Qualitative analysis of fractal-fractional order COVID-19 mathematical model with case study of Wuhan

Zeeshan Ali a, Faranak Rabiei a,*, Kamal Shah b, Touraj Khodadadi c

a School of Engineering, Monash University Malaysia, 47500 Selangor, Malaysia
b Department of Mathematics, University of Malakand, Dir(L), 18000 Khyber Pakhtunkhwa, Pakistan
c Department of Information Technology, School of Science and Engineering, Malaysia University of Science and Technology, 47810 Selangor, Malaysia

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Abstract In this manuscript, a qualitative analysis of the mathematical model of novel coronavirus (COVID-19) involving a new devised fractal-fractional operator in the Caputo sense having the fractional-order \( q \) and the fractal dimension \( p \) is considered. The concerned model is composed of eight compartments: susceptible, exposed, infected, super-spreaders, asymptomatic, hospitalized, recovery and fatality. When choosing the fractal order one we obtain fractional order, and when choosing the fractional order one a fractal system is obtained. Considering both the operators together we present a model with fractal-fractional. Under the new derivative the existence and uniqueness of the solution for considered model are proved using Schaefer’s and Banach type fixed point approaches. Additionally, with the help of nonlinear functional analysis, the condition for Ulam’s type of stability of the solution to the considered model is established. For numerical simulation of proposed model, a fractional type of two-step Lagrange polynomial known as fractional Adams-Bashforth (AB) method is applied to simulate the results. At last, the results are tested with real data from COVID-19 outbreak in Wuhan City, Hubei Province of China from 4 January to 9 March 2020, taken from a source (Ndaïrou, 2020). The Numerical results are presented in terms of graphs for different fractional-order \( q \) and fractal dimensions \( p \) to describe the transmission dynamics of disease infection.

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1. Introduction

A severe outbreak occurred in China’s Hubei Province at the end of 2019 caused by a virus known as the corona and have been named the novel COVID-19. This pandemic is ongoing, and over 7.38 million people almost in all countries of the
Infectious diseases are diseases that move from one body to another by modes of transmission, including water, food, through physical contact, air droplets, mother to newborn, etc. To understand the infectious disease dynamic effects, we necessitate some suitable mathematical tools like statistical estimation, differential equations, difference equations, etc. In the existing research work, researchers have proved that several infected cases claimed they had been working in a local fish and wild animal market in Wuhan from where they got the above-said infection. Afterward, the researchers have confirmed that the disease’s widespread nature is caused by direct contact between people. The city, mentioned above, is also a great trading center in China, where the infection transported to several countries around the globe via immigration, for detail, see [2–4]. We give some elementary statistical representation, so we have put the real data in line and histograms for the number of infected cases and deaths per day. The figures show the exponential growth of infected cases and deaths, as depicted in Figs. 1–4.

Infectious diseases are diseases that move from one body to another by modes of transmission, including water, food, through physical contact, air droplets, mother to newborn, etc. To understand the infectious disease dynamic effects, we necessitate some suitable mathematical tools like statistical estimation, differential equations, difference equations, etc. In the existing research work, researchers have proved that the mathematical models designed with the help of fractional calculus (FC) tools are often more accurate and stable compared to integer-order calculus due to its more degree of freedom. The spread of mentioned above disease maintains its memory extremely connected to non-Markovian dynamics. The utilization of non-local operators in the study of aforesaid models are justified in this respect, see [5–7] and the references cited therein. Nonetheless, most of the biological models are based on the classical method to get a system of nonlinear first-order differential equations. Therefore, there is still room to develop before-mentioned mathematical models by the tools of advanced FC. Several new investigations, including numerous biological models, have shown to be extremely helpful and more accurate than their equivalents. For example, authors [8] proposed a TB infection mathematical model for the Khyber Pakhtunkhwa province of Pakistan involving Caputo fractional-order derivative and tested it using the real data of the mentioned province shows the advantages of fractional-order model. In [9], the authors introduced a pine wilt disease model involving fractional-order derivative of Caputo-Fabrizio type and confirmed its productiveness by establishing its unique solution. While utilizing the idea beta-derivative, the authors in [10] designed a new fractional model for Rubella disease and proved its unique solutions and stability. In [11], the authors analyzed ordinary and fractional order models of dengue outbreak in the Cape Verde islands. They showed that fractional-order operators in the sense of Caputo-Fabrizio have the smallest squared sum of errors. In [12], the authors analyzed the mathematical model of an infectious disease called Ebola by Caputo, Caputo-Fabrizio and Atanagana-Baleanu operators.

Stability is an important branch of the qualitative theory of differential equations. As we know that, sometimes finding the exact solution is quite challenging to obtain. Therefore, various numerical techniques were developed to find a solution. In this regard, we check the stability of the given problem. We can find various types of stability in literature, including Lyapunov, Exponential and Asymptotic, etc. But the most important type of stability, which is first pointed out by Ulam in 1940 [13] is called Ulam stability. He posed a problem about the stability of functional equations. In the following year, Hyers answered to Ulam question partially in the context of Banach spaces [14]. He considered two real Banach spaces, say $B_1, B_2$ and defined an additive functions such that for $\epsilon > 0$ and for each mapping $\psi : B_1 \rightarrow B_2$ satisfying

$$||\psi(z + y) - \psi(z) - \psi(y)|| \leq \epsilon,$$

for all $z, y \in B_1$. There is a unique additive mapping $\phi : B_1 \rightarrow B_2$ with

$$||\psi(z) - \phi(z)|| \leq \epsilon,$$

for all $z \in B_1$. That is why the name of this stability is Ulam-Hyers stability. Lately, Hyers results are extended and generalized by researchers for difference and functional equations in different directions, for detail, see [15–19,21,20] and references cited therein. From a numerical and optimization point of view, Ulam-Hyers stability is essential because it provides a bridge between the exact and numerical solutions. It is also easy to check for the obtain solution. Recently, Khan et al.

![Fig. 1 Number of infected people per day in Wuhan.](image-url)
studied Ulam-Hyers stability of fractional-order advection-reaction diffusion system. In [22], Khan et al. studied the dynamical behavior and Ulam-Hyers stability of the HIV-TB coinfection model under the Atangana-Baleanu fractional derivative.

The area of mathematical modeling for the COVID-19 is an interesting area of research. Many studies have been recorded on mathematical modeling of an outbreak of the mentioned above disease with different direction and under ordinary and FC tools. Shah et al. [23] considered two compartments COVID-19 mathematical model and studied the existence theory, Ulam-Hyers stability and numerical solution under the Atanagana-Baleana-Caputo \(ABC\) fractional derivative. For the numerical solution, they have used the Laplace transform coupled with the Adomian decomposition method. Das and Samanta [24] considered four compartments COVID-19 mathematical model and studied the dynamical behavior of the considered compartments under the Caputo fractional
derivative with the case study of Japan. Baleanu et al. [25] studied six compartments COVID-19 mathematical model under the Caputo and Caputo-Fabrizio fractional derivatives. Shaikh et al. [26] considered six compartments COVID-19 mathematical model and studied the dynamics of transmission and control under the Caputo-Fabrizio fractional derivative with the case study of India. Alkahtani et al. [27] considered eight compartments COVID-19 mathematical model and investigated the dynamical behavior of the considered compartments under the Caputo-Fabrizio and ABC fractional derivatives. For the numerical solution, they have utilized the homotopy analysis transform method. In [28], Abdo et al. considered the fourteen compartments COVID-19 mathematical model and studied the existence theory, Ulam-Hyers stability and numerical results under ABC fractional derivative. For more detail, see [29–32].

Since, we know that there are complexities and false information, which makes it hard to provide a suitable mathematical model for COVID-19 with classical order differentiation. So in such a situation, nonlocal operators are ideal because they can capture non-localities and some memory effects depending on if there is a power law, crossover effects or fading memory. But in some situations, the power law, crossover and fading memory cannot replicate the more complex behavior of the mentioned above infectious decease. In this case, the newly introduced operators having both fractional and fractal orders can be more suitable mathematical tools to handle such behaviors [33]. In [34], Atangana introduced the fractal-fractional differential and integral operators and showed the connection between fractal calculus and FC. The newly introduced operators have two orders, the first is fractional order, and the second is the fractal dimension. The advanced studies show that the fractal-fractional order operators are the best tools to analyze the mathematical models for real-world data. Aguilar et al. [35] used the fractal-fractional operators and studied the transmission behavior of malaria disease. Li et al. [36] presented the comparative analysis of rural and commercial banks data of Indonesia for the years 2004–2014 through the fractal-fractional operator in the sense of Caputo derivative and showed the advantages of varying fractal and fractional orders. In [37], Atangana et al. considered the competition system and proposed the field data of banks for 2004–2014 of Indonesia banks of the type rural and commercial under the fractal-fractional Caputo-Fabrizio derivative. Qureshi and Atangana [38] studied the transmission dynamics of diarrhea disease with the help of fractal-fractional order operators. Ahmed and Khan [39] derived and studied the dynamics of the polluted lakes system under fractal-fractional operator in the sense of Atangana-Baleanu derivative. In [40], Atangana studied the novel COVID-19 mathematical model with the help of operators mentioned above and showed the efficiency of lockdown. Ali et al. [41] studied the existence theory, Ulam-Hyers stability and numerical results of the four compartments COVID-19 mathematical model under fractal-fractional operator in the sense of Caputo derivative.

This manuscript aims to develop a mathematical model of COVID-19 with fractal-fractional operators and to study the existence and numerical results with Wuhan’s real data. Therefore, we consider the ordinary first order mathematical model, which have proposed in [42]. The proposed mathematical model has eight compartments including susceptible population S(t), the exposed class E(t), the infected population I(t), the super-spreaders class P(t), the infectious but asymptomatic class A(t), hospitalized H(t), the recovery class R(t) and fatality class F(t) at time $t \in \mathcal{J} = [0, T]$ as:

$$
\begin{align*}
\frac{dS(t)}{dt} &= -\beta S(t)E(t) - \beta S(t)H(t) - \beta S(t)P(t), \\
\frac{dE(t)}{dt} &= \beta S(t)E(t) + \beta S(t)H(t) + \beta S(t)P(t) - \gamma E(t), \\
\frac{dI(t)}{dt} &= \gamma E(t) - (\mu + \xi)I(t) - \sigma I(t), \\
\frac{dA(t)}{dt} &= \gamma I(t) - (\mu + \xi)A(t) - \sigma A(t), \\
\frac{dP(t)}{dt} &= \gamma I(t) - (\mu + \xi)P(t) - \sigma P(t), \\
\frac{dH(t)}{dt} &= \gamma I(t) - (\mu + \xi)H(t) - \sigma H(t), \\
\frac{dR(t)}{dt} &= \gamma I(t) + \gamma P(t) + \gamma H(t), \\
\frac{dF(t)}{dt} &= \sigma I(t) + \sigma P(t) + \sigma H(t),
\end{align*}
$$

subject to initial conditions

$$
S(0) = S_0, \quad E(0) = E_0, \quad I(0) = I_0, \quad P(0) = P_0, \quad A(0) = A_0, \quad H(0) = H_0, \quad R(0) = R_0, \quad F(0) = F_0,
$$

where $\mathcal{J} = [0, T]$ and $\mathbb{N}$ is the total population. The parameters which involves in the model (1) are described as: $\beta$ is the transmission coefficient from infected individuals, $\ell$ is the relative transmissibility of hospitalized patients, $\gamma$ denote the rate at which exposed become infectious, $\rho_a$ is the rate at which exposed people become infected $I$, $\rho_p$ is the rate at which exposed people become super-spreaders, $\xi_a$ denotes the rate of being hospitalized, $\xi_r$ is the recovery rate without being hospitalized, $\sigma_a$ denotes the recovery rate of hospitalized patients, $\sigma_p$ is the disease induced death rate due to infected class, $\sigma_s$ is the disease induced death rate due to super-spreaders and $\sigma_h$ is the disease induced death rate due to hospitalized class. We impose some necessary assumptions on the model: all the involved parameters in the model (1) are nonnegative. Moreover, as the total population size $N$ is constant, the value of $S(t)$ is calculated by:

$$
S(t) = N - \lfloor E(t) + I(t) + P(t) + A(t) + H(t) + R(t) + F(t) \rfloor.
$$

In addition, the total population of death people due to the disease is given by (see Fig. 5):

$$
D(t) = \sigma_a I(t) + \sigma_p P(t) + \sigma_h H(t).
$$

The rest of the paper is organized as follows: In Section 2, we recall the definitions of fractal-fractional derivative and integral operators. In Section 3, Schaefer’s and Banach type fixed point approaches are used to build up some appropriate conditions for the existence and uniqueness of the solution. In Section 4, some necessary conditions are established for the Ulam-Hyers stability of the solution of the considered model (1) via nonlinear functional analysis. For the numerical simulation of model (1), the numerical algorithm is constructed in Section 5. Section 6 presents the numerical results and discussion after fitting the real data from Wuhan city of China [42] into the numerical algorithm and simulate it using Matlab. Section 7 is devoted to the conclusion.
2. Fundamental background

Definition 2.1. [34] Consider the continuous and differentiable function \( \chi(t) \) in \((a, b)\) with \( p \) order, then the fractal-fractional derivative of \( \chi(t) \) with \( q \) order Riemann-Liouville derivative is shown by:

\[
\mathbb{D}^{q,p}\chi(t) = \frac{1}{\Gamma(m-q)} \frac{d}{dt} \int_0^t (t-s)^{m-q-1} \chi(s)ds, \tag{4}
\]

with \( m-1 < q, p \leq m \), where \( m \in \mathbb{N} \) and \( \frac{\partial \chi(t)}{\partial \alpha} = \lim_{\alpha \to 0} \frac{\chi(t+\Delta t) - \chi(t)}{t^{\alpha}} \).

Definition 2.2. [34] Let \( \chi(t) \) is continuous in interval \((a, b)\) then the fractal-fractional integral of \( \chi(t) \) with order \( q \) is defined by:

\[
\mathbb{I}^{q,p}\chi(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \chi(s)ds. \tag{5}
\]

Note: we are defining Banach space \( \mathcal{U} = \mathcal{I} \times \mathcal{I} \times \mathcal{I} \times \mathcal{I} \times \mathcal{I} \times \mathcal{I} \times \mathcal{I} \times \mathcal{I} \times \mathcal{I} \times \mathcal{I} \) under the norm:

\[
\| \Phi \| = \| S(t) \| + \| E(t) \| + \| I(t) \| + \| P(t) \| + \| A(t) \| + \| H(t) \| + \| R(t) \| + \| F(t) \|.
\]

3. Qualitative analysis of model (1)

Before analyzing any biological model, it is natural to ask whether such dynamical problem really exist or not. This question is guaranteed by fixed point theory. Here, we will try to use the same theory for the proposed model (1) being part of this research.

Since the integral is differentiable we can reformulate the proposed problem (1) as:

\[
\begin{align*}
\mathbb{D}^q S(t) &= \beta(t) E(t) S(t) - \delta(t) S(t),
\mathbb{D}^q E(t) &= \gamma(t) R(t) E(t) - \beta(t) E(t) S(t) - \delta(t) E(t),
\mathbb{D}^q I(t) &= \beta(t) E(t) S(t) - \beta(t) I(t) E(t) - \gamma(t) I(t) E(t) - \delta(t) I(t),
\mathbb{D}^q P(t) &= \beta(t) E(t) S(t) - \beta(t) P(t) E(t) - \delta(t) P(t),
\mathbb{D}^q A(t) &= \beta(t) E(t) S(t) - \beta(t) A(t) E(t) - \delta(t) A(t),
\mathbb{D}^q H(t) &= \beta(t) E(t) S(t) - \beta(t) H(t) E(t) - \delta(t) H(t),
\mathbb{D}^q R(t) &= \beta(t) E(t) S(t) - \beta(t) R(t) E(t) - \delta(t) R(t),
\mathbb{D}^q F(t) &= \beta(t) E(t) S(t) - \beta(t) F(t) E(t) - \delta(t) F(t).
\end{align*}
\]

Now, transform the problem (1) into the fixed point problem. Let the operator \( T : \mathcal{U} \to \mathcal{U} \) defined by:

\[
T(\mathcal{X})(t) = \mathcal{X}(t) + \int_0^t \frac{1}{\Gamma(q)} (t-s)^{q-1} \Phi(s, \mathcal{X}(s))ds.
\]

For the existence theory, we use the following theorem [43].

Theorem 3.1. Assume that the operator \( T : \mathcal{U} \to \mathcal{U} \) is a completely continuous and the set define by:

\[
\mathcal{B}(T) = \{ \mathcal{X} \in \mathcal{U} : \mathcal{X} = \kappa T(\mathcal{X}), \kappa \in [0,1] \}.
\]
be bounded. Then $T$ has a fixed point in $\mathcal{U}$.

**Theorem 3.2.** Let $\Phi : \mathcal{I} \times \mathcal{U} \rightarrow \mathbb{R}$ is a continuous function. Then the operator $T$ is compact.

**Proof.** Let $B$ be bounded set in $\mathcal{U}$. Then there is $C_\Phi > 0$ with $|\Phi(t, \mathcal{I}(t))| \leq C_\Phi$, $\forall \mathcal{I} \in B$. So for any $\mathcal{I} \in B$, one can get

$$
\|T(\mathcal{I})\| \leq \frac{p \epsilon^q}{\Gamma(q)} \max_{t \in \mathcal{I}} \int_t^0 s^{q-1}(t - s)^{q-1} ds,
$$

$$
\leq \frac{p \epsilon^q}{\Gamma(q)} \max_{t \in \mathcal{I}} \int_t^0 z^{q-1}(1 - z)^{q-1} ds,
$$

$$
\leq \frac{p \epsilon^q}{\Gamma(q)} \max_{t \in \mathcal{I}} \|B(q, p)\|,
$$

where $B(q, p)$ is well known Beta function. Thus, $B(T)$ is uniformly bounded.

Next, for the equicontinuity of the operator $T$, for any $t_1, t_2 \in \mathcal{I}$ and $\mathcal{I} \in B$, we have

$$
\|T(\mathcal{I})(t_1) - T(\mathcal{I})(t_2)\| \leq \frac{p \epsilon^q}{\Gamma(q)} \max_{t \in \mathcal{I}} \int_t^0 |s^{q-1}(t_1 - s)^{q-1} ds - \int_t^0 s^{q-1}(t_2 - s)^{q-1} ds|,
$$

$$
\leq \frac{p \epsilon^q}{\Gamma(q)} \max_{t \in \mathcal{I}} |(t_1^{q-1} - t_2^{q-1})| = 0 \quad \text{as} \quad t_1, t_2 \rightarrow t.
$$

Therefore, $T$ is equicontinuous. Since, $T$ is a bounded operator and continuous as well, so by Arzelà-Ascoli theorem $T$ is relatively compact and so completely continuous. □

**Theorem 3.3.** Let for all $t \in \mathcal{I}$ and $\mathcal{I} \in \mathcal{U}$, there is a real number $C_\Phi > 0$ with $|\Phi(t, \mathcal{I}(t))| \leq C_\Phi$. Then the considered system (1) has at least one solution in the given space $\mathcal{U}$.

**Proof.** Consider a set $\mathcal{B} = \{ \mathcal{I} \in \mathcal{U} : \mathcal{I} = \kappa T(\mathcal{I}), \kappa \in [0, 1]\}$ and show that $\mathcal{B}$ is bounded. Suppose $\mathcal{I} \in \mathcal{B}$, then $\mathcal{I} = \kappa T(\mathcal{I})$. For $t \in \mathcal{I}$, one can easily obtain

$$
\|\mathcal{I}\| \leq \frac{\epsilon C_\Phi T^{q-1}}{\Gamma(q)} B(q, p).
$$

Hence $\mathcal{B}$ is bounded. So, by Theorem 3.2 and by Theorem 3.1, $T$ has at least one fixed point. Thus, the considered system (1) has at least one solution. □

For further analysis, let the following hypothesis:

(A) There is constant $L_\Phi > 0$ such that for every $\mathcal{I}, \mathcal{I} \in \mathcal{U}$, we have

$$
|\Phi(t, \mathcal{I}) - \Phi(t, \mathcal{I})| \leq L_\Phi|\mathcal{I} - \mathcal{I}|.
$$

For the uniqueness we use Banach’s Contraction Theorem [43].

**Theorem 3.4.** Under the hypothesis (A) and if $\Omega < 1$, then the solution of the considered system (1) is unique, where

$$
\Theta = \frac{p \epsilon^q L_\Phi T^{q-1}}{\Gamma(q)} B(q, p).
$$

**Proof.** Let we define $\max_{t \in \mathcal{I}} |\Phi(t, 0)| = M_\Phi < \infty$, such that $r \geq \frac{p \epsilon^q M_\Phi B(q, p) M_\Phi}{\Gamma(q) - p \epsilon^q M_\Phi B(q, p) L_\Phi}$.

We show that $T(\mathcal{B}) \subset \mathcal{B}$, where $\mathcal{B} = \{ \mathcal{I} \in \mathcal{U} : \|\mathcal{I}\| \leq r \}$.

For $\mathcal{I} \in \mathcal{B}$, we have

$$
||T(\mathcal{I})|| \leq \frac{p \epsilon^q}{\Gamma(q)} \max_{t \in \mathcal{I}} \int_t^0 s^{q-1}(t - s)^{q-1} |\Phi(t, \mathcal{I}(t)) - \Phi(t, 0)| + |\Phi(t, 0)| ds
$$

$$
\leq \frac{p \epsilon^q}{\Gamma(q)} \max_{t \in \mathcal{I}} \int_t^0 s^{q-1}(t - s)^{q-1} ds \cdot \left( \frac{p \epsilon^q}{\Gamma(q)} \max_{t \in \mathcal{I}} \int_t^0 s^{q-1}(t - s)^{q-1} |\Phi(t, \mathcal{I}(t)) - \Phi(t, 0)| ds \right)
$$

$$
\leq r.
$$

Let the operator $T : \mathcal{U} \rightarrow \mathcal{U}$ defined by (10). Then in view of (A), for every $t \in \mathcal{I}$ and for $\mathcal{I}, \mathcal{I} \in \mathcal{U}$, we have

$$
||T(\mathcal{I}) - T(\mathcal{I})|| \leq \frac{p \epsilon^q}{\Gamma(q)} \max_{t \in \mathcal{I}} \int_t^0 s^{q-1}(t - s)^{q-1} |\Phi(t, \mathcal{I}(t)) - \Phi(t, \mathcal{I}(t))| ds
$$

$$
\leq \epsilon ||\mathcal{I} - \mathcal{I}||.
$$

Hence, $T$ is contraction from (12). Thus, the integral Eq. (9) has a unique solution and so does system (1) has a unique solution. □

4. Ulam stability

In this section, we develop and present some results on stability of the model (1), we will consider a small perturbation $\Psi \in C(\mathcal{I})$ which depends only on the solution and $\Psi(0) = 0$. Further,

- $|\Psi(t)| \leq \epsilon, \text{ for } \epsilon > 0$
- $\text{FFF} D^\rho \mathcal{I}(t) = \Phi(t, \mathcal{I}(t)) + \Psi(t).

**Lemma 4.1.** The solution of the perturbed problem

$$
\text{FFF} D^\rho \mathcal{I}(t) = \Phi(t, \mathcal{I}(t)) + \Psi(t)
$$

$$
\mathcal{I}(0) = \mathcal{I}_0
$$

satisfies the following relation

$$
|\mathcal{I}(t) - \left( \mathcal{I}_0(t) + \frac{p c \epsilon^q}{\Gamma(q)} \int_0^t s^{q-1}(t - s)^{q-1} |\Phi(s, \mathcal{I}(s))| ds \right)\right| < \left( \frac{p c \epsilon^q}{\Gamma(q)} B(q, p) \right) t < \epsilon.
$$

**Proof.** The proof is easy so we skip it. □

**Theorem 4.1.** Under hypothesis (A) and Result 14 in Lemma 4.1, the solution of the integral Eq. (9) is Ulam-Hyers stable. Consequently, the considered system is Ulam-Hyers stable if $\Theta < 1$, where $\Theta$ is given by (11).

**Proof.** Suppose $\mathcal{I} \in \mathcal{U}$ be a unique solution and any solution (9), then

$$
\left| \mathcal{I}(t) - \mathcal{I}(t) \right| < \left| \mathcal{I}(t) - \left( \mathcal{I}_0(t) + \frac{p c \epsilon^q}{\Gamma(q)} \int_0^t s^{q-1}(t - s)^{q-1} |\Phi(s, \mathcal{I}(s))| ds \right)\right|
$$

$$
\leq \left| \mathcal{I}(t) - \left( \mathcal{I}_0(t) + \frac{p c \epsilon^q}{\Gamma(q)} \int_0^t s^{q-1}(t - s)^{q-1} |\Phi(s, \mathcal{I}(s))| ds \right)\right|
$$

$$
+ \left( \mathcal{I}_0(t) + \frac{p c \epsilon^q}{\Gamma(q)} \int_0^t s^{q-1}(t - s)^{q-1} |\Phi(s, \mathcal{I}(s))| ds \right) - \left( \mathcal{I}_0(t) + \frac{p c \epsilon^q}{\Gamma(q)} \int_0^t s^{q-1}(t - s)^{q-1} |\Phi(s, \mathcal{I}(s))| ds \right)
$$

$$
< \epsilon c \epsilon^q \left( \frac{p c \epsilon^q}{\Gamma(q)} B(q, p) \right) t < \epsilon.
$$

From which we have

$$
\|\mathcal{I} - \mathcal{I}\| < \epsilon c \epsilon^q \Theta \|\mathcal{I} - \mathcal{I}\|.
$$

From (15), we can write

$$
\|\mathcal{I} - \mathcal{I}\| < \epsilon c \epsilon^q \Theta \|\mathcal{I} - \mathcal{I}\|.
$$
Hence the result (16) concluded that the solution of (9) is Ulam-Hyers stable and consequently, the solution of the supposed problem is Ulam-Hyers stable. □

5. Numerical algorithm of model (1)

Here, we are going to construct a numerical procedure for the concerned model to perform simulation. In view of Eq. (9), we have
Fig. 9  Dynamical behavior of infected class (population) for model (1) corresponding to different value of $q$ and $p$.

Fig. 10  Dynamical behavior of super-spreader class (population) for model (1) corresponding to different value of $q$ and $p$.

Fig. 11  Dynamical behavior of asymptotic class (population) for model (1) corresponding to different value of $q$ and $p$. 
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Then we approximate the above obtained integrals to

\[ P(t) = p_0 + \int_0^t p(t-s)^{q-1} f(s, S(s), I(s), R(s), F(s))ds, \]

\[ A(t) = A_0 + \int_0^t A(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ H(t) = H_0 + \int_0^t H(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ R(t) = R_0 + \int_0^t R(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ F(t) = F_0 + \int_0^t F(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds. \]

We now present the numerical method of this system using a new approach at \( t_{m-1} \). The system becomes

\[ S_{m-1} = S_0 + \int_{t_{m-1}}^{t_m} s(s-t)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ E_{m-1} = E_0 + \int_{t_{m-1}}^{t_m} e(s-t)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ I_{m-1} = I_0 + \int_{t_{m-1}}^{t_m} i(s-t)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ P_{m-1} = P_0 + \int_{t_{m-1}}^{t_m} p(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ A_{m-1} = A_0 + \int_{t_{m-1}}^{t_m} a(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ H_{m-1} = H_0 + \int_{t_{m-1}}^{t_m} h(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ R_{m-1} = R_0 + \int_{t_{m-1}}^{t_m} r(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ F_{m-1} = F_0 + \int_{t_{m-1}}^{t_m} f(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds. \]

Then we approximate the above obtained integrals to

\[ S(t) = S_0 + \int_0^t s(s-t)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ E(t) = E_0 + \int_0^t e(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ I(t) = I_0 + \int_0^t i(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ P(t) = P_0 + \int_0^t p(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ A(t) = A_0 + \int_0^t a(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ H(t) = H_0 + \int_0^t h(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ R(t) = R_0 + \int_0^t r(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds, \]

\[ F(t) = F_0 + \int_0^t f(t-s)^{q-1} f(s, S(s), I(s), P(s), A(s), H(s), R(s), F(s))ds. \]

Within the finite interval \([t_m, t_{m+1}]\), we approximate the function \( s^{q-1}f(s, S, E, I, P, A, H, R, F) \) where \( j = 1, 2, \ldots, 8 \), using the Lagrange piece-wise interpolation and \( h = t_m - t_{m-1} \) such that

\[ S_k \approx \frac{1}{2}[(t_{m+1})^q f(s, S_k, E_k, I_k, P_k, A_k, H_k, R_k, F_k)] - \frac{1}{2}[(t_{m-1})^q f(s, S_k, E_k, I_k, P_k, A_k, H_k, R_k, F_k)]. \]
After evaluation of the integrals of (18), we get the following number of confirmed deaths per day corresponding to different value of \( m \):

\[
S_{n+1} = S_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} S_m(s) \, ds, \\
E_{n+1} = E_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} E_m(s) \, ds, \\
I_{n+1} = I_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} I_m(s) \, ds, \\
P_{n+1} = P_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} P_m(s) \, ds, \\
A_{n+1} = A_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} A_m(s) \, ds, \\
H_{n+1} = H_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} H_m(s) \, ds, \\
R_{n+1} = R_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} R_m(s) \, ds, \\
F_{n+1} = F_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} F_m(s) \, ds.
\]

After evaluation of the integrals of (18), we get the following dynamical behavior of fatality class (population) for model (1) corresponding to different value of \( H \).

\[
S_{n+1} = S_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} S_m(s) \, ds, \\
E_{n+1} = E_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} E_m(s) \, ds, \\
I_{n+1} = I_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} I_m(s) \, ds, \\
P_{n+1} = P_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} P_m(s) \, ds, \\
A_{n+1} = A_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} A_m(s) \, ds, \\
H_{n+1} = H_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} H_m(s) \, ds, \\
R_{n+1} = R_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} R_m(s) \, ds, \\
F_{n+1} = F_0 + \frac{\rho}{T} \sum_{t=0}^{n} \int_{t}^{t+1} s^{p-1} (t_{n+1} - s)^{q-1} F_m(s) \, ds.
\]
Wuhan. The following initial values are considered:

\[ N = 11,000,000/250 \]

This denominator has been determined because of restriction movements of individuals due to quarantine in the city of Wuhan. The total population of Wuhan city considered almost 11 million people, and for proposed model (1) here, the total population is considered as \( N = 11,000,000/250 \). This denominator has been determined because of restriction of movements of individuals due to quarantine in the city of Wuhan.

The following initial values are considered:

- \( S_0 = N - 7 \)
- \( E_0 = 0 \)
- \( I_0 = 1 \)
- \( P_0 = 5 \)
- \( A_0 = 0 \)
- \( R_0 = 0 \)
- \( F_0 = 0 \)
- \( I_a = 0 \)
- \( I_r = 0 \)
- \( R_r = 0 \)
- \( F_r = 0 \)
- \( k = 0.1 \)

The step size for evaluating the numerical results is \( h = 0.1 \).

We simulate the proposed model (1) for susceptible, exposed, infected, super-spreaders, asymptomatic, hospitalized, recovery and fatality compartments as well as confirmed infected and dead against the real data obtained from the source given in [42]. The numerical simulation of proposed model is given in Figs. 6–16 corresponding to different values of \( p, q = 0.7, 0.8, 0.9, 1.0 \). It is worth noting that the prediction depends on the fractal-fractional orders beside the theoretical parameters. From Fig. 6, we see that susceptible population has been rapidly decreased at different rate and became steady after almost 30 days that is caused due to individual movements restriction. The decay rate is faster on small fractional order. As susceptibility was decreasing, which caused the increase in exposure class population to become infective. The number of exposed people increased at a different rate due to fractal-fractional order to catch infection. This increase was faster on greater fractal-fractional order and vice versa (see Fig. 8) then after restriction on individual movement were applied, curve of the exposed population decreased and converges to zero. During the 66 days, the infection raised up with different fractional order, as it can be seen in Fig. 9.

In the initial few days, the infection was increasing slightly but after some time the flow of infection has increased rapidly which may be caused due to various reasons. After government of China ordered the individual movements restriction in Wuhan city, which has caused the slowing down the infection population. In Fig. 10, dynamical behavior of super-spreaders population has almost same pattern as infected population but there is a difference in initial value of super-spreaders population which is based on real data of Wuhan the initial super-spreaders population \( (P_0 = 5) \). (Table 1).

From Figs. 12, 14 and 16 the similar behavior of increment can be seen from asymptotic, recovered and fatality populations that may caused due to death or recovery of disease infection. Also, it is clear that the rate of growth for all three classes of asymptotic, fatality and recovered populations, is faster for lower fractional order rather than the higher order. The total population of confirmed death is calculated based on Eq. (3) and the result is presented in Fig. 16 for different values of \( q \) and \( p \). The graph shows that the confirmed case of death population increased first rapidly and after movement restriction was set up it decreased again. It can be observed.

### Table 1

| Parameters | Value | Units |
|------------|-------|-------|
| \( \beta \) | 2.55 | day\(^{-1} \) |
| \( \ell \) | 1.56 | dimensionless |
| \( \beta' \) | 7.65 | day\(^{-1} \) |
| \( \gamma \) | 0.25 | day\(^{-1} \) |
| \( \rho_1 \) | 0.580 | dimensionless |
| \( \rho_2 \) | 0.001 | dimensionless |
| \( x_o \) | 0.94 | day\(^{-1} \) |
| \( x_i \) | 0.27 | day\(^{-1} \) |
| \( x_r \) | 0.5 | day\(^{-1} \) |
| \( \sigma_i \) | 3.5 | day\(^{-1} \) |
| \( \sigma_r \) | 1 | day\(^{-1} \) |
| \( \sigma_h \) | 0.3 | day\(^{-1} \) |

### 6. Numerical results and discussion

In this study we use the real data of disease outbreak of Wuhan city of China for 66 days (from 4 January to 9 March 2020) See [42]. The total population of Wuhan city considered is almost 11 million people, and for proposed model (1) here, the total population is considered as \( N = 11,000,000/250 \). This denominator has been determined because of restriction of movements of individuals due to quarantine in the city of Wuhan. The following initial values are considered: \( S_0 = N - 7 \), \( E_0 = 0 \), \( I_0 = 1 \), \( P_0 = 5 \), \( A_0 = 0 \), \( R_0 = 0 \), \( F_0 = 0 \). The step size for evaluating the numerical results is \( h = 0.1 \).

We simulate the proposed model (1) for susceptible, exposed, infected, super-spreaders, asymptomatic, hospitalized, recovery and fatality compartments as well as confirmed infected and dead against the real data obtained from the source given in [42]. The numerical simulation of proposed model is given in Figs. 6–16 corresponding to different values of \( p, q = 0.7, 0.8, 0.9, 1.0 \). It is worth noting that the prediction depends on the fractal-fractional orders beside the theoretical parameters. From Fig. 6, we see that susceptible population has been rapidly decreased at different rate and became steady after almost 30 days that is caused due to individual movements restriction. The decay rate is faster on small fractional order. As susceptibility was decreasing, which caused the increase in exposure class population to become infective. The number of exposed people increased at a different rate due to fractal-fractional order to catch infection. This increase was faster on greater fractal-fractional order and vice versa (see Fig. 8) then after restriction on individual movement were applied, curve of the exposed population decreased and converges to zero. During the 66 days, the infection raised up with different fractional order, as it can be seen in Fig. 9.

In the initial few days, the infection was increasing slightly but after some time the flow of infection has increased rapidly which may be caused due to various reasons. After government of China ordered the individual movements restriction in Wuhan city, which has caused the slowing down the infection population. In Fig. 10, dynamical behavior of super-spreaders population has almost same pattern as infected population but there is a difference in initial value of super-spreaders population which is based on real data of Wuhan the initial super-spreaders population \( (P_0 = 5) \). (Table 1).

From Figs. 12, 14 and 16 the similar behavior of increment can be seen from asymptotic, recovered and fatality populations that may caused due to death or recovery of disease infection. Also, it is clear that the rate of growth for all three classes of asymptotic, fatality and recovered populations, is faster for lower fractional order rather than the higher order. The total population of confirmed death is calculated based on Eq. (3) and the result is presented in Fig. 16 for different values of \( q \) and \( p \). The graph shows that the confirmed case of death population increased first rapidly and after movement restriction was set up it decreased again. It can be observed.

### 7. Conclusion

We have studied a new type of model of COVID-19 with eight compartments including susceptible, exposed, infected, super-spreaders, asymptomatic, hospitalized, recovery and fatality classes with anew devised fractal-fractional operator in the sense of Caputo having the fractional-order \( q \) and the fractal dimension \( p \). First, we have successfully developed the existence theory for the solution of the proposed model via Banach and Schaefer’s type fixed point approaches, proving that the considered model has a unique solution. Additionally, we have
established the necessary conditions for Ulam-Hyers stability via nonlinear functional analysis, which shows the obtained solution is stable. By applying fractional type AB method, we have simulated the results for different values of $q$ and $p$ via the help of Matlab-16 and demonstrated that individual movements restriction has a significant impact on the transmission dynamics of the current outbreak. By adopting precautionary measures, the transmission of the disease in society can be reduced. Also, for such type of dynamical study, FC may be used as powerful tools to understand the global dynamics of the mentioned disease.

Declararion of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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