On periodic motions of a body with an internal moving mass on a rough horizontal plane in the case of anisotropic friction

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Abstract. We consider a mechanical system consisting of a body moving on a horizontal plane and a mobile internal mass, which moves in a circle with a constant velocity. Dry anisotropic friction force acts between the body and the plane. The body moves without jumping. Five periodic modes of motion are established and conditions of their existence are found. A qualitative analysis of each periodic mode has been performed. In the plane of parameters the domains corresponding to the above periodic modes are determined.

1. Introduction

We consider a mechanical system consisting of a rigid body, located on a horizontal rough plane, and a material point (internal mass) moving inside the body. We assume that the motion of the internal mass inside the body takes place in a vertical plane in a circle with the center at the mass center of the body. The internal mass does not interact with the external environment and the velocity’s absolute value of its relative circular motion is a constant. We suppose that the dry friction force acts between the body and the support plane.

The study of dynamics of bodies moving by means of displacements of internal elements is not only of theoretical interest, but may be also significant for the development of mobile devices (vibration robots). The advantage of these devices is that they do not require special movers (wheels, tracks, legs, etc.) and can be made in a closed shell. The above circumstance makes them resistant to aggressive environmental influences and allows to use them both on solid surfaces and in liquids.

Dynamics of bodies containing internal elements has been studied in many works. A bibliography on this subject can be found in [1,2]. For the mechanical system under consideration, a complete qualitative study of dynamics was performed in the case when the friction between the body and the plane is isotropic and is described by the classical Coulomb law [3,4]. An analysis was also made of motions of the rigid body in the presence of viscous friction [5,6] and on an inclined plane [7,8]. Problems of optimal control of body motion by means of internal mass were considered in [9,10].

In this paper we study the motions of the body under assumption that friction force acting on the body is anisotropic.

2. Equations of motion

We consider motions of the body in the vertical plane. To this end we introduce an absolute coordinate system $OXY$, directing the axis $OX$ along the direction of motion of the body (figure 1). By $X$ and $Y$ we...
denote the coordinates of body's mass center $O$. To describe the relative motion of the internal point $A$ we introduce coordinate system $O_1\xi\eta$, which is fixed in the body. The axis $O_1\xi$ is parallel to the horizontal plane. Then the coordinates $\xi, \eta$ determining the position of the internal mass with respect to the body read

$$\xi = R \sin \omega t, \quad \eta = -R \cos \omega t,$$

where $\omega = \text{const}$ is absolute value of the angular velocity of the radius vector $O_A$ ($|O_A| = R$).

Let us introduce the following notations. $M$ and $m$ are the masses of the body and the material point $A$ respectively. $N$ is the normal reaction acting on the body from the support plane, $g$ is the gravitational acceleration. By $k_u$ and $k_d$ we denote the dry friction coefficients when the body moves in the positive and negative direction, respectively. In what follows we suppose that $k_d > k_u$.

Figure 1. System

The equations of motion of the body have the form

$$\begin{align*}
M\ddot{x} + m(\ddot{x} + \ddot{\xi}) &= F_c, \\
M\ddot{y} + m(\ddot{y} + \ddot{\eta}) &= N - (M + m)g,
\end{align*}$$

where $\ddot{\xi}$ and $\ddot{\eta}$ are functions given by the formulas (1). The dry friction force $F_c$ reads

$$F_c = \begin{cases}
-k_u N, & \dddot{x} > 0 \cup \dddot{x} = 0, \quad m\dddot{x} < -k_u N, \\
k_d N, & \dddot{x} < 0 \cup \dddot{x} = 0, \quad m\dddot{x} > k_d N, \\
m\dddot{x}, & \dddot{x} = 0, \quad -k_u N \leq m\dddot{x} \leq k_d N.
\end{cases}$$

We introduce the dimensionless coordinates $x, y$ by the formulas

$$X = \frac{Rmx}{M + m}, \quad Y = \frac{Rmy}{M + m},$$

and pass to the new time $t \rightarrow \omega t$ keeping the old notation $t$.

Without loss of any generality we can assume that at $t = 0$ the internal mass is in the lowest point of the trajectory of its circular motion. Then the new time $t$ can be regarded as the angle between the radius vector $O_A$ and the negative direction of axis $O_1\eta$ (see figure 1).

The equations of motion in dimensionless variables take the form

$$\begin{align*}
\dddot{x} &= \sin t + f_c, \\
\dddot{y} &= -\cos t + n - \mu,
\end{align*}$$

where

$$f_c = \frac{F_c}{Rm\omega^2}, \quad n = \frac{N}{Rm\omega^2}, \quad \mu = \frac{(M + m)g}{Rm\omega^2}.$$
We suppose that the body moves without jumping, i.e. the equality $\ddot{y} = 0$ is satisfied. It means that the vertical component of the inertia force caused by the relative motion of the internal mass does not exceed the absolute value of the gravity force applied at the center of mass of the system. It is easy to see that the above condition is equivalent to the following inequality

$$\mu \geq 1.$$  \hspace{1cm} (7)

In this case the body performs rectilinear translational motion described by the following equation

$$\dot{u} = \sin t + f_c,$$  \hspace{1cm} (8)

where

$$f_c = \begin{cases} k_u (\mu + \cos t), & u > 0 \cup [u = 0, \sin t > k_u (\mu + \cos t)], \\ k_d (\mu + \cos t), & u < 0 \cup [u = 0, \sin t < -k_d (\mu + \cos t)], \\ -\sin t, & u = 0, -k_d (\mu + \cos t) \leq \sin t \leq k_u (\mu + \cos t). \end{cases}$$  \hspace{1cm} (9)

The new variable $u$ is dimensionless body’s velocity: $u = \dot{x}$.

The equations of motion of the body in the positive and negative directions have the form

$$\dot{u} = \sin t - k_u (\cos t + \mu)$$  \hspace{1cm} (10)

and

$$\dot{u} = \sin t + k_d (\cos t + \mu).$$  \hspace{1cm} (11)

3. Zones of deceleration

In the dynamics of the body the following three cases can take place.

Case I. The relative motion of the internal mass cannot cause the displacement of body from a state of rest.

Case II. Resting body can start to move in positive direction due to relative motion of the internal mass.

Case III. Resting body can start to move both in positive and negative direction due to relative motion of the internal mass.

To study the above cases in more detail we consider time intervals, where the acceleration of the body is opposite to its velocity or equal to zero. These intervals correspond to certain arcs of the circular trajectory of the relative motion of the internal mass. We call these arcs as deceleration zones, and we call the corresponding time intervals as deceleration intervals.

The deceleration zones play an important role for the dynamics of the body. If the body stops when the internal mass passes through the deceleration zone, it will remain at rest until the mass leaves the deceleration zone. This phenomenon is called body sticking [2].

The right-hand sides of the equations (10) and (11) are equal to zero at the boundaries of the deceleration intervals, i.e. at the moments of time when the internal mass enters or leaves the deceleration zone. Thus, the deceleration intervals can be found by equating the right-hand sides of the equations (10) and (11) to zero, i.e. by analyze of the following two trigonometric equations

$$\sin t - k_u (\mu + \cos t) = 0, \quad \sin t + k_d (\mu + \cos t) = 0.$$  \hspace{1cm} (12)

Let us replace in the equations (12) $\sin t$ and $\cos t$ by $\xi$ and $-\eta$ respectively and consider equivalent systems of algebraic equations

$$\begin{cases} \xi - k_u (\mu - \eta) = 0, \\ \xi^2 + \eta^2 = 1 \end{cases} \quad \begin{cases} \xi + k_d (\mu - \eta) = 0, \\ \xi^2 + \eta^2 = 1 \end{cases}.$$  \hspace{1cm} (13)

The values $\xi$ and $\eta$ can be interpreted as the relative motion’s coordinates of the internal mass normalized by the factor $R^{-1}$. We denote the real solutions of the systems (13) by $(\xi_1, \eta_1)$, putting $i = 1, 2$ for the solutions of the first system, and $i = 3, 4$ for the solutions of the second system.
In the plane of the variables \( \tilde{\xi} \) and \( \tilde{\eta} \), the real solutions of the systems (13) correspond to the intersection points of the circle \( \tilde{\xi}^2 + \tilde{\eta}^2 = 1 \) with two straight lines: \( \alpha_1 \), given by the equation \( \tilde{\xi} - k_u (\mu - \tilde{\eta}) = 0 \), and \( \alpha_2 \), given by the equation \( \tilde{\xi} + k_d (\mu - \tilde{\eta}) = 0 \) (figure 2).

![Figure 2. Zones of deceleration](image)

If the solutions of the systems are found, then the corresponding moments of time \( t_i \) can be determined from equations

\[
\sin t_i = \frac{\tilde{\xi}_i}{\xi_i}, \quad \cos t_i = -\frac{\tilde{\eta}_i}{\eta_i}.
\]

The direction of body acceleration depends on the position of the internal mass. If it is over the line \( \alpha_1 \), then the body acceleration is positive. Otherwise, if the internal mass is below the straight line \( \alpha_1 \), then the acceleration is negative or equal to zero. In particular, the acceleration can be equal to zero either when the internal mass passes the boundary points \( t_1, t_2 \), or in the case of body sticking. Similarly, when the internal mass is over the straight line \( \alpha_2 \), the acceleration of the body is negative. When it is below the straight line \( \alpha_2 \), the acceleration is positive or turns to zero. In fact, the acceleration turns to zero at the boundary points \( t_3, t_4 \), or in the case of body sticking. Thus, the deceleration zones are the arcs located below both the lines \( \alpha_1 \) and \( \alpha_2 \). In figure 2, this is the arc bounded by the points \( t_2, t_3 \) and the arc bounded by the points \( t_4, t_1 \). We call these arcs upper and lower zones of deceleration respectively.

The positions of the lines \( \alpha_1, \alpha_2 \) depend on the values of the parameters \( \mu, k_u, \) and \( k_d \). When changing the parameter \( \mu \), the point \( P \) of the intersection of the lines \( \alpha_1 \) and \( \alpha_2 \) moves along the vertical axis. The evolution of parameters \( k_u \) and \( k_d \) changes the angle of inclination of the lines \( \alpha_1 \) and \( \alpha_2 \), respectively. By taking this facts into account it is not difficult to establish the location of the lines \( \alpha_1 \) and \( \alpha_2 \) in the above three cases:

- in Case I the lines \( \alpha_1 \) and \( \alpha_2 \) are located above the circle, so that the line \( \alpha_2 \) has no points in common with it, and the line \( \alpha_1 \) has at most one common point with the circle;
- in Case II the line \( \alpha_2 \) is located above the circle and has at most one common point with it, and the line \( \alpha_1 \) intersects the circle at two points;
in Case III each of the lines $\alpha_1$ and $\alpha_2$ intersects with the circle at two points.

In Case I the whole circle is a deceleration zone. In Case II there is one deceleration zone, which is an arc of the circle. In Case III, there are two deceleration zones (see figure 3).

![Figure 3. Possible cases of $\alpha_1$ and $\alpha_2$ location](image)

In the next section we consider the above cases in detail.

4. Modes of periodic motions

4.1. On motion in Case I. It is easy to see that lines $\alpha_1$, $\alpha_2$ did not have intersections with the circle, if the following condition is satisfied

$$\mu > \left( k_u^2 + 1 \right)^{1/2} / k_u .$$

Under the condition (15) the right hand sides of both equations (10) and (11) do not change the sign and the Case I takes place. In this case the acceleration of the body is always directed opposite to its velocity. It follows that at any initial speed, the body stops after a finite period of time and then remain at rest. The periodic modes of motion are not possible in Case I.

4.2. On periodic motion in Case II. The Case II takes place if the inequalities

$$\left( k_u^2 + 1 \right)^{1/2} / k_d \leq \mu < \left( k_u^2 + 1 \right)^{1/2} / k_u$$

are hold. It follows that the first system (13) has two real solutions

$$\tilde{\eta}_{1,2} = \frac{k_u^2 \mu \pm \sqrt{1 + k_u^2 (1 - \mu^2)}}{k_u^2 + 1}, \quad \tilde{\xi}_{1,2} = k_u (\mu - \tilde{\eta}_{1,2}) ,$$

which correspond to moments of time $t_{1,2}$, and the second system (13) does not have real solutions.

In this case the solution of equation (8) with the initial condition $u(t_1) = 0$ is periodic. Indeed, in the interval $[t_1, t_2]$ the acceleration of the body is positive. Hence, it starts to move in positive direction with increasing velocity, which can be found from equation (10)

$$u(t) = \int_{t_1}^t \sin \tau - k_u (\cos \tau + \mu) \, d\tau .$$

The function $u(t)$ given by formula (18) monotonically increases and decreases in the intervals $[t_1, t_2]$ and $[t_2, t_1 + 2\pi]$ respectively. From (18) we have $u(t_1 + 2\pi) = -2\pi k_u < 0$. Hence, there exists $t_* \in [t_1, t_1 + 2\pi]$ such that $u(t_*) = 0$, i.e. the body stops at $t_*$ and stays in rest (sticking) in the
interval \([t_n, t_n + 2\pi]\). At \(t = t_n + 2\pi\) the body starts to move in positive direction and repeats the same motion in the interval \([t_n + 2\pi, t_n + 4\pi]\). Thus, if \(u(t_n) = 0\), then the body moves with periodically changing velocity sticking in the intervals\([t_n + 2\pi n, t_n + 2\pi(n+1)], n \in Z\).

4.3. On periodic motions in Case III. Let us suppose now that the following inequality is satisfied
\[
\mu < \left( k_n^2 + 1 \right)^{1/2} / k_d .
\] (19)

Then each of the lines \(\alpha_1\) and \(\alpha_2\) intersects the circle at two points and the Case III takes place. In this case the first system (13) has solutions (17) and the second system (13) has the following solutions
\[
\tilde{\eta}_{3,a} = \frac{k_a^2 \mu + \sqrt{1 + k_a^2 (1 - \mu^2)}}{k_a^2 + 1}, \quad \tilde{\xi}_{3,a} = -k_d (\mu - \tilde{\eta}_{3,a}) .
\] (20)

The moments \(t_i\) \((i = 1, 2, 3, 4)\) corresponding to the above solutions of systems (13) separate the time period from \(t_1\) to \(t_1 + 2\pi\) into four intervals. In the intervals \([t_1, t_2]\) and \([t_3, t_4]\) the acceleration of the body has positive and negative directions respectively. The intervals \([t_2, t_3]\) and \([t_4, t_1 + 2\pi]\) are deceleration intervals corresponding to upper and lower deceleration zones respectively.

Due to the presence of two zones of deceleration the dynamics of the body becomes more complicated. In particular, there are different types of periodic motions in this case. We note that type of periodic motion depends on parameters values and only one of these modes is possible for a fixed set of parameters values. Let us consider this question in more details.

4.3.1. Periodic motion with sticking in lower deceleration zone. As above we consider the solution of equation (8) with the initial condition \(u(t_1) = 0\). As we have already shown such a solutions has the form (18) on the interval \([t_1, t_n]\), where \(t_1\) is found from condition \(u(t_1) = 0\). Depending on parameters values the moment of time \(t_1\) can belong to one of intervals \([t_2, t_3], [t_3, t_4]\) or \([t_4, t_1 + 2\pi]\). First we suppose \(t_1 \in [t_3, t_1 + 2\pi]\). It is possible if the following inequality is satisfied
\[
-\cos t_4 + \cos t_1 - k_a \sin t_4 + k_a \sin t_1 - k_d \mu (t_4 - t_1) \geq 0 .
\] (21)

In this case the body stops in the lower deceleration zone and stays in rest till \(t = t_1 + 2\pi\). At the moment of time \(t_1 + 2\pi\) the body starts to move in positive direction and repeats the same motion in the interval \([t_1 + 2\pi, t_1 + 4\pi]\). That is it performs the motion with periodically changing velocity. This periodic mode of motion is similar to one considered in Case II.

4.3.2. Periodic motion with sticking in upper and lower deceleration zones. Let us now suppose that \(t_1\) is in \([t_2, t_3]\). In this case the body stops in the upper deceleration zone and stays in rest till \(t = t_3\). At \(t = t_3\) body starts to move in negative direction. It can be shown that if the body stops in upper deceleration zone, then it also stops in lower deceleration zone and stays in rest till \(t = t_3 + 2\pi\). At \(t = t_3 + 2\pi\) the body starts to move in positive direction and repeats the same motion in the interval \([t_3 + 2\pi, t_3 + 4\pi]\), i.e. it performs the motion with periodically changing velocity. From (18) it is easy to see that such a periodic motion mode exists if the following condition is satisfied.
described above periodic modes are determined.

4.3.4. Periodic motion with stops outside deceleration zones. Suppose now that $t_s \in [t_3, t_4]$. From (18) it is easy to see that it is possible if the following conditions hold

$$\begin{align*}
-\cos t_3 + \cos t_1 - k_u \sin t_1 + k_u \sin t_1 - k_u \mu(t_1 - t_1) &\leq 0, \\
-\cos t_4 + \cos t_1 - k_u \sin t_4 + k_u \sin t_1 - k_u \mu(t_4 - t_1) &< 0. 
\end{align*}$$

(22)

It can be shown that for any initial velocity the motion passes into this mode in a finite time interval.

4.3.3. Periodic motion with sticking in lower deceleration zone and a stop without sticking. Suppose that conditions (23) are satisfied, the body starts to move in negative direction at $t = t_*$ and stops at a certain moment $t = t_*$. Depending on parameters either $t_* \in [t_4, t_1 + 2\pi]$ or $t_* \in [t_1 + 2\pi, t_2 + 2\pi]$. First we suppose that the body stops in $[t_4, t_1 + 2\pi]$. That is, the following inequality holds.

$$-\cos t_4 + \cos t_1 + k_u \sin t_4 - k_u \sin t_1 + k_u \mu(t_1 - t_* + 2\pi) \geq 0$$

(24)

If conditions (23) and (24) are satisfied, then the body starts to move in positive direction at $t = t_1$, stops at $t = t_*$ and starts to move in negative direction till the stop at $t = t_* \in [t_4, t_1 + 2\pi]$. It stays at rest till $t = t_* + 2\pi$. In the interval $[t_1 + 2\pi, t_1 + 4\pi]$ the body and repeats the above-described motion. Thus, it performs the motion with periodically changing velocity. It can be shown that for any initial velocity the motion passes into this mode in a finite time interval.

4.3.4. Periodic motion with stops outside deceleration zones. Let us suppose that conditions (23) are satisfied and the condition (24) is fulfilled with opposite sign. Then $u(t_1 + 2\pi) < 0$ and this mode of motion is not periodic. However, numerical study has shown that there is periodic motion such that $u(t_a) = u(t_b) = 0$, where $t_a$ and $t_b$ do not belong to the deceleration intervals. In this case body starts to move in positive direction at $t_a$, stops at $t_b$ and starts to move in negative direction till the stop at $t_a + 2\pi$. In the interval $[t_a + 2\pi, t_a + 4\pi]$ the body and repeats the above-described motion. That is, it performs the motion with periodically changing velocity. Numerical study has shown that any motion of the body is asymptotic to this periodic mode of motion.

5. Domains of periodic modes of motion

In the space of parameters inequalities (15), (16), (19), (21), (22), (23) and (24) determine domains with different modes of motion. For $\mu = 1.6$ these domains are presented in figure 4 on plane of parameters $k_u, k_d$. Domain I corresponds to Case I. Domain II corresponds to Case II. Domains IIIa, IIIb, IIIc and IIId correspond to different modes of motion in case III. Domain IIIa corresponds to periodic motion with sticking in lower deceleration zone when inequality (21) is satisfied. Domain IIIb corresponds to periodic motion with sticking in upper and lower deceleration zones when inequality (22) is satisfied. Domain IIIc corresponds to periodic motion with sticking in lower deceleration zone and a stop without sticking when inequalities (23) and (24) are satisfied. Domain IIId corresponds to periodic motion with stops outside deceleration zones when inequalities (23) are satisfied and inequality (24) is satisfied with opposite sign.

6. Conclusions

In this paper, we have described all the possible periodic modes of motion of the considered mechanical system. The conditions for the existence of these modes were found. A qualitative analysis of the each periodic modes has been performed. In the plane of parameters the domains corresponding to the above periodic modes are determined.
Figure 4. Domains for $\mu = 1.6$

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