Master integrals with 2 and 3 massive propagators for the 2-loop electroweak form factor — planar case

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Abstract

We compute the master integrals containing 2 and 3 massive propagators entering the planar amplitudes of the 2-loop electroweak form factor. The masses of the $W$, $Z$ and Higgs bosons are assumed to be degenerate. This work is a continuation of our previous evaluation of master integrals containing at most 1 massive propagator. The $1/\epsilon$ poles and the finite parts are computed exactly in terms of a new class of 1-dimensional harmonic polylogarithms of the variable $x$, with $\epsilon = 2 - D/2$ and $D$ the space-time dimension. Since thresholds and pseudothresholds in $s = \pm 4m^2$ do appear in addition to the old ones in $s = 0, \pm m^2$, an extension of the basis function set involving complex constants and radicals is introduced, together with a set of recursion equations to reduce integrals with semi-integer powers. It is shown that the basic properties of the harmonic polylogarithms are maintained by the generalization. We derive small-momentum expansions $|s| \ll m^2$ of all the 6-denominator amplitudes. We also present large momentum expansions $|s| \gg m^2$ of all the 6-denominator amplitudes which can be represented in terms of ordinary harmonic polylogarithms. Comparison with previous results in the literature is performed finding complete agreement.

Key words: Feynman diagrams, Multi-loop calculations, Vertex diagrams, Electroweak Sudakov, Harmonic polylogarithms

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1 Introduction

We present in this paper the computation of the master integrals containing 2 and 3 massive propagators entering the planar amplitudes of the 2-loop electroweak form factor. This work is the natural continuation of our previous evaluation of master integrals containing at most 1 massive propagator. The model process we consider is:

\[ f(p_1) + \bar{f}(p_2) \rightarrow X(q), \] (1)

where \( f \bar{f} \) is an on-shell massless fermion pair, \( p_1^2 = p_2^2 = 0 \), and \( X \) is a singlet under the electroweak gauge group \( SU(2)_L \times U(1)_Y \).

At the 2-loop level, the annihilation in Eq. (1) involves the emission of 2 virtual bosons among \( \gamma, W, Z \) and \( H \)'s. The cases of emission of (i) 2 photons and (ii) 1 photon and 1 massive boson, have already been treated in our previous work \[1\]. Since the above amplitudes have only thresholds in \( s = 0, m^2 \) and pseudothreshold in \( s = -m^2 \), we succeeded to express them in terms of 1-dimensional harmonic polylogarithms \[2, 3\] with maximum weight up to 4 included.

In this paper we consider the amplitudes with 2 or 3 bosons of mass \( m \) exchanged: the masses of \( W, Z \) and Higgs bosons are assumed to be degenerate, \( m_W \approx m_Z \approx m_H \approx m \). In general, these amplitudes have thresholds in \( s = 0, m^2, 4m^2 \) and pseudothresholds in \( s = -m^2, -4m^2 \). As a consequence of this more complicated structure, the basis function set of the harmonic polylogarithms considered in \[2\] is no more sufficient and a generalization is presented. New basis functions such as \( 1/(4 \pm x), 1/\sqrt{x}(4 \pm x) \), etc. are introduced and recursion relations to reduce integrals with semi-integer powers coming from the evaluation of the master integrals are derived. The basic properties of the harmonic polylogarithms, such as the uniqueness of representation as repeated integration, the algebra structure, the closure under the inverse transformation \( x \rightarrow 1/x \), etc. are all maintained.

We use dimensional regularization \[4\] to regulate both ultraviolet and infrared divergences, which then appear as poles in \( 1/\epsilon \), with \( \epsilon = 2 - D/2 \) and \( D \) the space-time dimension. The ultraviolet poles are related to coupling constant renormalization and are subtracted with ordinary renormalization prescriptions, such as for example the \( \overline{MS} \) scheme. The infrared poles are not physical, and in the physical cross-section, are canceled by the corresponding ones appearing in the real photon emission contributions or are factorized into QED structure functions. The use of dimensional regularization as double regulator for both infrared and ultraviolet divergences makes a priori impossible to trace the dynamical origin of the \( 1/\epsilon \) poles. This is however possible by using power counting estimates for the ultraviolet and

\[ g^4(m_{Z,H}^2 - m_{W}^2)^n \] can be included by means of expansions of the denominators of the form

\[ \frac{1}{k^2 + m_{Z,H}^2} = \frac{1}{k^2 + m_{W}^2} \frac{m_{Z,H}^2 - m_{W}^2}{(k^2 + m_{W}^2)^2} + \cdots. \] (2)

The amplitudes with powers of the \( W \) denominators can be reduced to the master integrals by means of the integration-by-parts identities (see Section 2).
the infrared regions together with some physical intuition (see sec. (4)). In the ultraviolet region one can usually neglect masses and external momenta while in the infrared region massive lines can be shrunk to a point. Furthermore leading $1/\epsilon$ poles often originate from ordered regions such as $k_1^2 \ll k_2^2$, with $k_1$ and $k_2$ the loop momenta. This (rather qualitative) analysis offers some checks of the results.

The paper is organized as follows.

In Section 2 we outline the strategy for the exact analytical evaluation of the 2-loop electro-weak form factor. The presentation is rather sketchy and we refer to our previous work [1] for a detailed discussion of the various steps of the computation.

In Section 3 we discuss the properties of the harmonic polylogarithms. An overview of the ordinary harmonic polylogarithms is presented, in order to introduce later the generalization necessary to represent the master integrals. As already anticipated, the basic tool is a set of recursion equations to reduce the integrals coming from the evaluation of the master integrals to a unique form, specified by the choice of the basis functions.

Section 4 contains the main results of our work. We present exact analytical results for the 21 non-trivial master integrals containing 2 and 3 massive denominators. All the master integrals are represented in terms of generalized harmonic polylogarithms.

In Section 5 we give the results for the reducible 6-denominator diagrams. The latter are 3 amplitudes of vertex-insertion type (see later), which are interesting by themselves or for reference use.

In Section 6 the small momentum expansion $|s| \ll m^2$ of all the 6-denominator amplitudes is presented. We compare our results with those present in the literature finding complete agreement.

In Section 7 we compute the large momentum expansion $|s| \gg m^2$ of all the 6-denominator amplitudes which can be represented in terms of ordinary harmonic polylogarithms. We could not give the large momentum expansion of the amplitudes containing generalized harmonic polylogarithms $H(\vec{w}; x)$, because the expansion of the latter for $|x| \gg 1$ is still an open problem.

In Section 8 we draw our conclusions and we discuss the open problems in the computation of the master integrals.

In order to make the paper as clear as possible, we added 4 appendices.

In Appendix A we report the 1-loop master integrals with 2 massive propagators which occur in our computation. They are written in terms of generalized harmonic polylogarithms, like the 2-loop amplitudes.

In Appendix B we present the 4 factorized 2-loop master integrals, which are the product of one-loop master integrals. Their derivation is rather trivial and we give them for completeness.

In Appendix C we compute some interesting 2-loop amplitudes containing up to 5 denominators included which can be reduced to simpler topologies by means of the integration-by-parts identities.

Finally, in Appendix D we give the reducible 6-denominator scalar amplitudes of self-energy insertion type.
Figure 1: Vertex-correction diagrams with 2 and 3 massive propagators. The topologies related to these diagrams are real 6-denominator topologies (see text). The graphical conventions are the same as in our previous paper [1].
Figure 2: Self-energy correction diagrams with 2 and 3 massive propagators. The topologies related to these diagrams are 5 and 4 denominator topologies (see text).
2 The computation of the form factor

The planar diagrams containing 2 and 3 massive propagators entering the 2-loop electroweak form factor are plotted in Figs. 1 and 2. We have omitted 1-particle-reducible contributions, related to field and mass renormalization of the external particles $f, \bar{f}$ and $X$. The amplitudes can be obtained, for example, by means of a Dyson-Schwinger expansion of the $\bar{f}fX$ vertex function $\mathcal{B}$. Let us look at the general structure of the diagrams:

- 1-loop: one has a triangle diagram with a vector particle, i.e. a $\gamma$, a $W$ or a $Z^0$, exchanged between the fermions;
- 2-loops: the triangle diagram has its bare propagators and vertices replaced by the corresponding one-loop quantities.$^5$

According to the kind of correction, the 2-loop diagrams are then naturally classified as:

1. Vertex-correction amplitudes (see Fig. 1). They all involve 6 different denominators, the maximal number of denominators for 2-loop 3-point functions. According to the kind of vertex corrected, there are the following cases:

   (a) Ladder diagrams.
   The “hard” $\bar{f}fX$ vertex, where annihilation occurs, is corrected. These are the diagrams $(a)$ and $(b)$ in Fig. 1:
   - diagram $(a)$ represents the exchange in the $t$ channel of a pair of $W/Z$ particles between the fermions and is a “non-singlet” contribution to the process;
   - diagram $(b)$ represents the conversion of the initial fermion pair into a boson pair which convert back into a fermion pair; fermions are emitted in the $t$-channel. This diagram is a non-singlet contribution which actually cannot be obtained as correction of the basic one-loop amplitude.

   (b) Vertex-insertions.
   One of the 2 vertices involving the external fermions is corrected. There are 5 of such diagrams:
   - diagram $(c)$ containing an abelian correction with a $W$ or a $Z$ to the basic one-loop amplitude;
   - diagram $(d)$ and $(e)$ representing respectively the emission by an intermediate boson of a photon internally or externally to the triangle;
   - diagram $(f)$ containing the conversion of a photon into an intermediate boson pair, i.e. the process $\gamma^* \rightarrow W^*W^*$;

$^5$We neglect in this paper the crossed ladder topology, which is a non-planar one.
• diagram (g) containing the splitting $Z^* \rightarrow W^*W^*$; this diagram is the only one having 3 massive propagators.

The vertex-insertion diagrams are on the same footing as the ladders but are conventionally named in a different way.

2. Self-energy correction amplitudes (see Fig. 2).

There are different cases, according to the kind of line corrected and the type of interaction. Self-energy-corrections may indeed occur on:

(a) a fermion line.

This is the case of diagram (b) in Fig. 2 with a bubble insertion. This amplitude has 5 different denominators, the one representing the corrected line being squared;

(b) a boson line.

This case is more complicated than the previous one because one can have mixing and quartic interactions. Concerning the first point, one can have:

i. a “diagonal” correction.

By this we mean a correction to a $\gamma$, $W$ or $Z$ propagator, involving a virtual fermion or boson pair. These are the cases (b), (c), (e), (f) and (h) in Fig. 2

ii. $\gamma - Z^0$ mixing.

By this we mean a contribution to the $\gamma - Z^0$ propagator. In this case one has 2 propagators with the same momenta but with different masses, which can be disentangled with partial fractioning [1]. These are the cases (d) and (g) in Fig. 2.

Let us now consider the second point. Because of the presence of both cubic and quartic interactions, one can have:

i. bubble insertion.

These are the cases (a) – (f). There are 5 different denominators. In the “diagonal” case, the denominator representing the corrected line is squared. In the $\gamma - Z^0$ mixing case, the amplitude is a superposition of two 5-denominator amplitudes, one containing the photon propagator and the other the $Z^0$ propagator;

ii. tadpole insertion.

These are the cases (g) and (h). There are 4 different denominators. In the “diagonal” case, the denominator representing the corrected line is squared. In the $\gamma - Z^0$ mixing case, the amplitude is a superposition of two 4-denominator amplitudes, one containing the photon propagator and the other the $Z^0$ propagator. The structure of these amplitudes is very simple, as one loop is factorized into a momentum-independent expression.
The exact analytic computation of the 2-loop electro-weak form factor is a rather lengthy process, even though the general method is rather clear. As explained in detail in [1], it involves 2 basic steps:

1. the reduction of the original amplitudes considered before — obtained with usual Feynman rules — to a few independent scalar integrals, called master integrals (MIs);

2. the analytic evaluation of all the master integrals generated with the previous step with some technique, such as Feynman parameters, dispersion relations, small and large momentum expansions or, in our case, differential equations in the external kinematical invariants.

2.1 The reduction to master integrals

By reduction to master integrals we mean the reduction of all the Feynman diagrams to a minimal set of independent scalar integrals. This process involves, in turn, the following three steps:

1. **Projection on invariant form factors.**

   The amplitude associated to a given Feynman diagram is, in general, a tensor integral of the form:

   \[ F_{\mu
u ij \ldots} = \int \frac{N_{\mu
u ij \ldots}(p_1, p_2, k_1, k_2)}{D_{i_1}^{n_1} D_{i_2}^{n_2} D_{i_3}^{n_3} D_{i_4}^{n_4} D_{i_5}^{n_5} D_{i_6}^{n_6}}, \tag{3} \]

   where \( \mu\nu ij \ldots \) denotes collectively spinor indices, four-vector indices and so on. For a scalar probe, with an interaction for example of the form \( \bar{f}fX \), the diagram has 2 spinor indices, while for a vector probe, with an interaction for instance of the form \( \bar{f}\gamma \mu fX^\mu \), the diagram has also a 4-vector index. By usual form-factor decomposition of (3), one can limit himself to consider scalar integrals of the form:

   \[ S_D = \int \frac{N(S_1, S_2, S_3, S_4, S_5, S_6, S_7)}{D_{i_1}^{n_1} D_{i_2}^{n_2} D_{i_3}^{n_3} D_{i_4}^{n_4} D_{i_5}^{n_5} D_{i_6}^{n_6}}, \tag{4} \]

   where

   \[ S_1 = k_1^2, \quad S_2 = k_2^2, \quad S_3 = k_1 \cdot p_1, \quad S_4 = k_1 \cdot p_2, \]

   \[ S_5 = k_2 \cdot p_1, \quad S_6 = k_2 \cdot p_2, \quad S_7 = k_1 \cdot k_2. \tag{5} \]

   The function \( N(S_1, S_2, S_3, S_4, S_5, S_6, S_7) \) is a polynomial in the kinematical invariants \( S_i \).

2. **Rotation to independent scalar amplitudes.**

   The scalar amplitudes in Eq. (4) are not linearly independent on each other as they involve for instance 6 denominators and 7 scalar products, i.e. a total of 13 factors, while there are only 7 kinematical invariants depending on the loop momenta \( k_1 \) and \( k_2 \). There are two different methods to eliminate the redundant amplitudes:
(a) Auxiliary diagram or auxiliary denominator scheme \[6\].

We express the scalar products in the numerator of (4) in terms of the original denominators \(D_{ij}\) \((j = 1 \ldots 6)\) and of an auxiliary denominator \(D_{i7}\) by means of invertible relations of the form:

\[
S_k = \sum_{j=1}^{7} a_{kj} D_{ij}
\]

Our task is then shifted to evaluate independent scalar amplitudes containing formally only denominators\(^6\):

\[
S_I = \int d^Dk_1 d^Dk_2 \frac{1}{D_{i1}^{a_1} D_{i2}^{a_2} D_{i3}^{a_3} D_{i4}^{a_4} D_{i5}^{a_5} D_{i6}^{a_6} D_{i7}^{a_7}},
\]

with \(n_i \leq 1\) for \(i \leq 6\) and \(n_7 \leq 0\). Since the numerator \(N\) is expressed by means of (3) as a polynomial in the denominators, a term \(D_i\) in the denominator can be canceled by an equal term \(D_i\) in the numerator. This means that line \(i\) is shrunk to a point and the topology number \(t\) of the diagram is decremented by one\(^6\). A given Feynman diagram then generates a pyramid of sub-diagrams corresponding to all the possible contractions of \(i \geq 1\) internal lines. All the possible resulting diagrams are plotted in Figs. 3, 4, 5 and 6.

Since the contracted propagators may be massless as well as massive, an amplitude with for example 2 massive propagators will generally contain, in its decomposition, independent subamplitudes with 2,1 and 0 massive propagators. The amplitudes computed in our previous work \[1\], containing 0 or 1 massive denominators, then, usually appear in the decomposition of the present amplitudes;

(b) Shift scheme \[7, 8, 9\].

No auxiliary denominator is introduced. A routing of the amplitudes has to be initially assumed with one internal line carrying momentum \(k_1\) and another internal line carrying momentum \(k_2\). This way one can simplify powers of \(k_1^2\) and \(k_2^2\) in the numerator against denominators, by means of formulas like \(k_i^2/(k_i^2 + a) = 1 - a/(k_i^2 + a)\) with \(i = 1, 2\) and \(a = 0, m^2\). A set of cancellation rules between scalar products in the numerator and denominators is established. An example of such a cancellation is: \(k_1 \cdot p_1/(k_1^2 + 2k_1 \cdot p_1) = 1/2 - k_1^2/2/(k_1^2 + 2k_1 \cdot p_1)\).

We then make all the possible scalar-product denominator cancellations. After such cancellations, the amplitude may loose the denominators with momenta \(k_1\) and \(k_2\), which were initially present by construction. The latter are then reproduced with proper shifts of the loop momenta, hence the name of the method. In general, with the shifts, new denominators

\(^6\)We define the topology number \(t\) as the number of different denominators \(D_i\) occurring in a diagram with \(n_i \geq 1\), irrespective of the powers of the scalar products \[1\].
are generated and the table of scalar-product denominator cancellation rules has to be updated to include also the new denominators. We go on through steps of cancellations and shifts till no more cancellations are feasible. The amplitudes generated in the final step represent the independent scalar amplitudes. Since the total number of invariants is 7, an independent amplitude with $t$ denominators has in general $T = 7 - t$ scalar products in the numerator. As with the previous scheme, it is clear that sub-amplitudes with all the possible contractions of the internal lines of the original (linearly dependent) amplitude do occur in the decomposition.

The contractions of internal line(s) in an amplitude may lead to the following general simplifications:

(a) The internal line(s) between 2 external lines are contracted.

This implies that 2 external lines meet in a new effective vertex and the 3-point function effectively simplifies to a 2-point function. There are 2 possibilities:

- contraction between external fermion lines.
  Since we are computing a 3-point function with 1 general external momentum $q = p_1 + p_2$ and 2 light-cone momenta $p_1$ and $p_2$: $p_1^2 = p_2^2 = 0$, we obtain a 2-point function with the general momentum $q$ flowing through it;

- contraction between an external fermion line and the probe line.
  The simplification is even greater in this case. The resulting diagram depends only on a single light-cone momentum, i.e. on $p_1$ or on $p_2$, but not on both. A single light-cone momentum is equivalent to a null momentum, as is clearly seen with Wick rotation. The resulting diagram is then effectively a vacuum amplitude, in which we can set to zero the external light-cone momentum, i.e. $p_1 \rightarrow 0$ or $p_2 \rightarrow 0$. The above property is the basis of many relations between amplitudes;

(b) the internal line where loops overlap is contracted.

The amplitude factorizes into the product of two 1-loop amplitudes, whose computation is trivial.

3. Reduction of independent amplitudes to master integrals.

It is possible to reduce the independent amplitudes to a small subset of them by means of identities obtained with integrations by parts [10].

- In the auxiliary diagram scheme, these identities are obtained as:

$$0 = \int d^D k_1 d^D k_2 \frac{\partial}{\partial k_i^\mu} \left\{ \frac{\nu^\mu}{D_{i_1}^{n_1} D_{i_2}^{n_2} D_{i_3}^{n_3} D_{i_4}^{n_4} D_{i_5}^{n_5} D_{i_6}^{n_6} D_{i_7}^{n_7}} \right\}, \quad (8)$$
with \( i = 1, 2 \) and \( v = k_1, k_2, p_1, p_2 \). By explicitly performing the derivatives and re-expressing the generated amplitudes in terms of independent ones as described before, one obtains relations among independent amplitudes with shifted indices \( n_j \rightarrow n_j \pm 1 \).

• In the shift scheme, the identities are obtained as:

\[
0 = \int d^Dk_1 d^Dk_2 \frac{\partial}{\partial k_i} \left\{ \frac{v^\mu S_{i1}^t \cdots S_{t7-t}^t}{D_{i1}^n \cdots D_{it}^n} \right\}. \quad (9)
\]

Once the integration-by-parts identities and eventual symmetry relations \([8, 9]\) have been generated, the next step is to combine them to achieve the complete reductions to the MIs. At present, there are 3 techniques to solve the ibps:

(a) symbolic method \([10]\).

Historically, this has been the first method used, introduced together with the ibps identities themselves. The ibp identities are treated as formal recursion equations in the indices \( n_j \) of the denominators (and of the scalar products) to take them to some reference values such as typically 0, 1, 2, −1;

(b) numerical-indices method \([7]\).

Explicit numerical values are replaced for the indices \( n_j \). In the auxiliary diagram scheme, one takes for the indices of the denominators both positive and negative values: \( n_j = \cdots -2, -1, 0, 1, 2 \cdots \), while for the indices of the auxiliary denominators only negative values: \( n_j' = 0, -1, -2 \cdots \).

In the shift scheme, one takes positive values for both the indices of the denominators and the scalar products: \( n_j = 0, 1, 2 \cdots \), \( l_j = 0, 1, 2 \cdots \).

In either scheme, a homogeneous system of linear equations is then generated, whose unknowns are the amplitudes themselves. The system is then solved with the method of elimination of variables, after having established the order in which amplitudes and equations have to be solved. The amplitudes which remain on the r.h.s., after all equations have been used, are the MIs for the specific hierarchy assumed;

(c) integral method \([11]\).

An integral representation for the expansion coefficients of the independent amplitudes in terms of master integrals is derived. This technique has not been used up to now, as far as we know, for original computations but for general studies.

The reduction through the ibps of a given independent amplitude involves MIs with subsets of denominators of the starting amplitude. In other words, the reduction involves MIs in which 0, 1, 2 \cdots \ internal lines of the original amplitude have been shrunk to a point. All the MIs appearing in our reductions are given in Fig. 7.
2.2 The evaluation of the master integrals: the differential equation method

In the previous Section we outlined how to reduce the 2-loop Feynman diagrams of the electroweak form factor to the master integrals. Let us now sketch how to explicitly evaluate the master integrals with the differential equation method. We may identify 2 basic steps:

1. Generation of the differential equations.

   This step is rather “automatic” in the sense that it does not offer specific difficulties. It involves, in turn, the following steps:

   (a) We take the derivatives of the MIs with respect to $x$ at fixed $a$ by differentiating the integrand with respect to the external momenta $p_1$ or $p_2$; the derivatives of the MIs are then expressed as a superposition of linearly-dependent scalar amplitudes;

   (b) We rotate the dependent scalar amplitudes to independent ones, either in the auxiliary diagram scheme or in the shift scheme;

   (c) We reduce the independent amplitudes to MIs by using the ibps identities and other eventual symmetry relations. This way the system closes on the MIs themselves. Linear systems of first-order differential equations with variable coefficients are then generated, whose unknowns are the MIs themselves.
Figure 4: The set of 17 independent 5-denominator diagrams, with 2 and 3 massive propagators.
Figure 5: The set of 11 independent 4-denominator diagrams, with 2 and 3 massive denominators.

Figure 6: The set of 4 independent 3-denominator diagrams together with the single 2-denominator diagram, with 2 and 3 massive propagators.
Figure 7: The set of 25 MIs with 2 and 3 massive denominators. Out of them, 4 are the product of 1-loop MIs. The dot on a line indicates a square of the corresponding denominator.
2. Solution of the differential equations.

In order to have a unique solution — as it should — initial conditions can often be obtained by studying the behaviour of the MIs close to thresholds or pseudothresholds. The system of differential equations is then solved by recursion in $\epsilon$. Let us remark that this step is not “automatic”, in the sense that obtaining a solution is in some cases a matter of intuition and experience.

Let us make a few comments about specific characteristics of the present computation. The most complicated cases involve topologies with more than 1 MI:

- $t = 4$: we encounter 2 vertex topologies having 2 MIs;
- $t = 5$: we encounter 2 vertex topologies having 2 MIs and 1 topology having 3 MIs$^7$. $^8$

A convenient basis for topologies with 2 MIs consists of the following amplitudes:

1. the “basic” amplitude, i.e. the amplitude with unitary numerator and with all the denominators having unitary indices;
2. the amplitude with unitary numerator and with one of the denominators squared, or with an independent scalar product left on the numerator.

The resulting system of 2 differential equations is often triangular in $D = 4$, allowing for a simple solution. For the only case of 3 MIs, we found that a convenient basis comprises the following amplitudes:

1. the “basic” amplitude defined above;
2. the amplitude with unitary numerator and one denominator squared, as in the previous case;
3. the amplitude with an irreducible scalar product in the numerator and the denominators all with unitary indices.

With this choice, the system of 3 ordinary differential equations becomes triangular in $D = 4$, allowing for a simple recursive solution in $\epsilon$.

3 Harmonic Polylogarithms of one variable

As discussed in the introduction, amplitudes with 2 and 3 massive propagators may have thresholds in $s = 4m^2$ and pseudothresholds in $s = -4m^2$. As a consequence, a generalization of the usual harmonic polylogarithms of one variable is needed. In the next Section we summarize the salient features of the ordinary harmonic polylogarithms (HPLs) that we want to preserve with the generalization, while in Section 3.2 we discuss such a generalization.

$^7$The $t = 6$ crossed ladder topology also has 3 MIs.
$^8$We have also used the symbolic method to reduce the topology with 3 MIs, with similar results.
3.1 Ordinary Harmonic Polylogarithms

The basic idea of the HPLs is that of representing a given integral introducing a minimal set of transcendental functions by making only linear transformations, such as partial fractioning and integration by parts. Furthermore, the transcendental functions are defined once and for all. Let us begin with a simple example. The integration of the power function with an integer exponent $n$,

$$
\int_1^x x^n\,dx' = \frac{x^{n+1}}{n+1} - \frac{1}{n+1} \quad \text{if } n \neq -1
$$

$$
= \log x \quad \text{if } n = -1,
$$

is again a power function, except for the case $n = -1$ for which we have the logarithm. Integration then introduces only one transcendental function, plus the original set of elementary functions. Larger sets of transcendental functions are needed to represent double integrals, such as for instance

$$
\int_0^x \frac{dx'}{1-x'} \log x' = \int_0^x \frac{dx'}{1-x'} \int_1^{x'} \frac{dx''}{x''} = -\log x \log(1-x) - \text{Li}_2(x),
$$

which involves a new transcendental functions, the well-known dilogarithm:

$$
\text{Li}_2(x) \equiv -\int_0^x \frac{dx'}{x'} \int_0^{x'} \frac{dx''}{1-x''} = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad \text{for } |x| \leq 1. \tag{12}
$$

However, many double integrals do not involve new transcendental functions with respect to the ones needed to represent single integrals such as (10). Let us indeed consider:

$$
\int_0^x \frac{dx'}{(1-x')^2} \log x' = \int_0^x \frac{dx'}{(1-x')^2} \int_1^{x'} \frac{dx''}{x''} = \log(1-x) + \left( \frac{1}{1-x} - 1 \right) \log x. \tag{13}
$$

The reduction above is done with an integration by parts: we integrate $1/(1-x)^2$, whose integral is elementary, and differentiate $\log x$, obtaining a simpler, elementary function. We then make a partial fractioning of

$$
\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{(1-x)}, \tag{14}
$$

i.e. another linear transformation. In general, whenever we have to integrate the product of a transcendental function times an elementary function whose integral is also elementary, we do a by-parts integration in order to derive, i.e. to simplify, the transcendental function. This applies in particular to the integral of the transcendental function itself:

$$
\int_0^x dx' \log x' = x \log x - x \tag{15}
$$
and in general to all the integrals of the form:

\[ \int_0^x \frac{dx'}{(1-x')^n \log x'} \]  

(16)

with \( n \neq 1 \). The examples given above generalize to multiple repeated integrations, of the form:

\[ \int_0^x \frac{dx_1}{(x_1 + a_1)^{n_1}} \cdots \int_0^{x_2} \frac{dx_2}{(x_2 + a_2)^{n_2}} \int_0^{x_3} \frac{dx_3}{(x_3 + a_3)^{n_3}} \cdots \]  

(17)

with \( a_1 \cdots a_l \) some constants. One has to introduce transcendental functions given by the repeated integrations of the inverse linear functions \( 1/(x + a_i) \):

\[ \int_0^x \frac{dx_1}{x_1 + a_1} \cdots \int_0^{x_2} \frac{dx_2}{x_2 + a_2} \int_0^{x_3} \frac{dx_3}{x_3 + a_3} \cdots \]  

(18)

For many applications, such as our previous computation of 2-loop electroweak amplitudes with 1 massive propagator [1], it is sufficient to introduce the following set of basis functions:

\[ g(0, x) = \frac{1}{x}, \]  

(19)

\[ g(1, x) = \frac{1}{1-x}, \]  

(20)

\[ g(-1, x) = \frac{1}{1+x}. \]  

(21)

Note that \( g(0, x) \) has a non-integrable singularity for \( x \to 0 \), while the other functions are finite in the same limit. The HPLs of weight 1 are defined as integrals of the basis functions:

\[ H(0, x) = \int_1^x \frac{dt}{t} = \log x, \]  

(22)

\[ H(1, x) = \int_0^x \frac{dt}{1-t} = -\log (1-x), \]  

(23)

\[ H(-1, x) = \int_0^x \frac{dt}{1+t} = \log (1+x), \]  

(24)

Note the slight asymmetry related to the lower bound of integration for \( H(0, x) \).

Let us now define the general HPL \( H(\vec{w}; x) \), where \( \vec{w} \) is a vector with \( w \) components, consisting of a sequence of “1”, “0”, and “-1”. The weight of a HPL is the number of its indices, \( w \), coinciding with the number of the repeated integrations. The HPLs of weight \( w + 1 \) have the following integral recursive definition:

\[ H(a, \vec{w}; x) = \int_0^x f(a; x') H(\vec{w}, x'), \]  

(25)

except for the case with all the indices zero:

\[ H(\vec{0}_w; x) = \frac{1}{w!} \log^w x, \]  

(26)
or, if a recursive definition is preferred:

\[ H(0, \vec{0}_w; x) = \int_1^x f(0; x')H(\vec{0}_w, x') = \frac{1}{(w+1)!} \log^{w+1} x, \]

where \( \vec{0}_w = (0, 0, \ldots, 0) \) is a vector containing \( w \) zeroes.

### 3.2 Generalized Harmonic Polylogarithms

The basis functions for the GHPLs involve various kinds of extensions of the basis functions defining the HPLs.

A trivial extension concerns functions involving different real constants:

\[ g(-4, x) = \frac{1}{4 + x}, \]
\[ g(4, x) = \frac{1}{4 - x}. \tag{28} \]

The above functions are related to amplitudes with threshold/pseudothreshold at \( s = \pm 4m^2 \) respectively, just as the old functions \( g(\mp 1; x) \) represent amplitudes with threshold/pseudothreshold in \( s = \pm m^2 \). We will see in a moment however that the above extension in not sufficient.

To represent 3--point functions with 3 massive propagators one also needs to introduce basis functions involving a complex constant:

\[ g(c, x) = \frac{1}{x - \frac{1}{2} - i\frac{\sqrt{3}}{2}}, \]
\[ g(\overline{c}, x) = \frac{1}{x - \frac{1}{2} + i\frac{\sqrt{3}}{2}}. \tag{30} \]

Note that \( \frac{1}{2} \pm i\frac{\sqrt{3}}{2} = \exp(\pm i\pi/3) \) are the non trivial cubic roots of "-1" and are the inverse of each other.

The non trivial extension, however, involves:

- radicals of the form

\[ g(-r, x) = \frac{1}{\sqrt{x(4 + x)}}, \]
\[ g(r, x) = \frac{1}{\sqrt{x(4 - x)}}. \tag{32} \]

These functions also describe amplitudes with thresholds and pseudothresholds in \( s = \pm 4m^2 \) respectively. It is indeed well-known that amplitudes involving 2 particles with the same mass \( m \neq 0 \) contain square roots of similar form as a phase-space reduction effect;

\[ ^9 \text{An alternative representation uses only real functions at the price of introducing squares of the integration variable: } 1/(x^2 - x + 1) \text{ and } x/(x^2 - x + 1). \]
• products of radical with inverse linear functions:

\[
\begin{align*}
g(-1 - r, x) &= \frac{1}{\sqrt{x(4 + x)} (1 + x)}, \\
g(1 - r, x) &= \frac{1}{\sqrt{x(4 + x)} (1 - x)}, \\
g(-1 + r, x) &= \frac{1}{\sqrt{x(4 - x)} (1 + x)}, \\
g(1 + r, x) &= \frac{1}{\sqrt{x(4 - x)} (1 - x)}. \\
\end{align*}
\]

In total, we have added 10 functions to the old basis. All the new basis functions above are finite in \(x = 0\) or have at most an integrable singularity \(\sim 1/\sqrt{x}\) for \(x \to 0\).

The GHPLs of weight 1 are given by integrals of the basis functions and can be written, in general, in terms of logarithms of complex functions of \(x\):

\[
\begin{align*}
H(-4, x) &= \int_0^x \frac{dt}{4 + t} = \log (4 + x) - 2 \log 2, \\
H(4, x) &= \int_0^x \frac{dt}{4 - t} = -\log (4 - x) + 2 \log 2, \\
H(c, x) &= \int_0^x \frac{dt}{t - \frac{1}{2} - i \frac{\sqrt{3}}{2}}, \\
&= \log \left(x - 1/2 - i \frac{\sqrt{3}}{2}\right) - \log \left(-1/2 - i \frac{\sqrt{3}}{2}\right), \\
H(\overline{c}, x) &= \int_0^x \frac{dt}{t - \frac{1}{2} + i \frac{\sqrt{3}}{2}}, \\
&= \log \left(x - 1/2 + i \frac{\sqrt{3}}{2}\right) - \log \left(-1/2 + i \frac{\sqrt{3}}{2}\right), \\
H(-r, x) &= \int_0^x \frac{dt}{\sqrt{t(4 + t)}} = 2 \arcsinh \left(\frac{\sqrt{x}}{2}\right), \\
&= 2 \log (\sqrt{x + 4} + \sqrt{x}) - 2 \log 2, \\
H(r, x) &= \int_0^x \frac{dt}{\sqrt{t(4 - t)}} = 2 \arcsin \left(\frac{\sqrt{x}}{2}\right), \\
&= -i \log \left\{\frac{\sqrt{4 - x + i \sqrt{x}}}{\sqrt{4 - x - i \sqrt{x}}}\right\}, \\
H(-1 - r, x) &= \int_0^x \frac{dt}{\sqrt{t(4 + t)}(1 + t)} = \frac{2}{\sqrt{3}} \arctan \left(\frac{\sqrt{3}x}{4 + x}\right), \\
&= \frac{i}{\sqrt{3}} \log \left\{\frac{\sqrt{4 + x - i \sqrt{3}x}}{\sqrt{4 + x + i \sqrt{3}x}}\right\}, \\
\end{align*}
\]

\(^{10}\)Alternative basis functions having the same asymptotic limits \(\sim 1/x\) for \(x \to \infty\) of the previous ones can be taken for instance as \(\sqrt{(4 \pm x)/x}/(1 \pm x)\).
H(1−r, x) = \int_0^x \frac{dt}{\sqrt{t(4+t)(1−t)}} = \frac{1}{\sqrt{5}} \log \left\{ \frac{\sqrt{4+x + \sqrt{5x}}}{\sqrt{4+x - \sqrt{5x}}} \right\}, \quad (48)

H(−1 + r, x) = \int_0^x \frac{dt}{\sqrt{t(4−t)(1+t)}} = \frac{2}{\sqrt{5}} \arctan \left( \sqrt{\frac{5x}{4−x}} \right), \quad (49)

= \frac{i}{\sqrt{5}} \log \left\{ \frac{\sqrt{4−x − i\sqrt{5x}}}{\sqrt{4−x + i\sqrt{5x}}} \right\}, \quad (50)

H(1 + r, x) = \int_0^x \frac{dt}{\sqrt{t(4−t)(1−t)}} = \frac{1}{\sqrt{3}} \log \left\{ \frac{\sqrt{4−x + \sqrt{3x}}}{\sqrt{4−x − \sqrt{3x}}} \right\}. \quad (51)

Note that all the integrals above have zero as the lower limit of integration.

The GHPLs of general weight \( w > 1 \) are defined exactly in the same way as the HPLs, according to Eq. (25), where now the components of \( \vec{\bar{w}} \) can also take the values \( ±4, c, c, ±r, ±1 ± r \).

Let us now discuss how the integrals appearing in the evaluation of the MIs can be reduced to the GHPLs plus, of course, elementary functions. As far as the indices \( ±4, c \) and \( c \) are concerned, there is basically nothing new with respect to the HPLs: the only difference is that individual terms in the expressions for the MIs are in general complex and only their sum is, at it should, real.

When radicals, i.e. semi-integer powers are involved, the situation is more complicated. The integral of a semi-integer power,

\[
\int (x + a)^{n-1/2}dx = \frac{(x + a)^{n+1/2}}{n + 1/2} \quad (n \text{ is an integer}),
\]

is always a semi-integer power, i.e. an elementary function. There is no need, then, to introduce basis functions of this kind. On the other hand, integrals of products of radicals, such as

\[
I(\alpha, \beta) = \int (x + a)^\alpha (x + b)^\beta dx,
\]

with \( \alpha \) and \( \beta \) half integers, involve in general new transcendental functions\(^{11}\). By taking \( a = 0 \) and \( b = ±4 \) (or \( a = ±4 \) and \( b = 0 \)), we cover with \( I(\alpha, \beta) \) the cases related to the basis functions with indices \( ±r \). The idea is to shift the indices to reference values, such as \( \alpha = −1/2 \) and \( \beta = −1/2 \) in our (arbitrary) choice of the basis functions, by using recursive relations.

Let us consider the general case in which \( \text{both} \) indices differ from their target values: \( \alpha \neq −1/2 \) and \( \beta \neq −1/2 \). The integrand of (53) is conveniently rewritten as

\[
(x + a)^\alpha (x + b)^\beta = \frac{1}{\sqrt{(x + a)(x + b)}} (x + a)^{n}(x + b)^{l},
\]

\(^{11}\)On the contrary, products of integer powers do not require the introduction of new basis functions because partial fractioning completely disentangles the factors:

\[
\frac{1}{(x + a)^n} \frac{1}{(x + b)^k} = \frac{A_n}{(x + a)^n} + \cdots + \frac{A_1}{x + a} + \frac{B_n}{(x + b)^n} + \cdots + \frac{B_1}{x + b}.
\]
with \( n = \alpha + 1/2 \) and \( l = \beta + 1/2 \) general integers.

We first make an algebraic reduction on the integrand to take one of the two indices \( n \) and \( l \) to its target value zero. There are two possibilities:

- If 1 of the 2 indices is positive, let’s say \( n > 0 \), we expand \((x + a)^n\) in powers of \( x + b \) with the binomial formula:

\[
(x + a)^n = \sum_{s=0}^{n} \binom{n}{s} (a - b)^{n-s} (x + b)^s, \tag{56}
\]

so that the integrand takes the form

\[
(x + a)^\alpha (x + b)^\beta = \sum_{s=0}^{n} \binom{n}{s} (a - b)^{n-s} (x + a)^{-1/2}(x + b)^{s+l-1/2} \quad \text{(if } n > 0),
\]

in which the index \( \alpha \) reached its target value.

- If both indices are negative, \( n < 0 \) and \( l < 0 \), we do ordinary partial fractioning:

\[
\frac{1}{(x + a)^{|n|} (x + b)^{|l|}} = \frac{A_{|n|}}{(x + a)^{|n|}} + \frac{A_{|n|-1}}{(x + a)^{|n|-1}} + \cdots + \frac{A_1}{x + a} + \frac{B_{|l|}}{(x + b)^{|l|}} + \frac{B_{|l|-1}}{(x + b)^{|l|-1}} + \cdots + \frac{B_1}{x + b}, \tag{58}
\]

so that

\[
(x + a)^\alpha (x + b)^\beta = A_{|n|}(x + a)^{-1/2-|n|}(x + b)^{-1/2} + A_{|n|-1}(x + a)^{1/2-|n|}(x + b)^{-1/2} + \cdots + A_1 (x + a)^{-3/2} (x + b)^{-1/2} + B_{|l|}(x + a)^{-1/2}(x + b)^{-1/2-|l|} + B_{|l|-1}(x + a)^{-1/2}(x + b)^{1/2-|l|} + \cdots + B_1 (x + a)^{-1/2} (x + b)^{-3/2} \tag{59}
\]

(if \( n < 0 \) and \( l < 0 \)).

In any of the above terms on the l.h.s. one of the two indices reached its target value.

To take also the second index to its target value, we use the integral identity:

\[
I(\alpha, \beta) = \frac{1}{\alpha + \beta + 1} (x + a)^{\alpha + 1} (x + b)^\beta - \frac{\beta(a - b)}{\alpha + \beta + 1} I(\alpha, \beta - 1). \tag{60}
\]

The above relation is obtained by doing an integration by parts on \( I(\alpha, \beta) \): we integrate \((x + a)^\alpha\) and differentiate \((x + b)^\beta\) with respect to \( x \); we then simplify the ratio \((x + a)/(x + b)\) as \(1 + (a - b)/(x + b)\).

The first term on the r.h.s. is a finite one and is therefore known. The above equation can then be used to lower the index \( \beta \) by one unit. For example \( I(\alpha, 3/2) \) can
be transformed to a linear combination of $I(\alpha, 1/2)$ and $I(\alpha, -1/2)$, while $I(\alpha, 1/2)$ can be reduced to $I(\alpha, -1/2)$. In general, by recursively using equation (60), one can reduce any $\beta < -1/2$ to $\beta = -1/2$.

By solving Eq. (60) with respect to the last term on the r.h.s, $I(\alpha, \beta - 1)$, and shifting $\beta \to \beta + 1$, one obtains an identity to raise the index $\beta$ by one unit:

$$ I(\alpha, \beta) = \frac{(x + a)^{\alpha+1}(x + b)^{\beta+1}}{(\beta + 1)(a - b)} - \frac{\alpha + \beta + 2}{(\beta + 1)(a - b)} I(\alpha, \beta + 1). $$  \hspace{1cm} (61)

By using Eqs. (60) and (61) we can then take the index $\beta$ to any desired reference value, such as the value $-1/2$ of our basis. Then, we succeeded in reducing an integral of the form (53) to $H(\pm r; x)$ plus terms containing elementary functions only.

In general, we encounter integrals containing both radicals and GHPLs, of the form:

$$ J(\alpha, \beta) = \int (x + a)^\alpha (x + b)^\beta H(v, \vec{w}; x) dx, $$  \hspace{1cm} (62)

where $H(v, \vec{w}; x)$ is a GHPL whose first index $v$ has been separated out for a reason that will become clear soon. Since $\vec{w}$ has $w$ components $H(v, \vec{w}; x)$ has weight $w + 1$. As with the previous integral, we can transform one of the two indices $\alpha$ and $\beta$, let’s say $\alpha$, to its target value $\alpha = -1/2$. The identity to reduce the second index is:

$$ J(\alpha, \beta) = \frac{1}{\alpha + \beta + 1} (x + a)^{\alpha+1}(x + b)^\beta H(v, \vec{w}; x) - \frac{1}{\alpha + \beta + 1} \int (x + a)^{\alpha+1}(x + b)^\beta g(v; x) H(\vec{w}; x) dx - \frac{\beta(a - b)}{\alpha + \beta + 1} J(\alpha, \beta - 1). $$  \hspace{1cm} (63)

The second term on the r.h.s. involves the integration of a GHPL of smaller weight. We then consider the last term as the only relevant one in the recursion. As before, the above equation can be directly used to lower the index $\beta$ by one unit. Note that the indices inside the GHPL are not touched by the reduction. Analogously to the previous case, by solving Eq. (63) with respect to $J(\alpha, \beta - 1)$ and sending $\beta \to \beta + 1$, one obtains an identity to raise $\beta$ by one unit.

In some cases, it is also necessary to integrate expressions involving 3 factors, of the form:

$$ L(\alpha, \beta; k) = \int dx \ (x + a)^{\alpha} \ (x + b)^{\beta} \ (x + c)^{-k}, $$  \hspace{1cm} (64)

with $\alpha$ and $\beta$ half integers and $k$ a general integer. For our computation, the relevant cases are $a = 0$, $b = \pm 4$ and $c = \pm 1$ (or $a = \pm 4$, $b = 0$ and $c = \pm 1$).

12Since we can take $\alpha = -1/2$ and the equation is used for $\beta < -1/2$, it holds that $\alpha + \beta > -1$, so the singularity in $\alpha + \beta = -1$ is always avoided.

13Note the the singularity for $\beta = -1$ is never reached since $\beta$ takes half-integer values only.

14We have put a minus sign in front of $k$ just for practical convenience, in order to simplify the forthcoming results a little bit.
Let us consider the various possibilities for the $k$ index. For $k < 0$ one can reduce the integral (63) to the simpler form of $I(\alpha, \beta)$ in (56) by means of the binomial expansion of $(x + c)^{|k|}$ in powers of $x + a$ or of $x + b$:

$$(x + c)^{|k|} = \sum_{s=0}^{\lfloor |k| \rfloor} \binom{|k|}{s} (c - a)^{|k| - s} (x + a)^s = \sum_{s=0}^{\lfloor |k| \rfloor} \binom{|k|}{s} (c - b)^{|k| - s} (x + b)^s.$$  (65)

Therefore we just need to consider the case $k > 0$ from now on.

By means of the algebraic relations in (57) and (59), we can shift one of the half-integer indices $\alpha$ and $\beta$ to the target value $-1/2$; let us assume for instance that $\alpha = -1/2$. We can then assume the integral of the form:

$$L'(\beta; k) = L(-1/2, -1/2; k) = \int \frac{dx}{\sqrt{(x + a)(x + b)}} (x + b)^l (x + c)^{-k},$$  (66)

with $l = \beta + 1/2$ a general integer.

With an algebraic reduction analogous to the one in Eqs. (57) and (59), we can reduce the product $(x + b)^l (x + c)^{-k}$ to a linear combination of terms involving powers of either $(x + b)$ or $(x + c)$, but not the product. The terms not containing $(x + c)$ belong to the simpler class of the integrals $I(\alpha, \beta)$ defined in Eq. (53), whose reduction has already been discussed. The terms not containing $(x + b)$ have the indices $\alpha$ and $\beta$ both equal to their target value $-1/2$. Our task is then the evaluation of integrals of the form:

$$\tilde{L}(k) = L(-1/2, -1/2; k) = \int \frac{dx}{\sqrt{(x + a)(x + b)}} (x + c)^{-k},$$  (67)

with $k$ integer and strictly positive, $k \in \mathbb{N}$. The required identity is:

$$\tilde{L}(k) = -\frac{1}{k-1} (x + a)^{-1/2} (x + b)^{-1/2} (x + c)^{-k+1}$$

$$- \frac{1}{2(k-1)} \frac{1}{(c-a)^{k-1}} \int dx (x + a)^{-3/2} (x + b)^{-1/2}$$

$$- \frac{1}{2(k-1)} \frac{1}{(c-b)^{k-1}} \int dx (x + a)^{-1/2} (x + b)^{-3/2}$$

$$+ \frac{1}{2(k-1)} \sum_{t=1}^{k-1} \left[ \frac{1}{(c-a)^{k-t}} + \frac{1}{(c-b)^{k-t}} \right] \tilde{L}(l).$$  (68)

The second and the third terms on the r.h.s. do not contain any power of $(x + c)$, can be reduced with the previous identities and are therefore known terms. The last term contains integrals $\tilde{L}(l)$ of the same form as the one on the l.h.s.: $\tilde{L}(k)$. Eq. (68) is then to be used to relate $\tilde{L}$ integrals, treating the other terms as known functions. By using Eq. (68) for instance for $k = 3$ one can reduce $\tilde{L}(3)$ to a superposition of $\tilde{L}(2)$ and $\tilde{L}(1)$, while using it for $k = 2$ one can reduce $\tilde{L}(2)$ to $\tilde{L}(1)$. In general,
by recursively using Eq. (68), one can reduce any $k > 1$ to $k = 1$, i.e. to the basic integral\(^{15}\)
\[
\tilde{L}(1) = \int dx \frac{1}{\sqrt{(x + a)(x + b)(x + c)}}, \tag{69}
\]
which defines the $H(\pm r \pm 1; x)$’s.

The derivation of Eq. (68) is analogous to the ones of the previous identities: one has to integrate $(x + c)^{-k}$ and to differentiate $(x + a)^{-1/2}(x + b)^{-1/2}$ with respect to $x$. The resulting expressions have to be simplified using $(x + c)/(x + a) = 1 + (c - a)/(x + a)$ and an analogous equation with $b$ replacing $a$. One also needs the following result:

\[
\frac{1}{(x + c)^k} \frac{1}{x + a} = - \sum_{l=0}^{k-1} \frac{1}{(c - a)^l+1} \frac{1}{(x + c)^{k-l}} + \frac{1}{(c - a)^k} \frac{1}{x + a} \tag{70}
\]

and an analogous equation with $b$ replacing $a$.

Finally, the last class of integrals we need to consider is:

\[
F(\alpha, \beta; k) = \int dx \ (x + a)^\alpha (x + b)^\beta (x + c)^{-k} H(v, \bar{w}; x). \tag{71}
\]

As with the previous case, we can reduce ourselves to the case $\alpha = -1/2$, $\beta = -1/2$ and $k > 0$, i.e. to:

\[
\tilde{F}(k) = F(-1/2, -1/2; k) = \int dx \frac{dx}{\sqrt{(x + a)(x + b)(x + c)^k}} H(v, \bar{w}; x). \tag{72}
\]

The identity reads:

\[
\tilde{F}(k) = - \frac{1}{k-1} (x + a)^{-1/2}(x + b)^{-1/2}(x + c)^{-k+1} H(v, \bar{w}; x) \tag{73}
\]

\[- \frac{1}{2(k-1)} \frac{1}{(c - a)^{k-1}} \int dx \ (x + a)^{-3/2} (x + b)^{-1/2} H(v, \bar{w}; x) \]

\[- \frac{1}{2(k-1)} \frac{1}{(c - b)^{k-1}} \int dx \ (x + a)^{-1/2} (x + b)^{-3/2} H(v, \bar{w}; x) \]

\[+ \frac{1}{k-1} \int dx \ (x + a)^{-1/2} (x + b)^{-1/2} (x + c)^{-k+1} g(v; x)H(\bar{w}; x) \]

\[+ \frac{1}{2(k-1)} \sum_{l=1}^{k-1} \left[ \frac{1}{(c - a)^{k-l}} + \frac{1}{(c - b)^{k-l}} \right] \tilde{F}(l). \]

The fourth term on the r.h.s. involves the integration of an elementary function times a GHPL of weight $w$, while the l.h.s involves the integration of a GHPL of weight $w + 1$. This term then has a smaller recursive weight than $\tilde{F}$ and has to be considered as a known integral. For the rest, analogous considerations to the one of

\(^{15}\)Note that the singularity of the coefficients in Eq. (68) for $k = 1$ forbids any further reduction.
the previous identity hold. By recursively using Eq. (73), one can reduce any \( k > 1 \) to \( k = 1 \), i.e. to the basic integral
\[
\tilde{F}(1) = \int \frac{dx}{\sqrt{(x + a)(x + b)(x + c)}} H(v, \vec{w}; x),
\]
which defines the GHPLs of weight \( w + 2 \).

### 3.3 Closure under the transformation \( x \to 1/x \) — a further extension

The extended basis function set introduced in the previous Section preserves the following fundamental properties of the HPLs:

- the uniqueness of representation as repeated integration noted before [2];
- the fulfilling of an algebra. This means that the product of two GHPLs of weights \( w_1 \) and \( w_2 \) can be written as a sum of GHPLs of weight \( w_1 + w_2 \).

The ordinary HPLs, however, have also the property of “closure” under the inverse transformation \( x \to 1/x \), i.e. anyone of the functions \( g(\pm 1; x), g(0; x) \) transforms into a linear combination of the same functions under \( x \to 1/x \). This property, which is useful for the large momentum expansion of the MIs, is not preserved by the generalization discussed in the previous Section. One can impose this property on the GHPLs at the price of a second basis extension. Before doing that, however, let us discuss in detail how the expansion of the ordinary HPLs for large values of the argument \( x \) is performed. The followings steps are taken in order:

1. we set \( x = 1/y \) in the HPL under consideration:
\[
H(\vec{w}; x) = H(\vec{w}; 1/y);
\]
2. we use the identities which allow to reduce \( H(\vec{w}; 1/y) \) to a combination of \( H(\vec{w}'; y) \)'s;
3. we expand \( H(\vec{w}'; y) \) for \( y \to 0 \), i.e. for small value of the argument, as explained in some detail in Section 6;
4. we perform the inverse substitution \( y = 1/x \) in the final result.

As an example consider the simple weight-1 HPL
\[
H(-1; x) = \int_0^x \frac{dt}{1 + t} = \log (1 + x).
\]
We have:
\[
H(-1; x) = H(-1; 1/y) = \int_0^1 \frac{dt}{1 + t} + \int_1^{1/y} \frac{dt}{1 + t} = H(-1; 1) - \int_1^y \frac{dt'}{1 + t'} \left[ \frac{1}{t'} - \frac{1}{1 + t'} \right] = -H(0; y) + H(-1; y).
\]
where, moving from Eq. (78) to Eq. (79), we divided the integral into the sum of two integrals (the first between 0 and 1 and the second between 1 and 1/y) and we replaced $t$ by $1/t'$.

Expanding the r.h.s. of Eq. (80) in series of $y$ and re-expressing $y$ as $1/x$ we finally have:

$$H(-1; x) \xrightarrow{x \to \infty} \log x + \frac{1}{x} - \frac{1}{2x^2} + \mathcal{O}\left(\frac{1}{x^3}\right). \tag{81}$$

For the weight-2 HPL

$$H(0, -1; x) = \int_{0}^{x} \frac{dt}{t} H(-1; t), \tag{82}$$

we have:

$$H(0, -1; x) = H(0, -1; 1/y) \tag{83}$$

$$= H(0, -1; 1) + \int_{1}^{1/y} \frac{dt}{t} H(-1; t), \tag{84}$$

$$= H(0, -1; 1) - \int_{1}^{y} \frac{dt'}{t'} H(-1; 1/t'), \tag{85}$$

$$= H(0, -1; 1) - \int_{1}^{y} \frac{dt'}{t'} \left[-H(0; t') + H(-1; t')\right], \tag{86}$$

$$= 2H(0, -1; 1) + H(0, 0; y) - H(0, -1; y), \tag{87}$$

where, moving from Eq. (85) to Eq. (86) we used Eq. (80).

Expanding the r.h.s. of Eq. (87) in series of $y$ and re-expressing $y$ as $1/x$ we have finally:

$$H(0, -1; x) \xrightarrow{x \to \infty} 2H(0, -1; 1) + \frac{1}{2} \log^2 x - \frac{1}{x} + \frac{1}{4x^2} + \mathcal{O}\left(\frac{1}{x^3}\right). \tag{88}$$

Let us remark that performing the substitution $x = 1/y$ the basis functions $g(n, 1/y)$ are expressed in terms of the basis functions belonging to the same set, $g(n', y)$: this is the closure under the transformation $x \to 1/x$. Moreover, in the final result the constants $H(\vec{w}; 1)$ do appear. The latter have been systematically evaluated and tabulated in [17] up to weight 4 included.

Following exactly the same steps in the case of the GHPLs, we find that the set of basis functions given in the previous Section is too small to preserve the closure under the transformation $x \to 1/x$. We can understand it, for example, trying to expand the GHPL

$$H(4; x) = \int_{0}^{x} \frac{dt}{4 - t} = -\log (4 - x) - 2\log 2. \tag{89}$$

We have:

$$H(4; 1/y) = H(4; 1) + \int_{1}^{1/y} \frac{dt}{4 - t}, \tag{90}$$
\[ H(4; 1) - \frac{1}{4} \int_1^y \frac{dt'}{t' - \frac{1}{4}}. \]  
(91)

\[ H(4; 1) + \int_1^y \frac{dt'}{t' - \frac{1}{4}}. \]  
(92)

\[ H(4; 1) + H(0; y) - \int_1^y \frac{dt'}{t' - \frac{1}{4}}. \]  
(93)

The integral in Eq. (93) does not belong to the set of GHPLs of the previous Section. We then add to the basis functions defined in Eqs. (28–37) the following functions:

\[ g(\pm 1/4; x) = \frac{1}{x} + \frac{1}{4}, \]  
(94)

\[ g(\pm 1 \pm r/4; x) = \frac{1}{\sqrt{x \pm \frac{1}{4}(1 \mp x)}}, \]  
(95)

\[ g(r_0/4; x) = \frac{1 - 2i \sqrt{x - \frac{1}{4}}}{x \sqrt{x - \frac{1}{4}}}, \]  
(96)

\[ g(-r_0/4; x) = \frac{1 - 2\sqrt{x + \frac{1}{4}}}{x \sqrt{x + \frac{1}{4}}}. \]  
(97)

To avoid a pole in \( x = 0 \), we have subtracted \( g(0; x) = 1/x \) with a proper coefficient, the residue in \( x = 0 \) of the new functions. In taking the limit \( x \to 0 \) inside square roots one has to remember that \( x \to x - i\epsilon \), with \( \epsilon \) a small positive number. The related GHPLs of weight 1 are defined as usual:

\[ H(w; x) = \int_0^x g(w; t)dt, \]  
(98)

since all the functions (94–97) are integrable in \( x = 0 \).

As an example, the reader may verify that:

\[ H(4, -r; x) = H(4, -r; 1/y) \]  
(99)

\[ = -H(4; 1)H(r/4; 1) + \frac{1}{2}H(-1, -r_0/4; 1) \]

\[ + H(1, r/4; 1) - \frac{1}{2}H(-r_0/4, 4; 1) \]

\[ + \left[ H(r/4; 1) + \frac{1}{2}H(-r_0/4; 1) \right] \left[ H(0; y) + H(4; y) \right] \]

\[ - \frac{1}{2}H(4, -r_0/4; y) - \frac{1}{2}H(-1, -r_0/4; y). \]  
(100)

Expanding the GHPLs for \( y \to 0 \) and substituting in the final expression \( y = 1/x \) we finally obtain:

\[ H(4, -r; x) \xrightarrow{x \to \infty} -H(4; 1)H(r/4; 1) + \frac{1}{2}H(-1, -r_0/4; 1) \]

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\[ +H(1, r/4; 1) - \frac{1}{2}H(-r_0/4, 4; 1) \]
\[ - \left[ H(r/4; 1) + \frac{1}{2}H(-r_0/4; 1) \right] \log x + \mathcal{O}\left(\frac{1}{x}\right). \tag{101} \]

As in the case of the HPLs, the final expression contains the constants \(H(\vec{w}; 1)\), which have to be evaluated. The \(x = 1\) GHPLs of weight one, \(H(a; 1)\), can be expressed in terms of known transcendental constants in an elementary way. In Eq. (101), the coefficient of the leading logarithm is, for example:
\[ H(r; 1) + \frac{1}{2}H(-r_0/4; 1) = \frac{\pi}{3} - 2[\log (1 + \sqrt{5}) - \log 2]. \tag{102} \]

Similar reductions are also possible in other cases but, in general, it is non trivial to reduce the \(H(\vec{w}; 1)\)'s with a general weight \(\vec{w}\) to a minimal set containing known transcendental constants such as \(\zeta(n)\) and eventual new transcendental constants.

The reduction of the \(H(\vec{w}; 1)\)'s to a minimal set of transcendental constants is beyond the goal of the present paper \[18\]; we restrict ourselves to the evaluation of the asymptotic expansions of the diagrams involving only ordinary HPLs.

4 Results for the master integrals

In this Section we present the results of our computation of the MIs involving up to 6 denominators included, which constitute a necessary input for the calculation of the 2-loop vertex diagrams in Figs. 1 and 2. They are expanded in a Laurent series in
\[ \epsilon = 2 - D/2, \tag{103} \]
up to the required order in \(\epsilon\). The coefficients of the series are expressed in terms of GHPL's (see Section 3) of the variable \(x\), defined as:
\[ x = -\frac{s}{a}, \tag{104} \]
where \(s = -q^2\) is the c.m. energy squared\(^{16}\). It holds \(q = p_1 + p_2\) and we defined \(a = m^2\). We denote by \(\mu\) the mass scale of the Dimensional Regularization (DR) — the so-called unit of mass. We work in Minkowski space and we normalize the loop measures as:
\[ \mathcal{D}^D k = \frac{d^D k}{i\pi^{D/2} \Gamma(3 - \frac{D}{2})} = \frac{d^{4-2\epsilon} k}{i\pi^{2-\epsilon} \Gamma(1 + \epsilon)}. \tag{105} \]

This definition makes the expression of the 1-loop tadpole — the simplest of all loop diagrams — particularly simple \[11\]. The denominators appearing in the master integrals are listed below:

\(^{16}\)We define the scalar product of two 4-vectors as: \(a \cdot b = -a_0 b_0 + \vec{a} \cdot \vec{b}\).
\[ D_1 = k_1^2, \]  
\[ D_2 = k_2^2, \]  
\[ D_3 = (k_1 + k_2)^2, \]  
\[ D_4 = (p_1 - k_1)^2, \]  
\[ D_5 = (p_2 + k_1)^2, \]  
\[ D_6 = (p_2 - k_2)^2, \]  
\[ D_7 = (p_1 - k_1 + k_2)^2, \]  
\[ D_8 = (p_2 + k_1 - k_2)^2, \]  
\[ D_9 = (p_1 + p_2 - k_1)^2, \]  
\[ D_{10} = (p_1 + p_2 - k_2)^2, \]  
\[ D_{11} = (p_1 + p_2 - k_1 - k_2)^2, \]  
\[ D_{12} = k_1^2 + a, \]  
\[ D_{13} = k_2^2 + a, \]  
\[ D_{14} = (k_1 + k_2)^2 + a, \]  
\[ D_{15} = (p_1 - k_1)^2 + a, \]  
\[ D_{16} = (p_2 + k_1)^2 + a, \]  
\[ D_{17} = (p_2 - k_2)^2 + a, \]  
\[ D_{18} = (p_1 - k_1 + k_2)^2 + a, \]  
\[ D_{19} = (p_2 + k_1 - k_2)^2 + a, \]  
\[ D_{20} = (p_1 + p_2 - k_1)^2 + a, \]  
\[ D_{21} = (p_1 + p_2 - k_2)^2 + a, \]  
\[ D_{22} = (p_1 + p_2 - k_1 - k_2)^2 + a. \]

The MIs are listed according to the increasing number of the denominators, which corresponds more or less to the level of complexity.

### 4.1 Topology \( t = 3 \)

\[
\begin{align*}
\mathcal{D} &= \mathcal{D}_{12} \mathcal{D}_{13} \mathcal{D}_{14} \\
&= \mu^{2(4-D)} \int \mathcal{D}_{k_1} \mathcal{D}_{k_2}, \quad \frac{1}{D_{12} D_{13} D_{14}} \\
&= \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-2}^{2} \epsilon^i F^{(1)}_i + O(\epsilon^3),
\end{align*}
\]

where:

\[
\frac{F^{(1)}_{-2}}{a} = -\frac{3}{2},
\]
\[
\begin{align*}
F^{(1)}_{-1} &= -\frac{9}{2}, \\
F^{(1)}_0 &= -\frac{21}{2} - \sqrt{3}H(r, 0; 1), \\
F^{(1)}_1 &= -\frac{45}{2} - \sqrt{3}\left[3H(r, 0; 1) + H(r, 0, 0; 1) + H(4, r, 0; 1)\right], \\
F^{(2)}_2 &= -\frac{93}{2} - \sqrt{3}\left[7H(r, 0; 1) + 3H(r, 0, 0; 1) + 3H(4, r, 0; 1) \\
&\quad + H(r, 0, 0, 0; 1) + H(4, r, 0, 0; 1) + H(4, 4, r, 0; 1)\right].
\end{align*}
\] (131)

(132)

(133)

(134)

This diagram has been originally evaluated in ref. [19].

Even though the above MI is a vacuum amplitude, its computation is non-trivial because of the presence of 3 massive propagators. We have evaluated it with the following method. We consider a vacuum sunrise with 2 internal lines with equal mass \(m\) and the third internal line with the different mass (squared) \(m' = zm^2\). We then differentiate the vacuum sunrise with respect to \(z\) and rewrite the result in terms of MIs by using the ibps identities. The resulting differential equation represents the evolution in one of the masses and is solved as in usual cases. We set at the end \(z = 1\) to obtain the equal-mass case.

Let us make a few remarks about the above result:

- The finite part of the MI, i.e. the \(O(\epsilon^0)\), involves 1 transcendental constant: \(H(r, 0; 1)\), related to the Clausen function;
- the \(O(\epsilon)\) part involves 2 new independent transcendental constants: \(H(r, 0, 0; 1)\) and \(H(4, r, 0; 1)\);
- the \(O(\epsilon^2)\) part involves 3 new transcendental constants: \(H(r, 0, 0, 0; 1)\), \(H(4, r, 0, 0; 1)\) and \(H(4, 4, r, 0; 1)\).

To simplify the above expressions, we have used the following identities to move the “0” indices to the right:

\[
\begin{align*}
H(0, r; 1) &= -H(r, 0; 1), \\
H(0, r, 0; 1) &= -2H(r, 0, 0; 1), \\
H(0, r, 0, 0; 1) &= -3H(r, 0, 0, 0; 1), \\
H(0, 0, r, 0; 1) &= 3H(r, 0, 0, 0; 1), \\
H(0, 4, r, 0; 1) &= -H(4, 0, r, 0; 1) - 2H(4, r, 0, 0; 1).
\end{align*}
\] (135)

The above relations are obtained by transforming products of \(H\)’s into linear combinations of \(H\)’s, as for example in:

\[0 = H(0; 1)H(r; 1) = H(0, r; 1) + H(r, 0; 1).\] (136)
4.2 Topology $t = 4$

\[
\begin{align*}
F_{2(k)}^{(2)} & = \frac{1}{2}, \\
F_{-1}^{(2)} & = \frac{5}{2} - \frac{x + 4}{\sqrt{x(x + 4)}} H(-r; x), \\
F_0^{(2)} & = \frac{19}{2} + \zeta(2) - \frac{1}{2} H(0, -1; x) - \frac{x + 4}{\sqrt{x(x + 4)}} \left[ 4H(-r; x) - \frac{3}{2} H(-r, -1; x) ight] \\
& - H(-4, -r; x) \left[ \frac{1}{x} + 1 \right] H(-1; x), \\
F_1^{(2)} & = \frac{65}{2} + 5\zeta(2) - \zeta(3) - 3H(0, -1; x) + 2H(0, -1, -1; x) - \frac{1}{2} H(0, 0, -1; x) \\
& + \left[ \frac{1}{x} + 1 \right] \left\{ -7H(-1; x) + 4H(-1, -1; x) - H(0, -1; x) \right\} \\
& + \frac{x + 4}{\sqrt{x(x + 4)}} \left\{ -2(6 + \zeta(2)) H(-r; x) + 6H(-r, -1; x) - 6H(-r, -1, -1; x) \\
& + 3H(-r, 0, -1; x) + 4H(-4, -r; x) - \frac{3}{2} H(-4, -r, -1; x) \\
& - H(-4, -4, -r; x) \right\}, \\
F_2^{(2)} & = \frac{211}{2} + 19\zeta(2) + \frac{9}{5} \zeta^2(2) - 5\zeta(3) - \left[ 13 + \zeta(2) \right] H(0, -1; x) \\
& - 8H(0, -1, -1; x) + 3H(0, -1, 0, -1; x) + 12H(0, -1, -1; x) \\
& - 3H(0, 0, -1; x) + 2H(0, 0, -1, -1; x) - \frac{1}{2} H(0, 0, 0, -1; x) \\
& + \frac{x + 4}{\sqrt{x(x + 4)}} \left\{ 2(\zeta(3) - 4\zeta(2) - 16) H(-r; x) + (6 + \zeta(2)) [3H(-r, -1; x) \\
& + 2H(-4, -r; x)] - 24H(-r, -1, -1; x) + 24H(-r, -1, -1, -1; x) \\
& - 9H(-r, -1, 0, -1; x) + 12H(-r, 0, -1; x) - 12H(-r, 0, -1, -1; x) \\
& + 3H(-r, 0, 0, -1; x) - 6H(-4, -r, -1; x) + 6H(-4, -r, -1, -1; x) \\
& - 3H(-4, -r, 0, -1; x) - 4H(-4, -4, -r; x) + \frac{3}{2} H(-4, -4, -r, -1; x) \\
& + H(-4, -4, -4, -r; x) \right\} - \left[ \frac{1}{x} + 1 \right] \left\{ (33 + 2\zeta(2)) H(-1; x) \\
& - 8H(-1, -1; x) + 4H(-1, -1, -1; x) - 2H(-1, -1, -1, -1; x) \\
& - 3H(-1, 0, -1; x) + 6H(-1, -1, -1; x) - 6H(-1, -1, -1, -1; x) \\
& + 3H(-1, 0, 0, -1; x) - 4H(-1, -1, -1, -1; x) \right\}.
\end{align*}
\]
\[-28H(-1,-1;x) + 16H(-1,-1,-1;x) - 6H(-1,0,-1;x)
+7H(0,-1;x) - 4H(0,-1,-1;x) + H(0,0,-1;x)\}.

The above 2-point function is the simplest amplitude having thresholds both in $s = m^2$ and $s = 4m^2$. It does not have any pseudo-threshold. The indices appearing in the GHPLs are indeed only $0$, $-1$, $-4$ and $-r$. Note that the index $-r$ appears eventually only once in the $H$’s. The related terms have coefficients always containing radicals, in order to reproduce the right causality structure.

$$\frac{\mu^2(4-D)}{\mathcal{D}^D k_1 \mathcal{D}^D k_2} \frac{1}{\mathcal{D}_5^D \mathcal{D}_7^D \mathcal{D}_1^D \mathcal{D}_3^D}$$

$$= \left(\frac{\mu^2}{a}\right)^{2\epsilon} \sum_{i=-2}^{1} \epsilon^{i} F^{(3)}_{i} + \mathcal{O}(\epsilon^{3}),$$

where:

\begin{align*}
F^{(3)}_{-2} &= \frac{1}{2}, \\
F^{(3)}_{-1} &= \frac{3}{5}, \\
F^{(3)}_{0} &= \frac{5}{2} + H(-1,x) - H(0,-1,x) + \frac{1}{x} \left[ H(-1,x) - \zeta(2)H(1,x) + 2H(1,0,-1,x) \right], \\
F^{(3)}_{1} &= -\frac{1}{2} - \zeta(2) + 7H(-1,x) - \zeta(2)H(1,x) - 4H(-1,-1,x) - 2H(0,-1,x) + 4H(0,-1,-1,x) + H(0,0,-1,x) + 2H(1,0,-1,x) + \frac{1}{x} \left[ 7H(-1,x) - (\zeta(2) - \zeta(3))H(1,x) - 4H(-1,-1,x) + H(0,-1,x) - \zeta(2)H(0,1,x) + 2H(0,1,0,-1,x) + 2H(1,0,-1,x) - 8H(1,0,-1,-1,x) \right].
\end{align*}

$$\frac{\mu^2(4-D)}{\mathcal{D}^D k_1 \mathcal{D}^D k_2} \frac{p_2 \cdot k_2}{\mathcal{D}_5^D \mathcal{D}_7^D \mathcal{D}_1^D \mathcal{D}_3^D}$$

$$= \left(\frac{\mu^2}{a}\right)^{2\epsilon} \sum_{i=-2}^{2} \epsilon^{i} F^{(4)}_{i} + \mathcal{O}(\epsilon^{3}),$$

where:

\begin{align*}
F^{(4)}_{-2} &= -\frac{1}{8} x, \\
F^{(4)}_{-1} &= -\frac{5}{16} x,
\end{align*}
\[
\frac{F_0^{(4)}}{a} = \frac{1}{8} - \frac{1}{2} H(-1, x) - \frac{1}{8x} H(-1, x) - x \left[ \frac{7}{32} + \frac{3}{8} H(-1, x) - \frac{1}{4} H(0, -1, x) \right], \tag{154}
\]

\[
\frac{F_1^{(4)}}{a} = \frac{9}{16} + \frac{1}{4} \zeta(2) - 3 H(-1, x) + 2 H(-1, -1, x) - H(0, -1, x) - \frac{1}{x} \left[ \frac{7}{16} H(-1, x) + \frac{1}{4} H(1, 0, -1, x) \right] + \frac{1}{2} H(-1, -1, x) + \frac{1}{8} H(0, -1, x) - \frac{1}{2} H(1, 0, -1, x)
+ x \left[ \frac{123}{64} + \frac{3}{8} \zeta(2) - \frac{41}{16} H(-1, x) + \frac{1}{4} \zeta(2) H(1, x) + \frac{3}{2} H(-1, -1, x) \right] + \frac{1}{4} H(0, -1, x) - H(0, -1, -1, x) - \frac{1}{4} H(0, 0, -1, x) - \frac{1}{2} H(1, 0, -1, x) \right], \tag{155}
\]

\[
\frac{F_2^{(4)}}{a} = \frac{39}{32} + \frac{1}{4} \zeta(2) - \frac{25}{2} \frac{1}{4} H(-1, x) - \zeta(2) H(-1, x) - \frac{1}{2} \zeta(2) H(1, x)
+ 12 H(-1, -1, x) - \frac{21}{2} H(0, -1, x) - 8 H(-1, -1, -1, x) + 3 H(-1, 0, -1, x)
+ 4 H(0, -1, -1, x) - \frac{1}{2} H(0, 0, -1, x) + H(1, 0, -1, x) - \frac{1}{x} \frac{21}{32} H(-1, x)
+ \frac{1}{4} \zeta(2) H(-1, x) + \frac{1}{2} \zeta(2) H(1, x) - \frac{1}{4} \zeta(3) H(1, x) - \frac{7}{4} H(-1, -1, x)
+ \frac{7}{16} H(0, -1, x) + \frac{1}{4} \zeta(2) H(0, 1, x) + 2 H(-1, -1, x) - \frac{3}{4} H(-1, 0, -1, x)
- \frac{1}{2} H(0, -1, -1, x) + \frac{1}{8} H(0, 0, -1, x) - H(0, 0, -1, x) - \frac{1}{2} H(0, 1, -1, x)
+ 2 H(1, 0, -1, x) \right] + x \left[ \frac{1681}{128} + \frac{41}{16} \zeta(2) - \frac{3}{8} \zeta(3) - \frac{379}{32} H(-1, x)
- \frac{3}{4} \zeta(2) H(-1, x) + \zeta(2) H(1, x) - \frac{1}{4} \zeta(3) H(1, x) + \frac{41}{4} H(-1, -1, x)
+ \frac{17}{8} H(0, -1, x) + \frac{1}{2} \zeta(2) H(0, -1, x) - \frac{1}{4} \zeta(2) H(0, 1, x) - 6 H(-1, -1, -1, x)
+ \frac{9}{4} H(-1, 0, -1, x) - H(0, -1, -1, x) - \frac{7}{4} H(0, 0, -1, x) - 2 H(1, 0, -1, x)
+ 4 H(0, -1, -1, -1, x) - \frac{3}{2} H(0, -1, 0, -1, x) + H(0, 0, -1, -1, x)
+ \frac{3}{4} H(0, 0, 0, -1, x) + \frac{1}{2} H(0, 1, 0, -1, x) + 2 H(1, 0, -1, -1, x) \right]. \tag{156}
\]

The above 2 MIs, which contain 2 massive denominators, can be expressed in terms of ordinary HPLs. The reason is that the 2 massive lines, roughly speaking, are in different channels: one is in the $s$ channel while the other is in the $t$ channel. The amplitudes do not have thresholds/pseudothresholds in $s = \pm 4m^2$, but only in $s = \pm m^2$. Both the indices “1” and “−1” do indeed appear inside the HPLs. The presence of 2 massive denominators is then a necessary but not a sufficient condition in order to have thresholds or pseudothresholds in $s = \pm 4m^2$. 

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where:

\[ F_{-2}^{(5)} = \frac{1}{2}, \quad (159) \]
\[ F_{-1}^{(5)} = \frac{5}{2} - H(0, x), \quad (160) \]
\[ F_{0}^{(5)} = \frac{19}{2} - \zeta(2) - 5H(0, x) + H(0, 0, x) + \frac{4 - x}{x\sqrt{4-x}}H(r, 0, x) \]
\[ + \frac{2}{x}H(r, r, 0, x), \quad (161) \]
\[ F_{1}^{(5)} = \frac{65}{2} - 5\zeta(2) - 2\zeta(3) - 19H(0, x) + \zeta(2)H(0, x) - H(r, r, 0, x) \]
\[ + 5H(0, 0, x) - H(0, 0, 0, x) + \frac{4 - x}{\sqrt{x(4-x)}}\left[\zeta(2)H(r, x) + 5H(r, 0, x) \right. \]
\[ - H(r, 0, 0, x) - H(0, r, 0, x) + 2H(4, 0, 0, x) \left] + \frac{2}{x}\zeta(2)H(r, r, x) \right. \]
\[ + 3H(r, r, 0, x) - H(r, r, 0, 0, x) - H(r, 0, r, 0, x) + 2H(4, r, 0, x) \]
\[ + H(0, 0, r, 0, x) \right] \right] \right]. \quad (162) \]

The double and the simple poles in the MI above have ultraviolet origin. The amplitude has indeed an ultraviolet sub-divergence related to the integration of the bubble together with an over-all UV divergence.

\[ = \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_7 D_8 D_12 D_{13}} \quad (157) \]
\[ = \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-2}^2 \epsilon^i F_i^{(6)} + \mathcal{O} (\epsilon^3), \quad (158) \]

where:

\[ \frac{F_{-2}^{(6)}}{a} = -\frac{1}{x}, \quad (165) \]
\[ \frac{F_{-1}^{(6)}}{a} = \frac{1}{x} H(0, x), \quad (166) \]
\[ \frac{F_{0}^{(6)}}{a} = -\frac{1}{x} \left[ 4 - \zeta(2) - 2H(0, x) + H(0, 0, x) \right] - \frac{4 - x}{x\sqrt{x(4-x)}}H(r, 0, x), \quad (167) \]
\[
\frac{F_1^{(6)}}{a} = \frac{1}{x} \left[ 2\zeta(2) + 2\zeta(3) + 4H(0, x) - \zeta(2)H(0, x) - 2H(0, 0, x) + H(r, r, 0, x) + H(0, 0, 0, x) \right] - \frac{4 - x}{x\sqrt{x(4 - x)}} \left[ \zeta(2)H(r, x) + 2H(0, 0, x) - H(0, r, x) \right] - H(0, r, 0, x) + 2H(4, r, 0, x),
\]

\[
\frac{F_2^{(6)}}{a} = \frac{1}{x} \left[ 16 - 4\zeta(2) - \frac{9}{10} \zeta^2(2) - 4\zeta(3) - 2(4 - \zeta(2) - \zeta(3))H(0, x) + H(0, 0, x) - \zeta(2)H(r, x) - 2H(r, 0, x) + H(r, r, 0, x) + H(r, r, 0, 0) - 2H(4, r, 0, x) - 2H(0, 0, 0, x) + H(0, 0, 0, x) \right] - \frac{4 - x}{x\sqrt{x(4 - x)}} \left[ 2\zeta(2)H(r, x) + 2\zeta(3)H(r, x) + 4H(r, 0, x) + (4 - \zeta(2))H(0, 0, x) - \zeta(2)H(0, r, x) - 2H(0, r, 0, x) + H(0, r, 0, 0) - 2H(0, r, 0, x) - 2H(4, r, 0, x) + H(0, 0, 0, x) - 2H(4, r, 0, 0, x) - 2H(4, r, 0, 0, x) + 4H(4, r, 0, 0, x) \right].
\]

The double pole in \( \epsilon \) in the MI above is the product of a simple UV pole coming from the nested bubble and of an IR pole coming from the massless line squared.

The above 2 MIs have a pseudothreshold in \( s = -4m^2 \) related to the exchange of 2 massive particles in the \( t \) channel and consequently only GQPLs with indices 0, 4, and \( r \) do appear in the \( \epsilon \) expansion. These MIs have also been computed in [20] by means of a transformation well-known in QED eliminating the square roots:

\[
x = \frac{(1 + z)^2}{z}.
\]

In general, this change of variable is very convenient for amplitudes not having the pseudothreshold in \( s = -m^2 \).

4.3 Topology \( t = 5 \)

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{t5_diagram}
\end{array}
\]

\[
= \mu^{2(4-D)} \int \mathcal{D}k_1 \mathcal{D}k_2 \frac{1}{\mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_{11} \mathcal{D}_{12} \mathcal{D}_{20}}
\]

\[
= \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=0}^{1} \epsilon^i F_i^{(7)} + O(\epsilon^3),
\]

where:

\[
aF_0^{(7)} = \frac{1}{x} \left[ H(0, 0, -1, x) + 3H(-r, -r, -1, x) - 2H(-r, -r, 0, x) - 2H(-r, 0, -r, x) \right],
\]

(173)
The above 2-point function has thresholds in $s = 0, m^2$ as well as in $s = 4m^2$ and no pseudothresholds. The index $r$ always appears twice in the $H$’s and the related coefficients contain no radicals. The finite part $O(\epsilon^0)$ of $F^{(7)}$ has been computed by the authors of [21] by means of a resummed small momentum expansion. With the help of the Mathematica [22] we have computed the first 20 terms of the small momentum expansion of our result and compared with their result, finding complete agreement.

\[ a F^{(7)}_1 = \frac{1}{x} \left[ 2H(0, 0, -1, x) - 6\zeta(2)H(-r, -r, x) + 6H(-r, -r, -1, x) \\
-4H(-r, -r, 0, x) - 4H(-r, 0, -r, x) - 4H(0, 0, -1, -1, x) \\
+H(0, 0, 0, -1, x) - 12H(-r, -r, -1, -1, x) + 6H(-r, -r, 0, -1, x) \\
+2H(-r, -r, 0, 0, x) - 3H(-r, -4, -r, -1, x) + 2H(-r, -4, -r, 0, x) \\
+2H(-r, -4, 0, -r, x) + 2H(-r, 0, -r, 0, x) + 2H(-r, 0, -4, -r, x) \\
+2H(-r, 0, 0, -r, x) \right]. \tag{174} \]

where:

\[ F^{(8)}_0 = -\frac{1}{x} \left[ \zeta(2) \left( H(0, c, x) + 2H(0, 1, x) + H(0, \overline{c}, x) \right) - 3H(0, \overline{c}, 0, -1, x) \\
-3H(0, c, 0, -1, x) - 4H(0, 1, 0, -1, x) \right] + \frac{i}{x} H(r, 0, 1) \left[ H(0, c, x) \\
-H(0, \overline{c}, x) \right]. \tag{177} \]

where:

\[ (p_2 \cdot k_1) = \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{\mathcal{D}_4 \mathcal{D}_6 \mathcal{D}_12 \mathcal{D}_13 \mathcal{D}_14} \tag{178} \]

\[ = \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=0}^{1} \epsilon^i F^{(9)}_i + \mathcal{O}(\epsilon^3), \tag{179} \]

where:

\[ F^{(9)}_0 = -1 + H(-1, x) - \zeta(2)H(1, x) - \frac{1}{2} \zeta(2) \left[ H(\overline{c}, x) + H(c, x) \right] \\
-H(0, -1, x) + 2H(1, 0, -1, x) + \frac{3}{2} \left[ H(\overline{c}, 0, -1, x) + H(c, 0, -1, x) \right] \\
+ \frac{1}{x} \left\{ \frac{1}{4} \left( \sqrt{3}H(r, 0, 1) + \zeta(2) \right) \left[ H(c, x) + H(\overline{c}, x) \right] - \frac{3}{4} H(\overline{c}, 0, -1, x) \right\} \]
\[
\begin{aligned}
\frac{3}{4}H(c, 0, -1, x) + H(-1, x) + \zeta(2)H(1, x) - 2H(1, 0, -1, x) \\
+ \frac{i}{4}\left\{ H(r, 0, 1) \left( 2 - \frac{1}{x} \right) \left[ H(c, x) - H(\tau, x) \right] + \sqrt{\frac{3}{x}} \left[ \zeta(2) \left( H(c, x) - H(\tau, x) \right) \right] \right\} - H(\tau, x) - 3H(c, 0, -1, x) + 3H(\tau, 0, -1, x) \bigg) \right) 
\end{aligned}
\]

\( F_1^{(9)} = -7 - \zeta(2) + 6H(-1, x) - 2\zeta(2)H(1, x) + \zeta(3)H(1, x) - \frac{1}{2}(\zeta(2) - \zeta(3)) \left[ H(c, x) + H(\tau, x) \right] - 4H(-1, -1, x) - H(0, -1, x) - 3\zeta(2)H(0, 1, x) - \frac{3}{2}\zeta(2) \left[ H(0, c, x) + H(0, \tau, x) \right] + \zeta(2) \left[ H(c, c, x) + H(\tau, \tau, x) \right] + \frac{1}{2}\zeta(2) \left[ H(\tau, c, x) + H(c, \tau, x) \right] + 2\zeta(2) \left[ H(\tau, 1, x) + H(1, 1, x) \right] + 4H(0, -1, -1, x) + H(0, 0, -1, x) + 4H(1, 0, -1, x) + 3 \left[ H(\tau, 0, -1, x) + H(0, \tau, 0, -1, x) \right] - \frac{3}{2} \left[ H(c, \tau, 0, -1, x) \right] + \frac{3}{2} \left[ H(\tau, 0, -1, x) \right] - 6 \left[ H(\tau, 0, -1, x) + H(c, 0, -1, -1, x) \right] - \frac{1}{2} \left[ H(\tau, 0, 0, -1, x) + H(c, 0, 0, -1, x) \right] - 4 \left[ H(\tau, 1, 0, -1, x) + H(c, 1, 0, -1, x) \right] + 9 \left[ H(0, \tau, 0, -1, x) + H(0, c, 0, -1, x) \right] + 6H(0, 1, 0, -1, x) - 8H(1, 0, -1, -1, x) + \frac{1}{4x} \sqrt{3} \left[ \left[ H(r, 0, 1) \right] \left[ H(c, x) + H(\tau, x) \right] + H(r, 0, 1) \left[ H(c, \tau, x) \right] + H(\tau, c, x) - 2H(c, c, x) - 2H(\tau, \tau, x) + H(0, \tau, x) + H(0, c, x) \right] - H(0, r, 0, 1) \left[ H(c, x) + H(-c, x) + H(4, r, 0, 1) \left[ H(c, x) + H(\tau, x) \right] \right] \right\} + \frac{i}{2} \left\{ H(r, 0, 1) + H(r, 0, 0, 1) + H(4, r, 0, 1) \left[ H(c, x) - H(\tau, x) \right] \right\} + H(r, 0, 1) \left[ H(c, \tau, x) - H(\tau, c, x) - 2H(c, c, x) + 2H(\tau, \tau, x) \right] + 3H(0, c, x) - 3H(0, \tau, x) - \frac{1}{2x} \left[ H(r, 0, 1) + H(r, 0, 0, 1) \right] + H(4, r, 0, 1) \left[ H(c, x) - H(\tau, x) \right] + H(r, 0, 1) \left[ H(c, \tau, x) \right] - H(\tau, c, x) - 2H(c, c, x) + 2H(\tau, \tau, x) + H(0, c, x) - H(0, \tau, x) \right] + \frac{\sqrt{3}}{2x} \left[ \left( \zeta(2) - \zeta(3) \right) \left( H(c, x) - H(\tau, x) \right) \right] + \zeta(2)H(0, c, x) - 4\zeta(2)H(c, 1, x) + 4\zeta(2)H(\tau, 1, x) + 2\zeta(2)H(c, c, x) - \zeta(2)H(0, \tau, x) - 4\zeta(2)H(c, 1, x) + 4\zeta(2)H(\tau, 1, x) - 2\zeta(2)H(c, c, x) \right) 
\]
\[+2\zeta(2)H(\overline{c}, x) - \zeta(2)H(c, x) + \zeta(2)H(\overline{c}, c, x) - 3H(c, 0, -1, x)\
+3H(\overline{c}, 0, -1, x) + 8H(c, 1, 0, -1, x) - 8H(\overline{c}, 1, 0, -1, x)\
-3H(c, 0, -1, x) + 3H(\overline{c}, 0, -1, x) + 3H(c, 0, -1, x)\
-3H(c, 0, -1, x) + 6H(c, 0, -1, x) - 6H(\overline{c}, 0, -1, x)\
+H(c, 0, -1, x) - H(\overline{c}, 0, -1, x) - 3H(0, c, 0, -1, x)\
+3H(0, \overline{c}, 0, -1, x)\right\}.

(181)

\[
\mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_4 D_6 D_{12} D_{13} D_{14}^2}
= \left(\frac{\mu^2}{a}\right)^2 \sum_{i=0}^1 i^2 F_i^{(10)} + O(e^3),
\]

(182)

where:

\[a^2 F_0^{(10)} = \frac{1}{x} \left\{ \frac{\sqrt{3}}{3} H(r, 0, 1) \left[ H(c, x) + H(\overline{c}, x) \right] + i \frac{\sqrt{3}}{3} \left[ \zeta(2) \left( H(c, x) - H(\overline{c}, x) \right) - 3H(c, 0, -1, x) + 3H(\overline{c}, 0, -1, x) \right] \right\},
\]

(184)

\[a^2 F_1^{(10)} = \frac{1}{x} \left\{ \frac{\sqrt{3}}{3} \left[ H(r, 0, 0, 1) + H(4, r, 0, 1) \right] \left[ H(c, x) + H(\overline{c}, x) \right] + H(r, 0, 1) \left[ H(c, \overline{c}, x) + H(\overline{c}, c, x) - 2H(c, c, x) - 2H(\overline{c}, \overline{c}, x) \right] + H(0, c, x) + H(0, \overline{c}, x) \right\} - i \frac{\sqrt{3}}{3} \left[ \zeta(3) \left( H(c, x) - H(\overline{c}, x) \right) - \zeta(2) \left( H(0, c, x) - H(0, \overline{c}, x) - 4H(c, 1, x) + 4H(\overline{c}, 1, x) - 2H(c, c, x) + 2H(\overline{c}, \overline{c}, x) \right) + H(c, 0, -1, x) - 2H(\overline{c}, 0, -1, x) + 2H(\overline{c}, c, 0, -1, x) + H(\overline{c}, 0, -1, x) - H(\overline{c}, 0, 0, -1, x) - \frac{1}{3} H(c, 0, 0, -1, x) + \frac{1}{3} H(\overline{c}, 0, 0, -1, x) + \frac{8}{3} H(c, 1, 0, -1, x) + \frac{8}{3} H(\overline{c}, 1, 0, -1, x) \right\}.
\]

(185)

The above 3 MIs are real as complex \(H(\cdots c \cdots ; x)\)'s always appear in the combinations: \(H(\cdots c \cdots ; x) + H(\cdots \overline{c} \cdots ; x)\) and \(i[H(\cdots c \cdots ; x) - H(\cdots \overline{c} \cdots ; x)]\). As discussed in the previous Section, the above topology is the only one having 3 MIs.

The amplitudes have a threshold in \(s = m^2\) in agreement with Cutkowsky rule as well as a pseudo thresholds in \(s = -m^2\).
\[ = \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4 \mathcal{D}_5 \mathcal{D}_6 \mathcal{D}_7 \mathcal{D}_8} \]  

\[ = \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-1}^{1} \epsilon_i F_1^{(11)} + \mathcal{O} (\epsilon^3), \]  

(187)

where:

\[ aF_{-1}^{(11)} = -\frac{1}{x} H(1, 0, x), \]  

(188)

\[ aF_0^{(11)} = -\frac{1}{x} \left\{ \zeta(2) H(1, x) + 2H(1, 0, x) - H(r, r, 0, x) + H(0, 1, 0, x) 
- H(1, 0, x) + H(1, 1, x) - 3H(1 + r, r, 0, x) 
+ \sqrt{3} H(r, 0, 1) H(1, x) \right\}, \]  

(189)

\[ aF_1^{(11)} = \frac{1}{x} \left\{ 2\zeta(2) H(1, x) - \zeta(2) H(1, x) - \zeta(2) H(0, 1, x) - 4H(1, 0, x) 
+ \zeta(2) H(1, 0, x) - \zeta(2) H(1, 1, x) + \zeta(2) H(r, r, 0, x) + 3\zeta(2) H(1 + r, r, 0, x) 
- 2H(0, 1, 0, x) + 2H(1, 0, x) - 2H(1, 1, 0, x) + 2H(r, 0, x) 
+ 6H(1 + r, r, 0, x) - H(0, 0, 0, x) + H(0, 1, 0, x) - H(0, 1, 1, 0, x) 
- H(1, 0, 0, x) - H(1, 1, 0, x) + H(1, 1, 0, x) - H(1, 1, 0, x) 
- H(r, r, 0, x) - H(0, 0, 0, x) + 2H(r, 4, r, 0, x) + H(0, r, r, 0, x) 
+ 3H(1 + r, r, 0, x) - 3H(1 + r, 0, 0, x) + 6H(1 + r, 4, r, 0, x) 
+ 3H(0, 1 + r, r, 0, x) + 3H(1, 1 + r, r, 0, x) - 3 \left[ \left( 2H(r, 0, 1) 
+ H(r, 0, 0, 1) + H(4, r, 0, 1) \right) H(1, x) + H(r, 0, 1) \left( H(0, 0, 1) 
+ H(0, 1, 0) \right) \right] \right\}. \]  

(190)

The above amplitude has pseudothresholds in \( s = -m^2 \) and \( s = -4m^2 \) and is the only one containing GHPLs with the index \( 1 + r \). The latter is then related to the (virtual) transition of a particle with mass \( m \neq 0 \) into a pair of particles with the same mass, i.e., a bubble with 2 equal mass lines. The index \( r \) appears in the GHPLs only 0 or 2 times.

\[ = \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4 \mathcal{D}_5 \mathcal{D}_6 \mathcal{D}_7 \mathcal{D}_8} \]  

\[ = \left( \frac{\mu^2}{a} \right)^{\epsilon} F_0^{(12)} + \mathcal{O} (\epsilon^3), \]  

(192)

where:

\[ aF_0^{(12)} = \frac{2}{x} \zeta(2) H(0, 1, x) + H(0, 1, 0, x) \]  

(193)
\[
(p_2 \cdot k_1) = \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{p_2 \cdot k_1}{D_3 D_4 D_6 D_{12} D_{13}} \tag{194}
\]

\[
= \left( \frac{\mu^2}{a} \right)^2 \sum_{i=0}^1 \epsilon_i F_i^{(13)} + \mathcal{O}(\epsilon^2), \tag{195}
\]

where:

\[
\frac{F_0^{(13)}}{a} = -1 + \zeta(2) + H(0, x) + \zeta(2) H(1, x) + H(1, 0, x) + H(1, 1, 0, x)
\]

\[
+ \frac{1}{x} \left[-\zeta(2) H(1, x) - H(1, 0, x) - H(1, 1, 0, x)\right], \tag{196}
\]

\[
\frac{F_1^{(13)}}{a} = -7 + 4\zeta(2) - \zeta(3) + 6H(0, x) + 3\zeta(2) H(1, x) - \zeta(3) H(1, x)
\]

\[
- 2H(0, 0, x) + 3\zeta(2) H(0, 1, x) + 3H(1, 0, x) + 5\zeta(2) H(1, 1, x)
\]

\[
+ H(0, 1, 0, x) - 2H(1, 0, 0, x) + 3H(1, 1, 0, x) + H(1, 0, 1, 0, x)
\]

\[
- 2H(1, 1, 0, 0, x) + 5H(1, 1, 1, 0, x) + 3H(0, 1, 1, 0, x) + \frac{1}{x} \left[\zeta(3) H(1, x)\right]
\]

\[
- 3\zeta(2) H(1, x) - \zeta(2) H(0, 1, x) - 3H(1, 0, x) - 5\zeta(2) H(1, 1, x)
\]

\[
- H(0, 1, 0, x) + 2H(1, 0, 0, x) - 3H(1, 1, 0, x) - H(0, 1, 1, 0, x)
\]

\[
- H(1, 0, 1, 0, x) + 2H(1, 1, 0, 0, x) - 5H(1, 1, 1, 0, x)\right]. \tag{197}
\]

The above 2 MIs contain HPLs with indices “0” and “1” only. They represent the emission of a photon by a charged vector boson in the t channel, and therefore have only a pseudothreshold in s = −m^2. They are IR (as well as UV) finite because the photon is emitted internally to the basic 1-loop triangle.

\[
(p_1 \cdot k_1) = \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_1 D_3 D_6 D_{13} D_{15}} \tag{198}
\]

\[
= \left( \frac{\mu^2}{a} \right)^2 \epsilon F_0^{(14)} + \mathcal{O}(\epsilon), \tag{199}
\]

where:

\[
a F_0^{(14)} = \frac{1}{x} \left[\zeta(2) H(0, -1, x) + \zeta(2) H(0, 1, x) + H(0, -1, 0, -1, x)
\right]

\[
- 2H(0, 1, 0, -1, x)\right]. \tag{200}
\]
where:

\[
F_0^{(15)} = -\frac{3}{2} + \frac{1}{2} \left[ 3 + \zeta(2) \right] H(-1, x) + \frac{1}{2} \zeta(2) H(1, x) - \frac{3}{2} H(0, -1, x) \\
+ \frac{1}{2} H(-1, 0, -1, x) - H(1, 0, -1, x) + \frac{1}{x} \left\{ \frac{1}{2} \left[ 3 + \zeta(2) \right] H(-1, x) \\
- \frac{1}{2} \zeta(2) H(1, x) + \frac{1}{2} H(-1, 0, -1, x) + H(1, 0, -1, x) \right\},
\]

\[
F_1^{(15)} = -9 - \zeta(2) + \frac{1}{2} \left[ 15 - \zeta(3) \right] H(-1, x) - \left[ \zeta(2) + \frac{1}{2} \zeta(3) \right] H(1, x) \\
- \left[ 6 + \zeta(2) \right] H(-1, -1, x) + \frac{3}{2} \zeta(2) \left( H(0, -1, x) + H(0, 1, x) \right) \\
+ 6H(0, -1, -1, x) + \frac{1}{2} H(0, 0, -1, x) + 2H(1, 0, -1, x) \\
- H(-1, -1, 0, -1, x) - 2H(-1, 0, -1, -1, x) - \frac{1}{2} H(-1, 0, 0, -1, x) \\
+ \frac{3}{2} H(0, -1, 0, -1, x) - 3H(0, 1, 0, -1, x) + 4H(1, 0, -1, -1, x) \\
+ \frac{1}{x} \left\{ \frac{1}{2} \left[ 15 - \zeta(3) \right] H(-1, x) + \left[ \zeta(2) + \frac{1}{2} \zeta(3) \right] H(1, x) \\
- \left[ 6 + \zeta(2) \right] H(-1, -1, x) + \frac{1}{2} \left[ 3 + \zeta(2) \right] H(0, -1, x) - \frac{1}{2} \zeta(2) H(0, 1, x) \\
- 2H(1, 0, -1, x) - H(-1, -1, 0, -1, x) - 2H(-1, 0, -1, -1, x) \\
- \frac{1}{2} H(-1, 0, 0, -1, x) + \frac{1}{2} H(0, -1, 0, -1, x) + H(0, 1, 0, -1, x) \\
- 4H(1, 0, -1, -1, x) \right\}. 
\]

\[
\mathcal{O} = \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_1 D_2 D_3 D_4 D_5 D_6 D_7 D_8 D_9 D_{10}} 
\]

\[
= \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-1}^1 \epsilon^i F_i^{(16)} + \mathcal{O} \left( \epsilon^2 \right),
\]

where:

\[
aF_{-1}^{(16)} = \frac{2}{x} H(-r, -r, x),
\]

\[
aF_0^{(16)} = \frac{1}{x} \left[ 4H(-r, -r, x) - 3H(-r, -r, -1, x) - 2H(-r, -4, -r, x) \\
+ 2H(0, -r, -r, x) \right],
\]

\[
aF_1^{(16)} = \frac{1}{x} \left[ 4 \left( 2 + \zeta(2) \right) H(-r, -r, x) - 6H(-r, -r, -1, x) + 4H(0, -r, -r, x) \\
- 4H(-r, -4, -r, x) + 12H(-r, -r, -1, -1, x) - 6H(-r, -r, 0, -1, x) \right].
\]
\[
+3H(-r, -4, -r, -1, x) + 2H(-r, -4, -4, -r, x) \\
-3H(0, -r, -r, -1, x) - 2H(0, -r, -4, -r, x) \\
+2H(0, 0, -r, -r, x) \]

The above MI has a simple UV pole coming from the sub-divergence in the bubble.

\[
\mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_5 D_7 D_8 D_{12} D_{13}} \quad (210)
\]

\[
= \left( \frac{\mu^2}{a} \right)^{2\epsilon} F_0^{(17)} + \mathcal{O} (\epsilon) ,
\]

where:

\[
a F_0^{(17)} = -\frac{1}{x} \left[ \zeta(2) H(0, 1, x) - 2H(0, 1, 0, -1, x) + 2H(0, r, r, 0) \right].
\]

4.4 Topology  \( t = 6 \)

\[
\mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_4 D_5 D_6 D_{12} D_{13} D_{14}} \quad (213)
\]

\[
= \left( \frac{\mu^2}{a} \right)^{2\epsilon} F_0^{(18)} + \mathcal{O} (\epsilon^2) ,
\]

where:

\[
a F_0^{(18)} = \frac{1}{x} \left[ \zeta(2) \left( H(0, 1, x) + H(1, 1, x) + H(1, c, x) + H(1, \bar{c}, x) \right) \\
+ H(0, c, x) + H(0, \bar{c}, x) \right) - 2H(1, 1, 0, -1, x) - 2H(0, 1, 0, -1, x) \\
-3H(0, c, 0, -1, x) - 3H(0, \bar{c}, 0, -1, x) - 2H(1, r, r, 0, x) \\
-2H(0, r, r, 0, x) - 3H(1, c, 0, -1, x) - 3H(1, \bar{c}, 0, -1, x) \]

\[
-\frac{i}{x} H(r, 0, 1) \left[ H(0, c, x) - H(0, \bar{c}, x) + H(1, c, x) \\
-H(1, \bar{c}, x) \right].
\]

The index \( r \) appears in the GHPLs only 0 or 2 times, so the coefficients of the related terms do not contain radicals.

\[
\mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_2 D_4 D_5 D_6 D_{12} D_{17}} \quad (216)
\]

\[
= \left( \frac{\mu^2}{a} \right)^{2\epsilon} F_0^{(19)} + \mathcal{O} (\epsilon) ,
\]

42
where:
\[
\begin{align*}
    a^2 F_0^{(20)} &= \frac{1}{x} \left\{ \zeta(2) \left[ H(0, -1, x) - H(0, 1, x) - H(1, 1, x) + H(1, -1, x) \right] \\
    &\quad - 2H(0, -1, 0, -1, x) + H(0, -1, 0, 0, x) + 2H(0, 1, 0, -1, x) \\
    &\quad - 2H(1, -1, 0, -1, x) + H(1, -1, 0, 0, x) + 2H(1, 1, 0, -1, x) \right\}. \\
\end{align*}
\]  

(218)

Because of analogous considerations to the previous ones, the above MI is expressed in terms of ordinary HPLs.

\[
\begin{align*}
    \mu^2 (4-D) \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_7 \mathcal{D}_8 \mathcal{D}_{15} \mathcal{D}_{16}} \\
    = \left( \frac{\mu^2}{a} \right)^2 F_0^{(20)} + \mathcal{O} (\epsilon), \\
\end{align*}
\]  

(219)

where:

\[
\begin{align*}
    a^2 F_0^{(20)} &= \frac{1}{x \sqrt{x(x+4)}} \left\{ 12 \zeta(2) H(-r, -1, x) - 6H(-r, -1, 0, -1, x) \\
    &\quad + 6H(-r, -1, 0, 0, x) - 12H(-r, -r, -1, x) \\
    &\quad + 8H(-r, -r, -r, 0, x) + 8H(-r, -r, 0, -r, x) \\
    &\quad + 4H(-r, 0, -r, -r, x) - 2H(-r, 0, 0, -1, x) \right\}. \\
\end{align*}
\]  

(221)

\[
\begin{align*}
    \mu^2 (4-D) \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_7 \mathcal{D}_8 \mathcal{D}_{12} \mathcal{D}_{13}} \\
    = \left( \frac{\mu^2}{a} \right)^2 F_0^{(21)} + \mathcal{O} (\epsilon), \\
\end{align*}
\]  

(222)

where:

\[
\begin{align*}
    a^2 F_0^{(21)} &= \frac{1}{x^2} \left[ 6\zeta(2) H(1, 1, x) + 6H(1, r, r, 0, x) - 2H(1, 0, 0, -1, x) \\
    &\quad + H(1, 0, 1, 0, x) - 12H(1, 1, 0, -1, x) + 4H(1, 1, 0, 0, x) \right]. \\
\end{align*}
\]  

(224)

The masses of the vector bosons exchanged in the $t$ channel completely cut-off the infrared singularities, so the above MI is IR (as well as UV) finite. As noted in our previous work [1], a non-zero mass on the outer boson line is already sufficient to completely screen the IR singularities.

The above MI has been computed in [21] by fitting a small momentum expansion to an assumed form for the exact expression. We have compared the first 15 terms of the small momentum expansion in [21] with an analogous expansion of...
our expression, finding complete agreement. The first few terms of the small
momentum expansion of $F^{(21)}$ are given in the next section. In [23] a leading-twist
large-momentum expansion of the above MI has been presented, based on the separa-
tion of the loop space in leading IR regions and a related approximation on the
integrand. The above result is in agreement with a preliminary large momentum
expansion of our expression [18].

5 Reducible six-denominator amplitudes

\[\int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_4 D_5 D_7 D_8 D_12 D_13} = \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_4 D_5 D_7 D_8 D_12 D_13} \] (225)

\[= \left( \frac{\mu^2}{a} \right)^{2\epsilon} F_0^{(22)} + \mathcal{O}(\epsilon), \] (226)

where:

\[a^2 F_0^{(22)} = \frac{1}{x} \left[ 2 \zeta(2) H(0, 1, 1, x) + 2 \zeta(2) H(1, 1, x) - H(0, 1, 0, 0, x) \right. \]

\[+ H(0, 1, 1, 0, x) - H(1, 1, 0, 0, x) + H(1, 1, 1, x) \] . (227)

In the above amplitude, the photon is emitted by the $W$ boson internally to the
triangle. It is therefore “trapped” and cannot propagate for large distances. The
consequence is that the above amplitude is IR finite, as expected on the basis of
physical intuition.

\[\int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_4 D_5 D_7 D_8 D_12 D_13} = \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_4 D_5 D_7 D_8 D_12 D_13} \] (228)

\[= \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-1}^{0} \epsilon^i F_i^{(23)} + \mathcal{O}(\epsilon), \] (229)

where:

\[a^2 F_{-1}^{(23)} = \frac{1}{x} \left[ 2 H(0, 0, 0, 1, x) + H(0, 1, 0, 0, 1, x) \right. \]

\[+ H(1, 1, 0, x) \] , (230)

\[a^2 F_0^{(23)} = \frac{1}{x} \left[ 4 \zeta(2) (H(0, 1, 1, x) + H(1, 1, x)) - 8 H(0, 0, 1, -1, x) \right. \]

\[+ H(0, 0, 1, 1, x) - H(0, 1, 0, 0, x) + 4 H(0, 1, 1, 0, x) \]

\[+ 8 H(1, 0, -1, -1, x) + H(1, 0, 1, 0, x) - H(1, 1, 0, 0, x) \]

\[+ 4 H(1, 1, 1, 0, x) \] . (231)
The above amplitude corresponds to a photon emitted by a $W$ boson outside the triangle. In this case, the photon can propagate to large distances and a collinear singularity is generated, characterized by the presence of the simple $1/\epsilon$ pole. As anticipated in the introduction, a qualitative analysis of the infrared singularities can be done by shrinking all the internal massive lines to a point: in this limit the diagram factorizes into a massless bubble evaluated at the light-cone momentum $p_2$ times a massless bubble evaluated at the general momentum $q$. The collinear singularity originate from the former bubble, representing the evolution of a jet formed by an initial particle.

\[
\begin{align*}
\frac{a^2 F_{-2}^{(24)}}{a^2 F_{-1}^{(24)}} &= \frac{1}{x}, \\
\frac{a^2 F_{0}^{(24)}}{a^2 F_{1}^{(24)}} &= \frac{1}{x} \bigg[ 4 + 2\zeta(2) - 2\zeta(3) - 4H(0, x) + \zeta(2)H(0, x) + 2\zeta(2)H(1, x) \\
&+ 4H(-1, x) + 2\zeta(2)H(-1, x) - 2H(0, 0, x) - 8H(0, -1, x) \\
&- 6H(-1, 0, -1, x) - 16H(-1, -1, -1, x) + 2H(0, 0, -1, x) \\
&- 4H(1, 0, -1, x) + 3H(r, r, 0, x) \bigg] + \frac{1}{x^2} \bigg[ 4H(-1, x) + 2\zeta(2)H(-1, x) \\
&- 2\zeta(2)H(1, x) - 2\zeta(2)H(0, 1, x) + 4H(0, -1, x) + 2\zeta(2)H(0, -1, x) \\
&- 2\zeta(2)H(r, r, x) + 8H(-1, -1, x) + 4H(0, 0, -1, x) + 4H(1, 0, -1, x) \\
&+ 16H(-1, -1, -1, x) - 6H(-1, 0, -1, x) + 8H(0, -1, -1, x) \\
&+ 2H(r, r, 0, x) + 2H(r, 0, r, x) - 4H(r, 4, r, 0, x) \\
&+ 16H(0, -1, -1, -1, x) - 6H(0, -1, 0, -1, x) + 8H(0, 0, -1, -1, x) \\
&+ 4H(0, 0, -1, x) + 4H(0, 1, 0, -1, x) \bigg] - \frac{4 - x}{x \sqrt{x(4 - x)}} \zeta(2)H(r, x) \\
\end{align*}
\]
\[-2H(r, 0, x) - H(r, 0, 0, x) - H(0, r, 0, x) + 2H(4, r, 0, x)\] \hspace{1cm} (237)

The above amplitude has a double pole in \(\epsilon\) coming from the following region: in the inner triangle a large momentum \(k_2\) flows while in the external triangle a soft momentum \(k_1\) flows, or \(k_1^2 \ll k_2^2\). In this limit, the diagram factorizes into a massless triangle with the well-known double IR pole (soft x collinear), times a coefficient functions given by the inner triangle, which is effectively point-like.

6 Small momentum expansion of six denominator amplitudes

In this Section we present the small momentum expansions \(|s| \ll m^2\) of all the 6-denominator diagrams, i.e. expansions in powers of \(x\) and \(L = \log x\) up to first order in \(x\) included. The expansion of the GHPLs for a small value of the argument \(|x| \ll 1\) is obtained in the following way:

- we explicitly write the GHPL as a repeated integration over the basis functions, as for example:

\[
H(1, r, r, 0; x) = \int_0^x \frac{dx_1}{1 - x_1} \int_0^{x_1} \frac{dx_2}{\sqrt{x_2(4 - x_2)}} \int_0^{x_2} \frac{dx_3}{\sqrt{x_3(4 - x_3)}} \log x_3;
\] \hspace{1cm} (238)

- we expand the basis functions in powers of \(x\) up to the required order in \(x^{17}\), such as for instance:

\[
\frac{1}{\sqrt{x(4 - x)}} = \frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{16} + \cdots;
\] \hspace{1cm} (239)

- we integrate term by term the expanded functions. This involves in general the integration of functions of the form \(x^q \log x^k\), with \(q\) integer or half integer and \(k\) integer.

The expansions are given below.

\[
\begin{align*}
\text{Diagram} & = \left(\frac{\mu^2}{a}\right)^2 \sum_{j=0}^2 x^j A_j^0 + O(x^3), \\
a^2 A_0^0 & = 4 + \sqrt{3}H(r, 0, 1) - L, \\
a^2 A_1^0 & = \frac{19}{9} + \frac{1}{2}\zeta(2) + \frac{3\sqrt{3}}{4}H(r, 0, 1) - \frac{13}{24}L, \\
a^2 A_2^0 & = \frac{12509}{16200} + \frac{2}{3}\zeta(2) + \frac{\sqrt{3}}{2}H(r, 0, 1) - \frac{197}{540}L
\end{align*}
\] \hspace{1cm} (240)

\footnote{The factors \(1/x\) and \(1/\sqrt{x}\) are clearly not expanded.}
\[
\left( \frac{\mu^2}{a} \right)^2 \sum_{j=0}^{2} x^j B_0^0 + O(x^3),
\]  
(244)

where:

\[
a^2 B_0^0 = 3 - 2L + \frac{1}{2}L^2,
\]  
(245)

\[
a^2 B_1^0 = \frac{11}{16} - \frac{1}{2}\zeta(2) - \frac{1}{2}L + \frac{1}{8}L^2,
\]  
(246)

\[
a^2 B_2^0 = \frac{7}{8} - \frac{1}{3}\zeta(2) - \frac{41}{108}L + \frac{5}{36}L^2.
\]  
(247)

\[
\left( \frac{\mu^2}{a} \right)^2 \sum_{j=0}^{2} x^j C_0^0 + O(x^3),
\]  
(248)

where:

\[
a^2 C_0^0 = 1 + 2\zeta(2) - L + \frac{1}{2}L^2,
\]  
(249)

\[
a^2 C_1^0 = -\frac{17}{24} - \zeta(2) + \frac{5}{12}L - \frac{1}{4}L^2,
\]  
(250)

\[
a^2 C_2^0 = \frac{827}{2160} + \frac{1}{2}\zeta(2) - \frac{61}{360}L + \frac{1}{8}L^2.
\]  
(251)

\[
\left( \frac{\mu^2}{a} \right)^2 \sum_{j=0}^{2} x^j E_0^0 + O(x^3),
\]  
(252)

where:

\[
a^2 E_0^0 = -4 + 3\zeta(2) - L + L^2,
\]  
(253)

\[
a^2 E_1^0 = -\frac{341}{72} + 3\zeta(2) - \frac{5}{6}L + L^2,
\]  
(254)

\[
a^2 E_2^0 = -\frac{2617}{600} + \frac{11}{4}\zeta(2) - \frac{79}{120}L + \frac{11}{12}L^2.
\]  
(255)

\[
\left( \frac{\mu^2}{a} \right)^2 \sum_{j=0}^{2} x^j G_0^0 + O(x^3),
\]  
(256)

where:

\[
a^2 G_0^0 = 3 - \zeta(2) - 2L + \frac{1}{2}L^2,
\]  
(257)

\[
a^2 G_1^0 = \frac{25}{16} - \frac{3}{4}\zeta(2) - \frac{5}{4}L + \frac{3}{8}L^2,
\]  
(258)

\[
a^2 G_2^0 = \frac{251}{216} - \frac{11}{18}\zeta(2) - \frac{107}{108}L + \frac{11}{36}L^2.
\]  
(259)
\[\left(\frac{\mu^2}{a}\right)^2 \sum_{j=0}^{2x} x^j \left[ \sum_{i=-1}^{0} \epsilon^i I^i_j \right] + O(x^3), \quad (260)\]

where:

\[a^2 I^{-1}_{0} = L, \quad (261)\]
\[a^2 I^0_{0} = -6 + 4\zeta(2) + 3L - \frac{1}{2}L^2, \quad (262)\]
\[a^2 I^{-1}_{1} = -\frac{1}{4} + \frac{3}{4}L, \quad (263)\]
\[a^2 I^0_{1} = -\frac{11}{2} + 3\zeta(2) + \frac{21}{8}L - \frac{3}{8}L^2, \quad (264)\]
\[a^2 I^{-1}_{2} = -\frac{1}{12} + \frac{11}{18}L, \quad (265)\]
\[a^2 I^0_{2} = \frac{859}{216} + \frac{22}{9}\zeta(2) + \frac{22}{9}L - \frac{11}{36}L^2. \quad (266)\]

\[\left(\frac{\mu^2}{a}\right)^2 \sum_{j=-1}^{2x} x^j \left[ \sum_{i=-2}^{1} \epsilon^i J^i_j \right] + O(x^3), \quad (267)\]

where:

\[a^2 J^{-2}_{-1} = 1, \quad (268)\]
\[a^2 J^{-1}_{-1} = 1 - L, \quad (269)\]
\[a^2 J^0_{-1} = 1 - \zeta(2) - L + \frac{1}{2}L^2, \quad (270)\]
\[a^2 J^1_{-1} = 1 - \zeta(2) - 2\zeta(3) + \zeta(2)L - L + \frac{1}{2}L^2 - \frac{1}{6}L^3, \quad (271)\]
\[a^2 J^{-1}_{0} = \frac{1}{4}, \quad (272)\]
\[a^2 J^0_{0} = \frac{35}{72} + \frac{1}{12}L, \quad (273)\]
\[a^2 J^1_{0} = -\frac{359}{432} + \frac{13}{12}\zeta(2) + \frac{7}{24}L - \frac{1}{24}L^2, \quad (274)\]
\[a^2 J^{-1}_{1} = -\frac{1}{18}, \quad (275)\]
\[a^2 J^0_{1} = -\frac{209}{675} + \frac{1}{180}L, \quad (276)\]
\[a^2 J^1_{1} = -\frac{141091}{162000} + \frac{1}{180}\zeta(2) + \frac{23}{1080}L - \frac{1}{360}L^2, \quad (277)\]
\[a^2 J^{-1}_{2} = \frac{1}{48}, \quad (278)\]
\[a^2 J^0_{2} = \frac{11063}{78400} + \frac{1}{1680}L, \quad (279)\]
\[a^2 J^1_{2} = \frac{105455939}{296352000} + \frac{47}{560}\zeta(2) + \frac{17}{6720}L - \frac{1}{336}L^2. \quad (280)\]
7 Large momentum expansion of six denominator amplitudes

In this Section we give the asymptotic expansions for $|s| \gg m^2$ of all the 6-denominator scalar integrals, i.e. the expansion in powers of $1/x$ up to the order $1/x^4$ included. These results are relevant for the study of the structure of the infrared logarithms coming from multiple emission.

\[
\frac{1}{D4} = \left(\frac{\mu^2}{a}\right)^{2\varepsilon} \sum_{j=1}^{4} \frac{1}{x^j} B_{-j}^0 + \mathcal{O}\left(\frac{1}{x^5}\right), \quad (281)
\]

where:

\[
a^2 B_{-1}^0 = -8a_4 + \frac{19}{4}\zeta^2(2) + 2\zeta(2) \log^2 2 - \frac{1}{3} \log^4 2, \quad (282)
\]

\[
a^2 B_{-2}^0 = -3 - 2\zeta(2) + \zeta(3) - 2\zeta(2)L - 3L - \frac{3}{2}L^2 - \frac{3}{2}L^3, \quad (283)
\]

\[
a^2 B_{-3}^0 = \frac{21}{16} - \frac{1}{2}\zeta(2) + \frac{1}{2}\zeta(3) - L\zeta(2) + \frac{3}{8}L - \frac{1}{8}L^2 - \frac{1}{4}L^3, \quad (284)
\]

\[
a^2 B_{-4}^0 = -\frac{11}{24} + \frac{1}{9}\zeta(2) + \frac{1}{3}\zeta(3) + \frac{3}{4}L - \frac{2}{3}L\zeta(2) + \frac{1}{4}L^2 - \frac{1}{6}L^3, \quad (285)
\]

and where $a_4 = \text{Li}_4(1/2)$.

\[
\frac{1}{D5} = \left(\frac{\mu^2}{a}\right)^{2\varepsilon} \sum_{j=1}^{4} \frac{1}{x^j} G_{-j}^0 + \mathcal{O}\left(\frac{1}{x^5}\right), \quad (286)
\]

where:

\[
a^2 G_{-1}^0 = -2 - \zeta(2) - \zeta(3) - L\zeta(2) - 2L - L^2 - \frac{1}{3}L^3, \quad (287)
\]

\[
a^2 G_{-2}^0 = -\frac{13}{162} + \frac{7}{18}\zeta(2) - \frac{1}{3}\zeta(3) - \frac{1}{3}\zeta(2)L + \frac{55}{108}L + \frac{7}{18}L^2 - \frac{1}{9}L^3. \quad (288)
\]

where:

\[
a^2 I_{-1}^{-1} = -2\zeta(3), \quad (292)
\]

\[
a^2 I_{-1}^0 = \frac{1}{5}\zeta^2(2), \quad (293)
\]
\( a^2 I_{-2}^{-1} = 1 + L + \frac{1}{2} L^2, \)  
\( a^2 I_2^0 = -4 + 2\zeta(2) - 4\zeta(3) + 2\zeta(2)L - 4L - 2L^2 - \frac{2}{3}L^3, \)  
\( a^2 I_{-3}^{-1} = -\frac{1}{8} + \frac{3}{4}L + \frac{1}{4}L^2, \)  
\( a^2 I_{-3}^0 = \frac{41}{8} - \frac{3}{2}\zeta(2) - 2\zeta(3) + \zeta(2)L + \frac{5}{4}L - \frac{7}{4}L^2 - \frac{1}{3}L^3, \)  
\( a^2 I_{-4}^{-1} = \frac{13}{108} + \frac{11}{18}L + \frac{1}{6}L^2, \)  
\( a^2 I_{-4}^0 = \frac{3019}{648} - \frac{16}{9}\zeta(2) - \frac{4}{3}\zeta(3) + \frac{2}{3}\zeta(2)L + \frac{23}{27}L - \frac{53}{36}L^2 - \frac{2}{9}L^3. \)  

### 8 Conclusions

We have presented the exact analytic evaluation of the 25 master integrals containing 2 and 3 massive propagators entering the planar amplitudes of the 2-loop electroweak form factor. While the reduction to master integrals does not present any new element with respect to our previous computation and is done with the same algorithm, the analytic evaluation of the master integrals requires a non-trivial extension of the harmonic polylogarithm theory. The presence of 2 massive particles in the \( s \) or in the \( t \) channel opens indeed thresholds and pseudothresholds in \( s = \pm 4m^2 \) respectively, in addition to the old ones in \( s = 0, \pm m^2 \).

The generalization of the 1-dimensional harmonic polylogarithms has basically required:

- the introduction of new basis functions, in addition to the usual one, involving complex constants and radicals;
- a set of recursion relations to take the integrals with semi-integer powers coming from the evaluation of the master integrals to a unique form fixed by the basis function choice.

The basic properties of the ordinary harmonic polylogarithms are maintained by the generalization.

The small momentum expansion of all the 6-denominator amplitudes has been obtained by means of a series expansion of the basis functions.

We could also obtain the large momentum expansion of all the six-denominator amplitude involving only ordinary harmonic polylogarithms.

We compared our results with those present in the literature usually in the form of resummed small momentum expansions or truncated large momentum expansions, finding complete agreement.

In order to complete the evaluation of the master integrals, 3 steps are still to be taken:

- as explained in Section \[3\] the transformation \( x \to 1/x \) requires the knowledge of all the \( H(\bar{w}; x) \)'s in \( x = 1 \). The \( H(\bar{w}; 1) \)'s have to be expressed in terms of a minimal set of transcendental constants \[13\];
• the evaluation of the master integrals related to the crossed ladder topology. The reduction, using both the numerical-indices method and the symbolic method, shows that this topology has 3 master integrals. The resulting system of 3 differential equations cannot be completely triangularized by means of the techniques discussed in this work, but can be split into a second-order and a first-order differential equations [18].

• The numerical evaluation of the generalized harmonic polylogarithms, which does not seem to have specific difficulties with respect to the ordinary case.

9 Acknowledgement

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A One-loop master integrals

In this Appendix we present the results for the 1-loop master integrals containing 2 massive propagators. We have recomputed them with the method of the differential equations described in the main body of the paper in terms of GHPLs. In the case of the bubble, we found that it was necessary to push the $\epsilon$ expansion up to third order included. In our previous work [1] we gave the expressions of the 1-loop master integrals containing at most 1 massive propagator. The above amplitudes are necessary for the computation of the factorized 2-loop master integrals, which are presented in the next section.

A.1 Bubble

\[
\begin{aligned}
\begin{tikzpicture}
\filldraw[fill=black, draw=black] (0,0) circle (0.2cm);
\end{tikzpicture}
\quad &= \mu^{(4-D)} \int \mathcal{D}k \frac{1}{(k^2 + a) [(p-k)^2 + a]} \\
&= \left( \frac{\mu^2}{a} \right) \epsilon^3 \sum_{i=-1}^{3} \epsilon^i B_i + \mathcal{O}(\epsilon^4),
\end{aligned}
\]

where:

\[
\begin{alignat}{2}
B_{-1} &= 1, \\
B_0 &= 2 - \frac{x+4}{\sqrt{x(x+4)}} H(-r, x),
\end{alignat}
\]
\[ B_1 = 4 - \frac{x+4}{\sqrt{x(x+4)}} \left[ 2H(-r,x) - H(-4,-r,x) \right], \quad (303) \]

\[ B_2 = 8 - \frac{x+4}{\sqrt{x(x+4)}} \left[ 4H(-r,x) - 2H(-4,-r,x) + H(-4,-4,-r,x) \right], \quad (304) \]

\[ B_3 = 16 - \frac{x+4}{\sqrt{x(x+4)}} \left[ 8H(-r,x) - 4H(-4,-r,x) + 2H(-4,-4,-r,x) \right. \]
\[ \left. - H(-4,-4,-4,-r,x) \right]. \quad (305) \]

### A.2 Vertex

\[ \begin{array}{c}
\bigg\rangle \\
\end{array} = \mu^{(4-D)} \int D^D k \frac{1}{k^2 [(p_1 - k)^2 + a] [(p_2 + k)^2 + a]}
\]
\[ = \left( \frac{\mu^2}{a} \right) \epsilon \sum_{i=0}^{2} \epsilon^i V_i + \mathcal{O} (\epsilon^3), \quad (306) \]

where:

\[ aK_0 = \frac{2}{x} H(-r,-r,x), \quad (307) \]
\[ aK_1 = -\frac{2}{x} \left[ H(-r,-4,-r,x) - H(0,-r,-r,x) \right], \quad (308) \]
\[ aK_2 = \frac{2}{x} \left[ H(-r,-4,-4,-r,x) - H(-r,-4,-r,x) \right. \]
\[ \left. + H(0,0,-r,-r,x) \right]. \quad (309) \]

### B Factorized master integrals

In this Appendix we give the expressions of the factorized 2-loop master integrals, i.e. of the MIs in which the 2 loops do not have common propagators. One has only to multiply the 1-loop master integrals representing the separated subdiagrams and convert products of GHPLs \( H(\vec{a};x)H(\vec{b};x) \) into linear combinations of \( H \)'s by using the algebra-identity in [2].

### B.1 Topology \( t = 2 \)

\[ \begin{array}{c}
\bigg\rangle \\
\end{array} = \mu^{2(4-D)} \int D^D k_1 D^D k_2 \frac{1}{D_{12} D_{13}} \quad (310) \]
\[ = \left( \frac{\mu^2}{a} \right)^2 \epsilon \sum_{i=-2}^{2} \epsilon^i F_1^{(25)} + \mathcal{O} (\epsilon^3), \quad (311) \]
where:

\[ \frac{F^{(25)}_{-2}}{a^2} = 1, \quad (312) \]
\[ \frac{F^{(25)}_{-1}}{a^2} = 2, \quad (313) \]
\[ \frac{F^{(25)}_0}{a^2} = 3, \quad (314) \]
\[ \frac{F^{(25)}_1}{a^2} = 4, \quad (315) \]
\[ \frac{F^{(25)}_2}{a^2} = 5. \quad (316) \]

The above amplitude appears, in general, in the reduction of all the amplitudes having at least 1 massive propagator in anyone of the 2 loops.

### B.2 Topology \( t = 4 \)

\[
\begin{align*}
\begin{tikzpicture}
  \node [circle, draw, inner sep=1pt] (a) at (0,0) {}; \\
  \node [circle, draw, inner sep=1pt] (b) at (0.5,0) {}; \\
  \node [circle, draw, inner sep=1pt] (c) at (1,0) {}; \\
  \node at (1.5,0) {\ldots}; \\
  \node at (2,0) {\ldots}; \\
\end{tikzpicture}
\end{align*}
\]

\[ = \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_2 D_{10} D_{12} D_{20}} \]
\[ = \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-2}^{2} \epsilon^i F_i^{(26)} + \mathcal{O} (\epsilon^3), \quad (317) \]

where:

\[ F_{-2}^{(26)} = 1, \quad (319) \]
\[ F_{-1}^{(26)} = 4 - H(0; x) - \frac{4 + x}{\sqrt{x(4 + x)}} H(-r; x), \quad (320) \]
\[ F_0^{(26)} = 12 - \zeta(2) - 4H(0; x) + H(0, 0; x) - \frac{(4 + x)}{\sqrt{x(4 + x)}} \left[ -4H(-r; x) + H(-r, 0; x) + H(-4, -r; x) + H(0, -r; x) \right], \quad (321) \]
\[ F_1^{(26)} = 32 - 4\zeta(2) - 2\zeta(3) - 12H(0; x) + H(0; x)\zeta(2) + 4H(0, 0; x) - H(0, 0, 0; x) + \frac{(4 + x)}{\sqrt{x(4 + x)}} \left[ -12H(-r; x) + \zeta(2)H(-r; x) + 4H(-r, 0; x) - H(-r, 0, 0; x) + 4H(-4, -r; x) - H(-4, -4, -r; x) - H(-4, 0, -r; x) + 4H(0, -r; x) - H(0, -r, 0; x) - H(-4, 0, -r; x) - H(0, 0, -r; x) \right], \quad (322) \]
\[ F_2^{(26)} = 80 - 12\zeta(2) - \frac{9}{10} \zeta^2(2) - 8\zeta(3) - 32H(0; x) + 4\zeta(2)H(0; x) + 2\zeta(3)H(0; x) + 12H(0, 0; x) - \zeta(2)H(0, 0; x) - 4H(0, 0, 0; x) + H(0, 0, 0, 0; x) \]
\[
\begin{align*}
&+ \frac{(4 + x)}{\sqrt{x(4 + x)}} \left[-32 H(-r; x) + 2(2\zeta(2) + \zeta(3))H(-r; x) + 12H(-r, 0; x) \\
&- \zeta(2)H(-r, 0; x) - 4H(-r, 0, 0; x) + H(-r, 0, 0, 0; x) + 12H(-4, -r; x) \\
&- \zeta(2)H(-4, -r; x) - 4H(-4, -r, 0; x) + H(-4, -r, 0, 0; x) \\
&- 4H(-4, -4, -r; x) + H(-4, -4, -r, 0; x) + H(-4, -4, -4, -r; x) \\
&+ H(-4, -4, 0, -r; x) - 4H(-4, 0, -r, 0; x) + H(-4, 0, -r, 0, 0; x) \\
&+ H(-4, 0, 0, -r; x) + H(-4, 0, 0, 0, -r; x) + H(-4, 0, 0, -r; x) \\
&+ H(0, 0, 0, -r; x) \right].
\end{align*}
\]

\[= \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{\mathcal{D}_1 \mathcal{D}_5 \mathcal{D}_{12} \mathcal{D}_{13}} \]  

\[= \left(\frac{\mu^2}{a}\right)^{2e} \sum_{i=-1}^{1} e^i F_i^{(27)} + \mathcal{O}(\epsilon^3), \]

where:

\[F_{-1}^{(27)} = \frac{1}{x}H(1, 0, x), \]

\[F_{0}^{(27)} = \frac{1}{x} \left[\zeta(2)H(1, x) + H(1, 0, x) + H(0, 1, 0, x) - H(1, 0, 0, x) \\
+ H(1, 1, 0, x)\right], \]

\[F_{1}^{(27)} = \frac{1}{x} \left[\left(\zeta(2) + 2\zeta(3)\right)H(1, x) + \left(1 - \zeta(2)\right)H(1, 0, x) + \zeta(2)H(1, 1, x) \\
+ \zeta(2)H(0, 1, x) + H(1, 1, 0, x) + H(0, 1, 0, x) - H(1, 0, 0, x) \\
+ H(0, 0, 1, 0, x) - H(0, 1, 0, 0, x) + H(0, 1, 1, 0, x) + H(1, 0, 0, 0, x) \\
+ H(1, 0, 1, 0, x) - H(1, 1, 0, 0, x) + H(1, 1, 1, 0, x)\right]. \]

\[\textbf{B.3 \ Topology} \ t = 5 \]

\[= \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_{10} \mathcal{D}_{15} \mathcal{D}_{16}} \]  

\[= \left(\frac{\mu^2}{a}\right)^{2e} \sum_{i=-1}^{1} e^i F_i^{(28)} + \mathcal{O}(\epsilon^3), \]  

54
where:

\[ aF_{-1}^{(28)} = \frac{2}{x} H(-r, -r, x), \]  
\[ aF_0^{(28)} = \frac{2}{x} \left[ 2H(-r, -r, x) - H(-r, -r, 0, x) - H(-r, -4, -r, x) - H(-r, 0, -r, x) \right], \]  
\[ aF_1^{(28)} = \frac{2}{x} \left[ (4 - \zeta(2))H(-r, -r, x) - 2H(-r, -r, 0, x) - 2H(-r, -4, -r, x) - 2H(-r, 0, -r, x) + H(-r, -4, -r, x) + H(-r, -4, 0, -r, x) + H(-r, 0, -r, 0, x) + H(-r, 0, -4, -r, x) + H(-r, 0, 0, -r, x) \right]. \]

C Reducible two-loop amplitudes

In this Appendix we present the expressions of some interesting 2-loop amplitudes which can be reduced to the MIs given in the present paper and in [1].

C.1 Topology \( t = 3 \)

This amplitude reduces to the product of tadpoles coming from the contraction of its massless line.

\[ \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{amplitude}
\end{array}
\end{array} = \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_8 D_{12} D_{13}} \]  
\[ = \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-2}^{2} \epsilon^i (F_i^{(29)}) + \mathcal{O}(\epsilon^3), \]  

where:

\[ F_{-2}^{(29)} \frac{a}{a} = -1, \]  
\[ F_{-1}^{(29)} \frac{a}{a} = -3, \]  
\[ F_0^{(29)} \frac{a}{a} = -7, \]  
\[ F_1^{(29)} \frac{a}{a} = -15, \]  
\[ F_2^{(29)} \frac{a}{a} = -31. \]
C.2 Topology $t = 5$

\[
\begin{align*}
\left(\begin{array}{c}
\end{align*}
\right)
= \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_4 D_5 D_6 D_7 D_8 D_9 D_{10}} \\
= \left( \frac{\mu^2}{a} \right)^2 2 \varepsilon \sum_{i=-1}^{1} e^i F^{(30)}_i + \mathcal{O}(\varepsilon^2),
\end{align*}
\]

where:

\[
\begin{align*}
aF^{(30)}_{-1} &= - \frac{1}{x} H(1, 0, x), \\
aF^{(30)}_0 &= - \frac{1}{x} \left[ 2 \zeta(2) H(1, x) + H(0, -1, x) + H(0, 1, 0, x) + H(1, 0, x) \\
&\quad - 2H(1, 0, -1, x) - H(1, 0, 0, x) + H(1, 1, 0, x) \right] \\
\quad + \frac{1}{(1 - x)} \left[ \zeta(2) - 2H(0, -1, x) \right], \\
aF^{(30)}_1 &= - \frac{1}{x} \left[ (2) H(1, x) + \zeta(3) H(1, x) + 3H(0, -1, x) + H(1, 0, x) \\
&\quad - \zeta(2) H(1, 0, x) + \zeta(2) H(1, 1, x) + 2\zeta(2) H(0, 1, x) + H(0, 0, -1, x) \\
&\quad + H(0, 1, 0, x) - H(1, 0, 0, x) + H(1, 1, 0, x) - 4H(0, -1, -1, x) \\
&\quad + H(0, 0, 1, 0, x) - 2H(0, 1, 0, -1, x) - H(0, 1, 0, 0, x) \\
&\quad + H(0, 1, 1, 0, x) + H(1, 0, 0, 0, x) + H(1, 0, 1, 0, x) \\
&\quad - H(1, 1, 0, 0, x) + H(1, 1, 1, 0, x) + 8H(1, 0, -1, -1, x) \right] \\
\quad + \frac{1}{(1 - x)} \left[ 3\zeta(2) - \zeta(3) - 6H(0, -1, x) + 8H(0, -1, -1, x) \right].
\end{align*}
\]

\[
\begin{align*}
\left(\begin{array}{c}
\end{align*}
\right)
= \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_4 D_5 D_6 D_7 D_8 D_{10}} \\
= \left( \frac{\mu^2}{a} \right)^2 2 \varepsilon \sum_{i=-1}^{1} e^i F^{(31)}_i + \mathcal{O}(\varepsilon^2),
\end{align*}
\]

where:

\[
\begin{align*}
aF^{(31)}_{-1} &= - \frac{1}{x} H(1, 0, x), \\
aF^{(31)}_0 &= - \frac{1}{x} \left[ \zeta(2) H(1, x) + 2H(1, 0, x) - H(1, 0, 0, x) + H(1, 1, 0, x) \right], \\
aF^{(31)}_1 &= - \frac{1}{x} \left[ 2\zeta(2) H(1, x) + 2\zeta(3) H(1, x) + 4H(1, 0, x) + \zeta(2) H(1, 1, x) \\
&\quad - 2\zeta(2) H(0, 1, x) - \zeta(2) H(1, 0, x) + 2H(1, 1, 0, x) - 2H(1, 0, 0, x) \\
&\quad - 2H(0, 1, 1, 0, x) + H(1, 0, 0, 0, x) + H(1, 0, 1, 0, x) \\
&\quad - H(1, 1, 0, 0, x) + H(1, 1, 1, 0, x) \right].
\end{align*}
\]
The above 2 amplitudes have a simple ultraviolet pole coming from the nested bubble, containing 1 massive denominator. The only difference is that the bubble is inserted in the s-channel in the former diagram and in the t-channel in the latter. The coefficient of the simple pole is indeed the same.

\[
\mathcal{M} = \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_1 D_4 D_6 D_{12} D_{14}} (351)
\]

\[
= \left( \frac{\mu^2}{a} \right)^2 e^{\epsilon} F^{(32)}_i + \mathcal{O}(\epsilon^2), (352)
\]

where:

\[
aF^{(32)}_{-3} = \frac{1}{x}, (353)
\]

\[
aF^{(32)}_{-2} = -\frac{1}{x} H(0, x), (354)
\]

\[
aF^{(32)}_{-1} = -\frac{1}{x} \left[ \zeta(2) - H(0, 0, x) \right], (355)
\]

\[
aF^{(32)}_0 = \frac{1}{x} \left[ 8 - 2\zeta(3) + H(r, r, 0, x) - 4H(0, x) + \zeta(2)H(0, x) - H(0, 0, x) \right]
+ \frac{2(4-x)}{x \sqrt{x(4-x)}} H(r, 0, x), (356)
\]

\[
aF^{(32)}_1 = -\frac{1}{x} \left[ 16 + 4\zeta(2) + \frac{9}{10} \zeta^2(2) - 2\zeta(3)H(0, x) - (4 - \zeta(2))H(0, 0, x) \right]
- \zeta(2)H(r, r, x) + 4H(r, r, 0, x) - H(0, 0, 0, x) + H(r, r, 0, 0) + H(r, 0, 0, x)
+ \frac{2(4-x)}{x \sqrt{x(4-x)}} \left[ \zeta(2)H(r, x) - H(0, 0, x) - H(0, r, 0, x) \right]
+ 2H(4, r, 0, x) \right]. (357)
\]

In the IR limit, the above amplitude factorizes into the product of a massless 1-loop triangle times a vacuum bubble with 2 masses. The triple pole is the product of a double IR pole coming from the triangle and a simple UV pole coming from the bubble.

\[
\mathcal{M} = \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_2 D_4 D_6 D_{12} D_{14}} (358)
\]

\[
= \left( \frac{\mu^2}{a} \right)^2 \sum_{i=-1}^0 e^{\epsilon} F^{(33)}_i + \mathcal{O}(\epsilon), (359)
\]

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where:

\[ aF_{-1}^{(33)} = -\frac{1}{x}H(0, 0, -1, x), \]  
\[ aF_0^{(33)} = -\frac{1}{x}\left[\zeta(2)H(0, 1, x) - H(0, 0, 0, -1, x) - 4H(0, 0, -1, -1, x) - 2H(0, 1, 0, -1, x)\right]. \]  

(360)  

(361)

According to IR power counting, one can shrink all the massive lines in the amplitude. This shows that the above amplitude has a simple collinear pole associated to the evolution of the lower external line.

D Scalar diagrams of self-energy type

In this Appendix we present the results for the scalar diagrams of self-energy insertion type (see Fig. 2). As explained in detail in Section 2, these diagrams are effectively 5- and 4-denominator amplitudes and are all reducible to the MIs by means of the ibps identities. Their analytic expressions are given for completeness.

\[
\begin{align*}
\int &= \mu^{2(4-D)} \int D^D k_1 D^D k_2 \frac{1}{D_4 D_5 D_7 D_12 D_13} \\
&= \left(\frac{\mu^2}{a}\right)^{2\epsilon} \sum_{i=-2}^{2} \epsilon_i F_i^{(34)} + \mathcal{O}(\epsilon^3), 
\end{align*}
\]  

(362)  

(363)

where:

\[ a^2 F_{-2}^{(34)} = -\frac{1}{x}, \]  
\[ a^2 F_{-1}^{(34)} = -\frac{1}{x}\left[1 - H(0, x)\right] + \frac{1}{(1 - x)}H(0, x), \]  
\[ a^2 F_0^{(34)} = \frac{1}{2x^2}H(-1, x) - \frac{1}{x}\left[\frac{3}{2} - \zeta(2) - H(0, x) - H(-1, x) + H(0, 0, x)\right. \\
+ \frac{1}{2}H(0, -1, x) - \frac{3}{2}H(1, 0, x)\right] \left[\zeta(2) + H(0, x) + H(-1, x) \\
- H(0, 0, x) + H(1, 0, x)\right] + \frac{1}{(1 - x)}\left[1 - \frac{1}{(1 - x)}\right]\left[2\zeta(2) \\
- H(0, -1, x)\right], \]  
\[ a^2 F_1^{(34)} = \frac{1}{x^2}\left[\frac{7}{4}H(-1, x) - 2H(-1, -1, x)\right] - \frac{1}{x}\left[\frac{11}{4} - \zeta(2) - 2\zeta(3) - H(0, x)\right. \\
+ \zeta(2)H(0, x) - \frac{7}{2}H(-1, x) - \frac{5}{2}\zeta(2)H(1, x) + H(0, 0, x) - \frac{7}{4}H(1, 0, x) \\
+ 4H(-1, -1, x) - \frac{5}{4}H(0, -1, x) - H(0, 0, x) - 2H(0, -1, -1, x) \\
- \frac{5}{4}H(1, 0, x) - \frac{7}{4}H(0, 0, x) - 2H(0, -1, -1, x). 
\]  

(364)  

(365)  

(366)
\[ a^2 F_z^{(34)} = \frac{1}{x^2} \left[ \frac{35}{8} H(-1, x) - \frac{3}{2} \zeta(2) H(-1, x) - 7 H(-1, -1, x) + 8 H(-1, -1, -1, x) \right] \\
\]
\[ -3 H(-1, 0, -1, x) + \frac{1}{x} \left[ -\frac{43}{8} + \frac{9}{10} \zeta(2)^2 + 2 \zeta(3) + (1 - \zeta(2)) \right] \\
\]
\[ -2 \zeta(3) H(0, x) + \frac{35}{4} H(-1, x) + 2 \zeta(2) H(-1, x) + \frac{9}{4} \zeta(2) H(1, x) \]
\[ +2 \zeta(3) H(1, x) - (1 - \zeta(2)) H(0, 0, x) - 14 H(-1, -1, x) \]
\[ +\frac{49}{8} H(0, -1, x) - \zeta(2) H(0, -1, x) + \frac{5}{2} \zeta(2) H(0, 1, x) + \frac{15}{8} H(1, 0, x) \]
\[ -\frac{3}{2} \zeta(2) H(1, 0, x) + \frac{3}{2} \zeta(2) H(1, 1, x) + H(0, 0, 0, x) - 5 H(0, -1, -1, x) \]
\[ +\frac{5}{4} H(0, 0, -1, x) + \frac{7}{4} H(0, 1, 0, x) - H(1, 0, -1, x) - \frac{7}{4} H(1, 0, 0, x) \]
\[ +16 H(-1, -1, -1, x) - 6 H(-1, 0, -1, x) + \frac{7}{4} H(1, 1, 0, x) \]
\[ -H(0, 0, 0, x) + 2 H(0, 0, -1, -1, x) - 8 H(0, -1, -1, -1, x) \]
\[ +3 H(0, -1, 0, -1, x) - \frac{1}{2} H(0, 0, 0, -1, x) + \frac{3}{2} H(0, 0, 1, 0, x) \]
\[ -2 H(0, 1, 0, -1, x) - \frac{3}{2} H(0, 1, 0, 0, x) + \frac{3}{2} H(0, 1, 1, 0, x) \]
\[ +8 H(1, 0, -1, -1, x) + \frac{3}{2} H(1, 0, 0, 0, x) + \frac{3}{2} H(1, 0, 1, 0, x) \]
\[ -\frac{3}{2} H(1, 1, 0, 0, x) + \frac{3}{2} H(1, 1, 1, 0, x) \]
\[ -\frac{1}{(1 - x)} \left[ 6 \zeta(2) - \frac{9}{10} \zeta^2(2) \right] \]
\[ -4 \zeta(3) - H(0, x) + \zeta(2) H(0, x) + 2 \zeta(3) H(0, x) - 2 \zeta(3) H(1, x) \]
\[ -\frac{35}{4} H(-1, x) - 2 \zeta(2) H(-1, x) + H(0, 0, x) - \zeta(2) H(0, 0, x) \]
\[ -H(1, 0, x) + \zeta(2) H(1, 0, x) - \zeta(2) H(1, 1, x) - \zeta(2) H(0, 1, x) - \zeta(2) H(0, 0, x) \]
\[ +14 H(-1, -1, x) - 14 H(0, -1, x) - H(0, 0, 0, x) - H(0, 1, 1, x) - H(0, 0, 1, x) - H(0, 1, 0, x) \]
\[ -2 H(1, 0, -1, x) + H(1, 0, 0, x) - H(1, 1, 0, x) + 16 H(0, -1, -1, x) \]
\[ -2 H(0, 0, -1, x) - 16 H(-1, -1, -1, x) + 6 H(-1, 0, -1, x) \]
\[ +H(0, 0, 0, x) - H(0, 0, 1, 0, x) + H(0, 1, 0, 0, x) \]

59
\[-H(0, 1, 1, 0, x) - H(1, 0, 0, 0, x) - H(1, 0, 1, 0, x)
+ H(1, 1, 0, 0, x) - H(1, 1, 1, 0, x)]
\[\frac{1}{(1-x)} \left[ 1 - \frac{1}{(1-x)} \right] \left\{ 35 \frac{\zeta(2)}{8} + \frac{9}{10} \zeta^2(2) - \frac{7}{4} \zeta(3) + \zeta(2)H(0, 1, x)
- \frac{35}{4} H(0, -1, x) - 2\zeta(2)H(0, -1, x) + 14H(0, -1, -1, x)
- 16H(0, -1, -1, x) + 6H(0, -1, 0, -1, x) - 2H(0, 0, 0, -1, x)
- 2H(0, 1, 0, -1, x) \right\}. \quad (368)\]

\[\mathcal{O}(\mathcal{O}(4-D) \int \mathcal{D}^2 k_1 \mathcal{D}^2 k_2 \mathcal{D}^2 \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4 \mathcal{D}_5 \mathcal{D}_6 \mathcal{D}_7 \mathcal{D}_8 \mathcal{D}_9 \mathcal{D}_{10} \mathcal{D}_{11} \mathcal{D}_{12} \mathcal{D}_{13} \mathcal{D}_{14} \quad (369)\]
\[= \left( \frac{\mu^2}{a} \right)^{2x} \sum_{i=-2}^{2} a^i F_i^{(35)} + \mathcal{O} (\varepsilon^3), \quad (370)\]

where:

\[a^2 F_{-2}^{(35)} = \frac{2}{x^2} - \frac{1}{6x}, \quad (371)\]
\[a^2 F_{-1}^{(35)} = \frac{2}{x^2} \left[ 1 + H(0, x) \right] + \frac{1}{6x} H(0, x), \quad (372)\]
\[a^2 F_0^{(35)} = \frac{1}{x^2} \left[ \frac{14}{3} - 2\zeta(2) + \frac{2}{3} H(0, x) + 2H(0, 0, x) \right] + \frac{1}{x} \left[ \frac{26}{27} + \frac{1}{6} \zeta(2)
- \frac{4}{9} H(0, x) - \frac{1}{6} H(0, 0, x) \right] - \frac{1}{3 \sqrt{x(4-x)}} \left[ \frac{8}{x^2} - \frac{4}{x} + \frac{1}{2} \right] H(r, 0, x) \quad (373)\]
\[a^2 F_1^{(35)} = \frac{1}{x^2} \left[ \frac{14}{9} + \frac{2}{3} \zeta(2) - 4\zeta(3) - \frac{58}{9} H(0, x) + 2\zeta(2)H(0, x) - \frac{2}{3} H(0, 0, x)
- 2H(0, 0, 0, x) \right] - \frac{1}{x} \left[ \frac{320}{81} + \frac{4}{9} \zeta(2) - \frac{1}{3} \zeta(3) - \left( \frac{26}{27} - \frac{1}{6} \zeta(2) \right) H(0, x)
- \frac{4}{9} H(0, 0, x) - \frac{1}{6} H(0, 0, 0, x) + \frac{1}{2} H(r, 0, x) \right] - \frac{1}{3 \sqrt{x(4-x)}} \left\{ \left[ \frac{8}{x^2} - \frac{4}{x} + \frac{1}{2} \right]
\zeta(2) H(r, x) - H(r, 0, x) - H(0, r, 0, x) - 2H(4, r, 0, x) \right\}
+ \frac{4}{3} \left[ \frac{120}{x^2} - \frac{1}{x} - 1 \right] H(r, 0, x) \right\}, \quad (374)\]
\[a^2 F_2^{(35)} = \frac{1}{x^2} \left[ \frac{10}{27} + \frac{58}{9} \zeta(2) + \frac{9}{5} \zeta(2)^2 - \frac{4}{3} \zeta(3) + \left( \frac{10}{27} + \frac{2}{3} \zeta(2) - 4\zeta(3) \right) H(0, x)
- \left( \frac{58}{9} - 2\zeta(2) \right) H(0, 0, x) - \frac{2}{3} H(0, 0, 0, x) + 4H(r, r, 0, x) \right] \quad (374)\]
\[-2H(0, 0, 0, x) \right] + \frac{1}{x^2} \int_{x(4-x)}^{3} \left( \frac{80}{9} \zeta(2)H(r, x) + \frac{16}{3} \zeta(3)H(r, x) \right) \]

\[-\left( \frac{160}{81} - \frac{4}{9} \zeta(2) + \frac{3}{20} \zeta(2)^2 - \frac{8}{9} \zeta(3) \right)H(0, x) \right] - \left( \frac{26}{27} - \frac{1}{6} \zeta(2) \right)H(0, 0, x) \]

\[-\frac{1}{2} \zeta(2)H(r, 0, 0, 0) + \frac{4}{3} H(r, 0, 0, 0) - \frac{1}{6} H(0, 0, 0, 0, x) \right] + \frac{1}{2} H(r, 0, 0, x) - \frac{1}{2} H(0, r, 0, x) \]

\[-\frac{1}{2} H(r, 0, 0, x) - \frac{1}{2} H(0, r, 0, x) \]
\[ = \left( \frac{\mu^2}{a} \right)^{2e} \sum_{i=-1}^{2} e^{i F_3^{(36)}} + \mathcal{O} (\epsilon^3), \tag{377} \]

where:

\[ a^2 F_3^{(36)} = -\frac{1}{1-x} \left[ \zeta(2) + 2H(0, x) - 2H(0, 0, x) + H(1, 0, x) \right], \tag{378} \]

\[ a^2 F_0^{(36)} = -\frac{3}{x} H(1, 0, x) - \frac{1}{(1-x)} \left[ \zeta(2) + 3\zeta(3) + 4H(0, x) + \zeta(2)H(1, x) \right] - 4H(0, 0, x) + 2H(1, 0, x) + 4H(0, 0, 0, x) + H(0, 1, 0, x) - 2H(1, 0, 0, x) + H(1, 1, 0, x), \tag{379} \]

\[ a^2 F_1^{(36)} = -\frac{1}{x} \left[ 3\zeta(2)H(1, x) + 6H(1, 0, x) + 3H(0, 1, 0, x) - 6H(1, 0, 0, x) + 3H(1, 1, 0, x) - 12H(1, 0, 0, x) + 3H(0, 0, 0, 0, x) + 3H(0, 1, 1, 0, x) - 6H(1, 1, 0, 0, x) + 3H(1, 1, 1, 0, x) - 12H(1, 1, 0, 0, x) - 8\zeta(3)H(0, x) - 2\zeta(2)H(1, x) + 3\zeta(3)H(1, x) - 8H(0, 0, x) + 4H(1, 0, x) + \zeta(2)H(1, 1, x) + \zeta(2)H(0, 1, x) + 8H(0, 0, 0, x) + 2H(0, 1, 0, x) - 8H(0, 1, 0, x) + 2H(1, 1, 0, x) - 8H(0, 0, 0, x) + H(0, 0, 0, x) - 2H(0, 1, 0, x) + H(0, 1, 1, 0, x) + 4H(1, 0, 0, 0, x) + H(0, 1, 0, 0, x) - 2H(1, 1, 0, 0, x) + H(1, 1, 1, 0, x) - 2H(1, 1, 0, 0, x) + H(1, 1, 1, 0, x) \right]. \tag{380} \]

The following diagram is reducible to a combination of two different 5-denominator diagrams, as explained in Eqs. (5,6) of [1], by simple partial fractioning.

\[
\begin{align*}
\includegraphics[width=0.2\textwidth]{diagram.png} &= \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{D_1 D_4 D_5 D_6 D_7 D_8 D_9 D_{10}} \tag{382} \\
&= \left( \frac{\mu^2}{a} \right)^{2e} \sum_{i=-3}^{1} e^{i F_3^{(37)}} + \mathcal{O} (\epsilon^2), \tag{383} \\
\end{align*}
\]

where:

\[ a^2 F_{-3}^{(37)} = \frac{1}{x}, \tag{384} \]

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\[ a^2 F^{(37)}_{-2} = -\frac{1}{x} H(0, x), \]  
\[ a^2 F^{(37)}_1 = -\frac{1}{x} \left[ \zeta(2) - H(0, 0, x) - H(1, 0, x) \right], \]  
\[ a^2 F^{(37)}_0 = \frac{1}{x} \left[ 8 - 2\zeta(3) - \left( 4 - \zeta(2) \right) H(0, x) + \left( \zeta(2) + \sqrt{3} H(r, 0; 1) \right) H(1, x) \right. \]  
\[ + 2H(1, 0, x) - H(0, 0, x) + H(0, 1, 0, x) - H(1, 0, 0, x) \]  
\[ + H(1, 1, 0, x) - 3H(1 + r, r, 0, x) \left] + \frac{2(4 - x)}{x \sqrt{x(4 - x)}} H(r, 0, x), \right. \]  
\[ a^2 F^{(37)}_1 = \frac{1}{x} \left\{ 16 + 4\zeta(2) + \frac{9}{10} \zeta^2(2) - 2\zeta(3) H(0, x) - 2 \left( \zeta(2) + \zeta(3) \right) H(1, x) \right. \]  
\[ - \left( 4 - \zeta(2) \right) \left( H(0, 0, x) + H(1, 0, x) \right) - \zeta(2) H(0, 1, x) - \zeta(2) H(1, 1, x) \right. \]  
\[ + 3\zeta(2) H(1 + r, r, x) - 2H(0, 1, 0, x) + 2H(1, 0, 0, x) - 2H(1, 1, 0, x) \right. \]  
\[ + 6H(1 + r, r, 0, x) + 6H(r, r, 0, x) - 3H(1 + r, r, x) \right. \]  
\[ - 3H(1 + r, r, 0, x) + 6H(1 + r, 4, r, 0, x) + 3H(0, 1 + r, r, 0, x) \right. \]  
\[ - H(0, 0, 0, x) - H(0, 0, 0, x) + H(0, 1, 0, x) - H(0, 1, 0, x) \right. \]  
\[ + 3H(1, 1 + r, r, 0, x) - H(1, 0, 0, x) - H(1, 0, 0, x) + H(1, 1, 0, x) \right. \]  
\[ - H(1, 1, 0, x) - \sqrt{3} \left[ H(r, 0; 1) + 2H(r, 0; 1) + H(4, r; 0) \right] H(1, x) \]  
\[ + H(r, 0; 1) H(0, 1, x) + H(r, 0; 1) H(1, 1, x) \right\} \]  
\[ + \frac{2(4 - x)}{x \sqrt{x(4 - x)}} \left[ \zeta(2) H(r, x) - H(r, 0, 0, x) - H(0, r, 0, x) \right. \]  
\[ + 2H(4, r, 0, x) \left] \right. \]  

where:

\[ a^2 F^{(38)}_{-1} = -\frac{1}{(1 - x)} H(0, x), \]  
\[ a^2 F^{(38)}_0 = \frac{1}{x} \left[ \zeta(2) H(1, x) + H(0, 1, 0, x) - 2H(1, 0, x) + H(1, 1, 0, x) \right] \]  
\[ - \frac{1}{(1 - x)} \left[ \zeta(2) + 2H(0, x) - H(0, 0, x) + H(1, 0, x) \right] \]  
\[ a^2 F^{(38)}_1 = -\frac{1}{x} \left[ \zeta(3) H(1, x) + 4H(1, 0, x) - 4\zeta(2) H(0, 1, x) - 4\zeta(2) H(1, 1, x) \right] \]  
\[ = \mu^{2(4 - D)} \int D^D k_1 D^D k_2 \frac{1}{D_3 D_4 D_5 D^2_{12} D_{13}} \]  
\[ = \left( \frac{\mu^2}{a} \right)^{2e} \sum_{i=-1}^{1} e^i F_i^{(38)} + \mathcal{O} \left( e^2 \right), \]  

\[ 63 \]
\[ -2H(1, 0, 0, x) - H(0, 0, 1, 0, x) + H(0, 1, 0, 0, x) - 4H(0, 1, 1, 0, x) \]
\[ -H(1, 0, 1, 0, x) + H(1, 1, 0, 0, x) - 4H(1, 1, 1, 0, x) \]
\[ - \frac{1}{(1 - x)} \left[ 2\zeta(2) + 2\zeta(3) + (4 - \zeta(2))H(0, x) + \zeta(2)H(1, x) - 2H(0, 0, x) + 2H(1, 0, x) \right. \]
\[ + H(0, 0, 0, x) + H(0, 1, 0, x) - H(1, 0, 0, x) + H(1, 1, 0, x) \] . \quad (393)
The following two diagrams are reducible to a combination of two different 4-denominator diagrams, as explained in Eqs. (5,6) of [1], by simple partial fractioning.

\[
\begin{align*}
\text{Diagram 1:} & \quad \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{\mathcal{D}_1 \mathcal{D}_4 \mathcal{D}_5 \mathcal{D}_{12} \mathcal{D}_{13}} \\
= & \quad \left( \frac{\mu^2}{a} \right)^{2\varepsilon} \sum_{i=-3}^{1} \epsilon^i F_{\varepsilon}^{(40)} + \mathcal{O}(\epsilon^2),
\end{align*}
\]

where:

\[ a F_{-3}^{(40)} = -\frac{1}{x}, \]

\[ a F_{-2}^{(40)} = \frac{1}{x} \left[ 1 - H(0, x) \right], \]

\[ a F_{-1}^{(40)} = -\frac{1}{x} \left[ 1 - \zeta(2) - H(0, x) + H(0, 0, x) + H(1, 0, x) \right], \]

\[ a F_{0}^{(40)} = -\frac{1}{x} \left[ 1 - \zeta(2) - 2\zeta(3) - \left( 1 - \zeta(2) \right) H(0, x) + \zeta(2) H(1, x) + H(0, 0, x) - H(0, 0, x) + H(0, 1, 0, x) + H(1, 0, x) - H(1, 0, 0, x) + H(1, 1, 0, x) \right], \]

\[ a F_{1}^{(40)} = \frac{1}{x} \left[ 1 - \zeta(2) - \frac{9}{10} \zeta^2(2) - 2\zeta(3) - H(0, x) + \left( \zeta(2) + 2\zeta(3) \right) H(0, x) + \left( 1 - \zeta(2) \right) \left( H(0, 0, x) + H(1, 0, x) \right) \right].
\]

\[
\begin{align*}
\text{Diagram 2:} & \quad \mu^{2(4-D)} \int \mathcal{D}^D k_1 \mathcal{D}^D k_2 \frac{1}{\mathcal{D}_1 \mathcal{D}_5 \mathcal{D}_5^2 \mathcal{D}_{12} \mathcal{D}_{13}} \\
= & \quad \left( \frac{\mu^2}{a} \right)^{2\varepsilon} \sum_{i=-1}^{2} \epsilon^i F_{\varepsilon}^{(41)} + \mathcal{O}(\epsilon^3),
\end{align*}
\]

where:

\[ a F_{-1}^{(41)} = \frac{1}{(1 - x)} H(0, x), \]

\[ a F_{0}^{(41)} = \frac{2}{x} H(1, 0, x) + \frac{1}{(1 - x)} \left[ \zeta(2) + H(0, x) - H(0, 0, x) + H(1, 0, x) \right].
\]
\[ aF_1^{(41)} = \frac{1}{x} \left[ 2\zeta(2)H(1, x) + 2H(1, 0, x) + 2H(0, 1, 0, x) - 2H(1, 0, 0, x) + 2H(1, 1, 0, x) \right] + \frac{1}{1 - x} \left[ \zeta(2) + 2\zeta(3) + H(0, x) - \zeta(2)H(0, x) + \zeta(2)H(1, x) - H(0, 0, x) + H(1, 0, x) + H(0, 0, 0, x) + H(0, 1, 0, x) - H(1, 0, 0, x) + H(1, 1, 0, x) \right] \]

\[ aF_2^{(41)} = \frac{2}{x} \left[ \zeta(2)H(1, x) + 2\zeta(3)H(1, x) + H(1, 0, x) - \zeta(2)H(1, 0, x) + \zeta(2)H(1, 1, x) + H(0, 0, 0, x) + H(0, 1, 0, x) - H(1, 0, 0, x) + H(1, 1, 0, x) \right] + \frac{1}{1 - x} \left[ \frac{9}{10} \zeta(2) + \frac{9}{10} \zeta(3) + \frac{9}{10} \zeta(2) - \frac{9}{10} \zeta(3) \right] H(0, x) + \left( \frac{9}{10} \zeta(2) + \frac{9}{10} \zeta(3) \right) H(1, x) - \left( \frac{9}{10} \zeta(2) + \frac{9}{10} \zeta(3) \right) H(0, 0, x) + \zeta(2)H(0, 1, x) + H(1, 0, x) - \zeta(2)H(1, 0, x) + \zeta(2)H(1, 1, x) + H(0, 0, 0, x) + H(0, 1, 0, x) - H(1, 0, 0, x) + H(1, 0, 0, 0, x) + H(0, 1, 0, x) + H(1, 0, 0, x) - H(1, 1, 0, x) + H(1, 1, 0, 0, x) \right] . \]

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