Abstract

The baroclinic instability problem is considered in the framework of Laplacian tidal theory. The Hilbert space of the quasigeostrophic vorticity budget is spanned by spheroidal functions. The fluid is linearly stable against quasigeostrophic disturbances. As the essential source of irregular ocean-atmosphere motions, baroclinic instability is ruled out by tidal theory. The midlatitude $\beta$-plane budget of vorticity fluxes is inconsistent with basic laws of motion on the rotating spherical surface. Realistic numerical simulations of global wave dynamics and dynamical circulation instabilities require a covariant account of fluid motion on the spherical planet.

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I. Introduction

Major difficulties in numerical climate simulations result from the fact that the potential energy in the ocean-atmosphere system is closely associated with large-scale features of its density field while a considerable fraction of its kinetic energy resides on fairly small eddy scales [1]. State-of-the-art models for weather prediction and climate simulation capture large-scale features of the global circulation with some degree of realism if they are determined by large-scale features of topography and external forcing. However, simulation of the dynamical control of density by transfer processes between the small scales of kinetic energy and the large potential energy scales is less satisfactory [2].

The key to these energy- and vorticity-fluxes is fluid instability. Of the numerous instabilities in the climate system none is considered as fundamental as baroclinic instability [3]. This concept refers to the growth of weakly divergent Rossby waves in a stably stratified and vertically sheared fluid on the rotating spherical surface. The contemporary understanding of cyclogenesis, predictability-limits and the transition to chaos, turbulence and stochasticity in the climate system revolves essentially around this process. Hence, it also provides the paradigm for the design and interpretation of irregular fluid motion in numerical circulation models, ranging from resolved scales down to the parameterization of subscale processes.

Baroclinic instability theory invokes four sets of approximations. First: shallow water theory with constant mean layer-thicknesses and -velocities [4]. Second: quasigeostrophy. For tidal theory, Longuet-Higgins [4] has shown that this approximate closure of the vorticity budget yields a meaningful and elegant Rossby wave filter of Laplace’s tidal equation. In baroclinic fluids this approach governs the thermal wind. Third: the midlatitude $\beta$-plane. To avoid formal difficulties with spherical coordinates, a tangential cartesian plane is pinned to the sphere at some extratropical latitude in analogy to Kelvin’s well-established equatorial $\beta$-plane. Fourth: a number of scaling assumptions which neglect fluid velocities relative to planet rotation and stratification relative to barotropicity wherever it appears uncritical. Quantitatively, these assumptions seem generally to be justified [5].

The paper at hand considers the baroclinic instability problem of Laplace’s tidal theory, independent of the midlatitude $\beta$-plane and scaling assumptions. Tidal theory differs in two aspects from current circulation theories and models: it is Newtonian (namely: covariant) and accounts consistently for the globe’s sphericity [6]. The relationship of its analytical structure and the theory of Heun functions [6] is increasingly understood. While a transformation of the tidal equation into Heun’s equation is not known, general approximations and exact special cases can be expressed in terms of Heun functions. These concepts provide a sound foundation for Rayleigh stability analysis in the framework of global wave-circulation theory. On the basis of covariant fluid dynamics on the rotating
spherical surface it will be shown that Rossby wave growth in baroclinic fluids plays a far lesser role for irregular ocean-atmosphere motions than currently thought.

The geometrically and dynamically consistent way to avoid formal difficulties with spherical co-ordinates is the use of index notation and covariant differentiation. Here, indices \( m, n, \ldots = 1, 2 \) run over longitude \( \lambda \) and latitude \( \varphi \) while braced indices \( (\ell) = 1, 2 \) refer to the top and bottom layer of the fluid and are not subject to the summation convention. Covariant differentiation will be denoted by a semicolon. For details of the notation see [6]. In this formalism, equations in curvilinear coordinates look widely similar to corresponding equations in cartesian coordinates with geometrical details consistently absorbed into core symbols and indices. Thus, formulas emphasize the physical structure of the problem.

II. Tidal Equations

The problem is considered in terms of the bishallow water equations on the rotating spherical surface [6]. Rayleigh stability theory requires the linearization of such equations around the considered basic state. For both layers, the mean layer mass per unit area and hence the concentration are here assumed to be constant

\[
R = R_{(1)} + R_{(2)} = \text{const}, \quad r = R_{(1)}/R = \text{const}.
\]

Thus, the effective pressure \( P(R, r) \) and the interfacial potential \( \mu(R, r) \)

\[
P = \frac{1}{2} \gamma_{(2)} (1 + \delta r^2) R^2, \quad \mu = \gamma_{(2)} \delta Rr,
\]

are also constant where \( \gamma(\ell) = g/\rho(\ell) \), \( g \) the gravitational acceleration and

\[
\delta = (\rho_{(2)} - \rho_{(1)})/\rho_{(1)} > 0
\]

the positive definite stratification parameter. The spherical generalization of a constant mean barycentric velocity \( V_n \) and a constant mean vertical shear \( W_n \) are given by

\[
V_n = a^2 U_0 (\cos^2 \varphi, 0), \quad W_n = a^2 W (\cos^2 \varphi, 0)
\]

with Earth’s radius \( a \) and constant angular velocities

\[
U_0 = r U_{(1)} + (1 - r) U_{(2)}, \quad U_1 = (1 - r) U_{(1)} + r U_{(2)}, \quad W = U_{(1)} - U_{(2)}.
\]

The barycentric and baroclinic potential vorticities of this circulation, \( Z_0 \) and \( Z_1 \), are obtained as

\[
R Z_0 = F_0 = 2(\Omega + U_0) \sin \varphi, \quad R Z_1 = F_1 = 2(\Omega + U_1) \sin \varphi
\]

while \( S = W \sin \varphi \). This circulation is driven by an external, meridionally varying surface pressure and the value of the vertical shear \( W \) is determined by \( \Omega \), the stratification parameter \( \delta \) and the equator-to-pole gradient of the surface pressure [3]. For small-amplitude perturbations \((m, \eta, j_n, i_n)\) of the state vector: layer mass, concentration, barycentric and baroclinic mass flux, linearization of the bishallow water equations around this circulation leads to the tidal problem [3]

\[
d_0 m + j^m_{i_n} = -R W^n \partial_n \eta \quad (1)
\]

\[
R d_1 \eta + i^m_{j_n} = -r_{12} W^n \partial_n m \quad (2)
\]

\[
d_0 j_n + \epsilon_{mn} F_0 j^m + \partial_n p_s = -W^m \partial_m i_n - 2 \epsilon_{mn} S i^m \quad (3)
\]

\[
d_1 i_n + \epsilon_{mn} F_1 i^m + \partial_n p_s = -r_{12} (W^m \partial_m j_n + 2 \epsilon_{mn} S j^m) \quad (4)
\]
where \( r_{12} = r(1 - r) \) and \( d_{0/1} = \partial_t + U_{0/1} \partial_\lambda \). The linearized pressure emerges as
\[
p_* = (\partial_R P)_r, m + (\partial_r P)_R \eta = c^2 p_1 m + c^2 p_2 R \eta
\]
while one finds for the linearized interfacial potential
\[
\mu_* = r_{12} R(\partial_R \mu)_r, m + r_{12} R(\partial_r \mu)_R \eta = c^2 \mu_1 m + c^2 \mu_2 R \eta.
\]
Here, \( c^2 = \gamma_2 R \) and
\[
p_1 = 1 + \delta r^2, \quad \mu_1 = r \mu_2 = r_{12} p_2 = \delta r r_{12}.
\]
These coefficients satisfy
\[
c^2(p_1 + \mu_2) = c^2(1 + p_2) = c_0^2 + c_1^2, \quad c^4(p_1 \mu_2 - p_2 \mu_1) = c^4 \mu_2 = c_0^2 c_1^2
\]
where the intrinsic barycentric and baroclinic phase speeds are given by
\[
c_{0/1}^2 = \frac{1}{2} c^2 (1 + \delta r \pm \sqrt{(1 - \delta r)^2 + 4 \delta r^2}).
\]
Taking the curl of (3) and (4) one arrives at the perturbation vorticity budgets
\[
R^2 d_0 z + j^a \partial_a F_0 = -W^a \partial_a R^2 \zeta - 2i^a \partial_a S
\]
\[
R^2 d_1 \zeta + i^a \partial_a F_1 = -r_{12}(W^a \partial_a R^2 z + 2j^a \partial_a S)
\]
with barycentric perturbation vorticity
\[
R^2 z = \epsilon^{an} j_{n; a} - F_0 m - 2 S R \eta
\]
and baroclinic perturbation vorticity
\[
R^2 \zeta = \epsilon^{an} i_{n; a} - F_1 R \eta - 2r_{12} S m.
\]
Equations (1) through (8) pose the Rayleigh stability problem for a generic stably stratified and vertically sheared fluid on the rotating spherical surface. In spherical bishallow water theory, constant mean layer-thicknesses exclude a mean flow with available potential energy. The tidal problem for a circulation with finite available potential energy is discussed in the Appendix.

**III. Quasigeostrophic Stability Analysis**

In the barotropic 1-layer limit, equations (1) through (4) reduce to Laplace’s standard tidal equations. In special cases, exact analytical solutions of the tidal equation are known in terms of confluent Heun functions, namely spheroidal functions \[7, 8\]. In the entire wave number space, approximate analytical solutions can be expressed in terms of spheroidal functions and the asymptotics of tidal functions coincide with the asymptotic behaviour of prolate spheroidal functions: the Margules regime of globally defined Legendre polynomials at small Lamb parameters \[9\] and the Matsuno regime of Hermite polynomials on the equatorial \( \beta \)-plane at large Lamb parameters \[10\]. Longuet-Higgins’ quasigeostrophic Rossby wave filter \[4\] retains these functional characteristics. In the 1-layer limit, the perturbation vorticity budget becomes
\[
R^2 d_t z + j^a \partial_a F = 0
\]
with perturbation vorticity
\[
R^2 z = \epsilon^{an} j_{n; a} - F m.
\]
As in the strictly nondivergent case, quasigeostrophy assumes that the perturbation mass flux is sufficiently represented by a stream function $A$

$$j_n = c_{nm} O^m A$$

while the mass perturbation in (9) is not supposed to vanish, thus giving rise to the notion of weakly divergent perturbations. A closed expression for (9) is now obtained by invoking the geostrophic approximation

$$c^2 m = -FA$$

leading for the vorticity budget to the equation

$$(\Delta - \alpha^2 y^2 - M\tau)A = 0$$

where $y = \sin \varphi$, $\alpha = 2a(\Omega + U)/c$ the Lamb parameter, $M$ the zonal wave number, $\nu = a(\omega - UM)/c$ the Doppler-shifted frequency and $\tau = \alpha/\nu$. This is the prolate spheroidal wave equation. The dispersion relation for quasigeostrophic Rossby waves becomes

$$\nu = -\alpha M/\epsilon(N, M; \alpha)$$

with prolate spheroidal eigenvalue $\epsilon(N, M; \alpha)$. Comparison of this expression with numerical solutions of the complete tidal equation (fig.1) demonstrates that with the exception (of the gravity branch) of the Yanai wave (mode number $N=0$) quasigeostrophy provides a satisfactory approximation to all Rossby modes of the tidal problem. Also, this shows that quasigeostrophy is by no means a regional, e.g. extratropical approximation. Rather, it is globally valid and includes the Margules regime as well as the Matsuno regime. Physically, the equatorial $\beta$-plane approximation of the prolate spheroidal equation accounts for wave trapping in the Yoshida guide. A similar wave guide in midlatitudes does not exist and a midlatitude $\beta$-plane does not appear in the systematic approximation theory of spheroidal functions.

Figure 1: Tidal eigenfrequencies (solid lines) and quasigeostrophic approximation (dotted lines). Frequencies larger than $|M|$ correspond to gravity modes. Negative zonal wave numbers $M$ indicate westward propagation. Dashed-dotted line: $\nu = \alpha$. Tidal frequencies were calculated by [11] and spheroidal eigenvalues computed with NAG-Lib routine F02GJE.
The functional structure of the tidal problem uniquely determines the physical interpretation of the so-called “beta-effect” and the spectrum of the tidal wave operator. The Coriolis term of tidal theory represents the meridional shear of a mean zonal flow with uniform angular velocity \( \Omega + U \). Doppler-shifts only appear with respect to \( U \) since the observer corotates with \( \Omega \). While the corotating observer does not see a frequency shift with respect to \( \Omega \), the corresponding meridional shear does not vanish on this transformation. Physically, the “beta-effect” of tidal theory refers to such meridional shear and differs profoundly from topography on the f-plane.

The spectrum of the tidal wave operator represents free waves in an elastic medium. Such a medium has two types of excitations: longitudinal P (primary, pressure or sound) waves and transversal S (secondary or shear) waves. Long gravity waves of covariant shallow water theory are represented as second sound in a strictly 2-dimensional bifluid. Rossby waves, on the other hand, obey

\[
\text{the dynamics of (radially polarized) shear waves} \quad \text{[12]. The frequencies of S-waves are always lower than P-wave frequencies: all Rossby frequencies lie below gravity frequencies.}
\]

Rossby waves are essentially divergence-free and well represented by Margules’ approximation. High Rossby frequencies are limited by meridional trapping at low latitudes which induces a weak divergence: weak divergence is characteristic of equatorial Rossby waves. Tropically trapped, weakly divergent Rossby waves are well approximated by Matsuno’s theory. Longuet-Higges’ quasigeostrophy unifies both approaches. With two Lamé coefficients, elastic wave theory is inherently a two-parameter problem. This is also true for generic tidal theory: while a “compressibility” controls gravity wave dynamics, the Lamb parameter represents the mean meridional shear that governs Rossby wave dynamics. Quasigeostrophy is the shear wave filter of tidal theory.

In application to the present bi-shallow problem, quasigeostrophy represents the barycentric and baroclinic mass flux perturbations in terms of stream functions

\[
j_n = \epsilon_{nm} \partial^m A, \quad i_n = \epsilon_{nm} \partial^m \psi
\]

and determines mass- and concentration-perturbations in (7) and (8) from the thermal wind relation of (3) and (4)

\[
c^2 p_1 m + c^2 p_2 R_\eta = -F_0 A - 2S\psi
\]

\[
c^2 \mu_1 m + c^2 \mu_2 R_\eta = -F_1 \psi - 2r_{12} SA.
\]

Solving for \( m \) and \( R_\eta \) and inserting the result into (7) and (8) the perturbation vorticities become

\[
R^2 z = - (\Delta - h_0 y^2) A - h y^2 \psi
\]

\[
R^2 \zeta = - (\Delta - h_{1y} y^2) \psi - r_{12} h y^2 A
\]

with

\[
h_0 = (\mu_2 \alpha_{(2)}^2 + r_{12} \alpha_{12}^2)/\mu_2, \quad h_1 = (r p_2 \alpha_{(2)}^2 + \alpha_{1}^2)/\mu_2, \quad h = (p_2 \alpha_{(2)}^2 - \alpha_{12} \alpha_1)/\mu_2
\]

where \( \alpha_{(\ell)} = 2a(\Omega + U_{(\ell)})/c, \alpha_{12} = \alpha_{(1)} - \alpha_{(2)} \) and \( \alpha_1 = 2a(\Omega + U_1)/c \). With these expressions the vorticity budgets (5) and (6) assume the form

\[
(\omega_0 \Delta - H_{00} y^2 - f_0 M) A = (W M (\Delta + 2) - H_{01} y^2) \psi
\]

\[
(\omega_1 \Delta - H_{11} y^2 - f_1 M) \psi = r_{12} (W M (\Delta + 2) - H_{10} y^2) A.
\]

Here, \( \omega_{0/1} = \omega - U_{0/1} M \) and \( f_0/1 = 2(\Omega + U_{0/1}) / 1 \) while

\[
H_{00} = \omega_0 h_0 + r_{12} h W M, \quad H_{01} = \omega_0 h + h_1 W M
\]
and
\[ H_{10} = \omega_1 h + h_0 WM, \quad H_{11} = \omega_1 h_1 + r_{12} h WM. \]

Equations (10) and (11) are a system of coupled spheroidal equations and spheroidal functions form a complete set of eigensolutions. Eliminating \( \Delta \psi \) from these equations yields
\[ q \psi = (\Delta - h_0 y^2 - M \tau_0) A \]
with \( \tau(\ell) = 2(\Omega + U(\ell))/(\omega - U(\ell)M) \) and
\[ q = M \tau_{12} - h y^2, \quad \tau_{12} = \tau(1) - \tau(2), \quad \tau_0 = r \tau(1) + (1 - r) \tau(2), \quad \tau_1 = (1 - r) \tau(1) + r \tau(2). \]

Substituting this expression for the baroclinic stream function into (10) results in a single fourth-order equation
\[ (\Delta - h_1 y^2 - M \tau_1) \frac{1}{q} (\Delta - h_0 y^2 - M \tau_0) A = r_{12} q A \tag{12} \]
for the barycentric stream function. Utilizing now the spheroidal property: \( \Delta A = (\beta^2 y^2 - \epsilon) A \) with
\[ \epsilon + M \tau_0 - (\beta^2 - h_0) y^2 = X q \]
for constant \( X \), equation (12) requires the simultaneous validity of the two quadratic equations
\[ \tau_{12} (X^2 + (1 - 2r)X - r_{12}) = 0 \]
\[ X^2 h + (h_0 - h_1) X - r_{12} h = 0. \]

These two equations express the major difference between the cartesian and the spherical stability problem. In cartesian geometry, the dispersion relation determines admissible eigenfrequencies. On the rotating spherical surface, background inhomogeneities due to the planet’s sphericity, coordinate-dependent Coriolis forces and the mean circulation also determine admissible Lamb parameters. Rayleigh theory of spatially inhomogeneous systems accounts for wave trapping. The compatibility of both quadratic equations is determined by the relation
\[ h_0 - h_1 = (1 - 2r) h - \alpha(1) \alpha(2)/\mu_2 \tag{13} \]
which follows from the definition of \( h_0, h_1 \) and \( h \). Given (13), essentially two classes of solutions of (12) exist: either, the stratified background is free of vertical shear or Rossby waves are strictly nondivergent. In the first case: \( W = 0 \) and \( \tau_{12} = 0 \). Hence: \( \tau_0 = \tau_1 = \tau \), while \( q = -h y^2 \) and (12) becomes
\[ (\Delta - \beta_0^2 y^2 - M \tau)(\Delta - \beta_0^2 y^2 - M \tau) A = 0 \]
where the Lamb parameters are obtained as
\[ \beta_0 = 2a(\Omega + U)/c_0, \quad \beta_1 = 2a(\Omega + U)/c_1 \]
with intrinsic barycentric and baroclinic phase speeds \( c_{0/1} \). The dispersion relation is
\[ (\nu_0 \epsilon(N, M; \beta_0) + \beta_0 M)(\nu_1 \epsilon(N, M; \beta_1) + \beta_1 M) = 0 \]
with prolate spheroidal eigenvalue \( \epsilon(N, M; \beta) \) and \( \nu_{0/1} = a(\omega - UM)/c_{0/1} \). In the shear-free case, Rossby waves are weakly divergent, propagate as barycentric and baroclinic modes and eigenfrequencies are real. In the second case: \( c \to \infty \) and \( h_0 = h_1 = h = 0 \), while \( q = M \tau_{12} = \text{const} \) so that (12) reduces to
\[ (\Delta - M \tau(1))(\Delta - M \tau(2)) A = 0. \]
In the strictly divergence-free case, spheroidal functions degenerate into Legendre polynomials and the dispersion relation becomes

\[(\epsilon + M\tau_{(1)})(\epsilon + M\tau_{(2)}) = 0\]

with \(\epsilon(N,M) = N(N + 1) + (2N + 1)|M| + M^2\). The mean vertical shear traps nondivergent Rossby waves in individual layers and eigenfrequencies are real. This remains true if the mean circulation exhibits available potential energy \[13\].

There are two more solutions if in one of the layers \(\alpha_{(\ell)} = 0\), i.e. \(U_{(\ell)} = -\Omega\). For \(\alpha_{(j)} = 0\), Rossby waves exist only in the complementary layer \(\ell \neq j\) and from (12) one finds the dispersion relation

\[\epsilon(N,M;\beta_{(\ell)}) = -M\tau_{(\ell)}\]

with Lamb parameter

\[\beta^2_{(\ell)} = 4a^2(\Omega + U_{(\ell)})^2/g' H_{(\ell)}\]

where \(H_{(\ell)} = R_{(\ell)}/\rho_{(\ell)}\) is the mean layer-thickness and \(g' = g(\rho_{(2)} - \rho_{(1)})/\rho_{(2)}\) the reduced gravity acceleration. Both of these solutions are weakly divergent and stable. The condition \(U_{(j)} = -\Omega\) implies a westward flow circulating the globe in one day. Hence, the layer is at rest in a nonrotating, inertial system and such conditions do not admit Rossby wave propagation (the mean meridional shear is absent). In the rotating system, corresponding large-scale velocities are of a magnitude that is not met on this planet. For practical purposes these solutions are hence of little significance.

The same dispersion relations follow if \(A\) and \(\psi\) in (10) and (11) are replaced with spheroidal functions and the resulting system of 3 algebraic equations is solved for the unknown Lamb parameters and eigenfrequencies (see Appendix). As a further alternative, the layer representation of spherical linearized bishallow water \[13\] may be chosen as starting point for the stability analysis rather than the modal representation (1) through (4). It is readily seen that the dispersion relation from such an approach coincides with the results obtained above (see Appendix). The baroclinic instability problem of tidal theory does not assume the form of a spherical Taylor-Couette flow and all eigenfrequencies are real. Unlike baroclinic gravity waves, Rossby waves do not feed on the energy of a stably stratified and vertically sheared mean flow. Thus, the flow is linearly stable against quasigeostrophic disturbances.

For the system (1) through (4) isopycnals coincide with equipotential surfaces. This type of configuration is generally considered in baroclinic instability theory: isopycnals are assumed to be “flat” and a slope-parameter does not enter the problem \[3\]. On the other hand, observers and modellers are typically concerned with sloped isopycnals and their erosion by baroclinic instability \[2\]. The tidal equations for bishallow water with sloping isopycnals are well known and their stability against nondivergent Rossby waves has been demonstrated \[13\]. The stability of such a system against quasigeostrophic Rossby waves is shown in the Appendix.

The physical interpretation of these results is best considered in comparison to Kelvin-Helmholtz instability. For this instability, highly divergent baroclinic gravity waves continuously sample both layers of a stably stratified and vertically sheared fluid. The dynamics of baroclinic gravity waves are controlled by the competition of (stabilizing) stratification and (destabilizing) vertical shear. If the vertical shear becomes too large, baroclinic gravity waves grow. None of these mechanisms plays a role in tidal Rossby wave dynamics. Rossby waves are inseparably linked to a definite value of the mean meridional shear determined by \(\Omega + U_{(\ell)}\). Hence, they propagate in individual layers and a distinction of barycentric and baroclinic Rossby waves is meaningless (unless the vertical shear vanishes). Weak divergencies are associated with meridional trapping and do not accommodate the exploration of adjacent layers. None of the mean flow features of a stably stratified, vertically sheared flow competes with the restoring stresses. Thus, a transfer of energy or vorticity between this type of circulation and Rossby waves is excluded in the framework of tidal theory.

Although mean flow available potential energy may alter the system’s wave guide geography radically, it does not change its stability properties. For very low-frequency Rossby waves, isopycnal slopes modify the effective Lamb parameter (see Appendix). This parameter may now assume real or
imaginary values. In the event of imaginary Lamb parameters, spheroidal wave operators change from prolate to oblate and Rossby waves may be meridionally trapped in a polar wave guide. Although the Lamb parameter of oblate Rossby waves becomes imaginary, their frequencies remain real and Rossby wave amplitudes do not commence growing. In general, a stably stratified and vertically sheared fluid with or without available potential energy lacks the faculties of energy- and vorticity-transfer to Rossby waves. This statement may require modification for certain initial conditions or the extremely steep isopycnals associated with outcropping. Independent of the role, these and other special cases may take, Laplace’s tidal theory does not support the ubiquity, baroclinic instabilities or the extremely steep isopycnals associated with outcropping. 

IV. Discussion

The results of the previous section are in clear contrast to baroclinic instability theory on the midlatitude $\beta$-plane. To identify the source of this discrepancy evaluate the midlatitude $\beta$-plane approximation of (12). In this sense, the latitude $y$ is fixed at some (extratropical) value $y_*$ while $G_0^2 = h_0 y_*^2$, $G_1^2 = h_1 y_*^2$ and $G_2^2 = h y_*^2$ are considered as constant parameters. Adopting furthermore a cartesian Laplace operator and trigonometric eigenfunctions with $K^2 = a^2(k_1^2+k_2^2)$ and $M = a k_1 \cos \varphi_*$ one obtains from (12) the approximate dispersion relation

$$(K^2 + G_1^2 + M \tau_1)(K^2 + G_0^2 + M \tau_0) = r_{12} q^2.$$ 

A little algebra readily shows that this expression is equivalent to

$$(K^2 + F_{(1)}^2 + M \tau_{(1)})(K^2 + F_{(2)}^2 + M \tau_{(2)}) = s F_{(1)}^2 F_{(2)}^2$$

with $s = \rho_{(1)}/\rho_{(2)}$ and $F_{(\ell)}^2 = \beta_{(\ell)} y_*^2$. This equation is quadratic in frequency. Unlike the results of the previous section, its roots do not assume an easily interpreted form. This is indicative of the difficulties of the primarily geometrical midlatitude $\beta$-plane approximation to accommodate the system’s physical structure. Nevertheless, the reality of these roots is obvious and the midlatitude $\beta$-plane approximation to the final wave equation (12) comes qualitatively to the same result as the spherical analysis with respect to stability. Hence, the trigonometric approximation of wave functions in itself - though unsatisfactory - is uncritical. The source of discrepancies is therefore a fundamentally different account of vorticity fluxes by covariant shallow waters and baroclinic instability theory.

In the covariant case, the vorticity budget (12) is uniquely determined by the equations of motion (1) through (4) and the quasigeostrophic thermal wind approximation. The validity of quasigeostrophy for the barotropic fluid is well demonstrated by fig.1 and the particular form of the same argument for the baroclinic system is again an unambiguous consequence of the equations of motion. These equations are the direct and unique result of the application of covariance requirements to the formulation of spherical shallow water dynamics. Thus, the vorticity fluxes of covariant shallow water theory are in essence the expression of basic geometrical and physical consistency conditions for the hydrostatic flow of the stably stratified and vertically sheared fluid on the rotating spherical surface.

Standard baroclinic instability theory, on the other hand, takes the Primitive Equations as a starting point and invokes the midlatitude $\beta$-plane approximation to derive the vorticity budget. It has been shown that the Primitive Equations do not pose a covariant dynamical problem and involve ambiguous mass and momentum fluxes as a consequence of the violation of Newton’s first law. For the vorticity budget it now becomes important that the midlatitude $\beta$-plane is much more poorly defined than the equatorial $\beta$-plane. Effectively, it takes the role of a geometric closure assumption which introduces ill-defined vorticity fluxes. In the stability analysis this inconsistency resurfaces as spurious Rossby wave growth, i.e. baroclinic instability.

Vorticity fluxes are crucial in maintaining and changing the density field of the ocean-atmosphere system and a consistent representation of vorticity dynamics is indispensable for realistic numerical simulations of the global circulation. At this time, a large number of numerical circulation models is
based on the Primitive Equations. Moreover, most models are not formulated in terms of spherical coordinates but utilize a multi-β-plane approach to approximate the globe’s sphericity: Laplace operators, for instance, are coded as a sum of second order derivatives, ignoring first order contributions from nontrivial Christoffel symbols. Widely independent of spatio-temporal resolution and the quality of subscale parametrizations, such models cannot expect to simulate the large-scale circulation with geometric-dynamic integrity and realism.

Theoretically as well as numerically, large-scale ocean-atmosphere dynamics require a covariant dynamical framework including the acknowledgement of the spherical geometry of the planet’s surface. The linear stability of the baroclinic fluid against quasigeostrophic disturbances calls for a covariant reanalysis of the observational evidence on dynamical instabilities in the ocean-atmosphere system.

Appendix

The effect of available potential energy of the circulation is considered. With constant surface pressure, the mean layer mass per unit area is now assumed to vary with latitude according to

\[ R(\ell) = R_E(\ell) (1 - b(\ell) y^2) \]

where \( R_E \) is its equatorial and \( R_P \) its polar value. As a geostrophic solution of the nonlinear shallow water equations, its slope parameters

\[ b(1) = a^2 (2 \Omega + U(1) + U(2)) W / 2 \gamma R_E(1) \approx a^2 \Omega W / \gamma R_E(1) \]

\[ b(2) = a^2 [(2 \Omega + U(2))(\delta U(2) - W) - U(1) W] / 2 \gamma R_E(2) \approx a^2 \Omega (\delta U(2) - W) / \gamma R_E(2) \]

with \( \gamma = \gamma(2) \delta \). The associated mean vertical shear \( W = U(1) - U(2) \) is in thermal wind balance and the Coriolis parameter in layer \( \ell \) is given by

\[ F(\ell) = R(\ell) Z(\ell) = 2(\Omega + U(\ell))\sin\varphi. \]

With \( d(\ell) = \partial_t + U(\ell) \partial_\lambda \), the layer representation of this tidal problem assumes the form

\[ d(\ell) m(\ell) + j_n = 0 \]

\[ d(\ell) v_n + \epsilon mn F(\ell) v_n + \partial_n p = 0 \]

where the perturbation pressures are given by

\[ p(1) = \gamma(1)m(1) + \gamma(2)m(2), \quad p(2) = \gamma(2)(m(1) + m(2)) = \gamma(2)m. \]

The curl of the momentum budgets yields the vorticity budgets

\[ R(\ell) d(\ell) z(\ell) + j_n = 0 \]

for the perturbation vorticities

\[ R(\ell) z(\ell) = \epsilon mn v_n - Z(\ell) m(\ell). \]

The quasigeostrophic approximation with

\[ j_n = \epsilon mn \partial_m A(\ell) \]

and the thermal wind relation

\[ \gamma m(1) = Z(2) A(2) - Z(1) A(1), \quad s\gamma m(2) = sZ(1) A(1) - Z(2) A(2). \]
leads to the coupled wave equations
\[ \gamma [(R_1 \Delta - R_1^a \partial_a - F_1^2 / \gamma) d_1(1) - e^{an} R_1^2 \partial_a Z_1 \partial_n] A_1(1) = -R_1 Z_2(2) F_1 d_1 A_2(2) \]
\[ \gamma [(R_2 \Delta - R_2^a \partial_a - F_2^2 / s \gamma) d_2(2) - e^{an} R_2^2 \partial_a Z_2 \partial_n] A_2(2) = -R_2 Z_1(1) F_2 d_2 A_1(1) \]

For \( 0 \leq |b(\ell)| \ll 1 \) these equations are a coupled system of spheroidal equations. Physically, this condition on the slope parameters excludes outcropping of isopycnals. With \( \beta_2(\ell) = 4a^2(\Omega + U(\ell))^2 / g' H_E(\ell) \), \( h(\ell) = \beta_2(\ell) + 2b(\ell) M \tau(\ell) \)
the quasigeostrophic vorticity budgets assume the form
\[ (\Delta - h_1(1)y^2 - M \tau(1)) A_1(1) = -sk\beta_1(1)\beta_2 y^2 A_2(2) \]
\[ k(\Delta - h_2(2)y^2 - M \tau(2)) A_2(2) = -\beta_1(1)\beta_2 y^2 A_1(1) \]

where \( k = \sqrt{H_E^{(1)}/H_E^{(2)}} \). Note that \( h(\ell) \) will become negative for very low Rossby wave frequencies at negative \( b(\ell) M \). Using now \( \Delta A(\ell) = (\beta^2 y^2 - \epsilon) A(\ell) \), these equations reduce to a system of coupled algebraic equations for the amplitudes of the stream functions. Nontrivial solutions exist for vanishing coefficients of the polynomial \( X + Y y^2 + Z y^4 \). For the present problem, these coefficients have the form
\[ X = (\epsilon + M \tau(1))(\epsilon + M \tau(2)) = 0 \]
\[ Y = (\epsilon + M \tau(1))(\beta^2 - h_2(2)) + (\epsilon + M \tau(2))(\beta^2 - h_1(1)) = 0 \]
\[ Z = \beta^4 - (h_1(1) + h_2(2))\beta^2 + h_1(1)h_2(2) - s\beta_1^2(1)\beta_2^2(2) = 0 \]

The first of these equations states the reality of eigenfrequencies for stream functions which remain regular at the poles. Also, it demonstrates that Rossby waves propagate in layers unless the vertical shear vanishes. The second equation selects the Lamb parameter for the respective mode and the third equation determines the Lamb parameters. For \( b(\ell) = 0 \) the solutions of the main text emerge while the case \( c \to \infty \) and \( b(\ell) \neq 0 \) has been discussed in [13].

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