Strong-Coupling Bose-Einstein Condensation

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We extend the theory of Bose-Einstein condensation from Bogoliubov’s weak-coupling regime to arbitrarily strong couplings.

PACS numbers: 98.80.Cq, 98.80. Hw, 04.20.Jb, 04.50+h

1. For \( \phi^4 \)-theory in \( D < 4 \) Euclidean dimensions with \( O(N) \)-symmetry, a powerful strong-coupling theory has been developed in 1998 [1]. It has been carried to 7th order in perturbation theory in \( D = 3 \) [2], and to 5th order in \( D = 4 - \epsilon \) dimensions [3]. The theory is an extension of a variational approach to path integrals that was once set by R.P. Feynman and collaborator in 1989 [4]. The extension to high orders is described in the textbook [5]. It is called Variational Perturbation Theory (VPT). Originally, the theory was designed to convert only the divergent perturbation expansions of quantum mechanics into exponentially fast convergent expressions [6]. In the papers [1][2], it was extended from quantum mechanics to \( \phi^4 \)-theory with its anomalous dimensions and produced all critical exponents. This is called quantum field theoretic VPT. That theory is explained in the textbook [7] and a recent review [8].

Surprisingly, this successful theory has not yet been applied to the presently so popular phenomena of Bose-Einstein condensation. These have so far mainly been focused [9] on the semiclassical treatments using the good-old Gross-Pitaevskii equations, or to the weak-coupling theory proposed many years ago by Bogoliubov [10]. This is somewhat surprising since the subject is under intense study by many authors. So far, only the shift of the critical temperature has been calculated to high orders [11]. There are only a few exceptions. For instance, a simple extension of Bogoliubov’s theory to strong couplings was proposed in [12] and pursued further in [13]. But that had an unpleasant feature that it needed two different chemical potentials to maintain the long-wavelength properties of Nambu-Goldstone excitations required by the spontaneously broken U(1)-symmetry in the condensate. For this reason it remained widely unnoticed. Another notable exception is the theory in [14] which came closest to our approach, since it was also based on a variational optimization of the energy. But by following Bogoliubov in identifying \( \alpha_0 \) as \( \sqrt{\rho_0} \) from the outset, they ran into the notorious problem of violating the Nambu-Goldstone theorem. Another approach that comes close to ours is found in the paper [15]. Here the main difference lies in the popular use the Hubbard-Stratonovic transformation (HST) to introduce a fluctuating collective pair field [16]. But, as pointed out in [2] and re-emphasized in [25], this makes it impossible to calculate higher-order corrections

2. The Hamiltonian of the boson gas has a free term

\[
H_0 \equiv \sum_p a_p^\dagger (\varepsilon_p - \mu) a_p = \sum_p a_p^\dagger \xi_p a_p, \tag{1}
\]

where \( \varepsilon_p \equiv p^2/2M \) are the single-particle energies and \( \xi_p \equiv \varepsilon_p - \mu \) the relevant energies in a grand-canonical ensemble. As usual, \( a_p^\dagger \) and \( a_p \) are creation and annihilation operators defined by the canonical equal-time commutators of the local fields \( \psi(x) = \sum_p e^{i px/\hbar} a_p \). The local interaction is

\[
H_{\text{int}} = \frac{g}{2V} \sum_{p,p',q} a_p^\dagger q a_p^\dagger - q a_{p'}^\dagger a_{p'} a_p. \tag{2}
\]

Instead of following Bogoliubov in treating the \( p = 0 \) modes of the operators \( a_p \) classically and identifying with the square-root of the condensate density \( \rho_0 \) we introduce the field expectation \( \langle \psi \rangle \equiv \sqrt{\Sigma_0/g} \) as a variational parameter, and rewrite \( H_{\text{int}} \) as \( H^0_{\text{int}} = (V/2g)\Sigma_0^2 \) plus

\[
H'_{\text{int}} = \frac{1}{2} \sum_{p \neq 0} 2 \Sigma_0 \left( a_p^\dagger a_p + a_{-p}^\dagger a_{-p} \right) + \Sigma_0 \left( a_p^\dagger a_{-p}^\dagger a_p + h.c. \right), \tag{3}
\]

plus a fluctuation Hamiltonian \( H''_{\text{int}} \), which looks like [2], except that the sum contains only non-zero-momentum
modes. Now we proceed according to the rules of VPT and introduce dummy variational parameter \( \Sigma \) and \( \Delta \) via an auxiliary Hamiltonian

\[
\tilde{H}_{\text{trial}} = -\frac{1}{2} \sum_{p \neq 0} \left[ \Sigma \left( a_p^\dagger a_p + a_{-p}^\dagger a_{-p} \right) + \Delta a_{-p} a_p + \text{h.c.} \right],
\]

leading a harmonic Hamiltonian

\[
H_0' = -V \frac{\mu}{g} \Sigma_0 + \frac{V}{2g} \Sigma_0^2 + \sum_{p \neq 0} (\varepsilon_p - \mu + 2\Sigma_0) a_p^\dagger a_p + \frac{1}{2} \Sigma_0 \sum_{p \neq 0} \left( a_p^\dagger a_{-p}^\dagger + \text{h.c.} \right) + \tilde{H}_{\text{trial}},
\]

for which we have to calculate the energy order by order in perturbation theory considering

\[
H_{\text{var}} = H_{\text{int}}'' - \tilde{H}_{\text{trial}}.
\]

as the interaction Hamiltonian. The zeroth-order variational energy is \( W_0 = \langle H_0' \rangle \), and the lowest-order correction comes from the expectation value \( \Delta_1 W = \langle H_{\text{var}} \rangle \). If the energy is calculate to all orders in \( H_{\text{var}} \) the result will be independent of the variational parameters \( \Sigma_0, \Sigma, \) and \( \Delta \), but the energy to any finite order will depend on it. The optimal values of the parameters are found by optimization (usually extremization), and the results converge exponentially fast as a function of the order.

A Bogoliubov transformation with as yet undetermined coefficients \( u_p, v_p \), constrained by the condition \( u_p^2 - v_p^2 = 1 \), produces a ground state with vacuum expectation values \( \langle a_p a_p \rangle = v_p^2 \) and \( \langle a_p a_{-p} \rangle = u_p v_p \), so that

\[
W_0 = -V \frac{\mu}{g} \Sigma_0 + \frac{V}{2g} \Sigma_0^2 + \sum_{p \neq 0} \left\{ \left[ \varepsilon_p - \mu - \Sigma + 2\Sigma_0 \right] v_p^2 + \left( \Sigma_0 - \Delta \right) u_p v_p \right\}.
\]

The first-order variational energy \( W_1 \) contains, in addition, the expectation value \( \langle H_{\text{var}}' \rangle \). Of this, the first part, \( \Delta_{(1,0)} W = \langle H_{\text{var}}'' \rangle \), is found immediately with the help of the standard commutation rules as a sum of three pair terms

\[
\langle a_{p+q}^\dagger a_{-q}^\dagger a_{-p} a_p \rangle = \langle a_{p+q}^\dagger a_{-p}^\dagger a_{-q} a_p \rangle + \langle a_{p+q}^\dagger a_{-p} a_{-q}^\dagger a_p \rangle + \langle a_{p+q}^\dagger a_{-p} a_{-q} a_{-q}^\dagger \rangle.
\]

so that

\[
\Delta_{(1,0)} W = \langle H_{\text{var}}'' \rangle = \frac{g}{2V} \sum_{p,p', \neq 0} \left( 2v_p^2 v_{p'}^2 + u_p v_p u_{p'} v_{p'} \right).
\]

The second part \( -\langle \tilde{H}_{\text{trial}} \rangle \) adds to this the expectation value

\[
\Delta_{(1,1)} W = \sum_{p \neq 0} \left( \Sigma v_p^2 + \Delta u_p v_p \right).
\]

In order to fix the average total number of particles \( N \), we differentiate \( W_1 \equiv W_0 + \Delta_{(1,0)} W + \Delta_{(1,1)} W \) with respect to \(-\mu\) and set the result equal to \( N \) to find the density \( \rho = N/V \) as

\[
\rho = \frac{\Sigma_0}{g} + \sum_{p \neq 0} v_p^2.
\]

The momentum sum is the density of particles outside the condensate, the uncondensed density

\[
\rho_u = \sum_{p \neq 0} \langle a_p^\dagger a_p \rangle = \frac{1}{V} \sum_p v_p^2
\]

implying that \( \Sigma_0/g \) is the condensate density \( \rho_0 \):

\[
\frac{\Sigma_0}{g} = \rho_0 = \rho - \rho_u.
\]

Now we extremize \( W_1 \) with respect to the variational parameter \( \Sigma_0 \) which yields the equation

\[
\mu - \Sigma_0 = \sum_{p \neq 0} (2v_p^2 + u_p v_p) = 2\rho_u + \sum_{p \neq 0} u_p v_p = 2\rho_u + \delta.(14)
\]

We are now able to fix the size of the Bogoliubov coefficients \( u_p \) and \( v_p \). The original way of doing this is algebraic, based on the elimination of the off-diagonal elements of the transformed Hamiltonian operator. In the framework of our variational approach it is more natural to use the equivalent procedure of extremizing the energy expectation \( W_0 \) with respect to \( u_p \) and \( v_p \) under the constraint \( u_p^2 - v_p^2 = 1 \), so that \( \partial u_p / \partial v_p = v_p / u_p \). Varying \( W_0 \), we obtain for each nonzero momentum the equation

\[
2(\varepsilon_p - \mu + 2\Sigma_0 - \Sigma) v_p + (\Sigma_0 - \Delta) (u_p + v_p^2 / u_p) = 0.
\]

In order to solve this we introduce the constant

\[
\tilde{\Sigma} = -\mu + 2\Sigma_0 - \Sigma = -\mu + 2g(\rho - \rho_u) - \Sigma,
\]

the right-hand side emerging after using \( \Sigma_0 \) and \( \Sigma \). We further introduce the constant

\[
\tilde{\Delta} = \Sigma_0 - \Delta = r \tilde{\Sigma}.
\]

Then we rewrite \( 15 \) in the simple form

\[
2(\varepsilon_p + \tilde{\Sigma}) v_p + r \tilde{\Sigma} (u_p + v_p^2 / u_p) = 0,
\]

which is solved for all \( p \) by the Bogoliubov transformation coefficients

\[
u_p^2 = \frac{1}{2} \left( 1 + \frac{\varepsilon_p + \tilde{\Sigma}}{\tilde{\Sigma}^2} \right), \quad v_p^2 = -\frac{1}{2} \left( 1 - \frac{\varepsilon_p + \tilde{\Sigma}}{\tilde{\Sigma}^2} \right), \quad \tilde{\Sigma}^2 = \sqrt\left( \varepsilon_p + \tilde{\Sigma} \right)^2 - \frac{1}{r^2} \tilde{\Sigma}^2.
\]
Having determined the Bogoliubov coefficients, we can calculate the above momentum sums in Eqs. (12) and (14). We begin with the uncondensed particle density with $k_\Sigma = \sqrt{2M\Sigma}/\hbar$, so that we find
\[
\rho_u = k^2_\Sigma \int_\rho^{(r)} / 4\pi^2, \tag{22}
\]
where
\[
I^{(r)}_{\rho_u} = \int_0^\infty d\kappa \kappa^2 \left( \frac{\kappa^2 + 1}{\sqrt{(\kappa^2 + 1)^2 - r^2}} - 1 \right) = \frac{\sqrt{2}}{3} f^{(r)}_{\rho_u}. \tag{23}
\]
with $f^{(1)}_{\rho_u} = 1$. The second momentum sum in Eq. (14) reads, after inserting (19),
\[
\delta \equiv \sum_{p \neq 0} (a_p a_p) = \frac{1}{V} \sum_{p \neq 0} u_p v_p = -\frac{\Sigma}{2} \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{1}{\varepsilon_p}, \tag{24}
\]
In contrast to (21), this is a divergent quantity. As a consequence of the renormalizability of the theory, the divergence can be removed by absorbing it into the inverse coupling constant of the model defined by
\[
\frac{1}{g R} = \frac{1}{g} - \frac{1}{V} \sum_{p \neq 0} \frac{1}{2\varepsilon_p} = \frac{1}{g} - \frac{1}{2} \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{1}{\varepsilon_p} \tag{25}
\]
The renormalized coupling is finite and measurable in two-body scattering as an s-wave scattering length: $g R = 2\pi\hbar^2 a_s / M$. Thus we introduce the finite renormalized quantity
\[
\delta_R = \frac{1}{V} \sum_{p \neq 0} u_p v_p = -r \frac{\Sigma}{2} \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{1}{\varepsilon_p} - \frac{1}{\varepsilon_p} \tag{26}
\]
and write $\delta = \delta_R + \delta_{\text{div}}$, where the divergence is the momentum sum
\[
\delta_{\text{div}} = -\frac{r \sum}{V} \sum_{p \neq 0} \frac{1}{2\varepsilon_p} = -\frac{r \sum}{V} \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{1}{\varepsilon_p} \tag{27}
\]
If we denote this by $-\Sigma / V v$, we have
\[
\delta = \delta_R + \delta_{\text{div}} = \delta_R - \frac{r \Sigma}{V v}. \tag{28}
\]
Inserting this together with (12) into (14), we find
\[
\frac{\mu - \Sigma_0}{g} = 2\rho_u + \delta_R + \delta_{\text{div}}. \tag{29}
\]
Recalling (13), this implies
\[
\frac{\mu}{g} = \rho_0 + 2\rho_u + \delta_R + \delta_{\text{div}} = \rho + \rho_u + \delta_R + \delta_{\text{div}}. \tag{30}
\]
If we evaluate the momentum sum (26) in the same way as (21), it yields
\[
\delta_R = k^2_\Sigma I^{(r)}_{\rho_u} / 4\pi^2, \tag{31}
\]
where $I_\delta$ is given by the integral
\[
I^{(r)}_{\rho_u} = -\int_0^\infty d\kappa \kappa^2 \left( \frac{1}{\sqrt{(\kappa^2 + 1)^2 - r^2}} - 1 \right) = 2f^{(r)}_\delta, \tag{32}
\]
with $f^{(1)}_{\delta} = 1$.

We continue the discussion with Eq. (16), which we rewrite using (20) as
\[
\frac{\Sigma}{g} = \rho - 3\rho_u - \delta_R - \frac{\Sigma}{g}. \tag{33}
\]
As before in Eqs. (24), (25), and (26), the last, divergent term can be absorbed into the first by renormalizing the coupling constant, so that we obtain
\[
\frac{\Sigma}{g R} = \rho - 3\rho_u - \delta_R - \frac{\Sigma}{g}. \tag{34}
\]
Finally, we calculate the total variational energy $W_1$. Inserting the Bogoliubov coefficients (19) into $W_0$ of Eq. (7) and adding the action energies $\Delta_{(1,0)} W + \Delta_{(1,1)} W$ of and (9) and (10), we have
\[
W_1 = -\frac{V}{g} \mu \Sigma_0 + \frac{V}{2g} \Sigma_0^2 + w_0(\Sigma, r) - \frac{r^2 \Sigma^2}{4V v} + \Delta_{(1,0)} W + \Delta_{(1,1)} W, \tag{35}
\]
where $w_0(\Sigma)$ is the convergent momentum sum
\[
w_0(\Sigma, r) = \frac{1}{2} \sum_{p \neq 0} \left\{ \left( \varepsilon_p - \varepsilon_p - \Sigma + \frac{r^2 \Sigma^2}{2\varepsilon_p} \right) \right\}. \tag{36}
\]
This is evaluated as in (21) to
\[
w_0(\Sigma) = V \Sigma k_\Sigma^3 I^{(r)}_{E} / 4\pi^2, \tag{37}
\]
where
\[
I^{(r)}_{E} = \int_0^\infty d\kappa \kappa^2 \left[ \sqrt{(\kappa^2 + 1)^2 - r^2} - 1 - \frac{r^2}{2\kappa^2} \right] = \frac{8\sqrt{2}}{15} f^{(r)}_E, \tag{38}
\]
with $f^{(1)}_E = 1$. If we rename all $I_{E}/4\pi^2$ to $I_{E}$, the energy $W_1$ becomes
\[
W_1 = -\frac{V}{g} \mu \Sigma_0 + \frac{V}{2g} \Sigma_0^2 + \frac{V}{2} k_\Sigma^3 \bar{I}^{(r)}_E \tag{39}
\]
\[
+ \frac{V}{2} k_\Sigma^3 (2 \bar{I}^{(r)}_{\rho_u} + \Delta \bar{I}^{(r)}_\delta) + V k_\Sigma^3 (\Sigma \bar{I}^{(r)}_{\rho_u} + \Delta \bar{I}^{(r)}_\delta).
\]
The expression is renormalized most simply using dimensional regularization, that allows us to use Veltman’s rule [7] to set $1/v = 0$. 


We now prepared to extremize the variational energy Eq. (39) with respect to \( \Sigma \) and \( \Delta \). We insert \( \Sigma / g = \rho - 3\rho_u - \delta R - \Sigma / g \) from (40) and \( \Delta / g = \rho - \rho_u - \Delta / g \) from (41) and vary \( W_1 \) in \( \delta \Sigma \) and \( \delta \Delta \). This yields the equations

\[
(\Sigma / g - \rho (\Lambda)) S_{11} + (\Delta / g - \rho (\Lambda)) S_{12} = 0, \quad (40)
\]
\[
(\Sigma / g - \rho (\Lambda)) S_{21} + (\Delta / g - \rho (\Lambda)) S_{22} = 0, \quad (41)
\]
where \( \rho_u^{(\pm)} = \rho - \rho_u \pm \delta R \) and

\[
S_{11} = \tilde{I}_a^{(r)} \partial k_3^{(r)} / \partial \Sigma, \quad S_{12} = (k_3^{(r)} / \Sigma) \partial \tilde{I}_a^{(r)} / \partial \tau, \quad (42)
\]
\[
S_{21} = \tilde{I}_a^{(r)} \partial k_3^{(r)} / \partial \Sigma, \quad S_{22} = (k_3^{(r)} / \Sigma) \partial \tilde{I}_a^{(r)} / \partial \tau. \quad (43)
\]

These equations are solved for \( r = 1 \) with

\[
\frac{\Sigma}{g} = \frac{\Delta}{g} = \rho - \rho_u - \delta R. \quad (44)
\]

The reason is simply that both \( \tilde{I}_a^{(r)} \) and \( \tilde{I}_a^{(r)} \) behave near \( r = 1 \) like \( (1 - r)^{3/2} \) so that derivative at \( r = 1 \) vanishes and both equations (43) give (44).

The solution of these equations is \( r = 1 \), thus guaranteeing the Nambu-Goldstone nature of the quasiparticle energies (20).

3. To extract experimental consequences it is useful to re-express all equations in a dimensionless form by introducing the reduced variables

\[
s \equiv \frac{\Sigma}{\varepsilon_a}, \quad (45)
\]

where \( \varepsilon_a \equiv \hbar^2 / 2Ma^2 \) is the natural energy scale of the system. We also introduce the reduced s-wave scattering length

\[
\tilde{a}_s \equiv 8\pi a_s / a, \quad (46)
\]
in terms of which the renormalized coupling constant is

\[
g_R = \frac{4\pi \hbar^2}{M} a_s = 8\pi \varepsilon_a a^2 a_s = \varepsilon_a a^3 \tilde{a}_s, \quad (47)
\]

while

\[
k_\Sigma = \frac{\sqrt{s}}{a}, \quad \frac{\Sigma}{g_R} = \frac{s}{8\pi a^2 a_s} = \frac{s}{a^3 \tilde{a}_s}, \quad (48)
\]

and the second-sound velocity reads

\[
c = \sqrt{\frac{s}{2\varepsilon_a}}, \quad v_a = \frac{p_a}{M} = \frac{\hbar}{aM}. \quad (49)
\]

Let us also define the reduced quantity \( \tilde{\delta} \equiv s^{3/2} \tilde{I}_\delta \) and

\[
\tilde{\rho}_u \equiv s^{3/2} \tilde{I}_\rho_u. \quad (50)
\]

In terms of these we calculate the reduced variational energy \( w_1 \equiv W_1 / N\varepsilon_a \) from Eq. (39) for \( r = 1 \):

\[
w_1 = -\tilde{a}_s (1 + \tilde{\rho}_u + \tilde{\delta}) (1 - \tilde{\rho}_u) + \frac{\tilde{a}_s}{2} (1 - \tilde{\rho}_u)^2 + \frac{s^{5/2}}{2} \tilde{I}_E
\]
\[
+ \frac{\tilde{a}_s}{2} (2\rho_u^2 + \delta^2) + \tilde{a}_s (\sigma \rho_u + \sigma \tilde{\delta}), \quad (51)
\]

where \( \sigma \Sigma \equiv 1 - 3\tilde{\rho}_u - \tilde{\delta} - s/\tilde{a}_s \) from (42) and \( \sigma \Delta \equiv 1 - \tilde{\rho}_u - s/\tilde{a}_s \) from (14). Inserting here \( \tilde{\rho}_u \) and \( \delta \), and going from the grand-canonical to the true proper energies by adding \( \mu N \) to \( W_1 \) forming \( W = W_1 + \mu V \rho \), we obtain the reduced energy

\[
w_1^r = \frac{\tilde{a}_s}{2} - \frac{4\sqrt{2}}{15\pi^2} s^{3/2} + \frac{\sqrt{2}}{3\pi^2} \tilde{a}_s s^{3/2} + \frac{1}{72\pi^4} a^4. \quad (52)
\]

The relation between \( s \) and \( \tilde{a}_s \) is from (14)

\[
\frac{s}{\tilde{a}_s} = 1 - s^{3/2} (I_{\rho_u} + I_\delta). \quad (53)
\]

leading to expansion

\[
w_1^r = \frac{\tilde{a}_s}{2} - \frac{4\sqrt{2}}{15\pi^2} s^{3/2} + \frac{\sqrt{2}}{3\pi^2} s^{3/2} + \frac{1}{72\pi^4} a^4 + \ldots. \quad (54)
\]

Note that in the strong-coupling limit, \( s \rightarrow s^{sc} = (3\pi^2 / \sqrt{2})^{1/3} \), the maximal depletion is \( \tilde{\rho}_u = 1/4 \), and the energy behaves like \( w_1^r \rightarrow B + A \tilde{a}_s \), with \( B = -4 \times 2^{1/2} / (5 \times 3\sqrt{3}p_{1/3}) \approx -0.48 \), and \( A = 1/2 + 1/24\sqrt{2}p^2 + 2^{1/4} / \sqrt{3} \pi \approx 0.72 \). The sound velocity at infinite coupling is \( c = \sqrt{s^{sc} / 2\varepsilon_a} \).

Let us now study the temperature dependence of our results. For this we introduce the temperature-dependent version of the integral (23) at \( r = 1 \), where we omit the trivial superscript, to find

\[
\rho_u(t) = k_\Sigma I_{\rho_u}(t) / 4a^2, \quad k_\Sigma = \sqrt{2M/\varepsilon_a} = \sqrt{s/a}. \quad (55)
\]

where \( I_{\rho_u}(t) \) is defined by the integral

\[
I_{\rho_u}(t) \equiv \int_0^\infty d\kappa \kappa^2 \left[ -\frac{\kappa^2 + 1}{\sqrt{(\kappa^2 + 1)^2 - 1}} c_\Sigma(\kappa) - 1 \right], \quad (56)
\]

where \( c_\Sigma(\kappa) \equiv \coth \left( \sqrt{(\kappa^2 + 1)^2 - 1} / 2t \right) \), and \( t \) is the reduced temperature

\[
t \equiv k_BT / \varepsilon_\Sigma, \quad \varepsilon_\Sigma \equiv \sqrt{2M/\varepsilon_a}. \quad (57)
\]

To find the \( s \) at any temperature we need Eq. (51) for \( T \neq 0 \), where it reads

\[
\delta_R(t) = k_\Sigma^2 I_{\delta}(t) / 4a^2, \quad (58)
\]

with

\[
I_{\delta}(t) \equiv -\int_0^\infty d\kappa \kappa^2 \left[ -\frac{1}{\sqrt{(\kappa^2 + 1)^2 - 1}} c_\Sigma(\kappa) - 1 \right]. \quad (59)
\]
The relation between the integral, which is equal to \( -s/\hat{s} \), and the temperature of the free Bose gas, and small \( s \) is obtained by expanding the integrand in powers of \( s \), the first correction going like \( \sqrt{s} \). To see this, we must proceed as in the derivation of the Robinson expansion of the Bose-Einstein integral function \( \Pi_L \), writing \( \rho_u / \rho = s^{3/2} \Pi_2 / 4\pi^2 + \Delta \rho_u / \rho \), with the second term being the integral

\[
\frac{1}{4\pi^2} \int_0^\infty d\kappa \kappa^2 \left[ \frac{\kappa^2 + s}{\sqrt{\kappa^2 + s}} - \frac{2}{\kappa^2 \sqrt{\kappa^2 + s} - s^2} \right],
\]

(63)

where \( T_c^0 \equiv T_c^0 / T_0 = 4\pi \zeta(3/2)^{-2/3} \) is the reduced critical temperature of the free Bose gas, and \( \tau \) is the ratio \( T / T_c^0 \). The integral can be done immediately for \( s = 0 \) and yields the well-known result

\[
\Delta \rho_u / \rho \bigg|_{s \to 0} = \Delta \rho_u^0 / \rho = \tau^{3/2}.
\]

(64)

For small \( s \), there is an additional subtracted term

\[
\frac{\Delta \rho_u'}{\rho} = \frac{1}{4\pi^2} \int_0^\infty d\kappa \kappa^2 \left[ \frac{\kappa^2 + s}{\sqrt{\kappa^2 + s}} - \frac{2}{\kappa^2 \sqrt{\kappa^2 + s} - s^2} \right] \times e^{\sqrt{(\kappa^2 + s)^2 - s^2} / \tau T_c^0 - 1},
\]

(65)

The first term takes its leading small-\( s \) behavior from the linear Nambu-Goldstone momentum behavior of second sound, becoming

\[
\frac{\Delta \rho_u'}{\rho} \approx \frac{2\tau T_c^0}{4\pi^2} \int_0^\infty d\kappa \kappa^2 \left[ \frac{\kappa^2 + s}{\kappa^4 + 2\kappa^2 s^2} - \frac{1}{\kappa^2} \right],
\]

(66)

which is equal to \(-s/\hat{s}\) \( \sqrt{s} / \tau T_c^0 / 4\pi \). Thus we obtain for small \( s \) the leading terms of the uncondensed particle density

\[
\rho_u / \rho = s^{3/2} \frac{\sqrt{2}}{3 \cdot 4\pi^2} + \tau^{3/2} - \frac{\tau T_c^0}{4\pi} \frac{s}{2} + \ldots.
\]

(67)

The last term is dominant for small \( s \). Its negative sign has an interesting effect upon the phase diagram observed in earlier publications, that for small coupling constant, the critical temperature increases above the free Bose gas value \( T_c^0 \) to \( T_c = \tau_c T_c^0 \) with

\[
\tau_c = 1 + \frac{2\hat{T}_c}{3 \cdot 4\pi^2} \frac{s}{2} + O(s).
\]

(68)

A similar limit square-root limit appears in the relation \( \delta \rho / \rho = s^{3/2} \Pi_2 / 4\pi^2 + \sqrt{2} / 4\pi^2 s^{3/2} \Delta \rho_1 / \rho \), where the last term is equal to

\[
-\frac{s}{4\pi^2} \int_0^\infty d\kappa \kappa^2 \left[ \frac{1}{\sqrt{(\kappa^2 + s)^2 - s^2}} e^{\sqrt{(\kappa^2 + s)^2 - s^2} / \tau T_c^0 - 1} \right] \frac{2}{s / \sqrt{2}}.
\]

(70)

To lowest order in \( \tau \) this yields

\[
-\frac{s}{4\pi^2} \int_0^\infty d\kappa \left[ \frac{2\tau T_c^0}{\kappa^2 + 2s} - \frac{\tau T_c^0}{4\pi} \sqrt{\frac{s}{2}} \right]
\]

(71)

so that we find

\[
\frac{\delta \rho}{\rho} = s^{3/2} \Pi_1 / 4\pi^2 = s^{3/2} \sqrt{2} / 4\pi^2 - \frac{\tau T_c^0}{4\pi} \frac{s}{\sqrt{2}}.
\]

(72)

According to \( \zeta \), this is equal to \(-s/\hat{s}\), implying for small \( s \), where the \( s^{3/2} \)-term is dominant, the relation between \( s \) and \( \hat{s} \) along the phase transition line

\[
\sqrt{\frac{s}{2}} \approx \frac{\hat{T}_c}{8\pi} \hat{s}.
\]

(73)

Inserting this into (68), we obtain

\[
\tau_c = 1 + \frac{\hat{T}_c^2}{3 \cdot 4\pi^2} \hat{s} + \ldots,
\]

(74)

which becomes with \( \hat{T}_c = [3(3/2)^{-2/3}]^{4/3} \cdot \tau_c / \sqrt{2} \),

\[
\frac{T_c}{T_c^0} = 1 + \frac{\hat{s}}{3(3/2)^{4/3}} + \ldots = 1 + C \frac{\hat{s}}{a} + \ldots,
\]

(75)

where the constant is \( C = 8\pi / 3(3/2)^{4/3} \approx 2.33 \). This shows the initial increase of the critical temperature for small repulsion between the bosons discussed in \( \zeta \).

Numerically, the prefactor \( C \) of the linear term is twice as big as the value \( C_{\text{V}} / \sqrt{T} = 0.93 \pm 0.13 \) predicted from 5-loop variational perturbation theory in Ref. \( \zeta \), and 83\% larger than the value \( C = 1.27 \pm 0.11 \) from its extension to seven-loops \( \zeta \). It is the same as the value obtained from a large-\( N \) calculation \( \zeta \).

For strong coupling, there is no subtlety and the phase transition can be extracted from a numerical plot of the location where \( \hat{s} \) is equal to unity. Some plots...
are shown in Fig. 1. The limit of infinitely strong coupling is found from the vanishing of the right-hand side of Eq. (53). In particular we can easily calculate the temperature dependence of the second-sound velocity \(\sqrt{s/2v_u}\) as a function of temperature using Eq. (53).

The behavior of Eqs. (53) and (54) is shown in Figs. 2 and 3. An interesting feature of Fig. 2 is that the curves continue smoothly beyond the strong-coupling limit \(1/\bar{a}_s = 0\) to negative values. This should be observable in experiments through a Feshbach resonance.

4. All properties of the strongly interacting Bose gas determined by the above theory can be calculated in the presence of superflow of velocity \(v\) by simply generalizing the integrals (53) and (54) for \(I_{\rho_u}(t)\) and \(I_{\rho_a}(t)\) to \(I_{\rho_u}(t,v)\) and \(I_{\rho_a}(t,v)\), these being defined by interchanging in each integrand the terms \((\kappa^2 + 1)/(\kappa^2 + 1 + 2KV/s^{3/2})\), where \(\nu\) is the reduced velocity of the gas \(\nu \equiv v/v_u\). From the associated second derivative of the energy we can easily find the superfluid density \(\rho_u\) as a function of the velocity.

There is no problem to drive the accuracy to any desired level, with exponentially fast convergence, as was demonstrated by the calculation of critical exponents in all Euclidean \(\varphi^4\) theories with \(N\) components in \(D\) dimensions [5]. The procedural rules were explained in the paper [22]. We merely have to calculate higher-order diagrams using the harmonic Hamiltonian [3] as the free theory that determines the Feynman diagrams, and [4] as the interaction Hamiltonian that determines the vertices. At any given order, the results are optimized in the variational parameters \(\Sigma_0\), \(\Sigma\), and \(\Delta\). The theory is renormalizable, so that all divergencies can be absorbed in a redefinition of the parameters of the orginal action, order by order. This is the essential advantage of the present theory over any previous strong-coupling scheme published so far in the literature, in particular over those based on Hubbard-Stratonovic transformations of the interaction, which are applicable only in some large-\(N\) limit as explained in [23], and for which no higher-loop calculations are renormalizable.

Our results can be made much more reliable in the \(\bar{\Sigma} \neq 0\) regime by calculating the contribution of the still-missing second-two-loop diagram.1

Acknowledgement: I am grateful to Axel Pelster and Aristeu Lima for interesting discussions, and to Flavio Nogueira and Henk Stoof for useful comments.

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1 The second diagram in Eq. (3.741) of the textbook [8]. Its contribution would be the 3 + 1-dimensional version of the last term in Eq. (3.767), is essential in the \(X \neq 0\) phase. Without this term, the slope of the quantum-mechanical energy as a function of the coupling constant is missed by 25%, discussed in the heading of Fig. 5.24.
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