On Modular Invariance and 3D Gravitational Instantons

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Abstract

We study the modular transformation properties of Euclidean solutions of 3D gravity whose asymptotic geometry has the topology of a torus. These solutions represent saddle points of the grand canonical partition function with an important example being the BTZ black hole, and their properties under modular transformations are inherited from the boundary conformal field theory encoding the asymptotic dynamics. Within the Chern Simons formulation, classical solutions are characterised by specific holonomies describing the wrapping of the gauge field around cycles of the torus. We find that provided these holonomies transform in an appropriate manner, there exists an associated modular invariant grand canonical partition function and that the spectrum of saddle points naturally includes a thermal bath in $AdS_3$ as discussed by Maldacena and Strominger. Indeed, certain modular transformations can naturally be described within classical bulk dynamics as mapping between different foliations with a “time” coordinate along different cycles of the asymptotic torus.

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I. INTRODUCTION

One of the regimes in which the correspondence [1–3] between string theory on anti-de Sitter space (AdS) and conformal field theory is under some measure of control is in the case of AdS$_3$. In this example, there are no propagating gravitational modes in the bulk and it has been known for some time [4] that the asymptotic dynamics of the gravitational field, with appropriate boundary conditions, induces a local conformal symmetry with central charge,

$$c_{cl} = \frac{3l^2}{2G},$$

(1)

in which $G$ is Newton’s constant and $-2/l^2$ is the cosmological constant. The particular asymptotic fall-off required as a boundary condition for the metric, for preservation of the asymptotic isometry, is known [4] to be equivalent to the conformal treatment of infinity of Penrose, and which was discussed in the context of the AdS/CFT correspondence by Witten [3].

As a consequence of the trivial bulk dynamics, it is well known that under these particular boundary conditions, the asymptotic dynamics of the gravitational field effectively reduces to classical Liouville theory [5] (see also [6]). Then, as discussed by Martinec [7], the AdS/CFT correspondence for the metric–stress tensor sector may be rephrased in this example as the semiclassical relation $T_{zz}^{\text{Liouville}} = \langle T_{zz}^{\text{CFT}} \rangle$, or an appropriate generalisation to include correlators. Liouville theory then describes the effective action for the Weyl anomaly of the CFT arising from its coupling to the background geometry, encoded in the conformal Ward identities. In the approach of [7], the level density of Liouville theory, equivalent to that of a free boson, i.e. $c_{eff} = 1$, is only a thermodynamic description of the microstates underlying the CFT associated with (1), and which would presumably also describe the microscopic origin of Bekenstein-Hawking entropy of the BTZ black hole following the argument of [8].

The suggestion that the asymptotic gravitational dynamics encodes (or is induced by) the current sector of an underlying CFT presents a new avenue to investigate inter-relations between classical bulk geometries and 2D CFT notions such as uniformization theory and modular invariance. The first of these was investigated in [7]. Recall that classical 3D Euclidean geometries may be realised as a quotient of Euclidean AdS$_3$ by an element of $SL(2,\mathbb{C})$, and that these matrices are equivalent up to conjugation to the holonomies of the gauge field describing the same classical solution within the Chern Simons formulation of 3D gravity [9]. Since the asymptotic boundary is necessarily a Riemann surface, it was shown in [7] that there is a correspondence between the conjugacy classes of holonomies and particular uniformizing coordinates associated with classical Liouville fields living on the boundary.

The advantage of working with AdS$_3$ is that one may go beyond the semi-classical formulation of the AdS/CFT correspondence, working with small curvatures $l \gg 1$, and consider quantum corrections in the bulk, and thus subleading $O(1/c_{cl})$ corrections in the boundary CFT. The second point noted above, the manifestation of modular invariance in terms of classical bulk geometries, which will be the subject addressed in the present work, requires an extension of this kind. As motivation, we note that in Euclidean signature, the compactified boundary of AdS$_3$ may be interpreted as a Riemann sphere. In practice, one does not
need to assume this theory actually lives at the boundary, but keeping this analogy one may recall that the presence of a CFT on the plane generally implies the existence of a modular invariant theory on the torus \([10]\). Indeed, an asymptotic torus geometry is known to be the topology associated with the Euclidean BTZ black hole \([11]\). The question then arises as to whether the BTZ black hole has an associated modular invariant partition function, or indeed how modular transformations would manifest themselves in the bulk theory.

To explore this suggestion in more detail, we concentrate for the moment on the BTZ black hole \([11]\). We may write the action in terms of standard thermodynamic quantities, the average energy \(\langle E \rangle\), spin \(\langle J \rangle\), and entropy \(S\), as

\[
I = \beta \langle E \rangle + \Omega \langle J \rangle - S.
\]

The classical black hole partition function, \(e^{-I}\), then takes the form

\[
e^{-I_{BH}} = \exp \left( \frac{2i\pi c_{cl}}{24} \left( \frac{1}{\tau} - \frac{1}{\bar{\tau}} \right) \right),
\]

where \(\langle E \rangle = M\) is the black hole mass, \(\beta\) is the inverse temperature, \(\Omega\) is the angular velocity, and \(\tau\) is a modular parameter (complex structure), given by \([13]\)

\[
\tau = \frac{\beta}{2\pi} \left( \Omega + \frac{i}{l} \right).
\]

In a recent analysis, Maldacena and Strominger \([12]\) noted that another saddle point, which one may interpret as a thermal bath in \(\text{AdS}_3\), may be represented by the action \(I = \beta \langle E \rangle\), and has a classical partition function of the form

\[
e^{-I_{\text{AdS}}} = \exp \left( -\frac{2i\pi c_{cl}}{24} (\tau - \bar{\tau}) \right),
\]

where \(\langle E \rangle\) is now interpreted as the \(\text{AdS}_3\) mass, which importantly is negative. This re-interpretation naturally corresponds to the rewriting of the full ensemble above via use of the 3D Smarr formula \(-\beta M + \Omega J + S = \langle \beta M + \Omega J \rangle\). It was observed in \([12]\) that these two solutions are related by the modular transformation \(\tau \to -1/\tau\), and that this transformation corresponds to an analogous mapping in the element of \(SL(2,\mathbb{C})\) by which they were constructed, leading to a conjecture that one should have a full \(SL(2,\mathbb{Z})\) spectrum of such solutions. This is plausible since both solutions above have the geometry of a solid torus, while the radial coordinate actually decouples from the dynamics. Thus the effective theory is defined on a torus and, if conformal as understood above, one expects a modular invariant partition function to exist. Indeed it was noted in \([12]\), that the dual conformal field theory was defined on the complex plane with specific identifications analogous to the identifications used in constructing the bulk solutions.

Since this modular invariant structure is associated with transformations of the quotient \(SL(2,\mathbb{C})\) matrix, one expects that these features should also be apparent in the Chern-Simons formalism as particular transformations on the holonomies of classical solutions. In this paper we shall show that this is indeed the case, and that a natural way to picture the effect of modular transformations such as \(\tau \to -1/\tau\) at the classical level is as a mapping to new canonical 2+1 decompositions of the Chern-Simons action, where the “time” direction is associated with different cycles of the asymptotic torus. On the other hand, at the level of the CFT, the holonomies of the classical gauge field naturally enter via an appropriate choice of basis for the asymptotic symmetry algebra. The modular invariance of the grand
canonical partition function [13] is conveniently made manifest by working with the chiral WZW models associated with the gauge fixed Chern Simons action. This choice of basis then enters through the characters of the Kac-Moody algebra describing the WZW partition function.

The layout of the paper is as follows. In Section 2 we review the structure of asymptotically toroidal solutions in 3D gravity, and construct a generic 2+1 decomposition in first order form. In Section 3 we discuss the prominent role played by the defining holonomies in determining the modular invariance properties of the grand canonical ensemble. We finish in Section 4 with some additional remarks on the semi-classical limit.

II. GENERAL CANONICAL DECOMPOSITIONS

Since classical solutions of 3D gravity have constant curvature, they can be represented in terms of a quotient of the hyperbolic space (Euclidean $AdS_3$)

$$ l^{-2} ds^2 = e^{2\rho} dwd\bar{w} + d\rho^2, \tag{5} $$

by an element $H$ of $SL(2,\mathbb{C})$, where $w$ is a complex coordinate on the plane, the radial coordinate is $\exp(\rho)$, and $l$ denotes the scale of the geometry. In particular, we shall be concerned here with asymptotically toroidal geometries, so that $H = H(\tau)$ is diagonal, and in the case of the BTZ black hole, given by

$$ H = \begin{pmatrix} e^{-i\pi/\tau} & 0 \\ 0 & e^{i\pi/\tau} \end{pmatrix}. \tag{6} $$

This generates the identifications,

$$ \rho \sim \rho + i\pi \left( \frac{1}{\tau} + \frac{1}{\bar{\tau}} \right), \quad w \sim \exp \left( -2i\pi \frac{1}{\tau} \right) w. \tag{7} $$

In this representation, the identification of the black hole boils down to the relation between $\tau$ and the physical variables $\beta$ and $\Omega$. In particular, were one to invert the identification in (3), and identify these physical parameters with $-1/\tau$, then the classical solution we have just described via identification of the hyperbolic space would correspond to a thermal bath in $AdS_3$ [12] as discussed in the introduction. Of course, it is equivalent to retain the identification of $\tau$ in (3) and instead to redefine the identification matrix $H$. This ambiguity in interpretation re-enters when we consider the appropriate choice of boundary conditions below.

The matrix $H$ has a very natural interpretation in the Chern-Simons formulation. To see this, we now recall that the Einstein–Hilbert action for Riemannian geometries, discarding invertibility for the dreibein (see [14]), may be rewritten as the difference of two Chern-Simons actions, with gauge group $SL(2,\mathbb{C})$ [15],

$$ I_{EH} = \frac{1}{16\pi G} \int_M \sqrt{g} \left( R + \frac{2}{l^2} \right) = i \left( I[A] - \overline{I[A]} \right) + B, \tag{8} $$

where $B$ is a boundary term. As noted in [17], the first-order form has a sign ambiguity associated with the relation $\sqrt{g} = \pm e$, where $e$ is the determinant of the dreibein. The
choice fixes the relative orientation of the orthonormal and metric frames, and we shall find it convenient to choose the (unconventional) negative sign \((e < 0)\), so that the Chern-Simons level \(k\), to be introduced shortly, will be positive. The gauge fields are then defined in terms of the spin connection and dreibein as \(A^a = \omega^a + i e^a / l\), and \(\overline{A}^a = \omega^a - i e^a / l\). The gauge field configuration corresponding to the Euclidean black hole is given by

\[
A_a = -\frac{1}{2} \begin{pmatrix} dr & i e^\tau dz / \tau \\ i e^{-\tau} dz / \tau & -dr \end{pmatrix},
\]

where \(r\) is a proper radial coordinate, and \(z = \varphi + \tau x^0\) in terms of the conventional real, periodic, Euclidean black hole coordinates. We note that the coordinate \(z\) naturally lives on a cylinder and is related to the plane coordinate \(w\) by an exponential mapping, the explicit form of which we shall not require. The crucial feature we wish to emphasize, is that the global structure of the gauge field is encoded in the identification matrix \(H\) (6). This follows from the holonomies of (9) which are given by

\[
P \exp \left( \int_\gamma A \right) = MHM^{-1},
\]

where \(M\) is an \(SL(2, \mathbb{C})\) matrix. Consequently, we should expect that the modular structure of classical solutions will involve the gauge field holonomies in a direct manner.

As we consider spacetimes which have the geometry of a solid torus, there are two natural choices for the foliating coordinate to use in pursuing a 2+1 canonical decomposition, namely, the contractible and non-contractible cycles. For generality we shall denote the choice of foliation coordinate as \(u\), and the coordinate along the other cycle as \(v\). Now, in formulating a path integral representation for the grand canonical partition function, one needs to specify boundary conditions which ensure that variations of the action are well defined, while also defining the ensemble by fixing certain physical quantities. This issue has recently been considered in some detail in [18]. The main point to emphasize is that for the ensemble of interest, the covariant form of the Chern Simons action requires no boundary terms at all!

In order to specify the boundary conditions, we now concretely define \(\tau\) in terms of physical quantities via (3), and reflect different identifications of the hyperbolic space, via the \(SL(2, \mathbb{C})\) mapping \(\hat{\tau} = (a\tau + b) / (c\tau + d)\). Then we have \(z = v + \hat{\tau} u\) and the boundary conditions appropriate to this problem are chiral,

\[
A^a_u = 0, \quad \overline{A}^a_u = 0.
\]

An important consequence of these relations, as noted in [16] and fully revealed in [17], is that this condition links the real and imaginary parts of \(A\) at the boundary, where the nonzero components are given by

\[
A^a_z = 2\omega^a_z, \quad \overline{A}^a_z = 2\overline{\omega}^a_z.
\]

Consequently, the \(SL(2, \mathbb{C})\) gauge field reduces at the boundary to the two real \(SU(2)\) currents above.

Since the covariant Chern-Simons action requires no boundary terms, the general 2+1 decomposition with \(A = A_u du + A_i dx^i\) then takes the form,
\[ I[A, \tau] = \frac{k}{4\pi} \int du \int_{\Sigma} e^{ij} T r \left( -A_i \partial_u A_j + A_u F_{ij} \right) \pm \frac{k^2}{4\pi} \int_{\mathcal{T}^2} Tr A^2, \]  

(13)

where \( k = l/(4G) = c_{cl}/6 \) so that with \( e < 0 \) both \( k \) and \( c_{cl} \) are positive. In fact, we should mention that there is a subtlety \(^1\) if one chooses \( u = x^0 \), in that the foliation becomes degenerate at \( r = 0 \). This can be treated by removing a small disk \( r < \epsilon \) on which the covariant action is well defined. However, we shall find here that an additional source will necessarily arise at \( r = 0 \), and resolve the degeneracy. Thus, given this caveat, we shall work generically with (13).

The sign of the boundary term depends on the choice of \( u \), as a consequence of the antisymmetry of the boundary 2-form. In this form, one takes the positive sign when \( u \) denotes the non-contractible cycle.

This representation is simply convenient for compactly expressing results associated with different solutions and different foliating coordinates. In order to have some specific examples we quote below the boundary conditions for the on-shell BTZ black hole and also the thermal \( AdS \) bath \((TAdS)\),

\[
\text{BTZ} = \begin{cases} 
  u = x^0, & \hat{\tau} = \tau \\
  u = \varphi, & \hat{\tau} = \frac{1}{\tau}
\end{cases} \quad \text{TAdS} = \begin{cases} 
  u = x^0, & \hat{\tau} = -\frac{1}{\tau} \\
  u = \varphi, & \hat{\tau} = -\tau
\end{cases}.
\]

(14)

Henceforth we shall focus our attention on these two Euclidean saddle points as they have a natural interpretation. For a specific choice of \( u \), the transformation between boundary conditions for each saddle point is just \( S : \tau \rightarrow -1/\tau \) as one would expect from the discussion of section 1. Note also that if we switch between cycles in defining \( u \), we also switch between the two saddle points up to a sign. This sign change is accounted for by an analogous change in the boundary term, and thus the form of the action is equivalent. This is not surprising since, as discussed earlier, the geometry of these solutions as described by the identification matrix \( H \) is equivalent up to a modular transformation of \( \tau \). However, the semiclassical regime in each case is quite different, with large black holes corresponding to the limit \( |\tau| \ll 1 \), while a low-temperature thermal bath corresponds to \( |\tau| \gg 1 \). As a consequence it was conjectured in [12] that since these solutions are Euclidean saddle points in the partition function there is a phase transition at some temperature separating phases dominated by each saddle point. We shall show in the next section that there is a natural grand canonical partition function which is indeed modular invariant and thus contains all \( SL(2, \mathbb{Z}) \) related saddle points.

III. THE PARTITION FUNCTION AND MODULAR INVARIANCE

Working within the class of asymptotically toroidal geometries we have the crucial simplification that the \( SL(2, \mathbb{C}) \) gauge field reduces at the boundary to two real affine \( SU(2) \) currents at level \( k \), whose components we denote again by \( \{A^i\} \) and \( \{\overline{A}^i\} \), with \( i = 1, 2, 3 \) and the \( SU(2) \) conventions are \([T_a, T_b] = i \epsilon_{abc} T^c \) with \( Tr(T_a T_b) = \delta_{ab}/2 \).

\(^1\) We thank M. Bañados for remarks on this point.
For each of these fields, variation of $A_u$ implies the constraint $F_{ij} = 0$, the most general solution of which, for a manifold with nontrivial cycles such as the solid torus, is given by

$$A = g^{-1} dg + g^{-1} \theta(u,v) g \equiv (g g_0)^{-1} d(g g_0). \quad (15)$$

In this expression $g$ is a single-valued group element, and $\theta$ is a Lie algebra-valued one-form which need depend only on the coordinate parametrising the contractible cycle. With the choice of gauge used in [13] and consistent with the boundary conditions of [4], the $\rho$-dependence may be factored out of the group element and the exterior derivative above reduces to a single $\partial_v$ component. Note, however, that this is a weaker constraint than that required to reduce the asymptotic dynamics to Liouville theory [5]. Since $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ is commutative, the components of $\theta$ may be rotated into the Cartan subalgebra of the group $(T_3)$. Under “large” gauge transformations (modular transformations) $\theta$ can change since it encodes the holonomy of the gauge field, and thus measures the wrapping around the non-contractible cycle of the torus. This zero mode will play a vital role in determining the value of the saddle point, and indeed the modular invariance properties, so its now convenient to fix $u$ equal to the contractible cycle, so that $\theta$ is present in the “spatial” slices. We shall comment on the alternative choice in Section 4.

In the conventional, operatorial, quantisation of Chern-Simons theory the zero-mode factor $g_0$ (or $\theta(u)$) may be parametrised as an appropriately normalised Wilson line. As we noted above, the zero-mode sector of the gauge field may be rotated into the Cartan subalgebra, and thus this Wilson line may be represented as

$$g_0 = P \exp \left( \text{const} \times \int_{\gamma(u,v)} \omega(u,v) a \cdot T^3 \right), \quad (16)$$

where $a$ is a constant, and $\omega$ is the abelian differential of the torus. For concreteness, if one chooses the holomorphic coordinates $(z, \bar{z})$, then the Lie algebra element may be written as [20],

$$g_0^{-1} dg_0 = \frac{ia}{2 \text{Im} \tau} T^3 - \frac{i \bar{\sigma}}{2 \text{Im} \tau} \omega T^3. \quad (17)$$

The holomorphic abelian differential is normalised so that $\int_\alpha \omega = 2\pi$ and $\int_\beta \omega = 2\pi \tau$ where $\alpha$ and $\beta$ are the contractible and non-contractible cycles respectively on the solid torus.

For example, the presence of this zero-mode in the black hole solution may be observed by noting that the on-shell gauge field has a non-vanishing $T^3$ component at $r = 0$,

$$A^3|_{r=0} = \frac{i}{2 \text{Im} \tau} (dz - d\bar{z}), \quad (18)$$

which has precisely the form of (17) with $a = \bar{\sigma} = 1$. Note that the sign of $a$ is according to convention and can be ignored.

In the partition function we are considering, this zero-mode characterises the particular class of classical solutions we are interested in, and thus should be fixed\(^2\). Proceeding in this

\(^2\)We note that quantisation of the zero mode as a quantum mechanical system was initially discussed in [19,14]. This is of course important in theories without boundary, and see [21] (and references therein) for a recent discussion of modular invariance in this context.
manner, the calculation of the partition function reduces to the construction of a specific Chern-Simons state, i.e. with prescribed holonomies around the cycles of the torus [16]. Note that while \( A_0^3 \) can be set to zero by a globally well-defined (large) gauge transformation, we need to distinguish such modular transformations as they change the nature of the classical solution, as discussed in the introduction.

With the zero-mode held constant in this way, one finds after solving the constraint, two copies of a classical \( SU(2) \) chiral WZW (CWZW) model for the group element \( gg_0 \) at level \( k \). Thus the full partition function factorises as

\[
Z_{SL(2,C)}(\tau, \bar{\tau}) = |Z_{SU(2)}(\tau)|^2. \tag{19}
\]

As is well known, the CWZW Lagrangian has the non-canonical form

\[
L = l_a(a) \dot{x}^a - H(x),
\]

in which the symplectic structure is just the \( SU(2) \) Kac-Moody algebra,

\[
[T^a_n, T^b_m] = i\epsilon^{abc} T^c_{n+m} + \frac{k}{2} \delta^{ab} \delta_{n+m,0}, \tag{20}
\]

where \( T^a_n \) are the Fourier components of the gauge field,

\[
A_v = g^{-1} \partial_v g = 2 \sum_{n=\infty}^{\infty} T^a_n e^{inv}. \tag{21}
\]

The Hamiltonian has the form \( H(\dot{\tau}) = k\dot{\tau} \text{Tr}(g^{-1} \partial_v g)^2/(4\pi) \), and thus the partition function reduces to the expectation value of the above Wilson line, parametrising the zero mode, in the CWZW model characterised by the Hamiltonian \( H(\dot{\tau}) \),

\[
Z_{SU(2)}(\tau) = \langle W_R \rangle_{CWZW}, \quad \text{where} \quad W_R = \exp \left( \frac{k}{2} \int_{S^1_{\tau=0}} a \cdot T^3 \right), \tag{22}
\]

which depends only on the representation \( R \) since \( a \) is fixed. The normalisation follows by reference to the definition (21). One may now observe that this Wilson line is precisely what was interpreted as an additional boundary term in the analysis of [13]. We see here that it arises naturally in the parametrisation of the zero-modes of the gauge field.

The calculation of the partition function is then formally equivalent to the calculation of a Chern-Simons state with a prescribed Wilson line around the non-contractible cycle of the solid torus. For large \( k \) (i.e. small curvatures) the \( AdS/CFT \) correspondence relates the classical saddle point for \( Z_{SL(2,C)}(\tau, \bar{\tau}) \sim e^{-I_{\text{inst}}} \) to the boundary CFT partition function. However, we can now consider the generic structure for finite \( k \), and given the factorisation into \( SU(2) \) currents at the boundary, one expects the partition function to be given by

\[
Z_{SL(2,C)} = \sum_{2s=0}^{k} \chi_{2s,k}(a, -\dot{\tau}) \chi_{2s,k}(a, -\dot{\tau}), \tag{23}
\]

where \( \chi_{2s,k}(a, \dot{\tau}) \) are the characters for affine \( SU(2) \), and \( s \) labels the spin of the representation. The shift in the sign of \( \dot{\tau} \) arises from our initial choice of the frame orientation, \( \sqrt{g} = -e \). Recall that as a consequence of (11) we have \( A_z^a = 2ie^a_z/l \) at the boundary, and thus the in using (21) we are working with a minimum weight condition on states, \( T^a_n|0\rangle = 0 \) \( (n > 0) \), rather than the usual maximum weight condition, \( T^a_n|0\rangle = 0 \). The sign reversal
for $\hat{\tau}$ is simply a convenient means to re-orient the generators so that we may avoid this subtlety in working with $SU(2)$ characters.

That (23) indeed arises in this case may be seen explicitly from the structure of the Hamiltonian $H = H(\hat{\tau})$, and recalling that one may realise the conformal symmetry of the model through the Sugawara construction,

$$L_n = \frac{1}{k_q} \sum_{m=-\infty}^{\infty} : T^a_m T^a_{n-m} :.$$  

The Virasoro generators $L_n$ are the Fourier coefficients of the energy-momentum tensor $T(v)$ (or appear via a Laurent expansion $T(w) = \sum_n L_n w^{-n-1}$ in the coordinates $(w, \bar{w})$). We have introduced $k_q = k + c_v$, the quantum shifted level, where $c_v = 2$ for $SU(2)$, which arises in regularisation of the composite operator $A^2$. Then the $SU(2)$ partition function may be expressed [13] as a sum over representations of

$$\chi_{2s,k}(a, \hat{\tau}) = \text{Tr}_s(W_s)q^{L_0-c_q/24} = \text{Tr}_s e^{2\pi i a k q^{L_0-c_q/24}},$$  

where $\hat{q} = \exp(2\pi i \hat{\tau})$, which is indeed recognisable as the definition of an affine $SU(2)$ character, in a particular basis associated with the Weyl chamber of affine $SU(2)$. To see this, recall that (see e.g. [23,24]) for an affine group, the root space is spanned in addition to the root space of the finite Lie group, by two additional generators $\Lambda_0$ and $\delta$. This is because the maximal commuting set of generators now includes, along with the Cartan subalgebra of $SU(2)$, the canonical central element $c$, and the number operator $d = -L_0$. We denote the basis vectors associated with these generators as $\Lambda_0$ and $\delta$. Then, since $SU(2)$ has just one simple root, with a basis vector $\nu$, we can decompose any vector $V$ in root space as,

$$V = 2\pi i (a\nu - \tau \Lambda_0 - b\delta).$$  

The notation we have used is consistent with our construction above, since we recall that the character of a representation is given by $\chi_R = \sum_{\lambda} \text{dim} R_{\lambda} e^\lambda$ where $\text{dim} R_{\lambda}$ is the multiplicity of the weight $\lambda$ in the representation $R$. Thus we see that $a$ gives the coefficient of $T^a$, while $\tau$ (or $\hat{\tau}$ in this case) is the coefficient of the number operator, $d = -L_0$. The final vector $b\delta$ does not appear in the character above, but we shall comment on this shortly.

Expressions for affine $SU(2)$ characters are well known (see e.g. [23,13]) as are their properties under modular transformations [23]. The feature which will be of relevance here is the role played by the zero mode of the gauge field.

We consider the action of the generators of $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$ which we denote $S$ and $T$. Their action on $\tau$ is given by $S : \tau \to -1/\tau$, and $T : \tau \to \tau + 1$. $\chi_{2s,k}$ is known to transform under $T$ via a phase determined by the conformal dimensions of the primary fields of the model [24,20]. Of most interest here is the transformation under $S$. One finds, with the particular action on the basis (26) described in [23], that $\chi_{2s,k}(\hat{\tau}, a, b)$ transforms as follows:

$$\chi_{2s,k} \left(-\frac{1}{\hat{\tau}}, \frac{a}{\hat{\tau}}, 0 + \frac{a^2}{4\hat{\tau}}\right) = \sqrt{\frac{2}{k+2}} \sum_{2x=0}^{k} \sin \left[\frac{\pi(2s+1)(2x+1)}{k+2}\right] \chi_{2s,k}(\hat{\tau}, a, 0).$$  

The particular form of the modular transformations we have used deserves some comment. One sees that with $a = b = 0$, one recovers the standard action on $\hat{\tau}$. The transformation
on $a$ may be understood by recalling that it describes the holonomy of the gauge field around the non-contractible cycle of the torus. This is unchanged under $T$, but the cycles flip under $S$, and thus for the holonomy to remain invariant, recalling the normalisation of the abelian differential, one requires the transformation above. In the introduction we noted that the matrices generating classical solutions via a quotient of the hyperbolic space are expected to be equivalent, up to conjugation by group elements, to the holonomies of the gauge field. Thus it is not unexpected that the parameter $a$ must transform under modular transformations. Finally, the basis element associated with the final coordinate $b$ is associated with the lowest eigenvalue of $L_0$, and thus the above shift reflects a shift of this kind. We shall return to this again shortly.

We can now conclude that the structure of the transformation properties under $T$ and $S$ ensures that $\chi_{2s,k}$ form a unitary representation of the modular group and as a consequence $Z_{SL(2,\mathbb{C})}$, since it is given by (23) as a diagonal sum over the characters, will be modular invariant \[24,20,22\]. Consequently, the result necessarily contains an $SL(2,\mathbb{Z})$ spectrum of saddle points associated with its invariance under the choice of $\hat{\tau} = (a\tau + b)/(c\tau + d)$, provided we now restrict $a, b, c, d \in \mathbb{Z}$.

However, we should point out that this result relied on using a specific definition of the modular transformation associated with transforming the coordinate $b$ from zero to $a^2/(2\hat{\tau})$. This coordinate does not naturally appear in the black hole solution for example, and thus it is more natural to assume that its fixed to zero, and not modified by modular transformations. If this definition is pursued one finds that modular invariance does not hold due to an additional factor of

$$\exp \left( 2i\pi \frac{k a^2}{4} \hat{\tau} \right),$$

which arises in the transformation law of $\chi_{2s,k}(\tau, a)$. This is the factor that was previously cancelled by the transformation of $b$. In the operatorial approach to Chern-Simons theories, one usually recovers modular invariance by the addition of a prefactor $\exp(\pi k a^2/(4\text{Im}\hat{\tau}))$ which one may readily check restores modular invariance for the sum over representations of the combination,

$$\chi_{2s,k}^M(\hat{\tau}, a) = \exp \left( \frac{\pi k a^2}{4\text{Im}\hat{\tau}} \right) \chi_{2s,k}(\hat{\tau}, a).$$

However, one observes from the form of (28), that such a term would be cancelled if the character were to be defined in terms of $L_0 + k a^2/4$ with $a$ fixed, rather than $L_0$. With the value of $a = 1$ associated with the on-shell black hole, one sees that this is a shift $L_0 \rightarrow L_0 + c_{cl}/24$ (cf. (1)).

We can understand this relation as a straightforward ”improvement term” for the Virasoro generators. To see this we recall that given the Sugawara definition for the Virasoro generators above, it is then a standard argument based on conformal invariance that a modular invariant partition function may be defined by the diagonal sum of Virasoro characters,

$$Z_V = \sum_{R, \overline{R}} |\chi_V|^2,$$

where

$$\chi_V = \text{Tr}_R q^{L_0 - c/24}.$$
where $R$ represents a highest weight module of the algebra. Importantly, the latter relation corresponds to a specific specialisation of the character, corresponding to the choice of a basis with a component in the direction of $L_0$. Note that in general the character can be defined by taking the trace of the exponential of a given element of the maximal abelian subalgebra.

For the CWZW model, one finds that $c = c_q = k \dim(G)/(k + c_v) = 3k/(k + 2)$, and thus in the semi-classical limit we have $c_q \sim 3$ for $k \gg 1$. The asymptotic conformal symmetry algebra discussed in the introduction, and associated with our choice of boundary conditions, has $c_{cl} = 6k$ in terms of the $SU(2)$ level in the semi-classical limit, and thus does not directly correspond to the construction above. Indeed this is why quantising the WZW model (or alternatively Liouville theory) cannot directly explain the microstates associated with the asymptotic conformal symmetry.

The relevant shift in the definition of the Virasoro generators so that they realise the asymptotic conformal symmetry with central charge $\sim 6k$ was discussed in the analysis of [20]. This may be justified by demanding that the Kac-Moody currents transform with conformal weight zero. This is not currently the case, and may be corrected by adding an “improvement” term to the energy momentum tensor. This is always possible since the Virasoros are ambiguous up to the addition of diagonal elements of the algebra and derivatives thereof. The general shift for $L_n$ implies that $\Delta L_0 = c_{cl}/24$, and this boosts the total central charge appearing in the Virasoro algebra to $c_{tot} = 3k/(k + 2) + 6k \sim 6k$ which agrees with the asymptotic conformal symmetry of [4]. Of course, this modification has not changed the Hilbert space, as we have simply shifted the lowest eigenvalue of $L_0$ [27]. Thus we see that this shift is indeed the same as accounting for the zero mode contribution to $L_0$ associated with the classical black hole solution. In this sense this description does not resolve the microstates, and the asymptotic level density is still described by a central charge of $O(1)$.

In other words, in the basis $(\tau, a)$ which we are using, the definition of the characters under which they form a representation of the modular group corresponds to (29), and with reference to the Virasoro specialisation, it is then the presence of $a \neq 0$ which provides the shift of the central charge to $c_{tot} = c_q + c_{cl}$. In a basis in which $a$ is also set to zero one then recovers the Virasoro specialisation of the character (30) associated with the combination $L_0 - c_{tot}/(24)$. In other words recovering modular invariance via the prefactor (29) is analogous in this basis, to the shift $L_0 \rightarrow (L_0 - c/(24))$ in the Virasoro specialisation which is known to restore modular invariance. In particular, we see that the Euclidean saddle point is predominantly determined by the “classical” zero mode of the black hole gauge field configuration. This analysis also indicates the connection between the choice of $b$, and the shift of the lowest eigenvalue of $L_0$ as mentioned above.

**IV. DISCUSSION**

To summarise, we have shown that a generic 2+1 canonical decomposition in the Chern-Simons formulation for Euclidean toroidal geometries admits a natural modular invariant partition function arising from the affine symmetry of these solutions. The grand canonical ensemble thus naturally contains a modular invariant spectrum of saddle points whose dominance depends on the position in moduli space, i.e. the choice of $\tau$. Before making
some further concluding remarks on the semi-classical limit, we briefly return to the issue of the choice of foliation coordinate $u$, that was fixed to be the contractible cycle in Section 3 in order to focus on the role of the zero mode. If we choose this coordinate instead as the non-contractible cycle, then the “spatial” integral includes no non-contractible cycles, and it appears that one can set $g_0 = 1$ in (15). However, the global structure of the space essentially ensures that this zero mode is recovered. We may see this by noting that there is a class of gauge transformations, leaving the boundary conditions invariant, which have the following form [20]

$$\hat{g}_m = \exp \left\{ \frac{i}{2\text{Im}\tau} (m\tau dz - m\tau d\bar{z}) \right\}, \quad (31)$$

and act as $g \to \hat{g}_m^{-1}g$, $g_0 \to g_0\hat{g}_m$ on the components of the gauge field in (13). In this expression $m$ labels the winding around the contractible cycle of the solid torus which is appropriate for this angular foliation. Now, in general one should set $m = 0$ since otherwise these transformations would be singular at $r = 0$. However, this is not true if, as in our case, there is a Wilson line passing through the “spatial” slice [20], since in this case there is no way to apply the transformation at the origin, and thus there is no singularity. Consequently, one is able to take such transformations into account, and there is no constraint requiring $g_0 = 1$, as it can be modified by transformations such as (31). Indeed we see that if this choice is made initially, then $g_0 \to g_0\hat{g}_m$ (31) generates precisely the transform under $S : \tau \to -1/\tau, a \to a/\tau$ of the black hole zero mode (18) if we identify $m = a = \bar{a} = 1$.

Thus the full partition function is independent of the choice of foliation as one would expect. Of course, if one only considers particular saddle points of the partition function then different foliations are quite convenient. Indeed, we recall once more the intriguing point that this zero mode structure is encoded at the level of classical saddle points by the 3D Smarr formula, as illustrated in the introduction.

As an illustration of the utility of these results we now focus on the black hole sector, for which its convenient to set $\hat{\tau} = \tau$, and consider $|\tau| \to 0$. In the Virasoro specialisation, it is a standard argument that the semi-classical limit of a modular invariant partition function may be extracted by considering the transformation of coordinates, $z \to e^{2\pi iz/\tau}, \tau \to e^{-2\pi i/\tau}$, appropriate for the $|\tau| \to 0$ boundary of moduli space [11,28]. This leads to the saddle points discussed in the introduction. Its interesting that, although a similar argument based on modular invariance could be applied to the KM characters discussed here, a straightforward extraction of this result follows by studying the asymptotic behaviour of the characters for $|\tau| \to 0$ as follows from their explicit representation in terms of $\theta$–functions [25,24,20]. Recalling (29) and (30), we find

$$Z_{SL(2,C)}(\tau, a) \sim \exp \left( 2\pi i \frac{c_{cl}}{24} \left( \frac{1}{\tau} - \frac{1}{\bar{\tau}} \right) \right). \quad (32)$$

which is the expected semi-classical limit for the black hole, and we have recalled that $a = 1$, and $c_{cl} = 6k$. The calculation may be performed along the lines of that in [13], although we note that the difficulties associated with working with a negative level $k$, and a non-compact group, can now be circumvented [17]; details will appear elsewhere.

One may also observe that to recover thermal $AdS_3$ under a modular transformation, the holonomy $a = 1$ should not transform. However, this is quite consistent since the thermal
AdS$_3$ solution needs the holonomy flipped to the other cycle of the torus, and this is achieved by ensuring $a$ remains invariant.

Knowledge of the semi-classical saddle points for the partition function, and the on-shell value of the modular parameter $\tau$, then allows a straightforward extraction of the entropy, as $S = (1 - \beta \partial_\beta) \ln Z$, in agreement with the Bekenstein-Hawking value. However, we have emphasised that the semi-classical contribution to the central charge which is necessary to describe this density of states is implicit in the classical background solution, and we have no microscopic picture of the contributing states. Nonetheless, as we have seen, the WZW picture is still helpful in unearthing relations between different classical solutions, as a consequence of an underlying modular structure. We note in passing that there has been a recent suggestion [29] that a more general class of asymptotic boundary conditions may provide a purely gravitational description of the black hole microstates. This picture has some connections with the worldsheet string perspective [30], and it would be interesting to study the modular structure in this generalised context. The connection with string theory also requires an extension to consider the sectors associated with different fermionic boundary conditions, for which a related discussion has recently appeared in [31].

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