ON CORON’S PROBLEM FOR THE $p$-LAPLACIAN

CARLO MERCURI, BERARDINO SCIUNZI, AND MARCO SQUASSINA

Abstract. We prove that the critical problem for the $p$-Laplacian operator admits a nontrivial solution in annular shaped domains with sufficiently small inner hole. This extends Coron’s problem to a class of quasilinear problems.

1. Introduction

We want to extend the classical result of Coron [4]. Consider the problem

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $1 < p < N$, $p^* := Np/(N-p)$ is the critical Sobolev exponent, $\Delta_p u := \text{div}(\nabla |\nabla u|^{p-2}\nabla u)$ is the $p$-Laplace operator. Solutions on the whole space will be considered in $\mathcal{D}^{1,p}(\mathbb{R}^N) := \{ u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N; \mathbb{R}^N) \}$ endowed with the norm $\| u \| := \| \nabla u \|_{L^p(\mathbb{R}^N)}$.

We denote by $W^{1,p}_0(\Omega)$ the closure of $C^\infty_c(\Omega)$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and define on $W^{1,p}_0(\Omega)$ the functional

$$J(u) := \frac{1}{p} \int_\Omega |\nabla u|^p dx - \frac{1}{p^*} \int_\Omega |u|^{p^*} dx.$$ 

As it is well-known in tackling problem (1.1) with variational techniques, the main difficulty is due to the fact that the embedding $W^{1,p}_0(\Omega) \subset L^{p^*}(\Omega)$ is not compact. We refer to [14] for a sample of the extensive literature on semi-linear problems involving the critical Sobolev exponent, largely inspired by the pioneering paper of Brezis and Nirenberg [3]. We also define

$$S := \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^p dx, \ u \in \mathcal{D}^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{p^*} dx = 1 \right\}$$

the best Sobolev constant, attained by nowhere zero functions in $\mathbb{R}^N$, see e.g. [15]. Equivalently

$$(1.2) \quad S = \inf_{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^N} |u|^{p^*} dx\right)^{\frac{p}{p^*}}}.$$

where by a simple scaling argument the infimum remains unchanged if taken on competing functions supported in an arbitrary subdomain of $\mathbb{R}^N$. In light of the Pohozaev identity obtained by Guedda and Veron [9, Corollary 3.1], we know that problem (1.1) does not admit positive solutions on a strictly star-shaped domain.

The main result of the paper is the following

---

2000 Mathematics Subject Classification. 35J92, 58E05, 54D30.

Key words and phrases. Coron’s problem, quasi-linear equations, critical exponent.

The second and third authors were partially supported by the MIUR project: “Variational and Topological Methods in the Study of Nonlinear Phenomena.”
Theorem 1.1. Let $2N/(N+2) < p \leq 2$, $x_0 \in \mathbb{R}^N$ and radii $R_2 > R_1 > 0$ such that

\begin{equation}
R_1 \leq |x - x_0| \leq R_2 \subset \Omega, \quad \{ |x - x_0| \leq R_1 \} \not\subset \overline{\Omega}.
\end{equation}

Then problem (1.1) admits a positive solution for $R_2/R_1$ sufficiently large.

Theorem 1.1 is, mainly, a consequence of Lemma 2.2, in which the compactness result [11, Theorem 1.2] and the symmetry result of [5] play a key role. There are several difficulties arising in the present quasilinear setting which are partially highlighted in Lemma 2.2, which make the proof more delicate than for dealing with the semilinear case $p = 2$. One of those is the fact that the classification of all positive solutions of the critical problem in $\mathbb{R}^N$ is not yet available for all $p \in (1, N)$. We observe that an extension of Lemma 2.2 to a broader range of $p$ would immediately yield an extension of Theorem 1.1. We conjecture that the symmetry result of [5] and hence Lemma 2.2 and Theorem 1.1 hold for all values of $p \in (1, N)$. Another open problem, arising in the proof of Lemma 2.2, is the nonexistence of sign-changing solutions of the critical problem in the half-space for $p \neq 2$. Such a limiting problem arises because of the boundary of $\Omega$. We show that in fact only the nonexistence result of the positive solutions of the critical problem in the half-space [11, Theorem 1.1] is needed. In the case $N = 2$ Theorem 1.1 holds for all $1 < p < 2$, which is the desired range for a $p$-Laplacian extension of the classical result of Coron. We point out that Theorem 1.1 extends [10, Theorem 1.1], where problem (1.1) had been studied assuming that $\Omega$ is invariant under the action of a closed subgroup of $O(N)$. It is an open problem whether (1.1) has nontrivial solutions when a $\mathbb{Z}_2$-homology group of $\Omega$ is nontrivial. This is the case for $p = 2$, see the celebrated analysis done in [1]. In several contributions dealing with the semi-linear case $p = 2$, see e.g. [6, 7, 12], it is shown that the existence of a nontrivial solution is possible also in contractible domains, hence conditions on the homology of $\Omega$ are not necessary for problem 1.1 to have solutions. A very well-known and challenging problem, even in the case $p = 2$, would be to exploit the combined effect of both the topology and the geometry of $\Omega$ in order to characterize the existence of a positive solution to problem (1.1).

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1.

2.1. Palais-Smale condition. We define $\mathbb{R}^N_+ := \{ x \in \mathbb{R}^N : x_N > 0 \}$ and denote by $D^{1,p}_0(\mathbb{R}^N_+)$ the closure of $C^\infty_c(\mathbb{R}^N) \subset D^{1,p}(\mathbb{R}^N)$ after extending by zero on $\mathbb{R}^N \setminus \mathbb{R}^N_+$.

Lemma 2.1. Let $u \in W^{1,p}_0(\Omega)$ be a sign-changing solution to (1.1). Then $J(u) \geq 2S^{N/p}N$. Moreover, the same conclusion holds for the sign-changing solutions of $-\Delta_p u = |u|^{p-2}u$ in $D^{1,p}(\mathbb{R}^N)$ or in $D^{1,p}_0(\mathbb{R}^N_+)$.

Proof. If $u \in W^{1,p}_0(\Omega)$ is a sign-changing solution to (1.1), then $u^\pm \in W^{1,p}_0(\Omega) \setminus \{0\}$ and by testing (1.1) with $u^\pm$ yields

$$\int_\Omega |\nabla u^+|^p dx = \int_\Omega |u^+|^{p^*} dx, \quad \int_\Omega |\nabla u^-|^p dx = \int_\Omega |u^-|^{p^*} dx.$$

In turn, using the definition of (1.2), we obtain

$$J(u) = J(u^+) + J(u^-) = \frac{1}{N} |u^+|^{p^*} + \frac{1}{N} |u^-|^{p^*} \geq 2S^{N/p}N,$$

concluding the proof. The same argument works for the problem on $\mathbb{R}^N$ and on $\mathbb{R}^N_+$. \qed

Lemma 2.2. Assume that $2N/(N+2) < p \leq 2$. Then $J$ satisfies the Palais-Smale condition for all $c \in (S^{N/p}/N, 2S^{N/p}/N)$.
Proof. Assume that for some \( c \in (S^{N/p}/N, 2S^{N/p}/N) \), \((u_n) \in W^{1,p}_0(\Omega)\) is such that \( J(u_n) \to c \), and \( J'(u_n) \to 0 \) in \( W^{-1,p'}(\Omega) \). We define on \( D^{1,p}(\mathbb{R}^N) \)

\[
J_{\infty}(u) := \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{p} \, dx - \int_{\mathbb{R}^N} \frac{|u|^{p^*}}{p^*} \, dx.
\]

On \( D^{1,p}(\mathbb{R}^N_+) \) we define the same functional \( J_{\infty} \) extending by zero on \( \mathbb{R}^N \setminus \mathbb{R}^N_+ \). By applying \([11, \text{ proof of Theorem 1.2}]\), which extends \([13]\), passing if necessary to a subsequence, we can infer that there exists a (possibly trivial) solution \( v_0 \in W^{1,p}_0(\Omega) \) of

\[
-\Delta_p u = |u|^{p^*-2}u \quad \text{in} \quad \Omega, \quad k \in \mathbb{N} \cup \{0\}, \text{nontrivial solutions } \{v_1, ..., v_k\} \text{ of}
\]

\[
-\Delta_p u = |u|^{p^*-2}u \quad \text{in} \quad H_i, \quad i \in \{0, 1, ..., k\},
\]

where \( H_i \) is either \( \mathbb{R}^N \) or (up to rotation and translation) \( \mathbb{R}^N_+ \), with either \( v_i \in D^{1,p}(\mathbb{R}^N) \) or (respectively) \( v_i \in D^{1,p}(\mathbb{R}^N_+) \), and there exist \( k \) sequences \( \{y_n^i\} \subset \Omega \) and \( \{\lambda_n^i\} \subset \mathbb{R}_+ \), satisfying

\[
\frac{1}{\lambda_n^i} \text{dist}(y_n^i, \partial \Omega) \to \infty, \quad n \to \infty, \quad \text{if } H_i \equiv \mathbb{R}^N \text{ or}
\]

\[
\frac{1}{\lambda_n^i} \text{dist}(y_n^i, \partial \Omega) < \infty, \quad n \to \infty, \quad \text{if (up to rotation and translation) } H_i \equiv \mathbb{R}^N_+ \text{, and}
\]

\[
\|u_n - v_0 - \sum_{i=1}^k (\lambda_n^i)^{(p-N)/p} v_i ((\cdot - y_n^i)/\lambda_n^i)\| \to 0, \quad n \to \infty,
\]

\[
\|u_n\|^p \to \sum_{i=0}^k \|v_i\|^p, \quad n \to \infty,
\]

(2.1)

\[
J(v_0) + \sum_{i=1}^k J_{\infty}(v_i) = c.
\]

The restriction on the levels \( c \) and Lemma 2.1 immediately yields the bound \( k \leq 1 \). If \( k = 0 \) compactness holds and we are done. If instead \( k = 1 \), we have two cases, namely \( v_0 \equiv 0 \) or \( v_0 \not\equiv 0 \). If \( v_0 \not\equiv 0 \), since

\[
J(v_0) \geq S^{N/p}/N, \quad J_{\infty}(v_1) \geq S^{N/p}/N,
\]

(actually \( J(v_0) > S^{N/p}/N \), as the Sobolev constant is never achieved on bounded domains) we obtain a contradiction by combining (2.1) with the assumption \( c < 2S^{N/p}/N \). If, instead, \( v_0 \equiv 0 \), then formula (2.1) reduces to \( J(v_1) = c \). Again by Lemma 2.1 \( v_1 \) does not change sign and by the nonexistence result \([11, \text{ Theorem 1.1}]\) \( H_1 \equiv \mathbb{R}^N \), namely \( v_1 \in D^{1,p}(\mathbb{R}^N) \) solves

(2.2)

\[
-\Delta_p u = u^{p^*-1} \quad \text{in} \quad \mathbb{R}^N, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N.
\]

Now, by the symmetry result of \([5, \text{ Theorem 2.1}]\), which holds in the range \( 2N/(N+2) < p \leq 2 \), \( v_1 \) is radially symmetric about some point and, in turn, by using \([8, \text{ Theorem 2.1(iii)]}\) (see also \([2]\)), after translation in the origin, for a suitable value of \( a > 0 \) \( v_1 \) is a Talenti function

\[
v_1(x) = \left(Na^p \left(\frac{N-p}{p-1}\right)^{p-1}\right)^{(N-p)/p} (a + |x|^{p/(p-1)})^{(p-N)/p},
\]
whose associated energy is \( c = J_\infty(v_1) = S^{N/p}/N \) [15], since \( v_1 \) achieves the best Sobolev constant \( S \). This is a contradiction again, since \( c > S^{N/p}/N \). This concludes the proof. \( \square \)

2.2. Proof of Theorem 1.1 concluded. Let \( R_1, R_2 \) be the radii of the annulus as in the statement of Theorem 1.1. As observed in [4, 14], without loss of generality, we may assume that \( x_0 = 0, R_1 = 1/(4R) \) and \( R_2 = 4R \) where \( R > 0 \) will be chosen sufficiently large. Let us set \( \Sigma := \{ x \in \mathbb{R}^N : |x| = 1 \} \) and consider the family of functions

\[
\varphi_t(x) := \left[ \frac{1 - t}{(1-t)^p + |x - t\sigma|^{p-1}} \right]^{N-p} \in \mathcal{D}^{1,p}(\mathbb{R}^N), \quad \text{for } \sigma \in \Sigma \text{ and } t \in [0, 1).
\]

Moreover, let us now consider a function \( \varphi \in C^\infty_c(\Omega) \) be such that \( 0 \leq \varphi \leq 1 \) on \( \Omega \), \( \varphi = 1 \) on \( \{1/2 < |x| < 2\} \) and \( \varphi = 0 \) outside \( \{1/4 < |x| < 4\} \), then define

\[
\varphi_R(x) := \begin{cases} 
\varphi(Rx) & \text{on } 0 \leq |x| < \frac{1}{R}, \\
1 & \text{on } \frac{1}{R} \leq |x| < R, \\
\varphi(x/R) & \text{on } |x| \geq R.
\end{cases}
\]

Finally, let us set

\[
w_t^\sigma(x) := u_t^\sigma(x)\varphi_R(x) \in W_0^{1,p}(\Omega), \quad w_0(x) := u_0(x)\varphi_R(x), \quad u_0(x) := \left[ \frac{1}{1 + |x|^{p-1}} \right]^{N-p}.
\]

Then, we have the following

**Lemma 2.3.** For \( \sigma \in \Sigma \) and \( t \in [0, 1) \), \( \|u_t^\sigma\| = \|u_0\| \), \( \|u_t^\sigma\|_{p^*} = \|u_0\|_{p^*} \) and \( \|u_t^\sigma\|^p = S\|u_t^\sigma\|_{p^*}^p \). Furthermore, there holds

\[
\lim_{R \to \infty} \sup_{\sigma \in \Sigma, t \in [0, 1)} \|w_t^\sigma - u_t^\sigma\| = 0.
\]

**Proof.** The first properties of \( u_t^\sigma \) follow by [15]. In the following \( C \) will denote a generic positive constant, independent of \( \sigma \in \Sigma \) and \( t \in [0, 1) \), which may vary from line to line. We have the inequality

\[
\int_{\mathbb{R}^N} |\nabla(w_t^\sigma - u_t^\sigma)|^p dx \leq C \sum_{i=1}^4 \mathbb{I}_i,
\]

where we have set

\[
\mathbb{I}_1 := \int_{\mathbb{R}^N \setminus B_{2R}} |\nabla u_t^\sigma|^p dx, \\
\mathbb{I}_2 := \int_{B_{(2R)}^{-1}} |\nabla u_t^\sigma|^p dx, \\
\mathbb{I}_3 := \frac{1}{R^p} \int_{B_{4R} \setminus B_{2R}} |u_t^\sigma|^p dx, \\
\mathbb{I}_4 := R^p \int_{B_{(2R)}^{-1}} |u_t^\sigma|^p dx.
\]

Taking into account that

\[
|\nabla u_t^\sigma(x)| \leq \frac{C}{((1-t)^p + |x - t\sigma|^{p-1})^{p-2}} \leq C \quad |x| \leq \frac{1}{2}, \quad |\nabla u_t^\sigma(x)| \leq \frac{C}{|x|^{p-1}} \quad |x| \geq 2,
\]

we have

\[
\mathbb{I}_1 \leq \frac{C}{((1-t)^p + |x - t\sigma|^{p-1})^{p-2}} \int_{\mathbb{R}^N \setminus B_{2R}} \frac{dx}{|x|^{N-p}} = C \int_{\mathbb{R}^N \setminus B_{2R}} \frac{dx}{|x|^{N-p}} \\
\mathbb{I}_2 \leq C \int_{B_{(2R)}^{-1}} \frac{dx}{|x|^{N-p}} \\
\mathbb{I}_3 \leq \frac{C}{R^p} \int_{B_{4R} \setminus B_{2R}} \frac{dx}{|x|^{N-p}} \\
\mathbb{I}_4 \leq C \int_{B_{(2R)}^{-1}} \frac{dx}{|x|^{N-p}}.
\]
we obtain
\[ I_1 = \int_{\mathbb{R}^N \setminus B_{2R}} |\nabla u_t|^p dx \leq C \int_{\mathbb{R}^N \setminus B_{2R}} \frac{1}{|x|^{p(N-1)}} dx \leq \frac{C}{R^{p-1}}, \]
\[ I_2 = \int_{B(2R)^c} |\nabla u_t|^p dx \leq C \int_{B(2R)^c} dx \leq \frac{C}{R^N}. \]
Moreover, we have
\[ I_3 = \frac{1}{R^p} \int_{B_{4R} \setminus B_{2R}} \left[ \frac{1-t}{(1-t)^p + |x-t\sigma|^p} \right]^{N-p} dx \leq \frac{C}{R^p} \int_{B_{4R} \setminus B_{2R}} \frac{1}{|x|^{p(N-1)}} dx \leq \frac{C}{R^{p-1}}, \]
\[ I_4 = R^p \int_{B(2R)^c} \left[ \frac{1-t}{(1-t)^p + |x-t\sigma|^p} \right]^{N-p} dx \leq R^p C \int_{B(2R)^c} dx \leq \frac{C}{R^{N-p}}. \]
This concludes the proof.

Let us now define
\[ (2.3) \quad S(u) := \frac{\|\nabla u\|^p}{\|u\|^p_{L^p(\mathbb{R}^N)}}, \quad u \in D^{1,p}(\mathbb{R}^N) \setminus \{0\}, \]
with the understanding that
\[ (2.4) \quad S(u; \Omega) = \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega)}}, \quad u \in W^{1,p}_0(\Omega) \setminus \{0\}, \]
after extending by zero outside \( \Omega \).

As a consequence of Lemma 2.3, we have the following

Lemma 2.4. If \( v_t^\sigma(x) := \|w_t^\sigma\|^{1}_{L^p(\mathbb{R}^N)} w_t^\sigma(x) \) and \( v_0(x) = \|w_0\|^{-1}_{L^p(\mathbb{R}^N)} w_0(x) \), then
\[ \lim_{R \to \infty} S(v_t^\sigma; \Omega) = S(u_t^\sigma) = S, \]
uniformly with respect to \( \sigma \in \Sigma \) and \( t \in [0,1) \).

We observe that \( J \) satisfies the Palais-Smale condition between the levels \( S^{N/p}/N \) and \( 2S^{N/p}/N \). Therefore, as it can be readily verified, the functional \( S(\cdot; \Omega) \), constrained to
\[ \mathcal{M} = \{ u \in W^{1,p}_0(\Omega) : \|u\|^p_{L^p(\Omega)} = 1 \}, \]
satisfies the Palais-Smale condition between \( S \) and \( \varpi S \), for some \( \varpi > 1 \) depending upon \( p \) and \( N \). Then, taking Lemma 2.4 into account, and assuming by contradiction that the problem does not admit any positive solution, by arguing exactly as in [14, pp.191-193] one proves Theorem 1.1 by performing a well-established deformation argument on \( S(\cdot; \Omega) \) as restricted to \( \mathcal{M} \), yielding a contradiction with the geometrical properties (1.3) of \( \Omega \). We point out that under our assumption \( 2N/(N+2) < p \), it follows \( p^* > 2 \) so that \( \mathcal{M} \) is a \( C^{1,1} \) smooth manifold.

\[ \square \]

Acknowledgements

C.M. would like to thank Prof. Abbas Bahri for various discussions at Rutgers University on noncompact problems.
References

[1] A. Bahri, J.-M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain, Comm. Pure Appl. Math. 41 (1988), 253–294.
[2] M.F. Bidaut-Véron, Local and global behavior of solutions of quasilinear equations of Emden-Fowler type, Arch. Rational Mech. Anal. 107 (1989), 293–324.
[3] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36 (1983), 437–477.
[4] J.M. Coron, Topologie et cas limite del injections de Sobolev, C.R. Acad. Sc. Paris 299 (1984), 209–212.
[5] L. Damascelli, S. Merchán, L. Montoro, B. Sciunzi, Radial symmetry and applications for a problem involving the $-\Delta_p$ operator and critical nonlinearity in $\mathbb{R}^N$, preprint.
[6] E.N. Dancer, A note on an equation with critical exponent, Bull. Lond. Math. Soc. 20 (1988), 600–602.
[7] W.Y. Ding, Positive solutions of $-\Delta u + u^{(n+2)/(n-2)} = 0$ on contractible domains, J. Partial Differ. Equ. 2 (1989), 83–88.
[8] M. Guedda, L. Veron, Local and global properties of solutions of quasi-linear elliptic equations, J. Differential Equations 76 (1988), 159–189.
[9] M. Guedda, L. Veron, Quasilinear elliptic equations involving critical Sobolev exponents, Nonlinear Anal. 13 (1989), 879–902.
[10] C. Mercuri, F. Pacella, On the pure critical exponent problem for the $p$-laplacian, Calc. Var. Partial Differential Equat., to appear.
[11] C. Mercuri, M. Willem, A global compactness result for the $p$-laplacian involving critical nonlinearities, Discrete Contin. Dyn. Syst. 28 (2010), 469–493.
[12] D. Passaseo, Multiplicity of positive solutions of nonlinear elliptic equations with critical Sobolev exponent in some contractible domains, Manuscripta Math. 65 (1989), 147–165.
[13] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, Math. Z. 187 (1984), 511–517.
[14] M. Struwe, Variational methods, Springer, New York, 1990.
[15] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976), 353–372.