PARAMETER OPTIMIZATION OF MULTI-AGENT FORMATIONS
BASED ON LQR DESIGN*

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Abstract. In this paper we study the optimal formation control of multiple agents whose interaction parameters are adjusted upon a cost function consisting of both the control energy and the geometrical performance. By optimizing the interaction parameters and by the linear quadratic regulation (LQR) controllers, the upper bound of the cost function is minimized. For systems with homogeneous agents interconnected over sparse graphs, distributed controllers are proposed that inherit the same underlying graph as the one among agents. For the more general case, a relaxed optimization problem is considered so as to eliminate the nonlinear constraints. Using the subgradient method, interaction parameters among agents are optimized under the constraint of a sparse graph, and the optimum of the cost function is a better result than the one when agents interacted only through the control channel. Numerical examples are provided to validate the effectiveness of the method and to illustrate the geometrical performance of the system.

Key words. formation control, LQR, parameter optimization

AMS subject classifications. 15A15, 15A09, 15A23

1. Introduction. The study of formations for a group of agents is inspired by the behaviors of various animal species in nature. For instance, fish, birds and ants always work in a cooperative manner so as to accomplish tasks that are beyond the capability of single ones. The formation of agents has its wide range of applications both in civilian and military life, where there are generally three basic parts: the determination of the underlying graph, the development of cooperative algorithms among agents that concerns with an exact assignment and the deployment of controllers that are responsible for the stability of the overall system.

Upon the construction of the underlying communication graphs, [1, 24] discovered that a geometry with an underlying graph being a rigid one or a persistent one would not transform under smooth movements once settled. Meanwhile, other graph properties such as connectivity, strong connectivity or even the existence of a spanning tree are proved to be sufficient to the convergence of consensus algorithms [7, 18, 6, 21].

In general, the optimization objectives in a formation system include costs of either the relative formation errors [16], the absolute position errors [17, 19] or the minimal traveling distances [25]. On the other hand, [2] discussed the optimal relative layout of wheeled robots in terms that the kinetic energy is minimized. [4] focused on distributed control design of large-scale dynamically isolated systems using LQR optimization. In the trajectory planning of formation reconfiguration, the main concerns are the minimal executing time [20], the fuel consumption [12] and the kinetic energy expenditure [13]. However, nearly all of the studies made the assumptions that the local dynamics of agents were pre-specified ones and agents were only coupled through the control channels rather than being dynamically related in the open loop.
In real time applications such as sensor networks, it is always flexible for us to control the behavior of each individual agent by designating its interaction rules rather than simply setting to zero. Thus problem arises on how to utilize this flexibility on interaction parameters so as to improve the performance of a formation system? Here by performance we focus on the geometries’ deformations with respect to the desired one during convergence and the control energy cost.

Although the problem of parameter optimization is brand new in formation control, in recent years, a similar problem has been discussed in the field of consensus algorithms as seen in a limited number of literatures. To our best knowledge, the optimal weight design over a fixed network topology is addressed in [22] and [23]. The optimal weight design under random networks and switching networks are explored in [8] and [9] respectively.

In conventional formation control that concerns formation maintenance while moving towards a destination, the discussion is divided into two subsequential steps with the first one being the relative formation maintenance or formation stabilization irrespective of the destination. For the second step, all agents in formation move towards the destination led by one or two agents known as the formation leader. This process is always termed destination attainment. During the process, for the purpose of formation maintenance, the translational speed of the formation system is bounded due to the physical restrictions of each agent. In bearing-only sensor-target localization, when a fleet of UAVs aiming to locate a moving object accurately and as fast as possible, they are expected to maintain an optimal geometry[3] during the entire process. In such a case, formation maintenance is more important. On the contrary, in payload transportation, formation maintenance is not strictly demanded while a rapid task execution time is the major concern.

To this end, in this paper we study the optimal formation control by taking both the above two indicators into account simultaneously. With different ratios of attentions being paid to those two indicators, we define three kinds of cooperative performance and characterize each of them using a quadratic cost matrix. In order to find the optimal interaction parameters between agents such that the upper bound of the cost function is minimized, the LQR control strategy is considered. Although in some literatures distributed optimization was discussed over dynamically isolated agents, i.e., agents were interconnected only through the control channel[15, 3], we aim to discover that the optimum in those literatures could be further minimized through some carefully selected interaction parameters among agents.

The remainder of this paper is organized as follows: In Section II, we introduce the basic theories on LQR control into the optimal formation control. In Section III, a simple system with two agents is considered to illustrate the relationship between different kinds of cooperative performance and the cost function we propose. In Section IV, the optimization problem with multiple agents is investigated. For the special case with homogeneous agents, we propose distributed controllers design method by considering dynamically isolated agents. For the more general case with heterogeneous agents, in order to avoid the nonlinearity in the constraint conditions, a relaxed optimization problem is further proposed and the structured parameter matrix is calculated based on the subgradient method. In Section V simulations are presented to validate those theoretical results and to demonstrate the relationships between the cost function and the cooperative performance. Finally conclusions are given in Section VI.
2. Preliminary. Let $\mathbb{R}_S$ denote the set of real symmetric matrices, $\mathbb{R}_S^+$ the set of positive definite matrices and $\mathbb{R}_A$ the set of $n \times n$ diagonal matrices. The set of eigenvalues of matrix $A$ is denoted by $\lambda(A)$ where the largest one is $\bar{\lambda}$. The $n \times n$ identity matrix is $I_n$ and its $i$th column is denoted by $e_i$. $\mathbb{M}$ is the set of nonsingular matrices. Let $A_{i,j}$ be the $i,j$th entry, then the $i$th block of a matrix $A$ and the $i$th block of a vector $v$ are defined by

$$
A_{i,j} = \begin{bmatrix} A_{2i-1,2j-1} & A_{2i-1,2j} \\ A_{2i,2j-1} & A_{2i,2j} \end{bmatrix}, 
v_i = \begin{bmatrix} v_{2i-1} \\ v_{2i} \end{bmatrix}
$$

respectively. We use $\text{diag}(v), v \in \mathbb{R}^n$ or $\text{diag}([v_i])$ to denote a diagonal matrix with diagonal entries from vector $v$. Similarly, $\text{diag}(A)$ when $A \in \mathbb{R}^{n \times n}$ is a vector with entries from the diagonal of $A$. $\Lambda(A)$ is a diagonal matrix that has the same diagonal entries as matrix $A$. We have $\Lambda(A) = \text{diag}(\text{diag}(A))$. The 2-block diagonal of matrix $A$ is denoted by $\bar{\Lambda}(A)$ where $\bar{\Lambda}(A)_{ii} = A_{ii}$ and all the other blocks are zeros.

An undirected graph is denoted by $G = (V, E)$ with $|V|$ vertices and $|E|$ edges. Node $i$ is a neighbor of node $j$, denoted by $i \sim j$, if they are connected by an edge in $E$. We do not consider self-loops in a graph. The adjacency matrix of $G$ is $\mathcal{A}(G) \in \mathbb{R}_S^{|V| \times |V|}$ and is determined by

$$
\mathcal{A}(G)_{ij} = \begin{cases} 
1, & i \sim j \\
0, & \text{otherwise}
\end{cases}
$$

A Graph $G$ is uniquely determined by an adjacency matrix thus $G$ is also said to be generated from $\mathcal{A}$.

The incidence matrix of $G$ is the $|E| \times |V|$ matrix $H$ defined by $H_{i,j} = 1$ if vertex $j$ is an endpoint of edge $i$ and $H_{i,j} = 0$ otherwise. The oriented incidence matrix $\bar{H}$ is defined by replacing a 1 in each row by a $-1$.

For a symmetric matrix $A$ and an adjacency matrix $\mathcal{A}$ such that $A = A \circ \mathcal{A}$ where $\circ$ is the Hadamard product, the underlying graph of matrix $A$ is the one that is generated from $\mathcal{A}$.

Consider a system modeled by the standard state-space equation $\dot{x} = Ax + Bu$ where $A, B$ is controllable and the quadratic cost function

$$
J(x(0), u) : = \int_{0}^{\infty} x^T(t)Qx(t) + u^T(t)u(t)dt
$$

(2.1)

where $Q \in \mathbb{R}_S^+$ and $x(0)$ is the initial state. $\int_{0}^{\infty} x^T(t)Qx(t)$ is called the energy expenditure of $x$. Based on LQR control theory, if we consider the state-feedback controller

$$
u(t) = -B^TX^+x(t)
$$

(2.2)

where $X^+$ is the unique positive definite solution to the algebraic Riccati equation

$$
AX + XA^T - XBB^TX + Q = 0
$$

(2.3)

then the cost function $J$ in (2.1) achieves its minimum of

$$
\bar{J}(x(0)) = x(0)^T X^+x(0)
$$

(2.4)

In order to concentrate on the parameter design issue, throughout the paper it is assumed that $\|x(0)\|_2 = 1$. For the worst case, $\bar{J}$ is upper bound by $\bar{J} = \lambda(X^+)$, and minimizing this upper bound of $\bar{J}$ is the main concern in this paper.
However, $\bar{J}$ is the minimal value of $J$ only if the matrices $A, B$ and $Q$ in the ARE (2.3) are determined ones. In a formation system, the interactions between neighboring agents, i.e., the off-diagonal entries in the system matrix $A$, are not any intrinsic properties but rather some pending parameters to be designed. Therefore, we focus primarily the following optimization problem

$$J^* = \min_{A'} \sup_{A'} \bar{J}(A)$$

s.t. $u = -X^+x, \|x(0)\|_2 = 1$

$A = A_0 + A', A_0 = \text{diag}(A)$

This paper aims to discuss this problem so as to exploit the relationships between the interaction parameters $A'$ among agents and the cooperative performance, and to further understand how to design a formation system that meet the desired cooperative performance in an optimistic way.

The following assumption is made true throughout the paper

**Assumption 2.1.** The local dynamics of each individual agent is fixed and known a priori.

The system architecture we considered consists of two layers one of which is the communication topology among agents as depicted by graph $G_\alpha$. When agents are dynamically isolated, $G_\alpha$ is an empty graph. In the other layer, the controllers of individual agents communicate with one another over an underlying graph denoted by $G_\beta$.

### 3. Optimal control of two agents formations.

In this section we focus on a simple system with two identical agents to illustrate the novel optimization problem we consider. More specifically, apart from minimizing the control energy, we also dive into the convergence process during destination attainment and distinguish different kinds of cooperative performance that meet various tasks requirements.

We consider two agents on positions $x_1$ and $x_2$ moving towards their destinations $\bar{x}_1$ and $\bar{x}_2$ respectively along parallel lines and in a cooperative manner, as shown in Fig. 3.1. Generally there are two different cooperative behaviors for the two agents one of which is the convergence to the destination of each individual agent, as measured by the position error $\delta x_i = x_i - \bar{x}_i, i = 1, 2$. The other one is the formation maintenance between agents, as represented by the relative formation error of each agent $i$: $|\delta x_1 - \delta x_2|$. Three kinds of performance could be considered concerning these two behaviors:

- **Destination-first performance:** each of the agents concentrates on moving towards its own destination in a selfish way. In Fig. 3.1(a) agent 1 left agent 2 far behind although $x_1 = \bar{x}_1$ at $t = 1$.
- **Formation-first performance:** each of the agents pays too much attention to the other agent and tries to maintain the relative formation on its best efforts. In Fig. 3.1(b) agent 1 does not move until agent 2 catches up so that $|\delta x_1 - \delta x_2| = 0$.
- **Neutral performance:** each agents find itself a balance between the above two kinds of performance, i.e., efforts are put on formation maintenance and destination attainment simultaneously, as for example shown in Fig. 3.1(c).

For the two agents formations, those three kinds of performance could be captured by three parameters $\gamma_1, \gamma_2$ and $\gamma_3$ in

$$J_1 = \int_0^\infty \gamma_1(\delta x_1 - \delta x_2)^2 + \gamma_2 \delta x_1^2 + \gamma_3 \delta x_2^2 dt$$  \hspace{1cm} (3.1)
where $\gamma_1 > 0$ corresponds to the importance of formation maintenance and $\gamma_2 > 0, \gamma_3 > 0$ correspond to the importance of approaching the destination. Thus when $\gamma_1 \ll \gamma_2$ and $\gamma_1 \ll \gamma_3$, the destination-first performance is captured, and similarly the condition that $\gamma_1 \gg \gamma_2$ and $\gamma_1 \gg \gamma_3$ correspond to the formation-first performance. When $\gamma_1 \gamma_2 \gamma_3 \neq 0$, $J_1$ combines both of the two kinds of performance with weights $\gamma_i$. In this case, when $J_1$ reaches a stable value, $\delta x_1 = 0$, which indicates agent 1 and agent 2 has attained the expected geometry, and $\delta x_1 = \delta x_2 = 0$, which indicates the achievement of the destination.

If we set the entries of the quadratic matrix $Q$ in (2.1) as $q_{11} = \gamma_1 + \gamma_2$, $q_{12} = -\gamma_1$ and $q_{22} = \gamma_1 + \gamma_3$, $J_1$ then has the compact form of $\int_0^{\infty} \delta x^T Q \delta x dt$. Thus the optimization problem concerning the two-agent formation system could be stated as

**Problem 3.1.** Consider two agents with identical local dynamics

$$\dot{x}_i(t) = ax_i(t), i = 1, 2$$

(3.2)

and are interconnected through parameters $r_1, r_2 \in \mathbb{R}$. The overall system is modeled by

$$\bar{S} : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & r_1 \\ r_2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} := Ax + u$$

(3.3)

Assume the quadratic matrix $Q = \begin{bmatrix} q & m \\ m & q \end{bmatrix} > 0$. Find the solution to the optimization problem

$$\min_{r_i \in \mathbb{R}} \|X^+\|_2$$

s.t. $2ax_{11} - x_{11}^2 + 2r_2 x_{12} - x_{12}^2 + q = 0$  
(3.4)

$2r_1 x_{12} - x_{12}^2 + 2ax_{22} - x_{22}^2 + q = 0$  
(3.5)

$4ax_{12} - 2x_{11} x_{12} + 2r_2 x_{22} - 2x_{12} x_{22} + 2r_1 x_{11} + 2m = 0$  
(3.6)

$$X^+ = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{22} \end{bmatrix} \in \mathbb{R}^+_S$$

**Remark 3.1.** When $Q$ is a positive definite matrix and when $B = I$, the existence of the positive solution $X^+$ to the optimization problem is guaranteed.

The solution to the above optimization problem is discussed in details in Section 4. Here we use a numerical example to show that the optimal solution is obtained when $r_1 \neq 0$ and $r_2 \neq 0$. 

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**Fig. 3.1.** The three kinds of cooperative performance
Example 3.1. Consider the cost matrix
\[
Q = \begin{bmatrix}
2 & -1 \\
-1 & 2
\end{bmatrix}
\]
According to (3.1), \( Q \) indicates that the maintenance of formation is as important as the convergence to the destination.

The local dynamics of each agent is uniquely chosen as a single integral with \( a = 2 \). First we consider agents with no interactions, i.e., \( A_1 = 2 \text{diag}(I) \) and \( r_1 = r_2 = 0 \). By solving Problem 3.1, one can obtain the optimal upper bound of the cost value \( \bar{J} = 4.65 \).

On the other hand, if interconnections between agents are considered, for example
\[
A_2 = \begin{bmatrix}
2 & 1.2 \\
-1 & 2
\end{bmatrix},
\]
the 2-norm of the solution \( X^+ \) is \( \bar{\lambda}(X^+) = 4.47 \), which is better than the previous one. The corresponding closed-loop system matrix is
\[
A_c = A_2 - X^+ = \begin{bmatrix}
-2.46 & 1.21 \\
-0.98 & -2.44
\end{bmatrix}
\]

Conjecture 3.1. If the local dynamics of agents are set to be \( \delta\dot{x}_i(t) = 2\delta x_i(t), i = 1, 2 \), for the worst case, the system with interaction \( A_2 \) has better formation performance and consumes less control energy than the situation when the two agents are disjoint.

4. Optimal Formation Control of Multiple Agents. For a group of agents that is assigned with cooperative payload transport tasks, each agent aims to approach its destination while maintaining a certain geometry during the process. In most cases, one may have to choose either the destination-first performance which may result in formation failure during the convergence, or the formation-first performance which may increase the task execution time. Similar issues arise in other situations such as formation reconfiguration and obstacle avoidance. Thus a balance between those two kinds of performance is demanding and is worth of exploration.

Consider a set \( C = \{Q \in \mathbb{R}^+_S: q_{ii} > 0, q_{ij} < 0\} \). Then the three kinds of performance introduced in Section 3 could be represented by choosing appropriate entices of \( Q \in C \).

Under Assumption 2.1 we decouple the matrix into
\[
A = A_0 + A'
\]
where \( A_0 = \hat{\Lambda}(A) \) if the dynamics of each agent is in two dimensions and \( A_0 = \Lambda(A) \) if it is in one dimension. Specially, when \( A' = 0 \), all agents are dynamically isolated with each other in the open loop.

Based on the LQR control, when focusing on the worst case, the optimization problem is formalized as

**Problem 4.1.**
\[
\begin{align*}
\min_{A'} & \|X\|_2 \\
\text{s.t.} & \quad A^T X + X A - X B^T B X + Q = 0 \\
& \quad A = A_0 + A', \ A_0 = \hat{\Lambda}(A), \ (A, B) \text{ controllable} \\
& \quad Q \in C, \ X \in \mathbb{R}^+_S
\end{align*}
\]
When $A'$ solves the optimization problem, $(A, Q)$ is called a matched pair and the corresponding positive solution of the ARE (4.1) is denoted by $X^*(A, Q)$ which is sometimes abbreviated by $X^*$.

Following are two well-proved lemmas from [14] that motivate our research.

**Lemma 4.1.** Consider the algebraic Riccati equation (4.1). The unitary stable hermitian solution $X^+$ is a continuous function of the 3-tuple $(A, Q, B^T B) \in \Phi$ where $\Phi : \{(A, Q, B^T B) | Q \in \mathbb{R}^+_S, (A, B) \text{ stabilizable}\}$. When $A, B, Q$ are continuous functions of a vector $\gamma$, so does $X^+$.

**Lemma 4.2.** Let $\Omega_e(A, Q, B^T B) = \{(\bar{A}, \bar{Q}, \bar{B}^T \bar{B}) | \|\bar{A} - A\| + \|\bar{Q} - Q\| + \|\bar{B}^T \bar{B} - B^T B\| < \varepsilon\}$. For every $(A, Q, B^T B) \in \Phi$ there exist positive constants $K$ and $\epsilon$ such that

$$
\|\bar{X}^+(\bar{A}, \bar{Q}, \bar{B}^T \bar{B}) - X^+(A, Q, B^T B)\| \leq K(\|\bar{A} - A\| + \|\bar{Q} - Q\| + \|\bar{B}^T \bar{B} - B^T B\|)^{1/2}
$$

(4.2)

is satisfied for all $(\bar{A}, \bar{Q}, \bar{B}^T \bar{B}) \in \Omega_e(A, Q, B)$. $X^+$ is said to be a locally Lipschitz continuous function of order $1/2$. These properties of the solution $X^+$ guarantee the feasibility of the optimization problem (4.1).

Recall Problem (4.1) although the objective function is convex, the constraints of the ARE (4.1) on arguments $X^+$ and $A_0$ are nonlinear, and the variable $X^+$ is expected to be symmetric, which falls into the category of the nonlinear semidefinite programming (NSDP) problem. Meanwhile, this minimax problem is nonsmooth on $X^+$, thus most of the analytical optimization techniques are inapplicable. Although the nonlinear optimization toolbox in Matlab may sometimes give structured solutions on both $A'$ and $X^+$, generally special cares are required so as to accelerate the optimization process and it may sometimes fail.

On the other hand, even if there are some analytical iterative methods to solve the optimization problem (4.1), the optimal solution of $X^+$ and $A'$ are dense matrices which indicates that the underlying graphs $G_\alpha$ and $G_\beta$ are complete graphs. This is obviously not applicative especially for large-scale formation systems. In the remaining context, we will first consider a special case where agents are homogeneous ones. Distributed controllers are considered that inherit the communication graph $G_\alpha$ among agents: $G_\alpha = G_\beta$, as for example shown in Fig. 4.1(a). Further for the general situation, we discuss the optimization problem with topology constraints imposed in the agents layer. Examples of the underlying graphs $G_\alpha$ and $G_\beta$ are illustrated in Fig. 4.1(b).

### 4.1. Distributed Controllers Design for Homogeneous Agents

As mentioned in the preliminary, the formation system is broken down into two layers each
of which has its own communication topology. In practical scenarios, agents and controllers always share communication channels, therefore generally \( G_\alpha = G_\beta \) is more preferred. In this subsection, we consider formation systems consist of homogeneous agents and let \( G_\alpha = G_\beta = G_h \). The following assumptions are also made

**Assumption 4.1.** In a formation system

- the dynamics of each agents is decoupled along \( x \) and \( y \) axis such that we can focus on the one-dimensional model of each agent
- the local dynamics of agents are congruent:
  \[ \dot{x}_i = ax_i \]
- the interaction parameters among any pairs of neighboring agents are congruent:
  \[ \dot{x}_i = ax_i + b \sum_{j \sim i} x_j \quad (4.3) \]
- the input matrix \( B \) is an identical matrix. This assumption is realistic and is easy to be satisfied in a formation system
- the quadratic matrix \( Q \) has identical diagonal entries and identical nondiagonal entries, i.e.,
  \[ \text{diag}(Q) = qI, \quad Q - \Lambda(Q) = pA \]

where \( A \) is the adjacency matrix of the assigned graph \( G_h \).

Assumption 4.1 allows us to decouple the overall formation system into independent subsystems, and thus the design of distributed controllers is also broken down accordingly, which leads to the following controller design method.

**Theorem 4.3.** Consider system \((4.3)\) and the Assumption 4.1. The optimal solution \( X^* \) to Problem 4.1 is

\[ X^* = xI + yA \quad (4.4) \]

where

\[ x = a + \sqrt{a^2 + q}, \quad y = \frac{2p}{2x - 4a} \]

and is the optimal controller when when \( b = \frac{p}{2x - 4a} \).

**Proof.** Under Assumption 4.1, the system matrix \( A \) and quadratic matrix \( Q \) has the special form of

\[ A = aI + bA, \quad Q = qI + pA \quad (4.5) \]

Assume the optimal solution has the form \( X^* = xI + yA \) such that the ARE is rewritten as

\[ (aI + bA)(xI + yA) + (xI + yA)(aI + bA) - (xI + yA)^2 + (qI + pA) = 0 \]

which has the equivalent expression of

\[ 2(axI + bxA + ayA + byA^2) - (x^2I + 2xyA + y^2A^2) + (qI + pA) = 0 \quad (4.6) \]
Let \( P \) be the orthogonal permutation matrix of \( A \) such that
\[
\Lambda_A = P^T A P
\]
where the diagonal entries are the eigenvalues of the adjacency matrix \( A(G) \). Left and right multiplying the left hand side expression in equation (4.6) by \( P^T \) and \( P \) yields
\[
(2ax - x^2 + q)I + (2bx + 2ay - 2xy + p)\Lambda_A + (2by - y^2)\Lambda_A^2 = 0 \tag{4.7}
\]
Due to the feasibility of the ARE and the positive semidefinite constraints on the solution \( X^* \), the parameters \( x, y, b \) are
\[
x = -\frac{2a - \sqrt{4a^2 + 4q}}{-2} \\
b = \frac{p}{2x - 4a} \\
y = 2b \tag{4.8}
\]
Substituting \( b \) by \( y/2 \) back into equation (4.7) and with some trivial calculations, we obtain that
\[
(2a - x)(x + \lambda_i(A)y) + q + \lambda_i(A)p = 0 \tag{4.9}
\]
Due to the expression of \( x \) in (4.8), \( 2a - x < 0 \). As the quadratic matrix \( Q > 0 \), it is also true that \( q + \lambda_i(A)p > 0 \) for all \( i \). Thus equation (4.9) is satisfied if and only if
\[
xI + yA \in \mathbb{R}_s^+
\]
This infers that \( X^* \) with \( x \) and \( y \) given in (4.8) is the positive definite solution and is the optimal controller when \( b = \frac{p}{2x - 4a} \).

**Remark 4.1.** Controller \( K = -X^* \) is the stabilization controller with underlying graph generated from \( A \). Thus the communication structures among agents and controllers are congruent ones, i.e., \( G_\alpha = G_\beta \) as, for example, the one in Fig. 4.1(a).

**Remark 4.2.** The synthesis of controller for each agent is carried out in a distributed way and the control signal passed to each agent is generated using information from the neighbors. This property provides the formation system the feature of scalability.

In [4] the authors focused themselves on the scalable synthesis of distributed controllers for large-scale identical isolated subsystems, i.e. \( A' \equiv 0 \), and further explored the relationship between the local controllers and the stability of the overall system. Next we will show that, compared to the formation system where \( A' \equiv 0 \), a set of carefully designed interaction parameters would indeed give the overall system a better cooperative performance and require less control energy.

**Corollary 4.4.** Consider a formation system consists of \( N \) agents under Assumption 4.1. For a matched pair \( (A_0 + A', Q) \), \( A' \equiv 0 \) if and only if \( Q \in \mathbb{R}_\Lambda \).

**Proof.** Recall (4.7), if \( p = 0 \), due to the positiveness of \( q \), it is necessarily that \( y = 0 \) and \( b = 0 \), which indicate \( A' = 0 \). According to the solutions \( x, y \) and \( b \) in Theorem 4.3 when \( b = 0, q = 0 \), which proves the sufficient condition.

When \( q_{ij} \equiv 0, i \neq j \), it means no cooperative behavior is expected in a formation system. Corollary 4.4 indicates that in such cases system with disjoint agents can
perform better. However, due to the inverse negative proposition of Corollary 4.3, as long as there exists \( q_{ij} \neq 0 \), having agents communicate with others at some appropriate parameters in the open loop is a better choice. Note that these results only apply to systems with identical agents and with equivalent weights being put on the diagonal of the cost matrix \( Q \). Systems with heterogeneous agents are not eligible to this property. Actually, when \( a_{ii} \) is not identical, interactions are required even if \( q_{ij} = 0 \) so as to make up a matched pair.

For a set of homogeneous agents, by finding the appropriate parameter \( b \) between each pair of neighboring agents, we developed the optimal state-feedback controllers that inherits exactly the same underlying structure as the agents, and the upper bounds of the quadratic cost function of the formation system is minimized.

4.2. Heterogeneous Formation Systems under Structure Constraints. In this subsection, we further consider a more general case where all those restrictions in Assumption 4.1 no longer hold true. When the dynamics of agents is coupled along \( x \) axis and \( y \) axis, the dimension of the overall system model is \( 2n \) instead of \( n \) and thus the local dynamics as well as the interaction parameters between neighbors are captured by \( 2 \times 2 \) matrices. In order to find optimal interaction parameters for a formation system where agents communicate over an assigned graph \( G \), we define a set of structured matrices whose underlying graph is \( G \):

\[
T_G = \{ M \in \mathbb{R}^{2n \times 2n} : M_{ij} = M_{ji} = A_{ij} M_{ij}, i \neq j \}
\]

where \( A \) is the adjacency matrix of \( G \).

The state-space model of the overall system is

\[
\dot{p} = Ap + Bu
\]

where \( A \in T_G \cap \mathbb{R}^S \) and \( B \in T_G \cap \mathbb{M} \). Note that for the general case, we restrict ourselves to systems where interactions among agents are independent of the transmission direction, i.e., matrix \( A \) is symmetric. This is only for the purpose of simplicity and all the results below apply to nonsymmetric case as well. Meanwhile, the structure of the input matrix indicates that agents exchanges their states and their control inputs simultaneously with the neighbors.

As mentioned before, the most difficult part when dealing with Problem 4.1 is the nonlinear constraints in the ARE. This inspires us to exploit methodologies to eliminate this restriction.

What we focus on in this paper is the two norm of \( X^+ \). We noticed that the upper bound of the positive solution to the ARE (4.1) has been discussed widely by researchers in the related fields. Quite a few upper bounds are given in literatures among which the following result borrowed from [10] is known as a relatively tighter bound than many others.

Lemma 4.5. Consider ARE (4.1). The norm of the positive semidefinite solution \( X^+ \) is upper bounded by

\[
\|X^+\| \leq \|P\|^{1/2} \|APA + Q\|^{1/2} + \mu(AP)
\]

where \( \mu(AP) = 1/2 \lambda(A \in P^T A) \) and \( P = (B^T B)^{-1} \).

Remark 4.3. This upper bound of \( \|X^+\| \) also validates the fact that the minimal value of \( J \) is correlated with \( A \) and \( Q \), and when \( Q \) and \( B \) are not diagonal matrices, the minimal upper bound of \( \|X^+\| \) is the one when \( A \) is nondiagonal as well.
We then propose the following relaxed optimization problem that is parallel to Problem 4.1 but the nonlinear constraint is avoided. Note that as we consider formation on a plane, the local dynamics between each pair of agents is represented by a square matrix $A'_{ij}$.

Problem 4.2.

$$\min_{A'} \phi(A') = \|P\|^{1/2} \|APA + Q\|^{1/2} + \frac{1}{2} \tilde{\lambda}(AP + PA)$$  \hspace{1cm} (4.12)

$$s.t. \ A = A_0 + A', \ A_0 = \tilde{\Lambda}(A), \ A \in T_G \cap \mathbb{R}^S,$$

$$Q \in T_G \cap C, \ B \in T_G \cap \mathbb{M}, \ P = (B^T B)^{-1}$$

The minimum of $\phi(A')$ is calculated when the the argument $A'$ fits into the desired underlying graph $G$. This minimum could be further reduced when this structure restriction is abandoned, which is, however, accompanied by the increase on the communication cost.

**Theorem 4.6.** The optimization problem (4.12) is convex.

**Proof.** It is well-known that $\tilde{\Lambda}(A)$ is a convex function on $A$. Due to the positiveness and symmetry of $APA + Q$, $\|APA + Q\| = \tilde{\lambda}(APA + Q)$. The conclusion is then self-evident. Despite of the convexity, the function $\phi(A')$ is non-smooth and thus is non-differential, which requires some special optimization techniques. In our case, the constraints are linear matrices constraints, thus it is then an ordinary linear programming (LP) problem. Interior-point methods and subgradient methods are two of the frequently used techniques when dealing with LP, as in [8, 11, 5, 22]. More specifically, a similar problem is discussed in [8] where the objective function $\phi$ is the expectation of the system matrix. Due to this similarity, here we also adopt the subgradient algorithm as in [8] to deal with this structure constrained optimization problem.

Matrix $A$ is actually an affine function of the interaction parameters $A_{ij}, i \sim j$. Assume the underlying graph $G$ has $m$ edges indexed from 1 to $m$. Define a vector $x \in \mathbb{R}^{4m}$ that is partitioned into $m$ sub-vectors $x_k = [x_k^1 \ x_k^2 \ x_k^3 \ x_k^4]^T, k \in [1, m]$. The entries $x_k^p, p \in [1, 4]$ indicate the interactions between agent $i$ and agent $j$ that are connected by edge $l$:

$$\begin{bmatrix} x_k^1 \\ x_k^2 \\ x_k^3 \\ x_k^4 \end{bmatrix} = A_{ij}$$

Then the affine form of $A$ on vector $x$ is

$$A(x) = A_0 + \sum_{k=1,j>i}^{m} \sum_{j>i}^{m} (x_k^1 E_{ij}^1 + x_k^2 E_{ij}^2 + x_k^3 E_{ij}^3 + x_k^4 E_{ij}^4)$$  \hspace{1cm} (4.13)

where

$$E_{ij}^1 = e_{2i-1} e_{2j-1}^T + e_{2j-1} e_{2i-1}^T$$

$$E_{ij}^2 = e_{2i-1} e_{2j}^T + e_{2j} e_{2i-1}^T$$

$$E_{ij}^3 = e_{2i} e_{2j-1}^T + e_{2j-1} e_{2i}^T$$

$$E_{ij}^4 = e_{2i} e_{2j}^T + e_{2j} e_{2i}^T$$
The partial differential of $A'(x)$ or $A(x)$ with respect to $x_k$ is
\[ \frac{\partial A(x)}{\partial x_k} = E_{ij}^p, \quad p \in \{1, 2, 3, 4\} \] (4.14)
and further
\[ L(x_k) \triangleq \frac{\partial (APA + Q)}{\partial x_k} = APE_{ij}^p + E_{ij}^p PA, \quad p \in \{1, 2, 3, 4\} \] (4.15)
The Hessian of $\phi(A')$ is then
\[ H_{ij}^p|_{x=x_k} = M_{ij}^T \frac{\partial (APA + Q)}{\partial x_k} v + \frac{1}{2} u^T \frac{\partial (AP + PA)}{\partial x_k} u 
= M_{ij}^T L(x_k)v + \frac{1}{2} u^T (E_{ij}^p P + PE_{ij}^p) u \] (4.16)
where $M = \frac{\sqrt{\|P\|}}{2\sqrt{\|APA+Q\|}}$. Vector $v$ is the unit eigenvectors associated with the largest eigenvalue of $APA+Q$ and $u$ is the one associated with $\lambda(AP+PA)$. The optimization algorithm is then carried out by computing the subgradient of $\phi(A'^k)$ at $A^k$ on each step with stepsize $\gamma^k > 0$. The stepsize satisfies the diminishing rule:
\[ \lim_{k \to \infty} \gamma^k = 0, \sum_{1}^{\infty} \gamma^k = \infty \]
As pointed out in [22], for the convex optimization problem, the convergence of the algorithm is well-known.

**Algorithm 1: Optimal Parameters Design**

- **Initialize** $A^1 \in T_G$, $B \in T_G$ and $Q \in T_G \cap C$, $A, i = 1, \epsilon > 0$;
- **Repeat**:
  - Compute $M$;
  - Compute $v^i$ and $u^i$;
  - Compute $H^i$ at $x^i$;
  - Set $A'^{i+1} = A'^i - \gamma^i H^i$;
  - $i := i + 1$;
- **Until** $|\Delta \phi(A')| < \epsilon$;

The optimal parameters generated from Algorithm 1 minimize the upper bound of $\|X^*\|$ in the ARE (4.1). We point out that there indeed have certain cases where a lower upper bound may relate to a higher value of $\|X^*\|$. However the original Problem 4.1 is a great trouble while this relaxed optimization problem could be carried out using the reliable subgradient method. Meanwhile, in most of cases we experimented, lower upper bound of $\|X^*\|$ always corresponds to a smaller value of $\|X^*\|$. Substituting the structured matrix $A$ generated from Algorithm 1 into the ARE (4.1), the resulting LQR controller $K = -B^T X^*$ is possibly a dense matrix without any structure constraints because of the density of $X^*$. In general, $X^*$ is usually a matrix with nonzero entries even when $A$ is a sparse matrix, as for example the ones in Fig. 4.1(b). This result is consistent to [15] where when no communication expenditure was considered, the undirected underlying graph of the optimal interaction
Graph among the LQR controllers was typically a complete graph. This reveals that a system with distributed controllers consumes less energy compared with a decentralized one. However a complex topology is obviously not practical, especially when the scale of the system grows. In the above research, due to the uniqueness of the positive solution of the ARE (4.1), it is impossible to find any structured solution so as to reduce the communication burden among controllers. Other methodologies are required to deal with this deficiency, which is our ongoing works.

In this section, the optimization problem is built up with the objective function being nonsmooth and with nonlinear constraints. Structure restrictions on the underlying graphs are introduced for the purpose of realtime applications. To find an analytical solution to the optimization problem, a special formation system with homogeneous agents is considered. The system is decoupled into \( n \) independent subsystems and the desired underlying graphs are imposed both among agents and among controllers. For the more general case, a relaxed optimization problem is proposed that eliminates the nonlinear constraints and thus is solved based on the subgradient method. The underlying graph \( G_\alpha \) is designed to be the desired one.

5. Examples. The proposed algorithms are validated on systems with identical agents and systems with heterogeneous agents respectively. In order to observe the cooperative performance of the relative formations during convergence more intuitively, we consider the relative formation system in the edge space that is transformed through

\[
e = (\tilde{H} \otimes I_2)p
\]

where \( \tilde{H} \) is the oriented incidence matrix of a rigid graph \( \tilde{G} = (\tilde{V}, \tilde{E}) \) and \( p \) is the coordinates of agents. For the desired formation \( p_d \in \mathbb{R}^{2n} \) of \( n \) agents and the corresponding \( e_d \in \mathbb{R}^{2|\tilde{E}|} \), we define a relative formation error function with respect to \( e_d \) as

\[
f_e(t) = \sum \left\| \frac{e_i}{\|e_i\|} - \frac{e_d_i}{\|e_d_i\|} \right\|	ag{5.1}
\]

When \( f_e = 0 \), the geometry formed by the agents coincide with the desired geometry although their actual coordinates may differ. A larger relative formation error corresponds to greater deformation effects of a geometry with respect to the desired one.

5.1. Formations with homogeneous agents. For a formation system with identical agents and identical interaction parameters, Theorem [43] provides an analytical method to design distributed controllers so as to minimize the cost function under the worst case.
The underlying graph is the one in Fig. 5.1(a). The parameters in (4.5) are $a = 2, q = 8$ and $p = -2$. According to Theorem 4.3, the distributed controller $X^*$ has identical diagonal entries of 5.46 and nondiagonal entries of -1.37. The trajectories of the six agents are shown in Fig. 5.2 with snapshots in Fig. 5.3. Fig. 5.4(a) demonstrates the position errors of the six agents with respect to the desired positions. It takes approximately 8 seconds for them to achieve the desired positions. However, according to Fig. 5.4(b), at approximately $t \approx 2$ sec, the relative formation error converges to zero, which indicates that with high attentions being paid to the relative formations of the six agents, they attain the desired geometry before arriving at the destination.

In order to observe different kinds of cooperative performance, the six agents are initialized at the same spot as shown in Fig. 5.5. When $q = 8, p = -2$, relative formation is more important during the assignment, thus agents spread immediately after they set off and maintain the desired geometry when approaching the destination. On the other hand, if the destination-first performance is more impotent in the mission, we set $q = 3$ and $p = -0.2$. Then the six agents approach their own destinations radially and form the desired geometry at almost the same time as they achieve the destinations at $t \approx 3$ sec. The formation errors for the two kinds of performance are shown in Fig. 5.6 in blue solid lines and red dashed lines respectively. This infers that the parameters in the cost matrix $Q$ allow us to take into account the geometrical performance during convergence.

5.2. Formations with heterogeneous agents. When agents exhibit different local dynamics and when the interactions between neighboring agents various from one another, Algorithm II is proposed to solve for the set of optimal interaction parameters.

We consider six agents in a formation system that communicate over graph $G_b$ as shown in Fig. 5.1(b).
and the nonzero input matrices are

\[
B_{11} = \begin{bmatrix} 3.99 & 0 \\ 0 & -3.67 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} -0.15 & 0 \\ 0 & 0.19 \end{bmatrix}, \quad B_{13} = \begin{bmatrix} 3.53 & 0 \\ 0 & -1.33 \end{bmatrix}, \quad B_{16} = \begin{bmatrix} 1.06 & 0 \\ 0 & -0.15 \end{bmatrix},
\]

\[
B_{22} = \begin{bmatrix} -3.20 & 0 \\ 0 & 4.41 \end{bmatrix}, \quad B_{23} = \begin{bmatrix} 2.89 & 0 \\ 0 & -1.54 \end{bmatrix}, \quad B_{26} = \begin{bmatrix} -2.46 & 0 \\ 0 & 0.41 \end{bmatrix},
\]

\[
B_{33} = \begin{bmatrix} -4.69 & 0 \\ 0 & -2.32 \end{bmatrix}, \quad B_{34} = \begin{bmatrix} 1.57 & 0 \\ 0 & -0.63 \end{bmatrix}, \quad B_{35} = \begin{bmatrix} 1.49 & 0 \\ 0 & -0.94 \end{bmatrix},
\]

\[
B_{44} = \begin{bmatrix} -3.22 & 0 \\ 0 & -1.12 \end{bmatrix}, \quad B_{45} = \begin{bmatrix} -1.23 & 0 \\ 0 & -2.98 \end{bmatrix}, \quad B_{46} = \begin{bmatrix} 0.26 & 0 \\ 0 & -1.31 \end{bmatrix},
\]

\[
B_{55} = \begin{bmatrix} -3.45 & 0 \\ 0 & 4.49 \end{bmatrix}, \quad B_{56} = \begin{bmatrix} 0.90 & 0 \\ 0 & 0.08 \end{bmatrix},
\]

\[
B_{66} = \begin{bmatrix} -1.05 & 0 \\ 0 & -3.65 \end{bmatrix}
\] (5.3)

The quadratic matrix is

\[
Q_{11} = \begin{bmatrix} 13.50 & 0 \\ 0 & 22.50 \end{bmatrix}, \quad Q_{12} = \begin{bmatrix} -1.17 & 0 \\ 0 & -2.63 \end{bmatrix}, \quad Q_{13} = \begin{bmatrix} -1.74 & 0 \\ 0 & -1.48 \end{bmatrix}, \quad Q_{16} = \begin{bmatrix} -4.21 & 0 \\ 0 & -3.12 \end{bmatrix},
\]

\[
Q_{22} = \begin{bmatrix} 13.50 & 0 \\ 0 & 22.50 \end{bmatrix}, \quad Q_{23} = \begin{bmatrix} -5.48 & 0 \\ 0 & -2.59 \end{bmatrix}, \quad Q_{26} = \begin{bmatrix} -4.77 & 0 \\ 0 & -5.31 \end{bmatrix},
\]

\[
Q_{33} = \begin{bmatrix} 18 & 0 \\ 0 & 30 \end{bmatrix}, \quad Q_{34} = \begin{bmatrix} -0.97 & 0 \\ 0 & -0.89 \end{bmatrix}, \quad Q_{35} = \begin{bmatrix} -1.77 & 0 \\ 0 & -1.24 \end{bmatrix},
\]

\[
Q_{44} = \begin{bmatrix} 13.50 & 0 \\ 0 & 22.50 \end{bmatrix}, \quad Q_{45} = \begin{bmatrix} -4.84 & 0 \\ 0 & -5.17 \end{bmatrix}, \quad Q_{46} = \begin{bmatrix} -4.01 & 0 \\ 0 & -5.23 \end{bmatrix},
\]

\[
Q_{55} = \begin{bmatrix} 18 & 0 \\ 0 & 30 \end{bmatrix}, \quad Q_{56} = \begin{bmatrix} -2.67 & 0 \\ 0 & -2.29 \end{bmatrix},
\]

\[
Q_{66} = \begin{bmatrix} 13.50 & 0 \\ 0 & 22.50 \end{bmatrix}
\] (5.4)

and all the other blocks are zeros. Using Algorithm 1, the subgradient matrix converges to zero after approximately 63 times of iterations. The interaction parameters
Fig. 5.6. Formation errors for different kinds of cooperative performance

Fig. 5.7. Worst-case cost values of six-agent formations. The bottom dashed line is the optimized cost value $J^*$

in matrix $A'$ are

$$A'_{12} = \begin{bmatrix} 4.79 & 3.60 \\ -1.12 & -0.96 \end{bmatrix}, \quad A'_{13} = \begin{bmatrix} -3.30 & 3.43 \\ 6.41 & 2.21 \end{bmatrix}, \quad A'_{16} = \begin{bmatrix} 2.68 & -2.72 \end{bmatrix},$$

$$A'_{23} = \begin{bmatrix} 10.33 & 0.61 \\ -3.23 & 1.97 \end{bmatrix}, \quad A'_{26} = \begin{bmatrix} 2.31 & -1.82 \end{bmatrix},$$

$$A'_{34} = \begin{bmatrix} -3.86 & -0.23 \\ -0.10 & 1.07 \end{bmatrix}, \quad A'_{35} = \begin{bmatrix} -2.14 & 0.69 \end{bmatrix},$$

$$A'_{45} = \begin{bmatrix} 3.65 & 0.63 \end{bmatrix}, \quad A'_{46} = \begin{bmatrix} -0.38 & -2.40 \end{bmatrix},$$

$$A'_{56} = \begin{bmatrix} -0.54 & 3.93 \\ 0.14 & 3.23 \end{bmatrix},$$

and all the other blocks are zeros. According to the structure of matrix $A'$, the underlying graph is exactly the one in Fig. 5.1(b) and the cost function achieves the minimum of $J^* = 3.39$ for the worst case. We tested 50 samples of randomly selected $A'$ under the structure constraints. Greater cost values are observed on all of those samples. The data are recorded in Fig. 5.7. The dashed line at the bottom is the value of $\|X^*\| = 3.39$ and all the blue stars are different worst-case cost values $\bar{J}$ when $A' \in T_G$ changes. This validates the effectiveness of the subgradient algorithm. Fig. 5.8 is an example of formations of the six agents under the matched pair $(A_0 + A', Q)$. Snapshots during the process are shown in Fig. 5.9.

For comparison, we consider a formation system where agents are dynamically isolated. The local dynamics, the input matrix and the cost matrix are congruent to the ones in (5.2), (5.3) and (5.4) respectively. The optimal controllers are calculated by the ARE (4.1). For the worst case, the cost function has a minimal value of
$\bar{J} = 4.53$, as shown in sample number 50 in Fig. 5.7. We compare the performance of a system with isolated agents and a system with optimized couplings (6.5). For both situations, agents are initialized at the worst case, i.e., $p_0 = -3v$ with $v$ being the orthogonal eigenvector of $\bar{\lambda}(A_0)$ and $\bar{\lambda}(A_0 + A')$ respectively. The two plots in Fig. 5.10 show their trajectories during the stabilization processes. The formation errors for system under the matched pair $(A_0 + A', Q)$ and the pair $(A_0, Q)$ are shown in Fig. 5.11(a). For the two situations, the snapshots of the geometries at time slot $t = 0.21sec$ are shown in Fig. 5.11(b). According to Fig. 5.11, the geometry of agents under the matched pair is more similar to the desired geometry at the same time slot. Formation systems with isolated agents have relatively greater value of cost function, and thus its relative formation errors are consistently greater than the one with agents interacted over optimal parameters.

6. Conclusions. In this paper, we proposed an optimal LQR control strategy for a group of agents to maintain desired geometries while moving towards the destination. Upon the three kinds of cooperative performance that were characterized by the cost matrix, it was proved that, compared to system with agents communicated only through the control channel, the upper bound of the minimum cost value could be further reduced by adjusting the interaction parameters between the pairs of neighboring agents and by the LQR controllers. Distributed controllers that inherited the desired underlying graph were developed for the set of homogeneous agents. When agents were heterogeneous ones, the optimization problem was relaxed so as to avoid the nonsmoothness and nonlinearity in the constraints. Under the constraint of a sparse underlying graph, parameters among agents were generated by a reliable iterative algorithm. Numerical examples further illustrated the relationship between the cost function and the cooperative performance we considered.
Fig. 5.10. Trajectories of six agents being initialized at their worst cases.

Fig. 5.11. Formation errors and the snapshots of systems with isolated agents and optimal coupled agents

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\[ J^*(Q) \]
\[ J^*(\bar{A},Q) - J^*(A,Q) \]
