Fixed Point Results for Fractal Generation of Complex Polynomials Involving Sine Function via Non-Standard Iterations

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ABSTRACT Due to the uniqueness and self-similarity, fractals became most attractive and charming research field. Nowadays researchers use different techniques to generate beautiful fractals for a complex polynomial $z^n + c$. This article demonstrates some fixed point results for a sine function (i.e. $\sin(z^n) + c$) via non-standard iterations (i.e. Mann, Ishikawa and Noor iterations etc.). Since each two steps iteration (i.e. Ishikawa and S iterations) or each three steps iteration (i.e. Noor, CR and SP iterations) have same escape radii for any complex polynomial, so we use these results for S, CR and SP iterations also to apply for the generation of Julia and Mandelbrot sets with $\sin(z^n) + c$. At some fixed input parameters, we observe the engrossing behavior of Julia and Mandelbrot sets for different $n$.

INDEX TERMS Fixed points, sine function, fractals.

I. INTRODUCTION

To draw the graphs via escape time algorithms in the form of unique and self-similar images by using some electronic tools became an attractive field named as fractals. The work on fractals was started in early twentieth century when Pierre Fatou and Gaston Julia tried to find the successive approximations of $f : z \rightarrow z^2 + c$ where $z, c \in \mathbb{C}$. In 1919 Julia [1] was succeeded to iterate this function but failed to sketch it. Mandelbrot [2] is known as the father of fractals because he used the word fractal for the complex graphs of $f : z \rightarrow z^2 + c$. In 1985, he sketched the Julia set and studied its features. He observed that for different values of $c$ the Julia sets have diversity in their nature. Moreover, he changed the roles of $z$ and $c$ in Julia set and defined a new set called Mandelbrot set. In Julia set we study the behaviour of the iterates for each $z$, and in the Mandelbrot set we study connectedness of Julia set for each $c$ defining those sets. His work was then extended by Lakhtakia et al. [3] in 1987. They generalized the Mandelbrot set for $f : z \rightarrow z^p + c$ where $p \geq 2$. Crowe et al. [4] defined the anti Julia and anti Mandelbrot sets in 1989 and discussed their connected locus. They generated complex graphs for $\overline{z^2 + c}$ and later on these graphs called “tricorn” [5]. The property to have uniqueness and self-similarity caused that fractals found applications in image encryption [6] or compression [7], cryptography [8], art and design [9]. The applications of fractal theory in the fields of electrical and electronics engineering revolutionized the industry of security control system, radar system, capacitors, radio and antennae for wireless system [10], [11]. Moreover, architects and engineers sketched and designed the maps of different projects on the basis of fractal theory [12]. Many generalizations have been made in fractals via various fixed point iterations. Fractals for rational and transcendental complex functions were elaborated in [13]. Higher dimensional fractals were discussed in [14], [15] and [16]. Interesting generalized Julia and Mandelbrot sets visualized with various iterations can be found in the literature, e.g, Mann iteration [17], Ishikawa-iteration [18], S-iteration [19], Noor-iteration [20], CR-iteration [21] and SP-iteration in [22]. Moreover, the Jungck-type iterations were used in [23]–[28] and [29]. Various iterations...
were also used to generate biomorphs [30], [31] and multi-corns [19], [32].

Throughout the history of fractals, researchers proved escape criteria for complex polynomials $z^n + c$ and $z^2 + cz + c$. During this research, we found some images on Internet and papers in which authors just generated Julia and Mandelbrot sets with $\sin(z^n) + c$, but they could not prove the escape criteria for this complex function. In this article, we prove the escape criteria for Mann, Ishikawa and Noor iterations for the complex function $\sin(z^n) + c$. We use the proved criteria in algorithms to generate Julia and Mandelbrot sets. At some fixed input parameters we present the image comparisons of Julia and Mandelbrot sets in orbits of some nonstandard iterations (i.e. in Mann-orbit, Ishikawa-orbit, S-orbit, Noor-orbit, CR-orbit and in SP-orbit).

The rest of the paper is organized as follows. Some important definitions of Mandelbrot and Julia sets and various iterations are presented in Sec. II. Section III presents some fixed point results in the generation of fractals for a complex function $\sin(z^n) + c$. We discuss the behavior of Julia and Mandelbrot sets for different $n$ via proposed algorithms in Sec. IV. We conclude this article in Sec. V.

II. PRELIMINARIES

In this section we present definitions of sets and iterations used in this article.

Definition 1 (Julia set [1]): Let $T_c : \mathbb{C} \rightarrow \mathbb{C}$ be a complex mapping, where $c \in \mathbb{C}$ is a parameter. Then the set of points

$$J_{T_c} = \{z \in \mathbb{C} : (T^k_c(z))_{k=0}^{\infty} \text{ is bounded}\},$$

where $T^k_c(z)$ is the $k$-th iterate of $z$ is called the filled Julia set. The set of boundary points of $J_{T_c}$ is called simple Julia set.

Definition 2 (Mandelbrot set [33]): Let $T_c : \mathbb{C} \rightarrow \mathbb{C}$ be a complex mapping, where $c \in \mathbb{C}$ is a parameter. Then the collection of constants $c$ for which the corresponding Julia set $J_{T_c}$ is connected is called as Mandelbrot set $M$, i.e.,

$$M = \{c \in \mathbb{C} : J_{T_c} \text{ is connected}\},$$

Equivalently, Mandelbrot set can be defined as [34]:

$$M = \{c \in \mathbb{C} : (T^k_c(\theta)) \rightarrow \infty \text{ as } k \rightarrow \infty\},$$

where $\theta$ is any critical point of $T_c$, this is true for $T(z) = z^n + c$, because its critical point is $z = 0$ and $T(0) = c$, so after the first iteration we get $z_1 = c$, so we can omit the first iteration and take $c$ as the $z_0$. There are many fixed point iterations in literature, but the basic one was introduced by Charles Emile Picard.

Definition 3 (Picard-iteration): Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be a complex map. Then for any $z_0 \in \mathbb{C}$ the Picard-iteration in complex space is defined as:

$$z_{k+1} = T(z_k),$$

where $k = 0, 1, 2, \ldots$

W. R. Mann defined a one step fixed point iteration in 1953 named as Mann-iteration. The Mann-iteration in complex space is defined as follows:

Definition 4 (Mann-iteration [35]): Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be a complex map. Then for any $z_0 \in \mathbb{C}$ the Mann-iteration is defined as:

$$z_{k+1} = (1 - \alpha_1)z_k + \alpha_1T(z_k),$$

where $\alpha_1 \in (0, 1]$ and $k = 0, 1, 2, \ldots$

A two step fixed point iteration was introduced by S. Ishikawa in 1974 called Ishikawa iteration. In complex space this iteration is defined as:

Definition 5 (Ishikawa-iteration [36]): Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be a complex map. Then for any $z_0 \in \mathbb{C}$ the Ishikawa-iteration is defined as:

$$\begin{cases}
z_{k+1} = (1 - \alpha_1)z_k + \alpha_1T(y_k), \\
y_k = (1 - \alpha_2)z_k + \alpha_2T(z_k),
\end{cases}$$

where $\alpha_1, \alpha_2 \in (0, 1]$ and $k = 0, 1, 2, \ldots$

Rani et al. used Mann and Ishikawa iterations to visualize the superior Julia and Mandelbrot sets ( [37] and [38]).

Some two and three steps iterations were also introduced in [21], [39], [40] and [22]. Their versions in complex space for some constants $\alpha_1, \alpha_2, \alpha_3 \in (0, 1]$ are defined as follows:

Definition 6 (S-iteration [39]): Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be a complex map. Then for any $z_0 \in \mathbb{C}$ the S-iteration is defined as:

$$\begin{cases}
z_{k+1} = (1 - \alpha_1)z_k + \alpha_1T(z_k), \\
y_k = (1 - \alpha_2)z_k + \alpha_2T(z_k),
\end{cases}$$

where $\alpha_1, \alpha_2 \in (0, 1]$ and $k = 0, 1, 2, \ldots$

Definition 7 (Noor-iteration [40]): Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be a complex map. Then for any $z_0 \in \mathbb{C}$ the Noor-iteration is defined as:

$$\begin{cases}
z_{k+1} = (1 - \alpha_1)z_k + \alpha_1T(y_k), \\
y_k = (1 - \alpha_2)z_k + \alpha_2T(z_k),
\end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3 \in (0, 1]$ and $k = 0, 1, 2, \ldots$

Definition 8 (CR-iteration [21]): Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be a complex map. Then for any $z_0 \in \mathbb{C}$ the CR-iteration is defined as:

$$\begin{cases}
z_{k+1} = (1 - \alpha_1)z_k + \alpha_1T(y_k), \\
y_k = (1 - \alpha_2)z_k + \alpha_2T(z_k),
\end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3 \in (0, 1]$ and $k = 0, 1, 2, \ldots$

Definition 9 (SP-iteration [22]): Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be a complex map. Then for any $z_0 \in \mathbb{C}$ the SP-iteration is defined as:

$$\begin{cases}
z_{k+1} = (1 - \alpha_1)y_k + \alpha_1T(y_k), \\
y_k = (1 - \alpha_2)x_k + \alpha_2T(z_k),
\end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3 \in (0, 1]$ and $k = 0, 1, 2, \ldots$
The researchers used the modified versions of S, Noor, CR and SP iterations in [41], [42], [43] and [44] to generalize the fractals. The aim of this research is to prove fixed point results via iterations (4), (5), (6) and (8) for the generation of fractals. Since all one step iterations have same escape criteria and same argument for two and three steps iterations, so we apply our established results for iterations (7), (9) and (10) to compare the images.

To generate fractals, it is necessary to define the orbit of the iteration. The orbits of our proposed iterations are defined and generalized as follows:

Definition 10 (Orbit for any proposed iteration): Let \( T(z_k) = \sin(z^n) + c \) be a complex sine function with \( n \geq 2 \). Then the sequence of iterates \( \{z_k\}_{k \in \mathbb{N}} \) from any proposed iterations (i.e. from iterations (4)–(10)) is called orbit of that iteration.

III. FIXED POINT RESULTS

In this section we prove some fixed point results (i.e. escape criterion or limitations) for complex sine function \( T(z) = \sin(z^n) + c \) where \( n \geq 2 \) and \( c \in \mathbb{C} \) via proposed-iterations. Algorithms are important to generate fractals and escape limitations are the basic key to run the algorithms. Since \( |\sin(z^n)| \leq 1, \forall z \in \mathbb{C} \) and also

\[
|\sin(z^n)| = \left| z^n - \frac{z^{3n}}{3!} + \frac{z^{5n}}{5!} - \ldots \right| = |z^n| \left| 1 - \frac{z^{2n}}{3!} + \frac{z^{4n}}{5!} - \ldots \right|,
\]

\[
|\sin(y^n)| = \left| y^n - \frac{y^{3n}}{3!} + \frac{y^{5n}}{5!} - \ldots \right| = |y^n| \left| 1 - \frac{y^{2n}}{3!} + \frac{y^{4n}}{5!} - \ldots \right|,
\]

and

\[
|\sin(x^n)| = \left| x^n - \frac{x^{3n}}{3!} + \frac{x^{5n}}{5!} - \ldots \right| = |x^n| \left| 1 - \frac{x^{2n}}{3!} + \frac{x^{4n}}{5!} - \ldots \right|,
\]

where \( x, y, z \in \mathbb{C} \). Throughout this section we use \( T(z) \) as \( T_r(z), z_0 = z, y_0 = y \) and \( x_0 = x \). Furthermore, to generate Julia and Mandelbrot sets, we assume that the sums of series \( 1 - \frac{z^{2n}}{3!} + \frac{z^{4n}}{5!} - \ldots \geq |a_1|, \frac{z^{2n}}{3!} - \ldots \geq |a_2| \) and \( 1 - \frac{z^{2n}}{3!} + \frac{z^{4n}}{5!} - \ldots \geq |a_3| \) where \( |a_1|, |a_2|, |a_3| \in (0, 1) \) and \( x, y, z \in \mathbb{C} \) except those values of \( x, y, z \) wherefore \( |a_1| = |a_2| = |a_3| = 0 \).

Theorem 1: Assume that \( T_r(z) = \sin(z^n) + c \) where \( n \geq 2 \) and \( c \in \mathbb{C} \) be a complex sine function with \( |z| \geq |c| > \left( \frac{2}{|a_1|} \right)^{-\frac{1}{m}} \). If the sequence of iterates \( \{z_k\}_{k \in \mathbb{N}} \) for Picard-iteration is defined as follows:

\[
z_{k+1} = T(z_k),
\]

where \( k = 0, 1, 2, \ldots \), then \( |z_k| \rightarrow \infty \) as \( k \rightarrow \infty \).

Proof: Since \( T(z) = \sin(z^n) + c \) and \( z_0 = z \), then Picard-iteration is:

\[
|z_{k+1}| = |T(z_k)|.
\]

For \( k = 0 \), we have

\[
|z_1| = |T(z)|. \geq |\sin(z^n) + c| \geq |z^n||a_1| - |c|.
\]

Because \( |\sin(z^n)| = |z^n - \frac{z^{3n}}{3!} + \frac{z^{5n}}{5!} - \ldots | \) and we assumed that \( |z^n - \frac{z^{3n}}{3!} + \frac{z^{5n}}{5!} - \ldots | \geq |z^n||a_1| \) where \( |a_1| \in (0, 1) \) and \( z \in \mathbb{C} \) except those values of \( z \) wherefore \( |a_1| = 0 \).

\[
|z_1| \geq |z^n||a_1| - |z|, \therefore |z| \geq |c|,
\]

\[
|z_1| \geq \left( |z^n||a_1| - 1 \right).
\]

For \( k = 1 \), we have

\[
|z_2| \geq |z_1| \left( |z_1^{-1}||a_1| - 1 \right) \geq |z_1| \left( |z_1^{-1}||a_1| - 1 \right)^2. \geq |z_1|^2 |(z_1^{-1}||a_1| - 1)^3. \geq |z_1|^3 |(z_1^{-1}||a_1| - 1)^4 \frac{c}{a_1^n} \frac{1}{m}, \text{ where } |a_1| \in (0, 1). \text{ This yields } |z_1^{-1}||a_1| - 1 > 1. \therefore |z_k| \rightarrow \infty \text{ when } k \rightarrow \infty. \square
\]

Corollary 1: Assume that

\[
|z_m| > \max \left\{ |z|, \left( \frac{2}{|a_1|^m} \right)^{\frac{1}{m}} \right\},
\]

for some \( m \geq 0 \). Since \( |z^{-1}||a_1| - 1 > 1 \), therefore \( |z_{m+k}| > |z| \left( |z_{m-1}||a_1| - 1 \right)^{m+k} \). Hence \( |z_k| \rightarrow \infty \text{ when } k \rightarrow \infty. \]

Theorem 2: Assume that \( T_r(z) = \sin(z^n) + c \) where \( n \geq 2 \) and \( c \in \mathbb{C} \) be a complex sine function with \( |z| \geq |c| \). If the sequence of iterates \( \{z_k\}_{k \in \mathbb{N}} \) for Mann-iteration is defined as follows:

\[
z_{k+1} = (1 - \alpha_1)z_k + \alpha_1 T(z_k), \tag{12}
\]

where \( \alpha_1 \in (0, 1) \) and \( k = 0, 1, 2, \ldots \), then \( |z_k| \rightarrow \infty \text{ when } k \rightarrow \infty. \)
Proof: Since \( T(z) = \sin(z^a) + c \), then the Mann-iteration is:

\[
|z_{k+1}| = |(1 - \alpha_1)z_k + \alpha_1 T(z_k)|.
\]

For \( k = 0 \), we have

\[
|z_1| = |(1 - \alpha_1)z_0 + \alpha_1 T(z_0)|.
\]

\[
\geq |(1 - \alpha_1)z + \alpha_1 (\sin(z^a) + c)|
\]

\[
\geq |(1 - \alpha_1)z_0 + \alpha_1 (\sin(z^a) + c)|
\]

\[
\geq |(1 - \alpha_1)z + \alpha_1 |c|| - |(1 - \alpha_1)z_0 + \alpha_1 (\sin(z^a) + c)|
\]

\[
\geq |(1 - \alpha_1)z + \alpha_1 |c|| - |(1 - \alpha_1)z_0 + \alpha_1 (\sin(z^a) + c)|
\]

Because \( |\sin(z^a)| = |z^a - \frac{z^{3a}}{3!} + \frac{z^{5a}}{5!} - \ldots| \) and we assumed that \( |z^a - \frac{z^{3a}}{3!} + \frac{z^{5a}}{5!} - \ldots| \geq |z^a||a_1| \) where \( |a_1| \in (0, 1) \) and \( z \in \mathbb{C} \) except those values of \( z \) where \( |a_1| = 0 \).

\[
|z_1| = |z| \left( |a_1| |z_1| \right)^{-1}.
\]

For \( k = 1 \), we have

\[
|z_2| \geq |z| \left( |a_1| |z_1| \right)^{2}.
\]

Because \( |z_1| \geq |z| \left( |a_1| |z_1| \right)^{2} \) and \( |z_1| \geq |z| \geq |c| > \left( \frac{2}{|a_1|} \right)^{-\frac{1}{a_1}} \), this implies \( |a_1| |z_1| \geq |(1 - \alpha_1)z_1 + \alpha_1 T(z_1)| \).

Now iterating up to \( k \)th term, we have

\[
|z_3| \geq |z| \left( |a_1| |z_1| \right)^{-1}.
\]

\[
|z_4| \geq |z| \left( |a_1| |z_1| \right)^{-1}.
\]

\[
|z_k| \geq |z| \left( |a_1| |z_1| \right)^{-1}.
\]

Since \( |z| > \left( \frac{2}{|a_1|} \right)^{-\frac{1}{a_1}} \), this implies \( |a_1| |z_1| \geq |(1 - \alpha_1)z_1 + \alpha_1 T(z_1)| \).

Therefore \( |z_k| \to \infty \) as \( k \to \infty \).

Corollary 2: Assume that

\[
|z_m| > \max \left\{ |c|, \left( \frac{2}{|a_1|} \right)^{\frac{1}{a_1}} \right\}.
\]

for some \( m \geq 0 \). Since \( |a_1| |z_1| \geq |(1 - \alpha_1)z_1 + \alpha_1 T(z_1)| \), therefore \( |z_{m+k}| \geq |z| \left( |a_1| |z_1| \right)^{m+k} \). Hence \( |z_k| \to \infty \) when \( k \to \infty \).

Theorem 3: Assume that \( T_1(z) = \sin(z^a) + c \) where \( n \geq 2 \) and \( c \in \mathbb{C} \) be a complex sine function with \( |z| \geq |c| > \left( \frac{2}{|a_1|} \right)^{\frac{1}{a_1}} \) and \( |z_0| \geq |c| > \left( \frac{2}{|a_2|} \right)^{\frac{1}{a_2}} \), where \( |a_1|, |a_2| \in (0, 1) \). If the sequence of iterates \( \{z_k\}_{k \in \mathbb{N}} \) for Ishikawa-iteration is defined as follows:

\[
\begin{align*}
|z_{k+1}| &= |(1 - \alpha_1)z_k + \alpha_1 T(z_k)|, \\
|y_k| &= |(1 - \alpha_2)z_k + \alpha_2 T(y_k)|,
\end{align*}
\]

where \( \alpha_1, \alpha_2 \in (0, 1) \) and \( k = 0, 1, 2, \ldots \), then \( |z_k| \to \infty \) when \( k \to \infty \).

Proof: Since \( T(z) = \sin(z^a) + c \), then the first step of Ishikawa-iteration is:

\[
|y_0| = |(1 - \alpha_2)z_0 + \alpha_2 T(z_0)|.
\]

For \( k = 0 \), we have

\[
|y_0| = |(1 - \alpha_2)z_0 + \alpha_2 T(z_0)|.
\]

\[
\geq |(1 - \alpha_2)z + \alpha_2 (\sin(z^a) + c)|
\]

\[
\geq |(1 - \alpha_2)z + \alpha_2 |c|| - |(1 - \alpha_2)z_0 + \alpha_2 (\sin(z^a) + c)|
\]

\[
\geq |(1 - \alpha_2)z + \alpha_2 |c|| - |(1 - \alpha_2)z_0 + \alpha_2 (\sin(z^a) + c)|
\]

Because \( |\sin(z^a)| = |z^a - \frac{z^{3a}}{3!} + \frac{z^{5a}}{5!} - \ldots| \) and we assumed that \( |z^a - \frac{z^{3a}}{3!} + \frac{z^{5a}}{5!} - \ldots| \geq |z^a||a_1| \) where \( |a_1| \in (0, 1) \) and \( z \in \mathbb{C} \) except those values of \( z \) where \( |a_1| = 0 \).

\[
|y_0| \geq |z| \left( |a_2| |z_1| \right)^{-1}.
\]

Since \( |z| > \left( \frac{2}{|a_1|} \right)^{\frac{1}{a_1}} \), this implies \( |a_2| |z_1| \geq |(1 - \alpha_2)z_1 + \alpha_2 T(z_1)| \).

Therefore we have

\[
|y_0| \geq |a_2| |z_1|.
\]

(14)

Now for the next step of Ishikawa-iteration we have

\[
|z_{k+1}| = |(1 - \alpha_1)z_k + \alpha_1 T(y_k)|.
\]

Again for \( k = 0 \), we have

\[
|z_1| = |(1 - \alpha_1)z + \alpha_1 (\sin(y^a) + c)|
\]

\[
\geq |(1 - \alpha_1)z + \alpha_1 |c|| - |(1 - \alpha_1)z_0 + \alpha_1 (\sin(y^a) + c)|
\]

\[
\geq |(1 - \alpha_1)z + \alpha_1 |c|| - |(1 - \alpha_1)z_0 + \alpha_1 (\sin(y^a) + c)|
\]

\[
\geq |(1 - \alpha_1)z + \alpha_1 |c|| - |(1 - \alpha_1)z_0 + \alpha_1 (\sin(y^a) + c)|
\]

Because \( |y| \geq |a_2| |z_1| \). Therefore

\[
|z_{k+1}| = |(1 - \alpha_1)z_k + \alpha_1 T(y_k)|.
\]

Iterating up to \( k \)th term, we have

\[
|z_k| \geq |z| \left( |a_2| |z_1| \right)^{-1}.
\]

Since \( |z| > \left( \frac{2}{|a_2|} \right)^{\frac{1}{a_2}} \) and \( |z| > \left( \frac{2}{|a_2|} \right)^{\frac{1}{a_2}} \), this implies

\[
|z_k| \geq |z| \left( |a_2| |z_1| \right)^{-1}.
\]

Hence \( |z_k| \to \infty \) as \( k \to \infty \).

Corollary 3: Assume that

\[
|z_m| > \max \left\{ |c|, \left( \frac{2}{|a_1|} \right)^{\frac{1}{a_1}}, \left( \frac{2}{|a_2|} \right)^{\frac{1}{a_2}} \right\}.
\]
for some $m \geq 0$. Since $\alpha_1 \alpha_2 |a_1| |a_2| |z^{n-1}| - 1 > 1$, therefore $|z_{m+k}| > |z_1 (\alpha_1 \alpha_2 |a_1| |a_2| |z^{n-1}| - 1)^m$. Hence $|z_k| \to \infty$ when $k \to \infty$.

**Theorem 4:** Assume that $T_c(z) = \sin(x^n) + c$ where $n \geq 2$ and $c \in \mathbb{C}$ be a complex sine function with $|z| \geq |c| > \left(\frac{2}{\alpha_2 |a_2|}\right)^{\frac{1}{n-1}}$ and $|z| \geq |c|$ > \left(\frac{2}{\alpha_1 |a_1|}\right)^{\frac{1}{n-1}}$, where $|a_1|, |a_2|, |a_3| \in (0, 1)$. If the sequence of iterates $\{z_k\}_{k \in \mathbb{N}}$ for Noor-iteration is defined as follows:

\[
\begin{align*}
    z_{k+1} &= (1 - \alpha_1)z_k + \alpha_1 T(y_k), \\
    y_k &= (1 - \alpha_2)z_k + \alpha_2 T(x_k), \\
    x_k &= (1 - \alpha_3)z_k + \alpha_3 T(z_k),
\end{align*}
\]

where $\alpha_1, \alpha_2, \alpha_3 \in (0, 1)$ and $k = 0, 1, 2, \ldots$, the $|z_k| \to \infty$ when $k \to \infty$.

**Proof:** Since $T(z) = \sin(z^n) + c$, then in the first step of Noor-iteration we have

\[
|z_1| = |(1 - \alpha_3)z_0 + \alpha_3 T(z_0)|.
\]

For $k = 0$, we have

\[
|z_0| = |(1 - \alpha_3)z_0 + \alpha_3 T(z_0)|.
\]

Therefore we have

\[
|z_0| \geq |z|.
\]

The second step of Noor-iteration is:

\[
|y_0| = |(1 - \alpha_2)z_0 + \alpha_2 T(x_0)|.
\]

For $k = 0$, we have

\[
|y_0| = |(1 - \alpha_2)z_0 + \alpha_2 T(x_0)|.
\]

This yields

\[
|y_0| \geq |z|.
\]

Now for the last step of Noor-iteration we have

\[
|z_{k+1}| = |(1 - \alpha_1)z_k + \alpha_1 T(y_k)|.\]

For $k = 0$, we get

\[
|z_1| = |(1 - \alpha_1)z_0 + \alpha_1 (\sin(y^n) + c)| \\
\geq |\alpha_1 \alpha_2 \alpha_3 |a_1| |a_2| |a_3| |z^n| - |z| \\
|z_1| = |z| \left(|\alpha_1 \alpha_2 \alpha_3 |a_1| |a_2| |a_3| |z^{n-1}| - 1\right).
\]

Iterating up to $k^{th}$ term, we have

\[
|z_k| \geq |(1 - \alpha_1)z_0 + \alpha_1 (\sin(y^n) + c)| \geq |\alpha_1 \alpha_2 \alpha_3 |a_1| |a_2| |a_3| |z^n| - 1| \\
|z_k| \geq |z| \left(|a_1 \alpha_2 \alpha_3 |a_1| |a_2| |a_3| |z^{n-1}| - 1\right).
\]

Since $|z| > \left(\frac{2}{\alpha_2 |a_2|}\right)^{\frac{1}{n-1}}$, $|z| > \left(\frac{2}{\alpha_1 |a_1|}\right)^{\frac{1}{n-1}}$ and $|z| > \left(\frac{2}{\alpha_1 |a_1|}\right)^{\frac{1}{n-1}}$, this implies $|z| > \left(\frac{2}{\alpha_2 |a_2| |a_2| |a_2|}\right)^{\frac{1}{n-1}}$. Therefore $|z_k| \to \infty$ when $k \to \infty$.

**Corollary 4:** Assume that

\[
|z_m| = \max\left\{ |c|, \left(\frac{2}{\alpha_1 |a_1|}\right)^{\frac{1}{n-1}}, \left(\frac{2}{\alpha_2 |a_2|}\right)^{\frac{1}{n-1}}, \left(\frac{2}{\alpha_3 |a_3|}\right)^{\frac{1}{n-1}}\right\},
\]

for some $m \geq 0$. Since $\alpha_1 \alpha_2 \alpha_3 |a_1| |a_2| |a_3| |z^{n-1}| - 1 > 1$, therefore $|z_{m+k}| > |z| (\alpha_1 \alpha_2 \alpha_3 |a_1| |a_2| |a_3| |z^{n-1}| - 1)^{m+k}$. Hence $|z_k| \to \infty$ when $k \to \infty$.

**IV. APPLICATIONS IN FRACTALS**

There are many different classes of complex fractals (i.e. Julia sets, Mandelbrot sets, Multibrot sets, Multicorns sets, Biomorphs and root finding fractals etc.). To generate such fractals some methods needed to execute the algorithms. In literature the most popular methods to generate the fractals are as follows:

- Distance Estimator [45],
- Potential Function Algorithms [46] and
- Escape Criterion [47].

We use escape criteria in Algorithms 1 and 2 to fascinate the Julia and Mandelbrot sets in graphs. The specification of the computer used to generate the examples was the following:

- Processor: Intel(R) Core(TM) i5-3320 M CPU @ 2.60GHz,
- System type: 32-bit Operating System and
- Software: Mathematica 7.0.

Next, we discuss only Julia and Mandelbrot sets in Picard, Mann, Ishikawa, S. Noor, CR and SP orbits for a complex
Algorithm 1 Generation of Julia Set
Input: \( T(z) = \sin(z^n) + c \) with degree \( n \geq 2 \)–a complex sine function, \( A \)–covered area, \( K \)–the maximum number of iterations, \( \alpha_1, \alpha_2, \alpha_3, |\alpha_1|, |\alpha_2|, |\alpha_3| \in (0, 1] \)–fixed parameters and \( c \in \mathbb{C} \)–complex constants, \( \text{coloursmap}[0..M - 1] \) coloursmap with \( M \) colours.
Output: Julia set.

1. \( \text{for } z_0 \in A \text{ do} \)
2. \( \quad EL \)–escape limitation for proposed-iterations from established corollaries
3. \( \quad k = 0 \)
4. \( \quad \text{while } k \leq K \text{ do} \)
5. \( \quad \quad \text{Desire Proposed iteration} \)
6. \( \quad \quad \quad \text{if } |z_{k+1}| > EL \text{ then} \)
7. \( \quad \quad \quad \quad \text{break} \)
8. \( \quad \quad \quad k = k + 1 \)
9. \( \quad \quad i = \lfloor (M - 1) \frac{k}{K} \rfloor \)
10. \( \quad \quad \text{colour } z_0 \text{ with } \text{coloursmap}[i] \)

Algorithm 2 Generation of Mandelbrot Set
Input: \( T(z) = \sin(z^n) + c \) with degree \( n \geq 2 \)–a complex sine function, \( A \)–covered area, \( K \)–the maximum number of iterations, \( \alpha_1, \alpha_2, \alpha_3, |\alpha_1|, |\alpha_2|, |\alpha_3| \in (0, 1] \)–fixed parameters and \( c \in \mathbb{C} \)–complex constants, \( \text{coloursmap}[0..M - 1] \) coloursmap with \( M \) colours.
Output: Mandelbrot set.

1. \( \text{for } c \in A \text{ do} \)
2. \( \quad EL \)–escape limitation for proposed-iterations from established corollaries
3. \( \quad k = 0 \)
4. \( \quad z_0 \)– any one critical point of \( T_c \)
5. \( \quad \text{while } k \leq K \text{ do} \)
6. \( \quad \quad \text{Desire Proposed iteration} \)
7. \( \quad \quad \quad \text{if } |z_{k+1}| > EL \text{ then} \)
8. \( \quad \quad \quad \quad \text{break} \)
9. \( \quad \quad \quad k = k + 1 \)
10. \( \quad \quad i = \lfloor (M - 1) \frac{k}{K} \rfloor \)
11. \( \quad \quad \text{colour } c \text{ with } \text{coloursmap}[i] \)

sine function. Now we use these results in our algorithms to generalize the Julia and Mandelbrot sets for a complex sine function \( T(z) = \sin(z^n) + c \) via different proposed iterations.

A. JULIA SET
Julia set is the most studied type of fractals. In this part of the paper we discuss some examples of Julia sets of the function \( T(z) = \sin(z^n) + c \) with degree \( n \geq 2 \) in the proposed orbits. We set the maximum number of iterations to 10 and \( A = [-4.5, 4.5]^2 \) as area for all Julia sets in Algorithm 1.
Example 1: In this example we take $n = 2$. In Figs.1–7 the parameters are $\alpha_1 = \alpha_2 = \alpha_3 = 0.1$, $|a_1| = |a_2| = |a_3| = 0.1$, in Figs.8–14 the parameters are $\alpha_1 = \alpha_2 = \alpha_3 = 0.3$, $|a_1| = |a_2| = |a_3| = 0.3$, in Figs.15–21 the parameters are $\alpha_1 = \alpha_2 = \alpha_3 = 0.5$, $|a_1| = |a_2| = |a_3| = 0.5$, in Figs.22–28 the parameters are $\alpha_1 = \alpha_2 = \alpha_3 = 0.7$,
$|a_1| = |a_2| = |a_3| = 0.7$ and in Figs. 29–35 the parameters are $\alpha_1 = \alpha_2 = \alpha_3 = 0.9$, $|a_1| = |a_2| = |a_3| = 0.9$.

We use $c = 0.005i$ in all Figs. 1–35. We observe that all images in Figs. 1–35 are quadratic Julia sets in Picard, Mann, Ishikawa, S, Noor, CR and SP orbits. We also notice the Julia sets in Figs. 1–35 have four lashes, two along $x$-axis.
and two along y-axis. The lash along the positive x-axis have large bulb as compare to the lash along negative x-axis. The lashes along y-axis are same. The images for $n = 2$ via all proposed iterations have resemblance but not completely the same. Moreover, we analyze that the images in Figs.1–35 are changing with the change in input parameters.
FIGURE 29. Quadratic Julia set in Picard-Orbit with generation time = 17.52s.

FIGURE 30. Quadratic Julia set in Mann-Orbit with generation time = 24.69s.

FIGURE 31. Quadratic Julia set in Ishikawa-Orbit with generation time = 35.77s.

FIGURE 32. Quadratic Julia set in S-Orbit with generation time = 32.65s.

FIGURE 33. Quadratic Julia set in Noor-Orbit with generation time = 33.64s.

FIGURE 34. Quadratic Julia set in CR-Orbit with generation time = 42.41s.

FIGURE 35. Quadratic Julia set in SP-Orbit with generation time = 43.9s.

FIGURE 36. Cubic Julia set in Picard-Orbit with generation time = 32.02s.

(i.e. $\alpha_1, \alpha_2, \alpha_3, |a_1|, |a_2|, |a_3|$). The image generation time is also calculated for each iteration.

Example 2: In this example we generate the Julia sets $n = 3$. In Figs.36–42 the parameters are $\alpha_1 = \alpha_2 = \alpha_3 = 0.5$, $c = 0.8i$ and $|a_1| = |a_2| = |a_3| = 0.001$. The images in Figs.36–42 are cubic Julia sets in Picard, Mann, Ishikawa, S, Noor, CR and SP orbits. The image in Fig.36 has six lashes...
in which three are fatty while other three are smart. The images in Figs.37—42 have also six lashes but different in size. In each image from Figs.37—42, the fatty lash is symmetrical along negative x-axis and the smarter one is along positive x-axis. The Cubic Julia sets also change with the change of parameters. Moreover, we observe that the images for each iteration have a difference in plot points.
FIGURE 45. Quadric Julia set in Ishikawa-Orbit with generation time $= 30.6$ s.

Example 3: For quadric Julia set we use complex sine function $T(z) = \sin(z^2) + c$. In Figs.43–49 the parameters are $\alpha_1 = \alpha_2 = \alpha_3 = 0.9$, $c = 0.4$ and $|a_1| = |a_2| = |a_3| = 0.0001$. We observe that the images in Figs.43–49 are quadric Julia sets in Picard, Mann, Ishikawa, S, Noor, CR and SP orbits with eight lashes. In each image of quadric Julia set, two lashes are symmetrical along the $x$-axis and two

FIGURE 46. Quadric Julia set in S-Orbit with generation time $= 35.16$ s.

FIGURE 47. Quadric Julia set in Noor-Orbit with generation time $= 41.88$ s.

FIGURE 48. Quadric Julia set in CR-Orbit with generation time $= 60.92$ s.

FIGURE 49. Quadric Julia set in SP-Orbit with generation time $= 65.91$ s.

FIGURE 50. Quadratic Mandelbrot in Picard-Orbit with generation time $= 100.92$ s.

FIGURE 51. Quadratic Mandelbrot set in Mann-Orbit with generation time $= 39.97$ s.

FIGURE 52. Quadratic Mandelbrot set in Ishikawa-Orbit with generation time $= 52.50$ s.
symmetrical along the y-axis while other four are originate from each edge of rectangular shape body of quadric Julia set. The images in Figs.43–49 are identical and unique and

have same geometry but slightly differ in Julia points. Each iterations have also minor difference in shape with others.

B. MANDELBROT SETS

Mandelbrot coined the word fractal for a self-similar complex graph of $z^2 + c$. He determined the properties of quadratic complex fractal and he observed that a classical quadratic Mandelbrot set has two main parts, one is the primary part (i.e. Belly of the Mandelbrot set or Cardiod) and other is the secondary part that contains one large and two small bulbs. Later on Mandelbrot set was generalized in such a way that each Mandelbrot set has $n$ small bulbs, $n − 1$ cardioids and small bulbs for $n ≥ 2$. In this subsection we sketch some
graphs of Mandelbrot at some fixed parameters and different $n$ by using the proposed iterations and the developed escape criterion in Algorithm 2. Moreover, we compare the images and discuss their nature for each iteration. In this subsection we choose $K = 30$ and for each quadratic Mandelbrot set area $A = [-6, 6]^2$, for each cubic Mandelbrot set area...
FIGURE 68. Cubic Mandelbrot set in Noor-Orbit with generation time = 37.84 s.

FIGURE 69. Cubic Mandelbrot set in CR-Orbit with generation time = 35.70 s.

FIGURE 70. Cubic Mandelbrot set in SP-Orbit with generation time = 45.59 s.

FIGURE 71. Decade Mandelbrot set in Picard-Orbit with generation time = 21.63 s.

FIGURE 72. Decade Mandelbrot set in Mann-Orbit with generation time = 20.81 s.

FIGURE 73. Decade Mandelbrot set in Ishikawa-Orbit with generation time = 24.24 s.

FIGURE 74. Decade Mandelbrot set in S-Orbit with generation time = 23.10 s.

\[ A = [-1.5, 1.5] \times [-2, 2] \] and for higher degree Mandelbrot sets \( A = [-1.5, 1.5]^2 \).

Example 4: For quadratic Mandelbrot set of complex sine function \( T(z) = \sin(z^2) + c \) we fixed the parameters as \( \alpha_1 = \alpha_2 = \alpha_3 = 0.2 \) (i.e. in Figs. 50–56) and the parameters for Figs. 57–63 are \( \alpha_1 = \alpha_2 = \alpha_3 = 0.4 \). Also we adjust \( |a_1| = |a_2| = |a_3| = 0.1 \) for each image in Figs. 50–63. The graphs in Figs. 50 and 57 present a series of connected quadratic Mandelbrot sets symmetrical along the \( x \)-axis and each set has four lashes symmetrical along \( y \)-axis in the Picard-orbit. The images in Figs. 51, 52, 54, 56, 58, 59, 61 and 63 are like the butterflies, flying in the direction of negative \( x \)-axis. The Mandelbrot sets in Figs. 53, 55, 60 and 62 have different bodies. Mandelbrot sets in some images contain the branches with small bulbs. We also observe that for each iteration Mandelbrot set differ in shape.
in Figs. 71–77 resemble with each other but different in the shapes of bulbs.

V. CONCLUSION

In this research we proved fixed point results for complex sine function \( T(z) = \sin(z^n) + c \) where \( n \geq 2 \) and \( c \in \mathbb{C} \) via Picard, Mann, Ishikawa and Noor iterations. We applied these results in algorithms and generalized the Julia and Mandelbrot sets. We demonstrated some examples for Julia and Mandelbrot sets of a complex sine function by using the proved results. We also compared the images at some fixed parameters and calculated the image generation time in second for each image. Moreover, we analyzed that images of Julia and Mandelbrot sets slightly different for each iteration.

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