Liquid-gas coexistence in binary Bose-Einstein condensates

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As a common physical phenomenon in nature, liquid-gas coexistence (LGC) is generally associated with finite temperature and long-range interaction. Here we report a new mechanism in engineering LGC well beyond the traditional frame, i.e., for ground state bosons under contact interactions. It is facilitated by the mismatch of spin polarization, dubbed “spin twist”, between single-particle and interaction channels of bosons with spin degree of freedom. Such spin twist creates an effective repulsion for low-density bosons and uniquely stabilizes the gas phase, thereby enabling LGC in the presence of a quantum droplet with much larger density. We have demonstrated the scheme taking the example of binary bosons subject to Rabi coupling and magnetic detuning. The liquid-gas transition and coexistence therein can be conveniently tuned by single-particle potentials and spin-dependent interactions. To characterize LGC, we have shown the phase separation of liquid and gas in a trap and identified two universal exponents for the critical behavior of their densities. The spin twist scheme for LGC can be generalized to a wide class of quantum systems with competing single-particle and interaction orders.

Introduction. Liquid-gas coexistence (LGC) is a characteristic phenomenon in the first-order liquid-gas transition[1], which widely exists in nature and has got many practical applications in industry such as oil, natural gas, aerospace, chemical engineering, etc. In recent years the research of LGC has even extended to hot nuclei[2–4] and active matters[5, 6]. In all these systems, LGC has two common features. First, its driving force is usually the temperature ($T$), and only within a finite $T$-window it can sustain. Secondly, it is associated with long range interactions typically with a repulsive core and an attractive tail (e.g., Lennard-Jones potential), as is believed to be essential for the liquid stabilization. Indeed, a textbook model for LGC is based on the Van der Waals’ equation of state[7, 8], which exactly reflects the importance roles played by temperature and long-range interaction. Previous theories have also studied the possibility of LGC at zero temperature that is strongly affected by quantum statistics[9, 10]. However, there the long-range interaction is still required, and it was concluded that bosons in principle cannot host LGC at ground state ($T = 0$).

The recent realization of quantum droplet in ultracold gases[11–21] provides an unprecedented opportunity for exploring LGC in a new platform with high controllability. These ultracold droplets (resemblance of liquids) are stabilized by an attractive mean-field interaction and a repulsive force from quantum fluctuations[22], where the interaction is not necessarily long range but can be modeled by a contact potential as in binary boson mixtures. Experimentally, the liquid-gas transition has been observed in these short-range interacting systems [17–21], as driven by the quantum pressure of finite-size system when its number decreases due to intrinsic atom loss. Nevertheless, up to date the phenomenon of LGC has not been reported in such system. In fact, for all existing setups of binary bosons[17–21], the liquid and gas cannot coexist as ground state in the thermodynamic limit ($N, V \rightarrow \infty$ with fixed $n = N/V$). Meanwhile, since the quantum droplet is quite fragile to thermal effect [23–25], the LGC at finite $T$, even exists, can be quite difficult to detect given the expected narrow $T$-window in reality.

In this work, we unveil a new mechanism for LGC in bosonic system at zero temperature and with contact interaction, thereby well beyond the traditional frame as well as previous theories[9, 10]. Such mechanism is based on the “spin twist” (i.e., a mismatch of spin polarization) between the single-particle and the interaction channels, and is very easy to implement in ultracold atoms with hyperfine spin structure. To illustrate the idea clearly, we consider a simple setup of interacting binary bosons subject to Rabi coupling ($\Omega$) and magnetic detuning ($\delta$), see schematics in Fig.1(a), as recently explored in experiments[26, 27]. These external fields ($\Omega$, $\delta$) determine an optimal spin polarization in single-particle level, which can be tuned to mismatch the one determined by spin-dependent interactions (Fig.1(b)). Such spin twist leads to an effective repulsion for bosons in low density limit (Fig.1(c)), which uniquely stabilizes the gas phase and therefore renders LGC in the presence of a quantum droplet (liquid) at much larger density (Fig.1(d)). In this case, the resulted liquid-gas transition and coexistence can be conveniently tuned by $\Omega$, $\delta$ and interaction strengths, and more importantly, they all occur for thermodynamic systems instead of finite-size ones, in contrast to the transitions observed previously[17–21]. To characterize LGC, we have demonstrated the phase separation of liquid and gas with discontinuous densities in a harmonic trap, and further identified two universal exponents for the critical behavior of their densities.
where a stable quantum droplet can be supported in the
spin state of binary bosons, \((\psi_r, \psi_i) \propto (\cos(\theta/2), \sin(\theta/2))\),
can be mapped onto a Bloch sphere with polarization \(S \equiv \langle \sigma_z \rangle \equiv \cos \theta\). Single-particle potentials \(\{\Omega, \delta\}\) and spin- dependent interactions \(\{g_{\sigma \sigma'}\}\) respectively optimize \(\theta \) as \(\theta_{sp}\) and \(\theta_{int}\). A ‘spin twist’ occurs when \(\theta_{sp} \neq \theta_{int}\) and thus \(S_{sp} \neq S_{int}\). (b) Density-tuned polarization from \(S_{sp} = \langle \gamma \rangle\) to \(S_{int} = \langle \beta \rangle\). Dashed and solid lines respectively show mean-field and total (with LHY correction) results. (c) Effective interaction (black solid line) and its individual contribution from mean-field (blue dashed) and LHY (red dot) sectors. The spin twist leads to an additional mean-field repulsion in low-density regime \((\sim (\gamma - \beta)^2)\), as marked by blue vertical line, which uniquely stabilizes the gas state. (d) Energy per particle (shifted by single-particle energy \(-\sqrt{\beta^2 + \delta^2}\)) as a function of density, where the double minima indicate liquid-gas coexistence near their first-order transition. In (b,c,d) we take parameters \((\alpha, \beta, \gamma, \eta) = (-0.1, 0.2, 0.5, 0.0137),\) and scale the density and energy per particle respectively by \(\Omega/g_0\) and \(\Omega\).

Our results can be readily detected in current cold atoms experiments, and the spin twist scheme can serve as a general tool to engineer liquid-gas transition and coexistence in a wide class of quantum systems with competing single-particle and interaction orders.

**Model.** We consider the binary bosons \((\uparrow, \downarrow)\) with Hamiltonian \(H = H_0 + U: (h = 1)\)

\[
H_0 = \int d\vec{r} \sum_{\sigma \sigma'} \psi_{\sigma}^\dagger(\vec{r})(-\Delta_{\sigma \sigma'} - [\Omega_{\sigma x} + \delta_{\sigma z}]_{\sigma \sigma'}) \psi_{\sigma}(\vec{r});
\]

\[
U = \frac{1}{2} \int d\vec{r} \sum_{\sigma \sigma'} g_{\sigma \sigma'} \psi_{\sigma}^\dagger(\vec{r}) \psi_{\sigma}^\dagger(\vec{r}) \psi_{\sigma z}(\vec{r}) \psi_{\sigma}(\vec{r}).
\]

Here \(\psi_{\sigma}^\dagger(\vec{r})\) is the creation operator of spin-\(\sigma\), and \(\sigma_i (i = x, y, z)\) are Pauli matrices; \(\Omega\) and \(\delta\) are respectively the strengths of Rabi coupling and magnetic detuning; \(g_{\sigma \sigma'}\) is the contact coupling strength between \(\sigma\) and \(\sigma'\), and here we consider \(g_{\uparrow \uparrow}, g_{\downarrow \downarrow} > 0\) and \(\delta g \equiv g_{\uparrow \uparrow} + \sqrt{g_{\uparrow \downarrow} g_{\downarrow \uparrow}} < 0\), where a stable quantum droplet can be supported in the absence of \(\Omega\) and \(\delta[22]\). The multiple parameters in this problem can be recombined into four dimensionless ones:

\[
\alpha \equiv \frac{\delta g}{g_0}; \quad \beta \equiv \frac{g_{\uparrow \downarrow} - g_{\downarrow \uparrow}}{4g_0}; \quad \gamma \equiv \frac{\delta}{\sqrt{\delta^2 + \Omega^2}}; \quad \eta \equiv m^2 \Omega g_0^2;
\]

with \(g_0 \equiv (g_{\uparrow \uparrow} + g_{\downarrow \downarrow} - 2g_{\uparrow \downarrow})/4\). Here \(\alpha\) characterizes the strength of overall attractive interaction; \(\beta\) and \(\gamma\), as shown later, stand for the optimal spin polarizations in interaction and single-particle channels, respectively; \(\eta\) measures the Rabi field with respect to interaction strength. To simplify the discussions, in this work we shall mainly consider the effects of tunable \(\alpha\) and \(\gamma\) while keep other parameters fixed.

**Spin twist and the induced effective repulsion.** Under the mean-field treatment, we replace the field operators by classical numbers: \(\psi_r = \sqrt{n} \cos(\theta/2), \psi_i = \sqrt{n} \sin(\theta/2),\) where \(n\) is the total density, and \(\theta\) determines the spin polarization

\[
S \equiv \frac{n_{\uparrow} - n_{\downarrow}}{n_{\uparrow} + n_{\downarrow}} = \cos \theta. \quad (2)
\]

The mean-field energy per volume, \(\epsilon_{mf} = E_{mf}/V\), is given by

\[
\epsilon_{mf} = -(\sqrt{1 - S^2} \Omega + S \delta)n + g_0 n^2 \frac{1}{2} \frac{(S - \beta)^2 + g_{\uparrow \uparrow} g_{\downarrow \downarrow} - g_{\uparrow \downarrow}^2}{4g_0^2}.
\]

Clearly, the first term contributed from single-particle potentials favors spin polarization \(S_{sp} = \gamma\), while the second term from interactions favors \(S_{int} = \beta\). A “spin twist” occurs when the two polarizations mismatch, i.e., \(\beta \neq \gamma\). The overall mean-field polarization, as determined by \(\partial \epsilon_{mf}/\partial S = 0\), is shown by dashed line in Fig.1(b), which is density-dependent and changes from \(\gamma\) to \(\beta\) as \(n\) increases.

A remarkable effect of such spin twist is to induce an effective repulsion uniquely in low-density limit. Here we define the effective interaction as

\[
g_{eff} \equiv \frac{\partial^2 \epsilon}{\partial n^2}, \quad (4)
\]

where \(\epsilon\) is the energy density after optimizing \(S\). In the absence of spin twist, the two terms in Eq.3 both favor \(S = \beta = \gamma\), and then we have \(g_{eff}^{(0)} = (g_{\uparrow \uparrow} g_{\downarrow \downarrow} - g_{\uparrow \downarrow}^2)/(4g_0)\). This is the conventional case of binary bosons, whose mean-field stability is given by \(g_{\uparrow \uparrow} g_{\downarrow \downarrow} > g_{\uparrow \downarrow}^2\) for any density. However, it is no longer true when spin twist occurs (\(\beta \neq \gamma\)). In this case, the single-particle and interaction terms compete with each other and the resulted \(S\) and \(g_{eff, mf}\) are generally non-dependent. In the low \(n\) limit, the single-particle terms dominate and lead to \(S \sim \gamma\), which results in the effective interaction

\[
g_{eff, mf} = g_{eff}^{(0)} + g_0 (\gamma - \beta)^2. \quad (5)
\]

Here we can see that the spin twist leads to an additional repulsion \(\sim g_0 (\gamma - \beta)^2\) at the mean-field level. Physically,
such repulsion originates from an excited spin orientation in interaction channel, as pinned by single-particle potentials. As a result, it should only work for low densities but not for high ones, where the interactions dominate and recover $S \sim \beta$ and $g_{\text{eff}, \text{mf}} \sim g_{\text{eff}, \text{int}}^{(0)}$.

Beyond the mean-field treatment, we have further carried out the Bogoliubov analysis and extracted the Lee-Huang-Yang (LHY) energy $\epsilon_{\text{LHY}}$ from quantum fluctuations[28]. In the low $n$ limit, we have adopted a second-order perturbation theory to obtain the leading order ($\sim n^2$) contribution to $\epsilon_{\text{LHY}}$[28], which well reproduces previous expansion in the case of $\beta = \gamma[26]$.

In Fig.1(c), we plot out the typical effective interaction $g_{\text{eff}}$ obtained from the total $\epsilon = \epsilon_{\text{mf}} + \epsilon_{\text{LHY}}$ as a function of $n$, as well as its individual contributions from mean-field and LHY parts. As expected, the mean-field contribution $g_{\text{eff}, \text{mf}}$ is positive only in low $n$ limit, and gradually reduces to a negative value as $n$ increases. The reduction is exactly given by $\sim g_0(\gamma - \beta)^2$ due to spin twist (Eq.5). In comparison, the LHY contribution $g_{\text{eff}, \text{LHY}}$ is always positive and continuously grows with $n$. The total $g_{\text{eff}}$ then shows intriguing non-monotonic behavior: as increasing $n$, it turns from positive to negative, and back to positive again at large $n$. Consequently, the energy per particle $\epsilon/n$ as a function of $n$ displays double minima, see Fig.1(d). Here the second minimum at finite $n$ is a self-bound droplet with zero pressure that is balanced by mean-field attraction and LHY repulsion, sharing the same spirit as the quantum droplet in ordinary case[22]. While the first minimum at $n = 0$, which has not shown up in previous studies, stands for a stable gas phase. Its stabilization is purely due to the unique low-$n$ repulsion generated by spin twist effect (c.f. Eq.5).

The double minima structure in Fig.1(d) implies a first-order transition between liquid and gas as well as their coexistence under proper conditions, as discussed below. Alternatively, for the case without spin twist, i.e., $\beta = \gamma$, we have checked that $\epsilon/n \sim n$ displays no double minima[28].

**Ground state phase diagram.** The ground state of the system (gas or self-bound droplet) is given by the global minimum in $\epsilon/n \sim n$ curve, which is associated with the lowest chemical potential $\mu = \partial \epsilon/\partial n = \epsilon/n$. In Fig.2, we present the ground state phase diagram in $(\alpha, \gamma)$ plane for a given set of $\beta, \eta$. Four phases are shown, i.e., pure droplet (I), droplet with metastable gas (II), gas with metastable droplet (III), and pure gas (IV). Typical $\epsilon/n \sim n$ landscapes for different regions are given in the inset plot. One can see that the double minimum structure appears in regions II and III, and the liquid-gas transition occurs at the II-III boundary when the two phases have the same $\epsilon/n = \mu = -\sqrt{\Omega^2 + \delta^2}$, i.e., the single-particle shift. Clearly, the transition can be conveniently tuned by single-particle potentials ($\gamma$) and interaction strengths ($\alpha$).

We would like to remark a crucial difference between the liquid-gas transition here and those observed previously in binary bosons[17–21]. In previous cases, the transition is driven by the enhanced quantum pressure as the boson number $N$ decays due to atom loss, and therefore it occurs for finite-size systems when $N$ reaches a finite critical value. However, in our system the transition occurs in the thermodynamic limit ($N,V \to \infty$ with fixed $n = N/V$) and is driven by the competition between single-particle and interaction potentials. Therefore, the current case allows a highly tunable transition point for an arbitrarily large system, and moreover, allows the exploration of LGC in a considerably broad parameter regime, as shown below.

**Liquid-gas coexistence.** We now analyze the feasibility of LGC for bosons confined in a trap. We consider a realistic system of $^{39}$K atoms with hyperfine states $|F = 1, m_F = -1\rangle \equiv | \uparrow \rangle$, $|F = 1, m_F = 0\rangle \equiv | \downarrow \rangle$, which has been well studied in ultracold droplet experiments[17–19]. In this system, $a_{\uparrow \uparrow} = 35a_B$, $a_{\downarrow \downarrow} = -53a_B$ ($a_B$ is the Bohr radius), and $a_{\uparrow \downarrow}$ is highly tunable by magnetic field. For a concrete demonstration, here we take $a_{\uparrow \downarrow} = 64a_B$, $\Omega = 7\pi\text{kHz}$ (thus $\alpha, \beta, \eta$ are all fixed), and only focus on the LGC phenomenon tuned by $\delta$ (or $\gamma$).

The coexistence of liquid and gas requires

$$\mu(n_L) = \mu(n_G), \quad P(n_L) = P(n_G);$$

where $n_L$ ($n_G$) is the liquid (gas) density, $\mu$ is the chemical potential and $P = \mu n - \epsilon$ is the pressure. In Fig.3(a,b), we plot out $P \sim 1/n$ and $\mu \sim n$ for several typical $\gamma$. We can see that $P \sim 1/n$ curves in Fig.3(a) share similar features as classical $P$-$V$ isotherms hosting LGC[7, 8]. Here we have followed the Maxwell’s construction to identify equilibrium densities of liquid and gas at their coexistence, as marked respectively by squares and

![Figure 2](image-url)
LGC has higher phases when $\gamma < \gamma_0$, and gas and liquid phases are indistinguishable. We further summarize the results in $(\gamma, \mu)$ plane in Fig.3(c), where $\mu_{LGC}$ (black solid line) separates the liquid and gas phases in the regime $\gamma \in (\gamma_0, \gamma_c)$. While for $\gamma < \gamma_0$, only liquid state is present once $\mu$ is above a finite value (dashed line) to support a self-bound droplet at $P = 0$. To observe LGC in practice, we suggest measuring the density profile of bosons under an external harmonic trap, and here for brevity we consider an isotropic trap $V(r) = m\omega^2 r^2/2$. Under the local density approximation $\mu(r) = \mu(0) - V(r)$, we compute the phase and density profiles inside the trap for given atom number $N$ and finally map out a phase diagram in $(\gamma, N)$ plane as Fig.3(d). As expected, LGC occurs within $\gamma \in (\gamma_0, \gamma_c)$ once $N$ is above a critical $N_c$ (solid line), at which point $\mu(0)$ reaches $\mu_{LGC}$ and a liquid phase starts to emerge at the trap center. In this case, LGC is characterized by a discontinuous density jump from $n_L$ to $n_G$ inside the trap, as shown by the inset of Fig.3(d).

![FIG. 3. Liquid-gas coexistence (LGC) tuned by $\gamma$ at fixed $\alpha = -0.11, \beta = 0.141, \eta = 0.0157$. (a) Pressure $P$ as a function of $1/n$ for different $\gamma = 0.465(=\gamma_0), 0.6, 0.683(=\gamma_c)$ (from bottom to top). (b) Shifted chemical potential $\Delta \mu \equiv \mu + \sqrt{\Omega^2 + \delta^2}$ as a function of $n$ for different $\gamma$ as in (a). The intersections between these curves and horizontal lines in (a,b) give the equilibrium densities of liquid ($n_L$, squares) and gas ($n_G$, circles), which are connected by binodal lines (dashed). For each curve in (a), the two shadow regions have the same area following the Maxwell’s construction. (c) Phase diagram of liquid, gas and vacuum in the $(\gamma, L)$ plane. LGC occurs along the black line for $\gamma \in (\gamma_0, \gamma_0, \gamma_c)$. For $\gamma > \gamma_c$, the liquid and gas are indistinguishable. (d) Phases tuned by $\gamma$ and boson number $N$ in an isotropic harmonic trap with frequency $\omega = 50(2\pi)$Hz. LGC occurs in the colored region and can be characterized by a sharp discontinuity in the density profile (see inset for example). For all plots, we scale the density and energy per particle respectively by $\Omega/g_0$ and $\Omega$.](image)

![FIG. 4. Universal critical scaling for the relative (a) and averaged (b) densities of liquid and gas at their coexistence. $\beta, \eta$ are the same as in Fig.3. Discrete points show numerical data and dashed lines show linear fittings. Here we take ln-ln plot, and the slopes for all fitting lines in (a) are 1/2 and in (b) are 1, giving the according exponents $\lambda$ and $\xi$ defined in Eq.(7).](image)

Interestingly, the liquid and gas densities show universal scaling near the critical melting of LGC. Here we explore the asymptotic power-law behavior of their relative and mean densities near $\gamma \sim \gamma_c$ and $n_L \sim n_G \sim n_c$:

$$\frac{n_L - n_G}{n_c} \sim (\gamma_c - \gamma)^{\lambda}; \quad \frac{n_L + n_G}{2n_c} - 1 \sim (\gamma_c - \gamma)^{\xi}, \tag{7}$$

with $\lambda, \xi$ the according critical exponents. In Fig.4(a,b), we have numerically extracted the exponents as $\lambda = 1/2$ and $\xi = 1$, which are found to be universal for all different $\alpha$. In fact, these universal exponents persist when switching to other tunable parameters, such as changing $\gamma$ to $\alpha/2$. To explain such universal phenomenon, we have adopted a mean-field theory as in the classical treatment of liquid-gas transition at finite temperature[1], which well predicts the universal critical exponents as above[28].

Summary. We have revealed a new mechanism using spin twist to engineer liquid-gas transition and coexis-
ence in boson systems, which does not rely on finite temperature or long-range interaction and thus is well beyond the traditional frame. We have demonstrated the scheme for a specific model of binary bosons under Rabi coupling and magnetic detuning, and the proposed liquid-gas phase separation and the universal scaling of equilibrium densities can be easily tested in current cold atoms experiment. Finally, we remark that the spin twist in creating an effective low-density repulsion for gas stabilization is a very robust mechanism, which can be applied to a wide class of quantum systems with competing single-particle and interaction orders. For instance, it is expected to still work when change $\Omega, \delta$ to other single-particle potentials in altering spins, such as the spin-orbit coupling. In this regard, the spin twist can serve as a general principle for achieving liquid-gas transition and coexistence at ultra-low temperatures, which hopefully would promote the practical use of such phenomena in a fascinating quantum world in future.

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[28] See supplementary materials for more details.
Supplementary Materials

I. BEYOND-MEAN-FIELD CORRECTION

A. The Bogoliubov method and LHY correction

The Hamiltonian in the momentum space is $H = H_0 + U$, where

$$H_0 = \sum_k \left[ e_k^0 (\psi_{\uparrow, k}^\dagger \psi_{\uparrow, k} + \psi_{\downarrow, k}^\dagger \psi_{\downarrow, k}) + \Omega (\psi_{\uparrow, k}^\dagger \psi_{\downarrow, k, k} + \psi_{\downarrow, k}^\dagger \psi_{\uparrow, k, k}) - \delta (\psi_{\uparrow, k}^\dagger \psi_{\uparrow, k} - \psi_{\downarrow, k}^\dagger \psi_{\downarrow, k}) \right];$$

$$U = \frac{1}{2V} \sum_{q,p,k} \sum_{\sigma \sigma'} \left[ g_{\sigma \sigma'} \psi_{\sigma, q + k}^\dagger \psi_{\sigma', q - k}^\dagger \psi_{\sigma', q + p} \psi_{\sigma, q - p} \right].$$  \hspace{1cm} (S1)

According to the standard Bogoliubov method, we have the quadratic Hamiltonian

$$\frac{H}{V} = \frac{E_{mf}}{V} + \frac{1}{2V} \sum_{\sigma \sigma'} \sum_{k \neq 0} g_{\sigma \sigma'}^2 n_{\sigma} n_{\sigma'} \sum_{k \neq 0} \frac{1}{2 \epsilon_k} - \frac{1}{2V} \sum_{k \neq 0} \left( 2e_k^0 + \Omega \frac{(n_{\uparrow} + n_{\downarrow})}{\sqrt{n_{\uparrow} n_{\downarrow}}} + g_{\uparrow} n_{\uparrow} + g_{\downarrow} n_{\downarrow} \right) + \frac{1}{2V} \sum_{k \neq 0} A_k^\dagger H_{Bog} A, \hspace{1cm} (S2)$$

where $A_k^\dagger = (\psi_{\uparrow, k}^\dagger, \psi_{\downarrow, k}^\dagger, \psi_{\uparrow, -k}^\dagger, \psi_{\downarrow, -k}^\dagger)$, and

$$H_{Bog} = \begin{pmatrix}
    e_k^0 + \Omega \sqrt{\frac{n_{\uparrow}}{n_{\downarrow}}} + g_{\uparrow} n_{\uparrow} & g_{\downarrow} n_{\uparrow} & g_{\uparrow} \sqrt{n_{\uparrow} n_{\downarrow}} - \Omega & g_{\downarrow} \sqrt{n_{\uparrow} n_{\downarrow}} - \Omega \\
    g_{\downarrow} n_{\uparrow} & e_k^0 + \Omega \sqrt{\frac{n_{\uparrow}}{n_{\downarrow}}} + g_{\downarrow} n_{\downarrow} & g_{\uparrow} \sqrt{n_{\uparrow} n_{\downarrow}} - \Omega & g_{\downarrow} \sqrt{n_{\uparrow} n_{\downarrow}} - \Omega \\
    g_{\downarrow} \sqrt{n_{\uparrow} n_{\downarrow}} - \Omega & g_{\uparrow} \sqrt{n_{\uparrow} n_{\downarrow}} - \Omega & e_k^0 + \Omega \sqrt{\frac{n_{\downarrow}}{n_{\uparrow}}} + g_{\uparrow} n_{\downarrow} & g_{\downarrow} n_{\downarrow} \\
    g_{\downarrow} \sqrt{n_{\uparrow} n_{\downarrow}} - \Omega & g_{\uparrow} \sqrt{n_{\uparrow} n_{\downarrow}} - \Omega & g_{\downarrow} n_{\downarrow} & e_k^0 + \Omega \sqrt{\frac{n_{\downarrow}}{n_{\uparrow}}} + g_{\downarrow} n_{\downarrow}
\end{pmatrix}. \hspace{1cm} (S3)$$

Under the Bogoliubov transformation, $H_{Bog}$ is diagonal in the basis of quasi-particle operators $\{b_{\pm, k}\}$,

$$\frac{1}{2} A_k^\dagger H_{Bog} A = \frac{1}{2} \mathcal{E}_{+, k} \left( b_{+, k}^\dagger b_{+, k} + b_{-, k}^\dagger b_{-, k} \right) + \frac{1}{2} \mathcal{E}_{-, k} \left( b_{-, k}^\dagger b_{-, k} + b_{+, k}^\dagger b_{+, k} \right). \hspace{1cm} (S4)$$

The Bogoliubov modes $\mathcal{E}_{\pm}$ satisfy

$$\left\| H_{Bog} - \mathcal{E} \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} \right\| = 0. \hspace{1cm} (S5)$$

This gives

$$\mathcal{E}_{(\pm), k} = \sqrt{D_k \pm \sqrt{D_k^2 - c_k^0 \left( e_k^0 + \Omega \sqrt{\frac{n_{\uparrow}}{n_{\downarrow}}} \right) \left( (e_k^0 + 2g_{\uparrow} n_{\uparrow} + \Omega \sqrt{\frac{n_{\downarrow}}{n_{\uparrow}}}) (e_k^0 + 2g_{\downarrow} n_{\downarrow} + \Omega \sqrt{\frac{n_{\uparrow}}{n_{\downarrow}}}) - (2g_{\uparrow} \sqrt{n_{\uparrow} n_{\downarrow}} - \Omega)^2 \right)}} \hspace{1cm} (S6)$$

with

$$D_k = \frac{1}{2} \left( e_k^0 + g_{\uparrow} n_{\uparrow} + \Omega \sqrt{\frac{n_{\uparrow}}{n_{\downarrow}}} \right)^2 + \frac{1}{2} \left( e_k^0 + g_{\downarrow} n_{\downarrow} + \Omega \sqrt{\frac{n_{\downarrow}}{n_{\uparrow}}} \right)^2 + \left( g_{\uparrow} \sqrt{n_{\uparrow} n_{\downarrow}} - \Omega \right)^2 - \frac{1}{2} \sum_{\sigma \sigma'} g_{\sigma \sigma'}^2 n_{\sigma} n_{\sigma'}. \hspace{1cm} (S7)$$

The total energy density that includes the quantum fluctuation can be obtained from Eq.(S2) and (S4) as

$$\frac{E}{V} = \frac{E_{mf}}{V} + \frac{1}{2V} \sum_{k \neq 0} \left( \sum_{\sigma \sigma'} g_{\sigma \sigma'}^2 n_{\sigma} n_{\sigma'} \frac{1}{2 \epsilon_k} - \frac{2e_k^0 + \Omega \frac{(n_{\uparrow} + n_{\downarrow})}{\sqrt{n_{\uparrow} n_{\downarrow}}} + g_{\uparrow} n_{\uparrow} + g_{\downarrow} n_{\downarrow}}{2} + \mathcal{E}_{+, k} + \mathcal{E}_{-, k} \right). \hspace{1cm} (S8)$$
In the large $k$ limit, $E_{\pm,k}$ has the form $\sqrt{c_1^2 + c_0^2} + c_0 \pm \sqrt{c_1^2 + c_0^2 + c_3}$, where $c, c_0, c_1, c_2,$ and $c_3$ are lengthy expressions extracted from Eq. (S6), e.g., $c = g_{11}n_1 + g_{11}n_\uparrow + \Omega^{n_1n_\uparrow}/\sqrt{n_1n_\uparrow}$, $c_0 = (g_{11} + 2g_{11})\Omega^\sqrt{n_1n_\uparrow} + (n_1 + n_\uparrow)^2\Omega^2$, and $c_1 = 4c_0 - c^2 + 2\sum_{\sigma\sigma'}g_{\sigma\sigma'n_\sigma n_{\sigma'}}$. Then we have $E_{+k} + E_{-k} \sim 2E^0_0 + c + (c_0 - \frac{c^2}{4} - \frac{c_1}{4})/c_{\text{eff}} + O(\frac{1}{c_{\text{eff}}^2})$, and therefore the ultraviolet divergence for the integration of various terms in Eq. (S8) can be exactly cancelled.

Given $E(n_1, n_\uparrow) = E(n, S)$ (here $S = \cos \theta$ is the spin polarization), one can find the optimal spin polarization via $\partial E(n, S)/\partial S = 0$. In practice, we set $\delta \Omega = 0$ in numerically calculating quantum fluctuations in order to avoid the complex excitation spectra at small $k$.

**B. The effective two-body interaction at low-density limit**

In the low $n$ limit, the optimal spin polarization is given by the single-particle configuration, $\cos \theta = S_{\text{sp}} = \gamma$. Here we have the single-particle eigen-states:

$$
\begin{pmatrix}
| + \rangle \\
| - \rangle
\end{pmatrix} = \begin{pmatrix}
\cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\
-\sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{pmatrix} \begin{pmatrix}
| \uparrow \rangle \\
| \downarrow \rangle
\end{pmatrix},
$$

(S9)

and in such $\{+, -\}$ basis the original Hamiltonian (S1) can be translated into

$$
H = \sum_k \left[ \left( \epsilon_k^\uparrow - \sqrt{\Omega^2 + \Delta^2} \right) \psi^\dagger_{+k}\psi_{-k} + \left( \epsilon_k^\downarrow + \sqrt{\Omega^2 + \Delta^2} \right) \psi^\dagger_{-k}\psi_{+k} \right] + \frac{1}{2V}\sum_{\{k_1\}} \left\{ g_1 \psi^\dagger_{+k_1}\psi^\dagger_{-k_2}\psi_{+k_3}\psi_{-k_4} + g_2 \psi^\dagger_{+k_1}\psi^\dagger_{+k_2}\psi_{+k_3}\psi_{-k_4} + g_3 \psi^\dagger_{-k_1}\psi^\dagger_{-k_2}\psi_{+k_3}\psi_{-k_4} + g_4 \psi^\dagger_{-k_1}\psi^\dagger_{+k_2}\psi_{+k_3}\psi_{-k_4} + g_5 \psi^\dagger_{+k_1}\psi^\dagger_{-k_2}\psi_{-k_3}\psi_{+k_4} + g_6 \psi^\dagger_{+k_1}\psi^\dagger_{-k_2}\psi_{+k_3}\psi_{+k_4} + g_7 \psi^\dagger_{-k_1}\psi^\dagger_{+k_2}\psi_{+k_3}\psi_{+k_4} + g_8 \psi^\dagger_{-k_1}\psi^\dagger_{-k_2}\psi_{-k_3}\psi_{-k_4} + h.c. \right\},
$$

(S10)

where $\sum_{\{k_1\}}$ includes momentum conservation. The effective two-body interaction under the second-order perturbation is

$$
g^{(2)}_{\text{eff}} = \begin{pmatrix}
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & +
\end{pmatrix},
$$

(S11)

where the vertices are $\begin{pmatrix}+\end{pmatrix} = g_1 = g_{\text{eff, mf}}, \begin{pmatrix}-\end{pmatrix} = g_5 = g_0 \sin^2 \theta,$ and $\begin{pmatrix}\gamma\end{pmatrix} = g_6/\sqrt{2} = -\sqrt{2} \sin \theta (\cos \theta - \beta) g_0$.

The internal lines refer to the single-particle Green’s functions:

$$
G^0_{-+}(p) = 1/(p^0 - \epsilon_p^0 + i0^+) \quad \text{and} \quad G^0_{++}(p) = 1/(p^0 - \epsilon_p^0 - 2\sqrt{\Omega^2 + \Delta^2} + i0^+).
$$

(S12)

With the renormalization relations $g_{\sigma\sigma'} \rightarrow g_{\sigma\sigma'} + g_{\sigma\sigma'}^2 \int \frac{dk}{(2\pi)^3}$, and the fact $\sum_{\sigma\sigma'}g_{\sigma\sigma'n_\sigma n_{\sigma'}/n^2 = g_1^2 + g_2^2 + g_6^2/2$, we can straightforwardly perform

$$
g^{(2)}_{\text{eff}} = g_{\text{eff, mf}} + g_1^2 \int \frac{dk}{(2\pi)^3} \left( \int \frac{dk}{2\pi} G^0_{-+}(k)G^0_{-+}(-k) + \frac{1}{2\epsilon_k^0} \right)
$$

(S13)

and obtain the effective interaction

$$
g^{(2)}_{\text{eff}} = g_{\text{eff, mf}} + \frac{(1 - \gamma^2)^{3/2}}{2\pi} \left[ 1 - \gamma^2 + \sqrt{2(\gamma - \beta)^2} \right] n^{1/2} g_0.
$$

(S14)

The difference $g^{(2)}_{\text{eff}} - g_{\text{eff, mf}}$ is just contributed from the quantum fluctuations (or Lee-Huang-Yang(LHY) corrections), as shown by the red dot line in Fig.1(c) and Fig.S1 at $n \rightarrow 0$. In the special case $\beta = \gamma$, this part reproduces the LHY-induced two-body interaction as discussed in Ref.[26]. Here the existence of (meta)stable gas requires $g^{(2)}_{\text{eff}} > 0$, and the boundary $g^{(2)}_{\text{eff}} = 0$ is shown as the black dashed line in Fig.2 (separating regions I and II).
C. Spin twist and double minima structure

To demonstrate the vital role of spin twist played in the liquid-gas coexistence, in Fig. S1 we compare the $\epsilon/n \sim n$ curves with and without spin twist. Fig. S1(a,b,c) are for the case with spin twist ($\beta \neq \gamma$), which are identical to Fig. 1(b,c,d) in the main text. It shows that the spin twist can result in an additional mean-field repulsion in the low-$n$ regime, which uniquely stabilizes the gas state and facilitates the liquid-gas coexistence in view of the double minima structure of $\epsilon/n \sim n$ curve. In comparison, Fig. S1(d,e,f) are for the case without spin twist ($\beta = \gamma$), where the mean-field contributions to $S$ and $g_{\text{eff}}$ are both static for all densities and the resulted $\epsilon/n \sim n$ can only show one minimum (representing either liquid or gas) but not two. In the latter case, the liquid and gas cannot be (locally) stable simultaneously and thus cannot coexist.

Fig. S1. The polarization, effective interaction and energy per particle curves, with (upper panel) and without (lower panel) spin-twist. The case of spin twist is identical to that shown in Fig. 1(b,c,d). Without spin-twist, the total effective interaction $g_{\text{eff}}$ always increases monotonically, and the energy per particle $\epsilon/n$ presents only one minimum, namely gas or droplet state. $\alpha$ and $\eta$ are the same as in Fig. 1.

II. CRITICAL EXPONENTS NEAR THE MELTING OF LIQUID-GAS COEXISTENCE

Inspired by the classical treatment of temperature-driven liquid-gas coexistence in Ref. [1], here we introduce the mixed thermodynamic function $W(x, \mu, n) = \epsilon(x, \mu, n) - \mu n$, which determines the equilibrium state via $\partial W/\partial n = 0$ and the coexistence critical point via $\partial^2 W(x_c, \mu_c, n_c)/\partial n^2 = \partial^3 W(x_c, \mu_c, n_c)/\partial n^3 = 0$. The variable $x$ here can represent any tunable parameter we are interested in, such as $\gamma$ or $\alpha$ in this work.

We expand $W(x, \mu, n)$ near the critical point to the fourth-order of $(n - n_c)$:

$$W(x, \mu, n) = W(x, \mu, n_c) + \sum_{i=1}^{i=4} w_i (n - n_c)^i,$$

where $w_4(x, \mu, n_c)$ is finite, and $w_1, 2, 3(x, \mu, n_c)$ is close to zero and can be expanded in terms of $(x - x_c)$ and $(\mu - \mu_c)$,

$$w_1 = \frac{\partial W(x, \mu, n_c)}{\partial n} = \frac{\partial^2 \epsilon(x_c, n_c)}{\partial n \partial x}(x - x_c) - (\mu - \mu_c), \quad w_{i=2,3} = \frac{\partial^i W(x, \mu, n_c)}{\partial n^i} = \frac{\partial^{i+1} \epsilon(x_c, n_c)}{\partial n^{i+1}}(x - x_c).$$

Denoting $\phi = (n_L - n_G)/2$ and $\phi_0 = (n_L + n_G)/2 - n_c$, we have

$$n_L - n_c = \phi_0 + \phi \quad \text{and} \quad n_G - n_c = \phi_0 - \phi.$$

Substituting Eq. (S17) into Eq. (S15), and rearranging the expansion in terms of $\phi^i$, we have

$$W(x, \mu, n_L) = W(x, \mu, n_c) + \sum_{i=1}^{i=4} \left( w_i \phi_0^i + \hat{w}_i \phi^i \right),$$

$$W(x, \mu, n_G) = W(x, \mu, n_c) + \sum_{i=1}^{i=4} \left( w_i \phi_0^i + (-1)^i \hat{w}_i \phi^i \right).$$
where the coefficients are

\[
\begin{align*}
\tilde{w}_1 &= w_1 + 2w_2\phi_0 + 3w_3\phi_0^2 + 4w_4\phi_0^3, \\
\tilde{w}_2 &= w_2 + 3w_3\phi_0 + 6w_4\phi_0^2, \\
\tilde{w}_3 &= w_3 + 4w_4\phi_0, \\
\tilde{w}_4 &= w_4,
\end{align*}
\]  

(S19)

At coexistence, \(n_L\) and \(n_G\) share the same pressure \(P(n_L) = P(n_G) \equiv -W(x, \mu, n)_{\text{min}}\). This means that for Eq. (S18) with a given \(\phi_0\), we have \(W(x, \mu, n_L) = W(x, \mu, n_G)\). Therefore, the terms of the odd-order of \(\phi\) are absent in Eq. (S18), i.e., \(\tilde{w}_1 = \tilde{w}_3 = 0\), which leads to

\[
\phi_0 = \frac{-w_3}{4w_4} = \frac{-1}{4w_4} \frac{\partial^4 \epsilon(x_c, n_c)}{\partial n^4 \partial x} (x - x_c).
\]  

(S20)

The function \(W(x, \mu, n)\) in Eq. (S18) is then reduced to \(W_0(x, \mu, \phi_0) + \tilde{w}_2\phi^2 + w_4\phi^4\), whose minimum describes the equilibrium state with

\[
\phi^2 = \frac{-\tilde{w}_2}{2w_4} = \frac{-1}{2w_4} \frac{\partial^3 \epsilon(x_c, n_c)}{\partial n^3 \partial x} (x - x_c) + O \left( (x - x_c)^2 \right).
\]  

(S21)

Eq. (S20) and (S21) give the critical exponents \(\xi = 1\) and \(\lambda = 1/2\).

Note that above derivation applies to any tunable parameter \(x\). In the main text we have chosen \(x = \gamma\), while keeping other parameters fixed. In Fig. S2, we choose \(x = \alpha\) as the tunable parameter, and find that the two critical exponents \(\xi = 1\) and \(\lambda = 1/2\) still apply in this case. Therefore, these exponents are universal ones in characterizing the liquid-gas coexistence in our system.

![Fig. S2. Universal critical scaling for the relative (a) and averaged (b) densities of liquid and gas at their coexistence. Here we consider the liquid-gas coexistence tuned by \(\alpha\). The slopes of all fitting lines in the (a) are 1/2 and in (b) are 1, giving the universal exponents \(\lambda = 1/2\) and \(\xi = 1\). \(\beta, \eta\) are the same as in Fig.4.](image-url)