A Non-perturbative Evidence toward the Positive Energy Conjecture for asymptotically locally $AdS_5$ IIB Supergravity on $S^5$

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Abstract

We consider the classical solution of the type IIB supergravity spontaneously compactified on $S^5$, whose metric depends only on the radial coordinate and whose asymptotic geometry is locally that of $AdS_5$, i.e., $R \times S^1 \times T^2$. We solve the equations of motion to obtain the general solutions satisfying these conditions, and find that the only naked-singularity-free solutions are the $AdS$ black holes and $AdS$ solitons. The other solutions, that smoothly interpolate these two solutions, are shown to have naked singularities even though their Ricci tensor is proportional to the metric with a negative constant. Thus, among the possible solutions of this type, the $AdS$ solitons are the unique lowest energy solution; this result is consistent with the recently proposed positive energy conjecture for the IIB $AdS$ supergravity on $S^5$.

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I. INTRODUCTION

The conjectured correspondence between the string theory in an Anti de Sitter (AdS) space and the conformal field theory (CFT) on the boundary opens up a new possibility of studying the conformal phase of the Yang-Mills theory via the analysis of supergravity, which, in many cases, is a useful low energy approximation of the string theory \[1\] \[2\] \[3\]. From this point of view, one natural case of interest is the large \(N\), near-horizon geometry of the \(D\)-threebranes of the type IIB string theory. In this case, by the AdS/CFT correspondence, one relates the four-dimensional supersymmetric Yang-Mills theory on the boundary of the \(AdS_5\) to the bulk \(AdS_5\) IIB supergravity.

One ultimately hopes to understand the dynamics of the gauge theory when the supersymmetry of the four-dimensional Yang-Mills theory is broken. A practical path toward this goal was suggested by Witten \[4\] as follows. Instead of considering the asymptotically globally \(AdS_5\), one considers the asymptotically locally \(AdS_5\) whose asymptotic geometry allows a non-trivial one-cycle. On the boundary Yang-Mills theory side, one assigns the anti-periodic boundary condition for the fermions around the one-cycle. This choice of the boundary condition breaks the supersymmetry by making relevant scalars and fermions massive, resulting a non-supersymmetric Yang-Mills theory. On the gravity side, in the spirit of the AdS/CFT correspondence, therefore, we are led to consider the asymptotically locally \(AdS_5\) supergravity whose asymptotic geometry contains a circle. In other words, the asymptotic geometry of the five-dimensional manifolds that we are interested in should look like \(R \times S^1 \times M_2\), where the first \(R\) denotes a generic time, \(S^1\) is the circle along which a non-trivial holonomy for fermions can be imposed, and \(M_2\) is a two-manifold.\[1\] Recalling that the original gauge theory contained sixteen supercharges, we suppose that \(M_2\) is a Riemann surface. We thus consider the case when \(M_2 = M_g\) where \(M_g\) is an arbitrary Riemann surface with genus \(g\); on the Yang-Mills theory side, this results the (2+1)-dimensional

\[1\] Here, for simplicity, we assume \(S^1\) is orthogonal to \(M_2\).
non-supersymmetric Yang-Mills theory in a space-time whose space-like hypersurface is a
genus $g$ Riemann surface upon the compactification on the $S^1$.

Horowitz and Myers recently raised the issue of the stability of the asymptotically locally
$AdS_5$ supergravity asking the question of validity of the positive energy conjecture
\cite{5}. When combined with the AdS/CFT correspondence, the instability on the supergravity
side, if it exists, will either imply the instability of the non-supersymmetric Yang-Mills
theory or cast some doubts on the exactness of the AdS/CFT correspondence. In fact, the
positive energy theorem was proved for space-times whose asymptotic geometry is globally
$AdS_5$, consistent with the stability of the supersymmetric Yang-Mills theory with sixteen
supercharges \cite{6}. However, in the context of the asymptotically locally flat space-time ge-
ometries, there are some explicit examples that show the existence of the regular solutions
of the Einstein equations whose energy is zero \cite{7} or not bounded from below \cite{8,9}. In a
similar vein, it was proved in \cite{10} that the positive energy conjecture fails for the asymptot-
ically locally Euclidean space-time. Despite these aspects of the general relativity with the
vanishing cosmological constant, the intriguing conjecture of Horowitz and Myers is that
the negative energy solution of \cite{5}, $AdS_5$ solitons, is the lowest energy solution among the
possible asymptotically locally $AdS_5$ solutions of the $AdS_5$ IIB supergravity, i.e., essentially
a positive energy conjecture.

In this paper, we find an evidence that supports the Horowitz-Myers conjecture; we
obtain the general solutions of the IIB supergravity compactified on a five-sphere whose
five-dimensional metric is of the form $ds^2 = -\alpha(r)dt^2 + \beta(r)(dx_1^2 + dx_2^2) + \gamma(r)d\theta^2 + \delta(r)dr^2$, which include the static genus one sector of the space-times introduced in the above. We find
that the AdS solitons (negative energy) of \cite{5} and AdS black holes (positive energy) of \cite{4}
are the unique regular solutions whose asymptotic geometry is locally $AdS_5$, while all other
solutions with the same asymptotic geometry possess naked singularities that show up in the
square of the Riemann tensor $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. Our results span all the static, radial-dependent
solutions of the genus one sector satisfying the asymptotic condition; since propagating
gravity wave degrees of freedom are expected to contribute a positive amount to the energy
and since the perturbative analysis of [5] proves that the $AdS_5$ solitons are local minimum of the energy, our results suggest that the $AdS_5$ solitons are the minimum energy solution when we consider the space-time geometries whose asymptotic spatial part of the metric is a product of a circle and a Riemann surface of genus one, i.e., a two-torus. In section II, we start by reviewing the spontaneous compactification of the IIB supergravity on a five-sphere (see, for example, [11] and [12]), which is the near-horizon geometry of the $D$-threebrane solutions [1]. In section III, we explicitly solve the equations of motion to get the general solutions of the form introduced in the above. In section IV, we show that the $AdS_5$ solitons and black holes are the unique regular solutions. In fact, the solution space is parameterized by points on $R^2$ and, at a fixed radius of the solution space, the $AdS_5$ solitons and the $AdS_5$ black holes are two points on a solution space circle at which naked singularities are absent. We conclude by discussing the cases of an arbitrary genus and, especially, the most interesting case of the genus zero for the test of the Horowitz-Myers conjecture.

II. SPONTANEOUS COMPACTIFICATION OF IIB SUPERGRAVITY ON $S^5$

We start from the following action from the IIB supergravity

$$I_{10} = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g^{(10)}} \left[ e^{-2\phi}(R^{(10)} + 4(D\phi)^2) - \frac{1}{2 \cdot 5!} F^{(10)2} \right],$$

where $\phi$ is the dilaton field, $F^{(10)}$ is the RR self-dual five-form field strength and $g^{(10)}_{AB}$, the ten-dimensional metric. All other fields are assumed to be zero. Here $G_{10}$ is the ten-dimensional Newton’s constant. We note that when solving the equations of motion, we have to impose the self-duality condition for the $F^{(10)}$ field by hand [13]. In this paper, we consider the spontaneous compactification of the ten-dimensional theory [1] on a $S^5$ [11].

For the metric, this consideration amounts to setting it as

$$ds^2 = g^{(5)}_{\mu\nu} dx^\mu dx^\nu + e^{-2\psi} d\Omega_5^2$$

where the five-dimensional metric $g^{(5)}_{\mu\nu}$ on the $M^5$ and the radius of the sphere $\exp(-\psi)$ depend only on the coordinates $x^\mu$ on the $M^5$. The metric $d\Omega_5^2$ is the standard metric
on the unit $S^5$ with five angle coordinates $\theta_1, \theta_2, \theta_3, \theta_4$ and $\theta_5$. Imposing the spherical symmetry on $S^5$ requires that the dilaton field depends only on the $x^\mu$ coordinates. In addition, it is necessary that the five-form field strength be the scalar on the $S^5$ or be proportional to the pseudoscalar $\epsilon_{\theta_1\theta_2\theta_3\theta_4\theta_5}$, the volume form on the unit five-sphere. Thus, the five-form field strength $F^{(10)}$ has nonvanishing components $F^{(10)}_{\mu_1\mu_2\mu_3\mu_4\mu_5} = F^{(5)}_{\mu_1\mu_2\mu_3\mu_4\mu_5}$ and $F^{(10)}_{\theta_1\theta_2\theta_3\theta_4\theta_5} = F^{(5)}_{\theta_1\theta_2\theta_3\theta_4\theta_5}$ where $F^{(5)}$ is a zero-form field strength on the $M^5$. The integration on the $S^5$ after plugging in the fields into Eq. (1) yields the five-dimensional action
\begin{equation}
I_5 = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left\{ e^{-2\phi_1} \left[ R^{(5)} + 20e^{2\psi} + 4(D\phi_1)^2 - 5(D\psi)^2 \right] - \frac{1}{2} e^{5\psi}(F^{(1)}_1)^2 + F^{(2)}_2 \right\}
\end{equation}

where $2\phi_1 = 5\psi + 2\phi$ and $G_5$ is the five-dimensional Newton’s constant. In deriving Eq. (3), we additionally used an equivalent but more convenient description of the five-form field strength $F^{(5)}$ by taking the five-dimensional Hodge dual of the $F^{(5)}$. Namely, we take the Hodge dual transformation via
\begin{equation}
F^{(5)}_2 = -\frac{1}{5!} e^{-5\psi} \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5} F^{(5)}_{\mu_1\mu_2\mu_3\mu_4\mu_5} \rightarrow F^{(5)}_{\mu_1\mu_2\mu_3\mu_4\mu_5} = e^{5\psi} F^{(5)}_2 \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}
\end{equation}

where $F^{(5)}_2$ is the zero-form field strength on the $M^5$. At level of the action, this transformation also dictates that the term
\begin{equation}
\frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \frac{(-1)}{2 \cdot 5!} e^{-5\psi} F^{(5)}^2
\end{equation}
changes to
\begin{equation}
\frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \frac{(-1)}{2} e^{5\psi} F^{(2)}_2^2
\end{equation}
to implement Hodge duality, which ensures that the equations of motion on the $M^5$ are equivalent in both case. We notice that the equations of motion for the five-form field strength become the Bianchi identities in the dual picture. Just as the Bianchi identities for the five-form field strength in five dimensions are vacuous, the equations of motion for the zero-form field strength are vacuous.
The Bianchi identities for the zero-form field strengths $F_1^{(5)}$ and $F_2^{(5)}$ imply that they are constants. The self-duality of the RR five-form field strength furthermore gives a constraint $F_1^{(5)2} = F_2^{(5)2}$. For the spontaneous compactification on the $S^5$ where the ten-dimensional space-time is the direct product of a five-manifold and a five-sphere with a constant radius, we require that

$$D_\mu \psi = 0. \quad (5)$$

The equation of motion for $\psi$ from the five-dimensional action Eq. (3) becomes

$$16 e^{-2\phi_1} - e^{3\psi} (F_1^{(5)2} + F_2^{(5)2}) = 0 \quad (6)$$

upon using the condition Eq. (5). Eq. (5) implies that the field $\phi_1$ should also be a constant, $D_\mu \phi_1 = 0$. The equations of motion for $\phi_1$ field and the metric $g_{\mu\nu}^{(5)}$ then become

$$R^{(5)} + 20 e^{2\psi} = 0 \quad (7)$$

and

$$\left( \frac{1}{2} R^{(5)} + 6 e^{2\psi} \right) g_{\mu\nu} - R_{\mu\nu} = 0, \quad (8)$$

respectively under the conditions Eq. (5) and $D_\mu \phi_1 = 0$. Here $R_{\mu\nu}$ is the Ricci tensor for the metric. In fact, the equation of motion for the metric, Eq. (8), becomes the equation of motion for $\phi_1$ field, Eq. (7), if we take the trace of Eq. (8); Eq. (7) is a consistent consequence of Eq. (8). In summary, after the spontaneous compactification on $S^5$, the equations of motion become Eq. (5), $D_\mu \phi_1 = 0$, and Eq. (8). We are led to the following effective action which produces the resulting equations of motion

$$I = \int d^5x \sqrt{-g^{(5)}} \left( R^{(5)} + 12 e^{2\psi} \right) \quad (9)$$

where $\psi$ is constant. This is the five-dimensional Einstein equation with a negative cosmological constant.

\footnote{It is reasonable to suppose that this configuration has the lower energy compared to higher harmonic modes on the $S^5.$}
III. DERIVATION OF STATIC SOLUTIONS

The five-dimensional metric that we consider in this paper, as explained in the section I, is of the form

$$ds^2 = g^{(2)}_{\alpha\beta} dx^\alpha dx^\beta + e^{2\psi_1} d\theta^2 + e^{\psi_2} (dx_1^2 + dx_2^2)$$ (10)

where the two-dimensional metric $g^{(2)}_{\alpha\beta}$ on the $M_2$, the radius $\exp(\psi_1)$ of $S^1$ and the size $\exp(\psi_2)$ of $T^2$ depend only on the coordinates $x^\alpha$ of $M_2$. We will eventually consider only static solutions and this means that the metric fields will depend only on a space-like combination of the two coordinates $x^\alpha$. Under these assumptions, similar to the $s$-wave reduction of the four-dimensional general relativity to a two-dimensional dilaton gravity (see for example [14]), we can dimensionally reduce the five-dimensional problem to an equivalent two-dimensional problem. After rescaling of the metric, $g^{(2)}_{\alpha\beta} = e^{-(3\psi_1 + \psi_2)/2} g_{\alpha\beta}$, the resulting two-dimensional action is computed to be

$$I_2 = \int d^2x \sqrt{-g} e^{-2\bar{\psi}} \left[ R + 12e^{2\psi + \bar{\psi} - \psi_1} - \frac{3}{2}(D\psi_1)^2 \right]$$ (11)

where $-2\bar{\psi} = \psi_1 + \psi_2$. We choose a conformal gauge for the two-dimensional metric

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -e^{2\rho} dx^+ dx^-.$$ (12)

We get the static equations of motion from the two-dimensional action Eq. (11) under the choice of the conformal gauge by imposing all fields depend only on a spacelike coordinate $x \equiv x^+ - x^-$. The static equations of motion are then summarized by a one-dimensional action

$$I_1 = \int dx \left( \Omega' \rho' + \frac{3}{2} e^{2\psi} e^{-\psi_1} \Omega^{1/2} e^{2\rho} - \frac{3}{4} \Omega_1'' \right)$$ (13)

where $\Omega = e^{-2\bar{\psi}}$ and the prime denotes the differentiation with respect to $x$. The gauge constraint

$$\Omega'' - 2\rho' \Omega' + \frac{3}{2} \Omega_1'' = 0.$$ (14)
should be supplemented to the equations from Eq. (13). From here on, we follow the scheme of [15] to solve the equations of motion.

We find that there are three symmetries of the action (13)

(a) \( \psi_1 \to \psi_1 + \alpha, \ \rho \to \rho + \frac{1}{2} \alpha \)

(b) \( x \to x + \alpha \)

(c) \( x \to e^\alpha x, \ \Omega \to e^\alpha \Omega, \ \rho \to \rho - \frac{3}{4} \alpha \)

where \( \alpha \) is an arbitrary real parameter of each symmetry transformation for the three fields in our problem. Using these three symmetries we can integrate the coupled second order differential equations once to get coupled first order differential equations by constructing three Noether charges. They are given by

\[
\begin{align*}
\psi_0 &= 3 \Omega \psi_1' - \Omega', \quad (15) \\
c_0 &= \Omega' \rho' - \frac{3}{4} \Omega \psi_1'^2 - \frac{3}{2} e^{2\psi} e^{-\psi_1} \Omega^{1/2} e^{2\rho}, \quad (16) \\
s + c_0 x &= \Omega \rho' - \frac{3}{4} \Omega', \quad (17)
\end{align*}
\]

corresponding to each symmetry. The gauge constraint Eq. (14) and the equations of motion for \( \rho \)

\[
\Omega'' = 3 e^{2\psi} e^{-2\psi_1} \Omega^{1/2} e^{2\rho}
\]

determine \( c_0 = 0 \), and thus the total number of constants of motion will be reduced from six to five. To get the general static solutions of Eqs. (15)-(17), we introduce the following field \( A \) by

\[
QA = \Omega',
\]

where a positive number \( Q \) satisfies \( Q^2 = 3 e^{2\psi} \). Then from Eq. (18) we have

\[
Q = e^{\psi_1} \Omega^{-1/2} e^{-2\rho} A'.
\]

Using Eqs. (17), (18), (19) and (20), we get the following equation
\[
\left(\frac{8}{3}QA + 2s - \frac{1}{3}\psi_0\right)A' = Q \frac{d}{dx} \left(e^{2\rho}e^{-\psi_1}\Omega^{3/2}\right),
\]

which can be integrated to yield

\[Qe^{2\rho}e^{-\psi_1}\Omega^{3/2} = \frac{4}{3}QA^2 + (2s - \psi_0/3)A + c_1 \equiv P(A),\]  

(21)

where we introduced a function \(P(A)\) and \(c_1\) is a constant of integration. We note that the sign of \(P(A)\) should be positive definite. Combining Eqs. (13)-(17), (19) and (21), we find

\[c_1 = -\frac{1}{6} \frac{\psi_0^2}{Q}.
\]

(22)

Putting Eq. (21) into Eq. (20), we get

\[\Omega \frac{dA}{dx} = P(A) \rightarrow \Omega \frac{d}{dx} = P(A) \frac{d}{dA}.
\]

(23)

By changing the differentiation variable from \(x\) to \(A\) with the help of Eq. (23) and using Eq. (19), we immediately find that Eq. (15) can be integrated to give

\[\psi_1(A) = \frac{1}{8} \ln |P(A)| - \frac{1}{8}(2s - 3\psi_0)I(A) + \psi_{10}
\]

(24)

where \(I(A) = \int P^{-1}dA\) and \(\psi_{10}\) is a constant of integration. In a similar way, we can rewrite Eq. (19) as

\[\frac{1}{\Omega} \frac{d\Omega}{dA} = \frac{QA}{P(A)},
\]

which, upon integration, becomes

\[\Omega = |P(A)|^{3/8}e^{-(6s-\psi_0)I(A)/8}\Omega_0
\]

(25)

where \(\Omega_0\) is a positive definite constant of integration. The field \(\rho\) can be obtained from Eqs. (21), (24) and (25)

\[e^{2\rho} = |P(A)|^{9/16}e^{(14s+3\psi_0)I(A)/16}Q^{-1}e^{\psi_{10}}\Omega_0^{-3/2}.
\]

(26)

The field \(A\) in terms of \(x\) can be determined from Eq. (23)
\[ x - x_0 = \int \frac{\Omega(A)}{P(A)} dA, \]

where \( x_0 \) is a constant of integration. We have solved all the equations of motion; the fields are given by Eqs. (24), (25), (26) and (27) and there are five constants of motion, \( s, \psi_0, \psi_{10}, \Omega_0 \) and \( x_0 \). Since \( D \equiv 4s^2 - 4s\psi_0/3 + \psi_0^2 \) is positive semi-definite, which becomes zero only when \( s = \psi_0 = 0 \), the quadratic equation \( P(A) = 0 \) has two real roots \( A_{\pm} \) where

\[ A_{\pm} = \frac{\psi_0 - 6s \pm 3\sqrt{D}}{8Q}. \]

We thus write \( P(A) \) as

\[ P(A) = \frac{4}{3} Q(A - A_+)(A - A_-). \]

Since \( Q > 0 \), we see that \( A_+ - A_- = 3\sqrt{D}/(4Q) > 0 \). As can be seen from Eq. (24), we have \( P(A)/Q > 0 \), which restricts the range of \( A \) variable to \( A_+ < A \) or \( A_- > A \). It turns out that two regions describe the same space-time if we transform \( s \rightarrow -s \) and \( \psi_0 \rightarrow -\psi_0 \) (and if we appropriately change the relationship between \( r \) and \( A \) below). Avoiding redundancy, we thus choose the region \( A_+ < A \). A convenient choice of a radial coordinate \( r \) is

\[ \frac{r^4}{l^4} = \frac{1}{\sqrt{3}} (A - A_-) \]

where \( l \) is a constant satisfying \( l^2 = e^{-2\psi} \) and thus denotes the radius of the \( S^5 \). Using this radial coordinate, we can rewrite the ten-dimensional metric as

\[ ds^2 = \frac{r^2}{l^2} \left( 1 - \frac{r_0^4}{r^4} \right)^{(\sqrt{D} - \psi_0 - 2s)/(4\sqrt{D})} \left[ \left( 1 - \frac{r_0^4}{r^4} \right)^{\psi_0/\sqrt{D}} \left( 1 - \frac{r_0^4}{r^4} \right)^{2s/\sqrt{D}} \right] d\theta^2 + dx_1^2 + dx_2^2 - \left( 1 - \frac{r_0^4}{r^4} \right)^{-1} dr^2 + e^{-2\psi} d\Omega_5^2 \]

where

\[ r_0^4 = \frac{\sqrt{3D}}{4Q} l^4, \]

and we have fixed two constants of integration as
\[ e^{4\psi_0} = \frac{1}{2\sqrt{Q}}, \quad \Omega_0^4 = \frac{1}{8Q^{3/2}}, \]  

which corresponds to the scale choice for the circle and the two-torus. Of the original five constants of integration, these two scale choices fix two of them. Since \( x_0 \) can be deleted from the metric by an appropriate diffeomorphism, we can disregard it as a gauge dependent parameter. Thus we end up with two constants of integration that parameterize the solution space in a gauge-independent fashion. Hereafter, we will rewrite \( \psi_0 = \alpha_1 \) and \( 2s = \alpha_2 \).

**IV. DISCUSSIONS**

The five-dimensional metric that we obtain from the analysis in this paper is as follows

\[
ds^2 = g_\mu^\nu dx^\mu dx^\nu = \frac{r^2}{l^2} \left[ -\left(1 - \frac{r_0^4}{r^4}\right)^{a_t} dt^2 + \left(1 - \frac{r_0^4}{r^4}\right)^{a_1} dx_1^2 + \left(1 - \frac{r_0^4}{r^4}\right)^{a_2} dx_2^2 + \left(1 - \frac{r_0^4}{r^4}\right)^{a_\theta} d\theta^2 \right] + \frac{l^2}{r^2} \left(1 - \frac{r_0^4}{r^4}\right)^{-1} dr^2
\]

and the ten-dimensional metric is the tensor product of the above metric with a five-sphere of a fixed radius \( l \). Here \( a_t, a_1, a_2 \) and \( a_\theta \) are given by

\[
a_t = \frac{-\alpha_1 + 3\alpha_2 + \sqrt{D}}{4\sqrt{D}}, \quad a_1 = a_2 = \frac{-\alpha_1 - \alpha_2 + \sqrt{D}}{4\sqrt{D}}, \quad a_\theta = \frac{3\alpha_1 - \alpha_2 + \sqrt{D}}{4\sqrt{D}},
\]

\[
D = \alpha_1^2 - \frac{2}{3}\alpha_1\alpha_2 + \alpha_2^2
\]

in terms of \( \alpha_1 \) and \( \alpha_2 \); they satisfy \( a_t + a_1 + a_2 + a_\theta = 1 \) resulting det \( g = -r^6/l^6 \). Eq. (31) represent the general solutions (under the conditions that we imposed earlier) that are

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3 After the initial submission of this paper, R. Myers informed us of Ref. [16], where the same metric was independently obtained. The analysis in the previous section shows that this metric corresponds to the general static, radial-dependent solution consistent with the boundary conditions that we imposed.
asymptotically locally $AdS_5$. By direct computation of the Ricci tensor, we can confirm that the metric of Eq. (31) satisfies

$$R_{\mu\nu} = -\frac{4}{l^2} g_{\mu\nu} \rightarrow R = -\frac{20}{l^2}, \quad R_{\mu\nu} R^{\mu\nu} = \frac{80}{l^4}$$

(32)

for all values of $r_0$, $\alpha_1$ and $\alpha_2$.

For a given value of $l$, the physically distinct solutions given in Eq. (31) are parameterized by points on $R^2$. To see this, we first note that the parameter $r_0$ plays the role of a radial coordinate. When $r_0 = 0$, Eq. (31) reduces to the $AdS_5$ space-time regardless of the values of $\alpha_1$ and $\alpha_2$. Furthermore, the seemingly two parameters $\alpha_1$ and $\alpha_2$ combine to give a single angular coordinate; we observe that the solutions remain invariant under the transformation of the constants of motion $\alpha_1$ and $\alpha_2$ via $(\alpha_1, \alpha_2) \rightarrow (c\alpha_1, c\alpha_2)$ for an arbitrary positive number $c$. This turns $(\alpha_1, \alpha_2)$ into a one-dimensional real projective space, i.e., a circle $S^1$.

We observe the behavior of $a_t$, $a_1$, $a_2$ and $a_\theta$ as we traverse along $S^1$ for a fixed non-zero value of $r_0$ in Fig. 1. We recognize two interesting points on $S^1$. The first one is the $\alpha_1 = 0$ and $\alpha_2 = 1$ case, which corresponds to the $AdS_5$ black holes.

$$ds^2 = \frac{r^2}{l^2} \left[ -\left(1 - \frac{r_0^4}{r^4}\right) dt^2 + dx_1^2 + dx_2^2 + d\theta^2 \right] + \frac{l^2}{r^2} \left(1 - \frac{r_0^4}{r^4}\right)^{-1} dr^2$$

(33)

The second is the $\alpha_1 = 1$ and $\alpha_2 = 0$ case, which corresponds to the $AdS_5$ solitons that have negative energy relative to the $AdS_5$ vacuum.

$$ds^2 = \frac{r^2}{l^2} \left[ -dt^2 + dx_1^2 + dx_2^2 + \left(1 - \frac{r_0^4}{r^4}\right) d\theta^2 \right] + \frac{l^2}{r^2} \left(1 - \frac{r_0^4}{r^4}\right)^{-1} dr^2$$

(34)

The metric (34) of the $AdS_5$ solitons can be obtained from the metric (33) of the $AdS_5$ black holes by the double Wick rotation. In our language, rather similar to the $U(1)$ electric-magnetic duality, the $\pi/2$ rotation along the $S^1$ of the black hole metric (33) yields the soliton metric (34). In addition, the general solutions Eq. (31), that correspond to

4We note that the solution does not change as we take $r_0 \rightarrow -r_0$. Thus, we concentrate on the non-negative values of $r_0$. 12
the rotation along the $S^1$ by an arbitrary angle, reveal the existence of “dyonic” solutions, some of which might have, in principle, a lower energy than that of the $AdS_5$ soliton (34). However, we can show that all of these “dyonic” solutions have naked singularities. For this purpose, we directly compute the square of the Riemann tensor from the metric (31) to find

$$l^4 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = 40 + 72 \frac{r_0^8}{r^8} + 72 f(r/r_0, \alpha_1, \alpha_2) \frac{r_0^{16}}{r^8(r^4 - r_0^4)^2},$$

(35)

where the function $f(x, \alpha_1, \alpha_2)$ is given by

$$f(x, \alpha_1, \alpha_2) = \frac{[(\alpha_1 - \alpha_2)^2(\alpha_1 + \alpha_2)\sqrt{D} - D^2](1 - 2x^4) - 8\alpha_1^2\alpha_2^2/27}{2D^2}.$$

The first term in Eq. (35) is the contribution from the $AdS_5$ vacuum, and the second term is the contribution from the $AdS_5$ black hole (or the same contribution from the $AdS_5$ soliton). We note that the third term has the double pole singularity in $r^4$ variable at $r_0^4$. From the spatial infinity ($AdS$ boundary) to the point $r = r_0$, the position of the horizon in the case of the $AdS_5$ black holes, the radial space-time evolution is smooth. Due to the singularity at $r = r_0$, however, the space-time geometry of Eq. (31) has a naked singularity unless, at least, $f(1, \alpha_1, \alpha_2) = 0$. For the lack of singularity at $r = r_0$, in fact, both coefficients of the $x^4$ term and the constant term of $f(x, \alpha_1, \alpha_2)$ should vanish since the third term of Eq. (35) has a double pole singularity. The value of $f(1, \alpha_1, \alpha_2)$, which ranges from zero to one, as we traverse along the circle $S^1$ is shown in Fig. 2. We immediately find from the figures that there are only two points on $S^1$ at which the function $f(1, \alpha_1, \alpha_2)$ vanishes. They precisely correspond to the $AdS_5$ solitons and the $AdS_5$ black holes. For these two cases, the whole function $f(x, \alpha_1, \alpha_2)$ is identically zero and, as a result, the third term in Eq. (35) itself disappears. This consideration shows that all of the general solutions Eq. (31) other than the $AdS_5$ black holes and $AdS_5$ solitons have naked singularity at $r = r_0$.

To summarize, the general solutions of the five-dimensional IIB supergravity spontaneously compactified on a five-sphere satisfying the asymptotic condition (asymptotically locally $AdS_5$) and of the form $ds^2 = -\alpha(r)dt^2 + \beta(r)(dx_1^2 + dx_2^2) + \gamma(r)d\theta^2 + \delta(r)dr^2$ are parameterized by all points on $R^2$. Among these points, naked-singularity-free solutions are

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AdS\(_5\) black holes (the ray from the origin toward the positive \(x\)-axis of \(R^2\)), AdS\(_5\) solitons (the ray from the origin toward the positive \(y\)-axis of \(R^2\)) and the vacuum AdS\(_5\) space-time (the origin of \(R^2\)), and all other solutions possess naked singularities. This result points to the feasibility of the part of the Horowitz-Myers conjecture, which, in turn, via the AdS/CFT correspondence, will be related to the stability of the (2+1)-dimensional non-supersymmetric Yang-Mills theory on a space-time whose spatial section is a two-torus.

For the hermiticity of the gauge theory Hamiltonian, the fermions on the \(S^1\) should either be periodic (R sector) or anti-periodic (NS sector). It was pointed out in [5] that the negative energy of the AdS\(_5\) solitons is the same as the Casimir energy of the NS sector fermions on a circle up to a factor 3/4 (also see [17]). In the case of the R sector fermions, the Casimir energy vanishes and, for the AdS\(_5\) black holes, the energy comes solely from its mass. The gravity side analysis is consistent with this picture in a sense that for a given value of \(r_0\) there are only two discrete points on a circle \(S^1\) that correspond to the naked-singularity-free solutions. Since the invariant size of the Ricci tensor is bounded from the above (including the point \(r = r_0\)) for the interconnecting solutions, one can get the formal expression for the energy by evaluating the surface integral Eq. (2.1) of [5] as follows

\[
\frac{E}{\text{Volume}_{S^1 \times T^2}} = -\frac{r_0^4}{16\pi G_5 l^5} \frac{\alpha_1 - 3\alpha_2}{\sqrt{D}}, \tag{36}
\]

although its interpretation as the true energy is not clear due to the presence of the naked singularities of the Riemann tensor at \(r = r_0\). The quantity Eq. (36) calculated in this fashion smoothly increases monotonically, as we move along the circle \(S^1\) from the AdS\(_5\) solitons (negative energy \(-r_0^4/(16\pi G_5 l^5)\) for \(\alpha_1 = 1\) and \(\alpha_2 = 0\)) to the AdS\(_5\) black holes (positive energy \(3r_0^4/(16\pi G_5 l^5)\) due to the black hole mass for \(\alpha_1 = 0\) and \(\alpha_2 = 1\)). Pushing the AdS/CFT correspondence further, one might expect that the solutions that interconnect these two points and possess naked singularities can be related to non-unitary gauge theories with the fermions whose holonomy along the circle \(S^1\) multiplies the fermion field by an arbitrary \(U(1)\) phase factor. It remains to be seen if this expectation is true.

An interesting generalization of the analysis presented in this paper is to consider the
space-time whose asymptotic geometry is of the form $R \times S^1 \times M_g$ for $g \neq 1$. In the case of the higher genus, there will be extra positive contribution to the effective potential that results from the negative curvature of the Riemann surface. Therefore, we may expect that the regular solutions in this case will have higher energy than the energy of the $AdS_5$ solitons. The situation is more dangerous for the genus zero case. As an effective $s$-wave sector, we may expect that the solutions in this sector, \textit{if exist}, would have lower energy than the energy of the $AdS_5$ solitons; in fact, the explicit examples of \cite{8} and \cite{9}, the bubble type solutions, that have an energy which is not bounded from below belong to the $g = 0$ case with the vanishing cosmological constant. More careful analysis of this case will be necessary to prove the Horowitz-Myers conjecture and therefore to establish the stability of the (2+1)-dimensional non-supersymmetric Yang-Mills theory on a space-time whose spatial section is a sphere.

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FIGURES

FIG. 1. The powers $a_t, a_1, a_2, a_\theta$ in the metric (31). The point $k = \pm \infty$ in the left figure is identified with the point $k = \mp \infty$ in the right figure.

FIG. 2. The function $f(1, \alpha_1, \alpha_2)$ in (35). The point $k = \pm \infty$ in the left figure is identified with the point $k = \mp \infty$ in the right figure.