$S$-brane solutions with (anti-)self-dual parallel charge density form on a Ricci-flat submanifold

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Abstract

A $D$-dimensional cosmological model with several scalar fields and antisymmetric $(p + 2)$-form is considered. For dimensions $D = 4m + 1 = 5, 9, 13, \ldots$ and $p = 2m - 1 = 1, 3, 5, \ldots$ we obtain a family of new cosmological type solutions with $4m$-dimensional oriented Ricci-flat submanifold $N$ of Euclidean signature. These solutions are characterized by a self-dual or anti-self-dual parallel charge density form $Q$ of rank $2m$ defined on $N$. The (sub)manifold $N$ may be chosen to be Kähler, or hyper-Kähler one, or 8-dimensional manifold of $Spin(7)$ holonomy. The generalization of solutions to a chain of extra (marginal) Ricci-flat factor-spaces is also presented. Solutions with accelerated expansion of extra factor-spaces are singled out. Certain examples of new solutions for IIA supergravity and for a chain of $B_D$-models in dimensions $D = 14, 15, \ldots$ are considered.
1 Introduction

This paper is devoted to generalization of composite electric S-brane solutions with maximal number of branes obtained earlier in [1, 2, 3]. (For S-brane solutions see [4]-[22] and refs. therein.) These solutions exist in gravitational models in dimensions $D = 4m + 1 = 5, 9, 13, \ldots$ and containing $(p + 2)$-form with $p = 2m - 1 = 1, 3, 5, \ldots$. An interesting feature of these "maximal" solutions is the linear relations between charge densities [1].

Namely, it was shown in [1] that electric S-brane solutions with maximal number of branes in 5-dimensional model with 3-form and scalar field exists when the charge densities of six electric branes obey the following relations

$$Q_{12} = \mp Q_{34}, \quad Q_{13} = \pm Q_{24}, \quad Q_{14} = \mp Q_{23}. \quad (1.1)$$

or, equivalently,

$$Q_{ij} = \pm \frac{1}{2} \varepsilon_{ijkl} Q^{kl} = \pm (\ast Q)_{ij}. \quad (1.2)$$

Here $Q_{ij} = -Q_{ji}$. When all $Q_{ij}$ ($i \neq j$) are non-zero the configuration from (1.2) is the only possible one that follows just from non-diagonal part of Hilbert-Einstein equations.

Analogous (anti)-self-duality relations were used in constructing exact solutions in dimensions $D = 4m + 1 = 5, 9, 13, \ldots$ [1, 2, 3].

In the case when the scalar field is absent we are led in [3] to a solution for $D = 5$ gravity with 3-form and found the absence of oscillating behaviour in this model when cosmological electric S-brane solutions with diagonal metric are considered. This behaviour corresponds to the case of a "frozen" point in a billiard when we approach to the singularity (for billiard approach with branes see [24, 25]).

It should be noted that the relations between charge densities of $D$-branes play an important role in the string theory, see, for example, formulas between charge densities of electric and magnetic branes in ref. [26]. It was shown that such relations in a general case may be described by mathematical $K$-theory, see [27, 28, 29] and references therein. But these relations are of topological origin, while the relations proposed in our papers are of dynamical origin: they appear as solutions of quadratic constraints on charge densities following from non-diagonal part of Hilbert-Einstein equations.

Here, in Section 4, we obtain new cosmological type solutions with an oriented Ricci-flat factor-space $N$ of dimension $4m$ and self-dual or anti-self-dual
non-zero parallel (i.e. covariantly constant) "charge density" form $Q$ of rank $2m$ defined on $N$. The examples of Ricci-flat Riemannian manifolds of dimension $4m$ equipped with (anti-)self-dual parallel $2m$-form contain: Kähler manifolds of holonomy group $SU(2m)$, hyper-Kähler manifolds with holonomy group $Sp(m)$ and 8-dimensional Ricci-flat manifold of $Spin(7)$ holonomy [39]. In Section 5 we generalize the solutions from Section 4 to the case when a chain of extra Ricci-flat factor-spaces is added into consideration. Here we obtain a family of new solutions with accelerated expansion of extra factor-spaces. In Section 6 we present examples of new solutions for IIA supergravity and for a chain of the so-called $B_D$-models from [11] in dimensions $D = 14, 15, ...$. In Appendix we give an explicit derivation of exact solutions from Section 5.

2 The model

As in [2] we consider the model governed by the action

$$S = \int_M d^Dz \sqrt{|g|} \left[ R[g] - m_{ab} g^{MN} \partial_M \varphi^a \partial_N \varphi^b - \frac{1}{q!} \exp(2\lambda \varphi^a) F^2 \right], \quad (2.1)$$

where $g = g_{MN} dz^M \otimes dz^N$ is the metric, $\vec{\varphi} = (\varphi^a)$ is a set (vector) of scalar fields, $a = 1, \ldots, l$; $\vec{\lambda} = (\lambda_a) \in \mathbb{R}^l$ is a constant vector (set) of dilatonic couplings, $(m_{ab})$ is a symmetric non-degenerate $l \times l$ matrix,

$$F = dA = \frac{1}{q!} F_{M_1 \ldots M_q} dz^{M_1} \wedge \ldots \wedge dz^{M_q}, \quad (2.2)$$

is a $q$-form, $q = p + 2 \geq 1$, on a $D$-dimensional manifold $M$.

In (2.1) we denote $|g| = |\det(g_{MN})|$, and

$$F^2 = F_{M_1 \ldots M_q} F_{N_1 \ldots N_q} g^{M_1 N_1} \ldots g^{M_q N_q}. \quad (2.3)$$

The equations of motion corresponding to (2.1) are

$$R^M_N - \frac{1}{2} \delta^M_N R = T^M_N, \quad (2.4)$$

$$\triangle[g] \varphi^a - \frac{\lambda^a}{q!} e^{2\lambda \varphi^a} F^2 = 0, \quad (2.5)$$

$$\nabla_{M_1}[g](e^{2\lambda \varphi^a} F^{M_1 \ldots M_q}) = 0. \quad (2.6)$$
Here, and in what follows \( \lambda(\varphi) = \lambda_a \varphi^a \). In (2.3) and (2.6), \( \triangle[g] \) and \( \nabla[g] \) are Laplace-Beltrami and covariant derivative operators corresponding to \( g \). Equations (2.4), (2.5) and (2.6) are, respectively, the multidimensional Einstein-Hilbert equations, the "Klein-Fock-Gordon" equation for the scalar field and the "Maxwell" equations for the \( q \)-form.

The source terms in (2.4) can be split up as

\[
T^M_N = T^M_N[\varphi, g] + e^{2\lambda(\varphi)} T^M_N[F, g],
\]

(2.7)

with

\[
T^M_N[\varphi, g] = m_{ab} [g^{M_0} \partial_Q \varphi^a \partial_Q \varphi^b - \frac{1}{2} \delta^M_N g^{PQ} \partial_P \varphi^a \partial_Q \varphi^b],
\]

(2.8)

\[
T^M_N[F, g] = \frac{1}{q!} \left[ - \frac{1}{2} \delta^M_N F^2 + q F_{NM_2...M_q} F^{NM_2...M_q} \right],
\]

(2.9)

being the stress-energy tensor of the scalar field and \( q \)-form, respectively.

3 S-brane solutions with flat factor spaces

Now, we describe \( Sp \)-brane solutions from [2] in dimensions

\[
D = n + 1 = 4m + 1 = 5, 9, 13, \ldots
\]

(3.1)

with

\[
p = 2m - 1 = 1, 3, 5, \ldots
\]

(3.2)

and non-exceptional dilatonic coupling

\[
\lambda^2 \equiv m^{ab} \lambda_a \lambda_b \neq \frac{n}{4(n - 1)} \equiv \lambda_0^2.
\]

(3.3)

Here, and in what follows the matrix \( (m^{ab}) \) is inverse to \( (m_{ab}) \).

These solutions are defined on the manifold

\[
M = (u_-, u_+) \times \mathbb{R}^n
\]

(3.4)

and have the following form

\[
ds^2 = w e^{2n \phi(u)} du^2 + e^{2\phi(u)} \sum_{i=1}^{n} (dy^i)^2
\]

(3.5)
$$\varphi^a = -\frac{\lambda^a}{K} \{ f(u) + \lambda_b (C^b_2 u + C^b_1) \} + C^a_2 u + C^a_1, \quad (3.6)$$

$$F = e^{2f(u)} du \wedge Q, \quad (3.7)$$

$$Q = \frac{1}{(p+1)!} Q_{i_0...i_p} dy^{i_0} \wedge \ldots \wedge dy^{i_p}. \quad (3.8)$$

$$w = \pm 1, \lambda^a = m^{ab} \lambda_b, C^a_2, C^a_1 \text{ are integration constants},$$

$$K \equiv \lambda^2 - \frac{n}{4(n-1)}, \quad (3.9)$$

and $Q_{i_0...i_p}$ are constant components of charge density form $Q$. $Q$ is a self-dual or anti-self-dual in a flat Euclidean space $\mathbb{R}^n$, i.e.

$$Q_{i_0...i_p} = \pm \frac{1}{(p+1)!} \varepsilon_{i_0...i_p j_0...j_p} Q^{j_0...j_p} = \pm (*Q)_{i_0...i_p}. \quad (3.10)$$

The function $\phi(u)$ is defined as follows

$$\phi = \frac{1}{2(1-n)K} [\lambda_a (C^a_2 u + C^a_1) + f(u)], \quad (3.11)$$

and the function $f(u)$ is given by relations

$$f = -\ln |z| |KQ^2|^{1/2} \quad (3.12)$$

with

$$z = \frac{1}{\sqrt{C}} \sinh [(u - u_0) \sqrt{C}], \quad K < 0, \ C > 0; \quad (3.13)$$

$$z = \frac{1}{\sqrt{-C}} \sin [(u - u_0) \sqrt{-C}], \quad K < 0, \ C < 0; \quad (3.14)$$

$$z = u - u_0, \quad K < 0, \ C = 0; \quad (3.15)$$

$$z = \frac{1}{\sqrt{C}} \cosh [(u - u_0) \sqrt{C}], \quad K > 0, \ C > 0, \quad (3.16)$$

where

$$Q^2 \equiv \frac{1}{(p+1)!} \sum_{i_0,...,i_p} Q^2_{i_0...i_p} > 0, \quad (3.17)$$

and

$$C = (\lambda_a C^a_2)^2 - K m^{ab} C^a_2 C^b_2. \quad (3.18)$$
The solution presented above for $w = -1$ describes a collection of $k \leq C_{2m}^{2m}$ electric $Sp$-branes with non-zero charge densities $Q_{i_0 \ldots i_p} \neq 0$, $i_0 < \ldots < i_p$, $p = 2m - 1$.

The case of one scalar field was considered previously in \[1\] and in \[3\] the solutions without scalar fields were studied.

4 Generalization to Ricci-flat factor space

Here, we generalize the solution from the previous section to the case when the manifold (3.4) is replaced by the manifold

$$M = (u_-, u_+) \times N,$$

where $N$ is $n$-dimensional oriented manifold of dimension $n = 4m$, $m = 1, 2, \ldots$, equipped with the Ricci-flat metric $h = h_{ij}(y)dy^i \otimes dy^j$ of Euclidean signature.

Let

$$Q = \frac{1}{(p + 1)!} Q_{i_0 \ldots i_p}(y)dy^{i_0} \wedge \ldots \wedge dy^{i_p}$$

be a non-zero form of rank $2m$ defined on the manifold $N$.

The form $Q$ is supposed to be parallel, i.e. covariantly constant w.r.t. $h$

$$(i) \quad \nabla [h]Q = 0,$$

and also self-dual or anti-self-dual one

$$(ii) \quad Q = \pm \ast Q.$$  

Here $\ast = \ast[h]$ is the Hodge operator corresponding to the metric $h$.

It follows from (i) that the form $Q$ is closed ($dQ = 0$) and

$$Q^2 \equiv \frac{1}{(p + 1)!} h^{i_0j_0} \ldots h^{i_pj_p} Q_{i_0 \ldots i_p} Q_{j_0 \ldots j_p}$$

is constant.

Obviously,

$$Q^2 > 0,$$

since $Q$ is non-zero and $h$ has Euclidean signature.
For non-exceptional value of the dilatonic coupling (3.3) the following solution on the manifold (4.1) is valid

\begin{align}
 ds^2 &= w e^{2n\phi(u)}du^2 + e^{2\phi(u)}h_{ij}(y)dy^idy^j, \\
 \varphi^a &= -\frac{\lambda^a}{K}[f(u) + \lambda_b(C^b_2u + C^b_1)] + C^a_2u + C^a_1, \\
 F &= e^{2f(u)}du \wedge Q,
\end{align}

with the 2m-form (4.2) obeying (4.3), (4.4) and (4.6).

The function $\phi(u)$ is given by (3.11), where $K$ is defined in (3.9), $C_2, C_1$ are integration constants and the function $f(u)$ is presented by relations (3.12)-(3.16) with $Q^2$ defined in (4.5). The integration constants $C$ and $C_2$ are related by the formula (3.18).

Now, we show that the metric with the line element (4.7), scalar fields (4.8) and 2m-form (4.9) does satisfy the field equations (2.4)-(2.6). For scalar fields $\varphi^a = \varphi^a(u)$ we get from (2.5) for Ricci-flat $h$ the same ordinary differential equations as in the flat case, so eqs. (2.5) are satisfied.

The relation (4.3) reduces the verification of the “Maxwell” equation (2.6) just to the identity that holds for a flat-case solution from the previous section.

Now, we verify the Hilbert-Einstein eqs. (2.4).

First, we prove that

\[ T[F, g]_i^j = 0, \]

for all $i, j = 1, \ldots, n$, i.e. for $\lambda = 0$ the form contributes as a dust matter.

Let us prove relation (4.10) for $i \neq j$. In what follows we use the following notation

\[ C_i^j = \sum_{i_1, \ldots, i_p=1}^n Q_{ii_1 \cdots i_p} Q^{ji_1 \cdots i_p}. \]

For $i \neq j$, $T[F, g]_i^j$ is proportional to $C_i^j$. This follows from (2.9) and equation (4.9), written in components

\[ F_{i_1 \cdots i_{2m}} = e^{2f(u)}Q_{i_1 \cdots i_{2m}}. \]

Due to (anti-) self-duality of $Q$ we get

\[ C_i^j = \pm \sqrt{|h|} \sum_{i_1, \ldots, i_p=1}^n \sum_{j_0, \ldots, j_p=1}^n \frac{1}{(p + 1)!} \varepsilon_{i_1 \cdots i_p j_0 \cdots j_p} Q^{ji_1 \cdots i_p}, \]
where \( i \neq j \). This can be further rewritten as

\[
C_{ij} = \pm \sqrt{|h|} \sum_{i_1, \ldots, i_p = 1}^{n} \sum_{j_1, \ldots, j_p = 1}^{n} \frac{1}{p!} \varepsilon_{ij_1 \ldots i_p jj_1 \ldots j_p} Q^{j_1 \ldots j_p} Q^{i_1 \ldots i_p}
\]

\[
= \pm \sqrt{|h|} \sum_{i_1, \ldots, i_p = 1}^{n} \sum_{j_1, \ldots, j_p = 1}^{n} \frac{(-1)^p}{p!} \varepsilon_{ij_1 \ldots j_p jj_1 \ldots j_p} Q^{j_1 \ldots j_p} Q^{i_1 \ldots i_p}
\]

\[
= (-1)^p C_{ij}.
\]  

(4.14)

Note that \( j \) is not summed over in the two sums above. In going from (4.11) to the first line of (4.14) we have carried out \( p+1 \) identical sums with: \( j_0 = j, j_1 = j, \ldots, j_p = j \), respectively.

From (4.14) one finds that for odd \( p = 2m - 1 \)

\[
C_{ij} = -C_{ij} \Rightarrow C_{ij} = 0, \quad i \neq j
\]

and, hence, relation (4.10) is valid for \( i \neq j \).

Now, we prove relation (4.10) for \( i = j \), i.e.

\[
T_i^i[F, g] = 0,
\]  

(4.15)

(no summation in \( i \)) for all \( i = 1, \ldots, n \).

It follows from (2.9) and (3.7), that

\[
T_i^i[F, g] = B(u)\left[-\frac{1}{2} \sum_{k=1}^{n} C_i^k + (q - 1)C_i^i\right],
\]  

(4.16)

(no summation in \( i \)) for all \( i = 1, \ldots, n \), where \( B(u) \) is function of \( u \). The matrix \( T_j^j[F, g] \) is trace-less

\[
\sum_{k=1}^{n} T_k^k[F, g] = 0,
\]  

(4.17)

since \( n = 2(p+1) = 2(q - 1) \). To prove (4.15) it is sufficient to verify that

\[
T_1^1 = \ldots = T_n^n,
\]  

(4.18)

or, equivalently,

\[
C_1^1 = \ldots = C_n^n.
\]  

(4.19)
Let us prove without restriction of generality $C_1^1 = C_2^2$. Indeed, using (4.13) we get (the summation over repeated indices is understood)

$$C_1^1 = \pm \sqrt{|h|} \varepsilon_{i_1 j_1 \ldots j_p} Q^{j_0 j_1 \ldots j_p} Q^{1 i_1 \ldots i_p}$$

(4.20)

$$= \pm \frac{\sqrt{|h|}}{(p + 1)!} [p \varepsilon_{1i_2 \ldots i_p j_0 j_1 \ldots j_p} Q^{j_0 j_1 \ldots j_p} Q^{12i_2 \ldots i_p} + (p + 1) \varepsilon_{1i_1 \ldots i_p 2j_1 \ldots j_p} Q^{2j_1 \ldots j_p} Q^{1i_1 \ldots i_p}]$$

$$= \pm \frac{\sqrt{|h|}}{(p + 1)!} [p \varepsilon_{2i_2 \ldots i_p j_0 j_1 \ldots j_p} Q^{j_0 j_1 \ldots j_p} Q^{21i_2 \ldots i_p} + (p + 1) \varepsilon_{2j_1 \ldots j_p i_1 \ldots i_p} Q^{1i_1 \ldots i_p} Q^{2j_1 \ldots j_p}]$$

$$= \pm \frac{\sqrt{|h|}}{(p + 1)!} \varepsilon_{2i_1 \ldots i_p j_0 j_1 \ldots j_p} Q^{j_0 j_1 \ldots j_p} Q^{2i_1 \ldots i_p}$$

$$= C_2^2.$$

Here, we used that $p + 1 = 2m$ is even. Thus, relations (4.10) are proved.

Now we check the Hilbert-Einstein eqs. (2.4). These equations are satisfied for non-diagonal components ($M \neq N$), since the Einstein tensor in the left hand site of (2.4) for the metric (4.7) is diagonal (see, for example, the Appendix in ref. [13]) and stress-energy tensor (2.7) $T^M_N$ is also diagonal due to eqs. (4.10). For the diagonal part of Hilbert-Einstein eqs. (2.4) we obtain the same ordinary differential equations (ODE) for $\phi(u)$ with constant parameter $Q^2$ from (1.5) as in the flat case. This follows from Ricci-flatness of $h$ and (4.10). But the latter ODE were checked in our recent paper [2]. It means that we proved that relations (4.7), (4.8) and (4.9) with surrounding notations and assumptions do really define exact solutions to field equations in dimensions $D = 4m + 1$.

Examples of (anti-)self-dual parallel $2m$-forms on Ricci-flat Riemannian manifolds of dimension $4m$. It should be noted that such parallel form does exist when $4m$-dimensional Riemannian manifold $N$ is Kähler Ricci-flat manifold with holonomy group $SU(2m)$. Indeed the $m$-th wedge power of Kähler 2-form, i.e.

$$\alpha = \Omega^m,$$

(4.21)

gives an example of non-zero parallel (i.e. covariantly constant) form of rank $2m$ [39]. Splitting this form into a sum of self-dual and anti-self-dual parallel
forms:
\[ \alpha = \alpha_+ + \alpha_- \]  
(4.22)
where
\[ \alpha_\pm = \frac{1}{2}(1 \pm \ast)\alpha \]  
(4.23)
and \( \ast = \ast[h] \) is the Hodge operator on \( N \), we get that either \( \alpha_+ \) or \( \alpha_- \) is a non-zero parallel form. Thus, we get an example of either self-dual or anti-self-dual parallel \( 2m \)-form on a Kähler (Ricci-flat) manifold of dimension \( 4m \).

When \( N \) is hyper-Kähler Ricci-flat manifold of dimension \( 4m \) with holonomy group \( Sp(m) \) there are three Kähler 2-forms: \( \Omega_1, \Omega_2, \Omega_3 \). In this case we have more examples of parallel forms (e.g. self-dual or anti-self-dual ones), since any wedge product
\[ \alpha = \Omega_1^{m_1} \wedge \Omega_2^{m_2} \wedge \Omega_3^{m_3}, \]  
(4.24)
with \( m_1 + m_2 + m_2 = m \), is a parallel \( 2m \)-form [39]. We should also mention that there exists a parallel 4-form on 8-dimensional Ricci-flat manifold of \( Spin(7) \) holonomy. See item 10.124 (Table 1) in [39] and also [40, 41] (and references therein).

## 5 Generalization to a set of extra Ricci-flat spaces

Here, we suggest a generalization of the solution from the previous section when the manifold (4.1) is replaced by

\[ M = (u_-, u_+) \times N \times N_1 \times \ldots N_k, \]  
(5.1)
where \( N_r \) are Ricci-flat manifolds with the metric \( h^r \) of dimension \( d_r \), \( r = 1, \ldots, k \).

The solution reads

\[ ds^2 = \exp \left( \frac{4mf(u)}{K(2-D)} \right) \left\{ wc^{2\epsilon u+2\epsilon} du^2 \right. \]  
+ \[ \exp(K^{-1}f(u) + 2c^0 u + 2\epsilon^0)h_{ij}(y)dy^i dy^j + \sum_{r=1}^{k} e^{2c^r u+2\epsilon^r} ds_r^2 \right\}, \]  
(5.2)
\[ \varphi^a = -\frac{\lambda^a}{K}f(u) + c^a u + \bar{c}^a, \]  
(5.3)
\[ F = e^{2f(u)} du \wedge Q, \quad (5.4) \]

\( a = 1, \ldots, l. \)

Here \( ds^2_r = h^r_{\mu_r \nu_r}(z_r) dz^\mu_r dz^\nu_r \) is the line element corresponding to the metric \( h^r \), \( f(u) \) is given by (3.12) and (3.13), (3.14), (3.15), (3.16),

\[ c = 4mc^0 + \sum_{r=1}^{k} d_r c^r \quad \bar{c} = 4m\bar{c}^0 + \sum_{r=1}^{k} d_r \bar{c}^r \quad (5.5) \]

and

\[ K = \lambda^2 + m + \frac{4m^2}{2 - D} \neq 0. \quad (5.6) \]

The integration constants obey the following relations:

\[ CK^{-1} + m_{ab} c^a_{\varphi} c^b_{\varphi} + 4m(c^0)^2 \]

\[ + \sum_{r=1}^{k} (c^r)^2 d_r - (4mc^0 + \sum_{r=1}^{k} c^r d_r)^2 = 0 \]

\[ 2mc^0 = \lambda_a c^a_{\varphi} \quad 2mc^0 = \lambda_a \bar{c}^a_{\varphi}. \quad (5.8) \]

The solution is derived in the Appendix. More special solutions were obtained previously in [31, 32].

When internal spaces \( N_1, \ldots, N_k \) are omitted we get the solution from the previous subsection with the following identifications between constants:

\[ c^a_{\varphi} = -\frac{\lambda^a}{K} (\lambda_b C^b_{2}) + C_{2}^{a}, \quad \bar{c}^a_{\varphi} = -\frac{\lambda^a}{K} (\lambda_b C^b_{1}) + C_{1}^{a}. \quad (5.9) \]

**Solutions with acceleration.** Now, we show that among obtained solutions there exist some special solutions with accelerated expansion of certain subspaces. Indeed, let us consider the simplest cosmological type solution with vanishing integration constants \( C = c^\varphi = \bar{c}^\varphi = c^r = 0. \)

This solution takes place when \( K < 0. \) The metric (5.2) for this case reads

\[ ds^2 = w d\tau^2 + B_0 \tau^{2\nu_0} h_{ij}(y) dy^i dy^j + \tau^{2\nu_1} \sum_{r=1}^{k} B_r ds^2_r, \quad (5.10) \]

where \( \tau > 0 \) is "synchronous" variable, \( B_0, B_1, \ldots, B_r \) are positive constants and

\[ \nu_0 = \frac{4m + 2 - D}{2\Delta}, \quad \nu_1 = \frac{2m}{\Delta}. \quad (5.11) \]
Here
\[ \Delta = (D - 2)K + 2m. \] (5.12)

When \( \nu_1 > 1 \), or \(-2m/(D - 2) < K < 0\) we get an accelerated expansion of factor spaces \( M_1, \ldots, M_k \). This takes place when
\[ -(d + 1)m < \lambda^2(D - 2) < -(d - 1)m. \] (5.13)

where
\[ d = \sum_{r=1}^{k} d_r. \] (5.14)

It is possible only if \( \lambda^2 < 0 \) or, equivalently, some of scalar fields are phantom ones.

We note that solutions of this Section for \( \lambda = 0 \) are in agreement with perfect fluid solutions from [34].

6 Examples

Here we consider two examples of solutions from the previous section.

6.1 IIA supergravity

Let us take the \( D = 10 \) IIA supergravity with the bosonic part of the action
\[
S = \int d^{10}z \sqrt{|g|} \left\{ R[g] - (\partial \phi)^2 - \sum_{a=2}^{4} e^{2\lambda_a \phi} F_a^2 \right\} - \frac{1}{2} \int F_4 \wedge F_4 \wedge A_2, \quad (6.1)
\]

where \( F_a = dA_{a-1} + \delta_{a4} A_1 \wedge F_3 \) is an \( a \)-form and
\[ \lambda_3 = -2\lambda_4, \quad \lambda_2 = 3\lambda_4, \quad \lambda_4^2 = \frac{1}{8}. \] (6.2)

We consider the solutions with zero forms \( A_1, A_3 \) (and hence vanishing \( F_2 \) and \( F_4 \)), i.e. we are interested in the NS-NS sector of the model (NS means Neveu-Schwarz).

Thus, we consider the solution from the previous section describing for \( w = -1 \) a “collection” of electric S1-branes, i.e. S-fundamental strings (SFS).

In this case we get \( m = 1, K = 1 \) and \( k \leq 5 \).
The solution reads
\[ ds^2 = \exp(-f(u)/2) \left\{ w e^{2c_0 + 2\bar{c}_0} du^2 \right\} \] (6.3)

\[ + \exp(f(u) + 2c_0 u + 2\bar{c}_0) h_{ij}(y) dy^i dy^j + \sum_{r=1}^{k} e^{2c_r u + 2\bar{c}_r} ds_r^2 \],

\[ \varphi = -\lambda_3 f(u) + c_\varphi u + \bar{c}_\varphi, \] (6.4)

\[ F_3 = e^{2f(u)} du \wedge Q. \] (6.5)

The function \( f(u) \) is given by relation
\[ f = -\ln \left[ |z||Q^2|^{1/2} \right] \] (6.6)

with
\[ z = \frac{1}{\sqrt{C}} \cosh \left[ (u - u_0) \sqrt{C} \right], \] (6.7)

and the integration constants obey
\[ c = 4c_0 + \sum_{r=1}^{k} d_r c_r \quad \bar{c} = 4\bar{c}_0 + \sum_{r=1}^{k} d_r \bar{c}_r, \] (6.8)

and
\[ C = -(c_\varphi)^2 - 4(c_0)^2 - \sum_{r=1}^{k} (c_r)^2 d_r + (4c_0 + \sum_{r=1}^{k} c_r d_r)^2 > 0 \] (6.9)

\[ 2c_0 = \lambda_3 c_\varphi \quad 2\bar{c}_0 = \lambda_3 \bar{c}_\varphi. \] (6.10)

For \( w = -1 \) all metrics \( h^r \) should be of Euclidean signatures, and for \( w = +1 \) one metric should be of pseudo-Euclidean signature while the other ones should be of Euclidean signatures.

**6.2 Chain of \( B_D \)-models**

Now, we consider the chain of the so-called \( B_D \)-models in dimensions \( D = 11, 12, \ldots \) with a set of "phantom" scalar fields suggested in [11] and defined by the action

\[ S_D = \int_M d^Dz \sqrt{|g|} \left\{ R[g] + g^{MN} \partial_M \tilde{\varphi} \partial_N \varphi - \sum_{a=4}^{D-7} \frac{1}{a!} \exp(2\bar{\lambda}_a \tilde{\varphi})(F^a)^2 \right\}. \] (6.11)
where $\vec{\varphi} = (\varphi^1, \ldots, \varphi^l) \in \mathbb{R}^l$, $\vec{\lambda}_a = (\lambda_{a1}, \ldots, \lambda_{al}) \in \mathbb{R}^l$, $l = D - 11$, rank$F^a = a$, $a = 4, \ldots, D - 7$. Here vectors $\vec{\lambda}_a$ satisfy the relations

$$
\vec{\lambda}_a \vec{\lambda}_b = N(a, b) - \frac{(a - 1)(b - 1)}{D - 2},
$$

(6.12)

$$
N(a, b) = \min(a, b) - 3,
$$

(6.13)

$a, b = 4, \ldots, D - 7$.

The vectors $\vec{\lambda}_a$ are linearly dependent, since

$$
\vec{\lambda}_{D-7} = -2\vec{\lambda}_4.
$$

(6.14)

For $D > 11$ vectors $\vec{\lambda}_4, \ldots, \vec{\lambda}_{D-8}$ are linearly independent.

The model (6.11) contains $l$ scalar fields with negative kinetic term (i.e. $m_{ab} = -\delta_{ab}$ in (2.1)) coupled with $l + 1$ forms. For $D = 11$ ($l = 0$) the model (6.11) coincides with truncated bosonic sector of $D = 11$ supergravity ("truncated" means without Chern-Simons term). For $D = 12$ ($l = 1$) (6.11) coincides with truncated $D = 12$ model from [35] that is being used for a field description of $F$-theory [36].

The matrix (6.13) was called in [11] as a fundamental matrix of the theory since it describes the intersection rules for branes. The $B_D$-models ("beautiful models") have a full set of binary brane configurations: this means that $\vec{\lambda}_a$-vectors are chosen in such way that standard ("orthogonal") intersection rule formula [37, 38, 23, 11] for any two brains gives us a natural number for dimension of intersection [11].

Let us consider the sector of the $B_D$-model with the form $F_{2m+1}$, $m \geq 2$. It should be $D \geq 2m + 8$ in order for this form to appear. Now we could write the solution from the previous section describing the collection of electric $S(2m - 1)$-branes localized on $4m$-dimensional sub-manifold $N$. We should also put an obvious restriction $D \geq 4m + 1$.

The calculation of $K$-parameter gives us

$$
K = 2 - m.
$$

(6.15)

For $m = 2$ we get $K = 0$ and our solution (from the previous section) does not work, e.g. for $D = 12$ model from [35]. (We are not able also to rewrite our solution for IIB model, since 5-form $F_5$ should be self-dual in this case.) For $m > 2$ we get $K < 0$ and our solution does work for $D \geq 14$. 

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7 Conclusions and discussions

So, we considered a $D = (n + 1)$-dimensional cosmological type model with several scalar fields and antisymmetric $(p + 2)$-form. We generalized the composite electric $S$-brane solutions from [2] for $D = 4m + 1 = 5, 9, 13, ...$ and $p = 2m - 1 = 1, 3, 5, ...$ to the case when the form $Q$ of rank $2m$ is defined on $4m$-dimensional oriented Ricci-flat space $N$ of Euclidean signature. The Ricci-flat submanifold $N$ may be chosen to be either Kähler manifold of holonomy group $SU(2m)$, or hyper-Kähler manifolds with holonomy group $Sp(m)$, or 8-dimensional Ricci-flat manifold of $Spin(7)$ holonomy.

Here, the form $Q$ is arbitrary non-zero parallel self-dual or anti-self-dual $2m$-form on $N$. For flat $N = \mathbb{R}^{4m}$ the components of this form in canonical coordinates are proportional to charge densities of electric branes.

We also found generalizations of the solutions to the case when a chain of extra Ricci-flat factor-spaces is added. We have shown that these solutions contain as a special case solutions with accelerated expansion of extra factor-spaces.

We also considered certain examples of solutions: for IIA supergravity and a for chain of $B_D$-models in dimensions $D = 14, 15, ....$

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Appendix

Here, we present an explicit derivation of the solution from Section 5.

We start with the manifold

$$M = \mathbb{R} \times M_0 \times \ldots \times M_k$$  \hspace{1cm} (A.1)
equipped by the metric

\[ g = w e^{2\gamma(u)} du \otimes du + \sum_{i=0}^{k} e^{2\phi_i(u)} h^i, \quad (A.2) \]

where \( w = \pm 1 \), \( u \) is a distinguished coordinate; \( h^i \) is a Ricci-flat metric on the manifold \( M_i \) with dimension \( d_i = \dim M_i \), \( i = 0, \ldots, k \), and \( M_0 = N \) is oriented.

We put for potential form: \( A = \Phi(u) \omega \), where \( \omega \) is a non-zero parallel (anti-) self-dual form on \( M_0 \) of rank \( 2m \) and hence

\[ F = dA = \dot{\Phi}(u) du \wedge \omega \quad (A.3) \]

\((\dot{x} \equiv dx/du)\). Here we use identity \( d\omega = 0 \) for the parallel form \( \omega \).

We also put

\[ \varphi^a = \varphi^a(u), \quad (A.4) \]

\( a = 1, \ldots, l \).

The field equations corresponding to the action (2.1) are equivalent to equations of motion for the \( \sigma \)-model with the action

\[ S_\sigma = \frac{\mu}{2} \int du \mathcal{N} \left\{ G_{ij} \dot{\phi}^i \dot{\phi}^j + m_{ab} \dot{\varphi}^a \dot{\varphi}^b + \omega^2 \exp\left[-2(U(\phi, \varphi))\dot{\Phi}^2\right] \right\}, \quad (A.5) \]

where

\[ \mathcal{N} = \exp(\gamma_0 - \gamma) > 0 \quad (A.6) \]

is the lapse function with

\[ \gamma_0(\phi) \equiv \sum_{i=0}^k d_i \phi^i, \quad (A.7) \]

\[ U = U(\phi, \varphi) = -\lambda(\varphi) + \frac{1}{2} d_0 \phi^0, \quad (A.8) \]

is brane co-vector (linear form) and

\[ G_{ij} = d_i \delta_{ij} - d_i d_j \quad (A.9) \]
are components of the "pure cosmological" minisupermetric, \(i, j = 0, \ldots, k\), see \[30\]. We denote
\[
\omega^2 = \frac{1}{(2m)!} h^{i_1j_1} \ldots h^{i_mj_m} \omega_{i_1\ldots i_m} \omega_{j_1\ldots j_m} > 0,
\]
(A.10)
the square of the non-zero \(\omega\)-form on the Riemannian manifold \(M_0\).

For finite space volumes \(V_i\) of \(M_i\) the action (2.1) just coincides (up to a surface term irrelevant for classical consideration) with the action (A.5) if \(\mu = -wV_0\ldots V_n\). The non-diagonal part of Hilbert-Einstein equations is satisfied due to (anti-) self-duality condition on \(\omega\): this may be verified as it was done in section 4. Diagonal part of Hilbert-Einstein equations and other field equations are governed by the action (A.5).

Action (A.5) may be also written in a more condensed form
\[
S_\sigma = \frac{\mu}{2} \int duN\{\bar{G}_{AB} \dot{x}^A \dot{x}^B + \omega^2 \exp[-2U(x)]\},
\]
(A.11)
where \(x = (x^A) = (\phi^i, \varphi^a)\),
\[
\bar{G}_{AB} = \begin{pmatrix} G_{ij} & 0 \\ 0 & m_{ab} \end{pmatrix},
\]
(A.12)
\(U(x) = U_A x^A\) is defined in (A.8) and
\[
(U_A) = \left( \frac{1}{2} d_0 \delta_i^0, -\lambda_a \right).
\]
(A.13)

We put
\[
(U, U) = \bar{G}^{AB} U_A U_B \neq 0,
\]
(A.14)
where
\[
\bar{G}^{AB} = \begin{pmatrix} G^{ij} & 0 \\ 0 & m^{ab} \end{pmatrix}
\]
(A.15)
is the matrix inverse to (A.12). Here (as in \[30\])
\[
G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2-D},
\]
(A.16)
i, \(j = 0, \ldots, k\). The scalar product (A.14) is
\[
(U, U) = K,
\]
(A.17)
with $K$ defined in Section 5.

In what follows, we will use contra-variant components $U^A = \bar{G}^{AB} U_B$:

$$U^i = \bar{G}^{ij} U_j = \frac{1}{2} (\delta^0_i - \frac{d_0}{D-2}), \quad U^a = -\lambda^a. \quad (A.18)$$

The problem of integrability may be simplified if we integrate the generalized Maxwell equations

$$\frac{d}{du} \left( \exp(-2U) \dot{\Phi} \right) = 0 \iff \dot{\Phi} = q \exp(2U), \quad (A.19)$$

where $q \neq 0$ is a constant (we are not interested in the trivial case $q = 0$).

Let us denote

$$Q = q \omega \quad (A.20)$$

and fix the time-gauge to be harmonic one, i.e. we put

$$\gamma = \gamma_0. \quad (A.21)$$

Obviously, $Q^2 > 0$.

Then, the Lagrange equations for the action $\mathcal{A.11}$ corresponding to $(x^A) = (\phi^i, \varphi^a)$, when equations $(A.19)$ are substituted, are equivalent to the Lagrange equations for the Lagrangian

$$L_Q = \frac{1}{2} \bar{G}_{AB} \dot{x}^A \dot{x}^B - V_Q, \quad (A.22)$$

with the zero-energy constraint imposed

$$E_Q = \frac{1}{2} \bar{G}_{AB} \dot{x}^A \dot{x}^B + V_Q = 0. \quad (A.23)$$

Here,

$$V_Q = \frac{1}{2} Q^2 \exp[2U(x)]. \quad (A.24)$$

Then, the Euler-Lagrange equations for the Lagrangian $\mathcal{A.1}$ have the following general solutions $\mathcal{A.33}$

$$x(u) = -\frac{U}{(U, U)} \ln |\bar{f}(u-u_0)| + cu + c, \quad (A.25)$$

where $c$ and $\bar{c}$ are constant vectors obeying,

$$U(c) = U(\bar{c}) = 0, \quad (A.26)$$

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and

\[
\begin{align*}
\bar{f}(\tau) &= \left| \frac{Q^2}{2E_1} \right|^{1/2} \sinh(\sqrt{|C|} \tau), \ C > 0, \ (U, U) < 0, \quad (A.27) \\
\left| \frac{Q^2}{2E_1} \right|^{1/2} \sin(\sqrt{|C|} |\tau|), \ C < 0, \ (U, U) < 0, \quad (A.28) \\
\left| \frac{Q^2}{2E_1} \right|^{1/2} \cosh(\sqrt{|C|} \tau), \ C > 0, \ (U, U) > 0, \quad (A.29) \\
\left| Q^2(U, U) \right|^{1/2} \tau, \ C = 0, \ (U, U) < 0, \quad (A.30)
\end{align*}
\]

\( C = 2E_1(U, U), \ E_1, \ u_0 \) are constants.

For the energy corresponding to the solution we have

\[ E_Q = E_1 + \frac{1}{2} (c, c). \] (A.31)

Rewriting this solution in components \( x = (x^A) = (\phi^i, \varphi^a) \) and using the relations presented above and \( f = -\ln |\bar{f}| \) we are led to the solution from Section 5.

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