Data Reduction for Maximum Matching on Real-World Graphs: Theory and Experiments

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Abstract
Finding a maximum-cardinality or maximum-weight matching in (edge-weighted) undirected graphs is among the most prominent problems of algorithmic graph theory. For \( n \)-vertex and \( m \)-edge graphs, the best known algorithms run in \( \tilde{O}(m\sqrt{n}) \) time. We build on recent theoretical work focusing on linear-time data reduction rules for finding maximum-cardinality matchings and complement the theoretical results by presenting and analyzing (thereby employing the kernelization methodology of parameterized complexity analysis) new (near-)linear-time data reduction rules for both the unweighted and the positive-integer-weighted case. Moreover, we experimentally demonstrate that these data reduction rules provide significant speedups of the state-of-the-art implementations for computing matchings in real-world graphs: the average speedup factor is 4.7 in the unweighted case and 12.72 in the weighted case.

1 Introduction
In their book chapter on matching, Korte and Vygen \cite{Korte2012} write that “matching theory is one of the classical and most important topics in combinatorial theory and optimization”. Correspondingly, the design and analysis of (weighted) matching algorithms plays a pivotal role in algorithm theory as well as in practical computing. Complementing the rich literature on matching algorithms (see Coudert et al. \cite{Coudert2018} and Duan et al. \cite{Duan2016} for recent accounts, the latter also providing a literature overview), in this work we focus on efficient linear-time data reduction rules that may help to speedup superlinear-time matching algorithms. Notably, while recent breakthrough results on (weighted) matching (including linear-time approximation algorithms \cite{Kratsch2018}) focus on the theory side, we study theory and practice, thereby contributing to both sides.

To achieve our results, we follow and complement recent purely theoretical work \cite{Niedermeier2018} presenting and analyzing linear-time data reductions for the unweighted case. More specifically, on the theoretical side we provide and analyze further data reduction rules for the unweighted as well as weighted case. On the practical side, we demonstrate that these data reduction rules may serve to speedup various matching solvers (including state-of-the-art ones) due to Huang and Stein \cite{Huang2016}, Kececioglu and Pecqueur \cite{Kececioglu2018}, and Kolmogorov \cite{Kolmogorov2018}.

Formally, we study the following two problems; note that we formulate them as decision problems since this better fits with presenting our theoretical part where we prove kernelization results (thereby employing the framework of parameterized complexity analysis). However, all our data reduction rules are “parameter-oblivious” and thus also work and are implemented for the optimization versions where the solution size is not known in advance.

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Maximum-Cardinality Matching

Input: An undirected graph $G = (V, E)$ and $s \in \mathbb{N}$.

Question: Is there a size-$s$ subset $M \subseteq E$ of nonoverlapping (that is, pairwise vertex-disjoint) edges?

Maximum-Weight Matching

Input: An undirected graph $G = (V, E)$, non-negative edge weights $\omega: E \to \mathbb{N}$, and $s \in \mathbb{N}$.

Question: Is there a subset $M \subseteq E$ of nonoverlapping edges of weight $\sum_{e \in M} \omega(e) \geq s$?

We remark that all our results extend to the case of rational weights; however, natural numbers are easier to cope with.

Related work. Micali and Vazirani [33] were the first to announce an $O(\sqrt{nm})$-time algorithm for Maximum-Cardinality Matching on graphs with $n$ vertices and $m$ edges; see Vazirani [37] for the details of this algorithm. This time bound was previously achieved only for bipartite graphs [20]. While the classic matching algorithm of Hopcroft and Karp [20] is simple, elegant, and also very efficient in practice (in fact we use it in one kernelization algorithm as subroutine), the algorithm of Micali and Vazirani [33] is rather complicated and not (yet) competitive in practice.

In fact, the fastest solver for Maximum-Cardinality Matching seems to be still the one by Kececioglu and Pecqueur [27], with a worst-case running time of $O(nm \cdot \alpha(n, m))$ ($\alpha$ denotes the inverse of the Ackermann function).

The (theoretically) fastest algorithm for Maximum-Weight Matching in sparse graphs is by Duan et al. [12] with a running time of $O(\sqrt{nm} \log(nN))$ (here $N$ denotes the largest integer weight). In practice, the fastest solver we found is due to Kolmogorov [28], which is an implementation of Edmonds’ algorithm [14, 13] for a perfect matching of minimum cost combined with many heuristic speedups.

Providing parameterized algorithms or kernels for Maximum-Cardinality Matching has recently gained high interest [8, 30, 17, 32, 16, 15, 22]. For Maximum-Weight Matching, however, we are only aware of the work by Iwata et al. [22] who provided an algorithm with a running time $O(t(m + n \log n))$ (here $t$ is the tree-depth of the input graph).

In this work, we transfer some data reduction rules for Vertex Cover to Maximum-Cardinality Matching. To this end, we use the algorithm by Iwata et al. [23] that in $O(m\sqrt{n})$ time exhaustively applies an LP-based data reduction rule due to Nenhauser and Trotter [35]. We refer to Hespe et al. [18] for a brief overview of practically relevant data reduction rules for Vertex Cover.

Very recently, Kaya et al. [26] provided a fine-tuned implementation of degree-based data reduction rules for Maximum-Cardinality Matching which is on average three times faster than our implementation when considering the same data reduction rules (see Reduction Rules 2.1 and 2.2 in Section 2.1).

Our contributions. We extend kernelization results [32] for Maximum-Cardinality Matching and lift them to Maximum-Weight Matching. Our data reduction rules for Maximum-Cardinality Matching are well-known (as crown rule [24] and LP-based rule [35]) for the NP-hard Vertex Cover problem. Our theoretical contribution here is to show that the crown rule is also correct for Maximum-Cardinality Matching. Moreover, we prove that the exhaustive application of the crown rule and exhaustive application of the LP-based rule lead to the very same graph; thus these two known rules can be seen as equivalent. We provide algorithms to efficiently apply our data reduction rules (for the unweighted and the weighted case). Herein, we have a particular eye on exhaustively applying the data reduction rules in (near) linear time, which seems imperative in an effort to practically improve matching algorithms. Hence, our main theoretical contribution lies in developing efficient algorithms implementing the data reduction rules,

\footnote{The only implementation of the algorithm of Micali and Vazirani [33] we are aware of is due to Huang and Stein [21]. This solver was the slowest in our experiments.}
Table 1: Summary of the speedup factors gained by our kernelization on graphs from the SNP library [31] with various solvers. We refer to Section 4 for details.

| implemented by          | solver                  | algorithmic approach by | speedup average | speedup median |
|-------------------------|-------------------------|-------------------------|----------------|----------------|
| Kolmogorov [28] (unweighted) | Edmonds [14, 13]       |                         | 157.30         | 29.27          |
| Huang and Stein [21]    | Micali and Vazirani [33]|                         | 608.79         | 28.87          |
| Kececioglu and Pecqueur [27]| Edmonds [14]         |                         | 4.70           | 2.20           |
| Kolmogorov [28] (weighted)| Edmonds [14, 13]       |                         | 12.72          | 1.40           |

thereby also showing a purely theoretical guarantee on the amount of data reduction that can be achieved in the worst case (this is also known as kernelization in parameterized algorithms). We proceed by implementing and testing the data reduction algorithms for MAXIMUM-CARDINALITY MATCHING and MAXIMUM-WEIGHT MATCHING, thereby demonstrating their practical effectiveness. More specifically, combining them in form of preprocessing with various solvers [28, 21, 27] yields partially huge speedups on sparse real-world graphs (taken from the SNP library [31]). We refer to Table 1 for an overview over the various solvers (with the core algorithmic approach they implement) and the speedup factors obtained by applying our data reduction rules as a preprocessing.

**Notation.** We use standard notation from graph theory. All graphs considered in this work are simple and undirected. For a graph $G = (V, E)$, we denote with $E(G) = E$ the edge set. For a vertex subset $V' \subseteq V$, we denote with $G[V']$ the subgraph induced by $V'$. We write $uv$ to denote the edge $\{u, v\}$ and $G - v$ to denote the graph obtained from $G$ by removing $v$ and all its incident edges. A feedback edge set of a graph $G$ is a set $X$ of edges such that $G - X = (V, E \setminus X)$ is a tree or forest. The feedback edge number denotes the size of a minimum feedback edge set. A vertex cover in a graph is a set of vertices that has a nonempty intersection with each edge in the graph.

A matching in a graph is a set of pairwise disjoint edges. Let $G$ be a graph and let $M \subseteq E(G)$ be a matching in $G$. We denote by $\text{mm}(G)$ a maximum-cardinality matching respectively a maximum-weight matching in $G$, depending on whether we have edge weights or not. If there are edge weights $\omega : E \to \mathbb{N}$, then for a matching $M$ we denote by $\omega(M) := \sum_{e \in M} \omega(e)$ the weight of $M$. Moreover, we denote with $\omega(G)$ the weight of a maximum-weight matching $\text{mm}(G)$, i.e. $\omega(G) := \omega(\text{mm}(G))$. A vertex $v \in V$ is called matched with respect to $M$ if there is an edge in $M$ containing $v$, otherwise $v$ is called free with respect to $M$. If the matching $M$ is clear from the context, then we omit “with respect to $M$”.

**Kernelization.** A parameterized problem is a set of instances $(I, k)$ where $I \in \Sigma^*$ for a finite alphabet $\Sigma$ and $k \in \mathbb{N}$ is the parameter. We say that two instances $(I, k)$ and $(I', k')$ of parameterized problems $P$ and $P'$ are equivalent if $(I, k)$ is a yes-instance for $P$ if and only if $(I', k')$ is a yes-instance for $P'$. A kernelization is an algorithm that, given an instance $(I, k)$ of a parameterized problem $P$, computes in polynomial time an equivalent instance $(I', k')$ of $P$ (the kernel) such that $|I'| + k' \leq f(k)$ for some computable function $f$. We say that $f$ measures the size of the kernel, and if $f(k) \in k^{O(1)}$, then we say that $P$ admits a polynomial kernel. Typically, a kernel is achieved by applying polynomial-time executable data reduction rules. We call a data reduction rule $\mathcal{R}$ correct if the new instance $(I', k')$ that results from applying $\mathcal{R}$ to $(I, k)$ is equivalent to $(I, k)$. An instance is called reduced with respect to some data reduction rule if further application of this rule has no effect on the instance.

**Structure of this work.** In Sections 2 and 3, we provide the kernelization results for MAXIMUM-CARDINALITY MATCHING and MAXIMUM-WEIGHT MATCHING which we experimentally evaluate on real-world data sets in Section 4. In Section 2 we discuss the unweighted case by recalling
old and presenting new data reduction rules. In Section 3 we show how to extend some of the data reduction rules presented for MAXIMUM-CARDINALITY MATCHING to MAXIMUM-WEIGHT MATCHING. In Section 4, we describe our experimental results, discuss the effect of our data reduction rules on state-of-the-art solvers, and evaluate the prediction quality of our theoretical kernelization results. We conclude in Section 5 with a glimpse on future research challenges.

2 Maximum-Cardinality Matching

For MAXIMUM-CARDINALITY MATCHING we first recall in Section 2.1 simple data reduction rules for low-degree vertices due to a classic result of Karp and Sipser [25]. Then we improve the known kernel-size for MAXIMUM-CARDINALITY MATCHING parameterized by the feedback edge number when only these two data reduction rules are exhaustively applied [32].

In Section 2.2, we discuss the crown data reduction rule (designed for VERTEX COVER [24]) and show that it also works for MAXIMUM-CARDINALITY MATCHING. To this end, we briefly describe a classic LP-based data reduction due to Nemhauser and Trotter [35]. It was known that this LP-based data reduction also removes all crowns from the input graph [3, 23]. We note that this does not immediately imply that we can use the LP-based data reduction in the context of MAXIMUM-CARDINALITY MATCHING (the correctness is only known for VERTEX COVER). We prove that exhaustively applying the crown data reduction rule is equivalent to “exhaustively” applying the LP-based data reduction. This allows us to use the algorithm of Iwata et al. [23] that exhaustively applies the LP-based data reduction in order to remove all crowns and nothing else.

Finally, we show in Section 2.3 a generalization of the crown data reduction rule. However, we leave it open how to apply this generalized crown data reduction rule efficiently. Note that we have implemented all of these data reduction rules (except the generalized crown data reduction rule); see Section 4 for an evaluation.

2.1 Removing low-degree vertices

For MAXIMUM-CARDINALITY MATCHING two simple data reduction rules are due to a classic result of Karp and Sipser [25]. They deal with vertices of degree at most two.

Reduction Rule 2.1 ([25]). Let \( v \in V \). If \( \deg(v) = 0 \), then delete \( v \). If \( \deg(v) = 1 \), then delete \( v \) and its neighbor, and decrease the solution size \( s \) by one.

Reduction Rule 2.2 ([25]). Let \( v \) be a vertex of degree two and let \( u, w \) be its neighbors. Then remove \( v \), merge \( u \) and \( w \), and decrease the solution size \( s \) by one.

If the degree of the considered vertex \( v \) is zero, then the maximum matching size remains unchanged. Otherwise, we have \( |\text{mm}(G)| = |\text{mm}(G')| + 1 \), where \( G' \) is the instance resulting from one application of either Reduction Rule 2.1 or 2.2: When applying Reduction Rule 2.1, then \( v \) is matched to its only neighbor \( u \). For Reduction Rule 2.2 the situation is not so clear as \( v \) is matched to \( u \) or to \( w \) depending on how the maximum-cardinality matching in the rest of the graph looks like. Thus, one can only fix the matching edge with endpoint \( v \) (in the original graph) in a simple postprocessing step.

Each of the above data reduction rules can be exhaustively applied in linear time. While for Reduction Rule 2.1 this is easy to see, for Reduction Rule 2.2 the algorithm needs further ideas [4]. However, to exhaustively apply both data reductions rules together only algorithms with superlinear running times are known [5].

Using the above data reduction rules, one can show a kernel with respect to the parameter feedback edge number, that is, the size of a minimum feedback edge set. We refer to Section 4.2 for a practical evaluation of a theoretical upper bound on the kernel size.

Theorem 2.1 ([32]). MAXIMUM-CARDINALITY MATCHING admits a linear-time computable kernel with at most \( 2k - 1 \) vertices and at most \( 3k - 2 \) edges, when parameterized by the feedback edge number \( k \).
Mertzios et al. [32] originally proved a kernel with at most 12\(k\) vertices and 13\(k\) edges. However, we can tighten this upper bound in the following way.

**Proof of Theorem 2.1.** Our kernelization procedure consists of the following two steps:

1. Apply Reduction Rule 2.1 exhaustively in linear time.
2. Apply Reduction Rule 2.2 exhaustively in linear time [4].

Let \(G^{(1)} = (V^{(1)}, E^{(1)})\) and \(G^{(2)} = (V^{(2)}, E^{(2)})\) be the graphs obtained after Steps (1) and (2), respectively. Thus, \(G^{(2)}\) is the graph returned by the kernelization algorithm. Note that \(G^{(2)}\) might contain isolated vertices and degree-one vertices.

To evaluate the size of \(G^{(2)}\), we first analyze the structure of \(G^{(1)}\). Since \(G^{(1)}\) is an induced subgraph of the input graph \(G\), it follows that \(G^{(1)}\) has a feedback edge set \(F^{(1)} \subseteq E^{(1)}\) of size at most \(k\). We show that \(G^{(1)}\) contains at most \(2k - 1\) vertices of degree at least three. To this end, consider the graph \(G_F^{(1)} = (V_F^{(1)}, E_F^{(1)})\) obtained from \(G^{(1)}\) as follows. Remove the edges in \(F^{(1)}\).

For each edge \(e = uv \in F^{(1)},\) we remove \(uv\) and we introduce two degree-one vertices \(w_u^e\) and \(w_v^e\) adjacent to \(u\) and \(v\), respectively. Formally, we have

\[
V_F^{(1)} := (V^{(1)} \cup \left\{ w_{u}^e, w_{v}^e \mid e = uv \in F^{(1)} \right\}) \quad \text{and} \quad E_F^{(1)} := (E^{(1)} \setminus F^{(1)}) \cup \left\{ w_{u}^e, w_{v}^e \mid e = uv \in F^{(1)} \right\}.
\]

Observe that \(G_F^{(1)}\) is a forest where \(V_F^{(1)} \setminus V^{(1)}\) are the leaves and \(V^{(1)} \subseteq V_F^{(1)}\) are the internal vertices. Since \(|F^{(1)}| \leq k\), we have at most \(2k\) leaves. Since \(G_F^{(1)}\) is a forest, it follows that \(G_F^{(1)}\) has at most \(2k - 1\) vertices degree of at least three.

We partition the vertex set of \(G^{(1)}\) into \(V^{(1)} = V_2^{(1)} \cup V_{\geq 3}^{(1)},\) where \(V_2^{(1)}\) are the vertices of degree two and \(V_{\geq 3}^{(1)}\) are the vertices of degree at least three. Since Reduction Rule 2.1 is exhaustively applied, the minimum degree in \(G^{(1)}\) is at least two. Moreover, note that by construction of \(G_F^{(1)}\), it follows that the set of vertices with degree at least three is identical in \(G_F^{(1)}\) and \(G^{(1)}\). Thus, \(|V_{\geq 3}^{(1)}| \leq 2k - 1\). Note that exhaustively applying Reduction Rule 2.2 in Step (2) on \(G^{(1)}\) will remove all vertices in \(V_2^{(1)}\) (and possibly some of \(V_{\geq 3}^{(1)}\)). Thus, the resulting graph \(G^{(2)}\) of our kernelization procedure has at most \(2k - 1\) vertices and consequently at most \(3k - 2\) edges.

Applying the \(O(m\sqrt{n})\)-time algorithm for Maximum-Cardinality Matching [33] altogether yields an \(O(n + m + k^{1.5})\)-time algorithm, where \(k\) is the feedback edge number.

### 2.2 Crown and LP-based data reduction

Crowns are a classic data reduction tool for Vertex Cover (given an undirected graph find a smallest set of vertices that covers all edges) and can be seen as a generalization of Reduction Rule 2.1 [24]. A crown satisfies the following properties (see Figure 1 for a visualization):

**Definition 2.2.** A **crown** in a graph \(G = (V, E)\) is a pair \((H, I)\) such that

1. \(I \subseteq V\) is an independent set in \(G\) (no two vertices in \(I\) are adjacent in \(G\)),
2. \(H = N(I) := \bigcup_{v \in I} N(v)\), and
3. there is a matching \(M_{H,I}\) between \(H\) and \(I\) that matches all vertices in \(H\).

It is not hard to see that, given a crown \((H, I)\), there is a minimum vertex cover containing all vertices in \(H\): The matching \(M_{H,I}\) implies that a minimum size vertex cover in \(G[H \cup I]\) has size \(|M_{H,I}| = |H|\). Since \(I\) is an independent set and \(H = N(I)\), taking all vertices in \(H\) into a vertex cover is at least as good as taking some vertices of \(I\). Thus, we end up with the following data reduction rule.
Reduction Rule 2.3 ([24]). Let \((H, I)\) be a crown. Then remove all vertices in \(H \cup I\) and decrease the solution size \(s\) by \(|H|\).

Luckily, Reduction Rule 2.3 not only works for Vertex Cover but also for Maximum-Cardinality Matching: Simply adding to any maximum-cardinality matching in the reduced graph the matching \(M_{H,I}\) results in a maximum-cardinality matching for \(G\), as proven in the next lemma.

Lemma 2.3. Reduction Rule 2.3 is correct, that is, \(|\text{mm}(G)| = |\text{mm}(G')| + |H|\).

Proof. Let \((H, I)\) be a crown in the input graph \(G\) and \(G' := G - (H \cup I)\). Observe that \(\text{mm}(G') \cup M_{H,I}\) is a matching of cardinality \(|\text{mm}(G')| + |H|\) in \(G\), thus \(|\text{mm}(G)| \geq |\text{mm}(G')| + |H|\). Conversely, observe that \(|\text{mm}(G)| \leq |\text{mm}(G - H)| + |H|\) as each vertex in \(H\) can be matched at most once. However, we have \(|\text{mm}(G - H)| = |\text{mm}(G - (H \cup I))| = |\text{mm}(G')|\) as each vertex in \(I\) has degree zero in \(G - H\) and can thus be removed (see Reduction Rule 2.1). Thus, \(|\text{mm}(G)| = |\text{mm}(G')| + |H|\). □

Now that we established that Reduction Rule 2.3 can also be applied for Maximum-Cardinality Matching, it remains to do so as fast as possible. However, to find a crown \((H, I)\) we need to also find a matching \(M_{H,I}\). Moreover, the size of \(M_{H,I}\) depends on the size of the crown; a crown can be quite large (consider for example a complete bipartite graph \(K_{n,n}\); there is only one crown which is the whole graph). Thus, applying Reduction Rule 2.3 even once in linear time seems hard to do. Indeed, the best known algorithm to apply Reduction Rule 2.3 (even just once) runs in \(O(\sqrt{nm})\) time [23].

Since we can compute in \(O(\sqrt{nm})\) time also a maximum-cardinality matching for the input graph, Reduction Rule 2.3 seems to be not useful (why decrease the size of the input when one can solve the problem in the same time?). However, the algorithm of Iwata et al. [23] to exhaustively apply Reduction Rule 2.3 has only a single step that requires superlinear time: the computation of one maximum-cardinality matching in a bipartite graph with \(O(n + m)\) vertices and edges. To find such a matching, we used a straightforward implementation of the classic algorithm of Hopcroft and Karp [20]. It turned out in our experiments that even this implementation is faster than computing a maximum-cardinality matching in non-bipartite graphs with any of the implementations for Maximum-Cardinality Matching that we tested.

Exhaustively removing crowns. Subsequently, we briefly sketch the algorithm of Iwata et al. [23] and prove that it exhaustively applies Reduction Rule 2.3 in an input graph \(G = (V, E)\). Note that their algorithm efficiently applies a classic linear programming (LP)-based data reduction rule for Vertex Cover [35]. We first observe that a straightforward adaptation of this LP-based data reduction rule to Maximum-Cardinality Matching (working with the LP-relaxation for the Maximum-Cardinality Matching-ILP) does not seem to work (Figure 2 provides a counterexample). However, Akiba and Iwata [3] already observed that the LP-based data reduction rule of Iwata et al. [23] (working with the LP-relaxation for the Vertex Cover-ILP) also removes all crowns from the input graph. It was hence already clear that the LP-based data reduction rule is at least as powerful as exhaustively applying the crown data reduction rule (in the context of Vertex Cover). Below, we show that exhaustively applying the crown data reduction rule is exactly as powerful as the LP-based data reduction rule. To be precise, we prove that “exhaustively”
applying the LP-based data reduction rule (as Iwata et al. [23] did) and exhaustively applying the crown data reduction rule (Reduction Rule 2.3) results in exactly the same graph. Thus, we can use the algorithm of Iwata et al. [23] working with the LP-relaxation for the Vertex Cover-ILP to apply Reduction Rule 2.3 in the context of Maximum-Cardinality Matching.

We start with explaining the LP-based kernelization for Vertex Cover (refer to Cygan et al. [9, Chapter 2] for a more detailed description with all proofs). The standard integer linear program (ILP) for Vertex Cover is as follows.

\[
\begin{align*}
\text{Minimize} & \quad \sum_{v \in V} x_v \\
\text{Subject to} & \quad x_v + x_u \geq 1 \quad \forall uv \in E \\
& \quad x_v \in \{0, 1\} \quad \forall v \in V
\end{align*}
\]

As usual, the LP relaxation (subsequently called VC-LP) is obtained by replacing the constraints \(x_v \in \{0, 1\}\) by \(0 \leq x_v \leq 1\). This LP and its dual LP always have half-integral solutions, that is, there is always an optimal solution such that each variable is assigned a value in \(\{0, 1/2, 1\}\) (this is true for a more general class of LPs called BIP2 [19]). Given a half-integral solution for the VC-LP, define the following three sets of vertices corresponding to variables set to 0, \(1/2\), and 1, respectively:

- \(V_1 := \{v \mid x_v = 1\}\),
- \(V_{1/2} := \{v \mid x_v = 1/2\}\), and
- \(V_0 := \{v \mid x_v = 0\}\).

The following classic result of Nemhauser and Trotter [35] forms the basis for the LP-based kernelization for Vertex Cover.

**Theorem 2.4** ([35]). There is a minimum vertex cover \(S\) for \(G\) such that \(V_1 \subseteq S \subseteq V_1 \cup V_{1/2}\).

**Remark** 1. The dual of the VC-LP is the following relaxation of the ILP for Maximum-Cardinality Matching:

\[
\begin{align*}
\text{Maximize} & \quad \sum_{uv \in E} y_{uv} \\
\text{Subject to} & \quad \sum_{u \in N(v)} y_{uv} \leq 1 \quad \forall v \in V \\
& \quad 0 \leq y_{uv} \leq 1 \quad \forall uv \in E
\end{align*}
\]

Notably, a statement similar to Theorem 2.4 does not hold for Maximum-Cardinality Matching, see Figure 2 for two counterexamples.

The following data reduction rule is an immediate consequence of Theorem 2.4:

**Reduction Rule 2.4.** Compute a solution for the VC-LP, remove all vertices in \(V_0 \cup V_1\), and decrease the solution size by \(|V_1|\).

Abu-Khzam et al. [1], Chlebík and Chlebíková [7] independently showed that RR 2.4 removes a crown (see Definition 2.2 for its definition). We include the proof for the sake of completeness.

**Lemma 2.5** ([1, 7]). For any solution \(X\) for the VC-LP, \((V_1, V_0)\) is a crown.

**Proof.** It is easy to see that \(V_0\) is an independent set since the constraint \(x_v + x_u \geq 1\) for all \(uv \in E\) forbids to set two variables of adjacent vertices to zero. Thus, we have that \(N(V_0) \subseteq V_1\). It follows from the optimality of \(X\) that \(N(V_0) = V_1\) (any variable of a vertex \(v \in V_1 \setminus N(V_0)\) could be set to \(1/2\) instead). To show that there is a matching between \(V_0\) and \(V_1\) that matches all vertices
Figure 2: Examples in which a solution for the matching-LP does not help in finding a maximum-cardinality matching. In both graphs, bold edges indicate a perfect matching and the numbers next to the edges give a valid optimal solution to the LP for **Maximum-Cardinality Matching**.

In the left graph, the single edge (in the middle), whose variable is set to one, is not contained in any perfect matching. In the right graph, there is only one perfect matching that contains one edge (in the middle), whose variable is set to zero.

in $V_1$, we use the optimality of $X$ together with Hall’s marriage theorem$^2$: Assuming that there is no such matching, it follows from Hall’s marriage theorem that there is a subset $W \subseteq V_1$ such that $|N(W) \cap V_0| < |W|$. However, this implies that setting all variables corresponding to vertices in $W \cup (N(W) \cap V_0)$ to $\frac{1}{2}$ results in a better solution to the VC-LP, a contradiction to the optimality of $X$.

Subsequently, we first show how to efficiently compute a solution for the VC-LP. Then, we show that “exhaustively” applying Reduction Rule 2.4 and “exhaustively” applying Reduction Rule 2.3 leads to exactly the same kernel.

A half-integral solution for the VC-LP can be obtained by computing a minimum vertex cover in a bipartite graph $G$: The vertices $V$ of $G$ consist of two copies of $V$, called $V_L$ and $V_R$: for $i \in \{L, R\}$ we set $V_i := \{v_i \mid v \in V\}$ and $V := V_L \cup V_R$. The edges $E$ of $G$ are as follows: $E := \{v_Lu_R, v_Ru_L \mid uv \in E\}$. Thus, $G$ has $2n$ vertices and $2m$ edges. Then, a solution for the VC-LP can be constructed from a vertex cover $S$ for $G$ as follows $^3$:

$$x_v = \begin{cases} 1, & \text{if } v_L \in S \land v_R \in S, \\ 0, & \text{if } v_L \notin S \land v_R \notin S, \\ \frac{1}{2}, & \text{else}. \end{cases}$$

Hence, using König’s theorem$^3$, we can compute a solution for the VC-LP in $O(\sqrt{mn})$ time with the algorithm of Hopcroft and Karp $[20]$.

Of course, to have Reduction Rule 2.4 as strong as possible, we would like to have a solution for the VC-LP with the maximum number of variables set to 0 and 1. Note that the above solution for the VC-LP does not necessarily fulfill this condition. However, Iwata et al. $[23]$ provided an algorithm that, given any solution for the VC-LP, computes an optimal solution for the VC-LP that minimizes, over all half-integral optimal solutions, the number of variables set to $\frac{1}{2}$ in linear time. Hence, using the algorithm of Iwata et al. $[23]$, we can exhaustively apply Reduction Rule 2.4 in $O(\sqrt{mn})$ time. By exhaustively applying Reduction Rule 2.4, we mean that we apply Reduction Rule 2.4 until the solution setting all variables to $\frac{1}{2}$ is the unique optimal solution for the VC-LP.

It remains to show that exhaustively applying Reduction Rule 2.4 results in the same instance as exhaustively applying Reduction Rule 2.3. This extends the work of Abu-Khzam et al. $[1]$ who showed that applying Reduction Rule 2.4 exhaustively removes all crowns.

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$^2$Hall’s marriage theorem states that, in any bipartite graph $G = (X \uplus Y; E)$, there is an $X$-saturating matching if and only if for every subset $W \subseteq X$ we have $|W| \leq |N(W)|$. In other words: every subset $W \subseteq X$ has sufficiently many adjacent vertices in $Y$.

$^3$König’s theorem states that, in any bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover. Moreover, such a minimum vertex cover can be constructed in linear time given a maximum matching.
Lemma 2.6. Let \( G \) be a graph, let \( G_{\text{LP}} \) be the graph obtained from exhaustively applying Reduction Rule 2.4 on \( G \), and let \( G_{\text{crown}} \) be the graph obtained from exhaustively applying Reduction Rule 2.3. Then \( G_{\text{LP}} = G_{\text{crown}} \).

Proof. First we show that there exists a set \( V_{i/2}^* \) of vertices such that for every half-integral optimal solution for the VC-LP for \( G \) that minimizes the number of variables set to \( 1/2 \), the set of vertices whose variables are set to \( 1/2 \) is \( V_{i/2}^* \). Let \( X' \) and \( X'' \) be two half-integral solutions for the VC-LP with the minimum number of variables set to \( 1/2 \). For \( i \in \{1, 0, 1/2\} \), let \( V_i' \) and \( V_i'' \) be, as defined above, the set of vertices whose variables are set to \( i \) in \( X' \) and \( X'' \), respectively. Assume for the sake of contradiction that \( V_{i/2}' \neq V_{i/2}'' \). We claim that the following (call it \( X \)) is an optimal solution for the VC-LP:

\[
x_v = \begin{cases} 
1 & \text{if } v \in V_i' \cup (V_i' \cap V_i''), \\
0 & \text{if } v \in V_i' \cup (V_i' \cap V_i''), \\
1/2 & \text{else (that is, } v \in V_{i/2}' \setminus (V_{i/2}'' \cup V_i')).
\end{cases}
\]

We define \( V_1, V_0, \) and \( V_{i/2} \) analogously for \( X \).

To prove the claim, we first verify that \( X \) is a solution for the VC-LP. It suffices to show that \( N(v) \subseteq V_1 \) for each vertex \( v \in V_0 \). If \( v \in V_0 \), then we have \( N(v) \subseteq V_{i/2}' \subseteq V_1 \). Otherwise (that is, \( v \in V_{i/2}' \cap V_0' \)), we have \( N(v) \subseteq N(V_{i/2}' \cap V_0') \subseteq N(V_{i/2}'' \cap N(V_0')) \subseteq (V_{i/2}' \cap V_{i/2}'') \cap V_0' \subseteq V_1 \), because \( N(V_{i/2}') \subseteq V_i' \cup V_{i/2}' \) and \( N(V_0') \subseteq V_i' \). Thus, we see that \( X \) is a solution for the VC-LP.

To prove \( X \) is optimal for the VC-LP. By Lemma 2.5, there is a matching \( M \) in \( G[V_i' \cup V_{i/2}] \) that matches all vertices of \( V_i' \), and thereby, we have \( \sum_{v \in V_i' \cup V_{i/2}} x_v'' \geq \sum_{v \in M} x_v'' + x_v' \geq |M| \geq |V_i'| \). It follows that

\[
\sum_{v \in V} x_v'' = \sum_{v \in V_i' \cup V_0'} x_v'' + \sum_{v \in V_{i/2}} x_v'' \geq |V_i'| + |V_{i/2}'' \cap V_{i/2}'| + \frac{1}{2}|V_{i/2}' \setminus (V_0'' \cup V_1')| = |V_i'| + \frac{1}{2}|V_{i/2}'| + \frac{1}{2}(|V_{i/2}' \cap V_0'') - |V_{i/2}' \cap V_{i/2}'|).
\]

Note that \( \sum_{v \in V} x_v'' = \sum_{v \in V} x_v' = |V_i'| + \frac{1}{2}|V_{i/2}'| \) by the optimality of \( X' \) and \( X'' \). Thus, we obtain \( |V_{i/2}' \cap V_0''| - |V_{i/2}' \cap V_{i/2}''| \leq 0 \). Then the cost of \( X \) is

\[
\sum_{v \in V} x_v = (|V_i'| + |V_{i/2}' \cap V_{i/2}'|) + \frac{1}{2}|V_{i/2}' \setminus (V_{i/2}'' \cup V_0')| = |V_i'| + \frac{1}{2}|V_{i/2}'| + \frac{1}{2}(|V_{i/2}' \cap V_{i/2}'| - |V_{i/2}' \cap V_{i/2}''|) \leq |V_i'| + \frac{1}{2}|V_{i/2}'|,
\]

and hence \( X \) is an optimal solution for the VC-LP for \( G \). Since the number of variables set to \( 1/2 \) in \( X \) is smaller than that of \( X' \) and \( X'' \), this contradicts our assumption on \( X' \) and \( X'' \). Consequently, \( G_{\text{LP}} \) is well-defined—\( G_{\text{LP}} = G[V_{i/2}'] \).

It remains to show that if Reduction Rule 2.3 is applied exhaustively, then the VC-LP for the resulting graph \( G_{\text{crown}} \) has a unique optimal solution, in which all variables are set to \( 1/2 \). Assume towards a contradiction that the VC-LP for \( G_{\text{crown}} \) has a solution \( X \) with some variables not set to \( 1/2 \). Then by Lemma 2.5, \( G_{\text{crown}} \) has a crown, which is a contradiction. Hence, in \( G_{\text{crown}} \) the VC-LP has a unique solution: setting all variables to \( 1/2 \).

\[ \square \]

It is known that applying Reduction Rule 2.4 results in an instance with at most \( 2\tau \) vertices, where \( \tau \) is the vertex cover number \( (\text{the size of a minimum vertex cover}) \) [9]. Combining this, the algorithm of Iwata et al. [23], and Lemma 2.6 gives the following theorem.

Theorem 2.7. Reduction Rule 2.3 can be exhaustively applied in \( O(\sqrt{m}n) \) time and the resulting instance has at most \( 2\tau \) vertices.
We refer to Section 4.2 for a practical evaluation of the theoretical upper bound of Theorem 2.7 on the number of vertices in the reduced instance. However, note that from a theoretical point of view, the kernel given in Theorem 2.1 is incomparable to the upper bound of Theorem 2.7: In a large odd cycle $C_{2n+1}$, applying the kernel behind Theorem 2.1 yields a constant-size kernel (as the feedback edge number is one), that is, Theorem 2.1 essentially solves this instance. Reduction Rule 2.3, however, does not remove a single vertex in this case. Conversely, applying Reduction Rule 2.3 on a complete bipartite graph $K_{3,n}$ (here $\tau = 3$) solves the instance again, but the kernel behind Theorem 2.1 does not remove a single vertex.

Note that the algorithm behind Theorem 2.7 contains only one step requiring super-linear running time: The computation of a bipartite matching to compute an initial solution for the VC-LP. Since computing matchings in practice is much easier for bipartite than for general graphs (even though the known theoretical worst-case running times are the same [37], it is not surprising that our implementation that exhaustively applies Reduction Rule 2.3 is significantly faster than the state-of-the-art matching implementations on the respective input graph (see Section 4).

2.3 Relaxed Crowns

In this subsection, we provide a data reduction rule that generalizes Reduction Rule 2.2 (for degree-two vertices) in the same sense as Reduction Rule 2.3 (crown rule) generalizes Reduction Rule 2.1 (for degree-one vertices). However, in contrast to Reduction Rule 2.3 we decided not to implement this rule as we do not have a sufficiently fast algorithm to apply the rule. Our new rule uses a relaxed crown concept, defined as follows:

**Definition 2.8.** A relaxed crown in a graph $G = (V, E)$ is a pair $(H, I)$ such that

1. $I \subseteq V$ is an independent set in $G$ (no two vertices in $I$ are adjacent in $G$),
2. $H = N(I) := \bigcup_{v \in I} N(v)$, and
3. for every $v \in H$ there is a matching $M_{H,I,v}$ between $H \setminus \{v\}$ and $I$ that matches all vertices in $H \setminus \{v\}$.

Compared to the crown concept we relaxed Condition (3). In particular, note that in the relaxed crown, the set $H$ can be larger (by one vertex) than $I$! Our new data reduction rule is as follows (see Figure 3 for an illustration).

**Reduction Rule 2.5.** Let $G$ be a crown-free graph and let $(H, I)$ be a relaxed crown in $G$. Then remove all vertices in $H \cup I$, add a new vertex $w$ with $N(w) = \bigcup_{u \in H} N(u) \setminus (H \cup I)$ and decrease the solution size $s$ by $|H| - 1$.

**Lemma 2.9.** Reduction Rule 2.5 is correct.
Figure 4: Left: Input graph. Right: The graph after applying Reduction Rule 3.2 to vertex v. Bold edges indicate the unique maximum-weight matching in each graph.

Proof. (⇒): Let $M \subseteq E$ be a maximum-cardinality matching in the input graph $G$ and let $G'$ be the reduced graph. We split $M$ into three parts $M = M_0 \cup M_1 \cup M_2$ as follows (edges with zero, one, or two endpoints in the relaxed crown): $M_i := \{uv \in M \mid i = |uw \cap (H \cup I)|\}$. Since $I$ is an independent set, it follows that $|M_1| + |M_2| \leq |H|$. We now make a case distinction whether or not $M_1$ is the empty set.

Case 1. $M_1 = \emptyset$: Note that $|M_2| \leq |H|$. By assumption, $G$ does not contain a crown and, thus, we can strengthen this to $|M_2| \leq |H| - 1$. Hence, $M_0$ is a matching for $G'$ with $|M_0| = |M| - |M_2| \geq |M| - (|H| - 1)$.

Case 2. $M_1 \neq \emptyset$: Let $uv \in M_1$ be an arbitrary edge with $u \in H$. Then, $uv$ is in $G'$ and $M_0 \cup \{uv\}$ is a matching for $G'$ of size at least $|M_0| + 1 = |M| - (|M_1| + |M_2|) + 1 \geq |M| - (|H| - 1)$.

(⇐): Let $M'$ be a maximum-cardinality matching in $G'$. If $w$ is not matched in $M'$, then $M \cup M_{H,I,w}$ is a matching of size $|M'| + |H| - 1$ for an arbitrary vertex $v \in H$. So we may assume that $uw \in M'$. Let $v \in H$ be an arbitrary vertex adjacent to $u$. Then, $(M \setminus \{uw\}) \cup M_{H,I,v}$ is a matching of size $|M'| + |H| - 1$ for $G$. Hence, $\mm(G) = |\mm(G')| + |H| - 1$. \qed

3 Maximum-Weight Matching

In this section, we show how to lift Theorem 2.1 (dealing with Maximum-Cardinality Matching) to the weighted case. Reduction Rules 2.1 and 2.2 are based on the simple observation that for every vertex $v \in V$ of degree at least one, there exists a maximum-cardinality matching containing $v$: If $v$ is not matched, then take an arbitrary neighbor $u$ of $v$, remove the edge containing $u$ from a maximum-cardinality matching, and add the edge $uv$. This observation does not hold in the weighted case—see, e.g., Figure 4 (left-hand side) where the only maximum-weight matching $\{au, bc\}$ leaves $v$ free. Thus, we need new ideas to obtain efficient data reduction rules for the weighted case.

Vertices of degree at most one. We start with the simple case of dealing with vertices of degree at most one. Here, the following data reduction rule is obvious.

Reduction Rule 3.1. If $\deg(v) = 0$ for a vertex $v \in V$, then delete $v$. If $\omega(e) = 0$ for an edge $e \in E$, then delete $e$.

Next, we show how to deal with degree-one vertices, see Figure 4 for a visualization.

Reduction Rule 3.2. Let $G = (V, E)$ be a graph with non-negative edge weights $\omega: E \rightarrow \mathbb{N}$. Let $v$ be a degree-one vertex and let $u$ be its neighbor. Then delete $v$, set the weight of every edge $e$ incident with $u$ to $\max\{0, \omega(e) - \omega(uv)\}$, and decrease the solution value $s$ by $\omega(uv)$.

While proving the correctness of this rule (see next lemma) is relatively straightforward, the naive algorithm to exhaustively apply Reduction Rule 3.2 is too slow for our purpose: If the edge weights are adjusted immediately after deleting $v$, then exhaustively applying the rule to a star requires $\Theta(n^2)$ time. However, as we subsequently show, Reduction Rule 3.2 can be exhaustively applied in linear time.

Lemma 3.1. Reduction Rule 3.2 is correct.
Proof. Let \( v \) be a vertex of degree one and let \( u \) be its neighbor. Let \( M \) be a matching of weight at least \( s \) for \( G \). We assume without loss of generality that \( M \) is of maximum weight and, hence, \( u \) is matched. If \( uv \in M \), then the deletion of \( v \) decreases the weight of the matching by \( \omega(uv) \). Hence, the resulting graph \( G' \) (with adjusted weights) has a matching of weight at least \( s - \omega(uv) \). If \( uv \notin M \), then \( M \) is also contained in the resulting graph \( G' \). As \( v \) is not matched, \( M \) contains exactly one edge \( e \) with \( u \in e \). Thus, \( e \) has in \( G' \) weight \( \max\{0, w(e) - w(uv)\} \) and \( M \) has in \( G' \) weight at least \( s - \omega(uv) \).

Conversely, let \( M' \) be a matching in the reduced graph \( G' \) with weight at least \( s - \omega(uv) \). We construct a matching \( M'' \) for \( G \) as follows. First, consider the case that \( u \) is matched by an edge \( e \). If \( e \) has in \( G' \) weight more than zero, then set \( M'' := M' \). If \( e \) has in \( G' \) weight zero, then set \( M'' := (M' \setminus \{e\}) \cup \{uv\} \). Second, if \( u \) is free, then set \( M'' := M' \cup \{uv\} \). In all three cases \( M'' \) is a matching in \( G \) with weight at least \( s \).

Lemma 3.2. Reduction Rule 3.2 can be exhaustively applied in \( O(n + m) \) time.

Proof. The basic idea of the algorithm exhaustively applying Reduction Rule 3.2 in linear time is as follows: We store in each vertex a number indicating the weight of the heaviest incident edge removed due to Reduction Rule 3.2. Then, whenever we want to access the “current” weight of an edge \( e \), we then subtract from \( \omega(e) \) the two numbers stored in the two incident vertices. Once Reduction Rule 3.2 is no more applicable, then we update the edge weights to get rid of the numbers in the vertices in order to create a Maximum-Weight Matching instance.

The details of the algorithm are as follows. First, in \( O(n + m) \) time we collect all degree-one vertices in a list \( L \) and initialize for each vertex \( v \) a counter \( c(v) := 0 \). Then, we process \( L \) one by one. For a degree-one vertex \( v \in L \), let \( u \) be its neighbor. We decrease \( c(u) \) by \( \max\{0, w(uv) - c(u) - c(v)\} \), then set \( c(u) := c(u) + \max\{0, w(uv) - c(u) - c(v)\} \), and then delete \( v \). If after the deletion of \( v \) its neighbor \( u \) has degree one, then \( u \) is added to \( L \). Thus, after at most \( n \) steps, each one doable in constant time, we processed \( L \). When \( L \) is empty, then in \( O(m) \) time we update for each edge \( uv \) its weight \( w(uv) := \max\{0, w(uv) - c(u) - c(v)\} \). This finishes the description of the algorithm.

Observe that we have the following invariant when processing the list \( L \): the weight of an edge \( uv \) is \( \max\{0, w(uv) - c(u) - c(v)\} \). With this invariant, it is easy to see that the algorithm indeed applies Reduction Rule 3.2 exhaustively.

Note that after applying Reduction Rule 3.2 we can have weight-zero edges and thus Reduction Rule 3.1 might become applicable. We do not know whether Reduction Rules 3.1 and 3.2 together can be applied exhaustively in linear time. However, for the kernel we present at the end of this section it is sufficient to apply Reduction Rule 3.2 exhaustively.

Vertices of degree two. Lifting Reduction Rule 2.2 to the weighted case is more delicate than lifting Reduction Rule 2.1 to Reduction Rules 3.1 and 3.2. The reason is that the two incident edges might have different weights. As a consequence, we cannot decide locally what to do with a degree-two vertex. Instead, we process multiple degree-two vertices at once. To this end, we use the following notation.

Definition 3.3. Let \( G \) be a graph. A path \( P = v_0v_1 \ldots v_\ell \) is a maximal path in \( G \) if \( \ell \geq 3 \) and the inner vertices \( v_1, v_2, \ldots, v_{\ell-1} \) all have degree two in \( G \), but the endpoints \( v_0 \) and \( v_\ell \) do not, that is, \( \deg_G(v_1) = \ldots = \deg_G(v_{\ell-1}) = 2 \), \( \deg_G(v_0) \neq 2 \), and \( \deg_G(v_\ell) \neq 2 \).

Definition 3.4. Let \( G \) be a graph. A cycle \( C = v_0v_1 \ldots v_{\ell}v_0 \) is a pending cycle in \( G \) if at most one vertex in \( C \) does not have degree two in \( G \).

The reason to study maximal paths and pending cycles is that we can compute a maximum-weight matching on path and cycle graphs in linear time, as stated next. This allows us to preprocess all vertices in a maximal path or a pending cycle at once.

Observation 3.5. Maximum-Weight Matching can be solved in \( O(n) \) time on paths and cycles.
we introduce data reduction rules for maximal paths and pending cycles. The

---

**Reduction Rule 3.3**

Observation 3.5

Reduction Rules 3.1 and 3.2, we can compute a maximum-weight matching. Otherwise, if \( G \) is a cycle, then we take an arbitrary edge \( e \) and distinguish two cases. First, we take \( e \) into a matching and remove both endpoints from the graph. In the resulting path, we compute in linear time a maximum-weight matching \( M \). Second, we delete \( e \) and obtain a path for which we compute in linear time a maximum-weight matching \( M' \). We then simply choose between \( M \cup \{e\} \) and \( M' \) the heavier matching as the result.

Now, using Observation 3.5, we introduce data reduction rules for maximal paths and pending cycles. Both rules are based on a similar idea which is easier to explain for a pending cycle. Let \( C \) be a pending cycle and \( u \in C \) be the degree-at-least-three vertex in \( C \). Then there are two cases: \( u \) is matched with a vertex not in \( C \) or it is not. Let \( M \) be a maximum-weight matching for \( G \), and let \( M' \) be a maximum-weight matching with the constraint that \( u \) is matched to a vertex outside \( C \). Clearly, \( M \cap E(C) \) is at least as large as \( M' \cap E(C) \). Looking only at \( C \), all that we need to know is the difference of the weights of these two matchings. This can be encoded with one vertex \( z \) which replaces the whole cycle \( C \) (see Figure 5 for an illustration). Then, matching \( z \) corresponds to taking the matching in \( C \) and not matching \( z \) corresponds to taking the matching in \( C - u \). Formalizing this idea, we arrive at the following data reduction rule.

**Reduction Rule 3.3.** Let \( G \) be a graph with non-negative edge weights. Let \( C \) be a pending cycle in \( G \), where \( u \in C \) has degree at least three in \( G \). Then replace \( C \) by an edge \( uz \) with \( \omega(uz) = \omega(C) - \omega(C - u) \) and decrease the solution value \( s \) by \( \omega(C - u) \), where \( z \) is a new vertex.

**Lemma 3.6.** Reduction Rule 3.3 is correct.

**Proof.** Let \( C \) be a pending cycle in \( G \) where \( u \in C \) has degree at least three in \( G \) and let \( G' \) be the graph obtained by applying Reduction Rule 3.3 to \( C \). We show \( \omega(G') = \omega(G) - \omega(C - u) \).

Let \( M \) be a maximum-weight matching in \( G \). Let \( M_\overline{C} := M \setminus E(C) \). Observe that \( \omega(M_\overline{C}) = \omega(M) - \omega(M \cap E(C)) \geq \omega(G) - \omega(C) \). If \( u \) is matched with respect to \( M_\overline{C} \), then we have \( M_\overline{C} = M \setminus E(C - u) \). Hence, \( \omega(G') \geq \omega(M_\overline{C}) \geq \omega(G) - \omega(C - u) \). If \( u \) is free with respect to \( M_\overline{C} \), then \( M_\overline{C} \cup \{uz\} \) is a matching in \( G' \) with weight at least \( \omega(G) - \omega(C) + (\omega(C) - \omega(C - u)) = \omega(G) - \omega(C - u) \). Hence, in both cases we have \( \omega(G') \geq \omega(G) - \omega(C - u) \).

Conversely, let \( M' \) be a maximum-weight matching in \( G' \). Recall that, for an edge-weighted graph \( H \), \( mm(H) \) denotes a maximum-weight matching in \( H \). If \( uz \in M' \), then \( (M' \setminus \{uz\}) \cup mm(C) \) is a matching in \( G \) with \( \omega(G') - (\omega(C) - \omega(C - u)) + \omega(C) = \omega(G') + \omega(C - u) \). Hence, \( \omega(G) \geq \omega(G') + \omega(C - u) \). If \( uz \notin M' \), then \( M' \cup mm(C - u) \) is a matching in \( G \) with weight at least \( \omega(G') + \omega(C - u) \). Again, in both cases we have \( \omega(G) \geq \omega(G') + \omega(C - u) \). Combined with \( \omega(G') \geq \omega(G) - \omega(C - u) \), we arrive at \( \omega(G') = \omega(G) - \omega(C - u) \).

The basic idea for maximal paths is the same as for pending cycles. The difference is that we have to distinguish four cases depending on whether or not the two endpoints \( u \) and \( v \) of a maximal path \( P \) are matched within \( P \). To avoid some trivial case distinctions, we assume that \( \omega(ew) = 0 \) if the edge \( ew \) does not exist in \( G \). We denote by \( P - u - v \) the path obtained from removing in \( P \) the vertices \( u \) and \( v \).

Figure 6 visualizes the next data reduction rule.
Figure 6: Applying Reduction Rule 3.4 on a path $P$ with endpoints $u$ and $v$ (where $u$ and $v$ are not adjacent). The four choices for $u$ and $v$ on whether or not they are matched to a vertex within the path are reflected by the three (full) edges on the right where at most one can be taken into a matching. Since the edge $uv$ is not contained in the input graph the weight of the edge $uv$ in the reduced graph simplifies to the displayed value.

**Reduction Rule 3.4.** Let $G = (V, E)$ be a graph with non-negative edge weights $\omega: E \to \mathbb{N}$. Let $P$ be a maximal path in $G$ with endpoints $u$ and $v$. Then remove all vertices in $P$ except $u$ and $v$, then add a new vertex $z$ and, if not already existing, add the edge $uv$. Furthermore, set $\omega(uz) := \omega(P - v) - \omega(P - u - v)$, $\omega(vz) := \omega(P - u) - \omega(P - u - v)$, and $\omega(uv) := \max\{\omega(uv), \omega(P) - \omega(P - u - v)\}$, and decrease the solution value $s$ by $\omega(P - u - v)$.

**Lemma 3.7.** Reduction Rule 3.4 is correct.

**Proof.** Let $G$ be the input graph with a maximal path $P$ with endpoints $u$ and $v$. Furthermore, let $G'$ be the reduced instance with $z$ defined as in the data reduction rule. We show that $\omega(G') = \omega(G) - \omega(P - u - v)$.

Let $M$ be a maximum-weight matching for $G$. We define $M_{P} := M \setminus E(P)$. Observe that $\omega(M_{P}) = \omega(M) - \omega(M \cap E(P)) \geq \omega(G) - \omega(P)$. We consider four cases.

1. If both $u$ and $v$ are matched with respect to $M_{P}$, then $M_{P} = M \setminus E(P - u - v)$ and hence
   
   $\omega(M_{P}) = \omega(M) - \omega(M \cap E(P - u - v)) \geq \omega(G) - \omega(P - u - v). \quad (1)$

2. Let one vertex in $\{u, v\}$ be matched and let one be free. Without loss of generality, we assume that $u$ is matched and $v$ is free with respect to $M_{P}$. Then, we have that $M_{P} = M \setminus E(P - u)$ and hence $\omega(M_{P}) \geq \omega(G) - \omega(P - u)$. Thus, $M_{P} \cup \{vz\}$ is a matching of weight at least
   
   $(\omega(G) - \omega(P - u)) + (\omega(P - u) - \omega(P - u - v)) = \omega(G) - \omega(P - u - v).$

3. Finally, if both $u$ and $v$ are free with respect to $M_{P}$, then $M_{P} \cup \{uv\}$ is a matching of weight at least
   
   $(\omega(G) - \omega(P)) + (\omega(P) - \omega(P - u - v)) = \omega(G) - \omega(P - u - v).$

Thus in each case we have $\omega(G') \geq \omega(G) - \omega(P - u - v)$.

Conversely, let $M'$ be a maximum-weight matching for $G'$. We define $M_{P} := M' \setminus \{uz, vz, uv\}$. Again, we distinguish four cases.

1. If both $u$ and $v$ are matched with respect to $M_{P}$, then $M_{P} = M'$. Hence, $M_{P} \cup \text{mm}(P - u - v)$ is a matching in $G$ with weight at least $\omega(G') + \omega(P - u - v)$.

2. If $u$ is matched and $v$ is free with respect to $M_{P}$, then w.l.o.g. $vz \in M'$. Hence, $M_{P} \cup \text{mm}(P - u)$ is a matching in $G$ with weight at least $\omega(G') - (\omega(P - u) - \omega(P - u - v)) + \omega(P - u) = \omega(G') + \omega(P - u - v)$.
3. If $u$ is matched and $v$ is free with respect to $\overline{M}$, then w.l.o.g. $uz \in M'$. Hence, $\overline{M} \cup \text{mm}(P-v)$ is a matching in $G$ with weight at least $\omega(G') - (\omega(P-v) - \omega(P-u-v)) + \omega(P-v) = \omega(G') + \omega(P-u-v)$.

4. Finally, if both $u$ and $v$ are free with respect to $\overline{M}$, then w.l.o.g $uv \in M'$ as $\omega(uv) \geq \omega(uz)$ and $\omega(uv) \geq \omega(vz)$. Now, we encounter two subcases.

(a) If $\omega(uv) > \omega(P) - \omega(P-u-v)$, then the edge $uv$ is in $G$ and in $G'$, having the same weight in both graphs. Then, $M' \cup \text{mm}(P-u-v)$ is a matching in $G$ with weight at least $\omega(G') + \omega(P-u-v)$.

(b) Otherwise, $\overline{M} \cup \text{mm}(P)$ is a matching in $G$ with weight at least $\omega(G') - (\omega(P) - \omega(P-u-v)) + \omega(P) = \omega(G') + \omega(P-u-v)$.

Hence, in all cases we have $\omega(G) \geq \omega(G') + \omega(P-u-v)$. Combined with $\omega(G') \geq \omega(G) + \omega(P-u-v)$, we can infer that $\omega(G') = \omega(G) - \omega(P-u-v)$.

**Lemma 3.8.** Reduction Rules 3.3 and 3.4 can be exhaustively applied in $O(n+m)$ time.

**Proof.** First, we collect in $O(n+m)$ time all maximal paths and all pending cycles [6, Lemma 3]. Given a maximal path or a pending cycle on $\ell$ vertices due to Observation 3.5 one can compute the necessary maximum-weight matchings (at most four) in $O(\ell)$ time. Moreover, replacing the maximal path or the pending cycle by the respective structure is doable in $O(\ell)$ time. Applying Reduction Rules 3.3 and 3.4 does not create new maximal paths (recall that a maximal path needs at least two vertices of degree two) or pending cycles. Thus, as all maximal paths and pending cycles combined contain at most $n$ vertices, Reduction Rules 3.3 and 3.4 can be exhaustively applied in $O(n+m)$ time.

Each of Reduction Rules 3.1, 3.2, and 3.4 can be exhaustively applied in linear time; however, we do not know whether all these data reduction rules together can be exhaustively applied in linear time. Note that after applying Reduction Rule 3.3 Reduction Rule 3.2 might become applicable. For our problem kernel below it suffices to apply Reduction Rules 3.1, 3.2, and 3.4 in a specific order (using Lemmas 3.2 and 3.8). Note that we might output a problem kernel where Reduction Rules 3.1 and 3.2 are applicable. In our experimental part it turned out that it is beneficial to apply the rules exhaustively (in superlinear time) to reduce the input graph as much as possible.

**Theorem 3.9.** Maximum-Weight Matching admits a linear-time computable $13k$-vertex and $17k$-edge kernel with respect to the parameter feedback edge number $k$.

**Proof.** Let $G = (V,E)$ be the input instance and $F \subseteq E$ a feedback edge set of size at most $k$. Without loss of generality, one can assume that the input graph does not contain a cycle where each vertex has degree two, or a path where the endpoints have degree one and the internal vertices have degree two, because otherwise such a cycle or path can be solved independently in linear time (see Observation 3.5).

The kernelization algorithm works as follows: First, exhaustively apply Reduction Rule 3.1 in $O(n+m)$ time. Second, exhaustively apply Reduction Rule 3.2 in $O(n+m)$ time (see Lemma 3.2). Third, exhaustively apply Reduction Rules 3.3 and 3.4 in $O(n+m)$ time (see Lemma 3.8). Note that when applying the rules in this order, the resulting graph $\hat{G} = (\hat{V}, \hat{E})$ does not contain any maximal paths, or pending cycles. But for each pending cycle we introduced a new degree one vertex. However, for each pending cycle there is at least one (distinct) edge in $F$. Hence, $\hat{G}$ has at most $k$ degree one vertices. Let $V_1$ be the set of degree-one vertices in $\hat{G}$. Moreover, after the second step of our kernelization (Reduction Rule 3.2) the graph contains at most $3k$ maximal paths [6, Lemma 2]. Thus, a feedback edge set $\hat{F} \subseteq \hat{E}$ for $\hat{G}$ of minimum size contains at most $4k$ edges (each application of Reduction Rule 3.4 increases the feedback edge set by one).
In this section, we provide an experimental evaluation of the presented data reduction rules on real-world graphs ranging from a few thousand vertices and edges to a few million vertices and edges. We analyze the effectiveness and efficiency of the kernelization as well as the effect on the maximum-cardinality matching. Afterwards, in Section 4.4, we analyze the effect of applying Reduction Rules 3.1 to 3.4 in combination with a solver for MAXIMUM-WEIGHT MATCHING.

4 Experimental Evaluation

In this section, we provide an experimental evaluation of the presented data reduction rules on real-world graphs ranging from a few thousand vertices and edges to a few million vertices and edges. We analyze the effectiveness and efficiency of the kernelization as well as the effect on the subsequently used state-of-the-art solvers of Huang and Stein [21], Kececioglu and Pecqueur [27], and Kolmogorov [28].

In Section 4.1, we give details about our test scenario. Then we first focus in Section 4.2 on the evaluation of Reduction Rules 2.1 to 2.3 and 3.1 to 3.4 in terms of running time needed to apply them and size of resulting instances. In Section 4.3, we then analyze the effect of applying Reduction Rules 2.1 to 2.3 in combination with a solver for MAXIMUM-CARDINALITY MATCHING. Afterwards, in Section 4.4, we analyze the effect of applying Reduction Rules 3.1 to 3.4 in combination with a solver for MAXIMUM-WEIGHT MATCHING.

4.1 Setup and Implementation Details

Our program is written in C++14 and the source code is available from https://git.tu-berlin.de/akt-public/matching-data-reductions.git. One can replicate all experiments by following the manual provided with the source code. We ran all our experiments on an Intel(R) Xeon(R) CPU E5-1620 3.60 GHz machine with 64 GB main memory under the Debian GNU/Linux 7.0 operating system, where we compiled the program (including the solvers of [28, 27]) with GCC 7.3.0. For the solver of Huang and Stein [21] we used Python 2.7.15rc1.

Data set. All tested graphs are from the established SNAP [31] data set with a time limit of one hour per instance. See Table 2 for a sample list of graphs with their respective numbers of vertices and edges. The full list is given in Table 4 in the Appendix. The weighted graphs are generated from the unweighted graphs by adding edge-weights between 1 and 1000 chosen independently and uniformly at random.

| Graph           | n    | m    | Graph           | n    | m    |
|-----------------|------|------|-----------------|------|------|
| p2p-Gnutella08  | 6301 | 20,777 | p2p-Gnutella05  | 8,846 | 31,839 |
| soc-sign-Slashdot090216 | 81,868 | 3.5 · 10^5 | soc-Slashdot0811 | 77,360 | 4.7 · 10^5 |
| loc-gowalla-edges | 2 · 10^5 | 9.5 · 10^5 | ca-HepTh        | 9,877 | 25,973 |
| web-BerkStan     | 6.9 · 10^5 | 6.6 · 10^6 | roadNet-TX      | 1.4 · 10^6 | 1.9 · 10^6 |
| ca-HepPh         | 12,008 | 1.2 · 10^5 | twitter-combined| 81,306 | 1.3 · 10^6 |
| amazon0601       | 4 · 10^5 | 2.4 · 10^6 | amazon0302      | 2.6 · 10^5 | 9 · 10^5 |

Table 2: A selection of our test graphs from SNAP [31] with their respective size.
Table 3: Set of solvers we used in our experiments. Here, “MM⇝W-PM” is the folklore reduction from Maximum-Cardinality Matching to Minimum Weighted Perfect Matching and “W-M⇝W-PM” is the folklore reduction from Maximum-Weight Matching to Minimum Weighted Perfect Matching.

| acronym     | implementation by                               | core algorithm     | language |
|-------------|-------------------------------------------------|--------------------|----------|
| KP-Edm      | Kececioglu and Pecqueur [27]                    | Edmonds [14]       | C        |
| Kol-Edm     | Kolmogorov [28]                                | Edmonds [14, 13], MM⇝W-PM | C++      |
| Kol-Edm-W   | Kolmogorov [28]                                | Edmonds [14, 13], W-M⇝W-PM | C++      |
| HS-MV       | Huang and Stein [21]                           | Micali and Vazirani [33] | Python 2.7 |

Implementation details of our kernelization algorithms. We implemented kernelization algorithms for the unweighted and weighted case. The first kernelization is for **Maximum-Cardinality Matching**, which exhaustively applies Reduction Rules 2.1 and 2.2. Our implementation here is rather simplistic in the sense that it maintains a list of degree-one and degree-two vertices which are processed one after the other (in a straightforward manner). We apply the rules exhaustively although this gives in theory a super-linear running time. Note that one can (theoretically) improve our implementation of Reduction Rule 2.2 by a linear-time algorithm of Bartha and Krész [4]. Very recently, Kaya et al. [26] provided a fine-tuned algorithm that exhaustively applies Reduction Rules 2.1 and 2.2 on bipartite graphs roughly three times faster than our naive implementation; their general approach should be also applicable for general graphs. However, our naive (super-linear) implementation for exhaustively applying Reduction Rules 2.1 and 2.2 was at least two times faster than reading and parsing the input graph and at least three times faster than the fastest implementation for finding maximum-cardinality matchings. Thus, applying Reduction Rules 2.1 and 2.2 was not a bottleneck in our implementation and we did not optimize it further.

The second kernelization is also **Maximum-Cardinality Matching** and it exhaustively applies **Reduction Rule 2.3**. To this end, we used the algorithm described by Iwata et al. [23]. The main steps of this algorithm are:

1. compute a maximum-cardinality matching in a given bipartite graph $\overline{G}$ (to compute the initial LP-solution; see Section 2.2) and
2. determine the topological ordering of the DAG formed by the strongly connected components of a given digraph $\overline{D}$.

(The underlying undirected graph of $\overline{D}$ is $\overline{G}$; the matching computed in Step 1 determines how the edges in $\overline{G}$ are directed in $\overline{D}$. Each crown in the input graph $G$ corresponds to a strongly connected component in $\overline{D}$; refer to Iwata et al. [23] for details.)

Both steps can be solved using classic algorithms. We implemented for Step 1 the classic $O(\sqrt{nm})$-time algorithm of Hopcroft and Karp [20] for finding a bipartite matching in $\overline{G}$. For Step 2, we implemented Kosaraju’s algorithm [2] for finding the strongly connected components of $\overline{D}$ in reverse topological order.

The third kernelization is for **Maximum-Weight Matching**. We use the algorithms described in Lemmas 3.2 and 3.8 to apply Reduction Rules 3.2 to 3.4. Deviating from the algorithm described in Theorem 3.9, based on empirical observations our program applies Reduction Rules 3.1 to 3.4 as long as possible. Hence, the kernelization does not run in linear time but further shrinks the input graph.

**Used solvers.** To test the effect of our data reduction rules, we compare the running time of a solver on an input instance against the running time of our kernelization procedure plus the running time of the same solver on the output of our kernelization procedure. We refer to Table 3 for an overview of the tested solvers. For the data reductions rules for Maximum-Weight Matching.
we used the solver of Kolmogorov [28] (implemented in C++) which is a fine-tuned version of Edmonds’ algorithm for Minimum-Weight Perfect Matching [14, 13]. Note that the solver of Kolmogorov [28] finds perfect matchings of minimum weight. We thus use this solver on graphs obtained from applying the folklore reduction from Maximum-Weight Matching (and thus also from Maximum-Cardinality Matching) to Minimum-Weight Perfect Matching. Applied on a graph $G$ with $n$ vertices and $m$ edges, the reduction adds a copy of $G$ and adds a weight-zero edge between each vertex in $G$ and its added copy. Thus, the resulting graph can be computed in linear time and has $2n$ vertices and $2m + n$ edges. To the best of our knowledge, the solver of Kolmogorov [28] plus the folklore reduction yields the currently fastest algorithm for Maximum-Weight Matching. For the rest of this paper, Kol-Edm-W denotes the solver of Kolmogorov [28] plus the folklore reduction for Maximum-Weight Matching.

To test the data reduction rules for Maximum-Cardinality Matching, we used three different solvers. First, we used the solver (denoted by Kol-Edm) of Kolmogorov [28] plus the folklore reduction for Maximum-Cardinality Matching, which we get basically for free from the weighted case. Second, we used the solver (denoted by KP-Edm) of Kececioglu and Pecqueur [27] (implemented in C) which is a fine-tuned version of Edmonds’ algorithm [14]. To the best of our knowledge this is in practice still the fastest algorithm. In our experiments, KP-Edm was clearly the fastest solver. Third, we used the solver (denoted by HS-MV) of Huang and Stein [21] (implemented in Python). This is the only implementation of the Micali-Vazirani algorithm [33] we are aware of. The Micali-Vazirani algorithm has currently the best asymptotic worst-case running time. However, in our tests HS-MV (Python) was clearly outperformed by KP-Edm (C).

### 4.2 Efficiency and Effectiveness of our Data Reduction Rules

**Effectiveness of our rules.** The effectiveness of our kernelization algorithms is displayed in Figure 7: Few graphs remained almost unchanged while other graphs were essentially solved by the kernelization algorithm.

For the unweighted case the situation is as follows: On the 44 tested graphs, on average 72% of the vertices and edges are removed by the kernelization; the median is 82%. The least amenable graph was amazon0302 with a size reduction of only 7%. In contrast, on 16 out of the 44 graphs the kernelization algorithm reduces more than 99% of the vertices and edges. This is mostly due to Reduction Rules 2.1 and 2.2: if Reduction Rules 2.1 and 2.2 are exhaustively applied, then an application of the crown rule (Reduction Rule 2.3) further reduces the kernel size only on four instance substantially. Moreover, a closer look on how often the degree-based rules Reduction Rules 2.1 and 2.2 are applied reveals that on the majority of our tested graphs Reduction Rule 2.1 is applied twice as much as Reduction Rule 2.2.

While the data reduction rules are less effective in the weighted case (see Figure 7), they reduce the graphs on average still by 51% with the median value being a bit lower with 48%. The least amenable graph is again amazon0302 with a size reduction of only 3%.

**Efficiency of our rules.** In Figure 8, we compare the running times of the KP-Edm solver (the fastest state-of-the-art solver on our instances for the unweighted case) when applied directly on the input graph together with running times of our data reduction rules for the unweighted case (RR 2.1 to 2.3).

On some graphs it does take more time to apply Reduction Rules 2.1 to 2.3 than executing the KP-Edm solver directly. But if only Reduction Rules 2.1 and 2.2 are applied, then the running time of the kernelization stays far below the running time of the KP-Edm solver while the resulting kernel size is mostly the same (see Figure 7). Applying Reduction Rule 2.3 without Reduction Rules 2.1 and 2.2 is clearly not a good idea as of KP-Edm is faster in finding a maximum-cardinality matching.

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4In a few cases HS-MV returned an edge set that is not a matching. However, a maximum-cardinality matching was easily recoverable from the returned edge set by removing one or two edges. The authors (Huang and Stein) and are working on a fix for this issue.
The figure is similar to the unweighted case. On most graphs the data reduction rules are applied much faster than the solver (Kol-Edm-W), but there are a few exceptions.

In Figure 9, we compare the running times of the Kol-Edm-W solver (the state-of-the-art solver) when applied directly to the input graph against the running times of our data reduction rules. We tested all unweighted rules (RR 2.1 to 2.3), only the crown rule (Reduction Rule 2.3), and only the unweighted rules for low degree vertices (RR 2.1 and 2.2). The graphs are ordered in both plots by relative size of the kernel after applying Reduction Rules 2.1 to 2.3. The crosses show the number of applications of Reduction Rule 2.1 (100% = number of applications of Reduction Rule 2.1 to 2.3). The picture is similar to the unweighted case when applied directly to the input graph together with running times of the Kol-Edm-W solver (the state-of-the-art solver).
Advice on which reduction rules to apply. If one has to find maximum-cardinality matchings on large real-world graphs, then we advise to always apply Reduction Rules 2.1 and 2.2 before feeding the graph to a solver. Whether or not applying Reduction Rule 2.3, however, depends on the specific type of real-world data at hand and should be tested on a few test cases: In most of our test cases the benefit paid with the higher running times was rather small. For the weighted case, we advise to apply our data reduction rules, but maybe invest time in a more efficient implementation of the rules (our implementation of the weighted rules could probably profit from further optimizations).

Kernel size: theory versus practice. In theory, we are used to measure the effectiveness of data reduction rules in terms of provable upper bounds for the size of resulting graph (the kernel) in a function only depending on some parameter. If Reduction Rules 2.1 and 2.2 are not applicable, then Theorem 2.1 states that a resulting graph has at most $2k$ vertices and at most $3k$ edges, where $k$ is the feedback edge number. In Figure 10, we measure the gap between the actual size of the kernel and the proven upper bound from Theorem 2.1. As a matter of fact, we can clearly observe that the upper bound shown in Theorem 2.1 is not suitable to explain why Reduction Rules 2.1 and 2.2 perform so well in real-world graphs: in 41 of our tested 44 graphs the input size itself is already smaller than the guaranteed upper bound of Theorem 2.1. We exclude Theorem 3.9 from the discussion here, as the upper bounds given are even weaker than the ones in Theorem 2.1.

Reduction Rule 2.3 and Theorem 2.7 are better suitable to explain the results: As can be seen in Figure 11, this $2\tau$-bound on the number of vertices is not optimal. However, there is a clear similarity between the lines indicating theoretical upper bound and measured kernel size. Moreover, on roughly 2/3 of the instances the worst-case upper bound $2\tau$ is smaller than $n$, that is, the deletion of some vertices is guaranteed.

Overall, Theorem 2.7 seems to deliver the better theoretical explanation. However, Theorem 2.7 is based on Reduction Rule 2.3 and, as can be seen in Figure 7, just applying Reduction Rules 2.1 and 2.2 is almost always better than only applying Reduction Rule 2.3.
Figure 9: Running time of applying Reduction Rules 3.1 to 3.4 and the Kol-Edm-W solver on weighted graphs (without kernelization). To all values 1 millisecond was added to display 0-values. The graphs are ordered by running time for Reduction Rules 3.1 to 3.4 (the ordering is slightly different from the ordering in Figure 8.)

(the exceptions are the graphs “web-NotreDame”, “web-Stanford”, “web-Google”, and “web-BerkStan”). Thus, while Theorem 2.7 somewhat explains the effects of Reduction Rule 2.3, no explanation is provided for the good performance of Reduction Rules 2.1 and 2.2.

Summarizing, the current theoretical upper bounds for the kernel size need improvement. The most promising route seems to be a multivariate analysis in the sense that one should use more than one parameter in the analysis [36]. Of course, the challenging part here is finding the “correct” parameters.

4.3 Running times for Maximum-Cardinality Matching

In this section we evaluate the effect on the running time of state-of-the-art solvers (see Table 3) for Maximum-Cardinality Matching if Reduction Rules 2.1 to 2.3 are applied in advance.

Note that all reported running times involving Kol-Edm are averages over 100 runs where we randomly permute vertex indices in the input. Although this permutation yields an isomorphic graph, we empirically observed that in the unweighted case the running time of Kol-Edm heavily depends on the permutation. For example, choosing a “good” or a “bad” permutation for the same graph may yield speedup of factor 20 or more. Precise data on the spectrum of running time variation (for graphs where the time limit was not reached) are shown in Figure 12. The running time of all other solvers and our kernelization algorithm were only marginally affected by changing the permutation.

We noticed that the different implementations vary greatly in the time they need for parsing the input graph (especially in the smaller graphs HS-MV (Python) needs more time to parse the graph than KP-Edm (C) needs to find a maximum-cardinality matching). Moreover, we mainly care about the speedup of the respective algorithms (when run on the kernel instead of the original input) and not about the speedup of the graph parsing. Hence, we neglect the time to parse the input graph in all running-time measures and discussions. Note that we do not neglect the time the implementation needs for parsing the kernel, as this is something that needs only be done with
Figure 10: Sizes and bounds (in terms of number of vertices plus edges) of various structures and kernels. To all values 1 was added to display 0-values. The upper-bound from Theorem 2.7 is not displayed as it only bounds the number of vertices (and has no non-trivial bound on the number of edges). The graphs are ordered by relative size of the remaining graph after the data reduction rules (as in Figure 7).

data reduction but not without. More precisely, for runs without data reduction, we report the time the particular implementation needs after the graph was loaded; for runs with data reduction we report the time of our data reduction rules plus the total time of the implementation (including parsing the kernel).

In Figure 13, we compare the running times of the solvers when they are directly applied on the input instances against the running time of the kernelization plus the running time of the solvers on the resulting kernel. The running time of the Kol-Edm solver is improved on average by a factor of 157.30 (median: 29.27) when the kernelization is applied (left-side of Figure 13). The running time of the HS-MV solver is improved on average by a factor of 608.79 (median: 28.87) when the kernelization is applied (right-side of Figure 13). Hence, it is safe to say that these two algorithms clearly benefit from the kernelization. However, on the instance amazon0302 the HS-MV solver is 25% faster without the kernelization and on the instance facebook-combined the Kol-Edm solver is (on average over the hundred runs) 6% faster without the kernelization. On all other instances the running times of both solvers where improved by the kernelization.

In contrast, the message drawn by the results for the KP-Edm solver is less clear (right-hand side of Figure 13). The running time of the KP-Edm solver is impaired on 9 of our 44 instances by the kernelization. On average we still get an improvement of the running time by a factor of 4.70 (median: 2.20). The reason for the unclear result for the KP-Edm solver is that most of the instances are too easy for it, that is, they are solved very quickly with or without kernelization. On harder instances (like our largest four graphs) we could observe a more significant speedup gained due to the kernelization.

### 4.4 Running times for Maximum-Weight Matching

In this section, we evaluate Reduction Rules 3.1 to 3.4 for Maximum-Weight Matching. The weighted graphs we used for our tests were generated from the unweighted graphs by adding edge-
weights between 1 and 1000 chosen independently and uniformly at random. In the weighted case we tested our kernelization only together with the Kol-Edm-W solver since the KP-Edm and HS-MV solvers only work in the unweighted setting. In contrast to the unweighted case (see Figure 12), we could not observe that the running time of Kol-Edm-W is affected when the vertices are permuted. For consistency, however, we take the average running times also in the weighted case. Note that for different permutations the data reduction rules were applied in different order resulting in kernels slightly differing in size (see Figure 14 for an example).

In Figure 15, we illustrate the running time comparison of our kernelization for the weighted data reduction rules against various unweighted data reduction rules and the running time of Kol-Edm-W when applied without kernelization (the four largest graphs are missing since we could not solve them without kernelization). Our weighted kernelization algorithm becomes slower than in the unweighted case (here, we mean just Reduction Rules 2.1 and 2.2). This is not surprising as our algorithm for Reduction Rules 3.1 to 3.4 is more involved than the one for the Reduction Rules 2.1 and 2.2. Furthermore, the solver of Kolmogorov [28] is significantly faster in the weighted case (Kol-Edm-W) than in the unweighted case (Kol-Edm). On three graphs the Kol-Edm-W computes a maximum-weighted matching faster than we can produce the kernel. However, on most graphs, our kernelization algorithm reduces the overall running time of Kol-Edm-W (on average by a factor of 12.72; median: 1.40). Note that also in the weighted case the kernelization is more frequently beneficial than it is not.

5 Conclusion

Our work shows that it practically pays off to use (linear-time) data reduction rules for computing maximum (unweighted and weighted) matchings. Our current state of the theoretical (kernel size upper bounds) analysis, however, is insufficient to fully explain this success. Here, a multivariate approach in which more than one parameter is taken into consideration seems like the natural
Figure 12: The spectrum of the Kol-Edm running times over 100 runs with random vertex permutations on unweighted graphs. The lower (upper) whisker is the minimum (maximum) running time. The solid box shows the median and the lower and upper quartile. For each graph we have two datasets: left (red) Kol-Edm on the input graph and right (blue) on the kernel. We excluded input graphs where we could not perform 100 runs in one hour. To all values 1 millisecond was added to display 0-values.

Figure 13: Running time of the three state-of-the-art solvers with and without kernelization (each mark indicates one instance). The inclined solid/dashed/dash dotted/dotted lines indicate a factor of 1/2/5/25 difference in the running time. To all values 1 millisecond was added to display 0-values. Timeouts are counted as 1 h (solid vertical and horizontal line). For Kol-Edm, the timeout behavior is special due to taking the average over 100 runs (see Figure 12). If at least one of the 100 runs took more than 1 h, then we aborted the computation and count a timeout (although the average running time might be below 1 h). Hence, we separated the plot for Kol-Edm (left diagram): The red, filled diamonds indicate instances without any timeout (with or without kernelization). The blue, non-filled diamonds indicate instances where without kernelization there was at least one timeout in the 100 runs and with kernelization there was no timeout. Thus, the true values for the blue, non-filled diamonds might be below 1 h (hence the separation). (There was no instance where a timeout occurred with kernelization but not without kernelization.)
Figure 14: In the middle we see two weighted graphs where both Reduction Rule 3.2 and Reduction Rule 3.4 can be applied. On the left-hand side is the resulting graph when Reduction Rule 3.2 is applied first and the right-hand side displays the resulting graph when Reduction Rule 3.4 is applied first.

Figure 15: Running time comparison with and without using our kernelization (RR 3.1 to 3.4) algorithms before Kol-Edm-W. The solid/dashed/dash dotted/dotted lines indicate a factor of 1/2/5/25 difference in the running time. To all values 1 millisecond was added to display 0-values.

next step [36]. Finding the right parameters is the challenging part here. In fact, adding to any graph $G$ to each vertex a new degree-one vertex as neighbor results in a graph $G'$ where Reduction Rule 2.1 reduces everything, whereas the original graph $G$ might not be amenable at all to the data reduction rules. Many graph parameters (including feedback edge number) cannot differentiate between $G$ and $G'$ and, hence, are not suited to explain the practical effectiveness.

Future research for unweighted matchings. Through the connection between Vertex Cover and Maximum-Cardinality Matching one might be able to transfer further kernelization results from Vertex Cover to Maximum-Cardinality Matching. However, there are known limitations: obtaining a kernel for Vertex Cover with $O(\tau^{2-\varepsilon})$ edges ($\tau$ is the vertex cover number) for any $\varepsilon > 0$ is unlikely in the sense that the polynomial hierarchy would collapse [10]. Thus, obtaining a kernel with $O(\tau^{2-\varepsilon})$ edges for Maximum-Cardinality Matching requires a new approach that should not work for Vertex Cover. So far, the kernelization algorithms we discuss in this paper do work for both Maximum-Cardinality Matching and Vertex Cover. However, as Maximum-Cardinality Matching is polynomial-time solvable, an $O(\tau^{2-\varepsilon})$-sized kernel trivially exists for Maximum-Cardinality Matching. The challenge here is to find such a kernel that is (near-)linear-time computable.

In future research, one might also study the combination of data reduction with linear-time approximation algorithms for matching [11]. Furthermore, it would be interesting to know whether
there is an efficient way of applying Reduction Rule 2.5 or a variation of it.

The solver of Kolmogorov [28] is significantly faster on weighted graphs (Kol-Edm-W) than on unweighted graphs (Kol-Edm). We believe that the reason for this is that in unweighted graphs there is a lot of symmetries, and unlucky tie-breaking seems to have a strong impact on the solver of Kolmogorov [28]. In the weighted case, the performance of the solver of Kolmogorov [28] was much more consistent under permuting the vertices in the input graph. As a consequence, we believe that the following might speedup the algorithm: given an unweighted graph, introduce edge-weights such that a maximum-weight matching in the then weighted graph is also a maximum-cardinality matching in the unweighted graph. Using the famous Isolation Lemma [34] one might even enrich and support this with a theoretical analysis.

Future research for weighted matchings. While our naive implementation for the unweighted case proved to be quite fast, the algorithm for the weighted case could benefit from further tuning. Note that in the unweighted case Reduction Rules 2.1 and 2.2 only make changes in the local neighborhood of the affected vertices. This is not the case in the weighted case, where the application of Reduction Rules 3.2 to 3.4 involve iterations over all edges, see Lemmas 3.2 and 3.8. Hence, applying the data reduction rules exhaustively requires a larger overhead. Although some improvements in the implementation might be possible, an improved algorithmic approach to exhaustively apply the data reduction rules is needed. Is there a (quasi-)linear-time algorithm to exhaustively apply Reduction Rules 3.1 to 3.4? Furthermore, is there a variant of the crown data reduction (Reduction Rule 2.3) for MAXIMUM-WEIGHT MATCHING?

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## Appendix - Full Data Set

Table 4: A full list of graph from the SNAP [31] data set which we used in our test scenario; here $|V^{v=1}|$ is the number of degree-$i$ vertices, $\Delta$ the maximum degree, and “MCM” the cardinality of a maximum matching.

| Graph                  | $n$   | $m$   | $|V^{v=1}|$ | $|V^{v=2}|$ | $\Delta$ | degeneracy | MCM    |
|------------------------|-------|-------|-------------|-------------|---------|------------|--------|
| as20000102             | 6.474 | 12.572| 2.384       | 2.430       | 1.458   | 12         | 1.048  |
| p2p-Gnutella08         | 8.301 | 20.777| 1.746       | 876         | 97      | 10         | 2.054  |
| p2p-Gnutella09         | 8.114 | 26.013| 2.490       | 1.129       | 102     | 10         | 2.574  |
| p2p-Gnutella05         | 8.424 | 31.839| 1.989       | 1.073       | 88      | 9          | 3.428  |
| p2p-Gnutella06         | 8.717 | 31.525| 1.966       | 1.171       | 115     | 9          | 3.405  |
| p2p-Gnutella25         | 22.687| 54.705| 9.296       | 3.526       | 66      | 5          | 6.017  |
| p2p-Gnutella04         | 10.876| 39.994| 2.467       | 1.439       | 103     | 7          | 4.348  |
| twitter-combined       | 81    |       |             |             |         |            |        |
| soc-sign-epinions      | 1.3   | 10^{5} | 5.10^{5}  | 6.652         | 22.920  | 83.752     | 28.065 |
| soc-Epinions           | 75.879| 4.1    | 10^{5}     | 35.755       | 11.347  | 3.044      | 21.960 |
| as-skitter             | 1.7   | 10^{6} | 1.110^{7}  | 2.210^{5}   | 2.610^{5} | 35.455     | 111    |
| loc-gowalla-edges      | 2     | 10^{5} | 9.510^{5}  | 49.452       | 30.459  | 14.730     | 51     |
| soc-LiveJournal1      | 4.8   | 10^{5} | 4.310^{5}  | 1.110^{6}   | 5.510^{5} | 20.333     | 372    |
| ca-HepTh               | 9877  |       | 25.973     | 2.109        | 2.014   | 65         | 31     |
| com-amazon             | 3.3   | 10^{5} | 9.310^{5}  | 25.709       | 37.326  | 549        | 6      |
| email-Enron            | 36.92 | 1.8    | 10^{5}     | 11.211       | 3.800   | 1.383      | 43     |
| com-lj                 | 4     | 10^{5} | 3.510^{5}  | 7.910^{5}   | 4.310^{5} | 14.815     | 360    |
| wiki-topcats           | 1.8   | 10^{6} | 2.510^{7}  | 6.248        | 42.917  | 2.410^{5}  | 99     |
| web-Googlenews         | 8.8   | 10^{5} | 4.310^{6}  | 1.510^{5}   | 1.110^{5} | 6.332      | 44     |
| web-BerkStan           | 6.9   | 10^{5} | 6.610^{6}  | 40.122       | 63.726  | 84.230     | 201    |
| com-dblp               | 3.2   | 10^{5} | 1.10^{6}   | 43.181       | 58.858  | 343        | 113    |
| roadNet-TX             | 1.4   | 10^{6} | 1.910^{6}  | 2.510^{5}   | 1.210^{5} | 12         | 3      |
| web-Stanford           | 2.8   | 10^{5} | 2.10^{6}   | 13.991       | 30.965  | 38.625     | 71     |
| roadNet-PA             | 1.1   | 10^{6} | 1.510^{6}  | 1.910^{5}   | 90.740  | 9          | 3      |
| roadNet-CA             | 2     | 10^{6} | 2.810^{6}  | 3.210^{5}   | 210^{5}  | 12         | 3      |
| ca-GrQc                | 542   | 14.484 | 1.197      | 1.115        | 81      | 43         | 2.329  |
| web-NotreDame          | 3.3   | 10^{5} | 1.110^{6}  | 1.610^{5}   | 35.129  | 10.721     | 155    |
| ca-AstroPh             | 12.008| 1.210^{5} | 1.493 | 1.801       | 491     | 238        | 5.649  |
| ca-CondMat             | 23.133| 93.439| 2.373       | 3.209       | 279     | 25         | 10.970 |
| twitter-combined       | 81.306| 1.310^{6} | 4.996 | 4.857       | 3.383   | 96         | 37.003 |
| soc-pokec-relationships| 1.6   | 10^{6} | 2.210^{7}  | 1.610^{5}   | 1.110^{7} | 14.854     | 47     |
| email-Eu-core          | 1.005 | 16.004 | 95         | 36          | 345     | 34         | 479    |
| ca-AstroPh             | 18.772| 2.10^{6} | 1.282 | 1.720       | 504     | 56         | 9.150  |
| amazon0505             | 4.1   | 10^{6} | 2.410^{6}  | 19.410       | 16.758  | 2.760      | 10     |
| amazon0312             | 4     | 10^{6} | 2.310^{6}  | 18.254       | 15.450  | 2.747      | 10     |
| twitter-combined       | 80.399| 88.234| 75         | 98          | 1.045   | 115        | 1.979  |
| amazon0302             | 2.6   | 10^{5} | 910^{5}    | 6.069        | 6.148   | 420        | 6      |
