A refinement of Shor’s Algorithm for determining order is introduced, which determines a divisor of the order after any one run of a quantum computer with almost absolute certainty. The information garnered from each run is accumulated to determine the order, and for any $k$ greater than 1, there is a guaranteed minimum positive probability that the order will be determined after at most $k$ runs. The probability of determination of the order after at most $k$ runs exponentially approaches a value negligibly less than one, so that the accumulated information determines the order with almost absolute certainty. The probability of determining the order after at most two runs is more than 60%, and the probability of determining the order after at most four runs is more than 90%.

1 Introduction

In quantum computing, there are a few algorithms which can be performed more efficiently than their most efficient known classical counterparts. One such example is Grover’s algorithm which improves the efficiency of searching an unsorted list to the order of the theoretical limit of efficiency, at a cost of $O(\sqrt{N})$, where $N$ is the length of the list (see for example, [1, 2]). Another example is supplied by Shor’s algorithms for determining order and for determining discrete logarithms, both of which can be performed in polynomial time with the aid of both a quantum computer and a classical computer. A consequence of the fact that Shor’s algorithm determines order in polynomial time is that composite numbers can be factorized in polynomial time. Since Shor’s algorithms aid in factorizing composite numbers and in solving the discrete logarithm problem, both in polynomial time, then their implementation on a quantum computer would challenge the security of many of today’s cryptographic algorithms (e.g. RSA, ElGamal, DSA, ECC).

Shor’s original algorithm had the property that the number of runs on the quantum computer needed to determine the order of $x$ modulo $n$ was $O(\log \log n)$. In Knill’s modification [3], the probability of success was improved, but on any single run of the quantum computer, the probability that the value output by
the computers would be a divisor of the order may still be significantly less than 1. Knill did, however, introduce the concept of accumulating information from various runs of the quantum computer.

It is the purpose of this paper to refine the algorithm to the point that after any one run on the quantum computer, the probability that the value output by the computers is a divisor of the order is negligibly less than 1. When this refinement is combined with the accumulation of information, as discussed above, the number of required runs on the quantum computer is reduced to $O(1)$ (assuming ideal working of the quantum computer, including extra demands on the Quantum Fourier Transform). The refinement to the algorithm is introduced in §4, and it is demonstrated in §8 that the probability of finding the required order with not more than $k$ runs on the quantum computer is greater than $1/e(\zeta) - O(n^{-\epsilon})$ in the asymptotic limit as $n \to \infty$, where $\zeta$ is the Riemann zeta function, $\epsilon$ is a positive number, and the statement $f > g - O(h)$ means that there exists a function $F$ such that $f > g - F$ in the asymptotic limit, and $F/h$ is bounded in the same limit.

The refinement is effected by increasing the number of qubits in the first register by a factor of about 1.5, thus increasing the requirements of space and time on the quantum computer by a constant factor, and increasing the accuracy required in performing the Quantum Fourier Transform on the first register.

In §4, the modular metric, which measures distances between elements of $\mathbb{Z}/q\mathbb{Z}$ is introduced for all $q$. The purpose for introducing the modular metric is in order to obtaining a proper and invariant concept of proximity.

In §5, Shor’s original algorithm is discussed.

In §4, a refinement of Shor’s algorithm is introduced in which each run of the quantum computer determines a divisor of the required order with almost absolute certainty, and the number of required runs on the quantum computer is $O(1)$.

In §5, an analysis of the probabilities of the measured value of the first register falling in some specific subsets of $\{0, 1, \ldots, q - 1\}$ is given.

In §6, some facts about continued fractions (which are used in the classical part of the algorithm to determine information about the order) are given, with a new result determining sufficient conditions to guarantee that the classical part of the algorithm will yield a divisor of the required order.

In §8, the results of §6 and §8 are united to demonstrate that the refinement guarantees, with probability negligibly less than 1 that each run of the quantum yields a divisor of the required order, and the Section also specifies sufficient information to determine approximate probabilities for each divisor.

In §8, an idealized version of the probability distribution is investigated in order to determine the probability that the order will be known after at most $k$ runs of the quantum computer.

In §8, the properties of the idealized probability distribution are modified to the more concrete distribution associated with the refinement of Shor’s Algorithm.
2 Modular Metrics

For \( q \in \mathbb{Z}, q > 1 \), let \( I_q = \{0, 1, \ldots, q-1\} \).

**Theorem 2.1**

For \( q \in \mathbb{Z}, q > 1 \), define \( \rho_q : I_q \times I_q \to \mathbb{Z} \) by

\[
\rho_q(x, y) = \min(|x - y|, q - |x - y|),
\]

then \( \rho_q \) is a metric on \( I_q \), and \( \rho_q(x, y) \leq \frac{q}{2} \) for all \( x, y \in I_q \).

This is proven in Appendix A.

The modular metric \( \rho_q \) is equivalent to a metric \( s_q \) on \( \mathbb{Z}/q\mathbb{Z} \) determined by the smallest distance between representatives of the respective cosets:

\[
s_q(\bar{x}, \bar{y}) = \min\{|x - y| : x \in \bar{x}, y \in \bar{y}\},
\]

for \( \bar{x}, \bar{y} \in \mathbb{Z}/q\mathbb{Z} \).

The modular metric gives a distance function on \( \{0, 1, \ldots, q - 1\} \) which is invariant under cyclic symmetries, and can be thought of an arc length on a circle around which the elements have been evenly spaced.

3 Shor’s Algorithm

The purpose of the quantum part of Shor’s Algorithm is to determine the order \( r \) of \( x \) modulo \( n \), where \( 0 < x < n \), and \( x \) and \( n \) are relatively prime, in other words, \( r \) is the smallest positive integer such that \( x^r \equiv 1 \mod n \) (note that \( 0 < r < n \)). In Shor’s paper, this was achieved in the following manner.

1. The state vector of the system is set to an initial state of

\[
|\psi_0\rangle = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a\rangle \otimes |0\rangle,
\]

where \( q \) is an appropriate power of 2 (the first register is composed of \( l \) qubits, where \( q = 2^l \)). In Shor’s paper, \( q \) is taken to be that unique power of 2 such that \( n^2 \leq q < 2n^2 \). The state vector \( |\psi_0\rangle \) arises from the state \( |\phi_0\rangle = |0\rangle \otimes |0\rangle \) by taking a quantum Fourier transform on the first register, or alternatively by applying a gate of \( H^\otimes l \) to the first register (so \( H \) is applied individually to each qubit), where \( H \) is the Hadamard gate.
2. The next step is to perform a modular exponentiation, so that $|\psi_0\rangle$ is mapped to

$$|\psi_1\rangle = \frac{1}{q^{\frac{r}{2}}} \sum_{a=0}^{q-1} |a\rangle \otimes |x^a \mod n\rangle$$

$$= \frac{1}{q^{\frac{r}{2}}} \sum_{k=0}^{r-1} \sum_{b=0}^{q-1-k} |br + k\rangle \otimes |x^{br+k} \mod n\rangle$$

$$= \frac{1}{q^{\frac{r}{2}}} \sum_{k=0}^{r-1} \sum_{b=0}^{q-1-k} |br + k\rangle \otimes |x^k \mod n\rangle,$$

where for real $y$, $\lfloor y \rfloor$ is the greatest integer less than or equal to $y$. The final equality follows from the fact that $r$ is the order of $x$ modulo $n$.

3. The next step is to take the quantum Fourier transform on the first register, so that the state becomes

$$|\psi_2\rangle = \frac{1}{q^{\frac{r}{2}}} \sum_{c=0}^{q-1} \sum_{a=0}^{q-1} \exp \left( \frac{2\pi i ac}{q} \right) |c\rangle \otimes |x^a \mod n\rangle$$

$$= \frac{1}{q^{\frac{r}{2}}} \sum_{c=0}^{q-1} \sum_{k=0}^{r-1} \sum_{b=0}^{q-1-k} \exp \left( \frac{2\pi i (br + k)c}{q} \right) |c\rangle \otimes |x^k \mod n\rangle.$$

4. The final step is to measure the value $c$ of the first register. The value of $c$ is then input into a classical computer (which already has values for $q$ and $n$), and a value for the fraction $d'/r'$ satisfying the following conditions is found:

- $d'/r'$ is in lowest terms ($d'$ and $r'$ have no common factors);
- $0 \leq d'/r' \leq 1$;
- $0 < r' < n$;
- $d'/r'$ is the nearest fraction to $c/q$ which satisfies the other three conditions.

This is done with the use of continued fractions.

Shor noted that the probability that $c$ ($c \in \mathbb{Z}, 0 \leq c < q$) is some given value which varies from an integral multiple of $\frac{q}{2}$ by at most $\frac{1}{2}$ (this is equivalent to equation (5.11) of Shor’s paper [4]), and that the value of the second register is $x^k \mod n$ for some given $k$, is greater than $\frac{1}{q^{r^2}}$. This observation can be formally expressed as follows: let $X$ be the random variable denoting the result of the measurement of the first register, and let $Y$ be the random variable denoting the result of a measurement of the second register, then for any given $d = 0, 1, \ldots, r-1$ and $k = 0, 1, \ldots, r-1$,

$$P \left( \rho_q \left( X, \frac{dq}{r} \right) \leq \frac{1}{2} \text{ and } Y = x^k \mod n \right) > \frac{1}{3r^2}.$$

It follows that the probability that $c$ is the value as given above is greater than $\frac{1}{4r}$, and so the probability that there exists an integer $d$ such that $0 < d < r$, $d$ is relatively prime to $r$ ($d$ and $r$ have no common
factor), and \( c \) differs from \( \frac{dr}{r} \) by at most \( \frac{1}{2} \), is greater than \( \phi(r)/(3r) \), where \( \phi \) is Euler’s totient function, defined by

\[
\phi(r) = \#\{d \in \mathbb{Z} : 0 < d < r, \ d \text{ and } r \text{ are relatively prime}\}.
\]

A formula for \( \phi \) is given by

\[
\phi(r) = r \prod_{p \text{ prime, } p|r} \left(1 - \frac{1}{p}\right).
\]

The requirement that \( d \) and \( r \) be relatively prime comes from the fact that the only information about \( d/r \) that can be be derived from \( c \) is its expression in lowest terms (no common factor for numerator and denominator), so that in order for the denominator to be the order of \( x \), \( d \) and \( r \) can have no common factors. Shor used the theorem that \( \phi(r)/r > \delta_1/\log \log r \) for some \( \delta_1 \) to yield the result that the probability above is greater than \( \delta/\log \log r \), for some \( \delta \), so that the number of trials required on the quantum computer is \( O(\log \log n) \).

4 Refinement of Shor’s Algorithm

The refinement of Shor’s Algorithm to be introduced in this paper incorporates a modification of the value of the parameter \( q \), and an accumulation of information in a similar manner to that suggested by Knill.

Take a positive real number \( \epsilon \), and let \( w = n^\epsilon \). Under the refinement, the algorithm for determining \( r \) is as follows. All steps except step 2 are performed on a classical computer.

1. Set \( s := 1 \) and \( q \geq 2wn^3 \) (e.g. set \( q \) to be that unique power of 2 such that \( 2wn^3 \leq q < 4wn^3 \));
2. Perform the quantum algorithm on the quantum computer with \( q \) as specified in Step 1, and measure the value \( c \) of the first register;
3. Determine the continued fraction expansion for \( \frac{c}{q} \);
4. Determine all denominators of convergents of the continued fraction expansion up to the first denominator greater than or equal to \( n \);
5. Let \( r' \) be the last denominator less than \( n \), and set \( s := \text{lcm}(s, r') \);
6. Calculate \( x^s \mod n \);
7. If \( x^s \not\equiv 1 \mod n \), then go to Step 2;
8. Output \( s \).

Note that the algorithm accumulates the information garnered from each measurement of \( c \). Note also that only the denominators of the convergents are calculated. There is no need to calculate their numerators.
For the size of $n$ that would be typically used in RSA encryption, the algorithm above determines the order $r$ with probability negligibly less than one (the probability of determining a nontrivial multiple of $r$ instead of the correct value is $O(n^{-1})$). The probability that the correct value of $r$ will be found after at most 2 runs of the quantum computer is at least 60%, the probability after at most 4 runs is at least 90%, the probability after at most 6 runs is at least 98%, and the probability after at most 8 runs is at least 99.5%.

The rest of the paper is devoted to analysing the above algorithm in order to demonstrate the properties claimed for it.

5 Proportabilities of specified values for the first register

The essential feature of the profile of probabilities of the measured value $c$ of the first register is that when $q \gg r$, then the probability concentrates in the vicinities of $\frac{dq}{r}$, where $d$ is an integer with a probability of about $\frac{1}{r}$ in each vicinity. Further, if $\frac{dq}{r}$ is an integer, then the full probability of $\frac{1}{r}$ effectively concentrates itself at $c = \frac{dq}{r}$, and if $\frac{dq}{r}$ is not an integer, then the probability of $c$ in the vicinity of $\frac{dq}{r}$ is essentially inversely proportional to $(c - \frac{dq}{r})^2$. It follows that for $q \gg r$, the only dependence that the probability profile in the vicinity of $\frac{dq}{r}$ has on $q$ is on the fractional part of $\frac{q}{r}$ (i.e. the full set of profiles is determined completely by the fractional part of $\frac{q}{r}$). This means qualitatively that as $q$ increases, the concentrated areas of probability recede from each other, but the individual profiles do not “spread”. These observations are made more rigourous in this Section.

All results presented in section without proof will be proven in Appendix B.

Since Shor’s Algorithm relies on measuring the value in the first register, and then entering the result of the measurement into the classical computer, then it is useful to have information about the probability distribution for the values taken by the first register in order to determine the probabilities of various outputs of the classical computer.

The parameter $q$ will now be taken to be an arbitrary positive integer, and a measurement of the first register will be taken when the computer is in the state

$$|\psi_2\rangle = \frac{1}{q} \sum_{c=0}^{q-1} \sum_{k=0}^r \sum_{b=0}^{\left\lfloor \frac{q-1-k}{r} \right\rfloor} \exp \left( \frac{2\pi i (br + k)c}{q} \right) |c\rangle \otimes |x^k \mod n\rangle.$$  

Note that $|\psi_2\rangle$ is the final form of the state vector before measurement in the quantum algorithm in Shor’s algorithm. The parameter $q$ is generally taken to be a power of 2 as a result of the requirement of the usage of qubits in the quantum algorithms for addition, multiplication and modular exponentiation. Modification to qudits (with a higher number of levels) of the algorithms for addition, multiplication and modular exponentiation will allow for a wider range of values for $q$. Also, $q$ is typically taken to be larger than $n$, although the results below are true for all possible values of $q$.

Let $X$ be the random variable describing the result of the measurement of the first register in the final step of the algorithm on the quantum computer, then $X$ must take the value of an integer between 0 and
$q - 1$, inclusive, and for $0 \leq c \leq q - 1$, the probability that $X = c$ is given by $P(X = c) = \langle \chi_c | \chi_c \rangle$, where

$$|\chi_c\rangle = \frac{1}{q} \sum_{k=0}^{r-1} \sum_{b=0}^{\lfloor \frac{q-1-k}{r} \rfloor} \exp\left(\frac{2\pi i (br + k)c}{q}\right) |x^k \mod n\rangle,$$

and so, since $x^k \not\equiv x^{k'} \mod n$ for $k$ and $k'$ such that $0 \leq k < r$, $0 \leq k' < r$ and $k \neq k'$, then

$$P(X = c) = \frac{1}{q^2} \sum_{k=0}^{r-1} \left| \sum_{b=0}^{\lfloor \frac{q-1-k}{r} \rfloor} \exp\left(\frac{2\pi i (br + k)c}{q}\right) \right|^2,$$

where the last equality is obtained by substituting $r - 1 - k$ for $k$.

If $\frac{c}{q} \in \mathbb{Z}$, then

$$\exp\left(\frac{2\pi i cr}{q}\right) = 1,$$

so that

$$P(X = c) = \frac{1}{q^2} \sum_{k=0}^{r-1} \left| \sum_{b=0}^{\lfloor \frac{q+k}{r} \rfloor - 1} 1 \right|^2$$

(2)

On the other hand, if $\frac{c}{q} \notin \mathbb{Z}$, then

$$P(X = c) = \frac{1}{q^2} \sum_{k=0}^{r-1} \left| \sum_{b=0}^{\lfloor \frac{q+k}{r} \rfloor - 1} \exp\left(\frac{2\pi i bc}{q}\right) \right|^2.$$
\[ = \frac{1}{q^2} \sum_{k=0}^{r-1} \left| \frac{\exp \left( \frac{2\pi icr}{q} \left\lfloor \frac{q+k}{r} \right\rfloor \right) - 1}{\exp \left( \frac{2\pi icr}{q} \right) - 1} \right|^2 \]
\[ = \frac{1}{q^2} \sum_{k=0}^{r-1} \sin^2 \left( \frac{\pi cr}{q} \left\lfloor \frac{q+k}{r} \right\rfloor \right) \frac{\sin^2 \frac{2\pi cr}{q}}{\sin^2 \frac{2\pi cr}{q}}. \] 

The second equality above follows from the evaluation of the geometric progression

\[ \sum_{b=0}^{\left\lfloor \frac{q+k}{r} \right\rfloor - 1} \exp \left( \frac{2\pi ibcr}{q} \right) = \frac{\exp \left( \frac{2\pi icr}{q} \left\lfloor \frac{q+k}{r} \right\rfloor \right) - 1}{\exp \left( \frac{2\pi icr}{q} \right) - 1}. \]

Much, if not all, of this is already known (e.g. page 17 of [3]).

If \( \frac{q}{r} \in \mathbb{Z} \), then

\[ P(X = c) = \begin{cases} \frac{1}{r}, & \frac{cr}{q} \in \mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases} \]

so that \( c \) is guaranteed to be a multiple of \( \frac{q}{r} \). Since \( c = \frac{dq}{r} \) for some integer \( 0 \leq d < r \), then \( c/q \) is guaranteed to be equal to \( d/r \) for some \( 0 \leq d < r \), and all values of \( d \) occur with equal probability \( 1/r \).

### 5.1 The Case That \( q/r \) is not an Integer

The case where \( \frac{q}{r} \notin \mathbb{Z} \) is more difficult.

In the case that \( 1 \leq q \leq r \), then

\[ P(X = c) = \frac{1}{q} \]

for all \( c \), so that all possible values of the first register occur with equal probability, and so no useful information can be obtained, as the behaviour is independent of \( r \geq q \).

Since no useful information can be obtained if \( q \leq r \), then from now it will be assumed that \( q > r \).

If \( \frac{q}{r} \in \mathbb{Z} \), then

\[ \frac{1}{r} - \frac{2}{q} + \frac{r}{q^2} < P(X = c) < \frac{1}{r} + \frac{2}{q} + \frac{r}{q^2}. \]
Note that $P(X = c) = \frac{1}{r} + O\left(\frac{1}{q}\right)$. If $q$ is much larger than $r$, then it follows that $P(X = c)$ is very close to $\frac{1}{r}$.

Suppose $\frac{c r}{q} \notin \mathbb{Z}$, then

$$P(X = c) \leq \frac{r}{q^2 \sin^2 \frac{\pi c r}{q}}.$$  \hspace{1cm} (5)

This gives an upper bound for $P(X = c)$, and demonstrates that as the distance between $c$ and the nearest integral multiple of $\frac{c}{q}$ increases, the maximum possible probability that $X = c$ decreases. Specifically, the measured value of the first register is more likely to be in the neighbourhood of some multiple of $\frac{c}{q}$ than it is not to be in any such neighbourhood.

Suppose that $c = \frac{dq}{r} + \Delta$, where $d \in \mathbb{Z}$ and $0 < |\Delta| \leq \frac{q}{2r}$, so that

$$P(X = c) \leq \frac{r}{q^2 \sin^2 \left(\frac{\pi d}{q} + \frac{\pi \Delta r}{q}\right)}$$

by straightforward substitution for $c$ in (5).

If $\frac{dq}{r} \notin \mathbb{Z}$, then $\Delta \notin \mathbb{Z}$, so that

$$P(X = c) < \frac{r}{q^2}.$$  \hspace{1cm} (6)

Since $P(X = c) = O\left(\frac{1}{q^2}\right)$, then for $q$ much larger than $r$, $P(X = c)$ is approximately equal to zero.

If $\frac{dq}{r} \notin \mathbb{Z}$, so that $\Delta \notin \mathbb{Z}$, then

$$P(X = c) < \frac{1}{\left(1 - \frac{\sin^2 \frac{\pi \Delta r}{q}}{\sin^2 \frac{\pi d q}{r}}\right)^2} \left(\frac{\sin^2 \frac{\pi dq}{r}}{\pi^2 \Delta^2 r^2} + \frac{2 |\sin \frac{\pi dq}{r}|}{\pi |\Delta| q} + \frac{r}{q^2}\right).$$ \hspace{1cm} (7)

Further, if $|\Delta| \leq \frac{q}{2r} |\sin \frac{\pi dq}{r}|$, then

$$P(X = c) > \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2 \Delta^2 r^2} - \frac{2 |\sin \frac{\pi dq}{r}|}{\pi |\Delta| q} + \frac{r}{q^2}.$$  \hspace{1cm} (8)
It follows that

\[ P(X = c) = \sin^2 \frac{\pi dq}{r^2 \Delta^2 r} + O \left( \frac{1}{q^2} \right), \]

so that if \( q \) is much larger than \( r \), then

\[ P(X = c) \sim \frac{\sin^2 \frac{\pi dq}{r^2 \Delta^2 r}}{\pi^2 \Delta^2 r}, \]

and so \( P(X = c) \) is inversely proportional to \((c - \frac{dq}{r})^2\) in the asymptotic limit. Note that the asymptotic profile is dependent only on the fractional part of \( \frac{dq}{r} \), and not on the size of \( q \).

5.2 Probabilities for Certain Subsets

Since the fraction which interests us, as far as determining the order is concerned, is not \( \frac{c}{q} \) (where \( c \) is the measured value of the first register), but \( \frac{d}{r} \), then the probability that \( \frac{c}{q} \) falls in the proximity of \( \frac{d}{r} \) is important, and the probability that \( \frac{c}{q} \) falls within a certain distance of \( \frac{d}{r} \) (or, equivalently, that \( c \) falls within a certain distance of \( \frac{dq}{r} \)), will be determined for a certain range of distances.

If \( \frac{dq}{r} \in \mathbb{Z} \), then

\[ 1 - \frac{2}{q} + \frac{r}{q^2} < P \left( \rho_q \left( X, \frac{dq}{r} \right) \leq \frac{q}{2r} \right) < 1 - \frac{3}{q} + \frac{r}{q^2}, \]

(9)

where \( \rho_q \) is the modular metric [4]. Note that \( P(\rho_q(X, \frac{dq}{r}) \leq \frac{q}{2r}) = \frac{1}{r} + O\left( \frac{1}{q^2} \right) \). If \( q \) is much larger than \( r \), then it follows that \( P(\rho_q(X, \frac{dq}{r}) \leq \frac{q}{2r}) \) is very close to \( \frac{1}{r} \).

The value of the parameter \( q \) will now be restricted so that \( q > 2r \).

From now, for \( c \in \{0, 1, \ldots, q - 1\} \), \( d_c \) and \( \Delta_c \) will be uniquely determined by the following conditions:

1. \( d_c \in \{0, 1, \ldots, r\} \);
2. \( c = \frac{d_c}{r} + \Delta_c \);
3. \( -\frac{q}{2r} < \Delta_c \leq \frac{q}{2r} \).

For \( 0 < u \leq \frac{q}{2r} - 1 \),

\[ P(|\Delta X| \geq u + 1) < \frac{2}{\pi^2 u}. \]

(10)
This determines a hard upper bound independent of $q$ for the probability that the distance between the measured value of the first register and the nearest multiple of $\frac{q}{r}$ exceeds any given value greater than 1 but no greater than $\frac{q}{r^2}$, and demonstrates that the measured value of the first register will tend to be close to a multiple of $\frac{q}{r}$. Specifically, for any large fixed distance, the probability that the difference between the measured value of the first register and the nearest multiple of $\frac{q}{r}$ exceeds this distance is small, independent of the size of $q$.

Let $u$ now be fixed subject to $0 < u \leq \frac{q}{r^2} - 1$.

If $\frac{dq}{r} \in \mathbb{Z}$, then

$$
\frac{1}{r} - \frac{2}{q} + \frac{r}{q^2} < P \left( \rho_q \left( X, \frac{dq}{r} \right) < u + 1 \right) < \frac{1}{r} + \frac{2}{q} \frac{(2u + 3)r}{q^2}. 
$$

(11)

If $\frac{dq}{r} \notin \mathbb{Z}$, then it follows from the bounds already determined on $P(X = c)$ for $c$ such that

$$
\left| c - \frac{dq}{r} \right| < u + 1,
$$

that if

$$
u \leq \frac{q}{\pi r} \left| \sin \frac{\pi dq}{r} \right| - 1,
$$

then

$$
\frac{1}{r} - \frac{2}{\pi^2 ru} - \frac{2}{\pi q} \left( \frac{r^2}{r - 1} + \ln \frac{r^2(u + 1)^2}{r - 1} \right) + \frac{r(2u + 1)}{q^2} 
\begin{aligned}
&< P \left( \left| X - \frac{dq}{r} \right| < u + 1 \right) \\
&< \frac{1}{\left( 1 - \frac{\pi^2(u+1)^2r^2}{6q^2} \right)^2} \left( \frac{1}{r} + \frac{2}{\pi q} \left( \frac{r^2}{r - 1} + \ln \frac{r^2(u + 1)^2}{r - 1} \right) + \frac{r(2u + 3)}{q^2} \right). 
\end{aligned}
$$

(12)

Note that if $u$ is large, and if $q$ is much larger than $ru$, then $P(\left| X - \frac{dq}{r} \right| < u + 1)$ is very close to $\frac{1}{r}$.

The probability that the measured value of the first register will be in the proximity of any specified multiple of $\frac{q}{r}$ has been determined to be very close to $\frac{1}{r}$ for any given multiple, and so the probability that $\frac{c}{q}$ (where $c$ is the measured value of the first register) is close to $\frac{d}{r}$ is approximately $\frac{1}{r}$ for any given value of $d$.

In summary, for $q$ very large, the nett probability of 1 is equally divided amongst the vicinities of $\frac{dq}{r}$ for $d \in \mathbb{Z}$, with the probability effectively concentrated within vicinities of fixed maximum width, so that as $q$ increases, the vicinities recede from each other while maintaining their maximum widths.
6 Continued Fractions

The determination of an appropriate rational number \( \frac{d'}{r'} \) from the measured value of \( c \) is done on a classical computer with the use of continued fractions (see [4], for example). In the context of the refinement of Shor’s Algorithm, we are interested in the width of the vicinity of \( \frac{d'}{r'} \) which will, with certainty, identify \( \frac{d}{r} \) as the correct approximation to \( \frac{c}{q} \) where \( c \) is the measured value of the first register. The width is linearly dependent on \( q \).

The definition of a continued fraction is given here, along with some useful properties.

The definition of continued fractions and most of the consequences, as drawn below, can be found in [6] and [7].

For integers \( a_0, a_1, a_2, \ldots, a_N \), where \( a_0 \geq 0 \) and \( a_i > 0 \) for \( i = 1, 2, \ldots, N \), define the continued fraction \([a_0, a_1, a_2, \ldots, a_N]\) by

\[
[a_0, a_1, a_2, \ldots, a_N] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_N}}}},
\]

so that \([a_0, a_1, a_2, \ldots, a_N]\) is a rational number. Alternatively, a finite continued fraction can be defined by induction on the number of terms as follows. For a non-negative integer \( a_0 \), define \([a_0]\) = \( a_0 \), and for integers \( a_0, a_1, a_2, \ldots, a_N \), as above, define

\[
[a_0, a_1, a_2, \ldots, a_N] = a_0 + \frac{1}{[a_1, a_2, \ldots, a_N]}.
\]

For any \( 0 \leq k \leq N \), \( \xi_k = [a_0, a_1, \ldots, a_k] \) is called a convergent of the continued fraction expansion.

If \( \xi = \frac{p}{q} \) is rational, then define \( a_i \) and \( \zeta_i \) by induction on \( i \) by

\[
\begin{align*}
\zeta_i &= \begin{cases} 
\xi, & i = 0, \\
\frac{1}{\zeta_{i-1} - a_{i-1}}, & \text{otherwise, if } \zeta_{i-1} \neq a_{i-1},
\end{cases} \\
a_i &= \lfloor \zeta_i \rfloor,
\end{align*}
\]

terminating when \( \zeta_i = a_i \) (i.e. when \( \zeta_i \) is an integer). This gives a continued fraction expansion \( \xi = [a_0, a_1, a_2, \ldots, a_N] \), where \( a_N > 1 \). Alternatively, \( \xi = [a_0, a_1, a_2, \ldots, a_N - 1, 1] \), yielding two distinct continued fraction expansions for \( \xi \). It is known that for any rational number \( \xi \), these two continued fraction expansions are the only possible expansions (for irrational numbers, there is exactly one continued fraction expansion, which is infinite).

Define integers \( p_k \) and \( q_k \) for \( k \geq -1 \) by induction on \( k \) as follows. Let

\[
p_k = \begin{cases} 
1, & k = -1, \\
a_0, & k = 0, \\
p_k-2 + a_k p_{k-1}, & k > 0,
\end{cases}
\]

(13)
\[
q_k = \begin{cases} 
0, & k = -1, \\
1, & k = 0, \\
q_{k-2} + a_k q_{k-1}, & k > 0,
\end{cases} \tag{14}
\]

then by standard results from the theory of continued fractions,

- \( \xi_k = p_k/q_k \) for \( k = 0, 1, 2, \ldots \);
- \( p_{k-1}q_k - p_k q_{k-1} = (-1)^k \) for \( k = 0, 1, 2, \ldots \);
- \( \xi_{k+1} - \xi_k = (-1)^k \frac{q_k - q_{k+1}}{q_k q_{k+1}} \);
- \( \gcd(p_k, q_k) = 1 \).

The first two statements are very easily proved by induction, and the third and fourth statements are trivial consequences of the first two.

It is also well-known that if \( \xi \) is a positive real number, \( p \) and \( q \) are positive integers, and \( |\xi - \frac{p}{q}| \leq \frac{1}{2q^2} \), then \( \frac{p}{q} \) is a convergent of the continued fraction expansion for \( \xi \) (see for example, [6, 7]).

It is proven in [3] that

**Theorem 6.1**

For \( k > 1 \), let \( \frac{p_k}{q_k} \) be the corresponding convergent of the continued fraction expansion for \( \xi \), so that \( p_k \) and \( q_k \) are defined by (13) and (14), then for \( 0 < q \leq q_k \) and \( p \in \mathbb{Z} \) such that \( \frac{p}{q} \neq \frac{p_k}{q_k} \),

\[
|p - q\xi| \geq |p_k - q_k\xi|,
\]

and

\[
\left| \frac{\xi - p}{q} \right| > \left| \frac{\xi - p_k}{q_k} \right|.
\]

We now come to the principal result that will be of use in analysing the refinement of Shor’s Algorithm, since it gives a sufficient condition on \( c \) (the result of measuring the first register) that will guarantee that the nearest fraction to \( \frac{c}{q} \) with denominator less than \( n \) is \( \frac{d}{r} \) for some integer \( d \), and that \( \frac{d}{r} \) is a convergent of the continued fraction expansion for \( \frac{c}{q} \).

**Theorem 6.2**

Suppose \( r, n \in \mathbb{Z} \) and \( 0 < r < n \). For \( v > 1 \), let \( q \) be an integer greater than or equal to \( 2vn^2 \). Suppose \( d \in \{0, 1, \ldots, r\} \) and \( |c - \frac{d}{r}| \leq v \). Let \( \frac{d'}{r'} \) be the fraction satisfying the following conditions:

- \( \frac{d'}{r'} \) is in lowest terms (\( d' \) and \( r' \) have no common factors);
\[ 0 \leq d'/r' \leq 1; \]
\[ 0 < r' < n; \]
\[ d'/r' \text{ is the nearest fraction to } c/q \text{ which satisfies the other three conditions.} \]

Then \( d'/r' = d/r \), and \( d/r \) is a convergent of the continued fraction expansion for \( c/q \). Define \( p_k \) and \( q_k \) for \( k = 0, 1, \ldots \), by (13) and (14), respectively. Let \( N = \max\{k : q_k < n\} \), then \( \xi_N = p_N/q_N = d/r \), so that \( d/r \) is the last convergent of the expansion which has denominator less than \( n \).

**Proof:** Since \( \left| c - \frac{dq}{r} \right| \leq v \), then

\[ \left| \frac{c}{q} - \frac{d}{r} \right| \leq \frac{v}{q} < \frac{1}{2n^2} < \frac{1}{2r^2}, \]

so that \( d/r \) is a convergent of the continued fraction expansion for \( c/q \).

Suppose \( f, s \in \mathbb{Z}, 0 < s < n, \) and \( 0 \leq f \leq s \). If \( f/s \neq d/r \), then

\[ \left| \frac{f}{s} - \frac{d}{r} \right| = \left| \frac{fr - ds}{rs} \right| \geq \frac{1}{rs} > \frac{1}{n^2}, \]

so that

\[ \left| \frac{c}{q} - \frac{f}{s} \right| = \left| \frac{d}{r} - \frac{f}{s} \right| = \left| \frac{c}{q} - \frac{d}{r} \right| > \frac{1}{n^2} - \frac{1}{2n^2} = \frac{1}{2n^2} \geq \left| \frac{c}{q} - \frac{d}{r} \right|, \]

so that \( d/r \) is the nearest fraction to \( c/q \) which satisfies the requisite three conditions.

Since \( d/r = d'/r' \) is a convergent of the continued fraction expansion for \( c/q \), then there exists \( N \) such that \( p_N = d' \) and \( q_N = r' < n \). If \( q_{N+1} < n \), then, as a consequence of Theorem 6.1,

\[ \left| \frac{c}{q} - \frac{p_{N+1}}{q_{N+1}} \right| < \left| \frac{c}{q} - \frac{p_N}{q_N} \right| = \left| \frac{c}{q} - \frac{d'}{r'} \right|, \]

contradicting the fact that \( d/r \) is the nearest fraction to \( c/q \) with denominator less than \( n \). It follows that \( q_{N+1} \geq n \), and so \( d/r \) is the last convergent of the expansion which has denominator less than \( n \). \( \square \)

### 7 Some Analysis of the Refinement of Shor’s algorithm

Recall the refinement of the Shor’s Algorithm as given earlier. All steps except step 2 are performed on a classical computer.
1. Set $s := 1$ and $q \geq 2wn^3$ (e.g. set $q$ to be that unique power of 2 such that $2wn^3 \leq q < 4wn^3$);

2. Perform the quantum algorithm on the quantum computer with $q$ as specified in Step 1, and measure the value $c$ of the first register;

3. Determine the continued fraction expansion for $\frac{c}{q}$;

4. Determine all denominators of convergents of the continued fraction expansion up to the first denominator greater than or equal to $n$;

5. Let $r'$ be the last denominator less than $n$, and set $s := \text{lcm}(s, r')$;

6. Calculate $x^s \mod n$;

7. If $x^s \not\equiv 1 \mod n$, then go to Step 2;

8. Output $s$.

The significant results of the last two sections can be summarised as follows:

- For $q$ very large, the nett probability of 1 is essentially equally divided amongst vicinities of $\frac{da}{r}$ of fixed finite maximum width for $d \in \mathbb{Z}$;

- The width of the vicinity of $\frac{da}{r}$ which will, with certainty, identify $\frac{d}{r}$ as the correct approximation to $\frac{c}{q}$, is linearly dependent on $q$.

This means that if a large enough value for $q$ is taken, then the vicinity which will, with certainty, identify $\frac{c}{q}$ as the correct approximation to $\frac{c}{q}$, will encompass the entire vicinity of $\frac{da}{r}$ in which the probability is effectively concentrated. This is the raison d’être for choosing $q$ with the value as given in the refinement.

In the refinement of Shor’s Algorithm, then

$$P(\left|\Delta X\right| \geq wn) < \frac{2}{\pi^2(wn - 1)},$$

as a consequence of (10), so that

$$P(\left|\Delta X\right| < wn) > 1 - \frac{2}{\pi^2(wn - 1)}, \quad (15)$$

and so if $n$ is large, then $P(\left|\Delta X\right| \geq wn)$ is very small, and $P(\left|\Delta X\right| < wn)$ is very close to 1. If $\frac{da}{r} \in \mathbb{Z}$, then by (14),

$$\frac{1}{r} - \frac{1}{wn^3} < P\left(\rho_q \left(X, \frac{da}{r}\right) < wn\right) < \frac{1}{r} + \frac{1}{wn^3} + \frac{1}{2wn^4}. \quad (16)$$

This result is proven in Appendix 3.
If $\frac{dq}{r} \notin \mathbb{Z}$, then, by (12),

\[
\frac{1}{r} - \frac{2}{\pi^2(wn - 1)} - \frac{1}{\pi wn^3} \left( n + 1 + \ln \frac{w^2 n^4}{n - 1} \right) < P \left( \left| X - \frac{dq}{r} \right| < wn \right) < \frac{1}{(1 - \frac{2}{\pi^2(wn - 1)})^2} \left( \frac{1}{r} + \frac{1}{\pi wn^3} \left( n + 1 + \ln \frac{w^2 n^4}{n - 1} \right) + \frac{1}{2wn^4} \right).
\]

This result is also proven in Appendix C.

It follows that if $n$ is large, as in the case for any practical RSA encryption algorithm, then $P(\rho_q(\Delta X, \frac{dq}{r}) < wn)$ is close to $1/r$ for all $d$.

For the refinement, the probability that $|\Delta_c| < wn$, where $c$ is the measured value of the first register, is greater than $1 - \frac{2}{\pi^2(wn - 1)}$ by (13). By Theorem 6.2, if $|\Delta_c| < wn$, then the last convergent of the continued fraction expansion for $c/q$ with denominator less than $n$ is necessarily of the form $d/r$ for some $d \in \mathbb{Z}$ such that $0 \leq d \leq r$, so that, in the refinement, $r'$ necessarily divides $r$, as this is the convergent which is determined by the refinement (or rather, its denominator is determined by the refinement). It follows that after each run on the quantum computer, the probability that $r'$ divides $r$ is greater than $1 - \frac{2}{\pi^2(wn - 1)}$. Since the runs on the quantum computer are, in effect, independent random samples with replacement, then the probability that $s$ still divides $r$ after $k$ runs on the quantum computer is greater than $(1 - \frac{2}{\pi^2(wn - 1)})^k$. Specifically, for the size of $n$ that would typically be used in RSA encryption, the probability that $s$ will not divide $r$ after $k$ runs on the quantum computer is negligibly small (of the same order of magnitude as $\frac{k}{wn}$). Since the value of $s$ is almost guaranteed to be a divisor of $r$ after $k$ runs of the quantum computer, and $x^s \equiv 1 \pmod{n}$ iff $s$ is a multiple of $r$, then it is almost guaranteed that when the refinement terminates, $s$ will be equal to $r$ ($s$ is certainly a multiple of $r$ on termination, and it is almost certain to be a divisor of $r$).

This can be expressed formally as follows. Let $A_k$ denote the random variable describing the result of the measurement of the first register after the $k$-th run of the quantum computer, let $B_k$ denote the random variable describing the corresponding value of $r'$ calculated by the classical computer, and let $C_k$ be the random variable defined by

\[ C_k = \text{lcm}(B_1, \ldots, B_k), \]

so that $C_k$ describes the value of $s$ after $k$ runs of the quantum computer, then, by (13),

\[ P(|\Delta_{A_k}| < wn) > 1 - \frac{2}{\pi^2(wn - 1)}, \]

for all $k$, so that by Theorem 6.2

\[ P(B_k|r) > 1 - \frac{2}{\pi^2(wn - 1)}. \]
for all \( k \), and so

\[
P(C_k | r) > \left(1 - \frac{2}{\pi^2(wn - 1)} \right)^k,
\]

for all \( k \), by the independence of the random variables \( A_k \) (from which the independence of the random variables \( B_k \) follows).

8 Some Results in Probability

It was noted in §7 that the probability that \( d/r \) is the fraction with denominator less than \( n \) which is closest to \( c/q \) (where \( c \) is the measured value of the first register) is close to \( 1/r \) (and in fact, it approaches \( 1/r \) in the limit as \( q \to \infty \)). The properties of the probability distributions of certain random variables (which are analogues of important random variables related to the refinement) associated with the idealized distribution follow. The purpose here is to get some idea of the probability that the refinement of Shor’s Algorithm will terminate after at most \( k \) runs of the quantum computer and output the required order.

Let a natural number \( s \) have prime factorization

\[
s = \prod_{j \in J} p_j^{a_j},
\]

where \( J \) is some index set, \( p_j \) are distinct primes, and \( a_j \geq 1 \) for all \( j \in J \). Let \( Z_i, \ i = 1, 2, 3, \ldots, \) denote independent uniformly distributed random variables from the sample space \( \{0, 1, \ldots, s-1\} \), so that for all \( i \), and for all \( d \) in the sample space, \( P(Z_i = d) = \frac{1}{s} \). Let \( R_i \) be the random variable defined by

\[
R_i = \frac{s}{\gcd(Z_i, s)},
\]

so that \( R_i \) are independent random variables, and \( R_i \) is the denominator of \( Z_i/s \), when expressed in lowest terms. For \( k = 1, 2, 3, \ldots, \) define the random variable \( S_k \) by

\[
S_k = \text{lcm}(R_1, R_2, \ldots, R_k).
\]

Note that \( s \) is a parameter for the probability distributions of \( Z_i, R_i, \) and \( S_k \).

**Theorem 8.1**

For all values of the parameter \( s \),

\[
P(S_k = s) > \frac{1}{\zeta(k)}.
\]
for all $k \geq 2$, where $\zeta$ is the Riemann zeta function defined by

$$
\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^z}},
$$

for $\Re(z) > 1$.

Specifically,

$$
P(S_2 = s) > \frac{6}{\pi^2} > \frac{3}{5}, \quad P(S_4 = s) > \frac{90}{\pi^4} > \frac{9}{10},
$$

so that the probability that $S_2$ is equal to $s$ is greater than 60%, and the probability that $S_4$ is equal to $s$ is greater than 90%. Similarly,

$$
P(S_6 = s) > \frac{945}{\pi^6} > \frac{49}{50}, \quad P(S_8 = s) > \frac{9450}{\pi^8} > \frac{199}{200},
$$

so that the probability that $S_6$ is equal to $s$ is greater than 98%, and the probability that $S_8$ is equal to $s$ is greater than 99.5%.

The proof of Theorem 8.1 is given in Appendix D.

9 More Analysis of the Refinement of Shor’s Algorithm

As before, let $A_k$ denote the random variable describing the result of the measurement of the first register after the $k$-th run of the quantum computer, let $B_k$ describe the corresponding value of $r''$ as determined by the refinement, and let the random variable $C_k$ be defined by

$$
C_k = \text{lcm}(B_1, \ldots, B_k).
$$

Further, let the random variable $D_k$ be defined by

$$
D_k = \left\lfloor \frac{rA_k}{q} + \frac{1}{2} \right\rfloor,
$$

so that $D_k$ describes the nearest integer to $\frac{r}{q}$, where $c$ is the measured value of the first register after the $k$-th run of the quantum computer. Note that $A_k$, $k = 1, 2, 3, \ldots$, are independent random variables,
and that for \( c = 0, \ldots, q - 1 \),

\[
P(A_k = c) = \begin{cases} 
\frac{1}{q} \sum_{m=0}^{r-1} \left\lfloor \frac{q + m}{r} \right\rfloor^2, & \frac{cr}{q} \in \mathbb{Z}, \\
\frac{1}{q} \sum_{m=0}^{r-1} \sin^2 \left( \frac{\pi cr}{q} \right), & \frac{cr}{q} \notin \mathbb{Z},
\end{cases}
\]

and as noted previously, for the refinement,

\[
P(|\Delta A_k| < wn) > 1 - \frac{2}{\pi^2(wn - 1)},
\]

so that

\[
P(B_k|\rho) > 1 - \frac{2}{\pi^2(wn - 1)},
\]

by Theorem 6.2 and so

\[
P(C_k|\rho) > \left( 1 - \frac{2}{\pi^2(wn - 1)} \right)^k.
\]

It follows that for any given \( d \), by (14) and (17),

\[
\frac{1}{r} - \frac{1}{wn^3} < P \left( \rho_{\frac{dq}{r}} < wn \right) < \frac{1}{r} + \frac{1}{wn^3} + \frac{1}{2wn^4},
\]

if \( \frac{dq}{r} \in \mathbb{Z} \), and

\[
\frac{1}{r} - \frac{2}{\pi^2(wn - 1)} - \frac{1}{\pi wn^3} \left( n + 1 + \ln \frac{w^2n^4}{n - 1} \right)
\]

\[
< P \left( \left| A_k - \frac{dq}{r} \right| < wn \right)
\]

\[
< \frac{1}{\left( 1 - \frac{\pi^2}{24n^2} \right)^2} \left( \frac{1}{r} + \frac{1}{\pi wn^3} \left( n + 1 + \ln \frac{w^2n^4}{n - 1} \right) + \frac{1}{2wn^4} \right),
\]

if \( \frac{dq}{r} \notin \mathbb{Z} \).

This concludes the summary of what is already known.
Note that \( D_k = d \) if
\[
\rho_q \left( A_k, \frac{dq}{r} \right) < wn,
\]
for \( d = 1, \ldots, r - 1 \), and \( D_k = 0 \) or \( D_k = r \) if
\[
\rho_q(A_k, 0) < wn.
\]

Given \( \rho_q(A_k, \frac{dq}{r}) < wn \), for some \( d = 0, 1, \ldots, r \), then \( D_k = d \), and
\[
B_k = \frac{r}{\gcd(D_k, r)}.
\]

It follows that for all \( d \),
\[
P \left( \rho_q \left( A_k, \frac{dq}{r} \right) < wn \right. \text{ and } D_k = d) = \frac{1}{r} + O \left( \frac{1}{wn} \right),
\]
and
\[
P \left( \rho_q \left( A_k, \frac{dq}{r} \right) \geq wn \right. \text{ for all } d) = O \left( \frac{1}{wn} \right).
\]

This means that the probability distribution for \( D_k \) becomes uniform in the asymptotic limit, and the results of the last Section become exact in the asymptotic limit. Here, \( D_k \) plays the same role as \( Z_i \), \( B_k \) plays the same role as \( R_i \), and \( C_k \) plays the same role as \( S_k \). This means that the asymptotic limit of \( P(C_k = r) \) as \( n \) becomes large should be greater than \( \frac{1}{\zeta(k)} \) for \( k \geq 2 \).

By a similar argument to that used in the proof of Theorem 8.1 (in Appendix B), for \( k \geq 2 \) and \( k \) small,
\[
P(C_k = r) = \prod_{j \in J} \left( 1 - \frac{1}{p_j^k} \right) + O \left( \frac{1}{w} \right)
\]
\[
= \prod_{j \in J} \left( 1 - \frac{1}{p_j^k} \right) + O(n^{-\epsilon}),
\]
(18)
since \( w = n^\epsilon \). Finally, since
\[
\prod_{j \in J} \left( 1 - \frac{1}{p_j^k} \right) > \prod_{p \text{ prime}} \left( 1 - \frac{1}{p^k} \right) = \frac{1}{\zeta(k)},
\]
then

\[ P(C_k = r) > \frac{1}{\zeta(k)} - O(n^{-\epsilon}) , \]

where the statement \( f > g - O(h) \) means that there exists a function \( F \) such that \( f > g - F \) in the asymptotic limit, and \( F/h \) is bounded in the same limit.

The derivation of (18) is given in Appendix E.

This means that for the size of \( n \) that would typically be used in RSA encryption, the probability that the correct value for \( r \) will be found after at most 2 runs of the quantum computer is at least 60%, and the probability that the correct value for \( r \) will be found after at most 4 runs of the quantum computer is at least 90%, etc.

\section{10 Conclusion}

There are various advantages and disadvantages to the refinement of Shor's algorithm as detailed in this paper. The advantages include the facts that each run of the quantum computer is almost certain to evaluate \( r' \) as a divisor of \( r \), and that the probability that the actual value of \( r \) will be found after at most \( k \) runs of the quantum computer is greater than \( 1/\zeta(k) \), so that the probability is greater than 60% that no more than 2 runs will be necessary, and greater than 90% that no more than 4 runs will be necessary. On the other hand, the quantum computer requires more space and time to run the refinement (the space and time requirements are each multiplied by approximately a constant), and the Quantum Fourier Transform requires more delicate rotations of angles (of the order of \( \frac{\pi}{n^2} \), rather than the order of \( \frac{\pi}{n^3} \), which is all that Shor's original algorithm would require). Also, the number of runs needed by Shor's original algorithm is \( O(\log \log n) \), and \( \log \log n \) is a very slowly growing function. For a value of \( n = 10^{400} \), if the logarithms are to base 2, \( \log \log n \) is between 10 and 11. These are questions which will have to be investigated in greater detail if the case of which algorithm is preferable is to be decided.

\section{11 Acknowledgements}

I would like to thank Ming Yung and Tim Baker for many helpful suggestions.

I would also like to thank Tony Bracken and Michael Nielsen for their assistance.

\section{A Proof of Theorem 2.1}

\textbf{Proof:} There are three conditions to be checked in order to show that \( \rho_q \) is a metric.
1. Note that $|x - y| \geq 0$ for all $x, y \in I_q$. Since $0 \leq x \leq q - 1$ and $0 \leq y \leq q - 1$, then $|x - y| \leq q - 1$, and so $q - |x - y| \geq 1 > 0$, so that for all $x, y \in I_q$,

$$
\rho_q(x, y) = \min(|x - y|, q - |x - y|) \geq 0.
$$

For all $x \in I_q$, $|x - x| = 0$, so $q - |x - x| = q$, and so $\rho_q(x, x) = 0$. Conversely, suppose $x, y \in I_q$ and that $\rho_q(x, y) = 0$. Since $\rho_q(x, y) = \min(|x - y|, q - |x - y|)$, then either $|x - y| = 0$ or $q - |x - y| = 0$. If $|x - y| = 0$, then $x = y$. If $q - |x - y| = 0$, then $|x - y| = q$, contradicting $|x - y| \leq q - 1$.

It follows that $\rho_q(x, y) \geq 0$ for all $x, y \in I_q$, and that $\rho_q(x, y) = 0$ iff $x = y$.

2. Since $|x - y| = |y - x|$, then $\rho_q(x, y) = \rho_q(y, x)$, so that $\rho_q$ is symmetric.

3. If $\rho_q(x, y) = |x - y|$ and $\rho_q(y, z) = |y - z|$, then

$$
\rho_q(x, z) \leq |x - z| \leq |x - y| + |y - z| = \rho_q(x, y) + \rho_q(y, z).
$$

If $\rho_q(x, y) = |x - y|$ and $\rho_q(y, z) = q - |y - z|$, then, since

$$
|y - z| \leq |x - y| + |x - z|,
$$

it follows that

$$
\rho_q(x, z) \leq q - |x - z| \leq q + |x - y| - |y - z| = \rho_q(x, y) + \rho_q(y, z).
$$

Similarly, if $\rho_q(x, y) = q - |x - y|$ and $\rho_q(y, z) = |y - z|$, then $\rho_q(x, z) \leq \rho_q(x, y) + \rho_q(y, z)$.

If $\rho_q(x, y) = q - |x - y|$ and $\rho_q(y, z) = q - |y - z|$, then $q - |x - y| \leq |x - y|$ and $q - |y - z| \leq |y - z|$, so that $2|x - y| \geq q$ and $2|y - z| \geq q$, and so $|x - y| \geq \frac{q}{2}$ and $|y - z| \geq \frac{q}{2}$. There are two cases.

- **Case 1 (0 \leq y < \frac{q}{2})**: Since $|x - y| \geq \frac{q}{2}$, then $y + \frac{q}{2} \leq x \leq q - 1$. Similarly, $y + \frac{q}{2} \leq z \leq q - 1$. Since $|x - y| = x - y$ and $|y - z| = z - y$, then $\rho_q(x, y) = q + x - y \geq q - x > z - x$ and $\rho_q(y, z) = q + y - z \geq q - z > x - z$. It follows that $\rho_q(x, z) = |x - z| < \rho_q(x, y) + \rho_q(y, z)$.

- **Case 2 (\frac{q}{2} \leq y < q - 1)**: Since $|x - y| \geq \frac{q}{2}$, then $0 \leq x \leq y - \frac{q}{2} < \frac{q}{2}$. Similarly, $0 \leq z \leq y - \frac{q}{2} < \frac{q}{2}$. Since $|x - y| = y - x$ and $|y - z| = y - z$, then $\rho_q(x, y) = q + x - y > x \geq x - z$ and $\rho_q(y, z) = q + z - y > z \geq z - x$. It follows that $\rho_q(x, z) = |x - z| < \rho_q(x, y) + \rho_q(y, z)$.

It follows that the Triangle Inequality holds.

It follows from these three facts that $\rho_q$ is a metric on $I_q$, to be called the modular metric.

Recall that $|x - y| \leq q - 1$ for $x, y \in I_q$. If $|x - y| \leq \frac{q}{2}$, then $\rho_q(x, y) \leq |x - y| \leq \frac{q}{2}$. On the other hand, if $\frac{q}{2} \leq |x - y| \leq q - 1$, then $\rho_q(x, y) \leq q - |x - y| \leq \frac{q}{2}$. In either case, $\rho_q(x, y) \leq \frac{q}{2}$. \[\square\]
The first result to prove is the result that if $\frac{q}{r} \in \mathbb{Z}$, then

$$P(X = c) = \begin{cases} \frac{1}{r}, & \frac{cr}{q} \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

In this case, then

$$\left\lfloor \frac{q + k}{r} \right\rfloor = \frac{q}{r},$$

for $k = 0, \ldots, r - 1$. If $\frac{cr}{q} \in \mathbb{Z}$, then substitution of (19) into (2) immediately yields

$$P(X = c) = \frac{1}{r}.$$ 

On the other hand, in the case that $\frac{cr}{q} \notin \mathbb{Z}$, then $\sin(\frac{2\pi cr}{q}) = \sin(\pi c) = 0$ (as $c \in \mathbb{Z}$), so that substitution of (19) into (3) immediately yields

$$P(X = c) = 0.$$ 

The next result is that if $1 \leq q \leq r$, then

$$P(X = c) = \frac{1}{q},$$

for all $c$ (thus yielding no useful information).

In this case,

$$\left\lfloor \frac{q + k}{r} \right\rfloor = \begin{cases} 0, & 0 \leq k < r - q, \\ 1, & r - q \leq k \leq r - 1, \end{cases}$$

so that

$$P(X = c) = \frac{1}{q} \sum_{k=0}^{r-1} \left| \sum_{b=0}^{\left\lfloor \frac{q+k}{r} \right\rfloor - 1} \exp \left( \frac{2\pi ibc}{q} \right) \right|^2.$$
\[ \frac{q}{r} - 1 < \left\lfloor \frac{q+k}{r} \right\rfloor < \frac{q}{r} + 1. \]

(4) is now proven as follows. If \( \frac{c}{q} \in \mathbb{Z} \), then it follows from (2) that

\[ \frac{1}{q^2} \sum_{k=0}^{r-1} (q/r - 1)^2 < P(X = c) = \frac{1}{q^2} \sum_{k=0}^{r-1} \left\lfloor \frac{q+k}{r} \right\rfloor^2 < \frac{1}{q^2} \sum_{k=0}^{r-1} (q/r + 1)^2, \]

and so, expanding the squares,

\[ \frac{1}{q} - \frac{2}{q} + \frac{r}{q^2} < P(X = c) < \frac{1}{q} + \frac{2}{q} + \frac{r}{q^2}. \]

On the other hand, if \( \frac{c}{q} \notin \mathbb{Z} \), then it follows from (3) that

\[
\begin{align*}
P(X = c) &= \frac{1}{q^2} \sum_{k=0}^{r-1} \sin^2 \left( \frac{\pi cr}{q} \right) \left\lfloor \frac{q+k}{r} \right\rfloor \\
&\leq \frac{1}{q^2} \sum_{k=0}^{r-1} \sin^2 \frac{\pi cr}{q} \\
&= \frac{1}{q^2} \frac{q}{\sin^2 \frac{\pi cr}{q}},
\end{align*}
\]

thus demonstrating (5).
Suppose that $c = dq + \Delta$, where $d \in \mathbb{Z}$ and $0 < |\Delta| < \frac{q}{r}$, then (substituting for $c$ in (3)),

$$P(X = c) = \frac{1}{q^2} \sum_{k=0}^{r-1} \frac{\sin^2 \left( \pi d \frac{k+r}{q} + \pi d \frac{\Delta}{q} \left( \frac{k+r}{q} - \frac{q}{r} \right) \right)}{\sin^2 \left( \frac{\pi d + \pi \Delta}{q} \right)}$$

$$= \frac{1}{q^2} \sum_{k=0}^{r-1} \frac{\sin^2 \left( \frac{\pi \Delta}{q} \left( \frac{k+r}{q} - \frac{q}{r} \right) \right)}{\sin^2 \left( \frac{\pi \Delta}{q} \right)}$$

exploiting the periodicity of \( \sin \).

If \( dq \in \mathbb{Z} \), then \( \Delta \in \mathbb{Z} \), so that

$$P(X = c) = \frac{1}{q^2} \sum_{k=0}^{r-1} \frac{\sin^2 \left( \pi \Delta + \pi \Delta \frac{\Delta}{q} \left( \frac{k+r}{q} - \frac{q}{r} \right) \right)}{\sin^2 \left( \frac{\pi \Delta}{q} \right)}$$

$$= \frac{1}{q^2} \sum_{k=0}^{r-1} \frac{\sin^2 \left( \frac{\pi \Delta}{q} \left( \frac{k+r}{q} - \frac{q}{r} \right) \right)}{\sin^2 \left( \frac{\pi \Delta}{q} \right)}$$

In particular, since

$$\left| \left( \frac{q+k}{r} \right) - \frac{q}{r} \right| < 1,$$

for all $k = 0, \ldots, r - 1$, then if $0 < |\Delta| \leq \frac{q}{2r}$, then

$$\left| \left( \frac{\pi \Delta}{q} \left( \frac{k+r}{q} - \frac{q}{r} \right) \right) \right| < \left| \frac{\pi \Delta}{q} \right| \leq \frac{\pi}{2},$$

and so

$$P(X = c) = \frac{1}{q^2} \sum_{k=0}^{r-1} \frac{\sin^2 \left( \frac{\pi \Delta}{q} \left( \frac{k+r}{q} - \frac{q}{r} \right) \right)}{\sin^2 \left( \frac{\pi \Delta}{q} \right)} < \frac{1}{q^2} \sum_{k=0}^{r-1} 1 = \frac{r}{q^2},$$

as \( \sin^2 y \) is a monotonic increasing function on \( y \in [0, \frac{\pi}{2}] \), thus yielding (3). This means that the probability that $X$ will differ from $dq$, for some integer $d$ such that $dq$ is also an integer, by at most $\frac{q}{2r}$ and by more than 0, is negligible if $q$ is much larger than $r$.

If $\frac{dq}{r} \notin \mathbb{Z}$, then $\Delta \notin \mathbb{Z}$. Since

$$\frac{q}{r} - 1 < \left( \frac{q+k}{r} \right) < \frac{q}{r} + 1,$$
for \( k = 0, \ldots, r - 1 \), so that
\[
\left| \left\lfloor \frac{q + k}{r} \right\rfloor - \frac{q}{r} \right| < 1,
\]
and since \(|\cos y| \leq 1\) for \( y \in \mathbb{R} \), it follows that
\[
|\sin(\pi \Delta)| - \frac{\pi |\Delta|r}{q} < \left| \sin \left( \pi \Delta + \frac{\pi \Delta r}{q} \left( \left\lfloor \frac{q + k}{r} \right\rfloor - \frac{q}{r} \right) \right) \right| < |\sin(\pi \Delta)| + \frac{\pi |\Delta|r}{q},
\]
for \( k = 0, \ldots, r - 1 \), thus giving bounds on the square root of the numerator of the summand in (3).

Since \( \frac{dq}{r} + \Delta \) is an integer, then
\[
|\sin(\pi \Delta)| = \left| \sin \frac{\pi dq}{r} \right|,
\]
so that
\[
\left| \sin \frac{\pi dq}{r} - \frac{\pi |\Delta|r}{q} \right| \left| \sin \left( \pi \Delta + \frac{\pi \Delta r}{q} \left( \left\lfloor \frac{q + k}{r} \right\rfloor - \frac{q}{r} \right) \right) \right| < \left| \sin \frac{\pi dq}{r} \right| + \frac{\pi |\Delta|r}{q},
\]
for \( k = 0, \ldots, r - 1 \). Upon taking the square (so we now have the numerator of the summand), it follows that
\[
\sin^2 \left( \pi \Delta + \frac{\pi \Delta r}{q} \left( \left\lfloor \frac{q + k}{r} \right\rfloor - \frac{q}{r} \right) \right) < \left( \left| \sin \frac{\pi dq}{r} \right| + \frac{\pi |\Delta|r}{q} \right)^2
\]
\[
= \sin^2 \frac{\pi dq}{r} + 2\pi |\Delta|r \left| \sin \frac{\pi dq}{r} \right| + \frac{\pi^2 \Delta^2 r^2}{q^2},
\]
and that if \( |\Delta| \leq \frac{\pi}{2\pi} |\sin \frac{\pi dq}{r}| \), then
\[
\sin^2 \left( \pi \Delta + \frac{\pi \Delta r}{q} \left( \left\lfloor \frac{q + k}{r} \right\rfloor - \frac{q}{r} \right) \right) > \left( \left| \sin \frac{\pi dq}{r} \right| - \frac{\pi |\Delta|r}{q} \right)^2
\]
\[
= \sin^2 \frac{\pi dq}{r} - 2\pi |\Delta|r \left| \sin \frac{\pi dq}{r} \right| + \frac{\pi^2 \Delta^2 r^2}{q^2},
\]
and so, substituting into (3),

\[ P(X = c) < \frac{r}{q^2 \sin^2 \frac{\pi dq}{q}} \left( \left| \sin \frac{\pi dq}{r} \right| + \frac{\pi |\Delta r|}{q} \right)^2 \]

\[ < \frac{1}{\pi^2 \Delta^2 r \left( 1 - \frac{\pi \Delta^2 r^2}{6q^2} \right)^2} \left( \left| \sin \frac{\pi dq}{r} \right| + \frac{\pi |\Delta r|}{q} \right)^2 \]

\[ = \frac{1}{\pi^2 \Delta^2 r \left( 1 - \frac{\pi \Delta^2 r^2}{6q^2} \right)^2} \left( \sin^2 \frac{\pi dq}{r} + \frac{2\pi |\Delta r| |\sin \frac{\pi dq}{r}| + \frac{\pi^2 \Delta^2 r^2}{q^2} \right) \]

\[ = \frac{1}{\left( 1 - \frac{\pi \Delta^2 r^2}{6q^2} \right)^2} \left( \sin^2 \frac{\pi dq}{r} + \frac{2\pi |\Delta r| |\sin \frac{\pi dq}{r}| + \frac{\pi^2 \Delta^2 r^2}{q^2} \right), \]

thus yielding (20), if \( |\Delta| \leq \frac{q}{2r} \), since

\[ 0 < y \left( 1 - \frac{1}{6} y^2 \right) < \sin y < y, \]

for \( y \in (0, \frac{\pi}{2}] \). Similarly, if \( |\Delta| \leq \frac{q}{\pi r} |\sin \frac{\pi dq}{r}| \), then

\[ P(X = c) > \frac{r}{q^2 \sin^2 \frac{\pi dq}{q}} \left( \left| \sin \frac{\pi dq}{r} \right| - \frac{\pi |\Delta r|}{q} \right)^2 \]

\[ > \frac{1}{\pi^2 \Delta^2 r \left( 1 - \frac{\pi \Delta^2 r^2}{6q^2} \right)^2} \left( \left| \sin \frac{\pi dq}{r} \right| - \frac{\pi |\Delta r|}{q} \right)^2 \]

\[ = \frac{1}{\pi^2 \Delta^2 r \left( 1 - \frac{\pi \Delta^2 r^2}{6q^2} \right)^2} \left( \sin^2 \frac{\pi dq}{r} - \frac{2\pi |\Delta r| |\sin \frac{\pi dq}{r}| + \frac{\pi^2 \Delta^2 r^2}{q^2} \right) \]

\[ = \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2 \Delta^2 r} - \frac{2\sin \frac{\pi dq}{r} |\Delta r|}{\pi |\Delta| q} + \frac{r}{q^2}, \]

thus yielding (21).

If \( \frac{dq}{r} \in \mathbb{Z} \), then by (3) and (3),

\[ \frac{1}{r} - \frac{2}{q} + \frac{r}{q^2} \]

\[ < P \left( X = \frac{dq}{r} \right) \]

\[ \leq P \left( \rho_q \left( X, \frac{dq}{r} \right) \leq \frac{q}{2r} \right) \]
\[ P \left( X = \frac{dq}{r} \right) + P \left( 0 < \rho_q \left( X, \frac{dq}{r} \right) \leq \frac{q}{2r} \right) < \frac{1}{r} + \frac{2}{q} + \frac{r}{q^2} + 2 \left\lfloor \frac{q}{2r} \right\rfloor \frac{r}{q^2}, \]

where \( \rho_q \) is the modular metric \([\square]\), since \( P(X = c) < \frac{r}{q} \) for all \( c \) such that \( 0 < \rho_q(c, \frac{dq}{r}) \leq \frac{q}{2r} \) and

\[
\# \left\{ c : 0 < \rho_q(c, \frac{dq}{r}) \leq \frac{q}{2r} \right\} = 2 \left\lfloor \frac{q}{2r} \right\rfloor.
\]

It follows that

\[
\frac{1}{r} - \frac{2}{q} + \frac{r}{q^2} < P \left( \rho_q \left( X, \frac{dq}{r} \right) \leq \frac{q}{2r} \right) < \frac{1}{r} + \frac{2}{q} + \frac{r}{q^2} + 2 \frac{q}{2r} \frac{r}{q^2} = \frac{1}{r} + \frac{2}{q} + \frac{r}{q^2} + \frac{1}{q} = \frac{1}{r} + \frac{3}{q} + \frac{r}{q^2},
\]

this yielding \([\square]\).

The value of the parameter \( q \) will now be restricted so that \( q > 2r \).

We are interested in the probability that the measured value \( c \) of the first register will fall inside a specified distance from an integral multiple of \( \frac{q}{r} \), so we are also interested in the probability that it will fall outside the specified distance. This is the motivation behind the following calculations.

Since

\[ P(X = c) \leq \frac{r}{q^2 \sin^2 \frac{\pi c}{2}}, \]

by \([\square]\), if \( c \in \mathbb{Z}, 0 \leq c \leq q, \) and \( \frac{c}{q} \notin \mathbb{Z}, \) and since \( \sin^2 y \) is a monotonic increasing function on \( y \in [0, \frac{\pi}{2}] \), then the following hold by straightforward substitution.

- If \( d \in \mathbb{Z}, d \in \{0, 1, \ldots, r - 1\}, 1 < C \leq \frac{q}{r}, \) and \( \frac{dq}{r} + C \in \mathbb{Z}, \) then
  \[
  P \left( X = \frac{dq}{r} + C \right) \leq \frac{r}{q^2 \sin^2 \left( \pi d + \frac{\pi C r}{q} \right)}.
  \]
\[ r \frac{q^2 \sin^2 \frac{\pi Cr}{q}}{C-1} \] 

\[ < \int_{C-1}^{C} \frac{r}{q^2 \sin^2 \frac{\pi \xi r}{q}} d\xi, \]

the equality following from the fact that \( d \in \mathbb{Z} \).

- If \( d \in \mathbb{Z}, d \in \{1, \ldots, r\}, 1 < C \leq \frac{q}{2r}, \) and \( \frac{dq}{r} - C \in \mathbb{Z} \), then

\[
P \left( X = \frac{dq}{r} - C \right) \leq \frac{r}{q^2 \sin^2 \left( \frac{\pi d - \pi Cr}{q} \right)} = \frac{r}{q^2 \sin^2 \frac{\pi Cr}{q}} < \int_{C-1}^{C} \frac{r}{q^2 \sin^2 \frac{\pi \xi r}{q}} d\xi,
\]

the equality following from the fact that \( d \in \mathbb{Z} \).

If \( d \in \mathbb{Z}, d \in \{0, 1, \ldots, r-1\}, 1 < A \leq A' \leq \frac{q}{2r}, \) and \( \frac{dq}{r} + A, \frac{dq}{r} + A' \in \mathbb{Z} \), then by (22),

\[
P \left( \frac{dq}{r} + A \leq X \leq \frac{dq}{r} + A' \right) \leq \sum_{c=\frac{dq}{r}+A}^{\frac{dq}{r}+A'} \frac{r}{q^2 \sin^2 \frac{\pi cr}{q}} < \int_{A-1}^{A'} \frac{r}{q^2} \csc^2 \frac{\pi \xi r}{q} d\xi = -\frac{1}{\pi q} \left( \cot \frac{\pi \xi r}{q} \right)_{A-1}^{A'} = \frac{1}{\pi q} \left( \cot \frac{\pi (A-1)r}{q} - \cot \frac{\pi A' r}{q} \right). \tag{23}\]

This gives an upper bound on the probability that \( X \) will fall between \( \frac{dq}{r} + A \) and \( \frac{dq}{r} + A' \).

Similarly, if \( d \in \mathbb{Z}, d \in \{1, \ldots, r\}, 1 < B \leq B' \leq \frac{q}{2r}, \) and \( \frac{dq}{r} - B, \frac{dq}{r} - B' \in \mathbb{Z} \), then

\[
P \left( \frac{dq}{r} - B' \leq X \leq \frac{dq}{r} - B \right) < \frac{1}{\pi q} \left( \cot \frac{\pi (B-1)r}{q} - \cot \frac{\pi B' r}{q} \right). \tag{24}\]

If \( d \in \mathbb{Z}, d \in \{0, 1, \ldots, r-1\}, 1 < A \leq \frac{q}{2r}, 1 < B \leq \frac{q}{2r}, \) and \( \frac{dq}{r} + A, \frac{(d+1)q}{r} - B \in \mathbb{Z} \), the let

\[ C = \left\lfloor \frac{(2d+1)q}{2r} \right\rfloor - \frac{dq}{r}, \]
so that $C \leq \frac{q}{r} < C + 1$, and then, by (23) and (24),

$$P\left(\frac{dq}{r} + A \leq X \leq \frac{(d + 1)q}{r} - B\right) = P\left(\frac{dq}{r} + A \leq X \leq \frac{q}{r} + C\right) + P\left(\frac{(d + 1)q}{r} - \frac{q}{r} - (C + 1) \leq X \leq \frac{(d + 1)q}{r} - B\right) < \frac{1}{\pi q} \left(\cot \frac{\pi(A - 1)r}{q} - \cot \frac{\pi Cr}{q}\right) + \frac{1}{\pi q} \left(\cot \frac{\pi(B - 1)r}{q} - \cot \left(\pi - \frac{\pi(C + 1)r}{q}\right)\right)$$

$$= \frac{1}{\pi q} \left(\cot \frac{\pi(A - 1)r}{q} + \cot \frac{\pi(B - 1)r}{q} - \cot \frac{\pi Cr}{q} + \cot \frac{\pi(C + 1)r}{q}\right) < \frac{1}{\pi q} \left(\cot \frac{\pi(A - 1)r}{q} + \cot \frac{\pi(B - 1)r}{q}\right),$$

since $\frac{\pi Cr}{q} \leq \frac{\pi}{2} < \frac{\pi(C + 1)r}{q}$, so that $\cot \frac{\pi Cr}{q}$ is positive, and $\cot \frac{\pi(C + 1)r}{q}$ is negative. Since $\cot y < \frac{1}{y}$ for $0 < y \leq \frac{\pi}{2}$, then it follows that

$$P\left(\frac{dq}{r} + A \leq X \leq \frac{(d + 1)q}{r} - B\right) < \frac{1}{\pi^2 r} \left(\frac{1}{A - 1} + \frac{1}{B - 1}\right).$$

Suppose $0 < u \leq \frac{q}{2r} - 1$, then it follows that

$$P\left(\frac{dq}{r} + u + 1 \leq X \leq \frac{(d + 1)q}{r} - u - 1\right) < \frac{1}{\pi^2 r} \left(\frac{1}{u} + \frac{1}{u}\right) = \frac{2}{\pi^2 ru},$$

thus demonstrating (40).

Adopting the same definitions of $d_c$ and $\Delta_c$ that were used in §§ then it follows that

$$P(\lvert\Delta X\rvert \geq u + 1) = \sum_{d=0}^{r-1} P\left(\frac{dq}{r} + u + 1 \leq X \leq \frac{(d + 1)q}{r} - u - 1\right) < \sum_{d=0}^{r-1} \frac{2}{\pi^2 ru} = \frac{2}{\pi^2 u}.$$

Therefore it follows that the probability that the measured value of the first register falls outside a specified distance from a multiple of $\frac{q}{r}$ is bounded above by a quantity which inversely proportional to one less than the distance, with no dependence of the upper bound on the size of $q$, thus making the possibility unlikely if the specified distance is large.

For each value of $d$, we are interested in the values of

$$P\left(\rho_q \left(X, \frac{dq}{r}\right) < u + 1\right).$$
Upper and lower bounds can be easily determined for

\[ P \left( \rho_q \left( X, \frac{dq}{r} \right) < u + 1 \right), \]

if \( \frac{dq}{r} \in \mathbb{Z} \), specifically, by (4) and (6),

\[
\frac{1}{r} - \frac{2}{q} + \frac{r}{q^2} < P \left( X = \frac{dq}{r} \right)
\leq P \left( \rho_q \left( X, \frac{dq}{r} \right) < u + 1 \right)
= P \left( X = \frac{dq}{r} \right) + P \left( 0 < \rho_q \left( X, \frac{dq}{r} \right) < u + 1 \right)
< \frac{1}{r} + \frac{2}{q} + \frac{r}{q^2} + 2|u + 1| \frac{r}{q^2}
\leq \frac{1}{r} + \frac{2}{q} + \frac{r}{q^2} + \frac{2(u + 1)r}{q^2}
= \frac{1}{r} + \frac{2}{q} + \frac{(2u + 3)r}{q^2},
\]

thus leading to (11).

If \( \frac{dq}{r} \notin \mathbb{Z} \), then

\[
P \left( \left| X - \frac{dq}{r} \right| < u + 1 \right) = \sum_{c \in \mathbb{Z} \mid \left| c - \frac{dq}{r} \right| \leq u + 1} P \left( X = c \right),
\]

so that, by (21),

\[
P \left( \left| X - \frac{dq}{r} \right| < u + 1 \right)
< \sum_{c \in \mathbb{Z} \mid \left| c - \frac{dq}{r} \right| \leq u + 1} \frac{1}{\left( 1 - \frac{\pi^2(u + 1)^2 r^2}{6q^2} \right)^2} \left( \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2 \left( c - \frac{dq}{r} \right)^2} + \frac{2 \left| \sin \frac{\pi dq}{r} \right|}{\pi \left| c - \frac{dq}{r} \right| q} + \frac{r}{q^2} \right),
\]

since \( |\Delta_c| < u + 1 \) for all c in the sum, and by (21),

\[
P \left( \left| X - \frac{dq}{r} \right| < u + 1 \right)
\]
\[
> \sum_{c \in \mathbb{Z}, \ |c - dq| < q + 1 \sin \frac{\pi dq}{r}} \left( \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2(c - dq)^2r} - \frac{2}{\pi} \frac{\sin \frac{\pi dq}{r}}{c - dq} + \frac{r}{q^2} \right).
\]

It follows that
\[
\sum_{c \in \mathbb{Z}, \ |c - dq| < q + 1 \sin \frac{\pi dq}{r}} \left( \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2(c - dq)^2r} - \frac{2}{\pi} \frac{\sin \frac{\pi dq}{r}}{c - dq} + \frac{r}{q^2} \right)
\leq P \left( \left| X - \frac{dq}{r} \right| < u + 1 \right)
\leq \frac{1}{\left( 1 - \frac{\pi^2(u+1)^2r^2}{6q^2} \right)^2} \sum_{c \in \mathbb{Z}, \ |c - dq| < u + 1} \left( \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2(c - dq)^2r} + \frac{2}{\pi} \frac{\sin \frac{\pi dq}{r}}{c - dq} + \frac{r}{q^2} \right).
\]

Specifically, if
\[
u \leq \frac{q}{\pi r} \left| \sin \frac{\pi dq}{r} \right| - 1,
\]then
\[
\sum_{c \in \mathbb{Z}, \ |c - dq| < u + 1} \left( \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2(c - dq)^2r} - \frac{2}{\pi} \frac{\sin \frac{\pi dq}{r}}{c - dq} + \frac{r}{q^2} \right)
\leq P \left( \left| X - \frac{dq}{r} \right| < u + 1 \right)
\leq \frac{1}{\left( 1 - \frac{\pi^2(u+1)^2r^2}{6q^2} \right)^2} \sum_{c \in \mathbb{Z}, \ |c - dq| < u + 1} \left( \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2(c - dq)^2r} + \frac{2}{\pi} \frac{\sin \frac{\pi dq}{r}}{c - dq} + \frac{r}{q^2} \right).
\]

The terms inside the sums on the upper and lower bounds in (26) will now be investigated, one at a time.

By the Mittag-Leffler expansion into partial fractions for \( \csc^2(\pi z) \) from complex analysis (which can be found in many books on complex analysis, such as [8, 9], or by differentiating the Mittag-Leffler expansion for \( \cot(\pi z) \), which can also be found in books on complex analysis, such as [10, 11]),
\[
\sum_{n=-\infty}^{\infty} \frac{1}{\pi^2(z - n)^2} = \frac{1}{\sin^2(\pi z)}.
\]
it follows that

$$\sum_{c \in \mathbb{Z}} \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2 (c - \frac{dq}{r})^2 r} = \frac{1}{r},$$

so that

$$\sum_{|c - \frac{dq}{r}| \leq \frac{u + 1}{r}} \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2 (c - \frac{dq}{r})^2 r} = \frac{1}{r} - \sum_{|c - \frac{dq}{r}| \geq \frac{u + 1}{r}} \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2 (c - \frac{dq}{r})^2 r}.$$

For $z \in \mathbb{R} \setminus \mathbb{Z}$,

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n - z)^2} = \sum_{n \in \mathbb{Z} \geq z + u + 1} \frac{1}{(n - z)^2}$$

$$= \sum_{n = [z + u + 1]}^{\infty} \frac{1}{(n - z)^2}$$

$$< \int_{[z + u]}^{\infty} \frac{1}{(\xi - z)^2} d\xi$$

$$= - \left[ \frac{1}{\xi - z} \right]_{[z + u]}^{\infty}$$

$$= \frac{1}{[z + u]} - z$$

$$\leq \frac{1}{u},$$

where for real $y$, $[y]$ is the least integer greater than or equal to $y$. Similarly, for $z \in \mathbb{R} \setminus \mathbb{Z}$,

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n - z)^2} > \frac{1}{[z + u + 1] - z}$$

$$> \frac{1}{u + 2}.$$

It follows that

$$\frac{1}{u + 2} < \sum_{n \in \mathbb{Z} \geq z + u + 1} \frac{1}{(n - z)^2} < \frac{1}{u}.$$
Similarly,

\[
\frac{1}{u + 2} < \sum_{n \in \mathbb{Z}, n - z \leq -(u + 1)} \frac{1}{(n - z)^2} < \frac{1}{u},
\]

so that

\[
\frac{2}{u + 2} < \sum_{n \in \mathbb{Z}, |n - z| \geq u + 1} \frac{1}{(n - z)^2} < \frac{2}{u},
\]

and so

\[
\sum_{c \in \mathbb{Z}, |c - dq_{\mathbb{R}}| < u + 1} \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2 (c - dq_{\mathbb{R}})^2 r} = \frac{1}{r} - \sum_{c \in \mathbb{Z}, |c - dq_{\mathbb{R}}| \geq u + 1} \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2 (c - dq_{\mathbb{R}})^2 r}
\]

satisfies

\[
\frac{1}{r} - \frac{2 \sin^2 \frac{\pi dq}{r}}{\pi^2 ru} < \sum_{c \in \mathbb{Z}, |c - dq_{\mathbb{R}}| < u + 1} \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2 (c - dq_{\mathbb{R}})^2 r} < \frac{1}{r} - \frac{2 \sin^2 \frac{\pi dq}{r}}{\pi^2 r(u + 2)},
\]

since \(dq_{\mathbb{R}} \notin \mathbb{Z}\), and so

\[
\frac{1}{r} - \frac{2}{\pi^2 ru} < \sum_{c \in \mathbb{Z}, |c - dq_{\mathbb{R}}| < u + 1} \frac{\sin^2 \frac{\pi dq}{r}}{\pi^2 (c - dq_{\mathbb{R}})^2 r} < \frac{1}{r}. \tag{27}
\]

This accounts for the first term in the sums in the upper and lower bounds in (26).

Since \([2u + 1] \leq \# \{c \in \mathbb{Z} : |c - \frac{dq}{r}| < u + 1\} \leq [2u + 2]\), then

\[
\frac{r(2u + 1)}{q^2} \leq \sum_{c \in \mathbb{Z}, |c - dq_{\mathbb{R}}| < u + 1} \frac{r}{q^2} < \frac{r(2u + 3)}{q^2}. \tag{28}
\]

This accounts for the third term in the sums in the upper and lower bounds in (26).

All that remains is the second term in the sums. Since

\[
\sum_{c \in \mathbb{Z}, 0 < |c - dq_{\mathbb{R}}| < u + 1} \frac{1}{c - dq_{\mathbb{R}}}
\]
\[
\begin{align*}
&= \frac{1}{dq - \frac{dq}{r}} + \sum_{c \in \mathbb{Z}} \frac{1}{c - \frac{dq}{r}} \\
&= \frac{1}{dq - \frac{dq}{r}} + \sum_{c = \lfloor \frac{dq}{r} \rfloor}^{\lfloor u + \frac{dq}{r} \rfloor + 1} \frac{1}{c - \frac{dq}{r}} \\
&< \frac{1}{dq - \frac{dq}{r}} + \int_{\lfloor \frac{dq}{r} \rfloor}^{\lfloor u + \frac{dq}{r} \rfloor + 1} \frac{1}{\xi - \frac{dq}{r}} d\xi \\
&= \frac{1}{dq - \frac{dq}{r}} + \left[ \ln \left( \xi - \frac{dq}{r} \right) \right]_{\lfloor \frac{dq}{r} \rfloor}^{\lfloor u + \frac{dq}{r} \rfloor + 1} \\
&= \frac{1}{dq - \frac{dq}{r}} + \ln \left( \frac{u + \frac{dq}{r}}{\frac{dq}{r} - \frac{dq}{r}} + 1 - \frac{dq}{r} \right) \\
&\leq \frac{1}{dq - \frac{dq}{r}} + \ln \left( \frac{u + 1}{\frac{dq}{r} - \frac{dq}{r}} \right),
\end{align*}
\]

and similarly,

\[
\begin{align*}
&= \sum_{c \in \mathbb{Z}} \frac{1}{dq - c} \\
&< \frac{1}{dq - \lfloor \frac{dq}{r} \rfloor} + \ln \left( \frac{\frac{dq}{r} - \left\lfloor \frac{dq}{r} \right\rfloor - u + 1}{\left\lfloor \frac{dq}{r} \right\rfloor} \right) \\
&< \frac{1}{dq - \lfloor \frac{dq}{r} \rfloor} + \ln \left( \frac{u + 1}{\left\lfloor \frac{dq}{r} \right\rfloor - \left\lfloor \frac{dq}{r} \right\rfloor} \right),
\end{align*}
\]

then

\[
\begin{align*}
&= \sum_{c \in \mathbb{Z}} \frac{1}{|c - \frac{dq}{r}|} \\
&< \frac{1}{dq - \left\lfloor \frac{dq}{r} \right\rfloor} + \frac{1}{\left\lfloor \frac{dq}{r} \right\rfloor - \left\lfloor \frac{dq}{r} \right\rfloor} + \ln \left( \frac{u + \left\lfloor \frac{dq}{r} \right\rfloor + 1}{\frac{dq}{r} - \left\lfloor \frac{dq}{r} \right\rfloor} \right) + \ln \left( \frac{\frac{dq}{r} - \left\lfloor \frac{dq}{r} \right\rfloor + 1}{\frac{dq}{r} - \left\lfloor \frac{dq}{r} \right\rfloor} \right) \\
&\leq \frac{1}{dq - \left\lfloor \frac{dq}{r} \right\rfloor} + \frac{1}{\left\lfloor \frac{dq}{r} \right\rfloor - \left\lfloor \frac{dq}{r} \right\rfloor} + \ln \left( \frac{u + 1}{\left\lfloor \frac{dq}{r} \right\rfloor - \left\lfloor \frac{dq}{r} \right\rfloor} \right) + \ln \left( \frac{u + 1}{\left\lfloor \frac{dq}{r} \right\rfloor - \left\lfloor \frac{dq}{r} \right\rfloor} \right).
\end{align*}
\]

Since \(\frac{1}{r} \leq \frac{dq}{r} - \left\lfloor \frac{dq}{r} \right\rfloor \leq \frac{r-1}{r}\) (and equivalently, \(\frac{1}{r} \leq \left\lceil \frac{dq}{r} \right\rceil - \frac{dq}{r} \leq \frac{r-1}{r}\), noting that for \(y \in \mathbb{R} \setminus \mathbb{Z}\), \([y] - [y] = 1\),
then
\[ \frac{1}{r} \left| \frac{dq}{r} - \left\lfloor \frac{dq}{r} \right\rfloor \right| + \frac{1}{r} \left( \left\lceil \frac{dq}{r} \right\rceil - \frac{dq}{r} \right) \leq r + \frac{r}{r - 1} = \frac{r^2}{r - 1}. \]

Similarly,
\[ \left( \frac{dq}{r} - \left\lfloor \frac{dq}{r} \right\rfloor \right) \left( \left\lceil \frac{dq}{r} \right\rceil - \frac{dq}{r} \right) \geq \frac{1}{r} \frac{r - 1}{r} = \frac{r - 1}{r^2}, \]
so that
\[ \ln \left( \left( \frac{dq}{r} - \left\lfloor \frac{dq}{r} \right\rfloor \right) \left( \left\lceil \frac{dq}{r} \right\rceil - \frac{dq}{r} \right) \right) \geq \ln \frac{r - 1}{r^2}, \]
and so
\[ \sum_{c \in \mathbb{Z}} \frac{1}{\left| c - \frac{dq}{r} \right|} < \frac{r^2}{r - 1} + \ln \frac{r^2(u + 1)^2}{r - 1}. \tag{29} \]

Gathering all the information about individual terms, it follows that if
\[ u \leq \frac{q}{\pi r} \left| \sin \frac{\pi dq}{r} \right| - 1, \]
then
\[ \frac{1}{r} - \frac{2}{\pi^2 r u} - \frac{2}{\pi q} \left( \frac{r^2}{r - 1} + \ln \frac{r^2(u + 1)^2}{r - 1} \right) + \frac{r(2u + 1)}{q^2} \]
\[ < P \left( \left| X - \frac{dq}{r} \right| < u + 1 \right) \]
\[ < \frac{1}{\left( 1 - \frac{r^2(u + 1)^2}{q^2} \right)^2} \left( \frac{1}{r} + \frac{2}{\pi q} \left( \frac{r^2}{r - 1} + \ln \frac{r^2(u + 1)^2}{r - 1} \right) + \frac{r(2u + 3)}{q^2} \right), \]
as a consequence of (26), (27), (28) and (29), thus yielding (12).
C  Proof of Inequalities in the Analysis of the Refinement

If \( dq \notin Z \), then by (11) and the fact that \( q \geq 2wn^3 \),

\[
\frac{1}{r} - \frac{1}{ wn^3} < \frac{1}{r} - \frac{1}{ wn^3} + \frac{r}{4w^2n^6} \\
\leq \frac{1}{r} - \frac{2}{q} \frac{r}{q^2} \\
< P \left( \rho_q \left( X, \frac{dq}{r} \right) < wn \right) \\
< \frac{1}{r} + \frac{2}{q} + \frac{(2wn + 1)r}{q^2} \\
\leq \frac{1}{r} + \frac{1}{ wn^3} + \frac{(2wn + 1)r}{4w^2n^6} \\
< \frac{1}{r} + \frac{1}{ wn^3} + \frac{1}{2wn^3},
\]

the final statement following from the fact that \( r < n \), so that \( r \leq n - 1 \), and so

\[ (2wn + 1)r \leq (2wn + 1)(n - 1) = 2wn^2 - 2wn + n - 1 = 2wn^2 - (2w - 1)n - 1. \]

This demonstrates (16).

If \( dq \notin Z \), then, by (12) and the fact that \( q \geq 2wn^3 \),

\[
\frac{1}{r} \frac{2}{\pi^2(wn - 1)} - \frac{1}{\pi wn^3} \left( n + 1 + \ln \frac{w^2n^4}{n - 1} \right) \\
< \frac{1}{r} \frac{2}{\pi^2r(wn - 1)} - \frac{1}{\pi wn^3} \left( \frac{r^2}{r - 1} + \ln \frac{r^2w^2n^2}{r - 1} \right) \\
< \left( \frac{1}{r} \frac{2}{\pi^2r(wn - 1)} - \frac{1}{\pi wn^3} \left( \frac{r^2}{r - 1} + \ln \frac{r^2w^2n^2}{r - 1} \right) \right) + \frac{(2wn - 1)}{q^2} \\
< P \left( \left| X - \frac{dq}{r} \right| < wn \right) \\
< \left( \frac{1}{1 - \left( \frac{\pi^2w^2n^2}{w^2n^2} \right)^2} \left( \frac{1}{r} + \frac{1}{\pi wn^3} \left( \frac{r^2}{r - 1} + \ln \frac{r^2w^2n^2}{r - 1} \right) \right) \right) \\
\leq \left( \frac{1}{1 - \left( \frac{\pi^2w^2n^2}{w^2n^2} \right)^2} \left( \frac{1}{r} + \frac{1}{\pi wn^3} \left( \frac{r^2}{r - 1} + \ln \frac{r^2w^2n^2}{r - 1} \right) \right) \right) + \frac{(2wn + 1)}{4w^2n^6} \\
= \frac{1}{(1 - \left( \frac{\pi^2w^2n^2}{w^2n^2} \right)^2} \left( \frac{1}{r} + \frac{1}{\pi wn^3} \left( \frac{r^2}{r - 1} + \ln \frac{r^2w^2n^2}{r - 1} \right) \right) + \frac{(2wn + 1)}{4w^2n^6} \\
< \frac{1}{(1 - \left( \frac{\pi^2}{2wn^2} \right)^2} \left( \frac{1}{r} + \frac{1}{\pi wn^3} \left( n + 1 + \ln \frac{w^2n^4}{n - 1} \right) + \frac{1}{2wn^4} \right).
The first inequality above follows from

- the fact that
\[(n + 1)(r - 1) - r^2 = (n + 1)(r - 1) - (r + 1)(r - 1) - 1 = (n - r)(r - 1) - 1 \geq 0,\]
since \(n - r \geq 1\) and \(r > 1\) (which is required by the fact that \(\frac{da}{r} \notin \mathbb{Z}\)),
- the fact that
\[n^2(r - 1) - r^2(n - 1) = (n - r)(nr - n - r) = (n - r)[(n - 1)(r - 1) - 1] \geq 1(2 - 1) = 1,\]
since \(r \geq 2\) and \(n \geq r + 1 \geq 3\), and so
\[\frac{n^2}{n - 1} > \frac{r^2}{r - 1}.
\]
This demonstrates (17).

### D Proof of Theorem 8.1

**Lemma D.1**

For each \(j \in J\), define the random variable \(B_{ij}\) with sample space \(\{0, 1, 2, \ldots, a_j\}\) by setting

\[R_i = \prod_{j \in J} p_j^{B_{ij}}.\]

Specifically, \(B_{ij}\) is the power to which \(p_j\) is raised in the prime factorization of \(R_i\). Then:

- The probability distribution for \(B_{ij}\) is given by
\[P(B_{ij} = b) = \begin{cases} \frac{p_j^{b} - p_j^{b-1}}{p_j^{b}}, & b > 0, \\ \frac{1}{p_j^{b}}, & b = 0. \end{cases}\]
- \(B_{ij}\) for \(i = 1, 2, 3, \ldots,\) and \(j \in J,\) are independent random variables.

**Proof:** Since \(B_{ij} = b\) iff \(b\) is the power to which \(p_j\) is raised in the prime factorization of \(R_i\), then

\[P(B_{ij} = b) = P(p_j^{b} | R_i \land p_j^{b+1} \nmid R_i),\]
and so since $R_i = s/\gcd(Z_i, s)$, it follows that

$$P(B_{ij} = b) = P(p_j^{a_j-b}|\gcd(Z_i, s) \land p_j^{a_j-b+1} \nmid \gcd(Z_i, s))$$

= \begin{cases} 
P(p_j^{a_j-b}|Z_i \land p_j^{a_j-b+1} \nmid Z_i), & b > 0, \\
P(p_j^{a_j}|Z_i), & b = 0.
\end{cases}

The number of elements of $\{0, 1, 2, \ldots, s - 1\}$ which are divisible by $p_j^{a_j-b}$ is $s/p_j^{a_j-b}$ for all $b = 0, 1, 2, \ldots, a_j$, so that, since $Z_i$ is uniformly distributed,

$$P(p_j^{a_j-b}|Z_i) = \frac{1}{p_j^{a_j-b}} = \frac{p_j^b}{p_j^{a_j}},$$

and so

$$P(B_{ij} = b) = \begin{cases} 
\frac{p_j^b - p_j^{b-1}}{p_j^a}, & b > 0, \\
\frac{1}{p_j^{a_j}}, & b = 0.
\end{cases}$$

This proves the required formula for $P(B_{ij} = b)$.

For the independence of $B_{ij}$, one can invoke the Chinese Remainder Theorem (as Knill did in [3], for example). Alternatively, one can also take the following approach. For $j_1, \ldots, j_l \in J$, and for $b_m = 0, 1, \ldots, a_{j_m}$ for $m = 1, \ldots, l$, then

$$P(B_{ij_1} = b_1, B_{ij_2} = b_2, \ldots, B_{ij_l} = b_l) = P(p_j^{b_m}|R_i \land p_j^{b_m+1} \nmid R_i, \text{ for all } m = 1, \ldots, l)$$

= $P(p_j^{a_{j_m}-b_m}|\gcd(Z_i, s) \land p_j^{a_{j_m}-b_m+1} \nmid \gcd(Z_i, s), \text{ for all } m = 1, \ldots, l).$

For any divisor $t$ of $s$, then the number of elements of $\{0, 1, \ldots, s - 1\}$ which are divisible by $t$ is $\frac{s}{t}$, so that, for all $i$,

$$P(t|Z_i) = \frac{1}{t}.$$

It follows that for $b_m = 0, 1, \ldots, a_{j_m}$, $m = 1, \ldots, l$, then

$$P \left( \prod_{i=1}^{m} p_j^{a_{j_m}-b_m}|Z_i \right) = \prod_{m=1}^{l} \frac{1}{p_j^{a_{j_m}-b_m}} = \prod_{m=1}^{l} P(p_j^{a_{j_m}-b_m}|Z_i).$$ (32)
For \( m = 1, \ldots, l \), define \( f_m : \{-1, 0, 1, \ldots, a_{jm}\} \to \mathbb{R} \) by

\[
f_m(b) = \begin{cases} 
\frac{1}{p_{jm}^a}, & b \geq 0, \\
0, & b = -1,
\end{cases}
\]

then

\[
P\left(p_{jm}^{a_{jm} - b} \mid \text{gcd}(Z_i, s)\right) = f_m(b),
\]

for \( b = -1, 0, 1, \ldots, a_{jm} \), and so, with the aid of (32),

\[
P\left(\prod_{m=1}^l p_{jm}^{a_{jm} - b_m} \mid \text{gcd}(Z_i, s)\right) = \prod_{m=1}^l f_m(b_m),
\]

for \( b_m = -1, 0, 1, \ldots, a_{jm}, m = 1, \ldots, l \). It follows that

\[
P(B_{ij_1}, B_{ij_2}, \ldots, B_{ij_l}) = \prod_{m=1}^l P(B_{ij_m} = b_m),
\]

for \( b_m = 0, 1, \ldots, a_{jm}, m = 1, \ldots, l \). For example, one can use a proof by induction on \( q \) to demonstrate that for \( b_m = 0, 1, \ldots, a_{jm}, m = 1, \ldots, q \), and for \( b_m = -1, 0, 1, \ldots, a_{jm}, m = q + 1, \ldots, l \),

\[
P(F_m(b_m, Z_i)) = \prod_{m=1}^q P(F_m(b_m, Z_i)) \prod_{m=q+1}^l P(G_m(b_m, Z_i))
\]

where \( G_m(b, Z_i) \) denotes the proposition denoting that \( p_{jm}^{a_{jm} - b} \) divides \( \text{gcd}(Z_i, s) \), and \( F_m(b, Z_i) \) denotes the proposition \( G_m(b, Z_i) \land \neg G_m(b-1, Z_i) \), so that \( F_m(b, Z_i) \) is equivalent to the proposition that \( p_{jm}^{a_{jm} - b} \) divides \( \text{gcd}(Z_i, s) \) and \( p_{jm}^{a_{jm} - b+1} \) does not divide \( \text{gcd}(Z_i, s) \). It follows that \( B_{ij_1}, B_{ij_2}, \ldots, B_{ij_l} \) are independent random variables. Since the set \( \{j_1, j_2, \ldots, j_l\} \) was arbitrary, and since \( Z_i \) are independent random variables, it follows that \( B_{ij} \) for \( i = 1, 2, \ldots, \) and for \( j \in J \), are independent random variables. \( \square \)
The proof of Theorem 8.1 can now be given.

**Proof:** Let $B_{ij}$ be random variables as in the proof of the Lemma, and define random variables $C_{kj}$ by

$$C_{kj} = \max(B_{1j}, \ldots, B_{kj}),$$

then

$$S_k = \prod_{j \in J} p_j^{C_{kj}}.$$

Since the sample space for $B_{ij}$ is $\{0, 1, \ldots, a_j\}$ for all $i, j$, then the sample space for $C_{kj}$ is also $\{0, 1, \ldots, a_j\}$ for all $k, j$. From the result in the Lemma that

$$P(B_{ij} = a_j) = \frac{p_j^{-a_j} - p_j^{-a_j-1}}{p_j^{-a_j}} = 1 - \frac{1}{p_j},$$

for all $i, j$, then

$$P(B_{ij} < a_j) = \frac{1}{p_j},$$

for all $i, j$, and so

$$P(C_{kj} < a_j) = P(B_{ij} < a_j \text{ for } i = 1, \ldots, k) = \prod_{i=1}^{k} P(B_{ij} < a_j) = \frac{1}{p_j^k},$$

as a consequence of the independence of $B_{1j}, B_{2j}, \ldots, B_{kj}$ (which follows from the independence of $Z_i$ for $i = 1, \ldots, k$). It follows that

$$P(C_{kj} = a_j) = 1 - \frac{1}{p_j^k},$$

and so

$$P(S_k = s) = P(C_{kj} = a_j \text{ for all } j \in J) = \prod_{j \in J} P(C_{kj} = a_j) = \prod_{j \in J} \left(1 - \frac{1}{p_j^k}\right),$$
as a consequence of the independence of $C_{kj}$ for $j \in J$ (which follows from the independence of $B_{ij}$ for $i = 1, 2, \ldots, k$ and $j \in J$). Therefore

$$P(S_k = s) = \prod_{j \in J} \left( 1 - \frac{1}{p_j^k} \right) > \prod_{p \text{ prime}} \left( 1 - \frac{1}{p^k} \right) = \frac{1}{\zeta(k)},$$

for $k \geq 2$. \qed

E Proof of (18)

A similar method to the proof of Theorem 8.1 can be used. Let $r$ have prime factorization

$$r = \prod_{j \in J} p_j^{a_j},$$

where $J$ is some index set, $p_j$ are distinct primes, and $a_j \geq 1$ for all $j \in J$. For each prime $p$, define the random variable $E_{kp}$, with sample space $\{0, 1, 2, \ldots\}$, by setting

$$B_k = \prod_{p \text{ prime}} p^{E_{kp}}.$$

Specifically, $E_{kp}$ is the power to which $p$ is raised in the prime factorization of $B_k$. By similar arguments to the ideal case, treated in §8 and Appendix E, then:

- For a finite set $J_0$ of primes, and for non-negative integers $b_p$ for $p \in J_0$,

  $$P(E_{kp} = b_p \text{ for all } p \in J_0) = O\left( \frac{1}{wn} \right),$$

  if $b_p > 0$ for some $p$ not dividing $r$, or $b_p > a_j$ for some $j \in J$ such that $p_j \in J_0$;

- For a finite set $J_0$ of primes, and for non-negative integers $b_p$ for $p \in J_0$,

  $$P(E_{kp} = b_p \text{ for all } p \in J_0) = \prod_{p \in J_0} p_j^{b_{p_j}} \left( 1 - \frac{1}{p_j^{a_j}} \right) \prod_{p \in J_0} \frac{1}{p_j^{a_j}} \left( 1 + O\left( \frac{r}{wn} \right) \right),$$

  if $b_p = 0$ for all $p$ not dividing $r$, and $b_{p_j} \leq a_j$ for all $j \in J$ such that $p_j \in J_0$.

It follows that for any subset $J_0 \subseteq J$,

$$P(E_{kp_j} < a_j \text{ for all } j \in J_0, \text{ and } E_{kp} = 0 \text{ for all } p \nmid r) = \prod_{j \in J_0} \frac{1}{p_j} \left( 1 + O\left( \frac{r}{wn} \right) \right).$$
For all $k$ and primes $p$, define the random variable $F_{kp}$ by

$$F_{kp} = \max(E_{1p}, \ldots, E_{kp}),$$

so that

$$C_k = \prod_{p \text{ prime}} p^{F_{kp}}.$$

It follows that for a finite set $J_0$ of primes, and for non-negative integers $b_p$ for $p \in J_0$,

$$P(F_{kp} = b_p \text{ for all } p \in J_0) = O\left(\frac{k}{wn}\right),$$

if $b_p > 0$ for some $p$ not dividing $r$, or $b_{pj} > a_j$ for some $j \in J$ such that $p_j \in J_0$.

Since $A_k$ are independent random variables, then $B_k$ are independent random variables, so that for any subset $J_0 \subseteq J$,

$$P\left(F_{kp} < a_j \text{ for all } j \in J_0, \text{ and } F_{kp} = 0 \text{ for all } p \nmid r\right)$$

$$= \prod_{i=1}^{k} P\left(E_{ip} < a_j \text{ for all } j \in J_0, \text{ and } E_{ip} = 0 \text{ for all } p \nmid r\right)$$

$$= \prod_{i=1}^{k} \prod_{j \in J_0} \frac{1}{p_j} \left(1 + O\left(\frac{r}{wn}\right)\right)$$

$$= \prod_{j \in J_0} \frac{1}{p_j} \left(1 + O\left(\frac{kr}{wn}\right)\right).$$

Therefore, similarly to the idealized case,

$$P(C_k = r)$$

$$= \sum_{J_0 \subseteq J} (-1)^{\#(J_0)} P\left(F_{kp} = a_j \text{ for all } j \in J, \text{ and } F_{kp} = 0 \text{ for all } p \nmid r\right)$$

$$= \sum_{J_0 \subseteq J} (-1)^{\#(J_0)} \prod_{j \in J_0} \frac{1}{p_j} \left(1 + O\left(\frac{kr}{wn}\right)\right)$$

$$= \prod_{j \in J} \left(1 - \frac{1}{p_j^r}\right) + \prod_{j \in J} \left(1 + \frac{1}{p_j^r}\right) O\left(\frac{kr}{wn}\right).$$
For $k$ bounded and greater than or equal to 2, since

$$\prod_{j \in J} \left( 1 + \frac{1}{p_j^k} \right) < \sum_{n=1}^{\infty} \frac{1}{n^k} = \zeta(k),$$

and since $r < n$, then for $k \geq 2$ and $k$ small,

$$P(C_k = r) = \prod_{j \in J} \left( 1 - \frac{1}{p_j^k} \right) + O\left( \frac{1}{w} \right) = \prod_{j \in J} \left( 1 - \frac{1}{p_j^k} \right) + O(n^{-\epsilon}),$$

since $w = n^\epsilon$, thus demonstrating (18).

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