Neutrino Quasinormal Modes of a Kerr-Newman-de Sitter Black Hole

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Abstract

Using the Pöschl-Teller approximation, we evaluate the neutrino quasinormal modes (QNMs) of a Kerr-Newman-de Sitter black hole. The result shows that for a Kerr-Newman-de Sitter black hole, massless neutrino perturbation of large $\Lambda$, positive $m$ and small value of $n$ will decay slowly.

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1 Introduction

It is well-known that there are three stages during the evolution of the field perturbation in the black hole background: the initial outburst from the source of perturbation, the quasinormal oscillations and the asymptotic tails. The frequencies and damping time of the quasinormal oscillations called "quasinormal modes" (QNMs) are determined only by the black hole's parameters and independent of the initial perturbations. A great deal of efforts have been devoted to the black hole's QNMs for the possibility of direct identification of black hole existence through gravitational wave detectors in the near future \[1, 2\]. The study of black hole's QNMs has a long history. Most of the studies immersed in an asymptotically flat space time. The discovery of the AdS/CFT \[3, 4\] correspondence and the expanding universe motivated the investigation of QNMs in de Sitter \[5, 6\] and anti-de Sitter \[7, 8, 9, 10\] space time in the past several years.

Most methods in evaluating the QNMs are numerical in nature. Recently, using the third-order WKB approximation, Cho evaluated the Dirac field QNMs of a Schwarzschild black hole \[26\]. A powerful WKB scheme was devised by Schutz and Will \[15\], and was extended to higher orders in \[16\]. Konoplya \[17\] extended the WKB approximation to sixth-order and calculated the QNMs of a D-dimensional Schwarzchild black hole. Zhidenko \[18\] calculated low-laying QNMs of a Schwarzschild-de Sitter black hole by using sixth-order WKB approximation and the approximation by Pöschl-Teller potential. Cardoso \[19\] calculated QNMs of the near extremal Schwarzschild-de Sitter black hole by using Pöschl-Teller approximation, which was proved to be exactly in the near extreme regime \[20\]. Yoshida \[21\] numerically analyzed QNMs in nearly extremal Schwarzschild-de Sitter spacetimes. The Kerr black hole is a more general case. It is also important to note that the most important QNMs are the lowest ones which have smaller imaginary on the astrophysical aspect and the most important spacetimes are the asymptotically flat and now perhaps the asymptotically de Sitter which supported by the recent observation data. So we discuss the QNMs of a Kerr-Newman-de Sitter black hole in this paper. Leaver \[22\] developed a hybrid analytic-numerical method to calculate the QNMs of black holes and applied to the Kerr black hole. Seidel and Iyer \[23\] computed the low-laying QNMs of Kerr black holes for both scalar and gravitational perturbations by using third-order WKB approximation. Berti et.al dealt with highly damped QNMs of Kerr black holes in \[24, 25\].

In this paper, we evaluate the QNMs of Kerr-Newman-de Sitter black hole for neutrino perturbation. In Section 2 we consider the massless Dirac equations for massless neutrino in the Kerr-Newman-de Sitter black hole and reduced it into a set of Schrödinger-like equations with a particular effective potential. We analyse the properties of the particular potential in section 3 and use the Pöschl-Teller potential approximation to evaluate the QNMs of massless neutrino in Section 4. Conclusions and discussions are presented in Section 5. Throughout this paper we use units in which \( G = c = M = 1 \).
2 MASSLESS DIRAC FIELD EQUATION IN THE KERR-NEWMAN-DE SITTER BLACK HOLE

Generally speaking, neutrino is a kind of uncharged Dirac particles without rest mass or with tiny mass. In curved spacetime, the spinor representations of massless Dirac equations are \[11\]

\[
\nabla_{AB} P^A = 0, \\
\nabla_{AB} Q^A = 0,
\]

where \(P^A\) and \(Q^A\) are two two-component spinors, the operator \(\nabla_{AB}\) denotes the spinor covariant differentiation. \(\nabla_{AB} = \sigma^\mu_{AB} \nabla_\mu,\) and \(\sigma^\mu_{AB}\) are 2 \times 2 Hermitian matrices which satisfy \(g^{\mu\nu} \sigma^\mu_{AB} \sigma^\nu_{CD} = \epsilon_{AC} \epsilon_{BD},\) where \(\epsilon_{AC}\) and \(\epsilon_{BD}\) are antisymmetric Levi-Civita symbols, the operator \(\nabla_\mu\) is covariant differentiation.

The metric of the Kerr-Newman-de Sitter black hole in the Boyer-Lindquist coordinate system is

\[
ds^2 = \frac{1}{\bar{\varrho}^2 \Sigma^2} \left( \Delta_r - \Delta_\theta a^2 \sin^2 \theta \right) dt^2 - \frac{\varrho^2}{\Delta_r} dr^2 - \frac{\varrho^2}{\Delta_\theta} d\theta^2 \\
- \frac{1}{\varrho^2 \Sigma^2} \left[ \Delta_\theta \left( r^2 + a^2 \right)^2 - \Delta_r a^2 \sin^2 \theta \right] \sin^2 \theta d\varphi^2 \\
+ \frac{2a}{\varrho^2 \Sigma^2} \left[ \Delta_\theta (r^2 + a^2) - \Delta_r \right] \sin^2 \theta dt d\varphi,
\]

(2.3)

where \(\bar{\varrho} = r + ia \cos \theta,\) \(\varrho^2 = \bar{\varrho} \bar{\varrho}^*,\) \(\Delta_r = (r^2 + a^2) \left( 1 + \frac{\Lambda}{3} r^2 \right) - 2Mr + Q^2,\)

(2.4)

\(\Delta_\theta = 1 + \frac{1}{3} \Lambda a^2 \cos^2 \theta,\) \(\Sigma = 1 + \frac{1}{3} \Lambda a^2.\)

(2.5)

Here, \(a\) and \(Q\) are the angular momentum per unit mass and electric charge of the black hole, \(M\) is the black hole mass and \(\Lambda\) is the positive cosmological constant.

The contravariant component of metric tensor is

\[
g^{\mu\nu} = \begin{pmatrix}
\frac{\Sigma^2}{\varrho^2} \left[ \frac{(r^2 + a^2)^2}{\Delta_r} - a^2 \sin^2 \theta \right] & 0 & 0 & a^2 \Sigma^2 \left( \frac{r^2 + a^2}{\Delta_r} - \frac{1}{\Delta_\theta} \right) \\
0 & -\Delta_r \varrho^{-2} & 0 & 0 \\
a^2 \Sigma^2 \left( \frac{r^2 + a^2}{\Delta_r} - \frac{1}{\Delta_\theta} \right) & 0 & -\Delta_\theta \varrho^{-2} & 0 \\
\frac{\Sigma^2 \varrho^2}{\varrho^2} \left( \frac{r^2 + a^2}{\Delta_r} - \frac{1}{\Delta_\theta} \right) & 0 & 0 & -\Sigma^2 \varrho^2 \sin^2 \theta \left( \frac{1}{\Delta_\theta} - \frac{a^2 \sin^2 \theta}{\Delta_r} \right)
\end{pmatrix}. 
\]

(2.6)
Choose the null tetrad as follows:

\[
\begin{align*}
    l^\mu &= \left[ \frac{(r^2 + a^2) \Sigma}{\Delta}, 1, 0, \frac{a \Sigma}{\Delta} \right], \\
n^\mu &= \frac{1}{2q^2} \left[ \Sigma \left( r^2 + a^2 \right), -\Delta r, 0, a \Sigma \right], \\
m^\mu &= \frac{1}{\sqrt{2} \Delta \theta} \left[ ia \Sigma \sin \theta, 0, \Delta \theta \right], \\
\bar{m}^\mu &= \frac{1}{\sqrt{2} \Delta \theta^*} \left[ -ia \Sigma \sin \theta, 0, -\Delta \theta \right].
\end{align*}
\]

(2.7)

The above null tetrad consists of null vector, i.e.

\[ l_\mu l^\mu = n_\mu n^\mu = m_\mu m^\mu = 0. \]

(2.8)

The null vector satisfies the following pseudo-orthogonality relations

\[ l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1, \quad l_\mu m^\mu = l_\mu \bar{m}^\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0. \]

(2.9)

They also satisfy metric conditions

\[ g_{\mu \nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu. \]

(2.10)

Set spinor basis \( \zeta^A_a = \sigma^A_a \), in which \( A \) is the spinor component index, \( a \) is the spinor basis index, both indices are 0 or 1.

The covariant differentiation \( \nabla_{AB}^c \xi^A \) for an arbitrary spinor \( \xi^A \) can be repreaented as the component along the spinor basis \( \zeta^A_a \), i.e.

\[ \zeta^A_a \zeta^B_b \zeta^C_c \nabla_{AB}^c \xi^C = \nabla_{ab}^c \xi^c = \partial_{ab}^c \xi^c + \Gamma_{dc}^c \xi^d, \]

(2.11)

where \( \partial_{ab} \) are ordinary spinor derivatives, \( \Gamma_{dc}^c \) are spin coefficients.

Now let

\[ \partial_{00} = l^\mu l_\mu \equiv D, \quad \partial_{11} = n^\mu n_\mu \equiv \Delta, \]
\[ \partial_{01} = m^\mu m_\mu \equiv \delta, \quad \partial_{1\bar{0}} = \bar{m}^\mu \bar{m}_\mu \equiv \bar{\delta}. \]

(2.12)

Then the Dirac equations (2.1) and (2.2) can be rewritten as four coupled equations

\[ (D + \epsilon - \rho) F_1 + (\bar{\delta} + \pi - \alpha) F_2 = 0, \]
\[ (\Delta + \mu - \gamma) F_2 + (\delta + \beta - \tau) F_1 = 0, \]
\[ (D + \epsilon^* - \rho^*) G_2 - (\bar{\delta} + \pi^* - \alpha^*) G_1 = 0, \]
\[ (\Delta + \mu^* - \gamma^*) G_2 + (\delta + \beta^* - \tau^*) G_1 = 0. \]

(2.13)
where $F_1, F_2, G_1, G_2$ are four-component spinors with $F_1 = P^0, F_2 = P^1, G_1 = Q^1, G_2 = Q^0$, $\alpha, \beta, \gamma, \epsilon, \mu, \pi, \rho, \tau$ etc. are Newman-Penrose symbols, while $\alpha^*, \beta^*$ etc. are, respectively, the complex conjugates of $\alpha, \beta$ etc.. The Newman-Penrose symbols are

$$
\rho = l_{\mu\nu}m^\mu\bar{m}^\nu = -\frac{1}{\bar{q}^*}, \quad k = l_{\mu\nu}m^\mu\nu = 0, \quad \lambda = -n_{\mu\nu}\bar{m}^\mu\bar{m}^\nu = 0,$$
$$
\tau = l_{\mu\nu}m^\mu n^\nu = -\frac{i\sqrt{\Delta_\theta}}{\sqrt{2}\bar{q}^2} \sin \theta, \quad \sigma = l_{\mu\nu}m^\mu\nu = 0, \quad \nu = -n_{\mu\nu}\bar{m}^\mu n^\nu = 0,$$
$$
\mu = -n_{\mu\nu}\bar{m}^\mu m^\nu = -\frac{\Delta_\tau}{2\bar{q}^2}, \quad \pi = -n_{\mu\nu}\bar{m}^\mu\nu = \frac{i\sqrt{\Delta_\theta}}{\sqrt{2}(\bar{q}^*)^2} \sin \theta,$$
$$
\epsilon = \frac{1}{2} (l_{\mu\nu}m^\mu\nu - m_{\mu\nu}\bar{m}^\mu\nu) = 0,$$
$$
\gamma = \frac{1}{2} (l_{\mu\nu}n^\mu\nu - m_{\mu\nu}\bar{m}^\mu\nu) = \frac{1}{4\bar{q}^2} \frac{d\Delta_\tau}{dr} + \mu,$$
$$
\beta = \frac{1}{2} (l_{\mu\nu}n^\mu\nu - m_{\mu\nu}\bar{m}^\mu\nu) = \frac{1}{2\sqrt{2}\bar{q}\sin \theta} \frac{d}{d\theta} (\sqrt{\Delta_\theta} \sin \theta),$$
$$
\alpha = \frac{1}{2} (l_{\mu\nu}n^\mu\nu - m_{\mu\nu}\bar{m}^\mu\nu) = \pi - \beta^*.
$$

Take three transformations as follows:

$$
F_1 = e^{-i\omega t}e^{im\phi}f_1(r, \theta), \quad F_2 = e^{-i\omega t}e^{im\phi}f_2(r, \theta),
$$
$$
G_1 = e^{-i\omega t}e^{im\phi}g_1(r, \theta), \quad G_2 = e^{-i\omega t}e^{im\phi}g_2(r, \theta), \quad (2.15)
$$
$$
U_1(r, \theta) = \bar{q}^* f_1(r, \theta), \quad U_2(r, \theta) = f_2(r, \theta),
$$
$$
V_1(r, \theta) = g_1(r, \theta), \quad V_2(r, \theta) = \bar{q} g_2(r, \theta), \quad (2.16)
$$
$$
U_1 = R_{-\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta), \quad U_2 = R_{-\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta),
$$
$$
V_1 = R_{-\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta), \quad V_2 = R_{-\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta). \quad (2.17)
$$

Eqs. (2.13) turn into

$$
\mathcal{D}_0 R_{-\frac{1}{2}}S_{-\frac{1}{2}} + \sqrt{\frac{\Delta_\theta}{2}} \mathcal{L}_\frac{1}{2} R_{+\frac{1}{2}}S_{+\frac{1}{2}} = 0,
$$
$$
\Delta_\tau \mathcal{D}_\frac{1}{2} R_{-\frac{1}{2}}S_{+\frac{1}{2}} - 2\sqrt{2\Delta_\theta} \mathcal{L}_\frac{1}{2} R_{-\frac{1}{2}}S_{+\frac{1}{2}} = 0,
$$
$$
\mathcal{D}_\frac{1}{2} R_{+\frac{1}{2}}S_{-\frac{1}{2}} - \sqrt{\frac{\Delta_\theta}{2}} \mathcal{L}_\frac{1}{2} R_{+\frac{1}{2}}S_{-\frac{1}{2}} = 0,
$$
$$
\Delta_\tau \mathcal{D}_\frac{1}{2} R_{+\frac{1}{2}}S_{-\frac{1}{2}} + 2\sqrt{2\Delta_\theta} \mathcal{L}_\frac{1}{2} R_{+\frac{1}{2}}S_{+\frac{1}{2}} = 0. \quad (2.18)
$$
where
\[
D_s = \partial_r - \frac{i \Sigma K}{\Delta r} + \frac{s}{\Delta r} \frac{d \Delta r}{dr},
\]
\[
D_s^+ = \partial_r + \frac{i \Sigma K}{\Delta r} + \frac{s}{\Delta r} \frac{d \Delta r}{dr},
\]
\[
L_s = \partial_\theta - \frac{\Sigma H}{\Delta \theta} + \frac{s}{\sqrt{\Delta \theta} \sin \theta} \frac{d (\sqrt{\Delta \theta} \sin \theta)}{d \theta},
\]
\[
L_s^+ = \partial_\theta + \frac{\Sigma H}{\Delta \theta} + \frac{s}{\sqrt{\Delta \theta} \sin \theta} \frac{d (\sqrt{\Delta \theta} \sin \theta)}{d \theta},
\]
\[
K = \left( r^2 + a^2 \right) \omega - am,
\]
\[
H = a \omega \sin \theta - \frac{m}{\sin \theta}.
\]

By using separation of variables, Eqs. (2.18) become
\[
D_0 R - \frac{1}{2} = \lambda R + \frac{1}{2}, \quad \Delta_r D_0^+ R + \frac{1}{2} = 2 \lambda R - \frac{1}{2},
\]
\[
\Delta_r D_0^+ R + \frac{1}{2} = \lambda_2 R - \frac{1}{2}, \quad \Delta_r D_0^+ R + \frac{1}{2} = 2 \lambda_2 R - \frac{1}{2},
\]
\[
\Delta_r L_\theta S + \frac{1}{2} = - \sqrt{2} \lambda S - \frac{1}{2}, \quad \Delta_r L_\theta S + \frac{1}{2} = \sqrt{2} \lambda S + \frac{1}{2},
\]
\[
(2.19)
\]

where \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) are the separation constants, and
\[
\lambda_1 = \lambda_3 = \frac{1}{2} \lambda_2 = \frac{1}{2} \lambda_4 \equiv \lambda.
\]

Then Eqs. (2.20) can be written as
\[
D_0 R - \frac{1}{2} = \lambda R + \frac{1}{2}, \quad \Delta_r D_0^+ R + \frac{1}{2} = \lambda R - \frac{1}{2},
\]
\[
\Delta_r \frac{1}{2} L_\theta S + \frac{1}{2} = - \sqrt{2} \lambda S - \frac{1}{2}, \quad \Delta_r \frac{1}{2} L_\theta S + \frac{1}{2} = \sqrt{2} \lambda S + \frac{1}{2},
\]
\[
(2.22)
\]

Substituted \( \sqrt{2} \lambda \) with \( \lambda \), \( \sqrt{2} R - \frac{1}{2} \) with \( R - \frac{1}{2} \), Eqs. (2.22) can be written as
\[
\Delta_r \frac{1}{2} D_0 R - \frac{1}{2} = \lambda \Delta_r \frac{1}{2} R + \frac{1}{2}, \quad \Delta_r \frac{1}{2} D_0^+ \Delta_r \frac{1}{2} R + \frac{1}{2} = \lambda R - \frac{1}{2},
\]
\[
\Delta_r \frac{1}{2} L_\theta S + \frac{1}{2} = - \lambda S - \frac{1}{2}, \quad \Delta_r \frac{1}{2} L_\theta S + \frac{1}{2} = \lambda S + \frac{1}{2}.
\]
\[
(2.23)
\]

Set
\[
\Delta_r \frac{1}{2} R + \frac{1}{2} = P + \frac{1}{2}, \quad R - \frac{1}{2} = P - \frac{1}{2},
\]
\[
(2.24)
\]
Eqs. (2.23) reduce to
\[
\Delta^\frac{1}{2} D_0^+ P_{+\frac{1}{2}} = \lambda P_{-\frac{1}{2}}, \quad \Delta^\frac{1}{2} D_0 P_{-\frac{1}{2}} = \lambda P_{+\frac{1}{2}},
\]
(2.25)
\[
\Delta^\frac{1}{2} L_{\frac{1}{2}} S_{+\frac{1}{2}} = -\lambda S_{-\frac{1}{2}}, \quad \Delta^\frac{1}{2} L_{\frac{1}{2}}^+ S_{-\frac{1}{2}} = \lambda S_{+\frac{1}{2}}.
\]
(2.26)

Introducing the tortoise coordinate transformation from the radial variable \( r \) to the tortoise coordinate \( r_* \) which is given by
\[
\frac{d}{dr_*} = \frac{\Delta_r d}{\bar{\omega}^2 dr},
\]
(2.27)
where
\[
\bar{\omega}^2 = r^2 + a^2 - \frac{am}{\omega}.
\]
(2.28)

We set
\[
D_0 = \frac{\bar{\omega}^2}{\Delta_r} \left( \frac{d}{dr_*} + i\sigma \right),
\]
(2.29)
where
\[
\sigma = -\Sigma\omega,
\]
(2.30)
and
\[
D_0^+ = \frac{\bar{\omega}^2}{\Delta_r} \left( \frac{d}{dr_*} - i\sigma \right).
\]
(2.31)

Eqs. (2.25) is reduced to
\[
\left( \frac{d}{dr_*} - i\sigma \right) P_{+\frac{1}{2}} = \lambda \frac{\Delta^\frac{1}{2}}{\bar{\omega}^2} P_{-\frac{1}{2}},
\]
(2.32)
\[
\left( \frac{d}{dr_*} + i\sigma \right) P_{-\frac{1}{2}} = \lambda \frac{\Delta^\frac{1}{2}}{\bar{\omega}^2} P_{+\frac{1}{2}}.
\]
(2.33)

By setting
\[
Z_\pm = P_{+\frac{1}{2}} \pm P_{-\frac{1}{2}},
\]
(2.34)
eq (2.32) and (2.33) changed to be
\[
\left( \frac{d}{dr_*} - \lambda \frac{\Delta^\frac{1}{2}}{\bar{\omega}^2} \right) Z_+ = i\sigma Z_-,
\]
(2.35)
\[
\left( \frac{d}{dr_*} + \lambda \frac{\Delta^\frac{1}{2}}{\bar{\omega}^2} \right) Z_- = i\sigma Z_+.
\]
(2.36)
From eq. (2.35) and (2.36), we obtain the radial wave equation

\[
\left( \frac{d^2}{d\tau^2} + \sigma^2 \right) Z_{\pm} = V_{\pm} Z_{\pm},
\]

(2.37)

where

\[
V_{\pm} = \lambda^2 W^2 \pm \lambda \frac{dW}{d\tau} \\
W = \frac{\Delta^{\frac{1}{2}}}{\omega^2}
\]

(2.38)

\(\lambda\) in Eqs. (2.23)-(2.38) is a separation constant and \(\sigma\) is the QNMs of the black hole. We derived the expression of \(\lambda\) at the limit of small cosmological constant \(\Lambda\) and slow rotating black hole in Appendix A. It can be written as a function of \(l\) (or \(j\)), \(m\), \(a\) and \(\sigma\).

### 3 PROPERTIES OF MASSLESS DIRAC FIELD EFFECTIVE POTENTIAL

From the Schrödinger-like equations in eq. (2.37), we can evaluate the QNMs. The form of \(V_+\) and \(V_-\) shown in eq. (2.38) are super-symmetric partners derived from the same super-potential \(W\) [12]. Ref [13] has proved that potentials related in this ways have the same spectral of QNMs. Thus we deal with eq. (2.37) with potential \(V_+\) in evaluating the QNMs. The effective potential also depends on \(\sigma\).

The QNMs are decided by the effective potential. Here we analyze the dependence of the effective potential on parameters \(m, l, Q, \Lambda, a\) and \(\sigma\). The effective potential as a function of \(r\) is plotted for some configurations of \(m, l, Q, \Lambda, a\) and \(\sigma\) in Fig. 1-6. From these figures, we can see that the dependence of \(V\) on \(l, \Lambda, Q\) is stronger than on \(m, a\) and \(\sigma\). Because there exists cosmological a constant \(\Lambda\), the space time possesses two horizons: the black hole horizon \(r = r_e\) and the cosmological horizon \(r = r_c\). While \(r\) varies from \(r_e\) to \(r_c\), the effective potential \(V\) reduces to zero.

Because of the rotation of the Kerr black hole, the azimuthal degeneracy of magnetic quantum number \(m\) is destroyed. On the Fig. 1 we show the dependence of effective potential on \(m\), and find that negative \(m\) will increase the maximum values of the effective potential and decrease the position of the peak.

In the Fig. 2 we show the dependence of the effective potential on angular momentum number \(l\). It is clear the peak value and position of the potential increase with \(l\).

Fig. 3 shows the dependence of the effective potential on the electric charge \(Q\) of
Figure 1: Variation of the effective potential for massless neutrino with $\Lambda = 0.01, \sigma = 1, a = 0.1, Q = 0.1, l = 5, j = \frac{9}{2}, m = \frac{9}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{9}{2}$.

Figure 2: Variation of the effective potential for massless neutrino with $\Lambda = 0.01, \sigma = 1, a = 0.1, Q = 0.1, m = \frac{1}{2}, l = 1, 2, 3, 4, 5, j = l - 1/2$. 
Figure 3: Variation of the effective potential for massless neutrino with $\Lambda = 0.01, \sigma = 1, a = 0.1, m = \frac{9}{2}, l = 5, j = \frac{9}{2}, Q = 0, 0.1, 0.3, 0.6, 0.9$.

Figure 4: Variation of the effective potential for massless neutrino with $\sigma = 1, a = 0.1, m = \frac{9}{2}, l = 5, j = \frac{9}{2}, Q = 0.1, \Lambda = 0, 0.01, 0.02, 0.05, 0.1$. 


Figure 5: Variation of the effective potential for massless neutrino with $\Lambda = 0.01, \sigma = 1, Q = 0.1, m = \frac{9}{2}, l = 5, j = \frac{9}{2}, a = 0, 0.05, 0.1, 0.15, 0.2$.

Figure 6: Variation of the effective potential for massless neutrino with $\Lambda = 0.01, a = 0.1, Q = 0.1, m = \frac{9}{2}, l = 5, j = \frac{9}{2}, \omega = 1, 10, 20, 100, 200$. 
black hole. The electric charge of black hole will increase the peak value, decrease the position of the peak and change the position of black hole horizon.

Fig.4 shows the dependence of effective potential on the cosmological constant $\Lambda$. Increasing of $\Lambda$ reduces the peak value of the effective potential, decreases the cosmological horizon radius $r_c$ and increases the black hole horizon radius $r_e$.

In the Fig.5 we show the dependence of the effective potential on $a$. For the positive value of $m$, rotation of the black hole reduce the peak value and increase the position of the peak.

For the limit of small $a$ and the perturbation method used in Appendix A, the effective of rotation to the separation constant $\lambda$ is small. When we consider the dependence of the effective potential on $\sigma$, we neglect the change of $\lambda$. For $\lambda = 5$, the Fig.6 suggests the potential varies slowly as $\sigma$ increases and approaches a limiting position.

4 MASSLESS NEUTRINO QNMS OF A KERR-NEWMAN-DE SITTER BLACK HOLE

In this section, we evaluate the QNMs by using Pöschl-Teller potential approximation instead of more popular WKB approximation. Konoplya [17] used sixth-order WKB approximation to calculated the QNMs of a D-dimensional Schwarzschild black hole and compared with the result of third-order WKB approximation. Zhidenko [18] calculated QNMs of a Schwarzschild-de Sitter black hole by using sixth-order WKB approximation and the approximation by Pöschl-Teller potential. The results of Ref [17] and [18] show that for large $l$ and $\Lambda$ Pöschl-Teller potential approximation can give results agree well with the sixth-order WKB approximation.

Proposed by Ferrari and Mashhoon [14], the Pöschl-Teller potential approximation method use Pöschl-Teller approximate potential

$$V_{PT} = \frac{V_0}{\cosh^2 (r^*/b)}.$$  \hspace{1cm} (4.1)

It contains two free parameters ($V_0$ and $b$) which are used to fit the height and the second derivative of the potential $V(r^*)$ at the maximum

$$\frac{1}{b^2} = -\frac{1}{2V_0} \left[ \frac{d^2 V}{dr^2} \right]_{r\rightarrow r_0}.$$  \hspace{1cm} (4.2)

The QNMs of the Pöschl-Teller potential can be evaluated analytically:

$$\sigma = \frac{1}{b} \left[ \sqrt{V_0 b^2 - \frac{1}{4}} - \left( n + \frac{1}{4} \right) i \right],$$  \hspace{1cm} (4.3)
where \( n \) is the mode number and \( n < l \) for low-laying modes. From eq. (4.3), the real parts of the QNMs are independent of \( n \). This relates to how to approximates the effective potential.

The effective potential also depends on \( \sigma \). This will complicate matters in eq. (4.3) because there are \( \sigma \) dependence on both sides of the equation. Thus we cannot obtain \( \sigma \) directly. For slowly rotating black hole with little cosmological constant \( \Lambda \), we can expand separation constant \( \lambda \) as series of \( a \ll 1 \), and expand the effective potential in power series of \( a \).

Firstly we express the position of the peak of the effective potential as series up to order \( a^5 \),

\[
    r_{\text{max}} = r_0 + r_1 a + r_2 a^2 + r_3 a^3 + r_4 a^4 + r_5 a^5
\]

and

\[
    0 = V'(r_{\text{max}}) = V'(r_0) + \Sigma_0 V''(r_0) + \frac{1}{2} \Sigma_0^2 V'''(r_0)
    + \frac{1}{6} \Sigma_0^3 V^{(4)}(r_0)
    + \frac{1}{24} \Sigma_0^4 V^{(5)}(r_0)
    + \frac{1}{120} \Sigma_0^5 V^{(6)}(r_0),
\]

where \( r_0 \) is the position of the peak of effective potential for the non-rotating black hole. Eq. (2.38) for the non-rotating black hole case, show that the expression of effective potential \( V \) does not depend on \( \sigma \) and can be solved independently. We evaluate the coefficients \( r_i \)'s order by solving this equation. The expression of \( r_{\text{max}} \) contains \( a \) and unknown \( \sigma \). We also expand \( \sigma \) as \( \sigma = \sigma_0 + \sigma_1 a + \sigma_2 a^2 + \sigma_3 a^3 + \sigma_4 a^4 + \sigma_5 a^5 \) and plug in the expansion for \( r_{\text{max}} \), and then expand the derivation of the potential \( V^{(n)}_0 \) performed with respect to \( r_0 \). We plug all these expansions back to eq. (4.3) and solve the coefficients \( \sigma_i \)'s self-consistently order by order in \( a \).

Now we evaluate massless neutrino QNMs of a Kerr-Newman-de Sitter black hole by using Pöshl-Teller potential approximation for \( \Lambda = 0 \), \( Q = 0.1 \), and \( a = 0.1 \), plot the results in Fig. 7 and list them in Appendix B. The results show that the real parts of the QNMs increase with \( l \) and the magnitude of the imaginary parts increase with \( n \).

Figures of QNMs varying with \( a \) is plotted in the Fig. 8 and plot the real and imaginary parts of QNMs as a function of \( a \) on the Fig. 8. We vary \( a \) from 0 to 0.2 to satisfy the condition \( a \ll 1 \). Because of the spherical symmetry of a non-rotating black hole, the QNMs is the azimuthal degenerate which can be verified in Fig. 8 and Fig. 9. \( a = 0 \) means non-rotating case, which is Reissner-Nordström-de Sitter black hole here. Different values of \( m \) have the same QNMs in the Fig. 8 and Fig. 9. They also clearly display the split of azimuthal degeneracy as \( a \) increasing from 0 to 0.2.

From the Fig. 9 we see that for a rotating black hole, the real parts of QNMs is also
Figure 7: QNMs of massless neutrino for $\Lambda = 0.01, Q = 0.1, a = 0.1$.

Figure 8: QNMs of massless neutrino vary with $a$ for $\Lambda = 0.01, Q = 0.1, l = 3, j = \frac{5}{2}, m = \pm \frac{5}{2}, \pm \frac{3}{2}, \pm \frac{1}{2}$.
Figure 9: Real and imaginary parts of the massless neutrino QNMs as a function of $\alpha$ for $\Lambda = 0.01, Q = 0.1, l = 3, j = \frac{5}{2}, m = \pm \frac{5}{2}, \pm \frac{3}{2}, \pm \frac{1}{2}$. 
related to the $n$ though we use the Pöschl-Teller potential approximation. For a slowly rotating Kerr-Newman-de Sitter black hole with little value of cosmological constant $\Lambda$, the separation constant $\lambda$ can be written as eq. (A.17) and the function of $\sigma$ which relates to $n$. As to the increasing of $a$, the real parts of QNMs increase for positive $m$ and decrease for negative $m$, while the split of real parts for negative $m$ is bigger than the positive case with the same magnitude of $m$. The split of real parts for $n$ increase with $a$ and the magnitude of $m$. $n$ increases the real parts for the positive $m$ and decreases the imaginary parts. The real parts split of $n$ for positive $m$ is bigger than imaginary case. For the negative $m$, the larger magnitude of $m$ change the imaginary parts more. The rotation increases the magnitude of imaginary parts and the split of different values of $m$. For the positive $m$ case, it is more complex but the Fig. 10 shows that the imaginary parts of different positive $m$ for different $n$ trend to the same values and the tendency is more clearly for large $n$.

We plot the image of QNMs varying with $Q$ in the Fig. 10 and plot the real and imaginary parts of QNMs as a function of $Q$ on the Fig. 11. The real parts of QNMs increase with $Q$, the split of different values of $n$ decrease first and increase later. For example, while $m = -\frac{5}{2}$ the real parts of $n = 0$ is larger than $n = 2$ for $Q = 0$ and the other way round for $Q = 0.9$. The split of imaginary parts of different $m$ for the same $n$ increase with $Q$.

In Fig. 12 and Fig. 13 we plot image of QNMs varying with cosmological constant

![Figure 10: QNMs of massless neutrino vary with $Q$ for $\Lambda = 0.01, a = 0.1, l = 3, j = \frac{5}{2}, m = \pm \frac{5}{2}$.](image-url)
Figure 11: Real and imaginary parts of the massless neutrino QNMs as a function of $Q$ for $\Lambda = 0.01, a = 0.1, l = 3, j = \frac{5}{2}, m = \pm \frac{5}{2}$.

Figure 12: QNMs of massless neutrino vary with $\Lambda$ for $Q = 0.1, a = 0.1, l = 3, j = \frac{5}{2}, m = \pm \frac{5}{2}$. 

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Λ and the real and imaginary parts of QNMs as a function of Λ. Fig.4 shows that Λ influence the effective potential more than other parameters. The real parts of QNMs and the magnitude of imaginary parts decrease with Λ. Just like varying with Q the split of different values of n decrease first and increase later. The split of imaginary parts of different m for the same n increase with Q. For a sufficient large Λ, the imaginary parts split of different m will be larger than that of different n, which means that in the limit of the near extreme Λ term for a slowly rotating black hole, the imaginary parts of the QNMs are mostly determined by m rather than n. This is why in Fig.12 the lines of $m = \frac{5}{2}$ and $m = -\frac{5}{2}$ seem to approach two dots for different m.

![Figure 13: Real and imaginary parts of the massless neutrino QNMs as a function of Λ for $Q = 0.1, a = 0.1, l = 3, j = \frac{5}{2}, m = \pm \frac{5}{2}$.](image)

Figure 13: Real and imaginary parts of the massless neutrino QNMs as a function of Λ for $Q = 0.1, a = 0.1, l = 3, j = \frac{5}{2}, m = \pm \frac{5}{2}$.

## 5 CONCLUSIONS AND DISCUSSIONS

We have evaluated low-laying massless neutrino QNMs of a Kerr-Newman-de Sitter black hole by using Pöshl-Teller potential approximation. We adopt a further approximation by making perturbative expansions for all the quantities in powers of parameter $a$.

In general, the real parts of QNMs increase with $l$. The magnitude of the imaginary
parts increase with $n$. A character feature of the QNMs of rotating black holes is the split of the azimuthal degeneracy for different values $m$. This is clearly displayed in Figs. 7, 8, 9. For a Kerr-Newman-de Sitter black hole, massless neutrino perturbation of large $\Lambda$, positive $m$ and small value of $n$ will decay slowly.

We can expand this methods to others black holes. All these works will enrich our knowledge about QNMs of different kind of black holes and give direct identification to distinguish the kind of black holes, through gravitational wave detectors in future.

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\section{Angular Eigenfunctions and Eigenvalues}

The angular equations (2.26) can be combined into

$$\Delta_\theta^{\frac{1}{2}} L_\theta^{\frac{1}{2}} \left[ \Delta_\theta^{\frac{1}{2}} L_\theta^{\frac{1}{2}} S_{-\frac{1}{2}} \right] + \lambda^2 S_{-\frac{1}{2}} = 0, \quad (A.1)$$

and a similar equation for $S_{+\frac{1}{2}}$ obtained by replacing $\theta$ by $\pi - \theta$. For small value of cosmological constant $\Lambda$, we expand eq. (A.1) as series of $\Lambda$, say by first-order expansion and

$$\left[ \mathcal{D}_0 + \mathcal{D}_1 + \left( \frac{1}{3} a^2 \cos^2 \theta \mathcal{D}_0 + \mathcal{D}_2 \right) \lambda + \lambda^2 \right] S_{-\frac{1}{2}} = 0, \quad (A.2)$$

where

$$\begin{align*}
\mathcal{D}_0 & \equiv \frac{d^2}{d\theta^2} + \cot \theta \frac{d}{d\theta} - \left( \frac{m - 1}{2} \cos^2 \theta \right)^2 - \frac{1}{2}, \\
\mathcal{D}_1 & \equiv 2a\omega m - a\omega \cos \theta - a^2\omega^2 \sin^2 \theta, \\
\mathcal{D}_2 & \equiv \frac{2}{3} a^2 \cos \theta \sin \theta \frac{d}{d\theta} - \frac{2}{3} a^2 m^2 - \frac{1}{3} a^3 \omega \cos \theta \\
& - \frac{2}{3} a^4 \omega^2 \sin^2 \theta + \frac{4}{3} a^3 \omega m + \frac{1}{6} a^2 \sin^2 \theta \\
& - \frac{2}{3} a^3 \omega m \cos^2 \theta + \frac{1}{3} a^4 \omega^2 \sin^2 \theta \cos^2 \theta \\
& - \frac{1}{3} a^3 \omega \cos \theta \sin^2 \theta - \frac{1}{2} a^2 \cos^2 \theta. \quad (A.3)
\end{align*}$$
The operator $D_0$ has no dependence on $a$ and the solution for $S_{-\frac{1}{2}}$ of equation

$$D_0 S_{-\frac{1}{2}} = -E^2 S_{-\frac{1}{2}}$$

(A.4)

can be written in terms of the standard spin-weighted spherical harmonics (Newman and Penrose 1966 [27]; Goldberg et al. 1967 [28])

$$S_{-\frac{1}{2}} (\cos \theta) e^{im\varphi} = -\frac{1}{2} Y_{jm} (\theta, \varphi), \quad E^2 = \left(j + \frac{1}{2}\right)^2,$$

(A.5)

where $j = \frac{(2l-1)}{2}$ with positive integer $l$. In general, the function $s Y_{jm} (\theta, \varphi)$ are defined in the following ways:

$$s Y_{jm} (\theta, \varphi) = \left(\frac{2j+1}{4\pi}\right)^{\frac{1}{2}} D^{-}_{sm}(\varphi, \theta, 0)$$

$$= \left[ \left(\frac{2j+1}{4\pi}\right) \frac{(j+m)! (j-m)!}{(j+s)! (j-s)!} \right]^{\frac{1}{2}} \left(\frac{\theta}{2}\right)^{2j} e^{im\varphi}$$

$$\times \sum_n \left(\frac{j-s}{n}\right) \left(\frac{j+s}{n+s-m}\right) (-1)^{j-s-n} \left(\cot \frac{\theta}{2}\right)^{2n+s-m},$$

(A.6)

where $D^{-}_{sm}(\varphi, \theta, 0)$ are the elements of the matrix representations of the rotation group which satisfy

$$D^{j}_{\mu_1m_1}(\alpha, \beta, \gamma) D^{j}_{\mu_2m_2}(\alpha, \beta, \gamma)$$

$$= \sum_j \langle j_1 j_2 \mu_1 \mu_2 | j \mu_1 + \mu_2 \rangle \delta_j (j_1 j_2 m_1 m_2 | j m_1 + m_2 \rangle$$

$$D^{j}_{\mu_1+\mu_2,m_1+m_2}(\alpha, \beta, \gamma),$$

(A.7)

$$\int d\Omega' D^{j}_{\mu_1 m_1}(\alpha, \beta, \gamma) D^{j}_{\mu_2 m_2}(\alpha, \beta, \gamma) = \frac{8\pi^2}{2j_1 + 1} \delta_{j_1 j_2} \delta_{\mu_1 \mu_2} \delta_{m_1 m_2},$$

(A.8)

where $\langle j_1 j_2 m_1 m_2 | j m_1 + m_2 \rangle$ are the Clebsch-Gordon coefficients and

$$j = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|,$$

(A.9)

$$\int d\Omega' \equiv \int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} \sin \beta d\beta.$$

(A.10)

The function $s Y_{jm} (\theta, \varphi)$ satisfies the parity and the orthogonality relations

$$s Y^*_{jm} (\theta, \varphi) = (-1)^{s+m} s Y_{j-m}(\theta, \varphi),$$

(A.11)

$$\int s Y^*_{jm'} (\theta, \varphi) s Y_{jm}(\theta, \varphi) d\Omega = \delta_{jm'},$$

(A.12)

$$\int d\Omega = \int_0^{\pi} \int_{\varphi=0}^{2\pi} \sin \theta d\theta d\varphi.$$
With the customary definition,

\[ sY_{jm}(\theta, \varphi) = \left[ \frac{(2j + 1)(j - m)!}{4\pi(j + m)!} \right]^{\frac{1}{2}} sP_{jm}(\theta)e^{im\varphi}, \]  

(A.14)

the operators \( \mathcal{L}^+_s \) and \( \mathcal{L}^+_s \), for \( a\omega = 0 \), are 'raising' and 'lowering' operators:

\[
(\partial_\theta - m \csc \theta - s \cot \theta) sP_{jm} = -[(j - s)(j + s + 1)]^{\frac{1}{2}} s+1P_{jm} \]  

(A.15)

\[
(\partial_\theta + m \csc \theta + s \cot \theta) sP_{jm} = +[(j + s)(j - s + 1)]^{\frac{1}{2}} s-1P_{jm} . \]  

(A.16)

For small values of \( a \) and \( \Lambda \), we can view \( H' = \mathcal{D}_1 + \left( \frac{1}{3} a^2 \cos^2 \theta \mathcal{D}_0 + \mathcal{D}_2 \right) \Lambda \) as a perturbation operator and obtain the results by using ordinary perturbation theory, say by the first-order expansion

\[
\chi^2 = \left( j + \frac{1}{2} \right)^2 - \langle -\frac{1}{2}jm \mid \mathcal{D}_1 + \left( \frac{1}{3} a^2 \cos^2 \theta \mathcal{D}_0 + \mathcal{D}_2 \right) \Lambda \mid -\frac{1}{2}jm \rangle + \cdots . \]  

(A.17)

here

\[
\langle -\frac{1}{2}jm \mid \mathcal{D}_1 + \left( \frac{1}{3} a^2 \cos^2 \theta \mathcal{D}_0 + \mathcal{D}_2 \right) \Lambda \mid -\frac{1}{2}jm \rangle \\
= \int \left( -\frac{1}{2}Y_{jm}^* \left( \mathcal{D}_1 + \left( \frac{1}{3} a^2 \cos^2 \theta \mathcal{D}_0 + \mathcal{D}_2 \right) \Lambda \right) -\frac{1}{2}Y_{jm} \right) d\Omega \\
= \left[ 2a\omega m + \left( \frac{4}{3} a^3 m - \frac{2}{3} a^2 m^2 \right) \Lambda \right] \langle -\frac{1}{2}jm \mid -\frac{1}{2}jm \rangle \\
- \left[ a^2 \omega^2 + \left( \frac{2}{3} a^4 \omega^2 - \frac{1}{6} a^2 \right) \Lambda \right] \langle -\frac{1}{2}jm \mid \sin^2 \theta \mid -\frac{1}{2}jm \rangle \\
+ \left[ \frac{1}{3} a^2 \left( j + \frac{1}{2} \right)^2 - \frac{2}{3} a^3 m - \frac{1}{2} a^2 \right] \Lambda \langle -\frac{1}{2}jm \mid \cos^2 \theta \mid -\frac{1}{2}jm \rangle \\
- \left( a\omega + \frac{1}{3} a^3 \omega \Lambda \right) \langle -\frac{1}{2}jm \mid \cos \theta \mid -\frac{1}{2}jm \rangle \\
- \frac{2}{3} a^2 \Lambda \langle -\frac{1}{2}jm \mid \cos \theta \sin \theta \partial_\theta \mid -\frac{1}{2}jm \rangle \\
+ \frac{1}{3} a^4 \omega \Lambda \langle -\frac{1}{2}jm \mid \sin^2 \theta \cos^2 \theta \mid -\frac{1}{2}jm \rangle \\
- \frac{1}{3} a^3 \omega \Lambda \langle -\frac{1}{2}jm \mid \cos \theta \sin^2 \theta \mid -\frac{1}{2}jm \rangle . \]  

(A.19)
\[
\begin{align*}
\langle \frac{1}{2} jm | \cos \theta | -\frac{1}{2} jm \rangle &= \langle j1m0| jm \rangle \langle j1\frac{1}{2}| j\frac{1}{2} \rangle = \frac{m}{2} \frac{1}{j(j+1)}, \\
\langle \frac{1}{2} jm | \cos^2 \theta | -\frac{1}{2} jm \rangle &= \frac{1}{3} + \frac{2}{3} \langle j2m0| jm \rangle \langle j2\frac{1}{2}| j\frac{1}{2} \rangle \\
&= \frac{1}{3} + \frac{2}{3} \frac{3m^2 - j(j+1)}{(2j-1)j(j+1)(2j+3)}, \\
\langle \frac{1}{2} jm | \sin^2 \theta | -\frac{1}{2} jm \rangle &= 2 \frac{3m^2 - j(j+1)}{(2j-1)j(j+1)(2j+3)}, \\
\langle \frac{1}{2} jm | \cos \theta \sin \theta | -\frac{1}{2} jm \rangle \\
&= \frac{2}{15} + \frac{2}{21} \langle j2m0| jm \rangle \langle j2\frac{1}{2}| j\frac{1}{2} \rangle - \frac{8}{35} \langle j4m0| jm \rangle \langle j4\frac{1}{2}| j\frac{1}{2} \rangle \\
&= \frac{2}{15} + \frac{2}{21} \frac{3m^2 - j(j+1)}{(2j-1)j(j+1)(2j+3)} - \frac{8}{35} \frac{(2j+1)(2j-4)!}{(2j+5)!} \\
&\times \{(j+m)(j+m-1)(j+m-2)(6j-8) - (j-m)(j-m-1)(9j+m)(j+m) - (j-m-2)(6j+m+3)\} \\
&\times \left\{ \left( j + \frac{1}{2} \right) \left( j - \frac{1}{2} \right) \left[ \left( j - \frac{3}{2} \right) (6j-1) - 9 \left( j - \frac{1}{2} \right) \left( j - \frac{3}{2} \right) \right] \\
&- \left( j - \frac{1}{2} \right) \left( j - \frac{3}{2} \right) \left[ 9 \left( j + \frac{1}{2} \right) \left( j + \frac{3}{2} \right) - \left( j - \frac{5}{2} \right) (6j+7) \right] \right\}, \\
\langle \frac{1}{2} jm | \sin \theta \cos \theta \partial_{\theta} | -\frac{1}{2} jm \rangle \\
&= \frac{2}{5} \langle j1m0| jm \rangle \langle j1\frac{1}{2}| j\frac{1}{2} \rangle - \frac{2}{5} \langle j3m0| jm \rangle \langle j3\frac{1}{2}| j\frac{1}{2} \rangle \\
&= \frac{m}{5} \frac{1}{j(j+1)} - \frac{8}{5} \frac{(2j+1)(2j-3)!}{(2j+4)!} \\
&\times [(j+m)(j+m-1)(4j-5m+1) - (j-m)(j-m-1)(4j+5m+1)] \\
&\times \left[ \left( j + \frac{1}{2} \right) \left( j - \frac{1}{2} \right) \left( 4j - \frac{3}{2} \right) - \left( j - \frac{1}{2} \right) \left( j - \frac{3}{2} \right) \left( 4j + \frac{7}{2} \right) \right].
\end{align*}
\]
### QNMS of Kerr-Newman-de Sitter Black Hole

Table 1: QNMs of massless neutrino for $\Lambda = 0.01, Q = 0.1, a = 0.1$

| $l$ | $n$ | $m = \frac{1}{2}$ | $m = -\frac{1}{2}$ | $m = \frac{3}{2}$ | $m = -\frac{3}{2}$ | $m = \frac{5}{2}$ |
|-----|-----|------------------|------------------|------------------|------------------|------------------|
| 1   | 0   | 0.1851-0.1002$i$ | 0.1748-0.09909$i$ | 0.3792-0.09474$i$ | 0.3535-0.09454$i$ |                   |
| 2   | 0   | 0.3723-0.09453$i$ | 0.3639-0.09442$i$ | 0.3792-0.09474$i$ | 0.3535-0.09454$i$ |                   |
| 3   | 0   | 0.5561-0.09302$i$ | 0.5481-0.09301$i$ | 0.5632-0.09310$i$ | 0.5388-0.09310$i$ | 0.5694-0.09320$i$ |
| 4   | 0   | 0.7399-0.09251$i$ | 0.7320-0.09251$i$ | 0.7471-0.09254$i$ | 0.7232-0.09256$i$ | 0.7536-0.09260$i$ |
| 5   | 0   | 0.9238-0.09228$i$ | 0.9159-0.09228$i$ | 0.9310-0.09230$i$ | 0.9073-0.09232$i$ | 0.9377-0.09233$i$ |
| 1   | 0   | 0.9239-0.2768$i$ | 0.9158-0.2769$i$ | 0.9314-0.2768$i$ | 0.9069-0.2771$i$ | 0.9383-0.2769$i$ |
| 2   | 0   | 0.9241-0.4612$i$ | 0.9156-0.4616$i$ | 0.9320-0.4611$i$ | 0.9064-0.4621$i$ | 0.9393-0.4610$i$ |
| 3   | 0   | 0.9243-0.6469$i$ | 0.9154-0.6463$i$ | 0.9326-0.6451$i$ | 0.9058-0.6474$i$ | 0.9404-0.6447$i$ |
| 4   | 0   | 0.9246-0.8298$i$ | 0.9152-0.8312$i$ | 0.9333-0.8288$i$ | 0.9053-0.8329$i$ | 0.9415-0.8281$i$ |
| 3   | 0   | 0.5279-0.09336$i$ |                   |                   |                   |                   |
| 1   | 0   | 0.5266-0.2811$i$ |                   |                   |                   |                   |
| 2   | 0   | 0.5256-0.4699$i$ |                   |                   |                   |                   |
| 4   | 0   | 0.7132-0.09269$i$ | 0.7595-0.09267$i$ | 0.7020-0.09295$i$ |                   |                   |
| 1   | 0   | 0.7123-0.2785$i$ | 0.7606-0.2777$i$ | 0.7008-0.2795$i$ |                   |                   |
| 2   | 0   | 0.7113-0.4651$i$ | 0.7625-0.4619$i$ | 0.6995-0.4672$i$ |                   |                   |
| 3   | 0   | 0.7104-0.6523$i$ | 0.7645-0.6452$i$ | 0.6987-0.6554$i$ |                   |                   |
| 5   | 0   | 0.8979-0.09240$i$ | 0.9439-0.09238$i$ | 0.8875-0.09254$i$ | 0.9497-0.09244$i$ | 0.8761-0.09277$i$ |
| 1   | 0   | 0.8973-0.2774$i$ | 0.9447-0.2770$i$ | 0.8867-0.2780$i$ | 0.9507-0.2771$i$ | 0.8750-0.2788$i$ |
| 2   | 0   | 0.8964-0.4629$i$ | 0.9460-0.4610$i$ | 0.8855-0.4641$i$ | 0.9523-0.4611$i$ | 0.8736-0.4658$i$ |
| 3   | 0   | 0.8955-0.6489$i$ | 0.9476-0.6445$i$ | 0.8844-0.6509$i$ | 0.9543-0.6445$i$ | 0.8725-0.6535$i$ |
| 4   | 0   | 0.8948-0.8351$i$ | 0.9492-0.8275$i$ | 0.8836-0.8379$i$ | 0.9564-0.8272$i$ | 0.8717-0.8414$i$ |
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