Dunkl-spherical maximal function

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Abstract

In this paper, we study the $L^p$-boundedness of the spherical maximal function associated to the Dunkl operators.

Keywords: Dunkl operators; Dunkl transform; Dunkl translations; Spherical maximal function.

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1 Introduction and backgrounds

In [13], E. Stein introduced the spherical maximal function by

$$M(f)(x) = \sup_{r>0} \int_{S^{d-1}} f(x - ry) d\sigma(y),$$

where $d\sigma$ is the surface measure on $S^{d-1}$ and showed that $M$ is bounded from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for the optimal range $p > \frac{d}{d-1}$ with $d \geq 3$. The case $d = 2$ was proved by Bourgain in [1]. The aim of this work is to extend these results to the Dunkl setting.

To begin, we recall some results in Dunkl theory (see [3, 4, 6, 7, 9, 15]) and we refer for more details to the survey [8].

Let $G \subset O(\mathbb{R}^d)$ be a finite reflection group associated to a reduced root system $R$. For $\alpha \in R$, we denote by $H_\alpha$ the hyperplane orthogonal to $\alpha$. We denote by $k$ a nonnegative multiplicity function defined on $R$ with the property that $k$ is $G$-invariant. For a given $\beta \in \mathbb{R}^d \setminus \bigcup_{\alpha \in R} H_\alpha$, we fix a
positive subsystem $R_+ = \{ \alpha \in R : \langle \alpha, \beta \rangle > 0 \}$. We associate with $k$ the index $\gamma = \sum_{\xi \in R_+} k(\xi)$ and a weighted measure $\nu_k$ given by

$$d\nu_k(x) := w_k(x)dx \quad \text{where} \quad w_k(x) = \prod_{\xi \in R_+} |\langle \xi, x \rangle|^{2k(\xi)} , \quad x \in \mathbb{R}^d.$$ 

Further, we introduce the Mehta-type constant $c_k$ by

$$c_k = \left( \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2}} w_k(x)dx \right)^{-1}.$$ 

For every $1 \leq p \leq +\infty$, we denote by $L_p^k(\mathbb{R}^d)$, the spaces $L_p(\mathbb{R}^d, d\nu_k(x))$, and we use $\| \|_{p,k}$ as a shorthand for $\| \|_{L_p^k(\mathbb{R}^d)}$.

By using the homogeneity of degree $2\gamma$ of $w_k$, for a radial function $f$ in $L_k^p(\mathbb{R}^d)$, there exists a function $F$ on $[0, +\infty)$ such that $f(x) = F(\|x\|)$, for all $x \in \mathbb{R}^d$. The function $F$ is integrable with respect to the measure $r^{2\gamma+d-1}dr$ on $[0, +\infty)$ and we have

$$\int_{\mathbb{R}^d} f(x) d\nu_k(x) = \int_0^{+\infty} \left( \int_{S^{d-1}} f(ry)w_k(ry)d\sigma(y) \right)r^{d-1}dr \quad \text{for all} \quad x \in \mathbb{R}^d.$$ 

where $S^{d-1}$ is the unit sphere on $\mathbb{R}^d$ with the normalized surface measure $d\sigma$ and

$$d_k = \int_{S^{d-1}} w_k(x)d\sigma(x) = \frac{c_k^{-1}}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma + \frac{d}{2})}.$$ 

The Dunkl operators $T_j$, $1 \leq j \leq d$ are the following $k$-deformations of directional derivatives $\frac{\partial}{\partial x_j}$ given by:

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \frac{f(x) - f(\rho_\alpha(x))}{\langle \alpha, x \rangle} , \quad f \in C_c(\mathbb{R}^d) , \quad x \in \mathbb{R}^d ,$$

where $\rho_\alpha$ is the reflection on the hyperplane $H_\alpha$ and $\alpha_j = \langle \alpha, e_j \rangle$, $(e_1, \ldots, e_d)$ being the canonical basis of $\mathbb{R}^d$. 

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Notice that in the case \( k \equiv 0 \), the weighted function \( w_k \equiv 1 \), the measure \( \nu_k \) coincide with the Lebesgue measure and the operator \( T_k \) reduced to the corresponding partial derivatives \( \frac{\partial}{\partial x_j} \). Therefore Dunkl analysis can be viewed as a generalization of classical Fourier analysis.

For \( y \in \mathbb{C}^d \), the system
\[
\begin{cases}
T_j u(x, y) = y_j u(x, y), & 1 \leq j \leq d, \\
u(0, y) = 1.
\end{cases}
\]

admits a unique analytic solution on \( \mathbb{R}^d \), denoted by \( E_k(x, y) \) and called the Dunkl kernel. This kernel has a unique holomorphic extension to \( \mathbb{C}^d \times \mathbb{C}^d \).

We have for all \( \lambda \in \mathbb{C} \) and \( z, z' \in \mathbb{C}^d \), \( E_k(z, z') = E_k(z', z) \), \( E_k(\lambda z, z') = E_k(z, \lambda z') \) and for \( x, y \in \mathbb{R}^d \), \( |E_k(x, iy)| \leq 1 \).

The Dunkl transform \( \mathcal{F}_k \) is defined for \( f \in \mathcal{D}(\mathbb{R}^d) \) by
\[
\mathcal{F}_k(f)(x) = c_k \int_{\mathbb{R}^d} f(y) E_k(-ix, y) d\nu_k(y), \quad x \in \mathbb{R}^d.
\]

We list some known properties of this transform:

i) The Dunkl transform of a function \( f \in L^1_k(\mathbb{R}^d) \) has the following basic property
\[
\|\mathcal{F}_k(f)\|_{\infty, k} \leq \|f\|_{1, k}.
\]

ii) The Dunkl transform is an automorphism on the Schwartz space \( \mathcal{S}(\mathbb{R}^d) \).

iii) When both \( f \) and \( \mathcal{F}_k(f) \) are in \( L^1_k(\mathbb{R}^d) \), we have the inversion formula
\[
f(x) = \int_{\mathbb{R}^d} \mathcal{F}_k(f)(y) E_k(ix, y) d\nu_k(y), \quad x \in \mathbb{R}^d.
\]

iv) (Plancherel’s theorem) The Dunkl transform on \( \mathcal{S}(\mathbb{R}^d) \) extends uniquely to an isometric automorphism on \( L^2_k(\mathbb{R}^d) \).

The Dunkl translation operators \( \tau_x \), \( x \in \mathbb{R}^d \) is defined on \( L^2_k(\mathbb{R}^d) \) by
\[
\mathcal{F}_k(\tau_x(f))(y) = E_k(ix, y) \mathcal{F}_k(f)(y), \quad y \in \mathbb{R}^d.
\]
As an operator on $L^2_k(\mathbb{R}^d)$, $\tau_x$ is bounded. According to ([15], Theorem 3.7), the operator $\tau_x$ can be extended to the space of radial functions $L^p_k(\mathbb{R}^d)^{rad}$, $1 \leq p \leq 2$ and we have for a function $f$ in $L^p_k(\mathbb{R}^d)^{rad}$,

$$\|\tau_x(f)\|_{p,k} \leq \|f\|_{p,k}.$$ 

It was shown in [10] that if $f$ is a radial function in $S(\mathbb{R}^d)$ with $f(y) = \tilde{f}(\|y\|)$, then

$$\tau_x(f)(y) = \int_{\mathbb{R}^d} \tilde{f}(A(x, y, \eta))d\mu_x(\eta) \quad (1.1)$$

where $A(x, y, \eta) = \sqrt{\|x\|^2 + \|y\|^2 - 2 < y, \eta >}$ and $\mu_x$ is a probability measure supported in the convex hull $co(G.x)$ of the $G$-orbit of $x$ in $\mathbb{R}^d$. We observe that,

$$\eta \in co(G.x) \implies \min_{g \in G} \|g.x - y\| \leq A(x, y, \eta) \leq \max_{g \in G} \|g.x - y\|. \quad (1.2)$$

We collect below some useful facts:

i) For all $x, y \in \mathbb{R}^d$, $\tau_x(f)(y) = \tau_y(f)(x)$.

ii) For $f \in L^2_k(\mathbb{R}^d) \cap L^1_k(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \tau_x(f)(y)d\nu_k(y) = \int_{\mathbb{R}^d} f(y)d\nu_k(y). \quad (1.3)$$

iii) For all $x \in \mathbb{R}^d$ and $f, g \in L^2_k(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \tau_x(f)(y)g(y)d\nu_k(y) = \int_{\mathbb{R}^d} f(y)\tau_x(g)(y)d\nu_k(y). \quad (1.4)$$

The Dunkl convolution product $*_{k}$ of two functions $f$ and $g$ in $L^2_k(\mathbb{R}^d)$ is given by

$$(f *_{k} g)(x) = \int_{\mathbb{R}^d} \tau_x(f)(-y)g(y)d\nu_k(y), \quad x \in \mathbb{R}^d.$$ 

The Dunkl convolution product is commutative and for $f, g \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\mathcal{F}_k(f *_{k} g) = \mathcal{F}_k(f)\mathcal{F}_k(g). \quad (1.5)$$
It was proved in ([15], Theorem 4.1) that when \( g \) is a bounded radial function in \( L^1_k(\mathbb{R}^d) \), then the application \( f \mapsto f \ast_k g \) initially defined on the intersection of \( L^1_k(\mathbb{R}^d) \) and \( L^2_k(\mathbb{R}^d) \) extends to \( L^p_k(\mathbb{R}^d) \), \( 1 \leq p \leq +\infty \) as a bounded operator. In particular, \[
\|f \ast_k g\|_{p,k} \leq \|f\|_{p,k} \|g\|_{1,k}.
\] (see [10, 11]) The Dunkl transform of \( \sigma \) is given by
\[
\mathcal{F}_k(\sigma)(x) = \frac{1}{c_k} \int_{\mathbb{R}^{d-1}} E_k(-ix, y) \omega_k(y) d\sigma(y) = c_\gamma j_{\gamma + \frac{d}{2} - 1}(\|x\|),
\] (1.6)
where \( j_{\gamma + \frac{d}{2} - 1} \) is the Bessel function of the first type and \( c_\gamma = \frac{1}{\Gamma(\gamma + \frac{d}{2})} \).

In particular (see [14]), the function
\[
\mathcal{F}_k(\sigma)(r\xi) = O(r^{-\frac{2\gamma + d - 1}{2}}), \quad r \to +\infty.
\] (1.7)
\[
\frac{1}{r} \frac{\partial}{\partial r} \mathcal{F}_k(\sigma)(r\xi) = O(r^{-\frac{2\gamma + d + 1}{2}}), \quad r \to +\infty.
\] (1.8)

Along this paper we use \( C \) to denote a suitable positive constant which is not necessarily the same in each occurrence and we write for \( x \in \mathbb{R}^d, \|x\| = \sqrt{\langle x, x \rangle} \). Furthermore, we denote by
- \( \mathcal{E}(\mathbb{R}^d) \) the space of infinitely differentiable functions on \( \mathbb{R}^d \).
- \( \mathcal{S}(\mathbb{R}^d) \) the Schwartz space of functions in \( \mathcal{E}(\mathbb{R}^d) \) which are rapidly decreasing as well as their derivatives.
- \( \mathcal{D}(\mathbb{R}^d) \) the subspace of \( \mathcal{E}(\mathbb{R}^d) \) of compactly supported functions.

## 2 Dunkl-spherical maximal function

For \( f \in L^2_k(\mathbb{R}^d) \), we define the maximal function \( \mathcal{M}_k f \) by
\[
\mathcal{M}_k(f)(x) = \sup_{r > 0} \frac{1}{\nu_k(B(0, r))} \left| \int_{B(0, r)} \tau_x f(y) d\nu_k(y) \right|,
\] (2.1)
where \( B(0, r) \) is the ball of radius \( r \) centered at 0 and \( x \in \mathbb{R}^d \).

It was proved in [15] that the maximal function is bounded on \( f \in L^p_k(\mathbb{R}^d) \) for \( 1 < p \leq \infty \), and of weak type \((1,1)\), for \( f \in L^1_k(\mathbb{R}^d) \) that is \( a > 0 \),
\[
\int_{E(a)} d\nu_k(x) \leq \frac{C}{a} \|f\|_{1,k}
\] (2.2)
where \( E(a) = \{ x : \mathcal{M}_k f(x) > a \} \) and \( C \) is a constant independent of \( a \) and \( f \).

As in [5], we define the spherical mean operator on
\[ A_k(\mathbb{R}^d) = \{ f \in L^1_k(\mathbb{R}^d) : \mathcal{F}_k f \in L^1_k(\mathbb{R}^d) \}, \]
for \( x \in \mathbb{R}^d \) and \( r > 0 \) by
\[
S_r(f)(x) = \frac{1}{d_k} \int_{S^{d-1}} \tau_x(f(-ry)) \omega_k(y) d\sigma(y) = \frac{1}{d_k} \int_{S^{d-1}} \tau_x(f(-y)) \omega_k(\frac{r}{y}) d\sigma_r(y).
\]

Now, the Dunkl-spherical maximal function \( M(f) \) is given by
\[
M(f)(x) = \sup_{r > 0} |S_r(f)(x)|, \quad x \in \mathbb{R}^d.
\]

**Theorem 2.1** Let \( 2\gamma + d \geq 2 \) and \( \frac{2\gamma + d}{2\gamma + d - 1} < p < 2\gamma + d \). Then there exists a constant \( C > 0 \) such that for all \( f \in L^p_k(\mathbb{R}^d) \),
\[
\| M(f) \|_{p,k} \leq C \| f \|_{p,k}.
\] (2.3)

Before proving the theorem, we need to establish some useful results. In fact, fix a function \( \psi_0 \in S(\mathbb{R}^d) \) which is the Dunkl transform of a \( C^\infty \)-radial function with compact support such that:
\[
\psi_0(0) = 1, \quad \left( \frac{\partial^i}{\partial r^i} \psi_0 \right)(0) = 0,
\] (2.4)
for \( 1 \leq i < \frac{2\gamma + d}{2} \) and where \( \frac{\partial}{\partial r} \) denotes the derivation in the radial direction. To obtain a such function, taking \( \psi \in S(\mathbb{R}^d) \) with \( \mathcal{F}_k(\psi) \) is a \( C^\infty \)-radial function with compact support and
\[
\psi(0) \neq 0 \quad \text{and} \quad \psi_0(r\xi) = \left( \sum_{j=0}^{\lfloor \frac{2\gamma + d}{2} \rfloor} a_j r^j \right) \psi(r\xi),
\]
for \( \xi \in \mathbb{S}^{d-1} \). The coefficients \( a_j \) are solutions of triangular system given by the conditions (2.4) and \( \lfloor \frac{2\gamma + d}{2} \rfloor \) is the least integer not less than \( \frac{2\gamma + d}{2} \).

Now set the functions \( \psi_j \) with
\[
\psi_1(y) = \psi_0(\frac{y}{2}) - \psi_0(y) \quad \text{and} \quad \psi_j(y) = \psi_1(2^{-(j-1)}y), \quad j \geq 1.
\] (2.5)
Thus, $\psi_j$ is radial and $F_k(\psi_j)$ is a compact supported functions. It follows that for some constants $t,C > 0$ we have

$$|\psi_1(y)| \leq C\|y\|^{\frac{2\gamma + d}{2}}, \quad \text{if} \quad \|y\| \leq t,$$

(2.6)

and

$$\sum_{j=0}^{+\infty} \psi_j(y) = 1, \quad y \in \mathbb{R}^d. \quad \text{(2.7)}$$

Let $m_j = F_k(\sigma)\psi_j$ and let $\varphi_j$ the function in $S(\mathbb{R}^d)$ such that $F_k(\varphi_j) = m_j$. It follows that for $f \in S(\mathbb{R}^d)$,

$$S_r(f)(x) = \sum_{j=0}^{+\infty} f *_{k} \varphi_j.$$ 

In fact, this can be done because from (2.7), we have

$$F_k(\sigma)(y) = \sum_{j=0}^{+\infty} F_k(\sigma)(y)\psi_j(y) = \sum_{j=0}^{+\infty} F_k(\varphi_j)(y)$$

and we can write

$$F_k(S_r(f))(y) = \sum_{j=0}^{+\infty} F_k(f *_{k} \varphi_j)(y),$$

which implies with the inversion formula that

$$S_r(f)(y) = F_k^{-1}\left(\sum_{j=0}^{+\infty} F_k(f *_{k} \varphi_j)\right)(y) = \int_{\mathbb{R}^d} \sum_{j=0}^{+\infty} F_k(f *_{k} \varphi_j)(z)E_k(iy,z)d\nu_k(z).$$

To interchange the sum and the integral, we proceed as follows:

From (1.5), we have
\[
\sum_{j=1}^{+\infty} \left( \int_{\mathbb{R}^d} |\mathcal{F}_k(f \ast_k \varphi_j)(z)|E_k(iy, z)|d\nu_k(z) \right)
\]

\[
\leq \sum_{j=1}^{+\infty} \left( \int_{\mathbb{R}^d} |\mathcal{F}_k(\varphi_j)(z)||\mathcal{F}_k(f)(z)|d\nu_k(z) \right)
\]

\[
\leq \sum_{j=1}^{+\infty} \left( \int_{\mathbb{R}^d} |\psi_j(z)||\mathcal{F}_k(f)(z)|d\nu_k(z) \right)
\]

\[
\leq \sum_{j=1}^{+\infty} \left( \int_{\|z\| \leq 2^{j-1}t} |\psi_j(z)||\mathcal{F}_k(f)(z)|d\nu_k(z) + \int_{\|z\| \geq 2^{j-1}t} |\psi_j(z)||\mathcal{F}_k(f)(z)|d\nu_k(z) \right).
\]

By using (2.5) and (2.6), we obtain

\[
\int_{\|y\| \leq 2^{j-1}t} |\psi_j(z)||\mathcal{F}_k(f)(z)|d\nu_k(z) \leq c2^{-(j-1)2^{j-1}d} \int_{\|y\| \leq 2^{j-1}t} \|z\|^{2^{j-1}d} |\mathcal{F}_k(f)(z)|d\nu_k(z)
\]

\[
\leq c2^{-(j-1)2^{j-1}d} \int_{\mathbb{R}^d} \|z\|^{2^{j-1}d} |\mathcal{F}_k(f)(z)|d\nu_k(z),
\]

and

\[
\int_{\|z\| \geq 2^{j-1}t} |\psi_j(z)||\mathcal{F}_k(f)(z)|d\nu_k(z) = \int_{\|z\| \geq 2^{j-1}t} |\psi_j(z)||\mathcal{F}_k(f)(z)|d\nu_k(z)
\]

\[
\leq 2^{-(j-1)t} \int_{\mathbb{R}^d} \|z\| |\psi_j(z)||\mathcal{F}_k(f)(z)|d\nu_k(z)
\]

\[
\leq 2^{-(j-1)t} \int_{\mathbb{R}^d} \|z\| |\mathcal{F}_k(f)(z)|d\nu_k(z),
\]

therefore \( \sum_{j=1}^{+\infty} \left( \int_{\mathbb{R}^d} |\mathcal{F}_k(f \ast_k \varphi_j)(z)|E_k(iy, z)|d\nu_k(z) \right) \) converge.

For all \( j \geq 0 \) and \( r > 0 \), we define the function \( \varphi_{j,r}(x) = r^{-2\gamma-d} \varphi_{j}(\frac{x}{r}) \). Then we can write,

\[
S_r(f)(x) = \sum_{j=0}^{+\infty} f \ast_k \varphi_{j,r}.
\]

Hence for \( f \in \mathcal{S}(\mathbb{R}^d) \), we obtain

\[
M(f) \leq \sum_{j=0}^{+\infty} M_{\varphi_{j,r}}, \quad (2.8)
\]
where \( M_{\varphi_j}(f)(x) = \sup_{r>0} \| f \ast_k \varphi_{j,r}(x) \| = \sup_{r>0} \int_{\mathbb{R}^d} \tau_x(f)(y)\varphi_{j,r}(y) d\nu_k(y) \).

To prove the theorem, it suffices to establish an inequality of the form
\[
\| M_{\varphi_j}(f) \|_{p,k} \leq C_{j,p} \| f \|_{p,k},
\]
with \( \sum_{j=0}^{+\infty} C_{j,p} < +\infty. \)

**Lemma 2.1** There exists a constant \( C > 0 \) such that, for any \( x \in \mathbb{R}^d \) and \( j \geq 0, \)
\[
|\varphi_j(x)| \leq C \frac{2^j}{(1 + \|x\|)^{2\gamma+1}}.
\]  

(2.9)

**Proof. 2.1** Remember that \( \varphi_j = S_r(F^{-1}_k(\psi_j)) \) for \( j \geq 1 \). We have,
\[
F^{-1}_k(\psi_j)(x) = 2^{(j-1)(2\gamma+d)} F^{-1}_k(\psi_1)(2^{j-1} x).
\]

Since \( F^{-1}_k(\psi_0) \) and \( F^{-1}_k(\psi_1) \) are in \( \mathcal{S}(\mathbb{R}^d) \), then
\[
|F^{-1}_k(\psi_0)(x)| \text{ and } |F^{-1}_k(\psi_1)(x)| \text{ are bounded by } \frac{C}{(1 + \|x\|)^{2\gamma+1}}.
\]

(2.10)

Taking \( \phi_j(x, y, \xi) = F^{-1}_k(\psi_1)(2^{j-1} \sqrt{\|x\|^2 + \|y\|^2 - 2 <y, \xi>}) \) for \( j \geq 1 \) and using (1.1), we get
\[
|\varphi_j(x)| = \left| \int_{\mathbb{R}^{d-1}} \tau_x(F^{-1}_k(\psi_j))(-y)\omega_k(y) d\sigma(y) \right| \\
\leq 2^{(j-1)(2\gamma+d)} \int_{\mathbb{R}^{d-1}} \left( \int_{\mathbb{R}^d} |\phi_j(x, y, \xi)| d\mu_x(\xi) \right) \omega_k(y) d\sigma(y).
\]

(2.11)

For \( j = 0 \), we have by (1.1)
\[
|\varphi_0(x)| = \left| \int_{\mathbb{R}^{d-1}} \tau_x(F^{-1}_k(\psi_0))(-y)\omega_k(y) d\sigma(y) \right| \\
\leq \int_{\mathbb{R}^{d-1}} \left( \int_{\mathbb{R}^d} |F^{-1}_k(\psi_0)(\sqrt{\|x\|^2 + \|y\|^2 - 2 <y, \xi>})| d\mu_x(\xi) \right) \omega_k(y) d\sigma(y).
\]

(2.12)
From (1.2), (2.10), (2.11) and (2.12), we get for \( j \geq 0 \)
\[
|\varphi_j(x)| 
\leq C 2^{j(2\gamma+d)} \int_{\mathbb{R}^{d-1}} \left( \int_{\mathbb{R}^d} \frac{1}{(1 + 2^j \min_{g \in G} \|gx - y\|)^{2\gamma+d+1}} d\mu_x(\xi) \right) \omega_k(y) d\sigma(y)
\]
\[
\leq C 2^{j(2\gamma+d)} \int_{\mathbb{R}^{d-1}} \frac{1}{(1 + 2^j \min_{g \in G} \|gx - y\|)^{2\gamma+d+1}} \omega_k(y) d\sigma(y).
\]
If \( \|x\| > 2 \), then \( \|gx - y\| \geq \|x\| - 1 \geq \frac{\|x\|}{2} \) and we have
\[
|\varphi_j(x)| 
\leq \frac{C 2^{j(2\gamma+d)}}{(1 + 2^{j-1}\|x\|)^{2\gamma+d+1}}
\leq \frac{C 2^{-j}}{\|x\|^{2\gamma+d+1}}
\leq C \frac{2^j}{(1 + \|x\|)^{2\gamma+d+1}}.
\]
(2.13)
If \( \|x\| \leq 2 \) then,
\[
|\varphi_j(x)| 
\leq \int_{\mathbb{R}^{d-1}} \frac{2^{j(2\gamma+d)}}{(1 + 2^j \min_{g \in G} \|gx - y\|)^{2\gamma+d+1}} \omega_k(y) d\sigma(y)
\]
\[
\leq 2^{j(2\gamma+d)} \int_{\mathbb{R}^{d-1}} \{y \in \mathbb{R}^{d-1} \colon \min_{g \in G} \|gx - y\| \leq 2^{-j} \} \omega_k(y) d\sigma(y)
\]
\[
+ 2^{j(2\gamma+d)} \sum_{i=0}^{+\infty} 2^{-(2\gamma+d+1)i} \int_{\mathbb{R}^{d-1}} \{y \in \mathbb{R}^{d-1} \colon \min_{g \in G} \|gx - y\| \leq 2^{i+1-j} \} \omega_k(y) d\sigma(y)
\]
\[
\leq C \left( 2^j + 2^j \sum_{i=0}^{+\infty} 2^{-2i} \right) \leq C \frac{2^j}{(1 + \|x\|)^{2\gamma+d+1}}.
\]
(2.14)
From (2.13) and (2.14), we obtain
\[
|\varphi_j(x)| \leq C \frac{2^j}{(1 + \|x\|)^{2\gamma+d+1}}.
\]

**Lemma 2.2** There exists a constant \( C > 0 \) such that, for any \( f \in S(\mathbb{R}^d) \), \( j \geq 0 \) and \( \alpha > 0 \),
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} d\nu_k(x) \leq \frac{C 2^j}{\alpha \|f\|_{1,k}}.
\]
where \( \tilde{E}(\alpha) = \{ x \in \mathbb{R}^d ; M\varphi_j(f)(x) > \alpha \} \) and \( C \) is a constant depending only on \( d \).

**Proof. 2.2** Let first prove that, for \( f \in \mathcal{S}(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \),

\[
\int_{\mathbb{R}^d} \tau_x(|\varphi_{j,r}|)(y)|f(y)|d\nu_k(y) \leq C 2^j M_k(|f|)(x) \quad j \geq 0, \tag{2.15}
\]

where \( M_k(f) \) is given by (2.1).

We denote by \( A_i = \{ y \in \mathbb{R}^d , r2^i \leq \| y \| < r2^{i+1} \} \), then

\[
\tau_x(|\varphi_{j,r}|)(y) = \tau_x\left( \sum_{i=-\infty}^{+\infty} |\varphi_{j,r}| \cdot \chi_{A_i} \right)(y) = \sum_{i=-\infty}^{+\infty} \tau_x\left( |\varphi_{j,r}| \cdot \chi_{A_i} \right)(y).
\]

From (2.9), one has

\[
(|\varphi_{j,r}| \cdot \chi_{A_i})(y) \leq C 2^j \frac{r^{-(2\gamma+d)}}{(1+\|y\|)^{2\gamma+d+1}} \chi_{A_i}(y) \leq C 2^j \frac{r^{-(2\gamma+d)}}{(1+2^i)^{2\gamma+d+1}} \chi_{A_i}(y),
\]

and since \( |\varphi_{j,r}| \cdot \chi_{A_i} \) is a radial function, this implies that

\[
\tau_x\left( |\varphi_{j,r}| \cdot \chi_{A_i} \right)(y) \leq C 2^j \frac{r^{-(2\gamma+d)}}{(1+2^i)^{2\gamma+d+1}} \tau_x(\chi_{A_i})(y).
\]

Using (1.4), we obtain

\[
\int_{\mathbb{R}^d} \tau_x\left( |\varphi_{j,r}| \right)(y)|f(y)|d\nu_k(y)
\]

\[
\leq C 2^j \sum_{i=-\infty}^{+\infty} \int_{\mathbb{R}^d} \frac{r^{-(2\gamma+d)}}{(1+2^i)^{2\gamma+d+1}} \tau_x(\chi_{A_i})(y)|f(y)|d\nu_k(y)
\]

\[
\leq C 2^j \sum_{i=-\infty}^{+\infty} \frac{r^{-(2\gamma+d)}}{(1+2^i)^{2\gamma+d+1}} \int_{\mathbb{R}^d} \chi_{A_i}(y) \tau_x(|f|)(y)d\nu_k(y)
\]

\[
\leq C 2^j \sum_{i=-\infty}^{+\infty} \frac{r^{-(2\gamma+d)}}{(1+2^i)^{2\gamma+d+1}} \int_{B(0,r2^{i+1})} \tau_x(|f|)(y)d\nu_k(y)
\]

\[
\leq C 2^j \sum_{i=-\infty}^{+\infty} \frac{r^{-(2\gamma+d)}}{(1+2^i)^{2\gamma+d+1}} (r2^{i+1})^{2\gamma+d} M_k(|f|)(x)
\]

\[
\leq C 2^j M_k(|f|)(x).
\]
By the fact that,

\[ \left| \int_{\mathbb{R}^d} \tau_x(\varphi_{j,r})(y) f(y) d\nu_k(y) \right| \leq \int_{\mathbb{R}^d} |\tau_x(\varphi_{j,r})(y)||f(y)| d\nu_k(y) \]

\[ \leq \int_{\mathbb{R}^d} \tau_x(|\varphi_{j,r}|)(y)|f(y)| d\nu_k(y), \]

we deduce

\[ M_{\varphi_j}(f)(x) \leq C2^j M_k(|f|)(x). \]

From (2.2), we conclude the proof of the lemma.

**Lemma 2.3** There exists a constant \( C > 0 \) such that, for any \( f \in \mathcal{S}(\mathbb{R}^d) \), \( j \geq 1 \),

\[ \|M_{\varphi_j}(f)\|_{2,k} \leq C2^{-\frac{j^2+2d-2}{2}}\|f\|_{2,k}. \]

**Proof. 2.3** (See [13], [16]) For \( f \in \mathcal{S}(\mathbb{R}^d) \), we have from (1.5)

\[ \mathcal{F}_k(f *_{k} \varphi_{j,r})(x) = \mathcal{F}_k(f)(x) \mathcal{F}_k(\varphi_{j,r})(x) = \mathcal{F}_k(f)(x) m_j(rx). \]

Put

\[ g_j(f)(x) = \left( \int_0^{+\infty} |f *_{k} \varphi_{j,r}(x)|^2 \frac{dr}{r} \right)^{\frac{1}{2}}, \]

the Littlewood-Paley function associated to the function \( \varphi_{j,r} \).

Using the Plancherel theorem and by (1.5), we obtain

\[ \|g_j(f)\|_{2,k}^2 = \int_{\mathbb{R}^d} |g_j(f)(x)|^2 d\nu_k(x) \]

\[ = \int_{\mathbb{R}^d} \left( \int_0^{+\infty} |f *_{k} \varphi_{j,r}(x)|^2 \frac{dr}{r} \right) d\nu_k(x) \]

\[ = \int_0^{+\infty} \left( \int_{\mathbb{R}^d} |\mathcal{F}_k(f *_{k} \varphi_{j,r})(x)|^2 d\nu_k(x) \right) \frac{dr}{r} \]

\[ = \int_0^{+\infty} \left( \int_{\mathbb{R}^d} |\mathcal{F}_k(f)(x)|^2 |m_j(rx)|^2 d\nu_k(x) \right) \frac{dr}{r} \]

\[ \leq \|f\|_{2,k}^2 \sup_{x \neq 0} \int_0^{+\infty} |m_j(rx)|^2 \frac{dr}{r}. \]
Since the function $m_j$ is radial, the integral is independent of $x$. By using the definition of $m_j$, we have that
\[
\int_0^{+\infty} \left| m_j(rx) \right|^2 \frac{dr}{r} = \int_0^{+\infty} \left| \mathcal{F}_k(\sigma)(rx) \right|^2 \left| \psi_j(rx) \right|^2 \frac{dr}{r} = \int_0^{+\infty} \left| \mathcal{F}_k(\sigma)(rx) \right|^2 \left| \psi_1(2^{-j}rx) \right|^2 \frac{dr}{r} = \int_0^{+\infty} \left| \mathcal{F}_k(\sigma)(2^jrx) \right|^2 \left| \psi_1(rx) \right|^2 \frac{dr}{r}.
\]

From (1.6), (1.7) and (2.6), we obtain
\[
\int_0^{+\infty} \left| m_j(rx) \right|^2 \frac{dr}{r} \leq C 2^{-j(2\gamma+d-1)} \int_0^{+\infty} \left| \psi_1(rx) \right|^2 \frac{dr}{r^{2\gamma+d-1}} \leq C 2^{-j(2\gamma+d-1)},
\]
this gives
\[
\|g_j(f)\|_{2,k} \leq C 2^{-j(2\gamma+d-1)} \|f\|_2.
\] (2.16)

Put now $\tilde{\varphi}_{j,r}(x) = r \frac{d}{dx} \varphi_{j,r}(x)$ and $\tilde{g}_j(f)$ the Littlewood-Paley function associated to $\tilde{\varphi}_{j,r}$, then
\[
\tilde{g}_j(f)(x) = \left( \int_0^{+\infty} \left| f_k \varphi_{j,r}(x) \right|^2 \frac{dr}{r} \right)^{\frac{1}{2}} = \left( \int_0^{+\infty} r \left| \frac{d}{dr} \varphi_{j,r} * f(x) \right|^2 \frac{dr}{r} \right)^{\frac{1}{2}}.
\]

Similarly, with the use of (1.6) and (1.8), we have
\[
\|\tilde{g}_j(f)\|_{2,k} = \int_{\mathbb{R}^d} \left( \int_0^{+\infty} r \left| \frac{d}{dr} \varphi_{j,r} * f(x) \right|^2 \frac{dr}{r} \right) d\nu_k(x) = C 2^{-j(2\gamma+d-1)} \|f\|_{2,k}.
\] (2.17)

Since $\lim_{r \to +\infty} f_k \varphi_{j,r}(x) = 0$, we have
\[
\left| f_k \varphi_{j,r}(x) \right|^2 = -2 \text{Re} \int_0^{+\infty} \overline{\varphi_{j,s} * f(x)} \frac{d}{ds} \varphi_{j,s} * f(x) ds \lesssim 2 \int_0^{+\infty} \left| \varphi_{j,s} * f(x) \right| |\varphi_{j,s} * f(x)| \frac{ds}{s}.
\]
Using Cauchy-Schwartz’s inequality, we deduce that

$$\sup_{r>0} |f \ast_k \varphi_{j,r}(x)|^2 \leq 2g_j(f)(x)\tilde{g}_j(f)(x).$$

Integrating over $\mathbb{R}^d$ and using again the Cauchy-Schwartz inequality, we obtain from (2.16) and (2.17)

$$\|M_{\varphi_j}(f)\|_{2,k} \leq C2^{-j^2-\frac{2+2d}{2}j}\|f\|_{2,k}.$$  

Lemma 2.4 There exists a constant $C > 0$ such that, for $f \in S(\mathbb{R}^d)$

$$\|M_{\varphi_j}(f)\|_{\infty,k} \leq C2^j\|f\|_{\infty,k} \quad j \geq 0.$$  

Proof. 2.4 From (1.5) and (2.9), one has for $f \in S(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$|f \ast_k \varphi_{j,r}(x)| \leq \int_{\mathbb{R}^d} |f(y)||\tau_x \varphi_{j,r}(y)|d\nu_k(y)$$

$$\leq \|f\|_{\infty,k} \int_{\mathbb{R}^d} |\varphi_{j,r}(y)|d\nu_k(y)$$

$$\leq C2^j\|f\|_{\infty,k} \int_{\mathbb{R}^d} \frac{1}{(1+\|x\|)^{2+2+1}}d\nu_k(y)$$

$$\leq C2^j\|f\|_{\infty,k},$$

which gives the result.

Remark 2.1 We observe that for $j = 0$ and from Lemmas 2.2, 2.4, we obtain by interpolation (see [13]) that $M_{\varphi_j}(f)$ is of strong-type $(p,p)$ with $1 < p \leq +\infty$.

Proof of theorem 2.1. According to Remark 2.1, we deduce by interpolation from Lemmas 2.2, 2.3 that

$$\|M_{\varphi_j}(f)\|_{p,k} \leq C2^{-j\left(2\gamma+d-\frac{2+2d}{p}-1\right)}\|f\|_{p,k},$$

for $1 < p \leq 2$ and $j \geq 0$.

Similarly, according to Remark 2.1, we get by interpolation from Lemmas 2.3, 2.4 that

$$\|M_{\varphi_j}(f)\|_{p,k} \leq C2^{-j\left(2\gamma+d-\frac{1}{p}-1\right)}\|f\|_{p,k},$$
for $2 \leq p \leq +\infty$ and $j \geq 0$.

Since for $\frac{2\gamma + d}{2\gamma + d - 1} < p < 2\gamma + d$, we have

$$
\sum_{j \geq 0}^{+\infty} 2^{-j(2\gamma + d - \frac{2\gamma + d - 1}{p})} < +\infty \quad \text{and} \quad \sum_{j \geq 0}^{+\infty} 2^{-j\left(\frac{2\gamma + d}{p} - 1\right)} < +\infty.
$$

This yields using (2.8)

$$
\|M(f)\|_{p,k} \leq C \|f\|_{p,k},
$$

which completes the proof.

**Remark 2.2** The case $2\gamma + d = 2$ implies that $k \equiv 0$ and was proved by Bourgain in [1].

**References**

[1] J. Bourgain, *Averages in the plane over convex curves and maximal operators*, J. Analyse Math. 47 (1986), 69-85.

[2] F. Dai and H. Wang, *A transference theorem for the Dunkl transform and its applications*, Journal of Functional Analysis 258 (2010), no. 12, 4052–4074.

[3] C. F. Dunkl, *Differential–Difference operators associated to reflection groups*, Trans. Amer. Math. 311 (1989), no. 1, 167–183.

[4] M.F.E. de Jeu, *The Dunkl transform*, Invent. Math. 113 (1993), no. 1, 147-162.

[5] H. Mejjaoli and K. Trimche, *On a mean value property associated with the Dunkl Laplacian operator and applications*, Integral Transform. Spec. Funct. 12 (2001), 279-302.

[6] M. Rösler, *Generalized Hermite polynomials and the heat equation for Dunkl operators*, Comm. Math. Phys. 192 (1998), no. 3, 519–542.

[7] M. Rösler, *Positivity of Dunkl’s intertwining operator*, Duke Math. J. 98 (1999), 445-463.
[8] M. Rösler, Dunkl operators: theory and applications, Orthogonal polynomials and special functions (Leuven, 2002), Lect. Notes Math. 1817, Springer–Verlag (2003), 93–135.

[9] M. Rösler, Bessel-type signed hypergroup on $\mathbb{R}$, Probability measures on groups and related structures, XI (Oberwolfach, 1994), World Sci. Publ., River Edge, NJ, 1995, pp. 292–304.

[10] M. Rösler, A positive radial product formula for the Dunkl kernel, Trans. Amer. Math. Soc. 355 (2003), no. 6, 2413–2438.

[11] M. Rösler, Markov Processes Related With Dunkl Operators, Advances in Applied Mathematics 21, 575–643 (1998).

[12] E.M. Stein, Maximal functions: spherical mean, Proc. Nat. Acad. Sci. U.S.A. 73(1976), 2174-2175.

[13] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Mathematical Series 30. Princeton University Press, Princeton, NJ, 1970.

[14] E.M. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Mathematical Series 32. Princeton University Press, Princeton, NJ, 1971.

[15] S. Thangavelu and Y. Xu, Convolution operator and maximal function for the Dunkl transform, J. Anal. Math. 97 (2005), 25–56.

[16] A. Zygmund, Trigonometric Series, Vol. I, II. Third edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2002.