Modeling and numerical experiments of air pollution on a complex terrain.

Ekkachai Tawinan and Suttida Wongkaew
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand
E-mail: ekkachai.thawinan@cmu.ac.th

Abstract. The modeling of air contaminant dispersion extended from the Gaussian plume model is proposed. This model takes into account the transport of pollutants emitted from open biomass burning located on a complex terrain where the concentration behavior is modeled by the advection-diffusion equation in a stationary frame. In addition, the existence, uniqueness, and regularity of this problem are discussed. The discontinuous Galerkin method is employed for numerical treatments. Especially for numerical experiments, the propagation of air pollutants in the complex terrain areas in Chiang Mai province, Thailand, is investigated. Results of numerical experiments demonstrate the validity of the air pollution dispersion model.

Keywords: air pollution, atmospheric transport equation, advection-diffusion equation, discontinuous Galerkin method

1. Introduction
Over the past decades, there have been increasing concerns about human health and the environment caused by toxic gases and small particles such as polycyclic aromatic hydrocarbons (PAHs) emitted from open biomass burning in Southeast Asian countries, especially in Thailand; see, e.g., in [9, 11] for impacts of PAHs on human health and see in [8] for an example of air pollution problems in Southeast Asian. Thailand is predominantly an agricultural country where 41 percent of the total land area is used for agricultural purposes [13]. The most important crop in Thailand is Rice. In the last decades, corn has been added to the agricultural mix, where 71.33 percent of the nation’s total corn farmland area is located in the northern parts of Thailand [13]. In particular, the cultivated areas in the northern region are complex topographical features and are usually small and scattered around mountainous terrains. Recently, agricultural burning becomes more serious, especially during the harvest season, from February to May, as it causes haze pollution and smog across the region. In 2015, recorded amounts of PM10-bound polycyclic aromatic hydrocarbons in Chiang Mai, one of the most densely populated provinces in the north of Thailand, were as high as 299 µg/m³.

The effects of the air pollution problem are not only on public health but also being the factor that diminishes economic and the quality of life, see, e.g., in [14, 18]. Consequently, air pollution problems have encouraged many researchers to develop a mathematical model for forecasting the distribution of pollutants and identifying areas where emissions have a more significant impact. Regarding atmospheric modeling, we refer to the Gaussian plume model [1, 3, 6, 12, 17] where...
air contaminant dispersion is described by advection-diffusion equation with a continuous point source.

Our contribution to the mathematical investigation of transport of contaminants in the atmosphere focuses mainly on the modeling of pollutant dispersion released from the open biomass burning located in the complex terrain. In particular, we formulate an air pollution model based on the Gaussian model with a new set of modified assumptions appropriated to these circumstances. Comparing to the Gaussian plume model, we consider the air pollutant dispersion in $x, y$ plane instead of diffusion in $z$ direction. Moreover, we augment our model with a modified advection term related to the height of the terrain.

In this work, we theoretically and numerically investigate our air pollution model. In addition, we consider the advection-diffusion equation with suitable boundary conditions that models the evolution of air pollutants. The existence and characterization properties of solutions are proved by applying established results for diffusion-transport equations, which can be founded in [7, 16]. Further, numerical aspects of our air pollution model are addressed. The advection-diffusion equation is discretized with a discontinuous Galerkin scheme proposed in [2, 4, 10], whose stability and convergence properties were examined in [5, 19].

This work is organized as follows. Section 2 provides a description regarding mathematical modeling of contaminant transport in atmospheric addressed in this work. Starting with a general Gaussian plume model, we give a short description of this model and mention its limitations in applications to complex terrain. We then move into formulating mathematical modeling of air contaminant transport on a complex terrain described by the advection-diffusion equation. Subsequently, the unique solvability of this system is proved in Section 3. Section 4 deals with the discretization of the advection-diffusion equations, where the discontinuous Galerkin scheme is described and employed for numerical solutions. Additionally, the results of numerical experiments are presented to validate our model. A section of the conclusion completes this work.

2. Air contaminant transport model

The dispersion of air pollutants is a complex phenomenon that depends on the pollutant types, the air composition, and the specific meteorological situation. If we consider particular pollutants, such as the sulfur oxides ($SO_2$), the pollutant dispersion can be restricted to the transport of a single contaminant as in [1, 17] by considering a smooth scalar function of mass density $u(x, t): \Omega \times (0, T) \rightarrow \mathbb{R}$ where $x = (x, y, z)^T \in \mathbb{R}^3$ is location in an open and connected space domain $\Omega \subset \mathbb{R}^3$ and $t$ is time in a time domain $[0, T]$ with $T > 0$. According to the conservation law of mass density, the time evolution of pollutant concentration in air is governed by the following partial differential equations

$$\partial_t u + \nabla \cdot J = f,$$

where $f(x, t)$ is a source term of the air pollutants and $J(x, t)$ is a vector function representing the mass flux that combines effects of advection and diffusion. It can be written as $J = bu - \mathbb{K}\nabla u$. The first part of the mass flux models the linear advection by wind defined as a vector function $b(x, t)$, whereas the term $-\mathbb{K}\nabla u$ corresponds to turbulent diffusion based on Fick’s law, which states that the diffusive flux is proportional to the concentration gradient and the negative sign is used to ensure that contaminant flows from high to low densities. The diffusive coefficient $\mathbb{K}$ is in general defined as a second-order tensor of material properties, and it can simplify as the diagonal matrix $\mathbb{K}(x) = \text{diag}(K_x, K_y, K_z)$ where its components represent the turbulent eddy diffusivities in corresponding directions $x, y,$ and $z$, respectively. The resulting three-dimensional advection-diffusion equation is presented as the following,

$$\partial_t u + \nabla \cdot (bu - \mathbb{K}\nabla u) = f.$$
Note that the equation (2) equipped with appropriated initial and boundary conditions should be solved numerically to estimate and predict the transport of air pollutants. From the computational points of view, the discretization of this system is a fully three-dimensional problem that would lead to a very expensive computational task. Therefore, for simplicity, we consider the Gaussian plume model stated in [17]. The concept of this model and corresponding assumptions are briefly reviewed as follows.

### 2.1. Gaussian plume model

We follow [17] to discuss the Gaussian plume model. To simplify the equation (2) and to guarantee the analytical solution of the Gaussian plume model, the number of assumptions is given as the following:

A1. The source term is considered as a single point source at \( x = (0,0,H) \) with a constant rate \( Q \) and can be written as \( f(x) = Q \delta(x) \delta(y) \delta(z - H) \) where \( \delta(\cdot) \) is the Direct delta function.

A2. The wind velocity is constant, i.e. \( |v| = v \geq 0 \), and has one direction then it can be transformed to positive \( x \)-axis so that \( v = (v,0,0) \).

A3. The solution is steady state; in another word, the time scale of interest is long enough so that all parameters are independent of time.

A4. The eddy diffusivities are functions depended on the distance in direction of wind, which is \( x \) here, and the second order tensor \( K \) is isotropic so that \( K_x(x) = K_y(x) = K_z(x) =: K(x) \).

A5. the term \( K_x \partial_x^2 u \) can be neglected by assuming that the \( v \) is sufficiently large then the contribution of diffusion term in the \( x \)-direction is very small.

A6. The variations of ground surface are very small so that it can be taken as the flat area, say \( z = 0 \).

A7. The contaminant does not penetrate the ground.

By assumptions A1-A7, the equation (2) is reduced to

\[
v \partial_x u = K \partial_y^2 u + K \partial_z^2 u + Q \delta(x) \delta(y) \delta(z - H),
\]

within the domain \( x,z \in [0, \infty) \) and \( y \in (-\infty, \infty) \). To obtain a well-posed problem, we write the equations with the boundary conditions; see in [6] for more details, as follows

\[
\begin{align*}
v \partial_x u &= K \partial_y^2 u + K \partial_z^2 u, \\
u(0,y,z) &= \frac{Q}{u} \delta(y) \delta(z - H), \\
u(\infty,y,z) &= u(x, \pm \infty, z) = u(x, y, \infty) = 0, \\
K \partial_z u(x, y, 0) &= 0.
\end{align*}
\]

where the source term becomes the boundary condition at \( (0, y, z), \) and the condition (4d) is derived from assumption A7. To obtain the Gaussian plume solution, we introduce the new variable

\[
r = \frac{1}{v} \int_0^x K(\xi) \, d(\xi),
\]

By eliminating variable \( K \) and letting \( U(r(x), y, z) = u(x, y, z) \), the analytical solution

\[
U(r, y, z) = \frac{Q}{4\pi vr} \exp\left(-\frac{y^2}{4r}\right) \left[ \exp\left(-\frac{(z-H)^2}{4r}\right) + \exp\left(-\frac{(z+H)^2}{4r}\right) \right].
\]
The proof can be found in standard arguments [17], and this equation is, in general, referred to as the Gaussian plume solution. We remark that although the nice mathematical setting and the analytical solution of this model is finally derived, the assumptions A1–A7 are quite strong and restricted us to specific applications such as the contamination from factories which usually located on flat ground and the domains of consideration are only several kilometers away [17]. Notice that, in the case of the complex domain, for example, the city surrounded by mountains, some properties in the Gaussian plume model are not fulfilled. To correct the Gaussian plume model for this limitation, some modified assumptions are added. The modification of the Gaussian plume model is discussed in the following section.

2.2. Reduction to two-dimensional advection-diffusion equations

In this work, we focus on the dispersion of smoke from agriculture waste burning. The evolution of air pollutants can cover a much larger distance in the range of several hundred kilometers of the location with complex terrain such that cities surrounded by mountains. Therefore, a new set of assumptions comparing to the Gaussian plume model should be proposed. The main idea is that we focus on spreading the contaminant in the \((x, y)\) plane instead of diffusion in \(z\) direction. In addition, we assume that the pollution stays inside the thin atmosphere layer near the surface compared with the affected distance in \(x\) and \(y\) direction on a scale of hundred kilometers. Therefore, the domain of consideration is a thin layer and can be assumed for simplicity to have a homogeneous in \(z\) direction. According to these assumptions, the density \(u\) depends only on \((x, y)\), i.e., \(u(x, y) = u(x, y, z)\).

Furthermore, the assumptions concerning the velocity of wind are added. In our case, the vector field \(\mathbf{b}\) is taken into account by the airflow with different speeds related to the height of a terrain. From the mathematical points of view, it can be written in the form of a linear relation between the averaged velocity at position \((x, y)\) and the height of terrain on that position, i.e.,

\[ \mathbf{b}(x, y) = \mathbf{b}(x, y, z) = h(x, y)\mathbf{v}(x, y). \]

Additionally, turbulent diffusive coefficients are assumed to be homogeneous with a similar form as in the case of the Gaussian plume,

\[ \mathcal{K}(x, y) = \mathcal{K}(x, y, z) = \text{diag}(K_x(x, y), K_y(x, y), K_z(x, y)). \]

By setting the above assumptions, all derivative terms respect to \(z\) vanish, and the equation (2) is reduced to

\[ \hat{\nabla} \cdot (h(x, y)\mathbf{v}(x, y)u(x, y) - \hat{\mathcal{K}}(x, y)\hat{\nabla}u) = f(x, y), \] (7)

where \(\hat{\nabla} = (\partial_x, \partial_y)\) and two-dimensional second order tensor,

\[ \hat{\mathcal{K}}(x, y) = \text{diag}(K_x(x, y), K_y(x, y)). \]

With this model setting, we can adopt the Gaussian plume model to any flow direction on the two-dimensional domain and allows us to simulate the propagation of pollution densities on a terrain with different height. Numerical experiments and simulation results can be seen in section 4.

For a further simplicity of notation and consideration, the assumption over the homogeneous in all direction of \(\mathcal{K}\) can be made where \(K_x(x, y) = K_y(x, y) =: \mu(x, y)\). Throughout the remainder of the paper, variables and operators are restricted to two-dimensional domain such
that on space $x = (x, y) \in \mathbb{R}^2$, $\nabla = \mathbf{\hat{n}} = (\partial_x, \partial_y)^T$. As a consequence, the equation (7) can now be rewritten as

$$\nabla \cdot (bu - \mu \nabla u) = f, \quad (8)$$

where here $b = h(x, y)v(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector function of velocity with zero inflow and free flows for outflow boundary.

Next, we state the problem in a more proper mathematical setting by first letting $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz-continuous boundary $\Gamma = \partial \Omega$. We suppose that the boundary are split into disjoint subsets of non-zero measure according to $\Gamma = \Gamma^+ \cup \Gamma^-$ where $\Gamma^+ = \{ x \in \Gamma : b(x) \cdot n(x) < 0 \}$ is defined to be an inflow boundary and $\Gamma^- = \{ x \in \Gamma : b(x) \cdot n(x) \geq 0 \}$ to be an outflow boundary where $n(x)$ denotes a normal vector at the point $x$ on the surface oriented pointing outside the domain. Moreover, we denote that $\Gamma_D$ and $\Gamma_N$ are parts of boundary assigned for Dirichlet and Neumann condition, respectively. Hence, for our problem we imposed a homogeneous Dirichlet condition over the inflow boundary conditions and Neumann condition on the outflow boundary as follows

$$u = 0 \quad \text{on} \quad \Gamma_D, \quad (9a)$$

$$u \chi_{\Gamma^D} - \mu \nabla u \cdot n = 0 \quad \text{on} \quad \Gamma_N, \quad (9b)$$

In practice, we restrict our problem to the case where $\Gamma^+ = \Gamma_D$, then it follows that $\Gamma^- = \Gamma_N$. Therefore the boundary condition (11c) reduces to

$$\mu \nabla u \cdot n = 0 \quad \text{on} \quad \Gamma_N. \quad (10)$$

Summarizing, we obtain the following air pollution on a complex terrain model:

$$\nabla \cdot (bu - \mu \nabla u) = f \quad \text{in} \quad \Omega, \quad (11a)$$

$$u = 0 \quad \text{on} \quad \Gamma_D, \quad (11b)$$

$$\mu \nabla u \cdot n = 0 \quad \text{on} \quad \Gamma_N, \quad (11c)$$

In the next section, we investigate the properties of the air pollution in a complex terrain model based on the mass conservative form of the advection-diffusion equations (11). In particular, the existence and uniqueness of solution to (11) is proved. Moreover, the results of numerical simulations are discussed in Section 4.

### 3. Well-posedness of model problem

To prove the solvability of the system of advection-diffusion equations (11) we suppose the following hypothesis:

**Assumption 1.** we assume that

(i) The diffusivity parameter $\mu$ is positive number and $\mu \geq \mu_0 > 0$.

(ii) The velocity $b(x) \in (L^\infty(\Omega))^2$ is vector function with $\nabla \cdot b \in L^2(\Omega)$ and $0 < \gamma_0 \leq \nabla \cdot b \leq \gamma_1$.

(iii) $f \in L^2(\Omega)$

We define

$$H^1_D(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \}. \quad (12)$$

Let $v \in H^1_D(\Omega)$. By multiplying the differential equation (11a) with $v$ and integrating over $\Omega$, we obtain

$$\int_{\Omega} -\nabla \cdot (\mu \nabla u)v \, dx + \int_{\Omega} \nabla \cdot (bu)v \, dx = \int_{\Omega} f v \, dx.$$
By applying Green’s formula we get
\[
\int_{\Omega} \mu \nabla y \cdot \nabla v \, dx + \int_{\Omega} \nabla \cdot (b u) v \, dx = \int_{\Omega} f v \, dx + \int_{\partial \Omega} \mu \frac{\partial u}{\partial n} v \, ds,
\]
which can also be rewritten as
\[
\int_{\Omega} \mu \nabla u \cdot \nabla v \, dx + \int_{\Omega} \nabla \cdot (b y) v \, dx = \int_{\Omega} f v \, dx + \int_{\Gamma_D} \mu \frac{\partial u}{\partial n} v \, ds + \int_{\Gamma_N} \mu \frac{\partial u}{\partial n} v \, ds.
\]
We impose that on \( \Gamma_D \) the test function \( v \) is null and by applying boundary conditions, it follows that
\[
\int_{\Omega} \mu \nabla u \cdot \nabla v \, dx + \int_{\Omega} \nabla \cdot (b u) v \, dx = \int_{\Omega} f v \, dx.
\]
Therefore, the weak formulation for the problem (11) is presented as follows

**Definition 1.** A function \( y \in H^1_{\Gamma_D}(\Omega) \) is called weak solution of the boundary value problem (11) if it satisfies the weak formulation
\[
\int_{\Omega} \mu \nabla u \cdot \nabla v \, dx + \int_{\Omega} \nabla \cdot (b u) v \, dx = \int_{\Omega} f v \, dx,
\]
for all \( v \in H^1_{\Gamma_D}(\Omega) \).

Let \( V := H^1_{\Gamma_D}(\Omega) \). To write the weak formulation (13) in a more compact way, we introduce the bilinear form \( a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R} \),
\[
a(u,v) = \int_{\Omega} \mu \nabla u \cdot \nabla v \, dx + \int_{\Omega} \nabla \cdot (b u) v \, dx,
\]
and the following linear functional \( F : V \rightarrow \mathbb{R} \) for a given function \( f \in L^2(\Omega) \),
\[
F(v) = \int_{\Omega} f v \, dx.
\]
Thus, the weak formulation of problem (11) becomes
\[
\text{find } u \in V : \quad a(u,v) = F(v) \quad \text{for all } v \in V.
\]

**Lemma 1.** Let Assumption 1 hold. Then:

(i) the bilinear form defined in (14) satisfies
\[
|a(u,v)| \leq \beta_1 \|u\|_V \|v\|_V, \quad \forall u,v \in V,
\]
\[
a(u,v) \geq \beta_2 \|v\|_V^2, \quad \forall v \in V,
\]
with constants \( \beta_1, \beta_2 > 0 \).

(ii) the linear form defined in (15) is continuous functional.

**Proof.** First, we focus on the bilinear form \( a : V \times V \rightarrow \mathbb{R} \) defined in (14). This mapping is well defined as \( \mu \) is bounded, and the functions \( u, v \) as well as their gradients \( \nabla u \) and \( \nabla u \) are in \( L^2(\Omega) \), which has the existence of the first integral in (14) as a consequence. We know that
\[
\nabla \cdot (b u) = b \cdot \nabla u + u \nabla \cdot b.
\]
Since $b$ is essentially bounded and $\nabla \cdot b \in L^2(\Omega)$, the second term of bilinear form is also well defined.

Next, we prove that the bilinear form is bounded. Let $u, v \in V$.

By Cauchy-Schwarz inequalities and the inequality, it follows that

$$\|\nabla w\|_{L^2(\Omega)} \leq \|w\|_{H^1(\Omega)}, \quad \forall w \in H^1(\Omega).$$

The first term on the right-hand side of (14) can be bounded as follows

$$\left| \int_{\Omega} \mu \nabla u \cdot \nabla v \, dx \right| \leq \int_{\Omega} |\mu \nabla u \cdot \nabla v| \, dx \leq \|\mu\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq \|\mu\|_{L^\infty(\Omega)} \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

For the second term, we have

$$\left| \int_{\Omega} \nabla \cdot (bu) v \, dx \right| \leq \int_{\Omega} |\nabla \cdot (bu) v| \, dx \leq \int_{\Omega} |(b \cdot \nabla u + y \nabla \cdot (b)) v| \, dx \leq \int_{\Omega} |b \cdot \nabla u| \, dx + \int_{\Omega} |u \nabla \cdot (b) v| \, dx \leq \|b\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \gamma_1 \|y\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|b\|_{L^\infty(\Omega)} \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + \gamma_1 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

Therefore,

$$|a(u, v)| \leq (\|\mu\|_{L^\infty(\Omega)} + \|b\|_{L^\infty(\Omega)} + \gamma_1) \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

This shows that the bilinear form $a(\cdot, \cdot)$ is bounded.

For the coercivity estimate, we consider the first term of bilinear form

$$\int_{\Omega} \mu \nabla v \cdot \nabla v \, dx \geq \mu_0 \|\nabla v\|_{L^2(\Omega)}^2.$$

Since $v \in H^1_{\Gamma_D}(\Omega)$, the Poincaré inequality holds

$$\|v\|_{L^2(\Omega)}^2 \leq c_1 \|\nabla v\|_{L^2(\Omega)}^2,$$

for a suitable positive constant $C$ independent of $v$. Therefore

$$\|v\|_{H^1(\Omega)}^2 = \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 \leq (c_1 + 1) \|\nabla v\|_{L^2(\Omega)}^2,$$

and it follows that

$$\int_{\Omega} \mu \nabla v \cdot \nabla v \, dx \geq \frac{\mu_0}{1 + c_1} \|\nabla v\|_{H^1(\Omega)}^2.$$
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We now move to the advective term.

\[
\int_{\Omega} \nabla \cdot (b v) v dx = \int_{\Omega} \nabla \cdot (b) v^2 dx + \int_{\Omega} (b \cdot \nabla v) v dx
= \int_{\Omega} \nabla \cdot (b) v^2 dx + \frac{1}{2} \int_{\Omega} b \cdot \nabla (v^2) dx
= \int_{\Omega} \nabla \cdot (b) v^2 dx - \frac{1}{2} \int_{\Omega} \nabla \cdot (b) v^2 dx + \frac{1}{2} \int_{\partial \Omega} b \cdot n v^2 ds
= \frac{1}{2} \int_{\Omega} \nabla \cdot (b) v^2 dx + \frac{1}{2} \int_{\Gamma_D} b \cdot n v^2 ds + \frac{1}{2} \int_{\Gamma_N} b \cdot n v^2 ds.
\]

We know that \( v = 0 \) on \( \Gamma_D \) and \( b \cdot n > 0 \) on \( \Gamma_N \). Therefore

\[
\int_{\Omega} \nabla \cdot (b v) v dx = \frac{1}{2} \int_{\Omega} \nabla \cdot (b) v^2 dx + \frac{1}{2} \int_{\Gamma_N} b \cdot n v^2 ds \\
\geq \frac{1}{2} \mu_0 \| v \|_{L^2(\Omega)}^2.
\]

We can conclude that

\[
a(v, v) = \int_{\Omega} \mu \nabla v \cdot \nabla v dx + \int_{\Omega} \nabla \cdot (b v) v dx \geq \frac{\mu_0}{1 + c_1} \| v \|_{H^1(\Omega)}^2.
\]  

Finally, we show that \( F \) is bounded.

\[
\left| \int_{\Omega} f v dx \right| \leq \| f \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)} \leq \| f \|_{L^2(\Omega)} \| v \|_{H^1(\Omega)}.
\]

\[\tag{18}\]

**Theorem 1.** Suppose that Assumption 1 is satisfied. Then the weak problem (16) possesses a unique solution \( y \in V \) and this solution satisfies the following estimation

\[
\| u \|_{H^1(\Omega)} \leq \frac{1 + c_1}{\mu_0} \| f \|_{L^2(\Omega)}. \]

\[\tag{19}\]

**Proof.** Existence and uniqueness of solution of (16) follows directly by using the results from Lemma 1 and then apply the Lax-Milgram Lemma. For the estimation, we can use the results from (17) and (18) by choosing \( v = u \) such that

\[
\frac{\mu_0}{1 + c_1} \| u \|_{H^1(\Omega)}^2 \leq |a(u, u)| = |F(u)| \leq \| f \|_{L^2(\Omega)} \| u \|_{H^1(\Omega)},
\]

then the estimation (19) holds.  

**4. Numerical results**

This section aims to present numerical experiments that illustrate the validity of the computational framework addressed in this work. Numerical experiments are divided into two parts. In the first part, the example of advection-diffusion equations with the exact solution is tested to perform the efficiency of discontinuous Galerkin methods. In the second part, we discuss the experiment with the domain of consideration is non-smooth terrain. In this case, the domain around Chiang Mai with an artificial source representing the contaminate density is tested. Results of numerical experiments validate our mathematical model describing the evolution of air pollutants on location with the complexity of terrains.
4.1. Numerical evaluation with exact solution

We consider the advection-diffusion equation (11) in two-dimensional spatial domain $\Omega = (1, 1)^2$. For convenience, the diffusive parameter and the velocity field are chosen to be $\mu = 1$ and $b = (1, 1)^T$, respectively. Moreover, the source term on the right hand side is given by

$$f = 2\alpha e^{\alpha(x^2+y^2)} \left(-2\mu(1 + \alpha(x^2 + y^2) + b_1x + b_2y)\right), \quad (21)$$

where the parameter $\alpha = -20$. The corresponding exact solution is presented as the following

$$u(x,y) = e^{\alpha(x^2+y^2)}. \quad (22)$$

This exact solution is the Gaussian function centered at the origin point as shown in figures with mesh size $h = 3.125e-2$. The results with $P_2$ elements on the triangle elements gives the order of convergence $\approx 3$ as expected, i.e., $k + 1$ for $P_k$ elements.

4.2. Numerical experiment with complex terrain

In this part, numerical experiments demonstrate the propagation of air pollution on a complex domain chosen to be the area of Chiang Mai province with an approximate size of $135 \times 135$ km$^2$ shown in Fig.1. The vector wind field is assumed to be in the South-west direction with the maximum speed of 20 km/h, and a minimum speed of 2 km/h linearly depended on the height of the terrain. We assume to have an artificial source of pollution, for example, smoke from a forest fire or agricultural waste burning in the upper right corner of the domain. The Gaussian function is used as a normalized source function such that it has a maximum of 1. Fig. 2 show the pollutant concentration results with corresponding to diffusive parameters $\mu = 1e-3$, $\mu = 5e-3$, and $10e - 3$, respectively. It can be seen that mass densities are concentrated are high in the low area and low in the area such as a mountain.

| $h$     | $L_2$-error | order |
|---------|-------------|-------|
| 2.500e-1| 2.2178e-01  | -     |
| 1.250e-1| 5.3171e-02  | 2.05  |
| 6.250e-2| 3.5875e-03  | 3.88  |
| 3.125e-2| 4.3131e-04  | 3.05  |

Table of $h$-size, $L_2$-error and order of convergence.

Figure 1. Terrain profile of area around Chiang Mai surrounded by mountains
5. Conclusion

The mathematical modeling of air pollutant dispersion distributed from open burning biomass located in complex terrain is addressed. This model is the modified Gaussian plume model. The behavior of air pollutants is described by steady-state advection-diffusion equations. Both theoretical and numerical aspects of these systems are investigated. In particular, the existence and characterization properties of solutions are proved.

Compared to the Gaussian plume model, this model can account for the height profile of the terrain to give effect to the velocity on the transport term. The most advantage of this method is that we can drop the height variable, which usually shown in the 3D domain to the 2D problem. This can reduce a significant amount of computation. In this work, the linear dependence between the height and wind velocity is applied, and this assumption can be improved in the future work for better results. From a theoretical point of view, the necessary conditions are shown in the assumption 1, and the important condition is that the divergence term of velocity $\nabla \cdot \mathbf{b}$ must be greater than 0 and bounded. For the sake of keeping the well-posedness of the problem, numerical treatment needs to be handled in the domain where lacking this condition.

In addition, air pollutant dispersion problems are solved numerically with the discontinuous Galerkin method. Numerical simulations of air pollution on complex terrain domains in Chiang Mai, Thailand are tested. Results of numerical experiments are demonstrated and validated the model.

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