Geometry of Boiti-Leon-Manna-Pempinelli Equation

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**Abstract**

In this article, we reduce the Boiti-Leon-Manna-Pempinelli (BLMP) equation to an ordinary differential equation by the Lie symmetry method. Then, we calculate infinite structural equations from Maurer-Cartan’s infinite forms for the symmetrical pseudo-group of this equation, by invariantized defining equation method. Finally using the multiplier method and homotopy operators we calculate conservation law for this equation.

**Keywords:** Boiti-Leon-Manna-Pempinelli Equation, Conservation Law, (2+1)-Dimensional, Group-Invariant Solutions, Lie Symmetry Method

1. Introduction

Lie groups have various applications in the different branches of physics and mathematics. One of the applications of Lie symmetry is to find a relation between transformations, Lie Algebra and its infinitesimal generators. These relations can be used to study the group of invariant solution of a differential equation. A method for the study of Lie pseudo-groups is given by Élie Cartan\(^1\). His method is the basis of actions of Lie pseudo-groups and use some differential forms which are called Maurer-Cartan’s forms. The differentiation of these forms lead to the formation of Cartan’s structural equations. These structural equations contain all the information about the given Lie pseudo-group. We try to get a set of infinite structural equations from Maurer-Cartan’s infinite forms of (2+1)-dimensional Boiti-Leon-Manna-Pempinelli equation\(^2-6\):

\[
\text{BLMP: } u_{yt} - 3 u_x u_{xy} - 3 u_y u_{xx} + u_{xxy} = 0.
\]

A finite and complete set of pseudo-group elements are selected as a symmetry group of BLMP equation. One of the applications of symmetry group is classification of the solutions; that is, the solutions that can be transformed to each other with one element of symmetry group. It is possible to reduce the number of the independent variables of the equation using the symmetry group. In this article, we reduce the number of independent variables of BLMP equation and change it to an ODE. One of the advantages of the symmetry group is the computation of the conservation law which is used in mathematical physics.

2. Infinitesimal Determining Equations of BLMP Equation

Let an one parameter Lie group of infinitesimal transformation acting on the independent and dependent variables of the BLMP equation, as:

\[
(\xi, \phi) = \left(\begin{array}{c}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\phi
\end{array}\right) = \left(\begin{array}{c}
t_x \\
x_x \\
y_y \\
u
\end{array}\right) + s \left(\begin{array}{c}
\phi \xi_1 \\
\phi \xi_2 \\
\phi \xi_3 \\
\phi \phi
\end{array}\right) + O(s^2),
\]

in which ‘s’ is a group parameter and \(\xi_i, \phi\) are infinitesimal part of transformation with respect to \((t, x, y, u) \in M = J^0 \cap (x, y, u) \simeq \mathbb{R}^4\). Assume \(v = \xi_1 \partial_t + \xi_2 \partial_x + \xi_3 \partial_y + \phi \partial_u\) be the corresponding infinitesimal transformation group. The fourth order prolongation is computed as:

\[
\text{Pr}^4(v) = v + \phi \partial_{u_t} + \phi^2 \partial_{u_{xx}} + \phi^3 \partial_{u_{xxy}} + \phi^4 \partial_{u_{xxxx}} + \phi^{xxy}
\]

\(\partial_{u_{xxy}},\) with coefficients

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\[ \phi' = D_t Q + \sum_{j=1}^{3} \zeta^j u_{j,t}, \text{ (7, Theorem (2.16))} \]

where

\[ Q = \phi - \sum_{j=1}^{3} \zeta^j u^j, J = (j_1, \cdots, j_k), 1 \leq j_k \leq 4, 1 \leq k \leq 3, \quad 0 < \# J \leq n. \]

with \( u^j := \partial u^j / \partial x^i \) and \( u^{j, \alpha} := \partial u^j / \partial x^\alpha \). The invariance conditions of the functional coefficient in the above vector fields, as this selection is done by selecting simpler form for each of these generators. By solving such a system, the one parameter groups \( g_i (s) \in \text{Diff} (M) \) are produced by each of the \( v_i, i = 1, \cdots, 4 \). Therefore, if we have one solution of BLMP equation in the form \( u = f(t, x, y) \), then we can obtain infinite number of solutions by acting each of the \( g_i (s) \) upon this solution.

Theorem 2.1 For any arbitrary solution \( u = f(t, x, y) \) of BLMP equation, and arbitrary small \( s \)

\[ u_1 = e^{-s} f \left( t e^{-s}, x e^{-3}, y \right), \quad u_2 = f \left( t, x, ye^{-s} \right) \]

\[ u_3 = f \left( t, x, y - s \right), \quad u_4 = f \left( t, x - ts, y + \frac{ts^2}{6} - \frac{xs}{3} \right) \]

are also solutions, where \( u_i \) is the transformed solution \( u = f(t, x, y) \) by \( g_i (s) = 1, \cdots, 4 \).

3. Optimal System of Sub-Algebras of BLMP Equation

We obtain the optimal system of BLMP equation, then using this set and Lie algebra elements of infinitesimal generators reduce one of the variables of BLMP equation. Then we obtain Lie algebra and optimal system of reduced equations. Repeating the same procedure, we finally change these equations to ODE’s. In fact, the optimal system is the best way for the classification of one dimensional sub-algebras. In this classification, conjugate equivalents are in a same class. Also, the classification of one dimensional sub-algebras means the classification of their adjoint representation orbits. To compute the adjoint representation, we use the following Lie series:

\[ \text{Ad} \left( \exp (s v_j) \right) = \sum_{k=0}^{\infty} \frac{s^k}{k!} t^k \left[ v_j, v \right] \]

for the vector fields \( v_j, v \). They can be arranged in the Table 2.

We correspond every element of the Lie algebra \( v = a_{1} v_1 + \cdots + a_{4} v_4 \) with a vector \( (a_{1}, \cdots, a_{4}) \). The adjoint acting will be also correspond with a linear transformation of vector fields. We have the following theorem:

Theorem 3.1 The one dimensional optimal system of Lie algebra of the BLMP equation includes the sub-algebras

Table 1.

| \( [*,*,*] \) | \( v_1 \) | \( v_2 \) | \( v_3 \) | \( v_4 \) |
|----------------|--------|--------|--------|--------|
| \( v_1 \)     | 0      | 0      | (2/3)  | (4/3)  |
| \( v_2 \)     | 0      | 0      | 0      | 0      |
| \( v_3 \)     | -(2/3) | 0      | 0      | 0      |
| \( v_4 \)     | -(4/3) | 0      | 0      | 0      |
Table 2. Adjoint representation

| \( Ad(exp(s)v)\) | \( v_1 \) | \( v_2 \) | \( v_3 \) | \( v_4 \) |
|----------------|--------|--------|--------|--------|
| \( v_1 \)     | \( v_1 \) | \( v_2 \) | \( e^{-2(s/3)}v_3 \) | \( e^{-4(s/3)}v_4 \) |
| \( v_2 \)     | \( v_1 \) | \( v_2 \) | \( v_3 \) | \( v_4 \) |
| \( v_3 \)     | \( v_1 \) | \( v_2 \) | \((2/3)s_1v_1 + v_3 \) | \( v_4 \) |
| \( v_4 \)     | \( v_1 \) | \( v_2 \) | \((4/3)s_1v_1 + v_4 \) |

with generators: (i) \( v_i + av_j \) and (ii) \( av_2 + bv_3 + cv_4 \) where \( a, b, c \) are constants.

**Proof.** Define the operator \( F^i_k : \mathcal{G} \to \mathcal{G} \) in the form of a linear map \( v \mapsto Ad(exp(sv)v) \) for \( i = 1, 2, 3, 4 \). We have

the matrices \( M^i_j \) corresponding with each of the \( F^i_j \)'s in respect to the \( \{v_1, v_2, v_3, v_4\} \) basis in the following way:

\[
M^1_1 = E_{11} + E_{22} + e^{-2s/3}E_{33} + e^{-4s/3}E_{44}, \quad M^1_2 = I_4, \quad M^1_3 = M^1_4 = I_4 + \frac{4s}{3}E_{44}.
\]

where \( E_{ij} \)'s are \( 4 \times 4 \) elementary matrices whose \( (i, j) \) element is one and other are zero. Suppose \( v = a_1v_1 + \cdots + a_4v_4 \). Then the combination \( \{F_1^1 \circ \cdots \circ F_1^4\} \) of these maps is:

\[
v \mapsto a_1v_1 + a_2v_2 + \frac{1}{3} \left( 2s_1a_1 + 3a_2 e^{-2s/3} \right) e^{-2s/3}v_3 + \frac{1}{3} \left( 4s_4a_1 + 3a_4 e^{-4s/3} \right) v_4.
\]

We can reduce \( r \) as follows. If \( a_1 \neq 0 \), then we can van- ish the coefficients of \( v_1 \) and \( v_4 \) with maps \( F_2^1 \) and \( F_4^1 \) by substitution \( s_1 = (-3a_1)/(2a_2) \) and \( s_4 = (-3a_4)/(4a_4) \). We can suppose \( a_1 = 1 \) by scaling. In this case reduces to the case (i) and if \( a_1 = 0 \) then it reduces to case (ii).

### 4. Two-Dimensional Optimal System

In this case, we obtain two-dimensional optimal system for BLMP equation. Select \( V^1 \) or \( V^2 \), as one of the elements of one dimensional optimal system in theorem 3.1 and consider \( V = a_1V_1 \cdots + a_4v_4 \) as an optimal vector field in which \( a_i \)'s are smooth functions of \( (t, x, y, u) \). By this selection, we have \([V, V] = \lambda V^0 + \mu V\); Computation of both sides of this equation yields the following system of linear equations:

\[
C_{jk} a_k a_j = 2a_i + \mu a_i, \quad (i = 1, \cdots, 4).
\]

The elements of the two dimensional optimal system are obtained by solving this system of linear equations for each of the selections of the members of the one dimensional optimal system from the theorem 3.1. After computing these elements, we can simplify them as the one dimensional case through acting the adjoint matrices to each of them and propose the following theorem:

**Theorem 4.1** An optimal system of two-dimensional Lie algebra from the BLMP equation is:

(i) \(<v_1 + av_2, v_4> \) (ii) \(<v_1 + av_2, v_4>

### 5. Similarity Reduction of BLMP Equation

In this section, we want to find new variables such that if we write the BLMP equation with respect to new variables, one of the independent variables of the equation can be reduced. For example, we consider one of the invariants corresponding to the symmetry generators of the optimal system \( v_1 + v_2 = t \partial_t + (x/3) \partial_x - (u/3) \partial_u + y \partial_y \). This element of optimal system has a determining equation \( dt/t = 3 dx/t = -3 du/u = dy/y \). Solving this equation yields three invariants in the forms \( r = x/t^{1/3}, s = y/t, \) and \( h = ut^{1/3} \). Now if we regard as a function of \( r \) and \( s \), the BLMP equation is reduced to form \((r h' + 4 h)/3 + s h' + 3 h' h + h''' \). For the rest of the optimal system elements and symmetry group, the reduced equations will be as the Table 3.

Now we want to find new variables by using invariants corresponding with infinitesimal generators in two-dimensional optimal system elements. If we write the BLMP equation in new variables, two of the independent variables of the equation reduces and the equation reduces to ODE. For example \(<v_1 + av_2, v_4> \) is the element of two-dimensional optimal system. Solving the determining equation of this element yields two invariants in the forms \( p = y/t \) and \( f = (x/6)^{2/3} + ut^{1/3} \). Now if we regard \( f \) as a function of \( p \), then we can write the dependent and independent variables of BLMP equation based on the new variables obtained from invariants. This leads to the reduction of the BLMP equation to an equation in the form \( 3 pf_{pp} + f = 0 \); which is solved as \( f = c_1 + c_2 p^{2/3} \), where \( c_1, c_2 \) are constants. Therefore, we find an exact solution.
Table 3. Similarity reduced equation

| $C_i$ | $r_i$ | $s_i$ | $h_j$ | Reduced equation |
|-------|-------|-------|-------|------------------|
| $v_1$ | $x^{1/3}$ | $y$ | $u^{1/3}$ | $-(r_{h_n} + h)/3 - 3h_{r_n} - 3h_{h_n} + h = 0$ |
| $v_2$ | $t$ | $y$ | $x^2/6t + u$ | $h_{r_n} + rh = 0$ |
| $v_3 + v_2$ | $x^{1/3}$ | $y/t$ | $u^{1/3}$ | $-(r_{h_n} + h)/3 - sh_{r_n} - 3h_{h_n} + h = 0$ |
| $v_3 + v_4$ | $t$ | $ye^{1/3}$ | $-x + u + x^2/6t$ | $r^h_{r_n} + 4r^h_{h_n} + 3r^2 h_{r_n} - 6r s^h_{r_n} h_{h_n} - 6r s h_{h_n} - s^h_{h_n} - 6 s h_{h_n} - 7s^{h_{r_n}} - h - 0.$ |

$$u = \frac{2}{6t} \left( 6c_1 t^3 + 6c_2 y^3 - x^2 \right), \quad c_1, c_2 \in \mathbb{R}$$

6. Maurer-Cartan Forms for the Pseudo-group of Local Diffeomorphisms

To calculate the Maurer-Catan’s forms, there are several ways\textsuperscript{11,13}. In this section, we use the invariable defining equations method for Lie pseudo-group\textsuperscript{14,15}. Let’s $M$ be the space of the independent and dependent variables of BLMP equation as a manifold and $D(M)$ a pseudo-group of diffeomorphisms in the form $X = \phi(x)$ on $M$. Also, suppose $D^r(M)$ as a bundle of infinite jets of maps in $D(M)$. If we show the local coordinate of the base space $M$ and $D^r(M)$, as $x = (x^i)$ and $(x', x'_0)$ respectively in which $x = (x^0)$ are target coordinates for $\phi \in D(M)$, that $\phi(x) = X$ and $X'_0 = (\partial^j X^a) / \partial x^l = \delta^{j+l} x^a / \partial x^l (\partial x^1)^j \cdots (\partial x^m)^l b$, for $I = (i_1, i_2, \ldots, i_m)$, then, the general and infinite Maurer-Cartan's forms are as $\sigma^a = X^a_{r} dx^l$ and $\mu^a = dx^a - X^a_{r} dx^l$, and $\mu^a = \delta^j X^a_{r} \partial / \partial x^l$ for $I = (i_1, \ldots, i_m)$, the basis of Maurer-Cartan's forms corresponding to their infinitesimal forms. For simplicity, we use the following symbols\textsuperscript{15}. By lifting and minimizing the infinitesimal defining system of BLMP equation\textsuperscript{14} we will have:

$$\mu^a [H] = \left( \begin{array}{c} \mu^a [H] \\mu^a [H] \\vdots \end{array} \right) \left( \begin{array}{c} H^1 \\H^2 \\vdots \end{array} \right).$$

$$\nabla^a [H] = \left( \frac{\partial (\mu^a [H])}{\partial x^l} \right)$$

shows Jacobian matrix of $\mu [H]$ with power series of variable $H = (H)$. Now let’s suppose $\sigma = \phi^l \partial / \partial x^l$, is an infinitesimal generator of sub pseudo group $G \subset D(M)$ and determining equation of BLMP the same as 1. The invariant forms $\mu^a_i$ of the pseudo group also satisfy in the equations 1. By replacing $x'$ by $X'$ and $\phi^l$ by $\mu^a_i$, the same system is obtained\textsuperscript{11}. Also, the structural equations of invariant coframe for Lie pseudo group are obtained by restricting the diffeomorphism structure equations 2 to the solution space of BLMP equation. Now we obtain Maurer-Cartan’s structure equations for Maurer-Cartan's forms corresponding to their infinitesimal forms. By solving this system, we will obtain four independent functions that are the basis of Maurer-Cartan’s forms\textsuperscript{14}, $\mu^a_i, \mu^a_{i+1}, \mu^a_{i+k}, \mu^a_{i+k}$, with $0 \leq k$. Therefore, we can write the Maurer-Cartan’s Taylor series on these base elements as follows:

$$\mu^a [H] = \sum_{k=0}^{\infty} \frac{1}{k!} \mu^a_{i+k} H^k_i,$$
\[ \mu^x[H] = \sum_{k=0}^{\infty} \frac{1}{k!} \mu^x_{kk} H^k + \frac{1}{3} \mu^x_{tt} H^3 + \frac{1}{3} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \mu^x_{k(k+1)} H^k H^1, \]

\[ \mu^y[H] = \sum_{k=0}^{\infty} \frac{1}{k!} \mu^y_{kk} H^k, \]

\[ \mu^u[H] = \sum_{k=0}^{\infty} \frac{1}{k!} \mu^u_{kk} H^k - \frac{1}{3} \mu^u_{tt} H^3 H^1 + \cdots + \frac{1}{9} \frac{1}{(k+2)!} \mu^u_{kk+2} H^k H^1. \]

By the formula 2 we equalize the first column in this matrix expression with the solution of the multiplication right hand side, we will have Maurer-Cartan's structure equations:

\[ d \mu^x_{kk} = (a-b) \mu^x_{kk+1} \wedge \mu^x_{tt}, \]

\[ d \mu^y_{kk} = \frac{a-3b}{3} \mu^y_{kk+1} \wedge \mu^y_{tt}, \]

\[ d \mu^z_{kk} = (a-b) \mu^z_{kk+1} \wedge \mu^z_{tt}, \]

\[ d \mu^u_{kk} = \frac{3a+b}{3} \mu^u_{kk+1} \wedge \mu^u_{tt} - \frac{a-b}{3} \mu^u_{kk+1} \wedge \mu^u_{tt}. \]

where \( a = C(k, n), b = C(k, n-1), C(r, s) = r! s! (r-s)!, \)

and \( \mu^x_{kk}, \cdots \mu^z_{kk}, \) are dual to \( y_1 \)'s and \( \alpha(t), \beta(t), \xi(t), \gamma(y) \) are replaced by their Taylor series.

### 7. Conservation Laws for BLMP Equation

There are different methods (such as Noether's basis method, multiplier method, direct method and etc.,\(^7,16,17,18,19\)) To compute conservation laws of a differential equation. We obtain the conservation laws of BLMP equation by multiplier method. The conservation law (as a divergence expression) for an equation system is defined as follows:

\[ D_i \Phi^i[u] = D_i \Phi^i[u] + \cdots + D_n \Phi^i[u]. \]

It is valid for all of the solutions of the given system in the \( \Phi^i[u] = \Phi(x, u^{(i)}), i = 1, \cdots, n \) expression. \( \Phi^i[u] \) is called the fluxes of the conservation law. The maximum degree of the derivation in the flux expression is called the order of conservation law\(^16\). A set of multipliers \( \Lambda_i[x, U^r(t)] = \Lambda_i[x, U^r(t)] )_{r=1} \) results in the existence of the divergence expression for the system \( \Xi^i[u] = \Xi^i(x, u^{(i)}) \), in the case that the identity \( \Lambda_i[x, U^r(t)] \Xi^i[u] = D \Phi^i[u] \) holds for the arbitrary functions \( U(x) \). Then there will be one conservation law as \( \Lambda_i[x, U^r(t)] = D \Phi^i[u] = 0 \) for the solution of BLMP equation, that is written as \( U(x) = u(x) \); providing \( \Lambda_i[U] \) is non-singular. The Euler operator in respect to \( U \) is defined by:

\[ E_{U^j} = \partial^j_U - D_i \partial_{U^j} + \cdots + (1)^{j} D_i \cdots D_i \partial_{U^j} + \cdots, \]

for \( j = 1, \cdots, q^6. \)

The equations \( E_{U^j} F(x, U^{(s)}) = 0, j = 1, \cdots, q \) hold for every function \( U(x) \) if and only if the relation \( F(x, U^{(s)}) = D \Psi(x, U^{(s-1)}) \) holds for some of the functions \( \Psi(x, U^{(s-1)}), i = 1, \cdots, q^6 \). A set of non-singular local multipliers \( \Lambda_i[x, U^{(s)}] = \Lambda_i[x, U^{(s)}] )_{r=1} \) results in the production of the locally conservation law for the system \( \Xi^i(x, u^{(t)}) \) if and only if the set of identities

\[ E_{U^j} \left( \Lambda_i[x, U^{(s)}] \Xi^i(x, u^{(t)}) \right) = 0, j = 1, \cdots, q, \]

hold for every optional function \( U(x) \). The set of equation 3 results in the linear determining equations from the solution for which a set of locally conservation law multipliers for system \( \Xi^i(x, u^{(t)}) \) is produced. Now, we want to obtain the local multipliers of the conservation law in the form \( \Lambda = \xi(x, t, u) \) for BLMP equation. The determining equation 3 for the BLMP equation are:

\[ E_\xi^x \left( \xi(t, x, y, U) \left[ u_{zt} - 3u_{zt} u_{xy} - 3u_{zt} u_{xx} + u_{xxxy} \right] \right) = 0, \]

where \( U(t, x, y) \) is an optional function. The calculation of equation 4 yields the PDE system. By solving the determining equation produced from 4, we have the following solution \( x = (x, 6\mu_{n}(n(t) + p(y) + au_{n} + m(t)), in which n(t), m(t), p(y) are optional coefficients. So local multipliers are obtained in the following way:

\[ (i): (x + 6\mu_{n}) n(t), (ii): au_{n}, (iii): p(y), (iv): m(t) \]

Each of the local multipliers determine a non-trivial local conservation law \( D_i \Psi + D_i \Phi + D_i \Omega = 0 \) with a determining form

\[ D_i \Psi + D_i \Phi + D_i \Omega = \xi(t, x, y, U) \left( u_{zt} - 3u_{zt} u_{xy} - 3u_{zt} u_{xx} + u_{xxxy} \right). \]
To calculate the $\Psi$, $\Phi$ and $\Omega$ we should invert the operator. This involves getting multi-dimensions integral from the statement involving the optional function and its derivatives and this is practically difficult in direct manner. Here, we use the homotopy operators\textsuperscript{20}. The homotopy operator is powerful algorithmic device originated from homological algebra and variational bi-complexes\textsuperscript{20,18}. The three-dimensional homotopy operator is a vector operator with three components in the form $\left(\mathcal{H}^i_u f, \mathcal{H}^j_u f, \mathcal{H}^k_u f\right)$, where $t$-component is given by:

$$\mathcal{H}^i_u f = \frac{1}{2} \left( \sum_{j=1}^{q} T^j_u f \right) \left( \sum_{j=1}^{q} \sum_{a,b,c,d} B^{ab}_{i,j,k} \left( D_j \right)^{a-1} \left( D_k \right)^{b-1} \left( D_i \right)^{c-1} \right) \frac{d\Phi}{d\mu_{i,j,k}} .$$  

(5)

The $x$- and $y$-components are defined analogously. $T^i_u f$ is obtained as the following:

$$T^i_u f = \frac{1}{2} \left( \sum_{j=1}^{q} T^j_u f \right) \left( \sum_{j=1}^{q} \sum_{a,b,c,d} B^{ab}_{i,j,k} \left( D_j \right)^{a-1} \left( D_k \right)^{b-1} \left( D_i \right)^{c-1} \right) \frac{d\Phi}{d\mu_{i,j,k}} .$$  

(6)

Where $M_1^i, M_2^i, M_3^i$ are the order of $u^i$ with respect to $t$, $x$ and $y$ in $f$ respectively, which in the BLMP equation $j = 1, M_1^i = 1, M_2^i = 3, M_3^i = 1$, and combinatorial coefficient.

$$B^{(t)} = B \left( i_1, i_2, i_2, k_1, k_2, k_2 \right) = C \left( i_1 + i_2, i_1 \right) \left( i_2 + i_2, i_2 \right) \left( i_1 + i_2, i_1 \right) \left( i_2 + i_2, i_2 \right)$$  

$$+ C \left( i_1 + i_2, k_1 \right) \left( i_1 + i_2, k_2 \right) \left( i_1 + i_2, k_2 \right) .$$

The integrands $T^x_u f$ and $T^y_u f$ are defined analogously. By the permutations, they have combinatorial coefficients $B^{(x)} = B \left( i_1, i_1, i_2, k_1, k_2, k_2 \right)$ and $B^{(y)} = B \left( i_2, i_2, i_1, k_2, k_2, k_2 \right)$.

We apply homotopy operator to find conserved quantities $\Psi$, $\Phi$ and $\Omega$ which yield of multiplier $\xi = \mu(t)$. We have $f = m(u_y - 3u_x u_y - 3u_x u_{xx} + u_{xxx})$. The integrands 5 and 6 are:

$$T^x_u f = 4mu_y,$$

$$T^y_u f = 3m(6u_x u_y - 2uu_{xy} - u_{xy}),$$

$$T^z_u f = m \left( 2u_x + u_{xx} - 6u_x^2 + uu_{xx} \right) - 4m' u .$$  

(7)

Apply 5 to the integrands 7 and get:

$$\Psi := \mathcal{H}^t_u f = 4mu_y,$$

$$\Phi := \mathcal{H}^t_u f = 3m \left( 3u_x u_y - uu_{xy} - u_{xy} \right),$$

$$\Omega := \mathcal{H}^t_u f = m \left( 2u_x + u_{xx} - 3u_x^2 - 3uu_{xx} \right) - 2m' u .$$

So, we have the first conservation law of the BLMP equation respect to multiplier $\xi = \mu(t)$

$$D_t \left( 4m(t) u_y \right) + D_x \left( 3m(t) \left( 3u_x u_y - uu_{xy} - u_{xy} \right) \right)$$

$$+ D_y \left( m(t) \left( 2u_x + u_{xx} - 3u_x^2 - 3uu_{xx} \right) - 2m(t) u \right) = 0 .$$

Now we find conservation law respect to multiplier $\xi = au_y$. In this case we have

$$D_t \left( 2 \left( uu_{xy} - u_x u_y \right) \right) + D_x \left( 5u, u_{xy} - 12u_x^2 u_y - u_{xx} u_y - 4uu_{xy} \right) + D_y \left( u_x^2 - 2u_x u_{xx} + u_{xx}^2 - 3u_x u_{xx} + uu_{xxx} + 2uu_{tx} \right) = 0 .$$

$$D_t \left( 2 \left( p(y) u_y - p' y u_x \right) \right) + D_x \left( 3p(y) \left( 3u_x u_y - uu_{xy} - u_{xy} \right) \right) + D_y \left( u_x u_{xx} - 3u_x u_{xx} \right) = 0 .$$

8. Conclusion

In this paper we obtained the Lie symmetries of the Boiti-Leon-Manna-Pempinelli equation by using the Lie symmetry method. Also, the one and two dimensional optimal system are computed. This led to reducing the equation to ODE’s and computing the conservation laws and Maurer-Cartan forms for the pseudo-group of local diffeomorphisms of this equation.

9. References

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