GROWTH DIAGRAMS, DOMINO INSERTION AND SIGN-IMBALANCE

THOMAS LAM

Abstract. We study some properties of domino insertion, focusing on aspects related to Fomin’s growth diagrams [Fom1, Fom2]. We give a self-contained proof of the semistandard domino-Schensted correspondence given by Shimozono and White [SW], bypassing the connections with mixed insertion entirely. The correspondence is extended to the case of a nonempty 2-core and we give two dual domino-Schensted correspondences. We use our results to settle Stanley’s ‘2ⁿ/2’ conjecture on sign-imbalance [Sta] and to generalise the domino generating series of Kirillov, Lascoux, Leclerc and Thibon [KLLT].

1. Introduction

Recently in [SW] Shimozono and White described a semistandard generalisation of domino insertion giving a bijection between colored biwords and pairs of semistandard domino tableaux of the same shape. They connected domino insertion with Haiman’s mixed and left-right insertion algorithms [Hai] and via this obtained the semistandard analogue. They also made explicit a color-to-spin property of domino insertion. This property appears to have been used earlier by Kirillov, Lascoux, Leclerc and Thibon [KLLT] for some special colored involutions.

Earlier, van Leeuwen [vL] had described domino insertion in terms of Fomin’s growth diagrams. He connected Barbasch and Vogan’s left-right insertion description [BV] with Garfinkle’s traditional bumping description [Gar]. He also defines insertion in the presence of a 2-core.

Our first aim in this paper is to give a self contained proof of the semistandard domino-Schensted correspondence, using elementary growth diagram calculations to prove all the main properties of the bijection which we also extend to the nonempty 2-core case. Thus our approach allows us to avoid mention of mixed insertion completely. We also describe two dual domino-Schensted bijections. These are bijections between multiplicity free colored biwords and pairs of semistandard domino tableaux which have conjugate shapes. Finally, we perform a detailed analysis of symmetric growth diagrams for domino insertion.

The study of growth diagrams leads us to a number of applications. These include a number of enumerative results for domino tableaux, an application to sign-imbalance, and a collection of product expansions for generating series of domino functions.

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The sign $\text{sign}(T)$ of a standard Young tableaux $T$ is the sign of its reading word. The sign imbalance of a shape $\lambda$ is defined as

$$\sum_{\text{SYT}: \text{sh}(T)=\lambda} \text{sign}(T).$$

That sign-imbalance is related to domino tableaux has been made explicit in work of White [Whi] and Stanley [Sta]. In particular, White gives a formula for the sign of the Young tableaux $T(D)$ associated to a domino tableaux $D$:

$$\text{sign}(T) = (-1)^{\text{ev}(D)}$$

where $\text{ev}(D)$ is the number of vertical dominoes in even columns of $D$. Domino tableaux are in bijection with hyperoctahedral involutions and we prove that in fact $\text{ev}(D)$ is equal to the number of barred two-cycles of $\pi$, where $D = P_\pi(\pi)$ is the insertion tableaux of $\pi$. This allows us to prove Stanley’s conjecture on sign-imbalance, our Theorem 24, which is a 4-parameter generalisation of the following elegant result:

$$\sum_{\text{SYT}: \text{sh}(T)=m} \text{sign}(T) = 2^{\lfloor m/2 \rfloor}.$$ 

Carré and Leclerc [CL] and Kirillov, Lascoux, Leclerc and Thibon [KLLT] have studied certain generating functions $H_\lambda(X; q)$ for domino tableaux which we loosely call domino functions. More general domino functions $G_\lambda(X; q)$ were developed also in [LLT], where they were connected with the Fock space representation of $U_q(\hat{\mathfrak{sl}}_2)$. These are defined as

$$G_\lambda(X; q) = \sum_D q^{\text{spin}(D)} x^D$$

where the sum is over all semistandard domino tableaux of shape $D$. The $H_\lambda$ are defined by $H_\lambda(X; q) = G_{2\lambda}(X; q)$. Product expansions of the sums $\sum H_\lambda(X; q)$ and $\sum H_{\lambda\lambda}(X; q)$ were given in [KLLT].

By studying colored involutions we give a product expansion for a 3-parameter generalisation of the sum $\sum_\lambda G_\lambda(X; q)$. When the parameters of this sum is specialised, we obtain both of the product expansions of [KLLT].

We now briefly describe the organisation of this paper. In Section 2 we give some notation and definitions for domino tableaux and colored words. We also give a description of domino insertion bumping in an informal manner, following mostly [SW]. In Section 3 we introduce and study growth diagrams. This is followed by a proof of the semistandard domino-Schensted correspondence and a description of the dual domino-Schensted correspondences. The section ends with a study of symmetric growth diagrams and some enumerative results. In Section 4 we apply the results of Section 3 to sign-imbalance. In Section 5 we combine the results of Section 3 with a study of the standardisation of colored involutions. These lead to a number of product expansions for generating series of domino functions. In Section 6 we give some final remarks concerning possible generalisations to longer ribbons.

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conjecture for study. My work on generating series of domino functions was inspired by the sum \( \sum (-1)^{\text{ev}(\lambda)} G_\lambda(X; -1) \) suggested to me by him.

2. Preliminaries

2.1. Domino Tableaux. We will let \([n] = \{1, 2, \ldots, n\}\) throughout.

Let \(\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l(\lambda) > 0)\) be a partition of \(n\). We will often not distinguish between a partition \(\lambda\) and its diagram (often called \(D(\lambda)\)) but the meaning will always be clear from the context. The partition \(\lambda \cup \mu\) is obtained by taking the union of the parts of \(\lambda\) and \(\mu\) (and reordering to form a partition). We denote by \(\tilde{\lambda}\) and \((\lambda^{(0)}, \lambda^{(1)})\) the 2-core and 2-quotient of \(\lambda\) respectively (see [Mac]). Every 2-core has the shape of a staircase \(\delta_r = (r, r-1, \ldots, 0)\) for some integer \(r \geq 0\). As usual, when \(\lambda\) and \(\mu\) are partitions satisfying \(\mu \subset \lambda\) we will use \(\lambda/\mu\) to denote the shape corresponding to the set-difference of the diagrams of \(\lambda\) and \(\mu\).

We denote the set of partitions by \(P\) and the set of partitions with 2-core \(\delta_r\) by \(P_r\). The set of all partitions \(\lambda\) satisfying the two conditions: \(\tilde{\lambda} = \delta_r\), \(|\lambda| = \delta_r + 2n\) will be denoted \(P_r(n)\). Note that \(P = \bigcup_{r,n} P_r(n)\).

A (standard) domino tableaux (SDT) \(D\) of shape \(\lambda\) consists of a tiling of the shape \(\lambda/\tilde{\lambda}\) by dominoes and a filling of each domino with an integer in \([n]\) so that the numbers are increasing when read along either the rows or columns. Here, \(n = \frac{1}{2}|\lambda/\tilde{\lambda}|\). A domino is any \(2 \times 1\) or \(1 \times 2\) shape, or equivalently, two adjacent squares sharing a common edge. The value of a domino is the number written inside it. We will write \(\text{dom}_i\) to indicate the domino with the value \(i\) inside. We will also write \(\text{sh}(D) = \lambda\).

An alternative way of describing a standard domino tableaux of shape \(\lambda\) is by a sequence of partitions \(\{\tilde{\lambda} = \lambda^0 \subset \lambda^1 \subset \ldots \subset \lambda^n = \lambda\}\), where \(\text{sh}(\text{dom}_i) = \lambda_i/\lambda_{i-1}\).

A semistandard domino tableaux (SSDT) \(D\) of shape \(\lambda\) consists of a tiling of the shape \(\lambda/\tilde{\lambda}\) by dominoes and a filling of each domino with an integer, so that the numbers are non-decreasing when read along the rows and increasing when read along the columns. The weight of such a tableaux \(D\) is the composition \(wt(D) = (\mu_1, \mu_2, \ldots)\) where there are \(\mu_i\) occurrences of \(i\)'s in \(D\). Let \(v(D)\) be the number of vertical dominoes in a domino tableaux \(D\). The spin \(sp(D)\), is defined as \(v(D)/2\). The standardisation of a semistandard domino tableaux \(D\) of weight \(\mu\) is a standard domino tableaux \(D^{st}\) obtained from \(D\) by replacing the dominoes containing 1’s with 1, 2, \ldots, \(\mu_1\) from left to right, the dominoes containing 2’s by \(\mu_1 + 1, \mu_1 + 2, \ldots, \mu_1 + \mu_2\), and so on.

More general skew (semi)standard domino tableaux are defined in a similar manner.

We should remark that Littlewood’s 2-quotient map [Lit] gives a bijection between standard domino tableaux of shape \(\lambda\) and pairs of standard Young tableaux of shapes \(\lambda^{(0)}\) and \(\lambda^{(1)}\). This bijection generalises naturally to the semistandard case. See for example [CL]. We will, however, not be needing this bijection.
2.2. Colored Words. We will mostly follow the notation of [SW] in this subsection.

A letter will be an integer with possibly a bar over it.

A colored word is a word made of letters. A colored word \( w \) is a colored permutation if each integer of \([n]\) is used exactly once, for some \( n \). Such a word will also be called a hyperoctahedral permutation or a signed permutation. The set (in fact group) of all such words will be denoted \( B_n \). We define \( w^{neg} \) to be the word obtained from \( w \) by converting all the bars to negative signs. The weight of a word is defined in the usual way, with the bars ignored. The operation \( ev \) removes the bars from a colored word. Thus if \( w = (231) \) then \( w^{ev} = (231) \).

A biletter is an ordered pair of letters denoted \( \langle x, y \rangle \).

A doubly colored biword is a sequence of biletters \( \langle x, y \rangle \) ordered canonically in the following way. A biletter \( \langle x, y \rangle \) occurs before \( \langle k, l \rangle \) if and only if one of the following holds:

1. \( x < k \)
2. \( x = k \), both are unbarred, and \( y^{neg} < l^{neg} \)
3. \( x = k \), both are barred and \( l^{neg} < y^{neg} \).

A doubly colored biword is a colored biword if only the bottom row has bars. For doubly colored biwords \( w \), its inverse \( w^{inv} \) is obtained by swapping the top and bottom letters of each biletter and then reordering. For colored biwords \( w \) its inverse \( w^{inv} \) is obtained by first moving the bars to the top row and then performing the inverse normally. Note that both \( inv \) and \( inv_r \) are involutions.

The total color of a word or a colored word \( w \), denoted \( tc(w) \), is the number of barred letters in the word.

Define \( ev \) to be the operation which removes the bars from a letter, word, biletter or a biword.

Standardisation \( st \) is defined as follows for a colored biword \( w \). First define \( w^{st} \) by replacing the top row with \( \{1, 2, 3, \ldots\} \) from left to right when the biword is written in order. Then, define \( w^{st} \) by (see [SW])

\[ w^{st} = w^{st \ inv} st \ inv. \]

For example, let \( w \) be the colored biword

\[ w = \begin{pmatrix} 1 & 1 & 2 & 3 & 3 \\ \overline{2} & 3 & 4 & \mathbf{T} & \mathbf{T} \end{pmatrix}. \]
Then \( w \) has top weight \((2, 2, 1)\) and bottom weight \((2, 1, 1, 1)\). Its inverse \( w^{inv} \) is given by
\[
 w^{inv} = \begin{pmatrix}
 1 & 1 & 2 & 3 & 4 \\
 3 & 3 & 1 & 1 & 2
\end{pmatrix}.
\]
Its standardisation \( w^{st} \) is computed as follows
\[
 w^{st} = \begin{pmatrix}
 1 & 2 & 3 & 4 & 5 \\
 2 & 3 & 4 & \top & \top
\end{pmatrix}.
\]
\[
 w^{st \ inv} = \begin{pmatrix}
 \top & \top & 2 & 3 & 4 \\
 5 & 4 & 1 & 2 & 3
\end{pmatrix}.
\]
\[
 w^{st \ inv \ st} = \begin{pmatrix}
 \top & 2 & 3 & 4 & 5 \\
 5 & 4 & 1 & 2 & 3
\end{pmatrix}.
\]
\[
 w^{st \ inv \ inv} = \begin{pmatrix}
 1 & 2 & 3 & 4 & 5 \\
 3 & 4 & 5 & 2 & \top
\end{pmatrix}.
\]

**Lemma 1.** Let \( w \) be a colored biword. Then
\[
 w^{st} = w^{inv \ st \ inv},
\]
\[
 w^{st} = w^{st \ inv \ st \ inv},
\]
\[
 w^{inv \ st} = w^{st \ inv}.
\]

**Proof.** The first statement is \[SW\], Proposition 12]. The second statement is essentially mentioned in \[SW\], Proposition 39]. All three statements can be checked directly, which may be done along the lines of Lemma 27 (but is easier). \( \square \)

We will occasionally identify a colored word \( w \) or a hyperoctahedral permutation \( \pi \) with the associated colored biword obtained by filling the top row with \( \{1, 2, \ldots, n\} \) from left to right. In the latter case, \( \pi^{inv} \) will be identified with the lower row of the inverse of the resulting biword. This the usual inverse in the group \( B_n \).

### 2.3. Domino insertion.

The normal Robinson-Schensted algorithm gives a bijection between permutations of \( S_n \) and pairs \( (P, Q) \) of standard Young tableaux (SYT) of size \( n \) and the same shape. A semistandard generalisation of this was given by Knuth. This is a bijection between certain matrices with non-negative integer entries (or alternatively unbarred biwords) and pairs of semistandard Young Tableaux of the same shape. We refer the reader to \[EC2\] for further details. Henceforth, familiarity with usual Robinson-Schensted insertion will be assumed.

In this section we describe the corresponding bijection for domino tableaux in a traditional insertion ‘bumping’ procedure. We will follow the description given by Shimozono and White \[SW\] for the rest of this section where more details may be found. As the whole theory will be developed completely from the growth diagram point of view in Section 3, we will not be completely formal. The reader is referred to \[Gar\], \[SW\], \[vL\] for full details.

Let \( D \) be a domino tableaux with \( sh(D) = \lambda \), no values repeated, and \( i \) a value which does not occur in \( D \). We will describe how to insert both a vertical and
horizontal domino with value \( i \) into \( D \). Let \( A \subset D \) be the sub-domino tableaux containing values less than \( i \). If \( \lambda \) has a 2-core \( \bar{\lambda} = \bar{\lambda} \), then we will always assume that \( \bar{\lambda} \subset \text{sh}(A) \). We set \( B \) to be the domino tableaux containing \( A \) and an additional vertical domino in the first column or an additional horizontal domino in the first row labelled \( i \). Let \( C = D/D' \) be the skew domino tableaux containing values greater than \( i \). Now we recursively define a bumping procedure as follows.

Let \((B, C)\) be a pair of domino tableaux (with no values repeated) overlapping in at most a domino which contains the largest value of \( B \). The combined shape of \( B \) and \( C \) must be a valid skew shape and the values of \( C \) larger than those of \( B \).

Let \( i < j \) be the largest and smallest values of \( B \) and \( C \) respectively. Denote the corresponding dominoes by \( \gamma_i \) and \( \gamma_j \). We now distinguish four cases:

1. If \( \gamma_i \) and \( \gamma_j \) do not touch, then we set \( B' = B \cup \gamma_j \) and \( C' = C - \gamma_j \).
2. If \( \gamma_i \) and \( \gamma_j \) intersect in exactly one square, then we add a domino containing \( j \) to \( B \) so that the shape of \( B \) contains both \( \gamma_i \) and \( \gamma_j \) together with the unique additional box which is diagonally outwards (right and down) from \( \gamma_i \cap \gamma_j \). We set \( C' = C - \gamma_j \).
3. If \( \gamma_i = \gamma_j \) and both are horizontal, then we ‘bump’ the domino \( \gamma_j \) to the next row, by setting \( B' \) to be the union of \( B \) with an additional (horizontal) domino with value \( j \) one row below that of \( \gamma_i \). We set \( C' = C - \gamma_j \).
4. If \( \gamma_i = \gamma_j \) and both are vertical, then we ‘bump’ the domino \( \gamma_j \) to the next column, by setting \( B' \) to be the union of \( B \) with an additional (vertical) domino with value \( j \) one column to the right of \( \gamma_i \). We set \( C' = C - \gamma_j \).

This procedure is repeated with \((B, C)\) replaced by \((B', C')\) until the (skew) domino tableaux \( C \) becomes empty.

The resulting \( B \) tableaux will be denoted by \( D \leftarrow i \) for the insertion of a horizontal domino and \( D \leftarrow \bar{i} \) for a vertical domino.

Let \( \mathbf{w} = w_1 w_2 \cdots w_n \) be a colored permutation and \( \delta_r \) be a 2-core assumed to be fixed throughout. Then the insertion tableaux \( P_{d}^{\delta(r)}(\mathbf{w}) \) is defined as \( ((\ldots((\delta_r \leftarrow w_1) \leftarrow w_2)\ldots) \leftarrow w_n) \). The sequence of shapes obtained in the process defines another standard domino tableaux called the recording tableaux \( Q_{d}^{\delta(r)}(\mathbf{w}) \).

As an example, the domino tableaux \( P_{d}^{\delta(342)}(\mathbf{342}) \) is constructed as follows:

\[
\begin{array}{c}
3 \\
3 & 4
\end{array} \quad \begin{array}{c}
2 \\
3 & 4
\end{array} \quad \begin{array}{c}
1 \\
2 & 4
\end{array} \quad \begin{array}{c}
3
\end{array}
\]

**Figure 2.** Insertion of \( \mathbf{w} = 342 \) into \( \emptyset \).

The following theorem will be proven in Section 3.

**Theorem 2.** Fix \( r \geq 0 \). The above algorithm defines a bijection between signed permutations \( \pi \in B_n \) and pairs of domino tableaux \((P, Q)\) of the same shape \( \lambda \in \mathcal{P}_r(n) \). This bijection satisfies the equality

\[
tc(\pi) = sp(P_d(\pi)) + sp(Q_d(\pi)).
\]
The insertion algorithm is due to Barbasch and Vogan \cite{BV} in a different form (left-right insertion and jeu-de-taquin). The insertion described here in terms of bumping is essentially that of Garfinkle \cite{Gar}. Van Leeuwen \cite{vL} proves that the Barbasch-Vogan algorithm is the same as the bumping description, and also shows that the bijection holds in the presence of a 2-core. That this algorithm sends total color to the sum of spins seems to have been first used by Kirillov, Lascoux, Leclerc and Thibon in \cite{KLLT} for certain hyperoctahedral involutions, though no details or proofs are present. More recently, the color-to-spin property is made explicit by Shimozono and White \cite{SW}.

Shimozono and White \cite{SW} only prove the color-to-spin property in the absence of a 2-core. However, the color-to-spin property is proven by studying the spin change for all the ‘bumps’ in the insertion and these are unaffected by the presence of a 2-core. Thus the generalisation of the domino insertion bijection to the 2-core case is immediate. Shimozono and White also give a semistandard generalisation of this bijection which is the case \( r = 0 \) of the following theorem. Their theory of domino insertion is developed in conjunction with other combinatorial algorithms including Haiman’s mixed insertion and left-right insertion.

**Theorem 3.** Fix a 2-core \( \delta_r \). There is a bijection between colored biwords \( w \) of length \( n \) and pairs \( (P_r^d(w), Q_r^d(w)) \) of semistandard domino tableaux with the same shape \( \lambda \in \mathcal{P}_r(n) \) with the following properties:

1. The bijection has the color-to-spin property:
   \[ tc(w) = sp(P_r^d(w)) + sp(Q_r^d(w)). \]

2. The weight of \( P_r^d(w) \) is the weight of the lower word of \( w \). The weight of \( Q_r^d(w) \) is the weight of the lower word of \( w \).

3. The bijection commutes with standardisation in the following sense:
   \[ P_r^d(w)^{st} = P_r^d(w^{st}). \]
   \[ Q_r^d(w)^{st} = Q_r^d(w^{st}). \]

The proof of this will be left until the next section, where we give an alternative description of domino insertion in terms of growth diagrams.

### 3. Growth Diagrams and Domino Insertion

#### 3.1. Properties of Growth Diagrams.

The insertion algorithm of subsection 2.3 can also be phrased in terms of Fomin’s growth diagrams \cite{Fom1,Fom2} (also known as the poset-theoretic description, or language of shapes). This was first made explicit by van Leeuwen \cite{vL}. We will show how growth diagrams are relevant to the semistandard generalisation of domino insertion of \cite{SW}. Thus our aim will be to give a short, stand-alone proof of Theorem 3 using elementary considerations of growth diagrams only, bypassing the connection with mixed insertion used by Shimozono and White. Thus their lemma \cite[Lemma 33]{SW} is replaced by our Lemma 9. The use of growth diagrams make the generalisation to the case of nonempty 2-core immediate. In fact one could use growth diagrams to define the entire correspondence and develop the theory beginning from that.
Let $M(i, j)$ be a $n \times n$ matrix taking values from $\{0, 1, -1\}$ thought of as the matrix representing a hyperoctahedral permutation. Thus it has one non-zero value in each row or column. We will take the row and column indices to lie in $[n]$.

The growth diagram (of $M(i, j)$) is an array of partitions $\lambda_{(i, j)}$ for $1 \leq i, j \leq n + 1$. Two ‘adjacent’ partitions $\lambda_{(i, j)}$ and $\lambda_{(i+1, j)}$ or $\lambda_{(i, j)}$ and $\lambda_{(i, j+1)}$ are either identical or differ by exactly one domino. Initially, all the partitions $\lambda_{(1,j)}$ and $\lambda_{(i,1)}$ are set to the same partition $\mu$. For our purposes this will usually be a partition satisfying $\mu = \bar{\mu}$. The remainder of the growth diagram will be determined from $\mu$ and the data $M(i, j)$ according to the following local rules.

Let $\lambda = \lambda_{(i,j)}$, $\mu = \lambda_{(i+1,j)}$, $\nu = \lambda_{(i,j+1)}$, $\rho = \lambda_{(i+1,j+1)}$ be the corners of a ‘square’. Assume (inductively) that $\lambda, \mu$ and $\nu$ are known. Then $\rho$ is determined as follows:

1. If $M(i, j) = 1$ then it must be the case that $\lambda = \mu = \nu$. Obtain $\rho$ from $\lambda$ by adding two to the first row.
2. If $M(i, j) = -1$ then it must be the case that $\lambda = \mu = \nu$. Obtain $\rho$ from $\lambda$ by adding two to the first column.
3. If $M(i, j) = 0$ and $\lambda = \mu$ or $\lambda = \nu$ (or both) then $\rho$ is set to the largest of the three partitions.
4. Otherwise $M(i, j) = 0$ and $\nu$ and $\mu$ differ from $\lambda$ by dominoes $\gamma$ and $\gamma'$. If $\gamma$ and $\gamma'$ do not intersect then $\rho$ is set to be the union $\lambda \cup \gamma \cup \gamma'$. If $\gamma \cap \gamma'$ is a single square $(k, l)$, then $\rho$ is the union of $\lambda \cup \gamma \cup \gamma' \cup (k + 1, l + 1)$. If $\gamma = \gamma'$ is a vertical domino then $\rho$ is obtained from $\lambda \cup \gamma$ by adding two to the column immediately to the right of $\gamma$. If $\gamma = \gamma'$ is a horizontal domino then $\rho$ is obtained from $\lambda \cup \gamma$ by adding two to the row immediately below $\gamma$.

We will call these rules the local rules of the growth diagram.

**Proposition 4.** The above algorithm is well defined. The growth diagram models the insertion of the colored permutation $\pi$ corresponding to $M(i, j)$ into a 2-core $\delta_r$ (in fact more generally any initial partition).

The partition $\lambda_{(i,j)}$ is the shape of the tableaux obtained after the first $i$ insertions and restricted to values less than $j$. Thus $\{\lambda_{(n+1,j)} : j \in [n + 1]\}$ is a chain of partitions determining $P^\pi_\delta(\pi)$ and $\{\lambda_{(i,n+1)} : i \in [n + 1]\}$ is a chain of partitions determining $Q^\pi_\delta(\pi)$.

**Proof.** This is proven via induction, by comparing domino insertion with the local rules of the growth diagram. The details can be found in [V].

For example, Figure 3 is the growth diagram corresponding to the insertion procedure of Figure 2.

**Lemma 5.** The local rules of a growth diagram are reversible in the following sense. Let $\lambda = \lambda_{(i,j)}$, $\mu = \lambda_{(i+1,j)}$, $\nu = \lambda_{(i,j+1)}$, $\rho = \lambda_{(i+1,j+1)}$ be the corners of a ‘square’ of the growth diagram. Then $\rho$, $\mu$ and $\nu$ determine $\lambda$ and $M(i, j)$.

**Proof.** This is a simple verification of the local rules.

Note, that there can be two legitimate standard domino tableaux corresponding to $\{\lambda_{(i,n+1)} : i \in [n + 1]\}$ and $\{\lambda_{(n+1,j)} : j \in [n + 1]\}$ which do not give a growth diagram.
Figure 3. Growth diagram for the insertion of \( w = \overline{3421} \) into \( \emptyset \).

corresponding to an insertion procedure. For example if \( \lambda_{(1,2)} = (2) = \lambda_{(2,1)} \) and \( \lambda_{(2,2)} = (2,2) \) then \( \lambda_{(1,1)} \) must be \( \emptyset \). This is not a valid growth diagram for insertion as \( \lambda_{(1,1)} \neq \lambda_{(2,1)} \).

Lemma 6. The correspondence

\[
\pi \rightarrow (P_d^r(\pi), Q_d^r(\pi))
\]

is a bijection between \( \pi \in B_n \) and pairs of standard domino tableaux of the same shape \( \lambda \in \mathcal{P}_r(n) \).

Proof. The previous Lemma implies that this correspondence is injective. As no dominoes can be removed from \( \delta_r \), the ‘initial’ row and column of the growth diagram (\( \lambda_{(1,j)} \) and \( \lambda_{(i,1)} \)) will consist completely of partitions equal to \( \delta_r \). Thus setting \( \lambda_{(i,n+1)} : i \in [n+1] \) and \( \lambda_{(n+1,j)} : j \in [n+1] \) to two tableaux of the shape \( \lambda \in \mathcal{P}_r(n) \) will give a growth diagram corresponding to the insertion of some hyperoctahedral permutation \( \pi \). \( \square \)

Lemma 7. Let \( \pi \) be a hyperoctahedral permutation. Domino insertion possesses the symmetry property

\[
P_d^r(\pi) = Q_d^r(\pi^{\text{inv}}).
\]
Proof. This is a consequence of the fact that the growth diagram local rules are symmetric. □

**Lemma 8.** Domino insertion for hyperoctahedral permutations $\pi$ possesses the color-to-spin property:

$$tc(\pi) = sp(P_d(\pi)) + sp(Q_d(\pi)).$$

Proof. Let $\lambda = \lambda_{(i,j)}, \mu = \lambda_{(i+1,j)}$, $\nu = \lambda_{(i,j+1)}$, $\rho = \lambda_{(i+1,j+1)}$ be the corners of a square of the growth diagram. Then the Lemma follows from the observation that

$$sp(\rho/\mu) + sp(\rho/\nu) = sp(\mu/\lambda) + sp(\nu/\lambda) + \begin{cases} 1 \text{ if } M(i,j) = -1 \\ 0 \text{ otherwise.} \end{cases}$$

This can be checked by considering the local rules case by case. □

**Lemma 9.** Let $\pi = \pi_1 \cdots \pi_n$ be a colored permutation. Then $\pi_i^{\text{neg}} < \pi_{i+1}^{\text{neg}}$ if and only if dom$_i$ lies to the left of dom$_{i+1}$ in $Q_d(\pi)$.

Proof. The main idea is to analyze a $1 \times 2$ rectangle of the growth diagram. Let $\lambda_0 = \lambda_{(i,j)}, \lambda_1 = \lambda_{(i+1,j)}, \lambda_2 = \lambda_{(i+2,j)}, \mu_0 = \lambda_{(i,j+1)}, \mu_1 = \lambda_{(i+1,j+1)}$ and $\mu_2 = \lambda_{(i+2,j+1)}$ be the corners of a $1 \times 2$ rectangle of the growth diagram. We will call the two squares of the $1 \times 2$ rectangle the first and second squares. We further assume that $M(i,j) = M(i+1,j) = 0$.

Now suppose that $\alpha_0 = \lambda_1/\lambda_0$ and $\alpha_1 = \lambda_2/\lambda_1$ are both dominoes so that $\alpha_0$ lies to the left of $\alpha_1$. Then it is easy to check that $\beta_0 = \mu_1/\mu_0$ and $\beta_1 = \mu_2/\mu_1$ are both dominoes since $M(i,j) = M(i+1,j) = 0$. We claim that in fact $\beta_0$ lies to the left of $\beta_1$. If $\alpha_0 = \mu_0$ this is trivial and most of the cases of the local rules are a simple verification.

The only interesting case is when $\lambda_1 = \mu_1$ and $\alpha_0$ is a vertical domino. In this case, $\beta_0$ has moved to the right when compared to $\alpha_0$. The key observation is that $\beta_0$ is placed in the column immediately to the right of $\alpha_0$, so it is either still to the left of $\alpha_1$ or it overlaps $\alpha_1$. When overlap occurs, $\beta_1$ will be moved further to the right and $\beta_0$ will remain to the left of $\beta_1$. This proves our claim.

To show (one direction of) our lemma, we just need to check, case by case, that the initial condition ($\alpha_0$ lying to the left of $\alpha_1$) holds for $j = \max(\pi_i^{\text{ev}}, \pi_{i+1}^{\text{ev}}) + 1$. As adding a new domino to the first column will be furthest to the left, and adding a new domino to the first row will be the furthest right this is a simple verification. The claim implies inductively that the same will continue to hold when we get to $\lambda_{(i,n+1)}, \lambda_{(i+1,n+1)}$ and $\lambda_{(i+2,n+1)}$, which give exactly dom$_i$ and dom$_{i+1}$ of $Q_d(\pi)$.

The other direction of the lemma is proven in exactly the same way, or one could replace ‘left’ by ‘above’ and ‘row’ by ‘column’. □

**Lemma 10.** Let $\pi = \pi_1 \cdots \pi_n$ be a colored permutation. Then $(\pi^{\text{inv}})_i^{\text{neg}} < (\pi^{\text{inv}})_{i+1}^{\text{neg}}$ if and only if dom$_i$ lies to the left of dom$_{i+1}$ in $P_d(\pi)$.

Proof. This is a consequence of Lemma 9 and Lemma 7. □

We are now ready to prove the semistandard domino-Schensted correspondence.
Proof of Theorem. That the correspondence exists for standard biwords is Lemma 6. Then the color-to-spin property follows from Lemma 8.

For the semistandard case, fix two weights \( \mu \) and \( \lambda \) and let these be the weights of the upper and lower words of a colored biword \( w \). We define \( P_r^d(w) \) by standardising the top row first:

\[
P_r^d(w) = Q_r^d(w^\text{inv}^r).
\]

This is well defined since \( w^\text{inv}^r \) has a colored permutation as its lower word. It is a semistandard domino tableaux because of Lemma 9. This allows us to define \( Q_r^d(w) \) by

\[
Q_r^d(w) = P_r^d(w^{\text{inv}^r}).
\]

Next we show that these definitions commute with standardisation. For example,

\[
P_r^d(w)^{\text{st}} = Q_r^d(w^\text{inv}^r)^{\text{st}}
= Q_r^d(w^\text{inv}^r, \text{st})
= Q_r^d(w^{\text{inv}^r})
= P_r^d(w^{\text{st}}).
\]

We have used Lemmas 11 and 7. A similar calculation proves that \( Q_r^d(w)^{\text{st}} = Q_r^d(w^{\text{st}}) \).

Since standardisation is injective (for both words and tableaux) when the weights \( \mu \) and \( \lambda \) are fixed, this proves that the correspondence

\[
w \rightarrow (P_r^d(w), Q_r^d(w))
\]

is injective for colored biwords with fixed weights for the top and bottom rows. The color-to-spin property is also a consequence of the standardisation procedure, as \( tc(w) = tc(w^{\text{st}}) = sp(P_r^d(w^{\text{st}})) + sp(Q_r^d(w^{\text{st}})) = sp(P_r^d(w)) + sp(Q_r^d(w^{\text{st}})) \).

Finally, one can show that correspondence is a surjection as follows. Suppose we are given a pair \((P, Q)\) of semistandard domino tableaux of shape \( sh(P) = sh(Q) \in \mathcal{P}_r(n) \) such that \( wt(P) = \lambda \) and \( wt(Q) = \mu \). Then we may obtain a colored biword \( v \) with standardised lower word by performing the inverse correspondence (in the standard case) to \((Q^{\text{st}}, P^{\text{st}})\). That the upper word can be converted to have weight \( \lambda \) is a consequence of the ‘only if’ part of Lemma 9. Thus \( v \) satisfies \( P_r^d(v) = Q^r \) and \( Q_r^d(v) = P \). Now perform the inverse correspondence to \((P^{\text{st}}, Q^{\text{st}})\), using Lemma 9 to prove that we can change the upperword of \( v^{\text{inv}^r} \) into weight \( \mu \).

This completes the proof. \( \square \)

An alternative way of proving the surjectiveness of the correspondence is by enumerating both colored words and pairs of tableaux of the same shape. Littlewood’s 2-quotient map will accomplish the latter.

For the case \( r = 0 \), it is easy to see that the definition used in the proof agrees with that of Shimozono and White [SW].

Corollary 11. The semistandard domino correspondence possesses the symmetry property:

\[
P_r^d(w) = Q_r^d(w^{\text{inv}^r}).
\]
Proof. This is a consequence of the definition used in the proof. □

3.2. Dual domino-Schensted correspondence. In this section we give a description of two closely related dual domino-Schensted correspondences. They are bijections between certain words and pairs of tableaux of the same shape, one of which is semistandard and the other is column-semistandard. For a description of the dual RSK correspondence for Young tableaux see [EC2].

A domino tableau $D$ is column-semistandard if its transpose is semistandard.

A dual colored biword is a colored biword such that the top row is ordered as usual, but when the bottom row is used to order two biletters, the reverse ordering is chosen. Thus $(x^i_y)$ precedes $(k^l)$ if

1. $x < k$, or
2. $x = k$ and $y^{neg} > l^{neg}$.

The operator $\overline{st}$ is defined for dual colored biwords as usual by standardising the top row. The operation $inv_d$ changes dual colored biwords to colored biwords and vice versa. It swaps the two letters of each biletter, moving the bar to the lower letter if needed, and orders the biletters accordingly.

A colored biword or dual colored biword is called multiplicity-free if any biletter $(i^i_j)$ occurs at most once. The same numbers may appear up to twice, but one must be barred and the other non-barred. For multiplicity-free biwords we define the following new standardisation operation $std$ by

$$ w^{std} = w^{st} inv_d \overline{st} inv_d. $$

Lemma 12. Let $w$ be a multiplicity free dual colored biword or colored biword. Then

$$ w^{std} = w^{inv_d \overline{st} inv_d \overline{st}}. $$

$$ w^{std inv_d} = w^{inv_d st}. $$

Proof. The proof is a direct verification, and very similar to Lemma 11. □

We may now define the two dual domino-Schensted correspondences $\alpha$ and $\beta$. Let $w$ be a multiplicity-free dual colored biword. Then we define $Q^r_\alpha(w)$ via domino-Schensted applied to $w^{st}$ where $w^{st}$ is now treated as a colored biword. To see that $Q^r_\alpha(w)$ is a column-semistandard domino tableau, we use Lemma 9. Also define $P^r_\alpha(w) = P^r_d(v)$, where $v$ is the lower word of $w$.

Now let $w$ be a multiplicity-free colored biword. We define the correspondence $\beta$ in a similar way. Set $Q^r_\beta(w)$ to be $Q^r_d(w^{inv_d \overline{st} inv_d})$. We define $P^r_\beta(w)$ by turning $P^r_d(w^{inv_d \overline{st} inv_d})$ into a column-semistandard tableau of the same weight as the lower word of $w$. That this is possible is a consequence of Lemma 11.

Note that both correspondences agree with the usual domino correspondence when applied to hyperoctahedral permutations.

Theorem 13. Let $r \geq 0$ be fixed. The map $\alpha$

$$ \alpha : w \rightarrow (P^r_\alpha(w), Q^r_\alpha(w)) $$
is a weight preserving bijection between multiplicity-free dual colored biwords $w$ of length $n$ and pairs of tableaux $(P, Q)$ of the same shape $\lambda \in \mathcal{P}_r(n)$ such that $P$ is semistandard and $Q$ is column-semistandard.

The map $\beta$

$$\beta : w \rightarrow (P^r_\beta(w), Q^r_\beta(w))$$

is a weight preserving bijection between multiplicity-free dual colored biwords $w$ of length $n$ and pairs of tableaux $(P, Q)$ of the same shape $\lambda \in \mathcal{P}_r(n)$ such that $P$ is column-semistandard and $Q$ is semistandard.

These maps satisfy the following properties:

(1) They commute with standardisation. Thus

$$(P^r_\alpha(w)^{st}, Q^r_\alpha(w)^{st}) = (P^r_\beta(w)^{std}, Q^r_\beta(w)^{std})$$

and similarly for $\beta$.

(2) The maps $\alpha$ and $\beta$ are related by

$$(Q^r_\alpha(w), P^r_\alpha(w)) = (P^r_\beta(w)^{inv_d}, Q^r_\beta(w)^{inv_d})$$.

(3) Both maps have the color-to-spin property.

Proof. The proof is analogous to that of Theorem 3, requiring use of Lemmas 9 and 10. □

3.3. Statistics on Domino Tableaux. In this subsection we will introduce and study a number of statistics on partitions and domino tableaux. Let $\lambda$ be a partition with 2-core $\tilde{\lambda}$. Let $o(\lambda)$ be the number of odd rows of $\lambda$. Thus $o(\lambda')$ is the number of odd columns. Let

$$d(\lambda) = \sum_{i=1}^{l(\lambda/2)} \left\lfloor \frac{\lambda_{2i}}{2} \right\rfloor.$$ 

Note that $d(\lambda) = d(\lambda')$ (see for example [Sta]). Also let

$$v(\lambda) = \sum_{i=1}^{l(\lambda)} \left\lfloor \frac{\lambda_i}{2} \right\rfloor.$$ 

Now let $D$ be a domino tableaux of shape $\lambda$. As before $v(D)$ is the number of vertical dominoes in $D$ and $sp(D) = v(D)/2$. Let $ov(D)$ and $ev(D)$ be the number of vertical dominoes in odd and even columns respectively. Thus $sp(D) = (ov(D) + ev(D))/2$. Let $mspin(\lambda)$ be the maximum spin over all domino tableaux of shape $\lambda$. Similarly, let $ov(\lambda)$ be the maximum of $ov(D)$ over all domino tableau of shape $\lambda$. Define $ev(\lambda)$ similarly. The cospin of a domino tableaux $D$ is $cosp(D) = mspin(\lambda) - sp(D)$ (and is always an integer).

The following lemma is a strengthening of a lemma in [Whi].

Lemma 14. Let $D$ be a domino tableaux of shape $\lambda$ with 2-core $\tilde{\lambda}$. Then

(1) $$ov(D) - ev(D) = \frac{o(\lambda) - o(\tilde{\lambda})}{2}.$$
Proof. We proceed by induction on the size of $\lambda$, while keeping $\tilde{\lambda}$ fixed. When $D$ has shape $\tilde{\lambda}$ then both sides are 0. Now let $D$ have shape $\lambda$ and suppose the Lemma is true for all shapes $\mu$ that can be obtained from $\lambda$ by removing a domino. Let $\gamma$ be the domino with the largest value in $D$. Removing $\gamma$ from $D$ gives a domino tableaux $D'$ for which $\Box$ holds. If $\gamma$ is a horizontal domino then neither side changes. If $\gamma$ is a vertical domino in an odd row then both sides decrease by 1 (changing from $D$ to $D'$). If $\gamma$ is a vertical domino in an even row then both sides increase by 1. \hfill $\Box$

Note that this implies that a domino tableaux $D$ which has the maximum spin (amongst all domino tableaux of shape $\lambda$) will also have the most number of odd vertical and even vertical dominoes. Thus for example, $m\text{spin}(\lambda) = ev(\lambda) + ov(\lambda)$.

3.4. Symmetric Growth Diagrams. We now specialise to the case where the matrix $M_\pi(i, j)$ corresponds to a hyperoctahedral involution $\pi$. Thus $M_\pi(i, j)$ is symmetric and $\pi$ satisfies $\pi^2 = 1$ in the group $B_n$. The hyperoctahedral involution $\pi$ will consist of a number of fixed points, barred fixed points, two-cycles and barred two-cycles. For example, let $\pi = (1635427)$. Then $\pi$ has one fixed point, two barred fixed points, one two-cycle and one barred two-cycle.

In this case we obtain the following proposition, part of which was first observed by van Leeuwen [14].

**Proposition 15.** Let $\pi \in B_n$ be a hyperoctahedral involution. Suppose $\pi$ has a fixed points, $b$ barred fixed points, $c$ two-cycles and $d$ barred two-cycles. Fix a 2-core $\delta_r$. Let the insertion tableaux $P^r_d(\pi) = Q^r_d(\pi)$ of $\pi$ into $\delta_r$ have shape $\lambda = sh(P^r_d(\pi))$ (which satisfies $\tilde{\lambda} = \delta_r$). Then

\[
\begin{align*}
sp(P^r_d(\pi)) &= \frac{b}{2} + d \\
o(\lambda) - o(\delta_r) &= b \\
o(\lambda') - o(\delta_r) &= c \\
d(\lambda) - d(\delta_r) &= c + d.
\end{align*}
\]

Proof. Since $P^r_d(\pi) = Q^r_d(\pi)$ for a hyperoctahedral involution by Lemma 7, the first equation is a consequence of the color-to-spin property of Theorem 2. For the other statements, note that the symmetry of $M_\pi(i, j)$ and of the local rules of the growth diagram imply that the growth diagram $\lambda_{(i, j)}$ itself is symmetric. We focus our attention on the partitions $\lambda_{(i, j)}$. If $M_\pi(i, j) = 1$ then $\lambda_{(i+1, j+1)}$ has two boxes added to its first row, and so $o(\lambda'_{(i+1, j+1)}) = o(\lambda'_{(i, j)}) + 2$. Similarly, if $M_\pi(i, j) = -1$ then $o(\lambda_{(i+1, j+1)}) = o(\lambda_{(i, j)}) + 2$. In both cases $d(\lambda_{(i, j)}) = d(\lambda_{(i+1, j+1)})$.

If $M_\pi(i, j) = 0$ and $\lambda_{(i+1, j)} = \lambda_{(i, j)} = \lambda_{(i, j+1)}$ then $\lambda_{(i, j)} = \lambda_{(i+1, j+1)}$. The only remaining case is if $\lambda_{(i+1, j)}$ differs from $\lambda_{(i, j)}$ by a domino, in which case $\lambda_{(i+1, j+1)} = \lambda_{(i, j+1)}$ as well. This implies that $\lambda_{(i+1, j+1)}$ differs from $\lambda_{(i, j)}$ by two dominos in two adjacent columns or rows. Regardless, the number of odd columns and rows is unchanged while $d(\lambda_{(i+1, j+1)}) = d(\lambda_{(i, j)}) + 1$. \hfill $\Box$
**Corollary 16.** Let $D = P_d(\pi)$ correspond to a hyperoctahedral involution $\pi$ with $b$ barred fixed points and $d$ barred two-cycles. Then

\[
\text{ev}(D) = d, \\
\text{ov}(D) = b + d.
\]

**Proof.** As before, let $\pi$ have $b$ barred fixed points. Then by Proposition 15,

\[
\text{ev}(D) + \text{ov}(D) = 2\text{sp}(D) = b + 2d.
\]

Combining Lemma 14 with Proposition 15 again we have,

\[
\text{ov}(D) - \text{ev}(D) = \frac{o(\lambda) - o(\bar{\lambda})}{2} = b.
\]

Subtracting the two equations and dividing by two, we obtain the first result. Summing the two equations give the second result. \qed

The significance of this Corollary will become apparent in Section 4.

3.5. **Some Enumeration for Domino Tableaux.** Let $f^\lambda$ be the number of SYT of shape $\lambda$. The Robinson-Schensted algorithm for standard Young tableaux (SYT) leads to a number of enumerative results including the following well known result.

**Proposition 17.** Let $n \geq 1$. Then

(2) \[
\sum_{\lambda \vdash n} (f^\lambda)^2 = n!.
\]

(3) \[
\sum_{\lambda \vdash n} f^\lambda = t(n).
\]

We can easily generalise these to domino tableaux. Define

\[
d^\lambda(q) = \sum_{\text{SDT } D: \text{sh}(D) = \lambda} q^{\text{spin}(D)}.
\]

It is unlikely that a ‘hook-length’ formula holds for $d^\lambda(q)$. Note that $d^\lambda(q)$ depends on more than just the 2-quotient $(\lambda^{(0)}, \lambda^{(1)})$ of $\lambda$. For example, $(3, 1, 1)$ and $(2, 2)$ have the same 2-quotient but $d^{(3,1,1)}(q) = 2q^{1/2}$ and $d^{(2,2)}(q) = 1 + q$. A cospin version of $d^\lambda(q)$ for more general ribbon tableaux was studied by Schilling, Shimozono and White in [SSW].

We have the following analogue of (2):

**Proposition 18.** Let $n \geq 1$ and $r \geq 0$ be fixed. Then

\[
\sum_{\lambda \in P_r(n)} (d^\lambda(q))^2 = (1 + q)^n n!
\]

where the sum is over all partitions $\lambda \in P_r(n)$.

**Proof.** This is an immediate consequence of the bijection in Theorem 2. \qed
Now define \( h_r(n) \) as follows:

\[
h_r(n) = \sum_{\lambda \in \mathcal{P}_r(n)} a^{(o(\lambda) - o(\delta_r))/2} b^{(o(\lambda') - o(\delta_r))/2} c^{d(\lambda) - d(\delta_r)} d^\lambda(q).
\]

When \( a = b = c = q = 1 \), this is the number of hyperoctahedral involutions in \( B_n \) and thus a domino analogue of \( t(n) \).

**Proposition 19.** The function \( h(n) = h_r(n) \) does not depend on \( r \). It satisfies the recursion

\[
h(n + 1) = (b + a q^{1/2}) h(n) + n c (1 + q) h(n - 1).
\]

The exponential generating function defined as

\[
E_h = \sum n h(n) \frac{t^n}{n!}
\]

is given by the formula

\[
E_h = \exp \left( (b + a q^{1/2}) t + c (1 + q) \frac{t^2}{2} \right).
\]

**Proof.** That \( h_r(n) \) does not depend on \( r \) follows from the fact that the tableaux being enumerated are in bijection with hyperoctahedral involutions. Furthermore, the bijection preserves the appropriate weighting according to Proposition 15. Thus we are in fact enumerating hyperoctahedral involutions.

The recursion for \( h(n) \) is immediate from the construction of a hyperoctahedral involution from barred and non-barred fixed points and two-cycles.

For the exponential generating function, we can use the exponential formula (see [EC2, Corollary 5.1.6]). Thus we think of a hyperoctahedral involution as a partition of \([n]\) into one and two element subsets. The one element subsets can be given a weight of \( b \) or \( a q^{1/2} \) while the two element subsets can be given a weight of \( c \) or \( c q \).

\[\square\]

4. **Sign-Imbalance and Stanley’s Conjecture**

Sign Imbalance can be defined for posets in general, but we will only concern ourselves with the posets arising from partitions.

Let \( T \) be a standard Young tableaux. Its reading word \( \text{reading}(T) \), for our purposes, will be obtained by reading the first row from left to right, then the second row, and so on. We set \( \text{sign}(T) = \text{sign}(\text{reading}(T)) \) where \( \text{reading}(T) \) is treated as a permutation.

Let \( \lambda \) be a partition. Then we set

\[
I_\lambda = \sum_T \text{sign}(T)
\]

where the sum is over all standard Young tableaux \( T \) of shape \( \lambda \). We say \( I_\lambda \) is the sign-imbalance of \( \lambda \).

It is not difficult to see that \( I_\lambda \) is related to domino tableaux. Suppose \( \lambda \) has no 2-core, then define an involution on standard Young tableaux of shape \( \lambda \) by swapping
$2i - 1$ with $2i$ for the smallest possible value of $i$ where this is possible. If no such swap is possible the tableaux is fixed by the involution.

The fixed points correspond exactly to the standard domino tableaux of shape $\lambda$. We obtain a standard Young tableaux $T(D)$ from a standard domino tableaux $D$, by filling the domino with a 1 with the values 1 and 2, the domino with a 2, with the values 3 and 4, and so on.

When $\lambda$ has 2-core $\delta_1$ (a single box) then we use an involution which swaps $2i$ with $2i + 1$ for the smallest value of $i$ where it is possible. Again, the fixed points are the standard domino tableaux of shape $\lambda$.

It is easy to see that these involutions are sign-reversing on tableaux which are not fixed points and thus we obtain the following proposition.

**Proposition 20.** Let $r \in \{0, 1\}$, $n \geq 1$ and $\lambda \in P_r(n)$. Then

$$I_\lambda = \sum_{sh(D) = \lambda} \text{sign}(D)$$

where the sum is over standard domino tableaux of shape $\lambda$ and the sign of a domino tableaux $D$ is the sign of the corresponding standard Young tableaux $T(D)$.

For other values of $r$, we have the following result, see [Sta]:

**Proposition 21.** Let $\lambda$ have 2-core $\delta_r$ for $r > 1$, then

$$I_\lambda = 0.$$  

There is another natural involution on standard Young tableaux of which standard domino tableaux are the fixed points. This is Schützenberger’s involution $S$, also known as evacuation. The fixed points of this involution are exactly the domino tableaux of shape $\lambda$ satisfying $\overline{\lambda} = \delta_r$ for $r \in \{0, 1\}$ (see [vL]). For a fixed shape $\lambda$, Stanley [Sta] has shown that $S$ is either always parity-reversing or parity-preserving.

By analysing the positions of horizontal and vertical dominoes in a standard domino tableaux, White [Whi] proves the following proposition.

**Proposition 22.** Let $D$ be a domino tableaux of shape $\lambda$ which has 2-core $\emptyset$ or $\delta_1$. Then

$$\text{sign}(D) = (-1)^{\text{ev}(D)}.$$  

White has also given an explicit formula (in terms of shifted tableaux) for the sign-imbalance of partitions which have 'near-rectangular' shape.

Combining Proposition 22 with Corollary 16 we obtain the following theorem.

**Theorem 23.** Fix $r \in \{0, 1\}$. Let $\pi$ be a hyperoctahedral involution. Then the sign of its insertion tableaux $\text{sign}(P_d^r(\pi))$ is equal to the number of barred 2-cycles.

**Proof.** This follows immediately from Corollary 16 and Proposition 22. □

We can now prove the following conjecture of Stanley [Sta], known as the '2$^\lfloor n/2 \rfloor$' conjecture.
Theorem 24. Let $m \geq 1$ be an integer. Then
$$\sum_{\lambda \vdash m} x^{v(\lambda)} y^{r(\lambda')} q^{d(\lambda)} t^{d(\lambda')} I_\lambda = (x + y)^{\lfloor m/2 \rfloor}.$$ Note that $d(\lambda) = d(\lambda')$ so that one of $q$ and $t$ is not needed.

Proof. Since $I_\lambda = 0$ for $\lambda$ with a 2-core larger than $\delta_1$, we may assume the sum is over $\lambda \in \mathcal{P}_r(n)$, for the unique $r \in \{0, 1\}$ and $n$ satisfying $2n + r = m$. Note that $o(\delta_1) = o(\delta'_1) = 1$ and $d(\delta_1) = 0$.

The standard domino tableaux of such shape correspond exactly to hyperoctahedral involutions $\pi \in B_n$. We define an involution $\alpha$ on all such $\pi$ by turning the two-cycle $(i, j)$ with the smallest value of $i$ from barred to non-barred or vice versa, if such an $i$ exists. By Theorem 23, $\alpha$ is sign-reversing for domino tableaux which are not fixed points. Furthermore, by Proposition 15, all of the statistics $o(\lambda) - r$, $o(\lambda') - r$ and $d(\lambda)$ remain fixed by $\alpha$.

The fixed points of $\alpha$ are exactly the hyperoctahedral involutions without two-cycles. Hence we obtain, using Proposition 15
$$\sum a^{(o(\lambda) - r)/2} b^{(o(\lambda') - r)/2} c^{d(\lambda)} I_\lambda = (a + b)^n.$$ To change this into the form of Stanley’s conjecture, observe that $2v(\lambda) + o(\lambda) = m = 2n + r$ implying that $(o(\lambda) - r)/2 = n - v(\lambda)$ and similarly for $v(\lambda')$ and $o(\lambda')$. Now substitute this and also $x = 1/a$ and $y = 1/b$. Finally multiply both sides by $(xy)^n$. \hfill \Box

Note that the fixed points of $\alpha$ in the proof are exactly the domino tableaux which are hook shaped. That these give the right hand side of the conjecture was shown by Stanley [Sta]. When we set $x = y = q = 1$ we obtain the following signed analogue of (3):
$$\sum_{\text{SYTT}} \text{sign}(T) = 2^{\lfloor n/2 \rfloor}$$
where the sum is over all standard Young tableaux $T$ of size $n$.

5. Domino Generating Functions

Let $\Lambda$ denote the ring of symmetric functions in a set of variables $X = (x_1, x_2, \ldots)$ taking coefficients in $\mathbb{C}$ (though the coefficient field will not affect the results). Its completion, $\tilde{\Lambda}$ includes symmetric power series of unbounded degree (though the coefficient of a monomial $m_\lambda$ will always be well defined).

Carré and Leclerc have defined symmetric functions $H_\lambda(X; q)$ via semistandard domino tableaux, in the same way that Schur functions arise from semistandard Young tableaux. Slightly more general functions $G_\lambda(X; q)$ were used in [LLT] and the two are connected via $H_\lambda(X; q) = G_{2\lambda}(X; q)$.

Let $\lambda$ be a partition. Define
$$G_\lambda = \sum_D q^{w(D)} x^{wt(D)}$$
where the sum is over all semistandard domino tableaux of shape \( \lambda \) and \( x^\mu := x_1^{\mu_1} x_2^{\mu_2} \ldots \) for a partition \( \mu \). There is a cospin version of this function which we will not need. In the notation of \([LLT]\), our \( G_\lambda \) would be denoted \( G_{\lambda/\tilde{\lambda}} \).

That the \( G_\lambda \) are symmetric functions is a consequence of a combinatorial interpretation of their expansion into Schur functions given by Carré and Leclerc. As spin is not always integral, the \( G_\lambda \) lie in the ring \( \Lambda[q^{1/2}] \). We will call the \( G_\lambda \) domino functions. Theorem \( \text{3} \) leads immediately to the following domino Cauchy identity.

**Proposition 25.** Fix \( r \geq 0 \). Then

\[
\sum_{\lambda \in P_r} G_\lambda(X;q)G_\lambda(Y;q) = \frac{1}{\prod_{i,j} (1 - x_i y_j)(1 - q x_i y_j)}.
\]

The dual domino-Schensted correspondence of Theorem \( \text{13} \) leads to the following dual domino Cauchy identity.

**Proposition 26.** Fix \( r \geq 0 \). Then

\[
\sum_{\lambda \in P_r} q^{\lambda/\delta, l/2} G_\lambda(X;q)G_{\lambda'}(Y;q^{-1}) = \prod_{i,j} (1 + x_i y_j)(1 + q x_i y_j).
\]

**Proof.** This follows from the fact that a column-semistandard domino tableaux \( D \) with \( m \) dominoes is a semistandard domino tableaux \( D' \) of the conjugate shape with spin given by

\[
sp(D') = \frac{m}{2} - sp(D).
\]

In \([KLLT]\), Kirillov, Lascoux, Leclerc and Thibon give two product expansions for certain sums of the \( G_\lambda \). These will be seen as specialisations of our Theorem \( \text{28} \). As the paper \([KLLT]\) contains no proofs, our theorem can be considered both as a proof and as a generalisation.

We begin by studying closely the effect of standardisation on a semistandard colored involution.

A colored word \( w \) is said to be a colored involution if \( w = w^{\text{inv}}. \) Every such word is given by the number of fixed points \( (i) \), barred fixed points \( (\bar{i}) \), two-cycles \( (i_j) \ldots (i_j) \) and barred two-cycles \( (\bar{i}_j) \ldots (\bar{i}_j) \). Let there be \( a_i, b_i, c_{ij} \) and \( d_{ij} \) of these respectively. Thus \( c_{ij} = c_{ji} \) and \( d_{ij} = d_{ji} \).

**Lemma 27.** Let \( w \) be a colored involution. Then its standardisation \( w^{\text{st}} \) is a signed involution with a fixed points, \( b \) barred fixed points, \( c \) two-cycles and \( d \) barred two-cycles, where:

\[
a = \sum_i a_i,
\]

\[
b = \sum_i b_i - 2 \sum_i \left\lfloor \frac{b_i}{2} \right\rfloor.
\]
\[ c = \sum_{i<j} c_{ij} \]
\[ d = \sum_{i<j} d_{ij} + \sum_i \left\lfloor \frac{b_i}{2} \right\rfloor. \]

In other words, the only change that occurs is that of barred fixed points becoming barred two-cycles.

**Proof.** It is clear from Lemma 1 that \( w^{st} \) is also an involution.

Fix an integer \( i \). Then in the colored word \( w \), the fixed points of the form \( (i) \) have exactly
\[ A = \sum_{j<i} (a_j + b_j + c_{jk} + d_{jk}) + b_i + \sum_k d_{ik} + \sum_{k<i} c_{ki} \]
biletters in front. If we look at \( w^{st \ inv} \) the same formula holds using a different ordering for the top row. Thus when we standardise and take inverse and standardise again, this set of biletters will receive identical numbers for both the top and bottom row, and will give us \( a_i \) fixed points.

Now consider barred fixed points \( (i) \). There are
\[ A = \sum_{j<i} (a_j + b_j + c_{jk} + d_{jk}) + \sum_{k>i} d_{ik} \]
biletters in front. Now let us consider what happens when we standardise the top row and take the inverse. We will similarly get all (barred or otherwise) fixed points of \( j < i \) in front and so on. The only possible difference are the biletters involving \( i \). The fixed points clearly make no contribution. Since the ordering for the lower letter is reversed when the upper letter is barred, the biletters occuring in front are only those of the form \( (j) \) where \( j > i \). There are exactly \( \sum_{k>i} d_{ik} \) of these, thus the collection of barred fixed points \( (i) \) will get the same set of numbers for the upper and lower biletters. However, individually, the numbers assigned for the two rows will be reversals of each other due to the ordering on the bottom row induced by the bars on the upper row.

Now consider what happens to the collection of biletters of the form \( (i) \) and \( i \neq j \). We need only show that these all become two-cycles when \( w \) is standardised. Since \( w^{st} \) is an involution we only need to check that these biletters do not become fixed points. Such a biletter has between
\[ A = \sum_{l<i} (a_l + b_l + c_{lk} + d_{lk}) + b_i + \sum_k d_{ik} + \sum_{k<j} c_{ki} \]
and
\[ B = \sum_{l<i} (a_l + b_l + c_{lk} + d_{lk}) + b_i + \sum_k d_{ik} + \sum_{k<j} c_{ki} + c_{ij} - 1 \]
biletters in front. After standardisation, exactly the same formula holds with \( i \) swapped with \( j \). We see that the top and bottom letters will never get the same number via standardisation (in fact if \( i < j \) then \( i \) will become a smaller number than what \( j \) becomes).
Exactly the same analysis holds for a biletter of the form \((i, j)\) and \(i \neq j\). \(\square\)

As an example, let \(w\) be the colored involution

\[
  w = \left(\begin{array}{cccccccc}
  1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 5 \\
  3 & 3 & 2 & 2 & 2 & 1 & 1 & 5 & 4 \\
\end{array}\right)
\]

with 3 barred fixed points, 2 two-cycles and 1 barred two-cycle. Then its standardisation

\[
  w^\text{st} = \left(\begin{array}{cccccccc}
  1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
  6 & 7 & 5 & 3 & 2 & 1 & 2 & 9 & 8 \\
\end{array}\right)
\]

has 1 barred fixed point, 2 two-cycle and 2 barred two-cycles.

**Theorem 28.** Let \(r \geq 0\) be fixed. Let \(S(X; a, b, c, q^{1/2}) \in \tilde{\Lambda}[X][a, b, c, q^{1/2}]\) be the symmetric power series

\[
  S(X; a, b, c, q^{1/2}) = \sum_{\lambda \in P_r} a^{\omega(\lambda) - \omega(\delta_r)} b^{\omega(\lambda') - \omega(\delta_r)} c^{d(\lambda) - d(\delta_r)} G_\lambda(X; q).
\]

Then \(S(X; a, b, c, q^{1/2})\) does not depend on \(r\) and has a product formula given by

\[
  \prod_i (1 + a q^{1/2} x_i) \prod_i (1 - c q x_i^2) \prod_{i<j} (1 - c x_i x_j) \prod_{i < j} (1 - c x_i x_j).
\]

**Proof.** Semistandard domino tableaux are in one-to-one correspondence with colored involutions by Theorem 3 and Corollary 11. If \(w\) is a colored involution then the shape and spin of \(P^\text{sd}(w)\) is that of \(P^\text{sd}(w^\text{st})\) and thus we may use Proposition 15 and Lemma 27 to calculate the contributions each colored involution makes to the weights \(\omega(\lambda), \omega(\lambda'), d(\lambda)\) and \(\text{sp}(P^\text{sd}(w))\).

Such colored involutions consist of a number of fixed points \((i, i)\) corresponding to the product \(\prod_i 1/(1 - bx_i)\). The barred fixed points \((i, i)\) correspond to the product \(\prod_i (1 + a q^{1/2} x_i) / (1 - c q x_i^2)\) since according to Lemma 27 all but at most one of the barred fixed points of each weight will pair to become a two-cycle upon standardisation. The two-cycles correspond to \(\prod_i 1/(1 - cx_i x_j)\) and the barred two-cycles correspond to \(\prod_{i<j} 1/(1 - c q x_i x_j)\). \(\square\)

There are a number of interesting specialisations. We will set \(r = 0\) for the next few examples.

1. When \(a = b = c = q^{1/2} = 1\), we obtain the square of a well known identity:

\[
  \left(\sum_{\lambda \in P} s_\lambda(X)\right)^2 = \left(\prod_i \frac{1}{(1 - x_i)} \prod_{i<j} (1 - x_i x_j)\right)^2.
\]

2. Substituting \(q^{1/2} = 0\) and using the fact that \(G_\lambda(X; 0) = s_\mu(X)\) for \(\lambda\) which satisfy \(\lambda = 2\mu\) (see [CL]), while \(G_\lambda(X; q) = 0\) for other \(\lambda \in P_0\), we get

\[
  \sum_{\lambda \in P} b^{\omega(\lambda)} c^{\omega(\lambda)} s_\lambda(X) = \frac{1}{\prod_i (1 - bx_i) \prod_{i<j} (1 - cx_i x_j)}.
\]

This is another well known identity which can be proved using growth diagrams for normal RSK.
(3) The case $b = c = 1$ and $a = 0$ picks out the $G$ of the form $G_{2\mu} = H_{\mu}$ and we obtain the first formula of [KLLT]:

$$
\sum_{\lambda} H_{\lambda}(X; q) = \frac{1}{\prod_i (1-x_i) \prod_{i<j} (1-x_ix_j) \prod_{i\leq j} (1-qx_ix_j)}.
$$

(4) The case $a = b = 0$ and $c = 1$ picks out the partitions of the form $2\lambda \vee 2\lambda$ giving us the second formula of [KLLT]:

$$
\sum_{\lambda} H_{\lambda \vee \lambda}(X; q) = \frac{1}{\prod_{i<j} (1-x_ix_j) \prod_{i\leq j} (1-qx_ix_j)}.
$$

Note that while $\sum G_{\lambda}$ over $\lambda \in P_r(n)$ does not depend on $r$, the individual $G_{\lambda}$ can differ greatly. In particular, two partitions $\lambda$ and $\mu$ with the same 2-quotient but with $\bar{\lambda} \neq \bar{\mu}$ may not have the same $G$ function. For example, $G_{(2,2)} = qs_2 + s_{1,1}$ while $G_{(3,1,1)} = q^{1/2}(s_2 + s_{1,1})$. Both $(2,2)$ and $(3,1,1)$ have 2-quotient $\{(1),(1)\}$.

### 6. Ribbon Tableaux

In this last section we make a few remarks concerning which results might be generalised to ribbon tableaux. We refer the reader to [LLT] for the important definitions.

Shimozono and White [SW2] also give a spin-preserving insertion algorithm for standard ribbon tableaux. Unfortunately, they stop short of giving a (spin-preserving) bijection between words and pairs of semistandard tableaux. Nevertheless, the standard correspondence works. It is a spin-preserving bijection between pairs of standard ribbon tableaux and permutations $\pi$ of the wreath product $S_n \wr C_p$. Again the involutions are in bijection with standard ribbon tableaux and thus we obtain a $p$-ribbon analogue of Proposition 19 with an identical proof.

**Proposition 29.** Let $h(n)$ be the polynomial in $q$ defined as

$$
h(n) = \sum_T \text{spin}(T)
$$

where the sum is over all standard ribbon tableaux of size $n$ (and fixed $p$-core). Then $h(n)$ satisfies the recurrence

$$
h(n+1) = (1 + q^{1/2} + \ldots + q^{(p-1)/2})h(n) + n(1 + q + \ldots + q^{p-1})h(n-1)
$$

and has exponential generating function

$$
E_h(t) = \exp \left((1 + q^{1/2} + \ldots + q^{(p-1)/2})t + (1 + q + \ldots + q^{p-1})t^2 - \frac{t^2}{2}\right).
$$

The statistics $o(\lambda)$ and $d(\lambda)$ are no longer suitable for longer ribbons. It seems likely that the statistic

$$
o_k(\lambda) = \# \{i : \lambda_i \equiv k \mod p\}
$$

may be interesting, but we have been unable to find any applications.
As Shimozono and White’s ribbon correspondence can be phrased in terms of growth diagrams, one might hope that a Lemma similar to Lemma 9 can be shown in the same way – this would allow a semistandard ribbon correspondence to be developed. Unfortunately this appears not to be the case, as ribbons may well not ‘bump’ to the next column or row but quite far away. This phenomenon occurs for certain longer ribbons regardless of whether we insist upon column or row insertion/bumping.

Possibly more promising is the following potential generalisation. The sums over standard Young tableaux of size $n$

$$\sum_T 1 = t(n)$$

$$\sum_T \text{sign}(T) = 2^{\lfloor n/2 \rfloor}$$

suggest that we might consider the sum

$$\sum_T \chi(\text{reading}(T))$$

for some other character $\chi$ of $S_n$. If this were to be related to $p$-ribbon tableaux and the wreath product $S_n \wr C_p$ then $\chi$ should take $p^{th}$ roots of unity as its values. One possibility is the (virtual) character which on the conjugacy class of cycle type $\lambda$ takes the value

$$\chi(C_\lambda) = \omega^{\lambda - l(\lambda)}$$

for some $p^{th}$ root of unity $\omega$.

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Department of Mathematics, M.I.T., Cambridge, MA 02139

E-mail address: thomasl@math.mit.edu