Towards a characterization of stretchable aligned graphs

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Abstract. We consider the problem of stretching pseudolines in a planar straight-line drawing to straight lines while preserving the straightness and the combinatorial embedding of the drawing. We answer open questions by Mchedlidze et al. [9] by showing that not all instances with two pseudolines are stretchable. On the positive side, for \( k \geq 2 \) pseudolines intersecting in a single point, we prove that in case that some edge-pseudoline intersection-patterns are forbidden, all instances are stretchable. For intersection-free pseudoline arrangements we show that every aligned graph has an aligned drawing. This considerably reduces the gap between stretchable and non-stretchable instances.

1 Introduction

Every planar graph \( G = (V, E) \) has a straight-line drawing [8,11]. In a restricted setting one seeks a drawing of \( G \) that obeys given constraints, e.g., Biedl et al. [1,2] studied whether a bipartite planar graph has a drawing where the two sets of the partitions can be separated by a straight line. Da Lozzo et al. [4] generalized this result and characterized the planar graphs with a partition \( L \cup R \cup S = V \) of the vertex set that have a planar straight-line drawing such that the vertices in \( L \) and \( R \) lie left and right of a common line \( l \), respectively, and the vertices in \( S \) lie on \( l \); refer to Fig. 1a. In this case \( S \) is called collinear. In particular, they showed that \( S \) is collinear if and only if there is a drawing of \( G \) such that there is an open simple curve \( P \) that starts and ends in the outer face of \( G \), separates \( L \) from \( R \), collects all vertices in \( S \) and that either entirely contains or intersects at most once each edge. We refer to \( P \) as a pseudoline with respect to \( G \).

Dujmovic et al. [3] proved the following surprising result: If \( S \) is a collinear set, then for every point set \( P \) with \( |S| = |P| \) there is a straight-line drawing \( \Gamma \) of \( G \) such that \( S \) is mapped to \( P \). Another recent research stream considers the problem of drawing all vertices on as few lines as possible [3]. Eppstein [7] proved that for every integer \( l \) there is a cubic planar graph graph \( G \) with \( O(l^3) \) vertices such that not all vertices of \( G \) can lie on \( l \) lines.

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Mchedlidze et al. [9] generalized the concept of a single pseudoline with respect to an embedded graph to arrangements of pseudolines and introduced the notion of aligned graphs, i.e., a pair \((G, \mathcal{A})\) where \(G\) is a planar embedded graph and \(\mathcal{A} = \{L_1, \ldots, L_k\}\) is a set of pseudolines \(L_i\) with respect to \(G\) that intersect pairwise at most once. We cite the original definition of aligned drawings [9]. A tuple \((\Gamma, \mathcal{A})\) of a (straight-line) drawing \(\Gamma\) of \(G\) and line arrangement \(\mathcal{A}\) is an aligned drawing of \((G, \mathcal{A})\) if and only if the arrangement of the union of \(\Gamma\) and \(\mathcal{A}\) has same combinatorial properties as the union of \(G\) and \(\mathcal{A}\). In the following, we specify these combinatorial properties. Let \(\mathcal{A} = \{L_1, L_2, \ldots, L_k\}\), i.e., line \(L_i\) corresponds to pseudoline \(L_i\). A (pseudo)-line arrangement divides the plane into a set of cells \(C_1, C_2, \ldots, C_\ell\). If \(\mathcal{A}\) is homeomorphic to \(\mathcal{A}\), then there is a bijection \(\phi\) between the cells of \(\mathcal{A}\) and the cells of \(\mathcal{A}\). If \((\Gamma, \mathcal{A})\) is an aligned drawing of \((G, \mathcal{A})\), then it has the following properties: (i) the arrangement of \(\mathcal{A}\) is homeomorphic to the arrangement of \(\mathcal{A}\), (ii) \(\Gamma\) is a straight-line drawing homeomorphic to the planar embedding of \(G\), (iii) the intersection of each vertex \(v\) and each edge \(e\) with a cell \(C\) of \(\mathcal{A}\) is non-empty if and only if the intersection of \(v\) and \(e\) with \(\phi(C)\) in \((\Gamma, \mathcal{A})\), respectively, is non-empty, (iv) if an edge \(uv\) (directed from \(u\) to \(v\)) intersects a sequence of cells \(C_1, C_2, \ldots, C_r\) in this order, then \(uv\) intersects in \((\Gamma, \mathcal{A})\) the cells \(\phi(C_1), \phi(C_2), \ldots, \phi(C_r)\) in this order, and (v) each line \(L_i\) intersects in \(\Gamma\) the same vertices and edges as \(L_i\) in \(G\), and it does so in the same order.

Mchedlidze et al. observed that not every aligned graph has an aligned drawing. For example, the modification of the Pappus configuration in Fig. 1b does not have an aligned drawing. Note that one endpoint of the edge is anchored on some pseudolines and that the edge crosses three pseudolines. Hence, Mchedlidze et al. studied a restricted subclass of aligned graphs that only contains edges \(uv\) that are either (see Fig. 1c and Fig. 1d)

- **free**, i.e., the entire edge \(uv\) is in a single cell,
- **aligned**, i.e., the entire edge \(uv\) is on a single pseudoline,
- **one-sided anchored**, i.e., \(u\) or \(v\) is on a pseudoline but not both, and \(uv\) does not cross a pseudoline,
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- 1-crossed, i.e., \( u \) and \( v \) are in the interior of a cell and \( uv \) crosses one pseudoline.

For this restricted class Mchedlidze et al. proved that every aligned graph has an aligned drawing. For this purpose they reduced their instances to aligned graphs that do neither have free edges nor aligned edges nor separating triangles. Then the original instance has an aligned drawing if the reduced instance has an aligned drawing. Thus, the key to success is to characterize the reduced instances and to prove that every reduced instance has an aligned drawing. In the reduced setting, Mchedlidze et al. were able to show that each cell of the pseudoline arrangement contains at-most a single vertex. Since the union of two adjacent cells in the line arrangement is convex, any placement of the vertices that respects the ordering constraints along the lines induces a valid aligned drawing of the reduced aligned graph. If we additionally allow two-sided anchored edges, i.e., edges where both endpoints are on pseudolines but that do not cross a pseudoline, then it is possible to construct a family of aligned graphs such that each cell can contain a number of vertices that is not bounded by the number of pseudolines.

**Contribution.** We show that every aligned graph on \( k \geq 2 \) pseudolines intersecting in a single point with free, aligned, one-sided and two-sided anchored, and 1-crossed edges has an aligned drawing. If we allow an additional edge type, we show that there is an aligned graph on two pseudolines that does not have an aligned drawing. Note that in the example given in Fig. 1b no point in the green cell is visible from the red vertex within the polygon defined by union of the (colored) cells traversed by the edge. Hence, this instance trivially does not admit an aligned drawing. In contrast, each edge in Fig. 3a can be drawn independently as a straight-line segment. We show that the entire instance does not admit a straight-line drawing. Further, we show that every aligned graph \((G,A)\) has an aligned drawing, if \(A\) does not have crossings, i.e., \(A\) corresponds to an arrangement \(A\) of parallel lines. This couples aligned graphs to hierarchical (level) graphs. This significantly narrows the gap in the characterization of realizable and non-realizable aligned graphs.

2 Preliminaries

We first introduce some notation used for aligned graphs on \( k \) pseudolines intersecting in a single point. Let \( O \) be a point called the *origin*. Let \( \mathcal{X} = \{X_1, X_2, \ldots, X_k\} \) be a pseudoline arrangement where the pseudolines pairwise intersect in \( O \); refer to Fig. 2. We refer to an aligned graph \((G,\mathcal{X})\) as a *k-star aligned graph*. Correspondingly, we refer to \((I,X)\), with \(X = \{X_1, X_2, \ldots, X_k\}\) as an aligned drawing of \((G,\mathcal{X})\), where the lines in \(X\) pairwise intersect in the origin \(O\). The curves in \(\mathcal{X}\) divide the plane into a set of cells \(Q_1, \ldots, Q_{2k}\) in counterclockwise order. These cells naturally correspond to the regions \(Q_1, \ldots, Q_{2k}\) bounded by the lines in \(X\).

We refer to an edge (vertex) as *free* if it is entirely in the interior of a cell. An *aligned edge* (vertex) is entirely on a pseudoline. For each 1-crossed edge \(e\)
Fig. 2: (a,b) (Pesudo)-line arrangements of a 3-star aligned graph. The green region indicates a cell.

Fig. 3: (a) A 2-aligned graph that does not have an aligned drawing. (b) We have \( \lambda_1/\lambda_2 = \tan(\alpha) < \tan(\beta) = |y_1|/(\lambda_2 + |x_1|) \).

there are \( l \) but not \( l + 1 \) pseudolines that intersect \( e \) in its interior. An edge \( e \) is \( i \)-anchored if \( i \) of its endpoints lie on \( i \) distinct pseudolines. Mchedlidze et al. used a triple \( (l_0, l_1, l_2) \), with \( l_i \in \mathbb{N} \cup \{ \perp \} \) to describe the complexity of an aligned graph \((G, A)\). Let \( E_i \) be the set of \( i \)-anchored edges; note that, the set of edges is the disjoint union \( E_0 \cup E_1 \cup E_2 \). A non-empty edge set \( A \subset E \) is \( l \)-crossed if \( l \) is the smallest number such that every edge in \( A \) is at most \( l \)-crossed. An aligned graph \((G, A)\) has alignment complexity \( (l_0, l_1, l_2) \), if \( E_i \) is at most \( l_i \)-crossed or has to be empty, if \( l_i = \perp \). In particular, Mchedlidze et al. proved that every aligned graph of alignment complexity \((1, 0, \perp)\) has an aligned drawing. Our results can be restated as that every 2-star aligned graph of alignment complexity \((1, 0, 0)\) has an aligned drawing. Further, there is an aligned graph of alignment complexity \((\perp, 1, \perp)\) that does not have an aligned drawing.

3 Star aligned graphs

In this section, we study whether \( k \)-star aligned graphs have aligned drawings. We first prove that the 2-star aligned graph in Fig. 3a does not have an aligned drawing.
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Fig. 4: (a) This 2-aligned graph does not have an aligned drawing. (b,c) The green curve indicates the Jordan curve that completes the black edge. The edge in (b) is an edge of a ccw-aligned graph. The edge depicted in (c) is forbidden in ccw-aligned graphs. (d) A comb of edges e, f.

**Theorem 1.** There is a 2-star aligned graph of alignment complexity \((\perp, 1, \perp)\) that does not have an aligned drawing.

**Proof.** Assume that the aligned graph in Fig. 3a has an aligned drawing. For \(i = 1, \ldots, 4, 5\) with \(1 = 5\), let \((x_i, y_i)\) be the point for \(v_i\), let \(\lambda_i\) be the distance of \(u_i\) to the origin \(O\). Since \(u_2v_1\) intersects the \(y\)-axis above \(u_1\), edge \(u_2v_1\) has a steeper slope than the segment \(u_2u_1\); see Fig. 3b. We obtain \(\lambda_1/\lambda_2 < |y_1|/(\lambda_2 + |x_1|)\) and therefore \(|x_1| < \lambda_2/\lambda_1 \cdot |y_1|\). Analogously, we obtain

\[
|x_i| < \frac{\lambda_{i+1}}{\lambda_i} \cdot |y_i|, \quad i = 1, 3 \quad |y_i| < \frac{\lambda_{i+1}}{\lambda_i} \cdot |x_i|, \quad i = 2, 4.
\]

Since \(v_{i+1}w_i\) are embedded as straight lines, we further obtain estimation (2) that \(|y_i| < |y_{i+1}|\) for \(i = 1, 3\) and \(|x_i| < |x_{i+1}|\) for \(i = 2, 4\). By multiplying the left and the right sides we obtain \(|x_1| \cdot |y_2| \cdot |x_3| \cdot |y_4| < |y_1| \cdot |x_2| \cdot |y_3| \cdot |x_4| \cdot \frac{\lambda_2\lambda_3\lambda_4\lambda_1}{\lambda_1\lambda_2\lambda_3\lambda_4} = |y_1| \cdot |x_2| \cdot |y_3| \cdot |x_4| < |y_2| \cdot |x_3| \cdot |y_4| \cdot |x_1|\). A contradiction.

### 3.1 Aligned drawings of counterclockwise star aligned graphs

We now consider aligned drawings of \(k\)-star aligned graphs \((G, \mathcal{A})\) for \(k \geq 2\). Recall that the aligned graph in Figure 4a does not have an aligned drawing. The crux is that the source of the red edges are free and the source of green edges are aligned. In the following we introduce so-called **counterclockwise aligned graphs** and show that they have aligned drawings.

We orient each non-aligned edge \(uv\) of an aligned graph \((G, \mathcal{X})\) such that it can be extended to a Jordan curve, i.e., a closed simple curve, \(C_{uv}\) with the property that it intersects each pseudoline exactly twice and has the origin to its left. A **counterclockwise aligned (ccw-aligned)** graph is a \(k\)-star aligned graph of alignment complexity \((1, 1, 0)\) whose orientation does not contain 1-anchored 1-crossed edges with a free source vertex.

We prove that every ccw-aligned graph has an aligned drawing. To prove this statement we follow the same proof strategy as Mchedlidze et al. In particular,
we have to augment our aligned graph to a particular ccw-aligned triangulation. Further, we use that for each aligned graph \((G, \mathcal{X})\) there is a reduced aligned graph \((G_R, \mathcal{X})\) (i.e., it does neither contain (i) separating triangles, nor (ii) free edges, nor (iii) aligned edges that are not incident to the origin \(O\)) with the property that \((G, \mathcal{X})\) has an aligned drawing if \((G_R, \mathcal{X})\) has an aligned drawing; see Lemma 2. In contrast to aligned graphs of alignment complexity \((1, 0, \perp)\) the size of \((G_R, \mathcal{X})\) is not bounded by a constant. The aim of Lemma 3 and Lemma 4 is to describe the structure of the reduced instances. This helps to prove Lemma 5 that states that each reduced instance has an aligned drawing.

We first introduce further notations. A k-star aligned graph \((G, \mathcal{X})\) is a proper k-star aligned triangulation if each inner face is a triangle, the boundary of the outer face is a \(2k\)-cycle of 2-anchored edges, the outer face does not contain the origin and there is a degree-2k vertex \(o\) on the origin incident to 2k aligned edges. We refer to a reduced proper ccw-aligned triangulation as a reduced aligned triangulation. We refer to 1-anchored 1-crossed and 2-anchored edges as separating.

The following lemma is a consequence from the results by Mchedlitze et al. [9]. For further details we refer to the Appendix.

**Lemma 2.** Every k-star aligned graph has an aligned drawing, if every reduced k-star aligned triangulation has an aligned drawing.

Hence, our main contribution is to characterize reduced k-star aligned triangulations and then, to prove that every such instance has an aligned drawing.

**Lemma 3.** Let \((G_R, \mathcal{X})\) be a reduced aligned triangulation and let \(o\) be the vertex on the origin. Then in \((G_R - o, \mathcal{X})\) each pseudoline \(X_i\) alternately intersects vertices and edges, and each comb contains at most one vertex.

**Proof.** Assume that there are two consecutive aligned vertices \(u\) and \(v\). Since \(G_R\) is triangulated and \(u\) and \(v\) are consecutive, \(G_R\) contains the edge \(uv\). This contradicts the assumption that \((G_R, \mathcal{X})\) does not contain aligned edges.

The following modification helps us to prove that there are no two consecutive edges along a pseudoline and that no comb contains two free vertices.

![Fig. 5: The curve \(\rho_i\) (a) and its modification in (b).](image-url)
Let \( \rho_i \) be the parts of \( X_i \) and \( X_{i+1} \) that are on the boundary of the cell \( Q_i \), see Figure 5. We modify \( \rho_i \) as follows. We first, join the endpoints of \( \rho_i \) in the infinity such that it becomes a simple closed curve. Let \( u \) be a vertex that lies on \( \rho_i \). We reroute \( \rho_i \) such that \( u \) now lies outside of \( \rho_i \). Since \( G_R \) is triangulated and \( \rho_i \) only intersects edges, \( \rho_i \) corresponds to a cycle in \( G'_R \) and therefore to a cut \( C_i \) in \( G_R \). Note, each edge of a connected component in \( G - C_i \) is a free edge.

Now assume that there are two distinct edges \( e, f \) that consecutively cross a pseudoline \( X_i \in \mathcal{X} \). By the premises of the lemma there is a vertex that lies on the origin \( O \). Hence both \( e \) and \( f \) cross \( X_i \) on the same side with respect to \( O \). Since \( e \) and \( f \) are distinct and \((G_R, \mathcal{X})\) is ccw-aligned, there is a cell \( Q_j \) such that \( Q_j \) contains two distinct vertices \( u \) and \( w \) incident to \( e \) and \( f \), respectively. Since \( G \) is triangulated and \( e \) and \( f \) are consecutive along \( X_i \), \( u \) and \( w \) are vertices in the same connected component of \( G - C_j \). Therefore, \((G_R, \mathcal{X})\) contains a free edge. A contradiction.

Consider a comb \( C \) in a cell \( Q_i \) that contains two distinct vertices \( u \) and \( v \) in its interior. Since \( G \) is triangulated and \( C \) is inclusion-minimal (it does not contain another edge-region), \( u \) and \( v \) belong to the same connected component of \( G_R - C_i \). Therefore \((G_R, \mathcal{X})\) contains a free edge.

We call a comb closed if its two separating edges have the same source vertex.

**Lemma 4.** For every reduced aligned triangulation \((G_R, \mathcal{X})\) there is a reduced aligned triangulation \((G''_R, \mathcal{X})\) where no closed comb contains a vertex such that \((G_R, \mathcal{X})\) has an aligned drawing if \((G''_R, \mathcal{X})\) has an aligned drawing.

**Proof.** By Lemma 3 we know that each comb contains at most one vertex. We apply induction over the number of closed combs that contain a vertex. Let \( v \) be a free vertex in a closed comb with separating edges \( uv_1, uv_2 \). Then we obtain an aligned graph \((G'_R, \mathcal{X})\) by contracting edge \( uv \) in the embedding. Since \((G_R, \mathcal{X})\) is reduced ccw-aligned, all edges outgoing from the free vertex \( v \) are 1-anchored 0-crossed or 0-anchored 1-crossed. In \((G'_R, \mathcal{X})\) they are now 2-anchored 0-crossed or 1-anchored 1-crossed with free target vertex. Since there is no other vertex in the comb and the comb is closed, \( v \) only has \( uv \) as incoming edge which is contracted. Therefore \((G'_R, \mathcal{X})\) is ccw-aligned. Assume that \((G''_R, \mathcal{X})\) has an aligned drawing. Since \( v \) is a free vertex, we obtain an aligned drawing of \((G, \mathcal{X})\) by placing \( v \) close to \( u \) within its closed comb. By Lemma 4 we obtain a reduced aligned triangulation \((G''_R, \mathcal{X})\) from \((G', \mathcal{X})\) such that \((G''_R, \mathcal{X})\) has an aligned drawing if \((G''_R, \mathcal{X})\) has an aligned drawing. In the construction the number of closed combs that contain a vertex is not increased.

We can now show that each reduced instance has an aligned drawing.

**Lemma 5.** Every reduced ccw-aligned triangulation has an aligned drawing.

**Proof.** By Lemma 3 we can assume that in our triangulation \((G, \mathcal{X})\) the closed combs contain no vertices. By Lemma 3 we know that each comb contains at
most one vertex and no vertex if it is closed. The main problem is to draw the 1-crossed edges. For those, we place each free vertex close to the right boundary of its comb. This allows to draw the incoming edges. Since \((G, X)\) is ccw-aligned, the target of each 1-crossed edge \(vu\) is free and allows to draw \(vu\).

We construct the aligned drawing \((T, X)\) as follows. Let \(o\) be the vertex on the origin. We call the sources of separating edges corners. First place \(o\) and all corners on \(X\) in the order induced from \(X\). For \(i = 1, \ldots, 2|X|\), let \(H_i\) be the half-pseudoline that is the right boundary of cell \(Q_i\). Let \(m_i\) denote the vertex on \(H_i\) that is adjacent to \(o\) and let \(r_i\) denote the vertex incident to the outer face on \(H_i\). Note that \(m_i, r_i\) are corners. We write \(u <_i v\) if \(u\) lies between \(o\) and \(v\) on \(H_i\) where \(u, v\) may be vertices and intersections of edges with \(H_i\). Note that \(<_i\) is a linear order. Define \(H_i\) correspondingly for \(X\); see Figure 7. The indices for \(m_i, Q_i\), etc. are considered \(\mod 2|X|\). In the following, we denote by \(\overline{uv}\) the line through two distinct points \(u, v\). Now consider a free vertex \(v\) in some cell \(Q_i\); see Figure 6a. It lies in a comb that is bounded by two separating edges \(u_1w_1, u_2w_2\) with \(u_1 <_i u_2\) on \(H_i\). Note that we have \(u_1 \neq u_2\) since the comb contains \(v\) and is thus not closed. We place \(v\) within the triangle bounded by

Fig. 6: (a) Placement of a free vertex \(v\) in cell \(Q_2\). It may be placed within the gray triangle. (b) Example for the observations with \(u'_1 = x_3\) and \(u'_2 = x_4\).

Fig. 7: The vertex \(o\) and the half-lines \(H_i\) and the vertices \(m_i, r_i\) for \(i = 1, \ldots, 4\). All remaining edges and vertices lie in the green area.
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Let \( m_{i+1}w_2, r_{i+1}w_1 \), \( H_i \) and between \( m_{i-1}w_1, r_{i-1}w_2 \) (if these lines cross within \( Q_i \), then this means within the triangle bounded by \( m_{i-1}w_1, r_{i-1}w_2, H_i \)). Note that \( v \) lies in \( Q_i \). We will show that the intersections of 1-crossed edges with \( H_i \) and the corners on \( H_i \) respect the order \(<\). Finally, we place for \( i = 1, \ldots, 2|X| \) the vertices on \( H_i \) that are neither \( o \) nor a corner arbitrarily on \( H_i \) respecting the order \(<\). This finishes the construction (edges are placed accordingly).

We next show that the vertices and edges of \( G \) appear for \( 1 \leq i \leq |X| \) along \( X_i \) and \( X'_i \) in the same order. Consider the free vertex \( v \) and the separating edges \( u_1w_1, u_2w_2 \) as defined above. Let \( m_i = x_1 < x_2 \cdots < x_k = r_i \) denote the corners on \( H_i-1 \). The following three observations imply that all 1-crossed edges with target \( v \) cross \( H_i \) in the correct order between \( u_1 \) and \( u_2 \); refer to Figure 6b.

1. \( m_{i-1}u \) and \( r_{i-1}u \) cross \( H_i \) between \( u_1 \) and \( u_2 \).
2. \( x_1u, \ldots, x_kv \) intersect \( H_i \) in the same order as \( x_1, \ldots, x_k \) lie on \( H_i-1 \).
3. Let \( v' \) be a free vertex in \( Q_i \). Let \( u_1'w_1', u_2'w_2' \) be the separating edges of the comb containing \( v' \). Then \( v' \) crosses \( H_i \) between \( u_1'v \cap H_i \) and \( u_2'v \cap H_i \).

For Observation 1 note that \( v \) lies between \( m_{i-1}u_1, r_{i-1}u_2 \). For Observation 2 note that \( x_1u, \ldots, x_kv \) cross pairwise in \( v \) and thus not in section \( Q_i \). These two observations imply that \( x_1u, \ldots, x_kv \) cross \( H_i-1 \) between \( u_1 \) and \( u_2 \).

We now show that all 1-crossed edges with target \( v \) cross \( H_i \) in the correct order between \( u_1 \) and \( u_2 \). By Observations 2, 3 the 1-crossed edges with target \( v \) cross \( H_i \) between \( m_{i-1}u \cap H_i \) and \( r_{i-1}u \cap H_i \). With Observation 1 they cross \( H_i \) between \( u_1 \) and \( u_2 \). By Observation 2 we know that the 1-anchored 1-crossed edges with target \( v \) cross \( H_i \) in the correct order. By Observations 2, 3 we obtain that each pair of a 0-anchored 1-crossed and a 1-anchored 1-crossed edge cross \( H_i \) in the correct order. Since the sources of 0-anchored 1-crossed edges with target \( v \) lie in different combs, they lie pairwise on different sides of some edge \( x_iv \) by Observation 3. Observation 2 then yields their correct ordering.

Since the corners on \( H_i \) respect \(<\) and all 1-crossed edges have free target vertices (as the triangulation is ccw-aligned), this implies that the intersections of 1-crossed edges with \( H_i \) and the corners on \( H_i \) respect the order \(<\). By construction, we placed the vertices on \( H_i \) that are not corners such that they also respect order \(<\). Thus the lines \( X_i \) intersect the vertices and edges in the same order as \( X'_i \).

We next show that our embedding is planar by showing that there is no location where edges cross. Since the order of intersections with lines in \( X \) is correct, there are no crossings on \( X \). This leaves us with the cells. Since the separating edges of \( Q_i \) appear in the same order on \( H_i \) and \( H_{i+1} \), they also appear in the same order on \( H_i \) and \( H_{i+1} \). Thus, separating edges of the same cell do not cross each other. We further obtain the same combs for \((\Gamma, XY)\). Consider again a free vertex \( v \) in \( Q_i \) and the corresponding separating edges \( u_1w_1, u_2w_2 \);
see Figure 6a. Since $v$ lies in the triangle bounded by $H_i$, $T_1$ and $m_{i+1}w_2$, it also lies in the comb bounded by $u_1w_1$, $u_2w_2$. Hence, every free vertex lies in the correct comb. Let $e$ be an edge incident to $v$. Then its other end vertex does not lie within the comb of $v$. It must therefore intersect $H_i$ between $u_1$ and $u_2$ if it is incoming, and it must intersect $H_{i+1}$ between $u_1w_1 \cap H_{i+1}$ and $u_2w_2 \cap H_{i+1}$ if it is outgoing. Since we have the same order on $H_i$ and $H_{i+1}$ respectively, edge $e$ crosses neither $u_1w_1$ nor $u_2w_2$ and thus not the interior of any other comb in $Q_i$. This means that 1. There are no crossings on separating edges in the corresponding cells. And that 2. Only edges incident to the free vertex $v$ in a comb intersect the interior of that comb. Those edges are all adjacent in $v$ and do not cross. We obtain that there are no crossings on $X$, no crossings on separating edges in the corresponding cells and no crossings within combs. Hence, our embedding is planar.

Since there are no free edges and the order of intersections with lines in $X$ is fixed, the order of incident edges around a free vertex is also fixed. For a vertex $u$ on $X$ we note that each adjacent free vertex is in another comb and therefore the order of incident edges around $u$ is also fixed. Therefore, our embedding $\Gamma$ induces the same combinatorial embedding as the embedding of $G$.

From Lemma 2 and Lemma 5 we directly obtain our main theorem.

**Theorem 6.** Every ccw-aligned graph $(G, X)$ has an aligned drawing.

### 4 Parallel lines

In this section, we prove that every aligned graph $(G, A)$ has an aligned drawing, if $A$ is intersection free, i.e., the line arrangement $A$ is a set of parallel lines.

Our result uses a result of Eades at al. [6], and of Pach and Toth [10]. Eades et al. consider hierarchical plane graphs. A graph $G = (V, E)$ with a mapping of the vertices to a layer $L_i$ is a hierarchical graph, where a set of layers $L$ is a set of ordered parallel horizontal lines $L_i \in L$. A hierarchical plane drawing of a hierarchical graph is a planar drawing where each vertex is on its desired layer and each edge is drawn as a $y$-monotone curve. Two hierarchical drawings are equivalent if each layer, directed from $-\infty$ to $\infty$, crosses the same set of edges and vertices in the same order. Eades et al. [6] proved that for every hierarchical planar drawing of a graph there is an equivalent hierarchical planar straight-line drawing. Pach and Toth [10] proved a similar result stating that for every $y$-monotone drawing where no two vertices have the same $y$-coordinate there is an equivalent $y$-monotone straight-line drawing such that each vertex keeps its $y$-coordinate. In contrast to these two results, we have that the $y$-coordinate is only prescribed for a subset of the vertices, i.e., there are some (free) vertices that have to be positioned between two layers (lines). The proof strategy is to extend the initial pseudoline arrangement with an additional set of intersection-free pseudolines such that there are no free vertices.

Due to [9] (compare Lemma 2), we can assume that there are neither free nor aligned edges. For the purpose of this section, a reduced aligned graph is an
aligned graph that has no aligned edges and no free vertices. Note that previously only free edges were forbidden. Thus, the current definition is more restrictive. The following theorem is an immediate corollary from the results of Eades et al. [6], and Pach and Toth [10].

**Theorem 7.** For every intersection-free pseudoline arrangement, every reduced aligned graph \((G,A)\) has an aligned drawing.

**Lemma 8.** Let \(A\) be an intersection-free pseudoline arrangement and let \(A'\) be a line arrangement homeomorphic to \(A\). For every aligned graph \((G,A)\) there is a reduced aligned graph \((G,A')\) such that \(A \subseteq A'\) and \((G,A)\) has an aligned drawing if \((G,A')\) has an aligned drawing.

![Fig. 8: Construction of the new pseudoline \(L_v\) (red) that contains \(v\). The red-dotted pseudoline \(L'_v\) indicates the copy of \(L\) (bottom blue) that crossed the edges in \(E_L\) (green) in the same order as \(L\).](image)

**Proof.** We first insert for each free vertex \(v\) a new pseudoline \(L_v\) to \(A\) such that \(v\) is on \(L\). Thus, the aligned graph \((G,A')\) does not have free vertices.

Let \(L\) be a pseudoline that is on the boundary the region \(R_v\) of \(A\) that contains \(v\). Let \(E_L\) be the set of edges of \(G\) that are (partially) routed through \(R_v\) and that are either crossed by \(L\) or that have an endpoint on \(L\). We assume that \(L\) is directed. Then the direction of \(L\) induces a total order of the edges in \(E_L\). We obtain a curve \(L'_v\) that crosses the edges in \(E_L\) in this order and in their interior. Since \(v\) is free, \(G\) is triangulated and \((G,A)\) contains neither free nor aligned edges, there is at least one edge \(e \in E_L\) that is incident to \(v\). Denote by \(e_f\) and \(e_l\) in \(E_L\) the first and last edge incident to \(v\). We obtain a pseudoline \(L_v\) that contains \(v\) from \(L'_v\) by rerouting \(L'_v\) along \(e_f\) and \(e_l\) such that it does not cross these edges in their interior and such that \(v\) is on the line (Fig. 8).

Now, let \((G,A')\) be the aligned graph that is obtained by the previous procedure for each free vertex \(v\). Let \(A'\) be any set of parallel lines that contains \(A\) and corresponds to \(A'\). Clearly, \((Gamma, A)\) is an aligned drawing of \((G,A)\) if \((Gamma, A')\) is an aligned drawing of \((G,A')\). This finishes the proof.

Theorem 7 and Lemma 8 together prove the following theorem.

**Theorem 9.** Let \(A\) be an intersection-free pseudoline arrangement and let \(A\) be a (parallel) line arrangement homeomorphic to \(A\). Then every aligned graph \((G,A)\) has an aligned drawing \((G,A)\).
5 Conclusion

In the paper, we showed that every aligned graph \((G, A)\) has an aligned drawing if \((G, A)\) is either a ccw-aligned graph or if \(A\) is intersection-free. Further, we provided a non-trivial example of a 2-star aligned graph that does not admit an aligned drawing. Thus, in our opinion the most intriguing open question is whether every aligned graph of alignment complexity \((1, 0, 0)\) has an aligned drawing, for general stretchable pseudoline arrangements \(A\). Our example shows that this statement is not true for aligned graphs of alignment complexity \((1, 1, 0)\). Our stretchability proof of counterclockwise aligned graphs uses the fact that we can move each free vertex \(v\) to an aligned vertex \(u\) on the cell of \(v\). Performing this operation for all free vertices at once ensures that we do not introduce edges of a forbidden alignment complexity. Figure 9 indicates that for general aligned graphs of alignment complexity \((1, 0, 0)\) there is not always a consistent mapping of free vertices to aligned vertices such that the resulting graph has the same alignment complexity. Thus it is unclear whether the techniques used in the paper can be used to decide whether every aligned graph of alignment complexity \((1, 0, 0)\) has an aligned drawing.

Fig. 9: There is no mapping of free vertices to aligned vertices on the boundary of the same cell such that moving the free vertices onto their image results in an aligned graph of alignment complexity \((1, 0, 0)\).

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A Reducing k-star aligned graphs

In this section, we give further details on how to reduce a k-star aligned graph. We first recall the definition of proper and reduced triangulations. A k-star aligned graph \((G, X)\) is a proper k-aligned triangulation if each inner face is a triangle, the boundary of the outer face is a 2k-cycle of 2-anchored edges, the outer face does not contain the origin and there is a degree-2k vertex \(o\) on the origin incident to 2k aligned edges. We refer to a reduced proper ccw-aligned triangulation as a reduced aligned triangulation if it does neither contain (i) separating triangles, nor (ii) free edges, nor (iii) aligned edges that are not incident to the origin \(O\).

Mchedlidze et al. proved the following triangulation lemma.

**Lemma 10.** For every aligned graph \((G, X)\) of alignment complexity \((1, 0, \perp)\) there is an aligned triangulation \((G', X')\) of alignment complexity \((1, 0, \perp)\) such that \(G\) is a subgraph of \(G'\).
Fig. 10: (a) Illustration of the key properties of a proper $k$-star aligned graph. (b) Examples of allowed (black) edges in a reduced instance and forbidden (red) edges.

Fig. 11: (a) The (black) separating edges are isolated by the green edges. (b) The black edges are removed and the red edges are obtained by the triangulation. (c) Final graph, after removing edges in the interior of a quadrangle $u, w_1, v, w_2$ and reinserting the black edges.
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Since ccw-aligned graphs contain 2-anchored and 1-anchored 1-crossed edges, we cannot immediately apply this lemma. In the following, we show that our instances can be modified such that we can use the previous lemma. For simplicity, we assume that there is no aligned that crosses the origin.

**Lemma 11.** Let \((G, \mathcal{X})\) be a ccw-aligned graph. Then there is a ccw-aligned triangulation \((G', \mathcal{X}')\) that contains \((G, \mathcal{X})\) as a subgraph. Moreover, the outer face of \((G', \mathcal{X}')\) is bounded by 2k-cycle \(C\) of 2-anchored edges and the outer face does not contain the origin in its interior.

**Proof.** Let \((G_2, \mathcal{X})\) be the graph that is constructed from \((G, \mathcal{X})\) as follows. First, add a 2k-cycle \(C\) of 2-anchored edges in the outer face such that the new outer face does not contain the origin. For each separating edge \(uv\) of \(G\) add two vertices \(w_1, w_2\) and the edges \(uw_1, w_1v\) and \(uw_2, w_2v\). Route and direct the edges according to Figure 11a. Finally, remove the edge \(uv\). Eventually, we arrive at an aligned graph of alignment complexity \((1, 0, \perp)\). With the application of Lemma 10 we obtain a triangulated aligned graph \((G_3, \mathcal{X}')\) of alignment complexity \((1, 0, \perp)\). We remove edges in the interior of each quadrangle \(u, w_1, v, w_2\) and reinserted the original edge \(uv\). Finally, we remove all edges and vertices in the region bounded by \(C\) that does not contain the origin. This yields the desired aligned graph \((G', \mathcal{X}')\).

Since no free edge of an ccw-aligned graph is incident to a triangle that contains the intersection in its interior, the next lemma follows from the results of Mchedlize et al.

**Lemma 12.** Let \((G, \mathcal{X})\) be a ccw-aligned graph and let \(e\) be an interior free edge or an aligned edge that is neither an edge of a separating nor a chord and does not contain the origin, then \((G/e, \mathcal{X})\) is a ccw-aligned graph and \((G, \mathcal{X})\) has an aligned drawing if \((G/e, \mathcal{X})\) has an aligned drawing.

Thus, we can now prove the main reduction lemma and therefore Lemma 2.

**Lemma 13.** For every ccw-aligned graph \((G, \mathcal{X})\) there is a reduced aligned triangulation \((G_R, \mathcal{X}')\) such that \((G, \mathcal{X})\) has an aligned drawing if \((G_R, \mathcal{X}')\) has an aligned drawing.

**Proof.** By Lemma 11 there is a aligned triangulation \((G_T, \mathcal{X})\) of \((G, \mathcal{X})\) with the outer face bounded by 2k-cycle of 2-anchored edges. Moreover, an aligned drawing of \((G_T, \mathcal{X})\) contains an aligned drawing of \((G, \mathcal{X})\).

By Mchedlize et al, we obtain a reduced aligned triangulation \((G'_R, \mathcal{X})\) from \((G_T, \mathcal{X})\) by either splitting \((G_T, \mathcal{X})\) into two aligned graphs at a separating triangle \(T\), or by contracting free or aligned edges that are not incident to \(o\) (Lemma 12). Moreover, we have that that \((G_T, \mathcal{X})\) has an aligned drawing if \((G'_R, \mathcal{X})\) has an aligned drawing.

In order to obtain a proper aligned triangulation \((G_R, \mathcal{X})\) from \((G'_R, \mathcal{X})\) we perform the reduction depicted in Figure 12. If there is an aligned edge that contains the origin in its interior, we place a subdivision vertex on this edge.
and inserted edges as depicted in Figure 12a. Note that in this case an aligned drawing of \((G_R, X)\) contains an aligned drawing of \((G'_R, X)\).

Consider the case that there is a vertex \(v\) on the origin that is incident to a free vertex \(u\). We obtain a new aligned graph \((G_R, X)\) by exhaustively applying the reductions depicted in Figure 12b. Since the black polygon (compare Figure 12b) in an aligned drawing of \((G_R, X)\) is star-shaped and its kernel contains the vertex \(v\), \((G'_R, X)\) has an aligned drawing if \((G'_R, X)\) has an aligned drawing.