Finitely generated subgroups of branch groups and subdirect products of just infinite groups

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Abstract. The aim of this paper is to describe the structure of finitely generated subgroups of a family of branch groups containing the first Grigorchuk group and the Gupta–Sidki 3-group. We then use this to show that all the groups in this family are subgroup separable (LERF).

These results are obtained as a corollary of a more general structural statement on subdirect products of just infinite groups.

Keywords: just infinite groups, subdirect products, branch groups.

In memory of Sergei Ivanovich Adian

§ 1. Introduction

A group is branch if it acts faithfully on a spherically homogeneous rooted tree and has lattice of subnormal subgroups similar to the structure of the tree [1]–[3]. A group $G$ is self-similar if it has a faithful action on a $d$-regular rooted tree, $d \geq 2$, such that any section of any element $g \in G$ is again an element of the group modulo the canonical identification of the subtree and the original tree. Just infinite branch groups constitute one of three classes of just infinite groups (infinite groups all of whose proper quotients are finite) [3]. Self-similar groups appear naturally in holomorphic dynamics [4]. Both classes are also important in many other areas of mathematics. Although quite different, these two classes of groups have a large intersection, and many self-similar groups are branch. In the class of finitely generated branch self-similar groups, there are torsion groups and torsion-free groups; groups of intermediate growth and groups of exponential growth; non-elementary amenable and non-amenable groups. Branch self-similar groups have a very interesting subgroup structure. Precise definitions, more details, and related references can be found in [5], [4].

Among the most important examples of branch self-similar groups is the 3-generated 2-group $\mathfrak{G}$ of intermediate growth [6] (known as the Grigorchuk group) which...
is defined by its action on a rooted binary tree $T$. See [7] for an introduction to this
group and [8] for detailed information and a list of open problems about it. Much
is known about subgroups of $\mathfrak{G}$, in particular, about the stabilizers of the vertices
of $T$ and points on its boundary; the rigid stabilizers; the centralizers; certain
subgroups of small index [9], [10]; maximal subgroups [11] as well as weakly maximal
subgroups (subgroups of infinite index that are maximal with this property) [12].

In [13] the first- and third-named authors announced a structural result for the
finitely generated subgroups of $\mathfrak{G}$ in terms of the notion of block subgroups; see
Definition 2.5 and Example 2.6 illustrated in Fig. 1.

![Figure 1. A block subgroup of the first Grigorchuk group](image)

The main purpose of this article is to prove a weak version of this result in
a more general context, which applies in particular to the Gupta–Sidki 3-group $G_3$;
see Theorem 2.9. We will also derive some important consequences of it, notably
that the groups in question are subgroup separable; see Theorem 2.11. The desired
result for branch groups will follow from a more general result of independent
interest about subdirect products of just infinite groups; see Theorem 2.15.

The article is organized as follows. The next section contains definitions and
principal results. It is divided into two subsections, of which the first deals with
branch groups and the second with general subdirect products. §3 contains proofs
of general statements on subdirect products and notably of Theorem 2.15. Finally, §4
is devoted to proving results concerning branch groups, in particular, Theorem 2.9.

## §2. Definitions and results

### 2.1. Branch groups.
The main goal of this research is to understand the struc-
ture of subgroups that are closed in the profinite topology of the first Grigorchuk
group $\mathfrak{G}$, the Gupta–Sidki 3-group $G_3$, and some other branch groups (observe
that branch groups are residually finite). One class of such subgroups consists of
finitely generated subgroups, as proven in [14] and [15]. It was shown there that
every infinite finitely generated subgroup of $\mathfrak{G}$ (resp. $G_3$) is commensurable\(^1\) with $\mathfrak{G}$
(resp. $G_3$). This unusual property relies on the fundamental result of Pervova [11]
that every maximal subgroup of $\mathfrak{G}$ (resp. of the Gupta–Sidki $p$-group $G_p$) has finite
index, which is thus equal to 2 (resp. $p$). For strongly self-replicating (see Defini-
tion 2.1) just infinite groups with the congruence subgroup property, the property
of having all maximal subgroups of finite index is preserved when passing to com-
mensurable groups ([14], Lemma 4). Therefore all weakly maximal subgroups of $\mathfrak{G}$

\[^1\text{Recall that two abstract groups } G_1 \text{ and } G_2 \text{ are (abstractly) commensurable if there are sub-}
\text{groups } H_1 \text{ of finite index in } G_1 \text{ such that } H_1 \text{ and } H_2 \text{ are isomorphic. In particular, if } H \text{ is a subgroup of finite index in } G, \text{ then } H \text{ is commensurable with } G.\]
(resp. \(G_p\)) are closed in the profinite topology. For a branch group \(G\), the stabilizers of points on the boundary of the tree are examples of weakly maximal subgroups \cite{16}, but there are many more; see \cite{17}. See \cite{12} for a description of all weakly maximal subgroups of \(\mathcal{G}\) and \(G_p\).

Let \(T\) be a \(d\)-regular rooted tree. This is a connected graph without cycles, with a distinguished vertex of degree \(d\) (the root) such that all other vertices are of degree \(d + 1\). There is a natural bijection between the set of vertices of \(T\) and the free monoid \(\{0, \ldots, d-1\}^*\), which is the set of finite words in the alphabet \(\{0, \ldots, d-1\}\). Under this identification, the root corresponds to the empty word \(\emptyset\). A natural partial order on the vertices of \(T\) is defined by letting \(v \leq w\) if \(v\) is a prefix of \(w\) or, equivalently, if the unique path without backtracking from the root to \(w\) hits \(v\) before \(w\). The \(n\)th level \(L_n\) of \(T\) is the set of all vertices at distance \(n\) from the root or, equivalently, the set of all words of length \(n\). Given a vertex \(v\) of \(T\), we write \(T_v\) for the subtree of \(T\) consisting of all vertices \(w \geq v\). It is naturally rooted at \(v\). By \(\text{Aut}(T)\) we denote the automorphism group of \(T\), that is, the set of all graph isomorphisms from \(T\) to itself that preserve the root. Equivalently, \(\text{Aut}(T)\) can be seen as the set of bijections of the free monoid \(\{0, \ldots, d-1\}^*\) that preserve lengths and prefixes.\(^{2}\)

Let \(G\) be a subgroup of \(\text{Aut}(T)\). We write \(\text{Stab}_G(v)\) for the stabilizer of the vertex \(v\) and \(\text{Stab}_G(L_n)\) for the \(n\)th level stabilizer, that is, the pointwise stabilizer

\[
\text{Stab}_G(L_n) := \bigcap_{v \in L_n} \text{Stab}_G(v).
\]

More generally, if \(X\) is a subset of vertices of \(T\), the group \(\text{Stab}_G(X)\) is its pointwise stabilizer. Other important subgroups of \(G\) are the rigid stabilizer of the vertex \(v\),

\[
\text{Rist}_G(v) := \bigcap_{w \notin T_v} \text{Stab}_G(w),
\]

which consists of the elements acting trivially outside \(T_v\), and the rigid stabilizer of the \(n\)th level,

\[
\text{Rist}_G(n) := \prod_{v \in L_n} \text{Rist}_G(v) = (\text{Rist}_G(v) \mid v \in L_n).
\]

For every vertex \(v\) of \(T\) we have a natural homomorphism

\[
\varphi_v : \text{Stab}_{\text{Aut}(T)}(v) \rightarrow \text{Aut}(T_v),
\]

where \(\varphi_v(g) = g|_{T_v}\) is simply the restriction of \(g\), that is, the action of \(g\) on \(T_v\). The element \(\varphi_v(g)\) is called the section of \(g\) at \(v\). Given a subgroup \(G\) of \(\text{Aut}(T)\), we will sometimes refer to \(\varphi_v(\text{Stab}_G(v))\) as the section of \(G\) at \(v\) and denote it by \(\varphi_v(G)\). It is trivial that the restriction of \(\varphi_v\) to \(\text{Rist}_G(v)\) is an isomorphism onto its image.

\(^{2}\)A bijection \(\varphi\) of \(\{0, \ldots, d-1\}^*\) preserves prefixes if for any words \(u, v\) there is a word \(v'\) such that \(\varphi(uv) = \varphi(u)v'\).
Definition 2.1. A subgroup $G$ of $\text{Aut}(T)$ is **self-similar** if, for every vertex $v$, its section $\varphi_v(G)$ is a subgroup of $G$ under the natural identification of $T_v$ with $T$. It is **self-replicating** (or fractal) if, for every vertex $v$, its section $\varphi_v(G)$ is equal to $G$ under the natural identification of $T_v$ with $T$. Following [18], we say that $G$ is **strongly self-replicating** if $\varphi_v(\text{Stab}_G(\mathcal{L}_n)) = G$ for every vertex $v$ of level $n$.

Definition 2.2. A subgroup $G$ of $\text{Aut}(T)$ is **weakly branch** if it acts transitively on $\mathcal{L}_n$ for all $n$ and all the rigid stabilizers $\text{Rist}_G(v)$ are infinite (or, equivalently, they are all non-trivial). The group $G$ is **branch** if it acts transitively on $\mathcal{L}_n$ for all $n$ and all the rigid stabilizers $\text{Rist}_G(n)$ are of finite index in $G$. Finally, $G$ is **regular branch over a subgroup $K$** if it acts transitively on $\mathcal{L}_n$ for all $n$ and is self-replicating. $K$ is of finite index and, for every vertex $v$ of the first level, $K$ is a subgroup of finite index in $\varphi_v(\text{Stab}_K(\mathcal{L}_1))$.

Regularly branch groups are branch, and branch groups are weakly branch. A branch group $G$ has the **congruence subgroup property** if, for every subgroup $H$ of finite index, there is an $n$ such that $H$ contains the $n$th level stabilizer $\text{Stab}_G(\mathcal{L}_n)$. This is equivalent to the coincidence of the profinite topology on $G$ with the restriction to $G$ of the natural topology of $\text{Aut}(T)$.

The **first Grigorchuk group** $\mathfrak{G}$, which acts on the 2-regular rooted tree, is probably the best known and most studied branch group. It was also the first example of a group of intermediate growth [19]. Among other properties, $\mathfrak{G}$ is self-replicating, regularly branch and just infinite [21], and possesses the congruence subgroup property. See [7], [8] for a formal definition, references and further details.

Other well-studied examples of branch groups are the **Gupta–Sidki $p$-groups** $G_p$ acting on the $p$-regular rooted tree, where $p \geq 3$ is a prime [20]. They are also self-replicating, regularly branch and just infinite [21], and possess the congruence subgroup property [22]. On the other hand, the Gupta–Sidki groups have infinite width of associated Lie algebras and, in contrast to $\mathfrak{G}$, it is unknown whether they are of intermediate growth. See [20], [21], [15] for a formal definition, references and further details.

We now introduce a less standard terminology to be used in what follows.

We say that vertices $u$ and $v$ of $T$ are **orthogonal** if the subtrees $T_u$ and $T_v$ are disjoint, that is, we simultaneously have $v \nparallel w$ and $w \nparallel v$. A set $U$ of vertices is **orthogonal** if it consists of pairwise-orthogonal vertices. It is called a **transversal** if every infinite geodesic ray from the root of the tree intersects $U$ at a single point. It is clear that a transversal is a finite set. Sets $U$ and $V$ of vertices are **orthogonal** if every vertex of one is orthogonal to every vertex of the other.

Definition 2.3. Let $U = (u_1, \ldots, u_k)$ be an ordered orthogonal set. Suppose that $G \leq \text{Aut}(T)$, $L$ is an abstract group, and there is a family $(L_j)^{k}_{j=1}$ of finite-index subgroups of $\varphi_{u_j}(\text{Rist}_G(u_j))$ that are all isomorphic to $L$. Let $\Psi = (\psi_1, \ldots, \psi_k)$ be a $k$-tuple of isomorphisms $\psi_j : L \rightarrow L_j$. Then the quadruple $(U, L, (L_j)^{k}_{j=1}, \Psi)$ determines a **diagonal subgroup** in $G$,

$$D := \left\{ g \in \prod_{j=1}^{k} \text{Rist}_G(u_j) \mid \exists l \in L, \forall j : \varphi_{u_j}(g) = \psi_j(l) \right\},$$

which is abstractly isomorphic to $L$. We say that $U$ is the support of $D$. 
Observe that in the degenerate case when $U = \{u\}$ is a singleton, the diagonal subgroups of $G$ supported on $\{u\}$ are exactly the subgroups of finite index in $\text{Rist}_G(u)$.

**Example 2.4.** The first Grigorchuk group $G = \langle a, b, c, d \rangle$ is regular branch over $K = \langle abab \rangle^\ast = \langle abab, badabada, abadabada \rangle$, which is a normal subgroup of index 16. The diagonal subgroup $D$ shown in Fig. 2 is determined by the group $L := K$ (regarded as an abstract group), the set $U = \{000, 01, 10\}$, the three copies of $K$ lying in $\text{Aut}(T_{000})$, $\text{Aut}(T_{01})$ and $\text{Aut}(T_{10})$ respectively, and the isomorphisms $\psi_i$, $i = 1, 2, 3$, given by conjugation by $a, b$ and $c$. Thus, $\psi_1: K \to K \leq \text{Aut}(T_{000})$ is defined by $\psi_1(g) = g^a$, whence the notation $K^a$ in Fig. 2, and similarly for $\psi_2$ and $\psi_3$. Observe that this subgroup $D$ is ‘purely’ diagonal in the sense that each factor $K$ coincides with $\varphi_{u_j}(\text{Rist}_G(u_j))$ instead of being only a subgroup of finite index.

![Diagram](image)

**Figure 2. A diagonal subgroup of $G$**

**Definition 2.5.** Suppose that $G \leq \text{Aut}(T)$. A **block subgroup** of $G$ is a finite product $A = \prod_{i=1}^n D_i$ of diagonal subgroups $D_i$ whose supports are pairwise orthogonal.

Observe that since the supports of the subgroups $D_i$ are pairwise orthogonal, we have $D_i \cap D_j = \{1\}$ if $i \neq j$ and $\prod_{i=1}^n D_i = \langle D_1, \ldots, D_n \rangle \leq G$.

**Example 2.6.** A picture describing a specific block subgroup of $G$ is given in Fig. 1. Recall that $G = \langle a, b, c, d \rangle$ is regular branch over the normal subgroup $K = \langle abab \rangle^\ast$ and we have $K \leq B = \langle b \rangle^\ast < G$ with $[B: K] = 2$ and $[G : B] = 8$. Moreover, the section $\varphi_v(\text{Rist}_G(v))$ is equal to $B$ if $v \in L_1$, and to $K$ otherwise.

Here is a detailed explanation of this example. We have $U_1 = \{1\}$ and $D_1 = \text{Rist}_G(1) = \{1\} \times B$. On the other hand, $U_2 = \{000, 001\}$, $L_1 = K \leq \text{Aut}(T_{000})$ and $L_2 = K \leq \text{Aut}(T_{001})$ with the isomorphisms $\psi_1 = \text{id}: K \to L_1$ and $\psi_2 = a^a: K \to L_2$ (conjugation by $a$). This gives us

$$D_2 = \{(g, aga, 1, 1, 1, 1, 1, 1) \in \text{Rist}_G(000) \times \text{Rist}_G(001) \mid g \in K\}.$$

Finally, the block subgroup depicted in Fig. 1 is the product of $D_1$ and $D_2$ (these two subgroups have trivial intersection).

It follows from the definition that if $G$ is finitely generated and $A$ is a block subgroup of $G$, then $A$ is finitely generated, and so is every subgroup $H$ with $A \leq H \leq G$ and $[H : A] < \infty$. We shall show that the converse also holds under certain conditions. To do this, we need more definitions.
Definition 2.7. Let $G \leq \text{Aut}(T)$ be a self-similar group. A family $\mathcal{X}$ of subgroups of $G$ is said to be inductive if the following conditions hold.

I) The subgroups $\{1\}$ and $G$ belong to $\mathcal{X}$.

II) Let $H \leq L$ be subgroups of $G$ such that $[L : H]$ is finite. Then $L$ is in $\mathcal{X}$ if and only if $H$ is in $\mathcal{X}$.

III) If $H$ is a finitely generated subgroup of $\text{Stab}_G(1)$ and all first-level sections of $H$ are in $\mathcal{X}$, then $H \in \mathcal{X}$.

Definition 2.8. A self-similar group $G$ possesses the subgroup induction property if, for any inductive class $\mathcal{X}$ of its subgroups, all finitely generated subgroups of $G$ belong to $\mathcal{X}$.

We can now state our main result.

Theorem 2.9. Let $G$ be a finitely generated self-similar branch group. Then the following assertions are equivalent.

1) The group $G$ possesses the subgroup induction property.

2) A subgroup $H$ of $G$ is finitely generated if and only if it contains a block subgroup $A$ with $[H : A] < \infty$.

3) A subgroup $H$ of $G$ is finitely generated if and only if there is an $n$ such that, for every $v \in \mathcal{L}_n$, the section $\varphi_v(\text{Stab}_H(\mathcal{L}_n))$ is either trivial or of finite index in $\varphi_v(\text{Stab}_G(v))$.

The proof of Theorem 2.9 will be given at the end of §4.

It is known that the first Grigorchuk group ([14], Theorem 3) and torsion GGS-groups [23] possess the subgroup induction property. Hence Theorem 2.9 yields the following corollary.

Corollary 2.10. Let $G$ be either the first Grigorchuk group or a torsion GGS-group. Then a subgroup $H$ of $G$ is finitely generated if and only if it contains a block subgroup $A$ with $[H : A] < \infty$.

The property of a group to have all finitely generated subgroups closed in the profinite topology is quite rare. Such groups are said to be subgroup separable (or LERF-groups, which stands for locally extensively residually finite). This property holds in particular for finite groups, finitely generated Abelian groups, finitely generated free groups [24], surface groups [25], amalgamated product of two free groups over a cyclic subgroup [26] and, more generally, limit groups [27]. It was proved in [28] that a subset of a free group which is the product of finitely many finitely generated subgroups is closed in the profinite topology. This remarkable property is known (in the finitely generated case) only for free groups. We conjecture that it holds for every subset of $\mathcal{G}$ (or $G_3$). Our result may be regarded as positive evidence for this conjecture. Indeed, if $G \leq \text{Aut}(T)$ is a branch group with the congruence subgroup property, then every block subgroup of $G$ is closed in the profinite topology [12]. Together with Theorem 2.9, this yields the following assertion.

Theorem 2.11. Let $G$ be a finitely generated self-similar branch group with the congruence subgroup property. If $G$ possesses the subgroup induction property, then it is subgroup separable.
As a corollary, we obtain another proof of the fact that both $G$ [14] and $G_3$ [15] are subgroup separable.

The first Grigorchuk group [14] and the Gupta–Sidki 3-group [15] were the only groups known to possess the subgroup induction property until the very recent work [23] of the second-named author and Francoeur generalizing the results of [15] to the Gupta–Sidki $p$-groups with any prime $p \geq 3$ and, more generally, to torsion GGS-groups acting on the $p$-regular rooted tree.

2.2. Subdirect products. A subgroup $H \leq \prod_{i \in I} G_i$ that projects onto each factor is called a subdirect product, and we write $H \leq_s \prod_{i \in I} G_i$.

Subdirect products have been used, for example, to represent many small perfect groups [29]. On the other hand, understanding subdirect products is the first step in the comprehension of general subgroups of direct products and is therefore of interest in itself.

In this context, diagonal subgroups play an important role. We need to introduce the notion of a diagonal subgroup in the abstract setting of a direct product in a manner which is coherent with the notion of a diagonal subgroup introduced in Definition 2.3.

Definition 2.12. Let $I$ be a set and let $(G_i)_{i \in I}$ be a family of groups which are all isomorphic to a common group $G$. We say that a subgroup $H$ of $\prod_{i \in I} G_i$ is diagonal if there are isomorphisms $\psi_i : G \to G_i$ such that

$$H = \text{diag}\left(\prod_{i \in I} \psi_i(G)\right) := \{(\psi_i(g))_{i \in I} \mid g \in G\}.$$ 

It follows directly from this definition that diagonal subgroups of $\prod_{i \in I} G_i$ are subdirect products.

It will be shown in §3 that if $G$ is just infinite, then the diagonal subgroups and $G \times G$ itself are basically the only examples of subdirect products in $G \times G$.

Lemma 2.13. Suppose that $G_1$ and $G_2$ are just infinite and $H \leq_s G_1 \times G_2$. Then either there are subgroups $C_i \leq G_i$ of finite index such that $H$ contains $C_1 \times C_2$, or $G_1$ and $G_2$ are isomorphic and $H = \text{diag}(G_1 \times \psi(G_1))$ for some isomorphism $\psi : G_1 \to G_2$.

For products of just infinite groups with more than two factors, full products and diagonal subgroups are the building blocks of subdirect products. We need one last definition before stating our result.

Definition 2.14. Let $(G_i)_{i \in I}$ be a family of groups indexed by a set $I$. A subgroup $H$ of $\prod_{i \in I} G_i$ is virtually diagonal by blocks if there are a set $\Delta$, a family $(G_\alpha)_{\alpha \in \Delta}$ of abstract groups and a family $(D_\alpha)_{\alpha \in \Delta}$ of subgroups of $\prod_{i \in I_\alpha} G_i$ such that $I$ is partitioned into $I = \bigsqcup_{\alpha \in \Delta} I_\alpha$ with the following properties.

1) For every $i$ in $I_\alpha$, the group $G_i$ is isomorphic to $G_\alpha$.

2) The subgroup $D_\alpha$ is a diagonal subgroup of $\prod_{i \in I_\alpha} G_i$ for some subgroup $L_\alpha$ of finite index in $G_\alpha$.

3) $H$ contains $\prod_{\alpha \in \Delta} D_\alpha$ as a subgroup of finite index.
Observe that it follows from the definition that if \( I_\alpha \) is a singleton, then \( D_\alpha \) is a subgroup of finite index in \( G_\alpha \).

We can now state our main result about subdirect products.

**Theorem 2.15.** Let \( G_i, 1 \leq i \leq n \), be just infinite groups. Suppose that at most two of them are virtually Abelian. Then all the subdirect products in \( \prod_{i=1}^n G_i \) are virtually diagonal by blocks.

The restriction on the number of virtually Abelian factors is necessary, as demonstrated by Example 3.7. On the other hand, it is not too restrictive in the sense that nearly all just infinite groups (and, in particular, all torsion just infinite groups) are not virtually Abelian. Indeed, McCarthy [30] showed that every just infinite group \( G \) contains a maximal Abelian normal subgroup (either trivial or of finite index) which is equal to its Fitting subgroup\(^3\) \( \Psi(G) \) and is isomorphic to \( \mathbb{Z}^n \) for some integer \( n \). Such a group \( G \) is not virtually Abelian if and only if \( \Psi(G) = \{1\} \), which corresponds to the case \( n = 0 \). Moreover, for every \( n > 0 \) there are only finitely many non-isomorphic just infinite groups with \( \Psi(G) = \mathbb{Z}^n \) ([30], Proposition 9). Finally, (weakly) branch groups are never virtually Abelian ([32], Proposition 10.4).

If the groups \( G_i \) in Theorem 2.15 are pairwise non-isomorphic, then all the sets \( I_\alpha \) are singletons and there are subgroups \( L_i \leq G_i \) of finite index such that \( H \) contains \( \prod_{i=1}^n L_i \). This immediately yields the following result on the rigidity of subdirect products of non-isomorphic groups.

**Corollary 2.16.** Let \( G_i, 1 \leq i \leq n \), be pairwise non-isomorphic just infinite groups. Suppose that at most two of them are virtually Abelian. Then every subdirect product of \( \prod_{i=1}^n G_i \) is a subgroup of finite index.

### §3. Block structure in products of just infinite groups

An important tool for the study of subdirect products is the notion of a fibre product. Recall that if \( \psi_1 : G_1 \twoheadrightarrow D \) and \( \psi_2 : G_2 \twoheadrightarrow D \) are group epimorphisms, their direct product\(^4\) is the subgroup \( C := \{(g_1, g_2) \in G_1 \times G_2 \mid \psi_1(g_1) = \psi_2(g_2)\} \) of \( G_1 \times G_2 \). Every fibre product is naturally isomorphic to a subdirect product. The converse also holds. More precisely, applying Goursat’s lemma [33] to subdirect products, we obtain the following assertion.

**Lemma 3.1.** If \( H \leq G_1 \times G_2 \) is a subdirect product, then there are epimorphisms \( \psi_1 : G_1 \twoheadrightarrow D \) and \( \psi_2 : G_2 \twoheadrightarrow D \) onto a quotient \( D \) of \( H \) such that \( H \) is the fibre product of \( \psi_1 \) and \( \psi_2 \).

This result enables us to prove Lemma 2.13.

**Proof of Lemma 2.13.** By Lemma 3.1, \( H \) is the fibre product of \( \psi_1 : G_1 \twoheadrightarrow D \) and \( \psi_2 : G_2 \twoheadrightarrow D \), where \( D \) is a quotient of \( G \). It follows directly from the definition of

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\(^3\)The Fitting subgroup \( \Psi(G) \) is the subgroup generated by the nilpotent normal subgroups of \( G \). If any ascending chain of normal subgroups of \( G \) stabilizes after a finite number of steps, then \( \Psi(G) \) is the unique maximal normal nilpotent subgroup of \( G \). Indeed, in this case there are nilpotent normal subgroups \( K_1, \ldots, K_n \) of \( G \) such that \( \Psi(G) = \langle K_1, \ldots, K_n \rangle \), which is a normal nilpotent subgroup of \( G \) by Fitting’s theorem [31].

\(^4\)This is the categorical fibre product, also called the pullback, of \( \psi_1 \) and \( \psi_2 \) in the category of groups.
a fibre product that $H$ contains the subgroup $\ker(\psi_1) \times \ker(\psi_2)$. If $D$ is finite, then $\ker(\psi_1)$ and $\ker(\psi_2)$ are subgroups of finite index in $G_1$ and $G_2$ respectively, and the conclusion of the lemma holds. Suppose that $D$ is not finite. Since $G_1$ and $G_2$ are just infinite, it follows that the subgroups $\ker(\psi_1)$ and $\ker(\psi_2)$ are trivial. Hence $D$ is isomorphic to both $G_1$ and $G_2$. But in this case $\psi_1$ and $\psi_2$ are isomorphisms and we have

$$H = \{(g, h) \in G_1 \times G_2 \mid \psi_1(g) = \psi_2(h)\}$$

$$= \{(g, h) \in G_1 \times G_2 \mid h = \psi_2^{-1} \circ \psi_1(g)\} = \text{diag}(G_1 \times \psi_2^{-1} \circ \psi_1(G_1)).$$

\[\square\]

**Remark 3.2.** The conclusion of Lemma 2.13 is optimal in the sense that there is a subdirect product $C \leq G \times G$ such that $G$ is neither diagonal nor equal to $G \times G$. Indeed, let $G$ be any group that has two distinct subgroups of index 2, one of which is characteristic.\(^{5}\) Then there is a subdirect product $C \leq G \times G$ that is neither a diagonal subgroup nor $G \times G$.

More precisely, for any characteristic subgroup $H$ of index two and any subgroup $J \neq H$ of index two, the fibre product $C$ of $G$ and any subgroup $G \to G/H$ and $G \to G/J$ is neither a diagonal subgroup nor the whole $G \times G$. Indeed, we have

$$C := \{(f, g) \in G \times G \mid f \in H \iff g \in J\}.$$

Hence $C$ contains $H \times J$, which is a subgroup of index 4 in $G \times G$. On the other hand, by definition, $C$ is not equal to $G \times G$ (for example, $C$ does not contain $(f, 1)$ for any $f$ outside $H$). Since both projections of $C \leq G \times G$ are onto, $C$ strictly contains $H \times J$ and is a subgroup of index 2 in $G \times G$. We claim that $C$ does not contain any diagonal subgroup $\text{diag}(G \times \psi(G))$. Indeed, assume the opposite. On the one hand, $C$ contains $\text{diag}(H \times \psi(H))$. On the other hand, $C$ contains $H \times \{1\}$ and, therefore, also contains $\{1\} \times \psi(H) = \{1\} \times H$ since $H$ is characteristic. But $C$ also contains $\{1\} \times J$. In particular, $C$ contains $H \times \{1\}$ and $\{1\} \times G$ and is a subgroup of index 2 in $G \times G$. It follows that $C = H \times G$. But this is not a subdirect product in $G \times G$, and we arrive at the desired contradiction.

To prove Theorem 2.15 for subdirect products in $\prod_{i \in I} G_i$, we introduce an equivalence relation on the indices $i \in I$ in Lemma 3.5. To do this, we need some notation. Recall that $\pi_i : \prod_{i \in I} G_i \to G_i$ is the canonical projection.

**Definition 3.3.** Let $H \leq \prod_{i \in I} G_i$ be a subdirect product. For any $J \subseteq I$ we define a subgroup $H_J$ by the formula

$$H_J := \left\{ g \in \prod_{i \in I} G_i \mid \pi_i(g) = 1 \text{ if } i \not\in J \text{ and } \exists h \in H \text{ s.t. } \forall j \in J : \pi_j(h) = \pi_j(g) \right\}.$$

In particular, $H_i := H_{\{i\}} \cong G_i$.

Indices $i$ and $j$ are said to be *dependent* if either $i = j$, or $i \neq j$ but $H_i$ is isomorphic to $H_j$ and we have $H_{\{i, j\}} = \text{diag}(H_i \times \psi(H_i))$ for some isomorphism $\psi : H_i \to H_j$. They are said to be *independent* if $i \neq j$ and $H_{\{i, j\}}$ contains $L_i \times L_j$, where $L_i$, $L_j$ are finite-index subgroups of $H_i$, $H_j$.

\(^{5}\)For example, one can take $G = \mathcal{G}$ (the first Grigorchuk group) and let $H$ be the stabilizer of the first level and $J$ be any of the other six subgroups of index 2.
Informally, $H_J$ is the projection of $H$ to $\prod_{i \in J} H_i$, but viewed as a subgroup of $\prod_{i \in I} H_i$.

Note that, in general, being independent is not the negation of being dependent; see Example 3.7. However, it follows from Lemma 2.13 that this is the case when the factors are just infinite.

**Corollary 3.4.** If $H \leq \prod_{i \in I} G_i$ is a subdirect product with all the groups $G_i$ just infinite, then any indices $i$ and $j$ are either dependent or independent.

We now show that being dependent is an equivalence relation and, in the case of products of just infinite groups that are not virtually Abelian, the elements of any finite set $J \subseteq I$ of pairwise-independent indices are ‘simultaneously independent’.

**Lemma 3.5.** Let $H \leq \prod_{i \in I} G_i$ be a subdirect product. Then being dependent is an equivalence relation. Moreover, if all the elements of a subset $J \subseteq I$ are pairwise dependent, then $H_J$ is diagonal.

**Proof.** Let $J \subseteq I$ be a subset. Suppose that there is a $j_0 \in J$ such that every $j \in J$ is dependent with $j_0$. Then for any $j \in J$ we have an isomorphism $\psi_j: G_{j_0} \to G_j$ such that $\pi_j(g) = \psi_j(\pi_{j_0}(g))$ for all $g \in H$. It follows directly that

$$H_J = \{(\psi_i(g))_{i \in J} \mid g \in G_j\} = \text{diag}\left(\prod_{i \in J} \psi_i(G_j)\right).$$

This proves the second assertion of the lemma.

On the other hand, if we take $J = \{i, j, k\}$, where the pairs $(i, j)$ and $(j, k)$ are dependent, then the above (with $j_0 = j$) implies that $i$ and $k$ are dependent. This proves that being dependent is a transitive relation. It is obviously symmetric and reflexive. □

**Lemma 3.6.** Suppose that $H \leq \prod_{i \in I} G_i$, where all the groups $G_i$ are just infinite and at most two of them are virtually Abelian. Let $J \subseteq I$ be a finite set of pairwise independent indices. Then there are subgroups $L_j \leq G_j$, $j \in J$, of finite index such that $H_J$ contains $\prod_{j \in J} L_j$.

**Proof.** The proof is by induction on the cardinality of $J$. When $|J| = 1$, there is nothing to prove. When $|J| = 2$, this is Lemma 2.13. Now let $J \subseteq I$ be of cardinality $d$ with $d \geq 3$. Renaming the indices, we may assume that $J = \{1, \ldots, d\}$. We can also assume that $G_1$ is not virtually Abelian since $n \geq 3$ and at most two of the $G_i$ are virtually Abelian. We claim that the conclusion of the lemma holds if there are an index $i$ and a subgroup $L_i$ of finite index in $H_i$ such that $H_J$ contains $\{1\} \times \cdots \times L_i \times \cdots \times \{1\}$. Indeed, let $\tilde{H}_{J \setminus \{i\}}$ be the subgroup of $H_{J \setminus \{i\}}$ consisting of the projections of the elements $g$ in $H$ such that $\pi_i(g)$ belongs to $L_i$. Formally,

$$\tilde{H}_{J \setminus \{i\}} := H_{J \setminus \{i\}} \cap \left(\bigl(L_i \times \prod_{j \in J \setminus \{i\}} H_j\bigr)\right).$$

Since $L_i$ is of finite index in $H_i$, the subgroup $\tilde{H}_{J \setminus \{i\}}$ is of finite index in $H_{J \setminus \{i\}}$ and, by the induction hypothesis, it contains a subgroup $\tilde{L}_j$ of finite index in $H_j$. □
for every $j \in J \setminus \{i\}$. On the other hand, $H_J$ contains $L_i \times \tilde{H}_J \setminus \{i\}$. This completes the proof of the claim.

The subgroup $H_J$ projects onto the set $H_{\{1,3,\ldots,d\}}$. By the induction hypothesis, this set contains $L_1 \times L_3 \times \cdots \times L_d$ for some subgroups of finite index in $H_i$, $i \neq 2$. In particular, for every element $g \neq 1$ in $L_1$ there is an $x$ in $H_2 = G_2$ such that $(g, x, 1, \ldots, 1)$ is in $H_J$. Performing the same argument for $H_{\{1,2,4,\ldots,d\}}$, we obtain a subgroup $A$ of finite index in $G_1$ such that, for every $g$ and $h$ in $A$, the group $H_J$ contains $(g, x, 1, \ldots, 1)$ and $(h, 1, y, 1, \ldots, 1)$ for some $x$ and $y$. By taking the commutator and then conjugating by elements of $H_J$, we see that the subgroup $\langle [g, h] \rangle^{G_1} \times \{1\} \times \cdots \times \{1\}$ is contained in $H_J$ for all $g$ and $h$ in $A$. As soon as $[g, h]$ is non-trivial, $\langle [g, h] \rangle^{G_1}$ is a non-trivial normal subgroup of $G_1$ (and hence a subgroup of finite index), and we are done. Since $G_1$ is not virtually Abelian, it is always possible to find elements $g$ and $h$ in $A$ that do not commute. □

The following example clarifies the conditions imposed in Lemma 3.6 and Theorem 2.15.

**Example 3.7.** Let $G$ be an Abelian group (for example, $G = \mathbb{Z}$) and $H_n := \{(g_1, \ldots, g_n) \in G^n \mid \sum_i g_i = 0\}$. Then $H_n$ is a subdirect product in $G^n$ whose indices are pairwise independent but not simultaneously independent when $n \geq 3$. Moreover, if $n \geq 3$, then $H_n$ is not virtually diagonal by blocks. In particular, neither the conclusion of Lemma 3.6 nor the conclusion of Theorem 2.15 holds for $H_n$.

On the other hand, $H_n$ can be viewed as a subdirect product of $G_1 = G$ and $G_2 = G^{n-1}$. With this identification and for $n \geq 3$, the indices 1 and 2 are neither independent nor dependent.

We now have all the ingredients necessary to prove Theorem 2.15.

**Proof of Theorem 2.15.** Let $G_i$, $1 \leq i \leq n$, be just infinite groups such that at most two of them are virtually Abelian. Suppose that $H \leq \prod_{i=1}^n G_i$. By Lemma 3.5 we can find an integer $d$ and a partition $\{1, \ldots, n\} = I_1 \sqcup \cdots \sqcup I_d$ such that the sets $I_\alpha$ are equivalence classes of dependent indices. In particular, $H$ is a subdirect product in $\prod_{\alpha=1}^d H_{I_\alpha}$ and $H_{I_\alpha} \leq \prod_{i \in I_\alpha} G_i$ is a diagonal subgroup. Put $i_\alpha := \min\{i \in I_\alpha\}$ and $G_\alpha := G_{i_\alpha}$. Then all the groups $G_i$, $i \in I_\alpha$, are isomorphic to $G_\alpha$, and so is $H_{I_\alpha}$.

On the other hand, viewing $H$ as a subdirect product in $\prod_{\alpha=1}^d H_{I_\alpha}$, we see that the indices $(I_\alpha)_{\alpha=1}^d$ are pairwise independent. By Lemma 3.6, for every $\alpha$, $1 \leq \alpha \leq d$, the group $H$ contains $L_\alpha$, a subgroup of finite index in $H_{I_\alpha}$. Since $H_{I_\alpha}$ is a diagonal subgroup of $\prod_{i \in I_\alpha} G_i \cong G_\alpha^{|I_\alpha|}$, the subgroup $L_\alpha$ is a diagonal subgroup of $L^{|I_\alpha|}$ for some subgroup $L$ of finite index in $G_\alpha$. It is clear that $\prod_{\alpha=1}^d L_\alpha$ is of finite index in $H$ and, therefore, $H$ is a subgroup of $\prod_{i=1}^n G_i$ virtually diagonal by blocks. □

Finally, we show that the set of subgroups virtually diagonal by blocks is closed under taking finite-index subgroups.

**Lemma 3.8.** Let $H$ be a subgroup virtually diagonal by blocks of a finite product of groups $\prod_{i=1}^n G_i$. Then every finite-index subgroup $K$ of $H$ is also virtually diagonal by blocks.
Proof. Let $B := \prod_{\alpha \in \Delta} D_{\alpha}$ be the finite-index diagonal by blocks subgroup of $H$ in the notation of Definition 2.14, where $D_{\alpha}$ is a diagonal subgroup of $L_{\alpha}^{\lvert \alpha \rvert}$ for some finite-index subgroup $L_{\alpha}$ of $G_{\alpha}$. We claim that the subgroup $B' := \prod_{\alpha \in \Delta} K \cap D_{\alpha}$ of $\prod_{i=1}^{n} G_i$ is diagonal by blocks and has finite index in $K$. Indeed, since $K$ is of finite index in $H$, $K \cap D_{\alpha}$ is a subgroup of finite index in $D_{\alpha}$ for every $\alpha \in \Delta$. In particular, $K \cap D_{\alpha}$ is a diagonal subgroup of $L_{\alpha}^{\lvert \alpha \rvert}$ for some finite-index subgroup $L'_{\alpha}$ of $L_{\alpha}$. We conclude that $L'_{\alpha}$ is a finite-index subgroup of $G_{\alpha}$ and $B'$ is diagonal by blocks. Finally, the index of $B'$ in $B$ is bounded above by the finite product $\prod_{\alpha \in \Delta} \lvert D_{\alpha} : K \cap D_{\alpha} \rvert$ and is hence finite. It follows that $B'$ is of finite index in $H$ and hence of finite index in $K$. □

§ 4. Block structure in branch groups

We begin this section with a consequence of Lemma 2.13. While this result is not used in the proof of the main results, we include it as another application of the technique of subdirect products.

Lemma 4.1. Let $A$ be a subgroup of the first Grigorchuk group $G$. Suppose that both first-level sections of $\text{Stab}_A(L_1)$ are equal to $G$. Then $A$ contains $C \times C$ for some finite-index subgroup $C$ of $G$ and is thus of finite index.

Proof. Since $\text{Stab}_A(L_1)$ is of finite index in $A$, we can suppose that $A$ is a subgroup of $H = \text{Stab}_G(L_1)$ and hence of $G \times G$. By Lemma 2.13, if both sections of $A$ are equal to $G$, then either $A$ contains $C \times C$ for some subgroup $C$ of finite index, or $A$ is of the form $\text{diag}(G \times \psi(G))$ for some automorphism of $G$. We claim that the second case is impossible or, more precisely, $\text{diag}(G \times \psi(G))$ is never a subgroup of $G$. Indeed, otherwise $c \cdot (a, \psi(a)) = (1, d\psi(a))$ belongs to $G$. But then $d\psi(a)$ is in $\varphi_1(\text{Rist}_G(1)) = B$ and hence $\psi(a)$ belongs to $dB \subseteq H$. In particular, $a$ belongs to $\psi^{-1}(H) = H$ since $H$ is a characteristic subgroup of $G$. This is our desired contradiction. □

The following easy lemma will be used many times without explicit reference.

Lemma 4.2. Let $G$ be any subgroup of $\text{Aut}(T)$ and $H$ a subgroup of $G$. Then the following properties are equivalent.

1) There is an $n$ such that for every $v \in L_n$ the section $\varphi_v(\text{Stab}_H(L_n))$ is either trivial or of finite index in $\varphi_v(\text{Stab}_G(v))$.

2) There is a transversal $X$ of $T$ such that for every $v \in X$ the section $\varphi_v(\text{Stab}_H(X))$ is either trivial or of finite index in $\varphi_v(G)$.

Proof. It is clear that property 1 implies property 2. For the converse, let $H$ be a subgroup such that the conclusion of property 2 holds for some $X$ and let $n$ be the maximal level of a vertex in $X$. Then $\text{Stab}_H(L_n)$ is a finite-index subgroup of $\text{Stab}_H(X)$. Now let $w$ be a vertex of level $n$ and $v$ its unique ancestor in $X$. Then $\varphi_w(\text{Stab}_H(L_n))$ is of finite index in $\varphi_w(\text{Stab}_H(X)) = \varphi_w(\varphi_v(\text{Stab}_H(X)))$. It follows that $\varphi_w(\text{Stab}_H(L_n))$ is either a finite-index subgroup of the trivial group (and hence trivial) or a finite-index subgroup of a finite-index subgroup of $\varphi_w(\text{Stab}_G(w))$. □

The subgroup induction property admits an equivalent characterization that will be more suitable for our purposes.
Proposition 4.3. Let $G \leq \text{Aut}(T)$ be a self-similar group. Then the following assertions are equivalent.

1) $G$ possesses the subgroup induction property.

2) For every finitely generated subgroup $H \leq G$, there is a transversal $X$ of $T$ such that for every $v \in X$ the section $\varphi_v(\text{Stab}_H(X))$ is either trivial or of finite index in $\varphi_v(G)$.

Proof. We first show that the subgroup induction property implies the second assertion. Let $\mathcal{X}$ be the class of all subgroups $H$ of $G$ admitting a transversal $X$ such that $\varphi_v(\text{Stab}_H(X))$ is either trivial or of finite index in $\varphi_v(G)$ for every $v$ in $X$. We have to show that this class is inductive.

It is obvious that $\{1\}$ and $G$ belong to $\mathcal{X}$ (in both cases we can take $X$ to be the root of $T$).

Let $H$ and $L$ be subgroups of $G$ such that $H$ is a finite-index subgroup in $L$. We need to show that $H$ is in $\mathcal{X}$ if and only if $L$ is. For every $X$, the subgroup $\text{Stab}_H(X)$ is of finite index in $\text{Stab}_L(X)$ and hence $\varphi_v(\text{Stab}_H(X))$ is of finite index in $\varphi_v(\text{Stab}_L(X))$. In particular, if $L$ is in $\mathcal{X}$, then so is $H$, with the same transversal $X$. On the other hand, if $H$ is in $\mathcal{X}$ with a transversal $X$, then for every $v \in X$ the subgroup $\varphi_v(\text{Stab}_L(X))$ is either finite or of finite index in $\varphi_v(G)$. Let $v$ be a vertex of $X$ such that $\varphi_v(\text{Stab}_L(X))$ is finite. Then there is a transversal $X_v$ of $T_v$ such that $\text{Stab}_v(\text{Stab}_L(X))(X_v)$ is trivial. Let $X'$ be the transversal obtained from $X$ by removing all $v \in X$ such that $\varphi_v(\text{Stab}_L(X))$ is finite and replacing them by the corresponding $X_v$. Then, for every $w \in X'$, the group $\varphi_w(\text{Stab}_L(X'))$ is either trivial or of finite index in $\varphi_w(\text{Stab}_G(w))$.

Finally, let $\{1, \ldots , d\}$ be the first-level vertices of $T$ and let $H$ be a finitely generated subgroup of $\text{Stab}_G(L_1)$ such that all its sections $H_1, \ldots , H_d$ belong to $\mathcal{X}$. Each of the $H_i$ comes with a transversal $X_i$ of $T_i$. The union $X$ of these $X_i$ is a transversal for $T$. Put $H' := \text{Stab}_H(X)$. This is a finite-index subgroup of $H$ whose first-level sections $H'_i := \varphi_{i}(H')$ are of finite index in $H_i$. Therefore, for every $v$ in $X_i$, the section $\varphi_v(H'_i) = \varphi_v(H')$ is either trivial or of finite index in $\varphi_v(G)$. It follows that $H'$ belongs to $\mathcal{X}$, and so does $H$ since $\mathcal{X}$ is closed under finite extensions.

Now suppose that $G$ possesses property 2 and let $\mathcal{X}$ be an inductive class of subgroups. In particular, $\mathcal{X}$ contains $\{1\}$ and all finite-index subgroups of $G$. Let $H$ be a finitely generated subgroup of $G$ and $X$ the corresponding transversal. We need to show that $H$ is in $\mathcal{X}$. Put $H' := \text{Stab}_H(X)$. Since $H'$ is of finite index in $H$, it is finitely generated, and if it belongs to $\mathcal{X}$, so does $H$. If $X$ consists only of the root of $T$, then $H$ is either finite or of finite index in $G$. In both cases, it belongs to $\mathcal{X}$. Otherwise, let $v$ be a vertex not in $X$ all of whose children $w_1, \ldots , w_d$ are in $X$. Since $H'$ stabilizes $X$, all the subgroups $\varphi_{w_i}(\varphi_v(H')) = \varphi_{w_i}(H')$ are in $\mathcal{X}$ and $\varphi_{v}(H')$ is a finitely generated subgroup of $G$ stabilizing the first level of $T_v$. In particular, $\varphi_{v}(H')$ belongs to $\mathcal{X}$. Put $X_1 := v \cup X \setminus \{w_1, \ldots , w_d\}$. By the above, $H'$ pointwise stabilizes $X_1$ and all its sections along $X_1$ are in $\mathcal{X}$. By induction there is an $m$ such that $X_m$ is the root and $H'$ belongs to $\mathcal{X}$. □

Remark 4.4. The alternative characterization of the subgroup induction property in Proposition 4.3 is natural even in a non-self-similar context. From now on, we
will say that a group $G$ possesses the subgroup induction property if it satisfies this alternative characterization.

It is in fact possible to slightly modify the original definition of an inductive class $\mathcal{X}$ of subgroups (replacing it by an inductive family $(\mathcal{X}_v)_{v \in T}$ of classes of subgroups) to take into account not necessarily self-similar groups in Definition 2.8. Proposition 4.3 and its proof extend to this general context. Details are left to the interested reader.

We now derive some corollaries from this more general subgroup induction property. The first is about the rank of $G$.

**Corollary 4.5.** Let $G \leq \text{Aut}(T)$ be a branch group with the subgroup induction property. Then $G$ is either finitely generated or locally finite.

**Proof.** We can assume that $G$ is not finitely generated as otherwise there is nothing to prove.

If $H$ is a finitely generated subgroup of $G$, there is an $n$ such that, for every vertex $v$ of level $n$, the section $\varphi_v(\text{Stab}_H(\mathcal{L}_n))$ is either trivial or of finite index in $\varphi_v(G)$. Then $\text{Rist}_G(\mathcal{L}_n) \cap H$ is of finite index in $H$ and, for every $v \in \mathcal{L}_n$, the subgroup $\varphi_v(\text{Rist}_G(\mathcal{L}_n) \cap H)$ is either trivial or of finite index in $\varphi_v(G)$. Since $\text{Rist}_G(\mathcal{L}_n) \cap H$ is of finite index in $H$, it is finitely generated.

On the other hand, for any $v \in \mathcal{L}_n$, the number of generators of $\text{Rist}_G(\mathcal{L}_n) \cap H$ is at least the number of generators of $\varphi_v(\text{Rist}_G(\mathcal{L}_n) \cap H)$. Since $G$ is branch but not finitely generated, the rigid stabilizer $\text{Rist}_G(\mathcal{L}_n)$ is not finitely generated, and neither is $\text{Rist}_G(w)$ for any $w$. Therefore, for any $w \in T$, the subgroups $\varphi_w(\text{Rist}_G(w))$ and $\varphi_w(G)$ are not finitely generated. Altogether, this implies that the subgroup $\varphi_w(\text{Rist}_G(\mathcal{L}_n) \cap H)$ is trivial for all $w \in \mathcal{L}_n$. Hence the subgroup $\text{Rist}_G(\mathcal{L}_n) \cap H$ is itself trivial and $H$ is finite. $\square$

We also have a result about the finite subgroups of $G$.

**Corollary 4.6.** Let $G \leq \text{Aut}(T)$ be an infinite group with the subgroup induction property and let $H$ be a finitely generated subgroup of $G$. Then $H$ is finite if and only if no section $\varphi_v(H)$ is of finite index in $\varphi_v(G)$.

**Proof.** One direction is trivial. For the other, let $H$ be a finitely generated subgroup of $G$. Then there is a transversal $X$ such that, for all $v \in X$, the section $\varphi_v(\text{Stab}_H(X))$ is either trivial or of finite index in $\varphi_v(G)$. If no section of $H$ is of finite index, then all the sections of $\text{Stab}_H(X)$ along $X$ are trivial. Hence $\text{Stab}_H(X)$ is trivial and $H$ is finite. $\square$

This enables us to generalize Theorem 3 in [15]. A branch group $G \leq \text{Aut}(T)$ is said to have trivial branch kernel, [34], if every finite-index subgroup of $G$ contains the rigid stabilizer of a level or, equivalently, if the topology defined by $\text{Rist}_G(\mathcal{L}_n)$ coincides with the profinite topology. This property follows from the more well-known congruence subgroup property, which says that every finite-index subgroup of $G$ contains the stabilizer of a level. See [34] for more on these properties.

**Corollary 4.7.** Let $G \leq \text{Aut}(T)$ be regular branch over a subgroup $K$. Suppose that $G$ possesses the subgroup induction property and has trivial branch kernel. Suppose also that there is a vertex $v$ of $T$ with $\varphi_v(\text{Stab}_K(v)) = G$. 
Then a finitely generated subgroup \( H \) of \( G \) is finite if and only if no section of \( H \) is equal to \( G \).

**Proof.** By Corollary 4.6, if \( H \) is infinite, it has a section \( \varphi(v)(\text{Stab}_H(v)) \) of finite index in \( G \). Since every section of \( \varphi(v)(\text{Stab}_H(v)) \) is a section of \( H \), it suffices to show that every subgroup \( N \) of finite index in \( G \) has a section equal to \( G \).

By the triviality of the branch kernel, \( N \) contains the rigid stabilizer of a level and, therefore, it contains \( \text{Rist}_G(w) \) for all \( w \) deep enough in the tree. On the other hand, since \( G \) is regular branch over \( K \), the section \( \varphi(w)(\text{Rist}_G(w)) \) contains \( K \). \( \square \)

In the next two lemmas we prove that, under appropriate conditions, the subgroup \( H \leq G \) contains a block subgroup \( A \) with \( [H:A] < \infty \) if and only if there is a transversal \( X \) of \( T \) such that for every \( v \in X \) the section \( \varphi(v)(\text{Stab}_H(X)) \) is either trivial or of finite index in \( \varphi(v)(G) \).

**Lemma 4.8.** Let \( G \leq \text{Aut}(T) \) be a just infinite branch group. Let \( H \leq G \) be a subgroup. Suppose that there is a transversal \( X \) in \( T \) such that, for every \( v \in X \), the section \( \varphi(v)(\text{Stab}_H(X)) \) is either trivial or of finite index in \( \varphi(v)(G) \). Then \( H \) contains a block subgroup \( A \) with \( [H:A] < \infty \).

**Proof.** Let \( H \leq G \) be a subgroup and suppose that there is a transversal \( X \) of \( T \) such that for every \( v \in X \) the section \( \varphi(v)(\text{Stab}_H(X)) \) is either trivial or of finite index in \( \varphi(v)(G) \). Since \( G \) is just infinite and branch, so are the groups \( \varphi(v)(G) \) ([12], Lemma 5.1). Hence they are not virtually Abelian.

Since \( G \) is branch, the group \( L := \prod_{v \in X} \text{Rist}_H(v) \) is of finite index in \( H \) and, for every \( v \in X \), the section \( \varphi(v)(L) \) is either trivial or of finite index in \( \varphi(v)(G) \). A variation of theorem 2.15 for branch groups implies that \( L \) is virtually diagonal by blocks in the sense of Definition 2.14. More precisely, there is a partition \( U = \bigcup_{i=1}^n U_i \) into orthogonal and pairwise-orthogonal subsets of vertices of \( T \) and \( L \) contains \( L' = \prod_{i=1}^n D_i \) as a finite-index subgroup, where the \( D_i \) are the diagonal subgroups. Since \( L' \) is of finite index in \( L \) and hence in \( H \), it follows that for \( u \in U \) in the support of \( D_i \) the section \( \varphi_u(D_i) \) is of finite index in \( \varphi_u(H) \) and hence in \( \varphi_u(\text{Rist}_G(u)) \). Therefore the subgroups \( D_i \) are diagonal in the sense of Definition 2.3 and \( L' \) is a block subgroup (in the sense of Definition 2.5) of finite index in \( H \). \( \square \)

In order to prove the following lemma, we introduce a partial order on the set of transversals. For transversals \( Y \) and \( X \) in \( T \), we say that \( Y \geq X \) if every element \( y \) of \( Y \) is the descendant of some \( x \) in \( X \).

**Lemma 4.9.** Suppose that \( G \leq \text{Aut}(T) \). Let \( H \leq G \) be a subgroup containing a block subgroup \( A \) with \( [H:A] < \infty \). Then there is a transversal \( X \) of \( T \) such that for every \( v \in X \) the section \( \varphi(v)(\text{Stab}_H(X)) \) is either trivial or of finite index in \( \varphi(v)(G) \).

**Proof.** Let \( A = \prod_{i=1}^k D_i \leq G \) be a block subgroup and, for each \( i \), let \( U_i \) be the support of the diagonal subgroup \( D_i \). In particular, the sets \( U_i \) are orthogonal and pairwise orthogonal. Let \( X \) be the unique minimal transversal of \( T \) containing all the sets \( U_i \). In particular, \( \text{Stab}_A(X) = \text{Stab}_A(\bigcup_{i=1}^k U_i) \). Then \( \varphi(v)(\text{Stab}_A(X)) \) is either trivial or of finite index in \( \varphi(v)(G) \) for every \( v \in X \). If \( A \) is of finite index in some subgroup \( H \leq G \), the subgroup \( \varphi(v)(\text{Stab}_H(X)) \) is either finite or of finite
index in $\varphi_v(G)$ for every $v \in X$. Therefore there is another transversal $Y \supseteq X$ such that $\varphi_v(\text{Stab}_H(Y))$ is either trivial or of finite index in $\varphi_v(G)$ for every $v \in Y$. □

In order to prove our main result, we need the following proposition.

**Proposition 4.10** ([23]). Let $G \leq \text{Aut}(T)$ be a finitely generated branch group with the subgroup induction property. Then $G$ is torsion and just infinite.

We now can prove our main result.

**Theorem 2.9.** Let $G$ be a finitely generated branch group. Then the following assertions are equivalent.

1) A subgroup $H$ of $G$ is finitely generated if and only if it contains a block subgroup $A$ with $[H : A] < \infty$.

2) A subgroup $H$ of $G$ is finitely generated if and only if there is an $n$ such that for every $v \in L_n$, the section $\varphi_v(\text{Stab}_H(L_n))$ is either trivial or of finite index in $\varphi_v(\text{Stab}_G(v))$.

3) A subgroup $H$ of $G$ is finitely generated if and only if there is a transversal $X$ of $T$ such that, for every $v \in X$, the section $\varphi_v(\text{Stab}_H(X))$ is either trivial or of finite index in $\varphi_v(\text{Stab}_G(v))$.

If moreover $G$ is self-similar, then these are equivalent to the following.

4) The group $G$ possesses the subgroup induction property in the sense of Definition 2.8.

**Proof.** First of all, if $G$ is finitely generated, then so is every block subgroup and, therefore, if a subgroup $H \leq G$ contains a block subgroup with $[H : A] < \infty$, then $H$ is finitely generated. Hence Theorem 2.9 follows from the fact that the following properties are equivalent.

a) Every finitely generated subgroup $H$ of $G$ contains a block subgroup $A$ with $[H : A] < \infty$.

b) For every finitely generated subgroup $H$ of $G$ there is an $n$ such that, for every $v \in L_n$, the section $\varphi_v(\text{Stab}_H(L_n))$ is either trivial or of finite index in $\varphi_v(\text{Stab}_G(v))$.

c) For every finitely generated subgroup $H$ of $G$ there is a transversal $X$ of $T$ such that, for every $v \in X$, the section $\varphi_v(\text{Stab}_H(X))$ is either trivial or of finite index in $\varphi_v(\text{Stab}_G(v))$.

d) (If $G$ is self-similar) the group $G$ possesses the subgroup induction property.

Properties b and c are equivalent by Lemma 4.2. If $G$ is self-similar, they are equivalent to Property d by Proposition 4.3. Moreover, by Lemma 4.9, Property c follows from Property a. On the other hand, if $G$ possesses Property b, then it is just infinite by Proposition 4.10 and, therefore, possesses Property a by Lemma 4.8. □

We finally derive Theorem 2.11 from Theorem 2.9.

**Proof of Theorem 2.11.** Let $G$ be a finitely generated branch group with the congruence subgroup property and the subgroup induction property.

Let $H$ be a finitely generated subgroup of $G$. By Theorem 2.9, there is a block subgroup $A \leq H$ with $[H : A]$ finite. By Lemma 6.7 in [12], $A$ is closed in the profinite topology and so is $H$. This proves that $G$ is subgroup separable, as required. □

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