COMPUTATIONAL GEOMETRY IN $\text{Heis}^3$

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Abstract. Among eight possible geometric structures on three-dimensional manifolds less studied from the differential geometric point of view are those modelled on the Heisenberg group $\text{Heis}^3$. We consider the Heisenberg left-invariant metric and use some results on Levi-Civita connection and curvature tensor to present solutions of equations for geodesic lines in the Heisenberg group. Using "Mathematica" software package we also present drawings of geodesic lines and metric balls in the Heisenberg group.

1. Heisenberg group $\text{Heis}^{2n+1}$

Heisenberg group plays an important role in many branches of mathematics such as representation theory, harmonic analysis, PDEs or even quantum mechanics, where it was initially defined as a group of $3 \times 3$ matrices

$$\{m(x, y, t)\} = \left\{ \begin{pmatrix} 1 & y & t \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R} \right\}.$$ with the usual multiplication rule.

We will use the following complex definition of the Heisenberg group:

$$(1) \quad \text{Heis}^{2n+1} = \mathbb{C}^n \times \mathbb{R} = \{(z, t) : z \in \mathbb{C}, t \in \mathbb{R}\} \quad \text{with} \quad (z, t) \cdot (z', t') = (z + z', t + t' + \text{Im}(<z, z'>))$$

where $<, >$ is the usual Hermitian product in $\mathbb{C}^n$. See [A], [AX], [G], [Ma] for details and other normalizations.

The element zero $0 = (0, \ldots, 0)$ is the unit of this group structure and the inverse element for $(z, t)$ is $(z, t)^{-1} = (-z, -t)$. Let $a = (z, t)$, $b = (w, s)$ and $c = (z', t')$. The commutator of the elements $a, b \in \text{Heis}^3$ is equal to

$$[a, b] = aba^{-1}b^{-1} = (z, t) \cdot (w, s) \cdot (-z, -t) \cdot (-w, -s) = (z + w - z - w, t + s - t - s + \alpha) = (0, \alpha)$$

where $\alpha \neq 0$ in general. For example $[(1, 0), (i, 0)] = (0, 2) \neq (0, 0)$. Which shows that $\text{Heis}^3$ is not abelian. On the other hand, for any $a, b, c \in \text{Heis}^3$, their double commutator is

$$[[a, b], c] = [(0, \alpha), (z', t')] = (z' - z', \alpha + t' - \alpha - t) = (0, 0)$$

This implies that $\text{Heis}^{2n+1}$ is a nilpotent Lie group with nilpotency 2.
2. Metrics on Heisenberg group $Heis^{2n+1}$

There are various different metrics we can define on $Heis^{2n+1}$.

Let us consider $(n+1)$-dimensional complex hyperbolic space $X = H^n_{\mathbb{C}}$. Every point on the ideal boundary $p \in \partial X$ can be identified with the class of asymptotic geodesics, issuing from this point, which defines a natural fibration of the space. The dual fibration consists of horospheres, "centered" at $p$ and orthogonal to all such geodesics. A group of isometries of the space, which fix the point $p$, can be represented using Iwasawa decomposition as $Isom(X)_p = K \cdot A \cdot H$, where $K$ is a group of "rotations" around some geodesic, $A$ - group of "dilations" along geodesics, and $H$ is a group of all "translations" along the horosphere. It can be verified, that $H$ is actually isomorphic to $Heis^{2n+1}$. This way the usual complex hyperbolic metric in the hyperbolic space $H^n_{\mathbb{C}}$ naturally induces a metric on the Heisenberg group.

Another possible example of metric on $Heis^{2n+1}$ is a known Cygan’s metric, defined as:

$$\rho_c((z,t),(z',t')) = \sqrt{\|z - z'|^4 + (t - t' + Im < z, z'>)^2}$$

We can see from the definition, that when we increase the radius $\lambda$ of a ball in this metric, the ball grows linearly with $\lambda$ "horizontally", and as $\lambda^2$ in the vertical direction. Also, the ball represents a convex body (in the Euclidean sense).

Another metric on $Heis^3$ will be defined below.

3. Left-invariant metric on Heisenberg group $Heis^3$

Each point $(x,y,t) = (z,t) \in Heis^3 = \mathbb{C} \times \mathbb{R}$ can be viewed as a translation from $0$ to this point as $(x,y,t) \cdot (0,0,0) = (x,y,t)$. Then the Euclidean coordinate directions are translated to $(x,y,t) \cdot (s,0,0) = (x+s, y, t-sy)$, $(x,y,t) \cdot (0,s,0) = (x, y+s, t+sx)$, $(x,y,t) \cdot (0,0,s) = (x, y, t+s)$. Differentiating, we obtain the vector fields

$$X = (1,0,-y), \quad Y = (0,1,x), \quad T = (0,0,1)$$

which are left-invariant vector fields by construction. We define the left-invariant metric on $Heis^3$ by taking $X,Y,T$ as the orthonormal frame in each tangent space $T_{(x,y,t)} Heis^3$.

**Definition 1.** Denote by $g$ the left-invariant metric on $Heis^3$ such that vector fields $X$, $Y$ and $T$ are orthonormal ones. The corresponding scalar product we denote as usual by $(, )$.

Due to (2) the coordinate vectors are

$$\frac{\partial}{\partial x} = (X+y)T, \quad \frac{\partial}{\partial y} = (Y-x)T, \quad \frac{\partial}{\partial z} = T.$$

Since by our choice $\{X,Y,T\}$ is an orthonormal basis, we arrive at the following formula for the metric tensor of our left-invariant metric in coordinates $(x,y,z)$:

$$g = \begin{pmatrix}
1 + y^2 & -xy & y \\
-x y & 1 + x^2 & -x \\
y & -x & 1
\end{pmatrix}.$$  

The following was proved by V. Marenich [Ma].

**Proposition 1.** For the covariant derivatives of the Riemannian connection of the left-invariant metric, defined above the following is true:

$$\nabla = \begin{pmatrix}
0 & T & -Y \\
-T & 0 & X \\
-Y & X & 0
\end{pmatrix},$$

where the $(i,j)$-element in the table above equals $\nabla_{E_i} E_j$ for our basis $\{E_k, k = 1, 2, 3\} = \{X, Y, T\}$.  

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4. Geodesic lines in $Heis^3$

We can find equations of geodesics issuing from $0=(0,0,0)$ following well know results [Ma]. Let $c(t)$ be such a geodesics with a natural parameter $t$, and its vector of velocity given by

$$\dot{c}(t) = \alpha(t)X(t) + \beta(t)Y(t) + \gamma(t)T.$$  

Then the equation of a geodesic $\nabla_{\dot{c}(t)}\dot{c}(t) \equiv 0$ and our table of covariant derivatives (3) give:

$$(\alpha'(t) + 2\gamma\beta(t))X(t) + (\beta'(t) - 2\gamma\alpha(t))Y(t) + \gamma'(t)T = 0.$$  

Thus we easily obtain the following equations for coordinates of the vector of velocity of the geodesic $c(t)$ in our left-invariant moving frame:

$$\begin{cases}
\alpha'(t) + 2\gamma\beta(t) = 0 \\
\beta'(t) - 2\gamma\alpha(t) = 0 \\
\gamma'(t) = 0
\end{cases}$$  

or

$$\begin{cases}
(\alpha(t) + \beta(t))' - 2\gamma(\alpha(t) - \beta(t)) = 0 \\
(\alpha(t) - \beta(t))' + 2\gamma(\alpha(t) + \beta(t)) = 0 \\
\gamma'(t) = 0
\end{cases}$$  

Because the parameter $t$ is natural we have

$$\alpha^2(t) + \beta^2(t) + \gamma^2 \equiv 1,$$

and we could take $\gamma(t) \equiv \gamma$ where $|\gamma| \leq 1$ is the cos of the angle between $\dot{c}(0)$ and $T$-axis. For $|\gamma| = 1$ we have

"vertical" geodesic, coinciding with "z-axis" in $Heis^3$, which is an integral line of the left-invariant vector field $T$. For $\gamma = 0$ our equations are linear. For $\gamma \neq 0$ after some easy computation one could find that:

$$\begin{cases}
\alpha(t) = r \cos(2\gamma t + \phi) \\
\beta(t) = r \sin(2\gamma t + \phi)
\end{cases}$$

where $r = \sqrt{\alpha^2 + \beta^2}$. To find equations for geodesics $c(t) = (x(t), y(t), z(t))$ issuing from $0$ we note that if

$$\dot{c}(t) = \alpha(t)X(t) + \beta(t)Y(t) + \gamma(t)T$$

and our left-invariant vector fields are

$$X = (1, 0, -y), \quad Y = (0, 1, x), \quad T = (0, 0, 1),$$

then

$$\frac{\partial}{\partial x} = X + yT, \quad \text{and} \quad \frac{\partial}{\partial y} = Y - xT.$$  

Therefore we easily have:

$$\begin{cases}
\dot{x}(t) = \alpha(t) \\
\dot{y}(t) = \beta(t) \\
\dot{z}(t) = \gamma - \alpha(t)y(t) + \beta(t)x(t)
\end{cases}$$

After some computations this gives the following equations for geodesics issuing from zero (see [Ma]):
Proposition 2. Geodesic lines issuing from zero $0$ in the Heisenberg group $\text{Heis}^3$ satisfy to the following equations:

\[
\begin{align*}
    x(t) &= \frac{r}{\gamma} (\sin(2\gamma t + \phi) - \sin(\phi)) \\
    y(t) &= \frac{r}{\gamma} (\cos(\phi) - \cos(2\gamma t + \phi)) \\
    z(t) &= \frac{1+\gamma^2}{2\gamma^2} t - \frac{1-\gamma^2}{4\gamma^2} \sin(2\gamma t)
\end{align*}
\]

for some numbers $\phi, r$ which could be defined from the initial condition $\dot{c}(0) = (r \cos(\phi), r \sin(\phi), \gamma)$; or if $\gamma = 0$, then they are "horizontal" and satisfy the following equations:

\[
\begin{align*}
    x(t) &= \alpha(0)t \\
    y(t) &= \beta(0)t \\
    z(t) &= 0
\end{align*}
\]

To find equations of geodesics issuing from an arbitrary point $(x, y, z) \in \text{Heis}^3$ it is sufficient to use left translation to this point and apply to the equation above the multiplication rule (1).

5. Computer generated pictures of geodesic lines and metric balls in $H^3$

Here we produce several images of geodesics and metric spheres in the Heisenberg group with the left invariant metric. They are surprisingly different from their analogs in the metric induced in the Heisenberg group by the metric of negative sectional curvature in the complex hyperbolic 2-space and Cygan’s metric, see [AX] and [G]. The reader may compare our images with many computer generated images in the last geometry given in Goldman’s book [G].

In Figure 1, we present the exp-image of the $\{X, T\}$-coordinate plane.

In Figure 2, we present metric balls with the center at $0$ and radii 1 and 3. This shows already a big difference with the ball shapes in the Cygan’s metric, see Section 2.

In Figure 3, we present the half of the ball of radius 5.

In Figure 4, we present the amplified singular point of the sphere of radius 5 and the amplified neighborhood of this point of the same sphere.

In Figure 5, we present the singular point of the metric sphere of radius $t = 20$.

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