Reducibility for wave equations of finitely smooth potential with periodic boundary conditions

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Abstract
In the present paper, the reducibility is derived for the wave equations with finitely smooth and time-quasi-periodic potential subjects to periodic boundary conditions. More exactly, the linear wave equation
\[ u_{tt} - u_{xx} + Mu + \varepsilon(V_0(\omega t)u_{xx} + V(\omega t,x)u) = 0, \quad x \in \mathbb{R}/2\pi\mathbb{Z} \]
can be reduced to a linear Hamiltonian system of a constant coefficient operator which is of pure imaginary points spectrum set, where \( V \) is finitely smooth in \((t,x)\), quasi-periodic in time \( t \) with Diophantine frequency \( \omega \in \mathbb{R}^n \), and \( V_0 \) is finitely smooth and quasi-periodic in time \( t \) with Diophantine frequency \( \omega \in \mathbb{R}^n \). Moreover, it is proved that the corresponding wave operator possesses the property of pure point spectra and zero Lyapunov exponent.

Keywords: KAM theory; Reducibility; Quasi-periodic wave operator; Finitely smooth potential; Periodic boundary conditions; Pure-point spectrum
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1. Introduction
In the present paper, we investigate the reducibility of
\[ u_{tt} - u_{xx} + Mu + \varepsilon(V_0(\omega t)u_{xx} + V(\omega t,x)u) = 0, \quad x \in \mathbb{R}/2\pi\mathbb{Z}. \] (1.1)

To that end, we need the following conditions:

\textbf{Assumption A.} Assume \( M > 0 \) is a constant, and \( V_0, V_1 \) are \( C^N \)-smooth and quasi-periodic in time \( t \) with frequency \( \omega \in \mathbb{R}^n \); that is, there are hull functions \( \gamma_0(\theta) \in C^N(\mathbb{T}^n, \mathbb{R}), \gamma'(\theta,x) \in C^N(\mathbb{T}^n \times [0,2\pi], \mathbb{R}) \) such that
\[ V_0(\omega t) = \gamma_0(\theta)|_{\theta = \omega t}, \quad V(\omega t,x) = \gamma'(\theta,x)|_{\theta = \omega t}, \quad \mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n, \]
where \( N > 200n \).

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Assumption B. Assume \( \omega \in [1, 2]^n \subset \mathbb{R}^n \) satisfies Diophantine conditions:
\[
|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|^{n+1}}, \quad k \in \mathbb{Z}^n \setminus \{0\},
\]
where \( \gamma \) is a constant and \( 0 < \gamma \ll 1 \).

We recall the reducibility problem for a time dependent linear system
\[
\dot{x} = A(t)x, \quad x \in \mathbb{R}^n,
\]
where \( A(t) \) is an \( n \times n \) real or complex value matrix. If \( A(t) \) is time \( T \)-periodic and continuous, it follows from Floquet theory that there exists a continuous time \( T \)-periodic coordinate change
\[
x = P(t)y
\]
such that \( 1.3 \) is changed into a constant system
\[
\dot{y} = By,
\]
where \( B \) is an \( n \times n \) complex value matrix independent of time \( t \). However, there usually does not exist the change \( 1.4 \) such that \( 1.3 \) is reduced to \( 1.5 \) when \( A(t) \) is time quasi-periodic. See [19]. Let us consider a special case: \( A(t) = \Lambda + \varepsilon Q(t) \), where \( \Lambda \) is a constant, \( Q(t) \) is time quasi-periodic and \( \varepsilon \) is small. The well known KAM (Kolmogorov-Arnold-Moser) theory can be applied to this case. See [11, 18, 25, 29], for example. In recent decades, there have been many literatures dealing with the reducibility of time quasi-periodic, infinite dimensional linear systems via KAM technique. One model is the time-quasi-periodic Schrödinger operator
\[
i \dot{u} = (H_0 + \varepsilon W(\omega t, x, -i\nabla))u, \quad x \in \mathbb{R}^d \setminus T_d = \mathbb{R}^d / 2\pi \mathbb{Z}^d,
\]
where \( H_0 = -\Delta + V(x) \) or an abstract self-adjoint (unbounded) operator while the perturbation \( W \) is quasi-periodic in time \( t \) and it may or may not depend on \( x \) or/and \( \nabla \). See [2, 4, 15–17, 19, 36], and the references therein.

Another model is the time-quasi-periodic wave operator or linear wave equation
\[
\ddot{u}_{tt} = (-\Delta + \varepsilon V(\phi_0 + \omega t, x; \omega))u.
\]
Up to now, the reducibility of \( 1.7 \) has not been explicitly dealt with. Note that a reducibility procedure has been included in classical KAM for the existence of lower-dimensional invariant tori for infinite dimensional Hamiltonian partial differential equations. It can be implicitly derived from the classical KAM [14, 27, 33, 37] that \( 1.7 \) with \( d = 1 \) and subject to Dirichlet boundary condition or periodic boundary condition can be reduced to a constant coefficient equation for “most” frequency \( \omega \), provided that \( V \) is analytic. For \( d = 1 \) and \( 1.7 \) with a finitely smooth potential \( V \) and subject to Dirichlet boundary condition, it has been recently proved that \( 1.7 \) can still be reduced to a constant system for “most” frequency \( \omega \). See [28].

In this paper, we will prove the following reducibility theorem:

[1] Here the word “most” means that for a given set \( \Pi \subset \mathbb{R}^n \) with Lebesgue measure equals to 1, there exists a subset \( \Pi_\varepsilon \subset \Pi \) with measure \( \Pi \setminus \Pi_\varepsilon \to 0 \) as \( \varepsilon \to 0 \) such that for “any \( \omega \in \Pi_\varepsilon \).”
Theorem 1.1. With Assumptions A, B, for any given $0 < \gamma \ll 1$, there exists an $\epsilon^* \in \mathbb{R}^+$ with $0 < \epsilon^* = \epsilon^*(n, \gamma) \ll \gamma$, and there exists a subset $\Pi \subset [1, 2]^n$ with

$$\text{Meas} \Pi \geq 1 - O(\gamma^{1/3})$$

such that for any $0 < \epsilon < \epsilon^*$ and for any $\omega \in \Pi$, there is a quasi-periodic symplectic change such that

$$u_t - u_{xx} + Mu + \epsilon(V_0(\omega t)u_{xx} + V(\omega t, x))u = 0, \quad x \in \mathbb{R}/2\pi \mathbb{Z}$$

(1.8)
is reduced to a linear Hamiltonian system

$$\begin{cases}
\dot{\mathbf{q}} = (\Lambda + \epsilon \mathbf{Q}) \dot{\mathbf{p}}, \\
\dot{\mathbf{p}} = -(\Lambda + \epsilon \mathbf{Q}) \mathbf{q},
\end{cases}$$

(1.9)

where $\Lambda = \text{diag}(\Lambda_j: j = 0, 1, 2, \cdots)$, $\Lambda_0 = \rho \sqrt{\mathbf{I}}, \Lambda_j = \rho \sqrt{\mathbf{I} + ME_{2j}}, \rho$ is a constant close to 1, $E_{2j}$ is a $2 \times 2$ unit matrix, and $\mathbf{Q} = \text{diag}(\mathbf{Q}_i: i = 0, 1, 2, \cdots)$ is independent of time with $\mathbf{Q}_0 \in \mathbb{R}$, $\mathbf{Q}_i$ being a real $2 \times 2$ matrix, and $\|\mathbf{Q}_i\| \leq C/i, i = 1, 2, \cdots$. Here $\| \cdot \|$ denotes the sup-norm for real matrices.

The more exact statement of Theorem 1.1 can be found in Theorem 2.1 in Section 2. From Theorem 1.1, the following two corollaries can be obtained.

Corollary 1.1. With Assumptions A, B, for any $\omega \in \Pi$ and $0 < \epsilon < \epsilon^*$, the wave operator

$$\mathcal{L} u(t, x) = (\partial_t^2 - \partial_x^2 + M + \epsilon(V_0(\omega t)\partial_x^2 + V(\omega t, x))u(t, x), \quad x \in \mathbb{R}/2\pi \mathbb{Z}$$

is of pure point spectrum property and of zero Lyapunov exponent.

Corollary 1.2. With Assumptions A, B, for any $\omega \in \Pi$ and $0 < \epsilon < \epsilon^*$, there exists a unique solution $u(t, x)$ with initial values $(u(0, x), u_t(0, x)) = (u_0(x), v_0(x)) \in \mathcal{H}^N \times \mathcal{H}^N$, which is almost-periodic in time and

$$\frac{1}{C} (\|u_0\|_{\mathcal{H}^N} + \|v_0\|_{\mathcal{H}^N}) \leq \|u(t)\|_{\mathcal{H}^N} + \|u_t(t)\|_{\mathcal{H}^N} \leq C (\|u_0\|_{\mathcal{H}^N} + \|v_0\|_{\mathcal{H}^N}),$$

where $C > 0$ is a constant, $\mathcal{H}^N = \mathcal{H}^N(T)$ is the usual Sobolev space.

Remark 1.1. Since $V_0(\omega t)\partial_x^2$ appears in (1.1), the perturbation is unbounded one. This kind of unbounded perturbation, which is of the highest unboundedness, can come from the linearization of some quasi-linear perturbation. For quasi-linear KdV equations and quasi-linear Schrödinger equations, there has been a progress about KAM theory [5–8, 10, 21–23, 31, 38]. It is still an open problem whether or not there exists KAM theory for quasi-linear wave equations. In the present paper, the potential $V_0(\omega t)$ in (1.1) does not depend on the space variable $x$. We find that the methods of Baldi-Berti-Montalto [5, 31, 32] and Roberto-Michela [38] is still valid for the $V_0(\omega t)$ in (1.1).

Remark 1.2. Here we would like to compare the results of Theorem 1.1 with some existent results. As mentioned before, without $V_0(\omega t)$, when $d = 1$ and the potential $V$ is analytic, the reducibility of (1.7) can be implicitly derived from the classical KAM theorems. However, there are some differences between the analytic potential $V$ and the finitely smooth one, not to mention
the existence of $V_0$. In this paper, by several times elegant variable and symplectic changes, the wave equation (1.1) can be written as a linear Hamiltonian system with Hamiltonian

$$H = \langle \tilde{A}z, \bar{z} \rangle + \epsilon \left[ (\tilde{R}z^2(\theta)z, z) + (\tilde{R}z^2(\theta)z, \bar{z}) + (\tilde{R}z^2(\theta)z, \bar{z}) \right].$$

See (2.16) for more details. The basic task is to search a series of symplectic coordinate changes to eliminate the perturbations $\tilde{R}z^2(\theta)$, $\tilde{R}z^2(\theta)$ and $\tilde{R}z^2(\theta)$ except the averages of the diagonal of $\tilde{R}z^2(\theta)$. To this end, the symplectic coordinate changes are the time-1 map of the flow for the Hamiltonian $\epsilon F$ where $F$ is of the form

$$F = (Fz^2(\theta)z, z) + (Fz^2(\theta)z, \bar{z}) + (Fz^2(\theta)z, \bar{z}).$$

- When the potential $V(\theta)$ ($\theta = \omega t$) is analytic in some strip domain $|\text{Im}\theta| \leq s_0^\nu$, where $\nu$ is the KAM iteration step, the perturbations $\tilde{R}z^2(\theta)$, $\tilde{R}z^2(\theta)$ and $\tilde{R}z^2(\theta)$ are also analytic in $|\text{Im}\theta| \leq s_0$ at the $\nu$-th KAM step. However, the strip width $s_0^\nu$'s have non-zero below bound. Actually, $s_0$ goes to zero very rapidly:

$$s_0 = \epsilon^{1/N}, \quad \epsilon^{(4/3)^{\nu}}, \quad \nu = 1, 2, \ldots.$$
where $\tilde{\Lambda} = \text{diag}(\tilde{\lambda}_j : j = 1, 2, \cdots)$ and $\tilde{\lambda}_j = \sqrt{j^2 + M + \xi_j}$. Moreover, (1.1) can be reduced to
\[u_{tt} - u_{xx} + M\xi u = 0,\]
where $M\xi$ is a Fourier multiplier. However, for periodic boundary condition, the eigenvalues $\lambda_j$ ($j = 0, 1, \cdots$) are double:
\[\lambda^x_0 = 1, \quad \lambda^x_j = 2, \quad j = 0, 1, \cdots.\]
In this case, the Hamiltonian $H$ can be reduced to
\[H_\infty = \langle (\Lambda + \epsilon \tilde{Q})u, u \rangle,\]
where $\Lambda$ and $\tilde{Q}$ are matrices defined as (1.9), $u$ is a vector defined as (2.20). Although we can still get some dynamical behaviour from this reducibility, (1.1) cannot be reduced to a linear wave equation with a Fourier multiplier as in Dirichlet boundary condition.

Remark 1.4. Since $\lambda^x_j = 2$, the homological equations are no longer scalar. For example, in order to eliminate the term $\langle R^{\mathbb{R}^d}(\theta)u, \overline{F} \rangle$ (see (2.21)-(2.24) for more details), the homological equations have the form:
\[\omega \cdot \partial_\theta F - i(\Lambda F - F \Lambda) = R,\]
where $F = F(\theta)$ is the unknown matrix of order 2, $\Lambda$ is a $2 \times 2$ constant matrix, $R = R(\theta)$ is known matrix of order 2. It is more complicated to find the solution of this matrix equation (1.10) than that of scalar homological equations. In this case, the delicate small divisor problem becomes one dealing with the inverse of the matrix
\[A := -(k, \omega)(1 \otimes 1) + 1 \otimes \Lambda - \Lambda \otimes 1\]
(see (7.5) for more details). A usual method dealing with (1.11) is to investigate $\partial^2_{\theta \theta} \det A$. See [11] and [14], for example. In the present paper, we use the variation principle of eigenvalues to deal with the inverse $A^{-1}$. The advantage of the variation principle of eigenvalues is that the method dealing with scalar small divisor problem [33] can be recovered.

Remark 1.5. In [9], it is proved that there is a quasi-periodic solution for any $d$-dimensional nonlinear wave equation with a quasi-periodic in time nonlinearity,
\[u_{tt} - \Delta u - V(x)u = \epsilon f(\omega t, x, u), \quad x \in \mathbb{T}^d,\]
where the multiplicative potential $V$ is in $C^q(\mathbb{T}^d; \mathbb{R})$, $\omega \in \mathbb{R}^n$ is a non-resonant frequency vector and $f \in C^q(\mathbb{T}^n \times \mathbb{T}^d \times \mathbb{R}; \mathbb{R})$. Because of the application of multi-scale-analysis, it is not clear whether the obtained quasi-periodic solution is linear stable and has zero Lyapunov exponent. As a corollary of Theorem 1.1, we can prove that the quasi-periodic solution by [9] is linear stable and has zero Lyapunov exponent, when $d = 1$.

Remark 1.6. When $d > 1$, it is a well-known open problem that (1.7) subject to Dirichlet or periodic boundary condition is reduced to a linear Hamiltonian system with a constant coefficient linear operator. See the series of talks by L.H. Eliasson [41–43]. Also see a recent paper [32] where the perturbation is a finite rank operator.

This paper is organized as follows. In Section 2, we redescribe Theorem 1.1 as Theorem 2.1. In Section 3–10, to prove the main results of the paper, some preliminary work and many lemmas will be given. The proof of Theorem 2.1 is in the last section.
2. Passing to Fourier coefficients

Consider the differential equation:

\[ \mathcal{L}u = u_{tt} - u_{xx} + Mu + \varepsilon (V_0(\theta)u_{xx} + V(\omega t, x)u) = 0 \]  (2.1)

subject to the boundary condition

\[ u(t, x) = u(t, x + 2k\pi), \quad k \in \mathbb{Z}. \]  (2.2)

It is well-known that the Sturm-Liouville problem

\[ -y'' + My = \lambda y, \quad x \in \mathbb{R}/2\pi \mathbb{Z} \]

has the eigenvalues and eigenfunctions, respectively,

\[ \lambda_k = k^2 + M, \quad k \in \mathbb{Z}, \]
\[ \phi_k(x) = e^{ikx}, \quad k \in \mathbb{Z}. \]

Set \(-\partial_{xx} + M\) as \(D\), the wave equation can be seen as

\[ u_{tt} = -Du - \varepsilon V_0(\omega t)Du - \varepsilon V_1(\omega t, x)u, \]  (2.3)

where \(V_1(\omega t, x) = V(\omega t, x) + MV_0(\omega t)\). Let \(u_t = v\), we have

\[ v_t = -(1 - \varepsilon V_0(\omega t))Du - \varepsilon V_1(\omega t, x)u. \]  (2.4)

**Step 1:**
Rescale

\[
\begin{align*}
q_t &= \frac{1}{\beta(\theta)|D|^\frac{1}{2}} \frac{\partial}{\partial \theta} \beta(\theta) |D|^\frac{1}{2} p - \frac{\varepsilon}{\beta(\theta)} V_1(\omega t, x), \\
p_t &= -(1 - \varepsilon V_0(\omega t))\beta^2(\theta) |D|^\frac{1}{2} q + \frac{\varepsilon}{\beta(\theta)} V_1(\omega t, x) |D|^\frac{1}{2} q.
\end{align*}
\]

Choose a suitable \(\tilde{\beta}(\theta)\), such that \(\beta(\theta) = (1 - \varepsilon V_0(\omega t))^{-\frac{1}{2}}\). Then

\[
\frac{1}{\beta^2(\theta)} = (1 - \varepsilon V_0(\omega t)) \beta^2(\theta) = a_0(\theta).
\]

Also, set \(\frac{\partial}{\partial \theta} \beta(\theta) = \varepsilon a_1(\theta), \beta^2(\theta)V_1(\theta, x) = \tilde{V}_1(\theta, x)\), we have

\[
\begin{align*}
q_t &= a_0(\theta)|D|^\frac{1}{2} p - \varepsilon a_1(\theta) q, \\
p_t &= -a_0(\theta)|D|^\frac{1}{2} q + \varepsilon a_1(\theta) p - \varepsilon |D|^\frac{1}{2} \tilde{V}_1(\theta, x)|D|^\frac{1}{2} q.
\end{align*}
\]
Clearly, we can see $a_0, \tilde{V}_1 \in C^N(\mathbb{T}^n \times [0,2\pi], \mathbb{R})$ and $a_1 \in C^{N-1}(\mathbb{T}^n \times [0,2\pi], \mathbb{R})$.

**Step 2:**
Now we consider the complex variable

$$z = \frac{q - ip}{\sqrt{2}}, \quad \bar{z} = \frac{q + ip}{\sqrt{2}}.$$  

Then, we have

$$\omega \cdot \partial_\theta z = i[a_0(\theta)|D|^4z - \epsilon a_1(\theta)\bar{z} + \epsilon i|D|^{-\frac{1}{2}}\frac{\tilde{V}_1(\theta,x)}{2}|D|^{-\frac{1}{2}}(z + \bar{z}),$$  

$$\omega \cdot \partial_\theta \bar{z} = -i[a_0(\theta)|D|^4\bar{z} - \epsilon a_1(\theta)z - \epsilon i|D|^{-\frac{1}{2}}\frac{\tilde{V}_1(\theta,x)}{2}|D|^{-\frac{1}{2}}(z + \bar{z}).$$  

(2.5)

**Step 3:**
Now we introduce a time variable change, a diffeomorphism of the torus $\mathbb{T}^n$ of the form

$$\theta = \theta + \omega \bar{a}(\theta), \quad \theta = \theta + \omega a(\theta).$$  

(2.6)

For any function $h(\theta,x)$ and $\tilde{h}(\theta,x)$, we introduce operators $A$ and $A^{-1}$, where

$$h(\theta,x) = (A^{-1}h)(\theta,x) = [h(\theta,x) = h(\theta + \omega a(\theta),x),$$

$$\tilde{h}(\theta,x) = (Ah)(\theta,x) = \tilde{h}(\theta + \omega a(\theta),x).$$  

(2.7)

Our aim is to rewrite the equation (2.5) in the new time variable $\theta$. Thus, we can set

$$z(\theta,t) = z(\theta + \omega \bar{a}(\theta),x) = [z](\theta,x),$$

$$a_i(\theta) = a_i(\theta + \omega \bar{a}(\theta)) = [a_i](\theta), \quad i = 0,1,$$

$$\tilde{V}_1(\theta,x) = \tilde{V}_1(\theta + \omega \bar{a}(\theta),x) = [\tilde{V}_1](\theta,x),$$

$$1 + \omega \partial_0 a(\theta) = \omega \bar{a} a(\theta) + \omega \partial_0 a(\theta) = [1 + \omega \partial_0 a](\theta).$$

We want to choose a function $a$ so that $[a_0]$ is proportional to $[1 + \omega \partial_0 A]$. Thus, it is enough to solve the equation

$$\rho(1 + \omega \partial_0 a(\theta)) = a_0(\theta), \quad \rho \in \mathbb{R}.$$  

(2.9)

Integrating on $\mathbb{T}^n$ we fix the value of $\rho$ as

$$\rho = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} a_0(\theta)d\theta.$$  

(2.10)

By (2.9), we get

$$a(\theta) = (\omega \cdot \partial_\theta)^{-1}[\frac{a_0}{\rho} - 1](\theta).$$  

(2.11)
For notational simplicity, rename \( \theta \), \([z,\bar{z}]\), \([\theta,\bar{\theta}]\), \([a_1]_{1 \leq a_0 < m}, \frac{\partial a_0}{[a_1]}\) as \( \theta, \bar{z}, b_0, V \). Then, we have

\[
\mathcal{F} : \begin{cases} 
\dot{z}_t = i \rho |D|^{\frac{1}{2}} \bar{z} - e b_0 \bar{z} + e i |D|^{\frac{1}{2}} \bar{z} |D|^{-\frac{1}{2}}(z + \bar{z}), \\
\dot{\bar{z}}_t = -i \rho |D|^{\frac{1}{2}} \bar{z} - e b_0 \bar{z} - e i |D|^{\frac{1}{2}} \bar{z} |D|^{-\frac{1}{2}}(z + \bar{z}).
\end{cases}
\]

By Sobolev embedding theorem and inverse function theorem, we see \( a \in C^{N-2n-2}(\mathbb{R}^m \times [0,2 \pi]) \) and \( \bar{a} \in C^{N-2n-2}(\mathbb{R}^m \times [0,2 \pi]) \). Thus, we can get \( b_0, V \in C^{N-2n-3}(\mathbb{R}^m \times [0,2 \pi]) \). In the following section, we renamed \( N = 2n - 3 \) as \( N \) for notational simplicity.

Make the ansatz

\[
z(t,x) = \mathcal{F}(z_k) = \sum_{k \in \mathbb{Z}} z_k(t) \phi_k(x), \quad \bar{z}(t,x) = \mathcal{F}(\bar{z}_k) = \sum_{k \in \mathbb{Z}} \bar{z}_k(t) \phi_k(x)
\]

and

\[
V(\omega t,x) = \sum_{k \in \mathbb{Z}} v_k(\omega t) \phi_k(x).
\]

Then (2.1) can be transformed as

\[
\frac{dz_k}{dt} = i \rho \sqrt{\lambda_k} z_k - e b_0 z_k + i e \sum_{l \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} c_{ljk} v_j(z_l + \bar{z}_j),
\]

\[
\frac{d\bar{z}_k}{dt} = -i \rho \sqrt{\lambda_k} \bar{z}_k - e b_0 \bar{z}_k - i e \sum_{l \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} c_{ljk} v_j(z_l + \bar{z}_j),
\]

where

\[
c_{ljk} = \int_0^{2\pi} e^{i(j+l-k)x} dx = \begin{cases} 0, & j + l - k \neq 0, \\ 2\pi, & j + l - k = 0. \end{cases}
\]

Endowed a symplectic transformation with \(-idz \wedge d\bar{z}\). Thus (2.12) is changed into

\[
\begin{cases} 
\dot{z}_k = i \frac{\partial H}{\partial \bar{z}_k}, & k \in \mathbb{Z}, \\
\dot{\bar{z}}_k = -i \frac{\partial H}{\partial z_k}, & k \in \mathbb{Z},
\end{cases}
\]

where

\[
H(z,\bar{z}) = \sum_{k \in \mathbb{Z}} \rho \sqrt{\lambda_k} z_k \bar{z}_k + e i \sum_{k \in \mathbb{Z}} b_0 \left( \frac{z_k^2 - \bar{z}_k^2}{2} \right) + e \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} c_{ljk} v_j(\theta) \left( z_l + \bar{z}_j \right) (z_k + \bar{z}_k).
\]

For two sequences \( x = (x_j \in \mathbb{C}, j \in \mathbb{Z}) \), \( y = (y_j \in \mathbb{C}, j \in \mathbb{Z}) \), define

\[
(x,y) = \sum_{j \in \mathbb{Z}} x_j y_j.
\]

Then we can rewrite (2.15) as follows:

\[
H(\bar{z},z) = \langle \rho \bar{z},z \rangle + e i \frac{b_0}{2} \left( \langle \bar{z},z \rangle - \langle z,\bar{z} \rangle \right) + e \left[ \langle \bar{R}^\theta(\partial) z, z \rangle + \langle \bar{R}^\theta(\partial) z, \bar{z} \rangle + \langle \bar{R}^\theta(\partial) \bar{z}, z \rangle \right],
\]

(2.16)
where

\[ \tilde{\Lambda} = \text{diag} \left( \sqrt{\lambda_j} : j \in \mathbb{Z} \right), \quad \theta = \omega t, \]

\[ \tilde{R}^{\theta}(\theta) = \left( \tilde{R}^{\theta}_{j,l}(\theta) : k, l \in \mathbb{Z} \right), \quad \tilde{R}^{\theta}_{j,l}(\theta) = \frac{1}{2} \sum_{j \in \mathbb{Z}} \frac{c_{j,l}v_j(\theta)}{\sqrt{\lambda_k} \sqrt{\lambda_l}}. \]  \hspace{1cm} (2.17)

\[ \tilde{R}^{\theta}(\theta) = \left( \tilde{R}^{\theta}_{j,l}(\theta) : k, l \in \mathbb{Z} \right), \quad \tilde{R}^{\theta}_{j,l}(\theta) = \frac{1}{2} \sum_{j \in \mathbb{Z}} \frac{c_{j,l}v_j(\theta)}{\sqrt{\lambda_k} \sqrt{\lambda_l}}. \]  \hspace{1cm} (2.18)

\[ \tilde{R}^{\theta}(\theta) = \left( \tilde{R}^{\theta}_{j,l}(\theta) : k, l \in \mathbb{Z} \right), \quad \tilde{R}^{\theta}_{j,l}(\theta) = \frac{1}{2} \sum_{j \in \mathbb{Z}} \frac{c_{j,l}v_j(\theta)}{\sqrt{\lambda_k} \sqrt{\lambda_l}}. \]  \hspace{1cm} (2.19)

For the sequence \( z = (z_j \in \mathbb{C}, \ j \in \mathbb{Z}) \), we can rewrite \( z \) as

\[ z = (z_0, z_j, z_{-j} : j = 1, 2, \cdots) \triangleq u = (u_j : j = 0, 1, 2, \cdots), \]  \hspace{1cm} (2.20)

where \( u_0 = z_0, \ u_j = (z_j, z_{-j})^T, \ j = 1, 2, \cdots \). Here \((z_j, z_{-j})^T\) denotes the transpose of the vector \((z_j, z_{-j})\). Let \( \Lambda_0 = \sqrt{\lambda_0}, \ \Lambda_j = \left( \begin{array}{cc} \sqrt{\lambda_j} & 0 \\ 0 & \sqrt{\lambda_{-j}} \end{array} \right), \ j = 1, 2, \cdots \). Note that \( \lambda_j = \lambda_{-j} = j^2 + M, \ j = 1, 2, \cdots \). Then \( \Lambda_j = \sqrt{\lambda}_j E_{2 \times 2}, \ j = 1, 2, \cdots \), where \( E_{2 \times 2} \) is a \( 2 \times 2 \) unit matrix. For \( u_j = (z_j, z_{-j})^T \) and \( \tilde{u}_j = (\tilde{z}_j, \tilde{z}_{-j})^T \), define \( u_j, \tilde{u}_j = z_j \tilde{z}_j + z_{-j} \tilde{z}_{-j}, \ j = 1, 2, \cdots \). Then we can also rewrite (2.13) as

\[ \hat{H} = (\rho A u, \bar{u}) + \epsilon t \frac{b_0}{2} \left( (\bar{u}, \bar{u}) - (u, u) \right) + \epsilon \left[ (R^{uu}(\theta) u, u) + (R^{u\theta}(\theta) u, \bar{u}) + (R^{\theta u}(\theta) \bar{u}, \bar{u}) \right], \]  \hspace{1cm} (2.21)

where

\[ \Lambda = \text{diag} (\Lambda_j : j = 0, 1, 2, \cdots), \quad \theta = \omega t, \]

\[ R^{uu}(\theta) = (R^{uu}_{j,l}(\theta) : k, l = 0, 1, 2, \cdots), \quad R^{u\theta}(\theta) = (R^{u\theta}_{j,l}(\theta) : k, l = 0, 1, 2, \cdots), \]  \hspace{1cm} (2.22)

\[ R^{\theta u}(\theta) = (R^{\theta u}_{j,l}(\theta) : k, l = 0, 1, 2, \cdots), \quad R^{uu}_{j,l}(\theta) = R^{\theta u}_{j,l}(\theta) = \frac{1}{2} R^{\theta u}_{j,l}(\theta), \]  \hspace{1cm} (2.23)

where

\[ R^{uu}_{j,l}(\theta) = \begin{cases} R_{0,0}(\theta), & k = l = 0; \\ (R_{0,j}(\theta), R_{0,-j}(\theta)), & k = 0, l = 1, 2, \cdots; \\ (R_{l,j}(\theta), R_{l,-j}(\theta))^T, & l = 0, k = 1, 2, \cdots; \\ (R_{-j,l}(\theta), R_{-k,-l}(\theta)), & k, l = 1, 2, \cdots, \end{cases} \]  \hspace{1cm} (2.24)

and

\[ R_{k,j}(\theta) = \frac{1}{2} \sum_{j \in \mathbb{Z}} \frac{c_{j,k}v_j(\theta)}{\sqrt{\lambda_k} \sqrt{\lambda_l}}, \quad k, l \in \mathbb{Z}. \]

Define a Hilbert space \( h_N \) as follows:

\[ h_N = \{ x = (x_k \in \mathbb{C} : k \in \mathbb{Z}) : \| x \|_N^2 = \sum_{k \in \mathbb{Z}} |k|^{2N} |x_k|^2 \}. \]  \hspace{1cm} (2.25)
Similarly define a Hilbert space $h_N$ as follows:

$$h_N = \{ y = (y_k : k = 0, 1, \cdots) : \| y \|_N^2 = \sum_{k=0}^{\infty} |k|^{2N} |y_k|^2 \}, \quad (2.26)$$

where $y_0 \in \mathbb{C}$, $y_k = (z_k, z_{-k})^T$, $z_k, z_{-k} \in \mathbb{C}$, $k = 1, 2, \cdots$, and $|y_k|^2 = |z_k|^2 + |z_{-k}|^2$. In (2.25) and (2.26), we define $|k|^{2N} = 1$, if $k = 0$. For $z = (z_0, z_j, z_{-j} : j = 1, 2, \cdots) \in h_N$, $u = (u_j : j = 0, 1, 2, \cdots) \in h_N$, where $u_0 = z_0, u_j = (z_j, z_{-j})^T, j = 1, 2, \cdots$. It can be obtained that

$$\| u \|_N = \| z \|_N.$$  

Recall that

$$\mathcal{F}(\theta, x) \in C^N(T^n \times [0, 2\pi], \mathbb{R}).$$

Note that the Fourier transformation $\mathcal{F}$ is isometric from $u \in \mathcal{H}^N[0, 2\pi]$ to $(u_k : k = 0, 1, \cdots) \in h_N$, where $\mathcal{H}^N[0, 2\pi]$ is the usual Sobolev space.

Now we state a lemma, which is used in the next section.

**Lemma 2.1.**

$$\sup_{\theta \in T^n} \| \sum_{|\alpha| \leq N} \partial^{\alpha}_\theta J^{\mu\alpha}(\theta) J \|_{h_N \to h_N} \leq C,$$

$$\sup_{\theta \in T^n} \| \sum_{|\alpha| \leq N} \partial^{\alpha}_\theta J^{\mu\alpha}(\theta) J \|_{h_N \to h_N} \leq C,$$

$$\sup_{\theta \in T^n} \| \sum_{|\alpha| \leq N} \partial^{\alpha}_\theta J^{\mu\alpha}(\theta) J \|_{h_N \to h_N} \leq C,$$

where $\| \cdot \|_{h_N \to h_N}$ is the operator norm from $h_N$ to $h_N$, and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n), |\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$, $\alpha_j$'s are positive integers, and $J = \text{diag}(J_j : j = 0, 1, \cdots), J_0 = \sqrt{\lambda_0}, J_j = \sqrt{\lambda_j \mathcal{E}_{2^j}}, j = 1, 2, \cdots$.

**Proof.** By (2.22), (2.23) and (2.24), we have that

$$\partial^{\alpha}_\theta J^{\mu\alpha}(\theta) J \triangleq (A^{\mu\alpha}_{kl}(\theta) : k, l = 0, 1, \cdots),$$

where

$$A^{\mu\alpha}_{kl}(\theta) = \left\{ \begin{array}{ll} \frac{1}{2} \sum_{j \in \mathbb{Z}} c_{j, 0} \partial^{\mu}_{\theta} v_j(\theta), & k = l = 0; \\ \left( \frac{1}{2} \sum_{j \in \mathbb{Z}} c_{j, k} \partial^{\mu}_{\theta} v_j(\theta), \frac{1}{2} \sum_{j \in \mathbb{Z}} c_{j, 0} \partial^{\mu}_{\theta} v_j(\theta) \right), & k = 0, l = 1, 2, \cdots; \\ \left( \frac{1}{2} \sum_{j \in \mathbb{Z}} c_{j, k} \partial^{\mu}_{\theta} v_j(\theta), \frac{1}{2} \sum_{j \in \mathbb{Z}} c_{j, 0} \partial^{\mu}_{\theta} v_j(\theta) \right), & l = 0, k = 1, 2, \cdots; \\ \left( \frac{1}{2} \sum_{j \in \mathbb{Z}} c_{j, k} \partial^{\mu}_{\theta} v_j(\theta), \frac{1}{2} \sum_{j \in \mathbb{Z}} c_{j, 0} \partial^{\mu}_{\theta} v_j(\theta) \right), & k, l = 1, 2, \cdots. \end{array} \right.$$ 

For any $u = (u_k : k = 0, 1, \cdots) \in h_N$,

$$\left( \sum_{|\alpha| \leq N} \partial^{\alpha}_\theta J^{\mu\alpha}(\theta) J \right) u = \left( \sum_{k=0}^{\infty} \left( \sum_{|\alpha| \leq N} A^{\mu\alpha}_{kl} \right) u_k : l = 0, 1, \cdots \right). \quad (2.28)
Suppose $\tilde{J} = \text{diag}(\sqrt{A_j} : j \in \mathbb{Z})$. Then for any $z = (z_k \in \mathbb{C} : k \in \mathbb{Z}) \in h_N$, 

$$
\left( \sum_{|\alpha| \leq N} \partial_{\alpha}^0 \tilde{J} \mathcal{R}^z(\theta) \tilde{J} \right) z = \left( \frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} C_{jk} \left( \sum_{|\alpha| \leq N} \partial_{\alpha}^0 v_j(\theta) z_k : l \in \mathbb{Z} \right) \right)_l.
$$

(2.29)

A combination of (2.25), (2.26), (2.28) and (2.29) gives

$$
\left\| \left( \sum_{|\alpha| \leq N} \partial_{\alpha}^0 J \mathcal{R}^{z\alpha}_0(\theta) J \right) a \right\|_N^2
= \sum_{l=0}^{\infty} R^l \left( \sum_{k=0}^{\infty} \left( \sum_{|\alpha| \leq N} A_{\alpha k} \right) a_k \right) \left| \frac{l}{N} \right| \left( \sum_{j \in \mathbb{Z}} \frac{1}{2} \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} C_{jk} \left( \sum_{|\alpha| \leq N} \partial_{\alpha}^0 v_j(\theta) z_k \right) \right) \left| \frac{l}{N} \right|.
$$

(2.30)

Let

$$
\gamma_j = \frac{(l+j)j}{l}, \text{ where } l, j = 1, 2, \cdots.
$$

Note that

$$
c_{jk} = \begin{cases} 
0, & j + l - k \neq 0, \\
2\pi, & j + l - k = 0.
\end{cases}
$$

By (2.30), one has

$$
\left\| \left( \sum_{|\alpha| \leq N} \partial_{\alpha}^0 J \mathcal{R}^{z\alpha}_0(\theta) J \right) a \right\|_N^2
= \sum_{l=0}^{\infty} \left| \frac{l}{N} \right| \left( \sum_{j \in \mathbb{Z}} \frac{1}{2} \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} C_{jk} \left( \sum_{|\alpha| \leq N} \partial_{\alpha}^0 v_j(\theta) z_k \right) \right) \left| \frac{l}{N} \right|.
$$

(2.30)

Suppose $J = \text{diag}(\sqrt{A_j} : j \in \mathbb{Z})$. Then for any $z = (z_k \in \mathbb{C} : k \in \mathbb{Z}) \in h_N$,
where \( C \) is a universal constant which might be different in different places. It follows that

\[
\sup_{\theta \in \mathcal{T}} \| J \sum_{|u| \leq N} \partial_{\theta} J \sum_{|u| \leq N} \|_{h_N \rightarrow h_N} \leq C. \tag{2.31}
\]

The proofs of the last two inequalities in (2.27) are similar to that of (2.31). \( \square \)

Now our goal is to find a symplectic transformation \( \Psi \), such that the term \( \epsilon \mathbf{i} \mathbf{P} \left( \langle u, u \rangle + \langle \pi, \pi \rangle \right) \) disappear. To this end, let \( G \) be a linear Hamiltonian of the form

\[
G = b_1(\theta) \left( \langle \Lambda^{-1} u, u \rangle + \langle \Lambda^{-1} \bar{u}, \bar{u} \rangle \right), \tag{2.32}
\]

where \( \theta = \omega t \) and \( b_1(\theta) \) need to be specified. Moreover, let

\[
\Psi = X^1_{\epsilon G}|_{t=1}, \tag{2.33}
\]

where \( X^1_{\epsilon G} \) is the flow of Hamiltonian, \( X_{\epsilon G} \) is the vector field of the Hamiltonian \( \epsilon G \) with the symplectic \( \mathbf{i} du \wedge d\bar{u} \). Let

\[
H_0 = \tilde{H} \circ \Psi. \tag{2.34}
\]

Recall that

\[
\tilde{H} = \langle p \Lambda u, \pi \rangle + \epsilon \left[ \frac{b_0}{2} \left( \langle \pi, \pi \rangle - \langle u, u \rangle \right) \right] + \epsilon \left[ \langle R^{\epsilon u}(\theta)u, u \rangle + \langle R^{\epsilon u}(\theta)\bar{u}, \bar{u} \rangle \right].
\]

Then we have \( \tilde{H} = N + \epsilon Q + \epsilon R_0 \), where

\[
N = \langle p \Lambda u, \pi \rangle, \quad Q = \frac{b_0}{2} \left( \langle \pi, \pi \rangle - \langle u, u \rangle \right), \tag{2.35}
\]

\[
R_0 = \left[ \langle R^{\epsilon u}(\theta)u, u \rangle + \langle R^{\epsilon u}(\theta)\bar{u}, \bar{u} \rangle + \langle R^{\epsilon u}(\theta)\pi, \pi \rangle \right]. \tag{2.36}
\]

Since the Hamiltonian \( \tilde{H} = \tilde{H}(\omega t, u, \pi) \) depends on time \( t \), we introduce a fictitious action \( I = \) constant, and let \( \theta = \omega t \) be angle variable. Then the non-autonomous \( \tilde{H}(\omega t, u, \pi) \) can be written as

\[
\omega I + \tilde{H}(\theta, u, \pi)
\]

with symplectic structure \( dI \wedge d\theta + \mathbf{i} du \wedge d\bar{u} \). See Section 45 (B) in [1]. By Taylor formula, we have

\[
H_0 = \tilde{H} \circ X^1_{\epsilon G} = N + \epsilon Q + \epsilon \{N, G\} + \epsilon^2 \int_0^1 \{Q, G\} \circ X^1_{\epsilon G} d\tau \tag{2.37}
\]

\[
+ \epsilon^2 \int_0^1 (1 - \tau) \{\{N, G\}, G\} \circ X^1_{\epsilon G} d\tau + \epsilon R_0 \circ X^1_{\epsilon G}.
\]
Thus, we can see

\[ R = \varepsilon \omega \cdot \partial_\theta b_1 \left( (\Lambda^{-1} u, u) - (\Lambda^{-1} \bar{u}, \bar{u}) \right) \]  

(2.38)

\[ + \varepsilon^2 \int_0^1 \{ Q, G \} \circ X^{x \theta}_t d\tau \]  

(2.39)

\[ + \varepsilon^2 \int_0^1 (1 - \tau) \{ [N, G], \hat{G} \} \circ X^{x \theta}_t d\tau \]  

(2.40)

\[ + \varepsilon R_0 \circ X^{1}_t. \]  

(2.41)

The aim of the following section is to estimate \( R \).

- **Estimate of (2.38).**

Let

\[ \overline{G} = \left( \begin{array}{c} \omega \cdot \partial_\theta b_0 \Lambda^{-1} \\ 0 \end{array} \right), \quad \bar{u} = \left( \begin{array}{c} u/N \end{array} \right). \]

Then, we have \( \overline{G} = \langle \varepsilon \overline{G}, \bar{u}, \bar{u} \rangle \).

Obviously,

\[ \sup_{\theta \in \mathbb{T}^n} \| \sum_{|\alpha| \leq N-1} \partial_\theta^\alpha J \overline{G}^{(\theta) J} \|_{hN \to hN} \leq C. \]

- **Estimate of (2.41).**

Let

\[ \hat{R} = \left( \begin{array}{cc} R^{m \theta}(\theta, \omega) & \frac{1}{\varepsilon} R^{m \theta}(\theta, \omega) \\ \frac{1}{\varepsilon} R^{m \theta}(\theta, \omega) & R^{m \theta}(\theta, \omega) \end{array} \right), \quad \mathcal{J} = \left( \begin{array}{cc} 0 & -i\text{id} \\ i\text{id} & 0 \end{array} \right). \]

and

\[ \overline{G} = \left( \begin{array}{cc} \frac{b_0}{\varepsilon^2} \Lambda^{-1} & 0 \\ 0 & \frac{b_0}{\varepsilon^2} \Lambda^{-1} \end{array} \right). \]  

(2.42)

Then we have

\[ R_0 = (\hat{R}(\theta) \left( \begin{array}{c} u/N \end{array} \right), \left( \begin{array}{c} u/N \end{array} \right)). \]

It follows that

\[ \varepsilon^2 \{ R_0, G \} = 4\varepsilon^2 \langle \hat{R}(\theta) \mathcal{J} \overline{G}(\theta) \bar{u}, \bar{u} \rangle. \]  

(2.43)

Let \( \hat{G} = \mathcal{J} \overline{G}(\theta) \) and \( \hat{G} \hat{G} = \hat{R} \hat{G} + (\hat{R} \hat{G})^T \). By Taylor formula, we have

\[ (2.41) = \varepsilon \langle R_1^\alpha \bar{u}, \bar{u} \rangle, \]

where

\[ R_1^\alpha = \hat{R} + 2\varepsilon \hat{R} \hat{G} + \sum_{j=2}^N \frac{2^{j+1} \varepsilon^j}{j!} \sum_{\alpha \leq j-1} \langle \hat{R}, \hat{G}, \ldots, \hat{G} \rangle. \]  

Thus, we can see

\[ \sup_{\theta \in \mathbb{T}^n} \| \sum_{|\alpha| \leq N} \partial_\theta^\alpha J R_1^\alpha(\theta) \|_{hN \to hN} \leq C. \]
• Estimate of (2.39).

\[ \{ Q, G \} = \frac{2b^2_0}{\rho} (\Lambda^{-1} u, \bar{u}) = (K^* \bar{u}, \bar{u}), \]  

(2.45)

where

\[ K^* = \begin{pmatrix} 0 & \frac{2b^2_0}{\rho} \Lambda^{-1} \\ \frac{2b^2_0}{\rho} \Lambda^{-1} & 0 \end{pmatrix}. \]  

(2.46)

By Taylor formula, we have

\[ (2.39) = \varepsilon^2 (K^* \bar{u}, \bar{u}) \]

By directly calculation, we have

\[ \{ \{ N, G \}, G \} = \langle H^*_1 \bar{u}, \bar{u} \rangle, \]  

(2.48)

where

\[ H^*_1 = \begin{pmatrix} 0 & -\frac{b^2_0}{\rho} \Lambda^{-1} \\ -\frac{b^2_0}{\rho} \Lambda^{-1} & 0 \end{pmatrix}. \]  

(2.49)

By Taylor formula, we have

\[ (2.40) = \varepsilon^2 (H^* \bar{u}, \bar{u}), \]

where

\[ H^* = \frac{K^*_1}{2} + \sum_{j=2}^{\infty} \frac{2^{j-2} \varepsilon_j}{j!} [\cdots H^*_1, \cdots, \hat{G}] \hat{G}. \]  

(2.50)

Now we have

\[ \sup_{\theta \in T^n} \| \sum_{|\alpha| \leq N-1} \partial^\alpha_\theta J K^*_1 (\theta) J \|_{h_N \to h_N} \leq C. \]

• Estimate of (2.40).

By directly calculation, we have

\[ \sup_{\theta \in T^n} \| \sum_{|\alpha| \leq N} \partial^\alpha_\theta J H^*_1 (\theta) J \|_{h_N \to h_N} \leq C. \]

In conclusion,

\[ \sup_{\theta \in T^n} \| \sum_{|\alpha| \leq N-1} \partial^\alpha_\theta J R J \|_{h_N \to h_N} \leq C. \]

Now, Theorem 1.1 can be transformed into a more exact expression.

**Theorem 2.1.** With Assumptions A, B, for given \( 1 \gg \gamma > 0 \), there exists \( \varepsilon^* \) with \( 0 < \varepsilon^* = \varepsilon^* (n, \gamma) \ll \gamma \), and exists a subset \( \Pi \subset [1, 2]^n \) with

\[ \text{Measure } \Pi \geq 1 - O(\gamma^{1/3}) \]
such that for any $0 < \varepsilon < \varepsilon^*$ and any $\omega \in \Pi$, there is a time-quasi-periodic symplectic change

$$\begin{pmatrix} u \\ \bar{u} \end{pmatrix} = \Phi(\omega t) \begin{pmatrix} \hat{u} \\ \hat{u} \end{pmatrix}$$

such that the Hamiltonian system (2.21) is changed into

$$\begin{cases}
\dot{\hat{u}}_k = i \frac{\partial \tilde{H}}{\partial \hat{u}_k}, & k \in \mathbb{Z}, \\
\dot{\hat{u}}_k = -i \frac{\partial \tilde{H}}{\partial \hat{u}_k}, & k \in \mathbb{Z},
\end{cases}$$

where

$$\tilde{H}(\hat{u}, \bar{u}) = \Lambda_0^{\infty} \hat{u}_0 \bar{u}_0 + \sum_{j=1}^{\infty} (\Lambda_j^{\infty} \hat{u}_j) \cdot \bar{u}_j,$$

with

(i) $Q_0$ and $Q_k (k = 1, 2, \cdots)$ are independent of time $t$, and $Q_0 \in \mathbb{R}$, $Q_k$ is a $2 \times 2$ real matrix $(k = 1, 2, \cdots)$;

(ii) $\tilde{Q} = \text{diag}(Q_j)$ satisfies $\|J \tilde{Q} J^{-1}\|_{h_N \rightarrow h_N} \leq C$, $J = \text{diag}(J_j : j = 0, 1, \cdots)$, $J_0 = \sqrt{\lambda_0}$, $J_j = \sqrt{\lambda_j} E_{22}$, $j = 1, 2, \cdots$;

(iii) $\Phi = \Phi(\omega t)$ is quasi-periodic in time and close to the identity map:

$$\|\Phi(\omega t) - \text{id}\|_{h_N \rightarrow h_N} \leq C \varepsilon,$$

where $\text{id}$ is the identity map from $h_N \rightarrow h_N$.

3. Analytical Approximation Lemma

We need to find a series of operators which are analytic in some complex strip domains to approximate the operators $R^{\omega}(\theta), R^{\overline{\omega}}(\theta)$ and $R^{\overline{\omega}}(\theta)$. To this end, we cite an approximation lemma (see [24, 34, 35] for the details). This method is used in [39], too.

We start by recalling some definitions and setting some new notations. Assume $X$ is a Banach space with the norm $\| \cdot \|_X$. First recall that $C^\mu (\mathbb{R}^n, X)$ for $0 < \mu < 1$ denotes the space of bounded Hölder continuous functions $f : \mathbb{R}^n \mapsto X$ with the form

$$\|f\|_{C^\mu, X} = \sup_{0 < |x-y| < 1} \frac{\|f(x) - f(y)\|_X}{|x-y|^\mu} + \sup_{x \in \mathbb{R}^n} \|f(x)\|_X.$$ 

If $\mu = 0$ then $\|f\|_{C^0, X}$ denotes the sup-norm. For $\ell = k + \mu$ with $k \in \mathbb{N}$ and $0 \leq \mu < 1$, we denote by $C^\ell(\mathbb{R}^n; X)$ the space of functions $f : \mathbb{R}^n \mapsto X$ with Hölder continuous partial derivatives, i.e., $\partial^\alpha f \in C^\mu (\mathbb{R}^n; X)$ for all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ with the assumption that
We define the norm $||f||_{C^r} = \sup_{|\alpha| \leq r} |||D^\alpha f|||_{C^r}$.

Lemma 3.1. (Jackson-Moser-Zehnder) Let $f \in C^l(\mathbb{R}^n; X)$ for some $l > 0$ with finite $C^l$ norm over $\mathbb{R}^n$. Let $\phi$ be a radial-symmetric, $C^\infty$ function, having as support the closure of the unit ball centered at the origin, where $\phi$ is completely flat and takes value 1. Let $K = \hat{\phi}$ be its Fourier transform. For all $\sigma > 0$ define

$$f_\sigma(x) := K \ast f = \frac{1}{\sigma^n} \int_{\mathbb{R}^n} K(\frac{x-y}{\sigma}) f(y) dy.$$ 

Then there exists a constant $C \geq 1$ depending only on $l$ and $n$ such that the following holds: for any $\sigma > 0$, the function $f_\sigma(x)$ is a real-analytic function from $C^n(\pi\mathbb{Z})^n$ to $X$ such that if $\Delta_\sigma^n$ denotes the $n$-dimensional complex strip of width $\sigma$,

$$\Delta_\sigma^n := \{ x \in \mathbb{C}^n | |\text{Im} x_j| \leq \sigma, 1 \leq j \leq n \},$$

then for any $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq l$ one has

$$\sup_{x \in \Delta_\sigma^n} ||D^\alpha f_\sigma(x) - \sum_{|\beta| \leq |\alpha|} \frac{\partial^{\beta + \alpha} f(\text{Re} x)}{\beta!} (\sqrt{\text{Im} x})^\beta ||_{C^r} \leq C ||f||_{C^l} \sigma^{l-|\alpha|},$$

and for all $0 \leq s \leq \sigma$,

$$\sup_{x \in \Delta_\sigma^n} ||D^\alpha f_\sigma(x) - \partial^\alpha f_\sigma(x)||_{C^s} \leq C ||f||_{C^l} \sigma^{l-|\alpha|}.$$ 

The function $f_\sigma$ preserves periodicity (i.e., if $f$ is $T$-periodic in any of its variable $x_j$, so is $f_\sigma$). Finally, if $f$ depends on some parameter $\xi \in \Pi \subset \mathbb{R}^n$ and

$$||f(x, \xi)||_{C^r(X)} := \sup_{\xi \in \Pi} ||D_j f(x, \xi)||_{C^r(X)}$$

are uniformly bounded by a constant $C$, then all the above estimates hold true with $|| \cdot ||$ replaced by $|| \cdot ||_{C^r}$.

The proof of this lemma consists in a direct check which is based on standard tools from calculus and complex analysis. It is used to deal with KAM theory for finite smooth systems by Zehnder \[13\]. Also see \[39\] and references therein, for example. For simplicity of notation, we shall replace $|| \cdot ||_X$ by $|| \cdot ||$. Now let us apply this lemma to the perturbation $P(\phi)$.

Fix a sequence of fast decreasing numbers $s_0 \downarrow 0$, $\nu \geq 0$, and $s_0 \leq \frac{1}{2}$. For an $X$-valued function $P(\phi)$, construct a sequence of real analytic functions $P^{(u)}(\phi)$ such that the following conclusions hold:

(1) $P^{(u)}(\phi)$ is real analytic on the complex strip $\mathbb{T}_s^{\nu}$ of the width $s_0$ around $\mathbb{T}^n$. 

(2) The sequence of functions $P^{(u)}(\phi)$ satisfies the bounds:
\[
\sup_{\phi \in T_n} \|P^{(u)}(\phi) - P(\phi)\| \leq C\|P\|_{C^s_\ell},
\]
(3.1)
\[
\sup_{\phi \in T_{n+1}} \|P^{(u+1)}(\phi) - P^{(u)}(\phi)\| \leq C\|P\|_{C^s_\ell},
\]
(3.2)
where $C$ denotes (different) constants depending only on $n$ and $\ell$.

(3) The first approximate $P^{(0)}$ is “small” with the perturbation $P$. Precisely speaking, for arbitrary $\phi \in T_{s_0}$, we have
\[
\|P^{(0)}(\phi)\| \leq C\|P\|_{C^s_\ell},
\]
(3.3)
where the constant $C$ is independent of $s_0$, and the last inequality holds true due to the hypothesis that $s_0 \leq \frac{1}{2}$.

(4) From the first inequality (3.1), we have the equality below. For any arbitrary $\phi \in T^n$, we have
\[
P(\phi) = P^{(0)}(\phi) + \sum_{v=0}^{+\infty} (P^{(u+1)}(\phi) - P^{(u)}(\phi)).
\]
(3.4)
Now take a sequence of real numbers $\{s_v \geq 0\}_{v=0}^{+\infty}$ with $s_v > s_{v+1}$ going fast to zero. Let $R^{p,q}(\theta) = P(\theta)$ for $p,q \in \{u,\overline{u}\}$. Then by (3.4) and (2.27), for $p,q \in \{u,\overline{u}\}$, we have,
\[
R^{p,q}(\theta) = R^{p,q}_0(\theta) + \sum_{l=1}^{+\infty} R^{p,q}_l(\theta),
\]
(3.5)
where $R^{p,q}_0(\theta)$ is analytic in $T_{s_0}$ with
\[
\sup_{\theta \in T_{s_0}} \|R^{p,q}_0(\theta)\|_{h_{\overline{u}} \to h_{p,q}} \leq C,
\]
(3.6)
and $R^{p,q}_l(\theta)$ ($l \geq 1$) is analytic in $T_{s_l}$ with
\[
\sup_{\theta \in T_{s_l}} \|JR^{p,q}_l(\theta)J\|_{h_{\overline{u}} \to h_{p,q}} \leq Cs_l^N.
\]
(3.7)

4. Iterative parameters of domains
Let
\begin{itemize}
  \item $e_0 = \varepsilon, e_v = \varepsilon^{(4)}^v, v = 0, 1, 2, \cdots$, which measures the size of perturbation at $v-th$ step.
  \item $s_v = \varepsilon^{1/N}_v, v = 0, 1, 2, \cdots$, which measures the strip-width of the analytic domain $T^n_{s_V}$.
\end{itemize}
$T^n_{s_V} = \{ \theta \in \mathbb{C}^n/2\pi \mathbb{Z}^n : |Im\theta| \leq s_V \}$. 

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\[ C(v) = C_1 2^{C_2 v}, \]

where \( C_1, C_2 \) are constants.

- \( K_v = 100v^{-1}2^v |\log \epsilon| \).

- \( \gamma_v = \frac{\gamma}{v}, 0 < \gamma \ll 1 \).

- A family of subsets \( \Pi_v \subset [1, 2]^u \) with \( [1, 2]^u \supset \Pi_0 \supset \cdots \supset \Pi_v \supset \cdots \), and

\[ \text{mes} \Pi_v \geq \text{mes} \Pi_{v-1} - C_\gamma^{1/3}. \]

- For an operator-value (or a vector-value) function \( B(\theta, \omega) \), whose domain is \((\theta, \omega) \in T^n_{\Lambda, \nu} \times \Pi_v \). Set

\[ \| B \|_{T^n_{\Lambda, \nu} \times \Pi_v} = \sup_{(\theta, \omega) \in T^n_{\Lambda, \nu} \times \Pi_v} \| B(\theta, \omega) \|_{h_N \to h_N}, \]

where \( \| \cdot \|_{h_N \to h_N} \) is the operator norm, and set

\[ \| B \|_{T^n_{\Lambda, \nu} \times \Pi_v} = \sup_{(\theta, \omega) \in T^n_{\Lambda, \nu} \times \Pi_v} \| \partial_\omega B(\theta, \tau) \|_{h_N \to h_N}. \]

5. Iterative Lemma

In the following, for a function \( f(\omega) \), denote by \( \partial_\omega \) the derivative of \( f(\omega) \) with respect to \( \omega \) in Whitney’s sense.

**Lemma 5.1.** For \( p, q \in \{ u, \pi \} \), let \( R_{0,0}^{p,q} = R_{0}^{p,q}, R_{1,0}^{p,q} = R_{1}^{p,q} \), where \( R_{0}^{p,q}, R_{1}^{p,q} \) are defined by (3.5), (3.6) and (3.7). Assume that we have a family of Hamiltonian functions \( H_v \):

\[ H_v = \lambda^{(v)}_0 \mu u_0 + \sum_{j=1}^{\infty} \left( \lambda^{(v)}_j \mu_j \right) \cdot \nu_j + \sum_{j \geq 0} \epsilon_j \left( \langle R_{1,1}^{v} u, u \rangle + \langle R_{1,1}^{v} u, u \rangle + \langle R_{1,1}^{v} u, u \rangle \right), \]

\[ \theta = \omega\tau, \quad \omega = (\omega_1, \omega_2, \cdots, \omega_n), \]

(A1)

\[ \lambda^{(0)}_v = \rho \sqrt{\lambda_0}, \quad \lambda^{(v)}_0 = \rho \sqrt{\lambda_0 + \sum_{i=0}^{v-1} \epsilon_i \mu^{(i)}_0}, \quad v \geq 1; \]

\[ \lambda^{(0)}_j = \rho \sqrt{\lambda_j E_{22}}, \quad \lambda^{(v)}_j = \rho \sqrt{\lambda_j E_{22} + \sum_{i=0}^{v-1} \epsilon_j \mu^{(i)}_j}, \quad j = 1, 2, \cdots, \quad v \geq 1, \]

where
Then there exists a compact set \( \Pi \) and exists a symplectic coordinate change such that the Hamiltonian function \( H \) which is defined on the domain \( T(p, m, q) + F \) or \( p, \nu \in \{1\}^d \\mu \) for fixed \( \theta \), \( p, \omega, q \in \{1\}^d \\mu \) satisfy the assumptions \( (i) \) \( u \in \{1\}^d \\mu \) denote the absolute value of a function, \( (ii) \) \( \mu_j^{(i)} = \mu_j^{(i)}(\omega) \) \( (j = 1, 2, \cdots, 0 \leq i \leq v - 1, v \geq 1) \) are \( 2 \times 2 \) real symmetry matrices with

\[
|\mu_j^{(i)}|_{\Pi} := \sup_{\omega \in \Pi} |\mu_j^{(i)}(\omega)| \leq C(i)/j, \quad (5.6)
\]

\[
|\mu_j^{(i)}|_{\Pi} := \sup_{\omega \in \Pi} \max_{1 \leq i \leq n} |\partial_{\omega} \mu_j^{(i)}(\omega)| \leq C(i)/j. \quad (5.7)
\]

Here \( \cdot \) denotes the sup-norm for real matrices.

Then there exists a compact set \( \Pi_{m+1} \subset \Pi_m \) with

\[
\text{mes} \Pi_{m+1} \geq \text{mes} \Pi_m - C_m^{1/3}, \quad (5.10)
\]

and exists a symplectic coordinate change

\[
\Psi : T_{x_{m+1}}^{\Pi_{m+1}} \times \Pi_{m+1} \to T_{x_{m}}^{\Pi_{m}} \times \Pi_{m},
\]

such that the Hamiltonian function \( H_m \) is changed into

\[
H_{m+1} \triangleq H_m \circ \Psi_m \nonumber
\]

\[
= \Lambda_0^{(m+1)} u_0 \bar{u}_0 + \sum_{j=1}^{\infty} \Lambda_j^{(m+1)} u_j \cdot \bar{u}_j + \sum_{l \geq m+1} e_l \left( (R_{l,m+1}^u u, \bar{u}) + (R_{l,m+1}^u u, \bar{u}) \right), \quad (5.12)
\]

which is defined on the domain \( T_{x_{m+1}}^{\Pi_{m+1}} \times \Pi_{m+1} \), and \( \Lambda_j^{(m+1)} \) satisfy the assumptions \( (A1)_{m+1} \) and \( R_{l,m+1}^{u,q} \) \( (p, q \in \{u, \bar{u}\}) \) satisfy the assumptions \( (A2)_{m+1} \).
6. Derivation of homological equations

Our end is to find a symplectic transformation \( \Psi_v \) such that the terms \( R_{l,v}^{uu}, R_{l,v}^{ut}, R_{l,v}^{tu} \) (with \( l = v \)) disappear. To this end, let \( F \) be a linear Hamiltonian of the form

\[
F = (F_{\theta}(\theta, \omega)u, u) + \langle F_{\theta}(\theta, \omega)u, \pi \rangle + \langle F_{\theta}(\theta, \omega)u, \pi \rangle,
\]

where \( \theta = \omega \), \( (F_{\theta}(\theta, \omega))^T = F_{\theta}(\theta, \omega), (F_{\theta}(\theta, \omega))^T = F_{\theta}(\theta, \omega), (F_{\theta}(\theta, \omega))^T = F_{\theta}(\theta, \omega) \).

Moreover, let

\[
\Psi = \Psi_m = X_{\varepsilon_m F}^{l=1},
\]

where \( X_{\varepsilon_m F}^{l} \) is the flow of the Hamiltonian, \( X_{\varepsilon_m F}^{l} \) is the vector field of the Hamiltonian \( \varepsilon_m F \) with the symplectic structure \( i du \wedge d\pi \). Let

\[
H_{m+1} = H_m \circ \Psi_m.
\]

By \( (5.1) \), we have

\[
H_m = N_m + R_m,
\]

with

\[
N_m = \omega l + \Lambda_0^{(m)} u_0 \pi_0 + \sum_{j=1}^{\infty} (\Lambda_j^{(m)} u_j) \cdot \pi_j,
\]

\[
R_m = \sum_{l=m}^{\infty} \varepsilon_l R_{lm},
\]

\[
R_{lm} = (R_{lm}^{uu}(\theta)u, u) + (R_{lm}^{ut}(\theta)u, \pi) + (R_{lm}^{tu}(\theta)u, \pi),
\]

where \( (R_{lm}^{uu}(\theta))^T = R_{lm}^{uu}(\theta), (R_{lm}^{ut}(\theta))^T = R_{lm}^{ut}(\theta), (R_{lm}^{tu}(\theta))^T = R_{lm}^{tu}(\theta) \).

Recall that the sequence \( z = (z_j \in \mathbb{C}, j \in \mathbb{Z}) \) can be rewritten as

\[
z = (z_0, z_j, z_{-j} : j = 1, 2, \cdots) = u \cdot (u_j : j = 0, 1, 2, \cdots),
\]

where \( u_0 = z_0, u_j = (z_j, z_{-j})^T, j = 1, 2, \cdots \). Suppose \( \{ \cdot , \cdot \} \) is the Poisson bracket with respect to \( idz \wedge d\pi \), that is

\[
\{ H(z, \pi), F(z, \pi) \} = i \left( \frac{\partial H}{\partial z} \frac{\partial F}{\partial \pi} - \frac{\partial H}{\partial \pi} \frac{\partial F}{\partial z} \right).
\]

Define

\[
\frac{\partial H}{\partial u_0} = \frac{\partial H}{\partial z_0}, \quad \frac{\partial H}{\partial u_j} = \frac{\partial H}{\partial z_j}, \quad \frac{\partial H}{\partial z_{-j}}, \quad j = 1, 2, \cdots, \quad \sum_{j=0}^{\infty} \frac{\partial H}{\partial u_j} \cdot \frac{\partial F}{\partial \pi_j} = \frac{\partial H}{\partial u} \cdot \frac{\partial F}{\partial \pi}.
\]

We can verify that

\[
\{ H(z, \pi), F(z, \pi) \} = \{ H(u, \pi), F(u, \pi) \} = i \left( \frac{\partial H}{\partial u} \frac{\partial F}{\partial \pi} - \frac{\partial H}{\partial \pi} \frac{\partial F}{\partial u} \right).
\]
So \( \{ \cdot, \cdot \} \) is also the Poisson bracket with respect to \( idu \wedge d\pi \). By combination of (6.1)-(6.7) and Taylor formula, we have

\[
H_{m+1} = H_m \circ X_{m,F}^{1}
\]

\[
= N_m + \varepsilon_m \{ N_m, F \} + \varepsilon_m^2 \int_0^1 (1 - \tau) \{ \{ N_m, F \}, F \} \circ X_{m,F}^{1} d\tau + \varepsilon_m \omega \cdot \partial_0 F
\]

\[
+ \varepsilon_m R_{mm} + \left( \sum_{l=m+1}^\infty \varepsilon_l R_{lm} \right) \circ X_{m,F}^{1} + \varepsilon_m^2 \int_0^1 \{ R_{mm,F}, F \} \circ X_{m,F}^{1} d\tau.
\]

(6.8)

Let \( \Gamma_{K_m} \) be a truncation operator. For any

\[
f(\theta) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{i(k, \theta)}, \quad \theta \in \mathbb{T}^n.
\]

Define, for any given \( K_m > 0 \),

\[
\Gamma_{K_m} f(\theta) = (\Gamma_{K_m} f)(\theta) \triangleq \sum_{|k| \leq K_m} \hat{f}(k) e^{i(k, \theta)},
\]

\[
(1 - \Gamma_{K_m}) f(\theta) = ((1 - \Gamma_{K_m}) f)(\theta) \triangleq \sum_{|k| > K_m} \hat{f}(k) e^{i(k, \theta)}.
\]

Then

\[
f(\theta) = \Gamma_{K_m} f(\theta) + (1 - \Gamma_{K_m}) f(\theta).
\]

Let

\[
\{ N_m, F \} + \Gamma_{K_m} R_{mm} = \{ \tilde{R}_{mm}^\pi u, \pi \},\]

where

\[
[R_{mm}^\pi]_{mm} := \text{diag} \left( \tilde{R}_{mmj}^\pi(0) : j = 0, 1, 2, \cdots \right),
\]

(6.9)

and \( R_{mmj}^\pi(\theta) \) is the matrix element of \( R_{mm}^\pi(\theta) \) and \( \tilde{R}_{mmj}^\pi(k) \) is the \( k \)-Fourier coefficient of \( R_{mmj}^\pi(\theta) \). Then

\[
H_{m+1} = N_{m+1} + C_{m+1} R_{m+1},
\]

(6.11)

where

\[
N_{m+1} = N_m + \varepsilon_m \{ [R_{mm}^\pi] u, \pi \} = \Lambda^{(m+1)}_0 u_0 \pi_0 + \sum_{j=1}^\infty (\Lambda^{(m+1)}_j u_j) \cdot \pi_j,
\]

(6.12)

\[
\Lambda^{(m+1)}_j = \Lambda^{(m)}_j + \varepsilon_m \tilde{R}_{mmj}^\pi(0) = \Lambda^{(m)}_j + \sum_{l=0}^m \varepsilon_l \mu_j^{(l)}, \quad \mu_j^{(m)} := \tilde{R}_{mmj}^\pi(0),
\]

(6.13)

\[
C_{m+1} R_{m+1} = \varepsilon_m (1 - \Gamma_{K_m}) R_{mm}
\]

(6.14)

\[
+ \varepsilon_m^2 \int_0^1 (1 - \tau) \{ \{ N_m, F \}, F \} \circ X_{m,F}^{1} d\tau
\]

(6.15)

\[
+ \varepsilon_m^2 \int_0^1 \{ R_{mm,F}, F \} \circ X_{m,F}^{1} d\tau
\]

(6.16)

\[
+ \left( \sum_{l=m+1}^\infty \varepsilon_l R_{lm} \right) \circ X_{m,F}^{1}.
\]

(6.17)
The equation (6.9) is called the homological equation. Developing the Poisson bracket \( \{u_{ij}, u_{mn}\} \), we get

\[
\omega \cdot \partial_0 F^{au}_{ij}(\theta, \omega) + i(\Lambda^{(m)} F^{au}_{ij}(\theta, \omega) + F^{au}_{ij}(\theta, \omega) \Lambda^{(m)}) = \Gamma_{K_n} R^{au}_{mnij}(\theta),
\]
(6.18)

\[
\omega \cdot \partial_0 F^{pu}_{ij}(\theta, \omega) - i(\Lambda^{(m)} F^{pu}_{ij}(\theta, \omega) + F^{pu}_{ij}(\theta, \omega) \Lambda^{(m)}) = \Gamma_{K_n} R^{pu}_{mnij}(\theta),
\]
(6.19)

\[
\omega \cdot \partial_0 F^{pu}_{ij}(\theta, \omega) + i(\Lambda^{(m)} F^{pu}_{ij}(\theta, \omega) - \Lambda^{(m)} F^{pu}_{ij}(\theta, \omega)) = \Gamma_{K_n} R^{pu}_{mnij}(\theta) - [R_{mnij}],
\]
(6.20)

where

\[
\Lambda^{(m)} = \text{diag}(\Lambda_j^{(m)} : j = 0, 1, 2, \ldots),
\]
(6.21)

and we assume

\[
\Gamma_{K_n} F^{au}_{ij}(\theta, \omega) = F^{au}_{ij}(\theta, \omega), \Gamma_{K_n} F^{pu}_{ij}(\theta, \omega) = F^{pu}_{ij}(\theta, \omega), \Gamma_{K_n} F^{pu}_{ij}(\theta, \omega) = F^{pu}_{ij}(\theta, \omega).
\]

\( F^{au}_{ij}(\theta, \omega), F^{pu}_{ij}(\theta, \omega), F^{pu}_{ij}(\theta, \omega) \) are written as the matrix elements of \( F^{au}(\theta, \omega), F^{pu}(\theta, \omega), F^{pu}(\theta, \omega) \), respectively. More exactly, for \( p, q \in \{u, p, \} \),

\[
F^{pq}_{ij}(\theta) = \begin{cases} a_{0,0}(\theta), & i = j = 0; \\ (a_{0,1}(\theta), a_{0,-1}(\theta)), & i = 0, j = 1, 2, \ldots; \\ (a_{i,0}(\theta), a_{i,-0}(\theta))^T, & j = 0, i = 1, 2, \ldots; \\ \left( \begin{array}{c} a_{i,j}(\theta) \\ a_{i,-j}(\theta) \\ a_{-i,j}(\theta) \\ a_{-i,-j}(\theta) \end{array} \right), & i, j = 1, 2, \ldots, \end{cases}
\]

where \( a_{i,j}(\theta) : \mathbb{T}_m^q \to \mathbb{R}, i, j = 0, 1, 2, \ldots \).

Then (6.18)-(6.20) can be rewritten as:

\[
\omega \cdot \partial_0 F^{au}_{ij}(\theta) + i(\Lambda_j^{(m)} F^{au}_{ij}(\theta) + F^{au}_{ij}(\theta) \Lambda_j^{(m)}) = \Gamma_{K_n} R^{au}_{mnij}(\theta),
\]
(6.22)

\[
\omega \cdot \partial_0 F^{pu}_{ij}(\theta) - i(\Lambda_j^{(m)} F^{pu}_{ij}(\theta) + F^{pu}_{ij}(\theta) \Lambda_j^{(m)}) = \Gamma_{K_n} R^{pu}_{mnij}(\theta),
\]
(6.23)

\[
\omega \cdot \partial_0 F^{pu}_{ij}(\theta) - i(\Lambda_j^{(m)} F^{pu}_{ij}(\theta) - F^{pu}_{ij}(\theta) \Lambda_j^{(m)}) = \Gamma_{K_n} R^{pu}_{mnij}(\theta),
\]
(6.24)

\[
\omega \cdot \partial_0 F^{pu}_{ij}(\theta) - i(\Lambda_j^{(m)} F^{pu}_{ij}(\theta) - F^{pu}_{ij}(\theta) \Lambda_j^{(m)}) = \Gamma_{K_n} R^{pu}_{mnij}(\theta) - \tilde{R}_{mnij}(0),
\]
(6.25)

where \( i, j = 0, 1, 2, \ldots \).

7. Solutions of the homological equations

**Lemma 7.1.** There exists a compact subset \( \Pi^+_{m+1} \subset \Pi_m \) with

\[
\text{mes}(\Pi^+_{m+1}) \geq \text{mes}\Pi_m - C_{\text{mes}}^{1/3}
\]
(7.1)

such that for any \( \omega \in \Pi^+_{m+1} \), the equation (6.20) has a unique solution \( F^{pu}(\theta, \omega) \), which is defined on the domain \( \mathbb{T}_m^u \times \Pi^+_{m+1} \), with

\[
\|J F^{pu}(\theta, \omega)J\|_{\mathbb{T}_m^u \times \Pi^+_{m+1}} \leq C(m+1)\varepsilon_m \frac{2^{(m+4)}}{\sigma},
\]
(7.2)

\[
\|J F^{pu}(\theta, \omega)J\|_{\mathbb{T}_m^u \times \Pi^+_{m+1}} \leq C(m+1)\varepsilon_m \frac{2^{(m+4)}}{22}.
\]
(7.3)
Proof. By passing to Fourier coefficients, (6.24) can be rewritten as

$$-\langle k, \omega \rangle \hat{F}^{\sigma}_{ij}(k) + (\Lambda_i^{(m)} \hat{F}^{\sigma}_{ij}(k) - i \hat{F}^{\sigma}_{ij}(k) \Lambda_j^{(m)}) = i \hat{R}^{\sigma}_{mnij}(k),$$  \hspace{1cm} (7.4)

where \( i, j = 0, 1, 2, \ldots, i \neq j, k \in \mathbb{Z}^n \) with \( |k| \leq K_m \). In the following, we always by “1” denote the identity from some finite dimensional space to itself. By applying \( j \) to both sides of (7.4), we have

$$(-\langle k, \omega \rangle (1 \otimes 1) + 1 \otimes \Lambda_i^{(m)} - \Lambda_j^{(m)} (T \otimes 1)) \hat{F}^{\sigma}_{ij}(k) = \gamma (i \hat{R}^{\sigma}_{mnij}(k)), \hspace{1cm} (7.5)$$

where \( A \otimes B \) is the tensor product of \( A \) and \( B \). Let \( \mu_{kij}^{(m)} \) be the \( l \)-th eigenvalue of \( 1 \otimes \Lambda_i^{(m)} - (\Lambda_j^{(m)} (T \otimes 1), l = 1, 2, 3, 4 \). Let

$$A_k = |k|^{2n+4} + 8,$

and

$$Q_{kij}^{(m)} \triangleq \left\{ \omega \in \Pi_m \left| -\langle k, \omega \rangle + \mu_{kij}^{(m)} \leq \frac{(|i-j| + 1) \gamma_m}{A_k} \right. \right\}, \hspace{1cm} (7.6)$$

where \( i, j = 0, 1, 2, \ldots, l = 1, 2, 3, 4, k \in \mathbb{Z}^n \) with \( |k| \leq K_m \), and \( k \neq 0 \) when \( i = j \). Let

$$\Pi_{m+1}^{+} = \Pi_m \setminus \bigcup_{|k| \leq K_m} \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{4} Q_{kij}^{(m)}.$$

Then for any \( \omega \in \Pi_{m+1}^{+} \), we have

$$| -\langle k, \omega \rangle + \mu_{kij}^{(m)} \geq \frac{(|i-j| + 1) \gamma_m}{A_k}. \hspace{1cm} (7.7)$$

Then

$$||(-\langle k, \omega \rangle (1 \otimes 1) + 1 \otimes \Lambda_i^{(m)} - \Lambda_j^{(m)} (T \otimes 1))^{-1}||_2 \leq \frac{A_k}{(|i-j| + 1) \gamma_m}. \hspace{1cm} (7.8)$$

Here \( || \cdot ||_2 \) denotes the spectral norm of matrices. Recall that \( R^{\sigma}_{mnij}(\theta) \) is analytic in the domain \( T_{nm}^n \) for any \( \omega \in \Pi_m 

$$||\hat{R}^{\sigma}_{mnij}(k)||_2 \leq \frac{C(m)}{\sqrt{ij}} e^{-\gamma_n |k|}, \hspace{1cm} (7.9)$$

which implies that

$$||\gamma (i \hat{R}^{\sigma}_{mnij}(k))||_2 \leq \frac{C(m)}{\sqrt{ij}} e^{-\gamma_n |k|}.$$  

By (7.5), we have

$$||\hat{F}^{\sigma}_{ij}(k)||_2 \leq \frac{A_k}{(|i-j| + 1) \gamma_m} ||\gamma (i \hat{R}^{\sigma}_{mnij}(k))||_2 \leq \frac{A_k}{\gamma_m (|i-j| + 1)} \frac{C(m) e^{-\gamma_n |k|}}{\sqrt{ij}}. \hspace{1cm} (7.10)$$

Then

$$||\hat{F}^{\sigma}_{ij}(k)||_2 \leq \frac{|k|^{2n+4} + 8}{\gamma_m (|i-j| + 1)} \frac{C(m) e^{-\gamma_n |k|}}{\sqrt{ij}}, \hspace{1cm} i \neq j. \hspace{1cm} (7.10)$$

Now we need the following lemmas:
Lemma 7.2. \[11\] For \(0 < \delta < 1, \nu > 1\), one has
\[
\sum_{k \leq 2^n} e^{-2|k|\delta |k|^\nu} < \left(\frac{\nu}{e}\right)^{\nu(1+\nu)^n}.
\]

Lemma 7.3. If \(A = (A_{ij})\) is a bounded linear operator on \(h_N\), then also \(B = (B_{ij}: i, j = 0, 1, 2, \cdots)\) with
\[
\|B_{ij}\| \leq \frac{|A_{ij}|}{|i-j|}, i, j = 0, 1, 2, \cdots; \quad i \neq j;
\]
and \(\|B\| \leq C\|A\|\), where \(\|\cdot\|\) is \(h_N \to h_N\) operator norm, we have
\[
B_{ij} = \begin{cases}
0, & i = 0; \\
\frac{b_0}{j}, & i = 0, j = 1, 2, \cdots; \\
\frac{b_0}{j}, & j = 0, i = 1, 2, \cdots; \\
\frac{b_j, b_i - j}{b_i - j}, & i, j = 1, 2, \cdots,
\end{cases}
\]
with \(b_{ij} \in \mathbb{R}, i, j = 0, 1, 2, \cdots\).

The proof of this result is similar to the proof of Theorem A.1 of \[33\] and so is omitted. See \[33\] for the details.

Therefore, by (7.10), we have
\[
\sup_{\mathcal{P}_{\lambda}^{\ast} \times \mathcal{P}_{\lambda}^{+}} (\|F_{\lambda} u_{\mu} (\theta, \omega) J_{\lambda}^J\|_2) \leq \frac{C(m)}{\gamma_m(i-j+1)}.
\]

where \(C\) is a constant depending on \(n, s_m = \frac{s_m - s_m}{4}\). By Lemma 7.3, we have
\[
\|F\|_{\mathcal{T}_{\lambda}^{\ast} \times \mathcal{P}_{\lambda}^{+}} \leq C(m) e_m^{-1} \gamma_m^{-\frac{2(n+4)}{m}} \leq C(m) e_m^{-2(n+4)}.
\]

It follows \(s_m > s_{m+1}\) that
\[
\|F\|_{\mathcal{T}_{\lambda}^{\ast} \times \mathcal{P}_{\lambda}^{+}} \leq \|F\|_{\mathcal{T}_{\lambda}^{\ast} \times \mathcal{P}_{\lambda}^{+}} \leq C(m) e_m^{-2(n+4)}.
\]

Applying \(\partial_{\omega} (l = 1, 2, \cdots, n)\) to both sides of (7.12), we have
\[
-\langle k, \omega \rangle \partial_{\omega} \hat{F}_{ij} (k) + \langle \Lambda_i^{(n)} \partial_{\omega} \hat{F}_{ij} (k) - \partial_{\omega} \hat{F}_{ij} (k) \Lambda_j^{(n)} \rangle = i \partial_{\omega} \hat{R}_{m+n,i,j} (k) + (*),
\]
where \(\hat{R}_{m+n,i,j} (k)\) is the Fourier transform of \(R_{m+n,i,j} (x)\).
where
\[
(*) = k_i F_{ij}^{\sigma}(k) - \partial_{\omega_k} \Lambda_i^{(m)} F_{ij}^{\sigma}(k) + \hat{F}_{ij}^{\sigma}(k) \partial_{\omega_k} \Lambda_j^{(m)}.
\] (7.13)

By applying "*" to both sides of (7.12), we have
\[
(\langle k, \omega \rangle (1 \otimes 1) + (1 \otimes \Lambda_i^{(m)}) \hat{F}_{ij}^{\sigma}(k) = \gamma(1 i \partial_{\omega_k} \hat{F}_{ij}^{\sigma}(k), \omega) + (\ast)),
\] (7.14)

Recalling \(|k| \leq K_m = 100s_m^{-1}2^{m} \log e\), and using (5.2)-(5.7) with \(v = m\), using (7.13), we have, on \(\omega \in \Pi_{m+1}\),
\[
\|\tilde{\ast}\| \leq CK_m\|\tilde{F}_{ij}^{\sigma}(k)\|_2.
\] (7.15)

According to (5.9),
\[
\|\partial_{\omega_k} \tilde{F}_{ij}^{\sigma}(k)\|_2 \leq \frac{C(m)e^{-\gamma_m|k|}}{\gamma_m(i - j + 1)}.
\] (7.16)

By (7.10), (7.14), (7.15) and (7.16), we have
\[
\|J_f \partial_{\omega_k} \tilde{F}_{ij}^{\sigma}(k) J_f\|_2 \leq \frac{A_{j} C_{m} C(m) e^{-\gamma_m|k|}}{\gamma_m(i - j + 1)} + \frac{N}{\gamma_m(i - j + 1)}.
\] (7.17)

Note that \(s_m > s_m' > s_{m+1}\). Again using Lemma 7.2 and Lemma 7.3 we have
\[
\|J \partial_{\omega_k} F^{\sigma}(\theta, \omega) J\|_{\Psi_{m+1}^{\pm}} = \|J \partial_{\omega_k} F^{\sigma}(\theta, \omega) J\|_{\Psi_{m+1}^{\pm}} = C(m + 1)e^{-\gamma_m|k|}.
\] (7.18)

The proof of the measure estimate (7.1) will be postponed to Section 10. This completes the proof of Lemma 7.1.

**Lemma 7.4.** There exists a compact subset \(\Pi_{m+1}^+ \subset \Pi_m\) with
\[
\text{mes}(\Pi_{m+1}^+) \geq \text{mes} \Pi_m - C_{\gamma_m^{-1/3}}.
\] (7.19)

such that for any \(\omega \in \Pi_{m+1}^+\), the equation (6.13) has a unique solution \(F^{\nu}(\theta)\), which is defined on the domain \(\Psi_{m+1}^{\pm} \times \Pi_{m+1}^+\), with
\[
\|J F^{\nu}(\theta, \omega) J\|_{\Psi_{m+1}^{\pm} \times \Pi_{m+1}^+} \leq C(m + 1)e^{-\gamma_m|k| / \gamma_{m+1}^{1/3}},
\]
\[
\frac{2}{N(m+1)}
\]

\[
\|J F^{\nu}(\theta, \omega) J\|_{\Psi_{m+1}^{\pm} \times \Pi_{m+1}^+} \leq C(m + 1)e^{-\gamma_m|k| / \gamma_{m+1}^{1/3}}.
\]

**Lemma 7.5.** There exists a compact subset \(\Pi_{m+1}^- \subset \Pi_m\) with
\[
\text{mes}(\Pi_{m+1}^-) \geq \text{mes} \Pi_m - C_{\gamma_m^{-1/3}}.
\] (7.20)

such that for any \(\omega \in \Pi_{m+1}^-\), the equation (6.19) has a unique solution \(F^{\nu}(\theta)\), which is defined on the domain \(\Psi_{m+1}^{\pm} \times \Pi_{m+1}^-\), with
\[
\|J F^{\nu}(\theta, \omega) J\|_{\Psi_{m+1}^{\pm} \times \Pi_{m+1}^-} \leq C(m + 1)e^{-\gamma_m|k| / \gamma_{m+1}^{1/3}},
\]
\[
\frac{2}{N(m+1)}
\]

\[
\|J F^{\nu}(\theta, \omega) J\|_{\Psi_{m+1}^{\pm} \times \Pi_{m+1}^-} \leq C(m + 1)e^{-\gamma_m|k| / \gamma_{m+1}^{1/3}}.
\]
Moreover, for $t \in [0, 1]$, $\|\tilde{u}(t)\|_N \leq 1$,
\[
\|\tilde{u}(t) - \tilde{u}_0\|_N \leq \varepsilon_m C(m+1)\varepsilon_m^{\frac{2(3m+4)}{3}} + \int_0^t \varepsilon_m \|B_m(\theta_0 + \omega s)\| \|\tilde{u}(s) - \tilde{u}_0\|_N ds, \tag{8.6}
\]

By Lemmas 7.4 and 7.5, their proofs are simpler than that of Lemma 7.1, so we omit them.

Let
\[
\Pi_{m+1} = \Pi_{m+1}^+ \cap \Pi_{m+1}^- \cap \Pi_{m+1}^-.
\]

By (7.1), (7.19) and (7.20), we have
\[
\text{mes} \Pi_{m+1} \geq \text{mes} \Pi_m - C\gamma_3^{1/3}.
\]

8. Coordinate change $\Psi$ by $\varepsilon_m F$

Recall $\Psi = \Psi_m = X'_{\varepsilon_m F}|_{t=1}$, where $X'_{\varepsilon_m F}$ is the flow of the Hamiltonian $\varepsilon_m F$ and $X_{\varepsilon_m F}$ is the vector field with symplectic $\varepsilon_m F$.

Let $\tilde{u} = \begin{pmatrix} u \\ \theta \end{pmatrix}$,
\[
B_m = \begin{pmatrix}
-2iF^{\sigma}(\theta, \omega) & -2iF^{\sigma}(\theta, \omega) \\
2iF^{\sigma}(\theta, \omega) & iF^{\sigma}(\theta, \omega)
\end{pmatrix}.
\]

By Lemmas 7.1, 7.2, and 7.3
\[
\|J B_m(\theta) J\|_{T_{m+1}^{1} \times T_{m+1}^{-}} \leq C(m+1)\varepsilon_m^{\frac{2(3m+4)}{3}}, \tag{8.4}
\]
\[
\|J B_m(\theta) J\|_{T_{m+1}^{1} \times T_{m+1}^{-}} \leq C(m+1)\varepsilon_m^{\frac{6(3m+4)}{3}}. \tag{8.5}
\]

It follows from (8.3) that
\[
\tilde{u}(t) - \tilde{u}_0 = \int_0^t \varepsilon_m B_m(\theta_0 + \omega s)\tilde{u}_0 ds + \int_0^t \varepsilon_m B_m(\theta_0 + \omega s)(\tilde{u}(s) - \tilde{u}_0) ds.
\]

Moreover, for $t \in [0, 1]$, $\|\tilde{u}_0\|_N \leq 1$,
\[
\|\tilde{u}(t) - \tilde{u}_0\|_N \leq \varepsilon_m C(m+1)\varepsilon_m^{\frac{2(3m+4)}{3}} + \int_0^t \varepsilon_m \|B_m(\theta_0 + \omega s)\| \|\tilde{u}(s) - \tilde{u}_0\|_N ds. \tag{8.6}
\]

Let
\[
i\tilde{u} = \varepsilon_m \frac{\partial F}{\partial \nu}, \quad -i\tilde{\nu} = \varepsilon_m \frac{\partial F}{\partial u}, \quad \dot{\theta} = \omega.
\]

More exactly,
\[
\begin{cases}
i\tilde{u} = \varepsilon_m (F^{\sigma}(\theta, \omega)u + 2iF^{\sigma}(\theta, \omega)\nu), \quad \dot{\theta} = \omega t, \\
i\tilde{\nu} = \varepsilon_m (2F^{\sigma}(\theta, \omega)u + F^{\sigma}(\theta, \omega)\nu), \quad \dot{\theta} = \omega t,
\end{cases}
\]

Recall $\dot{\theta} = \omega t$. (8.1)

Then
\[
\frac{d\tilde{u}(t)}{dt} = \varepsilon_m B_m(\theta)\tilde{u}, \quad \dot{\theta} = \omega.
\]

Let $\tilde{u}(0) = \tilde{u}_0 \in h_N \times h_N$, $\theta(0) = \theta_0 \in T_{m+1}^{1}$ be initial value. Then
\[
\begin{cases}
\tilde{u}(t) = \tilde{u}_0 + \int_0^t \varepsilon_m B_m(\theta_0 + \omega s)\tilde{u}(s) ds, \\
\theta(t) = \theta_0 + \omega t.
\end{cases}
\]

By Lemmas 7.1, 7.2, and 7.3
where $\| \cdot \|$ is the operator norm from $h_N \times h_N \to h_N \times h_N$. By Gronwall’s inequality,

$$
\| \tilde{u}(t) - \tilde{u}_0 \|_N \leq C(m+1)\epsilon_m^{1/2} \exp \left( \int_0^t \epsilon_m \|B_m(\theta_0 + \omega s)\| ds \right) \leq \epsilon_m^{1/2}.
$$

(8.7)

Thus,

$$
\Psi_m : \mathbb{T}^n_{m+1} \times \Pi_{m+1} \to \mathbb{T}^n_{m} \times \Pi_{m},
$$

(8.8)

and

$$
\| \Psi_m - id \|_{h_N \to h_N} \leq \epsilon_m^{1/2}.
$$

(8.9)

Since (8.2) is linear, $\Psi_m$ is a linear coordinate change. According to (8.3), construct Picard sequence:

$$
\left\{ \begin{array}{l}
\tilde{u}_0(t) = \tilde{u}_0, \\
\tilde{u}_{j+1}(t) = \tilde{u}_0 + \int_0^t \epsilon_m B(\theta_0 + \omega s)\tilde{u}_j(s) ds, \ j = 0, 1, 2, \cdots.
\end{array} \right.
$$

By (8.9), this sequence with $t = 1$ goes to

$$
\Psi_m(u_0) = \tilde{u}(1) = (id + P_m(\theta_0))u_0.
$$

(8.10)

where $id$ is the identity from $h_N \times h_N \to h_N \times h_N$, and $P_m(\theta_0)$ is an operator form $h_N \times h_N \to h_N \times h_N$ for any fixed $\theta_0 \in \mathbb{T}^n_{m+1}$, $\omega \in \Pi_{m+1}$, and is analytic in $\theta_0 \in \mathbb{T}^n_{m+1}$, with

$$
\|P_m(\theta_0)\|_{\mathbb{T}^n_{m+1} \times \Pi_{m+1}} \leq \epsilon_m^{1/2}.
$$

(8.11)

Note that (8.2) is a Hamiltonian system, so $P_m(\theta_0)$ is a symplectic linear operator from $h_N \times h_N$ to $h_N \times h_N$.

9. Estimates of remainders

The aim of this section is devoted to the estimates of the remainders:

$$
C_{m+1}R_{m+1} = (6.14) + \cdots + (6.17).
$$

• Estimate of (6.14).

By (6.7), let

$$
\mathbf{R}_m = \mathbf{R}_m(\theta) = \left( \begin{array}{c}
\mathbf{r}_{uu}(\theta) \\
\mathbf{r}_{um}(\theta) \\
\mathbf{r}_{mu}(\theta) \\
\mathbf{r}_{mm}(\theta)
\end{array} \right),
$$

then

$$
R_{m+1} = \mathbf{R}_m \left( \begin{array}{c}
u \\
u
\end{array} \right).
$$

So

$$
(1 - \Gamma_{\kappa})R_{m+1} = (1 - \Gamma_{\kappa}) \mathbf{R}_m \left( \begin{array}{c}
u \\
u
\end{array} \right).
$$

By the definition of truncation operator $\Gamma_{\kappa}$,

$$
(1 - \Gamma_{\kappa})\mathbf{R}_{m+1} = \sum_{|k| > \kappa} \mathbf{R}_{m+1}(k)e^{i[k, \theta]}, \ \theta \in \mathbb{T}^n_{m+1}, \ \omega \in \Pi_{m}.
$$
Since $\tilde{R}_{mn} = \tilde{R}_{mn}(\theta)$ is analytic in $\theta \in \mathbb{T}^n_{m}$, 
\[
\sup_{(\theta, \omega) \in \mathbb{T}^n_{m+1} \times \Pi_{m+1}} \| J(1 - \Gamma_{K_{m}})\tilde{R}_{mn} J\|_{\mathbb{H}^{N}}^2 \leq \sum_{|k| > K_{m}} \| J\tilde{R}_{mn}(k) J\|_{\mathbb{H}^{N}}^2 e^{2|k|_{m+1}}
\]
\[
\leq \| J\tilde{R}_{mn} J\|_{\mathbb{H}^{N}}^2 \sum_{|k| > K_{m}} e^{-2(|\omega - x_{m+1})|} |k|
\]
\[
\leq C^2(m)\varepsilon_m^{-1} e^{-2K_{m}(x_{m+1})} \quad \text{(by (5.8))}
\]
\[
\leq C^2(m)\varepsilon_m^2,
\]
which leads to 
\[
\| J(1 - \Gamma_{K_{m}})\tilde{R}_{mn} J\|_{\mathbb{T}^n_{m+1} \times \Pi_{m+1}} \leq \varepsilon_m C(m + 1).
\]
Thus, 
\[
\| \varepsilon_m J(1 - \Gamma_{K_{m}})\tilde{R}_{mn} J\|_{\mathbb{T}^n_{m+1} \times \Pi_{m+1}} \leq \varepsilon_m^2 C(m + 1) \leq \varepsilon_m C(m + 1).
\]
Similarly, 
\[
\| \varepsilon_m J(1 - \Gamma_{K_{m}})\tilde{R}_{mn} J\|_{\mathbb{T}^n_{m+1} \times \Pi_{m+1}} \leq \varepsilon_m^2 C(m + 1) \leq \varepsilon_m C(m + 1).
\]

- Estimate of (6.10).

Let
\[
S_m = \left( \begin{array}{c} F^{mn}(\theta, \omega) \\ \frac{1}{2} F^{m\Pi}(\theta, \omega) \\ F^{mm}(\theta, \omega) \end{array} \right),
\]
Then we have
\[
F = \langle S_m(\theta) \left( \begin{array}{c} u \\ \eta \end{array} \right), \left( \begin{array}{c} u \\ \eta \end{array} \right) \rangle = \langle S_m \dot{u}, \ddot{u} \rangle = \langle \, u \, \rangle.
\]

Then
\[
\varepsilon_m^2 \{ R_{mn}, F \} = 4\varepsilon_m^2 \langle \tilde{R}_{mn}(\theta) \not\mathcal{J} S_m(\theta) \ddot{u}, \ddot{u} \rangle.
\]
Note $\mathbb{T}^n_{m} \times \Pi_{m} \supset \mathbb{T}^n_{m+1} \times \Pi_{m+1}$. By (5.8) and (5.9) with $l = m, v = m$
\[
\| \tilde{R}_{mn}(\theta) \|_{\mathbb{T}^n_{m+1} \times \Pi_{m+1}} \leq \| \tilde{R}_{mn}(\theta) \|_{\mathbb{T}^n_{m} \times \Pi_{m}} \leq C(m),
\]
\[
\| \tilde{R}_{mn}(\theta) \|_{\mathbb{T}^n_{m+1} \times \Pi_{m+1}} \leq C(m).
\]

Let $\tilde{S}_m(\theta) = \not\mathcal{J} S_m(\theta)$. Then by Lemmas 7.1, 7.4 and 7.5, we have
\[
\| J\tilde{S}_m(\theta) J \|_{\mathbb{T}^n_{m+1} \times \Pi_{m+1}} \leq C(m + 1)\varepsilon_m \frac{2(m+4)}{N},
\]
\[
\| J\tilde{S}_m(\theta) J \|_{\mathbb{T}^n_{m+1} \times \Pi_{m+1}} \leq C(m + 1)\varepsilon_m \frac{4(m+4)}{N},
\]
and
\[
\| \tilde{R}_{mn} \not\mathcal{J} S_m \|_{\mathbb{T}^n_{m+1} \times \Pi_{m+1}} = \| \tilde{R}_{mn}\tilde{S}_m \|_{\mathbb{T}^n_{m+1} \times \Pi_{m+1}} \leq C(m)C(m + 1)\varepsilon_m \frac{2(m+4)}{N}.
\]
Set

\[ [\tilde{R}_{nm}, \tilde{S}_m] = \tilde{R}_{nm}\tilde{S}_m + (\tilde{R}_{nm}\tilde{S}_m)^T. \]

Note that the vector field is linear. So, by Taylor formula, one has

\[ (6.16) = \frac{\partial^2}{\partial \theta^2} \left( \tilde{R}_m^* (\theta) \tilde{u}, \tilde{u} \right), \]

where

\[ \tilde{R}_m^* (\theta) = 2^2 \tilde{R}_{nm}\tilde{S}_m + \sum_{j=2}^{\infty} \frac{2j+1}{j!} \left( \cdots [\tilde{R}_{nm}\tilde{S}_m], \cdots \tilde{S}_m \right) \tilde{S}_m. \]

By (9.3) and (9.4),

\[ \| J^{\tilde{R}_m^*} (\theta) J \|_{\mathcal{T}^m_{n+1} \times \Pi_{m+1}} \leq C(m)C(m+1)\frac{2(3n+1)}{n} \sum_{j=1}^{\infty} \frac{C(m)C(m+1)\frac{2(3n+1)}{n}}{j!} \leq C(m)C(m+1)\frac{2(3n+1)}{n}. \]

By (9.3) and (9.5),

\[ \| J^{\tilde{R}_m^*} (\theta) J \|_{\mathcal{T}^m_{n+1} \times \Pi_{m+1}} \leq C(m)C(m+1)\frac{2(3n+1)}{n}. \]

Thus,

\[ \| \frac{\partial^2}{\partial \theta^2} \left( R_{nm}^* (\theta) \tilde{u}, \tilde{u} \right) \|_{\mathcal{T}^m_{n+1} \times \Pi_{m+1}} \leq C(m)C(m+1)\frac{2(3n+1)}{n} \leq C(m)C(m+1)^n. \]  \hfill (9.7)

and

\[ \| \frac{\partial^2}{\partial \theta^2} \left( R_{nm}^* (\theta) \tilde{u}, \tilde{u} \right) \|_{\mathcal{T}^m_{n+1} \times \Pi_{m+1}} \leq C(m)C(m+1)\frac{2(3n+1)}{n} \leq C(m)C(m+1)^n. \]  \hfill (9.8)

- **Estimate of (6.15)**

By (6.9),

\[ \{N_m, F\} = \{R_{nm}[u, \Pi] - \Gamma_{nm}R_{nm}, \hat{R}_{nm} = R_{nm}. \}

Thus,

\[ (6.15) = \frac{\partial^2}{\partial \theta^2} \left( R_{nm}^* (\theta) \tilde{u}, \tilde{u} \right) \|_{\mathcal{T}^m_{n+1} \times \Pi_{m+1}} \leq C(m)C(m+1)^n. \]  \hfill (9.9)

Note \( R_{nm}^* \) is a quadratic polynomial in \( u \) and \( \Pi \). So we write

\[ R_{nm}^* = \left( \langle \mathcal{R}_m (\theta, \omega) \tilde{u}, \tilde{u} \rangle \right) \| u \|_{\mathcal{U}}. \]  \hfill (9.10)

By (5.6) and (5.7) with \( l = v = m \), and using (9.4) and (9.5),

\[ J_m^* \|_{\mathcal{T}^m_{n+1} \times \Pi_{m+1}} \leq C(\epsilon_m)\frac{2(3n+1)}{n}, \quad \| J_m^* \|_{\mathcal{T}^m_{n+1} \times \Pi_{m+1}} \leq C(\epsilon_m)\frac{6(3n+1)}{n}. \]  \hfill (9.11)

where \( \| \cdot \| \) is the operator norm in \( h_N \times h_N \rightarrow h_N \times h_N \). Recall \( F = \langle S_m (\theta, \omega) \tilde{u}, \tilde{u} \rangle \). Set

\[ [\mathcal{R}_m, \tilde{S}_m] = \mathcal{R}_m\tilde{S}_m + (\mathcal{R}_m\tilde{S}_m)^T. \]  \hfill (9.12)
Using Taylor formula to (9.9), we get

\[
(6.15) \quad \frac{\varepsilon_m^2}{2!} (R_m^e, F) + \cdots + \frac{\varepsilon_m^j}{j!} \left[ \sum_{j=2}^{\infty} \frac{2j^2}{j!} \left( \sum_{m=1}^{\infty} R_m, \tilde{S}_m \right) \tilde{u}, \tilde{u} \right] \]

By (9.4), (9.11) and (9.12), we have

\[
\| J \mathcal{R}^{**} (\theta, \omega) J \|_{T_{\Pi_m+1}^1, \Pi_m} \leq \sum_{j=2}^{\infty} \frac{C(m)}{j!} \left( \varepsilon_m C(m+1) \varepsilon_m^{-2/(m+1)} \right)^j \leq C(m+1) \varepsilon_m^{4/3} = C(m+1) \varepsilon_{m+1}. \quad (9.13)
\]

Similarly,

\[
\| J \mathcal{R}^{**} (\theta, \omega) J \|_{T_{\Pi_m+1}^1, \Pi_m} \leq C(m+1) \varepsilon_{m+1}. \quad (9.14)
\]

- Estimate of (6.17)

\[
(6.17) \quad \sum_{j=m+1}^{\infty} \varepsilon_t (R_{lm} \circ X_{lm}^1, F).
\]

We can write

\[
R_{lm} = J \mathcal{R}_{lm} (\theta, \omega).
\]

Then, by Taylor formula, one has

\[
R_{lm} \circ X_{lm}^1 = R_{lm} + \sum_{j=1}^{\infty} \frac{1}{j!} \left[ J \mathcal{R}_{lm} (\theta, \omega) \right]^{j-1} \tilde{S}_m \varepsilon_m^j.
\]

where

\[
\tilde{S}_m = 2^j \left[ \sum_{m=1}^{\infty} R_m, \tilde{S}_m \right] \tilde{u}, \tilde{u} \]

By (5.8), (5.9),

\[
\| J \mathcal{R}_{lm} J \|_{T_{\Pi_m}^1, \Pi_m} \leq C(l), \quad \| J \mathcal{R}_{lm} J \|_{T_{\Pi_m+1}^1, \Pi_m} \leq C(l).\]

Combining the last inequalities with (9.4) and (9.5), one has

\[
\| J \mathcal{R}_{lm} J \|_{T_{\Pi_m+1}^1, \Pi_m} \leq \| J \mathcal{R}_{lm} J \|_{T_{\Pi_m}^1, \Pi_m} \left( \| J \mathcal{R}_{lm} J \|_{T_{\Pi_m}^1, \Pi_m} 4 \varepsilon_m \right)^j \leq C^2 \varepsilon_m \varepsilon_m^{-2/(m+1)} \varepsilon_m^{4/3} \leq \frac{C}{30} \varepsilon_m^{4/3}.
\]
where $|J^{-1}|_{\mathcal{T}_t^{
u} \times \Pi_{m+1}} \leq C$ is used, and

$$
|J \tilde{R}_{lm,j} S_{jj}^l|_{\mathcal{T}_t^{
u} \times \Pi_{m+1}} \leq |J \tilde{R}_{lm,j} S_{jj}^l|_{\mathcal{T}_t^{
u} \times \Pi_{m+1}} \left( |J \tilde{S}_{m,j}^l|_{\mathcal{T}_t^{
u} \times \Pi_{m+1}} 4 \epsilon_m \right)
$$

Thus, let

$$
\mathcal{R}_{l,m+1} := \tilde{R}_{lm} + \sum_{j=1}^{\infty} \frac{1}{j} \tilde{R}_{lmj},
$$

then

$$
\mathbf{(6.17)} = \sum_{l=m+1}^{\infty} s_{l} \left( \mathcal{R}_{l,m+1} \tilde{u}, \tilde{u} \right) \tag{9.16}
$$

and

$$
|J \mathcal{R}_{l,m+1} |_{\mathcal{T}_t^{
u} \times \Pi_{m+1}} \leq C^2 (m) \leq C (m+1), \quad |J \mathcal{R}_{l,m+1} |_{\mathcal{T}_t^{
u} \times \Pi_{m+1}} \leq C^2 (m) \leq C (m+1).
$$

As a whole, the remainder $R_{m+1}$ can be written as

$$
C_{m+1} R_{m+1} = \sum_{l=m+1}^{\infty} s_{l} (\mathcal{R}_{l,m+1} (\theta) u, u) + (\mathcal{R}_{l,m+1} (\theta) u, \nu) + (\mathcal{R}_{l,m+1} (\theta) \nu, \nu), \quad v = m+1,
$$

where, for $p, q \in \{ u, \nu \}$, $R_{l,m+1}^{pq}$ satisfies (5.8) and (5.9) with $v = m+1, l \geq m+1$. This shows that Assumption (A2)$_{v}$ with $v = m+1$ holds true.

By (6.13), we know

$$
\mu_{ij}^{(m)} = R_{mnj}^{mm}(0).
$$

Taking $p = u, q = \nu$ into (5.8) and (5.9), we have

$$
|\mu_{ij}^{(m)}|_{\Pi_{m+1}} \leq |R_{mnj}^{mm} (\theta, \omega) |/ j \leq C(m) / j,
$$

$$
|\mu_{ij}^{(m)}|_{\Pi_{m+1}} \leq |\partial_{\omega} R_{mnj}^{mm} (\theta, \omega)| / j \leq C(m) / j.
$$

This shows that Assumption (A1)$_{v}$ with $v = m+1$ holds true.

### 10. Estimate of measure

In this section, $C$ denotes a universal constant, which may be different in different places.

**Lemma 10.1.** If $|i|, |j| >> 1$, then

$$
\mu_{ij}^{(m)} = \rho \sqrt{\lambda_{l}} - \rho \sqrt{\lambda_{l} + O(\frac{\epsilon_{0}}{|i|})} + O(\frac{\epsilon_{0}}{|j|}),
$$

(10.1)

where $\lambda_{k} = k^2 + M$, $k \in \mathbb{Z}$, $\mu_{ij}^{(m)}$ is the $l$-th eigenvalue of $1 \otimes \Lambda_{i}^{(m)} - (\Lambda_{i}^{(m)})^T \otimes 1$, $i, j = 1, 2, \ldots$, $i \neq j$, $l = 1, 2, 3, 4$ (for more details, see Section 7, the proof of Lemma 7.1).
Proof. Recall that
\[ \Lambda^{(m)}_i = \rho \sqrt{\lambda_i} E_{22} + O(\frac{E_0}{|i|}), \quad i \neq 0. \]

By computation, we have
\[
1 \otimes \Lambda^{(m)}_i - (\Lambda^{(m)}_j)^T \otimes 1 = \rho \sqrt{\lambda_i} (E_{22} \otimes E_{22}) - \rho \sqrt{\lambda_j} (E_{22} \otimes E_{22}) + E_{22} \otimes G_i + G_j \otimes E_{22}
\]
\[ = \rho (\sqrt{\lambda_i} - \sqrt{\lambda_j}) E_{44} + E_{22} \otimes G_i + G_j \otimes E_{22}, \quad (10.2) \]

where \( G_i \) is a \( 2 \times 2 \) matrix such that \( |G_i| \leq C_0 \epsilon_0 |i| \).

Then
\[
|1 \otimes \Lambda^{(m)}_i - (\Lambda^{(m)}_j)^T \otimes 1 - \rho (\sqrt{\lambda_i} - \sqrt{\lambda_j}) E_{44}| \leq (C |i| + C |j|) \epsilon_0.
\]

Note that \( 1 \otimes \Lambda^{(m)}_i - (\Lambda^{(m)}_j)^T \otimes 1 \) is Hermitian. By the perturbation theory for eigenvalue of matrices, we obtain (10.1). \( \square \)

Now let us return to (7.6)
\[ Q^{(m)}_{kij} \triangleq \left\{ \omega \in \Pi_m \bigg| -\langle k, \omega \rangle + \mu_{kij}^m \right\} < \frac{(i - j + 1) \gamma_m}{A_k}, \quad A_k = |k|^{2n+4} + 8. \quad (10.3) \]

Case 1. \( i \neq j \). If \( Q^{(m)}_{kij} = \emptyset \), then \( \text{mes} Q^{(m)}_{kij} = 0 \). So we assume \( Q^{(m)}_{kij} \neq \emptyset \). Then there exists \( \omega \in \Pi_m \) such that
\[ | -\langle k, \omega \rangle + \mu_{kij}^m| < \frac{|i - j + 1| \gamma_m}{A_k}. \quad (10.4) \]

(1.1) \( k \neq 0 \).

By Lemma 10.1,
\[ |\mu_{kij}^m| = |\rho \sqrt{\lambda_i} - \rho \sqrt{\lambda_j} + O(\frac{E_0}{|i|}) + O(\frac{E_0}{|j|})| \geq \frac{1}{2} |\sqrt{\lambda_i} - \sqrt{\lambda_j}|. \quad (10.5) \]

Furthermore, it is easy to verify that
\[ |\sqrt{\lambda_i} - \sqrt{\lambda_j}| \geq \frac{4(|i - j + 1| \gamma_m}{A_k}. \quad (10.6) \]

Then by (10.4), (10.5) and (10.6), one has
\[ |\langle k, \omega \rangle| \geq \frac{|\mu_{kij}^m| - (i - j + 1) \gamma_m}{A_k} \geq \frac{1}{2} |\sqrt{\lambda_i} - \sqrt{\lambda_j}| - \frac{(i - j + 1) \gamma_m}{A_k} \]
\[ \geq \frac{1}{4} |\sqrt{\lambda_i} - \sqrt{\lambda_j}| \geq \frac{1}{C} |i - j|. \]

So
\[ |i - j| \leq C|\langle k, \omega \rangle|. \quad (10.7) \]
(1.1.1) \( i \geq i_0, \ j \geq j_0 \).

By (10.1), we have that, when \( \omega \in \Pi_m \) such that (10.4) holds true, the following inequality holds true:

\[
| - \langle k, \omega \rangle + \rho i - \rho j | = | - \langle k, \omega \rangle + \mu_{i,j}^m + (\rho i - \rho j - \mu_{i,j}^m) | \leq \frac{|i - j| + 1}{A_k} \gamma_m + \frac{C_1(M)}{i_0} + \frac{C_2(M)}{j_0},
\]

where \( C_1(M) > 0 \) and \( C_2(M) > 0 \) are constants.

Thus

\[
Q_{kj}^{(m)} \subset \left\{ \omega \in \Pi_m \mid | - \langle k, \omega \rangle + \tilde{l} | < \frac{|i + 1}{A_k} \gamma_m + \frac{C_1(M)}{i_0} + \frac{C_2(M)}{j_0} \right\} \triangleq \tilde{Q}_kl. \tag{10.9}
\]

By (10.7), one has

\[
\tilde{l} \leq C |\langle k, \omega \rangle| \leq C |k|. \tag{10.10}
\]

Note that \( k \neq 0 \). Then

\[
\frac{d(-\langle k, \omega \rangle + \rho \tilde{l})}{d\omega} > \frac{1}{2} |k| \geq \frac{1}{2}.
\]

It follows that

\[
\text{mes} \tilde{Q}_{kl} \leq 4 \left( \frac{|\tilde{l}| + 1}{A_k} \gamma_m + \frac{C_1(M)}{i_0} + \frac{C_2(M)}{j_0} \right). \tag{10.11}
\]

Take

\[
j_0 = i_0 = |k|^{n+2} \ell^{-1/3}. \tag{10.12}
\]

Then

\[
\text{mes} \bigcup_{1 \leq l \leq C|k|} \tilde{Q}_{kl} \leq \frac{C|k| \gamma_m}{A_k} + C \sum_{1 \leq |i| \leq C|k|} \left( \frac{C_1(M)}{i_0} + \frac{C_2(M)}{j_0} \right) \leq \frac{C|k| \gamma_m}{A_k} + \gamma_m^{1/3} \frac{C|k|}{|k|^{n+2}} \leq C \ell^{-1/3} |k|^{n+1}.
\]

It follows from (10.9) that

\[
\text{mes} \bigcup_{i \geq i_0} Q_{ki}^{(m)} \leq \frac{C \ell^{1/3}}{|k|^{n+1}}, \tag{10.13}
\]

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Let $i \leq i_0$ or $j \leq j_0$.

By (10.7), one has $|i - j| \leq C|k|$. In addition, $1 \otimes \Lambda_j^{(m)} - (\Lambda_j^{(m)})^T \otimes 1$ is obviously Hermitian. Then by the variation of eigenvalues for Hermitian matrix, we have

$$\left| \frac{d}{d\omega}(-\langle k, \omega \rangle + \mu_{ij}^{\text{ml}}) \right| \geq |k| - \left| \frac{d\mu_{ij}^{\text{ml}}}{d\omega} \right| \geq \frac{1}{2}.$$ 

Therefore,

$$\text{mes} \bigcup_{1 \leq i \leq i_0 \atop |i - j| \leq C|k|} Q_{kiij}^{(m)} \leq \sum_{1 \leq i \leq i_0 \atop |i - j| \leq C|k|} 4(|i - j| + 1)\gamma_m \leq \frac{C|k|\gamma_m i_0}{A_k} \leq C|k|^{n+3} \gamma_m \frac{1}{A_k} \leq \frac{C\gamma_m^{2/3}}{|k|^{n+1}}. \quad (10.14)$$

Similarly, one has

$$\text{mes} \bigcup_{1 \leq j \leq j_0 \atop |i - j| \leq C|k|} Q_{kiij}^{(m)} \leq \frac{C\gamma_m^{2/3}}{|k|^{n+1}}. \quad (10.15)$$

(1.2) $k = 0$.

By (10.5) and (10.6), one has $Q_{kiij}^{(m)} = \emptyset$, then

$$\text{mes} Q_{kiij}^{(m)} = 0. \quad (10.16)$$

Case 2. $i = j$, one has $k \neq 0$.

At this time, by Lemma 10.1,

$$-\langle k, \omega \rangle + \mu_{ij}^{\text{ml}} = -\langle k, \omega \rangle + O\left(\frac{\varepsilon_0}{|i|}\right). \quad (10.17)$$

(2.1) Suppose $|\langle k, \omega \rangle| \geq \frac{2\gamma_m^{2/3}}{A_k}$.

(2.1.1) $i > \frac{C\varepsilon_0 A_k}{\gamma_m^{2/3}}$.

By (10.17), one has

$$| - \langle k, \omega \rangle + \mu_{ij}^{\text{ml}} | \geq \frac{2\gamma_m^{2/3}}{A_k} - \frac{C\varepsilon_0}{i} > \frac{\gamma_m^{2/3}}{A_k}.$$ 

It follows from (10.4) that $Q_{kiil}^{(m)} = \emptyset$. Then

$$\text{mes} Q_{kiil}^{(m)} = 0. \quad (10.18)$$
\[
(2.1.2) \quad i \leq \frac{C_{0A_k}}{\beta} \triangleq \tilde{k}.
\]

Note that

\[
d(-\langle k, \omega \rangle + \mu_i^{m_i}) = |k| + O\left(\frac{E_0}{|l|}\right) \geq \frac{1}{2}.
\]

Then

\[
\text{mes} \bigcup_{i \leq \tilde{k}} Q^{(m)}_{ki} \leq \frac{4k \gamma_m}{A_k} \leq C \gamma_m^{1/3}.
\]

(10.19)

(2.2) Suppose \(|\langle k, \omega \rangle| < \frac{2\gamma^2}{3m A_k}.

Let

\[
\tilde{Q}_k = \left\{ \omega \in \Pi_{m_i} \big| |\langle k, \omega \rangle| < \frac{2\gamma^2}{3m A_k} \right\}.
\]

Note that \(|\frac{d(\langle k, \omega \rangle)}{d\omega}| = |k| \geq 1.

Then

\[
\text{mes} \tilde{Q}_k \leq \frac{4\gamma_m^{2/3}}{A_k},
\]

and

\[
\text{mes} \bigcup_{k \in \mathbb{Z} \setminus \{0\}} \tilde{Q}_k \leq \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{C \gamma_m^{2/3}}{A_k} \leq C \gamma_m^{1/3}.
\]

(10.20)

Combining (10.13), (10.14), (10.15), (10.16), (10.18), (10.19) and (10.20), we have

\[
\text{mes} \bigcup_{|k| \leq K_n} \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{4} Q^{(m)}_{ki} \leq C \gamma_m^{1/3}.
\]

(10.21)

Let

\[
\Pi_{m+1}^+ = \Pi_m \setminus \bigcup_{|k| \leq K_n} \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{4} Q^{(m)}_{ki}.
\]

Then we have proved the following Lemma 10.2.

**Lemma 10.2.**

\[
\text{mes} \Pi_{m+1}^+ \geq \text{mes} \Pi_m - C \gamma_m^{1/3}.
\]

11. **Proof of Theorems**

Theorem 2.1 is a more exact statement of Theorem 1.1. Let

\[
\Pi_{\infty} = \bigcap_{m=1}^{\infty} \Pi_m,
\]

and

\[
\Psi_{\infty} = \lim_{m \to \infty} \Psi_0 \circ \Psi_1 \circ \cdots \circ \Psi_m.
\]
By (5.1) and (5.11), one has
\[ \Psi_\infty : T^n \times \Pi_\infty \to T^n \times \Pi_\infty, \]
\[ ||\Psi_\infty - id|| \leq \epsilon^{1/2}, \]
and, by (5.12),
\[ H_\infty = H \circ \Psi_\infty = \sum_{j=0}^{\infty} \langle \Lambda_j^{(1)} u_j, \nu_j \rangle, \]
where \( \Lambda_j^{(1)} = \Lambda_j^{(0)} + Q_j^{(0)} \), and \( Q_j^{(0)} \) is independent of time, \( Q_0 \in \mathbb{R}, Q_j \in gl(\mathbb{R}, 2) \) with \( j \neq 0 \).

This completes the proof of Theorem 2.1.

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