Platonic polyhedra tune the 3-sphere: harmonic analysis on simplices

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Abstract
A spherical topological manifold of dimension $n - 1$ forms a prototile on its cover, the $(n - 1)$-sphere. The tiling is generated by the fixpoint-free action of the group of deck transformations. By a general theorem, this group is isomorphic to the first homotopy group. A basis for the harmonic analysis on the $(n - 1)$-sphere is given by the spherical harmonics that transform according to irreducible representations of the orthogonal group. Multiplicity and selection rules appear in the form of reduction of group representations. The deck transformations form a subgroup and so the representations of the orthogonal group can be reduced to those of this subgroup. Upon reducing to the identity representation of the subgroup, the reduced subset of spherical harmonics becomes periodic on the tiling and tunes the harmonic analysis on the $(n - 1)$-sphere to the manifold. A particular class of spherical 3-manifolds arises from the Platonic polyhedra. The harmonic analysis on the Poincaré dodecahedral 3-manifold was analyzed along these lines. For comparison we construct here the harmonic analysis on simplicial spherical manifolds of dimension $n = 1, 2, 3$. Harmonic analysis can be applied to the cosmic microwave background observed in astrophysics. Selection rules found in this analysis can detect the multiple connectivity of spherical 3-manifolds on the space part of cosmic space–time.

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1. Introduction
Viewed on its universal cover $S^{n-1}$, a spherical topological manifold $\mathcal{M}$ is the prototile of a tiling generated by the corresponding deck transformations. By a general theorem in topology given by Seifert and Threlfall [11] (pp 195–8), the group deck($\mathcal{M}$) of deck transformations is isomorphic to the first homotopy group $\pi_1(\mathcal{M})$ and hence is a topological invariant.

A particular class of spherical 3-manifolds arises from the five Platonic polyhedra. Everitt [2] discusses their topology and gives a graphical algorithm for their homotopy groups. In table 1.1, we list the five polyhedra and from [2] give the known order of their homotopy group.

Table 1.1. The five 3-manifolds arising from the Platonic polyhedra, the orders $|\pi_1(\mathcal{M})|$ of their first homotopy groups, and the volume fraction $\text{frac}(\mathcal{M}) = |\pi_1(\mathcal{M})|^{-1}$ of the prototile wrt the volume of the 3-sphere.

| $\mathcal{M}$ | tetrahedron | cube | octahedron | icosahedron | dodecahedron |
|----------------|-------------|-----|-----------|-------------|-------------|
| $|\pi_1(\mathcal{M})|$ | 5           | 8   | 8         | 120         |             |
| $\text{frac}(\mathcal{M})$ | 0.2         | 0.125 | 0.125     | 0.0083      |             |

Since the groups deck($\mathcal{M}$) and $\pi_1(\mathcal{M})$ are isomorphic and deck($\mathcal{M}$) acts fixpoint-free on $S^3$, the volume fraction of $\mathcal{M}$ taken as prototile on $S^3$ is equal to $|\pi_1(\mathcal{M})|^{-1}$. We see that the tetrahedron and the dodecahedron display extremal values of this fraction.

For general notions of topology we refer to [11, 12]. The harmonic analysis on these manifolds can be started from $S^{n-1}$. There its basis is the complete, orthonormal set $\{Y_{\lambda}\}$ of spherical harmonics, the square integrable eigenmodes of $S^{n-1}$. To pass to a 3-manifold $\mathcal{M}$ universally covered by $S^{n-1}$, consider the maximal subset $\{Y_{\lambda}^0\}$ of this basis periodic with respect to deck transformations. Due to the periodicity, it can be restricted to the prototile $\mathcal{M}$ and forms its eigenmodes. These periodic eigenmodes tune the sphere $S^{n-1}$ to the topology of $\mathcal{M}$.

To analyze in detail the periodic eigenmodes we turn to groups and their representations. Under the group $O(n, R)$ of isometries of $S^{n-1}$, the spherical harmonics $Y_{\lambda}^k$ transform according to a set $D^0_k$ of irreducible representations. The periodic subset $Y_{\lambda}^0$ transforms according to the identity representation $D^0$ of the group deck($\mathcal{M}$) on its universal cover $S^{n-1}$.
of deck transformations. In terms of the group/subgroup pair \( O(n, R) > \text{deck}(\mathcal{M}) \), we require the reduction of the irreducible representations \( D^s > D^0 \). Clearly, not all representations \( D^s \) will reduce to the representation \( D^0 \). The non-occurrence provides a selection rule on eigenmodes of \( S^{n-1} \) which is another mark for the topology of \( \mathcal{M} \).

Motivated by physics, harmonic analysis on topological 3-manifolds has been invoked in cosmological models of the space part of space–time [8, 9]. A direct experimental access to the topology from the autocorrelation of the cosmic matter distribution is difficult. As an alternative, the rich data from fluctuations of the cosmic microwave background (CMB) radiation were examined by harmonic analysis. It is hoped to find in this way the characteristic selection rules and tuning for a specific nontrivial topology, distinct from the standard simply connected one. Of the Platonic 3-manifolds, the Poincaré dodecahedral manifold of minimal volume fraction and its eigenmodes have found particular attention. Representation theory was applied to the harmonic analysis on Poincaré’s dodecahedral 3-manifold in [6]. A comparative study of the harmonic analysis, tuned to different topological 3-manifolds, can provide clues for future applications.

To initiate such a comparative study, we turn here to simplicial manifolds on \( S^{(n-1)} \), \( n-1 = 1, 2, 3 \). For illustration of the group and representation theory we start in sections 2 and 3 with the cases \( n-1 = 1, 2 \). Section 4 deals with the Platonic tetrahedral 3-manifold.

A regular simplex with \((n+1)\) vertices can be centrally projected to \( S^{(n-1)} \) to yield a tiling into \((n+1)\) spherical simplices. The simplical tiling can be generated by the fixpoint-free action of the cyclic group \( C_{n+1} \) acting as deck(\( S_0(n-1) \)) on a prototile. In figure 1, we illustrate symbolically the tilings and simplicial manifolds for \( n-1 = 1, 2, 3 \).

We require the identity representation \( D^0 \) of \( C_{n+1} \) in the reduction of representations of the groups

\[
O(n, R) > C_{n+1}. \tag{2}
\]

Working with this pair of groups will display the difference in the topology between \( S^{(n-1)} \) and \( S_0(n-1) \) as part of the harmonic analysis.

Now we note that the original n-simplex, of which \( S_0(n-1) \) forms a spherical face, has the symmetry group \( S(n+1) \), the symmetric group on \((n+1)\) variables. The representation theory of \( S(n+1) \) is well known [10] (pp 38–9), [3] (pp 214–31), and we shall make full use of it.

\( S(n+1) \) is a Coxeter group generated by reflections, see [4]. It has the Coxeter diagram with \( n \) nodes

\[
S(n+1) = o \circ o \cdots o. \tag{3}
\]

Each node of this diagram, [4] (pp 31–3), describes the generator of a Weyl reflection \( W_a \) determined by a Weyl or reflection vector \( a \) in the Euclidean space \( E^n \) embedding \( S^{(n-1)} \), with \( W_a \) the involutive map

\[
x \in E^n : (W_a \times x) \rightarrow x - 2 \frac{(x, a)}{(a, a)} a. \tag{4}
\]

Moreover, the diagram equation (3) implies definite relations and scalar products between the various Weyl reflection vectors. In section 4.3, we shall take \( S(5) \) as a Coxeter group and describe its Weyl reflection vectors.

Since \( C_{n+1} \) is generated by the cyclic permutation \((1, 2, \ldots, n+1) \in S(n+1) \), we can refine the group/subgroup pair equation (2) as

\[
O(n, R) > S(n+1) > C_{n+1}. \tag{5}
\]

The group \( S(n+1) \) here is taken as a subgroup of \( O(n, R) \) via the orthogonal irreducible representation \( D^{[n]} \) of partition \( f = [n] \) since (i) the Weyl reflections of the defining representation for the Coxeter group equation (3) are orthogonal transformations and (ii) the irreducible orthogonal representations of \( S(n+1) \) are characterized by Young diagrams \( f \). The orthogonal irreducible representation \( D^f \), \( f = [n] \) has dimension \( n \) and is equivalent to the defining representation of the Coxeter group.

The cyclic subgroup \( C_{n+1} \leq S(n+1) \) in equation (5) is generated by the product of all the Weyl reflection generators of the Coxeter group. This product is termed the Coxeter element, see [4] (pp 74, 174).

An advantage in using the scheme equation (5) is the fact that the subgroup \( S(n) < S(n+1) \) permuting the vertices \( 1, 2, \ldots, n \) naturally appears as a symmetry subgroup acting on the spherical simplex \( S_0(n) \). Borrowing the terminology from space groups in \( E^n \), we call \( S(n) \) the point group of \( S_0(n) \).

The reduction of irreducible representations for \( O(n, R) > S(n+1) \) was studied in [5] in full detail for \( n = 2, 3 \). This allows us to work out the harmonic analysis for these cases. We apply the representation theory of groups, following [5], in the following sections to the harmonic analysis on the simplicial manifolds for \( n-1 = 1, 2, 3 \). By a first step in the reduction of representations according to equation (5) we find multiplicities and selection rules for the representations of \( O(n, R) \) and \( S(n+1) \) which subduce to the identity representation of \( C_{n+1} \). This reduction we handle by character technique.

In the second step, we characterize and construct an explicit set of orthonormal eigenmodes which span a unitary linear space of functions \( L^2 \) for the harmonic analysis on the simplicial topological manifolds. The measure \( d\mu \) on \( L^2 \) in our analysis is taken over from the universal cover \( S^{(n-1)} \). There are two alternative but equivalent approaches differing in the choice of the domain:

(I) A given function \( \mathcal{F} \) with domain \( S_0(n-1) \) can be extended to a \( C_{n+1} \)-periodic function on the universal cover \( S^{(n-1)} \) and then analyzed exclusively in terms of a \( C_{n+1} \)-periodic basis.

(II) By \( C_{n+1} \)-periodicity, this analysis can be collapsed entirely to the domain \( S_0(n-1) \), which forms a spherical tile of \( S^{(n-1)} \) of volume \( \text{vol}(S^{(n-1)})/(n+1) \), with the range of integration restricted to the topological manifold as domain. The expansion coefficients of a given function \( \mathcal{F} \) with domain \( S_0(n) \) are found from the scalar products on \( L^2 \) of \( \mathcal{F} \) with the basis functions.

We focus on the choice (I) since the modification to (II) is straightforward. The main results of the following three sections are given in tables after each section.
2. The 1-simplex $S_0(1)$ on the circle $S^1$

As the simplest paradigm we treat here the spherical simplex $S_0(1)$, with universal cover the unit circle $S^1$ according to figure 1, left.

2.1. The symmetric group $S(3)$

Using orthogonal coordinates in the Euclidean plane $E^2$, the symmetry group $S(3)$ of the regular triangle is generated by two Weyl reflections with matrices

$$
(1, 2) : \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (2, 3) : \begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}.
$$

These matrices generate Young’s orthogonal representation $D^{[2]}$ with basis vectors corresponding to the two Young tableaux

$$
\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}.
$$

The two other irreducible representations of $S(3)$ are

$$
D^{[3]} : D^{[3]}(1, 2) = D^{[3]}(2, 3) = 1, \quad D^{[11]} : D^{[11]}(1, 2) = D^{[11]}(2, 3) = -1.
$$

The character table of $S(3)$ in terms of classes, denoted by the cycle structures $k = (1)^3, (2)(1), (3)$ is given as table 2.1.

2.2. The cyclic group $C_3$

The cyclic group as a subgroup $C_3 < S(3)$ is given in cycle notation by

$$
C_3 = \{(1, 2, 3), (1, 2, 3)^2 = (3, 2, 1), (1, 2, 3)^3 = e\}.
$$

In terms of the complex number $\lambda_3 = \exp(2\pi i/3)$, the three irreducible representations $D^\alpha, \alpha = 0, 1, 2$ of $C_3$ coincide with their characters and are given for the group elements in table 2.2.

2.3. The reduction $S(3) > C_3$

The multiplicity $m(f, \alpha)$ of the subgroup irreducible representation $D^\alpha$ in the group irreducible representation $D^f$ for $H < G$ is given from character technique by

$$
m(f, \alpha) = \frac{1}{|H|} \sum_{k \in H} n(k) \chi^f(k)\tilde{\chi}^\alpha(k). \quad (10)
$$

Here, $n(k)$ is the number of elements in class $k$ of the subgroup $H$. We are interested only in the identity representation $D^0$ of $H = C_3$, collect the characters $\chi^f(k)$ of the group elements equation (9) in the irreducible representations of $S(3)$ and $\chi^0(k) = D^0(k) = 1$, and compute the multiplicities $m(f, 0)$ from equation (10) with $|C_3| = 3, n(k) = 1$ in table 2.3.

As a result, the identity representation $D^0$ of $C_3$ is contained once in the irreducible representations $D^{[3]}, D^{[11]}$ of $S(3)$ but not in $D^{[2]}$.

2.4. The reduction $O(2, R) > S(3)$

Now we turn to the group/subgroup pair $O(2) > S(3)$. The harmonic analysis on the circle $S^1$ is given by the complex Fourier series, spanned by the complete orthonormal set of functions of $z = \exp(i\phi), 0 \leq \phi < 2\pi$,

$$
Y_m(z) = \frac{1}{\sqrt{2\pi}} e^{im\phi}, \quad m = 0, \pm 1, \ldots, \quad (11)
$$

$$
\int_0^{2\pi} d\phi Y_p^*(z)Y_q(z) = \delta_{pq},
$$

$$
T_{(1, 2)}Y_m(z) = (-1)^m Y_{-m}.
$$

We include the reflection $(1, 2) : \phi \to \pi - \phi$ by introducing the new basis

$$
Y_0, Y_{m, \epsilon} = \sqrt{\frac{1}{2}} \left[ Y_m + \epsilon(-1)^m Y_{-m} \right], \quad \epsilon = \pm 1, \quad m = 1, 2, \ldots, \quad (12)
$$
2.4 determines the harmonic analysis on O(2) to yield the full reduction of representations for the groups equation (5) in table 2.4.

Table 2.4 determines the harmonic analysis on S0(1) carried out on S4: any function F with domain S0(1) has a unique C3-periodic extension to S3 and can be expanded into C3-periodic basis functions. The alternative would be to restrict the integration in equation (11) to the sector S0 of angular range 2\pi/3 and to reduce the volume from 2\pi to 2\pi/3.

The C3-periodic basis is obtained from the irreducible representations of O(2, R) by restriction to the values \nu = 0, m \equiv 0 \mod 3 and \epsilon = \pm 1. This restricts the irreducible representations of S3 to f = [3], [111]. Finally, the representation of the point symmetry group S(2) of the spherical simplex S0(1) is given in terms of partitions f by

\begin{align}
\nu : m \equiv \nu \mod 3, \quad \nu = 0, 1, 2, \quad (13) \end{align}

Then the results from [5] (pp 255–7) can be combined with those from table 2.3 to yield the full reduction of representations for the groups equation (5) in table 2.4.

2.5. Tables for O(2, R) > S(3) > C3

Table 2.1. Characters for the three partitions f and classes k in cycle notation of the symmetric group S3.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
k & (1) & (2)(1) & (3) \\
\hline
[3] & 1 & 1 & 1 \\
[21] & 2 & 0 & -1 \\
[111] & 1 & -1 & 1 \\
\hline
\end{tabular}
\caption{Table 2.1. Characters for the three partitions f and classes k in cycle notation of the symmetric group S3.}
\end{table}

Table 2.2. Characters for the three irreducible representations of C3 with \lambda_3 = \exp(2\pi i/3).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\chi & e & (1, 2, 3) & (3, 2, 1) \\
\hline
D^0 & 1 & 1 & 1 \\
D^1 & 1 & \lambda_3 & \lambda_3^2 \\
D^2 & 1 & \lambda_3^2 & \lambda_3 \\
\hline
\end{tabular}
\caption{Table 2.2. Characters for the three irreducible representations of C3 with \lambda_3 = \exp(2\pi i/3).}
\end{table}

Table 2.3. Characters and multiplicity m(f, 0) in the reduction of representations S(3) > C3.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
k & e & (1, 2, 3) & (3, 2, 1) & m(f, 0) \\
\hline
\chi^{[3]}(k) & 1 & 1 & 1 & 1 \\
\chi^{[21]}(k) & 2 & -1 & -1 & 0 \\
\chi^{[111]}(k) & 1 & 1 & 1 & 1 \\
\chi^{[0]}(k) & 1 & 1 & 1 & 1 \\
\hline
\end{tabular}
\caption{Table 2.3. Characters and multiplicity m(f, 0) in the reduction of representations S(3) > C3.}
\end{table}

The 3-simplex with vertices 1, 2, 3 and 4 can be centrally projected to the sphere S2. The spherical triangle spanned by vertices 1, 2, 3 on S2 is taken as the simplicial topological manifold S0(2).

3. The 2-simplex S0(2) on the sphere S2

The 2-faces of the regular 3-simplex centrally projected to the sphere S2, figure 2, tiles it into four spherical triangles. The tiling has the symmetry S(4) with the Coxeter diagram

\begin{align}
S(4) : o - o - o. \quad (19) \end{align}

The triangle obtained by dropping the vertex labelled 4 we take as the simplicial topological manifold S0(2) seen on its universal cover S2. The generators (1, 2), (2, 3), (3, 4) of S(4) act as three reflection planes of the tetrahedron. The first two of them map S0(2) into itself and generate the point group S(3) of the simplicial manifold.

3.1. The group S(4)

The characters of the irreducible representations of S(4) are given from [3] (p 187) in table 3.1.

3.2. The cyclic group C4

The cyclic group C4 is generated as

\begin{align}
C_4 : \left\{ (1, 2, 3, 4), (1, 2, 3, 4)^2 = (1, 3)(2, 4), (1, 2, 3, 4)^3 = (4, 3, 2, 1), (1, 2, 3, 4)^4 = e \right\}. \quad (20) \end{align}

From these expressions it can be seen that the elements of C4 generate all the cosets of the subgroup S(3) < S(4). Hence, their actions produce the full tiling of S2 by triangles, are
fixpoint free and so fulfill all the properties required for deck($S_0(2)$).

With $\lambda_4 = \exp(2\pi i / 4) = i$, the four 1-dimensional representations of $C_4$ are generated by

\[
D^0(1, 2, 3, 4) = 1, \quad D^1(1, 2, 3, 4) = i, \\
D^2(1, 2, 3, 4) = i^2 = -1, \quad D^3(1, 2, 3, 4) = i^3 = -i,
\]

(21)

The characters of $C_4$ are given in table 3.2.

3.3. The reduction $S(4) > C_4$

We require the reduction of the representations $f$ of $S(4)$ to the identity representation $D^0$ of $C_4$. For this purpose, we write the characters $\chi^f$ of these representations and $\chi^0 = D^0$ for the classes $k$ of $C_4$ in the form of table 3.3 and compute the multiplicity $m(f, 0)$.

As a result, the identity representation $D^0$ of $C_4$ is contained once in the representations $f = [4], [22], [211]$ but not in $f = [31], [1111]$. Since the representations $f = [22], [211]$ are 2- and 3-dimensional, we must still find the 1-dimensional subspaces for the representation $D^0(C_4)$. This will be done by projection in subsections 3.5.1 and 3.5.2.

3.4. The reduction $O(3, R) > S(4)$

Next, we turn to the harmonic analysis of $O(3, R)$ acting on the sphere $S^2$. $O(3)$ is the direct product $SO(3, R) \times \mathcal{J}$, where $\mathcal{J}$ is the group generated by parity. Following [5], we characterize the irreducible representations by $(l, \kappa)$ where $l = 0, 1, \ldots$ is the integer angular momentum and $\kappa = \pm 1$ a parity label. The basis of the irreducible representation $D^l(SO(3, R))$ is spanned in spherical polar coordinates $\theta, \phi$ by the spherical harmonics, [1] (pp 19–25),

\[
Y_{lm}(\theta, \phi), \quad -l \leq m \leq l,
\]

(22)

which under the parity operation $P$ transform as

\[
PY_{lm} = (-1)^l Y_{lm}, \quad -l \leq m \leq l.
\]

(23)

The parity therefore is

\[
\kappa = (-1)^l.
\]

(24)

In table 3.4, we reproduce from [5] (pp 259–60) the decomposition of irreducible representations for $(O(3, R) \times \mathcal{J}) > S(4)$ and combine it with the results from table 3.3. Because of equation (24), parity $\kappa$ and $l$ are correlated.

This table can be extended by use of character technique given in [5].

Counting the total number of states up to $l = 4$ from table 3.4 one finds 25 basis states for the group $O(3, R)$. Only seven of them contribute to the $C_4$-periodic subset.

3.5. The explicit reduction $O(3, R) > S(4) > C_4$

For higher representations $(l, \kappa)$ the multiplicity $m((l, \kappa), f)$ of the partition $f$ will take values larger than 1. In [5], we devised a way that allows for a complete orthogonal basis in this case. The idea is to introduce an additional hermitian operator, termed a generalized Casimir operator, as a polynomial in the components of the angular momentum operator, which for fixed $(l, \kappa)$ by distinct eigenvalues separates repeated partitions $f$. We refer to [5] (pp 263–66) for the details.

We now determine the 1-dimensional subspaces for the partitions $f = [22], [211]$ belonging to the representation $D^0$ of $C_4$.

3.5.1. The partition $f = [211]$. Following [5], equation (6.9), we choose particular coordinates with respect to the 4-simplex figure 2,

\[
y_1 = \frac{1}{2} [x_1 + x_2 - x_3 - x_4], \\
y_2 = \frac{1}{2} [x_1 + x_3 - x_2 - x_4], \\
y_3 = \frac{1}{2} [x_1 + x_4 - x_2 - x_3].
\]

(25)

These coordinates are convenient and stem from the analysis of the tetrahedral group $T_d$ isomorphic to $S(4)$ [5] (p 258–68).

Expressed in these coordinates, the generators of $S(4)$ have the $3 \times 3$ matrix representation

\[
D^{[211]}(1, 2) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{bmatrix},
\]

\[
D^{[211]}(2, 3) = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

\[
D^{[211]}(3, 4) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix},
\]

(26)

which generate the representation $D^{[211]}$ used in relation with the isomorphic tetrahedral group $[5]$ (p 261). We use primes to distinguish this representation from the equivalent Young representation $D^{[311]}$ [3] (pp 225–6).

The representation with the associate partition $D^{[211]}$ is obtained by multiplying each matrix for a generator in equation (26) by $(-1)$, corresponding to the representation $D^{[311]}$. Within this representation we construct the matrix of the Coxeter element

\[
D^{[211]}(1, 2, 3, 4) = \begin{bmatrix}
0 & 0 & -1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

(27)

and by taking powers the representation for all four elements $h \in C_4$. The projection operator to the representation $D^0$ of $C_4$ becomes

\[
P^{[211]}0 = \frac{1}{4} \sum_{h \in C_4} D^{[211]}(h) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

(28)

It picks the second basis function of $D^{[211]}$ as the one that transforms according to $D^0$.

Alternatively, we describe the same representation $f = [211]$ in terms of the standard Young–Yamanouchi basis as given by Hamermesh [3] (pp 224–6). In contrast to the primed
In terms of the canonical group/subgroup sequence $S(4) > S(3) > S(2)$ underlying the Young representation, the three Young basis vectors in equation (29) correspond to the partitions $[21], [21], [111]$ of $S(3)$ in that order. We find the matrix representations of the reflection generators and of the generator of $C_4$ as

$$D^{[211]}(1, 2) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D^{[211]}(2, 3) := \begin{bmatrix} -\frac{i}{2} & \frac{i}{2} \sqrt{3} & 0 \\ \frac{i}{2} \sqrt{3} & \frac{i}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$D^{[211]}(3, 4) := \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \sqrt{2} \\ 0 & \frac{1}{2} \sqrt{2} & \frac{1}{2} \\ -\frac{1}{2} \sqrt{3} & \frac{1}{2} & \frac{1}{2} \sqrt{3} \end{bmatrix},$$

$$D^{[211]}(1, 2, 3, 4) := \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \sqrt{3} & \frac{1}{2} \sqrt{2} \\ \frac{i}{2} \sqrt{3} & \frac{i}{2} & -\frac{1}{2} \sqrt{2} \\ 0 & \frac{1}{2} \sqrt{2} & \frac{1}{2} \sqrt{3} \end{bmatrix}. $$

We can find the linear combination of the basis states belonging to $D_0(C_4)$ by constructing the eigenvector $\psi^{[211]}_0$ of the representation matrix $D^{[211]}(1, 2, 3, 4)$ with eigenvalue 1. This eigenvector is found in terms of the Young tableau labels equation (29) as

$$\psi^{[211]}_0 = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right).$$

From the linear combination of the basis states with the coefficients equation (31) we can draw a surprising conclusion. The Young representation basis equation (29) displays explicitly the subpart of $S(3)$ as part of the Young diagram occupied by the numbers 1, 2 and 3. The basis vector equation (31) of $D_0(C_4)$ in the Young representation $[211]$ of $S(4)$ is a generic mixture of the two inequivalent representations $[21], [111]$ of the group $S(3)$. Although we interpreted $S(3)$ as the point group of the simplicial manifold, the reduction to $D_0$ does not preserve a single representation of this subgroup. The cyclic group $C_4$ still provides four coset generators for the subgroup $S(3) < S(4)$, and so $S(4)$ is generated from all the products of elements from its subgroups $S(3), C_4$. Moreover, our result equation (31) shows that the use of the Young representation may not be optimal for the construction. Comparison of equation (31) with the simpler result equation (28) favors the primed tetrahedral form equation (26) of the representation.

3.5.2. The partition $f = [22]$. The second representation $D^{[22]}$ we treat in the Young orthogonal representation $[3]$ (p 226) with the Young tables

$$1 : \left[ \begin{array}{c} 1 \\ 3 \\ 2 \end{array} \right], \quad 2 : \left[ \begin{array}{c} 1 \\ 3 \\ 4 \end{array} \right], \quad 3 : \left[ \begin{array}{c} 1 \\ 2 \\ 4 \end{array} \right].$$

and with generators

$$D^{[22]}(1, 2) := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad D^{[22]}(2, 3) := \begin{bmatrix} -\frac{i}{2} & -\frac{i}{2} \sqrt{3} \\ \frac{i}{2} \sqrt{3} & \frac{i}{2} \end{bmatrix},$$

$$D^{[22]}(3, 4) := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad D^{[22]}(1, 2, 3, 4) := \begin{bmatrix} -\frac{i}{2} & \frac{i}{2} \sqrt{3} \\ \frac{i}{2} \sqrt{3} & \frac{i}{2} \end{bmatrix}. $$

By use of equation (20) we find from the second line of equation (33) the representation of the Coxeter element. The projection operator for the representation $D^0(C_4)$ becomes the matrix

$$p^{[22]}_0 = \frac{1}{4} \sum_{h \in C_4} D^{[22]}(h) = \frac{1}{4} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

and determines a unique linear combination in the Young tables,

$$\psi^{[22]}_0 = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

as the basis for $D^0(C_4)$. The representation of the point group $S(3)$ of the simplicial manifold is fixed by the partition $f = [21]$. This completes the group/subgroup basis construction of the harmonic analysis for a function $f$ on the simplicial topological manifold $S_0(2)$ when $C_4$ periodically extended to $S^2$. The new feature here is the appearance of multiplicity problems that require projections and the construction of appropriate operators. The $C_4$ -periodic basis for the harmonic analysis of $S_0(2)$ on the sphere $S^2$ is characterized by labels $(l, \kappa, f)$ of the irreducible representations of $O(3, R), S(4)$, with $f$ restricted to allowed values.

3.6. Tables for $O(3, R) > S(4) > C_4$

| Table 3.1. Characters for the group $S(4)$. |
|---------------------------------------------|
| $\chi^x(k)$ | (1)$^x$ | (4)$^y$ | (3)(1)$^z$ | (2)$^r$ | (2)(1)$^z$ |
|-----------------|-------------|-------------|-------------|-------------|-------------|
| $[4]$            | 1           | 1           | 1           | 1           | 1           |
| $[31]$           | 3           | -1          | 0           | -1          | 1           |
| $[22]$           | 2           | 0           | -1          | 2           | 0           |
| $[211]$          | 3           | 1           | 0           | -1          | 1           |
| $[111]$          | 1           | -1          | 1           | 1           | -1          |

| Table 3.2. Characters for the cyclic group $C_4$. |
|---------------------------------------------|
| $\chi^x$ | e | (1, 2, 3, 4)$^x$ | (1, 3)(2, 4)$^y$ | (4, 3, 2, 1)$^z$ |
|-----------------|-------------|-------------|-------------|-------------|
| $D^0$           | 1           | 1           | 1           | 1           |
| $D^1$           | 1           | i           | -1          | -i          |
| $D^2$           | 1           | -i          | 1           | -1          |
| $D^3$           | 1           | -i          | -1          | i           |

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Table 3.3. Characters and multiplicities in the reduction $S(4) \rightarrow C_4$.  

| $k$ | $e$ | $(1, 2, 3, 4)$ | $(1, 3)(2, 4)$ | $(4, 3, 2, 1)$ | $m(f, 0)$ |
|-----|-----|----------------|----------------|----------------|-----------|
| $\chi^{[1]}(k)$ | 1   | 1             | 1             | 1             | 1         |
| $\chi^{[3]}(k)$ | 3   | -1            | -1            | -1            | 0         |
| $\chi^{[2]}(k)$ | 2   | 0             | 0             | 2             | 1         |
| $\chi^{[1]}(k)$ | 3   | 1             | 1             | -1            | 1         |
| $\chi^{[1]}(k)$ | 1   | -1            | -1            | 1             | 0         |
| $\chi^{[0]}(k)$ | 1   | 1             | 1             | 1             | 1         |

(38)

Table 3.4. The multiplicities $m(l, \kappa, f)$ of representations $D^f$ of $S(4)$ and $m(l, \kappa, 0)$ of $C_4$-periodic eigenmodes in the general reduction $O(3) \rightarrow S(4) \rightarrow C_4$ for $(l, \kappa, l) \leq 4$. Because of equation (24), only the pairs $(l, \kappa)$ with $\kappa = (-1)^l$ contribute to the $C_4$-periodic modes.

| $(l, \kappa, f)$ | $m(l, \kappa, 0)$ | $m(l, \kappa, 4)$ | $m(l, \kappa, 22)$ | $m(l, \kappa, 211)$ | $m(l, \kappa, 1111)$ |
|-----------------|------------------|------------------|------------------|------------------|------------------|
| $(0, 0)$        | 0                | 0                | 0                | 0                | 0                |
| $(1, 1)$        | 0                | 0                | 0                | 0                | 0                |
| $(2, 1)$        | 0                | 0                | 0                | 0                | 0                |
| $(3, -1)$       | 1                | 1                | 0                | 0                | 0                |
| $(4, 1)$        | 1                | 1                | 1                | 0                | 0                |

(39)

4. The tetrahedral 3-simplex $S_0(3)$ on the sphere $S^3$

We consider the 3-sphere $S^3 < E^4$ and an inscribed regular 4-simplex with its vertices enumerated as 1, 2, 3, 4 and 5. The full point symmetry of the 4-simplex is $S(5)$. Central projection of the 3-faces of this simplex to $S^3$ yields a tiling with five tetrahedral tiles. We choose the tetrahedron obtained by dropping the vertex 5 as the simplicial manifold $S_0(3)$. Its internal point symmetry group is $S(4)$. The homotopy group $\pi_1(S_0(3))$ of the Platonic tetrahedron is described by Everitt [2] by a graph algorithm. Its prime dimension 5 identifies it and the group of deck transformations as the cyclic group $C_5$. In the following subsections, we implement the group/subgroup analysis in analogy to the previous sections with the goal to characterize the harmonic analysis on $S_0(3)$.

4.1. The reduction $S(5) > C_5$

The characters of the irreducible representations of $S(5)$ for all seven classes and irreducible representations from [3] (p 276) are given in table 4.1.

We shall need the characters of table 4.1 for the reduction of the rotation group.

The cyclic group $C_5$ has the elements

$$g = (1, 2, 3, 4, 5), \quad g^2 = (1, 3, 5, 2, 4),$$

$$g^3 = (1, 4, 2, 5, 3), \quad g^4 = (1, 5, 4, 3, 2) \quad g^5 = e. \quad (40)$$

They belong to the classes (4)(1) or (5) of $S(5)$. The computation of the multiplicity $m(f, 0)$ of the identity representation $D^0(C_5)$ is straightforward and we include it in the last column of table 4.1. The representation $D^0$ is contained once in the representations $f = [5], [11111], [32], [221]$, twice in the representation $f = [311]$, but not in the representations $f = [41], f = [2111]$. Again, we must determine the corresponding subspaces for the first set of representations.

4.2. $O(4, R)$ and Weyl reflections acting on $S^3$

We shall adopt from [6] the coordinate description of the sphere $S^3$, equivalent to an element $u \in SU(2, C)$, and relate it to the vector notation $x = (x_0, x_1, x_2, x_3)$ in $E^4$ by

$$u = \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}, \quad \det(u) = z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1,$$

$$z_1 = x_0 - ix_3, \quad z_2 = -x_2 + ix_1. \quad (41)$$

We use the isomorphism $SO(4, R) \sim (SU^1(2, C) \times SU^1(2, C))/Z_2$ with the action defined as

$$\left(\begin{array}{c} u \\ g \end{array}\right) \rightarrow \left(\begin{array}{c} u \\ g \end{array}\right) \rightarrow \left(\begin{array}{c} u \\ g \end{array}\right), \quad (g_1, g_2) \times u \rightarrow (g_2)^{-1}ug_1. \quad (42)$$

Here, $Z_2$ denotes the subgroup generated by the action of $(-e, -e) \in (SU^1(2, C) \times SU^1(2, C))$ in equation (42) which preserves any point $u \in S^3$. We stress that the half-angular parameters here must be used in their full range appropriate for $SU(2, C)$. The spherical harmonics on $S^3$ are shown in [6] to be equal to the Wigner representation functions $D^j$ of $SU(2, C)$ equation (85),

$$Y_{m_1 m_2}^j(u) = D_{m_1 m_2}^j(u), \quad j = 0, 1/2, 1, 3/2, \ldots, \quad -j \leq m \leq j, \quad i = 1, 2,$$

$$\int D_{m_1 m_2}^j(u) \bar{D}_{m_1 m_2}^j(u) d\mu(u) = \delta_{j,-j} \delta_{m_1 m_2} \delta_{m_1 m_2}. \quad (43)$$

The measure $d\mu(u)$ can be expressed by three Euler angle parameters [1] (pp 62-3). We denote the irreducible representation obtained by the action of rotations and Weyl reflections on the spherical harmonics equation (43) as $D^{(j, l)}$. In the appendix, we summarize some properties of the Wigner $D^j$ functions. In terms of the complex numbers $(z_1, z_2, \bar{z}_1, \bar{z}_2)$ in equation (41), the spherical harmonics are homogeneous polynomials of degree $(2j)$, see equation (85). Under the action of rotations they transform, due to their representations properties, irreducibly with a representation $D^{(j, l)}$ given by direct product matrices,

$$(T_{g_1 g_2}) D_{m_1 m_2}^j(u) = D_{m_1 m_2}^j(g_1^{-1}ug_2) = \sum D_{m_1' m_1'}(g_1^{-1}) D_{m_2 m_2}'(g_2), \quad (44)$$

$$D_{m_1 m_2}^{(j, l)}(g_1, g_2) = \langle (j, l)(m_1, m_2) | T_{g_1 g_2} | (j, l)(m_2, m_2) \rangle = D_{m_1 m_2}^j(g_1) D_{m_2 m_2}^j(g_2), \quad (45)$$

where finally we passed for convenience to a bracket notation. The defining action of $SO(4, R)$ on $E^4$ by real $4 \times 4$ rotation matrices is included in this notation as the representation $D^{(1/2, 1/2)}$. 

7
For the character of the irreducible representation $D^{(j,j)}$ of a rotation it follows from equation (44) that it is given by

$$
\chi^{(j,j)}(T_{(g,g)}) = \text{Trace}(D^{(j)}(g^{-1})D^{(j)}(g)) = \chi^{j}(g^{-1})\chi^{j}(g),
$$

(45)
in terms of pairs of characters $\chi^{j}(g)$ of $SU(2, C)$. The dimension of $D^{(j,j)}$ from equation (45) is $(\chi^{j}(e))^2 = (2j + 1)^2$.

To determine the action of the symmetric group $S(5)$ on $S^3$ we need the action of a Weyl reflection equation (4) in the coordinate description equation (41) and at the same time must extend the action from $SO(4, R)$ to $O(4, R)$. We start with the Weyl reflection for the particular Weyl vector $a_0 = (1, 0, 0, 0)$ which acts as

$$(W_{a_0} \times (x_0, x_1, x_2, x_3)) \rightarrow (−x_0, x_1, x_2, x_3).$$

(46)

Inserting this reflection into the complex coordinates of $u$ from equation (41) we find that the image of $u$ under $W_{a_0}$ can be rewritten in matrix notation as

$$W_{a_0}(u) = −u^T.$$  

(47)

In addition, we need the representations and characters for reflections from $O(4, R)$. For $u ∈ SU(2, C)$ we need the properties equations (83) and (87). This equation and the homogeneity equation (88) of degree $2j$ allow us to rewrite the Weyl operator $T_{a_0}$ acting on a spherical harmonic equation (43) as

$$(T_{a_0}D^{j}_{m_1m_2})(u) = D^{j}_{m_1m_2}(−u^T) = (−1)^{j}D^{j′}_{m_2m_1}(q^{−1}uq) = (−1)^{j}D^{j}_{m_2m_1}(u).$$

(48)

We used the homogeneity equation (88) to extract the phase factor ($−1)^{j}$. With respect an action equation (44) of $SO(4, R)$ following the reflection $W_{a_0}(u)$, we get from equation (48)

$$((T_{(g,g′)}T_{a_0})D^{j}_{m_1m_2})(u) = (−1)^{j}((T_{(g,g′)}T_{(g′,q)})D^{j}_{m′_2m′_1})(u).$$

(49)

For the Weyl operator $T_{a_0}$ equation (48) we find the involutive and conjugation properties

$$T_{a_0}T_{(g,g′)}T_{a_0} = T_{(g,g′)},$$

(50)

$$T_{a_0}T_{(g,g′)}T_{a_0} = T_{(g,g′)},$$

The Weyl vector $a_0$ in the coordinates equation (41) corresponds to $u = e$. For a general unit Weyl vector $a$ we can choose a rotation $R ∈ SO(4, R)$ such that $a = R_a e$. Then by the general operator relation between Weyl reflections and rotations

$$W_a = W_{R_a} = RW_{a_0}R^{-1},$$

(51)

general Weyl reflection $W_a$ can be expressed in terms of the particular reflection $W_{a_0}$. It remains to construct a rotation $R$ in the terms of the coordinates equation (41) we note that

$$((e, v) × e) = v,$$

(52)

which means that the element $(e, v) ∈ (SU^j(2, C) × SU^j(2, C))$ applied to $a_0$ gives the most general point $a ∈ S^3$ and so provides the rotation $R$.

With this choice of $R$, the action of a general Weyl reflection operator on a spherical harmonic can be written by use of equation (50) as

$$T_a = T_{(e,v^{−1})}T_{a_0}T_{(e,v)} = T_{(v_0,v^{−1})}T_{a_0}.$$  

(53)

Here, the matrix $v_0$ is obtained from the Weyl vector $a$ by inserting the vector components of $x = a$ into the matrix equation (41). From the factorization equation (53) of the general Weyl reflection operator and from equation (44) we find the matrix representation of $T_a$ as

$$((j, j)(m_1′m_2′)T_{(v_0,v^{−1})}T_{a_0}((j, j)(m_2m_1))$$

$$= (−1)^{2j}((j, j)(m_1′m_2′)T_{(e,v^{−1})}T_{(q,q)}((j, j)(m_2m_1)))$$

$$= (−1)^{2j}D^{j}_{m_2m_1}(q^{−1}v_0q^{-1})D^{j}_{m_1′m_1′}(v_0q).$$

(54)

Here, we used the complex conjugation property $q = q^{-1}uq$ of $SU(2, C)$, equation (83). The trace of this representation becomes

$$\chi^{(j,j)}(T_a) = (−1)^{2j}\chi^{j}(q^{−1}v_0q^{−1})(v_0q)^T$$

$$= \chi^{j}(v_0q^{-1}q)^T$$

$$\chi^{j}(e) = 2j + 1.$$  

(55)

This result is independent of $u$ as expected from equation (53) since all Weyl operators by equation (51) are conjugate to $T_{a_0}$ and the trace is independent of conjugations.

All operators that are not pure rotations can be written with the help of equation (50) in the form

$$T = T_{(g,g′)}T_{a_0}.$$  

(56)

The matrix elements of this operator are given similar to equation (54) by

$$((−1)^{2j}((j, j)(m_1′m_2′)T_{(g,g′,g′q)}((j, j)(m_2m_1)))$$

$$= (−1)^{2j}D^{j}_{m_2m_1}(q^{−1}g′1D^{j}_{m_1′m_1′}(g′q))$$

(57)

with $q$ given in equation (83). It follows from equation (57) that the character of this operator in the representation $D^{(j,j)}$ is given by

$$\chi^{(j,j)}(T_{(g,g′)}T_{a_0}) = (−1)^{2j}\sum_{m_1,m_2}D^{j}_{m_2m_1}(q^{−1}g′1D^{j}_{m_1′m_1′}(g′q))$$

$$= \chi^{j}(g′,q′)T = \chi^{j}(g′,g′g).$$

(58)

Here, we used from equation (83) $q′ = (−q)$, which leads to the cancellation of the phase factor $−1)^{2j}$, and $q^{−1}g′1q = q^{−1}g1 = g1$.  

4.3. The representations of $S(5)$

From the Coxeter diagram of $S(5)$, we can construct in $E^4$ a set of four Weyl vectors associated with four generators. The scalar products must form, [4] (pp 108–10), the matrix

$$\langle a_j, a_k \rangle = \begin{bmatrix}
1 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 1 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 1 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & 1
\end{bmatrix}. $$

(59)
Here, the off-diagonal scalar products are determined from the Coxeter diagram by \( (a_i, a_{i+1}) = \cos(\pi/3) = \frac{1}{2} \), \( i = 1, 2, 3 \). We fulfill equation (59) by the choice of four Weyl vectors given in table 4.2.

From equation (53), we compute for each Weyl vector \( w_i, i = 1, \ldots, 4 \) the matrices \( v_i \) that appear in the reflection operator and include them in the last column of table 4.2. From the generators of \( S(5) \) we can construct representatives \( g_k \) of the seven classes \( k \) appearing in table 4.2 and express them as products of the Weyl reflection operators \( W_i := W_{a_i} \) in table 4.3.

For products of up to four Weyl reflection operators equation (53) we find with equation (50)

\[
T_a = T_{(w_i w_j)} T_{a_i}, \\
T_b = T_{(w_i w_j w_k)} T_{a_i}, \\
T_c = T_{(w_i w_j w_k w_l)} T_{a_i}, \\
T_d = T_{(w_i w_j w_k w_l w_m)} T_{a_i}.
\]

Here, products with an even number of Weyl operators are rotations, those with an odd number become a product of a rotation with \( T_{a_i} \).

### 4.4. Multiplicity in the reduction \( O(4, R) > S(5) > C_5 \)

To write explicitly the action of the class representatives from table 4.3, we must convert the corresponding product of Weyl operators for each class representative with the help of equation (50) into an explicit action of \( SU^j(2, C) \times SU^j(2, C) \) on \( u \) or on \( \text{det} u \), respectively. Once we have computed these actions, the characters in the representation \( D^{(j, j)} \) are found by use of equations (44) and (58), respectively. It turns out that for five classes \( k \in S(5) \) the characters \( \chi^{(j, j)}(k) \) are periodic with periods 2, 3, 4 and 5, respectively wrt \( (2j) \).

The results and corresponding recursion relations are given in tables 4.7 and 4.8. The real scalar product

\[
m((j, j), f) = \frac{1}{5!} \sum_{k \in S(5)} n(k) \chi^{(j, j)}(k) \chi_f(k),
\]

with \( n(k) \) the number of elements in class \( k \in S(5) \), then yields the multiplicity \( m((j, j), f) \) in the reduction \( O(4, R) > S(5) \). In table 4.9, we write down the multiplicities computed according to equation (61) for the reduction of representations in \( O(4, R) > S(5) \). The multiplicity of \( C_5 \)-periodic modes of \( O(4, R) \) is then given by

\[
m((j, j), 0) = \sum_f m((j, j), f) m(f, 0)
\]

in the last column of table 4.9.

Of the 506 harmonic polynomials for \( O(4, R) \) up to degree \( 2j = 10 \), 101 are \( C_5 \)-periodic eigenmodes of \( S_0(3) \), and so demonstrate on average the topological selection rules for the simplicial manifold \( S^0(3) \). There are other specific selection rules like the absence of \( C_5 \)-periodic eigenmodes for \( 2j = 1 \). By using equation (61) and the characters given in tables 4.7, 4.8, table 4.9 can be extended to any value \( (2j) > 10 \). We note another recursion relation for the multiplicity \( m((j, j), f) \): for all except the first two classes, the characters \( \chi^{(j, j)}(k) \) from tables 4.7 and 4.8 obey

\[
k = (3(1)^2, (2)^2(1), (3)(2), (4)(1), (5) : \chi^{(j+30, j+30)}(k) = \chi^{(j, j)}(k).
\]

Using in equation (61) the characters for the first two classes from table 4.9 and for the others the periodicity equation (63) one finds for the multiplicity the recursion relation

\[
m((j+30, j+30), f) = m((j, j), f) + (2j + 36).
\]

#### 4.5. The explicit reduction \( S(5) > C_5 \)

For the partitions \( f = [32, [221], [311] \) we now compute the \( C_5 \)-periodic states \( \psi^{(j)} \) as linear combinations of the basis states \( \phi_j \) in the Young orthogonal representation. For each representation \( D^j \) that admits \( D^0 \), they are given as the single or at most two eigenvectors of eigenvalue 1 for the Coxeter element generating \( C_5 \) in the representations spaces for those partitions which from table 4.1 reduce to \( D^0 \). We give as a vector the coefficients \( \psi_j^{(j)} \) in the linear combinations

\[
\psi^{(j)} = \sum_j \phi_j \psi_j^{(j)}.
\]

The representation matrices for the generators of \( S(5) \) and \( C_5 \) and the coefficients in equation (65) are given in tables 4.4–4.6.

#### 4.6. Tables for \( O(4, R) > S(5) > C_5 \)

| \( \chi^{(j)}(k) \) | (1) | (2)(1) | (2)(2) | (3)(1) | (2)(2) | (3)(2) | (4)(1) | (5) | \( m(f, 0) \) |
|-----------------|-----|-------|-------|--------|-------|--------|--------|-----|--------|
| \( n(k) \)      | 1   | 10    | 15    | 20     | 20    | 30     | 24     |    |        |
| [5]             | 1   | 1     | 1     | 1      | 1     | 1      | 1      |    | 1      |
| [11111]         | 1   | 1     | 1     | 1      | 1     | 1      | 1      |    | 1      |
| [41]            | 4   | 2     | 0     | 1      | 1     | 0      | 1      |    | 0      |
| [2111]          | 4   | 2     | 0     | 1      | 1     | 0      | 1      |    | 0      |
| [32]            | 5   | 1     | 1     | 1      | 1     | 0      | 1      |    | 0      |
| [2221]          | 5   | 1     | 1     | 1      | 1     | 0      | 1      |    | 0      |
| [3311]          | 6   | 0     | 0     | 0      | 0      | 1      | 2      |    | 2      |

Table 4.2. Weyl vectors \( a_i \) and matrices \( v_i \) in the Weyl operators equation (53) for the generators \( (i, i+1) \) of the group \( S(5) \).
Table 4.3. Representatives for the seven classes $k$ of $S(5)$ in terms of Weyl reflections $W_i := W_{a_i}$ and matrices $g_k, r_k, g_i g_j$, appearing in equations (45), (56) and (58).

| $k$ : | (1)$^3$ | (2)$^3(1)$ | (2)$^2(1)$ | (3)$^1(1)^2$ |
|------|---------|---------|---------|-------------|
| $g_k$ | $e$     | $(1, 2)$| $(1, 2)(3, 4)$| $(1, 2)(2, 3)$ |
|       | $W_i$   | $W_i W_j$| $W_i W_j$| $W_i W_j$ |
| $g_l$ | $[0 \frac{\sqrt{2} - i}{\sqrt{3}} \frac{i \sqrt{3}}{2} - \frac{2}{2}]$| $\frac{i \sqrt{3}}{2} - \frac{2}{2}$| $\frac{i \sqrt{3}}{2} - \frac{2}{2}$| (68) |
| $g_r$ | $[\frac{\sqrt{2} - i}{\sqrt{3}} 0 \frac{\sqrt{2} + i}{\sqrt{3}}]$| $\frac{\sqrt{2} - i}{\sqrt{3}} 0 \frac{\sqrt{2} + i}{\sqrt{3}}$| $\frac{\sqrt{2} - i}{\sqrt{3}} 0 \frac{\sqrt{2} + i}{\sqrt{3}}$| (69) |

Table 4.4. Periodic state for the partition $f = [32]$. Young tableau basis:

$$1 = [\frac{1}{3} \frac{2}{5} \frac{3}{4}], \quad 2 = [\frac{1}{2} \frac{3}{5} \frac{4}{4}], \quad 3 = [\frac{1}{3} \frac{2}{5} \frac{4}{4}], \quad 4 = [\frac{1}{2} \frac{3}{4} \frac{5}{4}], \quad 5 = [\frac{1}{3} \frac{2}{4} \frac{5}{4}].$$

Representation $D^{[32]}$ for generators from [3] (pp 227–8):

$$D^{[32]}(1, 2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad D^{[32]}(3, 4) = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D^{[32]}(2, 3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{1}{2} \end{bmatrix}, \quad D^{[32]}(4, 5) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

(71)
The matrix $D^{[32]}(4, 5)$ given in [3] (p 228) has an error and was replaced. Representation of Coxeter element of $C_5$:

$$D^{[32]}((1, 2), (3, 4), (2, 3), (3, 4), (4, 5)) =
\begin{pmatrix}
-\frac{1}{5} & 0 & -\sqrt{\frac{2}{5}} & 0 & \sqrt{\frac{2}{5}} \\
-\sqrt{\frac{2}{5}} & \frac{1}{4} & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & -\frac{1}{4} \\
\sqrt{\frac{2}{5}} & \sqrt{\frac{1}{2}} & \frac{1}{4} & -\sqrt{\frac{1}{2}} & \frac{1}{4} \\
0 & \sqrt{\frac{1}{10}} & \frac{1}{4} & -\sqrt{\frac{1}{2}} & \frac{1}{4} \\
0 & -\frac{1}{4} & \sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{2}} & -\frac{1}{4}
\end{pmatrix}.
$$

(72)

Eigenvector of Coxeter element for the single eigenvalue 1 equation (65):

$$\psi^{[32]0} = \left(\sqrt{\frac{2}{5}}, -1, -\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}, 1\right).$$

(73)

Table 4.5. Periodic state for the partition $f = [221]$. The Young tableaux of the basis are the mirror images of those for the associate partition [32] equation (70). The representation matrices can then be computed from those of $D^{[32]}$ equation (71).

Representation of the Coxeter element

$$D^{[221]}((1, 2), (2, 3), (3, 4), (4, 5), (5, 6)) =
\begin{pmatrix}
-\frac{1}{3} & 0 & \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{2}{3}} \\
-\sqrt{\frac{2}{3}} & \frac{1}{4} & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & -\frac{1}{4} \\
\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{2}} & \frac{1}{4} & -\sqrt{\frac{1}{2}} & \frac{1}{4} \\
0 & -\sqrt{\frac{1}{10}} & \frac{1}{4} & -\sqrt{\frac{1}{2}} & \frac{1}{4} \\
0 & -\frac{1}{4} & \sqrt{\frac{1}{10}} & -\sqrt{\frac{1}{2}} & -\frac{1}{4}
\end{pmatrix}.
$$

(74)

Eigenvector of the Coxeter element for the eigenvalue 1

$$\psi^{[221]0} = \left(\sqrt{\frac{2}{3}}, -1, -\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, 1\right).$$

(75)

4.7. Harmonic analysis on $S_0(3)$

We summarize the basis construction for the harmonic analysis on $S_0(3)$ in terms of $C_5$-periodic states on the sphere $S^3$.

(i) The spherical harmonics for fixed degree $2j = 0, 1, 2, \ldots$ are the Wigner $D^j(\alpha)$-functions given in equation (85).

(ii) For the reduction $O(4, R) \supset S(5)$, the multiplicity of representations $D^j$ is known from table 4.9 and computable from equation (61) with the characters $\chi^{(j)}$ for $O(4, R)$ from tables 4.7, 4.8 and $\chi^{(i)}$ for $S(5)$ given in table 4.4. The partitions $f = [41], [2111]$ are forbidden. For allowed partitions $f$, the explicit state

Table 4.7. The angles $\phi$ and characters $\chi^{(j)}(k)$ for the classes $k \in S(5)$ and the representations $D^{(j)}$ of $O(4, R)$ with $(2j) = 0, 1, 2, 3, 4, 5$. General expressions and recursion relations are given in table 4.8.

| $k$ | $\phi$ |
|-----|--------|
| $(1)^3$ | $\phi(g_i)/2 = \phi(g_i)/2 = 0$ |
| $(2)(1)^2$ | $\phi(g_i)/2 = \phi(g_i)/2 = \pi/3$ |
| $(3)(1)^2$ | $\phi(g_i)/2 = \phi(g_i)/2 = 2\pi/3$ |
| $(2)^2(1)$ | $\phi(g_i)/2 = \phi(g_i)/2 = 2\pi/3$ |
| $(2)(1)$ | $\phi(g_i)/2 = 2\pi/3$ |
| $(4)(1)$ | $\phi(g_i)/2 = 2\pi/3$ |
| $(5)$ | $\phi(g_i)/2 = 2\pi/10, \phi(g_i)/2 = 6\pi/10$ |

$$\chi^{(j)}(k, (2j)) : 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5
\begin{pmatrix}
1 & 4 & 9 & 16 & 25 & 36 \\
12 & 3 & 4 & 5 & 6 \\
1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0
\end{pmatrix}.$$
Table 4.8. General expressions and recursion relations for the characters $\chi^{(j,p)}(k)$ as functions of classes $k$ of $S(5)$. The initial values are given in table 4.7.

| $k$ | $\chi^{(j,p)}(k)$ |
|-----|------------------|
| (1)$^3$ | $\chi^{(j,p)} = (2j+1)^2$ |
| (2)(1)$^2$ | $\chi^{(j,p)} = (2j+1)$ |
| (3)(1)$^2$ | $\chi^{(j+3,2j+2)} = \chi^{(j,p)}$ |
| (2)$^2$(1) | $\chi^{(j+1,2j)} = \chi^{(j,p)}$ |
| (3)(2) | $\chi^{(j+3,2j+2)} = \chi^{(j,p)}$ |
| (4)(1) | $\chi^{(j+2,2j+2)} = \chi^{(j,p)}$ |
| (5) | $\chi^{(j+5,2j+5)} = \chi^{(j,p)}$ |

Table 4.9. Multiplicities $m((j, f), f)$ in the reduction of representations $D^{(j,p)} = \sum m((j, f), f)D^f$ from $O(4, R)$ to $S(5)$ as function of $(2j) = 0, \ldots, 10$ and of all partitions $f$. The number $m((j, f), 0)$ in the last column denotes the total number of $C_5$-periodic modes for fixed $(2j)$, $v_0(f)$ in the last row the total number for a fixed partition $f$ up to $(2j) = 10$.

| $f$ | [5] [11111] [41] [2111] [32] [221] [311] | $m((j, f), 0)$ |
|-----|---------------------------------|------------------|
| 0 | 1 | 1 |
| 1 | 1 | 1 |
| 2 | 1 | 1 |
| 3 | 1 | 1 |
| 4 | 1 | 1 |
| 5 | 1 | 1 |
| 6 | 1 | 1 |
| 7 | 1 | 1 |
| 8 | 2 | 1 |
| 9 | 2 | 1 |
| 10 | 2 | 1 |

| $v_0(f)$ | 12 | 1 | 0 | 0 | 26 | 14 | 48 | 101 |

5. Conclusion

Methods of group theory allow one to construct and analyze the harmonic analysis on topological manifolds. This is demonstrated in section 4 for the simplicial manifold $S_0(3)$. The multiplicities provide the specific selection rules for the chosen simplex topology. The symmetric group $S(5)$ plays a key role. Its representations $f = [41], [2111]$ are eliminated from the harmonic analysis. The details for the basis construction are given in section 4.7.

In general, the harmonic analysis on two different manifolds $M, M'$ covered by the sphere $S^{n-1}$ is unified by the spherical harmonics and corresponding representations. The differences between topologies appear in the form of different subgroups of deck transformations. In the harmonic analysis these involve different group/subgroup representations and reductions in $O(n, R) \supset \text{deck}(M), O(n, R) \supset \text{deck}(M')$. Intermediate subgroups as $S(n+1)$ in equation (2) can dominate the harmonic analysis on spherical manifolds. The reduction $O(n, R) \supset S(n+1), n > 2$ for simplicial manifolds may require generalized Casimir operators as exemplified in [5] for $n = 3$. Selection rules for $S(n+1) \supset C_{n+1}$ eliminate complete representations $D^f$ of the group $S(n+1)$ from the harmonic analysis on the sphere $S^{n-1}$ when restricted to the simplicial manifold.

To see the topological variety of the harmonic analysis, compare the tetrahedral Platonic 3-manifold $M$ analyzed here with the dodecahedral Platonic 3-manifold $M'$. The homotopy group of Poincaré's dodecahedral 3-manifold $M$ is, compare [11] (pp 216–8), the binary icosahedral group. It was found in [6] that the isomorphic group deck$(M')$ acts exclusively as a subgroup of $SU(2, C)^3$ from the right on the sphere $S^3$ in the coordinates equation (41), with the consequence of a degeneracy of the dodecahedral eigenmodes. The multiplicity in the reduction from $O(4, R)$ to the subset of eigenmodes for the dodecahedral 3-manifold is completely resolved by a generalized Casimir operator. The multiplicity analysis in [6, 7] shows that the lowest dodecahedral eigenmodes are of degree $(2j) = 12$.

Comparison with the present harmonic analysis for the simplicial 3-manifold demonstrates a genuine dependence of the selection rules and the spectrum of eigenmodes on the topology and on the topologically invariant subgroups.
involved. Corresponding implications can be drawn for the use of harmonic analysis in the cosmic topology of 3-space.

Appendix A. Some properties of the group $SU(2, C)$ and its representations

The group elements $u \in SU(2, C)$ have the unitary, unimodular and complex conjugation properties

$$u^\dagger = u^{-1}, \quad \det(u) = 1,$$

$$\bar{u} = q^{-1} u q, \quad q^{-1} = -q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad q^T = q^{-1}. \quad (83)$$

The irreducible representations of $SU(2, C)$ are the Wigner $D^j$ functions

$$D^j_{m_1, m_2}(u), \quad j = 0, 1/2, 1, \ldots, \quad -j \leq m_1 \leq j, \quad i = 1, 2, \quad (84)$$

which are homogeneous polynomials with real coefficients of degree $2j$ in the complex matrix elements equation (41) of $u$. They are explicitly given, [13] (pp 163–6), [1] (pp 56–67), by

$$D^j_{m_1, m_2}(z_1, z_2, \bar{z}_1, \bar{z}_2) = \left[ rac{(j + m_1)! (j - m_1)!}{(j + m_2)!(j - m_2)!} \right]^{1/2} \times \sum_{\sigma} (j + m_2)! (j - m_2)! (-1)^{m_1 - m_2 + \sigma} \times \sum_{\sigma} \frac{(j + m_1 - \sigma)(m_2 - m_1 + \sigma)! (m_2 - m_1 - \sigma)!}{z_1^j z_2^j \bar{z}_1^{j+m_1-\sigma} \bar{z}_2^{j+m_2+\sigma}}, \quad j = 0, 1/2, 1, 3/2, \ldots. \quad (85)$$

Unitarity and reality relations of these polynomial representations imply

$$D^j_{m_1, m_2}(u^{-1}) = \overline{D^j_{m_1, m_2}(u)}, \quad (86)$$

$$\overline{D^j_{m_1, m_2}(u)} = D^j_{m_1, m_2}(\bar{u}).$$

From these equations follows the transposition property:

$$D^j_{m_1, m_2}(u^\dagger) = D^j_{m_1, m_2}(u). \quad (87)$$

The polynomial homogeneity from equation (85) reads

$$D^j_{m_1, m_2}(\lambda u) = \lambda^{(j)} \ D^j_{m_1, m_2}(u). \quad (88)$$

To pass from an element $g \in SU(2, C)$ to its character $\chi^j(g)$ in the representation $D^j(g)$ we first obtain the angle $\phi$ from its trace,

$$\frac{1}{2\pi} \chi^{1/2}(g) = \frac{1}{2} \text{Trace}(g) := \cos(\phi/2) \quad (89)$$

and then for general $j$ use

$$\chi^j(g) = \sum_{m=-j}^{j} \exp(im\phi) = \frac{\sin((2j+1)/2)}{\sin(\phi/2)}. \quad (90)$$

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References

[1] Edmonds A R 1957 Angular Momentum in Quantum Mechanics (Princeton, NJ: Princeton University Press)
[2] Everitt B 2004 3-manifolds from Platonic solids Topol. Appl. 138 253–63
[3] Hamermesh M 1962 Group Theory and its Application to Physical Problems (Reading, MA: Addison-Wesley)
[4] Humphreys J E 1990 Reflection Groups and Coxeter Groups (Cambridge: Cambridge University Press)
[5] Kramer P and Moshinsky M 1966 Group theory of harmonic oscillators, III. States with permutation symmetry Nucl. Phys. 82 241–74
[6] Kramer P 2005 An invariant operator due to F Klein quantizes H Poincare’s dodecahedral manifold J. Phys. A: Math. Gen. 38 3517–40
[7] Kramer P 2006 Harmonic polynomials on the Poincare dodecahedral 3-manifold J. Geom. Symmetry Phys. 6 55–66
[8] Lachieze-Rey M and Luminet J-P 1995 Cosmic topology Phys. Rep. 254 135–214
[9] Luminet J-P, Weeks J R, Riazuelo A, Lehoucq R and Uzan J-Ph 2003 Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background Nature 425 593–5
[10] Robinson G de B 1961 Representation Theory of the Symmetric Group (Toronto: University of Toronto Press)
[11] Seifert H and Threlfall W 1980 Lehrbuch der Topologie (Leipzig) Chelsea Reprint 1980
[12] Thurston W P 1997 Three-Dimensional Geometry and Topology (Princeton, NJ: Princeton University Press)
[13] Wigner E P 1959 Group Theory and its Application to the Quantum Mechanics of Atomic Spectra (New York: Academic)