GIAMBELLI AND DEGENERACY LOCUS FORMULAS FOR CLASSICAL $G/P$ SPACES

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ABSTRACT. Let $G$ be a classical complex Lie group, $P$ any parabolic subgroup of $G$, and $X = G/P$ the corresponding homogeneous space, which parametrizes (isotropic) partial flags of subspaces of a fixed vector space. In the mid 1990s, Fulton, Pragacz, and Ratajski [41, 46, 103] asked for global formulas which express the cohomology classes of the universal Schubert varieties in flag bundles – when the space $X$ varies in an algebraic family – in terms of the Chern classes of the vector bundles involved in their definition. This has applications to the theory of degeneracy loci of vector bundles and is closely related to the Giambelli problem for the torus-equivariant cohomology ring of $X$. We explain our recent explicit answer to these questions [122] in terms of combinatorial data coming from the Weyl group.

0. Introduction

The theory of degeneracy loci of vector bundles has its roots in the 19th century, motivated by questions in elimination theory and enumerative algebraic geometry. The modern subject began with the work of Thom and Porteous in topology, which was generalized and extended to the algebraic setting by Kempf, Laksov, and Lascoux [43, 67, 76]. The simplest example involves two complex vector bundles $E$, $F$ on a smooth algebraic variety $M$. Given a generic map of vector bundles $f : E \to F$ and $r$ any integer, the locus $M_r$ of points $m \in M$ where $\text{rank}(f_m) \leq r$ is called a degeneracy locus. Thom [125] showed that the homology class of $M_r$ must be Poincaré dual to a universal polynomial in the Chern classes of the vector bundles $E$ and $F$, and Porteous [97] later found this representing polynomial. Such degeneracy loci arise frequently in problems of algebraic geometry and singularity theory, therefore explicit Chern class formulas for these loci can be quite useful. We refer to [32, 46, 56, 101, 127] for surveys, and to [11, 25, 30, 31, 41, 45, 54, 65, 71, 98, 99, 103, 106, 107, 116, 118] for an incomplete list of applications.

In a series of papers in the 1990s, Fulton [39, 40, 41] generalized the work of Kempf-Laksov further, to a map of flagged vector bundles, and studied an analogue of the same problem for the other classical Lie groups. This involved degeneracy loci of isotropic flags of subbundles of a fixed vector bundle, which is equipped with a symplectic or orthogonal form. In all cases, the Schubert polynomials representing the cohomology classes of the loci were defined by an algorithm using divided difference operators (stemming from [0, 26, 27, 52]) applied to a ‘top polynomial’
which represented the class of the diagonal. Related computations were performed at much the same time by Pragacz and Ratajski [103], and other competing theories of Schubert polynomials in the Lie types B, C, and D were discovered [10, 37]. In [41] and [46, §9.5], Fulton and Pragacz asked for combinatorially explicit, global formulas for the classes of degeneracy loci, which have a similar shape for all the classical groups, and are determinantal whenever possible. The aim of this article is to describe the answer to this question which was obtained in [122], building on a series of earlier works, in terms of data coming from the Weyl group.

Graham [50] recast the above degeneracy locus problem using the language of Lie theory, and studied the universal case when the structure group $G$ of the fibre bundles involved is any complex reductive group (see also [13, §6.6]). He observed that the degeneracy locus question of [41] is essentially equivalent to the problem of obtaining a formula for the equivariant Schubert classes in the torus-equivariant cohomology ring of the flag variety $G/B$ (when $G$ is a classical group, there is also the twisted case, when the bilinear form takes values in a line bundle). Indeed, from the point of view of a Lie theorist, there seems to be no reason to exclude the exceptional groups from the degeneracy locus story. We will suggest a reason below why the classical groups appear to be special for this question.

In type A, the double Schubert polynomials of Lascoux and Schützenberger [77, 82] were characterized as the unique polynomials that satisfy the general degeneracy locus formula of [39]. Fomin and Kirillov [37] observed that this strong uniqueness property breaks down in type B, where in fact there is a plethora of available (single) theories. However, the Schubert polynomials of Billey and Haiman [10] impressed us as the most combinatorially explicit theory among those available in the other classical Lie types. These polynomials enjoyed most of the properties of the type A single Schubert polynomials, but their translation (as given in [10]) into Chern class formulas on $G/B$ involved a change of variables and an ensuing loss of combinatorial control. This problem was first addressed by the author [118, 119], using a more natural and geometric substitution of the variables. Ikeda, Mihalcea, and Naruse [59] later introduced double versions of the Billey-Haiman Schubert polynomials and extended the substitution of [118, 119] to this setting – expressing it in a better way, as a ring homomorphism (the geometrization maps of §7.3). With this work, the search for a satisfactory analogue of the Lascoux-Schützenberger theory in the other classical Lie types was finally over.

Although the decision of which theory of Schubert polynomials to use is clearly important, by construction they only provide formulas in terms of the Chern roots of the vector bundles involved. When the initial degeneracy locus problem carries the symmetries of a parabolic subgroup $P$ of $G$, we seek an answer which manifestly exhibits the same symmetries. This should generalize the Jacobi-Trudi determinants and Schur Pfaffians that appear when the Schubert polynomials are evaluated on (maximal) Grassmannian elements of the Weyl group, as in [41, 46, 67, 70, 103]. In other words, we desire formulas that are native to $G/P$, i.e., expressed in terms of Schubert classes that live in the cohomology ring of $G/P$. It turned out that a precise understanding of the Giambelli problem for $H^*(G/P)$, which is closely related to the degeneracy loci formulas above, was necessary for further progress.

The cohomology of $X = G/P$ is a free abelian group on the basis of Schubert classes, the cohomology classes of the Schubert varieties. When $G$ is a classical Lie group, there are certain special Schubert classes among these, which generate the
ring $H^\ast(X)$. This is one place where the fact that $G$ is classical is important: at present, we do not know how to define special classes for the exceptional groups. For classical $G$, one has a good definition of special Schubert varieties, which is uniform across the four types. In this case, the variety $X$ parametrizes partial flags of subspaces of a vector space, which in types B, C, and D are required to be isotropic with respect to an orthogonal or symplectic form. If $X$ is an (isotropic) Grassmannian, then the special Schubert varieties are defined as the locus of (isotropic) linear subspaces which meet a given (isotropic or coisotropic) linear subspace nontrivially, following [16, 90]. The special Schubert varieties on any partial flag variety $X$ are the inverse images of the special Schubert varieties on the Grassmannians to which $X$ projects. The special Schubert classes are the cohomology classes of the special Schubert varieties; in most examples, they are equal to the Chern classes of the universal quotient bundles over $X$, up to a factor of two.

The Giambelli problem challenges us to write a general Schubert class as an explicit polynomial in the above special classes. The papers [17, 19, 48] addressed this question for all (isotropic) Grassmannians, and [20, 122] extended the answer to any classical $G/P$ space. To do this, we had to go beyond the known hermitian symmetric, fully commutative examples, and invent a considerable body of new combinatorics. The Schubert classes are indexed by (typed) $k$-strict partitions, the Giambelli formulas are expressed using Young’s raising operators [88, 129] and studied using a new calculus of these operators [17, 120], the Schur polynomials are extended to theta and eta polynomials, and instead of Young tableaux, we count paths in $k$-transition trees. Ultimately, all of these objects can be understood purely in terms of the combinatorics of the Weyl group of (signed) permutations.

The degeneracy locus problem is equivalent to the Giambelli problem when the space $X$ varies in an algebraic family, and thus would appear to be more difficult. Indeed, in most cases where determinantal formulas for the double Schubert polynomials representing the loci were known, these formulas were significantly more complicated than their single versions – which address the Giambelli problem in that case. The type A paper [20] changed that paradigm: it established the surprising fact that if one uses the language of quiver polynomials, then the answer to the degeneracy locus problem has the same shape as that for the Giambelli problem, and indeed, a near identical proof! This picture was generalized to all classical types in [122], in a synthesis which used all of the above ingredients, and added some new ones. The results were combinatorial splitting formulas for the Schubert polynomials of [10, 59], and direct translations of these into degeneracy loci formulas, with the symmetries native to the appropriate $G/P$ space.

The goal of this paper is to explain the above story. The narrative combines elements from algebraic geometry, Lie theory, and combinatorics, and we have strived to keep the exposition as self-contained as possible. We include one original contribution: a new proof of the main result of [59], which states that the double Schubert polynomials in types B, C, and D represent the Schubert classes. The setup in [59] uses localization in equivariant cohomology, which we do not require here. The key idea – exploited in [120, 122, 123] – is to use the elegant approach to Schubert polynomials via the nilCoxeter algebra and the Yang-Baxter equation, pioneered in [36, 37, 38]. One of the advantages of this approach is that the Schubert polynomials are defined simply and directly in terms of reduced decompositions in the Weyl group, without requiring the use of a ‘top polynomial’. From this point of
view, one can also understand why the stability property of Schubert polynomials is needed: it is only in the stable equivariant cohomology ring that compatibility with divided differences alone (both left and right! – an important insight of [59]) is enough to characterize the universal Schubert classes, up to a scalar factor. Anderson and Fulton [2] have recently also given a different proof of the main theorem of [59], within the framework of degeneracy loci, using a geometric argument based on Kazarian’s Gysin formulas and multi-Schur Pfaffians.

We have made no attempt to provide a survey, and in particular the extensive literature on the Schubert calculus and the equivariant cohomology of homogeneous spaces is barely touched upon. In special cases, there are alternatives to the combinatorial formulas shown here: the reader may consult [2, 9, 11, 58, 60, 73, 79, 93, 124] for examples of what is known, and the papers [16, 17, 20, 59, 120, 122] for further references to related research. Throughout this article, we work with cohomology groups, at times with rational coefficients. However, from these, one can deduce results for cohomology with integer coefficients, and also in the algebraic category, for the Chow groups of algebraic cycles modulo rational equivalence. The necessary modifications to achieve this are explained in detail in [13, 41, 50].

This article is organized as follows. We begin in §1 and §2 with a discussion of Giambelli formulas for Grassmannians, expressing them using the language of raising operators. Section 3 contains general facts about the cohomology of $G/P$ spaces and the Giambelli problem in this context. The combinatorial data coming from the Weyl group and the algebraic objects necessary to state the general degeneracy locus formulas are given in §4 and §5 respectively. In particular, §5.3 contains splitting formulas for Schubert polynomials, which admit direct translations in §6 to Chern class formulas for degeneracy loci. Section 7 outlines the proofs of the main theorems, and §8 contains some questions for the future.

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1. The Giambelli formula of classical Schubert calculus

The main object of study in classical Schubert calculus is the Grassmannian $X = G(m, n)$, which is the set of all $m$-dimensional complex linear subspaces of $V = \mathbb{C}^n$. Given any subset $H$ of $V$, we let $\langle H \rangle$ denote the $\mathbb{C}$-linear span of $H$. Let $e_1, \ldots, e_n$ denote the canonical basis of $\mathbb{C}^n$, and $d = n - m$ be the codimension of the subspaces in $X$. The general linear group $GL_n(\mathbb{C})$ acts transitively on $X$, and the stabilizer of the point $\langle e_1, \ldots, e_m \rangle$ under this action can be identified with the subgroup $P$ of matrices in $GL_n(\mathbb{C})$ of the block form

$$
\begin{pmatrix}
* & * \\
0 & *
\end{pmatrix}
$$

where the 0 in the lower left corner denotes a $d \times m$ zero matrix. In this way we get a description of $X$ as a coset space

$$
X = GL_n(\mathbb{C})/P
$$

from which one can deduce that $X$ is a complex manifold of dimension $md$. The subgroup $P$ is a maximal parabolic subgroup of $GL_n(\mathbb{C})$. A similar analysis shows that the manifold $X$ is isomorphic to $U(n)/(U(m) \times U(d))$, and hence is a compact
manifold. In fact, $X$ is a projective algebraic variety, and may be described by a system of quadratic polynomial equations, known as the Plücker relations. For further details on this and other aspects of this section, we refer to [42, 91].

In the latter half of the 19th century, Hermann Schubert gave a first systematic treatment of enumerative projective geometry [105], in which the Grassmannian $X$ played a prominent part. The course of his study led him to introduce certain natural closed algebraic subsets of $X$, later known as the Schubert varieties [109]. To define them, set $F_i = \langle e_1, \ldots, e_i \rangle$ for each integer $i \in [1,n]$, and define a complete flag of subspaces

$$F_* : 0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n.$$ 

The stabilizer $B \subset GL_n(\mathbb{C})$ of $F_*$ is the Borel subgroup of upper triangular matrices in $GL_n(\mathbb{C})$. In modern language, the Schubert varieties are the closures of the $B$-orbits in $X$. Each $B$-orbit in $X$ is called a Schubert cell; there are finitely many such cells, and they induce a cell decomposition of the manifold $X$.

We call a subset $P \subset [1,n]$ of cardinality $m$ an index set. Any point $\Sigma \in X$ defines an index set $P(\Sigma)$ by

$$P(\Sigma) := \{ p \in [1,n] \mid \Sigma \cap F_p \supseteq \Sigma \cap F_{p-1} \}.$$ 

Observe that $P(\Sigma') = P(\Sigma)$ for any point $\Sigma'$ in the orbit $B.\Sigma \subset X$. On the other hand, given any subspace $\Sigma' \subset V$, one can easily construct a basis $\{g_1, \ldots, g_n\}$ of $V$ such that $F_i = \langle g_1, \ldots, g_i \rangle$ for each $i$ and $\Sigma' = \{\{g_1, \ldots, g_n\} \cap \Sigma\}$. It follows from this that any point $\Sigma' \in X$ such that $P(\Sigma') = P(\Sigma)$ must be in the orbit $B.\Sigma$. In other words, the $B$-orbits (or Schubert cells) in $X$ correspond 1-1 to the index sets $P$. We let $X^\circ_P(F_*)$ denote the Schubert cell given by $P$, that is, 

$$X^\circ_P(F_*) := \{ \Sigma \in X \mid P(\Sigma) = P \}.$$ 

The definition implies that we have a cell decomposition

$$G(m,n) = \coprod_P X^\circ_P(F_*).$$

Suppose that $P = \{ p_1 < \cdots < p_m \}$ is an index set. Any subspace $\Sigma \in X^\circ_P(F_*)$ is spanned by the rows of a unique $m \times n$ matrix $A = \{a_{ij}\}$ in a special reduced row echelon form: there is a pivot entry 1 in position $(i,p_i)$, all other entries in the $i$th row after the pivot are zero, and all entries below the pivot entries are zero. If $j < p_i$ and $j \neq p_r$ for all $r < i$, then then $a_{ij}$ is a free variable, and gives an affine coordinate for the Schubert cell $X^\circ_P$. For example, if $m = 4$, $n = 10$, and $P = \{3,5,6,9\}$, then

$$A = \begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & * & 1 & 0 \end{pmatrix}.$$ 

From this description we see that the $B$-orbit $X^\circ_P$ is isomorphic to affine space $\mathbb{C}^{|P|}$, where $|P|$ is the number of $*$’s in the reduced row echelon form of matrices in the cell, namely

$$|P| = \sum_{j=1}^{m} (p_j - j).$$
At this point it is convenient to introduce a different parametrization for the Schubert cells which makes their codimension apparent: let

$$\lambda_j := d + j - p_j, \ 1 \leq j \leq m.$$  

It is clear that there is a 1-1 correspondence between the vectors $\lambda = (\lambda_1, \ldots, \lambda_m)$ and index sets $P$ (for fixed $m$ and $n$); for example $P = \{3, 5, 6, 9\}$ corresponds to $\lambda = (4, 3, 3, 1)$.

The conditions on the index set $P$ imply that

$$d \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0,$$

equivalently, that $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a partition such that $\lambda_1 \leq d$ and the length $\ell(\lambda)$ (that is, number of non-zero parts $\lambda_j$) is at most $m$. Recall that any partition $\lambda$ can be represented by a Young diagram of boxes, arranged in left-justified rows, with $\lambda_j$ boxes in the $j$th row. The above conditions state that the diagram of $\lambda$ is contained in an $m \times d$ rectangle, which is the Young diagram of the partition $(d^m) = (d, \ldots, d)$. The example shown below corresponds to a Schubert cell in $G(4, 10)$ indexed by the partition $\lambda = (5, 4, 2)$.

We identify a partition with its Young diagram; an inclusion $\lambda \subset \mu$ of partitions corresponds to the containment of their respective diagrams. The weight of $\lambda$, denoted $|\lambda|$, is the total number of boxes in $\lambda$, hence $|\lambda| = \sum_{j=1}^m \lambda_j$, and $\lambda$ is a partition of the integer $|\lambda|$. For each $\lambda$ as above, we have a Schubert cell $X_\lambda^\circ$, which is equal to $X_\lambda^Z$ for the index set $P$ corresponding to $\lambda$. $X_\lambda^Z$ has (complex) dimension $\sum_{j=1}^m (d - \lambda_j) = md - |\lambda| = \dim X - |\lambda|$, and therefore codimension $|\lambda|$ in $X$.

The Schubert variety $X_\lambda(F^\ast)$ is the closure of the Schubert cell $X_\lambda^Z(F^\ast)$; it is an algebraic variety also of codimension $|\lambda|$ in $X$. We have

$$X_\lambda(F^\ast) = \coprod_{\mu \supset \lambda} X_\mu^Z(F^\ast) = \{ \Sigma \in X \mid \dim(\Sigma \cap F_{d+j-\lambda_j}) \geq j, \ 1 \leq j \leq m \}. $$

For each partition $\lambda$ contained in $(d^m)$, let $[X_\lambda] \in H^{2|\lambda|}(X, \mathbb{Z})$ denote the cohomology class Poincaré dual to the cycle defined by $X_\lambda(F^\ast)$. If $F^\ast$ is another complete flag, then there is an element $g$ in $\text{GL}_n(\mathbb{C})$ such that $g \cdot F^\ast = F^\ast$. It follows that $[X_\lambda(F^\ast)] = [X_\lambda(F^\ast)]$, and therefore that the Schubert class $[X_\lambda]$ only depends on the partition $\lambda$, and not on the flag $F^\ast$. The cell decomposition of $X$ implies that the classes of the Schubert varieties give a $\mathbb{Z}$-basis for $H^\ast(X, \mathbb{Z})$. In other words, there is a direct sum decomposition

$$H^\ast(X, \mathbb{Z}) = \bigoplus_{\lambda \subset (d^m)} \mathbb{Z}[X_\lambda].$$

Of course, the cohomology $H^\ast(X, \mathbb{Z})$ is also a ring under the cup product, and dually under the intersection product of homology cycles. It follows that the structure of this ring is determined by intersecting Schubert varieties in general position.

The simplest such varieties are the special Schubert varieties

$$X_r(F^\ast) = \{ \Sigma \in X \mid \Sigma \cap F_{d+1-r} \neq 0 \}.$$
for $1 \leq r \leq d$ (here the index $r$ is identified with the partition $(r, 0, \ldots, 0)$). Historically, it was natural to focus on the $X_r$ since these spaces are the easiest to work with geometrically. The corresponding classes $[X_r]$ are the *special Schubert classes*. These cohomology classes can be realized as characteristic classes of certain universal vector bundles over $G(m, n)$. Let $E'$ denote the tautological rank $m$ vector bundle over $X$, $E$ the trivial rank $n$ vector bundle, and $E'' = E/E'$ the quotient bundle, so that we have a short exact sequence

$$0 \to E' \to E \to E'' \to 0$$

(3)

of vector bundles over $X$. Then $[X_r]$ is by definition the $r$-th *Segre class* of $E'$, or equivalently, the $r$-th *Chern class* of $E''$, denoted $c_r(E'')$.

The work of Pieri [96] and Giambelli [48] established that the special classes $c_r = c_r(E'')$ generate the cohomology ring $H^*(X, \mathbb{Z})$. Giambelli proved the following explicit formula which writes a general Schubert class $[X_\lambda]$ as a polynomial in special classes:

$$[X_\lambda] = \det(c_{\lambda_i + j - i}(E''))_{1 \leq i,j \leq m}.$$  

(4)

In equation (4) and in the remainder of this paper, our convention is that $c_0 = 1$ and $c_r = 0$ whenever $r < 0$. Observe that there are relations among the $c_r$ in $H^*(X, \mathbb{Z})$, so that the right hand side of formula (4) is not unique. However, the natural inclusion $G(m, n) \hookrightarrow G(m + 1, n + 1)$ induces a surjection

$$H^*(G(m + 1, n + 1), \mathbb{Z}) \to H^*(G(m, n), \mathbb{Z}).$$

For a fixed partition $\lambda$ and codimension $d$, the Giambelli polynomial in (4) is the unique one that is preserved under the above map, for all $m$ greater than or equal to the number of (non-zero) parts of $\lambda$.

For our purposes here it will be important to rewrite formula (4) using A. Young’s *raising operators* [129]. An *integer sequence* is a sequence of integers $\alpha = (\alpha_1, \alpha_2, \ldots)$ only finitely many of which are non-zero. Given any integer sequence $\alpha$ and natural numbers $i < j$, we define

$$R_{ij}(\alpha) := (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots).$$

A raising operator $R$ is any monomial in these $R_{ij}$’s. If $(c_1, c_2, \ldots)$ is any ordered set of commuting independent variables and $\alpha$ is an integer sequence, we let $c_\alpha := \prod_{i \geq 1} c_{\alpha_i}$, with the understanding that $c_0 = 1$ and $c_r = 0$ if $r < 0$. For any raising operator $R$, set $R_{c_\alpha} = c_{R\alpha}$ (note that we slightly abuse the notation here and consider that the raising operator $R$ acts on the index $\alpha$, and not on the monomial $c_\alpha$ itself). Consider the raising operator expression

$$R^0 := \prod_{i<j} (1 - R_{ij})$$

which we expand as a formal power series in the $R_{ij}$. Then for any integer sequence $\alpha$, we have

$$R^0 c_\alpha = \det(c_{\alpha_i + j - 1})_{i,j}.$$  

(5)

Equation (5) is a formal consequence of the *Vandermonde identity*

$$\prod_{1 \leq i < j \leq m} (x_i - x_j) = \det(x_i^{m-j})_{1 \leq i,j \leq m};$$

for $1 \leq r \leq d$ (here the index $r$ is identified with the partition $(r, 0, \ldots, 0)$). Historically, it was natural to focus on the $X_r$ since these spaces are the easiest to work with geometrically. The corresponding classes $[X_r]$ are the *special Schubert classes*. These cohomology classes can be realized as characteristic classes of certain universal vector bundles over $G(m, n)$. Let $E'$ denote the tautological rank $m$ vector bundle over $X$, $E$ the trivial rank $n$ vector bundle, and $E'' = E/E'$ the quotient bundle, so that we have a short exact sequence

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Equation (5) is a formal consequence of the *Vandermonde identity*

$$\prod_{1 \leq i < j \leq m} (x_i - x_j) = \det(x_i^{m-j})_{1 \leq i,j \leq m};$$
for a proof of this see e.g. [21]. It follows that we may rewrite (4) as
\[ X_\lambda = R^0 c_\lambda(E''), \]
where \( c_\lambda(E'') = \prod_i c_{\lambda_i}(E'') \) denotes a monomial in the Chern classes of \( E'' \), and \( R^0 \) is applied to \( c_\lambda \) as above.

**Example 1.** We have

\[ [X_{(5,4,2)}] = (1 - R_{12})(1 - R_{13})(1 - R_{23}) c_{(5,4,2)} \]
\[ = c_{(5,4,2)} - c_{(6,3,2)} - c_{(6,4,1)} - c_{(5,5,1)} + c_{(7,3,1)} + c_{(6,4,1)} + c_{(6,5,0)} - c_{(7,4,0)} \]
\[ = c_5 c_4 c_2 - c_6 c_3 c_2 - c_5^2 c_1 + c_7 c_3 c_1 + c_6 c_5 - c_7 c_4 = \begin{vmatrix} c_5 & c_6 & c_7 \\ c_3 & c_4 & c_5 \\ 1 & c_1 & c_2 \end{vmatrix}. \]

Soon after he proved (4), Giambelli published a second paper [49] where he studied a parallel formalism in the theory of symmetric polynomials. For any integer \( r \), let \( e_r(Y_{(d)}) \) denote the \( r \)-th elementary symmetric polynomial in the commuting variables \( Y_{(d)} = (y_1, \ldots, y_d) \). Given a partition \( \mu \) with at most \( d \) non-zero parts, consider
\[ s_\mu(Y_{(d)}) = \det(y_i^{\mu_i + d-j})_{1 \leq i,j \leq d} / \det(y_i^{d-j})_{1 \leq i,j \leq d}. \]
The \( s_\mu(Y_{(d)}) \) for varying \( \mu \) may be identified with the polynomial characters of the general linear group \( GL_d(\mathbb{C}) \); this had been established a few years earlier by Schur in his 1901 thesis [119] (in fact, equation (7) is a special case of the Weyl character formula). For any partition \( \lambda \), let \( \bar{\lambda} \) be the conjugate partition, whose Young diagram is the transpose of the diagram of \( \lambda \). Then Jacobi and Trudi proved that the Schur polynomial \( s_{\bar{\lambda}}(Y_{(d)}) \) satisfies
\[ s_{\bar{\lambda}}(Y_{(d)}) = R^0 e_\lambda(Y_{(d)}) = \det(e_{\lambda_i + j - i}(Y_{(d)}))_{i,j} \]
for any \( \lambda \subset (d^m) \), where \( e_\lambda := \prod_i e_{\lambda_i} \). We may thus consider \( s_{\bar{\lambda}} \) as a polynomial in the algebraically independent variables \( e_r \), for \( 1 \leq r \leq d \). In the theory of characteristic classes, the variables \( y_1, \ldots, y_d \) represent the Chern roots of the quotient vector bundle \( E'' \). Using \( c(E'') \) to denote the total Chern class \( 1 + c_1(E'') + c_2(E'') + \cdots \) of \( E'' \), we obtain the following restatement of equations (4) and (6).

**Theorem 1** (Classical Giambelli, [48]). For any partition \( \lambda \) whose diagram fits inside an \( m \times (n - m) \) rectangle, we have
\[ [X_\lambda] = s_{\bar{\lambda}}(c(E'')) \]
in the cohomology ring of \( G(m,n) \).

For more on the connection between the representation theory of the general linear group and the classical Schubert calculus, see [3] [4] [177].

2. **Giambelli formulas for isotropic Grassmannians**

The study of homogeneous spaces of Lie groups was extended further during the first half of the twentieth century by the work of Elie Cartan [22] [23] and Ehresmann [29]. They considered the irreducible compact hermitian symmetric spaces, which generalize the Grassmannian \( G(m,n) \), and began exploring their cohomology rings. Rather than proceeding along the lines of the classical Schubert...
calculus, this work used Cartan’s theory of invariant differential forms. It was only in the 1980s that analogues of Pieri’s rule and Giambelli’s formula were obtained for all hermitian symmetric Grassmannians, in the work of Hiller and Boe \cite{55, 100}. More recently, Pragacz and Ratajski \cite{102, 104} proved Pieri type rules and Buch, Kresch, and the author \cite{16, 17, 19} generalized both the Pieri and Giambelli formulas of \cite{55, 100} to arbitrary symplectic and orthogonal Grassmannians, using different notions of special Schubert classes. We will follow the references \cite{17, 19} in this section.

Let \( V = \mathbb{C}^N \) and equip \( V \) with a non-degenerate skew-symmetric or symmetric bilinear form \( (\ , \) \). A subspace \( \Sigma \) of \( V \) is called isotropic if the restriction of \( (\ , \) \) to \( \Sigma \) vanishes identically. Since the form is non-degenerate, the dimension of any isotropic subspace is at most \( N/2 \). Given a nonnegative integer \( m \leq N/2 \), we let \( X \) denote the complex manifold which parametrizes all the isotropic subspaces of dimension \( m \) in \( V \). This space has a transitive action of the group \( G = \text{Sp}(V) \) or \( G = \text{SO}(V) \) of linear automorphisms preserving the form on \( V \), unless \( m = N/2 \) and the form is symmetric. In the latter case the space of isotropic subspaces has two isomorphic connected components, each a single \( \text{SO}(V) \) orbit.

An isotropic flag \( F_* \) is a complete flag \( 0 = F_0 \subset F_1 \subset \cdots \subset F_N = V \) of subspaces of \( V \) such that \( F_i = F_j^\bot \) whenever \( i + j = N \); in particular, \( F_i \) is an isotropic subspace for all \( i \leq N/2 \). Let \( B \subset G \) denote the Borel subgroup which is the stabilizer of the flag \( F_* \). The Schubert cells in \( X \) relative to the flag \( F_* \) are the orbit closures for the natural action of \( B \) on \( X \). We call a subset \( P \) of \( [1, N] \) of cardinality \( m \) an index set if for all \( i, j \in P \) we have \( i + j \neq N + 1 \). Any point \( \Sigma \in X \) defines an index set \( P(\Sigma) \) by the prescription \((\ref{1})\), since no vector in \( F_j \setminus F_{j-1} \) is orthogonal to a vector in \( F_{N+1-j} \setminus F_{N-j} \). In the same manner as in \((\ref{1})\) equation \((\ref{2})\) establishes a one to one correspondence between Schubert cells \( X^*_P(F_*) \) relative to \( F_* \) and index sets \( P \).

The closures of the Schubert cells are the Schubert varieties \( X_P(F_*) \), and their classes \([X_P]\) in \( H^*(X, \mathbb{Z}) \) are the Schubert classes, which form an additive basis of \( H^*(X, \mathbb{Z}) \). Moreover, there are special Schubert varieties, consisting of the locus of subspaces \( \Sigma \) in \( X \) which meet a given subspace \( F_j \) non-trivially, and corresponding special Schubert classes, which generate the cohomology ring of \( X \). In the following sections, we will see that the Schubert varieties and classes may equivalently be indexed by \( k \)-strict partitions and typed \( k \)-strict partitions. As in \((\ref{1})\) this is a convention which makes their codimension (or cohomological degree) apparent, and we will require in order to state the Giambelli formulas of this section.

2.1. Symplectic Grassmannians. Suppose that \( N = 2n \) is even and the form \((\ , \) \) is skew-symmetric, so that \( X = \text{IG}(m, 2n) \) is a symplectic Grassmannian. Write \( m = n-k \) for some \( k \) with \( 0 \leq k \leq n-1 \). If \( k = 0 \), then \( X \) is the Lagrangian Grassmannian \( \text{LG}(n, 2n) \), and if \( k = n-1 \), then \( X \) is projective \( 2n-1 \)-space \( \mathbb{P}^{2n-1} \), since every line through the origin in \( V \) is isotropic. These are the only hermitian symmetric examples. The space \( \text{IG}(n-k, 2n) \) may be identified with a quotient \( \text{Sp}_{2n}(\mathbb{C})/P_k \) of the symplectic group \( \text{Sp}_{2n}(\mathbb{C}) \) by a maximal parabolic subgroup \( P_k \).

For instance, the subgroup \( P_0 \) is known as the Siegel parabolic and consists of those matrices of the symplectic group whose lower left quadrant is an \( n \times n \) zero matrix (for the standard symplectic form).

Let \( \Delta^o = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i < j\} \) and define a partial order on \( \Delta^o \) by agreeing that \((i', j') \leq (i, j)\) if \( i' \leq i \) and \( j' \leq j \). A subset \( D \) of \( \Delta^o \) is an order ideal...
if \((i, j) \in D\) implies \((i', j') \in D\) for all \((i', j') \in \Delta^\circ\) with \((i', j') \leq (i, j)\). In the next figure, the pairs \((i, j)\) in a typical finite order ideal are displayed as positions in a matrix above the main diagonal.

\[
\begin{array}{cccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\end{array}
\]

A partition \(\lambda\) is \(k\)-strict if no part \(\lambda_j\) greater than \(k\) is repeated; if \(k = 0\) this means that \(\lambda\) is a strict partition, i.e., has distinct non-zero parts. To any \(k\)-strict partition \(\lambda\) we associate the order ideal

\[\mathcal{C}(\lambda) := \{(i, j) \in \Delta^\circ \mid \lambda_i + \lambda_j > 2k + j - i\}.\]

In general, for any fixed positive \(k\) and any finite order ideal \(D\) of \(\Delta^\circ\), one can construct a \(k\)-strict partition \(\lambda\) such that \(\mathcal{C}(\lambda) = D\).

The Schubert varieties on \(IG(n - k, 2n)\) are indexed by \(k\)-strict partitions whose diagrams fit in an \((n - k) \times (n + k)\) rectangle. Any such \(\lambda\) corresponds to an index set \(\mathcal{P}(\lambda) = \{p_1(\lambda) < \cdots < p_m(\lambda)\}\) by the prescription

\[p_j(\lambda) := n + k + j - \lambda_j - \#\{i < j \mid (i, j) \in \mathcal{C}(\lambda)\}.\]

If \(F_\bullet\) is a fixed isotropic flag of subspaces in \(V\), we obtain the Schubert cell

\[(10) \quad X_\lambda^\circ(F_\bullet) := \{\Sigma \in X \mid \mathcal{P}(\Sigma) = \mathcal{P}(\lambda)\}\]

and the Schubert variety \(X_\lambda(F_\bullet)\) is the closure of this cell. One can show that

\[(11) \quad X_\lambda(F_\bullet) = \{\Sigma \in X \mid \dim(\Sigma \cap F_{p_j(\lambda)}) \geq j \quad \forall 1 \leq j \leq \ell(\lambda)\},\]

where \(\ell(\lambda)\) denotes the number of (non-zero) parts of \(\lambda\); see e.g. [16, §4.2] and [19, App. A]. This variety has codimension \(|\lambda|\) and defines, using Poincaré duality, a Schubert class \([X_\lambda]\) in \(H^{2|\lambda|}(IG, \mathbb{Z})\).

The \textit{special Schubert varieties} are given by

\[X_r(F_\bullet) = \{\Sigma \in X \mid \Sigma \cap F_{n+k+1-r} \neq 0\}\]

for \(1 \leq r \leq n + k\), and their classes \([X_r]\) are the \textit{special Schubert classes}. Let \(E'\) denote the tautological rank \((n - k)\) vector bundle over \(IG(n - k, 2n)\), \(E\) the trivial rank \(2n\) vector bundle, and \(E'' = E/E'\) the quotient bundle. As in the example of the type A Grassmannian in [11] we have \([X_r] = c_r(E'')\) for \(1 \leq r \leq n + k\).

For any \(k\)-strict partition \(\lambda\), we define the operator

\[(12) \quad R^\lambda := \prod_{i < j}(1 - R_{ij}) \prod_{(i, j) \in \mathcal{C}(\lambda)} (1 + R_{ij})^{-1}\]

where the first product is over all pairs \(i < j\) and second product is over pairs \(i < j\) such that \(\lambda_i + \lambda_j > 2k + j - i\). If \(c = 1 + c_1 t + c_2 t^2 + \cdots\) is any formal power series in commuting variables \(c_r\), we define the \textit{theta polynomial} \(\Theta_\lambda\) by

\[(13) \quad \Theta_\lambda(c) := R^\lambda c_\lambda.\]
Theorem 2 (Giambelli for IG, [17]). For any k-strict partition λ whose diagram fits inside an \((n-k) \times (n+k)\) rectangle, we have
\[
[X_\lambda] = \Theta_\lambda(c(E''))
\]
in the cohomology ring of IG\((n-k, 2n)\).

Example 2. Let \(k = 1\). Then the following holds in the ring \(H^*(IG(4, 10), \mathbb{Z})\).
\[
[X_{(3,1,1)}] = \frac{1 - R_{12}}{1 + R_{12}} (1 - R_{13})(1 - R_{23}) c_{(3,1,1)} = (1 - 2R_{12} + 2R_{12}^2)(1 - R_{13} - R_{23}) c_{(3,1,1)}
\]
\[
= c_{(3,1,1)} - 2c_{(4,0,1)} - c_{(4,1)} + 2c_5 - c_{(3,2)} + 2c_{(4,1)} - 2c_5 = c_3 c_2^2 - c_4 c_1 - c_3 c_2.
\]

Comparing (14) with (16), we see that the polynomials \(\Theta_\lambda\) play the role of the Schur polynomials for the Giambelli problem on IG\((n-k, 2n)\). An important difference with the story for the type A Grassmannian is that there are relations among the \(c_r\) which persist even as \(n \to \infty\), namely:
\[
\frac{1 - R_{12}}{1 + R_{12}} c_{(r,r)} = c_r^2 + 2 \sum_{i=1}^{r} (-1)^i c_{r-i} c_{r-i} = 0 \quad \text{for all } r > k.
\]

Another difference is that the raising operator expressions \(R^\lambda\) which enter in (13) depend on the partition \(\lambda\) (see also Example 3 below).

We next specialize the above to the Lagrangian Grassmannian LG\((n, 2n)\), which is the case where \(k = 0\). The Schubert classes in \(H^*(LG(n, 2n), \mathbb{Z})\) are indexed by strict partitions \(\lambda\) whose diagrams fit inside a square of side \(n\). The theta polynomial (13) specializes to a \(Q\)-polynomial
\[
Q_\lambda(c) := \prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}} c_\lambda.
\]

The Giambelli formula of Theorem 2 for LG becomes
\[
[X_\lambda] = Q_\lambda(c(E'')).
\]

Following Pragacz [100], formula (16) may be expressed using a Schur Pfaffian, as follows. For partitions \(\lambda = (a, b)\) with only two parts, we have
\[
[X_{(a,b)}] = \frac{1 - R_{12}}{1 + R_{12}} c_{(a,b)} = c_a c_b - 2c_{a+1} c_{b-1} + 2c_{a+2} c_{b-2} - \cdots
\]
while for \(\lambda\) with 3 or more parts,
\[
[X_\lambda] = \text{Pfaffian}([X_{(\lambda_i, \lambda_j)}])_{1 \leq i < j \leq 2^\ell},
\]
where \(\ell\) is the least positive integer such that \(2^\ell \geq \ell(\lambda)\).

As we alluded to above, the identities (17) and (18) go back to the work of Schur on the projective representations of symmetric groups [50, 64, 111, 114], where he introduced a family of symmetric functions \(\{Q_\lambda(X)\}\) known as Schur \(Q\)-functions. We let \(X = (x_1, x_2, \ldots)\) be a list of variables, define \(q_r(X)\) by the equation
\[
\prod_{i=1}^{\infty} \frac{1 + x_i t}{1 - x_i t} = \sum_{r=0}^{\infty} q_r(X)t^r
\]
and then use the same relations (17) and (18) with \(q_r(X)\) in place of \(c_r\) to define \(Q_{(a,b)}(X)\) and then \(Q_\lambda(X)\), for each strict partition \(\lambda\). Once more we emphasize that there are relations among the \(q_r\), the simplest being \(q_1^2 = 2q_2\); hence, the above polynomials which define \(Q_\lambda(X)\) are not uniquely determined. However, the
equivalece of the raising operator and Pfaffian definitions of $Q_\lambda$ is based on the following Pfaffian identity from [13]:

\[
\prod_{1 \leq i < j \leq 2r} \frac{x_i - x_j}{x_i + x_j} = \text{Pfaffian} \left( \frac{x_i - x_j}{x_i + x_j} \right)_{1 \leq i, j \leq 2r}
\]

which holds in the quotient field of $\mathbb{Z}[x_1, \ldots, x_{2r}]$.

**Remark 1.** Although the definition (15) is not standard, it is in direct analogy with the usage of the term ‘Schur polynomial’ in type A. We reserve the name ‘$Q$-polynomial’ for the polynomial in the variables $c_r$ given in [13], and also for its principal specialization, when $c_r$ is replaced by $q_r(X)$ for each integer $r$. This nomenclature extends to the theta polynomials; compare (13) with (34) in §5.2

**Example 3.** Assume that $k > 0$, and let $\lambda$ be a $k$-strict partition. Then if $\lambda_i + \lambda_j \leq 2k + j - i$ for all $i < j$, we have

\[
\Theta_\lambda(c) = \prod_{i < j} (1 - R_{ij}) c_\lambda = \det(c_{\lambda_i + j - i})_{i, j},
\]

while if $\lambda_i + \lambda_j > 2k + j - i$ whenever $1 \leq i < j \leq \ell(\lambda)$, we have

\[
\Theta_\lambda(c) = \prod_{i < j} \frac{1 - R_{ij}}{1 + R_{ij}} c_\lambda = \text{Pfaffian}(c_{\lambda_i, \lambda_j})_{i < j}.
\]

We deduce that as $\lambda$ varies, the polynomial $\Theta_\lambda(c)$ interpolates between the Jacobi-Trudi determinant (19) and the Schur Pfaffian (20).

### 2.2. Orthogonal Grassmannians

Consider the case where $m < N/2$ and the form $(\ , \ )$ is symmetric, so that $X = \text{OG}(m, N)$ is an orthogonal Grassmannian. If $N = 2n + 1$ is odd, then the Schubert varieties in $\text{OG}(m, 2n + 1)$ are indexed by the same set of $k$-strict partitions that index the Schubert varieties in $\text{IG}(m, 2n)$.

For any $k$-strict partition $\lambda$, let $\ell_k(\lambda)$ denote the number of parts $\lambda_i$ of $\lambda$ which are strictly greater than $k$. Let $E''_{\text{IG}}$ and $E''_{\text{OG}}$ be the universal quotient vector bundles over $\text{IG}(n - k, 2n)$ and $\text{OG}(n - k, 2n + 1)$, respectively. One then knows (see e.g. [5] §3.1) that the map which sends $c_r(E''_{\text{IG}})$ to $c_r(E''_{\text{OG}})$ for all $r$ extends to an isomorphism of graded rings $H^*(\text{IG}, \mathbb{Q}) \to H^*(\text{OG}, \mathbb{Q})$, which sends a Schubert class $[X_\lambda]$ on $\text{IG}$ to $2^{r(\lambda)}$ times the corresponding Schubert class on $\text{OG}$. This isomorphism shows that the Schubert calculus on symplectic and odd orthogonal Grassmannians coincides, up to well determined powers of two. In particular, one can easily transfer the Giambelli formula of (2.1) to $\text{OG}(m, 2n + 1)$.

We next assume that $N = 2n$ is even, so that $m = n - k$ with $k > 0$. A typed $k$-strict partition is a pair consisting of a $k$-strict partition $\lambda$ together with an integer in $\{0, 1, 2\}$ called the *type* of $\lambda$, and denoted $\text{type}(\lambda)$, such that $\text{type}(\lambda) > 0$ if and only if $\lambda_i = k$ for some $i \geq 1$. To any typed $k$-strict partition $\lambda$ we associate the order ideal

\[
C'(\lambda) := \{(i, j) \in \Delta^\circ \mid \lambda_i + \lambda_j \geq 2k + j - i\}
\]

in $\Delta^\circ$. The Schubert cells in the cohomology of the even orthogonal Grassmannian $X = \text{OG}(n - k, 2n)$ are indexed by the typed $k$-strict partitions $\lambda$ whose diagrams are contained in an $(n - k) \times (n + k - 1)$ rectangle. For any such $\lambda$, define the index
function $p_j = p_j(\lambda)$ by

$$p_j(\lambda) := n + k + j - \lambda_j - \# \{ i < j \mid (i, j) \in C'(\lambda) \}$$

$$- \begin{cases} 1 & \text{if } \lambda_j > k, \text{ or } \lambda_j = k < \lambda_{j-1} \text{ and } n + j + \text{type}(\lambda) \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

We obtain an index set $\mathcal{P}(\lambda)$ associated to any typed $k$-strict partition $\lambda$ as above, and a Schubert cell $X^\lambda_p(F_\bullet)$ defined by \([10]\).

The Schubert variety $X_p(F_\bullet)$ is best defined as the closure of $X^\lambda_p(F_\bullet)$, since a geometric description of $X_p(F_\bullet)$ analogous to \([11]\) involves subtle parity conditions (see \([16]\) App. A)). Fix a maximal isotropic subspace $L$ of $V$, so that $\dim(L) = n$. Two maximal isotropic subspaces $E$ and $F$ of $V$ are said to be in the same family if $\dim(E \cap F) \equiv n \pmod{2}$. The cohomology classes $[X_p]$ of the Schubert varieties $X_p(F_\bullet)$ in OG are are independent of the choice of isotropic flag $F_\bullet$ as long as $F_n$ is in the same family as $L$. If $\mathcal{P}$ corresponds to $\lambda$, then the associated Schubert class $[X_\lambda] = [X_p]$ in $H^{[\lambda]}(X, \mathbb{Z})$ is said to have a type which agrees with the type of $\lambda$.

The special Schubert varieties in OG$(m, 2n)$ can be defined as before by a single Schubert condition, as the locus of $\Sigma$ is in $X$ which intersect a given isotropic subspace or its orthogonal complement non-trivially (see \([16\] §3.2)). The corresponding special Schubert classes

$$(21) \quad \tau_1, \ldots, \tau_{k-1}, \tau_k, \tau_k', \tau_{k+1}, \ldots, \tau_{n+k-1}$$

are indexed by the typed $k$-strict partitions with a single non-zero part, and generate the cohomology ring $H^*(X, \mathbb{Z})$. Here $\text{type}(\tau_k) = 1$, $\text{type}(\tau_k') = 2$, and if

$$0 \to E' \to E \to E'' \to 0$$

denotes the universal sequence of vector bundles over $X$, then we have

$$(22) \quad c_r(E'') = \begin{cases} \tau_r & \text{if } r < k, \\ \tau_k + \tau_k' & \text{if } r = k, \\ 2\tau_r & \text{if } r > k. \end{cases}$$

We set $c_\alpha = \prod_i c_{\alpha_i}$. Given any typed $k$-strict partition $\lambda$, we define the operator

$$(23) \quad R^\lambda := \prod_{i < j} (1 - R_{ij}) \prod_{(i, j) \in C'(\lambda)} (1 + R_{ij})^{-1}$$

where the first product is over all pairs $i < j$ and the second product is over pairs $i < j$ such that $\lambda_i + \lambda_j \geq 2k + j - i$. Let $R$ be any finite monomial in the operators $R_{ij}$ which appears in the expansion of the power series $R^\lambda$ in \((23)\). If $\text{type}(\lambda) = 0$, then set $R \ast c_\lambda := c_{R \lambda}$. Suppose that $\text{type}(\lambda) > 0$, let $d = \ell_k(\lambda) + 1$ be the index such that $\lambda_d = k < \lambda_{d-1}$, and set $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_{d-1}, \alpha_{d+1}, \ldots, \alpha_\ell)$ for any integer sequence $\alpha$ of length $\ell$. If $R$ involves any factors $R_{ij}$ with $i = d$ or $j = d$, then let $R \ast c_\lambda := \frac{1}{2} c_{R \lambda}$. If $R$ has no such factors, then let

$$R \ast c_\lambda := \begin{cases} \tau_k^i c_{\tilde{R} \lambda} & \text{if } \text{type}(\lambda) = 1, \\ \tau_k^i c_{\tilde{R} \lambda} & \text{if } \text{type}(\lambda) = 2. \end{cases}$$

We define the eta polynomial $H_\lambda$ by

$$(24) \quad H_\lambda(c) := 2^{-\ell_k(\lambda)} R^\lambda \ast c_\lambda.$$
Example 4. Let $H$ be the Weyl group of $G$, and let $\Delta$ be the set of positive simple roots $\alpha$ which generate the simple reflections $s_\alpha$, which are related to the variables $c_r$ by the formal equations \([22]\).

**Theorem 3** (Giambelli for OG, [19]). For every typed $k$-strict partition $\lambda$ whose diagram fits inside an $(n-k) \times (n+k-1)$ rectangle, we have
\[
[X_\lambda] = H_\lambda(c(E''))
\]
in the cohomology ring of $OG(n-k, 2n)$.

**Example 4.** Let $k = 1$ and consider the typed partition $\lambda = (3, 1, 1)$ with type$(\lambda) = 2$. Then the corresponding Schubert class in $H^*(OG(4, 10), \mathbb{Z})$ satisfies
\[
[X_{(3,1,1)}] = \frac{1}{2} \frac{1 - R_{12} - R_{13}}{1 + R_{12} + R_{13}} (1 - R_{23}) \ast c_{(3,1,1)}
\]
\[
= \frac{1}{2} (1 - 2R_{12} + 2R_{12}^2)(1 - 2R_{13} - R_{23}) \ast c_{(3,1,1)}
\]
\[
= \tau_3 \tau_1' (\tau_1 + \tau_1') - 2\tau_3 \tau_1' - \tau_3 \tau_2 + \tau_5.
\]
In general, the Giambelli formula \([25]\) expresses the Schubert class $[X_\lambda]$ as a polynomial in the special Schubert classes \([24]\) with integer coefficients.

To conclude this section, we consider the space of all maximal isotropic subspaces of an even dimensional orthogonal vector space $V$. This set is a disjoint union of two connected components, each giving one family (or $SO(V)$-orbit) of such subspaces. The (irreducible) algebraic variety $OG = OG(n, 2n)$ is defined by choosing one of these two components. If $H$ is a hyperplane in $V$ on which the restriction of the symmetric form is non-degenerate, then the map $\Sigma \rightarrow \Sigma \cap H$ gives an isomorphism between $OG(n, 2n)$ and $OG(n-1, 2n-1)$. The analysis for the (type B) odd orthogonal Grassmannian in the maximal isotropic case therefore applies to $OG$. In particular, the Schubert classes $X_\lambda$ may be indexed by strict partitions $\lambda$ with $\lambda_1 \leq n - 1$, and the Giambelli formula for $OG$ reads
\[
[X_\lambda] = P_\lambda(c(E''))
\]
where $E'' \rightarrow OG$ is the universal quotient bundle and the $P$-polynomial $P_\lambda$ is related to $Q_\lambda$ by the equation $P_\lambda = 2^{-\ell(\lambda)}Q_\lambda$.

### 3. Cohomology of $G/P$ spaces

In this section $G$ will denote a connected complex reductive Lie group. The main examples we will consider are the classical groups: the general linear group $GL_n(\mathbb{C})$, the symplectic group $Sp_{2n}(\mathbb{C})$, and the (odd and even) orthogonal groups $SO_N(\mathbb{C})$. A closed algebraic subgroup $P$ of $G$ is called a parabolic subgroup if the quotient $X = G/P$ is compact, or equivalently, a projective algebraic variety. We proceed to generalize the constructions of the previous section to the manifold $X$.

The Borel subgroups $B$ of $G$ are the maximal connected solvable subgroups; a subgroup $P$ of $G$ is parabolic if and only if it contains a Borel subgroup. Fix a Borel subgroup $B$, let $T \cong (\mathbb{C}^*)^r$ be the maximal torus in $B$, and $W = N_G(T)/T$ be the Weyl group of $G$. The simple reflections $s_\alpha$ which generate $W$ are indexed by the set $\Delta$ of positive simple roots $\alpha$, and are also in one-to-one correspondence with the vertices of the Dynkin diagram $D$ associated to the root system. The parabolic subgroups $P$ containing $B$ are in bijection with the subsets $\Delta_P$ of $\Delta$ (or $D$) – as in the figures of \([30]\). In particular each root $\alpha \in \Delta$ corresponds to a maximal
parabolic subgroup, which is associated to the subset $\Delta \setminus \{\alpha\}$. We let $W_P$ denote the subgroup of $W$ generated by all the simple reflections in $\Delta_P$.

The length $\ell(w)$ of an element $w$ in $W$ is equal to the least number of simple reflections whose product is $w$. We denote by $w_0$ the element of longest length in $W$. It is known that every coset in $W/W_P$ has a unique representative $w$ of minimal length; we denote the set of all minimal length $W_P$-coset representatives by $W^P$. The manifold $X$ has complex dimension equal to the length of the longest element in $W^P$. The set $W^P$ (or the coset space $W/W_P$) indexes the Schubert cells, varieties, and classes on $X$ as follows. First, the Bruhat decomposition $G = \bigsqcup_{w \in W} BwB$ of the group $G$ induces a cell decomposition

$$G/P = \bigsqcup_{w \in W^P} BwP/P$$

of $X = G/P$. The cell $BwP/P$ is isomorphic to the affine space $C^{\ell(w)}$. Since we prefer the length of the indexing element to equal the codimension of the cell in $X$, we define the Schubert cell $X_w$ to be $Bw_0wP/P$. The Schubert variety $X_w$ is defined as the closure of the Schubert cell $X_w^0$, and its cohomology class $[X_w]$ lies in $H^{\ell(w)}(X, \mathbb{Z})$. This completes the additive description of the cohomology of $X$, as a free abelian group on the basis of Schubert classes $[X_w]$:

$$H^*(X, \mathbb{Z}) = \bigoplus_{w \in W^P} \mathbb{Z}[X_w].$$

### 3.1. The Giambelli problem

We seek to generalize the classical Schubert calculus of $\mathbb{P}$ to the $G/P$ spaces in their natural cell decompositions described above. By this we mean to understand as explicitly as possible the multiplicative structure of the cohomology ring $H^*(G/P, \mathbb{Z})$, expressed in the basis of Schubert classes. However, it is clear that the extension of the classical Pieri and Giambelli formulas to $G/P$ depends on a choice of special Schubert classes which generate the cohomology ring. For arbitrary $G$, it is fortunate that there is one example of a parabolic $P$ where the choice of generating set is clear: the case when $P = B$ is a Borel subgroup of $G$, and $X = G/B$ is the (complete) flag variety of $G$. In this case, the cohomology ring $H^*(X, \mathbb{Q})$ is generated by the classes of the Schubert divisors $X_{\alpha_0}$, one for each simple root $\alpha$. Equivalently, we can form a generating set by taking any $\mathbb{Z}$-basis of $H^2(X, \mathbb{Z})$, which is related to the Schubert divisor basis by a linear change of variables. In a seminal paper, Borel [12], using the theory of group characters and associated characteristic classes of line bundles over $G/B$, gave an invariant, group theoretic approach to this question, which we recall below.

A character of the group $B$ is a homomorphism of algebraic groups $B \to \mathbb{C}^*$. We denote the abelian group of characters of $B$ by $\hat{B}$. Observe that any character $\chi$ of $B$ is uniquely determined by its restriction to $T$, since $B = T \ltimes U$ is the semidirect product of $T$ and the unipotent subgroup $U$ of $B$, and regular invertible functions on $U$ are constant. It follows that the character group $\hat{T}$ of $T$ is isomorphic to $\hat{B}$.

If $\chi$ is a character of $B$, then we get an induced free action of $B$ on the product $G \times \mathbb{C}$ by $b \cdot (g, z) = (gb^{-1}, \chi(b)z)$. The quotient space $L_\chi = (G \times \mathbb{C})/B$, also denoted by $G \times B \mathbb{C}$, projects to the flag manifold $G/B$ by sending the orbit of $(g, z)$ to $gB$. This makes $L_\chi$ into the total space of a holomorphic line bundle over $G/B$, which is the homogeneous line bundle associated to the weight $\chi$. Let $\text{Pic}(G/B)$ be the Picard group of isomorphism classes of line bundles on $G/B$, with
the group operation given by the tensor product. Then the map \( \chi \mapsto L_\chi \) is a group homomorphism \( \hat{B} \to \text{Pic}(G/B) \). Composing this with the first Chern class map \( c_1 : \text{Pic}(G/B) \to H^2(G/B, \mathbb{Z}) \) gives a group homomorphism

\[
\hat{B} \to H^2(G/B, \mathbb{Z})
\]

\[
\chi \mapsto c_1(L_\chi).
\]

If \( S(\hat{B}) \) denotes the symmetric algebra of the \( \mathbb{Z} \)-module \( B \), then the above map extends to a homomorphism of graded rings

\[
c : S(\hat{B}) \to H^*(G/B, \mathbb{Z})
\]

called the characteristic homomorphism.

For any abelian group \( A \), let \( A_\mathbb{Q} \) denote \( A \otimes \mathbb{Q} \), and let \( S := S(\hat{B})_\mathbb{Q} \). Borel [12] proved that the morphism \( c \) is surjective, after tensoring with \( \mathbb{Q} \), and that the kernel of \( c \) is the ideal generated by the \( W \)-invariants of positive degree in \( S \), denoted \( (S^W)^+ \). We thus obtain the classical Borel presentation of the cohomology ring of \( G/B \),

\[
H^*(G/B, \mathbb{Q}) \cong S/(S^W_+).
\]

Furthermore, for any parabolic subgroup \( P \supset B \), the projection map \( G/B \to G/P \) induces an injection \( H^*(G/P) \hookrightarrow H^*(G/B) \), and this inclusion is realized in the presentation \( (26) \) by taking \( W_P \)-invariants on the right hand side:

\[
H^*(G/P, \mathbb{Q}) \cong S^{W_P}/(S^W_+).
\]

One knows that if the algebraic group \( G \) is special in the sense of [112], then the isomorphisms \( (26) \) and \( (27) \) hold with \( \mathbb{Z} \)-coefficients. For the classical groups, this is the case if \( G \) is the general linear group \( \text{GL}_n \) or the symplectic group \( \text{Sp}_{2n} \).

**Example 5.** Suppose that \( G = \text{GL}_n(\mathbb{C}) \), and choose the Borel subgroup \( B \) of upper triangular matrices, as in [11]. The flag manifold \( G/B \) then parametrizes complete flags of subspaces

\[
E_* : 0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n = \mathbb{C}^n
\]

in \( \mathbb{C}^n \). The maximal torus \( T \subset B \) is the group of invertible diagonal matrices, and the characters of \( T \) are the maps

\[
\text{diag}(t_1, \ldots, t_n) \mapsto t_1^{\alpha_1} \cdots t_n^{\alpha_n},
\]

where \( \alpha_1, \ldots, \alpha_n \) are integers. In this way, the multiplicative group of characters \( \hat{T} \) (and \( \hat{B} \)) is identified with the additive group \( \mathbb{Z}^n \), and we have \( S(\hat{T}) = \mathbb{Z}[x_1, \ldots, x_n] \).

The Weyl group \( W \) is the symmetric group \( S_n \), and \( (26) \) implies that \( H^*(G/B, \mathbb{Z}) \) is isomorphic to \( \mathbb{Z}[x_1, \ldots, x_n]/I \), where \( I \) is the ideal generated by the non-constant symmetric symmetric polynomials in \( x_1, \ldots, x_n \).

In the case of the Grassmannian \( G(m, n) = \text{GL}_n(\mathbb{C})/P \), the parabolic subgroup of \( W = S_n \) is \( W_P = S_m \times S_{n-m} \), embedded in \( S_n \) in the obvious way. The \( W_P \)-coset representatives of minimal length are the permutations \( w \in S_n \) such that \( w_1 < \cdots < w_m \) and \( w_{m+1} < \cdots < w_n \). If \( \lambda = (\lambda_1, \ldots, \lambda_m) \) is a partition which indexes a Schubert variety \( X_\lambda \) in \( G(m, n) \), then the minimal length coset representative \( w = w_\lambda \) corresponding to \( \lambda \) is determined by the equations \( w_j = \lambda_{m+1-j} + j \), for \( 1 \leq j \leq m \). The ring presentation \( (27) \) assumes the form

\[
H^*(G(m, n), \mathbb{Z}) \cong (\mathbb{Z}[x_1, \ldots, x_m]^{S_m} \otimes \mathbb{Z}[x_{m+1}, \ldots, x_n]^{S_{n-m}})/I
\]
and the two groups of variables $x_1, \ldots, x_m$ and $x_{m+1}, \ldots, x_n$ are the Chern roots of the vector bundles $E'$ and $E''$ in the universal exact sequence \( (3) \), respectively.

Although the presentation \cite{27} is very natural from a Lie-theoretic point of view, in general there will be more than one way to identify special Schubert classes which generate the cohomology ring of $G/P$ among the $W_P$-invariants in $S$. However, when $G$ is a classical group, there is a good uniform choice of special Schubert class generators of $H^*(G/P)$, as explained in the introduction. Initially, the special Schubert classes on any Grassmannian are defined exactly as in \cite{11} and \cite{2}. If $P$ is arbitrary, then the partial flag variety $G/P$ admits projection maps to various Grassmannians $G/P_r$, defined by omitting all but one of the subspaces in the flags parametrized by $G/P$. The special Schubert classes on $G/P$ are defined to be the pullbacks of the special Schubert classes from the Grassmannians $G/P_r$. The Giambelli problem then asks for an explicit combinatorial formula which writes a general Schubert class in $H^*(G/P)$ as a polynomial in these special classes.

### 3.2. Equivariant cohomology and degeneracy loci.

The definition of equivariant cohomology begins with the construction of a contractible space $EG$ on which $G$ acts freely. If $BG = EG/G$ denotes the quotient space, which is a classifying space for the group $G$, then the map $EG \to BG$ is a universal principal $G$-bundle. In the topological category, this means that if $U \to M$ is any principal $G$-bundle, then there is a morphism $f : M \to BG$, which is unique up to homotopy, such that $U \cong f^*EG$. If $X$ is any topological space endowed with a $G$-action, then $G$ acts diagonally on $EG \times X$ (that is, $g(e, x) := (ge, gx)$) and the quotient $EG \times^GX := (EG \times X)/G$ exists. The $G$-equivariant cohomology ring $H^*_G(X)$ of $X$ is then defined by

$$H^*_G(X) := H^*(EG \times^GX)$$

where we usually take cohomology with $\mathbb{Q}$-coefficients. Note that any subgroup of $G$ also acts freely on $EG$, therefore we can similarly define classifying spaces $BB := EG/B$, $BT := EG/T$, etc., and equivariant cohomology rings $H^*_B(X)$, $H^*_T(X)$, respectively. The natural map $EG \times^TX \to EG \times^B X$ induces an isomorphism $H^*_B(X) \cong H^*_T(X)$. For more on equivariant cohomology, see e.g. \cite{14, 57}.

A **flag bundle** is a fibration with fibers isomorphic to the flag variety $X = G/B$. Motivated by the theory of degeneracy loci of vector bundles, Fulton, Pragacz, and Ratafjiki \cite{39, 40, 46, 103} studied the question of obtaining explicit Chern class formulas for the classes of universal Schubert varieties in flag bundles for the classical Lie groups. In effect, this is the Giambelli problem when the homogeneous space $X$ varies in a family. We give a description of the problem here assuming that we are in Lie type A, B, or C for simplicity. Suppose that $E \to M$ is a vector bundle over a variety $M$, which in type B or C comes equipped with a non-degenerate symmetric or skew-symmetric bilinear form $E \otimes E \to \mathbb{C}$, respectively. Assume that $E_r$ and $F_s$ are two complete flags of subbundles of $E$, taken to be isotropic in types B and C. For any $w$ in the Weyl group, we have the degeneracy locus

\begin{equation}
X_w := \{ b \in M \mid \dim(E_r(b) \cap F_s(b)) \geq d_w(r, s) \ \forall r, s \}
\end{equation}

where $d_w(r, s)$ is a rank function taking values in the nonnegative integers. The inequalities in \cite{28} are exactly those which define the Schubert variety $X_w(F_s)$ in the flag variety $X$. Assuming that $M$ is smooth and that the locus $X_w$ has pure codimension $\ell(w)$ in $M$ (hypotheses which can both be relaxed), one seeks a formula
for the cohomology class $[X_w] \in H^*(M)$ in terms of the Chern classes of the vector bundles which appear in \((28)\).

Graham [50] studied the above problem by placing it in a more general Lie-theoretic framework, as follows. The morphism $BB \to BG$ is a flag bundle, and he showed that the fiber product space $BB \times_{BG} BB$ is a classifying space for the question posed by Fulton et. al., because any other example will pull back from it. Therefore, all the desired formulas for $[X_w]$ occur in $H^*(BB \times_{BG} BB)$. Furthermore, it is known (see e.g. [14, \S1]) that $H^*(BB) = S$, $H^*(BG) = S^W$, and the Leray-Hirsch theorem implies that there is a natural isomorphism
\[(29)\]
\[H^*(BB \times_{BG} BB) \cong S \otimes S^w.\]

Another key observation in [50] is that the degeneracy locus problem is equivalent to the question of obtaining an equivariant Giambelli formula in $H^*_T(G/B)$. Indeed, there is a $(B \times B)$-equivariant isomorphism $EG \times G \to EG \times_{BG} EG$ sending $(e,g)$ to $(e,ge)$, and passing to the quotient spaces gives a natural isomorphism
\[(30)\]
\[EG \times_B (G/B) \cong BB \times_{BG} BB\]
which maps $EG \times_B X_w$ to $X_w$. In view of the isomorphism \((30)\), we will call $BB \times_{BG} BB$ the Borel mixing space associated to $G/B$. Taking the cohomology of both sides of \((30)\), we obtain an isomorphism
\[H^*_T(G/B) = H^*(EG \times_B (G/B)) \cong H^*(BB \times_{BG} BB)\]
which sends the equivariant Schubert class $[X_w]^T := [EG \times_B X_w]$ to $[X_w]$. Finally, we remark that an equivariant Giambelli formula for $[X_w]$ in $H^*_T(G/B)$ specializes to a Giambelli formula for $[X_w]$ in $H^*(G/B)$ when we set all the variables coming from the linear $T$-action equal to zero.

4. Weyl groups, Grassmannian elements, and transition trees

In this section we explain the combinatorial objects which enter into the algebraic and geometric formulas of later sections. We begin with a discussion of the Weyl groups for the classical root systems of type A, B, C, and D.

4.1. Weyl groups. Let $W_n$ denote the hyperoctahedral group of signed permutations on the set $\{1, \ldots, n\}$, which is the semidirect product $S_n \times \mathbb{Z}_2^n$ of the symmetric group $S_n$ with $\mathbb{Z}_2^n$. We adopt the notation where a bar is written over an entry with a negative sign; thus $w = (3, \bar{1}, 2)$ maps $(1,2,3)$ to $(3,-1,2)$. The group $W_n$ is the Weyl group for the root system $B_n$ or $C_n$, and is generated by the simple transpositions $s_i = (i, i + 1)$ for $1 \leq i \leq n - 1$ and the sign change $s_0(1) = \bar{1}$. The symmetric group $S_n$ is the subgroup of $W_n$ generated by the $s_i$ for $1 \leq i \leq n - 1$, and is the Weyl group for the root system $A_{n-1}$. The elements of the Weyl group $\widehat{W}_n$ for the root system $D_n$ is the subgroup of $W_n$ consisting of all signed permutations with an even number of sign changes. The group $\widehat{W}_n$ is an extension of $S_n$ by the element $s_0 = s_0 s_1 s_0$, which acts on the right by
\[(w_1, w_2, \ldots, w_n) s_0 = (\overline{w}_1, w_3, \ldots, w_n).\]

There are natural embeddings $W_n \hookrightarrow W_{n+1}$ and $\widehat{W}_n \hookrightarrow \widehat{W}_{n+1}$ defined by adjoining the fixed point $n + 1$. We let $S_\infty := \cup_n S_n$, $W_\infty := \cup_n W_n$, and $\widehat{W}_\infty := \cup_n \widehat{W}_n$.

Set $N_0 := \{0, 1, \ldots\}$ and $N_\Box := \{\Box, 1, 2, \ldots\}$. A reduced word of an element $w$ in $W_\infty$ (respectively $\widehat{W}_\infty$) is a sequence $a_1 \cdots a_\ell$ of elements in $N_0$ (respectively $N_\Box$)
such that \( w = s_{a_1} \cdots s_{a_t} \) and \( \ell \) is minimal, so (by definition) equal to the length \( \ell(w) \) of \( w \). We say that \( w \) has descent at position \( r \) if \( \ell(w_{s_r}) < \ell(w) \), where \( s_r \) is the simple reflection indexed by \( r \). For \( r \in \mathbb{N}_0 \), this is equivalent to the condition \( w_r > w_{r+1} \), where we set \( w_0 = 0 \) (so \( w \) has a descent at position 0 if and only if \( w_1 < 0 \)). Given any Weyl group elements \( u_1, \ldots, u_p, w \), we will write \( u_1 \cdots u_p = w \) if \( \ell(u_1) + \cdots + \ell(u_p) = \ell(w) \) and the product of \( u_1, \ldots, u_p \) is equal to \( w \). In this case we say that \( u_1 \cdots u_p \) is a reduced factorization of \( w \).

4.2. Grassmannian elements. A permutation \( w \in S_\infty \) is Grassmannian if there exists an \( m \geq 1 \) such that \( w_i < w_{i+1} \) for all \( i \neq m \). The shape of such a Grassmannian permutation \( w \) is the partition \( \lambda = (\lambda_1, \ldots, \lambda_m) \) with \( \lambda_{m+1-j} = \lambda_j - j \) for \( 1 \leq j \leq m \). Notice that there are infinitely many permutations of a given shape \( \lambda \). However, for each fixed \( m \) and \( n > m \), we obtain a bijection between the set of permutations in \( S_n \) with at most one descent at position \( m \) and the set of partitions \( \lambda \) whose diagram fits inside an \( m \times (n-m) \) rectangle.

Fix a nonnegative integer \( k \). An element \( w = (w_1, w_2, \ldots) \) in \( W_\infty \) is called \( k \)-Grassmannian if and only if we have \( \ell(ws_i) = \ell(w) + 1 \) for all \( i \neq k \). When \( k = 0 \), this says that \( w \) is increasing: \( w_1 < w_2 < \cdots \), while when \( k > 0 \), then \( w \) is \( k \)-Grassmannian if and only if

\[
0 < w_1 < \cdots < w_k \quad \text{and} \quad w_{k+1} < w_{k+2} < \cdots.
\]

There is an explicit bijection between \( k \)-Grassmannian elements of \( W_\infty \) and \( k \)-strict partitions, under which the elements \( w \) in \( W_n \) correspond to those partitions \( \lambda \) whose diagram fits inside an \( (n-k) \times (n+k) \) rectangle. This bijection is obtained as follows. The absolute value of the negative entries in \( w \) form a (possibly empty) \( k \)-strict partition \( \mu \) whose diagram gives the part of the diagram of \( \lambda \) which lies in columns \( k+1 \) and higher. The boxes in these columns which lie outside of the diagram of \( \mu \) are partitioned into south-west to north-east diagonals. For each \( i \) between 1 and \( k \), let \( d_i \) be the diagonal among these which contains \( w_i \) boxes. Then the bottom box \( B_i \) of \( \lambda \) in column \( k+1-i \) is \( k \)-related to \( d_i \), for \( 1 \leq i \leq k \). Here ‘\( k \)-related’ means that the directed line segment joining \( B_i \) with the lowest box of the diagonal \( d_i \) is north-west to south-east (this notion was used in \[16\] \[17\]). We will denote the Weyl group element associated to \( \lambda \) by \( w_\lambda \).

**Example 6.** (a) The 3-Grassmannian element \( w = (3, 5, 8, 4, 1, 2, 6, 7) \) corresponds to the 3-strict partition \( \lambda = (7, 4, 3, 1, 1) \).

\[
\lambda = \\
\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & & & & \\
\bullet & \bullet & \bullet & & & & & \\
\bullet & \bullet & \bullet & & & & & \\
\bullet & \bullet & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\bullet & & & & & & & \\
\end{array}
\]

(b) A permutation \( \sigma \in S_\infty \) with a unique descent at position \( m \) may be viewed as an \( m \)-Grassmannian element of \( W_\infty \), via the natural inclusion of \( S_\infty \) in \( W_\infty \). In this case, the \( m \)-strict partition associated to \( \sigma \) is the conjugate of the shape of \( \sigma \) as a Grassmannian permutation.
Fix a \( k \in \mathbb{N}_0 \) with \( k \neq 1 \). An element \( w \in \tilde{W}_\infty \) is called \( k \)-Grassmannian if \( \ell(ws_i) = \ell(w) + 1 \) for all \( i \neq k \). We say that \( w \) is 1-Grassmannian if \( \ell(ws_i) = \ell(w) + 1 \) for all \( i \geq 2 \). The \( \square \)-Grassmannian elements are therefore the ones whose values strictly increase, while if \( k > 0 \), then \( w \) is \( k \)-Grassmannian if and only if

\[
|w_1| < w_2 < \cdots < w_k \quad \text{and} \quad w_{k+1} < w_{k+2} < \cdots .
\]

There is a bijection between the \( k \)-Grassmannian elements of \( \tilde{W}_\infty \) and typed \( k \)-strict partitions, under which the elements \( w \) in \( \tilde{W}_n \) correspond to partitions \( \lambda \) whose diagram fits inside an \((n-k) \times (n+k-1)\) rectangle, obtained as follows. Consider the negative entries \( w_{k+j} \) for \( j \geq 1 \) and subtract one from each one, to obtain the parts of a strict partition \( \mu \). As above, the partition \( \mu \) gives the part of \( \lambda \) which lies in columns \( k+1 \) and higher, and the boxes in these columns outside of \( \mu \) are partitioned into south-west to north-east diagonals. Choose \( d_1, \ldots, d_k \) among these diagonals such that the number of boxes in \( d_i \) is given by \(|w_i| - 1\), for \( 1 \leq i \leq k \). The bottom boxes of \( \lambda \) in the first \( k \) columns are then \( k'-related \) to these \( k \) diagonals, with bottom box \( B_i \) in column \( k + 1 - i \) related to diagonal \( d_i \). Here ‘\( k'\)-related’ differs from ‘\( k\)-related’ by a shift down by one unit, as shown in the next figure. Finally, we have type(\( \lambda \)) > 0 if and only if \(|w_1| > 1\), and in this case type(\( \lambda \)) is equal to 1 or 2 depending on whether \( w_1 > 0 \) or \( w_1 < 0 \), respectively. We will denote the Weyl group element associated to \( \lambda \) by \( w_\lambda \), as above.

**Example 7.** The 3-Grassmannian element \( w = (\overline{2}, 6, 7, 5, 3, 1, 4, 8) \) corresponds to the 3-strict partition \( \lambda = (7, 5, 3, 2) \) of type 2.

![Diagram of a 3-Grassmannian element](image)

To each \( k \)-Grassmannian element \( w \) in \( W_\infty \) or \( \tilde{W}_\infty \), we attach a finite order ideal \( \mathcal{C}(w) \) in \( \Delta^\varrho \) by the prescription

\[
\mathcal{C}(w) := \{(i, j) \in \Delta^\varrho \mid w_{k+i} + w_{k+j} < 0\}.
\]

It is easy to check that if \( \lambda \) is a \( k \)-strict partition or a typed \( k \)-strict partition, then \( \mathcal{C}(w_\lambda) \) is equal to the ideal \( \mathcal{C}(\lambda) \) or \( \mathcal{C}'(\lambda) \) defined in [2] respectively.

4.3. **Transition trees and Stanley coefficients.** The notion of a transition was introduced by Lascoux and Schützenberger in [22, 24], where it was applied to achieve efficient, recursive computations of type A Schubert polynomials and a new form of the Littlewood-Richardson rule. Analogues of their transition equations for the other classical Lie types were obtained by Billey [3], using the Schubert polynomials found in [10]. We will require the extension of this theory given in [122], which includes the \( k \)-Grassmannian elements for all \( k > 0 \). For positive integers \( i < j \) we define reflections \( t_{ij} \in S_\infty \) and \( \overline{t}_{ij}, \overline{t}_{ii} \in W_\infty \) by their right actions

\[
(\ldots, w_i, \ldots, w_j, \ldots, w_{ij}) t_{ij} = (\ldots, w_j, \ldots, w_i, \ldots),
\]

\[
(\ldots, w_i, \ldots, w_j, \ldots, \overline{w}_j, \ldots, w_i, \ldots) \overline{t}_{ij} = (\ldots, \overline{w}_j, \ldots, \overline{w}_i, \ldots),
\]

and

\[
(\ldots, w_i, \ldots) \overline{t}_{ii} = (\ldots, \overline{w}_i, \ldots),
\]

respectively.
and let $t_{ji} = t_{ij}$.

Following Lascoux and Schützenberger [84], for any permutation $\varpi \in S_\infty$, we construct a rooted tree $T(\varpi)$ with nodes given by permutations and root $\varpi$ as follows. If $\varpi = 1$ or $\varpi$ is Grassmannian, then set $T(\varpi) := \{\varpi\}$. Otherwise, let $r$ be the last descent of $\varpi$, and set $s := \max(\{i > r \mid \varpi_i < \varpi_r\})$. The definitions of $r$ and $s$ imply that we have $\ell(\varpi t_{rs}) = \ell(\varpi) - 1$. Let

$$I(\varpi) := \{i \mid 1 \leq i < r \text{ and } \ell(\varpi t_{rs}) = \ell(\varpi)\}.$$ 

If $I(\varpi) \neq \emptyset$, then let $\Psi(\varpi) := \{\varpi t_{rs}t_{ir} \mid i \in I(\varpi)\}$; otherwise, let $\Psi(\varpi) := \Psi(1 \times \varpi)$. To recursively define $T(\varpi)$, we join $\varpi$ by an edge to each $v \in \Psi(\varpi)$, and attach to each $v \in \Psi(\varpi)$ its tree $T(v)$.

One can show that $T(\varpi)$ is a finite tree whose leaves are all Grassmannian permutations. The tree $T(\varpi)$ is the Lascoux-Schützenberger transition tree of $\varpi$. We define the Stanley coefficient $c_\lambda^{\varpi}$ to be the number of leaves of shape $\lambda$ in the transition tree $T(\varpi)$ associated to $\varpi$.

**Example 8.** The transition tree of $\varpi = 143265$ is displayed below. At all nodes where the branching rule is applied, the positions $r$ and $s$ are shown in boldface.

There exist alternative combinatorial formulas for the coefficients $c_\lambda^{\varpi}$ in terms of Young tableaux [28, 34, 105]; see also [87]. We note here the result of Fomin and Greene [34] that $c_\lambda^{\varpi}$ is equal to the number of semistandard tableaux $T$ of shape $\tilde{\lambda}$ such that the column word of $T$, obtained by reading the entries of $T$ from bottom to top and left to right, is a reduced word for $w$. We display below the three tableaux associated to the leaves of the tree for $143265 = s_3s_2s_3s_5$ in Example 8:

$$\begin{array}{c}
\begin{array}{c}
2 \\
3 \\
5
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
2 \\
3 \\
5
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
2 \\
3
\end{array}
\end{array}$$

A signed permutation $w \in W_\infty$ is said to be increasing up to $k$ if it has no descents less than $k$. This condition is automatically satisfied if $k = 0$, and for positive $k$ it means that $0 < w_1 < w_2 < \cdots < w_k$. If $k \geq 2$, we say that an element $w \in W_\infty$ is increasing up to $k$ if it has no descents less than $k$; this means

---

1The leaves of the transition trees of [84] are actually vexillary permutations.
that $|w_1| < w_2 < \cdots < w_k$. By convention we agree that every element of $\tilde{W}_\infty$ is increasing up to $\emptyset$ and also increasing up to 1.

For any $w \in W_\infty$ which is increasing up to $k$, we construct a rooted tree $T^k(w)$ with nodes given by elements of $W_\infty$ and root $w$ as follows. Let $r$ be the last descent of $w$. If $w = 1$ or $r = k$, then set $T^k(w) := \{w\}$. Otherwise, let $s := \max(i > r \mid w_i < w_r)$ and $\Phi(w) := \Phi_1(w) \cup \Phi_2(w)$, where

$$\Phi_1(w) := \{wt_{rs}t_{ir} \mid 1 \leq i < r \text{ and } \ell(wt_{rs}t_{ir}) = \ell(w)\},$$

$$\Phi_2(w) := \{wt_{rs}t_{ir} \mid i \geq 1 \text{ and } \ell(wt_{rs}t_{ir}) = \ell(w)\}.$$  

To recursively define $T^k(w)$, we join $w$ by an edge to each $v \in \Phi(w)$, and attach to each $v \in \Phi(w)$ its tree $T^k(v)$. We call $T^k(w)$ the $k$-transition tree of $w$.

**Example 9.** The 1-transition tree of $w = 3\overline{1}2645$ is displayed below, with positions $r$ and $s$ shown in boldface.

For any $w \in \tilde{W}_\infty$ which is increasing up to $k$, we construct the $k$-transition tree $\tilde{T}^k(w)$ with nodes given by elements of $\tilde{W}_\infty$ and root $w$ in a manner parallel to the above. Let $r$ be the last descent of $w$. If $w = 1$, or $k \neq 1$ and $r = k$, or $k = 1$ and $r \in \{\emptyset, 1\}$, then set $\tilde{T}^k(w) := \{w\}$. Otherwise, let $s := \max(i > r \mid w_i < w_r)$ and $\tilde{\Phi}(w) := \tilde{\Phi}_1(w) \cup \tilde{\Phi}_2(w)$, where

$$\tilde{\Phi}_1(w) := \{wt_{rs}t_{ir} \mid 1 \leq i < r \text{ and } \ell(wt_{rs}t_{ir}) = \ell(w)\},$$

$$\tilde{\Phi}_2(w) := \{wt_{rs}t_{ir} \mid i \neq r \text{ and } \ell(wt_{rs}t_{ir}) = \ell(w)\}.$$  

To define $\tilde{T}^k(w)$, we join $w$ by an edge to each $v \in \tilde{\Phi}(w)$, and attach to each $v \in \tilde{\Phi}(w)$ its tree $\tilde{T}^k(v)$.

According to [122, Lemma 3 and §6], the $k$-transition trees $T^k(w)$ and $\tilde{T}^k(w)$ are finite trees, all of whose nodes are increasing up to $k$. Moreover, the leaves of these trees are $k$-Grassmannian elements. For any $k$-strict (respectively, typed
$k$-strict) partition $\lambda$, the mixed Stanley coefficient $e^w_\lambda$ (respectively, $d^w_\lambda$) is defined to be the number of leaves of $T^k(w)$ (respectively, $\tilde{T}^k(w)$) of shape $\lambda$. For instance, from Example 9 we deduce that $e^{\tilde{T}2645}_{(4,1)} = e^{\tilde{T}2645}_{(3,2)} = 1$ and $e^{\tilde{T}2645}_{(3,1,1)} = 2$.

**Example 10.** When $k = 0$, there exist alternative combinatorial formulas for the integers $e^w_\lambda$ and $d^w_\lambda$ involving certain tableaux $[53, 68, 74, 75]$. For instance, let $w \in W_\infty$ and $\lambda$ be a Young diagram with $r$ rows such that $|\lambda| = \ell(w)$. We call a sequence $b = (b_1, \ldots, b_m)$ *unimodal* if for some index $j$ we have

$$b_1 > b_2 > \cdots > b_{j-1} > b_j < b_{j+1} < \cdots < b_m.$$ 

A subsequence of $b$ is any sequence $(b_{i_1}, \ldots, b_{i_p})$ with $1 \leq i_1 < \cdots < i_p \leq m$. A Kraśkiewicz tableau (also known as a standard decomposition tableau) for $w$ of shape $\lambda$ is a filling $T$ of the boxes of $\lambda$ with nonnegative integers in such a way that (i) if $t_i$ is the sequence of entries in the $i$-th row of $T$, reading from left to right, then the row word $t_r \ldots t_1$ is a reduced word for $w$, and (ii) for each $i$, $t_i$ is a unimodal subsequence of maximum length in $t_r \ldots t_{i+1} t_i$. Lam [75] proved that for any $w \in W_\infty$, $e^w_\lambda$ is equal to the number of Kraśkiewicz tableaux for $w$ of shape $\lambda$.

### 5. Schubert Polynomials and Symmetric Functions

#### 5.1. The nilCoxeter Algebra and Schubert Polynomials

Bernstein-Gelfand-Gelfand [6] and Demazure [26, 27] used divided difference operators to construct an algorithm that produces Gessel polynomials which represent the Schubert classes on the flag variety $G/B$ in the Borel presentation of the cohomology ring, starting from the choice of a representative for the class of a point. For the general linear group, Lascoux and Schützenberger [82, 84] applied this algorithm with a very natural choice of top degree polynomial to define Schubert polynomials, a theory which included both single and double versions.

In a series of papers [36, 37, 38, 74], Fomin, Stanley, Kirillov, and Lam developed an alternative approach to the theory of Schubert polynomials, which extended to all the classical types. The main idea was to use the nilCoxeter algebra of the Weyl group, an abstract algebra which is isomorphic to the algebra of divided difference operators. The work was complicated by the fact that in Lie types $B$, $C$, and $D$, there are many candidate theories of Schubert polynomials (see [10, 40, 80, 81, 118, 119] for examples), and it was not clear why any one of them should be preferred from the others. However, the theory that was best understood from a combinatorial point of view was that of Billey and Haiman [10], which incorporated both the Lascoux-Schützenberger type A Schubert polynomials and the classical Schur $Q$-functions. Double versions of the Billey-Haiman polynomials were recently introduced and studied by Ikeda, Mihalcea, and Naruse [59]. Their type $B$, $C$, and $D$ double Schubert polynomials will be key ingredients in our story, and we discuss their construction using the nilCoxeter algebra below.

**The nilCoxeter Algebra** $W_n$ of the hyperoctahedral group $W_n$ is the free associative algebra with unity generated by the elements $u_0, u_1, \ldots, u_{n-1}$ modulo the relations

$$u_i^2 = 0,$$ 
$$u_i u_{i+j} = u_j u_i, \quad |i - j| \geq 2,$$ 
$$u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}, \quad i > 0,$$ 
$$u_0 u_1 u_0 u_1 = u_1 u_0 u_1 u_0.$$
For any $w \in W_n$, choose a reduced word $a_1 \cdots a_k$ for $w$ and define $u_w = a_1 \cdots a_k$. Since the last three relations listed are the Coxeter relations for the Weyl group $W_n$, it follows that $u_w$ is well defined. Moreover, the $u_w$ for $w \in W_n$ form a free $\mathbb{Z}$-basis of $W_n$. We denote the coefficient of $u_w \in W_n$ in the expansion of the element $h \in W_n$ by $\langle h, w \rangle$; thus $h = \sum_{w \in W_n} \langle h, w \rangle u_w$ for all $h \in W_n$.

Let $t$ be an indeterminate and define
\[
A_i(t) := (1 + tu_{n-1})(1 + tu_{n+1}) \cdots (1 + tu_i);
\]
\[
\tilde{A}_i(t) := (1 - tu_i)(1 - tu_i+1) \cdots (1 - tu_{n-1});
\]
\[
C(t) := (1 + tu_{n-1}) \cdots (1 + tu_1)(1 + 2tu_0)(1 + tu_1) \cdots (1 + tu_{n-1}).
\]
Suppose that $X = (x_1, x_2, \ldots)$, $Y = (y_1, y_2, \ldots)$, and $Z = (z_1, z_2, \ldots)$ are three infinite sequences of commuting variables. Let $C(X) = C(x_1)C(x_2) \cdots$ and for $w \in W_n$, define
\[
(31) \quad \mathcal{C}_w(X; Y, Z) := \langle \tilde{A}_{n-1}(z_{n-1}) \cdots \tilde{A}_1(z_1)C(X)A_1(y_1) \cdots A_{n-1}(y_{n-1}), w \rangle.
\]
Set $\mathcal{C}_w(X; Y) := \mathcal{C}_w(X; Y, 0)$. The polynomials $\mathcal{C}_w(X; Y)$ are the type $C$ Schubert polynomials of Billey and Haiman and the $\mathcal{C}_w(X; Y, Z)$ are their double versions due to Ikeda, Mihalcea, and Naruse. Note that $\mathcal{C}_w$ is really a polynomial in the $Y$ and $Z$ variables, with coefficients which are formal power series in $X$. In fact, these formal power series are symmetric in the $X$ variables; this follows immediately from Fomin and Kirillov’s result [37, Prop. 4.2] that $C(s)C(t) = C(t)C(s)$, for any two commuting variables $s$ and $t$. We set $F_w(X) := \mathcal{C}_w(X; 0, 0)$ and call $F_w$ the type $C$ Stanley symmetric function indexed by $w \in W_n$.

For any $\varpi \in S_n$, the double Schubert polynomial $\mathfrak{S}_\varpi(Y, Z)$ of Lascoux and Schützenberger is given by
\[
\mathfrak{S}_\varpi(Y, Z) := \mathcal{C}_\varpi(0; Y, Z) = \langle \tilde{A}_{n-1}(z_{n-1}) \cdots \tilde{A}_1(z_1)A_1(y_1) \cdots A_{n-1}(y_{n-1}), \varpi \rangle.
\]
The polynomial $\mathfrak{S}_\varpi(Y) := \mathfrak{S}_\varpi(Y, 0)$ is the single Schubert polynomial. It is transparent from the definition that $\mathcal{C}_w$ is stable under the natural inclusion of $W_n$ in $W_{n+1}$; it follows that $\mathfrak{S}_w$ and $\mathcal{C}_w$ are well defined for $\varpi \in S_\infty$ and $w \in W_\infty$, respectively. Equation (31) is equivalent to the relations
\[
(32) \quad \mathcal{C}_w(X; Y, Z) = \sum_{u^{\prime}w = w} \mathfrak{S}_{u^{-1}}(-Z)\mathcal{C}_{u^{\prime}}(X; Y) = \sum_{u^{\prime}w = w} \mathfrak{S}_{u^{-1}}(-Z)F_{u^{\prime}}(X)\mathcal{C}_w(Y)
\]
summed over all reduced factorizations $uw = w$ and $uw^{\prime} = w$ (respectively) with $u, u^{\prime} \in S_\infty$ (compare with [59 Cor. 8.10]). In particular, we have
\[
(33) \quad \mathfrak{S}_\varpi(Y, Z) = \sum_{u^{\prime}w = \varpi} \mathfrak{S}_{u^{-1}}(-Z)\mathfrak{S}_{u^{\prime}}(Y)
\]
summed over all reduced factorizations $uu^{\prime} = \varpi$ in $S_\infty$.

**Example 11.** For $\varpi = 321 \in S_3$ we have that
\[
\mathfrak{S}_{321}(Y, Z) = \langle -(1 - z_2u_2)(1 - z_1u_1)(1 - z_1u_2)(1 + y_1u_2)(1 + y_1u_1)(1 + y_2u_2), 321 \rangle
\]
\[
= -z_2z_1^2 + z_2z_1y_1 + z_2z_1y_2 - z_2y_1y_2 + z_1^2y_1 - z_1y_1y_2 + y_2^2y_2
\]
\[
= (y_1 - z_1)(y_1 - z_2)(y_2 - z_1).
\]
Let \( w_0 = (n, n - 1, \ldots, 1) \) be the longest element of \( S_n \). Then one can show (see [58, Cor. 4.4]) that
\[
\mathfrak{G}_{w_0}(Y, Z) = \prod_{i+j \leq n} (y_i - z_j).
\]
There are more complicated formulas (see [59, Thm. 1.2]) for the double Schubert polynomials \( \mathfrak{C}_{w_0} \) and \( \mathfrak{D}_{w_0} \), where \( w_0 \) denotes the longest element in the respective Weyl group. We do not require these formulas in this article.

For the orthogonal types B and D we will work with coefficients in the ring \( \mathbb{Z}[\frac{1}{2}] \). For \( w \in W_\infty \), let \( s(w) \) denote the number of \( i \) such that \( w_i < 0 \). The type B double Schubert polynomials \( \mathfrak{B}_w \) are related to the type C polynomials by the equation \( \mathfrak{B}_w = 2^{-s(w)} \mathfrak{C}_w \). We will use the nilCoxeter algebra \( \mathfrak{W}_n \) of \( \mathfrak{W}_n \) to define type D Schubert polynomials. \( \mathfrak{W}_n \) is the free associative algebra with unity generated by the elements \( u_{\square}, u_1, \ldots, u_{n-1} \) modulo the relations
\[
\begin{align*}
&u_i^2 = 0, \\
&u_{\square} u_1 = u_1 u_{\square}, \\
&u_{\square} u_2 u_{\square} = u_2 u_{\square} u_2, \\
&u_{i+1} u_i = u_i u_{i+1}, \\
&u_i u_j = u_j u_i & j > i + 1, \text{ and } (i, j) \neq (\square, 2).
\end{align*}
\]

For any \( w \in \mathfrak{W}_n \), choose a reduced word \( a_1 \cdots a_\ell \) for \( w \) and define \( u_w = u_{a_1} \cdots u_{a_\ell} \). We denote the coefficient of \( u_w \in \mathfrak{W}_n \) in the expansion of the element \( h \in \mathfrak{W}_n \) by \( \langle h, w \rangle \). Let \( t \) be an indeterminate and, following Lam [74, 4.4], define
\[
D(t) := (1 + tu_{n-1}) \cdots (1 + tu_2)(1 + tu_1)(1 + tu_{\square})(1 + tu_2) \cdots (1 + tu_{n-1}).
\]
Let \( D(X) = D(x_1)D(x_2) \cdots \), and define
\[
\mathfrak{D}_w(X; Y, Z) := \langle \hat{A}_{n-1}(z_{n-1}) \cdots \hat{A}_1(z_1)D(X)A_1(y_1) \cdots A_{n-1}(y_{n-1}), w \rangle.
\]
The polynomials \( \mathfrak{D}_w(X; Y) := \mathfrak{D}_w(X; Y, 0) \) are the type D Billey-Haiman Schubert polynomials, and the \( \mathfrak{D}_w(X; Y, Z) \) are their double versions studied in [59].

5.2. Schur, theta, and eta polynomials. In this section, we use raising operators to construct the polynomials from [17, 19, 20, 22] which will appear later in splitting formulas for the double Schubert polynomials.

For \( m, n \geq 1 \), set \( Y_{(m)} = (y_1, \ldots, y_m) \) and \( Z_{(n)} = (z_1, \ldots, z_n) \). Define the complete supersymmetric polynomials \( h_r(Y_{(m)}/Z_{(n)}) \) by their generating function
\[
\prod_{i=1}^{m} (1 - y_i t)^{-1} \prod_{j=1}^{n} (1 - z_j t)^{-1} = \sum_{r=0}^{\infty} h_r(Y_{(m)}/Z_{(n)}) t^r.
\]
The supersymmetric Schur polynomial \( s_\lambda(Y_{(m)}/Z_{(n)}) \) is obtained by setting
\[
s_\lambda(Y_{(m)}/Z_{(n)}) := R^0 h_\lambda(Y_{(m)}/Z_{(n)}) = \det \left( h_{\lambda_i+j-i}(Y_{(m)}/Z_{(n)}) \right)_{i,j}
\]
for any partition \( \lambda \), where \( h_\lambda := \prod h_\lambda \). The usual Schur polynomials satisfy the identities \( s_\lambda(Y_{(m)}) = s_\lambda(Y_{(m)}/Z_{(n)})|_{Z_{(n)}=0} \) and
\[
s_\lambda(0/Z_{(n)}) = s_\lambda(Y_{(m)}/Z_{(n)})|_{Y_{(m)}=0} = (-1)^{|\lambda|} s_\lambda(Z_{(n)}).
\]
Fix an integer \( k \geq 0 \), and define formal power series \( \vartheta_r = \vartheta_r(X; Y) \) for \( r \in \mathbb{Z} \) by the equation
\[
\prod_{i=1}^{\infty} \frac{1 + x_i t}{1 - x_i t} \prod_{j=1}^{k} (1 + y_j t) = \sum_{r=0}^{\infty} \vartheta_r t^{r}.
\]
Set \( \vartheta_\lambda := \prod_i \vartheta_{\lambda_i} \) and
\[
\Theta_\lambda(X; Y) := R^\lambda \vartheta_\lambda
\]
for any \( k \)-strict partition \( \lambda \), where \( R^\lambda \) is the raising operator in \([12]\). Following \([17]\), we call \( \Theta_\lambda(X; Y) \) a \textit{theta polynomial}, although this is a slight abuse of language (compare with Remark \([4]\)). Define \( \eta_k = \frac{1}{2} \vartheta_k - \frac{1}{2} e_k(Y) \) and \( \eta'_k = \frac{1}{4} \vartheta_k - \frac{1}{2} e_k(Y) = \frac{1}{2} \sum_{i=0}^{k-1} q_{k-i}(X) e_i(Y) \). For any typed \( k \)-strict partition \( \lambda \), consider the \textit{eta polynomial}
\[
H_\lambda(X; Y) := 2^{-\ell_k(\lambda)} R^\lambda \ast \vartheta_\lambda.
\]
The raising operator expression in \((35)\) is defined in the same way as the analogous one in equation \((23)\), but using \( \vartheta_r \) and \( \eta_k, \eta'_k \) in place of \( c_r, \tau_k, \tau'_k \), respectively.

When \( k = 0 \), the indexing partitions \( \lambda \) are strict, we have that \( \vartheta_r(X) = q_r(X) \) and \( \Theta_\lambda(X; Y(0)) = Q_\lambda(X) \) are the classical Schur \( Q \)-functions \([111]\), and \( H_\lambda(X; Y(0)) = P_\lambda(X) \) is a Schur \( P \)-function. The ring \( \Gamma := \mathbb{Z}[q_1, q_2, \ldots] \) is the ring of Schur \( Q \)-functions and \( \Gamma' := \mathbb{Z}[P_1, P_2, \ldots] \) is the ring of Schur \( P \)-functions (see \([56]\) and \([89]\) \textsection 3.8). For our purposes here we need to know that \( \mathcal{C}_w(X; Y, Z) \) lies in \( \Gamma[Y, Z] \) and \( \mathcal{D}_w(X; Y, Z) \) lies in \( \Gamma'[Y, Z] \). These assertions follow from Example \([13]\) below, but we sketch a quick proof of the first one. Since \( C(t)C(-t) = 1 \), we deduce that the type \( C \) Stanley functions \( F_w \) satisfy the cancellation rule
\[
F_w(t, -t, x_1, x_2, \ldots) = F_w(x_1, x_2, \ldots).
\]
Any symmetric function satisfying the rule \((36)\) lies in \( \Gamma \), by a theorem of Pragacz \([100]\). We conclude from equation \((32)\) that \( \mathcal{C}_w(X; Y, Z) \in \Gamma[Y, Z] \).

**Theorem 4** \(([17] [19] [82])\). (a) If \( \varpi \in S_\infty \) is a Grassmannian permutation with a unique descent at \( m \) and \( \lambda \) is the corresponding partition of length at most \( m \), then we have
\[
\mathcal{S}_w(Y) = s_\lambda(Y_{(m)}).
\]
(b) If \( w_\lambda \in W_\infty \) is the \( k \)-Grassmannian element corresponding to the \( k \)-strict partition \( \lambda \), then we have
\[
\mathcal{C}_{w_\lambda}(X; Y) = \Theta_\lambda(X; Y_{(k)}).
\]
(c) If \( w_\lambda \in \tilde{W}_\infty \) is the \( k \)-Grassmannian element corresponding to the typed \( k \)-strict partition \( \lambda \), then we have
\[
\mathcal{D}_{w_\lambda}(X; Y) = H_\lambda(X; Y_{(k)}).
\]

**Remark 2.** Although the equality of polynomials \((37)\) in \( \mathbb{Z}[Y] \) is directly analogous to \((38)\) and \((39)\), the latter two equations are much harder to prove. One reason for this is that e.g. \((38)\) is an equality in the ring \( \mathbb{Z}[Y] = \mathbb{Z}[q_1(X), q_2(X), \ldots; y_1, y_2, \ldots] \), where there are relations among the \( q_r \), and these relations play a crucial role in the proof. Furthermore, the proofs of \((38)\) and \((39)\) in \([17]\) \textsection 3 \([19]\) rely on Theorems \([2]\) and \([3]\). We will generalize Theorem \([4]\) below to a result (Theorem \([4]\) below) which gives analogous formulas for arbitrary double Schubert polynomials.
Example 12. (a) Let $\varpi$ be a Grassmannian permutation with unique descent at $m$ and $\lambda = (\varpi_m - m, \ldots, \varpi_1 - 1)$ be the corresponding partition. The double Schubert polynomial $\mathfrak{S}_{\varpi}(Y, Z)$ is a ‘double Schur polynomial’ $s_{\lambda}(Y_m, Z)$, also known as a factorial Schur function. In this case, formula (33) can be made more explicit, by working as follows. In any reduced factorization $\varpi = uv$, the right factor $v$ is also Grassmannian, and corresponds to a partition $\mu$ whose diagram is contained in the diagram of $\lambda$; moreover, we have $\mathfrak{S}_v(Y) = s_{\mu}(Y_m)$. The left factor $u$ is a fully commutative element of $S_{\infty}$ (in the sense of [115]), and it follows from [11, §2] that the Schubert polynomial $\mathfrak{S}_{u^{-1}}(Z)$ in (33) is a flagged skew Schur polynomial. Setting $Z_{(r)} = (z_1, \ldots, z_r)$ for each $r$, we conclude (see also [69, Prop. 4.1]) that

$$s_{\lambda}(Y_m, Z) = \sum_{\mu \subseteq \lambda} s_{\mu}(Y_m) \det \left( e_{\lambda_i - \mu_j - i + j}(-Z_{(\lambda_i + m - i)}) \right)_{1 \leq i, j \leq m}.$$ 

(b) Let $\lambda$ be a $k$-strict partition. In any reduced factorization $w_\lambda = uv$ of the $k$-Grassmannian element $w_\lambda$, the right factor $v$ is also $k$-Grassmannian, and hence $v = w_\mu$ for some $k$-strict partition $\mu$. Assuming that $u \in S_{\infty}$, we note that $u$ may not be fully commutative (for example, let $k = 1$ and consider the factorization $231 = uv$ with $u = 321$ and $v = 213$), although it is a skew element of $W_{\infty}$, in the sense of [120, Def. 4]. Equations (32) and (38) give

$$(40) \quad \mathfrak{c}_{w_\lambda}(X; Y, Z) = \sum_{uw_\mu = w_\lambda} \Theta_{\mu}(X; Y_{(k)}) \mathfrak{S}_{u^{-1}}(-Z)$$

summed over all reduced factorizations $uw_\mu = w_\lambda$ with $u \in S_{\infty}$. Suppose next that $k = 0$ and $\lambda$ is a strict partition of length $\ell$. Then $w_\lambda$ is a fully commutative element and $\mathfrak{c}_{w_\lambda}(X; Y, Z) = Q_\lambda(X; Z)$ is a ‘double’ analogue of Schur’s $Q$-function introduced by Ivanov [62]. Equation (40) becomes

$$Q_\lambda(X; Z) = \sum_{\mu \subseteq \lambda} Q_{\mu}(X) \det \left( e_{\lambda_i - \mu_j}(-Z_{(\lambda_i - 1)}) \right)_{1 \leq i, j \leq \ell}$$

summed over all strict partitions $\mu \subset \lambda$ with $\ell(\mu) = \ell(\lambda) = \ell$.

5.3. Splitting formulas for Schubert polynomials. Following [20, 122], we say that an element $w \in W_{\infty}$ is compatible with the sequence $a : a_1 < \cdots < a_p$ of elements of $\mathbb{N}_0$ if all descent positions of $w$ are contained in $a$. We say that an element $w \in \tilde{W}_{\infty}$ is compatible with the sequence $a : a_1 < \cdots < a_p$ of elements of $\mathbb{N}_0$ if all descent positions of $w$ are listed among $\square, a_1, \ldots, a_p$, if $a_1 = 1$, or contained in $a$, otherwise. Let $w$ be an element of $W_{\infty}$ (respectively $\tilde{W}_{\infty}$) compatible with $a$ as above and let $b : b_1 < \cdots < b_q$ be a second sequence of elements of $\mathbb{N}_0$ (respectively $\mathbb{N}_0$) such that $w^{-1}$ is compatible with $b$. We say that a reduced factorization $u_1 \cdots u_{p+q-1} = w$ is compatible with $a$, $b$ if $u_j(i) = i$ whenever $j < q$ and $i \leq b_q - j$ or whenever $j > q$ and $i \leq a_j - q$, where we set $u_j(0) = 0$.

Suppose that $w = \varpi \in S_{\infty}$ and $a_1 > 0$. Given any sequence of partitions $\lambda = (\lambda^1, \ldots, \lambda^{p+q-1})$, we define

$$(41) \quad c^w_{\lambda} := \sum_{u_1 \cdots u_{p+q-1} = \varpi} c^{u_1} \cdots c^{u_{p+q-1}}.$$
Next suppose that \( w \in W_\infty \) and \( b_1 = 0 \). Given any sequence of partitions \( \lambda = (\lambda_1, \ldots, \lambda^{p+q-1}) \) with \( \lambda^q a_1 \)-strict, we define
\[
(42) \quad f^w_\lambda := \sum_{u_1 \cdots u_p+q-1 = w} c^u_{\lambda_1} \cdots c^{u_q}_{\lambda^{q-1}} c^{u_{q+1}}_{\lambda_q} \cdots c^{u_{p+q-1}}_{\lambda^{p+q-1}}.
\]

Finally, suppose that \( w \in \bar{W}_\infty \) and \( b_1 = \varnothing \). Given any sequence of partitions \( \underline{\lambda} = (\lambda_1, \ldots, \lambda^{p+q-1}) \) with \( \lambda^q a_1 \)-strict and typed, define
\[
(43) \quad g^w_\underline{\lambda} := \sum_{u_1 \cdots u_p+q-1 = w} c^u_{\lambda_1} \cdots c^{u_q}_{\lambda^{q-1}} c^{u_{q+1}}_{\lambda_q} \cdots c^{u_{p+q-1}}_{\lambda^{p+q-1}}.
\]

The sums in equations (41), (42), and (43) are over all reduced factorizations \( u_1 \cdots u_p+q-1 = w \) compatible with \( a, b \) such that \( u_i \in S_\infty \) for all \( i \neq q \). Moreover, the nonnegative integers \( c^u_{\lambda_i} \), \( c^u_{\lambda^{q-1}} \), and \( c^u_{\lambda^{p+q-1}} \) in these formulas are the (mixed) Stanley coefficients which were defined in (47).

Set \( Y_i := \{ y_{a_i-1+1}, \ldots, y_{a_i} \} \) for each \( i \geq 1 \) and \( Z_j := \{ z_{b_{j-1}+1}, \ldots, z_{b_j} \} \) for each \( j \geq 1 \). Notice in particular that \( Y_i = \emptyset \) if \( a_i = 0 \) or \( a_1 = \varnothing \).

**Theorem 5** (Splitting Schubert polynomials, [20] [22]). (a) Suppose that \( a_1 > 0 \) and that \( w \) and \( w^{-1} \) in \( S_\infty \) are compatible with the sequences \( a \) and \( b \), respectively. Then the Schubert polynomial \( S_w(Y, Z) \) satisfies
\[
(44) \quad S_w = \sum_{\underline{\lambda}} c^\underline{\lambda}_{\lambda_1} s_{\lambda_1}(0/Z_q) \cdots s_{\lambda^{q-1}}(0/Z_2) s_{\lambda_q}(Y_1/Z_1) s_{\lambda_q+1}(Y_2) \cdots s_{\lambda^{p+q-1}}(Y_p)
\]
summed over all sequences of partitions \( \underline{\lambda} = (\lambda_1, \ldots, \lambda^{p+q-1}) \).

(b) Suppose that \( b_1 = 0 \) and that \( w \) and \( w^{-1} \) in \( W_\infty \) are compatible with the sequences \( a \) and \( b \), respectively. Then the Schubert polynomial \( C_w(X; Y, Z) \) satisfies
\[
(45) \quad C_w = \sum_{\underline{\lambda}} f^w_\underline{\lambda} s_{\lambda_1}(0/Z_q) \cdots s_{\lambda^{q-1}}(0/Z_2) \Theta_{\lambda^q}(X; Y_1) s_{\lambda^q+1}(Y_2) \cdots s_{\lambda^{p+q-1}}(Y_p)
\]
summed over all sequences of partitions \( \underline{\lambda} = (\lambda_1, \ldots, \lambda^{p+q-1}) \) with \( \lambda^q a_1 \)-strict.

(c) Suppose that \( b_1 = \emptyset \) and that \( w \) and \( w^{-1} \) in \( \bar{W}_\infty \) are compatible with the sequences \( a \) and \( b \), respectively. Then the Schubert polynomial \( D_w(X; Y, Z) \) satisfies
\[
(46) \quad D_w = \sum_{\underline{\lambda}} g^w_\underline{\lambda} s_{\lambda_1}(0/Z_q) \cdots s_{\lambda^{q-1}}(0/Z_2) H_{\lambda^q}(X; Y_1) s_{\lambda^q+1}(Y_2) \cdots s_{\lambda^{p+q-1}}(Y_p)
\]
summed over all sequences of partitions \( \underline{\lambda} = (\lambda_1, \ldots, \lambda^{p+q-1}) \) with \( \lambda^q a_1 \)-strict and typed.

**Remark 3.** Theorem 5 includes multiple formulas for the double Schubert polynomials, depending on the choice of sequences \( a \) and \( b \). In each case there is a basic formula which involves the descents of \( w \) and \( w^{-1} \), defined by choosing \( a \) and \( b \) to be minimal, and all the other formulas are obtained from it by splitting the variables further. It would be desirable to eliminate the hypothesis on \( b_1 \) from parts (b) and (c); for some obstacles in the way of achieving this, see [22], Example 3.

**Example 13.** Suppose that \( a_1 = 0 \) and \( a_1 = \emptyset \) in parts (b) and (c) of Theorem 5. Then we obtain the splitting formulas
\[
(47) \quad C_w = \sum_{\underline{\lambda}} f^w_\underline{\lambda} s_{\lambda_1}(0/Z_q) \cdots s_{\lambda^{q-1}}(0/Z_2) Q_{\lambda^q}(X) s_{\lambda^q+1}(Y_2) \cdots s_{\lambda^{p+q-1}}(Y_p)
\]
and
\[
D_w = \sum_{\lambda} q^{w}_{\lambda}(0/Z_q) \cdots s_{\lambda_{r-1}}(0/Z_2) P_{\lambda^r}(X) s_{\lambda_{r+1}}(Y_2) \cdots s_{\lambda_{r+q-1}}(Y_p),
\]
respectively, where the sums are over all sequences of partitions \(\lambda = (\lambda^1, \ldots, \lambda^{r+q-1})\) with \(\lambda^i\) strict. Thus we see that in all cases, the Schubert polynomials can be expressed as a positive linear combination of products of Jacobi-Trudi determinants times (at most) a single Schur Pfaffian. Recall also that in this situation, there exist tableau-based combinatorial expressions for the coefficients \(f^{w}_{\lambda}\) and \(g^{w}_{\lambda}\). Single Schubert polynomial versions of equations (47) and (48) were given in [128].

**Example 14.** To simplify the notation, write \(\mathcal{C}_w\) for \(\mathcal{C}_w(X; Y, Z)\), \(Q_{\lambda}\) for \(Q_{\lambda}(X)\), and \(\Theta^{(k)}_{\lambda}\) for \(\Theta_{\lambda}(X; Y_{(k)})\).

(a) Let \(w = 3\overline{T} = s_9s_8s_1 \in W_3\) and set \(b = (0 < 1)\). The following equalities correspond to \(a = (1 < 2)\) and \(a = (0 < 1 < 2)\), respectively, in Theorem 5(b).

\[
\mathcal{C}_{\overline{T}} = \Theta^{(1)}_{(4, 2)} + \Theta^{(1)}_{(1, 3, 2)} y_2 - z_1 \Theta^{(1)}_{(3, 2)}
= Q_{(4, 2)} + Q_{(4, 1)} y_1 + Q_{(3, 2)} y_2 + (Q_{(3, 1)} y_1^2 + (Q_{(2, 2)} y_1 y_2) y_2
- z_1 (Q_{(3, 2)} + Q_{(3, 1)} y_1 + Q_{(2, 1)} y_2^2).
\]

(b) Let \(w = 2\overline{S} = s_2s_1s_0s_1s_2 \in W_3\), set \(b = (0 < 2)\), \(e_i^0 = e_i(y_1, y_2)\) and \(e_i^1 = e_i(z_1, z_2)\) for \(i = 1, 2\). The following equalities correspond to taking \(a\) to be \((2), (1 < 2), \) and \((0 < 2)\), respectively, in Theorem 5(b).

\[
\mathcal{C}_{2\overline{S}} = \Theta^{(2)}_{3} - e_1^0 \Theta_{4}^{(2)} + e_2^0 \Theta_{3}^{(2)}
= \Theta^{(1)}_{3} + \Theta^{(1)}_{4} y_2 - e_1^0 (\Theta^{(1)}_{4} + \Theta^{(1)}_{3} y_2) + e_2^0 (\Theta^{(1)}_{3} + \Theta^{(1)}_{2} y_2)
= Q_3 + Q_4 e_1^0 y_1 + Q_3 e_2^0 - e_1^0 (Q_4 + Q_3 e_1^0 y_1 + Q_2 e_2^0) + e_2^0 (Q_3 + Q_2 e_1^0 + Q_1 e_2^0).
\]

**6. Degeneracy Locci**

In this section, we show how the splitting formulas of Theorem 5 translate directly into Chern class formulas for degeneracy loci in the sense of [41] and [3.2] with the symmetries native to the corresponding \(G/P\) space.

Let \(c = (c_0, c_1, c_2, \ldots)\) and \(d = (d_1, d_1, d_2, \ldots)\) be two families of commuting variables, with \(c_0 = d_0 = 1\) as usual. We first extend the definitions of the polynomials \(s_{\lambda}(c)\), \(\Theta_{\lambda}(c)\), and \(H_{\lambda}(c)\) from [11] and [14] to obtain polynomials in the formal difference \(c - d\) (this notation stems from the theory of \(\lambda\)-rings). Define elements \(g_r\) and \(h_r\) for \(r \in \mathbb{Z}\) by the identities of formal power series

\[
\sum_{r = -\infty}^{+\infty} g_r t^r := \left( \sum_{i=0}^{\infty} c_i t^i \right) \left( \sum_{i=0}^{\infty} d_i t^i \right)^{-1};
\]

\[
\sum_{r = -\infty}^{+\infty} h_r t^r := \left( \sum_{i=0}^{\infty} (-1)^i c_i t^i \right)^{-1} \left( \sum_{i=0}^{\infty} (-1)^i d_i t^i \right).
\]

For any partition \(\lambda\), define the Schur polynomial \(s_{\lambda}(c - d)\) by

\[
s_{\lambda}(c - d) := R^0 h_{\lambda} = \det(h_{\lambda_i+j-i})_{i,j}.
\]
For any $k$-strict partition $\lambda$, define the theta polynomial $\Theta_\lambda(c - d)$ by

$$\Theta_\lambda(c - d) := R^\lambda g_\lambda$$

where $R^\lambda$ denotes the raising operator in Eq. (12). Finally, for any typed $k$-strict partition $\lambda$, define the eta polynomial $H_\lambda(c - d)$ by

$$H_\lambda(c - d) := 2^{-\ell(\lambda)} R^\lambda \star g_\lambda$$

where $R^\lambda$ denotes the raising operator in Eq. (23), and the action $\star$ is defined as in Eq. (2.2) replacing $c_1$ by $g_\lambda$ throughout. If $V$ and $V'$ are two vector bundles on an algebraic variety $M$ with total Chern classes $c(V)$ and $c(V')$, respectively, we denote the class $s_\lambda(c(V) - c(V'))$ by $s_\lambda(V - V')$, the class $\Theta_\lambda(c(V) - c(V'))$ by $\Theta_\lambda(V - V')$, and the class $H_\lambda(c(V) - c(V'))$ by $H_\lambda(V - V')$.

6.1. **Type A degeneracy loci.** Fix a sequence $a : a_1 < \cdots < a_p$ of positive integers with $a_p < n$, which gives a subset of the nodes of the Dynkin diagram for the root system of type $A_{n-1}$:

```
1 -- 2 -- 3 -- 4 -- 5 -- 6 -- \cdots -- n-1
```

Let $S^n_a$ be the set of permutations $\varpi \in S_n$ whose descent positions are listed among the integers $a_1, \ldots, a_p$, i.e., that are compatible with $a$. These elements are the minimal length coset representatives in $S_n/S_a$, where $S_a$ denotes the parabolic subgroup of $S_n$ generated by the simple transpositions $s_i$ for $i \not\in \{a_1, \ldots, a_p\}$.

Let $E \to M$ be a vector bundle of rank $n$ on an algebraic variety $M$, assumed to be smooth for simplicity. Consider a partial flag of subbundles of $E$

$$E_* : 0 \subset E_1 \subset \cdots \subset E_p \subset E$$

with rank $E_i = a_i$ for each $i$, and a complete flag $0 = F_0 \subset F_1 \subset \cdots \subset F_n = E$ of subbundles of $E$ (with rank $F_i = i$ for each $i$). For every $\varpi \in S^n_a$, we define the **degeneracy locus** $X_{\varpi} \subset M$ as the locus of $b \in M$ such that

$$\dim(E_r(b) \cap F_s(b)) \geq \# \{ i \leq a_r \mid \varpi_i > n - s \} \forall r, s.$$ 

A precise definition of $X_{\varpi}$ as a subscheme of $X$ can be obtained by pulling back from the universal case, which occurs on the partial flag bundle $F^n(E)$ (see [41] and [46, \S6.2 and App. A] for more details). Assume further that $X_{\varpi}$ has pure codimension $\ell(\varpi)$ in $M$, which is the case when the vector bundles are in general position. The next result will be a formula for the cohomology class $[X_{\varpi}]$ in $H^{2\ell(\varpi)}(M)$ in terms of the Chern classes of the bundles $E_r$ and $F_s$.

Consider a second sequence $b : 0 \leq b_1 < \cdots < b_q$ with $b_q < n$.

**Theorem 6** ([20]). Suppose that $\varpi \in S^n_a$ and that $\varpi^{-1}$ is compatible with $b$. Then we have

$$[X_{\varpi}] = \sum_{\lambda} c^w_\lambda s_{\lambda^1}(F_{n-b_{q-1}} - F_{n-b_q}) \cdots s_{\lambda^q}(E_{-E_1 - F_{n-b_1}}) \cdots s_{\lambda^{q-1}}(E_{p-1} - E_p)$$

in $H^*(M)$, where the sum is over all sequences of partitions $\lambda = (\lambda^1, \ldots, \lambda^{q+q-1})$ and the coefficients $c^w_\lambda$ are given by (11).
6.2. Symplectic degeneracy loci. Fix a sequence $a : a_1 < \cdots < a_p$ of nonnegative integers with $a_p < n$, which is a subset of the nodes of the Dynkin diagram for the root system of type $C_n$:

$$
\begin{array}{cccccccc}
a_1 & a_2 & \cdots & \cdots & a_p \\
0 & 1 & 2 & 3 & 4 & 5 & \cdots & n-1
\end{array}
$$

Denote by $W^a$ the set of signed permutations $w \in W_n$ whose descent positions are listed among the integers $a_1, \ldots, a_p$.

Let $E \rightarrow M$ be a vector bundle of rank $2n$ on a smooth algebraic variety $M$. Assume that $E$ is a symplectic bundle, so that $E$ is equipped with an everywhere nondegenerate skew-symmetric form $E \otimes E \rightarrow \mathbb{C}$. Consider a partial flag of sub-bundles of $E$

$$
E_w : 0 \subset E_p \subset \cdots \subset E_1 \subset E
$$

with rank $E_i = n - a_i$ and $E_1$ isotropic, and a (complete) isotropic flag $0 = F_0 \subset F_1 \subset \cdots \subset F_{2n} = E$ of sub-bundles of $E$ with rank $F_i = i$ for each $i$ and $F_n+i = F_{n+i}^\perp$ for $1 \leq i \leq n$. Fix a second sequence $b : 0 = b_1 < \cdots < b_q$ with $b_q < n$, and define quotient bundles

$$
\begin{align*}
Q_1 &:= E/E_1, \quad Q_2 := E_1/E_2, \quad \ldots, \quad Q_p := E_{p-1}/E_p \\
\hat{Q}_2 &:= F_n/F_{n-b_2}, \quad \ldots, \quad \hat{Q}_q := F_{n-b_{q-1}}/F_{n-b_q}.
\end{align*}
$$

(49)

There is a group monomorphism $\phi : W_n \rightarrow S_{2n}$ with image

$$
\phi(W_n) = \{ \varpi \in S_{2n} \mid \varpi_i + \varpi_{2n+1-i} = 2n + 1, \text{ for all } i \}.
$$

The map $\phi$ is determined by setting, for each $w = (w_1, \ldots, w_n) \in W_n$ and $1 \leq i \leq n$, $n$, $\phi(w)_i := \begin{cases}
 n + 1 - w_{n+1-i} & \text{if } w_{n+1-i} \text{ is unbarred,} \\
 n + \overline{w}_{n+1-i} & \text{otherwise.}
\end{cases}$

For every $w \in W^a$ we have the degeneracy locus $X_w \subset M$, which is the locus of $b \in M$ such that

$$
\dim(E_r(b) \cap F_s(b)) \geq \# \{ i \leq n - a_r \mid \phi(w)_i > 2n - s \} \quad \forall \ r, s.
$$

As in the type A case, assuming that $X_w$ has pure codimension $\ell(w)$ in $M$, we give a formula for the class $[X_w]$ in $H^{2\ell(w)}(M)$.

**Theorem 7 (122).** Suppose that $w \in W^a$ and that $w^{-1}$ is compatible with $b$. Then we have

$$
[X_w] = \sum_{\lambda} f^w_\lambda \cdot s^w_{\lambda^1}(Q_1) \cdots s^w_{\lambda_{q-1}}(Q_2) \Theta_{\lambda^1}(Q_1 - F_n)s_{\lambda_{q+1}}(Q_2) \cdots s_{\lambda_{p+q-1}}(Q_p)
$$

$$
= \sum_{\lambda} f^w_\lambda \cdot s^w_{\lambda^1}(F_{n+b_{q-1}} - F_{n+b_q}) \cdots \Theta_{\lambda^1}(E - E_1 - F_n) \cdots s_{\lambda_{p+q-1}}(E_{p-1} - E_p)
$$

in $H^*(M)$, where the sum is over all sequences of partitions $\lambda = (\lambda^1, \ldots, \lambda^{p+q-1})$ with $\lambda^q$ $a_1$-strict, and the coefficients $f^w_\lambda$ are given by [122].

6.3. Orthogonal degeneracy loci.
6.3.1. The odd orthogonal case. Let $E \to M$ be a vector bundle of rank $2n + 1$ on a smooth algebraic variety $M$. Assume that $E$ is an orthogonal bundle, i.e. $E$ is equipped with an everywhere nondegenerate symmetric form $E \otimes E \to \mathbb{C}$. Fix a sequence $a : a_1 < \cdots < a_p$ of nonnegative integers with $a_p < n$, as in the symplectic case. Consider a partial flag of subbundles of $E$

$$E_* : 0 \subset E_p \subset \cdots \subset E_1 \subset E$$

with rank $E_i = n - a_i$ and $E_1$ isotropic, and a complete isotropic flag $0 = F_0 \subset F_1 \subset \cdots \subset F_{2n+1} = E$ of subbundles of $E$. Let $b : b_1 < \cdots < b_q$ be a sequence of integers with $b_q < n$ and define the quotient bundles $Q_i$ and $\tilde{Q}_j$ using the same equations \([49]\) as in type C.

There is a group monomorphism $\phi' : W_n \to S_{2n+1}$ with image

$$\phi'(W_n) = \{ \varpi \in S_{2n+1} \mid \varpi_i + \varpi_{2n+2-i} = 2n + 2, \text{ for all } i \}. $$

The map $\phi$ is determined by setting, for each $w = (w_1, \ldots, w_n) \in W_n$ and $1 \leq i \leq n$,

$$\phi'(w)_i := \begin{cases} 
 n + 1 - w_{n+1-i} & \text{if } w_{n+1-i} \text{ is unbarred}, \\
 n + 1 + \overline{w}_{n+1-i} & \text{otherwise}.
\end{cases}$$

For every $w \in W^a$ we have the degeneracy locus $X_w \subset M$. This is the locus of $b \in M$ such that

$$\dim(E_r(b) \cap F_s(b)) \geq \# \{ i \leq n - a_r \mid \phi'(w)_i > 2n + 1 - s \} \forall r, s,$$

and we assume as before that $X_w$ has pure codimension $\ell(w)$ in $M$. For any $a_1$-strict partition $\lambda$, let $\tilde{\Theta}_\lambda = 2^{-s(\lambda)}\Theta_\lambda$, where $s(\lambda) = \# \{ i \mid w_i < 0 \}$. We then have the following analogue of Theorem 7

**Theorem 8 \([122]\).** Suppose that $w \in W^a$ and that $w^{-1}$ is compatible with $b$. Then we have

$$[X_w] = \sum_\lambda \tilde{f}_w^\lambda s_{\lambda_1}(\tilde{Q}_1) \cdots s_{\lambda_{p+q-1}}(Q_p) \tilde{\Theta}_\lambda(Q_1 - F_n) \cdots s_{\lambda_{p+q-1}}(Q_p)$$

$$= \sum_\lambda \tilde{f}_w^\lambda \Theta_\lambda(E - E_1 - F_{n+1}) \cdots s_{\lambda_{p+q-1}}(E_{p-1} - E_p)$$

in $H^*(M)$, where the sum is over all sequences of partitions $\lambda = (\lambda_1, \ldots, \lambda^{p+q-1})$ with $\lambda_q$ $a_1$-strict, and the coefficients $\tilde{f}_w^\lambda$ are given by \([122]\).

6.3.2. The even orthogonal case. Let $E \to M$ be an orthogonal vector bundle of rank $2n$ on a smooth algebraic variety $M$. Fix a complete isotropic flag $0 = F_0 \subset F_1 \subset \cdots \subset F_{2n} = E$ of subbundles of $E$, and a sequence $a : a_1 < \cdots < a_p$ of elements of $\mathbb{N}$ with $a_p < n$, which gives a subset of the vertices of the Dynkin diagram for the root system of type $D_n$:

\[
\begin{array}{ccccccccccc}
\bullet & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \bullet \\
1 & 2 & 3 & 4 & 5 & 6 & \cdots & n-1 & \bullet
\end{array}
\]

Consider a partial flag of subbundles of $E$

$$E_* : 0 \subset E_p \subset \cdots \subset E_1 \subset E$$
with rank $E_i = n - a_i$ and $E_1$ isotropic and in the same family as $F_n$ if $a_1 = \emptyset$. Fix a sequence $b : \emptyset = b_1 < \cdots < b_q$ with $b_q < n$, and define the quotient bundles $Q_i$ and $\bar{Q}_j$ as in type $C$, using the equations \[19\].

Recall that $\bar{W}_n$ is a subgroup of $W_n$, and hence we have a group monomorphism $\phi : \bar{W}_n \hookrightarrow S_{2n}$, defined by restricting the map $\phi$ of \[6.2\] to $\bar{W}_n$. Let $\bar{W}^a$ be the set of signed permutations $w \in \bar{W}_n$ which are compatible with $a$. For every $w \in \bar{W}^a$, we define the degeneracy locus $\mathfrak{X}_w \subset M$ as the closure of the locus of $b \in M$ such that

$$\dim(E_r(b) \cap F_s(b)) = \# \{ i \leq n - a_r \mid \phi(w_0w_0)_i > 2n - s \} \quad \forall \, r, s,$$

with the reduced scheme structure. Assume further that $\mathfrak{X}_w$ has pure codimension $\ell(w)$ in $M$, and consider its cohomology class $[\mathfrak{X}_w]$ in $H^{2\ell(w)}(M)$.

**Theorem 9 (122).** Suppose that $w \in \bar{W}^a$ and that $w^{-1}$ is compatible with $b$. Then we have

$$[\mathfrak{X}_w] = \sum_{\lambda} g^w_\lambda s_{\lambda_1}(\bar{Q}_1) \cdots s_{\lambda_{q-1}}(\bar{Q}_2) H_{\lambda_q}(Q_1 - F_n) s_{\lambda_{q+1}}(Q_2) \cdots s_{\lambda_{p+q-1}}(Q_p)$$

$$= \sum_{\lambda} g^w_\lambda s_{\lambda_1}(F_{n+b_{q-1}} - F_{n+b_q}) \cdots H_{\lambda_q}(E - E_1 - F_n) s_{\lambda_{q+1}}(E_{p-1} - E_p),$$

in $H^*(M)$, where the sum is over all sequences of partitions $\lambda = (\lambda^1, \ldots, \lambda^{q-1})$ with $\lambda^3 a_1$-strict and typed, and the coefficients $g^w_\lambda$ are given by \[13\].

**Remark 4.** Using the identification of degeneracy locus formulas with Giambelli polynomials in equivariant cohomology (see 50, 3.2 and 7.3), the theorems in this section translate easily into corresponding results for the $T$-equivariant cohomology ring of $G/P$. The Chern roots of the vector bundle $F_n$ (up to a sign) are identified with the weights of the linear $T$-action on the standard representation $V$ of $G$; specifically, the class $-c_1(F_{n+1-i}/F_{n-i})$ is mapped onto the variable $t_i$ in the notation of 50 \[10\] (and the variable $y_i$ in 7.3 below). The results may also be used to deduce formulas for the restriction of an equivariant Schubert class in $H^*_T(G/P)$ to a torus fixed point, as is done in op. cit.

7. PROOFS OF THE MAIN THEOREMS

We discuss here what is involved in proving the aforementioned theorems. We pay particular attention to the proofs of Theorems 6 and 9 and establish the connection between the Schubert polynomials of 59 and the geometry of degeneracy loci.

7.1. Proofs of IG and OG Giambelli. Although briefly mentioned in \[1\] and \[2\] so far there has been little discussion of Pieri rules for isotropic Grassmannians. The first such rules were proved by Hiller and Boe 55 for the maximal isotropic Grassmannians, and used by Pragacz 100 to prove his Giambelli formulas for these spaces. Pragacz and Ratajski 102, 104 obtained Pieri type rules for isotropic Grassmannians, but used a different notion of ‘special Schubert classes’ than ours (their special classes were the Chern classes of the universal subbundles $E'$ in \[2\].

The proofs of Theorems 2 and 3 from 17, 19 require the classical Pieri rules from 16, which hold for the special Schubert classes $c_r$ and $\tau_r$ defined in 2. For
the symplectic Grassmannian $IG(n - k, 2n)$, the Pieri rule has the form
\begin{equation}
(50) \quad c_p \cdot [X_\lambda] = \sum_{\lambda \rightarrow \mu, |\mu| = |\lambda| + p} 2^{N(\lambda, \mu)} [X_\mu].
\end{equation}
When $k > 0$, the Pieri relation $\lambda \rightarrow \mu$ and the definition of the exponents $N(\lambda, \mu)$ is more complicated than in the $k = 0$ case of the Lagrangian Grassmannian. However, the Pieri rule (50) can still be used recursively to show that a general Schubert class $[X_\lambda]$ may be written as a polynomial in the special classes. Therefore, in order to prove Theorem 2, it suffices to show that the expression $R^\lambda c_\lambda$ satisfies the same Pieri rule (50), and this is the approach taken in [17]. The complex argument that proves that $R^\lambda c_\lambda$ obeys (50) uses subtle alternating properties of $R^\lambda c_\lambda$ which depend on the order ideal $C_\lambda$, and a substitution algorithm which is a mathematical model of controlled evolution. The reader will find an exposition of the main ideas in [120, §3]. The proof of Theorem 3 is similar to that of Theorem 2. It exploits the weight space decomposition of $H^*(OG(m, 2n), \mathbb{Q})$ induced by the natural involution of the Dynkin diagram of type $D_n$, and a surprising relation between the Schubert calculus on even and odd orthogonal Grassmannians.

Currently, there is a second, simpler proof of Theorem 2, which however uses many of the same ideas from [17]. In [124], we employ raising operators to define double theta polynomials that lie in the ring $\Gamma[Y, Z]$, and are ultimately shown to agree with $\mathcal{C}_w(X; Y, Z)$ (modulo the ideal of relations in $\Gamma[Y, Z]$). The double theta polynomials are analogs of the usual double (or factorial) Schur polynomials, and give equivariant Giambelli expressions for the Schubert classes in $H^*_T(IG(m, 2n))$. Although the equivariant Giambelli polynomials are more complicated to define than the cohomological ones $R^\lambda c_\lambda$, to show that they represent the Schubert classes it suffices to check that they obey the equivariant Chevalley formula (see e.g. [92, Prop. 4.1]), which is easier to work with than the Pieri rule (50).

7.2. Proofs of Theorems 4 and 5. Following [10, 37, 82] or by specializing the arguments of [17, §3], one shows that the single Schubert polynomials represent the Schubert classes in $H^*(G/B)$, and that the algebra that they span is isomorphic to the stable cohomology ring $H(G/B)$ of the complete flag manifold as $n \to \infty$.

Therefore, to prove Theorem 4 it suffices to show that (a) the Schur, theta, and eta polynomials indexed by single row partitions agree with the corresponding single Schubert polynomials, and (b) the Giambelli formulas of [11] and [2] hold in $H(G/B)$. Assertion (a) is easy to check directly from the definition of the Schubert polynomials, and (b) is an immediate consequence of Theorems 11 and 2.

The proof of Theorem 5 is based on the transition equations for Schubert polynomials mentioned in [1, 3, 5], and some important actors that have remained largely in the background until now, the Stanley symmetric functions of [10, 37, 113] and the mixed Stanley functions of [122]. We will state the key result here for the type C Billey-Haiman Schubert polynomials $\mathcal{C}_w(X; Y)$. Let $\mathcal{C}_w(X; Y; k)$ denote the power series obtained from $\mathcal{C}_w(X; Y)$ by setting $y_j = 0$ for all $j > k$.

**Theorem 10** ([122]). Suppose that $w \in W_\infty$ is increasing up to $k$. Then we have
\begin{equation}
(51) \quad \mathcal{C}_w(X; Y; k) = \sum_{\lambda} \mathcal{C}^w_\lambda \Theta_\lambda(X; Y; k)
\end{equation}
where the sum is over all $k$-strict partitions $\lambda$ with $|\lambda| = \ell(w)$. 
For example, taking $k = 0$ in Theorem 10 gives the following expansion of the type C Stanley function $F_w$, which was first proved by Billey [3]:

\begin{equation}
F_w(X) = \sum_{\lambda: |\lambda| = \ell(w)} e^w \lambda Q_\lambda(X)
\end{equation}

with the sum over strict partitions $\lambda$ of $\ell(w)$.

When $w$ is increasing up to $k$, we can identify $\mathcal{C}_w(X; Y(k))$ in (51) with the restriction $J_w(X; Y(k))$ of a `mixed Stanley function' $J_w(X; Y)$ (see [122, §2] for the precise definitions). Furthermore, one shows directly from their definition in [51] that the Schubert polynomials satisfy splitting formulas which express them as sums of products of (mixed) Stanley functions in mutually disjoint groups of variables. Theorem 5 follows immediately by combining these two ingredients.

7.3. Divided differences and geometrization maps. We define an action of $W_\infty$ on $\Gamma[Y, Z]$ by ring automorphisms as follows. The simple reflections $s_i$ for $i > 0$ act by interchanging $y_i$ and $y_{i+1}$ and leaving all the remaining variables fixed. The reflection $s_0$ maps $y_1$ to $-y_1$, fixes the $y_j$ for $j \geq 2$ and all the $z_j$, and satisfies

\begin{equation}
\begin{align*}
    s_0(q_r(X)) &= q_r(y_1, x_1, x_2, \ldots) = q_r(X) + 2 \sum_{j=1}^{r} y_j q_{r-j}(X).
\end{align*}
\end{equation}

For each $i \geq 0$, define the divided difference operator $\partial^w_i$ on $\Gamma[Y, Z]$ by

\begin{equation}
\partial^w_i f := \frac{f - s_i f}{y_i - y_{i+1}}, \quad \partial^w_i f := \frac{f - s_i f}{y_i - y_{i+1}} \quad \text{for } i > 0.
\end{equation}

Consider the ring involution $\omega : \Gamma[Y, Z] \to \Gamma[Y, Z]$ determined by

\begin{equation}
\omega(y_j) = -z_j, \quad \omega(z_j) = -y_j, \quad \omega(q_r(X)) = q_r(X)
\end{equation}

and set $\partial^w_i := \omega \partial^w_i \omega$ for each $i \geq 0$.

**Theorem 11** (Uniqueness, [59]). The polynomials $\mathcal{C}_w(X; Y, Z)$ for $w \in W_\infty$ are the unique family of elements of $\Gamma[Y, Z]$ satisfying the equations

\begin{equation}
\partial^w_i \mathcal{C}_w = \begin{cases} \mathcal{C}_{ws_i} & \text{if } \ell(ws_i) < \ell(w), \\ \mathcal{C}_w & \text{if } \ell(s_i w) < \ell(w), \\ 0 & \text{otherwise,} \end{cases}
\end{equation}

for all $i \geq 0$, together with the condition that the constant term of $\mathcal{C}_w$ is 1 if $w = 1$, and 0 otherwise.

**Proof.** In [58] Lemma 3.5] and [37] Theorem 7.1] Fomin, Stanley, and Kirillov provide simple proofs (using the nilCoxeter relations) that the single Schubert polynomials $\mathcal{C}_w(X; Y)$ satisfy the equations in (53) which involve the $\partial^w_i$ operators. In view of equation (52), the arguments in loc. cit. extend easily to show that the double Schubert polynomials $\mathcal{C}_w(X; Y, Z)$ fulfill the entire list of conditions in the theorem. Following [59] [7.4], the uniqueness is shown as follows. If $\{\mathcal{C}_w\}$ for $w \in W_\infty$ is a second family of elements satisfying the displayed conditions, then by inducting on the length of $w$ one sees that $\partial^w_i (\mathcal{C}_w - \mathcal{C}_w) = \partial^w_i (\mathcal{C}_w - \mathcal{C}_w) = 0$ for all $i \geq 0$. We deduce that the difference $\mathcal{C}_w' - \mathcal{C}_w$ is invariant under the action of $s_i$ and $\omega s_i \omega$ for every $i$, and hence must be a constant. Finally, the condition on the constant term implies that $\mathcal{C}_w' = \mathcal{C}_w$, for all $w \in W_\infty$.  

\[\Box\]
The connection between Theorem 5 and Theorems 6, 7, 8, and 9 depends on an important ring homomorphism derived from §10 and 50, which we call the geometrization map. We will discuss this homomorphism in detail in the Lie types A, C, and D (leaving type B as an exercise for the reader), and use it to give a complete proof of Theorem 10.

Let $G$ denote the group $\text{GL}_n(\mathbb{C})$, $\text{Sp}_{2n}(\mathbb{C})$, or $\text{SO}_{2n}(\mathbb{C})$ with its standard representation $V = \mathbb{C}^n$ for $\text{GL}_n$ or $V = \mathbb{C}^{2n}$ in the latter two cases. In type A, equip $V$ with the zero form, and in types C and D equip $V$ with an antidiagonal symplectic or orthogonal form $(\cdot, \cdot)$, so that the standard basis $\{e_1, \ldots, e_{2n}\}$ of $V$ satisfies $(e_i, e_j) = 0$ for $i + j \neq 2n + 1$ and $(e_i, e_{2n+1-i}) = 1$, for $1 \leq i \leq n$. We obtain an induced vector bundle $E = EG \times^G V$ over $BG$ and bilinear form $E \otimes E \rightarrow \mathbb{C}$. Let $V_\bullet$ be the isotropic flag in $V$ with $V_i = \langle e_1, \ldots, e_i \rangle$ for each $i \in [1, n]$, and $B$ denote the stabilizer of $V_\bullet$. The pullback of the bundle $E$ to $BB$ has an isotropic flag $V'_\bullet = \{ EG \times^B V_i \}$, of subbundles of $E$. If $\pi_1$ and $\pi_2$ are the two projection maps $BB \times^G BG \rightarrow BB$, then we obtain the two isotropic flags of subbundles $V'_\bullet := \pi_1^* V'_\bullet$ and $F'_\bullet := \pi_2^* V'_\bullet$ of the pullback of $E$ to $BB \times^G BG$. This is the universal case of the degeneracy locus problems considered in [0.4.10, 0.6.2, 0.8.4], for the parabolic subgroup $P = B$.

**Type A.** Introduce two new sets of commuting variables $X = (x_1, x_2, \ldots)$, $Y = (y_1, y_2, \ldots)$, and let $X_n = (x_1, \ldots, x_n)$ and $Y_n = (y_1, \ldots, y_n)$. In this case $G = \text{GL}_n$ and it follows from the above discussion and [29] that there is a natural presentation

$$H^*(BB \times^G BG, \mathbb{Z}) \cong \mathbb{Z}[X_n, Y_n]/I_n$$

where $I_n$ denotes the ideal generated by the differences $e_i(x_1, \ldots, x_n) - e_i(y_1, \ldots, y_n)$ for $1 \leq i \leq n$. The inverse of the isomorphism sends the class of $x_i$ to $-c_1(V_i/V_{i-1})$ and of $y_i$ to $-c_1(F_{n+1-i}/F_{n-i})$ for each $i$ with $1 \leq i \leq n$.

The geometrization map is the ring homomorphism

$$\pi_n : \mathbb{Z}[Y, Z] \rightarrow \mathbb{Z}[X_n, Y_n]/I_n$$

defined by setting $\pi_n(y_i) := x_i$ and $\pi_n(z_i) := y_i$ for $1 \leq i \leq n$ and $\pi_n(y_i) = \pi_n(z_i) = 0$ for $i > n$. Fulton [39, 41] showed that for $\varpi \in S_n$, the homomorphism $\pi_n$ sends $\mathbb{S}_\varpi(Y, Z)$ to a polynomial which represents the universal Schubert class $[X_{\varpi}]$ in the presentation (54). A different way to establish this is obtained by arguing as in the proof of Theorem 12 below.

**Type C.** Here $G = \text{Sp}_{2n}$ and we have a natural presentation

$$H^*(BB \times^G BG, \mathbb{Z}) \cong \mathbb{Z}[X_n, Y_n]/J_n$$

where $J_n$ denotes the ideal generated by the differences $e_i(x_1^2, \ldots, x_n^2) - e_i(y_1^2, \ldots, y_n^2)$ for $1 \leq i \leq n$. The inverse of the isomorphism sends the class of $x_i$ to $-c_1(V_{n+1-i}/V_{n-i})$ and of $y_i$ to $-c_1(F_{n+1-i}/F_{n-i})$ for each $i$ with $1 \leq i \leq n$.

Recall that $h_j(Y_n) = s_j(Y_n)$ denotes the $j$-th complete symmetric polynomial, which is the sum of all monomials of total degree $j$ in the variables $Y_n$. The geometrization map is the ring homomorphism

$$\pi_n : \Gamma[Y, Z] \rightarrow \mathbb{Z}[X_n, Y_n]/J_n$$
determined by setting

\[
\pi_n(q_r(X)) := \sum_{i=0}^{r} e_i(X_n)h_{r-i}(Y_n) \quad \text{for all } r,
\]

\[
\pi_n(y_i) := \begin{cases} -x_i & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } i > n \end{cases} \quad \text{and} \quad \pi_n(z_j) := \begin{cases} y_j & \text{if } 1 \leq j \leq n, \\ 0 & \text{if } j > n. \end{cases}
\]

**Theorem 12** (Geometrization, [59]). For any \( w \in W_n \), the geometrization map \( \pi_n \) maps \( \mathcal{C}_w(X,Y,Z) \) to a polynomial which represents the universal Schubert class \([X_w] \in H^*(BB \times BG BB, \mathbb{Z})\) in the presentation (55).

**Proof.** The simple reflections \( s_i \) in \( W_\infty \) act on \( Z[X,Y] \) as follows. The reflection \( s_i \) interchange \( x_i \) and \( x_{i+1} \) for \( i > 0 \), while \( s_0 \) replaces \( x_1 \) by \(-x_1\); all other variables remain fixed. We have corresponding divided difference operators \( \partial^\ell_i : Z[X,Y] \rightarrow Z[X,Y] \). For each \( i \geq 0 \) and \( f \in Z[X,Y] \), they are defined by

\[
\partial^\ell_0 f := \frac{f - s_0 f}{2x_1}, \quad \partial^\ell_i f := \frac{f - s_i f}{x_{i+1} - x_i} \quad \text{for } i > 0.
\]

Let \( \hat{\varpi} \) denote the involution of \( Z[X,Y] \) obtained by interchanging the variable \( x_j \) with \( y_j \) for all \( j \geq 1 \). Geometrically, this corresponds to the automorphism of \( BB \times BG BB \) given by switching the two factors. Define the \( y \)-divided difference operators \( \partial^\ell_i \) on \( Z[X,Y] \) by \( \partial^\ell_i = \hat{\varpi} \partial^\ell_i \hat{\varpi} \) for each \( i \geq 0 \).

Restricting the above to \( W_n \) and \( 0 \leq i \leq n - 1 \), we obtain divided differences \( \partial^\ell_i, \partial^\ell_i \) acting on \( Z[X_n,Y_n]/J_n \) and hence on \( H^*(BB \times BG BB, \mathbb{Z}) \). In the mid-1990s, building on the work of Bernstein-Gelfand-Gelfand [6] and Demazure [26, 27], Fulton [40, 41] studied the action of the operators \( \partial^\ell_i \) on the universal Schubert classes \([X_w] \), constructing them geometrically using correspondences by \( \mathbb{P}^1 \)-bundles. It follows from this work that we have

\[
\partial^\ell_i [X_w] = \begin{cases} [X_w] & \text{if } \ell(ws_i) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \quad \partial^\ell_i [X_w] = \begin{cases} [X_{s_iw}] & \text{if } \ell(s_iw) < \ell(w), \\ 0 & \text{otherwise,} \end{cases}
\]

for all \( i \).

Write \( BB_n \) and \( BG_n \) for \( BB \) and \( BG \), respectively, to emphasize the dependence on the rank \( n \), and denote by \( BM_n \) the Borel mixing space \( BB_n \times BG_n BB_n \). The natural embedding of \( W_n \) into \( W_{n+1} \) induces a morphism \( \phi_n : BM_n \rightarrow BM_{n+1} \) and hence a map of cohomology rings

\[
\phi^\ell_n : H^*(BM_{n+1}, \mathbb{Z}) \rightarrow H^*(BM_n, \mathbb{Z})
\]

which in terms of the presentation (55) is given by sending \( x_{n+1} \) and \( y_{n+1} \) to zero. Let

\[
\mathbb{H}(BM_\infty) := \lim_{\leftarrow} H^*(BM_n, \mathbb{Z})
\]

be the stable cohomology ring of \( BM_n \), which is the inverse limit in the category of graded rings of the system of maps \( \{\phi^\ell_n\}_{n \geq 1} \) in (57). For each \( w \in W_\infty \), we have a stable Schubert class \( C_w \) in \( \mathbb{H}(BM_\infty) \), defined as the element \( \lim_{\leftarrow} [X_w] \). It follows from the equations (56) that

\[
\partial^\ell_i C_w = \begin{cases} C_{ws_i} & \text{if } \ell(ws_i) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \quad \partial^\ell_i C_w = \begin{cases} C_{s_iw} & \text{if } \ell(s_iw) < \ell(w), \\ 0 & \text{otherwise,} \end{cases}
\]
Remark 5.\ The degree zero component of $C_w$ is 1 if $w = 1$, and 0 otherwise. Moreover, using the presentation (55) and arguing as Theorem 11 it is easy to check that the family $\{C_w\}$ for $w \in W_\infty$ is uniquely determined by these conditions.

The composite homomorphism
\[ \Gamma[Y, Z] \xrightarrow{\cong} \mathbb{Z}[X_n, Y_n]/J_n \xrightarrow{\cong} H^*(BM_n, \mathbb{Z}) \]
is compatible with the maps $\phi^*_w$, therefore there is an induced ring homomorphism
\[ \pi_\infty : \Gamma[Y, Z] \rightarrow \mathbb{H}(BM_\infty). \]
One verifies that $\pi_\infty$ respects the actions of the divided differences on its domain and codomain. We deduce the theorem, since both $\mathcal{C}_w$ and $C_w$ are characterized by the equations (59) and (58), respectively, and the same degree zero condition. One can show that, in fact, the map $\pi_\infty$ is a canonical isomorphism of graded rings. $\square$

Remark 5. For any $w \in W_n$, the image of $\mathcal{C}_w$ under the geometrization map $\pi_n$ may be computed as follows. Use (52) and (53) to write
\[ \mathcal{C}_w(X; Y, Z) = \sum_{u, v, u'} e_v' \mathcal{G}_{u'-1}(-Z)Q_\lambda(X)\mathcal{G}_{u'}(Y) \]
where the sum is over all reduced factorizations $uu' = w$ and strict partitions $\lambda$ with $u, u' \in S_n$ and $|\lambda| = \ell(v)$. We then have
\[ \pi_n(\mathcal{C}_w(X; Y, Z)) = \sum_{u, v, u', \lambda} e_v' \mathcal{G}_{u'-1}(-Y)Q_\lambda(X_n/Y_n)\mathcal{G}_{u'}(-Z_n). \]

We call the polynomial $Q_\lambda(X_n/Y_n)$ a supersymmetric $Q$-polynomial; it is obtained from the $Q$-polynomial $Q_\lambda(c)$ in (15) by specializing $c_r$ to $\sum_i e_i(X_n)h_{r-i}(Y_n)$ for each $r$. The supersymmetric $Q$-polynomials have properties directly analogous to those of the $Q$-polynomials $Q_\lambda(X_n)$ of Pragacz and Ratafia (103), which satisfy the identity $Q_\lambda(X_n) = Q_\lambda(X_n/Y_n)|_{Y_n=0}$. Moreover, setting $Z = 0$ and $Y_n = 0$ in (59), one recovers the symplectic Schubert polynomials of (115).

Type D. Here $G = SO_{2n}$ and according to (29) the cohomology ring $H^*(BB \times_{BG} BB, \mathbb{Q})$ is presented as a quotient
\[ H^*(BB \times_{BG} BB, \mathbb{Q}) \cong \mathbb{Q}[X_n, Y_n]/J'_n \]
where $J'_n$ denotes the ideal generated by the differences $e_i(x_1^2, \ldots, x_n^2) - e_i(y_1^2, \ldots, y_n^2)$ for $1 \leq i \leq n - 1$ and $x_1 \cdots x_n - y_1 \cdots y_n$. The inverse of the isomorphism (60) sends the class of $x_i$ to $-c_1(Y_n+1-i)/Y_n-i$ and of $y_i$ to $-c_1(F_n+1-i/F_n-i)$ for each $i$ with $1 \leq i \leq n$. The geometrization map is the algebra homomorphism
\[ \pi'_n : \Gamma'[Y, Z] \rightarrow \mathbb{Q}[X_n, Y_n]/J'_n \]
defined by setting
\[ \pi'_n(P_r(X)) := \frac{1}{2} \sum_{i=0}^r e_i(X_n)h_{r-i}(Y_n) \quad \text{for all } r, \]
\[ \pi'_n(y_i) := \begin{cases} -x_i & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } i > n \end{cases} \quad \text{and} \quad \pi'_n(z_j) := \begin{cases} y_j & \text{if } 1 \leq j \leq n, \\ 0 & \text{if } j > n. \end{cases} \]

One can now prove a type D version of Theorem 12. For any $w \in W_n$, the map $\pi'_n$ sends $\mathcal{D}_w(X; Y, Z)$ to a polynomial which represents the universal Schubert class
Similarly the Chern roots of $F$ with arguments in types A–C are simpler, and the reader may also consult the references [20, §4.1] (for Theorem 6) and [122, Thm. 3] (for Theorems 7 and 8).

The equality in Theorem 9 is therefore obtained by applying $\pi'_n$ to formula (46).

7.4. Proof of Theorems 6–9. We will give the proof of Theorem 9 below; the arguments in types A–C are simpler, and the reader may also consult the references [20, §4.1] (for Theorem 6) and [122, Thm. 3] (for Theorems 7 and 8).

The variables $x_i$ for $1 \leq i \leq n$ give the Chern roots of the various vector bundles over $BB \times_{BG} BB$, which pull back to give the roots of the corresponding vector bundles over $M$. In particular the Chern roots of $Q_1$ are (the pullbacks of) $x_1, \ldots, x_n, -x_1, \ldots, -x_n$, while those of $Q_r$ for $r \geq 2$ are $-x_{a_{r-1}+1}, \ldots, -x_{a_r}$. Similarly the Chern roots of $F_{n+1-r}$ are represented by $-y_{r-1}, \ldots, -y_n$ for each $r$. With $k = a_1$, we have

$$
\vartheta_r(X; Y_1) = \sum_{i=0}^{r} q_{r-i}(X)e_i(y_1, \ldots, y_n)
$$

for each $r \geq 0$. Therefore, we compute that

$$
\pi'_n(\vartheta_r(X; Y_1)) = \sum_{i,j \geq 0} e_{r-i-j}(X)e_i(-x_1, \ldots, -x_{a_1})h_j(Y_n)
= \sum_{j \geq 0} e_{r-j}(x_1, \ldots, x_n, -x_1, \ldots, -x_{a_1})h_j(Y_n) = c_r(Q_1 - F_n).
$$

Moreover, we have $\pi'_n(e_{a_1}(Y_1)) = (-1)^{a_1}x_1 \cdots x_{a_1} = c_{a_1}(E_n - E_1)$, hence

$$
\pi'_n(\eta_{a_1}(X; Y_1)) = \frac{1}{2}(c_{a_1}(Q_1 - F_n) + c_{a_1}(E_n - E_1)) \quad \text{and}
$$

$$
\pi'_n(\eta_{a_1}(X; Y_1)) = \frac{1}{2}(c_{a_1}(Q_1 - F_n) - c_{a_1}(E_n - E_1)).
$$

Furthermore, for any partition $\mu$ and $r \geq 2$, we have

$$
\pi'_n(s_{\mu}(Y_r)) = s_{\mu}(-x_{a_{r-1}+1}, \ldots, -x_{a_r}) = s_{\mu}(Q_r),
$$

while

$$
\pi'_n(s_{\mu}(0/Z_r)) = s_{\mu}(-y_{b_{r-1}+1}, \ldots, -y_{b_r}) = s_{\mu}(Q_r) = s_{\mu}(F_{n+b_{r-1}} - F_{n+b_r}).
$$

The equality in Theorem 9 is therefore obtained by applying $\pi'_n$ to formula (46).

8. Suggestions for future research.

In this section we propose some natural directions to follow in future work.
8.1. Quantum cohomology of $G/P$. The last two decades have seen much exploration of the Gromov-Witten theory and quantum cohomology rings of homogeneous spaces. In particular, one seeks to extend the classical understanding of Schubert calculus to the quantum setting, with analogues of the theorems of Pieri, Giambelli, and computations of Schubert structure constants for $\text{QH}^*(G/P)$. One of the motivations for [122] was the fact that the known classical Giambelli formulas expressed in terms of the special Schubert classes in this paper have straightforward extensions to the small quantum cohomology ring of $G/P$; see [7, 18, 24, 33, 71, 72]. We expect that there should be quantum Schubert polynomials and Giambelli formulas in types B, C, and D which restrict to the results in the present paper when the quantum parameters are set equal to zero.

8.2. $K$-theory of $G/P$. The classes of the structure sheaves $\mathcal{O}_{X_w}$ of the Schubert varieties $X_w$, $w \in W_P$ provide a natural basis for the Grothendieck group of vector bundles on $G/P$. For $G = \text{GL}_n$, one has the Lascoux-Schützenberger theory of Grothendieck polynomials [44, 83], and many of the type A results for cohomology can be generalized [15, 21, 78]. Even for the Grassmannian $G(m,n)$, however, one does not yet have a Giambelli formula for $[\mathcal{O}_{X_\lambda}]$ in terms of special Schubert classes that clearly extends the raising operator expression (6) in §1 (see [15, Thm. 1] for a Jacobi-Trudi recursion). There should be versions of the double Grothendieck polynomials for the other classical Lie types, defined using the degenerate Hecke algebra in place of the nilCoxeter algebra; the type A theory is worked out in [35]. One could then seek analogues of Lascoux’s transition equations [78] and the splitting formula of [21, Thm. 4] for these objects. Some recent related work on equivariant $K$-theory in the other types may be found in [51, 52, 61].

8.3. Combinatorial questions. There is a large body of research on the combinatorial aspects of the Schubert calculus, most all of it in the situation where the underlying Weyl group elements are fully commutative in the sense of [115]. By contrast, the combinatorial theory developed in [16, 17, 19, 120, 122, 123] is in its infancy, with many questions worth exploring further. We would like a deeper understanding of the connections between the formulas that appear in special cases, such as in the works cited in the introduction, which differ from those given here.

We mentioned in §4.3 that the type A Stanley coefficients $c_\lambda^\nu$ enumerate Young tableaux as well as leaves in transition trees. However the mixed Stanley coefficients $e_\lambda^\nu$ and $d_\lambda^\nu$ are only known to be positive through transition when $k > 0$; are there alternative combinatorial formulas for them? In type A, Little [87] has studied the combinatorics of the Lascoux-Schützenberger tree; is there an analogue of his bijection for the $c_\lambda^\nu$ which involves the 0-transition trees of $\mathbb{S}$ and Kraskiewicz-Lam tableaux [68, 74]? Do the mixed Stanley coefficients include a rule which computes the Schubert structure constants on isotropic Grassmannians? Some partial results on this last question are obtained in [122, §2.4].

8.4. A theory for the exceptional types. It natural to ask whether the uniform choice of special Schubert classes on the Grassmannians for the classical groups from [16] and [11, 23] extends to the exceptional Lie types, and to look for canonical Giambelli expressions native to $G/P$ for arbitrary reductive groups $G$ and $P$. Anderson [11] has obtained degeneracy loci formulas for vector bundles with structure group $G_2$, and there are parallels between the combinatorics of Schubert calculus
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on the Grassmannians $G(m, n)$ and the hermitian symmetric (or cominuscule) quotients of $E_6$ and $E_7$ [126]. More remains to be understood however to obtain a generalization of the results discussed in this article to any $G/P$ space.

8.5. Connections with representation theory. Do the raising operator formulas of this paper appear in other areas, and in particular in representation theory? The Schur $S$- and $Q$-polynomials were studied by Schur (and his advisor Frobenius) in order to compute the characters and projective characters of the symmetric group and the polynomial characters of the general linear group. The discovery of formulas such as (8) and (17), (18) in the algebra of symmetric functions and associated group representations long preceded their realizations in the Schubert calculus. A representation-theoretic understanding of the raising operator expressions (34) and (35), parallel to the extensive theory of Schur polynomials, would certainly be desirable. Young’s raising operators are well known in algebraic combinatorics, but we suspect that their full potential has not yet been exploited. For a non-exhaustive list of references which apply them in various settings, see [47, 56, 85, 86, 88, 90, 94, 120, 121, 123, 124, 129].

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