Barut-Girardello coherent states for \( sp(N, C) \) and multimode Schrödinger cat states

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Abstract

Overcomplete families of states of the type of Barut-Girardello coherent states (BG CS) are constructed for noncompact algebras \( u(p, q) \) and \( sp(N, C) \) in quadratic bosonic representation. The \( sp(N, C) \) BG CS are obtained in the form of multimode ordinary Schrödinger cat states. A set of such macroscopic superpositions is pointed out which is overcomplete in the whole \( N \) mode Hilbert space (while the associated \( sp(N, C) \) representation is reducible). The multimode squared amplitude Schrödinger cat states are introduced as macroscopic superpositions of the obtained \( sp(N, C) \) BG CS.

1 Introduction

Recently an interest is shown in the literature \([1, 2, 3, 4, 5, 8]\) to applications and generalizations of the Barut-Girardello coherent states (BG CS) \([9]\). The BG CS were constructed as eigenstates of lowering Weyl operator of the algebra \( su(1, 1) \). The BG CS representation was used to construct explicitly squeezed states (SS) for the generators of the group \( SU(1, 1) \) which minimize the Schrödinger uncertainty relation for two observables \([3]\) and eigenstates of general element of the complexified \( su(1, 1) \) \([3, 4]\). These algebra related CS can be considered as states which generalize the eigenvalue property of BG CS to the case of linear combination of lowering and raising Weyl operators and even of all the \( SU(1, 1) \) generators. Passing to other algebras it is important first to construct the eigenstates of Weyl lowering operators, which is the extension of the BG definition of CS to the desired algebra.

Our aim in the present work is to construct BG CS for the symplectic algebra \( sp(N, C) \) and its subalgebras \( u(p, q) \), \( p + q = N \), in the quadratic bosonic representation. Here \( N \) is the dimension of Cartan subalgebra, while the dimension of \( sp(N, C) \) is \( N(2N + 1) \).

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functions \( f \) with the Casimir operator \( \mathfrak{su}(2) \) of the growth \((1,1)\). ± takes the values \( \nu \) and they resolve the identity operator, \( |z\rangle \),\( I \) of these states over the orthonormal basis of eigenstates \( z \).\( N = 1, 2, \ldots, [10] \). We establish that the \( sp(N, C) \) BG CS take the form of superpositions of multimode Glauber CS \([1] \) \(|\alpha\rangle\) and \(|-\alpha\rangle\) (Eq. \((12)\)), i.e. the form of multimode ordinary Schrödinger cat states. The set of these macroscopic superpositions of Glauber (or canonical \([12]\)) CS includes several subsets of states extensively studied in quantum optics (see e.g. \([13, 14]\)). We also introduce multimode squared amplitude Schrödinger cat states as superpositions of the constructed \( sp(N, C) \) BG CS.

In the recent E-print \([8]\) the BG CS have been constructed for the \( u(N-1,1) \) algebra. Here we construct overcomplete families of states for \( u(p,q) \).

2 The Barut-Girardello coherent states

The property of the Glauber CS \(|\alpha\rangle\) to be eigenstates of photon number lowering operator \( a, a|\alpha\rangle = \alpha|\alpha\rangle \) (\( \alpha \) is complex number, \( [a,a^\dagger] = 1 \)) was extended by Barut and Girardello \([3]\) to the case of Weyl lowering operator \( K_\pm \) of \( su(1,1) \) algebra. Here we briefly review some of their properties. The defining equation is

\[
K_-|z;k\rangle = z|z;k\rangle,
\]

where \( z \) is (complex) eigenvalue and \( k \) is Bargman index. For discrete series \( D(\pm)(k) \) \( k \) takes the values \( \pm 1/2, \pm 1, \ldots \). The Cartan-Weyl basis operators \( K_\pm = K_1 \pm iK_2, \) \( K_3 \) of \( su(1,1) \) obey the relations

\[
[K_3, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_3,
\]

with the Casimir operator \( C_2 = K_3^2 - (1/2)[K_-K_+ + K_+K_-] = k(k-1) \). The expansion of these states over the orthonormal basis of eigenstates \(|k+n,k\rangle\) of \( K_3 \) \( (K_3|n+k,k\rangle = (n+k)|n+k,k\rangle, \) \( n = 0, 1, 2, \ldots) \) is

\[
|z;k\rangle = \sum_{\nu=0}^\infty \frac{z^{\nu}}{\Gamma(2\nu+1)} |n+k,k\rangle,
\]

where \( I_\nu(z) \) is the first kind modified Bessel function, and \( \Gamma(z) \) is gamma function. The above BG states are normalized to unity. Their scalar product is

\[
\langle k; z_1|z_2;k\rangle = I_{2k-1}(2\sqrt{z_1z_2}) \left[I_{2k-1}(2|z_1|)I_{2k-1}(2|z_2|)\right]^{-1}
\]

and they resolve the identity operator,

\[
\int d\mu(z,k)|z;k\rangle \langle k; z| = I, \quad d\mu(z,k) = \frac{2}{\pi} K_{2k-1}(2|z|)I_{2k-1}(2|z|)d^2z,
\]

where \( K_\nu(x) \) is the modified Bessel function of the second kind. Due to this property any state \(|\psi\rangle\) can be correctly represented by the analytic function

\[
f_\psi(z) = \sqrt{I_{2k-1}(2|z|)}/(z^{k-1/2})\langle k; z^*|\psi\rangle,
\]

which is of the growth \((1,1)\). The operators \( K_\pm \) and \( K_3 \) act in the Hilbert space of analytic functions \( f_\psi(z) \) as linear differential operators.
\[ K_+ = z, \quad K_- = 2k \frac{d}{dz} + z \frac{d^2}{dz^2}, \quad K_3 = k + z \frac{d}{dz}. \]  

(5)

The \( SU(1,1) \) group related CS provide another (analytic in the unit disk) representation in Hilbert space which has been recently shown to be related to the BG representation through a Laplace transform.

3 The BG CS for \( sp(N, C) \)

The BG CS for semisimple Lie algebras are naturally defined as eigenstates of mutually commuting Weyl lowering (or raising) operators \( E_{\alpha'} (E_{\alpha'}^\dagger) \): \( E_{\alpha'} |z\rangle = z_{\alpha'} |z\rangle \). This can be extended to any algebra, where lowering/raising operators exist. We shall consider here the simple Lie algebra \( sp(N, C) \) (the symplectic algebra of rank \( N \) and dimension \( N(2N + 1) \)). We redenote the Cartan-Weyl basis as \( E_{ij}, E_{ij}^\dagger, H_{ij} \) (\( i, j = 1, 2, \ldots, N \), \( E_{ij} = E_{ji}, H_{ij}^\dagger = H_{ji} \)), and write the \( sp(N, C) \) commutation relations

\[
\begin{align*}
[E_{ij}, E_{kl}] &= [E_{ij}, E_{kl}^\dagger] = 0, \\
[E_{ij}, E_{kl}^\dagger] &= \delta_{jk}H_{il} + \delta_{il}H_{jk} + \delta_{ik}H_{jl} + \delta_{jl}H_{ik}, \\
[E_{ij}, H_{kl}] &= \delta_{il}E_{jk} + \delta_{jl}E_{ik}, \\
[E_{ij}^\dagger, H_{kl}] &= -\delta_{ik}E_{jl}^\dagger - \delta_{jk}E_{il}^\dagger, \\
[H_{ij}, H_{kl}] &= \delta_{il}H_{kj} - \delta_{jk}H_{il}.
\end{align*}
\]

(6)

The BG CS \( |\{z_{kl}\}\rangle \) for \( sp(N, C) \) are defined as

\[ E_{ij} |\{z_{kl}\}\rangle = z_{ij} |\{z_{kl}\}\rangle, \quad i, j = 1, 2, \ldots, N. \]

(7)

Let us note that the Cartan subalgebra is spanned by \( H_{ii} \) only and \( H_{ij}, j \neq i \) are also Weyl lowering and raising operators as all \( E_{ij} \) are: we have simply separated the mutually commuting lowering operators \( E_{ij} \). We shall construct the \( sp(N, C) \) BG CS for the quadratic bosonic representation, which is realized by means of the operators

\[ E_{ij} = a_i^{} a_j, \quad E_{ij}^\dagger = a_i^\dagger a_j^\dagger, \quad H_{ij} = \frac{1}{2} (a_i^\dagger a_j + a_j^\dagger a_i^{}), \]

(8)

where \( a_i^{}, a_i^\dagger \) are \( N \) pairs of boson annihilation and creation operators. These operators act irreducibly in the subspaces \( \mathcal{H}^\pm \) spanned by the number states \( |n_1, \ldots, n_N\rangle \) with even/odd \( n_{\text{tot}} \equiv n_1 + n_2 + \ldots + n_N \). The whole Hilbert space \( \mathcal{H} \) of the \( N \) mode system is a direct sum of \( \mathcal{H}^\pm \).

The \( sp(N, C) \) is the complexification of \( sp(N, R) \) and therefor the hermitian quadratures of the above operators span over \( R \) the \( sp(N, R) \) algebra. In this way for \( N = 1 \) one gets from (8) \( sp(1, R) \sim su(1, 1), \)

\[ K_1 = \frac{1}{4} (a^2 + a^{\dagger 2}), \quad K_2 = \frac{i}{4} (a^2 - a^{\dagger 2}), \quad K_3 = \frac{1}{4} (2a^{\dagger} a + 1). \]

(9)

with the quadratic Casimir operator \( C_2 = K_3^2 - K_1^2 - K_2^2 = -3/16 \). Eigenstates of \( a^2 \) were constructed in the first paper of ref. [13].
One general property of CS $|\{z_{kl}\}\rangle$ for the representation (8) is that they depend effectively on $N$ complex parameters $\alpha_j$ (not of $N^2 + N$ as one might expect). Indeed, using the boson commutation relations $[a_i, a_j] = 0$ and the definition (7) we easily get

$$z_{ij}z_{kl} = z_{ik}z_{jl} = z_{il}z_{jk},$$

wherefrom we get the factorization $z_{ij} = \alpha_i \alpha_j$. Therefor in the above bosonic representation the definition (7) is rewritten as

$$a_i a_j |\{ \alpha_k \alpha_l \}\rangle = \alpha_i \alpha_j |\{ \alpha_k \alpha_l \}\rangle, \quad i,j = 1, 2, \ldots, N.$$  

(11)

The general solution to this system of equations is most easily obtained in the Glauber CS representation. It reads

$$|\{ \alpha_k \alpha_l \}\rangle = C^+_\alpha |\alpha \rangle + C^-_\alpha | - \alpha \rangle \equiv |\alpha; C^+, C^-\rangle,$$

(12)

where $|\alpha\rangle$ are multimode Glauber CS, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ and $C^\pm(\alpha)$ are arbitrary functions, subjected to the normalization condition $| |\alpha\rangle|^2 = \alpha \cdot \alpha = |\alpha_1|^2 + \ldots + |\alpha_N|^2$.

$$|C^+_\alpha|^2 + |C^-_\alpha|^2 + 2\text{Re}(C^-_\alpha C^+_\alpha) N(|\alpha\rangle) = 1, \quad N(|\alpha\rangle) = \langle \pm \alpha | \mp \alpha \rangle = e^{-2|\alpha|^2}.  \quad (13)$$

Thus the families of states $|\alpha; C^+, C^-\rangle$ represent the whole set of $sp(N,C)$ BG CS for the representation (8). They have the form of macroscopic superpositions of Glauber multimode CS. Such superposition states are also called Schrödinger cat states.

The large family of $sp(N,C)$ CS (12) contains many known in quantum optics sets of states [13, 14, 15] and many yet not studied sets. Let us point out some of the well known particular subsets of (12). The limiting cases of $C^-_\alpha = 0$ or $C^+_\alpha = 0$ recover the overcomplete family of Glauber multimode CS, and $C^-_\alpha = \pm C^+_\alpha$ produces the multimode even/odd CS [14]. When $N = 2$ the ”pair CS” [13] and the ”two mode Schrödinger cat states” [16] are obtained in appropriate manner. Most of the superpositions (12) for the one mode case ($N = 1$) are thoroughly studied [13]. Nevertheless (as far as we know) even in the one dimensional case no family of Schrödinger cat states was pointed out which is overcomplete in the strong sense in whole Hilbert space $\mathcal{H}$. Here we provide such families.

Consider in (12) the choice of

$$C^+_\alpha = \cos \varphi, \quad C^-_\alpha = \pm i \sin \varphi,$$

which clearly satisfy the normcondition [13] for any angle $\varphi$,

$$|\alpha; \varphi, \pm\rangle = \cos \varphi |\alpha\rangle \pm i \sin \varphi | - \alpha\rangle.$$  

(15)

In Fock basis (number states $|n_1, \ldots, n_N\rangle$ we have the expansion

$$|\alpha; \varphi, \pm\rangle = e^{-|\alpha|^2/2} \sum_{n_i=0}^\infty \frac{\alpha_1^{n_1} \ldots \alpha_N^{n_N} e^{\pm i(1)^{n_1} + \ldots + n_N \varphi}}{\sqrt{n_1! \ldots n_N!}} |n_1, \ldots, n_N\rangle.$$  

(16)

By direct calculations we find that these states resolve the unity operator for any $\varphi$ and thereby provide an analytic representation in the whole $\mathcal{H}$.
where \( a_j = P_\varphi \alpha \), \( a_j^\dagger = P_\varphi \frac{d}{d\alpha} \), (18)

where \( P_\varphi \) acts as inversion operator with respect to \( \varphi \):

\[
P_\varphi f(\varphi) = f(-\varphi).
\]

At \( \varphi = 0, \pi \) the Glauber CS representation \( a = \alpha, a^\dagger = d/d\alpha \) is recovered. The subsets of the one mode states (12) corresponding to \( C^- = \pm C^+ \) (the even/odd CS \(|\alpha\rangle_\pm\)) and \( C^- = C_+ \exp(i\phi) \) (the Yurke-Stoler states), the nonclassical properties of which were extensively studied [13], do not resolve the unity in whole \( \mathcal{H} \). The multimode even/odd CS \(|\vec{\alpha}; C^+, \pm C^+ \rangle \equiv |\vec{\alpha}\rangle_\pm\) are overcomplete in the even/odd subspaces \( \mathcal{H}^\pm, \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \):

\[
\frac{1}{\pi^N} \int d^2\vec{\alpha} |\alpha\rangle_\pm \langle \alpha| = 1_\pm.
\] (19)

The notations of (7) enable us to introduce the squared amplitude Schrödinger cat states in the form

\[
|\{z_{kl}\}; D_+, D_-\rangle = D_+(\{z_{ij}\}|\{z_{kl}\}) + D_-(-\{z_{ij}\}|{-z_{kl}}),
\] (20)

where the functions \( D_\pm(\{z_{ij}\}) \) have to be subjected to the normalization condition (supposing \( \langle \{z_{ij}\}|\{z_{kl}\} = 1 \))

\[
|D_+|^2 + |D_-|^2 + D_-^* D_+(\{z_{kl}\}|{-z_{kl}}) + D_+^* D_-(-\{z_{kl}\}|\{z_{kl}\}) = 1.
\]

These superpositions are eigenstates of all \( E^2_{ij} \),

\[
E^2_{ij} |\{z_{kl}\}; \text{cats}\rangle = z^2_{ij}|\{z_{kl}\}; \text{cats}\rangle.
\] (21)

Evidently, the above two formulas are of quite general nature. But we note that the above squared amplitude cat’s states are explicitly given, since \(|\{z_{kl}\}\rangle\) are constructed (eq. (12)). In the two mode case the particular choice of \( D_- = D_+ \exp(i\phi) \) was considered in ref. [16] under the name “two mode Schrödinger cat states”.

4 **BG CS for the algebra \( u(p, q) \)**

The algebra \( u(p, q) \), \( p + q = N \), is real form of \( \mathfrak{sl}(N, \mathbb{C}) \) and it is a subalgebra of \( \mathfrak{sp}(N, \mathbb{C}) \). Therefore the BG CS for \( u(p, q) \) should be obtained from \( \mathfrak{sp}(N, \mathbb{C}) \) CS \(|\vec{\alpha}; C_-, C_+ \rangle\) by a suitable restrictions. In this section we consider these problems in greater detail in the bosonic representation (8).

The following subset of operators of (8) close the \( u(p, q) \) algebra [11].
\[
E_{\alpha\mu} = a_\alpha a_\mu, \quad E_{\alpha\mu}^\dagger = a_\alpha^\dagger a_\mu^\dagger, \quad H_{\alpha\beta} = \frac{1}{2}(a_\alpha^\dagger a_\beta + a_\beta a_\alpha^\dagger), \quad H_{\mu\nu} = \frac{1}{2}(a_\mu^\dagger a_\nu + a_\nu a_\mu^\dagger),
\]

(22)

where we adopted the notations \(\alpha, \beta, \gamma = 1, \ldots, p, \ \mu, \nu = p + 1, \ldots, p + q, \ \ p + q = N\) (while \(i, j, k, l = 1, 2, \ldots, N\)). For \(p = 1 = q\) the three standard \(su(1, 1)\) operators \(K_+, K_-, K_3\) are \(K_+ = E_{12}, K_- = E_{12}^\dagger, K_3 = (H_{11} + H_{22} + 1)/2\). The subsets of hermitian operators

\[
M^{(p)}_{\alpha\beta} = \frac{1}{2}(H_{\alpha\beta} + H_{\beta\alpha} - \delta_{\alpha\beta}), \quad \tilde{M}^{(p)}_{\alpha\beta} = i(H_{\beta\alpha} - H_{\alpha\beta})
\]

\[
M^{(q)}_{\mu\nu} = \frac{1}{2}(H_{\mu\nu} + H_{\nu\mu} - \delta_{\mu\nu}), \quad \tilde{M}^{(q)}_{\mu\nu} = i(H_{\nu\mu} - H_{\mu\nu})
\]

(23)

realize representations of compact subalgebras \(u(p)\) and \(u(q)\) correspondingly. The \(u(p, q)\) algebra acts irreducibly in the subspaces of eigenstates of the hermitian operator \(L\),

\[
L = \sum_\alpha M^{(p)}_{\alpha\alpha} - \sum_\mu M^{(q)}_{\mu\mu} = \sum_\alpha H_{\alpha\alpha} - \sum_\mu H_{\mu\mu}.
\]

(24)

This is the linear in generators Casimir operator and the higher Casimirs here are expressed in terms of \(L\) \((23)\). Denoting the eigenvalue of \(L\) by \(l\) we have the expansion \(H = \sum_{l=-\infty}^{\infty} \oplus H_l\). The representations corresponding to \(\pm l\) are equivalent (but the subspaces \(H_{\pm l}\) are orthogonal).

The commuting Weyl lowering generators of \(u(p, q)\) are \(E_{\mu\alpha} = a_\mu a_\alpha\). Therefor the \(u(p, q)\) BG CS are defined (in the above bosonic representation) as \(|\{\alpha_\beta\alpha_\gamma\}; l, p, q\rangle\),

\[
a_\mu a_\gamma |\{\alpha_\beta\alpha_\gamma\}; l, p, q\rangle = a_\mu a_\gamma |\{\alpha_\beta\alpha_\gamma\}; l, p, q\rangle,
\]

\[
\gamma = 1, \ldots, p, \ \mu = p + 1, \ldots, p + q.
\]

(25)

Solutions to these equations are \(|\tilde{\alpha}; l, p, q\rangle = |\{\alpha_\beta\alpha_\gamma\}; l, p, q\rangle\)

\[
|\tilde{\alpha}; l, p, q\rangle = \sum_{n_{\alpha_1, \ldots, \alpha_N}, n_{\beta_1, \ldots, \beta_N}} \frac{\alpha_{n_1} \cdots \alpha_{n_N}}{\sqrt{n_1! \cdots n_{N-1}!(n_p - n_q - l)!}} |n_1, \ldots, n_{N-1}; n_p - n_q - l\rangle,
\]

(26)

where \(\alpha_i, i = 1, \ldots, N\), are arbitrary complex parameters, \(n_p = \sum_\alpha n_\alpha, \ n_q = \sum_\mu n_\mu, \ n_{\beta_i} = n_1 + n_2 + \ldots + n_{N-1}\) and \(l = n_p - n_q\) \((|\psi\rangle\) denotes a nonnormalized state). If we multiply \(|\tilde{\alpha}; l, p, q\rangle\) by \(\exp(-|\tilde{\alpha}|^2/2\) and sum over \(l\) we evidently get the normalized Glauber multimode CS \(|\tilde{\alpha}\rangle\) (for any pair \(p, q\)),

\[
|\tilde{\alpha}\rangle = e^{-\frac{1}{2}|\tilde{\alpha}|^2} \sum_{l=-\infty}^{\infty} |\tilde{\alpha}; l, p, q\rangle.
\]

(27)

The last equality suggests that the states \(|\tilde{\alpha}; l, p, q\rangle\) form overcomplete families in \(H_l\) for every \(p, q\). And this is the case. Indeed, let \(|\psi_l\rangle\) be any states from \(H_l\). Using the overcompleteness of \(|\tilde{\alpha}\rangle\), the formula \((27)\) and the orthogonality relations

\[
\langle p, q, l'; \tilde{\alpha}| |\tilde{\alpha}; l, p, q\rangle = \delta_{ll'},
\]

(28)

one can get the resolution of unity in \(H_l\) in terms of the \(u(p, q)\) CS \(|\tilde{\alpha}; l, p, q\rangle\).
\[
\int d\mu(\alpha) |\alpha; l, p, q\rangle \langle q, p, l; \alpha| = 1_l,
\]
\[
d\mu(\alpha) = \frac{1}{\pi^N} d^2 \alpha e^{-|\alpha|^2}.
\] (29)

Now we note that in \(u(p, q)\) CS \([26]\) one complex parameter, say \(\alpha_N\), can be absorbed into the normalization factor by redefining the rest as \(z_1 = \alpha_1 \alpha_N, \ldots, z_p = \alpha_p \alpha_N, z_{p+1} = \alpha_{p+1}/\alpha_N, \ldots, z_{N-1} = \alpha_{N-1}/\alpha_N\). Then we can write \(|\alpha; l, p, q\rangle = |\tilde{z}; l, p, q\rangle\),

\[
|\tilde{z}; l, p, q\rangle = \sum_{\tilde{n}_p - \tilde{n}_q = l} \alpha_N^{-l} \frac{z_1^{n_1} \cdots z_{N-1}^{n_{N-1}}}{\sqrt{n_1! \cdots n_{N-1}!(\tilde{n}_p - \tilde{n}_q - l)!}} |n_1, \ldots, n_{N-1}, \tilde{n}_p - \tilde{n}_q - l\rangle,
\] (30)

where \(\tilde{z} = (z_1, \ldots, z_{N-1})\). In terms of variables \(\tilde{z}\) the resolution of unity reads \((d^2 \tilde{z} = d\text{Re} \tilde{z} d\text{Im} \tilde{z})\)

\[
1_l = \int d\mu(\tilde{z}; l, p, q) |\tilde{z}; l, p, q\rangle \langle q, p, l; \tilde{z}|,
\]
\[
d\mu(\tilde{z}; l, p, q) = \frac{1}{\pi^N} F(|\tilde{z}_p|, |\tilde{z}_q|; l, p, q) d^2 \tilde{z},
\] (31)

where \(|\tilde{z}_p|^2 = |z_1|^2 + \ldots + |z_p|^2, |\tilde{z}_q|^2 = |z_{p+1}|^2 + \ldots + |z_{N-1}|^2\), and

\[
F(|\tilde{z}_p|, |\tilde{z}_q|; l, p, q) = \int d^2 \alpha_N |\alpha_N|^2(q-p-l) \exp \left[ - \left( \frac{|\tilde{z}_p|^2}{|\alpha_N|^2} + |\tilde{z}_q|^2|\tilde{\alpha}_N|^2 + |\alpha_N|^2 \right) \right]
\] (32)

One can prove that the above measure is unique in the class of continuous functions (see the Appendix). In the particular case of \(q = 1\) (then \(p = N - 1, \tilde{n}_q = 0\) and \(\tilde{z}_q = 0\)) and negative \(l, -l \geq p\), the \(u(p, 1)\) BG CS were constructed in [8] with the resolution unity measure (in \(H_l\))

\[
d\mu'(\tilde{z}) = F'(|\tilde{z}|, l, p, 1) d^2 \tilde{z}, \quad F'(l) = \frac{2|\tilde{z}|^{-l-p}}{\pi^N} K_{l-p}(2|\tilde{z}|),
\] (33)

where \(K_{\nu}(z)\) is the modified Bessel function of the second kind. From continuity of \(F'(|\tilde{z}|, l, p, 1)\) and \(F(|\tilde{z}|, l, p, 1)\) we deduce that they coincide (see the Appendix). Then using the analyticity property of Bessel functions \(K_{\nu}(z)\) \([22]\) we establish the following integral representation for \(K_{\nu}(z)\) with \(\nu = 0, 1, \ldots\) and \(\text{Re} z \geq 0\),

\[
K_{\nu}(2z) = 2\pi(2z)^{-\nu} \int_0^\infty dx x^{1+\nu} e^{-(x+z^2)/x}.
\] (34)

For \(p = 1, q = 1\) our states \(|\tilde{z}; l, p, q\rangle\) recover (as the states of \([8]\) do) the BG CS for \(u(1, 1)\) \([7]\).

5 Discussion

We have constructed the Barut-Girardello type coherent states (BG CS) for the noncompact algebras \(u(p, q)\) and \(sp(N, C)\) in the \(N\) mode quadratic bosonic representation (Eq. \([8]\)). The general set of \(sp(N, C)\) CS is obtained in the form of macroscopic superpositions \(|\tilde{\alpha}; C_+, C_-\rangle\) (Eq. \([12]\)) of multimode Glauber CS. Such superpositions are called
ordinary multimode Schrödinger cat states. Several particular cases of these cat states are intensively studied in the literature, due to their nonclassical properties \[ \text{13, 14} \]. The new states (e.g. \( |\alpha; \varphi, \pm\rangle \), eq. (3)) can also exhibit interesting nonclassical properties, such as ordinary quadrature squeezing and subpoissonian photon statistics. They possess the intelligence property to minimize the Robertson multidimensional uncertainty relation \[ \text{20} \] for the hermitian quadratures \( X_{ij}, Y_{ij} \) of Weyl lowering operators \( E_{ij} = X_{ij} - iY_{ij} \), since they are eigenstates of all \( E_{ij} \) \[ \text{4} \]. The constructed \( u(p, q) \) and \( sp(N, C) \) BG CS are stable under the action of the free field evolution operator.

However, these states cannot exhibit squeezing of the observables \( X_{ij} \) and \( Y_{ij} \) since here the variances of \( X_{ij} \) and \( Y_{ij} \) are equal as a result of their eigenvalue property \[ 1 \]. In the representation \( E_{ij} = a_i a_j \) the \( X_{ij} \) (or \( Y_{ij} \)) squeezing is the squared amplitude squeezing.

Squeezing of the quadratures of \( a_i a_j \) for a given modes \( i, j \) can be achieved in two ways:

a) in the eigenstates of the complex combinations \( u_{ij} a_i a_j + v_{ij} a_i^\dagger a_j^\dagger \);

b) in the eigenstates \( a_i^2 a_j^2 \).

For \( i \neq j \) squeezed states (SS) of type a) were constructed in \[ \text{1} \] as eigenstates of \( u K_+ + v K_- \), \( K_\pm \) being generators of \( SU(1, 1) \sim Sp(1, R) \) in the representations with Bargman index \( k = 1/2, 1, \ldots \). Those eigenstates minimize the Schrödinger uncertainty relation for \( K_1 \) and \( K_2 \). By means of two boson operators \( a_i, a_j \) the realization of \( K_\pm \) is according to \[ \text{8} \] with fixed \( i, j \) and \( k = (1 - l)/2, l = n_i - n_j \). Eigenstates of other combinations of \( K_\pm \) and \( K_3 \) in the two mode case were studied in \[ \text{16, 18, 17} \]. In the one mode case Schrödinger intelligent states for two generators of \( SU(1, 1) \) were constructed and discussed in \[ \text{3, 5} \]. As it was noted in \[ \text{8} \] the passage from eigenstates of \( E_{ij} \) (i.e. from BG CS) to the eigenstates of combination \( u_{ij} a_i a_j + v_{ij} a_i^\dagger a_j^\dagger \) (which could exhibit squeezing of \( X_{ij}, Y_{ij} \)) can’t be performed by unitary squeeze operator \( S(u, v) \). Let us recall that the ordinary SS are associated to the complexified Heisenberg algebra, where the passage from eigenstates of \( a_i \) to the eigenstates of \( u_{ij} a_i a_j + v_{ij} a_i^\dagger a_j^\dagger \) is accomplished by unitary operator - the standard squeeze operator.

A family of states in which squeezing of quadratures of any product \( a_i a_j \), \( i, j = 1, 2, \ldots, N \), can occur should be called a family of multimode squared amplitude SS. Example of such multimode SS is given by the Robertson intelligent states, which should be eigenstates of \( u_{kl} a_i a_j + v_{kl} a_i^\dagger a_j^\dagger \) (summation over repeated indices) \[ \text{3} \]. These are multimode SS of type a). Multimode squared amplitude SS of type b) are eigenstates of all \( (a_i a_j)^2 \). The latter have been introduced here explicitly by means of eq. (20). For \( N = 2 \) a particular subsets of (20) have been studied in \[ \text{16} \], where it was shown that they exhibit ordinary squeezing and subpoissonian statistics. We note here that they can exhibit squared amplitude squeezing as well, which should be considered elsewhere.

As to the overcompleteness properties of the constructed CS it is worth noting that the families of \( sp(N, C) \) CS \( |\alpha; C_+ = \cos \varphi, C_- = \pm i \sin \varphi\rangle \equiv |\alpha; \varphi, \pm\rangle \) are overcomplete in the whole Hilbert space, while the representation \[ \text{8} \] is irreducible in the subspaces \( \mathcal{H}^\pm \) (spanned by Fock states with even/odd total number of bosons) only. Let us compare this property with the corresponding one of group related CS - by construction the latter are overcomplete in the space which is irreducible under the group (and therefor algebra) action: the \( Sp(N, R) \) group related CS in the representation \[ \text{8} \] are overcomplete in \( \mathcal{H}^\pm \), not in the whole \( \mathcal{H} \). The BG type of CS are in fact algebra related and enable
the resolution of unity in the whole $\mathcal{H}$. The resolution unity measure for the $sp(N, C)$ normalized BG CS $|\bar{\alpha}; \varphi \pm\rangle$ was obtained the same as for the multimode Glauber CS. The resolution unity measures for $u(p, q)$ CS $|\bar{z}; l, p, q\rangle$ generalize the BG CS measure [9] for $u(1, 1)$ and the recently obtained measure [8] for $u(p, 1)$ to the case of any $p, q$ and any value of the first Casimir $l$. Finally it worth noting high symmetry of Glauber multimode CS: these are simultaneously $H_W$ group related CS and $h_W$ and $sp(N, C)$ algebra related CS (where $H_W$ denotes the Heisenberg-Weyl group).

6 Appendix

A. Uniqueness of the resolution unity measures $d\mu(\bar{z}, l, p, q)$

Suppose that there exists another function $F'(|\bar{z}_p|, |\bar{z}_q|; l, p, q)$ such that the new measure $d\mu' = F'd^2\bar{z}$ resolves the unity $1_l$ as in eq. (31). Then we should have

$$0 = \int d^2\bar{z} [F(|\bar{z}_p|, |\bar{z}_q|; l, p, q) - F'(|\bar{z}_p|, |\bar{z}_q|; l, p, q)] |\bar{z}; l, p, q\rangle\langle q, p, l; \bar{z}|,$$

(35)

Substituting the expansion (33) of $|\bar{z}; l, p, q\rangle\langle$ and integrating with respect to angles $\varphi_i = \text{arg} z_i$ we obtain that the difference function

$$\Phi(r_1, r_2, \ldots, r_{N-1}) \equiv F(\tilde{r}_p, \tilde{r}_q; l, p, q) - F'(\tilde{r}_p, \tilde{r}_q; l, p, q),$$

where $\tilde{r}_p \equiv |\bar{z}_p| = \sqrt{r_1^2 + \ldots + r_p}$ and $\tilde{r}_q \equiv |\bar{z}_q| = \sqrt{r_{p+1}^2 + \ldots + r_{N-1}^2}$, should be orthogonal to the monomials

$$r_1^{2n_1+1} \ldots r_{N-1}^{2n_{N-1}+1}, \quad r_i = |z_i|, \: i = 1, \ldots, N - 1, \: n_i = 1, \ldots,$$

Redenoting $r_i^2$ again as $r_i$ one can write this orthogonality in the form

$$\int_0^\infty dr_1 \ldots dr_{N-1} \Phi(r_1, \ldots, r_{N-1}) r_1^{n_1} \ldots r_{N-1}^{n_{N-1}} = 0,$$

(36)

where $n_i = 1, 2, \ldots, \: i = 1, \ldots, N - 1$. Herefrom it follows that $\Phi$ is orthogonal to any function $f(r_1 \ldots r_{N-1})$ which admits Taylor expansion,

$$\int_0^\infty dr_1 \ldots dr_{N-1} \Phi(r_1, \ldots, r_{N-1}) f(r_1 \ldots r_{N-1}) = 0.$$

(37)

This implies that $\Phi = F - F' = 0$ almost everywhere. Indeed, if $\Phi \neq 0$ it must be nonpositive definite (in order to obey (36)) and if $\Phi$ is well behaved (it is sufficient to be continuous) we could find $f$ which is negative in the domains where $\Phi < 0$. But then we could not maintain (37), which proves that $F = F'$ almost everywhere. If $F$ and $F'$ are continuous they should coincide.

B. Proof of the representation (34) of $K_\nu(z)$
In case of $q = 1 \ (p = N - 1)$ and $-l \geq p$ our measure function $F$, eq. (32), depends on $r_1, \ldots, r_p$ through $|\vec{z}| = |z_1|^2 + \ldots + |z_p|^2|^{-1/2} \equiv \vec{r}$ and it is easily seen that $F$ is a continuous (and positive) function of $r_1, \ldots, r_p$. The measure function of ref. [8] is $F' \sim r^{-l-p}K_{-l-p}(2r)$ (in their case $-l > p$) is also continuous [22], therefor the difference $\Phi$ of these two functions is continuous and in view of (36) they have to coincide pointwise. This proves formula (34) for $\text{Im} z = 0, \text{Re} z > 0$.

The Bessel function $K_\nu(z)$ is analytic and regular everywhere except of the negative half of the real line [22]. Let us consider the right hand side of (34) as a definition of a new function $F(z; \nu)$, $z$ complex, $\nu$ real. The integral is convergent for $\text{Re} z > 0$ and the function $F(z; \nu)$ is evidently analytic. We proved in the above that the two analytic functions $F(z; \nu)$ and $K_\nu(2z)$ ($\nu = 0, 1, \ldots$) coincide on the positive part of the real line. Then they coincide in the whole domain of analyticity. We note that (34) does not hold for negative $\nu$: $K_{-\nu}(2z) = K_\nu(2z)$, but $F(z; -\nu) \neq F(z; \nu)$.

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