QUANTUM ALGEBRAS
AND LIE GROUPS

Enrico Celeghini

Dipartimento di Fisica, Università di Firenze
and I.N.F.N. Sezione di Firenze
L.go E.Fermi 2, I50125 Firenze, Italy.
e-mail CELEGHINI@FI.INFN.IT

INTRODUCTION

Quantum groups are born in Leningrad\[1\] in connection with the quantum inverse problem method as it arises in soliton theory. Other motivations came out after and the logic of the subject has been repeatedly reversed\[2\] and it is still not univocally established. The approach followed here attempts to use the link of the field with Lie groups as much as it is possible at the moment and, perhaps, a little more.

Quantum groups\[3\] are groups of matrices with not commuting elements. Only in simplest cases the quantum algebras quoted in the title have been shown to be the corresponding differential structures\[4\]. In general (and often only in principle, because the R-matrices also are not always known) the connection between quantum groups and quantum algebras is built using the algebraic approach to the quantum inverse scattering method i.e. the Yang-Baxter equation.

The first question on quantum algebras is about their general definition. It is not unreasonable to think that the work of Woronowicz can be extended and that, any way, we have a one to one correspondence among quantum algebras and quantum group, as it happens in the Lie limit. If this is true, the two properties universally ascribed to them, i.e. the fact that they are deformations of Lie algebras and Hopf algebras, are not enough to define quantum algebras. From the Truini\[5\] contribution to this symposium, it seems indeed that the class of structures so defined is too wide to hope for a q-group for each of them. In the same time it is not clear which of the properties we see in concrete examples (primitive Cartan subalgebra with unchanged commutators, coalgebra linear in not vanishing roots...) are really relevant. So we are reduced with a temporary operative definition: quantum algebras are quantum algebras i.e. the ones we can find in literature.
Let us consider now the rôle of Lie groups in physics: we give a physical meaning to the generators and to some simple polynomials of them (physical observables) and represent them as operators on the Hilbert space of physical states. In addition we use a (trivial) rule to construct the observables of a composed system from the ones of the components. In more mathematical terms we deal with the algebra, the representations and the coalgebra. But all these things are well known for quantum algebras also: the “generators” are not univocally defined, as in the Lie case, because we have not primitivity to identify them but we can choose them on physical basis; the representations coincide (at least for simple cases) with the ones of Lie algebras because the universal enveloping algebra (UEA) is the same; the coalgebra acting on the product space $V \otimes V$ and being isomorphic to the algebra on $V$ has the correct properties to extend to $q$-algebras the product of representations.

Actually Lie algebras have another fundamental property: they can be integrated to Lie groups by the exponential mapping and nothing similar exists for quantum algebras at least in commutative structures that can be realized in a Hilbert space. But in the following we will attempt to convince you that Lie structures are so close that we can use them in such a way that we do not really need anything more to perform with quantum algebras the same play we are used to do with Lie groups. Indeed the group has the canonical rôle given by the adjoint action i.e. the group is used to realize infinite duplicates of the algebra, each corresponding to one “observer” or to one “reference frame”. To the algebra, and to the algebra only, we reserve the part to describe quantities that can be measured. So we (at least I) will be content to have the same possibility to realize, in similar way, a group of transformations on the quantum algebra and to give to the infinite duplicates of the $q$-algebra so found the same interpretation as observables in different reference frames. We shall see that, at least on the representations, the play will be done exactly by the Lie group, integration of the Lie limit of the $q$-algebra.

To be more pedagogical let we discuss the argument in general but referring to an example. We hope that, in the end, you will admit that, at a formal level admissible for a physicist:

1) Quantum structures can be dealt more or less as the Lie one.

2) The $q$-coalgebra also plays a fundamental physical rôle as it establishes the rules for combining single elementary objects into composite systems. Its “generators” have, of course, the properties we require to physical observables i.e. they are hermitian and symmetric in the two components.

3) The adjoint action of the Lie group can by extended naturally to $q$-algebra and $q$-coalgebra.

4) Quantum structures are not, or not entirely, involved inventions of the mind but, exactly like Lie ones, express also fundamental invariances of the physical world. The “simple example” we introduce is, indeed, more than a simple example: it will show that, as in particle physics we define a particle as an unitary irreducible representation of the Poincaré group, in solid state physics we have to define the phonon as an unitary irreducible representation of the “Poincaré quantum group” as defined in the following.

**LIE GROUP APPLIED TO QUANTUM ALGEBRA**

It is well known that the universal enveloping algebra of a quantum algebra coincide, for $|q| \neq 1$, with the universal enveloping algebra of the corresponding Lie algebra ($UEA^q \equiv UEA$). This means that it is possible to realize a one to one mapping...
among the quantum “generators” $X_i^q$ and the corresponding Lie ones (i.e. the ones obtained in the Lie limit) $X_i$ and vice versa or, in formulas,

$$X_i^q \equiv X_i^q(\bar{X}) \quad X_i \equiv X_i(\bar{X})$$

where $\bar{X}$ ($\bar{X}^q$) means the full set of Lie (quantum) generators. As it is written in textbooks, the action of the group on the Lie generators is

$$X'_i \equiv g X_i' g^{-1} \equiv e^{i\bar{\alpha} \cdot \bar{X}} X_i e^{-i\bar{\alpha} \cdot \bar{X}}$$

and it saves the Lie commutation relations (CR). But this mapping generates a “natural” action of the Lie group on the quantum generators also. Indeed, we can define

$$X_i^{q'} \equiv e^{i\bar{\alpha} \cdot \bar{X}} X_i^q e^{-i\bar{\alpha} \cdot \bar{X}} = X_i^q(\bar{X} e^{-i\bar{\alpha} \cdot \bar{X}}) = X_i^q(\bar{X}')$$

in analogy with the action on the Lie generators: the quantum CR also

$$[X_i^q, X_j^q] = F_{ij}(\bar{X}^q(\bar{X}))$$

are saved by the mapping:

$$[X_i^{q'}, X_j^{q'}] = e^{i\bar{\alpha} \cdot \bar{X}} [X_i^q, X_j^q] e^{-i\bar{\alpha} \cdot \bar{X}} = F_{ij}(\bar{X}^q(\bar{X}')) = F_{ij}(\bar{X}^{q'}).$$

The relevant point is that this is still true for the coalgebra also. The coalgebra of the Lie algebra we are considering is not primitive because it is fixed by the coalgebra of the quantum algebra and the mapping with the Lie one:

$$\Delta(X_i) \equiv X_i(\Delta(X^q));$$

but, because we are dealing with an Hopf algebra, we have still the homomorphism:

$$[\Delta(X_i), \Delta(X_j)] = f_{ik}^j \Delta(X_k).$$

So we can repeat the preceding discussion: the same adjoint action of the Lie group does not change not only the Lie coalgebra CR,

$$[\Delta(X_i)' , \Delta(X_j)'] = f_{ik}^j \Delta(X_k)'$$

but also their quantum counterparts,

$$[\Delta(X_i)^q', \Delta(X_j)^q'] = F_{ij}(\bar{X}^q)'$$

where the action of the Lie group has been defined, as usual, by means of the coproduct

$$\Delta(X'_i) \equiv e^{i\bar{\alpha} \cdot \Delta(\bar{X})} \Delta(X_i) e^{-i\bar{\alpha} \cdot \Delta(\bar{X})} = \Delta(X'_i) \quad \Delta(X_i^q') \equiv e^{i\bar{\alpha} \cdot \Delta(\bar{X})} \Delta(X_i^q) e^{-i\bar{\alpha} \cdot \Delta(\bar{X})} = \Delta(\bar{X}_i^q').$$

In such a way, the adjoint action has been fully extended to the $q$-algebra.

To be more clear let us consider our example: $E_q(1, 1)$. $E_q(1, 1)$ is a deformation of $E(1, 1)$ and is generated, as its Lie counterpart, by three generators. In the following we call $\mathcal{E}, \mathcal{P}, \mathcal{B}$ the generators of $E(1, 1)$ and $\mathcal{E}_q, \mathcal{P}_q, \mathcal{B}_q$ the ones of $E_q(1, 1)$. We have:

$$[\mathcal{B}_q, \mathcal{P}_q] = i\mathcal{E}_q, \quad [\mathcal{B}_q, \mathcal{E}_q] = (i/w) \sinh(w\mathcal{P}_q), \quad [\mathcal{E}_q, \mathcal{P}_q] = 0.$$
where $w \equiv \log q$. The coproducts read

\[
\Delta(P_q) = P_q \otimes 1 + 1 \otimes P_q, \\
\Delta(E_q) = E_q \otimes e^{wP_q/2} + e^{-wP_q/2} \otimes E_q, \\
\Delta(B_q) = B_q \otimes e^{wP_q/2} + e^{-wP_q/2} \otimes B_q;
\]

and the antipodes

\[
\gamma(B_q) = -B_q + \left(\frac{i}{2}\right) wE_q, \quad \gamma(P_q) = -P_q, \quad \gamma(E_q) = -E_q.
\]

The Casimir of $E_q(1,1)$ is

\[
C = E_q^2 - 4/w^2 \sinh^2(wP_q/2),
\]

and, because of the Schur lemma, it identifies the irreducible representations of the UEA. It is straightforward to verify that this quantum algebra, obtained by contraction from the well known $SU_q(1,1)$ in ref. [7], satisfies the Hopf algebra axioms and to realize that the limit $w \to 0$ gives the Poincaré Lie algebra in one spatial dimension.

Looking to the expression of $C$ it is now trivial to realize a mapping of $E_q(1,1)$ on $E(1,1)$ and vice versa

\[
E = E_q, \quad P = 2/w \sinh(wP_q/2), \quad B = \text{sech}(wP_q/2) B_q
\]

from which the coalgebra associated to the Lie algebra is easy obtained

\[
\Delta(P) = 2/w \sinh(w\Delta(P_q)/2) = 1/w \left(e^{wP_q/2} \otimes e^{wP_q/2} - e^{-wP_q/2} \otimes e^{-wP_q/2}\right), \\
\Delta(E) = E_q \otimes e^{wP_q/2} + e^{-wP_q/2} \otimes E_q, \\
\Delta(B) = \text{sech}[w\Delta(P_q)/2] \Delta(B_q);
\]

where the r.h.s. can be obviously rewritten in terms of $E$, $P$ and $B$, showing that the coproduct associated by the mapping is not the primitive one.

We can now operate with the conventional Lorentz boost $e^{i\alpha B}$ to change the reference frame. The formulas for Lie algebra (and coalgebra) are written in textbooks; so we report the quantum ones only:

\[
E_q' = \cosh(\alpha) E_q - 2/w \sinh(\alpha) \sinh(wP_q/2), \\
P_q' = 2/w \arcsinh\{w/2[2/w \cosh(\alpha) \sinh(wP_q/2) - \sinh(\alpha) E_q]\},
\]

and, of course,

\[
\Delta(E_q)' = \Delta(E_q'), \quad \Delta(P_q)' = \Delta(P_q').
\]

Let look now at the representations. Because we are interested in the representation with $C = 0$, let we consider it only. From the irreducible representations for $E(1,1)$ in the plane-wave basis with $C$ and $P$ diagonal, the action of quantum variables is obtained as:

\[
C |0,p> = 0, \\
P_q |0,p> = p |0,p>, \\
E_q |0,p> = E |0,p>, \quad (E = 2/w \sinh(wp/2)).
\]
The full representation is generated by the Lorentz boost $e^{-iaB}$:

$$|0, p' > = e^{-iaB} |0, p >$$

and we have

$$\mathcal{P}_q|0, p' > = p' |0, p' > \quad (p' = 2/w \arcsinh[e^{-a}\sinh(wp/2)])$$

$$\mathcal{E}_q|0, p' > = E' |0, p' > \quad (E' = e^{-a}E)$$

and the Lie normalization become

$$<0, p'|0, p> = 4/w \tanh(wp/2) \delta(p' - p).$$

Discrete transformations act in the same way of the Lie case and will not be discussed.

### PHONONS

To have directions about physical applications of $q$-algebras, we must look to physical applications of Lie groups, in outline they are applied:

- in phenomenology and a good example can be $SU(3)$ (but, from it, we arrived to standard model, quite more that phenomenology...);
- in dynamics (think, for instance, to $SO(4)$ and the hydrogen atom...) and
- in kinematics as the Poincaré group.

$Q$-algebras applications also can, indeed, be classified in the same three fields; we have:

- papers in phenomenology (for instance the Biedenharn’s studies on $a_q$ and $a_q^+$) where $q$ is a new parameter to be fitted \(^9\);
- papers in dynamics (the best example is, to my knowledge, the XXZ model obtained as deformation of the XXX one, where $q$ measures the breaking of the rotational symmetry \(^10\)) and
- applications to kinematics, object of this talk, in which $q$ is connected to lattice spacing \(^11\).

Always we had to find the good “generators” and give them a physical meaning.

Roughly speaking $q$-algebras differ from Lie ones for some exponential of the generators of the kind $e^{wX}$ (where $w \equiv \log q$). In the usual representations, where $X$ is a differential operator, $e^{wX}$ is a finite difference operator: so $q$-groups seem related to finite difference problems and this suggests to look for applications in discrete physics. But, now, we have a problem: $e^{wX}$ must be a number and so $wX$; in consequence $[w] = [X]^{-1}$, but for simple groups CR are inhomogeneous, and, so, generators and consequently $w$ (or $q$) must be dimensionless and standard discrete physics have not dimensionless parameters. On the contrary solid state physics consider parameters of the dimension of a length: the way out are inhomogeneous $q$-algebras; their CR are, indeed, homogeneous in the abelian subalgebra that can acquire (and actually acquires) a dimension of $[l]^{-1}$ so that $[w] = [l]$.

From a technical point of view, inhomogeneous $q$-algebras have a strange peculiarity: contrary to their Lie limits, they cannot be obtained by standard contraction from simple algebras, where by standard we mean contraction with respect to a subalgebra. So to build inhomogeneous $q$-algebras from homogeneous ones we are forced to define a new kind of contraction that does not save a subalgebra and involves, in the limit, the quantum parameter $w$ also, using, in such a way, a same sort of analyticity in $w$. The
details can be found in §7, where, among other things, the contraction of \( SU_q(1,1) \) to \( E_q(1,1) \) is explicitly performed.

As we shall show, the kinematical invariance of crystal is, indeed, described by this Poincaré quantum group. In front of space-time of special relativity (and its mathematical description, the Poincaré Lie group), crystals (and Poincaré quantum group) have not in themselves a continuous translational invariance but only a discrete translational invariance as discussed before.

Let us so consider the linear chain of equal masses lying at a distance \( a \) from one another, with nearest neighbor harmonic interaction. The equations of motion are:

\[
\ddot{z}_j(t) = \omega^2 (z_{j-1}(t) + z_{j+1}(t) - 2z_j(t))
\]

where \( z_j(t) \) is the displacement of the \( j \)-th mass \((j = 0, 1, \ldots, N)\). Periodic boundary conditions are assumed and initial conditions \( z_j(0), \dot{z}_j(0) \) must be specified.

We embed the ordinary system for displacements into the partial differential equation (PDE)

\[
\left( \partial_t^2 + (2v/a)^2 \sin^2(-ia\partial_x/2) \right) z(x,t) = 0 ,
\]

where \( v = \omega a \). The periodic conditions are \( z(0,t) = z(Na,t) \) while the Cauchy data consist in the assignment of smooth functions \( z(x,0) \) and \( \partial_t z(x,0) \). When \( z(ja,0) = z_j(0), \partial_t z(ja,0) = \dot{z}_j(0) \) for all \( j \), it is easy to see that the solutions of the ordinary system are directly obtained as \( z_j(t) = z(ja,t) \) irrespectively of the behaviour of the solutions in the points \( x \neq ja \).

The continuum limit \( a \to 0 \) of PDE obviously reproduces the Klein–Gordon equation in dimension \((1+1)\) with velocity \( v \) and mass \( m = 0 \). This constitutes a differential realization of the Casimir of the \( E(1,1) \) algebra, which is actually the kinematical symmetry of the continuous system. Likewise, our PDE identifies a realization with Casimir \( C = 0 \) of the pseudoeuclidean quantum algebra \( E_q(1,1) \) described before which in its own right can be considered the kinematical symmetry of the harmonic crystals. The dimensional deformation parameter \( w \), also, has a simple physical interpretation as it is related to lattice spacing by \( w = i a \).

Because \( w \) results imaginary, \( q \) is on the unit circle: the topology of \( E_q(1,1) \) is now completely different but the essential results about the connection with Lie groups are still valid.

The related realization of the \( q \)-algebra is obtained from ref.\[7\] yielding

\[
\mathcal{E}_q = (i/v) \partial_t, \quad \mathcal{P}_q = -i\partial_x, \quad \mathcal{B}_q = i(x/v)\partial_t - (vt/a) \sin(-ia\partial_x).
\]

In the momentum representation, a realization of the \( E_q(1,1) \) in terms of the diagonal \( \mathcal{P}_q \) and the position operator \( X = i\partial/p \) is given by:

\[
\mathcal{E}_q = (2/a) \sin(ap/2), \quad \mathcal{B}_q = (1/a) \{ \sin(ap/2), X \}_+, \quad \mathcal{P}_q = p.
\]

Our notation has been chosen for its transparent physical meaning, but its is not the good one from mathematical point of view. As a matter of fact, mathematicians do not introduce \( \mathcal{P}_q \) but \( k = e^{i a \mathcal{P}_q} \). If we rewrite everything in terms of \( k \) the real topology appears and we see that \( \mathcal{P}_q \) is determined up to an integer multiple of \( 2\pi/a \). In such a way, the topology of the \( q \)-algebra with \(|q| = 1\) implies what in solid state physics is
called the reduction to the first Brillouin zone i.e. the limitation of the values of $P_q$ and $E_q$: $0 \leq p < 2\pi/a, E_q > 0$.

The expression for the generator $B_q$ can be inverted in $X$:

$$X = (1/2) \{ E_q^{-1}, B_q \}.$$

The time derivative of $X$ is given by $\dot{X} = iv \{ E_q, X \}$ and the commutator, evaluated from the $q$-algebra, gives the well known group velocity of the phonons

$$\dot{X} = v_g = v \cos(aP_q/2).$$

Let us show how the coproduct can be brought to bear to study composed systems and, in particular, the fusion of phonons. It is well known that, when the symmetry is given by a Lie algebra, the generators of the global symmetry of a composed system are obtained by summing the generators of the symmetry of the elementary constituents. This is related to the fact that each generator $G$ of a Lie algebra is a primitive element, i.e. $\Delta(G) = I \otimes G + G \otimes I$. Then $G^{(1)} \equiv G \otimes I$ acts on the vector space of the first elementary system and $G^{(2)} \equiv I \otimes G$ on the second. The algebras generated by $G^{(1)}$ and $G^{(2)}$ are both isomorphic to that generated by $G$ and since $\Delta$ is a homomorphism of algebras, then $G^{(1)} + G^{(2)}$ generates the same symmetry on the composed system. In the quantum group context we can have non primitive generators, but the very same considerations are still valid, after that the symmetries both of pseudo-particles and of $q$-algebras have been considered.

Phonons are bosons: to save their statistics and generate a correct composite system, we need symmetrical operators in $V \otimes V$. So the operators of the coalgebra we have found cannot describe the observables of the two phonons system: they are not symmetric (and not even hermitian). In the same time $q$-algebras have always a symmetry in themselves: $q \leftrightarrow q^{-1}$ (in our case equivalent to $a \leftrightarrow -a$) corresponding to the exchange of the two spaces in $V \otimes V$. So we can attempt to impose this symmetry by hand, substituting all the coalgebra with its symmetrized form under the transformation $a \leftrightarrow -a$.

Now $P_q^s, E_q^s$ and $B_q^s$ are all symmetric and hermitian and an explicit check shows that they still close the $E_q(1,1)$ algebra. They are, in such a way, good candidates to describe the observables for the system of two phonons. It must be stressed, in the same time, that they are not a coalgebra of our $E_q(1,1)$ because they do not satisfy to all the requirement of an Hopf algebra; in particular they do not allow for building, by iteration, the operators for three and more phonons: the global operators must be calculated from the original coproduct up to the required number of phonons and then completely symmetrized. It is easy to show that for each $n$ this procedure close again the $E_q(1,1)$ algebra.

Let us look, now, how the simplest composed system i.e. a phonon obtained by fusion of two phonons is described by the $q$-algebra. Because $P_q$ is defined up to $2\pi/a$, the composition of the momenta also has the same property: $P_q^s = P_q^{(1)} + P_q^{(2)} + 2\pi n/a$, showing that the Umklapp process is implied by the quantum group symmetry. The other global generators, that depend from the $P_q$’s through trigonometrical functions only, have not this problem.
In concrete, take two differently polarized phonons with the same direction of propagation, velocity parameters $v_1$ and $v_2$ and dispersion relations\cite{12}:

$$\Omega_1 = (2v_1/a) \sin(aP_q^{(1)}/2), \quad \Omega_2 = (2v_2/a) \sin(aP_q^{(2)}/2),$$

where $\Omega_1 = v_1 \mathcal{E}_q^{(1)}$ and $\Omega_2 = v_2 \mathcal{E}_q^{(2)}$ are the energies of the two phonons. The explicit coproduct of $\mathcal{E}_q$ reads

$$\mathcal{E}_q^s = \cos\left(\frac{aP_q^{(1)}}{2}\right) \mathcal{E}_q^{(2)} + \cos\left(\frac{aP_q^{(2)}}{2}\right) \mathcal{E}_q^{(1)} = (2/a) \sin\left(\frac{(P_q^s)}{2}\right).$$

To realize the fusion, the energy conservation implies the existence of a branch of the dispersion relation with velocity $v$ such that the global energy $\Omega = \Omega_1 + \Omega_2$ is related to $\mathcal{E}_q^s$ by $\Omega = v|\mathcal{E}_q^s|.$

Moreover, from the same definition of the position operator in terms of the algebra, we obtain the position operation of the two phonon system:

$$X^s = \frac{1}{2}(X^{(1)} + X^{(2)}) + \frac{1}{2}\left\{\frac{\sin(a(P_q^{(1)} - P_q^{(2)})/2)}{\sin(a(P_q^{(1)} + P_q^{(2)})/2)} , \frac{1}{2}(X^{(1)} - X^{(2)})\right\}_+,$$

which reproduces the Heisenberg algebra $[X^s, P_q^s] = i$ for the global variables. Finally the group velocity of the composite system $\dot{X}^s = i [\Omega, X^s] = v \cos(aP_q^s/2)$ appears formally identical to that of the elementary system, having performed the Umklapp process.

To conclude let we stress that there is nothing of peculiar in the phonon, the relevant point being its dispersion relation connecting energy and momentum: all the quasi-particles with the same dispersion relation can be handled with $E_q(1, 1).$ Analogously the quasi-particles with dispersion relation of the kind: $\mathcal{E}_q \approx \sin^2\left(\frac{aP_q}{2}\right)$ are unitary irreducible representations of the Galilei quantum group $\Gamma_q(1)$ as can be seen in the contribution of Tarlini\cite{13}.

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