RATIONAL HOMOLOGY BALLS IN 2-HANDLEBODIES

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Abstract. We prove that there are rational homology balls $B_p$ smoothly embedded in the 2-handlebodies associated to certain knots. Furthermore we show that, if we rationally blow up the 2-handlebody along the embedded rational homology ball $B_p$, then the resulting 4-manifold cannot be obtained just by a sequence of ordinary blow ups from the 2-handlebody under a certain mild condition.

1. Introduction

In the theory of smooth 4-manifolds, the rational blow-down (defined by Fintushel-Stern [4]) is a useful tool for producing exotic smooth structures: We first cut out a plumbing manifold $C_p$ smoothly embedded in a given 4-manifold $M$ and we then paste a rational homology ball $B_p$ along the boundary $\partial C_p (= \partial B_p)$ to obtain a new 4-manifold $\tilde{M} = (M - C_p) \cup_{\partial C_p} B_p$. We recall briefly the definitions of $C_p$ and $B_p$ in Section 1.3. The rational blow-down surgery increases the signature while it keeps $b_2^+$ fixed. So it and its generalization (J. Park [9]) are very useful to construct many important exotic 4-manifolds with small Euler numbers; refer J. Park [10], Stipsicz-Szabó [13], J. Park-Stipsicz-Szabó [11], for instance.

On the other hand, as a reverse surgery, the rational blow-up is defined by $M = (M - B_p) \cup_{\partial C_p} C_p$ for a 4-manifolds $M$ where $B_p$ is smoothly embedded. The rational blow-down would also give an intriguing performance for constructing interesting 4-manifolds. The rational blow-up will decrease the signature while it keeps $b_2^+$ fixed. Therefore it would be useful to construct minimal 4-manifolds with $c_1^2 < 0$ for example.

In this paper we first show in Theorem 1.3 that Fintushel-Stern’s rational homology ball $B_p$ is smoothly embedded into the special 2-handlebody $M(p, m)$ associated to a certain knot $K(p, m)$, where the knot $K(p, m)$ and the 2-handlebody $M(p, m)$ are defined in Definition 1.1. We then show in Corollary 1.6 that the rational blow-up along $B_p$ in $M(p, m)$ is not a sequence of ordinary blow-ups of $M(p, m)$.

So it would be an intriguing problem to find or construct 4-manifolds $X$ containing $M(p, m)$ associated the knot $K(p, m)$ and to investigate what happens after rationally blowing-up $B_p$ in $X$ in case of the resulting manifold is not a sequence of blow-ups. We leave it for further studies.

1.1. Smoothly embedded rational homology balls. At first, the knot $K(p, m)$ and the 2-handlebody $M(p, m)$ are defined as follows:

**Definition 1.1.** For $p \geq 2$ and $m \in \mathbb{Z}$, we denote by $K(p, m)$ the knot defined by Figure 1. The 2-handlebody $M(p, m)$ associated to the knot $K(p, m)$ is defined by attaching a 2-handle to $D^4$ along $K(p, m)$ in $\partial D^4$ with framing $p^2m - p - 1$.
Remark 1.2. The knot $K(p,m)$ is the twisted torus knot $T(p,p(m-1)+1)_{p-1,1}$ defined in Callahan-Dean-Weeks [2] and Dean [3]. According to Vafaee [14, Corollary 3.2] for example, the knot $K(p,m)$ is isotopic to a torus knot $T(p,mp-1)$. Hence the boundary of 2-handlebody $M(p,m)$ with framing $p^2m - p - 1$ is the lens space $L(p(mp-1)-1,p^2)$.

One of the main theorems is the following:

**Theorem 1.3.** For any $p \geq 2$ and $m \in \mathbb{Z}$, there is the smoothly embedded rational homology ball $B_p$ in the 2-handlebody $M(p,m)$ associated to the knot $K(p,m)$.

We prove it in Section 2. As a corollary, one can detect a rational homology ball $B_p$ embedded in a given 4-manifold $X$ by looking at its Kirby diagram. Precisely:

**Corollary 1.4.** If a Kirby diagram of $X$ contains the knot $K(p,m)$ with framing $(p^2m - p - 1)$ such that no dotted circles representing a 1-handle are linked with $K(p,m)$, then $B_p$ is embedded in $X$.

In Khodorovskiy [7], she proves that the rational homology ball $B_p$ is smoothly embedded in a neighborhood of a sphere $S^2$ with self-intersection number $(-p-1)$. Theorem 1.3 may be regarded as a generalization of her results because $K(p,m)$ is an unknot for $m = 0$. In PPS [12], the authors with J. Park generalize Khodorovskiy [7] for general rational homology balls $B_{p,q}$ by using techniques developed in HTU [6] from the minimal model program for 3-folds in algebraic geometry. They prove the existence of rational homology balls $B_{p,q}$ smoothly embedded in the plumbing of disk bundles over spheres according to a certain linear graph which is roughly speaking half of the negative-definite plumbing graph associated to $C_{p,q}$. Recently, Owens [8] further generalizes theorems in Khodorovskiy [7] and PPS [12] and gives relatively simple topological proofs of those theorems.

But the embeddings of the rational homology balls in Khodorovskiy [7], PPS [12], Owens [8] are simple; that is, the rational blow-ups of their embedded rational homology balls are just sequence of the ordinary blow-ups. It implies that their rational homology balls are not useful in general for constructing interesting 4-manifolds.

1.2. **Rational blow-up surgery.** In contrast, we show that the embedding of $B_p$ in Theorem 1.3 is not simple under certain mild conditions in Corollary 1.6. For this we first show:

**Theorem 1.5.** Let $\tilde{M}(p,m)$ be the rationally blown-up 4-manifold along $B_p$ from $M(p,m)$. Then $\tilde{M}(p,m)$ is the plumbing manifold with the plumbing graph in Figure 2.
We prove it in Section 3. We then show that the embedding of $B_p$ in Theorem 1.3 is not simple:

**Corollary 1.6.** The rationally blown-up 4-manifold $\tilde{M}(p,m)$ cannot be obtained from $M(p,m)$ by any sequence of ordinary blow-ups of $M(p,m)$ either if $p$ is a positive even integer and $m$ is an odd integer or if $p \geq 2$ and $m - 1 \leq -2$.

**Proof.** If $p$ is a positive even integer and $m$ is an odd integer, then $\tilde{M}(p,m)$ is a manifold with an even intersection form. So there are no $(-1)$-classes in the plumbing as Figure 2. On the other hand, if $m$ is sufficiently small, precisely, if $m - 1 \leq -2$, then there are also no $(-1)$-classes in the plumbing. □

1.3. **Notions.** The plumbing diagram of $C_p$ is given as below

\[-(p + 2) \quad -2 \quad -2 \quad m - 1 \quad p - 2\]

And the rational homology ball $B_p$ is given as follows: Let $F_{p-1}$ ($p \geq 2$) be the Hirzebruch surface having the negative section $s_0$ with $s_0 \cdot s_0 = -(p-1)$. Let $s_\infty$ be a positive section with $s_\infty \cdot s_\infty = p - 1$ and $f$ a fiber. Then $B_p$ is the complement of the pair of 2-spheres represented by the homology classes $s_\infty + f$ and $s_0$ in $F_{p-1}$. The Kirby diagram for $B_p$ is given in Figure 3 (cf. Gompf-Stipsicz [5, Figure 8.41, p.331]).

\[\text{Figure 3. A rational homology ball } B_p\]

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2. **Smoothly embedded rational homology balls**

This section is devoted to proof of Theorem 1.3.

**Theorem 1.3.** For any $p \geq 2$ and $m \in \mathbb{Z}$, there is a smoothly embedded rational homology ball $B_p$ in the 2-handlebody $M(p,m)$ associated to the knot $K(p,m)$. 
Proof. In Figure 4, the picture (B) is isotopic to (A). We apply $p$-multiple handle slide of the $(p - 1)$-framed 2-handle over the $m$-framed 2-handle to Figure 4(B) so that we get Figure 5. Then we cancel the 1-handle and the $m$-framed 2-handle in Figure 5. After all, we get the $(p^2m - p - 1)$-framed knot $K(p, m)$ in Figure 1. Note that there is a rational homology ball $B_p$ in Figure 4. Therefore $B_p$ is embedded in the 2-handlebody associated to the knot $K(p, m)$. \hfill \Box

3. Rational blow-ups along the rational homology balls

This section is devoted to proof of Theorem 1.5

Theorem 1.5. Let $\bar{M}(p, m)$ be the rationally blown-up 4-manifold from $M(p, m)$. Then $\bar{M}(p, m)$ is the plumbing manifold with the following plumbing graph:

\[
\begin{array}{cccc}
-p-2 & -2 & -2 & m-1 \\
p-2
\end{array}
\]

Proof. Note that Figure 4(A) is equal to Figure 6. We will show that Figure 7(A) can be obtained from Figure 6 by rationally blowing up $B_p$ in Figure 6.
At first, we claim that there is $C_p$ embedded in Figure 7(A): As in Gompf-Stipsicz [5, 12.67, 12.68, p.516], we do a handle slide to Figure 7(A) to get Figure 7(B), and then, we apply another handle slide to Figure 7(B) so that we get Figure 2, which implies that $C_p$ is embedded in Figure 7(A).

Next, it is clear that the complement of $B_p$ in the 4-manifold represented by Figure 6 and that of $C_p$ in Figure 7(A) are the same. Furthermore it is easy to see from the Kirby diagrams Figure 6 and Figure 7(A) that the two complements are glued to the boundary of $B_p$ in Figure 6 and that of $C_p$ in Figure 7(A), respectively, by the same gluing map. Therefore one can conclude that Figure 2 is obtained from Figure 4(A) by a rational blowing-up. □

Remark 3.1. The main part of the above proof is that the rational blow-up of Figure 4(A) is diffeomorphic to the plumbing 4-manifold with the plumbing graph of Figure 2. The referee informed us that this result appears also in the proof of Theorem 5.1 (3) in Akbulut-Yasui [1].

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