A Feasibility Study of Differentially Private Summary Statistics and Regression Analyses for Administrative Tax Data

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Abstract: Federal administrative tax data are invaluable for research, but because of privacy concerns, access to these data is typically limited to select agencies and a few individuals. An alternative to sharing microlevel data is a validation server, which allows individuals to query statistics without directly accessing the confidential data. This paper studies the feasibility of using differentially private (DP) methods to implement such a server. We provide an extensive study on existing DP methods for releasing tabular statistics, means, quantiles, and regression estimates. We also include new methodological adaptations to existing DP regression methods for using new data types and returning standard error estimates. We evaluate the selected methods based on the accuracy of the output for statistical analyses, using real administrative tax data obtained from the Internal Revenue Service. Our findings show that a validation server is feasible for simple, univariate statistics but struggles to produce accurate regression estimates and confidence intervals. We outline challenges and offer recommendations for future work on validation server frameworks. This is the first comprehensive statistical study of DP regression methodology on a real, complex dataset, that has significant implications for the direction of a growing research field and public policy.

1 Introduction

Federal tax data, derived from individuals’ and businesses’ tax and information returns, are invaluable resources for understanding the economic effects of policies, social and economic forces, and
more on individuals, families, and firms. For example, Chetty et al. (2014) used tax data to study economic mobility across generations and how elementary school teacher quality affects economic outcomes later in life. However, full access to these data is available only to select government agencies, to a very limited number of researchers working in collaboration with analysts in those agencies, or through highly selective programs within the Internal Revenue Service (IRS) Statistics of Income (SOI) Division. Additionally, the manual process of vetting each statistical release for disclosure risks is labor intensive and imperfect because it relies on subjective human review.

1.1 Background on Accessing Confidential Data

Typically, researchers access federal, confidential data in two ways: (1) direct access to the confidential data or, (2) public statistics and public data files. This framework for accessing data is normal for most federal statistical agencies in the United States, such as the U.S. Census Bureau and Bureau of Labor Statistics. However, both options have major drawbacks for accessing administrative tax data. At SOI, researchers must undergo an extensive clearance process to gain direct access to the confidential data. However, this clearance process can take several months and has eligibility requirements, such as being a U.S. citizen. When releasing SOI public statistics and public data files, SOI has progressively restricted and distorted the public information over the years due to general concern for protecting participants’ data privacy (Bryant et al., 2014).

The concepts of verification and validation servers provide a potential middle ground between these two extremes. We carefully delineate these terms, which are sometimes confused. A verification server is a system that provides information about the quality of statistical inference derived from publicly released synthetic data. In other words, the verification server might report whether inferences about the sign, statistical significance, magnitude of the estimates, and among others, derived from the synthetic data are consistent with the confidential data\(^1\). A validation server, the focus of this paper, is a system that allows users to submit and run statistical queries on the confidential (validation) data after the users have developed their queries using the synthetic data.

As an example of a validation server, the U.S. Census Bureau provides a framework to vali-

\(^1\)Though we do not consider verification servers in this paper, we note that Barrientos et al. (2018) created a pilot verification server for the U.S. Office of Personnel Management that allowed users to verify synthetic data estimates while satisfying differential privacy.
date analyses for two experimental synthetic databases via the Synthetic Data Server at Cornell University: the Synthetic Longitudinal Business Database and the Survey of Income and Program Participation’s (SIPP) Synthetic Beta Data Product (Benedetto et al., 2013; Drechsler and Vilhuber, 2014). After researchers ensure their code runs successfully and completely on the synthetic data, researchers can then submit it to the Census Bureau to be run on the confidential data. Disclosure control experts review outputs from the confidential data analyses before they are released back to the researchers. This process resembles a controlled remote access system except the researchers must also document the synthetic data usefulness and provide potential improvements for future synthetic data releases, such as the SIPP Synthetic Beta v7.0 (Benedetto et al., 2018).

This validation framework allows researchers to access otherwise restricted data. Though useful, staff must still manually review the results for disclosure risks, which can delay releasing the results to the researchers. To address the manual review issue, one might adopt methods that satisfy differential privacy (DP) (Dwork et al., 2006), which enables privacy-loss accounting and permits cumulative confidentiality risk control. Because DP provides privacy guarantees \textit{a priori}, privacy-loss accounting and is agnostic to the confidential data, a well-constructed DP validation server could remove the need for manual review. This would significantly increase the likelihood that statistical agencies would adopt validation servers, but as of yet no work has considered the practical feasibility of a DP validation server for common statistical queries.

1.2 Contributions from this Paper

We explore the idea of creating an automated validation server that releases statistics, such as regression estimates and their associated standard errors, that meet a pre-defined privacy guarantee within the DP framework. The automated validation server resembles the current validation framework from the U.S. Census Bureau but would reduce the burden of manual review and could allow research on a wider range of topics that use IRS tax data.

To understand the feasibility of creating an automated validation server, we conduct an extensive study on state-of-the-art DP mechanisms for releasing tabular statistics, mean statistics, quantile statistics, and statistics from regression analyses with cross-sectional data. Based on the current DP literature and informal interviews with tax experts, we prioritize these analyses as a
first stage of a potential validation server for administrative tax data. There are several other analyses, such as model selection or nonparametric regression discontinuity and kink designs, that have been identified as important but the current DP methodology does not support those methods or is at an early stage. In some cases, such as model selection or regression discontinuity design, the goal is still to carry out linear regression, but additional steps are needed to identify the correct design matrix or functional form of the model in a private way.

To measure feasibility, we test the DP methods on the SOI Public Use File (PUF). SOI annually develops and releases this database of sampled individual income tax returns after applying privacy protections. Several organizations, such as the American Enterprise Institute, the Urban-Brookings Tax Policy Center, and the National Bureau of Economic Research develop PUF-based microsimulation models that help inform the public on potential impacts of policy proposals (Bierbrauer et al., 2021; DeBacker et al., 2019; McClelland et al., 2019). But access to this public file is limited, and we cannot provide the full data for others to replicate our study results using the PUF. For this reason, the results presented in the main body of this paper use the 1994 to 1996 Current Population Survey Annual Social and Economic Supplements (CPS ASEC), which are publicly accessible through IPUMS USA (Ruggles et al., 2021). The CPS ASEC has similar variables as the PUF, allowing for similar queries to those we run on the PUF. We provide parallel results using the PUF in the Supplemental Materials. Additionally, we have a public repository containing all of our code, data, and results.²

Testing DP methods on this specific case study is a significant contribution because the data privacy and confidentiality community can assess how robust and accurate these methods are under conditions commonly encountered in real-life applications. Further, our work supports SOI’s goal of safely expanding access to the IRS data, which few researchers can currently use.

Prior to testing the methods, we develop a selection criteria that excludes methods which work in theory only under specific conditions that would not normally be met in practice or cannot be tested. We find that many of the proposed methods in the literature could not be practically used in our setting. When selecting DP algorithms, we provide descriptions of each method, consider

²GitHub repo website, https://github.com/UrbanInstitute/formal-privacy-comp-appendix
their ease of implementation, determine whether they require any additional tuning parameters, and assess their computational feasibility. This assessment will be useful for the statistical data privacy and confidentiality community in developing more applicable DP methods for the future.

We also contribute new methodology for obtaining estimates using DP regression. Specifically, we improve the sensitivity calculation for fitting the models with binary or categorical predictors and provide means of obtaining standard error estimates in addition to point estimates. Many of the existing methods for DP regression only provide point estimates, but we require that the validation server provides full inference. Further, obtaining standard errors or confidence intervals is crucial to most statistical analyses.

We define feasibility based on the impact of DP methods on analyses for making public policy decisions and their accuracy according to several utility metrics. We evaluate the methods using real data and identify how specific data features might challenge some of these methods. To the best of our knowledge, this paper provides the first comprehensive case study on DP methodology for various statistical analyses. DP is a rapidly growing and popular field of study, but the vast majority of the work has focused on theoretical developments. Our results and conclusions will be vital in informing the current state of DP methods, addressing practical problems, and identifying directions for future work, particularly for statistical inference.

We organize the remainder of the paper as follows. Section 2 provides the relevant definitions and theorems from DP and the fundamental DP mechanisms used in this paper. Section 3 examines several categories of DP methods, providing a thorough discussion on which methods met our selection criteria, the methodological extensions we provide, and which methods could not be implemented in practice. Section 4 discusses the case studies and assumptions we made about the data and data analysts’ knowledge. Section 5 covers how we define the data utility and what metrics we use to evaluate the DP methods. Sections 6 and 7 compare the selected DP methods on the CPS ASEC data and determines the feasibility of using these methods for a validation server. Concluding remarks and areas for future work are given in Section 8. Additional technical details on the methods and expanded results for our case studies can be found in the Supplementary Materials.
2 Differential Privacy

Differential privacy (DP) is a mathematical framework for providing a provable and quantifiable amount of privacy protection. Various definitions of DP exist, and given a particular definition, one can prove whether a method satisfies that type of DP. We refer to such methods as differentially private (DP\(^1\)) methods. Satisfying DP is a provable feature of a method, not the data—a common misconception.

In this section, we reproduce the different definitions of DP used in this study, and three key theorems that are used in many methods studied in this paper. We use the following notation: \(X \in \mathbb{R}^{n \times r}\) is the confidential dataset representing \(n\) data points and \(r\) variables and \(M : \mathbb{R}^{n \times r} \to \mathbb{R}^k\) denotes the statistical query, i.e., \(M\) is a function mapping \(X\) to \(k\) real numbers. We denote a randomized or noisy version of \(M\) using \(\mathcal{M} : \mathbb{R}^{n \times r} \to \mathbb{R}^k\), which is a function that satisfies DP.

### 2.1 Definitions of Differential Privacy

**Definition 1.** *Differential Privacy* ([Dwork et al., 2006]): A sanitization algorithm, \(\mathcal{M}\), satisfies \(\epsilon\)-DP if for all subsets \(S \subseteq \text{Range}(\mathcal{M})\) and for all \(X, X'\) such that \(d(X, X') = 1\),

\[
\frac{\Pr(\mathcal{M}(X) \in S)}{\Pr(\mathcal{M}(X') \in S)} \leq \exp(\epsilon)
\]

(1)

where \(\epsilon > 0\) is the privacy loss budget and \(d(X, X') = 1\) represents the possible ways that \(X'\) differs from \(X\) by one record.

Definition 1 is known as \(\epsilon\)-DP. At a high level, DP links the potential for privacy loss to how much the answer of a query (such as a statistic) is changed given the presence or absence of any possible person’s data from any possible data set. The role of \(\epsilon\) is to control the privacy loss. Intuitively, when \(\epsilon\) decreases, the maximum distance between the probability distributions of \(\mathcal{M}(X)\) and \(\mathcal{M}(X')\) become smaller, indicating that \(\mathcal{M}(X)\) and \(\mathcal{M}(X')\) are less distinguishable.

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3A note on terminology: In the context of DP, the terms mechanism, algorithm, and method are often used interchangeably to describe the process of releasing a private statistical output. Sometimes this refers to a simple process, such as adding noise directly to a computed statistic. Other times it refers to more complicated processes, such as using post-processing (explained later in this section). We do not see a clear delineation in the literature when using the three terms. More crucially is that anything referred to as a DP method, DP mechanism, or DP algorithm must provably satisfy the relevant definition of DP.

4Note that we use DP as an acronym for both “differential privacy” and “differentially private”.

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in distribution. Hence, users cannot determine whether the mechanism’s outputs are based on $X$ or $X’$, which in turn protects the confidential information of that record that distinguishes $X$ and $X’$. Thus, low values of $\epsilon$ indicate high privacy levels and vice versa. $\epsilon$-DP can also be interpreted from a more statistical perspective in the context of hypothesis testing and under both frequentist (Wasserman and Zhou, 2010) and Bayesian (Desfontaines, 2021; Kasiviswanathan and Smith, 2014) paradigms.

There are two definitions of what it means to differ by one record. One assumes the presence or absence of a record, where the dimensions of $X$ and $X’$ differ by one row. The other assumes the change of the value in one record, where $X$ and $X’$ have the same dimensions. Li et al. (2016) refers to these as unbounded DP for presence or absence of a record and bounded DP for the change of a record. They state that unbounded DP satisfies an important composition theorem, which we will discuss later in this section (see Theorem 1), whereas bounded DP does not. In this paper, we assume unbounded DP, because we rely on Theorem 1.

Several relaxations of $\epsilon$-DP have been developed in order to inject less noise into the output, such as $(\epsilon, \delta)$-DP (Dwork et al., 2006), probabilistic DP (Machanavajjhala et al., 2008), concentrated DP (Dwork and Rothblum, 2016), Rényi differential privacy (Mironov, 2017), and zero-concentrated DP (Bun and Steinke, 2016). Although these definitions use the same provable privacy framework, they utilize alternative parameters offering different privacy guarantees. In return, they allow more possibilities for the type of noise added.

**Definition 2.** $(\epsilon, \delta)$-**Differential Privacy** (Dwork et al., 2006): A sanitization algorithm, $M$, satisfies $(\epsilon, \delta)$-DP if for all $X, X’$ that are $d(X, X’) = 1$,

$$\Pr(M(X) \in S) \leq \exp(\epsilon) \Pr(M(X’) \in S) + \delta$$

where $\delta \in [0, 1]$.

Definition 2 provides a simple relaxation of Definition 1 by adding the parameter $\delta$, which allows the privacy loss associated with the $\epsilon$ bound to fail at a rate no greater than $\delta$. Definition 1 can also be defined as a special case of $(\epsilon, \delta)$-DP when $\delta = 0$. 

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Dwork and Rothblum (2016) proposed concentrated DP, which aimed to reduce the total privacy loss over multiple queries. Bun and Steinke (2016) later improved this definition of privacy with zero-concentrated DP (zCDP or $\rho$-zCDP), given in Definition 3.

**Definition 3. Zero-Concentrated Differential Privacy (Bun and Steinke, 2016):** A sanitization algorithm, $M$, satisfies $(\xi, \rho)$-zero-concentrated differential privacy if for all $X, X'$ that are $d(X, X') = 1$ and $\alpha \in (1, \infty)$,

$$D_\alpha(M(X)||M(X')) \leq \xi + \rho \alpha,$$

where $D_\alpha(M(X)||M(X'))$ is the $\alpha$-Rényi divergence between the distribution of $M(X)$ and $M(X')$. $\rho$-CDP is a special case of $(\xi, \rho)$-zCDP when $\xi = 0$.

Bun and Steinke (2016) provide results that allows us to relate $\epsilon$-DP and $(\epsilon, \delta)$-DP algorithms to $\rho$-zCDP equivalents. They show in their Proposition 4 that if $M$ satisfies $\epsilon$-DP, then $M$ satisfies $(\frac{1}{2} \epsilon^2)$-zCDP. They also show in their Proposition 3 that if $M$ satisfies $\rho$-zCDP, then $M$ is $(\rho + 2\sqrt{\rho \log(1/\delta)}, \delta)$-DP for any $\delta > 0$. In their Lemma 3.6, they present a slight strengthening of this result stating that if $M$ satisfies $(\xi, \rho)$-zCDP, then $M$ is $(\xi + \rho + \sqrt{4\rho \log(\sqrt{\pi \rho}/\delta)}, \delta)$-DP for any $\delta > 0$. We use the result of this Lemma to express any $\rho$-zCDP algorithm in terms of $(\epsilon, \delta)$-DP. For a given $(\epsilon, \delta)$, we must find the value of $\rho$ that satisfies $\epsilon = \rho + \sqrt{4\rho \log(\sqrt{\pi \rho}/\delta)}$.

### 2.2 Global Sensitivity and Differentially Private Mechanisms

In this section, we introduce the concept of global sensitivity and present three of the fundamental mechanisms that satisfy $\epsilon$-DP and $(\epsilon, \delta)$-DP and form the building blocks of many of the DP algorithms we test in this paper.

Independent of the values of $\epsilon$ and $\delta$, an algorithm that satisfies $\epsilon$-DP or $(\epsilon, \delta)$-DP must adjust the amount of noise added to the output based on the maximum possible change between any two databases that differ by one row. This is commonly referred to as the global sensitivity (GS), given in Definition 4.

**Definition 4. $l_1$-Global Sensitivity (Dwork et al., 2006):** For all $X, X'$ such that $d(X, X') = 1$,
the global sensitivity of a function \( M \) is

\[
\Delta_1(M) = \sup_{d(X,X')=1} \| M(X) - M(X') \|_1
\] (4)

We can calculate global sensitivity under different norms. For instance, \( \Delta_2(M) \) represents the \( l_2 \) norm GS (\( l_2 \)-GS) of the function \( M \). Although the definition is straightforward, calculating the GS can often be difficult in practice. For instance, we cannot calculate a finite GS of one of the most common statistics, the sample mean, if the variable is not bounded (or the bound is not known).

A commonly used mechanism satisfying \( \epsilon \)-DP is the Laplace mechanism, given in Definition 5. Dwork et al. (2006) proved it satisfies \( \epsilon \)-DP and uses the \( l_1 \)-GS. Another popular mechanism is the Gaussian mechanism, given in Definition 6, which uses the \( l_2 \)-GS of the statistical query. Dwork and Roth (2014) showed the Gaussian mechanism satisfies \((\epsilon,\delta)\)-DP.

**Definition 5. Laplace mechanism** (Dwork et al., 2006): Given any function \( M : \mathbb{R}^{n \times r} \rightarrow \mathbb{R}^k \), the Laplace mechanism is defined as:

\[
\mathcal{M}(X) = M(X) + (\eta_1, \ldots, \eta_k).
\] (5)

where \((\eta_1, \ldots, \eta_k)\) are i.i.d. \( \text{Laplace}(0, \frac{\Delta_1(M)}{\epsilon}) \).

**Definition 6. Gaussian mechanism** (Dwork and Roth, 2014): Given any function \( M : \mathbb{R}^{n \times r} \rightarrow \mathbb{R}^k \), the Gaussian mechanism is defined as:

\[
\mathcal{M}(X) = M(X) + (\eta_1, \ldots, \eta_k).
\] (6)

where \((\eta_1, \ldots, \eta_k)\) are i.i.d. \( N \left( 0, \sigma^2 \right) = \left( \frac{\Delta_2(M)}{\sqrt{2 \log(1.25/\delta)}} \right)^2 \).

Both the Laplace and Gaussian mechanisms are simple and quick to implement, but they apply only to numerical values (without additional post-processing). A more general mechanism which satisfies \( \epsilon \)-DP is the Exponential mechanism, given in Definition 7, which allows for the sampling of values from a noise-infused distribution rather than adding noise directly.
Definition 7. **Exponential mechanism** (McSherry and Talwar, 2007): The Exponential mechanism releases values with a probability proportional to

\[
\exp \left( \frac{e^u(X, \theta)}{2\Delta_1(u)} \right),
\]

where \( u(X, \theta) \) is a quality function that determines the values for each possible output, \( \theta \), on \( X \).

2.3 Composition and Post-Processing Theorems

Lastly, we introduce the important concepts of composition and post-processing. These enable the development of more complex DP methods which release multiple statistics and make additional, structural or noise-reducing, adjustments. The composition theorems given in Theorem 1 formalize the concept of totaling the privacy loss incurred across multiple queries or datasets.

**Theorem 1. Composition Theorems** (Bun and Steinke, 2016; Dwork and Rothblum, 2016; McSherry, 2009): Suppose a mechanism, \( M_j \), provides \((\epsilon_j, \delta_j)\)-DP or \((\xi_j, \rho_j)\)-zCDP for \( j = 1, \ldots, J \).

a) **Sequential Composition:** The sequence of \( M_j(X) \) applied on the same \( X \) provides \((\sum_{j=1}^J \epsilon_j, \sum_{j=1}^J \delta_j)\)-DP or \((\sum_{j=1}^J \xi_j, \sum_{j=1}^J \rho_j)\)-zCDP.

b) **Parallel Composition:** Let \( X_j \) be disjoint subsets of the dataset \( X \), \( j = 1, \ldots, J \). The sequence of \( M_j(X_j) \) provides \((\max_{j \in \{1, \ldots, J\}} \epsilon_j, \max_{j \in \{1, \ldots, J\}} \delta_j)\)-DP or \((\max_{j \in \{1, \ldots, J\}} \xi_j, \max_{j \in \{1, \ldots, J\}} \rho_j)\)-zCDP.

If we want to make \( J \) statistical queries on \( X \) and we want the total privacy loss to equal \( \epsilon \), the composition theorems state under what conditions we may allocate portions of the overall \( \epsilon \) to each statistic. Under sequential composition, a typical appropriation is dividing \( \epsilon \) and \( \delta \) equally by \( J \). For example, a data practitioner might want to query the mean and standard deviation of a variable. These two queries will require using the sequential composition, allocating an equal amount of privacy budget to each query. Dwork et al. (2010) and Bun and Steinke (2016) proposed other forms of sequential composition, but the methods we test do not rely on these works.

Conversely, parallel composition does not require splitting the budget because the noise is applied to disjoint subsets of the input domain. For example, privacy experts will often leverage parallel composition to sanitize histogram counts, assuming that the bins are disjoint subsets of
The data. Noise can then be added to each bin independently without needing to split \( \epsilon \) or \( \delta \).

Theorem 2 (post-processing) states that any function applied to the output of a DP mechanism also satisfies DP. Some DP methods, as will be shown later, use the post-processing theorem to correct any inconsistencies or values that are not possible and to compute additional summaries required to perform statistical inference.

**Theorem 2. Post-Processing Theorem** (Bun and Steinke, 2016; Dwork et al., 2006; Nissim et al., 2007): If \( M \) be a mechanism that satisfies \((\epsilon, \delta)\)-DP or \((\xi, \rho)\)-zCDP, and \( g \) be any function, then \( g(M(X)) \) also satisfies \((\epsilon, \delta)\)-DP or \((\xi, \rho)\)-zCDP.

### 2.4 Setting the Privacy Loss Budget

When considering a validation server that releases results from numerous statistical queries, we must accurately account for the total privacy loss across all queries. To accomplish this, the mechanisms must use common privacy parameters. Accordingly, we show that every DP method considered in this paper satisfies \((\epsilon, \delta)\)-DP. This means if we institute a validation server using the methods presented in this paper, the total privacy loss across all queries can be tracked by the total \( \epsilon \) and \( \delta \) used from each individual query.

The total privacy budget and the best way to allocate portions of the privacy budget to each query (or user) are areas for discussion that we will not attempt to answer in this paper. The scientific community still has no general consensus on what value of \( \epsilon \) should be used for practical implementation. Early DP research focused on \( \epsilon \) values that were less than or equal to one and suggested that an epsilon of two or three would release too much information (Dwork, 2008). Researchers have also proposed other technical interpretations relating to hypothesis testing (Wasserman and Zhou, 2010) or odds-ratios (Machanavajjhala et al., 2008) to interpret and set limits on \( \epsilon \).

More recently, many privacy researchers working in practical applications frame the decision as a social choice question. They interpret the parameter as a way to quantify the trade-off between accuracy and worst-case privacy loss (Abowd and Schmutte, 2019). This means that privacy experts should explain to policymakers ways of interpreting the privacy and utility trade-off. Ultimately
stakeholders will need to set the privacy parameter in ways that are relevant to their contexts, such as in keeping with laws that apply to the data.

Accordingly, many practical applications of DP use large values of $\epsilon$ (by theoretical standards). In 2008, for example, privacy researchers applied $(\epsilon, \delta)$-DP method with values at $(8.6, 10^{-5})$ to release a synthetic version of the OnTheMap data, a United States commuter dataset (Machanavajjhala et al., 2008). More recently, in 2020, Google’s COVID-19 Mobility Reports used 2.64-DP for the daily reports; a total of 79.22-DP monthly (Aktay et al., 2020). In the same year, LinkedIn revealed their LinkedIn’s Audience Engagement API that protected LinkedIn members’ content engagement data, which used $(\epsilon, \delta)$-DP with daily values of $(0.15, 10^{-10})$ or $(34.9, 7 \times 10^{-9})$ for monthly queries (Rogers et al., 2020). In 2021, the Census Bureau used privacy loss of $\rho = 2.56$ on person-level redistricting data and $\rho = 0.07$ on unit-level redistricting data for a total privacy loss budget on this product of $\rho = 2.63$. This converts to $\epsilon = 17.41$ and $\delta = 10^{-10}$.

Given these examples and the evolving understanding of $\epsilon$, we explore a wide range of $\epsilon$ values in this paper based on values seen in theoretical work and practical applications. By doing so, we gain a better sense of the methods’ feasibility at different levels of $\epsilon$. Specifically, we examine the effects of $\epsilon$ and $\delta$ on accuracy within the context of our study, which we hope will contribute to the conversation about how to set the privacy loss budget.

When considering the total privacy loss budget, it is important to understand that choosing the value of $\epsilon$ for any single query is sensitive to other factors, such as the sample size, the total number of desired queries, and the size of the population from which data are sampled. Those questions concern the overall framework of a fully implemented validation server, and we do not seek to answer those questions in this paper. We make this point to note that our goal is simply to identify which algorithms are desirable, if any, based on their relative privacy and utility trade-offs, rather than their absolute trade-offs.

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5The U.S. Census Bureau set the final privacy loss budget on June 9, 2021, which can be found at https://www.census.gov/newsroom/press-releases/2021/2020-census-key-parameters.html. The specific privacy budget allocation can be found at https://www2.census.gov/programs-surveys/decennial/2020/program-management/data-product-planning/2010-demonstration-data-products/ppnf20210608/2021-06-08-privacy-budgetallocation.pdf.
3 Differentially Private Algorithms for a Tax Data Validation Server

For this section, we review the DP methods that we implement for the feasibility study. We focus on methods that directly return the specific statistic. We do not consider alternative paradigms, such as creating bespoke DP histograms to calculate statistics (Foote et al., 2019). However, a validation server with sufficient computational capabilities could consider this in future work.

3.1 Tabular Statistics

The literature has shown that adding Laplace noise produces the most accurate estimates for single tabular queries. For example, Rinott et al. (2018), Bowen et al. (2021), Shlomo (2018), and Liu (2018) found the Laplace mechanism is still “hard to beat” for disseminating frequency tables when the data have a large number of observations, or there are a lot of parameters and attributes. One can query the counts for a histogram jointly or as separate counts. We explore both because under separate queries Mironov (2017) showed composition rules that reduce the overall variability of the Gaussian mechanism if the $l_2$-GS is 1 (which is true in the counts we consider). Ultimately, we found that joint queries outperformed separate queries, even with the advanced composition. Therefore, we only present results for joint tabular statistics using the Laplace and Gaussian mechanisms.

3.2 Quantile Statistics

The methods used for generating DP quantiles use either the Laplace mechanism or the Exponential mechanism. For instance, Smith (2011) proposed an algorithm, $IndExp$, for selecting individual quantiles using the Exponential mechanism that satisfies $\epsilon$-DP. $IndExp$ has since been implemented in the SmartNoise (2020) and IBM (2019) DP libraries. Gillenwater et al. (2021) recently extended this method to two other algorithms, $AppIndExp$ and $JointExp$. The former is the same as $IndExp$ but optionally satisfies $(\epsilon, \delta)$-DP using the composition theorem from Dong et al. (2020) to choose an optimal $\epsilon$ for multiple queries given a total $(\epsilon, \delta)$. $JointExp$ samples multiple quantiles jointly, satisfying $\epsilon$-DP, and does not need composition for multiple quantiles. Using the Laplace mechanism, Nissim et al. (2007) developed an approach satisfying $(\epsilon, \delta)$-DP for sampling median values using smooth sensitivity that can be extended to query any other quantile. We will hereafter refer to this method as $Smooth$. We do not test other proposed approaches from (Dwork and Lei, 2009; Tzamos et al., 2020) because they require fine-tuning based on distributional assumptions that a
researcher might not realistically have in a validation server setting.

3.3 Means and Confidence Intervals

For mean estimates, we review DP methods that release means with their associated confidence intervals (CI). We find that common approaches in the literature use the Laplace mechanism, Gaussian mechanism, or Exponential mechanism for releasing some means with CIs. Du et al. (2020) conducted a comprehensive research study that aimed to move the theory to practice for releasing private CIs. The authors developed five new methods, two based on directly applying Laplace noise named, NOISYVAR and NOISYMAD, and three based on querying quantiles from the Exponential mechanism to estimate the standard deviation called, CENQ, SYMQ, and MOD. Du et al. (2020) also compared their methods against methods developed by Karwa and Vadhan (2017), D’Orazio et al. (2015), and Brawner and Honaker (2018).

Another type of DP method for releasing mean estimates uses quantiles, but the methods assume the data are very nearly Gaussian. We choose to not test these methods on our heavily skewed data because our preliminary tests show poor performance. Other methods require more information than is realistic for our application. For instance, Karwa and Vadhan (2017), Bowen and Liu (2020), D’Orazio et al. (2015), and Biswas et al. (2020) require the researcher to set bounds on the standard deviation to calculate the GS. In a validation server setting, a user might not have any information on these bounds, so we do not test these methods for our study.

3.4 Differentially Private Regression Analyses

For this subsection, we begin with an overview of the current DP approaches for regression estimates. Specifically, we center our attention on the linear model $Y = X\beta + \epsilon$, where $Y$ is the vector representing the observations for the response, $X$ is the design matrix, and $\epsilon$ is the vector of independent and identically distributed normal errors. We then explain the criteria for including methods in this feasibility study, discuss the selected methods, and detail any adaptations required to include the methods in the experiments.
3.4.1 Traditional Differentially Private Approaches for Regression Analyses

We classify DP methods for regression analysis according to the outputs they produce: (1) point estimates only, (2) point and interval estimates, and (3) other outputs related to regression analysis, such as diagnostic plots. Because we are particularly interested in methods that provide full statistical inference, we focus our study on methods from category (2) and discuss these methods in greater detail. We review the other two types further in the Supplemental Materials.

Sheffet (2017) developed \((\epsilon, \delta)\)-differentially private algorithms that, with certain probability, output summary statistics used for either traditional linear regression or ridge regression. When the outputs are summaries for ridge regression, the penalization parameter is a function of the algorithm’s inputs instead of being predefined by the user. Sheffet (2017) derived CIs and t-statistics that account for the noise added to the confidential summaries. In addition, they included a second algorithm that adds Gaussian noise directly to the sufficient statistic \(S = [X, Y]'[X, Y]\) and showed that one could obtain CIs for the regression coefficients under certain conditions (i.e., the norm of the true regression coefficients is upper bounded). Despite this method’s promise, the paper does not provide practical guidance on defining the additional tuning parameters that are essential for guaranteeing the correct confidence or significance level of the CIs and t-statistics. The lack of obtainable code is another reason we exclude it from our study.

Sheffet (2019) proposed a \((\epsilon, \delta)\)-DP mechanism that provides estimates with random noise drawn from the Wishart distribution. The mechanism defines a noisy statistic that preserves the property of being positive-definite, a common problem when adding noise to the regression sufficient statistics. According to the author, this mechanism is only valid for \(\epsilon \in (0, 1)\) and \(\delta \in (0, \frac{1}{e})\).

Wang et al. (2019) also developed \(\rho\)-zCDP and \(\epsilon\)-DP methods that release noisy versions of the summary statistics while preserving positive definiteness. These methods add noise using either a normal distribution (for \(\rho\)-zCDP) or the spherical analogue of the Laplace distribution (for \(\epsilon\)-DP). The positive definiteness is achieved by using eigenvalue decomposition and censoring the eigenvalues falling below a given threshold. Although Sheffet (2019) and Wang et al. (2019) did not derive CIs or t-values under the proposed mechanisms for the normal linear model, these contributions...
paved the way to develop DP methods that allow full inference for regression coefficients.

Ferrando et al. (2020) developed a general approach to produce point and interval estimates for different DP mechanisms applied to linear regression queries. The paper outlines two $\epsilon$-DP strategies that employ a noisy version of the sufficient statistics. The first approach applies the noisy statistics in classic ordinary least squares point and interval estimators. This strategy’s accuracy and coverage is ensured by large-sample arguments. The second strategy also uses plug-in estimators, but computes CIs using a parametric bootstrap. The method accounts for both the injected noise and the underlying sampling distributions.

A drawback of these two approaches is the lack of clarity on computing the points and interval estimates of the coefficients when the inverse of the noisy covariance matrix is not positive-definite. We adapt the approach from Ferrando et al. (2020) for this study by applying a regularized version of the noisy sufficient statistic to handle the positive definiteness issue. We provide details on how we use regularization in Section 3.4.2. We acknowledge that a new version of Ferrando et al. (2020) was recently released, where the authors made the update after we completed our feasibility study (Ferrando et al., 2021). Although there are slight differences between the main algorithms in each version, the first version of the algorithm produces valid results. We provide a full description of the algorithm in the Supplementary Materials.

For Bayesian inference, the developments in DP regression models are still at an early stage. Bernstein and Sheldon (2019) and Wang (2018) proposed approaches that can be used for full inference in the context of regression. We now briefly describe them and discuss the reasons for excluding them from the feasibility study.

Bernstein and Sheldon (2019) offered an approach that relies on a noisy version of the sufficient statistics. The procedure uses a Bayesian framework and employs a large-sample distributional characterization of the sufficient statistics. Using a Bayesian framework allows the authors to draw from the regression coefficient’s posterior distribution and, thus, provide point and interval estimates. This approach requires prior knowledge of the predictors’ second and fourth crossed population moments. A researcher can overcome this issue by privately querying the corresponding sampling moments, eliciting a prior for the moments, or assigning a distribution to the predictors.
Privately querying the sample moments will require an exponentially increasing portion of the total privacy budget, because the number of required moments grows exponentially with the number of predictors. This leads to exponentially increasing the noise with respect to the number of parameters in the model, assuming the privacy budget remains fixed. Also, as stated by the authors, the approach does not account for the noise added to the moments and may introduce some miscalibration of uncertainty. The other two alternatives require eliciting a prior for the moments or specifying a distribution for the predictor. Neither is generally feasible for all models, so this approach would have limited use in the validation server framework. For these reasons, we decided not to consider Bernstein and Sheldon (2019)’s method in the feasibility study.

Wang (2018) also provided a method that draws from the posterior distribution of the regression coefficients. However, users must spend part of the privacy budget for each draw. This aspect limits the applicability of Wang (2018)’s algorithm, since any accurate Monte Carlo approximation would divide the privacy budget into many small values. Splitting the privacy parameter too much dramatically decreases the method’s statistical usefulness. For this reason, we ultimately do not include this approach in our study.

3.4.2 Selected Methods and Adaptations

When selecting methods to include in our study, we prioritize the feasibility of implementation before evaluating the statistical usefulness of the outputs. Specifically, we select methods that: can be used for linear regression models with normal errors, can handle multiple predictors, and provide full inference on the estimates. For example, we exclude methods that only provide t-values for regression coefficients without also providing point estimates. Finally, we exclude procedures that meet the criteria above yet are not possible to implement, i.e., if: (1) the manuscript lacks the information needed to implement the method it describes, (2) the pseudocode is absent and implementation requires non-trivial choices reflecting the theory underlying the proposed method, (3) the method has difficult-to-fix errors, (4) the authors failed to reply to inquiries about their method or its implementation, or (5) the method achieves differential privacy under non-testable assumptions.

Based on this selection criteria, we assess each of the methods discussed in Section 3.4.1.
Without additional adaptations, only one method, Ferrando et al. (2020), meets all the inclusion criteria. We include Brawner and Honaker (2018)’s method, because, even though it was not originally designed for linear regression, a small adaptation makes it eligible. We also modify both Ferrando et al. (2020)’s and Brawner and Honaker (2018)’s methods and obtain new versions of the methods to compare in the feasibility study.

We repurpose elements of the algorithm from Ferrando et al. (2020) to perform full inference with other mechanisms, increasing the number of testable methods from two to six. Ferrando et al. (2020)’s approach employs a parametric bootstrap to approximate the distribution of the coefficients’ estimator while accounting for the underlying data-generating distribution and the DP mechanism (see Algorithm 3 in Ferrando et al. (2020)). Although the original method uses the Laplace mechanism, we can employ this same technique with other DP mechanisms after making some simple adaptations. Specifically, we adapt Algorithm 3 to be compatible with the Analytic Gaussian mechanism from Balle and Wang (2018), Algorithm 2 from Sheffet (2019) (hereafter, the Wishart mechanism), and both approaches in Algorithm 2 from Wang et al. (2019) (hereafter, the Regularized Normal and Regularized Spherical Laplace mechanisms).

Ferrando et al. (2020)’s Algorithm 3 inputs are all functions of the noisy version of the summary statistics $S_H = S + H$, where $S = [X,Y]'[X,Y]$ is the sufficient statistic for the linear model $Y = X\beta + e$, $Y$ is the vector representing the observations for the response, $X$ is the design matrix, $e$ is the vector of independent and identically distributed normal errors, and the elements of matrix $H$ are random noise sampled to achieve differential privacy. The algorithm uses Monte Carlo simulation to approximate the sampling distribution of an estimator of $\beta$, say $\hat{\beta}_b$, based on $S_H$ that accounts for the randomness in both $e$ and $H$. This algorithm assumes that the sample size $n$ is known, which might not be true in many applied scenarios. We choose to include the intercept in the linear model, so we obtain a privatized version of the sample size from the entry $(1,1)$ of $S_H$. We use this as the noisy sample size when applying Ferrando et al. (2020)’s Algorithm 3. If we chose not to include the intercept in our model, we would need to spend part of our privacy budget querying a noisy estimate of the sample size.

Implementing Ferrando et al. (2020)’s algorithm requires that $S_H$ be positive definite, which
is not always guaranteed in practice. For this reason, we use regularization when needed, i.e.,
$S'_rH = S_H - rI_{(p+1)\times(p+1)}$, such that $r$ is equal to zero if $S_H$ is positive definite or less than the
minimum eigenvalue of $S_H$ otherwise. Thus, $S'_rH$ is guaranteed to be positive-definite. To find $r$
for the Wishart mechanism, we use Remark (2) of Sheffet (2019). The remark contains an analytic
expression to ensure that $S'_rH$ is positive definite with probability greater than $1-\delta$. For the Laplace
and Analytic Gaussian mechanisms implementations, we follow a similar idea and define $r$, such
that $P[H - rI_{p\times p}$ definite positive] $\approx p_0$, where $p_0$ is close to one. For the Regularized Normal and
Regularized Spherical Laplace mechanisms, we define $S'_rH$ as the regularized matrix resulting from
Wang et al. (2019)’s Algorithm 2.

Unfortunately, we only have an analytical expression of $r$ for the Wishart mechanism. Hence,
as proposed in Peña and Barrientos (2021), we employ simulations to approximate the distribution
of the smallest eigenvalue of $H$ and define $r$ to be the $p_0$-percentile of this distribution. Note that
$P[S'_rH$ definite positive] $> P[H - rI_{p\times p}$ definite positive]. When defining $r$ as before, $r$ sometimes
fails to make $S'_rH$ positive definite, with small probability. Thus, we define $r$ as equal to three
times the minimum eigenvalue of $S_H$, which we choose based on empirical testing. We provide our
adaptation of Ferrando et al. (2020)’s algorithm in the Supplemental Materials.

Since the effect of $H$ on the inferences should diminish as the sample size increases, Ferrando
et al. (2020) discusses another strategy to make inferences about $\beta$ that only accounts for the
randomness in $e$ but not in $H$. Under this strategy, the idea is to treat $S'_rH$ as if it was $S$. Let
$\hat{\beta}$ represent the maximum likelihood estimator based $S$ (i.e., without DP), and let $\hat{\beta}_H$ denote an
estimator of $\beta$ obtained by plugging-in $S'_rH$ in the formulas of the maximum likelihood estimator
without DP. Theorem 1 in Ferrando et al. (2020) shows that $\hat{\beta}_H$ and $\hat{\beta}$ share the same asymptotic
normal distribution and extends to mechanisms other than the Laplace mechanism, such as the
Analytic Gaussian and Wishart. For this reason and to provide a method alternative to that based
on $\hat{\beta}_b$ and its bootstrap-based distribution, we employ $\hat{\beta}_H$ and its asymptotic normal distribution
to compute point and interval regression coefficient estimates.

In addition to methods based on Ferrando et al. (2020)’s approach, we also implement the
bootstrap approach described in Section 6.3 of Brawner and Honaker (2018). We can directly apply
this method when the summary statistic of interest is an average or a sum, such as $S$. Unfortunately, Brawner and Honaker (2018)’s strategy to compute CIs cannot be used in a straightforward manner for the CIs of regression coefficients.

However, since this method allows drawing multiple realizations of the noisy $S$ (after splitting the privacy parameters $(\delta, \epsilon)$), we can approximate the sampling distribution of the regression coefficients relying on an asymptotic assumption. Specifically, let $\hat{\beta}_{BH,1}, \ldots, \hat{\beta}_{BH,K}$ represent $K$ sampled regression coefficients. Each of these coefficients is computed based on a realization of the noisy $S$ using the equivalent of $(\delta/K, \epsilon/K)$-differential privacy under Brawner and Honaker (2018)’s approach. Thus, we approximate the sampling distribution of the regression coefficients by using a multivariate normal distribution, where the mean and covariance matrix correspond to the sample mean and covariance matrix based on $\hat{\beta}_{BH,1}, \ldots, \hat{\beta}_{BH,K}$. We detail how we compute the sensitivity of $S$ for the Laplace mechanism in the Supplemental Materials.

In summary, we consider six methods for our feasibility study: the Laplace mechanism, the BHM, and adaptations to the Analytic Gaussian, Wishart, Regularized Normal, and Regularized Spherical Laplace mechanisms. Each method uses both of Ferrando et al. (2020)’s approaches except the BHM. We summarize the methods selected for evaluation in Table 1.

Table 1: Summary of the differentially private regression methods we test for our tax use case studies. All methods return two types of confidence intervals (bootstrap-based and plug-in-based asymptotic) estimated using an adaptation from Ferrando et al. (2020), with the exception of BHM.

| Method                              | Source                  | Privacy Definition |
|-------------------------------------|-------------------------|--------------------|
| Analytic Gaussian mechanism        | Balle and Wang (2018)   | $(\epsilon, \delta)$-DP |
| Laplace mechanism                   | Ferrando et al. (2020)  | $\epsilon$-DP      |
| Regularized Normal mechanism        | Algorithm 2 (Wang et al., 2019) | $\rho$-zCDP       |
| Regularized Spherical Laplace       | Algorithm 2 (Wang et al., 2019) | $\epsilon$-DP   |
| mechanism                           |                         |                    |
| Wishart mechanism                   | Algorithm 2 (Sheffet, 2019) | $(\epsilon, \delta)$-DP |
| Brawner-Honaker Method (BHM)        | Brawner and Honaker (2018) | $\rho$-zCDP       |

4 Data Used for Case Studies

We evaluate the DP methods using two different datasets. We use the 2012 PUF to present results that will be most similar to a real validation server implementation using IRS data. SOI created
this file based on a subsample of the confidential taxpayer data. The file contains 200 variables with 172,415 records and represents United States tax filers. The PUF contains many variables with skewed distributions or have values predominantly zero, a notable feature for this study.

Because the PUF is a restricted access file, we also test our DP algorithms using the 1994 to 1996 CPS ASEC data to ensure that other researchers can replicate the findings reported in this paper. The U.S. Census Bureau generates the CPS ASEC from a probability sample, which contains detailed information about income. Similarly to the PUF, the CPS ASEC data has skewed variables and variables that are predominantly zeros. The biggest difference between the PUF and CPS ASEC is that the latter reports information at the person level and household level instead of tax units. The CPS ASEC represents the U.S. civilian non-institutional population and has 91,500 households and 157,959 people. These differences should not have a meaningful impact on the feasibility study.

We present the CPS ASEC results in the main body of this paper and the PUF results in the Supplemental Materials. Only select results from our evaluations are presented in this paper due to space constraints and readability. We provide more extensive results along with the code to reproduce our publicly available results online.\textsuperscript{6}

4.1 Tax Policy Case Studies

We base our tests on the types of analyses that tax economists would normally query for the PUF data. For example, a tax expert would likely make one or more of the following queries:

- **Counting query**: How many tax returns have salary and wage income in excess of $100,000?
- **Mean statistic query**: What are the means for the total and subsets of the total population?
- **Quantile statistic query**: What is the income threshold for the top 10 percent of earners?
- **Regression query**: What is the relationship between education and annual earnings?

For regression analysis, we replicate a published cross-sectional multiple linear regression model from \textcite{Card1999}. We reproduce the models found from column (5) in Table 1 on page 1809 that aims to model the relationship between education and annual earnings across gender\textsuperscript{7}.

\textsuperscript{6}GitHub repo website, https://github.com/UrbanInstitute/formal-privacy-comp-appendix
\textsuperscript{7}The paper does not provide reproducible code, and our data from which we derive results differ by a few
4.2 Assumptions on the Data and Analysts’ Knowledge

We make some assumptions to deal with limitations to existing DP methodology. We assume there are no survey weights and, for queries other than counts, there are no empty subsets for categorical variables. To the best of our knowledge, there are no existing methods satisfying $(\epsilon, \delta)$-DP or $(\xi, \rho)$-zCDP that handle survey weights as generated by most federal statistical agencies, including SOI, because those systems extensively use the confidential data to generate the final survey weights (e.g., non-response adjustments). For the second assumption, if someone using a validation server applies their analyses to an empty subset, the query will result in an error message. If released, this error message will inform the user that there are no observations in the interested subset, violating DP. We acknowledge the reality of this issue for creating a validation server, but addressing this problem is beyond the scope of our paper.

In order to calculate the sensitivity, we also assume the users input bounds for the continuous variables and provides a list of the levels in categorical variables. We assume users would derive these inputs using prior knowledge or publicly available synthetic data. In our experiments, we use the observed minimum and maximum as bounds for continuous variables and observed levels for categorical variables. Using this strategy to set the bounds and levels implies that in practice results may be worse than our findings show. Our approach also implies that if a method has poor performance, we do not expect that altering the bounds or levels would lead to improvements. However, the proposed procedure can serve as a starting point and as a way to choose methods. Future work can focus on identifying strategies for determining input parameters.

5 Utility Metrics for Experimentation

We use several statistical measures to evaluate the methods for releasing statistics under DP. Because we use real sample data rather than simulated, we define utility as the empirical distance between the confidential estimates and the estimates from DP methods. We broadly categorize the metrics that: (1) measure the error of the DP statistic relative to the confidential estimate and (2) measure the similarity of inferences that are drawn using both the point estimates and uncertainty between the confidential and DP output.
The first category is straightforward and consists of computing the $l_1$ error of the simulated DP statistics with respect to the confidential statistics. For the histograms and quantiles, where we produce multiple statistics, we compute the average $l_1$ error across multiple bins or quantiles.

The second category of utility metrics is less familiar to a statistical audience, requiring more explanation. This criteria concerns statistics with uncertainty estimates, which are the mean and regression queries. The first metric we use to compare similarity of inference on the confidential data and the DP estimates is the confidence interval ratio (CIR). This is the ratio of width of the DP CI to the confidential CI, and this metric informs us how much increased uncertainty comes from using the DP estimate.

Along with the CIR, we calculate the percentage of times across the simulated DP regression estimates that the following three criteria are all achieved: (1) the signs match between the confidential data and DP estimates, (2) the significance (assuming 0.05 level) match between the confidential and DP estimates, and (3) the CIs for the confidential and DP estimates overlap. We term this metric as sign, significance, and overlap (SSO) match. The percentage of SSO match across simulated DP estimates gives a heuristic metric for how frequently the DP estimates are expected to provide similar inference to the confidential estimates. As we will show, there are cases where DP estimates give a high percentage of SSO match, but have dramatically wider CIs than the non-DP estimates. We provide CIR and SSO together to try to describe the empirical utility that aligns with data users’ analyses. If both the CIR is close to 1 and the SSO match percentage is high, this indicates that DP estimates from these mechanisms are expected to produce similar inference to the confidential estimates.

These measures are similar conceptually the confidence interval overlap (CIO) measure from Karr et al. (2006), but we found that measure hard to interpret in the regression results for our study. For example, when the confidential and DP CIs do not overlap, the CIO value is negative, but it is not clear how to interpret the value. Conversely, when the DP CI completely covers the confidential CI, the CIO cannot be less than 0.5, even as the DP CI becomes infinitely wide. The results shown in the Supplementary Material for the estimate of simple means and CIs give the CIO values, but we do not include them for the regression results.
Although the CIR and SSO measures are less familiar to a statistical audience, they give an interpretive way of assessing the inferential differences between the confidential and DP outputs. In a practical setting, this evaluation tells us how the DP methods could impact the public policy decisions for a particular tax case study.

6 Tabular, Quantile, and Mean Statistic Results

In this section, we present the evaluations of the DP methods for tabular, mean, and quantile estimates. For simplicity, we only describe the results at a high level, but result tables (along with results on the SOI PUF data) can be found in the Supplementary Material. When testing our methods, we run all DP methods 100 times for each of \( \epsilon = \{0.01, 0.1, 0.5, 1, 5, 10, 15, 20\} \), \( \delta = \{10^{-3}, 10^{-7}, 10^{-10}\} \), and use a bootstrap size of 10 and 25 for the Brawner-Honaker method (BHM) for means. We direct interested readers to our GitHub repo,\(^8\) which contains the code, data, and complete results from our study.

6.1 Histograms for Tabular Summaries

We test the Laplace and Gaussian mechanisms for providing counts of individuals within income categories.\(^9\) We calculate the maximum error over all cumulative sums as a percentage of the sum over the whole histogram. We also compute the mean error over all cumulative sums. These two metrics allow us to interpret the expected error given how a tax researcher might use the queried histograms. Discerning bunching or jumps in the distribution is useful for developing more complex models and relies on understanding the cumulative distribution function. The maximum relative error tells us the biggest change across the distribution in the total percentage of individuals who fall below or above a certain cutoff, informing the jumps. Therefore, the mean error tells us the average error across all subsets of the histogram and indicates where bunching occurs.

Table 7 in the Supplementary Material provides the results. Overall, the utility is high. We see that the max relative error and mean error are below 0.5\% for \( \epsilon = 1 \) in 99\% of the cases for the Laplace mechanism and below 5\% for \( \epsilon = 0.1 \). With lower error and a stronger privacy guarantee, the Laplace mechanism is preferable.

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\(^8\)GitHub repo website, https://github.com/UrbanInstitute/formal-privacy-comp-appendix

\(^9\)We considered making separate queries for each income bin, but the results were strictly worse than making one query for the whole histogram.
6.2 Quantiles

We test AppIndExp, JointExp, and Smooth for estimating quantiles. Using the CPS income variable, we estimate the income deciles from 10% to 90%. Researchers could use these values for producing tax policy tables\textsuperscript{10}. We compute the mean error across quantiles and the mean error across the ordered differences between quantiles. The latter would be of interest to researchers if they are trying to understand income disparities. Table 8 in the Supplementary Material shows the results.

We find that both AppIndExp and JointExp offer high utility, and they are generally preferable to Smooth for all but the highest values of $\epsilon$. For $\epsilon < 5$, Smooth performs much worse than the other two methods. This result is likely due to more extensive splitting of $\epsilon$ and $\delta$ required by the algorithm. We find JointExp performs better than AppIndExp at estimating the quantiles at lower levels of $\epsilon$. Both seem to quickly converge to their best solution by $\epsilon = 0.1$, but the error does not converge to zero even for higher levels of $\epsilon$. This suggests that the methods are not empirically unbiased and may be an artifact of the method for sampling from the Exponential mechanism, which is implemented in the source code.

6.3 Estimates of the Mean with Confidence Intervals

We test NOISYMAD, NOISYVAR, and BHM for computing the mean income with an estimate of the 95% confidence interval. NOISYMAD and NOISYVAR perform similarly and both provide highly accurate statistics. We find that all three methods for means are approximately unbiased. BHM performs comparably to the other two methods only when $\delta = 10^{-3}$ and when $\epsilon < 1$. Otherwise, the other two methods clearly outperform BHM. For $\epsilon \geq 1$, the relative error is almost always less than 1% for all methods.

We find more variation for the CI measures than point estimate bias, which reflects the differing approaches to estimating uncertainty. For $\epsilon \geq 0.5$, we find NOISYVAR provides CIR values close to 1 and SSO values of 100%. NOISYMAD performs well for $\epsilon = 0.1$, but as $\epsilon$ becomes larger, the widths of the CIs produced by NOISYMAD shrink and are consistently narrower than the confidential CIs. These results would produce overly confident inference for researchers performing hypothesis testing. For BHM, the intervals are much wider especially for $\epsilon < 5$, resulting in high

\textsuperscript{10}https://www.taxpolicycenter.org/statistics/household-income-quintiles
SSO match but also high CIR values that shrink towards 1 as $\epsilon$ grows more slowly than $NOISYVAR$. More details on these results can be found in the Supplementary Materials.

7 Regression Results

We evaluate the methods listed in Table 1 for producing DP estimates of linear regression coefficients. When testing our methods, we run all DP methods 100 times for $\epsilon = \{0.1, 0.5, 1, 5, 10, 15, 20\}$, $\delta = \{10^{-3}, 10^{-7}, 10^{-10}\}$, and a bootstrap size of 10 and 25 for the Brawner-Honaker method ($BHM$).

In order to improve readability, we present results only for selected methods and values of $\epsilon$ and $\delta$, depending on the range of the results. Certain methods clearly outperformed other methods, so we only present the results for the two best performing methods. We provide results for additional statistics and a model using the SOI PUF data in the Supplemental Materials. Complete results for additional methods (and SOI PUF results) can be found in our GitHub repo.\textsuperscript{11}

The regression model outcome is the log annual earnings, and the model aims to estimate the relationship between education and annual earnings by gender\textsuperscript{12}. The model contains six coefficients, including the intercept, and we display the results for two here. The “Non-White” coefficient is for a binary indicator variable, and on the confidential data it has a point estimate close to zero with a 95% confidence interval that covers zero. The “Years of Education” coefficient is for a count variable, and it has a strong positive effect with a tighter 95% confidence interval that does not cover zero. We select these two coefficients from the six to highlight the differences in utility depending on the strength of the relationship between the predictor and the outcome in the confidential data. The other three non-intercept predictors included in the model are quite skewed, so they present a different challenge. As can be seen in the Supplementary Material, their utility is very poor due to their large sensitivity (based on their sample bounds). We emphasize that the results presented here should be seen as a “good case.”

Figure 1 shows the distributions of simulated DP outputs from the Analytic Gaussian Mechanism ($AGM$) and Laplace Mechanism ($LM$) for three levels of $\epsilon$. We see that distributions of the outputs for both DP mechanisms get tighter around the confidential estimates (solid black lines)\textsuperscript{11}.

\textsuperscript{11}GitHub repo website, https://github.com/UrbanInstitute/formal-privacy-comp-appendix

\textsuperscript{12}In the paper, separate models are fit for men and women. We present results here for the model fit on women.
Figure 1: Distribution of simulated DP estimates for regression coefficients. Confidential estimates (black horizontal lines) and 95% confidence intervals (dotted lines) are shown. $\delta = 10^{-7}$ for AGM.

as $\epsilon$ grows. But, a large proportion of the simulated estimates fall outside the confidential 95% confidence intervals (dotted lines) even for $\epsilon = 5$. This result suggests that the noise added to protect privacy from these mechanisms is larger than what we would expect from the sampling variability. We remind readers that these results are from the two best performing methods. While many simulated queries returned for this model contain satisfactory results, there is a reasonably high probability that the query could return estimates that are far from the confidential estimates.

Table 2 summarizes the empirical error from these estimates with respect to the non-DP estimate. The table shows the 5th and 95th quantiles of the simulated outputs and the $l_1$ error for three levels of $\epsilon$. We see that the AGM outperforms LM and both mechanisms provide smaller errors as $\epsilon$ increases. The size of the errors are smaller for the “Years of Education” coefficient both in absolute terms and relative to the confidential estimate. Interpreting the coefficients as odds ratios may help better understand the potential impact on a user’s inference if they were provided DP estimates. Based on the confidential estimates, an increase for each additional one year of education corresponds to an estimated relative increase in the annual earnings by 17.5%. Under the AGM, the 90% simulated range of this estimated relative increase in annual earnings for each
Table 2: Summary of the regression results. Confidential (non-DP) estimate values for each coefficient are shown. 5th, 95th, quantiles, and $l_1$ Error for select coefficients, methods, and values of $\epsilon$. $\delta = 10^{-7}$ for AGM.

| Coefficient       | Epsilon | Method | Non-DP  | DP 5%  | DP 95% | $l_1$ |
|-------------------|---------|--------|---------|--------|--------|-------|
| Non-White         | 0.5     | AGM    | -0.015  | -0.144 | 0.105  | 0.064 |
| Non-White         | 0.5     | LM     | -0.015  | -0.233 | 0.225  | 0.109 |
| Non-White         | 1       | AGM    | -0.015  | -0.070 | 0.065  | 0.037 |
| Non-White         | 1       | LM     | -0.015  | -0.128 | 0.108  | 0.059 |
| Non-White         | 5       | AGM    | -0.015  | -0.084 | 0.009  | 0.018 |
| Non-White         | 5       | LM     | -0.015  | -0.063 | 0.021  | 0.025 |

| Years of Education | 0.5     | AGM    | 0.161   | 0.083  | 0.240  | 0.038 |
| Years of Education | 0.5     | LM     | 0.161   | 0.018  | 0.282  | 0.060 |
| Years of Education | 1       | AGM    | 0.161   | 0.113  | 0.211  | 0.026 |
| Years of Education | 1       | LM     | 0.161   | 0.077  | 0.212  | 0.032 |
| Years of Education | 5       | AGM    | 0.161   | 0.149  | 0.174  | 0.008 |
| Years of Education | 5       | LM     | 0.161   | 0.124  | 0.181  | 0.016 |

additional one year of education is 12.0% to 23.5% for $\epsilon = 1$ and 16.1% to 19.0% for $\epsilon = 5$.

Next, we evaluate the estimated private CIs compared to the confidential estimates. We show the AGM and LM results for the two different estimation methods for producing CIs, asymptotic and bootstrap. We use the CIR to compare how much the width of the CI increases due to the DP mechanism, and we measure the percentage of simulated DP estimates that match SSO. If both the CIR is close to 1 and SSO match percentage is high, this indicates that researchers would expect the DP estimates to produce similar inference to the confidential estimates.

Table 3 shows the results using the asymptotic and bootstrap methods for estimating private CIs. Using the asymptotic estimator, we see that the CIR is always close to 1, which we would expect because this method does not take the noise from the privacy mechanism into account. For SSO, when $\epsilon$ is 0.5 or 1, we see that the match rate is below 10 or 25%, respectively. At $\epsilon = 5$, the rate is roughly 50-60% for the AGM. Conversely, the bootstrap CIs are much wider, but they provide a higher SSO match percentage. When $\epsilon = 5$, the AGM using bootstrap CIs gives 80 to 90% for the SSO match, and the private CI is roughly 2 to 4 times wider than the confidential CI.

These results demonstrate that using either AGM or LM can produce regression estimates with either similar CI width or reasonable SSO percentage match but not both. At $\epsilon = 5$, the results may be good enough for some users. However, compared to other DP applications, data
Table 3: Summary of the private CI estimates. Median and 90th percentile (in parenthesis) CIR and percentage of SSO match for select coefficients, methods, and values of $\epsilon$. Both methods for estimating CI shown. $\delta = 10^{-7}$ for AGM.

| Coefficient          | Epsilon | Method | Asymptotic CIR | Asymptotic SSO% | Bootstrap CIR | Bootstrap SSO% |
|----------------------|---------|--------|----------------|-----------------|--------------|---------------|
| Non-White            | 0.5     | AGM    | 1.04 (1.15)    | 8%              | 9.74 (47.4)  | 53%           |
|                      | 0.5     | LM     | 1.11 (1.26)    | 8%              | 13.0 (72.3)  | 42%           |
| Non-White            | 1       | AGM    | 1.02 (1.08)    | 19%             | 4.50 (19.8)  | 54%           |
|                      | 1       | LM     | 1.03 (1.16)    | 7%              | 7.20 (25.5)  | 46%           |
| Non-White            | 5       | AGM    | 1.00 (1.02)    | 53%             | 1.78 (17.5)  | 82%           |
|                      | 5       | LM     | 1.01 (1.03)    | 43%             | 2.03 (28.9)  | 71%           |
| Years of Education   | 0.5     | AGM    | 1.03 (1.15)    | 4%              | 32.1 (102)   | 75%           |
|                      | 0.5     | LM     | 1.08 (1.21)    | 9%              | 42.7 (180)   | 58%           |
| Years of Education   | 1       | AGM    | 1.02 (1.07)    | 24%             | 14.8 (65.8)  | 86%           |
|                      | 1       | LM     | 1.04 (1.13)    | 15%             | 24.7 (101)   | 79%           |
| Years of Education   | 5       | AGM    | 1.00 (1.02)    | 62%             | 4.43 (26.6)  | 93%           |
|                      | 5       | LM     | 1.01 (1.03)    | 42%             | 6.03 (81.8)  | 84%           |

users would need to spend a high value of privacy loss for a single model. We note again that this is a “good case.” The other coefficients in the model and the SOI PUF results show worse utility (provided in the Supplementary Material).

Both asymptotic and bootstrap interval estimates can be provided without incurring additional privacy loss, so the validation server might give both to the user. This scenario would provide both an estimate of the variation due to the privacy mechanism and the variation in the confidential estimate without noise. Further work is needed to understand how data users might utilize both these sources of information together.

To help visualize the simulated point and CIs estimates under the two different methods, we plot all 100 simulated estimates with 95% CIs for the AGM. Each case is color-coded for whether they match sign, significance, and overlap (blue), or if they do not match at least one (red). We remind the readers that for the “Non-White” coefficient, the confidential estimate (black) has a 95% CI that covers zero, whereas “Years of Education” coefficient does not. This distinction is relevant, because, in the former case, SSO match is achieved with private CIs that cover zero. In the latter case, SSO is achieved with private CIs that do not cover zero. We also plot only the 90 best simulated estimates in order to aid plot readability.
Figure 2: Regression results showing the 90 best (out of 100) simulated DP estimates and confidence intervals for the Analytic Gaussian Mechanism and $\epsilon = 1$. Results shown for two coefficients with confidence intervals estimated using the asymptotic approach. $\delta = 10^{-7}$ for AGM.

Figure 2 shows the simulated estimates with the asymptotic CIs. In this case, we see, as expected, that the intervals are all roughly equal length to the confidential interval. As the point estimates move away from the confidential estimate the intervals stop overlapping. One might argue that this is not a large problem for many of the “Years of Education” DP estimates. However, the “Non-White” DP estimates that are significantly positive or negative would lead to substantially different inference from the confidential estimates.

Figure 3 shows the DP estimates using the bootstrap CIs. In contrast to the previous figure, the noisy CIs are much wider than the confidential estimate. They almost all overlap or cover the confidential CI, since they take into account the extra variation induced by the DP mechanism. For the “Non-White” coefficient, many estimates switch sign, but the intervals typically cover zero regardless. In the case of “Years of Education”, we see that, even with the increased uncertainty, most of the estimates are still significant, because the confidential estimate is strongly significant. As was seen in Table 3, the intervals are much wider, but the DP estimates using bootstrap CIs are more likely to match SSO.
8 Conclusions from the Case Study

In this paper, we survey and test the feasibility of the latest DP methods for summary statistics and regression analyses on real tax data. To the best of our knowledge, this is the first comprehensive evaluation of these DP methods for practical applications within a validation server framework for real-world data. We find that DP algorithms for summary statistics perform well for privacy loss budgets close to or in some cases smaller than 1. Conversely, even in the best case we tested, regression methods require at least $\epsilon = 5$ for full inference. Practical applications would likely require either larger sample sizes or allocating more $\epsilon$ on every query in order to return estimates with satisfactory levels of uncertainty. Based on our results, we identify a few challenges and avenues for future work.

8.1 Challenges

We find that existing DP regression methods are limited in applicability and often add high levels of noise. The only methods we found that met our inclusion criteria perturb the sufficient statistics
prior to computing the noisy regression estimates. We identify three primary shortcomings with perturbing the sufficient statistics. First, the privacy budget must be split \( \frac{(k+1)(k+2)}{2} \) times for \( k \) coefficients. Second, when perturbing the sufficient statistics, the added noise becomes multiplicative rather than additive. Third, most methods are designed for normal data, so, when applied to skewed data, the results worsen significantly because of increased sensitivity.

Besides methodological issues, we encounter challenges with coding the various algorithms. For instance, obtaining code for applications and ensuring it is error-free presents challenges. Although we do not expect academics to provide production-ready code, we often find the research code to be messy, hard to read, and difficult to alter for our use cases. In some cases, the issues are minor and were easily addressed through consultation with the authors. In other cases, we could not use the method due to a lack of functioning code. When no code was available and the author(s) did not respond, we attempted to code the methods ourselves. But, in some cases, the manuscript does not provide enough information for us to implement the method and is thus excluded from the feasibility study.

These issues emphasize the importance of open-source code to facilitate wider use and acceptance of DP algorithms in practice. For example, OpenDP from Harvard University and SmartNoise from Microsoft are developing a suite of open-source software tools to implement DP methods. These platforms are still under development, but they may offer improved solutions in the future.

8.2 Future Work

One vital area for improvement is developing DP algorithms for data that are not Gaussian. Many of the methods we test performed well in their original papers, because the authors tested on well-behaved or normally distributed data. Real data are often skewed, such as the 2012 PUF and CPS ASEC data, resulting in these same methods performing poorly.

Another area for improvement is developing DP algorithms with a focus on estimating the uncertainty of the estimates. Many data privacy experts create DP regression methods to output accurate predictions. But, as seen in Section 7, these methods perform poorly either by returning very large confidence intervals or by not reporting the standard error at all. This appears to be a significant gap in the DP literature that must be addressed.
Future work should focus on improving DP methods for ordinary least squares regression coefficient and interval estimates and then expand that research to address other important economic use cases. Our consultation with tax experts indicated that incorporating of survey weights for various summary statistics, regression discontinuity, and regression kink designs are important areas for future work. We hope this study provides the data privacy and confidentiality community with a better understanding of the capabilities, limitations, and challenges of current DP methods for summary statistics and regression analyses.

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Supplemental Materials for
“A Feasibility Study of Differentially Private Summary Statistics
and Regression Analyses for Administrative Tax Data”

This file contains the Supplemental Materials to accompany the paper “A Feasibility Study of
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9 Extended Review of Differentially Private Regression Methods

As stated in the main text, DP methods for regression can be classified according to the outputs
they produce: (1) point estimates only, (2) point estimates and interval estimates, and (3) other
outputs related to regression analysis, such as diagnostic plots. In this section, we cover categories
(1) and (3).

Most DP algorithms fall into the first category, returning only noisy point estimates. Methods
in this category frequently rely on objective-perturbation-based approaches, such as the functional
mechanism (Chaudhuri et al., 2011; Fang et al., 2019; Gong et al., 2019; Zhang et al., 2012). These
approaches provide DP coefficients estimates by maximizing a perturbed version of the objective
function. We can obtain the perturbed versions of the objective function by either adding noise
to the function or using a polynomial representation of the function, where we added noise to the
polynomial coefficients.

Other DP point estimate methods borrow ideas from robust statistics. For example, Avella-
Medina (2020) proposes $(\epsilon, \delta)$-differential private methods to obtain robust estimators for various
problems, including regression analysis. Chen et al. (2020) developed $\epsilon$-DP methods for median
regression, which can be seen as a robust version of ordinary least squares based regression. Finally,
some DP point estimate approaches add noise to sufficient statistics (Wang, 2018). For a list of
DP methods for simple linear regression, we refer the reader to Alabi et al. (2020).

Next, we review recent contributions that release outputs distinct from point and interval
estimates. Even though these methods are beyond the current scope of the validation server, they provide key information for regression analysis and could be fruitful additions to future versions of the validation server.

Barrientos et al. (2019) proposed an $\epsilon$-DP method to perform hypothesis testing for single coefficients. This approach has the advantage of not requiring the response and predictors to be bounded. However, the privacy loss budget may be costly when performing hypothesis testing for multiple coefficients using this method. The Bayesian procedure by Amitai and Reiter (2018) has similar capabilities and limitations, since it targets specific summaries of the posterior distributions of the regression coefficients, such as tail probabilities.

Residual analysis is another important task in regression and a crucial tool for model validation when finding solutions to problems, such as heterogeneity of errors and lack of linearity. Chen et al. (2016) describe $\epsilon$-DP methods that release a private version of the residuals versus fitted values plot. Model selection is also key for regression analysis, and Lei et al. (2016) has developed an $(\epsilon, \delta)$-DP algorithm for this purpose. Other contributions related to regression analysis involve algorithms developed for regularized regression, such as Lasso (Dandekar et al., 2018; Talwar et al., 2015).

9.1 Details on Differentially Private Bootstrap Regression

Ferrando et al. (2020) proposed a DP bootstrap-based algorithm that adds noise to the sufficient statistic for linear model $Y = X\beta + e$, where $Y$ is the vector representing the observations for the response, $X$ is the design matrix, and $e$ is the vector of independent and identically distributed normal errors. We assume that the reported noisy statistic is of the form

$$S_H^r = S + H - rI_{(p+1) \times (p+1)}$$

where $S = [X, Y]'[X, Y]$ is the sufficient statistic for linear model $Y = X\beta + e$, the matrix $H$ denotes the noise added to achieve differential privacy using either the Laplace, Analytic Gaussian, or Wishart mechanism, and the $r$ is defined as discussed in Section 3.4.2. For the Normal and Spherical Laplace mechanisms, we define $S_H^r$ as the already-regularized resulting matrix obtained from Wang et al. (2019)’s Algorithm 2. We denote $P_H$ as the corresponding probability distribution.
of \( H \) under a given mechanism. To define \( H \) for the Normal and Spherical Laplace mechanisms, we use a symmetric version of the added noise as in Wang et al. (2019)'s Algorithm 2, i.e., we define \( H = (\tilde{H} + \tilde{H}^T)/2 \) where the entries of \( \tilde{H} \) are drawn from the corresponding normal or spherical Laplace distribution. The up-to-date version of the algorithm proposed by Ferrando et al. (2020) assumes that \( r \) is equal to zero and only considers the Laplace mechanism. This algorithm also assumes that the sample size \( n \) is known. This is something we cannot assume and, for this reason, we replace the sample size by a DP version of it. Notice that when the intercept is included in the model and represented by the first column of \( X \), a privatized version of the sample size is available at the entry (1,1) of \( S'_r \). If the intercept is not included, users will need to spend part of their privacy budget querying this quantity. The algorithm below summarizes the employed algorithm after modifications and adaptations.

**Algorithm 1** DP bootstrap-based regression

**Input:**
- \( S'_r \): regularized noisy sufficient statistic
- \( B \): number of bootstrap samples
- \( P_H \): probability distribution of \( H \)
- \( p \): number of regression coefficients

1. \( \hat{n} \leftarrow S'_r[1,1] \) \( \triangleright \) assuming the intercept is part of the model
2. \( \hat{X}^T \hat{X} \leftarrow S'_r[1:p,1:p] \) \( \triangleright \) submatrix of \( S'_r \) with \( 1:p = (1, \ldots, p) \).
3. \( \hat{X}^T \hat{Y} \leftarrow S'_r[1:p,p+1] \)
4. \( \hat{Y}^T \hat{Y} \leftarrow S'_r[p+1,p+1] \)
5. \( \hat{\beta} \leftarrow (\hat{X}^T \hat{X})^{-1} \hat{X}^T \hat{Y} \)
6. \( \hat{\sigma}^2 \leftarrow (\hat{Y}^T \hat{Y} - (X^T Y)^T (\hat{X}^T X)^{-1} \hat{X}^T \hat{Y})/ (\hat{n} - p - 1) \)
7. for \( b \) in \( \{1, \ldots, B\} \) do
8. Sample \( h \) from \( P_H \)
9. Sample \( \hat{X}^T u \) from \( N(0, \hat{n} \hat{\sigma}^2 \hat{X}^T X) \)
10. \( \tilde{\beta}_b \leftarrow (\hat{X}^T \hat{X})^{-1}(\hat{X}^T \hat{X} - h[1:p,1:p])\hat{\beta} + (\hat{X}^T \hat{X})^{-1}(\hat{X}^T u + h[1:p,p+1]) \)
return: \( \tilde{\beta}_1, \ldots, \tilde{\beta}_B \)

**9.2 Sensitivity Calculations**

To compute the sensitivity of \( S \) for the Laplace mechanism described in Section 3.4.2, we assume that the response and predictors are bounded—a common assumption in differential privacy. Without loss of generality, we assume that the response and predictors take values \([0, 1]\). Because \( S \) is a symmetric matrix and under the previous assumption, its sensitivity is upper-bounded by the
number of entries in and above the diagonal, i.e., \((p + 1)(p + 2)/2\), where \(p\) is the number of regression coefficients. While using the upper bound \((p + 1)(p + 2)/2\) as sensitivity is a valid strategy, it is particularly inefficient in the presence of categorical predictors. To improve this upper bound, we implement some strategies when categorical predictors are part of the analysis. We assume that all categorical predictors are included in the model as dummy variables and fixing one level as the reference. The implemented strategies are listed below:

1. One or more entries in the diagonal of \(S\) are identical to entries in the off-diagonal. Hence, we only need to add noise to the unique entries.

2. If the categorical predictor has more than two levels, then some entries of \(S\) will be exactly zero, eliminating the need to add noise to those entries.

3. If the categorical predictor has more than two levels and the model has an intercept, then we multiply the set of the corresponding dummy variables by the column of ones in \(X\) (representing the intercept) results in a vector that counts the number of ones in the dummy variables. The vector is equal to a contingency table that counts the number of observations at each level, leaving out the reference one. Although this vector has a dimension greater than one, its sensitivity is equal to one. Hence, we take advantage of this fact to reduce the sensitivity of \(S\).

4. If the categorical predictor has more than two levels and the model has a numeric predictor, then we multiply the set of corresponding dummy variables by the column in \(X\) representing the numerical predictor results in a vector of partial sums. Each entry of this vector is equal to the summation of numerical predictor values across the observations that share the same level as the categorical predictor. Similar to the previous item, while this vector has a dimension greater than one, its sensitivity is equal to one. Hence, we take advantage of this fact to reduce the sensitivity of \(S\).

We follow the same strategy to define the sensitivity for the Analytic Gaussian mechanism and the approach proposed by Brawner and Honaker (2018). For the Wishart mechanism, the sensitivity is equal to the upper bound for \(l_2\)-norm of the rows in \([X, Y]\). We therefore only consider item 2, i.e., an entry of \(S_H\) is set to be equal to zero if the same entry in \(S\) is zero by definition.
We also realize that the magnitude of the sensitivity depends on the upper and lower bounds of the variables (response and predictors) involved in the analysis. Since the variables are often in different scales, users can easily face scenarios where the magnitude of the sensitivity is highly dominated by, for example, a single variable. This single variable could have an interval length (upper minus lower bound) that is much larger than those of the remaining variables. If users ignore this issue under such scenario, the mechanism will add too much noise to those summaries involving the remaining variables. To avoid this problem, we first use the provided bounds to rescale all variables to lie in the $[0, 1]$ interval. We then implement the DP method and compute the point and interval estimates of the regression coefficients. Finally, we use the provided bounds again to scale back and report the estimates in the original scale.

10 Candidate DP Methods for Histograms, Quantile, and Mean Statistics

Table 4: Summary of the DP tabular methods reviewed in Section 3.1.

| Tabular Statistics |
|--------------------|
| Method             | Privacy Definition | Off-the-Shelf vs. Hand-Coding | Selected for Case Study |
| Laplace mechanism  | $\epsilon$-DP     | off-the-shelf via R and Python code on GitHub | Yes |
| Gaussian mechanism | $(\epsilon, \delta)$-DP | off-the-shelf via R and Python code on GitHub | Yes |

11 CPS Experimental Results

In this section, we present tables and figures for the DP tabular, mean, and quantile results discussed in the main paper. We also provide additional results for the regression coefficients not shown in the main paper.

11.1 Histograms

We test the Laplace and Gaussian mechanisms for providing tax data histograms.

We utilize a couple utility metrics which allow us to interpret the error distributions in such a way that relates to how a tax researcher might use the queried histograms. In particular, discerning cut points or jumps in the distribution is important, which rely on understanding the cumulative distribution function. We measure utility on the CDF in two interpretable ways. First, we measure the maximum error of over all cumulative sums relative to the sum over the whole histogram.
Table 5: Summary of the DP quantile statistics reviewed in Section 3.2.

| Method                  | Privacy Definition | Off-the-Shelf vs. Hand-Coding | Selected or Not Selected for Case Study |
|-------------------------|--------------------|-------------------------------|-----------------------------------------|
| AppIndExp (Gillenwater et al., 2021; Smith, 2011) | \((\epsilon, \delta)\)-DP | off-the-shelf via Python code on GitHub | Yes                                      |
| JointExp (Gillenwater et al., 2021) | \(\epsilon\)-DP | off-the-shelf via Python code on GitHub | Yes                                      |
| Smooth (Nissim et al., 2007) | \((\epsilon, \delta)\)-DP | off-the-shelf via Python code on GitHub | Yes                                      |
| Concentrated Smooth (Gillenwater et al., 2021) | CDP | off-the-shelf via Python code on GitHub | No, requires fine tuning which is not realistic for our application |
| Propose-Test-Release (Dwork and Lei, 2009) | \((\epsilon, \delta)\)-DP | No code publicly available | No, requires fine tuning which is not realistic for our application |
| Tzamos et al. (2020) | \(\epsilon\)-DP | No code publicly available | No, requires fine tuning which is not realistic for our application |
Table 6: Summary of the DP mean confidence interval methods reviewed in Section 3.3.

| Method                        | Privacy Definition | Off-the-Shelf vs. Hand-Coding | Selected or Not Selected for Case Study |
|-------------------------------|--------------------|-------------------------------|-----------------------------------------|
| Brawner and Honaker (2018)    | zCDP ↔ (ϵ, δ)-DP   | off-the-shelf via R code on GitHub | Yes                                     |
| D’Orazio et al. (2015)        | ϵ-DP               | off-the-shelf via R code on GitHub | No, requires a priori bounds set on standard deviation |
| Karwa and Vadhan (2017)       | (ϵ, δ)-DP          | off-the-shelf via R code on GitHub | No, requires a priori bounds set on standard deviation |
| COINPRESS (Biswas et al., 2020) | zCDP               | off-the-shelf via R code on GitHub | No, requires a priori bounds set on variance/covariance |
| NOISYMAD (Du et al., 2020)    | ϵ-DP               | off-the-shelf via R code on GitHub | Yes                                     |
| NOISYVAR (Du et al., 2020)    | ϵ-DP               | off-the-shelf via R code on GitHub | Yes                                     |
| CENQ (Du et al., 2020)        | ϵ-DP               | off-the-shelf via R code on GitHub | Yes, for non-skewed data                |
| MOD (Du et al., 2020)         | ϵ-DP               | off-the-shelf via R code on GitHub | Yes, for non-skewed data                |
| SYMQ (Du et al., 2020)        | ϵ-DP               | off-the-shelf via R code on GitHub | Yes, for non-skewed data                |

Table 7: Summary of the histogram results. Median simulated values for the errors are provided and 99th percentile simulated values are in parenthesis. Select values of ϵ provided. For Gaussian mechanism, δ = 10^{-7}.

| Method            | Epsilon | Max Relative Error | Mean Error       |
|-------------------|---------|--------------------|------------------|
|                   |         |                    |                  |
| Laplace mechanism | 0.01    | 5.19% (11.5%)      | 30.7% (47.1%)    |
| Gaussian mechanism| 0.01    | 13.3% (28.5%)      | 84.1% (123.2%)   |
| Laplace mechanism | 0.1     | 0.54% (1.15%)      | 3.11% (4.94%)    |
| Gaussian mechanism| 0.1     | 2.11% (4.69%)      | 13.2% (19.0%)    |
| Laplace mechanism | 1       | 0.06% (0.12%)      | 0.30% (0.48%)    |
| Gaussian mechanism| 1       | 0.21% (0.42%)      | 1.32% (1.84%)    |
| Laplace mechanism | 5       | 0.01% (0.02%)      | 0.02% (0.07%)    |
| Gaussian mechanism| 5       | 0.04% (0.09%)      | 0.25% (0.38%)    |
Figure 4: Income histogram results: max relative error for two methods and varying levels of $\epsilon$ and $\delta$.

This tells the maximum percentage error for someone trying to determine the what percentage of individuals fall below or above a certain cutoff. Second, we measure the mean error over all cumulative sums for the histogram, which tells us the average error across all adjacent subsets of the histogram. Results are summarize in Table 7 and shown in Figures 4 and 5 respectively.

11.2 Quantiles

Users may also query certain quantiles as a way to understand the distribution of data, such as income, rather than querying a histogram or mean value. We tested three different methods for producing multiple quantiles, which is likely to be of interest for those making queries. We summarize the results in Table 8 and Figure 6. We find that the mean bias is similar as $\epsilon$ increases, but the distribution of output tightens. This suggests some small permanent bias in these methods.

11.3 Estimates of the Mean and Confidence Intervals

We compared NOISYVAR, NOISYMAD, and BHM for computing a DP estimate of the mean and CI. Three additional methods listed in Table 6 were not tested due to the heavy skew of the data. Our results indicate that they would only provide unbiased results for nearly perfectly Gaussian
Figure 5: Income histogram results: mean cumulative sums error for two methods and varying levels of $\epsilon$ and $\delta$.

Table 8: Summary of the quantile results. Median simulated values for the mean errors across 9 quantiles are provided and 99th percentile simulated mean errors are in parenthesis. Select values of $\epsilon$ provided. For AppIndExp and Smooth, $\delta = 10^{-7}$.

| Method   | Epsilon | Mean Absolute Bias ($) | Mean Ordered Differences Absolute Bias ($) |
|----------|---------|------------------------|------------------------------------------|
| AppIndExp | 0.01    | 3,136 (120,932)        | 739 (118,697)                             |
| JointExp | 0.01    | 919 (2,518)            | 88 (339)                                  |
| Smooth   | 0.01    | 553,500 (1,042,832)    | 124,667 (124,667)                         |
| AppIndExp | 0.1     | 1,644 (2,047)          | 108 (216)                                 |
| JointExp | 0.1     | 956 (1,421)            | 115 (204)                                 |
| Smooth   | 0.1     | 119,401 (303,380)      | 81,259 (124,667)                          |
| AppIndExp | 1       | 1,670 (1,795)          | 188 (221)                                 |
| JointExp | 1       | 1,080 (1,100)          | 179 (183)                                 |
| Smooth   | 1       | 2,589 (11,168)         | 892 (5,532)                               |
| AppIndExp | 5       | 1,652 (1,686)          | 205 (221)                                 |
| JointExp | 5       | 1,083 (1,423)          | 210 (220)                                 |
| Smooth   | 5       | 1,652 (2,027)          | 224 (294)                                 |

Table 9 and Figure 7 shows the simulated distributions of confidence interval overlap (CIO) and confidence interval ratio (CIR) values. We plot these two metrics against each other because...
Figure 6: Average absolute bias across quantiles for two methods and varying levels of $\epsilon$ and $\delta$.

Table 9: Summary of the confidence interval measures for the mean statistic results. Median simulated values for the metrics are provided and 90th percentile (worse) simulated values are in parenthesis. Select values of $\epsilon$ provided. For BHM, $\delta = 10^{-7}$.

| Method     | Epsilon | CIR     | CIO     |
|------------|---------|---------|---------|
| NOISYMAD   | 0.1     | 1.30 (1.37) | 0.87 (0.72) |
| NOISYVAR   | 0.1     | 1.51 (1.63) | 0.82 (0.70) |
| BHM        | 0.1     | 7.14 (11.26) | 0.57 (0.54) |
| NOISYMAD   | 0.5     | 0.73 (0.76) | 0.86 (0.85) |
| NOISYVAR   | 0.5     | 1.02 (1.07) | 0.97 (0.91) |
| BHM        | 0.5     | 1.92 (2.69) | 0.75 (0.68) |
| NOISYMAD   | 1       | 0.70 (0.73) | 0.85 (0.84) |
| NOISYVAR   | 1       | 0.99 (1.04) | 0.98 (0.94) |
| BHM        | 1       | 1.28 (1.78) | 0.85 (0.75) |
| NOISYMAD   | 5       | 0.69 (0.72) | 0.85 (0.84) |
| NOISYVAR   | 5       | 0.99 (1.04) | 0.99 (0.98) |
| BHM        | 5       | 0.99 (1.27) | 0.91 (0.83) |

taken together they greatly aid the interpretation of the results. The red lines plotted at $CIO = 0.5$ and $CIR = 1$ form four quadrants on the charts. We summarize the interpretation of each quadrant as follows:

- **Top left quadrant:** indicates noisy CIs that are more narrow than the confidential CI but...
Figure 7: Mean income confidence intervals ratio and overlap results for three methods and varying levels of $\epsilon$. BHM Results Only Shown for $\delta = 0.01$.

mostly contained within the confidential CI.

- **Top right quadrant**: indicates noisy CIs that are wider than the confidential CI but mostly encompass the confidential CI.
• **Bottom left quadrant**: indicates noisy CIs that are both more narrow than the confidential CI and biased away from the confidential point estimate.

• **Bottom right quadrant**: indicates noisy CIs that are both wider than the confidential CIs and biased away from the confidential point estimate.

### 11.4 Additional CPS Results for Regression Models

In this section we provide results for the other estimated coefficients in the CPS model. Results for other methods we evaluate can be found in our GitHub repository.

The linear regression model we fit on the CPS data is specified as:

\[
\log(\text{Annual Earnings}) = \beta_0 + \beta_1 \cdot I(\text{NonWhite}) + \beta_2 \cdot \text{Years of Education} + \\
\beta_3 \cdot \text{Potential Experience} + \beta_4 \cdot \text{Potential Experience}^2 + \beta_5 \cdot \text{Potential Experience}^3
\]  

Figure 8: Distribution of simulated DP estimates for regression coefficients. Confidential estimates (black horizontal lines) and 95% confidence intervals (dotted lines) are shown. \( \delta = 10^{-7} \) for AGM.

Figure 8 shows violin plots with the distribution of simulated estimates for the four coefficients not included in the main paper.

Table 10 provides the error rates and quantiles on the distribution of the DP estimates for
the additional coefficients. We note that the results do not clearly improve as $\epsilon$ grows for the non-intercept coefficients. These variables are highly skewed, particularly when the square or cube is taken.

*Table 10: Summary of the regression results. Confidential (non-DP) estimate values for each coefficient are shown. 5th, 95th, quantiles, and L1 Error for select coefficients, methods, and values of $\epsilon$. $\delta = 10^{-7}$ for AGM.*

| Coefficient | Epsilon | Method | Non-DP | DP 5% | DP 95% | L1     |
|-------------|---------|--------|--------|-------|--------|--------|
| Intercept   | 0.5     | AGM    | 6.19   | 5.35  | 7.44   | 0.596  |
| Intercept   | 0.5     | LM     | 6.19   | 4.80  | 8.00   | 0.943  |
| Intercept   | 1       | AGM    | 6.19   | 4.72  | 6.83   | 0.513  |
| Intercept   | 1       | LM     | 6.19   | 5.60  | 7.53   | 0.488  |
| Intercept   | 5       | AGM    | 6.19   | 5.35  | 6.56   | 0.338  |
| Intercept   | 5       | LM     | 6.19   | 4.51  | 6.59   | 0.510  |
| Potential Exp. | 0.5   | AGM    | 0.154  | -4.65e-02 | 0.214  | 0.104  |
| Potential Exp. | 0.5   | LM     | 0.154  | -4.83e-02 | 0.199  | 0.104  |
| Potential Exp. | 1     | AGM    | 0.154  | -1.72e-02 | 0.221  | 8.66e-02 |
| Potential Exp. | 1     | LM     | 0.154  | -1.08e-02 | 0.175  | 0.101  |
| Potential Exp. | 5     | AGM    | 0.154  | 4.70e-02  | 0.472  | 9.38e-02 |
| Potential Exp. | 5     | LM     | 0.154  | 4.51e-02  | 0.692  | 0.129  |
| Potential Exp.² | 0.5   | AGM    | -5.42e-03 | -1.01e-02 | 5.79e-03 | 5.46e-03 |
| Potential Exp.² | 0.5   | LM     | -5.42e-03 | -8.39e-03 | 3.63e-03 | 5.22e-03 |
| Potential Exp.² | 1     | AGM    | -5.42e-03 | -7.53e-03 | 5.00e-03 | 9.00e-03 |
| Potential Exp.² | 1     | LM     | -5.42e-03 | -5.54e-03 | 5.84e-03 | 5.76e-03 |
| Potential Exp.² | 5     | AGM    | -5.42e-03 | -2.27e-02 | 7.70e-04 | 5.28e-03 |
| Potential Exp.² | 5     | LM     | -5.42e-03 | -2.39e-02 | 2.50e-03 | 9.00e-03 |
| Potential Exp.³ | 0.5   | AGM    | 5.81e-05  | -1.02e-04 | 1.48e-04 | 7.70e-05 |
| Potential Exp.³ | 0.5   | LM     | 5.81e-05  | -6.66e-05 | 1.34e-04 | 7.24e-05 |
| Potential Exp.³ | 1     | AGM    | 5.81e-05  | -9.21e-05 | 7.97e-05 | 1.76e-04 |
| Potential Exp.³ | 1     | LM     | 5.81e-05  | -1.14e-04 | 6.28e-05 | 8.38e-05 |
| Potential Exp.³ | 5     | AGM    | 5.81e-05  | -3.41e-05 | 3.08e-04 | 7.74e-05 |
| Potential Exp.³ | 5     | LM     | 5.81e-05  | -6.01e-05 | 3.01e-04 | 1.48e-04 |

Figures 9 and 10 show the estimates and corresponding CIs estimated using the two different methods for the coefficients not included in the main paper. We see even at $\epsilon = 5$ that the results for the “Potential Experience” coefficients are still quite poor.

12 SOI PUF Experimental Results

In this section, we present tables and figures for the DP tabular, mean, quantile, and regression results on the private SOI PUF data.
Figure 9: Regression results showing the 90 best (out of 100) simulated DP estimates and confidence intervals for the Analytic Gaussian Mechanism and \( \epsilon = 5 \). Results shown for four coefficients with confidence intervals estimated using the asymptotic approach. \( \delta = 10^{-7} \) for AGM.

12.1 Histograms

We test the Laplace and Gaussian mechanisms for providing tax data histograms.

Table 11: Summary of the SOI histogram results. Median simulated values for the errors are provided and 99th percentile simulated values are in parenthesis. Select values of \( \epsilon \) provided. For Gaussian mechanism, \( \delta = 10^{-7} \)

| Method               | Epsilon | Max Relative Error | Mean Error       |
|----------------------|---------|--------------------|------------------|
|                      |         |                    |                  |
| Laplace mechanism    | 0.01    | 12.2% (19.0%)      | 110% (131%)      |
| Gaussian mechanism   | 0.01    | 16.6% (22.2%)      | 127% (147%)      |
| Laplace mechanism    | 0.1     | 3.15% (6.42%)      | 36.0% (44.0%)    |
| Gaussian mechanism   | 0.1     | 7.61% (13.8%)      | 87.4% (100%)     |
| Laplace mechanism    | 1       | 0.36% (0.75%)      | 3.85% (4.95%)    |
| Gaussian mechanism   | 1       | 1.34% (2.63%)      | 16.8% (19.8%)    |
| Laplace mechanism    | 5       | 0.08% (0.15%)      | 0.33% (0.61%)    |
| Gaussian mechanism   | 5       | 0.28% (0.57%)      | 3.28% (3.92%)    |
Figure 10: Regression results showing the 90 best (out of 100) simulated DP estimates and confidence intervals for the Analytic Gaussian Mechanism and $\epsilon = 5$. Results shown for four coefficients with confidence intervals estimated using the bootstrap approach. $\delta = 10^{-7}$ for AGM.

Figure 11: SOI Income histogram results: max relative error for two methods and varying levels of $\epsilon$ and $\delta$. 

49
Figure 12: SOI Income histogram results: mean cumulative sums error for two methods and varying levels of $\epsilon$ and $\delta$. 

*Figure 12: SOI Income histogram results: mean cumulative sums error for two methods and varying levels of $\epsilon$ and $\delta$.***
12.2 Quantiles

Table 12: Summary of the SOI quantile results. Median simulated values for the mean errors across 9 quantiles are provided and 99th percentile simulated mean errors are in parenthesis. Select values of $\epsilon$ provided. For AppIndExp and Smooth, $\delta = 10^{-7}$.

| Method      | Epsilon | Mean Absolute Bias ($) | Mean Ordered Differences Absolute Bias ($) |
|-------------|---------|------------------------|-------------------------------------------|
| AppIndExp   | 0.01    | 6,867 (2,822,716)      | 5,869 (2,826,831)                         |
| JointExp    | 0.01    | 49,865 (1,732,398)     | 26,753 (1,691,948)                        |
| Smooth      | 0.01    | 23,877,189 (41,592,581)| 5,897,289 (5,897,289)                     |
| AppIndExp   | 0.1     | 579 (2,659)            | 451 (3,012)                               |
| JointExp    | 0.1     | 384 (938)              | 244 (634)                                 |
| Smooth      | 0.1     | 278,636 (5,604,680)    | 213,595 (5,460,180)                       |
| AppIndExp   | 1       | 50 (304)               | 34 (238)                                  |
| JointExp    | 1       | 34 (74)                | 11 (51)                                   |
| Smooth      | 1       | 6,525 (46,023)         | 6,602 (44,974)                            |
| AppIndExp   | 5       | 31 (95)                | 16 (63)                                   |
| JointExp    | 5       | 35 (44)                | 5 (9)                                     |
| Smooth      | 5       | 316 (1,179)            | 322 (1,379)                               |

Figure 13: Average absolute bias across SOI quantiles for two methods and varying levels of $\epsilon$ and $\delta$. 
12.3 Estimates of the Mean and Confidence Intervals

Table 13: Summary of the SOI confidence interval measures for the mean statistic results. Median simulated values for the metrics are provided and 90th percentile (worse) simulated values are in parenthesis. Select values of $\epsilon$ provided. For BHM, $\delta = 10^{-7}$.

| Method     | Epsilon | CIR    | CIO     |
|------------|---------|--------|---------|
| NOISYMAD   | 0.1     | 2.24 (2.40) | 0.72 (0.69) |
| NOISYVAR   | 0.1     | 2.52 (2.75) | 0.69 (0.61) |
| BHM        | 0.1     | 13.5 (19.7) | 0.54 (0.53) |
| NOISYMAD   | 0.5     | 0.58 (0.61) | 0.79 (0.77) |
| NOISYVAR   | 0.5     | 1.09 (1.22) | 0.93 (0.82) |
| BHM        | 0.5     | 3.33 (4.74) | 0.65 (0.60) |
| NOISYMAD   | 1       | 0.45 (0.46) | 0.72 (0.71) |
| NOISYVAR   | 1       | 1.02 (1.09) | 0.97 (0.91) |
| BHM        | 1       | 1.92 (2.58) | 0.75 (0.69) |
| NOISYMAD   | 5       | 0.40 (0.42) | 0.70 (0.69) |
| NOISYVAR   | 5       | 1.00 (1.03) | 0.99 (0.97) |
| BHM        | 5       | 1.04 (1.34) | 0.90 (0.83) |
Figure 14: Mean SOI income confidence intervals ratio and overlap results for three methods and varying levels of $\epsilon$. BHM Results Only Shown for $\delta = 0.01$. 
12.4 Regression Models

Figure 15: Distribution of simulated DP estimates for SOI regression coefficients. Confidential estimates (black horizontal lines) and 95% confidence intervals (dotted lines) are shown. $\delta = 10^{-7}$ for AGM.
Table 14: Summary of the SOI regression results. Confidential (non-DP) estimate values for each coefficient are shown. 5th, 95th, quantiles, and L1 Error for select coefficients, methods, and values of $\epsilon$. $\delta = 10^{-7}$ for AGM.

| Coefficient | Epsilon | Method | Non-DP | DP 5% | DP 95% | L1  
|-------------|----------|--------|--------|-------|-------|-----
| Age 65+     | 0.5      | AGM    | -3.16  | -51.8 | 36.8  | 20.3
| Age 65+     | 0.5      | LM     | -3.16  | -91.8 | 66.4  | 31.5
| Age 65+     | 1        | AGM    | -3.16  | -29.8 | 27.4  | 14.1
| Age 65+     | 1        | LM     | -3.16  | -40.0 | 34.2  | 19.4
| Age 65+     | 5        | AGM    | -3.16  | -8.49 | 2.78  | 2.66
| Age 65+     | 5        | LM     | -3.16  | -10.5 | 5.01  | 3.9
| Log Dividends | 0.5     | AGM    | -3.53  | -82.8 | 77.6  | 42.1
| Log Dividends | 0.5    | LM     | -3.53  | -103  | 84.3  | 45.0
| Log Dividends | 1     | AGM    | -3.53  | -58.1 | 45.0  | 27.4
| Log Dividends | 1    | LM     | -3.53  | -70.1 | 67.9  | 35.4
| Log Dividends | 5     | AGM    | -3.53  | -19.0 | 10.4  | 7.80
| Log Dividends | 5    | LM     | -3.53  | -21.8 | 17.7  | 8.68
| Log Net AGI | 0.5      | AGM    | -0.433 | -12.7 | 10.6  | 5.57
| Log Net AGI | 0.5      | LM     | -0.433 | -19.6 | 16.4  | 7.96
| Log Net AGI | 1        | AGM    | -0.433 | -6.42 | 5.18  | 3.27
| Log Net AGI | 1        | LM     | -0.433 | -9.68 | 9.60  | 4.24
| Log Net AGI | 5        | AGM    | -0.433 | -2.34 | 1.26  | 0.878
| Log Net AGI | 5        | LM     | -0.433 | -2.35 | 1.70  | 0.943
| MCG First Dollar | 0.5  | AGM    | -28.4  | -2170 | 2290  | 1180
| MCG First Dollar | 0.5 | LM     | -28.4  | -3130 | 3860  | 1340
| MCG First Dollar | 1  | AGM    | -28.4  | -1370 | 1130  | 598
| MCG First Dollar | 1 | LM     | -28.4  | -1810 | 1710  | 840
| MCG First Dollar | 5  | AGM    | -28.4  | -346  | 343   | 175
| MCG First Dollar | 5  | LM     | -28.4  | -479  | 368   | 196
Table 15: Summary of the SOI private CI estimates. Median and 90th Percentile Values CIR and Percentage of Sign, Significance Agreement, and CI Overlap for select coefficients, methods, and values of $\epsilon$. Both methods for estimating CI shown. $\delta = 10^{-7}$ for AGM.

| Coefficient  | Epsilon | Method | Asymptotic CIR | SSO% | Bootstrap CIR | SSO% |
|--------------|---------|--------|----------------|------|--------------|------|
| Age 65+      | 0.5     | AGM    | 2.79 (3.95)    | 9%   | 39.6 (44.8)  | 0%   |
| Age 65+      | 0.5     | LM     | 4.02 (5.38)    | 4%   | 64.6 (74.0)  | 1%   |
| Age 65+      | 1       | AGM    | 2.00 (2.08)    | 9%   | 20.6 (21.3)  | 3%   |
| Age 65+      | 1       | LM     | 3.13 (3.82)    | 4%   | 32.4 (35.3)  | 0%   |
| Age 65+      | 5       | AGM    | 1.01 (1.39)    | 45%  | 4.88 (5.00)  | 11%  |
| Age 65+      | 5       | LM     | 1.17 (1.65)    | 46%  | 6.58 (6.73)  | 7%   |
| Log Dividends| 0.5     | AGM    | 2.59 (3.36)    | 2%   | 210 (359)    | 0%   |
| Log Dividends| 0.5     | LM     | 3.10 (4.26)    | 2%   | 244 (540)    | 0%   |
| Log Dividends| 1       | AGM    | 1.96 (2.47)    | 5%   | 124 (161)    | 0%   |
| Log Dividends| 1       | LM     | 2.65 (3.38)    | 2%   | 172 (267)    | 2%   |
| Log Dividends| 5       | AGM    | 1.02 (1.38)    | 10%  | 33.3 (34.6)  | 5%   |
| Log Dividends| 5       | LM     | 1.16 (1.60)    | 9%   | 45.3 (47.9)  | 1%   |
| Log Net AGI  | 0.5     | AGM    | 2.68 (3.80)    | 1%   | 120 (182)    | 0%   |
| Log Net AGI  | 0.5     | LM     | 3.85 (5.03)    | 1%   | 163 (310)    | 0%   |
| Log Net AGI  | 1       | AGM    | 1.98 (2.73)    | 7%   | 67.5 (80.8)  | 4%   |
| Log Net AGI  | 1       | LM     | 3.07 (3.68)    | 3%   | 100 (132)    | 2%   |
| Log Net AGI  | 5       | AGM    | 1.01 (1.37)    | 17%  | 16.6 (17.0)  | 8%   |
| Log Net AGI  | 5       | LM     | 1.15 (1.63)    | 24%  | 22.2 (23.1)  | 0%   |
| MCG First Dollar | 0.5   | AGM    | 2.53 (3.48)    | 3%   | 164 (290)    | 0%   |
| MCG First Dollar | 0.5   | LM     | 3.37 (4.44)    | 2%   | 214 (431)    | 0%   |
| MCG First Dollar | 1     | AGM    | 1.93 (2.53)    | 3%   | 99.5 (123)   | 0%   |
| MCG First Dollar | 1     | LM     | 2.73 (3.49)    | 1%   | 145 (208)    | 0%   |
| MCG First Dollar | 5     | AGM    | 1.01 (1.38)    | 7%   | 25.1 (26.1)  | 3%   |
| MCG First Dollar | 5     | LM     | 1.14 (1.63)    | 11%  | 34.5 (35.9)  | 0%   |
Figure 16: SOI regression results showing the 90 best (out of 100) simulated DP estimates and confidence intervals for the Analytic Gaussian Mechanism and $\epsilon = 1$. Results shown for two coefficients with confidence intervals estimated using the asymptotic approach. $\delta = 10^{-7}$ for AGM.

Figure 17: SOI regression results showing the 90 best (out of 100) simulated DP estimates and confidence intervals for the Analytic Gaussian Mechanism and $\epsilon = 1$. Results shown for two coefficients with confidence intervals estimated using the bootstrap approach. $\delta = 10^{-7}$ for AGM.
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58
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