CONTINUOUS WELCH BOUNDS WITH APPLICATIONS

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Abstract: Let $(\Omega, \mu)$ be a measure space and $\{\tau_\alpha\}_{\alpha \in \Omega}$ be a normalized continuous Bessel family for a finite dimensional Hilbert space $H$ of dimension $d$. If the diagonal $\Delta := \{(\alpha, \alpha) : \alpha \in \Omega\}$ is measurable in the measure space $\Omega \times \Omega$, then we show that

$$\sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} |\langle \tau_\alpha, \tau_\beta \rangle|^2 \geq \frac{1}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)} \left[ \frac{\mu(\Omega)^2}{(d+m-1)} - (\mu \times \mu)(\Delta) \right], \quad \forall m \in \mathbb{N}.\$$

This improves 47 years old celebrated result of Welch [IEEE Transactions on Information Theory, 1974]. We introduce the notions of continuous cross correlation and frame potential of Bessel family and give applications of continuous Welch bounds to these concepts. We also introduce the notion of continuous Grassmannian frames.

Keywords: Welch bound, continuous Bessel family, Grassmannian frames, Zauner’s conjecture.

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1. Introduction

In 1974, L. Welch proved the following milestone result which revolutionized the study of finite set of vectors in finite dimensional Hilbert spaces.

Theorem 1.1. [Welch bounds] Let $n \geq d$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{C}^d$, then

$$\sum_{j=1}^n \sum_{k=1}^n |\langle \tau_j, \tau_k \rangle|^{2m} \geq \frac{n^2}{(d+m-1)}, \quad \forall m \in \mathbb{N}.\$$

In particular,

$$\sum_{j=1}^n \sum_{k=1}^n |\langle \tau_j, \tau_k \rangle|^2 \geq \frac{n^2}{d}.$$

Further,

(1) (Higher order Welch bounds) $\max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle|^{2m} \geq \frac{1}{n-1} \left[ \frac{n}{(d+m-1)} - 1 \right], \quad \forall m \in \mathbb{N}.$

In particular,

(First order Welch bound) $\max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle|^2 \geq \frac{n - d}{d(n-1)}.$

A very powerful application of Welch bounds is the lower bound on root-mean-square (RMS) absolute cross relation of unit vectors $\{\tau_j\}_{j=1}^n$ which is defined as
Theorem 1.1 says that

\[ I_{\text{RMS}}(\{\tau_j\}_{j=1}^n) \geq \left( \frac{n - d}{d(n-1)} \right)^{\frac{1}{2}}. \]

Another powerful application of Theorem 1.1 is the lower bound for frame potential which is introduced by Benedetto and Fickus [4] and further studied in [6,8]. Let us recall that given a collection of unit vectors \( \{\tau_j\}_{j=1}^n \), the frame potential is defined as

\[ \text{FP}(\{\tau_j\}_{j=1}^n) := \sum_{j=1}^n \sum_{k=1}^n |\langle \tau_j, \tau_k \rangle|^2. \]

Theorem 1.1 directly tells

\[ \text{FP}(\{\tau_j\}_{j=1}^n) \geq \frac{n^2}{d}. \]

There are several practical applications of Theorem 1.1 such as correlations [28], codebooks [14], numerical search algorithms [40,41], quantum measurements [29], coding and communications [31,35], code division multiple access (CDMA) systems [22,23], wireless systems [27], compressed sensing [34], ‘game of Sloanes’ [20], equiangular tight frames [32], etc.

A decade ago, a continuous version of Theorem 1.1 appeared in the paper [12] which states as follows.

**Theorem 1.2.** [12] Let \( \mathbb{C}P^{n-1} \) be the complex projective space and \( \mu \) be a normalized measure on \( \mathbb{C}P^{n-1} \). If \( \{\tau_{\alpha}\}_{\alpha \in \mathbb{C}P^{n-1}} \) is a continuous frame for a \( d \)-dimensional subspace \( \mathcal{H} \) of a Hilbert space \( \mathcal{H}_0 \), then

\[ \int_{\mathbb{C}P^{n-1}} \int_{\mathbb{C}P^{n-1}} |\langle \tau_{\alpha}, \tau_{\beta} \rangle|^{2m} d\mu(\alpha) d\mu(\beta) \geq \frac{1}{(d+m-1)^m}, \quad \forall m \in \mathbb{N}. \]

Drawback of Theorem 1.2 is that it works only for the measures defined on complex projective spaces. Further, we need a generalization of Inequality (1) for measure spaces. Therefore it is desirable to improve Theorem 1.2 and to get a continuous version of Inequality (1) by replacing maximum by supremum. For the sake of completeness, we note that there are some further refinements of Theorem 1.1 see [9,13,37]. The goal of this article is to derive Theorem 1.1 for arbitrary measure spaces (Theorem 2.7). We give some applications of Theorem 2.7. We also ask some problems for further research.

## 2. Continuous Welch bounds

Our proof of the result stated in the abstract is using the theory of continuous frames. This is generalization of frames indexed by discrete sets to measurable sets. Continuous frames are introduced independently by Ali, Antoine and Gazeau [1] and Kaiser [21]. In the paper, \( K \) denotes \( \mathbb{C} \) or \( \mathbb{R} \) and \( \mathcal{H} \) denotes a finite dimensional Hilbert space.

**Definition 2.1.** [1,21] Let \( (\Omega, \mu) \) be a measure space. A collection \( \{\tau_{\alpha}\}_{\alpha \in \Omega} \) in a Hilbert space \( \mathcal{H} \) is said to be a **continuous frame** (or generalized frame) for \( \mathcal{H} \) if the following holds.

(i) For each \( h \in \mathcal{H} \), the map \( \Omega \ni \alpha \mapsto \langle h, \tau_{\alpha} \rangle \in K \) is measurable.
(ii) There are $a, b > 0$ such that
\[
\|h\|^2 \leq \int_{\Omega} |\langle h, \tau_\alpha \rangle|^2 d\mu(\alpha) \leq b \|h\|^2, \quad \forall h \in \mathcal{H}.
\]
If $a = b$, then the frame is called as a tight frame and if $\|\tau_\alpha\| = 1$, $\forall \alpha \in \Omega$, then we say that the frame is normalized. If $a = b = 1$, then the frame is called as a Parseval frame. If we do not demand the first inequality in (ii), then we say it is a **continuous Bessel family** for $\mathcal{H}$.

We first observe that there is an abundance of continuous frames for finite dimensional Hilbert spaces. Further, it is known that given any finite measure space $(\Omega, \mu)$ and a finite dimensional space $\mathcal{H}$, there exists a continuous frame $\{\tau_\alpha\}_{\alpha \in \Omega}$ for $\mathcal{H}$ [23]. Given a continuous Bessel family, the analysis operator
\[
\theta_\tau : \mathcal{H} \ni h \mapsto \theta_\tau h \in L^2(\Omega); \quad \theta_\tau h \ni \alpha \mapsto \langle h, \tau_\alpha \rangle \in \mathbb{K}
\]
is a well-defined bounded linear operator. Its adjoint, the synthesis operator is given by
\[
\theta_\tau^* : L^2(\Omega) \ni f \mapsto \int_{\Omega} f(\alpha) \tau_\alpha d\mu(\alpha) \in \mathcal{H}.
\]
By combining analysis and synthesis operators, we get the frame operator, defined as
\[
S_\tau := \theta_\tau^* \theta_\tau : \mathcal{H} \ni h \mapsto \int_{\Omega} \langle h, \tau_\alpha \rangle \tau_\alpha d\mu(\alpha) \in \mathcal{H}.
\]
Note that the integrals are weak integrals (Pettis integrals [23]). Following result captures the trace of frame operator using Bessel family.

**Theorem 2.2.** Let $\{\tau_\alpha\}_{\alpha \in \Omega}$ be a continuous Bessel family for $\mathcal{H}$. Then
\[
\text{Tra}(S_\tau) = \int_{\Omega} \|\tau_\alpha\|^2 d\mu(\alpha),
\]
\[
\text{Tra}(S_\tau^2) = \int_{\Omega} \int_{\Omega} |\langle \tau_\alpha, \tau_\beta \rangle|^2 d\mu(\alpha) d\mu(\beta).
\]

**Proof.** Let $\{\omega_j\}_{j=1}^d$ be an orthonormal basis for $\mathcal{H}$, where $d$ is the dimension of $\mathcal{H}$. Then
\[
\text{Tra}(S_\tau) = \sum_{j=1}^d \langle S_\tau \omega_j, \omega_j \rangle = \sum_{j=1}^d \left( \int_{\Omega} \langle \omega_j, \tau_\alpha \rangle \tau_\alpha d\mu(\alpha), \omega_j \right)
\]
\[
= \sum_{j=1}^d \int_{\Omega} \langle \omega_j, \tau_\alpha \rangle \langle \tau_\alpha, \omega_j \rangle d\mu(\alpha) = \int_{\Omega} \left( \sum_{j=1}^d \langle \tau_\alpha, \omega_j \rangle \omega_j , \tau_\alpha \right) d\mu(\alpha)
\]
\[
= \int_{\Omega} \|\tau_\alpha\|^2 d\mu(\alpha).
\]
Further,
\[ \text{Tra}(S_\tau^2) = \sum_{j=1}^{d} \langle S_\tau^2 \omega_j, \omega_j \rangle = \sum_{j=1}^{d} \langle S_\tau \omega_j, S_\tau \omega_j \rangle = \sum_{j=1}^{d} \left( \int_{\Omega} \langle \omega_j, \tau_\alpha \rangle \langle \tau_\alpha, S_\tau \omega_j \rangle \, d\mu(\alpha), S_\tau \omega_j \right) \]

\[ = \sum_{j=1}^{d} \int_{\Omega} \langle \omega_j, \tau_\alpha \rangle \langle \tau_\alpha, S_\tau \omega_j \rangle \, d\mu(\alpha) = \int_{\Omega} \left( \sum_{j=1}^{d} \langle \tau_\alpha, S_\tau \omega_j \rangle \omega_j, \tau_\alpha \right) \, d\mu(\alpha) \]

\[ = \int_{\Omega} \left( \sum_{j=1}^{d} \langle S_\tau^* \tau_\alpha, \omega_j \rangle \omega_j, \tau_\alpha \right) \, d\mu(\alpha) = \int_{\Omega} \langle S_\tau^* \tau_\alpha, \tau_\alpha \rangle \, d\mu(\alpha) \]

\[ = \int_{\Omega} \int_{\Omega} |\langle \tau_\alpha, \tau_\beta \rangle|^2 \, d\mu(\alpha) \, d\mu(\beta). \]

\[ \square \]

Note that a finite spanning set is a frame for finite dimensional Hilbert space \([19]\). Thus it is not required to assume any condition on set of vectors in the discrete case to derive Theorem 2.2. However, we need to assume the Besselness for continuous family of vectors to assure the existence of frame operator. With Lemma 2.3 we derive continuous Welch bounds. First we need a lemma.

**Lemma 2.3.** If \( \{\tau_\alpha\}_{\alpha \in \Omega} \) is a normalized continuous Bessel family for \( \mathcal{H} \) with bound \( b \), then \( \mu(\Omega) \leq b \text{dim}(\mathcal{H}) \). In particular, \( \mu(\Omega) < \infty \).

**Proof.** Let \( \text{dim}(\mathcal{H}) = d \) and \( \{\omega_j\}_{j=1}^{d} \) be an orthonormal basis for \( \mathcal{H} \). Then

\[ \mu(\Omega) = \int_{\Omega} \|\tau_\alpha\|^2 \, d\mu(\alpha) = \int_{\Omega} \sum_{j=1}^{d} |\langle x_\alpha, \omega_j \rangle|^2 \, d\mu(\alpha) = \sum_{j=1}^{d} \int_{\Omega} |\langle x_\alpha, \omega_j \rangle|^2 \, d\mu(\alpha) \]

\[ = \sum_{j=1}^{d} \int_{\Omega} |\langle \omega_j, x_\alpha \rangle|^2 \, d\mu(\alpha) \leq \sum_{j=1}^{d} b \|\omega_j\|^2 = bd. \]

\[ \square \]

**Theorem 2.4.** Let \( (\Omega, \mu) \) be a measure space and \( \{\tau_\alpha\}_{\alpha \in \Omega} \) be a normalized continuous Bessel family for \( \mathcal{H} \) of dimension \( d \). If the diagonal \( \Delta := \{(\alpha, \alpha) : \alpha \in \Omega\} \) is measurable in the measure space \( \Omega \times \Omega \), then

\[ \int_{\Omega \times \Omega} |\langle \tau_\alpha, \tau_\beta \rangle|^2 \, d(\mu \times \mu)(\alpha, \beta) = \int_{\Omega} \int_{\Omega} |\langle \tau_\alpha, \tau_\beta \rangle|^2 \, d\mu(\alpha) \, d\mu(\beta) \geq \frac{\mu(\Omega)^2}{d}. \]

Equality holds in Inequality (2) if and only if \( \{\tau_\alpha\}_{\alpha \in \Omega} \) is a tight continuous frame. Further, we have the first order continuous Welch bound

\[ \sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} |\langle \tau_\alpha, \tau_\beta \rangle|^2 \geq \frac{1}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)} \left[ \frac{\mu(\Omega)^2}{d} - (\mu \times \mu)(\Delta) \right]. \]

**Proof.** Let \( \lambda_1, \ldots, \lambda_d \) be eigenvalues of the frame operator \( S_\tau \). Then \( \lambda_1, \ldots, \lambda_d \geq 0 \). Now using the diagonalizability of \( S_\tau \), Cauchy-Schwarz inequality and Theorem 2.2 we get

\[ \mu(\Omega)^2 = \left( \int_{\Omega} \|\tau_\alpha\|^2 \, d\mu(\alpha) \right)^2 = (\text{Tra}(S_\tau))^2 = \left( \sum_{k=1}^{d} \lambda_k \right)^2 \leq \sum_{k=1}^{d} \lambda_k^2 \]

\[ = d \text{Tra}(S_\tau^2) = d \int_{\Omega} \int_{\Omega} |\langle \tau_\alpha, \tau_\beta \rangle|^2 \, d\mu(\alpha) \, d\mu(\beta). \]
Equality holds if and only if we have equality in Cauchy-Schwarz inequality if and only if the frame is tight. Since the measure is finite (Lemma 2.3), using Fubini’s theorem,

\[
\mu(\Omega)^2 = \int_{\Omega} \int_{\Omega} |\langle \tau(\alpha), \tau(\beta) \rangle|^2 \, d\mu(\alpha) \, d\mu(\beta) = \int_{\Omega \times \Omega} |\langle \tau(\alpha), \tau(\beta) \rangle|^2 \, d(\mu \times \mu)(\alpha, \beta)
\]

\[
= \int_{\Delta} |\langle \tau(\alpha), \tau(\beta) \rangle|^2 \, d(\mu \times \mu)(\alpha, \beta) + \int_{(\Omega \times \Omega) \setminus \Delta} |\langle \tau(\alpha), \tau(\beta) \rangle|^2 \, d(\mu \times \mu)(\alpha, \beta)
\]

\[
= (\mu \times \mu)(\Delta) + \int_{(\Omega \times \Omega) \setminus \Delta} |\langle \tau(\alpha), \tau(\beta) \rangle|^2 \, d(\mu \times \mu)(\alpha, \beta)
\]

\[
\leq (\mu \times \mu)(\Delta) + \sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} |\langle \tau(\alpha), \tau(\beta) \rangle|^2 (\mu \times \mu)(\Omega \setminus \Delta).
\]

which gives the required inequality after rearrangement. □

Under the stronger assumption that \{\tau(\alpha)\}_{\alpha \in \Omega} is a continuous frame for \mathcal{H}, Inequality (2) appears in Chapter 16 of [38]. We now illustrate Theorem 2.4 using the following example.

**Example 2.5.** Let \( \Omega := [0, 2\pi] \) and \( \mu \) be the Lebesgue measure on \( \Omega \). Define

\( \tau(\alpha) := (\cos \alpha, \sin \alpha), \ \forall \alpha \in \Omega. \)

Then

\[
\int_{\Omega} |\langle (x, y), \tau(\alpha) \rangle|^2 \, d\alpha = \int_{0}^{2\pi} |\langle (x, y), (\cos \alpha, \sin \alpha) \rangle|^2 \, d\alpha = \int_{0}^{2\pi} (x \cos \alpha + y \sin \alpha)^2 \, d\alpha
\]

\[
= \int_{0}^{2\pi} (x^2 \cos^2 \alpha + y^2 \sin^2 \alpha + 2xy \sin \alpha \cos \alpha) \, d\alpha
\]

\[
= \pi(x^2 + y^2) = \pi \| (x, y) \|^2, \quad \forall (x, y) \in \mathbb{R}^2.
\]

Therefore \( \{\tau(\alpha)\}_{\alpha \in \Omega} \) is a normalized continuous frame for \( \mathbb{R}^2 \) [38]. Next we verify inequalities in Theorem 2.4.

\[
\int_{\Omega} \int_{\Omega} |\langle \tau(\alpha), \tau(\beta) \rangle|^2 \, d\mu(\alpha) \, d\mu(\beta) = \int_{0}^{2\pi} \int_{0}^{2\pi} |\langle (\cos \alpha, \sin \alpha), (\cos \beta, \sin \beta) \rangle|^2 \, d\alpha \, d\beta
\]

\[
= \int_{0}^{2\pi} \int_{0}^{2\pi} (\cos \alpha \cos \beta + \sin \alpha \sin \beta)^2 \, d\alpha \, d\beta
\]

\[
= \left( \int_{0}^{2\pi} \cos^2 \alpha \, d\alpha \right) \left( \int_{0}^{2\pi} \cos^2 \beta \, d\beta \right) + 2 \left( \int_{0}^{2\pi} \cos \alpha \sin \alpha \, d\alpha \right) \left( \int_{0}^{2\pi} \cos \beta \sin \beta \, d\beta \right) + \left( \int_{0}^{2\pi} \sin^2 \alpha \, d\alpha \right) \left( \int_{0}^{2\pi} \sin^2 \beta \, d\beta \right)
\]

\[
= 2\pi^2 \left( \frac{(2\pi)^2}{2} \right) = \frac{\mu(\Omega)^2}{d}
\]

and
We execute the proof of Theorem 2.4 for the space $\text{Sym}^m$. Then using Theorem 2.6 we get

$$\tau^2 \leq \frac{1}{\mu(\Omega)} \right) \left[ \frac{\mu(\Omega)^2}{d} - (\mu \times \mu)(\Delta) \right].$$

Our next goal is to derive higher order continuous Welch bounds. We are going to use the following result.

**Theorem 2.6.** If $\mathcal{V}$ is a vector space of dimension $d$ and $\text{Sym}^m(\mathcal{V})$ denotes the vector space of symmetric $m$-tensors, then

$$\dim(\text{Sym}^m(\mathcal{V})) = \binom{d + m - 1}{m}, \quad \forall m \in \mathbb{N}.$$  

**Theorem 2.7.** Let $(\Omega, \mu)$ be a measure space and $\{\tau_\alpha\}_{\alpha \in \Omega}$ be a normalized continuous Bessel family for $\mathcal{H}$ of dimension $d$. If the diagonal $\Delta := \{(\alpha, \alpha) : \alpha \in \Omega\}$ is measurable in the measure space $\Omega \times \Omega$, then

$$\int_{\Omega \times \Omega} |\langle \tau_\alpha, \tau_\beta \rangle|^{2m} d(\mu \times \mu)(\alpha, \beta) = \int_{\Omega} \int_{\Omega} |\langle \tau_\alpha, \tau_\beta \rangle|^{2m} d\mu(\alpha) d\mu(\beta) \geq \frac{\mu(\Omega)^2}{(d + m - 1)}, \quad \forall m \in \mathbb{N}.$$  

Equality holds in Inequality (3) if and only if $\{\tau_\alpha\}_{\alpha \in \Omega}$ is a tight continuous frame. Further, we have the higher order continuous Welch bounds

$$\sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} |\langle \tau_\alpha, \tau_\beta \rangle|^{2m} \geq \frac{1}{(\mu \times \mu)(\Omega)} \left[ \frac{\mu(\Omega)^2}{(d + m - 1)} - (\mu \times \mu)(\Delta) \right], \quad \forall m \in \mathbb{N}.$$  

**Proof.** First note that $\{\tau_\alpha\}_{\alpha \in \Omega}$ is a normalized continuous Bessel family for the Hilbert space $\text{Sym}^m(\mathcal{H})$. We execute the proof of Theorem 2.6 for the space $\text{Sym}^m(\mathcal{H})$. Let $\lambda_1, \ldots, \lambda_{\dim(\text{Sym}^m(\mathcal{H}))}$ be eigenvalues of $S_\tau$. Then using Theorem 2.6 we get

$$\mu(\Omega)^2 = \left( \int_{\Omega} \|\tau_\alpha\|^2 d\mu(\alpha) \right)^2 = \left( \int_{\Omega} \|\tau_\alpha^{\otimes m}\|^2 d\mu(\alpha) \right)^2 = (\text{Tra}(S_\tau))^2$$

$$= \left( \sum_{k=1}^{\dim(\text{Sym}^m(\mathcal{H}))} \lambda_k \right)^2 \leq \dim(\text{Sym}^m(\mathcal{H})) \sum_{k=1}^{\dim(\text{Sym}^m(\mathcal{H}))} \lambda_k^2$$

$$= \left( d + m - 1 \right) \text{Tra}(S_\tau^2) = \left( d + m - 1 \right) \int_{\Omega} \int_{\Omega} |\langle \tau_\alpha^{\otimes m}, \tau_\beta^{\otimes m} \rangle|^{2m} d\mu(\alpha) d\mu(\beta)$$

$$= \left( d + m - 1 \right) \int_{\Omega} \int_{\Omega} |\langle \tau_\alpha, \tau_\beta \rangle|^{2m} d\mu(\alpha) d\mu(\beta)$$

and hence
\[
\frac{\mu(\Omega)^2}{(\frac{m}{d} + m - 1)^2} = \int_\Omega \int_\Omega |(\tau_\alpha, \tau_\beta)|^{2m} d\mu(\alpha) d\mu(\beta) = \int_\Omega \int_\Omega |(\tau_\alpha, \tau_\beta)|^{2m} d(\mu \times \mu)(\alpha, \beta) = \int_\Delta |(\tau_\alpha, \tau_\alpha)|^{2m} d(\mu \times \mu)(\alpha, \beta) + \int_{(\Omega \times \Omega) \setminus \Delta} |(\tau_\alpha, \tau_\beta)|^{2m} d(\mu \times \mu)(\alpha, \beta) = (\mu \times \mu)(\Delta) + \int_{(\Omega \times \Omega) \setminus \Delta} |(\tau_\alpha, \tau_\beta)|^{2m} d(\mu \times \mu)(\alpha, \beta) \leq (\mu \times \mu)(\Delta) + \sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} |(\tau_\alpha, \tau_\beta)|^{2m} (\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)
\]

which gives Inequality (4). \qed

**Corollary 2.8.** Theorem 1.1 is a corollary of Theorem 2.7.

**Proof.** Take \( \Omega = \{1, \ldots, n\} \) and \( \mu \) as the counting measure. \qed

**Corollary 2.9.** Theorem 1.3 is a corollary of Theorem 2.7.

**Proof.** Take \( \Omega = \mathbb{C}^{n-1} \) and \( \mu \) as the normalized measure on \( \mathbb{C}^{n-1} \). \qed

We observe that given a measure space \( \Omega \), the diagonal \( \Delta \) need not be measurable (see [15]). This is the reason behind the measurability of diagonal in Theorem 2.7. Further, we see that the measurability of the diagonal \( \Delta \) was used only in deriving Inequality (4) and not in Inequality (3).

In [36], Waldron generalized Welch bounds to vectors which need not be normalized. In the following result we state such a result for continuous Bessel family whose proof is similar to the proof of Theorem 2.7.

**Theorem 2.10.** Let \( (\Omega, \mu) \) be a \( \sigma \)-finite measure space and \( \{\tau_\alpha\}_{\alpha \in \Omega} \) be a continuous Bessel family for \( \mathcal{H} \) of dimension \( d \). If the diagonal \( \Delta := \{\alpha, \alpha : \alpha \in \Omega\} \) is measurable in the measure space \( \Omega \times \Omega \), then
\[
\int_{\Omega \times \Omega} |(\tau_\alpha, \tau_\beta)|^{2m} d(\mu \times \mu)(\alpha, \beta) = \int_\Omega \int_\Omega |(\tau_\alpha, \tau_\beta)|^{2m} d\mu(\alpha) d\mu(\beta) \geq \frac{1}{(d+m-1)} \left( \int_\Omega \|\tau_\alpha\|^{2m} d\mu(\alpha) \right)^2, \quad \forall m \in \mathbb{N}.
\]

Equality in Inequality (4) holds if and only if \( \{\tau_\alpha\}_{\alpha \in \Omega} \) is a tight continuous frame. Further, we have the generalized higher order continuous Welch bounds
\[
\sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} |(\tau_\alpha, \tau_\beta)|^{2m} \geq \frac{1}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)} \left[ \frac{1}{(d+m-1)} \left( \int_\Omega \|\tau_\alpha\|^{2m} d\mu(\alpha) \right)^2 - \int_\Delta \|\tau_\alpha\|^{4m} d(\mu \times \mu)(\alpha, \beta) \right],
\]

for all \( m \in \mathbb{N} \).

Note that we imposed \( \sigma \)-finiteness of measure in Theorem 2.10 to use Fubini’s theorem whereas we derived in Lemma 2.3 that measure is finite for normalized continuous Bessel family. Also note that Theorem 2.10 remains valid as long as Fubini’s theorem is valid (for instance, it is valid for complete measure spaces). Since Fubini’s theorem is not valid for arbitrary measure spaces, we are finally left with the following problem.

**Question 2.11.** Classify measure spaces \( (\Omega, \mu) \) such that Theorem 2.10 holds? In other words, given a measure space \( (\Omega, \mu) \), does the validity of Inequality (5) or Inequality (6) implies conditions on measure space \( (\Omega, \mu) \), say \( \sigma \)-finite?
In a recent work, Christensen, Datta and Kim derived Welch bounds for dual frames \[9\]. We now extend this result to continuous frames. For this we recall the notion of dual frame. A continuous frame \(\{\omega_\alpha\}_{\alpha \in \Omega}\) for \(H\) is said to be a dual for a continuous frame \(\{\tau_\alpha\}_{\alpha \in \Omega}\) for \(H\) if \(\theta_\omega \theta_\tau = I_H\) or \(\theta_\tau \theta_\omega = I_H\), the identity operator on \(H\). In terms of weak integrals, this is same as

\[
\int_\Omega \langle h, \tau_\alpha \rangle \omega_\alpha \, d\mu(\alpha) = h, \; \forall h \in H \quad \text{or} \quad \int_\Omega \langle h, \omega_\alpha \rangle \tau_\alpha \, d\mu(\alpha) = h, \; \forall h \in H.
\]

We now see that the frame \(\{S^{-1}_\tau \tau_\alpha\}_{\alpha \in \Omega}\) is always a dual to a frame \(\{\tau_\alpha\}_{\alpha \in \Omega}\) for \(H\). Further, if \(\{\omega_\alpha\}_{\alpha \in \Omega}\) is any dual for \(\{\tau_\alpha\}_{\alpha \in \Omega}\), then

\[
\int_\Omega |\langle h, \omega_\alpha \rangle|^2 \, d\mu(\alpha) \geq \int_\Omega |\langle h, S^{-1}_\tau \tau_\alpha \rangle|^2 \, d\mu(\alpha), \; \forall h \in H.
\]

We need two more results before we derive continuous Welch bounds for dual frames.

**Theorem 2.12.** If \(\{\tau_\alpha\}_{\alpha \in \Omega}\) is a continuous frame for \(H\), then for any linear operator \(T : H \to H\), we have

\[
\text{Tra}(T) = \int_\Omega \langle TS^{-\frac{1}{2}}_\tau \tau_\alpha, S^{-\frac{1}{2}}_\tau \tau_\alpha \rangle \, d\mu(\alpha).
\]

**Proof.** First we prove the theorem for Parseval frames. Assume that \(\{\tau_\alpha\}_{\alpha \in \Omega}\) is Parseval. Let \(\{\omega_j\}_{j=1}^d\) be an orthonormal basis for \(H\), where \(d\) is the dimension of \(H\). Then

\[
\text{Tra}(T) = \sum_{j=1}^d \langle T \omega_j, \omega_j \rangle = \sum_{j=1}^d \left( \int_\Omega \langle T \omega_j, \tau_\alpha \rangle \, d\mu(\alpha) \right) \omega_j
\]

\[
= \sum_{j=1}^d \int_\Omega \langle T \omega_j, \tau_\alpha \rangle \langle \tau_\alpha, \omega_j \rangle \, d\mu(\alpha) = \int_\Omega \left( \sum_{j=1}^d \langle \tau_\alpha, \omega_j \rangle T \omega_j, \tau_\alpha \right) \, d\mu(\alpha)
\]

\[
= \int_\Omega \langle T \tau_\alpha, \tau_\alpha \rangle \, d\mu(\alpha).
\]

Now the theorem follows by noting that \(\{S^{-\frac{1}{2}}_\tau \tau_\alpha\}_{\alpha \in \Omega}\) is a Parseval frame for \(H\). \(\square\)

**Theorem 2.13.** If \(\{\omega_\alpha\}_{\alpha \in \Omega}\) is a dual continuous frame for \(\{\tau_\alpha\}_{\alpha \in \Omega}\), then

\[
\int_\Omega \int_\Omega |\langle \tau_\alpha, \omega_\beta \rangle|^2 \, d\mu(\alpha) \, d\mu(\beta) \geq \text{dim}(H).
\]

**Proof.** Inequality \(\text{(7)}\) says that

\[
\int_\Omega \int_\Omega |\langle \tau_\alpha, \omega_\beta \rangle|^2 \, d\mu(\beta) \, d\mu(\alpha) \geq \int_\Omega \int_\Omega |\langle \tau_\alpha, S^{-\frac{1}{2}}_\tau \tau_\beta \rangle|^2 \, d\mu(\beta) \, d\mu(\alpha), \; \forall \alpha \in \Omega.
\]

Therefore

\[
\int_\Omega \int_\Omega |\langle \tau_\alpha, \omega_\beta \rangle|^2 \, d\mu(\beta) \, d\mu(\alpha) \geq \int_\Omega \int_\Omega |\langle \tau_\alpha, S^{-\frac{1}{2}}_\tau \tau_\beta \rangle|^2 \, d\mu(\beta) \, d\mu(\alpha).
\]

Now we simplify the right side and use Theorem 2.12 to get

\[
\int_\Omega \int_\Omega |\langle \tau_\alpha, S^{-\frac{1}{2}}_\tau \tau_\beta \rangle|^2 \, d\mu(\beta) \, d\mu(\alpha) = \int_\Omega \int_\Omega |\langle S^{-\frac{1}{2}}_\tau \tau_\alpha, S^{-\frac{1}{2}}_\tau \tau_\beta \rangle|^2 \, d\mu(\beta) \, d\mu(\alpha)
\]

\[
= \int_\Omega \|S^{-\frac{1}{2}}_\tau \tau_\alpha\|^2 \, d\mu(\alpha) = \int_\Omega \langle S^{-\frac{1}{2}}_\tau \tau_\alpha, S^{-\frac{1}{2}}_\tau \tau_\alpha \rangle \, d\mu(\alpha)
\]

\[
= \text{Tra}(I_H) = \text{dim}(H).
\]

\(\square\)
Theorem 2.14. Let \( \{\tau_\alpha\}_{\alpha \in \Omega} \) be a continuous frame for \( \mathcal{H} \) of dimension \( d \). Assume that \( \{\omega_\alpha\}_{\alpha \in \Omega} \) is a dual continuous frame for \( \{\tau_\alpha\}_{\alpha \in \Omega} \) and
\[
\langle \tau_\alpha, \omega_\alpha \rangle = \langle \tau_\beta, \omega_\beta \rangle, \quad \forall \alpha, \beta \in \Omega.
\]
If the diagonal \( \Delta := \{(\alpha, \alpha) : \alpha \in \Omega\} \) is measurable in the measure space \( \Omega \times \Omega \), then
\[
\sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} |\langle \tau_\alpha, \omega_\beta \rangle|^2 \geq \frac{d(\mu(\Omega)^2 - d(\mu \times \mu)(\Delta))}{\mu(\Omega)^2 (\mu \times \mu)(\Omega \times \Omega) \setminus \Delta}).
\]

Proof. Since \( \{\omega_\alpha\}_{\alpha \in \Omega} \) is a dual for \( \{\tau_\alpha\}_{\alpha \in \Omega} \) we have \( \theta_\alpha^* \theta_\beta = I_\mathcal{H} \). Let \( \{\rho_j\}_{j=1}^d \) be an orthonormal basis for \( \mathcal{H} \). Then
\[
d = \dim(\mathcal{H}) = \sum_{j=1}^d \langle \rho_j, \rho_j \rangle = \sum_{j=1}^d \left(\int_\Omega \langle \rho_j, \tau_\alpha \rangle \omega_\alpha d\mu(\alpha), \rho_j\right)
= \int_\Omega \sum_{j=1}^d \langle \rho_j, \tau_\alpha \rangle \langle \omega_\alpha, \rho_j \rangle d\mu(\alpha) = \int_\Omega \langle \omega_\alpha, \sum_{j=1}^d \langle \tau_\alpha, \rho_j \rangle \rho_j \rangle d\mu(\alpha)
= \int_\Omega \langle \omega_\alpha, \tau_\alpha \rangle d\mu(\alpha) = \int_\Omega \langle \tau_\alpha, \omega_\alpha \rangle d\mu(\alpha).
\]
Set \( \gamma := \langle \tau_\alpha, \omega_\alpha \rangle \) which is independent of \( \alpha \) by the assumption. Then
\[
\int_\Delta |\langle \tau_\alpha, \omega_\alpha \rangle|^2 d(\mu \times \mu)(\alpha, \beta) = \int_\Delta |\gamma|^2 d(\mu \times \mu)(\alpha, \beta) = \int_\Delta \left(\frac{1}{\mu(\Omega)} \int_\Omega \langle \tau_\alpha, \omega_\alpha \rangle d\mu(\alpha)\right)^2 d(\mu \times \mu)(\alpha, \beta)
= \int_\Delta \left(\frac{d}{\mu(\Omega)}\right)^2 d(\mu \times \mu)(\alpha, \beta) = \frac{d^2(\mu \times \mu)(\Delta)}{\mu(\Omega)^2}.
\]

Theorem 2.14 then gives
\[
\sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} |\langle \tau_\alpha, \omega_\beta \rangle|^2 \geq \frac{1}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)} \int_{(\Omega \times \Omega) \setminus \Delta} |\langle \tau_\alpha, \omega_\beta \rangle|^2 d(\mu \times \mu)(\alpha, \beta)
= \frac{1}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)} \left[ \int_\Omega \int_{(\Omega \times \Omega) \setminus \Delta} |\langle \tau_\alpha, \omega_\beta \rangle|^2 d\mu(\alpha) d\mu(\beta) - \int_\Delta |\langle \tau_\alpha, \omega_\alpha \rangle|^2 d(\mu \times \mu)(\alpha, \beta) \right]
\geq \frac{1}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)} \left[ \frac{d^2(\mu \times \mu)(\Delta)}{\mu(\Omega)^2} \right] = \frac{d}{\mu(\Omega)((\Omega \times \Omega) \setminus \Delta)} \left[ 1 - \frac{d(\mu \times \mu)(\Delta)}{\mu(\Omega)^2} \right].
\]

\[\Box\]

Corollary 2.15. Let \( \{\tau_\alpha\}_{\alpha \in \Omega} \) be a continuous frame for \( \mathcal{H} \) of dimension \( d \). Assume that \( \{\omega_\alpha\}_{\alpha \in \Omega} \) is a dual continuous frame for \( \{\tau_\alpha\}_{\alpha \in \Omega} \). If the diagonal \( \Delta := \{(\alpha, \alpha) : \alpha \in \Omega\} \) is measurable in the measure space \( \Omega \times \Omega \), then
\[
\sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} |\langle \tau_\alpha, \omega_\beta \rangle|^2 \geq \frac{1}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)} \left[ d - \int_\Delta |\langle \tau_\alpha, \omega_\alpha \rangle|^2 d(\mu \times \mu)(\alpha, \beta) \right].
\]

Higher order continuous Welch bounds leads to the following question which we do not have answer at present.

Question 2.16. Is there a higher order version of Theorem 2.14 like Theorem 2.7?
Theorem 2.17. Let \( \{\tau_\alpha\}_{\alpha \in \Omega} \) be a normalized continuous Bessel family for \( \mathcal{H} \) of dimension \( d \). Then
\[
\frac{1}{\mu(\Omega)} \operatorname{Tra}(\theta^*_r \theta_r) \geq \left( \frac{\mu(\Omega)}{d} \right)^{d-1}, \quad \forall r \in [1, \infty)
\]
and
\[
\frac{1}{\mu(\Omega)} \operatorname{Tra}(\theta^*_r \theta_r) \leq \left( \frac{\mu(\Omega)}{d} \right)^{d-1}, \quad \forall r \in (0, 1).
\]

Proof. Let \( \lambda_1, \ldots, \lambda_{\dim(Sym^n(\mathcal{H}))} \) be eigenvalues of \( S_r \). Let \( r \in [1, \infty) \). Using Jensen’s inequality
\[
\left( \frac{1}{d} \sum_{k=1}^{d} \lambda_k \right)^r \leq \frac{1}{d} \sum_{k=1}^{d} \lambda_k^r.
\]
Since \( S_r \) is diagonalizable we get
\[
\left( \frac{\mu(\Omega)}{d} \right)^r = \frac{1}{d} \int_{\Omega} \|\tau_\alpha\|^2 \, d\mu(\alpha) \leq \frac{1}{d} \operatorname{Tra}(S_r) \leq \frac{1}{d} \operatorname{Tra}(\theta^*_r \theta_r)^r.
\]
Similarly the case \( r \in (0, 1) \) follows by using Jensen’s inequality. \( \square \)

Theorem 2.18. Let \( 2 < p < \infty \). Let \( (\Omega, \mu) \) be a measure space and \( \{\tau_\alpha\}_{\alpha \in \Omega} \) be a normalized continuous Bessel family for \( \mathcal{H} \) of dimension \( d \). If the diagonal \( \Delta := \{(\alpha, \alpha) : \alpha \in \Omega\} \) is measurable in the measure space \( \Omega \times \Omega \), then
\[
\int_{\Omega \times \Omega} |(\tau_\alpha, \tau_\beta)|^p \, d(\mu \times \mu)(\alpha, \beta) = \int_{\Omega} \int_{\Omega} |(\tau_\alpha, \tau_\beta)|^p \, d\mu(\alpha) \, d\mu(\beta)
\]
\[
\geq \frac{1}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)^{\frac{p}{2}}} \left( \frac{\mu(\Omega)^2}{d} - (\mu \times \mu)(\Delta) \right)^{\frac{p}{2}} + (\mu \times \mu)(\Delta).
\]

Proof. Define \( r := 2p/(p-2) \) and \( q \) be the conjugate index of \( p/2 \). Then \( q = r/2 \). Using Theorem 2.14 and Holder’s inequality, we have
\[
\frac{\mu(\Omega)^2}{d} - (\mu \times \mu)(\Delta) \leq \int_{(\Omega \times \Omega) \setminus \Delta} |(\tau_\alpha, \tau_\beta)|^2 \, d(\mu \times \mu)(\alpha, \beta)
\]
\[
\leq \left( \int_{(\Omega \times \Omega) \setminus \Delta} |(\tau_\alpha, \tau_\beta)|^2 \, d(\mu \times \mu)(\alpha, \beta) \right)^{\frac{q}{2}} \left( \int_{(\Omega \times \Omega) \setminus \Delta} d(\mu \times \mu)(\alpha, \beta) \right)^{\frac{1}{q}}
\]
\[
= \left( \int_{(\Omega \times \Omega) \setminus \Delta} |(\tau_\alpha, \tau_\beta)|^p \, d(\mu \times \mu)(\alpha, \beta) \right)^{\frac{p}{q}} (\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)^{\frac{p}{2}}
\]
\[
= \left( \int_{(\Omega \times \Omega) \setminus \Delta} |(\tau_\alpha, \tau_\beta)|^p \, d(\mu \times \mu)(\alpha, \beta) \right)^{\frac{p}{q}} (\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)^{\frac{p^2}{2q}}
\]
which gives
\[
\left( \frac{\mu(\Omega)^2}{d} - (\mu \times \mu)(\Delta) \right)^{\frac{p}{2}} \leq \left( \int_{(\Omega \times \Omega) \setminus \Delta} |(\tau_\alpha, \tau_\beta)|^p \, d(\mu \times \mu)(\alpha, \beta) \right) (\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)^{\frac{p}{2}-1}.
\]
Therefore
\[
\frac{1}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)^{\frac{p}{2}-1}} \left( \frac{\mu(\Omega)^2}{d} - (\mu \times \mu)(\Delta) \right)^{\frac{p}{2}} + (\mu \times \mu)(\Delta)
\]
\[
\leq \int_{(\Omega \times \Omega) \setminus \Delta} |\langle \tau_\alpha, \tau_\beta \rangle|^p d(\mu \times \mu)(\alpha, \beta) + \int_\Delta |\langle \tau_\alpha, \tau_\beta \rangle|^p d(\mu \times \mu)(\alpha, \beta)
\]
\[
= \int_{\Omega \times \Omega} |\langle \tau_\alpha, \tau_\beta \rangle|^p d(\mu \times \mu)(\alpha, \beta).
\]

There are four more bounds which are in line with Welch bounds. To state them we need a definition.

**Definition 2.19.** [20] Given $d \in \mathbb{N}$, define Gerzon's bound
\[
Z(d, K) := \begin{cases} 
  d^2 & \text{if } K = \mathbb{C} \\
  \frac{d^2}{d(d+1)} & \text{if } K = \mathbb{R}.
\end{cases}
\]

**Theorem 2.20.** [7, 11, 17, 20, 24, 26, 30, 40] Define $m := \dim_{\mathbb{R}}(K)/2$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{K}^d$, then

(i) (Bukh-Cox bound)
\[
\max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \frac{Z(n-d, K)}{n(1+m(n-d-1)\sqrt{m^{-1}+n-d})-Z(n-d, K)} \quad \text{if } n > d.
\]

(ii) (Orthoplex/Rankin bound)
\[
\max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \frac{1}{\sqrt{d}} \quad \text{if } n > Z(d, K).
\]

(iii) (Levenstein bound)
\[
\max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq \frac{n(m+1) - d(md+1)}{(n-d)(md+1)} \quad \text{if } n > Z(d, K).
\]

(iv) (Exponential bound)
\[
\max_{1 \leq j, k \leq n, j \neq k} |\langle \tau_j, \tau_k \rangle| \geq 1 - 2n^{-\frac{1}{d}}.
\]

Theorem 2.20 leads to the following problem.

**Question 2.21.** Whether there is a continuous version of Theorem 2.20? In particular, does there exists a continuous version of

(i) Bukh-Cox bound?
(ii) Orthoplex/Rankin bound?
(iii) Levenstein bound?
(iv) Exponential bound?

3. Applications

Our first application of Theorem 2.7 is to the continuous version of RMS correlation of vectors which we define as follows.
**Definition 3.1.** Let \( \{\tau_\alpha\}_{\alpha \in \Omega} \) be a normalized continuous Bessel family for \( \mathcal{H} \). If the diagonal \( \Delta \) is measurable, then the **continuous root-mean-square** (CRMS) **absolute cross relation** of \( \{\tau_\alpha\}_{\alpha \in \Omega} \) is defined as

\[
I_{CRMS}(\{\tau_\alpha\}_{\alpha \in \Omega}) := \left( \frac{1}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)} \int_{(\Omega \times \Omega) \setminus \Delta} |\langle \tau_\alpha, \tau_\beta \rangle|^2 d(\mu \times \mu)(\alpha, \beta) \right)^{\frac{1}{2}}.
\]

Theorem 2.7 now gives the following estimate.

**Proposition 3.2.** Under the set up as in Definition 3.1, one has

\[
1 \geq I_{CRMS}(\{\tau_\alpha\}_{\alpha \in \Omega}) \geq \left( \frac{1}{(\mu \times \mu)((\Omega \times \Omega) \setminus \Delta)} \left[ \frac{\mu(\Omega)^2}{d} - (\mu \times \mu)(\Delta) \right] \right)^{\frac{1}{2}}.
\]

Our second application of Theorem 2.7 is to the continuous version of frame potential which is defined as follows.

**Definition 3.3.** Let \( \{\tau_\alpha\}_{\alpha \in \Omega} \) be a normalized continuous Bessel family for \( \mathcal{H} \). The **continuous frame potential** of \( \{\tau_\alpha\}_{\alpha \in \Omega} \) is defined as

\[
F P(\{\tau_\alpha\}_{\alpha \in \Omega}) := \int_{\Omega} \int_{\Omega} |\langle \tau_\alpha, \tau_\beta \rangle|^2 d\mu(\alpha) d\mu(\beta).
\]

Note that the order of integration does not matter in Definition 3.3. Further, finiteness of measure says that potential is finite. In general, it is difficult to find potential using Definition 3.3. Following theorem simplifies it to a greater extent.

**Theorem 3.4.** If \( \{\tau_\alpha\}_{\alpha \in \Omega} \) is a normalized continuous Bessel family for \( \mathcal{H} \), then

\[
F P(\{\tau_\alpha\}_{\alpha \in \Omega}) = Tr(S_\tau^2) = Tr((\theta^*_\tau \theta_\tau)^2).
\]

**Proof.** This follows from Theorem 2.2. 

Using Theorem 2.7 we have following estimates.

**Proposition 3.5.** Given a normalized continuous Bessel family \( \{\tau_\alpha\}_{\alpha \in \Omega} \) for \( \mathcal{H} \), one has

\[
\frac{\mu(\Omega)^2}{d} \leq F P(\{\tau_\alpha\}_{\alpha \in \Omega}) \leq \mu(\Omega)^2.
\]

Further, if the diagonal \( \Delta \) is measurable, then one also has

\[
(\mu \times \mu)(\Delta) \leq F P(\{\tau_\alpha\}_{\alpha \in \Omega}) \leq \mu(\Omega)^2.
\]

Proposition 3.5 and the study of paper [4] leads to the following problem.

**Question 3.6.** Is there a characterization of continuous frames using continuous frame potential (like Theorem 7.1 in [4])?

Our third application of Theorem 2.7 is to the continuous frame correlations defined as follows.

**Definition 3.7.** Let \( \{\tau_\alpha\}_{\alpha \in \Omega} \) be a normalized continuous frame for \( \mathcal{H} \). We define the **continuous frame correlation** of \( \{\tau_\alpha\}_{\alpha \in \Omega} \) as

\[
M(\{\tau_\alpha\}_{\alpha \in \Omega}) := \sup_{\alpha, \beta \in \Omega, \alpha \neq \beta} |\langle \tau_\alpha, \tau_\beta \rangle|.
\]

In discrete frame theory the notion which comes along with frame correlation is the notion of Grassmannian frames defined in [31]. We next set up the notion of continuous Grassmannian frames.
Definition 3.8. A normalized continuous frame \( \{ \tau_\alpha \}_{\alpha \in \Omega} \) for \( \mathcal{H} \) is said to be a continuous Grassmannian frame for \( \mathcal{H} \) if

\[
\mathcal{M}(\{ \tau_\alpha \}_{\alpha \in \Omega}) = \inf \{ \mathcal{M}(\{ \omega_\alpha \}_{\alpha \in \Omega}) : \{ \omega_\alpha \}_{\alpha \in \Omega} \text{ is a normalized continuous frame for } \mathcal{H} \}.
\]

Using compactness and continuity arguments it is known that Grassmannian frames exist in every dimension with any number of vectors (greater than or equal to dimension) \( [3] \). However we can not use this argument for measures. Therefore we state the following open problem.

Question 3.9. Classify measure spaces and (finite dimensional) Hilbert spaces so that continuous Grassmannian frames exist.

The notion which is associated to Grassmannian frames is the notion of equiangular frames (see \( [31] \)). For the continuous case, we set the definition as follows.

Definition 3.10. A continuous frame \( \{ \tau_\alpha \}_{\alpha \in \Omega} \) for \( \mathcal{H} \) is said to be \( \gamma \)-equiangular if there exists \( \gamma \geq 0 \) such that

\[
|\langle \tau_\alpha, \tau_\beta \rangle| = \gamma, \quad \forall \alpha, \beta \in \Omega, \alpha \neq \beta.
\]

There is a celebrated Zauner’s conjecture for equiangular tight frames (see \( [2] \)). For the purpose of record, we set the continuous version of Zauner’s conjecture as follows.

Conjecture 3.11. (Continuous Zauner’s conjecture) For a given measure space \( (\Omega, \mu) \) and for every \( d \in \mathbb{N} \), there exists a \( \gamma \)-equiangular tight continuous frame \( \{ \tau_\alpha \}_{\alpha \in \Omega} \) for \( \mathbb{C}^d \) such that \( \mu(\Omega) = d^2 \).

Theorem 3.12. Let \( \{ \tau_\alpha \}_{\alpha \in \Omega} \) be a normalized continuous frame for \( \mathcal{H} \). Then

\[
\mathcal{M}(\{ \tau_\alpha \}_{\alpha \in \Omega}) \geq \left( \frac{1}{(\mu \times \mu)(\Omega \times \Omega) \setminus (\mu \times \mu)(\Delta)} \left[ \frac{\mu(\Omega)^2}{d} - (\mu \times \mu)(\Delta) \right] \right)^\frac{1}{2} =: \gamma.
\]

If the frame is \( \gamma \)-equiangular, then we have equality in Inequality (8).

In the case of (discrete) Grassmannian frames, the converse statement of Theorem 3.12 is valid (see \( [31] \)). There are also relations between number of elements in the frame and dimension of the space (Theorem 2.3 in \( [31] \)). We do not know any such relation between measure of \( \Omega \) and the dimension of \( \mathcal{H} \).

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References

[1] S. Twareque Ali, J.-P. Antoine, and J.-P. Gazeau. Continuous frames in Hilbert space. Ann. Physics, 222(1):1–37, 1993.

[2] Marcus Appleby, Ingemar Bengtsson, Steven Flammia, and Dardo Goyeneche. Tight frames, Hadamard matrices and Zauner’s conjecture. J. Phys. A, 52(29):295301, 26, 2019.

[3] J.J. Benedetto and J.D Kolesar. Geometric properties of grassmannian frames for \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). EURASIP J. Adv. Signal Process. 2006, (049850), 2006.

[4] John J. Benedetto and Matthew Fickus. Finite normalized tight frames. Adv. Comput. Math., 18(2-4):357–385, 2003.

[5] Cristiano Bocci and Luca Chiantini. An introduction to algebraic statistics with tensors, volume 118 of Unitext. Springer, Cham, 2019.
[6] Bernhard G. Bodmann and John Haas. Frame potentials and the geometry of frames. *J. Fourier Anal. Appl.*, 21(6):1344–1383, 2015.

[7] Boris Bukh and Christopher Cox. Nearly orthogonal vectors and small antipodal spherical codes. *Israel J. Math.*, 238(1):359–388, 2020.

[8] Peter G. Casazza, Matthew Fickus, Jelena Kovacević, Manuel T. Leon, and Janet C. Tremain. A physical interpretation of tight frames. In *Harmonic analysis and applications*, Appl. Numer. Harmon. Anal., pages 51–76. Birkhäuser Boston, Boston, MA, 2006.

[9] Ole Christensen, Somantika Datta, and Rae Young Kim. Equiangular frames and generalizations of the Welch bound to dual pairs of frames. *Linear Multilinear Algebra*, 68(12):2495–2505, 2020.

[10] Pierre Comon, Gene Golub, Lek-Heng Lim, and Bernard Mourrain. Symmetric tensors and symmetric tensor rank. *SIAM J. Matrix Anal. Appl.*, 30(3):1254–1279, 2008.

[11] John H. Conway, Ronald H. Hardin, and Neil J. A. Sloane. Packing lines, planes, etc.: packings in Grassmannian spaces. *Experiment. Math.*, 5(2):139–159, 1996.

[12] S. Datta, S. Howard, and D. Cochran. Geometry of the Welch bounds. *Linear Algebra Appl.*, 437(10):2455–2470, 2012.

[13] Somantika Datta. Welch bounds for cross correlation of subspaces and generalizations. *Linear Multilinear Algebra*, 64(8):1484–1497, 2016.

[14] Cunsheng Ding and Tao Feng. Codebooks from almost difference sets. *Des. Codes Cryptogr.*, 46(1):113–126, 2008.

[15] Jozef Dravecký. Spaces with measurable diagonal. *Mat. Časopis Sloven. Akad. Vied*, 25(1):3–9, 1975.

[16] M. Ehler and K. A. Okoudjou. Minimization of the probabilistic $p$-frame potential. *J. Statist. Plann. Inference*, 142(3):645–659, 2012.

[17] John I. Haas, Nathaniel Hammen, and Dustin G. Mixon. The Levenstein bound for packings in projective spaces. *Proceedings*, volume 10394, *Wavelets and Sparsity XVII*, SPIE Optical Engineering+Applications, San Diego, California, United States of America, 2017.

[18] Marina Haikin, Ram Zamir, and Matan Gavish. Frame moments and Welch bound with erasures. *2018 IEEE International Symposium on Information Theory (ISIT)*, pages 2057–2061, 2018.

[19] Deguang Han, Keri Kornelson, David Larson, and Eric Weber. *Frames for undergraduates*, volume 40 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2007.

[20] John Jasper, Emily J. King, and Dustin G. Mixon. Game of sloanies: best known packings in complex projective space. *Proc. SPIE 11138, Wavelets and Sparsity XVIII*, 2019.

[21] Gerald Kaiser. *A friendly guide to wavelets*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2011.

[22] J. Kovacevic and A. Chebira. Life beyond bases: The advent of frames (part I). *IEEE Signal Processing Magazine*, 24(4):86–104, 2007.

[23] J. Kovacevic and A. Chebira. Life beyond bases: The advent of frames (part II). *IEEE Signal Processing Magazine*, 24(5):115–125, 2007.

[24] K.K. Mukkavilli, A. Sabharwal, E. Erkip, and B. Aazhang. On beamforming with finite rate feedback in multiple-antenna systems. *IEEE Transactions on Information Theory*, 49(10):2562–2579, 2003.

[25] A. Rahimi, B. Daraby, and Z. Darvishi. Construction of continuous frames in Hilbert spaces. *Azerb. J. Math.*, 7(1):49–58, 2017.

[26] R. A. Rankin. The closest packing of spherical caps in $n$ dimensions. *Proc. Glasgow Math. Assoc.*, 2:139–144, 1955.

[27] C. Rose, S. Ulukus, and R. D. Yates. Wireless systems and interference avoidance. *IEEE Transactions on Wireless Communications*, 1(3):415–428, 2002.

[28] Dilip V. Sarwate. Bounds on crosscorrelation and autocorrelation of sequences. *IEEE Trans. Inform. Theory*, 25(6):720–724, 1979.

[29] A. J. Scott. Tight informationally complete quantum measurements. *J. Phys. A*, 39(43):13507–13530, 2006.

[30] Mojtaba Soltanalian, Mohammad Mahdi Naghsh, and Petre Stoica. On meeting the peak correlation bounds. *IEEE Transactions on Signal Processing*, 62(5):1210–1220, 2014.

[31] Thomas Strohmer and Robert W. Heath, Jr. Grassmannian frames with applications to coding and communication. *Appl. Comput. Harmon. Anal.*, 14(3):257–275, 2003.

[32] Máté Sustik, Joel A. Tropp, Inderjit S. Dhillon, and Robert W. Heath, Jr. On the existence of equiangular tight frames. *Linear Algebra Appl.*, 426(2-3):619–635, 2007.

[33] Michel Talagrand. Pettis integral and measure theory. *Mem. Amer. Math. Soc.*, 51(307):ix+224, 1984.
[34] Yan Shuo Tan. Energy optimization for distributions on the sphere and improvement to the Welch bounds. *Electron. Commun. Probab.*, 22:Paper No. 43, 12, 2017.

[35] Joel A. Tropp, Inderjit S. Dhillon, Robert W. Heath, Jr., and Thomas Strohmer. Designing structured tight frames via an alternating projection method. *IEEE Trans. Inform. Theory*, 51(1):188–209, 2005.

[36] Shayne Waldron. Generalized Welch bound equality sequences are tight frames. *IEEE Trans. Inform. Theory*, 49(9):2307–2309, 2003.

[37] Shayne Waldron. A sharpening of the Welch bounds and the existence of real and complex spherical t-designs. *IEEE Trans. Inform. Theory*, 63(11):6849–6857, 2017.

[38] Shayne F. D. Waldron. *An introduction to finite tight frames*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, 2018.

[39] L. Welch. Lower bounds on the maximum cross correlation of signals. *IEEE Transactions on Information Theory*, 20(3):397–399, 1974.

[40] Pengfei Xia, Shengli Zhou, and Georgios B. Giannakis. Achieving the Welch bound with difference sets. *IEEE Trans. Inform. Theory*, 51(5):1900–1907, 2005.

[41] Pengfei Xia, Shengli Zhou, and Georgios B. Giannakis. Correction to: “Achieving the Welch bound with difference sets” [IEEE Trans. Inform. Theory 51 (2005), no. 5, 1900–1907]. *IEEE Trans. Inform. Theory*, 52(7):3359, 2006.