Nonequilibrium Josephson effect in mesoscopic ballistic multiterminal SNS junctions

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We present a detailed study of nonequilibrium Josephson currents and conductance in ballistic multiterminal SNS-devices. Nonequilibrium is created by means of quasiparticle injection from a normal reservoir connected to the normal part of the junction. By applying a voltage at the normal reservoir the Josephson current can be suppressed or the direction of the current can be reversed. For a junction longer than the thermal length, $L \gg \xi_T$, the nonequilibrium current increases linearly with applied voltage, saturating at a value equal to the equilibrium current of a short junction. The conductance exhibits a finite bias anomaly around $eV \sim \hbar v_F/L$. For symmetric injection, the conductance oscillates $2\pi$-periodically with the phase difference $\phi$ between the superconductors, with position of the minimum ($\phi = 0$ or $\pi$) dependent on applied voltage and temperature. For asymmetric injection, both the nonequilibrium Josephson current and the conductance becomes $\pi$-periodic in phase difference. Inclusion of barriers at the NS-interfaces gives rise to a resonant behavior of the total Josephson current with respect to junction length with a period $\sim \lambda_F$. Both three and four terminal junctions are studied.

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I. INTRODUCTION

The art of controlling Josephson current transport through mesoscopic superconducting junctions poses many challenges for theory and experiment from both fundamental and applied points of view. Control of Josephson current requires multi-terminal devices - superconducting transistors. One example is the Josephson field effect transistor (JOFET) where control of the Josephson equilibrium current is imposed via an electric gate. Another solution is to connect the normal region to a normal voltage biased reservoir. Recent progress in fabrication of superconducting junctions has brought forward a number of interesting multiterminal structures, e.g. 2DEG junctions, metallic junctions and high-Tc junctions.

Injection of electrons and holes allows nonequilibrium quasiparticle distributions to be maintained in the N-region, making it possible to control the nonequilibrium Josephson current. The problem of nonequilibrium current injection in ballistic junctions is of particular interest: the Josephson current is transported through bands of Andreev levels. This provides means for achieving a dramatic variation of the Josephson current.

The purpose of this paper is to provide a broad description of Josephson current transport through ballistic SNS junctions under conditions of nonequilibrium in the normal region due to contact with a voltage biased normal reservoir. Connection of the normal part of an SNS junction to a normal electron reservoir gives rise to broadening of the Andreev bound levels. Van Wees et al. were the first ones to consider this broadening in perfect SNS junctions and to describe essential aspects of the variation of the Josephson current with voltage, in terms of nonequilibrium population of Andreev levels. Moreover, Wendin and Shumeiko have predicted that nonequilibrium filling of Andreev levels may reveal very large Josephson currents with different directions, and that pumping between levels could reverse the direction of the Josephson current. This has been investigated in detail by Bagwell and coworkers and by Samuelsson et al.

Recently, Morpurgo et al. observed Andreev levels by using the injection lead as a spectroscopic probe. Suppression of the Josephson current due to injection has been demonstrated in both ballistic and diffusive SNS junctions. The physical mechanism of the effect in diffusive junctions is essentially the same as in ballistic junctions. Very recently, Baselmans et. al. were able also to reverse the direction of the Josephson current.

A decisive step beyond the work of van Wees et al. was taken by Samnelsson et al., who showed that an essential aspect is the ability of the scatterer at the injection point to shift the phases of the quasiparticles. In such a case, the connection to the injection lead also affects the form of the wave function of the Andreev resonances, and therefore affects Josephson currents flowing through the resonances. As a result, modification of the Josephson current under injection does not reduce to the effect of non-equilibrium population. This is particularly dramatic for long junctions, where the equilibrium Josephson current is exponentially small at finite temperature.

In contrast, this anomalous nonequilibrium Josephson current does not depend on the length of the junction (long-range Josephson effect). This means that, in principle, a dissipationless current of the order of the equilibrium Josephson current of a short junction can be restored under conditions of filling up all the Andreev levels in the gap. The effect is most pronounced in junctions with a small number of transport modes. This opens up the possibility for a new kind of Josephson transistor where the supercurrent is turned on when the gate voltage is switched from $eV = 0$ to $eV = \Delta$. 

[1]
The complete picture of the nonequilibrium current also includes the current injected into the junction. This injection current is dependent on the properties of the Andreev levels in the junction. It therefore provides information on the nonequilibrium Josephson current. We found that it is closely related to the anomalous current and has similar properties. The injection current has also in itself been at the focus of great interest in recent literature.

The paper is organized as follows. In section II we present a general discussion of the currents in a 3-terminal SNS device. In section III we describe our model based on the stationary BdG equation. We derive all currents in the case of a three terminal junction without barriers at the NS-interface in section IV. In section V we discuss the equilibrium and nonequilibrium Josephson currents, both in a short and long junctions. The effect of barriers at the NS-interfaces is discussed in section VI and the injection current and the conductance are analyzed in section VII. In section VIII we discuss the four terminal junction and how it differs from the three terminal one. Finally, in section IX we present our conclusions.

II. NONEQUILIBRIUM JOSEPHSON CURRENTS

We will consider two junction configurations: 3- and 4-terminal (see Fig. 1). The normal part of the junction is inserted between two superconducting electrodes. The superconducting electrodes are connected with each other to form a loop and the magnetic flux threading the loop allows us to control the phase difference \( \phi = \phi_R - \phi_L \) across the junction.

![Superconducting loop](image)

**FIG. 1.** A schematic picture of the three terminal SNS junction setup under consideration, with a normal reservoir attached to the normal part of the junction. The normal reservoir is connected to the superconducting loop (grounded) via a voltage source biased at \( V \). The right figure shows a close-up of the junction area with the arrows showing the direction of the current flow in the junction.

We consider a junction in the ballistic limit, i.e. when the length \( L = L_2 + L_3 \) of the normal part of the junction is shorter then both the elastic and inelastic scattering lengths, \( L \ll l_e, l_i \). The 3-terminal configuration is an elementary structure which gives all necessary information for understanding also the properties of the 4-terminal junction, to be discussed below. We use a simplified description of the connection point, modeling it by a scattering matrix \( S \) that connects ingoing and outgoing wave function amplitudes:

\[
\Psi_{out} = S \Psi_{in},
\]

with

\[
S = \begin{pmatrix}
\sqrt{1 - 2\epsilon} & \sqrt{\epsilon} & \sqrt{\epsilon} \\
\sqrt{\epsilon} & r & d \\
\sqrt{\epsilon} & d & r
\end{pmatrix},
\]

where \( r \) and \( d \) are reflection and transmission amplitudes for scattering between lead 2 to lead 3 and \( \sqrt{\epsilon} \) is the scattering amplitude from the injection lead 1 to lead 2 or 3. In a multichannel treatment, \( r, d \) and \( \epsilon \) become matrices describing the scattering between the channels. In this paper we however choose to consider a single-mode structure.

In the junction presented in Fig. 1, the current \( I_1 \) injected into the junction from the normal reservoir splits at the connection point. At the NS-interfaces, the normal current is converted into a supercurrent. The supercurrent flows around the loop and is drained at a point connected to the normal reservoir via a voltage source biased at voltage \( V \). There are two major questions about the currents: (i) what is the current \( I_1 \) in injection electrode 1 as function of the applied voltage, and (ii) how is the current split between the arms 2 and 3. The first problem has been discussed earlier [12], the picture is the following: due to Andreev quantization the problem is equivalent to a resonant transmission problem. For weak coupling to the normal reservoir, \( \epsilon \ll 1 \), the probability of an incoming electron to be reflected is large unless its energy coincides with an Andreev level. In such a case, the electron is back scattered as a hole which produces a current density peak. The current as a function of applied voltage between the normal reservoir and the junction (IVC) thus increases stepwise, typical for resonant transport, with position and height of the steps depending on the phase difference between the superconductors.

The current distribution among the left and right arms of the junction is also phase dependent. However, there is a less trivial aspect of the problem related to the Josephson current in the loop. There is no possibility to distinguish the Josephson current which flows along the loop (as the result of an applied phase difference) from the split injection current except in the limit of weak coupling to the external reservoir. In the limit \( \epsilon \ll 1 \) the injection current (\( \sim \epsilon \)) vanishes while the Josephson current remains finite. This allows us to separate the problem of the Josephson current under injection from the problem of splitting of the injection current.

The scattering states carrying the current can qualitatively be described as electrons or holes entering the SNS junction from the injection lead 1, being split at the connection point, scattered back and forth in the junction by
Andreev reflections at the NS-interfaces and normal reflections at the connection point, and then finally leaving the junction, having effectively transported current from one superconductor to the other. When the lifetime of the Andreev resonances is smaller than the inelastic scattering time in the junction, the quasiparticle distribution in the normal region is determined by the Fermi distribution function of the normal reservoir, and the current in the leads \( j = 2 \) or \( 3 \) from injected quasiparticles can be written

\[
I_j = \int_{-\infty}^{\infty} dE(i^e_j n^e + i^h_j n^h)
\]

where \( i^e(h) \) is the current density for injected electrons (holes) and \( n^e(h) = n_F(E \pm eV) \) are the Fermi distribution functions in the normal reservoir, with \( n_F = [1 + \exp(E/kT)]^{-1} \). This current can conveniently be rewritten

\[
I_j = \int_{-\infty}^{\infty} dE \left[ \frac{i^e_j}{2} (n^e + n^h) + \frac{i^h_j}{2} (n^e - n^h) \right] = I^e_j + I^h_j
\]

where \( i^e = i^e + i^h \) and \( i^- = i^e - i^h \). Quasiparticles are also injected from the superconductors for energies above the superconducting gap. Since the superconductors are grounded \( (V = 0) \), the current from the superconductors is an equilibrium current. This current plus the current \( I^+ = I^+ + I^h_2 \) injected from the normal reservoir in absence of applied voltage, is the total equilibrium current. Applying a bias voltage \( (V \neq 0) \), \( I^+ \) becomes the nonequilibrium current due to population of the empty Andreev levels, giving rise to current jumps when the injection energy \( eV \) equals the Andreev level energies (see Fig. 2). This makes it possible to probe the energy of the Andreev levels.

The \( I^- \) part of the current is entirely nonequilibrium current. It partly consists of the injection current; however, there is also a component which does not vanish in the limit of weak coupling to the reservoir: we call this the anomalous Josephson current. This current results from a different form of the Andreev resonance wave functions in the open junction compared with the wave functions of true Andreev bound states. The origin of the anomalous current can qualitatively be described by considering the lowest order quasiparticle classical paths which contribute to the resonances in transparent junctions \( (R \ll 1) \) with perfect NS interfaces.

Consider a resonant state where the most of the electrons move to the left and the holes to the right, only a fraction of them travelling in the opposite direction due to normal scattering at the connection point. An injected electron gives rise to a leftgoing electron in lead 2 with the amplitude \( 1 + e^{i\phi_0} d^* e^{-i\phi_L} \) with \( \phi = \phi_R - \phi_L \) (not taking electron and hole dephasing and the energy dependent phase picked up when Andreev reflecting into account) thus giving a contribution to the current of order \( 1 + RD + Re(rd^* e^{i\phi}) \) (see inset in Fig. 3). Correspondingly, an injected hole gives rise to a rightgoing hole in lead 3 with amplitude \( 1 + e^{-i\phi_0} d^* e^{i\phi_R} r^* \) and a contribution to the current of order \( 1 + RD + Re(rd^* e^{-i\phi}) \) (see right figure in inset in Fig. 3). The difference current \( i^- \) thus contains a part proportional to \( Re(rd^* (e^{i\phi} - e^{-i\phi})) = 2Im(rd^*) \sin(\phi) \), which is the leading term in the anomalous current. At a resonant state where the particles move in the opposite direction, i.e. the electrons to the

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**FIG. 3.** The charge current density for two resonant Andreev levels for injected electrons \( i^e \) (dotted) and holes \( i^h \) (dashed), their sum \( i^+ \) (solid) and difference \( i^- \) (dash-dotted). Note that the difference current \( i^- \) has the same sign for both resonances. Inset: Two lowest order paths for an injected electron (solid) or a hole (dashed) at a resonance. The grey ellipse denotes the effective scatterer due to the three lead connection. The difference of the currents due to these processes is proportional to \( Im(rd^*) \sin(\phi) \), the first order term of the anomalous current.

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**FIG. 2.** The current voltage characteristics (IVC) for \( I^+ \) (upper) and \( I^- \) (lower) for a junction with seven Andreev levels for \( 0 < E < \Delta \). The currents jump every time the voltage \( eV \) is equal to the energy of an Andreev level, typical for resonant transport.
right and the holes to the left, we find from the same arguments that the anomalous current is again proportional to \(2\text{Im}(rd^* - \sin(\phi))\), with the same sign. The anomalous current thus flows in the same direction for all resonances, in contrast to the equilibrium Josephson current which changes sign from one level to the next. The IVC for \(I^-\) is thus a staircase, as shown in Fig. 3 saturating at \(eV > \Delta\) due to the absence of sharp resonances for energies above the superconducting gap. This has a dramatic effect on the long range properties of the Josephson current.

For a long junction \((L \gg \xi_0 = \hbar v_F / \Delta)\), the IVC in Fig. 2 becomes dense, since there is a large number \(\sim L / \xi_0\) of Andreev levels in the junction. The spacing between the Andreev levels is \(\sim \hbar v_F / L\), so at temperatures exceeding the interlevel distance, the current \(I^+\) is averaged to zero while \(I^-\) is reduced to a smooth ramp function. We thus get a current \(I^-\) that increases linearly with voltage up to \(eV = \Delta\) and saturates at a level of the order of the equilibrium Josephson current of a short junction, \(I \sim e\Delta / \hbar\). This current is independent of the length of the junction, since there is a large number of levels \(\sim L\) each carrying a current \(\sim 1/L\).

### III. Calculation of the Current

#### A. General formulation

We consider a three-terminal junction with asymmetric current injection \((L_2 \neq L_3)\) and perfect transmission at the NS interfaces. The junction can be described by the stationary 1-D Bogoliubov-de Gennes (BdG) equation:

\[
\begin{bmatrix} \hat{H}_0 & \Delta \hat{\alpha} \\ \Delta^* \hat{\alpha} & -\hat{H}_0 \end{bmatrix} \Psi = E \Psi
\]

\[
\hat{H}_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - E_F
\]

which gives \(E\) as a departure from \(E_F\). We apply the approximation \((2)\) with \(\Delta(x)\) constant in the superconductors and zero in the normal region.

\[
\Delta(x) = \begin{cases} \Delta e^{i\phi_L} & x < -L_2 \\ 0 & -L_2 < x < L_3 \\ \Delta e^{i\phi_R} & x > L_3 \end{cases}
\]

where the phase difference between the superconductors is \(\phi = \phi_R - \phi_L\). We can then make an ansatz with plane waves in the different regions of the junction. For positive energies \(E > 0\) we put in the normal regions \(j = 1, 2, 3\),

\[
\Psi_j = c_{j,}^+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{ik_j^+x} + c_{2j,}^+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-ik_j^+x} + d_{j,}^- \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{ik_j^-x} + d_{2j,}^- \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-ik_j^-x}
\]

and in the superconductors \(j = L, R\)

\[
\Psi_j = d_{j,}^+ \begin{bmatrix} u e^{i\phi_j} \\ v \end{bmatrix} e^{ik_j^+x} + d_{2j,}^+ \begin{bmatrix} v e^{i\phi_j} \\ u \end{bmatrix} e^{-ik_j^+x} + d_{j,}^- \begin{bmatrix} w e^{i\phi_j} \\ v \end{bmatrix} e^{ik_j^-x} + d_{2j,}^- \begin{bmatrix} v e^{i\phi_j} \\ u \end{bmatrix} e^{-ik_j^-x}
\]

The coherence factors \(u\) and \(v\) are defined as

\[
u(+), v(-) = \begin{cases} \sqrt{\frac{1}{2}(1 \pm \xi/E)} & E > \Delta \\ \sqrt{\frac{1}{2}(E \pm \xi)/\Delta} & E < \Delta \end{cases}
\]

where \(\xi = \sqrt{E^2 - \Delta^2}\) for \(E > \Delta\) and \(\xi = i\sqrt{\Delta^2 - E^2}\) for \(E < \Delta\). The wavevectors are \(q_{e,h} = \sqrt{2m/\hbar^2 \sqrt{E_F \pm \xi}}\) in the superconductors and and \(k_{e,h} = \sqrt{2m/\hbar^2 \sqrt{E_F \pm E}}\) in the normal regions. The wavefunctions are matched at the NS-interfaces and at the injection point. The three-terminal injection point is modeled by the scattering matrix \((3)\) given by Eq. (8). The scattering amplitudes \(\epsilon (0 \leq \epsilon \leq 0.5)\), and \(d\) and \(r\) obey the relations \(R(rd^*) = -\epsilon/2\) and \(D + R = 1 - \epsilon (D = |d|^2, R = |r|^2)\) due to the unitarity of the scattering matrix. Moreover, \(\text{Im}(rd^*) = \sigma \sqrt{RD - \epsilon^2}/4\), with \(\sigma = \pm 1\) dependent on the phase of the scatterer. For simplicity the coupling parameter \(\epsilon\) is chosen real and positive. The scattering amplitudes are assumed to be energy independent, which gives the scattering matrix for hole wavefunction amplitudes \(S_h = S_e^*\).

Assuming \(\Delta \ll E_F\) we make the approximation \(q^e = q^h = k_F\) in the superconductors and \(k^e = k^h = k_F\) in the normal region except in exponentials where we put \(k^{e,h} = k_F \pm E/\hbar v_F\). At energies \(E < \Delta\), only electrons and holes from the normal reservoirs are injected in the junction. For \(E > \Delta\) quasiparticles from the superconductors are also injected. The current density in the three normal regions, which is what is needed to calculate all currents in the junction, are calculated using the quantum mechanical formula \((9)\):

\[
i_j(E) = \frac{e}{\hbar} (|c_j^+|^2 - |c_j^-|^2 - |c_j^{e,h}|^2) + |c_j^{e,h}|^2).
\]

We now define energy dependent phases \(\theta_{2,3} = \gamma - \beta_{2,3}\) in each of the leads 2 and 3, consisting of the phase \(\gamma = \text{arcos}(E/\Delta)\) picked up by the electrons and holes when Andreev reflecting, and the dephasing \(\beta_{2,3} = (k^e - k^h)L_{2,3} = 2E L_{2,3}/(\hbar v_F)\) of the electrons and holes while propagating ballistically through the normal region. Furthermore, it is convenient to separate out the specific features of asymmetry by introducing sum phases \(2\theta = \theta_2 + \theta_3\), \(\beta = \beta_2 + \beta_3\), and the difference phases \(\chi = \theta_2 - \theta_3\), defining essential phase parameters characterizing the junction,

\[
\theta = \gamma - \beta/2 = \text{arcos}(E/\Delta) - EL/(\hbar v_F) \quad 11
\]

\[
\chi = \beta_3 - \beta_2 = 2EL/(\hbar v_F) \quad 12
\]

where \(L = L_2 + L_3\) and \(l = L_3 - L_2\).
The current densities of the scattering states in leads 2 and 3 from electrons \( i_{2,3}^e \) and holes \( i_{2,3}^h \) are then given by

\[
i_{2,3}^{e,h} = - \frac{e}{h} \frac{\sigma}{Z} \left\{ 2D \sin \phi \sin 2\theta \\
+ \left[ \frac{\sigma^2}{4} \frac{D}{Z} - \epsilon^2 \right] \sin \phi (\cos \chi - \cos 2\theta) \\
+ \epsilon \left[ 1 - \cos (2\gamma - \beta_2) + \cos \phi (\cos \chi - \cos 2\theta) \right] \right\} \]

(13)

From Eqs. (13) and (14) it follows that the sum of the electron and hole current densities, \( i^+ = i^e + i^h \), are equal in leads 2 and 3, giving the sum current density

\[
i^+ = i_3^+ = i_2^+ = - \frac{4e}{h} \frac{\sigma}{Z} \left\{ D \sin \phi \sin 2\theta \right\} .
\]

(16)

The difference current densities \( i^- = i^e - i^h \) in leads 2 and 3 are not equal, however. We therefore define the anomalous current density \( i_a \) as that part of the difference current density which survives in the limit \( \epsilon \to 0 \),

\[
i_a = - \frac{4e}{h} \frac{\sigma}{Z} \left\{ \sqrt{RD} - \epsilon^2 / 4 \sin \phi (\cos \chi - \cos 2\theta) \right\} .
\]

(17)

The injection current density \( i_{in} = i_3^- - i_2^- \) is given by,

\[
i_{in} = \frac{4e}{h} \frac{\epsilon^2}{Z} \left\{ \sin^2 \chi + (\cos \chi + \cos \phi) (\cos \chi - \cos 2\theta) \right\}
\]

(18)

and splits asymmetrically between the two horizontal arms 2 and 3,

\[
i_{in,2,3} = \pm \frac{2e}{h} \frac{\epsilon^2}{Z} \left\{ 1 - \cos (2\theta - \beta_{2,3}) + \cos \phi (\cos \chi - \cos 2\theta) \right\}
\]

(19)

From the relations \( i^+ (E) = - i^+ (-E) \) and \( i^- (E) = i^- (-E) \) one can calculate the current densities for all energies inside the gap \( |E| < \Delta \). The continuum current density, for energies outside the gap \( |E| > \Delta \), is calculated in the same way. However, since the Andreev reflection probability decays very rapidly outside the gap, the Andreev resonances become very broad and contribute much less to the current. Only the quasiparticles injected from the superconductors contribute significantly to the current, as will be discussed below. The full formulas for the continuum current density for a symmetric junction \( l = 0 \) is presented in Appendix A.

### B. Weak coupling limit

Throughout the paper we will mainly discuss the situation when the normal reservoir is weakly coupled to the normal part of the junction, \( \epsilon \ll 1 \). In this limit the Andreev resonances are very sharp and the current densities are calculated by evaluating the expression \( \epsilon / Z \) appearing in the Eqs. (10)-(18), in the limit \( \epsilon \to 0 \). This is done in detail in Appendix B, and gives [see Eq. (19)]

\[
\lim_{\epsilon \to 0} \frac{\epsilon}{Z} = \sum_{n, \pm} \frac{\pi}{D} \sin \phi \sin 2\theta \left\{ \frac{dE}{d\phi} \right\} \delta (E - E_n^\pm).
\]

(20)

where \( E_n^\pm \) are the energies of the bound Andreev states. To calculate the current density, information about the bound state energies as well as the derivative of the energy with respect to phase difference is thus needed. The bound state energies are given by the zeros of the denominator \( Z \) [Eq. (13)] at \( \epsilon = 0 \), namely,

\[
\cos 2\theta = R \cos \chi + D \cos \phi.
\]

(21)

The energy of the Andreev levels as a function of phase difference \( \phi \) is plotted in Fig. 4. In the figure it is shown that the Andreev levels appear in pairs, labeled by \( n \), with an upper (+) and a lower (−) level (referring to \( E > 0 \)). The index \( n \) is zero for the pair of levels with positive energy closest to \( E_F \). In the case of one single bound state, the level is labeled by \( E_0^- \).

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**FIG. 4.** Andreev bound state energies as a function of phase difference \( \phi \) for different lengths \( L = 0 \) (left), \( L \sim \xi_0 \) (middle) and \( L \gg \xi_0 \) (right) of the junction with \( D = 0.7 \). Solid lines are for a symmetric junction \( l = 0 \), dashed for an asymmetric one. A gap opens up in the spectrum at \( \phi = 0 \) due to the asymmetry.

The derivative of the bound state energy with respect to phase is obtained by differentiating Eq. (21), giving

\[
\frac{dE_n^\pm}{d\phi} = \frac{D \sin \phi}{2 \sin 2\theta}
\]
\begin{equation}
\times \left( \frac{1}{\sqrt{\Delta^2 - (E_n^\pm)^2}} + \frac{L}{\hbar v_F} + \frac{l}{\hbar v_F} R \frac{\sin \chi}{\sin 2\theta} \right)^{-1}. \tag{22}
\end{equation}

The expression for the sum current density is given by inserting Eqs. (20)-(22) into Eq. (16), giving
\begin{equation}
i^+ = \frac{2e}{h} \sum_{n,\pm} \frac{dE}{d\phi} \delta(E - E_n^\pm), \tag{23}\end{equation}
where the relation \(\text{sgn}\left[(dE/d\phi) \sin \phi \sin 2\theta\right] = -1\) [see Eq. (23)] has been taken into account. The expression (23) coincides with the equation for the Andreev bound state current derived directly from the BdG equation. From the alternating slopes of the energy-phase relation \(E(\phi)\), plotted in Fig. 4, it is clear that the sum current density \((\sim dE/d\phi)\) changes sign between two subsequent Andreev resonances (see Fig. 3).

The anomalous current density \(i_a\) is given directly by inserting Eq. (20) into (17), namely
\begin{equation}
i_a = -\sigma \frac{2e}{h} \text{sgn}(\sin \phi) \sqrt{RD} \times \sum_{n,\pm} \frac{\cos \chi - \cos \phi}{|\sin 2\theta|} \left| \frac{dE}{d\phi} \right| \delta(E - E_n^\pm). \tag{24}\end{equation}

For a symmetric junction \(l = 0, \cos \chi = 1\), the anomalous current density does not change sign as a function of energy, opposite to the sum current density (see Fig. 3). For finite asymmetry, the anomalous current might change sign. However, this does not lead to strong suppression of the total anomalous current, as will be shown below in section VB.

The injection current \(i_{inj} = i_3^+ - i_2^-\) is proportional to \(\epsilon^2\) and thus goes to zero for \(\epsilon \ll 1\). We approximate the injection current in the weak coupling limit by the first order term in \(\epsilon\), given by inserting the expression for \(\epsilon/Z\) in the zero coupling limit into Eq. (20)
\begin{equation}
i_{inj} = \epsilon \frac{8e}{h} \sum_{n,\pm} \frac{\sin^2 \chi + D(\cos \chi - \cos \phi)^2}{|\sin 2\theta|} \left| \frac{dE}{d\phi} \right| \delta(E - E_n^\pm). \tag{25}\end{equation}

The injection current density is closely related to the anomalous current density \(i_a\), in the sense that the injection current density is positive for all energies and values of the phase difference \(\phi\).

C. Structure of the nonequilibrium current

Including the continuum contribution from the superconductors (Appendix A) in Eq. (4), we can finally write down the structure of the total current in each lead:
\begin{equation}
I_j = \int_{-\infty}^{\infty} dE \left[ i^+ + i^* \right] n_F \tag{26}\end{equation}
where \(i^*\) is the current density from the quasiparticles injected from the superconductors. The equilibrium current \((V = 0)\) flowing in leads 2 and 3 is given by
\begin{equation}
I_{eq} = \int dE \left[ i^+ + i^* \right] n_F, \tag{27}\end{equation}
while in lead 1 it is zero. Subtracting the equilibrium current from the total current we get the nonequilibrium current in the horizontal leads 2 and 3. We divide the nonequilibrium current into the the regular current \(I_r\) associated with the nonequilibrium population of the existing resonant states,
\begin{equation}
I_r = \int dE \left[ \frac{i^+}{2} (n^e + n^h) - n_F(E_n^\pm) \right], \tag{28}\end{equation}
the anomalous current \(I_a\) associated with the essential modification of the Andreev states due to the open normal lead,
\begin{equation}
I_a = \int dE \left[ i^+ (n^e - n^h) \right], \tag{29}\end{equation}
and the injected current \(I_1\)
\begin{equation}
I_1 = I_{inj} = \int dE \left[ \frac{i_{inj}}{2} (n^e - n^h) \right]. \tag{30}\end{equation}

With these definitions, the total currents in leads 2 and 3 may be written as
\begin{equation}
I_2 = I_{eq} + I_r + I_a - I_{inj,2}, \tag{31}\end{equation}
\begin{equation}
I_3 = I_{eq} + I_r + I_a + I_{inj,3}, \tag{32}\end{equation}
where \(I_{inj} = I_{inj,2} + I_{inj,3}\). As discussed in Section II, the separation of the anomalous current is arbitrary, and has physical meaning only in the weak coupling limit when \(I_{inj} \rightarrow 0\).

In the weak coupling limit, the integrals in Eqs. (28)-(30) become sums over resonant states
\begin{equation}
I_r = \frac{e}{h} \sum_{n,\pm} \frac{dE_n^\pm}{d\phi} \left[ n^e(E_n^\pm) + n^h(E_n^\pm) - 2n_F(E_n^\pm) \right], \tag{33}\end{equation}
where \(I_{inj} = I_{inj,2} + I_{inj,3}\). As discussed in Section II, the separation of the anomalous current is arbitrary, and has physical meaning only in the weak coupling limit when \(I_{inj} \rightarrow 0\).

The equilibrium current for energies \(|E| < \Delta\) is given by inserting Eq. (23) into (27),
\begin{equation}
I_{eq}^b = \frac{2e}{h} \sum_{n,\pm} \frac{dE_n^\pm}{d\phi} n_F(E_n^\pm), \tag{34}\end{equation}
For energies above the gap, the equilibrium current results from quasiparticles injected from the superconductors only, since this current is the only continuum current being finite in the weak coupling limit (see Appendix A).
IV. JOSEPHSON CURRENT OF A SHORT JUNCTION

For a short junction \( L = l = 0 \), there is exactly one resonance for positive energies \( 0 < E < \Delta \). For no coupling to the normal reservoir \( \epsilon = 0 \), this resonant Andreev state is converted into a bound Andreev state, with the dispersion relation \( E_0^- = \Delta \sqrt{1 - D \sin^2(\phi/2)} \). The equilibrium current of a short junction is thus given by the well known relation

\[
I_{eq} = \frac{e\Delta}{h} \frac{D \sin \phi}{2\sqrt{1 - D \sin^2(\phi/2)}} \tanh(E_0^-/2kT). \tag{35}
\]

The continuum current is zero, which can be seen by putting \( L = 0 \) (\( \beta = 0 \)) in the equations for the continuum current in Appendix A. At zero temperature and zero applied bias, only the level with negative energy \(-E_0^-\) is populated. For an applied voltage bias \( V > 0 \), the electron (hole) population is shifted upwards (downwards) in energy. When the voltage \( eV = E_0^- \), the energy of the resonant level, the level becomes populated and there is an abrupt jump of the current. The regular part of the current, \( I_r \), jumps an amount \( \delta I_r = -I_{eq} \), thus cancelling the equilibrium Josephson current. This has recently been observed in experiments.

The anomalous current jumps by the amount

\[
\delta I_a = \left| \frac{e\Delta}{h} \frac{D \sin \phi}{2|\sin(\phi/2)|\sqrt{1 - D \sin^2(\phi/2)}} \right| \tag{36}
\]

\[
\delta I_r = \left| \frac{e\Delta}{h} \frac{D \sin \phi}{\xi_0} \frac{\sqrt{D}\sin(\phi)}{2|\sin(\phi/2)|\sqrt{1 - D \sin^2(\phi/2)}} \right| \tag{37}
\]

Both jumps are proportional to \( \sqrt{D} \), and the magnitude of the total current at \( E_0^- < eV < E_0^+ \) is then much larger than the equilibrium current.

When the bias voltage is further increased to \( eV = E_0^+ \) there is a second current jump: the regular current jumps in the opposite direction and becomes equal to the small negative bound state equilibrium current \( I_r = -I_{eq} \). The anomalous current, however, again jumps \( \delta I_a \) in the same direction. For voltages \( eV > E_0^+ \) the total current in the junction is thus \( I_{eq}^+ + 2\delta I_a \). The full formulas for all the individual currents including temperature dependence is given in Appendix C.

FIG. 5. The currents \( I_{eq} + I_r \) (dash-dotted), \( I_a \) (dashed) and the total Josephson current \( I_{eq} + I_r + I_a \) (solid) in the horizontal leads 2 and 3 as a function of voltage \( V \) at \( T = 0 \) for a short junction \( L = 0 \) with \( D = 0.8, \phi = 3\pi/4, \epsilon = 0.01 \) and \( \sigma = -1 \). The total current is \( I_{eq} \) for \( eV < E_0^- \) and \( \delta I_a \) for \( eV > E_0^- \).

The effect of finite temperature in a zero length junction is merely to smear the steps in the IVC.

In the symmetric case (\( l = 0 \)) it is interesting to extend the discussion to a longer junction with two resonant levels (see Fig. 4), since the current distribution between the levels becomes nontrivial. In the limit \( D \ll 1 \), both resonances have energies close to the gap edge, \( E_0^\pm \approx \Delta \), and with the additional approximation \( \beta/2 > \sqrt{D} \) we obtain the expression for the derivative of energy with respect to phase [see Eq. (23)]

\[
\frac{dE_0^\pm}{d\phi} = \pm \frac{\Delta\sqrt{D} L}{4 \xi_0} \sin(\phi) \frac{\sin(\phi/2)}{|\sin(\phi/2)|\sqrt{1 - D \sin^2(\phi/2)}} \tag{38}
\]

The equilibrium bound state current becomes proportional to \( I_{eq}^b \sim dE_0^+/d\phi + dE_0^-/d\phi \sim D \) (taking terms of order \( D \) into account), but for the currents of the individual levels \( \sim \sqrt{D} \). The resonant levels thus carry opposite "giant" currents which almost cancel in equilibrium. For \( L > 0 \), we also have to take the continuum contribution into account. In has been shown that the continuum contribution to the equilibrium current is \( I_{eq}^c = -1/2I_{eq} \), thus giving the total equilibrium current \( I_{eq} = 1/2I_{eq}^c \).

At zero temperature, when a voltage equal to the lowest lying level \( eV = E_0^- \) is applied, the regular and anomalous current jumps

\[
\delta I_r = \left| \frac{e\Delta}{h} \frac{D \sin \phi}{\xi_0} \frac{\sqrt{D}\sin(\phi)}{2|\sin(\phi/2)|\sqrt{1 - D \sin^2(\phi/2)}} \right| \tag{37}
\]

\[
\delta I_a = \sigma \frac{e\Delta}{h} \frac{D \sin \phi}{\xi_0} \frac{\sqrt{D}\sin \phi}{\sqrt{1 - D \sin^2(\phi/2)}} \tag{38}
\]

Both jumps are proportional to \( \sqrt{D} \), and the magnitude of the total current at \( E_0^- < eV < E_0^+ \) is then much larger than the equilibrium current.

We now discuss the Josephson current in a long (\( L \gg \xi_0 \)) symmetric (\( l = 0 \)) junction, and we treat the effects of asymmetry below. In a long junction there are many \( N = [L/(\xi_0\pi)] \) pairs of resonances, as seen in Fig. 4. The width of each resonance is \( \Gamma = \epsilon h v_F/(2L) \). This width must not be too small if the quasiparticles are to be able to enter and leave the junction without being scattered inelastically (\( \Gamma > h v_F/L \)). This gives an upper limit for the length \( L < l \epsilon \).
The derivative of energy with respect to phase $dE/d\phi$ in Eq. (22), which determines the current in Eqs. (32)-(34), can be simplified in a long junction $L \gg \xi_0$,

$$\frac{dE^\pm}{d\phi} = \pm \frac{\hbar \nu_F}{L} \gamma \sqrt{D} \sin(\phi) \left| 2|\sin(\phi/2)|\sqrt{1 - D \sin^2(\phi/2)} \right|. \quad (39)$$

This expression holds everywhere except close to the gap edge, $\Delta - E_n \sim (\hbar \nu_F/L)(\xi_0/L)$, a distance much smaller than the energy distance $\pi \hbar \nu_F/L$ between the pairs of levels. Therefore, equation (39) can be used for calculation of the currents of all levels except the last pair of levels closest to the energy gap. The current from this last pair of levels must always be treated on a separate footing.

According to Eq. (39), each of the Andreev levels carry a current of the order of $1/L$. Furthermore, as follows from the exact Eq. (22), each pair of levels carries a small net current, $dE_+^n/d\phi - dE_-^n/d\phi$, of the order of $(1/L)^3$. The sum of the currents from all bound states is thus determined by the current $(\sim 1/L)$ from the last pair of levels.

A. Equilibrium current

For the equilibrium current of a long junction, the contribution from the Andreev bound states at $|E| < \Delta$ and from the continuum at $|E| > \Delta$ are of the same magnitude (see Fig. 6). The continuum current (see Appendix A) is given by

$$I_{eq}^c = \frac{e}{\hbar} \left( \int_{-\infty}^{-\Delta} + \int_{\Delta}^{\infty} \right) dE n_F$$

$$\times \left( \cos \beta \cosh 2\beta c - R - D \cos \phi \right)^2 + \left( \sin \beta \sinh 2\beta c \right)^2, \quad (40)$$

with $\beta c = \arccos(E/\Delta)$. Following the method by Ishii and Svidzinsky et al., one can rewrite this integral as a sum over the residues,

$$I_{eq}^c = -I_{eq}^b + 4kT \pi i \sum_{p=0}^{\infty} i(E_p), \quad (41)$$

where the first term results from the poles of the current density $i(E)$ in Eq. (10) and the second term from the poles of the distribution function $n_F(E)$, given by $E_p = \pm 2kT \pi (1/2 + p)$. The first term in (41) is the current carried by the bound states with negative sign. The total equilibrium current $I_{eq}$ is then just given by the second term. For zero temperature this sum over poles turns into an integral and the total equilibrium current is then given by

$$I_{eq}(T = 0) = \frac{e \hbar \nu_F}{\hbar L} \frac{\sqrt{D} \sin(\phi) \arccos(R + D \cos \phi)}{2\pi |\sin(\phi/2)|\sqrt{1 - D \sin^2(\phi/2)}}. \quad (42)$$

At high temperatures $\hbar \nu_F/L \ll kT \ll \Delta$, only the first term $(p = 0)$ in the sum (41) needs to be included, and the equilibrium current becomes

$$I_{eq}(kT \gg \hbar \nu_F/L) = \frac{e \Delta}{\hbar} \pi \left( \frac{4kT}{\Delta} \right)^2 D \sin \phi e^{-2\pi L/\xi_T}, \quad (43)$$

where $\xi_T = \hbar \nu_F/kT$. The equilibrium current for a long junction at finite temperature is thus exponentially small. Expressions (22) and (39) extend earlier results to the case of arbitrary transparency $D$ of the junction.

![Figure 6](image-url)

**FIG. 6.** The equilibrium bound state (a) and continuum (b) currents and their sum (dashed) as a function of length $L$ for finite $kT = 0.2\Delta$, $D = 0.8$, $\phi = 3\pi/4$ and $\epsilon = 0.01$. There is a cusp in both the bound state and continuum currents when a new bound state forms out of the continuum. The total equilibrium current, however, dies monotonically with increased length. Inset: The equilibrium bound state and continuum currents as a function of temperature for a long junction $L = 15\xi_0$ with $D = 0.8$, $\phi = 3\pi/4$ and $\epsilon = 0.01$. The bound state current (a) decreases from $I_{eq}^b(T = 0) = i^*$ to $-I_{eq}^b$, when the temperature is increased from zero to $kT \ll \hbar \nu_F/L$. The continuum current (b) is unaffected in this temperature regime.

We are also interested in analyzing the separate behavior of the bound state current, because this current is revealed in nonequilibrium, as will be discussed in detail below. Therefore, using relation (39) we can write Eq. (24) on the form

$$I_{eq}^b = \frac{e \hbar \nu_F}{\hbar} \frac{\sqrt{D} \sin(\phi) \arccos(R + D \cos \phi)}{2\pi |\sin(\phi/2)|\sqrt{1 - D \sin^2(\phi/2)}} \times \sum_{n=0}^{N-1} \left[ \tanh(E_n^-/2kT) + \tanh(E_n^+/2kT) \right] + i^* \tanh(\Delta/2kT). \quad (44)$$
The term $i^*$ results from the last pair of levels at $E \approx \Delta$, and is of the order $1/L$, as discussed above. At $T = 0$, the sum in Eq. (14) is zero, and we thus find that $i^* = I_{eq}^r(T = 0)$. When the temperature is increased, the sum in Eq. (14) starts to contribute with negative sign and the bound state current is decreased. The continuum current (and also $i^*$), however, is independent of temperature for $kT \ll \Delta$, since it is an integral over states with $|E| > \Delta$ [see Eq. (10)]. At $kT \gg \hbar v_F/L$ the total equilibrium current is exponentially small (see Eq. (13)) and has thus decreased an amount $I_{eq}(T = 0) - I_{eq}(kT \gg \hbar v_F/L) \approx I_{eq}(T = 0)$. This is thus solely due to decrease of the bound state current, as shown in the inset in Fig. 3.

B. Regular current

The regular current can be written, inserting relation (20) into Eq. (22), on the form

$$I_r = \frac{e \hbar v_F}{2L} \sqrt{D} \frac{\sin(\phi)}{2|\sin(\phi/2)|\sqrt{1 - D \sin^2(\phi/2)}} \times \sum_{n=0}^{N-1} \left[g(E_n^-) - g(E_n^+)\right] + \frac{i^*}{2} \delta(\Delta)$$

(45)

where $g(E) = \tanh[(E + eV)/2kT] + \tanh[(E - eV)/2kT] - 2 \tanh(E/2kT)$. The regular current $I_r$ jumps up or down every time $eV = E_n^\pm$ [see Fig. 3]. Each current jump has the magnitude

$$\delta I_r = \frac{e \hbar v_F}{L} \sqrt{D} \frac{\sin(\phi)}{2|\sin(\phi/2)|\sqrt{1 - D \sin^2(\phi/2)}}$$

(46)

at zero temperature. At voltages $eV > \Delta$, the regular current is the sum of all states in the range $0 < E < \Delta$, and is equal to the negative bound state current $-i^*$. 

![Fig. 7. The equilibrium current $I_{eq}$ plus the regular current $I_r$ vs voltage. $L = 10\xi_0$, $\phi = \pi/2$, $D = 0.8$, $\epsilon = 0.05$. Solid line - $T = 0$, dashed-dotted - $kT = 0.04\Delta$, dashed - $kT = 0.07\Delta$. The regular current jumps alternating by $\pm \delta I_r$ every time the voltage is equal to the energy of an Andreev resonance. For $kT \gg \hbar v_F/L$ and $eV > \Delta$ the current $I_{eq} + I_r = I_{eq}$. It is interesting to study the sum $I_{eq} + I_r$, plotted in Fig. 3, at temperatures $kT \gg \hbar v_F/L$. In this temperature regime the equilibrium current is exponentially small and also the regular current steps in the IVC in Fig. 3 are suppressed. For a voltage $eV \sim \Delta$, the last level, carrying the major part ($i^*$) of the bound state current, is populated and the current $I_{eq} + I_r$ jumps to $I_{eq}^*$, the value of the continuum current, since all bound states are populated. This current $I_{eq}^*$ is of the order of $1/L$ and the current $I_r + I_{eq}^*$ is increased from zero to be $\sim 1/L$ when increasing the voltage from $V = 0$ to $eV \sim \Delta$.]

C. Anomalous current

The anomalous current is given by inserting Eq. (39) into Eq. (33),

$$I_a = -\frac{e \hbar v_F}{4L} \sqrt{RD} \sin \phi \sum_{n=0}^{N-1} \left[h(E_n^+) + h(E_n^-)\right]$$

(47)

where $h(E) = \tanh[(E - eV)/2kT] - \tanh[(E + eV)/2kT]$. We have neglected the current from the last level close to $E = \Delta$, because the currents of all levels add up and the current from the last level is negligible. The IVC at zero temperature looks like a staircase, as shown in Fig. 3.

![Diagram showing the equilibrium current $I_{eq}$, the regular current $I_r$, and the anomalous current $I_a$. The diagram illustrates the behavior of the currents as a function of voltage $eV$. The equilibrium current $I_{eq}$ is plotted as a solid line, the regular current $I_r$ as a dashed line, and the anomalous current $I_a$ as a dotted line. The diagram shows the jumps in the current as the voltage changes.]
FIG. 8. The anomalous current as a function of voltage $V$ for (a) $\phi = \pi/4$ and (b) $\phi = 3\pi/4$ for $L = 10\xi_0$, $D = 0.8$, $\epsilon = 0.05$ and $\sigma = -1$. Temperature $T = 0$ (solid) and $T = 0.1\Delta$ (dashed line). The current steps with magnitude $\delta I_a$ are smeared to a straight line for $kT \gg \hbar v_F/L$. Upper inset: The critical anomalous current at $eV = \Delta$ as a function of transparency $D$ for coupling constant $\epsilon = 0.1$. Due to finite coupling $\epsilon$, the critical current always goes to zero for $R = 0$. Lower inset: The anomalous current $I_a(eV = \Delta, kT \gg \hbar v_F/L)$ as a function of phase difference $\phi$ for different transparencies $D = 0.1, 0.5$ and $0.9$. The highest amplitude corresponds to the highest transparency and vice versa.

The magnitude of the current step at zero temperature is given by

$$\delta I_a = \frac{e \hbar v_F}{\hbar} \frac{\sqrt{2RD \sin \phi}}{2L(1 - D \sin^2(\phi/2))}. \quad (48)$$

At temperatures larger then the interlevel distance, $kT \gg \hbar v_F/L$, the staircase IVC is smeared out to a straight slope, as shown in Fig. 8. The exact position of each level becomes irrelevant and we can write the sum over bound states in (17) as an integral, noting that the expression $dE/dn = \pi \hbar v_F/L$ holds for all levels in the sum (17).

$$\sum_{n=0}^{N} [\hbar(E_n^+ + \hbar(E_n^-)]$$

$$= \frac{2L}{\pi \hbar v_F} \int_{0}^{\Delta} dE [\tanh(E + eV) - \tanh(E - eV)]$$

$$= \frac{4L}{\hbar v_F \pi} f(V, T) \quad (49)$$

where

$$f(V, T) = kT \ln \left( \frac{\cosh(\Delta + eV)/kT}{\cosh(\Delta - eV)/kT} \right), \quad (50)$$

and the anomalous current takes the simple form

$$I_a = -\frac{e}{\hbar} \frac{\sqrt{2RD \sin \phi}}{\pi[1 - D \sin^2(\phi/2)]} f(V, T). \quad (51)$$

In the limit $\hbar v_F/L \ll kT \ll \Delta$, $f(V, T) = \min(eV, \Delta)$: the anomalous current thus scales linearly with applied voltage up to $\Delta$. It follows from Eq. (17) that $I_a$ is independent of the length of the junction, being the sum of $N \sim L$ levels which each carries a current $I_n \sim 1/L$. This gives that the anomalous current roughly is equal to the total equilibrium current of the short junction. The critical anomalous current is plotted with respect to transparency in the inset in Fig. 8. In the limit $D \ll 1$ it is given by $(I_a)_c = (e/\hbar)(\sqrt{D}/\pi) f(V, T)$. It is proportional to the first power of $\Delta$ for $T$ close to $T_c$, therefore surviving up to $kT \approx \Delta$. The anomalous current-phase relation (see inset in Fig. 8) is $2\pi$ periodic and resembles that of the equilibrium Josephson current. The direction of the anomalous current is however proportional to $\sigma$, i.e. dependent on the phase of the scatterer at the connection point, which is not the case for the equilibrium Josephson current.

To get the complete picture of the Josephson current in a long junction, $I = I_{eq} + I_r + I_a$ is plotted as a function of bias voltage for different temperatures in Fig. 9.

![FIG. 9. The total current $I = I_{eq} + I_r + I_a$ as a function of voltage. At zero temperature we have $I_r + I_{eq}$ (dash-dotted), $I_a$ (dashed) and the total current $I_r + I_{eq} + I_a$ (solid). The total current for temperatures $kT \gg \hbar v_F/L$, and in this limit the total current roughly coincides with the anomalous current, given by Eq. (51).](image)

D. Asymmetric junction

The effect of asymmetry is most pronounced in the long limit when the asymmetry is much larger than the coherence length but much smaller than the total length of the junction, $L \gg l \gg \xi_0$. In this limit, the derivative of energy with respect to phase $dE/d\phi$ in Eq. (22) reduces to the expression of a symmetric long junction (23), since $|\sin \theta | > |R| \sin \chi |$ (see Appendix B). The equilibrium current $I_{eq}$ and the regular current $I_r$ are not substantially changed in comparison to the symmetric case. In contrast, the anomalous current is modified in a nontrivial way, taking the form

$$I_a = -\frac{e}{\hbar} \frac{\hbar v_F}{L} |R|^{3/2} \sin \phi$$
× \sum_{n,\lambda} \frac{\cos \chi - \cos \phi}{1 - (D \cos \phi + R \cos \chi)^2} (n^e - n^h), \quad (52)

obtained by inserting Eq. (39) into (33). For \( T = 0 \) the step structure in the IVC is modified due to the change of Andreev levels as a result of the asymmetry (see Fig. 11). Already for small asymmetry \( l \sim \xi_0 \), the anomalous current might change dramatically. Depending on the phase difference of the junction, the IVC is renormalized and changes sign for \(-\pi/2 < \phi < \pi/2\).

When the temperature is increased beyond the inter-level distance \( kT \gg \hbar v_F / l \), the step structure becomes smeared and we get a periodic modulation of the IVC on the scale of \( eV \sim \hbar v_F / l \). This modulation arises from the factor \( \cos \chi \).

When the temperature is further increased to \( kT \gg \hbar v_F / l \) this periodic structure is smeared out and the IVC once again becomes a straight line, but with renormalized slope. In this high temperature limit the amplitude of the terms in the sum in Eq. (52) oscillates with a period \( \hbar v_F / l \). During this period, the filling factors \( n \) can be taken to be constant, and we can sum over one period to get the average value. Performing this summation in the continuum limit, we get

\[
\sum_{\text{one period}} \frac{\cos \chi - \cos \phi}{1 - (D \cos \phi + R \cos \chi)^2} \approx \frac{\hbar v_F}{2l} \int_0^{2\pi} \frac{\cos(\chi) - \cos(\phi)}{1 - (D \cos \phi + R \cos \chi)^2} d\chi
\]

\[
= \frac{L}{l} \frac{1}{8R\sqrt{D}} \left( \frac{\left| \sin(\phi/2) \right|}{\sqrt{1 - D\cos^2(\phi/2)}} - \frac{\left| \cos(\phi/2) \right|}{\sqrt{1 - D\sin^2(\phi/2)}} \right). \quad (53)
\]

This quantity is energy independent and we can then sum over the filling factors following the procedure from the symmetric case \( \xi_0 \)

\[
\sum_{\text{averaged periods}} (n^e - n^h) \approx \frac{4l}{\hbar v_F \pi} f(V,T). \quad (54)
\]

The anomalous current then becomes

\[
I_a = -\sigma^2 \frac{e}{\hbar} \frac{D}{\pi \sqrt{R}} \sin \phi \\
\times \left( \frac{\left| \sin(\phi/2) \right|}{\sqrt{1 - D\cos^2(\phi/2)}} - \frac{\left| \cos(\phi/2) \right|}{\sqrt{1 - D\sin^2(\phi/2)}} \right) f(V,T),
\]

(55)

which is independent of both the length \( L \) and the asymmetry \( l \). We also find that the renormalized anomalous current becomes \( \pi \)-periodic. This can qualitatively be explained by the fact that the \( 2\pi \)-periodic part of the anomalous current density is very sensitive to asymmetry, oscillating fast with energy on the scale of \( \hbar v_F / l \), becoming washed out during summation over bound states at high temperatures \( kT \gg \hbar v_F / l \). The \( 2\pi \) periodic part of the current does not have this sensitivity and is the only part of the anomalous current that survives. The asymmetric anomalous current-phase relation is shown in Fig. 11.

![Fig. 10](image1)

**Fig. 10.** The asymmetric anomalous current \( I_a \) vs voltage for different asymmetries (a) \( l = 0\xi_0 \), (b) \( l = 2\xi_0 \) and (c) \( l = 40\xi_0 \) for \( kT \gg \hbar v_F / l, D = 0.8, \epsilon = 0.05, L = 60\xi_0, \phi = \pi/4 \) and \( \sigma = -1 \). The IVC is changed dramatically already for as small asymmetry \( l \sim \xi_0 \), if the phase difference \(-\pi/2 < \phi < \pi/2\).

![Fig. 11](image2)

**Fig. 11.** The asymmetric anomalous current \( I_a \) at \( eV = \Delta \) and \( kT \gg \hbar v_F / l \) as a function of phase difference \( \phi \) for different transparency constants \( D = 0.1, 0.5 \) and 0.9. Inset: The critical anomalous current \( (I_a)_c \), for \( eV = \Delta \) and \( kT \gg \hbar v_F / l \), as a function of transparency \( D \) for coupling constant \( \epsilon = 0.1 \).

The \( \pi \)-periodicity and the zeros at \( \phi = n\pi/2 \) give the condition that the slope of the IVC must change sign due to asymmetry in the range \(-\pi/2 < \phi < \pi/2\), as shown in Fig. 11. The critical asymmetric anomalous current as a function of transparency \( D \) is shown in the inset in Fig. 11. The critical asymmetric anomalous current as a function of transparency \( D \) is shown in the inset in Fig. 11. The behavior is very similar to the critical anomalous...
current in the symmetric case, the main difference being that the amplitude is reduced by roughly a factor of two.

VI. INTERFACE BARRIERS

In any realistic experimental situation, normal reflection at the NS-interface, modeled by a barrier with reflection amplitude \( r_{n} \), must be taken into account. The general expression, considering both the interface barriers and the midpoint scatterer, becomes analytically intractable. We can however analyze the case where the midpoint scatterer is absent \( (R = 0) \) to get an understanding of the effect of NS-barriers on the junction properties, and then treat the general case with injection and midpoint scatterer numerically.

In the absence of the superconducting leads (a NININ-junction), the two barriers give rise to normal Breit-Wigner resonances for the electrons and holes. Understanding the properties of these resonances turns out to be crucial for describing the behavior of Andreev levels and current transport. The energies of the electron and hole resonances are calculated straightforwardly

\[
E_{n}^{e} = -2E_{F} \left[ 1 - \frac{\pi(n - \nu)}{k_{F}L} \right], \\
E_{m}^{h} = 2E_{F} \left[ 1 - \frac{\pi(m - \nu)}{k_{F}L} \right],
\]

where \( r_{b} = \sqrt{R_{b}}e^{i\pi/\nu} \) and \( n(m) \) are integers denoting the index of the electron (hole) resonances. The intersection between electron and hole resonances \( (E_{n}^{e} = E_{m}^{h}) \) is given by \( L_{n+m} = \lambda_{F}/4(m + n - 2\nu) \) with the Fermi wavelength \( \lambda_{F} = 2\pi/k_{F} \). These normal resonances are plotted in Fig. 12.

![Fig. 12. The Andreev levels (solid) and the normal electron and hole resonances (dotted) as a function of length \( L \) of the junction with four Andreev levels in (a) weak resonance limit \( R_{b} \ll 1 \) (b) strong resonance limit \( R_{b} \sim 1 \). The lengths of two subsequent intersections of normal resonances \( L_{n+m} \) and \( L_{n+m-1} \) are shown with arrows.](image)

For the junction with superconducting leads, one can in the same way as before calculate the equation for the bound Andreev states \( (|E| < \Delta) \), with the result

\[
D_{b}^{2} \cos \phi + 2R_{b} \cos \beta - \cos(2\gamma - \beta) - R_{b}^{2} \cos(2\gamma + \beta) + 4R_{b} \sin^{2} \gamma \cos(\beta_{0}) = 0
\]

where we have defined \( \beta_{0} = \pm 2E^{e,h}/(h\nu_{F}/L) \) and \( (+-) \) denotes hole(electron) resonance energies. One can draw some qualitative conclusions on how the Andreev levels are related to the normal resonances by looking at Fig. 2. In the limit of high barrier transparency \( R_{b} \ll 1 \), the Andreev levels are weakly modified by the barriers. In the opposite limit \( R_{b} \sim 1 \), the Andreev levels get pinned at the normal resonances, but there are no level crossings at the points where the normal electron and hole resonances intersect.

We find the same interlevel distance \( h\nu_{F}/L \) in the junction with the superconducting leads (SINIS junction) and normal leads (NININ-junction). The main difference is that the normal resonance move very quickly through the junction when the length \( L \) increases, while the Andreev levels oscillate up and down.

Considering Andreev state energies close to the Fermi level, \( E \ll \Delta \), one can derive a simplified dispersion relation

\[
\sin^{2}(\beta/2) = \frac{D_{b}^{2} \cos^{2}(\phi/2) + 4R_{b} \sin^{2}(\beta_{0}/2)}{(1 + R_{b})^{2}}.
\]

Using this relation we can study the bound state current in different length limits.

In the short limit, \( L \ll \xi_{0} \) there are two cases to be considered. For nearly transparent barriers \( D_{b} \sim 1 \), and thus broad resonances \( \Gamma = D_{b}\nu_{F}/L \gg \Delta \), one can neglect dephasing (putting \( \beta = 0 \)) and just get the total transparency of the junction \( D = D_{b}^{2}/(D_{b}^{2} + 4R_{b} \sin^{2}(\beta_{0}/2)) \) to be put into the standard zero length junction equilibrium current formula. In the strong barrier case \( D_{b} \ll 1 \) the resonances are sharp \( \Gamma \ll \Delta \) and one can not neglect the dephasing. Assuming that the resonance is close to Fermi energy \( E^{e,h} \ll \Delta \), we can put \( \beta \ll 1 \) in (58) and obtain

\[
E = \pm \sqrt{\Gamma^{2} \cos^{2}(\phi/2) + (E^{e,h})^{2}}.
\]

When the resonance is exactly at the Fermi energy \( E^{e,h} = 0 \), the Josephson current is given by

\[
I = \frac{e}{h} \sin(\phi/2) \tanh \left( \frac{\Gamma \cos(\phi/2)}{2kT} \right).
\]

The critical current at low temperatures \( (kT \ll \Gamma) \) is thus smaller than the critical current of a short, clean junction by a factor \( \Gamma/\Delta \).

For a long junction \( L \gg \xi_{0} \) we can calculate the derivative of energy with respect to phase,
\[ \frac{dE}{d\phi} = \pm \frac{\hbar v_F}{2L} \frac{D_b^2 \sin \phi}{\sqrt{(1 + R_b)^4 - [D_b \cos \phi - 4R_b \cos(\lambda b)]^2}} \]  
\hspace{1cm} (61)

In the weak barrier limit \( R_b \ll 1 \), this just causes oscillations with length around the clean junction \( (R_b = 0) \) result. In the strong barrier limit \( R_b \sim 1 \), one can distinguish two limits: When the length of the junction is far away from the length \( L_{n+m} \), the electron and hole resonances intersect, the junction is out of resonance. The second term in Eq. (61) is negligible and the current from the individual levels thus becomes

\[ I = \pm \frac{e v_F}{4L} D_b \sin \phi. \]  
\hspace{1cm} (62)

It is proportional to \( D_b^2 \) and thus strongly suppressed. In the opposite limit, when the length of the junction \( L = L_{n+m} = \lambda_F(m + n - 2\nu)/4 \), the junction is in resonance. When \( n + m \) is even we get the current carried by each level

\[ I = \pm \frac{e v_F}{4L} D_b \sin \phi \]  
\hspace{1cm} (63)

and when \( n + m \) is odd we get

\[ I = \pm \frac{e v_F}{L} D_b \sin \phi \]  
\hspace{1cm} (64)

We see that the current is proportional to \( D_b \), just as expected for the junction in resonance. The current carried is thus of the order of the single barrier junction current. An interesting feature is that the current is dependent on the parity of the sum of the electron and hole resonance indices \( n + m \). When the third lead is connected to the junction, the scattering at the connection point just splits the Breit Wigner resonances, and the qualitative picture for the bound states derived without the third lead connected survives.

To calculate the total equilibrium, regular or anomalous current, the currents carried by all individual levels have to be summed up. In the weak barrier limit \( R_b \ll 1 \) we just find that all properties calculated above for the symmetric junction without barriers hold, with a small length dependent modulation \( \sim R_b \) with a period \( \delta L \sim \lambda_F \). In the strong barrier limit \( R_b \sim 1 \), the result will depend on whether the junction is in or out of resonance.

Fig. 13 shows the resonant behavior of the equilibrium current as a function of length. The current has a peak around length \( L = \lambda_F/4(m + n + 2\nu) \). The phase dependence of the current at the resonant peaks is well described by the expressions for the single level currents (63) and (64).

The anomalous current is also strongly length dependent and when the junction is in resonance we have an anomalous current \( I_a \sim \sigma D_b \sqrt{R D} \) while when we are out of resonance \( I_a \sim \sigma D_b^2 \sqrt{R D} \). It turns out that there is an anomalous current even without scattering at the connection point, but it oscillates around zero as a function of length with the period \( \sim \lambda_F \).

VII. INJECTION CURRENT AND CONDUCTANCE

Although the nonequilibrium Josephson current is at the focus of this paper, the injection current that flows between the normal reservoir and the SNS-junction is also of great interest: it determines the conductance of the circuit. The conductance of SNS-structures has been studied intensely in recent years[13] and is an interesting quantity in itself. It can also be used to determine the direction of the Josephson current in the junction[14] or to detect a large Josephson current in the superconducting loop which changes the applied external flux vs phase dependence, thus modifying the phase dependence of the conductance.[\]

A. Injection current

We start by discussing the symmetric junction \( l = 0 \), and comment on the modifications due to asymmetry below. The current injected in lead 1 for energies \( |E| < \Delta \)
can be calculated to lowest order in $\epsilon$ by inserting Eq. (23) into Eq. (30)

$$I_1 = \epsilon \frac{e}{h} \sqrt{D} \frac{|\cos(\phi/2)|}{\sqrt{1 - D \sin^2(\phi/2)}} \times \sum_{n, \pm} \left| \frac{dE_n^\pm}{d\phi} \right| \left[ n^e(E_n^\pm) - n^h(E_n^\pm) \right]. \quad (65)$$

This current is proportional to $\epsilon$ (unlike the Josephson current discussed above), which is also true for the continuum contribution. For $|E| > \Delta$, the injection current density in Eq. (30), $i_{inj} = i_1 = i_3 - i_2$, is obtained from Eq. (A3). This current density is roughly described by the normal current density $4\epsilon e/h$, with oscillations around this value due to the resonances in the scattering states (see Fig. 14). These oscillations are strongest around $E \sim \Delta$ and decrease with increasing energy.

The injection current is proportional to the modulus $|dE/d\phi|$, just like the anomalous current, as discussed above. The IVC thus has the shape of a staircase, as shown in Fig. 14.

![Fig. 14. The injection current in lead 1 as a function of voltage for (a) $\phi = \pi/4$, (b) $\phi = 3\pi/4$ and (c) $\phi = \pi$. Zero T (solid lines) and $T = 0.05\Delta$ (dashed line) with $D = 0.8$, $L = 10\xi_0$ and $\epsilon = 0.05$. For $eV > \Delta$ the IVC approaches the value of a normal junction.](image)

FIG. 14. The injection current in lead 1 as a function of voltage for (a) $\phi = \pi/4$, (b) $\phi = 3\pi/4$ and (c) $\phi = \pi$. Zero $T$ (solid lines) and $T = 0.05\Delta$ (dashed line) with $D = 0.8$, $L = 10\xi_0$ and $\epsilon = 0.05$. For $eV > \Delta$ the IVC approaches the value of a normal junction.

We can derive expressions for the injection current in different length limits. A complete set of formulas are given in Appendix A. Here we only present the heights of the current steps in the IVC’s in some representative cases.

In the limit of zero length of the junction ($L = 0$) there is only one Andreev level (for $0 < E < \Delta$) and the magnitude of the step is

$$\delta I_1 = \epsilon \frac{e}{h} \sqrt{D} \frac{|\sin(\phi/2)| \cos^2(\phi/2)}{2(1 - D \sin^2(\phi/2))} \quad (66)$$

while for two levels close to $\Delta$ we get ($L \sim \xi_0$)

$$\delta I_1 = \epsilon \frac{e}{h} \frac{\Delta L}{\xi_0} \frac{\cos^2(\phi/2)}{\sqrt{1 - D \sin^2(\phi/2)}} \quad (67)$$

In the long junction limit ($L \gg \xi_0$) the current is given by putting Eq. (23) into (33).

$$I_1 = \epsilon \frac{e}{h} \frac{\hbar v_F}{2L} \frac{\cos^2(\phi/2)}{1 - D \sin^2(\phi/2)} \sum_{n=0}^N \left[ h(E_n^-) + h(E_n^+) \right]. \quad (68)$$

The current step at zero temperature is

$$\delta I_1 = \epsilon \frac{e}{h} \frac{\hbar v_F}{2L} \frac{\cos^2(\phi/2)}{1 - D \sin^2(\phi/2)} \quad (69)$$

For high temperatures $kT \gg \hbar v_F/L$, the sum (68) can be converted to an integral, just as for the anomalous current [10], which gives the current for $eV < \Delta$

$$I_1 = \epsilon \frac{4e}{h} \frac{\cos^2(\phi/2)}{1 - D \sin^2(\phi/2)} f(V, T). \quad (70)$$

where $f(V, T)$ is given by Eq. (61). The IVC thus becomes linear for $eV < \Delta$, with the slope independent of length and temperature, as is seen in Fig. 14. All information about individual Andreev levels is washed out.

The effect of asymmetry on the injection current is drastic in the limit of a long junction with large asymmetry $L \gg l \gg \xi_0$, just as for the asymmetric anomalous current. This shows the strong relationship between the two currents. The injection current is given by inserting Eq. (23) into (33)

$$I_1 = \epsilon \frac{e}{h} \frac{\hbar v_F}{L} \sum_{n=0}^N \frac{D \sin^2(\phi) + R \sin^2 \chi}{1 - (D \cos \phi + R \cos \chi)^2} (n^e - n^h). \quad (71)$$

Averaging over periods and summing up the filling factors, just as in the case of the anomalous current [see Eq. (23)], the injection current becomes

$$I_1 = \epsilon \frac{e}{h} \frac{8}{R \pi} \left[ 1 - \sqrt{D} \times \left( \frac{|\sin(\phi/2)|^3}{\sqrt{1 - D \cos^2(\phi/2)}} + \frac{|\cos(\phi/2)|^3}{\sqrt{1 - D \sin^2(\phi/2)}} \right) \right] f(V, T). \quad (72)$$

The injection current in this limit does not depend on either the length $L$ or the asymmetry $l$. It is $\pi$-periodic, $I_1(\phi + \pi) = I_1(\phi)$, for the same reasons as discussed above for the asymmetric anomalous current. The implications of the $\pi$ periodicity for the conductance is discussed below.
B. Conductance

For the conductance, we discuss the whole range of coupling parameters \( \epsilon \), not only the weak coupling limit. The conductance is defined as

\[
G(V, \phi) = \frac{dI}{dV}(n^e - n^h)
\]

\[
= \frac{e}{kT} \int_{-\infty}^{\infty} \frac{i_h}{2} \left[ \cosh^{-2} \frac{(E + eV)}{2kT} + \cosh^{-2} \frac{(E - eV)}{2kT} \right]. \tag{73}
\]

At zero temperature \( T = 0 \) the conductance for a symmetric junction \( l = 0 \) can be written for \( eV < \Delta \), noting that \( i_h(E) = i_h(-E) \),

\[
G(V, \phi) = e^2 \frac{4 \cos^2(\phi/2) \sin^2 \theta}{h} \left[ (1 - \epsilon) \cos 2\theta - R - D \cos \phi \right]^2 + \epsilon^2 \sin^2 2\theta. \tag{74}
\]

At a voltage fulfilling the relation \( 1 - \cos 2\theta = D(1 - \cos \phi)/\sqrt{1 - 2\epsilon} \), which is exactly at an Andreev resonance, and a phase difference \( \phi = 0 \mod 2\pi \), the conductance is \( G = 4e^2/h \), which is the conductance for a perfect NS-interface. This holds for any transparency \( D \), length \( L \) and coupling constant \( \epsilon \).

For \( L = 0 \) we get \( \cos \theta = eV/\Delta \) to be inserted into (74). This gives rise to a peak in the conductance at the voltage \( eV \approx E_0^r \), the energy of the single Andreev resonance. In the long limit \( (L \gg \xi_0) \) for \( E \ll \Delta \) we get \( \theta = \pi/2 - eV/E \) which results in conductance oscillations as a function of applied voltage (73) with a period \( \pi \hbar v_F/L \), the distance between the pairs of Andreev levels. This is made clear by taking the derivative of current with respect to voltage in Fig. 14.

The conductance vs phase relation \( G(V, \phi) \) is also altered when voltage is applied. For voltages around \( eV/(\hbar v_F/L) \approx n\pi \), i.e. at an energy between the pairs of Andreev resonances, the conductance has a maximum around \( \phi \approx \pi \) and a local minimum at \( \phi = 0 \) [apart from the absolute minimum at \( \phi = \pi \) due to the factor \( \cos^2(\phi/2) \) in Eq. (74)]. For \( eV/(\hbar v_F/L) \approx \pi/2 + n\pi \), i.e. at an energy between the two Andreev resonances in the pair, the maximum shifts to \( \phi = 0 \) and the minimum to \( \phi = \pi \) [see Fig. 15]. This behavior has recently been observed in both ballistic and quasiballistic junctions. It has also been predicted for diffusive junctions.

This voltage dependence of the conductance holds for \( kT \ll \hbar v_F/L \).

\[
\text{FIG. 15. The conductance as a function of } \phi \text{ for different voltages (a) } eV = 0, \text{ (b) } eV = 0.075\Delta \text{ and (c) } eV = 0.15\Delta \text{ at temperature } kT = 0.01\Delta. \ D = 0.9, \ \epsilon = 0.05 \text{ and } L = 20\xi_0.
\]

For the additional condition zero voltage \( V = 0 \), the expression (74) reduces to

\[
G(0, \phi) = e^2 \frac{2e^2}{h} \frac{2 \cos^2(\phi/2)}{R + D \cos^2(\phi/2)}. \tag{75}
\]

For \( D \ll R \) the maximum conductance has a universal magnitude \( G_{\text{max}}(0) = G_N \epsilon/(1 - \epsilon)^2 \) where \( G_N = e^2/h \) is the conductance of the junction in the normal state.

When the coupling is weak, \( \epsilon \ll 1 \), the maximum conductance is \( G_{\text{max}} \sim G_N \), i.e much smaller than the normal conductance. In this limit the Andreev resonances are sharp and there are no available Andreev states at \( E_F \), because the scattering at the three lead connection point opens up a gap in the Andreev spectrum at \( \phi = \pi \) (see Fig. 1). The conductance is thus strongly suppressed. The conductance rapidly increases with voltage and temperature and has a maximum at \( eV, kT \sim \hbar v_F/L \). This happens because electrons (holes) with energies \( E > 0 \) (\( E < 0 \)) tunnel into the first resonant Andreev state at finite energy. This gives rise to a finite energy conductance peak (see Fig. 15). The maximum conductance \( G_{\text{max}} \) is plotted as a function of temperature in Fig. 16 (note that the minimum conductance always is zero).
It increases with increasing $T$, reaches a maximum around $kT \sim h v_F/L$, drops again for $kT > h v_F/L$ but saturates at a constant value $G_{\text{max}} = G_N$ for $kT \gg h v_F/L$. Inset: The conductance as a function of $\phi$ for different temperatures $kT = 0, 0.01 \Delta, 0.05 \Delta, 0.05 \Delta$ and $0.1 \Delta$ at zero voltage $V = 0$. Increasing temperature from bottom to top at $\phi = 0$. $D = 0.9, \epsilon = 0.05$ and $L = 20 \xi_0$.

The conductance as a function of phase difference $\phi$ behaves differently for different temperatures $T$, as is shown in the inset in Fig. 16. For $kT \gg h v_F/L$, a limit only accessible for the long junction $L \gg \xi_0$, the maxima of the conductance around $\phi \approx \pi$ are shifted to a maximum at $\phi = 0 \mod 2\pi$. This holds independent of voltage applied. A similar effect in a quasi-ballistic system has been reported by Dimoulias et al.

In the weak coupling limit $\epsilon \ll 1$ we can use expression (76) to get the conductance in the long limit for $h v_F/L \ll kT \ll \Delta$ and $eV < \Delta$

$$G(\phi) = \epsilon e^2 h \cos^2(\phi/2)$$

which is independent of voltage, temperature and length of the junction and has a maximum $G_{\text{max}} = G_N$ at $\phi = 0 \mod 2\pi$.

In the same limit we get the conductance in the asymmetric junction from expression (77)

$$G(\phi) = \epsilon e^2 h \frac{8}{\pi} \left[1 - \sqrt{D}\right] \times \left(\frac{|\sin(\phi/2)|^3}{\sqrt{1 - D \cos^2(\phi/2)}} + \frac{|\cos(\phi/2)|^3}{\sqrt{1 - D \sin^2(\phi/2)}}\right).$$

It is $\pi$-periodic and has a maximum for $\phi = \pi(n + 1)/2$ and a minimum for $\phi = \pi n$. This $\pi$-periodicity can be qualitatively explained by considering the lowest order paths giving rise to the conductance.

FIG. 16. The maximum conductance $G_{\text{max}}/G_N$ as a function of $T$ for zero voltage. $L = 10 \xi_0$, $D = 0.9$ and $\epsilon = 0.05$. At $T = 0$ the maximum conductance is $G_{\text{max}} \ll G_N$. The conductance as a function of phase difference $\phi$ in the inset in Fig. 16. For $kT \ll h v_F/L$ but saturates at a constant value $G_{\text{max}} = G_N$ for $kT \gg h v_F/L$. Inset: The conductance as a function of $\phi$ for different temperatures $kT = 0, 0.01 \Delta, 0.025 \Delta, 0.05 \Delta$ and $0.1 \Delta$ at zero voltage $V = 0$. Increasing temperature from bottom to top at $\phi = 0$. $D = 0.9, \epsilon = 0.05$ and $L = 20 \xi_0$.

FIG. 17. The paths in the asymmetric junction giving the first order terms of the conductance. Electrons are drawn with solid lines, holes with dashed. The grey ellipse denotes the effective scatterer due to the three lead connection. The upper paths give rise to a $2\pi$ periodic component of the current, suppressed at finite temperature $kT \gg h v_F/L$. The lower paths, time reversed, give rise to a $\pi$ periodic component of the current, not suppressed by temperature.

The upper paths in Fig. 17, corresponding to an injected electron and giving rise to an outgoing hole, produce a current of the order $i_1 \sim |e^{i(\phi_1 + \beta_2)} + e^{i(\phi_2 + \beta_1)}|^2 = 2 + 2 \cos(\phi + \chi)$. This part of the current is $2\pi$ periodic in the phase, rapidly oscillating in energy with a period $h v_F/L$. It is washed out when summing up the levels in a long junction at finite temperature. The lower paths in Fig. 17, corresponding to an injected electron giving rise to an outgoing electron, produce a current of the order $i_2 \sim |d^* e^{i(\phi + \beta)} + d e^{i(-\phi + \beta)}|^2 = D(2 + 2 \cos 2\phi)$. This part of the current is $\pi$ periodic in phase and not sensitive to asymmetry.

The discussion about the periodicity of the conductance oscillations with respect to phase goes back to the early eighties. A $\pi$-periodic contribution to the weak localization correction to the conductance in a similar system was predicted by Spivak et al. and discussed further by Altshuler et al. It has been shown in numerical simulations for a structure similar to ours that the full conductance, i.e not only the weak localization contribution, might become $\pi$-periodic at finite temperatures. A large $\pi$-periodic conductance oscillation with phase was also observed in diffusive samples. Whether the explanation to the crossover from $2\pi$ to $\pi$ periodicity with increased temperature discussed above can account for these observations remains to be investigated.

VIII. FOUR TERMINAL JUNCTION

One problem with the three terminal junction is, as discussed above, that it is not possible to separate the injection current from the Josephson current in a clear way for arbitrary coupling $\epsilon$. In a four terminal junction, this is possible under certain conditions, which makes it interesting to discuss this configuration separately.

We consider two different types of junction configurations (see Fig. 18). The upper junction is a straight-
forward extension of the three terminal device pictured in Fig. 18. Two normal reservoirs are connected to the normal part of the junction. The reservoirs are then connected to the grounded superconducting loop via voltage sources biased at \( V_1 \) and \( V_4 \) respectively.

![Diagram](image)

**FIG. 18.** Two different setups of the four terminal junction. In (a) the normal reservoirs are biased independently at \( V_1 \) and \( V_4 \) with respect to the superconducting loop (grounded), in (b) only the potential difference between the normal reservoirs, \( V_1 \), is determined. In the right figure, a close up of the junction area is shown, with the direction of the currents showed with arrows.

The current injected from a normal reservoir is split at the connection point. One part of the current flows through the junction directly into the other reservoir and the other part of the current is divided between the leads 2 and 3. In the general case, the currents in lead 1 and 4 are not equal \( I_1 \neq I_4 \). However, by adjusting the potentials \( V_1 \) and \( V_4 \), the currents in the vertical lead can be put equal and from current conservation at the connection point it follows \( I_2 = I_3 \) and we have a clear separation between the injection current \( I_1 = I_4 \) and the Josephson current circulating in the superconducting loop \( I_2 = I_3 \).

In the lower junction, this separation follows directly from current conservation at the connection point, since \( I_2 = I_3 \). In this junction a bias \( V \) is applied between the normal reservoirs, which are only connected to the superconducting loop via the four lead connection point. We define the potential of the superconducting loop in this junction to be zero and the potentials of the normal reservoirs to \( V_1 \) and \( V_4 \) respectively, just as for the upper junction. Our biasing arrangement then gives \( V = V_1 - V_4 \). The condition of current conservation, \( I_h(V_1,V_4) = I_v(V_1,V_4) \), gives a second condition on \( V_1 \) and \( V_4 \). With these definitions we can calculate the current in both junctions in the same way.

The cross-shaped connection point is modelled by the scattering matrix

\[
S = \begin{pmatrix}
\sqrt{\epsilon} & \sqrt{\epsilon} & \frac{d_+}{\sqrt{\epsilon}} \\
\sqrt{\epsilon} & \frac{d_-}{\sqrt{\epsilon}} & \frac{d_+}{\sqrt{\epsilon}} \\
\frac{d_-}{\sqrt{\epsilon}} & \frac{d_+}{\sqrt{\epsilon}} & \sqrt{\epsilon}
\end{pmatrix}
\]

(78)

where the \( \epsilon \) describes the coupling of the SNS junction to the vertical normal lead \((0 \leq \epsilon \leq 0.25)\). The horizontal scattering amplitudes now obey the relations \( Re(rd^*) = -\epsilon \) and \( D + R = 1 - 2\epsilon \). The same holds for the vertical scattering amplitudes \( r_\perp \) and \( d_\perp \).

The current densities \( i^{(h),1(4)} \), with the upper index 1 or 4 denoting the lead from which the quasiparticles are injected, are calculated in the same way as in the case of the three terminal junction. Due to the symmetry of the scattering matrix, quasiparticles injected from leads 1 or 4 give rise to the same current density in leads 2 and 3, i.e \( i^{(h),1}_2 = i^{(h),4}_2 \).

The expressions for the sum and anomalous current densities become very similar to the three terminal expressions [see Eq. (62) and (64)], i.e. one just changes \( \epsilon \rightarrow 2\epsilon \) [also changing \( Z = Z(\epsilon \rightarrow 2\epsilon) \)] and divides by two, noting that \( \sqrt{\epsilon} = \sqrt{\epsilon} \) and \( \sqrt{\epsilon} = \sqrt{\epsilon} \). Neither the vertical transparency \( D_\perp \) nor the reflectivity \( R_\perp \), thus appear explicitly in these expressions. The factor one half simply reflects that there are two normal leads connected to the normal part of the junction. In the limit of weak coupling \( \epsilon \ll 1 \), the sum of the current densities from both normal reservoirs is equal to the current density from the single normal reservoir in the three terminal junction, \( i^{+1} + i^{+4} = i^+ \) and \( i^1_a = i^4_a \) (simply reflecting that one cannot create more Josephson current by adding more normal leads).

In this weak coupling limit the current in the horizontal lead \( I = I_2 = I_3 \) is given by

\[
I = I_{eq} + \frac{1}{2}[I_h(V_1) + I_r(V_4)] + \frac{1}{2}[I_a(V_1) + I_a(V_4)]
\]

(79)

with \( I_{eq}, I_r \) and \( I_a \) the same as in the three terminal case, given by Eq. (22)-(24). Noting the relations \( I_r(-V) = I_r(V) \) and \( I_a(-V) = -I_a(V) \), we see that (i) for bias \( V_1 + V_4 = 0 \) the anomalous current is zero, and (ii) for \( V_1 - V_4 = 0 \) the regular current is zero. We can thus control the regular and anomalous currents in the upper junction in Fig. 18 independently by adjusting either the potential difference \( V_1 - V_4 \) or the sum \( V_1 + V_4 \) between the normal reservoirs, keeping the other quantity constant.

### A. Injection current and conductance

The injection currents \( I_1 \) and \( I_4 \) in the four terminal device is qualitatively different from the injection current in the three terminal device, since in the four terminal junction the injected quasiparticles from one normal reservoir can travel directly through the junction to the other normal reservoir.

The current in leads \( j = 1, 4 \) can be written [see Eq. (26)]
\[ \mathcal{T} = \frac{1}{2} \left[ 1 - 2 \epsilon \cos 2\theta - D \cos \phi \right.
\]
\[ \left. - R \right] \left[ R_{\perp} - D_{\perp} - 2 \text{Re} \left[ d(d_{\perp} - r_{\perp}) \sin^2 \left( \frac{\phi}{2} \right) \right] + \left( \sin^2 2\theta - 1 + 2\epsilon + (D \cos \phi + R) \cos 2\theta \right) \left( R_{\perp} - D_{\perp} \right) \right]. \]  

Some general comments can be made about the conductance \( G = dI_{nj}/dV \). When the vertical and horizontal leads are decoupled \( (\epsilon \to 0) \), the conductance reduces to \( G = (e^2/h)D_{\perp} \), the conductance of the normal vertical channel. For finite coupling, an additional term is added to the conductance \( \delta G \sim \epsilon \). This additional term \( \delta G \) is dependent on the phase difference \( \phi \), but it is also, unlike for the three terminal junction, dependent on the scattering amplitudes \( r, d, r_{\perp} \) and \( d_{\perp} \), i.e. not only the scattering probabilities \( R, D, R_{\perp} \) and \( D_{\perp} \). This becomes clear when we note that we can rewrite the expression \( \text{Re} \left[ d(d_{\perp} - r_{\perp}) \right] = 1/2 \left[ R_{\perp} - D_{\perp} + ((R_{\perp} - D_{\perp})[R - D] - 4m|r_d^*| |m| |r^*|)/(1 - 4\epsilon) \right] \), i.e. dependent on \( \sigma_\perp \), just like the anomalous current. This contribution can be explained qualitatively by interference between quasiparticle paths where one path describes scattering in the vertical direction and the other one in the horizontal direction, thereafter leaving the junction.

In the case of zero temperature and voltage, the conductance becomes
\[ G = \frac{2e^2}{h} |D_{\perp}| + \epsilon \left[ 1 + \frac{D_{\perp} - R_{\perp} + \sin^2(\phi/2) \text{Re} \left[ d(d_{\perp} - r_{\perp}) \right]}{R + D \cos^2(\phi/2)} \right] \]  

which is independent of the length \( L \) of the junction. The conductance at zero phase difference given by \( G(\phi = 0) = (2e^2/h)|D_{\perp}|/(1 - 2\epsilon) \). From this value, the conductance then increases or decreases, depending on the phases of the scattering amplitudes, monotonically with \( \phi \to \pi \), as is seen in Fig. 19.
IX. CONCLUSIONS

We have analyzed the equilibrium and nonequilibrium Josephson currents and conductance in a ballistic, multi-terminal, single mode SNS junction. The nonequilibrium is created by means of quasiparticle injection from a normal reservoir connected to the normal part of the junction. By applying a voltage \( V \) to the normal reservoir, up to the superconducting gap \( \Delta \), the equilibrium current of a short junction \( L \ll \xi_0 \) can be suppressed. When the junction is longer \( L \geq \xi_0 \), the direction of the Josephson current changes sign as a function of applied voltage. For a junction longer then the thermal length \( L \gg \xi_T \), the equilibrium Josephson current is exponentially small. The nonequilibrium Josephson current in this regime is dominated by the anomalous current, arising from the modification of the current carrying Andreev states due to coupling to the normal reservoir. This anomalous current scales linearly with applied voltage and saturates at a magnitude of the order of the equilibrium current carried by a short junction, \( I \sim \epsilon \Delta / h \).

The conductance oscillates as a function of the phase difference \( \phi \) between the superconductors, with a period of \( 2\pi \) in a symmetric junction. The position of the conductance minima, \( \phi = 0 \) or \( \pi \), is dependent both on applied voltage and temperature. The conductance exhibits a finite bias anomaly, at \( eV \sim \hbar v_F / L \), the position of the first current carrying Andreev level.

Asymmetric injection gives rise to oscillations of all currents on the scale of \( eV \sim \hbar v_F / l \) where \( l \) is the length difference between the two leads. At temperatures above this energy, these oscillations are smeared and we get renormalized anomalous and injection currents that are \( \pi \)-periodic.

Introduction of barriers at the NS-interfaces give a strong length dependence of all currents, governed by the Breit Wigner resonances between the normal barriers. There are resonant current peaks at lengths where the normal electron and hole resonances cross.

Connecting a second normal reservoir to the normal part of the junction allows a clear separation between the injection current, flowing between the two normal reservoirs, and the Josephson current, flowing between the superconductors.

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APPENDIX A: CONTINUUM STATE CURRENT

Here we present formulas for the continuum current for a symmetric \((l = 0)\) three terminal junction without barriers at the NS-interfaces. The continuum current consists of particles injected from both the normal reservoir and the superconducting reservoirs. The current density in lead 2 from all injected quasiparticles from the superconductors is

\[
i^2_2 = \frac{e}{\hbar} \frac{2 \sin \beta \sinh \gamma_c}{Z_c} \left\{ \sin \phi \left[ (4D - 2D\epsilon - \epsilon^2) \cosh \gamma_c - 2D\epsilon e^{-\gamma_c} \right] - 4\sigma \sqrt{RD - \epsilon^2} \sin^2(\phi/2) \cosh \gamma_c \right\}, \tag{A1}\]

where \( Z_c = |\cos \beta (\cosh 2\gamma_c (1 - \epsilon) + \epsilon \sinh 2\gamma_c) - R - D \cos \phi|^2 + |\sin \beta (\sinh 2\gamma_c (1 - \epsilon) + \epsilon \cosh 2\gamma_c)|^2 \) and \( \gamma_c = \arccosh(E/\Delta) \). In lead 3 we get \( i^2_3(\phi) = -i^2_2(-\phi) \). This current density is an oscillating function of energy with largest amplitude for energies close to \( E = \Delta \) and is given at negative energies by \( i^2_2(E) = -i^2_2(-E) \).

For the particles injected from the normal reservoir, the sum current in lead 2 becomes

\[
i^2_2 = \frac{2e}{\hbar} \frac{\sin \beta \cosh \gamma_c}{Z_c} \left\{ \sin \phi \left[ 2D \cosh \gamma_c + \epsilon \sin \gamma_c \right] + 4\sigma \sqrt{RD - \epsilon^2} \sin^2(\phi/2) \cosh \gamma_c \right\}, \tag{A2}\]

with \( i^2_3(\phi) = -i^2_2(-\phi) \) in lead 3. The difference current in lead 2 has the form

\[
i^2_2 = \frac{2e}{\hbar} \frac{\cosh \gamma_c}{Z_c} \left\{ - \cosh \gamma_c [\epsilon \cos \phi + 2\sigma \sqrt{RD - \epsilon^2}/4 \sin \phi] + (1 - \epsilon) [\sin \gamma_c - \sin 3\gamma_c] + \epsilon \cosh 3\gamma_c \\
+ \cosh \gamma_c (R + D \cos \phi) + \cosh \gamma_c (\epsilon (1 + \cos \phi) + 2\sigma \sqrt{RD - \epsilon^2}/4 \sin \phi) \right\}, \tag{A3}\]

with \( i^2_3(\phi) = -i^2_2(-\phi) \) in lead 3. For negative energies we get \( i^2_2(E) = -i^2_2(-E) \) and \( i^2_3(E) = i^2_3(-E) \).

APPENDIX B: SPECTRAL DENSITIES OF CURRENTS IN THE WEAK COUPLING LIMIT

In this appendix we analyze the central quantity in the current density expressions \((19)-(23)\), given by

\[
\frac{\epsilon}{Z} = \frac{\epsilon}{(1 - \epsilon) \cos 2\theta - R \cos \chi - D \cos \phi} + \epsilon^2 \sin^2 2\theta \tag{B1}\]

in the limit of zero coupling \( \epsilon \to 0 \). We can conveniently rewrite

\[
\frac{\epsilon}{Z} = \frac{1}{\sin^2 2\theta \ F^2} + \epsilon^2 \tag{B2}\]

with \( F(E, \phi) = [(1 - \epsilon) \cos 2\theta - R \cos \chi - D \cos \phi] / \sin 2\theta \). In the limit of zero coupling the expression becomes

\[
\lim_{\epsilon \to 0} \frac{\epsilon}{F^2 + \epsilon^2} = \pi \delta(F) = \sum_{n, \pm} \left| \frac{\partial}{\partial E} \delta(E - E_{n, \pm}) \right|^{-1} \tag{B3}\]
with the energies $E_n^\pm$, given by
\[
\cos 2\theta - R \cos \chi - D \cos \phi = 0,
\] (B4)
being the energies of the bound Andreev states. By rewriting
\[
\frac{\partial}{\partial E} F = \frac{d\phi}{dE} \frac{\partial}{\partial \phi} F = \frac{d\phi}{dE} \frac{D \sin \phi}{\sin 2\theta}
\] (B5)
the expression (B3) becomes
\[
\lim_{\epsilon \to 0} \frac{\epsilon}{Z} = \sum_{n, \pm} \frac{\pi}{D} \sin \phi \sin 2\theta \left| \frac{dE}{d\phi} \right| \delta(E - E_n^\pm). 
\] (B6)

From Eq. (B4), the derivative of energy with respect to phase becomes
\[
\frac{dE}{d\phi} = -\frac{D \sin \phi}{2 \sin 2\theta \left( \frac{1}{\sqrt{D^2 - \phi^2}} + \frac{L}{\hbar v_F} + \frac{L}{\hbar v_F} R \frac{\sin \chi}{\sin \chi} \right)}.
\] (B7)

Using the relation (B4) and the fact that $R + D = 1$ we can rewrite
\[
|\sin 2\theta| = \sqrt{(D + R)^2 - (D \cos \phi + R \cos \chi)^2} = \sqrt{D^2 \sin^2 \phi + R^2 \sin^2 \chi + 2DR(1 - \cos \phi \cos \chi)}.
\] (B8)
which shows that $|\sin 2\theta| > |R| \sin \chi$. This gives that, since $L \geq l$ by definition, the factor in the parenthesis in the denominator in Eq. (B7) is always positive. The relation
\[
\operatorname{sgn} \left( \frac{dE}{d\phi} \sin \phi \sin 2\theta \right) = -1
\] (B9)
then follows from Eq. (B7). The Eqs. (B6) and (B9) are the technical result of this appendix.

APPENDIX C: CURRENTS FOR DIFFERENT LENGTHS

Here we list all expressions for the partial currents in different length limits. The junction considered is a symmetric ($l = 0$) three terminal junction without barriers at the NS-interfaces, in the weak coupling limit $\epsilon \ll 1$.

1. Short limit ($L = 0$)
\[
I_a^b = \frac{e \hbar v_F}{h} \frac{D \sin \phi}{2 \sqrt{1 - D \sin^2(\phi/2)}} \tanh(E_n^-/2kT), 
\] (C1)
\[
I_1^b = \frac{e \hbar v_F}{h} \frac{\sqrt{D} \sin(\phi/2)}{2 \sqrt{1 - D \sin^2(\phi/2)}} h(E_n^-), 
\] (C2)
\[
I_a = \frac{e \sigma D \sqrt{R} \sin \phi \sin(\phi/2)}{h} \frac{h(E_0)}{1 - D \sin^2(\phi/2)}, 
\] (C3)
\[
I_1 = \frac{e \epsilon \sqrt{D} \sin(\phi/2)}{h} \frac{h(E_0)}{1 - D \sin^2(\phi/2)}, 
\] (C4)

When there are two Andreev levels, with $1 \ll D$, $\beta/2 > \sqrt{D}$ and $E_0^+ \approx E_0^- \approx \Delta$, the currents become:
\[
I_a^b = \frac{e \Delta L}{h \xi_0} \frac{\sqrt{D} \sin(\phi)}{2 |\sin(\phi/2)| \sqrt{1 - D \sin^2(\phi/2)}} \left[ \tanh(E_n^-/2kT) - \tanh(E_n^+/2kT) \right],
\] (C5)
\[
I_1 = \frac{e \Delta L}{h \xi_0} \frac{\sqrt{D} \sin(\phi)}{4 |\sin(\phi/2)| \sqrt{1 - D \sin^2(\phi/2)}} [g(E_n^+) - g(E_n^-)],
\] (C6)
\[
I_a = \frac{e \Delta L}{h} \frac{\sqrt{R} \sin(\phi)}{2 \xi_0 (1 - D \sin^2(\phi/2))} [h(E_n^-) + h(E_n^+) + h(E_0^-)],
\] (C7)
\[
I_1 = \frac{e \Delta L}{h} \frac{\cos(\phi/2)}{\sqrt{1 - D \sin^2(\phi/2)}} [h(E_n^-) + h(E_n^+) + h(E_0^-)].
\] (C8)

2. Long limit ($L \ll \xi_0$)
\[
I_a^b = \frac{e \hbar v_F}{h} \frac{\sqrt{D} \sin(\phi)}{L} \frac{2 |\sin(\phi/2)| \sqrt{1 - D \sin^2(\phi/2)}} \left[ \tanh(E_n^-/2kT) - \tanh(E_n^+/2kT) \right] 
+ i^* \tanh(\Delta/2kT),
\] (C9)
\[
I_a = \frac{e \hbar v_F}{h} \frac{\sqrt{D} \cos(\phi/2)}{L} \left( \sum_{n=0}^{N} [g(E_n^-) - g(E_n^+)] \right) + i^* \frac{1}{2} g(\Delta),
\] (C10)
\[
I_a = \frac{e \hbar v_F}{h} \frac{\sigma \sqrt{D} R \sin \phi}{L} \left( \sum_{n=0}^{N} [h(E_n^-) + h(E_n^+)] \right),
\] (C11)
\[
I_1 = \frac{e \hbar v_F}{h} \frac{\cos(\phi/2)}{L} \left( \sum_{n=0}^{N} [h(E_n^-) + h(E_n^+)] \right),
\] (C12)
where $h(E) = \tanh[(E - eV)/2kT] - \tanh[(E + eV)/2kT]$ and $g(E) = \tanh[(E + eV)/2kT] + \tanh[(E - eV)/2kT] - 2\tanh(E/2kT)$.

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