Probing topological states through the exact non-Markovian decoherence dynamics of a spin coupled to a spin bath in real-time domain

Chuan-Zhe Yao\(^1\) and Wei-Min Zhang\(^1\)

\(^{1}\)Department of Physics and Center for Quantum information Science, National Cheng Kung University, Tainan 70101, Taiwan

In this paper, we explore the decoherence dynamics of a probing spin coupled to a spin bath, where the spin bath is given by a controllable 1D transverse-field Ising chain. The 1D transverse-field Ising chain with free-ends boundary condition is equivalent to a modified Kitaev model with non-local Majorana bound states in its topological phase. We find that the probing spin non-Markovian decoherence dynamics can manifest the topological structure of the spin chain. By controlling the external magnetic field on the Ising chain, we find the close relationships between the quantum phase transitions, the topological edge states, and the non-Markovian dynamics in real-time domain. We also investigate the corresponding quantum entanglement dynamics in this topological system.

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I. INTRODUCTION

In condensed matter physics, the transverse-field Ising model not only allows identification of quantum phase transitions \(1\), but also has been experimentally realized through the CoNb\(_2\)O\(_6\) compound \(2\), trapped ions \(3-6\), Mott insulator \(7\), and Rydberg atom \(8\) etc., and therefore it has been widely investigated. On the other hand, the recent experiments with polar molecules and ion chains provide a new direction for the dynamics of many-body systems in real-time domain. In particular, the transverse-field Ising model has been revisited in the investigation of non-equilibrium physics, such as dynamical quantum phase transition, through the time evolution of observables in the transverse-field Ising chain under dynamical quench \(9-12\), the Loschmidt echo of a probing spin homogeneously coupled to a transverse-field Ising chain \(13-15\), and the decoherence dynamics of a transverse-field Ising chain coupled to a thermal bath \(16\). However, most of these investigations are mainly considered for Markov processes, while the dynamics of many open systems are often non-Markovian dominated. In this paper, we shall investigate the real-time non-Markovian dynamics for a transverse-field Ising model in different quantum phases with different initial states, through its coupling to a probing spin.

As it is well known, through the JordanWigner transformation, the transverse-field Ising model can be mapped onto the Kitaev chain model \(15\). In the fermionic representation, the well-known quantum phase transition of the model can be understood as a transition from the weak-pairing BCS regime to the strong-pairing Bose-Einstein condensate regime \(22,23\). The phase diagram can be classified according to the topological order \(22,25\). The weak-pairing phase is topologically non-trivial, while the strong-pairing phase is topologically trivial. Moreover, the Kitaev model possesses Majorana zero modes non-locally separated at the two ends of the open chain in the topologically non-trivial phase. The dynamical behavior of quantum phase transition in the model must be associated with the topologically non-local property of the Majorana zero modes. However, the previous studies only consider the transverse-field Ising model with periodic boundary condition where the Majorana zero modes cannot be manifested, while the solution of the model with free-end boundary condition is determined by a transcendental equation which has not been analytically solved so far.

In this paper, we will modify the transverse-field Ising model such that the local magnetic field does not apply to the last spin of the Ising chain. We find that such modified model can be solved analytically with the free-end boundary condition. Moreover, we can derive the exact master equation of a probing spin coupled to the transverse-field Ising model \(20,34\). Through the investigation of the non-Markovian dynamics of the probing spin coupled to this modified transverse-filed Ising chain, one can probe experimentally how the non-trivially topological properties of the transverse-filed Ising model with free-end boundary condition can be manifested in the real-time domain. A great number of papers have been devoted to the study entanglement close to quantum phase transition, and there have been indications that entanglement is enhanced near the quantum critical point \(35,38\). We also numerically explore the non-Markovian dynamics of the entanglement entropy which shows a diagnostic tool for the study of quantum phase transitions.

The rest of the paper is organized as follows. In Sec.\(\text{II}\) we introduce our modified transverse-field Ising model and study its topological characterization. In Sec.\(\text{III}\) we derive the exact master equation of a probing spin coupled to the modified transverse-field Ising model using the path integral approach in coherent state representation \(26\). In Sec.\(\text{IV}\) we analyze in detail the non-Markovian decoherence dynamics of the probing spin coupled to the modified transverse-field Ising model by investigating the two-time spin-spin correlation function in different phases of the spin chain, different spin-spin chain coupling, different spin-flip energy, and different initial states. The effects of topologically non-local property on the non-Markovian dynamics are also clarified under
II. THE MODIFIED TRANSVERSE-FIELD ISING MODEL AND ITS TOPOLOGICAL STRUCTURE

To probe the topological structure and dynamical quantum phase transition through the non-Markovian decoherence dynamics and entanglement entropy in real-time domain, we couple a probing spin to a modified 1D transverse-field Ising chain as shown in Fig. 1. The Hamiltonian of the system is

\[ H = H_A + H_I + H_B \]

\[ = -\omega_0\sigma_0^z - \eta\sigma_0^z\sigma_1^z - \sum_{j=1}^{N-1} J_j\sigma_j^x\sigma_{j+1}^x - \sum_{j=1}^{N} h_j\sigma_j^z, \]  

(1)

where the first term is the Hamiltonian of the probing spin \(\sigma_0\), the second term is the coupling between the probing spin \(\sigma_0\) and the first spin \(\sigma_1\) in the modified transverse-field Ising chain. The last two terms are the Hamiltonian of the modified 1D transverse-field Ising model, which is an N-spin chain with the nearest coupling \(J_j\) and the local external magnetic field \(h_j\). Different from the previous works on the conventional transverse-field Ising chain with a period boundary condition, here we set the transverse-field Ising chain to have a free-end boundary condition so that its topological features can be presented on the edges. Usually, all the eigenfunctions of the transverse-field Ising model with the free-end boundary condition cannot be solved analytically. Hence, we also set \(h_N = 0\), namely, the local magnetic fields do not apply to the last spin \(\sigma_N\). This modification makes the free-boundary transverse-field Ising chain analytically solvable but the topological structure of the system can still be maintained as we will show later. The spin-flip energy \(\omega_0\) of the probing spin \(\sigma_0\) and the coupling energy \(\eta\) between \(\sigma_0\) and \(\sigma_1\) are controllable. For simplicity, we also set \(J_i = J, h_i = h\) \((i = 1, \cdots, N - 1)\).

By applying the Jordan-Wigner transformation

\[ \sigma_j^+ = (\sigma_j^x + i\sigma_j^y)/2 = c_j \prod_{m<j} e^{-i\pi c_m^\dagger c_m} \]  

(2a)

\[ \sigma_j^- = (\sigma_j^x - i\sigma_j^y)/2 = c_j \prod_{m<j} e^{i\pi c_m^\dagger c_m} \]  

(2b)

\[ \sigma_j^z = 2c_j^\dagger c_j - 1, \]  

(2c)

the total system can be transformed into a fermionic system:

\[ H = \omega_0(2\alpha^\dagger\alpha - 1) - \eta(\alpha^\dagger - \alpha)(c_1^\dagger + c_1) \]

\[ - \sum_{j=1}^{N-1} (Jc_j^\dagger c_{j+1} + Jc_j^\dagger c_{j+1}^\dagger + h c_j^\dagger c_j + H.c.). \]  

(3)

As it is shown, after the Jordan-Wigner transformation, the transverse-field Ising chain is reduced to the Kitaev chain where the hoping and pairing strength are the same \cite{18,23}, except that the on-site chemical potential of the last site is zero as an effect of the modification of the model. We take further a Bogoliubov transformation to the spin chain

\[ b_k = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (u_i k^i c_i + v_i k^i c_i^\dagger) \]  

(4a)

\[ b_k^\dagger = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (v_i^* k^i c_i + u_i^* k^i c_i^\dagger) \]  

(4b)

such that \(H_B\) is diagonalized,

\[ H_B = \sum_k \epsilon_k (b_k^\dagger b_k - b_k b_k^\dagger) \]  

(5)

where the \(b_k\) and \(b_k^\dagger\) are creation and annihilation operators of Bogoliubov quasi-particles (bogoliubons) with the spectrum

\[ \epsilon_k = \begin{cases} J\sqrt{1 + \lambda^2 - 2\lambda \cos \frac{k\pi}{N}} & k \neq 0 \\ 0 & k = 0 \end{cases} \]  

(6)

where \(\lambda = h/J\). The corresponding wavefunctions for the non-zero-energy bogoliubons can be analytically solved

\[ u_{kj} = q_k \left\{ \frac{-J}{\epsilon_k} \sin \left[ \frac{(j-1)k\pi}{N} \right] + \left( 1 - \frac{J\lambda}{\epsilon_k} \right) \sin \frac{jk\pi}{N} \right\}, \]  

(7a)

\[ v_{kj} = q_k \left\{ \frac{-J}{\epsilon_k} \sin \left[ \frac{(j-1)k\pi}{N} \right] - \left( 1 + \frac{J\lambda}{\epsilon_k} \right) \sin \frac{jk\pi}{N} \right\}, \]  

(7b)
and that of the zero-energy bogoliubon is

\[
\begin{aligned}
u_{0j} &= \begin{cases} 
q_0(-\lambda)^{j-1} + \frac{1}{\sqrt{2N}} \sum_{k=0}^{N-1} \frac{1}{\sqrt{\lambda^2 + \gamma^2}} (a_k^\dagger a_k + b_k^\dagger b_k) & j < N \\
q_0(-\lambda)^{N-1} + \frac{1}{\sqrt{2N}} \sum_{k=0}^{N-1} \frac{1}{\sqrt{\lambda^2 + \gamma^2}} (a_k^\dagger a_k + b_k^\dagger b_k) & j = N 
\end{cases} \\
u_{0j} &= \begin{cases} 
q_0(-\lambda)^{j-1} - \frac{1}{\sqrt{2N}} \sum_{k=0}^{N-1} \frac{1}{\sqrt{\lambda^2 + \gamma^2}} (a_k^\dagger a_k + b_k^\dagger b_k) & j < N \\
q_0(-\lambda)^{N-1} - \frac{1}{\sqrt{2N}} \sum_{k=0}^{N-1} \frac{1}{\sqrt{\lambda^2 + \gamma^2}} (a_k^\dagger a_k + b_k^\dagger b_k) & j = N 
\end{cases} 
\end{aligned}
\] (8a)

where \(q_k, q_0\) are the normalization constants. Thus, the Hamiltonian of Eq. (1) can be expressed as

\[
H = -\omega_0(2a^\dagger a - 1) - \sum_{k=0}^{N-1} V_k(a^\dagger a - a_k^\dagger a_k + b_k^\dagger b_k) + \sum_{k=0}^{N-1} \epsilon_k(2b_k^\dagger b_k - 1),
\] (9)

with

\[
V_k = \begin{cases} 
-2\eta \lambda \sin \frac{k\pi}{N} & k \neq 0 \\
\sqrt{N(1 + \lambda^2 + 2\lambda \cos \frac{k\pi}{N})} & k = 0
\end{cases}
\] (10)

The above analytical solutions, Eqs. (8), are in fact a consequence of the modification with \(h_N = 0\). In the case of \(h_N = h\), no such analytical solution has been found in the literature. Meanwhile, our modified transverse-field Ising chain has some different characters from the Kitaev chain. The difference is manifested first in the spectra of the model with and without setting \(h_N = 0\), as shown in Fig. 2(a) and Fig. 2(b), respectively. The resulting excited state spectra are similar for the two cases, but their ground state behavior is very different. The modified model always has zero-energy states, independent of the value of \(\lambda\), while the zero-energy states only exist in the region of \(\lambda < 1\) for the ordinary transverse-field Ising chain (or the equivalent Kitaev chain).

Secondly, the wavefunction distribution of the ground states in the two cases are also significantly different in the region of \(\lambda > 1\), as shown in Fig. 2(c) and (d). Note that the zero-energy bogoliubon state \((k = 0)\) is two-fold degenerate states, with the particle number \(b_0^\dagger b_0 = 0\) and 1, respectively. These states can be described by the Majorana zero modes \(\gamma_L = -i(b_0 - b_0^\dagger), \gamma_R = b_0 + b_0^\dagger\).

Figure 2(c) demonstrates the non-local separation of the two Majorana zero modes in the topologically non-trivial phase \((\lambda < 1)\). In particular, if \(\lambda\) equals 0, these two Majorana zero modes locate perfectly at end of the two sides of the spin chain, just the same as that in Kitaev model \([9, 18]\). As \(\lambda\) gets larger and larger, the left Majorana zero mode spreads into other sites, while the right Majorana zero mode remains unchanged due to the setting \(h_N = 0\) as shown in Fig. 2(c). At the critical point \(\lambda_c = 1\), the left Majorana zero mode is uniformly distributed in the all sites of the chain. However, for the Kitaev chain, the wavefunction of the left and right Majorana zero modes are symmetrically distributed over the chain for \(\lambda < 1\), see Fig. 2(d). Through comparing Fig. 2(c) and Fig. 2(d), we find that the wavefunction distributions of the left Majorana zero mode are the same for \(\lambda < 1\) in both cases. When \(\lambda > 1\), for the modified transverse-field Ising chain, the wavefunction of the left Majorana zero mode distributes more on the right-hand side than the left-hand side. With continuously increasing \(\lambda\), both the two Majorana zero modes \(\gamma_L, \gamma_R\) will eventually condense to the last site \(N\), and therefore the zero-energy state still exists but no longer has the topological non-local property. This solution \((\lambda > 1)\) of the modified Ising chain is very different from the Kitaev model in which there exists no longer zero-energy state for \(\lambda > 1\).

The above topological properties of the zero-energy states can be understood more comprehensively through calculating the winding number, which is used to identify topological phases of matter \([23, 24, 40, 41]\). To this end, we rewrite the Hamiltonian of the spin chain in Eq. (9)
in the pseudo spin representation:

\[ H_B = \sum_k (c_k^\dagger c_k) P^{-1} \left( \begin{array}{c} z(k) \\ x(k) \end{array} \right) \left( \begin{array}{c} c_k \\ c_k^\dagger \end{array} \right), \]

where

\[ P = \left( \begin{array}{cc} z(k) - \sqrt{z^2(k) + x^2(k)} & x(k) \\ z(k) + \sqrt{z^2(k) + x^2(k)} & x(k) \end{array} \right) \]

and

\[ x(k) = J \sin(k\pi/N) \]
\[ z(k) = J \cos(k\pi/N) - h + (h - j)\delta_{k,0}. \]

The winding number is defined as the line integral along a close curve on the \( z-x \) plane

\[ W = \frac{1}{2\pi} \int \frac{1}{x^2 + z^2} (zdx - xdz), \]

as the total number of times that the curve travels counterclockwise around the origin. Explicit results show that the winding number \( W = 1 \) for \( \lambda < 1 \), which means that the spin chain is in the topologically non-trivial phase, while it is in the topologically trivial phase with \( W = 0 \) for \( \lambda > 1 \), although there is still zero-energy ground state. This demonstrates a quantum phase transition in the modified transverse-field Ising chain associated with the topological non-local feature, namely, a transition from the topologically nontrivial phase to the topologically trivial phase occurs when \( \lambda \) passes through \( \lambda_c = 1 \).

In conclusion, the modified transverse-field Ising chain has the same topological structure as the ordinary model, even through the ground state energy and its wavefunctions behave so different in the two models. It also demonstrates explicitly that the topology of the system is determined by the detailed non-local properties of the zero-mode wavefunctions. Because the modified transverse-field Ising model can be analytically diagonalized, we can also use it to study the exact decoherence dynamics of the system through its coupling to a probing spin, from which the topological structure of the system in decoherence dynamics can be experimentally observed in real-time domain.

III. THE EXACT MASTER EQUATION

The topological structure and quantum phase transition of the modified Ising chain can be explored through the non-Markovian decoherence dynamics of the probing spin, which is determined from the time evolution of the reduced density matrix of the probing spin obtained from the total density matrix of the probing spin coupled with the spin chain by tracing out all possible states of the spin chain

\[ \rho_A(t) = \text{Tr}_B[U(t,t_0)\rho_{tot}(t_0)U^\dagger(t,t_0)], \]

where \( U(t,t_0) = \exp[-iH(t-t_0)] \) is the time evolution operator of the total system. Initially we assume that the two subsystem (spin \( \sigma_0 \) and the spin chain) are decoupled [42-43], that is, \( \rho_{tot}(t_0) = \rho_A(t_0) \otimes \rho_B(t_0) \). Then in the fermionic coherent state representation

\[ \langle \xi_f | \rho_A(t) | \xi_f' \rangle = \int d\mu(\xi_0)d\mu(\xi_0')\langle \xi_0 | \rho_A(0) | \xi_0' \rangle \mathcal{K}(\xi_f^*, \xi_f', t|\xi_0, \xi_0', t_0), \]

where \( \xi_0, \xi_0'^*, \xi_f, \xi_f^* \) are the eigenvalues of the fermionic coherent states and are Grassmann numbers. The propagator \( \mathcal{K}(\xi_f^*, \xi_f', t|\xi_0, \xi_0', t_0) \) is determined by the action of the probing spin \( \sigma_0 \) and the influence functional [42], which can be solved by applying the path integral approach to integrate out all the degree of freedom of the spin chain [26-31]. With a tedious derivation (see appendix A), the exact master equation for the probing spin coupled to the transverse-field Ising chain is obtained

\[ \dot{\rho}(t) = -i[\mathcal{H}(t), \rho(t)] + \gamma(\sigma(t) - \sigma(t)^\dagger) - \gamma(\sigma(t) - \sigma(t)^\dagger) + \gamma(\sigma(t) - \sigma(t)^\dagger) - \gamma(\sigma(t) - \sigma(t)^\dagger) + \Gamma(t)\sigma(t) - \Gamma(t)\sigma(t)^\dagger, \]

where all the time-dependent coefficients are determined by the generalized non-equilibrium Green functions incorporating the pairing dynamics as follows,

\[ \mathcal{A}(t,t_0) = -\mathcal{L}(U(t,t_0)U^{-1}(t,t_0))_{12} \]
\[ \mathcal{G}(t,t_0) = \frac{1}{2} \left[ \mathcal{L}(U(t,t_0)U^{-1}(t,t_0)) + \mathcal{H} \right]_{11}, \]
\[ \mathcal{F}(t,t_0) = \mathcal{V}(t,t_0) - \mathcal{L}(U(t,t_0)U^{-1}(t,t_0)) \mathcal{V}(t,t) + \mathcal{H}, \]
\[ \mathcal{E}(t,t_0) = \frac{i}{2} \mathcal{L}(U(t,t_0)U^{-1}(t,t_0)) - \mathcal{H}, \]

The Green functions \( \mathcal{U}(t,t_0) \) and \( \mathcal{V}(t,t) \) are \( 2 \times 2 \) matrix and satisfy the integro-differential equations [26-34]

\[ \frac{d}{dt} \mathcal{U}(t,t_0) - 2i\omega_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathcal{U}(t,t_0) + \int_{t_0}^{t} \mathcal{G}(t,\tau)\mathcal{U}(\tau,\tau_0) d\tau = 0 \]
\[ \mathcal{V}(t,\tau) = \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau} d\tau_2 \mathcal{U}(\tau,\tau_1)\mathcal{G}(\tau_1,\tau_2)\mathcal{U}(t,\tau_2) \]

with the initial condition \( \mathcal{U}(t_0,t_0) = I \). The integral memory kernels

\[ \mathcal{G}(t,t_0) = 2 \text{Re}[g(t,t_0)] \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \]
\[ \mathcal{G}(t,t_0) = 2 \text{Im}[g(t,t_0)] \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \]
where
\[ g(t, t_0) = \int \frac{d\omega}{2\pi} J(\omega) e^{-i\omega(t-t_0)}, \tag{21a} \]
\[ g_\beta(t, t_0) = \int \frac{d\omega}{2\pi} J(\omega) f(\omega) e^{-i\omega(t-t_0)}, \tag{21b} \]
and \( f(\omega) = [e^{i(\epsilon_k - \mu)} + 1]^{-1} \) is the Fermi-Dirac distribution of the spin chain at initial time \( t_0 \). The spectral density of the system \( J(\omega) = 2\pi \sum_{N=1}^{N} |V_k|^2 \delta(\omega - \epsilon_k) \), where \( V_k \) and \( \epsilon_k \) are given by Eq. \( \text{(10)} \) and Eq. \( \text{(6)} \), respectively, and its exact form is as follows,
\[ J(\omega) = \frac{\eta^2}{\omega} \sqrt{-[\omega^2/4 - J^2(1 - \lambda)^2][\omega^2/4 - J^2(1 + \lambda)^2]} + \left\{ \begin{array}{ll} \pi \eta^2(1 - \lambda^2)\delta(\omega, 0) & \lambda < 1 \\ 0 & \lambda \geq 1 \end{array} \right., \tag{22} \]
and the latter term is caused by the non-local zero modes. Note that when \( \lambda \geq 1 \), the zero-modes have no contribution to the spectral density because the zero-modes move to the right-hand side of the spin chain and therefore decoupled from the probing spin. From Eq. \( \text{(19a)} \) and \( \text{(20a)} \), it shows that the memory kernel is determined by the effective spectral density \( \tilde{J}(\omega) = 2 \Re\{J(\omega)\} \) plotted in Fig. \( \text{3} \). Notice that except for the case of \( \lambda = 1 \), there is a gap in the middle of the effective spectral density, which will induce the localized modes and prevent decoherence \( \text{[30]} \), as we will discuss in detail in the next section.

![FIG. 3: (Colour online) The effective spectral density \( \mathcal{J}(\lambda, \omega) \) from the topologically non-trivial phase \( \lambda < 1 \) to the topologically trivial phase \( \lambda > 1 \).](image)

### IV. THE EXACT NON-MARKOVIAN DYNAMICS

#### A. The analytical solution of the retarded and correlation Green functions

By coupling the probing spin \( \sigma_0 \) with the spin chain (see Fig. \( \text{1} \)), we find that the dynamics of the probing spin will manifest the topological structure of the spin chain. As one has seen, the renormalized Hamiltonian of the probing spin, the dissipation and fluctuation coefficients in its exact master equation Eq. \( \text{(17)} \) are all determined by the Green functions \( U(t, t_0) \) and \( V(t, t) \). The solutions of these two Green functions fully depend on the density of states of the spin chain as well as the coupling between the probing spin and the spin chain through the spectral density of Eq. \( \text{(22)} \). Their physical consequences can be seen more clearly in the Heisenberg picture. Using the Heisenberg equation of motion, after the Jordan-Wigner transformation, the dynamics of the probing spin \( \sigma_0 \) is described by the corresponding fermion operators \( a(t) \) and \( a^\dagger(t) \). Their Heisenberg equation of motions, after eliminating the degrees of freedom of the spin chain, lead to
\[ \frac{d}{dt} a(t) - i\omega_0 a(t) = 2 \int_{t_0}^{t} \Re\{g(t, \tau)\}[a^\dagger(\tau) - a(\tau)] d\tau \]
\[ = \sum_k iV_k [e^{-2i\omega_0} b_k(t_0) + e^{2i\omega_0} b_k^\dagger(t_0)] \tag{23} \]
which is the generalized quantum Langevin equation \( \text{[32]} \), where the third term is the damping and the right-hand side of the equation is the noise force. Due to the linearity of Eq. \( \text{(23)} \), its general solution has the form as
\[ \begin{pmatrix} a(t) \\ a^\dagger(t) \end{pmatrix} = U(t, t_0) \begin{pmatrix} a(t_0) \\ a^\dagger(t_0) \end{pmatrix} + \sum_k F_k(t, t_0) \begin{pmatrix} b_k(t_0) \\ b_k^\dagger(t_0) \end{pmatrix}, \tag{24} \]
where \( a(t_0), a^\dagger(t_0), b_k(t_0) \) and \( b_k^\dagger(t_0) \) are the initial annihilation and creation operators of the probing spin \( \sigma_0 \) and the spin chain, respectively.

From Eq. \( \text{(24)} \), it can easily be shown that
\[ U(t, t_0) = \begin{pmatrix} \langle\{a(t), a^\dagger(t)\}\rangle & \langle\{a(t), a(t)\}\rangle \\ \langle\{a^\dagger(t), a(t)\}\rangle & \langle\{a^\dagger(t), a^\dagger(t)\}\rangle \end{pmatrix} \tag{25} \]
which is indeed an extension of the usual retarded Green function incorporating with pairings. The equation of motion of \( U(t, t_0) \) is given by the integro-differential Eq. \( \text{(19a)} \), which can also be easily justified by substituting Eq. \( \text{(24)} \) into the Heisenberg equation of motion \( \text{(23)} \). As shown in our previous work \( \text{[30]} \), the modified Laplace transform \( \tilde{U}(s) = \int_{t_0}^{\infty} U(t, t_0)e^{i\omega(t-t_0)} \) of Eq. \( \text{(19a)} \) is
\[ \tilde{U}(s) = i \left( s + 2\omega_0 - \Sigma(s) \right) \left( s + 2\omega_0 + \Sigma(s) \right)^{-1} \tag{26} \]
where the self-energy correction \( \Sigma(s) \) is the Laplace transform of the integral kernel in Eq. \( \text{(19a)} \)
\[ \Sigma(s) = \int \frac{d\omega}{2\pi} J(\omega) \Delta(\omega) \equiv \frac{i}{2} \mathcal{J}(\omega), \tag{27} \]
and \( \Delta(\omega) = \mathcal{P} \int \frac{d\omega}{2\pi} \frac{J(\omega)}{s-\omega} \) is the principal value of the integral. Applying the inverse transformation to Eq. \( \text{(26)} \), we
can analytically solve \( U(t, t_0) \), which consists of a summation of dissipationless oscillations (or localized bound states determined by the real part of the self-energy correction) to the probing spin and non-exponential decays induced by the discontinuity in the imaginary part of the self-energy correction \([30]\)

\[
U(t, t_0) = \sum_{s_p} \left( \begin{array}{cc}
X(s_p) & Y(s_p) \\
Y(s_p) & X(-s_p)
\end{array} \right) e^{-is_p(t-t_0)} \\
+ \int_{-\infty}^{\infty} ds \left( \mathcal{J}(s) e^{-is(t-t_0)} + \mathcal{J}(s) e^{is(t-t_0)} \right) \left( \begin{array}{cc}
(s-2\omega) & s \\
-2s & s
\end{array} \right)
\]

where \( \{s_p\} \) is the set of the poles for the determinant of \( \mathcal{U}(s) \) located at the real axis, i.e. \( s - 2\omega - \Delta(s_p) = 0 \), and

\[
\begin{align}
X(s) &= \frac{(s - 2\omega - \Delta(s))^2}{(s - 2\omega - \Delta(s))^2 + \Delta(s)\Delta'(s)(s - 2\omega)^2}, \\
Y(s) &= \frac{\Delta^2(s)}{2\Delta(s)(\Delta(s) - s) + \Delta'(s)(s^2 - 4\omega^2)}.
\end{align}
\]

Both the dissipationless oscillations arose from the localized modes and the non-exponential decays in Eq. \((28)\) are closely related to the structure of the density of states of the spin chain.

On the other hand, \( \{F_k(t, t_0)\} \) in Eq. \((24)\) is the noise source characterizes the noise force, the right-hand side of Eq. \((23)\) associated with the initial operators \( \{b_k(t_0), b_k^\dagger(t_0)\} \) of the spin chain, and obeys the equation of the motion:

\[
\frac{d}{dt} F_k(t, t_0) - 2i\omega_k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F_k(t, t_0) + \int_{t_0}^t G(t, \tau) F_k(\tau, t_0) d\tau = iV_k \left( e^{-2ie_{k\tau}} \tau - e^{2ie_{k\tau}} \tau \right). 
\]

It is easy to find that its general solution is given by

\[
F_k(t, t_0) = iV_k \int_{t_0}^t U(\tau, t_0) \left( e^{-2ie_{k\tau}} \tau - e^{2ie_{k\tau}} \tau \right) d\tau,
\]

which generates the non-equilibrium correlation Green function \( V(t, \tau) \):

\[
\begin{align}
V(t, \tau) &= \sum_k \langle F_k^\dagger(\tau, t_0) \left( b_k(t_0) b_k^\dagger(t_0) \right) F_k(t, t_0) \rangle. \\
\end{align}
\]

The non-equilibrium correlation Green function \( V(t, \tau) \) describes the particle-hole and particle-particle correlations arose from the fluctuations of the spin chain. Notice that if the flipping energy of the probing spin \( \sigma_0 \) equals zero \( (\omega_0 = 0) \), then as shown in our previous work \([34]\), the non-equilibrium Green functions \( U(t, t_0) \) and \( V(\tau, t) \) obey the following identities

\[
U_{11}(t, t_0) = U_{22}(t, t_0), U_{12}(t, t_0) = U_{21}(t, t_0), V_{11}(\tau, t) = V_{22}(\tau, t) = -V_{12}(\tau, t) = -V_{21}(\tau, t),
\]

Thus, the time-dependent dissipation and fluctuation coefficients in the master equation Eq. \((17)\) are reduced to

\[
\gamma(t, t_0) = \gamma(0), \eta(t, t_0) = 0.
\]

that are solely determined by the retarded Green function \( U(t, t_0) \). As a consequence, \( V(t, \tau) \) is not an independent Green function, which indicates that the dynamics process will be independent of the initial state of the spin chain if \( \omega_0 = 0 \).

B. Decoherence dynamics for different phase of the spin chain

Through the relation between the time-dependent dissipation and the fluctuation coefficients in the exact master equation and the non-equilibrium retarded and correlation Green functions, we can analytically solve the non-Markovian dynamics of the probing spin \( \sigma_0 \), from which the topological dynamics of the spin chain can be manifested in the real-time domain that is experimentally measurable. As we also discussed earlier, the spin chain will undergo a quantum phase transition from the topologically non-trivial phase to the topologically trivial phase when the flipping energy \( \lambda \) changes across the critical point \( \lambda_c = 1 \). To understand the manifestation of the topological structure in the real-time non-Markovian decoherence dynamics of the probing spin \( \sigma_0 \), we study the two-time spin correlation

\[
\langle \sigma_0^z(t) \sigma_0^z(t_0) \rangle = \langle 4a^\dagger(t)a(t)a^\dagger(t)a(t_0) \rangle - \langle 2a^\dagger(t)a(t) \rangle + 1
\]

by varying different value of \( \lambda \). For simplicity, we first set \( \omega_0 = 0 \) and \( \eta = J \). The result of the correlation \( \langle \sigma_0^z(t) \sigma_0^z(t_0) \rangle \) with different \( \lambda \) is shown in Fig. \ref{fig:4}(a). Figure \ref{fig:4}(a) shows clearly a critical transition at \( \lambda = 1 \). The two-time correlation keeps oscillation between the positive and negative value in all the time for \( \lambda < 1 \), while although it also oscillates for \( \lambda < 1 \) in the beginning, it will eventually approach to a stationary value.

To understand the underlying mechanism of this transition, we plot the real part of the determinant of \( \mathcal{U}^{-1}(s) \) with different \( \lambda \) in Fig. \ref{fig:4}(b), which determines the localized modes of the probing spin. Notice that there are discontinuous parts (the flat lines) in the function due to the non-zero values of the imaginary part of \( \mathcal{U}^{-1}(s) \) in these regions, and the imaginary part is determined by
the spectral density $J(s)$ in Eq. (22). The discontinuous parts locate exactly in the regions where the spectral density has non-zero value. As we have discussed earlier, the poles $\{s_p\}$ that make $|\hat{U}(s_p)|^{-1} = 0$ form the localized modes and contribute the dissipationless term in Eq. (28). The locations of these poles are the intersection points of $\text{Re}[|\hat{U}(s)|^{-1}]$ and the horizontal axis with the spectral density $J(s) = 0$ (the imaginary part of $|\hat{U}(s)|^{-1}$ vanishes). In other words, the different decoherence dynamics associated with the topological phase transition is determined by these different dissipationless-localized modes. More specifically, we first consider the case of $\lambda = 0$ that the probing spin $\sigma_0$ is only coupled to the left Majorana zero mode because it perfectly locates at the ends of the spin chain, as we have shown in the sec. II. In this case, there are three localized modes (one pole located at 0 and two symmetrically located at the positive and negative sides, as shown in Fig. 3(b)). Then the energy keeps exchange between the probing spin $\sigma_0$ and the zero-energy bogoliubon of the spin chain through the left Majorana zero mode. This leads to the two-time correlation as a cosinusoidal oscillation for $\lambda = 0$, as we can see in Fig. 4(a). Once $\lambda > 0$, the probing spin $\sigma_0$ will couple to not only the left Majorana zero mode but also others with higher energy modes in the spin chain, so its energy will also dissipate to the non-zero continuous modes of the spin chain. This leads to a non-exponential decay given by the latter term in Eq. (28). In fact, in the topologically non-trivial phase ($\lambda < 1$), Fig. 4(b) shows that there are always three localized modes. Therefore, after a short-time decay, the two-time correlation will reduce to a dissipationless oscillation. As $\lambda$ increasing, the decay term will become more and more dominant. When it reaches to the critical point $\lambda = 1$, all the localized modes vanish (see Fig. 4(b)) and Fig. 4(c). Thus, the dissipationless term vanishes in Eq. (28), and the spin correlation shows the maximum decoherence effect. On the other hand, in the topologically trivial phase ($\lambda > 1$), Fig. 4(b) shows that only one localized mode occurs at $s_p = 0$. This leads to the two-time correlation eventually approach to a stationary value (no oscillation). Furthermore, if $\lambda$ keeps increasing, the coupling term $\eta\sigma_0\sigma_1$ between the two subsystems in the total Hamiltonian becomes relatively weak, which results in the two subsystems being loosely affected to each other. As a result, we can see that the dynamics of the two-time correlation becomes more and more stable as $\lambda$ gets lager and larger, as shown in Fig. 4(c).

C. Decoherence dynamics for different spin-spin chain coupling

Notice that the coupling $\eta$ between the two subsystems can significantly affect the non-Markovian decoherence dynamics which is fully determined by the density of states of the spin chain and the coupling between the probing spin and the spin chain through the spectral density Eq. (22). In Fig. 5(a), 5(b), and 5(c), we plot the two-time correlation $\langle \sigma_0(t)\sigma_0(t_0) \rangle$ with $\lambda = 0$ for different values of $\eta$. The critical point ($\lambda = 1$), and the topologically trivial phase ($\lambda = 2$), respectively. To understand these different behaviors of the correlations in different coupling region, we plot again the real part of the determinant of $\hat{U}^{-1}(s)$ with the different corresponding values of $\eta$ in Fig. 5(c), 5(e), and 5(f). Figure 5(d) shows that for the topologically non-trivial phase ($\lambda = 0$), there exist always three
localized modes, which are independent of the value of $\eta$ (except for the trivial case $\eta = 0$). The two-time correlation always shows a cosinusoidal oscillation. Figure 5(d) also shows that the change of $\eta$ will affect the locations of the localized modes, which determine the frequencies of the dissipationless oscillation. Because the stronger the coupling $\eta$ is, the easier it is to exchange energy between the two subsystems and the probing spin is affected from the topologically non-local state of the spin chain, the two-time correlation shows the oscillation with the higher frequency, as shown in Fig. 5(a). At the critical point ($\lambda = 1$), Fig. 5(e) shows that when $\eta < J$, the real part of $|\tilde{U}^{-1}(s)|$ vanishes only at $s = 0$ where the imaginary part has non-zero value (the spectral density $J(0) > 0$, see Fig. 3). Hence, there is no pole (localized mode) in this region, and the two-time correlation decays to zero monotonically, as a typical Markov process. On the other hand, we find from Fig. 5(e) that there are two localized modes when $\eta > J$, and thus the dissipationless oscillation terms get contribution in the two-time correlation. As a result, the dynamics of the probing spin $\sigma_0$ undergoes a transition from a Markovian process in the weak coupling region to a non-Markovian process in the strong coupling region, as shown in Fig. 5(b). For the topologically trivial phase ($\lambda = 2$), Fig. 5(f) show that there is only one localized mode at $s = 0$ in the weak coupling region as we mentioned in 4. But there are three localized modes in the strong coupling region. Note that the intersection points located between $2 < |s| < 6$ in Fig. 5(f) are not poles because the imaginary part of $|\tilde{U}^{-1}(s)|$ has non-zero value in this range, as shown in Fig. 3. In conclusion, the two subsystems exchange energy in the beginning in the weak coupling region ($\eta < J$), and then they reach the qualitatively different steady states for the different topological phases of the spin chain. When the coupling $\eta$ between the two subsystems gets stronger, the probing spin has to take longer time to reach the steady state. While, the two subsystems always maintain energy exchange in the both phases of the spin chain in the strong coupling region ($\eta > J$). Thus, the topological effect of the spin chain to the decoherence dynamics of the probing spin becomes insignificant.

**D. Decoherence dynamics for different spin-flip energy**

In the previous discussion, we only discuss about the case that the probing spin $\sigma_0$ has zero flipping energy ($\omega_0 = 0$), in which the time-dependent coefficients in the master equation are independent of the correlation Green function $V(t, t)$, as shown in Eq. (34). In other words, for $\omega_0 = 0$, the decoherence dynamics is independent of the environmental noise which is associated with the initial state of the spin chain. To have a further understanding of the effect of the spin-flip energy $\omega_0$ of the probing
of the localized modes located in these two regions have contrast behavior as $\omega_0$ increasing, the amplitudes of the localized modes decrease in the region $-|1-\lambda| < s < 2|1-\lambda|$ while they increase in the region $s < -2(1+\lambda)$ or $s > 2(1+\lambda)$, which can be seen clearly from Fig. 7 where we plot the summation of the amplitudes of all localized modes versus $\omega_0$.

Nevertheless, for $\omega_0 \neq 0$, the decoherence dynamics is no longer solely determined by the dissipation from $U(t,t_0)$, but also the fluctuations associated with the initial states of both the probing spin $\sigma_0$ and the spin chain, which is characterized by the correlation Green function $V(t,t)$. To obtain a more comprehensive physical picture, we plot the two-time correlation $\langle \sigma_0(t)\sigma_0(t_0) \rangle$ with different values of $\omega_0$ and different initial states of $\sigma_0$ for the spin chain in the topologically non-trivial phase ($\lambda = 1/2$), the critical point ($\lambda = 1$), and the topologically trivial phase ($\lambda = 3$) in Fig. 8(a), 8(b), and 8(c), respectively. The results show that if the probing spin $\sigma_0$ is initially in the high-energy (spin-down) state, its energy will be dissipated into the spin chain, and it tends to decay to the low-energy (spin-up) state. If the probing spin $\sigma_0$ is initially in the low-energy state, it exchanges little energy with the spin chain and will most likely remain in the low-energy state. The larger the value of $\omega_0$, the more apparent this phenomenon in the region $\omega_0 < \omega_1$, where the localized modes are all located between $-2|1-\lambda| < s < 2|1-\lambda|$. However, for $\omega_0 > \omega_2$, the amplitudes of the localized modes become large with $\omega_0$ increase, so that even the probing spin in the high-energy state becomes more hardly to dissipate its energy into the spin chain (see the dashed green lines in Fig. 8(a) and Fig. 8(b)). Moreover, for $\omega_0 \neq 0$, the initial state of the spin chain also affect on the decoherence dynamics of the probing spin $\sigma_0$. The spin chain is assumed initially in thermal equilibrium state. We plot again the two-time correlation for different phases with the initial finite tem-

$\begin{array}{cccc}
\lambda < J & \lambda = J & \lambda > J \\
\eta \leq J, \omega_0 = 0 & 3+0 & 0+0 & 1+0 \\
\eta \leq J, 0 < \omega_0 < \omega_1 & 3+0 & \times & 2+0 \\
\eta \leq J, \omega_1 < \omega_0 < \omega_2 & 1+0 & 0+0 & 0+0 \\
\eta \leq J, \omega_2 < \omega_0 & 1+2 & 0+2 & 0+2 \\
\eta > J, \omega_0 = 0 & 3+2 & 0+2 & 1+2 \\
\eta > J, 0 < \omega_0 < \omega_1 & 3+2 & \times & 2+2 \\
\eta > J, \omega_1 < \omega_0 & 1+2 & 0+2 & 0+2 \\
\end{array}$

TABLE I: The number of localized modes with different values of $\omega_0$, $\eta$, and $\lambda$. 

spin $\sigma_0$ and the consequence of the initial dependence of the spin chain, we first plot the real part of the determinant of $U^{-1}(s)$ again for topologically non-trivial phase, topologically trivial phase, and critical point in different coupling regions, with different values of $\omega_0$ in Fig. 6. Figure 6 shows that the increase of $\omega_−$ located between a zero which may affect the number of localized modes, makes the locations of the localized modes away from different coupling regions with different values of $\omega$ phase, topologically trivial phase, and critical point in $\omega_s<$. 

FIG. 6: (Colour online) The inverse of the determinant of $U(s)$ with different values of $\omega_0$ in both phases and different coupling regions.

FIG. 7: (Colour online) The summation of the amplitudes of all localized modes versus $\omega_0$ in each phases and coupling regions.
temperature $T = 5J/k_B$ of the spin chain in Fig. 8(d), 8(e), and 8(f). The results show that for the high temperature, the two diagonal terms of the correlation Green function $V_{11}(t,t)$ and $V_{22}(t,t)$ are similar. As a result, the two-time correlations of the two initial states become closer to each other, as shown in Fig. 8(d), 8(e), and 8(f). In other words, if the spin chain is initially at a relatively high temperature, the dependence of the dynamics on the initial state of $\sigma_0$ will diminish because the thermal fluctuation dominates the decoherence dynamics.

Putting all the above analyses together, we find that all the parameters $\lambda$, $\eta$, and $\omega_0$ can induce different number and amplitudes of the localized modes and therefore affect the non-Markovian memory effect. In particular, for $\omega_0 = 0$ and in the weak coupling region, the topological phase transition can be significantly manifested in the dissipation dynamics of the probing spin $\sigma_0$. In other words, the topological structure of the spin chain can be observed through the non-Markovian decoherence dynamics of the probing spin $\sigma_0$. On the other hand, in the strong region, the topological non-local state more strongly coupled to the probing spin so that the topological effect in the non-Markovian dynamics becomes more significant, as shown in Fig. 5. However, for $\omega_0 \neq 0$, the noise effect gets involved into the decoherence dynamics, which is strongly correlated with the initial state of spin $\sigma_0$ and the initial temperature of the spin chain. As a result, the manifestation of the topological structure of the spin chain on the non-Markovian decoherence dynamics of the probing spin $\sigma_0$ is merged. Hence, we propose a experimental probe of the topological structure of the spin chain through the decoherence dynamics of an external spin $\sigma_0$ coupling weakly to the spin chain at low temperature.

V. THE DYNAMICS OF ENTANGLEMENT ENTROPY

In this section, we study the dynamics of the quantum entanglement. In the static case, the behavior of the entanglement has a universal character that the entanglement of the system state would be enhanced near a quantum phase transition and reach the maximum at the critical point. Therefore, it can be used as an estimator of quantum correlations [44] and as a detector to classify quantum phase transitions [35,38,45,46]. It is also interesting to see how entanglement developed in time when the system is far away from the equilibrium state or ground state. Therefore, we would like to further investigate the relation between the entanglement and quantum phase transitions in the non-equilibrium region.

The entanglement between the probing spin $\sigma_0$ and the spin chain can be characterized by the von Neumann entropy $S_A(t) = -\text{Tr} \rho_A(t) \ln \rho_A(t)$. The reduced density matrix $\rho_A(t)$ of the probing spin $\sigma_0$ that obeys the master equation Eq. (17) can be expressed as

\begin{align}
(\rho_A)_{11}(t) &= V_{22}(t,t) + U_{12}(t,t_0)U_{21}(t,t_0)(a^\dagger(t_0)a(t_0))
+ U_{11}(t,t_0)U_{22}(t,t_0)(a(t_0)a^\dagger(t_0)) \\
(\rho_A)_{22}(t) &= V_{11}(t,t) + U_{11}(t,t_0)U_{22}(t,t_0)(a^\dagger(t_0)a(t_0))
+ U_{12}(t,t_0)U_{21}(t,t_0)(a(t_0)a^\dagger(t_0)) \\
(\rho_A)_{12}(t) &= (\rho_A)_{21}(t) = U_{11}(t,t_0)(a(t_0))
+ U_{12}(t,t_0)(a^\dagger(t_0)).
\end{align}

Moreover, we find that there is a relation between the entanglement entropy $S_A(t)$ and the two-time correlation $\langle \sigma_0^z(t)\sigma_0^z(t_0) \rangle$ if the initial state of $\sigma_0$ is a pure state,
For $\lambda$ maximum value of such entanglement entropy oscillation state due to the existence of localized modes \[30\]. The $\sigma$ spin always keeps oscillating, which means that the probing from 0 in Fig. 9(a). For $\lambda < S$ spin $\sigma$ no entanglement between $\sigma$ be initially in a pure state. In the beginning, there is entropy $S_{\sigma}$ mentioned in Sec. III. the probing spin independent of the initial state of the spin chain, as we we $V_{\lambda}$ topologically non-trivial phase ($\lambda < 1$) to the topologically trivial phase ($\lambda > 1$) is shown in Fig. 10(a), 10(b), and 10(c), respectively. The results show that the different initial states of the probing spin $\sigma_0$ result in total different behaviors of the entanglement entropy, particularly for the high spin-flip energy $\omega_0$. This is because the probing spin $\sigma_0$ with the high-energy initial state is more favor to decay than that with the low-energy initial state. For a small value of $\omega_0$, the probing spin $\sigma_0$ initially in the low-energy state tends to decay to the mixed state. As a result, we can see in Fig. 10 that for $\omega_0 = 0.5J$ (red lines) and $\omega_0 = 2J$ (green lines), the probing spin $\sigma_0$ with the low-energy initial state $| \uparrow \rangle$ is less favor to be entangled with the spin chain, while the probing spin $\sigma_0$ with the high-energy initial state $| \downarrow \rangle$ is more favor to entangle with the spin chain. On the other hand, for a large value of $\omega_0$, the probing spin $\sigma_0$ initially in the low-energy state still remains in its initial state, while the spin initially in the high-energy state tends to decay to the low-energy state. In both cases, the probing spin $\sigma_0$ tends to evolve toward the pure state so that the probing spin $\sigma_0$ and the spin chain are less entangled, as we can see from the result of $\omega_0 = 5J$ (yellow line) in Fig. 10. More importantly, we can see in Fig. 10 that for $\omega_0 \neq 0$, the critical point does not always has the maximum entanglement, i.e., the enhancement of the entanglement near the critical region is suppressed in this case. We also plot the entanglement entropy in different phases with the initial temperature of the spin chain $k_B T = 5J$ in Fig. 10(d), 10(e), and 10(f). The results show that the dependence of the initial state of $\sigma_0$ also diminish in the high temperature due to the thermalization. Meanwhile, the thermal fluctuation makes the entanglement entropy increase, particularly for the high-energy initial state $| \downarrow \rangle$ of $\sigma_0$. These results for $\omega_0 \neq 0$ show that the relation between the entanglement entropy and quantum phase transition is less obvious due to the initial state dependence and the thermal effect. Therefore, we find

![Image](image_url)

**FIG. 9**: (Colour online) (a) The entanglement entropy $S_A(t)$ for $\omega_0 = 0$ and $\eta = J$ with the spin chain varying from the topologically non-trivial phase ($\lambda < 1$) to the topologically trivial phase ($\lambda > 1$). (b) The long-time entanglement entropy versus $\lambda$, which is given by

$$S_A(t) = -\frac{1}{2} \left\langle \sigma_0^z(t) \sigma_0^z(t_0) \right\rangle \ln \frac{1}{2} \left\langle \sigma_0^z(t) \sigma_0^z(t_0) \right\rangle - \frac{1}{2} \left\langle \sigma_0^z(t) \sigma_0^z(t_0) \right\rangle \ln \frac{1}{2} \left\langle \sigma_0^z(t) \sigma_0^z(t_0) \right\rangle.$$  (38)

We plot $S_A(t)$ for $\omega_0 = 0$ and $\eta = J$ with different values of $\lambda$ in Fig. 9(a). In this case, we have $V_{12}(t, t) = V_{22}(t, t)$, then the entanglement entropy is independent of the initial state of the spin chain, as we mentioned in Sec. III, the probing spin $\sigma_0$ is assumed to be initially in a pure state. In the beginning, there is no entanglement between $\sigma_0$ and the spin chain, and the entropy $S_A(t_0)$ equals zero. When $t > t_0$, the probing spin $\sigma_0$ begins to entangle with the spin chain due to the coupling between them so that the entropy $S_A$ increases from 0 in Fig. 9(a). For $\lambda < 1$, the entanglement entropy always keeps oscillating, which means that the probing spin $\sigma_0$ and the spin chain never reach the equilibrium state due to the existence of localized modes \[30\]. The maximum value of such entanglement entropy oscillation is always $\ln 2$, while its lower bound rises as $\lambda$ increases. For $\lambda \geq 1$, the probing spin $\sigma_0$ will reach equilibrium with the spin chain in the long-time limit, and the entanglement entropy will approach to a stationary value which decreases as $\lambda$ increases. Note that the entanglement entropy approaches to $\ln 2$ at $\lambda = 1$, in agreement with the expectation that there is maximum entanglement at the critical point of quantum phase transitions. We further plot the entanglement entropy versus $\lambda$ under the long-time limit in Fig. 9(b) to see more clearly its closed relation with quantum phase transition. Figure 9(b) shows a qualitative change of the entanglement entropy when the quantum phase transition occurs. It shows that the entanglement entropy can be used to diagnose quantum phase transitions in the non-equilibrium regions.
VI. CONCLUSION

The energy eigenfunctions of the conventional transverse-field Ising model, or the equivalent Kitaev model, cannot be analytically solved with the free-ends boundary condition. We introduce a modified transverse-field Ising model which is analytically solvable. Its spectrum as well as its ground state wavefunction distribution with those of the ordinary transverse-field Ising model are comparable but also distinguishable. We show that different from the ground state of the original spin model which has zero energy only for \( \lambda < 1 \), the ground state energy of the modified model is always zero, and the modified transverse-field Ising model still has the phase transition at \( \lambda = 1 \), namely the zero energy ground state wavefunction have different topological properties for \( \lambda < 1 \) and \( \lambda > 1 \). We also prove that the quantum phase transition is associated with the change of the topological winding number. Moreover, the results of the ground state wavefunction distribution indicate that the distribution of the left Majorana zero mode in the modified model is same as that of the ordinary model for \( \lambda < 1 \). While, the left Majorana zero mode of the modified model moves to the right-hand side and merges with the right Majorana zero mode such that the zero energy Majorana modes become topologically trivial for \( \lambda > 1 \).

We then propose a scheme to experimentally measure the topological structure of the modified spin chain through the non-Markovian decoherence dynamics of a probing spin in the real-time domain by coupling the probing spin to the spin chain. We derive the exact master equation of a probing spin and analyzed in detail its non-Markovian decoherence dynamics by studying the two-time correlation function \( \langle \sigma^z_0(t)\sigma^z_0(t_0) \rangle \). We find that

(i) in the topologically non-trivial phase, the topological non-local property induces different localized modes from the case in the topologically trivial phase. These localized modes qualitatively change the non-Markovian memory effect so that the topological structure of the transverse-field Ising chain is manifested.

(ii) The coupling \( \eta \) between the probing spin and the spin chain and the flipping energy \( \omega_0 \) of the probing spin also affect the non-Markovian decoherence dynamics. For strong coupling \( \eta \), the non-Markovian oscillation is dominant in both phases so that the manifestation of the topological phase transition in the non-Markovian decoherence dynamics of the probing spin becomes weak. While for large spin-flip energy \( \omega_0 \), the manifestation is also suppressed due to the noise effects associated with the initial state of the probing spin and the temperature of the spin chain.

(iii) The dynamical entanglement entropy can be expressed in the two-time correlation \( \langle \sigma^z_0(t)\sigma^z_0(t_0) \rangle \) so that the entanglement entropy between the probing spin and the spin chain can characterize the quantum phase transition, which is equivalent to the description of the decoherence dynamics with
After integrating over all the degrees of freedom of the Science and Technology of Taiwan under Contract No. The results presented in this paper also provide small spin-flip energy and the system is at a low temperature. As a result, the topological structure of the transverse-spin chain by path integral approach, we obtain the exact form of the propagating function

$$\mathcal{K}(\xi_f^*, \xi_f^*, t|\xi_0, \xi_0^*, t_0)$$

$$= \mathcal{N}(t) \exp \left[ (\xi_f^*, \xi_f^*) \mathbf{J}_1(t, t_0) \left( \begin{array}{c} \xi_0^* \\ \xi_0^* \end{array} \right) + (\xi_f^*, \xi_f^*) \mathbf{J}_2(t, t_0) \left( \begin{array}{c} \xi_f^* \\ \xi_f^* \end{array} \right) + (\xi_f^*, \xi_f^*) \mathbf{J}_3(t, t_0) \left( \begin{array}{c} \xi_f^* \\ \xi_f^* \end{array} \right) \right]$$

(1)  

where $\mathcal{N}(t)$ is the normalization constant and $\mathbf{J}_1(t, t_0)$, $\mathbf{J}_2(t, t_0)$, and $\mathbf{J}_3(t, t_0)$ are functions of the $U(t, t_0)$ and $V(t, t)$, and their exact formulas are given in Ref. [34].

After substituting Eq. (1) into Eq. (1) and taking the time derivative on both sides, we have

The propagator $\mathcal{K}(\xi_f^*, \xi_f^*, t|\xi_0, \xi_0^*, t_0)$ acting on the grassmann numbers $\xi_0^*, \xi_0$ of the initial state can be transferred into functions which only depend on the grassmann numbers $\xi_f^*, \xi_f$ of the state at time $t$. Then Eq. (A2) becomes

$$\langle \xi_f|\rho_A(t)|\xi_f' \rangle = \mathcal{N}(t) \int d\mu(\xi_0)d\mu(\xi_0^*) \langle \xi_0|\rho_A|\xi_0^* \rangle \mathcal{K}(\xi_f^*, \xi_f^*, t|\xi_0, \xi_0^*, t_0) \left[ \frac{N(t)}{N(t)} + \langle \xi_f^*, \xi_f^* \rangle \mathbf{J}_1(t, t_0) \left( \begin{array}{c} \xi_0^* \\ \xi_0^* \end{array} \right) + \langle \xi_f^*, \xi_f^* \rangle \mathbf{J}_2(t, t_0) \left( \begin{array}{c} \xi_f^* \\ \xi_f^* \end{array} \right) + \langle \xi_f^*, \xi_f^* \rangle \mathbf{J}_3(t, t_0) \left( \begin{array}{c} \xi_f^* \\ \xi_f^* \end{array} \right) \right]$$

(A2)

The propagator $\mathcal{K}(\xi_f^*, \xi_f^*, t|\xi_0, \xi_0^*, t_0)$ acting on the grassmann numbers $\xi_0^*, \xi_0$ of the initial state can be transferred into functions which only depend on the grassmann numbers $\xi_f^*, \xi_f$ of the state at time $t$. Then Eq. (A2) becomes

$$\langle \xi_f|\rho_A(t)|\xi_f' \rangle = \langle \xi_f|\rho_A(t)|\xi_f' \rangle \left[ \frac{N(t)}{N(t)} + A(t) + B(t)\xi_f^* \xi_f' + C(t)\xi_f^* \frac{\partial}{\partial \xi_f'} + D(t)\xi_f^* \frac{\partial}{\partial \xi_f'} + E(t)\xi_f^* \frac{\partial}{\partial \xi_f'} + F(t)\xi_f^* \frac{\partial}{\partial \xi_f'} + G(t)\frac{\partial}{\partial \xi_f'} \right]$$

(A3)

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Appendix A

We begin with Eq. (16) to derive the master equation. After integrating over all the degrees of freedom of the

$$\mathcal{N}(t) \int d\mu(\xi_0)d\mu(\xi_0^*) \langle \xi_0|\rho_A|\xi_0^* \rangle \mathcal{K}(\xi_f^*, \xi_f^*, t|\xi_0, \xi_0^*, t_0) \left[ \frac{N(t)}{N(t)} + \langle \xi_f^*, \xi_f^* \rangle \mathbf{J}_1(t, t_0) \left( \begin{array}{c} \xi_0^* \\ \xi_0^* \end{array} \right) + \langle \xi_f^*, \xi_f^* \rangle \mathbf{J}_2(t, t_0) \left( \begin{array}{c} \xi_f^* \\ \xi_f^* \end{array} \right) + \langle \xi_f^*, \xi_f^* \rangle \mathbf{J}_3(t, t_0) \left( \begin{array}{c} \xi_f^* \\ \xi_f^* \end{array} \right) \right]$$

(A1)
$aa = 0$, and the eigenenergies of $A$ are symmetric in sign, Eq. (A1) can be reduced to

$$A = -\frac{\dot{N}(t)}{N(t)}$$  \hspace{1cm} (A6)$$

$$B = \dot{V}_{11}(t) - \left[ \dot{U}(t,t_0)U^{-1}(t,t_0) \right] V(t,t) + H.c.]_{11}$$  \hspace{1cm} (A7)$$

$$C = -F^* = \left[ \dot{U}(t,t_0)U^{-1}(t,t_0) \right]_{12}$$  \hspace{1cm} (A8)$$

$$D = E = \dot{V}_{11}(t) - \left[ \dot{U}(t,t_0)U^{-1}(t,t_0) \right] V(t,t) + H.c.]_{11} + \left[ U^{-1}(t,t_0)\dot{U}(t,t_0) \right]_{11}$$  \hspace{1cm} (A9)$$

$$G = \dot{V}_{11}(t) - \left[ \dot{U}(t,t_0)U^{-1}(t,t_0) \right] (V(t,t) + I) + H.c.]_{11}$$  \hspace{1cm} (A10)$$

Then using the D-algebra of the creation and annihilation operators, we obtain the exact master equation.

$$\dot{\rho}_A(t) = -i[\epsilon(t)\sigma^A_a, \rho_A(t)]$$

$$+ \gamma(2\rho_A(t)\sigma^A_a - a^\dagger a \rho_A(t) - \rho_A(t)a^\dagger a]$$

$$+ \tilde{\gamma}(a^\dagger \rho_A(t)\sigma^A_a^\dagger + \sigma^A_a \rho_A(t) - \rho_A(t)\sigma^A_a^\dagger - \rho_A(t)a^\dagger a^\dagger]$$

$$+ \Lambda(t)a^\dagger \rho_A(t)a^\dagger + \Lambda^*(t)a \rho_A(t)a,$$  \hspace{1cm} (A11)$$

The time-dependent dissipation and fluctuation coefficients in the master equation are

$$\Lambda(t) = -C(t)$$

$$= -\left[ \dot{U}(t,t_0)U^{-1}(t,t_0) \right]_{12},$$  \hspace{1cm} (A12a)$$

$$\gamma(t) = B(t)$$

$$= \frac{1}{2}\left[ \dot{U}(t,t_0)U^{-1}(t,t_0) + H.c.]_{11},$$  \hspace{1cm} (A12b)$$

$$\tilde{\gamma}(t) = \frac{B(t) - G(t)}{2}$$

$$= \dot{V}_{11}(t) - \left[ \dot{U}(t,t_0)U^{-1}(t,t_0) \right] V(t,t) + H.c.]_{11},$$  \hspace{1cm} (A12c)$$

$$\epsilon(t) = D(t) - \frac{B(t) + G(t)}{2}$$

$$= \frac{i}{2}\left[ \dot{U}(t,t_0)U^{-1}(t,t_0) - H.c.]_{11}.$$  \hspace{1cm} (A12d)$$

* Electronic address: wzhang@mail.ncku.edu.tw

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