The complementary polynomials and the Rodrigues operator. A distributional study

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Abstract

We can write the polynomial solution of the second order linear differential equation of hypergeometric-type

\[ \phi(x)y'' + \psi(x)y' + \lambda y = 0, \]

where \( \phi \) and \( \psi \) are polynomials, \( \deg \phi \leq 2 \), \( \deg \psi = 1 \) and \( \lambda \) is a constant, among others, by using the Rodrigues operator \( R_k(\phi, u) \) (see [3]) where \( u \) is certain linear operator which satisfies the distributional equation

\[ \frac{d}{dx}[\phi u] = \psi u, \quad (1) \]

as

\[ P_n(x) = B_n R_n(\phi, u)[1], \quad B_n \neq 0, \quad n = 0, 1, 2, \ldots \]

Taking this into account we construct the complementary polynomials. Among the key results is a generating functional function in closed form leading to derivations of recursion relations and addition theorem. The complementary polynomials satisfy a hypergeometric-type differential equation themselves, have a three-term recursion among others and Rodrigues formulas.

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1 Introduction

In this paper\textsuperscript{1} we will consider the polynomial solutions, $P_n$, of the second order linear differential equation of hypergeometric-type \textsuperscript{9}

$$\phi(x)P''_n(x) + \psi(x)P'_n(x) + \lambda_n P_n(x) = 0, \quad (2)$$

where $\phi$ and $\psi$ are polynomials, $\deg \phi \leq 2$, $\deg \psi \leq 1$, and

$$\lambda_n = -n\psi' - \frac{n}{2}(n - 1)\phi'', \quad n = 0, 1, 2, \ldots$$

In fact, it is well-known that such family of polynomials satisfies a property of orthogonality, i.e.

$$\langle u, P_n P_m \rangle = r_n \delta_{n,m}, \quad r_n \neq 0, \quad n, m = 0, 1, 2, \ldots$$

where $\delta_{n,m}$ is the Kronecker’s symbol and the linear functional $u$ satisfies the distributional equation

$$\frac{d}{dx}[\phi u] = \psi u. \quad (3)$$

Taking this into account is possible to write the such polynomials through the following Rodrigues functional formula:

$$P_n u = B_n \frac{d^n}{dx^n}[u_n], \quad B_n \neq 0, \quad n = 0, 1, 2, \ldots \quad (4)$$

where $u_0 := u, u_k := \phi u_{k-1}, k = 1, 2, 3, \ldots$.

Remark 1.1 Note that although is possible to find an integral representation for the linear functional $u$, since the polynomial solutions of such differential equation are the classical orthogonal polynomials (COP) \textsuperscript{9}, is important to take into account that the technique presented in this manuscript can be generalized to the semiclassical orthogonal polynomials where such integral representation is not known for each semiclassical functional (see, for example, \textsuperscript{6,2}).

Our first goal is to construct complementary polynomials for them by using their Rodrigues functional representation, Eq. (4), in a simple and natural way. The generating functional function of these complementary polynomials is obtained in closed form.

The paper is organized as follows. In the next section we introduce some notations and definitions useful for the next sections. In section 3 we construct the complementary polynomials. In Section 4 we establish their generating functional function. The second order linear differential equation associated to the complementary polynomials is derived in Section 5.

\textsuperscript{1} Note that this paper is a natural extension of \textsuperscript{10} in terms of linear functionals.
2 Preliminaries

In this section we will give a brief survey of the operational calculus that we will use in the rest of the paper.

Let \( P \) be the linear space of polynomial functions in \( \mathbb{C} \) (in the following we will refer to them as polynomials) with complex coefficients and \( P' \) be its algebraic dual space, i.e., \( P' \) is the linear space of all linear applications \( u : P \to \mathbb{C} \). In the following we will call the elements of \( P' \) as functionals.

Let \( \{ P_n \}_{n \geq 0} \) be a sequence of polynomials such that \( \deg P \leq n \) for all \( n \geq 0 \). A sequence defined in this way is said to be a basis or a basis sequence of \( P \) if and only if \( \deg P_n = n \) for all \( n \geq 0 \). Since the elements of \( P' \) are linear functionals, it is possible to determine them from their actions on a given basis \( \{ P_n \}_{n \geq 0} \) of \( P \).

In general, we will represent the action of a functional over a polynomial by

\[
\langle u, P \rangle, \quad u \in P', \quad P \in P.
\]

Therefore, a functional is completely determined by a sequence of complex numbers \( u_n := \langle u, x^n \rangle, n \geq 0 \), the so-called moments of the functional.

**Definition 2.1** Let \( u \in P' \) be a functional. We say that \( u \) is a quasi-definite (or regular) functional if and only if there exists a polynomial sequence \( \{ P_n \}_{n \geq 0} \), which is orthogonal with respect to \( u \).

Let us define the following operations in \( P' \). For any polynomial \( h \) and any \( c \in \mathbb{C} \), let \( u' := \frac{d}{dx} u, \ h u, \) and \( (x - c)^{-1} u \) be the linear functionals defined on \( P \) by (see [5][6])

\[
\begin{align*}
(1) & \quad \langle u', P \rangle := - \langle u, P' \rangle, \quad P \in P, \\
(2) & \quad \langle Qu, P \rangle := \langle u, Q P \rangle, \quad P, Q \in P, \\
(3) & \quad \langle (x - c)^{-1} u, P \rangle := \langle u, \theta_c(P) \rangle, \quad P \in P, \text{ where } \theta_c(P)(x) = \frac{P(x) - P(c)}{x - c}.
\end{align*}
\]

Furthermore, for any linear functional \( u \) and any polynomial \( P \) we get

\[
\frac{d}{dx} [Pu] := (Pu)' = Pu' + gu.
\]  

3 Complementary Polynomials

Before to introduce such polynomials let us introduce the Rodrigues operator.

**Definition 3.1** Given a polynomial \( \phi \) and a linear functional \( u \), such that there exists a polynomial \( \psi \) in such a way

\[
\frac{d}{dx} [\phi u] = \psi u.
\]
We define the $k$-th Rodrigues operator associated with the pair $(\phi, u)$, as:

$$
R_0(\phi, u) := I,
$$

$$
R_1(\phi, u)[p] = q \quad \text{where} \quad \frac{d}{dx}[pu_1] = q u,
$$

$$
R_k(\phi, u) := R_1(\phi, u) \circ R_{k-1}(\phi, u_1), \quad k = 2, 3, \ldots
$$

where $I$ represents the identity operator.

The pair $(\phi, u)$ satisfying the above condition we will call classical pair.

The following results are related with the Rodrigues operator.

**Lemma 3.1** [3, Lemma 4.2] Let $(\phi, u)$ be a classical pair, then for any positive integer $k$ and any polynomial $\pi$, the function

$$
R_1(\phi, u_k)[\pi]
$$

is a polynomial of degree $\deg(\pi) + 1$ with leading coefficient:

$$
-\frac{\lambda_{m+2k}}{m+2k} \neq 0, \quad k = 0, 1, 2, \ldots, \quad m = \deg \pi.
$$

**Remark 3.1** In the sequel we will denote by $R_{k, \ell}$ the $k$-th Rodrigues operator associated with the pair $(\phi, u_k)$, $R_k(\phi, u_\ell)$, and $R_k := R_{k, 1}$.

Let us now introduce the complementary polynomials, $P_\nu(x; n)$, defining them in terms of the Rodrigues operator.

**Definition 3.2** Let $n, \nu$ be two integers, with $0 \leq \nu \leq n$, the complementary polynomial of degree $\nu$ with respect to the $P_n$ is defined as

$$
P_n(x) = B_n P_{n-\nu}(x; n), \quad B_n \neq 0.
$$

**Remark 3.2** Note these polynomials for $\nu = 1$ are considered by M. Alfaro and R. Alvarez-Nodarse in [1]. More generally, it is easy to check that $P_{n-\nu}(x; n)$ is, up to a constant, the $\nu$-th derivative of $P_n$, i.e. $P_n^{(\nu)}(x)$ (see e.g. [3, Lemma 3.3]).

Taking into account this remark and the theory of COP, the following result holds.

**Lemma 3.2** $P_\nu(x; n)$ is a polynomial of degree $\nu$ that satisfies the recursive differential equation:

$$
P_{\nu+1}(x; n) = \phi(x) \frac{dP_\nu(x; n)}{dx} + \left(\psi(x) + (n - \nu - 1)\phi'(x)\right)P_\nu(x; n).
$$

By the Rodrigues functional formula (1), $P_0(x; n) \equiv 1$.

**Proof:** Equations (6); and (7) follow by induction. Case $\nu = 1$ is derived by carrying out explicitly the innermost differentiation in Eq. (1), which is a natural way of working with
the Rodrigues functional formula (4) that yields

\[ P_n(x)u = B_n \frac{d^{n-1}}{dx^{n-1}} \left[ \phi^n u \right] = B_n \frac{d^{n-1}}{dx^{n-1}} \left[ n \phi^{n-1} \phi' u + \phi^n u' \right]. \] (8)

showing, taking into account (3), that

\[ P_n(x) = B_n \mathcal{R}_{n-1}[(n-1)\phi'(x) + \psi(x)], \]

therefore

\[ \mathcal{P}_1(x; n) = (n-1)\phi'(x) + \psi(x). \] (9)

Assuming the validity of the Rodrigues functional formula (6) for \( \nu \) we carry out another differentiation in Eq. (6) obtaining

\[
\begin{align*}
P_n(x)u &= B_n \frac{d^{n-\nu-1}}{dx^{n-\nu-1}} \left\{ (n-\nu)\phi' \mathcal{P}_\nu(x; n) \phi^{n-\nu-1} u + \phi^{n-\nu} \mathcal{P}'_\nu(x; n) \right\} \\
&= B_n \frac{d^{n-\nu-1}}{dx^{n-\nu-1}} \left\{ (n-\nu)\phi' \mathcal{P}_\nu(x; n) (\psi - \phi') \mathcal{P}_\nu(x; n) + \phi \mathcal{P}'_\nu(x; n) \right\} \phi^{n-\nu-1} u \\
&= B_n \frac{d^{n-\nu-1}}{dx^{n-\nu-1}} \left( \mathcal{P}_{\nu+1}(x; n)u_{n-\nu-1} \right) .
\end{align*}
\] (10)

Comparing the right hand side of Eq. (9) proves Eq. (6) by induction along with the recursive differential equation (DE) (7).

\[ \square \]

In terms of a generalized Rodrigues functional representation we have

**Theorem 3.1** The polynomials \( \mathcal{P}_\nu(x; n) \) satisfy the Rodrigues functional formulas

\[
\begin{align*}
\mathcal{P}_\nu(x; n)u_{n-\nu} &= \frac{d^\nu}{dx^\nu} [u_n] \quad \Leftrightarrow \quad \mathcal{P}_\nu(x; n) = \mathcal{R}_{n-\nu}[1], \\
\mathcal{P}_\nu(x; n)u_{n-\nu} &= \frac{d^{\nu-\mu}}{dx^{\nu-\mu}} [\mathcal{P}_\mu(x; n)u_{n-\mu}] \quad \Leftrightarrow \quad \mathcal{P}_\nu(x; n) = \mathcal{R}_{n-\mu,n-\nu} \circ \mathcal{R}_{\mu,n-\mu}[1].
\end{align*}
\] (11) \hspace{1cm} (12)

Note that the proof of this result is straightforward taking into account the hypergeometric character of the DE (2) and Lemma 3.3 in [3].

4 generating Functional Function

**Definition 4.1** The generating functional function for the polynomials \( \mathcal{P}_\nu(x; n) \) is

\[ \mathcal{P}(y, x; n) = \sum_{\nu=0}^{\infty} \frac{y^\nu}{\nu!} \mathcal{P}_\nu(x; n). \] (13)

The series converges for \(|y| < \epsilon\) for some \( \epsilon > 0 \) and can be summed in closed form if the generating functional function is regular at the point \( x \).
Theorem 4.1  The generating functional function for the polynomials \( P_\nu(x; n) \) is given in closed form by

\[
P(y, x; n) u = \left( \frac{\phi(x + y\phi(x))}{\phi(x)} \right)^n \tilde{u}, \quad (14)
\]

where \( \tilde{u} = \exp(y\phi(x) \frac{d}{dx}) \circ u \).

**Proof:** Taking into account that \( u_k = \phi^k u \), equation (14) follows by multiplying the generatriz function by \( \phi^n \) and substituting the Rodrigues functional representation, Eq. (11) in Eq. (13) which yields

\[
\phi^n(x) P(y, x; n) u = \sum_{\nu=0}^{\infty} \frac{y^\nu}{\nu!} P_\nu(x; n) \phi^n(x) u = \sum_{\nu=0}^{\infty} \frac{(y\phi(x))^\nu}{\nu!} \frac{d^\nu}{dx^\nu} [u_n],
\]

converging for \(|y\phi(x)| < \epsilon\) for a suitable \( \epsilon > 0 \) if \( x \) is a regular point of the generating functional function.

In fact, at this point the series can be summed exactly, because the expression inside the derivatives is independent of the summation, obtaining

\[
\phi^n(x) P(y, x; n) u = \exp (y\phi(x) \frac{d}{dx}) [\phi^n(x) u] = \phi^n(x + y\phi(x)) \tilde{u}, \quad (16)
\]

and therefore

\[
P(y, x; n) u = \left( \frac{\phi(x + y\phi(x))}{\phi(x)} \right)^n \tilde{u}. \quad (17)
\]

Differentiating Eq. (13) and substituting the generalized Rodrigues functional formula (12) in this yields Eq. (15) similarly.

□

Remark 4.1  Note that if the linear \( u \) admits an integral representation as

\[
\langle u, P \rangle = \int_{\Omega} P(z) \rho(z) dz,
\]

where \( \Omega \) is certain contour in the complex plane, then we can rewrite (17) as

\[
P(y, x; n) = \left( \frac{\phi(x + y\phi(x))}{\phi(x)} \right)^n \frac{\rho(x + y\phi(x))}{\rho(x)}. \quad (18)
\]

Since we are going to consider recurrence relations we are going to translate the case \( \mu = 1 \) of Eq. (15) into partial differential equations (PDEs).

Theorem 4.2  The generating functional function satisfies the PDEs

\[
(1 + y\phi(x) + \frac{1}{2} y^2 \phi''(x)) \frac{\partial P(y, x; n)}{\partial y} = \left[ P_1(x; n) + y\phi(x) P_1'(x; n) \right] P(y, x; n), \quad (19)
\]
\[
\frac{\partial P(y, x; n)}{\partial y} = [(n - 1)\phi'(x + y\phi(x)) + \psi(x + y\phi(x))] P(y, x; n - 1), \quad (20)
\]

\[
(1 + y\phi'(x) + \frac{1}{2}y^2\phi''(x)) \frac{\partial P(y, x; n)}{\partial x} = \left\{ (1 + y\phi'(x)) P'(x; n) - \frac{1}{2}y\phi'' P_1(x; n) \right\} y P(y, x; n), \quad (21)
\]

\[
\phi(x) \frac{\partial P(y, x; n)}{\partial x} = (1 + y\phi'(x))[\psi(x) + (n - 1)\phi'(x) + y\phi(x)(\psi' + (n - 1)\phi'')] \times P(y, x; n - 1) - [\psi(x) + (n - 1)\phi'(x)] P(y, x; n). \quad (22)
\]

**Proof:** From Eq. (15) for \(\mu = 1\) in conjunction with Eq. (14) and taking into account that \(u\) is regular we obtain

\[
\phi(x + y\phi(x)) \frac{\partial P(y, x; n)}{\partial y} = \phi(x)[\psi(x + y\phi(x)) + (n - 1)\phi'(x + y\phi(x))] P(y, x; n). \quad (23)
\]

Substituting in Eq. (23) the Taylor series-type expansions

\[
\begin{align*}
\phi(x + y\phi(x)) &= \phi(x)(1 + y\phi'(x) + \frac{1}{2}y^2\phi''(x)), \\
\phi'(x + y\phi(x)) &= \phi'(x) + y\phi''(x), \\
\psi(x + y\phi(x)) &= \psi(x) + y\psi'(x)\phi(x),
\end{align*}
\]

since \(\phi\) and \(\psi\) are polynomials of degree, at most, 2 and 1, respectively, we verify Eq. (19). Using the exponent \(n - 1\) instead of \(n\) of the generating functional function we can obtain Eq. (20).

By differentiation of the generating functional function, Eq. (16), with respect to the variable \(x\) we find Eq. (22). Using the exponent \(n\) instead of \(n - 1\) of the generating functional function in conjunction with Eq. (22) leads to Eq. (21) where in any case we were assume the linear functional is regular. \(\square\)

In fact the polynomials \(P_\nu(x; n)\) satisfy different recursions and other relations which are the same that H. J. Weber obtained in [10] hence we will omit here.

5 The second order linear differential equation of hypergeometric-type

**Theorem 5.1** The polynomials \(P_\nu(x; n)\) satisfy the Sturm-Liouville differential equation

\[
\frac{d}{dx} \left( \frac{dP_\nu(x; n)}{dx} u_{n-\nu+1} \right) = -\mu_{n, \nu} P_\nu(x; n) u_{n-\nu}, \quad (25)
\]


which is equivalent to, assuming \( u \) is regular,

\[
\phi(x) \frac{d^2 \mathcal{P}_\nu(x; n)}{dx^2} + [(n - \nu)\phi'(x) + \psi(x)] \frac{d \mathcal{P}_\nu(x; n)}{dx} = -\mu_{n,\nu} \mathcal{P}_\nu(x; n),
\]

(26)

and the eigenvalues are given by

\[
\mu_{n,\nu} = -\nu((n - \frac{\nu+1}{2})\phi'' + \psi'), \quad \nu = 0, 1, \ldots
\]

(27)

Taking into account the hypergeometric character of the DE (1), and Theorem 3.1 this result is very well-known and straightforward.

6 Conclusions

We have used a natural way of working with the Rodrigues operator and the Rodrigues functional formula of a given set polynomials solutions of the second order differential equation (1) which leads to a set of closely related complementary polynomials that satisfy their own Rodrigues formulas, always have a generating functional function that can be summed in closed form leading to numerous recursion relations and addition theorems.

Furthermore, since the differential equation (1) is of hypergeometric-type, the complementary polynomials satisfy a second order linear differential equation similar to that of the original polynomials.

In fact, it is a straightforward calculation to verify that this method generates all the basics of the COP, \( \Delta \)-classical OP, and \( q \)-polynomials (see, for example, [3],[7],[8] and references therein).

Moreover this method, considered in a similar way by H. J. Weber, is not restricted to the COP and can be applied to semiclassical orthogonal polynomials.

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