Solving the Odd Perfect Number Problem: Some New Approaches

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Abstract—A conjecture predicting an injective and surjective mapping $X = \frac{\sigma(p^k)}{p^k}, Y = \frac{\sigma(m^2)}{m^2}$ between OPNs $N = p^k m^2$ (with Euler factor $p^k$) and rational points on the hyperbolic arc $XY = 2$ with $1 < X < 1.25 < 1.6 < Y < 2$ and $2.85 < X + Y < 3$, is disproved. We will show that if an OPN $N$ has the form above, then $p^k < \frac{5}{4} m^2$. We then give a somewhat weaker corollary to this last result ($m^2 - p^k \geq 8$) and give possible improvements along these lines. We will also attempt to prove a conjectured improvement to $p^k < m$ by observing that $\frac{\sigma(p^k)}{m} \neq 1$ and $\frac{\sigma(p^k)}{m} \neq \frac{\sigma(m)}{p^k}$ in all cases. Lastly, we also prove the following generalization: If $N = \prod_{i=1}^{r} p_i^{\alpha_i}$ is the canonical factorization of an OPN $N$, then $\sigma(p_i^{\alpha_i}) \leq \frac{2N}{3p_i^{\alpha_i}}$ for all $i$. This gives rise to the inequality $N^{2-r} \leq (\frac{5}{4})(\frac{2}{3})^{r-1}$, which is true for all $r$, where $r = \omega(N)$ is the number of distinct prime factors of $N$.

Index Terms—Odd perfect number, Euler factor, inequalities, OPN components, non-injective and non-surjective mapping

I. INTRODUCTION

A perfect number is a positive integer $N$ such that the sum of all the positive divisors of $N$ equals $2N$, denoted by $\sigma(N) = 2N$. The question of the existence of odd perfect numbers (OPNs) is one of the longest unsolved problems of number theory. This paper gives some new and nontraditional attempts and approaches to solving the Odd Perfect Number (OPN) Problem.

Hereinafter, we shall let $N = p^k m^2$ denote an OPN with Euler (prime-power) factor $p^k$ (with $p \equiv k \equiv 1 \pmod{4}$ and $\gcd(p, m) = 1$), assuming at least one such number exists.

II. SOME PREPARATORY LEMMAS

The following lemmas will be very useful later on:

Lemma II.1. $0 < \rho_2 \leq \frac{5}{2} < 1 < \rho_1 < \frac{9}{4} < \frac{5}{3} < \mu_1 < 2 < 3 \leq \mu_2$ where:

$\rho_1 = \frac{\sigma(p^k)}{p^k}$

$\rho_2 = \frac{\sigma(p^k)}{m^2}$

$\mu_1 = \frac{\sigma(m^2)}{m^2}$

$\mu_2 = \frac{\sigma(m^2)}{p^k}$

Proof: We refer the interested reader to the author’s master’s thesis completed in August of 2008 [2].

Lemma II.2. $\frac{57}{20} < \rho_1 + \mu_1 < 3 < \frac{11}{4} \leq \rho_2 + \mu_2$

Proof: Again, the interested reader is referred to [2], where it is shown that “any further improvement to the lower bound of $\frac{57}{20}$ for $\rho_1 + \mu_1$ would be equivalent to showing that there are no OPNs of the form $5m^2$ which would be a very major result. Likewise, any further improvement on the upper bound of 3 would have similar implications for all arbitrarily large primes and thus would be a very major result.” (These assertions, which are originally Joshua Zelinsky’s, were readily verified by the author using Mathematica.)

Lemma II.3. Define $\rho_3 = \frac{\sigma(p^k)}{m} \text{ and } \mu_3 = \frac{\sigma(m)}{p^k}$. Then $\rho_3 \neq \mu_3$, and the following statements hold:

- If $\rho_3 < 1$, then $p^k < m$.
- Suppose that $1 < \rho_3$:
  - If $\rho_3 < \mu_3$, then $\frac{5}{4} m < p^k < \sqrt{2} m$.
  - If $\mu_3 < \rho_3$, then $m < p^k$.

Proof: The interested reader is again referred to [2]. The crucial part of the argument is in showing that $\frac{\sigma(p^k)}{p^k} < \frac{\sigma(m)}{m}$.

III. MAIN RESULTS

First, we prove that a conjectured one-to-one correspondence is actually both not surjective and not injective.

Conjecture III.1. For each $N = p^k m^2$ an OPN with $N > 10^{900}$, there corresponds exactly one ordered pair of rational numbers $(\frac{\sigma(p^k)}{p^k}, \frac{\sigma(m^2)}{m^2})$ lying in the region

$1 < \frac{\sigma(p^k)}{p^k} < \frac{5}{4}, \frac{8}{5} < \frac{\sigma(m^2)}{m^2} < 2,$ and

$\frac{57}{20} < \frac{\sigma(p^k)}{p^k} + \frac{\sigma(m^2)}{m^2} < 3,$ and vice-versa.
Proof: (Note that this is actually a refutation of the conjecture.) First, we note that the equation \( \frac{\sigma(x)}{x} = \frac{\sigma(p^k)}{p^k} \) has the sole solution \( x = p^k \) since prime powers are solitary. Next, let \( X = \frac{\sigma(p^k)}{p^k} \) and \( Y = \frac{\sigma(m^2)}{m^2} \). It is straightforward to observe that, since the abundancy index is an arithmetic function, then for each \( N = p^km^2 \) an OPN (with \( N > 10^{360} \)), there corresponds exactly one ordered pair of rational numbers \((X, Y)\) lying in the hyperbolic arc \( XY = 2 \) bounded as follows: 1 < \( X < 1.25 \), 1.6 < \( Y < 2 \), and 2.85 < \( X + Y < 3 \). (Note that these bounds are the same ones obtained in Lemma [II.1] and Lemma [II.2].)

We now disprove the backward direction of the conjecture. We do this by showing that the mapping \( X = \frac{\sigma(p^k)}{p^k} \) and \( Y = \frac{\sigma(m^2)}{m^2} \) is not surjective nor injective in the specified region.

\((X, Y)\) is not surjective. We prove this claim by producing a rational point \((X_0, Y_0)\) lying in the specified region, and which satisfies \( X_0 = \frac{\sigma(pq)}{pq} \) where \( p \) and \( q \) are primes satisfying 5 < \( p < q \). Notice that

\[
1 < X_0 = \left( \frac{p+1}{p} \right) \left( \frac{q+1}{q} \right) = \left( 1 + \frac{1}{p} \right) \left( 1 + \frac{1}{q} \right) \leq \frac{812}{77} = \frac{96}{77}
\]

where \( \frac{96}{77} < 1.2468 < 1.25 \). Now, by setting \( Y_0 = \frac{2}{X_0} \), the other two inequalities for \( Y_0 \) and \( X_0 + Y_0 \) would follow. Thus, we now have a rational point \((X_0, Y_0)\) in the specified region, and which, by a brief inspection, satisfies \( X_0 \neq \frac{\sigma(p^k)}{p^k} \) for all primes \( p \) and positive integers \( k \) (since prime powers are solitary). Consequently, the mapping defined in the backward direction of the conjecture is not surjective.

Remark. Since the mapping is not onto, there are rational points in the specified region which do not correspond to any OPN.

\((X, Y)\) is not injective. It suffices to construct two distinct OPNs \( N_1 = p_1^{k_1}m_1^2 \) and \( N_2 = p_2^{k_2}m_2^2 \) that correspond to the same rational point \((X, Y)\). Since it cannot be the case that \( p_1^{k_1} \neq p_2^{k_2}, m_1^2 \neq m_2^2 \), we consider the scenario \( p_1^{k_1} = p_2^{k_2}, m_1^2 \neq m_2^2 \). Thus, we want to produce a pair \((m_1, m_2)\) satisfying \( \frac{\sigma(m_1^2)}{m_1^2} = \frac{\sigma(m_2^2)}{m_2^2} \). (A computer check by a foreign professor using Maple produced no examples for this equation in the range \( 1 \leq m_1 < m_2 \leq 300000 \). But then again, in pure mathematics, absence of evidence is not evidence of absence.) Now, from the inequalities \( p^k < m^2 \) and \( N = p^km^2 > 10^{300} \), we get \( m_2^2 > 10^{150} \). A nonconstructive approach to finding a solution to \( \frac{\sigma(m_1^2)}{m_1^2} = \frac{\sigma(m_2^2)}{m_2^2} \) would then be to consider \( 10^{150} < m_1^2 < m_2^2 \) and Erdos’ result that “The number of solutions of \( \frac{\sigma(a)}{a} = \frac{\sigma(b)}{b} \) satisfying \( a < b \leq x \) equals \( Cx + o(x) \) where \( C \geq \frac{8}{147} \)” ([3], [1]) (Note that C here is the same as the (natural) density of friendly integers.) Given Erdos’ result then, this means that eventually, as \( m_2 \to \infty \), there will be at least \( 10^{150} \frac{8}{147} \) solutions \( (m_1, m_2) \) to \( \frac{\sigma(m_1^2)}{m_1^2} = \frac{\sigma(m_2^2)}{m_2^2} \), a number which is obviously greater than 1. This finding, though nonconstructive, still proves that the mapping defined in the backward direction of the conjecture is not injective.

Next, we show how the components \( p^k \) and \( m^2 \) of an OPN are related.

Theorem III.1. \( p^k < \frac{2}{3}m^2 \)

Proof: This theorem follows from the inequalities \( \rho_1 > 1 \) and \( \rho_2 \leq \frac{2}{3} \).

A somewhat weaker result than Theorem III.1 is the following:

Theorem III.2. \( m^2 - p^k \geq 8 \)

Proof: From Theorem III.1 we know that \( m^2 - p^k > \frac{2}{3}m^2 \). So in particular, we are sure that \( m^2 - p^k > 0 \). But m odd implies that \( m^2 \equiv 1 \pmod{4} \), and we also know that \( p^k \equiv 0 \pmod{4} \). Thus, \( m^2 - p^k \equiv 0 \pmod{4} \), which is equivalent to saying that \( 4|(m^2 - p^k) \). Since \( m^2 - p^k > 0 \), then this implies that \( m^2 - p^k \geq 4 \). Suppose \( m^2 - p^k = 4 \).

Then \( p^k = m^2 - 4 = (m+2)(m-2) \). Hence, we have the simultaneous equations \( p^{k-x} = m + 2 \) and \( p^x = m - 2 \) where \( k \geq 2x + 1 \). Consequently, we have \( p^{k-x} = p^x = 4 \), which implies that \( p^x | (p^{k-2x} - 1) = 4 \) where \( k - 2x \) is odd. Since \( (p-1)|(p^k - 1) \forall y \geq 1 \), this last equation implies that \( (p-1)/4 \). Likewise, the congruence \( p \equiv 1 \pmod{4} \) implies that \( 4|(p-1) \). These two divisibility relations imply that \( p-1 = 4 \), or \( p = 5 \). Hence, \( 5^x(5^{k-2x} - 1) = 4 \). Since 5 does not divide 4, \( x = 0 \) and thus \( 5^k - 1 = 4 \), which means that \( k = 1 \).

Therefore, \( p = 5, k = 1, x = 0 \). Consequently, \( p^{k-x} = 5^{1-0} = 5 = m + 2 \) and \( p^x = 5^0 = 1 = m - 2 \). Either way, we have \( m = 3 \).

All of these computations imply that \( N = p^km^2 = 5^3 \cdot 3^2 = 45 \) is an OPN. But this contradicts the fact that \( \frac{\sigma(N)}{N} = \frac{26}{15} < 2 \) (i.e., \( N = 45 \) is deficient).

Thus, \( m^2 - p^k \geq 8 \).

Lastly, we prove the following generalization to the inequality \( \sigma(p^k) \leq \frac{2}{3}m^2 = \frac{2N}{3p^k} \).

Theorem III.3. Let \( N = \prod_{i=1}^{l} p_i^{\alpha_i} \) be the canonical factorization of an OPN \( N \), where \( p_1 < p_2 < \ldots < p_l \) are primes,
t = \omega(N) and \alpha_i > 0 \text{ for all } i. \text{ Then } \sigma(p_i^{\alpha_i}) \leq \frac{2}{3} \frac{N}{p_i^{\alpha_i}} \text{ for all } i.

Proof: Let an OPN be given in the form \(N = p_1^{\alpha_1} M\) for a particular \(i\). Since \(p_i^{\alpha_i} || N\) and \(N\) is an OPN, then

\[
\sigma(p_i^{\alpha_i})\sigma(M) = 2p_i^{\alpha_i} M.
\]

Since \(p_i^{\alpha_i}\) and \(\sigma(p_i^{\alpha_i})\) are always relatively prime, we know that \(p_i^{\alpha_i}\sigma(M)\) and we have \(\sigma(M) = hp_i^{\alpha_i}\) for some positive integer \(h\). Assume \(h = 1\). Then \(\sigma(M) = p_i^{\alpha_i}\), forcing \(\sigma(p_i^{\alpha_i}) = 2M\). Since \(N\) is an OPN, \(p_i^{\alpha_i}\) is odd, whereupon we have an odd \(\alpha_i\) by considering parity conditions from the last equation. But this means that \(p_i^{\alpha_i}\) is the Euler’s factor of \(N\), and we have \(p_i^{\alpha_i} = p^{k}\) and \(M = m^2\). Consequently, \(\sigma(m^2) = \sigma(M) = p_i^{\alpha_i} = p^{k}\), which contradicts the fact that \(\mu_2 \geq 3\). Now suppose that \(h = 2\). Then we have the equations \(\sigma(M) = 2p_i^{\alpha_i}\) and \(\sigma(p_i^{\alpha_i}) = M\). (Note that, since \(M\) is odd, \(\alpha_i\) must be even.) Applying the \(\sigma\) function to both sides of the last equation, we get \(\sigma(\sigma(p_i^{\alpha_i})) = \sigma(M) = 2p_i^{\alpha_i}\), which means that \(p_i^{\alpha_i}\) is an odd superperfect number. But Kanold [4] showed that odd superperfect numbers must be perfect squares (no contradiction at this point, since \(\alpha_i\) is even), and Suryanarayana [5] showed in 1973 that “There is no odd perfect number of the form \(p^{2\alpha}\) (where \(p\) is prime). Thus, \(h = \frac{\sigma(M)}{p_i^{\alpha_i}} \geq 3\),

whereupon we have the result \(\sigma(p_i^{\alpha_i}) \leq \frac{2}{3} M = \frac{2}{3} \frac{N}{p_i^{\alpha_i}}\) for the chosen \(i\). Since \(i\) was arbitrary, we have proved our claim in this theorem.

The following corollary is a direct consequence of Theorem III.3

**Corollary III.1.** Let \(N\) be an OPN with \(r = \omega(N)\) distinct prime factors. Then \(N^{2-r} \leq \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{r-1}\).

**IV. CONCLUSIONS**

It is hoped that the new mathematical ideas presented in this paper would serve as a “spark plug” for future number theorists who would consider doing “serious research” on OPNs and would pave the way for the eventual resolution of the OPN Problem.

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