Bernstein Polynomial Inequality on a Compact Subset of the Real Line

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Running head: Bernstein inequality

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Abstract

We prove an analogue of the classical Bernstein polynomial inequality on a compact subset $E$ of the real line. The Lipschitz continuity of the Green function for the complement of $E$ with respect to the extended complex plane and the differentiability at a point of $E$ of a special, associated with $E$, conformal mapping of the upper half-plane onto the comb domain play crucial role in our investigation.

*Key Words:* Polynomial inequality, Bernstein inequality, Green’s function.

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1. Introduction and the main result

Let $E \subset \mathbb{R}$ be a non-polar compact set, i.e., there exists the Green function $g_{\Omega}(z, \infty)$ of $\Omega := \mathbb{C} \setminus E$ with pole at infinity. Denote by $P_n, n = 1, 2, \ldots$ the set of all (real) polynomials of degree at most $n$ and let $\| \cdot \|_E$ be the supremum norm on $E$. The classical Bernstein inequality states that for $p_n \in P_n$,

$$|p'_n(x)| \leq \frac{n}{\sqrt{1-x^2}} \|p_n\|_{[-1,1]}, \quad x \in (-1,1). \tag{1.1}$$

Recently, Totik [20] found the conditions on $E$ and $x_0 \in E$ under which the analogue of (1.1), i.e., the inequality

$$|p'_n(x_0)| \leq c(x_0, E)n\|p_n\|_E \tag{1.2}$$

is true. In particular, it follows directly from [17, Theorem 3.3] and [5, Lemma 3] (for details, see [20, Theorem 2]) that (1.2) is equivalent to the fact that $g_{\Omega}$ is Lipschitz continuous at $x_0$, i.e.,

$$\limsup_{\Omega \ni z \to x_0} \frac{g_{\Omega}(z)}{|z-x_0|} < \infty. \tag{1.3}$$

For the properties of $E$ with (1.3) we refer the reader to [8, 18, 9, 3, 4] and the many references therein.

Baran [7, p. 489] and Totik [17, Theorems 3.2 and 3.3] independently found the exact value of the constant $c(x_0, E)$ in the case where $x_0$ is the interior (with respect to $\mathbb{R}$) point of $E$. Our objective is to extend their result to the general case of $E$ and $x_0$ satisfying (1.3). To achieve this, we prove Theorem 1, which is of independent interest.

**Theorem 1** Let $E$ and $x_0 \in E$ satisfy (1.3), then

(i) there exists a finite nonzero normal derivative

$$\frac{\partial g_{\Omega}(x_0)}{\partial n} =: h(x_0, E).$$

Also if $E$ is regular for the Dirichlet problem in $\Omega$, then

(ii) for $E_\delta := E \cup [x_0 - \delta, x_0 + \delta], \delta > 0$,

$$\lim_{\delta \to 0^+} h(x_0, E_\delta) = h(x_0, E). \tag{1.4}$$

A straightforward application of Theorem 1 and [17, Theorems 3.2 and 3.3] yields the following statement.
Theorem 2. Let $E$ and $x_0 \in E$ satisfy (1.3), then

(i) for $p_n \in \mathbb{P}_n$,
\[ |p'_n(x_0)| \leq h(x_0, E)n||p_n||_E. \]

Also if $E$ is regular for the Dirichlet problem in $\Omega$, then

(ii) for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for $n > N$, there is $p_n \in \mathbb{P}_n, p_n \not\equiv 0$ satisfying
\[ |p'_n(x_0)| \geq (1 - \varepsilon)h(x_0, E)n||p_n||_E. \]

Proof. Since by Theorem 1(i), [17, Theorem 3.2], and by the monotonicity of the Green function with respect to the region for $p_n \in \mathbb{P}_n$,
\[ |p'_n(x_0)| \leq h(x_0, E_\delta)n||p_n||_{E_\delta} \leq h(x_0, E)n||p_n||_E, \]
taking the limit as $\delta \to 0$, we obtain (i).

Next, according to Theorem 1(ii), for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ with
\[ h(x_0, E_\delta) \geq \left(1 - \frac{\varepsilon}{2}\right)h(x_0, E). \]

Moreover, by [17, Theorem 3.3], there exists $N = N(\delta, \varepsilon)$ such that for any $n > N$, there is $p_n \in \mathbb{P}$ satisfying
\[ |p'_n(x_0)| \geq \left(1 - \frac{\varepsilon}{2}\right)h(x_0, E_\delta)||p_n||_{E_\delta} \geq (1 - \varepsilon)h(x_0, E)||p_n||_E, \]
which yields (ii).

\[ \square \]

It was shown in [19, Corollary 2.3] that the Bernstein factor $h(x_0, E)n$ found in Theorem 2 can also be stated in another form as a limit of equilibrium densities.

2. Proof of Theorem 1(i)

Let $E$ and $x_0 \in E$ satisfy (1.3). Consider the analytic in $\mathbb{H} := \{z : \Im z > 0\}$ function
\[ f(z) = f(z, E) := i \left( \int_E \log(z - \zeta)d\mu_E(\zeta) - \log \text{cap } E \right), \quad z \in \mathbb{H}, \quad (2.1) \]
where $\text{cap } E$ is the logarithmic capacity of $E$ and $\mu_E$ is the equilibrium measure for $E$ (see [14] or [16] for the basic notions of the potential theory).

Note that
\[ \Im f(z) = g_\Omega(z) > 0, \quad z \in \mathbb{H}. \quad (2.2) \]
It is well known and easy to see that $S := f(\mathbb{H}) \subset \mathbb{H}$ is a “comb domain”, i.e, the boundary of $S$ consists of $\mathbb{R}$ and at most a countable number of closed vertical intervals with one endpoint on $\mathbb{R}$. Conformal mappings of $\mathbb{H}$ onto such domains play significant role in several areas of analysis (see, for example [10]). We need only some elementary properties of $S$ which are discussed below.

Repeating the reasoning from [2, pp. 222-223] one can show that $f$ is a univalent function with the following properties.

First, since by $(1.3)$ $x_0$ is a regular point, according to the Monotone Convergence Theorem (see [15, p. 21]) there exists
\[
\lim_{y \to 0^+} f(x_0 + iy) = -\pi \mu_E(E \cap (-\infty, x_0)) =: w_0.
\]
Second, for any $z \in S$,
\[
\{ \zeta \in S : \Re \zeta = \Re z, \Im \zeta > \Im z \} \subset S.
\]
Third, function $f$ satisfies the boundary correspondence $f(\infty) = \infty$.

Moreover, $(1.3)$ and $(2.1)$ imply that
\[
\{ w = w_0 + i\eta : \eta > 0 \} \subset S.
\]
Indeed, assume that $(2.3)$ does not hold, then for some $d > 0$ we have $[w_0, w_0 + id] \subset \mathbb{H} \setminus S$. To get a contradiction, we use the notion of the module of a family of curves. We refer to [1, Chapter 4], [11, Chapter IV] or [6, pp. 341-360] for the definition and basic properties of the module such as conformal invariance, comparison principle, composition laws, etc. We use these properties without further references.

Consider the Jordan curve $\gamma := f((x_0, x_0 + i]) \cup \{w_0\}$. Without loss of generality we can assume that $\gamma$ “approaches $w_0$ from the right”, i.e., there is $0 < \varepsilon < d$ such that
\[
\gamma \cap \{ w : |w - w_0| = \varepsilon, \pi/2 < \arg(w - w_0) < \pi \} = \emptyset.
\]
Denote by $D \subset \{ w \in \mathbb{H} : 0 < \arg(w - w_0) < \pi/2 \}$ a Jordan domain bounded by $[w_0, w_0 + id]$, a subarc of $\gamma$ and a subarc of a circle $\{ w : |w - w_0| = d \}$. For $0 < \delta < d$, denote by $w_\delta \in D$ any point satisfying
\[
|w_\delta - w_0| = \delta, \quad \frac{\pi}{4} < \arg(w_\delta - w_0) < \frac{\pi}{2},
\]
\[
\{ w : |w - w_0| = \delta, \arg(w_\delta - w_0) < \arg(w - w_0) < \frac{\pi}{2} \} \subset D.
\]
Let $z_\delta := f^{-1}(w_\delta)$. We also assume that $\delta$ is so small that $|z_\delta - x_0| < 1$. Let $z_1 := x_0 + iu, w_1 := f(z_1)$, where a sufficiently large but fixed number $u \geq 1$ is chosen as follows. Denote by $\Gamma_1 = \Gamma_1(z_\delta)$ the family of all crosscuts of $\mathbb{H}$ which
separate \( z_\delta \) and \( x_0 \) from \( z_1 \) and \( \infty \). Let \( \Gamma_2 = \Gamma_2(w_\delta) \) be the family of all crosscuts of the quadrilateral
\[
\{ w : |w_\delta - w| < |w - w_0| < d, 0 < \arg(w - w_0) < \pi/2 \}
\]
which separate its boundary circular components. We choose \( u \) so large that for each \( \gamma_2 \in \Gamma_2 \) there exists \( \gamma_1 \in f(\Gamma_1) \) with the property \( \gamma_1 \subset \gamma_2 \), which yields that \( m(\Gamma_2) \leq m(f(\Gamma_1)) \).

According to [5, (2.5)], we obtain
\[
\frac{1}{\pi} \log \frac{u}{|z_\delta - x_0|} + 2 \geq m(\Gamma_1) = m(f(\Gamma_1)) \geq m(\Gamma_2) = \frac{2}{\pi} \log \frac{d}{\delta},
\]
which implies
\[
g_{\Omega}(z_\delta) = \Im w_\delta > \frac{\delta}{2} \geq \frac{d}{2e\pi \sqrt{u}} |z_\delta - x_0|^{1/2}.
\]
Therefore,
\[
\frac{g_{\Omega}(z_\delta)}{|z_\delta - x_0|} \to \infty \quad \text{as} \ \delta \to 0,
\]
which contradicts (1.3). This completes the proof of (2.3).

Next, for \( l > 0 \) and \( v > 0 \), consider the quadrilateral
\[
Q_{l,v} := \{ z \in \mathbb{H} : v < |z| < v\sqrt{1+l^2}, \Re z < v \}.
\]
Denote by \( \Gamma_{l,v} \) the family of all crosscuts of \( Q_{l,v} \) that separate its circular boundary components.

We claim that there exists a positive constant \( c = c(l) \) such that
\[
m(\Gamma_{l,v}) - \frac{1}{\pi} \log \sqrt{1+l^2} \geq c. \quad (2.4)
\]
Indeed, applying the transformation \( z \to z/v \) and using conformal invariance of the module, we can reduce the proof of (2.4) to the case where \( v = 1 \). Since for \( r \in (1, \sqrt{1+l^2}) \),
\[
\{ \theta : re^{i\theta} \in Q_{l,1} \} = (\cos^{-1}(1/r), \pi),
\]
for the module of \( \Gamma_{l,1} \) we have
\[
m(\Gamma_{l,1}) - \frac{1}{\pi} \log \sqrt{1+l^2}
\geq \int_{1}^{\sqrt{1+l^2}} \frac{dr}{r(\pi - \cos^{-1}(1/r))} - \frac{1}{\pi} \log \sqrt{1+l^2}
\]
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\[
\int_1^{\sqrt{1+l^2}} \cos^{-1}(1/r) dr \
\geq \frac{1}{\pi^2} \int_1^{\sqrt{1+l^2}} \cos^{-1}(1/r) dr =: c,
\]

which yields (2.4).

In the proofs of some of the lemmas below we use the following family of curves and its module. For \( w_1 = w_0 + i\eta \) and \( w_2 = w_0 + i \) with \( 0 < \eta < 1 \) we introduce the family \( \Gamma_0 = \Gamma_0(\eta) \) of all crosscuts of \( S \) that separate points \( w_0 \) and \( w_1 \) from \( w_2 \) and \( \infty \) in \( S \). Let \( z_k := f^{-1}(w_k), k = 1, 2. \) According to [5, (2.5)], if \( |z_1 - x_0| < |z_2 - x_0| \), then for the module of \( \Gamma_0 \) we have

\[
m(\Gamma_0) = m(f^{-1}(\Gamma_0)) \leq \frac{1}{\pi} \log \left| \frac{z_2 - x_0}{z_1 - x_0} \right| + 2. \quad (2.5)
\]

Following [11, p. 173], for \( \beta \in (0, \pi/2) \) and \( \varepsilon > 0 \), define the truncated cone

\[
S_\beta^\varepsilon = S_\beta^\varepsilon(w_0) := \{ w : |\arg(w - w_0) - \pi/2| < \varepsilon, 0 < |w - w_0| < \varepsilon \}.
\]

We say that \( \partial S \) has an inner tangent (with inner normal \( i \)) at \( w_0 \) if for every \( \beta \in (0, \pi/2) \) there is an \( \varepsilon = \varepsilon(\beta) > 0 \) so that \( S_\beta^\varepsilon \subset S \).

**Lemma 1** \( \partial S \) has an inner tangent at \( w_0 \).

**Proof.** Assume that \( \partial S \) does not have an inner tangent at \( w_0 \). Then, there exist \( \varepsilon > 0 \) and a sequence of real numbers \( \{b_n\}_1^\infty \) such that

\[
d_n := |b_n - w_0| < 1, \quad \lim_{n \to \infty} b_n = w_0,
\]

\[
[b_n, b_n + id_n] \subset \mathbb{H} \setminus S,
\]

\[
\sqrt{1 + \varepsilon^2} d_1 < 1, \quad \sqrt{1 + \varepsilon^2} d_{n+1} < d_n. \quad (2.6)
\]

For \( r > 0 \), denote by \( \gamma(r) = \gamma(w_0, S, r) \subset \{ w : |w - w_0| = r \} \) the crosscut of \( S \) which has a nonempty intersection with the ray \( \{ w \in \mathbb{H} : \Re w = w_0 \} \). For \( 0 < r < R \), denote by \( D(r, R) = D(w_0, S, r, R) \subset S \) the bounded simply connected domain whose boundary consists of \( \gamma(r), \gamma(R) \), and two connected parts of \( \partial S \). Let \( m(r, R) = m(w_0, S, r, R) \) be the module of the family \( \Gamma(r, R) = \Gamma(w_0, S, r, R) \) of all crosscuts of \( D(r, R) \) which separate circular arcs \( \gamma(r) \) and \( \gamma(R) \) in \( D(r, R) \). Note that

\[
m(r, R) \geq \frac{1}{\pi} \log \frac{R}{r}, \quad 0 < r < R. \quad (2.8)
\]

Moreover, according to (2.4), (2.6), and (2.7),

\[
m(d_n, d_n\sqrt{1 + \varepsilon^2}) - \frac{1}{\pi} \log \sqrt{1 + \varepsilon^2} \geq c = c(\varepsilon) > 0.
\]


Therefore, by virtue of (2.8) we obtain
\[ m(Γ_0(d_n)) \geq m(d_n, 1) \]
\[ \geq m(\sqrt{1 + \varepsilon^2 d_1}, 1) + \sum_{k=1}^{n} m(d_k, \sqrt{1 + \varepsilon^2 d_k}) + \sum_{k=2}^{n} m(\sqrt{1 + \varepsilon^2 d_k}, d_{k-1}) \]
\[ \geq \frac{1}{\pi} \log \frac{1}{d_n} + \sum_{k=1}^{n} \left( m(d_k, \sqrt{1 + \varepsilon^2 d_k}) - \frac{1}{\pi} \log \sqrt{1 + \varepsilon^2} \right) \]
\[ \geq \frac{1}{\pi} \log \frac{1}{d_n} + nc. \]

Comparing the last inequality with (2.5) for \( η = d_n \), we have
\[ \frac{g_{Ω}(z_1)}{|z_1 - x_0|} \geq \frac{e^{πcn}}{|z_2 - x_0|} \rightarrow ∞ \quad \text{as} \quad n \rightarrow ∞, \]
which contradicts (1.3).

\[ \Box \]

According to Lemma 1 and the Ostrowski Theorem (see [11, p. 177, Theorem 5.5]) for every \( β \in (0, \pi/2) \) there exist nontangential limits
\[ \lim_{S_β(x_0) ; z \to x_0} f(z) = w_0, \]
\[ \lim_{S_β(x_0) ; z \to x_0} \arg \frac{f(z) - w_0}{z - x_0} = 0. \]

Lemma 2 The function \( f \) has a positive angular derivative at \( x_0 \), i.e.,
\[ f'(x_0) := \lim_{S_β(x_0) ; z \to x_0} \frac{f(z) - w_0}{z - x_0} > 0 \quad (2.9) \]
exists for every \( β \in (0, \pi/2) \).

Proof. Assume that (2.9) is not true. By the Jenkins-Oikawa-Rodin-Warschawski Theorem (see [11, p. 180, Theorem 5.7]) there exists \( ε > 0 \) such that for every \( δ > 0 \) there are \( 0 < s < r < δ \) satisfying
\[ m(s, r) - \frac{1}{\pi} \log \frac{r}{s} \geq ε. \]

Therefore, we can find sequences of real numbers \( \{s_n\}_1^∞ \) and \( \{r_n\}_1^∞ \) such that
\[ 0 < r_n < s_{n-1} < r_{n-1} < 1 \]
and
\[ \lim_{n \to ∞} s_n = \lim_{n \to ∞} r_n = 0, \]

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Furthermore, by (2.8)

\[ m(\Gamma_0(s_n)) \geq m(s_n, 1) \geq m(r_1, 1) + \sum_{k=1}^{n} m(s_k, r_k) + \sum_{k=1}^{n-1} m(r_{k+1}, s_k) \]

\[ \geq \frac{1}{\pi} \log \frac{1}{s_n} + \sum_{k=1}^{n} \left( m(s_k, r_k) - \frac{1}{\pi} \log \frac{r_k}{s_k} \right) \]

\[ \geq \frac{1}{\pi} \log \frac{1}{s_n} + n\varepsilon. \]

Comparing the last inequality with (2.5) for \( \eta = s_n \), we obtain

\[ g_\Omega(z_1) \geq \frac{1}{|z_2 - x_0|} e^{\pi(\varepsilon n^2)} \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty, \]

which contradicts (1.3).

\[ \square \]

Note that Lemma 2 and (2.2) imply Theorem 1(i).

3. Proof of Theorem 1(ii)

In this section we assume that \( E \) is a regular set satisfying (1.3). Hence, the extension of the Green function by letting \( g_\Omega(z) := 0, z \in E \) produces a continuous function in \( \mathbb{C} \).

**Lemma 3** For the Green function we have

\[ \int_{-\infty}^{\infty} \frac{g_\Omega(x)dx}{(x - x_0)^2} < \infty. \] (3.1)

**Proof.** Consider \( E^* := \{1/(x - x_0) : x \in E\} \subset \mathbb{R} \cup \{\infty\} \setminus \{0\} \). Note that

\[ g_\Omega(z) = g_{\mathbb{C}\setminus E^*}(1/(z - x_0), 0), \quad z \in \overline{\mathbb{C}}. \]

Denote by \( \mathcal{P}_\infty \) the cone of positive harmonic functions on \( \mathbb{C} \setminus E^* \) which have vanishing boundary values at every point of \( E^* \setminus \{\infty\} \). Since (1.3) implies

\[ \limsup_{\mathbb{C}\setminus E^* \ni \zeta \to \infty} g_{\mathbb{C}\setminus E^*}(\zeta, 0)|\zeta| < \infty, \]

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according to [4, Theorem 1] or [9, Theorem 1] the dimension of \( P_\infty \) is 2. Therefore, by virtue of [12, Theorem 3] (cf. [13, Section VIII A.2]), there exists a Phragmén-Lindelöf function for \( \mathbb{C} \setminus E^* \) which yields
\[
\int_{-\infty}^{\infty} g_{\mathbb{C} \setminus E^*}(t,0)dt < \infty
\]
and (3.1) follows.

\[\square\]

According to Lemma 3 and Monotone Convergence Theorem (see [15, p. 21]), for any \( \varepsilon > 0 \) there exists \( \delta_1 = \delta_1(\varepsilon) > 0 \) with the property
\[
\int_{x_0-\delta_1}^{x_0+\delta_1} \frac{g_\Omega(x)dx}{(x-x_0)^2} < \varepsilon. \tag{3.2}
\]
Next, let \( 0 < \delta = \delta(\varepsilon, \delta_1) < \delta_1 \) be such that for \( x_0 - \delta \leq x \leq x_0 + \delta \) the inequality \( g_\Omega(x) < \varepsilon \delta_1 \) holds. Consider the function
\[
u_\delta(z) := g_\Omega(z) - g_{\Omega_\delta}(z), \quad z \in \Omega_\delta := \mathbb{C} \setminus E_\delta,
\]
which is harmonic in \( \Omega_\delta \).

By the maximum principle
\[
u_\delta(x) \leq \varepsilon \delta_1, \quad x \in \mathbb{R}.
\]
Moreover, (3.2) implies that
\[
\int_{x_0-\delta_1}^{x_0+\delta_1} \frac{u_\delta(x)dx}{(x-x_0)^2} \leq \int_{x_0-\delta_1}^{x_0+\delta_1} \frac{g_\Omega(x)dx}{(x-x_0)^2} < \varepsilon.
\]
Applying the Poisson formula (see [11, p. 4]), for \( y > 0 \) we obtain
\[
u_\delta(x_0 + iy) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{u_\delta(x)dx}{(x-x_0)^2 + y^2}
\]
\[
\leq \frac{y}{\pi} \left( \int_{|x-x_0| \geq \delta_1} \frac{u_\delta(x)dx}{(x-x_0)^2} + \int_{|x-x_0| < \delta_1} \frac{g_{\Omega_\delta}(x)dx}{(x-x_0)^2} \right)
\]
\[
\leq \frac{y}{\pi} \left( \frac{2}{\delta_1} \varepsilon \delta_1 + \varepsilon \right) < y\varepsilon.
\]

Therefore,
\[
\frac{\partial u_\delta(x_0)}{\partial n} := \lim_{y \to 0^+} \frac{u_\delta(x_0 + iy)}{y} \leq \varepsilon. \tag{3.3}
\]
Monotonicity of the Green function and (3.3) yield
\[
\frac{\partial g_{\Omega}(x_0)}{\partial n} \leq \frac{\partial g_{\Omega}(x)}{\partial n} \leq \frac{\partial g_{\Omega}(x_0)}{\partial n} + \varepsilon,
\]
which completes the proof of (1.4).

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References

[1] L.V. Ahlfors, Conformal Invariants, McGraw-Hill, New York, 1973.

[2] V.V. Andrievskii, The highest smoothness of the Green function implies the highest density of a set, Ark. Mat. 42 (2004) 217–238.

[3] V.V. Andrievskii, On sparse sets with the Green function of the highest smoothness, Comp. Met. and Fun. Theory, 5 (2005) 301–322.

[4] V.V. Andrievskii, Positive harmonic functions on Denjoy domains in the complex plane, J. d’Analyse Math. 104 (2008) 83–124.

[5] V.V. Andrievskii, Polynomial approximation on a compact subset of the real line, Journal of Approximation Theory 230 (2018) 24–31.

[6] V.V. Andrievskii, H.-P. Blatt, Discrepancy of Signed Measures and Polynomial Approximation, Springer-Verlag, Berlin/New York, 2002.

[7] M. Baran. Complex equilibrium measure and Bernstein type theorems for compact sets in \(\mathbb{R}^n\), Proc. Amer. Math. Soc. 123(2) (1995) 485–494.

[8] L. Carleson, V. Totik, Hölder continuity of Green’s functions, Acta Sci. Math. (Szeged) 70 (2004) 557–608.

[9] T. Carroll, S. Gardiner, Lipschitz continuity of the Green function in Denjoy domains, Ark. Mat. 46 (2008) 271–283.

[10] A. Eremenko, P. Yuditskii, Comb functions, Contemp. Math. 578 (2012) 99–118.

[11] J.B. Garnet, D.E. Marshal, Harmonic Measure, Cambridge University Press, Cambridge, 2005.
[12] P.P. Kargaev, Existence of the Phragmén-Lindelöf function and certain quasianalyticity conditions, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 126 (1983) 97–108 (Russian); English translation in J. Soviet Math. 27 (1984) 2486-2495.

[13] P. Koosis, The Logarithmic Integral, I, Cambridge University Press, Cambridge, 1988.

[14] T. Ransford, Potential Theory in the Complex Plane, Cambridge University Press, Cambridge, 1995.

[15] W. Rudin, Real and Complex Analysis, 3rd ed., McGraw-Hill, New York, 1987.

[16] E.B. Saff , V. Totik, Logarithmic Potentials with External Fields, Springer-Verlag, New York/Berlin, 1997.

[17] V. Totik, Polynomial inverse images and polynomial inequalities, Acta Math. 187 (2001) 139–160.

[18] V. Totik, Metric Properties of Harmonic Measures, Mem. Amer. Math. Soc. 184 (2006), no 867.

[19] V. Totik, Bernstein- and Markov-type inequalities for trigonometric polynomials on general sets, International Mathematics Research Notices, rnu030, 35 pages. doi:10.1093/imrn/rnu030

[20] V. Totik, Reflection on a theorem of V. Andrievskii, manuscript, 2018.