Weighted networks are more synchronizable: how and why

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Most real-world networks display not only a heterogeneous distribution of degrees, but also a heterogeneous distribution of weights in the strengths of the connections. Each of these heterogeneities alone has been shown to suppress synchronization in random networks of dynamical systems. Here we review our recent findings that complete synchronization is significantly enhanced and becomes independent of both distributions when the distribution of weights is suitably combined with the distribution of degrees. We also present new results addressing the optimality of our findings and extending our analysis to phase synchronization in networks of non-identical dynamical units.

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I. INTRODUCTION

The recent explosion in the study of complex networks of dynamical systems [1] has provided compelling evidence that the synchronization of coupled oscillators is drastically influenced by the structure of the underlying network of couplings [2, 3, 4, 5, 6, 7, 8, 9, 10]. Previous works on synchronization have focused mainly on the influence of the topology of the connections by assuming that the coupling strength is uniform. These works have shown that the ability to synchronize is generally enhanced in both small-world networks (SWNs) [11] and random scale-free networks (SFNs) [12] as compared to regular networks [13], indicating that synchronizability is strongly related to the average distance between oscillators. More recently, it has been shown that synchronizability also depends critically on the heterogeneity of the degree distribution and that this can lead to counter-intuitive behaviors [14]. In particular, random networks with strong heterogeneity in the degree distribution, such as SFNs, are more difficult to synchronize than random homogeneous networks [14], despite the fact that heterogeneity reduces the average distance between nodes [15].

However, synchronization, as many other dynamical processes, is influenced not only by the topology, but also by the strength of the connections. Very recently, it has been shown that synchronization is suppressed in networks with heterogeneous distribution of weights in the connection strengths [15], even when the distribution of degrees is homogeneous. This is an important result because most complex networks where synchronization is relevant are indeed weighted and display a highly heterogeneous distribution of both degrees and weights [16, 17]. Examples include brain networks, both at the neuronal and cortical level [18], and airport networks [19], which underlie the synchronization of epidemic outbreaks in different cities [19]. The identification and study of complex weighted networks with improved synchronizability properties is thus of great interest.

In this paper, we review this fundamental problem in the context of complete synchronization of identical oscillators [20, 21]. We introduce a weighted coupling scheme and we show that, for a given network topology, the synchronizability is maximum when the network of couplings is weighted and directed. For large sufficiently random networks, the maximum synchronizability is primarily determined by the mean degree of the network and does not depend on the degree distribution and system size. This contrasts with the case of unweighted (and undirected) coupling, where the synchronizability is strongly suppressed as the degree heterogeneity or number of oscillators is increased. We also show that the total cost involved in the weighted couplings is significantly reduced as compared to the case of unweighted coupling and appears to be minimum when the synchronizability is maximum. Furthermore, we present new results that extend our findings to the synchronization of phase oscillators and provide evidence for the optimality of our weighted model.

The paper is organized as follows. The problem of complete synchronization is formulated in Sec. II and is analyzed in Sec. III. The problem of phase synchronization is considered in Sec. IV. In Sec. V we discuss the distribution of weights that would maximize synchronizability for a given network topology. The conclusions are presented in the last section.

II. COMPLETE SYNCHRONIZATION

We introduce a weighted model of linearly coupled identical oscillators and we present a condition for the linear stability of the completely synchronized states in terms of the eigenvalues of the coupling matrix.

The dynamics of a weighted network of N identical
oscillators is described by:

\[ \dot{x}_i = F(x_i) + \sigma \sum_{j=1}^{N} A_{ij} [H(x_j) - H(x_i)], \quad (1) \]

\[ = F(x_i) - \sigma \sum_{j=1}^{N} G_{ij} H(x_j), \quad i = 1, \ldots, N, \quad (2) \]

where \( F = F(x) \) governs the dynamics of each individual oscillator, \( H = H(x) \) is the output function, and \( \sigma \) is the overall coupling strength. Matrix \( A = (A_{ij}) \) is the adjacency matrix of the underlying network of couplings, where \( A_{ij} = w_{ij} \) if there is a link of strength \( w_{ij} > 0 \) from node \( j \) to node \( i \), and 0 otherwise. Matrix \( G = (G_{ij}) \), defined as \( G_{ij} = -A_{ij} + \delta_{ij} \sum_{j=1}^{N} A_{ij} \), is the coupling matrix. The rows of matrix \( G \) have zero sum and this ensures that the completely synchronized state \( \{x_i(t) = s(t), \forall i \} \) is a solution of Eq. (2).

In the case of symmetrically coupled oscillators with uniform coupling strength, the network of couplings is unweighted and undirected, and \( G \) is the usual (symmetric) Laplacian matrix \( L = (L_{ij}) \): the diagonal entries are \( L_{ii} = k_i \), where \( k_i \) is the degree of node \( i \), and the off-diagonal entries are \( L_{ij} = -1 \) if nodes \( i \) and \( j \) are connected and \( L_{ij} = 0 \) otherwise. For \( G_{ij} = L_{ij} \), heterogeneity in the degree distribution suppresses synchronization in important classes of networks \([7]\). The synchronizability may be easily enhanced if we modify the topology of the network of couplings. Here, however, we address the problem of enhancement of synchronizability for a given network topology.

In order to enhance the synchronizability of networks with heterogeneous degree distribution, we propose to scale the coupling strength by a function of the degree of the nodes. For specificity, we take

\[ G_{ij} = L_{ij}/k_i^\beta, \quad (3) \]

where \( \beta \) is a tunable parameter. We say that the network or coupling is weighted when \( \beta \neq 0 \) and unweighted when \( \beta = 0 \). Networks with \( \beta = 0 \) and \( \beta > 0 \) are depicted in Figs. (a) and (b), respectively.

The underlying network associated with the Laplacian matrix \( L \) is undirected and unweighted, but for \( \beta \neq 0 \), the network of couplings becomes not only weighted but also directed because the resulting matrix \( G \) is in general asymmetric. This is a special kind of directed network where the number of in-links is equal to the number of out-links in each node, and the directions are encoded in the strengths of in- and out-links [Fig. (b)]. In spite of the possible asymmetry of matrix \( G \), all the eigenvalues of matrix \( G \) are nonnegative reals and can be ordered as \( 0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \), as shown below.

In matrix notation, Eq. (3) can be written as \( G = D^{\beta} L \), where \( D = \text{diag}\{k_1, k_2, \ldots, k_N\} \) is the diagonal matrix of degrees. (We recall that the degree \( k_i \) is the number of oscillators coupled to oscillator \( i \).) From the identity \( \det(D^{\beta} L - \lambda I) = \det(D^{-\beta/2} L D^{-\beta/2} - \lambda I) \), valid for any \( \lambda \), we have that the spectrum of eigenvalues of matrix \( G \) is equal to the spectrum of a symmetric matrix defined as \( H = D^{-\beta/2} L D^{-\beta/2} \). As a result, all the eigenvalues of matrix \( G \) are real, as anticipated above. Moreover, because \( H \) is positive semidefinite, all the eigenvalues are nonnegative and, because the rows of \( G \) have zero sum, the smallest eigenvalue \( \lambda_1 \) is always zero. If the network is connected, \( \lambda_2 > 0 \) for any finite \( \beta \). Naturally, the study of complete synchronization of the whole network only makes sense if the network is connected. For \( \beta = 1 \), matrix \( H \) is the normalized Laplacian matrix \([22]\). In this case, if \( N \geq 2 \) and the network is connected, then \( 0 < \lambda_2 \leq N/(N - 1) \) and \( N/(N - 1) \leq \lambda_N \leq 2 \). For spectral properties of unweighted complex networks, see Refs. \([22, 23, 24, 25]\).

The variational equations governing the linear stability of a synchronized state \( \{x_i(t) = s(t), \forall i\} \) of the system in Eqs. (2) and (3) can be diagonalized into \( N \) blocks of the form

\[ \frac{d\eta}{dt} = [DF(s) - \alpha DH(s)] \eta, \quad (4) \]

where \( D \) denotes the Jacobian matrix, \( \alpha = \sigma \lambda_i \), and
\(\lambda_i\) are the eigenvalues of the coupling matrix \(G\). The largest Lyapunov exponent \(\Lambda(\alpha)\) of this equation can be regarded as a master stability function, which determines the linear stability of the synchronized state \(2\): the synchronized state is stable if \(\Lambda(\sigma\lambda_i) < 0\) for \(i = 2, \ldots N\). (The eigenvalue \(\lambda_1\) corresponds to a mode parallel to the synchronization manifold.)

For many oscillatory dynamical systems \(2, 26\), the master stability function \(\Lambda(\alpha)\) is negative in a single, finite interval \((\alpha_1, \alpha_2)\). Therefore, the network is synchronizable for some \(\alpha\) when \(\alpha_1 < \sigma\lambda_2 \leq \ldots \leq \sigma\lambda_N < \alpha_2\), as depicted in Fig. 2. This is equivalent to the condition

\[
R \equiv \frac{\lambda_N}{\lambda_2} < \frac{\alpha_2}{\alpha_1},
\]

where \(\alpha_2/\alpha_1\) depends only on the dynamics \((F, H, s)\), while the eigenratio \(R\) depends only on the coupling matrix \(G\). The problem of synchronization is then reduced to the analysis of eigenvalues of the coupling matrix \(G\): the smaller the eigenratio \(R\), the larger the synchronizability of the network and vice versa. Alternatively, the eigenratio \(R\) can also be regarded as a measure of the speed to synchronize, where the speed is defined by the smallest transverse Lyapunov exponent.

### III. ENHANCEMENT OF SYNCHRONIZABILITY IN WEIGHTED NETWORKS

We study the system in Eqs. (2) and (3) and we argue that the synchronizability is maximum \((R\) is minimum) and the coupling cost is minimum for \(\beta = 1\).

#### A. Mean Field Approximation

A mean field approximation provides insight into the effects of degree heterogeneity and the dependence of \(R\) on \(\beta\).

The dynamical equations (2) can be rewritten as:

\[
\frac{d\mathbf{x}_i}{dt} = \mathbf{F}(\mathbf{x}_i) + \sigma k_i^{1-\beta} [\mathbf{H}_i - \mathbf{H}(\mathbf{x}_i)],
\]

where

\[
\mathbf{H}_i = \frac{1}{k_i} \sum_j A_{ij} \mathbf{H}(\mathbf{x}_j)
\]

is the local mean field from all the nearest neighbors of oscillator \(i\). If the network is sufficiently random and the system is close to the synchronized state \(s\), we may assume that \(\mathbf{H}_i \approx \mathbf{H}(s)\) and we may approximate Eq. (6) as

\[
\frac{d\mathbf{x}_i}{dt} = \mathbf{F}(\mathbf{x}_i) + \sigma k_i^{1-\beta} [\mathbf{H}(s) - \mathbf{H}(\mathbf{x}_i)],
\]

indicating that all the oscillators are decoupled and forced by a common oscillator with output \(\mathbf{H}(s)\).

From a variational equation analogous to Eq. (4), we have that all oscillators in Eq. (8) will be synchronized by the common forcing when

\[
\alpha_1 < \sigma k_i^{1-\beta} < \alpha_2 \quad \forall i.
\]

For \(\beta \neq 1\), it is enough to have a single node with degree very different from the others for this condition not to be satisfied for any \(\alpha\). In this case, the complete synchronization becomes impossible because the corresponding oscillator cannot be synchronized. This explains the results of Ref. (2) on the suppression of synchronizability due to heterogeneity in unweighted networks.

Within the mean field approximation, the eigenratio is \(R = (k_{\text{max}}/k_{\text{min}})^{1-\beta}\) for \(\beta \leq 1\) and \(R = (k_{\text{min}}/k_{\text{max}})^{1-\beta}\) for \(\beta > 1\), where \(k_{\text{min}} = \min\{k_i\}\) and \(k_{\text{max}} = \max\{k_i\}\), and reaches its minimum at \(\beta = 1\). This suggests that the maximum synchronizability in weighted networks is achieved when \(\beta = 1\). As shown in Ref. (21), the same indication is provided by the analysis of a diffusion process related to the spread of information in the network. We thus expect the (un-approximated) system in Eq. (2) to exhibit maximum synchronizability at \(\beta = 1\).

We now test this prediction numerically on random SFNs. (24) The networks are generated as follows. Each node is assigned to have a number \(k_i \geq k_{\text{min}}\) of “half-links” according to the probability distribution \(P(k) \sim k^{-\gamma}\), where \(\gamma\) is a scaling exponent and \(k_{\text{min}}\) is a constant integer. The network is generated by randomly connecting these half-links to form links, prohibiting self- and repeated links. We discard the networks that are not connected. In the limit \(\gamma = \infty\), all nodes have the same degree \(k = k_{\text{min}}\).

Our numerical computations confirm that the eigenratio \(R\) has a well defined minimum at \(\beta = 1\), as shown in Fig. (k) for various \(\gamma\). The only exception is the class of homogeneous networks, where all the nodes have the same degree. When the network is homogeneous, the weights \(k^{-\beta}\) can be factored out and the eigenratio \(R\) is independent of \(\beta\) (Fig. (k) solid line). Random homogeneous networks correspond to random SFNs with \(\gamma = \infty\). In SFNs, the heterogeneity increases as the scaling exponent \(\gamma\) is reduced. As shown in Fig. (k) the minimum of the eigenratio \(R\) becomes more pronounced as the heterogeneity
Therefore, for $\beta$ matrix

Moreover, the semicircle law holds and the spectrum of
tions (11) are expected to hold true for any large, suffi-
quence and sufficiently large minimum degree

Each curve corresponds to an average over 50 realizations of

of the degree distribution is increased. A pronounced
minimum for the eigenratio $R$ at $\beta = 1$ is also observed
in various other models of complex networks [20, 21].

B. Mean Degree Approximation

In what follows, we present an approximation for the
eigenratio $R$. Here we focus on the case of maximum
synchronizability ($\beta = 1$).

Based on results of Ref. 24 for random networks with
given expected degrees, we have

$$\max\{1 - \lambda_2, \lambda_N - 1\} = [1 + o(1)] \frac{2}{\sqrt{k}}.$$  (10)

Moreover, the semicircle law holds and the spectrum of
matrix $H$ is symmetric around 1 for $k_{\text{min}} \gg \sqrt{k}$ in the
thermodynamic limit [24]. These results are rigorous for
ensembles of networks with a given expected degree se-
dence and sufficiently large minimum degree $k_{\text{min}}$, but
our numerical computation supports the hypothesis that the
approximate relations

$$\lambda_2 \approx 1 - \frac{2}{\sqrt{k}}, \quad \lambda_N \approx 1 + \frac{2}{\sqrt{k}},$$  (11)

hold under much milder conditions. In particular, relations
(11) are expected to hold true for any large, suffi-
ciently random network with $k_{\text{min}} \gg 1$.

Under the assumption that $1 - \lambda_2 \approx \lambda_N - 1 \approx 2/\sqrt{k}$,
the eigenratio can be written as

$$R \approx \frac{1 + 2/\sqrt{k}}{1 - 2/\sqrt{k}}.$$  (12)

Therefore, for $\beta = 1$, the eigenratio $R$ is primarily deter-
dined by the mean degree and does not depend on the
number of oscillators and the details of the degree distri-
ution. This is a remarkable result because, regardless of
the degree distribution, the network at $\beta = 1$ is just as
synchronizable as a random homogeneous network with
the same mean degree, and random homogeneous net-
works appear to be asymptotically optimal in the sense that
$R$ approaches the absolute lower bound in the ther-
moscopic limit for large enough $k$.  

We now test our predictions on random SFNs. As
shown in Fig. 4 in unweighted SFNs, the eigenratio $R$
increases with increasing heterogeneity of the degree dis-
tribution (see also Ref. 7). But, as shown in the same
figure, the eigenratio does not increase with heterogeneity
when the coupling is weighted at $\beta = 1$. The differ-
ce is particularly large for small scaling exponent $\gamma$,
where the variance of the degree distribution is large and
the network is highly heterogeneous (note the logarith-
mic scale in Fig. 4). For $\beta = 1$, the eigenratio $R$ is well
approximated by the relation in Eq. (12) [Fig. 4 solid
line]. For $\beta = 1$, the eigenratio of the SFNs is also very
well approximated by the eigenratio of random homoge-
nous networks with the same number of links [Fig. 4
circle]. Therefore, for $\beta = 1$, the variation of the eigenratio
$R$ with the heterogeneity of the degree distribution in
SFNs is mainly due to the variation of the mean degree of
the networks, which increases in random SFNs as the
scaling exponent $\gamma$ is reduced.

In Fig. 5 we show the eigenratio $R$ as a function of the
system size $N$. In unweighted SFNs, the eigenratio in-
creases strongly as the number of oscillators is increased.
Therefore, it may be very difficult or even impossible
to synchronize large unweighted networks. However, for
$\beta = 1$, the eigenratio of large networks appears to be in-
dependent of the system size, as shown in Fig. 5 for ran-
dom SFNs. Similar results are observed in many other
models of complex networks. Altogether, these provide
strong evidence for relation (12) and shows that synchro-
C. Coupling Cost

We now turn to the problem of the cost involved in the network of couplings. The total cost \( C \) is naturally defined as the total input strength of the connections of all nodes at the synchronization threshold:

\[
C = \sigma_{\text{min}} \sum_{i=1}^{N} k_i^{1-\beta},
\]

where \( \sigma_{\text{min}} = \alpha_1/\lambda_2 \) is the minimum coupling strength for the network to synchronize. We recall that \( \alpha_1 \) is the point where the master stability function first becomes negative. For \( \beta = 1 \), we have \( C = N \alpha_1/\lambda_2 \).

As a function of \( \beta \), the cost has a broad minimum at \( \beta = 1 \), as shown in Fig. 6 for random SFNs. This result is important because it shows that maximum synchronizability and minimum cost occur exactly at the same point. The cost for random SFNs at \( \beta = 1 \) is very well approximated by the cost for random homogeneous networks with the same mean degree [Fig. 7, ◦], in agreement with our analysis in Sec. III B that, at \( \beta = 1 \), the eigenvalue \( \lambda_2 \) is fairly independent of the degree distribution. The cost at \( \beta = 1 \) is significantly reduced as compared to the case of unweighted coupling (\( \beta = 0 \)), as shown in Fig. 7. The difference becomes more pronounced when the scaling exponent \( \gamma \) is reduced and the degree distribution becomes more heterogeneous. Similar results are observed in other models of complex networks. Therefore, cost reduction is another important advantage of suitably weighted networks.

IV. PHASE SYNCHRONIZATION

The above analysis on identical oscillators can serve as a good approximation even when the oscillators are not fully identical. In some realistic situations [28], however, the interacting oscillators are significantly different, especially in the oscillation frequencies [29]. Here we extend the enhancement of synchronizability to networks of non-identical oscillators.

We consider phase synchronization in complex networks of limit cycle oscillators,

\[
\dot{x}_i = \omega_i F(x_i) - \sigma \sum_{j=1}^{N} G_{ij} H(x_j), \quad i = 1, \ldots, N,
\]

where \( \dot{x} = F(x) \) represents a Van der Pol oscillator, \( \dot{u} = v, \dot{v} = 0.3(1-u^2)v - u \), and the couplings are through both variables \( u \) and \( v \) such that \( H(x) = x \). We assume that the frequencies \( \omega_i \) follow a uniform distribution in an interval \([1 - \Delta \omega, 1 + \Delta \omega]\). In our simulations we set \( \Delta \omega = 0.2 \).

Now we study the effects of the weighted coupling on phase synchronization. For an arbitrary \( \beta \), on average each oscillator receives an effective input of strength
pronounced for the networks with weighted networks, but the oscillations are much more collective oscillations emerge for both weighted and unweighted networks. As the coupling strength is increased, neither of the networks display significant collective behavior [Fig. 8(a)]. As the coupling strength is increased, the overall behavior is similar for weighted and unweighted networks but, again, the amplitude is increased. The oscillations are much more pronounced for the networks with weighted networks than with unweighted networks but, again, the amplitude is increased. The overall behavior is similar for weighted and unweighted networks but, again, the amplitude is increased.

\[
\sigma^* = \sigma \sum_{i}^{N} k_i^{1-\beta} / N
\]

We can compare the synchronization behavior at different \(\beta\) for the same effective strength \(\sigma^*\). In Fig. 8 we show time series of the mean field \(X = \sum_{i=1}^{N} x_i / N\) for unweighted \((\beta = 0)\) and weighted random SFNs \((\beta = 1)\). For small coupling strength \(\sigma^*\), neither of the networks display significant collective behavior [Fig. 8(a)]. As the coupling strength is increased, collective oscillations emerge for both weighted and unweighted networks, but the oscillations are much more pronounced for the networks with \(\beta = 1\) [Fig. 8(b)].

In Fig. 9 we show the amplitude of the mean field \(X\) as a function of \(\sigma^*\). Here the amplitude is defined as the standard deviation of \(X\) over time. The amplitude is approximately zero for small coupling strength, increases sharply as \(\sigma^*\) is increased beyond a certain critical value, and saturates for large \(\sigma^*\) (Fig. 9). (This transition becomes sharper as the number \(N\) of oscillators is increased). The overall behavior is similar for weighted and unweighted networks but, again, the amplitude is significantly larger for networks with \(\beta = 1\) at the same value of \(\sigma^*\). Moreover, the amplitude of random SFNs at \(\beta = 1\) is well approximated by the amplitude of random homogeneous networks with the same mean degree (Fig. 9), which are networks that exhibit good phase synchronization properties.

All of these provide strong evidence that our weighted coupling scheme strongly enhances the synchronizability of networks of non-identical oscillators. This is expected to be relevant for realistic networks.

Within the weighted coupling scheme of Eq. (4), the networks that exhibit maximum synchronizability and minimum cost are those with \(\beta = 1\) (Sec. III). Physically, this seems to be related to the fact that the synchronization of the network is determined by the input signal at each node (as opposed to the output signal) and that the sum \(\sum_{j} A_{ij} = k_i^{1-\beta}\) of the strengths of all the in-links of a node becomes uniformly equal to 1 for all the nodes exactly at \(\beta = 1\). This leads to the natural question of whether there is any other distribution of weights that can further improve the synchronizability and cost of the network for a given topology of connections. As we show, the answer is positive for finite size networks, but seems to be negative in the thermodynamic limit.

In order to address this question we introduce the following weighted coupling scheme. If there is a link between a node \(i\) with degree \(k_i\) and a node \(j\) with degree \(k_j\), the strength \(A_{ij} = w_{ij}\) of link \(j \rightarrow i\) is taken to be

\[
w_{ij} = (k_i k_j)^\theta / N_r
\]

where \(\theta\) is a tunable parameter and \(N_r = \sum_{j \sim i} (k_i k_j)^\theta\) is a normalization factor that ensures that \(\sum_{j} A_{ij} = 1\) (the sum \(\sum_{j \sim i}\) is over all the neighbors of node \(i\)). For \(\theta = 0\) we recover the model in Eq. (4) for \(\beta = 1\).

This new weighted model is partially motivated by observations in real networks, including scientific collaboration networks [15], airport networks [16] [16], and metabolic networks [14], which have been found to exhibit a strong correlation between the weight \(w_{ij}\) and product of the corresponding degrees as \((w_{ij}) \sim (k_i k_j)^\theta\).

A similar model has been considered in the study of coherence in networks of chaotic maps [14].

**V. WHAT ARE THE MOST SYNCHRONIZABLE NETWORKS?**
unweighted networks \( \beta \) oscillators. In Fig. 10, we show the eigenratio in the context of complete synchronization of identical oscillators, as indicated in the figure. The eigenratio and the cost reach a minimum for some \( \theta \), \( \gamma \), and \( C \) different values of the scaling exponent. In each case, both the normalized cost \( C/(N\alpha) \) as functions of \( \theta \) for random SFNs with various choices for the number \( N \) of oscillators, as indicated in the figure.

We now study the weighted model \( \beta \) numerically in the context of complete synchronization of identical oscillators. In Fig. 11, we show the eigenratio \( R \) and the normalized cost \( C/(N\alpha) \) as functions of \( \theta \) for different values of the scaling exponent. In each case, both the eigenratio and the cost reach a minimum for some \( \theta^*(N, \gamma) > 0 \), which is approximately the same for \( R \) and \( C \) and indicates that these networks synchronize better than those of the weighted model \( \beta = 1 \) \( (\theta = 0) \). However, as compared to the difference between unweighted networks \( (\beta = 0) \) and networks weighted for \( \beta = 1 \) in model \( \beta = 1 \), the further reduction of \( R \) and \( C \) within model \( \beta = 1 \) is very small.

Indeed, \( R(\beta = 0) - R(\beta = 1) \) and \( C(\beta = 0) - C(\beta = 1) \) are orders of magnitude larger than \( R(\theta = 0) - R(\theta = \theta^*) \) and \( C(\theta = 0) - C(\theta = \theta^*) \) even when the networks are strongly heterogeneous (i.e., \( \gamma \) is small). Moreover, the difference between the synchronizability at \( \theta = 0 \) and \( \theta = \theta^* \) reduces when the number \( N \) of oscillators is increased, as shown in Fig. 11. Since \( \theta^* \) approaches \( \theta = 0 \) for increasing \( N \), our numerics are consistent with the hypothesis that the minimum of \( R \) and \( C \) will be at \( \theta = 0 \) for \( N \to \infty \). That is, the weighted model \( \beta = 1 \) may be optimal in the thermodynamic limit. The verification of this hypothesis is currently an interesting open problem.

VI. CONCLUSIONS

Complex networks with strong heterogeneity in the distributions of either degrees or weights are poorly synchronizable. This suppression of synchronizability is due to the different intensities of the input signals received by different nodes in the network. We have shown that the synchronizability is significantly improved when the distribution of weights is constrained by the distribution of degrees in such a way that all the nodes receive the same intensity of input signal. In this case, the distributions of degrees and weights as well as the distribution of the total strength of out-links per node, \( \sum_j A_{ij} \), may be highly heterogenous, but the distribution of the total strength of in-links, \( \sum_j A_{ij} \), is the same for all the nodes in the network. In particular, our analysis shows that properly weighted complex networks are much more synchronizable than the corresponding unweighted networks.

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