FLAT BUNDLES AND HYPER-HODGE DECOMPOSITION ON
SOLVMANIFOLDS

HISASHI KASUYA

Abstract. We study rank 1 flat bundles on solvmanifolds whose cohomologies are non-trivial. By using Hodge theoretical properties for all topologically trivial rank 1 flat bundles, we represent the structure theorem of Kähler solvmanifolds as extensions of Hasegawa’s result and Benson-Gordon’s result for nilmanifolds.

1. Introduction

Let $M$ be a compact manifold and $(\mathcal{A}^\ast_{\mathbb{C}}(M), d)$ (resp. $(\mathcal{A}^\ast_{\mathbb{R}}(M), d)$) the $\mathbb{C}$-valued (resp. $\mathbb{R}$-valued) de Rham complex and $\mathcal{A}^\ast_{\mathbb{C}, cl}(M)$ (resp. $\mathcal{A}^\ast_{\mathbb{R}, cl}(M)$) the subspace of $\mathbb{C}$-valued (resp. $\mathbb{R}$-valued) closed forms. We denote by $F(M)$ the set of isomorphism classes of flat $\mathbb{C}$-line bundles $E_\phi = (M \times \mathbb{C}, d + \phi)$ where $M \times \mathbb{C}$ is a topologically trivial line bundle and $\phi \in \mathcal{A}^1_{\mathbb{C}, cl}(M)$. We denote by $\mathcal{C}(\pi_1(M))$ the space of characters of $\pi_1(M)$ which can be factored as

$$\pi_1(M) \to H_1(\pi_1(M), \mathbb{Z})/(\text{torsion}) \to \mathbb{C}^\ast.$$ 

Then we have a correspondence $\iota: F(M) \to \mathcal{C}(\pi_1(M))$ such that $\iota(E_\phi)(\gamma) = e^{\int_{\gamma} \phi}$ for $\gamma \in \pi_1(M)$. By this, the map $A^\ast_{\mathbb{C}, cl}(M) \ni \phi \mapsto E_\phi \in F(M)$ induces a surjection $H^p(M, \mathbb{C}) \ni [\phi] \mapsto E_\phi \in F(M)$. We consider $\mathbb{R}^\ast$-action on $F(M)$ such that

$$\mu_t(E_\phi) = E_{t\Re \phi + \Im \phi}$$

for $t \in \mathbb{R}^\ast$. By the definition, a unitary flat bundle $E_\phi \in F(M)$ (i.e. $\iota(E_\phi)$ is a unitary character) is fixed by the $\mathbb{R}^\ast$-action. We can consider the cochain complex $(A^\ast_{\mathbb{C}}(M), d + \phi)$ as the de Rham complex with values in a flat bundle $E_\phi$. We denote by $H^p(M, d + \phi)$ the cohomology of $(A^\ast_{\mathbb{C}}(M), d + \phi)$. We define $J^p(M) = \{E_\phi \in F(M) | H^p(M, d + \phi) \neq 0\}$ and denote $\overline{J}(M) = \bigcup J^p(M)$.

The main objects of this paper are solvmanifolds. Solvmanifolds are compact homogeneous spaces $G/\Gamma$ of simply connected solvable Lie groups $G$ by lattices (i.e. cocompact discrete subgroups) $\Gamma$. For solvmanifolds $G/\Gamma$, in this paper, for each $p$, we describe the set $J^p(G/\Gamma)$ by using adjoint representations of $G$ (see Section 5).

Remark 1. Let $\overline{J}^p(M)$ denote the set of the isomorphism classes of (not necessarily topologically trivial) flat $\mathbb{C}$-line bundles over a compact manifold $M$ whose cohomologies are non-trivial. The set $\overline{J}^p(M)$ is called the cohomology jump locus of $M$. The set $\overline{J}^p(M)$ was studied by several authors (for examples [4], [37], [28], [29].

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and \([19]\). For solvmanifolds \(G/\Gamma\), we have \(\mathcal{J}_p(G/\Gamma) = \mathcal{J}'_p(G/\Gamma)\) (see Corollary \([5,2]\).

**Definition 1.1.** \(M\) has the \(\mu_{R^*}\)-symmetry on cohomologies if for each \(E_\phi \in F(M)\) and \(t \in \mathbb{R}^*\), we have

\[
\dim H^r(M, d + \phi) = \dim H^r(M, d + t \text{Re}\phi + \text{Im}\phi)
\]

for each \(r\).

If \(M\) has the \(\mu_{R^*}\)-symmetry on cohomologies and there exists a non-unitary flat bundle \(E_\phi\) such that \(H^r(M, d + \phi) \neq 0\), then \(\mathcal{J}(M)\) is an infinite set. Let

\[
\overline{A}^r(M) = \bigoplus_{E_\phi \in F(M)} (A^r_\phi(M), d + \phi).
\]

Then by isomorphism \(E_\phi \otimes E_\phi \cong E_{\phi + \varphi}\), the \(\overline{A}^r(M)\) is a differential graded algebra.

**Definition 1.2.** \(M\) is hyper-formal if the differential graded algebra \(\overline{A}(M)\) is formal in the sense of Sullivan \([35]\).

**Definition 1.3.** We suppose \(M\) admits a symplectic form \(\omega\). \((M, \omega)\) is hyper-hard-Lefschetz if for each \(E_\phi \in F(M)\) for the linear map

\[
[\omega]^{n-i} \wedge : H^i(M, d + \phi) \to H^{2n-i}(M, d + \phi)
\]

is an isomorphism for any \(i \leq n\) where \(\dim M = 2n\).

Let \((M, J)\) be a compact complex manifold. Consider the double complex \((A^*\cdot(M), \partial, \bar{\partial})\) and the Dolbeault cohomology \(H^\bullet\cdot(M, \bar{\partial})\). We also consider the Bott-Chern cohomology \(H^\bullet\cdot(M, \partial\bar{\partial})\) defined by

\[
H^\bullet\cdot(M, \partial\bar{\partial}) = \frac{\text{ker} \partial \cap \text{ker} \bar{\partial}}{\text{im} \partial \bar{\partial}}.
\]

We say that a \((M, J)\) admits the strong-Hodge-decomposition if the natural maps

\[
H^\bullet\cdot(M, \partial\bar{\partial}) \to H^\bullet\cdot(M, \bar{\partial}), \quad \text{Tot}^* H^\bullet\cdot(M, \partial\bar{\partial}) \to H^\bullet(M, d)
\]

are isomorphisms (see \([18]\)). This condition is equivalent to the \(\partial\bar{\partial}\)-lemma as in \([17]\) (see \([3]\)) and hence this condition implies the formality in the sense of Sullivan. Let \(E_\phi \in F(M)\) be a unitary flat bundle. Then we have \([\phi] \in \sqrt{-1} H^\bullet(M, \mathbb{R})\) and hence we can take \(\phi \in \sqrt{-1} A^\bullet_\phi(M)_R\). We consider the decomposition \(d + \phi = (\partial + \phi^{1,0}) + (\bar{\partial} + \phi^{0,1})\) where \(\phi^{1,0}\) and \(\phi^{0,1}\) are \((1,0)\) and \((0,1)\) components of \(\phi\) respectively. Then \((A^r_\phi(M), \bar{\partial} + \phi^{0,1})\) is considered as the Dolbeault complex with values in a flat holomorphic bundle \(E_\phi\). For a holomorphic 1-form \(\theta \in H^{1,0}(M, \bar{\partial})\), we consider the differential operator \(\bar{\partial} + \phi^{0,1} + \theta\). We denote by \(H^\bullet(M, \bar{\partial} + \phi^{0,1} + \theta)\) the cohomology of \((A^\bullet_\phi(M), \bar{\partial} + \phi^{0,1} + \theta)\). We assume \((M, J)\) admits the strong-Hodge-decomposition. Then we have \(H^1(M, d) \cong \text{ker} \partial \cap \text{ker} \bar{\partial} = H^{1,0}(M, \bar{\partial}) \oplus \bar{H}^{1,0}(M, \bar{\partial})\). By this for holomorphic 1-forms \(\theta\) and \(\varphi\), we have the flat bundle \(E_{\delta - \varphi + \theta + \bar{\theta}} \in F(M)\) and the decomposition

\[
d + \varphi - \bar{\varphi} + \theta + \bar{\theta} = (\partial + \varphi + \bar{\varphi}) + (\bar{\partial} - \bar{\varphi} + \theta)
\]

of differential operators. Moreover each element in \(F(M)\) can be written as \(E_{\delta - \varphi + \theta + \bar{\theta}}\). We denote by

\[
H^\bullet(M, (\varphi + \theta + (\bar{\varphi} - \bar{\theta})
\]

\([\text{Gr} - \mathbb{R}^*]\)
the vector space
\[
\frac{\ker(\partial + \vartheta + \bar{\theta}) \cap \ker(\bar{\partial} - \bar{\vartheta} + \vartheta)}{\im(\partial + \vartheta + \bar{\theta})(\bar{\partial} - \bar{\vartheta} + \vartheta)}
\]

**Definition 1.4.** Let \((M, J)\) be a compact complex manifold admitting the strong-Hodge-decomposition.

- \((M, J)\) satisfies the hyper-strong-Hodge-decomposition if for each holomorphic 1-forms \(\theta\) and \(\vartheta\) the natural maps
  \[
  H^*(M, (\partial + \vartheta + \bar{\theta})(\bar{\partial} - \bar{\vartheta} + \vartheta)) \to H^*(M, \bar{\partial} - \bar{\vartheta} + \vartheta),
  \]
  \[
  H^*(M, (\partial + \vartheta + \bar{\theta})(\bar{\partial} - \bar{\vartheta} + \vartheta)) \to H^*(M, \partial - \vartheta + \bar{\theta})
  \]
  are isomorphisms.

We have the following relations like [17].

**Proposition 1.5.** Let \((M, J)\) be a compact complex manifold admitting the strong-Hodge-decomposition. Then if \((M, J)\) satisfies hyper-strong-Hodge-decomposition, \(M\) has the \(\mu^R\)-symmetry on cohomologies and \(M\) is hyper-formal.

In case \((M, J)\) admits a Kähler structure, for holomorphic 1-forms \(\theta\) and \(\vartheta\), the pair \((E_{\bar{\partial} - \bar{\vartheta}}, \vartheta)\) is considered as a Higgs bundle in Simpson’s sense [36]. By using the harmonic metric on Higgs bundle, we can show the Kähler identity (see [36, Section 2]). As similar to the proof of ordinary strong-Hodge-decomposition (see [18]), we have:

**Theorem 1.6.** ([36, Section 2]) Let \((M, J, \omega)\) be a compact Kähler manifold. Then \((M, J)\) satisfies hyper-strong-Hodge-decomposition and \((M, \omega)\) is hyper-hard-Lefschetz.

In this paper, we study the above properties on solvmanifolds.

**Theorem 1.7.** Let \(M\) be a \(2n\)-dimensional solvmanifold. Then the following conditions are equivalent

1. \(M\) admits a complex structure \(J\) and \((M, J)\) admits strong-Hodge-decomposition and hyper-strong-Hodge-decomposition.
2. \(\dim H^1(M, \mathbb{R})\) is even, \(M\) has the \(\mu^R\)-symmetry on cohomologies and \(M\) is hyper-formal.
3. \(M\) has the \(\mu^R\)-symmetry on cohomologies, \(M\) admits a symplectic form \(\omega\) and \((M, \omega)\) is hyper-hard-Lefschetz.
4. \(M\) is written as \(G/\Gamma\) where \(G = \mathbb{R}^{2k} \times \varphi \mathbb{R}^{2l}\) such that the action \(\varphi : \mathbb{R}^{2k} \to \text{Aut}(\mathbb{R}^{2l})\) is semi-simple and for any \(x \in \mathbb{R}^{2k}\) the all eigenvalues of \(\varphi(x)\) are unitary.
5. \(M\) admits a Kähler structure.

**Remark 2.** In [21], Hasegawa showed that formal nilmanifolds are tori and in particular nilmanifolds admitting strong-Hodge-decomposition are tori. In [8], Benson-Gordon showed that hard-Lefschetz symplectic nilmanifolds are tori. These results give the structure theorem for Kähler nilmanifolds. Theorem 1.7 can be regarded as extensions of Hasegawas result and Benson-Gordons result for nilmanifolds.
Remark 3. Equivalence of (4) and (5) in Theorem [1,7] were already proved by Hasegawa in [21] by using Arapura-Nori’s results in [5]. But it is not clear to consider the result in [21] as extensions of Hasegawa result and Benson-Gordons result for nilmanifolds.

Remark 4. Arapura-Nori’s results in [5] follows from Arapura’s earlier work in [4]. The proof of Theorem [1,7] is similar to the Arapura’s idea in [3]. But the proof of Theorem [1,7] is independent of [4].

Remark 5. By Theorem [1,7] there exists no examples of non-Kähler solvmanifolds satisfying hyper-strong-Hodge-decomposition. But there exist examples of non-Kähler solvmanifolds satisfying strong-Hodge-decomposition, hyper-formality and hyper-hard-Lefschetz property (see Section [8]).

We suggest new problem for non-Kähler geometry.

Problem 1. Provide non-Kähler manifolds $M$ such that $M$ admit complex structures $J$ and $(M, J)$ satisfy strong-Hodge-decomposition and hyper-strong-Hodge-decomposition.

2. HYPER-STRONG-HODGE-DECOMPOSITION AND $\mu_{\mathbb{R}}-$SYMMETRY ON COHOMOLOGIES

Let $(M, J)$ be a compact complex manifold. Let $E_{\phi} \in F(M)$ be a unitary flat bundle with $\phi \in \sqrt{-1} A^*_{\mathbb{R}}(M)$. For a holomorphic form $\theta \in H^{1,0}(M, \bar{\partial})$, we consider the differential operator $\bar{\partial} + \phi^{0,1} + \theta$ on $A^*_C(M)$ and the cohomology $H^*(M, \bar{\partial} + \phi^{0,1} + \theta)$.

Lemma 2.1. For any $t \in \mathbb{C}^*$, we have

$$\dim H^*(M, \bar{\partial} + \phi^{0,1} + \theta) = \dim H^*(M, \bar{\partial} + \phi^{0,1} + t\theta).$$

Proof. Considering bi-grading $A^*_C(M) = A^{p, q}(M)$. Then the cohomology $H^*(M, \bar{\partial} + \phi^{0,1} + \theta)$ is the total cohomology of the double complex $(A^{*,*}(M), \bar{\partial} + \phi^{0,1}, \theta)$. Consider the spectral sequence $E^{*,*}_{r}$ of the double complex $(A^{*,*}(M), \bar{\partial} + \phi^{0,1}, \theta)$. Then as [11] or [13], we have

$$E^{p, q}_{r} \cong \frac{X^{p, q}_{r}}{Y^{p, q}_{r}}$$

where for $r \geq 2$,

$$X^{p, q}_{r} = \{ \psi_{p, q} \in A^{p, q}(M) | \bar{\partial}\psi_{p, q} + \phi^{0,1} \wedge \psi_{p, q} = 0, \exists \psi_{p+i, q+i} \in A^{p+i, q-i}(M),$$

$$s.t. \theta \wedge \psi_{p+i-1, q-i+1} + \bar{\partial}\psi_{p+i-1, q-i+1} + \phi^{0,1} \wedge \psi_{p+i-1, q-i+1} = 0, 1 \leq i \leq r-1\},$$

$$Y^{p, q}_{r} = \{ \theta \wedge \omega_{p-1, q} + \bar{\partial}\omega_{p-1, q} + \phi^{0,1} \wedge \omega_{p-1, q} \in A^{p, q}(M),$$

$$s.t. \theta \wedge \omega_{p-i, q+i} + \bar{\partial}\omega_{p-i, q+i} + \phi^{0,1} \wedge \omega_{p-i, q+i} = 0, 2 \leq i \leq r-1\}.$$

For $t \in \mathbb{C}^*$, we also consider the spectral sequence $E^{*,*}_{r}(t)$ of the double complex $(A^{*,*}_C(M), \bar{\partial} + \phi^{0,1}, t\theta)$. Then we have

$$E^{p, q}_{r}(t) \cong \frac{X^{p, q}_{r}(t)}{Y^{p, q}_{r}(t)}$$
where for $r \geq 2$,
\[
\begin{align*}
X^p,q_r(t) &= \{ \psi_{p,q}^t \in A^{p,q}(M) | \bar{\partial}\psi_{p,q}^t + \phi^{0,1} \wedge \psi_{p,q}^t = 0, \\
& \exists \psi_{p+i,q+i}^t \in A^{p+i,q+i}(M), \\
& s.t. \theta \wedge \psi_{p+i-1,q+i-1}^t + \bar{\partial}\psi_{p+i,q-i}^t + \phi^{0,1} \wedge \psi_{p+i,q-i}^t = 0, \ 1 \leq i \leq r-1 \}, \\
Y^p,q_r(t) &= \{ t\theta \wedge \omega_{p-1,q}^t + \bar{\partial}\omega_{p,q-1}^t + \phi^{0,1} \wedge \omega_{p,q-1} \in A^{p,q}(M) | \\
& \exists \omega_{p-i,q+i-1}^t \in A^{p-i,q+i-1}(M), \\
& s.t. \theta \wedge \omega_{p-i,q+i-1}^t + \bar{\partial}\omega_{p,q-i-1}^t + \phi^{0,1} \wedge \omega_{p,q-i-1}^t = 0, \ 2 \leq i \leq r-1 \}.
\end{align*}
\]
For $\psi_{p,q} \in X^{p,q}$, considering $\psi_{p+i,q-i}^t = t^i \psi_{p+i,q-i}$, we can say $\psi_{p,q} \in X^{p,q}(t)$. For $\theta \wedge \omega_{p-1,q} + \bar{\partial}\omega_{p,q-1} + \phi^{0,1} \wedge \omega_{p,q-1} \in Y^{p,q}$, considering $\omega_{p-i,q+i-1}^t = t^{-i} \omega_{p-i,q+i-1}$ we can say
\[
\theta \wedge \omega_{p-1,q} + \bar{\partial}\omega_{p,q-1} + \phi^{0,1} \wedge \omega_{p,q-1} \in Y^{p,q}(t),
\]
By these relations we have $X^{p,q} = X^{p,q}(t)$ and $Y^{p,q} = Y^{p,q}(t)$ and hence $E^{p,q}_r \cong E^{p,q}(t)$. Hence for sufficiently large $r$, we have
\[
E_\infty = E_r \cong E^{p,q}_\infty(t) \cong E^{p,q}_\infty(t).
\]
Since the spectral sequences $E^{*,*}_r$ and $E^{*,*}_r(t)$ converge to the cohomologies $H^*(M, \bar{\partial} + \phi^{0,1} + \theta)$ and $H^*(M, \bar{\partial} + \phi^{0,1} + t\theta)$ respectively, we have the lemma follows.

**Proposition 2.2.** Let $(M, J)$ be a compact complex manifold admitting a strong-Hodge-decomposition. Then if $(M, J)$ satisfies hyper-strong-Hodge-decomposition, $M$ has the $\mu_R$-symmetry on cohomologies and $M$ is hyper-formal.

**Proof.** By the definition of hyper-strong-Hodge-decomposition, for each 1-forms $\theta$ and $\bar{\theta}$, we have an isomorphism
\[
H^*(M, d + \theta - \bar{\theta} + \theta + \bar{\theta}) \cong H^*(M, \bar{\partial} - \bar{\partial} + \theta + \bar{\theta}) \cong H^*(M, \bar{\partial} - \partial + \bar{\theta} - \theta)
\]
and we can easily check that the inclusion
\[
(\ker(\partial + \partial + \theta), \bar{\partial} - \partial + \bar{\theta}) \to (A^*_c(M), d + \bar{\partial} - \partial + \theta + \bar{\theta})
\]
is a quasi-isomorphism. We consider the quotient map
\[
(\ker(\partial + \partial + \bar{\theta}), \bar{\partial} - \bar{\partial} + \theta) \to (H^*(M, \partial - \partial + \theta), \bar{\partial} - \bar{\partial} + \theta).
\]
Then since the map
\[
H^*(M, (\partial + \partial + \bar{\theta})(\bar{\partial} - \bar{\theta} + \theta)) \to H^*(M, \partial - \partial + \bar{\theta})
\]
is isomorphism, each cohomology class in $H^*(M, \partial - \partial + \bar{\theta})$ is represented by $(\bar{\partial} - \bar{\theta} + \theta)$-closed element and hence we have
\[
(H^*(M, \partial - \partial + \bar{\theta}), \bar{\partial} - \bar{\partial} + \theta) \cong (H^*(M, \partial - \theta + \bar{\theta}), 0)
\]
and we can easily check that the map
\[
(\ker(\partial + \partial + \bar{\theta}), \bar{\partial} - \bar{\partial} + \theta) \to (H^*(M, d + \partial - \bar{\partial} + \theta + \bar{\theta}), 0)
\]
is quasi-isomorphism. By the isomorphism $H^*(M, d + \partial - \bar{\partial} + \theta + \bar{\theta}) \cong H^*(M, \partial - \partial + \theta + \bar{\theta})$, we have the quasi-isomorphism
\[
(\ker(\partial + \partial + \bar{\theta}), \bar{\partial} - \bar{\partial} + \theta) \to (H^*(M, \partial - \partial + \bar{\theta}), 0).
\]
By the isomorphism $H^*(M, d + \vartheta - \bar{\vartheta} + \theta + \bar{\theta}) \cong H^*(M, \bar{\vartheta} - \bar{\vartheta} + \theta)$ and Lemma 21 for $t \in \mathbb{R}^*$, we have
\[
\dim H^*(M, d + \vartheta - \bar{\vartheta} + \theta + \bar{\theta}) = \dim H^*(M, \bar{\vartheta} - \bar{\vartheta} + \theta) = \dim H^*(M, d + \vartheta - \bar{\vartheta} + t\theta) = \dim H^*(M, d + \vartheta - \bar{\vartheta} + t\theta + t\bar{\theta}).
\]
Hence the $\mu_G$-symmetry on cohomologies holds.

We prove hyper-formality by similar idea in [28]. The direct sum
\[
\bigoplus_{E_{\vartheta - \bar{\vartheta}, \theta}} \ker(\vartheta + \bar{\vartheta}, \bar{\vartheta} - \vartheta + \theta)
\]
is a sub-differential graded algebra of $\mathcal{A}(M) = \bigoplus_{E_{\vartheta - \bar{\vartheta}, \theta}} \mathcal{A}(M)$, where $\mathcal{A}(M)$ is the direct sum of sub-differential graded algebras $\mathcal{A}(M)$ and $\mathcal{A}(M)$. We have the differential graded algebra quasi-isomorphisms
\[
\bigoplus_{E_{\vartheta - \bar{\vartheta}, \theta}} \ker(\vartheta + \bar{\vartheta}, \bar{\vartheta} - \vartheta + \theta) \to \mathcal{A}(M)
\]
and
\[
\bigoplus_{E_{\vartheta - \bar{\vartheta}, \theta}} \ker(\vartheta + \bar{\vartheta}, \bar{\vartheta} - \vartheta + \theta) \to \bigoplus_{E_{\vartheta - \bar{\vartheta}, \theta}} H^*(M, d + \vartheta - \bar{\vartheta} + \theta + \bar{\theta}).
\]
Hence $\mathcal{A}(M)$ is formal. \hfill $\square$

3. Algebraic hulls

We review the algebraic hulls.

**Proposition 3.1.** ([31 Proposition 4.40]) Let $G$ be a simply connected solvable Lie group (resp. torsion-free polycyclic group). Then there exists a unique $\mathbb{R}$-algebraic group $H_G$ with an injective group homomorphism $\psi : G \to H_G(\mathbb{R})$ so that:
- $\psi(G)$ is Zariski-dense in $H_G$.
- $Z_{H_G}(U(H_G)) \subset U(H_G)$.
- $\dim U(H_G) = \dim G$ (resp. rank $G$).

Such $H_G$ is called the algebraic hull of $G$.

We denote $U_G = U(H_G)$ and call $U_G$ the unipotent hull of $G$.

In [23] or [24], the author showed:

**Proposition 3.2.** Let $G$ be a simply connected solvable Lie group. Then $U_G$ is abelian if and only if $G = \mathbb{R}^n \ltimes \mathbb{R}^m$ such that the action $\varphi : \mathbb{R}^n \to \text{Aut}(\mathbb{R}^m)$ is semi-simple.

In this paper we show:

**Proposition 3.3.** Let $M$ be a solvmanifold with a fundamental group $\Gamma$. If $U_\Gamma$ is abelian, $M$ is written as $G/\Gamma$ where $G = \mathbb{R}^n \ltimes \mathbb{R}^m$ for $n = \dim H^1(M, \mathbb{R})$ such that the action $\varphi : \mathbb{R}^n \to \text{Aut}(\mathbb{R}^m)$ is semi-simple.

**Proof.** For a simply connected solvable Lie group $G$ which contains $\Gamma$ as a lattice. Consider the algebraic hulls $H_G$ and $H_\Gamma$. Then we have $H_\Gamma \subset H_G$ and $U_G = U_\Gamma$ (see [31 Proof of Theorem 4.34]). Let $N$ be a nilpotent subgroup in $H_G$. Then we have the direct product $N = N_s \times N_u$ where $N_s$ (resp. $N_u$) is the set of semi-simple (resp. unipotent) part of the Jordan decompositions of elements of $N$ (see [35]).
Since \( N_s \) is contained in an algebraic torus and \( N_u \subset U_G \) (see [10] Theorem 10.6), \( N = N_s \times N_u \) is abelian. Hence any nilpotent subgroup in \( \Gamma \) or \( G \) is abelian.

As [27], we have a normal nilpotent subgroup \( \Delta \) in \( \Gamma \) such that [\( \Gamma, \Gamma \)] is a finite index subgroup in \( \Delta \) and \( \Gamma/\Delta \) is free abelian. By the above argument, \( \Delta \) is abelian. Hence for \( n = \dim H^1(M, \mathbb{R}) \), we have

\[
\begin{array}{c}
0 \rightarrow \Delta \rightarrow \Gamma \rightarrow \mathbb{Z}^n \rightarrow 0.
\end{array}
\]

Since \( \Delta \) is abelian, as [6], we have a simply connected solvable Lie group \( G \) such that

\[
\begin{array}{c}
0 \rightarrow \Delta \otimes \mathbb{R} \rightarrow G \rightarrow \mathbb{Z}^n \otimes \mathbb{R} \rightarrow 0.
\end{array}
\]

By using [16, Theorem 2.2], we have a simply connected nilpotent Lie-subgroup \( C \) such that \( G = C : (\Delta \otimes \mathbb{R}) \). By the above argument, \( C \) is abelian. For \( p : C \rightarrow C/C \cap (\Delta \otimes \mathbb{R}) \cong G/(\Delta \otimes \mathbb{R}) \cong \mathbb{Z}^n \otimes \mathbb{R} \), we have a homomorphism \( q : C/C \cap (\Delta \otimes \mathbb{R}) \rightarrow C \subset G \) such that \( p \circ q = 1d \) and hence we have a splitting \( G = \mathbb{Z}^n \otimes \mathbb{R} \ltimes_\varphi \Delta \otimes \mathbb{R} \).

By Proposition 3.2, \( \varphi \) is semi-simple. Hence the proposition follows. \( \square \)

4. Cohomology of solvmanifolds

Let \( G \) be a simply connected solvable Lie group with a lattice \( \Gamma \) and \( g \) the Lie algebra of \( G \). Let \( N \) be the nilradical (i.e. maximal connected nilpotent normal subgroup) of \( G \). Then \( N \cap \Gamma \) is a lattice in \( N \) and \( \Gamma/(\Gamma \cap N) \) is a lattice in \( G/N \).

We consider the cochain complex \( \bigwedge g^*_C \) with the derivation \( d \) which is the dual to the Lie bracket of \( g \).

Let \( C(G, N) = \{ \alpha \in \text{Hom}(G, \mathbb{C}^* ) | \alpha|_N = 1 \} \) and \( C(G, N, \Gamma) \) the set of characters of \( \Gamma \) given by the restrictions of \( \alpha \in C(G, N, \Gamma) \). Since \( \Gamma/(\Gamma \cap N) \) is a lattice in \( G/N \), the set \( C(G, N, \Gamma) \) of identified with \( \text{Hom}(\Gamma/(\Gamma \cap N), \mathbb{C}^* ) \). For \( \alpha \in C(G, N) \), for a 1-dimensional vector space \( V_\alpha \) with \( G \)-action via \( \alpha \), we consider the cochain complex \( \bigwedge g^*_C \otimes V_\alpha \) such that for \( x \otimes v_\alpha \in \bigwedge g^*_C \otimes V_\alpha \) the differential \( d_\alpha \) on \( \bigwedge g^*_C \otimes V_\alpha \) is given by

\[
d_\alpha(x \otimes v_\alpha) = dx \otimes v_\alpha + \alpha^{-1} d\alpha \wedge x \otimes v_\alpha.
\]

We consider \( \bigwedge g^*_C \otimes V_\alpha \) as the space of \( \alpha \)-twisted left-invariant differential forms with values in the flat bundle \( \mathcal{E}_{\alpha^{-1}d\alpha} \). We have the inclusion

\[
\bigwedge g^*_C \otimes V_\alpha, d_\alpha \subset (A^*_C(G/\Gamma), d + \alpha^{-1} d\alpha).
\]

We consider the direct sum

\[
\bigoplus_{\alpha \in C(G, N)} \left( \bigwedge g^*_C \otimes V_\alpha, d_\alpha \right).
\]

Let \( F(G/\Gamma, N) = \{ E_{\alpha^{-1}d\alpha} \in F(G/\Gamma) | \alpha \in C(G, N) \} \). We notice that the map \( C(G, N) \ni \alpha \mapsto E_{\alpha^{-1}d\alpha} \in F(G/\Gamma, N) \) is not injective but this map gives the 1-1 correspondence \( C(G, N, \Gamma) \rightarrow F(G/\Gamma, N) \). We also consider the direct sum

\[
\bigoplus_{E_{\phi} \in F(G/\Gamma, N)} (A^*(G/\Gamma), d + \phi).
\]

Then we have the inclusion

\[
\bigoplus_{\alpha \in C(G, N)} \left( \bigwedge g^*_C \otimes V_\alpha d_\alpha \right) \hookrightarrow \bigoplus_{E_\phi \in F(G/\Gamma, N)} (A^*(G/\Gamma), d + \phi).
\]
Theorem 4.1. ([26] Theorem 1.3]) This inclusion induces a cohomology isomorphism
\[ \bigoplus_{\alpha \in \mathcal{C}(G,N)} H^*(g, V_\alpha) \cong \bigoplus_{E_\phi \in F(G/\Gamma, N)} H^*(G/\Gamma, d + \phi). \]

Take a simply connected nilpotent subgroup $C \subset G$ such that $G = C \cdot N$ as [15, Proposition 3.3]. Since $C$ is nilpotent, the map
\[ \Phi : C \ni c \mapsto \bigoplus_{\alpha \in \mathcal{C}(G,N)} (Ad_c)_s \otimes \alpha(c) \in \text{Aut} \left( \bigoplus_{\alpha \in \mathcal{C}(G,N)} \wedge g_c^* \otimes V_\alpha \right), \]

is a homomorphism where $(Ad_c)_s$ is the semi-simple part of the Jordan decomposition of $(Ad_c)$. We denote by
\[ \left( \bigoplus_{\alpha \in \mathcal{C}(G,N)} \wedge g_c^* \otimes V_\alpha \right)^{\Phi(C)} \]

the subcomplex consisting of the $\Phi(C)$-invariant elements.

Lemma 4.2. ([26] Lemma 5.2]) The inclusion
\[ \left( \bigoplus_{\alpha \in \mathcal{C}(G,N)} \wedge g_c^* \otimes V_\alpha \right)^{\Phi(C)} \subset \bigoplus_{\alpha \in \mathcal{C}(G,N)} \wedge g_c^* \otimes V_\alpha \]

induces a cohomology isomorphism.

We have a basis $X_1, \ldots, X_n$ of $g_C$ such that $(Ad_c)_s = \text{diag}(\alpha_1(c), \ldots, \alpha_n(c))$ for $c \in C$. Let $x_1, \ldots, x_n$ be the basis of $g_C^*$ which is dual to $X_1, \ldots, X_n$. Let $v_\alpha$ be a basis of $V_\alpha$ for each character $\alpha \in \mathcal{C}(G, N)$. By $G = C \cdot N$, we have $G/N = C/C \cap N$ and hence we have $\mathcal{C}(G, N) = \{ \alpha \in \text{Hom}(C, C^*) | \alpha|_{C \cap N} = 1 \}$. We have
\[ \left( \bigoplus_{\alpha \in \mathcal{C}(G,N)} \wedge g_c^* \otimes V_\alpha \right)^{\Phi(C)} = \bigwedge \langle x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n} \rangle. \]

Hence we have:

**Corollary 4.3.** The inclusion
\[ \bigwedge \langle x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n} \rangle \hookrightarrow \bigoplus_{E_\phi \in F(G/\Gamma, N)} (A^*(G/\Gamma), d + \phi) \]

induces a cohomology isomorphism.

**Corollary 4.4.** For $E_\phi \in F(G/\Gamma, N)$, let
\[ A^*_\phi = \left\{ x_I \otimes v_{\alpha_I} \in \bigwedge \langle x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n} \rangle | I \subset \{1, \ldots, n\}, \ E_{\phi^{-1}d\alpha_I} = E_\phi \right\} \]

where for a multi-index $I = \{i_1, \ldots, i_p\} \subset \{1, \ldots, n\}$ we write $x_I = x_{i_1} \wedge \cdots \wedge x_{i_p}$ and $\alpha_I = \alpha_{i_1} \cdots \alpha_{i_p}$. Then the inclusion $A^*_\phi \subset (A^*_C(M), d + \phi)$ induces a cohomology isomorphism.
5. \( \mathcal{J}(M) \) on solvmanifolds

Let \( M \) be a compact manifold. We consider the sets \( \mathcal{J}^p(M) = \{ E_\phi \in F(M) | H^p(M, d+\phi) \neq 0 \} \) and \( \mathcal{J}(M) = \bigcup \mathcal{J}^p(M) \) as in Introduction.

If \( M \) is an aspherical manifold with \( \pi_1(M) \cong \Gamma \), then we can identified \( \mathcal{J}^p(M) \) with the set \( \mathcal{J}^p(\mathcal{C}(\Gamma)) = \{ \alpha \in \mathcal{C}(\Gamma) | H^p(\Gamma, V_\alpha) \neq 0 \} \) where \( H^*(\Gamma, V_\alpha) \) is the group cohomology with values in the module \( V_\alpha \) given by \( \alpha \). We also consider the set \( \mathcal{J}^p(\mathcal{C}(\Gamma)) = \{ \alpha \in \text{Hom}(\Gamma, \mathbb{C}^*) | H^p(\Gamma, V_\alpha) \neq 0 \} \) which is identified with the set \( \mathcal{J}^p(M) \) on a aspherical manifold as in Remark 1.

**Lemma 5.1** ([29]). Let \( \Gamma \) be a torsion-free finitely generated nilpotent group. Then we have \( \mathcal{J}(\Gamma) = \bigcup \mathcal{J}^p(\Gamma) = \{ 1_\Gamma \} \) where \( 1_\Gamma \) is the trivial character of \( \Gamma \).

Let \( G \) be a simply connected solvable Lie group with a lattice \( \Gamma \) and \( g \) the Lie algebra of \( G \). Then the solvmanifold \( G/\Gamma \) is an aspherical manifold with \( \pi_1(G/\Gamma) \cong \Gamma \). We have:

**Corollary 5.2.** For \( E_\phi \in \mathcal{J}^p(G/\Gamma) \), we have \( E_\phi \in F(G/\Gamma, N) \). Hence, we have 

\[
\mathcal{J}^p(G/\Gamma) = \mathcal{J}^p(G/\Gamma) \subset F(G/\Gamma, N).
\]

**Proof.** Consider the character \( \alpha \in \mathcal{J}^p(\Gamma) \) which corresponds to \( E_\phi \in \mathcal{J}^p(G/\Gamma) \). If the restriction \( \alpha|_{\Gamma/N} \) is non-trivial, then considering the Hochshild-Serre spectral sequence \( E_\ast^\ast \), by Lemma 5.1 we have

\[
E_2^{p,q} = H^p(\Gamma \cap N, H^q(\Gamma \cap N, V_\alpha)) = 0
\]

and hence we have \( H^*(G/\Gamma, E_\phi) = H^*(\Gamma, V_\alpha) = 0 \) since the Hochshild-Serre spectral sequence converges to \( H^*(\Gamma, V_\alpha) \). Hence the restriction \( \alpha|_{\Gamma/N} \) is trivial and \( \alpha \) induces a character on \( \Gamma/\Gamma \cap N \). Since \( \Gamma/\Gamma \cap N \) is a lattice in the abelian Lie group \( G/N \), we can extend \( \alpha \) to a character of \( G \) whose restriction on \( N \) is trivial. Hence we can say \( E_\phi \in F(G/\Gamma, N) \). □

Now we use the same setting as in Section 4. By Corollary 5.2, the inclusion

\[
\bigoplus_{E_\phi \in F(G/\Gamma, N)} (A^*(G/\Gamma), d+\phi) \subset \overline{\mathcal{A}}(M) = \bigoplus_{E_\phi \in F(M)} (A^*_\Gamma(M), d+\phi)
\]

induces a cohomology isomorphism and hence considering the inclusions

\[
\bigwedge \langle x_1 \otimes v_{a_1}, \ldots, x_n \otimes v_{a_n} \rangle \subset \bigoplus_{E_\phi \in F(G/\Gamma, N)} (A^*(G/\Gamma), d+\phi) \subset \overline{\mathcal{A}}(G/\Gamma)
\]

we have:

**Theorem 5.3.** The inclusion

\[
\bigwedge \langle x_1 \otimes v_{a_1}, \ldots, x_n \otimes v_{a_n} \rangle \rightarrow \overline{\mathcal{A}}(G/\Gamma)
\]

induces a cohomology isomorphism.

Let \( K^p(G) = \{ \alpha_{i_1 \ldots i_p} \in \text{Hom}(G, \mathbb{C}^*) | 1 \leq i_1 \leq \cdots \leq i_p \leq n \} \) and \( K^p(G, \Gamma) = \{ E_{\alpha_{i_1 \ldots i_p}} \in F(G/\Gamma) | \alpha \in K^p(G) \} \).

By Corollary 5.2 and Corollary 124, we have:

**Proposition 5.4.** We have \( \mathcal{J}^p(G/\Gamma) \subset K^p(G, \Gamma) \).
We have a differential graded algebra isomorphism
\[ \bigwedge (x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n}) \cong \bigwedge u^* \]
where \( u \) is the Lie algebra of the unipotent hull \( U_G \) of \( G \) as Section 3 (see [26, Remark 4.] and [24]). By Theorem 5.3 we have:

Corollary 5.5. We have a quasi-isomorphism
\[ \bigwedge u^* \to A^* (G/\Gamma) \]
and hence \( \bigwedge u^* \) is the Sullivan minimal model of \( A^* (G/\Gamma) \).

In [21], Hasegawa proved that for a nilpotent Lie algebra \( n \), the differential graded algebra \( \bigwedge n^* \) is formal if and only if \( n \) is abelian. Hence we have:

Corollary 5.6. A solvmanifold \( G/\Gamma \) is hyper-formal if and only if the unipotent hull \( U_G \) is abelian.

Suppose we have \( [\omega] \in H^2(n) \) such that \( [\omega]^n \neq 0 \) where \( 2n = \dim n \). In [8] (see also [20, Section 4.6.4]), Benson and Gordon proved that for any \( 0 \leq i \leq n \) the linear operator
\[ [\omega]^{n-i} \wedge : H^i(n) \to H^{2n-i}(n) \]
is an isomorphism if and only if \( n \) is abelian. Hence we have:

Corollary 5.7. Suppose a solvmanifold \( (G/\Gamma, \omega) \) admits a symplectic structure \( \omega \). Then \( (G/\Gamma, \omega) \) is hyper-hard-Lefschetz if and only if the unipotent hull \( U_G \) is abelian.

We have the action of \( C \) on the differential graded algebra \( \bigwedge u^* \cong \bigwedge (x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n}) \).
\[ c \cdot (x_i \otimes v_{\alpha_i}) = \alpha_i x_i \otimes v_{\alpha_i} \]
Consider the induced action of \( C \) on the cohomology \( H^* (u) \). We take the weight decomposition \( H^p (u) = \bigoplus V^p_{\nu i} \) of this \( C \)-action.

Then we have:

Theorem 5.8. Let \( \mathcal{L}^p (G, \Gamma) = \{ E_{\nu i-1}^p, \mu_j \in F(G/\Gamma)|V^p_{\nu i} \neq 0 \} \). Then we have \( \mathcal{J}^p (M) = \mathcal{L}^p (G, \Gamma) \)

Proof. Consider the \( C \)-action
\[ \Phi : C \ni c \mapsto (Ad_c)_s \in \text{Aut} (\bigwedge g^*_C) \]
on \( \bigwedge g^*_C \) and the weight decomposition
\[ \bigwedge g^*_C = \bigoplus W^*_{\nu i} \]
of this action. Then we have
\[ \bigwedge u^* \cong \bigwedge (x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n}) \cong \bigoplus W^*_{\nu i} \otimes \langle v_{\nu i-1} \rangle. \]
We have
\[ H^p (W^*_{\nu i} \otimes \langle v_{\nu i-1} \rangle) = V^p_{\nu j}. \]
We have
\[ A^*_\phi = \bigoplus_{E_{\nu i-1}^p, \mu_j = E_\phi} W^*_{\nu i} \otimes \langle v_{\nu i-1} \rangle \]
Proposition 5.10. Suppose that one of the characters $G/\mu$ is non-unitary. Then for a solvmanifold $G/\Gamma$, there exists a non-unitary flat bundle $E_\phi \in F(M)$ such that

$$H^1(G/\Gamma, d + \phi) \neq 0.$$  

Proof. By Theorem 5.8, it is sufficient to prove that there exists a non-unitary weight $\mu_j$ of the above action of $C$ on the cohomology $H^1(u)$ such that $V^\mu_{\mu_j} \neq 0$. This follows from the following lemma.

Lemma 5.11. Let $u$ be a $C_n$-nilpotent Lie algebra. Let $A \in \text{Aut}(u)$ be a semi-simple automorphism. Consider the automorphism $\wedge A^*$ on $\wedge u^*$ induced by $A$ and its restriction on $H^1(u) = \ker(d|_{\wedge^1 u^*})$. If there exists a non-unitary eigenvalue of $A$, then the action on $H^1(u) = \ker(d|_{\wedge^1 u^*})$ induced by $A$ has a non-unitary eigenvalue.

Proof. We will show inductively on nilpotency $s$ of $u$ (i.e. $s$ is the number such that $C^s u \neq 0$ and $C^s u = 0$ where $C^s u$ is the $s$-th term of the lower central series of $u$.)

In case $s = 1$, since $H^1(u) = \ker(d|_{\wedge^1 u^*}) = u^*$, there is nothing to prove.

We assume that the statement holds in the case $s \leq n$ and $u$ has nilpotency $n + 1$. Then we have the decomposition

$$\wedge^s u^* = \wedge^2 (u/C^n u^*) \otimes \wedge (C^n u)^*$$

such that $d(C^n u)^* \subset \wedge^2 (u/C^n u)^*$. Since $A(C^n u) \subset (C^n u)$, we have $A^* ((C^n u)^*) \subset (C^n u)^*$ and $A^* ((u/C^n u)^*) \subset (u/C^n u)^*$. By the assumption of $A$, there exists a non-unitary eigenvalue of $A^*$ on $(C^n u)^*$ or $(u/C^n u)^*$. If there exists a non-unitary eigenvalue of $A^*$ on $(u/C^n u)^*$, then by the induction assumption, we can prove the statement. If there exists a non-unitary eigenvalue of $A^*$ on $(C^n u)^*$, then by $d(C^n u)^* \subset \wedge^2 (u/C^n u)^*$, we have a non-unitary eigenvalue of $A^* \wedge^2 (u/C^n u)^*$ and hence we have a non-unitary eigenvalue of $A^*$ on $(u/C^n u)^*$. Hence the lemma follows.

Theorem 5.12. Let $G/\Gamma$ be a solvmanifold. $G/\Gamma$ has the $\mu_\infty^*$-symmetry on cohomologies if and only if the characters $\alpha_1, \ldots, \alpha_n$ are all unitary characters.

Proof. Suppose that $G/\Gamma$ has the $\mu_\infty^*$-symmetry on cohomologies. By Proposition 5.10 if one of the characters $\alpha_1, \ldots, \alpha_n$ is non-unitary, then by Proposition 5.10 there exists a non-unitary flat bundle $E_\phi \in F(M)$ such that

$$H^1(G/\Gamma, d + \phi) \neq 0.$$
We have \( \{ \mu_t(E_\phi) = E_{Rt\phi + \text{Im}\phi} | t \in \mathbb{R}^* \} \subset J(G/\Gamma) \) and hence \( J(G/\Gamma) \) is a finite set. But since \( J(G/\Gamma) \) is a finite set by Corollary 5.9 the characters \( \alpha_1, \ldots, \alpha_n \) are all unitary characters.

Suppose that the characters \( \alpha_1, \ldots, \alpha_n \) are all unitary characters. Then by Proposition 5.4 \( J(G/\Gamma) \) consists of unitary flat bundles. Since any unitary flat bundle is fixed by the \( \mathbb{R}^n \)-action via \( \mu \), \( G/\Gamma \) has the \( \mu_{\mathbb{R}^n} \)-symmetry on cohomologies. \( \square \)

6. PROOF OF THEOREM 1.7

Proof. (1)\( \Rightarrow \) (2) follows from Proposition 2.2.

We prove (2)\( \Rightarrow \) (4). Write \( M = G/\Gamma \). By Corollary 5.6 the unipotent hull \( U_G = U_\Gamma \) is abelian. By Proposition 5.3 we can write \( G = \mathbb{R}^{2k} \ltimes \varphi \mathbb{R}^{2l} \) such that the actions \( \varphi : \mathbb{R}^{2l} \to \text{Aut}(\mathbb{R}^{2l}) \) is semi-simple. By using Theorem 5.12 we can easily check that all eigencharacters of \( \phi \) are unitary. Hence (2)\( \Rightarrow \) (4) follows.

Noting that the dimensions of cohomologies of odd degree of hard Lefschetz symplectic manifolds are even (see [9]), as similar to the proof of (2)\( \Rightarrow \) (4), we can prove (3)\( \Rightarrow \) (4) by Corollary 5.7.

We prove (4)\( \Rightarrow \) (5). By the assumption, we have a \( \varphi(\mathbb{R}^{2k}) \)-invariant flat Kähler metric \( h \) on \( \mathbb{R}^{2l} \). For a flat Kähler metric \( g \) on \( \mathbb{R}^{2k} \), we have the left-invariant Kähler metric \( g \times h \) on \( G \) and this induces a Kähler metric on \( G/\Gamma \). Hence (4)\( \Rightarrow \) (5) follows.

(5)\( \Rightarrow \) (1) and (5)\( \Rightarrow \) (3) follow from Theorem 1.6.

Now we have (1)\( \Rightarrow \) (2)\( \Rightarrow \) (4)\( \Rightarrow \) (5)\( \Rightarrow \) (1) and (3)\( \Rightarrow \) (4)\( \Rightarrow \) (5)\( \Rightarrow \) (3). Hence the theorem follows. \( \square \)

7. COMPUTATIONS OF THE COHOMOLOGY \( H^*(M, \partial + \phi^{0.1} + \theta) \) ON CERTAIN SOLVMANIFOLDS

Let \( \mathfrak{g} \) be a Lie algebra with a complex structure \( J \). We consider the differential bi-graded algebra \( \Lambda^{*,*} \mathfrak{g}_C \) with the differential \( \partial \). Let \( \theta \in H^{1,0}(\Lambda^{*,*} \mathfrak{g}_C, \partial) = \ker \partial_{\Lambda^{1,0} \mathfrak{g}_C} \) such that \( \theta \neq 0 \). We consider the cohomology \( H^*(\Lambda^{*,*} \mathfrak{g}_C, \partial + \theta) \) which is the total cohomology of the double complex \( (\Lambda^{*,*} \mathfrak{g}_C, \partial, \partial) \). Then we have the spectral sequence \( E_1^{*,*} \) of the double complex \( (\Lambda^{*,*} \mathfrak{g}_C, \partial, \partial) \) such that the first term \( E_1^{*,*} \) is the cohomology of \( (\Lambda^{*,*} \mathfrak{g}_C, \theta) \) and \( E_1^{*,*} \) converges to \( H^*(\Lambda^{*,*} \mathfrak{g}_C, \partial + \theta) \). By simple computation, we have \( E_1^{0,*} = 0 \) and hence we have \( H^*(\Lambda^{*,*} \mathfrak{g}_C, \partial + \theta) = 0. \)

**Theorem 7.1.** ([34] Theorem 1), ([14] Main Theorem), ([12] Theorem 2, Remark 4), ([32] Theorem 1.10), ([33] Corollary 3.10), ([11] Theorem 3.8) Let \( G \) be a simply connected nilpotent Lie group with a lattice \( \Gamma \) and left-invariant complex structure \( J \). Let \( \mathfrak{g} \) be the Lie algebra of \( G \) Suppose that one of the following conditions holds:

- \( (G, J) \) is a complex Lie group;
- \( J \) is an Abelian complex structure;
- \( J \) is a nilpotent complex structure;
- \( J \) is a rational complex structure;
- \( \mathfrak{g} \) admits a torus-bundle series compatible with \( J \) and with the rational structure induced by \( \Gamma \).
Then the inclusion
\[ \left( \bigwedge^{*,*} g^*_C, \bar{\partial}, \partial \right) \hookrightarrow \left( A^{*,*}(G/\Gamma), \partial, \bar{\partial} \right), \]
induces an isomorphism
\[ H^{*,*}(\bigwedge^{*,*} g^*_C, \bar{\partial}) \cong H^{*,*}(N/\Gamma'', \bar{\partial}). \]

Let \((G, \Gamma, J)\) as in assumption of this theorem. For a non-trivial holomorphic 1-form \(\theta \in H^{1,0}(G/\Gamma, \bar{\partial})\), we have \(\theta \in \bigwedge^{1,0} g_C\) we have the injection
\[ \left( \bigwedge^{*,*} g^*_C, \theta, \bar{\partial} \right) \hookrightarrow \left( A^{*,*}(G/\Gamma), \theta, \bar{\partial} \right) \]
between double complexes which induces a cohomology isomorphism
\[ H^{*,*}(\bigwedge^{*,*} g^*_C, \bar{\partial}) \cong H^{*,*}(G/\Gamma, \bar{\partial}). \]
By using spectral sequences of double complexes, we can easily check the isomorphism
\[ H^{*,*}(\bigwedge^{*,*} g^*_C, \bar{\partial}) \cong H^{*,*}(G/\Gamma, \bar{\partial}). \]
(see [2, Proposition 1.1]). By the above argument, we have
\[ H^*(\bigwedge g^*_C, \bar{\partial} + \theta) = 0 \]
and hence we have:

**Corollary 7.2.** For any non-trivial holomorphic 1-form \(\theta \in H^{1,0}(G/\Gamma, \bar{\partial})\), we have
\[ H^*(G/\Gamma, \bar{\partial} + \theta) = 0. \]

Next we consider a solvable Lie group \(G\) with the following assumption.

**Assumption 7.3.** \(G\) is the semi-direct product \(\mathbb{C}^n \ltimes_{\varphi} N\) so that:
(1) \(N\) is a simply connected nilpotent Lie-group with a left-invariant complex structure \(J\).
Let \(a\) and \(n\) be the Lie algebras of \(\mathbb{C}^n\) and \(N\) respectively.
(2) For any \(t \in \mathbb{C}^n\), \(\varphi(t)\) is a holomorphic automorphism of \((N, J)\).
(3) \(\varphi\) induces a semi-simple action on the Lie-algebra \(n\) of \(N\).
(4) \(G\) has a lattice \(\Gamma\). (Then \(\Gamma\) can be written by \(\Gamma = \Gamma' \ltimes_{\varphi} \Gamma''\) such that \(\Gamma'\) and \(\Gamma''\) are lattices of \(\mathbb{C}^n\) and \(N\) respectively and for any \(t \in \Gamma'\) the action \(\varphi(t)\) preserves \(\Gamma''\).)
(5) The inclusion \(\bigwedge^{*,*} n^*_C \subset A^{*,*}(N/\Gamma'')\) induces an isomorphism
\[ H^{*,*}(\bigwedge^{*,*} n^*_C, \bar{\partial}) \cong H^{*,*}(N/\Gamma'', \bar{\partial}). \]

Consider the set \(\mathcal{C}(G, N) = \{ \alpha \in \text{Hom}(G, \mathbb{C}^*) | \alpha|_N = 1 \}\). Let \(HL(G/N)\) be the set of the isomorphism classes of holomorphic \(\mathbb{C}\)-line bundles. We define the subset \(HL(G, N, \Gamma) = \{ [E_{\alpha - i} \alpha]_{\text{hol}} \in HL(G/N) | \alpha \in \mathcal{C}(G, N) \}\) where \([E_{\alpha - i} \alpha]_{\text{hol}}\) is the holomorphically isomorphism class containing a flat bundle \(E_{\alpha - i} \alpha\). We consider
the bi-graded cochain complex \((A^{*,*}(G/\Gamma), \bar{\partial} + \phi^{0,1})\) as the Dolbeault complex with values in a holomorphic flat bundle \(E_{\phi}\). We consider the direct sum
\[
\bigoplus_{[E_{\phi}]_{hol} \in HL(G,N,\Gamma)} \big(A^{*,*}(G/\Gamma), \bar{\partial} + \phi^{0,1}\big).
\]
Then by the wedge products and the tensor products, this direct sum is a differential bigraded algebra.

**Theorem 7.4** ([25]). There exists a differential bigraded sub-algebra \(A^{*,*}\) of
\[
\bigoplus_{[E_{\phi}]_{hol} \in HL(G,N,\Gamma)} \big(A^{*,*}(G/\Gamma), \bar{\partial} + \phi^{0,1}\big)
\]
such that we have a differential bigraded algebra isomorphism \(\iota\) : \(\bigwedge^{*,*}(a \oplus n)_{\mathbb{C}}^{\bullet} \cong A^{*,*}\) and the inclusion
\[
A^{*,*} \hookrightarrow \bigoplus_{[E_{\phi}]_{hol} \in HL(G,N,\Gamma)} \big(A^{*,*}(G/\Gamma), \bar{\partial} + \phi^{0,1}\big)
\]
induces a cohomology isomorphism.

For a non-trivial holomorphic 1-form \(\theta \in H^{1,0}(G/\Gamma, \bar{\partial})\), by Theorem 7.4 we have \(\theta \in A^{1,0}\) and by the differential bigraded algebra isomorphism \(\iota\) : \(\bigwedge^{*,*}(a \oplus n)_{\mathbb{C}}^{\bullet} \cong A^{*,*}\), we have the injection
\[
\bigwedge^{*,*}(a \oplus n)_{\mathbb{C}}^{\bullet}, \bar{\partial} \mapsto \bigoplus_{[E_{\phi}]_{hol} \in HL(G,N,\Gamma)} \big(A^{*,*}(G/\Gamma), \bar{\partial} + \phi^{0,1}\big)
\]
between double complexes which induces a cohomology isomorphism
\[
H^{*,*}(\bigwedge^{*,*}(a \oplus n)_{\mathbb{C}}^{\bullet}, \bar{\partial}) \cong \bigoplus_{[E_{\phi}]_{hol} \in HL(G,N,\Gamma)} H^{*,*}(G/\Gamma, \bar{\partial} + \phi^{0,1}).
\]
By using spectral sequences of double complexes, we can easily check the isomorphism
\[
H^{*}(\bigwedge(a \oplus n)_{\mathbb{C}}^{\bullet}, \bar{\partial} + \theta) \cong \bigoplus_{[E_{\phi}]_{hol} \in HL(G,N,\Gamma)} H^{*,*}(G/\Gamma, \bar{\partial} + \phi^{0,1} + \theta).
\]
(see [2] Proposition 1.1)). By the above argument, we have
\[
H^{*}(\bigwedge(a \oplus n)_{\mathbb{C}}^{\bullet}, \bar{\partial} + \theta) = 0
\]
and we have
\[
\bigoplus_{[E_{\phi}]_{hol} \in HL(G,N,\Gamma)} H^{*,*}(G/\Gamma, \bar{\partial} + \phi^{0,1} + \theta) = 0.
\]
Now we have:

**Corollary 7.5.** For a flat holomorphic bundle \([E_{\phi}]_{hol} \in HL(G,N,\Gamma)\) and a non-trivial holomorphic 1-form \(\theta \in H^{1,0}(G/\Gamma, \bar{\partial})\), we have
\[
H^{*}(G/\Gamma, \bar{\partial} + \phi^{0,1} + \theta) = 0.
\]
8. **Examples**

Let $G = \mathbb{C} \ltimes \mathbb{C}^2$ such that $\phi(z_1) = \begin{pmatrix} e^{\frac{z_1 + z_2}{2}} & 0 \\ 0 & e^{\frac{-z_1 - z_2}{2}} \end{pmatrix}$. Then for some $a \in \mathbb{R}$ the matrix $\begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix}$ is conjugate to an element of $SL(2, \mathbb{Z})$. Hence for any $0 \neq b \in \mathbb{R}$ we have a lattice $\Gamma = (a\mathbb{Z} + b\sqrt{-1}\mathbb{Z}) \ltimes \Gamma''$ such that $\Gamma''$ is a lattice of $\mathbb{C}^2$. Then for a coordinate $(z_1, z_2, z_3) \in \mathbb{C} \ltimes \mathbb{C}^2$, for the Lie algebra $g$ of $G$, we have

$$\langle g_C \rangle = \langle d\bar{z}_1, e^{-\frac{z_1 + z_2}{2}}d\bar{z}_2, e^{\frac{z_1 + z_2}{2}}d\bar{z}_3 \rangle \oplus \langle d\bar{z}_1, e^{-\frac{z_1 + z_2}{2}}d\bar{z}_2, e^{\frac{z_1 + z_2}{2}}d\bar{z}_3 \rangle.$$

We have a left-invariant symplectic structure

$$\omega = \sqrt{-1}dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + d\bar{z}_3 \wedge d\bar{z}_3.$$

Take $b \notin \pi\mathbb{Z}$, then $G/\Gamma$ satisfy the strong-Hodge-decomposition (see [2]). Moreover by Corollary 5.6 and Corollary 5.7 $G/\Gamma$ is hyper-formal and hyper-hard-Lefschetz. On the other hand, by Theorem 1.7 $G/\Gamma$ does not satisfy the hyper-strong-Hodge-decomposition.

We observe in more detail. By Remark 6 we have

$$\mathcal{J}(G/\Gamma) = \left\{ E_0, E_{dz_1 + dz_2}, E_{dz_1 + d\bar{z}_1}, E_{d\bar{z}_1 + d\bar{z}_2}, E_{-dz_1 - d\bar{z}_1} \right\}$$

Hence we have

$$H^*(G/\Gamma, d + \frac{dz_1 + d\bar{z}_1}{2}) \neq 0.$$

But by Corollary 4.5 we have

$$H^*(G/\Gamma, \partial + \frac{dz_1}{2}) = 0$$

and hence

$$H^*(G/\Gamma, d + \frac{dz_1 + d\bar{z}_1}{2}) \neq H^*(G/\Gamma, \partial + \frac{dz_1}{2}).$$

This implies that $G/\Gamma$ does not satisfy the hyper-strong-Hodge-decomposition (see the proof of Proposition 5.2).

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(Hisashi Kasuya) Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, JAPAN
E-mail address: kasuya@math.titech.ac.jp