SEMI-INVARIANTS FOR CONCEALED-CANONICAL ALGEBRAS

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Abstract. In the paper is we generalize known descriptions of rings of semi-invariants for regular modules over Euclidean and canonical algebras to arbitrary concealed-canonical algebras.

Throughout the paper \( \mathbb{k} \) is a fixed algebraically closed field. By \( \mathbb{Z} \), \( \mathbb{N} \) and \( \mathbb{N}_+ \), we denote the sets of the integers, the non-negative integers and the positive integers, respectively. Finally, if \( i, j \in \mathbb{Z} \), then \([i, j] := \{k \in \mathbb{Z} | i \leq k \leq j\}\) (in particular, \([i, j] = \emptyset \) if \( i > j \)).

Introduction

Concealed-canonical algebras have been introduced by Lenzing and Meltzer [22] as a generalization of Ringel’s canonical algebras [26]. An algebra is called concealed-canonical if it is isomorphism to the endomorphism ring of a tilting bundle over a weighted projective line. The concealed-canonical algebras can be characterized as the algebras which posses sincere separating exact subcategory [23] (see also [28]). Together with tilted algebras [7,20], the concealed-canonical algebras form two most prominent classes of quasi-tilted algebras [19]. Moreover, according to a famous result of Happel [18], every quasi-tilted algebra is derived equivalent either to a tilted algebra or to a concealed-canonical algebra.

Despite investigations of a structure of the categories of modules over concealed-canonical algebras, geometric problems have been studied for this class of algebras (see for example [2,3,6,14,15,17,29]). Often these problem were studied for canonical algebras only and sometimes the authors restrict their attention to the concealed-canonical algebras of tame representation type.

In the paper we study a problem, which has been already investigated in the case of canonical algebras. Namely, given a concealed-canonical algebra \( \Lambda \) and a module \( R \), which is a direct sum of modules from of sincere separating exact subcategory of \( \text{mod} \Lambda \), we want to describe a structure of the ring of semi-invariants associated to \( \Lambda \) and the dimension vector of \( R \). This problem has been solved provided \( \Lambda \) is a canonical algebra and \( R \) comes from a distinguished sincere separating exact subcategory.

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exact subcategory of mod $\Lambda$ (the answers have been obtained independently by Skowroński and Weyman [29] and Domokos and Lenzing [14, 15]). This problem has also been solved for another class of concealed-canonical algebras, namely the path algebras of Euclidean quivers [30] (see also [12, 27]). The obtained results are very similar, although the methods used in the proof are completely different. The aim my paper is to obtain a unified proof of the above results, which would generalize to an arbitrary concealed-canonical algebra. This aim is achieved if the characteristic of $k$ equals 0. If char $k > 0$, then we show that an analogous result is true if we study the semi-invariants which are the restrictions of the semi-invariants on the ambient affine space. The precise formulation of the obtained results can be found in Section 6. In particular we prove that the studied rings of semi-invariants are always complete intersections, and are polynomial rings if the considered dimension vector is “sufficiently big”.

The paper is organized as follows. In Section 1 we introduce a setup of quivers and their representations, which due to a result of Gabriel [16] is an equivalent way of thinking about algebras and modules. Next, in Section 2 we gather facts about concealed-canonical algebras (equivalently, quivers). In Section 3 we introduce semi-invariants and present their basic properties. Next, in Section 4 we study the semi-invariants in the case of concealed-canonical quivers more closely. Section 5 is devoted to presentation of necessary facts about the Kronecker quiver, which is the minimal concealed-canonical quiver. Finally, in Section 6 we present and proof the main result.

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1. Quivers and their representations

By a quiver $\Delta$ we mean a finite set $\Delta_0$ (called the set of vertices of $\Delta$) together with a finite set $\Delta_1$ (called the set of arrows of $\Delta$) and two maps $s, t : \Delta_1 \to \Delta_0$, which assign to each arrow $\alpha$ its starting vertex $s\alpha$ and its terminating vertex $t\alpha$, respectively. By a path of length $n \in \mathbb{N}_+$ in a quiver $\Delta$ we mean a sequence $\sigma = (\alpha_1, \ldots, \alpha_n)$ of arrows such that $s\alpha_i = t\alpha_{i+1}$ for each $i \in [1, n - 1]$. In the above situation we put $t\sigma := n$, $s\sigma := s\alpha_n$ and $t\sigma := t\alpha_1$. We treat every arrow in $\Delta$ as a path of length 1. Moreover, for each vertex $x$ we have a trivial path $1_x$ at $x$ such that $t1_x := 0$ and $s1_x := x := t1_x$. For the rest of the paper we assume that the considered quivers do not have oriented cycles, where by an oriented cycle we mean a path $\sigma$ of positive length such that $s\sigma = t\sigma$.

Let $\Delta$ be a quiver. We define its path category $k\Delta$ to be the category whose objects are the vertices of $\Delta$ and, for $x, y \in \Delta_0$, the morphisms from $x$ to $y$ are the formal $k$-linear combinations of paths starting at
and terminating at \( y \). If \( \omega \) is a morphism from \( x \) to \( y \), then we write \( s\omega := x \) and \( t\omega := y \). By a representation of \( \Delta \) we mean a functor from \( k\Delta \) to the category \( \text{mod} k \) of finite dimensional vector spaces. We denote the category of representations of \( \Delta \) by \( \text{rep} \Delta \). Observe that every representation of \( \Delta \) is uniquely determined by its values on the vertices and the arrows. Given a representation \( M \) of \( \Delta \) we denote by \( \dim M \) its dimension vector defined by the formula \( (\dim M)(x) := \dim_k M(x) \), for \( x \in \Delta_0 \). Observe that \( \dim M \in \mathbb{N}^{\Delta_0} \) for each representation \( M \) of \( \Delta \). We call the elements of \( \mathbb{N}^{\Delta_0} \) dimension vectors. A dimension vector \( d \) is called sincere if \( d(x) \neq 0 \) for each \( x \in \Delta_0 \).

By a relation in a quiver \( \Delta \) we mean a \( k \)-linear combination of paths of lengths at least 2 having a common starting vertex and a common terminating vertex. Note that each relation in a quiver \( \Delta \) is a morphism in \( k\Delta \). A set \( \mathfrak{R} \) of relations in a quiver \( \Delta \) is called minimal if \( \langle \mathfrak{R} \setminus \{ \rho \} \rangle \neq \langle \mathfrak{R} \rangle \) for each \( \rho \in \mathfrak{R} \), where for a set \( \mathfrak{X} \) of morphisms in \( \Delta \) we denote by \( \langle \mathfrak{X} \rangle \) the ideal in \( k\Delta \) generated by \( \mathfrak{X} \). Observe that each minimal set of relations is finite. By a bound quiver \( \Delta \) we mean a quiver \( \Delta \) together with a minimal set \( \mathfrak{R} \) of relations. Given a bound quiver \( \Delta \) we denote by \( k\Delta \) its path category, i.e. \( k\Delta := k\Delta / \langle \mathfrak{R} \rangle \). By a representation of a bound quiver \( \Delta \) we mean a functor from \( k\Delta \) to \( \text{mod} k \). In other words, a representation of \( \Delta \) is a representation \( M \) of \( \Delta \) such that \( M(\rho) = 0 \) for each \( \rho \in \mathfrak{R} \). We denote the category of representations of a bound quiver \( \Delta \) by \( \text{rep} \Delta \). Moreover, we denote by \( \text{ind} \Delta \) the full subcategory of \( \text{rep} \Delta \) consisting of the indecomposable representations. It is known that \( \text{rep} \Delta \) is an abelian Krull–Schmidt category.

An important role in the study of representations of quivers is played by the Auslander–Reiten translations \( \tau \) and \( \tau^- \) [1, Section IV.2], which assign to each representation of a bound quiver \( \Delta \) another representation of \( \Delta \). In particular, we will use the following consequences of the Auslander–Reiten formulas [1, Theorem IV.2.13]. Let \( M \) and \( N \) be representations of a bound quiver \( \Delta \). If \( \text{pdim} \Delta \leq 1 \), then

\[
(1.1) \quad \dim_k \text{Ext}^1_{\Delta}(M, N) = \dim_k \text{Hom}_{\Delta}(N, \tau M).
\]

Dually, if \( \text{idim} \Delta \leq 1 \), then

\[
(1.2) \quad \dim_k \text{Ext}^1_{\Delta}(M, N) = \dim_k \text{Hom}_{\Delta}(\tau^- N, M).
\]

Let \( \Delta \) be a bound quiver. We define the corresponding Tits form \( \langle -,- \rangle_{\Delta} : \mathbb{Z}^{\Delta_0} \times \mathbb{Z}^{\Delta_0} \to \mathbb{Z} \) by the formula

\[
\langle d', d'' \rangle_{\Delta} := \sum_{x \in \Delta_0} d'(x) \cdot d''(x) - \sum_{\alpha \in \Delta_1} d'(s\alpha) \cdot d''(t\alpha) + \sum_{\rho \in \mathfrak{R}} d'(s\rho) \cdot d''(t\rho),
\]

for \( d', d'' \in \mathbb{Z}^{\Delta_0} \). Bongartz [8, Proposition 2.2] has proved that

\[
\langle \dim M, \dim N \rangle_{\Delta} = \dim_k \text{Hom}_{\Delta}(M, N) - \dim_k \text{Ext}^1_{\Delta}(M, N) + \dim_k \text{Ext}^2_{\Delta}(M, N)
\]
for any $M, N \in \text{rep} \Delta$ provided $\text{gldim} \Delta \leq 2$.

2. SEPARATING EXACT SUBCATEGORIES

In this section we present facts about sincere separating exact subcategories, which we use in our considerations. For the proofs we refer to [23, 26].

Let $\Delta$ be a bound quiver and $\mathcal{X}$ a full subcategory of $\text{ind} \Delta$. We denote by $\text{add} \mathcal{X}$ the full subcategory of $\text{rep} \Delta$ formed by the direct sums of representations from $\mathcal{X}$. We say that $\mathcal{X}$ is an exact subcategory of $\text{ind} \Delta$ if $\text{add} \mathcal{X}$ is an exact subcategory of $\text{rep} \Delta$, where by an exact subcategory of $\text{rep} \Delta$ we mean a full subcategory $\mathcal{E}$ of $\text{rep} \Delta$ such that $\mathcal{E}$ is an abelian category and the inclusion functor $\mathcal{E} \hookrightarrow \text{rep} \Delta$ is exact.

We put $\mathcal{X}_- := \{ X \in \text{ind} \Delta : \text{Hom}_\Delta(\mathcal{X}, X) = 0 \}$ and $\mathcal{X}_+ := \{ X \in \text{ind} \Delta : \text{Hom}_\Delta(X, \mathcal{X}) = 0 \}$.

Let $\Delta$ be a bound quiver. Following [23] we say that $\mathcal{R}$ is a sincere separating exact subcategory of $\text{ind} \Delta$ provided the following conditions are satisfied:

1. $\mathcal{R}$ is an exact subcategory of $\text{ind} \Delta$ stable under the actions of the Auslander–Reiten translations $\tau$ and $\tau^-$. 
2. $\text{ind} \Delta = \mathcal{R}_- \cup \mathcal{R} \cup \mathcal{R}_+$. 
3. $\text{Hom}_\Delta(X, \mathcal{R}) \neq 0$ for each $X \in \mathcal{R}_-$ and $\text{Hom}_\Delta(\mathcal{R}, X) \neq 0$ for each $X \in \mathcal{R}_+$. 
4. $P \in \mathcal{R}_-$, for each indecomposable projective representation $P$ of $\Delta$, and $I \in \mathcal{R}_+$, for each indecomposable injective representation $I$ of $\Delta$.

Lenzing and de la Peña [23] have proved that there exists a sincere separating exact subcategory $\mathcal{R}$ of $\text{ind} \Delta$ if and only if $\Delta$ is concealed-canonical, i.e. $\text{rep} \Delta$ is equivalent to the category of modules over a concealed-canonical algebra. In particular, if this is the case, then $\text{gldim} \Delta \leq 2$.

For the rest of the section we fix a concealed-canonical bound quiver $\Delta$ and a sincere separating exact subcategory $\mathcal{R}$ of $\text{ind} \Delta$. Moreover, we put $\mathcal{P} := \mathcal{R}_-$ and $\mathcal{Q} := \mathcal{R}_+$. Finally, we denote by $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{Q}$ the dimension vectors of the representations from $\text{add} \mathcal{P}$, $\text{add} \mathcal{R}$ and $\text{add} \mathcal{Q}$, respectively.

It is known that $\text{pdim}_\Delta P \leq 1$ for each $P \in \mathcal{P}$ and $\text{idim}_\Delta Q \leq 1$ for each $Q \in \mathcal{Q}$. Next, $\text{pdim}_\Delta R = 1$ and $\text{idim}_\Delta R = 1$ for each $R \in \mathcal{R}$. The categories $\mathcal{P}$ and $\mathcal{Q}$ are closed under the actions of $\tau$ and $\tau^-$, hence using the Auslander–Reiten formulas (1.1) and (1.2) we obtain...
that $\text{Ext}^1_{\Delta}(\mathcal{P}, \mathcal{R}) = 0 = \text{Ext}^1_{\Delta}(\mathcal{R}, \mathcal{Q})$. In particular,

$$\langle d', d \rangle_{\Delta} \geq 0 \quad \text{and} \quad \langle d, d'' \rangle_{\Delta} \geq 0$$

for all $d' \in P, d \in R$ and $d'' \in Q$.

We have $\mathcal{R} = \bigsqcup_{\lambda \in P_1} \mathcal{R}_\lambda$ for connected uniserial categories $\mathcal{R}_\lambda$, $\lambda \in \mathbb{P}_k^1$. For $\lambda \in \mathbb{P}_k^1$ we denote by $r_\lambda$ the number of the pairwise non-isomorphic simple objects in $\text{add} \mathcal{R}_\lambda$. Then $r_\lambda < \infty$ for each $\lambda \in \mathbb{P}_k^1$. Moreover, $\sum_{\lambda \in \mathbb{P}_k^1} (r_\lambda - 1) = |\Delta_0| - 2$. In particular, if $\mathcal{X}_0 := \{ \lambda \in \mathbb{P}_k^1 : r_\lambda > 1 \}$, then $|\mathcal{X}_0| < \infty$.

Fix $\lambda \in \mathbb{P}_k^1$. If $R_{\lambda, 0}, \ldots, R_{\lambda, r_\lambda - 1}$ are chosen representatives of the isomorphisms classes of the simple objects in $\text{add} \mathcal{R}_\lambda$, then we may assume that $\tau R_{\lambda, i} = R_{\lambda, i - 1}$ for each $i \in [0, r_\lambda - 1]$, where we put $R_{\lambda, i} := R_{\lambda, i \mod r_\lambda}$, for $i \in \mathbb{Z}$. For any $i \in \mathbb{Z}$ and $n \in \mathbb{N}_+$ there exists a unique (up to isomorphism) representation in $\mathcal{R}_\lambda$ whose socle and length in $\text{add} \mathcal{R}_\lambda$ are $R_{\lambda, i}$ and $n$, respectively. We fix such representation and denote it by $R_{\lambda, i}^{(n)}$ and its dimension vector by $e^{n}_{\lambda, i}$. Then the composition factors of $R_{\lambda, i}^{(n)}$ are (starting from the socle) $R_{\lambda, 0}, \ldots, R_{\lambda, i + n - 1}$. Consequently, $e^{n}_{\lambda, i} = \sum_{j \in [i, i+n-1]} e_{\lambda, j}$, where $e_{\lambda, j} := \text{dim} R_{\lambda, j}$, for $j \in \mathbb{Z}$. Moreover, for all $i \in \mathbb{Z}$ and $n, m \in \mathbb{N}_+$ there exists an exact sequence

$$0 \to R_{\lambda, i}^{(n)} \to R_{\lambda, i}^{(n+m)} \to R_{\lambda, i+n}^{(m)} \to 0.$$ 

Obviously, for each $R \in \mathcal{R}_\lambda$ there exist $i \in \mathbb{Z}$ and $n \in \mathbb{N}_+$ such that $R \simeq R_{\lambda, i}^{(n)}$. Moreover, it is known that the vectors $e_{\lambda, 0}, \ldots, e_{\lambda, r_\lambda - 1}$ are linearly independent. Consequently, if $R \in \text{add} \mathcal{R}_\lambda$, then there exist uniquely determined $q_{i0}^R, \ldots, q_{i(r_\lambda - 1)}^R \in \mathbb{N}$ such that $\text{dim} R = \sum_{i \in [0, r_\lambda - 1]} q_i^R e_{\lambda, i}$. We put $q_i^R := q_i^R \mod r_\lambda$, for $i \in \mathbb{Z}$. Observe that for each $i \in \mathbb{Z}$ the number $q_i^R$ counts the multiplicity of $R_{\lambda, i}$ as a composition factor in the Jordan–Hölder filtration of $R$ in the category $\text{add} \mathcal{R}_\lambda$.

Let $R = \bigoplus_{\lambda \in \mathbb{P}_k^1} R_{\lambda}$, for $R_{\lambda} \in \text{add} \mathcal{R}_\lambda$, $\lambda \in \mathbb{P}_k^1$. Then we put $q_{i\lambda}^R := q_i^R \lambda$ for $\lambda \in \mathbb{P}_k^1$ and $i \in \mathbb{Z}$. Next, we put $p_i^R := \min\{q_{i\lambda}^R : i \in \mathbb{Z}\}$, for $\lambda \in \mathbb{P}_k^1$, and $p_{i\lambda}^R := q_{i\lambda}^R - p_i^R$, for $\lambda \in \mathbb{P}_k^1$ and $i \in \mathbb{Z}$. Then

$$\text{dim} R = \sum_{\lambda \in \mathbb{P}_k^1} p_i^\lambda \cdot h_\lambda + \sum_{\lambda \in \mathbb{P}_k^1} \sum_{i \in [0, r_\lambda - 1]} p_{i\lambda}^R \cdot e_{\lambda, i},$$

where $h_\lambda := \sum_{i \in [0, r_\lambda - 1]} e_{\lambda, i}$, for $\lambda \in \mathbb{P}_k^1$. It is known that $h_\lambda = h_\mu$ for any $\lambda, \mu \in \mathbb{P}_k^1$. We denote this common value by $h$. Then

$$\text{dim} R = p^R \cdot h + \sum_{\lambda \in \mathbb{P}_k^1} \sum_{i \in [0, r_\lambda - 1]} p_{i\lambda}^R \cdot e_{\lambda, i},$$

where $p^R := \sum_{\lambda \in \mathbb{P}_k^1} p_{i\lambda}^R$. It is known that if $R, R' \in \text{add} \mathcal{R}$ and $\text{dim} R = \text{dim} R'$, then $p^R = p^{R'}$ and $p_{i\lambda}^R = p_{i\lambda}^{R'}$ for any $\lambda \in \mathbb{P}_k^1$ and $i \in [0, r_\lambda - 1]$.
Consequently, for each \(d \in \mathbb{R}\) there exist uniquely determined \(p^d \in \mathbb{N}\) and \(p^d_{\lambda,i} \in \mathbb{N}\) for \(\lambda \in \mathbb{P}^1_k\) and \(i \in [0, r_\lambda - 1]\), such that
\[
d = p^d \cdot h + \sum_{\lambda \in \mathbb{P}^1_k} \sum_{i \in [0, r_\lambda - 1]} p^d_{\lambda,i} \cdot e_{\lambda,i}
\]
and for each \(\lambda \in \mathbb{P}^1_k\) there exists \(i \in [0, r_\lambda - 1]\) with \(p^d_{\lambda,i} = 0\). Again we put \(p^d_{\lambda,i} := p^d_{\lambda,i \text{ mod } r_\lambda}\), for \(d \in \mathbb{R}\), \(\lambda \in \mathbb{P}^1_k\) and \(i \in \mathbb{Z}\).

It is known that \(h\) is sincere. Moreover, \(h\) can be used in order to distinguish between representations from \(\mathcal{P}\), \(\mathcal{Q}\) and \(\mathcal{R}\). Namely, if \(X\) is an indecomposable representation of \(\Delta\), then
\[
\langle \dim X, h \rangle_{\Delta} > 0.
\]
Dually, if \(X\) is an indecomposable representation of \(\Delta\), then
\[
\langle h, \dim X \rangle_{\Delta} > 0.
\]

Let \(\lambda, \mu \in \mathbb{P}^1_k\), \(i, j \in \mathbb{Z}\) and \(m, n \in \mathbb{N}_+\). Then
\[
\dim_k \text{Hom}_\Delta(R_{\lambda,i}^{(n)}, R_{\mu,j}^{(m)}) = \min \{q_{\lambda,i+n-1}, q_{\mu,j}^{(m)}\}
\]
in particular, \(\text{Hom}_\Delta(R_{\lambda,i}^{(n)}, R_{\mu,j}^{(m)}) = 0\) if \(\lambda \neq \mu\). The above formula, together with the Auslander–Reiten formula (1.1), implies that
\[
\langle e_{\lambda,i}^n, d \rangle_{\Delta} = p^d_{\lambda,i+n-1} - p^d_{\lambda,i-1}
\]
for any \(\lambda \in \mathbb{P}^1_k\), \(i \in \mathbb{Z}\), \(n \in \mathbb{N}_+\) and \(d \in \mathbb{R}\). In particular,
\[
\langle h, d \rangle_{\Delta} = 0 = \langle d, h \rangle_{\Delta}
\]
for each \(d \in \mathbb{R}\).

An important role in the proofs will be played by ext-minimal representations. We call a representation \(V\) ext-minimal if there is no decomposition \(V = V_1 \oplus V_2\) with \(\text{Ext}^1_\Delta(V_1, V_2) \neq 0\). We recall facts on ext-minimal representations belonging to add \(\mathcal{R}\).

First assume that \(d \in \mathbb{R}\) and \(p^d = 0\). In this case there is a unique (up to isomorphism) ext-minimal representation \(W \in \text{add} \mathcal{R}\) with dimension vector \(d\), which is constructed inductively in the following way. For \(\lambda \in \mathbb{P}^1_k\), let \(I_\lambda := \{i \in [0, r_\lambda - 1]: p^d_{\lambda,i} \neq 0\}\). For \(\lambda \in \mathbb{P}^1_k\) and \(i \in I_\lambda\), we denote by \(m_{\lambda,i}\) the minimal \(m \in \mathbb{N}_+\) such that \(p^d_{\lambda,i+m} = 0\). By induction there exists (unique up to isomorphism) ext-minimal representation \(W' \in \text{add} \mathcal{R}\) with dimension vector \(d - \sum_{\lambda \in \mathbb{P}^1_k} \sum_{i \in I_\lambda} e_{\lambda,i}^{m_{\lambda,i}}\). Then \(W := W' \oplus \bigoplus_{\lambda \in \mathbb{P}^1_k} \bigoplus_{i \in I_\lambda} R_{\lambda,i}^{(m_{\lambda,i})}\) is ext-minimal.

We will use the following property of the above representation.

**Lemma 2.1.** Assume \(d \in \mathbb{R}\) and \(p^d = 0\). Let \(W \in \text{add} \mathcal{R}\) be an ext-minimal representation with dimension vector \(d\). If \(\lambda \in \mathbb{P}^1_k\), \(i \in \mathbb{Z}\), \(n \in \mathbb{N}_+\), \(p^d_{\lambda,i} = p^d_{\lambda,i+n}\) and \(p^d_{\lambda,j} \geq p^d_{\lambda,i}\) for each \(j \in [i, i+n]\), then \(\text{Hom}_\Delta(R_{\lambda,i+1}^{(n)}, W) = 0\).
Proof. Observe that $\text{Hom}_R(R_{\lambda,k}^{(n)}, R_{\lambda,k}^{(m,k)}) = 0$ for each $k \in I_\lambda$, since one easily checks that either $q_{\lambda,i}^{d_{k,i}} = 0$ (if $p_{\lambda,i}^d = 0$) or $p_{\lambda,k}^{d_{k,i}+1} = 0$ (if $p_{\lambda,i}^d > 0$). Now the claim follows by induction. \qed

Now let $d \in R$ be arbitrary. The description of the ext-minimal representations with dimension vector $d$, which belong to add $R$, has been given in [25, Theorem 3.5] (this theorem has been formulated in the case $\Delta = (\Delta, \emptyset)$ for a Euclidean quiver $\Delta$, but its proof translates to an arbitrary concealed-canonical bound quiver). We will not repeat the formulation here, but only mention some consequences. First, if $W \in \text{add } R$ and $\text{dim } W = d$, then $W$ is ext-minimal if and only if $\text{dim}_k \text{End}_\Delta(W) = p^d + \langle d, d \rangle_\Delta$. In particular,

$$\text{(2.7)} \quad p^d + \langle d, d \rangle_\Delta = \min \{ \text{dim}_k \text{End}_\Delta(W) : W \in \text{add } R \text{ such that } \text{dim } W = d \}$$

(here we use also [25 Lemma 2.1]). Next, if $W \in \text{add } R$ is an ext-minimal representation with dimension vector $d$ and $W' \in \text{add } R$ is an ext-minimal representation with dimension vector $d - p^d \cdot h$, then there exists an exact sequence $0 \to \bigoplus_{\lambda \in \mathbb{P}_k} R_\lambda \to W \to W' \to 0$ with $R_\lambda \in R_\lambda$ (in particular, indecomposable) for each $\lambda \in \mathbb{P}_k$ (obviously, $\text{dim } R_\lambda$ is a multiplicity of $h$ for each $\lambda \in \mathbb{P}_k$).

3. Semi-invariants

Let $\Delta$ be a bound quiver and $d$ a dimension vector. By $\text{rep}_\Delta(d)$ we denote the set of the representations $M$ of $\Delta$ such that $M(x) = \mathbb{k}^{d(x)}$ for each $x \in \Delta_0$. We may identify $\text{rep}_\Delta(d)$ with a Zariski-closed subset of the affine space $\text{rep}_\Delta(d) := \prod_{\alpha \in \Delta_1} \text{M}_{d(\alpha) \times d(\alpha)}(\mathbb{k})$, hence it has a structure of an affine variety. The group $\text{GL}(d) := \prod_{x \in \Delta_0} \text{GL}(d(x))$ acts on $\text{rep}_\Delta(d)$ by conjugation: $(g \cdot M)(\alpha) := g(\alpha) \cdot M(\alpha) \cdot g(\alpha)^{-1}$, for $g \in \text{GL}(d)$, $M \in \text{rep}_\Delta(d)$ and $\alpha \in \Delta_1$. The set $\text{rep}_\Delta(d)$ is a $\text{GL}(d)$-invariant subset of $\text{rep}_\Delta(d)$ and the $\text{GL}(d)$-orbits in $\text{rep}_\Delta(d)$ correspond to the isomorphism classes of the representations of $\Delta$ with dimension vector $d$. If $\mathcal{X}$ is a full subcategory of $\text{ind } \Delta$, then we denote by $\mathcal{X}(d)$ the set of $V \in \text{rep}_\Delta(d)$ such that $V \in \text{add } \mathcal{X}$.

Let $\Delta$ be a quiver and $\theta \in \mathbb{Z}_{\Delta_0}$. We treat $\theta$ as a $\mathbb{Z}$-linear function $\mathbb{Z}_{\Delta_0} \to \mathbb{Z}$ in a usual way. If $d$ is a dimension vector, then by a semi-invariant of weight $\theta$ we mean every function $f \in \mathbb{k}[	ext{rep}_\Delta(d)]$ such that $f(g^{-1} \cdot M) = \chi^\theta(g) \cdot f(M)$ for any $g \in \text{GL}(d)$ and $M \in \text{rep}_\Delta(d)$, where $\chi^\theta(g) := \prod_{x \in \Delta_0} (\text{det } g(x))^{\theta(x)}$ for $g \in \text{GL}(d)$.

Now let $\Delta$ be a bound quiver and $d$ a dimension vector. If $\theta \in \mathbb{Z}_{\Delta_0}$, then a function $f \in \mathbb{k}[	ext{rep}_\Delta(d)]$ is called a semi-invariant of weight $\theta$ if $f$ is the restriction of a semi-invariant of weight $\theta$ from $\mathbb{k}[	ext{rep}_\Delta(d)]$. This definition differs from the definition used in other papers on the subject
(see for example [5][11][13][15]), however these are the semi-invariants which one needs to understand in order to study King’s moduli spaces for representations of bound quivers [21]. Moreover, the two definitions coincide if the characteristic of \( k \) equals 0. We denote the space of the semi-invariants of weight \( \theta \) by \( \text{SI}[\Delta, d]_\theta \). If \( d \) is sincere, then we put \( \text{SI}[\Delta, d] := \bigoplus_{\theta \in \mathbb{Z}_{\Delta_0}} \text{SI}[\Delta, d]_\theta \) and call it the algebra of semi-invariants for \( \Delta \) and \( d \) (we assume sincerity of \( d \), since under this assumption \( \mathbb{Z}_{\Delta_0} \) is isomorphic with the character group of \( \text{GL}(d) \)).

We recall a construction from [13]. Let \( \Delta \) be a bound quiver. Fix a representation \( V \) of \( \Delta \) and define \( \theta^V : \mathbb{Z}_{\Delta_0} \to \mathbb{Z} \) by the condition:

\[
\theta^V(\dim M) = \dim_k \text{Hom}_\Delta(V, M) - \dim_k \text{Hom}_\Delta(M, \tau V)
\]

for each representation \( M \) of \( \Delta \). The formula (1.1) implies that \( \theta^V = \langle \dim, - \rangle_\Delta \) if \( \text{pdim}_{\Delta} V \leq 1 \). Dually, if \( V \) has no indecomposable projective direct summands (i.e. \( \tau^\ast \tau V \simeq V \) [11, Theorem IV.2.10]) and \( \text{idim}_\Delta \tau V \leq 1 \), then \( \theta^V = -\langle - , \dim \tau V \rangle_\Delta \) by the formula (1.2).

Now let \( d \) be a dimension vector. If \( \theta^V(d) = 0 \), then we define a function \( c^V_d \in \mathbb{k}[\text{rep}_\Delta(d)] \) in the following way. Let \( P_1 \xrightarrow{f} P_0 \to V \to 0 \) be the minimal projective presentation of \( V \). One shows that

\[
\dim_k \text{Ker} \text{Hom}_\Delta(f, M) = \dim_k \text{Hom}_\Delta(V, M)
\]

and

\[
\dim_k \text{Coker} \text{Hom}_\Delta(f, M) = \dim_k \text{Hom}_\Delta(M, \tau V),
\]

hence

\[
(3.1) \quad \dim_k \text{Hom}_\Delta(P_0, M) - \dim_k \text{Hom}_\Delta(P_1, M) = \dim_k \text{Hom}_\Delta(V, M) - \dim_k \text{Hom}_\Delta(M, \tau V) = \theta^V(d) = 0,
\]

for each \( M \in \text{rep}_\Delta(d) \). Thus, we may define \( c^V_d \in \mathbb{k}[\text{rep}_\Delta(d)] \) by the formula \( c^V_d(M) := \text{det} \text{Hom}_\Delta(f, M) \) for \( M \in \text{rep}_\Delta(d) \). Note that \( c^V_d \) is defined only up to a non-zero scalar. If \( M \in \text{rep}_\Delta(d) \), then \( c^V_d(M) = 0 \) if and only if \( \text{Hom}_\Delta(V, M) \neq 0 \). Moreover, if \( \text{pdim}_\Delta V \leq 1 \) and \( M \in \text{rep}_\Delta(d) \), then \( c^V_d(M) = 0 \) if and only if \( \text{Ext}_1^\Delta(V, M) \neq 0 \). It is known that \( c^V_d \in \text{SI}[\Delta, d]_{\theta^V} \). This function depends on the choice of \( f \), but the functions obtained for different \( f \)’s differ only by non-zero scalars.

In fact, we could start with an arbitrary \( \Delta \)-admissible projective presentation, where, for a representation \( V \) of a bound quiver \( \Delta \) and a dimension vector \( d \), we call a projective representation \( P_1' \to P_0' \to V \to 0 \) of \( V \) \( \Delta \)-admissible if \( \dim_k \text{Hom}_\Delta(P_0', M) = \dim_k \text{Hom}_\Delta(P_1', M) \) for any (equivalently, some) \( M \in \text{rep}_\Delta(d) \).

**Lemma 3.1.** Let \( \Delta \) be a bound quiver, \( d \) a dimension vector and \( P_1' \xrightarrow{f'} P_0' \to V \to 0 \) a \( \Delta \)-admissible projective presentation of a representation \( V \) of \( \Delta \).
(1) If \( \theta^V(d) = 0 \), then there exists \( \xi \in k \) such \( \xi \neq 0 \) and \( c_d^V(M) = \xi \cdot \det \Hom_\Delta(f', M) \) for each \( M \in \rep_\Delta(d) \).

(2) If there exists \( M \in \rep_\Delta(d) \) such that \( \det \Hom_\Delta(f', M) \neq 0 \), then \( \theta^V(d) = 0 \).

**Proof.** Let \( P_1 \xrightarrow{f} P_0 \to V \to 0 \) be the minimal projective presentation of \( V \). There exists projective representations \( P \) and \( Q \) of \( \Delta \) and isomorphisms \( g_1 : P' \to P \oplus P \oplus Q \) and \( g_0 : P'_0 \to P_0 \oplus P \) such that
\[
f' = g_0^{-1} \circ \begin{bmatrix} f & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \circ g_1.
\]
Consequently,
\[
(3.2) \quad \Hom_\Delta(f', M) = \Hom_\Delta(g_1, M)
\]
\[
\circ \begin{bmatrix} 
\Hom_\Delta(f, M) & 0 \\
0 & \Hom_\Delta(Id_P, M) \\
0 & 0
\end{bmatrix} \circ \Hom_\Delta(g_0^{-1}, M)
\]
for each \( M \in \rep_\Delta(d) \). Since the presentation \( P'_1 \xrightarrow{f'} P'_0 \to V \to 0 \) is \( d \)-admissible, \((3.1)\) implies that the condition \( \theta^V(d) = 0 \) is equivalent to the condition \( \dim_k \Hom_\Delta(Q, M) = 0 \) for each \( M \in \rep_\Delta(d) \). Together with \((3.2)\) this implies our claims. \( \square \)

As an immediate consequence we obtain the following.

**Corollary 3.2.** Let \( \Delta \) be a bound quiver, \( d \) a dimension vector and \( 0 \to V_1 \to V \to V_2 \to 0 \) an exact sequence such that \( \theta^{V_1}(d) = 0 = \theta^{V_2}(d) \).

(1) If \( \theta^V(d) = 0 \), then (up to a non-zero scalar) \( c_d^V = c_d^{V_1} \cdot c_d^{V_2} \).

(2) If \( c_d^{V_1} \cdot c_d^{V_2} \neq 0 \), then \( \theta^V(d) = 0 \) and (up to a non-zero scalar)
\[
c_d^V = c_d^{V_1} \cdot c_d^{V_2}.
\]

**Proof.** Let \( P'_1 \xrightarrow{f'} P'_0 \to V_1 \to 0 \) and \( P''_1 \xrightarrow{f''} P''_0 \to V_2 \to 0 \) be the minimal projective presentations of \( V_1 \) and \( V_2 \), respectively. Then there exists a projective presentation of \( V \) of the form
\[
P'_1 \oplus P''_1 \xrightarrow{f} P'_0 \oplus P''_0 \to V \to 0,
\]
where \( f = \begin{bmatrix} f' & g \\ 0 & f'' \end{bmatrix} \) for some \( g \in \Hom_\Delta(P''_1, P'_0) \). One easily sees that
\[
\det \Hom_\Delta(f, M) = c_d^{V_1}(M) \cdot c_d^{V_2}(M)
\]
for each \( M \in \rep_\Delta(d) \), hence the claims follows from Lemma \((3.1)\). \( \square \)

The following fact is an extension of \([10], \text{Lemma 1(a)}\) to the setup of bound quivers.

**Lemma 3.3.** Let \( \Delta \) be a bound quiver and \( d \) a dimension vector. If \( 0 \to V_1 \to V \to V_2 \to 0 \) is an exact sequence, \( \theta^V(d) = 0 \) and \( c_d^V \neq 0 \), then \( \theta^{V_2}(d) \leq 0 \).
Proof. If \( \theta^V(d) > 0 \), then
\[
\dim_k \text{Hom}_{\Delta}(V_2, M) \geq \theta^V(d) > 0
\]
for each \( M \in \text{rep}_{\Delta}(d) \). This immediately implies that \( \text{Hom}_{\Delta}(V, M) \neq 0 \) for each \( M \in \text{rep}_{\Delta}(d) \), hence \( c_d^V = 0 \), contradiction. \( \square \)

We have the following multiplicative property.

**Lemma 3.4.** Let \( \Delta \) be a bound quiver and \( d \) a dimension vector. If \( V_1 \) and \( V_2 \) are representations of \( \Delta \), \( V := V_1 \oplus V_2 \), \( \theta^V(d) = 0 \) and \( c_d^V \neq 0 \), then \( \theta^{V_1}(d) = 0 = \theta^{V_2}(d) \) and \( c_d^{V_1} \cdot c_d^{V_2} \) (up to a non-zero scalar).

**Proof.** See [13, Lemma 3.3]. \( \square \)

We will also use another multiplicative property.

**Lemma 3.5.** Let \( \Delta \) be a bound quiver and \( V \) a representation of \( \Delta \). If \( d' \) and \( d'' \) are dimension vectors and \( \theta^V(d') = 0 = \theta^V(d'') \), then
\[
c_d^V(W' \oplus W'') = c_{d'}^V(W') \cdot c_{d''}^V(W'')
\]
for all \( (W', W'') \in \text{rep}_{\Delta}(d') \times \text{rep}_{\Delta}(d''). \)

**Proof.** Let \( P_1 \xrightarrow{f} P_0 \to V \to 0 \) be the minimal projective presentation of \( V \). If \( (W', W'') \in \text{rep}_{\Delta}(d') \times \text{rep}_{\Delta}(d'') \), then
\[
\text{Hom}_{\Delta}(f, W' \oplus W'') = \begin{bmatrix}
\text{Hom}_{\Delta}(f, W') & 0 \\
0 & \text{Hom}_{\Delta}(f, W'')
\end{bmatrix}
\]
and both \( \text{Hom}_{\Delta}(f, W') \) and \( \text{Hom}_{\Delta}(f, W'') \) are square matrices, hence the claim follows. \( \square \)

The following result follows from the proof of [13, Theorem 3.2] (note that the assumption about the characteristic of \( k \) made in [13, Theorem 3.2] is only necessary to prove surjectivity of the restriction morphism, which we have for free with our definition of semi-invariants).

**Proposition 3.6.** Let \( \Delta \) be a bound quiver, \( d \) a dimension vector and \( \theta \in \mathbb{Z}^{\Delta_0} \).

1. If \( \theta(d) \neq 0 \), then \( \text{SI}[\Delta, d]_{\theta} = 0 \).
2. If \( \theta(d) = 0 \), then the space \( \text{SI}[\Delta, d]_{\theta} \) is spanned by the functions \( c_d^V \) for \( V \in \text{rep} \Delta \) such that \( \theta^V = \theta \) and \( c_d^V \neq 0 \). \( \square \)

In fact we may take a smaller spanning set.

**Corollary 3.7.** Let \( \Delta \) be a bound quiver and \( d \) a dimension vector. If \( \theta \in \mathbb{Z}^{\Delta_0} \) and \( \theta(d) = 0 \), then the space \( \text{SI}[\Delta, d]_{\theta} \) is spanned by the functions \( c_d^V \) for \( \text{Ext}_{\Delta}^1(V, V_2) \neq 0 \) and there is a decomposition \( V = V_1 \oplus V_2 \) with \( \text{Ext}_{\Delta}^1(V_1, V_2) \neq 0 \). Lemma 3.4 implies that \( \theta^{V_1}(d) = 0 = \theta^{V_2}(d) \) and \( c_d^{V_1} \cdot c_d^{V_2} \neq 0 \).

**Proof.** Assume that \( V \) is a representation of \( \Delta \) such that \( \theta^V = \theta \), \( c_d^V \neq 0 \) and there is a decomposition \( V = V_1 \oplus V_2 \) with \( \text{Ext}_{\Delta}^1(V_1, V_2) \neq 0 \). Lemma 3.4 implies that \( \theta^{V_1}(d) = 0 \) and \( c_d^{V_1} \cdot c_d^{V_2} \neq 0 \). If \( 0 \to V_2 \to W \to V_1 \to 0 \) is a non-split exact sequence, then...
Corollary 3.2 and Lemma 3.4 imply that (up to a non-zero scalar) 
\( c_d^W = c_{d_1}^{x_1} \cdot c_{d_2}^{x_2} = c_d^V \). Since \( \dim_k \text{End}_{\Delta}(W) < \dim_k \text{End}_{\Delta}(V) \) (see for example [25 Lemma 2.1]), the claim follows by induction. \( \square \)

We may even take a smaller set, if we are only interested in generators of \( \text{SI}[\Delta, d] \). Namely, we have the following.

**Corollary 3.8.** Let \( \Delta \) be a bound quiver and \( d \) a sincere separating exact subcategory. Then the algebra \( \text{SI}[\Delta, d] \) is generated by the semi-invariants \( c_d^V \) for \( V \in \text{rep}_\Delta(d) \) such that \( \theta^V(d) = 0 \), \( c_d^V \neq 0 \) and \( V \) is indecomposable.

**Proof.** This follows from Proposition 3.6 and Lemma 3.4 (this is also the content of [13 Corollary 3.4]). \( \square \)

4. Preliminary results

Throughout this section we fix a concealed-canonical bound quiver \( \Delta \) and a sincere separating exact subcategory \( R \) of \( \text{ind} \Delta \). We will use notation introduced in Section 2. We also fix \( d \in R \) such that \( p := p^d > 0 \). Notice that this implies that \( d \) is sincere.

First we prove that the algebra \( \text{SI}[\Delta, d] \) is controlled by the representations from \( \text{add} R \).

**Lemma 4.1.** Let \( V \) be a representation of \( \Delta \) such that \( \theta^V(d) = 0 \). If \( c_d^V \neq 0 \), then \( V \in \text{add} R \) and \( \theta^V = \langle \text{dim} V, - \rangle_\Delta \).

**Proof.** Assume that \( P \in \mathcal{P} \) is a direct summand of \( V \). Since \( \text{pdim}_\Delta P \leq 1 \), (2.1) and (2.3) imply that 
\[ \theta^P(d) = \langle \text{dim} P, d \rangle_\Delta \geq \langle \text{dim} P, h \rangle_\Delta > 0. \]
Consequently, \( c_d^V = 0 \) by Lemma 3.4 contradiction. Dually, \( V \) cannot have a direct summand from \( Q \). Finally, since \( \text{pdim}_\Delta V = 1 \), \( \theta^V = \langle \text{dim} V, - \rangle_\Delta \). \( \square \)

Together with Corollary 3.7 this lemma immediately implies the following.

**Corollary 4.2.** Let \( \theta \in \mathbb{Z}^{\Delta_0} \) be such that \( \text{SI}[\Delta, d]_\theta \neq 0 \). Then there exists \( r \in R \) such that \( \theta = \langle r, - \rangle_\Delta \) and \( \langle r, d \rangle_\Delta = 0 \). \( \square \)

Taking into account Corollary 3.8 we need to identify \( V \in \text{ind} \Delta \) such that \( \theta^V(d) = 0 \) and \( c_d^V \neq 0 \). The first step in this direction is the following.

**Lemma 4.3.** Let \( V \) be an indecomposable representation of \( \Delta \). If \( \theta^V(d) = 0 \) and \( c_d^V \neq 0 \), then \( V = P_{\lambda, i+1}^{(n)} \) for some \( \lambda \in \mathbb{P}_{k,i}, i \in \mathbb{Z} \) and \( n \in \mathbb{N}_+ \) such that \( p^d_{\lambda, i} = p^d_{\lambda, i+n} \) and \( p^d_{\lambda, j} \geq p^d_{\lambda, i} \) for each \( j \in [i+1, i+n-1] \).

**Proof.** We know from Lemma 4.1 that \( V \in \mathcal{R} \), hence there exists \( \lambda \in \mathbb{P}_{k,i}, i \in \mathbb{Z} \) and \( n \in \mathbb{N}_+ \) such that \( V = P_{\lambda, i+1}^{(n)} \). Then \( \theta^V(d) = p^d_{\lambda, i+n} - p^d_{\lambda, i} \) by (2.5), thus the condition \( \theta^V(d) = 0 \) means that \( p^d_{\lambda, i} = p^d_{\lambda, i+n} \). Finally,
the condition \( e_d^V \neq 0 \) and Lemma 3.3 imply that \( \theta^{V'}(d) \leq 0 \) for each factor representation \( V' \) of \( V \). The sequence (2.2) implies that \( R_{\lambda,j+1}^{(n_i+j-j)} \) is a factor representation of \( V \) for each \( j \in [i+1, i+n-1] \), hence the claim follows. \( \square \)

Now we show that the representations described in the above lemma give rise to non-zero semi-invariants.

**Lemma 4.4.** Let \( \lambda \in \mathbb{P}^1_k \), \( i \in \mathbb{Z} \) and \( n \in \mathbb{N} \) be such that \( p_{\lambda,i}^d = p_{\lambda,i+n}^d \) and \( p_{\lambda,j}^d \geq p_{\lambda,i+n}^d \) for each \( j \in [i+1, i+n-1] \). If \( V := R_{\lambda,i+1}^{(n)} \), then \( \theta^V(d) = 0 \) and there exists \( R \in \mathcal{R}(d) \) such that \( e_d^V(R) \neq 0 \).

**Proof.** We only need to show that there exists \( R \in \mathcal{R}(d) \) such that \( c_d^V(R) \neq 0 \). Let \( W \in \mathcal{R} \) be an ext-minimal representation for \( d - p \cdot h \) and fix \( \mu \in \mathbb{P}^1_k \) different from \( \lambda \) such that \( r_{\mu} = 1 \). If \( R := W \oplus R_{\mu,0}^{(p)} \), then \( R \in \text{rep}_{\lambda}(d) \) and \( \text{Hom}_{\lambda}(V, R) = \text{Hom}_{\lambda}(V, W) = 0 \) by Lemma 2.1, hence the claim follows. \( \square \)

As a consequence we present a smaller generating set of \( \text{SI}[\Delta, d] \). First we introduce some notation. For \( \lambda \in \mathbb{P}^1_k \) we denote by \( I_{\lambda} \) the set of \( i \in [0, r_{\lambda}-1] \) such that there exists \( n \in \mathbb{N}_+ \) with \( p_{\lambda,i}^d = p_{\lambda,i+n}^d \) and \( p_{\lambda,j}^d > p_{\lambda,i}^d \) for each \( j \in [i+1, i+n-1] \) (such \( n \), if exists, is uniquely determined by \( \lambda \) and \( i \), and we denote it by \( n_{\lambda,i} \)). Observe that \( I_{\lambda} = \{0\} \) and \( n_{\lambda,0} = 1 \) if \( r_{\lambda} = 1 \).

**Corollary 4.5.** The algebra \( \text{SI}[\Delta, d] \) is generated by the semi-invariants \( c_d^{R_{\lambda,i+1}^{(n)}} \) for \( \lambda \in \mathbb{P}^1_k \) and \( i \in I_{\lambda} \).

**Proof.** For \( \lambda \in \mathbb{P}^1_k \) we denote by \( I_{\lambda} \) the set of all pairs \((i, n)\) such that \( (i, n) \in [0, r_{\lambda}-1] \times \mathbb{N}_+ \) such that \( p_{\lambda,i}^d = p_{\lambda,i+n}^d \) and \( p_{\lambda,j}^d \geq p_{\lambda,i+n}^d \) for each \( j \in [i+1, i+n-1] \). Observe that if \( \lambda \in \mathbb{P}^1_k \) and \((i, n) \in I_{\lambda} \), then \( i \in I_{\lambda} \). Corollary 3.8 and Lemma 4.3 imply that the algebra \( \text{SI}[\Delta, d] \) is generated by the semi-invariants \( c_d^{R_{\lambda,i+1}^{(n)}} \) for \( \lambda \in \mathbb{P}^1_k \) and \((i, n) \in I_{\lambda} \). Now, let \( \lambda \in \mathbb{P}^1_k \) and \((i, n) \in I_{\lambda} \). Obviously, \( n \geq n_{\lambda,i} \). If \( n > n_{\lambda,i} \), then (up to a non-zero scalar) \( c_d^{R_{\lambda,i+1}^{(n)}} = c_d^{R_{\lambda,i+1}^{(n_{\lambda,i})}} \cdot c_d \) by Corollary 3.3(1), as according to (2.2) we have an exact sequence

\[
0 \to R_{\lambda,i+1}^{(n_{\lambda,i})} \to R_{\lambda,i+1}^{(n)} \to R_{\lambda,i+1}^{(n-n_{\lambda,i})} \to 0.
\]

Since \( R_{\lambda,i+n_{\lambda,i}+1}^{(n-n_{\lambda,i})} = R_{\lambda,(i+n_{\lambda,i}+1)\mod r_{\lambda}}^{(n-n_{\lambda,i})} \) and \((i+n_{\lambda,i}) \mod r_{\lambda}, n-n_{\lambda,i}) \in I_{\lambda} \), the claim follows by induction. \( \square \)

At the later stage we will prove that for each non-zero semi-invariant \( f \) there exists \( R \in \mathcal{R}(d) \) such that \( f(R) \neq 0 \). At the moment we formulate the following versions of this fact.
Lemma 4.6. Let \( V \) be a representation of \( \Delta \) such that \( \theta^V(d) = 0 \) and \( c^V_d \neq 0 \). Then there exists \( R \in \mathcal{R}(d) \) such that \( c^R_d(R) \neq 0 \).

Proof. Let \( X \) be an indecomposable direct summand of \( V \). Lemma 3.4 implies that \( c^X_d \neq 0 \). Consequently, Lemmas 4.3 and 4.4 imply that there exists \( R_X \in \mathcal{R}(d) \) such that \( c^X_d(R_X) \neq 0 \). Since \( \mathcal{R}(d) \) is an irreducible and open subset of \( \text{rep}_\Delta(d) \) \([15\, Section 4]\), there exists \( R \in \mathcal{R}(d) \) such that \( c^X_d(R) \neq 0 \) for each indecomposable direct summand \( X \) of \( V \). Using once more Lemma 3.4 we obtain that \( c^R_d(R) \neq 0 \). \( \square \)

Lemma 4.7. If \( q \in \mathbb{N} \) and \( f \in \text{SI}[\Delta, d]_{(q \cdot h, -)\Delta} \) is non-zero, then there exists \( R \in \mathcal{R}(d) \) such that \( f(R) \neq 0 \).

Proof. If \( q = 0 \), then the claim is obvious, since \( \text{SI}[\Delta, d]_0 = k \). Thus assume \( q > 0 \). We know that \( \text{SI}[\Delta, d]_{(q \cdot h, -)\Delta} \) is spanned by the functions \( c^V_d \) for \( V \in \text{add} \mathcal{R} \) with dimension vector \( q \cdot h \). It is enough to prove that \( c^V_d(M) = 0 \) for all \( V \in \text{add} \mathcal{R} \) and \( M \in \text{rep}_\Delta(d) \) such that \( \dim V = q \cdot h \) and \( M \not\in \mathcal{R}(d) \). Every such \( M \) has an indecomposable direct summand \( Q \) from \( Q \). Indeed, since \( M \not\in \mathcal{R}(d) \), it has an indecomposable direct summand \( X \) which belongs to \( \mathcal{P} \cup Q \). If \( X \in Q \), then we take \( Q := X \). If \( X \in \mathcal{P} \), then \( \langle \dim M - \dim X, h \rangle_\Delta < 0 \) by \([2.3]\) and \([2.0]\). Consequently, \( M \) has an indecomposable direct summand \( Q \) with \( \langle \dim Q, h \rangle_\Delta < 0 \). Using again \([2.3]\) and \([2.6]\) we get \( Q \in Q \).

Then

\[
\dim_k \text{Hom}_\Delta(V, M) \geq \dim_k \text{Hom}_\Delta(V, Q) = \langle q \cdot h, \dim Q \rangle_\Delta > 0
\]

by \([2.4]\) and the claim follows. \( \square \)

Recall from Corollary 4.2 that the possible weights are of the form \( \langle r, - \rangle_\Delta \) for \( r \in \mathbb{R} \) such that \( \langle r, d \rangle_\Delta = 0 \). Our next aim is to show that it is enough to understand those which are for the form \( \langle q \cdot h, - \rangle_\Delta \) for \( q \in \mathbb{N} \).

We start with the following easy lemma.

Lemma 4.8. Let \( W \in \text{add} \mathcal{R} \) be such that \( \theta^W(d) = 0 \) and \( c^W_d \neq 0 \). If \( q \in \mathbb{N} \) and \( f \in \text{SI}[\Delta, d]_{(q \cdot h, -)\Delta} \) is non-zero, then there exists \( R \in \mathcal{R}(d) \) such that \( c^W_d(R) \cdot f(R) \neq 0 \).

Proof. Since \( \mathcal{R}(d) \) is an open irreducible subset of \( \text{rep}_\Delta(d) \), the claim follows from Lemmas 4.6 and 4.7. \( \square \)

Proposition 4.9. Let \( r \in \mathbb{R}, \langle r, d \rangle_\Delta = 0 \) and \( W \in \text{add} \mathcal{R} \) be an ext-minimal representation for \( r - p^r \cdot h \).

1. If \( c^W_d = 0 \), then \( \text{SI}[\Delta, d]_{(r, -)\Delta} = 0 \).
2. If \( c^W_d \neq 0 \), then the map

\[
\text{SI}[\Delta, d]_{(p^r \cdot h, -)\Delta} \to \text{SI}[\Delta, d]_{(r, -)\Delta}, f \mapsto c^W_d \cdot f,
\]

is an isomorphism of vector spaces.
Proof. Let \( \Phi : \text{SI}[\Delta, d]_{(\gamma, h, -)\Delta} \to \text{SI}[\Delta, d]_{(\varepsilon, -)\Delta} \) be the map given by \( \Phi(f) := c^W_d \cdot f \), for \( f \in \text{SI}[\Delta, d]_{(\gamma, h, -)\Delta} \).

It follows from Corollary 3.7 and Lemma 4.1 that \( \text{SI}[\Delta, d]_{(\varepsilon, -)\Delta} \) is spanned by the functions \( c^W_d \) for ext-minimal \( V \in \text{add } R \) such that \( \text{dim } V = r \). If \( V \in \text{add } R \) is ext-minimal and \( \text{dim } V = r \), then there exists an exact sequence \( 0 \to R \to V \to W \to 0 \), where \( R \in \text{add } R \) and \( \text{dim } R = p^x \cdot h \). Thus Corollary 3.2 implies that \( (\text{up to a non-zero scalar}) c^W_d = c^W_d \cdot c^R_d = \Phi(c^R_d) \). This shows that \( \Phi \) is an epimorphism. In particular, \( \text{SI}[\Delta, d]_{(\varepsilon, -)\Delta} = 0 \) if \( c^W_d = 0 \). On the other hand, if \( c^W_d \neq 0 \), then \( \Phi \) is a monomorphism (hence an isomorphism) by Lemma 4.8. \( \square \)

In the previous papers on the subject the authors have studied either the semi-invariants on the whole variety \( \text{rep}_{\Delta}(d) \) \([14, 15]\) or on the closure of \( \mathcal{R}(d) \) only \([29]\). However, the answers they have obtained did not differ. We have the following explanation of this phenomena.

**Proposition 4.10.** If \( f \in k[\text{rep}_{\Delta}(d)] \) is a non-zero semi-invariant, then there exists \( R \in \mathcal{R}(d) \) such that \( f(R) \neq 0 \).

**Proof.** Fix \( r \in \mathbb{R} \) such that \( f \in \text{SI}[\Delta, d]_{(\varepsilon, -)\Delta} \). The previous lemma implies that \( f = c^W_d \cdot f' \), where \( W \in \text{add } R \) is an ext-minimal representation with dimension vector \( r - p^x \cdot h \) and \( f' \in \text{SI}[\Delta, d]_{(\gamma, h, -)\Delta} \).

Consequently, the claim follows from Lemma 4.8. \( \square \)

Observe that this proposition means in particular, that \( \text{SI}[\Delta, d] \) is a domain, hence the product of two non-zero semi-invariants is non-zero again.

Proposition 4.9 implies that the subalgebra \( \bigoplus_{\gamma \in \mathbb{N}} \text{SI}[\Delta, d]_{(\gamma, h, -)\Delta} \) of \( \text{SI}[\Delta, d] \) plays a crucial role. In Section 6 we show that the study of this subalgebra can be reduced to the case of the Kronecker quiver. Thus in the next section we recall facts about semi-invariants for the Kronecker quiver.

## 5. The Kronecker Quiver

Our aim in this section is to collect necessary facts about representations and semi-invariants for the Kronecker quiver \( K_2 \), i.e. the quiver

\[
\begin{array}{c}
\bullet \\
\alpha \\
\beta \\
\bullet
\end{array}
\]

with the empty set of relations. In this case a sincere separating exact subcategory is uniquely determined. Let \( \mathcal{T} = \bigsqcup_{\lambda \in \mathbb{P}_k} \mathcal{T}_\lambda \) by the sincere separating exact subcategory of \( \text{ind } K_2 \).

For \( \zeta, \xi \in k \) let \( N_{\zeta, \xi} \) be the representation \( \begin{array}{c} \zeta \\ \xi \end{array} \). Then the simple objects in \( \mathcal{T} \) are precisely the representations \( N_{\zeta, \xi} \) for \( (\zeta : \xi) \in \mathbb{P}^1_k \). Moreover, if \( (\zeta : \xi), (\zeta' : \xi') \in \mathbb{P}^1_k \), then \( N_{\zeta, \xi} \simeq N_{\zeta', \xi'} \) if and only if \( (\zeta : \xi) = (\zeta' : \xi') \). Consequently, by abuse of notation, we will denote
appropriately we may assume that $N_\lambda \in \mathcal{T}_\lambda$ for each $\lambda \in \mathbb{P}_k$. In particular, $\tau N_\lambda = N_\lambda$ for each $\lambda \in \mathbb{P}_k$.

The Kronecker quiver can be viewed as the minimal concealed-canonical bound quiver. Namely, we can embed the category $\text{rep} K_2$ into the category of representations of an arbitrary concealed-canonical quiver. We describe a construction of such an embedding more precisely.

Let $\Delta$ be a concealed-canonical bound quiver with a sincere separating exact subcategory $\mathcal{R}$ of $\text{ind} \Delta$. Let $R := \bigoplus_{\lambda \in \mathbb{P}_k} \bigoplus_{i \in I_\lambda} R_{\lambda, i}$ for subsets $I_\lambda \subseteq [0, r_\lambda - 1)$ such that $|I_\lambda| = r_\lambda - 1$ (in particular, $I_\lambda = \emptyset$ if $r_\lambda = 1$), where we use notation introduced in Section 4.2. Let $R^\perp$ denote the full subcategory of $\text{rep} \Delta$, whose objects are $M \in \text{rep} \Delta$ such that $\text{Hom}_\Delta(R, M) = 0 = \text{Ext}^1_\Delta(R, M)$. Lenzing and de la Peña [23, Proposition 4.2] have proved that there exists a fully faithful exact functor $F : \text{rep} K_2 \to \text{rep} \Delta$ which induces an equivalence between $\text{rep} K_2$ and $R^\perp$. Moreover, $F$ induces an equivalence between $\mathcal{T}$ and $R^\perp \cap \mathcal{R}$. The simple objects in $R^\perp \cap (\text{add} \mathcal{R})$, which are the images of the simple objects in $\text{add} \mathcal{T}$, are of the form $R^{(r_\lambda)}_{\lambda, i}$ for $\lambda \in \mathbb{P}_k$, where for $\lambda \in \mathbb{P}_k$ we denote by $i$ the unique element of $[0, r_\lambda - 1] \setminus I_\lambda$. Consequently, (if we choose appropriate parameterization) $F(N_\lambda) \simeq R^{(r_\lambda)}_{\lambda, i}$ for each $\lambda \in \mathbb{P}_k$.

Let $p \in \mathbb{N}$. We define the functions $f^{(0)}_{(p,p)}, \ldots, f^{(p)}_{(p,p)} \in \mathbb{k}[\text{rep} K_2(p, p)]$ by the condition: if $V \in \text{rep} K_2(p, p)$, then

$$\det(S \cdot V_\alpha - T \cdot V_\beta) = \sum_{i \in [0, p]} S^i \cdot T^{p-i} \cdot f^{(i)}_{(p,p)}(V).$$

Note that $f^{(0)}_{(p,p)}, \ldots, f^{(p)}_{(p,p)}$ are semi-invariants of weight $(-1, 1)$. If $(\zeta : \xi) \in \mathbb{P}_k$, then (by choosing a projective presentation of $N_{\zeta, \xi}$ in an appropriate way) we get

$$c^{N_{\zeta, \xi}}_{(p,p)}(V) = \det(\xi \cdot V_\alpha - \zeta \cdot V_\beta) = \sum_{i \in [0, p]} \xi^i \cdot \zeta^{p-i} \cdot f^{(i)}_{(p,p)}(V).$$

It is well known (see for example [30]) that $\text{SI}[K_2, (p, p)]$ is the polynomial algebra in $f^{(0)}_{(p,p)}, \ldots, f^{(p)}_{(p,p)}$. In particular,

$$\dim_{\mathbb{k}} \text{SI}[K_2, (p, p)]_{(-q, q)} = \binom{q + p}{q}$$

for each $q \in \mathbb{N}$.

We will need the following lemma.

**Lemma 5.1.** If $f_1, f_2 \in \text{SI}[K_2, (p, p)]_{(-1, 1)}$ and

$$\{V \in \text{rep} K_2(p, p) : f_1(V) = 0\} = \{V \in \text{rep} K_2(p, p) : f_2(V) = 0\},$$

then (up to a non-zero scalar) $f_1 = f_2$. 
**Proof.** From the description of $SI[K_2, (p, p)]$ it follows that $f_1$ and $f_2$ are irreducible, hence the claim follows. \(\square\)

6. The main result

Throughout this section we fix a concealed-canonical bound quiver $\Delta$ and a sincere separating exact subcategory $\mathcal{R}$ of $\text{ind} \Delta$. We use freely notation introduced in Section 2. We also fix $d \in \mathbb{R}$ such that $p := p^d > 0$.

First we investigate the algebra $\bigoplus_{q \in \mathbb{N}} SI[\Delta, d]_{(q \cdot h, -)} \Delta$. We introduce some notation. For $\lambda \in \mathbb{P}^1_k$ we denote by $I^0_{\lambda}$ the set of $i \in [0, r \lambda - 1]$ such that $p^d_{\lambda,i} = 0$. Observe that $I^0_{\lambda} \subseteq I_{\lambda}$ for each $\lambda \in \mathbb{P}^1_k$ (the sets $I_{\lambda}$ for $\lambda \in \mathbb{P}^1_k$ were introduced before Corollary 4.5). Recall that, for $\lambda \in \mathbb{P}^1_k$ and $i \in I_{\lambda}$, $n_{\lambda,i}$ denotes the minimal $n \in \mathbb{N}_+$ such that $p^d_{\lambda,i} + n = 0$.

We put $c^\lambda_d := \prod_{i \in I^0_{\lambda}} c^R_{\lambda,i}$ . An iterated application of Corollary 3.2(1) to exact sequences of the form (2.2) implies that $c^\lambda_d = c^R_{\lambda,i}$ for each $i \in I^0_{\lambda}$.

We have the following fact.

**Lemma 6.1.** The algebra $\bigoplus_{q \in \mathbb{N}} SI[\Delta, d]_{(q \cdot h, -)} \Delta$ is generated by the semi-invariants $c^\lambda_d$ for $\lambda \in \mathbb{X}$.

**Proof.** This fact has been proved in [4], but for completeness we include its (shorter) proof here.

Fix $q \in \mathbb{N}$. Proposition 3.7 and Lemma 4.1 imply that $SI[\Delta, d]_{(q \cdot h, -)} \Delta$ is spanned by the semi-invariants $c^V_d$ for ext-minimal $V \in \text{add} \mathcal{R}$ with dimension vector $q \cdot h$. Fix such $V$. Since $V$ is ext-minimal with dimension vector $q \cdot h$, $V = \bigoplus_{\lambda \in \mathbb{X}} R^{(k_{\lambda}, r_{\lambda})}$, where $\mathbb{X} \subseteq \mathbb{P}^1_k$ and $i_{\lambda} \in [0, r_{\lambda} - 1]$ and $k_{\lambda} \in \mathbb{N}_+$ for each $\lambda \in \mathbb{X}$. Moreover, Lemma 4.3 implies that $i_{\lambda} \in I^0_{\lambda}$ for each $\lambda \in \mathbb{X}$. An iterated application of Corollary 3.2(1) to exact sequences of the form (2.2) implies that $c^V_d = (c^\lambda_d)^{k_{\lambda}}$ for each $\lambda \in \mathbb{X}$. Consequently, $c^V_d = \prod_{\lambda \in \mathbb{X}} (c^\lambda_d)^{k_{\lambda}}$ by Lemma 3.4, hence the claim follows. \(\square\)

The following fact is crucial.

**Proposition 6.2.** There exists a regular map

$$\Phi : \text{rep}_{K_2}(p, p) \rightarrow \text{rep}_{\Delta}(d)$$

such that $\Phi^*$ induces an isomorphism

$$\bigoplus_{q \in \mathbb{N}} SI[\Delta, d]_{(q \cdot h, -)} \rightarrow \bigoplus_{q \in \mathbb{N}} SI[K_2, (p, p)](-q)$$

of $\mathbb{N}$-graded rings and (up to a non-zero scalar) $\Phi^*(c^\lambda_d) = c^{N_{(p, p)}}_{(p, p)}$ for each $\lambda \in \mathbb{P}^1_k$. 
Proof. For each \( \lambda \in \mathbb{P}^1_k \) we fix \( i_\lambda \in T^0_\lambda \). From Section 5 we know that there exists a fully faithful exact functor \( F : \text{rep} K_2 \to \text{rep} \Delta \) such that \( F(N_\lambda) \approx R^{(\lambda)}_{\lambda} \) for each \( \lambda \in \mathbb{P}^1_k \). Observe that for each \( R \in \text{add} (\bigcup_{\lambda \in \mathbb{P}^1_k \setminus X_0} R_\lambda) \) (recall that \( X_0 \) is the set of all \( \lambda \in \mathbb{P}^1_k \) such that \( r_\lambda > 1 \)) there exists \( N \in T \) with \( F(N) \approx R \).

Put \( E_1 := F(S_1) \) and \( E_2 := F(S_2) \), where \( S_i \) is the simple representation of \( K_2 \) at \( i \), for \( i \in \{1, 2\} \), i.e.

\[
S_1 := k \oplus 0 \quad \text{and} \quad S_2 := 0 \oplus k.
\]

Then [24, Proposition 2.3] (see also [9, Proposition 5.2]) implies that there exists a regular map \( \Phi' : \text{rep} K_2(p, p) \to \text{rep} \Delta(p \cdot h) \) such that \( \Phi'(N) \approx F(N) \) for each \( N \in \text{rep} K_2(p, p) \). Moreover, there exists a morphism \( \varphi : \text{GL}(p, p) \to \text{GL}(p \cdot h) \) of algebraic groups such that \( \Phi'(g * N) = \varphi(g) * \Phi'(N) \), for all \( g \in \text{GL}(p, p) \) and \( N \in \text{rep} \Delta(p \cdot h) \), and

\[
\chi_\theta(\varphi(g)) = (\det(g(1)))^{\theta(\dim E_1)} \cdot (\det(g(2)))^{\theta(\dim E_2)},
\]

for all \( g \in \text{GL}(p, p) \) and \( \theta \in \mathbb{Z}^{\Delta_0} \).

Let \( W \in \text{add} R \) be an ext-minimal representation for \( d' := d - p \cdot h \).

We define \( \Phi : \text{rep} K_2(p, p) \to \text{rep} \Delta(d) \) by \( \Phi(N) := \Phi'(N) \oplus W \) for \( N \in \text{rep} K_2(p, p) \).

Let \( q \in \mathbb{N} \). We show that \( \Phi^*(f) \) is a semi-invariant of weight \((-q, q)\) for each \( f \in \text{SI}[\Delta, d]_{(q, h, -)} \). Using Proposition 3.6 and Lemma 4.1 it suffices to show that \( \Phi^*(c^V) \) is a semi-invariant of weight \((-q, q)\) for each representation \( V \) of \( \Delta \) with dimension vector \( q \cdot h \).

Now, if \( g \in \text{GL}(p, p) \) and \( N \in \text{rep} K_2(p, p) \), then

\[
(\Phi^*(c^V))(g^{-1} * N) = c^V_d(W \oplus \Phi'(g^{-1} * N)) = c^V_{d'}(W) \cdot c^V_{p \cdot h}(\varphi(g^{-1}) * \Phi'(N)) = c^V_{d'}(W) \cdot \chi^{(q, h, -)}_{(q, h, -)}(\varphi(g)) \cdot c^V_{p \cdot h}(\Phi'(N)) = \chi^{(q, h, -)}_{(q, h, -)}(\varphi(g)) \cdot (\Phi^*(c^V_d))(N),
\]

where the second and the last equalities follow from Lemma 3.5. Using (6.1) we get

\[
\chi_{(q, h, -)}^{(q, h, -)}(\varphi(g)) = (\det(g(1))^{-q} \cdot (\det(g(2)))^q,
\]

since

\[
(h, \dim E_i)_{\Delta} = ((1, 1), \dim S_i)_{\Delta} = (-1)^i
\]

for each \( i \in \{1, 2\} \) (we use here that \( F \) is exact).

The above implies that \( \Phi^* \) induces a homomorphism

\[
\bigoplus_{q \in \mathbb{N}} \text{SI}[\Delta, d]_{(q, h, -)} \to \bigoplus_{q \in \mathbb{N}} \text{SI}[K_2, (p, p)]_{(-q, q)}
\]

of \( \mathbb{N} \)-graded rings. We need to show that this is an isomorphism.
First we show $\Phi^*(f) \neq 0$ for each non-zero semi-invariant $f$ (in particular, this will imply that \([6.2]\) is a monomorphism). Let
\[ Z := \{ M \in \text{rep}_\Delta(d) : \text{there exists } N \in \text{rep}_{K_2}(p,p) \text{ such that } M \simeq W \oplus \Phi(N) \}. \]
In other words, $Z$ in the closure of the image of $\Phi$ under the action of $\text{GL}(d)$. Using Proposition \([4.10]\) it suffices to show that $Z$ contains a non-empty open subset of $\mathcal{R}(d)$. Let
\[ \mathcal{U} := \{ M \in \mathcal{R}(d) : c_d^\lambda(M) \neq 0 \text{ for each } \lambda \in \mathbb{X}_0 \} \]
and $\dim_k \text{End}_\Delta(M) = p + \langle d, d \rangle_\Delta$.

Since the function
\[ \text{rep}_\Delta(d) \ni M \mapsto \dim_k \text{End}_\Delta(M) \in \mathbb{Z} \]
is upper semi-continuous, \([2.7]\) implies that $\mathcal{U}$ is a non-empty open subset of $\mathcal{R}(d)$, which consists of ext-minimal representations. In particular, if $M \in \mathcal{U}$, then there exists an exact sequence of the form $0 \to R \to M \to W \to 0$ with $R \in \text{add} \mathcal{R}$ such that $\dim R = p \cdot h$. If $p^R_\lambda \neq 0$ for some $\lambda \in \mathcal{X}_0$, then $\text{Hom}_\Delta(R_{\lambda,i}^{(r)}, M) \neq 0$. Consequently, $\text{Hom}_\Delta(R_{\lambda,i}^{(r)}, M) \neq 0$, hence $c_d^\lambda(M) = 0$, contradiction. Thus $p^R_\lambda = 0$ for each $\lambda \in \mathcal{X}_0$, hence $M \simeq W \oplus R$ and $R \in \text{add}(\bigcup_{\lambda \in \mathbb{P}^1 \setminus \mathcal{X}_0} \mathcal{R}_\lambda)$. In particular, there exists $N \in \text{rep} \mathcal{T}$ such that $F(N) \simeq R$, hence $M \in \mathcal{Z}$.

Now we fix $\lambda \in \mathbb{P}^1$. We show that (up to a non-zero scalar) $\Phi^*(c_d^\lambda) = c_{(p,p)}^{N,\lambda}$ for each $\lambda \in \mathbb{P}^1_k$. According to Lemma \([6.1]\), this will imply that \([6.2]\) is an epimorphism, hence finish the proof. Fix $N \in \text{rep}_{K_2}(p,p)$. Then
\[ (\Phi^*(c_d^\lambda))(N) = 0 \text{ if and only if } \text{Hom}_\Delta(R_{N,i}^{(r)}, F(N)) \neq 0. \]
Since $R_{N,i}^{(r)} \simeq F(N)$ and $F$ is fully faithful,
\[ (\Phi^*(c_d^\lambda))(N) = 0 \text{ if and only if } \text{Hom}_{K_2}(N, N) \neq 0. \]
Similarly, if $N \in \text{rep}_{K_2}(p,p)$, then
\[ c_{(p,p)}^{N,\lambda}(N) = 0 \text{ if and only if } \text{Hom}_{K_2}(N, N) \neq 0. \]
Consequently, the claim follows from Lemma \([5.1]\). \(\square\)

**Corollary 6.3.** If $r \in R$ and $\text{SI}([\Delta, d]_{(r,-)}_\Delta) \neq 0$, then
\[ \dim_k \text{SI}([\Delta, d]_{(r,-)}_\Delta) = \binom{p^r + p}{p^r}. \]

**Proof.** Proposition \([4.9][2]\) implies that
\[ \dim_k \text{SI}([\Delta, d]_{(r,-)}_\Delta) = \dim_k \text{SI}[\Delta, d]_{(p^r, h,-)_\Delta}. \]
Next,
\[ \dim_k \text{SI}[\Delta, d]_{(p^r, h,-)_\Delta} = \dim_k \text{SI}[K_2, (p, p)]_{(-p^r, p^r)} \]
by Proposition \([6.2]\), hence the claim follows from \([5.2]\). \(\square\)
Let $\Phi : \text{rep}_K[p,p] \to \text{rep}_\Delta(d)$ be a regular map constructed in Proposition 6.2. For $j \in [0,p]$ we denote by $f_d^{(j)}$ the inverse image of $f_{(p,p)}^{(j)}$ under $\Phi^*$. Then (5.1) implies that (up to a non-zero scalar)

$$(6.3) c_d^{(\zeta,\xi)} = \sum_{j \in [0,p]} \xi^j \cdot \zeta^{p-j} \cdot f_d^{(j)}$$

for each $(\zeta : \xi) \in \mathbb{P}_k^1$. As the first application we get the following (smaller) set of generators of $SI[\Delta, d]$.

**Proposition 6.4.** The algebra $SI[\Delta, d]$ is generated by the semi-invariants $f_d^{(0)}, \ldots, f_d^{(p)}$ and $c_{d_i}^{R_{\lambda,i}}$ for $\lambda \in \mathbb{X}_0$ and $i \in I_\lambda$.

**Proof.** Recall from Corollary 4.5 that the algebra $SI[\Delta, d]$ is generated by the semi-invariants $c_{d_i}^{R_{\lambda,i}}$ for $\lambda \in \mathbb{X}_0$ and $i \in I_\lambda$. Thus we only need to express, for each $\lambda \in \mathbb{P}_k^1 \setminus \mathbb{X}_0$ and $i \in I_\lambda$, $c_d^{R_{\lambda,i}}$ as the polynomial in the semi-invariants listed in the proposition. However, if $\lambda \in \mathbb{P}_k^1 \setminus \mathbb{X}_0$ and $i \in I_\lambda$, then $c_d^{R_{\lambda,i}} = c_d$, hence the claim follows from (6.3). \(\square\)

We give another formulation of Proposition 6.4. Let $A$ be the polynomial algebra in the indeterminates $S_0, \ldots, S_p$ and $T_{\lambda,i}$ for $\lambda \in \mathbb{X}_0$ and $i \in I_\lambda$. Proposition 6.4 says that the homomorphism $\Psi : A \to SI[\Delta, d]$ given by the formulas: $\Psi(S_j) := f_d^{(j)}$, for $j \in [0,p]$, and $\Psi(T_{\lambda,i}) := c_d^{R_{\lambda,i}}$, for $\lambda \in \mathbb{X}_0$ and $i \in I_\lambda$, is an epimorphism. Our last aim is to describe its kernel.

First, we introduce an $R$-grading in $A$ by specifying the degrees of the indeterminates as follows: $\deg(S_j) := h$ for $j \in [0,p]$ and $\deg(T_{\lambda,i}) := e^{\alpha_{\lambda,i}}$, for $\lambda \in \mathbb{X}_0$ and $i \in I_\lambda$. Note that $\Psi$ is a homogeneous map, i.e. $\Psi(\mathcal{A}_r) = SI[\Delta, d]_{(r,-)\Delta}$ for each $r \in R$.

Let $R_0$ be the submonoid of $R$ generated by the elements $h$ and $e^{\alpha_{\lambda,i}}$, for $\lambda \in \mathbb{X}_0$ and $i \in I_\lambda$. Obviously, if $r \in R$, then $\mathcal{A}_r \neq 0$ if and only if $r \in R_0$. Similarly, Corollary 4.5 implies that $SI[\Delta, d]_{(r,-)\Delta} \neq 0$ if and only if $r \in R_0$ (recall that $SI[\Delta, d]$ is a domain).

**Lemma 6.5.** If $r \in R_0$, then

$$\text{dim}_k \mathcal{A}_r = \left( \frac{p^r + p + |\mathbb{X}_0|}{p^r} \right).$$

**Proof.** One easily observes that there is an isomorphism $\mathcal{A}_{p^r \cdot h} \to \mathcal{A}_r$ of vector spaces (induced by multiplying by the unique monomial of degree $r - p^r \cdot h$). Moreover, $\bigoplus_{q \in R_0} \mathcal{A}_{p^q \cdot h}$ is the polynomial algebra generated by $S_0, \ldots, S_p$ and $\prod_{i \in I_\lambda} T_{\lambda,i}$ for $\lambda \in \mathbb{X}_0$. Now the claim follows. \(\square\)
The formula (6.3) implies that for each \( \lambda \in X_0 \) there exist \( \zeta_\lambda, \xi_\lambda \in k \) such that

\[
\prod_{i \in I_0^0} R_{\lambda,i}^{(\xi_{\lambda,i})} = \sum_{j \in [0,p]} \xi_\lambda^j \cdot \zeta_\lambda^{p-j} \cdot f_d^{(j)}.
\]

Obviously, \((\zeta_\lambda, \xi_\lambda) \neq (0,0)\) and \((\zeta_\lambda : \xi_\lambda) = \lambda\).

**Proposition 6.6.** We have

\[
\text{Ker } \Psi = \left( \sum_{j \in [0,p]} \xi_\lambda^j \cdot \zeta_\lambda^{p-j} \cdot S_j - \prod_{i \in I_0^0} T_{i,\lambda} : \lambda \in X_0 \right).
\]

**Proof.** Let

\[
\mathcal{J} := \left( \sum_{j \in [0,p]} \xi_\lambda^j \cdot \zeta_\lambda^{p-j} \cdot S_j - \prod_{i \in I_0^0} T_{i,\lambda} : \lambda \in X_0 \right).
\]

Obviously, \( \text{Ker } \Psi \subseteq I \). Observe that both \( \text{Ker } \Psi \) and \( \mathcal{J} \) are graded ideals (with respect to the grading introduced above). Consequently, in order to prove our claim it suffices to show that \( \dim_k \mathcal{J}_r = \dim_k \text{Ker } \Psi_r \) for each \( r \in R_0 \).

We already know from Lemma 6.5 and Corollary 6.3 that

\[
\dim_k \text{Ker } \Psi_r = \dim_k A_r - \dim_k SI[\Delta, r]_{(r, -)_{\Delta}}
\]

\[
= \binom{p^r + p + |X_0|}{p^r} - \binom{p^r + p}{p^r}
\]

for each \( r \in R_0 \). On the other hand, similarly as in the proof of Lemma 6.5, we show that \( \dim_k \mathcal{J}_r = \dim_k J_{p^r h} \) for each \( r \in R_0 \). Moreover, the algebra \( \bigoplus_{q \in \mathbb{N}} (A/\mathcal{J})_{q h} \) is obviously the polynomial algebra in \( p^r + p \) indeterminates. This, together with Lemma 6.5, immediately implies our claim. \( \square \)

We may summarize our considerations in the following theorem (compare [29, Theorem 1.1]).

**Theorem 6.7.** We have the isomorphism

\[
SI[\Delta, d] \simeq A/ \left( \sum_{j \in [0,p]} \xi_\lambda^j \cdot \zeta_\lambda^{p-j} \cdot S_j - \prod_{i \in I_0^0} T_{i,\lambda} : \lambda \in X_0 \right).
\]

If

\[
i(d) := \{ \lambda \in X_0 : |I_{\lambda}| > 1 \},
\]

then \( SI[\Delta, d] \) is a complete intersection given by \( \max(0, i(d) - p - 1) \) equations. In particular, \( SI[\Delta, d] \) is polynomial algebra if and only if \( i(d) \leq p + 1 \).
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