quad a = 1, 2; \tag{5.12} \]
\[ B^{(3)} \equiv \frac{\alpha^{(3)I} \mathbb{P}^I}{\beta_3} - \frac{\beta_3}{2 \beta_3} m_3 \alpha^{(3)} , \tag{5.13} \]
\[ Q^{(12)} \equiv \alpha^{(12)} - \frac{2}{m_3^2} B^{(1)} B^{(2)} , \tag{5.14} \]
\[ Q^{(23)} \equiv \alpha^{(23)} - \alpha^{(3)} m_3 B^{(2)} + \frac{2}{m_3^2} B^{(2)} B^{(3)} , \tag{5.15} \]
\[ Q^{(31)} \equiv \alpha^{(31)} + \alpha^{(3)} m_3 B^{(1)} + \frac{2}{m_3^2} B^{(3)} B^{(1)} , \tag{5.16} \]

and \( \alpha^{(ab)} \) are defined in \( (4.8) \).

We note that, with exception of \( K^{(12)} \), all forms in \( (5.8) \), \( (5.9) \) that depend on \( \mathbb{P}^I \) (the fermionic form \( F \), the linear form \( B^{(3)} \) and the quadratic forms \( Q^{(aa+1)} \)) are non-homogeneous polynomials in \( \mathbb{P}^I \). Therefore, as seen from \( (5.11)-(5.16) \), the dependence on the fermionic form \( F \), the linear forms \( B^{(3)} \), and the quadratic forms \( Q^{(aa+1)} \) leads to the cubic vertices that are non-homogeneous polynomials in \( \mathbb{P}^I \). The appearance of massive field interaction vertices involving different powers of derivatives is a well-known fact (see e.g. [60, 61]). Thus, we see that the light-cone formalism gives a very simple explanation to this phenomenon by means of the fermionic form \( F \), the linear form \( B^{(3)} \) and the quadratic forms \( Q^{(aa+1)} \). To understand the remaining characteristic properties of solution \( (5.8), (5.9) \), we consider vertices for fields with fixed spin values.

\(^{[1]}\) We recall that the short notation like \( p_{[3]}^- (Q^{(aa+1)}) \) is used to indicate a dependence of \( p_{[3]}^- \) on \( Q^{(12)}, Q^{(23)}, Q^{(31)} \).
5.1.1 Cubic interaction vertices for fields with fixed but arbitrary spin values

In this section, we restrict ourselves to cubic vertices for fields with fixed spin values. Vertices (5.8), (5.9) describe an interaction of the towers of bosonic massive fields in (2.12) and fermionic massless fields in (2.33). We now obtain vertex for two massless spin \( s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2} \) fermionic fields and one massive spin-\( s^{(3)} \) bosonic field. This, we consider vertex involving two massless fermionic spin \( s^{(1)} + \frac{1}{2} \) and \( s^{(2)} + \frac{1}{2} \) fields and one massive bosonic spin-\( s^{(3)} \) field having mass parameter \( m_3 \):

\[
m_1 = 0, \quad m_2 = 0, \quad m_3 \neq 0, \quad s^{(1)} + \frac{1}{2}, \quad s^{(2)} + \frac{1}{2}, \quad s^{(3)}. \tag{5.17}
\]

The massless spin-(\( s^{(1)} + \frac{1}{2} \)) and -(\( s^{(2)} + \frac{1}{2} \)) fermionic fields are described by respective ket-vectors \( |\psi_{s^{(1)}}^{m_1=0}\rangle \) and \( |\psi_{s^{(2)}}^{m_2=0}\rangle \), while the massive spin-\( s^{(3)} \) bosonic field is described by a ket-vector \( |\phi_{s^{(3)}}\rangle \).

The ket-vectors of massless fields \( |\psi_{s^{(a)}}^{m_a=0}\rangle \), \( a = 1, 2 \), can be obtained from (2.27) by the replacement \( s \rightarrow s^{(a)}, \alpha^I \rightarrow \alpha^{(a) I} \), \( a = 1, 2 \), in (2.27), while the ket-vector of the massive field \( |\phi_{s^{(3)}}\rangle \) can be obtained from (2.9) by the replacement \( s \rightarrow s^{(3)}, \alpha^I \rightarrow \alpha^{(3) I}, \alpha \rightarrow \alpha^{(3)} \) in (2.9). Taking into account that the ket-vectors \( |\psi_{s^{(a)}}^{m_a=0}\rangle \), \( a = 1, 2 \), are the respective degree \( s^{(a)} \) homogeneous polynomials in the oscillators \( \alpha^{(a) I} \), while the ket-vector \( |\phi_{s^{(3)}}\rangle \) is a degree \( s^{(3)} \) homogeneous polynomial in the oscillators \( \alpha^{(3) I}, \alpha^{(3)} \), it is easy to understand that the vertex we are interested in must satisfy the equations

\[
(\alpha^{(a) I} \alpha^{(a) I} - s^{(a)}) |p^{(a)}_{[3]}\rangle = 0, \quad a = 1, 2, \tag{5.18}
\]

\[
(\alpha^{(3) I} \alpha^{(3) I} + \alpha^{(3)} \alpha^{(3)} - s^{(3)}) |p^{(3)}_{[3]}\rangle = 0. \tag{5.19}
\]

These equations tell us that the vertex must be a degree-\( s^{(a)} \) homogeneous polynomial in the respective oscillators. Taking into account that the forms \( F, B^{(3)} \) and \( Q^{(aa+1)} \) are the respective degree 1 and 2 homogeneous polynomials in the oscillators we find the general solution of Eqs. (5.18), (5.19) as:

\[
p^{(a)}_{[3]}(s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2}, s^{(3)}; x) = K^{(12)}(B^{(3)})^x \prod_{a=1}^{3}(Q^{(aa+1)})^{y^{(a+2)}}, \quad \text{K vertex}, \tag{5.20}
\]

\[
p^{(3)}_{[3]}(s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2}, s^{(3)}; x) = F(B^{(3)})^x \prod_{a=1}^{3}(Q^{(aa+1)})^{y^{(a+2)}}, \quad \text{F vertex}, \tag{5.21}
\]

where integers \( y^{(a)} \) are expressible in terms of the \( s^{(a)} \) and an integer \( x \) by the relations

\[
y^{(a)} = \frac{s - x}{2} - s^{(a)}, \quad a = 1, 2, \quad \text{for K vertex}; \tag{5.22}
\]

\[
y^{(3)} = \frac{s + x}{2} - s^{(3)}, \tag{5.23}
\]

The integer \( x \) expresses the freedom of the solution and labels all possible cubic interaction vertices that can be constructed for the fields under consideration. For vertices (5.20), (5.21) to be sensible,
we impose the restrictions
\[ x \geq 0, \quad y^{(a)} \geq 0, \quad a = 1, 2, 3; \]
\[ s - x \quad \text{even integer}, \quad \text{for K vertex}; \quad (5.24) \]
\[ x \geq 0, \quad y^{(a)} \geq 0, \quad a = 1, 2, 3; \]
\[ s - x \quad \text{odd integer}, \quad \text{for F vertex}; \quad (5.25) \]
which amount to the requirement that the powers of all forms in (5.20), (5.21) be non–negative integers. We note that using relations (5.22), (5.23) allows rewriting the restrictions (5.24), (5.25) as:
\[ \max(0, s^{(3)} - s^{(1)} - s^{(2)}) \leq x \leq s^{(3)} - |s^{(1)} - s^{(2)}|, \quad \text{for K vertex}; \quad (5.26) \]
\[ \max(0, s^{(3)} - s^{(1)} - s^{(2)} - 1) \leq x \leq s^{(3)} - 1 - |s^{(1)} - s^{(2)}|, \quad \text{for F vertex}. \quad (5.27) \]
Compared to vertices for three massless fields (4.12), (4.13), vertices (5.20), (5.21) are a non–homogeneous polynomial in \( P^I \). An interesting property of vertices (5.20), (5.21) is that the maximal number of powers of the momentum \( P^I \), denoted by \( k_{\text{max}} \), is independent of \( x \) and is determined only by \( s \):
\[ k_{\text{max}} = s + 1, \quad \text{for K vertex}; \quad (5.28) \]
\[ k_{\text{max}} = s, \quad \text{for F vertex}. \quad (5.29) \]

5.2 Cubic vertices for one massless fermionic field, one massless bosonic field and one massive fermionic field

We proceed with the cubic interaction vertex for three fields with the mass values
\[ m_1|_{B} = m_2|_{F} = 0, \quad m_3|_{F} \neq 0, \quad (5.30) \]
i.e. the massless bosonic field carries external line index \( a = 1 \), the massless fermionic field carries external line index \( a = 2 \), and the massive fermionic field corresponds to \( a = 3 \). Equations for the vertex involving two massless fields can be obtained from Eqs. (3.41) in the limit as \( m_1 \to 0, m_2 \to 0 \). The general solution for vertex is found to be
\[ p^{(3)} = K^{(23)} V^K(B^{(3)}; Q^{(aa+1)}), \quad \text{K vertex}, \quad (5.31) \]
\[ p^{(3)} = F^{(1)} V^F(B^{(3)}; Q^{(aa+1)}), \quad \text{F vertex}, \quad (5.32) \]
where we use the notation
\[ K^{(23)} = \frac{1}{\beta_2 \beta_3} p_{\theta_2} (\mathbb{P}^I \gamma^I \gamma_s + \beta_2 m_3) \eta_3, \quad (5.33) \]
\[ F = F^{(1)} - \frac{2}{m_3} K^{(23)} B^{(1)}, \quad (5.34) \]
\[ F^{(1)} = \frac{1}{\beta_2 \beta_3} p_{\theta_2} \left( \beta_1 \mathbb{P}^I \gamma^I - \gamma^I \mathbb{P}^I + \beta_2 m_3 \gamma^I \gamma_s \right) \eta_3 \alpha^{(1)} I, \quad (5.35) \]

\[ \text{If } x = 0, \text{ then restrictions } (5.26) \text{ become the restrictions well known in the angular momentum theory: } |s^{(1)} - s^{(2)}| \leq s^{(3)} \leq s^{(1)} + s^{(2)}, \text{ while restriction } (5.27) \text{ takes the form } |s^{(1)} - s^{(2)}| \leq s^{(3)} - 1 \leq s^{(1)} + s^{(2)}. \]

\[ \text{Expressions for } K^{(12)}, F, \text{ and } Q^{(aa+1)} \text{ imply that } k_{\text{max}} = 1 + x + 2 \sum_{a=1}^{3} y^{(a)}. \text{ Taking expressions for } y^{(a)} \text{ into account we find } (5.28), (5.29). \]
These equations tell us that the vertex must be a degree-

tive oscillators. Taking into account that the forms

and homogeneous polynomials in the oscillators we find the general solution of Eqs. (5.18),
is easy to understand that the vertex we are interested in must satisfy the equations

The massless spin-

the respective ket-vectors

while the ket-vector

that describe an interaction of the towers of bosonic and fermionic fields. We now obtain

In this section, we restrict ourselves to cubic vertices for fields with fixed spin values. Vertices

Cubic interaction vertices for fields with fixed but arbitrary spin values

In this section, we restrict ourselves to cubic vertices for fields with fixed spin values. Vertices

(5.31),(5.32) describe an interaction of the towers of bosonic and fermionic fields. We now obtain

vertices for fields with fixed spin values.

The massless spin-

bosonic field and massless spin-

fermionic field are described by

The ket-vectors of massless fields

while the ket-vector of the massive field

are defined in (4.8).

To understand the remaining characteristic properties of solution (5.31),(5.32), we consider vertices for fields with fixed spin values.
\[ p_3^- (s^{(1)}, s^{(2)} + \frac{1}{2}, s^{(3)} + \frac{1}{2}; x) = K^{(23)} (B^{(3)})^x \prod_{a=1}^{3} (Q^{(a+1)}) y^{(a+2)}, \quad \text{K vertex}, \quad (5.44) \]

\[ p_3^- (s^{(1)}, s^{(2)} + \frac{1}{2}, s^{(3)} + \frac{1}{2}; x) = F(B^{(3)})^x \prod_{a=1}^{3} (Q^{(a+1)}) y^{(a+2)}, \quad \text{F vertex}, \quad (5.45) \]

where integers \( y^{(a)} \) are expressible in terms of the \( s^{(a)} \) and an integer \( x \) by the relations

\[ y^{(a)} = \frac{s - x}{2} - s^{(a)}, \quad a = 1, 2, \quad \text{for K vertex}; \quad (5.46) \]

\[ y^{(3)} = \frac{s + x}{2} - s^{(3)}, \quad \text{for K vertex}; \]

\[ y^{(1)} = \frac{s - x + 1}{2} - s^{(1)}, \quad \text{for F vertex}; \]

\[ y^{(2)} = \frac{s - x - 1}{2} - s^{(2)}, \quad \text{for F vertex}; \]

\[ y^{(3)} = \frac{s + x - 1}{2} - s^{(3)}, \quad \text{for F vertex}; \quad (5.47) \]

The integer \( x \) expresses the freedom of the solution and labels all possible cubic interaction vertices that can be constructed for the fields under consideration. For vertices (5.20), (5.21) to be sensible, we impose the restrictions

\[ x \geq 0, \quad y^{(a)} \geq 0, \quad a = 1, 2, 3; \]

\[ s - x \quad \text{even integer}, \quad \text{for K vertex}; \quad (5.48) \]

\[ x \geq 0, \quad y^{(a)} \geq 0, \quad a = 1, 2, 3; \]

\[ s - x \quad \text{odd integer}, \quad \text{for F vertex}; \quad (5.49) \]

which amount to the requirement that the powers of all forms in (5.44), (5.45) be non-negative integers. We note that using relations (5.46), (5.47) allows rewriting the restrictions (5.48), (5.49) as

\[ \max(0, s^{(3)} - s^{(1)} - s^{(2)}) \leq x \leq s^{(3)} - |s^{(1)} - s^{(2)}|, \quad \text{for K vertex}; \quad (5.50) \]

\[ \max(0, s^{(3)} - s^{(1)} - s^{(2)} + 1) \leq x \leq s^{(3)} - |s^{(1)} - s^{(2)} - 1|, \quad \text{for F vertex}. \quad (5.51) \]

The maximal number of powers of the momentum \( P^I \), denoted by \( k_{\text{max}} \), is independent of \( x \) and is determined only by \( s \),

\[ k_{\text{max}} = s + 1, \quad \text{for K vertex}; \quad (5.52) \]

\[ k_{\text{max}} = s, \quad \text{for F vertex}. \quad (5.53) \]

\[ \text{If } x = 0, \text{ then restrictions (5.50) becomes the restrictions well known in the angular momentum theory: } |s^{(1)} - s^{(2)}| \leq s^{(3)} \leq s^{(1)} + s^{(2)}, \text{ while restriction (5.51) takes the form } |s^{(1)} - s^{(2)} - 1| \leq s^{(3)} \leq s^{(1)} + s^{(2)} - 1 \]
5.3 Cubic vertices for one massless bosonic field and two massive fermionic fields with the same mass values

The case under consideration is most interesting because it involves the Yang-Mills and gravitational interactions of massive arbitrary spin fields as particular cases. We now consider the cubic interaction vertex for one massless bosonic field and two massive fermionic fields with the same mass values,

\[ m_1 |_\nu = m_2 |_\nu \equiv m \neq 0, \quad m_3 |_\beta = 0, \quad (5.54) \]

i.e. the massive fermionic fields carry external line indices \( a = 1, 2 \), while the massless bosonic field corresponds to \( a = 3 \). The analysis of equations for the vertex is straightforward and the general solution is found to be

\[ p_{[3]}^- = K^{(12)} V^K (L^{(1)}, L^{(2)}, B^{(3)}; Q^{(12)}; Z), \quad \text{K vertex}; \quad (5.55) \]
\[ p_{[3]}^- = F_0^{(3)} V^F (L^{(1)}, L^{(2)}, B^{(3)}; Q^{(12)}; Z), \quad \text{F vertex}; \quad (5.56) \]

where we use the notation

\[ K^{(12)} = \frac{1}{\beta_1 \beta_2} p_{\theta_1} \left( \beta_3 P^I \gamma^I \gamma_\ast + m \bar{\beta}_3 \right) \eta_2 \quad (5.57) \]
\[ F_0^{(3)} = \frac{1}{\beta_1 \beta_2} p_{\theta_1} \left( \frac{\bar{\beta}_3 \beta_1 P^I}{\beta_3} - \gamma^I P^I - m \bar{\beta}_3 \gamma^I \gamma_\ast \right) \eta_2 \alpha^{(3)I}, \quad (5.58) \]
\[ L^{(1)} \equiv B^{(1)} - \frac{1}{2} m \alpha^{(1)}, \quad L^{(2)} \equiv B^{(2)} + \frac{1}{2} m \alpha^{(2)}, \quad (5.59) \]
\[ B^{(a)} \equiv \alpha^{(a)I} P^I - \frac{\bar{\beta}_a}{2 \beta_a} m \alpha^{(a)}, \quad a = 1, 2; \quad (5.60) \]
\[ B^{(3)} \equiv \alpha^{(3)I} P^I \beta_3, \quad (5.61) \]
\[ Q^{(12)} \equiv \alpha^{(12)} - \frac{\alpha^{(2)}}{m} B^{(1)} + \frac{\alpha^{(1)}}{m} B^{(2)}, \quad (5.62) \]
\[ Z \equiv L^{(1)} \alpha^{(22)} + L^{(2)} \alpha^{(31)} + B^{(3)} \left( \alpha^{(12)} - \alpha^{(1)} \alpha^{(2)} \right), \quad (5.63) \]

and \( \alpha^{(ab)} \) are defined in (4.8). Thus, we see that vertices \((5.55),(5.56)\) depend, among other things, on linear form \( B^{(3)} \) (5.61), which is degree-1 homogeneous polynomial in the momentum \( P^I \). This implies that cubic interaction vertices that are homogeneous polynomials in \( P^I \) can be constructed for certain fields. This also implies that the minimal number of powers of \( P^I \) in \((5.55),(5.56)\) is not equal to zero in general (for example, the dependence on \( B^{(3)} \) leads to an increasing number of powers of the momentum \( P^I \)). All the remaining forms that depend on the momentum \( P^I \) and enter the cubic vertex (the fermionic forms \( K^{(12)} \), \( F \), the linear forms \( L^{(1)}, L^{(2)} \) and the quadratic forms \( Q^{(12)} \) are non-homogeneous polynomials in \( P^I \). To discuss the remaining important properties of solution \((5.55),(5.56)\) we restrict attention to cubic vertices for fields with fixed spin values.

5.3.1 Cubic interaction vertices for fields with fixed but arbitrary spin values

In this section, we restrict ourselves to cubic interaction vertices for the fields with mass values given in (5.54). Vertices \((5.55),(5.56)\) describe interaction of the towers of massive fermionic fields.
Two massive spin fields and one massless bosonic field. This, we consider vertex involving two massive fermionic fields and massless bosonic fields (2.16). We next obtain the vertex for two massive fermionic bosonic field:

\[ m_1 = m, \quad m_2 = m, \quad m \neq 0, \quad m_3 = 0, \]  
(5.64)

Two massive spin fields \( s^{(1)} + \frac{1}{2} \) and \( s^{(2)} + \frac{1}{2} \) fermionic fields are described by the respective ket-vectors \( |\psi_{s^{(1)}}\rangle \) and \( |\psi_{s^{(2)}}\rangle \), while one massless spin-\( s^{(3)} \) bosonic field is described by a ket-vector \( |\phi_{s^{(3)}}\rangle \). The ket-vectors of massive fields \( |\psi_{a(a)}\rangle, a = 1, 2, \) can be obtained from (2.20) by the replacement \( s \rightarrow s^{(a)}, \alpha^I \rightarrow \alpha^{(a)I}, \alpha \rightarrow \alpha^{(a)} \), \( a = 1, 2 \), while the ket-vector of massless field \( |\phi_{s^{(3)}}\rangle \) can be obtained from (2.13) by the replacement \( s \rightarrow s^{(3)}, \alpha^I \rightarrow \alpha^{(3)I} \). Taking into account that the ket-vectors \( |\psi_{a(a)}\rangle, a = 1, 2, \) are the respective degree-\( s^{(a)} \) homogeneous polynomials in the oscillators \( \alpha^{(a)I}, \alpha^{(a)} \), while the ket-vector \( |\phi_{s^{(3)}}\rangle \) is a degree-\( s^{(3)} \) homogeneous polynomial in the oscillator \( \alpha^{(3)I} \) it is easy to understand that the vertex we are interested in must satisfy the equations

\[ (\alpha^{(a)I} \tilde{\alpha}^{(a)I} + \alpha^{(a)} \tilde{\alpha}^{(a)} - s^{(a)}) |p_{[3]}^-\rangle = 0, \quad a = 1, 2, \]  
(5.65)

\[ (\alpha^{(3)I} \tilde{\alpha}^{(3)I} - s^{(3)}) |p_{[3]}^-\rangle = 0. \]  
(5.66)

These equations tell us that the vertex must be a degree-\( s^{(a)} \) homogeneous polynomial in the respective oscillators. Taking into account that the forms \( F_{0}^{(3)}, L^{(3)} \), \( B^{(3)} \) (5.58)-(5.61) are degree 1 homogeneous polynomials in the oscillators, while forms \( Q^{(12)} \) (5.62) and \( Z \) (5.63) are respective degree 2 and 3 homogeneous polynomials in the oscillators we find the general solution of Eqs. (5.65), (5.66) as

\[ p_{[3]}^- (s^{(1)} + \frac{1}{2} s^{(2)} + \frac{1}{2} s^{(3)}; k_{\text{min}}, k_{\text{max}}) \]
\[ = K^{(12)} (L^{(1)}) x^{(1)} (L^{(2)}) x^{(2)} (B^{(3)}) x^{(3)} (Q^{(12)}) y^{(3)} Z y, \quad \text{K vertex;} \]  
(5.67)

\[ = F_{0}^{(3)} (L^{(1)}) x^{(1)} (L^{(2)}) x^{(2)} (B^{(3)}) x^{(3)} (Q^{(12)}) y^{(3)} Z y, \quad \text{F vertex,} \]  
(5.68)

where the parameters \( x^{(1)}, x^{(2)}, x^{(3)}, y^{(3)}, y \) are given by

\[ x^{(1)} = k_{\text{max}} - k_{\text{min}} - s^{(2)} - 1, \]
\[ x^{(2)} = k_{\text{max}} - k_{\text{min}} - s^{(1)} - 1, \]
\[ x^{(3)} = k_{\text{min}}, \]
\[ y^{(3)} = s - 2s^{(3)} - k_{\text{max}} + 2k_{\text{min}} + 1, \]
\[ y = s^{(3)} - k_{\text{min}}, \forall \text{K vertices;} \]  
(5.69)

\[ x^{(1)} = k_{\text{max}} - k_{\text{min}} - s^{(2)} - 1, \]
\[ x^{(2)} = k_{\text{max}} - k_{\text{min}} - s^{(1)} - 1, \]
\[ x^{(3)} = k_{\text{min}}, \]
\[ y^{(3)} = s - 2s^{(3)} - k_{\text{max}} + 2k_{\text{min}} + 2, \]
\[ y = s^{(3)} - k_{\text{min}} - 1, \forall \text{F vertices;} \]  
(5.70)
and \( s \) is defined in \( 4.14 \). New integers \( k_{\text{min}} \) and \( k_{\text{max}} \) in \( 5.67 \)–\( 5.70 \) are the freedom in our solution. In general, vertices \( 5.67 \)–\( 5.68 \) are non-homogeneous polynomials in the momentum \( \mathbb{P}^I \) and the integers \( k_{\text{min}} \) and \( k_{\text{max}} \) are the respective minimal and maximal numbers of powers of the momentum \( \mathbb{P}^I \) in \( 5.67 \)–\( 5.68 \). As noted above, the minimal number of powers of the momentum \( \mathbb{P}^I \) is not equal to zero in general. For vertices \( 5.67 \)–\( 5.68 \) to be sensible, we should impose the restrictions

\[
x^{(a)} \geq 0, \quad a = 1, 2, 3; \quad y^{(3)} \geq 0, \quad y \geq 0,
\]

which amount to requiring the powers of all forms in \( 5.67 \)–\( 5.68 \) to be non-negative integers. Using \( 5.69 \)–\( 5.70 \), restrictions \( 5.71 \) can be rewritten in a more convenient form as

\[
k_{\text{min}} + \max_{a=1,2} s^{(a)} + 1 \leq k_{\text{max}} \leq s - 2s^{(3)} + 2k_{\text{min}} + 1,
\]

for \( K \) vertices; \( 5.72 \)

\[
k_{\text{min}} + \max_{a=1,2} s^{(a)} + 1 \leq k_{\text{max}} \leq s - 2s^{(3)} + 2k_{\text{min}} + 2,
\]

for \( F \) vertices. \( 5.73 \)

5.3.2 Interaction of massive arbitrary spin fermionic field with massless scalar field

Before to study of Yang-Mills and gravitational interactions of fermionic fields we consider interaction vertices of fermionic fields with scalar field. This, we present the list of all cubic vertices for the massive arbitrary spin-\( (s + \frac{1}{2}) \) fermionic field interacting with the massless scalar field, i.e. we consider vertices \( 5.67 \), \( 5.68 \) with

\[
s^{(1)} = s^{(2)} = s, \quad s^{(3)} = 0.
\]

\( 5.74 \)

Allowed spin values and powers of derivatives \( k_{\text{min}}, k_{\text{max}} \) should satisfy restrictions in \( 5.72 \), \( 5.73 \). Plugging \( s^{(3)} = 0 \) in the last restriction in \( 5.73 \) we immediately learn that F-vertices are not allowed. Plugging \( s^{(3)} = 0 \) in the last restriction in \( 5.72 \) we learn that all allowed K-vertices take \( k_{\text{min}} = 0 \). Plugging \( k_{\text{min}} = 0 \) in the 1st restrictions in \( 5.72 \) leads to the following allowed value of \( k_{\text{max}} \):

\[
s + 1 \leq k_{\text{max}} \leq 2s + 1.
\]

\( 5.75 \)

Relations \( 5.67 \), \( 5.69 \) lead to the following interactions vertices

\[
p_{(3)}^{-} (s + \frac{1}{2}, s + \frac{1}{2}, 0; 0, k_{\text{max}}) = K^{(12)} (L^{(1)} L^{(2)})^{k_{\text{max}} - s - 1} (Q^{(12)})^{2s + 1 - k_{\text{max}}}.
\]

\( 5.76 \)

Vertices \( 5.76 \) supplemented with inequalities for allowed values for \( k_{\text{max}} \) \( 5.75 \) constitute complete list of vertices that can be built for two massive spin-\( (s + \frac{1}{2}) \) fermionic fields and one massless scalar field.

For the case of \( s = 0 \) we obtain only one allowed value of \( k_{\text{max}} = 1 \) and the corresponding vertex

\[
p_{(3)}^{-} (\frac{1}{2}, \frac{1}{2}, 0; 0, 1) = K^{(12)}
\]

\( 5.77 \)

\footnote{This can be checked by taking into account that the forms \( K^{(12)}, F^{(3)}, L^{(1)}, L^{(2)}, Q^{(12)} \) and \( Z \) are degree 1 polynomials in \( \mathbb{P}^I \), while the form \( B^{(3)} \) is degree 1 homogeneous polynomial in \( \mathbb{P}^I \) (see \( 5.57 \)–\( 5.63 \)).}
describes interaction of spin-$\frac{1}{2}$ fermionic field with scalar field. Appropriate covariant vertex is well known, $\mathcal{L} = \bar{\psi} \gamma^\mu \psi \phi$. For the case of $s > 0$ we obtain $s + 1$ interaction vertices labelled by allowed values of $k_{\text{max}}$ (5.75). Note that form $Q^{(12)}$ (5.62) does not have a smooth massless limit ($m \to 0$). This implies that the interaction of the massive spin $s + \frac{1}{2} \geq \frac{3}{2}$ field (5.81) does not admit a sensible massless limit when $k_{\text{max}} < 2s + 1$. In other words there are $s$ vertices with $s + 1 \leq k_{\text{max}} < 2s + 1$ which do not have a smooth massless limit and one vertex with $k_{\text{max}} = 2s + 1$ which has smooth massless limit. For example covariant vertex of scalar field with spin-$\frac{1}{2}$ fermionic field which has smooth massless limit takes the form $\mathcal{L} = \bar{\Psi}^{AB} \Psi^{AB} \phi$, where $\Psi^{AB}$ is defined in (4.25).

5.3.3 Yang-Mills interaction of massive arbitrary spin fermionic field

We now apply our results in Section 5.3.1 to the discussion of the Yang-Mills interaction of the massive arbitrary spin fermionic fields (16). It turns out that Yang-Mills interaction of massive fermionic fields are described by $F$-vertices. Therefore we restrict our attention to the $F$-vertices in what follows. We first present the list of all cubic $F$-vertices for the massive arbitrary spin-$s + \frac{1}{2}$ fermionic field interacting with the massless spin-1 field (Yang-Mills field). This is, we consider the $F$-vertices (5.68) with

$$ s^{(1)} = s^{(2)} = s, \quad s^{(3)} = 1. \quad (5.78) $$

The 2nd restrictions in (5.73) lead to one allowed value of $k_{\text{min}} = 0$. Substituting this value of $k_{\text{min}}$ in the 1st inequalities in (5.73), we obtain

$$ s + 1 \leq k_{\text{max}} \leq 2s + 1, \quad k_{\text{min}} = 0, \quad s \geq 0; \quad (5.79) $$

We now discuss those vertices from the list in (5.79) that correspond to the Yang-Mills interaction of the massive arbitrary spin field. We consider various spin fields in turn.

a) Spin-$\frac{1}{2}$ field ($s = 0$). Plugging $s = 0$ in (5.79) we obtain $k_{\text{max}} = 1$ and therefore the cubic vertex of the Yang-Mills interaction of the massive spin-$\frac{1}{2}$ fermionic field is a degree 1 non-homogeneous polynomial in derivatives. Relations (5.68),(5.70) lead to the Yang-Mills interaction of the massive spin-$\frac{1}{2}$ field

$$ p_{[3]}^{-}(\frac{1}{2}, \frac{1}{2}, 1; 0, 1) = F_0^{(3)}(Q^{(12)}). \quad (5.80) $$

b) Spin $s + \frac{1}{2} \geq \frac{3}{2}$ field. All vertices given in (5.79) are candidates for the Yang-Mills interaction of the fermionic spin $s + \frac{1}{2} \geq \frac{3}{2}$ field. We therefore impose an additional requirement, which allows us to choose one suitable vertex: given value of $s$, we look for the vertex with the minimal value of $k_{\text{max}}$. It can be seen that such a vertex is given by $k_{\text{max}} = s + 1$. Taking into account that $k_{\text{min}} = 0$ we obtain from (5.68),(5.70) the Yang-Mills interaction of the massive spin $s + \frac{1}{2} \geq \frac{3}{2}$ field (17),

$$ p_{[3]}^{-}(s + \frac{1}{2}, s + \frac{1}{2}, 1; 0, s) = F_0^{(3)}(Q^{(12)})^s, \quad s \geq 1. \quad (5.81) $$

A few remarks are in order.

16 Discussion of gauge invariant formulation of massive arbitrary spin fermionic fields in constant electromagnetic field may be found in (72).

17 A gauge invariant description of the electromagnetic interaction of the massive spin-2 field was obtained in (63). The derivation of the electromagnetic interaction of massive spin $s = 2, 3$ fields from string theory is given in (64, 65). In these references, the electromagnetic field is treated as an external (non-dynamical) field.
i) The forms $F_0^{(3)}$ has smooth massless limit ($m \to 0$). Therefore, the Yang-Mills interaction of the massive spin-$\frac{1}{2}$ fermionic field given in (5.80) has a smooth massless limit, as this should be. This interaction in the massless limit coincides with the respective interaction of the massless spin-$\frac{1}{2}$ fermionic field in Table I.

ii) The form $Q^{(12)}$ does not have a smooth massless limit ($m \to 0$). This implies that the Yang-Mills interaction of the massive spin-$s + \frac{1}{2} > \frac{3}{2}$ field (5.81) does not admit a sensible massless limit; in light-cone approach, it is contribution of $Q^{(12)}$ that explains why the Yang-Mills interaction of the massive spin-$s + \frac{1}{2} > \frac{3}{2}$ field does not admit the massless limit. As was expected, the Yang-Mills interaction of the massive spin-$s + \frac{1}{2} > \frac{3}{2}$ field (5.81) involves higher derivatives.

To make contact with covariant vertices we now present some covariant vertices corresponding to above given light-cone vertices. This is, the minimal YM interaction of spin-$\frac{3}{2}$ field corresponding to the light-cone vertex (5.81) with $s = 1$ takes the form

$$
\mathcal{L}_{YM} = (\bar{\psi}^B \gamma^A \psi^B + \bar{\psi}_0 \gamma^A \psi_0) \phi^A + \frac{1}{m} (\frac{d-2}{d-1})^{1/2} (\bar{\psi}^A \gamma^B \psi_0 - \bar{\psi}_0 \gamma^A \psi^B) F^{AB} + \frac{1}{(d-1)m} \bar{\psi}_0 \gamma^{AB} \psi_0 F^{AB},
$$

(5.82)

where $\psi^A$ and $\psi_0$ are the respective spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ fields which are used in the framework of gauge invariant formulation of massive spin-$\frac{3}{2}$ field\(^{18}\).

Another covariant vertex that corresponds to light-cone K-vertex (see (5.67))

$$
p_{[3]}(\frac{3}{2}, \frac{3}{2}, 1; 0, 2) = K^{(12)} Z,
$$

(5.83)

is given by

$$
\mathcal{L}_{high-der} \equiv (\bar{\psi}^A \psi^B - \frac{1}{4} \bar{\psi}^C \gamma^{AB} \psi^C) F^{AB} + \frac{d-3}{\sqrt{(d-1)(d-2)}} (\bar{\psi}^A \gamma^B \psi_0 - \bar{\psi}_0 \gamma^A \psi^B) F^{AB}
$$

$$
- \frac{d^2 - 9d + 16}{4(d-1)(d-2)} \bar{\psi}_0 \gamma^{AB} \psi_0 F^{AB}.
$$

(5.84)

5.3.4 Gravitational interaction of massive arbitrary spin fermionic field

We proceed with the discussion of the gravitational interaction of the massive arbitrary spin fermionic field. We first present the list of all cubic vertices for the massive spin-$(s + \frac{1}{2})$ fermionic field interacting with the massless spin-2 field. This is, we consider vertices (5.68) with

$$
s^{(1)} = s^{(2)} = s, \quad s^{(3)} = 2.
$$

(5.85)

The 2nd restrictions in (5.73) lead to two allowed values of $k_{min}$: $k_{min} = 0, 1$. Plugging these values of $k_{min}$ in the 1st restrictions in (5.73), we obtain two families of vertices

$$
k_{min} = 1, \quad s + 2 \leq k_{max} \leq 2s + 2, \quad s \geq 0; \quad (5.86)
$$

$$
k_{min} = 0, \quad s + 1 \leq k_{max} \leq 2s, \quad s \geq 1; \quad (5.87)
$$

\(^{18}\) We derived covariant Lagrangian (5.82) by using the on-shell gauge invariant formulation described in Ref.[69] (for discussion of off-shell gauge invariant formulation of arbitrary spin massive fermionic fields see [70].

31
We now discuss those vertices from the list given in (5.86), (5.87) that correspond to the gravitational interaction of the massive arbitrary spin fermionic fields. We consider various spin fields in turn.

a) Spin-1/2 field ($s = 0$). The gravitational interaction of the massive spin-1/2 fermionic field is given by (5.86). Plugging $s = 0$ in (5.86), we obtain the well-known relation $k_{\text{max}} = 2$, which tells us that the cubic vertex of the gravitational interaction of the massive spin-1/2 fermionic field is a degree 2 non-homogeneous polynomial in the derivatives. Formulas (5.68), (5.70) lead to the gravitational interaction of the massive spin-1/2 fermionic field,

$$p_{[3]}(\frac{1}{2}, \frac{1}{2}; 1, 2) = F_0^{(3)} B^{(3)}.$$  \hspace{1cm} (5.88)

b) Spin 3/2 field ($s = 1$). The gravitational interaction vertex of the massive spin-3/2 is given in (5.87). Plugging $s = 1$ in (5.87) we obtain $k_{\text{max}} = 2$. Formulas (5.68), (5.70) then lead to the gravitational interaction of the massive spin-3/2 field

$$p_{[3]}(\frac{3}{2}, \frac{3}{2}; 0, 2) = F_0^{(3)} Z.$$ \hspace{1cm} (5.89)

c) Higher-spin fermionic fields, ($s + \frac{1}{2} \geq \frac{5}{2}$). All vertices given in (5.86), (5.87) are candidates for the gravitational interaction of the higher-spin fermionic fields. We should impose some additional requirement that would allow us to choose one suitable vertex. Our additional requirement is that given a spin-($s + \frac{1}{2}$), we look for vertex with the minimal value of $k_{\text{max}}$. It can be seen that such a vertex is given by (5.87) with $k_{\text{max}} = s + 1$. We note that $k_{\text{min}} = 0$ and relations (5.68), (5.70) lead to the gravitational interaction of the massive higher-spin fermionic field,

$$p_{[3]}(s + \frac{1}{2}, s + \frac{1}{2}; 0, s + 1) = F_0^{(3)} Z(Q^{(12)})^{s-1}, \quad s \geq 2.$$ \hspace{1cm} (5.90)

A few remarks are in order.

i) Since forms $F_0^{(3)}$ (5.58), $B^{(3)}$ (5.61), and $Z$ (5.63) have a smooth massless limit ($m \to 0$), the gravitational interactions of the massive low spin $\frac{1}{2}$, $\frac{3}{2}$ fermionic fields (5.88), (5.89) have a smooth massless limit, as they should. These gravitational interactions in the massless limit reduce to the corresponding interactions of the massless spin $\frac{1}{2}$, $\frac{3}{2}$ fermionic fields given in Table I.

ii) Since the form $Q^{(12)}$ (5.62) does not have a smooth massless limit ($m \to 0$), the gravitational interaction of the massive higher-spin fermionic field (5.90) does not admit a sensible massless limit; it is the form $Q^{(12)}$ that explains why the gravitational interaction of the massive higher spin field does not admit the massless limit. Higher derivatives in the gravitational interaction of the massive higher-spin fermionic field are related to the contribution of $Q^{(12)}$ (5.62).  \[19\]

5.3.5 Interaction of arbitrary spin massless bosonic field with two massive spin-$\frac{1}{2}$ fermionic fields

We finish this section with the discussion of the interaction vertices for arbitrary spin-$s$ bosonic massless field and massive spin-$\frac{1}{2}$ fermionic fields.\[20\] This is, we consider vertices (5.68) with

$$s^{(1)} = s^{(2)} = 0, \quad s^{(3)} = s.$$ \hspace{1cm} (5.91)

\[19\] Gauge invariant formulations of the gravitational interaction of massive fields are studied e.g. in [66]. Interesting discussion of various aspects of the massive spin 2 field in gravitational background may be found in [67, 68].

\[20\] Discussion of vertices for arbitrary spin massless bosonic field and two massive scalar fields in framework of BRST invariant approach may be found in [71].
Plugging these spin values in restrictions (5.72), (5.73) we find that \( k_{\text{min}} \) and \( k_{\text{max}} \) turn out to be fixed uniquely and there are only one allowed values of \( k_{\text{min}} \) and \( k_{\text{max}} \) for K-vertices and one allowed values of \( k_{\text{min}} \) and \( k_{\text{max}} \) for F-vertices:

\[
\begin{align*}
  k_{\text{min}} &= s, & k_{\text{max}} &= s + 1, & \text{for K vertices,} & (5.92) \\
  k_{\text{min}} &= s - 1, & k_{\text{max}} &= s, & \text{for F vertices.} & (5.93)
\end{align*}
\]

Making use of relations (5.67)-(5.70) leads to the respective K- and F- interaction vertices

\[
\begin{align*}
  p_{[3]} \left( \frac{1}{2}, \frac{1}{2}, s; s, s + 1 \right) &= K^{(12)} (B^{(3)})^s, \quad (5.94) \\
  p_{[3]} \left( \frac{1}{2}, \frac{1}{2}, s; s - 1, s \right) &= F^{(3)}_0 (B^{(3)})^{s-1}. \quad (5.95)
\end{align*}
\]

Both these vertices have smooth massless limit.

5.4 Cubic vertices for one massless fermionic field, one massive fermionic field and one bosonic field with the same mass value

We now consider the cubic interaction vertex for one massless field and two massive fields with the same mass values,

\[
m_1|_a = m_2|_F \equiv m \neq 0, \quad m_3|_F = 0, \quad (5.96)
\]

i.e. the massive bosonic field carries external line index \( a = 1 \), one massive fermionic field carries external line index \( a = 2 \), while the massless fermionic field corresponds to \( a = 3 \). The analysis of equations for the vertex is straightforward and the general solution is found to be

\[
\begin{align*}
  p_{[3]}^- &= K^{(23)} V^K (L^{(1)}, L^{(2)}, B^{(3)}; Q^{(12)}; Z), \quad \text{K vertex;} \quad (5.97) \\
  p_{[3]}^- &= F V^F (L^{(1)}, L^{(2)}, B^{(3)}; Q^{(12)}; Z), \quad \text{F vertex;} \quad (5.98)
\end{align*}
\]

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where we use the notation

\[
K^{(2\ell)} = \frac{1}{\beta_2 \beta_3} p\theta_2 \left( \mathbb{P} \gamma^I \gamma_s - m \beta_3 \right) \eta_3
\]

\[
F = \frac{1}{\beta_2 \beta_3} p\theta_2 \left( \frac{\beta_1}{\beta_1} \mathbb{P} I - \gamma^I \mathbb{P} J + m \beta_3 \gamma^I \gamma_s \right) \eta_3 \alpha^{(1)I} ,
\]

\[
+ \frac{1}{\beta_2 \beta_3} p\theta_2 \left( - \mathbb{P} I \gamma^I \gamma_s + \frac{\beta_1}{\beta_1} \beta_3 m \right) \eta_3 \alpha^{(1)} ,
\]

\[
L^{(1)} \equiv B^{(1)} - \frac{1}{2} m \alpha^{(1)}, \quad L^{(2)} \equiv B^{(2)} + \frac{1}{2} m \alpha^{(2)},
\]

\[
B^{(a)} \equiv \frac{\alpha^{(a)I} \mathbb{P} I}{\beta_a} - \frac{\beta_a}{2 \beta_a} m \alpha^{(a)}, \quad a = 1, 2;
\]

\[
B^{(3)} \equiv \frac{\alpha^{(3)I} \mathbb{P} I}{\beta_3},
\]

\[
Q^{(12)} \equiv \alpha^{(12)} - \frac{\alpha^{(2)}}{m} B^{(1)} + \frac{\alpha^{(1)}}{m} B^{(2)},
\]

\[
Z \equiv L^{(1)} \alpha^{(23)} + L^{(2)} \alpha^{(31)} + B^{(3)} (\alpha^{(12)} - \alpha^{(1)} \alpha^{(2)}),
\]

To discuss the remaining important properties of solution (5.97), (5.98) we restrict attention to cubic vertices for fields with fixed spin values.

### 5.4.1 Cubic interaction vertices for fields with fixed but arbitrary spin values

We now restrict ourselves to cubic vertices for the fields with fixed but arbitrary spin values and with mass values given in (5.96). Vertices (5.97), (5.98) describe interaction of the towers of fermionic and bosonic fields. We next obtain the vertex for one massive bosonic field, one massive fermionic field and one massless fermionic field. This, we consider vertex involving one massive bosonic spin-$s^{(1)}$ field, one massive fermionic spin-$(s^{(2)} + \frac{1}{2})$ field and one massless spin-$(s^{(3)} + \frac{1}{2})$ fermionic field, where bosonic and fermionic fields have the same mass parameter $m$:

\[
m_1 = m, \quad m_2 = m, \quad m \neq 0, \quad m_3 = 0,
\]

\[
s^{(1)}, \quad s^{(2)} + \frac{1}{2}, \quad s^{(3)} + \frac{1}{2}.
\]

Massive bosonic spin-$s^{(1)}$ and fermionic spin-$(s^{(2)} + \frac{1}{2})$ fields are described by the respective ket-vectors $|\phi_{s^{(1)}}\rangle$, $|\psi_{s^{(2)}}\rangle$, while the massless spin-$(s^{(3)} + \frac{1}{2})$ fermionic field is described by ket-vector $|\psi_{s^{(3)}=0}\rangle$. The ket-vectors $|\phi_{s^{(1)}}\rangle$, $|\psi_{s^{(2)}}\rangle$ can be obtained from the respective expressions in (2.9), (2.20) by the replacement $s \rightarrow s^{(a)}$, $\alpha^I \rightarrow \alpha^{(a)I}$, $\alpha \rightarrow \alpha^{(a)}$, $a = 1, 2$, while the ket-vector $|\psi_{s^{(3)}=0}\rangle$ can be obtained from (2.27) by the replacement $s \rightarrow s^{(3)}$, $\alpha^I \rightarrow \alpha^{(3)I}$. Taking into account that the ket-vectors $|\phi_{s^{(1)}}\rangle$, $|\psi_{s^{(2)}}\rangle$ are the respective degree-$s^{(1)}$ homogeneous polynomials in the oscillators $\alpha^{(a)I}$, $\alpha^{(a)}$, while the ket-vector $|\psi_{s^{(3)}=0}\rangle$ is a degree-$s^{(3)}$ homogeneous polynomial in the oscillator $\alpha^{(3)I}$ it is easy to understand that the vertex we are interested in must satisfy the equations

\[
(\alpha^{(a)I} \tilde{\alpha}^{(a)I} + \alpha^{(a)} \tilde{\alpha}^{(a)} - s^{(a)}) |\tilde{p}_{s^{(a)}}\rangle = 0, \quad a = 1, 2,
\]

\[
(\alpha^{(3)I} \tilde{\alpha}^{(3)I} - s^{(3)}) |\tilde{p}_{s^{(3)}}\rangle = 0.
\]
These equations tell us that the vertex must be a degree-$s^{(a)}$ homogeneous polynomial in the respective oscillators. Taking into account that the forms $F$, $L^{(1)}$, $L^{(2)}$, $B^{(3)}$, $Q^{(12)}$, $Z$ are degree-1 homogeneous polynomials in the oscillators, while forms $Q^{(12)}$, $Z$ are respective degree 2 and 3 homogeneous polynomials in the oscillators we find the general solution of Eqs. (5.107), (5.108) as

$$p[3](s^{(1)}, s^{(2)} + \frac{1}{2}, s^{(3)} + \frac{1}{2} ; k_{\text{min}}, k_{\text{max}})$$

$$= K^{(23)}(L^{(1)})x^{(1)}(L^{(2)})x^{(2)}(B^{(3)})x^{(3)}(Q^{(12)})y^{(3)}Zy, \quad \text{K vertex; (5.109)}$$

$$= F(L^{(1)})x^{(1)}(L^{(2)})x^{(2)}(B^{(3)})x^{(3)}(Q^{(12)})y^{(3)}Zy, \quad \text{F vertex, (5.110)}$$

where the parameters $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, $y^{(3)}$, $y$ are given by

$$x^{(1)} = k_{\text{max}} - k_{\text{min}} - s^{(2)} - 1,$$

$$x^{(2)} = k_{\text{max}} - k_{\text{min}} - s^{(1)} - 1,$$

$$x^{(3)} = k_{\text{min}},$$

$$y^{(3)} = s - 2s^{(3)} - k_{\text{max}} + 2k_{\text{min}} + 1,$$

$$y = s^{(3)} - k_{\text{min}}, \quad \text{for K vertices; (5.111)}$$

$$x^{(1)} = k_{\text{max}} - k_{\text{min}} - s^{(2)} - 1,$$

$$x^{(2)} = k_{\text{max}} - k_{\text{min}} - s^{(1)},$$

$$x^{(3)} = k_{\text{min}},$$

$$y^{(3)} = s - 2s^{(3)} - k_{\text{max}} + 2k_{\text{min}},$$

$$y = s^{(3)} - k_{\text{min}}, \quad \text{for F vertices; (5.112)}$$

and $s$ is defined in (4.14). New integers $k_{\text{min}}$ and $k_{\text{max}}$ in (5.109)-(5.112) are the freedom in our solution. In general, vertices (5.109), (5.110) are non-homogeneous polynomials in the momentum $P^I$ and the integers $k_{\text{min}}$ and $k_{\text{max}}$ are the respective minimal and maximal numbers of powers of the momentum $P^I$ in (5.109), (5.110). As noted above, the minimal number of powers of the momentum $P^I$ is not equal to zero in general. For vertices (5.109), (5.110) to be sensible, we should impose the restrictions

$$x^{(a)} \geq 0, \quad a = 1, 2, 3; \quad y^{(3)} \geq 0, \quad y \geq 0, \quad (5.113)$$

which amount to requiring the powers of all forms in (5.109), (5.110) to be non–negative integers. Using (5.111), (5.112), restrictions (5.113) can be rewritten in a more convenient form as

$$k_{\text{min}} + \max_{a=1,2} s^{(a)} + 1 \leq k_{\text{max}} \leq s - 2s^{(3)} + 2k_{\text{min}} + 1,$$

$$0 \leq k_{\text{min}} \leq s^{(3)}, \quad \text{for K vertices; (5.114)}$$

$$k_{\text{min}} + \max(s^{(1)} - 1, s^{(2)}) + 1 \leq k_{\text{max}} \leq s - 2s^{(3)} + 2k_{\text{min}},$$

$$0 \leq k_{\text{min}} \leq s^{(3)}, \quad \text{for F vertices. (5.115)}$$

\footnote{This can be checked by taking into account that the forms $F$, $K^{(23)}$, $L^{(1)}$, $L^{(2)}$, $Q^{(12)}$ and $Z$ are degree-1 polynomials in $P^I$, while the form $B^{(3)}$ is degree-1 homogeneous polynomial in $P^I$ (see (5.99)-(5.105)).}
5.5 Cubic vertices for one massless bosonic field and two massive fermionic fields with different mass values

We now consider the cubic interaction vertices for fields with the following mass values:

$$m_1|_F \neq 0, \quad m_2|_F \neq 0, \quad m_1 \neq m_2, \quad m_3|_B = 0,$$

(5.116)
i.e. the massive fermionic fields carry external line indices $a = 1, 2$, while the massless bosonic field corresponds to $a = 3$. Equations for the vertex involving one massless field can be obtained from Eqs.(3.41) in the limit as $m_3 \to 0$. The general solution for vertex takes the form

$$p_{[3]} = K^{(12)} V^K (L^{(1)}, L^{(2)}; Q^{(aa+1)}),$$  

(5.117)

$$p_{[3]} = F V^F (L^{(1)}, L^{(2)}; Q^{(aa+1)}),$$  

(5.118)

where we use the notation

$$K^{(12)} \equiv \frac{1}{\beta_1 \beta_2} p_{\theta_1} \left( \mathbb{P}^I \gamma^I \gamma^*_a - m_1 \beta_2 + m_2 \beta_1 \right) \eta_2,$$  

(5.119)

$$F \equiv F_0^{(3)} - \frac{2}{m_1 + m_2} K^{(12)} B^{(3)},$$  

(5.120)

$$F_0^{(3)} \equiv \frac{1}{\beta_1 \beta_2} p_{\theta_1} \left( \frac{\bar{\beta}_1}{\beta_3} \mathbb{P}^I - \gamma^I \gamma^*_J + (m_1 \beta_2 + m_2 \beta_1) \gamma^I \gamma^*_a \right) \eta_2 \alpha^{(3)I},$$  

(5.121)

$$L^{(1)} \equiv B^{(1)} - \frac{m_2^2}{2m_1} \alpha^{(1)}, \quad L^{(2)} \equiv B^{(2)} + \frac{m_1^2}{2m_2} \alpha^{(2)},$$  

(5.122)

$$B^{(a)} \equiv \frac{\alpha^{(a)I} \mathbb{P}^I}{\beta_a} - \frac{\bar{\beta}_a}{2 \beta_a} m_a \alpha^{(a)}, \quad a = 1, 2;$$  

(5.123)

$$B^{(3)} \equiv \frac{\alpha^{(3)I} \mathbb{P}^I}{\beta_3},$$  

(5.124)

$$Q^{(12)} \equiv \alpha^{(12)} - \frac{\alpha^{(2)}}{m_2} B^{(1)} + \frac{\alpha^{(1)}}{m_1} B^{(2)},$$  

(5.125)

$$Q^{(23)} \equiv \alpha^{(23)} - \frac{m_2 \alpha^{(2)}}{m_1^2 - m_2^2} B^{(3)} - \frac{2}{m_1^2 - m_2^2} B^{(2)} B^{(3)},$$  

(5.126)

$$Q^{(31)} \equiv \alpha^{(31)} - \frac{m_1 \alpha^{(1)}}{m_1^2 - m_2^2} B^{(3)} + \frac{2}{m_1^2 - m_2^2} B^{(3)} B^{(1)},$$  

(5.127)

and $\alpha^{(ab)}$ are defined in (4.8). An interesting property of the solution obtained is the appearance of expressions like $m_1^2 - m_2^2$ in the denominators of the quadratic forms $Q^{(23)}$, (5.126) and $Q^{(31)}$, (5.127); the forms $Q^{(23)}$, $Q^{(31)}$ are therefore singular as $m_1 \to m_2$. For this reason, we considered the case of $m_1 = m_2$ separately in Section 5.3.

As can be seen from (5.117)-(5.127), it is impossible to construct a cubic vertex that would be a homogeneous polynomial in the momentum $\mathbb{P}^I$. All forms that depend on $\mathbb{P}^I$ and enter the vertex (i.e. $K^{(12)}$, $F$, $L^{(1)}$, $L^{(2)}$, and $Q^{(aa+1)}$) are non-homogeneous polynomials in $\mathbb{P}^I$. This implies that the cubic vertex is a non-homogeneous polynomial in $\mathbb{P}^I$ in general. To understand the remaining characteristic properties of the solution obtained, we consider the vertices for fields with fixed spin values.
5.5.1 Cubic interaction vertices for fields with fixed but arbitrary mass values

Discussion of cubic interaction vertices for two massive fermionic fields with different mass values and one massless bosonic field largely follows that in Section 5.3.1.

The vertices for two massive spin \( s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2} \) fermionic fields \(|\psi^{s(1)}\rangle, |\psi^{s(2)}\rangle\) with different mass values and one massless bosonic spin-\(s^{(3)}\) field \(|\phi^{m_{(3)}}_{s^{(3)}}\rangle\) can be obtained by solving Eqs. (5.65), (5.66) with \( p_{[3]}^{-}\) given in (5.117), (5.118). We then obtain the cubic vertices

\[
p_{[3]}^{-}(s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2}, s^{(3)}; x^{(1)}, x^{(2)})
= K^{(12)}(L^{(1)})x^{(1)}(L^{(2)})x^{(2)} \prod_{a=1}^{3}(Q^{(a+1)})y^{(a+2)}, \quad K \text{ vertex; (5.128)}
\]

\[
= F(L^{(1)})x^{(1)}(L^{(2)})x^{(2)} \prod_{a=1}^{3}(Q^{(a+1)})y^{(a+2)}, \quad F \text{ vertex; (5.129)}
\]

where the parameters \( y^{(a)} \) are given by

\[
y^{(1)} = \frac{1}{2}(s^{(2)} + s^{(3)} - s^{(1)} + x^{(1)} - x^{(2)}),
\]

\[
y^{(2)} = \frac{1}{2}(s^{(1)} + s^{(3)} - s^{(2)} - x^{(1)} + x^{(2)}),
\]

\[
y^{(3)} = \frac{1}{2}(s^{(1)} + s^{(2)} - s^{(3)} - x^{(1)} - x^{(2)}), \quad \text{for K vertex; (5.130)}
\]

\[
y^{(1)} = \frac{1}{2}(s^{(2)} + s^{(3)} - s^{(1)} + x^{(1)} - x^{(2)} - 1),
\]

\[
y^{(2)} = \frac{1}{2}(s^{(1)} + s^{(3)} - s^{(2)} - x^{(1)} + x^{(2)} - 1),
\]

\[
y^{(3)} = \frac{1}{2}(s^{(1)} + s^{(2)} - s^{(3)} - x^{(1)} - x^{(2)} + 1), \quad \text{for F vertex. (5.131)}
\]

Two integers \( x^{(1)}, x^{(2)} \) are the freedom of our solution. For fixed spin values \( s^{(1)}, s^{(2)}, s^{(3)} \), these integers label all possible cubic interaction vertices that can be built for the fields under consideration. For vertices (5.128), (5.129) to be sensible we impose the restrictions

\[
x^{(1)} \geq 0, \quad x^{(2)} \geq 0, \quad y^{(a)} \geq 0, \quad a = 1, 2, 3; \quad \text{for K vertex; (5.132)}
\]

\[
s - x^{(1)} - x^{(2)} \quad \text{even integer, for K vertex; (5.133)}
\]

\[
x^{(1)} \geq 0, \quad x^{(2)} \geq 0, \quad y^{(a)} \geq 0, \quad a = 1, 2, 3; \quad \text{for F vertex, (5.133)}
\]

which amount to the requirement that the powers of all forms in (5.128), (5.129) be non–negative integers. The maximal number of powers of \( P^I \) in (5.128), (5.129), which is denoted by \( k_{max} \), is
given by

\[
    k_{\max} = \frac{1}{2} (s^{(1)} + s^{(2)} + 3s^{(3)} + x^{(1)} + x^{(2)}) + 1, \quad \text{for K vertex; (5.134)}
\]

\[
    k_{\max} = \frac{1}{2} (s^{(1)} + s^{(2)} + 3s^{(3)} + x^{(1)} + x^{(2)} + 1), \quad \text{for F vertex. (5.135)}
\]

We note that using (5.130), (5.131) allows rewriting restrictions (5.132), (5.133) in the equivalent form

\[
    |s^{(1)} - s^{(2)} - x^{(1)} + x^{(2)}| \leq s^{(3)} \leq s^{(1)} + s^{(2)} - x^{(1)} - x^{(2)}, \quad \text{for K vertex; (5.136)}
\]

\[
    |s^{(1)} - s^{(2)} - x^{(1)} + x^{(2)}| + 1 \leq s^{(3)} \leq s^{(1)} + s^{(2)} - x^{(1)} - x^{(2)} + 1, \quad \text{for F vertex; (5.137)}
\]

### 5.6 Cubic vertices for one massless fermionic field, one massive fermionic field and one massive bosonic field with different mass values

We now consider the cubic interaction vertices for fields with the following mass values:

\[
    m_1|_B \neq 0, \quad m_2|_F \neq 0, \quad m_1 \neq m_2, \quad m_3|_F = 0, \quad (5.138)
\]

i.e. the massive bosonic field carries external line index \(a = 1\), the massive fermionic field carries external line index \(a = 2\), while the massless fermionic field corresponds to \(a = 3\). Equations for the vertex involving one massless field can be obtained from Eqs. (3.41) in the limit as \(m_3 \to 0\). The general solution for vertex takes the form

\[
    p^-_{[3]} = K^{(3)} V^K (L^{(1)}, L^{(2)}; Q^{(a=1)}), \quad \text{K vertex; (5.139)}
\]

\[
    p^-_{[3]} = F V^F (L^{(1)}, L^{(2)}; Q^{(a=1)}), \quad \text{F vertex; (5.140)}
\]

where we use the notation

\[
    K^{(2)} = \frac{1}{\beta_2 \beta_3} \eta_2 \left( \eta^I \gamma^I \gamma^*_s - m_2 \beta_3 \right) \eta_3, \quad (5.141)
\]

\[
    F = \frac{1}{\beta_2 \beta_3} \eta_2 \left( \frac{\beta_1}{\beta_2} \gamma^I \eta^I + \beta_2 \beta_3 \gamma^I \eta^I \right) \eta_3 \alpha^{(1)} \alpha^{(1)} I,
\]

\[
    \quad + \frac{1}{\beta_2 \beta_3} \eta_2 \left( -m_2 \gamma^I \eta^I \gamma^*_s + \beta_3 \gamma^I \eta^I \right) \eta_3 \alpha^{(1)} \alpha^{(1)} I
\]

\[
    L^{(1)} \equiv B^{(1)} = \frac{m_2^2}{2m_1} \alpha^{(1)}, \quad L^{(2)} \equiv B^{(2)} = \frac{m_3^2}{2m_2} \alpha^{(2)}, \quad (5.143)
\]

\[
    B^{(a)} = \frac{\alpha^{(a)} I \eta^I}{\beta_a} - \frac{\beta_a}{2 \beta_a} m_a \alpha^{(a)}, \quad a = 1, 2; \quad (5.144)
\]

\[
    B^{(3)} = \frac{\alpha^{(3)} I \eta^I}{\beta_3}, \quad (5.145)
\]

---

22 Expressions for \(K^{(12)}, F, L^{(a)}\) and \(Q^{(a=1)} \) imply that \(k_{\max} = x^{(3)} + x^{(2)} + y^{(1)} + y^{(2)} + y^{(3)} + 1\) for K vertex and \(k_{\max} = x^{(3)} + x^{(2)} + 2y^{(1)} + 2y^{(2)} + y^{(3)} + 2\) for F vertex. Relations for \(y^{(a)} \) then lead to (5.134), (5.135).

23 If \(x^{(1)} = x^{(2)} = 0\), then restrictions (5.136) becomes the restrictions well known in the angular momentum theory: \(|s^{(1)} - s^{(2)}| \leq s^{(3)} \leq s^{(1)} + s^{(2)}\), while restriction (5.137) takes the form \(|s^{(1)} - s^{(2)}| \leq s^{(3)} - 1 \leq s^{(1)} + s^{(2)}\).
\[ Q^{(12)} \equiv \alpha^{(12)} - \frac{\alpha^{(2)}}{m_2} B^{(1)} + \frac{\alpha^{(1)}}{m_1} B^{(2)} , \]  
\[ Q^{(23)} \equiv \alpha^{(23)} - \frac{m_2 \alpha^{(2)}}{m_1^2 - m_2^2} B^{(3)} - \frac{2}{m_1^2 - m_2^2} B^{(2)} B^{(3)} , \]  
\[ Q^{(31)} \equiv \alpha^{(31)} - \frac{m_1 \alpha^{(1)}}{m_1^2 - m_2^2} B^{(3)} + \frac{2}{m_1^2 - m_2^2} B^{(3)} B^{(1)} , \]  
(5.146)
(5.147)
(5.148)

and \( \alpha^{(ab)} \) are defined in (4.8). To understand the remaining characteristic properties of the solution obtained, we consider the vertices for fields with fixed spin values.

### 5.6.1 Cubic interaction vertices for fields with fixed but arbitrary mass values

Discussion of cubic interaction vertices for one massless fermionic field, one massive fermionic field and one massive bosonic field with different mass values largely follows that in Section 5.4.1.

The vertices for one massive bosonic spin-\( s^{(1)} \) field \( |\phi_{s^{(1)}}\rangle \), one massive fermionic spin-(\( s^{(2)} + \frac{1}{2} \)) field \( |\psi_{s^{(2)}}\rangle \) with different mass values and one massless fermionic spin-(\( s^{(3)} + \frac{1}{2} \)) field \( |\psi_{s^{(3)}}^{m=0}\rangle \) can be obtained by solving Eqs. (5.65), (5.66) with \( p^{[3]} \) given in (5.139), (5.140). We then obtain the cubic vertices

\[ p^{[3]}_{[q]}(s^{(1)}, s^{(2)} + \frac{1}{2}, s^{(3)} + \frac{1}{2}; x^{(1)}, x^{(2)}) \]

\[ = K^{(23)} (L^{(1)}) x^{(1)} (L^{(2)}) x^{(2)} 3 \prod_{a=1} (Q^{(a+1)}) y^{(a+2)} , \quad K \text{ vertex; } \]  
(5.149)
\[ = F (L^{(1)}) x^{(1)} (L^{(2)}) x^{(2)} 3 \prod_{a=1} (Q^{(a+1)}) y^{(a+2)} , \quad F \text{ vertex; } \]  
(5.150)

where the parameters \( y^{(a)} \) are given by

\[ y^{(1)} = \frac{1}{2} (s^{(2)} + s^{(3)} - s^{(1)} + x^{(1)} - x^{(2)} ) , \]
\[ y^{(2)} = \frac{1}{2} (s^{(1)} + s^{(3)} - s^{(2)} - x^{(1)} + x^{(2)} ) , \]
\[ y^{(3)} = \frac{1}{2} (s^{(1)} + s^{(2)} - s^{(3)} - x^{(1)} - x^{(2)} ) , \quad \text{for K vertex; } \]  
(5.151)
\[ y^{(1)} = \frac{1}{2} (s^{(2)} + s^{(3)} - s^{(1)} + x^{(1)} - x^{(2)} + 1) , \]
\[ y^{(2)} = \frac{1}{2} (s^{(1)} + s^{(3)} - s^{(2)} - x^{(1)} + x^{(2)} - 1) , \]
\[ y^{(3)} = \frac{1}{2} (s^{(1)} + s^{(2)} - s^{(3)} - x^{(1)} - x^{(2)} - 1) , \quad \text{for F vertex. } \]  
(5.152)

Two integers \( x^{(1)}, x^{(2)} \) are the freedom of our solution. For fixed spin values \( s^{(1)}, s^{(2)}, s^{(3)} \), these integers label all possible cubic interaction vertices that can be built for the fields under consideration.
For vertices (5.149), (5.150) to be sensible we impose the restrictions

\[ x^{(1)} \geq 0, \quad x^{(2)} \geq 0, \quad y^{(a)} \geq 0, \quad a = 1, 2, 3; \]

\[ s - x^{(1)} - x^{(2)} \text{ even integer}, \quad \text{for K vertex; (5.153)} \]

\[ x^{(1)} \geq 0, \quad x^{(2)} \geq 0, \quad y^{(a)} \geq 0, \quad a = 1, 2, 3; \]

\[ s - x^{(1)} - x^{(2)} \text{ odd integer}, \quad \text{for F vertex. (5.154)} \]

which amount to the requirement that the powers of all forms in (5.149), (5.150) be non-negative integers. The maximal number of powers of \( P_I \) in (5.149), (5.150), which is denoted by \( k_{\text{max}} \), is given by

\[ k_{\text{max}} = \frac{1}{2} (s^{(1)} + s^{(2)} + 3s^{(3)} + x^{(1)} + x^{(2)}) + 1, \quad \text{for K vertex; (5.155)} \]

\[ k_{\text{max}} = \frac{1}{2} (s^{(1)} + s^{(2)} + 3s^{(3)} + x^{(1)} + x^{(2)} + 1), \quad \text{for F vertex. (5.156)} \]

We note that using (5.151), (5.152) allows rewriting restrictions (5.153), (5.154) in the equivalent form

\[ |s^{(1)} - s^{(2)} - x^{(1)} + x^{(2)}| \leq s^{(3)} \leq s^{(1)} + s^{(2)} - x^{(1)} - x^{(2)}, \quad \text{for K vertex; (5.157)} \]

\[ |s^{(1)} - s^{(2)} - x^{(1)} + x^{(2)} - 1| \leq s^{(3)} \leq s^{(1)} + s^{(2)} - x^{(1)} - x^{(2)} - 1, \quad \text{for F vertex; (5.158)} \]

6 Cubic interaction vertices for two massive fermionic fields and one massive bosonic field

We finally consider the cubic interaction vertex for three massive fields:

\[ m_1|_F \neq 0, \quad m_2|_F \neq 0, \quad m_3|_B \neq 0. \quad (6.1) \]

The general solution for vertex is found to be

\[ p_{[3]}^K = K^{(12)} V^K (L^{(a)}; Q^{(a+1)}), \quad \text{K vertex; (6.2)} \]

\[ p_{[3]}^F = F V^F (L^{(a)}; Q^{(a+1)}), \quad \text{F vertex; (6.3)} \]

\[ \begin{align*}
24 \quad & \text{Expressions for } K^{(12)}, F, L^{(a)}\text{ and } Q^{(a+1)} \quad (5.141)-(5.148) \text{ imply that } k_{\text{max}} = x^{(1)} + x^{(2)} + 2y^{(1)} + 2y^{(2)} + y^{(3)} + 1 \text{ for K vertex and } k_{\text{max}} = x^{(1)} + x^{(2)} + 2y^{(1)} + 2y^{(2)} + y^{(3)} + 1 \text{ for F vertex. Relations for } y^{(a)} \quad (5.151),(5.152) \text{ then lead to (5.155),(5.156).} \\
25 \quad & \text{If } x^{(1)} = x^{(2)} = 0, \text{ then restriction (5.157) becomes the restrictions well known in the angular momentum theory, } |s^{(1)} - s^{(2)}| \leq s^{(3)} \leq s^{(1)} + s^{(2)}, \text{ while restriction (5.158) takes the form } |s^{(1)} - s^{(2)} - 1| \leq s^{(3)} \leq s^{(1)} + s^{(2)} - 1. \end{align*} \]
where we use the notation

\[ K^{(12)} = \frac{1}{\beta_1 \beta_2} p_{\theta_1} \left( \Pi^I J^I g_{s} - m_{12} \right) \eta_2 \] (6.4)

\[ F = \frac{1}{\beta_1 \beta_2} p_{\theta_1} \left( \frac{\beta_2}{\beta_3} \Pi^I J^I g_{s} + (m_1 \beta_2 + m_2 \beta_1) J^I g_{s} \right) \eta_2 \alpha^{(n)} I \]

\[ + \frac{1}{\beta_1 \beta_2} p_{\theta_1} \left( -\tilde{m}_2 \Pi^I J^I g_{s} + \tilde{m}_3 m_{12} + \frac{2 \beta_1 \beta_2}{\beta_3} m_3^2 \right) \eta_2 \alpha^{(n)} I \] (6.5)

\[ L^{(a)} \equiv B^{(a)} - \frac{m_{a+1}^2 - m_{a+2}^2}{2m_a} \alpha^{(a)} \] (6.6)

\[ B^{(a)} \equiv \frac{\alpha^{(a)} I \Pi^I}{\beta_a} - \frac{\tilde{\beta}_a m_a \alpha^{(a)} I}{2 \beta_a} \] (6.7)

\[ Q^{(aa+1)} \equiv \alpha^{(aa+1)} - \frac{\alpha^{(a+1)}}{m_{a+1}} B^{(a)} + \frac{\alpha^{(a)}}{m_a} B^{(a+1)} - \frac{m_{a+2}^2}{2m_a m_{a+1}} \alpha^{(a)} \alpha^{(a+1)} \] (6.8)

\[ m_{ab} \equiv m_a \beta_b - m_b \beta_a \] (6.9)

and \( \alpha^{(ab)} \) are defined in (4.8). From the expressions for \( F \) (6.5), \( L^{(a)} \) (6.6) and the quadratic forms \( Q^{(aa+1)} \) (6.8), it follows that the cubic vertex for massive fields is singular as \( m_a \to 0 \), \( a = 1, 2, 3 \). We now restrict attention to vertices for fields with fixed spin values.

### 6.1 Cubic vertices for massive fields with fixed but arbitrary spin values

Vertices (6.2), (6.3) describe interaction of the towers of massive totally symmetric fields (2.12), (2.26). We next obtain vertex for massive spin \( s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2} \) and \( s^{(3)} \) fields. This, we consider vertex involving two spin \( s_1 + \frac{1}{2} \) and \( s_2 + \frac{1}{2} \) fermionic fields having the respective mass parameters \( m_1 \) and \( m_2 \) and one spin-\( s_3 \) bosonic field having mass parameter \( m_3 \):

\[ m_1|_F \neq 0, \quad m_2|_F \neq 0, \quad m_3|_F \neq 0, \]

\[ s^{(1)} + \frac{1}{2}, \quad s^{(2)} + \frac{1}{2}, \quad s^{(3)}. \] (6.10)

The massive spin-(\( s^{(1)} + \frac{1}{2} \)) and -(\( s^{(2)} + \frac{1}{2} \)) fermionic fields are described by the respective ket-vectors \(| \psi_{s^{(1)}} \rangle, | \psi_{s^{(2)}} \rangle \), while the massive spin-\( s^{(3)} \) bosonic field is described by ket-vector \(| \phi_{s^{(3)}} \rangle \). The ket-vectors of massive fields \(| \psi_{s^{(a)}} \rangle, a = 1, 2 \) and \(| \phi_{s^{(3)}} \rangle \), can be obtained from the respective expressions in (2.20) and (2.9), by replacing \( s \to s^{(a)} \), \( \alpha^I \to \alpha^{(a)} I \), \( \alpha \to \alpha^{(a)} \). Because \(| \psi_{s^{(1)}} \rangle \), \(| \psi_{s^{(2)}} \rangle \), \(| \phi_{s^{(3)}} \rangle \) are respective degree-\( s^{(a)} \) homogeneous polynomials in \( \alpha^{(a)} I \), \( \alpha^{(a)} \), it is obvious that the vertex we are interested in must satisfy the equations

\[ (\alpha^{(a)} I \bar{\alpha}^{(a)} I + \alpha^{(a)} \bar{\alpha}^{(a)} - s^{(a)}) | \vec{p}^{(a)} \rangle = 0, \quad a = 1, 2, 3, \] (6.11)

which tell us that the vertex \( \vec{p}^{(a)} \) must be a degree-\( s^{(a)} \) homogeneous polynomial in the oscillators \( \alpha^{(a)} I \), \( \alpha^{(a)} \). Taking into account that the forms \( F \), \( L^{(a)} \) and \( Q^{(aa+1)} \) are respective degree 1 and 2
homogeneous polynomials in oscillators we obtain the general solution of Eqs. (6.11) as

$$p^-_3(s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2}, s^{(3)}; x^{(1)}, x^{(2)}, x^{(3)} = K^{(12)} \prod_{a=1}^{3} (L^{(a)})^{x^{(a)}} (Q^{(a+1)})^{y^{(a+2)}}, \quad \text{K vertex} ; (6.12)$$

$$p^-_3(s^{(1)} + \frac{1}{2}, s^{(2)} + \frac{1}{2}, s^{(3)}; x^{(1)}, x^{(2)}, x^{(3)} = F \prod_{a=1}^{3} (L^{(a)})^{x^{(a)}} (Q^{(a+1)})^{y^{(a+2)}}, \quad \text{F vertex} ; (6.13)$$

where integers $y^{(a)}$ are expressible in terms of $s^{(a)}$ and three integers $x^{(a)}$ labeling the freedom of our solution,

$$y^{(a)} = \frac{1}{2}(s + x^{(a)} - x^{(a+1)} - x^{(a+2)}) - s^{(a)}, \quad a = 1, 2, 3, \quad \text{for K vertex} ; (6.14)$$

$$y^{(a)} = \frac{1}{2}(s + x^{(a)} - x^{(a+1)} - x^{(a+2)} - 1) - s^{(a)}, \quad a = 1, 2;$$

$$y^{(3)} = \frac{1}{2}(s + x^{(3)} - x^{(1)} - x^{(2)} + 1) - s^{(3)}, \quad \text{for F vertex} ; (6.15)$$

and $s$ is given in (4.14). The maximal number of powers of $\mathbb{P}^3$ in (6.12), (6.13), denoted by $k_{max}$, is given by

$$k_{max} = \frac{1}{2}(s + \sum_{a=1}^{3} x^{(a)}) + 1, \quad \text{for K vertex} ; (6.16)$$

$$k_{max} = \frac{1}{2}(s + \sum_{a=1}^{3} x^{(a)} + 1), \quad \text{for F vertex} . (6.17)$$

Requiring the powers of the forms $L^{(a)}$ and $Q^{(a+1)}$ in (6.12), (6.13) to be non–negative integers gives the restrictions

$$x^{(a)} \geq 0, \quad y^{(a)} \geq 0, \quad a = 1, 2, 3 ;$$

$$s + \sum_{a=1}^{3} x^{(a)} \quad \text{even integer}, \quad \text{for K vertex} ; (6.18)$$

$$x^{(a)} \geq 0, \quad y^{(a)} \geq 0, \quad a = 1, 2, 3 ;$$

$$s + \sum_{a=1}^{3} x^{(a)} \quad \text{odd integer}, \quad \text{for F vertex}. (6.19)$$

Using relations (6.14), (6.15) allows rewriting restrictions (6.18), (6.19) as

$$s^{(3)} - s^{(1)} - s^{(2)} + x^{(1)} + x^{(2)} \leq x^{(3)} \leq s^{(3)} - |s^{(1)} - s^{(2)} - x^{(1)} + x^{(2)}|, \quad \text{for K vertex} ; (6.20)$$

$$s^{(3)} - s^{(1)} - s^{(2)} + x^{(1)} + x^{(2)} - 1 \leq x^{(3)} \leq s^{(3)} - |s^{(1)} - s^{(2)} - x^{(1)} + x^{(2)}| - 1 , \quad \text{for F vertex} . (6.21)$$

26 Expressions for $K^{(12)}, F, L^{(a)}$ and $Q^{(a+1)}$ in (6.4)–(6.8) imply that $k_{max} = 1 + \sum_{a=1}^{3} (x^{(a)} + y^{(a)})$. Taking $y^{(a)}$ in (6.14), (6.15) into account we then find (6.16), (6.17).

27 If $x^{(a)} = 0, a = 1, 2, 3$, then restrictions (6.20) becomes the restrictions well known in the angular momentum theory: $|s^{(1)} - s^{(2)}| \leq s^{(3)} \leq s^{(1)} + s^{(2)}$, while restriction (6.21) takes the form $|s^{(1)} - s^{(2)}| \leq s^{(3)} - 1 \leq s^{(1)} + s^{(2)}$. 42
We now give examples of vertices for particular cases and make comment concerning hermitian properties of our vertices.

Interaction vertices for two massive spin-$\frac{1}{2}$ and spin-$\frac{1}{2}$ fermionic fields having the respective mass parameters $m_1$, $m_2$ and one massive spin-$s$ bosonic field with mass parameter $m_3$ are given by

\[
p_{\bar{w}}(\frac{1}{2}, \frac{1}{2}, s; 0, 0, s) = K^{(12)}(L^{(3)})^s, \tag{6.22}
\]

\[
p_{\bar{w}}(\frac{1}{2}, \frac{1}{2}, s; 0, 0, s - 1) = F(L^{(3)})^{s-1}. \tag{6.23}
\]

Consider vertices (6.22), (6.23) for $s = 1$. Covariant vertex corresponding to our light-cone $F$-vertex with $s = 1$ in (6.23) is given by

\[
\mathcal{L} = \bar{\psi}_1 \gamma^A \psi_2 (\phi^A + \frac{\partial^A}{m_3} \phi), \tag{6.24}
\]

while covariant vertex corresponding to our light-cone $K$-vertex with $s = 1$ in (6.22) takes the form

\[
\mathcal{L} = \bar{\psi}_1 \gamma^{AB} \psi_2 F^{AB}. \tag{6.25}
\]

In (6.24), $\phi^A$ and $\phi$ are vector and scalar fields of the Lorentz algebra to be used in gauge invariant formulation of spin-1 massive field with mass parameter $m_3$.

The light-cone vertex (6.23) and the covariant vertex (6.24) are not hermitian when $m_1 \neq m_2$, i.e., when $\psi_1 \neq \psi_2$. We can obtain hermitian light-cone and covariant vertices in a standard way: by adding or subtracting appropriate hermitian conjugated expression. In this way we obtain the standard covariant vertices

\[
\mathcal{L} = (\bar{\psi}_1 \gamma^A \psi_2 + \bar{\psi}_2 \gamma^A \psi_1) (\phi^A + \frac{\partial^A}{m_3} \phi), \tag{6.26}
\]

\[
i\mathcal{L} = (\bar{\psi}_1 \gamma^A \psi_2 - \bar{\psi}_2 \gamma^A \psi_1) (\phi^A + \frac{\partial^A}{m_3} \phi), \tag{6.27}
\]

The same holds true for the vertex in (6.25), i.e., the light-cone vertex and covariant vertex (6.25) are not hermitian in general. As before, we can obtain hermitian light-cone and covariant vertices by adding or subtracting appropriate hermitian conjugated expression. In this way we obtain the standard covariant vertices

\[
\mathcal{L} = (\bar{\psi}_1 \gamma^{AB} \psi_2 + \bar{\psi}_2 \gamma^{AB} \psi_1) F^{AB}, \tag{6.28}
\]

\[
i\mathcal{L} = (\bar{\psi}_1 \gamma^{AB} \psi_2 - \bar{\psi}_2 \gamma^{AB} \psi_1) F^{AB}. \tag{6.29}
\]

To summarize, in order to obtain hermitian vertices we should supplement our vertices by appropriate hermitian conjugated expressions. Note also that, for the space-time dimensions when Dirac fields allow restriction to Majorana fields, our vertices are hermitian as they stand.

Interaction vertices for two massive spin-$\frac{3}{2}$ and spin-$\frac{3}{2}$ fermionic fields and one massive spin-$s$
bosonic field are given by

\[ p^{-\[3\]}(s; 0, 0, s - 2) = K^{(12)}(L^{(3)})^{s-2}Q^{(31)}, \]  
\[ p^{-\[3\]}(s; 0, 0, s) = K^{(12)}(L^{(3)})^{s}Q^{(12)}, \]  
\[ p^{-\[3\]}(s; 0, 1, s - 1) = K^{(12)}L^{(2)}(L^{(3)})^{s-1}Q^{(31)}, \]  
\[ p^{-\[3\]}(s; 1, 1, s) = K^{(12)}L^{(1)}L^{(2)}(L^{(3)})^{s}, \]  
\[ p^{-\[3\]}(s; 0, 0, s - 3) = F(L^{(3)})^{s-3}Q^{(31)}, \]  
\[ p^{-\[3\]}(s; 0, 0, s - 1) = F(L^{(3)})^{s-1}Q^{(12)}, \]  
\[ p^{-\[3\]}(s; 0, 1, s - 2) = FL^{(2)}(L^{(3)})^{s-2}Q^{(31)}, \]  
\[ p^{-\[3\]}(s; 1, 1, s - 1) = FL^{(1)}L^{(2)}(L^{(3)})^{s-1}. \]  

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\section*{Appendix A \quad Notation}

We use \(2^{[d/2]} \times 2^{[d/2]}\) Dirac gamma matrices \(\gamma^A\) in \(d\)-dimensions, \(\{\gamma^A, \gamma^B\} = 2\eta^{AB}, \gamma^A \gamma^B = \gamma^0 \gamma^A \gamma^0\), where \(\eta^{AB}\) is mostly positive flat metric tensor and flat vectors indices of the \(so(d-1, 1)\) algebra take the values \(A, B = 0, 1, \ldots, d-1\). To simplify our expressions we drop \(\eta_{AB}\) in scalar products, i.e. we use \(X^A Y^A \equiv \eta_{AB} X^A Y^B\).

Matrices \(\Gamma_*\) and \(\gamma_*\) are defined by

\[ \Gamma_* \equiv \epsilon \gamma^0 \gamma^1 \ldots \gamma^{d-1}, \quad \gamma_* \equiv \epsilon \gamma^1 \ldots \gamma^{d-2}, \]  
\[ \Gamma_*^2 = 1, \quad \gamma_*^2 = 1. \]  

The matrices \(\Gamma_*\) and \(\gamma_*\) are related by

\[ \Gamma_* \equiv \gamma^+ - \gamma_*^-, \quad \gamma^+ = \gamma^0 \gamma^{d-1}. \]  

Antisymmetrized product of two \(\gamma\)-matrices is normalized as

\[ \gamma^{AB} = \frac{1}{2} \gamma^A \gamma^B - (A \leftrightarrow B). \]  

We use notation for projector operators
\[ \Pi^\alpha \equiv \frac{1}{2} \gamma^- \gamma^+, \quad \Pi^\alpha \equiv \frac{1}{2} \gamma^+ \gamma^- . \] (A.5)

For bosonic field we use Fourier transformation
\[ |\phi(x, \alpha)\rangle = \int \frac{d^{d-1}p}{(2\pi)^{(d-1)/2}} e^{i(x^\beta - x^\beta p^I)}(1-N_\alpha)|\phi(x^+, p, \alpha)\rangle \] (A.6)

while for fermionic fields
\[ |\psi_s(x, \alpha)\rangle = (p_\theta \psi_s(x, \alpha) + \psi_s^\dagger(x, \alpha)p_\eta)|0\rangle , \] (A.7)

we use
\[ |\psi(x, \alpha)\rangle = \int \frac{d^{d-1}p}{(2\pi)^{(d-1)/2}} e^{i(x^\beta - x^\beta p^I)} i^{-N_\alpha} \hat{U} |\psi(p, \alpha)\rangle , \] (A.8)

where we use the notation
\[ N_\alpha \equiv \alpha \bar{\alpha} , \] (A.9)

\[ \hat{U} \equiv p_\theta U_\theta + (U^\dagger p_\eta) \cdot p_\eta , \] (A.10)

\[ ((U^\dagger p_\eta) \cdot p_\eta) \equiv U^{\tau \alpha} p_\eta^{\beta} p_{\eta \alpha} , \] (A.11)

\[ U \equiv \frac{1}{\sqrt{2}}(1 + i\Gamma^\alpha) , \] (A.12)

and the scalar oscillators \( \alpha \) are defined by (2.7).
References

[1] P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949).
[2] S. Weinberg, Phys. Rev. 150, 1313 (1966).
[3] J. B. Kogut and D. E. Soper, Phys. Rev. D 1, 2901 (1970).
[4] P. Goddard, J. Goldstone, C. Rebbi and C. B. Thorn, Nucl. Phys. B 56, 109 (1973).
[5] M. Kaku and K. Kikkawa, Phys. Rev. D 10, 1110 (1974).
[6] J. F. L. Hopkinson, R. W. Tucker and P. A. Collins, Phys. Rev. D 12, 1653 (1975).
[7] M. B. Green and J. H. Schwarz, Nucl. Phys. B 218, 43 (1983).
[8] M. B. Green, J. H. Schwarz and L. Brink, Nucl. Phys. B 219, 437 (1983).
[9] M. B. Green and J. H. Schwarz, Nucl. Phys. B 243, 475 (1984).
[10] L. Brink, O. Lindgren and B. E. W. Nilsson, Nucl. Phys. B 212, 401 (1983).
[11] S. Mandelstam, Nucl. Phys. B 213, 149 (1983).
[12] L. Brink and A. Tollsten, Nucl. Phys. B 249, 244 (1985).
[13] M. B. Green and J. H. Schwarz, Phys. Lett. B 122, 143 (1983).
[14] L. Brink, M. B. Green and J. H. Schwarz, Nucl. Phys. B 223, 125 (1983).
[15] H. Hata, K. Itoh, T. Kugo, H. Kunitomo and K. Ogawa, Phys. Rev. D 34, 2360 (1986).
[16] H. Hata, K. Itoh, T. Kugo, H. Kunitomo and K. Ogawa, Phys. Rev. D 35, 1318 (1987).
[17] W. Siegel and B. Zwiebach, Nucl. Phys. B 282, 125 (1987).
[18] W. Siegel and B. Zwiebach, Phys. Lett. B 184, 325 (1987).
[19] W. Siegel, “Introduction To String Field Theory,” arXiv:hep-th/0107094.
[20] A. K. H. Bengtsson, I. Bengtsson and L. Brink, Nucl. Phys. B 227, 31 (1983).
[21] A. K. H. Bengtsson, I. Bengtsson and L. Brink, Nucl. Phys. B 227, 41 (1983).
[22] A. K. H. Bengtsson, I. Bengtsson and N. Linden, Class. Quant. Grav. 4, 1333 (1987).
[23] E. S. Fradkin and R. R. Metsaev, Class. Quant. Grav. 8, L89 (1991).
[24] R. C. Brower, C. I. Tan and C. B. Thorn, Phys. Rev. D 73, 124037 (2006) [arXiv:hep-th/0603256].
[25] S. J. Brodsky and G. F. de Teramond, arXiv:0709.2072 [hep-ph].
[26] S. J. Brodsky and G. F. de Teramond, arXiv:0707.3859 [hep-ph].
[27] B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. B 305, 545 (1988).
[28] E. Bergshoeff, E. Sezgin, Y. Tani and P. K. Townsend, Annals Phys. 199, 340 (1990).
[29] O. Bergman and C. B. Thorn, Phys. Rev. D 52, 5980 (1995) [arXiv:hep-th/9506125].
[30] E. S. Fradkin and M. A. Vasiliev, Phys. Lett. B 189, 89 (1987).
[31] M. A. Vasiliev, Phys. Lett. B 243, 378 (1990).
[32] M. A. Vasiliev, Phys. Lett. B 567, 139 (2003) [arXiv:hep-th/0304049].
[33] M. A. Vasiliev, Comptes Rendus Physique 5, 1101 (2004) [arXiv:hep-th/0409260].
[34] X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, arXiv:hep-th/0503128.
[35] M. A. Vasiliev, Sakharov memorial lectures in physics, vol.1, 427-445; Moscow 1991.
[36] R. R. Metsaev, Nucl. Phys. B 759, 147 (2006) [arXiv:hep-th/0512342].
[37] R. R. Metsaev, Nucl. Phys. B 563, 295 (1999) [arXiv:hep-th/9906217].
[38] R. R. Metsaev, Phys. Lett. B 531, 152 (2002) [arXiv:hep-th/0201226].
[39] R. R. Metsaev, Phys. Lett. B 590, 95 (2004) [arXiv:hep-th/0312297].
[40] F. A. Berends, G. J. H. Burgers and H. Van Dam, Z. Phys. C 24, 247 (1984).
[41] F. A. Berends, G. J. H. Burgers and H. van Dam, Nucl. Phys. B 260, 295 (1985).
[42] X. Bekaert, N. Boulanger and S. Cnockaert, JHEP 0601, 052 (2006) [arXiv:hep-th/0508048].
[43] N. Boulanger, S. Leclercq and S. Cnockaert, Phys. Rev. D 73, 065019 (2006) [arXiv:hep-th/0509118].
[44] S. Deser and Z. Yang, Class. Quant. Grav. 7, 1491 (1990).
[45] F. A. Berends, G. J. H. Burgers and H. van Dam, Nucl. Phys. B 271, 429 (1986).
[46] S. Weinberg, Phys. Rev. 181, 1893 (1969).
[47] S. Weinberg, Phys. Rev. 133, B1318 (1964).
[48] S. Weinberg, Phys. Rev. 134, B882 (1964).
[49] J. K. Barrett and G. Savvidy, Phys. Lett. B 652, 141 (2007) [arXiv:0704.3164 [hep-th]].
[50] J. E. Paton and H. M. Chan, Nucl. Phys. B 10, 516 (1969).
[51] R. R. Metsaev, Mod. Phys. Lett. A 6, 2411 (1991).
[52] M. Goroff and J. H. Schwarz, Phys. Lett. B 127, 61 (1983).
[53] T. Hori, Nucl. Phys. B 274, 401 (1986).
[54] C. Aragone and A. Khoudeir, Class. Quant. Grav. 7, 1291 (1990).
[55] A. A. Tseytlin, Phys. Lett. B 185, 59 (1987).
[56] R. R. Metsaev, Class. Quant. Grav. 10, L39 (1993).
[57] R. R. Metsaev, Phys. Lett. B 309, 39 (1993).
[58] R. R. Metsaev, Mod. Phys. Lett. A 8, 2413 (1993).
[59] R. R. Metsaev and A. A. Tseytlin, Phys. Lett. B 185, 52 (1987).
[60] S. N. Gupta and W. W. Repko, Phys. Rev. 165, 1415 (1968).
[61] S. Ferrara, M. Porrati and V. L. Telegdi, Phys. Rev. D 46, 3529 (1992).
[62] R. R. Metsaev, Phys. Rev. D 71, 085017 (2005) [arXiv:hep-th/0410239].
[63] S.M.Klishchevich and Y.M.Zinovev, Phys.Atom.Nucl. 61, 1527 (1998); hep-th/9708150.
[64] P. C. Argyres and C. R. Nappi, Phys. Lett. B 224, 89 (1989).
[65] S. M. Klishchevich, Int. J. Mod. Phys. A 15, 395 (2000) [arXiv:hep-th/9805174].
[66] A. Cucchieri, M. Porrati and S. Deser, Phys. Rev. D 51, 4543 (1995) [arXiv:hep-th/9408073].
[67] I. L. Buchbinder, D. M. Gitman, V. A. Krykhtin and V. D. Pershin, Nucl. Phys. B 584, 615 (2000) [arXiv:hep-th/9910188].
[68] I. L. Buchbinder, D. M. Gitman and V. D. Pershin, Phys. Lett. B 492, 161 (2000) [hep-th/0006144]
[69] R. R. Metsaev, arXiv:hep-th/0612279.
[70] R. R. Metsaev, Phys. Lett. B 643, 205 (2006) [arXiv:hep-th/0609029].
[71] A. Fotopoulos, N. Inges, A. C. Petkou and M. Tsulaia, JHEP 0710, 021 (2007) [arXiv:0708.1399 [hep-th]].
[72] S. M. Klishchevich, Int. J. Mod. Phys. A 15, 609 (2000) [arXiv:hep-th/9811030].