Holomorphic Chern-Simons theory coupled to off-shell Kodaira-Spencer gravity

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Abstract

We construct an action for holomorphic Chern-Simons theory that couples the gauge field to off-shell gravitational backgrounds, comprising the complex structure and the $(3,0)$-form of the target space. Gauge invariance of the off-shell action is achieved by enlarging the field space to include an appropriate system of Lagrange multipliers, ghost and ghost-for-ghost fields. Both the BRST transformations and the BV action are compactly and neatly written in terms of superfields which include fields, backgrounds and their antifields. We show that the anti-holomorphic target space derivative can be written as a BRST-commutator on a functional space containing the anti-fields of both the dynamical fields and the gravitational backgrounds. We derive from this result a Ward identity that determines the anti-holomorphic dependence of physical correlators.


1 Introduction

Holomorphic Chern-Simons theory (HCS) [1] was introduced by Witten as the target space field theory describing the dynamics of a stack of $N$ 5-branes of topological string theory of the $B$ type living on a Calabi-Yau complex 3-fold $X$. The action of HCS

$$\Gamma = \int_X \Omega \wedge \text{Tr} \left( \frac{1}{2} A \bar{\partial}_Z A + \frac{1}{3} A^3 \right)$$

(1.1)

is a 6-dimensional analogue of the topological 3-dimensional Chern-Simons action [2]. The gauge field $A$, encoding the open string degrees of freedom, is a one-form with values in the Lie algebra of $SU(N)$ of type $(0,1)$ with respect to the chosen complex structure on $X$

$$A = dZ^i A_i(Z, \bar{Z}) = dZ^i A^a_i(Z, \bar{Z}) T^a.$$  

(1.2)

In the formula above $T^a$ are the $SU(N)$ generators and Tr is the trace in its fundamental representation. $\bar{\partial}_Z$ is the Dolbeault operator relative to complex coordinates $(Z^i, \bar{Z}^\bar{i})$ compatible with the chosen complex structure

$$\bar{\partial}_Z = dZ^i \frac{\partial}{\partial Z^i}.$$  

(1.3)

The HCS action (1.1) depends therefore on two different classical geometrical data. One of them is the complex structure that one picks on $X$. The other is $\Omega$, the globally defined holomorphic $(3,0)$-form on $X$

$$\Omega = \Omega_{ijk}(Z, \bar{Z}) dZ^i \wedge dZ^j \wedge dZ^k = \rho(Z, \bar{Z}) \epsilon_{ijk} dZ^i \wedge dZ^j \wedge dZ^k,$$

(1.4)

which, for Calabi-Yau three-folds, is unique up to a rescaling. $\Omega$ and the complex structure on $X$ are in correspondence with the closed moduli parametrizing the closed string vacuum in which the 5-branes live. Since the $(3,0)$-form $\Omega$ depends on the complex structure on $X$, the moduli space of closed strings is the total space of a complex line bundle whose base is the moduli space of complex structures on $X$ and whose fiber is the holomorphic $(3,0)$-form.

To exhibit explicitly the dependence of the theory on the complex structure of $X$ it is convenient to introduce the Beltrami parametrization of the differentials $dZ^i$

$$dZ^i = \Lambda^i_j \left( dz^j + \mu^j \bar{d}z^j \right),$$

(1.5)

where $(z^i, z^\bar{i})$ is a fixed system of complex coordinates. The Beltrami differential

$$\mu \equiv \mu^i \frac{\partial}{\partial z^i} \equiv \mu^j dz^j \frac{\partial}{\partial z^i}$$

(1.6)

is a $(0,1)$-form with values in the holomorphic tangent $T^{(1,0)} X$. The action (1.1) rewrites in the system of coordinates $(z^i, z^\bar{i})$ as follows

$$\Gamma_0(\Omega, \mu) = \int_X \Omega \wedge \left( \frac{1}{2} A \nabla A + \frac{1}{3} A^3 \right),$$

(1.7)
where
\[ \nabla \equiv \bar{\partial} - \mu^i \partial_i, \quad \bar{\partial} \equiv dz^i \frac{\partial}{\partial z^i}. \] (1.8)

In this formulation, the dependence of the theory on the closed moduli is captured by the two classical backgrounds fields — \( \Omega \) and \( \mu \).

The original action (1.1) is invariant under \( \Omega \)-preserving holomorphic reparametrizations. The coupling of \( A \) to the classical background \( \mu \) promotes this global invariance into a local symmetry under which \( \mu \) transforms as a gauge field
\[ s_{\text{diff}} \mu^i = -\bar{\partial} \xi^i + \xi^j \partial_j \mu^i - \partial_j \xi^i \mu^j. \] (1.9)

In (1.9), \( \xi^i \) is the ghost of \( \Omega \)-preserving local diffeomorphisms
\[ \partial i_\xi(\Omega) = 0, \] (1.10)
where \( i_\xi \) is the contraction of a form with the vector field \( \xi^i \partial_i \).

The backgrounds \( \Omega \) and \( \mu \) in (1.7) must satisfy the classical equations of motion of the closed topological string theory:
\[ F^i \equiv \bar{\partial} \mu^i - \mu^j \partial_j \mu^i = 0, \] (1.11)
\[ \hat{\nabla} \Omega \equiv \nabla \Omega + \partial_i \mu^i \Omega = 0. \] (1.12)

The first of such equations is the celebrated Kodaira-Spencer equation [3] which expresses the integrability of the Beltrami differential; the second equation expresses the holomorphicity of \( \Omega \) in the complex structure associated to \( \mu^i \). Indeed the action (1.7) is invariant under the gauge BRST symmetry\(^1\)

\[ s A = -\nabla c - [A, c]_+, \]
\[ s c = -c^2, \] (1.13)

where \( c = c^a T^a \) is the anti-commuting ghost associated to \( SU(N) \) gauge transformations, only if the closed string equation of motions (1.11) and (1.12) are satisfied. It should be kept in mind that \( A \) and \( c \) are the dynamical variables of HCS while \( \mu^i \), \( \Omega \) and \( \xi^i \) are classical non-dynamical fields.

For the purpose of investigating the quantum properties of HCS field theory, like its renormalization and its anomalies, it is useful to extend both gravitational backgrounds \( \mu \) and \( \Omega \) to be generic off-shell functions. Hence in this article we will write down the appropriate generalization of the action (1.7) valid also when \( \mu \) and \( \Omega \) do not satisfy their equations of motion (1.11) and (1.12). Nevertheless, as mentioned above, the closed string fields will still be treated as non-dynamical backgrounds. In the context of string theory our result could help understanding the

\(^1\)In this paper we will adopt the convention that fields and operators carrying odd ghost number anti-commute with fields and operators carrying odd form degree. In particular, the BRST operator \( s \) and the Dolbeault differential \( \nabla \) anti-commute.
back-reaction of the 5-branes on the closed string vacuum, since the presence of branes modifies the equation of motions (1.11) and (1.12) and puts the backgrounds off-shell.

The standard method to go “off-shell” is to introduce new fields acting as Lagrange multipliers whose equations of motions are precisely the closed string equations (1.11) and (1.12) and whose gauge transformation properties are such that the action is gauge invariant even for off-shell backgrounds. This strategy has been adopted by the authors of [4], who were able to solve, so-to-say, half of the problem: they introduced a Lagrange multiplier whose equation of motion is the Kodaira-Spencer equation (1.11), but they did not reformulate the second equation (1.12) in the same way. We achieve this task in the present article.

The reason why the authors of [4], whose main focus is the closed target space field theory, have restricted \( \Omega \) to be holomorphic, has to do with the different status that equations (1.11) and (1.12) enjoy in the Kodaira-Spencer field theory: Eqs. (1.11), which are the classical equations of motion derived from the Kodaira-Spencer action [3], are equivalent to the BRST-invariance of the closed vertex operators associated to the complex structure moduli. This is the standard relation between the second quantized classical equations of motion and first-quantized vertex operators.

Eq. (1.12), instead, is not an equation of motion of Kodaira-Spencer field theory. The \( \Omega \) which enters the Kodaira-Spencer action must be holomorphic and hence it is a parameter and not a dynamical field of Kodaira-Spencer theory. From this point of view, Kodaira-Spencer theory does not provide the second quantized formulation for the first-quantized vertex operator (of non-standard world-sheet ghost number) associated to \( \Omega \).

On the other hand, in the open string field theory it seems to be perfectly sensible to treat \( \Omega \) and \( \mu \) on the same footing: we will therefore introduce Lagrange multipliers whose equations of motion coincide with both (1.11) and (1.12) and will determine their gauge transformation properties. We will find it necessary to enlarge the \( SU(N) \) gauge symmetry to include a number of new ghost (and ghost-for-ghost) fields which can be thought of as “descendants” of the Lagrange multipliers and which ensure the nilpotency of the full BRST transformations.

Since \( \Omega \) becomes, in our construction, an off-shell background, the HCS action that we will derive enjoys a larger reparametrization invariance than the original action (1.7). This invariance include reparametrizations which are not \( \Omega \) preserving:

\[
\begin{align*}
\text{s}_{\text{diff}} \mu^i &= -\ddbar \xi^i + \xi^j \partial_j \mu^i - \partial_j \xi^i \mu^j, \\
\text{s}_{\text{diff}} \Omega &= \partial i_\xi(\Omega), \\
\text{s}_{\text{diff}} A &= \xi^i \partial_i A, \\
\text{s}_{\text{diff}} \xi^i &= \xi^j \partial_j \xi^i,
\end{align*}
\]

(1.14)

together with analogous transformations for all the other dynamical fields that we will introduce. We will refer to the reparametrization invariance (1.14) acting on off-shell \( \mu^i \) and \( \Omega \) as chiral diffeomorphism invariance. Chiral diffeomorphisms will be further discussed in Section 2.

\[\text{A different method to couple HCS to off-shell gravitational backgrounds has been put forward in [5]. Contrary to our approach, the (3,0)-form \( \Omega \) is not treated in [5] as a background independent of the complex structure \( \mu \) and, correspondingly, the \( \Omega \) equation (1.12) is still implicitly assumed, much like in the treatment of [4]. Moreover, the strategy employed to lift the Kodaira-Spencer constraint (1.11) entails the inclusion among the dynamical fields of the (1,0) component of the gauge field, together with a series of satellite fields, thus introducing a large gauge redundancy and making the dependence on the complex structure \( \mu \) fairly implicit.}\]
In Section 3 we write down the HCS gauge-invariant action coupled to off-shell gravitational backgrounds $\mu$ and $\Omega$ and the nilpotent BRST transformations acting on Lagrange multipliers and ghost for ghosts.

In Section 4 we show that all fields and backgrounds of the theory, together with their anti-fields, belong in superfields which are the sum of fields with different form and ghost degree and have simple and compact BRST transformations rules.

In Section 5 we rewrite also the BV action of the theory in terms of superfields: we find that the full BV action is obtained from the classical HCS action by promoting both fields and backgrounds to the superfield that each of them belong to.

In the last Section of this paper, building on the superfield formulation of the theory, we uncover an extended $N=2$ supersymmetric structure which underlies the off-shell HCS theory. We show that the anti-holomorphic target space derivative $\partial_{\bar{\jmath}}$ can be written as the (anti)-commutator of the gauge BRST operator with a supersymmetry charge $G_{\bar{\jmath}}$ which acts on the space of all the dynamical fields and the gravitational backgrounds together with their anti-fields. From this we derive a Ward identity which controls the anti-holomorphic dependence of physical correlators: the detailed analysis of the implications of this identity for the quantum properties of HCS is left to the future.

## 2 Chiral reparametrization invariance

The coupling of HCS to the holomorphic Beltrami differentials (1.5) is determined by requiring invariance under chiral reparametrizations. Chiral reparametrizations act on the Beltrami differentials as follows

$$s_{\text{diff}} \mu^i = -\partial \xi^i + \xi^j \partial_j \mu^i - \partial_j \xi^i \mu^j,$$  \hspace{1cm} (2.1)

where $\xi^i$ is the anti-commuting ghost field of chiral diffeomorphisms:

$$s_{\text{diff}} \xi^i = \xi^j \partial_j \xi^i.$$ \hspace{1cm} (2.2)

It is important to keep in mind that $s_{\text{diff}}$ is nilpotent for generic $\mu^i$, independently of the validity of the Kodaira-Spencer equation (1.11). On the space of Beltrami differentials $\mu^i$ which do satisfy Eq. (1.11) there exists a natural action of non-chiral (standard) reparametrizations which follows from the definition (1.5): one can show [6] that the actions of chiral and non-chiral reparametrizations coincide on such space if one identifies the chiral ghost $\xi^i$ with the following combinations of the ghosts ($c^i, c^\jmath$) of standard diffeomorphisms

$$\xi^i = c^i + \mu^i c^\jmath.$$ \hspace{1cm} (2.3)

There is no notion of standard reparametrizations of “off-shell” Beltrami differentials, i.e. of $\mu^i$’s which do not satisfy the Kodaira-Spencer equation: invariance under chiral diffeomorphisms (2.1) represents the extent of reparametrization invariance appropriate for off-shell $\mu^i$.

Matter fields with only anti-holomorphic indices transform under chiral diffeomorphisms as scalars

$$s_{\text{diff}} \phi_{ij...} = \xi^i \partial_i \phi_{ij...}.$$ \hspace{1cm} (2.4)
For example, the transformation law under chiral reparametrizations of the gauge field $A = A_i \, dx^i$ is
\[ s_{\text{diff}} A_i = \xi^i \partial_i A_i. \] (2.5)

The action of chiral diffeomorphisms on tensors with holomorphic indices is instead
\[ s_{\text{diff}} \phi_{i\bar{j}\ldots k} = \xi^j \partial_j \phi_{i\bar{j}\ldots k} - \partial_j \xi^i \phi_{i\bar{j}\ldots k} + \partial_k \xi^i \phi_{i\bar{j}\ldots j} + \cdots. \] (2.6)

For example, in the following Section we will introduce the Lagrange multiplier $C_i = C_{i\bar{k}} \, dx^{\bar{k}}$ which transforms under chiral reparametrizations as follows
\[ s_{\text{diff}} C_{i\bar{k}} = \xi^j \partial_j C_{i\bar{k}} + \partial_i \xi^j C_{j\bar{k}}. \] (2.7)

For chiral reparametrizations there is a natural definition of covariant anti-holomorphic derivative
\[ \hat{\nabla}_{\bar{k}} \phi_{i\bar{j}\ldots j} = \nabla_{\bar{k}} \phi_{i\bar{j}\ldots j} + \partial_j \mu^i_{\bar{k}} \phi_{i\bar{j}\ldots j} - \partial_k \mu^i_{\bar{j}} \phi_{i\bar{j}\ldots j} + \cdots. \] (2.8)

There is instead no natural notion of covariant holomorphic derivative. However the holomorphic derivative of a tensor with no holomorphic indices is a tensor with one holomorphic lower index.\(^3\)

We will use the notation
\[ \hat{\nabla} \equiv dx^{\bar{k}} \hat{\nabla}_{\bar{k}} \equiv \nabla + \hat{\Gamma}, \] (2.9)

where the connection $\hat{\Gamma}$ denotes the appropriate tensor product of matrices with holomorphic indices
\[ (\hat{\Gamma})^i_j = dx^{\bar{k}} \partial_j \mu^i_{\bar{k}} \] (2.10)
acting on holomorphic tensors in the usual way. For example
\[ \nabla V_i \equiv \nabla V_i - \partial_i \mu^j V_j. \] (2.11)

### 3 Gauge invariance

The variation of the HCS action
\[ \Gamma_0 = \frac{1}{2} \int X \Omega \text{Tr} (A \nabla A + \frac{2}{3} A^3) \] (3.1)
under the BRST gauge transformations
\[ s A = -\nabla c - [A,c]_+, \]
\[ s c = -c^2 \] (3.2)

\(^3\) “Natural” in this context means that the connection in Eq. (2.8) depends only on $\mu^i$ and not on the choice of a metric.
is:

\[ s \Gamma_0 = \frac{1}{2} \int_X \Omega \text{Tr}(\nabla c \nabla A + A \nabla^2 c) = \]

\[ = \frac{1}{2} \int_X \Omega \nabla \text{Tr}(c \nabla A) + \Omega \text{Tr}(c \nabla^2 A + A \nabla^2 c) \]

\[ = \frac{1}{2} \int_X \hat{\nabla}(\Omega) \text{Tr}(c \nabla A) + \Omega \text{Tr}(c \nabla^2 A + A \nabla^2 c) , \]  

(3.3)

where

\[ \hat{\nabla} \Omega \equiv \nabla \Omega + \partial_i \mu^i \Omega , \]

\[ \nabla \equiv \bar{\partial} - \mu^i \partial_i \equiv dx^i \nabla_i \equiv dx^i (\partial_i - \mu_i^i \partial_i) . \]

(3.4)

The curvature of the \( \nabla \)-differential is

\[ \nabla^2 = dx^i dx^j (\partial_i - \mu_i^i \partial_i) (\partial_j - \mu_j^j \partial_j) = -dx^i dx^j (\partial_i \mu_j^j - \mu_i^i \partial_i \mu_j^j) \partial_j = -F^i \partial_i , \]  

(3.5)

where

\[ F^i \equiv \bar{\partial} \mu^i - \mu^j \partial_i \mu^j \]

(3.6)

is the Kodaira-Spencer (0,2)-form with values in the holomorphic tangent.

Eq. (3.3) shows that \( \Gamma_0 \) is gauge-invariant only if both \( \Omega \) and \( \mu^i \) are “on-shell”, i.e. if they satisfy the equations

\[ F^i \equiv \bar{\partial} \mu^i - \mu^j \partial_i \mu^j = 0 , \quad \hat{\nabla} \Omega \equiv \nabla \Omega + \partial_i \mu^i \Omega = 0 . \]

(3.7)

The first equation is equivalent to the nilpotency of \( \nabla \) while the second expresses the holomorphicity of \( \Omega \) in the complex structure defined by \( \mu \). Let us introduce the Lagrange multipliers

\[ C_i \equiv C_{i \bar{j}} dx^\bar{j} , \]

(3.8)

a (0,1)-form with values in the holomorphic cotangent, in correspondence with the first of (3.7), and

\[ B \equiv dx^i dx^j B_{i j} , \]

(3.9)

a (0,2)-form, in correspondence with the second equation.

If their BRST variations are taken to be

\[ s B = -\text{Tr}(c \nabla A) , \]

\[ s C_i = \text{Tr}(-c \partial_i A + \partial_i c A) , \]

(3.10)

the action

\[ \Gamma = \frac{1}{2} \int_X [\Omega \text{Tr}(A \nabla A + \frac{2}{3} A^3) + \Omega (\nabla B + F^i C_i)] \]  

(3.11)
Henceforth the BRST transformations with parameters $\mu$ are not nilpotent when acting on the Lagrange multipliers

$$
\begin{align*}
\bar{s}^2 B &= \text{Tr}(c \nabla^2 c) - \nabla \text{Tr} (A c^2) = -\mathcal{F}^i \text{Tr}(c \partial_i c) - \nabla \text{Tr} (A c^2), \\
\bar{s}^2 C_i &= \text{Tr} (-c \partial_i \nabla c + \partial_i c \nabla c) - \partial_i \text{Tr} (A c^2) = \nabla \text{Tr} (c \partial_i c) + \text{Tr} c [\nabla, \partial_i] c - \partial_i (\text{Tr} A c^2) = \\
&= \nabla \text{Tr} (c \partial_i c) - \partial_i \text{Tr} (A c^2),
\end{align*}
$$

(3.12)

where we made use of the relation

$$[\partial_i, \nabla_i] = -\partial_i \mu^j \partial_j. \quad (3.13)$$

The lack of nilpotency of $(3.10)$ is due to the existence of new local symmetries of the action $(3.11)$

$$B \rightarrow B' = B + \mathcal{F}^i f_i + \nabla d, \quad C_i \rightarrow C_i' = C_i - \nabla f_i + \partial_i d, \quad (3.14)$$

with parameters $d \equiv d_i dx^i$ and $f_i$ which are, respectively, a $(0,1)$-form and a section of the holomorphic cotangent. The transformations $(3.14)$ are symmetries of the action $(3.11)$ since they leave invariant the combination

$$\nabla B + \mathcal{F}^i C_i \rightarrow \nabla B + \mathcal{F}^i C_i + (\nabla (\mathcal{F}^i f_i) - \mathcal{F}^i \nabla f_i) + (\nabla^2 d + \mathcal{F}^i \partial_i d) =$$

$$= \nabla B + \mathcal{F}^i C_i. \quad (3.15)$$

In the equation above we made use of $(3.5)$ and of the Bianchi identity for $\mathcal{F}^i$:

$$0 = \epsilon^{ijk} [\nabla_i, [\nabla_j, \nabla_k]] = -\epsilon^{ijk} \nabla_i (\mathcal{F}^j_{\partial_k} \partial_i) + \epsilon^{ijk} \mathcal{F}^j_{\partial_k} \partial_i (\nabla_i) =$$

$$= -\epsilon^{ijk} \nabla_i \mathcal{F}^j_{\partial_k} \partial_i + \epsilon^{ijk} \mathcal{F}^i_{\partial_j} \partial_i (\nabla_i) = -\epsilon^{ijk} \nabla_i \mathcal{F}^j_{\partial_k} + \mathcal{F}^i_{\partial_j} \partial_j \mu^j \partial_j, \quad (3.16)$$

which can equivalently be written as

$$\nabla \mathcal{F}^i = 0. \quad (3.17)$$

Henceforth the BRST transformations

$$\begin{align*}
s A &= -\nabla c - [A, c]_+, \\
s c &= -c^2, \\
s B &= -\text{Tr}(c \nabla A) - \mathcal{F}^i f_i - \nabla d, \\
s C_i &= \text{Tr} (-c \partial_i A + \partial_i c A) - \nabla f_i + \partial_i d, \\
s d &= \text{Tr} (A c^2), \\
s f_i &= -\text{Tr}(c \partial_i c), \quad (3.18)
\end{align*}$$

$^4$The gauge transformation laws of $C_i$ in Eq. (3.10) differ from those given in [4] but they are equivalent to them when $\Omega$ is on-shell.
where \( f_i \) and \( d \) are *anti-commuting* fields with ghost number +1, are nilpotent when acting on \( A, c, B \) and \( C_i \). The transformations (3.18) are however *not* nilpotent when acting on \( d \) and \( f_i \):

\[
\begin{align*}
  s^2 d &= -\frac{1}{3} \nabla \text{Tr} c^3, \\
  s^2 f_i &= -\frac{1}{3} \partial_i \text{Tr} c^3.
\end{align*}
\]  

(3.19)

The reason why the BRST rules (3.18) are not nilpotent on \( d \) and \( f_i \) can be traced back to the fact that the replacements

\[
\begin{align*}
  d \rightarrow d' &= d + \nabla e, \\
  f_i \rightarrow f'_i &= f_i + \partial_i e
\end{align*}
\]  

(3.20)

leave unchanged the transformations of \( B \) and \( C_i \) in (3.18). Therefore, by introducing a scalar *commuting* ghost-for-ghost field \( e \) of ghost number +2, we obtain at last the fully nilpotent BRST transformations of the action (3.11)

\[
\begin{align*}
  s A &= -\nabla c - [A, c]_+, \\
  s c &= -c^2, \\
  s B &= -\text{Tr} (c \nabla A) - \mathcal{F}^i f_i - \nabla d, \\
  s C_i &= \text{Tr} (-c \partial_i A + \partial_i c A) - \hat{\nabla} f_i + \partial_i d, \\
  s d &= \text{Tr} (A c^2) - \nabla e, \\
  s f_i &= -\text{Tr} (c \partial_i c) + \partial_i e, \\
  s e &= \frac{1}{3} \text{Tr} c^3.
\end{align*}
\]  

(3.21)

The structure of these BRST transformations is possibly made more transparent by the remark that the \( c \)-dependent terms in the BRST variations of \( B, d \) and \( e \) are precisely the forms which appear in the BRST descent equations that are generated by the holomorphic Chern-Simons (0,3)-form and the on-shell \( \nabla \):

\[
\begin{align*}
  s \Gamma^{(0,3)} &= -\nabla \Gamma^{(0,2)}, \\
  s \Gamma^{(0,2)} &= -\nabla \Gamma^{(0,1)}, \\
  s \Gamma^{(0,1)} &= -\nabla \Gamma^{(0,0)}, \\
  s \Gamma^{(0,0)} &= 0 \text{ if } \nabla^2 = 0,
\end{align*}
\]  

(3.22)

where

\[
\begin{align*}
  \Gamma^{(0,3)} &= \text{Tr} (A \nabla A + \frac{2}{3} A^3), \\
  \Gamma^{(0,2)} &= \text{Tr} (c \nabla A), \\
  \Gamma^{(0,1)} &= -\text{Tr} (A c^2), \\
  \Gamma^{(0,0)} &= -\frac{1}{3} \text{Tr} c^3.
\end{align*}
\]  

(3.23)

Therefore, when \( \mathcal{F}^i = 0 \), the cocycle

\[
\begin{align*}
  \tilde{\Gamma}^{(0,3)} &= \Gamma^{(0,3)} + \nabla B, \\
  \tilde{\Gamma}^{(0,2)} &= \Gamma^{(0,2)} + s B + \nabla d = 0, \\
  \tilde{\Gamma}^{(0,1)} &= \Gamma^{(0,1)} + s d + \nabla e = 0, \\
  \tilde{\Gamma}^{(0,0)} &= \Gamma^{(0,0)} + s e = 0.
\end{align*}
\]  

(3.24)
is a solution of the descent equations (3.22) which is BRST equivalent to the Chern-Simons cocycle (3.23) and whose (0, 3) component is precisely the form which appears in the off-shell action (3.11).

Summarizing, the (0,3)-form which appears in the off-shell Chern-Simons action is the representative of the solution of the cohomology problem (3.22) which is characterized by the vanishing of the components of lower form-degree: its top-form component is, when $\nabla^2 = 0$, s-invariant — not just $s$-invariant modulo $\nabla$. The terms in (3.21) involving $f_i$ and $C_i$ are necessary to make $\tilde{\Gamma}^{(0,3)} + \mathcal{F}^i C_i$ s-invariant even when $\nabla^2 \neq 0$.

The action (3.11) contains only covariant anti-holomorphic derivatives and therefore is manifestly invariant under chiral diffeomorphisms of both fields and backgrounds

$$s_{\text{diff}} A = \xi_i \partial_i A, \quad s_{\text{diff}} c = \xi_i \partial_i c,$$

$$s_{\text{diff}} B = \xi_i \partial_i B, \quad s_{\text{diff}} C_i = \xi_j \partial_j C_i + \partial_i \xi^j C_j,$$

$$s_{\text{diff}} d = \xi^i \partial_i d, \quad s_{\text{diff}} f = \xi^i \partial_i f, \quad s_{\text{diff}} e = \xi^i \partial_i e,$$

$$s_{\text{diff}} \mu^i = -\nabla^i \xi^i, \quad s_{\text{diff}} \xi^i = \xi^j \partial_j \xi^i, \quad s_{\text{diff}} \Omega = \partial i \xi(\Omega).$$

Moreover the gauge BRST transformations (3.21) are also manifestly covariant, since they are expressed in terms of anti-holomorphic derivatives and holomorphic derivative of chiral reparametrizations scalars. Therefore the off-shell action (3.11) is invariant under the nilpotent total BRST operator $s_{\text{tot}}$

$$s_{\text{tot}} \equiv s_{\text{diff}} + s,$$

which encodes both the $SU(N)$ gauge symmetry and the global $\Omega$-preserving holomorphic reparametrization symmetry of the original action (1.1).

4 Anti-fields and the chiral N=2 structure of the BRST transformations

It is known [7] that the structure of the BRST symmetry of 3-dimensional (real) CS theory becomes considerably more transparent when one considers, together with the gauge connection $A$ and the ghost field $c$, also their corresponding anti-fields $A^*$ and $c^*$, which are, respectively, a 2-form of ghost number -1 and a 3-form of ghost number -2. All these fields can be collected in one single superfield, a polyform:

$$\mathcal{A} = c + A + A^* + c^*,$$

whose total grassmannian degree $f = n_{\text{ghost}} + n_{\text{form}}$, the sum of ghost number $n_{\text{ghost}}$ and anti-holomorphic form degree $n_{\text{form}}$, is $f = +1$. The BRST transformations of both fields and anti-fields of the 3-dimensional CS theory write nicely in terms of $\mathcal{A}$ as follows

$$(s + d) \mathcal{A} + \mathcal{A}^2 = 0.$$
In this Section we will see that a similar strategy of collecting fields in polyforms of given grassmann parity also elucidates the geometrical content of the BRST transformations of the HCS theory coupled to off-shell gravitational backgrounds.

Let us first write down the BRST transformations of the anti-fields of the dynamical fields.\(^5\) The anti-field of a \((0,1)\)-form \(A = A_1 dx^i\) is naturally a \((3,2)\)-form, \(A^*\), whose BRST variation is

\[
s A^* = -\Omega \nabla A - \frac{1}{2} \hat{\nabla} \Omega A + \cdots ,
\]

where the dots denote the contribution from fields other than \(A\). In order to obtain an anti-field which sits in the same superfield (4.1) as \(c\) and \(A\), it is convenient to introduce the holomorphic density \(\Omega = \rho \epsilon_{ijk} dz^i dz^j dz^k\)

and to pull out a factor of \(\rho\) from the definition of the anti-field \(A^*\):

\[
A^* \to \rho A^* \quad c^* \to \rho c^* .
\]

The redefined \(A^*\) becomes a \((0,2)\)-form and (4.3) gets replaced by

\[
s A^* = -\nabla A - \frac{1}{2} \frac{\hat{\nabla} \rho}{\rho} A + \cdots .
\]

We will use the notation

\[
\hat{\nabla} \frac{\rho}{\rho} = \frac{\nabla \rho - \partial_i \mu^i \rho}{\rho} \equiv \hat{\nabla} \log \rho .
\]

Redefining both \(A^*\) and \(c^*\) in this way, we obtain for their BRST transformations the expressions

\[
s A^* = -\nabla A - A^2 - [c, A^*]_+ + 2 C^{*\ i} \partial_i c +
\]

\[
- \frac{1}{2} \left( \nabla \log \rho \right) A - B^* \nabla c - \left( \nabla B^* + \left( \nabla \log \rho \right) B^* \right) c +
\]

\[
+ \left( \partial_i C^{*\ i} + \left( \partial_i \log \rho \right) C^{*\ i} \right) c + c^2 d^* ,
\]

\[
s c^* = -[c, c^*]_+ - \nabla A^* - [A, A^*]_+ + 2 C^{*\ i} \partial_i A + 2 f^{*\ i} \partial_i c +
\]

\[
- \left( \nabla \log \rho \right) A^* - B^* \nabla A + \left( \partial_i C^{*\ i} + \left( \partial_i \log \rho \right) C^{*\ i} \right) A +
\]

\[
+ \left( \partial_i f^{*\ i} + \left( \partial_i \log \rho \right) f^{*\ i} \right) c + [A, c]_+ d^* + c^2 e^* .
\]

The Lagrange multipliers \(B, d, e\) sit in a single superfield with \(f = 2\). Therefore the corresponding anti-fields \(B^*, d^*, e^*\) are holomorphic densities with \(f = 0\). The multipliers \(C_i\) and \(f_i\) have \(f = +1\), and thus \(C^{*\ i}\) and \(f^{*\ i}\) are holomorphic densities with \(f = +1\). We will find it convenient to redefine \(C^{*\ i}\) and \(f^{*\ i}\) by pulling out a factor of \(\rho\), as we did with \(A^*\) and \(c^*\),

\[
C^{*\ i} \to \rho C^{*\ i} , \quad f^{*\ i} \to \rho f^{*\ i} ,
\]

\(^5\)For a condensed introduction to anti-fields and the Batalin-Vilkovisky (BV) formalism see [8].
so that the new anti-fields $C^*i$ and $f^*i$ are forms of anti-holomorphic degree 2 and 3 respectively. We obtain therefore for the BRST transformation laws of the anti-fields of the Lagrange multipliers:

$$sB^* = -\frac{1}{2} \hat{\nabla}\rho,$$
$$sd^* = \partial_i (\rho C^{*i}) - \hat{\nabla}B^*,$$
$$se^* = \partial_i (\rho f^{*i}) - \nabla d^*,$$
$$sC^{*i} = -\frac{1}{2} \mathcal{F}^i,$$
$$sf^{*i} = -\hat{\nabla}C^{*i} - \frac{B^* \mathcal{F}^i + (\hat{\nabla}\rho) C^{*i}}{\rho}.\quad(4.10)$$

The BRST transformation laws (4.10) make clear that the three anti-fields $B^*, d^*, e^*$ can be put together with the holomorphic density $\rho$ to form a “complete” BRST multiplet

$$B^* \equiv \rho + 2 B^* + 2 d^* + 2 e^*\quad(4.11)$$

containing components of all degrees $n_{\text{form}} = 0, 1, 2, 3$ which transform as follows:

$$s(2 \rho) = 0,$$
$$s(2 B^*) = -\hat{\nabla}\rho,$$
$$s(2 d^*) = -\hat{\nabla}(2 B^*) + \partial_i (2 \rho C^{*i}),$$
$$s(2 e^*) = -\nabla(2 d^*) + \partial_i (2 \rho f^{*i}).\quad(4.12)$$

To form a complete multiplet out of $C^{*i}$ and $f^{*i}$ we need a (0,0)-form of ghost number 1 and a (0,1)-form of ghost number 0 with values in the holomorphic tangent: The natural candidates are $\xi^i$, the chiral reparametrizations ghost, and $\mu^i$, the Beltrami differentials. This motivates considering the total BRST operator

$$s_{\text{tot}} = s_{\text{diff}} + s,\quad(4.13)$$

which encodes both the chiral reparametrizations invariance and the gauge symmetry of HCS theory. Indeed, one can check that by defining

$$M^i \equiv \xi^i + \mu^i + 2 C^{*i} + 2 (f^{*i} - \frac{2}{\rho} B^* C^{*i}) \equiv \xi^i + \mu^i + 2 C^{*i} + 2 f^{*i},\quad(4.14)$$

the transformation rules for $C^{*i}$ and $f^{*i}$ in (4.10) assume the form

$$s_{\text{tot}} M^i = - (\hat{\partial} M^i - M^j \partial_j M^i).\quad(4.15)$$

This equation also reproduces the correct BRST transformations for $\xi^i$ and $\mu^i$. From the same equation it also follows that the anti-holomorphic derivative acting on super-fields $\Phi_{ij...k...}$

$$\hat{\nabla}_k (M^i) \Phi_{ij...k...} \equiv \partial_k \Phi_{ij...k...} - M^j \partial_j \Phi_{ij...k...} + \partial_j M^l \Phi_{ij...l...k...} - \partial_k M^l \Phi_{ij...l...k...} + \cdots\quad(4.16)$$
is covariant under the transformations (4.15). Moreover the covariant differential
\[ \nabla(M) \equiv dx^k \hat{\nabla}_k(M) \] (4.17)
satisfies
\[ \{s_{\text{tot}}, \nabla(M)\} + \nabla(M)^2 = 0. \] (4.18)
This means that the operator
\[ \delta \equiv s_{\text{tot}} + \nabla(M) \] (4.19)
is nilpotent:
\[ \delta^2 = 0. \] (4.20)
It is easily seen that the transformations (4.12) rewrite in terms of this super-covariant anti-holomorphic derivative as
\[ s_{\text{tot}}^* B = -\nabla(M) B^*. \] (4.21)

The introduction of the flat super-Beltrami \( M_i \) allows one to recast the BRST transformations of the gauge supermultiplet \( c, A, A^*, c^* \) in a form which is analogous to the transformations (4.3) of the three-dimensional theory. Defining the modified anti-fields
\[ A_n^* = A^* - \frac{B^*}{\rho} A - \frac{d^*}{\rho} c, \quad c_n^* = c^* - 2 \frac{B^*}{\rho} A_n^* - \frac{d^*}{\rho} A - \frac{e^*}{\rho} c \] (4.22)
and the superfield
\[ \mathcal{A} \equiv c + A + A_n^* + c_n^*, \] (4.23)
the transformations of the gauge multiplet in (3.21) and (4.8) write as
\[ s_{\text{tot}} \mathcal{A} = -\nabla(M) \mathcal{A} - \mathcal{A}^2. \] (4.24)

Let us turn to the BRST transformations of the Lagrange multipliers. To form a complete BRST multiplet \( B \) out of \( B, d, e \) we need to introduce the anti-field \( \rho^* \), with ghost number -1 and anti-holomorphic form degree 3, corresponding to the background \( \rho \).

Let us comment on the significance of BRST transformations of the backgrounds and of their anti-fields. Backgrounds (or coupling constants) can appear both in the classical action and in the gauge-fixing term. Backgrounds which appear only in the gauge-fixing term are of course unphysical. It is convenient in various contexts to extend the action of the BRST operator on the unphysical backgrounds by introducing corresponding fermionic super-partners to form trivial BRST doublets (see [9] and references therein). The BRST variation of physical backgrounds (or coupling constants) must instead be put to zero since varying a physical coupling constant is, by definition, not a symmetry. Indeed in HCS theory the gauge BRST transformations of the
(physical) backgrounds $\rho$ and $\mu^i$ vanish, as indicated in (4.12) and (4.21). However in the BV formalism it is natural to consider also the anti-fields corresponding to physical backgrounds. Anti-fields of backgrounds do not appear in the BV action since the BRST variation of the physical backgrounds vanish. Their BRST variations are naturally defined in the BV formalism by the derivatives of the BV action with respect to the backgrounds. For HCS theory the BRST variations of the anti-fields of $\rho$ and $\mu^i$ can be defined to be

$$s\rho^* = -\frac{\partial \Gamma_{BV}}{\partial \rho} = -\text{Tr} \left( \frac{1}{2} A^* \nabla A + \frac{1}{3} A^3 \right) - \frac{1}{2} \nabla B - \frac{1}{2} F^i C_i,$$

$$\frac{1}{\rho} s\mu^*_i = -\frac{1}{\rho} \frac{\partial \Gamma_{BV}}{\partial \mu^i} = \frac{1}{2} \left( -\text{Tr} (A \partial_i A) + \partial_i B - \hat{\nabla} C_i - (\hat{\nabla} \log \rho) C_i \right) + \text{Tr} \left( A^* \partial_i c \right) - \frac{B^*}{\rho} \left( \text{Tr} (c \partial_i A) - \partial_i d + \hat{\nabla} f_i \right) - \nabla \left( \frac{B^*}{\rho} \right) f_i + C^{*j} \partial_i \left( f_j \right) + (\partial_j C^{*j} + (\partial_j \log \rho) C^{*j}) f_i + \frac{d^*}{\rho} \partial_i e,$$ (4.25)

where $\Gamma_{BV}$ is the BV action.

The content of the relation (4.25) is that the variations of the action with respect to the physical backgrounds are BRST-closed: since the BRST transformations do depend on the backgrounds this is not self-evident but it is ensured by the general BV formula. In the enlarged field space which includes anti-fields of backgrounds such variations are BRST-trivial.

The superfield which collects together $B, d, e$ and $\rho^*$ and has nice BRST transformation laws turns out to be

$$B = e + d + B_n + 2 \rho_n^*,$$ (4.26)

where

$$B_n \equiv B - 2 C^{*i} f_i - \text{Tr} (A^*_n c),$$

$$2 \rho_n^* \equiv 2 \rho^* - 2 C^{*i} C_i - 2 f^*_n f_i - \text{Tr} (A^*_n A + c^*_n c).$$ (4.27)

One can check that the BRST transformation laws for $B, d, e$ rewrite in terms of $B$ as follows

$$s_{\text{tot}} B = -\nabla (M) B + \frac{1}{3} \text{Tr} A^3.$$ (4.28)

The Lagrange multipliers $C_i$ and $f_i$ sit in a superfield which contains also a 2-form of ghost number -1 and a 3-form of ghost number -2 with values in the holomorphic cotangent. Looking at (4.14) one sees that these should be identified with the anti-fields $\mu^*_i$ and $\xi^*_i$ of the backgrounds $\mu^i$ and $\xi^i$. Since $M^i$ is valued in the holomorphic tangent, $M^i_*$ is naturally a holomorphic density. Choosing its components to be

$$M^*_i = \rho f_i + (\rho C_i + 2 B^* f_i) + 2 \mu^*_i + 2 \xi^*_i,$$ (4.29)

---

\footnote{In Eq. (4.25) we defined the functional derivative of $\Gamma_{BV}$ with respect to $\rho$ by keeping constant the true anti-fields $A^*, e^*, C^{*i}$ and $f^{*i}$, and not the redefined ones in (4.5),(4.9).}
its BRST transformation writes
\[ s_{\text{tot}} M^i_* = -\hat{\nabla} (M) M^i_* + B^* \partial_i B - B^* \text{Tr} A \partial_i A. \] (4.30)

The BRST transformations of all fields and backgrounds and their anti-fields write in a nice compact form in terms of the coboundary operator \( \delta \):
\[
\begin{align*}
\delta M_i &+ M_i \partial_j M^j = 0, \\
\delta A + A^2 & = 0, \\
\delta B & = \frac{1}{3} \text{Tr} A^3, \\
\delta M^*_i & = B^* \partial_i B - B^* \text{Tr} A \partial_i A, \\
\delta B^* & = 0. 
\end{align*}
\] (4.31)

Let us comment on the geometrical interpretation of the BRST transformations (4.31). The first of (4.31) tells us that the super-Beltrami field \( M_i \) has flat Kodaira-Spencer curvature with respect to the differential \( \delta \). The second equation expresses the flatness of the gauge super-connection \( A \). Since \( A \) is flat, the Chern-Simons polyform
\[ \Gamma_{CS} = \text{Tr} (A \delta A + \frac{2}{3} A^3) = -\frac{1}{3} \text{Tr} A^3 \] (4.32)
is a \( \delta \)-cocycle. The third equation in (4.31) says that such cocycle is \( \delta \)-exact, being the \( \delta \)-variation of \( B \). Taking the \( \partial_i \) derivative of this equation one obtains
\[ \delta \partial_i B = \text{Tr} \partial_i A A^2 = \delta \text{Tr} A \partial_i A. \] (4.33)
This means that \( \Omega_i \equiv \partial_i B - \text{Tr} A \partial_i A \) is a \( \delta \)-cocycle
\[ \delta (\partial_i B - \text{Tr} A \partial_i A) = \delta \Omega_i = 0. \] (4.34)
The fourth equation in (4.31)
\[ B^* \Omega_i = \delta M^*_i \] (4.35)
implies therefore
\[ \delta B^* \Omega_i = 0. \] (4.36)
This is consistent with the fifth equation in (4.31) and implies that \( \Omega_i \) is also \( \delta \)-trivial
\[ \Omega_i = \delta C_i, \quad M^*_i \equiv B^* C_i. \] (4.37)
5 The action

Not only the BRST transformations but also the action rewrites in a neat form in terms of superfields. The BV action corresponding to the gauge invariant action (3.11) is

\[ 2 \Gamma_{BV} = \rho \text{Tr} \left( A \nabla A + \frac{2}{3} A^3 \right) + \rho \nabla B + \rho F^i C_i - 2 \rho A^* s A - 2 \rho e^* s e + -2 B^* s B - 2 d^* s d - 2 e^* s e - 2 \rho C^{*i} s C_i - 2 \rho f^{*i} s f_i , \]  

(5.1)

where we chose to think of \( \Gamma_{BV} \) as a \((0,3)\)-form with values in the holomorphic densities rather than a \((3,1)\)-form as the notation in Eq. (3.11) implies.

We have seen that when working with the superfields it is natural to promote the gauge BRST operator to the total \( s_{\text{tot}} \) which includes the chiral diffeomorphisms, by introducing the chiral reparametrization ghost \( \xi^i \) which should be thought of as a background, in the same way as \( \rho \) and \( \mu^i \). The corresponding BV action has extra terms with respect to the gauge BV action (5.1) which are proportional to the background \( \xi^i \). It is this extended action which writes most simply in terms of superfields. Of course one can always recover the gauge action (5.1) by putting \( \xi^i \) to zero.

A direct computation shows that (the extended) \( \Gamma_{BV} \) is the \((0,3)\)-component of the following polyform with values in the holomorphic densities

\[ 2 \Gamma_{BV} = -B^* s_{\text{tot}} B - M^*_i s_{\text{tot}} M^i - B^* \text{Tr} (A s_{\text{tot}} A) = -B^* \text{Tr} (A \nabla (\bar{M}) A + \frac{2}{3} A^3) + B^* \nabla (\bar{M}) B + M^*_i \left( \bar{\partial} M^i - M^j \partial_j M^i \right) . \]  

(5.2)

We see therefore that, in much the same way as it happens for 3d CS theory [7], the BV action is obtained from the classical action (3.11) by replacing every field and background with the superfield to which it belongs

\[ A \rightarrow A , \quad B \rightarrow B , \quad \rho C_i \rightarrow M^*_i , \quad \mu^i \rightarrow M^i , \quad \rho \rightarrow B^* . \]  

(5.3)

6 Anti-holomorphic dependence of physical correlators

The stress-energy tensor of a topological quantum field theory is a BRST anti-commutator

\[ T_{\mu\nu} = \{ s, G_{\mu\nu} \} , \]  

(6.1)

where \( G_{\mu\nu} \) is the supercurrent. If both \( T_{\mu\nu} \) and \( G_{\mu\nu} \) are conserved one obtains a corresponding relation for the charges

\[ P_\mu = \{ s, G_\mu \} , \]  

(6.2)

where \( P_\mu \) is the generator of translations and \( G_\mu \) is a vector supersymmetry. Since \( P_\mu \) is implemented on local fields by space-time derivatives

\[ \partial_\mu = \{ s, G_\mu \} , \]  

(6.3)
the relation (6.2) proves that correlators of local observables of topological field theories are space-time independent.

HCS theory is, in a sense, semi-topological: it does not depend on the full space-time metric but only on the Beltrami differential $\mu^i$. Consequently we expect that a holomorphic version of the relation (6.3) holds for HCS:

$$\hat{\nabla}_{\bar{i}} = \{s, G_{\bar{i}}\}. \quad (6.4)$$

In this section we want to explore the validity of such a relation. We will find that a suitable $G_{\bar{i}}$ does indeed exist if we enlarge the functional space upon which $G_{\bar{i}}$ acts to include the anti-fields of both the dynamical fields and the backgrounds $\mu^i$ and $\Omega$.

It is convenient to introduce a field $\gamma^i(\bar{z})$, which depends only on the anti-holomorphic coordinates $\bar{z}^i$ and define the scalar operator

$$G_{\bar{\gamma}} = \gamma^i G_{\bar{i}}, \quad (6.5)$$

which carries ghost number -1. It turns out that a suitable $G_{\bar{\gamma}}$ which satisfies (6.4) is defined by the following simple action on the superfields that we introduced in Section 4

$$G_{\bar{\gamma}} A = i_{\bar{\gamma}}(A), \quad G_{\bar{\gamma}} B = i_{\bar{\gamma}}(B), \quad G_{\bar{\gamma}} B^* = i_{\bar{\gamma}}(B^*), \quad G_{\bar{\gamma}} M^i = i_{\bar{\gamma}}(M^i), \quad G_{\bar{\gamma}} M^*_i = i_{\bar{\gamma}}(M^*_i), \quad (6.6)$$

where $i_{\bar{\gamma}}$ is the contraction of a form with the antiholomorphic vector field $\gamma^i \partial_i$. $G_{\bar{\gamma}}$ so defined is easily seen to satisfy the relation

$$\{s_{\text{tot}}, G_{\bar{\gamma}}\} = \{i_{\bar{\gamma}}, \bar{\partial}\}, \quad (6.7)$$

where $s_{\text{tot}}$ is the BRST operator which include both gauge transformations and chiral diffeomorphisms:

$$s_{\text{tot}} = s_{\text{diff}} + s. \quad (6.8)$$

Note that the gauge BRST operator $s$ acts trivially on the gravitational backgrounds ($\mu^i$, $\rho$, $\xi^i$). Let us show that (6.7) implies (6.4) for the dynamical fields. Indeed, let $\Phi$ be a field which is neither $\mu^i$ nor $\xi^i$. We have

$$G_{\bar{\gamma}} s_{\text{diff}}(\Phi) = G_{\bar{\gamma}} (L_\xi \Phi) = L_\xi G_{\bar{\gamma}}(\Phi), \quad s_{\text{diff}} G_{\bar{\gamma}}(\Phi) = L_\xi G_{\bar{\gamma}}(\Phi), \quad \{s_{\text{diff}}, G_{\bar{\gamma}}\} = L_\xi(i_{\bar{\gamma}}(\mu^i)), \quad (6.9)$$

where $L_\xi$ denotes the action of chiral diffeomorphisms with parameter $\xi^i$. Hence

$$\{s, G_{\bar{\gamma}}\} \Phi = \{i_{\bar{\gamma}}, \bar{\partial}\} \Phi - \{s_{\text{diff}}, G_{\bar{\gamma}}\} \Phi = \{i_{\bar{\gamma}}, \hat{\nabla}\} \Phi, \quad (6.10)$$

which is equivalent to (6.4). Note that on the backgrounds, we have instead

$$\{s, G_{\bar{\gamma}}\} \xi^i = 0, \quad \{s, G_{\bar{\gamma}}\} \mu^i = i_{\bar{\gamma}}(\mathcal{F}^i). \quad (6.11)$$
When writing down explicitly $G_\gamma$ on the component fields one verifies that its action on the sector which does not include the Lagrange multipliers $B$ and $C_i$ does not involve the antifields of $\mu_i^*$ and $\rho^*$:

\[
\begin{align*}
G_\gamma c & = i_\gamma(A), \\
G_\gamma A & = i_\gamma(A^*) - i_\gamma \left( \frac{B^*}{\rho} A \right) - i_\gamma \left( \frac{d^*}{\rho} \right) c, \\
G_\gamma A^* & = i_\gamma(c^*) - 2 i_\gamma \left( \frac{B^*}{\rho} \right) A^* - \frac{B^*}{\rho} i_\gamma(A^*) + \\
& \quad + i_\gamma \left( \frac{B^*}{\rho} \right) B^* \overline{A} + \frac{B^*}{\rho} i_\gamma \left( \frac{d^*}{\rho} \right) c + \left( \frac{B^*}{\rho} \right)^2 i_\gamma(A), \\
G_\gamma c^* & = -2 i_\gamma \left( \frac{B^*}{\rho} \right) c^* - \frac{d^*}{\rho} i_\gamma(A^*) - i_\gamma \left( \frac{d^*}{\rho} \right) B^* A, \\
G_\gamma (\rho) & = 2 i_\gamma(B^*), \\
G_\gamma B^* & = i_\gamma(d^*), \\
G_\gamma d^* & = i_\gamma(e^*), \\
G_\gamma e^* & = 0, \\
G_\gamma \mu_i^* & = 2 i_\gamma(C^{*i}), \\
G_\gamma C^{*i} & = i_\gamma(f^{*i}) - 2 i_\gamma \left( \frac{B^*}{\rho} \right) C^{*i} - 2 \frac{B^*}{\rho} i_\gamma(C^{*i}), \\
G_\gamma f^{*i} & = -2 i_\gamma \left( \frac{B^*}{\rho} \right) f^{*i} - 2 \frac{d^*}{\rho} i_\gamma(C^{*i}), \\
G_\gamma e & = i_\gamma(d), \\
G_\gamma d & = i_\gamma(B) - 2 i_\gamma(C^{*i}) f_i - \text{Tr} \left( i_\gamma(A^*) c \right) + i_\gamma \left( \frac{B^*}{\rho} \text{Tr}(A c) \right), \\
G_\gamma f_i & = i_\gamma(C_i). 
\end{align*}
\]

The action of $G_\gamma$ on $B$ and $C_i$ involves instead the anti-fields $\mu_i^*$ and $\rho^*$ whose BRST transformations we introduced in (4.25):

\[
\begin{align*}
G_\gamma B & = 2 i_\gamma(\rho^*) - 2 i_\gamma(C^{*i}) C_i - i_\gamma \left( \frac{d^*}{\rho} \right) \text{Tr}(A c) + \frac{B^*}{\rho} \text{Tr}(A i_\gamma(A)) + \\
& \quad - \text{Tr}(i_\gamma(A^*) A), \\
G_\gamma C_i & = 2 i_\gamma \left( \frac{\mu_i^*}{\rho} \right) - 2 i_\gamma \left( \frac{B^*}{\rho} C_i \right) - 2 i_\gamma \left( \frac{d^*}{\rho} \right) f_i. 
\end{align*}
\]

The existence of $G_\gamma$ therefore reflects the semi-topological character of the theory. Since the relation (6.3) valid for topological theories is replaced in HCS by (6.4), the correlators of physical local observables $O(z, \bar{z})$

\[
F(z, \bar{z}) = \langle O(z, \bar{z}) \cdots \rangle \quad \text{with} \quad s O(z, \bar{z}) = 0,
\]

where the dots denote insertions of physical observables at space-time points other than $(z, \bar{z})$, satisfy the identity

\[
\hat{\nabla}_i F(z, \bar{z}) = \langle s \left( G_\gamma O(z, \bar{z}) \right) \cdots \rangle.
\]
One cannot immediately conclude, from this Ward identity (and BRST invariance) that $F(z, \bar{z})$ is a holomorphic function (tensor) on $X$. This for two reasons.

First of all we have seen that $G_\gamma$ when acting on $B$ and $C_i$ produces the $\mu_i^*$ and $\rho^*$: since $\rho$ and $\mu^i$ are not dynamical (one does not integrate over them) the Ward identity (6.15) says that the $\bar{z}$-dependence of physical correlators involving $B$ and $C_i$ can be expressed in terms of derivative of correlators with respect to the moduli $\rho$ and $\mu^i$.

Secondly, even if restricted to observables which do not involve the Lagrange multipliers $B$ and $C_i$, the Ward identity (6.15) “almost” implies the holomorphicity of $F(z, \bar{z})$, but not quite. Indeed, the $G_\gamma$ variations (6.12) of fields other than $B$ and $C_i$ contain the dynamical anti-fields, and the functional averages of the BRST variation of operators which depend on the anti-fields are, in general, zero only up to contact terms.

At any rate it is clear that the Ward identity (6.15) strongly constrains the anti-holomorphic dependence of physical correlators. This equation should therefore play for the Green functions of physical observables of HCS field theory the role that the holomorphic anomaly equation plays for the open-closed topological string amplitudes [10]. For example, it is conceivable that one could determine, to a large extent, the space-time dependence of physical correlators of HCS using the identity (6.15) together with assumptions about the behavior of correlators at infinity. An analogous approach to compute topological open and closed string amplitudes by integrating the holomorphic anomaly equation has been quite successful [3], [11].

The study of the full implications for the quantum properties of HCS field theory is left to the future. Here we will limit ourselves to few brief comments. First of all there is the issue of anomalies: the chiral diffeomorphism symmetry (3.25) can, in principle, suffer from anomalies, and, indeed, it does [12]. Chiral diffeomorphism invariance can be restored at the price of introducing a dependence on the anti-holomorphic Beltrami differentials and, possibly, on the Kähler metric. The chiral diffeomorphism invariant theory should display an anomalous Ward identity which controls the anti-holomorphic dependence on the backgrounds very much like (6.15) does for the space-time anti-holomorphic dependence.

But, of course, the real question which remains to be addressed is the ultraviolet completeness of the HCS quantum field theory. Being a 6-dimensional gauge theory, HCS theory is superficially not renormalizable. On the other hand its string interpretation suggests the opposite. We believe that the extended supersymmetry structure (6.4) capturing the semi-topological character of the theory and the identity (6.15) restricting the space-time dependence of quantum correlators should be instrumental in ensuring that the physical sector of the theory is indeed free of ultra-violet divergences.

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