Casimir effect for curved boundaries in Robertson–Walker spacetime

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Abstract
Vacuum expectation values of the energy–momentum tensor and the Casimir forces are evaluated for scalar and electromagnetic fields in the geometry of two curved boundaries on the background of the Robertson–Walker spacetime with negative spatial curvature. The boundaries under consideration are conformal images of the flat boundaries in Rindler spacetime. Robin boundary conditions are imposed in the case of the scalar field and perfect conductor boundary conditions are assumed for the electromagnetic field. We use the conformal relation between the Robertson–Walker and Rindler spacetimes and the corresponding results for two parallel plates moving with uniform proper acceleration through the Fulling–Rindler vacuum. For the general scale factor the vacuum energy–momentum tensor is decomposed into the boundary-free and boundary-induced parts. The latter is non-diagonal. The Casimir forces are directed along the normals to the boundaries. For the Dirichlet and Neumann scalars and for the electromagnetic field these forces are attractive for all separations.

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1. Introduction
An interesting topic in the investigation of the Casimir effect (for a review see [1]) is the dependence of the vacuum properties on the geometry of background spacetime. Analytic solutions can usually be found only for highly symmetric bulk and boundary geometries. In particular, motivated by the braneworld scenarios, the investigations of the Casimir effect in anti-de Sitter (AdS) spacetime have attracted a great deal of attention. The Casimir effect provides a natural mechanism for stabilizing the radion field in these models, as required
for a complete solution of the hierarchy problem. In addition, the Casimir energy gives a contribution to both the brane and bulk cosmological constants and, hence, has to be taken into account in the self-consistent formulation of the braneworld dynamics. The recent measurements of the Casimir forces between macroscopic bodies provide a sensitive test for constraining the parameters of long-range interactions predicted by modern unification theories [2]. The Casimir energy and corresponding Casimir forces for two parallel branes in AdS spacetime have been investigated in [3] and the local Casimir densities were considered in [4]. The Casimir effect in higher dimensional generalizations of AdS spacetime with compact internal spaces has been considered in [5].

Another popular background in gravitational physics is de Sitter (dS) spacetime. Previously the Casimir effect on this background, described in planar coordinates, is investigated in [6] for a conformally coupled massless scalar field. In this case the problem is conformally related to the corresponding problem in Minkowski spacetime and the vacuum characteristics are generated from those for the Minkowski counterpart multiplying by the conformal factor. The Casimir densities for a massive scalar field with an arbitrary curvature coupling parameter are considered in [7]. In [8] the vacuum expectation value of the energy–momentum tensor for a conformally coupled scalar field is investigated in dS spacetime with static coordinates in the presence of curved branes on which the field obeys the Robin boundary conditions with coordinate-dependent coefficients. In these papers the conformal relation between dS and Rindler spacetimes and the results for the Rindler counterpart were used.

Continuing our previous work [9], in this paper we study an exactly solvable problem with bulk and boundary polarizations of the vacuum on the background of less symmetric (compared to maximally symmetric dS bulk) Robertson–Walker spacetime with negative spatial curvature. The vacuum expectation values of the energy–momentum tensor and the Casimir forces are considered for both scalar and electromagnetic fields in the geometry of two curved boundaries. The corresponding vacuum densities are obtained by making use of the relation between the vacuum expectation values in conformally related problems (see, for instance, [10]) and the results for two infinite plane boundaries moving with uniform proper accelerations through the Fulling–Rindler vacuum [11, 12]. A closely related problem for the evaluation of the energy–momentum tensor of a Casimir apparatus in a weak gravitational field has been recently considered in [13, 14] (see also [15]). In these papers it has been shown that the Casimir energy for a configuration of parallel plates gravitates according to the equivalence principle.

The organization of this paper is as follows. In section 2 the bulk and boundary geometries are specified for the problem under consideration. The vacuum expectation value of the energy–momentum tensor and the Casimir forces are investigated for a scalar field in the geometry of two curved boundaries with Robin boundary conditions. The case of the electromagnetic field with perfect conductor boundary conditions on the boundaries is discussed in section 3. The main results are summarized in section 4.

2. Scalar field

We consider a conformally coupled massless scalar field $\varphi(x)$ on the background of $(D + 1)$-dimensional Robertson–Walker (RW) spacetime with negative spatial curvature. In the hyperspherical coordinates $(r, \theta_1 = \theta, \theta_2, \ldots, \theta_{D-1})$, the corresponding line element has the form

$$ds^2 = g_{ik} dx^i dx^k = a^2(\eta)(d\eta^2 - \gamma^2 dr^2 - r^2 d\Omega_{D-1}^2),$$

(1)
where \( \gamma = 1/\sqrt{1 + r^2} \) and \( d\Omega^2_{D-1} = d\theta^2 + \sum_{k=2}^{D-1} \left( \prod_{l=1}^{k-1} \sin^2 \theta_l \right) d\theta_k^2 \) is the line element on the \((D - 1)\)-dimensional unit sphere in Euclidean space. In the case of conformal coupling, the field equation takes the form

\[
\left( \nabla^i \nabla_i + \frac{D-1}{4D} R \right) \phi(x) = 0, \tag{2}
\]

where \( R \) is the Ricci scalar for the RW spacetime. We assume that the field obeys Robin conditions

\[
\left( \beta_j + n^i_j \nabla_i \right) \phi(x) = 0, \quad x \in S_j, \quad j = a, b, \tag{3}
\]
on two boundaries described by the equations

\[
\sqrt{1 + r^2} \cos \theta = c_j, \tag{4}
\]
with positive constants \( c_a \) and \( c_b \), \( c_a > c_b \). For the region between the boundaries, the corresponding normal has the components

\[
n^i_j = \delta_j \left[ c_j r \alpha(\eta) \right]^{-1}(0, \gamma^{-2}(\gamma - c_j), -\sin \theta, 0, \ldots, 0), \tag{5}
\]
with \( \delta_a = 1 \) and \( \delta_b = -1 \). The coefficients \( \beta_j \) in (3) will be specified below. We are interested in the vacuum expectation value (VEV) of the energy–momentum tensor in the region between two boundaries. In figure 1 we have plotted these boundaries on the \((r, \theta)\) plane by thick lines for the values \( c_a = 2 \) and \( c_b = 0.5 \). Note that for points on boundary (4) one has \( 0 \leq \theta \leq \arccos \left( \sqrt{1 - c_j^2} \right) \) for \( c_j \leq 1 \) and \( 0 \leq \theta \leq \pi \) for \( c_j > 1 \).

In order to evaluate the VEV of the energy–momentum tensor, we present the RW line element in the form conformal to the Rindler metric. This can be done by making use of the coordinate transformation

\[
x^i = (\eta, r, \theta_1, \ldots, \theta_{D-1}) \rightarrow x'^i = (\eta, x'), \tag{6}
\]
with \( x' = (x'^1, x'^2, \ldots, x'^D) \), defined by the relations (for the case \( D = 3 \) see [10])

\[
x'^1 = \xi = \xi_0 \Omega, \quad x'^2 = \xi_0 \Omega \cos \theta_2 \sin \theta_3, \ldots, \quad x'^D = \xi_0 \Omega \prod_{l=1}^{D-2} \sin \theta_l \tag{7}
\]

\[
x'^{D-1} = \xi_0 \Omega \cos \theta_{D-1} \prod_{l=1}^{D-2} \sin \theta_l, \quad x'^D = \xi_0 \Omega \prod_{l=1}^{D-1} \sin \theta_l.
\]
Here \( \xi_0 \) is a constant having the dimension of length and
\[
\Omega = \gamma / (1 - ry \cos \theta).
\] (8)

In coordinates \( x^i \) the RW line element becomes
\[
dx^2 = g^R_{ik} \, dx^i \, dx^k = a^2(\eta) \xi^{-2} g^R_{ik} \, dx^i \, dx^k.
\] (9)

The latter is manifestly conformal to the metric \( g^R_{ik} \) of the Rindler spacetime:
\[
g^R_{ik} = \text{diag}(\xi^{2}, -1, \ldots, -1).
\] (10)

From (7) and (8) it follows that in coordinates \( x^i \) the boundaries (4) are described by simple equations \( \xi = \xi_j = \xi_0/\epsilon_j \). In figure 1 we have plotted the coordinate lines \( \xi = \text{const} \) in the plane \((r, \theta)\) for several values of the constant (numbers near the curves). The thick lines correspond to the boundaries. The boundaries divide the space into three regions. In the regions \( \xi < \xi_a \) and \( \xi > \xi_b \), the VEVs are the same as in the geometry with a single boundary and were investigated in [9]. In what follows we shall be concerned with the region between the boundaries, \( \xi_a < \xi < \xi_b \). Using the standard transformation formula for the VEVs of the energy–momentum tensor in conformally related problems, we can generate the results for the RW spacetime from the corresponding results for a scalar field \( \varphi_R \) in the Rindler spacetime for two infinite plates [11, 12] on which the field obeys the boundary conditions
\[
(\beta_{Rj} + n^R_{ij} \nu_j) \varphi_R(x') = 0, \quad \xi = \xi_j, \quad n^R_{ij} = \delta_j \delta_i.
\] (11)

with a constant coefficients \( \beta_{Rj} \). Note that in the Rindler problem \( \xi_j^{-1} \) is the proper acceleration of the plate. The corresponding VEV of the energy–momentum tensor we shall denote by
\[
\langle 0_R | T^i_{jk} [g_{Rm}, \varphi_R] | 0_R \rangle.
\]

In order to find the relation between the coefficients in boundary conditions (4) and (11), we note that under the conformal transformation \( g'_{ik} = [a(\eta)/\xi]^{2} \frac{g^R_{ik}}{2} \), the field \( \varphi_R \) transforms in accordance with
\[
\varphi'(x') = [\xi/a(\eta)]^{(D-1)/2} \varphi_R(x').
\] (12)

Now, comparing boundary conditions (3), (11) and taking into account equation (12), we find
\[
a(\eta) \beta_j = \xi_j \beta_{Rj} + \delta_j (1 - D)/2.
\] (13)

Note that with these Robin coefficients the boundary conditions (3) are time independent. From (13) it follows that the Dirichlet boundary condition in the problem on the RW bulk corresponds to the Dirichlet condition in the Rindler counterpart. For the case of the Neumann boundary condition in the RW bulk the corresponding problem in the Rindler spacetime is of the Robin type with \( \xi_j \beta_{Rj} = \delta_j (D - 1)/2 \).

The VEV of the energy–momentum tensor in coordinates \((\eta, x')\) is found by using the transformation formula for conformally related problems (see, e.g., [10]):
\[
\langle 0_{Rw} | T^i_{jk} [g^R_{lm}, \varphi_R] | 0_{Rw} \rangle = [\xi/a(\eta)]^{D+1} \langle 0_R | T^i_{jk} [g^R_{lm}, \varphi_R] | 0_R \rangle + \langle 0_R | T^i_{jk} [g^R_{lm}, \varphi_R] | 0_R \rangle^{(an)}
\] (14)

where the second term on the right-hand side is determined by the trace anomaly. In odd spacetime dimensions this term is absent. For a scalar field \( \varphi_R(x') \) with boundary conditions (11), the VEV in the corresponding Rindler problem is presented in the form:
\[
\langle 0_R | T^i_{jk} [g^R_{lm}, \varphi_R] | 0_R \rangle = \langle 0_R | T^i_{jk} [g^R_{lm}, \varphi_R] | 0_R \rangle + \langle T^i_{jk} |^{(b)} \rangle,
\] (15)

where \( \langle 0_R \rangle \) is the vacuum state for the Rindler spacetime in the absence of boundaries. The term [12]
\[
\langle T^i_{jk} |^{(b)} \rangle = \sum_{D=0}^{D-1/2} \frac{2^{2-D} \pi^{-(D+1)/2}}{\Gamma((D-1)/2)} \tilde{g}^k \int_0^\infty d\nu \lambda^D \int_0^\infty d\omega \lambda \Omega_{\phi a}(\lambda \alpha, \lambda \beta) F^{(a)}(\alpha, \beta) K_{\phi a}(\lambda \alpha) L_{\phi a}(\lambda \beta)
\] (16)
represents the correction due to the presence of the boundaries. In this formula, \( I_\omega (z) \) and \( K_\omega (z) \) are the modified Bessel functions and for a given function \( f(z) \) the barred notations stand for
\[
\bar{f}^{(j)} (z) = \delta_{j} \xi_j \beta_{j} f(z) + zf'(z),
\]
with \( j = a, b \). Other notations are as follows:
\[
\Omega_{aw}(u, v) = \frac{\tilde{K}_{aw}^{(b)} (v) / \tilde{K}_{aw}^{(a)} (u)}{\tilde{I}_{aw}^{(b)} (v) \tilde{K}_{aw}^{(a)} (u) - \tilde{K}_{aw}^{(b)} (v) \tilde{I}_{aw}^{(a)} (u)},
\]
\[
\Omega_{bw}(u, v) = \frac{\tilde{I}_{bw}^{(a)} (u) / \tilde{I}_{bw}^{(b)} (v)}{\tilde{I}_{bw}^{(b)} (u) \tilde{K}_{bw}^{(a)} (u) - \tilde{K}_{bw}^{(b)} (v) \tilde{I}_{bw}^{(a)} (u)}.
\]

For a given function \( g(z) \), the functions \( F^{(i)} [g(z)] \) have the form
\[
F^{(i)} [g(z)] = -Dg^2(z) - \frac{D - 1}{z} g(z)g'(z) + 2 \left( 1 + \frac{\omega^2}{z^2} \right) g^2(z),
\]
\[
F^{(i)} [g(z)] = g^2(z) + \left( \frac{\omega^2}{z^2} - \frac{D + 1}{D - 1} \right) g^2(z), \quad i = 2, \ldots, D,
\]
and \( F^{(0)} [g(z)] = -F^{(1)} [g(z)] - \sum_{j=1}^{D} F^{(i)} [g(z)] \). In (16), the term \( T^{k}_{(R)j} \) is the part induced by a single plate at \( \xi = \xi_{j} \) when the second plate is absent and the second term on the right-hand side is the correction due to the second boundary.

Formula (16) with \( j = a \) and \( j = b \) provides two equivalent representations for the VEV. Note that the surface divergences on the boundary at \( \xi = \xi_{j} \) are contained in the first term on the right-hand side of this formula and the second term is finite. The single boundary-induced part in the region \( \xi > \xi_{j} \) is given by the formula [16, 17]
\[
\langle T^{k}_{(R)j} \rangle = \frac{-2^{1-D} \rho^k}{\pi^{(D+1)/2} D \Gamma \left( \frac{D-1}{2} \right)} \int_{0}^{\infty} d\lambda\lambda^{D-1} \int_{0}^{\infty} d\omega J_{\omega}(\lambda \xi_{j}) F^{(i)} [K_{\omega}(\lambda \xi_{j})].
\]

The corresponding expression in the region \( \xi < \xi_{j} \) is obtained from (21) by the replacements \( I_{\omega} \rightarrow K_{\omega} \).

For the RW background, similar to (15), the VEV of the energy–momentum tensor in coordinates \( x^i \) is presented in the form:
\[
\langle 0_{\text{RW}} | T^{k}_{li} [g'_{im}, \varphi] | 0_{\text{RW}} \rangle = \langle 0_{\text{RW}} | T^{k}_{li} [g'_{im}, \varphi] | 0_{\text{RW}} \rangle + \langle T^{k}_{li} [g'_{im}, \varphi] \rangle^{(b)},
\]
where \( \langle 0_{\text{RW}} | T^{k}_{li} [g'_{im}, \varphi] | 0_{\text{RW}} \rangle \) is the vacuum expectation value in the boundary-free RW spacetime and the part \( \langle T^{k}_{li} [g'_{im}, \varphi] \rangle^{(b)} \) is the correction due to boundaries (4). For the separate terms on the right-hand side of (22) we have
\[
\langle 0_{\text{RW}} | T^{k}_{li} [g'_{im}, \varphi] | 0_{\text{RW}} \rangle = [\xi / \alpha(\eta)]^{D+1} \langle 0_{R} | T^{k}_{li} [g'_{im}, \varphi] | 0_{R} \rangle + \langle T^{k}_{li} [g'_{im}, \varphi] \rangle^{(am)},
\]
\[
\langle T^{k}_{li} [g'_{im}, \varphi] \rangle^{(b)} = [\xi / \alpha(\eta)]^{D+1} \langle T^{k}_{(R)li} \rangle^{(b)}.
\]

Now, the VEV of the energy–momentum tensor in coordinates \( x' \) is obtained from (22) by the coordinate transformation \( x^i \rightarrow x'. \) As before, this VEV may be written in the form of the sum of purely RW and boundary parts:
\[
\langle 0_{\text{RW}} | T^{k}_{li} [g'_{im}, \varphi] | 0_{\text{RW}} \rangle = \langle 0_{\text{RW}} | T^{k}_{li} [g'_{im}, \varphi] | 0_{\text{RW}} \rangle + \langle T^{k}_{li} \rangle^{(b)}. \]
Using the expression for \( \langle 0 \rangle_{\text{RW}}|T^k_i[g_{lm}, \varphi]|0_{\text{RW}} \rangle \) from [17], for the purely RW part one finds (for the vacuum polarization in RW spacetimes see [10, 18, 19] and references therein)

\[
\langle 0 \rangle_{\text{RW}}|T^k_i[g_{lm}, \varphi]|0_{\text{RW}} \rangle = \langle T^k_i[g_{lm}, \varphi] \rangle^{(\text{an})} + \frac{2[a(\eta)]^{-D-1}}{(4\pi)^{D/2} \Gamma(D/2)} \times \text{diag}(-D, 1, \ldots, 1) \int_0^\infty \frac{\omega^D d\omega}{e^{2\pi \omega} + (-1)^D} \sum_{l=1}^{l_m} \left( \frac{D - 1 - 2l}{2\omega} \right)^2 + 1, \tag{26}
\]

where \( l_m = D/2 - 1 \) for even \( D \geq 2 \) and \( l_m = (D - 1)/2 \) for odd \( D > 1 \), and the value of the product should be taken 1 for \( D = 1, 2 \). In particular, in \( D = 3 \) for the anomaly part, we have

\[
\langle T^k_i[g_{lm}, \varphi] \rangle^{(\text{an})} = \frac{(1/3) H^k_i - (1/6) H^2_i/6}{2880\pi^2}, \tag{27}
\]

where the expressions for the tensors \( H^k_i \) are given in [10]. Now it can be easily checked that for the static case, \( a(\eta) = \text{const} \), one has \( \langle 0 \rangle_{\text{RW}}|T^k_i[g_{lm}, \varphi]|0_{\text{RW}} \rangle = 0 \).

After the coordinate transformation, for the boundary-induced energy–momentum tensor in coordinates \( x^i \) one has (no summation over \( l \)):

\[
\begin{align*}
&T^{(b)}_1 = [\xi/a(\eta)]^{D+1} \langle T^{(b)}_1 \rangle_{\text{Rindler}}, \quad l = 0, 3, \ldots, D, \\
&T^{(b)}_i = [\xi/a(\eta)]^{D+1} \left[ \langle T^{(b)}_i \rangle_{\text{Rindler}} + (-1)^l \Omega^2 \sin^2 \theta \left( \langle T^{(b)}_1 \rangle_{\text{Rindler}} - \langle T^{(b)}_2 \rangle_{\text{Rindler}} \right) \right], \quad l = 1, 2,
\end{align*}
\]

with the tensor \( \langle T^{(b)}_i \rangle_{\text{Rindler}} \) given by (16). Similar to the case of the Rindler problem, the tensor \( \langle T^{(b)}_i \rangle_{\text{Rindler}} \) is decomposed into the single boundary and the second boundary-induced parts. The resulting energy–momentum tensor is non-diagonal due to the anisotropy of the vacuum stresses in the Rindler counterpart. The time dependence of the boundary-induced part in the VEV of the energy–momentum tensor appears in the form \( a^{-D-1}(\eta) \). In addition this VEV depend on the radial coordinate \( r \) and on the angle \( \theta \). In the components \( \langle T^{(b)}_i \rangle_{\text{Rindler}} \) with \( l = 0, 3, \ldots, D \), the dependence on \( r \) and \( \theta \) enters in the combination \( \xi = \xi(r, \theta) \). So, for a fixed \( \eta \), these components are constant on the lines \( \xi = \text{const} \) (see figure 1). This is not the case for the diagonal components \( \langle T^{(b)}_i \rangle_{\text{Rindler}} \) with \( l = 1, 2 \), and for the off-diagonal component. In the model with \( D = 3 \) and with a power-law scale factor \( a(t) = \alpha t^\gamma, t = [\alpha(1 - c)\eta]^{1/(1-c)} \) being the proper time, for the boundary-free part one has

\[
\langle 0 \rangle_{\text{RW}}|T^k_i[g_{lm}, \varphi]|0_{\text{RW}} \rangle = \frac{c(c^2 - 6c + 3)(3c - 4)}{2880\pi^2 t^4} \text{diag} \left( \frac{3c}{3c - 4}, 1, 1, 1 \right). \tag{29}
\]

Note that the second term on the right-hand side of (26), with the time dependence \( t^{-4c} \), is cancelled by a similar term coming from the anomaly part. In this case, at early (late) stages of the cosmological expansion the boundary-induced part dominates for \( c > 1 \) (\( c < 1 \)).

The \( k \)th component of the Casimir force acting per unit surface of the boundary at \( \xi = \xi_j \) is determined by the expression \( \langle T^{(b)}_i \rangle_{\xi = \xi_j, \text{Rindler}} \), where the normal to the boundary is defined by relation (5). Using this expression and formulae (28), it can be seen that the force is present in the form

\[
\langle T^{(b)}_i \rangle_{\xi = \xi_j} = n^j_\xi \langle \xi_j/a(\eta) \rangle^{D+1} \langle T^{(b)}_1 \rangle_{\xi = \xi_j}, \tag{30}
\]

Hence, we conclude that the Casimir force is directed along the normal to the boundary and does not depend on the point of the boundary. Note that the quantity \( \rho^{(b)}_{\xi_j} = -\langle T^{(b)}_1 \rangle_{\xi = \xi_j} \) determines the effective pressure on the plate at \( \xi = \xi_j \) in the corresponding Rindler problem.
As was shown in [11, 12], the latter is presented in the form
\[ p^{(j)} = p^{(j)}_1 + p^{(j)}_{(\text{int})}, \]
where the first term on the right is the pressure for a single plate at \( \xi = \xi_j \) when the second plate is absent and \( p^{(j)}_{(\text{int})} \) is the pressure induced by the presence of the second plate. The latter can be termed as the interaction part of the effective pressure. The surface divergences are contained in the single boundary parts (for a recent discussion of the surface divergences see [15]) and the interaction parts are finite for all non-zero distances between the boundaries. By using the expression of \( p^{(j)}_{(\text{int})} \), for the corresponding pressure in RW bulk, we find
\[ p^{(j)}_{(\text{int})} = \left[ \xi_j / a(\eta) \right] D + 1 p^{(j)}_{R(\text{int})} = \frac{2^{1-D} \pi^{-(D+1)/2}}{\Gamma((D - 1)/2) a^{D+1}(\eta)} \int_0^\infty dx x^{D-2} \int_0^\infty d\omega \times \left[ x^2 + \omega^2 + (1 - 1/D) \delta_j \beta R_j - \xi_j^2 \beta R_j^2 \right] \Omega_{j\omega}(x \xi_a / \xi_j, x \xi_b / \xi_j). \] (31)

In dependence of the coefficients in the boundary conditions the corresponding forces can be either attractive or repulsive. In particular, for Dirichlet boundary conditions (\( \beta R_j = \infty \)) and for Neumann boundary conditions in the Rindler spacetime problem (\( \beta R_j = 0 \)) the interaction forces are always attractive. Note that the latter corresponds to the Robin-type problem in the RW background (see (13)). For Neumann boundary conditions in the RW problem we have \( \delta_j \xi_j \beta R_j = (D - 1)/2 \). The quantity \( p^{(j)}_{(\text{int})} \) determines the force by which the scalar vacuum acts on the plate due to the modification of the spectrum for the zero-point fluctuations by the presence of the second plate. As the vacuum properties depend on a spatial point, there is no \emph{a priori} reason for the forces to be equal for \( j = a \) and \( j = b \), and they in general are different. In figure 2, the interaction parts in the corresponding vacuum pressures are plotted as functions of the ratio \( c_a/c_b = \xi_b/\xi_a \). As is seen from the graphs, the forces are attractive in this case as well.

At small separations between the boundaries, corresponding to \( c_a/c_b - 1 \ll 1 \), for the Dirichlet condition on one of the boundaries and the non-Dirichlet boundary condition on the other, the leading term in the asymptotic expansion has the form
\[ p^{(j)}_{(\text{int})} \approx \frac{(1 - 2^{-D}) D \xi_b (D + 1)}{(4\pi)^{(D+1)/2} a^{D+1}(\eta)} \Gamma((D + 1)/2), \] (32)
with $\zeta_R(x)$ being the Riemann zeta function. The corresponding forces are repulsive. For all other combinations of the boundary conditions the leading term is given by

$$p^{(j)}_{(\text{inh})} \sim -\frac{D\zeta_R(D+1)}{(4\pi)^{(D+1)/2}a^{D+1}(\eta)} \Gamma((D+1)/2),$$

and the forces are attractive. At large separations, $c_a/c_b \gg 1$, the interaction parts decay as

$$p^{(a)}_{(\text{inh})} \propto -\frac{a}{a^{D+1}(\eta)} \ln^3(c_a/c_b),$$

$$p^{(b)}_{(\text{inh})} \propto \frac{1}{a^{D+1}(\eta)} \ln^3(c_a/c_b).$$

The nature of the corresponding forces depends on the coefficients in Robin boundary conditions. In particular, the vacuum forces acting on the boundaries can be repulsive for small distances and attractive for large distances. This provides a possibility for the stabilization of the interplate distance by vacuum forces. For example, in the case $D = 3$ this type of situation is realized with the coefficients $\beta_R = 0$, $\beta_R = 1/5$.

### 3. Electromagnetic field

In this section we consider the electromagnetic field in the region between two boundaries described by (4) with $j = a, b$. We assume that these boundaries are perfect conductors with the boundary conditions of vanishing of the normal component of the magnetic field and the tangential components of the electric field, evaluated at the local inertial frame. Since the electromagnetic field is conformal in $D = 3$ we restrict ourselves to this case. As in the previous section, we use the conformal relation between the problems in RW and Rindler spacetimes. The VEV of the energy–momentum tensor for the electromagnetic field in the region between two conducting plates moving with uniform proper acceleration in the Fulling–Rindler vacuum is investigated in [11]. Similar to the scalar case, this expectation value is presented in the form (15), where the boundary-free part is given by the formula

$$\langle \tilde{0}_R | T_{k\ell}^{R} \rangle = \frac{11}{240\pi^2 \xi^4} \text{diag}(-1, 1/3, 1/3, 1/3).$$

For the correction due to the boundaries we have the expression

$$\langle T_{(R)}^{k\ell} \rangle = \frac{8k}{4\pi^2 \xi^4} \int_0^{\infty} dk^3 \int_0^{\infty} d\omega \sum_{\sigma = 0, 1} \Omega_{j\omega}^{(\sigma)}(k_\omega a, k_\omega j \xi) F^{(i)}_{\text{em}}[Z_{j\omega}^{(\sigma)}(k_\omega, k_\omega j \xi)].$$

where the notations are defined as

$$Z_{j\omega}^{(\sigma)}(u, v) = \frac{\text{d}_u}{\partial_u} [I_{\omega}(v)K_{\omega}(u) - K_{\omega}(v)I_{\omega}(u)]$$

and

$$\Omega_{j\omega}^{(\sigma)}(u, v) = \frac{K_{\omega}(v)/K_{\omega}(u)}{\text{d}_u Z^{(\sigma)}_{j\omega}(u, v)}, \quad \Omega_{\omega}^{(\sigma)}(u, v) = \frac{I_{\omega}(v)/I_{\omega}(u)}{\text{d}_u Z^{(\sigma)}_{\omega}(u, v)}.$$

For the functions $F^{(i)}_{\text{em}}[g(z)]$ we have

$$F^{(i)}_{\text{em}}[g(z)] = (-1)^i \frac{g^{(2)}(z) + [1 - (-1)^i \omega^2/z^2]g^2(z)}{z^2}, \quad i = 0, 1,$$

and $F^{(i)}_{\text{em}}[g(z)] = -g^2(z)$ for $i = 2, 3$. In (38), for the modified Bessel functions $f^{(0)}(u) = f(u)$ and $f^{(1)}(u) = f'(u)$ with $f = I_{\omega}, K_{\omega}$. In the region $\xi > \xi_j$ for the single boundary-induced part one has the expression [16, 17]

$$\langle T_{(R)}^{k\ell} \rangle = -\frac{8k}{4\pi^2 \xi_j^4} \int_0^{\infty} dx x^3 \int_0^{\infty} d\omega \sum_{\sigma = 0, 1} \frac{I_{\omega}^{(\sigma)}(x)}{K_{\omega}^{(\sigma)}(x)} F^{(i)}_{\text{em}}[K_{\omega}(x\xi/x_j\xi)].$$
The corresponding formula in the region $\xi < \xi_j$ is obtained from (40) by the replacements $I_\omega \leftrightarrow K_\omega$. In formulae (36) and (40), the terms with $\sigma = 0$ and $\sigma = 1$ correspond to the contributions of the transverse electric (TE) and transverse magnetic (TM) waves, respectively.

The VEV of the energy–momentum tensor in the RW bulk is presented in the form (25), where the boundary-free part is given by the expression

$$\langle \tilde{0}_{\text{RW}} | T^i_j [g_{lm}, \varphi] | \tilde{0}_{\text{RW}} \rangle = \frac{11a^{-4}(q)}{240\pi^2} \text{diag}(-1, 1/3, 1/3, 1/3) + \frac{62^{3}H^{3} + 3^{1}H^{3}}{2880\pi^2}. \quad (41)$$

The correction due to the boundaries is given by formulae (28) with the Rindler components from (36). On the basis of (36), these corrections are decomposed into single boundary- and second boundary-induced parts. The time-dependence of the first term on the right-hand side of (41) is the same as that for the boundary-induced part and the relative role of the boundary-free and boundary-induced terms depend on the behaviour of the last term in (41).

In particular, for the power-law expansion with $a(t) = \alpha t^\gamma$, we have

$$\langle \tilde{0}_{\text{RW}} | T^i_j [g_{lm}, \varphi] | \tilde{0}_{\text{RW}} \rangle = \frac{c(31c^2 + 54c - 27)(3c - 4)}{1440\pi^2 t^4} \text{diag} \left( \frac{3c}{3c - 4}, 1, 1, 1 \right)$$

$$- \frac{c(c - 2)}{18\pi^2 \alpha t^{2\gamma}} \text{diag} \left( \frac{3c}{c - 2}, 1, 1, 1 \right). \quad (42)$$

The boundary-induced VEV behaves as $t^{-4c}$ and for $c < 1$ the second term on the right-hand side of (42) dominates at late stages of the cosmological expansion. The corresponding energy density and effective pressures are negative.

Now we turn to the investigation of the vacuum forces acting on boundaries. As in the case of a scalar field, the electromagnetic Casimir force is directed along the normal to the boundary. This force is decomposed into self-action and interaction parts. The latter is induced by the presence of the second boundary and is given by the expression

$$P_{\text{em(int)}}^{(j)} = -\frac{a^{-D-1}(q)}{4\pi^2} \int_0^\infty dx \int_0^\infty d\omega \sum_{\sigma = 0, 1} (-1)^{\sigma} (1 + \omega^2/x^2)^{\sigma} \Omega_{j_{\text{em}}}^{(\sigma)}(x\xi_a/\xi_j, x\xi_b/\xi_j). \quad (43)$$

The corresponding forces are attractive. The interaction parts of the vacuum pressures on the boundaries are plotted in figure 3 as functions of the ratio $c_a/c_b$. Note that $P_{\text{em(int)}}^{(j)}$ is the
sum of the corresponding quantities for the Dirichlet and Neumann (in the Rindler spacetime problem) scalars and the asymptotics at small and large distances directly follow from those for the scalar case.

4. Conclusion

In this paper we have considered an exactly solvable problem for the Casimir effect with curved bulk and boundary geometries. In obtaining the VEVs of the energy–momentum tensor for conformally coupled massless scalar and electromagnetic fields, we use the conformal relation between the RW spacetime with negative spatial curvature and Rindler spacetime. Boundaries in the problem on the RW background are described by (4). They are the conformal images of two parallel plates moving with constant proper acceleration through the Fulling–Rindler vacuum. We assumed Robin boundary conditions in the case of the scalar field and perfect conductor boundary conditions for the electromagnetic field. For the corresponding Rindler problem the Casimir densities are given by expressions (16) and (36) for the scalar and electromagnetic fields, respectively.

In order to generate the VEVs for the RW problem, first we use the coordinate transformation (7) which presents the RW metric in the form manifestly conformal to the Rindler metric. As the next step, we obtain the VEVs in new coordinates by the conformal transformation from the results of the corresponding problem in the Rindler spacetime. For the scalar field the coefficients in Robin boundary conditions are related by (13). At the final stage, we transform the VEVs of the energy–momentum tensor to the initial coordinates. In this way the VEVs are decomposed into boundary-free and boundary-induced parts. The latter are given by expressions (28). The vacuum stresses in the Rindler spacetime problem are anisotropic and as a consequence of this the boundary-induced part in the vacuum energy–momentum tensor for the RW problem is non-diagonal.

Having the VEVs of the energy–momentum tensor we have investigated the Casimir forces acting on the boundaries. These forces are decomposed into self-action and interaction parts. The interaction forces are directed along the normal to the boundary and the corresponding effective pressures are given by expressions (31) and (43) for the scalar and electromagnetic fields, respectively. These pressures are independent of the point on the boundary. For the scalar field, in dependence of the Robin coefficients, the interaction forces can be either attractive or repulsive. For Dirichlet and Neumann boundary conditions they are attractive for all separations between the boundaries. In the case of the electromagnetic field the force is the sum of the forces for Dirichlet and Neumann scalars and is attractive.

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