ANISOTROPIC PERIMETER AND ISOPERIMETRIC QUOTIENT OF INNER PARALLEL BODIES

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Abstract. The aim of this note is twofold: to give a short proof of the results in [S. Larson, A bound for the perimeter of inner parallel bodies, J. Funct. Anal. 271 (2016), 610–619] and [G. Domokos and Z. Lángi, The isoperimetric quotient of a convex body decreases monotonically under the eikonal abrasion model, Mathematika 65 (2019), 119–129]; and to generalize them to the anisotropic case.

1. Introduction

Let $\Omega, K \subset \mathbb{R}^n$ be two convex bodies (i.e., compact convex sets) with non-empty interior, and let

$$\Omega \sim \lambda K := \{x \in \mathbb{R}^n : x + \lambda K \subset \Omega\} \quad \lambda \geq 0,$$

be the family of inner parallel sets of $\Omega$ relative to $K$, where $A \sim C := \bigcap_{x \in C} (A - x)$ denotes the Minkowski difference of two convex bodies $A$ and $C$ (see [5, §3.1]). Let

$$r_{\Omega,K} := \max\{\lambda \geq 0 : \lambda K + x \subset \Omega \text{ for some } x \in \mathbb{R}^n\}$$

be the inradius of $\Omega$ relative to $K$, that is, the greatest number $\lambda$ for which $\Omega \sim \lambda K$ is not empty.

For every convex body $C \subset \mathbb{R}^n$, let $P_K(C)$ denote its anisotropic perimeter relative to $K$, defined by

$$P_K(C) := \int_{\partial C} h_K(\nu_C) \, d\mathcal{H}^{n-1},$$

where $h_K(\xi) := \sup\{\langle x, \xi \rangle : x \in K\}$ is the support function of $K$, $\nu_C$ denotes the exterior unit normal vector to $C$, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. If $C$ is a convex body with non-empty interior, $P_K(C)$ coincides with the anisotropic Minkowski content

$$\frac{d}{dt} V_n(C + t K) \bigg|_{t=0} = \lim_{t \to 0} \frac{V_n(C + t K) - V_n(C)}{t},$$

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where $V_n$ denotes the $n$-dimensional volume (see [5, Lemma 7.5.3]). Furthermore, in the Euclidean setting (i.e., when $K$ is the unit ball $B$ of $\mathbb{R}^n$), then $P_B(C) = \mathcal{H}^{n-1}(\partial C)$.

The main results of the present note are Theorems 1.1 and 1.2 below, that have been proved in the Euclidean setting in [4, Thm. 1.2] and [3, Thm. 1.1] respectively. We refer the reader to these papers for motivations and applications.

**Theorem 1.1.** (i) Let $\Omega, K \subset \mathbb{R}^n$ be two convex bodies with non-empty interior. Then it holds that

$$P_K(\Omega \sim \lambda K) \geq \left(1 - \frac{\lambda}{r_{\Omega,K}}\right)^{n-1} P_K(\Omega), \quad \forall \lambda \geq 0. \tag{3}$$

(ii) Equality holds in (3) for some $\lambda^* \in (0, r_{\Omega,K})$ if and only if $\Omega$ is homothetic to a tangential body of $K$. If this is the case equality holds for all $\lambda \geq 0$ and every parallel set $\Omega \sim \lambda K$ is homothetic to $\Omega$ for every $\lambda \in [0, r_{\Omega,K})$.

(We postpone to Section 2 the definition of tangential body.)

**Theorem 1.2.** Let $\Omega, K \subset \mathbb{R}^n$ be two convex bodies with non-empty interior, and let

$$I(\lambda) := \frac{V_n(\Omega \sim \lambda K)}{P_K(\Omega \sim \lambda K)^{\frac{n}{n-1}}}, \quad \lambda \in [0, r_{\Omega,K})$$

denote the anisotropic isoperimetric quotient of $\Omega \sim \lambda K$ relative to $K$.

Then, either $I$ is strictly decreasing on $[0, r_{\Omega,K})$, or there is some value $\lambda^* \in [0, r_{\Omega,K})$ such that $I$ is strictly decreasing on $[0, \lambda^*)$ and constant on $[\lambda^*, r_{\Omega,K})$. Furthermore, in the latter case, for any $\lambda \in [\lambda^*, r_{\Omega,K})$, $\Omega \sim \lambda K$ is homothetic both to $\Omega \sim \lambda^* K$ and to a tangential body of $K$ (more precisely, to an $(n-2)$-tangential body of $K$).

Both results can be interpreted in terms of the level sets of the anisotropic distance function from the boundary of $\Omega$, defined by

$$\delta_{\Omega,K}(x) := \inf\{\rho_K(y-x) : y \in \Omega^c\}, \quad x \in \Omega \tag{4}$$

where $\rho_K(x) := \max\{\lambda \geq 0 : \lambda x \in K\}$ is the gauge function of $K$ and we assume that $K$ contains 0 as an interior point (see [2] for a detailed analysis of $\delta_{\Omega,K}$). Specifically, since $\rho_K(x) \leq 1$ if and only if $x \in K$, it is not difficult to check that $\Omega \sim \lambda K = \{x \in \Omega : \delta_{\Omega,K}(x) \geq \lambda\}$. We remark that related results in the Euclidean setting are contained in [1 §3], where, in particular, one can find the proof of [1 Thm. 1.2] (see p. 104 and Lemma 3.7 therein).
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2. Proof of Theorem 1.1

In the following we shall use the notations of [5]. Let \( C \subset \mathbb{R}^n \) be a convex body. We say that \( x \in \partial C \) is a regular point of \( \partial C \) if \( C \) admits a unique support plane at \( x \). Given two convex bodies \( C, K \subset \mathbb{R}^n \), we say that \( C \) is a tangential body of \( K \) if, for each regular point \( x \) of \( \partial C \), the support plane of \( C \) at \( x \) is also a support plane of \( K \) (see [5, §2.2]). From [5, Thm. 2.2.10] it follows that \( C \) is a tangential body of a ball if and only if it is homothetic to its form body, defined by

\[
C_* := \bigcup_{\nu \in S} \{ x \in \mathbb{R}^n : \langle x, \nu \rangle \leq 1 \},
\]

where \( S \) is the set of outward unit normal vectors to \( \partial C \) at regular points of \( \partial C \).

The definition of \( p \)-tangential body is more involved. Since it is not of primary importance for the exposition of the paper, we refer to [5, §2.2]. In connection with the statement of Theorem 1.2 we limit ourselves to recall that, if \( C \) is a \( p \)-tangential body of \( K \) for some \( p \in \{0, \ldots, n-1\} \), then it is also a tangential body of \( K \).

Given the convex bodies \( K_1, \ldots, K_n \subset \mathbb{R}^n \), we denote by \( V(K_1, \ldots, K_n) \) their mixed volume (see [5, §5.1]). Moreover, for every pair \( C, K \) of convex bodies we define

\[
V_i(C, K) := V(C, \ldots, C, \underbrace{K, \ldots, K}_{\text{i times}}, \ldots, K), \quad i \in \{0, \ldots, n\}.
\]

From now on we shall assume that \( \Omega, K \subset \mathbb{R}^n \) are two convex bodies with non-empty interior. To simplify the notation, we denote by \( r := r_{\Omega,K} \) the inradius of \( \Omega \) relative to \( K \), and we define the functions

\[
v_i(\lambda) := V_i(\Omega \sim \lambda K, K), \quad \lambda \in [0, r], \quad i \in \{0, \ldots, n\}.
\]

We recall that, by [5, Lemma 7.5.3], \( v_0 \) is differentiable and

\[
v'_0(\lambda) = -n v_1(\lambda), \quad \forall \lambda \in [0, r].
\]

Theorem 2.1. (i) The functions

\[
f_i(\lambda) := v_i(\lambda)^{\frac{1}{i}}, \quad i \in \{0, \ldots, n-1\},
\]

are concave in \([0, r]\).

(ii) Assume that there exists \( \lambda^* \in [0, r) \) such that, for \( i = 0 \) or \( i = 1 \),

\[
f_i(\lambda) = \frac{r-\lambda}{r-\lambda^*} f_i(\lambda^*), \quad \forall \lambda \in [\lambda^*, r].
\]

Then, for every \( \lambda \in [\lambda^*, r) \), \( \Omega \sim \lambda K \) is homothetic both to \( \Omega \sim \lambda^* K \), and to a tangential body of \( K \).

Proof. (i) The claim is a direct consequence of the concavity property of the family \( \lambda \mapsto \Omega \sim \lambda K \) (see [5, Lemma 3.1.13]) and of the Generalized Brunn–Minkowski inequality (see [5, Theorem 7.4.5]).
(ii) Since, by (5), \( v_0' = -n v_1 = -nf_1^{n-1} \) and \( v_0(r) = 0 \), if (7) holds for \( i = 1 \) then it holds also for \( i = 0 \). Hence, it is enough to prove the claim only in the case \( i = 0 \).

Therefore, assume that (7) holds for \( i = 0 \) and let \( \lambda \in [\lambda^*, r) \). After a translation, we can assume that \( rK \subseteq \Omega \), so that \( (r - \lambda)K \subseteq \Omega \sim \lambda K =: \Omega^* \). Hence
\[
\begin{align*}
\frac{r - \lambda}{r - \lambda^*} \Omega^* &= \left[ \frac{r - \lambda}{r - \lambda^*} \Omega^* + (\lambda - \lambda^*)K \right] \sim (\lambda - \lambda^*)K \\
&\subseteq \Omega^* \sim (\lambda - \lambda^*)K = \Omega \sim \lambda K.
\end{align*}
\]

On the other hand, (7) implies that the sets \( \frac{r - \lambda}{r - \lambda^*} \Omega^* \) and \( \Omega \sim \lambda K \) have the same volume, so that they must coincide, and the conclusion follows. \( \square \)

The proof of Theorem 1.1(i) is a direct consequence of Theorem 2.1(i), once we recall that \( \mathcal{P}_K(C) = n V_{(i)}(C, K) \) (see [5, (5.34)]). Specifically,
\[
\mathcal{P}_K(\Omega \sim \lambda K) = n v_1(\lambda),
\]

is a concave (non-negative) function in \([0, r]\), so that (3) follows.

Let us prove part (ii). Assume that equality holds in (3) for some \( \lambda_0 \in (0, r) \). By the concavity of \( f_1 \) it follows that the equality holds in (3) for every \( \lambda \in [0, r] \). Hence, the conclusion follows from Theorem 2.1(ii).

3. Proof of Theorem 1.2

Using the notation of Section 2, we recall that
\[
v(\lambda) := V_n(\Omega \sim \lambda K) = v_0(\lambda), \quad p(\lambda) := \mathcal{P}_K(\Omega \sim \lambda K) = n v_1(\lambda), \quad \lambda \in [0, r].
\]

By (5), \( v \) is differentiable everywhere with \( v'(\lambda) = -p(\lambda) \), whereas \( p \) is differentiable almost everywhere and admits left and right derivatives at every point, since \( p^{\frac{1}{n-1}} \) coincides, up to a constant factor, with the concave function \( f_1 \).

Hence, \( I \) is right-differentiable at every point of \([0, r]\), and a direct computation shows that its right derivative is given by
\[
I_+'(\lambda) = -p(\lambda) \frac{p_+'}{n-1} \xi(\lambda), \quad \lambda \in [0, r),
\]
where
\[
(8) \quad \xi(\lambda) := p(\lambda)^2 + \frac{n}{n-1} v(\lambda)p_+'(\lambda).
\]

The proof of Theorem 1.2 is then an easy consequence of the following result.

Lemma 3.1. The function \( \xi \), defined in (8), is non-negative and non-increasing in \([0, r]\). Furthermore, if \( \xi \) vanishes at some point \( \lambda^* \in [0, r) \), then (7) holds for \( i = 0 \) and \( i = 1 \), and, in addition, \( \Omega \sim \lambda^* K \) is homothetic to an \((n-2)\)-tangential body of \( K \).
Proof. The function $\xi(−\lambda)/n^2$ coincides with the function $\Delta(\lambda)$ defined in the proof of Theorem 7.6.19 in [5], where all the stated properties are proved. □

Remark 3.2. In the planar case $n = 2$, Theorem 1.2 gives the stronger conclusion that the isoperimetric quotient is strictly decreasing in $[0, r)$ unless $\Omega$ is homothetic to $K$, in which case it is constant. Specifically, assume that $\xi(\lambda^*) = 0$ for some $\lambda^* \in [0, r)$; the stated property will follow if we can prove that $\Omega = rK$. Since the only $0$-tangential body to $K$ is $K$ itself, from Lemma 3.1 we deduce that, for every $\lambda \in [\lambda^*, r)$, $\Omega \sim \lambda K$ is homothetic to $K$. After a translation we can assume that $\Omega \sim \lambda K = (r - \lambda^*)K$. The concavity property of the family of parallel sets (see [5, Lemma 3.1.13]), together with the fact that $\Omega \sim \lambda K = (r - \lambda)K$ for every $\lambda \in [\lambda^*, r]$, imply that

$$(1 - t)\Omega \subseteq (1 - t)rK \quad \forall t \in [\lambda^*/r, 1].$$

For $t = \lambda^*/r$ we get the inclusion $\Omega \subseteq rK$; on the other hand, the opposite inclusion $\Omega \supseteq rK$ follows from the definition of inradius.

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