WHAT IS ACTUALLY A METRIC GRAPH?

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Abstract. Metric graphs are often introduced based on combinatorics, upon “associating” each edge of a graph with an interval; or else, casually “gluing” a collection of intervals at their endpoints in a network-like fashion. Here we propose an abstract, self-contained definition of metric graph. Being mostly topological, it doesn’t require any knowledge from graph theory and already determines uniquely several concepts that are commonly and unnecessarily defined in the literature. Nevertheless, many ideas mentioned here are folklore in the quantum graph community: we discuss them for later reference. They can easily be extended to related settings, like hypergraphs and simplicial complexes.

1. Metric graphs as quotient spaces

Let $E$ be a countable set. Given some $(\ell_e)_{e \in E} \subset (0, \infty)$, we consider the family $[0, \ell_e]_{e \in E}$ of metric measure subspaces of $\mathbb{R}$ (wrt Euclidean metric $d_e$ and Lebesgue measure $\lambda_e$) and their disjoint union

$$
E := \bigsqcup_{e \in E} [0, \ell_e]:
$$

we adopt the usual notation $(x, e)$ for the element of $E$ with $x \in [0, \ell_e]$ and $e \in E$.

We endow $E$ with the disjoint union topology: by definition, this means that a subset $U$ of $E$ is open if and only if its preimage $\varphi_e^{-1}(U)$ is open in $[0, \ell_e]$ for each $e \in E$, where $\varphi_e$ is the canonical injection $\varphi_e : [0, \ell_e] \ni x \mapsto (x, e) \in E$. Hence a set $U$ is open if and only if each $\varphi_e^{-1}(U)$ is a union of sets of the form $[0, \varepsilon_i], (\varepsilon_2, \ell_e]$, or $(\varepsilon_3, \varepsilon_4]$, for $\varepsilon_i \in (0, \ell_e)$. Disjoint unions of such sets thus form a basis of the topology of $E$.

The disjoint union topology of $E$ is metrizable and indeed it agrees with the topology induced by the (generalized) metric defined by setting

$$
d_E((x, e), (y, f)) := \begin{cases} 
d_e(x, y) = |x - y|, & \text{if } e = f \text{ and } x, y \in [0, \ell_e], \\ \infty, & \text{otherwise}. \end{cases}
$$

Consider the set

$$
\mathcal{V} := \bigsqcup_{e \in E} \{0, \ell_e\}
$$

of endpoints of $E$. Given any equivalence relation $\sim$ on $\mathcal{V}$, we extend it to an equivalence relation on $E$ by equality: i.e., two elements $(x_1, e_1), (x_2, e_2) \in E$ belong to the same equivalence class if and only if $(x_1, e_1) = (x_2, e_2)$ or else $(x_1, e_1), (x_2, e_2) \in \mathcal{V}$ and $(x_1, e_1) \sim (x_2, e_2)$. With an abuse of notation we denote this equivalence relation on $E$ again by $\sim$: this allows us to introduce quotient sets.

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Definition 1.1. We call \( \mathcal{G} := \mathcal{E} / \sim \) a metric graph and \( \mathcal{V} := \mathcal{V} / \sim \) its set of vertices.

Remark 1.2. This setting can be slightly generalized by considering an additional countable set \( E_\infty \) and replacing the sets \( E \) and \( V \) studied above by

\[
\mathcal{E} := \bigcup_{e \in E} [0, \ell_e] \sqcup \bigcup_{e \in E_\infty} [0, \infty)
\]

and

\[
\mathcal{V} := \bigcup_{e \in E} \{0, \ell_e\} \sqcup \bigcup_{e \in E_\infty} \{0\},
\]

respectively. In this way we can add semi-infinite leads to a metric graph consisting of a “countable core of bounded edges”.

According to this definition, a metric graph is uniquely determined by a family \( (\ell_e)_{e \in E} \) and an equivalence relation on \( \mathcal{V} \). Its vertices are the cells of the partition of \( \mathcal{V} \) induced by \( \sim \). Two vertices \( v, w \in \mathcal{V} \) are said to be adjacent if there exists some (not necessarily unique) \( e \in E \) such that \( \{x, y\} = \{0, \ell_e\} \) for representatives \( x \) of \( v \) and \( y \) of \( w \); in this case we write \( v \sim w \) and, with an abuse of notation, also \( v \sim e \). The cardinality \( \deg(v) \) of the set \( \{w \in \mathcal{V} : w \text{ is adjacent to } v\} \) is called degree of \( v \in \mathcal{V} \); \( \mathcal{G} \) is called combinatorially locally finite if \( \deg(v) < \infty \) for all \( v \in \mathcal{V} \), and metrically locally finite if \( \sum_{e \in E} \ell_e < \infty \) for all \( v \in \mathcal{V} \).

To justify Definition 1.1, we are going to show how \( \mathcal{G} \) can be canonically endowed with a metric. Following [BII01, Def. 3.1.12] we introduce the quotient pseudo-metric defined by

\[
d_\mathcal{G}(\xi, \theta) := \inf \sum_{i=1}^k d_e(\xi_i, \theta_i), \quad \xi, \theta \in \mathcal{G},
\]

where the infimum is taken over all \( k \in \mathbb{N} \) and all pairs of \( k \)-tuples \( (\xi_1, \ldots, \xi_k) \) and \( (\theta_1, \ldots, \theta_k) \) with \( \xi = \xi_1, \theta = \theta_k, \) and \( \xi_i \sim \xi_{i+1} \) for all \( i = 1, \ldots, k-1 \). We call \( d_\mathcal{G} \) the path pseudo-metric of \( \mathcal{G} \).

Definition 1.3. A metric graph is connected if the path pseudo-metric doesn’t attain the value \( \infty \).

Remark 1.4. While \( d_\mathcal{G} \) is a priori only a pseudo-metric, it is actually a (generalized) metric (i.e., \( d_\mathcal{G}(\xi, \theta) = 0 \) implies \( \xi = \theta \); but the value \( \infty \) can still be attained), which we call the path metric of \( \mathcal{G} \), if \( E \) is finite or, more generally, if \( \inf_{e \in E} \ell_e > 0 \).

Alternatively, consider the doubly connected part \( \mathcal{G}_d \) of \( \mathcal{G} \), i.e., the set of all \( (x, e), x \in (0, \ell_e) \), whose removal doesn’t turn \( \mathcal{G} \) into a disconnected metric graph. Let us assume \( \mathcal{G}_d \neq \emptyset \) and denote by \( E_d \) the set of its edges. Then \( d_\mathcal{G} \) is a metric if \( \inf_{e \in E_d} \ell_e > 0 \).

A connected metric graph is hence a metric space. Furthermore, \( d_E \) is the disjoint union length (pseudo-)metric, in the sense of [BII01, Def. 3.1.15]; hence \( d_\mathcal{G} \) is actually a length (pseudo-)metric, thus any connected metric graph is a length metric space (in the sense of [Stu06]); and even a geodesic space (again in the sense of [Stu06]) whenever \( \inf_{e \in E} \ell_e > 0 \).

In the latter case, the metric space \( \mathcal{G} \) is also complete, hence a Polish space.

The topology of \( \mathcal{G} \) induced by the pseudo-metric \( d_\mathcal{G} \) is easily described: a basis of this topology consists of open balls wrt to \( d_\mathcal{G} \), i.e., of sets that are either open subintervals of
let $[0, \ell_e]'s ("open subsets of edges") or – up to gluing wrt $\sim$ – disjoint unions of semi-open subintervals of $[0, \ell_e]'s ("open stars centred at vertices").

Remark 1.5. $\mathcal{G}$ canonically becomes a topological space whenever endowed with the quotient topology\footnote{i.e., a subset of $\mathcal{G}$ is open if and only if it consists of equivalence classes whose union is open in $\mathcal{E}$; it also follows that a subset of $\mathcal{G}$ is closed if and only if it consists of equivalence classes whose union is closed in $\mathcal{E}$.}. A basis of the topology of $\mathcal{G}$ is then given by images under the canonical surjection of elements of a basis of $\mathcal{E}$: in particular, disjoint unions of open subintervals of $[0, \ell_e]'s ("open subsets of edges") and – up to gluing wrt $\sim$ – disjoint unions of semi-open subintervals of $[0, \ell_e]'s ("open stars centred at vertices")\footnote{It is known that given $q : \mathcal{E} \to \mathcal{G}$ and given a basis $\mathcal{B}$ of the topology of $\mathcal{E}$, $q(\mathcal{B})$ is a basis of the topology of $\mathcal{G}$ if and only if $q$ is open, which is of course especially the case if $q$ is the canonical surjection.}. Hence, the canonical quotient topology on $\mathcal{G}$ coincides with the topology induced by the path pseudo-metric on $\mathcal{G}$.

Finally, $\mathcal{G}$ is clearly a measure space with respect to the direct sum measure $\mu = \bigoplus_{e \in \mathcal{E}} \lambda_e$ \cite{Fre03, 214K}; this measure space is finite if $\mathcal{G}$ has finite volume, i.e., if $\mu(\mathcal{G}) = \sum_{e \in \mathcal{E}} \ell_e < \infty$. A sufficient condition for $\mathcal{G}$ to be a metric measure space in the sense of \cite[§ 3]{Stu06} is that $\inf_{e \in \mathcal{E}} \ell_e > 0$.

Remark 1.6. Let $\iota := q \circ \partial$, where $\partial : \mathcal{E} \ni e \mapsto (0, \ell_e) \in \mathcal{V}^2$ and $q$ is the canonical extension to $\mathcal{V}^2$ of the canonical surjection $q : \mathcal{V} \to \mathcal{V}$ defined by $q(x, y) := \{q(x), q(y)\}$: the latter set may thus consist of either one or two elements of $\mathcal{V}$.

Then the triple $\mathcal{G} := (\mathcal{V}, \mathcal{E}, \iota)$ is a multigraph (recall that, by definition, a multigraph may have loops and parallel edges, s. \cite[§ 1.10]{Die05}): we call it the combinatorial multigraph underlying $\mathcal{G}$. The (pseudo)metric on $\mathcal{G}$ induces a (pseudo)metric on $\mathcal{G}$ – in fact, the canonical one commonly used in graph theory.

2. Graph surgery

Definition 2.1. Let $\mathcal{G}$ be a metric graph. Given $\mathcal{E}_0 \subset \mathcal{E}$, consider $\mathcal{E}_0 := \bigcup_{e \in \mathcal{E}_0} [0, \ell_e]$ and the set $\mathcal{V}_0 := \bigcup_{e \in \mathcal{E}_0} \{0, \ell_e\}$ of endpoints of $\mathcal{E}_0$. The metric graph $\mathcal{G}_0 := \mathcal{E}_0 / \sim_0$ is said to be a metric subgraph of $\mathcal{G} := \mathcal{E} / \sim$, where $\sim_0$ is the restriction of $\sim$ to $\mathcal{V}_0$.

A connected component of a metric graph $\mathcal{G}$ is a metric subgraph $\mathcal{G}_0$ of $\mathcal{G}$ that is maximal (wrt to $\subset$ for $\mathcal{E}_0$) among connected ones.

Definition 2.2. Let $\mathcal{G}$ be a metric graph. Let $\approx$ be a further equivalence relation on $\mathcal{V}$. Then $\mathcal{G} := \mathcal{E} / \approx$ is called a rewiring of $\mathcal{G} = \mathcal{E} / \sim$; and a cut (resp., non-trivial cut) of $\mathcal{G}$ if $\approx$ is coarser (resp., strictly coarser) than $\sim$.

Any function $f : \mathcal{G} \to \mathcal{K}$ canonically induces a function $\hat{f} : \mathcal{E} \to \mathcal{K}$, or equivalently a family $\hat{f} = (f_e)_{e \in \mathcal{E}}$ with $f_e : [0, \ell_e] \to \mathcal{K}$ for all $e \in \mathcal{E}$ (and hence on each rewiring of $\mathcal{G}$, and especially on each of its cuts); the converse is wrong, though, since the boundary values of $f$ in $\mathcal{V}$ may conflict with the equivalence relation $\sim$ that defines $\mathcal{G}$.
We regard a cut \( \hat{\mathcal{G}} \) as a new metric graph obtained by cutting through some vertices of \( \mathcal{G} \). By definition, any cut of \( \mathcal{G} \) shares with \( \mathcal{G} \) its edge set \( \mathcal{E} \): this can be limiting in certain situations and suggests to introduce the following.

**Definition 2.3.** Let \( \mathcal{G} \) be a metric graph and \( \hat{\mathcal{E}} \) be a countable set such that there exists a surjection \( \varsigma : \hat{\mathcal{E}} \to \mathcal{E} \). Given a vector \( (\ell_e)_{e \in \hat{\mathcal{E}}} \) and an equivalence relation \( \sim \) on \( \hat{\mathcal{V}} = \bigcup_{e \in \hat{\mathcal{E}}} \{0, \ell_e\} \), consider the natural extension of \( \sim \) wrt an induced surjection \( \varsigma : \hat{\mathcal{E}} := \bigcup_{e \in \hat{\mathcal{E}}} [0, \ell_e] \to \mathcal{E} \): given \( x, y \in \hat{\mathcal{E}}, x \sim y \) if \( \varsigma(x) = \varsigma(y) \).

Then the metric graph \( \hat{\mathcal{G}} := \hat{\mathcal{G}} / \sim \) is called a **subdivision** of \( \mathcal{G} \) if for all \( e \in \mathcal{E} \) the set \( \varsigma^{-1}(e) \) can be enumerated in such a way, say \( \varsigma^{-1}(e) = \{e_1, \ldots, e_{k_e}\} \), that

- \( (0, e) = (0, e_1), (\ell_{e_1}, e_1) \sim (0, \ell_{e_2}), \ldots, (\ell_{k_e-1}, k_e - 1) \sim (0, k_e), (\ell_{k_e}, k_e) = (\ell_e, e) \)
- \( \sum_{j=1}^{k_e} \ell_{e_j} = \ell_e \)

Given a connected metric graph \( \mathcal{G} \), any two subdivisions of \( \mathcal{G} \) are isometric metric spaces.

Roughly speaking, \( \varsigma(\hat{e}) = e \) if \( \hat{e} = e_i \) for some \( i \in \{1, \ldots, k_e\} \) (i.e., if \( e \) is the edge that has been split to produce \( e_1, \ldots, e_{k_e} \), one of which is precisely \( \hat{e} \)); and \( x \sim y \) if \( x, y \) are representatives of a new vertex that has been created in \( \hat{\mathcal{G}} \) inside the edge \( [0, \ell_e] \). Again, each function \( f = \bigoplus_{e \in \mathcal{E}} f_e : \mathcal{G} \to \mathbb{K} \) canonically induces a new function \( \hat{f} = \bigoplus_{e \in \varsigma^{-1}(\mathcal{E})} f_e \) on any subdivision \( \hat{\mathcal{G}} \), but the converse is generally wrong.

**Remark 2.4.** Given a subdivision \( \mathcal{G}' \) of a metric graph \( \mathcal{G} \), the equivalence relation \( \sim \) that defines \( \mathcal{G} \) can be canonically identified with the the equivalence relation \( \sim' \) that defines \( \mathcal{G}' \); hence, the set of all equivalence relations on \( \mathcal{V} \) can be canonically embedded in the class of all equivalence relations on \( \mathcal{V}' \). Therefore, given two different subdivisions \( \mathcal{G}', \mathcal{G}'' \) of \( \mathcal{G} \), there is always a new subdivision whose vertex set contains all vertices of both \( \mathcal{G}' \) and \( \mathcal{G}'' \) (this defines a partial ordering on the set of subdivisions of \( \mathcal{G} \)).

In the literature, surgery of metric graphs has been frequently performed according to these rules: metric graphs arising by cutting through vertices of \( \mathcal{G} \) in the sense of [BKKM19, Def. 3.2] are non-trivial cuts of subdivisions of \( \mathcal{G} \), in the language of the present note; whereas metric graphs arising by transplantation (and especially unfolding) as in [BKKM19, Def. 3.15 and Def. 3.16] are rewirings of subdivisions of \( \mathcal{G} \). (Non-trivial) symmetrisations of edges in [BKKM19, Def. 3.17], on the other hand, can not be described in terms of a subdivision’s rewirings or cuts.

**Definition 2.5.** Let \( \mathcal{G} \) be a metric graph. We call any metric graph arising from a rewiring or cut of a subdivision of \( \mathcal{G} \) as a **rearrangement** of \( \mathcal{G} \).

While comparing rearrangements \( \mathcal{G}_1, \mathcal{G}_2 \) of \( \mathcal{G} \) we can certainly assume without loss of generality that they are rewiring or cuts of the same subdivision \( \hat{\mathcal{G}} = \hat{\mathcal{G}} / \sim \) of \( \mathcal{G} \). While rearrangements of \( \mathcal{G} \) generally have a different metric, given any two equivalence relations \( \approx_1 \) and \( \approx_2 \) on \( \hat{\mathcal{V}} \) and the associated canonical surjections \( q_1 : \hat{\mathcal{E}} \to \mathcal{G}_1 \) and \( q_2 : \hat{\mathcal{E}} \to \mathcal{G}_2 \), the set-valued map

\[
Q_{12} : q_1 \circ q_2^{-1} : \mathcal{G}_2 \to \mathcal{G}_1
\]
allows us to identify points in the metric graphs $G_1 := \hat{E}/\approx_1$ and $G_2 := \hat{E}/\approx_2$.

3. Function spaces

Given two metric graphs $G_1, G_2$, we write $G_1 \equiv G_2$ if both $G_1, G_2$ are subdivisions of the same metric graph $\mathfrak{G}$. Now, $\equiv$ is an equivalence relation on the set of all metric graphs; we call the corresponding equivalence classes $\mathfrak{G} = [\mathfrak{G}]$ primitive metric graphs: i.e., a primitive metric graph is a metric graph modulo removing vertices of degree 2. Whenever considering a continuous function $f$ on a metric graph $G$, there is a uniquely determined continuous function induced by $f$ on any further metric graph belonging to $G = [G]$. It would be appropriate to consider the space of continuous functions $C(G)$, yet in practice the notation $C(G)$ is customary in the literature: this space is isometrically isomorphic to the space of continuous functions supported on any other representative of $[G]$.

Similarly, two functions on $G$ can be identified if they agree up to a Lebesgue null set. Accordingly, any measurable $f : G \to \mathbb{K}$ can – up to a Lebesgue null set – be canonically identified with a unique function defined on any rearrangement of $G$: accordingly, the Lebesgue space $L^p(G)$ is isomorphic to $L^p(G')$ for any $p \in [1, \infty]$.

Summing up, we can introduce the function spaces

$$C(G) \text{ and } L^p(G), \quad 1 \leq p \leq \infty$$

and then, recursively, for all $k \in \mathbb{N}$ the spaces

$$C_k(G) := \left\{ f = \bigoplus_{e \in E} f_e \in \bigoplus_{e \in E} C^k([0, \ell_e]) : f^{(h)} := \bigoplus_{e \in E} f_e^{(h)} \in C(G) \text{ for all } 1 \leq h \leq k \right\}$$

and

$$W^{k,p}(G) := \{ f \in L^p(G) : f^{(h)} \in C(G) \text{ for all } 0 \leq h \leq k - 1 \text{ and } f^{(j)} \in L^p(G) \text{ for all } 0 \leq j \leq k \}, \quad 1 \leq p \leq \infty.$$  

Observe that $W^{k,p}(G)$ is the closure of $C_k(G)$ with respect to the norm $\|f\|_{k,p} := \sum_{h=0}^k \|f^{(h)}\|_p$.

4. Graph operations

If two metric graphs $G_1, G_2$ are defined upon the same $E$, they are completely characterized by the equivalence relations $\sim_1, \sim_2$. Accordingly, we can easily define binary operations on metric graphs by means of operations involving $\sim_1, \sim_2$. Recalling that given any binary relation $A \subset \mathcal{V} \times \mathcal{V}$, the equivalence relation generated by $A$ is by definition the intersection of the equivalence relations on $\mathcal{V}$ that contain $A$, we can, e.g., consider

- the intersection of $G_1, G_2$ is the metric graph on $E$ obtained by taking $\sim$ to be $\sim_1 \cap \sim_2$ (this is automatically an equivalence relation!);
- the union of $G_1, G_2$ is the metric graph on $E$ obtained by taking $\sim$ to be the equivalence relation generated by $\sim_1 \cup \sim_2$ (the latter is automatically reflexive and symmetric).
Example 4.1. Take $\mathcal{G}_1$ to be a cycle consisting of two edges; and $\mathcal{G}_2$ to be the disconnected graph consisting of two loops, each consisting of one edge. In the above formalism, they are modeled by taking $E = \{1, 2\}$ and, for any $\ell_1, \ell_2 \in (0, \infty)$, by the equivalence relations

\[
\begin{array}{cccc}
(0, 1) & (\ell_1, 1) & (0, 2) & (\ell_2, 2) \\
(\ell_1, 1) & \times & \times & \\
(0, 2) & & \times & \times \\
(\ell_2, 2) & \times & \times & \\
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
(0, 1) & (\ell_1, 1) & (0, 2) & (\ell_2, 2) \\
(\ell_1, 1) & \times & \times & \\
(0, 2) & & \times & \times \\
(\ell_2, 2) & \times & \times & \\
\end{array}
\]

respectively. Their intersection and union are given by the equivalence relations

\[
\begin{array}{cccc}
(0, 1) & (\ell_1, 1) & (0, 2) & (\ell_2, 2) \\
(\ell_1, 1) & \times & \times & \\
(0, 2) & & \times & \times \\
(\ell_2, 2) & \times & \times & \\
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
(0, 1) & (\ell_1, 1) & (0, 2) & (\ell_2, 2) \\
(\ell_1, 1) & \times & \times & \\
(0, 2) & & \times & \times \\
(\ell_2, 2) & \times & \times & \\
\end{array}
\]

respectively, i.e., they correspond to two disjoint intervals and to the figure-8 graph, respectively.

Remark 4.2. Following the above path, we can also define the complement of $\mathcal{G}_2$ in $\mathcal{G}_1$ as the metric graph on $E$ obtained by taking $\sim$ to be the equivalence relation generated by $\sim_1 \setminus \sim_2$; and, canonically, the complement of $\mathcal{G} := \mathcal{G}_2$ obtained by taking $\mathcal{G}_1$ to be the flower graph (much like in the discrete graph setting, where the canonical ambient graph is the complete one).

Complements of metric graph tend to be trivial, though. Take e.g. a lasso graph: formally, it is given by $E = \{1, 2\}$ and, for any $\ell_1, \ell_2 \in (0, \infty)$, by the equivalence relation

\[
\begin{array}{cccc}
(0, 1) & (\ell_1, 1) & (0, 2) & (\ell_2, 2) \\
(\ell_1, 1) & \times & \times & \times \\
(0, 2) & \times & \times & \\
(\ell_2, 2) & \times & \times & \\
\end{array}
\]

on $\{(0, 1), (\ell_1, 1), (0, 2), (\ell_2, 2)\}$; the equivalence relation generated by its complement yields

\[
\begin{array}{cccc}
(0, 1) & (\ell_1, 1) & (0, 2) & (\ell_2, 2) \\
(\ell_1, 1) & \times & \times & \times \\
(0, 2) & \times & \times & \times \\
(\ell_2, 2) & \times & \times & \times \\
\end{array}
\]

i.e., the complement of the lasso graph is the figure-8 graph. Likewise, the figure-8 graph is also the complement of the cycle (formally consisting of two edges) as well as complement of the disconnected graphs consisting of either two intervals or of two loops.

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