Variational analysis of deconfinement in gluodynamics

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The deconfinement transition in 3+1 dimensional gluodynamics is studied using the gauge invariant variational method introduced by Kogan and Kovner a few years ago. We identify a first order phase transition, characterized by a discontinuous jump in the entropy of the system, resulting in a transparent picture of the mechanism of deconfinement. The calculation of the ratio of the transition temperature to the mass of the lightest glueball in the model yields 0.18 in complete agreement with the lattice estimate.

1 Introduction

In the almost 25 years since the pioneering work of Polyakov \cite{1} and Susskind \cite{2} much effort has been devoted to attempts to understand both the basic physics and quantitave features of the deconfining phase transition of QCD.

The high temperature phase, i.e. for temperatures well above the critical temperature \( T_c \), is becoming well understood, and is widely believed to resemble a plasma of almost free quarks and gluons. However, the transition region, \( T_c < T < 2T_c \), is very poorly understood. This region is the most interesting one since it is there that the transition between ‘hadronic’ and ‘partonic’ degrees of freedom occurs.

The study of the transition region is a complicated and inherently non-perturbative problem which has mostly evaded treatment by analytical methods. The variational method introduced several years by Kogan and Kovner \cite{3} appears well suited to a discussion of such a problem.

Recently, the method was applied to the study of the phase transition in \( SU(N) \) gluodynamics. As a result of these studies \cite{4,5}, we identified a first order phase transition at a transition temperature, in units of the lightest state in the theory, of \( T_c = 0.18 \), in complete agreement with the lattice estimate for \( SU(3) \) \cite{6}.

We minimize the relevant thermodynamic potential at finite temperature, i.e. the Helmholtz free energy, on a set of suitably chosen trial density matrix functionals. As with any variational calculation there is an element of guesswork involved. However, the choice of the trial states is dictated by rather general principles. As in the quantum mechanical case, the trial state ought to be simultaneously rich enough to span the interesting physics and to lead to calculable results. This last requirement is particularly stringent, since only functional integrals of the gaussian form can be explicitly evaluated. Specific to the quantum field case are the requirements of gauge invariance and of correct UV behaviour. To understand this last statement, it suffices to mention first, that the free energy is dominated by UV modes while we are interested in the IR physics and, second, that in an asymptotically free theory such as gluodynamics, low and high momentum modes are, in general, non-separable. The method of \cite{3,4,5} to be discussed below accommodates all these requirements, leading to what is, to the author’s best knowledge, the only first principle analytical analysis of the deconfinement phase transition in gluodynamics.

2 The variational Ansatz

We consider density matrices which, in the field basis, have Gaussian matrix elements and where gauge invariance is explicitly imposed by projection onto the gauge-invariant sector of the Hilbert space

\[
\rho[A, A'] = \int DU \exp \left\{ -\frac{1}{2} \int_{x,y} \left[ A^a_i(x) G^{-1}_{ij} A^b_j(y) + A^a_i(x) U^a_{ij} A^b_j(y) - 2A^a_i(x) H^{ab}_{ij}(x,y) A^b_j(y) \right] \right\},
\]

where \( \int_{x,y} = \int d^3x \, d^3y \), \( DU \) is the \( SU(N) \) group-invariant measure, and under an \( SU(N) \) gauge transformation \( U \)

\[
A^a_i(x) \rightarrow A^{Ua}_{ij}(x) = S^{ab}(x) A^b_j(x) + \lambda^a_i(x),
\]
with
\[ S^{ab} = \frac{1}{2} \text{tr}(\sigma^a U^\dagger \sigma^b U), \] (3)
\[ \lambda^a_i = \frac{i}{g} \text{tr}(\sigma^a U^\dagger \partial U), \] (4)
and \( \frac{\sigma^a}{2} \) form an \( N \times N \) Hermitian representation of \( SU(N) \): \( [\frac{\sigma^a}{2}, \frac{\sigma^b}{2}] = \frac{if^{abc}}{2} \sigma^c \) with normalization \( \text{tr}(\sigma^a \sigma^b) = 2 \delta^{ab} \).

We take the variational functions diagonal in both colour and Lorentz indices, and translationally invariant
\[ G^{-1}_{ij}(x, y) = \delta^{ab} \delta_{ij} G^{-1}(x - y), \]
\[ H_{ij}^a(x, y) = \delta^{ab} \delta_{ij} H(x - y). \] (5)
Further, we split the momenta into high and low modes with \( k \leq M \) and restrict the kernels \( G^{-1} \) and \( H \) to the one parameter momentum space forms
\[ G^{-1}(k) = \begin{cases} M, & k < M \\ M^2, & k > M \end{cases}, \]
\[ H(k) = \begin{cases} M, & k < M \\ 0, & k > M \end{cases}. \] (6)

The logic behind this choice of ansatz is the following. At finite temperature we expect \( H(k) \) to be roughly proportional to the Boltzmann factor \( \exp[-E(k)/\beta] \). In our ansatz, the role of one particle energy is played by the variational function \( G^{-1}(k) \) and its form is motivated by the propagator of a massive scalar field, i.e. \( (k^2 + M^2)^{1/2} \). We will be interested only in temperatures close to the phase transition, and those we anticipate to be small, \( T_c \leq M \). For those temperatures one particle modes with momenta \( k > M \) are not populated, and we thus put \( H(k) = 0 \). For \( k \leq M \) the Boltzmann factor is non-vanishing, but small. Further, it depends only very weakly on the value of the momentum. With the above restrictions on the kernels, only two variational parameters, \( M \) and \( H \), remain.

Importantly, the density matrix functional in eq. (1) describes, for \( H = 0 \), a pure state \( \rho = |\Psi[A]\rangle \langle \Psi[A]| \) where \( \Psi[A] \) are Gaussian wave functionals. For \( H \neq 0 \), eq. (1) describes a mixed state with \( |H| \) proportional to the entropy of the trial density matrix.

The expectation value of a gauge invariant operator in the variational state eq. (1) is then given by
\[ \langle O \rangle_{A, A'} = Z^{-1} \text{Tr}(\rho O) \]
\[ = Z^{-1} \int DU D\lambda A O(A, A') \]
\[ \exp \left\{ -\frac{1}{2} \left[ AG^{-1} A + A' U G^{-1} A' U - 2 A H A' U \right] \right\} \bigg|_{A' = A}, \] (7)
where \( Z \) is the normalization of the trial density matrix \( \rho \), i.e.
\[ Z = \text{Tr} \rho \]
\[ = \int DU D\lambda A O(A, A') \]
\[ \exp \left\{ -\frac{1}{2} \left[ AG^{-1} A + A' U G^{-1} A' U - 2 A H A' U \right] \right\} \] (8)
To evaluate the above expressions we first perform, for fixed \( U(x) \), the gaussian integration over the vector potential \( A \). For \( Z \) we get, in leading order in \( H \),
\[ Z = \int DU \exp \left\{ -\frac{1}{2} \lambda \left( \frac{G^{-1}}{2} + \frac{H}{4}(S + S^T) \right) \right\} + \frac{3}{4} H G \text{tr}(S + S^T) \}. \] (9)
We now integrate out the high momentum, \( k^2 > M^2 \), modes of \( U \) perturbatively to one-loop order. This effects a renormalization group transformation on the low modes, replacing the bare coupling \( g^2 \) by the running coupling \( g^2(M) \) [3, 7]. To one-loop accuracy, the coupling \( g^2(M) \) runs identically to the Yang-Mills coupling [7].

3 The effective \( \sigma \)-model

The normalization \( Z \) can be then interpreted as the generating functional
\[ Z = \text{Tr} \rho = \int DU e^{-S(U)} \] (10)
for an effective non-linear \( \sigma \)-model for the low momentum modes \( (k^2 < M^2) \) defined by the action
\[ S(U) = \frac{M}{2g^2} \text{tr}(\partial U \partial U^\dagger) \]
\[ - \frac{H}{8g^2} \text{tr}\left[(U^\dagger \partial U - \partial U^\dagger U)(\partial U U^\dagger - U \partial U^\dagger)\right] \]
\[ - \frac{1}{4\pi^2} HM^2 \text{tr} U U^\dagger \text{tr} U, \] (11)
where \( U \) independent pieces have been dropped.

The matrix \( U \) plays the same role as Polyakov’s loop \( P \) at finite temperature — the functional integration over \( U \) projects out the physical subspace of the large Hilbert space on which the Hamiltonian of glueodynamics is defined.

This \( \sigma \)-model has a phase transition [3] at the critical point (for \( \Lambda_{QCD} = 150 \text{MeV}, N = 3 \) and with the one-loop Yang-Mills \( \beta \) function)
\[ M_c = \Lambda_{QCD} e^{2\beta} = 8.86 \Lambda_{QCD} = 1.33 \text{Gev}. \] (12)
For \( M < M_c \), the \( \sigma \)-model is in a disordered, \( SU(N)_L \otimes SU(N)_R \) symmetric, phase with massive excitations and where \( \langle U \rangle = 0 \). Since \( U \) is the Polyakov loop, this corresponds to a confined state. When \( M > M_c \), the \( \sigma \)-model is in an ordered, \( SU(N)_V \) symmetric, phase with massless Goldstone bosons for which \( \langle U \rangle \neq 0 \), corresponding to a deconfined state.

With this analysis we have established a correspondence between the \( \sigma \)-model phase transition and the deconfinement transition in \( SU(N) \) gluodynamics.

In fact, this correspondence can be argued in rather general terms. Instead of restricting ab initio the density matrix to the form eq. (1), imagine that we take some arbitrary gauge-invariant density matrix ansatz that depends on the \( A \) fields and is integrated over the \( U \) fields. We allow this new ansatz (and whatever kernels it may contain) to remain arbitrary until we have no choice but to restrict it. Then we integrate out the \( A \) fields to obtain a partition function of \( \sigma \) fields as before. This effects a renormalisation of the \( \sigma \)-model. In fact, this correspondence can be argued in rather general terms. Instead of restricting ab initio the density matrix to the form eq. (1), imagine that we take some arbitrary gauge-invariant density matrix ansatz that depends on the \( A \) fields and is integrated over the \( U \) fields. We allow this new ansatz (and whatever kernels it may contain) to remain arbitrary until we have no choice but to restrict it. Then we integrate out the \( A \) fields to obtain a partition function of \( \sigma \) fields as before. This effects a renormalisation of the \( \sigma \)-model. In fact, this correspondence can be argued in rather general terms. Instead of restricting ab initio the density matrix to the form eq. (1), imagine that we take some arbitrary gauge-invariant density matrix ansatz that depends on the \( A \) fields and is integrated over the \( U \) fields. We allow this new ansatz (and whatever kernels it may contain) to remain arbitrary until we have no choice but to restrict it. Then we integrate out the \( A \) fields to obtain a partition function of \( \sigma \) fields as before. This effects a renormalisation of the \( \sigma \)-model.

Thus, in order to study deconfinement in \( SU(N) \), our aim should be to model the physics of each \( \sigma \)-model phase as accurately as possible and calculate the transition scale \( M_c \). We then calculate the free energy of \( SU(N) \) in each phase, including any possible contribution from the high modes, at temperature \( T \) and extract the minimal free energy. The deconfinement transition occurs at the temperature for which the free energies calculated in the ordered and disordered phases of the low mode \( \sigma \)-model coincide.

Although we will take eq. (1) as the ansatz for the density matrix, we shall keep the kernels \( G^{-1} \) and \( H \) arbitrary until we have no choice but to restrict them.

### 4 Calculation of the free energy

The Helmholtz free energy \( F \) of the density matrix \( \rho \) is given by

\[
F = \langle H \rangle - T \langle S \rangle,
\]

where \( H \) is the standard Yang-Mills Hamiltonian

\[
H = \int d^3x \left[ \frac{1}{2} F_{ai}^2 + \frac{1}{2} B_{ai}^2 \right],
\]

with

\[
E_i^a(x) = i \frac{\delta}{\delta A_{ai}^a(x)},
\]

\[
B_i^a(x) = \frac{1}{2} \epsilon_{ijk} \{ \partial_j A_{bk}^a(x) - \partial_k A_{bj}^a(x) + g f^{abc} A_{bj}^a(x) A_{bk}^c(x) \},
\]

\( S \) is the entropy, and \( T \) is the temperature.

Thus

\[
F = \frac{1}{2} \left( \text{Tr}(E^2 \rho) + \text{Tr}(B^2 \rho) \right) + T \cdot \text{Tr}(\rho \ln \rho).
\]

In the disordered phase, no progress seems possible without restricting the arbitrary kernels. Following [Ref. 4], we adopt the forms eq. (15). For small \( H \), we consider only the first non-trivial order in \( H \), that is a term of \( o(H \ln H) \) in the entropy. This term can be written as a product of left \( SU(N) \) and right \( SU(N) \) currents and does, therefore, vanish in the disordered, \( SU(N)_L \otimes SU(N)_R \) symmetric, phase [Ref. 4]. The remaining contribution to the free energy, the average of the Hamiltonian, is evaluated in the mean field approximation [Ref. 5]. The free energy is minimized for \( M = M_c \approx 1.33 \text{GeV} \)

\[
F_{\text{dis}} = -\frac{N^2 M_c^4}{30 \pi^2}.
\]
The simplest option to evaluate the free energy in the disordered phase is to use perturbation theory. Perturbation is certainly appropriate for large enough values of $M$, where the expectation value of the $U$ field is close to unity. From numerical studies it is known that the transition occurs when the expectation value of $U$ is greater than .5. We can thus expect perturbation theory to be qualitatively reliable down to the transition point. In the leading order perturbation theory approximation to the ordered phase of the $\sigma$-model, however, minimisation with respect to arbitrary kernels $G^{-1}$ and $H$ for both high and low modes is possible. Further, the analysis can be carried out to all orders in the thermal disorder kernel $H$.

In this approximation, the $U$ matrices can be parameterised in the standard exponential form and expanded in the coupling $g$

$$\begin{align*}
    U &= \exp \left\{ ig\varphi^a \pi^a \right\} = 1 + ig\varphi^a \pi^a / 2 + \ldots \\
    4T &= \frac{N^2 - 1}{45} \\
    H &= 2p \left( \frac{1 + e^{-2p}}{1 - e^{-2p}} \right) \end{align*}$$

Hence at leading order one can take

$$\begin{align*}
    U &\approx 1, \\
    \partial_i U &\approx ig\partial_i \varphi^a \pi^a / 2.
\end{align*}$$

Thus, the gauge transformations reduce to

$$A_i^a \to A_i^a - \partial_i \varphi^a$$

and the Hamiltonian reduces to

$$\mathcal{H} = \frac{1}{2} \left[ E_i^{a2} + (\epsilon_{ijk} \partial_j A_k^a)^2 \right].$$

These last two equations describe the theory $U(1)^{N^2-1}$; in the leading order of $\sigma$-model perturbation theory, the $SU(N)$ Yang–Mills theory reduces to the $U(1)^{N^2-1}$ free theory. Moreover, the density matrix eq. becomes Gaussian again, because the gauge transformations are linear. One has

$$\rho[A, A^\prime] = \int D\varphi \ \exp \left\{ - \frac{1}{2} \left[ AG^{-1} A^\prime + (A^\prime - \partial \varphi) G^{-1} (A^\prime - \partial \varphi) - 2AH(A^\prime - \partial \varphi) \right] \right\}. $$

Now the theory of $N^2 - 1$ $U(1)$ free fields in 3+1 dimensions is completely tractable; the variational analysis for the $U(1)$ theory (with Gaussian ansatz) was discussed in [10]. The free energy in momentum space in terms of the arbitrary kernels $G^{-1}$ and $H$ is

$$F = \frac{N^2 - 1}{2} \int \frac{d^3 p}{(2\pi)^3} \left[ G^{-1}(1 + GH) + p^2 G(1 - GH)^{-1} \right. \right.$$}

$$- 4T \left[ \log \left\{ \frac{GH}{(1 - (GH)^2)^{1/2} - (1 - GH)} \right\} \right. \right.$$}

$$\left. - \log \left\{ \frac{1 - (1 - (GH)^2)^{1/2}}{GH} \right\} \right] \right.

\left. \left. \cdot \right( \frac{1 - (1 - (GH)^2)^{1/2}}{(1 - (GH)^2)^{1/2} - (1 - GH)} \right) \right).$$

The kernels which minimise the free energy are

$$G^{-1} = p \left( \frac{1 + e^{-2p}}{1 - e^{-2p}} \right),$$

$$H = 2p \left( \frac{1 + e^{-2p}}{1 - e^{-2p}} \right)$$

and the minimal value of the free energy at temperature $T$ is

$$F = -\frac{\pi^2 N^2 T^4}{45}.$$}

So we see that the free energy of $SU(N)$ is minimised with $M = M_c$ in the disordered phase of the $\sigma$-model for temperatures from zero up to a temperature $T_c$ where

$$F = -\frac{N^2 M_c^4}{30\pi^2} = -\frac{\pi^2 N^2 T_c^4}{45},$$

which in turn implies

$$T_c = \left( \frac{3}{2} \right)^{1/4} \frac{M_c}{\pi} \simeq 470\text{MeV}.$$}

5 Conclusions

We find that the deconfinement phase transition is strongly first order with a transition temperature of 470MeV. The transition is due to an abrupt jump in the entropy. In the disordered phase it is zero since glueballs are heavy and, therefore, their contribution is suppressed by the Boltzmann factor $\exp(-M_g/T)$. In the ordered phase, the entropy is non-vanishing and proportional to the number of coloured states.

The major uncertainty in the calculation is due to the fact that the scale $M_c$ is very sensitive to the mean field approximation. It is, therefore, only meaningful to compare the results of our approach with lattice estimates for $M_c$ independent quantities. For one such quantity, the transition temperature in units of the
mass of the lightest glueball in the model, i.e. \(2M_c\) \[\text{[2]}\], we find

\[
\frac{T_c}{2M_c} = \frac{1}{2\pi} \left(\frac{3}{2}\right)^{1/4} \simeq 0.18. \quad (28)
\]

which is in agreement with the lattice result.

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