ENTROPY VS VOLUME FOR PSEUDO-ANOSOV MAPS

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Abstract. We will discuss theoretical and experimental results concerning comparison of entropy of pseudo-Anosov maps and volume of their mapping tori. Recent study of Weil-Petersson geometry of the Teichmüller space tells us that they admit linear inequalities for both sides under some bounded geometry condition. We construct a family of pseudo-Anosov maps which violates one side of inequalities under unbounded geometry setting, present an explicit bounding constant for a punctured torus, and provide several observations based on experiments.

1. Introduction

Let $\Sigma = \Sigma_{g,p}$ be an orientable surface of genus $g$ with $p$ punctures, $M(\Sigma)$ the mapping class group of $\Sigma$, and assume that $3g - 3 + p \geq 1$. According to Thurston [23], the element of $M(\Sigma)$ is classified into three classes, namely, periodic, pseudo-Anosov and reducible. A pseudo-Anosov element $\phi$ of $M(\Sigma)$ defines two natural numerical invariants. One is the entropy $\text{ent}(\phi)$ which is the logarithm of a stretching factor of invariant foliation of $\phi$ (often called a dilatation of $\phi$). The other is, thanks to the fibration theorem of Thurston [24], the hyperbolic volume $\text{vol}(\phi)$ of its mapping torus,

$$T(\phi) = \Sigma \times [0, 1]/\sim$$

where $\sim$ identifies $(x, 1)$ with $(f(x), 0)$ for some representative $f$ of $\phi$.

Our study is motivated by experiments of the last author in his 2000 thesis [21] on comparison of $\text{ent}(\phi)$ and $\text{vol}(\phi)$. To see this, we let $M^{\text{pA}}(\Sigma)$ be the set of pseudo-Anosov mapping classes of $M(\Sigma)$ and put

$$E(\Sigma) = \{(\text{vol}(\phi), \text{ent}(\phi)) \mid \phi \in M^{\text{pA}}(\Sigma)\}.$$  

Figure 1 is the plot of $E(\Sigma_{2,0})$ for all pseudo-Anosov classes represented by words of length at most 7 with respect to the Lickorish generator.

Although the samples may not be sufficiently many, it would be conceivable to predict that $\text{ent}(\phi)/\text{vol}(\phi)$ are bounded from both sides, namely, one may expect to have a constant $C$ depending only on the topology of $\Sigma$ satisfying

$$C^{-1}\text{vol}(\phi) \leq \text{ent}(\phi) \leq C\text{vol}(\phi).$$  \hspace{1cm} (1.1)

This expectation turns out to be true for bounded geometry case, which can be derived straightforwardly from recent works by Brock [3], Minsky [16] and Brock-Mazur-Minsky [4] together with a comparison of Teichmüller and Weil-Petersson...
metrics on the Teichmüller space. Moreover, the left inequality holds without assuming boundedness in geometry by a comparison of two metrics above. However, the theory does not say very much about accurate value of the constant $C$.

From computing viewpoints, it is rather easy to work with not closed but punctured disk cases, which have nice description in terms of braid data. Let $D_n$ be an $n$-punctured disk. See Figure 11 for more accurate plots for the case of $D_6$.

The purpose of this paper is to present two theoretical results and to provide some observations based on our experiments for pseudo-Anosov maps on punctured disks.

More precisely, we explicitly construct a family of pseudo-Anosov maps $\phi_k$ such that $\text{ent}(\phi_k)$ goes to infinity as $n$ goes to infinity while $\text{vol}(\phi_k)$ remains bounded from above in Theorem 3.3 which violates the inequality of the right hand side of (1.1), where geometry in this family is unbounded. We also give an explicit constant for $\Sigma_{1,1}$. Namely, we prove in Theorem 5.2 that for each $\phi \in M^{pA}(\Sigma_{1,1})$, we have

$$\frac{\text{ent}(\phi)}{\text{vol}(\phi)} > \frac{\log(\frac{3+\sqrt{5}}{2})}{2v_8} \approx 0.1313,$$

where $v_8 \approx 3.6638$ is the volume of a regular ideal octahedron. This bound is not best possible unfortunately. However, if we restrict our attention to mapping classes of block length 1, we obtain the best possible lower bound

$$\frac{\log(\frac{3+\sqrt{5}}{2})}{2v_3} \approx 0.4741,$$

where $v_3 \approx 1.0149$ is the volume of a regular ideal tetrahedron in Proposition 5.3.

The organization of this paper is as follows. After some preliminaries in the next section, we present what can be known from recent results and prove Theorem 3.2 in section 3. We describe our experiments and observations derived from them in section 4, and then discuss sharp bounds in section 5.

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2. Preliminaries

2.1. Perron-Frobenius theorem. Let $M = (m_{ij})$ and $N = (n_{ij})$ be matrices with the same size. We shall write $M \geq N$ (resp. $M > N$) whenever $m_{ij} \geq n_{ij}$ (resp. $m_{ij} > n_{ij}$) for each $ij$. We say that $M$ is positive (resp. non-negative) if $M > 0$ (resp. $M \geq 0$), where $0$ is the zero matrix.

For a square and non-negative matrix $T$, let $\lambda(T)$ be its spectral radius, that is the maximal absolute value of eigenvalues of $T$. We say that $T$ is irreducible if for every pair of indices $i$ and $j$, there exists an integer $k = k_{ij} > 0$ such that the $(i, j)$ entry of $T^k$ is strictly positive. The matrix $T$ is primitive if there exists an integer $k > 0$ such that the matrix $T^k$ is positive. By definition, a primitive matrix is irreducible. A primitive matrix $T$ is Perron-Frobenius if $T$ is an integral matrix.

The following theorem is commonly referred to as the Perron-Frobenius theorem, see for example Theorem 1.1 in [20].

Theorem 2.1 (Perron-Frobenius). Let $T$ be a primitive matrix. Then, there exists an eigenvalue $\lambda > 0$ of $T$ such that

1. $\lambda$ has strictly positive left and right eigenvectors $\hat{x}$ and $y$ respectively, and
2. $\lambda > |\lambda'|$ for any eigenvalue $\lambda' \neq \lambda$ of $T$.

If $T \geq B \geq 0$ and $\beta$ is an eigenvalue of $B$, then $\lambda \geq |\beta|$. Moreover $\lambda = |\beta|$ implies $T = B$.

2.2. Pseudo-Anosov mapping classes. The mapping class group $\mathcal{M}(\Sigma)$ is the group of isotopy classes of orientation preserving homeomorphisms of $\Sigma$, where the group operation is induced by composition of homeomorphisms. An element of the mapping class group is called a mapping class.

A homeomorphism $\Phi : \Sigma \to \Sigma$ is pseudo-Anosov if there exists a constant $\lambda = \lambda(\Phi) > 1$ called the dilatation of $\Phi$ and there exists a pair of transverse measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that

$$\Phi(\mathcal{F}^s) = \frac{1}{\lambda} \mathcal{F}^s$$

and

$$\Phi(\mathcal{F}^u) = \lambda \mathcal{F}^u.$$ 

A mapping class which contains a pseudo-Anosov homeomorphism is called pseudo-Anosov. We define the dilatation of a pseudo-Anosov mapping class $\phi$, denoted by $\lambda(\phi)$, to be the dilatation of a pseudo-Anosov homeomorphism of $\phi$. Fixing $\Sigma$, the dilatation $\lambda(\phi)$ for $\phi \in \mathcal{M}^{PA}(\Sigma)$ is known to be an algebraic integer with a bound on its degree depending only on $\Sigma$. Thus the set

$$\text{Dil}(\Sigma) = \{\lambda(\phi) > 1 \mid \phi \in \mathcal{M}^{PA}(\Sigma)\}$$

is discrete and in particular achieves its infimum $\lambda(\Sigma)$.

The (topological) entropy $\text{ent}(f)$ is a measure of the complexity of a continuous self-map $f$ on a compact metric space, see for instance [25]. For a pseudo-Anosov homeomorphism $\Phi$, the equality $\text{ent}(\Phi) = \log(\lambda(\Phi))$ holds [3] and $\text{ent}(\Phi)$ attains the minimal entropy among all homeomorphisms $f$ which are isotopic to $\Phi$.

Choosing a representative $f : \Sigma \to \Sigma$ of $\phi \in \mathcal{M}(\Sigma)$, we form the mapping torus

$$\mathbb{T}(\phi) = \Sigma \times [0, 1]/ \sim,$$

where $\sim$ identifies $(x, 0)$ with $(f(x), 1)$. Then $\phi$ is pseudo-Anosov if and only if $\mathbb{T}(\phi)$ admits a complete hyperbolic structure of finite volume [24] [17]. Since such a
structure is unique up to isometry, it makes sense to speak of the volume \( \text{vol}(\phi) \) of \( \phi \), the hyperbolic volume of \( \mathbb{T}(\phi) \).

### 2.3. Generating sets of mapping class groups.

Let \( G \) be a group with a symmetric generating set \( \mathcal{G} \) so that if \( h \in \mathcal{G} \) then \( h^{-1} \in \mathcal{G} \). The word length of \( h \) relative to \( \mathcal{G} \) is defined by \( \min \{ k \mid h = h_1 h_2 \cdots h_k, \ h_i \in \mathcal{G} \} \).

We introduce a generating set of the mapping class group \( \mathcal{M}(D_n) \) by \( \mathcal{G}(D_n) = \{ \hat{t}_{c_1} \pm 1, \cdots , \hat{t}_{c_n} \pm 1 \} \), where \( \hat{t}_{c_i} \) denotes the mapping class which represents the positive half twist about the arc \( c_i \) from the \( i \)th puncture to the \((i + 1)\)st (Figure 3).

![Figure 3](image-url)

**Figure 3.** (left) arc \( c_i \), (right) positive half twist \( \hat{t}_{c_i} \).

The \( n \)-braid group \( B_n \) and the mapping class group \( \mathcal{M}(D_n) \) are related by the surjective homomorphism

\[
\Gamma : B_n \to \mathcal{M}(D_n) \quad \sigma_i \mapsto \hat{t}_{c_i}
\]

where \( \sigma_i \) for \( i \in \{ 1, \cdots , n-1 \} \) is the Artin generator (Figure 4, left). The kernel of \( \Gamma \) is the center of \( B_n \) which is generated by a full twist braid \( (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n \). Note that \( \mathcal{M}(D_n) \) is isomorphic to a subgroup of \( \mathcal{M}(\Sigma_{0,n+1}) \) by replacing the boundary of \( D_n \) with the \((n + 1)\)st puncture. In the rest of the paper we regard a mapping class of \( \mathcal{M}(D_n) \) fixing the \((n + 1)\)st puncture.

We say that a braid \( b \in B_n \) is \textit{pseudo-Anosov} if \( \Gamma(b) \in \mathcal{M}(D_n) \) is pseudo-Anosov, and when this is the case, \( \text{vol}(\Gamma(b)) \) equals the hyperbolic volume of the link complement \( S^3 \setminus \overline{b} \) in the 3-sphere \( S^3 \), where \( \overline{b} \) is the braided link of \( b \) which is a union of the closed braid of \( b \) and the braid axis (Figure 4, right). Hereafter we represent a mapping class of \( \mathcal{M}(D_n) \) by a braid and we denote \( \Gamma(b) \in \mathcal{M}(D_n) \) by \( b \).

Given a simple closed curve \( \alpha \) on \( \Sigma \), let \( t_{\alpha} \) be a mapping class which represents the positive Dehn twist about \( \alpha \). Then \( \mathcal{G}(\Sigma_{1,1}) = \{ t_{a}^{\pm 1}, t_{b}^{\pm 1} \} \) is a generating set for \( \mathcal{M}(\Sigma_{1,1}) \), where \( a \) and \( b \) are the meridian and the longitude of a once-punctured torus.

### 3. Theory for entropy vs volume

#### 3.1. Linear inequalities in bounded geometry.

We here briefly review what could be known about entropy vs volume for \( \Sigma \) from the existing theory.
To see this more precisely, let us introduce two norms for a pseudo-Anosov $\phi$ with respect to metrics on the Teichmüller space. Let $||\phi||_*$ be the translation distance of $\phi$ with respect to the metric $*$, where $*$ is the Teichmüller metric which we denote by $* = T$ or the Weil-Petersson metric by $* = WP$. This is nothing but the minimal distance of the orbit of the action of $\phi$ on the Teichmüller space with respect to corresponding metrics. Notice here that

$$\text{ent}(\phi) = ||\phi||_T$$

Starting point is a seminal result by Brock [3], which shows that there is a constant $D$ depending only the topology of $\Sigma$ so that

$$D^{-1} \text{vol}(\phi) \leq ||\phi||_W P \leq D \text{vol}(\phi)$$

for any pseudo-Anosov $\phi$ on $\Sigma$. Thus we would like to compare two norms. The Teichmüller distance is originally studied as a distance derived from quasi-conformal maps between two Riemann surfaces, and Linch [14] succeeded to obtain comparison of two metrics directly. The modern theory introduces an infinitesimal interpretation of Teichmüller metric, see for instance [7], and it leads us to Linch’s inequality

$$||\phi||_W P \leq -2\pi \chi(\Sigma) ||\phi||_T$$

just based on the Cauchy-Schwarz inequality between infinitesimal forms of two metrics, see for instance in [19]. This together with Brock’s inequality immediately implies the first half of Theorem 3.2 which will be stated below.

The inequality of the right hand side in (1.1) does not hold in general as we will see explicitly in Theorem 3.3. However we would like to have control under some bounding condition on the geometry of $\phi$. The deep analysis carried out for Teichmüller metric by Minsky [16] and for Weil-Petersson metric by Brock-Mazur-Minsky [4] are the ones we are looking for. Just one of conclusions of their works could be stated as follows for our purpose.

**Theorem 3.1** (Minsky [16], Brock-Mazur-Minsky [4]). For any $\varepsilon > 0$, there exists $\delta > 0$ such that both Teichmüller and Weil-Petersson geodesics invariant by the action of a pseudo-Anosov $\phi$ has no intersection with the subset of the Teichmüller space consisting of hyperbolic surfaces with closed geodesic of length $\leq \delta$ if $T(\phi)$ contains no closed geodesics of length $\leq \varepsilon$.

Since the part of the Teichmüller space by thick surfaces appeared above is invariant by the action of the mapping class group, and moreover the quotient is compact
There exists a constant $B = B(\Sigma)$ depending only on the topology of $\Sigma$ such that
\[ B \text{vol}(\phi) \leq \text{ent}(\phi) \]
holds for any pseudo-Anosov $\phi$ on $\Sigma$. Furthermore, for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, \Sigma) > 1$ depending only on $\varepsilon$ and the topology of $\Sigma$ such that the inequality
\[ \text{ent}(\phi) \leq C \text{vol}(\phi) \]
holds for any pseudo-Anosov $\phi$ on $\Sigma$ whose mapping torus $T(\phi)$ has no closed geodesics of length $\leq \varepsilon$.

3.2. Entropy wins Volume in unbounded geometry. The following asserts that the inequality of the right hand side of (1.1) does not hold without bounded geometry condition.

**Theorem 3.3.** There exists a sequence of $\phi_k \in \mathcal{M}^{pA}(\Sigma)$ such that $\frac{\text{ent}(\phi_k)}{\text{vol}(\phi_k)}$ is unbounded.

We first review a construction of pseudo-Anosov mapping classes by Penner [18] and how to compute their dilatation. We follow the notation in [13]. Let $S(\Sigma)$ be the set of isotopy classes of essential simple closed curves on $\Sigma$. Let $S'(\Sigma)$ denote the set of isotopy classes of essential, closed 1-manifolds embedded on $\Sigma$. We refer to the components of $A \in S'(\Sigma)$ as elements of $S(\Sigma)$ and we write $A = a_1 \cup \cdots \cup a_m$ for $a_i \in S(\Sigma)$. For simplicity, we identify elements of $S'(\Sigma)$ with 1-manifolds on $\Sigma$, and we assume that elements $A, B \in S'(\Sigma)$ meet transversely with minimal intersection number. We say that $A \cup B$ fills $\Sigma$ if each component of $\Sigma \setminus (A \cup B)$ is topologically a disk or a disk with a puncture.

For $A = a_1 \cup \cdots \cup a_m \in S'(\Sigma)$ and $B = b_1 \cup \cdots \cup b_n \in S'(\Sigma)$, suppose that $A \cup B$ fills $\Sigma$. Let $\mathcal{R}(A, B)$ be the free semigroup of $\mathcal{M}(\Sigma)$ generated by the Dehn twists
\[ \{ t_{a_i}^{+1} \mid a_i \in A \} \cup \{ t_{b_j}^{-1} \mid b_j \in B \}. \]
Penner shows that $\phi \in \mathcal{R}(A, B)$ is pseudo-Anosov if each $t_{a_i}^{+1}$ for $i \in \{ 1, \ldots, m \}$ and each $t_{b_j}^{-1}$ for $j \in \{ 1, \ldots, n \}$ occurs at least once in $\phi$ [18].

We now construct a (bigon) train track $\tau$ obtained from $A \cup B$ by a deformation of each intersection $a_i \cap b_j$ as in Figure 5.

In a neighborhood $N(c)$ of a simple closed curve $c \in A \cup B$, we take a push-off $c'$ of $c$ in either side in $N(c)$ (Figure 4(left)). We represent $\phi$ using $t_{a_i'}$ and $t_{b_j'}^{-1}$ rather than $t_{a_i}$ and $t_{b_j}^{-1}$. Note that $\tau$ carries each $t_{a_i'}(\tau)$ (Figure 5) and each $t_{b_j'}^{-1}(\tau)$. Thus $\tau$ carries each $\phi(\tau)$ for each $\phi \in \mathcal{R}(A, B)$. This implies that each $\phi$ induces a graph map $\phi_* : \tau \rightarrow \tau$. The non-negative integral matrix $M = (m_{tu})$, called the incident matrix for $\phi_*$, can be defined as follows: each $m_{tu}$ equals the number of the times the image of the $u$th edge of $\tau$ under $\phi_*$ passes through the $t$th edge. Then $\lambda(\phi)$ for a pseudo-Anosov mapping class $\phi \in \mathcal{R}(A, B)$ is equal to the spectral radius of $M$. Proof of Theorem 3.3
$M$ \[18\]. (Here $M$ is in fact a Perron-Frobenius matrix, and hence $\lambda(\phi)$ equals the largest eigenvalue of $M$ strictly greater than 1.)

**Proof of Theorem 3.3.** For the proof, it suffices to show that there exists a sequence of $\phi_k \in \mathcal{M}^{PA}(\Sigma)$ such that $\lim_{k \to \infty} \text{ent}(\phi_k) = \infty$ and $\text{vol}(\phi_k)$ remains bounded above. Let $\phi = t_{a_1} \cdots t_{a_n}$ and consider a family of pseudo-Anosov mapping classes $\phi_k = t^k_{a_1}$ for each $k > 0$. We show that this is a desired family. Let $\hat{a}$ denote the knot $a_1 \times \{1/2\}$ in $\mathbb{T}(\phi)$ and let $W$ denote the closure of $\mathbb{T}(\phi) \setminus N(\hat{a})$, where $N(\hat{a})$ is a regular neighborhood of $\hat{a}$. Then $W(m/n)$ denotes the manifold obtained from $\mathbb{T}(\phi)$ by an $(m, n)$-Dehn filling on $W$, relative to a suitable choice of the longitude in $\partial N(\hat{a})$. We notice that $W(1/k)$ is homeomorphic to $\mathbb{T}(t^k_{a_1})$. Hence by [22 Proposition 6.5.2], we have

$$\frac{\text{vol}(\mathbb{T}(t^k_{a_1} \phi))}{v_3} \leq ||[W, \partial W]||,$$

where $[W, \partial W]$ is the relative fundamental class, $\| \cdot \|$ is the Gromov norm for a homology class $\cdot \in H_*(W, \partial W)$. The volume of $\phi_k$ is thus bounded above.

The proof is completed once one shows that $\lambda(\phi_{k'}) > \lambda(\phi_k)$ for $k' > k$, since the set $\text{Dil}(\Sigma)$ is discrete. Consider a vertex $v$ of $\tau$. Let $\beta$ be an edge of $\tau$ emanating from $v$ such that $\beta$ is contained in some $b_j$ and $\beta$ intersects with the push-off $a'_1$. Let $\alpha_u$ be any edge of $\tau$ contained in $a_1$ (Figure 7). The incident matrix for the mapping class $t^k_{a_1}$ is of the form

$$N_k = I + P_k,$$

where $I$ is the identity matrix and $P_k$ is a non-negative integral matrix such that the $(u, t)$ entry equals $k$. If $k' > k$ we have $N_{k'} > N_k$, and in particular $N_{k'} M > N_k M$, and
where $M$ is the incident matrix for $\phi$. Note that $N_k M$ is the incident matrix for $\phi_k = t^k_{a_1} \phi$, and we have $\lambda(\phi_{k'}) > \lambda(\phi_k)$ since $N_{k'} M > N_k M$. □

![Figure 7. $\beta_t$ and $\alpha_u$.](image)

4. Experiments and Observations

4.1. Experiments. For a computation of the braid dilatation, we use a program by T. Hall [9]. For a computation of the volume of links in the 3-sphere $S^3$, we use the program “SnapPea” by J. Weeks [26]. Here we exhibit the computation for $\Sigma \in \{D_3, D_4, D_5, D_6\}$.

We have

$$(\text{vol}(\phi^m), \text{ent}(\phi^m)) = (m \text{vol}(\phi), m \text{ent}(\phi))$$

and hence for any mapping class $\phi \in \mathcal{M}^{pA}(\Sigma)$, the line in $\mathbb{R}^+ \times \mathbb{R}^+$ of the slope $\text{ent}(\phi)/\text{vol}(\phi)$ passing through the origin must intersect $\mathcal{E}(\Sigma)$ in infinitely many points. Let

$$\mathcal{E}_k(\Sigma) = \{(\text{vol}(\phi), \text{ent}(\phi)) \mid \phi \in \mathcal{M}^{pA}(\Sigma) \text{ up to word length } k\}.$$ 

The following is the plots of $\mathcal{E}_{15}(D_3)$, $\mathcal{E}_{12}(D_4)$, $\mathcal{E}_{10}(D_5)$ and $\mathcal{E}_{11}(D_6)$ (Figures 8–11).

4.2. Observations. Recall that $\lambda(\Sigma)$ is the minimal dilatation among $\lambda(\phi)$ for $\phi \in \mathcal{M}^{pA}(\Sigma)$. We introduce the following notation:

$$\lambda_k(\Sigma) = \min\{\lambda(\phi) \mid \phi \in \mathcal{M}^{pA}(\Sigma) \text{ up to word length } k\},$$

$$\lambda(\Sigma; c) = \min\{\lambda(\phi) \mid \phi \in \mathcal{M}^{pA}(\Sigma), \mathcal{T}(\phi) \text{ has } c \text{ cusps}\},$$

$$\lambda_k(\Sigma; c) = \min\{\lambda(\phi) \mid \phi \in \mathcal{M}^{pA}(\Sigma) \text{ up to word length } k, \mathcal{T}(\phi) \text{ has } c \text{ cusps}\}.$$

In the case of $\Sigma = D_n$, the number of the cusps of the mapping torus $\mathcal{T}(b)$ of $b \in \mathcal{M}^{pA}(D_n) < \mathcal{M}^{pA}(\Sigma_{0,n+1})$ equals the number of the components of the link $\overline{b}$, since $\mathcal{T}(b) = S^3 \setminus \overline{b}$.

The minimal dilatation $\lambda(\Sigma)$, and the minimal entropy $\text{ent}(\Sigma)$, are known for the following surfaces.
Figure 8. $\mathcal{E}_{15}(D_3)$.  

Figure 9. $\mathcal{E}_{12}(D_4)$.  

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Figure 10. $\mathcal{E}_{10}(D_5)$. 

Figure 11. $\mathcal{E}_{11}(D_6)$. 
We now turn to the volume. The set

\[ \{ v > 0 \mid v \text{ is the volume of a hyperbolic 3-manifold} \} \]

called the *volume spectrum*, is a well-ordered closed subset of the set of real numbers \( \mathbb{R} \) of order type \( \omega^\omega \). In particular any subset of the volume spectrum achieves its infimum. We set

\[
\begin{align*}
\text{vol}(\Sigma) &= \min \{ \text{vol}(\phi) \mid \phi \in \mathcal{M}^{\text{pA}}(\Sigma) \}, \\
\text{vol}_k(\Sigma) &= \min \{ \text{vol}(\phi) \mid \phi \in \mathcal{M}^{\text{pA}}(\Sigma) \text{ up to word length } k \}, \\
\text{vol}(\Sigma; c) &= \min \{ \text{vol}(\phi) \mid \phi \in \mathcal{M}^{\text{pA}}(\Sigma), \ T(\phi) \text{ has } c \text{ cusps} \}, \text{ and} \\
\text{vol}_k(\Sigma; c) &= \min \{ \text{vol}(\phi) \mid \phi \in \mathcal{M}^{\text{pA}}(\Sigma) \text{ up to word length } k, \ T(\phi) \text{ has } c \text{ cusps} \}.
\end{align*}
\]

A question is which mapping class reaches \( \lambda(\Sigma) \), and which one reaches \( \text{vol}(\Sigma) \). For the case \( \Sigma \in \{ D_3, D_5 \} \), there exists a mapping class simultaneously reaching both \( \lambda(\Sigma) \) and \( \text{vol}(\Sigma) \). The 3-braid \( \beta_3 \) with minimal dilatation realizes \( \text{vol}(D_3) \) (F. Guérinat and D. Futer [8, Theorem B.1]). For the 5-braid \( \beta_5 \) with minimal dilatation, the link \( \beta_5 \) equals the \((−2, 3, 8)\)-pretzel link (Figure 12). It is shown that the \((−2, 3, 8)\)-pretzel link complement and the Whitehead link complement have the minimal volume among orientable 2-cusped hyperbolic 3-manifolds (Agol [2]). Hence \( \beta_5 \) also realizes \( \text{vol}(D_5) \approx 3.66339 \). One may ask that whether there exists a mapping class simultaneously reaching both \( \lambda(\Sigma) \) and \( \text{vol}(\Sigma) \). This question seems to be false in general. In our experiment, for the case \( D_6 \), \( \text{vol}_1(D_6) \) and \( \lambda_1(D_6) \) are not reached by the same mapping class. It may be caused by the fact that the mapping torus reaching \( \text{vol}_1(D_6) \) and the one reaching \( \lambda_1(D_6) \) have different number of cusps. It would be natural to revise the question by:

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure12.png}
\caption{link \( \beta_5 \) (left) is equal to \((−2, 3, 8)\)-pretzel link (right).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure13.png}
\caption{chain-link with 3 components.}
\end{figure}
Question 4.1. Does there exist a mapping class of $\mathcal{M}^{pA}(\Sigma)$ simultaneously reaching both $\lambda(\Sigma; c)$ and $\text{vol}(\Sigma; c)$?

The computation together with theoretical results shows this is likely to be positive. More precisely:

1-a. The 3-braid $\beta_3$ reaches both $\lambda(D_3) \approx 2.61803$ and $\text{vol}(D_3) \approx 4.05976$. Thus $\lambda(D_3) = \lambda(D_3; 2)$ and $\text{vol}(D_3) = \text{vol}(D_3; 2)$.

1-b. It is easy to verify that the 3-braid $\sigma_1^2\sigma_2^{-1}$ reaches $\lambda(D_3; 3)$. This 3-braid also reaches $\text{vol}_{15}(D_3; 3) = \text{vol}(S^3 \setminus C_3) \approx 5.33348$, where $C_3$ is the chain-link with 3 components (Figure 13). Among orientable 3-cusped hyperbolic 3-manifolds, $S^3 \setminus C_3$, which is called the magic manifold, is the one with the smallest known volume.

2-a. The 4-braid $\beta_4$ reaches both $\lambda(D_4) = \lambda(D_4; 2) \approx 2.29663$ and $\text{vol}_{12}(D_4) \approx 4.85117$.

2-b. The 4-braid $\sigma_1^5\sigma_2\sigma_3^{-1}$ reaches both $\lambda_{12}(D_4; 3) \approx 2.61803$ and $\text{vol}_{12}(D_4; 3) = \text{vol}(S^3 \setminus C_3)$.

3-a. The 5-braid $\beta_5$ reaches both $\lambda(D_5) = \lambda(D_5; 2) \approx 1.72208$ and $\text{vol}(D_5) = \text{vol}(D_5; 2) \approx 3.66386$.

3-b. The 5-braid $\sigma_1\sigma_2^2\sigma_3\sigma_4$ reaches both $\lambda_{10}(D_5; 3) \approx 2.08102$ and $\text{vol}_{10}(D_5; 3) = \text{vol}(S^3 \setminus C_3)$.

4-a. The 6-braid $\sigma_1^4\sigma_2\sigma_3^2\sigma_4\sigma_5$ reaches both $\lambda_{11}(D_6; 2) \approx 1.8832$ and $\text{vol}_{11}(D_6; 2) \approx 4.41533$.

4-b. The 6-braid $\sigma_1^2\sigma_2\sigma_1^2\sigma_3\sigma_2\sigma_4\sigma_5$ reaches both $\lambda_{11}(D_6; 3) = \lambda(\beta_5)$ and $\text{vol}(D_6; 3) = \text{vol}(S^3 \setminus C_3)$.

These observations are straightforward from the following plots of $E_{15}(D_3)$, $E_{12}(D_4)$, $E_{10}(D_5)$ and $E_{11}(D_6)$, restricted to the range of the volume $< 5.334$ (Figure 14).

We can check that in the plots all the 3-cusped mapping tori have the same volume $\approx 5.33348$ and other mapping tori have 2 cusps and smaller volumes than the one with 3 cusps.

Remark 4.2. In the experiment, all braids in (1-b), (2-b), (3-b) and (4-b) have the same volume $\approx 5.33348 \approx \text{vol}(S^3 \setminus C_3)$. Actually we can verify these mapping tori are homeomorphic to $S^3 \setminus C_3$, see (11).

Remark 4.3. The 5-braids $\beta_5$ and $\beta'_5 = \sigma_1^4\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_4$ both realize the minimal dilatation $\lambda(D_5)$, but the inequality $\text{vol}(D_5) = \text{vol}(\beta_5) < \text{vol}(\beta'_5)$ holds. This example says that it is not true that the mapping class with minimal dilatation realizes the minimal volume.

In Theorem 3.2 it is shown that there exists a constant $B = B(\Sigma)$ such that $B \text{ vol}(\phi) \leq \text{ent}(\phi)$ holds. However it is not quite obvious to find accurate value of $B$. We set

\[ I(\Sigma) = \inf \left\{ \frac{\text{ent}(\phi)}{\text{vol}(\phi)} \middle| \phi \in \mathcal{M}^{pA}(\Sigma) \right\} \] and

\[ I_k(\Sigma) = \min \left\{ \frac{\text{ent}(\phi)}{\text{vol}(\phi)} \middle| \phi \in \mathcal{M}^{pA}(\Sigma) \text{ up to word length } k \right\}. \]

It is natural to ask:

Question 4.4. Does there exist a mapping class $\phi \in \mathcal{M}^{pA}(\Sigma)$ which attains $I(\Sigma)$?
To study Question 4.4, we compute $I_k(\Sigma)$ for $\Sigma \in \{D_3, D_4, D_5, D_6\}$ (Figure 15). We see that $I_k(D_3)$ is achieved by the mapping class $\sigma_1\sigma_2^{-1}$ up to $k = 15$. On the other hand for any other surfaces, $I_k(\Sigma)$ decreases as $k$ increases. We will study $I(D_3)$ and $I(\Sigma_{g,0})$ in the next section.

**Question 4.5.** Is it true that $I(D_n) > I(D_{n+1})$ for all $n \geq 3$? Is it true that $I(\Sigma_{g,0}) > I(\Sigma_{g+1,0})$ for all $g \geq 2$?

5. **A lower bound of $I(\Sigma_{1,1})$**

Let $a$ and $b$ be the meridian and the longitude of a once-punctured torus. We set $L = t_a$ and $R = t_b^{-1}$, noting that $\{a, b\}$ fills $\Sigma_{1,1}$. We first recall the following well known result:

**Lemma 5.1.** Each $\phi \in M^{PA}(\Sigma_{1,1})$ is conjugate to a mapping class $L^{m_1}R^{n_1} \cdots L^{m_\ell}R^{n_\ell} \in R(\{a\}, \{b\})$ (5.1)

where $\ell, m_i$ and $n_i$ are positive integers, and the mapping class $L^{m_1}R^{n_1} \cdots L^{m_\ell}R^{n_\ell}$ is unique up to cyclic permutations. Conversely, every mapping class of the form (5.1) is pseudo-Anosov.

The integer $\ell$ in the above is called the block length of $\phi$. 
Figure 15. minimal ratio up to some word length for $\Sigma \in \{ D_3, D_4, D_5, D_6 \}$.

**Theorem 5.2.** For each $\phi \in M^{pA}(\Sigma_{1,1})$, we have

$$\frac{\text{ent}(\phi)}{\text{vol}(\phi)} > \frac{\log(\frac{3+\sqrt{5}}{2})}{2v_8} \approx 0.1313,$$

where $v_8 \approx 3.6638$ is the volume of a regular ideal octahedron.

**Proof.** Let $M_L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $M_R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. The matrix $M_L$ is the incident matrix for $L$ and the matrix $M_R$ is the incident matrix for $R$. Suppose that $\phi = L^{m_1}R^{n_1} \cdots L^{m_\ell}R^{n_\ell}$ of block length $\ell$. Then

$$M' = M_L^{m_1}M_R^{n_1}M_L^{m_2}M_R^{n_2} \cdots M_L^{m_\ell}M_R^{n_\ell}$$

is the incident matrix for $\phi$. Since $M' \geq (M_LM_R)^\ell$, the largest eigenvalue of $M'$ is greater than that of $(M_LM_R)^\ell$. We thus have

$$\lambda(\phi) \geq \lambda((LR)^\ell) = \left(\frac{3+\sqrt{5}}{2}\right)^\ell.$$

On the other hand, using a result [1 Corollary 2.4] of Agol,

$$\text{vol}(\phi) < 2\ell v_8.$$  \hfill (5.2)

Hence we have

$$\frac{\text{ent}(\phi)}{\text{vol}(\phi)} > \frac{\ell \cdot \log(\frac{3+\sqrt{5}}{2})}{2\ell v_8} = \frac{\log(\frac{3+\sqrt{5}}{2})}{2v_8} \approx 0.1313.$$
With the aid of SnapPea, one can show:

**Proposition 5.3.** For each \( \phi \in M^{pA}(\Sigma_{1,1}) \) of block length 1

\[
\frac{\text{ent}(\phi)}{\text{vol}(\phi)} \geq \frac{\text{ent}(LR)}{\text{vol}(LR)} = \frac{\log(\frac{3+\sqrt{5}}{2})}{2v_3} \approx 0.4741.
\]

**Proof.** Let \( c_{1,1} = \frac{\text{ent}(LR)}{\text{vol}(LR)} \). If \( \text{ent}(\phi) \geq c_{1,1} \cdot 2v_8 \), then \( \frac{\text{ent}(\phi)}{\text{vol}(\phi)} > c_{1,1} \) since \( \text{vol}(\phi) < 2v_8 \) (see (5.2)). Let

\[ Y = \{ \phi \in M^{pA}(\Sigma_{1,1}) \text{ of block length 1} \mid \text{ent}(\phi) < c_{1,1} \cdot 2v_8 < 3.4748 \}. \]

This is a finite set.

If \( \phi \) is written as \( L^m R^n \), then \( \lambda(\phi) \) is the largest eigenvalue of \( \begin{pmatrix} 1 + mn & m \\ n & 1 \end{pmatrix} \).

Thus

\[ \lambda(\phi) = \frac{2 + mn + \sqrt{4mn + (mn)^2}}{2}. \]

If \( L^m R^n \in Y \), then \( 2 + mn + \sqrt{4mn + (mn)^2} < e^{3.4748} < 33 \). Hence we have \( mn \leq 31 \). We compute \( \frac{\text{ent}(\phi)}{\text{vol}(\phi)} \) for each \( \phi = L^m R^n \) with \( mn \leq 31 \) by SnapPea, and we see that it is greater than or equal to \( c_{1,1} \).

In the case of \( \Sigma_{1,1} \) and \( D_3 \), we propose the following conjectures:

**Conjecture 5.4.**

\[ I(\Sigma_{1,1}) = \frac{\text{ent}(LR)}{\text{vol}(LR)} \approx 0.4741 \quad \text{and} \quad I(D_3) = \frac{\text{ent}(\sigma_1 \sigma_2^{-1})}{\text{vol}(\sigma_1 \sigma_2^{-1})} \approx 0.2370. \]

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