On Bures-Distance and \(^\ast\)-Algebraic Transition Probability between Inner Derived Positive Linear Forms over \(W^\ast\)-Algebras

Peter M. Alberti (peter.alberti@itp.uni-leipzig.de) * and Armin Uhlmann (armin.uhlmann@itp.uni-leipzig.de)
Institute of Theoretical Physics
University of Leipzig
Augustusplatz 10, D-04109 Leipzig, Germany

Abstract. On a \(W^\ast\)-algebra \(M\), for given two positive linear forms \(\nu, \varrho \in M^\ast_+\) and algebra elements \(a, b \in M\) a variational expression for the Bures-distance \(d_B(\nu^a, \varrho^b)\) between the inner derived positive linear forms \(\nu^a = \nu(a^\ast \cdot a)\) and \(\varrho^b = \varrho(b^\ast \cdot b)\) is obtained. Along with the proof of the formula also some earlier result of S. Gudder on non-commutative probability will be slightly extended. Also, the given expression of the Bures-distance nicely relates to some system of seminorms proposed by D. Buchholz and which occurred along with the problem of estimating the so-called 'weak intertwiners' in algebraic quantum field theory. In the last part some optimization problem will be considered.

Keywords: \(W^\ast\)-algebras, positive linear forms, Bures-distance, inner operations

MSC codes: 46L89, 46L10, 58B10, 58B20

1. Introduction

1.1. Basic settings on Bures-distance

Throughout the paper the Bures-distance function \(d_B\) \cite{11} and related metric concepts on the positive cone \(M^\ast_+\) of the bounded linear forms \(M^\ast\) over a \(W^\ast\)-algebra \(M\) will be considered. Start with defining the Bures-distance \(d_B(M|\nu, \varrho)\) between \(\nu, \varrho \in M^\ast_+\).

Definition 1. \(d_B(M|\nu, \varrho) = \inf_{\{\pi, K\}, \varphi \in S_{\pi,M}(\nu), \psi \in S_{\pi,M}(\varrho)} \| \psi - \varphi \|\).

Instead of \(d_B(M|\nu, \varrho)\) often also \(d_B(\nu, \varrho)\) will be used. For unital \(*\)-representation \(\{\pi, K\}\) of \(M\) on a Hilbert space \(\{K, < \cdot, \cdot >\}\) and for \(\mu \in M^\ast_+\) we let \(S_{\pi,M}(\mu) = \{\xi \in K : \mu(\cdot) = \langle \pi(\cdot)\xi, \xi \rangle\}\). Then, the infimum within the defining formula for \(d_B(\nu, \varrho)\) extends over all those \(\pi\) relative to which \(S_{\pi,M}(\nu) \neq \emptyset\) and \(S_{\pi,M}(\varrho) \neq \emptyset\) simultaneously hold, and within each such representation the vectors \(\varphi\) and \(\psi\) may be varied through all of \(S_{\pi,M}(\nu)\) and \(S_{\pi,M}(\varrho)\), respectively. The scalar product

\* Partially supported by ‘Deutsche Forschungsgemeinschaft’.
$\mathcal{K} \times \mathcal{K} \ni \{\chi, \eta\} \mapsto \langle \chi, \eta \rangle \in \mathbb{C}$ on the representation Hilbert space by convention is supposed to be linear with respect to the first argument $\chi$, and antilinear in the second argument $\eta$, and maps into the complex field $\mathbb{C}$. Let $\mathbb{C} \ni z \mapsto \bar{z}$ be the complex conjugation, and be $\Re z$ and $|z|$ the real part and absolute value of $z$, respectively. Greek letters and their labelled derivates (except for $\pi$, which is reserved for representations only) will be used to label elements of the complex Hilbert spaces on which the concrete $\mathbb{C}^*$-algebras $\pi(M)$ are supposed to act. The norm of $\chi \in \mathcal{K}$ is given by $\|\chi\| = \sqrt{\langle \chi, \chi \rangle}$. Relating operator and $\mathbb{C}^*$-algebra theory, refer the reader to standard monographs, e.g. \cite{13, 23, 19}. For simplicity, for the $\mathbb{C}^*$-norm of an element $x \in M$ as well as for the operator norm of a concrete bounded linear operator $x \in \mathcal{B}(\mathcal{K})$ the same notation $\|x\|$ will be used. In both these cases, the involution ($^*$-operation) respectively the taking of the hermitian conjugate of an element $x$ is indicated by the transition $x \mapsto x^*$. The notions of hermiticity and positivity for elements are defined as usual in $\mathbb{C}^*$-algebra theory, and $M_h$ and $M_+$ are the hermitian and positive elements of $M$, respectively. With view to the above, to make these settings more unambiguous, agree that Greek letters will not be used as symbols for linear operators over $\mathcal{K}$ or elements of $M$. The null and the unit element/operator in $M$ and $\mathcal{B}(\mathcal{K})$ will be denoted by $0$ and $1$.

For notational purposes mainly, in short recall some fundamentals relating (bounded) linear forms which subsequently might be of concern in context of Definition 1. Remind that the topological dual space $M^*$ of $M$ is the set of all those linear functionals (linear forms) which are continuous with respect to the operator norm topology. Equipped with the dual norm $\|\cdot\|_1$, which is given by $\|f\|_1 = \sup\{|f(x)| : x \in M, \|x\| \leq 1\}$ and which is referred to as the functional norm, $M^*$ is a Banach space. For each given $f \in M^*$, the hermitian conjugate functional $f^* \in M^*$ is defined by $f^*(x) = \overline{f(x^*)}$, for each $x \in M$. Remind that $f \in M^*$ is hermitian if $f = f^*$ holds, and $f$ is termed positive if $f(x) \geq 0$ holds, for each $x \in M_+$. Also remind that a bounded linear form over $M$ is positive if, and only if, $\|f\|_1 = f(1)$ is fulfilled. For positive linear forms one has the following fundamental estimate (Cauchy-Schwarz inequality)

$$\forall g \in M^*_+ : |g(y^*x)|^2 \leq g(y^*y)g(x^*x), \forall x, y \in M,$$  \hspace{1cm} (1-1a)

which accordingly also holds on $\mathbb{C}^*$-algebras. From this it is easily inferred that for each $g \in M^*_+ \\backslash \{0\}$ the subset $I_g \subset M$ defined by

$$I_g = \{x \in M : g(x^*x) = 0\}$$  \hspace{1cm} (1-1b)

is a (proper) left ideal in $M$. Provided this ideal is trivial, $I_g = \{0\}$, the positive linear form $g \in M^*_+$ is called faithful (positive linear form).
The most important consequence of positivity and (1-1a) is that, for each \( g \in M^*_+ \), there exists a cyclic *-representation \( \pi_g \) of \( M \) on some Hilbert space \( \mathcal{K}_g \), with cyclic vector \( \Omega \in \mathcal{K}_g \), and obeying \( g(x) = \langle \pi_g(x)\Omega, \Omega \rangle \), for all \( x \in M \). This fact usually is referred to as the Gelfand-Neumark-Segal theorem (GNS). Such a representation (which is unique up to unitary isomorphisms) will be referred to as with \( g \) associated cyclic representation, or GNS-representation of \( g \), respectively. Note that considering such construction in the special case with \( g = \nu + \rho \) will provide a unital *-representation \( \pi = \pi_g \) such that \( S_{\pi,M}(\nu) \neq \emptyset \) and \( S_{\pi,M}(\rho) \neq \emptyset \) hold (we omit the details, all of which are standard). It is exactly this fact which makes that the expression in Definition 1 makes sense even in the C*-algebraic case.

Apart from the functional norm topology, mention also the \( w^* \)-topology on \( M^*_+ \), which is the weakest locally convex topology generated by the seminorms \( \rho_x, x \in M \), with \( \rho_x(f) = |f(x)| \), for each \( f \in M^* \). Remind that according to a basic result of Banach space theory (Alaoglu-Banach theorem) each closed, bounded subset of the dual Banach space \( M^* \) has to be \( w^* \)-compact.

Along with Definition 1 an auxiliary metric structure arises which can be compared to the metric structure given by the ‘natural’ distance \( d_1(\nu, \rho) = \|\nu - \rho\|_1 \) on \( M^*_+ \). The relevant basic facts will be stated here without proof and read as follows.

**Proposition 1.** Let \( d_B : M^*_+ \times M^*_+ \ni \{\nu, \rho\} \mapsto d_B(M|\nu, \rho) \in \mathbb{R}_+ \) be given in accordance with Definition 1. Then the following hold:

1. \( d_B \) is a distance function on the points of \( M^*_+ \);
2. \( d_B \) is topologically equivalent with \( d_1 \) on bounded subsets of \( M^*_+ \).

Especially, for \( \{\nu, \rho\} \in M^*_+ \times M^*_+ \setminus \{0,0\} \) one has

\[
c(\nu, \rho)^{-1} d_1(\nu, \rho) \leq d_B(M|\nu, \rho) \leq \sqrt{d_1(\nu, \rho)},
\]

with \( c(\nu, \rho) = \sqrt{\|\nu\|_1} + \sqrt{\|\rho\|_1} \).

Remark that item (1) and ‘one half’ of the estimate (1-2), from which (2) obviously can be followed, were anticipated and proved by D. Bures in [11], whereas the other half of (1-2) can be seen by arguments given by H. Araki in [6, 7] e.g.; omit any details on this matter but remark that D. Bures refers to the state space of \( M \), \( \mathcal{S}(M) = \{f \in M^*_+ : f(1) = 1\} \). This simplifies matters insofar that in restriction to \( \mathcal{S}(M) \) then \( d_B \) gets unconditionally topologically equivalent with \( d_1 \).
1.2. Prerequisites, useful estimates and examples

In conjunction with the Bures-distance $d_B$ one has the functor $P$ of the (*-algebraic) transition probability \[25\]. For given \(W^*-\)algebra $M$ and positive linear forms $\nu, \varrho \in M_+^*$ the definition reads as follows:

**Definition 2.** \(P_M(\nu, \varrho) = \sup_{\{\pi, \varphi\} \in S_{\pi, M}(\nu), \psi \in S_{\pi, M}(\varrho)} |\langle \psi, \varphi \rangle|^2.\)

Thereby, the range of variables over which the supremum has to be extended is the same as in **Definition 1**. With the help of $P_M$ one then gets a (uniquely solvable) expression for the Bures-distance:

\[
d_B(M|\nu, \varrho)^2 = \left\{ \|\nu\|_1 - \sqrt{P_M(\nu, \varrho)} \right\} + \left\{ \|\varrho\|_1 - \sqrt{P_M(\nu, \varrho)} \right\}. \tag{1-3}
\]

Remark that $P$ is of importance on its own rights (and independent from the just mentioned appearance within (1-3)) since it can be well-adapted to several applications in (algebraic) quantum physics, non-commutative probability and estimation theory. The latter also was the heuristic intention behind the introduction of this functor in \[25\]. For a particular range of applications see e.g. \[4, 27\].

Many properties of $P$ are known. In the following only some very few of these properties will be referred to explicitly. For instance, essentially by means of the Cauchy-Schwarz inequality from the definition of $P$ the following fundamental estimates can be obtained:

\[
|f(1)|^2 \leq P_M(\nu, \varrho) \leq \nu(a) \varrho(a^{-1}), \tag{1-4}
\]

where $f$ can be any linear form of the set

\[
\Gamma_M(\nu, \varrho) = \left\{ f \in M^* : |f(y^*x)|^2 \leq \nu(y^*y)\varrho(x^*x), \forall x, y \in M \right\}. \tag{1-5}
\]

and $a$ can be any invertible, positive element $a \in M_+$. Note that $\Gamma_M(\nu, \varrho)$ obviously is $w^*$-closed and bounded ($\sqrt{\|\nu\|_1 \|\varrho\|_1}$ is a common upper bound), and thus is a $w^*$-compact subset of $M^*$.

For the estimate from above see eq. (16) in \[25\]. Relating the estimate from below, suppose a unital *-representation $\{\pi, \mathcal{K}\}$ of $M$ on $\mathcal{K}$ with $S_{\pi, M}(\nu) \neq \emptyset$ and $S_{\pi, M}(\varrho) \neq \emptyset$ to be given. By standard facts one then infers that for given $\varphi \in S_{\pi, M}(\nu)$, $\psi \in S_{\pi, M}(\varrho)$

\[
\Gamma_M(\nu, \varrho) = \left\{ \langle \pi(\cdot)k\psi, \varphi \rangle : k \in \langle \pi(M)\rangle_1 \right\}. \tag{1-6}
\]

has to be fulfilled. In this formula $\langle \pi(M)\rangle_1$ is the unit ball within the commutant $vN$-algebra $\pi(M)'$. From this and **Definition 2** with the
help of the Theorem of B. Russo and H. Dye [15] then also the validity of the estimate from below in (1-4) follows, see eq. (3) in [1].

Apply (1-4) to the special case of two vector states, which is heuristically important in a quantum physical context of two wave functions:

**Example 1.** Let \( M = \mathcal{B}(\mathcal{H}) \) be the algebra of bounded linear operators on a Hilbert space \( \mathcal{H} \). Let \( \mu_\psi = \langle (\cdot)\psi, \psi \rangle \) be the vector form generated by \( \psi \in \mathcal{H} \) on \( M \), and be \( p_\varphi \) the orthoprojection onto the span of \( \varphi \in \mathcal{H} \). Then, considering \( f = \langle (\cdot)\psi, \varphi \rangle \in \Gamma_M(\mu_\varphi, \mu_\psi) \) and \( a = p_\varphi + \varepsilon^{-1}p_\perp^\varphi \), for \( \varepsilon \in \mathbb{R}_+ \setminus \{0\} \), and inserting this into (1-4) provides

\[
|\langle \psi, \varphi \rangle|^2 \leq P_M(\mu_\varphi, \mu_\psi) \leq |\langle \psi, \varphi \rangle|^2 + \varepsilon\|\varphi\|^2\mu_\psi(p_\perp^\varphi).
\]

For \( \varepsilon \to 0 \) from this follows, in any case of two vectors \( \psi, \varphi \in \mathcal{H} \).

Also, constellations among the positive linear forms \( \nu, \varrho \in M_+^* \) are known such that for some \( a \geq 0 \) the upper estimate within (1-4) turns into an equality. This then provides an expression for \( P_M(\nu, \varrho) \).

To explain such stuff, fix some notation first. In all that follows for \( x \in M \) and \( \mu \in M_+^* \) a positive linear form \( \mu \) will be defined by

\[
\mu_x(y) = \mu(x^*yx),
\]

for each \( y \in M \). If this situation occurs the positive linear form \( \mu_x \) will be referred to as an inner derived (from \( \mu \)) positive linear form. The main result of [25] refers to this and reads as follows:

**THEOREM 1.** \( \forall \mu \in M_+^*, a, b \in M, a^*b \geq 0 : \sqrt{P_M(\mu^a, \mu^b)} = \mu(a^*b) \).

For instance, in choosing \( a \geq 0, b = 1 \) the premises of the previous result are fulfilled in a trivial manner and one thus arrives at the formula

\[
\forall \mu \in M_+^*, a \in M_+ : P_M(\mu^a, \mu) = \mu(a)^2.
\]  

(1-7)

Remark that **Example 1** in the case of non-orthogonal vectors can be seen as a special case of (1-7) as well. It is interesting that the seemingly very special situation with the premises of (1-7) addresses itself to a wide range of characteristic applications. One of these reads as follows:

**Example 2.** By the Radon-Nikodym theorem of S. Sakai [22] we are always in such a situation if amongst two normal positive linear forms \( \nu, \varrho \in M_+^* \) a relation of domination \( \varrho \leq \lambda \nu \), with \( \lambda \in \mathbb{R}_+ \setminus \{0\} \), takes place, in which situation also the notation \( \varrho \ll \nu \) will be used. That is, for \( \varrho \ll \nu \) there is \( a \in M_+ \) with \( \varrho = \nu^a \). In view of the above in such situation then especially \( P_M(\varrho, \nu) = \nu(a)^2 \) follows. It is known that \( a \) gets unique if \( s(a) \leq s(\nu) \) is required to hold, with the supports of the operator \( a \) and normal positive linear form \( \nu \), respectively. To this unique \( a \) one usually refers as the Sakai’s Radon-Nikodym operator of \( \varrho \) relative to \( \nu \), and then also the notation \( a = \sqrt{d\varrho/d\nu} \) will be used.
Finally, it is interesting that in any case with the help of the bounds appearing along with (1-4) the value of $P_M(\nu, \varrho)$ can be approximated to an arbitrary degree of precision from both sides. This and some other relevant informations are the content of the following result.

**THEOREM 2.** Let $M$ be a $\mathcal{W}^*$-algebra, and be $\nu, \varrho \in M^*_+$. Then, the following facts hold:

1. $\sqrt{P_M(\nu, \varrho)} = \inf_{x>0} \sqrt{\nu(x)\varrho(x^{-1})}$;
2. $\sqrt{P_M(\nu, \varrho)} = \sup_{f \in \Gamma_M(\nu, \varrho)} |f(1)|$.

The infimum in (1) extends over all positive invertible elements of $M$. Moreover, if $\{\pi, \mathcal{K}\}$ is any unital $^*$-representation of $M$ over some Hilbert space $\mathcal{K}$ such that $S_{\pi,M}(\nu) \neq \emptyset$ and $S_{\pi,M}(\varrho) \neq \emptyset$ are fulfilled, then the following is fulfilled:

3. $\sqrt{P_M(\nu, \varrho)} = \sup_{\psi \in S_{\pi,M}(\varrho)} |\langle \psi, \varphi \rangle|, \ \forall \varphi \in S_{\pi,M}(\nu)$.

Also, the supremum in (2) is a maximum and is attained at some $f \in \Gamma_M(\nu, \varrho)$, and some maximizing $f$ can be chosen as $f = \langle \pi(\cdot)\psi_0, \varphi_0 \rangle$, for some $\psi_0 \in S_{\pi,M}(\varrho)$, $\varphi_0 \in S_{\pi,M}(\nu)$.

For proofs of (1)–(3) see Corollary 1, Corollary 3 and Theorem 3 in [1], for the additional informations on the attainability of the supremum in (2), see [7] and [2]. The previous result remains valid even if $M$ is supposed to be a unital $\mathcal{C}^*$-algebra.

**Remark 1.** The question arises whether the functor $P$ in a reasonable manner (i.e. such that a relation of type (1-3) with a metric distance $d_B$ remained true on its domain of definition) could be extended to some yet more general category of $^*$-algebras (including some unbounded operator algebras showing up in relativistic quantum field theory e.g.), see [24, 26]. Besides the just mentioned $\mathcal{C}^*$-algebraic cases the answer seems to be in the negative.
1.3. THE MAIN RESULT

Under the premises of Theorem 1, let us suppose now that some unital \(\ast\)-representation \(\{\pi, K\}\) has been chosen in accordance with \(S_{\pi,M}(\mu) \neq \emptyset\). Then, for \(\Omega \in S_{\pi,M}(\mu)\) one has \(\pi(a)\Omega \in S_{\pi,M}(\mu^a)\) and \(\pi(b)\Omega \in S_{\pi,M}(\mu^b)\). Hence, in making use of (1-6) in the special case of \(\Gamma_M(\mu, \mu)\), with \(\varphi = \psi = \Omega\), and in the special case of \(\Gamma_M(\mu^a, \mu^b)\) with \(\psi = \pi(b)\Omega\) and \(\varphi = \pi(a)\Omega\), and respecting positivity of \(a^\ast b\), one easily infers that

\[\mu(a^\ast b) = \|\pi(\sqrt{a^\ast b})\Omega\|^2 = \sup_{g \in \Gamma_M(\mu, \mu)} |g(a^\ast b)| = \sup_{f \in \Gamma_M(\mu^a, \mu^b)} |f(1)|\]

has to be fulfilled. The formula of Theorem 1 and Theorem 2 (2) together with the previous then show that the following is valid.

**COROLLARY 1.**

\[\forall \mu \in M^+\ast, \ a, b \in M, \ a^\ast b \geq 0 : \sqrt{P_M(\mu^a, \mu^b)} = \sup_{f \in \Gamma_M(\mu, \mu)} |f(a^\ast b)|.\]

The first goal of the paper will be to extend the assertion of Corollary 1 as to hold true under much weaker premises. More precisely, instead of considering two positive linear forms \(\nu, \varrho\) which both are inner derived positive linear forms \(\nu = \mu^a\) and \(\varrho = \mu^b\) from one and the same positive linear form \(\mu\) via operators \(a, b \in M\) which obey the positivity assumption \(a^\ast b \geq 0\), subsequently two arbitrarily chosen inner derived positive linear forms are permitted to be considered, without any further restriction. Based on this, under the same premises on the positive linear forms a variational expression for the Bures-distance function will be derived.

**THEOREM 3.** Let \(M\) be a \(W\ast\)-algebra, and be \(\nu, \varrho \in M^+_\ast\), and \(a, b \in M\). Then, the following facts hold true:

1. \[\sqrt{P_M(\nu^a, \varrho^b)} = \sup_{f \in \Gamma_M(\nu, \varrho)} |f(a^\ast b)|;\]
2. \[d_B(\nu^a, \nu^b)^2 = \sup_{a^\ast b = y^\ast x} \{\nu(a^\ast a - y^\ast y) + \varrho(b^\ast b - x^\ast x)\}.\]

Obviously, (1) is the announced extension of the assertion of Corollary 1, whereas by (2), which will be shown to be a consequence of (1), the mentioned variational expression for the distance \(d_B\) between two inner derived from a given pair \(\{\nu, \varrho\}\) positive linear forms is given.

Foremost, such expression as given in (2) can be useful since it allows for estimating the behavior of the Bures-distance at \(\{\nu, \varrho\}\) if this pair is undergoing an inner perturbation towards another pair \(\{\nu^a, \varrho^b\}\) of positive linear forms. As it comes out, the geometry of submanifolds
of mutually coordinated (via inner operations) positive linear forms of $W^*$-algebras to a great deal can be based on this formula. We will not elaborate on this in this paper, but instead within Section 3 we will be concerned with one particular aspect of this geometry more in detail.

In the course of the derivation of the main result several further characterizations of $P$ (and thus of $d_B$ as well) will be obtained.

2. Results and proofs

2.1. Further characterizations of transition probability

In all what follows $M$ is a $W^*$-algebra, and $\nu, \varrho \in M_+^*$ are fixed but can be arbitrarily chosen positive linear forms. Start with some sequences from Theorem 2. Relating notations, when occurring in conjunction with inf or sup, in each case of occurrence the variables $x > 0$, $\{x\}$, $\{e\}$ and $\{y, x\}$ are thought to extend over all positive invertible elements $x$, all finite decompositions $\{x\} = \{x_1, \ldots, x_n\}$ of the unity into positive elements, all finite decompositions $\{e\} = \{e_1, \ldots, e_n\}$ of the unity into orthoprojections, and all finite double systems $\{y, x\} = \{y_1, x_1, \ldots, y_n, x_n\}$ of elements obeying $\sum_j y_j^* x_j = 1$, respectively, within $M$, where $n$ can range through the naturals, $n \in \mathbb{N}$.

**COROLLARY 2.** The following properties hold:

1. $\sqrt{P_M(\nu, \varrho)} = \inf_{\{x\}} \sum_j \sqrt{\nu(x_j)} \varrho(x_j)$;
2. $\sqrt{P_M(\nu, \varrho)} = \inf_{\{e\}} \sum_j \sqrt{\nu(e_j)} \varrho(e_j)$;
3. $\sqrt{P_M(\nu, \varrho)} = \inf_{\{y, x\}} \frac{1}{2} \sum_j \{\nu(y_j^* y_j) + \varrho(x_j^* x_j)\}$;
4. $\sqrt{P_M(\nu, \varrho)} = \inf_{\{1 = y^* x\}} \frac{1}{2} \{\nu(y^* y) + \varrho(x^* x)\}$;
5. $\sqrt{P_M(\nu, \varrho)} = \inf_{x > 0} \frac{1}{2} \{\nu(x) + \varrho(x^{-1})\}$.

**Proof.** Note that according to (1-5) for each $f \in \Gamma_M(\nu, \varrho)$ and any finite positive decomposition $\{x\}$ of the unity one has $|f(1)| \leq \sum_j |f(x_j)| = \sum_j |f(\sqrt{x_j} \sqrt{x_j})| \leq \sum_j \sqrt{\nu(x_j)} \varrho(x_j)$. According to Theorem 2 (2) therefore

$$\sqrt{P_M(\nu, \varrho)} \leq \inf_{\{x\}} \sum_j \sqrt{\nu(x_j)} \varrho(x_j) \leq \inf_{\{e\}} \sum_j \sqrt{\nu(e_j)} \varrho(e_j) \quad (*)$$

can be followed. That is, validity of (2) will imply that also (1) is true. To see that (2) holds, let $\varepsilon > 0$. According to Theorem 2 (1) there
exists invertible \( x \in M_+ \) and obeying \( \nu(x)g(x^{-1}) < P_M(\nu, g) + \varepsilon \). Since the map \( y \mapsto y^{-1} \) in restriction to the invertible elements of \( M_+ \) is normcontinuous, and since we are in a \( W^* \)-algebra, in addition we may even suppose that \( x \) satisfying the above estimate is chosen with finite spectrum, that is, \( x = \sum_{j=1}^{n} \lambda_j e_j \) is fulfilled, with \( \lambda_j > 0 \), and some finite decomposition \( \{ e_1, \ldots, e_n \} \) of the unity into mutually orthogonal orthoprojections of \( M \). Using this spectral decomposition, one arrives at the expression

\[
\nu(x)g(x^{-1}) = \sum_{j} \nu(e_j)g(e_j) + \sum_{j \neq k} \{ \lambda_j \lambda_k^{-1} \nu(e_j)g(e_k) + \lambda_k \lambda_j^{-1} \nu(e_k)g(e_j) \}.
\]

Owing to strict positivity of \( \lambda \)'s and non-negativity of \( \nu(e_j) \)'s one has

\[
\lambda_j \lambda_k^{-1} \nu(e_j)g(e_k) + \lambda_k \lambda_j^{-1} \nu(e_k)g(e_j) \geq 2 \sqrt{\nu(e_j)g(e_j)} \sqrt{\nu(e_k)g(e_k)},
\]

for each \( j > k \). In fact, for \( \sqrt{\nu(e_j)g(e_j)} \sqrt{\nu(e_k)g(e_k)} = 0 \) this is trivial, whereas in the other case the estimate follows from minimizing the positive function \( F(t) = t \nu(e_j)g(e_k) + t^{-1} \nu(e_k)g(e_j) \) over \( \mathbb{R}_+ \backslash \{0\} \), which problem has a solution, since in this case both coefficients of \( t \) and \( t^{-1} \) are strictly positive. By means of this estimate and the above one finally arrives at

\[
P_M(\nu, g) + \varepsilon \geq \nu(x)g(x^{-1}) \geq \left\{ \sum_{j} \sqrt{\nu(e_j)g(e_j)} \right\}^2. \quad (**)\]

From this inf \( \{ p \} \) \( \sum_{j} \sqrt{\nu(p_j)g(p_j)} \leq \sqrt{P_M(\nu, g) + \varepsilon} \) is seen. Since \( \varepsilon > 0 \) could have been chosen at will, \( \sqrt{P_M(\nu, g)} \geq \inf \{ p \} \sum_{j} \sqrt{\nu(p_j)g(p_j)} \) follows, with \( \{ p \} \) extending over the finite decompositions of the unity into orthoprojections of \( M \). From this and (*) then (1) and (2) follow.

In order to prove (3), to given \( \varepsilon > 0 \), for each \( \delta > 0 \) by means of the decomposition \( \{ e_1, \ldots, e_n \} \) of the unity into orthoprojections \( e_j \) obeying (**) let us define a double system \( \{ y(\delta), x(\delta) \} \subset M \) by setting \( x_j(\delta) = \mu_j(\delta) e_j, y_j(\delta) = \mu_j(\delta)^{-1} e_j \), with

\[
\mu_j(\delta) = \sqrt{\frac{\nu(e_j) + \delta}{g(e_j) + \delta}}
\]

for each \( j \leq n \). Then, also \( \sum_{j} y_j^*(\delta)x_j(\delta) = 1 \) holds, and therefore the double system \( \{ y(\delta), x(\delta) \} \) is a special case of those double systems considered in context of the infimum in (3). Hence, one has

\[
\frac{1}{2} \inf \{ y, x \} \sum_{j} \{ \nu(y_j^*y_j) + g(x_j^*x_j) \} \leq F(\delta), \quad \text{for each } \delta > 0,
\]

with the
auxiliary function \( \delta \mapsto F(\delta) \) defined by \( F(\delta) = \frac{1}{2} \sum_j \{ \nu(y_j(\delta)^* y_j(\delta)) + g(x_j(\delta)^* x_j(\delta)) \} \). Since with this choice one easily infers that \( F(\delta) \) may be expressed as

\[
F(\delta) = \sum_{j, \nu(e_j) \neq 0} \frac{1}{2} \sqrt{\{g(e_j) + \delta\} \nu(e_j)} \sqrt{\frac{\nu(e_j)}{\nu(e_j) + \delta}} + \sum_{j, \nu(e_j) \neq 0} \frac{1}{2} \sqrt{\{\nu(e_j) + \delta\} g(e_j)} \sqrt{\frac{\nu(e_j)}{\nu(e_j) + \delta}},
\]

in view of the previous and (**) then

\[
\lim_{\delta \to 0} F(\delta) = \sum_j \sqrt{\nu(e_j) g(e_j)} \leq \sqrt{P_M(\nu, g) + \varepsilon} \quad (\ast')
\]

can be followed. Therefore \( \sqrt{P_M(\nu, g) + \varepsilon} \geq \frac{1}{2} \inf_{\{y, x\}} \sum_j \{\nu(y_j^* y_j) + g(x_j^* x_j)\} \) is seen. Since such procedure can be performed for each \( \varepsilon > 0 \), one can be assured that \( \sqrt{P_M(\nu, g) \geq \frac{1}{2} \inf_{\{y, x\}} \sum_j \{\nu(y_j^* y_j) + g(x_j^* x_j)\}} \) is fulfilled, where \( \{y, x\} \) is allowed to run through all finite double systems obeying \( \sum_j y_j^* x_j = 1 \). On the other hand, for each such double system and \( f \in \Gamma_M(\nu, g) \) one has

\[
|f(1)| \leq \sum_j |f(y_j^* x_j)| \leq \sum_j \sqrt{\nu(y_j^* y_j) g(x_j^* x_j)}.
\]

Now, for each two elements \( x, y \in M \), from \( \{\sqrt{\nu(y^* y) - \sqrt{g(x^* x)}}\}^2 \geq 0 \) the estimate \( \sqrt{\nu(y^* y) g(x^* x)} \leq \frac{1}{2} \{\nu(y^* y) + g(x^* x)\} \) is inferred. Hence, the above estimate relating double systems can be continued accordingly and results in \( |f(1)| \leq \frac{1}{2} \sum_j \{\nu(y_j^* y_j) + g(x_j^* x_j)\} \). This has to hold for each \( f \in \Gamma_M(\nu, g) \) and finite double system \( \{y, x\} \) obeying \( \sum_j y_j^* x_j = 1 \). Thus also \( \sqrt{P_M(\nu, g) \leq \frac{1}{2} \inf_{\{y, x\}} \sum_j \{\nu(y_j^* y_j) + g(x_j^* x_j)\}} \) is seen. In view of the above then equality follows, that is, (3) is seen to hold. Note in context of (\ast') that if an element \( a(\delta) \in M \) is defined by means of the above \( y_j(\delta) \) through the setting \( a(\delta) = \sum_j y_j(\delta)^* y_j(\delta) \), one has \( a(\delta) > 0 \), invertible with \( a(\delta)^{-1} = \sum_j x_j(\delta)^* x_j(\delta) \), and then (\ast') under the above premises on \( \varepsilon \) equivalently also shows that

\[
\lim_{\delta \to 0} \frac{1}{2} \{\nu((a(\delta)) + g(a(\delta)^{-1})) = \sum_j \sqrt{\nu(e_j) g(e_j)} \leq \sqrt{P_M(\nu, g) + \varepsilon}
\]

has to be fulfilled. Since \( \varepsilon > 0 \) can be arbitrarily chosen, from the previous then even an estimate

\[
\sqrt{P_M(\nu, g) \geq \inf_{x > 0} \frac{1}{2} \{\nu(x) + g(x^{-1})\}} \quad (\ast'')
\]
can be seen to be fulfilled, where now the infimum extends over all invertible, positive elements of $M$. On the other hand, for each invertible, positive element $x \in M$, one has the identity

$$\frac{1}{2} \left\{ \sqrt{\nu(x) - \nu(x^{-1})} \right\}^2 + \sqrt{\nu(x) \nu(x^{-1})} = \frac{1}{2} \{ \nu(x) + \nu(x^{-1}) \}. \quad (2\text{-}1a)$$

Taking the infimum over the invertible positive $x \in M$ on both sides and respecting non-negativity of $(1/2) \{ \sqrt{\nu(x) - \nu(x^{-1})} \}^2$ then will show that the following estimate has to be fulfilled:

$$\inf_{x > 0} \sqrt{\nu(x) \nu(x^{-1})} \leq \inf_{x > 0} \frac{1}{2} \left\{ \sqrt{\nu(x) - \nu(x^{-1})} \right\}^2 + \inf_{x > 0} \sqrt{\nu(x) \nu(x^{-1})} \leq \inf_{x > 0} \frac{1}{2} \{ \nu(x) + \nu(x^{-1}) \}. \quad (2\text{-}1b)$$

Hence, from Theorem 2 (1) one can conclude that $\sqrt{P_M(\nu, \varrho)} \leq \inf_{x > 0} (1/2) \{ \nu(x) + \nu(x^{-1}) \}$ has to hold. From this in view of ($\ast''$) the validity of (5) follows.

Finally, for each $\varepsilon > 0$ by the just proven (5) there exists an invertible $a > 0$ obeying $\sqrt{P_M(\nu, \varrho)} + \varepsilon \geq (1/2) \{ \nu(a) + \nu(a^{-1}) \}$. In defining $y_\varepsilon = \sqrt{a}$ and $x_\varepsilon = \sqrt{a^{-1}}$ one has $1 = y_\varepsilon^* x_\varepsilon$, and the above estimate then turns into $(1/2) \{ \nu(y_\varepsilon^* y_\varepsilon) + \nu(x_\varepsilon^* x_\varepsilon) \} \leq \sqrt{P_M(\nu, \varrho)} + \varepsilon$. On the other hand, according to (3) one has

$$\sqrt{P_M(\nu, \varrho)} \leq \inf_{\{y = y^* y\}} (1/2) \{ \nu(y^* y) + \nu(x^* x) \} \leq (1/2) \{ \nu(y_\varepsilon^* y_\varepsilon) + \nu(x_\varepsilon^* x_\varepsilon) \}.$$ 

From these estimates, and since $\varepsilon > 0$ can be taken at will, validity of (4) then gets evident. This completes the proof of all the assertions.

### 2.2. Miscellaneous comments

In the following we will comment on the facts coming along with Corollary 2, and will supplement them with further useful auxiliary results and remarks.

#### 2.2.1. Comments on Corollary 2 (1)–(2) : quadratic means

For normal states, $P_M(\nu, \varrho)$ is the same as the generalized transition probability $T_M(\nu, \varrho)$ as given in [12].

The definition of V. Cantoni refers to the two probability measures $\nu(E_x(d\lambda))$ and $\varrho(E_x(d\lambda))$ over the Borel sets of $\mathbb{R}^1$ that can be naturally associated with two normal states $\nu$, $\varrho$ on $M$ through the projection valued measure $E_x(d\lambda)$ of a selfadjoint element, say $x \in M$, with spectral
representation \( x = \int_{\mathbb{R}^1} \lambda E_x(d\lambda) \) (remind that in a quantum mechanical context the hermitian elements are the candidates of bounded observables). In line with some proposal of G. Mackey, see Chapter 2, 2-2, 2-6 in [20], and in accordance with some physically motivated axioms saying what properties of a ‘transition probability’ should be considered as indispensable at all, see [21, 17, 16] e.g., in [12] one defines a generalized transition probability by

\[
T_M(\nu, \varrho) = \inf_{x \in M} \left\{ \int_{\mathbb{R}^1} QM_x(\nu, \varrho)(d\lambda) \right\}^2 ,
\]

(2-2)

with the quadratic means \( QM_x(\nu, \varrho)(d\lambda) = \sqrt{\nu(E_x(d\lambda))\varrho(E_x(d\lambda))} \) of these measures, which is a Borel measure on the line again. On carefully analyzing the quadratic means in the special case of two normal states and one of which is faithful at least, the proof that \( P_M(\nu, \varrho) \) of Definition 2 equals the expression (2-2) was given in [8].

As has been yet remarked by S. Gudder, see Theorem 1 in [16], mathematically (2-2) amounts to \( \sqrt{T_M(\nu, \varrho)} = \inf_{\{e\}} \sum_j \sqrt{\nu(e_j)\varrho(e_j)} \), which is (2) in this special case.

In summarizing from all that, the news added through COROLLARY 2 to that subject are in the following:

- the expression in COROLLARY 2 (2) reflects those aspects behind (2-2) which remain valid for any positive linear forms (not only normal ones) on a \( W^*\)-algebra;

- the expression in COROLLARY 2 (1) can be taken as the common general \( C^*\)-algebraic essence of the matter around quadratic means.

2.2.2. Comments on COROLLARY 2 (3)–(4) : some seminorms on \( M \)

For normal states, (3) had been conjectured by D. Buchholz, motivated by some application to relativistic quantum field theory, and has been proved in the special case of \( B(H) \) in [10], see eq. (2.10) there.

But note that there the intention was to deal even with certain vector states of some \( *\)-algebras of (unbounded) operators. In contrast to this, in the following we will strictly adhere to the (bounded) context of a \( W^*\)-algebra \( M \) and positive linear forms.

To start discussions around COROLLARY 2 (3)–(4), for given \( \nu, \varrho \in M_+^* \) let us consider two realvalued functions on \( M \), \( \tau_{\nu, \varrho} \) and \( \upsilon_{\nu, \varrho} \), which are defined at \( z \in M \) by

\[
\tau_{\nu, \varrho}(z) = \inf_{\{y, x\} \subset M, z = \sum_j y_j^* x_j} \frac{1}{2} \sum_j \left\{ \nu(y_j^* y_j) + \varrho(x_j^* x_j) \right\} ,
\]

(2-3a)

\[
\upsilon_{\nu, \varrho}(z) = \inf_{z = y^* x} \frac{1}{2} \left\{ \nu(y^* y) + \varrho(x^* x) \right\} .
\]

(2-3b)
Thereby, within the former expression the infimum is to be taken over all finite double systems \( \{y, x\} \) of operators of \( M \) obeying \( z = \sum_{j \leq n} y_j^* x_j \), with \( n \in \mathbb{N} \) arbitrarily chosen. For notational simplicity subsequently use the shortcut notation \( z = \{y, x\} \) whenever such type of relation occurs. If we want to consider only minimal systems of that kind \( (n = 1) \), to which e.g. within (2-3b) is referred to, the condition \( z = y^* x \) will be explicitly used.

Note that the assertions of COROLLARY 2 (3)–(4) then read as

\[
\nu_{\nu, \varrho}(1) = \tau_{\nu, \varrho}(1) = \sqrt{P_M(\nu, \varrho)}.
\]  
(2-3c)

Also it is obvious from the structure of the expression within definition (2-3a) that \( \tau_{\nu, \varrho} \) is a seminorm, whereas from (2-3b) it is obvious that \( \tau_{\nu, \varrho} \) is a lower bound for \( \nu_{\nu, \varrho} \):

\[
\tau_{\nu, \varrho}(z) \leq \nu_{\nu, \varrho}(z).
\]  
(2-3d)

Remark that in relativistic quantum field theory there was some hope that seminorms of \( \tau \)-type should be useful in proving existence of non-trivial (weak) intertwiners between so-called standard representations [29, 10], these standard representations roughly corresponding to the cyclic *-representations of \( \nu \) and \( \varrho \) accordingly, in our bounded context (for the context see also [18], especially Definition 2.2.14). Clearly, this in a specific setting is the (highly non-trivial) analog over unbounded observable algebras of the (comparably trivial) task of analyzing the structure of the set \( \Gamma_M(\nu, \varrho) \) in the bounded case. In the bounded case, the above idea reduces to the inquiry for upper bounds of \( f \in \Gamma_M(\nu, \varrho) \) which read in terms of the seminorm \( \tau_{\nu, \varrho} \), that is, one is looking for estimates by \( \tau_{\nu, \varrho} \) from above

\[
\forall z \in M : |f(z)| \leq c \tau_{\nu, \varrho}(z),
\]  
(2-3e)

for some real constant \( c > 0 \), for instance.

More precisely, the news around COROLLARY 2 (3)–(4) to be annotated by our comments will be in the following:

- with respect to the seminorm (2-3a) the estimate (2-3e) holds, with \( c = 1 \), and this estimate being the best possible in favor of the above task, that is, \( \Gamma_M(\nu, \varrho) \) appears to be trivial, \( \Gamma_M(\nu, \varrho) = \{0\} \), if, and only if, \( \tau_{\nu, \varrho} \) is trivial, \( \tau_{\nu, \varrho} \equiv 0 \);

- the seminorm \( \tau_{\nu, \varrho} \) can be calculated exactly even if \( \{y, x\} \) under the infimum in (2-3a) is bent to be varied only through minimal double systems with \( z = y^* x \), that is, according to this and (2-3b) one has \( \tau_{\nu, \varrho} = \nu_{\nu, \varrho} \) to hold;
Thus, the following estimate has been established:

\[ \tau_{\nu,\varrho}(z) = v_{\nu,\varrho}(z) = \sup_{f \in \Gamma_{M}(\nu,\varrho)} |f(z)| = \sqrt{P_{M}(\nu,\varrho^{2})} = \sqrt{P_{M}(\nu^a, \varrho^b)}. \quad (2-3f) \]

**Proof.** First note that each finite double system \( \{y, x\} \) obeying \( 1 = \{y, x\} \) through setting \( \tilde{y}_j = y_j a \) and \( \tilde{x}_j = x_j b \), respectively, provides another finite double system of the same length \( \{\tilde{y}, \tilde{x}\} \) with \( a^*b = \{\tilde{y}, \tilde{x}\} \) (especially, minimal double systems will be transformed into minimal ones again). Hence, in view of **COROLLARY 2** (3)–(4) and (2-3a)–(2-3b) one can conclude as follows:

\[
\sqrt{P_{M}(\nu^a, \varrho^b)} = \left(1/2\right) \inf_{1=\{y, x\}} \sum_{j} \nu^a(y_j^* y_j) + \varrho^b(x_j^* x_j)
\]

\[
\geq \left(1/2\right) \inf_{a^*b=\{y, x\}} \sum_{j} \nu(y_j^* y_j) + \varrho(x_j^* x_j)
\]

\[
= \tau_{\nu,\varrho}(a^*b).
\]

Thus, the following estimate has been established:

\[ \tau_{\nu,\varrho}(a^*b) \leq \sqrt{P_{M}(\nu^a, \varrho^b)}. \quad (\circ) \]

Also, if to the pair \( \{\nu, \varrho\} \) a representation \( \{\pi, \mathcal{K}\} \) as in the premises of **THEOREM 2** (3) is chosen, with fixed \( \varphi \in \mathcal{S}_{\pi,M}(\nu) \) and \( \psi \in \mathcal{S}_{\pi,M}(\varrho) \) then obviously also \( \pi(a) \varphi \in \mathcal{S}_{\pi,M}(\nu^a) \) and \( \pi(b) \psi \in \mathcal{S}_{\pi,M}(\varrho^b) \) are fulfilled. Application of (1-6) with respect to \( \{\nu, \varrho\} \), \{\nu^a, \varrho^b\} and \{\nu, \varrho^2\} will yield that \( \langle \pi(\cdot)k\psi, \varphi \rangle \), \( \langle \pi(\cdot)k\pi(b)\psi, \pi(a)\varphi \rangle \) and \( \langle \pi(\cdot)k\pi(z)\psi, \varphi \rangle \), respectively, will be running through all of \( (\pi(M)' )_1 \). Now, for each \( k \in (\pi(M)' )_1 \) one has \( \langle k\pi(b)\psi, \pi(a)\varphi \rangle = \langle k\pi(z)\psi, \varphi \rangle = \langle \pi(z)k\psi, \varphi \rangle \). Hence, in line with **THEOREM 2** (2), when
the latter accordingly is applied to these three special situations at hand, under the premise of $z = a^*b$ the estimate $(c)$ can be continued as follows:

$$\tau_{\nu, \varrho}(z) \leq \sqrt{P_M(\nu^a, \varrho^b)} = \sqrt{P_M(\nu, \varrho^z)} = \sup_{f \in \Gamma_M(\nu, \varrho)} |f(z)|. \quad (c')$$

Now, suppose $z = \{y, x\}$ in context of $\{\nu, \varrho\}$. By definition of $\Gamma_M(\nu, \varrho)$, for $f \in \Gamma_M(\nu, \varrho)$ one has

$$|f(z)| \leq \sum_j |f(y_j^*x_j)| \leq \sum_j \sqrt{\nu(y_j^*y_j) \varrho(x_j^*x_j)}$$

$$\leq \frac{1}{2} \sum_j \{\nu(y_j^*y_j) + \varrho(x_j^*x_j)\}.$$ 

From this in view of $(2-3a)$ $\sup_{f \in \Gamma_M(\nu, \varrho)} |f(z)| \leq \tau_{\nu, \varrho}(z)$ follows, which with the help of $(2-3d)$ can be turned into

$$\sup_{f \in \Gamma_M(\nu, \varrho)} |f(z)| \leq \tau_{\nu, \varrho}(z) \leq \nu_{\nu, \varrho}(z). \quad (c'')$$

On the other hand, for $\varepsilon > 0$, COROLLARY 2 (4) can be applied to the pair $\{\nu, \varrho^z\}$ and yields invertible $a > 0$ obeying $\sqrt{P_M(\nu, \varrho^z)} + \varepsilon \geq \{1/2\} \{\nu(a) + \varrho^z(a^{-1})\}$. Let us define $y = \sqrt{a}$ and $x = \sqrt{a^{-1}}z$. Then, $z = y^*x$ and $\{\nu(a) + \varrho^z(a^{-1})\} = \{\nu(y^*y) + \varrho(x^*x)\}$ are fulfilled. Hence, in view of above $\nu_{\nu, \varrho}(z) \leq \sqrt{P_M(\nu, \varrho)} + \varepsilon$ can be followed. Since $\varepsilon > 0$ can be taken at will, from the latter in accordance with $(2-3b)$ we get $\nu_{\nu, \varrho}(z) \leq \sqrt{P_M(\nu, \varrho^z)}$. Upon taking together this with $(c'')$ and $(c')$ we can conclude that in fact equality has to occur within $(c''')$ and $(c')$, that is, $(2-3f)$ holds. This closes the proof of COROLLARY 3.

**Proof** (of THEOREM 3). Formula THEOREM 3 (1) is given by one of the particular subequations coming along with $(2-3f)$. Moreover, according to another subequation of the latter $\nu_{\nu, \varrho}(z) = P_M(\nu^a, \varrho^b)^{1/2}$ holds. Inserting this into $(1-3)$ in view of $(2-3b)$ yields $d_B(\nu^a, \varrho^b)^2 = \nu(a^*a) + \varrho(b^*b) - \inf_{z = y^*x} \{\nu(y^*y) + \varrho(x^*x)\} = \sup_{z = y^*x} \{\nu(a^*a - y^*y) + \varrho(b^*b - x^*x)\}$, and which is THEOREM 3 (2).

Remark 2. (1) Without proof remark that $P_M(\nu, \varrho) = 0$ is equivalent with $\nu \perp \varrho$, see e.g. [5]. Recall that orthogonality of two $C^*$-algebraic positive linear forms $\nu, \varrho$ is defined as $\|\nu - \varrho\|_1 = \|\nu\|_1 + \|\varrho\|_1$.

(2) Especially for states $\nu, \varrho$ occurring along with quantum physical problems over an algebra of observables $M$, one is inclined to give
$P_M(\nu, \varrho)$ a (quantum) probabilistic interpretation. COROLLARY 3 in such context will tell us that an interpretation which reads in terms of the transition probability, but now between the 'perturbed' states $\nu^a$ and $\varrho^b$, extends also to the value of the rather abstractly defined seminorms $M \ni z \mapsto \tau_{\nu, \varrho}(z)$ at $z = a^*b$. Thus, if to given pair $\{\nu, \varrho\}$ of states and in accordance with (2-3f) and the previous item (1) those operators $a, b$ are considered which are solutions of the equation $\tau_{\nu, \varrho}(a^*b) = 0$ (and for which both $\nu^a$ and $\varrho^b$ are states again), then these might be interpreted as all possible elementary 'operations' (i.e. inner implementable perturbations) driving $\{\nu, \varrho\}$ into mutually orthogonal states.

(3) Due to the mentioned interpretation of the values of the seminorm $\tau_{\nu, \varrho}$ in terms of $\sqrt{P_M}$, which manifests itself by (2-3f), some subadditivity property of $\sqrt{P_M}$ in respect to inner derived positive linear forms can be followed:

$$a^*b = \sum_{j \leq n} a_j^*b_j \implies \sqrt{P_M(\nu^a, \varrho^b)} \leq \sum_{j \leq n} \sqrt{P_M(\nu^{a_j}, \varrho^{b_j})}.$$ 

(4) The fact that $\tau_{\nu, \varrho} = v_{\nu, \varrho}$ holds is mainly due to our restriction to bounded operator algebras and cannot be expected to extend simply to a context with $^*$-algebras of unbounded operators.

2.2.3. Comments on COROLLARY 2 (5): minimizing abelian algebras

That COROLLARY 2 (5) is a notable result on its own rights - and is not something to be easily abandoned - has been recognized only recently, and as such will be discussed here (and more detailed in the next section) for the first time.

In comparing the item in question to THEOREM 2 (1) one at once notices that the essential difference with the latter result lies in the fact that under the infimum instead of a geometrical means now the arithmetical means of the same two expressions enters.

Quite naturally, in context of COROLLARY 2 (5) (and in context of THEOREM 2 (1) as well) a main interest will be in describing the structure of those invertible $x \in M_+$ at which by the expression of $\frac{1}{2}\{\nu(x) + \varrho(x^{-1})\}$ (or $\sqrt{\nu(x) \varrho(x^{-1})}$, respectively) the (common) infimum $\sqrt{P_M(\nu, \varrho)}$ is nearly attained. Such problems and related questions we are going to discuss now. Thereby, for these purposes of estimation theory COROLLARY 2 (5) seems to be better suited than THEOREM 2 (1). For instance, the map $x \mapsto \frac{1}{2}\{\nu(x) + \varrho(x^{-1})\}$ is more sensitive to certain variations of the positive invertible operator...
x ∈ M than the map x ↦→ \sqrt{\nu(x) \varrho(x^{-1})} is (compare the behavior of both under the change x ↦→ \lambda x, for real \lambda > 0, simply).

Relating the quality of the mentioned approximation one has the following simple facts (cf. also Theorem 4.4 in [2]).

**COROLLARY 4.** Let \( \nu, \varrho \in M^+_1 \), and be \( \{x\} \subset M^+ \) a sequence of invertible elements. The following facts are equivalent:

1. \( \sqrt{P_M(\nu, \varrho)} = \lim_{n \to \infty} \frac{1}{2} \{\nu(x_n) + \varrho(x_n^{-1})\} \);
2. \( \sqrt{P_M(\nu, \varrho)} = \lim_{n \to \infty} \nu(x_n) = \lim_{n \to \infty} \varrho(x_n^{-1}) \).

Moreover, if \( \text{Comm}[M] \) is the family of all abelian \( W^* \)-subalgebras of \( M \) with the same unity as \( M \), then one has

3. \( P_M(\nu, \varrho) = \inf_{R \in \text{Comm}[M]} P_R(\nu|_R, \varrho|_R) \).

**Proof.** In view of eqs. (2-1) the asserted equivalence immediately follows from Theorem 2 (1) and Corollary 2 (5). Also (3) can be seen as an obvious consequence from each of these items.

Now, for given pair \( \{\nu, \varrho\} \) of positive linear forms a set \( \text{Min}_M(\nu, \varrho) \) will be defined as follows:

\[
\text{Min}_M(\nu, \varrho) = \left\{ x \in M^+ : \sqrt{P_M(\nu, \varrho)} = \frac{1}{2} \{\nu(x) + \varrho(x^{-1})\} \right\}.
\]

The elements of \( \text{Min}_M(\nu, \varrho) \) will be called minimizing (positive invertible) elements to the pair \( \{\nu, \varrho\} \), where in this notation tacitly to the context with Corollary 2 (5) is referred to.

Note that since the set of all invertible positive elements is neither compact nor closed, it is a non-trivial problem to decide from a concrete pair \( \{\nu, \varrho\} \) of positive linear forms whether or not the infimum within Corollary 2 (5) is a minimum.

In fact, in general this cannot happen, as the following simple counterexample shows.

**Example 3.** According to elementary spectral theory for invertible \( y \in M^+ \) one has \( y \geq \|y^{-1}\|^{-1} 1 \). Hence, for each pair \( \{\nu, \varrho\} \neq \{0, 0\} \) of positive linear forms and for each invertible \( x \in M^+ \) one infers that \( \{\nu(x) + \varrho(x^{-1})\}/2 \geq \{\|\nu\|/\|x^{-1}\| + \|\varrho\|/\|x\|\}/2 > 0 \) has to be fulfilled. On the other hand, according to Remark 2 (1), in the special case of \( \nu \perp \varrho \) one has \( \sqrt{P_M(\nu, \varrho)} = 0 \). Thus, in view of the previous estimate in case of a nontrivial pair of mutually orthogonal positive linear forms \( \text{Min}_M(\nu, \varrho) = \emptyset \) holds.
On the other hand, there exist also classes where this question can be answered affirmatively. A criterion relating this matter is easily obtained from Corollary 4 (1)–(2) and reads as follows:

\[ x \in M_+, \sqrt{P_M(\nu, \rho)} = \nu(x) = \rho(x^{-1}) \iff x \in \text{Min}_M(\nu, \rho). \quad (2-4) \]

Example 4. Suppose \( \rho = \nu^a \), with \( a \in M_+ \) being invertible. Then, in view of Theorem 1 the criterion (2-4) gets applicable with \( x = a \) and yields that the infimum in Corollary 2 (5) is a minimum.

Let us refer to an abelian \( W^* \)-subalgebra \( R \subset M \) with \( 1 \in R \) as minimizing abelian subalgebra if the infimum within Corollary 4 (3) is a minimum and is attained at \( R \). For instance, if \( \text{Min}_M(\nu, \rho) \neq \emptyset \) is fulfilled then in line with the above the infimum is attained at each subalgebra \( R \) which is generated by \( 1 \) and some particular \( x \in \text{Min}_M(\nu, \rho) \). Thus, in generalizing from the problem on existence of minimizing elements the more general question on existence of minimizing abelian subalgebras naturally arises.

3. Special subjects

3.1. Minimizing elements

In this paragraph we inquire for existence and uniqueness of minimizing positive invertible elements, and we are going to derive some results on the structure of \( \text{Min}_M(\nu, \rho) \). Let \( x, z \in M_+ \) be any two invertible positive elements. Let \( \delta = (z-x) \). Then, the following algebraic identity can be easily checked to hold:

\[ z^{-1} = x^{-1} - x^{-1}\delta x^{-1} + \Delta(z, x), \quad (3-1a) \]

where \( \Delta(z, x) = m(z, x)^* m(z, x) \) holds, and \( m(z, x) \) is defined by

\[ m(z, x) = (x^{-1/2}\delta x^{-1/2})(x^{-1/2}zx^{-1/2})^{-1/2}x^{-1/2}. \quad (*) \]

By construction of \( \Delta(z, x) \) and by invertibility of \( z, x \) from \( (*) \) then

\[ \Delta(z, x) \in M_+, \text{ with } \left\{ \Delta(z, x) = 0 \iff \delta = 0 \right\} \quad (3-1b) \]

can be followed. Also, since \( x^{-1/2}\delta x^{-1/2} \) is commuting with \( x^{-1/2}zx^{-1/2} \), from \( (*) \) yet another expression for \( m(z, x) \) can be obtained, and which reads as

\[ m(z, x) = (x^{-1/2}zx^{-1/2})^{-1/2}x^{-1/2}\delta x^{-1}. \quad (3-1c) \]
With the help of (3-1a) and in the previous notations one then finds
\[
\frac{1}{2} \left\{ \nu(z) + \varrho(z^{-1}) \right\} - \frac{1}{2} \left\{ \nu(x) + \varrho(x^{-1}) \right\} = \\
\frac{1}{2} \left\{ \nu(\delta) - \varrho(x^{-1} \delta x^{-1}) \right\} + \frac{1}{2} \varrho(\Delta(z, x)) . \tag{3-1d}
\]

Note that the set \( M^\text{inv} \) of all invertible positive elements of \( M \) is an open non-pointed subcone within the real Banach space \( \{ M_h, \| \cdot \| \} \) of the hermitian portion of \( M \). Hence, for a particular \( x \in M^\text{inv} \) and given \( y \in M_h \), for all \( t \in \mathbb{R} \) sufficiently small \( z_t = x + ty \in M^\text{inv} \) has to hold (one might take \( |t| < \|x^{-1}yx^{-1}\|^{-1} \), e.g.). In this special situation the formula (3-1d) at such parameter \( t \) reads as
\[
\frac{1}{2} \left\{ \nu(z_t) + \varrho(z_t^{-1}) \right\} - \frac{1}{2} \left\{ \nu(x) + \varrho(x^{-1}) \right\} = \\
\frac{t^2}{2} \left\{ \nu(y) - \varrho(x^{-1}yx^{-1}) \right\} + \frac{t^2}{2} \varrho(\Delta_t(y|x)) , \tag{3-2}
\]
where \( \Delta_t(y|x) = t^{-2} \Delta(z_t, x) \) is defined for \( t \neq 0 \), and at \( t = 0 \) we let \( \Delta_0(y|x) = \| \cdot \| - \lim_{t \to 0} t^{-2} \Delta(z_t, x) = x^{-1}yx^{-1}yx^{-1} \). We are now ready for the following redefinition of \( \text{Min}_M(\nu, \varrho) \).

**PROPOSITION 2.** For any \( \nu, \varrho \in M^+_+ \) the following holds:
\[
\text{Min}_M(\nu, \varrho) = \left\{ x \in M^\text{inv}_+ : \nu(y) = \varrho(x^{-1}yx^{-1}), \forall y \in M_h \right\} . \tag{3-3}
\]

**Proof.** Suppose \( x \in \text{Min}_M(\nu, \varrho) \). Then, for each fixed \( y \in M_h \), and all \( t \in \mathbb{R} \setminus \{0\} \) sufficiently small, in accordance with (3-2)
\[
-\left| \nu(y) - \varrho(x^{-1}yx^{-1}) \right| \geq -|t| \varrho(\Delta_t(y|x))
\]
has to hold. Having in mind that according to the above \( t \mapsto \Delta_t(y|x) \) is norm-continuous at \( t = 0 \), one then has \( \lim_{t \to 0} |t| \varrho(\Delta_t(y|x)) = 0 \). In view of the previous estimate from this \( \nu(y) = \varrho(x^{-1}yx^{-1}) \) is seen.

On the other hand, assume \( x \in M^\text{inv}_+ \) such that, for each \( y \in M_h \), \( \nu(y) = \varrho(x^{-1}yx^{-1}) \) is satisfied. For each other \( z \in M^\text{inv}_+ \), let \( \delta = (z - x) = y \). One then especially has \( \{ \nu(\delta) - \varrho(x^{-1} \delta x^{-1}) \} = 0 \). Hence, (3-1d) can be applied and owing to positivity of \( \Delta(z, x) \) and \( \varrho \) yields \( \frac{1}{2} \left\{ \nu(z) + \varrho(z^{-1}) \right\} - \frac{1}{2} \left\{ \nu(x) + \varrho(x^{-1}) \right\} \geq 0 \). Hence, since \( z \) can be arbitrarily chosen from \( M^\text{inv}_+ \), \( x \in \text{Min}_M(\nu, \varrho) \) follows. This completes the proof of (3-3).

After these preliminaries we may now summarize as follows.
THEOREM 4. Let $M$ be a $W^*$-algebra. For $\nu, \varrho \in M^*_+$ one has:

1. $\operatorname{Min}_M(\nu, \varrho) \neq \emptyset \iff \exists a \in M^*_+ \ni \varrho = \nu^a$;

2. $\operatorname{Min}_M(\nu, \varrho) = \{x + I_\nu\} \cap M^*_+; \forall x \in \operatorname{Min}_M(\nu, \varrho)$;

3. $\# \operatorname{Min}_M(\nu, \varrho) = 1 \iff \exists a \in M^*_+ \ni \varrho = \nu^a; \nu$ is faithful.

Proof. According to Example 4, for $\varrho = \nu^a$ with $a \in M^*_+$ one has $a \in \operatorname{Min}_M(\nu, \varrho)$. On the other hand, if $\operatorname{Min}_M(\nu, \varrho) \neq \emptyset$ is supposed, in line with formula (3-3) and since linear forms on a $C^*$-algebra are uniquely determined through their values on the hermitian portion, $\nu = \varrho(x^{-1}(\cdot)x^{-1})$ has to be fulfilled, for some $x \in M^*_+$. That is, $\varrho = \nu^a$ holds, with $a = x$. In summarizing, (1) is valid.

To see (2), suppose $x \in \operatorname{Min}_M(\nu, \varrho)$ and be $z \in M^*_+$. According to the previous then $\varrho = \nu^x$, and therefore from (3-3) and (3-1d) one infers that $z \in \operatorname{Min}_M(\nu, \varrho)$ happens if, and only if, $\nu(x\Delta(z, z)x) = 0$ is fulfilled. By construction of $\Delta(z, z)$ the latter is equivalent with $m(z, x)x \in I_\nu$, see (1-1b). According to (3-1c) the latter is the same as $(x^{-1/2}zx^{-1/2})^{-1/2}x^{-1/2}\delta \in I_\nu$, with $\delta = (z - x)$. Since $I_\nu$ is a left ideal and $(x^{-1/2}zx^{-1/2})^{-1/2}x^{-1/2}$ is invertible, from this we finally conclude that for $z \in M^*_+$ the condition $z \in \operatorname{Min}_M(\nu, \varrho)$ has to be equivalent with $\delta \in I_\nu$. Owing to $\operatorname{Min}_M(\nu, \varrho) \subset M^*_+$ this is (2).

In order to see (3), remark first that for faithful $\nu$ one has $I_\nu = \{0\}$. Hence, from the just proved (2) uniqueness of a minimizing element evidently follows. On the other hand, for an eventually existing $r \in I_\nu \setminus \{0\}$ owing to $r^*r = |r|^2$ also $|r| \in I_\nu \setminus \{0\}$ follows, see (1-1b). Hence, since $a \in \operatorname{Min}_M(\nu, \varrho)$ is invertible, by standard facts and owing to $z \geq a$ also $z \sqsubset M^*_+$ follows, for $z = a + |r|$. By (2) this however then implies $z \in \operatorname{Min}_M(\nu, \varrho)$. Since $z \neq a$ holds we therefore have $\# \operatorname{Min}_M(\nu, \varrho) > 1$, for non-faithful $\nu$. Taking together this with the previous yields (3).

Since $\operatorname{Min}_M(\varrho, \nu) = \{x^{-1} : x \in \operatorname{Min}_M(\nu, \varrho)\}$ holds, from THEOREM 4 (2) for $\operatorname{Min}_M(\nu, \varrho) \neq \emptyset$ one infers that both positive linear forms have to be faithful or not, only simultaneously. In reversing this another class of counterexamples is easily obtained.

Example 5. Let $\nu, \varrho \in M^*_+$. Suppose exactly one of the two forms to be faithful. Then, the infimum in COROLLARY 2 (5) cannot be attained on the invertible positive elements of $M$.

Remark 3. According to THEOREM 4 (1) minimizing elements can exist if, and only if, each one of the two positive linear forms of a pair $\{\nu, \varrho\}$ can be inner derived by means of some positive invertible element from the other one in quest. That is, in fact all these cases are yet covered by Example 4.
As announced at the end of 2.2.3, the next best question to be raised is to inquire for existence of a commutative $W^*$-subalgebra $R$ of $M$, with $1 \in R$, such that the infimum in Corollary 4 (3) could be attained.

3.2. Minimizing commutative subalgebras

Start with examples where minimizing abelian subalgebras exist but which are found slightly beyond of Example 4.

**Example 6.** Suppose $\varrho = \nu^a$, for $a \in M_+$. By functional calculus (use the spectral representation theorem within the $W^*$-algebra $M$) one infers $a(a + \varepsilon 1)^{-1}a \leq a$ to hold, for each real $\varepsilon > 0$. Hence, $a_\varepsilon = (a + \varepsilon 1) \in M_{+}^{+\nu}$ with $\varrho(a_\varepsilon^{-1}) \leq \nu(a)$. Owing to this, Theorem 2 (1) and Theorem 1 (or (1-7), equivalently) then $\nu(a) = \sqrt{P_M(\nu, \varrho)} \leq \nu(a_\varepsilon) = \nu(a) + \varepsilon \|\nu\|_1$ as well as $\nu(a)^2 = P_M(\nu, \varrho) \leq \varrho(a_\varepsilon^{-1}) \nu(a_\varepsilon) \leq \nu(a)^2 + \varepsilon \nu(a)\|\nu\|_1$ are obtained. Upon performing the limit $\varepsilon \to 0$ in both relations and regarding Corollary 4 (1)-(2) will give that the $W^*$-subalgebra generated by $a$ and $1$ can be chosen as minimizing commutative subalgebra $R$.

The fact that a subalgebra $R$ be minimizing for a given pair $\{\nu, \varrho\}$ implies that some very specific additional conditions have to be fulfilled. An important instance of such conditions occurs in context of those minimizing subalgebras which come along with Example 6.

**Lemma 1.** Suppose $\nu, \varrho \in M_+^*$ and let $R$ be a $W^*$-subalgebra of $M$ such that $\varrho|_R = (\nu|_R)^a$ holds, for some $a \in R_+$. Then, whenever $R$ is minimizing for $\{\nu, \varrho\}$ the relation

$$\nu^p(a) - \nu^{p^\perp}(a) = \nu(a)$$

holds, for each orthoprojection $p \in M$ obeying $p^\perp \in I_\varrho$.

**Proof.** Let $P = P_M(\nu, \varrho)$. The assumption that $R$ be minimizing together with the reasoning of Example 6 when applied in respect of $\{\nu|_R, \varrho|_R\}$ over $R$ prove that, for $a_\varepsilon = a + \varepsilon 1$ with $\varepsilon > 0$, one has $\sqrt{P} = \nu(a) = \lim_{\varepsilon \to 0} \nu(a_\varepsilon) = \lim_{\varepsilon \to 0} \varrho(a_\varepsilon^{-1})$. Now, let $u = p + \lambda p^\perp$, with real $\lambda \neq 0$. Define $a_\varepsilon(\lambda) = u a_\varepsilon u$. Then, for each $\varepsilon > 0$ one has $a_\varepsilon(\lambda) \in M_{+}^{inv}$. Note also that the assumption on $p$ saying that $p^\perp \in I_\varrho$ be fulfilled together with the special structure of $u$ imply $\varrho(y) = \varrho(pyp) = \varrho(u^*yu) = \varrho(u^{-1}yu^{-1}x)$ to be fulfilled, for each $y \in M$. Hence, by construction of $a_\varepsilon(\lambda)$ then especially also $\lim_{\varepsilon \to 0} \varrho(a_\varepsilon(\lambda)^{-1}) = \lim_{\varepsilon \to 0} \varrho(a_\varepsilon^{-1}) = \sqrt{P}$ follows. On the other hand, since $\nu(a_\varepsilon(\lambda)) = \nu^p(a_\varepsilon) + 2\lambda \Re \nu(p^\perp a_\varepsilon p) + \lambda^2 \nu^{p^\perp}(a_\varepsilon)$ is fulfilled, in view of the above one arrives at $\lim_{\varepsilon \to 0} \nu(a_\varepsilon(\lambda)) = \nu^p(a) +$
2\lambda \Re \nu(p^+ap) + \lambda^2 \nu^+(a). Note that according to Theorem 2 (1) the estimate \(\lim_{\epsilon \to 0} \nu(\alpha(\epsilon)) \rho(\alpha(\epsilon)^{-1}) \geq P\) has to be fulfilled, which condition in view of the previous amounts to requiring \(\sqrt{P} \{\nu^p(a) + 2\lambda \Re \nu(p^+ap) + \lambda^2 \nu^+(a)\} \geq P = \sqrt{P} \{\nu^p(a) + 2 \Re \nu(p^+ap) + \nu^+(a)\}\), for all reals \(\lambda \neq 0\). That is,

\[2\sqrt{P}(\lambda - 1) \left\{ \Re \nu(p^+ap) + \frac{1}{2}(\lambda + 1) \nu^+(a) \right\} \geq 0\]

has to be fulfilled, for each real \(\lambda \neq 0\).

Suppose \(P \neq 0\) first. In considering the previous estimate for \(\lambda > 1\) one infers that \(\Re \nu(p^+ap) + \frac{1}{2}(\lambda + 1) \nu^+(a) \geq 0\) has to be fulfilled, whereas for \(\lambda < 1\) we see that \(\Re \nu(p^+ap) + \frac{1}{2}(\lambda + 1) \nu^+(a) \leq 0\) has to be fulfilled. Upon performing the limits \(\lambda \searrow 1\) and \(\lambda \nearrow 1\) within the mentioned relations for \(\lambda > 1\) and \(\lambda < 1\), respectively, and then comparing the results will show that \(\nu^+(a) = -\Re \nu(p^+ap)\) has to be fulfilled. By means of this then \(\nu(a) = \nu^p(a) + 2 \Re \nu(p^+ap) + \nu^+(a) = \nu^p(a) - \nu^+(a)\) is seen. This proves the result in case of \(P \neq 0\).

Finally, for \(P = 0\) one has \(\nu(a) = 0\). Owing to \(a \geq 0\) then \(a \in I_\nu\). Hence also \(0 = \nu(pa) = \nu(ap)\), and therefore from \(\nu^+(a) = \nu(a) - 2 \Re \nu(ap) + \nu^p(a)\) one then gets \(\nu^p(a) - \nu^+(a) = 0\) which is in accordance with (3-4) in this special case.

Having in mind Example 5 remark that for faithful \(\nu\) and \(\rho = \nu^\alpha\), with \(a \in \mathcal{M}_+\) and \(\ker a \neq \{0\}\), the most simple situations arise where Example 6 provides cases which go beyond of Example 4. Less trivial situations of that kind arise from generalizing Example 2 and modifying those arguments, along which we have been following within Example 6. The result in question, which however will be proved here only in some sketchy way, reads as follows.

**PROPOSITION 3.** Let \(\{\nu, \rho\}\), with normal \(\nu, \rho \in \mathcal{M}_+\), and support orthoprojections which are mutually \(\leq\)-comparable, say \(s(\rho) \leq s(\nu)\) be fulfilled. Then a minimizing commutative \(\mathcal{W}\)-subalgebra \(\mathcal{R}\) of \(M\) exists.

**Sketch of proof.** Remark first that for normal positive linear forms \(\nu, \rho\) with supports obeying \(s(\rho) \leq s(\nu)\) the problem in quest via some appropriately chosen normal \(*\)-representation \(\{\pi, \mathcal{K}\}\), which obeys \(\mathcal{S}_{\pi,M}(\nu) \not= \emptyset\) and \(\mathcal{S}_{\pi,M}(\rho) \not= \emptyset\), always can be reduced to the analogous problem over the \(\nu\mathcal{N}\)-algebra \(N = \pi(M)^{\prime\prime}\). In this setting, with given \(\varphi \in \mathcal{S}_{\pi,M}(\nu)\), the assumption about the supports can be shown to ensure existence of some (possibly unbounded) selfadjoint positive linear operator \(A\), which is affiliated with \(N\) and which obeys \(\psi = A\varphi \in \mathcal{S}_{\pi,M}(\rho)\).
Note that since \( A \) is affiliated with \( N \), the operator \( A \) can be chosen to be independent from the particularly chosen \( \varphi \) within \( S_{\pi,M}(\nu) \). Let \( \nu_{\pi} \) and \( \varrho_{\pi} \) be the vector functionals generated by \( \varphi \) and \( \psi \) over the \( \nu N \)-algebra \( N \). Extending the notion ‘inner derived positive linear form’ slightly to include at least also such situations with vector forms on \( N \) and (unbounded) positive selfadjoint linear operators affiliated with \( N \), for \( \varrho_{\pi} = \nu_{\pi} A \) one easily proves that formula (1-7) remains true in the sense of \( \sqrt{P_N(\nu_{\pi},\varrho_{\pi})} = \nu_{\pi}(A) = \langle A\varphi, \varphi \rangle \). Since then also the arguments raised in context of Example 6 are easily justified to remain valid with \( A_{\varepsilon} = A + \varepsilon 1 \) instead of \( a_{\varepsilon} \), following along the same line of conclusions as in Example 6 will provide \( P_N(\nu_{\pi},\varrho_{\pi}) = P_R(\nu_{\pi}|R,\varrho_{\pi}|R) \), with \( R \) being the commutative \( \nu N \)-subalgebra of \( N \) generated by the spectral resolution of \( A \). Finally, since always \( P_N(\nu_{\pi},\varrho_{\pi}) = P_M(\nu,\varrho) \) is fulfilled (note that \( \nu_{\pi} \circ \pi = \nu \) and \( \varrho_{\pi} \circ \pi = \varrho \) hold), in view of normality of \( \pi \), which implies that even \( N = \pi(M) \) holds, the just mentioned result about \( \nu_{\pi}, \varrho_{\pi} \) over \( N \) can be rewritten easily into one over \( M \).

3.3. LEAST MINIMIZING COMMUTATIVE SUBALGEBRA

3.3.1. Generalities on the problem

It is plain to see (from each of the items of Corollary 2, e.g.) that the map \( R \mapsto \sqrt{P_R(\nu|R,\varrho|R)}, \nu, \varrho \in M^*_+, \) with respect to the inclusion \( \subset \) between \( W^*-\)subalgebras of \( M \) behaves \( \leq \)-(anti-)monotone. Hence, if there is a minimizing commutative subalgebra \( R \), then also each larger than this commutative subalgebra has to be minimizing.

Going the other way around in this context is less trivial. For instance, one might ask for existence of a least minimizing commutative \( W^*-\)subalgebra of \( M \) with the same unit. In case of existence of a least minimizing subalgebra the latter will be denoted by \( R_M(\nu,\varrho) \).

Note that a least minimizing subalgebra must not exist in either case of a pair \( \{\nu,\varrho\} \) where a minimizing commutative subalgebra exists. To formulate a result on this, for the following agree to make use of \( R[x] \) as notation for the commutative \( W^*-\)subalgebra of \( M \) which is generated by \( 1 \) and the hermitian element \( x \in M_h \). Then, the simplest counterexamples against existence of a least minimizing algebra can be generated along the following auxiliary construction.

**Lemma 2.** Suppose \( \varrho = \nu^x \) holds, with \( x \in M_+ \). Then, for each \( k \in I_{\nu} \cap M_+ \), \( R[x+k] \) is a minimizing abelian subalgebra to \( \{\nu,\varrho\} \). In case \( \varrho \not\in \mathbb{R}_+ \nu \), \( \bigcap_{k \in I_{\nu} \cap M_+} R[x+k] = \mathbb{C} \cdot 1 \) \( (3-5) \) is fulfilled, there cannot exist a least element among all minimizing commutative subalgebras to the pair \( \{\nu,\varrho\} \).
Proof. It is easily inferred from (1-1a) and (1-1b) that also $q = \nu x + k$ holds, for each $k \in I_\nu \cap M_+$. By Example 6 then $R[x + k]$ will be a special minimizing commutative subalgebra. Also, from Definition 2 with the help of known properties of the Cauchy-Schwarz inequality one easily infers that for each pair $\{\nu, q\}$ of positive linear forms

$$\sqrt{P_M(\nu, q)} \leq \sqrt{\|\nu\|_1 \|q\|_1}$$

is fulfilled, with equality occuring if, and only if, $q = \lambda \cdot \nu$ happens for some non-negative real $\lambda$. On the other hand, from the structure of Corollary 2 (5) it is easily seen that $\sqrt{P_M(\nu, q)} = \sqrt{\|\nu\|_1 \|q\|_1}$ is equivalent with the fact that $C \cdot 1$ be among the minimizing subalgebras. Now, assume $\nu, q$ as in (3-5). Then, according to the first of the previously mentioned facts the second condition in (3-5) in case of existence of a least minimizing subalgebra implied the latter to be trivial, whereas by the first condition in (3-5) and owing to the second of the above mentioned facts the trivial algebra $C \cdot 1$ is excluded from being a minimizing subalgebra. Thus, a least minimizing subalgebra cannot exist in this case.

Unfortunately, the condition (3-5) can be satisfied easily, e.g. it can be shown to be fulfilled for any two non-commuting pure states (the following $2 \times 2$-case exemplarily can stand for any situation of this kind; we omit the details).

Example 7. Let $M = M_2(\mathbb{C})$ be the full algebra of $2 \times 2$-matrices with complex entries, $p, q \in M$ one-dimensional orthoprojections, with $[p, q] = pq - qp \neq 0$. Let $x = p + \varepsilon p^\perp$, with $0 < \varepsilon < 1$, and be $\nu \in M^+_+ \setminus \{0\}$ with $\nu(q) = 0$ (such positive linear form trivially exists). Define $q = \nu x$. Then, $q \in I_\nu \cap M_+$, and in line with the first part of Lemma 2 for both $x$ and $y = x + q$ one has that $R[x]$ and $R[y]$ are minimizing commutative subalgebras, which owing to the assumptions obey $[x, y] \neq 0$, and therefore both have to be non-trivial as well as cannot be the same, $R[x] \neq R[y]$. Since each non-trivial commutative subalgebra of $M_2(\mathbb{C})$ can be generated by exactly two atoms, from the previous then $R[x] \cap R[y] = C \cdot 1$ has to be followed. This especially means that condition (3-5) is fulfilled, and thus in accordance with the other assertion of Lemma 2 a least minimizing subalgebra cannot exist.

The above negative result and the previous counterexample together with some view on the structure of the condition (3-5) indicate that existence of a least minimizing abelian subalgebra seems to depend sensitively from the size as well as from the mutual position of the kernel-ideals $I_\nu$ and $I_\nu$ to each other (cf. also Lemma 1). Remind that the kernel-ideal $I_\nu$ in a $W^*$-algebra gets manageable especially if $\nu$ is supposed to be normal. In this case $I_\nu = Ms(\nu)^\perp$ holds, where $s(\nu)$ is the support orthoprojection of the normal positive linear form $\nu$ (be
careful about the context; the same notation \( s(x) \) will be also used for the support of an hermitian element \( x \in M_+ \), which subsequently also will play a rôle). Unfortunately, even in the normal case only very few answers are known on this subject, except we are in the special case with \( \rho \ll \nu \) which relates to Example 2, and where sufficiently many examples of minimizing abelian subalgebras are known. Before going into the details some auxiliary notion relating a general pair \( \{ \nu, \rho \} \) of normal positive linear forms will be introduced:

**Definition 3.** Let \( R \subset M \) be a \( \mathcal{W}^* \)-subalgebra of \( M \), which contains the unity of \( M \). \( R \) is called \( \{ \nu, \rho \} \)-projective provided the condition

\[
\forall y \in R : \quad \nu^{s(\rho)}(y) = \nu(ys(\rho)) \tag{3-6}
\]

is fulfilled (\( R \) will be simply referred to as projective subalgebra if the ordered pair is unambiguously given by the context).

**Example 8.** For a normal positive linear form \( \nu \) the unital subalgebra \( M_\nu \) defined by

\[
M_\nu = \{ x \in M : \nu(xy) = \nu(yx), \forall y \in M \}
\]

is a \( \mathcal{W}^* \)-subalgebra of \( M \), which usually is called \( \nu \)-centralizer. Obviously, if the support \( s(\rho) \) of another normal positive linear form \( \rho \) obeys \( s(\rho) \in M_\nu \), then relation (3-6) is automatically fulfilled, for each \( \mathcal{W}^* \)-subalgebra \( R \) of \( M \). Hence, in this case each such \( R \) is \( \{ \nu, \rho \} \)-projective.

**Remark 4.** (1) Since for each normal positive linear form \( \nu \) one has \( s(\nu) \in M_\nu \), according to Example 8 in case of normal \( \nu, \rho \in M_+^* \) with equal supports, \( s(\nu) = s(\rho) \), each subalgebra \( R \) of \( M \) is both \( \{ \nu, \rho \} \)- and \( \{ \rho, \nu \} \)-projective.

(2) Obviously, for given \( \{ \nu, \rho \} \) the set of all \( \{ \nu, \rho \} \)-projective subalgebras of \( M \) is non-void, and each subalgebra of a projective subalgebra is projective again. Also, the set of all projective subalgebras of \( M \) is closed with respect to intersections.

(3) Suppose \( \rho = \nu^2 \), for a pair \( \{ \nu, \rho \} \) of normal positive linear forms, with \( x \in M_+ \) obeying \( xs(\rho) = s(\rho)x \). Then, according to Example 6 and since then obviously (3-6) is fulfilled for \( R = R[x] \), the latter subalgebra is an example of a minimizing abelian projective subalgebra of \( M \) for \( \{ \nu, \rho \} \).

(4) Suppose under the conditions of the previous (3) that a least minimizing abelian subalgebra \( \mathcal{R}_M(\nu, \rho) \) exists. According to the previous two items it follows that \( \mathcal{R}_M(\nu, \rho) \) has to be projective, too.
3.3.2. **Radon-Nikodym theorem and minimizing projective subalgebras**

For the following recall that in case of \( g \ll \nu \) the Radon-Nikodym operator \( x = \sqrt{dg/d\nu} \) of \( g \) relative to \( \nu \) is understood to be the unique element \( x \in M_+ \) which obeys both \( g = \nu^x \) and \( s(x) \leq s(\nu) \).

**Lemma 3.** Suppose \( \nu, g \in M^*_+ \) are normal, with \( g \ll \nu \). Let \( R \) be any minimizing abelian projective subalgebra of \( M \) for \( \{ \nu, g \} \). Then, the following facts are valid:

1. \( \forall k \in s(\nu) \downarrow M_+ s(\nu) \downarrow : R[\sqrt{dg/d\nu} + k] \) is minimizing, projective;
2. \( \exists k \in s(\nu) \downarrow M_+ s(\nu) \downarrow : R[\sqrt{dg/d\nu} + k] \subset R \).

**Proof.** According to *Example 2* and *Example 6* one knows that the assumptions ensure that minimizing abelian subalgebras in fact have to exist. Since \( \nu \) is normal, as mentioned above \( I_\nu = M s(\nu) \downarrow \) holds. Hence \( I_\nu \cap M_+ = s(\nu) \downarrow M_+ s(\nu) \downarrow \) holds, and then by **Lemma 2** we know that the formula in (1) provides minimizing abelian subalgebras. Moreover, since \( g \ll \nu \) implies \( s(\sqrt{dg/d\nu}) = s(g) \leq s(\nu) \), one obviously has that each of \( \sqrt{dg/d\nu} + k \), with \( k \in s(\nu) \downarrow M_+ s(\nu) \downarrow \), commutes with \( s(g) \). Hence, by **Remark 4 (3)** all the subalgebras given in accordance with (1) also are projective. Thus, it remains to be shown that each minimizing abelian projective subalgebra \( R \) has a subalgebra as given in line with (1). Note that for \( g = 0 \) the assertion holds since then \( C \cdot 1 \) is minimizing. In line with this, we are going to prove the previous assertion in the non-trivial case with \( \nu, g \neq 0 \).

Let \( R \) be any minimizing abelian projective subalgebra to the given pair \( \{ \nu, g \} \). Note that by their very definitions the conditions of normality for a positive linear form, as well as the relation \( \ll \) among normal positive linear forms, are hereditary conditions when considered in restriction to \( W^* \)-subalgebras of \( M \). Thus especially we also find \( \nu |_R \ll \nu |_R \) on \( R \). Therefore we have unique Radon-Nikodym operators \( x = \sqrt{dg/d\nu} \) and \( z = \sqrt{dg|_R/d\nu|_R} \). As mentioned above we then especially have \( s(x) = s(g) \leq s(\nu) \), and since \( g \neq 0 \) is supposed in this case, we also have \( z \neq 0 \). The assumption that \( R \) be minimizing together with the reasoning of *Example 6* when applied for \( \{ \nu, g \} \) over \( M \), and for \( \{ \nu |_R, g |_R \} \) over \( R \), respectively, prove that for \( x_\varepsilon = x + \varepsilon \cdot 1 \) and \( z_\varepsilon = z + \varepsilon \cdot 1 \), with \( \varepsilon > 0 \), one has \( \lim_{\varepsilon \to 0} \nu(x_\varepsilon) = \lim_{\varepsilon \to 0} g(x^{-1}_\varepsilon) = \nu(x) = \sqrt{P_M(\nu, g)} = \sqrt{P_R(\nu |_R, g |_R)} = \nu(z) = \lim_{\varepsilon \to 0} \nu(z_\varepsilon) = \lim_{\varepsilon \to 0} g(z^{-1}_\varepsilon) \).

Hence, since \( \delta = (z_\varepsilon - x_\varepsilon) = (z - x) \) and \( g = \nu^x \) hold, upon taking the limit \( \varepsilon \to 0 \) within the relations which occur if (3-1d) is considered for \( z_\varepsilon, x_\varepsilon \) instead of \( z, x \) we will arrive at

\[
0 = - \lim_{\varepsilon \to 0} \nu(x x_\varepsilon^{-1} \delta x^{-1}_\varepsilon x) + \lim_{\varepsilon \to 0} \nu(x x_\varepsilon^{-1} \delta z^{-1}_\varepsilon \delta x_\varepsilon^{-1} x),
\]  

(3-7a)
where also the special form of $m(z, x)$ arising along with (3-1c) has been taken into account. Also note that by elementary facts on spectral theory $s_\varepsilon = xx^{-1} \equiv x^{-1}x$ is positive for each $\varepsilon$. Also, if positive reals are regarded as a directed set in its descending ordering, then $\{s_\varepsilon : \varepsilon > 0\}$ is fulfilled, which has the support orthoprojection $s(x)$ of $x$ as least upper bound, that is, $\text{l.u.b.}\{s_\varepsilon : \varepsilon > 0\} = s(x)$ is fulfilled. In passing note that the assertion on monotonicity can be understood as a special consequence of the fact saying that the function $\mathbb{R}_+ \setminus \{0\} \ni t \mapsto t^{-1}$ is operator-(anti)monotoneous over $\mathbb{M}^+_\mathbb{R}$ (for generalities on that and related, see [9, 14]). Since $s(x) = s(\varrho)$ holds, from the previous with the help of (1-1a) for each $y \in M$ one then easily concludes that $|\nu^s(\varrho)(y) - \nu(s_\varrho y s_\varrho)| \leq |\nu^s(\varrho)(y) - \nu(y s_\varrho)| + |\nu(y s_\varrho) - \nu(s_\varrho y s_\varrho)| \leq 2\|y\| \sqrt{\nu(s(x) - s_\varrho)}\|\nu\|_1$ must be fulfilled. From this owing to normality of $\nu$ and $\text{l.u.b.}\{s_\varepsilon : \varepsilon > 0\} = s(x) = s(\varrho)$ follows. From this in view of (3-7a) especially also follows that both limits within (3-7a) really exist. Now, remember that $R$ by assumption is both minimizing and projective. Hence, in view of LEVMA 1 and Definition 3 both, (3-4) with $a = z$ and $p = s(\varrho)$, as well as the particular case of the relation in (3-6) at $y = z$ hold. That is, $\nu(z) = \nu^s(\varrho)(z) - \nu^s(\varrho)^{{\perp}}(z)$ and $\nu(s(\varrho)^{{\perp}} z s(\varrho)) = 0$ are fulfilled. From the latter $\nu(z) = \nu^s(\varrho)(z) + 2\Re \nu(s(\varrho)^{{\perp}} z s(\varrho)) + \nu^s(\varrho)^{{\perp}}(z) = \nu^s(\varrho)(z) + \nu^s(\varrho)^{{\perp}}(z)$ is obtained. This together with the former provides the following relation:

$$\forall y \in M : \nu^s(\varrho)(y) = \lim_{\varepsilon \to 0} \nu(s_\varepsilon y s_\varepsilon) \quad (3-7b)$$

But then, since owing to $s(x) = s(\varrho)$ also $\nu^s(\varrho)(x) = \nu(x)$ must be fulfilled, $\nu^s(\varrho)(\delta) = \nu(\delta)$ can be followed. Remind that $\nu(\delta) = 0$ holds. In specializing $y = \delta$ within (3-7b), in line of the previous (3-7a) can be also read as

$$\lim_{\varepsilon \to 0} \nu(s_\varepsilon \delta z^{-1} \delta s_\varepsilon) = 0. \quad (3-7d)$$

Also note that by the estimate $z_\varepsilon \leq (\|z\| + \varepsilon)\mathbf{1}$, which is valid by triviality, $(\|z\| + \varepsilon)^{-1} \mathbf{1} \leq z_\varepsilon^{-1}$ is implied. But then, since the linear map $M \ni y \mapsto s_\varepsilon y \delta s_\varepsilon \in M$ is positive, from the previous and by positivity of $\nu$ one infers $\nu(s_\varepsilon \delta z^{-1} \delta s_\varepsilon) \geq (\|\delta\| + \varepsilon)^{-1} \nu(s_\varepsilon \delta^2 s_\varepsilon) \geq 0$. Regarding the limit of the latter as $\varepsilon \to 0$, and respecting that $\|\delta\| \neq 0$ holds, in view of (3-7d) yields $\nu^s(\varrho)(\delta^2) = 0$, finally. Owing to $s(\varrho) \leq s(\nu)$ from this $\nu(\varrho) = \mathbf{0}$ follows. Hence, since $s(\varrho) = s(x)$ and $z \in \mathbb{R}_+ \subset \mathbb{M}_\mathbb{R}$ hold, the conclusion is that $z = x + k$ has to be fulfilled, with $k = z s(\varrho)^{{\perp}} = s(\varrho)^{{\perp}} z \in s(\varrho)^{{\perp}} M_\mathbb{R} s(\varrho)^{{\perp}}$. But note that by $\nu(\delta) = 0$ then also $\nu(k) = 0$.
follows. By positivity of $k$ and $ks(\varrho)^\perp = k$ from this we conclude to $s(\nu)s(\varrho)^\perp ks(\varrho)^\perp s(\nu) = 0$, which is equivalent with $ks(\varrho)^\perp s(\nu) = 0$, and thus $k$ even must obey $k \in s(\nu)^\perp M_+ s(\nu)^\perp$. This together with the obvious relation $R[x+k] = R[z] \subset R$ is the assertion of (2).

**THEOREM 5.** Suppose $\varrho \ll \nu$ is fulfilled, for normal positive linear forms $\nu, \varrho \in M_+^*$, with faithful $\nu$. The following facts hold:

1. provided $\mathcal{R}_M(\nu, \varrho)$ exists it obeys
   \[ \mathcal{R}_M(\nu, \varrho) = R\left[\sqrt{d\varrho/d\nu}\right]; \quad (3-8) \]

2. if also $\varrho$ is faithful then $\mathcal{R}_M(\nu, \varrho)$ exists.

**Proof.** By Lemma 3(1) one knows that $R = R[\sqrt{d\varrho/d\nu}]$ is minimizing and projective. Hence, if $\mathcal{R}_M(\nu, \varrho)$ is assumed to exist then by Remark 4 (3)–(4) the minimizing subalgebra $\mathcal{R}_M(\nu, \varrho) \subset R[\sqrt{d\varrho/d\nu}]$ has to be also projective (occasionally remark that this conclusion does not rely on the premise on faithfulness of $\nu$). Hence, Lemma 3(2) can be applied to $R = \mathcal{R}_M(\nu, \varrho)$. By faithfulness of $\nu$ one has $s(\nu)^\perp = 0$ and then the mentioned application yields $R \subset R[\sqrt{d\varrho/d\nu}]$, and in view of the above the formula (3-8) then is seen to hold, that is, (1) is valid. To see (2), note that in this case $1 = s(\nu) = s(\varrho)$ holds, which via Remark 4 (1) implies that Lemma 3(2) can be applied to each minimizing subalgebra of each minimizing $R$. Thus it is the least one of this sort.

3.3.3. $\mathcal{R}_M(\nu, \varrho)$ as a projective subalgebra

Suppose $\varrho \ll \nu$ such that a least minimizing subalgebra exists. As has been remarked in line of the previous proof the algebra $\mathcal{R}_M(\nu, \varrho)$ then has to be a minimizing projective subalgebra. Application of Lemma 3 then yields that provided $\mathcal{R}_M(\nu, \varrho)$ exists the latter has to equal to

\[ R_\infty(\nu, \varrho) = \bigcap_{k \in s(\nu)^\perp M_+ s(\nu)^\perp} R[\sqrt{d\varrho/d\nu} + k]. \quad (3-9a) \]

From Lemma 3(2) even $\mathcal{R}_M(\nu, \varrho) = R[\sqrt{d\varrho/d\nu} + k_\infty]$ can be seen to hold, for some $k_\infty \in s(\nu)^\perp M_+ s(\nu)^\perp$. In line with (3-9a) the latter especially means that $R[\sqrt{d\varrho/d\nu} + k_\infty] \subset R[\sqrt{d\varrho/d\nu} + \lambda s(\nu)^\perp]$ has to be fulfilled, for each $\lambda \in \mathbb{R}_+$. Therefore $k_\infty \in \mathbb{R}_+ s(\nu)^\perp$ has to hold. In summarizing from the latter and (3-9a), in the general case of $\varrho \ll \nu$
the conclusion of THEOREM 5 (1) and formula (3-8) generalize to the following implication, which must be fulfilled for some $\gamma \in \mathbb{R}_+$:

$$\mathcal{R}_M(\nu, \varrho) \text{ exists } \Rightarrow \mathcal{R}_M(\nu, \varrho) = \bigcap_{\lambda \in \mathbb{R}_+} R[\sqrt{d\varrho/d\nu} + \lambda s(\nu)\perp]$$

$$= R[\sqrt{d\varrho/d\nu} + \gamma s(\nu)\perp]$$

$$= R_\infty(\nu, \varrho).$$

To summarize from this, for given \{\nu, \varrho\} obeying $\varrho \ll \nu$ the algebra $R_\infty(\nu, \varrho)$ can be regarded to be the only candidate for $\mathcal{R}_M(\nu, \varrho)$. Thereby, the $\gamma$ within (3-9b) will be made more explicit later.

Note that in the special case of $\varrho \ll \nu$ with $s(\varrho) \in M^\nu$ one can go a step further. Then, since owing to Example 8 the assertion of LEMMA 3 (2) can be applied to any minimizing subalgebra $R$, the above can be strengthened to the assertion that, depending from whether or not $R_\infty(\nu, \varrho)$ is minimizing, either a least minimizing abelian subalgebra will exist and then obeys $\mathcal{R}_M(\nu, \varrho) = R_\infty(\nu, \varrho)$, or a least minimizing abelian subalgebra cannot exist at all.

**LEMMA 4.** Suppose $\varrho \ll \nu$, with $s(\varrho) \in M^\nu$ one can go a step further. Then, since owing to Example 8 the assertion of LEMMA 3 (2) can be applied to any minimizing subalgebra $R$, the above can be strengthened to the assertion that, depending from whether or not $R_\infty(\nu, \varrho)$ is minimizing, either a least minimizing abelian subalgebra will exist and then obeys $\mathcal{R}_M(\nu, \varrho) = R_\infty(\nu, \varrho)$, or a least minimizing abelian subalgebra cannot exist at all.

**LEMMA 4.** Suppose $\varrho \ll \nu$, with $s(\varrho) \in M^\nu$. Then $R_\infty(\nu, \varrho)$ is minimizing if, and only if, a least minimizing abelian subalgebra exists.

Having in mind these facts, and knowing that the special case of faithful $\nu$ has been dealt with yet in THEOREM 5, with providing a complete answer for faithful $\varrho$, we are now going to analyze the family of algebras occurring under the intersection within (3-9b) more thoroughly in the remaining cases (in particular, those with non-faithful $\nu$) which are not yet covered by the premises of THEOREM 5. To this sake some auxiliary technical facts on hereditary subalgebras and elementary spectral theory will be needed. Recall some standard fact from $W^*$-theory first.

**Remark 5.** If $R[y, y^*]$ is the smallest $W^*$-subalgebra of $M$ generated by $y \in M$ and 1, then this is the $\sigma(M, M_*)$-closure of all polynomials in $y, y^*$ (including the constants as $\mathbb{C} \cdot 1$). Here, $M_*$ is the predual of $M$, which is the Banach (sub)space of $M^*$ (with respect to the functional norm) which is generated by all normal positive linear forms (refer also to the elements of $M_*$ as normal (linear) forms). The $\sigma(M, M_*)$-topology is the weakest locally convex topology on $M$ such that all the seminorms $p_f, f \in M_*$, with $p_f(x) = |f(x)|$ for $x \in M$, are continuous.

Suppose now $\varrho \ll \nu$, and let an orthoprojection $q$ be defined by $q = s(\varrho) + s(\nu)^\perp$. On the hereditary $W^*$-subalgebra $qMq$ define another normal positive linear forms $\nu_q, \varrho_q$ by $\nu_q = \nu|_{qMq}$ and $\varrho_q = q|_{qMq}$, respectively. Then $\varrho_q \ll \nu_q$ is fulfilled, with supports in $qMq$ obeying
In fact, since owing to Remark 5 showing that provided (3-9a) and (3-9b) one has

\[ \text{spec}_p(x_q) \cup \{0\} = \text{spec}_p(x) \]  

(3-10a)

this can be easily seen to hold. For \( y \in (qMq)_h \subset M_h \) we let \( R_q[y] \) be the \( W^* \)-subalgebra of \( qMq \) generated by \( y \) and the unity \( q \) of \( qMq \). In view of Remark 5 it is plain to see that \( R_q[y] = qR[y]q \) holds. We are going to show that provided \( R_M(\nu, q) \) exists then \( R_{qMq}(\nu_q, q_q) \) exists and obeys

\[ R_{qMq}(\nu_q, q_q) = qR_M(\nu, q)q. \]  

(3-10b)

In fact, owing to \( s(x) = s(q) \) for each \( k \in s(\nu)^{\perp} M s(\nu)^{\perp} \) also \( x+k \in qMq \) holds, one has \( R_q[x_q + k] = qR[x + k]q \). Hence, in accordance with (3-9a) and (3-9b) one has \( qR_M(\nu, q)q = \cap_{\lambda > 0} R_q[x_q + \lambda s(\nu)^{\perp}] = R_q[x_q + \gamma s(\nu)^{\perp}] = \cap_k R_q[x_q + k] \), for some real \( \gamma \geq 0 \). We may apply formula (3-9a) with respect to the hereditary algebra \( qMq \) and normal positive linear forms \( \nu_q, q_q \). The result is \( R_\infty(\nu_q, q_q) = \cap_k R_q[x_q + k] \), with \( k \) running through \( s(\nu_q)^{\perp} M_+ s(\nu_q)^{\perp} = s(\nu)^{\perp} M_+ s(\nu)^{\perp} \) (see above). Hence, in view of the previous one has \( qR_M(\nu, q)q = R_q[x_q + \gamma s(\nu)^{\perp}] = R_\infty(\nu_q, q_q) \). Especially, application of Lemma 3 (1) for \( \nu_q, q_q \) on \( qMq \) then shows that \( R_\infty(\nu_q, q_q) \) is minimizing. But then, since \( s(\nu_q) = s(q_q) \) and \( q_q \ll \nu_q \) hold, when considering Lemma 4, Remark 4 (1) and (3-9b) for \( \nu_q, q_q \) on \( qMq \), one gets \( R_\infty(\nu_q, q_q) = R_{qMq}(\nu_q, q_q) \). From this in view of the above (3-10b) follows.

Close our preliminaries with the following auxiliary result which matters some elementary spectral theory.

**LEMMA 5.** Suppose \( x \in M_+ \), \( s(x) < 1 \), with point spectrum \( \text{spec}_p(x) \).

Depending from the latter, the following cases may occur for the commutative \( W^* \)-subalgebra \( R_0(x) = \bigcap_{\lambda \in \mathbb{R}_+} R[x + \lambda s(x)^{\perp}] \), where \( \gamma \) can stand for any non-negative real:

\[
R_0(x) = \begin{cases}  
R[x] & \text{if } \text{spec}_p(x) \setminus \{0\} = \emptyset, \\
R[x + \lambda_0 s(x)^{\perp}] & \text{if } \text{spec}_p(x) \setminus \{0\} = \{\lambda_0\}, \\
\neq R[x + \gamma s(x)^{\perp}] & \text{if } \# \text{spec}_p(x) \setminus \{0\} \geq 2.
\end{cases}
\]

Especially, \( R_0(x) = R[x] \) holds if, and only if, \( \text{spec}_p(x) \setminus \{0\} = \emptyset \) is fulfilled.
Proof. Some preliminary results will be derived first. Let \( \{ E_x(t) : t \in \mathbb{R} \} \) be the spectral resolution of \( x \) within the projection lattice of \( M \). Then, the eigenprojection of the positive element \( x + \lambda s(x)^{\perp} \) to the spectral value \( \lambda \in \mathbb{R}_+ \) is given by

\[
E_{x+\lambda s(x)^{\perp}}(\{\lambda\}) = \begin{cases} s(x)^{\perp} + E_x(\{\lambda\}) & \text{for } \lambda \in \mathbb{R}_+ \setminus \{0\}, \\ s(x)^{\perp} & \text{for } \lambda = 0. \end{cases} \tag{*}
\]

In fact, by assumption \( E_x(\{0\}) = s(x)^{\perp} \) holds, and thus the part of \((*)\) relating to \( \lambda = 0 \) is valid. Also, for \( \lambda \in \mathbb{R}_+ \setminus \{0\} \) it is clear from \( E_x(\{0\})E_x(\{\lambda\}) = 0 \) and the above that \( p = s(x)^{\perp} + E_x(\{\lambda\}) \) is an orthoprojection in \( M \), and which obeys \( (x + \lambda s(x)^{\perp})p = \lambda p \). Note in this context that \( E_x(\{\lambda\}) \) is non-vanishing iff \( \lambda \in \text{spec}_p(x) \). Also, for an orthoprojection \( q \geq p \) one has \((q-p)s(x)^{\perp} = 0 \) and \((q-p)E_x(\{\lambda\}) = 0 \).

Hence, assuming \( (x + \lambda s(x)^{\perp})q = \lambda q \) yields \( x(q-p) = \lambda(q-p) \), which according to spectral theory necessarily implies \((q-p) E_x(\{\lambda\}) \). In view of the above then \((q-p) = 0 \). Thus, there is no larger than \( p \) orthoprojection \( q \) in \( M \) with \( x + \lambda s(x)^{\perp} = \lambda q \), which means \( p = E_{x+\lambda s(x)^{\perp}}(\{\lambda\}) \). This is \((*)\).

Next, it is useful to take notice that the following alternatives exist:

\[
R[x + \lambda s(x)^{\perp}] \begin{cases} = R[x] & \text{if } \lambda \notin \text{spec}_p(x) \setminus \{0\} \text{ or } \lambda = 0, \\ \subsetneq R[x] & \text{else.} \end{cases} \tag{**}
\]

To see \((**i)\), note first that obviously \( R[x + \lambda s(x)^{\perp}] \subset R[x] \). Since for \( \lambda \notin \text{spec}_p(x) \setminus \{0\} \) one has \( E_x(\{\lambda\}) = 0 \), from \((*)\) then \( E_{x+\lambda s(x)^{\perp}}(\{\lambda\}) = s(x)^{\perp} \) is seen, and thus both \( x + \lambda s(x)^{\perp} \) and \( s(x)^{\perp} \) have to belong to \( R[x + \lambda s(x)^{\perp}] \), and thus \( x \) does, too. In view of the above then \( R[x + \lambda s(x)^{\perp}] = R[x] \), which for \( \lambda = 0 \) is trivially valid, is seen to hold for \( \lambda \notin \text{spec}_p(x) \setminus \{0\} \). In case of \( \lambda \in \text{spec}_p(x) \setminus \{0\} \), the element \( x + \lambda s(x)^{\perp} \) has full support, and according to \((*)\) \( s(x)^{\perp} \) is a proper sub-projection of the eigen-orthoprojection \( E_{x+\lambda s(x)^{\perp}}(\{\lambda\}) \) to the spectral value \( \lambda \in \text{spec}_p(x + \lambda s(x)^{\perp}) \). Since each spectral eigenprojection has to be a minimal orthoprojection of the generated commutative \( W^* \)-algebra \( R[x + \lambda s(x)^{\perp}] \), from the previous \( s(x)^{\perp} \notin R[x + \lambda s(x)^{\perp}] \) has to be followed. Hence, in this case then \( R[x + \lambda s(x)^{\perp}] \subsetneq R[x] \), which completes the proof of \((**i)\).

After these preparations, we are going to prove the assertions of our results on \( R_0(x) \). Note that the validity in case of \( \text{spec}_p(x) \setminus \{0\} = \emptyset \) or \( \text{spec}_p(x) \setminus \{0\} = \{\lambda_0\} \) is straightforward from \((**)\). Thus, we have to consider explicitly only the case with \( \# \text{spec}_p(x) \setminus \{0\} \geq 2 \). From \((**)\) then obviously \( R_0(x) \subsetneq R[x] \) follows. Especially this also means
that the assertion is valid for $\gamma = 0$. Now, in line with this, but in contrast with the assertion, assume we had $R_0(x) = R[x + \gamma s(x)^\perp]$, with $\gamma > 0$. Then, since $\#\text{spec}_p(x)\{0\} \geq 2$ is fulfilled, there has to exist $\lambda \in \text{spec}_p(x)\{0\}$ with $\lambda \neq \gamma$. Thus $E_{x+\lambda s(x)^\perp}(\{\lambda\}) \in R[x + \lambda s(x)^\perp]$, and $E_{x+\gamma s(x)^\perp}(\{\gamma\}) \in R_0(x)$ by assumption. Since by definition of $R_0(x)$ one has $R_0(x) \subset R[x + \lambda s(x)^\perp]$, both $E_{x+\lambda s(x)^\perp}(\{\lambda\})$ and $E_{x+\gamma s(x)^\perp}(\{\gamma\})$ have to be in $R[x + \lambda s(x)^\perp]$. From ($\ast$) and since $\gamma \neq \lambda$ is fulfilled, we see $s(x)^\perp = E_{x+\gamma s(x)^\perp}(\{\gamma\}) E_{x+\lambda s(x)^\perp}(\{\lambda\}) \in R[x + \lambda s(x)^\perp]$, and therefore also $x \in R[x + \lambda s(x)^\perp]$ holds. From this and $R[x + \lambda s(x)^\perp] \subset R[x]$ then $R[x + \lambda s(x)^\perp] = R[x]$ had to be followed. Owing to the choice of $\lambda$ in accordance with $\lambda \in \text{spec}_p(x)\{0\}$ this is in contradiction with ($\ast\ast$). Thus, also in case of $\gamma > 0$ a relation $R_0(x) = R[x + \gamma s(x)^\perp]$ cannot happen. Finally, note that by the just proven allowance is made for any situations with $R_0(x)$ that might occur. Particularly, from this and ($\ast\ast$) one also infers that $R_0(x) = R[x]$ cannot happen unless $\text{spec}_p(x)\{0\} = \emptyset$, whereas in the latter case this then in fact occurs. Thus, also the final assertion is seen to be true.

3.3.4. The main result for $\varrho \ll \nu$ and with $s(\varrho) \in M^{\nu}$

Suppose $\varrho \ll \nu$ such that $\mathcal{R}_M(\nu, \varrho)$ exists. Then, we derive a formula of $\mathcal{R}_M(\nu, \varrho)$ which generalizes (3-8) to this context. In addition also partial answers on the existence problem for $\mathcal{R}_M(\nu, \varrho)$ will be given.

**THEOREM 6.** Let $M$ be a $W^*$-algebra, and let two normal positive linear forms $\nu, \varrho$ be given on $M$ and obeying $\varrho \ll \nu$. Let a non-negative real $\lambda_0$ be defined by

$$
\lambda_0 = \sup \left\{ \lambda : \lambda \in \text{spec}_p(\sqrt{d\varrho/d\nu}) \cup \{0\} \right\}. \tag{3-11a}
$$

The following facts hold true.

1. Provided $\mathcal{R}_M(\nu, \varrho)$ exists then it obeys

$$
\mathcal{R}_M(\nu, \varrho) = R\left[\sqrt{d\varrho/d\nu} + \lambda_0 s(\nu)^\perp\right], \tag{3-11b}
$$

with the additional condition

$$
\#\text{spec}_p(\sqrt{d\varrho/d\nu}) \begin{cases} 
\leq 2 & \text{if } s(\varrho) = s(\nu), \\
= 1 & \text{else}
\end{cases} \tag{3-11c}
$$

fulfilled in case of non-faithful $\nu$. 
(2) Assume \{\nu, \varrho\} with \(s(\varrho) \in M^\nu\). Then, if \(\nu\) is faithful, or in all cases with non-faithful \(\nu\) obeying \(\dim \overline{s(\nu)} < \infty\) and \(\varrho\) respecting (3-11c), a least \ minimizing abelian subalgebra exists.

Proof. Let \(x = \sqrt{d\varrho/d\nu}\) and assume \(\mathcal{R}_M(\nu, \varrho)\) exists. Then, (3-9b) yields that \(\mathcal{R}_M(\nu, \varrho) = R[x + \gamma s(\nu)^\perp]\) has to be fulfilled, for some \(\gamma \in \mathbb{R}_+\). We are going to determine the real \(\gamma\) in terms of \(x\). Let \(q = s(\varrho) + s(\nu)^\perp\). According to (3-10b) and in using the notations introduced in context of eqs. (3-10), with respect to the hereditary \(W^*\)-subalgebra \(\mathcal{M}_q\) and normal positive linear forms \(\nu_q, \varrho_q\) then also \(\mathcal{R}_{q_M}(\nu_q, \varrho_q)\) exists and obeys \(\mathcal{R}_{q_M}(\nu_q, \varrho_q) = R_q[x_q + \gamma s(\nu)^\perp]\). On the other hand, an application of (3-9b) on \(q_M\) with \(\nu_q, \varrho_q\) yields \(\mathcal{R}_{q_M}(\nu_q, \varrho_q) = R_q(x_q)\), with the algebra \(R_q(x_q)\) constructed as in Lemma 5 in terms of \(x_q = \sqrt{d\varrho_q/d\nu_q}\) and with respect to \(q_M\). Since both \(s(x_q) = s(\varrho_q) = s(\nu_q) = s(\varrho)\) and \(s(\nu_q)^\perp = s(\nu)^\perp\) hold on \(q_M\), in view of the above we therefore conclude that, provided \(\mathcal{R}_M(\nu, \varrho)\) has been assumed to exist, then \(R_q(x_q) = R_q[x_q + \gamma s(\nu)]\) has to be fulfilled, for some \(\gamma \in \mathbb{R}_+\). But then, in case of \(s(\varrho) = s(x_q)\), Lemma 5 can be applied on \(q_M\) and gives that \#spec\(p(x_q)\), \(\{0\} < 2\), has to be fulfilled, with \(\gamma = \sup\{\lambda : \lambda \in \text{spec}_{p}(x_q) \cup \{0\}\}\). Note that the condition \(s(\varrho) = s(x_q) < q\) is equivalent with \(s(\nu) < 1\), and that in this case then \(0 \in \text{spec}_{p}(x)\) holds. Hence, by (3-10a) in this case \#spec\(p(x_q)\), \(\{0\} = \#spec_{p}(x)\), \(\{0\}\). Especially, the previously given \(\gamma\) then obeys \(\gamma = \lambda_0\), with \(\lambda_0\) as given in accordance with (3-11a). Thus, in summarizing from this and the previous, assuming that \(\mathcal{R}_M(\nu, \varrho)\) exists for non-faithful \(\nu\) implies that (3-11b) and \#spec\(p(x)\), \(\leq 2\) hold. Now, suppose \(s(\varrho) < s(\nu) < 1\). Then, assuming \(\lambda_0 > 0\) would imply \(q^\perp \in R[x + \lambda_0 s(\nu)^\perp]\), for \(q^\perp\) is the eigenprojection of \(x + \lambda_0 s(\nu)^\perp\) to eigenvalue 0. But at the same time certainly \(q^\perp \not\in R[x]\) since by supposition of this case \(q^\perp < s(\varrho)^\perp\) has to hold and \(s(\varrho)^\perp\) has to be a minimal orthoprojection of \(R[x]\). Thus, \(R[x + \lambda_0 s(\nu^\perp)]\) cannot be a subalgebra of \(R[x]\) in this case. In view of the meaning of \(\mathcal{R}_M(\nu, \varrho)\) and since \(R[x]\) is minimizing the latter contradicts the just derived formula (3-11b) in the case of non-faithful \(\nu\). Hence, for \(s(\varrho) < s(\nu) < 1\) one must have \(\lambda_0 = 0\). In view of (3-11a) and since for non-faithful \(\nu\) one has \(0 \in \text{spec}_{p}(x)\) one then infers that \(\text{spec}_{p}(x) = \{0\}\) holds. This completes the proof of (3-11c). That (3-11b) remains true also for faithful \(\nu\) follows since then owing to \(s(\nu)^\perp = 0\) formula (3-11b) simply reduces to formula (3-8), which according to Theorem 5 (1) is true, however, and which completes the proof of (1).

To see (2), note that for faithful \(\nu\) formula (3-9a) yields \(R_\infty(\nu, \varrho) = R[\sqrt{d\varrho/d\nu}]\). Hence, according to Lemma 3, the algebra \(R_\infty(\nu, \varrho)\) is minimizing. But then, since \(\varrho\) obeys \(s(\varrho) \in M^\nu\), from Lemma 4 we
may also conclude that $\mathcal{R}_M(\nu, \varrho)$ exists. This proves the part of (2) relating to a faithful $\nu$.

Suppose now $\nu$ to be non-faithful, but with $\dim s(\nu)^\perp M s(\nu)^\perp < \infty$ fulfilled, and $\varrho$ such that $s(\varrho) \notin M^\nu$ holds and condition (3-11c) is respected. Note that in this case $0 \in \text{spec}_p(x)$ holds. Also, by the assumption of finite dimensionality then $\text{spec}(k) = \text{spec}_p(k)$ holds, for each $k \in s(\nu)^\perp M s(\nu)^\perp$, and if $p_\lambda$ is the eigenprojection of $k$ to $\lambda \in \text{spec}_p(k)$, we have $\sum_{\lambda \in \text{spec}_p(k)} p_\lambda = s(\nu)^\perp$. By the same kind of auxiliary arguments from elementary spectral theory, which have been yet using in line of the proof of Lemma 5 in some special case, in literally the same way (the details of which therefore will not be mentioned) can be also applied in order to compare the spectral structures of $x + k$ and $x$ (below these facts will be tacitly made use of). Suppose $\lambda_0 = 0$ first. Then, zero is the only eigenvalue of $x$, and therefore one infers that $\text{spec}_p(x + k) = \text{spec}_p(k) \cup \{0\}$ for $s(\varrho) < s(\nu)$, and $\text{spec}_p(x + k) = \text{spec}_p(k)$ for $s(\varrho) = s(\nu)$. Owing to this and $s(x) \leq s(\nu)$, whereas each of the above $p_\lambda$ for $\lambda \in \text{spec}_p(k) \setminus \{0\}$ will be also the corresponding eigenprojection to the same $\lambda \in \text{spec}_p(x + k)$ with respect to $x + k$, the projection $p_0 + \{s(\nu) - s(\varrho)\}$, or $\{s(\nu) - s(\varrho)\}$ respectively, will be the eigenprojection of $x + k$ to eigenvalue zero in case of $0 \in \text{spec}_p(x + k) \cap \text{spec}_p(k)$, and in case of $0 \notin \text{spec}_p(x + k)$ but with $0 \notin \text{spec}_p(k)$, respectively. Therefore, $p_\lambda \in R[x + k]$ for each $\lambda \in \text{spec}_p(k) \setminus \{0\}$, and $p_0 + \{s(\nu) - s(\varrho)\} \in R[x + k]$. Then in view of the above in each case also their sum $s(\nu)^\perp + \{s(\nu) - s(\varrho)\}$ has to be in $R[x + k]$, that is, $s(\varrho) \in R[x + k]$ has to hold. From this owing to $s(x) = s(\varrho) \leq s(\nu)$ then $x = s(\nu)^\perp x \in R[x + k]$ is seen. Hence, $R[x + k] \supset R[x]$ follows, for each $k \in s(\nu)^\perp M s(\nu)^\perp$, and therefore one has $R_\infty(\nu, \varrho) = R[x]$. From Lemma 3 follows that $R_\infty(\nu, \varrho)$ is minimizing. Thus, since $\varrho$ obeys $s(\varrho) \in M^\nu$, by Lemma 4 we may conclude that $\mathcal{R}_M(\nu, \varrho)$ exists. Hence, for non-faithful $\nu$ and $\# \text{spec}_p(x) = 1$ the assertion of (2) is true.

Suppose $s(\varrho) = s(\nu)$ and $\# \text{spec}_p(x) = 2$, with non-faithful $\nu$. Then, $\lambda_0 > 0$, and for each $k \in s(\nu)^\perp M s(\nu)^\perp$ one has $p_\lambda \in R[x + k]$, for $\lambda \in \text{spec}_p(k) \setminus \{\lambda_0\}$. If $\lambda_0 \notin \text{spec}_p(k)$ from this $s(\nu)^\perp \in R[x + k]$ follows, from which $R[x] \subset R[x + k]$ is seen. For $\lambda_0 \in \text{spec}_p(k)$, however, $p_\lambda + E_x(\{\lambda_0\})$ is the $\lambda_0$ corresponding eigenprojection of $x + k$, and therefore instead of $p_\lambda \in R[x + k]$ one finds $p_\lambda + E_x(\{\lambda_0\}) \in R[x + k]$. Summing up then yields $s(\nu)^\perp + E_x(\{\lambda_0\}) \in R[x + k]$ instead. But then also $k + \lambda_0 E_x(\{\lambda_0\}) = (s(\nu)^\perp + E_x(\{\lambda_0\}))(x + k) \in R[x + k]$. Hence, since $x + \lambda_0 s(\nu)^\perp$ can be combined together from the mentioned elements as $x + \lambda_0 s(\nu)^\perp = (x + k) - (k + \lambda_0 E_x(\{\lambda_0\})) + \lambda_0(s(\nu)^\perp + E_x(\{\lambda_0\}))$, 

Burin.tex; 28/03/2022; 20:56; p.34
$x + \lambda_0 s(\nu) \perp \in R[x + k]$ is seen. Note that owing to $s(\nu) \perp \in R[x]$ in any case one has $x + \lambda_0 s(\nu) \perp \in R[x]$. We may summarize these facts and conclude that, for non-faithful $\nu$ with $s(\nu) = s(\nu)$ and $\# \text{spec}_p(x) = 2$, $R[x + \lambda_0 s(\nu) \perp] \subset R[x + k]$ holds, for each $k \in s(\nu) \perp M_+ s(\nu) \perp$. Hence, $R_\infty(\nu, g) = R[x + \lambda_0 s(\nu) \perp]$, and thus according to LEMMA 3 also in this case the algebra $R_\infty(\nu, g)$ is minimizing. Since $s(\nu) = s(\nu) \in M''$ holds LEMMA 4 can be applied once more again and yields that $\mathcal{R}_M(\nu, g)$ exists. This closes the proof of (2), and at the same time also completes the proof of the theorem.

3.3.5. \textit{Examples and consequences}  
Start with discussing THEOREM 6 in the finite dimensional case.

\textbf{Example 9.} Suppose $2 \leq \dim M < \infty$, and $\nu, g$ two non-zero positive linear forms obeying $g \ll \nu$, but which are not mutually proportional. Then, the corresponding Radon-Nikodym operator cannot be proportional to the support of $\nu$, $\sqrt{d\nu/d\nu} \notin \mathbb{R}_+ s(\nu)$. Since $s(\nu) \leq s(\nu)$ is the support of $\sqrt{d\nu/d\nu}$, from these facts $\# \text{spec}(\sqrt{d\nu/d\nu}) \geq 2$ follows. Hence, since by finite-dimensionality one has $\text{spec}_p(\sqrt{d\nu/d\nu}) = \text{spec}(\sqrt{d\nu/d\nu})$, the condition (3-11c) in case of non-faithful $\nu$ could be satisfied only if $\# \text{spec}(\sqrt{d\nu/d\nu}) = 2$ and $s(g) = s(\nu) < 1$ were fulfilled. But then $\sqrt{d\nu/d\nu}$ as a Radon-Nikodym operator had to be proportional with $s(\nu) = s(\nu)$, which however contradicts to the above mentioned fact. Thus, in view of THEOREM 6 (1) for non-faithful $\nu$ and under the above premises a least minimizing algebra cannot exist in the finite dimensional case. Especially, from the latter and by formula (3-11b) one also infers that provided a least minimizing algebra exists then $\mathcal{R}_M(\nu, g) = R[\sqrt{d\nu/d\nu}]$ will occur, in any case. From THEOREM 6 (2) one infers that the latter case really can happen, e.g. in case of faithful $\nu$ and $g$ obeying $g \ll \nu$ and $s(g) \in M''$.

As the previous example shows the deviation from the law (3-8) as indicated by (3-11b) could be observed only for $\dim M = \infty$. That this deviation really can occur is seen by the following example.

\textbf{Example 10.} Let $M = L^\infty(I, m')$, where $\{I, m'\}$ is the unit interval $I = [0, 1]$ with a measure $m' = (m + \delta_0)/2$, where $m$ is the Lebesgue measure and $\delta_0$ is concentrated on $\{0\}$, with $\delta_0(\{0\}) = 1$. Let $\nu$ correspond to the class of the characteristic function $\chi_{(0, 1)}$ of $(0, 1]$ via $\nu(\cdot) = \int_{(0, 1]}(\cdot) dm'$, and be $f$ a strictly increasing function, which is continuous on $[0, 1]$, except for one point $t_0 > 0$ where it is only left-continuous with $f(t_0) = \lambda_0 > 0$, and which obeys $0 < f(t) \leq 1$ for
Along with Theorem 6 (3-11b) then yields \( R \) \( M \) \( \Theta \) \( s \) \( t > 0 \), and \( f(0) = 0 \). Define \( \varrho(\cdot) = \int f(\cdot) \, dm' \). Then, \( \varrho \ll \nu \) (even \( \varrho \leq \nu \) holds) and \( s(\nu) = s(\varrho) = \chi_{[0,1]} < \chi_{[0,1]} = 1 \), with Radon-Nikodym operator \( x = f \) obeying \( \{0, \lambda_0\} = \text{spec}_p(x) \). Hence condition (3-11c) is fulfilled in this case. Since owing to \( M^\nu = M \) one has \( s(\varrho) \in M^\nu \) to be fulfilled by triviality, THEOREM 6 (2) can be applied and formula (3-11b) then yields \( R_M(\nu, \varrho) = R[f + \lambda_0 \chi_{\{0\}}] \).

Along with THEOREM 6 (1) comes another necessary condition for \( R_M(\nu, \varrho) \) to exist which often will be useful. To explain this, in the following let \( \text{Aut}(M) \) denote the group of all *-automorphisms of \( M \), and for \( y \in M \) we let \( \text{Aut}_y(M) \) be those *-automorphisms which leave the element \( y \) fixed. Clearly, since we have to do with *-automorphisms one has \( \text{Aut}_y(M) = \text{Aut}_{y^*}(M) \), for each \( y \in M \).

**Remark 6.** Remind that a *-isomorphism \( \Phi \) from one \( W^* \)-algebra \( M \) onto another \( W^* \)-algebra \( N \) automatically is \( \sigma(M, M_\nu) \sigma(N, N_\nu) \) continuous. From this and Remark 5 then follows that \( \Phi \in \text{Aut}_y(M) \iff \Phi \in \text{Aut}_x(M), \forall x \in R[y, y^*] \), is valid for each \( y \in M \).

**COROLLARY 5.** For the pair \( \{\nu, \varrho\} \) of normal positive linear forms suppose \( \varrho \ll \nu \), with Radon-Nikodym operator \( x = \sqrt{d\varrho/d\nu} \), and let \( \lambda_0 \) be defined in accordance with formula (3-11a). Then, existence of \( R_M(\nu, \varrho) \) implies the following to hold:

\[
\forall k \in s(\nu)^\perp M_\perp s(\nu)^\perp : \text{Aut}_{x+k}(M) \subset \text{Aut}_{x+\lambda_0 s(\nu)^\perp}(M) \tag{3-12}
\]

**Proof.** In view of (3-9a) and THEOREM 6 (1) the premises imply \( R[x + \lambda_0 s(\nu)^\perp] \subset R[x + k] \) to be fulfilled, for each \( k \in s(\nu)^\perp M_\perp s(\nu)^\perp \). From this it is evident that by each *-automorphisms \( \Phi \) leaving pointwise invariant all elements of \( R[x + k] \) in particular also each element of \( R[x + \lambda_0 s(\nu)^\perp] \) is left invariant. This is (3-12).

We will show that among the assumptions in THEOREM 6 (2) also the condition \( \dim s(\nu)^\perp M s(\nu)^\perp < \infty \) is a sensitive one. For simplicity this will be demonstrated by such an example, which by its construction and owing to the procedure applied can stand for a whole class of analogous (even non-commutative) situations where (3-12) fails and thus a least minimizing subalgebra cannot exist then.

**Example 11.** Let \( M = L^\infty(I, m) \), where \( \{I, m\} \) is the unit interval \( I = [0,1] \) with Lebesgue-measure \( m \). Let \( \tau \in M_\perp^\perp \) be the standard tracial state given on \( M \) by \( \tau(x) = \int_I dm \, x \), for \( x \in M \). Suppose \( \nu = \tau(\chi_0(\cdot)) \), where \( \chi_0 \) corresponds to the class of the characteristic function of the interval \([0,1/2]\). Assume \( \varrho = \tau(f(\cdot)) \), where we let \( f \) correspond...
to the class of some continuous, monotoneous function \( f \) on \([0, 1]\), with \( 1 \geq f(t) > 0 \) for \( t < 1/2 \) and \( f(t) = 0 \) else. We then have \( \varrho \ll \nu \), \( s(\nu) = \chi_0 < 1 \) and \( x = \sqrt{d\varrho/d\nu} = f \). Let us consider the \(*\)-automorphism \( \Phi_\varrho \) which is induced on \( M \) by the measure preserving point-transformation \( g : I \ni t \mapsto (1 - t) \in I \) of the unit interval, that is, in the sense of equivalence of functions, \( \Phi_\varrho(x) = x \circ g \) is fulfilled. Obviously, \( \Phi_\varrho \) is idempotent, that is, a symmetry. Note that \( \Phi_\varrho(\chi_0) = \chi_1 \) holds, where \( \chi_1 \) stands for the class of the characteristic function of the interval \([1/2, 1] \) within \( M \), that is, \( \Phi_\varrho(\chi_0) = \chi_0^1 \) is fulfilled. From \( 0 \leq f \leq \chi_0 \) then \( \Phi_\varrho(f) \in \chi_0^1 M_+ \chi_0^1 \) follows. Let us define \( k = \Phi_\varrho(f) \). Owing to idempotency of \( \Phi_\varrho \) then \( \Phi_\varrho \in \text{Aut}_{x+k}(M) \) follows. On the other hand, according to the above and since \( \chi_0 \in R[x] \) holds we certainly must have \( \Phi_\varrho \not\in \text{Aut}_{x}(M) \). In fact, otherwise according to the equivalence mentioned on in Remark 6, in contrast to the above we also had \( \chi_0 \) to be a fixed point of \( \Phi_\varrho \), a contradiction. Now, the Radon-Nikodym operator \( x = f \) by choice of \( f \) obeys \( \text{spec}_\varrho(x) = \{0\} \). Hence, \( \lambda_0 = 0 \). But then existence of the above constructed \( \Phi_\varrho \) proves that condition (3-12) is violated, and thus in view of COROLLARY 5 this means that \( \mathcal{R}_M(\nu, \varrho) \) cannot exist in the case at hand.

3.3.6. Does each minimizing subalgebra dominate a minimizing projective subalgebra?

Note that according to THEOREM 6 (1) and LEMMA 3 (1) existence of the least minimizing subalgebra especially also means that each minimizing subalgebra \( R \) possesses a minimizing projective subalgebra. One finds the following useful auxiliary characterization of this fact.

COROLLARY 6. Let \( \nu, \varrho \) be normal positive linear forms, with \( \varrho \ll \nu \) and Radon-Nikodym operator \( x \). Let \( R \) be a minimizing abelian \( \mathcal{W}^* \)-subalgebra, and be \( z \in R_+ \) the \( R \)-relative Radon-Nikodym operator achieving \( \varrho|_R = \nu|_R^z \). The following items are mutually equivalent:

1. \( R_1 \subset R \), for some minimizing projective subalgebra \( R_1 \);
2. \( \nu^{s(\varrho)}(z) = \nu(z) \).

In the latter case \( R_1 = R[x + k] \) can be chosen in (1), for some \( k \in s(\nu)^\perp M_+ s(\nu)^\perp \).

Proof. For a minimizing \( R \) the condition \( \nu^{s(\varrho)}(z) = \nu(z) \) implies existence of \( k \in s(\nu)^\perp M_+ s(\nu)^\perp \) with \( R[x + k] \subset R \). This can be seen exactly along the same way as demonstrated in the course of the proof of LEMMA 3 (2) (see from (3-7c) onward). In view of LEMMA 3 (1) then \( R_1 = R[x + k] \) can be chosen in (1). To see the other direction,
assume $R_1 \subset R$ with some minimizing projective subalgebra $R_1$. By Lemma 3 (2) one knows that $k \in s(\nu)^{-1}M_+s(\nu)^{-1}$ exists with $R[x+k] \subset R_1$. Then also $R[x+k] \subset R$ holds, and thus $x+k \in R$. Owing to $s(\varrho|_R) \in R$ and since $R$ is commutative, one has $y = s(\varrho|_R)(x+k) = (x+k)s(\varrho|_R) \in R_+$. From this and $\varrho = \nu^x = \nu^{(x+k)}$ then $\varrho|_R = \nu^{(x+k)}|_R = \nu^{y}|_R = \nu|_R$ is seen. In view of $s(y) \leq s(\varrho|_R)$ and by uniqueness of the Radon-Nikodym operator $z$ in $R$ then $z = y$ follows. Now, $s(\varrho|_R) \geq s(\varrho)$ and $s(x) = s(\varrho) \leq s(\nu)$ hold. Hence, $s(\varrho)y = s(\varrho)s(\varrho|_R)(x+k) = s(\varrho)(x+k) = x$ must be fulfilled, and therefore also $s(\varrho)z = zs(\varrho) = s(\varrho)zs(\varrho)$. Since $R$ is minimizing, from the previous together with Lemma 1 (put $p = s(\varrho)$ and $a = z$ in (3-4)) by literally the same arguments which led us to see (3-7c) within the proof of Lemma 3 (2) then the desired relation $\nu^{\delta(\varrho)}(z) = \nu(z)$ is seen to hold also in the situation at hand.

Remark 7. (1) The condition $s(\varrho) \in M^\nu$ within Theorem 6 (2) makes that Corollary 6 (2) is trivially satisfied, and then in line with Remark 4 (1) each minimizing subalgebra is projective.

(2) Suppose $\varrho \ll \nu$ but with $s(\varrho) \notin M^\nu$ (thus $M$ cannot be commutative). It is an open question whether other minimizing subalgebras than those respecting Corollary 6 (2) could exist at all.

(3) Note that $M = M_2(\mathbb{C})$ is the least case where the previous question might be non-trivial (cf. Example 9). But in this case, the characteristic configuration of a pair $\{\nu, \varrho\}$ to be dealt with for a decision in the usual canonical manner may be reduced to pairs $\{a, p\}$ of $2 \times 2$-matrices, with positive definite $a$ and one-dimensional orthoprojection $p$ obeying $pa \neq ap$. Thus calculations can be carried out explicitly (we omit the details), and in fact show that $R = R[x] = R[p]$ is the only minimizing subalgebra. This also completes the analysis of Example 9 in the $2 \times 2$-case: for $\nu, \varrho$ which are not mutually proportional and which obey $\varrho \ll \nu$ the least minimizing subalgebra exists iff $\nu$ is faithful. In view of Example 7 then even follows that, for a general pair of mutually non-proportional positive linear forms on $M = M_2(\mathbb{C})$, $\mathcal{R}_M(\nu, \varrho)$ exists if, and only if, one of the two forms is faithful at least. Thus, in this case we have a complete solution of the problem for a non-commutative $M$, even without imposing the condition $\varrho \ll \nu$.

(4) Suppose $\{\nu, \varrho\}$ such that Corollary 6 (2) be fulfilled in each case of a minimizing subalgebra. Then, the problem on existence of a least minimizing subalgebra will be reduced to the question whether or not $R_\infty(\nu, \varrho)$ were equal to $R[x + \lambda_0 s(\nu)^{-1}]$ (see Lemma 4 for
a special case). As Example 11 shows, for the latter to happen both (3-11c) and (3-12) are necessary conditions, which are rather independent from each other.

(5) The method by means of which the assertion on equality of the intersection algebra $R_{\infty}(\nu, \varrho)$ of (3-9a) to one of the intersecting minimizing subalgebras $R[x + \lambda_0 s(\nu)^{\perp}]$ has been disproved, and which is based on considering symmetries, seems to be very effective and in a modified form is a common method to disprove uniqueness of optimizing elements (algebras, decompositions, . . . ) in similar $^*$-algebraic optimization problems, see e.g. [28].

**Notes**

1. Most of the material of section 2 as well as some parts of section 3, especially 3.2, are reproduced from the part 'foundational material' of the manuscript [3].

**References**

1. P. M. Alberti. A note on the transition probability over $C^*$-algebras. *Lett. Math. Phys.*, 7:25–32, 1983.
2. P. M. Alberti. A Study on the Geometry of Pairs of Positive Linear Forms, Algebraic Transition Probability and Geometrical Phase over Non-Commutative Operator Algebras (I). *Z. Anal. Anw.*, 11(3):293–334, 1992.
3. P. M. Alberti. Bures Geometry in the State Space of von-Neumann Algebras. (unpublished manuscript), 1998.
4. P. M. Alberti and V. Heinemann. Bounds for the $C^*$-algebraic transition probability yield best lower and upper bounds to the overlap. *J. Math. Phys.*, 30(9):2083–2089, September 1989.
5. P. M. Alberti and A. Uhlmann. Transition probabilities on $C^*$- and $W^*$-algebras. In H. Baumgärtel, G. Laßner, A. Pietsch, and A. Uhlmann, editors, *Proceedings of the Second International Conference on Operator Algebras, Ideals, and Their Applications in Theoretical Physics, Leipzig 1983*, pages 5–11, Leipzig, 1984. BSB B.G. Teubner Verlagsgesellschaft. Teubner-Texte zur Mathematik, Bd.67.
6. H. Araki. A remark on Bures Distance Function for normal states. *Publ. RIMS (Kyoto)*, 6:477–482, 1970/71.
7. H. Araki. Bures distance function and a generalization of Sakai’s non-commutative Radon-Nikodym theorem. *Publ. RIMS (Kyoto)*, 8:335–362, 1972.
8. H. Araki and G. Raggio. A remark on transition probability. *Lett. Math. Phys.*, 6:237–240, 1982.
9. J. Bendat and S. Sherman. Monotone and convex operator functions. *Trans. Amer. Math. Soc.*, 79:58–71, 1955.
10. D. Buchholz. On quantum fields that generate local algebras. *J. Math. Phys.*, 31(8):1839–1846, August 1990.

11. D. Bures. An extension of Kakutani’s theorem on infinite product measures to the tensor product of semifinite $W^*$-algebras. *Trans. Amer. Math. Soc.*, 135:199–212, 1969.

12. V. Cantoni. Generalized transition probability. *Comm. Math. Phys.*, 44(2):125–128, 1975.

13. P. Dixmier. *Les $C^*$-Algèbres et leurs Représentations*. Gauthier-Villars, Paris, 1964.

14. W. Donogue. *Monotone matrix functions and analytic continuation*, volume 207 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-Heidelberg-NY, 1974.

15. H. Dye and B. Russo. A note on unitary operators in $C^*$-algebras. *Duke Math. J.*, 33:413–416, 1966.

16. S. Gudder. Expectation and transition probability. *Intern. J. Theor. Phys.*, 20:383–395, 1981.

17. S. P. Gudder. Cantoni’s Generalized Transition Probability. *Comm. Math. Phys.*, 63:265–267, 1978.

18. R. Haag. *Local Quantum Physics*. Texts and Monographs in Physics. Springer-Verlag, NY-Heidelberg-Berlin, 1992.

19. R. Kadison and J. Ringrose. *Fundamentals of the Theory of Operator Algebras*, volume I Elementary Theory. Academic Press, Inc., NY–London–Paris, 1983.

20. G. Mackey. *The Mathematical Foundation of Quantum Mechanics*. The Mathematicals Physics Monograph Series. W.A.Benjamin, Inc., New York–Amsterdam, 1963.

21. B. Mielnik. Theory of filters. *Comm. Math. Phys.*, 15(1):1–46, 1969.

22. S. Sakai. A Radon-Nikodym theorem in $W^*$-algebras. *Bull. Amer. Math. Soc.*, 71:149–151, 1965.

23. S. Sakai. *C$^*$-Algebras and $W^*$-Algebras*. Springer-Verlag, Berlin-Heidelberg-New York, 1971.

24. A. Uhlmann. An Introduction to the Algebraic Approach to some Problems of Theoretical Physics. In: *série des cours et conférences sur la physique des hautes énergies*, No. 3, 1974. Centre de Recherches Nucléaires, Strasbourg.

25. A. Uhlmann. The transition probability in the state space of a $^*$-algebra. *Rep. Math. Phys.*, 9:273–279, 1976.

26. A. Uhlmann. The transition probability for states of $^*$-algebras. *Annalen der Physik*, 42:524–531, 1985.

27. A. Uhlmann. Geometric phases and related structures. *Rep. Math. Phys.*, 36(2/3):461–481, 1995.

28. A. Uhlmann. Entropy and Optimal Decomposition of States Relative to a Maximal Commutative Subalgebra. *Open Sys. & Information Dyn.*, 5:209–227, 1998.

29. J. Yngvason. On the Algebra of Test Functions for Field Operators (Decomposition of linear Functionals into Positive Ones). *Comm. Math. Phys.*, 34:315–333, 1973.

Address for Offprints: Institute of Theoretical Physics, University of Leipzig, Augustusplatz 10, D-04109 Leipzig, Germany