CONTRACTIONS OF SUBCURVES OF LOG SMOOTH CURVES

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Abstract. Let $C$ be a nodal curve, and let $E$ be a union of semistable subcurves of $C$. We consider the problem of contracting the connected components of $E$ to singularities in a way that preserves the genus of $C$ and makes sense in families. In order to do this, we introduce the notion of mesa curve, a nodal curve with a logarithmic structure and a nice subcurve. We then show that such a contraction exists for families of mesa curves. Resulting singularities include the elliptic Gorenstein singularities.

1. Introduction

Let $C$ be a nodal curve, and let $E$ be a union of semistable subcurves of $C$. Our goal is to contract the connected components of $E$ to singularities in a way that preserves the genus of $C$ and is compatible with families.

To that end we introduce the notion of a mesa curve (Definition 3.1, below). A mesa curve consists of a log curve $\pi : C \to S$ of any genus and a section $\lambda$ of the characteristic sheaf of $C$. This section encodes a subcurve $E$ of $C$, which may consist of several connected subcurves of any genus, which we call mesas. The mesas are subject to a cohomological condition, namely $H^1(E, O_E(-\lambda)) = 0$. Our main theorem is that mesa curves admit a contraction map to a flat family of singular curves with good properties.

Theorem 1.1. For each mesa curve $\pi : C \to S$, there is a commutative triangle

$$
\begin{array}{c}
C \\
\downarrow \pi \\
S
\end{array} \xrightarrow{\tau} \begin{array}{c}
\overline{C} \\
\downarrow \psi
\end{array}
$$

so that

(i) $\psi$ is flat and proper with reduced 1-dimensional fibers;
(ii) $\tau$ is surjective;
(iii) $\tau$ restricts to an isomorphism $C - E \to \overline{C} - \tau(E)$;
(iv) in each geometric fiber, $C_\pi \xrightarrow{\tau_\pi} \overline{C_\pi} \to \overline{\pi}$, $\tau_\pi$ takes each connected component of $E \times_S \overline{\pi}$ of genus $g$ to a distinct singular point of $\overline{C}$ of genus $g$;
(v) if a connected component of $E \times_S \overline{\pi}$ is a small mesa of genus one, its image is an elliptic Gorenstein singularity;
(vi) for each morphism of schemes $T \to S$, there is a natural isomorphism $\overline{C \times_S T} \cong \overline{C} \times_S T$, compatible with the maps from $C$ and to $S$.

(We will define “small mesa” in Section 3.)

Part (vi) implies that our construction will induce maps between suitable moduli spaces of curves. The construction of the contracted curve is given locally in Definition 4.3.

For a mesa curve over an algebraically closed field, we will also be able to explicitly describe the regular functions in a neighborhood of the image of $E$ (see Proposition 8.2). We pause...
for a moment to describe the resulting ring of functions independently of log structures: let $C$ be a nodal curve and $E$ a connected semistable subcurve of genus $g$. Let $Z$ be the union of the irreducible components of $C$ not contained in $E$, and let $x_1, \ldots, x_m$ be local parameters on $Z$ of the points $p_1, \ldots, p_m$ where $Z$ meets $E$. Then for any log structure making $(C, E)$ into a mesa curve, there is an open neighborhood $U$ of $\tau(E)$ in $\overline{C}$ so that

$$
\Gamma(U, \mathcal{O}_C) = \left\{ f \in \mathcal{O}_Z(U \cap Z) \mid f(p_i) = f(p_j) \text{ for all } i, j \text{ and } \left[ \frac{\partial f}{\partial x_i}(p_i) \right]_{i=1}^m \in V \right\}
$$

where $V$ is a linear subspace of $k^m$ of codimension $g$. Possible singularities include all of the elliptic Gorenstein singularities (such as cusps and tacnodes), elliptic Gorenstein singularities meeting an ordinary $n$-fold point transversally, and a union of cusps meeting transversally.

In [10], in the discussion after Corollary 1.14, Smyth notes that contracting an elliptic bridge to a tacnode requires more data than the subcurve to be contracted. It is interesting to see that in our situation the log structure provides the necessary information: see Example 8.5

We are deeply motivated by a similar contraction map, constructed by Keli Santos-Parker in his thesis [7], then later refined in [8], in which a subcurve of a family of centrally aligned log curves of genus one is collapsed to an elliptic Gorenstein singularity. This construction is limited to collapsing a single genus one subcurve of a genus one log curve; generalizing this map to one capable of collapsing multiple subcurves of a higher genus curve is the central goal of this paper.

Our strategy for overcoming these limitations is to describe both the data of the contraction and the contracted curve itself more locally on $C$. The method of encoding the subcurve $E$ in [8] requires the dual graph of $C$ to be a rooted tree; we replace this data with the section $\lambda$, which is indifferent to the complement of $E$. The contraction in [8] is constructed by applying Proj to certain ring of global sections; we construct an affine local neighborhood of the image of $E$ instead.

1.1. Future work. In [8, Proposition 4.6.3.1] Ranganathan, Santos-Parker, and Wise use their contraction map to give a modular interpretation of the Vakil-Zinger desingularization of $\overline{M}_{1,n}(Y, \beta)$. More specifically, they show that the Vakil-Zinger desingularization $VZ_{1,n}(Y, \beta)$ parametrizes stable maps admitting a factorization through the contraction, that is, diagrams

$$
\begin{array}{ccc}
\tilde{C} & \longrightarrow & C \\
\downarrow & & \downarrow \\
\overline{C} & \longrightarrow & Y
\end{array}
$$

where

(i) $C$ is centrally aligned;
(ii) $C \rightarrow Y$ is a stable map of homology class $\beta$;
(iii) $\tilde{C}$ is a canonical log modification of $C$ with a distinguished subcurve $E$, where $E$ is the largest “disc” around the core collapsed by $\tilde{C} \rightarrow Y$;
(iv) $\tilde{C} \rightarrow \overline{C}$ is the contraction map of [8], sending $E$ to an elliptic Gorenstein singularity.

An obstacle to building this space of stable maps with factorizations in higher genus is that $C \rightarrow Y$ may collapse multiple subcurves of genus greater than 0, so one would want a
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contraction map \( \tilde{C} \rightarrow \overline{C} \) collapsing multiple subcurves. With the results of this paper, we now have such a contraction map.

We therefore hope to show in future work that the same construction may now be applied to the moduli of stable maps of genus \( g \geq 2 \) which collapse at most a union of disjoint elliptic subcurves. This would yield a modular partial desingularization of the locus of these maps in \( \overline{M}_{g,n}(Y, \beta) \).

1.2. Outline of the paper. We begin in Section 2 with background material on logarithmic structures and log curves. The most important bits for the remainder of the paper are an exposition of a local structure theorem for log curves, the notion of associated bundle, and the description of the associated bundle for a log curve over an algebraically closed field.

With these fundamental notions in hand, we can define the notion of mesa curve in Section 3. We then make some general remarks and make explicit the way that mesa curves generalize the centrally aligned curves of [8].

In Section 4 we motivate and state the local construction of \( \overline{C} \). We then indicate how to reduce the proof of Theorem 1.1 to a local, “standard” situation, where the subcurve to be contracted is connected.

In Section 5, we consider the problem of finding sections of \( \mathcal{O}_E(-\lambda) \) with certain values and reduce it to a Mittag-Leffler problem.

In Section 6, we show that the contracted family of curves is flat and commutes with base change. Besides being part of the statement of Theorem 1.1, these facts are used frequently in subsequent proofs.

Section 7 consists of a couple of well-definedness checks, specifically showing that an affine local description of the singularity glues with the rest of \( C \) and that the particular choice of affine open neighborhood is inconsequential. In addition, this will show that the contraction map is an isomorphism in the complement of \( E \).

In Section 8, we give a description of the regular functions in a neighborhood of the contracted singularity in the case that the base is the spectrum of an algebraically closed field. A corollary is the reducedness of the resulting curve. We then give several examples of the resulting singularities.

Finally, in Section 9, we argue that the contraction map is surjective and the contracted family of curves is proper.

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2. Log structures and log curves

For a more comprehensive introduction to log structures, the reader may wish to consult [4] or [5]. All monoids are assumed commutative.

Let \( X \) be an algebraic space. Regard \( \mathcal{O}_X \) as a sheaf of monoids on the small étale site of \( X \) with the operation induced by the multiplication of \( \mathcal{O}_X \). A log structure on an algebraic space \( X \) consists of an étale sheaf of monoids \( M_X \), together with a morphism of sheaves of monoids \( \epsilon : M_X \rightarrow \mathcal{O}_X \) so that \( \epsilon \) restricts to an isomorphism \( \epsilon : \epsilon^{-1}(\mathcal{O}_X^*) \rightarrow \mathcal{O}_X^* \). The pair \( (X, \epsilon : M_X \rightarrow \mathcal{O}_X) \) is called a log algebraic space. We will often abbreviate \( (X, \epsilon : M_X \rightarrow \mathcal{O}_X) \)
to $(X, M_X)$ or just $X$. It is convenient to identify $\epsilon^{-1}(\mathcal{O}_X^*)$ with $\mathcal{O}_X^*$. The quotient sheaf of monoids $\overline{M}_X := M_X / \mathcal{O}_X^*$ is called the characteristic sheaf of $X$. Throughout this section, when we take a stalk of an étale sheaf at a geometric point, we intend the étale notion of stalk.

Given log algebraic spaces $(X, \epsilon : M_X \to \mathcal{O}_X)$ and $(Y, \eta : M_Y \to \mathcal{O}_Y)$, a morphism of log algebraic spaces from $X$ to $Y$ is a morphism of algebraic spaces $f : X \to Y$ and a morphism $f^\flat : f^{-1}M_Y \to M_X$ so that

\[
\begin{array}{ccc}
f^{-1}M_Y & \xrightarrow{f^\flat} & M_X \\
f^{-1}\epsilon & \downarrow & \downarrow \epsilon \\
f^{-1}\mathcal{O}_Y & \xrightarrow{f^\flat} & \mathcal{O}_X
\end{array}
\]

commutes.

It is convenient to have a way to specify log structures without explicit reference to $\mathcal{O}_X^*$. If $e : M \to \mathcal{O}_X$ is any morphism from a sheaf of monoids $M$ to $\mathcal{O}_X$, we may define the associated log structure to be the dashed arrow in the pushout diagram of étale sheaves of monoids on $X$ below:

\[
\begin{array}{ccc}
e^{-1}(\mathcal{O}_X^*) & \longrightarrow & M \\
\downarrow \epsilon & & \downarrow \\
\mathcal{O}_X^* & \longrightarrow & M^a \quad \epsilon
\end{array}
\]

Given a morphism of algebraic spaces $f : X \to Y$ and a log structure $\eta : M_Y \to Y$, the inverse image log structure on $X$, denoted $f^*M_Y$, is the log structure associated to the composition $f^{-1}M_Y \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. There is an induced morphism of sheaves $f^*M_Y \to M_X$; if this morphism is an isomorphism, we say $f$ is strict.

A chart for a log structure $\epsilon : M \to \mathcal{O}_X$ is a morphism of monoids $e : P \to \Gamma(X, \mathcal{O}_X)$ so that when we take the induced morphism $e : P \to \mathcal{O}_X$ from a constant sheaf of monoids, then the associated log structure of this morphism, we arrive at a log structure isomorphic to $e : M \to \mathcal{O}_X$. We think of a chart as a choice of global generators of the log structure. A log structure is said to be quasicoherent if it admits a chart étale locally.

For each monoid $P$, we may give $\text{Spec } \mathbb{Z}[P]$ a log structure associated to the usual map $P \to \mathbb{Z}[P]$. A chart $e : P \to \Gamma(X, \mathcal{O}_X)$ is then equivalent to a strict map of algebraic spaces $X \to \text{Spec } \mathbb{Z}[P]$. Analogously, a chart of a morphism $f : X \to Y$ of log algebraic spaces is a commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \text{Spec } \mathbb{Z}[Q] \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f} & \text{Spec } \mathbb{Z}[P]
\end{array}
\]

with strict horizontal arrows where the right vertical map is induced by a morphism $P \to Q$ of monoids.

A monoid $P$ is called finitely generated if there is a surjective morphism $\mathbb{N}^r \to P$ for some integer $r$. There is a groupification functor $P \mapsto P^{gp}$ from monoids to groups, computed by...
formally adding inverses to all elements of $P$. A monoid $P$ is said to be \textit{integral} if the natural morphism $P \to P^{\text{op}}$ is injective. If $P$ is finitely generated and integral, we say $P$ is \textit{fine}. An integral monoid $P$ is \textit{saturated} if, for any $x \in P^{\text{op}}$ and $n \in \mathbb{N}$ so that $n \cdot x \in P$, we have $x \in P$. A monoid that is both fine and saturated is called \textit{fs}. A log structure $\epsilon : M_X \to \mathcal{O}_X$ is said to be \textit{coherent} (resp. \textit{integral}, \textit{fine}, \textit{saturated}, \textit{fs}) if it locally has a chart by finitely generated (resp. integral, fine, saturated, fs) monoids.

A morphism $i : T' \to T$ of log algebraic spaces is called an \textit{exact closed immersion} if it is strict and the underlying map of algebraic spaces is a square-zero extension. A morphism $f : X \to Y$ of fine log algebraic spaces is called \textit{log smooth} if

(i) The underlying map of algebraic spaces is locally of finite presentation;

(ii) For any diagram of solid arrows

\[
\begin{array}{ccc}
T' & \longrightarrow & X \\
\downarrow i & & \downarrow f \\
T' & \longrightarrow & Y
\end{array}
\]

where $i : T' \to T$ is an exact closed immersion of affine log schemes, there is a morphism of log algebraic spaces completing the diagram.

The underlying morphism of algebraic spaces of a log smooth map need not be smooth.

A homomorphism of integral monoids $h : Q \to P$ is \textit{integral} if the induced map of rings $\mathbb{Z}[Q] \to \mathbb{Z}[P]$ is flat (cf. \cite[Proposition (4.1)]{4}). A morphism $f : X \to Y$ of integral log algebraic spaces is \textit{integral} if, for any $x \in X$, the induced morphism $\overline{M}_{Y,f(x)} \to \overline{M}_{X,x}$ is integral.

\textbf{Definition 2.1.} (cf. \cite[Definition 1.2]{3}) Let $S$ be an fs log scheme. A \textit{log curve} over $S$ is a log smooth and integral morphism $\pi : C \to S$ of fs log algebraic spaces such that every geometric fiber of $\pi$ is a reduced and connected curve.

In other words, a log curve $\pi : C \to S$ is a relative curve so that $\pi$ is locally of finite presentation on rings (from log smoothness) and on monoids (from the fs hypothesis), flat on rings (implied by log smoothness) and on monoids (from integrality), and that satisfies a log version of the infinitesimal lifting criterion for smoothness.

F. Kato has shown that the underlying algebraic space of a log curve is nodal \cite[Theorem 1.3]{3}, and the log structure of a log curve is constrained in such a way that it encodes data very similar to that of a pointed nodal curve. We adapt his statement in \cite[Subsection 1.8]{3} for our purposes.

\textbf{Theorem 2.2.} Let $\pi : C \to S$ be a family of proper log curves. If $\overline{x} \in C$ is a geometric point with image $\overline{s} \in S$, let $Q = \overline{M}_{C,\overline{x}}$ and $P = \overline{M}_{S,\overline{s}}$. We may find connected strict \'{e}tale neighborhoods $V$ of $\overline{x}$ and $U$ of $\overline{s}$ fitting into a diagram

\[
\begin{array}{ccc}
C & \overset{\pi}{\leftarrow} & V \\
\downarrow & & \downarrow \\
S & \overset{\pi}{\leftarrow} & U
\end{array}
\]

in which all horizontal arrows are strict and the labelled arrows are \'{e}tale. There are three possibilities for the right square, depending on $Q$:
• (the smooth germ) $Q \cong P$, and the right square is isomorphic to

$$\begin{array}{c}
\text{Spec } O_U[x] \\
\downarrow \\
U
\end{array} \longrightarrow \begin{array}{c}
\text{Spec } Z[P] \\
\downarrow \\
\text{Spec } Z[P]
\end{array}$$

where the right vertical arrow is induced by the identity map on $P$

• (the germ of a marked point) $Q \cong P \oplus \mathbb{N}$, and the right square is isomorphic to

$$\begin{array}{c}
\text{Spec } O_U[x] \\
\downarrow \\
U
\end{array} \longrightarrow \begin{array}{c}
\text{Spec } Z[P \oplus \mathbb{N}] \\
\downarrow \\
\text{Spec } Z[P]
\end{array}$$

where the right vertical arrow is induced by the inclusion $P \to P \oplus \mathbb{N}$, and the top map sends the generator of $\mathbb{N}$ to $x$.

• (the node) There is an element $\delta_x \in P$ mapping to an element $t$ of $\Gamma(U, O_U)$, the monoid $Q$ is isomorphic to $\{(p, p') \in P \oplus P \mid p - p' \in \mathbb{Z} \delta_x\}$, and the right square is isomorphic to

$$\begin{array}{c}
\text{Spec } O_U[x, y]/(xy - t) \\
\downarrow \\
U
\end{array} \longrightarrow \begin{array}{c}
\text{Spec } Z[Q] \\
\downarrow \\
\text{Spec } Z[P]
\end{array}$$

The right vertical arrow is induced by the diagonal map $P \to Q$, and the top horizontal arrow sends $(\delta_x, 0)$ to $x$ and $(0, \delta_x)$ to $y$.

**Proof.** We sketch how to obtain our statement from [3, Subsection 1.8]. First we reduce to the case that $S$ is affine Noetherian, since F. Kato assumes all schemes are locally Noetherian.

We may assume $S$ is affine. Since the defining equations of $C$ can only involve finitely many elements of $O_S$ and the log structure of $C$ is finitely generated, $C$ can be obtained by pulling back a log curve $C' \to S'$, where the underlying scheme of $S'$ is the spectrum of a Noetherian ring. Since our étale local description is stable under pullback, we may assume $S$ is affine Noetherian.

Let $A = \text{Spec } O_{S, \pi}$, and consider the pullback $C_A$ of $C$ to $\text{Spec } A$. For each case, F. Kato gives an étale neighborhood $V_A$ of $\pi$, a strict étale map $V_A \to V'_A$, where $V'_A$ is either $\text{Spec } A[x]$ or $\text{Spec } A[x, y]/(xy - t)$, and a chart for $V_A$ in such a way that we obtain a diagram with strict horizontal arrows

$$
\begin{array}{c}
C_A \\
\downarrow \\
\text{Spec } A
\end{array} \leftarrow \begin{array}{c}
V_A \\
\downarrow \\
\text{Spec } A
\end{array} \longrightarrow \begin{array}{c}
V'_A \\
\downarrow \\
\text{Spec } A
\end{array} \longrightarrow \begin{array}{c}
\text{Spec } Z[Q] \\
\downarrow \\
\text{Spec } Z[P]
\end{array}.
$$

By standard arguments, we can extend this diagram to an étale neighborhood $U$ of $\pi$ in $S$, giving our statement.

We will say that a geometric point $\pi \in C$ is smooth, marked, or a node according to the kind of neighborhood it has. If $\pi$ is marked, the stalk of $M_C$ at $\pi$ is naturally isomorphic
over $\mathcal{M}_{S,\pi(\overline{\pi})}$ to $\mathcal{M}_{S,\pi(\overline{\pi})} \oplus \mathbb{N}$; given a global section $\sigma$ of $\mathcal{M}_C$, we define the slope of $\sigma$ at $\overline{\pi}$ to be the value of $\sigma_x$ in the $N$ coordinate. If $\overline{\pi}$ is nodal, the element $\delta_x$ of $\mathcal{M}_{S,\pi}$ is called the smoothing parameter of the node $\overline{\pi}$. We note that the monoid $Q$ in the case of the germ of a node may also be presented as $\mathbb{N} \oplus \mathbb{N} \beta \oplus P/(\alpha + \beta \sim \delta_x)$, with $\alpha$ mapping to $x$ and $\beta$ mapping to $y$.

We will need to consider the stalks of sections of $\mathcal{M}_C$ later. Let $U$ and $V$ be neighborhoods as in Theorem 2.2. For each geometric point $r$ of $S$ factoring through $U$, let $P_r = P/\eta^{-1}(\mathcal{O}_C^*)$. Denote by $-r : \Gamma(U, \mathcal{M}_S) \rightarrow \mathcal{M}_{S,r}$ the map to the stalk at $r$; this is the quotient map $P \rightarrow P_r$. Consider a geometric point $t$ of $C$ factoring through $V$.

If $V$ is the germ of a smooth point, the stalk map $\Gamma(V, \mathcal{M}_C) \rightarrow \mathcal{M}_{C,t}$ to the stalk at $t$ is:

(i) If $(0, 1)$ does not map to a unit in $\mathcal{O}_{C,t}$,
$$P \oplus \mathbb{N} \rightarrow P_{\pi(t)} \oplus \mathbb{N},$$
$$(p, n) \mapsto (p_{\pi(t)}, n);$$

(ii) If $(0, 1)$ does map to a unit,
$$P \oplus \mathbb{N} \rightarrow P_{\pi(t)},$$
$$(p, n) \mapsto p_{\pi(t)};$$

If $V$ is the germ of a node, let $\alpha = (\delta_x, 0)$ and let $\beta = (0, \delta_x)$. One computes that the map $Q = \Gamma(V, \mathcal{M}_C) \rightarrow \mathcal{M}_{C,t}$ to the stalk at $t$ is:

(i) If neither $\alpha$ nor $\beta$ maps to a unit in $\mathcal{O}_{C,t}$,
$$Q \rightarrow \{(p, q) \in P_{\pi(t)} \oplus P_{\pi(t)} \mid p - q \in \mathbb{Z}(\delta_x)_{\pi(t)}\},$$
$$(p, q) \mapsto (p_{\pi(t)}, q_{\pi(t)});$$

(ii) If $\alpha$ maps to a unit in $\mathcal{O}_{C,t}$, but $\beta$ does not,
$$Q \rightarrow P_{\pi(t)},$$
$$(p, q) \mapsto q_{\pi(t)};$$

(iii) If $\beta$ maps to a unit in $\mathcal{O}_{C,t}$, but $\alpha$ does not,
$$Q \rightarrow P_{\pi(t)},$$
$$(p, q) \mapsto p_{\pi(t)};$$

(iv) And if both $\alpha$ and $\beta$ map to units (or equivalently, $\delta_x$ maps to a unit in $\mathcal{O}_{S,\pi(\overline{\pi})}$),
$$Q \rightarrow P_{\pi(t)},$$
$$(p, q) \mapsto p_{\pi(t)} = q_{\pi(t)}.$$

In other words, when $V$ is the germ of a node $\overline{\pi}$, a section of $\mathcal{M}_C(V)$ may be regarded as a choice of an element of $P$ on each branch through the node, differing by an integral multiple of the smoothing parameter, $\delta_x$. If $\overline{t}$ lies in one of these branches, but not the other, the restriction of the section to the stalk at $\overline{t}$ only retains the value on the relevant branch. Where the node is smoothed out—where $\delta_x$ maps to a unit—the values on the two branches become equal, and the stalk at $\overline{t}$ can be taken to be either value.
A typical log curve and its dual graph.

Given a log smooth curve \( \pi: C \to S \) where the underlying scheme of \( S \) is an algebraically closed field, we can define the dual graph \( \Gamma(C) \) of \( C \) to be the triple of sets

(i) A set of vertices, \( V(\Gamma(C)) \), equal to the set of components of \( C \);
(ii) A set of edges, \( E(\Gamma(C)) \), equal to the set of nodes of \( C \);
(iii) A set of half-edges, \( H(\Gamma(C)) \), equal to the set of marked points of \( C \),

together with incidence functions

(i) \( i_E \), taking an edge \( e \in E(\Gamma(C)) \) to the unordered pair of components of the two branches through \( e \);
(ii) \( i_H \), taking a half-edge \( h \in H(\Gamma(C)) \) to the the component to which \( h \) belongs.

The smoothing parameter \( \delta_e \) of an edge \( e \) is thought of as its length. Note that the vertices incident to an edge need not be distinct: consider the nodal cubic. This definition of the dual graph has the deficiency that it is not easy to describe which direction a path traverses a loop, but this will not be an issue for our purposes.

A piecewise linear function on \( \Gamma(C) \) consists of the data

(i) For each \( v \in V(\Gamma(C)) \), an element \( f_v \in \Gamma(S, M_S) \);
(ii) For each \( h \in H(\Gamma(C)) \), a natural number \( n_h \in \mathbb{N} \),

such that whenever \( v, w \) are a pair of vertices incident to a common edge, \( e \), \( f_v - f_w \) is an integer multiple of \( \delta_e \). The set of all piecewise linear functions on \( \Gamma(C) \) will be denoted \( PL(\Gamma(C)) \).

An integral log scheme \( X \) may regarded as roughly a scheme with a distinguished family of generalized Cartier divisors parametrized by \( M_X \). More precisely, let \( (X, \epsilon: M_X \to X) \) be a log scheme, let \( U \to X \) be an étale map, and let \( \sigma \in \Gamma(U, \overline{M}_X) \). Since the log structure on \( X \) is integral, the action of \( \mathcal{O}_X^* \) on \( M_X \) is free, so the sheaf of lifts of \( \sigma \) to \( M_U \) is an \( \mathcal{O}_U^* \)-torsor, \( T \). The structure map \( \epsilon: M_X \to \mathcal{O}_X \) restricts to a map \( T \to \mathcal{O}_U \). Taking the invertible sheaf associated to \( T \), then dualizing, we arrive at an invertible sheaf \( \mathcal{O}_U(\sigma) \). Dualizing the map \( T \to \mathcal{O}_U \) gives us a canonical section of \( \mathcal{O}_U(\sigma) \).

It can be helpful to introduce coordinates. Since Zariski and étale \( \mathcal{O}_U^* \)-torsors coincide, there exists a Zariski cover \( \{U_i \to U\}_{i \in I} \) of \( U \) and sections \( \sigma_i \in \Gamma(U_i, M_X) \) so that \( \mathcal{O} = \mathcal{O}_U(\sigma) \). Since the log structure is integral, for each \( i, j \) there is a unique \( \xi_{i,j} \in \mathcal{O}_X^*(U_i \cap U_j) \) so that \( \sigma_i|_{U_i \cap U_j} = \xi_{i,j} \cdot \sigma_j|_{U_i \cap U_j} \). Since \( \epsilon \) is the identity on \( \mathcal{O}_U^* \), it follows that \( \epsilon(\sigma_i) \) and \( \epsilon(\sigma_j) \) differ by \( \xi_{i,j} \) on \( U_i \cap U_j \). In the case that \( \epsilon(\sigma_i) \) as a non–zero divisor for each \( i \), it follows that \( (\epsilon(\sigma_i))_{i \in I} \) defines an effective Cartier divisor on \( X \). A different choice of \( \sigma_i \)s gives a rationally equivalent Cartier divisor. Whether \( (\epsilon(\sigma_i))_{i \in I} \) defines a Cartier divisor or not, the tuple \( (\xi_{i,j}) \) automatically satisfies the cocycle condition, and therefore it gives us gluing data for the
line bundle $\mathcal{O}_U(\sigma)$. Explicitly,

$$\mathcal{O}_U(\sigma)(V) = \left\{ (f_i)_{i \in I} \in \prod_{i \in I} \mathcal{O}_X(U_i \cap V) \mid f_i|_{U_i \cap U_j \cap V} = \xi_{i,j}|_V \cdot f_j|_{U_i \cap U_j \cap V} \text{ for all } i, j \in I \right\}.$$ 

We will abbreviate this formula in the future to the more mnemonic, if notationally abusive,

$$\mathcal{O}_U(\sigma) = \left\{ (f_i) \mid f_i = \epsilon \left( \frac{\sigma_i}{\sigma_j} \right) f_j \right\}.$$ 

With respect to this presentation of $\mathcal{O}_U(\sigma)$, the canonical section is $(\epsilon(\sigma))$. When $(\epsilon(\sigma))_{i \in I}$ represents a Cartier divisor $D$, the line bundle $\mathcal{O}_U(\sigma)$ is the same as $\mathcal{O}_U(D)$, and the canonical sections of each agree.

The isomorphism class of the pair $(\mathcal{O}_U(\sigma), (\epsilon(\sigma)))$ does not depend on the choice of lifts of $\sigma$, so defines a morphism of monoids from $\Gamma(U, M_X)$ to the group of isomorphism classes of line bundles with sections on $U$. Under sufficiently nice hypotheses, such as integrality, the specification of such a morphism for each $U$ is equivalent to the specification of a log structure [5].

The notion of associated line bundle extends to sections of $M^\text{gp}$: if $\sigma, \tau \in \Gamma(U, M_X)$, the line bundle associated to $\sigma - \tau \in \Gamma(U, M^\text{gp}_X)$ is given by

$$\mathcal{O}_U(\sigma - \tau) = \left\{ (f_i) \mid f_i = \epsilon \left( \frac{\sigma_i}{\tau_j} \cdot \frac{\tau_j}{\sigma_i} \right) f_j \right\}.$$ 

If $\sigma \in \Gamma(U, M^\text{gp}_X)$ and $\tau \in \Gamma(U, M_X)$, we also have an induced morphism $\mathcal{O}_U(\sigma) \to \mathcal{O}_U(\sigma + \tau)$, given by $(f_i) \mapsto (\epsilon(\tau_i)f_i)$. All of the associated line bundle maps are functorial in $X$.

**Example 2.3.** Suppose $X$ is an integral log scheme. If $\sigma \in \Gamma(X, M_X)$, we have a map $\mathcal{O}_X(-\sigma) \to \mathcal{O}_X$ which is injective if and only if $(\epsilon(\sigma))$ represents an effective Cartier divisor. If so, then $\mathcal{O}_X(-\sigma)$ is the ideal sheaf of this Cartier divisor.

Even if $(\epsilon(\sigma))$ is not a Cartier divisor, the image of the map $\mathcal{O}_X(-\sigma) \to \mathcal{O}_X$ is an ideal sheaf for a subscheme $\Sigma$ of $X$, and we have an exact sequence

$$\mathcal{O}_X(-\sigma) \to \mathcal{O}_X \to \mathcal{O}_\Sigma \to 0.$$ 

In this way, each section of the characteristic sheaf of $X$ determines a subscheme which behaves like an effective Cartier divisor. This point of view will be important for our work.

We now give a description of the sections of the characteristic sheaf of a logarithmic curve which slightly generalizes [2, Remark 7.3].

**Proposition 2.4.** Let $\pi : C \to S$ be a log smooth curve, and let $v$ be a geometric point of $S$. Let $\Gamma$ be the dual graph of the fiber $C_v$ of $C$ over $v$. Then there is an étale neighborhood $U$ of $v$ so that $\Gamma(\pi^{-1}(U), M_C)$ is in one-to-one correspondence with the set of piecewise linear functions on $\Gamma$.

If $\sigma \in \Gamma(\pi^{-1}(U), M_C)$, the corresponding element of $\text{PL}(\Gamma)$ is determined by taking $f_v$ to be the stalk of $\sigma$ at the generic point of $v$ for all $v \in V(\Gamma)$, and taking $n_h$ to be the slope of $\sigma$ at $h$ for all $h \in H(\Gamma)$.
Proof. For each edge $e$ of $\Gamma$ (that is, each node of $C_\sigma$), use Theorem 2.2 to choose a standard neighborhood

\[
C \xleftarrow{\text{ét}} V_e \longrightarrow \text{Spec } \mathbb{Z}[Q_e]
\]

\[
\downarrow \quad \pi \quad \downarrow
\]

\[
S \xleftarrow{\text{ét}} U_e \longrightarrow \text{Spec } \mathbb{Z}[P]
\]

of $e$. Similarly, for each half edge $h$ of $\Gamma$, use Theorem 2.2 to choose a neighborhood

\[
C \xleftarrow{\text{ét}} V_h \longrightarrow \text{Spec } \mathbb{Z}[Q_h]
\]

\[
\downarrow \quad \pi \quad \downarrow
\]

\[
S \xleftarrow{\text{ét}} U_h \longrightarrow \text{Spec } \mathbb{Z}[P]
\]

of $h$, where $Q_h = P \oplus \mathbb{N}_\gamma h$. Finally, for each smooth geometric point $p$ of $C_\sigma$, we may choose a neighborhood

\[
C \xleftarrow{\text{ét}} V_p \longrightarrow \text{Spec } \mathbb{Z}[Q_p]
\]

\[
\downarrow \quad \pi \quad \downarrow
\]

\[
S \xleftarrow{\text{ét}} U_p \longrightarrow \text{Spec } \mathbb{Z}[P]
\]

where $Q_p = P$.

By quasicompactness, we may choose a finite set $X$ of these points $p$, so that $\{V_e \to C|U\}_{e \in E(\Gamma)} \cup \{V_h \to C|U\}_{h \in H(\Gamma)} \cup \{V_p \to C|U\}_{p \in X}$ jointly cover $C_\sigma$. For brevity, let $I = E(\Gamma) \cup H(\Gamma) \cup X$.

Taking the fiber product over all the $U_i$s, then choosing a connected component, we may assume that all of the $U_i$s are equal to a single connected étale neighborhood $U$ of $\sigma$. Let $Z$ be the complement of the union of the images of the $V_i$s in $C$. Since $Z$ is closed and $\pi$ is proper, $\pi(Z)$ is closed and does not contain $\sigma$. Replacing $U$ with the complement of $(U \to S)^{-1}(\pi(Z))$ in $U$, we may assume that the $V_i$s cover $C|U$.

Now, the sections of $\Gamma(\pi^{-1}(U), \overline{M}_C)$ may be identified with tuples

\[
(\sigma_i)_{i \in I} \in \prod_{i \in I} \Gamma(V_i, M_C) = \prod_i Q_i
\]

so that whenever $\vec{t}$ is a geometric point in the intersection of the images of $V_i$ and $V_j$, the stalks of $\sigma_i$ and $\sigma_j$ agree at $\vec{t}$.

Fix such a tuple $\sigma = (\sigma_i)$, and if $\vec{t}$ is a geometric point of $C|U$, denote by $\sigma_\vec{t}$ the common value of the stalks of $\sigma_i$ at $\vec{t}$ where $\vec{t}$ factors through $V_i$. We now construct a corresponding piecewise linear function. Choose any two smooth points $\vec{t}$ and $\vec{\pi}$ of a component $v$ of $C_\sigma$. The stalks $M_{C,\vec{t}}$ and $M_{C,\vec{\pi}}$ are each naturally isomorphic to $P$, so naturally isomorphic to each other. Since $\vec{t}$ and $\vec{\pi}$ have overlapping neighborhoods, we may deduce from the various specialization maps that $\sigma_\vec{t} = \sigma_\vec{\pi}$. Therefore we can assign a well-defined value $f_v$: take it to be the common value of $\sigma_\vec{t}$ when $\vec{t}$ is a smooth geometric point of $v$. If $v, w$ are components of $C_\sigma$ joined by an edge $e$, the description of $Q_e$ ensures that $f_v - f_w$ is an integral multiple of $\delta_e$. Looking at the coefficient of $\gamma_h$ in $\sigma_h$, we get a slope $n_h$ for each half edge $h \in H(\Gamma)$. This gives the mapping from $\Gamma(\pi^{-1}(U), M_C) \to PL(\Gamma)$.

Before we embark on the converse, we note that this cover makes it easy to compute the dual graph of any other fiber of $C$ over a point in $U$. Let $\tau$ be a geometric point of $U$. 
Then for any $i \in I$, the fiber of $V_i$ over $\tau$ consists of one component unless $i$ is an edge and $(\delta_i)_{\tau} \neq 0$. It follows that contracting the edges $e$ of $\Gamma$ so that $(\delta_e)_{\tau} = 0$ gives the dual graph $\Gamma(C_\tau)$.

Conversely, given a piecewise linear function $((f_v), (n_h))$ we can choose corresponding sections $\sigma_i \in Q_i$ in an obvious way: for each $e \in E(\Gamma)$, set $\sigma_e = (f_v, f_w)$ where $v, w$ are the vertices incident to $e$; for each $h \in H(\Gamma)$, set $\sigma_h = f_v + n_h \gamma_h$ where $v$ is the vertex incident to $h$; and finally, for each $p \in X$, set $\sigma_p = f_v$, where $v$ is the component to which $p$ belongs. We need to check that these sections agree on all stalks at geometric points. Let $\ell$ be any geometric point of $C|_U$. If $\ell$ is a node or a marked point, then $\ell$ factors through a unique $V_i$ so $\sigma_\tau$ is well-defined. If $\ell$ is a smooth point, let $V_i$ and $V_j$ be two neighborhoods through which $\ell$ factors. Then $(\sigma_i)_{\tau}$ is the stalk $(f_v)_{\pi(\ell)}$ in $P_{\pi(\ell)}$ for some component $v$ of $C_\tau$ meeting $V_i$. Similarly, $(\sigma_j)_{\tau}$ is the stalk $(f_w)_{\pi(\ell)}$ for some component $w$. Since $V_i$ and $V_j$ both intersect the component of the fiber of $C$ containing $\ell$, it follows that there is a path from $v$ to $w$ in $\Gamma$ so that all the smoothing parameters along the path become 0 at $\pi(\ell)$. Since the difference $f_v - f_w$ is an integral linear combination of these smoothing parameters, $(\sigma_i)_{\tau} = (\sigma_j)_{\tau}$, as required. □

Finally, we recall a description of associated line bundles for log curves over geometric points.

**Proposition 2.5.** (\cite[Proposition 2.4.1]{E}) Let $\pi : C \to S$ be a log curve over $S$, where the underlying scheme of $S$ is the spectrum of an algebraically closed field. Let $\sigma$ be a global section of $\overline{M}_C$ with corresponding piecewise linear function $((f_v), (n_h))$. Let $v$ be a component of $C$. Then

$$O_C(\sigma)|_v = O_v \left( \sum_p \mu_p P \right) \otimes \pi^* O_S(f_v)$$

where the sum is over the edges and half-edges $p$ incident to $v$, and $\mu_p$ is the “outgoing slope” of the piecewise linear function at the point $p$: when $p$ is a half-edge, this is the integer $n_p$; when $p$ is a node joining $v$ to $w$, this is $(f_w - f_v)/\delta_p$.

3. **Mesa curves**

Let $C$ be a log smooth curve over the spectrum of an algebraically closed field. By a subcurve of $C$, we mean a union of some of its irreducible components. A path $P$ in the dual graph of $C$ is a sequence $v_0 e_1 v_1 e_2 \cdots e_k v_k$ of vertices and edges in $\Gamma(C)$ so that the vertices $v_i$ are distinct and $v_{i-1}$ and $v_i$ are the ends of the edge $e_i$ for all $i$. The length of a path $P$ is the sum $\lambda(P) = \sum_{i=1}^k \delta_{e_i}$ of the smoothing parameters along the path.

Let $E$ be any connected subcurve of $C$, and let $F$ be any connected subcurve of $E$ with the same arithmetic genus as $E$. Let $v$ be any component of $E$. Note that the graph $\Gamma(E)/\Gamma(F)$, obtained by contracting the subgraph $\Gamma(F)$ of $\Gamma(E)$ to a point, is a tree. It follows that there is a unique path $P_v = v_1 e_1 v_2 e_2 \cdots v_i e_i v$ in the dual graph of $E$ where $v_1 \in F$, and $v_i \notin F$ for $i > 1$. We may use this to define a piecewise linear distance function $\lambda_E$ on the dual graph of $E$ by taking the value of $\lambda_E$ on a vertex $v$ to be $\lambda_E(v) = \lambda(P_v)$, and taking the slope of $\lambda_E$ to be 1 on all marked points. If

(i) $E$ has no marked points,

(ii) any path from $F$ to a component of $C$ adjacent to $E$ has the same length, $\rho_E$,

(iii) and every component of $E - F$ lies on a path from $F$ to a component outside of $E$, then...
we say the pair \((E, F)\) is a \emph{mesa} and define \(\overline{\lambda}_E\) to be the section of \(\overline{M}_C\) with piecewise linear function \(\rho_E - \lambda_E\) on \(E\) and 0 elsewhere. We call \(\rho_E\) the \emph{radius} of \((E, F)\). Note that \(E\) is the support of the subscheme of \(C\) with sheaf of ideals the image of \(\mathcal{O}_C(-\overline{\lambda}) \to \mathcal{O}_C\), and \(F\) is the closure of the locus where \(\overline{\lambda} = \rho_E\), so a mesa is determined by its \(\overline{\lambda}\) function. Motivated by the shape of the graph of \(\overline{\lambda}\), we tend to say \(E\) is a mesa and \(F\) is its top. If, in addition, \(H^1(E, \mathcal{O}_E(-\overline{\lambda})) = 0\), we say that \(E\) is \(\overline{\lambda}\)-acyclic.

When \(E\) has positive genus, there is a minimal connected subcurve with the same arithmetic genus as \(E\), which we will denote by \(\text{core}(E)\). A mesa \((E, F)\) in which \(F = \text{core}(E)\) will be called \emph{small}.

A mesa curve extends these notions to families.

**Definition 3.1.** A \emph{mesa curve} over a scheme \(S\) consists of

(i) a proper log curve \(\pi : C \to S\),
(ii) a section \(\overline{\lambda} \in \Gamma(C, \overline{M}_C)\),

such that, for each geometric point \(x\) of \(S\), the pullback of \(\overline{\lambda}\) to \(C \times_S x\) is of the form \(\sum_{i=1}^k \overline{\lambda}_E_i\), for some \(k \geq 0\), where the \(E_i\) are disjoint \(\overline{\lambda}\)-acyclic mesas of \(C \times_S x\).

Given a mesa curve, \((\pi : C \to S, \overline{\lambda})\), we may define a global version of \(E\) as the subscheme of \(C\) with sheaf of ideals given by the image of \(\mathcal{O}_C(-\overline{\lambda}) \to \mathcal{O}_C\). If the pullback of \(E\) to each geometric fiber of \(C\) has at most one connected component, we say that \(C\) is a \emph{simple} mesa curve.

**Example 3.2.**

![Diagram](image)

A typical dual graph of a genus 3 simple mesa curve, \(C \to \text{Spec } k\). We have labeled the vertices with the degree of the restriction of \(\mathcal{O}_C(-\overline{\lambda})\) to the corresponding components, using Proposition 2.5. Filled dots are rational components. Empty dots are genus 1 components. \(E\) is displayed in red and \(F\) is the core of \(E\). There are two paths from the core to the vertex with degree \(-2\): since \(E\) is a mesa, they have the same length.

Note that the underlying nodal curve of \(C\) is not stable: the rational component on the right is missing a special point.

Mesa curves are a generalization of the centrally aligned curves of [8]. For comparison, we recall the definition of centrally aligned curve. Let \(\pi : C \to S\) be a log curve of genus one. Give \(\overline{M}_S^\beta\) the ordering induced by taking \(\overline{M}_S\) as the positive cone. Note that there is a global section \(\lambda \in \Gamma(C, \overline{M}_C)\) that measures distance from the core of \(C\) in each fiber. Then a centrally aligned curve consists of (c.f. [8, Definition 4.6.2.1])

(i) a proper genus 1 log curve \(\pi : C \to S\) and
(ii) a section \( \rho \in \Gamma(S, \overline{M}_S) \)

such that for each geometric point \( \bar{s} \) of \( S \),

(i) \( \rho_{\bar{s}} \) is comparable to \( \lambda(v) \) for each vertex \( v \in \Gamma(C_{\bar{s}}) \),

(ii) the subcurve \( E_{\bar{s}} \) of \( C_{\bar{s}} \) where \( \lambda < \rho \) is stable, and

(iii) the lengths \( \lambda(v) \) for \( v \in \Gamma(E_{\bar{s}}) \) are totally ordered.

We may interpret a centrally aligned curve \((C, \rho)\) as a simple mesa curve by taking \( \lambda = \max\{\rho - \lambda, 0\} \). The resulting mesa is necessarily small, and we will see in Lemma 3.6 that \( \lambda \)-acyclicity holds.

Justifying our casual use of \( E \) rather than \((E, F)\) to describe a mesa, the choice of \( F \) is uniquely determined when \( E \) is \( \lambda \)-acyclic.

**Lemma 3.3.** Suppose \( E \) has positive genus. Then there is at most one subcurve \( F \) of \( E \) so that \((E, F)\) is a \( \lambda \)-acyclic mesa.

**Proof.** Suppose that \((E, F)\) is a \( \lambda \)-acyclic mesa.

In order that \((E, F)\) be \( \lambda \) acyclic, \( \mathcal{O}_E(-\lambda) \) must have positive total degree on the core of \( E \). This implies that \( \text{core}(E) \) must be incident to an edge of \( \Gamma(C) \) not in \( F \). Find a path \( P \) from \( \text{core}(E) \) to a component adjacent to \( E \) starting with this edge. Then \( \lambda(P) = \rho_E \). That is, the radius of \((E, F)\) is solely determined by \( E \). Then, walking inwards from the boundary of \( E \), there is at most one subcurve \( F \) with this distance from every component adjacent to \( E \).

When \( C \) is a simple mesa curve, there is only one radius \( \rho \) to worry about in each fiber. In fact, these \( \rho \)s glue together to a global section of \( \overline{M}_S \).

**Lemma 3.4.** If \( \pi : C \to S \) is a simple mesa curve, there is a global section \( \rho \in \overline{M}_S(S) \) restricting to the radii \( \rho_{E_{\bar{s}}} \) for each geometric point \( \bar{s} \) of \( S \).

**Proof.** For each geometric point \( \bar{s} \), construct an étale neighborhood \( U_{\bar{s}} \) of \( \bar{s} \) and a cover \( \{V_{\bar{s}} \to C|_{U_{\bar{s}}}\}_{i \in I} \) as in Proposition 2.4. Then following the isomorphisms \( \overline{M}_{S,\bar{s}} \cong P \cong \Gamma(U_{\bar{s}}, \overline{M}_S) \) gives a unique extension of \( \rho_{E_{\bar{s}}} \in \overline{M}_{S,\bar{s}} \) to a section \( \rho^\bar{s} \in \Gamma(U_{\bar{s}}, \overline{M}_S) \). For any geometric point \( \bar{t} \) in \( U_{\bar{s}} \), the stalk of \( \rho^\bar{s} \) at \( \bar{t} \) must agree with \( \rho_{E_{\bar{t}}} \). Then the sections \( \rho^\bar{s} \) glue to the required global section.

If \( \pi : C \to S \) is a simple mesa curve and \( \bar{s} \) is a geometric point of \( S \), we note that \( \rho_{\bar{s}} = 0 \) if and only if the fiber of \( E \) over \( \bar{s} \) is empty. Note that then \( \rho : \mathcal{O}_S(-\rho) \to \mathcal{O}_{\bar{s}} \) is an isomorphism if and only if \( \rho_{\bar{s}} = 0 \), and \( \mathcal{O}_S(-\rho) \to \mathcal{O}_{\bar{s}} \) is the zero map otherwise. In other words, the image of \( \mathcal{O}_S(-\rho) \) in \( \mathcal{O}_S \) is an ideal sheaf for the locus in \( S \) over which \( E \) is non-empty.

We recall the definition of genus of a curve singularity as stated in [10, Definition 1.7].

**Definition 3.5.** Let \( p \in C \) be a closed point on a curve and let \( \nu : \tilde{C} \to C \) denote the normalization of \( C \) at \( p \). The \( \delta \)-invariant \( \delta(p) \) and the number of branches \( m(p) \) are defined by the formulas

\[
\delta(p) = \dim_v(\nu_* \mathcal{O}_{\tilde{C}}/\mathcal{O}_C),
\]

\[
m(p) = |\nu^{-1}(p)|,
\]

and we define the **genus** \( g(p) \) by

\[
g(p) = \delta(p) - m(p) + 1.
\]
Also of interest to us are elliptic Gorenstein singularities. We will use the following characterization of Smyth.

**Lemma 3.6.** [M. Lemma 2.2] Let \( \pi : C \to \text{Spec } k \) be a reduced curve and let \( \nu : \tilde{C} \to C \) be the normalization. Let \( p \in C \) be a point with pre-images \( p_1, \ldots, p_m \) in \( \tilde{C} \). Then \( p \) is an elliptic Gorenstein singularity if and only if \( \nu^* : \mathcal{O}_{\tilde{C}, p} \to \mathcal{O}_{\tilde{C}, \nu^{-1}(p)} \) satisfies

(i) \( \nu^*(m_p/m_p^2) \subseteq \bigoplus_{i=1}^{m} m_{p_i}/m_{p_i}^2 \) is a codimension-one subspace.

(ii) \( \nu^*(m_p/m_p^2) \nsubseteq m_{p_i}/m_{p_i}^2 \) for any \( i = 1, \ldots, m \).

(iii) \( \nu^*(m_p) \supseteq \bigoplus_{i=1}^{m} m_{p_i} \).

4. **Construction of the contraction map: intuition and strategy**

Start by assuming that \( C \) is simple. A tempting way forward is to construct \( \tilde{C} \) as a topological space by collapsing \( E \) to a point fiberwise, then to take \( \mathcal{O}_{\tilde{C}} = \tau_* \mathcal{O}_C \). In general, this fails to produce a curve with singularities of the correct genus.

**Example 4.1.** Consider a stable curve \( C \to \text{Spec } k \) consisting of a smooth elliptic component, \( E \), with two attached rational components. Give the base the log structure associated to the chart \( \mathbb{N} \delta \to k \) sending \( \delta \to 0 \), and give \( C \) the log structure with both smoothing parameters equal to \( \delta \). With this log structure, \( (E, E) \) is a mesa. If we let \( \tau : C \to \tilde{C} \) be the quotient map of topological spaces collapsing \( E \) to a point, then \( (\tilde{C}, \tau_* \mathcal{O}_C) \) is a genus 0 curve with two components meeting at an ordinary double point.

**Example 4.2.** Let \( S = \text{Spec } k[t]/(t^2) \) and give \( S \) the log structure associated to the chart \( \mathbb{N} \delta \to k[t]/(t^2) \) sending \( \delta \to t \). Let \( C \) be a mesa curve over \( S \) restricting to the curve of the previous example, in which the nodes are beginning to smooth out. Let \( \mathcal{O}_{C_0} = \mathcal{O}_C \otimes k \).

Consider the sequence

\[
0 \to \mathcal{O}_{C_0} \xrightarrow{\ell} \mathcal{O}_C \to \mathcal{O}_{C_0} \to 0,
\]

and its pushforward,

\[
0 \to \tau_* \mathcal{O}_{C_0} \to \tau_* \mathcal{O}_C \to \tau_* \mathcal{O}_{C_0}.
\]

We would like to compute \( (\tau_* \mathcal{O}_C) \otimes k \). This is the same as the image of \( \tau_* \mathcal{O}_C \) in \( \tau_* \mathcal{O}_{C_0} \). Let \( E \) be the elliptic component and let \( Z_1, Z_2 \) be the rational components. Let \( p_1, p_2 \) be the respective points where \( E \) meets \( Z_1 \) and \( Z_2 \). Let \( U = C - (Z_1 \cup Z_2) \). Since \( \tau \) is an isomorphism in the complement of \( E \), we may restrict our attention to the geometric stalk of the sequence \( 0 \to \tau_* \mathcal{O}_{C_0} \to \tau_* \mathcal{O}_C \to \tau_* \mathcal{O}_{C_0} \) at \( \tau(E) \). Let \( f \in (\tau_* \mathcal{O}_{C_0})_{\tau(E)} \). Since \( U \) is affine, finding lifts of \( f \) to \( \tau_* \mathcal{O}_C \) is equivalent to finding compatible lifts of the restriction \( f|_U \) to \( \mathcal{O}_C(U) \) and of the stalks \( f_{\mathfrak{m}} \) to \( \mathcal{O}_{C, \mathfrak{m}} \).

There are units \( \alpha_i \) so that the étale stalk \( (\mathcal{O}_C)_{\mathfrak{m}} \simeq (\mathcal{O}_S[x_i, y_i]/(x_i y_i - \alpha_i t))_{(x_i, y_i)} \), where \( x_i \) vanishes on \( Z_i \) and \( y_i \) vanishes on \( E \). A lift of \( f_{\mathfrak{m}} \) is of the form

\[
f_{\mathfrak{m}}(x_i, y_i) + t g_{\mathfrak{m}}(x_i, y_i).
\]

On \( U \), \( \mathcal{O}_C(U) = \mathcal{O}_{C_0}(U)[t]/(t^2) \), so a lift of \( f|_U \) is of the form

\[
f|_U + t g|_U.
\]

Write \( f_{\mathfrak{m}} = a^i_{0,0} + a^i_{1,0} x_i + a^i_{2,0} x_i^2 + \cdots + a^i_{0,1} y_i + a^i_{0,2} y_i^2 + \cdots \). Consider the restriction of \( f_{\mathfrak{m}}(x_i, y_i) + t g_{\mathfrak{m}}(x_i, y_i) \) to the complement of \( Z_i \). Then \( x_i \) is invertible in the complement of \( Z_i \), so we have

\[
y_i = \alpha_i t/x_i
\]
on this locus. The restriction of \( f_{\mathfrak{m}}(x_i, y_i) + tg_{\mathfrak{m}}(x_i, y_i) \) to the complement of \( Z_i \) may then be written

\[
a_{i,0}^0 + a_{1,0}^1 x_i + \ldots + t(\alpha_i a_{i,1}^0 x_i^{-1} + g_{\mathfrak{m}}(x,0))
\]

Therefore the problem of finding compatible lifts of the \( f_i \) is the same as that of finding a rational function \( g \) on \( E \otimes k \) with pole \( \alpha_1 a_{1,0}^1 \) at \( p_1 \) and pole \( \alpha_2 a_{1,0}^2 \) at \( p_2 \). This is the Mittag-Leffler problem; it is classical that a solution exists if and only if \( c_1 \alpha_1 a_{1,0}^1 + c_2 \alpha_2 a_{1,0}^2 = 0 \), for some nonzero constants \( c_1, c_2 \). That is, \((\overline{C}, (\tau_*, \mathcal{O}_C) \otimes k)\) has a tacnode at \( \tau(E) \). This is a genus 1 singularity, as desired.

These examples suggest that the issue is the failure of \( \tau_* \) to commute with base change in \( S \) when applied to \( \mathcal{O}_C \), so our strategy will be to find constituent parts of the desired structure sheaf whose formation commutes with base change in \( S \). Let \( \pi : C \to S = \text{Spec} A \) be a simple mesa curve, let \( E \) be the subscheme of \( C \) defined by the sheaf of ideals \( \text{im}\{\mathcal{O}_C(-\overline{\lambda}) \to \mathcal{O}_C\} \), and let \( \rho \) be the global section of \( \overline{M}_S \) in Lemma 3.4.

When \( \mathcal{O}_C(-\overline{\lambda}) \to \mathcal{O}_C \) is injective, \( \mathcal{O}_C(-\overline{\lambda}) \) is the ideal sheaf of \( E \) in \( C \). It is reasonable to expect that \( \tau_* \mathcal{O}_C(-\overline{\lambda}) \) will be the ideal sheaf of the point \( \tau(E) \) in \( \overline{C} \). If \( U \) is a neighborhood of \( E \) small enough that \( \tau(U) \) is an affine neighborhood of \( \tau(E) \), this ideal sheaf will have values \( \Gamma(U, \mathcal{O}_C(-\overline{\lambda})) \). We will be able to conclude from \( \overline{\lambda} \)-acylicity that the formation of this module commutes with base change in \( S \). This will be the first constituent of \( \mathcal{O}_C \).

Since \( \Gamma(U, \mathcal{O}_C(-\overline{\lambda})) \) is supposed to be the ideal sheaf of a point, the rest of \( \mathcal{O}_C \) should be generated by the constant functions, i.e. \( \psi^{-1} \mathcal{O}_S \). On a sufficiently small neighborhood of \( \tau(E) \), the sections of this sheaf should be identifiable with \( \Gamma(S, \mathcal{O}_S) \). This will be the second constituent of \( \mathcal{O}_C \).

At \( \tau(E) \), we expect the constant functions and the ideal sheaf of \( \tau(E) \) to intersect in the constant functions that are zero on \( E \), i.e., \( \Gamma(S, \mathcal{O}_S(-\rho)) \).

We now turn this intuition into the construction of a suitable ring. Let \( U \) be any open subset of \( C \) containing \( E \). (We will be more particular about \( U \)'s identity later, but we leave it variable here so that we will have the language to compare different choices of \( U \).)

There is a non-unital \( \mathcal{O}_C \)-bilinear multiplication on \( \mathcal{O}_C(-\overline{\lambda}) \) induced by the log structure on \( C \):

\[
\mathcal{O}_C(-\overline{\lambda}) \otimes_{\mathcal{O}_C} \mathcal{O}_C(-\overline{\lambda}) \cong \mathcal{O}_C(-2\overline{\lambda}) \xrightarrow{\overline{\lambda}} \mathcal{O}_C(-\overline{\lambda}).
\]

It may be helpful to be more explicit: recall that the sections of \( \mathcal{O}_C(-\overline{\lambda}) \) on \( U \) can be presented as

\[
\Gamma(U, \mathcal{O}_C(-\overline{\lambda})) = \left\{ (f_i) \mid f_i = \epsilon \left( \frac{\overline{\lambda}_i}{\lambda_i} \right) j_i \right\}.
\]

In these coordinates, the multiplication is \( (f_i) \cdot (g_i) = (\epsilon(\overline{\lambda}_i)f_i g_i) \). Given sections \( f, g \in \mathcal{O}_C(-\overline{\lambda}) \), we will denote this product by \( \overline{\lambda}(fg) \). Note that \( \overline{\lambda}(fg) = \overline{\lambda}(f)g = f\overline{\lambda}(g) \), rather than \( \overline{\lambda}(f)\overline{\lambda}(g) \).

Let \( B(U) = \Gamma(U, \mathcal{O}_C(-\overline{\lambda})) \oplus \Gamma(S, \mathcal{O}_S) \). Since \( \mathcal{O}_C(-\overline{\lambda}) \) is a \( \tau^{-1}\mathcal{O}_S \) module, we can give \( B(U) \) a ring structure by the rule

\[
(f, c) \cdot (g, d) = (\overline{\lambda}(fg) + df + cg, cd).
\]
Now give $\Gamma(S, O_S(-\rho))$ a $B(U)$-module structure by $(f, c) \cdot g = cg$. The map
\[
\Gamma(S, O_S(-\rho)) \to B(U)
g \mapsto ((\rho - \bar{\lambda})(g), -\rho(g))
\]
is a $B(U)$-module homomorphism:
\[
(f, c) \cdot ((\rho - \bar{\lambda})(g), -\rho(g)) = (\bar{\lambda}(f(\rho - \bar{\lambda})(g)) - f\rho(g) + c(\rho - \bar{\lambda})(g), -c\rho(g)) = (f(\bar{\lambda} + \rho - \bar{\lambda})(g) - f\rho(g) + (\rho - \bar{\lambda})(cg), -\rho(cg)) = ((\rho - \bar{\lambda})(cg), -\rho(cg)).
\]
In particular, its image is an ideal of $B(U)$. Let $\overline{B}(U)$ be the quotient ring.

We now have an exact sequence of $A$-modules
\[
\Gamma(S, O_S(-\rho)) \to B(U) \to \overline{B}(U) \to 0
\]
in which the last two modules are $A$-algebras. We will call this the $\overline{B}$-sequence. (In Proposition 6.5 we will see that this sequence is short exact for a good choice of $U$.)

Let $U = \text{Spec } \overline{B}(U)$. There is a map of $A$-algebras $B(U) \to \Gamma(U, O_C)$ induced by $\bar{\lambda}$ on the first summand and $\pi^*$ on the second. The composite
\[
\Gamma(S, O_S(-\rho)) \to B(U) \to \Gamma(U, O_C)
\]
is zero, so the morphism $B(U) \to \Gamma(U, O_C)$ descends to an $A$-algebra morphism $\overline{B}(U) \to \Gamma(U, O_C)$. Taking Spec, we obtain a commutative triangle
\[
\begin{array}{ccc}
U & \xrightarrow{\pi} & \overline{U} \\
\downarrow \pi & & \downarrow \psi \\
S & \xrightarrow{\psi} & \overline{S}.
\end{array}
\]

We now state the construction of $\overline{C}$ in what we will call the “standard situation.” We will be able to show by the end of the section that this determines $\overline{C}$ in general.

**Standard Situation:**

(i) $S = \text{Spec } A$ is a Noetherian, finite dimensional, affine scheme.
(ii) $(\pi : C \to S, \bar{\lambda})$ is a simple mesa curve so that $C$ is a projective scheme.
(iii) $E$ is the subscheme of $C$ defined by the sheaf of ideals $\text{im}\{O_C(-\bar{\lambda}) \to O_C\}$.
(iv) $\rho$ is the global section of $\overline{M}_S$ from Lemma 3.4.
(v) $\sigma_1, \ldots, \sigma_h$ are sections of $\pi$ so that each component of each geometric fiber of $C - E \to S$ meets at least one $\sigma_i$.
(vi) $U$ is the complement of the $\sigma_i$s in $C$.

**Definition 4.3.** Assume that we are in the standard situation. Let $U = \text{Spec } \overline{B}(U)$, as above. The restriction of $\tau$ to $U - E$ will turn out to be an open immersion (Proposition 7.2). We define the contracted curve $\overline{C}$ as the pushout
\[
\begin{array}{ccc}
U - E & \xrightarrow{\tau} & C - E \\
\downarrow & & \downarrow \\
U & \xrightarrow{\pi} & \overline{C}.
\end{array}
\]
When we wish to emphasize the sections $\sigma_i$ in use, we will write $(C, \sigma_i)$ instead of $C$.

Note that the commutative triangle (1) induces a commutative triangle

$$
\begin{array}{ccc}
C & \xrightarrow{\tau} & \overline{C} \\
\pi \downarrow & & \downarrow \psi \\
S & &
\end{array}
$$

4.1. **Reduction to the standard situation.** We first reduce to the case that $C \to S$ is simple. Suppose that Theorem [1.1] holds for simple mesa curves, and let $\pi : C \to S$ be an arbitrary mesa curve. We begin by constructing $C \to \overline{C}$ in an étale neighborhood of each geometric point $\overline{s}$ of $S$. Suppose that $S$ is an étale neighborhood of $\overline{s}$ as in Proposition 2.4. Then the decomposition $\overline{\lambda} = \sum_{i=1}^k \overline{E_i}$ holds globally. Construct contractions $\tau_i : C \to \overline{C}_i$ of the simple mesa curves $(C, \overline{E}_i)$. Note that $C - E_i \to \overline{C}_i$ is an open immersion, so we can identify $E_j$ with its image in $\overline{C}_i$, $j \neq i$. Then we can define

$$
\overline{C} = \colim \left\{ \begin{array}{ccc}
C - \bigcup_i E_i \\
\overline{C}_1 - \bigcup_{i \neq 1} E_i & \overline{C}_2 - \bigcup_{i \neq 2} E_i & \ldots & \overline{C}_k - \bigcup_{i \neq k} E_i \\
\end{array} \right. 
$$

where each arrow is the open immersion induced by $\tau_i$. This satisfies the conclusions of Theorem [1.1]. Since the contraction construction commutes with base change for simple mesa curves, étale descent gives us the contraction for general mesa curves.

We next claim that by working locally enough in $S$, we may work in the standard situation.

**Lemma 4.4.** Let $\pi : C \to S$ be a simple mesa curve, and let $\overline{s}$ be a geometric point of $S$.

Then there is an étale neighborhood $S'$ of $\overline{s}$ so that

(i) $S' = \Spec A$ is affine;
(ii) $C' = C \times_S S'$ is a projective scheme;
(iii) There are sections $\sigma_1, \ldots, \sigma_h : S' \to C'$ of $\pi|_{S'}$ meeting each component of $(C - E) \cap \pi^{-1}(\overline{s})$;
(iv) For all geometric points $\overline{t} \in S'$ and components $Z$ of $(C - E) \cap \pi|_{S'}^{-1}(\overline{t})$, one of the sections $\sigma_i$ meets $Z$.

**Proof.** Note that properties (ii), (iii), (iv), are stable under étale localization of $S$. It is well-known that there is an étale neighborhood $U_1$ of $\overline{s}$ so that (ii), and (iii) hold.

Construct an étale neighborhood $U$ of $\overline{s}$ and a cover $\{V_i \to C|_{U}\}_{i \in I}$ as in Proposition 2.4. Suppose $Z$ is a component of $(C - E) \cap \pi^{-1}(\overline{t})$, where $\overline{t}$ is a geometric point of $U$. Let $\overline{u}$ be any smooth geometric point of $Z$. The description of $\mathcal{M}_C$ in Proposition 2.4 implies that there is a component $v$ of $C_{\overline{u}}$ so that $(\overline{\lambda}_v)_{\overline{t}} = \overline{\lambda}_{\overline{s}}$. On the other hand, since $Z$ is not a component of $E \cap \pi^{-1}(\overline{t})$, the stalk of $\overline{\lambda}$ at $\overline{u}$ is zero. By definition of $\overline{\lambda}$, $\overline{\lambda}_v$ is a sum of smoothing parameters on a path from $v$ to a component $w$ of $C_{\overline{s}}$ outside of $E$. Since $(\overline{\lambda}_v)_{\overline{t}} = 0$, all of the nodes on this path smooth out over $\overline{t}$. This implies that $Z$ meets the section $\sigma_j$ going through $w$, so we have (iv).

Finally, take an affine neighborhood $U_2$ of $\overline{s}$ inside $U \times_S U_1$. This gives us the desired étale neighborhood. \qed
Lemma 4.5. It suffices to construct $C \to \overline{C} \to S$ when $S$ is Noetherian and finite dimensional.

Proof. Every affine $S$ is a projective limit of Noetherian and finite dimensional rings. All of the properties we ask of $C \to \overline{C} \to S$ are preserved under projective limits. \qed

Next, we need to be able to glue our local contracted curves together. We state a couple of results early. Proposition 4.6 will follow immediately from Corollary 6.7 and Proposition 7.2. Proposition 4.7 is Corollary 7.5.

Proposition 4.6. Assume we are in the standard situation and let $f : T \to S$ be a morphism of affine, Noetherian, finite-dimensional schemes. Pull back the data of the standard situation along $f$ to obtain $\pi_T : C_T \to T$, $E_T$, $\rho_T$, $\sigma_{T,1}, \ldots, \sigma_{T,h}$, and $U_T$ over $T$. These data are again in the standard situation, so we may apply Definition 4.3 to obtain a triangle

$$
\begin{array}{ccc}
C_T & \longrightarrow & \overline{C}_T \\
\downarrow & & \downarrow \\
T & & \\
\end{array}
$$

Then there is a uniquely specified isomorphism $\phi_{S,T} : \overline{C}_T \to \overline{C} \times_S T$ so that

$$
\begin{array}{ccc}
C_T & \longrightarrow & \overline{C} \times_S T \\
\downarrow & \phi_{S,T} & \downarrow \\
\overline{C}_T & \longrightarrow & \overline{C} \times_S T \\
\downarrow & & \downarrow \\
C_T & \longrightarrow & \overline{C}_T \\
\downarrow & & \downarrow \\
T & & T \\
\end{array}
$$

commutes. Moreover, these isomorphisms satisfy the cocycle condition in the sense that if $g : W \to T$ is a second morphism of affine, Noetherian, finite-dimensional schemes,

$$
\phi_{S,W} = \phi_{T,W} \circ \phi_{S,T}|_W.
$$

Proposition 4.7. Suppose we are in the standard situation and $\sigma'_1, \ldots, \sigma'_k$ are a second choice of sections. There is a uniquely specified isomorphism $\phi_{\sigma_\bullet, \sigma'_\bullet} : (\overline{C}, \sigma_\bullet) \to (\overline{C}, \sigma'_\bullet)$ so that

$$
\begin{array}{ccc}
C_T & \longrightarrow & (\overline{C}, \sigma'_\bullet) \\
\downarrow & \phi_{\sigma_\bullet, \sigma'_\bullet} & \downarrow \\
(\overline{C}, \sigma_\bullet) & \longrightarrow & (\overline{C}, \sigma'_\bullet) \\
\downarrow & & \downarrow \\
T & \longrightarrow & T \\
\end{array}
$$

commutes. Moreover, if $\sigma''_1, \ldots, \sigma''_l$ is a third choice of sections, the cocycle condition

$$
\phi_{\sigma_\bullet, \sigma''_\bullet} = \phi_{\sigma'_\bullet, \sigma''_\bullet} \circ \phi_{\sigma_\bullet, \sigma'_\bullet}
$$

holds.
Now, suppose that \( \pi : C \to S \) is a simple mesa curve where \( S = \text{Spec } A \) is Noetherian and finite-dimensional. Then there is an étale cover \( S' \to S \) and sections of \( \pi|_{S'} \) as in Lemma 4.4. The two preceding results imply that the map \( C' \to \overline{C'} \) over \( S' \) descends to a map over \( S \). Taking projective limits, we have contractions of mesa curves over all affine schemes in a manner that commutes with base change, so étale descent gives contractions of mesa curves over a general base.

This leaves a great deal to be verified. We must check that in the standard situation,

(i) The construction commutes with pullback in \( S \).
(ii) \( \overline{C} \) is flat over \( S \).
(iii) \( \tau \) is surjective.
(iv) \( \tau : U - E \to \overline{U} - \tau(E) \) is an isomorphism.
(v) \( \overline{C} \) does not depend on the choice of sections \( \sigma_i : S \to C \).
(vi) If \( S \) is a geometric point, \( C \) has a genus \( g \) singularity at \( \tau(E) \), which is elliptic Gorenstein if \( E \) is a small mesa of genus one.
(vii) \( C \) has reduced geometric fibers.
(viii) \( \psi : C \to S \) is proper.

5. Values of \( \mathcal{O}_E(-\lambda) \)

In this section, we examine the values that sections of \( \mathcal{O}_E(-\lambda) \) on \( E \) can take on the attachment points of \( E \) to the rest of \( C \). These turn out to be constrained by a Mittag-Leffler problem in a way that will later imply that \( \tau(E) \) is a genus \( g \) singularity.

For each curve \( D \) over an algebraically closed field \( k \), sheaf \( \mathcal{L} \in \text{Pic}(D) \), and point \( p \in D \), fix an isomorphism \( \text{ev}_p : \mathcal{L} \otimes k(p) \to k \). Given a section \( \sigma \in \mathcal{L}(V) \) where \( p \in V \), denote by \( \sigma(p) \) the image of \( \sigma \) under the composition \( \mathcal{L}(V) \to \mathcal{L} \otimes k(p) \xrightarrow{\text{ev}_p} k \).

The following lemmas are well-known. The statements about nonzero coefficients may seem out of place; they become useful when verifying that \( C \) has Gorenstein singularities when \( E \) is a small mesa of genus one.

**Lemma 5.1.** Let \( p_0, \ldots, p_n, q \) be \( n + 2 \) distinct closed points of \( \mathbb{P}^1_k \). Then

(i) For any \( a_0, \ldots, a_n \in k \), there is a unique global section \( \sigma \) of \( \mathcal{O}_{\mathbb{P}^1_k}(p_0 + \cdots + p_n - q) \) satisfying the equations

\[ \sigma(p_i) = a_i, \quad 0 \leq i \leq n. \]

(ii) There are nonzero constants \( c_0, \ldots, c_n \), so that for all global sections \( \sigma \) of \( \mathcal{O}_{\mathbb{P}^1_k}(p_0 + \cdots + p_n - q) \)

\[ \sigma(q) = c_0\sigma(p_0) + \cdots + c_n\sigma(p_n) \]

**Proof.** Omitted. \( \square \)

**Lemma 5.2.** Suppose \( D \) is either a smooth elliptic curve or a ring of rational curves. Let \( Z \) be an effective divisor of positive degree on \( D \). Then \( H^1(D, \mathcal{O}_D(Z)) = 0 \).

**Proof.** In either case, \( D \) is Gorenstein with dualizing sheaf \( \omega_D \cong \mathcal{O}_D \). By Serre duality,

\[ H^1(D, \mathcal{O}_D(Z)) \cong H^0(D, \mathcal{O}_D(-Z))^\vee. \]

If \( D \) is smooth elliptic, the latter group is zero since \( \mathcal{O}_D(-Z) \) has negative degree.
If $D$ is a ring of rational curves, let $D_1, \ldots, D_r$ be its components, and let $p_1, \ldots, p_r$ be its nodes, so that $p_i$ joins $D_i$ to $D_{i+1}$ for $i < r$ and $p_r$ joins $D_r$ to $D_1$. Then we may identify $H^0(D, \mathcal{O}_D(Z))$ with the set of tuples of sections $(\sigma_i) \in \prod_i \Gamma(D_i, \mathcal{O}_D(-Z|_{D_i}))$ satisfying $\sigma_i(p_i) = \sigma_{i+1}(p_i)$ for all $1 \leq i < r$ and $\sigma_r(p_r) = \sigma_1(p_r)$.

Now, each $\mathcal{O}_{D_i}(-Z|_{D_i})$ has non-positive degree. For those $i$ where $-Z|_{D_i} = 0$, if for either point $p_j \in D_i$ we have $\sigma_i(p_j) = 0$, then $\sigma_i$ is zero. For those $i$ where $\mathcal{O}_{D_i}(-Z|_{D_i})$ has negative degree, $\sigma_i = 0$. Since there is at least one $i$ with $\deg(-Z|_{D_i}) < 0$, it follows by working our way around the ring that the only global section of $\mathcal{O}_D(-Z)$ is zero. \hfill $\square$

**Lemma 5.3.** Suppose $D$ is a nodal curve. Let $p_1, \ldots, p_n$ be distinct smooth closed points of $D$, $n \geq 1$. For each $i$, $1 \leq i \leq n$, let

$$\delta_i : k(p_i) \to H^1(D, \mathcal{O}_D)$$

be the connecting homomorphism associated to the short exact sequence

$$0 \to \mathcal{O}_D \to \mathcal{O}_D(p_i) \to k(p_i) \to 0.$$ 

Then $\delta : \bigoplus_{i=1}^n k(p_i) \to H^1(D, \mathcal{O}_D)$, the connecting homomorphism associated to the short exact sequence

$$0 \to \mathcal{O}_D \to \mathcal{O}_D(p_1 + \cdots + p_n) \to \bigoplus_{i=1}^n k(p_i) \to 0,$$

is the direct sum of the maps $\delta_i$.

**Proof.** Notice that for each $i$ there is a diagram of exact sequences

$$0 \to \mathcal{O}_D \to \mathcal{O}_D(p_i) \to k(p_i) \to 0$$

in which the rightmost map is the inclusion into the $i$th coordinate. Taking associated long exact sequences, we arrive at the commutative square

$$\begin{array}{ccccccccc}
\cdots & \to & k(p_i) & \xrightarrow{\delta_i} & H^1(D, \mathcal{O}_D) & \to & \cdots \\
& & \downarrow & & \downarrow & & \\
\cdots & \to & \bigoplus_{j=1}^n k(p_j) & \xrightarrow{\delta} & H^1(D, \mathcal{O}_D) & \to & \cdots.
\end{array}$$

This implies the result. \hfill $\square$

**Lemma 5.4.** Let $D$ be a connected nodal curve over $k$ of genus $g$. Let $p_1, \ldots, p_n$ be distinct smooth closed points of $D$, $n \geq 1$, so that $H^1(\mathcal{O}_D(p_1 + \cdots + p_n)) = 0$. Consider the following problem:

> Given $a_1, \ldots, a_n \in k$, find a global section $\sigma$ of $\mathcal{O}_D(p_1 + \cdots + p_n)$ so that $\sigma(p_i) = a_i$, $1 \leq i \leq n$.

There is a codimension $g$ condition on the $a_i$’s under which a solution exists. Moreover, if there is a solution, then there is a 1-dimensional family of solutions.
When $D$ is a ring of rational curves or a smooth elliptic curve, there are nonzero constants $c_1, \ldots, c_n$ so that the problem admits a solution if and only if
\[ c_1a_1 + \cdots + c_na_n = 0. \]

**Proof.** There is a short exact sequence
\[ 0 \to \mathcal{O}_D \to \mathcal{O}_D(p_1 + \cdots + p_n) \xrightarrow{\text{ev}} \bigoplus_{i=1}^n k(p_i) \to 0 \]
where the right map is induced by evaluation at $p_1, \ldots, p_n$.

Taking global sections, we obtain
\[ 0 \to k \to \Gamma(D, \mathcal{O}_D(p_1 + \cdots + p_n)) \xrightarrow{\text{ev}} k^n \to k^g \to 0. \]

Our problem can now be identified with the problem of finding a lift of $(a_1, \ldots, a_n) \in k^n$ to $\Gamma(D, \mathcal{O}_D(p_1 + \cdots + p_n))$. By exactness, there is a codimension $g$ condition under which a lift exists. Exactness also implies that solutions form a 1-dimensional family.

Now suppose that $D$ is a ring of rational curves or a smooth elliptic curve. By Lemma 5.2, $H^1(D, \mathcal{O}_D(p_i)) = 0$ for each $i$, so the map $\delta_i : k(p_i) \to H^1(D, \mathcal{O}_D) \cong k$ associated to the short exact sequence
\[ 0 \to \mathcal{O}_D \to \mathcal{O}_D(p_i) \to k(p_i) \to 0 \]
is surjective. In particular, there is a nonzero constant $c_i$ so that $\delta_i : a \mapsto c_ia$. By Lemma 5.3, the map $k^n \to k^g$ is $(a_1, \ldots, a_n) \mapsto c_1a_1 + \cdots + c_na_n$. This proves the result.

**Lemma 5.5.** Suppose that we are in the standard situation with $S = \text{Spec } k$. Let $q_1, q_2, \ldots, q_m$ be the points where $E$ meets $\overline{C - E}$. Consider the following problem:

Given $a_1, \ldots, a_m \in k$, find a section $\sigma \in \Gamma(E, \mathcal{O}_E(-\lambda))$ so that
\[ \sigma(q_i) = a_i, \quad 1 \leq i \leq m. \]

There is a codimension $g$ condition on the $a_i$s under which a solution exists. Moreover, if there is a solution, then there is a 1-dimensional family of solutions.

If $E$ is a small mesa of genus one, there are nonzero constants $c_1, \ldots, c_m$ so that a solution exists if and only if
\[ c_1a_1 + \cdots + c_ma_m = 0. \]

**Proof.** Let $F$ denote the top of the mesa $E$. Let $Z$ be any component of $\overline{E - F}$. Then $Z$ is rational, and by Proposition 2.5, $\mathcal{O}_Z(-\overline{\lambda})|_Z$ is of the form $\mathcal{O}_Z(p_0 + \cdots + p_n - q)$ where $p_0, \ldots, p_n$ are the points of $Z$ “heading away” from $F$ and $q$ is the point of $Z$ “heading towards” $F$. Similarly, the restriction of $\mathcal{O}_Z(-\overline{\lambda})$ to $F$ is of the form $\mathcal{O}_F(p_1 + \cdots + p_n)$ where $p_1, \ldots, p_n$ are the points where $F$ meets $\overline{C - F}$. Therefore, we may inductively apply Lemma 5.1 to reduce to the problem of Lemma 5.4. If $E$ is a small mesa of genus one, then $F = \text{core}(E)$ is either a ring of rational components or a smooth elliptic curve. The result follows.

**Lemma 5.6.** Suppose $D$ is a proper log curve over an algebraically closed field and $E \subseteq D$ is a small mesa of genus 1. Then $E$ is $\lambda$-acyclic.

**Proof.** Let $q_1, q_2, \ldots, q_m$ be the points where $E$ meets $\overline{D - E}$. Consider the problem of the previous lemma:
Given \( a_1, \ldots, a_m \in k \), find a section \( \sigma \in \Gamma(E, \mathcal{O}_E(-\lambda)) \) so that
\[
\sigma(q_i) = a_i, \quad 1 \leq i \leq m.
\]

The restriction of \( \mathcal{O}_E(-\lambda) \) to \( \text{core}(E) \) is of the form \( \mathcal{O}_{\text{core}(E)}(p_1 + \cdots + p_n) \) where \( p_1, \ldots, p_n \) are the points where \( \text{core}(E) \) meets \( D - \text{core}(E) \). By Lemma 5.2, \( H^1(\text{core}(E), \mathcal{O}_{\text{core}(E)}(p_1 + \cdots + p_n)) = 0 \). Then, as in the previous lemma, we may inductively apply Lemma 5.1 to reduce to the problem of Lemma 5.4 on the core of \( E \). Therefore there is a codimension 1 condition on the \( a_i \)'s under which a solution to the problem exists, and if a solution exists, then there is a 1 dimensional family of solutions.

It follows that \( \dim H^0(E, \mathcal{O}_E(-\lambda)) = m \). Since \( \deg \mathcal{O}_E(-\lambda) = m \) and \( E \) has genus one, we conclude by Riemann-Roch that \( H^1(E, \mathcal{O}_E(-\lambda)) = 0 \).

\[ \square \]

6. Flatness and commutativity with base change

In this section we show that \( \mathcal{U} \) is flat over \( S \) and the formation of \( \mathcal{U} \) commutes with base change in \( S \). It is interesting how closely related the two questions are.

**Lemma 6.1.** Let
\[
C^\bullet : 0 \to C^0 \to \cdots \to C^m \to 0
\]
be a cochain complex of flat \( A \)-modules so that \( H^i(C^\bullet) = 0 \) for \( i > j \). Then \( C^\bullet \) is quasi-isomorphic to its truncation \( \tau_{\leq j} C^\bullet \), and \( \tau_{\leq j} C^\bullet \) is a cochain complex of flat \( A \)-modules.

**Proof.** Omitted. \[ \square \]

**Theorem 6.2.** Let
\[
\begin{array}{ccc}
U_T & \xrightarrow{g} & U \\
\downarrow \pi_T & & \downarrow \pi \\
T & \xrightarrow{f} & S
\end{array}
\]
be a cartesian diagram of schemes with \( \pi \) quasicompact and separated, and let \( \mathcal{F} \) be a quasi-coherent sheaf on \( U \), flat over \( S \). Then

(i) If \( R^q \pi_* \mathcal{F} = 0 \), for \( q > i \), then the natural map \( f^*(R^i \pi_* \mathcal{F}) \to R^i \pi_{T,*}(g^* \mathcal{F}) \) is an isomorphism.

(ii) If \( R^q \pi_* \mathcal{F} = 0 \) for \( q > 0 \), then \( \pi_* \mathcal{F} \) is flat over \( S \).

**Proof.** The assertion is local on \( S \) and \( T \), so we may assume that \( S = \text{Spec} \, A \), \( T = \text{Spec} \, B \). Then what we have to show is

(i) If \( H^q(U, \mathcal{F}) = 0 \) for \( q > i \), then \( H^i(U, \mathcal{F}) \otimes_A B \to H^i(U_T, \mathcal{F} \otimes_A B) \) is an isomorphism.

(ii) If \( H^q(U, \mathcal{F}) = 0 \) for \( q > 0 \), then \( H^0(U, \mathcal{F}) \) is a flat \( A \)-module.

Since \( \pi \) is quasicompact and separated, we can find a cover \( \mathcal{U} = \{U_1, \ldots, U_n\} \) of \( U \) by open affines with affine intersections. Then \( H^i(U, \mathcal{F}) \) is computed by the Čech complex
\[
\check{C}^p(\mathcal{F}, \mathcal{U}) = \bigoplus_{i_1 < i_2 < \cdots < i_p} \Gamma(U_{i_1} \cap \cdots \cap U_{i_p}, \mathcal{F})
\]
in the sense that \( H^i(\check{C}^\bullet(\mathcal{U}, \mathcal{F})) \cong H^i(U, \mathcal{F}) \). We note that since \( \mathcal{F} \) is \( S \)-flat, \( \check{C}(\mathcal{U}, \mathcal{F}) \) is a complex of flat \( A \)-modules.
Proof. We have an exact sequence which establishes (i).

Taking the associated long exact sequence and noting \( H(\Gamma(C, \mathcal{F})) \) of Lemma 6.3. Suppose it has dimension \( d \), then \( H(S) \) commutes with base change in degree 0, and let \( \mathcal{F} \) be the pullback cover of \( \mathcal{F} \) via the evaluation map \( \Gamma(C, \mathcal{F}) \). Since \( \mathcal{F} \) is affine locally this is the isomorphism \( \Gamma(C, \mathcal{F}) = 0 \) for all sufficiently large \( \mathfrak{m} \).

For (ii), we note that if \( i = 0 \), \( H^0(U, \mathcal{F}) = H^0(C^\bullet) = C^0 \) is A-flat.

Lemma 6.3. Suppose \( C \) is a smooth curve of genus \( g \). Let \( p_1, \ldots, p_s \) be distinct closed points of \( C \), and let \( \mathcal{L} \) be a locally free sheaf of rank one on \( C \) with \( \deg(\mathcal{L}) > 2g - 2 + s \). Then the evaluation map \( \Gamma(C, \mathcal{L}) \rightarrow \bigoplus_{i=1}^s k(p_i) \) is surjective.

Proof. We have an exact sequence

\[
0 \rightarrow \mathcal{L}(-p_1 - \cdots - p_s) \rightarrow \mathcal{L} \rightarrow \bigoplus_{i=1}^s k(p_i) \rightarrow 0.
\]

Taking the associated long exact sequence and noting \( H^1(C, \mathcal{L}(-p_1 - \cdots - p_s)) = 0 \) gives the result.

Proposition 6.4. Assume that we are in the standard situation. For \( q \geq 1 \),

\[
H^q(U, \mathcal{O}_U(-\lambda)) = 0.
\]

Proof. Let \( \Sigma \) be the sum of the divisors of the sections \( \sigma_i \). Note that

\[
\mathcal{O}_U(-\lambda) \cong \varinjlim_{n \in \mathbb{N}} \mathcal{O}_C(-\lambda + n\Sigma).
\]

Affine locally this is the isomorphism \( \Gamma(C, \mathcal{O}_C)[f^{-1}] \cong \varinjlim_{n \in \mathbb{N}} f^{-n}\Gamma(C, \mathcal{O}_C) \) where \( f \) is a local equation for \( \Sigma \).

Since cohomology commutes with colimits, it is enough to show that \( H^q(C, \mathcal{O}_C(-\lambda + n\Sigma)) = 0 \) for all sufficiently large \( n \). We have assumed that \( S \) is finite dimensional; suppose it has dimension \( d \). Then \( C \) is \( d + 1 \) dimensional, so \( H^{d+2}(C, \mathcal{O}_C(-\lambda + n\Sigma)) = 0 \). We now work by descending induction on \( q \).

Suppose \( H^{q+1}(C, \mathcal{O}_C(-\lambda + n\Sigma)) = 0 \) for all sufficiently large \( n \). Then by Theorem 6.2 cohomology commutes with base change in degree \( q \) for \( \mathcal{O}_C(-\lambda + n\Sigma) \). Since \( \pi \) is proper and \( \mathcal{O}_C(-\lambda + n\Sigma) \) is coherent, \( H^q(C, \mathcal{O}_C(-\lambda + n\Sigma)) \) is finitely generated. Let \( \tilde{t} \) be any geometric point of \( S \), and let \( C_{\tilde{t}}, E_{\tilde{t}} \) be the pullbacks of \( C \) and \( E \) to \( \tilde{t} \) respectively. If \( q \geq 2 \), then \( H^q(C_{\tilde{t}}, \mathcal{O}_{C_{\tilde{t}}}(\lambda + n\Sigma)) \) vanishes for dimension reasons. If \( q = 1 \), let \( Z_1, \ldots, Z_r \) be the
components of $C_T$ not contained in $E_T$, and let $p_1, \ldots, p_s$ be the intersection points of the $Z_i$s with each other and with $E_T$. Consider the short exact sequence

$$0 \to \mathcal{O}_{C_T} \to \mathcal{O}_{E_T} \oplus \bigoplus_{i=1}^{r} \mathcal{O}_{Z_i} \to \bigoplus_{i=1}^{s} \mathcal{O}_{k(p_i)} \to 0.$$  

Twisting by $-\lambda + n\Sigma$ and then taking the associated long exact sequence gives

$$H^0(E_T, \mathcal{O}_{E_T}(-\lambda)) \oplus \bigoplus_{i=1}^{r} H^0(Z_i, \mathcal{O}_{Z_i}(-\lambda + n\Sigma|Z_i)) \to k^s \to H^1(C_T, \mathcal{O}_{C_T}(-\lambda + n\Sigma)).$$

By Lemma 4.3(iv), $\mathcal{O}_{Z_i}(-\lambda + n\Sigma|Z_i)$ has degree at least $n$. Applying Lemma 6.3 to each $Z_i$, we can make the first map surjective by choosing $n$ greater than $2g - 2 + s$. Looking a little further to the right in the same long exact sequence we now have

$$0 \to H^1(C_T, \mathcal{O}_{C_T}(-\lambda + n\Sigma)) \to H^1(E_T, \mathcal{O}_{E_T}(-\lambda)) \oplus \bigoplus_{i=1}^{m} H^1(Z_i, \mathcal{O}_{Z_i}(n\Sigma|Z_i)).$$

Choosing $n$ larger than $2g - 2 + s$, we can also make $H^1(Z_i, \mathcal{O}_{Z_i}(n\Sigma|Z_i))$ vanish. By $\lambda$-acyclicity, $H^1(E_T, \mathcal{O}_{E_T}(-\lambda))$ is also zero, so $H^1(C_T, \mathcal{O}_{C_T}(-\lambda + n\Sigma)) = 0$.

Altogether, for $n \geq 2g - 2 + s$,

$$H^q(C_T, \mathcal{O}_{C_T}(-\lambda + n\Sigma)) \cong H^q(C, \mathcal{O}_C(-\lambda + n\Sigma)) \otimes_A k(\mathcal{T}) = 0.$$  

The number of nodes in a fiber is bounded above, so we may choose $n$ uniformly for all $T$. By Nakayama’s Lemma, $H^q(C, \mathcal{O}_C(-\lambda + n\Sigma)) = 0$. The result follows. \hfill \square

**Proposition 6.5.** Assume we are in the standard situation. Then

(i) the $\overline{B}$-sequence

$$0 \to \Gamma(S, \mathcal{O}_S(\rho)) \to B(U) \to \overline{B}(U) \to 0$$

is short exact.

(ii) The terms of the $\overline{B}$-sequence are each flat over $S$.

(iii) If $f : T = \text{Spec} R \to S$ is a morphism of affine Noetherian, finite-dimensional schemes, we may pull back the data of the standard situation to obtain $\pi_T : C_T \to T$, $\lambda_T$, $\rho_T$, sections $\sigma_T$, $\ldots$, $\sigma_{T,k}$, and their complement $U_T$. These data are still in the standard situation, so we may form $\overline{B}(U_T)$. Then there is a natural isomorphism of short exact sequences of $R$-modules

$$0 \to \Gamma(T, \mathcal{O}_{T}(\rho_T)) \to B(U_T) \to \overline{B}(U_T) \to 0.$$  

Proof. Let us begin with (i). We have to show that the map $\Gamma(S, \mathcal{O}_S(\rho)) \to B(U)$ is injective; the rest of the sequence is exact by definition. Recall that $B(U) = \Gamma(U, \mathcal{O}_C(-\lambda)) \oplus \Gamma(S, \mathcal{O}_S)$ and the first map $\Gamma(S, \mathcal{O}_S(\rho)) \to B(U)$ is given by $d \mapsto ((\rho - \lambda)(d), -\rho(d))$.

We always have (iii) for flat maps $T \to S$, so we may work étale locally on $S$. Therefore we may assume that there is a section $\sigma : S \to C$ of $\pi$ so that $\sigma$ goes through the locus in $C$ where $\lambda = \rho$. (For each geometric point $\overline{\sigma}$ of $S$, choose a neighborhood as in Proposition
The map \( \sigma \) is in fact a map of log schemes, since the log structure of \( C \) along the image of \( \sigma \) is that pulled back from \( S \). Now, consider the composite

\[
\pi^{-1} \mathcal{O}_S(-\rho) \rightarrow \mathcal{O}_C(-\rho) \overset{e}{\rightarrow} \mathcal{O}_C(-\lambda).
\]

Taking global sections on \( U \) and considering the pullback maps induced by \( \sigma \) gives us a commutative square

\[
\begin{array}{ccc}
\Gamma(U, \pi^{-1} \mathcal{O}_S(-\rho)) & \longrightarrow & \Gamma(U, \mathcal{O}_C(-\lambda)) \\
\sigma^* \downarrow & & \downarrow \sigma^* \\
\Gamma(S, \mathcal{O}_S(-\rho)) & \longrightarrow & \Gamma(S, \mathcal{O}_S(-\rho)).
\end{array}
\]

where the left vertical arrow is the canonical isomorphism, and the lower horizontal arrow is the identity, since it is induced by the zero element of \( \mathcal{M}_S \). The composite map from the lower left to upper left to upper right corner is the map from \( \Gamma(S, \mathcal{O}_S(-\rho)) \) to the first coordinate of \( B(U) \). Therefore \((f, c) \mapsto \sigma^*(f)\) is a splitting of the \( B \)-sequence, which implies (i).

Now consider (iii). We have natural isomorphisms \( \Gamma(S, \mathcal{O}_S(-\rho)) \otimes_A R \rightarrow \Gamma(S, \mathcal{O}_T(-\rho_T)) \) and \( \Gamma(S, \mathcal{O}_S) \otimes_A R \rightarrow \Gamma(S, \mathcal{O}_S) \). By Proposition 6.4 and Theorem 6.2, the base change map \( \Gamma(U, \mathcal{O}_C(-\lambda)) \otimes_A R \rightarrow \Gamma(U_T, \mathcal{O}_{C_T}(-\lambda_T)) \) is also an isomorphism. This gives us a solid diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \Gamma(S, \mathcal{O}_S(-\rho)) \otimes_A R \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma(T, \mathcal{O}_T(-\rho_T))
\end{array}
\]

\[
\begin{array}{ccc}
B(U) \otimes_A R & \longrightarrow & \overline{B}(U) \otimes_A R \\
\downarrow & & \downarrow \\
B(U_T) & \longrightarrow & \overline{B}(U_T)
\end{array}
\]

in which the solid square commutes since the maps between sheaves associated to sections of the characteristic sheaf commute with pullback. The universal property of the cokernel gives us a unique isomorphism \( \overline{B}(U) \otimes_A R \rightarrow \overline{B}(U_T) \) so that the diagram remains commutative. Since all vertical arrows are isomorphisms, we may fill in a zero in the upper left. This proves (iii).

This leaves (ii). It is clear that \( \Gamma(S, \mathcal{O}_S(-\rho)) \) and \( \Gamma(S, \mathcal{O}_S) \) are each flat over \( A \). By Proposition 6.4 and Theorem 6.2, \( \Gamma(U, \mathcal{O}_C(-\lambda)) \) is also \( A \)-flat, so the \( B \)-sequence is a flat resolution of \( B(U) \). Now, the possible rings \( R \) in (iii) include the rings \( A/I \) for all ideals \( I \) of \( A \), so the top row of diagram (2) shows \( \text{Tor}_1^A(A/I, \overline{B}(U)) = 0 \) for all \( I \). It follows \( \overline{B}(U) \) is flat over \( A \), which concludes the proof. \( \square \)

**Corollary 6.6.** In the standard situation, \( \psi : \overline{U} \rightarrow S \) is flat.

**Corollary 6.7.** Assume we are in the standard situation and let \( f : T = \text{Spec} R \rightarrow S \) be a morphism of affine, Noetherian, finite-dimensional schemes. Pull back the data of the standard situation along \( f \) to obtain \( \pi_T : C_T \rightarrow T, U_T, E_T, \rho_T, \) and \( \sigma_{T,1}, \ldots, \sigma_{T,h} \) over \( T \). These data are again in the standard situation, so we may apply Definition 4.3 to obtain a
triangle

Then there is a uniquely specified isomorphism \( \phi_{S,T} : U_T \to U \times_S T \) so that

\[
\begin{array}{ccc}
U_T & \to & \overline{U}_T \\
\downarrow & \downarrow & \downarrow \\
T & \to & \overline{T} \\
\end{array}
\]

commutes. Moreover, these isomorphisms satisfy the cocycle condition in the sense that if \( g : W \to T \) is a second morphism of affine, Noetherian, finite-dimensional schemes,

\[
\phi_{S,W} = \phi_{T,W} \circ \phi_{S,T} |_W.
\]

Proof. We get the isomorphism \( \phi_{S,T} \) by taking Spec of the map \( \overline{B}(U) \otimes_A R \to \overline{B}(U_T) \) from Proposition [6.5](#). The commutativity of \( \phi_{S,T} \) with the map from \( U \) follows from a straightforward diagram chase. The commutativity of \( \phi_{S,T} \) with the maps to \( T \) holds since \( \overline{B}(U) \otimes_A R \to \overline{B}(U_T) \) is a map of \( R \)-modules. The cocycle condition holds since it holds of the cohomology–and–base change map. \( \square \)

7. Well-definedness of the contracted curve

We will assume throughout the remainder of the paper that we are in the standard situation.

Our first task will be to see that \( \tau \) restricts to an isomorphism in the complement of \( E \). This will ensure that the pushout defining \( C \) exists.

**Lemma 7.1.** Let \( I_{\tau(E)} \subseteq \overline{B}(U) \) be the ideal of the scheme theoretic image of \( E \) under \( \tau \). Let \( \phi : B(U) \to \overline{B}(U) \) be the quotient map. Then

\[
\phi^{-1}(I_{\tau(E)}) = \left\{ (f,c) \mid f \in \Gamma(U, \mathcal{O}_U(-\overline{\lambda})) \text{ and } c \in \rho(\Gamma(S, \mathcal{O}_S(-\rho))) \right\}
\]

and

\[
I_{\tau(E)} = \left\{ \phi(f,0) \mid f \in \Gamma(U, \mathcal{O}_U(-\overline{\lambda})) \right\}.
\]

Proof. If \( E \) is empty, then both \( \rho \) and \( \overline{\lambda} \) are zero. This allows us to identify \( \Gamma(S, \mathcal{O}_S(-\rho)) \) with \( \Gamma(U, \mathcal{O}_U(-\overline{\lambda})) \) and \( \Gamma(U, \mathcal{O}_C(-\overline{\lambda})) \) with \( \Gamma(U, \mathcal{O}_C) \), which allows us to identify \( \overline{B}(U) \) with \( \Gamma(U, \mathcal{O}_C) \):

\[
\begin{array}{ccccccccc}
\Gamma(S, \mathcal{O}_S(-\rho)) & \to & \Gamma(U, \mathcal{O}_C(-\overline{\lambda})) \oplus \Gamma(S, \mathcal{O}_S) & \to & \overline{B}(U) & \to & 0 \\
\sim & & \sim & & \sim & & \\
\Gamma(S, \mathcal{O}_S) & \xrightarrow{\pi^S_{\oplus-\text{id}}} & \Gamma(U, \mathcal{O}_C) \oplus \Gamma(S, \mathcal{O}_S) & \to & \Gamma(U, \mathcal{O}_C) & \to & 0
\end{array}
\]

It is then clear that the formula given for \( \phi^{-1}(I_{\tau(E)}) \) reduces to the unit ideal of \( B(U) \) and the formula given for \( I_{\tau(E)} \) reduces to the unit ideal of \( \overline{B}(U) \), as desired.
If \( E \) is nonempty, let \( \phi(f, c) \) be an arbitrary element of \( \mathcal{B}(U) \). By definition of \( \tau \), the image of \( \phi(f, c) \) in \( \Gamma(U, \mathcal{O}_U) \) is \( \overline{X}(f) + \pi^*c \). It is clear that \( \overline{X}(f) \) vanishes on \( E \), so \( \overline{X}(f) + \pi^*c \) vanishes on \( E \) if and only if \( \pi^*c \) does. We now want to show that \( \pi^*c \) vanishes on \( E \) if and only if \( c \in \rho(\Gamma(S, \mathcal{O}_S(-\rho))) \).

We may check this étale locally on \( S \), so we may assume that there is a section \( \sigma : S \to C \) going through the locus where \( \overline{X} = \rho \). Note that \( \sigma^*(\overline{X}) = \rho \). Now, if \( \pi^*c \) reduces to 0 in \( \mathcal{O}_E \), there is an open cover \( U_i \) of \( U \) and elements \( d_i \in \Gamma(U_i, \mathcal{O}_C(-\overline{X})) \) so that \( \pi^*c|_{U_i} = \overline{X}(d_i) \). Pulling back along \( \sigma \), we obtain an open cover \( V_i = \sigma^{-1}(U_i) \) of \( S \) and elements \( \sigma^*d_i \in \Gamma(V_i, \mathcal{O}_S(-\rho)) \) so that

\[
c|_{V_i} = \sigma^*(\pi^*c|_{U_i}) = \rho(\sigma^*d_i).
\]

That is, \( c \) is locally in the image of \( \mathcal{O}_S(-\rho) \xrightarrow{\rho} \mathcal{O}_S \). Since \( S \) is affine, \( c \in \rho(\Gamma(S, \mathcal{O}_S(-\rho))) \). Conversely, if \( c \in \rho(\Gamma(S, \mathcal{O}_S(-\rho))) \), then since \( \rho = \overline{X} + (\rho - \overline{X}) \),

\[
\pi^*c \in \overline{X}\left((\rho - \overline{X})(\Gamma(U, \mathcal{O}_C(-\rho)))\right) \subseteq \overline{X}(\Gamma(U, \mathcal{O}_C(-\overline{X}))),
\]

so \( \pi^*c \) vanishes on \( E \). Therefore

\[
\phi^{-1}(I_{\tau(E)}) = \left\{(f, c) \mid f \in \Gamma(U, \mathcal{O}_U(-\overline{X})) \text{ and } c \in \rho(\Gamma(S, \mathcal{O}_S(-\rho)))\right\}
\]

Now, for any \( (f, c) \in \phi^{-1}(I_{\tau(E)}) \), we may find an element \( d \in \Gamma(S, \mathcal{O}_S(-\rho)) \) so that \( c = -\rho(d) \). Using the \( \mathcal{B} \)-sequence, we may rewrite \( \phi(f, c) \) as \( \phi(f - (\rho - \overline{X})(d), 0) \). This shows

\[
I_{\tau(E)} = \left\{\phi(f, 0) \mid f \in \Gamma(U, \mathcal{O}_U(-\overline{X}))\right\}.
\]

\[\square\]

**Proposition 7.2.** The map \( \tau : U \to \overline{U} \) restricts to an isomorphism \( U - E \to \overline{U} - \tau(E) \). In particular, the pushout defining \( \overline{C} \) exists and the map \( \tau : C - E \to \overline{C} - \tau(E) \) is also an isomorphism.

**Proof.** Since \( \pi : E \to S \) is proper and \( \overline{U} \to S \) is affine (hence separated), \( \tau : E \to \overline{U} \) is proper. In particular, \( \tau(E) \) is closed, so \( \tau(E) \) coincides with the scheme theoretic image of \( E \).

As above, let \( I_{\tau(E)} \subseteq \mathcal{B}(U) \) be the ideal of the scheme theoretic image of \( E \) under \( \tau \), and let \( \phi : B(U) \to \mathcal{B}(U) \) be the quotient map. Let \( \phi(f, 0) \in I_{\tau(E)} \). Since \( \phi(f, 0) \) maps to \( \overline{X}(f) \) in \( \Gamma(U, \mathcal{O}_C) \), we have a cartesian square

\[
\begin{array}{ccc}
D(\overline{X}(f)) & \longrightarrow & D(\phi(f, 0)) \\
\downarrow & & \downarrow \\
U & \xrightarrow{\tau} & \overline{U}.
\end{array}
\]

As \( f \) varies over the elements of \( \Gamma(U, \mathcal{O}_C(-\overline{X})) \), the open subsets \( D(\phi(f, 0)) \) vary over an open cover of \( U - \tau(E) \), and the \( D(\overline{X}(f)) \) vary over an open cover of \( U - E \). Therefore it is enough to show that the top arrow is an isomorphism for each element \( \phi(f, 0) \).

We start by showing that the top row of \((3)\) induces an isomorphism on global regular functions. Note that \( \Gamma(S, \mathcal{O}_S(-\rho)) \otimes_{B(U)} B(U)[(f, 0)^{-1}] = 0 \), as \( (f, 0) \) acts as 0 on \( \Gamma(S, \mathcal{O}_S(-\rho)) \). Then, tensoring the \( \mathcal{B} \)-sequence

\[
0 \to \Gamma(S, \mathcal{O}_S(-\rho)) \to B(U) \xrightarrow{\phi} \mathcal{B}(U) \to 0
\]
with the flat $B(U)$-module $B(U)[(f,0)^{-1}]$, we get
\[ 0 \to B(U)[(f,0)^{-1}] \xrightarrow{\sim} \mathcal{O}_T(D(\phi(f,0))) \to 0. \]

Now consider the map
\[ B(U)[(f,0)^{-1}] \to \Gamma(U, \mathcal{O}_C)[\overline{\lambda}(f)^{-1}] \cong \Gamma(D(\overline{\lambda}(f)), \mathcal{O}_C) \]
induced by tensoring $B(U) \to \Gamma(U, \mathcal{O}_C)$ with $B(U)[(f,0)^{-1}]$. We’d like to show that this map is an isomorphism. For injectivity, suppose $(g,c) \cdot (f,0)^{-n}$ maps to 0. Then $\overline{\lambda}(f)^k(\overline{\lambda}(g)+c) = 0$ for some $k$. Now, multiplying by a unit,
\[ (g,c) \cdot (f,0)^{-n} \cdot (f,0)^{n+k+1} = (g,c) \cdot ((\overline{\lambda}(f))^k f,0) \]
\[ = (\overline{\lambda}(g) + c) \cdot (\overline{\lambda}(f)^k f,0) \]
\[ = 0 \]
so $(g,c) \cdot (f,0)^{-n}$ must have been zero to begin with in $B(U)[(f,0)^{-1}]$.

For surjectivity, suppose $g \cdot (\overline{\lambda}(f))^{-n} \in \Gamma(U, \mathcal{O}_C)[\overline{\lambda}(f)^{-1}]$. This has preimage $(f \cdot g,0) \cdot (f,0)^{-(n+1)}$ since
\[ (f \cdot g,0) \cdot (f,0)^{-(n+1)} \mapsto \overline{\lambda}(f)g \overline{\lambda}(f)^{-(n+1)} \]
\[ = g \cdot \overline{\lambda}(f)^{-n} \]

The map on global regular functions induced by $\tau$ is the composite isomorphism $\mathcal{O}_T(D(\phi(f,0))) \to \mathcal{O}_C(D(\overline{\lambda}(f)))$. The same argument shows that if $T \to S$ is an étale cover,
\[ D(\overline{\lambda}(f)) \times_S T \cong D(\overline{\lambda}(f)|_{C \times_S T}) \to D(\phi(f|_{C \times_S T},0)) \cong D(\phi(f,0)) \times_S T \]
also induces an isomorphism on global regular functions.

To complete the proof, we check that $D(\overline{\lambda}(f))$ is affine étale locally on $S$. Let $\Sigma$ be the sum of the divisors of the sections $\sigma_i$. Replacing $S$ by an étale cover if necessary, we can find a finite union of sections $\Sigma'$, passing through the smooth locus of each component of each fiber of $E$ and disjoint from each other and the $\sigma_i$’s. Consider the sheaves $\mathcal{O}_C(n(\Sigma + \Sigma'))$. Arguing as in Proposition 6.4, we see that for a sufficiently large $n$, $H^1(C, \mathcal{O}_C(n(\Sigma + \Sigma'))) = 0$, since this holds on fibers. Then we can check that $\mathcal{O}_C(n(\Sigma + \Sigma'))$ is relatively very ample on fibers; this is clear for sufficiently large $n$. Consider the associated embedding of $C$ into projective space; the locus $U' = C - (\Sigma \cup \Sigma')$ is a closed subset of the complement of some hyperplane, so affine. Then $D(\overline{\lambda}(f)) = U - V(\overline{\lambda}(f)) = U' - V(\overline{\lambda}(f)|_{U'})$ is a distinguished open subset of $U'$, so it is affine too. \(\square\)

**Corollary 7.3.** The map $\tau : C \to \overline{C}$ is surjective.

**Proof.** We have $\tau$ sends $U - E$ onto $U - \tau(E)$. We noted in the course of the previous proof that the image of $E$ is $\tau(E)$. Gluing with the rest of $C$, $\tau$ is surjective. \(\square\)

Next, we must verify is that $\overline{C}$ does not depend on our choice of sections $\sigma_i$. This follows quickly from the next lemma.
Lemma 7.4. Assume we are in the standard situation. For any open subset \( V \) of \( U \) containing \( E \), define \( V \to V = \text{Spec } B(V) \) as in the discussion preceding Definition 4.3. Then

\[
\overline{C} = \text{colim} \begin{cases} \quad U - E \to C - E \\ \uparrow \downarrow \\ \overline{U} \end{cases} \cong \text{colim} \begin{cases} \quad \quad \quad \quad V - E \to C - E \\ \downarrow \quad \downarrow \\ \overline{V} \quad \end{cases}
\]

Proof. It suffices to show that

\[
\begin{array}{ccc}
V & \to & U \\
\downarrow & & \downarrow \\
\overline{V} & \to & \overline{U}
\end{array}
\]

is cocartesian, since then the tall commutative square in

\[
\begin{array}{ccc}
V - E & \to & U - E \\
\downarrow & & \downarrow \\
\overline{V} & \to & \overline{U}
\end{array}
\]

is cocartesian, implying that both the right square and wide square in

\[
\begin{array}{ccc}
V - E & \to & U - E \to C - E \\
\downarrow & \quad & \downarrow \\
\overline{V} & \to & \overline{U} \to \overline{C}
\end{array}
\]

are cocartesian, which implies the result.

Since \( \overline{U} \) is affine, it suffices to check that it has the universal property of a pushout for maps to affine schemes. This is equivalent to checking that the right square of

\[
\begin{array}{ccc}
\Gamma(U, \mathcal{O}_C(-\overline{\lambda})) \oplus \Gamma(S, \mathcal{O}_S) & \to & \overline{B}(U) \to \Gamma(U, \mathcal{O}_C) \\
\downarrow & & \downarrow \\
\Gamma(V, \mathcal{O}_C(-\overline{\lambda})) \oplus \Gamma(S, \mathcal{O}_S) & \to & \overline{B}(V) \to \Gamma(V, \mathcal{O}_C)
\end{array}
\]

is cartesian. Since the kernels of the leftmost horizontal maps coincide, this is equivalent in turn to checking that the wide square above is cartesian.

To that end, let \((f, c) \in \Gamma(V, \mathcal{O}_C(-\overline{\lambda})) \oplus \Gamma(S, \mathcal{O}_S)\) and \(g \in \Gamma(U, \mathcal{O}_C)\) be elements so that \(g|_V = \overline{\lambda}(f) + c\). Let \(W = U - E\). Note that \(\overline{\lambda}\) acts by multiplication by a unit on \(W\), so there is a unique \(h \in \Gamma(W, \mathcal{O}_C(-\overline{\lambda}))\) such that \((g - c)|_W = \overline{\lambda}(h)\). Restricting to \(V \cap W\), we have

\[
\overline{\lambda}(f|_{V \cap W}) = (g - c)|_{V \cap W} = \overline{\lambda}(h|_{V \cap W}).
\]

Since \(\overline{\lambda}\) acts as multiplication by a unit on \(W\), \(f|_{V \cap W} = h|_{V \cap W}\). Gluing \(f\) with \(h\), we obtain a unique element

\[
\overline{h} \in \Gamma(V \cup W, \mathcal{O}_C(-\overline{\lambda})) = \Gamma(U, \mathcal{O}_C(-\overline{\lambda}))
\]
Corollary 7.5. Suppose we are in the standard situation and \( \sigma'_1, \ldots, \sigma'_k \) are a second choice of sections. There is a uniquely specified isomorphism \( \phi_{\sigma'_1, \ldots, \sigma'_k} : (C, \sigma'_1) \to (C, \sigma'_k) \) so that

\[
\begin{array}{c}
C_T \\
\downarrow \phi_{\sigma'_1, \ldots, \sigma'_k} \\
(C, \sigma'_1) \\
\downarrow \\
T
\end{array}
\]

commutes. Moreover, if \( \sigma''_1, \ldots, \sigma''_l \) is a third choice of sections, the cocycle condition

\[\phi_{\sigma'_1, \ldots, \sigma'_k} = \phi_{\sigma''_1, \ldots, \sigma''_l} \circ \phi_{\sigma'_1, \ldots, \sigma'_k}\]

holds.

Proof. Let \( U, U', U'' \) be the complements of the \( \sigma_i \)'s, \( \sigma'_i \)'s, and \( \sigma''_i \)'s, respectively. Then applying Lemma 7.4 to \( U \) and \( U' \) with \( V = U \cap U' \) gives the unique isomorphism \( \phi_{\sigma_1, \sigma'_1} \), as \( (C, \sigma) \) and \( (C, \sigma'_1) \) satisfy the same universal property. Applying the lemma to each of \( U, U', \) and \( U'' \) with \( V = U \cap U' \cap U'' \) gives the cocycle condition. \( \square \)

8. Regular functions near the singular point

Our goal in this section will be to describe the regular functions in a neighborhood of \( \tau(E) \) — that is, the ring \( \overline{B}(U) \) — when the base is the spectrum of an algebraically closed field. This provides us with a fairly general picture of the sorts of singularities that can arise in \( \overline{C} \), and we give several examples to illustrate some of the possibilities. A corollary is that \( \overline{C} \) is reduced, as \( \overline{B}(U) \) is reduced.

We start with a description of \( \Gamma(U, \mathcal{O}_C(-\overline{\lambda})) \).

Proposition 8.1. Suppose \( S = \text{Spec} \ k \) is a geometric point and \( E \) is nonempty with arithmetic genus \( g \). Let \( Z \) be the union of the irreducible components of \( U \) not contained in \( E \), and let \( p_1, \ldots, p_m \) be the points in which \( E \) meets \( Z \). Finally, let \( V \) be the set of tuples \( \{a_1, \ldots, a_m\} \subseteq \mathbb{K}^m \) so that there exists \( \sigma \in \Gamma(E, \mathcal{O}_E(-\overline{\lambda})) \) with \( \sigma(p_i) = a_i \) for all \( i \), with \( 1 \leq i \leq m \).

Then there is an exact sequence

\[
0 \to \Gamma(S, \mathcal{O}_S(-\rho)) \overset{\pi}{\to} \Gamma(U, \mathcal{O}_C(-\overline{\lambda})) \to \Gamma(Z, \mathcal{O}_Z(-\overline{\lambda})) \to \mathbb{K}^m/V \to 0,
\]

where the second map is induced by restriction from \( U \) to \( Z \), and the third map is induced by evaluation at the points \( p_i \). Moreover, the first map admits a splitting, so that

\[
\Gamma(U, \mathcal{O}_C(-\overline{\lambda})) \cong \Gamma(S, \mathcal{O}_S(-\rho)) \oplus \left\{ f \in \Gamma(Z, \mathcal{O}_Z(-\overline{\lambda})) \mid [f(p_i)]_{i=1}^m \in V \right\}.
\]

Proof. Let \( F \) be the top of the mesa \( E \). Note that \( \rho = \overline{\lambda} \) on the smooth locus of \( F \) and \( \rho > \overline{\lambda} \) elsewhere on \( C \) (in the sense that for \( v \) a vertex of \( \Gamma(C) \) outside of \( F \), there is a nonzero section \( \alpha \in \Gamma(S, \mathcal{M}_S \overline{\lambda}) \) so that \( \rho = \lambda_v + \alpha \)). Then the map \( \pi^{-1} \mathcal{O}_S(-\rho) \to \mathcal{O}_U(-\overline{\lambda}) \overset{\rho}{\to} \mathcal{O}_U(-\overline{\lambda}) \)
is induced by multiplication by a unit on the smooth locus of $F$ and multiplication by zero elsewhere.

Now, we have a short exact sequence

$$0 \to \mathcal{O}_U(\overline{\lambda}) \to \mathcal{O}_E(\overline{\lambda}) \oplus \mathcal{O}_Z(\overline{\lambda}) \to \bigoplus_{i=1}^{m} k(p_i) \to 0.$$ 

This gives us an exact sequence

$$0 \to \Gamma(U, \mathcal{O}_U(\overline{\lambda})) \to \Gamma(E, \mathcal{O}_E(\overline{\lambda})) \oplus \Gamma(Z, \mathcal{O}_Z(\overline{\lambda})) \to k^m.$$ 

That is, a section of $\mathcal{O}_U(\overline{\lambda})$ is the same as a pair of sections of $\mathcal{O}_E(\overline{\lambda})$ and $\mathcal{O}_Z(\overline{\lambda})$ that agree at the points $p_i$.

Let $f_Z \in \Gamma(Z, \mathcal{O}_Z(\overline{\lambda}))$, and consider the problem of extending $f_Z$ to $\Gamma(U, \mathcal{O}_C(\overline{\lambda}))$. This is equivalent to finding a section $f_E \in \Gamma(E, \mathcal{O}_E(\overline{\lambda}))$ so that $f_E(p_i) = f_Z(p_i)$ for all $i$. By definition, such an $f_E$ exists if and only if $[f_Z(p_i)]_{i=1}^{m}$ lies in $V$. By Lemma 5.5, for each $f_Z$ satisfying this condition, there is precisely a 1-dimensional family of such $f_E$s. In fact, since the image $I$ of $k \cong \Gamma(S, \mathcal{O}_S(\rho)) \xrightarrow{\rho - \overline{\lambda}} \mathcal{O}_E(\overline{\lambda})$ is 1-dimensional and any $h \in I$ satisfies $h(p_i) = 0$ for all $i$, every choice of $f_E$ in the 1-dimensional family is obtained by adding an element of $I$. That is,

$$0 \to \Gamma(S, \mathcal{O}_S(\rho)) \xrightarrow{\rho - \overline{\lambda}} \Gamma(U, \mathcal{O}_C(\overline{\lambda})) \to \Gamma(Z, \mathcal{O}_Z(\overline{\lambda})) \to k^m/V \to 0$$

is exact.

To show that we have a splitting of the first map, let $\sigma : S \to C$ be any section through the smooth locus of $F$. Then $f \mapsto \sigma^*f$ gives a splitting of $\Gamma(S, \mathcal{O}_S(\rho)) \to \Gamma(U, \mathcal{O}_C(\overline{\lambda}))$, since $\rho - \overline{\lambda} = 0$ on the smooth locus of $F$.

**Proposition 8.2.** Suppose $S = \text{Spec} \ k$ is a geometric point and $E$ is nonempty with arithmetic genus $g$. Let $Z$ be the union of the irreducible components of $U$ not contained in $E$, and let $p_1, \ldots, p_m$ be the points in which $E$ meets $Z$. Finally, let $V$ be the set of tuples $\{(a_1, \ldots, a_m)\} \subseteq k^m$ so that there exists $\sigma \in \Gamma(E, \mathcal{O}_E(\overline{\lambda}))$ with $\sigma(p_i) = a_i$ for all $i$, with $1 \leq i \leq m$.

Then

(i) $V$ is a subspace of $k^m$ of codimension $g$;

(ii) We may identify $\overline{B}(U)$ with the ring

$$\left\{ (f, c) \in \Gamma(Z, \mathcal{O}_Z(\overline{\lambda})) \oplus k \mid [f(p_i)]_{i=1}^{m} \in V \right\}$$

where the multiplication is given by $(f, c) \cdot (f', c') = (\overline{\lambda}(ff') + fc + f'c, cc')$;

(iii) With respect to this presentation, the ideal $I_{\tau(E)} \subseteq \overline{B}(U)$ is the subset of $\overline{B}(U)$ where $c = 0$;

(iv) $\tau(E)$ is a singularity of genus $g$;

(v) If $E$ is a small mesa of genus one, then $\tau(E)$ is elliptic Gorenstein with $m$ branches.

In particular, for an arbitrary mesa curve $\pi : C \to S$, the geometric fibers of $\psi : \mathcal{C} \to S$ are reduced and one-dimensional.

**Proof.** Part (i) is immediate from Lemma 5.5.

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Consider the truncation

\[ 0 \to \Gamma(S, \mathcal{O}_S(-\rho)) \xrightarrow{\rho - \lambda} \Gamma(U, \mathcal{O}_C(-\lambda)) \to \left\{ f \in \Gamma(Z, \mathcal{O}_Z(-\lambda)) \mid [f(p_i)]_{i=1}^m \in V \right\} \to 0 \]

of the long exact sequence of the previous proposition. Note that \( \rho : \Gamma(S, \mathcal{O}_S(-\rho)) \to \Gamma(S, \mathcal{O}_S) \) is the zero map, since \( E \) is nonempty and \( S \) is a geometric point. Tack on summands of \( \Gamma(S, \mathcal{O}_S) \approx k \) to the last two terms, it follows

\[ 0 \to \Gamma(S, \mathcal{O}_S(-\rho)) \xrightarrow{\rho - \lambda} B(U) \to \left\{ (f, c) \in \Gamma(Z, \mathcal{O}_Z(-\lambda)) \oplus k \mid [f(p_i)]_{i=1}^m \in V \right\} \to 0 \]

is exact. Then, by definition of \( \overline{B}(U) \), we may identify \( \overline{B}(U) \) with the ring

\[ \left\{ (f, c) \in \Gamma(U, \mathcal{O}_Z(-\lambda)) \oplus k \mid [f(p_i)]_{i=1}^m \in V \right\} \]

with multiplication \((f, c) \cdot (f', c') = (\overline{\lambda}(ff') + f'c + f'c, cc')\). This proves (ii).

(iii) is clear from the expression for \( I_{\tau(E)} \) in Lemma 7.1

Now, consider the factorization

\[ \xymatrix{ U \ar[r] \ar[d]_{\tau} & \text{Spec} \Gamma(U, \mathcal{O}_U) \ar[d]^\tau \cr \overline{U} } \]

of \( \tau \). Note that

\[ \Gamma(U, \mathcal{O}_U) \cong \left\{ f \in \Gamma(Z, \mathcal{O}_Z) \mid f(p_i) = f(p_j) \text{ for all } i, j \right\} \]

\[ \cong \left\{ (f, c) \in \Gamma(Z, \mathcal{O}_Z) \oplus k \mid f(p_i) = 0 \text{ for all } i \right\} \]

\[ \cong \left\{ (f, c) \in \Gamma \left( Z, \mathcal{O}_Z \left( -\sum p_i \right) \right) \oplus k \right\} \]

\[ \cong \left\{ (f, c) \in \Gamma(Z, \mathcal{O}_Z(-\lambda)) \oplus k \right\}, \]

with multiplication \((f, c) \cdot (f', c') = (\overline{\lambda}(ff') + f'c + f'c, cc')\). Written this way, we see that \( \tau \) identifies \( \overline{B}(U) \) with the subring of \( \Gamma(U, \mathcal{O}_U) \) cut out by the condition \([f(p_i)]_{i=1}^m \in V\).

It is clear that \( Z \) is the normalization of both \( \text{Spec} \Gamma(U, \mathcal{O}_U) \) and \( \overline{U} \). Since \( \overline{B}(U) \) has codimension \( g \) in \( \Gamma(U, \mathcal{O}_U) \), \( \tau(E) \) has genus \( g \). This shows (iv).

In order to show (v), suppose that \( E \) is a small mesa of genus one. Lemma 5.5 tells us that there are nonzero constants \( c_1, \ldots, c_m \) so that \([f(p_i)]_{i=1}^m \in V \iff \sum_{i=1}^m c_if(p_i) = 0\). One verifies easily that the conditions of Lemma 3.6 hold: (i) holds since we impose the condition \( \sum_{i=1}^m c_if(p_i) = 0 \), (ii) holds since each \( c_i \) is nonzero, and (iii) holds since the condition \( \sum_{i=1}^m c_if(p_i) = 0 \) holds if each \( f(p_i) \) is zero. Therefore \( \tau(E) \) is an elliptic Gorenstein singularity when \( E \) is a small mesa of genus one.

\[ \square \]

**Example 8.3.** Let \( C \) be a nodal curve over \( S = \text{Spec} k \), \( k \) algebraically closed, so that \( C \) consists of a smooth genus \( g \) component, \( E \), and \( g \) rational curves \( Z_1, \ldots, Z_g \) attached to \( E \) at distinct points \( p_1, \ldots, p_g \), so that \( H^1(E, \mathcal{O}_E(p_1 + \cdots + p_g)) = 0 \). (A generic choice of \( p_1, \ldots, p_g \in E \) will do the job.)

We can give \( S \) the log structure associated to the chart \( N\delta \to k \) sending \( \delta \to 0 \). Then \( C \) may be given a compatible log structure where \( \delta_{p_i} \) is \( \delta \) for each \( i \). Then

\[ \lambda_v = \begin{cases} \delta & \text{if } v = E \\ 0 & \text{otherwise} \end{cases} \]
defines a piecewise linear function on $\Gamma(C)$ so that $(C, \overline{\lambda})$ is a mesa curve. Choose the open set $U$ as the complement of the points at infinity on the $Z_i$s, and let $Z = U \cap \bigcup_{i=1}^{9} Z_i$. By Proposition 8.2,

$$\overline{B}(U) \cong \left\{ f \in \Gamma(Z, \mathcal{O}_Z(-\overline{\lambda})) \oplus k \left| \frac{[f(p_i)]_{i=1}^g = 0}{} \right. \right\},$$

since $V$ is a codimension $g$ subspace of $k^g$. Simplifying further,

$$\overline{B}(U) \cong \left\{ f \in \Gamma(Z, \mathcal{O}_Z(-p_1 - \cdots - p_g)) \oplus k \left| f(p_i) = 0 \text{ for all } i \right. \right\} \cong \left\{ f \in \Gamma(Z, \mathcal{O}_Z(-2p_1 - \cdots - 2p_g)) \oplus k \right\} \cong \left\{ (f_1, \ldots, f_g) \in k[x_1^2, x_1^3] \times \cdots \times k[x_g^2, x_g^3] \left| f_1(0) = \cdots = f_g(0) \right. \right\},$$

where $x_1, \ldots, x_g$ are local parameters of $p_1, \ldots, p_g$ on $Z_1, \ldots, Z_g$, respectively. That is, the singularity at $\tau(E)$ consists of $g$ cusps glued transversally at their singular points.

**Example 8.4.** When a genus one mesa is not small (when the top is not the core), the resulting singularity need not be Gorenstein. Consider a mesa curve over $S = \text{Spec} \ k$, $k$ algebraically closed, with dual graph

```
     ___________
    /             /
   /               /
  /                 /
```

where the filled dots have genus zero, the empty dot has genus one, and the red components make up both $E$ and the top of $E$. From left to right, name the components $Z_1, E_1, E_2,$ and $Z_2$. Let $p_1$ be the point where $Z_1$ meets $E_1$, and let $p_2$ be the point where $Z_2$ meets $E_2$. Choose $U$ as the complement of the points at infinity of $Z_1$ and $Z_2$, and let $Z = U \cap (Z_1 \cup Z_1)$.

By Proposition 8.2, we know

$$\overline{B}(U) \cong \left\{ f \in \Gamma(Z, \mathcal{O}_Z(-\overline{\lambda})) \oplus k \left| \frac{[f(p_1)]}{f(p_2)} \in V \right. \right\}$$

where $V$ is some codimension 1 subspace of $k^2$.

Now, $V$ is the set of pairs $(a_1, a_2) \in k^2$ so that there exists a global section $\sigma \in \Gamma(E, \mathcal{O}_E(-\overline{\lambda}))$ with $\sigma(p_1) = a_1$ and $\sigma(p_2) = a_2$. As a start, we may ask if there exists $\sigma_1 \in \Gamma(E_1, \mathcal{O}_{E_1}(-\overline{\lambda})) \cong \Gamma(E_1, \mathcal{O}_{E_1}(p_1))$ with $\sigma_1(p_1) = a_1$. The exact sequence

$$0 \to \mathcal{O}_{E_1} \to \mathcal{O}_{E_1}(p_1) \to k(p_1) \to 0$$

gives rise to the exact sequence

$$0 \to k \to \Gamma(E_1, \mathcal{O}_{E_1}(p_1)) \to k \to 0,$$

which tells us that every $\sigma_1 \in \Gamma(E_1, \mathcal{O}_{E_1}(p_1))$ satisfies $\sigma_1(p_1) = 0$. This same condition must hold for any $\sigma \in \Gamma(E, \mathcal{O}_E(-\overline{\lambda}))$, so the codimension 1 condition cutting out $V$ is $a_1 = 0$. 


Now, simplifying,

\[
\mathcal{B}(U) \cong \left\{ f \in \Gamma(Z, \mathcal{O}_Z(-\lambda)) \oplus k \mid \begin{bmatrix} f(p_1) \\ f(p_2) \end{bmatrix} \in V \right\}
\]
\[
\cong \left\{ f \in \Gamma(Z, \mathcal{O}_Z(-p_1 - p_2)) \oplus k \mid f(p_1) = 0 \right\}
\]
\[
\cong \left\{ (f, g) \in k[x^2, x^3] \times k[y] \mid f(0) = g(0) \right\},
\]

where \(x\) and \(y\) are local parameters of \(p_1\) and \(p_2\) on \(Z_1\) and \(Z_2\), respectively. That is, \(\overline{C}\) consists of a rational curve with a cusp glued to a smooth rational curve.

More generally, if \(E_1\) were attached to \(k\) rational curves and \(E_2\) were attached to \(l\) rational curves, \(\tau(E)\) would consist of an elliptic \(k\)-fold Gorenstein singularity glued transversally to an ordinary \(l\)-fold point.

**Example 8.5.** (Variation of log structure in a contraction to a tacnode)

Suppose \(E\) is a smooth elliptic curve over an algebraically closed field \(k\), and \(p, q\) are two closed points of \(E\). Form a curve \(C_0\) by attaching rational components \(Z_p\) and \(Z_q\) to \(E\) at \(p\) and \(q\), respectively. If \(C_0\) is given the structure of a mesa curve so that \(E\) is the mesa, then we expect \(\overline{C_0}\) to consist of two rational components meeting in an elliptic Gorenstein singularity with 2 branches: a tacnode.

Consider the Stein factorization of the contraction map,

\[
C_0 \longrightarrow C_0^+ = (\overline{C_0}, \tau_* \mathcal{O}_{C_0})
\]
\[
\tau \downarrow \downarrow \tau
\]
\[
\overline{C_0}.
\]

The intermediate curve \(C_0^+\) consists of two rational components meeting in an ordinary double point, and the map \(\tau\) collapses the ordinary double point to the tacnode. There is a one parameter family of such maps. To see this, choose a local parameter \(y_p\) of \(p\) in \(Z_p\) and a local parameter \(y_q\) of \(q\) in \(Z_q\). Near the node, \(C_0^+\) has functions

\[
\{(f(y_p), g(y_q)) \mid f(0) = g(0)\}
\]

and \(\tau\) could be the inclusion of any of the subring defined by \(a \frac{df}{dy_p}(0) = \frac{dg}{dy_q}(0)\) for some choice of \(a \in k^*\).

Each possible contraction is realized for a different choice of log structure on \(C_0\). Let us see how.

Let \(S = \text{Spec } k[z, z^{-1}]\), \(C = C_0 \times S\), and let \(\pi : C \rightarrow S\) be the projection onto the second factor. Give \(S\) the log structure associated to the chart \(\mathbb{N}\delta \rightarrow k[z, z^{-1}]\) where \(\delta \mapsto 0\). Choose an étale neighborhood \(U_p\) of \(p\) in \(C_0\) so that there is an étale map \(U_p \rightarrow \text{Spec } k[x_p, y_p]/(x_py_p)\). We can pull this back to an étale neighborhood of \(p \times S\):

\[
C \leftarrow \text{ét} \quad U_p \times S \quad \text{ét} \quad \text{ét}
\]
\[
S \leftarrow \text{Spec } k[z, z^{-1}, x_p, y_p]/(x_py_p)
\]
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Given Spec $k[z, z^{-1}, x_p, y_p]/(x_py_p)$ the log structure associated to the chart

$$\begin{align*}
\lambda_{\alpha_p} \oplus \lambda_{\beta_p} \oplus \lambda_{\delta}/(\alpha_p + \beta_p \sim \delta) & \longrightarrow k[z, z^{-1}, x_p, y_p]/(x_p, y_p) \\
\alpha_p & \mapsto x_p \\
\beta_p & \mapsto y_p \\
\delta & \mapsto 0,
\end{align*}$$

then give $U_p \times S$ the pulled back log structure. (We are taking advantage of the isomorphism

$$\begin{align*}
\lambda_{\alpha} \oplus \lambda_{\beta} \oplus P/(\alpha + \beta \sim \delta) & \longrightarrow \{(p, q) \in P \oplus P \mid p - q \in \mathbb{Z}\delta\} \\
(n, m, p) & \mapsto (p + n\delta, p + m\delta)
\end{align*}$$

to specify the chart.)

We may repeat the process for $q$. This time, we give $U_q \times S$ the log structure pulled back from the chart (note the image of $\beta_q$)

$$\begin{align*}
\lambda_{\alpha_q} \oplus \lambda_{\beta_q} \oplus \lambda_{\delta}/(\alpha_q + \beta_q \sim \delta) & \longrightarrow k[z, z^{-1}, x_q, y_q]/(x_q, y_q) \\
\alpha_q & \mapsto x_q \\
\beta_q & \mapsto y_q \\
\delta & \mapsto 0.
\end{align*}$$

Given $C - \{(p, q) \times S\}$ the log structure pulled back from $S$. These local descriptions glue together in the evident way to give $C$ a log structure. Letting $E$ be the mesa, $C \to S$ gains the structure of a mesa curve.

For $a \in k^*$, let $C_a$ be the fiber of $C$ over $z = a$. $C_a$ is isomorphic to $C_0$ as a scheme in a canonical way. The piecewise linear function associated to $\overline{x}$ is constant with respect to $a$, and the sheaf $\mathcal{O}_{C_a}(-\overline{x})$ has the same isomorphism class in all fibers. What varies is the map $\mathcal{O}_{C_a}(-\overline{x}) \to \mathcal{O}_{C_a}$.

To see this, we write the sections of $\mathcal{O}_{C_a}(-\overline{x})$ with respect to the the cover $U_p, U_q, E_a - \{(p, q)\}, C_a - E_a$ on which $\overline{x}$ lifts to $\beta_p, \beta_q, \delta$, and 0 respectively. Let $U$ be the complement of suitably chosen points of $C_a$, as usual. Note that the chosen lifts of $\overline{x}$ all coincide on overlaps, so $\Gamma(U, \mathcal{O}_{C_a}(-\overline{x}))$ is equal to

$$\left\{(h_p, h_q, h_E, h_{C-E}) \in (\mathcal{O}_{U_p} \times \mathcal{O}_{U_q} \times \mathcal{O}_{E_a \setminus \{p,q\}} \times \mathcal{O}_{C_a \setminus E_a})(U) \mid h_i = h_j \text{ for all } i, j\right\},$$

where the equation on the right is to be interpreted as equality of each pair of functions on the intersection of their domains.

Since there is no twisting, $h_p, h_q, h_E$ are constant on $E_a$. In particular, $f_p(p) = f_q(q)$. The map to $\Gamma(U, \mathcal{O}_{C})$ is induced by multiplying $h_p, h_q, h_E, h_{C-E}$ by $y_p, zy_q, 0, 1$, respectively. Then $\mathcal{O}_{\overline{C_a}}(U)$ is the subring of

$$\{(f(y_p), g(y_q)) \mid f(0) = g(0)\}$$

determined by $a \frac{df}{dy_p}(0) = ah_p(p) = ah_q(q) = \frac{dg}{dq}(0)$.

9. Properness

The goal of this section is to show that $\psi : \overline{C} \to S$ is proper. We do this in two steps: first we show that $\psi : \overline{C} \to S$ is separated. It then follows that $\tau : C \to \overline{C}$ is proper. Then we show that $\tau$ has $\overline{C}$ as its scheme-theoretic image. Since the image of a proper map is proper, we get that $\psi : \overline{C} \to S$ is proper as desired.
Proposition 9.1. Then $\psi : \overline{C} \to S$ is separated.

Proof. Choose an affine open cover $\{V_i\}_{i \in I}$ of $C - E$. We will identify the $V_i$s with their isomorphic images in $\tau(C - E) = \overline{C} - \tau(E)$. Then $\mathcal{U} = \{\overline{U}\} \cup \{V_i\}_{i \in I}$ is an affine open cover of $\overline{C}$. It suffices to show that for each $V, W \in \mathcal{U}$, $V \cap W \to V \times_S W$ is a closed immersion. When $V = W = \overline{U}$, this is clear since affine maps are separated. If $V = V_i$ and $W = V_j$, we have that $V \cap W \to V \times_S W$ is a closed immersion, since $\pi : C \to S$ is separated.

It remains to check that each map $\overline{U} \cap V_i \to \overline{U} \times_S V_i$ is a closed immersion for each $i$. Note that since $V_i$ is disjoint from $E$, $(U - E) \cap V_i = U \cap V_i$ and $\tau : U \cap V_i \to \overline{U} \cap V_i$ is an isomorphism. Consider the diagram

\[
\begin{array}{ccc}
(U - E) \cap V_i & \longrightarrow & (U - E) \times_S V_i \\
\downarrow & & \downarrow \\
U \cap V_i & \longrightarrow & U \times_S V_i \\
\downarrow & & \downarrow \\
\overline{U} \cap V_i & \longrightarrow & \overline{U} \times_S V_i
\end{array}
\]

It is now enough to show that the image of $(U - E) \cap V_i$ in $\overline{U} \times_S V_i$ is closed. We know that the image of $(U - E) \cap V_i \to (U - E) \times_S V_i$ is closed, since $\pi : C \to S$ is separated. Since $(U - E) \times V_i \to \overline{U} \times V_i$ is an open immersion and $E \to \overline{U}$ surjects onto the complement of $U - E$ in $\overline{U}$, it suffices to show that the closure of the image of $(U - E) \cap V_i \to \overline{U} \times V_i$ does not intersect $\tau(E) \times V_i$.

Let $I$ be the ideal of $\tau(E) \times V_i$ in $\overline{U} \times V_i$ and let $J$ be the ideal of the scheme theoretic image of $(U - E) \cap V_i \to \overline{U} \times V_i$. Now, the closure of the image of $(U - E) \cap V_i \to \overline{U} \times V_i$ has empty intersection with $\tau(E) \times V_i$ if and only if $I + J$ is the unit ideal of $\Gamma(\overline{U} \times V_i, \mathcal{O}_{\overline{U} \times V_i})$. This holds if and only if there exist elements $g_j$ of the ideal of $\tau(E) \times V_i$ in $\overline{U} \times V_i$ and a cover $\{U_j\}$ of $U - E \cap V_i$ so that $g_j$ restricts to a unit on $U_j$ for each $j$.

Since the image of $E$ agrees with the scheme theoretic image of $E$ in $\overline{U}$, which has ideal $I_{\tau(E)}$, $\overline{U} - \tau(E) \cong U - E$ has a cover by the open sets $D(\overline{\lambda}(f))$ for $\phi(f,0) \in I_{\tau(E)}$. Pulling back these functions to $\overline{U} \times V_i$ gives the result. \hfill $\square$

Proposition 9.2. The morphism $\tau : C \to \overline{C}$ has $\overline{C}$ as its scheme theoretic image.

Proof. Since $\tau : C - E \to \overline{C} - \tau(E)$ is an isomorphism, this holds away from $E$. Therefore it suffices to check that $\tau : U \to \overline{U}$ has $\overline{U}$ as its scheme-theoretic image, which is equivalent in turn to showing that $\overline{B}(U) \to \Gamma(U, \mathcal{O}_U)$ is injective.

To that end, suppose $\phi(f, c) \in \overline{B}(U)$ and $\overline{\lambda}(f) + \pi^*c = 0$. We want to show that $(f, c) = ((\rho - \overline{\lambda})(d), -\rho(d))$ for some $d \in \Gamma(S, \mathcal{O}_S(-\rho))$.

Working étale locally on $S$, we may find a section $\sigma : S \to C$ so that the image of $\sigma$ is contained in the locus where $\overline{\lambda} = \rho$. Set $d = \sigma^*(f) \in \Gamma(S, \mathcal{O}_S(-\rho))$. Our goal is now to show that $(f, c) - ((\rho - \overline{\lambda})(\sigma^*(f)), -\rho(\sigma^*(f)))$ is zero. Consider the second coordinate first:

$$c + \rho(\sigma^*(f)) = \sigma^*(\pi^*c + \overline{\lambda}(f)) = 0.$$  

It remains to show that $f - (\rho - \overline{\lambda})(\sigma^*(f)) = 0$. Note that on the complement of $E$ in $U$, $\rho - \overline{\lambda} = \rho$ and $\rho(\sigma^*f) = -\pi^*c = \overline{\lambda}(f) = f$. It follows that the restriction of $f - (\rho - \overline{\lambda})(\sigma^*(f))$
to $U - E$ is zero, as desired. We also have that the difference is zero on the image of $\sigma$:

$$\sigma^*(f - (\rho - \lambda)(\sigma^*(f))) = \sigma^*(f) - \sigma^*(f) = 0.$$ 

Our strategy is now to localize enough that we can use these two properties to conclude that $f - (\rho - \lambda)(\sigma^*(f))$ is zero.

First, since $\Gamma(U, \mathcal{O}_C(-\lambda)) = \text{colim}_{n \in \mathbb{N}} \Gamma(C, \mathcal{O}_C(-\lambda + n\Sigma))$, where $\Sigma$ is the union of the divisors of the sections $\sigma_1, \ldots, \sigma_h$, there is some $n$ so that $f - (\rho - \lambda)(\sigma^*(f))$ admits a lift $g$ to $\Gamma(C, \mathcal{O}_C(-\lambda + n\Sigma))$. Since $\Sigma$ is a Cartier divisor, this lift is unique, so it is equivalent to show that $g$ is zero. In addition, as in the proof of Proposition 6.4, we may choose $n$ so that $H^1(C, \mathcal{O}_C(-\lambda + n\Sigma)) = 0$. In particular, by Theorem 6.2, taking global sections of $\mathcal{O}_C(-\lambda + n\Sigma)$ commutes with arbitrary base change in $S$.

We now proceed by contradiction. Suppose that $g$ is nonzero. Then there is some point $s$ of $S$ so that the Zariski stalk of $g$ at $s$ is nonzero. Denote by $A_s$ the Zariski stalk of $A$ at $s$, by $\hat{A}$ the completion of $A_s$ at $m_s$, and let $C_s$ and $\hat{C}$ denote the respective pullbacks of $C$ to these rings. (Note that $C_s$ is not the fiber of $C$ over $s$, contrary to our earlier usage.) Since $\Gamma(C_s, \mathcal{O}_{C_s}(-\lambda + n\Sigma))$ is finitely generated and $A_s$ is Noetherian, [1, Corollary 10.19] implies that the map

$$\Gamma(C_s, \mathcal{O}_{C_s}(-\lambda + n\Sigma)) \to \Gamma(C_s, \mathcal{O}_{C_s}(-\lambda + n\Sigma)) \otimes_{A_s} \hat{A} \cong \Gamma(\hat{C}, \mathcal{O}_{\hat{C}}(-\lambda + n\Sigma))$$

is injective: if $g_s$ is nonzero, its restriction to $\hat{A}$ must also be nonzero. This implies in turn that there is some integer $k$ so that the restriction of $g_s$ to $A_s/m_s^k$ is nonzero.

Now, $A_s/m_s^k$ is local Artinian, so admits a finite composition series $0 = M^{(r)} \subset M^{(r-1)} \subset \cdots \subset M^{(0)} = A_s/m_s^k$ with filtration quotients isomorphic to $k^s = A_s/m_s$. By flatness of $C$, we have an exact sequence

$$0 \to M^{(i+1)} \mathcal{O}_C(-\lambda + n\Sigma) \to M^{(i)} \mathcal{O}_C(-\lambda + n\Sigma) \to \mathcal{O}_{C \times S k^s}(-\lambda + n\Sigma) \to 0$$

for each $i$. Since $\mathcal{O}_C(-\lambda + n\Sigma)$ is flat, commutes with base change, and has no $H^1$, we have an induced sequence

$$0 \to M^{(i+1)} \Gamma(\hat{C}, \mathcal{O}_{\hat{C}}(-\lambda + n\Sigma)) \to M^{(i)} \Gamma(\hat{C}, \mathcal{O}_{\hat{C}}(-\lambda + n\Sigma)) \to \Gamma(C \times_S k, \mathcal{O}_{C \times S k}(-\lambda + n\Sigma)) \to 0$$

Since $g$ is nonzero, there is a first $i$ so that $g \notin M^{(i+1)} \Gamma(\hat{C}, \mathcal{O}_{\hat{C}}(-\lambda + n\Sigma))$. Consider the image of $g$ in $\Gamma(C \times_S k, \mathcal{O}_{C \times S k}(-\lambda + n\Sigma))$. We may consider this as an element of $\Gamma(U \times_S \hat{k}^s, \mathcal{O}_{U \times_S \hat{k}^s}(-\lambda))$. Now the description of Proposition 8.1 applies. The component of $g$ in the $\Gamma(S, \mathcal{O}_S(-\rho))$ component must be zero since $g$ is zero on $\sigma$. The part of $g$ in $\Gamma(Z, \mathcal{O}_Z(-\lambda))$ must also be zero, since $g$ restricts to 0 on the complement of $E$ and $Z$ is reduced. This contradicts that $g$ is nonzero, and we have the result.

□

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