Vortex Penetration into a Type II Superconductor due to a Mesoscopic External Current

Eran Sela and Ian Affleck

Department of Physics and Astronomy, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z1

Applying the London theory we study curved vortices produced by an external current near and parallel to the surface of a type II superconductor. By minimizing the energy functional we find the contour describing the hard core of the flux line, and predict the threshold current for entrance of the first vortex. We assume that the vortex entrance is allowed due to surface defects, despite the Bean-Livingston barrier. Compared to the usual situation with a homogeneous magnetic field, the main effect of the present geometry is that larger magnetic fields can be applied locally before vortices enter the superconducting sample. It is argued that this effect can be further enhanced in anisotropic superconductors.

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I. INTRODUCTION

Surface barrier effects in type II superconductors have been predicted by Bean and Livingston and de Gennes. The entry of flux lines into a planar type II superconductor situated in an external magnetic field $H_{\text{ext}}$ parallel to its surface is opposed by a strong surface barrier when $H_{\text{ext}} = H_{c1}$, the first critical field. Therefore the entry of flux lines could occur at a field value $H_{\text{ext}} = H_S \sim H_{c2} \gg H_{c1}$, where $H_{c2}$ is the second critical field. These surface barrier effects have been observed experimentally in the 60's on lead thallium alloys and on niobium metal, and make it difficult to measure directly thermodynamic properties of the superconductor.

Typically surface barriers are reduced due to surface disorder, which creates large local magnetic fields and allows for nucleation of vortices. Suppression of surface barriers for flux penetration was observed in YBaCuO and in BiSrCaCuO whiskers due to heavy ion irradiation. In ellipsoid-shaped YBaCuO it has been argued that due to roughness of submicrometer order the surface barrier does not push the penetration field $H_S$ above $H_{c1}$ but only lowers the rate of vortex entry.

Another source for the delay of the entry of flux lines into superconductors is the “geometrical barrier” which is particularly important in thin films of constant thickness (i.e., rectangular cross section). This effect is absent only when the superconductor is of exactly ellipsoidal shape or is tapered like a wedge with a sharp edge where flux penetration is facilitated. The resulting absence of hysteresis in wedge-shaped samples was nicely shown by Morozov et al.

In this paper we study another source for delay of entrance of flux lines, due to inhomogeneity of the external magnetic field. In particular we consider magnetic field produced by an external current $I$ flowing parallel to the surface of a type II superconductor, see Fig. 1. The magnetic field produced by the external current enters the sample as curved vortices at sufficiently large current. We find that the entrance of the first line occurs when the induced magnetic field at the surface at the position closest to the wire already exceeds the bulk critical field $H_{c1}$. This delay in entrance of the curved vortices occurs due to geometrical reasons: The entry and outlet points are associated with an energy cost $\sim \frac{\phi_0^2}{I \mu_0}$, where $\lambda$ is the penetration depth, $\phi_0$ is the flux quantum, and $\mu_0$ is the free permeability. Note that $\phi_0 / \mu_0 k_B = 0.2464 \times 10^6 K/\mu m$, implying that in typical superconductors this is a large energy scale. In addition the spatially averaged magnetic field experienced by the vortex is lower than the the maximal one occurring closest to the wire. Considering those effects in an actual calculation we find how large a magnetic field can be applied locally without introducing vortices into the sample.

This implies that application of magnetic field by an external current near the SC can be convenient for experiments demanding sizable magnetic fields in the vortex-free state. As such an experiment we mention the London-Hall effect. Whereas this effect was observed in regular superconductors, it is now interesting to measure it in high temperature superconductors. Typically $H_{c1}$ is quite low in these materials and therefore vortices penetrate the sample at very low homogeneous magnetic fields; hence our geometry can be useful. How-
ever other surface effects seem to be an additional obstacle for the observation of the London effect in high temperature superconductors.\textsuperscript{20}

A parameter which we leave out of consideration in this work is anisotropy of the superconductor, which is particularly important in high temperature layered superconductors. In the case of strong anisotropy additional complications enter the problem even in the case of uniform magnetic field, where the direction of the vortices deviates from the direction of the external magnetic field.\textsuperscript{22} For certain (elliptical) treatment of the short distance cutoff the vortices can have two different directions, corresponding to two degenerate minima in the free energy.\textsuperscript{22}

We argue that strong anisotropy is expected to have important effects in our geometry, increasing further the maximal local magnetic field allowed before curved flux lines penetrate the sample. Consider the case where the $c$ axis of a uniaxially anisotropic superconductor corresponds to the direction $x$ in our geometry, $c \parallel x$. In this case the surface of the superconductor, parallel to the external wire, corresponds to an ab-plane. In the limit of strong anisotropy $\lambda_{ab} \ll \lambda_c$, the bulk critical field parallel to the surface $H_{c1} \approx \frac{\phi_0}{\pi \mu_0 \lambda_{ab} x}$, where $\frac{\phi_0}{\lambda_c}$ becomes very small.

On the other hand, the entry and outlet points of the flux tube deviates from the direction of the external magnetic field. However here we shall concentrate on small currents and a single flux line.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{\textit{\textbf{FIG. 2:}} $\Gamma$ is the axial line of the vortex line. The closed contour $\gamma$ is $\Gamma + \Gamma_1$.}
\end{figure}

In the type II limit, where the coherence length $\xi$ is much shorter than the penetration depth $\lambda_c$, the total free energy at zero temperature is given by\textsuperscript{2}

\begin{equation}
F[\Gamma] = \frac{\mu_0}{2} \int_{r>\xi} d^3r \left[ \tilde{H}^2 + \theta(x)\lambda^2 (\nabla \times \tilde{H})^2 \right] - \frac{\mu_0}{2} \int d^3r \vec{A} \cdot \vec{j}_{\text{ext}}.
\end{equation}

Here $\vec{j}_{\text{ext}} = -I\delta(x+d)\delta(z)\hat{y}$, $I$ is the applied current through the wire, $\vec{A}$ is the vector potential $\vec{H} = \nabla \times \vec{A}$ and $\theta(x)$ is the unit step function. The integral $\int_{r>\xi}$ is carried out in all space outside of the vortex “hard core” $\Gamma$. We assume that the radius of curvature of $\Gamma$ is larger than $\xi$ at any point in $\Gamma$. Note that at $x = 0$ there is an apparent kink in $\gamma$, however this should be thought of as a kink only for length scales large compared to $\xi$.

The corresponding equations for the magnetic field $\vec{H}$ are the Maxwell equation, $\nabla \times \vec{H} = \vec{j}_{\text{ext}}$, for $x < 0$, and the London equation, $(1 - \lambda^2 \nabla^2)\vec{H}(\vec{r}) = \frac{\phi_0}{\mu_0} \int d^3r' \delta(\vec{r} - \vec{r}')$ for $x > 0$. For all $x$ we also have $\nabla \cdot \vec{H} = 0$. In addition we impose appropriate boundary conditions at $x = 0$: the magnetic field is continuous, and no supercurrents flow perpendicular to the surface: $\vec{j}_x = (\nabla \times \vec{H})_x = 0$. To construct a solution we use the functions

\begin{align}
\vec{H}_x^{\text{hom}}(\vec{k}_2, B(\vec{k}_2)) &= \int \frac{d^2k_2}{(2\pi)^2} e^{i\vec{k}_2 \cdot \vec{r}} \\
&\times \begin{cases} A(\vec{k}_2) [-k_2^2 \hat{x} + i\vec{k}_2 \tau(k_2)] e^{-\tau(k_2)x} & x > 0 \\
B(\vec{k}_2) [k_2 \tau(k_2) \hat{x} + i\vec{k}_2 \tau(k_2)] e^{k_2x} & x < 0, \end{cases} \\
\tilde{H}_y(\vec{r}) &= \frac{\phi_0}{\mu_0} \int d^3r' \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} \frac{1}{1 + \lambda^2 k^2}, \\
\tilde{H}_y(x, d') &= \frac{I'}{2\pi} \frac{(-z, 0, x + d')}{(x + d')^2 + z^2}.
\end{align}

\text{II. FORMULATION}

Suppose that a type-II superconductor (SC) occupies the region $x > 0$ and magnetic field is induced by an external current $I$ flowing along a wire of zero cross-section at $(x, z) = (-d, 0)$, as seen in Fig. 1. Our main object under consideration is a curved flux line lying in the plane $y = 0$. Let $\gamma$ denote the closed contour in Fig. 2 consisting of the axial line of the flux line $\Gamma$ and a line $\Gamma_1$ symmetric to $\Gamma$ with respect to the plane $x = 0$, corresponding to an image vortex. Upon further increasing the current a lattice of curved vortices is expected to form along the wire. However here we shall concentrate on small currents and
Here $\vec{k}_2 = k_y \hat{y} + k_z \hat{z}$, $k_2 = \sqrt{k_y^2 + k_z^2}$, and $\tau(k) = \sqrt{k^2 + \lambda^2}$. For any $A(\vec{k}_2), B(\vec{k}_2)$, the function $\vec{H}_\text{hom}$ satisfies the homogeneous equations $\nabla \times \vec{H}_\text{hom} = 0$ for $x < 0$, and $(1 - \lambda^2 \nabla^2) \vec{H}_\text{hom} = 0$ for $x > 0$. The function $\vec{H}_v$ satisfies the London equation $(1 - \lambda^2 \nabla^2) \vec{H}_v(\vec{r}) = \frac{\Phi_0}{\mu_0} \int d\vec{r}' \delta(\vec{r} - \vec{r}')$ in all space. The function $\vec{H}_{v',d'}$ satisfies Maxwell equation $\nabla \times \vec{H}_{v',d'} = \vec{J}_\text{ext}$ for $\vec{J}_\text{ext} = \int \delta(x + d') \delta(z) \hat{y}$ for all space.

Defining the surface 2-dimensional Fourier transform $\vec{H}_{\mu}^{\text{surf}}(\vec{k}_2) = \int dxdy e^{-ik_2 \cdot \vec{r}} \vec{H}_\mu(x, y, z)$, for $\mu = \gamma, \{1', d', \}$, one finds

$$\vec{H}_\gamma^{\text{surf}}(\vec{k}_2) = \frac{\Phi_0}{2\mu_0 \lambda^2} \int d\vec{r}' \frac{e^{-ik_2 \cdot \vec{r}' - \tau(k_2)||\vec{r}'||}}{\tau(k_2)},$$
$$\vec{H}_{1,d}^{\text{surf}}(\vec{k}_2) = \delta(k_2) \pi I e^{-|k_2|d} (i \text{ sgn } k_2, 0, \text{ sgn } d).$$

The solution of the equations satisfying the desired boundary conditions is obtained by adding together the functions in Eq. (2), and solving for $A(\vec{k}_2)$ and $B(\vec{k}_2)$ to give continuity. It is convenient to include an image current at $x = -d$. The total magnetic field is

$$\vec{H} = \vec{H}_0 + \vec{H}_e + \vec{H}_s,$$
$$\vec{H}_0 = \theta(-x)(\vec{H}_{1,d} + \vec{H}_{-1,-d}) + \vec{H}_{s0}, \quad \vec{H}_{s0} = \vec{H}_{\text{hom}}^{\text{A}_0, B_0},$$
$$\vec{H}_e = \theta(x) \vec{H}_\gamma, \quad \vec{H}_s = \vec{H}_{\text{hom}}^{\text{A}_1, B_1},$$

where

$$A_0(\vec{k}_2) = \frac{2[\vec{H}_{1,d}^{\text{surf}}(\vec{k}_2)]_x}{ik_2(\tau(k_2) + k_2)}; \quad B_0(\vec{k}_2) = -A_0(\vec{k}_2) \frac{k_2}{\tau(k_2)},$$
$$A_1(\vec{k}_2) = B_1(\vec{k}_2) = \left[\frac{\vec{H}_{1,d}^{\text{surf}}(\vec{k}_2)}{\vec{k}_2(k_z + \tau(k_2))}\right].$$

In the absence of vortices the magnetic field is given by $\vec{H}_0$. It is plotted in Fig. 3 for $d = 5\lambda$.

![Image of field lines](image)

**FIG. 3:** Field lines of the vortex-free solution $\vec{H}_0(x, z)$ for $d = 5\lambda$ [direction of field lines correspond to anticlockwise rotation around $(x, z) = (-d, 0)$].

The total free energy as function of $\Gamma$ is obtained by substituting the magnetic field Eq. (1) into the free energy Eq. (1). We obtain

$$F = F_0 + F_v + F_s + F_{\text{ext}}.$$

Here

$$F_0 = \frac{\mu_0 I}{2} \int d^3r [\vec{H}_0^2 + \theta(x)\lambda^2 (\nabla \times \vec{H}_0)^2 - 2\vec{A}_0 \cdot \vec{j}_{\text{ext}}],$$
$$F_v = \frac{\mu_0 I}{2} \int d^3r [\vec{H}_v^2 + \theta(x)\lambda^2 (\nabla \times \vec{H}_v)^2], \quad i = v, s,$$
$$F_{\text{ext}} = -\mu_0 \int d^3r (\vec{A}_e \times \vec{A}_s) \cdot \vec{j}_{\text{ext}}.$$

The contour of integration $(x, y, z) = (0, -\infty, 0) \to (0, \infty, 0)$ corresponds to the external current. Physically the wire should be closed into a loop, and we may close the contour of integration e.g. in the $xy$ plane from the $x \to -\infty$ side. Then, using Green’s theorem we obtain

$$F_{\text{ext}} = \mu_0 I \int_{-\infty}^{-d} dx \int_{-\infty}^{\infty} dy [\vec{H}_e(x, y, 0) + \vec{H}_s(x, y, 0)]z.$$
The term \( F \) account for the cutoff we restrict the contour integration \( \mathcal{V}(\mathbf{r}) \) is no need to regulate space in which we search for the minimum of line having 2 equal length sides (\( \pi \lambda \)). In all our calculations \( \xi \) and for the specified currents. We assume that \( \Gamma \) has the reflection symmetry \( z \rightarrow -z \), and plot \( \Gamma \) only for \( z \geq 0 \).

\[
 \mathcal{V}(\mathbf{r}) = e^{ikr}J_1(|r|)F \frac{1}{\pi \lambda} \exp\left(-\frac{|\mathbf{r}|}{\lambda}\right) e^{-\tau(k)r}, \tau(k) = \frac{k}{\tau(k)}J_0(k|\mathbf{r}|),
\]

where \( J_0(x) \) is a Bessel function, and this integral can be done and expressed in terms of other Bessel functions. Note that \( \mathcal{V}(s)(r_x \rightarrow 0, r_z \rightarrow 0) = (2\pi \lambda)^{-1} \), hence there is no need to regulate \( F_s \) with a cutoff.

Different than the usual case with a uniform magnetic field, in our problem the energy \( F = F_0 + F_v + F_s + F_{ext} \) is a function of the contour \( \Gamma \) and is minimized for a particular contour which we need to find. To this end we minimize \( F[\Gamma] \) numerically, approximating \( \Gamma \) by a polyline having 2M equal length sides (\( M = 8 \) in most simulations). We assume that \( \Gamma \) has the reflection symmetry \( z \rightarrow -z \). This leads to a \( M + 1 \)-dimensional parameter space in which we search for the minimum of \( F \). For an example see Fig. (5). In all our calculations \( \xi = 0.001\lambda \).

\[
 f(x_0) = \phi_0 [H_{ext} e^{-x_0/\lambda} - \frac{1}{2} h(2x_0) + H_{c1} - H_{ext}] .
\]

Here \( h(r) = \frac{\phi_0}{2\pi \lambda^2} K_0(\frac{r}{\lambda}) \) is the function giving the field at distance \( r \) of a single straight flux line, \( H_{c1} = \frac{1}{2} h(\xi) \approx \frac{\phi_0}{4\pi \lambda^2} \log \frac{\xi}{\lambda} \), and \( K_0 \) is the zero-order Bessel function.

The term \( \phi_0 H_{ext} e^{-x_0/\lambda} \) describes the interaction of the line with the external field and associated screening currents. It is a repulsive term. The term \( -\phi_0 h(2x_0)/2 \) represents the attraction between the line and its image. When \( H_{0} \sim H_{c1} \) there is a strong barrier opposing the entry of a line. We can understand this barrier as follows: When \( H_{ext} = H_{c1} \), \( f(x_0 = 0) = f(x_0 = \infty) = 0 \). If we start from \( x_0 \) large and bring the line closer to the surface, the repulsive term \( \sim e^{-2x_0/\lambda} \) dominates the image term \( \sim e^{-2x_0/\lambda} \). Thus \( f \) becomes positive and we have a barrier. The barrier disappears, however, in high fields. When \( H > H_{S} = \phi_0/4\pi \xi \), the slope \( \partial f/\partial x_0 \) becomes negative. \( H_{S} \) is of the order of the thermodynamic critical field \( H_{c2} \). The conclusion is that, at field \( H > H_{S} \), the lines cannot enter in an ideal specimen (although their entry is thermodynamically allowed as soon as \( H > H_{c1} \)). However this picture is modified in experiment due to surface inhomogeneities producing local large magnetic fields, and allowing vortices to enter the sample above \( H_{c1} \).
A. Results for wire with zero width

We find a similar energy barrier for the entrance or exit of a curved vortex in our geometry with an external current rather than an homogeneous external magnetic field. This barrier can be visualized in the curves in Fig. 6 (except for the diamonds). Note that typically the barrier height $\Delta$ is of order $\Delta \sim \phi_0^2/\mu_0 \lambda \gg T_c$, where $T_c$ is the critical temperature of the SC. This implies rather small tunneling probabilities $e^{-\Delta/T} \ll 1$ which prevents entry of vortices for clean surfaces. However for strong disorder, vortices can enter more efficiently via nucleation at impurity sites. The contours corresponding to the minimum of the curves with stars and squares are plotted in Fig. 5. In all our calculations $\xi = .001\lambda$.

![Graph showing evolution of surface barrier as function of external current](image)

**FIG. 6:** Evolution of surface barrier as function of external current for $d/\lambda = 10$. When $I < I_{c0}$ (diamonds, $I = 19\phi_0/\lambda$) the force on the line points always towards the surface. When $I_{c0} < I < I_{c1}$ (stars, $I = 22\phi_0/\lambda$) there exists a meta-stable minima with positive energy. When $I_{c1} < I < I_S$ (squares and triangles $I = 26, 80 \times \phi_0/\lambda$) the minimum energy is negative, but a barrier opposes the entry of the flux line. In each point in this plot we have minimized numerically $F$ with respect to the contour $\Gamma$ at fixed $x_0$.

Figure 6 implies the following picture. For infinitesimal current there is no stable vortex configuration. As the current increases we identify three threshold currents $I_{c0} < I_{c1} < I_S$: When the current exceeds $I_{c0}$ a meta-stable minima with $F > 0$ occurs. When the current exceeds $I_{c1}$, the minimum energy changes sign, $F < 0$, but still there is an energy barrier for the entrance of a flux line. When the current exceeds $I_S$ the barrier disappears.

In Fig. 7 we investigated the dependence of $I_{c0}$ and $I_{c1}$ on the distance to the wire $d$. In the limit $d \gg \lambda$ the results for $I_{c1}$ are consistent with the formula $I_{c1} \rightarrow \pi d H_{ext}$ [see diagonal dashed line in Fig. 7]. The behavior of $I_{c0}$ in that limit shows that the region of metastability $I_{c0} < I < I_{c1}$ is very narrow. This behavior appears in sharp contrast to the case of uniform magnetic field even in the limit $d \gg \lambda$: We recall that Eq. (12) predicts metastable solutions for infinitesimal homogeneous magnetic field $H_{ext}$. These states live far from the surface as $H_{ext}$ becomes smaller. This effect is absent in our geometry both due to the fact that the effective external magnetic field created by the wire decays at long distances from the surface and due to the line energy for penetration a long distance into the SC. In the other extreme limit $d \ll \lambda$ we observed from the numerical solution that the contour $\gamma$ can be approximated by a circle centered at the origin. Making this assumption we can calculate $I_{c0}^{\text{circle}} = 10^{6.6981} \phi_0/\lambda$ ($x_0 \sim 0.72\lambda$, $F \mu_0 \lambda/\phi_0^2 = 0.1715$), and $I_{c1}^{\text{circle}} = 10^{0.749} \phi_0/\lambda$ ($x_0 = 1.27\lambda$, $F = 0$) in the limit $d \rightarrow 0$. This approximation is in reasonable agreement with the actual solution as the horizontal dashed lines show.

![Graph showing dependence of threshold currents $I_{c0}$ and $I_{c1}$ on $d/\lambda$.](image)

**FIG. 7:** Dependence of threshold currents $I_{c0}$ and $I_{c1}$ on $d/\lambda$.

The shape of the contour changes as function of $d$. In Fig. 5 we plot the extension of the contour in the $x$ and $z$ directions. We fitted the numerical results for $x_0$ with an empirical formula $x_0/\lambda = c + \log(d/\lambda)$ with $c \sim 1$, implying that the penetration of the vortex is of order $\lambda$ for all $d$. On the other hand it appears that $z_0$ grows linearly as function of $d$. In the limit $d \rightarrow 0$ we have $x_0/\lambda \rightarrow 1.26$ and $z_0/\lambda \rightarrow 1.43$ (implying that the circular contour is only an approximation).

![Graph showing extensions of curved flux line along $x$ and $z$](image)

**FIG. 8:** Extensions of the curved flux line along $x$ and $z$ as function of $d/\lambda$, at $I = I_{c1}$.

For disordered surfaces, the present geometry can be useful for application of large magnetic fields on a SC sample in a vortex free state. The maximal magnetic field that can be applied in a vortex free state using the wire...
geometry is $\tilde{H}_0[x = z = 0]$ [see Eq. (1)] at current $I_{c1}$. In the limit $d \gg \lambda$ this magnetic field coincides with the bulk first critical field $H_{c1} \approx \frac{\phi_0}{4\pi \mu_0 \lambda} \log(\lambda/\xi)$, however at smaller $d$ the magnetic field at the surface increases. This is shown in Fig. (1) where we plot the magnetic field $H_{\text{surface}} = [H_0(0, 0, 0)]_z = [\tilde{H}_0(0^+, 0, 0)]_z$ given in Eq. (1) at the current $I_{c1}$, which we calculated above as function of $d$. Note that the field enhancement is small for $d > 3\lambda$ ($H_{\text{surface}} \approx 2H_{c1}$ for $d = 3\lambda$).

We turn to an estimation of the threshold current $I_S$ at which the barrier disappears. A more precise calculation would involve the Ginzburg-Landau theory. We follow the above analysis of $H_S$. Since the London theory is applicable at distances $\gg \xi$ we estimate $I_S$ using

$$\frac{\partial F}{\partial x_0} \bigg|_{x_0 \sim \xi} = 0. \quad (13)$$

We find numerically that at $x_0 \ll \lambda$ the closed contour $\gamma$ is well approximated by a circle with radius $x_0$ centered at $z = 0$. In the limit $x_0 \sim \xi \ll \lambda$ we can evaluate the functional $F$ analytically as function of $x_0$. In Eq. (13) for $F_v$ we can set $\exp(-|x - x'|/\lambda) \rightarrow 1$, hence

$$F_v(x_0) \sim \frac{\phi_0^2 x_0}{32\pi \mu_0 \lambda^2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \frac{\cos(\theta_1 - \theta_2)}{|\sin(\frac{\theta_1 - \theta_2}{2})|} \times \theta(2x_0) \left| \frac{\theta_1 - \theta_2}{2} - \xi \right|. \quad (14)$$

Compared to $F_s$, the stray term is negligibly small, $F_s \sim \frac{\phi_0^2}{\mu_0 \lambda} (\frac{x_0}{\lambda})^2$. The interaction energy with the external current reads

$$F_{\text{ext}}(x_0) = -\frac{I \phi_0 x_0^2}{2\lambda^2} \tilde{f}(d/\lambda), \quad \tilde{f}(x) = \int_0^\infty dy e^{-y^2} \left( \sqrt{y^2 + 1} - y \right). \quad (15)$$

Using these formulas for $F_v$ and $F_{\text{ext}}$ we obtain from Eq. (13) the estimate

$$I_S \sim \frac{\phi_0}{8\mu_0 \xi \tilde{f}(d/\lambda)}. \quad (16)$$

The dependence of $I_S$ on $d/\lambda$ is hidden in the function $\tilde{f}(x)$, with $\tilde{f}(x) \rightarrow x^{-1}$, $\tilde{f}(x) \rightarrow \log(x^{-1/2}) + c$, where $c \sim 0.3$. For $d \gg \lambda$ we have $I_S \sim \frac{\phi_0}{\mu_0 \lambda}$. In this case the magnetic field due to the external current at $x = z = 0$ is $[H_0(0^+) = 0]_z \rightarrow \frac{\phi_0}{\xi}$. It is of the order of the second critical field $H_{c2}$. In the other limit $d \ll \lambda$ we have $I_S \sim \frac{\phi_0}{4\mu_0 \xi \log(\xi/\lambda)}$. Note that this behavior holds for $\xi \ll d \ll \lambda$. In this regime we have $I_S \gg I_{c1} \sim \frac{\phi_0}{\mu_0 \lambda}$. In Fig. (10) we plot the phase diagram.

B. Finite wire cross section

Experimentally the wire carrying the external current has a finite cross section. Therefore it is important to include this effect in our calculations. Consider the rectangular cross section as shown in Fig. (11), and assume current $I$ flows uniformly in this cross section. We can write the external current as

$$\tilde{j}_{\text{ext}} = -\frac{I}{x_w} \int_{d - \frac{z}{x_w}}^{d + \frac{z}{x_w}} d\tilde{d} \int_{-\frac{\tilde{z}}{x_w}}^{\frac{\tilde{z}}{x_w}} d\tilde{z} (x - \tilde{d}) \delta(z - \tilde{z}) \hat{y}. \quad (17)$$

The modification to the magnetic field $\tilde{H} = \tilde{H}_0 + \tilde{H}_v + \tilde{H}_s$ occurs only in the first term,

$$\tilde{H}_0(\tilde{r}) = \frac{1}{x_w} \int_{d - \frac{z}{x_w}}^{d + \frac{z}{x_w}} d\tilde{d} \int_{-\frac{\tilde{z}}{x_w}}^{\frac{\tilde{z}}{x_w}} d\tilde{z} \left[ \tilde{H}_0(x, y, z - \tilde{z}) \right]_{d - \tilde{d}},$$

where $\tilde{H}_0(x, y, z)$ is given in Eq. (11). Next we focus on the modification of the vortex dependent part of the free
energy $F = F_e + F_s + F_{ext}$. Only the term $F_{ext}$ is modified. Using Eq. (17) it is easy to find that

$$F_{ext} \rightarrow -\frac{I\phi_0}{\pi} \int_{z}^{z+w} d\zeta \int_{0}^{\infty} dk e^{-kd} \cos(kr_z)(1-e^{-\tau(k)r_x}) \times$$

$$\times \left(1 - \frac{k}{\tau(k)} \left(\frac{\sin k\frac{z+w}{2}}{\frac{k^2}{4}} - \frac{\sin k\frac{z-w}{2}}{\frac{k^2}{4}}\right)\right).$$

Let us first specialize to the case of square cross section where the wire touches the SC, $x_w = z_w = 2d$, and compare this with a point like cross section $x_w = z_w \rightarrow 0$ which we considered until now (We ignore any electron or Cooper pair tunneling between the SC and wire). First we repeated the calculation of $I_{c0}$ and $I_{c1}$. The results are roughly the same for both cross sections for $d < \lambda$, and deviations up to 10% are obtained for $d > \lambda$ up to $d = 100\lambda$. In Fig. (12) we compare the contours at $I_{c1}$ as function of $d$ for the two cross sections. We can see that $z_0$ changes by a factor of up to 1.6 for $d \leq 90\lambda$. The

$z_w \rightarrow \infty$ can be treated analytically since the external field $\vec{H}_0$ is uniform at all $x > (d + \frac{z_w}{2})$. In this limit the maximal magnetic field at the surface before vortices penetrate is $H_{surface}(I_{c1}) \rightarrow H_{c1}$. For finite $z_w$ we calculated $H_{surface}(I_{c1})$ numerically with the result plotted in Fig. (14). Accordingly $z_w$ should not be too large in order to obtain the effect discussed here including the enhancement of the surface field in the vortex free state for a disordered surface.

In this work we studied solutions of London theory in a geometry where an external mesoscopic current flows parallel to a surface of a SC. Only above a threshold current $I_{c0}$ there exist solutions with curved flux lines entering and leaving the SC at the surface. At a larger threshold current, $I_{c1}$, these solutions become energetically favorable, however an energy barrier separates them from a vortex free solution. At a third threshold current, $I_{c2}$, this barrier disappears. To determine the current at which vortices actually penetrate the sample one has to account for the degree of disordered of the surface. For strong surface disorder the vortex can penetrate at

![FIG. 11: Rectangular wire cross section.](image1)

![FIG. 12: Dependence on $d$ of the lengths $x_0$ and $z_0$, characterizing the vortex contour, for zero versus finite wire cross section [$x_w = z_w = 0$ (stars) and $x_w = z_w = 2d$ (squares)].](image2)

![FIG. 13: Magnetic field at the surface (see definition in the text) just before the entry of the first vortex at $I \rightarrow I_{c1}$, for either zero wire cross section ($x_w = z_w = 0$, stars) or finite cross section ($x_w = z_w = 2d$, squares).](image3)

![FIG. 14: Magnetic field at the surface at current approaching $I_{c1}$ as function of $z_w$, for $x_w = 2d$, $d = \lambda/2$.](image4)

**IV. CONCLUSIONS**

In this work we studied solutions of London theory in a geometry where an external mesoscopic current flows parallel to a surface of a SC. Only above a threshold current $I_{c0}$ there exist solutions with curved flux lines entering and leaving the SC at the surface. At a larger threshold current, $I_{c1}$, these solutions become energetically favorable, however an energy barrier separates them from a vortex free solution. At a third threshold current, $I_{c2}$, this barrier disappears. To determine the current at which vortices actually penetrate the sample one has to account for the degree of disordered of the surface. For strong surface disorder the vortex can penetrate at
we have \( F_{(H_0, H_s)} = \mu_0 \int d^3 r (H_0 \cdot \vec{H}_s + \theta(x) \lambda^2 (\vec{\nabla} \times H_0) \cdot (\vec{\nabla} \times H_s)) = 0 \). (A1)

From Eq. (A1) we have \( \vec{H}_0 = \vec{H}_0 + \vec{H}_{s0} \) where \( \vec{H}_0 = \theta(-x)(\vec{H}_{l,d} + \vec{H}_{l,-d}) = \vec{\nabla} \times \vec{A}_1 \) and

\[
\vec{A}_1 = \frac{I \hat{y}}{4\pi} \log \frac{(x + d)^2 + z^2}{(x - d)^2 + z^2}, \quad x < 0.
\]

(A2)

Correspondingly, we have \( F_{(H_0, H_s)} = F_{(H_0, H_s)} + F_{(H_{s0}, H_s)} \). Consider the term \( F_{(H_0, H_s)} = \mu_0 \int_{x < 0} d^3 r \vec{H}_0 \cdot \vec{H}_s \). We will use the vector identity \((\vec{\nabla} \times \vec{A}) \cdot \vec{B} = \vec{A} \cdot (\vec{\nabla} \times \vec{B}) + \vec{\nabla} \cdot (\vec{A} \times \vec{B})\), with \( \vec{A} = \vec{A}_1, \vec{B} = \vec{H}_s \), and the fact that \( \vec{\nabla} \times \vec{H}_s = 0 \). Then the volume integral can be transformed to an integral on the surface \( x = 0^+ \). However this integral vanishes because \( \vec{A}_1(0^-, y, z) = 0 \), hence \( F_{(H_0, H_s)} = 0 \). Now let us consider the term \( F_{(H_{s0}, H_s)} \) and define \( \vec{H}_s = \vec{\nabla} \times \vec{A}_s \). For the integral in the region \( x < 0 \) we use the above vector identity with \( \vec{A} = \vec{A}_s, \vec{B} = \vec{H}_{s0} \), and for \( x > 0 \) we use the vector identity with \( \vec{A} = \vec{H}_{s0}, \vec{B} = \vec{\nabla} \times \vec{H}_s \). Taking into account that \( \vec{H}_s \) and \( \vec{H}_{s0} \) satisfy the homogeneous equations we obtain

\[
\int_{x < 0} d^3 r (\vec{\nabla} \times \vec{A}_s) \cdot \vec{H}_{s0} = \int dS (\vec{A}_s^- \times \vec{H}_{s0}^-),
\]

\[
\int_{x > 0} d^3 r (\vec{H}_{s0} \cdot \vec{H}_s + \lambda^2 (\vec{\nabla} \times \vec{H}_{s0}) \cdot (\vec{\nabla} \times \vec{H}_s)) = -\lambda^2 \int dS (\vec{H}_{s0}^+ \times (\vec{\nabla} \times \vec{H}_s^+)).
\]

(A3)

Here \( dS = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} d z, \) and \( \vec{H}_s^\pm = \vec{H}(x = 0^\pm, y, z) \). For \( x > 0 \) we can use \( \vec{A}_s = -\lambda^2 \vec{\nabla} \times \vec{H}_s \), which follows from London equation for \( \vec{H}_s \). Next we use the fact that, by construction, \( \vec{H}_s = \vec{H}_{s0}^+ + \vec{H}_e^+ \). This allows us to express \( \vec{A}_s^- = -\lambda^2 \vec{\nabla} \times (\vec{H}_{s0}^+ + \vec{H}_e^+) \), and combine the two equations of Eq. (A3) as

\[
F_{(H_{s0}, H_s)} = -\mu_0 \lambda^2 \int dS \left[ (\vec{\nabla} \times \vec{H}_{s0}^-) \times (\vec{H}_{s0}^+ - \vec{H}_0^+) + (\vec{\nabla} \times \vec{H}_e^+) \times \vec{H}_{s0}^+ \right] x.
\]

(A4)

Now we use explicit forms of these factors: \((\vec{H}_{s0}^+ - \vec{H}_0^+) = \frac{Id_\gamma}{z + \lambda z^2}, (\vec{H}_{s0}^-) = 0 \),

\[
(\vec{\nabla} \times \vec{H}_{s0}^+) = \frac{\phi_0}{2\mu_0 \lambda^2} \int d^3 r \int \frac{d^2 k_2}{(2\pi)^2} e^{-i k_2 \cdot \vec{r} + i(k_2 y + k_2 z) - i(k_2)(\vec{r} \cdot \hat{z})} \frac{-i k_2 (\tau(k_2) - k_2)}{k_2 \tau(k_2)},
\]

\[
(\vec{\nabla} \times \vec{H}_e^+) = \frac{\phi_0}{2\mu_0 \lambda^2} \int d^3 r \int \frac{d^2 k_2}{(2\pi)^2} e^{-i k_2 \cdot \vec{r} + i(k_2 y + k_2 z) - i(k_2)(\vec{r} \cdot \hat{z})} \frac{i k_2}{\tau(k_2)} \left( 1 - \frac{\tau^2(k_2)}{k_2^2} \right),
\]

\[
(\vec{H}_{s0}^+) = -I \int \frac{d^2 k_2}{(2\pi)^2} e^{ik_2 \cdot \vec{r} - |k_2|d k_2 2\pi \delta(k_y)} \frac{\tau(k_2)}{\tau(k_2) + k_2}.
\]

(A5)

Plugging these expressions in Eq. (A3), one can readily obtain \( F_{(H_{s0}, H_s)} = 0 \) (without performing any integration), completing the proof for \( F_{(H_0, H_s)} = 0 \).
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