An Extended Solution to the Equations Describing a 3-Conductor Transmission Line

GEORGE ANGELOV
Department of Microelectronics
Faculty of Electronics Engineering and Technologies
Technical University of Sofia
8 Kliment Ohridski Blvd., 1756 Sofia
BULGARIA

Abstract: - The full derivation of the generalized and extended solution to the equations describing three-conductor transmission line is given in this paper; the brief results are presented in a previous paper. The considerations proceed from the C. Paul formulation of lossless transmission lines terminated by linear loads. In contrast to C. Paul, the conjoint interaction between the two lines is considered here and the influence of the receptor line is not neglected, that is the weak-coupling approximation is not applied. In result, an extended and generalized mathematical model compared the original model of C. Paul is obtained. In particular, a mixed problem for the hyperbolic system describing the three-conductor transmission line is formulated. It is shown that the formulated mixed problem is equivalent to an initial value problem for a functional system on the boundary of hyperbolic system’s domain with voltages and currents as the unknown functions in this system are the lines’. The system of functional equations can be resolved by a fixed-point method that enables us to find an approximated but explicit solution. The method elaborated in this paper might be applied also for linear as well as nonlinear boundary conditions.

Key-Words: three-conductor transmission line, electromagnetic compatibility, fixed point method, linear hyperbolic system, initial-boundary problem, mixed problem for hyperbolic system

Received: November 20, 2019. Revised: April 4, 2020. Accepted: April 19, 2020, Published: May 2, 2020.

1 Introduction

VLSI systems and their electromagnetic compatibility (EMC) aspects have attracted a lot of research interest (cf. [1]-[8]). In this paper an EMC model of a 3-conductor transmission line is considered taking into account the results of Clayton R. Paul [9]. In contrast to the considerations in [9], a generalized approach for solving the above problem is proposed. It is also proved that the weak coupling assumptions introduced in [9] are a particular case of the more general handling.

We obtain a general solution of the system by modeling pairwise the interacting 3-conductor transmission line introduced in [9] by keeping to the methods in [11]-[13] that were also used in other solutions such as [14], [15]. Our starting circuit is given in Fig. 1 (cf. [9]). The reference conductor for the line voltages is denoted by the ground symbol. Although it could represent an infinite ground plane, a wire, an overall shield, in our setup it is a land of a printed circuit board. The rest conductors are also lands of a printed circuit board, nevertheless they could be other objects as well. We presume the line to be an uniform and lossless line (cf. [7], [8]).

The top circuit is the generator circuit. It is terminated by a resistive load \( R_L \) and it is driven by a voltage source with open-circuit voltage \( U_S(t) \) and source resistance \( R_S \). The bottom circuit is the receptor circuit. It is terminated by a resistive load \( R_{NE} \) at the near end and by a resistive load \( R_{FE} \) at the far end. At the terminals of the receptor circuit, the electric and magnetic fields originating by the voltage and current of the generator circuit, interact with the receptor circuit producing crosstalk voltages.
We aim at finding a solution for the crosstalk voltages based on a system that is more than the one in [9]. That is, we proceed from the hyperbolic system (1) obtained in accordance to the TEM mode of propagation (cf. [1]-[8]). The voltages with respect to the reference conductor \( u_k(x,t) (k=1,2) \) and the currents of each circuit \( i_k(x,t) (k=1,2) \) are functions of position \( x \) and time \( t \).

\[
\begin{align*}
\frac{\partial u_G(x,t)}{\partial x} + L_G \frac{\partial i_G(x,t)}{\partial t} &= -L_m \frac{\partial i_R(x,t)}{\partial t} \\
\frac{\partial i_G(x,t)}{\partial x} + (C_G + C_m) \frac{\partial u_G(x,t)}{\partial t} &= C_m \frac{\partial u_R(x,t)}{\partial t} \\
\frac{\partial u_R(x,t)}{\partial x} + L_R \frac{\partial i_R(x,t)}{\partial t} &= -L_m \frac{\partial i_G(x,t)}{\partial t} \\
\frac{\partial i_R(x,t)}{\partial x} + (C_R + C_m) \frac{\partial u_R(x,t)}{\partial t} &= C_m \frac{\partial u_G(x,t)}{\partial t}
\end{align*}
\]  

(1)

with the following boundary

\[
\begin{align*}
u_G(0,t) &= U_S(t) - R_{SG}i_G(0,t), & U_{NE} &= u_R(0,t) = -R_{NE}i_R(0,t) \\
u_G(\Lambda,t) &= R_{IG}i_G(\Lambda,t), & U_{FE} &= u_R(\Lambda,t) = R_{FEG}i_R(\Lambda,t)
\end{align*}
\]  

(2)

and initial conditions:

\[
\begin{align*}
u_G(x,0) &= u_{G0}(x), & u_R(x,0) &= u_{R0}(x) \\
i_G(x,0) &= i_{G0}(x), & i_R(x,0) &= i_{R0}(x), & x \in [0,\Lambda]
\end{align*}
\]  

(3)

Before going further, we would stress upon the fact that in our considerations, we do not apply the weak coupling assumption as opposite to [9] where this assumption is applied. This means that we do not neglect the right-hand side of (1). Therefore, our method is a more general case of (1).

Rewrite the above system (1) in the form

\[
\begin{align*}
(C_G + C_m) \frac{\partial u_G(x,t)}{\partial t} - C_m \frac{\partial u_R(x,t)}{\partial x} + \frac{\partial i_G(x,t)}{\partial t} &= 0 \\
-C_m \frac{\partial u_G(x,t)}{\partial t} + (C_R + C_m) \frac{\partial u_R(x,t)}{\partial x} + \frac{\partial i_R(x,t)}{\partial t} &= 0 \\
L_G \frac{\partial i_G(x,t)}{\partial t} + L_m \frac{\partial i_R(x,t)}{\partial x} + \frac{\partial u_G(x,t)}{\partial x} &= 0 \\
L_m \frac{\partial i_G(x,t)}{\partial t} + L_R \frac{\partial i_R(x,t)}{\partial x} + \frac{\partial u_R(x,t)}{\partial x} &= 0
\end{align*}
\]

and introduce denotations

\[
\begin{align*}
&u_1(x,t) = u_G(x,t); \quad u_2(x,t) = u_R(x,t); \\
&i_1(x,t) = i_G(x,t); \quad i_2(x,t) = i_R(x,t) \\
&C_{11} = C_G + C_m, & C_{12} = C_R + C_m, & C_{21} = -C_m, & C_{22} = C_G + C_m \\
&L_1 = L_G, & L_2 = L_R \\
&C = \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}, & L = \begin{pmatrix} L_1 & L_2 \\ L_2 & L_2 \end{pmatrix}
\end{align*}
\]

Then we reach the following mixed problem in the new denotations

\[
\begin{align*}
&\frac{\partial u_1(x,t)}{\partial t} + \frac{\partial u_2(x,t)}{\partial x} + \frac{\partial i_1(x,t)}{\partial t} = 0, \\
&\frac{\partial u_1(x,t)}{\partial t} + \frac{\partial u_2(x,t)}{\partial t} + \frac{\partial i_2(x,t)}{\partial x} = 0, \\
&\frac{\partial i_1(x,t)}{\partial t} + \frac{\partial i_2(x,t)}{\partial t} + \frac{\partial u_1(x,t)}{\partial x} = 0, \\
&\frac{\partial i_1(x,t)}{\partial t} + \frac{\partial i_2(x,t)}{\partial t} + \frac{\partial u_2(x,t)}{\partial x} = 0
\end{align*}
\]

2 Hyperbolic system transformation

In a matrix form the above system (4) is
\[
\begin{bmatrix}
C_{11} & C_{12} & 0 & 0 \\
C_{12} & C_{22} & 0 & 0 \\
0 & 0 & L_{11} & L_{12} \\
0 & 0 & L_{12} & L_{22}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u_1}{\partial t} \\
\frac{\partial u_2}{\partial t} \\
\frac{\partial i_1}{\partial t} \\
\frac{\partial i_2}{\partial t}
\end{bmatrix} +
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
L_{22} & -L_{12} \\
-L_{12} & L_{11}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u_1}{\partial t} \\
\frac{\partial u_2}{\partial t} \\
\frac{\partial i_1}{\partial t} \\
\frac{\partial i_2}{\partial t}
\end{bmatrix} = 0
\] (5)

Since
\[
\Delta_c = C_i C_{22} - C_{12}^2 = (C_0 + C_m)(C_0 + C_m) - C_{12}^2 = C_0 C_R + C_0 C_m + C_R C_m > 0
\]
we have to assume

**Assumption (L):** \( \Delta_L = L_0 L_R - L_m^2 = L_0 L_{22} - L_{12}^2 \neq 0 \).

This implies
\[
[A] = 
\begin{bmatrix}
C_{11} & C_{12} & 0 & 0 \\
C_{12} & C_{22} & 0 & 0 \\
0 & 0 & L_{11} & L_{12} \\
0 & 0 & L_{12} & L_{22}
\end{bmatrix} = \Delta_c \Delta_L \neq 0
\]
and therefore \( A^{-1} \) does exist:
\[
A^{-1} = \frac{1}{\Delta_c \Delta_L} 
\begin{bmatrix}
C_{22} \Delta_L & -C_{12} \Delta_L & 0 & 0 \\
-C_{12} \Delta_L & C_{11} \Delta_L & 0 & 0 \\
0 & 0 & L_{22} \Delta_C & -L_{12} \Delta_C \\
0 & 0 & -L_{12} \Delta_C & L_{11} \Delta_C
\end{bmatrix}
\]

Multiplying (5) from the left by \( A^{-1} \) and in view of
\[
B = 
\begin{bmatrix}
\frac{C_{22}}{\Delta_c} & \frac{-C_{12}}{\Delta_c} & 0 & 0 \\
\frac{-C_{12}}{\Delta_c} & \frac{C_{11}}{\Delta_c} & 0 & 0 \\
0 & 0 & \frac{L_{22}}{\Delta_L} & \frac{-L_{12}}{\Delta_L} \\
0 & 0 & \frac{-L_{12}}{\Delta_L} & \frac{L_{11}}{\Delta_L}
\end{bmatrix} \times
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]
we obtain
\[
\begin{bmatrix}
\frac{\partial u_1}{\partial t} \\
\frac{\partial u_2}{\partial t} \\
\frac{\partial i_1}{\partial t} \\
\frac{\partial i_2}{\partial t}
\end{bmatrix} +
\begin{bmatrix}
\frac{L_{22}}{\Delta_L} -\frac{L_{12}}{\Delta_L} & 0 & 0 \\
\frac{-L_{12}}{\Delta_L} & \frac{L_{11}}{\Delta_L} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u_1}{\partial t} \\
\frac{\partial u_2}{\partial t} \\
\frac{\partial i_1}{\partial t} \\
\frac{\partial i_2}{\partial t}
\end{bmatrix} = 0
\] (6)

Rewrite (6) in a matrix form
\[
\frac{\partial U(x,t)}{\partial t} + B \frac{\partial U(x,t)}{\partial x} = 0
\] (7)

Substitute \( U(x,t) = HZ(x,t) \) in (7):
\[
H \frac{\partial Z(x,t)}{\partial t} + BH \frac{\partial Z(x,t)}{\partial x} = 0
\]
and multiplying by \( H^{-1} \) we obtain
\[
\frac{\partial Z(x,t)}{\partial t} + H^{-1}BH \frac{\partial Z(x,t)}{\partial x} = 0
\]

We have to find \( H \) such that \( H^{-1}BH = B^{can} \), where
\[
B^{can} = 
\begin{bmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_4
\end{bmatrix}
\]
and \( \lambda_k (k = 1, 2, 3, 4) \) are the eigen values of \( B \), i.e. the roots of the equation
\[
|B - \lambda I| =
\begin{bmatrix}
-\lambda & 0 & \frac{C_{22}}{\Delta_c} & \frac{-C_{12}}{\Delta_c} \\
0 & -\lambda & \frac{-C_{12}}{\Delta_c} & \frac{C_{11}}{\Delta_c} \\
\frac{L_{22}}{\Delta_L} -\frac{L_{12}}{\Delta_L} & -\lambda & 0 \\
-\frac{L_{12}}{\Delta_L} & \frac{L_{11}}{\Delta_L} & 0 & -\lambda
\end{bmatrix}
\]
\[
\begin{align*}
\lambda^4 - L_1 C_{11} + 2 L_1 C_{12} + L_2 C_{22} - \lambda^2 + \frac{1}{\Delta_c} = 0 \\
= \Delta_c \Delta_L (L_1 C_{11} + 2 L_1 C_{12} + L_2 C_{22}) - \lambda^2 + 1 = 0
\end{align*}
\]

We suppose

**Assumption (D):**

\[
D = (L_1 C_{11} + 2 L_1 C_{12} + L_2 C_{22})^2 - 4 (C_{11} C_{22} - C_{12}^2) (L_1 L_2 - L_1^2) > 0
\]

For the characteristic roots we obtain

\[
\lambda_1 = \sqrt{\frac{[L_1 C_{11} + 2 L_1 C_{12} + L_2 C_{22} + \sqrt{D}]}{2 (C_{11} - C_{12}^2) (L_1 L_2 - L_1^2)}},
\]

\[
\lambda_2 = \sqrt{\frac{[L_1 C_{11} + 2 L_1 C_{12} + L_2 C_{22} - \sqrt{D}]}{2 (C_{11} - C_{12}^2) (L_1 L_2 - L_1^2)}},
\]

\[
\lambda_3 = -\sqrt{\frac{[L_1 C_{11} + 2 L_1 C_{12} + L_2 C_{22} + \sqrt{D}]}{2 (C_{11} - C_{12}^2) (L_1 L_2 - L_1^2)}},
\]

\[
\lambda_4 = -\sqrt{\frac{[L_1 C_{11} + 2 L_1 C_{12} + L_2 C_{22} - \sqrt{D}]}{2 (C_{11} - C_{12}^2) (L_1 L_2 - L_1^2)}}.
\]

**Remark.** For the sake of simplicity, we could find the eigenvectors of

\[
\begin{bmatrix}
B^{-1} - \mu_k \mathbf{I}
\end{bmatrix} \mathbf{H}^{(k)} = 0;
\]

\[
\mu_k = 1/\lambda_k, \quad \mathbf{H}^{(k)} = \begin{pmatrix}
\xi_{k1}, \xi_{k2}, \xi_{k3}, \xi_{k4}
\end{pmatrix}^T
\]

(\text{instead of } (B - \lambda_k \mathbf{I}) \mathbf{H}^{(k)} = 0)

Because

\[
B^{-1} = \begin{bmatrix}
0 & 0 & L_{11} & L_{12} \\
0 & 0 & L_{12} & L_{22} \\
C_{11} & C_{12} & 0 & 0 \\
C_{12} & C_{22} & 0 & 0
\end{bmatrix}
\]

has a simpler form than \(B\).

Now we have to solve the following systems in order to obtain 4 eigenvectors \(\mathbf{H}^{(k)} (k = 1, 2, 3, 4)\)

\[
\begin{align*}
(B - \lambda_1 \mathbf{I}) \mathbf{H}^{(1)} &= 0, \\
(B - \lambda_2 \mathbf{I}) \mathbf{H}^{(2)} &= 0, \\
(B - \lambda_3 \mathbf{I}) \mathbf{H}^{(3)} &= 0, \\
(B - \lambda_4 \mathbf{I}) \mathbf{H}^{(4)} &= 0
\end{align*}
\]

corresponding to eigenvector

\[
\mathbf{H}^{(k)} = \begin{pmatrix}
\xi_{k1}, \xi_{k2}, \xi_{k3}, \xi_{k4}
\end{pmatrix}^T; \quad (k = 1, 2, 3, 4)
\]

To solve (9) we have to assume:

\[
L_{12} C_{11} + L_{22} C_{12} = L_m (C_G + C_m) - L_R C_m \neq 0
\]

and

\[
L_{12} C_{22} + L_1 C_{12} = L_m (C_R + C_m) - L_G C_m \neq 0
\]

Therefore

\[
\xi_{1k} = \frac{\lambda_k^2 \Delta_c C_{12} + L_{12}}{\lambda_k (L_2 C_{11} + L_2 C_{12})}, \quad \\
\xi_{2k} = \frac{L_{22} - \lambda_k^2 \Delta_c C_{11}}{\lambda_k (L_2 C_{11} + L_2 C_{12})}, \quad \\
\xi_{3k} = 1, \quad \\
\xi_{4k} = \frac{L_{22} C_{12} + L_{22} C_{22} - \lambda_k^2 \Delta_c}{(L_2 C_{22} + L_2 C_{12})}
\]

Introduce denotations:

\[
\gamma_k = \frac{\left(\frac{1}{\lambda_k^2}\right) - (L_1 C_{11} + L_2 C_{12})}{L_2 C_{11} + L_2 C_{12}} = (k = 1, 2).
\]

Note that

\[
\lambda_1 > \lambda_2 > 0; \quad \lambda_3 = -\lambda_1; \quad \lambda_4 = -\lambda_2;
\]

\[
\sqrt{D} \Rightarrow \gamma_2 - \gamma_1 = \frac{\sqrt{D}}{L_2 C_{11} + L_2 C_{12}}.\]

Then we obtain the following eigenvectors:

\[
\mathbf{H}^{(1)} = \begin{pmatrix} p_1, q_1, 1, \gamma_1 \end{pmatrix}^T, \quad \mathbf{H}^{(2)} = \begin{pmatrix} p_2, q_2, 1, \gamma_2 \end{pmatrix}^T,
\]

\[
\mathbf{H}^{(3)} = \begin{pmatrix} -p_1, -q_1, 1, \gamma_1 \end{pmatrix}^T, \quad \mathbf{H}^{(4)} = \begin{pmatrix} -p_2, -q_2, 1, \gamma_2 \end{pmatrix}^T
\]

where

\[
p_k = \frac{L_{12} + \lambda_k^2 \Delta_c C_{12}}{\lambda_k (L_2 C_{11} + L_2 C_{12})} = \lambda_k (L_1 + L_2 \gamma_k),
\]

\[
q_k = \frac{L_{22} - \lambda_k^2 \Delta_c C_{11}}{\lambda_k (L_2 C_{11} + L_2 C_{12})} = \lambda_k (L_2 + L_2 \gamma_k), \quad (k = 1, 2)
\]

Thus transformation matrix becomes
3 Boundary conditions derivation
with respect to the new variables

Introduce new variables $U = HZ$ and $Z = H^{-1}U$, where $U = (u_1, u_2, i_1, i_2)^T$, $Z = (l_1, l_2, l_3, l_4)^T$.

Then

$$u_1(x,t) = p_1 l_1(x,t) + p_2 l_2(x,t) - p_1 l_3(x,t) - p_2 l_4(x,t)$$
$$u_2(x,t) = q_1 l_1(x,t) + q_2 l_2(x,t) - q_1 l_3(x,t) - q_2 l_4(x,t)$$
$$i_1(x,t) = l_1(x,t) + l_3(x,t) + l_4(x,t)$$
$$i_2(x,t) = \gamma_1 l_1(x,t) + \gamma_2 l_2(x,t) + \gamma_1 l_3(x,t) + \gamma_2 l_4(x,t)$$

and

$$l_1(x,t) = \frac{1}{2(\gamma_2 - \gamma_1)} \left( q_1 \sqrt{\frac{\Delta_c}{\Delta_L}} u_1(x,t) + \gamma_1 \dot{i}_1(x,t) + i_2(x,t) \right)$$
$$l_2(x,t) = \frac{1}{2(\gamma_2 - \gamma_1)} \left( -q_1 \sqrt{\frac{\Delta_c}{\Delta_L}} u_1(x,t) + \gamma_2 \dot{i}_1(x,t) - i_2(x,t) \right)$$
$$l_3(x,t) = \frac{1}{2(\gamma_2 - \gamma_1)} \left( -p_1 \sqrt{\frac{\Delta_c}{\Delta_L}} u_2(x,t) + \gamma_1 \dot{i}_2(x,t) + i_2(x,t) \right)$$
$$l_4(x,t) = \frac{1}{2(\gamma_2 - \gamma_1)} \left( p_1 \sqrt{\frac{\Delta_c}{\Delta_L}} u_2(x,t) - \gamma_2 \dot{i}_2(x,t) - i_2(x,t) \right)$$

Then the mixed problem (1)-(3) becomes as follows: to find a solution of the system

$$\frac{\partial I_1(x,t)}{\partial t} + \lambda_1 \frac{\partial I_1(x,t)}{\partial x} = 0,$$
$$\frac{\partial I_2(x,t)}{\partial t} + \lambda_2 \frac{\partial I_2(x,t)}{\partial x} = 0,$$
$$\frac{\partial I_3(x,t)}{\partial t} - \lambda_1 \frac{\partial I_3(x,t)}{\partial x} = 0,$$
$$\frac{\partial I_4(x,t)}{\partial t} - \lambda_2 \frac{\partial I_4(x,t)}{\partial x} = 0$$

with initial conditions and boundary conditions in the new variables:

$$I_1(x,0) = \frac{1}{2(\gamma_2 - \gamma_1)} \left( q_1 \sqrt{\frac{\Delta_c}{\Delta_L}} u_{i_1}(x) + \gamma_1 \dot{i}_{i_1}(x) + i_{i_2}(x) \right) = I_{i_0}(x)$$
$$I_2(x,0) = \frac{1}{2(\gamma_2 - \gamma_1)} \left( -q_1 \sqrt{\frac{\Delta_c}{\Delta_L}} u_{i_1}(x) + \gamma_2 \dot{i}_{i_2}(x) - i_{i_2}(x) \right) = I_{i_0}(x)$$
$$I_3(x,0) = \frac{1}{2(\gamma_2 - \gamma_1)} \left( -p_1 \sqrt{\frac{\Delta_c}{\Delta_L}} u_{i_2}(x) - \gamma_1 \dot{i}_{i_1}(x) + i_{i_2}(x) \right) = I_{i_0}(x)$$
$$I_4(x,0) = \frac{1}{2(\gamma_2 - \gamma_1)} \left( p_1 \sqrt{\frac{\Delta_c}{\Delta_L}} u_{i_2}(x) + \gamma_2 \dot{i}_{i_2}(x) - i_{i_2}(x) \right) = I_{i_0}(x)$$

To obtain the boundary conditions

$$u_1(0,t) = U_s(t) - R_{i_1}(0,t), \quad u_1(x,\Lambda) = R_{i_1}(\Lambda,t),$$
$$U_{NE} = u_2(0,t) = -R_{i_2}(0,t), \quad U_{NE} = u_2(\Lambda,t) = R_{i_2}(\Lambda,t)$$

with respect to the new variables we take into account.
4 Obtaining functional equations equivalent to the mixed problem

Proceeding as in [11], we find the characteristics of the system (10) which are forming four families of curves

\[ \frac{dx}{dt} = \lambda_1, \quad \frac{dx}{dt} = \lambda_2, \quad \frac{dx}{dt} = -\lambda_1, \quad \frac{dx}{dt} = -\lambda_2 \]  \hspace{1cm} (15)

Through each point \((x, t) \in \Pi = \{(x, t) \in [0, \Lambda] \times [0, T]\}\) there are 4 characteristics: \(C_1, C_2\) with positive slopes and \(C_3, C_4\) with negative slopes. A characteristic \(C_k\) \((k = 1, 2)\) through a point \((0, t_k)\) intersects the boundary \(x = \Lambda\) at some point \((\Lambda, t_k + T_k)\) where \(T_k\) can be found by integration of \(\frac{dx}{dt} = \lambda_k\). Since the characteristic \(C_k\) is \(x - \lambda_k t = \text{const}\), then the straight line through \((0, t_k)\) is

\[ x - \lambda_k t = -\lambda_k t_k \Rightarrow t = \frac{x}{\lambda_k} + t_k. \]

Setting \(x = \Lambda\) and \(t = t_k + T_k\) we obtain

\[ \Lambda - \lambda_k (t + T_k) = -\lambda_k t \Rightarrow T_k = \frac{\Lambda}{\lambda_k}. \]

Similarly, a characteristic \(C_p\) \((p = 3, 4)\) is \(x + \lambda_p T = \text{const}\) (with \(\lambda_3 = -\lambda_1, \lambda_4 = -\lambda_2\)) and the straight line through a point \((\Lambda, t_k)\) is \(x + \lambda_p t = \Lambda + \lambda_p t_k\). It intersects \(x = 0\) at a point \((0, t_p + T_p)\). Therefore

\[ \lambda_p (t_p + T_p) = \Lambda + \lambda_p t_k \Rightarrow T_p = \frac{\Lambda}{\lambda_p} \quad (p = 3, 4), \]

i.e. \(T_3 = T_1, T_4 = T_2\).

Introduce directional derivatives

\[ D_k = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \frac{d}{dt} \frac{\partial}{\partial x} \right) + \lambda_k \right) = \left( p_k + R_k \right) I_k (0, t) + \left( p_k + R_k \right) I_k (0, t) =

Then system (10) can be written in the form:

\[ D_k I_k = 0 \quad (k = 1, 2) \] \hspace{1cm} (16)

\[ D_k I_k = 0 \quad (k = 3, 4) \] \hspace{1cm} (17)

Integrating each equation from (16) along the characteristic \(C_k\) from \((0, t)\) to \((\Lambda, t + T_k)\) (where the integration is a line integral along \(C_k\)), we obtain

\[ I_k (\Lambda, t + T_k) = I_k (0, t) \quad (t \geq 0). \]

In the same way, by integrating in (17) from \((0, t+T_k)\) to \((\Lambda,t)\), we get

\[ I_k (\Lambda, t) = I_k (0, t + T_k) \quad (t \geq 0). \]

Present (12) in the form:

\[ \left( p_1 + R_s \right) I_1 (0, t) + \left( p_2 + R_s \right) I_2 (0, t) + \left( q_1 + R_{SE} \gamma_1 \right) I_1 (0, t) + \left( q_2 + \gamma_2 R_{SE} \right) I_2 (0, t) = \]

\[ \left( q_2 - R_{SE} \gamma_2 \right) I_1 (0, t) + \left( q_1 - R_{SE} \gamma_1 \right) I_2 (0, t) \]

\[ \left( R_3 + p_2 \right) I_3 (\Lambda, t) + \left( p_1 + R_L \right) I_4 (\Lambda, t) = \]

\[ \left( p_1 - R_k \right) I_1 (\Lambda, t) + \left( p_2 - R_k \right) I_2 (\Lambda, t) \]

\[ \left( q_2 + R_{FE} \gamma_2 \right) I_3 (\Lambda, t) + \left( q_1 + R_{FE} \gamma_1 \right) I_4 (\Lambda, t) = \]

Since

\[ \Delta_{12} = \left| \begin{array}{cc} p_1 + R_s & p_2 + R_s \\ q_1 + R_{SE} \gamma_1 & q_2 + R_{SE} \gamma_2 \end{array} \right| = \left( p_1 q_2 - p_2 q_1 \right) + \left( \gamma_2 - \gamma_1 \right) R_{SE} + \]

\[ \left( p_1 \gamma_2 - p_2 \gamma_1 \right) R_{SE} + \left( q_2 - q_1 \right) R_{SE} = \]
\[ (\gamma_2 - \gamma_1) \sqrt{\Delta_L / \Delta_C} + (\gamma_2 - \gamma_1) R_s R_{NE} + \left( \lambda_1 - \lambda_2 \right) \frac{C_{22} \Delta_L + L_{11} \Delta_C}{L_{11} C_{11} + L_{22} C_{12}} R_{NE} + (q_2 - q_1) R_s = \]
\[ = \sqrt{D} \left( \frac{\Delta_L / \Delta_C + R_s R_{NE}}{L_{11} C_{11} + L_{22} C_{12}} \right) \left( \lambda_1 - \lambda_2 \right) \times \left( C_{22} \Delta_L + L_{11} \Delta_C \right) R_{NE} + \left( \lambda_2 + \lambda_1 \lambda_2 \Delta_C C_{11} \right) R_s \]
\[ \neq 0 \]

Hence, we can solve first two equations with respect to \( I_1(0,t) \) which leads to
\[ I_1(0,t) = A_{i0}(t) + A_{i1} I_3(0,t) + A_{i2} I_4(0,t) , \]
where
\[ A_{i0}(t) = \frac{(q_3 + R_{NE} \gamma_3) U_{i}(t)}{\Delta_{i2}} ; \]
\[ A_{i1} = \frac{2(p_2 \gamma_R R_{NE} - q_2 R_s)}{\Delta_{i2}} ; \]
\[ A_{i2} = \frac{p_1 q_2 - p_2 q_1 - (q_2 + q_1) R_s + (p_1 \gamma_R + p_2 \gamma_R) R_{NE} + (\gamma_1 - \gamma_2) R_s R_{NE}}{\Delta_{i2}} ; \]
\[ I_2(0,t) = A_{i0}(t) + A_{i1} I_3(0,t) + A_{i2} I_4(0,t) , \]
where
\[ A_{i0}(t) = \frac{(q_1 + R_{NE} \gamma_3) U_{i}(t)}{\Delta_{i2}} ; \]
\[ A_{i1} = \frac{p_1 q_2 - p_2 q_1 - (q_2 + q_1) R_s}{\Delta_{i2}} - \frac{(p_1 \gamma_R + p_2 \gamma_R) R_{NE} + (\gamma_1 - \gamma_2) R_s R_{NE}}{\Delta_{i2}} ; \]
\[ A_{i2} = \frac{-2p_1 \gamma_R + 2q_2 R_s}{\Delta_{i2}} ; \]

Similarly
\[ \Delta_{i4} = \begin{vmatrix} p_2 + R_s & p_1 + R_L \\ q_2 + \gamma_R R_{FE} & q_1 + \gamma_R R_{FE} \end{vmatrix} = \]
\[ = (p_2 + R_s)(q_1 + \gamma_R R_{FE}) - (q_2 + \gamma_R R_{FE})(p_1 + R_L) = \]
\[ = -\sqrt{D} \left( \frac{\Delta_L / \Delta_C + R_{FE} R_{L}}{L_{11} C_{11} + L_{22} C_{12}} \right) \frac{\lambda_1 - \lambda_2}{L_{12} C_{11} + L_{22} C_{12}} \times \left( C_{22} \Delta_L + L_{11} \Delta_C \right) R_{FE} + \left( L_{22} + \lambda_1 \lambda_2 \Delta_C \right) R_s \]
\[ \neq 0 \]

Consequently
\[ I_3(\Lambda,t) = B_{i1} I_1(\Lambda,t) + B_{i2} I_2(\Lambda,t) , \]
where
\[ B_{i1} = \frac{2p_1 \gamma_R R_{FE} - 2q_2 R_s}{\Delta_{i4}} ; \]
\[ B_{i2} = \frac{p_1 q_1 - p_2 q_2 - (q_1 + q_2) R_s}{\Delta_{i4}} + \frac{(p_1 \gamma_R + p_2 \gamma_R) R_{FE} + (\gamma_1 - \gamma_2) R_s R_{FE}}{\Delta_{i4}} ; \]
\[ I_4(\Lambda,t) = B_{i3} I_1(\Lambda,t) + B_{i4} I_2(\Lambda,t) , \]
where
\[ B_{i3} = \frac{p_1 q_1 - p_2 q_2 + (q_1 + q_2) R_s}{\Delta_{i4}} + \frac{(p_1 \gamma_R + p_2 \gamma_R) R_{FE} - (p_1 \gamma_R + p_2 \gamma_R) R_{FE}}{\Delta_{i4}} ; \]
\[ B_{i4} = \frac{2q_4 R_s - 2p_2 \gamma_R R_{FE}}{\Delta_{i4}} . \]

So, we have obtained a system of functional equations
\[ I_1(0,t) = A_{i0}(t) + A_{i1} I_3(0,t) + A_{i2} I_4(0,t) , \]
\[ I_3(\Lambda,t) = B_{i1} I_1(\Lambda,t) + B_{i2} I_2(\Lambda,t) , \]
\[ I_4(\Lambda,t) = B_{i3} I_1(\Lambda,t) + B_{i4} I_2(\Lambda,t) , \]
\[ (k = 3,4) \]

Taking into account \( I_4(\Lambda,t-T_k) = I_4(0,t) \), we can rewrite the first two equations in the following way:
\[ I_1(0,t) = A_{i0}(t) + A_{i1} I_3(0,t) + A_{i2} I_4(0,t) , \]
\[ I_2(0,t) = A_{i0}(t) + A_{i1} I_3(0,t) + A_{i2} I_4(0,t) , \]
\[ I_3(\Lambda,t) = B_{i1} I_1(\Lambda,t) + B_{i2} I_2(\Lambda,t) , \]
\[ I_4(\Lambda,t) = B_{i3} I_1(\Lambda,t) + B_{i4} I_2(\Lambda,t) , \]
\[ (k = 1,2) \]

Similarly
\[ I_3(\Lambda,t) = I_4(0,t-T_k) \]
\[ I_4(\Lambda,t) = B_{i3} I_1(0,t-T_k) + B_{i4} I_2(0,t-T_k) , \]
\[ I_3(\Lambda,t) = I_4(0,t-T_k) \]
\[ I_4(\Lambda,t) = B_{i3} I_1(0,t-T_k) + B_{i4} I_2(0,t-T_k) , \]
Denoting the unknown functions by
\[ I_1(t, t) = I_1(t), \quad I_2(t, t) = I_2(t), \]
\[ I_3(t, t) = I_3(A, t), \quad I_4(t, t) = I_4(A, t) \]
and taking into account \( T_1 = T_3, \ T_2 = T_4 \) we obtain the following system:
\[
\begin{align*}
I_1(t) &= A_{10}(t) + A_{11}I_1(t - T_1) + A_{12}I_2(t - T_2) \\
I_2(t) &= A_{20}(t) + A_{13}I_1(t - T_1) + A_{22}I_4(t - T_2) \\
I_3(t) &= B_{11}I_1(t - T_1) + B_{12}I_2(t - T_2) \\
I_4(t) &= B_{21}I_1(t - T_1) + B_{22}I_2(t - T_2)
\end{align*}
\]

To obtain initial conditions on the intervals \([-T_1, 0], [-T_2, 0]\) one can shifted the initial functions \(I_{10}(t), I_{20}(t), I_{30}(t), I_{40}(t)\) from the interval \([0, \infty)\) along the characteristics to the intervals \([-T_1, 0], [-T_2, 0]\) (cf. [12]).

The obtained functions after the above transformation on the boundary we denote by \(I_{10}(t), I_{20}(t), I_{30}(t), I_{40}(t)\).

If \(u_{10}(x), u_{20}(x), i_{10}(x), i_{20}(x)\) are periodic functions then \(I_{10}(t), I_{20}(t), I_{30}(t), I_{40}(t)\) are periodic functions too.

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The obtained functions after the above transformation on the boundary we denote by
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5 Operator presentation of the periodic problem

Introduce the sets
\[
M_1 = \left\{ I_1(t) \in C_{[0, \infty)} : |I_1(t)| \leq I_{01} e^{\rho(t - T_0)}, \,
|t| \in [kT_0, (k + 1)T_0], \,
I_1(t) = I_{10}(t), t \in [-T_1, 0] \right\},
\]
\[
M_2 = \left\{ I_2(t) \in C_{[0, \infty)} : |I_2(t)| \leq I_{02} e^{\rho(t - T_0)}, \,
|t| \in [kT_0, (k + 1)T_0], \,
I_2(t) = I_{20}(t), t \in [-T_2, 0] \right\},
\]
\[
M_3 = \left\{ I_3(t) \in C_{[0, \infty)} : |I_3(t)| \leq I_{03} e^{\rho(t - T_0)}, \,
|t| \in [kT_0, (k + 1)T_0], \,
I_3(t) = I_{30}(t), t \in [-T_3, 0] \right\},
\]
\[
M_4 = \left\{ I_4(t) \in C_{[0, \infty)} : |I_4(t)| \leq I_{04} e^{\rho(t - T_0)}, \,
|t| \in [kT_0, (k + 1)T_0], \,
I_4(t) = I_{40}(t), t \in [-T_4, 0] \right\},
\]

\((k = 0, 1, 2, \ldots)\) \((k = 0, 1, 2, \ldots)\), where \(C_{[0, \infty)}\) is the set of all continuous \(T_0\)-periodic functions and \(I_{00}, T_0, \mu\) are positive constants and \(\mu T_0 = \mu_0 = \text{const}.

We use the technique of fixed point theory in uniform spaces (cf. [13]). For that purpose, we introduce a saturated family of pseudo-metrics in the Cartesian product
\[
M = M_1 \times M_2 \times M_3 \times M_4 :
\]
\[
\rho^\infty(I_\alpha, T_\alpha) = \max \left\{ \left| I_\alpha(t) - T_\alpha(t) e^{-\rho(t - kT_0)} : t \in [kT_0, (k + 1)T_0] \right| \right\}
(n = 1, 2, 3, 4; \, k = 0, 1, 2, \ldots)
\]

where the index set of this family consists of all ordered fours
\[
(p_1, p_2, p_3, p_4) \in N_0 \times N_0 \times N_0 \times N_0; \, N_0 = \{0, 1, 2, \ldots\}
\]
corresponding to the initial points of the intervals
\[
[kT_0, (p_1 + 1)T_0] \times [p_2T_0, (p_2 + 1)T_0] \times [p_3T_0, (p_3 + 1)T_0] \times [p_4T_0, (p_4 + 1)T_0].
\]

Introduce maps \(j_n(k) : N_0 \rightarrow N_0\) \((n = 1, 2)\) in the following way:
\[
j_1 : N_0 \rightarrow N_0, \, [kT_0, (k + 1)T_0] \rightarrow [kT_0 - T_1, (k + 1)T_0 - T_1].
\]

We suppose that \(T_p = m_p T_0, \, (p = 1, 2)\). Therefore
\[
[kT_0 - T_p, (k + 1)T_0 - T_p] = [\left( k - m_p \right) T_0, \left( k + 1 - m_p \right) T_0] - \]
and then \(j_p(k) : k \rightarrow k - m_p\) provided \(k - m_p \geq 0\). \(j_p^n(k) = j_p(j_p^{n-1}(k))\), \(j_p^0(k) = k\). The definition of \(j_p\) implies that \(j_p^n(k) \in N_0\) only for finite \(m\).

Define
\[
\begin{align*}
&j(p_1, p_2, p_3, p_4) = \left( j_1(p_1), j_2(p_2), j_3(p_3), j_4(p_4) \right) \\
&j^2(p_1, p_2, p_3, p_4) = \left( j_1^2(p_1), j_2^2(p_2), j_3^2(p_3), j_4^2(p_4) \right)
\end{align*}
\]

In particular,
\[
\begin{align*}
&j(p, p, p, p) = \left( j_1(p), j_2(p), j_3(p), j_4(p) \right)
\end{align*}
\]

The set \(M\) turns out into a complete uniform space with a saturated family of pseudometrics (cf. [13]).
6 Existence–uniqueness of periodic solution

The main result is:

**Theorem 1.** Let the following conditions be fulfilled:

\[ I_{10}(t), I_{30}(t) \in C_{r_1}^1[-T_1, 0], \quad I_{20}(t), I_{40}(t) \in C_{r_2}^1[-T_2, 0] \]

\[ U_S(t) \in C_{r_3}[0, \infty), \quad \bar{U}_S = \max \{U_S(t) : t \in [0, T_0]\} \]

Assumptions (D) and (L) are valid;

\[ T_1 = m_1 T_0, T_2 = m_2 T_0 \] for positive integers \( m_1, m_2 \)

\[
\frac{q_2 + R_{SE} T_2}{\lambda_2 L_{12} + \left( L_2 \lambda_2 + R_{SE} \right) \times \lambda_2^2 \left( L_2 C_{11} + L_2 C_{12} \right)} < \min \{I_{10}, I_{20}\}
\]

\[
\frac{q_1 + R_{SE} T_1}{\lambda_2 L_{12} + \left( L_2 \lambda_2 + R_{SE} \right) \times \lambda_2^2 \left( L_2 C_{11} + L_2 C_{12} \right)} < \min \{I_{10}, I_{20}\}
\]

Then there exists a unique \( T_0 \)-periodic solution of (19).

**Proof:** The set \( M_1 \times M_2 \times M_3 \times M_4 \) is a uniform space with the above saturated family of pseudometrics. In view of \( \lambda_1 > \lambda_2 \Rightarrow \lambda / \lambda_2 > \lambda / \lambda_1 \Rightarrow T_2 > T_1 \) we show that \( B \) maps \( M_1 \times M_2 \times M_3 \times M_4 \) into itself. It is easy to verify that all components of the operator \( B \) are periodic functions.

For \( t \in [k T_0, (k+1) T_0] \) and sufficiently large \( \mu > 0 \) , having in mind (12), (13) we obtain:

\[
\left| B^{(0)}(I_1, I_2, I_3, I_4)(t) \right| \leq \left| A_{10}(t) \right| + \left| A_{11} I_{1}(t-T_1) \right| + \left| A_{12} I^2_{1}(t-T_2) \right| \\
\leq \left| A_{10}(t) \right| + \left| A_{11} I_{1}(t-T_1) \right| + \left| A_{12} I^2_{1}(t-T_2) \right| \leq e^{\mu(t-T_0)} \left| I_{1}(t) \right| + \left| I_{1}(t) \right| e^{\mu(t-T_1)} + \left| I_{1}(t) \right| e^{\mu(t-T_2)}
\]

\[
\left| B^{(1)}(I_1, I_2, I_3, I_4)(t) \right| \leq e^{\mu(t-T_0)} \left| I_{1}(t) \right| + \left| I_{1}(t) \right| e^{\mu(t-T_1)} + \left| I_{1}(t) \right| e^{\mu(t-T_2)}
\]

\[
\left| B^{(2)}(I_1, I_2, I_3, I_4)(t) \right| \leq e^{\mu(t-T_0)} \left| I_{1}(t) \right| + \left| I_{1}(t) \right| e^{\mu(t-T_1)} + \left| I_{1}(t) \right| e^{\mu(t-T_2)}
\]

\[
\left| B^{(3)}(I_1, I_2, I_3, I_4)(t) \right| \leq e^{\mu(t-T_0)} \left| I_{1}(t) \right| + \left| I_{1}(t) \right| e^{\mu(t-T_1)} + \left| I_{1}(t) \right| e^{\mu(t-T_2)}
\]

\[
\left| B^{(4)}(I_1, I_2, I_3, I_4)(t) \right| \leq e^{\mu(t-T_0)} \left| I_{1}(t) \right| + \left| I_{1}(t) \right| e^{\mu(t-T_1)} + \left| I_{1}(t) \right| e^{\mu(t-T_2)}
\]
\[ |B_1^{(i)}(I_1, I_2, I_3, I_4)(t) \| \leq |A_{21}(t) + A_{22}||I_2(t) - T_1(t)| + |A_{23}||I_3(t) - T_2(t)| \leq \frac{[1 + R_{02}]M_{\bar{U}S}}{|\alpha_3|} + |A_{22}|r_{02}e^{\mu_0(t-T_1(t))} + |A_{23}|r_{03}e^{\mu_0(t-T_2(t))} \leq e^{\mu_0(t-T_1(t))} \left( K_{\alpha_3} + |A_{22}|r_{02}e^{\mu_0(t-T_1(t))} + |A_{23}|r_{03}e^{\mu_0(t-T_2(t))} \right) \leq e^{\mu_0(t-T_2(t))} \]
\[ p^{(k)} (B^{(k)}(I, J), B^{(k)}(\bar{T}, \bar{T})) \leq \]
\[ \leq |B_{11}| e^{-\mu T} p^{(k)} (I, \bar{T}) + |B_{22}| e^{-\mu T} p^{(k)} (I, \bar{T}) \leq \]
\[ \leq \left( |B_{11}| e^{-\mu T} + |B_{22}| e^{-\mu T} \right) \left( p^{(k)} (I, \bar{T}) + p^{(k)} (I, \bar{T}) \right) \leq \]
\[ \leq K(\mu) \left( p^{(k)} (I, \bar{T}) + p^{(k)} (I, \bar{T}) + p^{(k)} (I, \bar{T}) \right) \]

Finally, we have
\[ \left| B^{(k)}(I, J)(t) - B^{(k)}(\bar{T}, \bar{T})(t) \right| \leq \]
\[ \leq \left| B_{22} \left| f_{1}(t-T) - \bar{T}_{1}(t-T) \right| + |B_{22}| \left| f_{2}(t-T) - \bar{T}_{2}(t-T) \right| \right| + \]
\[ + |B_{22}| \left| e^{-\mu T} p^{(k)} (I, \bar{T}) + e^{-\mu T} p^{(k)} (I, \bar{T}) \right| \leq \]
\[ \leq e^{-\mu T} \left| B_{22} \right| p^{(k)} (I, \bar{T}) + |B_{22}| e^{-\mu T} p^{(k)} (I, \bar{T}) \]

which implies
\[ p^{(k)} (B^{(k)}(I, J), B^{(k)}(\bar{T}, \bar{T})) \leq \]
\[ \leq |B_{22}| e^{-\mu T} p^{(k)} (I, \bar{T}) + |B_{22}| e^{-\mu T} p^{(k)} (I, \bar{T}) \leq \]
\[ \leq \left( |B_{11}| e^{-\mu T} + |B_{22}| e^{-\mu T} \right) \left( p^{(k)} (I, \bar{T}) + p^{(k)} (I, \bar{T}) \right) \leq \]
\[ \leq K(\mu) \left( p^{(k)} (I, \bar{T}) + p^{(k)} (I, \bar{T}) + p^{(k)} (I, \bar{T}) \right) \]

Therefore
\[ p^{(k)} \left( \left( B^{(k)}(I, J), B^{(k)}(I, J), B^{(k)}(I, J), B^{(k)}(I, J), B^{(k)}(I, J), B^{(k)}(I, J) \right) \right) \leq \]
\[ \leq 4K(\mu) \left( p^{(k)} (I, \bar{T}) + p^{(k)} (I, \bar{T}) + p^{(k)} (I, \bar{T}) \right) \]

It is easy to see that \( j^{n}(k, k, k, k) < Q(k) < \infty \)
\( n=1, 2, \ldots \) is uniformly bounded by \( n \); \( Q \) is a positive constant not depending on \( n \). Indeed, every interval goes to the left from the initial point after a finite number \( n \) of iteration of \( j \). This means that the operator \( B \) is contractive one in the sense of definition from [13] and has a unique fixed point \((I(t), J(t), J(t), J(t))\), which is a solution of (19).

Finally, we note that the solution can be approximated by a sequence of successive approximations with advanced prescribed accuracy.

Theorem 1 is thus proved.

7 Results validation

Since our goal is to find \( U_{NE} = u_{2}(0, t); U_{FE} = u_{2}(\Lambda, t) \) we have (cf. (12), (13)):
\[ u_{2}(0, t) = q_{1}I_{1}(0, t) + q_{2}J_{2}(0, t) - q_{1}J_{1}(0, t) - q_{2}J_{2}(0, t) = \]
\[ q_{1}I_{1}(t) + q_{2}J_{2}(t) - q_{1}J_{1}(t) - q_{2}J_{2}(t) \]
\[ u_{2}(\Lambda, t) = q_{1}I_{1}(\Lambda, t) + q_{2}J_{2}(\Lambda, t) - q_{1}J_{1}(\Lambda, t) - q_{2}J_{2}(\Lambda, t) \]

where \((I(t), J(t), J(t), J(t))\) is the solution obtained in the above theorem.

We have to check the conditions of our Theorem 1 referring to the data from [9]:
\[ L_{11} = L_{12} = L_{22} = 0.8529 \mu H/m; \]
\[ L_{12} = 0.3725 \mu H/m; \]
\[ L_{12} = L_{21} = L_{m}; \]
\[ L_{12} = L_{22} = L_{m} \]
\[ C_{11} = C_{G} + C_{m} = C_{R} + C_{m} = 24.762 \mu F/m; \]
\[ C_{12} = C_{21} = C_{m} = -18.036 \mu F/m; \]
\[ L_{12} = L_{22} = L_{m} \]
\[ C_{11} = C_{12} = C_{22} = 0.3725 \times \]
\[ 4 \times 0.8529 \times 18.036 = 2.036 \mu F/m; \]
\[ C_{12} = 2.036 \mu F/m; \]
\[ \Delta_{C} = C_{11} - C_{22} = 0.3725 \times \]
\[ 4 \times 0.8529 \times 18.036 = 2.036 \mu F/m; \]
\[ \Delta_{C} = 2.036 \mu F/m; \]
\[ \Delta_{L} = L_{12} - L_{22} = L_{12} \]
\[ L_{1}^{2} - L_{2}^{2} = \]
\[ 0.8529 \times 2.036 \times 18.036 = 0.0588 > 0; \]
\[ \lambda_{1} = \sqrt{\frac{L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22}}{2\Delta_{L}}} \approx \]
\[ \approx 0.0321157 \approx 0.1792 \]
The inequalities from the main theorem are:

\[ |\lambda_1 \lambda_2 + (L_{22} \lambda_2 + R_{NE}) \gamma_2| U_S = \Delta_{12} \]

\[ \approx \frac{|0.0628 + 1.038 (0.1437 + R_{NE})|}{0.427 + (24.051 R_S + 3.5014) R_{NE} + 19.6 R_S} U_S \leq \min \{I_{10}, I_{20}\} \]

In what follows we verify how the same data satisfy the conditions generated by the particular case under weak coupling assumptions. Indeed, the main system becomes

\[ \frac{\partial u_g(x,t)}{\partial t} + \frac{\partial \dot{u}_g(x,t)}{\partial x} = 0 \]

\[ -\frac{\partial u_r(x,t)}{\partial t} + \frac{\partial \dot{u}_r(x,t)}{\partial x} = 0 \]

\[ \Delta \phi = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \dot{u}_g \\ \dot{u}_r \end{bmatrix} \]

\[ u_i(x,t) = u_g(x,t) \quad i(x,t) = i_g(x,t) \]

\[ C_{11} = C_G + C_m, \quad C_{12} = -C_m, \quad C_{22} = C_R + C_m \]

\[ L_{11} = L_G + L_{21} = L_m, \quad L_{22} = L_R \]

or in a matrix form

\[ \begin{bmatrix} C_{11} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 \\ 0 & 0 & L_{11} & 0 \\ 0 & 0 & L_{12} & L_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial u_g}{\partial t} \\ \frac{\partial \dot{u}_g}{\partial x} \\ \frac{\partial u_r}{\partial t} \\ \frac{\partial \dot{u}_r}{\partial x} \end{bmatrix} = 0. \]

Since

\[ \Delta_c = C_{11} C_{22} = (C_G + C_m)(C_R + C_m) > 0 \]

\[ (L) \Delta_c = L_{11} L_{22} - L_{12} L_{21} > 0 \]

it follows \[ |A| = \Delta_c \Delta_L \neq 0 \]

and therefore
In view of $L_{12}C_{11} + L_{22}C_{12} \neq 0$ and $L_{12}C_{22} + L_{11}C_{12} \neq 0$ we obtain the roots of

$$B^{-1} - \mu I = (L_{11}C_{11} - \mu^2)(L_{22}C_{22} - \mu^2) = 0$$

From the matrix

$$\begin{bmatrix}
-\mu & 0 & L_{11} & 0 \\
0 & -\mu & L_{12} & L_{22} \\
C_{11} & 0 & -\mu & 0 \\
C_{12} & C_{22} & 0 & -\mu
\end{bmatrix} \approx
\begin{bmatrix}
-\mu & 0 & L_{11} & 0 \\
0 & -\mu & L_{12} & L_{22} \\
0 & 0 & L_{12}C_{11} - \mu^2 & 0 \\
0 & 0 & L_{11}C_{12} + L_{22}C_{22} & L_{22}C_{22} - \mu^2
\end{bmatrix}$$

we obtain for $\mu_1 = -\mu_5 = \sqrt{L_{11}C_{11}}$ the eigenvectors

$$\begin{bmatrix}
\xi_{12} = L_{12}C_{11} + L_{22}C_{12} \\
L_{11}C_{12} + L_{12}C_{22} \sqrt{C_{11}} \\
L_{11}C_{11} - L_{22}C_{22} \\
L_{11}C_{11} + L_{12}C_{22}
\end{bmatrix}, \quad \xi_{31} = L_{12}C_{11} - L_{22}C_{22} \sqrt{C_{11}},
$$

$$\begin{bmatrix}
\xi_{32} = L_{12}C_{11} + L_{22}C_{12} \\
L_{11}C_{12} + L_{12}C_{22} \sqrt{C_{11}} \\
L_{12}C_{11} - L_{22}C_{22} \\
L_{12}C_{11} + L_{12}C_{22}
\end{bmatrix}, \quad \xi_{33} = L_{12}C_{11} - L_{22}C_{22} \sqrt{C_{11}}.$$

Similarly for $\mu_2 = -\mu_4 = \sqrt{L_{22}C_{22}}$ we have

$$\begin{bmatrix}
\xi_{22} = \sqrt{L_{22}C_{22}}/C_{22}, \\
\xi_{22} = \sqrt{L_{22}C_{22}}/C_{22} \\
\xi_{34} = 0, \\
\xi_{44} = \sqrt{L_{22}C_{22}}/C_{22}
\end{bmatrix}, \quad \xi_{43} = 1, \quad \xi_{42} = 1.$$

Consequently, the transformation matrix is:

$$\begin{bmatrix}
L_{12}C_{11} + L_{12}C_{22} \\
L_{11}C_{12} + L_{12}C_{22} \sqrt{C_{11}} \\
L_{11}C_{11} - L_{12}C_{22} \\
L_{11}C_{12} + L_{12}C_{22}
\end{bmatrix} \approx
\begin{bmatrix}
L_{12}C_{11} - L_{22}C_{22} \\
L_{11}C_{12} + L_{12}C_{22} \sqrt{C_{11}} \\
L_{12}C_{11} - L_{12}C_{22} \\
L_{12}C_{11} + L_{12}C_{22}
\end{bmatrix}. \quad \xi_{31} = L_{12}C_{11} - L_{22}C_{22} \sqrt{C_{11}},
$$

$$\begin{bmatrix}
L_{12}C_{11} + L_{22}C_{12} \\
L_{11}C_{12} + L_{12}C_{22} \sqrt{C_{11}} \\
L_{12}C_{11} - L_{22}C_{22} \\
L_{12}C_{11} + L_{12}C_{22}
\end{bmatrix} \approx
\begin{bmatrix}
L_{12}C_{11} - L_{22}C_{22} \\
L_{11}C_{12} + L_{12}C_{22} \sqrt{C_{11}} \\
L_{12}C_{11} - L_{12}C_{22} \\
L_{12}C_{11} + L_{12}C_{22}
\end{bmatrix}.$$
And then in view of (12) we have
\[ u_i(x,t) = \frac{L_{11}C_{11} - L_{22}C_{22}}{L_{11}C_{12} + L_{12}C_{22}} \sqrt{L_{11}} \mathbf{I}_1(x,t) - \\
\frac{L_{11}C_{11} - L_{22}C_{22}}{L_{11}C_{12} + L_{12}C_{22}} \sqrt{L_{11}} \mathbf{I}_2(x,t); \]
\[ u_j(x,t) = \frac{L_{12}C_{11} + L_{22}C_{12}}{L_{11}C_{12} + L_{12}C_{22}} \sqrt{L_{12}} \mathbf{I}_1(x,t) + \sqrt{L_{22}} \mathbf{I}_2(x,t) - \\
\frac{L_{12}C_{11} + L_{22}C_{12}}{L_{11}C_{12} + L_{12}C_{22}} \sqrt{L_{12}} \mathbf{I}_3(x,t) - \sqrt{L_{22}} \mathbf{I}_4(x,t); \]
\[ i_j(x,t) = \frac{L_{11}C_{11} - L_{22}C_{22}}{L_{11}C_{12} + L_{12}C_{22}} \mathbf{I}_1(x,t) + \frac{L_{12}C_{11} - L_{22}C_{22}}{L_{11}C_{12} + L_{12}C_{22}} \mathbf{I}_2(x,t); \]
\[ i_j(x,t) = I_1(x,t) + I_2(x,t) + I_3(x,t) + I_4(x,t). \]

If we take the specific parameters again from [9]
\[ L_{11} = L_G = L_R = 0.8529 \, \text{µH/m}; \]
\[ C_{11} = C_G + C_m = C_R + C_m = 46.762 \, \text{pF/m} \]
it is obvious that \( L_{11} C_{11} - L_{22} C_{22} = 0 \). This implies \( u_i(x,t) = u_j(x,t) = 0 \). The contradiction obtained shows the advantages of our method.

8 Conclusion

In this paper we presented the full derivation of the equations describing a 3-conductor transmission line terminated by linear loads. In such way, we extended the general method from [12] by shrinking the mixed problem for the hyperbolic system expressing TEM propagation lengthwise the lines to a functional system on the boundary. In result, by applying the fixed point method we can obtain in an explicit form the solution to the system of functional equations by successive approximations beginning with simple initial approximation. Our method is applicable to nonlinear boundary conditions too. Besides, in this paper we prove existence-uniqueness of a more general periodic solution and demonstrated the benefits of our method on the samples related to examinations of cross-talks. It should be noted that the method elaborated here can be applied to nonlinear boundary conditions.

Acknowledgment

This work is part of the research of L10S7 SynChaLab supported by Project “National Center of Mechatronics and Clean Technologies”, Contract No. BG05M2OP001-1.001-0008/28.02.2018, funded by the Bulgarian Operational Program “Science and Education for Smart Growth 2014-2020”.

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