Abstract

It is shown that each integrable mapping is connected with a hierarchical completely integrable system of equations of evolution type which are invariant with respect to the transformation described by this mapping.

1 Introduction

Since the second half of the previous century it has been known that some equations (or systems of equations) possess some nonlinear symmetries which permit the generation of new solutions from known ones. These substitutions are known as Bäcklund Transformations. The reader can find details of the history of this subject in [1]. Comparatively recently it has been realised that such symmetries may be considered as a powerful method for the construction of explicit solutions (including solitons, and doubly periodic solutions) of...
all known completely integrable systems. The essence of this construction consists in finding a suitable partial solution of the nonlinear system under consideration and then in integration of its Bäcklund transformation under the given boundary conditions. For the equations of Self-Duality in four dimensions this approach was first employed in the work of Fairlie, Corrigan, Goddard and Yates [2]. For systems of more common physical application, the explicit form of the Bäcklund transformation was written down and solved in [3]. The generalization to the case of equations in a space of dimension 1+2 may be found in [4].

This paper attempts to answer the question ‘What conditions must a mapping fulfil in order that it can be considered as the group of inner symmetries of some integrable system of equations?’ In other words we write down the system of equations determining the Bäcklund transformations of some integrable system. We cannot as yet give the general solution — this procedure involves functional equations, but in principle the general solution (or maybe classification thereof) will provide the answer to the question of describing all systems which possess a Bäcklund transformation.

2 Condition of Integrability

In what follows the term ‘integrable mapping’ will play the most important role. The simplest way to understand this is from the point of view of integrable equations. Suppose we have an equation for a undetermined function \( \theta \) in the form

\[
F(\theta, \theta_i, \theta_{ij}, \theta_{ijk}, \ldots) = 0
\]  

(2.1)

where subscripts denote multiple differentiations with respect to the \( N \) independent variables \( x_j, \ j = 1, \ldots, N \). Imagine that the solution to this equation depends upon some parameter \( \tau \) and differentiate the system (2.1) with respect to it., and call the result \( \theta_\tau = \Theta \)

\[
\frac{\partial F}{\partial \theta} \Theta + \sum \frac{\partial F}{\partial \theta_i} \Theta_i + \cdots = 0.
\]  

(2.2)

This is the symmetry equation for (2.1). If some solution of the equation of symmetry may be represented in terms of \( \theta \) and its derivatives up to a fixed order, then equation (2.1) is called integrable. In this case, (2.1) possess some form of exact solution. It may happen that (2.2) may admit a general
solution. In this case (2.1) is exactly integrable. For example the universal
equation proposed in [4] is of this kind. This equation in the simplest case
in two dimensions is the so-called Bateman equation
\[ \theta^2_{yy} - 2\theta_x\theta_y\theta_{xy} + \theta^2_{xx} = 0. \] (2.3)
The symmetry equation for \( \Theta \) is
\[ 2(\theta_x\theta_{yy} - \theta_y\theta_{xy})\Theta_x + 2(\theta_y\theta_{xx} - \theta_x\theta_{xy})\Theta_y + \theta^2_x\Theta_{yy} - 2\theta_x\theta_y\Theta_{xy} + \theta^2_y\Theta_{xx} = 0. \] (2.4)
The general solution of this equation depending upon two arbitrary functions
may be directly verified to take the form
\[ \Theta = \theta_x F(\frac{\theta_y}{\theta_x}) + G(\frac{\theta_y}{\theta_x}) \] (2.5)
Thus the Bateman equation is exactly integrable, and its general solution is
well known to be given by the solution for \( \theta \) of the implicit equation
\[ xf(\theta) + yg(\theta) = c \] (2.6)
where \( f, g \) are arbitrary functions and \( c \) is a constant, possibly zero. The
restriction to two dimensions is inessential.

3 Discrete Integrable Substitution

In the present section we shall consider local transformations acting on a
vector function of several independent arguments. This means that this
substitution may be written down as
\[ \tilde{u} = \varphi[u] \equiv \varphi(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}...) \] (3.1)
where the vector function \( \varphi \) depends on \( \tilde{u} \) and its derivatives up to order
\( m \) taken at the same point. This restriction of locality is not obligatory. The
discrete symmetry transformation of the equations of self duality in 4
dimensions is non-local, but all our considerations also apply to this case.

The question is how to construct the system of equations which will be
invariant respect to the substitution (2.6). In other words, how to construct
the dynamical system, the group of inner symmetry of which corresponds
to the substitution (2.6); and to determine when this is possible. We shall
denote by \( \vec{\varphi} \) the matrix operator in the tangent space. Let us assume that
the vector function \( \vec{u} \) depends on some parameter \( \tau \) and ask ourselves how
the derivatives of \( \tilde{u}_\tau \) and \( u_\tau \) are related to each other. The answer is contained in
the following definition of the operator \( \vec{\varphi} \) which acts as a matrix operator
on the column vector \( u_\tau \):

\[
\tilde{u}_\tau = \vec{\varphi}(u_\tau) = (\vec{\varphi}_u + \vec{\varphi}_{u_i}D_i + \vec{\varphi}_{u_{ij}}D^2_{ij} + \ldots)(u_\tau) \tag{3.2}
\]

It is useful to rewrite (2) in a purely matrix form. Let us consider the matrices
\( \varphi_{i\ldots k}^\alpha \) with elements

\[
(\varphi_{i\ldots k}^\alpha)_{\beta} = \frac{\partial\varphi^\alpha}{\partial u_\beta} - \sum \partial^i \frac{\partial\varphi^\alpha}{\partial u^i_\beta} + \sum \partial^i \partial_j \frac{\partial\varphi^\alpha}{\partial u_{ij}^i} - \ldots
\]

\[
(\varphi_{i}^\alpha)_{\beta} = \frac{\partial\varphi^\alpha}{\partial u_\beta} - 2 \sum \partial^i \frac{\partial\varphi^\alpha}{\partial u^i_\beta} + 3 \sum \partial^k \partial_j \frac{\partial\varphi^\alpha}{\partial u_{ijk}^i} - \ldots
\]

\[
(\varphi_{ij}^\alpha)_{\beta} = \frac{\partial\varphi^\alpha}{\partial u_\beta} - 6 \sum \partial^k \frac{\partial\varphi^\alpha}{\partial u^{ik}_\beta} + 10 \sum \partial^k \partial_l \frac{\partial\varphi^\alpha}{\partial u_{ijkl}^i} - \ldots
\]

and so on. In terms of this notation the relation (3.2) may be written as:

\[
\vec{F}(\varphi(u)) = (\varphi_\ast \vec{F}(u)) + \sum \partial_i((\varphi_i^\alpha \vec{F}(u))) + \sum \partial_i\partial_j((\varphi_{ij}^\alpha \vec{F}(u))) + \ldots \tag{3.3}
\]

where \( \vec{F}(u) \equiv \vec{u}_\tau, \vec{F}(u) \equiv \vec{u}_\tau \).

Definition

The substitution (2.6) will be called integrable if the vector function \( \vec{F} \) in
(3.3) is a function of its arguments coincides with \( \vec{F} \). If it is possible to find
the general solution of (3.3), then the equation (2.6) is exactly integrable.

If the substitution (2.6) is integrable, then (3.3) becomes the equation
determining the vector function \( \vec{F} \). From this construction it is clear that
each system of the form

\[
\vec{u}_\tau = \vec{F}(u), \tag{3.4}
\]

where \( \vec{F}(u) \) is any solution of (3.3), is invariant under the discrete substitution
(2.6).

The system (3.3) is self consistent since for each substitution there exists
one obvious solution, viz., \( \vec{F}(u) = D_\tau \vec{u} \), where \( D_\tau \) denotes differentiation with
respect to the independent argument $x_i$. Now for this solution, the system (3.4) takes the simple form $u_t = u_{x_1}$, which is indeed manifestly invariant under every substitution of the form (2.4).

4 Example of (1+2)d Davey Stewartson Equation

We can take any of the known integrable systems, but here confine ourselves to the case of the Davey Stewartson equation ([7] for the purposes of illustration. Consider the Toda lattice substitution in two dimensional space which can be written in two equivalent forms:

$$Sq \equiv \tilde{q} = \frac{1}{r}; \quad Sr \equiv \tilde{r} = r(rq - (\ln r)_{xy})$$

or

$$Su \equiv \tilde{u} = u - v_{xy}; \quad Su = rq; \quad Sv \equiv \tilde{v} = v + \ln \tilde{u}; \quad v = \ln r$$

The general equation (3.3) in the variables $u, v$ takes the form:

$$F^+ - 2F + F^- = \left(\frac{F}{u}\right)_{xy}$$

or explicitly:

$$F(u - v_{xy}, v + \ln(u - v_{xy})) - 2F(uv) + F(u + (v - \ln u)_{xy}, v - \ln u) = \left(\frac{F(u, v)}{u}\right)_{xy}$$

We may directly check that apart from the obvious solutions $F = u_x, u_y$ the functional equation has the solution

$$F = \alpha \partial_x(u_x - 2uv_x) + \beta \partial_y(u_y - 2uv_y)$$
where $\alpha, \beta$ are arbitrary constants. For $F_2$ we immediately obtain

$$(F_2)_{xy} = \alpha [-\partial_x \partial_y (v_{xx} + v_x^2) + 2u_{xx}] + \beta [-\partial_x \partial_y (v_{yy} + v_y^2) + 2u_{yy}]$$

which yields the system of two equations invariant with respect to the above (Toda lattice) substitution:

$$u_t = \alpha \partial_x (u_x - 2uv_x) + \beta \partial_y (u_y - 2uv_y)$$

$$\partial_x \partial_y [v_t + \alpha (v_{xx} + v_x^2) + \beta (v_{yy} + v_y^2)] = 2(\alpha u_{xx} + \beta u_{yy})$$

This is the Davey-Stewartson equation. If we return to variables $r, q$, it takes its original non-local form.

5 Conclusion

In the process of iteration of a discrete transformation of a given solution, (say a multisoliton solution) to an integrable system both backwards and forwards the general feature which arises is that after a finite number of steps the iteration becomes singular in either direction. An analogous situation occurs in the construction of the general solution of the $CP(N)$ models [8] where the repeated action of the transformation upon an instanton solution produces the anti-instanton solution after a finite number of steps. This situation is strongly reminiscent of what happens in the representation theory of continuous groups; the finite dimensional representations are obtained by the repeated action of ladder operators on a highest weight state, and this action admits only a finite number of repetitions before annihilation of the state vector. We are led to conjecture that something analogous to a group action lies at the foundation of all integrable systems, the properties of which we are only beginning to discern.

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