An analogue of Serre’s conjecture for a ring of distributions

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Abstract: The set $A := \mathbb{C}\delta_0 + D'$, obtained by attaching the identity $\delta_0$ to the set $D'$ of all distributions on $\mathbb{R}$ with support contained in $(0, \infty)$, forms an algebra with the operations of addition, convolution, multiplication by complex scalars. It is shown that $A$ is a Hermite ring, that is, every finitely generated stably free $A$-module is free, or equivalently, every tall left-invertible matrix with entries from $A$ can be completed to a square matrix with entries from $A$, which is invertible.

Keywords: Hermite ring, Serre’s conjecture, algebraic $K$-theory, Schwartz distribution theory

MSC: Primary 46F10; Secondary 19B10, 19K99, 46H05

1 Introduction

The aim of this note is to show that the ring $A$ is a Hermite ring. The relevant definitions are recalled below. For preliminaries on the distribution theory of L. Schwartz, we refer the reader to [1] and [2]. For the commutative algebraic terminology used below, we refer to [3] and [4].

Let $D'$ denote the set of all distributions $T \in D'(\mathbb{R})$ having their distributional support contained in the half line $(0, \infty)$. Then $D'$ is an algebra with pointwise addition, pointwise multiplication by scalars, and with convolution taken as multiplication in the algebra. However, $D'$ lacks the identity element with respect to multiplication. We can adjoin the identity element to the algebra $D'$, hence obtaining the larger algebra $A := \mathbb{C}\delta_0 + D'$, whose elements are of the form $\alpha \delta_0 + T$, where $\alpha \in \mathbb{C}$ and $T \in D'$. $A$ is also an algebra with the same operations. We will denote the convolution operation henceforth by $\ast$.

Serre’s question from 1955 was if, for the ring $R = \mathbb{F}[x_1, \cdots, x_d]$ ($\mathbb{F}$ a field), every finitely generated projective $R$-module is free. This was eventually settled in 1976, independently, by Quillen and by Suslin, and the considerations over this question gave rise to the subject of algebraic $K$-theory. In light of the Hilbert-Serre theorem, Serre’s question for $R = \mathbb{F}[x_1, \cdots, x_d]$ can be reduced to the question of whether every finitely generated stably free $R$-module is free. A commutative unital ring $R$ having the property that every finitely generated stably free $R$-module is free is called a Hermite ring. In terms of matrices over the ring $R$, one has the following characterisation of Hermite rings, see for example [3, p.VIII], [5, p.1029], [6, Lemma 8.1.20, p.290].

Let $R$ be a commutative unital ring with multiplicative identity denoted by 1. For $m, n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, we denote by $R^{m \times n}$ the matrices with $m$ rows and $n$ columns having entries from $R$. The identity element in $R^{k \times k}$ having 1s on the diagonal and zeroes elsewhere will be denoted by $I_k$. A tall matrix $F \in R^{k \times k}$ is said to be left-invertible if there exists a $g \in R^{k \times k}$ such that $gf = I_k$. The ring $R$ is Hermite if and only if
for all $k$ and $K \in \mathbb{N}$ such that $k < K$, and 
for all $f \in R^{K \times K}$ such that there exists a $g \in R^{k \times K}$ so that $gf = I_k$, 
there exists an $f_c \in R^{K \times (K-k)}$ and there exists a $G \in R^{K \times K}$ 
such that $G \begin{bmatrix} f & f_c \end{bmatrix} = I_K$.

In other words, the ring $R$ is Hermite if every left invertible matrix over $R$ can be extended to an invertible one.

**Example 1.1.** $R = \mathbb{C}$ is a Hermite ring. Indeed, suppose that $f \in C^{K \times k}$ is left-invertible, and that $gf = I_k$ for some $g \in C^{k \times K}$. Then if $v \in C^K$ is such that $f v = 0$, it follows that $v = f g v = g 0 = 0$. So the columns $v_1, \ldots, v_k$ of $f$ are linearly $C^K$, and hence we can find $v_{k+1}, \ldots, v_K \in C^K$ such that \{ $v_1, \ldots, v_K, v_{k+1}, \ldots, v_K$ \} forms a basis for $C^K$. Defining $f_c = \begin{bmatrix} v_{k+1} & \cdots & v_K \end{bmatrix} \in C^{K \times (K-k)}$, we have that \[ \begin{bmatrix} f & f_c \end{bmatrix} \in C^{K \times K} \] is invertible in $C^{K \times K}$.

The following example is well-known, see e.g. [6, Example 8.1.27, p.292].

**Non-example 1.2.** Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, and let $C(S^2, \mathbb{R})$ be the ring of real-valued continuous functions on $S^2$, with pointwise operations. Then $C(S^2, \mathbb{R})$ is not a Hermite ring. Indeed, taking $f \in (C(S^2, \mathbb{R}))^{3 \times 1}$ as the map sending the point $x$ to the normal vector $x \times$ to the manifold $S^2$, that is, $x \mapsto \hat{n}(x) = x : S^2 \to \mathbb{R}^3$, we see that $f$ is left invertible, thanks to the fact that $(x, x) = 1$ in $\mathbb{R}^3$ for each $x \in S^2$. But if $C(S^2, \mathbb{R})$ were a Hermite ring, then $f$ could be extended to an invertible matrix $\begin{bmatrix} f & g & h \end{bmatrix} \in (C(S^2, \mathbb{R}))^{3 \times 3}$. This results in continuous maps $v_1, v_2 : S^2 \to \mathbb{R}^3$ so that $\{ f(x) = n(x), v_1(x), v_2(x) \}$ forms an orthonormal basis for $\mathbb{R}^3$.\footnote{This is impossible since the normal vector $\hat{n}(x)$ is a continuous function on $S^2$.}

In particular, $v_1 : S^2 \to \mathbb{R}^3$ would be a continuous tangent vector field on $S^2$ which is nowhere vanishing, contradicting the Hairy Ball Theorem [7].

Our result is the following:

**Theorem 1.3.** $(C \delta_0 + D', +, *)$ is a Hermite ring.

**Proof.** Let $A$ be the ring $C \delta_0 + D'$. Let $f \in A^{k \times k}$ be left invertible, with $gf = I_k \delta_0$ for some $g \in A^{k \times K}$. Write 
\[ f = \delta_0 f_0 + f_* , \]
\[ g = \delta_0 g_0 + g_* , \]
where $f_* \in (D')^{k \times k}, f_0 \in C^{K \times k}$, and $g_* \in (D')^{k \times K}, g_0 \in C^{k \times K}$. From $gf = I_k \delta_0$, we obtain that 
$g_0 f_0 \delta_0 + (g_0 f_* + g_* f_0 + g_* f_*) = \delta_0 I_k$.

As the entries of $f_*, g_*$ belong to $D'$, it follows that there exists an $\varepsilon > 0$ such that each of the entries of $g_0 f_* + g_* f_0 + g_* f_*$ has its support in $(\varepsilon, \infty)$. So if we act both sides (entry-wise) on a test function $\varphi \in D'(|\mathbb{R})$ such that $\text{supp}(\varphi) \subset (-\infty, \varepsilon)$, then we obtain 
$g_0 f_0 \varphi(0) = I_k \varphi(0)$.

Choosing $\varphi(0) \neq 0$, this now shows that $g_0 f_0 = I_k$. But as $C$ is Hermite, we can now find a $f_c \in C^{K \times (K-k)}$ and a $G_0 \in C^{K \times K}$ such that 
$G_0 \begin{bmatrix} f_0 & f_c \end{bmatrix} = I_K$.

that is, 
$(G_0 \delta_0) \begin{bmatrix} f_0 \delta_0 + f_* & f_c \delta_0 \end{bmatrix} = I_K \delta_0 - (G_0 \delta_0) \begin{bmatrix} -f_* & 0 \delta_0 \end{bmatrix}$. 

As $f_* \in (D')^{K \times k}$, it follows that $T \in (D')^{K \times K}$. Suppose that $\varepsilon' > 0$ is such that each entry of $T$ has its support contained in $(\varepsilon', \infty)$. We claim that $I_K \delta_0 - T$ is invertible in $(A)^{K \times K}$. Define the “geometric series” 
$S = I_K \delta_0 + T + T^2 + T^3 + \cdots$. 


We will now show that $S$ is well-defined. We recall the theorem on supports for convolution of distributions [1, Theorem 8, p.120], namely that

$$\text{supp}(T_1 * T_2) \subset \text{supp}(T_1) + \text{supp}(T_2)$$

for any two distributions $T_1, T_2 \in \mathcal{D}'(\mathbb{R})$ whose supports satisfy the convolution condition. It follows that in our case, each entry $T_n$ has its support contained in $n \cdot \text{supp}(T) = n[e', \infty) = [ne', \infty)$. So it follows that the series for $S$ converges. Indeed, given any test function $\phi \in \mathcal{D}(\mathbb{R})$, the series (with the action $\langle T_n^0, \phi \rangle$ understood to be entry-wise)

$$\sum_{n=1}^{\infty} \langle T_n^0, \phi \rangle$$

contains only finitely many nonzero terms. Now if $S_n$ denotes the $n$th partial sum of the series $I_K\delta_0 + T + T^2 + T^3 + \cdots$, we have

$$(I_K\delta_0 - T)S = (I_K\delta_0 - T) \lim_{n \to \infty} S_n = \lim_{n \to \infty} (I_K\delta_0 - T)S_n = \lim_{n \to \infty} (I_K\delta_0 - T^{n+1}) = I_K\delta_0.$$ 

The second equality above follows from the continuity of convolution in $\mathcal{D}'$; see [1, Theorem 7, p.120]. Now, setting

$$G = S(G_0\delta_0) = (I_K\delta_0 - T)^{-1}(G_0\delta_0) \in \mathcal{A}^{K \times K},$$

we have

$$G \left[ f \mid f_c\delta_0 \right] = (I_K\delta_0 - T)^{-1}(G_0\delta_0) \left[ f_0\delta_0 + f, \mid f_c\delta_0 \right] = (I_K\delta_0 - T)^{-1}(I_K\delta_0 - T) = I_K\delta_0.$$ 

This completes the proof.

\[\square\]

2 Remarks

2.1 A conjecture

Another natural convolution algebra is the algebra $\mathcal{E}'(\mathbb{R})$ of all compactly supported distributions, again the usual pointwise addition and convolution taken as multiplication. We have the following:

Conjecture 2.1. $(\mathcal{E}'(\mathbb{R}), +, *)$ is a Hermite ring.

2.2 A corona-type condition for left invertibility

The famous Carleson corona theorem [8] answered Kakutani’s 1942 question of whether the ‘corona’ $M(H^\infty)\downarrow D$ is empty. Here $H^\infty$ denotes the Banach algebra of bounded holomorphic functions in the unit disc $D := \{ z \in \mathbb{C} : |z| < 1 \}$, with pointwise operations and the supremum norm $\|f\|_{\infty} := \sup\{|f(z)| : z \in D\}$. Also, $M(H^\infty)$ denotes the maximal ideal space of $H^\infty$ (the set of all multiplicative linear functionals $\phi : H^\infty \to \mathbb{C}$, endowed with the Gelfand topology, that is the topology induced from the dual space $\mathcal{L}(H^\infty, \mathbb{C})$ equipped with the weak-* topology). From the elementary theory of Banach algebras (see e.g. [9, Lemma 9.2.6]), the answer to this question in the affirmative is equivalent to the following result (the matricial version given below is attributed to [10], and is a consequence of [8]).
Theorem 2.2. Let $f \in (H^{\infty})^{k-k}$, where $K \geq k$. Then the following are equivalent:
1. There exists a $g \in (H^{\infty})^{k-k}$ such that $gf = I_k$.
2. There exists an $\epsilon > 0$ such that for all $z \in D$, $f(z)^*f(z) \geq \epsilon^2 I_k$.

(Here $f(z)^*f(z) \geq \epsilon^2 I_k$ means that $\|f(z)v\|_{C^k} \geq \epsilon^2\|v\|_{C^k}^2$ for all $v \in C^k$, and $\| \cdot \|_{C^k}$ denotes the usual Euclidean norm on $C^k$.)

Theorem 2.3. Let $A$ be the ring $(\mathbb{C}\delta_0 + \mathcal{D}', +, *)$. Then the following are equivalent for $f = f_0\delta_0 + f_* \in A^{k-k}$, where $f_0 \in C^{k-k}$ and $f_* \in (\mathcal{D}')^{k-k}$, $K, k \in \mathbb{N}$ with $K \geq k$:
1. There exists a $g \in A^{k-k}$ such that $gf = I_k\delta_0$.
2. There exists an $\epsilon > 0$ such that $f_0^*f_0 \geq \epsilon^2 I_k$.

Proof. (1)$\Rightarrow$(2): Write $g = g_0\delta_0 + g_*$, where $g_0 \in C^{k-k}$ and $g_* \in (\mathcal{D}')^{k-k}$. Then
$$I_k\delta_0 = gf = g_0f_0\delta_0 + (g_0f_* + g_*f_0 + g_*f_*),$$
and since the bracketed expression has support in $(0, \infty)$, it follows that $I_k = g_0f_0$. Then with $\epsilon := \|g_0^*\|^2$, where $\|g_0^*\|$ denotes the induced operator norm of the multiplication map $\nu \mapsto g_0^*\nu : C^k \to C^k$, and $C^k$, $C^k$ are equipped with the usual Euclidean 2-norms.

(2)$\Rightarrow$(1): If (2) holds, then $\ker(f_0^*f_0) = \{0\}$, and so $f_0^*f_0$ is invertible. Taking $g_0 := (f_0^*f_0)^{-1}f_0^*$, we then have $g_0f_0 = (f_0^*f_0)^{-1}f_0^*f_0 = I_k$. We have
$$g_0f = I_k\delta_0 + g_0f_*,$$
and since the support of $T := g_0f_*$, is contained in $(0, \infty)$, it follows that $I_k\delta_0 + T$ is invertible as an element of $A^{k-k}$, with the inverse
$$(I_k\delta_0 + g_0f_*)^{-1} = (I_k\delta_0 + T)^{-1} = I_k\delta_0 - T + T^2 - T^3 + \cdots.$$ 
So $g := (I_k\delta_0 + g_0f_*)^{-1}g_0 \in A^{k-k}$ and
$$gf = (I_k\delta_0 + g_0f_*)^{-1}g_0 = (I_k\delta_0 + g_0f_*)^{-1}(I_k\delta_0 + g_0f_*) = I_k\delta_0.$$
This completes the proof. $\square$

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