SOME RELATIONS BETWEEN THE SPECTRA OF SIMPLE AND NON-BACKTRACKING RANDOM WALKS

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Abstract. We establish some relations between the spectra of simple and non-backtracking random walks on non-regular graphs, generalizing some well-known facts for regular graphs. Our two main results are 1) a quantitative relation between the mixing rates of the simple random walk and of the non-backtracking random walk 2) a variant of the “Ihara determinant formula” which expresses the characteristic polynomial of the adjacency matrix, or of the laplacian, as the determinant of a certain non-backtracking random walk with holomorphic weights.

1. Results

It has been noted by many authors that non-backtracking random walks on graphs looking locally like trees are simpler, from a combinatorial point of view, than usual random walks. This is for instance an ingredient of the proof of Alon’s conjecture on random regular graphs by Friedman [10] or, more recently, by Bordenave [6]. The study of the spectrum of non-backtracking random walks is also at the heart of the “community detection” problem and of the solution to the “spectral redemption conjecture” for various models of random graphs [7]. Non-backtracking random walks are known to mix faster than the usual ones [1]. In [20], the non-backtracking random walk is used to prove a cut-off phenomenon for the usual random walk on regular graphs. In geometric group theory, the “cogrowth” is directly related to the leading eigenvalue of non-backtracking random walks: this is discussed in [22], where a possibility to use this to extend the notion of co-growth to non-regular graphs is suggested. The non-backtracking random walk may also be used as an analog of “classical dynamics” in the field of quantum chaos on discrete graphs: see the papers by Smilansky [24, 23], which were a source of inspiration for this work. This note is a contribution to the study of the relation between the spectra of simple and non-backtracking random walks, for non-regular graphs. It originates in the work [3], where we used non-backtracking random walks to study the quantum ergodicity problem for eigenfunctions of Schrödinger operators on non-regular expander graphs.

Let $G = (V, E)$ be a graph without multiple edges and self-loops. We assume that the degree $D(x)$ of a vertex $x$ is bounded above and below: $2 \leq D(x) \leq D$. In the results about spectral gaps, we actually have to assume that $D(x) \geq 3$. We are interested in relating the spectrum of various operators, such as the laplacian, the adjacency matrix, and certain weighted non-backtracking random walks. Such relations are well-known for regular graphs (i.e. those for which $D(x)$ is constant), and the goal of this note is to partially extend what is known to non-regular graphs. Our two main results are 1) a relation between the mixing rates of the simple random walk and of the non-backtracking random walk 2) a variant of the “Ihara determinant formula” [17, 16] which expresses the characteristic polynomial of the adjacency matrix, or of the laplacian, as the determinant of a certain non-backtracking transfer matrix with holomorphic coefficients.

We will write $y \sim x$ to mean that $y$ is a neighbour of $x$ in $G$.

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The first operator we are interested in is the adjacency matrix \( A \). It acts on \( \mathbb{C}^V \) by the formula \( Af(x) = \sum_{y \sim x} f(y) \), and is self-adjoint on \( \ell^2(V, u) \) if \( V \) is endowed with the uniform measure \( u \).

We are also interested in the spectrum of the laplacian, defined by:

\[
P : \mathbb{C}^V \longrightarrow \mathbb{C}^V \quad \quad Pf(x) = \frac{1}{D(x)} \sum_{y \sim x} f(y).
\]

(1)

If we endow the set of vertices \( V \) with the measure \( \pi(x) = D(x) \), then \( P \) is self-adjoint on \( \ell^2(V, \pi) \). Note that this implies \( \sum_x Pf(x) \pi(x) = \sum_x f(x) \pi(x) \). The space \( \ell^2_0(V, \pi) \) of functions orthogonal to constants in \( \ell^2(V, \pi) \) is preserved by \( P \).

Let \( B \) be the set of oriented edges, endowed with the uniform measure \( U \) (each edge has weight 1). Denoting \( Q(x) = D(x) - 1 \), the “transfer operator” is defined by

\[
S : \ell^2(B, U) \longrightarrow \ell^2(B, U) \quad \quad Sf(e) = \frac{1}{Q(o(e))} \sum_{e' \sim e} f(e')
\]

(2)

where \( e' \sim e \) means that \( t(e') = o(e) \) and \( e \) is not the reverse of \( e' \). The operator \( S \) is stochastic, it is the generator of the non-backtracking random walk. It is not self-adjoint, but we have \( \sum_x Sf(e) = \sum_\pi f(e) \). This is equivalent to saying that \( S^* : \ell^2(B, U) \longrightarrow \ell^2(B, U) \) is also stochastic. When \( G \) is finite, this implies that \( S \) preserves the space \( \ell^2_0(B, U) \) of functions orthogonal to constants in \( \ell^2(B, U) \).

We will finally use the non-stochastic operator

\[
\mathcal{B} : \mathbb{C}^B \longrightarrow \mathbb{C}^B \quad \quad \mathcal{B}f(e) = \sum_{e' \sim e} f(e')
\]

(3)

If the graph \( G \) is finite and \( (q + 1) \)-regular, then \( A \) and \( P \) (resp. \( B \) and \( S \)) are the same operator up to a homothety, and there is an explicit relation between the characteristic polynomials of \( A \) and \( B \):

\[
\det(I^{[B]} - uB) = (1 - u^2)^{r-1} \det((1 + u^2 q)I^{[V]} - uA)
\]

(4)

where \( r = |E| - |V| + 1 \) is the rank of the fundamental group. This is the contents of the Ihara determinant formula \( [17, 18] \), generalised in stages by Hashimoto, Bass and Kotani–Sunada \( [11, 15, 12, 14, 13, 5, 19] \). For finite non-regular graphs the relation reads

\[
\det(I^{[B]} - uB) = (1 - u^2)^{r-1} \det(I^{[V]} - uA + u^2 Q)
\]

(5)

where \( Q \) is the diagonal matrix with components \( Q(x) = D(x) - 1 \). Note that the right-hand side in (5) is not directly related to the characteristic polynomial of \( A \). The identity (5) relates eigenvalues (and eigenvectors) of \( S \) to solutions of \( (I^{[V]} - uA + u^2 Q)\nu = 0 \), which are not eigenvectors of \( A \). Our goal is twofold:

- for finite graphs, compare the mixing rates of \( S \) and \( P \). For regular graphs, there is an exact relation between eigenvalues of \( S \) and \( P \), which implies that the spectral gap of \( P \) on \( \ell^2_0(V, \pi) \) is explicitly related to the spectral gap of \( S \) on \( \ell^2_0(B, U) \).
- extend formula (4) to non-regular graphs in a way different of (5), by finding an identity involving the characteristic polynomial of \( A \) on the right-hand side.
The result about spectral gaps also holds for infinite graphs. We recover, in a less geometric but more quantitative way, the result of Ortner and Woess [22] saying that $P$ has a spectral gap on $ℓ^2(V, π)$ if and only if the spectral radius of $S$ on $ℓ^2(B, U)$ is strictly less than 1.

We do not discuss the “spectral gap” of $A$ as this is not as properly defined as for $P$ (the top eigenvalue and eigenvector of $A$ are not explicit in general).

1.1. Spectral gap and mixing rate for $P$ and $S$. Let us first assume that $G$ is finite, connected and non-bipartite. This is equivalent to assuming that 1 is a simple eigenvalue of $P^2$. In other words, the spectrum of $P^2$ in $ℓ^2(V, π)$ is contained in $[0, 1 − β]$, for some $β > 0$ which measures the mixing rate of the simple RW on $G$.

Our first result is the following:

**Theorem 1.1.** Assume that $G$ is finite and that $D(x) ≥ 3$ for all $x$. Assume that the spectrum of $P^2$ on $ℓ^2(V, π)$ is contained in $[0, 1 − β]$. Then the spectrum of $S^2S^2$ on $ℓ^2(B, U)$ is contained in $[0, 1 − c(D, β)]$, where $c(D, β)$ depends only on $D$ and $β$, and is positive if $β$ is so.

Note that $∥S∥_{ℓ^2(B, U)} ≤ 1$, as $∥Sf∥ = ∥f∥$ as soon as $f$ is a function on $B$ that is constant on edges having the same terminus. However, our theorem says that $∥S^2∥_{ℓ^2(B, U)} ≤ (1 − c(D, β))^{1/2}$. The value of $c(D, β)$ is given in (10).

**Corollary 1.2.** For all $n ≥ 1$,

$$∥S^n∥_{ℓ^2(B, U)} ≤ (1 − c(D, β))^{n/4}.$$  

This gives the rate of mixing of the non-backtracking RW.

The converse is easier: in the course of the proof, we will also see that if the spectrum of $S^2S^2$ on $ℓ^2(B, U)$ is contained in $[0, 1 − c]$, then the spectrum of $P^2$ on $ℓ^2(V, π)$ is contained in $[0, 1 − D^{-2}c]$ (Remark 2.2).

We were primarily interested in finite graphs in view of the application to quantum ergodicity [3], but the result also holds for infinite graphs:

**Theorem 1.3.** Assume that $G$ is infinite and that $D(x) ≥ 3$ for all $x$.

(i) If the spectrum of $S^2S^2$ on $ℓ^2(B, U)$ is contained in $[0, 1 − c]$, then the spectrum of $P^2$ on $ℓ^2(V, π)$ is contained in $[0, 1 − D^{-2}c]$.

(ii) If the spectrum of $P^2$ on $ℓ^2(V, π)$ is contained in $[0, 1 − β]$, then the spectrum of $S^2S^2$ on $ℓ^2(B, U)$ is contained in $[0, 1 − c(D, β)]$, where $c(D, β)$ is given by (16).

**Remark 1.4.** It is well-known that $G$ is amenable iff the spectral radius of $P$ is 1 (see [18, 8, 9]). Thus Theorem 1.3 says that $G$ is amenable iff $∥S^2∥_{ℓ^2(B, U)} = 1$. It was proven before by Ortner and Woess that $G$ is amenable iff the spectral radius of $S$ is 1 [22]. This can be recovered by our methods: indeed, one direction results from our Theorem 1.3 in the other direction, one can follow the same lines as in our Remark 2.2 to show that if $∥S^{n}S^{n}∥_{ℓ^2(B, U)} < 1$ then $∥P^{2(n−1)}∥_{ℓ^2(V, π)} < 1$ (with an explicit bound).

Our method is very down-to-earth and gives, by basic manipulations, a quantitative relation between the spectral gap of $P$ and $∥S^2∥_{ℓ^2(B, U)}$. The method in [22] is less direct and more geometric: it starts from the general fact that $G$ is amenable iff SOLG (the symmetrized oriented line graph) is amenable. And then it is shown that SOLG is amenable iff the spectral radius of $S$ is 1.

1.2. Determinant relation. We now assume that $G$ is finite.

Let $T = (V(T), E(T))$ be the universal cover of $G$: $T$ is a tree, and there exists a subgroup $Γ$ of of automorphism group of $T$, acting without fixed points on $V(T)$, such
that $G = \Gamma \backslash T$. Let $\tilde{A}$ be the adjacency matrix of $T$. The Green function on $T$ will be denoted by
\[
G(x, y; z) = \langle \delta_x, (\tilde{A} - z)^{-1}\delta_y \rangle_{\ell^2(V(T))}
\]
for $z \in \mathbb{C} \setminus \mathbb{R}$.

Given $v, w \in T$ with $v \sim w$, we denote by $T^{(v|w)}$ the tree obtained by removing from $T$ the branch emanating from $v$ that passes through $w$. We define the restriction $H^{(v|w)}(x, y) = H(x, y)$ if $v, w \in T^{(v|w)}$ and zero otherwise. We then denote $G^{(v|w)}(\cdot, \cdot; z)$ the corresponding Green function.

Given $z \in \mathbb{C} \setminus \mathbb{R}$, $v \in V$, $w$ a neighbour of $v$ we denote
\[
G^z(v) = G(\tilde{v}, \tilde{v}; z) \quad \text{and} \quad \zeta^z(v, v) = -G^{(\tilde{v}|\tilde{v})}(\tilde{v}, \tilde{v}; z),
\]
where $(\tilde{v}, \tilde{w})$ is a lift of the edge $(v, w)$ in $T$. This definition does not depend on the choice of the lifts. If $e = (w, v) \in B$, we also use the notation $G^z(e) = G(\tilde{w}, \tilde{v}; z)$, $\zeta^z(e) = \zeta^z(w, v)$. Note that $G^z(e)$ is invariant under edge-reversal, whereas $\zeta^z(e)$ is not. In the formula below, the function $\zeta^z$ on $B$ acts on $\mathbb{C}^B$ as a multiplication operator.

**Theorem 1.5.** For all $z \in \mathbb{C} \setminus \mathbb{R}$,
\[
(6) \quad \prod_{e \in E} (-G^z(e)) \cdot \det \left( (\zeta^z)^{-1} I^{|B|} - B \right) = \det \left( z I^{|V|} - A \right) \cdot \prod_{x \in V} (-G^z(x))
\]

**Remark 1.6.** In the case of a $(q+1)$-regular graph, $\zeta^z$ is a constant function, which solves the quadratic equation
\[
(7) \quad z = q\zeta^z + \frac{1}{\zeta^z}
\]
See Lemma 3.1 below. We also have $G^z(x) = \frac{\zeta^z}{(\zeta^z)^2 - 1}$ and $G^z(e) = \frac{(\zeta^z)^2}{(\zeta^z)^2 - 1}$ for all $x$ and all $e$. It can be checked that Theorem 1.5 reduces to (1) by setting $u = \zeta^z$. It is, however, different from (5) for non-regular graphs. Although extensions of the Ihara formula (4) have been studied by many authors, the variant (6) seems to be new.

Note that (6) holds for any functions $\zeta^z$ that are solutions of the system of algebraic equations appearing in Lemma 3.1. These are by no means unique : for instance, in the regular case, there are 2 solutions to equation (7). It is nice, however, to know an explicit solution of this system that can be expressed in terms of Green functions.

**Remark 1.7.** The theorem generalizes to the case where $\mathcal{A}$ is replaced by a discrete "Schrödinger operator" of the form $\mathcal{A}_p + W : \mathbb{C}^V \rightarrow \mathbb{C}^V$, $(\mathcal{A}_p + W)f(x) = \sum_{y \sim x} p(x, y)f(y) + W(x)f(x)$ where $W$ is a real-valued function on $V$, and $p$ is such that $p(x, y) = p(y, x) \in \mathbb{R}$ and $p(x, y) \neq 0$ if $x \sim y$. The definitions of the Green functions $G^z$ and $\zeta^z$ should be modified (in the obvious manner) to incorporate the weights $p$ and the potential $W$. The definition of $B$ should be modified to
\[
B_p f(e) = \sum_{e' \sim e} p(e') f(e')
\]
and (6) becomes
\[
\prod_{e \in E} \frac{(-G^z(e))}{p(e)} \cdot \det \left( (\zeta^z)^{-1} I^{|B|} - B_p \right) = \det \left( z I^{|V|} - \mathcal{A}_p - W \right) \cdot \prod_{x \in V} (-G^z(x)).
\]
This remark, in particular, allows to cover the case of $\det(z I^{|V|} - P)$, noting that $P$ is conjugate to $\mathcal{A}_p$ with $p(x, y) = (D(x)D(y))^{-1/2}$.

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2. Proof of Theorem 1.1

We start by noting that if \( f \in \ell^2_o(V, \pi) \), then

\[
\frac{1}{2} \sum_{x \in V} \frac{1}{D(x)} \sum_{y, y' \sim x} |f(y) - f(y')|^2 \geq \beta \|f\|_{\ell^2(V, \pi)}^2.
\]

This just comes from the identity

\[
\frac{1}{2} \sum_{x \in V} \frac{1}{D(x)} \sum_{y, y' \sim x} |f(y) - f(y')|^2 = \sum_x D(x)|f(x)|^2 - \sum_x D(x)|Pf(x)|^2 = \langle f, (I - P^2)f \rangle_{\ell^2(V, \pi)}
\]

To prove Proposition 1.1 we use the following decomposition of the space of functions on \( B \):

\[
\ell^2_o(B, U) = O(\ell^2_o(V, \pi)) \oplus T(\ell^2_o(V, \pi)) \oplus (O(\ell^2_o(V, \pi))^\perp \cap T(\ell^2_o(V, \pi))^\perp).
\]

The space \( O(\ell^2_o(V, \pi)) \) is the image of \( \ell^2_o(V, \pi) \) under the map \( Of(e) = f(o(e)) \). Thus \( O(\ell^2_o(V, \pi)) \) is the space of functions (orthogonal to constants) such that \( f(e) \) depends only on the origin of \( e \). Similarly, \( T(\ell^2_o(V, \pi)) \) is the space of functions such that \( f(e) \) depends only on the terminus of \( e \). It is the image of \( \ell^2_o(V, \pi) \) under the map \( Tf(e) = f(t(e)) \). If \( G \) is non-bipartite, we have \( O(\ell^2_o(V, \pi)) \cap T(\ell^2_o(V, \pi)) = \{0\} \). Note that each space \( O(\ell^2_o(V, \pi)), T(\ell^2_o(V, \pi)) \) is orthogonal to \( \mathbb{1} \) in \( \ell^2(B, U) \), but that the two spaces are NOT orthogonal to each other. The space \( O(\ell^2_o(V, \pi))^\perp \cap T(\ell^2_o(V, \pi))^\perp \) is of dimension \( 2|E| - 2|V| + 1 = r - 1 \), where \( r \) is the rank of the fundamental group of \( G \).

By definition, it is the space of functions \( f : B \rightarrow \mathbb{C} \) such that, for all \( x \in V \),

\[
\sum_{e, o(e) = x} f(e) = 0 \quad \text{and} \quad \sum_{e, t(e) = x} f(e) = 0.
\]

\[\textbf{Remark 2.1.}\] In the infinite case, the proof of Theorem 1.3 will be similar, using the decomposition

\[
\ell^2(B, U) = O(\ell^2(V, \pi)) \oplus T(\ell^2(V, \pi)) \oplus (O(\ell^2(V, \pi))^\perp \cap T(\ell^2(V, \pi))^\perp).
\]

To start the proof we use the Dirichlet identity for \( f \in \ell^2(B, U) \):

\[
\langle f, (I - S^2S^2)f \rangle_{\ell^2(B, U)} = \frac{1}{2} \sum_{e, e'} |f(e) - f(e')|^2 S^2S^2(e, e').
\]

Let us decompose \( f \) according to (9): \( f = F + G + H \) where \( F \in O(\ell^2_o(V, \pi)), G \in T(\ell^2_o(V, \pi)), H \in (O(\ell^2_o(V, \pi))^\perp \cap T(\ell^2_o(V, \pi))^\perp) \).

We are first going to prove that

\[
\langle f, (I - S^2S^2)f \rangle_{\ell^2(B, U)} \geq Q^{-4} \beta \|F + H\|_{\ell^2(B, U)}^2
\]

where \( Q = D - 1 \) (and, recall, \( D \) is an upper bound on the degree).

In order to have \( S^2S^2(e, e') > 0 \), there must exist \( e_1, e_1', e_2 \in B \) such that \( e \sim e_1 \sim e_2 \) and \( e' \sim e_1' \sim e_2 \). Counting the number of possibilities, we see that \( S^2S^2(e, e') \geq
\((D(t(e)) - 2)Q^{-4} \geq Q^{-4}\) if \(t(e) = t(e')\). Here we use the assumption that \(D(t(e)) \geq 3\). Thus,

\[
\langle f, (I - S^2S^2) f \rangle_{\ell^2(B,U)} \geq \frac{Q^{-4}}{2} \sum_{\substack{e,e', t(e) = t(e')}} |f(e) - f(e')|^2 \geq \frac{Q^{-4}}{2} \sum_{\substack{e,e', t(e) = t(e')}} \frac{1}{D(t(e))} |f(e) - f(e')|^2 = \frac{Q^{-4}}{2} \sum_{\substack{e,e', t(e) = t(e')}} \frac{1}{D(t(e))} |(F + H)(e) - (F + H)(e')|^2.
\]

Let us fix a vertex \(x \in V\). Using the fact that \(F\) depends only on the origin, and that \(H\) satisfies \(\|H\|_{L^2(B,U)} \geq 2\beta\),

\[
\sum_{\substack{e,e', t(e) = t(e')}\sim x} |(F + H)(e) - (F + H)(e')|^2 = \sum_{y,y' \sim x} |F(y) - F(y')|^2 + \sum_{e,e', t(e) = t(e')} |H(e) - H(e')|^2 + 4 \text{Re} \sum_{e,e', t(e) = t(e')} \bar{F}(e)(H(e) - H(e')) = \sum_{y,y' \sim x} |F(y) - F(y')|^2 + \sum_{e,e', t(e) = t(e')} |H(e)|^2 + |H(e')|^2 + 4 \text{Re} \sum_{e,e', t(e) = t(e')} F(e)H(e)
\]

Summing now over \(x\), and using the fact that \(F\) and \(H\) are orthogonal,

\[
\sum_{\substack{e,e', t(e) = t(e')}} \frac{1}{D(t(e))} |(F + H)(e) - (F + H)(e')|^2 = \sum_{x} D(x)^{-1} \sum_{y,y' \sim x} |F(y) - F(y')|^2 + 2 \text{Re} \sum_{e} F(e)H(e) \geq \sum_{x} D(x)^{-1} \sum_{y,y' \sim x} |F(y) - F(y')|^2 + 2 \|H\|_{L^2(B,U)}^2 \geq 2 \beta \|F\|^2 + 2 \|H\|^2 \geq 2 \beta \|F + H\|^2.
\]

On the last line, we have used \(\|S^2S^2\| \geq 2\beta\). This concludes the proof of \(\|G\|^2\).

Now, let \(G \in T(\ell^2_0(V, \pi))\). Then again, by looking at what it means to have \(S^2S^2(e,e') > 0\), we see that if \(y,y'\) are two vertices such that \(\text{dist}(y,y') = 2\) (in other words, \(y\) and \(y'\) have a common neighbour \(x\)), then we can find edges \(e,e'\) such that \(t(e) = y, t(e') = y'\) and \(S^2S^2(e,e') \geq Q^{-4}\). Indeed, we may choose \(e,e'\) such that \(t(e) = y\) and \(o(e) \neq x\), \(t(e') = y'\) and \(o(e') \neq x\), and \(S^2S^2(e,e') \geq (D(x) - 2)Q^{-4} \geq Q^{-4}\).

Thus

\[
\langle G, (I - S^2S^2) G \rangle_{\ell^2(B,U)} = \frac{1}{2} \sum_{e,e'} |G(e) - G(e')|^2 S^2S^2(e,e') \geq \frac{Q^{-4}}{2} \sum_{x} \sum_{y,y' \sim x} |G(y) - G(y')|^2 \geq Q^{-4} \beta \|G\|^2.
\]
Remark 2.2. We can also write (using the fact that \(S^*S^2\) is stochastic)

\[
\langle G, (I - S^*S^2)G \rangle_{\mathcal{E}(B,U)} = \frac{1}{2} \sum_{y,y':d(y,y')=2} |G(y) - G(y')|^2 \sum_{e,e':d(e)=y,f(e)=y'} S^*S^2(e, e')
\]

\[
\leq D^2 \frac{1}{2} \sum_{x \in V} \frac{1}{D(x)} \sum_{y,y':y \sim y'} |G(y) - G(y')|^2 = D^2 \langle G, (I - D^2)G \rangle_{\mathcal{E}(V,x)}
\]

and this proves part (i) of Theorem 1.3.

Let \(A > 1\) (to be chosen later, depending on \(\beta\) and \(D\)). Let \(f = F + G + H\) as before. Assume first that \(A\|F + H\| \geq \|G\|\). Then by the triangular inequality \(\|f\| \leq (1 + A)\|F + H\|\). In addition, as we have seen,

\[
\langle f, (I - S^*S^2)f \rangle_{\mathcal{E}(B,U)} \geq Q^{-4}\beta \|F + H\|_2^2 \geq Q^{-4}\beta (1 - A)^{-2} \|f\|^2.
\]

Otherwise, \(A\|F + H\| \leq \|G\|\), and \(\|f\| \leq (1 + A)\|G\|\). Noting that the operator norm of \(I - S^*S^2\) is less than 1, we write for all \(f = F + G + H\),

\[
\langle f, (I - S^*S^2)f \rangle_{\mathcal{E}(B,U)} \geq \langle G, (I - S^*S^2)G \rangle_{\mathcal{E}(B,U)} - 2A^{-1}\|G\|^2 - A^{-2}\|G\|^2
\]

\[
\geq (Q^{-4}\beta - 3A^{-1})\|G\|^2 \geq \frac{(Q^{-4}\beta - 3A^{-1})}{(1 + A^{-2})^2} \|f\|^2.
\]

Choosing \(A\) such that \(A^{-1} = Q^{-4}\beta / 6\), and gathering (15) and (14) we get the result with

\[
c(D, \beta) = \min \left( \frac{Q^{-4}\beta}{2(1 + Q^{-4}\beta/6)^2}, \frac{Q^{-4}\beta}{(1 + 6Q^4/\beta)^2} \right).
\]

3. Proof of the determinant relation

The relations in the next lemma follow from the resolvent identity, and are proven (for instance) in [3]. For a vertex \(v\) of \(T\), \(\mathcal{N}_v\) stands for the set of neighbouring vertices.

Lemma 3.1. For any \(v \in V(T)\), \(z = E + i\eta \in \mathbb{C}^+\), if we let \(2m^\#(v) = -\frac{1}{G(v,v,z)}\), we have

\[
z = \sum_{w \sim v} \zeta^z(v,u) + 2m^\#(v)\quad \text{and} \quad z = \sum_{u \in \mathcal{N}_v \setminus \{w\}} \zeta^z(v,u) + \frac{1}{\zeta^z(w,v)}.
\]

For any non-backtracking path \((v_0, \ldots, v_k)\) in \(T\),

\[
G(v_0, v_k; z) = -\prod_{j=0}^{k-1} \zeta^z(v_{j+1}, v_j) = -\prod_{j=0}^{k-1} \zeta^z(v_j, v_{j+1}) \frac{1}{2m^\#(v_k)}.
\]

Also, for any \(w \sim v\), we have

\[
\zeta^z(w,v) = \frac{m(w)^2}{m(v)^2} \zeta^z(v,w), \quad \frac{1}{\zeta^z(w,v)} - \zeta^z(w,v) = 2m^\#(v),
\]

Remark 3.2. We can note that (17) may be written as

\[
((\zeta^z)^{-1})^{B} - B)^{-1}(e,e') = \delta_{x=y} + \sum_{k=0}^{+\infty} \zeta^z(e') (\zeta^z B)^k(e,e') = -2m^\#(x)(A - z)^{-1}(x,y)
\]

for all \(e,e' \in B\) and \(x = o(e'), y = t(e)\).

In the case of regular graphs, this is formula (2.4) in [22], where it is attributed to Grigororchuk, with various proofs published by Woess, Szwarc [26], Northshield, Bartholdi [21, 4].
3.1. **Operator relations.** In this section $z \in \mathbb{C}^+$ is fixed, so we write $\zeta(x,y)$ instead of $\zeta^z(x,y)$, $m(x)$ instead of $m^z(x)$. If $e = (x,y) \in B$, we write $m_1(e) = m(x)$ and $m_2(e) = m(y)$. A function on $B$ defines a multiplication operator on $\ell^2(B)$ (i.e. an operator which is diagonal in the canonical basis). We use the same notation for a function and the associated operator.

Let us introduce the notation

$$\mathcal{P} f(x) = \frac{1}{D(x)} \sum_{y \sim x} f(x,y).$$

This is a projector on the space of functions depending only on the origin, which may be identified with $\ell^2(V, \pi)$, isometrically embedded into $\ell^2(B)$ by the map $\psi \mapsto O(\psi)$ defined in the previous section.

Let $L = D(2m_1)^{-1}\mathcal{P}$. Let

$$Hg(x) = \sum_{y,y \sim x} \frac{1}{2m(y)} (\zeta(y,x)g(y,x) - g(x,y))$$

Theorem 1.5 is based on the following exact relation:

**Proposition 1.**

$$H \circ (\zeta^{-1}I - B) = (A - zI) \circ L$$

**Proof.** Let $\phi = Lf$ and $g = -(\zeta^{-1}I - B)f$. The latter relation implies that for any $y \sim x$,

$$\phi(x) = (2m(x))^{-1}\left( f(x,y) + g(x,y) + \frac{f(x,y)}{\zeta(x,y)} \right).$$

We then calculate

$$\mathcal{A} \phi(x) = \sum_{y,y \sim x} \phi(y) = \sum_{y,y \sim x} \frac{1}{2m(y)} \left( f(y,x) + g(x,y) + \frac{f(x,y)}{\zeta(x,y)} \right).$$

We now use Lemma 3.1 to write

$$\sum_{y,y \sim x} \frac{1}{2m(y)} f(y,x) = \sum_{y,y \sim x} \frac{1}{2m(y)} (\zeta(y,x)2m(x)\phi(x) - \zeta(y,x)f(x,y) - \zeta(y,x)g(y,x))$$

$$= \sum_{y,y \sim x} \zeta(x,y)\phi(x) + \sum_{y,y \sim x} \frac{1}{2m(y)} (-\zeta(y,x)f(x,y) - \zeta(y,x)g(y,x))$$

$$= (z - 2m(x))\phi(x) + \sum_{y,y \sim x} \frac{1}{2m(y)} (-\zeta(y,x)f(x,y) - \zeta(y,x)g(y,x)).$$

Altogether,

$$\mathcal{A} \phi(x) = (z - 2m(x))\phi(x) + \sum_{y,y \sim x} \frac{1}{2m(y)} \left( \frac{1}{\zeta(x,y)} \right) f(x,y) - \zeta(y,x)g(y,x) + g(x,y)$$

$$= (z - 2m(x))\phi(x) + \sum_{y,y \sim x} f(x,y) + \sum_{y,y \sim x} \frac{1}{2m(y)} (-\zeta(y,x)g(y,x) + g(x,y))$$

$$= (z - 2m(x))\phi(x) + 2m(x)\phi(x) + \sum_{y,y \sim x} \frac{1}{2m(y)} (-\zeta(y,x)g(y,x) + g(x,y))$$

$$= z\phi(x) + \sum_{y,y \sim x} \frac{1}{2m(y)} (-\zeta(y,x)g(y,x) + g(x,y))$$

which is the desired relation. \qed
We note that \( Hg = -D \left( \mathcal{P}(2m_2)^{-1} g \right) + \mathcal{P}(2m_2)^{-1} \nu(\zeta g) \) where \( \nu f(x, y) = f(y, x) \) is the edge reversal involution. So \( H \) itself is of the form \( H = D \mathcal{P} \circ K \) where \( K = (2m_2)^{-1}(\iota \zeta - I) \). We have proven that

\[
D \mathcal{P} \circ K \circ (\zeta^{-1} I - B) = (A - z) \circ (2m_1)^{-1} D \mathcal{P}.
\]

This is equivalent to the two relations

\[
\mathcal{P} \circ K \circ (\zeta^{-1} I - B) \circ P = D^{-1}(A - z) \circ (2m_1)^{-1} D \mathcal{P}
\]

and

\[
\mathcal{P} \circ K \circ (\zeta^{-1} I - B) \circ (I - \mathcal{P}) = 0.
\]

The latter implies that \( K \circ (\zeta^{-1} I - B) \) sends \( \text{Ker} \mathcal{P} \) to itself.

We use the decomposition \( \mathbb{C}^B = \text{Im} \mathcal{P} \oplus \text{Ker} \mathcal{P} \). The two relations above tell us that

\[
\det[K \circ (\zeta^{-1} I - B)] = \det[(A - z) \circ (2m_1)^{-1}] \times \det[K \circ (\zeta^{-1} I - B)_{\text{Ker} \mathcal{P} \rightarrow \text{Ker} \mathcal{P}}.
\]

But \( f \in \text{Ker} \mathcal{P} \) is equivalent to \( Bf = -\nu f \). Thus if \( f \in \text{Ker} \mathcal{P} \),

\[
K \circ (\zeta^{-1} I - B)f(x, y) = (2m_1)^{-1} f(x, y) \left( \zeta(x, y) - \frac{1}{\zeta(x, y)} \right) = -f(x, y).
\]

Finally we obtain

\[
\det K \det(\zeta^{-1} I - B) = \det(A - z) \prod_{x \in V} (2m(x))^{-1} (-1)^{\dim \text{Ker} \mathcal{P}} = \det(A - z) \prod_{x \in V} (-G(x)) (-1)^{\dim \text{Ker} \mathcal{P}}
\]

\[
= \det(A - z) \prod_{x \in V} (-G(x)) (-1)^{|B||V|} = \det(z - A) \prod_{x \in V} (-G(x))
\]

since \( |B| = 2|E| \) is even.

To prove the determinant relation, there remains to compute \( \det K \). But \( K \) is diagonal by blocks of size 2 in the canonical basis of \( \mathbb{C}^B \). More precisely, denoting by \( \delta_e \) the element of \( \mathbb{C}^B \) that takes the value 1 on \( e \) and 0 elsewhere,

\[
K \delta_e = -\frac{1}{2m(y)} \delta_e + \frac{\zeta(x, y)}{2m(x)} \delta_e
\]

if \( e = (x, y) \). Thus

\[
\det K = \prod_{e = (x, y) \in E} (2m(x))^{-1}(2m(y))^{-1}(1 - \zeta(x, y)\zeta(y, x)) = \prod_{e = (x, y) \in E} (-G(x, y)).
\]

This yields the announced relation.

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