Frequency Dispersion of the Signal in the Recursive Digital Section of the Second Order

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Abstract: The problems of modeling the signal and dispersion properties of a second order recursive section in the integer parameter space are considered. The formulation and solution of the section synthesis problem by selective and dispersive criteria using the methods of integer nonlinear mathematical programming are given. The availability of obtaining both positive and negative frequency dispersion of a signal in a recursive section, as well as the possibility of minimizing dispersion distortions in the system, is shown.

1. Introduction

Digital filtering is one of the most popular digital signal processing algorithms. Digital filters, as devices for frequency discrimination of an input signal, are usually developed based on the requirements for their frequency responses, such as amplitude-frequency response (AFR), phase-frequency (PFR) response, required phase or group delay characteristics. Such synthesis is called a multifunctional, that is, the synthesis of filters with all the required characteristics, in contrast to the synthesis with only frequency response.

As is well known, recursive digital filters (IIR filters) have the highest capabilities in meeting complex selective requirements. However, a characteristic feature of linear IIR filters is the fact that they are dispersive discrete systems. The strong frequency dispersion is due to the different time (dispersion) of processing the spectral components of the input broadband signal by a digital filter. As is well known, the processing of spectral components in the system determines formally the average time delay or group delay time (GDT), as a frequency derivative of the filter phase characteristic:

\[ \tau_{gr(v)} = -\frac{\partial \phi}{\partial \omega} = -\frac{1}{2\pi} \frac{\partial \phi}{\partial \nu}, \quad \omega = 2\pi \nu \]  

At the same time, for a formal estimate of the frequency dispersion of a wideband signal in a digital filter, the dispersion coefficient \( D(\nu) \), can be entered, as the rate of change of the group delay time vs frequency:

\[ D(\nu) = \frac{\partial \tau_{gr}}{\partial \omega} = \frac{1}{2\pi} \frac{\partial \tau_{gr}}{\partial \nu} \quad [s/Hz] \]  

Thus, the group delay and the dispersion are a convenient quantitative measure of the change, modification of the filter phase response:

\[ \varphi(\omega) = \arg H(e^{j\omega}) \],

where \( H(e^{j\omega}) \) is the filter frequency gain.

As can be seen, the dispersion coefficient can quantify very small phase deviations.

It is obvious that the introduction of the frequency dispersion coefficient can be justified only if it is possible to efficiently control the dispersion properties of a digital filter at the design stage. At the level of the standard classical calculation, when an analytical representation of functional relationships is necessary at each stage, starting naturally with an analytical approximation of the required
characteristics, it is impossible to design an IIR filter according to a given law of variation of its
dispersion (or a given dispersion characteristic) due to extreme complexity of analytical calculations.
Thus, an analytical calculation of digital filters or phase correctors, taking into account the possibility
of providing the required phase linearity, minimum phase $\Delta \varphi(\omega)$ distortions already results in
considerable mathematical difficulties. If it is necessary to implement the complex law of phase
variation, then it is impossible to achieve this by analytical calculation. Therefore, IIR filters designed
by analytical calculation, for example, by the bilinear transformation of an analog prototype, can have
phase distortions in the filter bandwidth in tens of degrees [1, 2], which, in terms of dispersion,
determines its extreme value, reaching for the audio-frequency band of some ms/Hz. Such a high
dispersion, of course, will result in a very significant distortion of the signal waveform of the broadband
signal processed by the filter. It may also be noted that the real data presentation format requires
quantizing their values, which leads to very undesirable consequences for such an IIR filter, such as
frequency response distortion, quantization distortion, essential scaling of real filter coefficients, and
possible small limit cycles when quantizing the results of internal intermediate calculations. In addition,
the real design solution can only be implemented using specialized signal processors, whereas the most
promising today programmable logic, as well as microprocessor controllers, require fundamentally
integer solutions and integer calculation arithmetic.

It is obvious that both analyzing and controlling the dispersion characteristic of a digital filter can be
effective only by using modern numerical methods of computer simulation, allowing to work not with
an analytical, but with a discrete representation of the characteristics of the filter, when both the initial
required and current characteristics are tabulated with a given discreteness of their representation in the
frequency domain and in the computing system represented by two-dimensional real arrays (vectors).

One of the most effective numerical methods of designing is discrete programming [3,4]. Discrete
programming methods allow to:
- significantly improve the accuracy of the representation of the desired filter characteristics at the
  initial stage of the synthesis by appropriately selecting the frequency sampling rate, followed by simple
digitization in the functional editor of the synthesis program. The accuracy of the representation means
here the value compared with the error of the analytical approximation of the characteristic: because
with a complex form of the required characteristic, the error of its approximation by polynomials of an
acceptable order can reach tens of percent, while the error of its frequency sampling and simple
digitization can be very small. It must also be remembered that the analytical approximation of a
complex frequency response usually also results in a sharp increase of phase distortion;
- provide multifunctional synthesis of a device, since there may be a number of required
  characteristics, whereas in analytical calculation it is possible to fulfill the requirements for only one
approximated characteristic of the IIR filter;
- the discrete representation makes it easy to calculate all the required characteristics using numerical
  methods of integration or derivation, including the dispersion characteristic of the filter. And this, in
  turn, makes it possible to set the task of designing digital filters, taking into account the required
dispersion properties, including digital phase correctors and frequency dispersion compensators;
- discrete programming methods allow the synthesis of digital IIR filters directly in the integer state
  space. In this case, the state space is understood primarily as the multidimensional space of integer
parameters (filter coefficients), input and output signals, i.e. integer time sequences, as well as basic
integer operations on data in the digital filtering algorithm. For example, the method of integer nonlinear
programming (INP) allows you to design effectively integer recursive filters with a given data
representation word length and maximum fulfillment of the requirements for the set of frequency
characteristics of the filter with an arbitrary form of their specification [5, 6]. An important advantage
of integer digital filters (IDF) is the absence of a quantization procedure (both for filter coefficients and
intermediate integer calculations) in the course of integer calculating the filter response in real time, and,
therefore, the absence of negative data quantization effects listed above.

This paper discusses the issues of modeling signal and dispersion properties of a recursive section of
the second order, as well as the formulation and solution of the problem of its synthesis according to
selective and dispersive criteria in the integer space of filter coefficients.
Simulation of a recursive integer section and formulation of the synthesis problem

For a recursive integer section of the second order, the transfer function has the following form [5]:

\[ H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{a_0 + a_1 z^{-1} + a_2 z^{-2}}. \]  

(1)

where the complex variable

\[ Z = e^{j\omega}, \quad \omega = \frac{2\pi f}{f_s} \]

is the digital frequency, and \( f_s \) is the sampling rate of the input signal.

All the coefficients of the system function (1) are integer, and their variation range is determined by the assigned word length of filter coefficients. The standard condition of stability is the absence of \( p_i \) poles of the transfer function outside the unit circle in the \( z \)-plane:

\[ |Zp_i| < 1. \]  

(2)

The difference equation of a filter section:

\[ y_n = \left( b_0 x_n + b_1 x_{n-1} + b_2 x_{n-2} - a_1 y_{n-1} - a_2 y_{n-2} \right)/a_0, \]  

(3)

where \( x_n \) is the input integer time sequence, \( y_n \) is the output sequence.

As can be seen, when calculating the filter response, the division by the integer coefficient \( a_0 \) should be performed, which can be implemented by the bit-shift operation provided that each \( i \)-th normalizing factor belongs to the integer binomial series (series of \( 2^q \)):

\[ a_{0i} \in \left\{ 2^q \right\}, \quad q = 0, W_k - 1 \quad i = 1, m, \]  

(4)

where \( W_k \) is the bit word length of filter integer coefficients, sign including.

Figure 1 shows a typical section structure of a recursive integer filter of the second order corresponding to the difference equation (3). As is seen, when its implementing, the calculation of the filter response \( y_n \) uses (besides conventional addition, multiplication and clock period delay operations) a shift by \( B = \log_2 a_0 \) bits, used to implement integer division by the normalizing factor \( a_0 \).

Fig. 1. Structure of a recursive integer section of the second order

It can be noted that the very final shift operation in the algorithm for calculating the response of a recursive integer section that violates the commutativity of the computational procedure for implementing zeros and poles of its transfer function (1), which does not allow, in turn, to implement an integer recursive section in canonical form.

In general, the problem of integer nonlinear programming for computer synthesis of a recursive integer section can be written as:
\[ F^* (IX^*) = \min F (IX) \quad IX \in I^6 \]

\[-2^{w_{k-1}} < a_i < 2^{w_{k-1}} \]

\[-2^{w_{k-1}} < b_d < 2^{w_{k-1}} \]  

\[ a_0 \in \{2^q\}, \quad q = 0, W_k - 1 \]  

\[ |Z_{ji}| < 1 \]  

where \( IX \) is a vector in a multidimensional integer parameter space (coefficients), \( F(IX) \) is an objective function.

The synthesis extremum problem (5) is written for the integer space \( f^* \) (filter coefficients) of dimension 6. Constraints (6) set the limits of variation of integer coefficients, and relation (7) determines the belonging of the \( a_0 \) coefficients to the binomial series. Functional constraints (8) control the condition of stability of the recursive filter in the synthesis process by the two poles of the transfer coefficient.

The multicriteria objective function is formed as weighted sum (9) of single objective functions \( f(IX) \), which determine fulfilling functional requirements for a particular kind of frequency response of the section:

\[ F(IX) = \sum \beta_i \cdot f_i(IX). \]

The coefficient \( \beta \) sets the significance (weight) of the characteristic. The single objective functions \( f(IX) \) themselves are most often formed by the criterion of mean-square error

\[ f_i(IX) = \frac{1}{P} \sum_{n=1}^{P} \left[ Y_n(IX) - Y_n^T \right]^2, \]

where \( Y_n(IX) \) is the current value of the filter frequency response on the \( n \)-th discrete frequency, and \( Y_n^T \) is the required value of the frequency response.

The iterative search solution of the extremal INP problems (2.10) and (2.16) in a given parameter space is performed by a software algorithmic complex, referring to the model program block for calculating the current functional characteristics of the integer section. The vector \( IX^0 \), which minimizes the scalar objective function \( F(IX) \) on the set of admissible integer solutions (6), is a Pareto-effective solution to the problem of INP synthesizing a recursive integer section of the second order from a set of contradictory characteristics.

Let us consider some typical examples of the synthesis of a recursive section of the second order according to the criteria of the selective requirements of a low-pass filter (LPF) and a given frequency dispersion of the signal in the passband of the filter.

**Synthesis of a recursive section with positive signal dispersion**

During the synthesis of a recursive section of the low-pass filter with the maximum positive signal dispersion in the passband of 0–4 kHz, the objective function was formed as a weighted sum of the single objective functions \( f_{\text{AFR}}(IX) \) and \( f_{\text{DISP}}(IX) \), respectively satisfying the requirements for the required filter selection (Fig. 2a, highlighted in red) and to the signal dispersion in the section bandwidth:

\[ F(IX) = \beta_1 f_{\text{AFR}}(IX) + \beta_2 f_{\text{DISP}}(IX), \]

The single objective function \( f_{\text{AFR}}(IX) \) was formed according to the criterion (10) of the minimum of the root-mean-square error to ensure a given frequency selection, and the dispersion criterion provided the maximum positive signal dispersion in the passband

\[ f_{\text{DISP}}(IX) = (D_{\text{max}})^{-1}, \quad D_{\text{max}} = \max_i D_i, \quad D > 0 \]
Regarding the objective function (11), the problem of integer programming for the synthesis of a recursive section with a data word length of $W_k = 10$ bits is written as:

$$ F^*(IX^*) = \min F(IX) \quad IX \in I^5 $$

$$ -511 \leq a_i \leq 511 $$

$$ -511 \leq b_i \leq 511 $$

$$ a_0 = 512 $$

$$ |Z_{pf}| < 1 $$

Thus, the minimization of the objective functional was carried out on a 5-dimensional integer parameter space in the allowable domain (14) with fixed normalizing coefficient (15) and the functional stability constraints of the recursive section over all transfer coefficient poles (16).

| Optimal coefficients for section with positive dispersion | Table 1 |
|--------------------------------------------------------|--------|
| $a_0$ | $a_1$ | $a_2$ | $b_0$ | $b_1$ | $b_2$ |
| 1024 | -266 | 363 | -247 | -273 | -236 |

Table 1 shows the optimal values of 10-bit integer coefficients in the synthesis of the recursive section of the low-pass filter according to the criterion of maximum positive dispersion. The graphs of its characteristics for a sampling frequency of 10 kHz are shown in Fig. 2.

![Graphs of characteristics](image)

Fig. 2. Responses of the synthesized section a) frequency response, b) phase response c) GDT, d) dispersion
As can be seen from the graphs presented, the error in the implementation of selective requirements was RMS = 0.016 with phase distortions $\Delta \phi(\omega) = 14$ degrees and a monotonous increase in the GDT in the passband, which caused a positive frequency dispersion of the signal at 23 ns/Hz.

**Synthesis of a recursive section with negative signal dispersion**

During the synthesis of a recursive section of the low-pass filter with the maximum negative signal dispersion in the passband, the objective function was formed as a weighted sum (11) of the single objective functions $f_{AFR}(\mathbf{X})$ and $f_{DISP}(\mathbf{X})$, respectively satisfying the requirements for the required filter selection (Fig. 2a, highlighted in red) and to the maximum negative signal dispersion in the bandwidth.

$$f_{MAX}(\mathbf{X}) = (D_{\text{max}})^{-1}, \quad D_{\text{max}} = \max_i |D_i| \quad D < 0$$

(17)

The problem of integer programming for the synthesis of a section with a data representation word length of $W_z = 10$ bits was written similarly to relations (16) - (19).

Optimal coefficients for section with negative dispersion Table 2.

|   | $a_0$ | $a_1$ | $a_2$ | $b_0$ | $b_1$ | $b_2$ |
|---|---|---|---|---|---|---|
|   | 1024 | -908 | 213 | -146 | -289 | 146 |

Table 2 shows the optimal values of the integer 10-bit coefficients of the section's transfer function. The graphs of its characteristics for a sampling frequency of 10 kHz are shown in Fig. 3.

Fig. 3. Responses of the synthesized section a) frequency response, b) phase response c) GDT, d) dispersion
As can be seen from the graphs presented, the RMS error in the implementation of selective requirements was $\text{RMS} = 0.01$ with phase distortions $\Delta \varphi(\omega)=8.4$ degrees and a monotonous decrease in GDT in the passband, which caused a positive frequency dispersion of the signal at 9 ns/Hz.

**Synthesis of a recursive section with minimum signal dispersion**

During the synthesis of a recursive section with minimal dispersion, the modulus of the maximum dispersion coefficient in the passband of a low-pass filter was minimized

$$f_{\text{MAX}}(\mathbf{IX}) = \max |D_i|,$$

and the objective function $f_{\text{AFR}}(\mathbf{IX})$ was formed by the criterion (10) to ensure a given frequency selection.

Optimal coefficients for section with near to zero dispersion Table 3.

| $a_0$ | $a_1$ | $a_2$ | $b_0$ | $b_1$ | $b_2$ |
|------|------|------|------|------|------|
| 1024 | -871 | 248  | -171 | -278 | 139  |

Table 3 shows the optimal values of the integer 10-bit coefficients of the section's transfer function. The graphs of its characteristics for a sampling frequency of 10 kHz are shown in Fig. 4.

As can be seen from the graphs, the MSE of the implementation of the selective requirements was 0.084 with phase distortions $\Delta \varphi(\omega)=3.4$ degrees and the alternating frequency dispersion of the signal with a maximum value of 7 ns/Hz in the 2nd order filter passband.
Conclusion
The results presented in the paper confirm the possibility of synthesis of recursive digital filtering systems with both positive and negative frequency signal dispersion in the operating frequency range by methods of integer nonlinear programming, as well as the possibility of minimizing dispersion distortions in the system. As is well known, the fundamental feature of INP-synthesis is the use of modern numerical methods of machine design, allowing to work not with an analytical, but with a discrete representation of the characteristics of an IIR filter, when the filter characteristics are tabulated with a given discreteness of their representation in the frequency domain and in the computing system represented by two-dimensional arrays (vectors). This makes it possible to calculate the dispersion characteristics of an IIR filter with a given accuracy using numerical methods of differentiating the phase response. On the other hand, this allows to apply effective search methods of discrete programming for the synthesis of a technical solution, allowing to design directly in the integer parameter space. The search criterion for this is that the current functioning of the synthesized filter complies with the required functioning in terms of the set of selective and dispersion responses. It is characteristic that the ability to a given amplitude selection of a signal is maximum with a high positive signal dispersion in the passband of an IIR filter and significantly decreases with a high negative signal dispersion.
Thus, the application of the multifunctional search design method allows to obtain effective design solutions taking into account also the dispersion requirements for a digital IIR or FIR filter [8,9].

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