Abstract. Following ideas in [MV], we prove that the Busemann function of the parabolic homotetic motion for a minimal central configuration of the N-body problem is a viscosity solution of the Hamilton-Jacobi equation and that its calibrating curves are asymptotic to the homotetic motion.

1. Introduction

We consider the N-body problem with potential $U : (\mathbb{R}^d)^N \to [0, \infty]$ given by

$$U(x) = \sum_{i<j} \frac{m_i m_j}{r_{ij}}$$

where $x = (r_1, \ldots, r_N) \in (\mathbb{R}^d)^N$ is a configuration of $N$ punctual positive masses $m_1, \ldots, m_N$ in Euclidean space $\mathbb{R}^d$, and $r_{ij} = |r_i - r_j|$. We will adopt the variational point of view and consider the Lagrangian $L : (\mathbb{R}^d)^{2N} \to [0, \infty]$ given by

$$L(x, v) = T(v) + U(x) = \frac{1}{2} \sum_{i=1}^N m_i |v_i|^2 + U(x)$$

as well as the action of an absolutely continuous curve $\gamma : [a, b] \to (\mathbb{R}^d)^N$ given by

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt$$

taking values in $[0, \infty]$. We will denote by $\mathcal{C}(x, y, \tau)$ the set of curves binding two given configurations $x, y \in (\mathbb{R}^d)^N$ in time $\tau > 0$, that is to say,

$$\mathcal{C}(x, y, \tau) = \{ \gamma : [a, b] \to (\mathbb{R}^d)^N \text{ absolutely continuous } |b-a = \tau, \gamma(a) = x, \gamma(b) = y \}$$

and $\mathcal{C}(x, y)$ will denote the set of curves binding two configurations $x, y \in (\mathbb{R}^d)^N$ without any restriction on time,

$$\mathcal{C}(x, y) = \bigcup_{\tau > 0} \mathcal{C}(x, y, \tau) .$$

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In what follows we will consider curves which minimize the action on these sets. We define the function \( \phi: (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \times (0, +\infty) \to \mathbb{R} \),

\[
\phi(x, y; \tau) = \inf\{ A(\gamma) | \gamma \in \mathcal{C}(x, y, \tau) \},
\]

and the Mañé critical action potential

\[
\phi(x, y) = \inf\{ A(\gamma) | \gamma \in \mathcal{C}(x, y) \} = \inf\{ \phi(x, y, \tau) | \tau > 0 \}.
\]

defined on \((\mathbb{R}^d)^N \times (\mathbb{R}^d)^N\). In the first definition, the infimum is achieved for every pair of configurations \(x, y \in (\mathbb{R}^d)^N\). In the second one the infimum is achieved if and only if \(x \neq y\). These facts are essentially due to the lower semicontinuity of the action.

**Definition 1.** A free time minimizer defined on an interval \(J \subset \mathbb{R}\) is an absolutely continuous curve \(\gamma: J \to (\mathbb{R}^d)^N\) which satisfies \(A(\gamma|[a,b]) = \phi(\gamma(a), \gamma(b))\) for all compact subinterval \([a, b] \subset J\).

There is a relatively easy way to give an example of a free time minimizer defined on an unbounded interval via the minimal configurations of the problem. Recall that the moment of inertia (about the origin) of a given configuration \(x \in (\mathbb{R}^d)^N\) is

\[
I(x) = \sum_{i=1}^{N} m_i |r_i|^2.
\]

We will use the norm in \((\mathbb{R}^d)^N\) given by \(\|x\|^2 = I(x)\). We say that \(x_0 \in (\mathbb{R}^d)^N\) with \(I(x_0) = 1\) is a (normal) minimal configuration of the problem if \(U(x_0) = \min\{U(x) | x \in (\mathbb{R}^d)^N, I(x) = 1 \}\). Also recall that a central configuration is a configuration \(a \in (\mathbb{R}^d)^N\) which admits homothetic motions i.e. of the form \(x(t) = \lambda(t)a\). This happens if and only if \(a\) is a critical point of \(x \mapsto \|x\|U(x)\) and \(\lambda\) satisfies the Kepler equation \(\ddot{\lambda} \lambda = -U(a)/I(a)\). Thus minimal configurations are in particular central configurations. For a given central configuration \(a\), choosing \(\mu\) that satisfies \(\mu^3 = U(a)/I(a)\) we have that \(x(t) = \mu t^{2/3}a\) is a parabolic homothetic motion. By Proposition 19 in [DM], if \(a\) is a minimal configuration such motions are free time minimizers, and we don’t know if there are other central configurations with this property.

Our main result is

**Theorem.** Let \(x_0\) be a minimal central configuration with \(\|x_0\| = 1\), \(U(x_0) = U_0\), and consider the the homotetic motion \(\gamma_0(t) = ct^2/2 x_0\), \(c = (2U_0)^{1/2}\). Then the Busemann function

\[
u(x) = \sup_{t>0} [\phi(0, \gamma_0(t)) - \phi(x, \gamma_0(t))] = \lim_{t \to +\infty} [\phi(0, \gamma_0(t)) - \phi(x, \gamma_0(t))]
\]

is a viscosity solution of the Hamilton-Jacobi equation

\[
\|Du(x)\|^2 = 2U(x).
\]
Moreover, for any $x \in (\mathbb{R}^d)^N$ there is a calibrating curve $\alpha : [0, \infty) \to (\mathbb{R}^d)^N$ of $u$ with $\alpha(0) = x$ and

$$\lim_{t \to \infty} \|\alpha(t)t^{-\frac{1}{3}} - cx_0\| = 0.$$ 

2. Preliminaries

We start recalling two Theorems of E. Maderna

**Theorem 1.** [M] There are constants $\alpha, \beta > 0$ such that for all $T > 0$,

$$\phi(x, y; T) \leq \alpha \frac{R^2}{T} + \beta \frac{T}{R}$$

whenever $x$ and $y$ are contained in a ball of radius $R > 0$ in $(\mathbb{R}^d)^N$.

**Theorem 2.** [M] There is $\eta > 0$ such that for all $y, z \in (\mathbb{R}^d)^N$

$$\phi(y, z) \leq \eta \|y - z\|^{\frac{1}{2}}$$

**Proposition 1.** Denote by $S(a, b)$ the action potential for the one-dimensional Kepler’s problem with potential energy $U_0/r$ and by $S(a, b; t)$ the action of the only solution to Kepler’s problem on the half line that goes from $a$ to $b \geq a$ in time $t$.

(a) The function

$$G(r) = S(0, r; 1) - S(0, r) = S(0, r; 1) - (8U_0r)^{\frac{1}{2}}$$

is decreasing on $]0, c[\), increasing on $]c, \infty[\) $G(c) = G'(c) = 0, G''(c) = 5/3$. It follows that if $\varepsilon > 0$ is sufficiently small, there is $\delta(\varepsilon) > 0$ such that $|r - c| \leq \varepsilon$ if $G(r) \leq \delta(\varepsilon)$.

(b) For $\bar{\varepsilon} > 0$ we have

$$S(0, r; 1 + \varepsilon) = \frac{r^2}{2(1 + \varepsilon)} + o(r^2)$$

as $r \to \infty$ uniformly on $\varepsilon \in [0, \bar{\varepsilon}]$.

(c) As $\sigma \to 1, s \to \infty$

$$S(r, cs^2, \sigma s) = (6U_0^2s)^{\frac{1}{2}}(2 + \frac{5}{9}(\sigma - 1)^2 + o((\sigma - 1)^2)) - (8U_0r)^{\frac{1}{2}} + O(s^{-\frac{1}{2}})$$

uniformly on $r \in [0, s^{1/3}]$. Thus

$$S(r, u, \sigma \left(\frac{2u^3}{U_0}\right)^{\frac{1}{2}}) = (8U_0)^{\frac{1}{2}}(u^{\frac{1}{2}}(1 + \frac{5}{18}(\sigma - 1)^2 + o((\sigma - 1)^2)) - r^{\frac{1}{2}}) + O(u^{-\frac{1}{2}})$$

as $\sigma \to 1, u \to \infty$ uniformly on $r \in [0, u^{1/2}]$. 

Proof. Items (a), (b) are proved in [MV]. We prove item (c) following also [MV]. Let $h(r, s, \sigma)$ be the energy of the only solution that goes from $r$ to $cs^\frac{2}{3}$ in time $\sigma s$. Then

$$\sigma s = \int_r^{cs^\frac{2}{3}} \frac{du}{\sqrt{2(h + U_0/u)}} = \frac{3s}{2} \int_{rs^{-\frac{1}{2}}/c}^{1} \frac{dv}{\sqrt{cs^\frac{2}{3}h/U_0 + 1/v}}$$

Define

$$F(x, y, k) = \int_{x^2}^{1} \sqrt{\frac{v}{1 + kv}}dv - \frac{2}{3}(1 + y)$$

then

$$F\left(\left(\frac{r}{c}\right)^{\frac{1}{2}}s^{-\frac{1}{4}}, \sigma - 1, \frac{c}{U_0} s^{\frac{2}{3}}h(r, s, \sigma)\right) = 0.$$ 

Since $F(0, 0, 0) = 0$, $F_k(0, 0, 0) = -\frac{1}{3}$, the implicit function theorem can be applied to solve $F(x, y, k(x, y)) = 0$. Implicit differentiation gives

$$k(x, y) = -\frac{10}{3}y + \frac{125}{21}y^2 + o(x^2 + y^2)$$

$$h(r, s, \sigma) = \left(\frac{2}{9}U_0^2\right)^{\frac{1}{2}} s^{-\frac{2}{3}} k\left(\left(\frac{r}{c}\right)^{\frac{1}{2}}s^{-\frac{1}{4}}, \sigma - 1\right)$$

$$= -\frac{10U_0}{3c}s^{\frac{2}{3}}((\sigma - 1) + \frac{125}{21}(\sigma - 1)^2 + o(s^{-\frac{2}{3}} + (\sigma - 1)^2))$$

$$S(r, cs^{\frac{2}{3}}, \sigma s) = \int_r^{cs^{\frac{2}{3}}} \frac{h + 2U_0/u}{\sqrt{2(h + U_0/u)}}du$$

$$= \int_r^{cs^{\frac{2}{3}}} \sqrt{2(h + U_0/u)}du - h\sigma s$$

$$= (6U_0^2s^{\frac{2}{3}}A(x, k) - \left(\frac{2}{9}U_0^2s\right)^{\frac{1}{2}} k\sigma$$

where $k$ is given by (6), with $x = (r/c)^{\frac{1}{2}}s^{-\frac{1}{4}}$, $y = \sigma - 1$, and

$$A(x, k) = \int_{x^2}^{1} \sqrt{k + \frac{1}{v}}dv = A_0(k) - B(x, k),$$

$$A_0(k) = \int_{0}^{1} \sqrt{k + \frac{1}{v}}dv, \quad B(x, k) = \int_{0}^{x^2} \sqrt{k + \frac{1}{v}}dv$$

We have as in [MV]

$$A_0(k) = 2 + \frac{k}{3} - \frac{k^2}{20} + o(k^2), \quad B(x, k) = 2|x| + O(k|x|^3).$$
Thus
\[ S(r, cs^\frac{2}{3}, \sigma s) = (6U_0^2 s)\frac{4}{3} (2 + \frac{k}{3}(1 - \sigma) - \frac{k^2}{20} + o(k^2) + O(k|x|^3) - 2|x|) \]
\[ = (6U_0^2 s)\frac{4}{3} (2 + \frac{5}{9}(\sigma - 1)^2 + o(s^{-\frac{4}{3}} + (\sigma - 1)^2) + O(\frac{\sigma - 1}{s})) - (8U_0 r)^{\frac{1}{2}} \]
□

3. Proof of the result

Consider the Kepler’s problem on \((\mathbb{R}^d)^N\) with Lagrangian
\[ L_0(x, v) = \frac{\|v\|^2}{2} + \frac{U_0}{\|x\|} \]
and action potential
\[ \phi_0(x, y) = \inf_{T > 0} \phi_0(x, y; T). \]
We have
\[ S(\|x\|, \|y\|; T) \leq \phi_0(x, y; T) \leq \phi(x, y; T), \]
\[ S(0, \|x\|; T) = \phi_0(0, x; T) = \phi(0, \|x\| x_0; T). \]

**Lemma 1.** Busemann function \(u\) is well defined and a viscosity subsolution of the Hamilton-Jacobi equation \([1]\)

**Proof.** We start by showing that the function
\[ \delta(t) = [\phi(0, \gamma_0(t)) - \phi(x, \gamma_0(t)) - \phi(0, \gamma_0(t)) - \phi(x, \gamma_0(t)) - \phi(\gamma_0(s), \gamma_0(t)) \]
is increasing. If \(s < t\), then
\[ \delta(s) - \delta(t) = \phi(x, \gamma_0(t)) - \phi(x, \gamma_0(s)) + [\phi(0, \gamma_0(s)) - \phi(0, \gamma_0(t))] \]
\[ = \phi(x, \gamma_0(t)) - \phi(x, \gamma_0(s)) - \phi(\gamma_0(s), \gamma_0(t)) \]
\[ \leq 0, \]
where the last inequality follows from the triangle inequality applied to the triple \((x, \gamma_0(s), \gamma_0(t))\). By the triangle inequality, \(\delta(t) \leq \phi(\gamma_0(0), x)\), hence \(\lim_{t \uparrow \infty} \delta(t) = \sup_{t > 0} \delta(t)\) and this limit is finite.

Since
\[ u(y) = \sup_{t > 0} \phi(0, \gamma_0(t)) - \phi(y, \gamma_0(t)) \]
\[ \geq \sup_{t > 0} \phi(0, \gamma_0(t)) - \phi(y, x) - \phi(x, \gamma_0(t)) \]
\[ = u(x) - \phi(y, x), \]
u is a viscosity subsolution. □
Let \( x \in (\mathbb{R}^d)^N \). By Theorem 6 in [DM], for \( cT^{2/3} > \|x\| \) there is \( y_T : [0, \tau_T] \rightarrow (\mathbb{R}^d)^N \) with \( y_T(0) = x, y_T(\tau_T) = \gamma_0(0) \) such that
\[
A_L(y_T) = \phi(x, \gamma_0(0)).
\]

**Proposition 2.** \( \lim_{T \rightarrow \infty} \frac{T}{\tau_T} = 1 \).

**Proof.** From (10), for all \( 0 < t < T \) we have
\[
A_L(y_T | [0, t]) + \phi(y_T(t), \gamma_0(T)) = \phi(x, \gamma_0(T)).
\]

We know that
\[
(\frac{3}{2}U_0)^{1/3}T^{2/3} - \|x\| \leq \|x - \gamma(T)\| \leq \int_0^{\tau_T} \|\dot{y}_T\| \leq \sqrt{\tau_T} \left( \int_0^{\tau_T} \|\dot{y}_T\|^2 \right)^{1/2}
\]

Thus
\[
\lim_{T \rightarrow \infty} \sup_{T \rightarrow \tau_T} \frac{T}{\tau_T} \leq \frac{8}{3}.
\]

From (8) and (12)
\[
S(\|x\|, cT^{\frac{2}{3}}; \tau_T) \leq \phi(x, \gamma_0(T); \tau_T) \leq \phi(x, 0) + 2(6U_0^2T)^{1/3}
\]
\[
S\left(\frac{\|x\|}{\tau_T^{\frac{2}{3}}}, c\left(\frac{T}{\tau_T}\right)^{\frac{2}{3}}; 1\right) \leq \frac{\phi(x, 0)}{\tau_T^{\frac{2}{3}}} + 2\left(\frac{6U_0^2T}{\tau_T}\right)^{\frac{1}{3}}
\]

Consider a sequence \( T^j \rightarrow \infty \) such that \( \frac{T^j}{\tau_T} \rightarrow t \). Then
\[
S(0, cs^{\frac{2}{3}}; 1) \leq 2(6U_0^2s)^{1/3} = (8U_0cs^{\frac{2}{3}})^{1/3}.
\]

By Proposition I(a), \( cs^{\frac{2}{3}} = c \) and so \( s = 1 \). □

Let \( \beta : [0, 1] \rightarrow (\mathbb{R}^d)^N \) be a curve joining 0 and \( x \) and \( t \leq \tau_T \), then
\[
\phi(0, y_T(t); t + 1) \leq A_L(\beta) + A_L(y_T | [0, t])
\]
\[
A_L(y_T) = \phi(x, \gamma_0(T)) \leq A_L(\beta) + \phi(0, \gamma_0(T))
\]
\[
\phi(0, y_T(t); t + 1) + A_L(y_T | [t, \tau_T]) \leq 2A_L(\beta) + \phi(0, \gamma_0(T))
\]
\[
\phi(0, y_T(t); t + 1) + \phi(y_T(t), \gamma_0(T); \tau_T - t) \leq 2A_L(\beta) + \phi(0, \gamma_0(T))
\]
Inequalities (14) and (15) give (16), which can be written as (17).

Letting \( s = s(T, t) = T/t, \sigma = \sigma(T, t) = (\tau_T - t)/T, \) from (17) we have
\[
\phi(0, y_T(t)t^{-\frac{2}{3}}; 1 + 1/t) + \phi(y_T(t)t^{-\frac{2}{3}}, \gamma_0(s); \sigma s) \leq 2t^{-\frac{4}{3}}A_L(\beta) + \phi(0, \gamma_0(s)).
\]
Defining \( r_T(t) = \|y_T(t)\| t^{-\frac{2}{3}} \), from (8) and (9) we get

\[
\phi_0(0, y_T(t)t^{-\frac{2}{3}}; 1 + 1/t) + \phi_0(y_T(t)t^{-\frac{2}{3}}, \gamma_0(s); \sigma s) \leq 2 t^{-\frac{1}{3}} A_L(\beta) + \phi_0(0, \gamma_0(s))
\]

(19)

\[
S(0, r_T(t); 1 + 1/t) + S(r_T(t), cs^{\frac{2}{3}}; \sigma s) \leq 2 t^{-\frac{1}{3}} A_L(\beta) + S(0, cs^{\frac{2}{3}}).
\]

(20)

**Proposition 3.** There are constants \( K, \bar{t} > 0 \) and \( \bar{s} > 1 \) such that for every \( t \geq \bar{t}, T \geq t\bar{s} \) we have \( r_T(t) \leq K \).

**Proof.** Suppose the Proposition is false, then there are sequences \( K_n \to \infty, t_n \to \infty, T_n \) such that \( s_n = T_n/t_n \to \infty, r_n = r_{T_n}(t_n) \to \infty \). Note that \( \sigma_n = \sigma(T_n, t_n) \to 1 \).

If \( r_n \leq s_n^{\frac{1}{3}} \), inequality (20) and items (b) and (c) of Proposition 12 give

\[
\frac{r_n^2 t_n}{2(1 + t_n)} + o(r_n^2) + (6U_0^2 s_n)\frac{1}{3}(5/9)(\sigma_n - 1)^2 - o((\sigma_n - 1)^2)) - (8U_0 r_n^2)\frac{2}{3} + O(s_n^{\frac{2}{3}}) \leq 2 t_n^{-\frac{1}{3}} A_L(\beta)
\]

which is impossible for \( n \) large. If \( r_n > s_n^{\frac{1}{3}} \), from item (b) of Proposition 12 we have

\[
2 t_n^{-\frac{1}{3}} A_L(\beta) \geq S(0, r_n; 1 + \frac{1}{t_n}) - S(0, s_n^{\frac{2}{3}})
\]

\[
\geq \frac{s_n^3 t_n}{2(1 + t_n)} + o(s_n^{\frac{2}{3}}) - 2(6U_0^2 s_n)^\frac{1}{3}
\]

which is impossible for \( n \) large as well. \( \square \)

**Lemma 2.** Busemann function \( u \) is a viscosity solution of the Hamilton-Jacobi equation (11)

**Proof.** For \( x \in (\mathbb{R}^d)^N \), let \( y_T : [0, \tau_T] \to (\mathbb{R}^d)^N \) be such that (10) holds.

By Theorem 11 and Proposition 3 there are constants \( K, \bar{t} > 0 \) and \( \bar{s} > 1 \) such that for every \( t \geq \bar{t}, T \geq t\bar{s} \) we have

\[
A_L(y_T|[0, t]) = \phi(x, y_T(t); t) \leq K(t^{\frac{2}{3}} t^{-1} + tt^{-\frac{2}{3}}) = 2K^t^{\frac{1}{3}}
\]

(21)

We claim that the family

\[
\{y_T|[0, t]\}
\]

(22)

is equicontinuous. Indeed, by (21) we have

\[
\int_0^t \|\dot{y}_T\|^2 ds \leq 2A_L(y_T|[0, t]) \leq 4K^t^{\frac{1}{3}}
\]

Thus, for each \( 0 < s < s' \leq t \)

\[
\|y_T(s) - y_T(s')\| \leq \int_s^{s'} \|\dot{y}_T(v)\| dv \leq \sqrt{s' - s} \left( \int_s^{s'} \|\dot{y}_T(v)\|^2 \right)^{\frac{1}{2}} \leq 2(K^t^{\frac{1}{3}})^{\frac{1}{2}} \sqrt{s' - s},
\]

showing the equicontinuity of (22). Since \( y_T(0) = x \) the family is also equibounded. From Ascoli’s Theorem, there is a sequence \( T_n \to \infty \) satisfying \( T_n \geq t\bar{s} \) such that \( y_{T_n}|[0, t]\) converges uniformly. Applying this argument to an increasing sequence
\( t_k \to \infty \), by a diagonal trick one obtains a sequence \( T_n \to \infty \) such that \( y_{T_n} \mid_{[0,t]} \to \alpha \mid_{[0,t]} \) uniformly for each \( t > 0 \). By lower semi-continuity

\[
\liminf_{n \to \infty} A_L(y_{T_n} \mid_{[0,t]}) \geq A_L(\alpha \mid_{[0,t]}).
\]

From Theorem 2

\[
|\phi(\alpha(t), \gamma(T)) - \phi(y_{T}(t), \gamma(T))| \leq \phi(\alpha(t), y_{T}(t)) \leq \eta \|\alpha(t) - y_{T}(t)\|^{\frac{1}{2}}.
\]

Using (11), (23), (24) we have for all \( t > 0 \)

\[
\begin{align*}
u(\alpha(t)) &= \limn \phi(\gamma(0), \gamma(T_n)) - \phi(\alpha(t), \gamma(T_n)) \\
&= \limn \phi(\gamma(0), \gamma(T_n)) - \phi(x, \gamma(T_n)) + A_L(y_{T_n} \mid_{[0,t]}) \\
&\geq u(x) + A_L(\alpha \mid_{[0,t]}) \geq u(\alpha(t)).
\end{align*}
\]

Then \( \alpha \) calibrates \( u \). \( \square \)

We now address the proof of equality (2).

**Proposition 4.** For any \( \varepsilon > 0 \) there is \( t_\varepsilon > 0 \) such that for \( t \geq t_\varepsilon \), \( T \geq t\bar{s} \)

\[
|r_{T}(t) - c| < \varepsilon.
\]

**Proof.** For \( \xi : [0,1 + \frac{1}{t}] \to \mathbb{R}^+ \) with action \( A(\xi) = S(0, r_{T}(t); 1 + \frac{1}{t}) \) consider the reparametrization \( \xi^* : [0,1] \to \mathbb{R}^+ \) given by \( \xi^*(s) = \xi(s(1 + t)/t) \). Then

\[
S(0, r_{T}(t); 1) \leq A(\xi^*) = \left(1 + \frac{1}{t}\right) \frac{1}{2} \int_0^{(t+1)/t} |\dot{\xi}(s)|^2 ds + \frac{t}{(1 + t)} \int_0^{(t+1)/t} U(\xi(s)) ds
\]

\[
\leq (1 + \frac{1}{t})S(0, r_{T}(t); 1 + \frac{1}{t})
\]

Let \( \bar{t} > 0, \bar{s} > 1 \) be given by Proposition 1. From inequalities (18), (19), (20) and (26), there is constant \( K_1 \) such that for \( t \geq \bar{t}, T \geq t\bar{s} \) we have

\[
\phi(0, y_{T}(t)t^{-\frac{3}{2}}; 1) + \phi(y_{T}(t)t^{-\frac{3}{2}}; \gamma_0(s); \sigma s) \leq \phi(0, \gamma_0(s)) + K_1t^{-\frac{1}{2}},
\]

\[
\phi_0(0, y_{T}(t)t^{-\frac{3}{2}}; 1) + \phi_0(y_{T}(t)t^{-\frac{3}{2}}; \gamma_0(s); \sigma s) \leq \phi_0(0, \gamma_0(s)) + K_1t^{-\frac{1}{2}}
\]

\[
S(0, r_{T}(t); 1) + S(r_{T}(t), cs^{\frac{3}{2}}; \sigma s) \leq S(0, cs^{\frac{3}{2}}) + K_1t^{-\frac{1}{2}}
\]

Inequality (29) and triangle inequality

\[
S(0, u) \leq S(0, r) + S(r, u)
\]

imply

\[
S(0, r_{T}(t); 1) \leq S(0, r_{T}(t)) + K_1t^{-\frac{1}{2}}.
\]

Proposition 4 then follows from Proposition 1(a). \( \square \)

**Proposition 5.** Given \( \varepsilon > 0 \) there are \( \bar{t}_\varepsilon > 0, \bar{s}_\varepsilon > 1 \) such that for \( t > \bar{t}_\varepsilon, T \geq t\bar{s}_\varepsilon \) we have \( |r_{T}(t) - c| < \varepsilon \) and \( \angle(r_{T}(t), x_0) < \varepsilon \)
Proof. Consider the Kepler’s problem on \((\mathbb{R}^d)^N\). Let \(z\) be a configuration satisfying \(\|z\| - c < \varepsilon\) and \(\tau > 0\). The minimizer (for \(L_0\)) \(\xi : [0, \tau] \to (\mathbb{R}^d)^N\) joining \(z\) to \(\gamma_0(s)\) in time \(\tau\) is a collision-free Keplerian arc; hence it is contained in the plane generated by \(z\) and \(\gamma_0(s)\). Introducing polar coordinates in this plane, one can identify \(z\) with \(re^{i\theta}\) and \(\gamma_0(s)\) with \(cs^{2/3}\) \(\varepsilon\in\mathbb{R}\subset\mathbb{C}\), where \(|r - c| < \varepsilon, |\theta| \leq \pi\).

The path \(\xi\) can be written in polar coordinates as

\[\xi(v) = \rho(v) e^{i\omega(v)}, \quad u \in [0, \tau]\]

where

\[\rho(0) = r, \quad \omega(0) = \theta, \quad \rho(\tau) = cs^{2/3}, \quad \omega(\tau) \in 2\pi\mathbb{Z}.

We claim that \(\xi\) is a direct path, that is to say, the total variation of the polar angle \(\omega\) is less than or equal to \(\pi\). Assume for the contrary that \(|\omega(\tau) - \theta| > \pi\). Changing the orientation of the plane if necessary, we can assume that \(\omega(\tau) \geq 2\pi\); hence there exists a unique integer \(k \geq 1\) such that \(\omega(\tau) = 2k\pi\).

The path \(\tilde{\xi}(v) = \rho(v) e^{i\tilde{\omega}(v)}\) with

\[\tilde{\omega}(v) = \theta - \frac{\theta}{2k\pi - \theta}(\omega(v) - \theta)\]

has the same ends as \(\xi\) and

\[A_{L_0}(\tilde{\xi}) - A_{L_0}(\xi) = \frac{1}{2} \left( \rho^2 \omega^2(v) dv - 1 \right) \int_0^\tau \rho^2 \omega^2(v) dv < 0,

which is a contradiction. Lambert’s theorem (see [A]), states that if \(x_1\) and \(x_2\) are two configurations and \(\tau > 0\), the action \(A_{L_0}(x_1, x_2; \tau)\) of the direct Keplerian arc joining \(x_1\) to \(x_2\) in time \(\tau\) is a function of three parameters only: the time \(\tau\), the distance \(|x_1 - x_2|\) between the two ends and the sum of the distances between the ends and the origin (i.e. \(|x_1| + |x_2|\)). Thus

\[(31) \quad \phi_0(re^{i\theta}, \gamma_0(s); \tau) = S(d_1(r, \theta, s), d_2(r, \theta, s); \tau)\]

where

\[2d_1(r, \theta, s) = r + cs^{2/3} - |re^{i\theta} - cs^{2/3}| = r(1 + \cos \theta) - l(r, s, \theta)\]

\[2d_2(r, \theta, s) = r + cs^{2/3} + |re^{i\theta} - cs^{2/3}| = 2cs^{2/3} + r(1 - \cos \theta) + l(r, s, \theta)\]

\[l(r, s, \theta) = O(s^{-\frac{3}{2}}), \quad s \to \infty\]

uniformly for \(r\) bounded.

Letting \(\sigma^* = \left(\frac{c}{d_2}\right)^{\frac{3}{2}}\sigma s\), from (5) we have for \(r\) bounded, \(s\) large and \(\sigma\) close to 1

\[S(d_1, d_2; \sigma s) = (8U_0)^{\frac{1}{2}}(d_2^2)\left(1 + \frac{5}{18}(\sigma^* - 1)^2 + o((\sigma^* - 1)^2)\right) - d_1^{\frac{1}{2}} + O(s^{-\frac{1}{2}})\]

\[\geq 2(6U_0^2s)^{\frac{1}{2}} - 2(U_0r(1 + \cos \theta))^{\frac{1}{2}} + O(s^{-\frac{1}{2}}).\]
From equation (4) for $\sigma = 1$ we have

$$S(r, cs_3^2) \leq S(r, cs_2^2; s) = 2(6U_0^2 s)^{\frac{1}{2}} - (8U_0 r)^{\frac{1}{2}} + O(s^{-\frac{1}{3}}).$$

Thus, for $r$ bounded, $s$ large and $\sigma$ close to 1, we have

$$S(d_1, d_2; \sigma s) \geq S(r, cs_2^2) + 2(U_0 r(1 - \cos \theta))^{\frac{1}{2}} + O(s^{-\frac{1}{3}}).$$

Recalling (9), inequalities (28), (30) imply for $s = T/t$, $\sigma = \sigma(T, t)$

$$\phi_0(y_T(t)t^{-\frac{1}{3}}, \gamma_0(s); \sigma s) \leq 2t^{-\frac{1}{2}}A_L(\beta) + S(r_T(t), cs_2^2)$$

From (25), (31), (32), (33) we have

$$2t^{-\frac{1}{2}}A_L(\beta) \geq 2(U_0 r_T(t)(1 - \cos \theta_T(t)))^{\frac{1}{2}} + O((T/t)^{-\frac{1}{3}})$$

It follows that given $\varepsilon > 0$ there are $\bar{t}_\varepsilon > 0$, $\bar{s}_\varepsilon > 1$ such that for $t \geq \bar{t}_\varepsilon$, $T \geq t\bar{s}_\varepsilon$ we have $|\theta_T(t)| < \varepsilon$. \(\square\)

For $\bar{t}_\varepsilon > 0$, $\bar{s}_\varepsilon > 1$ as in Proposition 5, $t \geq \bar{t}_\varepsilon$, $T \geq t\bar{s}_\varepsilon$,

$$\|y_T(t)t^{-\frac{1}{3}} - cx_0\| \leq 2\varepsilon.$$

Taking $T = T_n$, $n \to \infty$ we have for $t \geq \bar{t}_\varepsilon$

$$\|\alpha(t)t^{-\frac{1}{3}} - cx_0\| \leq 2\varepsilon.$$

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