Introduction

The purposes of this note are

• To propose a direct and “elementary” proof of the main result of [3], namely that the semi-classical spectrum near a global minimum of the classical Hamiltonian determines the whole semi-classical Birkhoff normal form (denoted the BNF) in the non-resonant case. I believe however that the method used in [3] (trace formulas) are more general and can be applied to any non degenerate non resonant critical point provided that the corresponding critical value is “simple”.

• To present in the completely resonant case a similar problem which is NOT what is done in [3]: there, only the non-resonant part of the BNF is proved to be determined from the semi-classical spectrum!

1 A direct proof of the main result of [3]

1.1 The Theorem

Let us give a semi-classical Hamiltonian $\hat{H}$ on $\mathbb{R}^d$ (or even on a smooth connected manifold of dimension $d$) which is the Weyl quantization of the symbol $H \equiv H_0 + \hbar H_1 + \hbar^2 H_2 + \cdots$.

Let us assume that $H_0$ has a global non degenerate non resonant minimum $E_0$ at the point $z_0$: it means that after some affine symplectic change of variables
\[ H_0 = E_0 + \frac{1}{2} \sum_{j=1}^{d} \omega_j (x_j^2 + \xi_j^2) + \cdots \] where the \( \omega_j \)'s are > 0 and independent over the rationals. We can assume that \( 0 < \omega_1 < \omega_2 < \cdots < \omega_d \). We will denote \( E_1 = H_1(z_0) \).

We assume also that
\[
\lim_{(x,\xi) \to \infty} \inf H(x, \xi) > E_0.
\]

Let us denote by \( \lambda_1(\hbar) < \lambda_2(\hbar) \leq \cdots \leq \lambda_N(\hbar) \leq \cdots \) the discrete spectrum of \( \hat{H} \). This set can be finite for a fixed value of \( \hbar \), but, if \( N \) is given, \( \lambda_N(\hbar) \) exists for \( \hbar \) small enough.

**Definition 1.1** The semi-classical spectrum of \( \hat{H} \) is the set of all \( \lambda_N(\hbar) \) (\( N = 1, \cdots \)) modulo \( O(\hbar^{\infty}) \). **NO uniformity with respect to** \( N \) **in the** \( O(\hbar^{\infty}) \) **is required.**

**Definition 1.2** The semi-classical Birkhoff normal form is the following formal series expansion in \( \Omega = (\Omega_1, \cdots, \Omega_d) \) and \( \hbar \):
\[
\hat{B} \equiv E_0 + \hbar E_1 + \sum_{j=1}^{d} \omega_j \Omega_j + \sum_{l+|\alpha| \geq 2} c_{l,\alpha} \hbar^{l} \Omega^{\alpha}
\]
with \( \Omega_j = \frac{1}{2} (-\hbar^2 \partial_j^2 + x_j^2) \). The series \( \hat{B} \) is uniquely defined as being the Weyl quantization of some symbol \( B \) equivalent to the Taylor expansion at \( z_0 \) of \( H \) by some automorphism of the semi-classical Weyl algebra (see [2]).

The main result is the

**Theorem 1.1 [3]** Assume as before that the \( \omega_j \)'s are linearly independent over the rationals. Then the semi-classical spectrum and the semi-classical Birkhoff normal form determine each other.

The main difficulty is that the spectrum of \( \hat{B} \) is naturally labelled by \( d \)-uples \( k \in \mathbb{Z}_+^d \) while the semi-classical spectrum is labelled by \( N \in \mathbb{N} \). We will denote by \( \psi \) the bijection \( \psi : N \to k \) of \( N \) onto \( \mathbb{Z}_+^d := \{ k = (k_1, \cdots, k_d) | \forall j, k_j \in \mathbb{Z}, k_j \geq 0 \} \) given by ordering the numbers \( \langle \omega | k \rangle \) in increasing order: they are pair-wise distincts because of the non-resonant assumption.

## 2 From the semi-classical Birkhoff normal form to the semi-classical spectrum

We have the following result
Theorem 2.1 The semi-classical spectrum is given by the following power series in $\hbar$:

$$\lambda_N(\hbar) \equiv E_0 + \hbar \left( E_1 + \frac{1}{2} \langle \omega | \psi(N) \rangle + \frac{1}{2} \right) + \sum_{j=2}^{\infty} \hbar^j P_j(\psi(N)) \quad (1)$$

where the $P_j$'s are polynomials of degree $j$ given by

$$P_j(Z) = \sum_{\ell + |\alpha| = j} c_{\ell,\alpha} \left( Z + \frac{1}{2} \right)^\alpha.$$

This result is an immediate consequence of results proved by B. Simon [5] and B. Helffer-J. Sjöstrand [4] concerning the first terms, and by J. Sjöstrand in [6] (Theorem 0.1) where he proved a much stronger result.

3 From the semi-classical spectrum to the $\omega_j$'s

3.1 Determining the $\omega_j$'s

Because $E_0 = \lim_{\hbar \to 0} \lambda_1(\hbar)$, we can subtract $E_0$ and assume $E_0 = 0$.

By looking at the limits, as $\hbar \to 0$, $\mu_N := \lim \lambda_N(\hbar)/\hbar$ ($N$ fixed), we know the set of all $E_1 + \sum_{j=1}^d \omega_j(k_j + \frac{1}{2})$, $(k_1, \ldots, k_d) \in \mathbb{Z}_+^d$.

Let us give 2 proofs that the $\mu_N$'s determine the $\omega_j$'s.

1. Using the partition function: from the $\mu_N$'s, we know the meromorphic function

$$Z(z) := \sum e^{-z \mu_N}.$$

$$Z(z) := e^{-z(E_1 + \frac{1}{2} \sum_{j=1}^d \omega_j)} \sum_{k \in \mathbb{Z}_+^d} e^{-z \langle \omega | k \rangle},$$

We have

$$Z(z) = e^{-z(E_1 + \frac{1}{2} \sum_{j=1}^d \omega_j)} \prod_{j=1}^d (1 - e^{-z \omega_j})^{-1}.$$

The poles of $Z$ are $\mathcal{P} := \bigcup_{j=1,\ldots,d} \left( \frac{2\pi i \mathbb{Z}}{\omega_j} \right)$. The set of $\omega_j$ is hence determined up to a permutation. We fix now $\omega = (\omega_1, \ldots, \omega_d)$ with $\omega_1 < \omega_2 < \cdots$.

From the knowledge of the $\omega_j$'s, we get the bijection $\psi$.

2. A more elementary proof: substract $\mu_1 = E_1 + \frac{1}{2} \sum \omega_j$ from the whole sequence and denote $\nu_N = \mu_N - \mu_1$. Then $\omega_1 = \nu_2$. Then remove the multiples of $\omega_1$. The first remaining term is $\omega_2$. Remove all integer linear combinations of $\omega_1$ and $\omega_2$, the first remaining term is $\omega_3$, $\cdots$
3.2 Determining the $c_{l,\alpha}$’s

Let us first fix $N$: from Equation (1) and the knowledge of $\lambda_N \mod O(\hbar^\infty)$ we know the $P_j(\psi(N))’s$ for all $j$’s.

Doing that for all $N$’s and using $\psi$ determine the restriction of the $P_j$’s to $\mathbb{Z}_+^d$ and hence the $P_j$’s.

4 A natural question in the resonant case

4.1 The context

For simplicity, we will consider the completely resonant case $\omega_1 = \cdots = \omega_d = 1$ and work with the Weyl symbols. Let us denote by $\Sigma = \frac{1}{2} \sum (x_j^2 + \xi_j^2)$.

The (Weyl symbol of the) QBNF is then of the form

$$B \equiv \Sigma + \hbar P_{0,1} + \sum_{n=2}^{\infty} \sum_{j+l=n} \hbar^j P_{2l,j}$$

where $P_{2l,j}$ is an homogeneous polynomial of degree $2l$ in $(x, \xi)$, Poisson commuting with $\Sigma$: $\{ \Sigma, P_{2l,j} \} = 0$.

For example, the first non trivial terms are:

- for $n = 2$: $P_{4,0} + \hbar P_{2,1} + \hbar^2 P_{0,2}$
- for $n = 3$: $P_{6,0} + \hbar P_{4,1} + \hbar^2 P_{2,2} + \hbar^3 P_{0,3}$.

The semi-classical spectrum splits into clusters $C_N$ of $N+1$ eigenvalues in an interval of size $O(\hbar^2)$ around each $\hbar(N + \frac{1}{2}d + P_{0,1})$ with $N = 0, 1, \cdots$.

The whole series $B$ is however NOT unique, contrary to the non-resonant case, but defined up to automorphism of the semi-classical Weyl algebra commuting with $\Sigma$.

Let $G$ be the group of such automorphisms (see [2]). The natural question is roughly:

**Is the QBNF determined modulo $G$ from the semi-classical spectrum, i.e. from all the clusters?**

4.2 The group $G$

The linear part of $G$ is the group $M$ of all $A$’s in the symplectic group which commute with $\hat{H}_2$, i.e. the unitary group $U(d)$.

We have an exact sequence of groups:

$$0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0.$$

\footnote{The Moyal bracket of any $A$ with $\hat{H}_2$ reduces to the Poisson bracket}

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Let us describe $K$ (the “pseudo-differential” part): 
Let $S = S_3 + \cdots$ in the Weyl algebra (the formal power series in $(\hbar, x, \xi)$ with the Moyal product and the usual grading degree $(\hbar^j x^\alpha \xi^\beta) = 2j + |\alpha| + |\beta|$)

$$g_S(H) = e^{iS/\hbar} H \ast e^{-iS/\hbar}$$

preserves $\Sigma$ iff $\{S_n, \Sigma\} = 0$. This implies that $n$ is even and $S_n$ is a polynomial in $z_j \overline{z}_k$ ($z_j = x_j + i\xi_j$). Then $K$ is the group of all $g_S$’s with $\{S, \Sigma\} = 0$.

References

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