New Wilf-equivalence results for dashed patterns

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Abstract
We give a sufficient condition for the two dashed patterns \( \tau^{(1)} - \tau^{(2)} - \cdots - \tau^{(\ell)} \) and \( \tau^{(\ell)} - \tau^{(\ell-1)} - \cdots - \tau^{(1)} \) to be (strongly) Wilf-equivalent. This permits to solve in a unified way several problems of Heubach and Mansour on Wilf-equivalences on words and compositions, as well as a conjecture of Baxter and Pudwell on Wilf-equivalences on permutations. We also give a better explanation of the equidistribution of the parameters \( \text{MAK} + b\text{MAJ} \) and \( \text{MAK}' + b\text{MAJ} \) on ordered set partitions. These results can be viewed as consequences of a simple proposition which states that the set valued statistics “descent set” and “rise set” are equidistributed over each equivalence class of the partially commutative monoid generated by a poset \((X, \leq)\).

1 Introduction and Main results
The main purpose of this paper is to establish equidistribution properties about dashed patterns in permutations, words and ordered set partitions.

1.1 Contribution to the Wilf-classification of dashed patterns
Let \( \mathbb{P}^* \) denote the free monoid generated by the set of positive integers \( \mathbb{P} \). A dashed (sometimes called generalized or vincular) pattern \( p \) of length \( m \) is a word in \( \mathbb{P}^* \) of length \( m \) such that \( p \) contains all letters in \([\ell] := 1, 2, \ldots, \ell\) for a certain positive integer \( \ell \) and in which two adjacent letters may or may not be separated by a dash. A dashed pattern \( p = \tau^{(1)} - \tau^{(2)} - \cdots - \tau^{(\ell)} \) is said to be of type \((j_1, j_2, \ldots, j_\ell)\) if the \( \tau^{(i)} \) have lengths \( j_i \). For example, \( 13 - 2 \) and \( 11 - 3 - 4 \) are two dashed patterns of type \((2, 1)\) and \((2, 1, 2)\), respectively.

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Let \( w = w_1w_2 \ldots w_n \) be a word in \( P^* \) and \( p = p_1 \ldots p_{j_1} - p_{j_1+1} \ldots p_{j_1+j_2} - \cdots - p_m \) be a dashed pattern of length \( m \). We say that the subsequence \( w_{i_1}w_{i_2} \ldots w_{i_m}, 1 \leq i_1 < i_2 < \cdots < i_m \leq n \), is an occurrence of the pattern \( p \) in \( w \) if

- it is order isomorphic to \( p \), that is \( w_{i_s} > w_{i_t} \) if and only \( p_s > p_t \) for \( 1 \leq s, t \leq m \), and
- the absence of a dash between two adjacent letters in the pattern \( p \) indicates that the corresponding letters in the subsequence \( w_{i_1}w_{i_2} \ldots w_{i_m} \) must be adjacent in the word \( w \).

For example, \( w = 24135 \) contains only two copies of \( p = 1 - 23 \), namely \( 235 \) and \( 135 \); although the subsequence \( 245 \) of \( w \) is isomorphic to \( p \), it does not form a copy of \( 1 - 23 \) because \( 4 \) and \( 5 \) are not adjacent in \( w \).

The main concern of this paper is the Wilf-equivalence on the set of dashed patterns. Let \( \mathcal{F} = \bigcup_{n \geq 0} \mathcal{F}_n \subseteq P^* \) be a collection of words such that \( \mathcal{F}_n \), the set which consists of all words in \( \mathcal{F} \) of length \( n \), is finite for \( n \geq 0 \). The main collections we consider in this paper are:

- The collection \( \mathcal{S} = \bigcup_{n \geq 0} \mathcal{S}_n \) of permutations, where \( \mathcal{S}_n \) is the symmetric group of order \( n \);
- For \( \ell \) a positive integer, the collection \( \mathcal{W}_\ell = \bigcup_{n \geq 0} \mathcal{W}_{\ell,n} \) of \( \ell \)-ary words (i.e., words the letters of which are \( \leq \ell \)), where \( \mathcal{W}_{\ell,n} \) stands for the set of words in \( \mathcal{W}_\ell \) of length \( n \);
- For a positive integer \( s \) and a set \( A \subseteq P \), the collection \( \mathcal{C}_{s,A} = \bigcup_{n \geq 0} \mathcal{C}_{s,A;n} \) of integer compositions of \( s \) all the parts of which are in \( A \), where \( \mathcal{C}_{s,A;n} \) is the set of compositions in \( \mathcal{C}_{s,A} \) of length \( n \).

If \( k \) is a nonnegative integer and \( p \) is dashed pattern, we let \( \mathcal{F}_n^{(k)}(p) \) denote the set of all words in \( \mathcal{F}_n \) containing exactly \( k \) occurrences of the pattern \( p \) and set \( f_n^{(k)}(p) = |\mathcal{F}_n^{(k)}(p)| \). Then, two patterns \( p \) and \( q \) are said to be \( \mathcal{F} \)-Wilf-equivalent, which is denoted \( p \sim \mathcal{F} q \), if \( f_n^{(0)}(p) = f_n^{(0)}(q) \) for all \( n \geq 0 \). More generally, patterns \( p \) and \( q \) are said to be strongly \( \mathcal{F} \)-Wilf-equivalent which is denoted \( p \sim_{\mathcal{F}} q \) (or just \( p \sim q \) if there is no confusion), if \( f_n^{(k)}(p) = f_n^{(k)}(q) \) for all \( n, k \geq 0 \). The \( \mathcal{F} \)-Wilf-equivalence is obviously an equivalence relation and a natural problem is to determine its equivalence classes. While for classical patterns (i.e., dashed patterns all the letters of which are separated by a dash), much is known about \( \mathcal{S} \)-Wilf classes for patterns of length up to \( 7 \), for the dashed patterns, much less is known. Actually, we don’t even know \( \mathcal{S} \)-Wilf classes for dashed patterns of length \( 4 \) (see e.g. [16]). The situation is similar for \( \mathcal{W}_\ell \)-Wilf equivalence and \( \mathcal{C}_{s,A} \)-Wilf equivalence.

A related fundamental problem in the theory of patterns is to find necessary and sufficient conditions for two patterns to be Wilf-equivalent. In this paper, we give sufficient conditions for the two dashed patterns \( \tau^{(1)} = \tau^{(2)} = \cdots = \tau^{(\ell)} \) and \( \tau^{(\ell)} = \tau^{(\ell-1)} = \cdots = \tau^{(1)} \) to be Wilf-equivalent. This leads to new Wilf-equivalences on compositions, words and permutations, and permits to answer several problems listed by Heubach and Mansour about \( \mathcal{W}_\ell \)- and \( \mathcal{C}_{s,A} \)-Wilf equivalences as well as a conjecture of Baxter and Pudwell about \( \mathcal{S} \)-Wilf-equivalence. Before to present our result, we need some additional terminology.
Recall that the descending runs of a word \( w = w_1w_2 \ldots w_n \in \mathbb{P}^* \) are the maximal contiguous decreasing subsequences of \( w \). For example, the word \( w = 3541655365 \) have five descending runs: 3, 5, 4, 65, 53 and 65. Given a multiset \( M = \{ D_1, D_2, \ldots, D_r \} \) of decreasing sequences, we let \( W(M) \) denote the set of words the descending runs of which are \( D_1, D_2, \ldots, D_r \). For example,

- if \( M_1 = \{ 321, 64, 75 \} \), then \( W(M_1) = \{ 3216475, 3217564 \} \) but the permutation \( 6432175 \notin W(M_1) \) since its descending runs are 64321 and 75;
- if \( M_2 = \{ 21, 21, 53 \} \), then \( W(M_2) = \{ 212153 \} \).

The reverse image \( r \) is the classical involutive transformation of \( \mathbb{P}^* \) that maps each word \( w = x_1x_2 \cdots x_n \in \mathbb{P}^* \) onto \( rw = x_nx_{n-1} \cdots x_1 \).

**Definition 1.1.** A collection of words \( F \subseteq \mathbb{P}^* \) is said to be:

- run-complete if for any multiset \( M \) of decreasing sequences, we have \( W(M) \subseteq F \) or \( W(M) \cap F = \emptyset \), i.e., if \( F \) contains a word in \( W(M) \) then \( F \) contains every word in \( W(M) \);
- reverse-complete if the reverse word \( rw \) is in \( F \) whenever \( w \in F \).

It is easy to check that the collections of permutations \( \mathcal{S} \), of \( \ell \)-ary words \( W[\ell] \), of compositions \( C_{\ell, A} \), are run-complete and reverse-complete. In this paper, a special kind of dashed patterns will play an important role. Before to define them, we need to introduce a partial order on \( \mathbb{P}^* \).

**Definition 1.2.** Given two words \( w \) and \( w' \) in \( \mathbb{P}^* \), \( w \) is said to be below \( w' \) if every letter in \( w \) is smaller than every letter in \( w' \). The partial order \( \ll \) on \( \mathbb{P}^* \) is defined as follows:

\[
\text{if } w \ll w' \iff w = w' \text{ or } w \text{ is below } w'.
\]  

(1.1)

Two words \( w \) and \( w' \) are said to be comparable if \( w \ll w' \) or \( w \gg w' \), otherwise they are incomparable.

**Definition 1.3.** A dashed pattern \( p = \tau^{(1)} - \tau^{(2)} - \cdots - \tau^{(\ell)} \) is said to be

- connected if, for \( i = 1, 2, \ldots, \ell - 1 \), \( \tau^{(i)} \) and \( \tau^{(i+1)} \) are incomparable or equal;
- piecewise decreasing (resp., increasing) if each \( \tau^{(i)} \) is (strictly) decreasing (resp., increasing).

For example, the pattern \( p = 234 - 1 - 124 \) is piecewise increasing but not connected since the two first sequences are comparable (234 \( \gg 1 \)). The pattern \( q = 52 - 14 - 3 \) is connected but not piecewise increasing (resp., decreasing). The pattern \( q = 52 - 41 - 3 \) is piecewise decreasing and connected.

If \( p_1, p_2, \ldots, p_m \) are dashed patterns, the \( m \)-statistic which associates to a word \( w \in \mathbb{P}^* \) the vector \( (a_1, a_2, \ldots, a_m) \) where \( a_i \) is the number of occurrences of \( p_i \) in \( w \) will be denoted by \( (p_1, \ldots, p_m) \). We say that two vectors \( (p_1, p_2, \ldots, p_m) \) and \( (q_1, q_2, \ldots, q_m) \) of dashed patterns are strongly \( F \)-Wilf-equivalent, which is denoted \( (p_1, p_2, \ldots, p_m) \sim_F (q_1, q_2, \ldots, q_m) \).
(q_1, q_2, \ldots, q_m), if and only if the m-statistics (p_1, \ldots, p_m) and (q_1, \ldots, q_m) have the same distribution over \mathcal{F}_n for all integers n \geq 0, that is for all integers n, k_1, k_2, \ldots, k_m \geq 0, we have

\left( f_n^{(k_1)}(p_1), f_n^{(k_2)}(p_2), \ldots, f_n^{(k_m)}(p_m) \right) = \left( f_n^{(k_1)}(q_1), f_n^{(k_2)}(q_2), \ldots, f_n^{(k_m)}(q_m) \right).

Consider the two transformations R and \langle r \mid \cdot \rangle of dashed patterns defined for p = \tau^{(1)} - \tau^{(2)} - \cdots - \tau^{(\ell)} by

R p := \tau^{(\ell)} - \tau^{(\ell-1)} - \cdots - \tau^{(1)}, \quad (1.2)
\langle r \mid p \rangle := r\tau^{(1)} - r\tau^{(2)} - \cdots - r\tau^{(\ell)}. \quad (1.3)

For example, if p = 2 5 4 - 3 - 1 2, we have R p = 1 2 - 3 - 2 5 4 and \langle r \mid p \rangle = 4 5 2 - 3 - 2 1.

The following is the main result of this section.

**Theorem 1.4.** Let \mathcal{F} \subseteq P^* be a run-closed collection of words. Then, for any piecewise decreasing and connected patterns p_1, p_2, \ldots, p_m, we have

(p_1, p_2, \ldots, p_m) \sim_{\mathcal{F}} (R p_1, R p_2, \ldots, R p_m). \quad (1.4)

In particular, the joint distribution of (p_1, R p_1) over \mathcal{F}_n is symmetric for any integer n \geq 0. If, in addition, \mathcal{F} is reverse-closed, then

(p_1, p_2, \ldots, p_m) \sim_{\mathcal{F}} (\langle r \mid p_1 \rangle, \langle r \mid p_2 \rangle, \ldots, \langle r \mid p_m \rangle). \quad (1.5)

The above result permits to unify previous results on Wilf-equivalences as well as to solve several problems on Wilf-equivalences. For instance, since the collection of compositions C_{s,A} is run- and reverse-complete, we obtain the following result as an immediate consequence of Theorem 1.4.

**Corollary 1.5.** For any set A \subseteq \mathcal{P} and any positive integer s, we have the following (strong) C_{s,A}-Wilf-equivalences:

(1) type (2, 1) patterns:

(a) 12 - 2 \sim_s 21 - 2 \quad (b) 13 - 2 \sim_s 31 - 2.

(2) type (3, 1) patterns:

(a) 123 - 1 \sim_s 321 - 1 \quad (b) 123 - 2 \sim_s 321 - 2 \quad (c) 123 - 3 \sim_s 321 - 3
\quad (d) 124 - 3 \sim_s 421 - 3 \quad (e) 134 - 2 \sim_s 431 - 2.

(3) type (2, 2) patterns:

(a) 13 - 12 \sim_s 12 - 13 \quad (b) 12 - 23 \sim_s 21 - 32 \quad (c) 13 - 23 \sim_s 23 - 13
\quad (d) 14 - 23 \sim_s 23 - 14 \quad (e) 13 - 24 \sim_s 24 - 13.
(4) type (1, 1, 2) patterns:

(a) $1 - 1 - 12 \sim_s 1 - 1 - 21$  
(b) $2 - 2 - 12 \sim_s 2 - 2 - 21$

(c) $2 - 2 - 13 \sim_s 2 - 2 - 31$.

(5) type (1, 2, 1) patterns:

(a) $1 - 12 - 2 \sim_s 1 - 21 - 2$  
(b) $1 - 13 - 2 \sim_s 1 - 31 - 2$

(c) $2 - 13 - 3 \sim_s 2 - 31 - 3$  
(d) $2 - 14 - 3 \sim_s 2 - 41 - 3$.

It is worth noting that the above $C_{s,A}$-Wilf-equivalences for (2, 1) patterns were first obtained by Heubach and Mansour (see Theorem 5.38 and Theorem 5.19 in [10]) and their proof is far to be obvious. All the other $C_{s,A}$-Wilf-equivalences in Corollary 1.5 settle problems of Heubach and Mansour (see Questions (1-a), (2), (3) and (4) in [10, pp. 177–178]).

Since the collections of $\ell$-ary words $W_\ell$ and of permutations $G$ are run- and reverse-complete, all the Wilf-equivalences given in Corollary 1.5 are also $W_\ell$-Wilf-equivalences and $G$-Wilf-equivalences. However, several of these equivalences are trivial. Recall that the symmetry class of a dashed pattern $p$ is the set of patterns $\{p, r p, c p, r c p\}$, where $r$ is the reverse map (that we have defined previously) and $c$ is the complement transformation, defined for a word $w = w_1 w_2 \ldots w_n$ by $c w = y_1 y_2 \ldots y_n$ where $y_i = \max(M + 1 - w_i)$, with $M$ being the maximum of the letters of $w$. For example, the patterns $2 - 31, 2 - 13, 13 - 2$ and $31 - 2$ form a symmetry class. It is easy to see (and well-known) that two dashed patterns in the same symmetry class are (strongly) $G$- and $W_\ell$-Wilf equivalent.

We summarize the non-trivial $G$- and $W_\ell$-Wilf equivalences of length 4 which can be deduced from Theorem 1.4 (or Corollary 1.5) in the two following results.

**Corollary 1.6.** For any positive integer $\ell$, we have the following (strong) $W_\ell$-Wilf-equivalences:

(i) $13 - 12 \sim_s 12 - 13$  
(ii) $12 - 23 \sim_s 21 - 32$  
(iii) $13 - 24 \sim_s 24 - 13$  
(iv) $123 - 1 \sim_s 321 - 1 \sim_s 123 - 3$  
(v) $124 - 3 \sim_s 421 - 3 \sim_s 134 - 2$

(vi) $1 - 12 - 2 \sim_s 1 - 21 - 2$  
(vii) $1 - 13 - 2 \sim_s 1 - 31 - 2$

(viii) $2 - 14 - 3 \sim_s 2 - 41 - 3$  
(ix) $12 - 1 - 1 \sim_s 21 - 1 - 1 \sim_s 12 - 2 - 2$

where parentheses indicate trivial equivalences (obtained from symmetry classes).

All the $W_\ell$-Wilf-equivalences in Corollary 1.6 settle problems of Heubach and Mansour (see Questions (2)–(5) in [10, pp. 238–239]). We also have 3 new (strong) $G$-Wilf-equivalences for patterns of length 4.

**Corollary 1.7.** We have the following (strong) $G$-Wilf-equivalences:

(a) $124 - 3 \sim_s 421 - 3$  
(b) $2 - 14 - 3 \sim_s 2 - 41 - 3$  
(c) $13 - 24 \sim_s 24 - 13$. 

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It is worth noting that the $\mathcal{S}$-Wilf-equivalence $(a)$ in Corollary 1.7 settles Conjecture 17 (b) in [3]. Baxter also obtained a proof which will be published in a upcoming paper (see the remarks after Conjecture 17 in [3]).

Before to conclude this section, we want to notice that Theorem 1.4 can be obtained from Clarke, Steingrímsson and Zeng’s transformation $\epsilon$ on words [6, Section 6], which can be seen as a particular case of the transformation $\theta$ of Foata and Randrianarivony [7]. It was shown in [6] that this transformation $\epsilon$ has two interesting properties: it preserves the descending runs and it exchanges the combinatorial parameters $\text{les}$ and $\text{res}$ (it is easy to see that we have $\text{les} = (31 - 2)$ and $\text{res} = (2 - 31)$). Actually, it is not difficult to see that the transformation $\epsilon$ also exchanges the parameters $(p)$ and $(R\, p)$ for any piecewise decreasing and connected pattern $p$ (see Theorem 3.4). This leads to the following refinement of Theorem 1.4.

**Theorem 1.8.** For any finite multiset $\mathcal{M} \subseteq B$ and any piecewise decreasing and connected patterns $p_1, p_2, \ldots, p_m$, the $m$-tuple $(p_1, p_2, \ldots, p_m)$ and $(R\, p_1, R\, p_2, \ldots, R\, p_m)$ have the same distribution over $\mathcal{W}(\mathcal{M})$.

We will show in this paper that Theorem 1.8 is a particular case of a result on partially commutative monoid which also permits to obtain a more simple and deeper explanation of an equidistribution result on ordered set partitions.

### 1.2 Statistics on ordered set partitions

Recall that an ordered set partition $\pi = (B_1, B_2, \ldots, B_k)$ of $\mathbb{N}_n := \{1, 2, \ldots, n\}$ is a sequence of disjoint and nonempty subsets $B_i$, called blocks, whose union is $\mathbb{N}_n$. The set of all ordered set partitions of $\mathbb{N}_n$ into $k$ blocks is denoted $\mathcal{OP}_k^n$. By convention, a block of a partition will be represented by the decreasing rearrangement of its elements and the blocks of a partition will be separated by a vertical line. For instance, the partition $\{(\{1\}, \{5, 8\}, \{2, 6, 9\}, \{3\}, \{4\})\} \in \mathcal{OP}_3^5$ will be written as $1 | 85 | 62 | 3 | 74$. It is well-known that $|\mathcal{OP}_k^n| = k! \, S(n, k)$ where $S(n, k)$ is the $(n, k)$-th Stirling number of the second kind (see e.g. [1]). The natural $q$-analogue $[k]_q! \, S_q(n, k)$ of the counting function $k! \, S(n, k)$ have arised as the generating function for the distribution of several statistics on ordered set partitions (see e.g. [15, 11, 12]). Here, $[k]_q!$ is the usual $q$-factorial $[1]_q[2]_q \cdots [k]_q$, where $[j]_q = 1 + q + q^2 + \cdots + q^{j-1}$, and $S_q(n, k)$ is the $q$-Stirling of the second kind defined by

$$S_q(n, k) = q^{k-1} \, S_q(n - 1, k - 1) + [k]_q \, S_q(n - 1, k)$$

for $n \geq k \geq 1$, and $S_q(n, k) = \delta_{n,k}$ if $n$ or $k$ are 0. The systematic study of statistics on ordered set partitions has its origins in the work of Steingrímsson [15]. Following Steingrímsson, a statistic $\text{stat}$ on ordered set partitions such that

$$\sum_{\pi \in \mathcal{OP}_k^n} q^{\text{stat} \, \pi} = [k]_q! \, S_q(n, k) \quad (n \geq k \geq 1),$$
is said to be Euler-Mahonian. Steingrímsson [15] found several Euler-Mahonian statistics on ordered set partitions, most of them reflect quite naturally the recursion for the q-Stirling numbers, but some others inspired by permutations statistics don’t have simple recursive structures; they are qualified as “hard”. We are mainly concerned in this paper with the two hard statistics $MAK + bMAJ$ and $MAK' + bMAJ$. They are defined as follows. Suppose we are given a partition $\pi = \pi_1 \pi_2 \cdots \pi_k \in \mathcal{OP}_n^k$.

- An integer $i$, $1 \leq i \leq k - 1$, is said to be a block descent (resp., block rise) if $B_i \gg B_{i+1}$ (resp., $B_i \ll B_{i+1}$), where $\gg$ is the partial order introduced in Definition 1.2. The sets of block descents and rises of $\pi$ will be denoted $bDES(\pi)$ and $bRISE(\pi)$. The block major index of $\pi$, denoted by $bMAJ \pi$, is defined as the sum of the block descents in $\pi$.

- The opener of a block is its least element and the closer is its greatest element. We will denote by $\text{Open}(\pi)$ and $\text{Clos}(\pi)$ the sets of openers and closers of the blocks of $\pi$, respectively.

- For an integer $i$, $1 \leq i \leq n$, we let $rsb_i \pi$ (resp., $lsb_i \pi$) denote the number of blocks $B$ in $\pi$ to the right (resp., left) of the block containing $i$ such that the opener of $B$ is smaller than $i$ and the closer of $B$ is greater than $i$. We then define the statistics $rsb$ and $lsb$ as the sum of their coordinate statistics, i.e.

$$rsb \pi = \sum_{i=1}^{n} rsb_i \pi \quad \text{and} \quad lsb \pi = \sum_{i=1}^{n} lsb_i \pi.$$

- The partition statistics $MAK$ and $MAK'$ are then defined by

$$MAK \pi = rsb \pi + \sum_{i \in \text{Clos}(\pi)} (n - i) \quad \text{and} \quad MAK' \pi = rsb \pi + \sum_{i \in \text{Open}(\pi)} (i - 1).$$

For example, if $\pi = 8 5 | 1 | 9 6 2 | 7 4 | 3$, we have

- $bDES(\pi) = \{1, 4\}$ and $bRISE(\pi) = \{2\}$, whence $bMAJ \pi = 1 + 4 = 5$;

- $\text{Open}(\pi) = \{1, 2, 3, 4, 5\}$ and $\text{Clos}(\pi) = \{1, 3, 7, 8, 9\}$;

- $(rsb_i \pi)_{1 \leq i \leq n} = (0, 0, 0, 0, 2, 1, 0, 1, 0)$ and $(lsb_i \pi)_{1 \leq i \leq n} = (0, 0, 1, 1, 0, 1, 2, 0, 0)$, whence $rsb \pi = 4$ and $lsb \pi = 5$;

- $MAK \pi = 5 + (8 + 6 + 2 + 1 + 0) = 19$ and $MAK' \pi = 5 + (0 + 1 + 2 + 3 + 4) = 15$.

**Result A.** For $n \geq k \geq 1$, the partition statistics $MAK + bMAJ$ and $MAK' + bMAJ$ are equidistributed over $\mathcal{OP}_n^k$.

Result A, originally conjectured by Steingrímsson, was proved by Zeng and the author (see Theorem 3.3 in [12]). An interesting fact is that $MAK$ and $MAK'$ can be seen as extension of the Mahonian permutation statistics $mak$ (introduced by Foata and Zeilberger [8]) and its variant $mak'$ (introduced by Clarke, Steingrímsson and Zeng [6]). More precisely, if $\sigma \in S_n$, let $D_1, D_2, \ldots, D_k$ be the descending runs of $\sigma$ (listed from left to right) and recall that $des(\sigma)$ is the number of descents of $\sigma$, where a descent is as usual an integer $i$ such that $\sigma(i) > \sigma(i + 1)$. Then the statistics $mak$ and $mak'$ can be defined
by

\[ \text{mak} \sigma := \text{MAK}(D_1 | D_2 | \cdots | D_k) + \left( \frac{n+1}{2} \right) - kn \quad (1.6) \]
\[ \text{mak}' \sigma := \text{MAK}'(D_1 | D_2 | \cdots | D_k) + \left( \frac{n+1}{2} \right) - kn. \quad (1.7) \]

For example, we have \( \text{mak}(185962374) = \text{MAK}(1|85|962|3|74) = 19. \)

**Result B.** For \( n \geq 1 \), the permutation statistics \((\text{des}, \text{mak})\) and \((\text{des}, \text{mak}')\) are equidistributed over \( S_n \).

Result B was obtained by Clarke, Steingrímsson and Zeng (see Proposition 16 in [6]) in their study of Euler-Mahonian statistics on permutations and its proof relies on a simple transformation on \( S_n \). This contrasts with the proof of Result A in [12] which is based on a non trivial path model for ordered set partitions (see Sections 8 and 9 in [12]) and showed no connection with Result B. Altogether, this leads to two natural questions:

- Since \( \text{MAK} \) and \( \text{MAK}' \) are extensions of \( \text{mak} \) and \( \text{mak}' \), can Result A and Result B be unified?
- Is there a simple proof of Result B?

We will answer these questions by giving a simple proof of a refinement of Result B which generalizes Result A.

**Theorem 1.9.** The 3-tuple of parameters

\[(\text{MAK}, \text{MAK}', b\text{DES}) \quad \text{and} \quad (\text{MAK}', \text{MAK}, b\text{DES})\]

are equidistributed over \( \mathcal{OP}_n^k \).

Notice that in order to see that Theorem 1.9 implies Result A, it suffices to observe that via its descending runs, a permutation can be seen as an ordered set partition with no block descent. Before to conclude this section, we want to indicate to the reader that the “connections” between Euler-Mahonian partition and permutation statistics is far to be understood. In the last section of this paper, we present a conjectured equidistribution result on ordered set partitions that generalizes an important equidistribution property over the set of permutations.

### 1.3 The key result

Although it seems a priori that the two topics we discuss are disconnected, they are in fact strongly related if we work in the partially commutative monoid generated by the poset \((\mathcal{P}_f(\mathcal{P}), \ll)\), where \(\mathcal{P}_f(\mathcal{P})\) is the collection of all finite subsets of \(\mathcal{P}\) and \(\ll\) is the partial order introduced in Definition 1.2. All the results presented in this paper can be derived from the following result on partially commutative monoid.
Theorem 1.10. For any poset \((X, \leq)\), the set-valued statistics “set of descents” and “set of rises” are equidistributed over each equivalence class of the partially commutative monoid \(L(X, \leq)\).

In Section 2, we recall some definitions in the theory of partially commutative monoid and we see how Theorem 1.10 implies all the results presented in this paper. In Section 3, we prove Theorem 1.10. We end this paper with some remarks and problems.

2 The partially commutative monoid \(L(B, \ll\)

2.1 Partially commutative monoids

We first recall the construction of the partially commutative monoid \(L(X, \leq)\) generated by a nonempty poset \((X, \leq)\). Let \(X^*\) be the free monoid generated by \(X\). Two words \(w\) and \(w'\) in \(X^*\) are said to be adjacent if there exist two words \(u\) and \(v\) and an ordered pair \((a, a') \in X^2\) of distinct and comparable elements, i.e. \(a < a'\) or \(a > a'\), such that \(w = uaa'v\) and \(w' = ua'av\). They are said to be equivalent if they are equal, or if there exists a sequence of words \(u_0, u_1, \ldots, u_p\) such that \(w_0 = w, w_p = w'\) and \(w_{i-1}\) and \(w_i\) are adjacent for \(1 \leq i \leq p\). This defines an equivalence relation \(R_\leq\) on \(X^*\), compatible with the multiplication in \(X^*\). Then, \(L(X, \leq)\) is defined as the quotient monoid \(X^*/R_\leq\). The equivalence class of a word \(w \in X^*\) will be denoted by \([w]\).

We now recall the definition of descent and rise (sometimes called ascent) in a word. Let \(w = x_1x_2 \cdots x_n\) be a word of length \(n\) in \(X^*\). Then, the integer \(i, 1 \leq i \leq n - 1\), is said to be a descent (resp., rise) if \(x_i > x_{i+1}\) (resp., \(x_i < x_{i+1}\)). The descent set \(DES(w)\) and rise set \(RISE(w)\) of \(w\) are

\[
DES(w) = \{i : 1 \leq i \leq n - 1 \text{ and } x_i > x_{i+1}\},
RISE(w) = \{i : 1 \leq i \leq n - 1 \text{ and } x_i < x_{i+1}\}.
\]

A word \(w = x_1x_2 \cdots x_n \in X^*\) is said to be minimal (resp., maximal) if \(DES(w) = \emptyset\) (resp., \(RISE(w) = \emptyset\)), that is for each \(i = 1, 2, \ldots, n - 1\) the following property holds:

if \(x_i\) and \(x_{i+1}\) are distinct and comparable, then \(x_i < x_{i+1}\) (resp., \(x_i > x_{i+1}\)). \(\text{ (2.1)}\)

The following result is due to Foata and Randrianarivony [7].

Proposition 2.1 (Proposition 2.2, [7]). Each equivalence class in \(L(X, \leq)\) contains one and only one minimal (resp., maximal) word.

Foata and Randrianarivony gave an explicit construction (see Section 2 in [7]) of the (unique) bijection \(\theta\) that sends each minimal word in \(X^*\) onto the maximal word that belongs to the same equivalence class. We now recall Theorem 1.10 which can be seen as an extension of Proposition 2.1 and is the key (and main) result of the paper.

Theorem 2.2. For any poset \((X, \leq)\), the set-valued statistics \(DES\) and \(RISE\), are equidistributed over each equivalence class \([w] \in L(X, \leq)\), i.e., for any set \(S\), there are as many words \(w' \in [w]\) satisfying \(DES(w') = S\) as those satisfying \(RISE(w') = S\).
2.2 The partially commutative monoid $L(B, \ll)$

Consider the poset $(B, \ll)$ where $B \subseteq \mathbb{P}^*$ is the set of (finite) decreasing sequences (or equivalently, nonempty finite subsets) of positive integers and $\ll$ is the partial order on $\mathbb{P}^*$ introduced in Definition 1.2, that it is for $D, D' \in B$, we have

$$D \ll D' \iff D = D' \text{ or } D \text{ is below } D'.$$

By convenience, the letters of a word in $B^*$ will be separated by vertical lines, and the equivalence class in $L(B, \ll)$ of an element $\pi \in B^*$ will be denoted by $[\pi]$. For example, the word $\pi$ the letters of which are from left to right $6 \, 5 \, 3$, $2 \, 1$ and $3$ is written as $\pi = 6 \, 5 \, 3 \, | \, 2 \, 1 \, 3 \, | \, 6 \, 5 \, 3 \, | \, 3 \, | \, 2 \, 1$.

Note that $\mathcal{OP}_n^k$ is just a subset of $B^*$. In particular, we can extend the notion of block descent (resp., block rise) to $B^*$. For $\pi = D_1 \, | \, D_2 \, | \cdots \, | \, D_k \in B^*$, we set

$$b\text{DES}(\pi) = \{i : 1 \leq i \leq k - 1 \text{ and } D_i \gg D_{i+1}\}$$

$$b\text{RISE}(\pi) = \{i : 1 \leq i \leq k - 1 \text{ and } D_i \ll D_{i+1}\}.$$

As an immediate consequence of Theorem 2.2, we obtain the following result.

**Corollary 2.3.** For any $\pi \in B^*$, the set valued statistics $b\text{DES}$ and $b\text{RISE}$ have the same distribution over the equivalence class $[\pi] \in L(B, \ll)$.

It will be convenient to extend the definition of pattern containment in $\mathbb{P}^*$ to $B^*$ (and thus to $\mathcal{OP}_n^k$).

**Definition 2.4.** Let $\pi = D_1 \, | \, D_2 \, | \cdots \, | \, D_k$ be an element of $B^*$ and $p = \tau^{(1)} - \tau^{(2)} - \cdots - \tau^{(\ell)}$ be a piecewise decreasing pattern of type $(j_1, j_2, \cdots, j_\ell)$.

We say that a word $w^{(1)} \, | \, w^{(2)} \, | \cdots \, | \, w^{(\ell)} \in B^*$ is an occurrence of $p$ in $\pi$ if there exist indices $1 \leq t_1 < t_2 < \cdots < t_\ell \leq k$ such that

- for $i = 1, 2, \ldots, \ell$, $w^{(i)}$ is a contiguous subsequence of $D_{t_i}$ of length $j_i$;

- the word $w^{(1)} \cdot w^{(2)} \cdot \cdots \cdot w^{(\ell)}$ is order isomorphic to $\tau^{(1)} \cdot \tau^{(2)} \cdots \cdot \tau^{(\ell)}$.

For example, the word $5 \, 3 \, 2 \, | \, 6 \, 4 \, 1 \, | \, 5 \, 4$ does not contain the pattern $3 \, 1 \, - \, 4 \, 2 \, - \, 4$, but it contains exactly one occurrence of the pattern $3 \, 1 \, - \, 4 \, 2 \, - \, 3$, namely the subword the letters of which are boldfaced in $5 \, 3 \, 2 \, | \, 6 \, 4 \, 1 \, | \, 5 \, 4$. If $p_1, p_2, \ldots, p_m$ are dashed patterns, we let $(p_1, \ldots, p_m)$ denote the $m$-statistic which associates to a word $\pi \in B^*$ the vector $(a_1, a_2, \cdots, a_m)$ where $a_i$ is the number of occurrences of $p_i$ in $\pi$.

**Proposition 2.5.** For any word $\pi$ in $B^*$ and any piecewise decreasing and connected pattern $p$, the parameter $(p)$ is constant on the equivalence class $[\pi] \in L(B, \ll)$.
Proof. Let \( \pi \) and \( \pi' \) be two adjacent words in \( \mathcal{B}^* \); then, there exist \( D_1, D_2, \ldots, D_k \) in \( \mathcal{B} \) and a positive integer \( j < k \) such that \( D_j \) and \( D_{j+1} \) are distinct and comparable (i.e., \( D_j \gg D_{j+1} \) or \( D_j \ll D_{j+1} \)) and

\[
\pi = D_1 \mid D_2 \mid \cdots \mid D_k \quad \text{and} \quad \pi' = D_1 \mid D_2 \mid \cdots \mid D_{j-1} \mid D_j \mid D_{j+1} \mid D_j \mid D_{j+2} \cdots \mid D_k.
\]

Let \( p = \tau^{(1)} - \tau^{(2)} - \cdots - \tau^{(\ell)} \) be a piecewise decreasing and connected dashed pattern. We have to show that \( (\tau(p)) = (\tau(\pi')) \). Clearly, it suffices, by symmetry, to prove that \( (\tau(p)) \leq (\tau(\pi')) \). Suppose that \( w^{(1)} \mid w^{(2)} \mid \cdots \mid w^{(\ell)} \) is an occurrence of \( p \) in \( \pi \) such that \( w^{(i)}, i = 1, 2, \ldots, \ell, \) is a contiguous subsequence of \( D_t \) for \( 1 \leq t_1 < t_2 < \cdots < t_\ell \leq k \).

1. If \( |\{j, j + 1\} \cap \{t_1, t_2, \ldots, t_\ell\}| \leq 1 \), it is obvious to see that \( w^{(1)} \mid w^{(2)} \mid \cdots \mid w^{(\ell)} \) is still an occurrence of \( p \) in \( \pi' \); thus \( (\tau(\pi)) \leq (\tau(\pi')) \).

2. We cannot have \( |\{j, j + 1\} \cap \{t_1, t_2, \ldots, t_\ell\}| = 2 \). Suppose the contrary. Then, \( j = t_m \) and \( j + 1 = t_{m+1} \) for a certain integer \( m \). Since the pattern \( p \) is connected, the sequences \( w^{(m)} \) and \( w^{(m+1)} \) are equal or incomparable. But, \( w^{(m)} \) and \( w^{(m+1)} \) are contiguous subsequences of \( D_j \) and \( D_{j+1} \) respectively, whence \( D_j \) and \( D_{j+1} \) are equal or incomparable. This contradicts our assumption.

Altogether, this implies that \( (\tau(\pi)) \leq (\tau(\pi')) \).

Combining Corollary 2.3 and Proposition 2.5, we arrive at the following result.

**Theorem 2.6.** For any word \( \pi \in \mathcal{B}^* \) and any piecewise decreasing and connected patterns \( p_1, p_2, \ldots, p_m \), the parameters \( (bDES,(p_1, p_2, \ldots, p_m)) \) and \( (bRISE,(p_1, p_2, \ldots, p_m)) \) have the same distribution over the class \([\pi] \in L(\mathcal{B}, \ll)\).

Given a multiset \( \mathcal{M} = \{D_1, D_2, \ldots, D_r\} \subseteq \mathcal{B} \), let \( \mathcal{R}(\mathcal{M}) \) denote the set of words in \( \mathcal{B}^* \) of length \( r \) the letters of which are \( D_1, D_2, \ldots, D_r \), that is

\[
\mathcal{R}(\mathcal{M}) = \{ D_{\sigma(1)} \mid D_{\sigma(2)} \mid \cdots \mid D_{\sigma(r)} \mid \sigma \in \mathcal{S}_r \}.
\]

For example, we have \( \mathcal{R}\{21, 21, 53\} = \{21|21|53, 21|53|21, 53|21|21\} \). It is clear that for all \( \pi \in \mathcal{R}(\mathcal{M}) \), we have \([\pi] \in \mathcal{R}(\mathcal{M}) \). The following result is immediate from Theorem 2.6.

**Corollary 2.7.** For any multiset \( \mathcal{M} \subseteq \mathcal{B} \) and any piecewise decreasing and connected patterns \( p_1, p_2, \ldots, p_m \), the parameters \( (bDES,(p_1, p_2, \ldots, p_m)) \) and \( (bRISE,(p_1, p_2, \ldots, p_m)) \) have the same distribution over \( \mathcal{R}(\mathcal{M}) \).

### 2.3 Applications

#### 2.3.1 Wilf classification of dashed patterns: Proof of Theorem 1.8

It is often convenient to identify a word \( w = w_1 w_2 \ldots w_n \in \mathcal{P}^* \) with the element \( \pi^w \) of \( \mathcal{B}^* \) the letters of which are from left to right the descending runs of \( w \), i.e., \( \pi^w \) is obtained

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from $w$ by placing a vertical line between $w_j$ and $w_{j+1}$ whenever $w_j \leq w_{j+1}$. For example, we have the correspondence

$$w = 3541655365 \leftrightarrow \pi^w = 3|541|65|53|65.$$  

It is easy to see that the map $w \rightarrow \pi^w$ establishes a bijection between $P^*$ and the set of minimal elements in $B^*$ (i.e., those which have no block descent). In particular, by Proposition 2.1, the map $w \rightarrow [\pi^w]$ establishes a bijection between $P^*$ and $L(B, \ll)$.  

Let $p = \tau^{(1)} - \tau^{(2)} - \cdots - \tau^{(l)}$ be a piecewise decreasing pattern. By definition, in an occurrence of $p$ in a word $w \in P^*$, the letters corresponding to a “bloc” $\tau^{(i)}$ have to form a contiguous subsequence of a descending run of $w$. This leads to

**Fact 2.8.** For any piecewise decreasing pattern $p$, the number of occurrences of $p$ in a word $w \in P^*$ is equal to the number of occurrences of $p$ in $\pi^w \in B^*$ (in the sense of Definition 2.4).

In the remainder of this section, we suppose we are given $m$ decreasing and connected patterns $p_1, p_2, \ldots, p_m$. It is easy to check that, for all multisets $\mathcal{M} \subseteq B$, the reverse map $R$ that sends each word $\pi = D_1 | D_2 | \cdots | D_k$ in $R(\mathcal{M})$ onto the word $R \pi = D_k | D_{k-1} | \cdots | D_1$ is an involution on $R(\mathcal{M})$ such that, for all $\pi \in R(\mathcal{M})$, we have

$$|bDES(R \pi)| = |bRISE(\pi)|, (p_1, p_2, \ldots, p_m)(R \pi) = (R p_1, R p_2, \ldots, R p_m)(\pi).$$  

(2.2)

Let $R_\emptyset(\mathcal{M})$ be the subset of $R(\mathcal{M})$ which consists of the minimal elements in $R(\mathcal{M})$, i.e.,

$$R_\emptyset(\mathcal{M}) = \{\pi \in R(\mathcal{M}) : bDES(\pi) = \emptyset\}.$$  

Recall that for a multiset $\mathcal{M} = \{D_1, D_2, \ldots, D_r\} \subseteq B$, we let $W(\mathcal{M})$ denote the set of words in $P^*$ the descending runs of which are $D_1, D_2, \ldots, D_r$. The map $w \in P^* \rightarrow \pi^w \in B^*$ sends bijectively $W(\mathcal{M})$ onto the set $R_\emptyset(\mathcal{M})$. For example, if $\mathcal{M} = \{421, 65, 75\}$, the reader can check that

$$W(\mathcal{M}) = \{4216575, 4217565\} \quad \text{and} \quad R_\emptyset(\mathcal{M}) = \{421|65|75, 421|75|65\}$$

Use of Fact 2.8, (2.2) and Corollary 2.7, we see that the parameters $(p_1, p_2, \ldots, p_m)$ and $(R p_1, R p_2, \ldots, R p_m)$ are equidistributed over $R_\emptyset(\mathcal{M})$, and thus over, $W(\mathcal{M})$. This ends the proof of Theorem 1.8 which refines Theorem 1.4.

### 2.3.2 Statistics on ordered set partitions

**Proof of Theorem 1.9.** It is easy to check that for any ordered set partition $\pi$, the value $\text{rsb} \pi$ (resp., $\text{lsb} \pi$) is equal to the number of occurrences, in the sense of Definition 2.4, of the pattern $2 - 3 1$ (resp., $3 1 - 2$) in $\pi$.

**Fact 2.9.** For any ordered set partition $\pi \in OP^h_n$, we have $\text{rsb} \pi = (2 - 3 1)(\pi)$ and $\text{lsb} \pi = (3 1 - 2)(\pi)$. 

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Since $\mathcal{OP}_n^k \subset B^*$ and any rearrangement of the blocks of a partition $\pi \in \mathcal{OP}_n^k$ is still in $\mathcal{OP}_n^k$, we see that the equivalence class $[\pi] \in L(\mathcal{B}, \ll)$ is contained in $\mathcal{OP}_n^k$. For example, if $\pi = 53|4|21 \in \mathcal{OP}_5^3$, we have

$$[\pi] = \{53|4|21, 53|21|4, 21|53|4\} \subseteq \mathcal{OP}_5^3.$$ 

Moreover, it is obvious that the set valued statistics Open and Clos are constant on each equivalence class $[\pi]$. This, combined with Fact 2.9 and Theorem 2.6, leads immediately to the following result.

**Corollary 2.10.** For any ordered set partition $\pi \in \mathcal{OP}_n^k$, the parameters

$$(b\text{DES}, \text{Open}, \text{Clos}, rsb, lsb) \quad \text{and} \quad (b\text{RISE}, \text{Open}, \text{Clos}, rsb, lsb)$$

have the same distribution over the equivalence class $[\pi]$, and thus over $\mathcal{OP}_n^k$.

It follows immediately from the definition of the parameters $\text{MAK}$ and $\text{MAK}'$ and Corollary 2.10 that the 3-statistics $(\text{MAK}, \text{MAK}', b\text{DES})$ and $(\text{MAK}, \text{MAK}', b\text{RISE})$ have the same distribution over $\mathcal{OP}_n^k$. Consider the transformation complement $c$ that maps each ordered set partition $\pi$ of $\mathbb{N}_n$ onto the ordered set partition $c\pi$ of $\mathbb{N}_n$ obtained from $\pi$ by complementing each of the letters in $\pi$, that is, by replacing $i$ by $n + 1 - i$. For example, if $\pi = 53|4|21$, then $c\pi = 13|2|45 \equiv 31|2|54$. It is easily checked that the complement $c$ sends the parameter $(\text{MAK}, \text{MAK}', b\text{RISE})$ onto $(\text{MAK}', \text{MAK}, b\text{DES})$. Since the complement map is an involution on $\mathcal{OP}_n^k$, we obtain that $(\text{MAK}, \text{MAK}', b\text{DES})$ and $(\text{MAK}', \text{MAK}, b\text{DES})$ are equidistributed over $\mathcal{OP}_n^k$. This concludes the proof of Theorem 1.9.

**A new Euler-Mahonian statistic.** It is worth noting that Corollary 2.10 permits to get new hard Euler-Mahonian statistics from known ones. For example, using the following result of Zeng and the author [12]

**Result C.** For $n \geq k \geq 1$, the partition statistic $\text{lsb} - b\text{MAJ} + k(k-1)$ is Euler-Mahonian on $\mathcal{OP}_n^k$.

and elementary properties of the reverse map $R$, we obtain the following result.

**Theorem 2.11.** Given a partition $\pi = B_1 \mid B_2 \mid \cdots \mid B_k$, let $\text{nbDes}(\pi)$ be the number of block non-descents of $\pi$, i.e. $\text{nbDes}(\pi) = k - 1 - |\text{bDES}(\pi)|$. Then, the partition statistic $\text{STAT} := rsb + k \cdot \text{nbDes} + b\text{MAJ}$ is Euler-Mahonian on $\mathcal{OP}_n^k$.

**Proof.** By result C, it suffices to prove that the parameter $\text{STAT}$ is equidistributed with the parameter $\text{lsb} - b\text{MAJ} + k(k-1)$ over $\mathcal{OP}_n^k$. It is easy to check that reverse map $R$ is an involutive transformation of $\mathcal{OP}_n^k$ and sends the parameter $(\text{lsb}, b\text{RISE})$ onto the parameter $(rsb, k - b\text{DES})$. This, combined with Corollary 2.10, implies that the parameters $(\text{lsb}, b\text{DES})$ and $(rsb, k - b\text{DES})$, and thus the parameters $\text{STAT} = rsb - (k \cdot b\text{DES} - b\text{MAJ}) + k(k-1)$ and $\text{STAT}' = \text{lsb} - b\text{MAJ} + k(k-1)$, have the same distribution over $\mathcal{OP}_n^k$. □

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3 Descent and rise sets in equivalence classes of a partially commutative monoid

This section is dedicated to the proof of Theorem 1.10 which asserts that the set-valued statistics \( DES \) and \( RISE \) are equidistributed over each equivalence class \([w] \in L(X, \leq)\) for any nonempty poset \((X, \leq)\). We shall give an “algebraic proof” as well as a bijective one.

3.1 A proof by the inclusion-exclusion principle

Let \( w \) be a word in \( X^* \). For a set \( S \subseteq \mathbb{P} \), we set

\[
\begin{align*}
  f_\geq(S) &:= |\{ w' \in [w] : DES(w') \subseteq S \}| \\
  g_\geq(S) &:= |\{ w' \in [w] : RISE(w') \subseteq S \}|
\end{align*}
\]

We have to show that \( f_\geq(S) = g_\geq(S) \). For a set \( T \subseteq \mathbb{P} \), define

\[
\begin{align*}
  f_\geq(T) &:= |\{ w' \in [w] : DES(w') \subseteq T \}| \\
  g_\geq(T) &:= |\{ w' \in [w] : RISE(w') \subseteq T \}|
\end{align*}
\]

so that we have \( f_\geq(S) = \sum_{T \subseteq S} f_\geq(T) \) and \( g_\geq(S) = \sum_{T \subseteq S} g_\geq(T) \). Then, by the principle of inclusion-exclusion (see e.g. Chapter 5 in [1]), we have

\[
f_\geq(S) = \sum_{T \subseteq S} (-1)^{|S| - |T|} f_\geq(T) \quad \text{and} \quad g_\geq(S) = \sum_{T \subseteq S} (-1)^{|S| - |T|} g_\geq(T). \quad (3.1)
\]

Suppose that \( T = \{ 1 \leq t_1 < t_2 < \cdots < t_{k-1} < n \} \). A moment’s thought will convince the reader that \( f_\geq(T) \) (resp., \( g_\geq(T) \)) is exactly the number of \( k \)-tuples \((w^{(1)}, w^{(2)}, \cdots, w^{(k)})\) in \((\mathbb{P}^*)^k\) such that

- the word \( w' = w^{(1)} \cdot w^{(2)} \cdots w^{(k)} \) obtained by concatenation of the \( w^{(i)} \)’s belongs to the class \([w]\);
- \(|w^{(i)}| = t_i - t_{i-1}\) for \( 1 \leq i \leq k \), with \( t_0 = 0 \) and \( t_k = n \), where \( n \) is the length of \( w \) and \(|w^{(i)}|\) is the length of \( w^{(i)} \);
- \( w^{(i)} \) is minimal (resp., maximal) for \( 1 \leq i \leq k \).

Recall that \( \theta \) denotes the (unique) bijection that sends each minimal word in \( X^* \) onto the maximal word that belongs to the same equivalence class. Then, it is easy to see that the bijection that sends each \( k \)-tuple of minimal words \((w^{(1)}, w^{(2)}, \cdots, w^{(k)})\) onto the \( k \)-tuple of maximal words \((\theta w^{(1)}, \theta w^{(2)}, \cdots, \theta w^{(k)})\) leads to the identity \( f_\geq(T) = g_\geq(T) \), from which we deduce, by (3.1), that \( f_\geq(S) = g_\geq(S) \). This concludes the proof of Theorem 2.2.
3.2 A bijective proof

3.2.1 An extremal case: \((X, \leq)\) is a totally ordered set

Suppose \((X, \leq)\) is a totally ordered set. Without loss of generality, we can assume \(X = \{1, 2, \ldots, r\}\), the set of the \(r\) first positive integers, with the natural order \(1 < 2 < \cdots < r\). In this case, the equivalence class of a word \(w \in X^r\) is just its rearrangement class (i.e., the set of words obtained by permuting the letters of \(w\)). Denote by \(R(n_1, n_2, \ldots, n_r)\) the rearrangement class of the word \(1^{n_1}2^{n_2} \cdots r^{n_r}\), or equivalently the set of words which have exactly \(n_1\) occurrences of the "letter" 1, \(n_2\) occurrences of 2, \ldots, \(n_r\) occurrences of \(r\). In this particular case, Theorem 1.10 just asserts that for any \(r\)-tuple \((n_1, n_2, \ldots, n_r)\) of nonnegative integers, the set-valued statistics \(DES\) and \(RISE\) are equidistributed over \(R(n_1, n_2, \ldots, n_r)\), or by abuse of notation,

\[
\sum_{w \in R(n_1, n_2, \ldots, n_r)} q^{DES(w)} = \sum_{w \in R(n_1, n_2, \ldots, n_r)} q^{RISE(w)}.
\]

(3.2)

It is not hard to give a bijective proof of (3.2). Indeed, for \(i = 1, 2, \ldots, r - 1\), there is a simple bijection \(\gamma_i : R(n_1, n_2, \ldots, n_r) \mapsto R(n_1, n_2, \ldots, n_{i-1}, n_i+1, n_i, n_{i+2}, \ldots, n_r)\) which preserves the descent set (see e.g. the proof of Theorem 10.2.1 in [14]): "consider a word \(w \in R(n_1, n_2, \ldots, n_r)\) and write all its factors of the form \((i+1)i\) in bold-face; then replace all the maximal factors \(i^a(i+1)^b\) with \(a \geq 0\), \(b \geq 0\), that do not involve any bold-face letters by \(i^b(i+1)^a\). Finally, rewrite all the bold-face letters in roman type.” It follows that the map \(\rho = \gamma_1 \gamma_2 \gamma_1 \cdots \gamma_{r-2} \gamma_2 \gamma_1 \gamma_{r-1} \cdots \gamma_2 \gamma_1\) establishes a bijection from \(R(n_1, n_2, \ldots, n_r)\) to \(R(n_r, n_{r-1}, \ldots, n_1)\) which preserves the descent set. Let \(c\) be the involutive transformation complement that maps each word \(w = w_1 w_2 \cdots w_n\) onto \(c\ w = (r + 1 - w_1)(r + 1 - w_2)\ldots(r + 1 - w_n)\). Then, it is easily checked that the map \(c \circ \rho : R(n_1, n_2, \ldots, n_r) \mapsto R(n_1, n_2, \ldots, n_r)\) is a bijection which sends the descent set onto the rise set.

3.2.2 The general case: a bijection based on the involution principle

The general involution principle, introduced by Garsia and Milne [9], permits to convert non-bijective proofs (notably, proofs by inclusion-exclusion principle) into bijective ones.

Let \((X, \leq)\) be a nonempty poset. For any (finite) subset \(S \subseteq \mathbb{P}\), let \(D_S\) and \(A_S\) be the subsets of \(X^\ast\) defined by

\[
D_S := \{w \in X^\ast : DES(w) = S\} \quad \text{and} \quad A_S := \{w \in X^\ast : RISE(w) = S\}.
\]

Using the general involution principle, we construct a bijection \(\Gamma_S : D_S \mapsto A_S\) for any (finite) subset \(S \subseteq \mathbb{P}\). In order to invoke the general involution principle (see e.g. [9] or Chapter 5 in [1]), we need (we assume that the reader is familiar with the used terminology)

- two signed sets \(Y = Y^+ \sqcup Y^-\) and \(Z = Z^+ \sqcup Z^-\), with \(D_S \subseteq Y^+\) and \(A_S \subseteq Z^+\);
- an alternating involution \(\psi\) on \(Y = Y^+ \sqcup Y^-\) with \(\text{Fix}(\psi) = D_S\), where \(\text{Fix}(\psi)\) stands for the set of fixed points of \(\psi\);
• an alternating involution $\phi$ on $Z = Z^+ \sqcup Z^-$ with $\text{Fix}(\phi) = A_S$;
• a sign-preserving bijection $F$ from $Y$ to $Z$.

The signed sets. Let $Y = Y^+ \sqcup Y^-$ be the signed set

$$Y = \{(w,T) : w \in X^*, S \subseteq T \subseteq \text{DES}(w)\}$$

with $(w,T) \in Y^+$ if $|T| - |S|$ is even, and $(w,T) \in Y^-$ if $|T| - |S|$ is odd. By identifying an element $w \in D_S$ with the pair $(w,S) \in Y^+$, we have $D_S \subseteq Y^+$. Similarly, let $Z = Z^+ \sqcup Z^-$ be the signed set

$$Z = \{(w,T) : w \in X^*, S \subseteq T \subseteq \text{RISE}(w)\}$$

with $(w,T) \in Z^+$ if $|T| - |S|$ is even, and $(w,T) \in Z^-$ if $|T| - |S|$ is odd. By identifying an element $w \in A_S$ with the pair $(w,S) \in Z^+$, we have $A_S \subseteq Z^+$.

The alternating involution $\phi$ on $Y = Y^+ \sqcup Y^-$. For $(w,T) \in Y$, let $d(w) = \max(\text{DES}(w) \setminus S)$ whenever $\text{DES}(w) \neq S$. Then define $\phi : Y \mapsto Y$ by

$$\phi(w,T) = \begin{cases} (w,T \setminus \{d(w)\}), & \text{if } d(w) \in T, \\ (w,T \cup \{d(w)\}), & \text{if } d(w) \notin T, \end{cases}$$

whenever $\text{DES}(w) \neq S$, and $\phi(w,S) = (w,S)$ if $\text{DES}(w) = S$. The mapping $\phi$ is clearly an alternating involution on $Y = Y^+ \sqcup Y^-$, and the only fixed points are the elements $(w,S)$ with $\text{DES}(w) = S$ which were identified with the elements of $D_S$.

The alternating involution $\psi$ on $Z = Z^+ \sqcup Z^-$. For $(w,T) \in Z$, let $a(w) = \max(\text{RISE}(w) \setminus S)$ whenever $\text{RISE}(w) \neq S$. Then define $\psi : Z \mapsto Z$ by

$$\psi(w,T) = \begin{cases} (w,T \setminus \{a(w)\}), & \text{if } a(w) \in T, \\ (w,T \cup \{a(w)\}), & \text{if } a(w) \notin T, \end{cases}$$

whenever $\text{RISE}(w) \neq S$, and $\psi(w,S) = (w,S)$ for $\text{RISE}(w) = S$. The mapping $\psi$ is clearly an alternating involution on $Z$, and the only fixed points are the elements $(w,S)$ with $\text{RISE}(w) = S$ which were identified with the elements of $A_S$.

The sign-preserving bijection $F$ from $Y$ to $Z$. Recall that, for $T \subseteq P$, the $T$-factorization of a word $w = x_1 x_2 \cdots x_n$ in $X^*$ is the (unique) factorization $w = w^{(1)} \cdot w^{(2)} \cdot \ldots \cdot w^{(k)}$ of $w$ defined by $x_i$ is the last letter of a factor $w^{(j)}$ if and only if $i \notin T$ or $i = n$. Then define $F : Y \mapsto Z$ as follows: if $(w,T) \in Y$ and the word $w$ has $T$-factorization $w = w^{(1)} \cdot w^{(2)} \cdots w^{(k)}$, we set $F(w,T) = (r w^{(1)} \cdot r w^{(2)} \cdots r w^{(k)}, T)$, where $r$ is, as usual, the reverse image.

In order to see that $F$ is well defined from $Y$ to $Z$, it suffices (since $S \subseteq T$) to prove that $T \subseteq \text{RISE}(r w^{(1)} \cdot r w^{(2)} \cdots r w^{(k)})$. By definition of $Y$, we have $T \subseteq \text{DES}(w)$ and thus, by definition of the $T$-factorization, the factors $w^{(j)}$, $1 \leq j \leq k$, are decreasing words. This implies that the words $r w^{(j)}$, $1 \leq j \leq k$, are increasing words, from which it
is immediate to see that \( T \subseteq RISE(r w^{(1)} \cdot r w^{(2)} \cdots r w^{(k)}) \). Moreover, since the letters of an increasing (or decreasing) word are pairwise comparable, the words \( w^{(j)} \) and \( r w^{(j)} \), \( 1 \leq j \leq k \), are equivalent, whence \( r w^{(1)} \cdot r w^{(2)} \cdots r w^{(k)} \in [w] \). It is clear that \( F \) is sign-preserving. To see that \( F \) is bijective, we construct its inverse. Define \( G : Z \mapsto Y \) as follows: if \( (w, T) \in Z \) and the word \( w \) has \( T \)-factorization \( w = w^{(1)} \cdot w^{(2)} \cdots w^{(k)} \), we set \( G(w, T) = (r w^{(1)} \cdot r w^{(2)} \cdots r w^{(k)}, T) \). It is easy to check that \( G \) is well-defined and that \( G = F^{-1} \). Summarizing, we have obtained the following result.

**Proposition 3.1.** The mapping \( F : Y \rightarrow Z \) is a sign-preserving bijection such that if \( (w, T) \in Y \) and \( F(w, T) = (w', T) \), then \( w \) and \( w' \) are equivalent, i.e., \( w' \in [w] \).

Applying the general involution principle (see e.g. Chapter 5 in [1]), we arrive at

**Theorem 3.2.** Let \( w \equiv (w, S) \in D_S \), i.e., \( w \in X^* \) and \( DES(w) = S \). There is a least integer \( o(w) \) such that

\[
F(\psi F^{-1} \phi F)^{o(w)} (w, S) \in A_S.
\]

If we set \( (w', S) := F(\psi F^{-1} \phi F)^{o(w)} (w, S) \), then the mapping \( \Gamma_S : w \mapsto w' \) is a bijection from \( D_S \) to \( A_S \) such that \( w' \in [w] \).

If we by \( \Gamma \) the transformation of \( X^* \) whose restriction on \( D_S \) is equal to \( \Gamma_S \), then \( \Gamma \) is a bijective transformation of \( X^* \) such that \( DES(w) = RISE(\Gamma w) \) and \( \Gamma w \in [w] \) for all \( w \in X^* \).

It follows from Proposition 2.1 that there exists a unique bijection \( \theta : D_\emptyset \mapsto A_\emptyset \) which satisfies the following condition:

for all \( w \in D_\emptyset \), \( \theta(w) \) and \( w \) are equivalent, i.e. \( \theta(w) \in [w] \).

Therefore, by Theorem 3.2, \( \theta \) is the restriction of \( \Gamma \) on \( D_\emptyset \), i.e., \( \theta = \Gamma_\emptyset \). Foata and Randriarivony [7] gave a more direct description of the mapping \( \theta \), which can be defined recursively as follows (see [6, Section 6]).

The mapping \( \theta : D_\emptyset \mapsto A_\emptyset \).

- if \( |w| \leq 1 \), \( \theta(w) = w \);
- if \( |w| = n \geq 2 \): suppose \( w = w' \cdot x \) with \( w' = x_1 x_2 \cdots x_{n-1} \) and \( x \) in \( X \). Let \( t \) be the largest integer \( \leq n - 1 \) such that \( x_t \) is incomparable and distinct from \( x \). Then \( \theta(w) \) is obtained from \( \theta(w') \) by inserting \( x \) between \( x_t \) and \( x_{t+1} \). If there is no such \( t \), then we set \( \theta(w) = x \cdot \theta(w') \).

Illustrations are given in the next subsection.
3.2.3 Illustration: the transformation $\Gamma : B^* \mapsto B^*$ and Clarke et al.’s transformation $\epsilon : \mathbb{P}^* \mapsto \mathbb{P}^*$

Theorem 3.2 leads to the following result.

**Theorem 3.3.** Consider the monoid $B^*$ generated by $(B, \ll)$. Then, the transformation $\Gamma : B^* \mapsto B^*$ described in Theorem 3.2 is a bijection which sends the parameter $bDES$ onto $bRISE$, and such that, for all $w \in B^*$, $\Gamma w$ belongs to the equivalence class $[w] \in L(B, \ll)$.

For example, if $w = 3 \mid 96 \mid 54 \mid 21 \mid 87$, then $S = bDES(w) = \{2, 3\}$. Running the algorithm for $\Gamma_S$, we obtain

$$w \equiv (w, S) = (3 \mid 96 \mid 54 \mid 21 \mid 87, \{2, 3\}) \xrightarrow{F} (3 \mid 21 \mid 54 \mid 96 \mid 87, \{2, 3\}) \in A_S,$$

whence $\Gamma(3 \mid 96 \mid 54 \mid 21 \mid 87) = 3 \mid 21 \mid 54 \mid 96 \mid 87$.

If $w = 21 \mid 96 \mid 54 \mid 3 \mid 87$, then $S = bDES(w) = \{2, 3\}$. Running the algorithm for $\Gamma_S$, we obtain

$$w \equiv (w, S) = (21 \mid 96 \mid 54 \mid 3 \mid 87, \{2, 3\}) \xrightarrow{F} (21 \mid 3 \mid 54 \mid 96 \mid 87, \{2, 3\}) \notin A_S$$

$$\xrightarrow{\psi} (21 \mid 3 \mid 54 \mid 96 \mid 87, \{1, 2, 3\}) \xrightarrow{F^{-1}} (96 \mid 54 \mid 3 \mid 21 \mid 87, \{1, 2, 3\})$$

$$\xrightarrow{\phi} (96 \mid 54 \mid 3 \mid 21 \mid 87, \{2, 3\}) \xrightarrow{F} (96 \mid 21 \mid 3 \mid 54 \mid 87, \{2, 3\}) \notin A_S$$

$$\xrightarrow{\psi} (96 \mid 21 \mid 3 \mid 54 \mid 87, \{2, 3, 4\}) \xrightarrow{F^{-1}} (96 \mid 87 \mid 54 \mid 3 \mid 21, \{2, 3, 4\})$$

$$\xrightarrow{\phi} (96 \mid 87 \mid 54 \mid 3 \mid 21, \{2, 3\}) \xrightarrow{F} (96 \mid 3 \mid 54 \mid 87 \mid 21, \{2, 3\}) \in A_S,$$

whence $\Gamma(21 \mid 96 \mid 54 \mid 3 \mid 87) = 96 \mid 3 \mid 54 \mid 87 \mid 21$.

If $w = 31 \mid 542 \mid 76$, then $S = bDES(w) = \emptyset$. Running the algorithm for $\Gamma_S$, we obtain

$$w \equiv (w, S) = (31 \mid 542 \mid 76, \emptyset) \xrightarrow{F} (31 \mid 542 \mid 76, \emptyset) \notin A_S$$

$$\xrightarrow{\psi} (31 \mid 76 \mid 542, \{2\}) \xrightarrow{F^{-1}} (31 \mid 76 \mid 542, \emptyset) \xrightarrow{\phi} (31 \mid 76 \mid 542, \emptyset) \notin A_S$$

$$\xrightarrow{\psi} (31 \mid 76 \mid 542, \{1\}) \xrightarrow{F^{-1}} (76 \mid 31 \mid 542, \{1\}) \xrightarrow{\phi} (76 \mid 31 \mid 542, \emptyset)$$

$$\xrightarrow{F} (76 \mid 31 \mid 542, \emptyset) \in A_S,$$

whence $\Gamma(31 \mid 542 \mid 76) = \theta(31 \mid 542 \mid 76) = 76 \mid 31 \mid 542$. When $bDes(w) = \emptyset$, it is more convenient to use the recursive description of $\theta$. For instance, if $w$ is as above, i.e. $w = 31 \mid 542 \mid 76$, we have $\theta(31 \mid 542) = 31 \mid 542$ and

$$\Gamma(w) = \theta(w) = \theta(31 \mid 542 \mid 76) = 76 \mid 31 \mid 542.$$
Clarke et al.’s involution $\epsilon : P^* \to P^*$. This transformation (see [6, Section 6]) is a variation of the transformation $\theta = \Gamma_0$. If $w = x_1 x_2 \ldots x_n \in P^*$, then $\epsilon(w) \in P^*$ is the word obtained from $R (\Gamma_0 (\pi^w) ) = R (\theta (\pi^w) )$ by deleting the vertical bars. For example, if $w = 3 6 4 5 3 5 3 1 7 6$, we have

$\pi^w = 3 \mid 6 4 \mid 5 3 \mid 5 3 1 \mid 7 6 \mapsto 6 4 \mid 7 6 \mid 3 \mid 5 3 \mid 5 3 \mid 3 \mid 7 6 \mid 6 4 \mapsto 5 3 \mid 1 \mid 5 3 \mid 3 \mid 7 6 \mid 6 4$.

whence $\epsilon(3 6 4 5 3 5 3 1 7 6) = 5 3 1 5 3 3 7 6 6 4$. Combining Theorem 3.3, Proposition 2.5 and Fact 2.8 leads to the following result.

**Theorem 3.4.** The map $\epsilon$ is a bijective transformation of $P^*$ such that for any piecewise decreasing and connected patterns $p_1, p_2, \ldots, p_m$, we have

$$(p_1, p_2, \ldots, p_m)(w) = (R p_1, R p_2, \ldots, R p_m)(\epsilon(w))$$

and $[\pi^w] = [\pi^{\epsilon(w)}]$ in $L(B, \ll)$.

It is worth noting that it is easy to give bijective proofs of Theorem 1.9 and Theorem 2.11 by using the map $\Gamma$.

4 Concluding remarks and open problems

4.1 Wilf-classification of dashed patterns in subclasses of permutations

In this paper, we focused our attention primarily on the (run- and reverse-complete) collections of permutations, of words and of compositions. Other interesting run-complete (but not reverse-complete) collections are:

- the collection $A^{(2)} = \bigcup_{n \geq 0} A_{2n}$ of reverse alternating permutations of even length,
  (and more generally, for any positive integer $k$, the collection $A^{(k)}$ of permutations the descending runs of which are of length $k$);

- the collection $D^{(1)} = \bigcup_{n \geq 0} D_n^{(1)}$ (resp., $D^{(3)} = \bigcup_{n \geq 0} D_n^{(3)}$) of Dumont permutations of the first (resp., third) kind, where $D_n^{(i)}$ is the set of Dumont permutations of the $i$-th kind of length $n$ (see e.g. [4] for a precise definition).

Note that there has been recent interest in the study of patterns in alternating permutations and Dumont permutations (see e.g. [13, 4]). Theorem 1.4 leads to wilf-classification results for these classes. For instance, as an immediate consequence of Theorem 1.4, we see that the joint distribution of the pair $((2 \prec -1), (1 \prec 3 \prec 2))$ over $D_n^{(1)}$ is symmetric, which is not obvious since $D_n^{(1)}$ has no apparent symmetries (see the remark after Corollary 4.5 in [4]). Note that Clarke et al.’s bijection leads to a simple bijective proof of this symmetry.
4.2 Euler-Mahonian partition statistics and permutation statistics

In this paper, we have shown that the equidistributions of the permutation statistics $(\text{des}, \text{mak})$ and $(\text{des}, \text{mak}')$ over the symmetric group, and on the other part, of the statistics $\text{MAK} + b\text{MAJ}$ and $\text{MAK}' + b\text{MAJ}$ over ordered set partitions have a very natural generalization in the context of ordered set partition statistics.

It seems that there is a strong connection between the so called Euler-Mahonian permutations statistics and Euler-Mahonian partitions statistics, but the link is far to be understood. As an illustration, we present a conjecture which generalizes an important equidistribution result over the symmetric group as well as an equidistribution result over set partitions.

Recall that a descent in a permutation $\sigma \in S_n$ is an integer $i$, $1 \leq i \leq n - 1$, such that $\sigma(i) > \sigma(i + 1)$. The Major index, denoted $\text{maj}$, is a well known Mahonian permutation statistic. For a permutation $\sigma \in S_n$, $\text{maj}_{\sigma}$ is defined as the sum of the descents in $\sigma$. For example, if $\sigma = 3 \ 2 \ 1 \ 7 \ 5 \ 6 \ 4 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1$ $\in S_{12}$, then $1, 2, 4, 6$ are the descents of $\sigma$ whence $\text{maj}_{\sigma} = 1 + 2 + 4 + 6 = 13$. The following result is due to Foata and Zeilberger [8].

Result C. For $n \geq 1$, the bistatistics $(\text{des}, \text{maj})$ and $(\text{des}, \text{mak})$ are equidistributed over $S_n$.

The Milne’s statistic $\text{MIL}$ is defined for an ordered set partition $\pi = B_1 \mid B_2 \mid \cdots \mid B_k \in \mathcal{OP}_k^n$ by $\text{MIL} \ \pi = |B_2| + 2|B_3| + \cdots + (k - 1)|B_k|$. For example, $\text{MIL}(8 \mid 9 \mid 7 \mid 5 \mid 6 \mid 4 \mid 3) = 1 \cdot 1 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1 = 17$. Steingrímsson [15] observed that the Major index is closely related to the partition statistic $\text{MIL}$. More precisely, he showed that if $\sigma \in S_n$ has descending runs $D_1, D_2, \ldots, D_k$ (i.e., $\pi^{\sigma} = D_1 \mid D_2 \mid \cdots \mid D_k$), then we have

$$\text{maj}_{\sigma} = \text{MIL}(D_1 \mid D_2 \mid \cdots \mid D_k) + \binom{n + 1}{2} - kn. \hspace{1cm} (4.1)$$

Steingrímsson [15] proved that the statistic $b\text{maj}MIL := \text{MIL} + b\text{MAJ}$ is Euler-Mahonian on $\mathcal{OP}_k^n$, while Zeng and the author [12] proved that the statistics $\text{MAK} + b\text{MAJ}$ (and $\text{MAK}' + b\text{MAJ}$) are Euler-Mahonian on $\mathcal{OP}_k^n$. Therefore, we have:

Result D. For $n \geq k \geq 1$, the partition statistics $\text{MIL} + b\text{MAJ}$ and $\text{MAK} + b\text{MAJ}$ are equidistributed over $\mathcal{OP}_k^n$.

We suspect that actually more is true. The following conjecture has been verified for $n \leq 11$ with the help of Einar Steingrímsson.

Conjecture 4.1. For $n \geq k \geq 1$, the bistatistics

$$(\text{bDes}, \text{MIL} + b\text{MAJ}) \quad \text{and} \quad (\text{bDes}, \text{MAK} + b\text{MAJ})$$

are equidistributed over $\mathcal{OP}_k^n$.

This conjecture is interesting since it permits, in view of (4.1) and (1.6), to unify Result C and Result D, and its solution will probably lead to a better understanding of the link between (Euler-Mahonian) permutation and partition statistics.
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