AN ENDPOINT ESTIMATE OF THE BILINEAR PARABOLOID RESTRICTION OPERATOR

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Abstract. In Fourier restriction problems, a cone and a paraboloid are model surfaces. The sharp bilinear cone restriction estimate was first shown by Wolff, and later the endpoint was obtained by Tao. For a paraboloid, the sharp $L^2$ bilinear restriction estimate was obtained by Tao, but the endpoint was remained open. In this paper we prove the endpoint $L^2$ bilinear restriction estimate for a paraboloid.

1. Introduction

Fix $n \geq 2$, let $\Sigma$ be a hypersurface defined by $\Sigma = \{(\xi, s(\xi)) : \xi \in \mathbb{R}^n}\). Then the (adjoint) Fourier restriction operator $R_{\Sigma} f$ for the hypersurface $\Sigma$ can be defined by

$$R_{\Sigma} f(x,t) := \int e^{2\pi i (x \cdot \xi + ts(\xi))} f(\xi) a(\xi) d\xi,$$

where $a(\xi)$ is a smooth cut-off function.

The (adjoint) restriction estimate $R_{\Sigma}^* f$ for $1 \leq p, q \leq \infty$ is of the form

$$\|R_{\Sigma} f\|_q \leq C_{p,q,\Sigma} \|f\|_p,$$  (1.1)

and the restriction problem is to determine $(p, q)$ for which the estimate $R_{\Sigma}^* f$ holds. There are two representative model hypersurfaces. One is a cone $\Sigma_{cone} = \{(\xi, |\xi|) : \xi \in \mathbb{R}^n\}$, and the other is a paraboloid $\Sigma_{parab} = \{(\xi, -|\xi|^2/2) : \xi \in \mathbb{R}^n\}$. For these two surfaces the restriction operators $R_{\Sigma_{cone}} f$ and $R_{\Sigma_{parab}} f$ are related to other problems such as the Bochner-Riesz conjecture, Kakeya conjecture and Sogge’s local smoothing conjecture, see [2, 12, 17, 20, 22, 28]. Moreover, they are also connected to the wave and Schrödinger equations because $R_{\Sigma_{cone}} \hat{f}$ and $R_{\Sigma_{parab}} \hat{f}$ are the solutions to the free wave equation $u_{tt} - \Delta u = 0$ and the free Schrödinger equation $4\pi i \partial_t u - \Delta u = 0$, respectively, see [15, 22, 23].

The bilinear restriction estimate $R_{\Sigma_1, \Sigma_2}^* f R_{\Sigma_2} g$ for $1 \leq p, q \leq \infty$ is of the form

$$\|R_{\Sigma_1, \Sigma_2} f R_{\Sigma_2} g\|_q \leq C_{p,q,\Sigma_1,\Sigma_2} \|f\|_p \|g\|_p,$$

where $\Sigma_1, \Sigma_2$ are two compact subsets of $\Sigma$ such that the set of unit normal vectors of $\Sigma_1$ are separated by a non-zero distance from the set of unit normal vectors of $\Sigma_2$. This bilinear restriction estimate $R_{\Sigma_1, \Sigma_2}^* f R_{\Sigma_2} g$ was used to improve the linear restriction estimate $R_{\Sigma}^* f$. (The restriction estimate have been improved further by Bourgain–Guth [4], Guth [5, 6], Wang [20], Hickman–Rogers [8].) In addition, as the relation between the Stein-Tomas restriction theorem and the Strichartz estimate, the bilinear restriction estimates $R_{\Sigma_1, \Sigma_2}^* (2 \times 2, q)$ lead to the corresponding bilinear estimates applied to null form estimates for the relevant dispersive equations, see [7, 9, 10, 16, 22, 23].

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The $L^2$ bilinear restriction estimate is based on the argument of Wolff [27] for a cone. His arguments are roughly composed of two steps. One is to use induction to avoid some critical case of the Kakeya set. The other is to deal with the remaining relaxed Kakeya set by utilizing some geometrical observation as follows:

The union $\Lambda(x_0)$ of all lines passing through a point $x_0$ and of direction normal to $\Sigma_{cone}$ becomes a cone.

We can see that if a line $\ell$ passes through $\Lambda(x_0)$, then $\ell \cap \Lambda(x_0)$ has at most $O(1)$ points, which is the key to obtain the sharp bilinear cone restriction.

However, in a paraboloid the analogous property does not hold. Specifically, the union $\Lambda(x_0)$ of all lines passing through $x_0$ and of direction normal to $\Sigma_{parab}$ does not contained in a hyper-surface. The reason is that while the cone has one vanishing principle curvature, the paraboloid has non-vanishing Gaussian curvature. Thus, $\ell \cap \Lambda(x_0)$ may have infinitely many points. Because of this difference, Wolff’s argument cannot be directly applied to the paraboloid case.

This difficulty was resolved by Tao [24] who used a kind of orthogonality due to the non-vanishing curvature. Such an orthogonality was first observed in the proof of the 2-dimensional restriction theorem by Fefferman and Córdoba. By combining Wolff’s arguments with the orthogonality Tao obtained the sharp bilinear restriction estimate by Fefferman and C´ordoba. By combining Wolff’s arguments with the orthogonality Tao obtained the sharp bilinear restriction estimate $R^*_{\Xi_1, \Sigma_2}(2 \times 2, p)$, $p > \frac{n+3}{n+1}$ for a paraboloid.

It is natural to ask whether the endpoint bilinear estimate $R^*_{\Xi_1, \Sigma_2}(2 \times 2, \frac{n+3}{n+1})$ is valid or not. Since the Kakeya example does not work in the endpoint $L^2$ bilinear restriction estimate, we can expect it. Tao [23] obtained the endpoint bilinear cone restriction estimate by exploring energy concentrations and the geometric observation (1.2). If one makes an attempt to prove the endpoint bilinear restriction estimate for a paraboloid, it is reasonable, first of all, to apply Tao’s arguments, but the geometric observation (1.2) does not hold for the paraboloid restriction operator. However, it still seems to have the $L^2$ bilinear paraboloid restriction estimate because the Kakeya example does not work.

In this paper we will prove the endpoint estimate $R^*_{\Sigma_1, \Sigma_2}(2 \times 2, \frac{n+3}{n+1})$ for a paraboloid. To state more explicitly, let $\Sigma = \Sigma_{parab}$ and

$$\Sigma_j = \{ (\xi, \tau) \in \Sigma : 1 < |\xi| < 2, \angle(\xi, (-1)^{j-1}e_1) < \pi/8 \}, \quad j = 1, 2,$$

where $e_1 \in \mathbb{R}^n$ is a standard unit vector. We define the operator $\mathcal{U}_j$ by $\mathcal{U}_jf = \mathcal{R}_{\Sigma_j}f$ for $j = 1, 2$;

$$\mathcal{U}_j f(x, t) := \int e^{2\pi i (x \cdot \xi - \frac{1}{2}t|\xi|^2)} \hat{f}(\xi) a_j(\xi) d\xi,$$

where $a_j$ is a smooth function which is equal to 1 on

$$\Xi_j = \{ \xi \in \mathbb{R}^n : (\xi, \tau) \in \Sigma_j \}$$

and supported in

$$\tilde{\Xi}_j = \{ \xi : 1/2 \leq |\xi| \leq 4, \angle(\xi, (-1)^{j-1}e_1) \leq \pi/4 \}.$$

**Theorem 1.1.** For $\frac{n+3}{n+1} \leq p \leq 2$, the estimate

$$\|\mathcal{U}_1 f_1 \mathcal{U}_2 f_2\|_{L^p(\mathbb{R}^n \times \mathbb{R})} \leq C_p \|f_1\|_{L^2(\Xi_1)} \|f_2\|_{L^2(\Xi_2)}$$

holds for all $f_1 \in L^2(\Xi_1)$ and $f_2 \in L^2(\Xi_2)$.

To obtain the endpoint we basically follow the arguments in [23]. We first reprove the sharp bilinear restriction estimate for a paraboloid without dyadic pigeonholing. Next we use an energy concentration argument. But, as mentioned above, we cannot directly apply Tao’s endpoint argument to the paraboloid problem because of the lack of (1.2). To get around this we will devise a new energy concentration argument where the dispersiveness of $\mathcal{U}_j f$ and the transversality between $\Sigma_1$ and $\Sigma_2$ play a crucial role instead of the geometric observation (1.2).
The Fourier restriction operator can be generalized to some oscillatory integral operator. The cone restriction operator is generalized to the oscillatory integrals satisfying the cinematic curvature condition, and similarly the paraboloid one generalized to the oscillatory integrals with the Carleson–Sjölin condition, see [13, 14, 18, 19, 25]. It was shown by S. Lee [14] that these curvature condition, and similarly the paraboloid one generalized to the oscillatory integrals with the cinematic curvature condition, the endpoint bilinear estimate was obtained by the author [11]. It is likely that the oscillatory integral operators with the Carleson–Sjölin condition have the endpoint bilinear estimate, too.

**Notation.** Let \( N > 1 \) be a large integer depending only on the dimension \( n \), which is used as a large exponent of the error terms. Let \( C_0 \) be an integer much larger than \( N \).

We use \( C \) to denote various large numbers which vary each time. It may depend on \( N \) but not depend on \( C_0 \). The notation \( A \lesssim B \) or \( A = O(B) \) implies \( A \leq CB \). If \( A \lesssim B \) and \( B \lesssim A \) we write \( A \sim B \).

When \( \phi(x, t) \) is a space-time function, let \( \phi(t) \) denote the spatial function \( \phi(t)(x) = \phi(x, t) \). The hat \( \widehat{\cdot} \) notation is used for the Fourier transform
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx.
\]

Let a spacetime cube \( Q(x_Q, t_Q; R_Q) \) be an \((n + 1)\)-dimensional cube in \( \mathbb{R}^{n+1} \) of side-length \( R_Q \) and centered at \((x_Q, t_Q) \in \mathbb{R}^n \times \mathbb{R} \) with sides parallel to the axes. Let \( D(x_D, t_D; r_D) \) denote an \( n \)-dimensional disc at time \( t_D \) of the form
\[
D(x_D, t_D; r_D) = \{(x, t_D)\in \mathbb{R}^n \times \mathbb{R} : |x - x_D| \leq r_D\}.
\]

For any compact subset \( \pi \subset \mathbb{R}^n \) we define a conic set \( \Lambda_{\pi}(x_0, t_0) \) with vertex \((x_0, t_0) \in \mathbb{R}^n \times \mathbb{R} \) by
\[
\Lambda_{\pi}(x_0, t_0) = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : x = x_0 + (t - t_0)w, \ w \in \pi, \ t \in \mathbb{R}\}, \tag{1.5}
\]

and let \( \Lambda_{\pi}(x_0, t_0; r) \) be the \( r \)-neighborhood of \( \Lambda_{\pi}(x_0, t_0) \). Briefly we write
\[
\Lambda_j(x_0, t_0) = \Lambda_{2^{-j}}(x_0, t_0) \tag{1.6}
\]

for \( j = 1, 2, \) and
\[
\Lambda_{\cup}(x_0, t_0) = \Lambda_1(x_0, t_0) \cup \Lambda_2(x_0, t_0).
\]

If \( Q = Q(x_Q, t_Q; R_Q) \) and \( c > 0 \) the \( cQ \) is defined by \( Q(x_Q, t_Q; cr_Q) \). Similarly, \( cD \) is defined.

Let \( \eta \) be a nonnegative Schwartz function on \( \mathbb{R}^n \) with \( \int \eta = 1 \) and whose Fourier transform is supported on the unit disc. By the Poisson summation formula we may have
\[
\sum_{k \in \mathbb{Z}^n} \eta(x - k) = 1. \tag{1.7}
\]

We define \( \eta_r \) for \( r > 0 \) by
\[
\eta_r(x) = r^{-n} \eta(x/r). \tag{1.8}
\]

2. Proof of Theorem 1.1

In this section we state some propositions and using them we prove Theorem 1.1. The proof of propositions are given in next sections.

We denote by
\[
\phi_j(x, t) = U_j f_j(x, t),
\]
and define the energy \( E(\phi_j) \) by
\[
E(\phi_j) = \|\phi_j(t)\|_2^2 \tag{2.1}
\]
where \( t \in \mathbb{R} \) is arbitrary. It makes sense by Plancherel’s theorem and
\[
\phi_j(t)(\xi) = e^{-\pi |t|} f_j(\xi) a_j(\xi). \tag{2.2}
\]
Using these notations we rewrite Theorem 1.1 as follows.

**Theorem 2.1.** For \( \frac{n+3}{n+1} \leq p \leq 2 \), the estimate

\[
\|\phi_1\phi_2\|_p \leq C_p E(\phi_1)^{1/2} E(\phi_2)^{1/2}
\]

(2.3)
holds for all \( \phi_1 \) and \( \phi_2 \) whose Fourier transforms are supported in \( \Sigma_1 \) and \( \Sigma_2 \) respectively.

The estimate (2.3) for \( p = 2 \) is well known. Thus, by interpolation it suffices to prove the theorem for

\[
p := \frac{n+3}{n+1}.
\]

**Definition 2.2.** For any \( R \geq C_0 \) we define a constant \( K(R) \) to be the best constant for which the estimate

\[
\|\phi_1\phi_2\|_{L^p(Q_R)} \leq K(R) E(\phi_1)^{1/2} E(\phi_2)^{1/2}
\]

(2.4)
holds for all spacetime cubes \( Q_R \) of sidelength \( R \) and all \( \phi_1, \phi_2 \) of which Fourier transforms are supported in \( \Sigma_1 \) and \( \Sigma_2 \) respectively and satisfy

\[
\text{marg}(\phi_1), \text{marg}(\phi_2) \geq 1/100 - R^{-1/N},
\]

(2.5)
where the margin \( \text{marg}(\phi_j) \) is defined by

\[
\text{marg}(\phi_j) := \text{dist(supp} \hat{\phi}_j, \Sigma \setminus \Sigma_j).
\]

Note that the margin condition can be removed by partitioning both \( x \)-space and \( \xi \)-space and some Lorentz transforms, see [23]. By the above definition it suffices to show

\[
K(R) \leq 2^{CC_0}.
\]

(2.6)

We may assume that

\[
E(\phi_1) = E(\phi_2) = 1.
\]

(2.7)

By some trivial estimates it follows

\[
\|\phi_1\phi_2\|_{L^p(Q_R)} \lesssim R^C.
\]

(2.8)

Thus we see that

\[
K(R) \lesssim R^C.
\]

(2.9)

By this estimate we may assume \( R \geq 2^{C_0} \). Let

\[
\overline{K}(R) := \sup_{2^{C_0} \leq R' \leq R} K(R').
\]

**Proposition 2.3.** Let \( R \geq 2^{C_0} \) and \( 0 < c \leq 2^{-C_0} \). Suppose that \( \phi_1, \phi_2 \) are Fourier supported in \( \Sigma_1 \) and \( \Sigma_2 \) respectively and satisfy the margin condition

\[
\text{marg}(\phi_j) \geq 1/100 - 2R^{-1/N}, \quad j = 1, 2.
\]

(2.10)

Then,

\[
\|\phi_1\phi_2\|_{L^p(Q_R)} \leq \left( (1 + Cc)\overline{K}(R/C_0) + c^{-C} \right) E(\phi_1)^{1/2} E(\phi_2)^{1/2},
\]

(2.11)

for all cubes \( Q_R \) of sidelength \( R \).

We will prove this proposition in Section 5. It is obtained by refining the proof of [23]. Technically the constant \( (1 + Cc) \) is important to obtain the endpoint. Note that the estimate (2.11) implies \( K(R) \leq (1 + Cc)\overline{K}(R/C_0) + c^{-C} \). By iterating this estimate it follows \( \overline{K}(R) \leq C_3 R^\varepsilon \) for all \( \varepsilon > 0 \).

For other propositions we define an energy concentration. We first introduce several constants relevant to the conic sets \( \Lambda_1(0) \) and \( \Lambda_2(0) \) defined as (1.5).

- Let \( A_w \) be the maximum angle between two lines \( L_j \) and \( L'_j \) passing through the origin and contained in \( \Lambda_j(0) \) for \( j = 1, 2 \).
• Let $A_d$ be the minimum angle between two lines $L_1 \subset A_1(0)$ and $L_2 \subset A_2(0)$ passing through the origin.
• Define the constant $A_*$ by
  \[ A_* := 4A_w A_d^{-1} + C. \]  
\[ \tag{2.12} \]

**Definition 2.4.** Let $\phi_1, \phi_2$ be Fourier supported in $\Sigma_1$ and $\Sigma_2$ respectively. For $0 \leq \varepsilon < 1$, $r > 0$ and $t \in \mathbb{R}$, let $D_\varepsilon^t(\phi_1, \phi_2)$ be the collection of discs $D = D(x_D, t_D; r_D)$ in $\mathbb{R}^n \times \{t\}$ with $r_D \geq C_1$ such that
  \[ \|\phi_1\|_{L^2(D(x_D, t_D; \mathbb{R}^n))} \|\phi_2\|_{L^2(D(x_D, t_D; \mathbb{R}^n))} \geq \varepsilon E(\phi_1)^{1/2} E(\phi_1)^{1/2}. \]  
\[ \tag{2.13} \]

We define the energy concentration $E_{r,t,s}^\varepsilon(\phi_1, \phi_2)$ of $\phi_1$ and $\phi_2$ at time $t$ by
  \[ E_{r,t,s}^\varepsilon(\phi_1, \phi_2) = \max \left( \frac{1}{2} E(\phi_1)^{1/2} E(\phi_1)^{1/2}, \sup_{D \in D^\varepsilon_\delta(\phi_1, \phi_2)} \sup_{D_1, D_2 \subset N(D)} \|\phi_1\|_{L^2(D_1)} \|\phi_2\|_{L^2(D_2)} \right) \]
  where $r_D$ denotes the radius of $D$ and $N(D)$ denotes the $A_*$-dilated disc of $D$. (See Figure 1.)

The condition (2.13) is a technical thing to handle errors. Since $\phi_1, \phi_2$ are compactly Fourier supported, they have Schwartz tails, and by using the Paley–Wiener theorem we can see that for any proper disc $D$, the $\|\phi_j\|_{L^2(D)}$ is nonzero for $j = 1, 2$.

![Figure 1. Energy concentration](image)

**Definition 2.5.** Let $R \geq 2C_0$, $r > 0$ and $0 \leq \varepsilon < 1$. We define $K_\varepsilon(R, r, \hat{r})$ to be the best constant for which the estimate
  \[ \|\phi_1 \phi_2\|_{L^p(Q_R)} \leq K_\varepsilon(R, r, \hat{r}) E(\phi_1)^{1/2} E(\phi_1)^{1/2} E_{r, \hat{r}, t, \varepsilon}(\phi_1, \phi_2)^{1/p'} \]  
\[ \tag{2.14} \]
holds for all spacetime cube $Q_R$ of sidleth $R$, all $t \in \mathbb{R}$ and all $\phi_1, \phi_2$ whose Fourier transforms are supported in $\Sigma_1$ and $\Sigma_2$ respectively and satisfy the margin condition (2.48).

From the definitions of $K(R)$ and $K_\varepsilon(R, r, \hat{r})$ it immediately follows
  \[ K(R) \leq K_\varepsilon(R, r, \hat{r}), \]
  \[ K_\varepsilon(R, r, \hat{r}) \leq 2^{1/p'} K(R). \]

By the dispersive property of $\phi_1, \phi_2$ the above estimates are further improved under certain circumstances.
Proposition 2.6. For any $R \geq 2^{C_0}$ and $0 < \varepsilon \ll 1$,  
\[
    \mathcal{K}(R) \leq (1 - C_0^{-C}) \sup_{r \geq C_0^{-CR}} \mathcal{K}_\varepsilon(R, r, \hat{r}).
\]  
(2.15)

This proposition will be proven in Section 7. In the above estimate, the constant $(1 - C_0^{-C})$ is crucial for closing an induction. The supremum condition $r \geq C_0^{-CR}$ is also important, which prevents a loss caused by iteration. Because of this condition we need only $O(1)$ iteration.

Proposition 2.7. Let $R \geq 2^{C_0}$. If $R^{-N/4} \leq \varepsilon \ll 1$, $r \geq C_0^{-C} R$ and $r/100 < \hat{r} \leq (2A_wA_d^{-1} + C_0^{-C})r$ then  
\[
    \mathcal{K}_\varepsilon(R, r, \hat{r}) \leq (1 + Cc)\overline{\mathcal{K}}(R) + 2^{CC_0}
\]
for any $0 < c \leq 2^{-C_0}$.

We will prove this in Section 8. In the above estimate, the constant $(1 + Cc)$ is crucial. Because of the condition $r \geq C_0^{-C} R$ in the above proposition, we can combine this with the previous one. To resolve it we use the following recursive estimate.

Proposition 2.8. Let $R \geq 2^{C_0}$. If $0 < \varepsilon \ll 1$, $C_0^{-C} R \leq r \leq C_0^{-C} R$ and $\hat{r} > 0$,  
\[
    \mathcal{K}_\varepsilon(R, r, \hat{r}) \leq (1 + Cc)\mathcal{K}_\varepsilon(R/C_0, r_2, \hat{r}_2) + c^{-C}
\]
for any $0 < c \leq 2^{-C_0}$, where $r_2 := r(1 - C_1^{-1/3N})$ and $\hat{r}_2 := (\hat{r})_2$.

We will prove this in Section 9. The above recursive estimate is obtained by modifying the proof of Proposition 2.3.

To prove (2.9) we combine the above three propositions. From Proposition 2.7 and Proposition 2.8 it follows that if $R^{-N/4} \leq \varepsilon \ll 1$, $r \geq C_0^{-C} R$ and $\hat{r} \leq (2A_wA_d^{-1} + C_0^{-C})r$, then  
\[
    \mathcal{K}_\varepsilon(R, r, \hat{r}) \leq (1 + Cc)\overline{\mathcal{K}}(R) + 2^{CC_0} + c^{-C}.
\]  
(2.16)

Indeed, by Proposition 2.7 we may assume that $C_0^{-C} R \leq r \leq C_0^{-C} R$. Let $r_0 := r$, $r_{j+1} := (r_j)_2$ and $\hat{r}_{j+1} := (\hat{r}_j)_2$ for $j = 0, 1, 2, \ldots$. Then if $\hat{r} \leq (2A_wA_d^{-1} + C_0^{-C})r$ then $\hat{r}_2 \leq (2A_wA_d^{-1} + C_0^{-C})r_2$. We take $J$ as the smallest integer such that $r \geq C_0^{-C} R$. Because of the condition $r \geq C_0^{-C} R$, we have $J = O(1)$. From Proposition 2.8 it follows that  
\[
    \mathcal{K}_\varepsilon(R/C_0^j, r_j, \hat{r}_j) \leq (1 + Cc)\mathcal{K}_\varepsilon(R/C_0^{j+1}, r_{j+1}, \hat{r}_{j+1}) + c^{-C}.
\]

By iteration we have  
\[
    \mathcal{K}_\varepsilon(R, r, \hat{r}) \leq (1 + Cc)^J \mathcal{K}_\varepsilon(R/C_0^J, r_J, \hat{r}_J) + 2^J c^{-C}.
\]

Since $r \geq C_0^{-C} R \geq C_0^{-C} 2^{-C_0}$ and $C_0$ is very large, we see that $r_J$ is comparable to $r$. By Proposition 2.7  
\[
    \mathcal{K}_\varepsilon(R, r, \hat{r}) \leq (1 + Cc)^{J+1} \overline{\mathcal{K}}(R/C_0^J) + 2^{CC_0} + 2^{J+1} c^{-C}.
\]

Since $\overline{\mathcal{K}}(R/C_0^J) \leq \overline{\mathcal{K}}(R)$, this estimate implies (2.16).

Combine Proposition 2.9 with (2.16). Then,  
\[
    \mathcal{K}(R) \leq (1 - C_0^{-C})(1 + Cc)\overline{\mathcal{K}}(R) + 2^{CC_0} + c^{-C}.
\]

If we set $c = 2^{-C_0}$ then $\overline{\mathcal{K}}(R) \leq (1 - C_0^{-C})\overline{\mathcal{K}}(R) + 2^{CC_0}$. By rearranging we obtain $\overline{\mathcal{K}}(R) \leq C_0^{2CC_0} \leq 2^{CC_0}$, which implies (2.9).

3. Preliminaries for the bilinear restriction estimate

Let $R \geq 2^{C_0}$, $0 < c \leq 2^{-C_0}$ and $\kappa > 0$ an integer such that  
\[
    r := 2^{-\kappa} R \sim R^{1/2}.
\]  
(3.1)
3.1. A wave packet decomposition. Let $L = c^{-2r} \mathbb{Z}^n$ and $V_j$ be a maximal $r^{-1}$-separated subset of $\tilde{\Xi}_j$ for $j = 1, 2$. For each $(x_0, v_j) \in L \times V_j$, we define a tube $T_j = T_j^{(x_0, v_j)}$ with initial position $x_0$ and direction $v_j$ by

$$T_j = \{ (x, t) \in \mathbb{R}^n \times \mathbb{R} : |t| \leq R, \ |(x - x_0) - tv_j| \leq r \},$$

and let $\mathbf{T}_j$ denote the collection of these tubes. We denote by $x(T_j) = x_0$ the position of $T_j$ and $v(T_j) = v_j$ the velocity of $T_j$.

Now we decompose $\phi_j$ into wave packets essentially supported on tubes $T_j$. To partition $\mathbb{R}^n$ into cubes of sidelength $c^{-2r}$, we set

$$\eta^{x_0}(x) := \eta \left( \frac{x - x_0}{c^{-2r}} \right).$$

(3.2)

Then,

$$\sum_{x_0 \in L} \eta^{x_0}(x) = 1.$$

(3.3)

To partition $\tilde{\Xi}_j$, let $B_{v_j}$ be a neighborhood of $v_j \in V_j$ with pairwise disjoint interiors so that

$$\tilde{\Xi}_j = \bigcup_{v_j \in V_j} B_{v_j}.$$

For each $\omega \in D(0; 1/r)$, we define a map $\Omega_\omega : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\Omega_\omega(v) = v + w$. Let $G$ be the set of these maps, and define $d\Omega$ by

$$\int_G F(\Omega)d\Omega = \frac{1}{|D(0; 1/r)|} \int_{D(0; 1/r)} F(\Omega_\omega)dw.$$

For each $\Omega \in G$ and $v_j \in V_j$, we define the multiplier $P_{\Omega, v_j}$ by

$$\hat{P_{\Omega, v_j}f} = \chi_{\Omega(B_{v_j})}\hat{f},$$

where $\chi$ denotes a characteristic function. For each $T_j = T_j^{(x_0, v_j)} \in \mathbf{T}_j$, we define a function $f_{T_j}$ by

$$f_{T_j}(x) := \eta^{x_0}(x) \int_G P_{\Omega, v_j}f_j(x)d\Omega.$$

(3.4)

We define a wave packet $\phi_{T_j}$ as

$$\phi_{T_j} := \mathcal{U}_j f_{T_j}.$$

(3.5)

Then by the linearity of Fourier transform,

$$\phi_j(x, t) = \sum_{T_j \in \mathbf{T}_j} \phi_{T_j}(x, t).$$

(3.6)

**Lemma 3.1** (Properties of the wave packets). Suppose that $\phi_j$ has a Fourier support in $\Sigma_j$ and a margin

$$\text{marg}(\phi_j) \geq Cr^{-1}$$

(3.7)

for $j = 1, 2$. Let $T_j = T_j^{(x_0, v_j)}$. We define a constant $h_{T_j}$ by

$$h_{T_j} := r^{n/2} \mathcal{M} \left( \int_G P_{\Omega, v_j}f_jd\Omega \right)(x_0),$$

(3.8)

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator. Then we have the followings.

- The margin of $\phi_{T_j}$ satisfies

$$\text{marg}(\phi_{T_j}) \geq \text{marg}(\phi_j) - Cr^{-1}.$$

(3.9)
• For \((x,t) \in \mathbb{R}^n \times [-CR,CR]\),

\[
|\phi_{T_j}(x,t)| \leq C_M c^{-C_r-n/2}h_{T_j} \left( 1 + \frac{\text{dist}(T_j, (x,t))}{r} \right)^{-M}, \quad \forall M > 0.
\]

•

\[
\left( \sum_{T_j \in T_j} h_{T_j}^2 \right)^{1/2} \lesssim c^{-C} \|f_j\|_2.
\]

Proof. Consider (3.9). From the definition of \(f_{T_j}\), we can see that the Fourier support of \(f_{T_j}\) is contained in a \(O(r^{-1})\)-neighborhood of \(v_j\). So, the spacetime Fourier transform of \(\phi_{T_j}\) is supported in a \(O(r^{-1})\)-neighborhood of the spacetime Fourier support of \(\phi_j\). From this we have (3.9).

Consider (3.10). If \(\rho\) is a smooth bump function supported on a \(O(1)\)-neighborhood of the origin and \(\rho^{v_j}(\xi) := \rho(4^{-1}r(\xi - v_j))\) then we may replace \(f_{T_j}\) with \(\rho^{v_j}f_{T_j}\). By interchanging the integrals we may write

\[
\phi_{T_j}(x,t) = \int K_{v_j}(x - y, t)f_{T_j}(y)dy,
\]

where

\[
K_{v_j}(x,t) = e^{2\pi i (x - \xi)} \rho(v_j)(\xi)\cdot d\xi.
\]

By integration by parts, if \(|t| \lesssim R\) then

\[
|K_{v_j}(x,t)| \leq C_mr^{-n} \left( 1 + \frac{|x - tv_j|}{r} \right)^{-M}, \quad \forall M > 0.
\]

Indeed, let \(\delta = r^{-1}\) and \(\Psi(x,t,\xi) = 2\pi(x \cdot (\delta\xi + v_j) - \frac{1}{2}t|\delta\xi + v_j|^2)\). We rewrite as

\[
K_{v_j}(x,t) = \delta^n \int e^{i\psi(x,t,\xi)} \rho(\xi)\cdot d\xi.
\]

Suppose that \(|x - tv_j| \geq C\delta^{-1}\). Then we have \(|\nabla_\xi \Psi(x,t,\xi)| \geq C\delta|x - tv_j|\), since \(|\delta^2 t\xi| \lesssim 1\). By integrating by parts,

\[
|K_{v_j}(x,t)| \leq C_M \delta^n \delta(x - tv_j)^{-M}, \quad \forall M > 0.
\]

On the other hand, we have a trivial estimate \(|K_{v_j}(x,t)| \lesssim \delta^n\). By combining these two estimates we have (3.13). Thus,

\[
|\phi_{T_j}(x,t)| \leq C_M r^{-n} \int \left( 1 + \frac{|(x - y) - tv_j|}{r} \right)^{-M} \eta^{v_j}(y) \left| \int_P P_{\Omega,v_j} f_j(y) d\Omega \right| dy
\]

\[
\leq C_M c^{-C} \left( 1 + \frac{|(x - x_0) - tv_j|}{r} \right)^{-M} \mathcal{M} \left( \int_P P_{\Omega,v_j} f_j d\Omega \right)(x_0)
\]

\[
\leq C_M c^{-C} r^{-n/2} h_{T_j} \left( 1 + \frac{\text{dist}(T_j, (x,t))}{r} \right)^{-M}.
\]

Consider (3.11). By the uncertainty principle, if \(|x - x_0| \lesssim c^{-2}r\) then

\[
\mathcal{M} \left( \int P_{\Omega,v_j} f_j d\Omega \right)(x_0) \lesssim c^{-C} \mathcal{M} \left( \int P_{\Omega,v_j} f_j d\Omega \right)(x).
\]

Thus,

\[
\sum_{T_j \in T_j} h_{T_j}^2 \lesssim c^{-C} \sum_{v_j \in V_j} \int \mathcal{M} \left( \int P_{\Omega,v_j} f_j d\Omega \right)(x)^2 dx.
\]
By the Hardy–Littlewood maximal theorem and Minkowski’s inequality,

\[ \left( \sum_{T_j \in T_j} \frac{h_{T_j}^2}{2} \right)^{1/2} \lesssim c \left( \sum_{v_j \in V_j} \left\| P_{\Omega,v_j} f_j d\Omega \right\|_{L^2} \right)^{1/2} \]

\[ \leq c \int \left( \sum_{v_j \in V_j} \left\| P_{\Omega,v_j} f_j \right\|_{L^2} ^2 \right)^{1/2} d\Omega. \]

By Plancherel’s theorem and orthogonality,

\[ \sum_{v_j \in V_j} \left\| P_{\Omega,v_j} f_j \right\|_{L^2} ^2 \lesssim \| f_j \|_{L^2} ^2. \]

Inserting this into the previous integral we have (3.11). \( \square \)

### 3.2. Estimates on a light conic set.

We define a kernel \( K_{j,t} \) by

\[ K_{j,t}(x) = \int e^{2\pi i (x - \frac{1}{2} t |\xi|^2)} a_j(\xi) d\xi. \]

Then \( \phi_j \) is written as

\[ \phi_j(x, t) = \mathcal{U}_j f(x, t) = K_{j,t} * f(x). \] (3.15)

**Lemma 3.2.** Let \( \Lambda_j \) be \( \{ x \in \mathbb{R}^n : (x, t) \in \Lambda_j(0) \} \) where \( \Lambda_j(0) \) is defined as in (3.10). Then,

\[ |K_{j,t}(x)| \leq C_M (1 + \text{dist}(\Lambda_{j,t}, x))^{-M}, \quad \forall M > 0. \] (3.16)

**Proof.** If \( \text{dist}(\Lambda_{j,t}, x) = 0 \), by a trivial estimate we have

\[ |K_{j,t}(x)| \lesssim 1. \]

Suppose that \( \text{dist}(\Lambda_{j,t}, x) > 0 \). The \( \xi \)-derivative of the phase \( x \cdot \xi - \frac{1}{2} t |\xi|^2 \) has

\[ \left| \nabla_{\xi} \left( x \cdot \xi - \frac{1}{2} t |\xi|^2 \right) \right| \geq \text{dist}(\Lambda_{j,t}, x) \]

for all \( \xi \in \tilde{\Xi}_j \). So, using integration by parts we obtain

\[ |K_{j,t}(x)| \leq C_M \text{dist}(\Lambda_{j,t}, x)^{-M}, \quad \forall M > 0. \]

Thus we have (3.11). \( \square \)

**Lemma 3.3.** Let \( Q \) be a cube of sidelength \( R^e \). Let \( \epsilon > 0 \) and \( j, k = 1, 2 \) but \( j \neq k \). Let \( \pi_k \subset \Xi_k \) be a hypersurface in \( \mathbb{R}^n \). Suppose that there is \( 0 \leq \varepsilon_0 < 1 \) such that for any \( v \in \Xi_j \) and any \( w, w' \in \pi_k \) with \( w \neq w' \),

\[ \left| \frac{v - w'}{w - w'} - \frac{w' - w}{w - w'} \right| \leq \varepsilon_0. \] (3.17)

Then for any \( r \geq R^e \),

\[ \| \phi_j \|_{L^2(Q \cap \Lambda_{\pi_k}(z_0; r))} \lesssim r^{1/2} E(\phi_j)^{1/2}. \]

**Proof.** It suffices to show

\[ \int \left\| \chi_{\Lambda_k^Q}(z_0; r) [\mathcal{U}_j(t)] f \right\|_{L^2(\mathbb{R}^n)}^2 dt \lesssim r^2 \| f \|_{L^2(\mathbb{R}^n)}^2, \]

where \( \Lambda_k^Q = \Lambda_{\pi_k} \cap Q \). By duality this is equivalent to

\[ \left\| \int \mathcal{U}_j(t)^\ast (\chi_{\Lambda_k^Q}(z_0; r) g(t)) dt \right\|_{L^2(\mathbb{R}^n)}^2 \lesssim r^2 \| g \|_{L^2(\mathbb{R}^{n+1})}^2. \] (3.18)

The left side of the above estimate is written as

\[ \int \int \langle [\mathcal{U}_j(s)]^\ast (\chi_{\Lambda_k^Q}(z_0; r) g(t)), (\chi_{\Lambda_k^Q}(z_0; r) g(s)) \rangle dt ds. \]
Let
\[ G_k(x, t) := \chi_{A_{k}^{Q}(z_0;r)}(x, t)g(x, t) \]
and
\[ I_j f(s, t) := \langle [\mathcal{U}_j(s)[\mathcal{U}_j(t)^*] f(t), f(s) \rangle. \]
The previous integral is divided into two parts
\[ \iint I_j G_k(s, t) dt ds = \iint_{|s-t| \leq Cr} I_j G_k(s, t) dt ds + \iint_{|s-t| \geq Cr} I_j G_k(s, t) dt ds. \]
To show (3.18) it suffices to prove
\[ \left| \iint_{|s-t| \geq Cr} I_j G_k(s, t) dt ds \right| \lesssim r^{-N} \|g\|_2^2 \tag{3.19} \]
and
\[ \left| \iint_{|s-t| \leq Cr} I_j G_k(s, t) dt ds \right| \lesssim r \|g\|_2^2. \tag{3.20} \]
Consider (3.19). If we set
\[ K_{j,t}(x) = \int e^{2\pi i(x \cdot \xi - \frac{1}{2} t |\xi|^2)} \varphi_j^2(\xi) d\xi \]
then
\[ [\mathcal{U}_j(s)[\mathcal{U}_j(t)^*] f(x) = \int K_{j,s-t}(x-y) f(y) dy. \]
We rewrite \( I_j G_k \) as
\[ I_j G_k = \iint K_{j,s-t}(x-y) \chi_{A_{k}^{Q}(z_0;r)}(y, t) \chi_{A_{k}^{Q}(z_0;r)}(x, s) g(y, t) \bar{g}(x, s) dx dy. \]
We divide
\[ K_{j,t} = \chi_{A_{j,t}(0;r)} K_{j,t} + (1 - \chi_{A_{j,t}(0;r)}) K_{j,t}. \]
From Lemma 3.2 it follows that
\[ (1 - \chi_{A_{j,s-t}(0;r)})(x)) |K_{j,s-t}(x)| \lesssim r^{-M}, \quad \forall M > 0. \]
Using this we have
\[ \left| \iiint ((1 - \chi_{A_{j,s-t}(0;r)}) K_{j,s-t}(x-y) \times \chi_{A_{k}^{Q}(z_0;r)}(y, t) \chi_{A_{k}^{Q}(z_0;r)}(x, s) g(y, t) \bar{g}(x, s) dx dy dt ds \right| \]
\[ \lesssim r^{-M} \|\chi_{A_{k}^{Q}(z_0;r)} g\|_2^2 \]
\[ \lesssim r^{-M} R C \|g\|_2^2 \]
\[ \lesssim r^{-M} \|g\|_2^2, \quad \forall M > 0, \]
where the last line follows from \( r \geq R^c \). Now, to show (3.19) it suffices to show
\[ \iint_{|s-t| \geq Cr} \chi_{A_{j,s-t}(0;r)}(x-y) K_{j,s-t}(x-y) \times \chi_{A_{k}^{Q}(z_0;r)}(y, t) \chi_{A_{k}^{Q}(z_0;r)}(x, s) g(y, t) \bar{g}(x, s) dx dy dt ds = 0. \]
It is enough to show that for \( s, t \in \mathbb{R} \) with \( |s-t| \geq Cr \), the equation
\[ \chi_{A_{j,s-t}(0;r)}(x-y) \chi_{A_{k}^{Q}(z_0;r)}(y, t) \chi_{A_{k}^{Q}(z_0;r)}(x, s) \]
(3.21)
vanishes. Consider the contrapositive statement that if \((3.21)\) is nonzero then one has \(|s-t| \lesssim r\). Suppose that \((3.21)\) is nonzero. Then, from the characteristic function \(\chi_{\Lambda_{s-t}(0,r)}(x-y)\) we can restrict ourselves to the case
\[
x - y = (s-t)v + O(r)
\]
for some \(v \in \Xi_j\). On the other hands, from \(\chi_{\Lambda_{\pi_{x-r}(0,r)}(y,t)}\) and \(\chi_{\Lambda_{\pi_{x-r}(0,r)}}(x,s)\) we also have
\[
x - x_0 = (s-t_0)w + O(r)
y - x_0 = (t-t_0)w' + O(r)
\]
for some \(w, w' \in \pi_k\). By combining \((3.22)\) and \((3.23)\) it follows that
\[
(s-t)(v - w) + (t-t_0)(w' - w) = O(r).
\]
If \(w = w'\), then we have \(|s-t| \lesssim r\). Otherwise, from \((3.17)\) we can see that there exists a unit vector \(u\) such that \((v-w) \cdot u \neq 0\) but \((w'-w) \cdot u = 0\). By taking inner product with such \(u\) for \((3.21)\), we have \(|s-t| \lesssim r\).

Consider \((3.20)\). By the Cauchy–Schwarz inequality and Plancherel’s theorem it follows that
\[
|I_j f(s, t)| \lesssim \|f(s)\| L^2(s) \|f(t)\| L^2(t).
\]
By this and the Hardy–Littlewood–Sobolev inequality,
\[
\left| \int \int_{|s-t| \leq r} I_j G_k(s, t) dtds \right| \lesssim \int \int_{|s-t| \leq r} \|g(s)\| L^2(s) \|g(t)\| L^2(t) dtds
\]
\[
\lesssim r \|g\| L^2.
\]
Thus we have \((3.20)\). \(\square\)

3.3. A basic bilinear restriction estimate. Let \(\psi\) be a nonnegative Schwartz function on \(\mathbb{R}^{n+1}\) with \(\int \psi = 1\) such that \(\tilde{\psi}\) is supported in the unit ball and
\[
\sum_{k \in \mathbb{Z}^{n+1}} \psi^4(z - k) = 1
\]
where \(z = (x, t) \in \mathbb{R}^{n+1}\). If \(q\) is a cube of side length \(r_q\) with center \(z_q\), we define
\[
\psi_q(z) := \psi(r_q^{-1}(z - z_q)).
\]
For convenience, we use the notations \(\int \psi \cdot \) and \(\| \cdot \|_{L^2(\psi)}\) to denote
\[
\int \psi := \int f \tilde{\psi} \quad \text{and} \quad \|f\|_{L^2(\psi)} := \|f \tilde{\psi}\|_2.
\]

Lemma 3.4. Let \(q\) be a cube of side-length \(R^{1/2}\). Suppose that \(\phi_{T_1}\) and \(\phi_{T_2}\) are wave packets defined as \((3.5)\). Then
\[
\|\phi_{T_1} \phi_{T_2}\|_{L^2(\psi_q)} \lesssim R^{-\frac{n+1}{4}} \|\phi_{T_1}\|_{L^2(\psi_q)} \|\phi_{T_2}\|_{L^2(\psi_q)}.
\]

Proof. By Plancherel’s theorem the estimate \((3.20)\) is equivalent to
\[
\|\psi_q \phi_{T_1} \ast \psi_q \phi_{T_2}\|_2 \lesssim R^{-\frac{n+1}{4}} \|\psi_q \phi_{T_1}\|_2 \|\psi_q \phi_{T_2}\|_2.
\]
By interpolation it suffices to show the following two estimates:
\[
\|\psi_q \phi_{T_1} \ast \psi_q \phi_{T_2}\|_1 \lesssim \|\psi_q \phi_{T_1}\|_1 \|\psi_q \phi_{T_2}\|_1,
\]
\[
\|\psi_q \phi_{T_1} \ast \psi_q \phi_{T_2}\|_\infty \lesssim R^{-\frac{n+1}{4}} \|\psi_q \phi_{T_1}\|_\infty \|\psi_q \phi_{T_2}\|_\infty.
\]
By Young’s inequality the first one is easily obtained. Consider the second one. Observe that the Fourier support of $\psi_0 \phi_0$ is contained in a ball $B(v(T_j), -|v(T_j)|^2/2; CR^{-1/2})$. Let $\chi_j := X_B(v(T_j), -|v(T_j)|^2/2; CR^{-1/2})$ be a characteristic function. Then,

$$\| \psi_0 \phi_0 \|_{L^1} \lesssim \| \chi_1 * \chi_2 \|_{L^1} \| \psi_0 \phi_0 \|_{L^\infty}.$$  

Simple computation gives $\| \chi_1 * \chi_2 \|_{L^1} \lesssim R^{-\frac{10}{n}}$. Thus we have the second estimate. □

4. Refining the proof of the sharp bilinear restriction estimate

In this section we refine the proof of the sharp bilinear paraboloid restriction estimate due to Tao [24].

4.1. Decomposition for bilinear estimates. Let $Q$ be the cube of sidelength $CR$ and centered at the origin. We decompose $Q$ into subcubes $\Delta$ of sidelength $2^{-C_0} CR$. For any integer $l$, we define $Q_l(Q)$ to be the collection of subcubes of sidelength $2^{-l} CR$ which are obtained by dividing the sides of $Q$.

Let

$$\Psi_{D(x_D, r_D)}(x) := \left(1 + \frac{|x - x_D|}{r_D}\right)^{-N\cdot 100}.$$  

and for each tube $T_j = T_j^{(x_0, v_j)}$,

$$\Psi_{T_j}(x, t) := \Psi_{D(x_0 - tv_j, r)}(x). \quad (4.1)$$

For each $2^{-C_0} CR$-cube $\Delta \in Q_{C_0}(Q)$ and each $T_j \in T_j$, we define

$$m_{\Delta, T_j} := \sum_{q \in Q_{C_0}(Q)} \sum_{T_i \in T_j} \| \psi_q \phi_{T_i} \|_{L^2(\psi_q)}^2 + R^{-10n} E(\phi_i) \quad (4.2)$$

and

$$m_{T_j} := \sum_{q \in Q_{C_0}(Q)} \sum_{T_i \in T_j} \| \psi_q \phi_{T_i} \|_{L^2(\psi_q)}^2 + R^{-10n} 2^{(n+1)C_0} E(\phi_i) \quad (4.3)$$

for $i \neq j$. Then,

$$\sum_{\Delta \in Q_{C_0}(Q)} m_{\Delta, T_j} = 1. \quad (4.4)$$

We now define $\Phi_j(\Delta)$ for each $\Delta \in Q_{C_0}(Q)$ by

$$\Phi_j(\Delta)(z) := \sum_{T_j \in T_j} m_{\Delta, T_j} \phi_{T_j}(z). \quad (4.5)$$

Then,

$$\phi_j(z) = \sum_{\Delta \in Q_{C_0}(Q)} \Phi_j(\Delta)(z). \quad (4.6)$$

We define a function $[\Phi_j]$ by

$$[\Phi_j](z) := \sum_{\Delta \in Q_{C_0}(Q)} \Phi_j(\Delta)(z) \chi_\Delta(z). \quad (4.7)$$

The main proposition of this section is as follows.
Proposition 4.1. Let $R \geq 2^{C_0}$ and $0 < c \leq 2^{-C_0}$. For any cube $Q$ we define a set $X(Q)$ by

$$X(Q) := \bigcup_{\Delta \in \mathcal{Q}_{c_0}(Q)} (1 - c)\Delta.$$

Suppose that $\phi_1, \phi_2$ have Fourier supports in $\Sigma_1$ and $\Sigma_2$ respectively which satisfy the normalization (2.14) and the relaxed margin condition (2.10). Then,

$$\|\phi_1\phi_2\|_{L^p(Q_R)} \leq (1 + Cc)\|\Phi_1\|_{L^p(X(Q))} + c^{-C},$$

where $Q$ is a cube of sidelength $CR$ contained in $C^2Q_R$ and

$$\text{marg}(\Phi_j) \geq \frac{1}{100} - \left(\frac{2^{C_0}}{R}\right)^{1/N}.$$

In the remaining parts of this section we will prove the above proposition. Consider the margin (4.9). From (4.9) and (4.5) it follows that

$$\text{marg}(\Phi_j) \geq \text{marg}(\phi_j) - CR^{-1/2} \geq \frac{1}{100} - 2\left(\frac{1}{R}\right)^{1/N} - C\left(\frac{1}{R}\right)^{1/2} \geq \frac{1}{100} - \left(\frac{2^{C_0}}{R}\right)^{1/N}.$$

The proof of (4.8) is accomplished through many steps. We begin with the following averaging lemma.

Lemma 4.2 (Lemma 6.1 in [23]). Let $R > 0$, $0 < c \leq 2^{-C_0}$, and let $Q_R$ be a cube of sidelength $R$. If $f$ is a smooth function, then there exists a cube $Q$ of sidelength $CR$ contained in $C^2Q_R$ such that

$$\|f\|_{L^p(Q_R)} \leq (1 + Cc)\|f\|_{L^p(X(Q))}.$$ 

By this lemma,

$$\|\phi_1\phi_2\|_{L^p(Q_R)} \leq (1 + Cc)\|\phi_1\phi_2\|_{L^p(X(Q))}.$$ 

Using the triangle inequality we divide $\|\phi_1\phi_2\|_{L^p(X(Q))}$ into three parts:

$$\|\phi_1\phi_2\|_{L^p(X(Q))} \leq \|\Phi_1\|_{L^p(X(Q))} + \|\Phi_1\|_{L^p(X(Q))} + \|\Phi_1\|_{L^p(X(Q))}.$$ 

To prove (4.8) it suffices to show

$$\|\phi_1 - [\Phi_1]\|_{L^p(X(Q))} \lesssim c^{-C}, \quad (4.10)$$

$$\|\Phi_1\phi_2 - [\Phi_2]\|_{L^p(X(Q))} \lesssim c^{-C}.$$ 

Since these two estimates are similarly obtained, we will consider only the first one.

4.2. $L^1$-bilinear estimates.

Lemma 4.3.

$$\|\phi_1 - [\Phi_1]\|_{L^1(X(Q))} \lesssim c^{-C}R.$$ 

Proof. By the Cauchy–Schwarz inequality it suffices to show

$$\|\phi_j\|_{L^2(Q)} \lesssim R^{1/2},$$

$$\|\phi_j - [\Phi_j]\|_{L^2(Q)} \lesssim c^{-C}R^{1/2},$$

for $j = 1, 2$. Consider (4.11). We have

$$\|\phi_j\|_{L^2(Q)}^2 \leq \int_{-CR}^{CR} \|\phi_j(t)\|_{L^2(Q)}^2 dt \lesssim RE(\phi_j) \leq R.$$
Thus (4.11) follows. Consider (4.12). By the triangle inequality and (4.11) it suffices to show
\[ \| [\Phi_j] \|_{L^2(Q)} \lesssim c^{-C} R^{1/2}. \] (4.13)

We have that
\[ \| \Phi_j^{(\Delta)} \|_{L^2(\Delta)}^2 \lesssim \int_{t_{\Delta} - c_2 - c_0 R}^{t_{\Delta} + c_2 - c_0 R} \| \Phi_j^{(\Delta)}(t) \|_2^2 \, dt \lesssim 2^{-C_0} R E(\Phi_j^{(\Delta)}). \]

By using (3.10),
\[ \sum_{\Delta \in \mathcal{Q}_{c_0}(Q)} E(\Phi_j^{(\Delta)}) = \sum_{\Delta \in \mathcal{Q}_{c_0}(Q)} \left\| \sum_{T_j \in T_j} \frac{m \Delta T_j}{m T_j} \phi_{T_j}(0) \right\|_2^2 \lesssim c^{-C} \sum_{\Delta \in \mathcal{Q}_{c_0}(Q)} \sum_{T_j \in T_j} \left( \frac{m \Delta T_j}{m T_j} \right)^2 h_{T_j}^2. \]

By (4.6), (4.7) and the triangle inequality,
\[ \sum_{\Delta \in \mathcal{Q}_{c_0}(Q)} E(\Phi_j^{(\Delta)}) \lesssim c^{-C} \sum_{T_j \in T_j} h_{T_j}^2 \lesssim c^{-C} E(\phi_j). \]

By the above estimates,
\[ \| [\Phi_j] \|_{L^2(Q)}^2 = \sum_{\Delta \in \mathcal{Q}_{c_0}(Q)} \| \Phi_j^{(\Delta)} \|_{L^2(\Delta)}^2 \lesssim 2^{-C_0} c^{-C} R E(\phi_j) \leq c^{-C} R. \]

Thus we have (4.13). \( \square \)

4.3. Orthogonality. By interpolation it now suffices to show
\[ \| (\phi_1 - [\Phi_1])\phi_2 \|_{L^2(X(Q))} \lesssim c^{-C} R^{-\frac{m+1}{4}}. \] (4.14)

By (4.6), (4.7) and the triangle inequality,
\[ \| (\phi_1 - [\Phi_1])\phi_2 \|_{L^2(X(Q))} \leq \sum_{\Delta \in \mathcal{Q}_{c_0}(Q)} \| \Phi_1^{(\Delta)} \phi_2 \|_{L^2(X(Q) \setminus \Delta)}. \]

Since the number of cubes in \( \mathcal{Q}_{c_0}(Q) \) is \( 2^{(n+1)C_0} \lesssim c^{-C} \), it is reduced to showing
\[ \| \Phi_1^{(\Delta)} \phi_2 \|_{L^2(X(Q) \setminus \Delta)} \lesssim c^{-C} R^{-\frac{m+1}{4}}. \] (4.15)

Observe that if \( q \in \mathcal{Q}_\kappa(Q) \) meets \( X(Q) \setminus \Delta \), then \( \text{dist}(q, \Delta) \geq cR \). By using (3.20), (3.26) and (4.5),
\[ \| \Phi_1^{(\Delta)} \phi_2 \|_{L^2(X(Q) \setminus \Delta)} \lesssim \sum_{q \in \mathcal{Q}_\kappa(Q)} \| \Phi_1^{(\Delta)} \phi_2 \|_{L^2(q)} \]
\[ = \sum_{q \in \mathcal{Q}_\kappa(Q)} \int_{\psi_q} \left( \sum_{T_1 \in T_1} \frac{m \Delta T_1}{m T_1} \phi_{T_1}(z) \right) \left( \sum_{T_2 \in T_2} \frac{m \Delta T_2}{m T_2} \phi_{T_2}(z) \right) \, dz \]
\[ \lesssim \sum_{T_1, T_1' \in T_1, T_2 \in T_2} \int_{\psi_q} \left( \frac{m \Delta T_1}{m T_1} \phi_{T_1}(z) \phi_{T_2}(z) \right) \left( \frac{m \Delta T_2}{m T_2} \phi_{T_1'}(z) \phi_{T_2'}(z) \right) \, dz. \] (4.16)

where \( z \) denotes \( (x, t) \). We write the integral in the above equation as
\[ \sum_{T_1, T_1' \in T_1, T_2 \in T_2} \int_{\psi_q} \left( \frac{m \Delta T_1}{m T_1} \phi_{T_1}(z) \phi_{T_2}(z) \right) \left( \frac{m \Delta T_2}{m T_2} \phi_{T_1'}(z) \phi_{T_2'}(z) \right) \, dz. \] (4.17)
We define $S$ to be the set of $(v_1, v_1', v_2, v_2') \in V_1 \times V_1 \times V_2 \times V_2$ such that

$$v_1 + v_2 = v_1' + v_2' + O(r^{-1}),$$

$$|v_1|^2 + |v_2|^2 = |v_1'|^2 + |v_2'|^2 + O(r^{-1}).$$

Lemma 4.4. Let $S^c = V_1 \times V_1 \times V_2 \times V_2 \setminus S$ and let $q$ be a cube of side-length $R^{1/2}$. Suppose that $T_1, T_1' \in T_1, T_2, T_2' \in T_2$ satisfy $(v(T_1), v(T_1'), v(T_2), v(T_2')) \in S$. Then,

$$\left| \int_{\psi_q^2} \phi_{T_1}(z) \phi_{T_2}(z) \overline{\phi_{T_1}'}(z) \overline{\phi_{T_2}'}(z) \, dz \right| = 0. \quad (4.19)$$

Proof. By Parseval’s formula the left side of (4.19) equals to

$$\left| \langle \psi_q \hat{\phi}_{T_1} \ast \psi_q \hat{\phi}_{T_2}, \ \psi_q \hat{\phi}_{T_1} \ast \psi_q \hat{\phi}_{T_2} \rangle \right|,$$

where the hat denotes the spacetime Fourier transform.

By the construction of wave packets $\phi_{T_i}$, we see that $\psi_q \hat{\phi}_{T_i}$ is supported on a $O(r^{-1})$-neighborhood of $(v(T_i), -|v(T_i)|^2/2)$, and that $\psi_q \hat{\phi}_{T_1} \ast \psi_q \hat{\phi}_{T_2}$ is supported on a $O(r^{-1})$-neighborhood of $(v(T_1) + v(T_2), -(|v(T_1)|^2 + |v(T_2)|^2)/2)$. Thus we can see that if $(v(T_1), v(T_1'), v(T_2), v(T_2'))$ do not satisfy (4.18) then the supports of $\psi_q \hat{\phi}_{T_1} \ast \psi_q \hat{\phi}_{T_2}$ and $\psi_q \hat{\phi}_{T_1} \ast \psi_q \hat{\phi}_{T_2}'$ are disjoint, so we have (4.19).

By the above lemma,

$$\left(4.17\right) \leq \sum_{(T_1, T_1', T_2, T_2') \in S} \left| \int_{\psi_q^2} \left( \frac{m \Delta T_1}{m T_1} \phi_{T_1} \phi_{T_2} \right) \left( \frac{m \Delta T_1'}{m T_1'} \phi_{T_1} \phi_{T_2} \right) \, dz \right| \quad (4.20)$$

where

$$S = \{(T_1, T_1', T_2, T_2') \in T_1 \times T_1 \times T_2 \times T_2 : (v(T_1), v(T_1'), v(T_2), v(T_2')) \in S\}.$$

Using the arithmetic-geometric mean inequality, we divide the integral of the above estimate as follows:

$$\left| \int_{\psi_q^2} \left( \frac{m \Delta T_1}{m T_1} \phi_{T_1}(z) \phi_{T_2}(z) \right) \left( \frac{m \Delta T_1'}{m T_1'} \phi_{T_1}'(z) \phi_{T_2}'(z) \right) \, dz \right|$$

$$\leq \int_{\psi_q^2} \left| \frac{m \Delta T_1}{m T_1} \phi_{T_1}(z) \phi_{T_2}(z) \right|^2 \, dz$$

$$+ \int_{\psi_q^2} \left| \frac{m \Delta T_1'}{m T_1'} \phi_{T_1}'(z) \phi_{T_2}'(z) \right|^2 \, dz,$$

where $z_q$ is the center of $q$. The two integrals of the right side in the above equation are of the same form. By combining (4.18), (4.20) and the above estimate, the estimate (4.17) is reduced to showing

$$\sum_{q \in Q, (Q) : \ \text{dist}(q, \Delta) \geq R} \sum_{(T_1, T_1', T_2, T_2') \in S} \int_{\psi_q^2} \left| \frac{m \Delta T_1}{m T_1} \phi_{T_1}(z) \phi_{T_2}(z) \right|^2 \, dz \leq c^{-2} R^{-2 \frac{3}{2}}. \quad (4.21)$$

We separate the summation $\sum_{(T_1, T_1', T_2, T_2') \in S}$ into two parts

$$\sum_{(T_1, T_1', T_2, T_2') \in S} = \sum_{T_1, T_1' \in T_1, T_2 \in T_2} \sum_{T_1' \in T_1, T_2' \in T_2} \sum_{(v(T_1), v(T_1'), v(T_2), v(T_2')) \in S}.$$
By rearranging the left side of (4.21) is bounded by
\[
\sum_{q \in \mathcal{Q}_n(Q): \frac{\Delta}{mT_1}, \frac{\Delta}{T_2} \leq cR} \sum_{T_1 \in T_1, T_2 \in T_2} \left( \frac{1}{mT_1} \int_{\psi_q^T} \left| \frac{\phi_{T_1}(z) \phi_{T_2}(z)}{\Psi_{T_1}(z_q)} \right| \right)^2 dz \sum_{T_1' \in T_1, T_2' \in T_2} m_{\Delta, T_1} \Psi_{T_1}^2(z_q),
\]
where \( \left( \frac{\Delta}{mT_1} \right)^2 \leq \left( \frac{\Delta}{mT_2} \right)^2 \) is used. To show (4.21) it suffices to prove the following two estimates:
\[
\max_{q \in \mathcal{Q}_n(Q): \frac{\Delta}{mT_1}, \frac{\Delta}{T_2} \leq cR} \sum_{T_1 \in T_1, T_2 \in T_2} m_{\Delta, T_1} \Psi_{T_1}^2(z_q) \lesssim c^{-C} R^{1/2}
\]
and
\[
\sum_{q \in \mathcal{Q}_n(Q): \frac{\Delta}{mT_1}, \frac{\Delta}{T_2} \leq cR} \sum_{T_1 \in T_1, T_2 \in T_2} \frac{1}{mT_1} \int_{\psi_q^T} \left| \frac{\phi_{T_1}(z) \phi_{T_2}(z)}{\Psi_{T_1}(z_q)} \right|^2 dz \lesssim c^{-C} R^{-n/2}.
\]

4.4. **Proof of the estimate (4.22)**. We take a close look at the condition
\[(v(T_1), v(T_1'), v(T_2), v(T_2')) \in S.\]
From the first equation of (4.18) we have
\[v_2' = v_1 + v_2 - v_1' + O(r^{-1}).\]
Inserting this into the second equation of (4.18), we have
\[|v_1|^2 - |v_1 + v_2 - v_1'|^2 = |v_1'|^2 - |v_2|^2 + O(r^{-1}),\]
which is equivalent to
\[(v_1' - v_1) \cdot (v_1' - v_2) = O(r^{-1}).\]
Let \(\sigma(v_1, v_2) \subset \mathbb{R}^n\) be the sphere of radius \(\frac{|v_1 - v_2|}{2}\) with center \(\frac{v_1 + v_2}{2}\). Then the above equation means that \(v_1'\) lies in the \(O(r^{-1})\)-neighborhood \(\sigma(v_1, v_2; Cr^{-1})\) of the sphere \(\sigma(v_1, v_2)\). Thus, if \(v_1, v_2\) are given then one has \(v_1' \in \sigma(v_1, v_2; Cr^{-1})\). Also, if \(v_1, v_2, v_1', v_2'\) are given then by (4.24) we see that \(v_2'\) is contained in a ball \(B(v_1 + v_2 - v_1'; Cr^{-1})\).

Now we use this observation. For given \(T_1, T_2\), we have that \(v(T_1)\) is contained in \(\sigma(v(T_1), v(T_2); Cr^{-1})\).

If \(v(T_1), v(T_2), v(T_1')\) is determined then \(v(T_2')\) has \(O(1)\) choices, and if \(v(T_2')\) is determined then the number of \(T_2''\) passing through \(z_0\) is only one. Thus to show (4.22) it suffices to show that for given \(T_1 \in T_1, T_2 \in T_2, q \in \mathcal{Q}_n(Q)\) with \(\text{dist}(q, \Delta) \geq cR\),
\[
\sum_{T_1' \in T_1: \text{dist}(q, \Delta) \geq cR} m_{\Delta, T_1} \Psi_{T_1}^2(z_q) \lesssim c^{-C} R^{1/2},
\]
where \(v_1 = v(T_1)\) and \(v_2 = v(T_2)\). By (4.2), we have
\[
\sum_{T_1' \in T_1: \text{dist}(q, \Delta) \geq cR} m_{\Delta, T_1} \Psi_{T_1}^2(z_q) \lesssim \sum_{T_2 \in T_2} \int_{4Q} |\phi_{T_2}(z)|^2 \Gamma_{\Delta}(z) dz + R^{-C},
\]
where
\[
\Gamma_{\Delta}(z) := \sum_{T_1' \in T_1: \text{dist}(q, \Delta) \geq cR} \psi_{T_1'}(z) \Psi_{T_1'}^2(z_q).
\]
Consider $\Gamma_\beta(q)$ passing through $z_q$ with $v(T'_1) \in V_1 \cap \sigma(v_1, v_2; C r^{-1})$. The union of $T'_1$ passing through $z_q$ forms a conic set $\Lambda_{\pi_1}(z_q; C r)$ where

$$\pi_1 := \sigma(v_1, v_2) \cap \Xi.$$

From $\text{dist}(\Delta, q) \geq c R$ it follows that the tubes $T'_1$ passing through $z_q$ can overlap at most $O(1)$ times on $\Delta$. Thus,

$$\int_{4q} |\phi T_2(z)|^2 \Gamma_q(z) dz \lesssim c^{-C} \int_{4q} |\phi T_2(z)|^2 \left(1 + \frac{\text{dist}(z, \Lambda_{\pi_1}(z_q))}{r}\right)^{-N/10} dz.$$

By combining this with (4.26),

$$\sum_{T'_1 \in T_1: v(T'_1) \in V_1 \cap \sigma(v_1, v_2; C r^{-1})} m_{\Delta, T'_1} \Psi_{T'_1}^2(z_q) \lesssim c^{-C} \sum_{T_2 \in T_2} \int_{4q} |\phi T_2(z)|^2 \left(1 + \frac{\text{dist}(z, \Lambda_{\pi_1}(z_q))}{r}\right)^{-N/10} dz + R^{-C}.$$

By a dyadic decomposition, to prove (4.25), it suffices to show

$$\sum_{T'_2 \in T_2} \|\phi T_2\|_{L^2(\Delta; \Lambda_{\pi_1}(z_q; Cr))}^2 \lesssim c^{-C} r. \quad (4.27)$$

We observe that for any $w_2 \in \Xi_2$ and $w_1, w'_1 \in \pi_1$ with $w_1 \neq w'_1$, there exists $0 < \varepsilon_0 < 1$ such that

$$\left(\frac{w_2 - w_1}{|w_2 - w_1|}, \frac{w'_1 - w_1}{|w'_1 - w_1|}\right) \leq \varepsilon_0. \quad (4.28)$$

Indeed, if we take $\xi_2 \in \sigma(v_1, v_2)$ such that $w'_1 + \xi_2 = v_1 + v_2$, that is, $(w'_1 + \xi_2)/2$ is the center of $\sigma(v_1, v_2)$, then $(w_1 - \xi_2, w'_1 - w_1) = 0$. Using this we have

$$\text{the left side of (4.28)} = \left(\frac{w_2 - \xi_2}{|w_2 - w_1|}, \frac{w'_1 - w_1}{|w'_1 - w_1|}\right).$$

So the above equation is bounded by $|w_2 - \xi_2|/|w_2 - w_1|$. From the definition of $\Xi_1$ and $\Xi_2$, we can see that there is $0 < \varepsilon_0 < 1$ such that $|w_2 - \xi_2|/|w_2 - w_1| \leq \varepsilon_0$. Thus we have (4.28). By Lemma 3.3

the left side of (4.27) $\lesssim r \sum_{T_2 \in T_2} E(\phi T_2).$ 

By (3.10), (3.11) and (2.7),

$$\sum_{T_2 \in T_2} E(\phi T_2) \lesssim c^{-C} \sum_{T_2 \in T_2} h_{T_2}^2 \lesssim c^{-C}. \quad (4.29)$$

By combining these two estimates we obtain (4.27).

\[ \blacksquare \]

**4.5. Proof of the estimate (4.23).** Consider the integral in the left side of (4.23). By applying Lemma 3.4 to the left side of (4.23) it suffices to show

$$\sum_{q \in Q_\omega(Q)} \sum_{T_1 \in T_1, T_2 \in T_2} \frac{1}{m_{T_1}} \frac{\|\phi T_1\|_{L^2(\psi_q)}^2}{\Psi_{T_1}^2(z_q)} \frac{\|\phi T_2\|_{L^2(\psi_q)}^2}{\Psi_{T_2}^2(z_q)} \lesssim c^{-C} R^{1/2}. \quad (4.30)$$

The left side of the above equation is written as

$$\sum_{T_1 \in T_1} \left(\sup_{q \in Q_\omega(Q)} \frac{\|\phi T_1\|_{L^2(\psi_q)}^2}{\Psi_{T_1}^2(z_q)}\right) \left(\frac{1}{m_{T_1}} \sum_{q \in Q_\omega(Q)} \sum_{T_2 \in T_2} \Psi_{T_2}^2(z_q) \|\phi T_2\|_{L^2(\psi_q)}^2\right).$$
Consider the inner summand. Since both the width of $T$ and the sidelength of $q$ are $\sim r$, by some basic estimates it follows that for any tube $T \in T_1 \cup T_2$ and $q \in Q_n$,

$$\Psi_T(z)\psi_q^{1/2}(z) \sim \Psi_T(z_q).$$  \hspace{1cm} (4.31)

Using this equation we have

$$\sum_{q \in Q_n(Q)} \sum_{T_2 \subset T_2} \Psi_{T_1}(z_q)\|\phi_{T_2}\|_{L^2(\psi_q)}^2 \lesssim \sum_{q \in Q_n(Q)} \sum_{T_2 \in T_2} \|\Psi_{T_1}\|_{L^2(\psi_q^{3/2})}^2,$$

which is $\lesssim m_{T_1}$ by (4.3). Now to prove (4.30) it is enough to show

$$\sum_{T_1 \in T_1} \sup_{q \in Q_n(Q)} \frac{\|\phi_{T_1}\|_{L^2(\psi_q)}}{\Psi_{T_1}(z_q)} \lesssim c^{-C}R^{1/2}.$$

By (3.10) and (4.31),

$$\frac{\|\phi_{T_1}\|_{L^2(\psi_q)}}{\Psi_{T_1}(z_q)} \lesssim c^{-C}r^{-n}h_{T_1}^2 \quad \frac{\|\Psi_{T_1}\|_{L^2(\psi_q)}}{\Psi_{T_1}(z_q)} \lesssim c^{-C}h_{T_1}^2.$$

Therefore, by (3.11)

$$\sum_{T_1 \in T_1} \sup_{q \in Q_n(Q)} \frac{\|\phi_{T_1}\|_{L^2(\psi_q)}}{\Psi_{T_1}(z_q)} \lesssim c^{-C}r \sum_{T_1 \in T_1} h_{T_1}^2 \lesssim c^{-C}R^{1/2}.$$

$\square$

5. Proof of Proposition 2.3

By Proposition 4.1 it suffices to show

$$\|[[\Phi_1][\Phi_2]]\|_{L^p(Q)} \leq (1 + Cc)\overline{K}(R/C_0)E(\phi_1)^{1/2}E(\phi_2)^{1/2}. \hspace{1cm} (5.1)$$

By (4.7),

$$\|[[\Phi_1][\Phi_2]]\|_{L^p(Q)} \leq \sum_{\Delta \in Q_{c_0}(Q)} \|\Phi_1(\Delta)\Phi_2(\Delta)\|_{L^p(\Delta)}.$$

From Proposition 1.1 we see that marg($\Phi_j(\Delta)$) $\geq 1/100 - (2C^n/R)^{1/N}$. By Definition 2.2 the right side of the above estimate is bounded by

$$\overline{K}(R/C_0) \sum_{\Delta \in Q_{c_0}(Q)} E(\Phi_1(\Delta))^{1/2}E(\Phi_2(\Delta))^{1/2}.$$

By the Cauchy–Schwarz inequality, this is bounded by

$$\overline{K}(R/C_0)\left(\sum_{\Delta \in Q_{c_0}(Q)} E(\Phi_1(\Delta))\right)^{1/2} \left(\sum_{\Delta \in Q_{c_0}(Q)} E(\Phi_2(\Delta))\right)^{1/2}.$$

Thus, to show (5.1) it suffices to show

$$\left(\sum_{\Delta \in Q_{c_0}(Q)} E(\Phi_j(\Delta))\right)^{1/2} \leq (1 + Cc)E(\phi_j)^{1/2}$$

for $j = 1, 2$. 
Lemma 5.1. Let $Q$ be a finite index set. Suppose that $m_{q,T_j}$ are non-negative numbers such that

$$\sum_{q \in Q} m_{q,T_j} \leq 1$$  \hspace{1cm} (5.2)

for all $T_j \in T$. Then,

$$\left( \sum_{q \in Q} E \left( \sum_{T_j \in T} m_{q,T_j} \phi_{T_j} \right) \right)^{1/2} \leq (1 + Cc)E(\phi_j)^{1/2}. \hspace{1cm} (5.3)$$

**Proof.** To get the constant $(1 + Cc)$, we need to consider the Fourier support of $f_{T_j}$. Let

$$Y_j = \bigcup \{ \xi \in B_{v_j} : \text{dist}(\xi, \Xi_j \setminus B_{v_j}) > Cc^2R^{-1/2} \}.$$  

We define the operator $P_{\Omega(Y)}$ by

$$P_{\Omega(Y)}f = \hat{\chi}_{\Omega(Y)} \hat{f}.$$

By Minkowski’s inequality,

$$\left( \sum_{q \in Q} E \left( \sum_{T_j \in T} m_{q,T_j} \phi_{T_j} \right) \right)^{1/2} = \left( \sum_{q \in Q} \left\| \sum_{T_j \in T} m_{q,T_j} \phi_{T_j}(0) \right\|_2 \right)^{1/2}$$

$$= \left( \sum_{q \in Q} \left\| \sum_{T_j \in T} m_{q,T_j} \int_{G} \eta^{x(T_j)} P_{\Omega\wedge(T_j)} \phi_j(0) d\Omega \right\|_2 \right)^{1/2}$$

$$\leq \int_{G} \left( \sum_{q \in Q} \left\| \sum_{T_j \in T} m_{q,T_j} \eta^{x(T_j)} P_{\Omega\wedge(T_j)} \phi_j(0) \right\|_2 \right)^{1/2} d\Omega,$$

which is less than or equal to the sum of

$$\int \left( \sum_{q \in Q} \left\| \sum_{T_j \in T} m_{q,T_j} \eta^{x(T_j)} P_{\Omega\wedge(T_j)} \phi_j(0) \right\|_2 \right)^{1/2} d\Omega \hspace{1cm} (5.4)$$

and

$$\int \left( \sum_{q \in Q} \left\| \sum_{T_j \in T} m_{q,T_j} \eta^{x(T_j)} P_{\Omega\wedge(T_j)} (1 - P_{\Omega\wedge(Y_j)}) \phi_j(0) \right\|_2 \right)^{1/2} d\Omega. \hspace{1cm} (5.5)$$

To prove (5.3), it suffices to show that

$$\text{(5.3)} \leq E(\phi_j)^{1/2} \hspace{1cm} \text{and} \hspace{1cm} \text{(5.5)} \leq CcE(\phi_j)^{1/2}.$$

Consider (5.4). From (4.2), we see that the Fourier transform of $\eta^{x_v}$ is supported in $D(0; c^2R^{-1/2})$. So, the Fourier support of $\eta^{x_v} P_{\Omega\wedge(Y_j)} f$ is contained in $B_{v_j}$. By orthogonality, (5.4) is bounded by

$$\int \left( \sum_{q \in Q} \sum_{v_j \in V_j} \left\| \sum_{x_0 \in L} m_{q,T_j(x_0,v_j)} \eta^{x_v} P_{\Omega\wedge(Y_j)} \phi_j(0) \right\|_2 \right)^{1/2} d\Omega.$$

By rearranging, it is equal to

$$\int \left( \sum_{v_j \in V_j} \left| P_{\Omega\wedge(Y_j)} \phi_j(x,0) \right|^2 \sum_{q \in Q} \left| \sum_{x_0 \in L} m_{q,T_j(x_0,v_j)} \eta^{x_v}(x) \right|^2 dx \right)^{1/2} d\Omega,$$

which is bounded by

$$\int \left( \sum_{v_j \in V_j} \left| P_{\Omega\wedge(Y_j)} \phi_j(x,0) \right|^2 \sum_{q \in Q} \sum_{x_0 \in L} m_{q,T_j(x_0,v_j)} \eta^{x_v}(x) \right)^{1/2} d\Omega.$$
By (3.3) and (5.2), this is bounded by
\[
\int \left( \sum_{v_j \in V} \int |P_{\Omega,v_j} P_{\Omega} \phi_j(x,0)|^2 \, dx \right)^{1/2} \, d\Omega.
\]
By orthogonality, the above is bounded by
\[
\int \|P_{\Omega} \phi_j(0)\|_2 \, d\Omega.
\]
Since \( \|P_{\Omega} \phi_j(0)\|_2 \leq E(\phi_j)^{1/2} \), we have that (5.4) \( \leq E(\phi_j)^{1/2} \).

Consider (5.5). Apply the previous arguments but using almost orthogonality instead of orthogonality. Then we have
\[
(5.5) \lesssim \int \|(1 - P_{\Omega} \phi_j(0))\|_2 \, d\Omega.
\]
By the Cauchy–Schwarz inequality this is bounded by
\[
\left( \int \|(1 - P_{\Omega} \phi_j(0))\|_2^2 \, d\Omega \right)^{1/2}.
\]
By Plancherel’s theorem and rearranging the integrals, this is equal to
\[
\int 1 - \chi_{\Omega(Y_j)}(\xi) \, d\Omega = \frac{1}{|D(0; CR^{-1/2})|} \int_{D(0; CR^{-1/2})} 1 - \chi_{Y_j}(\xi + w) \, dw \lesssim c^2.
\]
Inserting this into the previous, we obtain that (5.5) \( \lesssim c^2 E(\phi_j) \).

\[\square\]

### 6. A Localization Operator

In this section we introduce a localization operator and state some relevant basic estimates. When exploiting energy concentrations, the localization operator is used as a tool.

By (2.2) we have \( \phi_j(t)(\xi) = e^{-\pi i(t - t_0)|\xi|^2} \phi_j(t_0)(\xi) \), which is written as
\[
\phi_j(t) = U_j[\phi_j(t_0)](t - t_0).
\]

**Definition 6.1.** Let \( D = D(x_D, t_D; r) \) be a disc. We define an operator \( P_D \phi_j \) by
\[
P_D \phi_j(t) = U_j[\chi_{D} \ast \eta_{r^{-1/2N}} \phi_j(t_D)](t - t_D)
\]
where \( \eta_r \) is defined as (1.8).

**Lemma 6.2.** Let \( r \geq C_0, D = D(x_D, t_D; r) \) and
\[
D^\pm := D(x_D, t_D; r(1 \pm r^{-1/2N})).
\]
Suppose that \( \phi_j \) satisfies that \( \text{marg}(\phi_j) \geq C_0 r^{-1+1/N} \) for \( j = 1, 2 \). Then,
\[
\text{marg}(P_D \phi_j) \geq \text{marg}(\phi_j) - C_0 r^{-1+1/N}
\]
and
\[
\|P_D\phi_j(t_D)\|_{L^2(\mathbb{R}^n \setminus D^+)} \lesssim r^{-N} E(\phi_j)^{1/2}, \quad (6.4)
\]
\[
\|(1 - P_D)\phi_j\|_{L^2(D^-)} \lesssim r^{-N} E(\phi_j)^{1/2}, \quad (6.5)
\]
\[
E(P_D\phi_j) \leq \|\phi_j\|_{L^2(D^+)}^2 + Cr^{-N} E(\phi_j), \quad (6.6)
\]
\[
E((1 - P_D)\phi_j) \leq \|\phi_j\|_{L^2(\mathbb{R}^n \setminus D^-)}^2 + Cr^{-N} E(\phi_j). \quad (6.7)
\]

**Proof.** Consider (6.3). Observe that the size of \(\text{supp} \hat{\phi}_j\) is comparable to that of \(\text{supp} \hat{\phi}_j(0)\). Similarly, the size of \(\text{supp} P_D\hat{\phi}_j\) and \(\text{supp} P_D\hat{\phi}_j(t_D)\) is comparable. From (6.2) we have
\[
P_D\hat{\phi}_j(t_D) = (\hat{\chi}_D \hat{\eta}_{r^{1-1/N}}) \ast \hat{\phi}_j(t_D). \quad (6.8)
\]
Since \(\eta_{r^{1-1/N}}(\xi) = \hat{\eta}(r^{1-1/N} \xi)\) is supported on \(D(0; r^{-1+1/N})\), the Fourier support of \(P_D\hat{\phi}_j(t_D)\) is expanded \(O(r^{-1+1/N})\) more than that of \(\hat{\phi}_j(t_D)\). Thus we have (6.3).

We have that \(1/2 D \subset D^- \subset D \subset D^+ \subset 2D\). From this relation it follows that
\[
0 \leq \chi_D \ast \eta_{r^{1-1/N}} \leq 1, \quad (6.9)
\]
\[
1 - \chi_D \ast \eta_{r^{1-1/N}}(x) \lesssim r^{-N} \quad \text{for } x \in \mathbb{R}^n \setminus D^+, \quad (6.9)
\]
\[
1 - \chi_D \ast \eta_{r^{1-1/N}}(x) \lesssim r^{-N} \quad \text{for } x \in D^- . \quad (6.10)
\]

Indeed, the first one is trivial. Consider (6.9). We have that
\[
\chi_D \ast \eta_{r^{1-1/N}}(x) \lesssim \left(1 + \frac{\text{dist}(x,D)}{r^{1-1/N}}\right)^{-M}, \quad \forall M > 0. \quad (6.11)
\]

If \(x \in \mathbb{R}^n \setminus D^+\) then
\[
\text{dist}(x,D) \geq r(1 + r^{-1/2N}) - r = r^{1-1/2N}.
\]
By inserting this into the previous inequality we can obtain (6.9).

Consider (6.10). We have that
\[
1 - \chi_D \ast \eta_{r^{1-1/N}}(x) \lesssim \left(1 + \frac{\text{dist}(x,\mathbb{R}^n \setminus D)}{r^{1-1/N}}\right)^{-M}, \quad \forall M > 0. \quad (6.12)
\]

If \(x \in D^-\), then
\[
\text{dist}(x,\mathbb{R}^n \setminus D) \geq r - r(1 - r^{-1/2N}) = r^{1-1/2N}.
\]
Thus we have (6.10).

Now consider from (6.4) to (6.7). By (6.2) and (6.9) it follows that
\[
\|P_D\phi_j(t_D)\|_{L^2(\mathbb{R}^n \setminus D^+)} = \|(\chi_D \ast \eta_{r^{1-1/N}})\phi_j(t_D)\|_{L^2(\mathbb{R}^n \setminus D^+)} \lesssim r^{-N} E(\phi_j)^{1/2}.
\]

So we have (6.3). Similar arguments also give (6.5).

From (6.4) it follows that
\[
E(P_D\phi_j) = \|P_D\phi_j(t_D)\|_2^2
\]
\[
\leq \|P_D\phi_j\|_{L^2(D^+)}^2 + Cr^{-N} E(\phi_j)^{1/2}
\]
\[
\leq \|\phi_j\|_{L^2(D^+)}^2 + Cr^{-N} E(\phi_j)^{1/2}.
\]

So we have (6.6). Analogously we have (6.7). \(\square\)

Now we consider some properties of \(P_D\phi_j\) in \(\mathbb{R}^n \times \mathbb{R}^n\).
By the Cauchy–Schwarz inequality,
\[ |P_D\phi_j(x,t)| \leq C_M r^{n/2} \left( 1 + \frac{\text{dist}((x,t), A_j(x_D, t_D; r))}{r^{1-1/N}} \right)^{-M} E(\phi_j)^{1/2}; \quad \forall M > 0 \tag{6.13} \]
and
\[ \| (1 - P_D)\phi_j \|_{L^\infty(\mathbb{D})} \lesssim r^{-N} E(\phi_j)^{1/2}. \tag{6.14} \]

**Proof.** Consider (6.14). By (6.2) and (3.15), we have that for \((x,t)\) by Lemma 3.2.

\[ \int \left( (1 - \chi_D * \eta_{r,1-N})^2 \right) \phi_j(y,t_D) \, dy. \]

By Lemma 3.2 and (6.11),
\[ |K_{j,t-t_D}(x-y)(\chi_D * \eta_{r,1-N})^2(y)| \leq C_M \left( 1 + \frac{\text{dist}((x,t), A_j(x_D, t_D; r))}{r^{1-1/N}} \right)^{-M} \int (\chi_D * \eta_{r,1-N})^2(y) \phi_j(y,t_D) \, dy \]
for any \(M > 0\). Thus,
\[ \int |K_{j,t-t_D}(x-y)(1 - \chi_D * \eta_{r,1-N})^2(y)\phi_j(y,t_D) \, dy. \]

By the Cauchy–Schwarz inequality,
\[ |(1 - P_D)\phi_j(x,t)| \leq \left( \int |K_{j,t-t_D}(x-y)(1 - \chi_D * \eta_{r,1-N})^2(y)\phi_j(y,t_D) \, dy \right)^{1/2} E(\phi_j)^{1/2}. \tag{6.15} \]

By Lemma 3.2 and (6.12) we have that for \((x,t) \in Q(x_D, t_D; r/4),\)
\[ \int |K_{j,t-t_D}(x-y)(1 - \chi_D * \eta_{r,1-N})^2(y)|^2 \, dy \lesssim r^{-M} \int |K_{j,t-t_D}(x-y)\phi_j(y,t_D) \, dy \lesssim r^{-M} \]
for any \(M > 0\). Substituting this in (6.15) we can obtain (6.14).}

**Lemma 6.4.** Let \(r \geq C_0\) and \(D = D(x_D, t_D; r)\). Then, for each \(t_0 \in \mathbb{R}\) there is a disc \(G_{j,t_0}(D)\) of radius \(A_w|t_D-t_0|/2 + 4r\) in \(\mathbb{R}^n \times \{t_0\}\) such that \(G_{j,t_0}(D)\) contains \(A_j(x_D, t_D; r) \cap (\mathbb{R}^n \times \{t_0\})\) and
\[ E(P_D\phi_j) \leq \| \phi_j \|_{L^2(G_{j,t_0}(D))} + Cr^{-N} E(\phi_j). \]

**Proof.** By (6.1) and (6.2),
\[ P_D\phi_j(x,t_D) = (\chi_D * \eta_{r,1-N})(x) \mathcal{H}_j(\phi_j(t_0))(x,t_D-t_0) \]
\[ = \int (\chi_D * \eta_{r,1-N})(x) K_{j,t-t_D}(x-y)\phi_j(y,t_0) \, dy. \tag{6.16} \]

If we ignore Schwarz tails, the equation \(K_{j,t-t_D}(x-y)\) implies that \(x-y\) is contained in \(A_j,t_D-t_0\) by Lemma 3.2. The \(A_j,t_D-t_0\) is contained in a disc of radius \(A_w|t_D-t_0|/2 + C\), so \(x-y\) is contained in a disc of radius \(A_w|t_D-t_0|/2 + C\). Since \(x\) is contained in \(D\), we see that that \(y\) is contained in
a disc $D_j^* := D(x_j, t_0; A_w |t_D - t_0|/2 + 2r)$ for some $x_j$. Using the symmetric property of $\Lambda_j$ about the origin, we also have that $y - x$ is contained in $\Lambda_j(x, t_0 - t_D)$, which implies $(y, t_0) \in \Lambda_j(x, t_D)$. So, we can see that $D_j^*$ contains $\Lambda_j(x_D, t_D; r) \cap (\mathbb{R}^n \times \{t_0\})$. Using this observation we have

$$\|(x_D * \eta_{1-1/n})^{1/2}(x)K_{j,t_D-t_0}(x-y)\|_2 \leq C_M \left(1 + \frac{\text{dist}(y, D_j^*)}{r^{1-1/N}}\right)^{-M}, \quad \forall M > 0.$$ 

Let $\tilde{D}_j^* := D(x_j, t_0; A_w |t_D - t_0|/2 + 2r(1 + r^{-1/2N}))$. By the above estimate,

$$\|(x_D * \eta_{1-1/n})^{1/2}(x)K_{j,t_D-t_0}(x-y)(1 - \chi_{\tilde{D}_j^*}(y))\|_2 \leq C_M r^{-M}, \quad \forall M > 0.$$ 

We divide $\phi_j(t_0) = \chi_{\tilde{D}_j^*}\phi_j(t_0) + (1 - \chi_{\tilde{D}_j^*})\phi_j(t_0)$ and insert it into (6.16). Then,

$$\|P_D\phi_j(t_D)\|_2 \leq \|\mathcal{U}_j(\chi_{\tilde{D}_j^*}\phi_j(t_0))(t_D - t_0)\|_2 + Cr^{-N}E(\phi_j)^{1/2} \leq \|\chi_{\tilde{D}_j^*}\phi_j(t_0)\|_2 + Cr^{-N}E(\phi_j)^{1/2} \leq \|\phi_j\|_{L^2(D(x_j, t_0; A_w |t_D - t_0|/2 + 4r))} + Cr^{-N}E(\phi_j)^{1/2}.$$ 

If we take $G_{j,t_0}(D) := D(x_j, t_0; A_w |t_D - t_0|/2 + 4r)$ then we have the desired estimate. Since $D_j^*$ contains $\Lambda(x_D, t_D; r) \cap (\mathbb{R}^n \times \{t_0\})$, the $G_{j,t_0}(D)$ also contains $\Lambda_j(x_D, t_D; r) \cap (\mathbb{R}^n \times \{t_0\})$. □

7. PROOF OF PROPOSITION 2.4

Let $\phi_1, \phi_2$ satisfy the margin (2.5) and the normalization (2.7). Let $Q = Q(x_Q, t_Q, R)$ be a cube of sidelength $R$ and centered at $(x_Q, t_Q)$ and let $I_Q = [t_Q - R/2, t_Q + R/2]$ be the time interval of $Q$.

By Definition 2.2 it suffices to show that for each $0 < \varepsilon < 1$

$$\|\phi_1\phi_2\|_{L^p(Q)} \leq (1 - C_0^{-C}) \sup_{r \geq C_0^{-C}R, \varepsilon \leq \frac{t_Q}{100} \leq \frac{t_Q + 2r}{100} \leq 2(A_w A_d^{-1} + C_0^{-C})r} K_c(R, r, \frac{t_Q}{100}).$$

By Definition 2.4 we have

$$\|\phi_1\phi_2\|_{L^p(Q)} \leq K_c(R, r, \frac{t_Q}{100})E_{r,\frac{t_Q}{100}}(\phi_1, \phi_2)^{1/p}.$$ 

It suffices to show the following proposition.

**Proposition 7.1.** Let $\phi_1, \phi_2$ be the same as described above and $R \geq 2C_0$. If $0 < \delta \leq C_0^{-C}$, then for $0 < \varepsilon < 1$, there exist $t_\varepsilon \in I_Q$, $r \geq C_0^{-C}R$ and $r \varepsilon R \leq (2(A_w A_d^{-1} + C_0^{-C}))r$ such that

$$E_{r,\frac{t_Q}{100}}(\phi_1, \phi_2) \leq 1 - \delta.$$ 

**Proof.** Let $t \in I_Q$ and let $D^*_t(\phi_1, \phi_2)$ and $\mathcal{N}(D)$ be defined as in Definition 2.4. We define $\mathcal{D}_t^*(\delta)$ to be the collection of discs $D \in \mathcal{D}_t^*(\phi_1, \phi_2)$ such that there exist $D_1, D_2 \subset \mathcal{N}(D)$ of radius $(2A_w A_d^{-1} + C_0^{-C})r_D$ satisfying

$$\|\phi_1\|_{L^2(D_1)} \|\phi_2\|_{L^2(D_2)} \geq 1 - \delta.$$ 

For a disc $D$ we define $\hat{r}_D$ to be the infimum of the radii of the discs $D_1, D_2 \subset \mathcal{N}(D)$ satisfying the above inequality, that is,

$$\hat{r}_D := \inf\{r : \|\phi_1\|_{L^2(D_1)} \|\phi_2\|_{L^2(D_2)} \geq 1 - \delta \text{ for } D_1, D_2 \subset \mathcal{N}(D) \text{ with } r_{D_1} = r_{D_2} = r\}.$$ 

Then for $D \in \mathcal{D}_t^*(\delta)$,

$$\hat{r}_D \leq (2A_w A_d^{-1} + C_0^{-C})r_D.$$ (7.1)
Since \( \hat{r}_D \) is the infimum, we have
\[
\sup_{D_1, D_2 \subset \mathcal{N}(D)} \| \phi_1 \|_{L^2(D_1)} \| \phi_2 \|_{L^2(D_2)} \leq 1 - \delta.
\]
In fact, since \( \phi_1 \) and \( \phi_2 \) are smooth, we have the equality
\[
\sup_{D_1, D_2 \subset \mathcal{N}(D) \atop \varepsilon_D = r_D} \| \phi_1 \|_{L^2(D_1)} \| \phi_2 \|_{L^2(D_2)} = 1 - \delta. \tag{7.2}
\]
Let
\[
\hat{r}_\delta(t) := \inf_{D \in \mathcal{D}_\delta(t)} r_D. \tag{7.3}
\]
Since \( \phi_1 \) and \( \phi_2 \) are smooth, there is a disc \( D \in \mathcal{D}_\delta(t) \) of radius \( r_\delta(t) \). Let \( \hat{r}_\delta(t) \) is the infimum of radii \( \hat{r}_D \) for \( D \in \mathcal{D}_\delta(t) \) with radius \( r_\delta(t) \), i.e.,
\[
\hat{r}_\delta(t) := \inf_{D \in \mathcal{D}_\delta(t) \atop r_D = \hat{r}_D} \hat{r}_D. \tag{7.4}
\]
Then from (7.1) it follows that for each \( t \in I_Q \),
\[
\hat{r}_\delta(t) \leq (2A_w A_q^{-1} + C_0^{-C}) r_\delta(t). \tag{7.5}
\]
To show that \( \hat{r}_\delta(t) \geq r_\delta(t)/100 \), let \( D \in \mathcal{D}_\delta(t) \) be a disc of radius \( r_\delta(t) \) with \( \hat{r}_D = \hat{r}_\delta(t) \), and let \( D_1, D_2 \subset \mathcal{N}(D) \) be the discs of radius \( r_\delta(t) \). If we suppose \( \hat{r}_\delta(t) < r_\delta(t)/100 \), then by (7.3) the distance between \( D_1 \) and \( D_2 \) is larger than \( C^{-1} r_\delta(t) \). Since the Fourier transform of \( \phi_j(t) \) is compactly supported, for any proper disc \( D \subset \mathbb{R}^n \times \{ t \} \) we have \( \| \phi_j \|_{L^2(D)} \geq 0 \). Thus, there is a disc \( D' \) of radius \( \leq r_\delta(t) \) such that
\[
\| \phi_1 \|_{L^2(N(D'))} \| \phi_2 \|_{L^2(N(D'))} = 1 - \delta.
\]
This implies \( D' \in \mathcal{D}_\delta(t) \) but it contradicts (7.3). Thus we have \( \hat{r}_\delta(t) \geq r_\delta(t)/100 \).

We also have that for each \( t \in I_Q \),
\[
E_{r_\delta(t)}(\hat{r}_\delta(t), t) (\phi_1, \phi_2) = 1 - \delta. \tag{7.6}
\]
Indeed, let \( D \) be a disc of radius \( r_\delta(t) \) in \( \mathbb{R}^n \times \{ t \} \). If \( D \in \mathcal{D}_\delta(t) \) then by (7.2) and (7.4),
\[
\sup_{D_1, D_2 \subset \mathcal{N}(D) \atop r_{D_1} = r_{D_2} = r_\delta(t)} \| \phi_1 \|_{L^2(D_1)} \| \phi_2 \|_{L^2(D_2)} = 1 - \delta.
\]
If \( D \notin \mathcal{D}_\delta(t) \) then from the definition of \( \mathcal{D}_\delta(t) \) it follows that for any discs \( D_1, D_2 \subset \mathcal{N}(D) \) of radius \( (2A_w A_q^{-1} + C_0^{-C}) r_\delta(t) \),
\[
\| \phi_1 \|_{L^2(D_1)} \| \phi_2 \|_{L^2(D_2)} < 1 - \delta.
\]
Thus we have (7.6).

We choose a time \( t_e \in I_Q \) such that
\[
\frac{1}{2} \sup_{t \in I_Q} r_\delta(t) - r_\delta(t_e) \leq \sup_{t \in I_Q} r_\delta(t). \tag{7.7}
\]
By (7.5) and (7.3), to prove the proposition it suffices to show that if \( 0 < \delta \leq C_0^{-C} \) then
\[
r_\delta(t_e) \geq C_0^{-C} R. \tag{7.8}
\]
Since \( \phi_1 \) and \( \phi_2 \) are smooth, from (7.1) it follows that for each \( t \in I_Q \), there exist discs \( D^I_{\delta} \subset \mathbb{R}^n \times \{ t \} \) of radius \( r_\delta(t) \) and \( D_1, D_2 \subset \mathcal{N}(D^I_{\delta}) \) of radius \( r_\delta(t) \) such that
\[
\| \phi_1 \|_{L^2(D_1)} \| \phi_2 \|_{L^2(D_2)} = 1 - \delta. \tag{7.9}
\]
Let \((x_e, t_e)\) be the center of \(D^\delta_{t_e}\),
\[ r_e = r^\delta(t_e) + C_0 \]
and
\[ A_{j,e} := A_j(x_e, t_e; C_0^2 A_* r_e). \]
To have \((7.8)\) it is enough to show
\[
\bigcup_{t \in I_Q} D^\delta_t \subset \bigcap_{j=1,2} C A_{j,e}. \tag{7.10}
\]
Indeed, since \(\Lambda_{1,e}\) and \(\Lambda_{2,e}\) meet transversely, the union \(\bigcup_{t \in I_Q} D^\delta_t\) is contained in a ball \(B(x_e, t_e; C_0^C r_e)\).
By comparing the length of \(I_Q\) with the radius \(C_0^C r_e\), we have \(r_e \geq C_0^{-C} R\), which implies \((7.8)\) because \(R \geq 2C_0\).
To show \((7.10)\), by \((7.7)\) it suffices to prove that for each \(t \in I_Q\), the disc \(\mathcal{N}(D^\delta_t)\) intersects both \(\Lambda_{1,e}\) and \(\Lambda_{2,e}\). Suppose for contradiction that there exists \(\mathcal{N}(D^\delta_t)\) contained in \(\mathbb{R}^{n+1} \setminus \Lambda_{j,e}\) for some \(j = 1, 2\). Let
\[ D_e := D(x_e, t_e; C_0 A_* r_e). \]
We decompose
\[
\|\phi_j\|^2_{L^2(\mathcal{N}(D^\delta_t))} \leq 4 \|P_{D_e} \phi_j\|^2_{L^2(\mathcal{N}(D^\delta_t))} + 4 \|(1 - P_{D_e}) \phi_j\|^2_{L^2(\mathcal{N}(D^\delta_t))}. \tag{7.11}
\]
From \(\mathcal{N}(D^\delta_t) \subset \mathbb{R}^{n+1} \setminus \Lambda_{j,e}\), one has dist\((\mathcal{N}(D^\delta_t), \Lambda_j(x_e, t_e; C_0 A_* r_e)) \geq C_0 A_* r_e\). Since \(r_e \geq C_0\) and the radius of \(\mathcal{N}(D^\delta_t)\) is \(\leq A_* r_e\), by \((6.13)\)
\[
\|P_{D_e} \phi_j\|_{L^2(\mathcal{N}(D^\delta_t))} \lesssim r_e^{n/2}(C_0 A_* r_e^{1/N})^{-N^{10}} E(\phi_j)^{1/2} |\mathcal{N}(D^\delta_t)|^{1/2} \lesssim C_0^{-C}. \tag{7.12}
\]
From \((6.4)\) and \(r_e \geq C_0\) it follows that
\[
\|(1 - P_{D_e}) \phi_j\|^2_{L^2(\mathcal{N}(D^\delta_t))} \leq E((1 - P_{D_e}) \phi_j)
\leq \|\phi_j(t_e)\|^2_{L^2(\mathbb{R}^n \setminus D_e^-)} + C_0^{-C}. \tag{7.13}
\]
Since \(\mathcal{N}(D^\delta_t) \subset \frac{1}{2} D_e \subset D_e^-\), we have
\[
\|\phi_j(t_e)\|^2_{L^2(\mathbb{R}^n \setminus D_e^-)} \leq 1 - \|\phi_j\|^2_{L^2(D_e^-)} \leq 1 - \|\phi_j\|^2_{L^2(\mathcal{N}(D^\delta_t))}. \]
By \((2.7)\) and \((7.9)\),
\[
\|\phi_j\|_{L^2(\mathcal{N}(D^\delta_t))} \geq 1 - \delta. \tag{7.14}
\]
Combining the above two estimates we have
\[
\|\phi_j\|^2_{L^2(\mathbb{R}^n \setminus D_e^-)} \leq 2\delta.
\]
Substituting this in \((7.13)\) we have
\[
\|(1 - P_{D_e}) \phi_j\|^2_{L^2(\mathcal{N}(D^\delta_t))} \leq 2\delta + C_0^{-C}.
\]
By inserting this and \((7.12)\) into \((7.11)\), it follows that
\[
\|\phi_j\|^2_{L^2(\mathcal{N}(D^\delta_t))} \leq 8\delta + 8C_0^{-C}.
\]
Comparing this with \((7.14)\) we have \(1 - 8C_0^{-C} \leq 10\delta\). However, since \(0 < \delta \leq C_0^{-C}\) is small, this is a contradiction. \(\square\)
8. Proof of Proposition 2.7

Let \( \phi_1, \phi_2 \) satisfy the margin requirement (2.5) and the energy normalization (2.7). Let \( Q \) be a cube of sidelength \( R \). By Definition 2.5 we may assume that \( \phi_1, \phi_2 \) satisfy

\[
\|\phi_1 \phi_2\|_{L^p(Q)} = K_c(R, r, \hat{r})E^\varepsilon_{r, \hat{r}, t_e}(\phi_1, \phi_2)^{1/p'}.
\]  

(8.1)

It suffices to show

\[
\|\phi_1 \phi_2\|_{L^p(Q)} \leq (1 + Cc)E^\varepsilon_{r, \hat{r}, t_e}(\phi_1, \phi_2)^{1/p'}K(R) + 2^{CC_0}.
\]  

(8.2)

We may assume that

\[
E^\varepsilon_{r, \hat{r}, t_e}(\phi_1, \phi_2) = \sup_{D \in D^\varepsilon_{r, \hat{r}, t_e}(\phi_1, \phi_2)} \sup_{D_1, D_2 \subset N(D)} \left( \|\phi_1\|_{L^2(D_1)}\|\phi_2\|_{L^2(D_2)} \right).
\]

Indeed, if \( E^\varepsilon_{r, \hat{r}, t_e}(\phi_1, \phi_2) = 1/2 \), we can take \( \hat{r} \geq r \) and \( \hat{r} \leq \tilde{r} \leq (2A_cA_d^{-1} + C_0)\hat{r} \) such that

\[
1/2 = \sup_{D \in D^\varepsilon_{r, \hat{r}, t_e}(\phi_1, \phi_2)} \sup_{D_1, D_2 \subset N(D)} \left( \|\phi_1\|_{L^2(D_1)}\|\phi_2\|_{L^2(D_2)} \right).
\]

Thus, in (8.2) we can replace \( E^\varepsilon_{r, \hat{r}, t_e}(\phi_1, \phi_2) \) with \( E^\varepsilon_{\tilde{r}, \hat{r}, t_e}(\phi_1, \phi_2) \).

By the smoothness of \( \phi_1, \phi_2 \), there exists a disc \( D_e \in D^\varepsilon_{\tilde{r}, \hat{r}, t_e}(\phi_1, \phi_2) \) of radius \( r \) such that

\[
\sup_{D_1, D_2 \subset N(D_e)} \left( \|\phi_1\|_{L^2(D_1)}\|\phi_2\|_{L^2(D_2)} \right) = E^\varepsilon_{\tilde{r}, \hat{r}, t_e}(\phi_1, \phi_2).
\]  

(8.3)

Set

\[
\Lambda_j^{[e]} := \Lambda_j(x_e, t_e; r'/2)
\]

for \( j = 1, 2 \) where \( (x_e, t_e) \) is the center of \( D_e \) and \( r' := \frac{r}{A_e}(1 + (\frac{r}{A_e})^{-1/2N}) \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Intersection of \( \Lambda_1 \) and \( \Lambda_2 \)}
\end{figure}

8.1. Consider the case that \( Q \) intersects both \( \Lambda_1^{[e]} \) and \( \Lambda_2^{[e]} \). Let \( D_Q = D(x_Q, t_Q; 4R) \) be the disc of radius \( 4R \) with the same center as \( Q \). We decompose

\[
\|\phi_1 \phi_2\|_{L^p(Q)} \leq \|P_{D_Q} \phi_1 P_{D_Q} \phi_2\|_{L^p(Q)} + \|P_{D_Q} \phi_1 (1 - P_{D_Q}) \phi_2\|_{L^p(Q)} + \|(1 - P_{D_Q}) \phi_1 \phi_2\|_{L^p(Q)}.
\]
From (6.14) and Hölder’s inequality it follows that
\[ \|(1 - P_{D_Q})\phi_1\phi_2\|_{L^p(Q)} \lesssim R^{-N+C}, \]
\[ \|P_{D_Q}\phi_1(1 - P_{D_Q})\phi_2\|_{L^p(Q)} \lesssim R^{-N+C}. \]
Both \(P_{D_Q}\phi_1\) and \(P_{D_Q}\phi_2\) satisfy the relaxed margin condition (2.10), so by Proposition 2.3
\[ \|P_{D_Q}\phi_1P_{D_Q}\phi_2\|_{L^p(Q)} \leq (1 + Cc)E(P_{D_Q}\phi_1)^{1/2}E(P_{D_Q}\phi_2)^{1/2}\kappa(R) + 2^{CC_0} \]
\[ \leq (1 + Cc)(E(P_{D_Q}\phi_1)^{1/2}E(P_{D_Q}\phi_2)^{1/2})^{1/\nu}\kappa(R) + 2^{CC_0}. \]
By (2.9), to prove (8.2) it suffices to show
\[ E(P_{D_Q}\phi_1)^{1/2}E(P_{D_Q}\phi_2)^{1/2} \leq E_{R,r,t,e}^\varepsilon(\phi_1, \phi_2) + CR^{-N}. \]
By Lemma 6.4 there are discs \(G_{1,t,e}(D_Q), G_{2,t,e}(D_Q)\) of radius
\[ R_e := A_w|t_Q - t_e|/2 + 16R \]
at time \(t_e\) such that the \(G_{j,t,e}(D_Q)\) contains \(\Lambda_j(x_Q, t_Q; 4R) \cap (\mathbb{R}^n \times \{t_e\})\) and
\[ E(P_{D_Q}\phi_j)^{1/2} \leq \|\phi_j\|_{L^2(G_{j,t,e}(D_Q))} + CR_e, \quad j = 1, 2. \]
To show
\[ \|\phi_j\|_{L^2(G_{1,t,e}(D_Q))}\|\phi_j\|_{L^2(G_{2,t,e}(D_Q))} \leq E_{R,r,t,e}^\varepsilon(\phi_1, \phi_2), \] (8.4)
we consider a geometric property of \(\Lambda_1^e \cap \Lambda_2^e\). Since \(\Lambda_1^e \cap \Lambda_2^e\) is a conic set (see Figure 2), we can see that \(A_d|t_Q - t_e| \leq 2r' \leq 4r/A_s\), and so
\[ R_e \leq 2A_wA_d^{-1}r/A_s + 16R \leq (2A_wA_d^{-1} + 2000C_{0^-}^e)\hat{r}/A_s \leq \hat{r}. \]
Since \(Q\) intersects \(\Lambda_j^e\) and the \(G_{j,t,e}(D_Q)\) contains \(\Lambda_j(x_Q, t_Q; 4R) \cap (\mathbb{R}^n \times \{t_e\})\), the \(G_{j,t,e}(D_Q)\) intersects \(\Lambda_j^e \cap (\mathbb{R}^n \times \{t_e\})\). To show that \(G_{j,t,e}(D_Q)\) is contained in \(N(D_e)\) it suffices to show
\[ r' + 2R_e \leq 2\hat{r}/A_s + 2(2A_wA_d^{-1} + 2000C_{0^-}^e)\hat{r}/A_s \leq A_s r. \]
Therefore we have (8.4).

8.2. Consider the case that \(Q\) is contained in \(\mathbb{R}^{n+1} \setminus \Lambda_j^e\) for some \(j = 1, 2\). We only consider the case that \(Q\) is contained in \(\mathbb{R}^{n+1} \setminus \Lambda_j^e\), because the other case is similar. Let \(D_e := D(x_e, t_e; 400A_e)\). By the triangle inequality,
\[ \|\phi_1\phi_2\|_{L^p(Q)} \leq \|P_{D_e}\phi_1\phi_2\|_{L^p(Q)} + \|(1 - P_{D_e})\phi_1\phi_2\|_{L^p(Q)}, \] (8.5)
Since \(Q\) is contained in \(\mathbb{R}^{n+1} \setminus \Lambda_j^e\) and \(\frac{\nu}{2} \geq \frac{1}{400A_e}\), from Lemma 6.3 it follows that
\[ \|P_{D_e}\phi_1\phi_2\|_{L^p(Q)} \leq Cr^{-N^3}. \] (8.6)
By Definition 2.3,
\[ \|(1 - P_{D_e})\phi_1\phi_2\|_{L^p(Q)} \leq K(R, r)E((1 - P_{D_e})\phi_1)^{1/2p}E_{R,r,t,e}((1 - P_{D_e})\phi_1, \phi_2)^{1/2p} \]
\[ \leq K(R, r)E((1 - P_{D_e})\phi_1)^{1/2p}E_{R,r,t,e}((\phi_1, \phi_2)^{1/2p}. \]
By (8.1),
\[ \|(1 - P_{D_e})\phi_1\phi_2\|_{L^p(Q)} \leq E((1 - P_{D_e})\phi_1)^{1/2p}\|\phi_1\phi_2\|_{L^p(Q)}. \]
By (6.7),
\[ E((1 - P_{D_e})\phi_1) \leq 1 - \|\phi(t_e)\|_{L^2(D_e/2)} + C \rho^{-N}. \]
By (2.13) and (2.7),
\[ \|\phi(t_e)\|_{L^2(D_e/2)} \geq \varepsilon \]
By combining the above three inequalities,
\[ \|(1 - P_{D_n})\phi_1 \phi_2\|_{L^p(Q)} \leq \kappa \|\phi_1 \phi_2\|_{L^p(Q)} \]
where \( \kappa := (1 - \varepsilon^2 + C r^{-N})^{1/2p}. \)
By applying the triangle inequality to the right side of the above inequality,
\[ \|(1 - P_{D_n})\phi_1 \phi_2\|_{L^p(Q)} \leq \kappa \|P_{D_n} \phi_1 \phi_2\|_{L^p(Q)} + \kappa \|(1 - P_{D_n})\phi_1 \phi_2\|_{L^p(Q)}. \]
By rearranging,
\[ \|(1 - P_{D_n})\phi_1 \phi_2\|_{L^p(Q)} \leq \frac{\kappa}{1 - \kappa} \|P_{D_n} \phi_1 \phi_2\|_{L^p(Q)}. \]
By inserting this estimate into \( (8.5) \),
\[ \|\phi_1 \phi_2\|_{L^p(Q)} \leq \frac{1}{1 - \kappa} \|P_{D_n} \phi_1 \phi_2\|_{L^p(Q)}. \]
From \( \varepsilon \geq R^{-N/4} \) and \( r \geq C_0^C R \), we have \( \frac{1}{1 - \kappa} \lesssim R^C N \), and
\[ \|\phi_1 \phi_2\|_{L^p(Q)} \leq R^C N \|P_{D_n} \phi_1 \phi_2\|_{L^p(Q)}. \]
By \( (8.6) \) we thus have \( (8.2) \).

9. PROOF OF PROPOSITION 2.8

Suppose that \( \phi_1, \phi_2 \) obey the margin condition (2.5). We may assume the normalization (2.7). It suffices to show that
\[ \|\phi_1 \phi_2\|_{L^p(Q)} \leq (1 + C c) K c_0(0, r, r_1, r_2) E_{r, r_1, r_2}(\phi_1, \phi_2)^{1/p'} + c^{-C}. \]
We apply Proposition 4.1. Then it suffices to show
\[ \|[\Phi_1][\Phi_2]\|_{L^p(Q)} \leq (1 + C c) K c_0(0, r, r_1, r_2) E_{r, r_1, r_2}(\phi_1, \phi_2)^{1/p'} + c^{-C}, \]
where the cube \( Q \) is of side-length \( CR \). By (4.1),
\[ \| [\Phi_1][\Phi_2]\|_{L^p(Q)} = \left( \sum_{\Delta \in \mathcal{Q}_{c_0}(Q)} \|\Phi_1^{(\Delta)} \Phi_2^{(\Delta)}\|_{L^p(\Delta)}^p \right)^{1/p}. \]
Since the sidelength of \( \Delta \) is \( 2^{-C_0} CR \), there is a cube of sidelength \( R/C_0 \) containing \( \Delta \). So, by the margin of \( \Phi_1 \) shown in Proposition 4.1 and Definition 2.5,
\[ \|\Phi_1^{(\Delta)} \Phi_2^{(\Delta)}\|_{L^p(\Delta)} \leq K c_0(0, r, r_1, r_2) E_{r_1, r_2, t}(\phi_1^{(\Delta)}, \phi_2^{(\Delta)})^{1/p'} \left( E(\Phi_1^{(\Delta)})^{1/2} E(\Phi_2^{(\Delta)})^{1/2} \right)^{1/p}. \]
By combining the above two equations,
\[ \| [\Phi_1][\Phi_2]\|_{L^p(Q)} \leq K c_0(0, r, r_1, r_2) \left( \sum_{\Delta \in \mathcal{Q}_{c_0}(Q)} E_{r_1, r_2, t}(\phi_1^{(\Delta)}, \phi_2^{(\Delta)})^{1/p'} \left( E(\Phi_1^{(\Delta)})^{1/2} E(\Phi_2^{(\Delta)})^{1/2} \right)^{1/p} \right)^{1/p}. \]
By the Cauchy–Schwarz inequality and Lemma 5.1, it is bounded by
\[ (1 + C c) K c_0(0, r, r_1, r_2) \sup_{\Delta \in \mathcal{Q}_{c_0}(Q)} E_{r_1, r_2, t}(\phi_1^{(\Delta)}, \phi_2^{(\Delta)})^{1/p'}. \]
Now it suffices to show
\[ \sup_{\Delta \in \mathcal{Q}_{c_0}(Q)} E_{r_1, r_2, t}(\phi_1^{(\Delta)}, \phi_2^{(\Delta)}) \leq (1 + C c) E_{r, r_1, r_2}(\phi_1, \phi_2) + C r^{-N}, \]
because $K(R, r, \hat{r}) \lesssim K(R) \lesssim R^C$. By Lemma 5.1 we have $E(\Phi^A_j)^{1/2} \leq (1 + Cc)E(\phi_j)^{1/2}$. Thus it is enough to show
\[ \|\Phi^A_j\|_{L^2(D(z_0, r))} \leq (1 + Cc)\|\phi_j\|_{L^2(D(z_0, r))} + Cr^{-N} \]
for all $\Delta \in Q_{C_0}(Q)$ and all $z_0 \in \mathbb{R}^{n+1}$. We have this estimate from the following lemma:

**Lemma 9.1.** Let $Q$ be a finite index set. Suppose that $m_{q,T_j}$ are non-negative numbers with \( \sum_j m_{q,T_j} \leq (\sum_j m_{q,T_j}^2)^{1/2} = (\sum_j m_{q,T_j}^2)^{1/2} \). Then, for any $r \geq 2C_0$ and $z_0 \in \mathbb{R}^{n+1}$,
\[ \left( \sum_{q \in Q} \left( \sum_{T_j \in T_j} m_{q,T_j} \phi_{T_j} \right)^2 \right)^{1/2} \leq (1 + Cc)\|\phi_j\|_{L^2(D(z_0, r))} + r^{-N}E(\phi_j)^{1/2}. \]

**Proof.** Let $\bar{D} = D(x_0, t_0; r(1 - 2r^{-1/3N}))$, $\bar{D}' = D(x_0, t_0; r(1 - r^{-1/2N}))$ and $\bar{D}'' = D(x_0, t_0; r)$. Then we have $\bar{D} \subset \bar{D}' \subset \bar{D}''$.
We divide
\[ \sum_{T_j \in T_j} m_{q,T_j} \phi_{T_j} = \sum_{T_j \in T_j} m_{q,T_j} P_{D'} \phi_{T_j} + \sum_{T_j \in T_j} m_{q,T_j}(1 - P_{D'}) \phi_{T_j}. \]
Consider the first summation in the right side. We have
\[ \left( \sum_{q \in Q} \left( \sum_{T_j \in T_j} m_{q,T_j} P_{D'} \phi_{T_j} \right)^2 \right)^{1/2} \leq \sum_{q \in Q} \left( \sum_{T_j \in T_j} m_{q,T_j} P_{D'} \phi_{T_j} \right)^2 \]
Applying Lemma 5.1 we have
\[ \sum_{q \in Q} \left( \sum_{T_j \in T_j} m_{q,T_j} P_{D'} \phi_{T_j} \right)^2 \leq (1 + Cc)E(P_{D'} \phi_j). \]
From (6.6) we can see that
\[ E(P_{D'} \phi_j) \leq \phi_j \|_{L^2(D') + Cr^{-N}E(\phi_j)} \]
\[ \leq \phi_j \|_{L^2(D'')} + Cr^{-N}E(\phi_j), \]
where $\bar{D}' \subset \bar{D}'' \subset \bar{D}''$ is used. Thus we obtain
\[ \left( \sum_{q \in Q} \left( \sum_{T_j \in T_j} m_{q,T_j} P_{D'} \phi_{T_j} \right)^2 \right)^{1/2} \leq (1 + Cc)\|\phi_j\|_{L^2(D'')} + Cr^{-N}E(\phi_j)^{1/2}. \]

To prove (9.2) it now suffices to show
\[ \left( \sum_{q \in Q} \left( \sum_{T_j \in T_j} m_{q,T_j} (1 - P_{D'}) \phi_{T_j} \right)^2 \right)^{1/2} \leq r^{-N}E(\phi_j)^{1/2}. \]
Since the operator $P_{D}$ is linear, we have
\[ \sum_{T_j \in T_j} m_{q,T_j} (1 - P_{D'}) \phi_{T_j} = (1 - P_{D'}) \left( \sum_{T_j \in T_j} m_{q,T_j} \phi_{T_j} \right). \]
By $\bar{D} \subset \bar{D}'' \subset \bar{D}'$ and (6.5),
\[ \left( \sum_{T_j \in T_j} m_{q,T_j} (1 - P_{D'}) \phi_{T_j} \right)^2 \leq \left( \sum_{T_j \in T_j} m_{q,T_j} \phi_{T_j} \right)^2 \]
\[ \leq r^{-N}E(\sum_{T_j \in T_j} m_{q,T_j} \phi_{T_j})^{1/2}. \]
Using this we have
\[
\left( \sum_{q \in \mathcal{Q}} \left\| \sum_{T_j \in \mathcal{T}_j} m_{q,T_j}(1 - \mathcal{P}_{\mathcal{D}'} \phi) T_j \right\|_{L^2(\mathcal{D})}^2 \right)^{1/2} \lesssim r^{-N} \left( \sum_{q \in \mathcal{Q}} E \left( \sum_{T_j \in \mathcal{T}_j} m_{q,T_j} \phi T_j \right) \right)^{1/2}.
\]

Thus, by Lemma 5.1 we obtain (9.3). \qed

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