A N-body problem with weak force potential through Hamilton-Jacobi equation approach

Putian Yang\textsuperscript{a} and Shiqing Zhang\textsuperscript{a}

\textsuperscript{a}Department of mathematics, Sichuan University

May 13, 2022

Abstract

This paper we consider for the $N$-body problem with potential $1/r^{\alpha}$ ($0 < \alpha < 1$) the existence of hyperbolic motions for any prescribed limit shape and any given initial configuration of the bodies. Here $E$ is the Euclidean space where the bodies moving and $\| \cdot \|_E$ is the norm induced by the inner product. The energy level $h > 0$ of the motion can also be chosen arbitrarily. We use the global viscosity solutions for the Hamilton-Jacobi equation $H(x, d_x u) = h$ and geodesics.

Keywords: Hamilton-Jacobi equations, weak force potential, N-body problem, geodesics

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\textsuperscript{*}Corresponding author: yangputian@stu.scu.edu.cn
1 Introduction

Before we describe the background and main issue of this paper, we firstly present some of notations about our working space \( E = \mathbb{R}^n, (n \geq 2) \). \( \langle \cdot, \cdot \rangle \) is the standard inner product of \( E \), \( |\cdot| \) is its induced norm. We also write the inner product of configuration space \( E^N \) as

\[
\langle x, y \rangle = \left( \frac{1}{2} \sum_{i=1}^{N} \sqrt{m_i m'_i \langle x_i, y_i \rangle} \right)^{1/2}
\]

for \( x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N) \in E^N \) where \( m_i \) and \( m'_i \) are mass of \( i \)-th body of configurations \( x \) and \( y \) respectively, and \( x_i, y_i \in E, (1 \leq i \leq N) \). The induced norm of configuration space \( E^N \) is denoted as \( \| \cdot \| \) and defined as

\[
\| x \| = \left( \frac{1}{2} \sum_{i=1}^{N} m_i |x_i|^2 \right)^{1/2}.
\]

Write

\[
S_\lambda := \{(x, y) \in E \times E \mid |x - y| < \lambda \}.
\]

We study \( N \)-body problem in \( E \) with potential

\[
U(x) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|x_i - x_j|^\alpha}
\]

where \( x = (x_1, \ldots, x_N) \in E^N, (x_i \in E) \) is the configuration, \( m_i \) is the mass of \( i \)-th body of \( x \).

Our main task is to find the hyperbolic motions of our stated \( N \)-body problem by measure of construction, and the motion satisfies

\[
\ddot{x}_i = \nabla_{x_i} U(x) = \alpha \sum_{j=1, j \neq i}^{N} \frac{x_j - x_i}{|x_j - x_i|^\alpha+2}.
\]

i.e.,

\[
\ddot{x} = \nabla U(x)
\]

(1.1)

Now we give some notations frequently used in this paper.

1. Denote \( \Omega \) be the subset of configurations with non-collision, i.e.,

\[
\Omega = \{x \in E^N \mid x_i \neq x_j, \forall 1 \leq i < j \leq N\}.
\]

write \( \sum := E^N \setminus \Omega \).

2. Denote \( r(x) = \min\{|x_i - x_j|\}_{1 \leq i < j \leq N} \), this is a essential data for the judgement of collision of a configuration is the minimal distance among its bodies, meaning that for \( x \in E^N \), \( r(x) > 0 \) if and only if \( x \in \Omega \). Similarly we denote \( R(x) = \max\{|x_i - x_j|\}_{1 \leq i < j \leq N} \).

3. Denote several line segment as follows:

\[
[xy] = \{tx + (1-t)y \mid x, y \in E^N; 0 \leq t \leq 1\}
\]

(1.2)
\[
\overline{xy} = \{tx + (1-t)y \mid x, y \in E^N; 0 < t < 1\} \quad (1.3)
\]

\[
\overline{xy\infty} = \{x + ty \mid x, y \in E^N; t > 0\}
\]

\[
\overline{|xy\infty|} = \{x + ty \mid x, y \in E^N; 0 \leq t \leq +\infty\}. \quad (1.5)
\]

4. Denote \(d(x, W) = \inf \{\|x - y\| \mid y \in W\}\) as the distance of a configuration \(x \in E^N\) to some subset \(W \subset E^N\).

5. Denote

\[
C(x, y, T) := \{\gamma : [\alpha, \beta] \to E^N \text{ for some } [\alpha, \beta] \subset \mathbb{R} \mid \gamma(\alpha) = x, \gamma(\beta) = y, \beta - \alpha = T\}
\]

and

\[
C(x, y) = \bigcup_{T > 0} C(x, y, T),
\]

without loss of generality, we can always assume \(\alpha = 0, \beta = T > 0\) for \(T\) under determined.

6. Denote

\[
l(\gamma) := \int_0^T |\dot{\gamma}(t)| dt
\]

as the Euclidean length of \(\gamma \in C(x, y)\).

7. Denote \(\angle(x, y)\) as the angle between \(x, y \in E\), i.e.,

\[
\cos \angle(x, y) := \frac{\langle x, y \rangle}{\|x\|\|y\|}.
\]

Similarly, if \(x, y \in E^N\),

\[
\cos \angle(x, y) := \frac{\langle x, y \rangle}{\|x\|\|y\|}.
\]

8. Denote \(S^{nN-1} = \{x \in E^N \mid \|x\| = 1\}\) where \(n = \dim E\), we find that for \(x, y \in S^{nN-1}\),

\[
\frac{1}{2}\|x - y\| = \sin \frac{1}{2}\angle(x, y)
\]

9.

We give several assumption that is for simplicity throughtout the artical.

1. We assume \(\min\{m_i\}_{1 \leq i \leq N} = 1\).

Now we can present the main thereom as our main issue
When a satisfies a collision, i.e., if

Theorem 1.3. (2002, Marchal [3]). If \( \gamma \in C(x, y) \) is defined on some interval \( [a, b] \), and satisfies \( A_L(\gamma) = \phi(x, y, b - a) \), then \( \gamma(t) \in \Omega \) for all \( t \in (a, b) \).

Here in our paper we need to know the absence of collision of potential with homogeneity \(-\alpha < 0\), they are studied, see[] for the convenience we restate here.

Theorem 1.4. When \( U(x) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|x_i - x_j|^\alpha} \) All minimizers of \( \phi_E \) are experience no collision, i.e., if \( A_E(\gamma) = \phi_E(x, y) \) where \( \gamma \in A(x, y, T) \) for some \( T > 0 \), then \( \gamma(t) \in \Omega, t \in (0, T) \)
We also have to list the very important Hamilton’s principle of least action as the following theorem which we will use and its proof is put later.

**Theorem 1.5.** For any \( E > 0 \), suppose \( \gamma \in \mathbf{C}(x, y, T) \) (\( T \) can be \(+\infty\) is a free time minimizer of \( A_E \) and \( \gamma|_{(0,T)} \subset \Omega \), then \( \gamma \) is a solution of 1.1 in \((0, T)\).

There is another result of \( \phi_E(\cdot, \cdot) \).

**Proposition 1.6.** \( \phi_E(\cdot, \cdot) \) is a distance function in \( \Omega \)

**Proof.** We first notice \( \phi_E \) meets triangular inequality.

For any \( x, y, z \in \Omega \), any \( \gamma_1, \gamma_2 \in \mathbf{C}(x, y) \) defined in \([0, T_1]\) and \([0, T_2]\) respectively, we set

\[
\gamma(t) := \begin{cases} 
\gamma_1(t) & 0 \leq t \leq T_1 \\
\gamma_2(t - T_1) & T_1 \leq t \leq T_1 + T_2 
\end{cases} 
\]  

(1.14)

thus

\[
\phi_E(x, z) = \inf \{ A_E(\eta) \mid \eta \in \mathbf{C}(x, z) \} \leq A_E(\gamma) \leq A_E(\gamma_1) + A_E(\gamma_2). 
\]  

(1.15)

Since \( \gamma_1, \gamma_2 \) are arbitrary, we have \( \phi_E(x, z) \leq \phi_E(x, y) + \phi_E(y, z) \).

Second, we verify that \( \phi_E(x, y) = 0 \) makes \( x = y \).

For any \( x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N) \in E^N \) and \( \gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbf{C}(x, y) \) defined on \([0, T]\). There is a \( T' \in (0, T] \) s.t.

\[
\max \{|\gamma_i(T') - x_i| \mid 1 \leq i \leq N\} = \max \{|x_i - y_i| \mid 1 \leq i \leq N\},
\]

and

\[
\max \{|\gamma_i(t) - x_i| \mid 1 \leq i \leq N\} \leq \max \{|x_i - y_i| \mid 1 \leq i \leq N\}
\]

for \( t \in [0, T'] \). Then there exists \( i_0 \) s.t.

\[
\max \{|x_i - y_i| \mid 1 \leq i \leq N\} = |\gamma_{i_0}(T') - x_{i_0}| \leq \int_0^{T'} |\dot{\gamma}_{i_0}| dt \leq \sqrt{T'} \left( \int_0^{T'} |\dot{\gamma}_{i_0}|^2 dt \right)^{1/2}.
\]  

(1.16)

Hence

\[
A_E(\gamma) \geq A_E(\gamma|_{[0,T']}) \geq \int_0^{T'} \|\dot{\gamma}(t)\|^2 + U(\gamma(t)) + Edt \geq \frac{1}{2} \int_0^{T'} \sum m_i |\dot{\gamma}_i(t)|^2 dt \geq \frac{m_{i_0}}{2} \int_0^{T'} |\dot{\gamma}_{i_0}(t)|^2 dt \geq \frac{1}{2T'} \max \{|x_i - y_i| \mid 1 \leq i \leq N\}.
\]  

(1.17)

So eventually \( \phi_E(x, y) = 0 \) makes \( \max \{|x_i - y_i|\}_i = 0 \), which means \( x = y \).

It is not difficult to see \( \phi_E(x, x) = 0 \). \qed
2 The existence and properties of free-time minimizers

3 Some preparations for geometric objects

Based on the elementary computing, we can verify the following essential geometric facts in Euclidean space $E^N$.

**Lemma 3.1.** For any $x \in E^N$, we must have $|x_i| \leq \sqrt{2} \|x\|$ and $|x_i - x_j| < 3 \|x\|$ for $1 \leq i < j \leq N$, hence $r(x) < 3 \|x\|$.

**Proof.** By the definition of $\|x\|$, we have $\sum |x_i|^2 = \sum m_i |x_i|^2 = 2 \|x\|^2$, thus $|x_i| \leq \sqrt{2} \|x\|$ for any $i$ and $|x_i - x_j| \leq 2 \sqrt{2} \|x\| < 3 \|x\|$.

**Theorem 3.2.** For $a \in S^{nN-1}$, $r(a) > 0$, $x, x' \in E^N$, the following statements are valid.

1. $\frac{\|x + ta\|}{\|x + ta\|} - a \leq \frac{1}{30} r(a)$

   \[ r(x + ta) \geq \frac{67}{70} r(a) t > 67 \]  \hfill (3.2)

   for any $t > \frac{1 + \|x\|}{r(a)}$

2. $r(x') > r(a) - 2 \|x' - a\| \geq (1 - 3\lambda)r(a)$

   \[ \cos \angle(x'_i - x'_j, a_i - a_j) \geq 1 - 6\lambda, 1 \leq i < j \leq N \]  \hfill (3.4)

   for $\|x' - a\| \leq \lambda r(a), \lambda \in (0, 1/2)$

3. If $\|x'\| = 1$ then

   \[ \cos \angle(a, x') = \langle a, x' \rangle \geq 1 - \frac{9}{2} \lambda^2 \]  \hfill (3.5)

**Corollary 3.3.**

**Proof.** Since $t > 70 \frac{1 + \|x\|}{r(a)}$, we have $t > 70 \frac{\|x\|}{r(a)} t < \frac{r(a)}{70}$ and $r(a) t > 70$.

1. Since $t > 70 \frac{\|x\|}{r(a)}$, and by lemma 1.1 we know that $r(a) < 3 \|a\| = 3$, we have $t - \|x\| > \frac{70 - 3 \|x\|}{\frac{r(a)}{70} \|x\|}$ hence $\frac{\|x\|}{t - \|x\|} < \frac{r(a)}{60}$.

   \[ \frac{\|x + ta\|}{\|x + ta\|} - a \leq \frac{1}{\|x - ta\|} \|x + ta - \|x + ta\| a\| \]  \hfill (3.6)

   \[ \leq \frac{1}{\|x + ta\|} (\|x\| + \|\|ta\| - \|x + ta\|\|) \leq \frac{2 \|x\|}{\|x + ta\|} \]  \hfill (3.7)

   \[ \leq \frac{2 \|x\|}{t - \|x\|} \leq \frac{r(a)}{30}. \]  \hfill (3.8)
\[ |x_i + ta_i - (x_j + ta_j)| = |t(a_i - a_j) + x_i - x_j| \]  
\[ \geq t|a_i - a_j| - |x_i - x_j| \]  
\[ \geq r(a)t - 3\|x\| > r(a)t - 2\frac{r(a)}{70}t = 67\frac{r(a)}{70}t > 67. \]  

2. First we have
\[ |a_i - a_j| \leq |a_i - x'_i| + |x'_i - x'_j| + |x'_j - a_j|. \]  
thus
\[ |x'_i - x'_j| \geq |a_i - a_j| - |x'_i - a_i| - |x'_j - a_j| > |a_i - a_j| - 3\|x' - a\|, \]
hence
\[ r(x') > r(a) - 3\|x' - a\| > (1 - 3\lambda)r(a). \]  
On the other hand
\[ \langle a_i - a_j, x'_i - x'_j \rangle = \langle a_i - a_j, a_i - a_j \rangle + \langle a_i - a_j, x'_i - a_i - x'_j + a_j \rangle \]  
\[ = |a_i - a_j|^2 + \langle a_i - a_j, (x'_i - a_i) - (x'_j - a_j) \rangle \]  
\[ \geq |a_i - a_j|^2 - |a_i - a_j||(x'_i - a_i) - (x'_j - a_j)| \]  
\[ \geq |a_i - a_j|^2 - |a_i - a_j|3\|x' - a\| \]  
\[ = |a_i - a_j|(|a_i - a_j| - 3\|x' - a\|). \]  
\[ |x'_i - x'_j| \leq |x'_i - a_i| + |a_i - a_j| + |a_j - x'_j| \leq |a_i - a_j| + 3\|x' - a\|. \]  
So we eventually have
\[ \frac{\langle a_i - a_j, x'_i - x'_j \rangle}{|a_i - a_j||x'_i - x'_j|} \geq |a_i - a_j| - 3\|x' - a\| \]  
\[ \geq |a_i - a_j| + 3\|x' - a\| \]  
\[ \geq 1 - \frac{6\|x' - a\|}{r(a)} \geq 1 - 6\lambda. \]  

3. For \( \|x'\| = 1 \)
\[ 2 - 2\langle x', a \rangle = \|x' - a\|^2 \leq \lambda^2 r(a)^2. \]  

hence
\[ \langle x', a \rangle = 1 - \frac{1}{2}\|x' - a\|^2 \geq 1 - \frac{\lambda^2}{2}r(a)^2 = 1 - \frac{9}{2}\lambda^2 \]  
\[ \square \]
References

[1] Gonzalo Contreras and Renato Iturriaga. Global minimizers of autonomous lagrangians. 1999.

[2] Adriana da Luz and Ezequiel Maderna. On the free time minimizers of the newtonian n-body problem. In Mathematical Proceedings of the Cambridge Philosophical Society, volume 156, pages 209–227. Cambridge University Press, 2014.

[3] Ch Marchal. How the method of minimization of action avoids singularities. Celestial Mechanics and Dynamical Astronomy, 83(1):325–353, 2002.

[4] John N Mather. Action minimizing invariant measures for positive definite lagrangian systems. Mathematische Zeitschrift, 207(1):169–207, 1991.