Research Article

A Few Inequalities Established by Using Fractional Calculus and Their Applications to Certain Multivalently Analytic Functions

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By making use of different techniques given in Miller and Mocanu (2000) (and also in Jack (1971)), some recent results consisting of certain multivalently analytic functions given both in Irmak (2005) and in Irmak (2010) are firstly restated and some of their applications are then pointed out.

1. Introduction, Definitions, and Notations

Let \( A(p) \) denote the class of functions \( f(z) \) of the form:

\[
f(z) = z^p + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \cdots + a_{p+n} z^{p+n} + \cdots
\]

\((a_{p+n} \in \mathbb{C}; n, p \in \mathbb{N} = \{1, 2, 3, \ldots \})\),

which are analytic and multivalent in the open unit disk \( U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\} \), where \( \mathbb{C} \) is the set of complex numbers.

For some useful implications of the main results, there is a need to recall certain well-known definitions relating to geometric function theory. As is known, a function \( f(z) \) belonging to the general class \( A(p) \) is said to be multivalently starlike function, multivalently convex function, and multivalently close-to-convex function (with respect to the origin \( (z = 0) \)), if it satisfies \( \Re e[zf'(z)/f(z)] > 0 \), \( \Re e[1 + zf''(z)/f'(z)] > 0 \), and \( \Re e[z^{-1} f'(z)] > 0 \) in the open unit disk \( U \), respectively. For the details of the definitions above, one may look over the works in [1–3].

Here and also throughout this paper, the symbol \( D_z^\mu \) denotes an operator of fractional calculus (i.e., that fractional derivative(s)), which is defined as follows (cf., e.g., [4] and see also [2, 5, 6]).

**Definition 1.** Let \( f(z) \) be an analytic function in a simply connected region of the \( z \)-plane containing the origin. The fractional integral of order \( \mu (\mu > 0) \) is defined by

\[
D_z^{-\mu} \{ f(z) \} = \frac{1}{\Gamma(\mu)} \int_0^z f(\xi)(z-\xi)^{\mu-1} d\xi,
\]

and the fractional derivative of order \( \mu (0 \leq \mu < 1) \) is defined by

\[
D_z^\mu \{ f(z) \} = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z f(\xi)(z-\xi)^{-\mu} d\xi,
\]

where the multiplicity of \( (z-\xi)^{\mu-1} \) involved in (2) and that of \( (z-\xi)^{-\mu} \) in (2) are removed by requiring \( \log(z-\xi) \) to be real when \( z-\xi > 0 \).

**Definition 2.** Under the hypotheses of Definition 1, the fractional derivative of order \( m+\mu \) \((m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; 0 \leq \mu < 1) \) is defined by

\[
D_z^{m+\mu} \{ f(z) \} = \frac{d^m}{dz^m} \{ D_z^\mu \{ f(z) \} \}.
\]

The aim of this investigation is first to restate some recent results relating to certain multivalently analytic functions and then to point a number of their applications out. For proofs...
of the main results, the well-known assertions in [7] and [8, p. 33–35] are used (see, also, for similar proofs, [9–11]). The main results include fractional calculus and the main purpose of using fractional calculus is also to extend the scope of the main results and to reveal certain complex inequalities which can be associated with (analytic and) geometric function theory (see, for their details, [1–4, 9]). For certain results determined by fractional calculus, for example, one may refer to the works in [5, 6, 12, 13]. For some results between certain complex inequalities and analytic functions, one can also check the papers, for instance, in the references in [2, 12, 13].

2. Main Results and Certain Consequences

The following assertions (Lemmas 3 and 4 below) will be required for proving the main results.

\[
\Re\left(\frac{z(z^{1+\mu}f'(z)/z^{\mu}f(z))}{1 + (z^{1+\mu}f(z)/z^{\mu}f(z))^\delta} - (p - \mu)^\delta\right)^\lambda > 0,
\]

where the value of each one of the above complex terms and their certain applications is taken to be its principal value and \(\Re\) denotes the set of real numbers.

**Proof.** First of all, Definition 1 readily provides us with the following fractional derivative formula for a power function:

\[
D_z^{\mu} z^\kappa = \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - \mu + 1)} z^{\kappa-\mu} \quad (\kappa > \mu - 1; \ 0 \leq \mu < 1).
\]

With the help of (7), let us define the function \(w(z)\) by

\[
\left[1 + \left(\frac{z^{1+\mu}f(z)}{D_z^{\mu} f(z)}\right)^\delta - (p - \mu)^\delta\right]^\lambda = 1 + w(z)
\]

(\(z \in \mathbb{U}\)).

**Lemma 3** (see [7]). Let \(w(z)\) be an analytic function in the unit disk \(\mathbb{U}\) with \(w(0) = 0\) and let \(0 < r < 1\). If \(|w(z)|\) attains its maximum value on the circle \(|z| = r\), then \(z_0 w'(z_0) = cw(z_0)\) \((c \geq 1)\).

**Lemma 4** (see [8]). Let \(\Omega \subset \mathbb{C}\) and suppose that the function \(\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}\) satisfies \(\psi(Me^{i\theta}, Ke^{i\theta}, z) \notin \Omega\) for all \(\theta \in \mathbb{R}\) and \(z \in \mathbb{U}\) with \(K \geq M\) \((n \in \mathbb{N})\). If the function \(p(z)\) is in the class \(H[a, n] = \{ p(z) \in H(\mathbb{U}) : p(z) = a + az^n + \cdots \ (z \in \mathbb{U})\}\) and \(\psi(p(z), zp'(z); z) \in \Omega\) for all \(z \in \mathbb{U}\), then \(|p(z)| < M\) \((z \in \mathbb{U})\).

By using Lemma 3, we first prove the following theorem.

**Theorem 5.** Let \(z \in \mathbb{U}\), \(\delta \in \mathbb{R} \setminus \{0\} : = \mathbb{R}^*, \lambda \in \mathbb{R}^*, p \in \mathbb{N}\), and \(0 \leq \mu < 1\), and let the function \(f(z) \in \mathfrak{A}(p)\) satisfy any one of the following inequalities:

\[
\Re \left\{ \frac{\lambda \delta z(z^{1+\mu}f'(z)/z^{\mu}f(z))^{\delta - 1}}{1 + (z^{1+\mu}f(z)/z^{\mu}f(z))^\delta} - (p - \mu)^\delta \right\} < \frac{1}{2\lambda \delta} \quad \text{if } \lambda \cdot \delta > 0
\]

\[
> \frac{1}{2\lambda \delta} \quad \text{if } \lambda \cdot \delta < 0.
\]

Clearly, the function \(w(z)\) is analytic in \(\mathbb{U}\) and \(w(0) = 0\). Differentiating both sides of the above identity, and if we make use of (8) once again, we easily arrive at

\[
\Re \left\{ \mathfrak{S}(z) \right\} = \Re \left\{ \frac{ce^{i\theta}}{1 + ce^{i\theta}} \right\}
\]

Putting \(z = z_0\) and then putting \(w(z_0) = e^{i\theta}\) in (9), we obtain

\[
\Re \left\{ \mathfrak{S}(z_0) \right\} = \Re \left\{ \frac{c(1 + e^{i\theta})}{1 + e^{i\theta}e^{i\theta}} \right\} = \frac{c^2}{2} \geq \frac{1}{2}.
\]
where $\theta \neq \pi$ and $c \geq 1$. Hence, (10) is a contradiction to the conditions in (5). Consequently, we conclude from definition (8) that

$$
\left| 1 + \left( \frac{zD^\mu[z]}{D^\mu_z[f(z)]} \right)^\delta - (p - \mu)^\delta \right|^\lambda - 1
$$

$$
= |w(z)| < 1,
$$

(z \in U, \delta \in \mathbb{R}^+, \lambda \in \mathbb{R}^+, p \in \mathbb{N}, 0 \leq \mu < 1),

which implies (6). Thus, the proof of the Theorem 5 is completed.

The second theorem (below) is given without proof since its proof is similar to the proof of the Theorem 5.

**Theorem 6.** Let $z \in U$, $\delta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}^+$, $p \in \mathbb{N}$, and $0 \leq \mu < 1$. If the function $f(z)$ in the class $A(p)$ satisfies any one of the following conditions:

$$
\frac{z(D^\mu[f(z)]/z^\mu)'(D^\mu[f(z)]/z^\mu)^{\delta-1}}{1 + (D^\mu[f(z)]/z^\mu)^\delta - (\Gamma(p+1)/\Gamma(p-\mu+1))^\delta}
$$

$$
< \frac{1}{2\lambda \delta} \quad \text{if } \lambda \cdot \delta > 0
$$

$$
> \frac{1}{2\lambda \delta} \quad \text{if } \lambda \cdot \delta < 0,
$$

then

$$
\Re \left( 1 + \left( \frac{D^\mu_z[f(z)]}{z^\mu} \right)^\delta - \left( \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} \right)^\delta \right)^\lambda > 0,
$$

where the value of each one of the above complex terms and their certain applications is taken to be its principal value.

**Theorem 7.** Let $z \in U$, $\delta \in \mathbb{R}^+$, $\lambda \in \mathbb{R}^+$, $M \geq 1$, $p \in \mathbb{N}$, and $0 \leq \mu < 1$. If $f(z) \in A(p)$ satisfies the following condition:

$$
\frac{z(D^\mu[f(z)]/z^\mu)'(D^\mu[f(z)]/z^\mu)^{\delta-1}}{1 + (D^\mu[f(z)]/z^\mu)^\delta - (\Gamma(p+1)/\Gamma(p-\mu+1))^\delta}
$$

$$
= \Delta[\mu, \delta; f](z),
$$

then

$$
\Re \left( 1 + \left( \frac{D^\mu_z[f(z)]}{z^\mu} \right)^\delta - \left( \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} \right)^\delta \right)^\lambda < \frac{M}{1+M},
$$

where $\psi(r, s; z) = \frac{s}{1+r}$, $\Omega = \{w \in \mathbb{C} : \Re \{w\} < \frac{M}{1+M} \}$;

$$
\psi(\lambda z, \lambda p(z); z) = \Delta[\mu, \delta; f](z) \in \Omega
$$

for all $z$ in $U$. Further, for any $\theta \in \mathbb{R}$, $K > nM \geq M$, and $z \in U$, because of $M \geq 1$ and $n \in \mathbb{N}$, we then obtain that

$$
\Re \left( \psi\left( \frac{Ke^\theta}{1+Me^\theta} \right) \right) = \Re \left( \frac{Ke^\theta}{1+Me^\theta} \right) \geq \frac{nM}{1+M} \geq \frac{M}{1+M};
$$

that is, $\psi(Ke^\theta, Ke^\theta; z) \notin \Omega$. Therefore, because of the Lemma 4, the function $p(z)$ defined in (16) follows the inequality in (15). This completes the desired proof. \qed
Since the proof of the following theorem is similar to the proof of the Theorem 7, the details are here omitted.

**Theorem 8.** Let \( z \in \mathbb{U} \), \( \delta \in \mathbb{R} \setminus \{0\} =: \mathbb{R}^* \), \( M \geq 1 \), \( \lambda \in \mathbb{R}^* \), \( p \in \mathbb{N} \), and \( 0 \leq \mu < 1 \). If the function \( f(z) \) in the class \( \mathcal{A}(p) \) satisfies the following inequality:

\[
\Re \left\{ \lambda \delta \cdot \left( z \left( \frac{D_z^p \{ f(z) \}}{z^{p-\mu}} \right)^\delta \cdot \left( \frac{D_z^p \{ f(z) \}}{z^{p-\mu}} \right)^{-1} \right) \times \left( 1 + \left( \frac{D_z^p \{ f(z) \}}{z^{p-\mu}} \right)^\delta - \left( \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} \right)^{-1} \right) \right\} < \frac{M}{1+M},
\]

then

\[
\left| 1 + \left( \frac{D_z^p \{ f(z) \}}{z^{p-\mu}} \right)^\delta - \left( \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} \right)^\lambda \right| < M.
\]

(21)

It is clear that, by suitably choosing the parameters \( \delta, \lambda, \mu, M \) and also \( p \) in the all theorems, one may easily receive several interesting and/or important results for analytic and geometric function theory (see, [1, 3], and also the others in the references). Since it is not possible to list all of them, we want to reveal only the useful certain consequences belonging to Theorems 5–8. They are in the following forms.

Letting \( \delta = 1 = \lambda = 1 = \mu = 0 \) and \( \delta = \lambda = 1, \mu \to 1 \) in the Theorem 5, we first get Corollaries 9 and 10, respectively.

**Corollary 9.** If a function \( f(z) \in \mathcal{A}(p) \) satisfies the inequality:

\[
\Re \left\{ \left( z \left[ f'(z) + z f'''(z) / f(z) \right] / f(z) \right) - \left( z f'(z) / f(z) \right)^2 \right\} < \frac{1}{2}
\]

\[
\quad \left( z \in \mathbb{U} \right),
\]

(23)

then

\[
\Re \left( \frac{z f''(z)}{f(z)} \right) > p - 1 \quad (z \in \mathbb{U});
\]

(24)

that is, \( f(z) \) is multivalently starlike in \( \mathbb{U} \).

**Corollary 10.** If a function \( f(z) \in \mathcal{A}(p) \) satisfies the inequality:

\[
\Re \left\{ \left( z \left[ f''(z) + z f''''(z) / f'(z) \right] / f'(z) \right) - \left( z f''(z) / f'(z) \right)^2 \right\} < \frac{1}{2}
\]

\[
\quad \left( z \in \mathbb{U} \right),
\]

(25)

then

\[
\Re \left( 1 + z f''''(z) / f'(z) \right) > p - 1 \quad (z \in \mathbb{U});
\]

(26)

that is, \( f(z) \) is multivalently convex in \( \mathbb{U} \).

Setting \( \delta = \lambda = 1 \) and \( \mu \to 1 \) in the Theorem 6, we second obtain Corollary 11 (below).

**Corollary 11.** If a function \( f(z) \in \mathcal{A}(p) \) satisfies the inequality:

\[
\Re \left\{ \frac{z f''''(z) - (p-1) f'(z)}{f'(z) - (p-1) z f''(z)} \right\} < \frac{1}{2}
\]

\[
\quad \left( z \in \mathbb{U} \right),
\]

(27)

then

\[
\Re \left( \frac{f'(z)}{z f''(z)} \right) > p - 1 \quad (z \in \mathbb{U});
\]

(28)

that is, \( f(z) \) is multivalently close-to-convex in \( \mathbb{U} \).

**Remark 12.** The above three consequences may be comparable with the results given by the papers in [12, 13].

**Remark 13.** Theorems 7 and 8 are equal to Theorems 5 and 6, respectively, when one takes \( M = 1 \) in Theorems 7 and 8.

**Conflict of Interests**

The author declares that there is no conflict of interests.

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