A Note on Pretzelosity TMD Parton Distribution

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Abstract

We show that the transverse-momentum-dependent parton distribution, called as Pretzelosity function, is zero at any order in perturbation theory of QCD for a single massless quark state. This implies that Pretzelosity function is not factorized with the collinear transversity parton distribution at twist-2, when the struck quark has a large transverse momentum. Pretzelosity function is in fact related to collinear parton distributions defined with twist-4 operators. In reality, Pretzelosity function of a hadron as a bound state of quarks and gluons is not zero. Through an explicit calculation of Pretzelosity function of a quark combined with a gluon nonzero result is found.

Transverse-Momentum-Dependent(TMD) parton distributions contain novel information of three-dimensional structure inside a hadron. These parton distributions can be extracted from high energy processes like Drell-Yan- and SIDIS processes, where differential cross-sections are predicted with TMD parton distributions according to TMD factorization theorems studied in \cite{1,2,3,4}. With recent progresses in \cite{5,6} it is possible to calculate the TMD parton distributions with Lattice QCD.

In general TMD parton distributions can not be calculated with perturbative theory of QCD. However, one can predict properties of TMD paron distributions with perturbative QCD if the struck quark has a large transverse momentum $k_\perp$. E.g., the $k_\perp$-dependence of the TMD unpolarized parton distribution of an unpolarized hadron can be predicted with perturbative QCD in the case of $k_\perp \gg \Lambda_{QCD}$ in \cite{1,3}. In this case, the TMD parton distribution takes a factorized form as a convolution of a perturbative coefficient function with the corresponding parton distribution in collinear factorization. The $k_\perp$-dependence is contained only in the perturbative coefficient function. Assuming the so called Pretzelosity TMD parton distribution of a transversely polarized hadron can be factorized with the transversity parton distribution in collinear factorization, it is found in \cite{7,8} that the perturbative coefficient function is zero at the leading order of $\alpha_s$. From \cite{9} the function is still zero at next-to-leading order.

In this letter we show that the perturbative coefficient function in the matching of Pretzelosity function to the transversity is zero at all orders of perturbative QCD. The matching calculation is in fact the calculation of Pretzelosity function of a transversely polarized quark. Our result also implies that Pretzelosity function of a transversely polarized quark is zero at any order. The interpretation of our result depends on how collinear divergences are regularized. Although Pretzelosity function of a single quark is zero, it does not imply that the function of a hadron is zero. Through an explicit calculation, we show that the function of a state consisting of a quark combined with a gluon is not zero. We also show that Pretzelosity function is factorized with parton distributions defined with twist-4 operators.

We will use the light-cone coordinate system, in which a vector $a^\mu$ is expressed as $a^\mu = (a^+, a^-, \vec{a}_\perp) = ((a^0 + a^3)/\sqrt{2}, (a^0 - a^3)/\sqrt{2}, a^1, a^2)$ and $a^2_\perp = (a^1)^2 + (a^2)^2$. Two light-cone vectors are introduced:
\( n^\mu = (0, 1, 0, 0) \) and \( l^\mu = (1, 0, 0, 0) \). With the two vectors one can define the metric in the transverse space as \( g_{ij} = \delta^{\mu
u} - n^\mu n^\nu - l^\mu l^\nu \).

It is well-known that there are light-cone singularities if one defines TMD parton distributions with gauge links along light-cone directions. We regularize the singularities as in \([11, 13]\) by introducing the gauge link slightly off light-cone direction:

\[
\mathcal{L}_u(\xi) = P \exp \left( -ig_s \int_0^\infty d\lambda \nu \cdot G(\lambda \nu + \xi) \right),
\]

with \( u^\mu = (u^+, u^-, 0, 0) \) and \( u^- \gg u^+ \). With the small- but finite \( u^+ \) light-cone singularities are regularized. There are different regularizations of the singularities, e.g., those in \([10, 11, 12]\). Different regularizations will not affect on our results. The classifications of TMD parton distributions have been studied in \([3, 13, 14, 15, 16, 17]\). We are interested in two TMD parton distributions at leading power or regularizations will not affect on our results.

In the collinear factorization, only one parton distribution at twist-2 is related to the transverse spin. In singular gauges transverse gauge links at \( \sin(3\phi_h - \phi_s) \) asymmetry in SIDIS. Relevant activities in experiment and modelling the function can be found in \([22]\) and references within.

In general, the defined TMD parton distributions in the transverse momentum space contain not only collinear divergences, but also I.R. divergences. The I.R. divergences can be subtracted by introducing a soft factor as shown explicitly in \([11, 13]\). One can also consider the TMD parton distributions in the transverse space or \( b \)-space by Fourier transformations:

\[
h_1(x, b) = \int d^2 k_\perp e^{ib \cdot k_\perp} h_1(x, k_\perp), \quad h_{1T}(x, b) = \int d^2 k_\perp e^{ib \cdot k_\perp} h_{1T}(x, k_\perp),
\]

where \( b \) is a two-dimensional vector \( b^\mu = (b^1, b^2) \). The I.R. divergences in TMD parton distributions defined in \( b \)-space are cancelled.

In the collinear factorization, only one parton distribution at twist-2 is related to the transverse spin. It is the transversity distribution \( q_{T} \) introduced in \([20, 21]\):

\[
\int \frac{d\lambda}{2\pi} e^{-ix\lambda P^+} \langle h(P, S_\perp) | \bar{\psi}(\lambda n) \mathcal{L}_{h}^\dagger(\lambda n) | \mathcal{L}_u(0) \psi(0) | h(P, S_\perp) \rangle = \frac{1}{2N_c} \left( i\gamma_5 \sigma^- S_{\mu} q_T(x) + \cdots \right),
\]

where \( \cdots \) denotes terms which are not related to the transverse spin or beyond twist-2. The gauge link \( \mathcal{L}_n \) here is along the light-cone direction \( n^\mu \). For small \( b \) one expects the factorization, in which TMD parton distributions can be factorized as a convolution of a perturbative coefficient functions with collinear parton distributions. In our case, it is:

\[
h_1(x, b) = \left[ C_1(b) \otimes q_T \right](x), \quad h_{1T}(x, b) = \left[ C_{1T}(b) \otimes q_T \right](x)
\]
where $C_1$ and $C_{1T}^\perp$ are perturbative coefficient functions. They are free from collinear singularities. The matching or the factorization is given in the $b$-space. One can also formulate it in the momentum space. In the case of $k_\perp \gg \Lambda_{QCD}$, one has:

$$h_1^\perp(x, k_\perp) = \left[C_1(k_\perp) \otimes q_T\right](x), \quad h_{1T}^\perp(x, k_\perp) = \left[C_{1T}^\perp(k_\perp) \otimes q_T\right](x).$$

(6)

It is noted that the perturbative coefficient functions $C_1(x, k_\perp)$ and $C_{1T}^\perp(x, k_\perp)$ here are free from collinear singularities but contain I.R. divergences. One can define in the momentum space the subtracted TMD parton distributions by introducing a soft factor. The subtracted TMD parton distributions have the same factorizations as in the above and the corresponding perturbative coefficient functions are finite. We will call them as subtracted perturbative coefficient functions.

Figure 1: Diagrams for contributions from collinear parton distributions to TMD parton distributions. The double lines represent gauge links.

It is interesting to note that the function $C_{1T}^\perp(x, b)$ was found to be zero at leading order of $\alpha_s$ in [8]. Recently it is found that it is still zero at next-to-leading order in [9]. We notice that the definition of Pretzelosity function in [8,9] is slightly different than that in Eq.(2). This difference will not change our conclusions. At tree-level, the contribution to $h_{1T}^\perp(x, k_\perp)$ is from Fig.1, where the black box represents the density matrix in Eq.(4). With this density matrix parton lines connecting the black box, or the struck partons, have zero transverse momenta. It is easy to find that the contribution is zero. Because the projection from the bottom of diagrams in Fig.1 represented by the black box in Eq.(4) is with $i\gamma^5\sigma^{-\mu}S_\perp\mu$, the matching calculation is essentially the calculation of Pretzelosity function of a single transversely polarized quark with the transverse spin $S_\perp$. From the result of the calculation we obtain the perturbative coefficient function by subtracting collinear divergences.

We replace the hadron $h(P, S_\perp)$ in Eq.(2) with a quark $q(p, S_\perp)$. Sandwiching the sum of all intermediate states $\sum_X |X\rangle\langle X| = 1$, we need to calculate the amplitude $\langle X|L^\perp(0)q(0)|q(p, S_\perp)\rangle$ to obtain TMD parton distributions of a single quark state. The amplitude takes the form:

$$\langle X|L^\perp(0)\psi(0)|q(p, S_\perp)\rangle = \Gamma(x, k_\perp, l, n, X) u(p, S_\perp),$$

(7)

where $\Gamma$ is a matrix in Dirac-spinor space. It depends on the vector $k_\perp$, $l$ and $n$. The intermediate state $X$ can contain gluons and pairs of quark and antiquark. Hence, $\Gamma$ also depends on momenta, polarization vectors and spinors of particles in the intermediate state $X$. We denote the dependence on these momenta and wave functions of particles in $|X\rangle$ collectively as $X$. With the amplitude, we obtain the vector $M^\mu$ in Eq.(2):

$$M^\mu(x, k_\perp) = \sum_X \delta(p^+ - x_{p^+} - P^+_X)\delta^2(k_\perp - P_{X\perp})p^+ S_\perp\nu$$
\[ \text{Tr} \left[ i\gamma_5 \sigma^{-\nu} \Gamma(x, k_\perp, l, n, X)i\gamma_5 \sigma^{+\mu} \Gamma(x, k_\perp, l, n, X) \right]. \] (8)

There are contributions to the amplitude in which the initial quark becomes one of partons in \(|X|\) after interactions. These contributions do not contribute to \(M^\mu\) because of helicity conservation of perturbative QCD and that the helicity of the initial quark is flipped. From the perturbation theory of QCD the matrix \(\Gamma\) defined in Eq.(7) contains only terms which are products of \(\gamma\)-matrices of even numbers. Therefore, it can be expanded in the form:

\[
\Gamma(x, k_\perp, l, n, X) = IA(x, k_\perp, l, n, X) + \gamma_5 B(x, k_\perp, l, n, X) + i\sigma^{\alpha\beta} C_{\alpha\beta}(x, k_\perp, l, n, X). \tag{9}
\]

where \(A\) and \(B\) are scalar functions, \(C_{\alpha\beta}\) is a tensor function. In order to have nonzero \(h_{1T}\), the trace in Eq.(5) must have nonzero contributions proportional to \(k_\perp^\mu k_\perp^\nu\). Using Eq.(9), it is easy to find that the term with \(A\) or \(B\) will not give contributions to the pretzelosity \(h_{1T}\). They can contribute to \(h_1\). Only the \(i\sigma^{\alpha\beta}\)-term gives a possible contribution to \(h_{1T}\). The contribution involves the sum over \(X\) for the product \(C_{\alpha\beta}C_{\alpha\beta}^\dagger\). The sum can be denoted as

\[
H_{\alpha\beta\sigma\rho}(x, k_\perp, l, n) = p^+ \sum_X \delta(p^+ - xp^+) \delta^2(k_\perp - P_X) C_{\alpha\beta}(x, k_\perp, l, n, X) C_{\alpha\beta}^\dagger(x, k_\perp, l, n, X). \tag{10}
\]

Only from this tensor \(h_{1T}\) obtains nonzero contributions. The tensor \(H_{\alpha\beta\sigma\rho}\) only depends on the vector \(k_\perp, l\) and \(n\) and can be decomposed with tensors built with \(k_\perp, l\) and \(n\), \(g^{\mu\nu}\) and \(\epsilon^{\mu\nu\alpha\beta}\). In order to obtain nonzero \(h_{1T}\), two indices of \(H_{\alpha\beta\sigma\rho}\) must be carried by two \(k_\perp\)’s respectively, e.g., the term in the decomposition:

\[
H_{\alpha\beta\sigma\rho}(x, k_\perp, l, n) = k_\perp^\beta k_\perp^\rho D^{\alpha\sigma}(x, k_\perp, l, n) + \cdots, \tag{11}
\]

where the indices of \(D^{\alpha\sigma}\) do not carried by \(k_\perp\). One may decompose the tensor \(D^{\alpha\sigma}\) with \(g^{\alpha\sigma}\) or tensors built with \(l\) and \(n\). Using the decomposition one can show Pretzelosity function is zero. Or, one direct project out the function and obtain:

\[
k_\perp^2 h_{1T}(x, k_\perp) = \frac{2}{k_\perp^2} D_{\alpha\sigma}(x, k_\perp, l, n) \left( k_{\perp\mu} k_{\perp\nu} + \frac{1}{2} k_\perp^2 g_{\perp\mu\nu} \right) \text{Tr} \left[ \gamma_5 \sigma^{-\nu} \sigma^{\rho\sigma} \gamma_5 \sigma^{+\mu} \sigma^{\alpha\beta} k_\perp^\beta k_\perp^\rho \right] = 0. \tag{12}
\]

For terms with the two \(k_\perp\)’s in Eq.(11) carrying indices other than \(\beta\) or \(\rho\) the same result is obtained. Therefore, we find that Pretzelosity function \(h_{1T}\) of a transversely polarized quark is zero.

The above result is derived in the space-time dimension \(d = 4\). If one uses dimensional regularization with \(d = 4 - \epsilon\), \(h_{1T}\) in Eq.(12) is proportional to \(\epsilon\). In general \(D_{\alpha\sigma}\) can contain pole terms in \(\epsilon\). In the limit \(\epsilon = 0\) one may obtain nonzero \(h_{1T}\). In this case the interpretation of the above result can be changed. This needs to be discussed in detail.

It should be noted that U.V. poles in \(D_{\alpha\sigma}\) are already cancelled by counter terms. Therefore, \(D_{\alpha\sigma}\) contains only pole terms representing collinear- and I.R. divergences. If we consider the subtracted TMD parton distribution \(h_{1T}(x, k_\perp)\) as discussed before or \(h_{1T}(x, b)\) in b-space, then the corresponding tensor of \(D_{\alpha\sigma}\) contain only collinear divergences, because I.R. divergences are subtracted in the subtracted \(h_{1T}(x, k_\perp)\) or cancelled in \(h_{1T}(x, b)\). One can use a small quark mass \(m\) to regularize collinear divergences associated with quarks. At the leading power of \(m\), i.e., neglecting \(m\) in nominators of quark propagators, \(\Gamma\) in Eq.(7) still consists of products of \(\gamma\)-matrices of even numbers. But, this does not regularize all collinear divergences in our case. Because there is already one gluon with fixed momentum in \(|X|\) at the leading order, there are collinear divergences associated with gluons of \(|X|\). We regularize these divergences by taking all gluons in \(|X|\) off-shell. The off-shellness is denoted as \(k_g^2\). At the end, we take
first the limit of \( m \to 0 \) and then limit of \( k_g^2 \to 0 \). With this regularization one can safely take the limit \( \epsilon \to 0 \) after the subtraction of U.V. poles. In this case, \( h_{1T}^+(x, b) \) and the subtracted \( h_{1T}^+(x, k_\perp) \) of a single quark is zero in the limit \( m \to 0 \). They are suppressed at least by \( m^2 \).

In the case of using the dimensional regularization for collinear divergences, the interpretation of our result is different. In this case we can not conclude that \( h_{1T}^+(x, b) \) or the subtracted \( h_{1T}^+(x, k_\perp) \) is zero. They are in general formally divergent. One needs to carefully study how collinear divergences are subtracted in the factorization in Eq.\([5, 6]\) when \( h_{1T}^+ \) is proportional to \( \epsilon \). A case similar to ours has been studied in \([23]\). From this study we can conclude that the perturbative coefficient function \( C_{1T}(x, b) \) is zero, or the perturbative coefficient function in the matching of the subtracted \( h_{1T}^+(x, k_\perp) \) to \( q_T(x) \) is zero. The main reason for this is the following: To obtain the perturbative coefficient function in momentum space, one needs to subtract collinear divergences in \( D^{\sigma\sigma} \). After the subtraction and the subtraction of I.R. divergences by the soft factor, one can safely take the limit \( \epsilon \to 0 \) and obtains the perturbative part of \( D^{\sigma\sigma} \). This part gives the contribution to the perturbative coefficient function \( C_{1T} \) in Eq.\([6]\). Because the contribution from this part is always multiplied with the factor \( \epsilon \) from the trace term in Eq.\([12]\), it is zero in the limit \( \epsilon \to 0 \). Therefore, one can not factorize Pretzelosity function with the collinear transversity parton distribution.

Although Pretzelosity function of a single massless quark is zero, it does not mean that the function is zero for a hadron as a bound state of quarks and gluons. In the matching of Pretzelosity function to the twist-2 transversity parton distribution, the transverse momentum of the struck quark connected to the black box in Fig.1 is neglected. Taking the momentum into account, one obtains nonzero result of \( h_{1T}^+ \). The contribution to the defined \( M^\mu \) in Eq.\([2]\) from Fig.1 is:

\[
M^\mu(x, k_\perp) = \int d^4\ell H_{\mu\nu}(\ell) \int \frac{d^4\xi}{(2\pi)^4} e^{-i\xi(\ell + k_g)} \langle h(P, S_\perp) | \bar{\psi}(\xi) i\gamma_5 \sigma^{\mu\nu} \psi(0) | h(P, S_\perp) \rangle, \tag{13}
\]

where the Fourier- transformed matrix element is represented by the black box in Fig.1. \( H^{\mu\nu} \) is given by the upper-part of diagrams in Fig.1. It is

\[
H^{\mu\nu}(\ell) = \text{Tr}\left\{ i\gamma_5 \sigma^{\mu\nu} \left[ i\gamma_\perp \cdot (\ell - k_g) \left( -ig_a T^a \gamma_\rho \right) + \frac{i}{u \cdot k_g} ( -ig_a u_\rho ) \right] \frac{1}{2N_c} \gamma_5 \sigma^{\nu\rho} \left[ (ig_a T^a \gamma_\rho \right) \left( -\frac{i\gamma_\perp \cdot (\ell - k_g)}{(\ell - k_g)^2} + \frac{-i}{u \cdot k_g} ( ig_a u_\rho ) \right] \frac{1}{2k_g^\perp}, \right. \tag{14}
\]

with \( k_g \) as the momentum of the gluon crossing the cut. Its components are given by \( k_g^+ = \ell^+ - xP^+ \) and \( k_g^\perp = \ell^\perp - k_\perp^\perp \). To obtain the contribution from twist-2 transversity, one only keeps the leading order of the collinear expansion by expanding \( H^{\mu\nu}(\ell) \) around \( \hat{\ell} = (yP^+, 0, 0, 0) \). At the leading order, as we have already shown, the contribution is zero at any order of \( \alpha_s \). But, beyond the leading order of the collinear expansion the contribution is not zero. We write the expansion as:

\[
H^{\mu\nu}(\ell) = H^{\mu\nu}(\hat{\ell}) + \frac{1}{2} \varepsilon^\alpha \varepsilon^\beta \frac{\partial^2 H^{\mu\nu}}{\partial \ell_{\perp\alpha} \partial \ell_{\perp\beta}}(\hat{\ell}) + \cdots, \tag{15}
\]

where \( \cdots \) denote irrelevant terms or higher orders. Taking the second term in the expansion, one obtains nonzero contribution to Pretzelosity function. The second term is given by:

\[
\frac{\partial^2 H^{\mu\nu}}{\partial \ell_{\perp\alpha} \partial \ell_{\perp\beta}}(\hat{\ell}) = g_{\perp}^{\mu\nu} \frac{8g_5^2 C_F}{P^+ (k_\perp^2)^3} \left[ \left( k_\perp^\alpha k_\perp^\beta + \frac{1}{2} k_\perp^2 g_\perp^{\alpha\beta} \right) \left[ \frac{4x^3}{y^2(y - x)_+} + \delta(x - y) \left( 2\ln \frac{y_2^2 \zeta_2}{k_\perp^2} + 1 \right) \right] + \frac{g^{\alpha\beta} k_\perp^2}{y^2(y - x)_+} - \frac{1}{2} \delta(x - y) \left( \ln \frac{y_2^2 \zeta_2}{k_\perp^2} + 1 \right) \right]. \tag{16}
\]
where we have taken the limit $\zeta_u^2 \to \infty$. The divergent terms with $\ln \zeta_u^2$ represent light-cone singularities. The terms in the first line of the above equation gives the contribution of $h_{IT}^{+}$ at leading power and leading order of $\alpha_s$. The terms in the second line gives a part of contribution to $h_1$ at the next-to-leading power. We parameterize the matrix element involved here as:

$$
\frac{1}{4\pi} \int d\lambda e^{-ixP^+\lambda} (h(P, S_{\perp}) | \bar{\psi}(\lambda n) \mathcal{L}_n^\dagger(\lambda n) i\gamma_\nu \sigma^{+\nu} \tilde{\partial}^\alpha \tilde{\partial}^\beta \mathcal{L}_n(0) \psi(0) | h(P, S_{\perp}))
$$

$$
= S_{\perp}^\alpha g_\perp^{\alpha\beta} T_1(x) + \left( S_{\perp}^\alpha g_\perp^{\alpha\nu} + S_{\perp}^\nu g_\perp^{\nu\beta} - g_\perp^{\alpha\beta} S_{\perp}^\nu \right) T_2(x), \quad (17)
$$

with the derivative $\tilde{\partial}^\mu$ defined as:

$$
f_\perp(x) \tilde{\partial}^\mu g(x) = \frac{1}{2} \left[ f_\perp(x) \partial^\mu g(x) - \left( \partial^\mu f_\perp(x) \right)^\dagger g(x) \right]. \quad (18)
$$

The matrix element is defined with twist-4 operators. $T_{1,2}(x)$ are twist-4 parton distributions. At the leading power and leading order of $\alpha_s$ we have the contribution to $h_{IT}^{+}$:

$$
h_{IT}(x, k_{\perp}) = -\frac{8g_2^2C_F}{(k_{\perp}^2)^3} \int_0^1 dy \left[ \frac{4x^3}{y^4(y-x)_+} + \delta(x-y) \left( 2 \ln \frac{y^2e^2}{k_{\perp}^2} + 1 \right) \right] T_2(y) + \cdots. \quad (19)
$$

In the above equation only the contribution from Fig.1, or from the twist-4 parton distribution $T_2$, is given explicitly. Hence, Pretzelosity function is matched to twist-4 parton distributions as expected in [24]. In Fig.1 only those twist-4 matrix elements consisting a pair of quark fields are considered. There are twist-4 matrix elements consisting a pair of quark fields combined with one- or two gluon field strength operators. The contributions from these twist-4 operators are denoted as $\cdots$. A complete matching by including all contributions is beyond the scope of this letter. We leave this for a future work.

Instead of giving a complete matching, we show here that Pretzelosity function of a quark combined with a gluon is nonzero through an explicit calculation. Such a state with the superposition of a single quark state is useful for studying and understanding single transverse-spin asymmetries as shown in [25] [26]. Following [25] [26] we construct the following state:

$$
|\lambda\rangle = |q(p, \lambda_q)\rangle + c_1 |g(p_1, \lambda_g)g(p_2, \lambda_g)\rangle, \quad \text{(20)}
$$

with $p_1 + p_2 = p$. The helicity of the system is denoted as $\lambda$ in $[\cdots]$. In the first term $\lambda_g = \lambda$. The $qq$-state has the total helicity $\lambda = \lambda_q + \lambda_g$. The state is in the fundamental representation in the sense that its wave function is given by:

$$
\langle 0 | G^{\mu,\nu}(x) \psi(y) | qg \rangle = T^{\mu} e^\mu(\lambda_g) u(p, \lambda_q) e^{-ip_1 \cdot y - ip_2 \cdot x}, \quad |qg\rangle = |q(p_1, \lambda_q)g(p_2, \lambda_g)\rangle. \quad \text{(21)}
$$

We specify the momentum as:

$$
p^{\mu} = (p^+, 0, 0, 0), \quad p_1^{\mu} = x_0 p^{\mu}, \quad p_2^{\mu} = (1 - x_0) p^{\mu} = \bar{x}_0 p^{\mu}. \quad \text{(22)}
$$

Now we consider Pretzelosity function of such a multi-parton state. The result shows some interesting aspects of Pretzelosity function. We describe the spin of the state by a spin density-matrix in the helicity space. The non-diagonal part of the density-matrix corresponds to the state with transverse spin. Because the operator $\mathcal{O}$ used to define Pretzelosity function is chirality-odd, the helicity of the quark has to be flipped. Then we find that only the $qg$-component in the state of Eq. (20) gives possible contributions,
in the case that the single quark state does no contribute as shown in the above. We need to calculate those matrix elements:

\[
\langle q(p_1, -)g(p_2, +)[+]|O|g(p_1, +)g(p_2, -)[-]\rangle, \quad \langle q(p_1, +)g(p_2, -)[-]|O|g(p_1, -)g(p_2, +)[+]\rangle.
\]

From these matrix elements one can extract Pretzelosity function.

The contributions to our Pretzelosity function are divided into two classes. One class of contributions are given by diagrams in Fig. 2. In Fig. 2 the diagrams represent the amplitude of the transition of the $qq$-state into one gluon through the operator $\mathcal{L}_u(0)\psi(0)$. In this amplitude, the initial gluon participants interactions. From Fig. 2 we obtain the non-diagonal part of the spin density-matrix hence Pretzelosity function. The calculation is straightforward. We obtain:

\[
h_{1T}^{1\perp}(x, k_\perp) \bigg|_{\text{Fig. 2}} = |c_1|^2 g_s^4 \left( \frac{1}{k_\perp^2} \right)^3 \frac{1-x}{1-x_0} \left[ -\frac{4CF}{N_c} (1-x_0)^2 + N_c^2 \frac{x_0(x_0 + x + xx_0 - x_0^2)}{(1-x_0)(x-x_0)} \right. \\
+ \frac{2x - x_0^2 + xx_0}{x-x_0} - 2N_c CFx_0(1-2x_0) \bigg] + \mathcal{O}(\zeta_u^{-2}),
\]

where we have taken the limit $\zeta_u^2 \to \infty$ or $u^+ \to 0$. This contribution is for $0 \leq x \leq 1$. Another class of contributions is from interferences of amplitudes where the initial gluon in one of amplitudes is a spectator. There are too many diagrams involved. But they give contributions only for $0 \leq x \leq x_0$. Therefore, our calculation in the above shows that Pretzelosity function is not zero for a state of a single quark combined with a gluon. In reality, quark-hadron correlations inside a hadron will generate in general nonzero Pretzelosity function.

From the result of calculating twist-4 contributions in Eq. (19) the large-$k_\perp$ behaviour of Pretzelosity function is predicted as $1/(k_\perp^2)^3$. This behaviour is also consistent with our result for the quark-gluon state in Eq.(24). The behaviour is expected in the absence of twist-2 contribution, as discussed in [7]. In this letter we have shown that the twist-2 contribution is in fact zero.

To summary: We have shown that the TMD Pretzelosity parton distribution of a single quark is zero at any order of QCD perturbation theory in the limit of zero quark mass. If one uses dimensional regularization for collinear divergences, the perturbation coefficient function in the matching of Pretzelosity function to the collinear transversity parton distribution is zero. Pretzelosity function should be matched to collinear parton distributions defined with twist-4 operators. We have shown through an explicit calculation that Pretzelosity function of a quark combined with a gluon is not zero. From our analysis of contributions only involving twist-4 operators of quark fields or our explicit calculation, the large-$k_\perp$-behaviour of Pretzelosity function is obtained.

Acknowledgments
The work is supported by National Nature Science Foundation of P.R. China(No.11675241). The work of K.B. Chen is supported by China Postdoctoral Science Foundation(No.2018M631588). The partial support from the CAS center for excellence in particle physics(CCEPP) is acknowledged.

References

[1] J.C. Collins, D.E. Soper and G. Sterman, Nucl. Phys. B250 (1985) 199, Nucl. Phys. B261, 104 (1985).
[2] X.D. Ji, J.P. Ma and F. Yuan, Phys. Lett. B597 (2004) 299, e-Print: [hep-ph/0405085]
[3] X.D. Ji, J.P. Ma and F. Yuan, Phys. Rev. D71 (2005) 034005, e-Print: [hep-ph/0404183]
[4] J.C. Collins and A. Metz, Phys. Rev. Lett. 93 252001, e-Print: [hep-ph/0408249]
[5] X. Ji, Phys. Rev. Lett. 110 (2013) 262002, e-Print: arXiv:1305.1539 [hep-ph], Sci. China Phys. Mech. Astron. 57, no. 7, 1407 (2014), e-Print: arXiv:1404.6680 [hep-ph].
[6] X. Ji, P. Sun, X. Xiong and F. Yuan, Phys.Rev. D91 (2015) 074009, e-Print: arXiv:1405.7640 [hep-ph].
[7] A. Bacchetta, D. Boer, M. Diehl, P.J. Mulders, JHEP 0808 (2008) 023, e-Print: arXiv:0803.0227 [hep-ph].
[8] D. Gutierrez-Reyes, I. Scimemi and A. Vladimirov, Phys.Lett. B769 (2017) 84, e-Print: arXiv:1702.06558 [hep-ph].
[9] D. Gutierrez-Reyes, I. Scimemi and A. Vladimirov, JHEP 1807 (2018) 172, e-Print: arXiv:1805.07243 [hep-ph].
[10] M.G. Echevarria, A. Idilbi and I. Scimemi, JHEP 1207 (2012) 002, e-Print arXiv:1111.4996 [hep-ph].
[11] M.G. Echevarria, A. Idilbi and I. Scimemi, Phys. Lett.B726 (2013) 795, e-Print arXiv:1211.1947 [hep-ph].
[12] J-Y. Chiu, A. Jain, D. Neill and I.Z. Rothstein, JHEP 1205 (2012) 084, e-Print: arXiv:1202.0814
[13] P.J. Mulders and R.D. Tangerman, Nucl.Phys. B461 (1996) 197, Erratum-ibid. B484 (1997) 538, e-Print: hep-ph/9510301
[14] A. Bacchetta, P. J. Mulders and F. Pijlman, Phys.Lett. B595 (2004) 309, e-Print: hep-ph/0405154
[15] A. Bacchetta et al., JHEP 0702 (2007) 093 e-Print: hep-ph/0611265.
[16] D. Bore and P.J. Mulders, Phys. Rev. D57 (1998) 5780, e-Print: hep-ph/9711485.
[17] K. Goeke, A. Metz and M. Schlegel, Phys. Lett. B618 (2005) 90, e-Print: hep-ph/0504130
[18] X.D. Ji and F. Yuan, Phys. Lett. B543 (2002) 66, e-Print: hep-ph/0206057, A.V. Belitsky, X.D. Ji and F. Yuan, Nucl. Phys. B656 (2003) 165, e-Print: hep-ph/0208038.
[19] D. Boer, P.J. Mulders and F. Pijlman, Nucl. Phys. B667 (2003) 201, e-Print: hep-ph/0303034.

[20] R.L. Jaffe, X.-D. Ji, Phys. Rev. Lett. 67 (1991) 552, Nucl. Phys. B375 (1992) 527-560.

[21] X. Artru and M. Mekhfi, Z. Phys. C 45, 669 (1990), J.L. Cortes, B. Pire and J.P. Ralston, Z. Phys. C 55, 409 (1992).

[22] C. Lefky and A. Prokudin, Phys. Rev. D91 (2015) 034010, e-Print: arXiv:1411.0580 [hep-ph].

[23] J. Collins and M. Diehl, Phys.Rev. D61 (2000) 114015, arXiv:hep-ph/9907498.

[24] I. Scimemi and A. Vladimirov, e-Print: arXiv:1804.08148 [hep-ph].

[25] H.G. Cao, J.P. Ma and H.Z. Sang, Commun. Theor. Phys. 53 (2010) 313-324, e-Print: arXiv:0901.2966 [hep-ph].

[26] J.P. Ma and H.Z. Sang, JHEP 1104:062, 2011, e-Print: arXiv:1102.2679 [hep-ph]. J.P. Ma, H.Z. Sang and S.J. Zhu, Phys. Rev. D85 (2012) 114011, e-Print: arXiv:1111.3717[hep-ph].