Holographic complexity of boosted black brane and Fisher information

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Abstract

In this paper, we have studied the holographic complexity for boosted black brane for strip like subsystem. Complexity has been computed for a subsystem chosen along and perpendicular to the boost direction. We have observed that there is an asymmetry in the result due to the boost parameter. The Fisher information metric and the fidelity susceptibility have also been computed using bulk dual prescriptions. It is noted that the two metrics computed holographically are different for both the pure black brane as well as the boosted black brane.

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1 Introduction

The discovery of gauge/gravity duality which relates conformal field theories living on the boundary of anti-de Sitter (AdS) spacetime to the bulk theory living in one extra spatial dimension has proved to be a remarkable progress in theoretical physics [1, 2, 3]. It has enabled us in getting deep insights in strongly coupled condensed matter systems and has also been the focus of recent developments in the field of string theory [4]-[11]. Holographic computation of quantum information theoretic quantities such as entanglement entropy (EE) and complexity of a subsystem living on the boundary conformal field theory (CFT) has also been an intense area of research [12]-[30]. Investigations in these directions have led to significant understanding of the basic laws governing such systems. It has been realized that small perturbations in the density matrix in the boundary field theory obey thermodynamic like relations which are similar to the black hole thermodynamical relations [31]-[46]. Interestingly such kind of relations have also been observed in the case of holographic complexity [47, 48]. The calculation prescription, for instance, for holographic EE involves the evaluation of the geometric area of spatial extremal surfaces embedded in asymptotically AdS spacetime whose boundary ends on the boundary of the subsystem at a fixed time [12, 13].

In this paper, the computation of HC for (d + 1)-dimensional boosted black brane has been carried out for a strip like subsystem. The prescription to compute the HC was proposed in [17]-[19]. It says that for subsystem A in the boundary, if $V(\gamma)$ denotes the maximal codimension one volume enclosed by the codimension two minimal RT surface in the bulk, then the holographic subregion complexity can be calculated from the following formula

$$C_V = \frac{V(\gamma)}{8\pi RG}$$

(1)

where $R$ is the radius of curvature of the spacetime.

Yet another proposal for computing the HC of a system exists in the literature. The content of the proposal is the following. The HC of a system can be obtained from the bulk action evaluated on the Wheeler-DeWitt patch [20]

$$C_W = \frac{A(W)}{\pi \hbar}$$

(2)

where $A(W)$ is the action evaluated on the Wheeler-DeWitt patch $W$ with a suitable boundary time. In this paper, we shall use the first prescription, which is the so called complexity-volume prescription in the entire investigation.

With the thin strip approximation one may consider that the bulk extension only penetrates the ultra violet region of the spacetime under consideration. The boosted black brane geometry in the ultra violet limit can be considered as a perturbation around AdS spacetime. In this paper, we have computed the HC using the complexity-volume proposal for the boosted black brane. In particular, we have carried out our computation up to both first and second orders in the perturbation parameter. Upto first order in perturbation, the HC has been computed for the subsystem in both parallel and perpendicular directions of boost. We have then defined a ratio which can be identified as the holographic complexity asymmetry and compute this ratio. We have then extended our analysis to second order in perturbation. In this case we have computed the HC for the subsystem which is perpendicular to the direction of boost. We have then moved on to compute holographically the Fisher information metric and the fidelity...
susceptibility for the boosted black brane. In this context, we would like to mention that there are two well known notions of distances in the quantum information literature, one is the Fisher information metric and the other is the Bures metric, also known as the fidelity susceptibility. It is also known that the two distances are not the same in general [35]. In this paper, we set out to establish this fact in the context of holography. The holographic computation of Fisher information metric is carried out using the proposal in [36] for both the pure black brane as well as the boosted black brane. We then follow the proposal in [41] to compute the Fisher information metric holographically. The proposal involves computing the change in subregion holographic complexity up to second order in perturbation about the pure AdS spacetime and multiplying it by a dimensionless constant $C_d$. The constant is fixed by comparing it with the result obtained from the relative entropy [36, 45, 43]. The other notion of distance, the fidelity susceptibility has also been computed using the prescription given in [44]. It has been argued that the gravity dual of the fidelity susceptibility is approximately given by the volume of the maximal time slice in AdS spacetime when the perturbation is exactly marginal. It was also generalized to incorporate mixed states also. The prescription was then applied to compute the fidelity susceptibility of the black brane. In this paper, this computation has been carried out for the boosted black brane.

Another aspect that has been looked at in this investigation is the following. Modifications to the holographic entanglement entropy (HEE) first law of thermodynamics have been obtained in AdS spacetimes carrying gauge charges [33, 46]. In particular, the boosted AdS black branes have given rise to modifications of the first law of holographic entanglement thermodynamics (HET) [46, 49]. It has been observed that the boosted black branes leads to an asymmetry in the first law of HET. In this work, two cases were investigated, namely, strip subsystem parallel to the boost direction and also the other perpendicular to the boost direction. It was found that $\Delta S_\perp \geq \Delta S_\parallel$ and the entanglement asymmetry ratio was also computed. The asymmetry was found to be dependent on the boost parameter and was bounded from above. As mentioned in the earlier preceding paragraph that the computation of the HC has also been done for two cases, namely, strip subsystem parallel to the boost direction and also the other perpendicular to the boost direction. These studies also allow us to find asymmetry in the HC for the two cases. Further, the investigations also puts some light on the dependence of the HC with the holographic entanglement entropy.

This paper is organized as follows. In section (2), computation of HC for $(d+1)$-dimensional AdS spacetime for a strip like subsystem has been done. The computation of HC for $(d+1)$-dimensional boosted black brane has been carried out in section (3). In section (4), we give a detailed analysis of the Fisher information metric and the fidelity susceptibility for boosted black brane. We have concluded in section (5). This paper has also an appendix.

### 2 Holographic complexity for $(d+1)$-dimensional AdS spacetime

In this section, we shall present a review of the computation of the HC for a strip like entangling surface in $(d+1)$-dimensional AdS spacetime [50]. The $AdS_{d+1}$ metric is given by

$$ds^2 = \frac{-dt^2 + dx_1^2 + \cdots + dx_{d-1}^2 + dz^2}{z^2}$$

where we have set the $AdS$ radius $R = 1$. To compute the holographic subregion complexity we embed a strip like surface in this background given by $t = \text{constant}$, $x_1 = x_1(z)$. The boundaries
of the extremal bulk surface coincide with the two ends of the interval \((-\frac{l}{2} \leq x_1 \leq \frac{l}{2}\)). The regulated size of the rest of the coordinates is taken to be large with \(0 \leq x_i \leq L_i\). The area of the strip like surface is given by

\[
A(0) = 2V_{(d-2)} \int_0^{z_*(0)} \frac{dz}{z^{d-1}} \sqrt{1 + x'_1(z)^2}
\]

where \(z_*(0)\) is the turning point of the surface and \(V_{(d-2)} = L_2 L_3 \cdots L_{d-1}\). The minimal surface is obtained by minimizing the area functional. On minimizing we get

\[
x'_1(z) = \frac{1}{\sqrt{\left(\frac{z_*(0)}{z}\right)^{2(d-1)} - 1}}.
\]

The identification of the boundary \(x_1(0) = l/2\) leads to the integral relation

\[
\frac{l}{2} = \int_0^{z_*(0)} \frac{dz}{\sqrt{\left(\frac{z_*(0)}{z}\right)^{2(d-1)} - 1}} = z_*(0) \int_0^1 t^{d-1} \frac{dt}{\sqrt{1 - t^{2(d-1)}}} = z_*(0) b_0
\]

where \(t = \frac{z}{z_*(0)}\). The volume of the minimal surface is given by

\[
V(0) = 2V_{(d-2)} \int_0^{z_*(0)} \frac{dz}{z^{d-1}} \int_0^{x_1(z)} dx_1(z)
\]

where \(\delta\) is the UV cutoff. Now using eq. \((5)\) we can write eq. \((7)\) as

\[
V(0) = \frac{V_{(d-2)}}{(d-1)\delta^{d-1}} \frac{2^{d-2}\pi^{(d-1)/2}}{(d-1)^2} \left( \frac{\Gamma \left( \frac{d}{2(d-1)} \right)}{\Gamma \left( \frac{1}{2(d-1)} \right)} \right)^{d-3} \frac{V_{(d-2)}}{l^{d-2}}
\]

where we have used eq. \((6)\) in writing the second line of the above equation. Hence the HC for pure AdS spacetime is given by

\[
C(0) = \frac{V(0)}{8\pi G_{(d+1)}} = \frac{V_{(d-2)}}{8\pi G_{(d+1)}(d-1)\delta^{d-1}} \frac{l}{8\pi G_{(d+1)}(d-1)^2} \frac{2^{d-2}\pi^{(d-1)/2}}{\Gamma \left( \frac{1}{2(d-1)} \right)} \left( \frac{\Gamma \left( \frac{d}{2(d-1)} \right)}{\Gamma \left( \frac{1}{2(d-1)} \right)} \right)^{d-3} \frac{V_{(d-2)}}{l^{d-2}}.
\]

Note that the first term in the holographic complexity is divergent (volume law) whereas the second term is finite. In the next section, we proceed to investigate the subregion holographic complexity of the boosted black brane.
3 Holographic subregion complexity for boosted $\text{AdS}_{d+1}$ black brane

The boosted $\text{AdS}_{d+1}$ black brane metric is given by

$$ds^2 = \frac{1}{z^2} \left( -\frac{fdt^2}{K} + K (dy - \omega)^2 + dx_1^2 + \cdots + dx_{d-2}^2 + \frac{dz^2}{f} \right)$$

with

$$K(z) = 1 + \beta^2 \gamma^2 \frac{z^d}{z_0^d}, \quad f(z) = 1 - \frac{z^d}{z_0^d}.$$  \hspace{1cm} (11)

where $0 \leq \beta \leq 1$ is the boost parameter, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $z_0$ is the horizon of the black brane. It is clear from the metric that the boost is taken along $y$ direction and the radius of curvature of $\text{AdS}$ spacetime has been set to one. The Kaluza-Klein one form $\omega$ reads

$$\omega = \beta^{-1} (1 - \frac{1}{K}) dt.$$ \hspace{1cm} (12)

The metric (10) has an anisotropy due to the boost along $y$ direction. This motivates us to investigate the effect of anisotropy on subregion holographic complexity. To carry out this investigation, we compute the subregion complexity for two cases, firstly, for a strip along the direction of boost ($y$-direction) and secondly, for a strip in the direction perpendicular to the boost ($x$-direction).

3.1 Strip parallel to the direction of boost

To compute the subregion complexity for a strip-like subregion we consider that the boundaries of the extremal bulk surface coincide with the two ends of the interval $-l/2 \leq y \leq l/2$ and $0 \leq x^i \leq L_i$, with $L_i \gg l$. The extremal surface is parametrized as $y = y(z)$. Furthermore we have taken the strip to be thin, so that the bulk extension can only penetrate the UV geometry.

Using the Ryu-Takayanagi prescription the entanglement entropy for this strip like subsystem is given by

$$S_\parallel \equiv \frac{\text{Area}(\gamma_A^{\text{min}})}{4G_{(d+1)}} = \frac{V_{(d-2)}}{2G_{(d+1)}} \int_\delta^{z_{\parallel}} \frac{dy}{z^{d-1}} \frac{1}{\sqrt{K(z) + \frac{(\partial_y z)^2}{f(z)}}}$$

where $G_{(d+1)}$ is $(d + 1)$-dimensional Newton’s constant, $\delta$ is the UV cutoff and $V_{(d-2)} \equiv L_1 L_2 L_3 \cdots L_{d-2}$. As we are interested in computing the change in complexity from pure $\text{AdS}$ spacetime, we can choose $L_i$ in such a way so that this $V_{(d-2)}$ has same value as in pure $\text{AdS}$ case. Here $z_{\parallel}$ is the turning point of the extremal surface inside the bulk geometry.

Now using the standard procedure of minimization we obtain from eq. (13) the following expression for the extremal surface

$$\frac{dy}{dz} \equiv \left( \frac{z}{z_{\parallel}} \right)^{d-1} \frac{1}{\sqrt{K(z) f(z) K(z)}} \frac{1}{\sqrt{\frac{K(z)}{K_{\ast}} - \left( \frac{z}{z_{\parallel}} \right)^{2d-2}}}$$

(14)
where \( K_s = K(z)|_{z=\bar{z}} \). To find an expression for the turning point in terms of the strip length we make the identification \( y(0) = l/2 \). This gives

\[
\frac{l}{2} = \int_0^{z_\parallel} dz \left( \frac{z}{z_\parallel} \right)^{d-1} \frac{1}{\sqrt{f(z) K(z) \left( \frac{K(z)}{K_s} - \left( \frac{z}{z_\parallel} \right)^{2d-2} \right)}}.
\]

For small subsystem the turning point of the RT surface will be near to the pure \( AdS \) boundary region \((z_\parallel \ll z_0)\). Now for finite boost we can evaluate the above integral by expanding it around the pure \( AdS \) such that the condition

\[
(z_\parallel/z_0)^d \ll 1, \quad \beta^2 \gamma^2 \left( \frac{z_\parallel}{z_0} \right)^d \ll 1 \quad (15)
\]

is always preserved. Under this approximation we can write the above integral as follows

\[
\frac{l}{2} = z_\parallel \int_0^1 dt \left( \frac{1}{\sqrt{R}} \left[ 1 + \frac{1}{2} p^d t^d - \frac{1}{2} q^d t^d + \frac{1}{2} q^d - \frac{1}{2} q^d - \frac{1}{2} q^d + \cdots \right] \right) = z_\parallel \left( b_0 + \frac{1}{2} (p^d b_1 - q^d b_1 + q^d I_1) \right) + \cdots \quad (16)
\]

where we have introduced \( R \equiv 1 - l^{2d-2} \), \( t = \frac{l}{z_\parallel} \), \( q^d = \beta^2 \gamma^2 \left( \frac{z_\parallel}{z_0} \right)^d \), \( p = \frac{z_\parallel}{z_0} \) and the dots indicate terms of higher order in \( (z_\parallel/z_0)^d \). The coefficients \( b_0, b_1 \), and \( I_1 \) are provided in the Appendix.

As we are using the metric to compute subregion HC keeping the strip length \( l \) same as in the case of pure \( AdS \) spacetime, hence the turning point of the extremal surface will change. To express the new turning point \( z_\parallel^* \) in terms of \( z_s(0) \), which is the turning point in \( AdS \) spacetime, we invert eq. (16) and use eq. (6) to get

\[
\tilde{z}_\parallel = \frac{l/2}{b_0 + \frac{1}{2} (p^d b_1 - q^d b_1 + q^d I_1)} \approx \frac{z_s(0)}{1 + \frac{1}{2} (p^d b_1 - q^d b_1 + q^d I_1)} \quad (17)
\]

where we have kept terms only up to \( (z_\parallel/z_0)^d \) under the thin strip approximation and \( \tilde{p} = \frac{z_s(0)}{z_0} \) and \( \tilde{q}^d = \beta^2 \gamma^2 (\frac{z_s(0)}{z_0})^d \).

Now the volume of the bulk extension under the RT minimal surface is given by

\[
V_\parallel = 2V_{(d-2)} \int_{\delta}^{z_\parallel} \frac{dz}{z_\parallel} \sqrt{\frac{K(z)}{f(z)}} \int_{\delta}^{z_\parallel} dz \left( \frac{z}{z_\parallel} \right)^{d-1} \frac{1}{\sqrt{f^2(u)K(u) \left( \frac{K(u)}{K_s} - \left( \frac{u}{z_\parallel} \right)^{2d-2} \right)}}
\]

\[
= 2V_{(d-2)} \int_{\delta}^{1} \frac{dt}{t^{d-2}} \int_{\delta}^{1} \frac{dz}{z^{d-2}} \int_{\delta}^{1} dw \ w^{d-1} \frac{1}{\sqrt{f^2(w)K(w) \left( \frac{K(w)}{K_s} - w^{2d-2} \right)}} \quad (18)
\]

where \( w = \frac{u}{z_\parallel} \), \( K(w) = 1 + (wq)^d \), \( f(w) = 1 - (tp)^d \). Now in the limit (15) one can make an expansion of the functions \((K, f)\) and keep terms up to linear order. This enables us to write the volume enclosed by the RT as a series around the pure \( AdS \) volume.
Under these approximations we can expand the volume as
\[
V_{\parallel} = \frac{2V_{(d-2)}}{z_s^{d-2}} \int_\frac{\delta}{z_s}^1 \frac{dt}{t^2} \left(1 + \frac{t^d}{2}(p^d + q^d)\right) \int_t^1 dw \, w^{d-1} \frac{1}{\sqrt{f(w)K(w)}\sqrt{\frac{K(w)}{K_+}} - w^{2d-2}}
\]
\[
= \frac{2V_{(d-2)}}{z_s^{d-2}} \int_\frac{\delta}{z_s}^1 \frac{dt}{t^2} \int_t^1 dw \, w^{d-1} \frac{1}{\sqrt{f(w)K(w)}\sqrt{\frac{K(w)}{K_+}} - w^{2d-2}}
\]
\[
+ \frac{V_{(d-2)}}{z_s^{d-2}} \left(p^d + q^d\right) \int_\frac{\delta}{z_s}^1 dt \int_t^1 dw \, w^{d-1} \frac{1}{\sqrt{f(w)K(w)}\sqrt{\frac{K(w)}{K_+}} - w^{2d-2}}.
\]

(19)

After evaluating these straight forward integrals, we use eq. (17) to obtain the minimal volume \(V_{\parallel}\) in terms of the minimal volume of pure AdS spacetime (eq. (6)) to get
\[
V_{\parallel} = V_{(0)} - \frac{V_{(d-2)}p^d}{(d-1)z_s^{(0)d-2}} \left(\frac{(d-2)\pi b_1}{2(d-1)b_0^2} + (2 - d)c_0\right)
\]
\[- \frac{V_{(d-2)}q^d}{(d-1)z_s^{(0)d-2}} \left(\frac{(d-2)\pi}{2(d-1)^2b_0^2} + \frac{2b_1}{b_0} - 1 + c_2 - c_0d\right).
\]

(20)

where we have kept terms up to linear order in \(p^d\) and \(q^d\). We can recast the change in volume using eq. (6) in terms of the length \(l\) of the strip as
\[
\Delta V_{\parallel} = V_{\parallel} - V_{(0)}
\]
\[
= - \frac{V_{(d-2)}l^2}{4b_0^2(d-1)z_0^d} \left[\left(\frac{(d-2)\pi b_1}{2(d-1)b_0^2} + (2 - d)c_0\right)
\right.
\]
\[
\left. + \beta^2 \gamma^2 \left(\frac{(d-2)\pi}{2(d-1)^2b_0^2} + \frac{2b_1}{b_0} - 1 + c_2 - c_0d\right)\right].
\]

(21)

Hence the change in complexity for a strip perpendicular to the direction of boost is given by
\[
\Delta C_{\parallel}^{(1)} = \frac{\Delta V_{\parallel}}{8\pi G_{(d+1)}}
\]
\[
= - \frac{V_{(d-2)}l^2}{32\pi G_{(d+1)}b_0^2(d-1)z_0^d}
\]
\[
\times \left[\left(\frac{(d-2)\pi b_1}{2(d-1)b_0^2} + (2 - d)c_0\right) + \beta^2 \gamma^2 \left(\frac{(d-2)\pi}{2(d-1)^2b_0^2} + \frac{2b_1}{b_0} - 1 + c_2 - c_0d\right)\right].
\]

(22)

It can be clearly seen that the change in holographic complexity depends on the boost parameter. Note that in the \(\beta \to 0\) limit, the result agrees with the pure black brane result obtained in [50].

This can be recast in the following form
\[
\Delta C_{\parallel}^{(1)} = - \frac{\pi(d-2)}{2(d-1)^3b_0^3} \left[\Delta S_{\parallel} - \frac{b_0^2}{(d+1)b_1^2} \Delta S_{\perp}\right]
\]

(23)
where the expression for $\Delta S_\parallel$ and $\Delta S_\perp$ are given by [49],

$$
\Delta S_\parallel = \frac{V_{(d-2)}l^2b_1(d+1)}{32G_{(d+1)}b_0^2z_0^d} \left( \frac{d-1}{d+1} + \frac{2}{d+1}\beta^2\gamma^2 \right)
$$

$$
\Delta S_\perp = \frac{V_{(d-2)}l^2b_1(d+1)}{32G_{(d+1)}b_0^2z_0^d} \left( \frac{d-1}{d+1} + \beta^2\gamma^2 \right)
$$

are the change in entanglement entropies up to first order in perturbation.

The interesting point to note in the above result is that the change in holographic complexity in the parallel direction contains information of the changes in the holographic entanglement entropy in both the parallel and the perpendicular directions of the boost with the respect to the subsystem.

### 3.2 Strip perpendicular to the direction of boost

In this subsection, we essentially follow the analysis similar to the earlier section to compute the holographic complexity of the strip like subsystem with the strip being perpendicular to the direction of boost. Using the Ryu-Takayanagi prescription the entanglement entropy for this strip like subsystem is as follows

$$
S_\perp \equiv \frac{\text{Area}(\gamma^\text{min}_A)_\perp}{4G_{(d+1)}} = \frac{V_{(d-2)}l^2b_1(d+1)}{2G_{(d+1)}} \int_\delta^{z_\perp^*} dz \frac{dz}{z^{d-1}} \sqrt{K(z)} \sqrt{\frac{1}{f(z)} + (\partial_z x^1)^2}.
$$

Here $z_\perp^*$ is the turning point of the extremal surface inside the bulk geometry. Now using the standard procedure of minimization we obtain from eq. (25) the following expression for extremal surface

$$
\frac{dx^1}{dz} = \left( \frac{z}{z_\perp^*} \right)^{d-1} \frac{1}{\sqrt{f(z)} \sqrt{\frac{K(z)}{K_*} - \left( \frac{z}{z_\perp^*} \right)^{2d-2}}}.
$$

where $K_* = K(z)|_{z=z_\perp^*}$. To find an expression for the turning point in terms of the strip length we make the identification $x^1(0) = l/2$. This yields

$$
\frac{l}{2} = \int_0^{z_\perp^*} dz \left( \frac{z}{z_\perp^*} \right)^{d-1} \frac{1}{\sqrt{f(z)} \sqrt{\frac{K(z)}{K_*} - \left( \frac{z}{z_\perp^*} \right)^{2d-2}}}
$$

where we have taken the same subsystem size as in the parallel case and is assumed to be small. Hence the turning point will lie near the asymptotic region ($z_\perp^* \ll z_0$). Thus for finite boost we can expand the above integral around pure AdS preserving the following condition

$$
\left( \frac{z_\perp^*}{z_0} \right)^d \ll 1, \quad \beta^2\gamma^2 \left( \frac{z_\perp^*}{z_0} \right)^d \ll 1
$$

(27)
Thus in this limit, expanding the above integral gives

\[
\frac{l}{2} = z_*^\perp \int_0^1 dt \frac{t^{d-1}}{\sqrt{R}} \left[ 1 + \frac{1}{2} x^d b_1 + \frac{1}{2} y^d I_1 + \cdots \right]
\]

\[
= z_*^\perp \left( b_0 + \frac{1}{2} (x^d b_1 + y^d I_1) \right) + \cdots
\]

(28)

where we have introduced \( R \equiv 1 - t^{2d-2} \), \( t = \frac{z}{z_0} \), \( y^d = \beta^2 \gamma^2 (z_*^\perp)^d \), \( x = (\frac{z}{z_0})^d \) and the dots indicate terms of higher order in \((\frac{z}{z_0})^d\).

To express the new turning point \( z_*^\perp \) in terms of \( z_*^{(0)} \), we invert eq. (28) and use eq. (6) to get

\[
z_*^\perp = \frac{l/2}{b_0 + \frac{1}{2} (x^d b_1 + y^d I_1)} \approx \frac{z_*^{(0)}}{1 + \frac{1}{2} (x^d b_1 + y^d I_1)}
\]

(29)

where we have kept the terms only up to \((\frac{z}{z_0})^d\) under the thin strip approximation. Note that the two turning points, the perpendicular and the parallel, reduces to the same result in the \( \beta \to 0 \) limit.

Now the volume of the bulk extension under RT minimal surface is given by

\[
V_\perp = 2V_{(d-2)} \int_{\delta}^{z_*^\perp} \frac{dz}{z^d} \sqrt{\frac{K(z)}{f(z)}} \int_{\delta}^{z_*^\perp} \frac{dz}{z^d} \sqrt{\frac{u}{z_*^d}} \frac{1}{\sqrt{f(u)K_*}} \frac{1}{\sqrt{K(u)K_* - (\frac{u}{z_*^d})^{2d-2}}}
\]

(30)

Once again in the limit \( \frac{z_*^\perp}{z_0} \ll 1 \), one can make an asymptotic expansion of the defining functions \((K, f)\) in terms of this parameter up to linear order. Under these approximations we can write the volume as

\[
V_\perp = \frac{2V_{(d-2)}(\delta)}{z_*^{(0)}^{d-2}} \int_{\delta}^{1} \frac{dt}{t^d} \left( 1 + \frac{t^d}{2} (x^d + y^d) \right) \int_{t}^{1} dw \sqrt{\frac{1}{f(w)K_*} - w^{2d-2}}
\]

\[
= \frac{2V_{(d-2)}(\delta)}{z_*^{(0)}^{d-2}} \int_{\delta}^{1} \frac{dt}{t^d} \int_{t}^{1} dw \sqrt{\frac{1}{f(w)K_*} - w^{2d-2}}
\]

\[
+ \frac{V_{(d-2)}(\delta)}{z_*^{(0)}^{d-2}} (x^d + y^d) \int_{\delta}^{1} \frac{dt}{t^d} \int_{t}^{1} dw \sqrt{\frac{1}{f(w)K_*} - w^{2d-2}}.
\]

(31)

Evaluating these straightforward integrals we use eq. (29) to obtain the minimal volume in terms of the minimal volume of pure AdS spacetime. This leads to

\[
V_\perp = V_0 - \frac{V_{(d-2)} x^d}{(d-1)z_*^{(0)}^{d-2}} \left( \frac{(d-2)\pi b_1}{2(d-1)b_0^2 + c_0} \right) - \frac{V_{(d-2)} y^d}{(d-1)z_*^{(0)}^{d-2}} \left( \frac{(d-2)\pi I_1}{2(d-1)b_0^2 + c_2} \right)
\]

\[
+ \frac{V_{(d-2)} c_0}{z_*^{(0)}^{d-2}} (x^d + y^d)
\]

(32)
where we have kept terms up to linear order in $\bar{x}^d$ and $\bar{y}^d$. In terms of the length $l$ of the strip, the change in volume can be recast in the form

$$\Delta V_\perp \equiv V_\perp - V(0)$$

$$= -\frac{V_{(d-2)}l^2}{4b_0^2(d-1)z_0^d} \left[ \frac{(d-2)\pi b_1}{2(d-1)b_0^2} + (2 - d)c_0 \right]$$

$$+ \beta^2 \gamma^2 \left[ \frac{(d-2)\pi I_1}{2(d-1)b_0^2} + e_2 - (d - 1)c_0 \right]. \quad (33)$$

Hence the change in complexity for a strip perpendicular to the direction of boost is given by

$$\Delta C^{(1)}_\perp = \frac{\Delta V_\perp}{8\pi G_{(d+1)}}$$

$$= -\frac{V_{(d-2)}l^2}{32\pi G_{(d+1)}b_0^2(d-1)z_0^d} \times \left[ \frac{(d-2)\pi b_1}{2(d-1)b_0^2} + \beta^2 \gamma^2 \left( \frac{(d-2)\pi I_1}{2(d-1)b_0^2} + e_2 - (d - 1)c_0 \right) \right]. \quad (34)$$

which can be recast in the following form

$$\Delta C^{(1)}_\perp = \Delta C^{(1)}_\parallel - \frac{V_{(d-2)}l^2\beta^2 \gamma^2 c_0}{32\pi G_{(d+1)}(d-1)b_0^{2-d}z_0^d} \left[ 1 + (d - 2)(d + 1)\frac{b_1}{b_0^2} \right]. \quad (35)$$

Note that there is an asymmetry in the holographic complexities in both the directions which owes its origin to the boost parameter. In the $\beta \to 0$ limit, this asymmetry vanishes and reassuringly both the changes agree with each other.

We now proceed to investigate the asymmetry in the result in HC for the parallel and perpendicular directions of the boost with respect to the subsystem size. For this let us define a quantity

$$R_C = \frac{\Delta C^{(1)}_\perp - \Delta C^{(1)}_\parallel}{\Delta C^{(1)}_\perp + \Delta C^{(1)}_\parallel}. \quad (36)$$

This can be called the holographic complexity ratio. To understand the effect of anisotropy on the holographic complexity. We use eqs. (23, 35) to get

$$R_C = \frac{1}{1 + r} \quad (37)$$

where

$$r = \frac{2\pi}{d-1} \frac{2 - \frac{b_1^2}{b_0^2}}{\beta^2 \gamma^2} \left( 1 - \frac{b_1^2}{b_0^2(d+1)} \right). \quad (38)$$

It is easy to check that $r$ is positive definite which implies that for $\beta \geq 0$, $R_C \geq 0$. Hence, there is an asymmetry in the holographic complexities for the perpendicular and parallel directions of the boost with respect to the system.

In the $\beta \to 0$ limit $R_C = 0$. The $\beta \to 0$ limit implies that the complexity is same ($\Delta C^{(1)}_\perp = \Delta C^{(1)}_\parallel$) in both the parallel or perpendicular directions of the subsystem.
3.3 Complexity upto second order in perturbation

In this section we have computed the subregion complexity for strip like subregion in boosted black brane \([10]\) background with the perturbation up to second order in \((\bar{z}/z_0)^d\) and \(\beta^2 \gamma^2 (\bar{z}/z_0)^d\) around the pure AdS background. The strip has been chosen to be in a direction perpendicular to the direction of boost. As we are interested in second order perturbation, therefore the expression for the turning point and volume will receive some corrections. As the volume under minimal surface depends upon the turning point of the minimal surface, hence we shall first compute the change in turning point. To begin with let us first compute the length \(l\) of the subsystem perturbatively in the same limit as in eq.\,(27) up to second order. The expression for length of the subsystem when the strip is perpendicular to the direction of boost is given by

\[
\frac{l}{2} = \int_0^{z_*} \frac{dz}{\sqrt{1 - \left(\frac{z}{z_0}\right)^d \sqrt{\frac{K(z)}{K_*}}} - \left(\frac{z}{z_*}\right)^2(d-1)}
\]

\[
= z_* \left[ \int_0^1 \frac{dt}{\sqrt{R}} + \frac{x^d}{2} \int_0^1 \frac{t^{d-1}}{\sqrt{R}} \left( t^d + \beta^2 \gamma^2 \frac{1-t^d}{R} \right) \right.
+ x^{2d} \int_0^1 \frac{dt}{\sqrt{R}} \left( \frac{3}{8} t^{2d} + \frac{\beta^2 \gamma^2}{4} t^d \left( \frac{1-t^d}{R} \right) + \beta^4 \left( \frac{3}{8} \frac{(1-t^d)^2}{R^2} - \frac{11-t^d}{R} \right) \right)
\]

\[
= z_* \left[ b_0 + \frac{x^d}{2} (b_1 + \beta \gamma^2 I_I) + x^{2d} \left( \frac{3}{8} b_2 + J_I \right) \right],
\]

where the coefficients \(b_0, b_1, I_I\) and \(J_I\) are given in the Appendix. In order to express this turning point \(z_*\) in terms of the turning point \(z_*^{(0)}\) of pure AdS background, we use eq. \((6)\) to get

\[
z_* = \frac{z_*^{(0)}}{1 + \frac{1}{2b_0} (b_1 + \beta^2 \gamma^2 I_I) \bar{x}^d + \left( \frac{3b_2}{8b_0} + \frac{J_I}{b_0} - d \frac{(b_1 + \beta^2 \gamma^2 I_I)^2}{2b_0} \right) \bar{x}^{2d}}
\]

where \(\bar{x} = z_*^{(0)}/z_0\). Now the expression for volume is given by

\[
V \simeq \frac{2V_{(d-2)}}{z_{*}^{d-2}} \int_{\bar{x}}^{1} \frac{dt}{\sqrt{f(t)}} \left( 1 + \frac{x^d + y^d}{2} t^d + \frac{3}{8} x^{2d} t^d + \frac{x^d y^d}{4} - \frac{y^d}{8} t^{2d} \right)
\]

\[
\times \int_1^t dw \frac{1}{\sqrt{f(w)}} \frac{w^{d-1}}{\sqrt{\frac{K(w)}{K_*} - w^{2(d-1)}}}.
\]

After some lengthy calculations, we obtain the volume from the above expression to be

\[
V = V(0) - \frac{V_{(d-2)} \bar{x}^d}{(d-1)z_*^{d-2}} \left( \frac{d}{d-1} \frac{\pi b_1}{2b_0} + (2-d) c_0 \right) - \frac{V_{(d-2)} \bar{y}^d}{(d-1)z_*^{d-2}} \left( \frac{d}{d-1} \frac{\pi I_I}{2b_0} + c_2 - (d-1) c_0 \right)
\]

\[
- \frac{V_{(d-2)} \bar{x}^{2d} \bar{v}_{00}}{z_*^{d-2}} - \frac{V_{(d-2)} \bar{x}^d \bar{y}^d \bar{v}_{01}}{z_*^{d-2}} + \frac{V_{(d-2)} \bar{y}^{2d} \bar{v}_{11}}{z_*^{d-2}}.
\]

\[
(42)
\]
where $V(0)$ is the volume under RT surface for pure AdS given in eq. (8) with

\[
v_{00} = \left( \frac{3\pi b_2 d - 2}{8b_0^2(d - 1)^2} - \frac{\pi b_1^2 (d - 2)(d + 3)}{8b_0^2(d - 1)^2} + \frac{c_0 b_1 (d - 2) - c_1 d^2 - 4}{b_0(d - 1)} \right)
\]

\[
v_{01} = \left( \frac{b_1}{b_0}(c_0 - \frac{c_2}{d - 1}) - \frac{c_3}{2} + \frac{(d + 2)c_1}{2(d + 1)} + \frac{d - 2 c_0 I_l}{d - 1} b_0 + \frac{2K_1}{d - 1} \right)
\]

\[
+ \frac{d - 2}{(d - 1)^2} \frac{\pi J_1}{b_0} - \frac{(d - 2)(d + 3) \pi b_1 I_l}{(d - 1)^2 b_0^2}
\]

\[
v_{11} = \left( \frac{c_4}{2} - \frac{c_1}{4(d + 1)} + \frac{2K_1}{d - 1} - \frac{c_3}{2d - 1} \right)
\]

\[
\frac{1}{b_0} - \frac{2}{(d - 1)^2} \frac{\pi J_1}{b_0^2} + \frac{(d - 2)(d + 3) \pi I_l^2}{(d - 1)^2 b_0^2}
\]

Now using the expression given in the earlier section for subregion HC for strip perpendicular to the direction of boost with perturbation up to first order, we write the HC upto second order in the perturbation to be

\[
\Delta C = \Delta C^{(1)}_\perp + \Delta C^{(2)}_\perp
\]

where

\[
\Delta C^{(2)}_\perp = -\frac{V(d - 2)I^{d + 2}}{8\pi G(d + 1)x_0^2(2b_0)^{d + 2} [v_{00} + \beta^2 \gamma^2 v_{01} - \beta^4 \gamma^4 v_{11}]}. \tag{45}
\]

This is the second order change in the holographic complexity. We will use this expression in the next section to calculate the Fisher Information metric. It is important to note that the boosted black brane is a stationary spacetime. For such spacetimes one should use the covariant HRT proposal instead of the static RT proposal. However it can be shown that at first order of the perturbative expansion it is sufficient to take the $t = constant$ slicing. At first order, the sole contribution comes from the metric perturbations $[37]-[39]$. Deviations of the minimal surface only contribute at the second order. Thus at second order one cannot work with the same $t = constant$ embedding for stationary asymptotically AdS spacetimes. But as shown in $[40]$ one can still work with the $t = constant$ slice in the boosted black brane spacetime but only for the perpendicular case. This is due to the fact that the minimal surface still remains in the same time slice. The deviations contribute in other spatial directions.

## 4 Fisher information metric and Fidelity susceptibility

In this section, we shall compute the Fisher information metric and the fidelity susceptibility for the boosted black brane from the proposals existing in the literature. Before we begin our analysis we would like to briefly mention about the quantities in the context of quantum information theory. Two well known notion of distance between two quantum states exist in the literature. One is the Fisher information metric and the other is the Bures metric or fidelity susceptibility. The Fisher information metric is defined as $[41]$

\[
G_{F,\lambda\lambda} = \langle \delta \rho \delta \rho \rangle^{(\sigma)}_{\lambda\lambda} = \frac{1}{2} tr \left( \delta \rho \frac{d}{d(\delta \lambda)} \log(\sigma + \delta \lambda \delta \rho)|_{\delta \lambda = 0} \right). \tag{46}
\]
where $\delta \rho$ is a small deviation from the density matrix $\sigma$.

A second notion of distance between two states is known as fidelity susceptibility and reads

$$G_{\lambda \lambda} = \partial^2 \lambda F; \quad F = tr \sqrt{\sigma_{\lambda + \delta \lambda} \sigma_{\lambda}}$$

(47)

where $\sigma$ and $\rho$ are the initial and final density matrices, $F$ is called the fidelity.

The first holographic computation of the Fisher information metric was carried out in [36], with the Fisher information metric defined as

$$G_{F,mm} = \frac{\partial^2}{\partial m^2} S_{rel}(\rho_m \parallel \rho_0); \quad S_{rel}(\rho_m \parallel \rho_0) = \Delta \langle H_{\rho_0} \rangle - \Delta S$$

(48)

where $m$ is a perturbation parameter, $\Delta \langle H_{\rho_0} \rangle$ is the change in modular Hamiltonian and $\Delta S$ is the change in entanglement entropy from the vacuum state. It has been shown in [36] that at first order in $m$ the relative entropy vanishes (Entanglement First Law) and in second order in $m$ the relative entropy is given by $S_{rel} = -\Delta S^{(2)}$. With this basic background in place we first compute the Fisher information metric for the black brane.

The inverse of the lapse function can be written as

$$\frac{1}{f(z)} = \frac{1}{1 - \frac{z^d}{z_0^d}} = 1 + mz^d + m^2 z^{2d} + \cdots$$

(49)

where $m = 1/z_0^d$, which is the perturbation parameter in the bulk. In terms of this parameter the change in area of the minimal surface upto second order is given by [45]

$$A - A_0 = \left[ \frac{V_{(d-2)a_1b_2}^2}{4b_0^d} \frac{d - 1}{d + 1} m + \frac{V_{(d-2)a_1h_0b_2}^2}{(2b_0)^{d+2}} m^2 \right]$$

(50)

where

$$h_0 = \frac{d - 1}{d + 1} \left( -\frac{b_1}{2b_0} + \frac{3(d + 1)}{4(2d + 1)} a_2 \right) .$$

(51)

The relative entropy in this case becomes

$$S_{rel} = -\frac{1}{4G_{(d+1)}} \left[ \frac{V_{(d-2)a_1h_0b_2}^2}{(2b_0)^{d+2}} m^2 \right] .$$

(52)

It is to be noted that $h_0$ has negative values for all $d$. Hence $S_{rel}$ is a positive quantity.

From eq. (48), the Fisher information metric therefore reads

$$G_{F,mm} = \frac{\partial^2}{\partial m^2} S_{rel} = \frac{V_{(d-2)a_1h_0b_2}^2}{2G_{(d+1)}(2b_0)^{d+2}} .$$

(53)

In [41], a proposal for computing the above quantity was given. The proposal is to consider the difference of two volumes yielding a finite expression

$$\mathcal{F} = C_d (V_{(m^2)} - V_{(0)})$$

(54)

where $V_{(m^2)}$ is evaluated for a second order fluctuation about $AdS$ spacetime. $C_d$ is a dimensionless constant which cannot be fixed from the first principles of the gravity side. We shall
now apply this proposal to compute the Fisher information metric for the black brane. The change in volume under Ryu-Takayanagi minimal surface at second order in perturbation takes the form (for $\beta = 0$ case)

$$V - V(0) = - \left[ \frac{V_{(d-2)l^2}}{4b_0^2(d-1)} m v_0 + \frac{V_{(d-2)l^{d+2}}}{(2b_0)^{d+2}} m^2 v_{00} \right].$$

(55)

The holographic dual of Fisher information metric is now defined as

$$G_{F,mm} = \partial^2 m \mathcal{F}; \quad \mathcal{F} = C_d (V - V(0))$$

(56)

with the constant $C_d$ to be determined by requiring that the holographic dual Fisher information metric from the above equation must agree with that obtained from the relative entropy \textsuperscript{[33]}. The constant $C_d$ is therefore given by

$$C_d = \frac{h_0 a_1}{4G(d+1)v_{00}}.$$  

(57)

On the other hand the relative entropy for boosted black brane ($\beta \neq 0$) is given by \textsuperscript{[16]}

$$S_{rel} = - \frac{1}{4G(d+1)} \left[ \left( \frac{l}{2b_0} \right)^{d+2} \left( h_0 + h_1 \beta^2 \gamma^2 + h_2 \beta^4 \gamma^4 \right) m \right].$$

(58)

where

$$h_1 = \left( -\frac{b_1}{b_0} + \frac{a_2}{2a_1} \right),$$

$$h_2 = \frac{d + 1}{d - 1} \left( \frac{b_1}{2b_0} + \frac{3a_2}{4a_1(d+1)} \right).$$

(59)

It is to be noted that $h_1$ and $h_2$ have negative values for all $d$. Hence $S_{rel}$ is a positive quantity. Thus the Fisher information metric reads

$$G_{F,mm} = \partial^2 m S_{rel} = - \frac{1}{2G(d+1)} \left( \frac{l}{2b_0} \right)^{d+2} \left( h_0 + h_1 \beta^2 \gamma^2 + h_2 \beta^4 \gamma^4 \right).$$

(60)

Now the change in volume under Ryu-Takayanagi minimal surface at second order in perturbation takes the form (for $\beta \neq 0$ case)

$$V - V(0) = - \frac{V_{(d-2)l^2}}{4b_0^2(d-1)} \left( \frac{d - 2 \pi b_1}{d - 1} \frac{2b_0^2}{2b_0^2} + (2 - d)c_0 \right) m - \frac{V_{(d-2)l^{d+2}}}{4b_0^2(d-1)} \left( \frac{d - 2}{d - 1} \frac{2b_0^2}{2b_0^2} + c_2 - (d - 1)c_0 \right) m$$

$$- \frac{V_{(d-2)l^{d+2}}}{(2b_0)^{d+2}} \left( v_{00} + \beta^2 \gamma^2 v_{01} - \beta^4 \gamma^4 v_{11} \right).$$

(61)

The holographic dual of the Fisher information metric can be defined as eq.(56) with the constant $C_d$ as follows

$$C_d = \frac{a_1}{4G(d+1)} \left( \frac{h_0 + h_1 \beta^2 \gamma^2 + h_2 \beta^4 \gamma^4}{v_{00} + \beta^2 \gamma^2 v_{01} - \beta^4 \gamma^4 v_{11}} \right).$$

(62)
It is obvious from the above expression that the constant $C_d$ matches with eq. (57) in the $\beta \to 0$ limit. We would like to mention that the constant $C_d$ in this case depends on the boost parameter $\beta$ which in the case of the pure black brane was independent of any physical parameter and depend only on the dimensionality of the spacetime.

We now look at the other holographic proposal [44] to compute the fidelity susceptibility. For pure states the expression for fidelity (47) reduces to

$$\langle \Psi(\lambda)|\Psi(\lambda + \delta \lambda)\rangle = 1 - G_{\lambda\lambda}(\delta \lambda)^2 + \cdots$$

where for simplicity we assume that the states depend on a single parameter $\lambda$. Therefore one can say that $G_{\lambda\lambda}$ measures the distance between two quantum states. $G_{\lambda\lambda}$ is called fidelity susceptibility. In [44] it has been proposed that for a $d$- dimensional CFT deformed by a perturbation, the fidelity susceptibility can be computed holographically by the formula

$$G_{\lambda\lambda} = n_{d-1} \frac{Vol(\Sigma_{\text{max}})}{R^d}$$

where $n_{d-1}$ is a $O(1)$ constant and $R$ is the radius of curvature of $AdS$ spacetime. $\Sigma_{\text{max}}$ is the maximum volume in the $AdS$ that ends at the $AdS$ boundary at a fixed time slice. Though the above formula has been derived for the case of pure states, it has been also applied for mixed states [44]. Therefore we can apply this formula to calculate the fidelity susceptibility for boosted black brane.

Let us first calculate fidelity susceptibility for the $AdS$ black brane. The metric for $AdS$ black brane in $(d + 1)$- dimensions is given by

$$ds^2 = \frac{1}{z^d} \left( -f dz^2 + dx_1^2 + \cdots + dx_{d-1}^2 + \frac{dz^2}{f} \right)$$

with

$$f = 1 - \frac{z^d}{z_0^d}.$$  

Using the formula (64), the fidelity susceptibility reads

$$G_{\lambda\lambda} = n_{d-1} L^{d-1} \int_0^{z_0} \frac{dz}{z^d} \frac{1}{\sqrt{1 - \frac{z^d}{z_0^d}}}$$

$$= n_{d-1} L^{d-1} \int_0^{z_0} \frac{dz}{z_0^{d-1}} \left[ 1 - \frac{1}{d} B\left( \frac{1}{d}, \frac{1}{2} \right) + \frac{z_0^{d-1}}{(d-1)\delta^{d-1}} \right].$$  

(67)

We see that the above expression for the fidelity susceptibility does not agree with the Fisher information metric obtained in eq. (53). We now proceed to calculate the fidelity susceptibility for the boosted black brane metric (10). In this case the fidelity susceptibility takes the form

$$G_{\lambda\lambda} = n_{d-1} L^{d-1} \int_0^{z_0} \frac{dz}{z_0^{d-1}} \sqrt{\frac{K(z)}{f(z)}}$$

$$= n_{d-1} L^{d-1} \left[ \frac{1}{z_0^{d-1}} \frac{1}{\sqrt{1 - \beta^2}} \int_0^1 \frac{dt}{t^d} \sqrt{\frac{1 - \beta^2(1-t^d)}{1-t^d} + \frac{1}{(d-1)\delta^{d-1}}} \right].$$

(68)
where \( t = z/z_0 \). Now making a transformation \( 1 - t^d = p \) yields

\[
G_{\lambda\lambda} = n_{d-1} L^{d-1} \left[ \frac{1}{d \sqrt{1 - \beta^2 z_0}} \int_0^1 dp \frac{\sqrt{1 - \beta^2 p}}{\sqrt{p (1 - p)^{(d-1)/2}}} + \frac{1}{(d-1) \delta^{d-1}} \right]
\]

\[
= n_{d-1} L^{d-1} \left[ \frac{1}{d \sqrt{1 - \beta^2 z_0}} B \left( \frac{1}{2}, \frac{1 - d}{d} \right) {}_2 F_1 \left( -\frac{1}{2}, \frac{1}{2} ; \frac{2 - d}{2d}; \beta^2 \right) + \frac{1}{(d-1) \delta^{d-1}} \right].
\]

It is clear from eq. (69) that in the \( \beta = 0 \) limit, the result matches with that of the \( AdS \) black brane given in eq. (67). We see that the above expression for the fidelity susceptibility does not agree with the Fisher information metric obtained in eq. (60).

5 Conclusion

In this paper, we have computed the holographic complexity for a boosted black brane for two cases where the boost direction is in the parallel and perpendicular direction to the strip length comprising the subsystem. The computation have been carried out upto both first and second orders in the boost parameter. An asymmetry has been found in the holographic complexity upto first order. The results upto second order in the boost parameter have been used to compute the Fisher information metric upto an undetermined constant. This constant has been fixed by equating this result with the holographic computation of Fisher information metric from the relative entropy. This has been done for both the pure black brane as well as the boosted black brane. The fidelity susceptibility has also been obtained by using the proposal in [44]. It is observed that the expressions for the Fisher information metric and the fidelity susceptibility are different from each other [35].
6 Appendix

Some useful Beta function integrals we have used are given here

\[
\begin{align*}
    b_0 &= \int_0^1 dt \ t^{d-1} \frac{1}{\sqrt{R}} = \frac{1}{2(d-1)} B\left(\frac{d}{2d-2}, \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma\left(\frac{3d}{2d-2}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)} \\
    b_1 &= \int_0^1 dt \ t^{2d-1} \frac{1}{\sqrt{R}} = \frac{1}{2(d-1)} B\left(\frac{d}{d-1}, \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma\left(\frac{d}{d-1}\right)}{(d+1) \Gamma\left(\frac{1}{2} + \frac{1}{d-1}\right)} \\
    b_2 &= \int_0^1 dt \ t^{3d-1} \frac{1}{\sqrt{R}} = \frac{1}{2(d-1)} B\left(\frac{3d}{2d-2}, \frac{1}{2}\right) \\
    I_1 &= \int_0^1 dt \ t^{d-1} (1-t^d) \frac{1}{R^2} = \frac{d+1}{d-1} b_1 - \frac{1}{d-1} b_0 \\
    c_0 &= \int_0^1 dt \ \frac{t^d}{\sqrt{R}} = \frac{1}{2(d-1)} B\left(\frac{d+1}{2(d-1)}, \frac{1}{2}\right) = \frac{\pi}{2(d^2-1)b_1} \\
    c_1 &= \int_0^1 dt \ \frac{t^{2d}}{\sqrt{R}} = \frac{1}{2(d-1)} B\left(\frac{2d+1}{2(d-1)}, \frac{1}{2}\right) = \frac{\pi}{2(2d+1)(d-1)b_2} \\
    c_2 &= \int_0^1 dt \ \frac{(1-t^d)^{R^2}}{R^2} = \frac{2}{d-1} c_0 + \frac{d-2}{2(d-1)^2} B\left(\frac{1}{2(d-1)}, \frac{1}{2}\right) = \frac{\pi}{(d+1)(d-1)^2 b_1} + \frac{\pi(d-2)}{2(d-1)^2 b_0} \\
    c_3 &= \int_0^1 dt \ \frac{t^d(1-t^d)}{R^2} = \frac{2}{d-1} c_0 + \frac{d+2}{d-1} c_1 \\
    J_1 &= \int_0^1 dt \ t^{d-1} \left(\frac{\beta^2 \gamma^2}{4} t^d + \beta^4 \gamma^4 \left(\frac{3(1-t^d)}{8(1-t^d)} - \frac{1}{2}\right)\right) \frac{1-t^d}{R^2} = \beta^2 \gamma^2 J_1 + \beta^4 \gamma^4 J_2 \\
    K_1 &= \int_0^1 dt \ \left(\frac{\beta^2 \gamma^2}{4} t^d + \beta^4 \gamma^4 \left(\frac{3(1-t^d)}{8(1-t^d)} - \frac{1}{2}\right)\right) \frac{1-t^d}{R^2} = \beta^2 \gamma^2 K_1 - \beta^4 \gamma^4 K_2 \\
\end{align*}
\]

with

\[
\begin{align*}
    J_1 &= \frac{1}{4(d-1)} \left((2d+1)b_2 - (d+1)b_1\right) \\
    J_2 &= \frac{1}{8(d-1)^2} \left((3-2d)b_0 - 2(d+1)(3-d)b_1 + 3(2d+1)b_2\right) - \frac{I_1}{2} \\
    K_1 &= \frac{1}{2(d-1)} c_0 + \frac{d+2}{4(d-1)} c_1 \\
    K_2 &= -\frac{d-4}{4(d-1)^2} c_0 + \frac{(d+2)(d-4)}{8(d-1)^2} c_1 + \frac{d}{8(d-1)} c_2 \\
\end{align*}
\]

where \( B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \) are the Beta-functions and we have used the identity \( B(x, \frac{1}{2}) B(x +
\[ \frac{1}{2}, \frac{1}{2} = \frac{\pi}{\tau} \] Further integrals are

\[ a_0 = \int_0^1 dt \frac{t^{d+1}}{\sqrt{a}} \frac{1}{2(d-1)} B \left( \frac{1-d}{d-1}, \frac{1}{2} \right) \]
\[ a_1 = \int_0^1 dt \frac{t^{d+1}}{d} \frac{1}{\sqrt{a}} \frac{1}{2(d-1)} B \left( \frac{1-d}{d-1}, \frac{1}{2} \right) \]
\[ a_2 = \int_0^1 dt \frac{t^{2d+1}}{\sqrt{a}} \frac{1}{2(d-1)} B \left( \frac{1+d/2}{d-1}, \frac{1}{2} \right) \]
\[ I_a = \int_0^1 dt \frac{1}{t^{d-1} \left( 1 - t^{2d} \right)} \frac{1}{R^{3/2}} = \frac{2d+1}{d-1} b_2 - \frac{1}{d-1} b_0 . \] (72)

Some identities we have used are

\[ b_0 = (2-d)a_0, \quad b_1 = \frac{2}{d+1} a_1, \quad b_2 = \frac{2+d}{2d+1} a_2 . \] (73)

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