STABILITY PROPERTIES OF THE REGULAR SET FOR THE NAVIER–STOKES EQUATION

PIERO D’ANCONA AND RENATO LUCÀ

Abstract. We investigate the size of the regular set for small perturbations of some classes of strong large solutions to the Navier–Stokes equation. We consider perturbations of the data that are small in suitable weighted $L^2$ spaces but can be arbitrarily large in any translation invariant Banach space. We give similar results in the small data setting.

1. Introduction and main results

We consider the 3D Navier–Stokes equation on $(t,x) \in (0, \infty) \times \mathbb{R}^3$:

$$
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - \Delta u &= -\nabla P \\
\nabla \cdot u &= 0 \\
u|_{t=0} &= u_0,
\end{aligned}
$$

(1.1)

describing the free motion of a viscous incompressible fluid with velocity $u$ and pressure $P$. For simplicity, we have settled the kinematic viscosity equal to one and we will use the same notation for the norm of scalar, vector or tensor quantities, thus we write

$$
\|P\|_{L^2}^2 := \int P^2 dx, \quad \|u\|_{L^2}^2 := \sum_j \|u_j\|_{L^2}^2, \quad \|\nabla u\|_{L^2}^2 := \sum_{j,k} \|\partial_k u_j\|_{L^2}^2,
$$

and $L^2(\mathbb{R}^3)$ instead of $[L^2(\mathbb{R}^3)]^3$, etc.

Global weak solutions to (1.1) are known to exist for any divergence free initial velocity in $L^2$, since the pioneering work of Leray [40]. The uniqueness and the persistence of regularity of Leray’s solutions are long standing open problems.

On the other hand, several global well-posedness results have been proved for small initial data and for large data with suitable symmetry. We will work in this simpler framework.

Earliest results in the small data setting go back to Fujita and Kato [21] for $u_0 \in \dot{H}^{1/2}$ and to Kato [31] for $u_0 \in L^3$. Then several functional spaces have been considered like Morrey spaces [28], [52], [32], [19], [35], Besov spaces [6], [45], and others. This approach culminated in the $BMO^{-1}$ well-posedness result of Koch and Tataru [34], that is the most general one in this direction.

In the large data framework, global solutions have been constructed imposing some additional symmetry on the data, e.g., $u_0$ axisymmetric, helical or two dimensional. Axisymmetric data have been studied by Ukhovskii and Iudovich [54] and Ladyzhenskaya [36] under a zero swirl assumption (see also [37], [8]; we recall that the swirl is the angular component of the velocity field with respect to the axis of symmetry). The interesting case of non zero swirl is still open. Additional results on axisymmetric solutions are [24], [23]. Helical data, which means invariant under the composition of a rotation and a translation along a fixed direction, have been considered in [44]. Further interesting large data results are [20], [22], [30], [9], [10].

Once a large solution, with good properties, has been constructed (with or without symmetry) a natural and important question concerns its stability for small
perturbations of the data. This perturbative approach was followed in a systematic way by Ponce, Racke, Sideris and Titi [46], for small $H^1$ perturbations. As for the small data problem, this was then extended to small perturbations in weaker norms, for instance $L^3$ [33], Besov spaces [26] and $BMO^{-1}$ [2].

In both small data and perturbative results it is customary to consider functional spaces that are scaling and translation invariant. More precisely, since the Navier–Stokes equation is invariant under the family of symmetries
\[ u \mapsto \lambda u(\lambda^2 t, \lambda(x - \bar{x})), \quad \lambda \in (0, \infty), \quad \bar{x} \in \mathbb{R}^3, \]
(1.2) it is natural consider initial data in Banach spaces invariant under
\[ u_0(x) \mapsto \lambda u_0(\lambda(x - \bar{x})), \quad \lambda \in (0, \infty), \quad \bar{x} \in \mathbb{R}^3. \]
(1.3)
On the other hand, if we are only interested in certain local regularity properties, it may suffices to require invariance with respect to the scaling but not necessarily under translations: that is to say, the norm of the data is invariant under (1.3) for all $\lambda > 0$ and a fixed $\bar{x}$. In the classical work [5] the authors prove, among other, the smoothness, in time-increasing neighborhoods of a point $\bar{x} \in \mathbb{R}^3$, of weak solutions with initial data $u_0$ such that
\[ \int_{\mathbb{R}^3} |u_0(x)|^2 |x - \bar{x}|^{-1} dx \ll 1. \]
Namely, the weighted $L^2$ norm has to be sufficiently small if we center an homogeneous weight of degree $-1$ at the point $\bar{x}$; see Theorem 1.1 below for the precise statement. The aim of this paper is to give some extensions and improvements of this result, in both the perturbative and small data frameworks.

First of all, we recall a classical notion of regularity for weak solutions to the Navier–Stokes equation.

**Definition 1.1.** Let $u(t, x)$ be a weak solution of (1.1). A point $(t, x) \in (0, \infty) \times \mathbb{R}^3$ is **regular** if $u$ is bounded on a neighborhood of $(t, x)$. In particular, this implies that $u$ is smooth, in the space variables, in a neighborhood of $(t, x)$; see [49]. A subset of $(0, \infty) \times \mathbb{R}^3$ is **regular** if all its points are regular.

**Definition 1.2.** We write
\[ \Pi_{\alpha, x} := \left\{(t, x) \in (0, \infty) \times \mathbb{R}^3 : t > \frac{|x - \bar{x}|^2}{\alpha}\right\} \]
for the region above the paraboloid of aperture $\alpha$ in the upper half space $(0, \infty) \times \mathbb{R}^3$, with vertex at $(t, x) = (0, \bar{x})$. When $\bar{x} = 0$ we also write $\Pi_{\alpha}$, instead of $\Pi_{\alpha, 0}$. Note that these sets are increasing with $\alpha$.

The following statement [5, Theorem D] applies to suitable weak solutions; we refer to Section 3 for the definition of suitability. In particular, the weak solutions given by the Leray approximation procedure are suitable [48, Theorem 2.3]. We use the notation $L^p_t L^q_x$, $H^k_x$ when the integration is over the time $t \in (0, \infty)$ and space $x \in \mathbb{R}^3$ variables, respectively. We write $\|f\|_{XY} := \|\|f\|_Y\|_X$ for nested norms and $XY$ for the associated normed spaces.

**Theorem 1.1** (Caffarelli–Kohn–Nirenberg). There exists a constant $\varepsilon_0 > 0$ such that the following holds. The set
\[ \Pi_{\varepsilon_0 - \varepsilon, x} := \left\{(t, x) : t > \frac{|x - \bar{x}|^2}{\varepsilon_0 - \varepsilon}\right\} \]
is regular for any suitable weak solution $u \in L^\infty_t L^2_x \cap L^2_t H^k_x$ to the Navier–Stokes equation with divergence free initial data $u_0 \in L^2$ such that
\[ \|x - \bar{x}|^{-1/2} u_0\|_{L^2} =: \varepsilon < \varepsilon_0. \]
In other words, if the weighted $L^2$ norm of $u_0$ is small enough, once we centre the weight in $\bar{x}$, then the solution is smooth in the region above a space-time paraboloid with vertex at $(t, x) = (0, \bar{x})$. The existence of suitable weak solutions in $L_t^\infty L^2_x \cap L^2_t \dot{H}^1_x$ is ensured by the Leray theory, see [48, Theorem 2.3], for any divergence free $u_0 \in L^2$.

It is important to stress out that condition (1.4) allows $u_0$ to be large at $x$ sufficiently far from $\bar{x}$. Thus (1.4) is not comparable with any translation invariant smallness assumption on $u_0$. This is quantified in the following remark.

Remark 1.1. There are data $u_0$ arbitrarily large in $\dot{B}^{-1}_r$, or any translation invariant Banach space, but such that the norms $\|x - \bar{x}\|^{-1/2} u_0$ are arbitrarily small. We recall that $\dot{B}^{-1}_r$ contains any Banach space invariant under $(1.3)$; see [7].

Indeed, assume for simplicity $\bar{x} = 0$ and let $\phi \in C_c^\infty (\mathbb{R}^3)$ be a divergence free vector field. Letting $\phi_K (x) := \phi (x - K \xi)$ for the translate of $\phi$ by the vector $K \xi$, with $|\xi| = 1$ and $K \gg 1$, since

$$\|x\|^{-1/2} \phi_K \|_{L^2} \simeq K^{-1/2},$$

recalling the translation invariance of $\dot{B}^{-1}_r$, we get as $K \to \infty$:

$$\|x\|^{-1/2} \phi_K \|_{L^2 (\mathbb{R}^3)} \to 0 \quad \text{and} \quad \|\phi_K\|_{\dot{B}^{-1}_r} = \text{const}.$$

Our main result is a perturbative version of Theorem 1.1. We need some preliminary definitions.

Definition 1.3. A couple $(r, q)$ is admissible if $2 \leq r < \infty$ and $2/r + 3/q = 1$.

Definition 1.4. A weak solution $w \in L_t^\infty L^2_x \cap L^2_t \dot{H}^1_x$ to the Navier–Stokes equation with divergence free initial data $w_0 \in L^2$ is a reference solution of size $K$ if

$$\int_0^\infty \left( \int_{\mathbb{R}^3} |w(t, x)|^q \, dx \right)^{\frac{r}{q}} \, dt =: \|w\|_{L_t^r L^q_x} =: K < \infty,$$

for an admissible couple $(r, q)$. We furthermore assume $w_0 \in L^2 (|x - x'|^{-1} \, dx)$, for some $x' \in \mathbb{R}^3$.

As well known, the boundedness assumption (1.5) ensures the smoothness and uniqueness of reference solutions; see Remark 3.1 for more details.

We are now ready to state the main result of the paper.

Theorem 1.2. Given $(r, q)$ admissible, there exists a constant $\delta_0 > 0$ such that the following holds. Let $w$ be a reference solution of size $K$ to the Navier–Stokes equation with divergence free initial data $w_0$. Then the set

$$\Pi_{\delta_0, x} := \left\{ (t, x) \, : \, t > \frac{|x - \bar{x}|^2}{\delta_0} \right\}$$

is regular for every suitable weak solution $u \in L_t^\infty L^2_x \cap L^2_t \dot{H}^1_x$ to the Navier–Stokes equation with divergence free data $u_0 \in L^2$ such that

$$\|x - \bar{x}\|^{-1/2} (u_0 - w_0) \|_{L^2} \leq \delta_0 e^{-K/\delta_0}.$$

Thus, if we perturb, in the weighted $L^2$ norm, the data of a reference solution, the corresponding weak solutions are still regular in the region above a paraboloid with vertex at $(t, x) = (0, \bar{x})$, where $\bar{x}$ is the center of the weight. We insist that the smallness assumption (1.7) allows for large perturbations, far from the point $\bar{x}$; see Remark 1.1.

---

1 As customary, we write $A \lesssim B$ if $A \leq CB$ for a certain constant $C > 0$ and $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.
In Section 2 we give a few applications of Theorem 1.2 to some classes of reference solutions. In Proposition 2.1, we consider large axisymmetric data $w_0$ with zero swirl. In Proposition 2.2, the reference data $w_0$ are assumed to fit a non-linear smallness assumption (see (2.2)) that can be satisfied by arbitrarily large $w_0 \in \dot{B}^{-1}_\infty$, thus escaping the hypothesis of the known small data results. These solutions have been studied in [10].

Further applications requires a slightly more general version of Theorem 1.2, since we also want to consider reference solutions with infinite energy. This will be allowed by Theorem 4.1, that generalizes Theorem 1.2. Thus, in Proposition 2.3, we can handle small 3D (three dimensional) perturbations of large 2D initial data $W_0 = (W_{0,1}, W_{0,2}) \in L^2 \cap L^1$, that we can consider 3D objects via the natural extension

$$\tilde{W}_0(x_1, x_2, x_3) := (W_{0,1}(x_1, x_2), W_{0,2}(x_1, x_2), 0).$$

(1.8)

In Proposition 2.4, we focus on Beltrami fields, namely initial data $w_0$ that are eigenvectors of the curl operator $\nabla \times w_0 = \lambda w_0$ on $\mathbb{R}^3$, with $\lambda \neq 0$. These vector fields give rise to explicit solutions to the Navier–Stokes equation with very rich topological structures.

In the second part of the paper we work in the small data setting. We investigate how the size of the regular set depends on the size of the initial data. Notice that the regular set $\Pi_{\alpha-p\cdot}\bar{x}$ considered in Theorem 1.1 converges to a maximal one:

$$\Pi_{\alpha-p\cdot}\bar{x} \to \Pi_{\alpha,\bar{x}} \quad \text{as} \quad \varepsilon \to 0.$$ 

In other words, we are not able to prove smoothness in the region below the limit paraboloid $\Pi_{\alpha,\bar{x}}$, even if the size $\varepsilon$ of the initial data is arbitrarily small.

In Theorem 6.1 we will prove that if the size $K$ of the reference solution is smaller than a threshold value, then the regular set is actually larger and invades the whole half space $\{t > 0\}$ in the limit

$$\| |x - \bar{x}|^{1/2}(u_0 - w_0)\|_{L^2} \to 0,$$

improving in this way Theorem 1.2 in the case of very small reference solutions.

Using Theorem 6.1, we are also able to cover the gap between the regularity Theorem 1.1 and the Kato well-posedness theory, which works for small $L^3$ initial data. To do so, we consider $u_0$ such that the critical\(^2\) weighted $L^p$ norms

$$\| |x - \bar{x}|^{\alpha}u_0\|_{L^p}, \quad 2 < p < 3, \quad \alpha = 1 - 3/p,$$

are sufficiently small. Then we show that the size of the regular set improves as $p$ increases. We recover full regularity in the limit $p \to 3^- \text{ (namely } \alpha \to 0^-)$, as expected in light of the $L^3$ well-posedness.

Noting that $\alpha < 0$ when $p < 3$, the same argument of Remark 1.1 allows to construct initial data that are arbitrarily large in any translation invariant Banach space but arbitrarily small in $L^p(|x - \bar{x}|^{\alpha}pdx)$, so that also the following theorem is not implied by the various known small data results. Let

$$\theta_1(p) := \left( \frac{p-2}{3-p} \right)^{1-p/3}, \quad \theta_2(p) := \left( \frac{p-2}{3-p} \right)^{1-p/2}.$$

(1.9)

It is easy to check that

$$\lim_{p \to 2^+} \theta_1(p) = 0, \quad \lim_{p \to 3^-} \theta_1(p) = 1,$$

while $\theta_2$ behaves in the opposite way

$$\lim_{p \to 2^+} \theta_2(p) = 1, \quad \lim_{p \to 3^-} \theta_2(p) = 0.$$

\(^2\) Notice that these norms are invariant under the natural scaling (1.3), with $\bar{x}$ fixed, that is why we refer to them as critical.
Theorem 1.3. There exists a constant $\delta_1 > 0$ such that the following holds. Let $2 < p < 3$, $\alpha = 1 - 3/p$ and $M > 1$. The set $\Pi_{M\delta_1, x}$ is regular for any suitable weak solution $u \in L^p L^2 \cap L^2 \mathcal{H}^1_2$ to the Navier–Stokes equation with divergence free initial data $u_0 \in L^p \cap L^2(\{x - x'\}^{-1} dx)$, for some $x' \in \mathbb{R}^3$, such that
\[
\theta_1 \|x - x'|u_0\|_{L^p}^{p/3} \leq \delta_1, \quad \theta_2 \|x - x'|u_0\|_{L^p}^{p/2} \leq \delta_1 e^{-M^2/\delta_1}.
\]

This can be interpreted in the following way. Since $\theta_2(p) \to 0$ as $p \to 3^-$, we can choose $p = p_M$ as a function of $M$ in such a way that
\[
e^{M^2/\delta_1} \cdot \theta_2(p_M) \to 0 \quad \text{as} \quad M \to \infty \quad (p_M \to 3^-),
\]
so that, since $\theta_1(p_M) \to 1$, we have proved:
\[
\|x - x'|u_0\|_{L^p} \leq \delta_1/2, \quad \Rightarrow \quad \Pi_{M\delta_1, x} \text{ is a regular set for } u,
\]
for all sufficiently large $M$. Notice that $\Pi_{M\delta_1, x}$ increases indefinitely as $M \to \infty$. In other words, if $M \to \infty$ and the $L^{pM}(\{x - x'|^{pM})$ norm of $u_0$ is less than $\delta_1/2$, the regular set invades the whole half space $\{t > 0\}$ in the limit.

It is worth noting that all the results presented in the paper rely upon the algebraic structure of the nonlinearity $N(u) := (u \cdot \nabla) u$. Indeed, like in other perturbative results [46, 33, 26, 2], we exploit the cancellation $\int_{\mathbb{R}^3} N(u) \cdot u = 0$. The novelty here is that also the behavior of $N(u)$ under the change of variables
\[(t, y) = (t, x - \xi t), \quad u_{\xi}(t, y) = u(t, x), \quad \xi \in \mathbb{R}^3,
\]
amely
\[N(u_\xi) = (u_\xi \cdot \nabla) u_\xi - (\xi \cdot \nabla) u_\xi,
\]
plays a key role.

The rest of the paper is organized as follows. In the next section we state some applications of this perturbative theory, namely Propositions 2.1 - 2.4. In Section 3 we fix our setting, recalling the definition of suitable solutions and the fundamental Caffarelli–Kohn–Nirenberg regularity criterion from [5]. In Section 4 we prove our main Theorem 1.2, in fact we prove the slightly more general Theorem 4.1. As a consequence, in Section 5, we deduce Propositions 2.1 - 2.4.

Section 6 is devoted to the small data theory. In Theorem 6.1 we improve the perturbative result for reference solutions of sufficiently small size $K$. This also allows to prove Theorem 1.3.

2. Applications: perturbative solutions

We consider solutions with bounded energy in Propositions 2.1, 2.2 and unbounded energy in Propositions 2.3, 2.4.

2.1. Axisymmetric solutions with zero swirl. Let \{(r \cos \Theta, r \sin \Theta, x_3), \Theta \in T := \mathbb{R}/2\pi \mathbb{Z}, r \in [0, \infty), x_3 \in (-\infty, \infty)\} cylindrical polar coordinates on $\mathbb{R}^3$. We say that a vector field $f$ is axisymmetric (with respect to the $x_3$-axis) if it is independent of $\Theta$, namely
\[f = f_r(r, x_3) e_r + f_{\Theta}(r, x_3) e_\Theta + f_{x_3}(r, x_3) e_{x_3}, \quad \text{(2.1)}
\]
Here $(e_r, e_\Theta, e_{x_3})$ is a positively oriented orthonormal frame with $e_r$ in the radial direction and $e_{x_3}$ in the direction of increasing $x_3$. The swirl of $f$ is the scalar function $f_{\Theta}$. By rotation invariance of the problem, the choice of the symmetry axis is clearly unimportant.

Proposition 2.1. There exists $\delta_2 > 0$ such that the following holds. Let $w_0$ be an axisymmetric divergence free vector field with zero swirl, which belongs to $H^2 \cap L^2(\{x - x'|^{-1} dx)$, for some $x' \in \mathbb{R}^3$. Then the set $\Pi_{\delta_2, x}$ is regular for any suitable
we have defined a 3D extension by 

\[ \mathcal{L} \in \mathcal{L}_2 \mathcal{H}_x^{1} \] to the Navier–Stokes equation with divergence free data \( w_0 \in L^2 \) such that

\[ \|x - \bar{x}\|^{-1/2}(u_0 - w_0)\|L^2 \leq \delta_2 e^{-\delta_2^{-1}(1+\|w_0\|_{B^{2.2}})} . \]

2.2. A nonlinear smallness assumption in the Koch–Tataru space. Following \([10]\), we consider initial data \( w_0 \) such that

\[ \|\mathcal{P}(e^{i\Delta}w_0 \cdot \nabla e^{i\Delta}w_0)\|_E \leq \sigma \exp \left(-\sigma^{-1}\|w_0\|_{B^{-1.2}}^4\right) , \] (2.2)

for a sufficiently small constant \( \sigma \). Here \( \mathcal{P} \) is the projection on the linear subspace of the divergence free vector fields, while

\[ \|f\|_E := \|f\|_{L_t^1B^{-1.1}_x(R^3)} + \sum_{j \in \mathbb{Z}} 2^{-j} \left( \int_0^\infty \|\Delta_j f(t)\|_{L_x^\infty(R^3)}^2 dt \right)^{1/2} \]

and \( \Delta_j \) is the frequency projection onto the annulus \( 2^{j-1} \leq |\xi| \leq 2^j \) (\( \xi \) is the Fourier conjugate variable of \( x \)).

The well-posedness for these initial data have been proved in \([10]\). The authors also provide concrete examples of divergence free vector fields that satisfies (2.2) and are arbitrarily large in \( \dot{B}^{-1,\infty}_\infty \). Indeed, they consider the family

\[ w_{0,\varepsilon} := (\partial_2 \phi_\varepsilon, -\partial_1 \phi_\varepsilon, 0) , \]

where

\[ \phi_\varepsilon := \left( \frac{-\ln \varepsilon}{e^{1-\alpha}} \right)^{1/5} \cos(\varepsilon^{-1}x_3)\phi(x_1, \varepsilon^{-\alpha} x_2, x_3) , \quad \alpha, \varepsilon \in (0, 1) , \]

and \( \phi \) is a Schwartz function. Letting \( \varepsilon \to 0 \) (see \([10, \text{Theorem } 2]\))

\[ \|\mathcal{P}(e^{i\Delta}w_{0,\varepsilon} \cdot \nabla e^{i\Delta}w_{0,\varepsilon})\|_E \to 0 , \quad \|w_{0,\varepsilon}\|_{\dot{B}^{-1,\infty}_\infty} \to \infty . \]

Proposition 2.2. Let \((r, q)\) be an admissible couple (see Definition 1.3) with \( q \neq \infty \). There exists \( \delta_0 > 0 \) such that the following holds. Let \( w_0 \in H^{1/2} \cap L^2(|x - x'|^{-1} dx \), for some \( x' \in \mathbb{R}^3 \), with zero divergence such that (2.2) holds with \( \sigma > 0 \) sufficiently small. Then the set \( \Pi_{\delta_0,2} \) is regular for any suitable weak solution \( u \in L_\infty^\infty \mathcal{L}_2 \mathcal{H}_x^{1} \) to the Navier–Stokes equation with divergence free data \( w_0 \in L^2 \) such that

\[ \|x - \bar{x}\|^{-1/2}(u_0 - w_0)\|L^2 \leq \delta_0 e^{-\delta_0^{-1}\|w\|_{L_x^1L_x^2}} , \]

where \( w \) is the (unique) solution to the Navier–Stokes equation with data \( w_0 \).

2.3. 2D solutions. For any 2D vector field \( F(x_1, x_2) = (F_1(x_1, x_2), F_2(x_1, x_2)) \), we have defined a 3D extension by

\[ \tilde{F}(x_1, x_2, x_3) := (F_1(x_1, x_2), F_2(x_1, x_2), 0) , \] (2.3)

If \( \Phi(x_1, x_2) \) is a 2D scalar field, the 3D extension is \( \tilde{\Phi}(x_1, x_2, x_3) := \Phi(x_1, x_2) \).

In the following proposition we analyze 3D weak solutions with initial data that are close (in the weighted \( L^2 \) norm) to the 3D extension of a 2D vector field \( W_0 \). Indeed, it is well known that the 2D Navier stokes equation is well-posed as long as \( W_0 \in L^2(R^2) \) is divergence free. Given a 2D solution \( W \) with initial data \( W_0 \) and pressure \( \tilde{P}_W \), it is immediate to check that the 3D extension \( \tilde{W} \) is a solution to the 3D Navier–Stokes equation with initial data \( \tilde{W}_0 \) and pressure \( \tilde{P}_W := \tilde{P}_W \). Thus we may wonder whether small perturbations of \( \tilde{W}_0 \) still give rise to weak solutions with some additional regularity. This is positively addressed in the following proposition.

Proposition 2.3. There exists a constant \( \delta_0 > 0 \) such that the following holds. Let \( \tilde{W} \) be the (unique) solution to the 2D Navier–Stokes equation with initial data \( W_0 \in \)
Suitable solutions. Let \( P \) is a solution to the Navier–Stokes equation with pressure 
\[
|\mathbf{x} - \overline{x}|^{-1/2}(u_0 - \overline{W}_0) |_{L^2(\mathbb{R}^3)} \leq \delta_0 e^{-\delta_0^{-1} \|W\|_{L^2(\mathbb{R}^3)}} ,
\]
there exists a suitable weak solution \( u \) to the 3D Navier–Stokes equation for which the set \( \Pi_{u_0,x} \) is regular.

2.4. Beltrami fields. A Beltrami field \( w_0 \) is an eigenfunction of the curl operator, relative to a non zero eigenvalue \( \lambda \). These vector fields are automatically divergence free and analytic. Indeed, they also satisfies \( \Delta w_0 = -\lambda^2 w_0 \). Letting \( P = -\frac{1}{2}|w_0(x)|^2 \), they are also stationary solutions to the Euler equation, namely \((w_0 \cdot \nabla)w_0 = -\nabla P\). Keeping this in mind, it is easy to check that the family of rescaled fields 
\[
w(t,x) = e^{-\lambda t} w_0(x) ,
\]
is a solution to the Navier–Stokes equation with pressure \( P(t,x) = -\frac{1}{2} |w(x)|^2 \).

Simple examples of Beltrami fields are the so called ABC flows, namely 
\[
(A \sin x_3 + C \cos x_2, B \sin x_1 + A \cos x_3, C \sin x_2 + B \cos x_1) , \quad A,B,C \in \mathbb{R} .
\]
Clearly, these are bounded vector fields with infinite energy. In fact, any Beltrami field on \( \mathbb{R}^3 \) has infinite \( L^2 \)-norm, since it is a nonzero eigenfunction of the laplacian. The interest of Beltrami fields is that, as a consequence of the Arnold structure theorem [1], they are natural candidates to possess rich topological structures. This means that their stream and vortex lines (that in this case coincides), can be knotted and linked in very complicated ways. Indeed, any (locally bounded) link can be realized by the vortex lines of a suitable Beltrami field on \( \mathbb{R}^3 \), as proved in [15]. Also vortex tubes linked and knotted in arbitrarily complicated way can be realized using Beltrami fields; see [16]. Here, one can moreover assume that these Beltrami fields are bounded (in fact, bounded by \( C(1 + |x|)^{-1} \) with all their derivatives). Similar results hold on the three dimensional torus [17]. In this setting, small perturbations of Beltrami fields are interesting since they may realize the well known physical phenomenon of vortex reconnection [14]. Whether a similar result may hold on \( \mathbb{R}^3 \) is still open. In the following proposition we observe that small perturbations of bounded Beltrami fields on \( \mathbb{R}^3 \) can be also analyzed in the framework of this paper.

**Proposition 2.4.** There exists \( \delta_3 > 0 \) such that the following holds. Let \( w_0 \in L^\infty(\mathbb{R}^3) \) such that \( \nabla \times w_0 = \lambda w_0 \), for some \( \lambda \neq 0 \). For any divergence free initial datum \( u_0 \in L^2_{\text{loc}} \), such that \( u_0 - w_0 \in L^2 \) and 
\[
\|\mathbf{x} - \overline{x}\|^{-1/2}(u_0 - w_0) |_{L^2} \leq \delta_3 e^{-\delta_3^{-1} \lambda^{-2} \|w_0\|_{L^\infty}} ,
\]
there exists a suitable weak solution \( u \) to the Navier–Stokes equation for which the set \( \Pi_{u_0,x} \) is regular.

3. Set up and preliminaries

**Suitable solutions.** Let \( u_0 \in L^2_{\text{loc}}(\mathbb{R}^3) \) be a divergence free vector field. Following [5, Section 7], [38, Chapter 30] and [41], we say that \( u \) is a suitable weak solution to the Navier–Stokes with initial data \( u_0 \) if:

1. there exists \( P \in L^{1/2}_{\text{loc}}((0, \infty) \times \mathbb{R}^3) \) such that \( (u, P) \) satisfies the first two equations in (1.1) in the sense of distributions;
2. \( u(t) \to u_0 \) weakly in \( L^2 \), as \( t \to 0^+ \).
3. For any compact set \( K \subset \mathbb{R}^3 \):
\[
\text{ess sup}_{t > 0} \int_K |u(t,x)|^2 \ dx < \infty, \quad \int_0^\infty \int_K |\nabla u(t,x)|^2 \ dxdt < \infty ;
\]
(4) The following local energy inequality is valid
\[
\int_{\mathbb{R}^3} |u|^2 \phi(t) + 2 \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \phi \leq \int_{\mathbb{R}^3} |u_0|^2 \phi(0)
\]
(3.1)
\[
+ \int_0^T \int_{\mathbb{R}^3} |u|^2 (\phi_0 + \Delta \phi) + \int_0^T \int_{\mathbb{R}^3} (|u|^2 + 2P) u \cdot \nabla \phi,
\]
for all non negative \( \phi \in C^\infty_c(\mathbb{R} \times \mathbb{R}^3) \) and for all \( t > 0 \).

Suitable weak solutions are \( L^2 \)-weakly continuous as functions of time [53, pp. 281–282], thus the initial condition (2) makes sense. The solutions considered by Leray in [40] are suitable for any divergence free \( u_0 \in L^2(\mathbb{R}^3) \); see [48, Theorem 2.3] or [38, Proposition 30.1 (A)]. Moreover, these solutions belong to \( L^\infty_t L^2_x \cap L^2_t H^1_x \) and satisfy the energy inequality
\[
\int_{\mathbb{R}^3} |u(t)|^2 + \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \leq \int_{\mathbb{R}^3} |u(0)|^2, \quad \text{for all } t > 0.
\]
(3.2)

**Suitable solutions with bounded energy.** We point out that any suitable weak solution which belongs to \( L^\infty_t L^2_x \cap L^2_t H^1_x \) satisfies the inequality (3.2), as consequence of the local inequality (3.1), via a simple limiting argument. From (3.2) and the weak convergence to the data, also the strong \( L^2 \) convergence \( u(t) \to u_0 \) can be easily deduced. Moreover, thanks to the suitability, the energy inequality can be ‘restarted’
\[
\int_{\mathbb{R}^3} |u(t)|^2 + \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \leq \int_{\mathbb{R}^3} |u(t_0)|^2, \quad \text{for all } t > t_0,
\]
(3.3)
at almost any time \( t_0 > 0 \). From this fact and the weak continuity, also the strong \( L^2 \) continuity follows, at almost any time \( t_0 > 0 \). In the proof of Lemma 7.1 we will show how to deduce a family of ‘restarted’ local energy inequalities from (3.1), the same argument allows to deduce (3.3) by (3.2).

By Sobolev’s embedding and interpolation, any function in \( L^\infty_t L^2_x \cap L^2_t H^1_x \) also belongs to
\[
L^\infty_T L^2_x \quad \text{if } 2/\bar{r} + 3/\bar{q} = 3/2, \quad \bar{q} \leq 6, \quad L^\infty_T L^2_x \quad \text{if } 2/\bar{r} + 3/\bar{q} \geq 3/2, \quad 2 \leq \bar{q} \leq 6,
\]
(3.4)
for all \( T > 0 \), where \( L^p_T \) means that the time integration is restricted to the interval \((0, T)\). In particular, this allows to make sense to the following representation formula for the pressure
\[
P = -\Delta^{-1} \nabla \otimes \nabla \cdot (u \otimes u) = \mathcal{R} \otimes \mathcal{R} \cdot (u \otimes u),
\]
(3.5)
at almost any time \( t > 0 \). Here \( \mathcal{R} := (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3) \) and \( \mathcal{R}_j \) is the \( j \)-th coordinate oriented Riesz transform. Indeed, using the \( L^{p+1} \) boundedness of \( \mathcal{R}_j \), the pressure automatically belongs to
\[
L^\infty_T L^2_x \quad \text{if } 2/\bar{r} + 3/\bar{q} = 3, \quad 1 < \bar{q} \leq 3, \quad L^\infty_T L^2_x \quad \text{if } 2/\bar{r} + 3/\bar{q} \geq 3, \quad 1 < \bar{q} \leq 3.
\]
(3.6)

**Regularity properties.** The following regularity criterion [5, Proposition 2], applies to suitable weak solutions. This criterion will be fundamental in the rest of the paper. We denote \( B(x, r) \subset \mathbb{R}^3 \) the open ball of radius \( r \) centred at \( x \). Let
\[
Q_r(t, x) := (t - r^2, t) \times B(x, r)
\]
the (space-time) parabolic cylinder of radius \( r \) with top point \((t, x)\)
\[
Q^*_r(t, x) := Q_r(t + r^2/8, x) = (t - 7r^2/8 < s < t + r^2/8) \times B(x, r).
\]

**Lemma 3.1.** (Caffarelli–Kohn–Nirenberg). There is an absolute constant \( \varepsilon^* \) such that the following holds. A point \((t, x)\) is regular (see Definition 1.1) for any suitable weak solution \( u \) to the Navier–Stokes equation such that
\[
\limsup_{r \to 0} \frac{1}{r} \int_{Q^*_r(t, x)} |\nabla u|^2 < \varepsilon^*.
\]
(3.7)
Remark 3.1. It is well known (see for instance [38, Proposition 14.2]) that reference solutions \( w \) to the Navier–Stokes equation (see Definition 1.4) satisfy the energy identity

\[
\|w(t)\|_{L^2}^2 + \int_0^t \|\nabla w\|_{L^2}^2 = \|w_0\|_{L^2}^2.
\]

Moreover, the Prodi–Serrin uniqueness result [47, 50], tells us that these solutions are unique in the class of weak solutions \( w' \) which satisfies the relative energy inequality. Thus \( w \) must coincide with the solutions given by the Leray approximation procedure, that are, in particular, suitable. Moreover \( w \in C_b([0, \infty); L^2(\mathbb{R}^3)) \), see [38, Proposition 14.3], namely it also satisfies \( \|w(s) - w(t)\|_{L^2} \to 0 \) as \( s \to t \), for all \( t \geq 0 \). Finally, all the points \((t, x) \in (0, \infty) \times \mathbb{R}^3\) are regular for \( w \). In fact, the regularity condition (3.7) is satisfied at any \( (t, x) \in (0, \infty) \times \mathbb{R}^3 \); see for instance Lemma 4.2. For a more direct argument we refer to [18], [27].

4. Proof of the main Theorem 1.2

We prove it as a consequence of the more general Theorem 4.1. The advantage is that this will allow us to consider reference solutions with infinite energy, like in the following definition.

Definition 4.1. We say that a suitable weak solution \( w \) to the Navier–Stokes equation with divergence free initial data \( w_0 \in L^2_{loc}(\mathbb{R}^3) \) is a generalized reference solution of size \( K \) if (1.5) holds, for some admissible couple, and the regularity condition (3.7) is satisfied at any \((t, x) \in (0, \infty) \times \mathbb{R}^3\). We also require \( w \in C_b([0, \infty); L^2(K)) \), for any compact set \( K \subset \mathbb{R}^3 \), and that the pressure can be represented as \( P = T \cdot (w \otimes w) \), where \( T \) is a \( 3 \times 3 \) symmetric matrix of operators that are bounded on \( L^p(|x|^\alpha dx) \), for all \( 1 < p < \infty \) and \(-3/p < \alpha < 3 - 3/p\).

The difference is that in the generalized notion we are not assuming \( w_0 \in L^2(\mathbb{R}^3) \) and \( w \in L^2_t L^2_x \cap L^2_t H^1_x \). This makes necessary to require a priori the suitability, the regularity condition (3.7), the continuity of \( t \to w(t) \in L^2(K) \) and the representation formula for the pressure.

Notice that all these properties are shared by reference solutions; see Remark 3.1. In this case, the representation formula for the pressure is satisfied letting \( T = R \otimes R \), where \( R := (R_1, R_2, R_3) \) and \( R_j \) is the \( j \)-th coordinate oriented Riesz transform; see (3.5). Notice that \( R \otimes R \) satisfies the weighted estimates we have required for \( T \). These family of estimate indeed holds for even more general singular integrals of convolution type [51].

Definition 4.2. Let \( v_0 \in L^2(\mathbb{R}^3) \). We say that \( v \in L^\infty_t L^2_x \cap L^2_t H^1_x \) is a suitable weak solution to the perturbed Navier–Stokes equation, around the solution \( w \), with initial data \( v_0 \), when there exists \( P_v \in L^3_{loc}((0, \infty) \times \mathbb{R}^3) \) such that

\[
\begin{align*}
\partial_t v + (v \cdot \nabla)v + (v \cdot \nabla)w + (w \cdot \nabla)v - \Delta v &= -\nabla P_v, \\
\nabla \cdot v &= 0,
\end{align*}
\]

in the sense of distributions, and \( v(t) \to v_0 \) weekly in \( L^2 \) as \( t \to 0^+ \). Moreover, given any non negative \( \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^3) \), the perturbed (local) energy inequality

\[
\begin{align*}
\int_{\mathbb{R}^3} |v(t,x)|^2 \phi(t,x) dx &+ 2 \int_{t_0}^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi \leq \int_{\mathbb{R}^3} |v(t_0,x)|^2 \phi(t_0,x) dx \\
&+ \int_{t_0}^t \int_{\mathbb{R}^3} |v|^2 (\phi_t + \Delta \phi) + \int_{t_0}^t \int_{\mathbb{R}^3} (|v|^2 + 2P_v)v \cdot \nabla \phi \\
&+ \int_{t_0}^t \int_{\mathbb{R}^3} |v|^2 w \cdot \nabla \phi + 2 \int_{t_0}^t \int_{\mathbb{R}^3} (v \cdot \nabla v) v \cdot \nabla \phi + (v \cdot \nabla)v \cdot w \phi
\end{align*}
\]

is satisfied for all \( t > t_0 \), where \( t_0 \) may be zero or almost any real number in \((0, \infty)\), and there exists a \( 3 \times 3 \) symmetric matrix \( T \) of operators, that are bounded on
Thus, our main theorem is now the following:

**Theorem 4.1.** Given $(r, q)$ admissible, there exists a constant $\delta_0 > 0$ such that the following holds. Let $w$ be a generalized reference solution of size $K$ to the Navier–Stokes equation with divergence free data $u_0 \in L^q_{\text{loc}}$. The set $\Pi_{\delta_0, 2}$ is regular for every suitable weak solution $u$ to the Navier–Stokes equation with divergence free data $u_0 \in L^q_{\text{loc}}$ such that

$$\|x - \bar{x}\|^{-1/2} (u_0 - w_0) \| L^2 \leq \delta_0 e^{\frac{K}{8} \delta_0},$$

and such that $u - w$ is a suitable weak solution to the perturbed Navier–Stokes equation (4.1), around the solution $w$, with data $u_0 - w_0 \in L^2$.

We have already observed that that reference solutions of size $K$ are, in particular, generalized reference solutions of size $K$; see also Lemma 4.2. Moreover, it is straightforward to check that if $w \in L^\infty_0 L^3_2 \cap L^q_1 H^2_2$ is a suitable weak solution to the Navier–Stokes equation with pressure $P_u$ and data $u_0 \in L^q(\mathbb{R}^3)$ and $w$ is a reference solution with pressure $P_w$ and data $w_0 \in L^q(\mathbb{R}^3)$, then the difference $u - w$ is a suitable weak solution to the perturbed Navier–Stokes equation 4.1, around the solution $w$, with pressure $P_{u - w} := P_u - P_w$ and data $u_0 - w_0$. We refer to Lemma 7.1 for details. Here we only remark that, since we have the representation formulas $P_u = R \otimes R \cdot (u \otimes u)$ and $P_w = R \otimes R \cdot (w \otimes w)$, see (3.5), (4.3) holds taking $T = R \otimes R$. In conclusion, Theorem 4.1 implies Theorem 1.2.

Moreover, keeping this in mind, the existence of suitable solutions $u$ that satisfy the assumption of Theorem 4.1 is ensured by the Leray theory, once we consider data $u_0 \in L^2$ and $w$ is a reference solution. This is not obvious when when $w$ is a generalized reference solution, since we may need to handle unbounded energies, for instance $u_0$ only locally square integrable. However, there are some relevant situations in which these (infinite energy) suitable solutions can be actually constructed by a simple adaptation of the Leray theory; see Proposition 5.3. This is the case when we consider the admissible couple $(r, q) = (2, \infty)$, for which Theorem 4.1 is substantially more efficient than (the simpler version) Theorem 1.2.

**Idea of the proof.** The main idea behind Theorem 1.1 is to use the local energy inequality (3.1) and the weighted $L^2$ smallness assumption on $u_0$ to prove that the regularity condition (3.7) is satisfied at any point inside the regular set.

A natural way to attack the perturbative case is trying to do the same, using the perturbed energy inequality (4.2), with initial data $v_0 := u_0 - w_0$. The difficulty is that the new terms in the perturbed energy inequality (4.2) contain the reference solution $w$, so that they can not be handled in a perturbative way, since $w$ may be large. To avoid this, we distinguish two time regimes $t \leq t^*$, $t > t^*$ and choose $t^*$ in such a way that these hard terms can be controlled for $t > t^*$, using a cancelation in the energy inequality that is analogous to the one used in the proof of Theorem 1.1. Then we use the (exponential) smallness assumption (1.6) on $u_0 - v_0$ to control the weighted $L^2$ norm of the solution $u - w$ up to the time $t^*$, so that we can ‘restart’ the small data problem at the time $t^*$. An appropriate choice of $t^*$ permits to conclude the proof.

**Proof.** We can assume $\bar{x} = 0$ since the general case follows by translation. We divide the proof into three steps.
4.1. First step: the perturbed equation.

Let \( v_0 := u_0 - w_0 \) and \( v := u - w \). Fix \( \xi \in \mathbb{R}^3 \). We have assumed \( v \) to be a weak solution to the perturbed Navier–Stokes equation (4.1), that, after the change of variables

\[
(t, y) = (t, x - \xi t), \quad v_\xi(t, y) := v(t, x), \quad w_\xi(t, y) := w(t, x),
\]

becomes

\[
\begin{aligned}
\partial_t v_\xi + ((v_\xi - \xi) \cdot \nabla) v_\xi + (v_\xi \cdot \nabla) w_\xi + (w_\xi \cdot \nabla) v_\xi - \Delta v_\xi &= -\nabla P_{v_\xi} \\
\nabla \cdot v_\xi &= 0 \\
\mathcal{T} \cdot (v_\xi \otimes v_\xi) + 2 \mathcal{T} \cdot (v_\xi \otimes w_\xi) &= P_{v_\xi}(t, y) - P_{v_\xi}(t, x),
\end{aligned}
\]

where \( v_\xi(0, \cdot) = v_0 \) means that we have \( L^2 \) weak convergence as \( t \to 0^+ \). Let

\[
\sigma_\mu(y) := (\mu + |y|^2)^{-\frac{1}{2}}, \quad \mu > 0
\]

and define

\[
t^*(\xi, \mu) := \sup\{ t \in [0, \infty) : \int_0^t \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(\tau, y)|^2 \, dy \, d\tau \leq \int_0^t \| w_\xi \|_{L^2_y}^r(\tau) \, d\tau \}.
\]

The integral \( \int_0^t \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(\tau, y)|^2 \, dy \, d\tau \) is finite, for any \( t > 0 \), since it is smaller than \( \mu^{-\frac{1}{2}} \int_0^t \int_{\mathbb{R}^3} |\nabla v(\tau, y)|^2 \, dy \, d\tau \), that is bounded by assumption. However, our ultimate goal is to obtain a bound that is uniform in \( \mu > 0 \). Since the functions \( t \to \int_0^t \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(\tau, y)|^2 \, dy \, d\tau \) and \( t \to \int_0^t \| w_\xi \|_{L^2_y}^r(\tau) \, d\tau \) are continuous and non decreasing in \( t > 0 \), it is easy to show that

\[
(i) \quad \int_0^{t^*} \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(t, y)|^2 \, dy \, d\tau = \int_0^{t^*} \| w_\xi \|_{L^2_y}^r(\tau) \, d\tau;
\]

\[
(ii) \quad \int_0^{t^*} \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(t, y)|^2 \, dy \, d\tau > \int_0^{t^*} \| w_\xi \|_{L^2_y}^r(\tau) \, d\tau \quad \text{for all} \quad t > t^*,
\]

provided \( t^* < \infty \). If \( t^* = \infty \) either there exists a divergent sequence \( t_n \to \infty \) such that

\[
\int_0^{t_n} \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(\tau, y)|^2 \, dy \, d\tau = \int_0^{t_n} \| w_\xi \|_{L^2_y}^r(\tau) \, d\tau,
\]

or there exists a single (possibly equal to zero) \( t_n \) which satisfies this and such that

\[
\int_0^{t_n} \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(\tau, y)|^2 \, dy \, d\tau < \int_0^{t_n} \| w_\xi \|_{L^2_y}^r(\tau) \, d\tau \quad \text{for all} \quad t > t_n.
\]

In both cases we have

\[
\int_0^{\infty} \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(\tau, y)|^2 \, dy \, d\tau \leq \int_0^{\infty} \| w_\xi \|_{L^2_y}^r(\tau) \, d\tau =: K,
\]

that, recalling the property (i) in (4.7), implies

\[
\int_0^{t^*} \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(t, y)|^2 \, dy \, d\tau \leq K,
\]

for any possible \( t^* \in [0, \infty] \).

For any \( t > t^* \) we consider the (space-time) segment

\[
L(t, \xi) := \{(s, \xi s) : s \in (0, t)\}.
\]

We will investigate for which \((t, \xi)\) this set is regular. Notice that the change of variables (4.5) maps \( L(t, \xi) \) into \((0, t) \times \{0\}\), a vertical segment above the origin of the space-time.
4.2. Second step: estimates for \( t \leq t^* \). We have assumed \( v \) to satisfy the perturbed energy inequality (4.2), that after the change of variables (4.5) becomes
\[
\int_{R^3} |\xi|^2 \phi(t, y) dy + 2 \int_0^t \int_{R^3} |\nabla v_\xi|^2 \phi \leq \int_{R^3} |v_0|^2 \phi(0, y) dy
\]
(4.9)
\[
+ \int_0^t \int_{R^3} |v_\xi|^2 (\phi_t - \xi \cdot \nabla \phi + \Delta \phi) + (|v_\xi|^2 + 2P_{v_\xi}) v_\xi \cdot \nabla \phi
+ \int_0^t \int_{R^3} |v_\xi|^2 w_\xi \cdot \nabla \phi + 2 \int_0^t \int_{R^3} (v_\xi \cdot w_\xi) v_\xi \cdot \nabla \phi + (v_\xi \cdot \nabla) v_\xi \cdot w_\xi \phi,
\]
valid for all \( t > 0 \) and \( \phi \in C_0^\infty (R \times R^3) \) non negative. Thus
\[
\int_{R^3} |\xi|^2 \phi(t, y) dy + 2 \int_0^t \int_{R^3} |\nabla v_\xi|^2 \phi \leq \int_{R^3} |v_0|^2 \phi(0, y) dy
\]
(4.10)
\[
+ \int_0^t \int_{R^3} |v_\xi|^2 (\phi_t - \xi \cdot \nabla \phi + \Delta \phi) + (|v_\xi|^2 + 2P_{v_\xi}) v_\xi \cdot \nabla \phi
+ \int_0^t \int_{R^3} 3|v_\xi|^2 w_\xi ||\nabla \phi| + 18 |v_\xi||\nabla v_\xi||\phi|.
\]
By a standard approximation argument (see the proof of Lemma 8.3 in [5]) this still holds for any test function of the form
\[
\phi(t, y) := \psi(t) \phi_1 (y)
\]
with \( \phi_1 \in C_0^\infty (R^3) \) non negative and
\[
\psi : [0, \infty) \rightarrow [0, \infty) \text{ absolutely continuous with } \dot{\psi} \in L^1 (0, \infty).
\]
We shall choose here
\[
\psi(t) := 1, \quad \phi_1 = \sigma_{\mu}(y) \chi(\delta y),
\]
where \( \sigma_{\mu}(y) \) has been defined in (4.6), \( \delta > 0 \) and \( \chi : [0, \infty) \rightarrow [0, \infty) \) is a smooth non increasing function such that
\[
\chi = 1 \text{ on } [0, 1], \quad \chi = 0 \text{ on } [2, \infty].
\]
Recalling \( P \in L^{3/2}_{loc} ((0, \infty) \times R^3), \) \( w \in L^1_t L^3_x, \) \( v \in L^\infty_t L^2_x \cap L^2_t H^1_x, \) so that \( v \) also belongs the mixed spaces in (3.4), since the same clearly holds for \( P_{v_\xi}, w_\xi, v_\xi, \) we can easily pass to the limit \( \delta \rightarrow 0 \) so that
\[
\int_{R^3} |v_\xi|^2 \sigma_{\mu}^4 + 2 \int_0^t \int_{R^3} |\nabla v_\xi|^2 \sigma_{\mu} \leq
\]
(4.11)
\[
\leq \int_0^t \int_{R^3} |v_\xi|^2 (-\xi \cdot \nabla \sigma_{\mu} + \Delta \sigma_{\mu}) + (|v_\xi|^2 + 2P_{v_\xi}) \sigma_{\mu} \cdot \nabla \sigma_{\mu}
+ 18 \int_0^t \int_{R^3} |v_\xi||\nabla v_\xi||\sigma_{\mu} + 3|v_\xi|^2 w_\xi ||\nabla \sigma_{\mu}|.
\]
Then, since
\[
|\nabla \sigma_{\mu}| < (\mu + |y|^2)^{-1} = \sigma_{\mu}^2, \quad \Delta \sigma_{\mu} < 0,
\]
we arrive to the inequality
\[
\int_{R^3} \sigma_{\mu}^4 |v_\xi|^2 \]
(4.12)
\[
+ 2 \int_0^t \int_{R^3} \sigma_{\mu}(|\xi||\nabla v_\xi|^2 \leq |\xi| \int_0^t \int_{R^3} \sigma_{\mu}^2 |v_\xi|^2
+ 18 \int_0^t \int_{R^3} \sigma_{\mu}^2 (|v_\xi|^3 + 2P_{v_\xi} ||v_\xi| + 3|v_\xi|^2 w_\xi) + 18 \sigma_{\mu} |v_\xi||\nabla v_\xi||\sigma_{\mu}|.
\]
The next goal is to deduce an integral inequality for the functions
\[
\alpha_{\mu}(t) := \int_{R^3} \sigma_{\mu}(y) |v_\xi(t, y)|^2 dy, \quad B_{\mu}(t) := \int_0^t \int_{R^3} \sigma_{\mu}(y) |\nabla v_\xi(\tau, y)|^2 dyd\tau.
\]
In order to bound the terms on the right hand side of (4.13) we use the Stein weighted estimates for singular integrals (7.7) and the Caffarelli–Kohn–Nirenberg interpolation inequalities (7.2). For brevity we will refer to these inequalities as SS and CKN, respectively.

We first bound the terms involving the pressure \( P_{v_\xi} = T \cdot (v_\xi \otimes v_\xi) + 2T \cdot (v_\xi \otimes w_\xi). \) We have
\[
2 \int_{R^3} \sigma_{\mu}^2 |P_{v_\xi} ||v_\xi| \leq 2 \int_{R^3} \sigma_{\mu}^2 |v_\xi||T \cdot (v_\xi \otimes v_\xi)|
+ 4 \int_{R^3} \sigma_{\mu}^2 |v_\xi||T \cdot (v_\xi \otimes w_\xi)| =: I + II.
\]
Here and in the following $Z \geq 1$ denotes several constants, only depending on $(r, q)$, possibly increasing from line to line. By the SS inequality (7.7) we have

$$I \leq 2\|\sigma, T \cdot (v_\xi \otimes w_\xi)\|_{L^2} \|\sigma_\mu v_\xi\|_{L^2} \leq Z \|\sigma_\mu v_\xi\|_{L^2} \|\sigma, v_\xi\|_{L^2} = Z \|\sigma_\mu^{1/2} v_\xi\|_{L^2} + \|\sigma_\mu v_\xi\|_{L^2}$$

then using the CKN inequality (7.2) we obtain

$$I \leq Z \|\sigma_\mu^{1/2} \nabla v_\xi\|_{L^2}^{1/2} \|\sigma_\mu^{1/2} v_\xi\|_{L^2} \|\sigma_\mu v_\xi\|_{L^2}^{1/2} = Z \dot{B}_\mu a_\mu^{1/2} \leq \frac{1}{6} \dot{B}_\mu + Z \dot{B}_\mu a_\mu.$$  

In a similar way

$$II \leq 4\|\sigma_\mu T \cdot (v_\xi \otimes w_\xi)\|_{L^{2q/(q-1)}} \|\sigma_\mu v_\xi\|_{L^{2q/(q-1)}} \leq Z \|\sigma_\mu v_\xi\|_{L^{2q/(q-1)}} \|\sigma_\mu v_\xi\|_{L^{2q/(q-1)}} \leq Z \|\sigma_\mu v_\xi\|_{L^q} \|\sigma_\mu v_\xi\|_{L^q}$$

and, again by CKN with $2\theta = 1 + 3/q$, which implies $r = 1/(1 - \theta)$, where $\theta$ is the interpolation parameter in (7.2), we get

$$II \leq Z \|\sigma_\mu^{2} \nabla v_\xi\|_{L^2}^{1/2} \|\sigma_\mu^{2} v_\xi\|_{L^2} = Z \|\sigma_\mu^{2} \nabla v_\xi\|_{L^2} = Z \|\sigma_\mu^{2} \nabla v_\xi\|_{L^2} \|\dot{B}_\mu a_\mu\|^2 \leq \frac{1}{6} \dot{B}_\mu + Z \|\sigma_\mu\|^2 a_\mu.$$  

We now consider the other terms in the right hand side of (4.13). As above, we use CKN to bound

$$|\xi| \int_{R^3} \sigma_\mu^2 |v_\xi| \leq Z |\xi| \|\sigma_\mu^{1/2} \nabla v_\xi\|_{L^2} \|\sigma_\mu^{1/2} v_\xi\|_{L^2} = Z |\xi| (\dot{B}_\mu a_\mu)^{1/2} \leq \frac{1}{6} \dot{B}_\mu + Z |\xi|^2 a_\mu;$$

and

$$\int_{R^3} \sigma_\mu^2 |v_\xi|^3 \leq Z \|\sigma_\mu v_\xi\|_{L^2} \|\sigma_\mu v_\xi\|_{L^2} \|\sigma_\mu^{1/2} v_\xi\|_{L^2} = Z \|\sigma_\mu v_\xi\|_{L^2} \|\sigma_\mu^{1/2} v_\xi\|_{L^2} \|\dot{B}_\mu a_\mu\|^2 \leq \frac{1}{6} \dot{B}_\mu + Z \|\sigma_\mu v_\xi\|_{L^2} a_\mu.$$  

In order to bound the terms with $w_\xi$ we use CKN with $2\theta = 1 + 3/q$

$$3 \int_{R^3} \sigma_\mu^2 |w_\xi|^2 |w_\xi| = 3 \|\sigma_\mu^{1/2} \nabla v_\xi\|_{L^2} \|\sigma_\mu^{1/2} v_\xi\|_{L^2} \|\dot{B}_\mu a_\mu\|^2 \leq \frac{1}{6} \dot{B}_\mu + Z \|\sigma_\mu v_\xi\|_{L^2} a_\mu.$$  

and CKN with $\theta = 1 - 2/r$, which implies $2/(1 - \theta) = r$.

$$18 \int_{R^3} \sigma_\mu^2 |v_\xi|^3 \|\nabla v_\xi\|_{L^2} \|\dot{B}_\mu a_\mu\|^2 \leq Z \|\sigma_\mu^{1/2} \nabla v_\xi\|_{L^2} \|\sigma_\mu^{1/2} v_\xi\|_{L^2} \|\dot{B}_\mu a_\mu\|^2 \leq \frac{1}{6} \dot{B}_\mu + Z \|\sigma_\mu v_\xi\|_{L^2} a_\mu.$$  

Now, recalling (4.13), summing all these inequalities and absorbing the resulting term $\int_0^t \dot{B}_\mu(\tau) d\tau = B_\mu(t)$ from the right hand side into the left hand side, we obtain

$$a_\mu(t) + B_\mu(t) \leq a_\mu(0) + Z \int_0^t \left( |\xi|^2 + \dot{B}_\mu(\tau) + \|w_\xi(\tau, \cdot)\|_{L^2(R^3)} \right) a_\mu(\tau) d\tau.$$  

Let

$$a(t) := \int_{R^3} |y|^{-1} |v_\xi(t, y)|^2 dy.$$  

Since $a_\mu(0) \leq a(0)$, we have obtained the estimate

$$a_\mu(t) + B_\mu(t) \leq a(0) + Z \int_0^t \left( |\xi|^2 + \dot{B}_\mu(\tau) + \|w_\xi(\tau, \cdot)\|_{L^2(R^3)} \right) a_\mu(\tau) d\tau,$$

for some constant $Z$. By Grönwall’s lemma we get, for $0 \leq t \leq t^*$:

$$a_\mu(t) \leq a(0) e^{Z A}, \quad A := B_\mu(t^*) + \|w_\xi|_{L_tL^r_x}^r + t^* |\xi|^2.$$  

---

3In Definition 4.1 we have required $T$ to satisfy the family of inequalities (7.7) only in the case $\mu = 0$, however the general case can be deduced as shown in [5, Lemma 7.2].
Recalling (4.8), namely that the quantity \( B_\mu(t^*) \) is smaller than \( K \), and since\(^4\)
\[
\|w_\xi\|_{L^q_\tau L^r_x} = \|w\|_{L^q_\tau L^r_x} =: K,
\]
we have
\[
A \leq 2K + t^*|\xi|^2.
\]
Thus, if we restrict to the vectors \( \xi \) such that
\[
|\xi|^2t^* \leq 1
\]
the estimate (4.15) gives
\[
a_\mu(t) \leq e^{2eZ(2K+1)} \quad \text{for all} \quad 0 \leq t \leq t^*,
\]
where we have denoted
\[
\epsilon := \sqrt{a(0)} = \|x^{-1/2}v_0\|_{L^2}.
\]
Taking a suitably larger constant \( Z \), this implies
\[
a_\mu(t) \leq Ze^{2K\epsilon^2} \quad \text{for all} \quad 0 \leq t \leq t^*.
\]

4.3. Third step: estimates for \( t > t^* \). Here the idea is to repeat the previous argument starting by the point \( (t^*, t^*\xi) \) of the segment \( L(t, \xi) \times \{1\} \), rather than the origin. To do so we want to use the inequality (4.9), but the time integration has to be over \([t^*, t]\) rather than \([0, t]\). Since we know that \( v \) satisfies the perturbed energy inequality (4.2), changing variables as in (4.5), we can actually do this on intervals \([t_n^*, t]\), where \( t_n^* \) is a sequence of times, smaller or equal than \( t^* \), and such that \( t_n^* \to t^* \). Notice that this is possible since in the inequality (4.2) the integration is over the time interval \([t_0, t]\) where \( t_0 \) is allowed to be zero or almost any real number in \((0, \infty)\).

Choosing as test functions \( \phi(t, y) := \psi_\mu(t|\sigma_\mu(y)\chi(\delta|y)|) \) where \( \chi \) and \( \sigma_\mu \) are as before\(^5\), while
\[
\psi_\mu(t) := e^{-kB_\mu(t)}, \quad B_\mu(t) := \int_{t_n}^t \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(\tau, y)|^2 d\tau dy,
\]
with \( k \) a positive constant to be specified, and proceeding as before, we arrive to
\[
[f_{\mathbb{R}^3} \psi_\mu|v_\xi|^2]_{t_n^*} + 2 \int_{t_n^*}^t \int_{\mathbb{R}^3} \psi_\mu|\nabla v_\xi|^2 \leq \]
\[
\leq \int_{t_n^*}^t \int_{\mathbb{R}^3} \psi_\mu|v_\xi|^2(-kB_\mu(t_\mu)\sigma_\mu - \xi \cdot \nabla \sigma_\mu + \Delta \sigma_\mu) + \int_{t_n^*}^t \int_{\mathbb{R}^3} \psi_\mu(|v_\xi|^2 + 2P_\xi v_\xi \cdot \nabla \sigma_\mu + 18 \int_{t_n^*}^t \int_{\mathbb{R}^3} \sigma_\mu|v_\xi||\nabla v_\xi||w_\xi| + 3 \int_{t_n^*}^t \int_{\mathbb{R}^3} \psi_\mu|v_\xi|^2|w_\xi||\nabla \sigma_\mu|.
\]
This implies, recalling (4.12),
\[
[f_{\mathbb{R}^3} \psi_\mu|v_\xi|^2]_{t_n^*} + 2 \int_{t_n^*}^t \int_{\mathbb{R}^3} \psi_\mu|\nabla v_\xi|^2 \leq \]
\[
\leq \int_{t_n^*}^t \int_{\mathbb{R}^3} \psi_\mu|v_\xi|^2((|\sigma_\mu|^2 - kB_\mu(t_\mu)\sigma_\mu) + \int_{t_n^*}^t \int_{\mathbb{R}^3} \sigma_\mu|v_\xi|^2(|v_\xi|^2 + 2P_\xi |v_\xi| + 3|v_\xi|^2|w_\xi|) + 18 \sigma_\mu|v_\xi||\nabla v_\xi||w_\xi|.
\]
Again, our goal is to prove an integral inequality for the functions
\[
a_\mu(t) := \int_{\mathbb{R}^3} \sigma_\mu(y)|v_\xi(t, y)|^2 dy, \quad B_{t_n^*, \mu}(t) := \int_{t_n^*}^t \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(\tau, y)|^2 d\tau dy.
\]
\( ^4 \) At fixed \( t \), is simply a translation of \( w \).
\( ^5 \) To be precise we have to consider two vanishing sequences \( \delta_{n, \mu} \) instead of \( \delta_\mu \), in order to be sure that the exceptional times, starting from which (4.2) may not be satisfied for at least one of our test functions, has measure zero. However, we keep writing \( \delta_\mu \) for simplicity.
We need to bound the terms in the right hand side of (4.18). Recalling the decomposition (4.14), with the same computations of the second step, we obtain

\[ I \leq \frac{1}{5} \dot{B}_{t_n, \mu} + Z \dot{B}_{t_n, \mu} a_\mu. \]

While, using the SS and CKN inequality, the last one with \(2\theta = 1 + 3/q\), that implies \(r = 1/(1 - \theta)\),

\[ II \leq Z \| w_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2}^{2(1 - \theta)} \| \sigma_\mu^{1/2} \nabla v_\xi \|_{L^2}^{2\theta} = Z \| w_\xi \|_{L^2} \| \sigma_\mu^{1 - \theta} \dot{B}_{t_n, \mu} \|
\]

\[ = \| w_\xi \|_{L^2} (a_\mu \dot{B}_{t_n, \mu})^{1 - \theta} \dot{B}_{t_n, \mu}^{2\theta - 1} \leq \frac{1}{5} \dot{B}_{t_n, \mu} + Z \dot{B}_{t_n, \mu} a_\mu + \frac{1}{15} \| w_\xi \|_{L^2}. \]

Exactly as in the second step,

\[ \int_{R^3} \sigma_\mu^2 |v_\xi|^3 \leq \frac{1}{5} \dot{B}_{t_n, \mu} + Z \dot{B}_{t_n, \mu} a_\mu, \]

while the next terms has to been estimated differently. Using CKN

\[ |\xi| \int_{R^3} \sigma_\mu^2 |v_\xi|^2 = \| \sigma_\mu v_\xi \|_{L^2}^2 \leq Z \| \sigma_\mu v_\xi \|_{L^2} \| \sigma_\mu^{1/2} \nabla v_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2}
\]

\[ = Z \| w_\xi \|_{L^2} \| \sigma_\mu^{1/2} \nabla v_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2} \]

and, still using CKN with \(2\theta = 1 + 3/q\), that implies \(r = 1/(1 - \theta)\),

\[ 3 \int_{R^3} \sigma_\mu^2 |v_\xi|^2 |w_\xi| \leq 3 \| w_\xi \|_{L^2} \| \sigma_\mu v_\xi \|_{L^2} \| \sigma_\mu^{1/2} \nabla v_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2}
\]

\[ \leq Z \| w_\xi \|_{L^2} \| \sigma_\mu^{1/2} \nabla v_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2}
\]

\[ = Z \| w_\xi \|_{L^2} \| \sigma_\mu v_\xi \|_{L^2} \| \sigma_\mu^{1/2} \nabla v_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2}
\]

\[ = Z \| w_\xi \|_{L^2} \| \sigma_\mu v_\xi \|_{L^2} \| \sigma_\mu^{1/2} \nabla v_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2}
\]

\[ \leq \frac{1}{5} \dot{B}_{t_n, \mu} + Z \dot{B}_{t_n, \mu} a_\mu + \frac{1}{15} \| w_\xi \|_{L^2}. \]

By CKN with \(\theta = 1 - 2/r\), that implies \(r = 2/(1 - \theta)\),

\[ 18 \int_{R^3} \sigma_\mu v_\xi \| \nabla v_\xi \|_{L^2} \| w_\xi \|_{L^2} \leq 18 \| \sigma_\mu^{1/2} \nabla v_\xi \|_{L^2} \| w_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2}
\]

\[ \leq Z \| w_\xi \|_{L^2} \| \sigma_\mu v_\xi \|_{L^2} \| \sigma_\mu^{1/2} \nabla v_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2}
\]

\[ = Z \| w_\xi \|_{L^2} \| \sigma_\mu v_\xi \|_{L^2} \| \sigma_\mu^{1/2} \nabla v_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2}
\]

\[ = Z \| \sigma_\mu v_\xi \|_{L^2} \| \sigma_\mu^{1/2} \nabla v_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2}
\]

\[ \leq \frac{1}{5} \dot{B}_{t_n, \mu} + Z \dot{B}_{t_n, \mu} a_\mu + \frac{1}{15} \| w_\xi \|_{L^2}. \]

We can now plug these inequalities in (4.18) so that

\[ a_\mu(t) \psi_\mu(t) - a_\mu(t_n) + 2 \int_{t_n}^{t} \dot{B}_{t_n, \mu}(s) \psi_\mu(s) ds \leq \]

\[ \leq \int_{t_n}^{t} \psi_\mu(s)(B_{t_n, \mu}(s) + 6Z \dot{B}_{t_n, \mu} a_\mu(s) + |\xi|^2 + \frac{1}{5} \| w_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2} - k \dot{B}_{t_n, \mu} a_\mu(s)) \| ds. \]

Now we subtract the first term of the right hand side from the left hand side and choose \(k = 6Z\) in order to cancel each other out the second and the last term on the right hand side. Thus, noting

\[ \int_{t_n}^{t} \dot{B}_{t_n, \mu}(s) \psi_\mu \]

\[ = \frac{1}{6Z} \int_{t_n}^{t} \psi_\mu - \psi_\mu(t_n) \psi_\mu(t) = \frac{1 - \psi_\mu(t)}{6Z} \]

we have proved

\[ a_\mu(t) \psi_\mu(t) - a_\mu(t_n) + \frac{1 - \psi_\mu(t)}{6Z} \leq (4.19) \]

\[ \leq |\xi|^2 \int_{t_n}^{t} \psi_\mu(s) ds + \frac{1}{5} \int_{t_n}^{t} \| w_\xi \|_{L^2} \| \sigma_\mu^{1/2} v_\xi \|_{L^2} \psi_\mu(s) ds =: I + II, \]

for \(t > t_n\). Since \(\psi_\mu \leq 1\), the term \(I\) is immediately bounded by

\[ I \leq |\xi|^2 t. \]
The bound of $II$ is more delicate and requires the property (ii) in (4.7). Indeed, letting

$$B_{t,\mu}(t) := \int_{t^*}^{t} \int_{\mathbb{R}^3} \sigma_\mu(y) |\nabla v_\xi(\tau, y)|^2 dyd\tau,$$

the property (ii) becomes

$$B_{t,\mu}(t) > \int_{t^*}^{t} \|w_\xi\|_{L^q}(\tau) d\tau, \quad \text{for} \quad t > t^*$$

and, since $B_{t^*,\mu}(t) \geq B_{t,\mu}(t)$, we can bound

$$5II = \int_{t_n^*}^{t} \|w_\xi(s, \cdot)\|_{L^q(\mathbb{R}^3)} \psi_\mu(s) ds = \int_{t_n^*}^{t} \|w_\xi(s, \cdot)\|_{L^q(\mathbb{R}^3)} e^{-6ZB_{t^*,\mu}(s)} ds$$

$$< \int_{t_n^*}^{t} \|w_\xi(s, \cdot)\|_{L^q(\mathbb{R}^3)} ds + \int_{t^*}^{t} \|w_\xi(s, \cdot)\|_{L^q(\mathbb{R}^3)} e^{-6Zf^*_\mu \psi_\mu(\tau)} ds$$

$$= \int_{t_n^*}^{t} \|w_\xi(s, \cdot)\|_{L^q(\mathbb{R}^3)} ds - \frac{1}{6Z} \left[ e^{-6Zf^*_\mu \psi_\mu(\tau)} \|w_\xi(s, \cdot)\|_{L^q(\mathbb{R}^3)} ds \right]_{s=t^*}$$

$$< \int_{t_n^*}^{t} \|w_\xi(s, \cdot)\|_{L^q(\mathbb{R}^3)} ds + \frac{1}{6Z}. \quad (4.21)$$

Thus, since $\|w_\xi(s, \cdot)\|_{L^q} = \|w(s, \cdot)\|_{L^q}$ is integrable, we can choose $t_n^*$ close enough to $t^*$ in such a way that

$$II \leq \frac{1}{30Z}. \quad (4.21)$$

Plugging the estimates (4.20), (4.21) into the inequality (4.19), we obtain

$$(a_\mu(t) - \frac{1}{10Z}) \psi_\mu(t) + \frac{1}{10Z} - |\xi|^2 |t| \leq 0 \quad (4.22)$$

In order to handle the term $a_\mu(t_n^*)$ we use $a_\mu(t_n^*) \leq Ze^{ZK} \epsilon^2$, which we have proved in (4.17), and we assume

$$\epsilon \leq 1 \quad \Rightarrow \quad a_\mu(t_n^*) \leq Ze^{ZK} \epsilon. \quad (4.23)$$

In fact we assume that $\epsilon$ is so small that

$$a_\mu(t_n^*) \leq Ze^{ZK} \epsilon \leq \frac{1}{30Z}. \quad (4.23)$$

in such a way that (4.22) gives

$$(a_\mu(t) - \frac{1}{10Z}) \psi_\mu(t) + \frac{1}{10Z} - |\xi|^2 |t| \leq 0,$$

or equivalently (we recall that $\psi_\mu(t) = e^{-6ZB_{t^*,\mu}(t)}$)

$$a_\mu(t) + \left( \frac{1}{10Z} - |\xi|^2 |t| \right) e^{6ZB_{t^*,\mu}(t)} \leq \frac{1}{10Z}, \quad (4.24)$$

for all $t > t^*$. Let us finally assume

$$\left( \frac{1}{10Z} - |\xi|^2 |t| \right) > 0 \quad \text{i.e.} \quad |\xi|^2 |t| < \frac{1}{10Z}. \quad (4.25)$$

Note that this assumption is stronger than (4.16), namely $|\xi|^2 t^* \leq 1$, since $Z \geq 1$ and $t^* < t$. The inequality (4.24) immediately gives

$$B_{t,\mu}(t) \leq B_{t^*,\mu}(t) < \infty, \quad \text{uniformly in} \ \mu > 0,$$

for $t > t^*$ such that (4.25) holds. Here we mean that $B_{t^*,\mu}(t)$ is bounded by a constant independent on $\mu > 0$. This constant may become arbitrarily large as $|\xi|^2 t$ approaches $\frac{1}{10Z}$, but this does not affect our argument. On the other hand, recalling (4.8), we also know that

$$B_\mu(t^*) = \int_{0}^{t^*} \int_{\mathbb{R}^3} \sigma_\mu(y) |\nabla v_\xi(\tau, y)|^2 dyd\tau < \infty, \quad \text{uniformly in} \ \mu > 0,$$
Thus, since \( B_\mu(t) = B_\mu(t^*) + B_{\tau;\mu}(t) \), we have proved
\[ B_\mu(t) < \infty, \quad \text{uniformly in } \mu > 0, \]
for \( t > t^* \) such that (4.25) holds. Since the weights \( \sigma_\mu(y) \) are increasing as \( \mu \to 0 \), and they converge to \( |y|^{-1} \), we can pass to the limit and come back to the old variables \((t, x)\), so that
\[
\lim_{\mu \to 0} B_\mu(t) = \int_0^t \int_{\mathbb{R}^3} |y|^{-1} |\nabla v_\xi(\tau, y)|^2 \, dy \, d\tau = \int_0^t \int_{\mathbb{R}^3} \frac{|\nabla v(\tau, x)|^2}{|x - \xi\tau|} \, dx \, d\tau < \infty,
\]
provided that (4.25) is satisfied.

This implies that the regularity condition (3.7) is satisfied at any \((s, \xi s) \in L(t, \xi)\).
Indeed, let \( 0 < s < t \), and let \( r > 0 \) be so small that \( 0 < s - 7r^2/8 < s + r^2/8 < t \) and \( |\xi|r \leq 1 \). For all \((\tau, x) \in Q_{r\xi}(s, \xi s)\),
\[
|x - \xi\tau| \leq |x - \xi s| + |\xi||s - \tau| \leq r + r^2|\xi| \leq 2r,
\]
from which we deduce
\[
\frac{1}{r} \int_{\mathbb{R}^3} (s - \xi^2)^2 \int_{Q_{r\xi}(s, \xi s)} |\nabla v(\tau, x)|^2 \, dx \, d\tau \leq \frac{2}{s - \xi^2} \int_{\mathbb{R}^3} |\nabla v(\tau, x)|^2 \, dx \, d\tau.
\]

Using this and (4.26) is clear that the quantity on the left hand side converges to zero as \( r \to 0 \). On the other hand, we already know that
\[
\limsup_{r \to 0} \frac{1}{r} \int_{\mathbb{R}^3} (s - \xi^2)^2 \int_{Q_{r\xi}(s, \xi s)} |\nabla w(\tau, x)|^2 \, dx \, d\tau < \varepsilon^*,
\]
since this is one of the requirement to be a generalized reference solution. Thus \( u = v + w \) satisfies the regularity condition (3.7) at any point \((s, \xi s) \in L(t, \xi)\) with \((t, \xi)\) satisfying (4.25). This implies, by Lemma 3.1, the regularity of \( L(t, \xi) \).

### 4.4. Conclusion of the proof

Summing up we have shown that there exists a constant \( Z \geq 1 \), that only depends on \((r, q)\), such that the following holds: if \( \varepsilon \) is sufficiently small enough to satisfy (4.23), then the segment \( L(t, \xi) \) is a regular set for \( u \), for any \( \xi \in \mathbb{R}^3 \) and \( t > 0 \) such that (4.25) holds. If we set
\[
\delta_0 = \frac{1}{30Z^2},
\]
then (4.23) follows by
\[
\varepsilon \leq \delta_0 e^{-K/\delta_0}
\]
and (4.25) follows by \(|\xi|^2 t < \delta_0\), which is equivalent to \( t > \frac{|\xi|^2}{\delta_0} \), namely
\[
(t, t\xi) \in \Pi_{\delta_0}, \quad \Pi_{\delta_0} := \left\{(t, x) \in (0, \infty) \times \mathbb{R}^3 : t > \frac{|\xi|^2}{\delta_0}\right\}.
\]
Thus \( L(t, t\xi) \) is regular provided that \( \varepsilon \) satisfies (4.28) and \((t, t\xi) \in \Pi_{\delta_0}\). As a consequence, we conclude that the paraboloid \( \Pi_{\delta_0} \), that is the union of these segments for arbitrary \( t > 0 \) and \( \xi \in \mathbb{R}^3 \), is a regular set for \( u \) provided that (4.28), namely our smallness assumption (4.4), holds. This concludes the proof.

□

Here we prove that reference solutions are generalized reference solutions.

**Lemma 4.2.** If \( w \) is a reference solution of size \( K \) to the Navier–Stokes equation (see Definition 1.4), then it is also a generalized reference solution of size \( K \) (see Definition 4.1).
Proof. Recalling Remark 3.1 and (3.5), we only need to show that the regularity condition (3.7) is satisfied at any \((t, x) \in (0, \infty) \times \mathbb{R}^3\). Recalling the argument at the end of the previous proof (after the inequality (4.26)) it suffices to prove
\[
\int_0^t \int_{\mathbb{R}^3} \left| \nabla w(\tau, x) \right|^2 \frac{dx}{|x - \xi \tau|} \, d\tau < \infty, \quad \text{for all } t > 0 \text{ and } \xi \in \mathbb{R}^3 \tag{4.29}
\]
By translation invariance we can assume \(x' = 0\), namely \(w_0 \in L^2(\mathbb{R}^3) \cap \mathcal{L}^2(|x|^{-1} \, dx)\). We change variables
\[
(t, y) = (t, x - \xi t), \quad w_\xi(t, y) = w(t, x),
\]
so that the local energy inequality becomes
\[
\int_{\mathbb{R}^3} \left| w_\xi \right|^2 \tilde{\phi} \, dy + 2 \int_0^t \int_{\mathbb{R}^3} \left| \nabla w_\xi \right|^2 \phi \leq \int_{\mathbb{R}^3} |w_0|^2 \phi(0, y) \, dy + \int_0^t \int_{\mathbb{R}^3} \left| w_\xi \right|^2 \left( \phi_t - \xi \cdot \nabla \phi + \Delta \phi \right) + (|w_\xi|^2 + 2P_{w_\xi}) w_\xi \cdot \nabla \phi, \tag{4.30}
\]
where \(P_{w_\xi} = \mathcal{R} \otimes \mathcal{R} \cdot (w_\xi \otimes w_\xi)\). We choose \(\phi(t, y) := \sigma_\mu(y)(\chi(0)|y|)\) where \(\sigma_\mu\) and \(\chi\) are as in the second step of the previous proof. Exactly as before, taking the limit \(\delta \to 0\) and, using the inequalities (4.12), (4.30) becomes
\[
\int_{\mathbb{R}^3} \sigma_\mu(y)\left| w_\xi \right|^2 \, dy \leq \left| \mathcal{R} \otimes \mathcal{R} \cdot (w_\xi \otimes w_\xi) \right| \int_{\mathbb{R}^3} \sigma_\mu(y) \left| \nabla w_\xi \right|^2 \, dy + \int_{\mathbb{R}^3} \left| w_\xi \right|^2 + \int_{\mathbb{R}^3} \left( (|w_\xi|^2 + 2P_{w_\xi}) |w_\xi| \right). \tag{4.31}
\]
We desire an integral inequality for the functions
\[
a_\mu(t) = \int_{\mathbb{R}^3} \sigma_\mu(y)\left| w_\xi \right|^2 \, dy, \quad B_\mu(t) = \int_0^t \int_{\mathbb{R}^3} \sigma_\mu(y) \left| \nabla w_\xi(\tau, y) \right|^2 \, d\tau, d\tau.
\]
To bound the pressure term we use the SS and CKN inequalities, the last one with \(29 = 1 + 3/q\), that implies \(r = 1/(1 - \theta)\), so that
\[
2 \int_{\mathbb{R}^3} \sigma_\mu^2 |P_{w_\xi}||w_\xi| = 2 \int_{\mathbb{R}^3} \sigma_\mu^2 |w_\xi||\mathcal{R} \otimes \mathcal{R} \cdot (w_\xi \otimes w_\xi)| \leq \|\sigma_\mu \mathcal{R} \otimes \mathcal{R} \cdot (w_\xi \otimes w_\xi)\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \|\sigma_\mu w_\xi\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \leq \|w_\xi\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \|\sigma_\mu w_\xi\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \leq \left\|w_\xi\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \left\|\sigma_\mu^2 \mathcal{R} \otimes \mathcal{R} \cdot (w_\xi \otimes w_\xi)\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \leq \left\|w_\xi\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \left\|\sigma_\mu^2 \mathcal{R} \otimes \mathcal{R} \cdot (w_\xi \otimes w_\xi)\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \leq \frac{1}{\delta} \hat{B}_\mu + C \left\|w_\xi\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} a_\mu. \tag{4.32}
\]
In a similar way, using CKN with \(29 = 1 + 3/q\)
\[
\int_{\mathbb{R}^3} \sigma_\mu^2 |w_\xi|^3 \leq \left\|w_\xi\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \left\|\sigma_\mu^2 \mathcal{R} \otimes \mathcal{R} \cdot (w_\xi \otimes w_\xi)\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \leq \left\|w_\xi\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \left\|\sigma_\mu^2 \mathcal{R} \otimes \mathcal{R} \cdot (w_\xi \otimes w_\xi)\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \leq \frac{1}{\delta} \hat{B}_\mu + C \left\|w_\xi\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} a_\mu, \tag{4.33}
\]
and, again by CKN,
\[
\left\|w_\xi\right\|^3 |w_\xi|^2 \leq \left\|w_\xi\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \left\|\sigma_\mu \mathcal{R} \otimes \mathcal{R} \cdot (w_\xi \otimes w_\xi)\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \left\|\sigma_\mu^2 \mathcal{R} \otimes \mathcal{R} \cdot (w_\xi \otimes w_\xi)\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} = \left\|w_\xi\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \left\|\sigma_\mu \mathcal{R} \otimes \mathcal{R} \cdot (w_\xi \otimes w_\xi)\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} \leq \frac{1}{\delta} \hat{B}_\mu + C \left\|w_\xi\right\|_{2\mathcal{L}^{2/3}(\mathbb{R}^3)} a_\mu.
\]
Using these, the inequality (4.31) becomes
\[
\alpha_\mu(t) + B_\mu(t) \leq a_\mu(0) + C \int_0^t \left( |\xi|^2 + 3 |w_\xi(\tau, \cdot)|_{2\mathcal{L}^2(\mathbb{R}^3)} \right) a_\mu(\tau) \, d\tau,
\]
for some \(C > 0\).
Letting \(a(t) := \int_{\mathbb{R}^3} |y|^{-1} |w_\xi(t, y)|^2 \, dy\) and using \(a_\mu(0) \leq a(0)\), we arrive to
\[
\alpha_\mu(t) + B_\mu(t) \leq a(0) + C \int_0^t \left( |\xi|^2 + |w_\xi(\tau, \cdot)|_{2\mathcal{L}^2(\mathbb{R}^3)} \right) a_\mu(\tau) \, d\tau. \tag{4.35}
\]
Since \( a(0) = \| x^{-1/2} w_0 \|_{L^2_\delta}^2 \) and \( \| w_\xi \|_{L^r_t L^q_x} \equiv \| w \|_{L^r_t L^q_x} =: \mathcal{K} \), the Grönwall inequality gives
\[
a_\mu(t) < a(0) e^{C(t(\xi^2 + \mathcal{K}))} \quad \text{for all} \quad t > 0.
\]
Plugging this into the right hand side of (4.35), since \( \| w_\xi \|_{L^r_t L^q_x} \equiv \mathcal{K} \), we find out that \( B_\mu(t) \) is bounded, for any time \( t > 0 \), uniformly in \( \mu \). Thus, since the weights \( \sigma_\mu(y) \) are increasing as \( \mu \to 0 \), we can pass to the limit
\[
\lim_{\mu \to 0} B_\mu(t) = \int_0^t \int_{\mathbb{R}^3} |y|^{-1} |\nabla w_\xi(\tau, y)|^2 \, dy \, d\tau < \infty, \quad \text{for all} \quad t > 0.
\]
In particular, coming back to the \((t, x)\) variables, we see that (4.29) is satisfied. This concludes the proof.

\[\square\]

5. Proof of Propositions 2.1 - 2.4

**Proof of Proposition 2.1.** Let \( w \) be the solution to the Navier–Stokes equation constructed in [37], for the zero swirl initial data \( w_0 \). This solution satisfies the energy inequality (3.2), in particular it belongs to \( L^\infty_t L^2_\delta \cap L^2_t H^1_\delta \), and moreover
\[
\| w \|_{L^\infty_t H^1_\delta} \lesssim 1 + \| w_0 \|^{5/3}_{H^2_\delta}.
\]
Indeed, as we will show at the end of the proof, this can be deduced following the proof of Lemma 5 in [37]. Thus \( w \in L^4_t H^1_\delta \), by interpolation, and \( w \in L^4_t L^6_\delta \), by Sobolev embedding. More precisely
\[
\| w \|_{L^4_t L^6_\delta} \lesssim 1 + \| w_0 \|^{4/3}_{H^2_\delta}.
\]
Since \((r, q) = (4, 6)\) is an admissible couple, we have shown that \( w \) is a reference solution of size \( \mathcal{K} \leq C(1 + \| w_0 \|^{16/3}_{H^2_\delta}) \), for some absolute constant \( C > 1 \); see Definition 1.4. The statement then follows by Theorem 1.2, taking \( \delta_2 = C^{-1} \delta_0 \).

It remains to prove (5.1). First of all we notice that, since \( \nabla \cdot w = 0 \), we have, by orthogonality,
\[
\| \nabla w \|_{L^2_\delta} = \| \nabla \times w \|_{L^2_\delta}.
\]
Thus, we can equivalently show
\[
\| \nabla \times w \|_{L^4_t L^6_\delta} \lesssim 1 + \| w_0 \|^{5/3}_{H^2_\delta}.
\]
Following the proof of Lemma 5 in [37], we have
\[
\begin{align*}
\| \nabla \times w \|_{L^2_\delta}^2(t) &+ k \int_0^t \| \nabla^2 w \|_{L^2_\delta}(s) \, ds \\
&\leq \| \nabla \times w_0 \|_{L^2_\delta}^2 + \int_0^t \| w \|_{L^2_\delta} \left\| \frac{\nabla \times w}{r} \right\|_{L^2_\delta} \| \nabla \times w \|_{L^2_\delta}(s) \, ds,
\end{align*}
\]
for a certain constant \( k > 0 \). We recall that \( r \) is the radial variables in a cylindrical polar coordinate system; see (2.1). Thus, using the inequality \( \| w \|_{L^\infty(\mathbb{R}^3)} \lesssim \| \nabla w \|^{1/2}_{L^2(\mathbb{R}^3)} \| \nabla^2 w \|^{1/2}_{L^2(\mathbb{R}^3)} \), (5.3), and the Young inequality,
\[
\| \nabla \times w \|_{L^2_\delta}^2(t) + k \int_0^t \| \nabla^2 w \|_{L^2_\delta}^2(s) \, ds \\
\leq \| \nabla \times w_0 \|_{L^2_\delta}^2 + \int_0^t \| \nabla^2 w \|_{L^2_\delta}^{1/2} \left\| \frac{\nabla \times w}{r} \right\|_{L^2_\delta} \| \nabla w \|_{L^2_\delta}^{1/2}(s) \, ds,
\]
\[
\leq \| \nabla \times w_0 \|_{L^2_\delta}^2 + \frac{k}{2} \int_0^t \| \nabla^2 w \|_{L^2_\delta}^2(s) \, ds + C \left( \| \nabla \times w \|_{L^2_t L^6_\delta}^4 \int_0^t \| \nabla w \|_{L^2_\delta}^2(s) \, ds \right),
\]
for some $C$ that only depends on $k$. Since $\int_0^1 \| \nabla w \|^2_{L^2_2}(s) \, ds \leq \| w_0 \|^2_{L^2_2}$ and, by [54, Lemma 1.2], [37, Lemma 3(ii)],
\[ \left\| \frac{\nabla \cdot w}{r} \right\|_{L^\infty_t L^2_3} \leq \left\| \frac{\nabla \cdot w_0}{r} \right\|_{L^2_3} \lesssim \| w_0 \|_{L^2_3}, \]
(5.6) implies (5.4) and the proof is concluded.

\[ \square \]

**Proof of Proposition 2.2.** Recalling Theorem 1 in [10], given $\sigma$ sufficiently small in the assumption (2.2), there exists a unique solution $w$ to the Navier–Stokes equation with initial data $w_0$. As the authors point out, in order to prove this fact it is actually sufficient assume $w_0 \in \dot{B}^{-1}_{\infty,2}(\mathbb{R}^3)$. On the other hand, the extra assumption $w_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$ ensures that $w$ belongs to $\in \dot{C}([0, \infty); \dot{H}^{1/2})$, via a standard propagation of regularity argument; see [38, Theorem 18.3]. This also implies $w \in L^2_t \dot{H}^{3/2}$; see [25, Theorem 2]. Thus, by interpolation, we see that $w$ belongs to $L^r_t \dot{H}^s$ provided $r \geq 2$ and $s = 1/2 + 2/r$ and, by Sobolev embedding, we have $w \in L^r_t L^s$ for every admissible couple $(r, q)$ provided $q \neq \infty$. (we recall that $(r, q)$ is admissible when $2/r + 3/q = 1$). Since we have required $w_0 \in L^2(\mathbb{R}^3)$, we also have $w \in \dot{C}([0, \infty); L^2(\mathbb{R}^3))$, again by the propagation of regularity argument. In conclusion, $w$ is a reference solution of size $\mathcal{K} = \| w \|_{L^r_t L^s_t}$, see Definition 1.4, and the statement follows by Theorem 1.2.

\[ \square \]

**Proof of Proposition 2.3.** The statement follows combining the forthcoming Propositions 5.1 and 5.3.

**Proposition 5.1.** There exists a constant $\delta_0 > 0$ such that the following holds. Let $W_0$ be a 2D divergence free vector field which belongs to $L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and let $W$ be the (unique) solution to the 2D Navier–Stokes equation with initial data $W_0$. The set $\Pi_{k,s}$ is regular for any suitable weak solution $u$ to the 3D Navier–Stokes equation, with divergence free initial data $u_0 \in L^2_{\text{loc}}(\mathbb{R}^3)$, such that:
\begin{enumerate}
  \item $u_0 - \overline{W_0} \in L^2(\mathbb{R}^3)$;
  \item $u - \overline{W} \in L^r_t L^2(\mathbb{R}^3) \cap L^2_t \dot{H}^1(\mathbb{R}^3)$ is a suitable weak solution to the perturbed 3D Navier–Stokes equation (4.1), around the solution $\overline{W}$, with data $u_0 - \overline{W_0}$; and
\end{enumerate}
\[ \| \varphi - \overline{x} \|_{L^2(\mathbb{R}^3)}^{-1/2} (u_0 - \overline{W_0}) \|_{L^2(\mathbb{R}^3)} \leq \delta_0 e^{-\delta_0^{-1}\| w \|_{L^2_{\text{loc}}(\mathbb{R}^3)}}. \]
It is not immediately clear that suitable weak solutions which satisfy the condition (2) of the statement actually exist. In Proposition 5.3 we will prove that this is indeed the case, for any initial datum $u_0 \in L^2_{\text{loc}}(\mathbb{R}^3)$, as long as (1) is satisfied and $\overline{W} \in L^2_t L^\infty$; see (5.7). Thus, this Proposition has to be considered the completion of Proposition 5.1 and, for the same reason, of the forthcoming Proposition 5.2.

**Proof of Proposition 5.1.** The 2D solution $W$ belongs to $L^\infty_t L^2(\mathbb{R}^2) \cap L^2_t \dot{H}^1(\mathbb{R}^2)$. This is a well know fact, indeed it is unique in the class of solutions which satisfies the 2D energy inequality, and thus it is, in particular, suitable. Moreover, as observed in the proof of Theorem 4 in [46], $W$ also belongs to $L^2_t L^\infty(\mathbb{R}^2)$. By the definition of $\overline{W}$, we immediately have
\[ \overline{W} \in L^2_t L^\infty(\mathbb{R}^3) \cap L^\infty_t L^2(K) \cap L^2_t \dot{H}^1(K), \]
(5.7) for any compact set $K \subset \mathbb{R}^3$. Since $(r, q) = (2, \infty)$ is admissible (namely $2/r + 3/q = 1$), in order to show that $\overline{W}$ is a generalized reference solution (see Definition 4.1), we need to check that it is suitable, that satisfies the regularity condition (3.7) at
any \((t,x) \in (0, \infty) \times \mathbb{R}^3\), that \(t \to \widetilde{W}(t) \in C_0((0, \infty); L^2(K))\), \(K \subset \mathbb{R}^3\) compact, and that the representation formula for the pressure is satisfied. The suitability follows straightforwardly by that of \(W\). We omit the obvious details. We only notice that given any \(\phi \in C_0^\infty((0, \infty) \times \mathbb{R}^3)\), we can define a family of 2D functions \(\phi_x(s, x_2, x_3) := \phi(x_1, x_2, x_3)\), so that \(W\) solves the 2D Navier–Stokes equation, in the weak sense, when we test against \(\phi_x(x_1, x_2)\), and similarly, given \(0 \leq \phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)\), the local energy inequality

\[
\int_{\mathbb{R}^3} |W|^2 \phi_{x_3}(t) + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla W|^2 \phi_{x_3} \leq \int_{\mathbb{R}^3} |W_0|^2 \phi_{x_3}(0) + \int_0^t \int_{\mathbb{R}^3} |W|^2 (\partial_t \phi_{x_3} + \Delta \phi_{x_3}) + \int_0^t \int_{\mathbb{R}^3} (|W|^2 + 2P_W) W \cdot \nabla \phi_{x_3},
\]

holds, where the space integration is over \((x_1, x_2) \in \mathbb{R}^2\). Thus, integrating over \(x_3 \in \mathbb{R}\), we see that \(\tilde{W}\) satisfies the weak 3D equation, letting \(P_{\tilde{W}} = \tilde{P}_W\), and

\[
\int_{\mathbb{R}^3} |\tilde{W}|^2 \phi(t) + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla \tilde{W}|^2 \phi \leq \int_{\mathbb{R}^3} |\tilde{W}_0|^2 \phi(0) + \int_0^t \int_{\mathbb{R}^3} |\tilde{W}|^2 (\partial_t \phi + \Delta \phi) + \int_0^t \int_{\mathbb{R}^3} (|\tilde{W}|^2 + 2P_{\tilde{W}}) \tilde{W} \cdot \nabla \phi;
\]

here we have used \(\tilde{W}_3 = (\tilde{W}_0)_3 = 0, \partial_t \tilde{W}_j = 0\) if \(i = 3\) or if \(j = 3\), and \(\partial_t P_{\tilde{W}} = 0\). Similarly, since \(W \in C_0((0, \infty); L^2(\mathbb{R}^3))\) (see [38, Proposition 14.3]), integrating with respect to \(x_3\), we see that \(t \to \tilde{W}(t) \in C_0((0, \infty); L^2(K))\), for any compact set \(K \subset \mathbb{R}^3\). Moreover, since \(P_W = \sum_{i,j=1}^3 \mathcal{R}_i \mathcal{R}_j \tilde{W}_i \tilde{W}_j\) and \(W_3 = 0\), the representation formula for the 3D pressure \(P_{\tilde{W}} = \sum_{i,j=1}^3 \mathcal{R}_i \mathcal{R}_j \tilde{W}_i \tilde{W}_j\) is valid. Indeed, letting \(F\) the Fourier transform, we have

\[
F(\sum_{i,j=1}^3 \mathcal{R}_i \mathcal{R}_j \tilde{W}_i \tilde{W}_j) = - \sum_{i,j=1}^3 \frac{\xi_i \xi_j}{\xi_i^2 + \xi_j^2 + \xi_3^2} F(\tilde{W}_i \tilde{W}_j) = - \sum_{i,j=1}^3 \frac{\xi_i \xi_j}{\xi_i^2 + \xi_j^2 + \xi_3^2} F(\tilde{W}_i \tilde{W}_j)
\]

\[
= - \delta_{\xi_3=0} \sum_{i,j=1}^2 \frac{\xi_i \xi_j}{\xi_i^2 + \xi_j^2} F(\tilde{W}_i \tilde{W}_j) = \delta_{\xi_3=0} F \sum_{i,j=1}^3 \mathcal{R}_i \mathcal{R}_j \tilde{W}_i \tilde{W}_j
\]

\[
= \delta_{\xi_3=0} F(P_W) = F(P_{\tilde{W}}) = F(P_{\tilde{W}}).
\]

We remark that there is no trouble to make sense of \(\mathcal{R}_i \mathcal{R}_j \tilde{W}_i \tilde{W}_j\) as long as \(\tilde{W}_i \tilde{W}_j(t)\) is bounded, see for instance [29, Remark 8.1.18], and so for almost any \(t \in (0, \infty)\), since \(\tilde{W}\) belongs to \(L^2 L^\infty(\mathbb{R}^3)\).

Finally, by definition of \(\tilde{W}\), one see that

\[
\int_{B(x,r)} |\nabla \tilde{W}|^2 \leq 2r \int_{\mathbb{R}^2} |\nabla W|^2,
\]

so that

\[
\frac{1}{r} \int \int_{Q_r(t,x)} |\nabla \tilde{W}|^2 \leq \int_{t-r^2/8}^{t+r^2/8} \int_{\mathbb{R}^2} |\nabla W|^2(s,x_1,x_2)dx_1dx_2ds \rightarrow 0 \quad \text{as} \quad r \rightarrow 0,
\]

because of the (space-time) integrability of \(|\nabla W|^2|.

In conclusion, we have shown that \(\tilde{W}\) is a generalized reference solution of size \(K = ||W||^2_{L^2 L^\infty(\mathbb{R}^2)}\), so that the statement follows by Theorem 4.1. 

\[\Box\]

5.1. Proof of Proposition 2.4. The statement follows combining the forthcoming Propositions 5.2 and 5.3. Notice that, given a Beltrami field \(u_0 \in L^\infty(\mathbb{R}^3)\), the family of rescaled Beltrami fields \(t \to e^{-t\lambda^2} u_0\) solves the Navier–Stokes equation and belongs to \(L^2_t L^\infty_x\); see Subsection 2.4 and (5.10).
Proposition 5.2. There exists a constant δ₁ > 0 such that the following holds. Let \( w_0 \in L^\infty(\mathbb{R}^3) \) such that \( \nabla \times w_0 = \lambda w_0 \) for some \( \lambda \neq 0 \). The set \( \Pi_{3,0} \) is regular for any suitable weak solution \( u \) to the Navier–Stokes equation, with divergence free initial data \( u_0 \in L^2_{loc} \), such that:

1. \( u_0 - w_0 \in L^2 \);
2. \( u - e^{-t\lambda^2}w_0 \in L^\infty_t L^2_x \cap L^2_t H^1_x \) is a suitable weak solution to the perturbed Navier–Stokes equation 4.1, around the solution \( e^{-t\lambda^2}w_0 \), with initial data \( u_0 - w_0 \);

and

\[
\| |x - \tilde{x}|^{-1/2}(u_0 - w_0)\|_{L^2} \leq \delta_1 e^{-\tilde{x}^2/\lambda^2}\|w_0\|^2_{L^\infty_x}.
\]

Proof of Proposition 5.2. Recalling Subsection 2.4, since \( w_0 \) is analytic (it is a solution of \( \Delta w_0 = -\lambda^2 w_0 \)), it is clear that \( w(t, x) := e^{-t\lambda^2}w_0 \) is a suitable weak solution to the Navier–Stokes equation that also satisfies the regularity condition (3.7) at any \( (t, x) \in (0, \infty) \times \mathbb{R}^3 \). Moreover, it is clear that \( t \in [0, \infty) \to w(t) \in L^2(K) \) is continuous for any compact \( K \subset \mathbb{R}^3 \) and, since the pressure can be chosen to be \( P = -\frac{1}{2}|w(t, x)|^2 \), the representation formula required by Definition 4.1 trivially holds taking \( T = -\frac{1}{2}Id \). Finally

\[
\| w \|^2_{L^\infty_t L^2_x} = \int_0^\infty e^{-2\lambda^2 s}\|w_0\|^2_{L^2_x} \, ds = \frac{1}{2}\lambda^{-2}\|w_0\|^2_{L^\infty_x},
\]

so that \( w \) is a generalized reference solution of size \( \mathcal{K} = \frac{1}{2}\lambda^{-2}\|w_0\|^2_{L^\infty_x} \), and the statement follows by Theorem 4.1, taking \( \delta_1 = 2\delta_0 \).

\[\square\]

5.2. Weak solutions with unbounded energy. In the following statement we assume the initial data \( w_0 \) to be only locally square integrable, but, since the difference \( u_0 - w_0 \) belongs to \( L^2 \), with \( w_0 \) the initial datum of a generalized reference solution \( w \in L^2_t L^\infty_x \), we are still able to prove the existence of suitable weak solutions.

Proposition 5.3. Let \( w \in L^2_t L^\infty_x \) be a generalized reference solution to the Navier–Stokes equation (see Definition 4.1), with divergence free data \( w_0 \in L^2_{loc} \). For any divergence free initial datum \( u_0 \) such that \( u_0 - w_0 \in L^2 \), there exists a suitable weak solution \( u \) to the Navier–Stokes equation, such that \( v := u - w \in L^\infty_t L^2_x \cap L^2_t H^1_x \) is a suitable weak solution to the perturbed Navier–Stokes equation (4.1), around \( w \), with data \( v_0 := u_0 - w_0 \).

Proof of Proposition 5.3. Once we have found \( v \), it is straightforward to check that the full statement follows taking \( u := w + v \) and \( P_u := P_w + P_v \); see Lemma 7.2 for more details. In order to prove the existence of \( v \), we first consider, as usual, the family of mollified problems \( 0 < \varepsilon < 1 \):

\[
\begin{aligned}
\partial_t v^{(\varepsilon)} + ((v^{(\varepsilon)} \ast \rho^{\varepsilon}) \cdot \nabla) v^{(\varepsilon)} + (v^{(\varepsilon)} \cdot \nabla) w + (w \cdot \nabla) v^{(\varepsilon)} - \Delta v^{(\varepsilon)} &= -\nabla P_{v^{(\varepsilon)}}, \\
v^{(\varepsilon)}(0) &= v_0,
\end{aligned}
\]

where \( 0 \leq \rho \in C_c^\infty(\mathbb{R}^3) \), \( \int_{\mathbb{R}^3} \rho = 1 \), \( \rho^\varepsilon(x) := \varepsilon^{-3}\rho(\varepsilon^{-1} x) \). Since all the vector fields involved are divergence free, the first equation can be rewritten as

\[
\partial_t v^{(\varepsilon)} = \nabla \cdot [(v^{(\varepsilon)} \ast \rho^{\varepsilon}) \otimes v^{(\varepsilon)}] - v^{(\varepsilon)} \otimes w - w \otimes v^{(\varepsilon)} - P_{v^{(\varepsilon)}} Id + \nabla v^{(\varepsilon)}.
\]

A standard fixed point argument allows to find, for any \( 0 < \varepsilon < 1 \), a unique solution \( v^{(\varepsilon)} \) in \( C_b([0, T); L^\infty_t L^2_x) \cap L^2_t H^1_x \) such that, for all \( t < T \):

\[
\int_{\mathbb{R}^3} |v^{(\varepsilon)}(t)|^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla v^{(\varepsilon)}|^2 \leq \int_{\mathbb{R}^3} |v_0|^2 + 2 \int_0^t \int_{\mathbb{R}^3} (v^{(\varepsilon)} \cdot \nabla) v^{(\varepsilon)},
\]
we write $L^r_T$ when the time integration is restricted to $t \in (0,T)$. The local existence time $T$ depends, in principle, on $\|v_0\|_{L^2}$, but it can be taken arbitrarily large via a standard continuation argument. Indeed, using Hölder’s and Young’s inequalities, we can estimate the last term of (5.13) as

$$2 \int_0^T \int_{\mathbb{R}^3} (v(\varepsilon) \cdot \nabla) v(\varepsilon) \cdot w \leq \|\nabla v(\varepsilon)\|_{L^2_T L^2_x}^2 + \|v(\varepsilon)\|_{L^2_T L^2_x}^2 \leq \int_0^T \int_{\mathbb{R}^3} |\nabla v(\varepsilon)|^2 + \int_0^T \|w\|_{L^2_x}^2 (s) \left( \int_{\mathbb{R}^3} |v(\varepsilon)(s,x)|^2 dx \right) ds,$$

plugging this into the (5.13) and absorbing the term $\int_0^T \int_{\mathbb{R}^3} |\nabla v(\varepsilon)|^2$ into the left hand side, we obtain

$$\int_{\mathbb{R}^3} |v(\varepsilon)(t)|^2 + \int_0^T \int_{\mathbb{R}^3} |\nabla v(\varepsilon)|^2 \leq \int_{\mathbb{R}^3} |v_0|^2 + \int_0^T \|w\|_{L^\infty_x}^2 (s) \left( \int_{\mathbb{R}^3} |v(\varepsilon)(s,x)|^2 dx \right) ds.$$

Thus, since $\|w\|_{L^\infty_x}^2 (t)$ is integrable by assumption, the Grönwall’s inequality gives

$$\int_{\mathbb{R}^3} |v(\varepsilon)(t,x)|^2 dx \leq Ae^K, \quad \int_0^T \int_{\mathbb{R}^3} |\nabla v(\varepsilon)(s,x)|^2 dx ds \leq A(1 + Ke^K),$$

for all $t < T$, where $A := \|v_0\|_{L^2}^2$ and $K := \|w\|_{L^\infty_x}^2$. Then, the continuation argument allows to extend the local theory to any $T > 0$, in such a way that

$$\sup_{t \geq 0} \int_{\mathbb{R}^3} |v(\varepsilon)(t,x)|^2 dx \leq A e^K, \quad \int_0^T \int_{\mathbb{R}^3} |\nabla v(\varepsilon)(s,x)|^2 dx ds \leq A(1 + Ke^K).$$

Since these bounds hold uniformly in $0 < \varepsilon < 1$, we can extract, for any $T > 0$, a subsequence, that with a little abuse of notations we still denote $v^{(\varepsilon)}$, that converges weakly to some $v$ in $L^2((0,T) \times \mathbb{R}^3)$, and such that $\nabla v^{(\varepsilon)}$ converges to $\nabla v$ in $L^2((0,\infty) \times \mathbb{R}^3)$, as $\varepsilon \to 0$. In the following we will repeat the same abuse a few times. By the weak lower semicontinuity of the norm and (5.15), we also have that $v$ belongs to $L^2_t H^{1/2}_x$. Now, it is well known that in the context of the Navier–Stokes equation the weak convergence of $v^{(\varepsilon)}$ to $v$ can be promoted to strong convergence in $L^2_t H^{1/2}_x$, for any $T > 0$. This is also the case here, by a straightforward adaptation of the standard argument, that we will recall at the end of the proof. This also implies that, on a certain subsequence, we have

$$\|v^{(\varepsilon)}(t) - v(t)\|_{L^2(\mathbb{R}^3)} \to 0 \quad \text{for almost every } t \in (0,\infty).$$

Using this and the first inequality in (5.15), we immediately see that $v \in L^\infty_t L^2_x$.

Since we already observed that $v \in L^2_t\dot{H}^{1/2}_x$, using again (5.15), Sobolev embedding and interpolation, we see that, for all $T > 0$, the vector fields $v, v^{(\varepsilon)}$ and $\rho^{(\varepsilon)}$ are $\varepsilon$-uniformly bounded in $L^2_t L^4_x$, provided $2/\tilde{p} + 3/\tilde{q} \geq 3/2, 2 \leq \tilde{q} \leq 6$; see (3.4). Using this fact, triangle inequality and interpolation, the $L^2_t L^4_x$ (strong) convergence of $v^{(\varepsilon)}$ (and of $v^{(\varepsilon)} \ast \rho^{(\varepsilon)}$) can be promoted to (strong) convergence in $L^2_t L^6_x$, provided $2/\tilde{p} + 3/\tilde{q} > 3/2, 2 < \tilde{q} < 6$, for any $T > 0$. Now we consider the pressure. Taking the divergence of the first equation in (5.11), we have

$$\Delta P_{v^{(\varepsilon)}} = -\nabla \cdot \nabla \cdot \left[ (v^{(\varepsilon)} \ast \rho^{(\varepsilon)}) \otimes v^{(\varepsilon)} + 2v^{(\varepsilon)} \otimes w \right],$$

or equivalently (we recall that $\mathcal{R}$ is the Riesz transform)

$$P_{v^{(\varepsilon)}} = \mathcal{R} \otimes \mathcal{R} \cdot \left[ (v^{(\varepsilon)} \ast \rho^{(\varepsilon)}) \otimes v^{(\varepsilon)} + 2v^{(\varepsilon)} \otimes w \right].$$

(5.17)

Since we have also assumed $w \in L^2_t L^\infty_x$, using the $L^{q \geq 1}$ boundedness of the Riesz transform, we have that $P_{v^{(\varepsilon)}}$ is $\varepsilon$-uniformly bounded in $L^2_t L^2_x$, provided $2/\tilde{p} + 3/\tilde{q} \geq 3, 2 \leq \tilde{q} \leq 3$; comparing to (3.6), the restriction $2 \leq \tilde{q}$ arises since we also need to estimate the contribution of $w$ to the pressure. Thus we can extract a subsequence, that we still denote with $P_{v^{(\varepsilon)}}$, which converges weakly to a function $P_v$ in $L^2_t L^4_x$, $2/\tilde{p} + 3/\tilde{q} \geq 3, 2 \leq \tilde{q} \leq 3$. Then the limit $P_v$ has to coincide with

$$P_v := \mathcal{R} \otimes \mathcal{R} \cdot (v \otimes v + 2v \otimes w).$$

(5.18)
We are now ready to show that the couple \((v, P_\varepsilon)\) is a suitable weak solution to the perturbed Navier–Stokes equation \((4.1)\), around \(w\); see Definition \((4.2)\). It is already clear that the equation \((4.1)\) is satisfied in the sense of distribution, that the representation formula \((4.3)\) for the pressure holds taking \(T = \mathcal{R} \otimes \mathcal{R}\), and we already know \(v \in L^2_tL^4_x \cap L^2_tH^1_x\), \(P_\varepsilon \in L^2_{loc}(0, \infty) \times \mathbb{R}^3\). Now we prove the perturbed local energy inequality \((4.2)\). We first consider the case \(t_0 = 0\). For any non negative test function \(\phi\), we take the scalar product of the first equation in \((5.11)\) with \(2\phi v^{(\varepsilon)}\), integrate over \((0, t) \times \mathbb{R}^3\) and by parts, so that

\[
\int_{\mathbb{R}^3} |v^{(\varepsilon)}|^2 \phi(t,x)dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla v^{(\varepsilon)}|^2 \phi = \int_{\mathbb{R}^3} |v_0|^2 \phi(0,x)dx \tag{5.19}
\]

\[
+ \int_0^t \int_{\mathbb{R}^3} |v^{(\varepsilon)}|^2 (\phi_t + \Delta \phi) + |v^{(\varepsilon)}|^2 (\rho^{(\varepsilon)} \cdot \nabla \phi + 2 P_\varepsilon v^{(\varepsilon)} \cdot \nabla \phi)
\]

\[
+ \int_0^t \int_{\mathbb{R}^3} |v^{(\varepsilon)}|^2 w \cdot \nabla \phi + 2 (v^{(\varepsilon)} \cdot w) v^{(\varepsilon)} \cdot \nabla \phi + 2 (v^{(\varepsilon)} \cdot \nabla) v^{(\varepsilon)} \cdot w \phi,
\]

Using this and the integrability and convergence properties of the functions involved, it is now straightforward to pass to the limit \(\varepsilon \to 0\) so that

\[
\int_{\mathbb{R}^3} |v|^2 \phi(t,x)dx + 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} |\nabla v^{(\varepsilon)}|^2 \phi \leq \int_{\mathbb{R}^3} |v_0|^2 \phi(0,x)dx + \int_0^t \int_{\mathbb{R}^3} |v|^2 (\phi_t + \Delta \phi) + |v|^2 w \cdot \nabla \phi + 2 P_\varepsilon v \cdot \nabla \phi
\]

\[
+ \int_0^t \int_{\mathbb{R}^3} |v|^2 (w \cdot \nabla \phi) + 2 (v \cdot w) v \cdot \nabla \phi + 2 (v \cdot \nabla) v \cdot w \phi,
\]

for almost every \(t > 0\). We omit the details of this straightforward fact. We only remark that, in order to handle the term \((v \cdot \nabla w) \cdot w \phi\), we need to notice that by \((5.16)\) end Egoroff’s theorem, we have, for any \(0 < \delta < 1\), \(t > 0\), that

\[
\sup_{v \in \Omega_\varepsilon} \|v^{(\varepsilon)}(s) - v(s)\|_{L^2(\mathbb{R}^3)} < \delta, \text{ for all } \varepsilon \text{ sufficiently small, where } \Omega_\varepsilon \subset (0, t) \text{ with } |(0, t) \setminus \Omega_\varepsilon| < \delta.
\]

We have already proved \((5.20)\), for almost any \(t > 0\), since

\[
\int_0^t \int_{\mathbb{R}^3} \phi |\nabla v|^2 \leq \lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{R}^3} \phi |\nabla v^{(\varepsilon)}|^2,
\]

the \((4.2)\) has been proved for almost every \(t > 0\) and \(t_0 = 0\). In fact, we can extended it to any \(t > 0\) once we modify \(v\) on a set of zero Lebesgue measure, in order to make the function \(t \in (0, \infty) \to v(t) \in L^2(\mathbb{R}^3)\) weakly continuous. This is a standard argument in the theory of vector valued time dependent functions, once we notice that \(\partial_t u \in L^1_tH^1_x\), for all \(T > 0\), that is immediately clear if we rewrite the equation as

\[
\partial_t v = \nabla \cdot \left[ -(v \otimes v - v \otimes w - w \otimes v - P_\varepsilon \text{Id} + \nabla v) \right],
\]

as we may taking advantage of the fact that \(v\) and \(w\) are divergence free. Then, the case \(t_0 > 0\) of the inequality can be deduced, for almost any \(t_0 > 0\), by the case \(t_0 = 0\), as explained in the forthcoming Lemma \(7.1\).

Let now \(\Phi\) be a \(C^2_b(\mathbb{R}^3)\) vector field. Recalling the momentum equation \((5.11)\), we have

\[
\int_0^t \int_{\mathbb{R}^3} \left( (v^{(\varepsilon)}(t) \cdot \Phi) dt = \int_0^t \int_{\mathbb{R}^3} \left( (v^{(\varepsilon)}(t) \cdot \rho^{(\varepsilon)} \otimes v^{(\varepsilon)}(t) + v^{(\varepsilon)}(t) \otimes w + w \otimes v^{(\varepsilon)}(t) + P_\varepsilon v^{(\varepsilon)}(t)) \cdot \nabla \Phi + v^{(\varepsilon)}(t) \cdot \Delta \Phi \right) ds,
\]

for all \(t' > 0\). Thus, recalling the \((\varepsilon\text{-uniform})\) integrability properties of \(v^{(\varepsilon)}\), \(P_\varepsilon v^{(\varepsilon)}\), and \(w \in L^2_tL^\infty_x\), we easily arrive to

\[
\lim_{t' \to 0^+} \sup_{0 < \varepsilon < 1} \left| \int_{\mathbb{R}^3} (v^{(\varepsilon)}(t') - v_0) \cdot \Phi \right| = 0 \tag{5.21}
\]

The \(L^2\)-weak convergence \(v(t) \to v_0\) as \(t \to 0^+\) is now a consequence of the following observation. Given any \(0 < \delta < 1\), for any sufficiently small time \(t > 0\), we can find a time \(t'\) at which \((5.16)\) holds and such that \(|t - t'|\) is also sufficiently small that we have these three facts: \(|\int_{\mathbb{R}^3} (v(t') - v(t)) \cdot \Phi| < \delta\), as a consequence of the
L^2\text{-weak continuity of the function } t \in (0, \infty) \rightarrow v(t), \ |\int_{\mathbb{R}^3}(v^{(c)}(t') - v_0) \cdot \Phi| < \delta, \text{ as a consequence of (5.21), and, taking } \varepsilon \text{ sufficiently small, } |\int_{\mathbb{R}^3}(v(t') - v^{(c)}(t')) \cdot \Phi| < \delta, \text{ as a consequence of (5.16). In conclusion, we have shown that } v \text{ is a suitable solution to the perturbed equation (4.1).}

In the proof we have used the strong } L^2((0, T) \times \mathbb{R}^3) \text{ convergence of } v^{(c)} \text{ to } v, \text{ as } \varepsilon \rightarrow 0. \text{ We conclude proving this fact. Using (5.12) and the } (\varepsilon\text{-uniform}) \text{ integrability properties of } v^{(c)}, P(v^{(c)}) \text{ and } w \in L_2^2 L_\infty^2, \text{ we can immediately check that } \sup_{\varepsilon > 0} \|\partial_t v^{(c)}\|_{L_2^2 H_{-1}^2} \text{ is finite. Thus, recalling (5.15), we are allowed to use the Aubin–Lions lemma (see for instance [39, Theorem 12.1]) to extract a subsequence}

\begin{equation}
\tag{5.22}
v^{(c)} \rightarrow v, \text{ strongly in } L^2_{\text{loc}}((0, \infty) \times \mathbb{R}^3).
\end{equation}

Define now } \gamma_{\leq R}(x) := \gamma(x/R), \text{ where } R > 1 \text{ and } \gamma \geq 0 \text{ is a smooth cut-off function of the unit ball in } \mathbb{R}^3. \text{ Let then } \eta_{R,2} := 1 - \gamma_{\leq R} \text{ and split}

\begin{equation}
\tag{5.23}
v^{(c)} - v = \eta_{R,2}^2(v^{(c)} - v) + \eta_{R,2}^3(v^{(c)} - v).
\end{equation}

Using the strong } L^2_{\text{loc}} \text{ convergence (5.22) and}

\begin{equation}
\limsup_{R \rightarrow \infty} \sup_{0 < \varepsilon < 1} \int_0^T \int_{\mathbb{R}^3} \eta_{R,2}^3(x)(v^{(c)} - v)(t, x)^2 dx dt = 0,
\end{equation}

we can immediately deduce strong convergence in } L^2((0, T) \times \mathbb{R}^3). \text{ In order to prove (5.23), since } v \in L^2((0, T) \times \mathbb{R}^3), \text{ it suffices to show}

\begin{equation}
\limsup_{R \rightarrow \infty} \sup_{0 < \varepsilon < 1} \int_0^T \int_{\mathbb{R}^3} \eta_{R,2}^3(x)|v^{(c)}(t, x)|^2 dx dt = 0.
\end{equation}

This will be deduced by an appropriate energy-type inequality. We take the scalar product of the first equation in (5.11) with the vector field } 2\eta_{R,2}^3 v^{(c)}. \text{ After elementary manipulations we obtain}

\begin{equation}
\tag{5.25}
\partial_t |v_{R,2}^{(c)}|^2 + 2|\nabla v_{R,2}^{(c)}|^2 = \nabla \cdot Z_{R,\varepsilon} +
\end{equation}

\begin{equation}
\tag{5.26}
+ \left[|v^{(c)}|^2(v^{(c)} \ast \rho^c) + 2P(v^{(c)}):v^{(c)} + |v^{(c)}|^2 w \right] \cdot \nabla \eta_{R,2}^3 + 2|v^{(c)}|^2 |\nabla \eta_{R,2}^3|^2 +
\end{equation}

\begin{equation}
\tag{5.27}
+ 2\eta_{R,2}^3(v^{(c)} \cdot w)v^{(c)} \cdot \nabla \eta_{R,2}^3 + 2(v^{(c)} \cdot \nabla)v^{(c)} \cdot w,
\end{equation}

where

\begin{equation}
\tag{5.28}
v_{R,2}^{(c)} := \eta_{R,2}^3 v^{(c)},
\end{equation}

and

\begin{equation}
\tag{5.29}
Z_{R,\varepsilon} := \eta_{R,2}^3 \left[ \nabla(|v^{(c)}|^2) - |v^{(c)}|^2(v^{(c)} \ast \rho^c) + 2P(v^{(c)}):v^{(c)} - 2(v^{(c)} \cdot w)v^{(c)} - |v^{(c)}|^2 w \right].
\end{equation}

We refer to [38, Proposition 14.1] for the detailed computation in the case } w = 0. \text{ Since } Z_{R,\varepsilon} \text{ is integrable on } (0, T) \times \mathbb{R}^3, \text{ uniformly in } \varepsilon, \text{ we arrive to } (0 < t < T):

\begin{equation}
\tag{5.30}
\int_{\mathbb{R}^3} v_{R,2}^{(c)}(t)^2 + \int_0^t \int_{\mathbb{R}^3} |v_{R,2}^{(c)}|^2 \leq \int_{\mathbb{R}^3} v_{R,2}(0)^2 +
\end{equation}

\begin{equation}
\tag{5.31}
+ \int_0^t \int_{\mathbb{R}^3} \left[|v^{(c)}|^2(v^{(c)} \ast \rho^c) + 2P(v^{(c)}):v^{(c)} + |v^{(c)}|^2 w \right] \cdot \nabla \eta_{R,2}^3 + 2|v^{(c)}|^2 |\nabla \eta_{R,2}^3|^2 +
\end{equation}

\begin{equation}
\tag{5.32}
+ 2\int_0^t \int_{\mathbb{R}^3} \eta_{R,2}^3(v^{(c)} \cdot w)v^{(c)} \cdot \nabla \eta_{R,2}^3 + (v^{(c)} \cdot \nabla)v^{(c)} \cdot w,
\end{equation}

Finally, since } |\nabla \eta_{R,2}^3|, |\nabla \eta_{R,2}^3| \lesssim R^{-1} \text{ and}

\begin{equation}
\tag{5.33}
2\int_0^t \int_{\mathbb{R}^3} (v_{R,2}^{(c)} \cdot \nabla)^2 v_{R,2}^{(c)} \leq \int_0^t \int_{\mathbb{R}^3} |\nabla v_{R,2}^{(c)}|^2 + \int_0^t |w|^2 \mathcal{P}_{x,s}(\int_{\mathbb{R}^3} |v_{R,2}^{(c)}(s, x)|^2 dx) ds,
\end{equation}

after absorbing } \int_0^t \int_{\mathbb{R}^3} |\nabla v_{R,2}^{(c)}|^2 \text{ into the left hand side of (5.26), we obtain}

\begin{equation}
\tag{5.34}
\int_{\mathbb{R}^3} v_{R,2}^{(c)}(t, x)^2 dx \lesssim_T R^{-1} + \int_{\mathbb{R}^3} v_{R,2}(0, x)^2 dx + \int_0^t |w|^2 \mathcal{P}_{x,s}(\int_{\mathbb{R}^3} |v_{R,2}^{(c)}(s, x)|^2 dx) ds.
\end{equation}
Since \( \|w\|_{L^2_\infty}(t) \) is integrable, the Grönwall inequality gives
\[
\int_{\mathbb{R}^3} |v_{\geq R}|^2(t,x) dx \lesssim_T Ae^K, \quad A := R^{-1} + \|\eta_{\geq R} v_0\|_{L^2}^2, \quad K := \|w\|_{L^2_\infty}^2 (5.27)
\]
for any \( 0 < t < T \). Since the right hand side of (5.27) does not depend on \( \varepsilon \) anymore and goes to zero as \( R \to \infty \), we obtain (5.24), so that the proof is concluded.

\[ \square \]

6. Small data

When \( K \) is sufficiently small, we can improve the size of the regular set of small perturbations of reference solutions.

**Theorem 6.1.** There exists a constant \( \delta_4 > 0 \) such that the following holds. Let \( w \) be a reference solution of size \( K \leq \delta_4 \) to the Navier–Stokes equation with divergence free initial data \( w_0 \) (see Definition 1.4). For any \( M > 1 \), the set
\[
\Pi_{M\delta_4,x} := \left\{ (t,x) : t > \frac{|x|}{M\delta_4} \right\} \quad (6.1)
\]
is regular, for every suitable weak solution \( u \in L^\infty L^2_2 \cap L^2 L^2_3 \) to the Navier–Stokes equation with divergence free initial data \( u_0 \in L^2 \) such that
\[
\| |x|^{-1/2}(u_0 - w_0) \|_{L^2} \leq \delta_4 e^{-M^2/\delta_4}. \quad (6.2)
\]
The size of the regular set (6.1) increases indefinitely as long as we consider smaller perturbations of \( w_0 \). More precisely, if we take a sequence \( w_0^n \) such that
\[
\| |x|^{-1/2}(u_0^n - w_0) \|_{L^2} \to 0 \quad \text{as} \quad n \to \infty,
\]
then we can find a divergent sequence of real numbers \( M_n \) such that (6.2) holds, so that \( \Pi_{M_n\delta_4,x} \) is a regular set for the corresponding weak solutions \( u^n \). Notice that we clearly have \( \Pi_{M_n\delta_4,x} \to \{ t > 0 \} \) as \( n \to \infty \) (since \( M_n \to \infty \)). The case \( K = 0 \) of Theorem 6.1, namely \( w = w_0 = 0 \), has been proved in [12, Corollary 1.6]. For a direct argument we refer to [42].

6.1. Proof of Theorem 6.1.

**Idea of the proof.** Comparing with Theorem 1.2, when we look at the perturbed energy inequality, we have no more trouble with the terms in which the reference solution \( w \) is involved, since \( w \) has been assumed to be small (\( K \ll 1 \)). This gives us enough freedom to improve the size of the regular sets. We again distinguish two time regimes \( t \leq t^*, t > t^* \), but now we choose \( t^* \) in such a way that the term contributing to the size of the regular set becomes very small for \( t > t^* \). Then we use the (exponential) smallness assumption (6.1) on \( u_0 - v_0 \) to control the weighted \( L^2 \) norm of \( u - w \), up to the time \( t^* \). This permits to conclude the proof.

**Proof.** We restrict to \( \bar{x} = 0 \); the general case follows by translation. Let \( v_0 := u_0 - w_0, v := u - w \) and denote \( \varepsilon := \| |x|^{-1/2} v_0 \|_{L^2(\mathbb{R}^3)} \). For all \( \xi \in \mathbb{R}^3 \) and \( T > 1 \) we investigate when
\[
L(T,\xi) := \{ (s,\xi s) : s \in (0,T) \}
\]
is a regular set. We again change variables
\[
(\xi, y) = (t, x - \xi t), \quad v_\xi(t,y) = v(t,x), \quad w_\xi(t,y) = w(t,x), \quad (6.3)
\]
and set
\[
\sigma_\mu(y) := (\mu + |y|^2)^{-\frac{1}{2}}, \quad \mu > 0. \quad (6.4)
\]
We define, for any $M > 1$:

$$
\Gamma(M, T, \xi, \mu) := \left\{ s \in (0, T] : \int_0^{s + T/M} \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(\tau, y)|^2 \, dy \, d\tau > M \right\}
$$

and

$$
t^*(M, T, \xi, \mu) := \begin{cases} 
\inf \Gamma(M, T, \xi, \mu) & \text{if } \Gamma(M, T, \xi, \mu) \neq \emptyset \\
T & \text{otherwise}.
\end{cases}
$$

(6.5)

The following estimate is an immediate consequence of the definition of $t^*$,

$$
B_{\mu}(t^*) := \int_0^{T} \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(\tau, y)|^2 \, dy \, d\tau \leq M(M + 1) \leq 2M^2.
$$

(6.6)

If there exists $\mu^*$ such that $t^*(\cdot, \mu) = T$ for all $0 < \mu < \mu^*$, taking the limit $\mu \to 0$ into (6.6), that we are allowed since the weights $\sigma_\mu(y)$ are increasing as $\mu \to 0$, we get

$$
B(T) := \int_0^{T} \int_{\mathbb{R}^3} |y|^{-1}|\nabla v_\xi(\tau, y)|^2 \, dy \, d\tau = \int_0^{T} \int_{\mathbb{R}^3} |v(\tau, x)|^2 \, dx \, d\tau \leq 2M^2.
$$

Using this bound and the argument at the end of Subsection 4.3, after the inequality (4.26), we show that $L(T, \xi)$ is regular.

Thus the hard case turns out to be $0 \leq t^*(\cdot, \mu) < T$ for a vanishing sequence of $\mu$. Noting that the quantity $t^*(\cdot, \mu)$ can not increase as $\mu \to 0$, this simply means that there exists some $\mu > 0$ such that $0 \leq t^*(\cdot, \mu) < T$. When this happens, we will write for simplicity $0 \leq t^* < T$. We set

$$
a_\mu(t) := \int_{\mathbb{R}^3} \sigma_\mu(y)|v_\xi(t, y)|^2 \, dy = \int_{\mathbb{R}^3} |v(t, x)|^2 \, dx.
$$

As in Subsection 4.2, see (4.15), we have

$$
a_\mu(t) \leq \epsilon^2 e^{ZA}, \quad A := B_{\mu}(t^*) + t^*|\xi|^2, \quad \text{for all } 0 \leq t \leq t^*,
$$

for a certain constant $Z > 1$. We assume

$$
|\xi|^2 t^* \leq M^2, \quad \epsilon \leq 1,
$$

so that, using also the (6.6), we obtain

$$
a_\mu(t) \leq e^{Z(M^2 + K)} \epsilon, \quad \text{for all } 0 \leq t \leq t^*,
$$

(6.7)

(6.8)

where $Z$ is a suitably larger constant.

Let $t^*_n \to t^*$, with $t^*_n \leq t^*$, so that the perturbed energy inequality

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |v_\xi(t, y)|^2 \phi(t, y) \, dy + 2 \int_{t_n}^{t} \int_{\mathbb{R}^3} |\nabla v_\xi|^2 \phi \leq \int_{\mathbb{R}^3} |v_\xi(t_n, y)|^2 \phi(t_n, y) \, dy \\
&+ \int_{t_n}^{t} \int_{\mathbb{R}^3} |v_\xi|^2 (\phi_t - \xi \cdot \nabla \phi + \Delta \phi) + \int_{t_n}^{t} \int_{\mathbb{R}^3} (|v_\xi|^2 + 2P_\xi)v_\xi \cdot \nabla \phi \\
&+ \int_{t_n}^{t} \int_{\mathbb{R}^3} |v_\xi|^2 (w_\xi \cdot \nabla \phi) + 2 \int_{t_n}^{t} \int_{\mathbb{R}^3} (v_\xi \cdot w_\xi)(v_\xi \cdot \nabla \phi) + (v_\xi \cdot \nabla) v_\xi \cdot w_\xi \phi,
\end{aligned}
$$

(6.9)

holds for the sequence of test functions we are going to define. Recall that this is the perturbed energy inequality (4.2), after the change of variables (6.3). We choose

$$
\phi(t, y) := \psi_\mu(t) \sigma_\mu(y) \chi(\delta|y|),
$$

where $\delta > 0$, $\chi : [0, \infty) \to [0, \infty)$ is a smooth non increasing function such that

$$
\chi = 1 \text{ on } [0, 1], \quad \chi = 0 \text{ on } [2, \infty],
$$

$\sigma_\mu(y)$ has been defined in (6.4) and

$$
\psi_\mu(t) := e^{-kB_{t_n\mu}(t)}, \quad B_{t_n\mu}(t) := \int_{t_n}^{t} \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(\tau, y)|^2 \, dy \, d\tau,
$$
with $k$ a positive constant to be specified. Again, we should consider vanishing sequences $\delta_n, \mu_n^*$, instead of $\delta, \mu$, in order to be sure that the exceptional times, starting from which (4.2), and so (6.9), may not be satisfied, for at least one of our test functions, has measure zero. We keep writing $\delta, \mu$, for simplicity. As in Subsection 4.3, by a repeated use of the Stein bound for singular integrals and of the Caffarelli–Kohn–Nirenberg inequality, and appropriate cancellations between the left hand side and the right hand side of (6.9), we obtain, taking $k = 6Z$, \[ a_\mu(t)\psi_\mu(t) - a_\mu(t^*_n) + \frac{1 - \psi_\mu(t)}{6Z} \leq |\xi|^2 \int_{t_n^*}^t \psi_\mu(s)ds + \frac{1}{5} \int_{t_n^*}^t \|u(s, \cdot)\|_{L^\infty(\mathbb{R}^2)}ds; \quad (6.10) \] compare with the inequality (4.19).

The second term of the right hand side is immediately bounded by $\frac{1}{3}K$, that we have assumed to be small. In order to bound the first term of the right hand side in an efficient way, we need to take advantage of the key definition (6.5) which implies \[ B_{t^*, \mu}(t) := \int_0^t \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(\tau, y)|^2 dyd\tau \geq M \] for all $t \geq t^* + T/M$. Thus (we recall that $\psi_\mu(t) = e^{-6ZB_{t^*, \mu}(t)}$ and $M, Z > 1$) \[ \int_{t_n^*}^{t^* + T} \psi_\mu(s) ds = \int_{t_n^*}^{t^* + T} e^{-6ZB_{t^*, \mu}(s)} ds \leq \int_{t_n^*}^{t^* + T} e^{-6ZB_{t^*, \mu}(s)} ds \leq \int_{t_n^*}^{t^* + T/M} e^{-6ZB_{t^*, \mu}(s)} ds + \int_{t^* + T/M}^{t^* + T} e^{-6ZB_{t^*, \mu}(s)} ds \leq t^* - t_n^* + \frac{T}{M} + e^{-6ZM} \left( T - \frac{T}{M} \right) < t^* - t_n^* + \frac{2T}{M} - e^{-6ZM} \frac{T}{M}. \] Thus, since $t_n^* \to t^*$, we can choose $n$ large enough so that \[ \int_{t_n^*}^{t^* + T} \psi_\mu(s) ds \leq \frac{2T}{M}. \] Consequently \[ a_\mu(t) + \frac{1}{6Z} - \frac{1}{3}K - e^{Z(M^2 + K)} - 2|\xi|^2 T e^{ZB_{t^*, \mu}(t)} \leq \frac{1}{6Z}, \] for $t^* \leq t \leq t^* + T$. Indeed, this follows by the inequality (6.10), once we recall $a_\mu(t_n^*) \leq e^{Z(M^2 + K)}$, see (6.8), and we use the estimates we have just proved for its right hand side.

We take $K$ and $\epsilon$ such that \[ K \leq \frac{1}{6Z}, \quad e^{Z(M^2 + K)} \leq \frac{1}{30Z}, \quad (6.11) \] thus \[ a_\mu(t) + \left( \frac{1}{6Z} - 2|\xi|^2 T \right) e^{ZB_{t^*, \mu}(t)} \leq \frac{1}{6Z}. \quad (6.12) \] We furthermore assume that \[ \left( \frac{1}{6Z} - 2|\xi|^2 T \right) > 0 \quad \text{namely,} \quad |\xi|^2 T < \frac{M}{20Z}, \quad (6.13) \] which is stronger than (6.7), i.e. $|\xi|^2 t^* \leq M^2$, since $t^* \leq T$ and $M, Z > 1$. In this way, the inequality (6.12) immediately implies \[ B_{t^*, \mu}(t) \leq B_{t^*, \mu}(t) < \infty, \quad \text{uniformly in } \mu > 0, \] for all $t^* \leq t \leq t^* + T$, as long as (6.13) holds. On the other hand, we already know that $B_{t^*}(t^*) \leq 2M^2$, see (6.6), so that, since $B_{t^*}(t) = B_{t^*}(t^*) + B_{t^*, \mu}(t)$, we have proved \[ B_{t^*}(t) := \int_0^t \int_{\mathbb{R}^3} \sigma_\mu(y)|\nabla v_\xi(\tau, y)|^2 dyd\tau < \infty, \quad \text{uniformly in } \mu > 0, \quad (6.14) \]
for all $0 \leq t \leq t^* + T$, as long as (6.13) holds. Again, we mean that $B_\mu(t)$ is bounded by a constant that is independent on $\mu$. This constant may however it may increase indefinitely as $|\xi|^2T$ approaches $\frac{M}{20\pi}$, but this does not affect our argument. Since the weights $\sigma_\mu(y)$ are increasing as $\mu \to 0$, and they converges to $|y|^{-1}$, we can pass to the limit in (6.14), so that

$$B(t) := \int_0^t \int_{\mathbb{R}^3} |y|^{-1} |\nabla v_\xi(\tau, y)|^2 \, dy \, d\tau < \infty,$$

under the same conditions. In particular, going back to the old variables,

$$B(T) = \int_0^T \int_{\mathbb{R}^3} \frac{|\nabla v(\tau, x)|^2}{|x - s\xi|} \, dx \, d\tau < \infty,$$

and we have already observed (see Subsection 4.3) that this implies that $L(T, \xi)$ is a regular set.

Summing up, if we assume (6.11) and (6.13), then $L(T, \xi)$ is regular. Notice that the condition (6.11) is ensured by

$$\mathcal{K} \leq \delta_4, \quad \epsilon \leq \delta_4 e^{-M^2/\delta_4},$$

once we choose $\delta_4 = 1/(90\pi^2)$. Under this choice, the condition (6.13) is implied by

$$(T, T') \in \Pi_{M \delta_4}, \quad \Pi_{M \delta_4} := \{ (t, x) \in (0, \infty) \times \mathbb{R}^3 : t > \frac{|x|^2}{M \delta_4} \},$$

and the proof is completed because $\Pi_{M \delta_4}$ is the union of such segments for arbitrary $T > 1$.

\[\square\]

6.2. Proof of Theorem 1.3. We first recall (a partial version of) the Kato $L^3$ theorem, which is needed in order to prove Theorem 1.3.

**Theorem 6.2 ([31]).** There exists a constant $\varepsilon_1 > 0$ such that the following holds. If $w_0 \in L^2 \cap L^3$ is a divergence free vector field such that

$$\|w_0\|_{L^3} < \varepsilon_1, \quad (6.15)$$

then there exists a global unique smooth solution $w \in C_b([0, \infty); L^2) \cap L^2_t H^1_x$ to the Navier–Stokes equation with data $w_0$, and

$$\|w\|_{L^3_t L^2_x} \leq C\|w_0\|_{L^3}. \quad (6.16)$$

**Remark 6.1.** If we also have $w_0 \in L^2(\{x \neq x'\}^{-1} dx)$, for some $x' \in \mathbb{R}^3$, then $w$ is a reference solution of size $\mathcal{K} = \|w\|_{L^3_t L^2_x}^3$; see Definition 1.4.

Theorem 1.3 covers the gap between the regularity Theorem 1.1 and the full regularity of solutions with small $L^3$ initial data. A similar phenomenon has been observed in [12] where some additional angular integrability to the data has been required in order to gain regularity. The connection between smoothness and higher angular integrability of solutions to the Navier–Stokes equation has also been observed in [43]. This is not surprising since the basic inequalities that are used to handle the local energy estimates, like weighted bound for the Riesz transform and the Caffarelli–Kohn–Nirenberg inequality (7.2), improves under additional angular integrability assumptions; see [3], [11], [13].

We prove Theorem 1.3 when $\hat{x} = 0$, the general case follows by translation. In order to apply Theorem 6.1, an appropriate decomposition of the initial data is required.

For any $s > 0$, we split

$$w_0 = \mathbb{P}_{u_0, \leq s} + \mathbb{P}_{u_0, > s} =: w_0 + v_0,$$
where
\[ u_{0, s}(x) := \begin{cases} u(x) & \text{if } |x| u(x)| \leq s, \\ 0 & \text{otherwise.} \end{cases} \]

Let us recall that \( \mathbb{P} \) is the projection onto the divergence free vector fields subspace, and it can be represented as \( \mathbb{P} = Id + (\mathcal{R} \otimes \mathcal{R}) \). This decomposition satisfies the key estimates
\[
\|w_0\|_{L^p(\mathbb{R}^3)} \leq C s^{1-p/3} \|x|^{\alpha} u_0\|_{L^p(\mathbb{R}^3)}^{p/3} \tag{6.17}
\]
\[
\|x|^{-1/2} v_0\|_{L^2(\mathbb{R}^3)} \leq C s^{1-p/2} \|x|^{\alpha} u_0\|_{L^p(\mathbb{R}^3)}^{p/2} \tag{6.18}
\]

where \( 2 < p < 3 \) and \( \alpha = 1 - 3/p \). Indeed, since the Riesz transform \( \mathcal{R} \), and so \( \mathbb{P} \), are bounded on \( L^3 \) and on \( L^2(|x|^{-1}dx) \), see for instance [51], the estimates (6.17) are consequence of the elementary estimates
\[
\|u_{0, s}\|_{L^p(\mathbb{R}^3)} \leq C s^{1-p/3} \|x|^{\alpha} u_0\|_{L^p(\mathbb{R}^3)}^{p/3} \tag{6.19}
\]
for some \( C > 1 \), where \( \theta_1(p) \), \( \theta_2(p) \) have been defined in (1.9).

Then we choose \( \delta_1 = C^{-1} \min(1, \delta_4, \varepsilon_1) \), where \( \delta_4 \) is the small constant in Theorem 6.1, \( \varepsilon_1 \) is the small constant in the Kato’s Theorem 6.2 and \( C^{1/10} \) is a constant larger than the ones in (6.19), (6.16). By (6.19) and the first assumption in (1.10),
\[
\|w_0\|_{L^3} \leq C^{1/10} \theta_1(\delta_1) \|x|^{\alpha} u_0\|_{L^p(\mathbb{R}^3)}^{p/3} \leq C^{1/10} \delta_1 \tag{6.20}
\]
By (6.16) and (6.20),
\[
\|w\|_{L^4 X L^3}^5 \leq C^{1/2} \|w_0\|_{L^3}^5 \leq C \delta_1 \leq C \delta_1 \leq \delta_1. \tag{6.21}
\]
By (6.19) and the second assumption in (1.10),
\[
\|x|^{-1/2} v_0\|_{L^2} \leq C^{1/10} \theta_2(p) \|x|^{\alpha} u_0\|_{L^p(\mathbb{R}^3)}^{p/2} \leq C^{1/10} \delta_1 e^{-M^2/\delta_4} \leq \delta_4 e^{-M^2/\delta_4}. \tag{6.22}
\]

Thanks to (6.21), (6.22) and recalling Remark 6.1\(^6\) we can apply Theorem 6.1 to conclude that \( \Pi_{M_4} \) is a regular set for \( u \). Since \( \Pi_{M_4} \subset \Pi_{M_4} \), the proof is complete.

7. Appendix

**Lemma 7.1.** If \( u \in L^\infty_x L^2_x \cap L^2_x H^1_x \) is a suitable weak solution to the Navier–Stokes equation with pressure \( P_u \) and divergence free data \( u_0 \in L^2 \) and \( w \) is a reference solution (see Definition 1.4) with pressure \( P_w \) and divergence free data \( w_0 \in L^2 \), then the difference \( u - w \) is a suitable weak solution to the perturbed Navier–Stokes equation (4.1), around the solution \( w \), with pressure \( P_u - w := P_u - P_w \) and data \( u_0 - w_0 \).

\(^6\) Notice that \( u_0 \in L^2(|x - x'|^{-1}dx) \). Indeed we have assumed \( u_0 \in L^2(|x - x'|^{-1}dx) \) and we can estimate \( \|x - x'|^{-1/2} u_0\|_{L^2} = \|x - x'|^{-1/2} u_0\|_{L^2} \leq \|x - x'|^{-1/2} u_0\|_{L^2} \).
Proof. The only non trivial thing to prove is that the perturbed local energy inequality (4.2) is satisfied. This is formally justified once we take the scalar product of the momentum equation for \( v \), namely the first equation in (4.1), against the vector field \( 2 \phi w \), we integrate over \( (t_0, t) \times \mathbb{R}^3 \) and then by parts. A rigorous proof requires the suitability of \( u \) and the fact that \( w \) is a reference solution.

We first consider the case \( t_0 = 0 \). Since \( (u, P_u) \) and \( (w, P_w) \) satisfy the local energy inequality (3.1), taking the difference, is straightforward to check that (4.2) is a consequence of

\[
0 = 2 \int_{\mathbb{R}^3} u \cdot w \phi(t, x) dx - 2 \int_{\mathbb{R}^3} u_0 \cdot w \phi(0, x) dx - 2 \int_{t_0}^t \int_{\mathbb{R}^3} u \cdot w \phi \nonumber \\
- 2 \int_{t_0}^t \int_{\mathbb{R}^3} (u - w) + 2 \int_{t_0}^t \int_{\mathbb{R}^3} (u \cdot \nabla) w - 2 \int_{t_0}^t \int_{\mathbb{R}^3} w \cdot \nabla u - 2 \int_{t_0}^t \int_{\mathbb{R}^3} w \cdot \nabla \phi \\
+ 4 \int_{t_0}^t \int_{\mathbb{R}^3} \phi \nabla u \cdot \nabla w - 2 \int_{t_0}^t \int_{\mathbb{R}^3} u \cdot w \phi \\
- 2 \int_{t_0}^t \int_{\mathbb{R}^3} P_w u \cdot \nabla \phi - 2 \int_{t_0}^t \int_{\mathbb{R}^3} P_w w \cdot \nabla \phi.
\]

Again, this identity is formally justified once we take the scalar product of the momentum equation for \( u \) against the vector field \( 2 \phi w \), the scalar product of the momentum equation for \( w \) against \( 2 \phi u \), we integrate over \( (0, t) \times \mathbb{R}^3 \) and then by parts. Since \( u \) is a suitable solution, this procedure is rigorous once we consider, rather than the field \( 2 \phi u \), the mollified field \( 2 \phi u \), letting eventually \( \varepsilon \to 0 \). The mollification has to be in the space-time variables, namely \( f^s(t, x) := \int_{\mathbb{R}^2} f(s, y) \rho^s(t - s, x - y) dyds \), where \( \rho^s := \varepsilon^{-4} \rho(\varepsilon^{-4}t) \), \( 0 \leq \rho \in C_c^\infty(\mathbb{R}^4) \) and \( \int_{\mathbb{R}^2} \rho = 1 \). The limit \( \varepsilon \to 0 \) can be easily justified recalling that \( u \in L^q_t L^2_x \cap L^q_t \dot{H}^1_x \), \( w \in C_b([0, \infty); L^q_t \cap L^q_t \dot{H}^1_x) \), with \( (r, q) \) admissible, and \( P_u, P_w \in L^{3/2}((0, \infty) \times \mathbb{R}^3) \). Notice that the function \( t \mapsto \int_{\mathbb{R}^3} u(t, x) \cdot w(t, x) \phi(t, x) dx \), that appears in the first line of the identity, is continuous, at any \( t \geq 0 \), since \( t \in [0, \infty) \to u(t) \) is \( L^2 \)-weakly continuous and \( w \) belongs to \( C_b([0, \infty); L^q_x) \); see Section 3 and Remark 3.1. Notice also that the integrability of the term \( (w \nabla) u w \phi \), which is the hardest to be handled, is guaranteed by \( u \in L^r_t L^2_x \cap L^q_t \dot{H}^1_x, 2/r + 3/q = 3/2, 2 \leq q \leq 6 \) (see (3.4)) and \( w \in L^r_t L^2_x \) with \( (r, q) \) admissible, that is \( 2/r + 3/q = 1, 3 < q \leq \infty \).

Once we have proved (4.2) for \( t_0 = 0 \), the inequality for almost any \( t_0 > 0 \) can be deduced in the following way. For any \( \varepsilon > 0 \), we consider the auxiliary test functions \( \phi^\varepsilon(t, x) := \eta^\varepsilon(t) \phi(t, x) \) where \( \eta^\varepsilon(t) \) is an \( \varepsilon \)-mollification of the step function with jump in \( t_0 \), namely \( \eta^\varepsilon(t) := \chi_{[t_0, \infty)} \ast \rho^\varepsilon \), and \( \rho^\varepsilon := \varepsilon^{-1} \rho(\varepsilon^{-1}t) \), with \( 0 \leq \rho \in C_c^\infty(\mathbb{R}) \) and \( \int_{\mathbb{R}} \rho = 1 \). We have \( \partial_t \phi^\varepsilon(t, x) = \rho^\varepsilon(t - t_0) \phi(t, x) + \eta^\varepsilon(t) \partial_t \phi(t, x) \), and, as \( \varepsilon \to 0 \):

\[
\eta^\varepsilon \to \chi_{[t_0, \infty)}, \quad \phi^\varepsilon \to \chi_{[t_0, \infty]} \phi, \quad \nabla \phi^\varepsilon \to \chi_{[t_0, \infty]} \nabla \phi, \quad \Delta \phi^\varepsilon \to \chi_{[t_0, \infty]} \Delta \phi.
\]

We now apply the inequality (4.2), proved in the case \( t_0 = 0 \), with the test function \( \phi^\varepsilon \). Taking the limit \( \varepsilon \to 0 \), we get

\[
\int_{\mathbb{R}^3} |v|^2 \phi(t) + 2 \int_{t_0}^t \int_{\mathbb{R}^3} |\nabla \phi|^2 ds(t) \leq \liminf_{\varepsilon \to 0} \int_{t_0}^t \rho^\varepsilon(s - t_0) \left( \int_{\mathbb{R}^3} |v|^2 \phi \right) (s) ds + \left( \int_{\mathbb{R}^3} |v|^2 \phi(t_0) \right) + 2 \int_{t_0}^t \int_{\mathbb{R}^3} (|v|^2 + 2 P_v) v \cdot \nabla \phi + 2 \int_{t_0}^t \int_{\mathbb{R}^3} (v \cdot \nabla) w \phi + 2 \int_{t_0}^t \int_{\mathbb{R}^3} (v \cdot w) (v \cdot \nabla \phi) + \phi(v \cdot \nabla) w \phi.
\]

Here we have used the integrability properties of \( v, P_v, w \) that we have recalled above. Thus, the required inequality is satisfied for any \( t_0 \) such that

\[
\lim_{\varepsilon \to 0} \int_{t_0}^t \rho^\varepsilon(s - t_0) \left( \int_{\mathbb{R}^3} |v|^2 \phi \right) (s) ds = \int_{\mathbb{R}^3} |v|^2 \phi(t_0),
\]

that is actually true for all the Lebesgue points of the function \( t \mapsto \int_{\mathbb{R}^3} |v|^2 \phi(t) \), and so for almost every \( t_0 > 0 \), since \( v \in L^q_t L^2_x \). This concludes the proof. \( \square \)
Lemma 7.2. Let \( v \in L^2_t L^\infty_x \) be a generalized reference solution to the Navier–Stokes equation (see Definition 4.1) with divergence free data \( w_0 \in L^2_{\text{loc}} \). If \( v \) is a suitable weak solution to the perturbed equation (4.1), around \( w \), with divergence free data \( v_0 \in L^2(\mathbb{R}^3) \), then \( u := v + w \) is a suitable weak solution to the Navier–Stokes equation with data \( u_0 := v_0 + w_0 \) and pressure \( P_u := P_v + P_w \).

Recalling that (2, \( \infty \)) is an admissible couple, the proof is analogous to that of Lemma 7.1. Here \( v \), which plays the role played by \( u \) in the previous lemma, enjoys the same integrability property, while \( w \) belongs to \( L^\infty_t L^2(K) \cap L^3_t \tilde{H}^1(K) \cap L^2_t L^\infty_x \), \( K \subset \mathbb{R}^3 \) compact, and \( t \in [0, \infty) \to w(t) \in L^2(K) \), \( K \subset \mathbb{R}^3 \) compact, is continuous. However, since in the proof all the vector fields are multiplied by a compactly supported test function, this does not affect the argument.

Proposition 7.3 (Caffarelli–Kohn–Nirenberg [4]). Assume that

1. \( r > 0 \), \( 0 < \theta \leq 1 \), \( \gamma < 3/r \), \( \alpha < 3/2 \), \( \beta < 3/2 \);
2. \( -\gamma + 3/r = \theta(-\alpha + 1/2) + (1 - \theta)(-\beta + 3/2) \);
3. \( \theta \alpha + (1 - \theta)\beta \leq \gamma \);
4. when \( -\gamma + 3/r = -\alpha + 1/2 \), assume also \( \gamma \leq \theta(\alpha + 1) + (1 - \theta)\beta \).

Then

\[
\|\sigma^0_\mu \nabla f\|_{L^r(\mathbb{R}^3)} \leq C\|\sigma^0_\mu |f|\|_{L^3(\mathbb{R}^3)} \|\sigma^0_\mu |f|\|_{L^2(\mathbb{R}^3)},
\]

where \( \sigma_\mu := (\mu + |x|^2)^{-1/2} \), \( \mu \geq 0 \). The constant \( C \) is independent of \( \mu \).

The (7.2) has been proved in [4] in the case \( \mu = 0 \). The general case can be obtained by the following standard argument. First notice that, by rescaling, it suffices to prove (7.2) in the case \( \mu = 1 \). Then we notice that

\( \sigma_1 \simeq 1 \) if \( |x| \leq 1 \), \( \sigma_1 \simeq |x|^{-1} \) if \( |x| \geq 1 \).

Split \( f = f_1 + f_2 := \chi_1 f + \chi_2 f \), where \( \chi_1 \) is a smooth cut-off function of the unit ball, namely \( 0 \leq \chi_1 \leq 1 \), \( \chi_1 = 1 \) if \( |x| \leq 1 \), \( \chi_1 = 0 \) if \( |x| \geq 2 \) (\( \chi_2 := 1 - \chi_1 \)). Using the inequality (7.2) with \( \mu = 0 \) we have

\[
\|\sigma^0_1 f_2\|_{L^r(\mathbb{R}^3)} \lesssim \|\sigma^0_1 \nabla f_2\|_{L^3(\mathbb{R}^3)} \|\sigma^0_1 f_2\|_{L^2(\mathbb{R}^3)},
\]

where we have used that \( \sigma_1 \simeq |x|^{-1} \) in the support of \( f_2 \).

If the parameters \( (\theta, r, \alpha, \beta, \gamma) \) satisfies the conditions (1–4), then \( (\theta, r, 0, 0, 0) \) satisfies the same conditions once we replace (2) with

\( -\gamma + 3/r \geq \theta(-\alpha + 1/2) + (1 - \theta)(-\beta + 3/2) \).

Thus, there exists \( s \geq r \) so that \( (\theta, s, 0, 0, 0) \) satisfies (1–4), and we can use again the inequality (7.2) with \( \mu = 0 \), so that

\[
\|f_1\|_{L^r(\mathbb{R}^3)} \lesssim \|\sigma^0_1 f_1\|_{L^3(\mathbb{R}^3)} \lesssim \|\nabla f_1\|_{L^3(\mathbb{R}^3)} \|\sigma^0_1 f_1\|_{L^2(\mathbb{R}^3)},
\]

recalling that \( f_1 \) is compactly supported. This is actually the Gagliardo–Nirenberg inequality. Since \( \sigma_1 \simeq 1 \) in the support of \( f_1 \) this implies

\[
\|\sigma^0_1 f_1\|_{L^r(\mathbb{R}^3)} \lesssim \|\sigma^0_1 \nabla f_1\|_{L^3(\mathbb{R}^3)} \|\sigma^0_1 f_1\|_{L^2(\mathbb{R}^3)},
\]

We finally observe that, for \( j = 1, 2 \),

\[
\|\sigma^0_j \nabla f_j\|_{L^2(\mathbb{R}^3)} \lesssim \|\sigma^0_j f \nabla \chi_j\|_{L^2(1 \leq |x| \leq 2)} + \|\sigma^0_j \chi_j \nabla f\|_{L^2(\mathbb{R}^3)} \lesssim \|\sigma^0_j f\|_{L^4(1 \leq |x| \leq 2)} + \|\sigma^0_j f\|_{L^2(\mathbb{R}^3)} \lesssim \|\sigma^0_j \nabla f\|_{L^2(\mathbb{R}^3)},
\]

where in the last inequality we have used the embedding \( \tilde{H}^1 \hookrightarrow L^6 \). Then (7.2) follows by (7.3), (7.5), (7.6).

The family of inequalities (7.7) has been proved in [51] in the case \( \mu = 0 \). The general case can be then deduced as shown in [5, Lemma 7.2].
Proposition 7.4 (Stein). Let $1 < p < \infty$ and $-3 + 3/p < \alpha < 3/p$. Let $T$ be a singular operator. Then
\[
\|\sigma^\alpha T f\|_{L^p} \leq C\|\sigma^\alpha f\|_{L^p},
\] where $\sigma_\mu := (\mu + |x|^2)^{-1/2}$, $\mu \geq 0$. The constant $C$ is independent of $\mu$.

8. Acknowledgments
The authors would like to thank Jean-Yves Chemin for suggesting to use the solutions of [10] as examples of reference solutions. Renato Lucà is supported by the ERC Starting Grant 676675 FLIRT.

References
[1] V. I Arnold, B. A. Khesin. Topological methods in hydrodynamics. Applied Mathematical Sciences, 125 (1998), Springer-Verlag, New York.
[2] P. Auscher, S. Dubois, P. Tchamitchian. On the stability of global solutions to Navier–Stokes equations in the space. J. Math. Pures Appl., (9) 83 (2004), no. 6, 673–697.
[3] F. Cacciafesta, R. Lucà. Singular integrals with angular integrability. Proc. Amer. Math. Soc., 144 (2016), no. 8, 3413–3418.
[4] L. Caffarelli, R. Kohn and L. Nirenberg. First order interpolation inequalities with weights. Compositio Math., 53 (1984), no. 3, 259–275.
[5] L. A. Caffarelli, R. Kohn and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier–Stokes equations. Comm. Pure Appl. Math., 35 (1982), no. 6, 771–831.
[6] M. Cannone. A generalization of a theorem by Kato on Navier–Stokes equations. Rev. Mat. Iberoamericana, 13 (1997), no. 3, 515–541.
[7] M. Cannone. Harmonic analysis tools for solving the incompressible Navier–Stokes equations. Handbook of mathematical fluid dynamics., Vol. III, 161–244, North-Holland, Amsterdam, 2004.
[8] D. Chae, J. Lee. On the regularity of the axisymmetric solutions of the Navier–Stokes equations. Math. Z., 239 (2002), no. 4, 645–671.
[9] J.-Y. Chemin, I. Gallagher. On the global wellposedness of the 3-D Navier-Stokes equations with large initial data. Ann. Sci. École Norm. Sup., (4) 39 (2006), no. 4, 679–698.
[10] J.-Y. Chemin, I. Gallagher. Wellposedness and stability results for the Navier–Stokes equations in $\mathbb{R}^3$. Ann. Inst. H. Poincaré Anal. Non Linéaire, 26 (2009), no. 2, 599–624.
[11] P. D’Ancona and R. Lucà. Stein–Weiss and Caffarelli–Kohn–Nirenberg inequalities with angular integrability. J. Math. Anal. Appl., 388 (2012), no. 2, 1061–1079.
[12] P. D’Ancona and R. Lucà. On the regularity set and angular integrability for the Navier–Stokes equation. Arch. Ration. Mech. Anal. 221 (2016), no. 3, 1255–1284.
[13] P.L. De Nápoli, I. Drelichman and R. G. Durán. Improved Caffarelli–Kohn–Nirenberg and trace inequalities for radial functions. Commun. Pure Appl. Anal., 11 (2012), no. 5, 1629–1642.
[14] A. Enciso, R. Lucà, D. Peralta-Salas. Vortex reconnection in the three dimensional Navier–Stokes equations. Adv. Math., 309 (2017), 452–486.
[15] A. Enciso, D. Peralta-Salas. Knots and links in steady solutions of the Euler equation. Ann. of Math., (2) 175 (2012), no. 1, 345–367.
[16] A. Enciso, D. Peralta-Salas. Existence of knotted vortex tubes in steady Euler flows. Acta Math., 214 (2015), no. 1, 61–134.
[17] A. Enciso, D. Peralta-Salas, F. Torres de Lizaur. Knotted structures in high-energy Beltrami fields on the torus and the sphere. Ann. Sci. Éc. Norm. Sup., to appear (arXiv:1505.01605).
[18] E. Fabes, B. Jones and N. Riviere The initial value problem for the Navier–Stokes equation with data in $L^p$. Arch. Ratton. Mech. Anal., 45 (1972), 222–240.
[19] P. Federbush. Navier and Stokes meet the wavelet. Commun. Math. Phys., 155 (1993), no. 2, 219–248.
[20] C. Foias, J.-C. Saut. Asymptotic behavior, as $t \to +\infty$, of solutions of Navier-Stokes equations and nonlinear spectral manifolds. Indiana Univ. Math. J., 33 (1984), no. 3, 459–477.
[21] H. Fujita and T. Kato. On the Navier–Stokes initial value problem I. Arch. Rational Mech. Anal., 16 (1964) 269–315.
[22] I. Gallagher. The tridimensional Navier–Stokes equations with almost bidimensional data: stability, uniqueness and life span Internat. Math. Res. Notices, (1997), no. 18, 919–935.
[23] I. Gallagher. Stability and weak-strong uniqueness for axisymmetric solutions of the Navier–Stokes equations Differential Integral Equations, 16 (2003), no. 5, 557–572.
I. Gallagher, S. Ibrahim and M. Majdoub. Existence et unicité de solutions pour le système de Navier-Stokes axissymétrique. (French) [Existence and uniqueness of solutions for an axisymmetric Navier-Stokes system]. **Comm. Partial Differential Equations**, 26 (2001), no. 5-6, 883–907.

I. Gallagher, D. Iftimie and F. Planchon. Non-explosion en temps grand et stabilité de solutions globales des équations de Navier-Stokes. **C. R. Math. Acad. Sci. Paris**, 334 (2002), no. 4, 289–292.

I. Gallagher, D. Iftimie and F. Planchon. Asymptotics and stability for global solutions to the Navier–Stokes equations. **Ann. Inst. Fourier**, 53 (2003), no. 5, 1387–1424.

Y. Giga. Solutions for semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system. **J. Diff. Eq.**, 62 (1986), no. 2, 186–212.

Y. Giga and T. Miyakawa. Navier–Stokes flow in $R^3$ with measures as initial vorticity and Morrey Spaces. **Comm. Partial Differential Equations**, 14 (1989), no. 5, 577–618.

L. Grafakos. Modern Fourier analysis. Second edition. **Graduate Texts in Mathematics**, 250 (2009), Springer, New York.

D. Iftimie. The 3D Navier–Stokes equations seen as a perturbation of the 2D Navier–Stokes equations. **Bull. Soc. Math. France**, 127 (1999), no. 4, 473–517.

T. Kato. Strong $L^p$-solutions of the Navier–Stokes equation in $R^m$, with applications to weak solutions. **Math. Z.**, 187 (1984), no. 4, 471–480.

T. Kato. Strong solutions of the Navier-Stokes equation in Morrey spaces. **Bol. Soc. Brasil. Mat. (N.S.)**, 22 (1992), no. 2, 127–155.

T. Kawanago. Stability estimate for strong solutions of the Navier–Stokes system and its applications. **Electron. J. Differential Equations**, (1998), no. 15, 23 pp. (electronic).

H. Koch and D. Tataru. Well-posedness for the Navier–Stokes equations. **Adv. Math.**, 157 (2001), no. 1, 22–35.

H. Kozono, M. Yamazaki. Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data. **Comm. Partial Differential Equations**, 19 (1994), no. 5-6, 959–1014.

T. Kato. Strong solutions of the Navier-Stokes equations in the presence of axial symmetry. (Russian) **Zap. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)**, 7 (1968), 155–177.

S. Leonardi, J. Málek, J. Nečas, M. Pokorný. On axially symmetric flows in $R^3$. **Z. Anal. Anwendungen** 18 (1999), no. 3, 639–649.

P. G. Lemarié-Rieusset. Recent developments in the Navier–Stokes problem. Chapman and Hall/CRC Research Notes in Mathematics, 431. Chapman and Hall/CRC, Boca Raton, FL, 2002.

P. G. Lemarié-Rieusset. The Navier–Stokes problem in the 21st century. Chapman and Hall/CRC, Boca Raton, FL, 2016.

J. Leray. Sur le mouvement d’un liquide visqueux emplissant l’espace. **Acta Math.**, 63 (1934), no. 1, 193–248.

F. Lin. A new proof of the Caffarelli–Kohn–Nirenberg theorem. **Comm. Pure Appl. Math.**, 51 (1998), no. 3, 241–257.

R. Lucà. On the size of the regular set of suitable weak solutions of the Navier–Stokes equation. **Journées équations aux dérivées partielles**, (2015), Exp. no. 5, 14 p., doi: 10.5802/jedp.634

R. Lucà. Regularity criteria with angular integrability for the Navier–Stokes equation. **Nonlinear Anal.**, 105 (2014), 24–40.

A. Mahalov, E. S. Titi, S. Leibovich. Invariant helical subspaces for the Navier–Stokes equations. **Arch. Rational Mech. Anal.**, 112 (1990), no. 3, 193–222.

F. Planchon. Global strong solutions in Sobolev or Lebesgue spces to the incompressible Navier-Stokes equations in $R^3$. **Ann. Inst. Henry Poincare, Anal. Non Lineaire**, 13 (1996), 319–336.

G. Ponce, R. Racke, T. C. Sideris and E. S. Titi. Global stability of large solutions to the 3D Navier–Stokes equations. **Comm. Math. Phys.**, 159 (1994), no. 2, 329–341.

G. Prodi. Un teorema di unicità per le equazioni di Navier–Stokes (Italian). **Ann. Mat. Pura Appl.**, 48 (1959), no. 4, 173–182.

V. Scheffer. Hausdorff measure and the Navier–Stokes equations. **Comm. Math. Phys.**, 55 (1977), no. 2, 97–112.

J. Serrin. On the interior regularity of weak solutions of the Navier–Stokes equations. **Arch. Rational Mech. Anal.**, 9 (1962) 187–195.

J. Serrin. The initial value problem for the Navier–Stokes equations. 1963 **Nonlinear problems (Proc. Sympos., Madison, Wis., 1962)**, pp. 69–98, Univ. of Wisconsin Press, Madison, Wis.
[52] M. E. Taylor. Analysis on Morrey spaces and applications to Navier–Stokes and other evolution equations. *Comm. Partial Differential Equations*, 17 (1992), no. 9-10, pp. 1407–1456.

[53] R. Temam. Navier–Stokes equations, Theory and Numerical Analysis. Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[54] M. R. Ukhovskii and V. I. Iudovich. Axially symmetric flows of ideal and viscous fluids filling the whole space. *Prikl. Mat. Meh.*, 32 59–69 (Russian). Translated as *J. Appl. Math. Mech.*, 32 (1968) 52–61.

Piero D’Ancona: SAPIENZA — UNIVERSITÀ DI ROMA, DIPARTIMENTO DI MATEMATICA, PIAZZALE A. MORO 2, I-00185 ROMA, ITALY

E-mail address: dancona@mat.uniroma1.it

Renato Luca: DEPARTEMENT MATHEMATIK UND INFORMATIK, UNIVERSITÄT BASEL, SPIEGELGASSE 1, 4051 BASEL, SWITZERLAND.

E-mail address: renato.luca@unibas.ch