On a Watson-like Uniqueness Theorem and Gevrey Expansions.

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Abstract

We present a maximal class of analytic functions, elements of which are in one-to-one correspondence with their asymptotic expansions. In recent decades it has been realized (B. Malgrange, J. Ecalle, J.-P. Ramis, Y. Sibuya et al.), that the formal power series solutions of a wide range of systems of ordinary (even non-linear) analytic differential equations are in fact the Gevrey expansions for the regular solutions. Watson’s uniqueness theorem belongs to the foundations of this new theory. This paper contains a discussion of an extension of Watson’s uniqueness theorem for classes of functions which admit a Gevrey expansion in angular regions of the complex plane with opening less than or equal to $\frac{\pi}{k}$, where $k$ is the order of the Gevrey expansion. We present conditions which ensure uniqueness and which suggest an extension of Watson’s representation theorem. These results may be applied for solutions of certain classes of differential equations to obtain the best accuracy estimate for the deviation of a solution from a finite sum of the corresponding Gevrey expansion.

- $\mathbb{C}$ stands for the complex plane;
- $S(\alpha, \beta)$ stands for the sector
  \[ S(\alpha, \beta) = \{ z \in \mathbb{C} : 0 < |z| < \infty, \alpha < \arg z < \beta \} \]
  where the number $\beta - \alpha$ is said to be the opening of the sector $S(\alpha, \beta)$.

Introduction. In 1912 G. Watson published the following result, [Wat1], section 8, Theorem V.

**Watson’s uniqueness theorem.** Let $\{p_0, p_1, \ldots\}$ be a given sequence of complex numbers, and $P(z)$ be a function satisfying the conditions:

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(i) $P(z)$ is analytic and single-valued in the sector $S(\alpha, \beta)$;

(ii) $P(z)$ admits the following series of estimates for all $z \in S(\alpha, \beta)$ except $|z| < \sigma$

\[
|P(z) - \sum_{k=0}^{n-1} \frac{p_k}{z^{k+1}}| \leq M \frac{n!}{a^n |z|^{n+1}}, \quad n = 0, 1, \ldots
\]  

(1)

where positive constants $M$, $a$ and $\sigma$ do not depend on $z$ or $n$.

If the opening of the sector $S(\alpha, \beta)$ satisfies the condition $\beta - \alpha > \pi$, then the function $P(z)$ is uniquely determined by conditions (i) and (ii): two functions $P_1(z)$ and $P_2(z)$ satisfying conditions (i) and (ii) with the same sequence $\{p_0, p_1, \ldots\}$ must coincide on $S(\alpha, \beta)$.

Set

\[
\hat{P}(z) = \sum_{k=0}^{\infty} \frac{p_k}{z^{k+1}}.
\]  

(2)

The formal power series $\hat{P}(z)$ is known as the Gevrey expansion of order 1 for a function $P(z)$ satisfying conditions (i) and (ii).

In recent times it has been realized (B. Malgrange, J. Ecalle, J.-P. Ramis, Y. Sibuya et al.), that the formal power series solutions of a wide range of systems of ordinary (even non-linear) analytic differential equations are in fact the Gevrey expansions for the regular solutions, see, for example, [Ra1], [RaSt] and [RaSt1]. Watson’s uniqueness theorem belongs to the foundations of this new theory.

To prove Watson’s uniqueness theorem we introduce the function

\[
P(z) = P_1(z) - P_2(z)
\]

which satisfies the inequalities

\[
|P(z)| \leq 2M \frac{n!}{a^n |z|^{n+1}}, \quad n = 0, 1, \ldots, \quad z \in S(\alpha, \beta).
\]  

(3)

Minimizing the right-hand side of (3), for fixed $z$, $|z| > \frac{1}{a}$, with respect to $n$ yields the inequality

\[
|P(z)| \leq M_a e^{-a|z|}
\]  

(4)

where $M_a = 4M \sqrt{2\pi a}$.

Thus, Watson’s uniqueness theorem can be derived from the following fact, a proof of which may be found in [Har], section 8.11.

**Lemma 1.** Let $P(z)$ be an analytic function in the sector $S_\varepsilon = S\left(-\frac{\pi}{2}(1+\varepsilon), \frac{\pi}{2}(1+\varepsilon)\right)$, $0 < \varepsilon < 1$, satisfying the following estimate

\[
|P(z)| \leq M e^{-a|z|} \quad \text{for all } z \in S_\varepsilon,
\]  

(5)
where \( a \) is a positive constant. Then \( P(z) \equiv 0 \).

We note that, since the estimate (4) holds for \( z \in S(\alpha, \beta) \backslash \{ z : |z| < \frac{1}{a} \} \), it holds also in the shifted sector \( S(\alpha, \beta) + de^{i\frac{\alpha + \beta}{2}} \) where \( d = \frac{1}{a \cos \frac{\pi}{2}} \). Thus, strictly speaking, lemma 1 should be applied rather to the shifted function \( P ((z + d) e^{i\frac{\alpha + \beta}{2}}) \).

If the opening of the sector \( S(\alpha, \beta) \) satisfies the condition \( \beta - \alpha < \pi \), then the above uniqueness result is indisputably false. Indeed, given \( \delta, 0 < \delta < \frac{\pi}{2} \), every function \( P(z) \) of the form

\[
P(z) = \frac{\varphi(z)}{z} e^{-z},
\]

(6)

where \( \varphi(z) \) is analytic and bounded in the sector \( S\left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right) \), satisfies the conditions (i) with

\[
M = \sup_{S\left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right)} |\varphi(z)|,
\]

\( a = \sin \delta \) and \( p_k = 0, k = 0, 1, \ldots \). This can be easily derived using the elementary inequalities

\[
e^{-|z|} < \frac{n!}{|z|^n}, \quad n = 0, 1, \ldots.
\]

Thus, in this case the set of functions which satisfy conditions (i) and (ii) of Watson’s theorem with \( \{ p_0, p_1, \ldots \} = \{0, 0, \ldots\} \) is rather large.

The two opposed cases, \( \beta - \alpha > \pi \) and \( \beta - \alpha < \pi \), discussed above lead us to regard the case \( \beta - \alpha = \pi \) as the critical case and the value \( \pi \) as the critical value of the opening \( \beta - \alpha \). If the opening of \( S(\alpha, \beta) \) takes this critical value the situation is much more delicate.

However, the statement of Lemma 1 can be extended to this critical case.

**Lemma 2.** Let \( P(z) \) be an analytic function in the sector \( S\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \) satisfying the following estimate

\[
|P(z)| \leq Me^{-a|z|} \quad \text{for all } z \in S\left(-\frac{\pi}{2}, \frac{\pi}{2}\right),
\]

(7)

where \( a \) is a positive constant. Then \( P(z) \equiv 0 \).

**Proof of Lemma 2.** We demonstrate three different proofs of Lemma 2.

1. Although Hardy’s argument in [Har], section 8.11 for the proof of lemma 1 is not applicable in this case it is still possible to apply the Phragmén-Lindelöf theorem to prove the result, and the proof can be made even simpler. Given \( k, k > 0 \), consider the function \( P_k(z) = P(z)e^{kz} \) inside the sector \( S(-\text{arctan} \frac{k}{a}, \text{arctan} \frac{k}{a}) \) where \( P(z) \) is a function satisfying the conditions of lemma 2. On the boundary of this sector we have
a|Imz| − kRez = 0 and so |P_k(z)| ≤ M. The standard Phragmén-Lindelöf theorem yields the same estimate inside the sector. Thus, for z > 0 we have |P(z)| ≤ Me^{-kz}. As k is independent of z we can let k tend to +∞ to obtain P(z) ≡ 0. ▲

(2) Alternatively, the condition (7) allows one to use the following well-known theorem of complex analysis.

**Theorem.** If P(z) is analytic and bounded in the sector S(−\(\frac{\pi}{2}\), \(\frac{\pi}{2}\)) and for some c > 0

\[
\int_{c-\infty}^{c+i\infty} \log |P(z)| \frac{1}{1+|z|^2} d|z| = -\infty,
\]

then P(z) ≡ 0.

Indeed, if P(z) satisfies (7) then P(z) satisfies (8). ▲

(3) It is also possible to use other arguments to prove Lemma 2; the idea of these arguments will be employed and extended later to prove the main result of our paper. Let P(z) be a function satisfying the conditions of lemma 2. Introduce the function

\[
F(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} P(z) e^{tz} dz.
\]

Condition (7) ensures the absolute convergence of the integral in (9) for every complex t, −a < Im t < a. Thus F(t) is an analytic function in the strip \{t : |Im t| < a\} of \(\mathbb{C}_t\). On the other hand, for t ∈ (−∞, a) the function \(P_t(z) = P(z)e^{tz}\) is analytic and bounded, |\(P_t(z)\)| ≤ M, in the half-plane \{z : Rez > 0\} of \(\mathbb{C}_z\). Moreover, \(P(z)e^{tz} \to 0\) as \(|z| \to \infty\). Using the Cauchy theorem this yields \(F(t) = 0\) for t < a. From the uniqueness theorem for analytic functions it follows that F(t) ≡ 0. Thus, P(z) ≡ 0. ▲

Lemma 2 shows that the conclusion of Watson’s uniqueness theorem, as stated above, still remains valid for the critical sector, that is the sector \(S(\alpha, \beta)\) with the critical values of its opening satisfying \(\beta - \alpha = \pi\).

However, and this will be explained below, a certain weakening of condition (ii) of Watson’s uniqueness theorem leads to a loss of the uniqueness property.

**Comment 1.** It is worthwhile noting that Watson’s uniqueness theorem can be transplanted to any sector \(S(\alpha, \beta)\) of the Riemann surface of log z with opening greater than or equal to \(\frac{\pi}{k}\), where k is a positive number. The following extension of Watson’s uniqueness theorem is valid.

\[\text{This theorem follows from the so-called Jensen inequality on the summability of the logarithmic integral of a bounded analytic function. The integral on the left-hand side of (8) generated an extensive literature, culminating in the two-volume monograph by Paul Koosis, The logarithmic integral, I and II, Cambridge Study in Advance Mathematics, Volumes 12 and 21, Cambridge University Press, (1992), (1998).}\]
Watson’s uniqueness theorem for order $k$. Let $k$ be a positive number. Let $\{p_0, p_1, \ldots\}$ be a given sequence of complex numbers, and $P(z)$ a function satisfying the conditions:

(i) $P(z)$ is analytic and single-valued in the sector $S(\alpha, \beta)$;

(ii) $P(z)$ admits the following series of estimates for all $z \in S(\alpha, \beta)$ except $|z| < \sigma$

\[
\left| P(z) - \sum_{j=0}^{n-1} \frac{p_j}{z^{j+1}} \right| \leq M \frac{(n!)^{\frac{1}{k}}}{(ka|z|)^{n+1}}, \quad n = 0, 1, \ldots, \text{for all,} \quad (10)
\]

where positive constants $M$, $a$ and $\sigma$ do not depend on $z$ or $n$.

If the opening of the sector $S(\alpha, \beta)$ satisfies the condition $\beta - \alpha \geq \frac{\pi}{k}$, then the function $P(z)$ is uniquely determined by the conditions (i) and (ii). ▲

This theorem follows immediately from the corresponding extension of Lemma 2 using the map $\zeta = \frac{z}{k}$. Comparing again the two opposed cases $\beta - \alpha > \frac{\pi}{k}$ and $\beta - \alpha < \frac{\pi}{k}$ shows that for this situation a sector $S(\alpha, \beta)$ of the Riemann surface of log $z$ will be a critical sector if its opening $\beta - \alpha$ is equal to $\frac{\pi}{k}$.

Expansions satisfying conditions similar to (10) are known as Gevrey expansions of order $k$, (see [Gev], [Ra2] and [Ra1].)

Watson’s uniqueness theorem was also extended by T. Carleman, see [Ca]. Carleman replaced $n!$ by a sequence of positive numbers $m_n$ and, under certain regularity conditions on the growth of $m_n$ as $n \to \infty$, he gave necessary and sufficient conditions for uniqueness.

The main theorem. We develop Watson’s result in a different direction, keeping in mind the possible application of this development to the analytic theory of differential equations in the complex plane. Given $\delta$, $0 < \delta < \frac{\pi}{2}$, we consider a function $P(z)$ which satisfies almost all the conditions of Watson’s uniqueness theorem, the only difference being that in [1] the condition $z \in S(\alpha, \beta)$ is replaced by $z \in S\left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right)$ and the positive constant $M$ in [1] is replaced by a positive function $M(\delta)$ which may depend on $\delta$. We plan to consider later the case in which the constant $a$ in [1] is replaced by a positive function $a(\delta)$ which may also depend on $\delta$.

Consider a positive function $M(\delta)$, defined on the interval $0 < \delta < \frac{\pi}{2}$, where, for simplicity, we assume that $\log M(\delta) > 1$. Our aim is to clarify under what conditions on $M(\delta)$ the set of functions satisfying [1] in every sub-sector $S\left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right)$ of $S\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, with $M = M(\delta)$, possesses a uniqueness property as stated in Watson’s uniqueness theorem.
The following extension of Watson's uniqueness theorem for the critical sector \( S \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \) is valid.

**Theorem 1.** Let \( \{p_0, p_1, \ldots\} \) be a given sequence of complex numbers, and \( P(z) \) be a function satisfying the conditions:

(i) \( P(z) \) is analytic and single-valued in the sector \( S \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); \)

(ii) For every \( \delta, 0 < \delta < \frac{\pi}{2} \), \( P(z) \) admits the following series of estimates

\[
|P(z) - \sum_{k=0}^{n-1} \frac{p_k}{z^{k+1}}| \leq K_P M(\delta) \frac{n!}{\alpha^n |z|^{n+1}}, \quad n = 0, 1, \ldots, \quad (11)
\]

for all \( z \in S\left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right), |z| > \sigma, \)

where \( \alpha, \sigma \) and \( K_P \) are positive constants which do not depend on \( z, n \) or \( \delta \), but may depend on \( P(z) \).

Assume that \( M(\delta) \) satisfies the estimate

\[
\int_{0}^{\frac{\pi}{2}} \log \log M(\delta)d\delta < \infty, \quad (12)
\]

then the function \( P(z) \) is uniquely determined by the formal power series \( \hat{P}(z) = \sum_{k=0}^{\infty} p_k/z^{k+1} \): two functions \( P_1(z) \) and \( P_2(z) \) satisfying conditions (i) and (ii) with the same sequence \( \{p_0, p_1, \ldots\} \) must coincide on \( S \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \)

**Corollary.** If the function \( M(\delta) \) in theorem 1 is of the form

\[
M(\delta) = M \exp \left( \frac{b}{\delta^\gamma} \right) \quad (13)
\]

for some positive numbers \( M, b, \) and \( \gamma \) which do not depend on \( \delta \), then the conclusion of the theorem remains true.

**Stirling’s formula.** Analytic functions, satisfying (11), (10) or (11) have been known in analysis for a long time.

We recall an example based upon Stirling’s Formula for Euler’s Gamma Function, \( \Gamma (z) \), which can be defined as

\[
\Gamma (z) = \int_{0}^{\infty} t^{z-1}e^{-t}dt.
\]

This function is analytic in \( S \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \), but it can be continued analytically to the whole of \( \mathbb{C} \) cut along the negative ray, that is to \( S (-\pi, \pi) \), since \( \frac{1}{\Gamma(z)} \) is an entire function with no zeros in the cut plane.
Stirling showed that
\[
\ln \Gamma(z) = \left( z - \frac{1}{2} \right) \ln z - z + \frac{1}{2} \ln 2\pi + o(1), \; z \to +\infty. \tag{14}
\]

We consider the so-called Binet function
\[
P(z) = \ln \Gamma(z) - \left( z - \frac{1}{2} \right) \ln z + \frac{1}{2} \ln 2\pi.
\tag{15}
\]
and we will associate with this function a sequence \(\{p_0, p_1, \ldots\}\) such that the relations (14) are valid for some choice of \(M\) and \(a\).

According to the first Binet formula we have
\[
P(z) = \int_0^\infty e^{-zt} F(t) dt \tag{16}
\]
where
\[
F(t) = \frac{1}{t} \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right). \tag{17}
\]
As \(F(t)\) is analytic in the disc \(|t| < 2\pi\) it can be represented inside the disc by its Taylor series
\[
F(t) = \sum_{k=0}^\infty f_k t^k.
\tag{18}
\]
Substituting \(F(t)\) for \(F(t)\) in (16) yields a formal power series
\[
\hat{P}(z) = \sum_{k=0}^\infty \frac{p_k}{z^{k+1}}, \; p_k = f_k k!.
\tag{19}
\]
Since \(F(t)\) is an even function, \(F(-t) = F(t)\), it follows from (13) that \(p_{2k-1} = 0\), for \(k = 1, 2, \ldots\), and further analysis shows that
\[
p_{2k-2} = \frac{B_{2k}}{2k(2k - 1)}, \; k = 1, 2, \ldots. \tag{20}
\]
Here \(B_{2k}\), \(k = 0, 1, \ldots\), are the Bernoulli numbers, which are defined as the coefficients of the Taylor series
\[
\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{k=2}^\infty \frac{B_{2k}}{(2k)!} t^{2k}, \tag{21}
\]
see, for example, formulas 23.1.1-23.1.3 and 6.1.42 of [Abr]. Thus \(P(z)\) can be re-written in the form
\[
\hat{P}(z) = \sum_{k=1}^\infty \frac{B_{2k}}{2k(2k - 1)z^{2k-1}}, \tag{22}
\]
and this series is known as Stirling’s series.\(^2\)

Using (16) one can show that \(\hat{P}(z)\) is the Gevrey expansion for \(P(z)\) in \(S\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\). Moreover, the following series of estimates is valid

\[
|P(z)| \leq K(z) \frac{|B_2|}{2|z|},
\]

\[
|P(z) - \sum_{k=1}^{n} \frac{p_{2k-2}}{z^{2k-1}}| \leq K(z) \frac{|B_{2(n+1)}|}{(2n+2)(2n+1)|z|^{2n+1}},
\]

(23)

\[
n = 1, \ldots, 0 < |z| < +\infty, z \in S\left(-\frac{\pi}{2}, \frac{\pi}{2}\right),
\]

where

\[
K(z) = \max_{u \geq 0} \left| \frac{z^2}{u^2 + z^2} \right|,
\]

(24)

(see, for example, formula 6.1.42 of [Abr]).

Using known asymptotics for the Bernoulli numbers, see formula 23.1.15 of [Abr], we have

\[
B_{2n+2} = (-1)^{n+2} \frac{2(2n+2)!}{(2\pi)^{2n+2}} (1 + o(1)), k \to \infty,
\]

(25)

and it follows that \(23\) is of the form given in (1) with \(a = 2\pi\). Thus, the Stirling series \(22\) is the Gevrey expansion for \(P(z)\) in the sector \(S\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\).

**Comment 2.** The importance of Stirling’s example for this subject lies in the following observations:

(i) It follows from (25) that the estimate (23), with the sequence \(\{p_0, p_2, \ldots\}\) given by (24), ensures uniqueness in the sector \(S\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\), with opening equal to \(\pi\). This is despite the fact that if \(z\) belongs to the boundary of \(S\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\) then the value of \(K(z)\) in the right-hand side of (23) is equal to \(\infty\).

Since the function \(K(z)\) in (26) can be represented as

\[
K(z) = \begin{cases} 
1 & \text{if } -\frac{\pi}{4} < \arg z < \frac{\pi}{4}, \\
\frac{1}{\sin(2\pi \arg z)} & \text{if } \frac{\pi}{4} \leq |\arg z| < \frac{\pi}{2}.
\end{cases}
\]

(26)

it follows that the function \(P(z)\) given by (15) satisfies all the conditions of Theorem 1, with \(M(\delta) = \frac{1}{\delta}\) and \(\sigma = 0\).

(ii) Substituting \(n_{opt}(|z|) = [\pi |z| - 1]\) for \(n\) in the inequality (23)\(^3\) yields

\[
|P(z) - \sum_{k=1}^{[\pi |z| - 1]} \frac{p_{2k-2}}{z^{2k-1}}| \leq K(z) \frac{2\sqrt{2\pi |z|}}{2\pi |z| - 1} e^{-2\pi |z|}.
\]

(27)

\(^2\)A similar definition for Stirling’s series is given in [Whi], but the notation \(B_k\) used there refers to \((-1)^{k+1}B_{2k}\), where \(B_{2k}\) is defined as above in (21).

\(^3\)This value of \(n = n_{opt}(|z|)\) can be guessed by minimizing the expression on the right-hand side of (23) for given \(|z|\).
For given fixed \( z \in S \left( -\frac{\pi}{4}, \frac{\pi}{4} \right) \) the estimate (27) gives the best possible accuracy when replacing \( P(z) \) by the optimal finite sum of Stirling’s series, and it follows from (26) that, for \( |z| > 1 \), the error is less than

\[
M e^{-a|z|},
\]

where \( M = .94891 \) and \( a = 2\pi \).

(iii) It also follows from (16)-(17) that \( \hat{P}(z) \) is a Gevrey expansion of \( P(z) \) in the whole region \( S(-\pi, \pi) \). However, the estimate (23) with \( a = 2\pi \) is valid in \( S \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \) only. It can be shown that in a sector \( S \left( -\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon \right) \) estimates similar to (11) are valid but the constant \( a \) in these estimates is less then \( 2\pi \). Moreover, the constant \( a \) for the sector \( S \left( -\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon \right) \) depends on \( \varepsilon \). For this example it can be proved that

\[
a(\varepsilon) = 2\pi \cos \varepsilon.
\]

One can derive a necessary and sufficient condition for the uniqueness following, for example, the technique of Ahlfors’ distortion theorem, [Ahl], and its complement and refinement by Warschawski, [War]. We intend to return to this question in a later publication. Keeping in mind that we will consider later the case in which the parameter \( a \) in (11) also depends on \( \delta \), and to apply our technique to the reconstruction problem, see Watson’s representation theorem and theorem 2, pages 15-16, we prefer another approach which we demonstrate below.

**The main Lemma.** Theorem 1 follows from the following generalization of Lemma 2.

**Lemma 3.** Assume that \( P(z) \) is analytic in the right half-plane of the \( z \)-plane. Assume further that for some \( a > 0 \), for all \( \delta > 0 \), and for all \( z \), \( z \in S \left( -\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta \right) \), the function \( P(z) \) satisfies the following estimate

\[
|P(z)| < M(\delta)e^{-a|z|},
\]

where \( M(\delta) \), \( \log M(\delta) > 1 \), \( \delta \in (0, \frac{\pi}{2}) \) satisfies condition (12). Then

\[
P(z) \equiv 0.
\]

**Proof of Lemma 3.** Unfortunately we cannot integrate the expression \( e^{zt}P(z) \) along the imaginary ray as we did in (11), nor, in general, can we integrate along a line parallel to it. This obstacle is quite typical of such situations and we will show a way in which it may be overcome.

Given \( \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2} \), introduce the Laplace transform

\[
F_\theta(t) = \int_{t_0} e^{zt}P(z) \, dz,
\]

where \( t \) is in the complex plane. If \( P(z) \) is analytic in \( S \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \) and \( t \) is in the region \( S \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \), then

\[
F_\theta(t) = \int_{t_0} e^{zt} P(z) \, dz.
\]

We need to show that

\[
F_\theta(t) \equiv 0.
\]
where

\[ l_\theta = \{ z : \arg z = \theta \} . \]

In what follows we will show that:

(i) The function \( F_\theta (t) \) can be continued analytically to an entire function \( F(t) \) which does not depend on \( \theta \).

(ii) The function \( F(t) \) is bounded outside any sector \( S_\delta, 0 < \delta < \frac{\pi}{2} \), with angle \( 2\delta \), which has its apex at a point \( \frac{a}{2 \sin \delta} \), contains the interval \( (\frac{a}{2 \sin \delta}, +\infty) \), and is symmetric with respect to the real line.

(iii) On the boundary \( \partial S_\delta \) of the sector \( S_\delta \) we have \( |F(t)| \leq \frac{2M(\delta)}{a} \).

(iv) The function \( F(t) \) satisfies all the conditions of Carleman’s theorem, see \cite{Ca1}, which yields \( F(t) \equiv 0 \).

**Remark 3.** Victor Havin, to whom we have shown our result, called our attention to the following fact. If one replaces the condition (30) with \( M(\delta) \) satisfying (12) by the stronger condition

\[ |P(z)| < M \exp \left( \frac{b}{\delta} \right) e^{-a|z|}, \]  

(33)

for some \( b > 0 \), then for any \( c, 0 < c < a \), there exists \( h > 0 \) and \( M_h > 0 \) such that

\[ |P(h + iy)| < M_h e^{-c|y|}. \]

(34)

Using the conditions of Lemma 2, the inequality (34) yields immediately \( P(z) \equiv 0 \).

To prove (34) one needs only to note that if \( z = x + iy \) and

\[ \frac{x}{|y|} = \tan \delta, \]

then the inequality (33) can be re-written

\[ |P(z)| < M \exp \left( \frac{b}{\arctan \frac{x}{|y|}} \right) e^{-a|z|}. \]

(35)

In turn, for

\[ |y| \gg x > 0, \]

(33) can be rewritten in the form

\[ |P(z)| < M' \exp \left( \frac{b|y|}{x} \right) e^{-a|y|}, \]

(36)

for some constant \( M' > 0 \), which clearly yields (34), after an appropriate choice of \( x = h \), for example,

\[ h = \frac{b}{a - c}. \]
Unfortunately, this beautiful argument does not cover the case of faster growth as $\delta \to 0$, as allowed in (12), and also, for example, in (13). ▲

We return now to the proof.

**Region of analyticity of** $F_\theta (t)$. To find the region of analyticity of $F_\theta (t)$ we provide the following estimate using (30),

$$|F_\theta (t)| < M(\delta) \int_{l_\theta} |e^{zt}| e^{-a|z|} |z| .$$  \hspace{1cm} (37)

Since

$$|e^{zt}| = e^{\frac{|z|}{2} (\sigma \cos \theta - \tau \sin \theta)}, \quad t = \sigma + i\tau,$$  \hspace{1cm} (38)

the integral exists and represents an analytic function in the half-plane

$$\Pi_{\theta,a} = \{ t \in \mathbb{C}_t : \sigma \cos \theta - \tau \sin \theta < a \} .$$  \hspace{1cm} (39)

In fact, the line

$$L_{\theta,a} = \{ t \in \mathbb{C}_t : \sigma \cos \theta - \tau \sin \theta = a \}$$  \hspace{1cm} (40)

divides the $t-$plane into two half-planes, and $\Pi_{\theta,a}$ is the half-plane containing the origin $t = 0$, see Figure 1, where we assume that $0 < \theta < \frac{\pi}{2}$.

Moreover, from (37) and (38) we have for $t \in \Pi_{\theta,a}$ the estimate

$$|F_\theta (t)| < M(\delta) \frac{1}{a - (\sigma \cos \theta - \tau \sin \theta)}$$

where

$$\delta = \left\{ \begin{array}{c} \frac{\pi}{2} - \theta, \theta > 0 \\ \frac{\pi}{2} + \theta, \theta < 0 \end{array} \right\}$$

\hspace{1cm} (41)

In what follows we will show that the functions $F_\theta (t), |\theta| < \frac{\pi}{2}$, are all elements of a single analytic function $F(t)$ which does not depend on $\theta$.

**Relationship between** $F_\theta (t)$ and $F_{-\theta} (t)$. Given $\delta, 0 < \delta < \frac{\pi}{2}$, consider the pair of functions $F_{\frac{\pi}{2} - \delta} (t)$ and $F_{\frac{\pi}{2} + \delta} (t)$ of the form (32) which are analytic in the half-planes $\Pi_{\frac{\pi}{2} - \delta,a}$ and $\Pi_{\frac{\pi}{2} + \delta,a}$ respectively. Now

$$\Pi_{\frac{\pi}{2} - \delta,a} = \{ t \in \mathbb{C}_t : \sigma \sin \delta - \tau \cos \delta < a \}$$

and

$$\Pi_{\frac{\pi}{2} + \delta,a} = \{ t \in \mathbb{C}_t : \sigma \sin \delta + \tau \cos \delta < a \}$$

and these half-planes are symmetric with respect to the real line of the $t-$plane.
On Figure 2 we show the two lines $L_{\frac{\pi}{2}-\delta,a}$ and $L_{\frac{\pi}{2}+\delta,a}$ crossing the positive ray of the $t$-plane at the point

$$t_{\delta,a} = \frac{a}{\sin \delta}, \quad (44)$$

at angles $\delta$ and $-\delta$ respectively.

These lines define two sectors, $S_l(\delta, a)$ (the left-hand sector) and $S_r(\delta, a)$ (the right-hand sector), both with their apexes at $t_{\delta,a}$ and with angle equal to $2\delta$.

Set

$$S_1(\delta, a) = \Pi_{\frac{\pi}{2}-\delta,a} \cap \Pi_{-\frac{\pi}{2}+\delta,a}$$

and

$$S_2(\delta, a) = \Pi_{\frac{\pi}{2}-\delta,a} \cup \Pi_{-\frac{\pi}{2}+\delta,a}. \quad (45)$$

The left-hand sector $S_l(\delta, a)$ is exactly $S_1(\delta, a)$, while the closure of the right-hand sector $S_r(\delta, a)$ is the complement of $S_2(\delta, a)$.

Using the representation (42) and Cauchy’s theorem yields

$$F_{\frac{\pi}{2}-\delta}(t) = F_{-\frac{\pi}{2}+\delta}(t), \quad t \in S_1(\delta, a).$$

Thus, $F_{\frac{\pi}{2}-\delta}(t)$ and $F_{-\frac{\pi}{2}+\delta}(t)$ can be considered as elements of the function $F(t, \delta)$ where

$$F(t, \delta) = \begin{cases} F_{\frac{\pi}{2}-\delta}(t), & t \in \Pi_{\frac{\pi}{2}-\delta,a} \\ F_{-\frac{\pi}{2}+\delta}(t), & t \in \Pi_{-\frac{\pi}{2}+\delta,a} \end{cases}. \quad (46)$$
and this function is analytic in $S_2(\delta, a)$.

However, $S_2(\delta, a)$ still does not give the maximal region of analyticity of the function $F(t)$ referred to in (i) of page 10.

**The maximal region of analyticity.** We require the following lemma.

**Lemma 4.** The functions $F(t, \delta)$ given by (46) are all elements of a single analytic function $F(t)$ which does not depend on $\delta$ and which is an entire function.

**Proof.** Consider $\delta' > \delta'' > 0$ and the corresponding functions $F(t, \delta')$ and $F(t, \delta'')$.

We have

$$S_r(\delta', a) \supset S_r(\delta'', a),$$

and so

$$C_t \setminus S_r(\delta', a) \subset C_t \setminus S_r(\delta'', a).$$

Using the same argument as that used above we see that

$$F(t, \delta') = F(t, \delta''), t \in C_t \setminus S_r(\delta', a),$$

and $F(t, \delta'')$ can be considered as an analytical continuation of $F(t, \delta')$ to the larger region $C_t \setminus S_r(\delta'', a)$.

The expression (44) for $t_{\delta, a}$ shows that

$$t_{\delta, a} \to \infty, \delta \to 0,$$

which yields

$$\bigcap_{0 < \delta < \frac{\pi}{2}} S_r(\delta', a) = \emptyset. \quad (47)$$

Thus the functions $F(t, \delta), 0 < \delta < \frac{\pi}{2}$, are all elements of a single function $F(t)$, which is an entire function, and so statement (i) is verified. ▲

**Estimates for $F(t)$ in the $t-$plane.** Using the estimate (41) for the region

$$\Pi_{\theta, a} = \Pi_{\theta, \frac{a}{2}} \cup L_{\theta, \frac{a}{2}}, \quad (48)$$

where $\Pi_{\theta, \frac{a}{2}}$ and $L_{\theta, \frac{a}{2}}$ are defined by (39) and (40) respectively, with $a$ replaced by $\frac{a}{2}$, yields

$$|F_{\theta}(t)| < \frac{2M(\delta)}{a}, t \in \Pi_{\theta, \frac{a}{2}}. \quad (49)$$

Hence, from (46) we have

$$|F(t, \delta)| < \frac{2M(\delta)}{a}, t \in S_2(\delta, \frac{a}{2}). \quad (50)$$
We now apply this estimate for the function \( F(t) \) given in (i). Now in the region \( S_2(\delta, \frac{\phi}{2}) \) the function \( F(t, \delta) \), and hence also the function \( F(t) \), is bounded, which verifies (ii). Moreover, it follows from (50) that \( F(t) \) satisfies the estimate
\[
|F(t)| \leq \frac{2M(\delta)}{a}.
\] (51)
in the sector \( S_2(\delta, \frac{\phi}{2}) \), and hence also on its boundary. This proves (iii).

The inequality (51) implies
\[
|F(|t| e^{\pm i\delta})| < \frac{2M(\delta)}{a}
\] (52)
which shows that the conditions of the following theorem of Carleman are satisfied.

Carleman’s Theorem, see [Ca1]. Let \( M(\phi) \) be a positive function (finite or infinite), for which \( \log M(\phi) > 1 \) and the following integral,
\[
\int_0^{2\pi} \log \log M(\phi)d\phi,
\] (53)
exists. Every entire function \( f(z) \) which satisfies the inequality
\[
|f(z)| < M(\phi), \ \phi = \arg z, \ \ 0 < \phi < 2\pi,
\] (54)
is a constant.4

Thus \( F(t) = C \) for some constant \( C \). To derive (iv) it remains to note that since for every \( \delta \) we have
\[
F(t, \delta) \to 0, \ t \to -\infty,
\]
it follows that
\[
F(t) \to 0, \ t \to -\infty.
\]
Thus
\[
F(t) \equiv 0.
\]
This proves (iv) and then
\[
P(z) \equiv 0
\]
follows immediately. ▲

Comment 4. (i) Our proof shows that it is possible to extend our result to the case in which \( a \) depends on \( \delta \). From (43) it follows that (47) holds only if
\[
\frac{a(\delta)}{\delta} \to +\infty \text{ as } \delta \to 0.
\] (55)

4If the function \( M(\phi) \) satisfies the stronger condition [38] this fact follows immediately from the Phragmén-Lindelöf theorem.
Thus (55) is a necessary condition for the uniqueness result to hold.

(ii) Carleman’s result was further developed by N. Levinson and N. Sjoberg (see the survey by M. Sodin, [Sod], for details) who obtained a more general result, from which there follows, in particular, an extension of Carleman’s theorem in which the sector is replaced by a half-strip. We believe that this extension may be of assistance in further investigating the restoration problem which we discuss below.

Watson’s uniqueness theorem shows that there is the possibility of reconstructing, in principle, a function $P(z)$ satisfying the conditions (i) and (ii) of this theorem from the formal power series $\hat{P}(z)$. This possibility can be realized using the Borel summation method. The corresponding result is known as Watson’s representation Theorem. The proof of this Theorem can be extracted from [Wat1], Section 9, though this statement has not been represented there in the form of a theorem. See also [Wat2], page 68, formulae (20), (21) and (22), and [Har], section 8.11.

**Watson’s representation (reconstruction) theorem.** Assume that the function $P(z)$ satisfies all the conditions of Watson’s uniqueness theorem in the sector

$$S \left( -\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon \right),$$

except $|z| < \sigma$, for some $\varepsilon, 0 < \varepsilon < \frac{\pi}{2}$.

Set

$$\hat{F}(t) = \sum_{n=0}^{\infty} \frac{p_n}{n!} t^n.$$

Then

(i) the formal power series $\hat{F}(t)$ is the Taylor series for a function $F(t)$, which is analytic in a disc $D_a$ of radius $a$, centered at the origin.

(ii) the function $F(t)$ can be continued analytically from the disc $D_a$ to the region

$$D_a \cup \{t \in \mathbb{C}_t : |\arg t| < \varepsilon \}.$$

(iii) the function $P(z)$ can be represented in the form

$$P(z) = \int_{0}^{+\infty} F(t) e^{-zt} dt, \ z \in S \left( -\frac{\pi}{2}, \frac{\pi}{2} \right),$$

and the integral is absolutely convergent.

**Remark 5.** Two problems immediately suggests themselves: (a) what happens to the region (58) of analyticity of $F(t)$ as $\varepsilon \to 0$? (b) given that $P(z)$ satisfies (55) in $S \left( -\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon \right)$ with fixed $a$ and $\varepsilon$, can the region of analyticity be improved?
The problem (a) was answered by F. Nevanlinna in 1918, see \[\text{Nev}\] even for the more general case of Gevrey expansions of an arbitrary order \(k\). The improvement of Watson’s representation theorem (problem (b)) was obtained 62 years later \(^5\) by Alan D. Sokal who applied it to the perturbation expansion in the \(\phi^2_4\) quantum field theory, see \[\text{Sok}\]. Sokal’s improvement gives a necessary and sufficient characterization for a large class of Borel-summable functions. It seems that one of the first reappearances of Watson’s theorems was in connection with the theory of anharmonic oscillators, see \[\text{GGS}\].

Our theorem 1 suggests a corresponding result for a function \(P(z)\), satisfying the inequalities \(\|\|\) in the critical sector \(S\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\) with \(M(\delta)\) satisfying \(\|\|\). Consider the simplest case, for which \(M(\delta)\) is a positive constant. For this case we have the extension of Watson’s uniqueness theorem alluded to in Lemma 2. This suggests the following representation theorem for \(P(z)\).

Let \(D_a\) be the disc of radius \(a\) centered at \(t = 0\) and let \(L^+_a\) be the half-stripe of the form
\[
L^+_a = \{z : \Re z > 0, |\Im z| < a\}.
\]

**Theorem (F. Nevanlinna).** Assume that the function \(P(z)\) is analytic in the right half-plane of the \(z\)-plane and satisfies the conditions (i) and (ii) of Theorem 1. Assume further that the function \(M(\delta)\) in \(\|\|\) is a bounded function in the interval \((0, \frac{\pi}{2})\). Let \(F(t)\) be a function analytic in the disc \(D_a\) which is represented in \(D_a\) as a sum of the power series in \(t\) given by \(\|\|\). Then \(F(t)\) can be analytically continued from the disc \(D_a\) to the “half-strip” \(D_a = D_a \cup L^+_a\) \(\|\|\) with the following bound in \(D_a\)
\[
|F(t)| \leq K_a' \exp (\sigma|t|)
\]
in every subregion \(D_a'\), \(a' < a\), where the positive constant \(K_a'\) depends on \(a'\). Moreover, for \(\Re z > \sigma\) the function \(P(z)\) can be represented in the form \(\|\|\) as the Laplace transform of \(F(t)\) in which the integral is absolutely convergent.

It is our aim to extend this result to more general uniqueness classes of functions satisfying \(\|\|\) with \(M(\delta)\) unbounded. We conjecture that it is possible to restore \(P(z)\) satisfying the conditions of Theorem 1 from its Gevrey expansion \(\hat{P}(z)\) using Borel summation as above if \(\|\|\) is replaced by the stronger condition
\[
M(\delta) < M \exp \left(\frac{b}{\delta}\right).
\]

\(^5\)Ramis’ book, \[\text{Ra2}\], explains why Watson’s theory was further developed only recently.
For classes of functions satisfying (12) or (13) we need to look for an alternative summation method. The part (iii) of our comment 4 and the recent publication [BLS] may be of assistance here.

For a discussion and a further extension of the results of F. Nevanlinna and A. D. Sokal, see [GG1].

**Conclusion.** Two points have motivated this work. The first is the discovery by Ramis and Sibuya that, roughly speaking, if a formal power series satisfies an analytical (non-linear) differential equation then there exists $k > 0$ such that for each sector with (critical) opening $\frac{\pi}{k}$ there exists a regular solution of the equation for which the formal solution is a Gevrey expansion of order $k$. Moreover, they proved the uniqueness of such a solution, and it is this which lead to a re-examination of Watson’s theorem for critical openings of the sector. If $P(z)$ is the regular solution of the Ramis-Sibuya theorem then we believe that $M(\delta)$ will satisfy (12) and, probably, the stronger condition (13). However, this is an open question.

The second point relates to the calculation of the best estimate of a function (and of the associated error) using an optimal finite sum of its Gevrey expansion. This can be done using estimates for $M(\delta)$ and $a^{-1}(\delta)$ in a given sector just as was done above in (28) for Stirling’s series.

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