Symmetry issue in Galileons

Davood Momeni¹ and Ratbay Myrzakulov¹
¹Eurasian International Center for Theoretical Physics
and Department of General & Theoretical Physics,
Eurasian National University, Astana 010008, Kazakhstan.

The symmetry issue for Galileons has been studied. In particular we address scaling (conformal) and Noether symmetrized Galileons. We have been proven a series of theorems about the form of Noether conserved charge (current) for irregular (not quadratic) dynamical systems. Special attentions have been made on Galileons. We have been proven that for Galileons always is possible to find a way to "symmetrized" Galileo’s field.

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I. INTRODUCTION

Canonical scalar fields are now so popular in theoretical physics because of their simplicity and easy way to interpret. Naturally if we use of the Kaluza-Klein reduction on Einstein-Hilbert action, the reduced lower dimensional action is equal to a scalar theory which is coupled to an abelian gauge field. As an example, we can obtain Brans-Dicke model as a alternative theory for gravity. If we apply this reduction scheme on a more generalized model of gravity, higher-dimensional Lovelock gravity, we will meet more terms of scalar fields, which are now coupled to the gravity or to its second order invariants. Technically, as we know when the brane model of Dvali-Gabadadze-Porrati (DGP) [1] is decoupled, the resulted model is the scalar theory but with nonlinear terms of interaction [2]. But the idea of nonlinear scalar models are older than this new motivated idea and indeed, it was Horndeski who proposed the most general scalar field theory which its equation of motion (Euler-Lagrange (EL)) remains second order [3]. As a fully covariant extension of the original Horndeski models, recently the idea of Galileon was introduced as the scalar theory with Galileon symmetry [4]. This idea has been extended and developed through recent years to explain different aspects of gravitational theory from black hole physics to cosmology [5]-[21]. The models are written in such a way that they remain invariant under a local transformation of fields $\phi \rightarrow \phi + \partial_\mu b$, here $b$ is gauge field. A remarkable note about Galileon models is that they are represented the most generalized form of any other modified theory in the literature. The idea of Galileon is proposed in [4] and later it was extended to covariant form [21]. Other extensions have been followed [22].

In particular, the first two terms of Galileon model are very intereted to study. The terms which we construct by $(\nabla_\mu \phi)^2$ and $(\nabla_\mu \phi)^2 \nabla_\nu \nabla^\nu \phi$. Traditionally, we write their Lagrangian
densities as following forms \[7, 8\]:

\begin{align*}
  \mathcal{L}_2 &= k(\phi, X), \\
  \mathcal{L}_3 &= -G(\phi, X)\nabla_\mu \nabla^\mu \phi,
\end{align*}

Here \(k\) and \(G\) are arbitrary functions of field \(\phi\) and its kinetic part \(X := -\partial_\mu \phi \partial^\mu \phi / 2\). Other higher order terms can be constructed using different geometric quantities like \(R\) (the Ricci tensor), \(G_{\mu\nu}\) (the Einstein tensor), and higher derivatives of field. Furthermore, we know that \(G = X\), we obtain covariant Galileons [21]. In this paper, we consider a model of Galileon, which is presented by the following action

\[S_{\text{tot}} = \int \frac{R}{2} \sqrt{-g} d^4x + \sum_{i=2}^5 \int d^4x \sqrt{-g} \mathcal{L}_i,\]

Where different Lagrangian densities have been defined by the following:

\begin{align*}
  \mathcal{L}_2 &= G_2(\phi, X), \\
  \mathcal{L}_3 &= G_3(\phi, X)\nabla_\mu \nabla^\mu \phi \\
  \mathcal{L}_4 &= G_{4,X}(\phi, X) \left\{ \nabla_\mu \nabla^\mu \phi \right\}^2 - \nabla_\alpha \nabla_\beta \phi \nabla^\alpha \nabla^\beta \phi + RG_4(\phi, X), \\
  \mathcal{L}_5 &= G_{5,X}(\phi, X) \left\{ \nabla_\mu \nabla^\mu \phi \right\}^3 - 3 \nabla_\mu \nabla^\mu \phi \nabla_\alpha \nabla_\beta \phi \nabla^\alpha \nabla^\beta \phi + 2 \nabla_\alpha \nabla_\beta \phi \nabla^\alpha \nabla^\mu \phi \nabla_\rho \nabla^\beta \phi.
\end{align*}

Here \(G_{i,X}(\phi, X) \equiv \frac{\partial G_i(\phi, X)}{\partial X}\). \(R\) is the Ricci tensor, \(G_{\mu\nu}\) is the Einstein tensor, also we set \(\kappa^2 = 8\pi G = 1, c = 1\). Our aim here is to address symmetry issue for Galileons. We have been investigated conformally invariant Galileons. Also we have been studied all possible Noether symmetries of such models in the cosmological FLRW model. Our plan in this work is as the following:

In Sec. [II] we’ll prove a theorem about conformal invariant Galileons. In Sec. [III] we review the fundamental theory of Noether symmetry for regular dynamical systems. In Sec. [IV] Noether symmetry is considered for k-inflation. In Sec. [V] we are considering higher order Galileons with Noether symmetries. we are considering higher order Galileons with Noether symmetries. In Sec. [VI] we have been proven a sequence of theorems about Noether symmetries for higher order derivatives models, including Galileons. We conclude in [VII].

II. CONFORMALLY INVARIANT GALILEONS

Conformal symmetry plays a significant role in quantum gravity [22]. In the absence of Lorentz symmetry and if two physical frames have different aspects, we need to find a way to know that they are invariant under conformal transformations or not. A type of conformal transformations is scaling symmetry in which the physical scalar degree of freedom of the system is re-scaled to \(\phi \to \lambda \phi\). We prove a theorem about conformal(scaling) symmetry of fifth order Galileon models:
Theorem: A type of Galileons in the following form:

\[
\mathcal{L} = F_2 \left( \frac{X}{\phi^2} \right) \nabla_\mu \nabla^\mu \phi + F_4 \left( \frac{X}{\phi^2} \right) \left[ \left( \nabla_\mu \nabla^\mu \phi \right)^2 - \nabla_\alpha \nabla_\beta \phi \nabla^\alpha \nabla^\beta \phi \right] + R F_4 \left( \frac{X}{\phi^2} \right) \]

is invariant under conformal (scaling) transformation.

**Proof:** It is straightforward to show that under scaling transformation, \( X \rightarrow \lambda^2 X \). The key point is to show that there exist solutions for \( G_i(\phi, X) \), \( i = 2, 4 \) such that:

\[
G_{2,4}(\phi, X) = G_{2,4}(\lambda \phi, \lambda^2 X),
\]

\[
\lambda G_{3,5}(\phi, X) = G_{3,5}(\phi, X).
\]

To solve (14,15) we derive w.r.t \( \lambda \), so we obtain:

\[
x \frac{\partial}{\partial x} G_{2,4}(x, y) + 2y \frac{\partial}{\partial y} G_{2,4}(x, y) = 0,
\]

\[
G_{3,5}(x, y) + x \frac{\partial}{\partial x} G_{3,5}(x, y) + 2y \frac{\partial}{\partial y} G_{3,5}(x, y) = 0.
\]

Here \( \{x, y\} \equiv \{\lambda \phi, \lambda^2 X\} \). By solving this couple of P.D.E, we obtain:

\[
G_{2,4}(\phi, X) = F_{2,4} \left( \frac{X}{\phi^2} \right)
\]

\[
G_{3,5}(\phi, X) = \frac{F_{3,5} \left( \frac{X}{\phi^2} \right)}{\phi}.
\]

This is our Q.E.D.

Consequently, scaling symmetry provides a way to symmetrize Galileons.

### III. NOETHER SYMMETRY: A QUICK REVIEW

Let us consider a dynamical system with \( N \) configurational coordinates \( q_i \) is defined by the Lagrangian \( L \equiv L(q_i, \dot{q}_i; t), \ 1 \leq i \leq N \). The set of EL equations for this dynamical system is written as \( \dot{p}_i - \frac{\partial L}{\partial q_i} = 0, \ p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \). We mention here that up or down indices have the same meaning since we are worked on flat space. What we called it as *Noether Symmetry Approach* [27] is the existence of a vector, Noether vector \( \vec{X} \)

\[
X = \sum_{i=1}^{N} \alpha_i(q) \frac{\partial}{\partial q_i} + \dot{\alpha}_i(q) \frac{\partial}{\partial \dot{q}_i},
\]

and a set of non-singular functions \( \alpha_i(q) \), in a such way that the Lie derivative of Lagrangian vanishes on all points of the manifold (the tangent space of configurations \( TQ \equiv \{q_i, \dot{q}_i\} \)):

\[
L_X \mathcal{L} = 0
\]
The mentioned condition can be written in the following expanded form:

\[
L_X \mathcal{L} = X \mathcal{L} = \sum_{i=1}^{N} \alpha^i(q) \frac{\partial \mathcal{L}}{\partial \dot{q}^i} + \dot{\alpha}^i(q) \frac{\partial \mathcal{L}}{\partial \dot{q}^i}.
\]  

From the phase-space point of view, existence of \( \vec{X} \) implies that the total phase flux enclosed in a region of space, is conserved along \( X \). In fact, it looks straightforward to show that (by taking into account the EL equations):

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0, \quad 1 \leq i \leq N.
\]  

So, we have:

\[
\sum_{i=1}^{N} \frac{d}{dt} \left( \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = L_X \mathcal{L}.
\]  

If we can find \( \alpha_i \) by vanishing the coefficients of all poewers of \( \dot{q}^i \), then we will show that there exist a **global** conserved charge as the following:

\[
\Sigma_0 = \sum_{i=1}^{N} \alpha^i p_i
\]  

In other words, the existence of Noether symmetry implies that the **Lie derivative of the Lagrangian** on a given vector field \( \mathbf{X} \) vanishes, i.e.

\[
\mathcal{L}_X \mathcal{L} = 0.
\]  

It has been proven that Noether symmetry is a powerful tool to study cosmological models in different models \[28\]-\[42\]. In our article we explore Noether symmetries \[22\] for the minimal Galileon model, given by \[3\].

### IV. K-INFLATION MODELS THROUGH NOETHER SYMMETRY

Let us to firstly focus on the term \( \mathcal{L}_2 \), which is called k-inflation in the literature, with the following Lagrangian \[43\]:

\[
\mathcal{L}_2 = k(\phi, X).
\]  

In the FLRW background with the metric \( ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \), the model has the following effective energy density \( \rho \) and pressure \( p \):

\[
\rho = -T^0_0 = 2Xk_{,X} - k, \quad p = k.
\]  

The point-like Lagrangian reads:

\[
L = 3M_G^2 a \dot{a}^2 + a^3 \left( k(\phi, X) - \lambda(X - \frac{1}{2} \dot{\phi}^2) \right), \quad M_G^2 = \frac{M_{\text{pl}}}{\sqrt{8\pi}}.
\]  

Where we introduced Lagrange multiplier \( \lambda \). To fix it, we perform variation w.r.t \( X \), so we obtain:

\[
\lambda = k_{,X}.
\]
Thus, the reduced form of the point-like Lagrangian is written as follows:

\[ L(a, \phi, X; \dot{a}, \dot{\phi}) = 3M_G^2 a \dot{a}^2 + a^3 (k(\phi, X) - k_{,X}X + \frac{1}{2} k_{,X} \dot{\phi}^2). \] (27)

The set of coordinates for configuration space is \( q_i = \{a, \phi, X\} \). Noether symmetry vector field is \( \vec{X} = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \phi} + \gamma \frac{\partial}{\partial X} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{\phi}} \) where all functions \( \alpha^i = \{\alpha, \beta, \gamma\} \) are functions of \( q^i \). Note that \( p_X = \frac{\partial L}{\partial \dot{X}} = 0 \) consequently we have the following constraint \( k_{,X}X(X - \frac{1}{2} \dot{\phi}^2) = 0 \) where we assume that \( k_{,X} \neq 0 \) and we obtain our constraint \( X = \frac{1}{2} \dot{\phi}^2 \).

Noether system is obtained as the following PDE system:

\[ \alpha - 2a \alpha_a = 0, \] (28)
\[ \alpha k_{,X} + \beta k_{,X,\phi} = 0 \] (29)
\[ k_{,X} \beta_{,\phi} + \gamma k_{,X}X/2 = 0 \] (30)
\[ 6M_G^2 \alpha_{,\phi} + k_{,X}a^2 \beta_{,a} = 0 \] (31)
\[ \alpha_{,X} = 0, \] (32)
\[ k_{,X} \beta_{,X} = 0. \] (33)

Furthermore we have constraint \( k_{,a} = 0 \). The most general non-degenerate solution with finite speed of sound which is defined by

\[ c_s^2 = \frac{p_X}{\rho_X} = \frac{k_{,X}}{k_{,X} + 2Xk_{,X,X}}. \] (34)

are given by the following:

\[ k_1(\phi, X) = XF_1(\phi) + F_2(\phi), \quad \alpha = \beta = 0, \quad c_s^2 = 1. \] (35)

\[ k_2(\phi, X) = XF_3(\phi) + C_1, \quad \alpha = 0, \beta = \frac{C_2}{F_3(\phi)}, \quad c_s^2 = 1. \] (36)

In both cases \( \gamma = \text{arbitrary} \) is overdetermined. Conserved charges for (35,36) read:

\[ Q_1 = 0, \] (37)
\[ Q_2 = C_2a^3 \dot{\phi}. \] (38)

Using (38) we obtain \( \dot{\phi} = \frac{Q_2}{a^2} \), so \( X = \frac{Q_2^2}{4a^2} \). Effective equations of state \( w = \frac{k}{2Xk_{,X,X}} \) read:

\[ w_1 = \frac{XF_1(\phi) + F_2(\phi)}{XF_1(\phi) - F_2(\phi)}, \] (39)
\[ w_2 = \frac{XF_3(\phi) + C_1}{XF_3(\phi) - C_1}. \] (40)

To have inflation we should have \( 2XF_1(\phi) < -F_2(\phi) \) for model (35) and \( XF_3(\phi) < -C_1 \) for model (36). Because (36) is just a special case of (35) when \( F_2 = C_1 \), so we study the physical results of (35) for inflation. First of all we show that it has the Sitter solution.
This solution corresponds to $H = \text{constant}$, $\phi \equiv \phi_{dS}$ and $\dot{\phi}|_{dS} = \ddot{\phi}|_{dS} = 0$. The modified Klein-Gordon equation for $\phi$ reads:

$$(k_{,X} + 2k_{,XX}X)\ddot{\phi} + 3Hk_{,X}\dot{\phi} + (2Xk_{,X} - k_{,\phi}) = 0. \quad (41)$$

Which is obtained as continuity equation $\dot{\rho} + 3H(\rho + p) = 0$ for $\rho, p$ presented by (24). We perturb (41) in slow-roll scheme, $\phi = \phi_{dS} + \delta \phi$, $F'_{2}(\phi_{dS}) = 0$, $\ddot{\delta \phi} \ll 3H\dot{\delta \phi}$, we obtain:

$$3H\ddot{\delta \phi} - \frac{F''(\phi_{dS})}{F'(\phi_{dS})}\delta \phi = 0. \quad (42)$$

The solution is in the form of $\delta \phi = (\delta \phi)_0 e^{pt}$ where $p = \frac{F''(\phi_{dS})}{3HF'(\phi_{dS})}$ which is quasi-stable if $\frac{F''(\phi_{dS})}{F'(\phi_{dS})} > 0$ which is generally valid for model given by (35) and for (36) we should have $F(\phi_{dS}) < 0$. So inflation can be terminated with such model of k-inflation which we obtained by Noether symmetry. To see why it was happened we mention here that the models (35,36) can be reduced to a canonical scalar models with a redefinition of the field $\psi = \pm \int \sqrt{F_{1}(\phi_{dS})}d\phi$ to the following forms:

$$\mathcal{L} = -\frac{1}{2} \partial_{\mu} \psi \partial^{\mu} \phi - U(\phi), \quad U_{1,2}(\phi) = -\{F_{2}(\phi), C_{1}\}. \quad (43)$$

The only possible model of $\mathcal{L}_{2}$ with Noether symmetry (Noether symmetrized) is the canonical scalar field with self-interaction.

**V. NOETHER SYMMETRY FOR THIRD ORDER MODEL**

To have a more comprehensive result, let us to consider the following extension of k-inflation which was proposed as minimal G-inflation [25]:

$$\mathcal{L}_{3} = k(\phi, X) - G(\phi, X)Y. \quad (44)$$

Where we denote by $Y = \nabla_{\mu}\nabla^{\mu}\phi$. There is no simple way to reduce this Lagrangian to a simpler quadratic form, due to the highly nonlinear term term $Y$. To resolve this problem, we propose a couple of Lagrange multipliers $\{\lambda, \mu\}$ as the following:

$$L = 3a\dot{a}^{2} + a^{3}\left[k(\phi, X) - G(\phi, X)Y\right] - a^{3}\left(\lambda(X - \frac{1}{2}\dot{\phi}^{2}) + \mu(Y - \nabla_{\mu}\nabla^{\mu}\phi)\right). \quad (45)$$

By varying the Lagrangian $L$ w.r.t to the $\{X, Y\}$ we obtain $\lambda = k_{,X} - YG_{,X}, \mu = -G$, so the reduced Lagrangian is written as the following:

$$L(a, \phi, X, Y; \dot{a}, \dot{\phi}) = 3a\dot{a}^{2} + a^{3}\left[k(\phi, X) - G(\phi, X)Y\right] - a^{3}(X - \frac{1}{2}\dot{\phi}^{2})(k_{,X} - YG_{,X}). \quad (46)$$

The appropriate set of coordinates for configuration space is $q^{i} = \{a, \phi, X, Y\}$. We define a vector field $\vec{X} = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \phi} + \gamma \frac{\partial}{\partial X} + \theta \frac{\partial}{\partial Y} + \dot{\alpha} \frac{\partial}{\partial a} + \dot{\beta} \frac{\partial}{\partial \phi}$, here functions $\alpha^{i} = \{\alpha, \beta, \gamma, \theta\}$ are...
defined on configuration space, so we have the following system of PDEs as a result of (22) for above point-like Lagrangian:

\[
\frac{\partial G}{\partial a} = \frac{\partial G}{\partial Y} = 0, \\
\frac{\partial k}{\partial a} = \frac{\partial k}{\partial Y} = 0,
\]

\[3\alpha a^{-1}(k - X(k, X - YG, X)) - YG) + \beta(k, \phi - X(k, X - YG, X) - YG, \phi) + \gamma(-X(k, X - YG, X) + YG, X) - \theta(XG, X + G) = 0 \tag{47}\]

\[
\alpha + 2a\alpha_a = 0 \\
3\alpha a^{-1}(k, X - YG, X) + \beta(k, X - YG, X) + \gamma(k, X - YG, X) \\
-\theta G, X + 2\beta, \phi(k, X - YG, X) = 0 \\
6\alpha, \phi + a^2\beta a(k, X - YG, X) = 0 \\
\alpha, X = \alpha, Y = 0 \\
\beta, X(k, X - YG, X) = 0 \\
\beta, Y(k, X - YG, X) = 0. \tag{50}\]

We know that \( p_a = 6a\dot{a}, \ p_\phi = a^3\dot{\phi}(k, X - YG, X) \) and the corressponding Noether charge is written as the following:

\[\Sigma = 6\alpha a\dot{a} + \beta a^3\dot{\phi}(k, X - YG, X) = \Sigma_0. \tag{56}\]

The system of PDEs has three major class of exact solutions.

**Class A**: The system has the following exact solutions if we impose \( k, X - YG, X = 0, \ G, X \neq 0 \):

\[
\alpha = \frac{\alpha_0}{\sqrt{a}}, \quad \beta = \frac{\alpha_0 Y \beta_0 (ch(\phi) - 3f(\phi))}{a\sqrt{a} f'(\phi)}, \quad \gamma = -\frac{\alpha_0 ch(\phi)}{a\sqrt{a} G, X}, \\
\theta = 0. \tag{57, 58, 59}\]

Where \( \{h(\phi), f(\phi)\} \) are arbitrary functions of \( \phi \). The associated conserved Noether charge is \( \Sigma_A = 6\alpha_0\sqrt{a}\dot{a} \) from here we find \( a(t) = \left[\frac{a_0^{3/2} + \Sigma A}{4\alpha_0} (t - t_0)\right]^{2/3} \). So we conclude here that:

**The third order action of G-inflation presented by (44) has Noether symmetry vector field**:

\[\dot{X} = \frac{\alpha_0}{\sqrt{a}} \frac{\partial}{\partial a} + Y \frac{\alpha_0}{a\sqrt{a}} \frac{\beta_0 (ch(\phi) - 3f(\phi))}{f'(\phi)} \frac{\partial}{\partial \phi} - \frac{\alpha_0 ch(\phi)}{a\sqrt{a} G, X} \frac{\partial}{\partial X} + \]

\[\frac{d}{dt} \left[ \frac{\alpha_0}{\sqrt{a}} \frac{\partial}{\partial a} \right] + \frac{d}{dt} \left[ Y \frac{\alpha_0}{a\sqrt{a}} \frac{\beta_0 (ch(\phi) - 3f(\phi))}{f'(\phi)} \frac{\partial}{\partial \phi} \right]. \tag{60}\]

So, the action is in the following form:

\[S = \int \sqrt{-g} d^4x \left( \frac{R}{2} + f(\phi) \nabla \mu \nabla \nu \phi \right) = \int \sqrt{-g} d^4x \left( \frac{R}{2} + 2X f'(\phi) \right). \tag{61}\]
It is important to mention here that the above Noether symmetrized model is written in the following equivalent form:

\[
S = \int \sqrt{-g} d^4x \left( \frac{R}{2} - \frac{1}{2} \nabla_\mu \psi \nabla^\mu \psi \right) \tag{62}
\]

Where \( \psi = \pm \int \sqrt{f} d\phi \). Equation of motion of a scalar field is obtained:

\[
\ddot{\psi} + 3H \dot{\psi} = 0 \tag{63}
\]

Which can be solved by \( \psi = \psi_0 \left[ \frac{a}{a_0} + \sum A_\alpha \left( t - t_0 \right) \right]^2 \). So, the Noether symmetrize \( \mathcal{L}_3 \) is completely integrable.

**Class B:** Now, we suppose that \( k, X = 0, G, X = 0 \). Consequently we have

\[
\alpha = \alpha(a, \phi), \quad \beta = \beta(a, \phi), \quad G = G(\phi), \quad k = k(\phi). \tag{64}
\]

Exact solution for PDEs are given by:

\[
\alpha = \alpha(a) = \frac{\alpha_0}{\sqrt{a}} \tag{65}
\]

\[
\beta = \beta(a) = \frac{\alpha(a)}{a} \tag{66}
\]

\[
\theta = \theta_0 g(\phi) \left[ \frac{\alpha(a)}{a} \right] \tag{67}
\]

Where

\[
k(\phi) = k_0 e^{-3\phi/\beta_0}, \tag{68}
\]

\[
G(\phi) = C e^{-3\phi/\beta_0} + \frac{\theta_0}{\beta_0} e^{-3\phi/\beta_0} \int d\phi g(\phi) e^{3\phi/\beta_0}. \tag{69}
\]

Noether vector is:

\[
\vec{X} = \frac{\alpha_0}{\sqrt{a}} \frac{\partial}{\partial a} a + \frac{\beta_0}{a} \frac{\partial}{\partial \phi} a + \theta_0 g(\phi) Y \frac{\alpha(a)}{a} \frac{\partial}{\partial Y} + \frac{d}{dt} \left[ \frac{\alpha_0}{\sqrt{a}} \right] \frac{\partial}{\partial a} + \frac{d}{dt} \left[ \frac{\beta_0}{\alpha(a)} \right] \frac{\partial}{\partial \phi} \tag{70}
\]

So, the following third order Galileon Lagrangian has Noether symmetry:

\[
\mathcal{L} = k(\phi) - G(\phi) \nabla_\mu \nabla^\mu \phi. \tag{71}
\]

or equivalently:

\[
S = -\frac{1}{2} \int \sqrt{-g} \left( XG, \phi - \frac{1}{2} k(\phi) \right) \tag{72}
\]

This form is the k-inflation model in the standard canonical form.

**Case C:** There is another interesting class of solutions when we put \( \gamma = \theta = 0, k X - Y G, X = \Psi, X \). In this case, we have the following solutions:

A: \( \alpha = \beta = 0, \Psi = F_1(a, \phi) \) This class of solutions is fully potential driven models in which the scalar field has no kinetic term. So, physically we are not able to use it in our descriptions.
**B:** \( \alpha = 0, \beta = F_2(a, \phi), \Psi = F_3(a) \) This model corresponds to Einstein-Hilbert action plus a type of fluid with pressure \( p \equiv \Psi \). Such perfect fluid models are completely integrable.

**C:** \( \alpha = \beta = 0, \Psi = \Psi(a, \phi, X) \) This solution corresponds to a generalized k-inflationary model in which the scalar field is coupled non minimally to the background through the function \( = \Psi(a, \phi, X) \).

**D:** \( \alpha = 0, \beta = c_1, \Psi = F_4(a, X) \) The model is a purely kinetic k-inflation which is coupled non minimally to the background.

**E:** \( \alpha = 0, \beta = F_5(\phi), \Psi = X F_6(a) + F_7(a) \) This case is similar to B, since we have a type of fluid in the cosmological background. It is completely integrable.

**F:** An interesting case has happened when we consider the following solutions:

\[
\alpha = \frac{F_9(\phi)}{\sqrt{a}}, \beta = \frac{C_1 c_0 - 3 \int F_9^2(\phi) d\phi}{a^{3/2} F_{10}(\phi)}, \quad \Psi = \frac{-2 X F_9(\phi)^2}{a (C_1 c_0 - 3 \int F_9^2(\phi) d\phi)} - F_{10}(\phi) \left( C_1 c_0 - 3 \int F_9^2(\phi) d\phi \right)
\]  

Among all these solutions, the last two ones are very interesting. Solution numbs E, is a purely kinetic model in the following form:

\[
S = \int \sqrt{-g} d^4x (X \frac{F_5(a)}{F_7(a)^2} + F_8(a))
\]

It describes a fluid coupled with a measles scalar field or equivalently a fluid with self interaction term \( U = U(a(\phi)) \).

But solution given by F, is a type of k-inflation, and it can be transformed into a canonical scalar model with the self-interaction term if \( \phi \rightarrow \dot{\phi} \).

**VI. NOETHER SYMMETRY FOR HIGHER DERIVATIVE LAGRANGIANS**

As we observed in previous sections, modified gravities always have nonlinearity, due to the higher order derivatives of the fields. Historically Ostrogradski [44] was the first one who studied canonical formalism (Hamiltonian formalism) for a class of models with higher order derivatives (see [45] for a comprehensive review). In the case of Galileon, even if we work at the level of minimal models, with \( L_3 \), there is no simple way to reduce point-like Lagrangian to the standard quadratic form \( L = L(\phi, X^m) \). A way is to introduce a set of appropriate Lagrange multipliers. But in this section we introduce an alternative to work with higher derivative Lagrangians. The simplest case which we are interested particularly, is a class of Lagrangian functions \( L(q_i, \dot{q}_i, \partial_t^n[q_i]), \quad n \leq 2 \). It is possible to extend it to \( n \leq 3 \), but such cases have not simple physical interpretations. We want to see how we can find generalized Noether symmetry for a Lagrangian which it contains second order time derivatives , namely \( L = L(q_i, \dot{q}_i, \ddot{q}_i), \quad 1 \leq i \leq N \). We are interested in the cases in which an integration part-by-part can not reduce the Lagrangian to the standard quadratic form \( L' = L'(q_i, \dot{q}_i) \). The first simple example is point-like Lagrangian of standard GR, which it contains \( \ddot{a} \) due to the curvature term \( R \sim (\dddot{a} + \frac{a^2}{\dot{a}^2}) \) but we fortunately can pass from it by integration. But for G-inflation models if you pass to the higher terms, we need a
generalized Noether symmetry. This is one of the most important motivations for us to write this section.

We define a couple of conjugate momentum:

\[ p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad r_i = \frac{\partial L}{\partial \ddot{q}_i}. \] (76)

The generalized EL equation is given by:

\[ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \ddot{q}_i} \right) = 0. \] (77)

Or equivalently

\[ \frac{\partial L}{\partial q_i} - \dot{p}_i + \ddot{r}_i = 0. \] (78)

We define a vector field:

\[ \vec{X} = \sum_{i=1}^{N} \left( \alpha_i \frac{\partial}{\partial q_i} + \dot{\alpha}_i \left[ \frac{\partial}{\partial \dot{q}_i} - 2 \frac{d}{dt} \left( \frac{\partial}{\partial \ddot{q}_i} \right) \right] - \ddot{\alpha}_i \frac{\partial L}{\partial \dddot{q}_i} \right). \] (79)

We call it as generalized Noether symmetry if and only if it satisfies:

\[ L_{\vec{X}}L = 0, \] (80)

In this case you find the following polynomial to be vanishing:

\[ \sum_{i=1}^{N} \alpha_i \frac{\partial L}{\partial q_i} + \sum_{j=1}^{N} \sum_{j=1}^{N} (\alpha_i)_{i,j} \dot{q}_j \left[ \frac{\partial L}{\partial \dot{q}_i} - 2 \frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{q}_i} \right) \right] - \ddot{\alpha}_i \frac{\partial L}{\partial \dddot{q}_i} = 0. \] (81)

If we collect terms of different powers of \( \{\dot{q}_i, \ddot{q}_j\} \) we obtain a system of 2nd order PDEs for \( \{a_i(q^k)\} \). Then the associated Noether charge is obtained by the following theorem:

**Theorem:** For Lagrangian \( L = L(q_i, \dot{q}_i, \ddot{q}_i) \), there exists a Noether vector symmetry given by (79) and a Noether conserved charge:

\[ K = \sum_{i=1}^{N} (p_i, r_i). \] (82)

**Proof:** Using (78) it is easy to show that \( K = 0 \), because we’ve:

\[ \begin{align*}
K &= \sum_{i=1}^{N} (\dot{\alpha}_i p_i - \dot{\alpha}_i p_i) \\
&= \sum_{i=1}^{N} (\alpha_i \dot{p}_i + \dot{\alpha}_i p_i - (\alpha_i \ddot{r}_i + 2\dot{\alpha}_i \dot{r}_i + \ddot{\alpha}_i r_i)) \\
&= \sum_{i=1}^{N} \left( \alpha_i \frac{\partial L}{\partial q_i} + \dot{\alpha}_i \left[ \frac{\partial L}{\partial \dot{q}_i} - 2 \frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{q}_i} \right) \right] - \ddot{\alpha}_i \frac{\partial L}{\partial \dddot{q}_i} \right) \\
&= L_{\vec{X}}L = 0.
\end{align*} \] (83)

This is our Q.E.D.

It is adequate to present the following generalized theorem for the dynamical system in the form \( L = L(q_i, \dot{q}_i^{(a)}; t), \quad q_i^{(a)} = (\partial t)^a[q_i], \quad 1 \leq i \leq N, \quad 1 \leq a \leq s, \quad s \leq N \). We are
remembering to the mind that in this class of models, the generalized EL equation is written as the following:

\[
\frac{\partial L}{\partial q_i} + \Sigma_{a=1}^s (-1)^a (\partial t)^a (p_i^a) = 0,
\]

(84)

Here \(p_i^a \equiv \frac{\partial L}{\partial \dot{q}_i^a}\) is the new set of conjugate momentum.

**Theorem:** For Lagrangian \(L = L(q_i, q_i^{(a)}; t)\), there exists a generalized conserved Noether current given by:

\[
K = \Sigma_{i=1}^N \Sigma_{a=1}^s (-1)^{a+1} (\partial t)^{a-1} \left[ \alpha_i p_i^a \right].
\]

(85)

**Proof:** Using the Leibniz rule for derivatives we’ve:

\[
\partial_t (K) = \Sigma_{i=1}^N \Sigma_{a=1}^s (-1)^{a+1} (\partial t)^a \left[ \alpha_i p_i^a \right]
\]

(86)

\[
\Rightarrow \Sigma_{i=1}^N \Sigma_{a=1}^s \Sigma_{k=1}^a \left( (-1)^{a+1} \frac{a!}{k!(a-k)!} (\partial t)^k [\alpha_i] (\partial t)^{a-k} \frac{\partial}{\partial q_i^{(a)}} \right) L
\]

\[
\Rightarrow L \bar{X} L = 0
\]

Where

\[
\bar{X} = \Sigma_{i=1}^N \Sigma_{a=1}^s \Sigma_{k=1}^a \left( (-1)^{a+1} \frac{a!}{k!(a-k)!} (\partial t)^k [\alpha_i] (\partial t)^{a-k} \frac{\partial}{\partial q_i^{(a)}} \right)
\]

(87)

is the Noether vector symmetry.

**Corollary:** For a general Galileon model with the following Lagrangian

\[
L = L(\phi, (\nabla)^a \phi), 1 \leq a < s.
\]

(88)

The following vector is conserved:

\[
K^\mu = \Sigma_{a=1}^s (-1)^{a+1} (\nabla)^{a-1} \left[ \alpha(\phi) \frac{\partial L}{\partial (\nabla)^a \phi} \right].
\]

(89)

i.e. \(\nabla^\mu K^\mu = 0\). In the above theorem if we use the conventional Galileon’s notations we should identify \(\nabla \equiv \nabla_\mu\), \(X \sim (\nabla_\mu \phi)^2\), \(Y \sim (\nabla_\mu)^2 \phi = \nabla_\mu \nabla_\nu \phi \) and etc.

**Illustrative example:** If we consider the minimal model of Galileon theory, \(\mathcal{L}_2 + \mathcal{L}_3 = k(\phi, X) - G(\phi, X) \nabla_\mu \nabla^\mu \phi\), we find:

\[
K^\mu = \nabla^\mu \left[ \alpha(\phi) G(\phi, X) \right] - \alpha(\phi) (k_{,X} G_{,X} \nabla_\mu \nabla^\mu \phi) \nabla^\mu \phi.
\]

(90)

Which is trivially conserved if we fix \(\alpha\) by Noether conservation vector condition \([22]\).
VII. CONCLUSIONS

Symmetry issue has been investigated for Galileons. In particular cases, we have been symmetrized Galileons under scaling (conformal) symmetry and Noether symmetry. We have proven theorem in which we have been shown that Galileons can be scaled (conformal) symmetrized if it has been made in a very special case. Furthermore, we have been studying Noether symmetries of Point-like Lagrangians for second and third orders Galileons. For second order, k-inflation models, we have been proven that the particular form of \( k(\phi, X) = XF(\phi) + G(\phi) \) has been Noether symmetrized. This result has been used in inflation scenario. This particular form has been reduced to single inflation model in Einstein frame. For third order Galileons, there are several Noether symmetrized models. Single minimally coupled scalar field and the single field coupled to a specific fluid have been obtained. Since Galileons contain higher order (second order) derivatives of a scalar field, we need to extend Noether symmetry to the case in which the Lagrangian is "irregular". We have been presented and proven theorem about the Noether conserved charge (current) of higher derivatives Lagrangians. We extended the idea of the Lie generator of the normal tangent space. We have been proven that a vector field:

\[
X^a = \sum_{s=1}^N \sum_{a=1}^s \sum_{k=1}^a \left( (-1)^{a+1} \frac{a!}{k! (a-k)!} (\partial t)^k (\partial t)^{a-k} \left[ \frac{\partial}{\partial q_i^{(a)}} \right] \right)
\]

is the Noether vector symmetry for \( L = L(q, q^{(a)}_t, t) \). For a general Galileon model \( L_2 + L_3 \), we have been proven that there exists conserved current.

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