SVD inversion for the bi-dimensional Conical Radon Transform

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Abstract.
In nuclear medicine, the Conical Radon Transforms family includes 2D and 3D models for scatter radiation imaging. The underlying idea is taking advantage of Compton effect in order to provide image reconstruction without rotating the detector system. Several techniques currently applied for the classical Radon transform may also be useful in this new context. Nevertheless, since the physical model is rather different, it is necessary to assess the performance of each method. In previous works, we have studied inversion by filtered-backprojection and adaptive algebraic reconstruction techniques. In this paper we study the performance of a method based in the factorization of the projection matrix in order to invert the projections. Particularly, we test our algorithms on the bi-dimensional Conical Radon Transform, a simple member of the Conical Radon family. We obtain useful results on the performance of the three methods which should be valuable in the more realistic 3D context.

1. Introduction
Although gamma radiation is required as means of investigation thanks to its privileged properties of penetration, it suffers severely from Compton scattering phenomenon which degrades the quality of images acquired by the detectors. The Compton scattering radiation is due to the presence of electric charges in the propagation medium. This event causes a lose of the energy of the incident radiation and also a deviation of its trajectory characterized by an angle $\omega$ with respect to its initial direction angle. It is probably the most important cause of degradation in gamma-ray imaging.

A new imaging principle that takes advantage of Compton scattering to perform reconstructions is based on the fact that the energy measured in a detected photon contains information about its direction after a first-order Compton scattering. The Conical Radon Transform family is a mathematical model of this new principle for camera gamma imaging [1–5]. Image reconstruction from its projections is guaranteed since the Conical Radon Transforms are analytically invertible [6]. This model provides a theoretical basis for the development of a nonmoving camera gamma with a fixed collimator.

Some aspects of the bi-dimensional Conical Radon Transform (TV) have been already studied and both, analytical [1, 7, 8] and algebraic [9] inversion have been obtained. In particular, it was shown that the TV can be inverted by a filtered back-projection technique (FBP) completely adapted to the geometry of the problem. The FBP method is widely used in applications of the classical Radon transform as it provides the most efficient analytical algorithm for inversion.
However, this method performs well when many projections are available but the image quality is significantly reduced when a limited number of projections is used. It was shown in [6] that an iterative algebraic reconstruction technique (ART) is superior when the number of projections is limited. Another drawback of FBP for the TV case is the upper V-shaped artifact in the image probably due to the discretization grid. Reconstructions carried out with ART exhibit the same artifact suggesting that the problem is linked to the model itself. We focus here on an alternative reconstruction technique, the Singular Value Decomposition (SVD) [10, 11] which has been previously used in related problems of image reconstruction from projections [12–14]. In particular, we study the inversion of the TV transform using SVD, and we compare its reconstruction performance with our previous results.

2. The bi-dimensional Conical Radon Transform

In this context, the object under study is thought as a flat three-dimensional object, i.e., of very low thickness, in which a non uniform radioactivity source distribution exists. A collimated linear detector, set parallel to the plane of the object, absorbs the photons coming from a perpendicular direction, after Compton scattering. The situation is represented in Figure 1.

The source of radiations is represented by a nonnegative integrable function \( f(x, y) \) with bounded support. Let \( E_0 \) be the energy of primary photons emitted by the object. We consider the total number of photons arriving on the detector with an energy \( E \leq E_0 \) resulting from a single Compton scattering. Calling \( D \) a detecting site on the \( x \) axis, the collision points \( M \) are on a line perpendicular to the detector at point \( D \). When the detector is fixed to detect exactly one photon energy, the scattering angle is fixed and the point sources \( N \) which will contribute to the measures lie on two lines that form an angle \( \omega \) with the line \( MD \). The photon flux density measured at a detecting site \( D \) is the sum of scattered radiation flux densities outgoing from the set of scattering sites \( M \) lying along the axis of the collimator at \( D \).

Then, the data acquisition may be formulated as follows:

\[
\tilde{g}(\xi, \omega) = K(\omega) \int l \frac{d\eta}{\eta} \int_0^\infty \frac{dr}{r} [f(\xi r \sin \omega, \eta + r \cos \omega) + f(\xi r \sin \omega, \eta - r \cos \omega)]
\]

(1)

where \( \xi, \eta, l \) and \( r \in \mathbb{R} \) and \( 0 < \omega < \pi \). Here \( \tilde{g}(\xi, \omega) \) is the measured photon flux density at \( D \) under a scattering angle \( \omega \), using the Cartesian coordinates of Figure 1. For ease of notation, we included all physical factors resulting from Compton scattering into one variable \( K(\omega) \). This variable contains the square of the classical electron radius, the average electron density, and the Klein-Nishina scattering probability function.

The inner integral with \( \eta = 0 \) is called the TV transform.

\[
g(\xi, \omega) = [TVf](\xi, \omega) = K(\omega) \int_0^\infty \frac{dr}{r} [f(\xi r \sin \omega, r \cos \omega) + f(\xi - r \sin \omega, r \cos \omega)]
\]

(2)

So the TV is a particular case of (1) where \( D \) and \( M \) are coincident. While this assumption leads to a less realistic model since the detection and collision sites are coincident, the model is mathematically simple and useful to study issues concerning the discretization without being obscured by other problems appearing in (1) such as multiple integrals or divergences (\( r = 0 \) case) that make necessary to regularize the integral by applying further mathematical procedures.

The value of \( g(\xi, \omega) \) gives the detected photon flux density at a detector labeled by \( \xi \in \mathbb{R} \) from angle \( 0 \leq \omega < \pi/2 \). This transformation has an analytic inverse [2] which rests on the action of a cosine-Fourier transform. We have also established the inversion by FBP in [7, 8]. In what follows, we will eventually refer to \( g(\xi, \omega) \) as the projections.
2.1. The Point Spread function

A very useful formulation of the \( T^V \) transform reads as follows

\[
g(\xi, \omega) = \int \int dx dy \text{PSF}_V(\xi, \omega | x, y) f(x, y) \tag{3}
\]

where the Point Spread function \( \text{PSF}_V \) is the impulse response of the operator \( T^V \), i.e., the image of a source point

\[
f(x, y) = \delta(x - x_0)\delta(y - y_0). \tag{4}
\]

Some easy calculations show that

\[
\text{PSF}_V(\xi, \omega | x_0, y_0) = K(\omega) \frac{1}{y_0} [\delta(x_0 - \xi + y_0 \tan \omega)\delta(x_0 - \xi - y_0 \tan \omega)]. \tag{5}
\]

The \( \text{PSF}_V \) is also called the kernel of the \( T^V \) transform.

3. Analytic and algebraic reconstruction

There are different ways to recover the image of the source object from projections. We focus here on two of the most relevant approaches, analytic techniques, which we mention succinctly, and algebraic techniques, which are our main interest in this work.

3.1. Analytic methods

Analytical methods are methods of reconstruction in the ideal case of continuous variables. They rely on individual analytical relations between \( f(x, y) \) and its projections \( g(\xi, \omega) \). Detailed implementations for FBP in the \( T^V \) case can be found elsewhere [1, 7, 8].
3.2. Algebraic methods

In this approach, the projection formula is discretized leading to a system of linear equations. Then, we search for \( f(x, y) \) in a space of finite dimension \( n = N \times N \), as a linear combination of basis functions \( \Phi_i \):

\[
f(x, y) = \sum_{i=0}^{n} f_i \Phi_i(x, y).
\]

where \( \Phi_i \) are the characteristic function of the pixels, defined by:

\[
\Phi_i(x, y) = \begin{cases} 
1, & \text{if } (x, y) = \text{pixel } i \\
0, & \text{otherwise}
\end{cases}
\]

Now, the value of the projection at one point is

\[
g_j = g(\xi_l, \omega_k) = [T^V f](\xi_l, \omega_k) = \sum_{i=1}^{n} f_i [T^V \Phi_i](\xi_l, \omega_k) = \sum_{i=1}^{n} A_{ji} f_i,
\]

where \( g_j \) is the value of the projection with angle \( \omega_k \) at detector \( \xi_l \).

Then, the matrix formulation of the image formation process reads

\[
Af = g
\]  

(6)

Here \( g \) is the projection vector (observed data), each of its components is a projection value, and its size is \( m \times 1 = DP \times 1 \), i.e., the number of detectors \( D \) times the number of angles \( P \); \( f \) is the image vector, each component is the intensity of a pixel of the source, its size \( n \) is equal to the number of pixels \( N \times N \), and finally \( A \) is the projection matrix, of size \( m \times n = DP \times N^2 \).

This algebraic formulation of the problem is, in some sense, the discrete relative of equation (3). Since the algebraic problem is sometimes ill-posed (noise usually puts in evidence ill-posedness of the problem), a straightforward inversion of matrix \( A \) is not always possible. Algebraic methods provide reconstruction using either iterative [6, 9, 14–17] or single step [14] algorithms. In scattered radiation problems, we have already tested the adaptive algebraic reconstruction technique with a random permutation scheme (RPS-AAAR) [9]. We now present an alternative inversion method, often used in the context of ill-posed problems, the SVD.

3.2.1. Singular value decomposition

The SVD [10, 11] is a non iterative technique that enables algebraic inversion for the problem involving a matrix \( A_{m \times n} \) with \( m \geq n \). Singular values are the square roots of the eigenvalues of the square matrix \( AA^t \). SVD allows factorization for a matrix in the form:

\[
A = U_{m \times m} \times S_{m \times n} \times V^t_{n \times n}
\]  

(7)

where \( U \) and \( V \) are orthogonal matrices whose columns are eigenvectors of \( AA^t \) and \( A^t A \) respectively and \( S \) is a diagonal matrix containing the singular values of \( A \). This factorization allows us to write the pseudo inverse \( A^t \) for \( A \):

\[
A^t = V \times S^{-1} \times U^t
\]

and thus the inversion problem is solved as:

\[
f = (V \times S^{-1} \times U^t) g
\]  

(9)

SVD allows accurate inversion and provides a practical way to solve the problem of instabilities due to limited numerical resolution by truncating the smallest singular values [10].
Table 1: Projection matrices for different angular discretizations with $N = 64$ and $D = 64$.

| Number of angles (P) | rows   | columns | Condition number | PSF     | $R$          |
|----------------------|--------|---------|------------------|---------|--------------|
| 295                  | 18880  | 4096    | $1.82 \times 10^4$ | —       |              |
| 64                   | 4096   | 4096    | $3.20 \times 10^9$ | $3.53 \times 10^{14}$ |              |
| 32                   | 2048   | 4096    | $2.92 \times 10^4$ | $5.33 \times 10^6$ |              |
| 16                   | 1024   | 4096    | $5.82 \times 10^3$ | $1.73 \times 10^5$ |              |

4. Numerical results

In this work, all matrices, projections and reconstructions were implemented in a machine with an $\epsilon = 2.2204 \times 10^{-16}$. This parameter conditions the problem numerically. We compare reconstructions made with SVD with those previously obtained using FBP and RPS-AART.

4.1. Projection matrices

We considered two different ways of generating the projection matrix $[T \mathbf{V} \Phi](\xi, \omega) = A_{ji}$

(i) by discretizing the analytical formula of the kernel of the TV transform from equation (5).

Each column of the projection matrix is the discretization of $PSF_{\mathbf{v}}(\xi, \omega|x_0, y_0)$ for a given point $(x_0, y_0)$. We call this matrix $PSF$.

(ii) by using an algorithm that performs the numerical calculation of the $TV$ defined by (2) of every basis element $\Phi_i$, $i = 1, ..., n$. Each column of $A$ is the $TV$ of a basis element. We call this matrix $R$.

In both cases, matrix dimension depends on the discretization of spacial and angular parameters: if $\xi$ in (2) is discretized by $D$ values and $\omega$ by $P$ values, the size of $A$ is $DP \times N^2$.

We considered $64 \times 64$ pixel images and different number of angles ($P$) between 16 and 295. The number of detectors was fixed to $D = N = 64$. For the case (ii), where the projection matrix $R$ is generated by transforming each element of the basis, we use a $dr = 0.01$ in the discretization of equation (2). Table 1 summarizes the properties and dimensions of the projection matrices. Notice that, for the 295 angles case, only the $PSF$ was calculated because the computational cost of the $R$ matrix is very high. Figures 2 and 3 show the SVD spectra of these matrices, i.e., the diagonal elements of matrix $S$ in (7).

Our data $g(\xi, \omega)$, are the projections of a $64 \times 64$ Shepp-Logan phantom simulated either by direct transformation using (2) or by multiplying the projection matrix $PSF$ times the image in vector form.

4.2. Reconstructions

We compare results using three different reconstruction techniques: FBP and 20 iterations RPS-AART (both presented in previous works and included here for comparison) and SVD. We used different angular resolutions ($P = 16, 32$ and $64$), for both types of projection matrices, $R$ and $PSF$ (Figure 4). Figure 5 shows the three reconstructions for a fixed angular resolution ($P = 295$ angles), using $PSF$ matrix in the case of algebraic methods. Notice that there is no $R$ matrix for $P = 295$ because its calculation takes too much time. Two figures of merit were considered in order to assess the results [9]: mean square error (MSE) and correlation. Table 2 shows their values for all the reconstructions. Notice that the SVD reconstruction for the $PSF$ with $P = 64$ is very poor: from Table 2 we see that the MSE is almost 3 and Figure 5 m) shows the reconstructed image. Nevertheless, truncating the SVD at the $4046^{th}$ singular
value, the following figures of merit are obtained: $\text{MSE} = 0.1730$ and $\text{Corr} = 0.9862$, and the reconstructed image is shown in Figure 6 a). Figures 6 b)\textemdash} c)\textemdash} d) show the MSE as a function of the number of singular values kept after regularization. For example, a truncation index of 500 means that the first 500 singular values were used in the reconstruction, while the rest was discarded. Figure 6 b) illustrates the MSE of a well-conditioned matrix in the sense of [10] (SVD decomposition of a square $R$ matrix for a $32 \times 32$ case with $D = P = 32$ and condition number $= 9.56 \times 10^{11}$) where a $MSE = 0$ is achieved without truncation. Figure 6 c) shows the MSE for four truncated SVD reconstructions with a small number of angular projections for both types of matrices, the truncation index $T$ moves between 1 and $DP$. Figure 6 d) shows the MSE for the SVD reconstruction using the $PSF$ ($P = 64$) and truncating at different levels between 3840 and 4096 (256 last singular values).
Figure 4: Shepp-Logan phantom reconstructions with different angular resolutions (number of angles $P = 16, 32$ and 64) and methods. Projections were generated using equation (2) (a-i) and equation (6) with $A = PSF$ (j-o).
5. Discussion

The problem of inverting the $TV$ transform using a singular value decomposition is numerically conditioned by the machine epsilon, $\epsilon = 2.2204 \times 10^{-16}$ in our case. The condition numbers of the matrices we used are shown in Table 1. In Figure 4 m) it can be seen that the reconstruction with the PSF matrix for the $P = 64$ case is not stable. This fact is explained by the high condition number of the matrix. The SVD spectra for this case exhibits the smallest singular values ($< 10^{-12}$) and an important gap, more than five orders of magnitude, between consecutive singular values at the 4046th value (Figure 2). On the other hand, truncating the smallest
singular values improves significantly the result. The gap indicates the optimal truncation index for regularization. We have an ill-conditioned matrix [10] with a well-determined optimal truncation index ($T = 4046$) making it possible to regularize the solution. This fact is confirmed by the graph showing the $MSE$ for reconstructions using different number of singular values (Figure 6 d)) where it can be seen that for indexes higher than 4046, the $MSE$ suddenly increases. Graphical and numerical results indicate that the SVD method (with or without truncation, depending on the condition number) clearly outperforms FBP and RPS-AART when enough angular projections are available. This is explained by the SVD factorization general hypothesis which states that the number of rows in the matrix should be greater than or equal to the number of columns: since the number of detectors was kept constant ($D = 64$), the number of angular projections determines the final dimension of the matrix. On the other hand, results are better for the RPS-AART when a limited number of projections is used.

### Table 2: MSE and correlation for each reconstruction technique without truncation in SVD

| Angular resolution [rad] | Number of angles (P) | PSF R | PSF R |
|--------------------------|----------------------|-------|-------|
| 0.005                    | 295                  | 0.98  | 0.11  |
| 0.023                    | 64                   | 1.10  | 0.27  | 0.28  | 2.99  | 0.00  |
| 0.050                    | 32                   | 1.73  | 0.32  | 0.32  | 0.58  | 0.78  |
| 0.100                    | 16                   | 2.44  | 0.34  | 0.45  | 0.93  | 1.56  |
| MSE                      |                      |       |       |
| 0.005                    | 295                  | 0.68  | 0.93  | 1.00  |
| 0.023                    | 64                   | 0.66  | 0.80  | 0.82  | 0.04  | 1.00  |
| 0.050                    | 32                   | 0.54  | 0.80  | 0.79  | 0.79  |
| 0.100                    | 16                   | 0.39  | 0.75  | 0.70  | 0.69  | 0.66  |

### 6. Conclusion
We tested the SVD technique for inverting the TV-transform. The method outperforms RPS-AART and FBP when the number of projections is enough to ensure the $m > n$ case, required in the SVD theory. With a low number of projections, RPS-AART is superior. Matrices were well-conditioned with the exception of one case. The instability can be overcome by a proper truncation of the smallest singular values: the SVD spectra shows that, although the matrix is ill-conditioned, the optimal truncation index is well-determined. Since our findings can be extended to other Conical Radon Transform family members, the results obtained provide a useful basis for an inversion problem including 2D and 3D reconstructions. Some concern appears when analyzing a possible 3D implementation of the reconstruction problem since its dimensions could make the SVD factorization computationally expensive. Additional work should be done in order to determine the prospect of using this technique in that context.

### References
[1] Morvidone M, Nguyen M K, Truong T T and Zaidi H 2010 *International Journal of Biomedical Imaging*, doi:10.1155/2010/208179 (Special Issue on Mathematical Methods for Images and Surfaces)
[2] Nguyen M K and Truong T T 2006 *Imagerie par Rayonnement Gamma Diffusé* (France: Hermes Science Publishing)
[3] Nguyen M K, Truong T T, Bui H D and Delarbre J L 2004 *Inverse Problems in Science and Engineering* 12

9
[4] Nguyen M K and Truong T T 2002 *Inverse Problems* **18** 265
[5] Delarbre J L 2005 *Imagerie d’émission gamma : Nouvelles méthodes d’exploitation du rayonnement diffusé* Ph.D. thesis Université de Cergy-Pontoise
[6] Herman G and Mayer L 1993 *IEEE Trans. on Med. Imaging* **12** 600
[7] Morvidone M, Nguyen M K, Truong T T and Zaidi H 2010 *Proc. 17th IEEE Int. Conf. on Image Processing* (Hong Kong: IEEE) p 629
[8] Morvidone M, Nguyen M K, Truong T T and Zaidi H 2011 *International Journal of Biomedical Imaging*, doi:10.1155/2011/913893
[9] Cebeiro J, Morvidone M and Rubio D 2013 *Proc. 4th Congress on Industrial Computational and Applied Mathematics* (Buenos Aires: ASAMCI)
[10] Hansen C 1986 *The truncated SVD as a method for regularization* (Stanford University Stanford, CA, USA: Stanford University)
[11] Burden R and Faires D 2010 (Boston, MA, USA: Brooks/Cole CENGAGE Learning)
[12] Selivanov V and Lecomte R 2001 *IEEE, Trans. Nucl. Sci.* **48** 761
[13] Shim Y S and Cho Z H 1981 *IEEE Transactions on Acoustics, Speech and Signal Processing* **29** 904
[14] Driol C 2008 *Imagerie par rayonnement Gamma diffusé à haute sensibilité* Ph.D. thesis Université de Cergy-Pontoise
[15] Guan H and Gordon R 1994 *Phys. Med. Biol.* **2005**
[16] Lu W and Yin F F 2004 *Medical Physics* **12** 3222
[17] Andersen A H and Kak A C 1984 *Ultrasonic Imaging* 81