A NOTE ON MULTIPLICITY OF SOLUTIONS NEAR RESONANCE OF SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. In this paper we are concerned with the multiplicity of solutions near resonance for the following nonlinear equation:

\[-\Delta u = \lambda u + f(x, u)\]

associated with the Dirichlet boundary condition, where \(f\) satisfies some appropriate conditions. We will treat this problem in the framework of dynamical systems. It will be shown that there exist a one-sided neighborhood \(\Lambda_+\) of the eigenvalue \(\mu_k\) of the Laplacian operator and a dense subset \(\mathcal{D}\) of \(\mathbb{R}\) such that the equation has at least four distinct nontrivial solutions generically for \(\lambda \in \Lambda_+ \cap \mathcal{D}\).

1. Introduction. This note is concerned with the following equation:

\[
\begin{cases}
-\Delta u = \lambda u + f(x, u), & x \in \Omega, \\
u(x) = 0, & x \in \partial\Omega,
\end{cases}
\]

where \(\Omega \subset \mathbb{R}^n\) is a bounded domain with smooth boundary \(\partial\Omega\), \(f\) is a bounded function, and \(\lambda \in \mathbb{R}\) is a parameter. We are mainly interested in the bifurcation from infinity and multiplicity of solutions near resonance of (1).

The bifurcation and multiplicity of semilinear elliptic equations near resonance is an interesting topic and has attracted much attention in the past decades, see [2, 7, 19, 24, 15, 14, 6, 22], etc. This problem (1) can be traced back to the earlier work [15] by Mawhin and Schmitt, where the authors considered the case when \(\lambda\) crosses an eigenvalue of odd multiplicity. Later Schmitt and Wang [22] developed a theory on bifurcation from infinity for potential operators, through which they extended the results in [15] to the case when \(\lambda\) crosses any eigenvalue \(\mu_k\) of the Laplacian. More specifically, under an abstract Landesman-Lazer type condition on the Nemitski operator \(\tilde{f} : H_0^1(\Omega) \to L^2(\Omega)\) corresponding to the function \(f(x, s)\), the authors proved the following result and its “dual” version: there exists \(\delta > 0\)

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such that for each $\lambda \in [\mu_k - \delta, \mu_k] =: \Lambda_-$ the equation (1) has at least two solutions, and for each $\lambda \in [\mu_k, \mu_k + \delta]$ at least one.

Note that a solution of (1) can be regarded as a stationary solution of the following evolution equation:

$$
\begin{cases}
  u_t - \Delta u = \lambda u + f(x, u), & x \in \Omega, \\
  u(x) = 0, & x \in \partial \Omega.
\end{cases}
$$

Moreover, there exists a Lyapunov function for system (2). Then the problem (1) can be studied via the method of dynamical systems.

Very recently, the equation (2) was studied in [9], where the authors established a general theorem on dynamic bifurcation from infinity in the framework of local semiflows on a Banach space. As an application of their theoretical results, they considered the equation (2), and gave some static bifurcation and multiplicity results of (2) under the following Landesman-Lazer type conditions on $f$:

$$
\liminf_{s \to +\infty} f(x, s) \geq \overline{f} > 0, \quad \limsup_{s \to -\infty} f(x, s) \leq -\underline{f} < 0
$$

uniformly for $x \in \Omega$. Specifically, they proved that for each eigenvalue $\mu_k$ of the Laplacian (associated with the Dirichlet boundary condition), there exists $\delta = \delta_k > 0$ such that for each $\lambda \in \Lambda_- = [\mu_k - \delta, \mu_k)$, the equation has at least two distinct stationary solutions $e^1_\lambda, e^\infty_\lambda$ with

$$
\lim_{\lambda \to \mu_k^-} \|e^\infty_\lambda\| = \infty,
$$

whereas $e^1_\lambda$ remains bounded on $\Lambda_-$. Furthermore, they also paid some attention to the case where

$$
f(x, s) = o(|s|) \quad \text{as} \quad s \to 0
$$

uniformly for $x \in \Omega$. They showed that there is a one-sided neighborhood $\Lambda_1$ of $\mu_k$ such that (2) has at least three distinct equilibria. For such a case on $f$, Chiappinelli, Mawhin and Nugari[3] studied the multiplicity of solutions of the problem near the first eigenvalue $\mu_1$. (Note that the nonlinearity in [3] was allowed to be unbounded.)

Motivated by the above works, we further discuss the bifurcation and multiplicity of the stationary solutions of (2). Roughly speaking, for any eigenvalue $\mu_k$, we will prove that there exist a dense subset $D$ of $\mathbb{R}$ and a one-sided neighborhood $\Lambda_1$ of $\mu_k$ such that the equation (1) has at least four distinct solutions for $\lambda \in D \cap \Lambda_1$ under the Landesman-Lazer type conditions and (4) on $f$.

It is worth mentioning that dual versions of our results mentioned above hold true if, instead of (3), we assume that

$$
\limsup_{s \to +\infty} f(x, s) \leq -\overline{f} < 0, \quad \liminf_{s \to -\infty} f(x, s) \geq \underline{f} > 0
$$

uniformly for $x \in \Omega$.

This work is organized as follows. In Section 2 we make some preliminaries. Section 3 is devoted to our main results.

2. Preliminaries. In this section we introduce some basic concepts and results. The interested reader is referred to [9, 11, 20] for details.
2.1. Basic topological notions and results. Let $X$ be a complete metric space with metric $d(\cdot,\cdot)$.

Let $A$ and $B$ be two subsets of $X$. The Hausdorff semidistance and Hausdorff distance between $A$ and $B$ are defined by

$$d_H(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y), \quad \delta_H(A, B) = \max\{d_H(A, B), d_H(B, A)\}.$$ 

The $\varepsilon$-neighborhood of $A$, denoted by $B(A, \varepsilon)$ or $B_X(A, \varepsilon)$, is defined as $\{y \in X : d(y, A) < \varepsilon\}$.

We call a subset $U$ of $X$ a neighborhood of $A$, if $\overline{A} \subset \text{int} U$.

**Lemma 2.1.** (see [1]). Let $X$ be a compact metric space. Denote $\mathcal{K}(X)$ the family of compact subsets of $X$ which is equipped with the Hausdorff metric $\delta_H(\cdot,\cdot)$. Then $\mathcal{K}(X)$ is a compact metric space.

2.2. Wedge/smash product of pointed spaces. Let $(X, x_0)$ and $(Y, y_0)$ be two pointed spaces. The smash product $(X, x_0) \wedge (Y, y_0)$ is defined as

$$(X, x_0) \wedge (Y, y_0) = [(X \times Y)/W, [W]],$$ 

where $W = X \times \{y_0\} \cup \{x_0\} \times Y$.

We denote by $[(X, x_0)]$ the homotopy type of a pointed space $(X, x_0)$. Since the operation “$\wedge$” preserves homotopy equivalence relations, it can be naturally extended to the homotopy types of pointed spaces. Let $\emptyset$ and $\Sigma^0$ be the homotopy types of the pointed spaces $\{p\}, p$ and $\{(p, q), q\}$, respectively, where $p$ and $q$ are two distinct points. By $\Sigma^n$ we denote the homotopy type of pointed $m$-dimensional sphere. One easily shows that

$$\Sigma^m \wedge \Sigma^n = \Sigma^{m+n}, \quad \forall m, n \geq 0.$$ 

2.3. Local semiflows on metric spaces. For the reader’s convenience, in this subsection we collect some fundamental notions and facts concerning local semiflows on metric spaces.

2.3.1. Local semiflows. Let $X$ be a complete metric space.

A local semiflow $\Phi$ on $X$ is a continuous mapping from an open set $\mathcal{D}(\Phi) \subset \mathbb{R}_+ \times X$ to $X$ and satisfies the following properties:

1. for each $x \in X$, there exists $0 < T_x \leq \infty$, called the escape time of $\Phi(t, x)$, such that

$$(t, x) \in \mathcal{D}(\Phi) \iff t \in [0, T_x).$$

2. $\Phi(0, \cdot) = \text{id}_X$, and

$$\Phi(t + s, x) = \Phi(t, \Phi(s, x))$$

for all $x \in X$ and $t, s \in \mathbb{R}_+$ with $t + s \leq T_x$.

Let $\Phi$ be a given local semiflow on $X$. For notational simplicity, we usually rewrite $\Phi(t, x)$ as $\Phi(t)x$.

Let $I \subset \mathbb{R}$ be an interval. A trajectory of $\Phi$ on $I$ is a continuous mapping $\gamma : I \to X$ that satisfies

$$\gamma(t) = \Phi(t - s)\gamma(s), \quad \forall t, s \in I, t \geq s.$$ 

A trajectory $\gamma$ on $\mathbb{R}$ is called a full trajectory.

Let $\gamma$ be a full trajectory. The $\omega$-limit set $\omega(\gamma)$ and $\omega^*$-limit set of $\gamma$ are defined, respectively, by

$$\omega(\gamma) = \{y \in X : \text{there exists } t_n \to \infty \text{ such that } \gamma(t_n) \to y\},$$
We call \( \text{orb}(\gamma) \) the orbit of \( \gamma \) on \( I \). The orbit of a full trajectory is simply called a full orbit.

For the sake of convenience, given \( U \subset X \), denote \( K_\infty(\Phi, U) \) the union of all bounded full orbits in \( U \). In the case \( U = X \), we will simply rewrite

\[ K_\infty(\Phi, X) = K_\infty(\Phi). \]

2.3.2. Asymptotic compactness of semiflows. Let \( N \subset X \). We say that \( \Phi \) does not explode in \( N \) if \( \Phi([0, T_x])x \subset N \) implies that \( T_x = \infty \).

**Definition 2.2.** (see [20]). \( N \subset X \) is said to be admissible, if for any sequences \( x_n \in N \) and \( t_n \to \infty \) with \( \Phi([0, t_n])x_n \subset N \) for all \( n \), the sequence \( \Phi(t_n)x_n \) has a convergent subsequence.

\( N \) is said to be strongly admissible, if it is admissible and moreover, \( \Phi \) does not explode in \( N \).

**Definition 2.3.** \( \Phi \) is said to be asymptotically compact on \( X \), if each bounded set \( B \subset X \) is strongly admissible.

Now let \( \Phi_\lambda (\lambda \in \Lambda) \) be a family of semiflows on \( X \), where \( \Lambda \) is a metric space. We say that \( \Phi_\lambda \) depends on \( \lambda \) continuously, if for any \( \lambda \in \Lambda \), \( x \in X, t \in \mathbb{R}_+ \) and any sequence \( (t_n, x_n, \lambda_n) \), whenever \( (t_n, x_n, \lambda_n) \to (t, x, \lambda) \) as \( n \to \infty \) and \( \Phi_\lambda(t)x \) is defined, then \( \Phi_{\lambda_n}(t_n)x_n \) is also defined for all \( n \) sufficiently large, and furthermore,

\[ \Phi_{\lambda_n}(t_n)x_n \to \Phi_\lambda(t)x \quad \text{as} \quad n \to \infty. \]

Suppose the family \( \Phi_\lambda (\lambda \in \Lambda) \) depends on \( \lambda \) continuously. Set

\[ \Pi(t)(x, \lambda) = (\Phi_\lambda(t)x, \lambda), \quad (x, \lambda) \in X \times \Lambda. \]

Then \( \Pi \) is a local semiflow on the product space \( X \times \Lambda \). For convenience, we call \( \Pi \) the skew-product flow of the family \( \Phi_\lambda (\lambda \in \Lambda) \).

The family \( \Phi_\lambda (\lambda \in \Lambda) \) is said to be \( \lambda \)-locally uniformly asymptotically compact (\( \lambda \)-u.a.c. in short), if the skew-product flow \( \Pi \) is asymptotically compact.

2.3.3. Invariant sets, attractors, and Morse decompositions. Let \( \Phi \) be a local semiflow on \( X \).

\( S \subset X \) is said to be positively invariant (resp. invariant), if \( \Phi(t)S \subset S \) (resp. \( \Phi(t)S = S \) for all \( t \geq 0 \).

A compact invariant set \( A \) is called an attractor of \( \Phi \), if it attracts a neighborhood \( U \) of itself, that is,

\[ \lim_{t \to \infty} d_H(\Phi(t)U, A) = 0. \]

**Definition 2.4.** Let \( S \) be a compact invariant set of \( \Phi \). An ordered collection \( \mathcal{M} = \{M_1, \cdots, M_l\} \) of disjointed compact invariant subsets of \( S \) is called a Morse decomposition of \( S \), if for any full trajectory \( \gamma \) contained in \( S \setminus \left( \bigcup_{1 \leq k \leq l} M_k \right) \), there exist \( i \) and \( j \) with \( i < j \) such that

\[ \omega^*(\gamma) \subset M_j, \quad \omega(\gamma) \subset M_i. \]  

**Remark 1.** A full trajectory satisfying (6) will be referred to as a connecting trajectory between \( M_i \) and \( M_j \).
Remark 2. One can also use equivalent definitions of Morse decompositions; see e.g. [20], Chap. III.

2.4. Conley index. In this subsection we briefly recall the definition of Conley index. The interested reader is referred to [4, 17] and [20], etc. for details.

Let $\Phi$ be a local semiflow on $X$. Since $X$ may be an infinite dimensional space, we will always assume $\Phi$ is asymptotically compact.

Let $N, E$ be two closed subsets of $X$. $E$ is called an exit set of $N$, if the two properties hold:
1. $E$ is $N$-positively invariant, that is, for any $x \in E \cap N$ and $t \geq 0$,
   \[ \Phi([0, t])x \subset N \implies \Phi([0, t])x \subset E; \]
2. for any $x \in N$, if $\Phi(t_1)x \notin N$ for some $t_1 > 0$, then there exists $t_0 \in [0, t_1]$ such that $\Phi(t_0) \in E$.

A compact invariant set $S$ of $\Phi$ is said to be isolated, if there is a neighborhood $N$ of $S$ such that $S$ is the maximal compact invariant set in $N$. Correspondingly, $N$ is called an isolating neighborhood of $S$.

An important example for isolating neighborhoods is the so called isolating block, which plays a crucial role in the Conley index theory.

Let $B \subset X$ be a bounded closed set. $x \in \partial B$ is called a strict egress (resp. strict ingress, bounce-off) point of $B$, if for every trajectory $\gamma : [-\tau, s] \to X$ with $\gamma(0) = x$, where $\tau \geq 0$, $s > 0$, the following properties hold:
1. there exists $0 < \varepsilon < s$ such that
   \[ \gamma(t) \notin B \ (\text{resp. } \gamma(t) \in \text{int} B, \ \text{resp. } \gamma(t) \notin B), \quad \forall t \in (0, \varepsilon); \]
2. if $\tau > 0$, then there exists $0 < \delta < \tau$ such that
   \[ \gamma(t) \in \text{int} B \ (\text{resp. } \gamma(t) \notin B, \ \text{resp. } \gamma(t) \notin B), \quad \forall t \in (-\delta, 0). \]

Denote by $B^e$ (resp. $B^i$, $B^b$) the set of all strict egress (resp. strict ingress, bounce-off) points of the closed set $B$, and set $B^- = B^e \cup B^b$.

A closed set $B \subset X$ is called an isolating block [20] if $B^-$ is closed and $\partial B = B^i \cup B^b$.

Let $S$ be a compact isolated invariant set. A pair of closed sets $(N, E)$ is said to be an index pair of $S$, if the following properties hold:
1. $N \setminus E$ is an isolating neighborhood of $S$; and
2. $E$ is an exit set of $N$.

If $B$ is a bounded isolating block, then one can deduce from [20] that $(B, B^-)$ is an index pair of the maximal compact invariant set $S = K_\infty(\Phi, B)$ in $B$.

Definition 2.5. Let $(N, E)$ be an index pair of $S$. Then the homotopy Conley index pair of $S$ is defined to be the homotopy type $\left[(N/E, [E])\right]$ of the pointed space $(N/E, [E])$, denoted by $h(\Phi, S)$.

Concerning the basic properties of Conley index, we refer the reader to the references mentioned above. Here we omit the details.

Remark 3. For convenience, if $U$ is an isolating neighborhood of a compact invariant set $S$ ($U$ need not be bounded), we also write
\[ h(\Phi, U) = h(\Phi, S), \]
hoping that this will not cause any confusion.
Let $S$ be a compact isolated invariant set. Denote $H_*$ and $H^*$ the singular homology and cohomology theories with coefficient group $\mathbb{Z}$, respectively. Applying $H_*$ and $H^*$ to $h(\Phi, S)$ one immediately obtains the homology and cohomology Conley indices of $S$.

The Poincaré polynomial of $S$, denoted by $p(t, S)$, is the formal polynomial

$$p(t, S) = \sum_{q=0}^{\infty} \beta_q t^q$$

with $\beta_q = \text{rank } H_q(h(\Phi, S))$.

Suppose that $S$ has a Morse decomposition $M = \{M_1, \cdots, M_l\}$. Then the following Morse equation

$$p(t, M_1) + \cdots + p(t, M_l) = p(t, S) + (1 + t)Q(t)$$

holds for some formal polynomial $Q(t) = \sum_{q=0}^{\infty} d_q t^q$ with $d_q \in \mathbb{Z}_+$.

3. Multiplicity of stationary solutions of evolution equations. In this section we consider the following boundary value problem:

$$\begin{cases}
  u_t - \Delta u - \lambda u = f(x, u), & x \in \Omega; \\
  u(x, t) = 0, & x \in \partial \Omega,
\end{cases}$$

(7)

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $\lambda \in \mathbb{R}$, and $f \in C^1(\overline{\Omega} \times \mathbb{R})$. We are basically interested in the static bifurcation and multiplicity of the stationary solutions of the equation under the following assumptions on $f$:

(A1) $f$ is a bounded function on $\overline{\Omega} \times \mathbb{R}$ with

$$\liminf_{s \to +\infty} f(x, s) \geq \overline{f} > 0, \quad \limsup_{s \to -\infty} f(x, s) \leq -\underline{f} < 0$$

uniformly for $x \in \overline{\Omega}$ (where $\overline{f}$ and $\underline{f}$ are independent of $x$);

(A2) $f(x, s) = o(|s|)$ as $|s| \to 0$ uniformly with respect to $x \in \overline{\Omega}$.

3.1. Mathematical setting. Let $H = L^2(\Omega)$ and $V = H^1_0(\Omega)$. By $(\cdot, \cdot)$ and $|\cdot|$ we denote the usual inner product and norm on $H$, respectively. The norm $|| \cdot ||$ on $V$ is defined as

$$||u|| = \left( \int_{\Omega} |\nabla u|^2 dx \right)^{1/2}, \quad u \in V.$$

Denote $A$ the operator $-\Delta$ associated with the homogenous Dirichlet boundary condition. $A$ is a sectorial operator and has a compact resolvent. Denote

$$0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots$$

the eigenvalues of $A$.

The equation (7) can be reformulated in an abstract form on $V$:

$$u_t + Au = \lambda u + \tilde{f}(u), \quad u = u(t) \in V,$$

(8)

where $\tilde{f}(u)$ is the Nemitski operator from $V$ to $H$ defined by

$$\tilde{f}(u)(x) = f(x, u(x)), \quad u \in V.$$

By the basic theory on evolution equations, see\cite{8, 23}, the equation (8) is well-posed on $V$. Specifically, for any $u_0 \in V$, there exists a unique solution $u(t)$ of (8) on the interval $[0, T)$ for some $T > 0$. 
Denote $\Phi_\lambda$ the semiflow generated by (8). Then it follows from (A1) that $\Phi_\lambda$ is a global semiflow. Namely, for each $u_0$, 
\[ u(t) = \Phi_\lambda(t)u_0 \]
is the solution of the equation on $\mathbb{R}_+$ with initial value $u(0) = u_0$. It is trivial to see that $\Phi_\lambda$ depends on $\lambda$ continuously. Moreover, using a standard argument (see [20, Chapter 1, Theorem 4.4]), one can easily verify that the family $\Phi_\lambda$ is $\lambda$-u.a.c.

Let $L = A - \mu_k$, where $\mu_k$ is an eigenvalue of $A$. The space $H$ can be decomposed into the orthogonal direct sum of its subspaces $H^-, H^0$ and $H^+$ corresponding to the negative, zero and positive eigenvalues of $L$, respectively. It is trivial to see that both $H^-$ and $H^0$ are finite-dimensional. Denote $P^\sigma$ ($\sigma \in \{0, \pm\}$) the projection from $H$ to $H^\sigma$.

Set
\[ V^\sigma = V \cap H^\sigma, \quad \sigma \in \{0, \pm\}. \]
Then one can see that $V^-$ and $V^0$ coincide with $H^-$ and $H^0$, respectively, as $H^-$ and $H^0$ are both finite-dimensional. We also have
\[ V = V^- \oplus V^0 \oplus V^+. \]
Set $W := V^- \oplus V^0$, and let 
\[ P_W = P^- + P^0 \]
be the projection from $V$ to $W$.

**Lemma 3.1.** (see [9]). Assume $\lambda \leq \mu_k + \eta$, where $\eta = (\mu_{k+1} - \mu_k)/2$. Then there exists $\rho_0 > 0$ (independent of $\lambda$) such that for any solution $u = u(t)$ of (8) on $\mathbb{R}_+$,
\[ \|u^+(t)\|^2 \leq \|u_0^+\|^2 e^{-\eta t} + \rho_0^2(1 - e^{-\eta t}), \quad \forall t \geq 0. \]
Here $u^+ = P^+u$.

**Remark 4.** By Lemma 3.1, we easily deduce that $K_\infty(\Phi_\lambda) \subset \Xi_\rho$ for all $\lambda \leq \mu_k + \eta$, where
\[ \Xi_\rho = \{v \in V : \|P^+v\| \leq \rho\}. \]
Furthermore, if $\rho > \rho_0$ then $\Xi_\rho$ is positively invariant under the semiflow $\Phi_\lambda$.

For any $0 \leq a < b \leq \infty$, denote 
\[ \Xi_\rho[a, b] = \{u \in \Xi_\rho : a \leq |w| \leq b\}, \]
where $w = P_Wu$.

**Lemma 3.2.** (see [9]). Let $\eta$ and $\rho_0$ be as in Lemma 3.1, and let $\rho > \rho_0$. Then there exist $R_0, c_0 > 0$ such that
1. if $\lambda \in [\mu_k, \mu_k + \eta]$, then for each solution $u(t)$ of (8) in $\Xi_{\rho}[R_0, \infty]$, we have
\[ \frac{d}{dt}|w(t)|^2 \geq c_0|w(t)|, \quad (9) \]
where $w(t) = P_Wu(t)$; and
2. for any $R > R_0$, there exists $0 < \theta \leq \eta$ such that the differential inequality (9) holds true for each $\lambda \in [\mu_k - \theta, \mu_k]$ and any solution $u(t)$ of (8) in $\Xi_{\rho}[R_0, R]$.

**Lemma 3.3.** (see [9]). Let the condition (A1) hold. Then $K_\infty(\Phi_\lambda)$ is uniformly bounded in $V$ for $\lambda \in [\mu_k, \mu_k + \eta]$, and
\[ h(\Phi_\mu, K_\infty(\Phi_\mu)) = h(\Phi_\lambda, K_\infty(\Phi_\lambda)) = \Sigma^{\rho + r}, \quad (10) \]
where $p$ is the sum of the multiplicities of the eigenvalues $\mu_i$ ($0 \leq i \leq k - 1$) of $A$, and $r$ the multiplicity of $\mu_k$. 

MULTICIPITY OF SOLUTIONS NEAR RESONANCE 3357
Lemma 3.4. (see [9]). Let the condition (A1) hold. Then \( S_\lambda := K_\infty(\Phi_\lambda) \) is nonvoid for all \( \lambda \in \mathbb{R} \). Furthermore, there exists \( \delta > 0 \) such that the following assertions hold.

1. For each \( \lambda \in \Lambda_- := [\mu_k - \delta, \mu_k) \), \( S_\lambda \) has a Morse decomposition \( \mathcal{M} = \{M_\lambda^\infty, M_\lambda^1\} \). Furthermore, there is at least one connecting trajectory \( \gamma \) between \( M_\lambda^1 \) and \( M_\lambda^\infty \).

2. \( M_\lambda^1 \) remains uniformly bounded on \( \Lambda_- \), whereas

\[
\lim_{\lambda \to \mu_k^-} \min_{v \in M_\lambda^\infty} ||v|| = \infty. \tag{11}
\]

3. Each of the sets \( \mathcal{C}^1 \) and \( \mathcal{C}^\infty \) has a connected component \( \Gamma \) with \( \Gamma[\lambda] \neq \emptyset \) for all \( \lambda \in \Lambda_- \) where

\[
\mathcal{C}^1 = \bigcup_{\lambda \in \Lambda_-} (M_\lambda^1 \times \{\lambda\}), \quad \mathcal{C}^\infty = \bigcup_{\lambda \in \Lambda_-} (M_\lambda^\infty \times \{\lambda\}).
\]

3.2. Static bifurcation and multiplicity of stationary solutions. Now we pay our attention to the static bifurcation and multiplicity of stationary solutions of (8). Note that the equation (7) has a natural Lyapunov function \( J(u) \) defined by

\[
J(u) = \frac{1}{2}((|u|^2 + |\lambda||u|^2) - \int_{\Omega} F(x,u)dx), \quad u \in V,
\]

where \( F(x,s) = \int_0^1 f(x,t)dt \). Then this problem can be treated via the method of dynamical systems.

Lemma 3.5. Let the condition (A1) hold. Then there exists \( \delta > 0 \) and an open dense subset \( \mathcal{D} \) of \( \mathbb{R} \) such that if \( \lambda \in \Lambda_- = [\mu_k - \delta, \mu_k) \) \( \cap \mathcal{D} \), then \( \Phi_\lambda \) has at least three distinct equilibria.

Proof. In fact, the above result is just a consequence of conclusion 3 in [9, Theorem 5.8]. Here we give a sketch of the proof for the reader’s convenience.

Take an isolating neighborhood \( N_1 \) of \( S_{\mu_k} \) with

\[
\Xi_\mu[0,R_0] \subset N_1, \tag{12}
\]

where \( R_0 \) is the number given in Lemma 3.2. We may restrict \( \delta \) such that \( N_1 \) is also an isolating neighborhood of \( \Phi_\lambda \) for all \( \lambda \in \Lambda := [\mu_k - \delta, \mu_k + \delta] \). Let \( M_\lambda^1 = K_\infty(\Phi_\lambda, N_1) \). Then

\[
h(\Phi_\lambda, M_\lambda^1) \equiv \text{const.}, \quad \lambda \in \Lambda. \tag{13}
\]

Thus by Lemma 3.3 we deduce that

\[
h(\Phi_\lambda, M_\lambda^1) = h(\Phi_{\mu_k}, M_{\mu_k}^1) = h(\Phi_{\mu_k}, S_{\mu_k}) = \Sigma^{p+r}, \quad \lambda \in \Lambda. \tag{14}
\]

By slightly modifying the proof of [21, Theorem 2.1], it can be shown that there is an open dense subset \( \mathcal{D} \) of \( \mathbb{R} \) such that all the equilibria of \( \Phi_\lambda \) are hyperbolic if \( \lambda \in \mathcal{D} \). Now assume \( \lambda \in \Lambda_- \cap \mathcal{D} \). Let \( \mathcal{M} = \{M_\lambda^\infty, M_\lambda^1\} \) be the Morse decomposition of \( S_\lambda \) as in Lemma 3.4. Hence, there is at least one equilibrium \( e_\lambda^\infty \) in \( M_\lambda^\infty \). We will show that there is another equilibrium \( z_\lambda^\infty \) in \( M_\lambda^\infty \) with \( z_\lambda^\infty \neq e_\lambda^\infty \).

We argue by contradiction and suppose \( M_\lambda^\infty \) consists of exactly one hyperbolic stationary solution \( e_\lambda^\infty \). Then \( p(t, M_\lambda^\infty) = t^m \) for some \( m \geq 0 \). Now we consider the Morse equation of \( \mathcal{M} \):

\[
p(t, M_\lambda^1) + p(t, M_\lambda^\infty) = p(t, S_\lambda) + (1 + t)Q(t).
\]
MULTIPLECTY OF SOLUTIONS NEAR RESONANCE

Since $h(\Phi_\lambda, M^1_\lambda) = \Sigma^{p+r}$ and $h(\Phi_\lambda, S_\lambda) = \Sigma^p$ (see (65) in [9]), one has
\[ t^{p+r} + t^m = t^p + (1 + t)Q(t). \]
But this is impossible for any formal polynomial $Q(t)$ with coefficients in $\mathbb{Z}_+$, as the sum of the coefficients of the left-hand side does not equal that of the right-hand side.

Since $\Phi_\lambda$ has at least two distinct equilibria $e^c_\lambda$ and $e^\infty_\lambda$ with $e^c_\lambda \in M^1_\lambda$ and $e^\infty_\lambda \in M^\infty_\lambda$, we deduce that $\Phi_\lambda$ has at least three distinct equilibria.

**Remark 5.** By the choice of $N_1$ and the definition of $M^1_\lambda$, we deduce that for each $\lambda \in \Lambda_\cdot \cap D$, $\Phi_\lambda$ has at least two distinct equilibria outside the domain $N_1$.

Now we show our main results on some new multiplicity results of stationary solutions for the equation (8) near each eigenvalue $\mu_k$.

Denote $\Lambda_\cdot = [\mu_k - \delta, \mu_k)$, $\Lambda_+ = (\mu_k, \mu_k + \delta]$, and let $D$ be the open dense subset of $\mathbb{R}$ given in Lemma 3.5.

**Theorem 3.6.** Assume $f$ satisfies the hypotheses (A1)-(A2). Denote $W^c_{loc}(0)$ the local center manifold of $\Phi_{\mu_k}$ at the equilibrium point 0, and let $\phi$ be the restriction of $\Phi_{\mu_k}$ on $W^c_{loc}(0)$. Suppose 0 is an isolated equilibrium of $\Phi_{\mu_k}$ (i.e., an isolated stationary solution of (8) at $\lambda = \mu_k$). Then there exists $\delta > 0$ such that one of the following assertions holds:

1. 0 is an attractor of $\phi$. In this case, for each $\lambda \in \Lambda_-$ (resp., $\lambda \in \Lambda_+$), $\Phi_\lambda$ has at least two (resp., three) distinct nontrivial equilibria; and for each $\lambda \in (\Lambda_- \cup \Lambda_+) \cap D$, it has at least four distinct nontrivial equilibria; see Figure 1.

2. 0 is a repeller of $\phi$ (i.e., an attractor of the inverse flow $\phi^{-1}$). When this occurs, for each $\lambda \in \Lambda_-$ (resp., $\lambda \in \Lambda_+ \cap D$) $\Phi_\lambda$ has at least three (resp., four) distinct nontrivial equilibria; see Figure 2.

3. 0 is neither an attractor nor a repeller of $\phi$. In this case, for each $\lambda \in \Lambda_-$ (resp., $\lambda \in \Lambda_+$) $\Phi_\lambda$ has at least three (resp., two distinct) nontrivial equilibria; and for each $\lambda \in \Lambda_- \cap D$, it has at least four distinct nontrivial equilibria; see Figure 3.

**Theorem 3.7.** Assume $f$ satisfies the hypotheses (A1)-(A2). Denote $W^c_{loc}(0)$ the local center manifold of $\Phi_{\mu_k}$ at the equilibrium point 0, and let $\phi$ be the restriction of $\Phi_{\mu_k}$ on $W^c_{loc}(0)$. Suppose 0 is an isolated equilibrium of $\Phi_{\mu_k}$ (i.e., an isolated stationary solution of (8) at $\lambda = \mu_k$). Then there exists $\delta > 0$ such that one of the following assertions holds:
1. 0 is an attractor of \( \phi \). In this case, for each \( \lambda \in (\Lambda_- \cup \Lambda_+) \cap D \), it has at least four distinct nontrivial equilibria;

2. 0 is a repellor of \( \phi \) (i.e., an attractor of the inverse flow \( \phi^{-1} \)). When this occurs, for each \( \Lambda_- \cap D \), \( \Phi_\lambda \) has at least four distinct nontrivial equilibria;

3. 0 is neither an attractor nor a repellor of \( \phi \). In this case, for each \( \lambda \in \Lambda_- \cap D \), it has at least four distinct nontrivial equilibria.

**Proof.** In the following argument, we always assume that \( \delta > 0 \) is sufficiently small so that Lemma 3.4 and 3.5 remain valid.

1. Let \( N_1 \) be the isolating neighborhood of \( S_{\mu_k} \) given in the proof of Lemma 3.6. Then by Remark 5, for each \( \lambda \in \Lambda_- \cap D \), the system \( \Phi_\lambda \) always has at least two distinct equilibria outside \( N_1 \).

Pick an isolating neighborhood \( N_0 \) of 0 with 

\[
N_0 \subset B_V(\beta) \subset N_1
\]

for some \( \beta > 0 \), where \( B_V(\beta) \) denotes the ball in \( V \) centered at 0 with radius \( \beta \). We may restrict \( \delta \) so that both \( N_0 \) and \( N_1 \) are isolating neighborhoods of \( \Phi_\lambda \) for all \( \lambda \in \Lambda := [\mu_k - \delta, \mu_k + \delta] \). Let

\[
K_\lambda^1 = K_\infty(\Phi_\lambda, N_1), \quad i = 0, 1.
\]

Then for each \( i \),

\[
h(\Phi_\lambda, K_\lambda^i) = \text{const.}, \quad \lambda \in \Lambda. \quad \tag{15}
\]

As \( N_0 \) is an isolating neighborhood of 0, one easily verifies that

\[
d_H(K_\lambda^0, \{0\}) \to 0 \quad \text{as} \quad \lambda \to \mu_k. \quad \tag{16}
\]

We may assume \( N_0 \) is chosen sufficiently small so that the product formula of Conley index given in [20, Chap. II, Theorem 3.1] holds true. Therefore, we have

\[
h(\Phi_{\mu_k}, \{0\}) = \Sigma^p \wedge h(\phi, \{0\}). \quad \tag{17}
\]

where \( p \) is given in Lemma 3.3. By Example 2.10 in [9], one sees that

\[
h(\phi, \{0\}) = \Sigma^0.
\]

Thereby

\[
h(\Phi_{\mu_k}, \{0\}) = \Sigma^p \wedge \Sigma^0 = \Sigma^p.
\]

Then we infer from (15) that

\[
h(\Phi_\lambda, K_\lambda^0) = h(\Phi_{\mu_k}, K_{\mu_k}^0) = h(\Phi_{\mu_k}, \{0\}) = \Sigma^p, \quad \lambda \in \Lambda. \quad \tag{18}
\]

By (15) and Lemma 3.3 we also have

\[
h(\Phi_\lambda, K_\lambda^1) = h(\Phi_{\mu_k}, K_{\mu_k}^1) = h(\Phi_{\mu_k}, S_{\mu_k}) = \Sigma^{p+r}, \quad \lambda \in \Lambda. \quad \tag{19}
\]

Therefore \( K_\lambda^1 \neq K_{\lambda}^0 \), and hence

\[
K_\lambda^1 \setminus N_0 \neq \emptyset, \quad \lambda \in \Lambda. \quad \tag{20}
\]

For each \( \lambda \in \Lambda \), pick a \( v_\lambda \in K_\lambda^1 \setminus N_0 \). Let \( u_\lambda(t) \) be a bounded full trajectory of \( \Phi_\lambda \) in \( K_\lambda^1 \) with \( u_\lambda(0) = v_\lambda \). We claim that if \( \delta \) is chosen small enough then either \( \omega(u_\lambda) \setminus N_0 \neq \emptyset \) or \( \omega^*(u_\lambda) \setminus N_0 \neq \emptyset \). \( \tag{21} \)

Indeed, if not, there would exist a sequence \( \lambda_n \to \mu_k \) (as \( n \to \infty \)) such that both \( \omega(u_n) \) and \( \omega^*(u_n) \) are contained in \( N_0 \) and hence in \( K_{\lambda_n}^0 \), where \( u_n = u_{\lambda_n} \). Thus by (16) we deduce that

\[
\lim_{n \to \infty} \max_{v \in \omega(u_n)} |J(v)| = 0 = \lim_{n \to \infty} \max_{v \in \omega^*(u_n)} |J(v)|. \quad \tag{22}
\]
Set
\[ \Gamma_n = \text{orb}(u_n) = \text{orb}(u_n) \cup \omega(u_n) \cup \omega^*(u_n). \]
Then
\[ \min_{v \in \Gamma_n} J(v) = \min_{v \in \omega(u_n)} J(v), \quad \max_{v \in \Gamma_n} J(v) = \max_{v \in \omega^*(u_n)} J(v). \]
It follows by (22) that
\[ \max_{v \in \Gamma_n} |J(v)| \to 0 \quad \text{as } n \to \infty. \quad (23) \]

We infer from the asymptotic compactness of the skew-product flow of the family \( \{ \Phi_\lambda \} \) and \( \Gamma_n \subset K^1 \subset N_1 \) that \( \bigcup_{\lambda \in \Lambda} \Gamma_n \) is precompact. Hence by Lemma 2.1 it can be assumed that \( \Gamma_n \) converges to a nonempty compact invariant set \( K \) of \( \Phi_{\mu_k} \) (in the sense of Hausdorff distance). Clearly 0 is in \( K \). Since each \( \Gamma_n \) is connected, \( K \) is connected as well. (23) implies that \( J(v) \equiv 0 \) on \( K \). Thereby \( K \) consists of equilibrium points of \( \Phi_{\mu_k} \). On the other hand, because
\[ u_n(0) \in \Gamma_n \setminus N_0 \quad (24) \]
for all \( n \), we deduce that \( K \setminus \text{int} N_0 \neq \emptyset \). Further by the connectedness of \( K \) one concludes that \( K \cap \partial V \neq \emptyset \) for any small neighborhood \( V \) of 0, which contradicts the hypothesis that 0 is an isolated equilibrium of \( \Phi_{\mu_k} \), which completes the proof of our claim.

By virtue of (21), for each \( \lambda \in \Lambda \) we can pick an equilibrium
\[ e^*_\lambda \in (\omega(u_\lambda) \cup \omega^*(u_\lambda)) \setminus N_0 \]
of \( \Phi_\lambda \). Note that
\[ e^*_\lambda \in K^1 \subset N_1 \setminus N_0. \]
Hence for each \( \lambda \in \Lambda_- \), we conclude that \( \Phi_\lambda \) has at least two nontrivial equilibria \( e^*_\lambda \) and \( e^*_\lambda \).

We also infer from the attractor bifurcation theory (see e.g. Ma and Wang [13, Theorem 4.3], [12, Theorem 6.1] or Li and Wang [11, Theorem 4.2]) that \( K^0_\lambda \) contains at least two distinct equilibrium points \( e^1_\lambda \) and \( e^0_\lambda \) for \( \lambda \in \Lambda_+ \), provided \( \delta \) is sufficiently small. In conclusion, \( \Phi_\lambda \) has at least three distinct nontrivial equilibria for \( \lambda \in \Lambda_+ \).

We now assume \( \lambda \in \Lambda \cap D \) and prove that if \( \delta > 0 \) is small enough, then \( \Phi_\lambda \) has at least two distinct nontrivial equilibrium points in \( N_1 \setminus N_0 \).

Suppose the contrary. Then there would exist a sequence \( \lambda_n \to \mu_k \) such that for each \( n \), \( \Phi_\lambda \) has exactly one equilibrium \( e_n = e^0_{\lambda_n} \) in \( N_1 \setminus N_0 \). There are two possibilities.

(i) \( \lim_{n \to \infty} |J(e_n)| = 0 \). When this occurs we first show that there are no connecting orbits between \( e_n \) and \( K^0_\lambda = K^0_{\lambda_n} \), provided \( n \) is sufficiently large. Based on this fact we further verify that
\[ K^1_n := K^1_{\lambda_n} = \{ e_n \} \cup K^0_n, \quad (25) \]
and hence \( M = \{ e_n \}, \ K^0_1 \) forms a Morse decomposition of \( K^1_n \).

Suppose that there is a subsequence of \( \{ n \}_{n=1}^\infty \) still denoted by \( \{ n \}_{n=1}^\infty \), such that for each \( n \), there is a connecting orbit \( \gamma_n \) between \( e_n \) and \( K^0_\lambda \). Let \( \Gamma_n = \tau_n \). Then \( \Gamma_n \) is a connected compact invariant set with
\[ e_n \in \Gamma_n, \quad \Gamma_n \cap K^0_n \neq \emptyset. \]
We observe that
\[ \min_{v \in \tilde{K}_n} J(v) \leq \min_{v \in \Gamma_n} J(v) \leq \max_{v \in \Gamma_n} J(v) \leq \max_{v \in \tilde{K}_n} J(v), \tag{26} \]
where \( \tilde{K}_n = \{e_n\} \cup K^0_n \). By (16) and the assumption that \( \lim_{n \to \infty} |J(e_n)| = 0 \), one can easily see that \( \max_{v \in \tilde{K}_n} |J(v)| \to 0 \) as \( n \to \infty \). Thus we conclude by (26) that
\[ \lim_{n \to \infty} \max_{v \in \Gamma_n} J(v) = 0. \tag{27} \]
Repeating the same argument below (23) with \( u_n(0) \) in (24) replaced by \( e_n \), one can obtain a contradiction.

We now check the validity of (25) for sufficiently large \( n \). Again we argue by contradiction and suppose there is a subsequence of \( \{n\}^\infty_{n=1} \), still denoted by \( \{n\}^\infty_{n=1} \), such that \( K^1_n \not= \{e_n\} \cup K^0_n \) for each \( n \). Then one necessarily has
\[ K^1_n \setminus (N_0 \cup \{e_n\}) \not= \emptyset, \quad n \geq 1. \]
For each \( n \), pick a \( v_n \in K^1_n \setminus (N_0 \cup \{e_n\}) \). Let \( u_n(t) \) be a bounded full trajectory of \( \Phi_{\gamma_n} \) in \( K^1_n \) with \( u_n(0) = v_n \). Because there is no connecting orbit between \( e_n \) and \( K^0_n \), we deduce that
\[ \text{either } \omega(u_n) = \omega^*(u_n) = \{e_n\}, \quad \text{or } \omega(u_n) \cup \omega^*(u_n) \subset K^0_n. \]
Clearly the first case cannot occur. Hence both \( \omega(u_n) \) and \( \omega^*(u_n) \) are contained in \( K^0_n \). Let \( \Gamma_n = \text{orb}(u_n) \). Then as in the argument leading to (27) it can be shown that \( \lim_{n \to \infty} \max_{v \in \Gamma_n} |J(v)| = 0 \). Further by repeating the argument following (23) one can obtain a contradiction.

Now since \( e_n \) is hyperbolic, we have \( p(t, e_n) = t^m \) for some \( m \geq 0 \). By (18) and (19) we also have
\[ p(t, K^0_n) = t^p, \quad p(t, K^1_n) = t^{p+r}. \]
The Morse equation of \( M \) then reads
\[ t^m + t^p = t^{p+r} + (1 + t)Q(t) \]
for some formal polynomial \( Q(t) \), which leads to a contradiction!

(ii) Either \( \limsup_{n \to \infty} J(e_n) > 0 \), or \( \liminf_{n \to \infty} J(e_n) < 0 \).

Without loss of generality we may assume
\[ \limsup_{n \to \infty} J(e_n) := 2J_+ > 0. \]
Hence there is a subsequence of \( e_n \), still denoted by \( e_n \), such that
\[ J(e_n) > J_+, \quad n \geq 1. \]
We claim that if \( n \) is sufficiently large, then for any full trajectory \( \gamma(t) \) in \( K^1_n \setminus (\{e_n\} \cup K^0_n) \), we have
\[ \omega^*(\gamma) = \{e_n\}, \quad \omega(\gamma) \subset K^0_n, \]
hence \( M = \{K^0_n, \{e_n\}\} \) forms a Morse decomposition of \( K^1_n \). Indeed, if this was false, there would exist a subsequence of \( \{n\}^\infty_{n=1} \), still denoted by \( \{n\}^\infty_{n=1} \), such that for each \( n \), there is a full trajectory \( \gamma_n(t) \) in \( K^1_n \setminus (\{e_n\} \cup K^0_n) \) such that
\[ \text{either } \omega(\gamma_n) = \omega^*(\gamma_n) = \{e_n\}, \quad \text{or } \omega(\gamma_n) \cup \omega^*(\gamma_n) \subset K^0_n. \]
The first case in fact cannot occur. Hence both $\omega(\gamma_n)$ and $\omega^*(\gamma_n)$ are contained in $K_n^0$. Since $\gamma_n$ is not contained in $K_n^0 = K_\infty (\Phi_{\lambda_n}, N_0)$, we deduce that there exists $t_0 \in \mathbb{R}$ such that $\gamma_n(t_0) \notin N_0$.

Let $\Gamma_n = \text{orb}(\gamma_n)$. Then using some similar argument as in (i), one can easily verify that $\max_{v \in \Gamma_n} |J(v)| = 0$ as $n \to \infty$. Further repeating the same argument below (23) with minor modifications, one can get a contradiction and completes the proof of the claim.

Now as in the first case (i), by checking the Morse equation of the Morse decomposition $M$ of $K_\lambda^1$, we immediately obtain a contradiction. Therefore for each $\lambda \in \Lambda \cap D$, $\Phi_\lambda$ has at least two distinct nontrivial equilibria in $N_1 \setminus N_0$. As the equation has at least two distinct nontrivial equilibria outside $N_1$ when $\lambda \in \Lambda_- \cap D$ and in $N_0$ when $\lambda \in \Lambda_+ \cap D$, respectively, we finally conclude that it has at least four distinct nontrivial equilibria for each $\lambda \in (\Lambda_- \cup \Lambda_+) \cap D$.

2. Assume 0 is a repeller of $\Phi$. Then as in 1, we infer from the attractor bifurcation theory (see e.g. Ma and Wang [13, Theorem 4.3], [12, Theorem 6.1] or Li and Wang [11, Theorem 4.2]) that $K_\lambda^0$ contains at least two distinct equilibria $e_1^\lambda$ and $e_2^\lambda$ for $\lambda \in \Lambda_-$. Since $\Phi_\lambda$ has at two distinct nontrivial equilibrium points outside $N_1$ for $\lambda \in \Lambda_- \cap D$, we conclude that it has at least four distinct nontrivial equilibria for each $\lambda \in \Lambda_- \cap D$.

3. Now we consider the case where 0 is neither an attractor nor a repeller of $\Phi$, in which we deduce by Li and Wang [11, Theorem 4.4] that the system $\Phi_\lambda$ bifurcates at each side of $\mu_k$ a nonempty compact invariant set $K_\lambda \subset N_0$ with $0 \not\in K_\lambda$ and $d_H (K_\lambda, \{0\}) \to 0$ as $\lambda \to \mu_k$.

$K_\lambda$ contains at least one nontrivial equilibrium $e_1^\lambda$.

We show that $h(\Phi_{\mu_k}, \{0\}) \neq \Sigma^{p+r}$, (28)

which fact will yield another equilibrium $e_2^\lambda \in N_1 \setminus N_0$ at both sides of $\mu_k$.

Consider the local center-unstable manifold $W_{cu}^{loc}(0)$ of $\Phi_{\mu_k}$ at 0. Denote $\psi$ the restriction of $\Phi_{\mu_k}$ on $W_{cu}^{loc}(0)$. Then $h(\Phi_{\mu_k}, \{0\}) = h(\psi, \{0\})$. (29)

Thus to prove (28), it suffices to check that $H_*(h(\psi, \{0\})) \neq H_*(\Sigma^{p+r})$, (30)

where (and below) $H_*(\cdot)$ and $H^*(\cdot)$ denote the singular homology and cohomology functors, respectively. We argue by contradiction and suppose the contrary. Then $H_{p+r}(h(\psi, \{0\})) = H_{p+r}(\Sigma^{p+r}) = \mathbb{Z}$.

Therefore by the Poincaré-Lefschetz duality theory of the Conley index (see McCord [16, Theorem 2.1] and Mrozek and Srzednicki [18, p. 164]),

$H^0(h(\psi^{-1}, \{0\})) = H_{p+r}(h(\psi, \{0\})) = \mathbb{Z}$.

On the other hand, pick a path-connected isolating block $B \subset W_{cu}^{loc}(0)$ of $S_0 = \{0\}$ with respect to the inverse flow $\psi^{-1}$ (such an isolating block is always available due to [5, Theorem 1.5]). Then since $S_0$ is not an attractor of $\psi^{-1}$ on the center-unstable manifold $W_{cu}^{loc}(0)$ (recall that $S_0$ is not an attractor of $\psi^{-1}$ on $W_{loc}(0)$), we
necessarily have $B^- \neq \emptyset$. Thus by the basic knowledge in the theory of algebraic topology one easily deduces that $H^0(B, B^-) = 0$. Hence

$$H^0(h(\psi^{-1}, \{0\})) = H^0(B, B^-) = 0,$$

which leads to a contradiction and justifies the validity of (28).

Recall that (see (19))

$$h(\Phi, K) = \sum^{p+r}, \lambda \in \Lambda = [\mu_k - \delta, \mu_k + \delta].$$

Note that

$$h(\Phi, K^0) = h(\Phi_{\mu_k}, K^0_{\mu_k}) = h(\Phi_{\mu_k}, \{0\}) \neq \sum^{p+r}, \forall \lambda \in \Lambda.$$

Therefore $K^1_\lambda \neq K^0_\lambda$, and hence

$$K^1_\lambda \setminus N_0 \neq \emptyset, \lambda \in \Lambda.$$

We are now in a position as in (20). Repeating the same argument below (20), it can be shown that the system has an equilibrium $e^*_\lambda$ in $N_1 \setminus N_0$.

In conclusion, there are at least two distinct nontrivial equilibria in $N_1$ for $\lambda \in \Lambda \setminus \{\mu_k\}$. Combining this result with what we have proved at the beginning of the proof of the first conclusion 1, one immediately concludes the validity of 3.

**Remark 6.** It is interesting to note that the equation always has four distinct nontrivial stationary solutions if $\lambda \in \Lambda_+ \cap D$.

**Remark 7.** Dual versions of all the results in this section hold true if, instead of (A1) (or (3)), we assume that

$$\limsup_{s \to +\infty} f(x, s) \leq -\overline{f} < 0, \quad \liminf_{s \to -\infty} f(x, s) \geq \underline{f} > 0 \quad (31)$$

uniformly for $x \in \overline{\Omega}$.

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