Spectrum of the supersymmetric t-J model with non-diagonal open boundaries

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Abstract

In this work we diagonalize the double-row transfer matrix of the supersymmetric t-J model with non-diagonal boundary terms by means of the algebraic Bethe ansatz. The corresponding reflection equations are studied and two distinct classes of solutions are found, one diagonal solution and other non-diagonal. In the non-diagonal case the eigenvalues in the first sectors are given for arbitrary values of the boundary parameters.

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1 Introduction

The t-J model is frequently invoked as a model for strongly correlated electrons systems, in particular, for high-$T_c$ cuprate superconductors [1, 2] as well as for heavy-fermion systems [3, 4]. Although the mechanisms proposed to explain high-$T_c$ superconductors usually invoke properties of two-dimensional systems [1, 5, 6], it has been argued [6, 7] that due to strong quantum fluctuations one-dimensional systems may share some features of the two-dimensional ones. The t-J Hamiltonian contains nearest-neighbor hopping and nearest-neighbor spin exchange terms and it can be derived from the Hubbard Hamiltonian for a band occupation close to half-filling and a large Coulomb repulsion by means of a canonical transformation which eliminates doubly occupied states [8]. For a review on this subject see for instance the Ref. [9]. The Hilbert space of this model is constrained to forbid double occupancy of single sites, leading to only three possible states at each lattice site. At the supersymmetric point, the one-dimensional t-J model becomes $sl(2|1)$ invariant and its integrability was first stated by Lai [10] and Sutherland [11], and subsequently reported by other authors [3, 12, 13]. Within the Quantum Inverse Scattering Method, the integrability of the supersymmetric t-J model was established in [14] for the case of periodic boundary conditions.

Though boundary conditions are not expected to influence the infinite volume properties, it can modify the finite-size corrections of massless systems in a strip of width $L$ which provides fundamental informations concerning the underlying conformal field theories [15]. On the other hand, boundary conditions also provides a mechanism to relate the critical behaviour of a variety of lattice systems such as the Heisenberg spin chain, the Ashkin-Teller and Potts models [16].

The study of integrable systems with arbitrary boundary conditions gained a tremendous impulse with Sklyanin’s [17] generalization of the Quantum Inverse Scattering Method [18] to accommodate the case of open boundaries. In Sklyanin’s approach, the construction of such models is based on solutions of the so-called reflection equations [17, 19] for a given bulk system. For the supersymmetric t-J model, two classes of diagonal solutions of the reflection equations were found in [20] corresponding to boundary chemical potentials and boundary magnetic fields. Physical properties like ground state structure and boundary susceptibilities have been studied in [21].

However, general non-diagonal open boundaries for the supersymmetric t-J model have not been considered so far in the literature. To our knowledge, the $Z_N$ Belavin model [22], the $SU(N)$ vertex model [23] and the spin-S Heisenberg chain [24] are the only multistates systems investigated so far with non-diagonal open boundaries. The recent progresses on this matter are mostly concentrated on the eight [25] and six [26, 27, 28] vertex models when the boundary parameters satisfy certain constraints.

Non-diagonal solutions of the reflection equations are known for a variety of integrable models based on $q$-deformed Lie algebras [29, 30, 31] but similar results concerning superalgebras, where the supersymmetric t-J model is inserted, are still concentrated on the $U_q[osp(1|2)]$ symmetry [32] and on diagonal solutions [33]. The main result of this paper is to show that the covering transfer matrix of the supersymmetric t-J model built from a general non-diagonal so-
olution of the reflection equation possess a trivial reference state needed to initiate an algebraic Bethe ansatz analysis.

This paper is organized as follows. In the next section we derive general solutions of the reflection equations for the supersymmetric t-J model. Two classes of solutions were found, a non-diagonal one with five free parameters and a diagonal solution with only one free parameter. In section 3 we perform the algebraic Bethe ansatz analysis and conclusions and remarks are presented in section 4.

2 Solutions of the Reflection Equation

In Sklyanin’s approach \cite{17}, the first step towards the construction of integrable models with open boundaries is to search for solutions of the reflection equations for a given integrable bulk system. This equation governs the integrability at the boundaries and it reads

\[ R_{21}(\lambda - \mu)K^-_2(\lambda)R_{12}(\lambda + \mu)K^+_1(\mu) = K^-_1(\mu)R_{21}(\lambda + \mu)K^-_2(\lambda)R_{12}(\lambda - \mu). \] (1)

The matrix \( K^-_1(\lambda) \) describes the reflection at one of the ends of an open chain, and a similar equation should also hold for a matrix \( K^+_1(\lambda) \) describing the reflection at the opposite boundary. In its turn the \( R \)-matrix entering in the reflection equation \((1)\) is a solution of the Yang-Baxter equation, namely

\[ R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu) \] (2)

defined in \( V \otimes V \otimes V \). Here \( V \) is a finite dimensional \( Z_2 \) graded space and \( R_{ij}(\lambda) \) consist of \( R(\lambda) \in \text{End}[V \otimes V] \) acting non trivially in the \( i \)th and \( j \)th spaces of \( V \otimes V \otimes V \).

The tensor products appearing in Eqs. \((1)\) and \((2)\) are defined as \([A \otimes B]_{ij}^{lk} = A_{ij}^l B_{k}^{j} (-1)^{(p_i+p_j)p_k}\) for generic matrices \( A \) and \( B \). This tensor product is equipped with Grassmann parities \( p_i \) assuming values on the group \( Z_2 \) which enable us to distinguish bosonic and fermionic degrees of freedom.

For the supersymmetric t-J model, the covering \( R \)-matrix consists of the rational \( sl(2|1) \) invariant solution of the Yang-Baxter equation,

\[ R(\lambda) = \sum_{i=1}^{3} a_i(\lambda) \hat{e}_{ii} \otimes \hat{e}_{ii} + b(\lambda) \sum_{i \neq j} (-1)^{p_j} \hat{e}_{jj} \otimes \hat{e}_{ii} + \sum_{i \neq j} \hat{e}_{ji} \otimes \hat{e}_{ij}, \] (3)

where the Boltzmann weights are explicitly given by \( a_i(\lambda) = 1 + (-1)^{p_i} \lambda \) and \( b(\lambda) = \lambda \). The elements \( \hat{e}_{ij} \) denotes usual \( 3 \times 3 \) Weyl matrices with components \( (\hat{e}_{ij})_{a\beta} = \delta_{ia}\delta_{j\beta} \) and in what follows we have adopted the grading \( p_1 = 1 \) and \( p_2 = p_3 = 0 \).

The \( R \)-matrix given in \((3)\) satisfies important symmetries relations namely,

\[
\begin{align*}
\text{PT-symmetry} & : \quad P_{12} R_{12}(\lambda) P_{12} = R_{12}^{\text{st1}}(\lambda) \\
\text{Unitarity} & : \quad R_{12}(\lambda) R_{12}(-\lambda) = (1 - \lambda^2) \text{Id} \otimes \text{Id} \\
\text{Cross-Unitarity} & : \quad R_{12}^{\text{st1}}(\lambda) M_1 R_{21}^{\text{st1}}(-\lambda - 2\rho) M_1^{-1} = \zeta(\lambda) \text{Id} \otimes \text{Id}
\end{align*}
\]
In the above relations $\text{Id}$ is the $3 \times 3$ identity matrix, $\zeta(\lambda)$ is a convenient normalization function and the matrix $M$ is a symmetry of the $R$-matrix, i.e.

$$[R(\lambda), M \otimes M] = 0.$$  (4)

The symbol $st_k$ stands for the supertransposition in the space labeled by the index $k$ and $P_{12} = \sum_{i,j=1}^{3} (-1)^{p_i p_j} \hat{e}_{ij} \otimes \hat{e}_{ji}$ denotes the graded permutator.

When these properties are fulfilled one can follow the scheme devised in [35, 36]. In this way the matrix $K^-(\lambda)$ is obtained by solving (1) and the matrix $K^+(\lambda)$ follows from the isomorphism

$$K^-(\lambda) \mapsto K^+(\lambda)^{st} = K^-(-\lambda - \rho)M.$$  (5)

In the case considered here we have $\rho = \frac{1}{2}$ and the matrix $M$ is the identity matrix.

Now we shall look for solutions of the reflection equation (1) in the most general form

$$K^{-}((l)) = \sum_{ij=1}^{3} k^{-}((l))_{ij} e_{ij}.$$  (6)

Substituting (6) and the $R$-matrix (3) in (1) we are left with a system of functional equations for the matrix elements $k^{-}((l))_{ij}$. A brute force analysis of these equations allow us to identify two branches of solutions

$$K^{-}(1) = \begin{pmatrix}
    k_{11}^{-}(1)(\lambda) & 0 & 0 \\
    0 & k_{22}^{-}(1)(\lambda) & 0 \\
    0 & 0 & k_{33}^{-}(1)(\lambda)
\end{pmatrix} K^{-}(2) = \begin{pmatrix}
    k_{11}^{-}(2)(\lambda) & 0 & 0 \\
    0 & k_{22}^{-}(2)(\lambda) & 0 \\
    0 & 0 & k_{33}^{-}(2)(\lambda)
\end{pmatrix}.  \quad (7)
$$

The upper index $(l)$ is aimed to distinguish these two branches and their non-null elements are determined from the following equations:

- Branch (1):

\[
2\lambda k_{11}^{-}(1)(\lambda) \frac{k_{32}^{-}(1)(\mu)}{k_{32}^{-}(1)(\lambda)} \left[ k_{11}^{-}(1)(\mu) - k_{33}^{-}(1)(\mu) \right]
= \left( \lambda + \mu \right) k_{11}^{-}(1)(\mu)^2 - 2\lambda k_{33}^{-}(1)(\mu) k_{11}^{-}(1)(\mu) \\
+ \left( \lambda - \mu \right) k_{23}^{-}(1)(\mu) k_{32}^{-}(1)(\mu)
\]

\[
2\lambda k_{22}^{-}(1)(\lambda) \frac{k_{32}^{-}(1)(\mu)}{k_{32}^{-}(1)(\lambda)} \left[ k_{11}^{-}(1)(\mu) - k_{33}^{-}(1)(\mu) \right]
= \left( \lambda + \mu \right) k_{11}^{-}(1)(\mu)^2 - 2\lambda k_{33}^{-}(1)(\mu) k_{11}^{-}(1)(\mu) \\
+ \left( \lambda - \mu \right) k_{23}^{-}(1)(\mu) k_{32}^{-}(1)(\mu)
\]

- Branch (2):

\[
2\lambda k_{11}^{-}(2)(\lambda) \frac{k_{32}^{-}(2)(\mu)}{k_{32}^{-}(2)(\lambda)} \left[ k_{11}^{-}(2)(\mu) - k_{33}^{-}(2)(\mu) \right]
= \left( \lambda + \mu \right) k_{11}^{-}(2)(\mu)^2 - 2\lambda k_{33}^{-}(2)(\mu) k_{11}^{-}(2)(\mu) \\
+ \left( \lambda - \mu \right) k_{23}^{-}(2)(\mu) k_{32}^{-}(2)(\mu)
\]
the other hand, the solution $q$ in the rational limit of one of the solutions presented in [20] for the
while the solution of Eqs. (13–14) turns out to be

\[
2k_{33}^{-1}(\lambda)k_{32}^{-1}(\mu) \left[ k_{11}^{-1}(\mu) - k_{33}^{-1}(\mu) \right] = -(\lambda - \mu)k_{11}^{-1}(\mu)^2 + 2\lambda k_{33}^{-1}(\mu)k_{11}^{-1}(\mu) - (\lambda + \mu)k_{33}^{-1}(\mu)^2 + (\lambda - \mu)k_{33}^{-1}(\mu)k_{32}^{-1}(\mu)
\]

(10)

\[
k_{22}^{-1}(\mu) \left[ k_{11}^{-1}(\mu) - k_{33}^{-1}(\mu) \right] = k_{11}^{-1}(\mu)^2 - k_{33}^{-1}(\mu)k_{11}^{-1}(\mu) - k_{33}^{-1}(\mu)k_{32}^{-1}(\mu)
\]

(11)

\[
k_{23}^{-1}(\lambda)k_{32}^{-1}(\mu) = k_{23}^{-1}(\mu)k_{32}^{-1}(\lambda)
\]

(12)

- Branch (2):

\[
k_{11}^{(2)}(\lambda) \left[ (\lambda - \mu)k_{11}^{(2)}(\mu) - (\lambda + \mu)k_{22}^{(2)}(\mu) \right] = -k_{22}^{(2)}(\lambda) \left[ (\lambda + \mu)k_{11}^{(2)}(\mu) + (\mu - \lambda)k_{22}^{(2)}(\mu) \right]
\]

(13)

\[
k_{22}^{(2)}(\lambda) = k_{33}^{(2)}(\lambda)
\]

(14)

The solution of Eqs. (13–14) associated to the non-diagonal branch is given by

\[
k_{11}^{(1)}(\lambda) = 1
\]

\[
k_{22}^{(1)}(\lambda) = \frac{h_0^- (h_3^- h_4^- + h_1^- h_2^-) + \lambda (h_5^- h_4^- + h_1^- h_2^-)}{(h_0^- - \lambda)(h_3^- h_4^- - h_1^- h_2^-)}
\]

\[
k_{33}^{(1)}(\lambda) = \frac{h_0^- (h_3^- h_4^- - h_1^- h_2^-) - \lambda (h_5^- h_4^- + h_1^- h_2^-)}{(h_0^- - \lambda)(h_5^- h_4^- - h_1^- h_2^-)}
\]

\[
k_{25}^{(1)}(\lambda) = \frac{2\lambda h_5^- h_3^-}{(h_0^- - \lambda)(h_3^- h_4^- - h_1^- h_2^-)}
\]

\[
k_{32}^{(1)}(\lambda) = \frac{2\lambda h_1^- h_4^-}{(h_0^- - \lambda)(h_1^- h_2^- - h_3^- h_4^-)}
\]

(15)

while the solution of Eqs. (13–14) turns out to be

\[
k_{11}^{(2)}(\lambda) = \frac{h_0^- + \lambda}{h_0^- - \lambda}
\]

\[
k_{22}^{(2)}(\lambda) = k_{33}^{(2)}(\lambda) = 1.
\]

(16)

The diagonal solution $K^{(2)}(\lambda)$ possess only one free parameter \{h_0^-\} and it is contained in the rational limit of one of the solutions presented in [20] for the $q$-deformed t-J model. On the other hand, the solution $K^{(1)}(\lambda)$ has altogether five free parameters \{h_0^-, h_1^-, h_2^-, h_3^-, h_4^-\} and it reduces to the second known diagonal solution by setting $h_3^- = h_4^- = 0$. Moreover, the non-diagonal solution $K^{(1)}(\lambda)$ has null entries in suitable positions for an algebraic Bethe ansatz study that will be explored in the next section. The question if this is the general scenario for the $sl(m|n)$ symmetry deserves to be further studied.
Next we turn our attention to the matrices $K^{+ (l)}(\lambda)$ which follows immediately from the isomorphism \([\mathbf{5}]\). Thus we have

\[
K^{+ (1)}(\lambda) = \left( \begin{array}{ccc}
k^{+ (1)}_{11}(\lambda) & 0 & 0 \\
0 & k^{+ (1)}_{22}(\lambda) & k^{+ (1)}_{23}(\lambda) \\
0 & k^{+ (1)}_{32}(\lambda) & k^{+ (1)}_{33}(\lambda) \end{array} \right) \quad K^{+ (2)}(\lambda) = \left( \begin{array}{ccc}
k^{+ (2)}_{11}(\lambda) & 0 & 0 \\
0 & k^{+ (2)}_{22}(\lambda) & k^{+ (2)}_{23}(\lambda) \\
0 & k^{+ (2)}_{32}(\lambda) & k^{+ (2)}_{33}(\lambda) \end{array} \right)
\]

where

\[
\begin{align*}
k^{+ (1)}_{11}(\lambda) &= 1 \\
k^{+ (1)}_{22}(\lambda) &= \frac{h_0^+ (h_3^+ h_4^+ - h_1^+ h_2^+)}{(h_0^+ + \frac{1}{2} + \lambda)(h_3^+ h_4^+ - h_1^+ h_2^+)} \\
k^{+ (1)}_{33}(\lambda) &= \frac{h_0^+ (h_3^+ h_4^+ - h_1^+ h_2^+)}{(h_0^+ + \frac{1}{2} + \lambda)(h_3^+ h_4^+ - h_1^+ h_2^+)} \\
k^{+ (1)}_{23}(\lambda) &= \frac{2(\lambda + \frac{1}{2}) h_2^+ h_3^+}{(h_0^+ + \frac{1}{2} + \lambda)(h_3^+ h_4^+ - h_1^+ h_2^+)} \\
k^{+ (1)}_{32}(\lambda) &= \frac{2(\lambda + \frac{1}{2}) h_1^+ h_3^+}{(h_0^+ + \frac{1}{2} + \lambda)(h_1^+ h_3^+ - h_3^+ h_4^+)}
\end{align*}
\]

and

\[
\begin{align*}
k^{+ (2)}_{11}(\lambda) &= \frac{h_0^+ - \frac{1}{2} - \lambda}{h_0^+ + \frac{1}{2} + \lambda} \\
k^{+ (2)}_{22}(\lambda) &= k^{+ (2)}_{33}(\lambda) = 1.
\end{align*}
\]

Here we remark that the matrices $K^{+ (l)}(\lambda)$ defined by Eqs. \([\mathbf{15}], [\mathbf{16}]\) consist of regular solutions, i.e. $K^{+ (l)}(0) = I$. One can also verify that

\[
\left[ K^{\pm (l)}(\lambda), K^{\pm (l)}(\mu) \right] = 0,
\]

which implies that the matrices $K^{\pm (l)}(\lambda)$ can be made diagonal by a similarity transformation independent of the spectral parameter $\lambda$. This later property has been used in \([\mathbf{23}], [\mathbf{24}]\).

Now we have the basic ingredients to build an integrable model with open boundary conditions. Following \([\mathbf{17}]\), the covering transfer matrix of the supersymmetric t-J model with open boundaries can be written as the following supertrace over the 3 x 3 auxiliary space $\mathcal{A}$,

\[
T^{(l,m)}(\lambda) = \text{str}_{\mathcal{A}} \left[ K^{+ (l)}_{\mathcal{A}}(\lambda) T_{\mathcal{A}}(\lambda) K^{-(m)}_{\mathcal{A}}(\lambda) \bar{T}_{\mathcal{A}}(\lambda) \right] \quad l, m = 1, 2
\]

where $T_{\mathcal{A}}(\lambda) = R_{\mathcal{A}L}(\lambda) R_{\mathcal{A}L-1}(\lambda) \ldots R_{\mathcal{A}1}(\lambda)$ and $\bar{T}_{\mathcal{A}}(\lambda) = R_{\mathcal{A}1}(\lambda) R_{\mathcal{A}2}(\lambda) \ldots R_{\mathcal{A}L}(\lambda)$ are the standard monodromy matrices that generate the corresponding closed t-J model with $L$ sites \([\mathbf{14}], [\mathbf{18}]\). We have also introduced the label $(l,m)$ in order to distinguish the $K$-matrices we
are considering since we can take \( K^{\pm(l)}(\lambda) \) either from \( K^{\pm(1)}(\lambda) \) or \( K^{\pm(2)}(\lambda) \). Thus altogether we have four different double-row transfer matrices.

Associated to each transfer matrix \( T^{(l,m)}(\lambda) \) we have an hamiltonian \( \mathcal{H}^{(l,m)} \) with open boundary conditions given by

\[
\mathcal{H}^{(l,m)} = \sum_{j=1}^{L-1} \sum_{\sigma=\pm} c_{j,\sigma}^\dagger (1 - n_{j,-\sigma}) c_{j+1,\sigma} (1 - n_{j+1,-\sigma}) + c_{j+1,\sigma}^\dagger (1 - n_{j+1,-\sigma}) c_{j,\sigma} (1 - n_{j,-\sigma})
\]

\[
+ 2 \sum_{j=1}^{L-1} \left[ S_j^z S_{j+1}^z + \frac{1}{2} (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) - \frac{1}{4} n_{j+1} n_{j+1} \right] + \sum_{j=1}^{L-1} \left( n_j + n_{j+1} \right) + \mathcal{H}^{B}_B^{(l,m)}
\]

(22)

where the integrable boundary terms \( \mathcal{H}^{B}_B^{(l,m)} \) are

\[
\mathcal{H}^{B,(1,1)} = \frac{- (h_0^+ h_2^+ + h_3^+ h_4^+)}{h_0 (h_1^+ h_2^+ h_3^+ h_4^+)} S_1^z - \frac{h_3^+ h_4^+}{h_0 (h_1^+ h_2^+ h_3^+ h_4^+)} S_1^z + \frac{h_1^+ h_4^+}{h_0 (h_1^+ h_2^- h_3^- h_4^-)} S_1^- + \frac{1}{2h_0} n_1
\]

\[
+ \frac{(h_0 - \frac{1}{2}) (h_1^+ h_2^- h_3^- h_4^-) S_L^z + h_0^+ h_3^+}{(h_0 - \frac{1}{2}) (h_1^+ h_2^- h_3^- h_4^-) S_L^z} - \frac{1}{2(h_0^+ - \frac{1}{2})} n_L
\]

(23)

\[
\mathcal{H}^{B,(1,2)} = - \frac{1}{h_0} n_1 + \frac{(h_0^+ h_2^+ + h_3^+ h_4^+)}{(h_0 - \frac{1}{2}) (h_1^+ h_2^+ h_3^+ h_4^+)} S_L^z + \frac{h_3^+ h_4^+}{(h_0 - \frac{1}{2}) (h_1^+ h_2^- h_3^- h_4^-) S_L^- - \frac{1}{2(h_0^+ - \frac{1}{2})} n_L}
\]

(24)

\[
\mathcal{H}^{B,(2,1)} = \frac{(h_0^+ h_2^+ + h_3^+ h_4^+)}{h_0 (h_1^+ h_2^- h_3^- h_4^-) S_1^z - \frac{h_3^+ h_4^+}{h_0 (h_1^+ h_2^- h_3^- h_4^-)} S_1^z + \frac{h_1^+ h_4^+}{h_0 (h_1^+ h_2^- h_3^- h_4^-)} S_1^- + \frac{1}{2h_0} n_1}
\]

\[
+ \frac{1}{(h_0^+ + \frac{1}{2})} n_L
\]

(25)

\[
\mathcal{H}^{B,(2,2)} = - \frac{1}{h_0} n_1 + \frac{1}{(h_0^+ + \frac{1}{2})} n_L.
\]

(26)

The hamiltonians \( \mathcal{H}^{(l,m)} \) are identified with the supersymmetric t-J model with open boundary conditions and, omitting terms proportional to the identity, they are proportional to \( \frac{d}{d\lambda} T^{(l,m)}(\lambda) \mid_{\lambda=0} \). We have expressed the hamiltonians (22-26) in terms of fermionic creation and annihilation operators \( c_{j,\sigma}^\dagger \) and \( c_{j,\sigma} \) acting on the site \( j \) and carrying spin index \( \sigma = \pm \). The spin operators \( S_j^+ \), \( S_j^- \) and \( S_j^z \) form an \( su(2) \) algebra, and the number operators are denoted \( n_j = n_{j,+} + n_{j,-} \) where \( n_{j,\sigma} = c_{j,\sigma}^\dagger c_{j,\sigma} \). Except for the case \( \mathcal{H}^{(2,2)} \), the remaining hamiltonians contain non-diagonal boundary terms. In the next section we will discuss the diagonalization of the transfer matrices \( T^{(l,m)}(\lambda) \) through the algebraic Bethe ansatz.
3 Algebraic Bethe Ansatz

The purpose of this section is to determine the eigenvalues and eigenvectors of the transfer matrices defined in Eq. (21) built from the two branches of solution of the reflection equations. For instance, when the $K$-matrices are diagonal this problem have been solved by the nested algebraic Bethe ansatz [20]. Despite of the recent efforts in solving commuting transfer matrices with general open boundary conditions [22]-[27], the progresses are modest when compared with the literature known for the diagonal case [33, 34]. Usually the diagonalization of double-row transfer matrices with non-diagonal $K$-matrices is a tantalizing problem due to the difficulty in finding a suitable reference state to perform a Bethe ansatz analysis. Here we note that the non-diagonal $K$-matrices for the supersymmetric t-J model permit us to use the usual ferromagnetic state as pseudovacuum state $\Psi_0$.

Next we consider the action of the elements of $U_{A}^{(l)}(\lambda)$ on the pseudovacuum state $|\Psi_0\rangle$ defined as

$$|\Psi_0\rangle = \bigotimes_{j=1}^{L} |0\rangle_j \quad |0\rangle_j = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$  \hfill (30)

The definitions (21,27,30) allow us to show that the elements of $U_{A}^{(l)}(\lambda)$ satisfy the relations

$$A^{(l)}(\lambda) |\Psi_0\rangle = k_{11}^{(l)}(\lambda)a_1^{2L}(\lambda) |\Psi_0\rangle$$

$$D_{11}^{(l)}(\lambda) |\Psi_0\rangle = \left\{ \frac{k_{11}^{(l)}(\lambda)}{a_1(2\lambda)}a_1^{2L}(\lambda) + \left[ k_{22}^{(l)}(\lambda) - \frac{k_{11}^{(l)}(\lambda)}{a_1(2\lambda)} \right] b^{2L}(\lambda) \right\} |\Psi_0\rangle$$

$$D_{22}^{(l)}(\lambda) |\Psi_0\rangle = \left\{ \frac{k_{11}^{(l)}(\lambda)}{a_1(2\lambda)}a_1^{2L}(\lambda) + \left[ k_{33}^{(l)}(\lambda) - \frac{k_{11}^{(l)}(\lambda)}{a_1(2\lambda)} \right] b^{2L}(\lambda) \right\} |\Psi_0\rangle$$

$$D_{12}^{(l)}(\lambda) |\Psi_0\rangle = k_{23}^{(l)}(\lambda)b^{2L}(\lambda) |\Psi_0\rangle$$

$$D_{21}^{(l)}(\lambda) |\Psi_0\rangle = k_{32}^{(l)}(\lambda)b^{2L}(\lambda) |\Psi_0\rangle$$

$$B_i^{(l)}(\lambda) |\Psi_0\rangle = \dagger$$

$$C_i^{(l)}(\lambda) |\Psi_0\rangle = 0 \quad i = 1, 2 \hfill (31)$$

\footnote{The author thanks M.J. Martins for pointing out this possibility.}
where the symbol $\dagger$ stands for a non-null value.

The relations (31) together with the Eq. (29) imply that $|\Psi_0\rangle$ is an eigenvector of $T^{(l,m)}(\lambda)$ whose respective eigenvalue is

$$\Lambda_0^{(l,m)}(\lambda) = \left\{ -k_{11}^{(l)}(\lambda)k_{11}^{-(m)}(\lambda) + \frac{k_{11}^{-(m)}(\lambda)}{a_1(2\lambda)} \sum_{i=2}^{3} k_{ii}^{+(l)}(\lambda) \right\} a_{2L}^{2L}(\lambda)$$

$$+ \left\{ \sum_{i,j=2}^{3} k_{ij}^{+(l)}(\lambda)k_{ji}^{-(m)}(\lambda) - \frac{k_{11}^{-(m)}(\lambda)}{a_1(2\lambda)} \sum_{i=2}^{3} k_{ii}^{+(l)}(\lambda) \right\} b_{2L}^{2L}(\lambda). \quad (32)$$

In the framework of the algebraic Bethe ansatz we now seek for other eigenvectors of $T^{(l,m)}(\lambda)$ in the multiparticle state form

$$|\Psi\rangle = B_{a_1}^{(l)}(\lambda_1^{(1)})B_{a_2}^{(l)}(\lambda_2^{(1)}) \ldots B_{a_{n_1}}^{(l)}(\lambda_{n_1}^{(1)}) \mathcal{F}^a_{1a_2 \ldots a_{n_1}} |\Psi_0\rangle. \quad (33)$$

In order to accomplish this task, the next step is to write appropriate commutation relations for the elements of $U_{\lambda}^{(l)}(\lambda)$ which also satisfies the quadratic relation (11) with $K(\lambda)$ replaced by $U_{\lambda}^{(l)}(\lambda)$. From Eqs. (11) and (27) it follows that three of those commutation relations are of great use, namely

$$A^{(l)}(\lambda)B_j^{(l)}(\mu) = \frac{a_1(\mu - \lambda)}{b(\mu - \lambda)} \frac{b(\mu + \lambda)}{a_1(\mu + \lambda)} B_j^{(l)}(\mu)A^{(l)}(\lambda) - \frac{1}{a_1(\lambda + \mu)} B_j^{(l)}(\lambda)\tilde{D}_j^{(l)}(\mu) \quad (34)$$

$$\tilde{D}_{ij}^{(l)}(\lambda)B_k^{(l)}(\mu) = \frac{r_{id}^{(l)}(\lambda + \mu - 1)r_{id}^{(l)}(\lambda - \mu - 1)}{b(\lambda + \mu - 1)b(\lambda + \mu - 1)} B_k^{(l)}(\mu)\tilde{D}_j^{(l)}(\lambda) + \frac{r_{id}^{(l)}(\lambda + \mu - 1)}{a_1(\lambda + \mu)} B_k^{(l)}(\lambda)\tilde{D}_j^{(l)}(\mu) \quad (35)$$

$$B_i^{(l)}(\lambda)B_j^{(l)}(\mu) = B_k^{(l)}(\mu)B_i^{(l)}(\lambda) \frac{r_{ik}^{(l)}(\lambda - \mu)}{a_1(\lambda - \mu)}. \quad (36)$$

In the above relations we have defined the operator

$$\tilde{D}_j^{(l)}(\lambda) = D_j^{(l)}(\lambda) - \frac{\delta_{ij}}{a_1(2\lambda)} A^{(l)}(\lambda) \quad (37)$$

and $r_{ij}^{(l)}(\lambda)$ denotes the matrix elements of the rational $sl(2)$ $R$-matrix,

$$r(\lambda) = \begin{pmatrix} 1 + \lambda & 0 & 0 & 0 \\ 0 & \lambda - 1 & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & 1 + \lambda \end{pmatrix}. \quad (38)$$

$^2$We have used the convention $r(\lambda) = \sum_{i,j,k,l}^{2} r_{ij}^{kl}(\lambda)^* c_{ij} \otimes c_{kl}$, where now $c_{ij}$ denotes $2 \times 2$ Weyl matrices.
By carrying on the fields $A^{(l)}(\lambda)$ and $D_{ij}^{(l)}(\lambda)$ over the multiparticle state $|\Psi\rangle$ we generate terms that are proportional to $|\Psi\rangle$ and others that are not. The terms proportional to $|\Psi\rangle$ contribute to the eigenvalue while the other terms are usually denominated unwanted terms. The eigenvalue $\Lambda^{(l,m)}(\lambda)$ is obtained from the first terms of the commutation relations (34,35), together with the requirement that $F_{\alpha_1\alpha_2...\alpha_{n_2}}$ are components of the eigenstates of an auxiliary double-row operator $\tilde{T}^{(l,m)}(\lambda, \{\lambda^{(1)}_j\})$ whose eigenvalue equation reads

$$\tilde{T}^{(l,m)}(\lambda, \{\lambda^{(1)}_j\}) |\mathcal{F}\rangle = \tilde{\Lambda}^{(l,m)}(\lambda, \{\lambda^{(1)}_j\}) |\mathcal{F}\rangle. \quad (39)$$

This auxiliary operator is given by the following trace over the $2 \times 2$ auxiliary space $\tilde{A}$

$$\tilde{T}^{(l,m)}(\lambda, \{\lambda^{(1)}_j\}) = \text{Tr}_{\tilde{A}} \left[ K_{\tilde{A}}^{(+l)}(\lambda) r_{\tilde{A}l} (\lambda + \lambda^{(1)}_1 - 1) \ldots r_{\tilde{A}n_2} (\lambda + \lambda^{(1)}_{n_2} - 1) \bar{K}_{\tilde{A}}^{(m)}(\lambda) r_{\tilde{A}n_1} (\lambda - \lambda^{(1)}_{n_1}) \ldots r_{\tilde{A}1} (\lambda - \lambda^{(1)}_1) \right]$$

and the associated $K$-matrices are given by

$$K^{(+l)}(\lambda) = \begin{pmatrix} k_{22}^{(+l)}(\lambda) & k_{23}^{(+l)}(\lambda) \\ k_{32}^{(+l)}(\lambda) & k_{33}^{(+l)}(\lambda) \end{pmatrix}, \quad \bar{K}^{(-l)}(\lambda) = \begin{pmatrix} k_{22}^{(-l)}(\lambda) - \frac{k_{11}^{(+l)}(\lambda)}{a_1(2\lambda)} & k_{23}^{(-l)}(\lambda) \\ k_{32}^{(-l)}(\lambda) & k_{33}^{(-l)}(\lambda) - \frac{k_{11}^{(-l)}(\lambda)}{a_1(2\lambda)} \end{pmatrix}. \quad (40)$$

For instance, the unwanted terms are eliminated provided the set of rapidities $\{\lambda^{(1)}_j\}$ satisfy the Bethe ansatz equations

$$\left[ \frac{a_1(\lambda^{(1)}_j)}{b(\lambda^{(1)}_j)} \right]^{2L} \left[ \frac{1}{a_1(2\lambda^{(1)}_j)} \sum_{i=2}^{3} k_{ii}^{(+l)}(\lambda^{(1)}_j) - k_{11}^{(+l)}(\lambda^{(1)}_j) \right] k_{11}^{(-m)}(\lambda^{(1)}_j) b(2\lambda^{(1)}_j) = (-1)^{n_1} \prod_{i \neq j}^{n_2} \frac{a_1(\lambda^{(1)}_i + \lambda^{(1)}_j)}{b(\lambda^{(1)}_i + \lambda^{(1)}_j) a_1(\lambda^{(1)}_i - \lambda^{(1)}_j) b(\lambda^{(1)}_i + \lambda^{(1)}_j - 1)} \tilde{\Lambda}^{(l,m)}(\lambda = \lambda^{(1)}_1, \{\lambda^{(1)}_j\}). \quad (42)$$

Thus we are left with the following expression for the eigenvalues $\Lambda^{(l,m)}(\lambda)$,

$$\Lambda^{(l,m)}(\lambda) = \left[ \frac{1}{a_1(2\lambda)} \sum_{i=2}^{3} k_{ii}^{(+l)}(\lambda) - k_{11}^{(+l)}(\lambda) \right] k_{11}^{(-m)}(\lambda) a_1(\lambda)^{2L} \prod_{i=1}^{n_1} \frac{a_1(\lambda^{(1)}_i - \lambda)}{b(\lambda^{(1)}_i - \lambda)} \frac{b(\lambda^{(1)}_i + \lambda)}{a_1(\lambda^{(1)}_i + \lambda)} + b(\lambda)^{2L} \prod_{i=1}^{n_1} \frac{1}{b(\lambda - \lambda^{(1)}_i) b(\lambda + \lambda^{(1)}_i - 1)} \tilde{\Lambda}^{(l,m)}(\lambda, \{\lambda^{(1)}_j\}). \quad (43)$$

This completes only the first step of the Bethe ansatz analysis since we still need to determine the eigenvalues $\Lambda^{(l,m)}(\lambda, \{\lambda^{(1)}_j\})$.

The auxiliary transfer matrix $\tilde{T}^{(l,m)}(\lambda, \{\lambda^{(1)}_j\})$ defined in Eq. (40) corresponds to that of an inhomogeneous $sl(2)$ vertex model with non-diagonal open boundaries. Typically, this problem
is carried out by a second Bethe ansatz. Here we note that the eigenvalues of \( \bar{T}^{(l,m)}(\lambda, \{\lambda_j^{(1)}\}) \) for the case \( n_1 = 1 \) can be obtained by conventional methods.

For the case \( n_1 = 1 \) the auxiliary double-row operator reads

\[
\bar{T}^{(l,m)}(\lambda, \{\lambda_1^{(1)}\}) = \Tr \left[ \bar{K}^{(l)}_A(\lambda)r_A(\lambda + \lambda_1^{(1)} - 1)\bar{K}^{-(m)}_A(\lambda)r_A(\lambda - \lambda_1^{(1)}) \right],
\]

which consist of a 2\( \times \)2 matrix. The secular equation, \( \det \left[ \bar{T}^{(l,m)}(\lambda, \{\lambda_1^{(1)}\}) - \bar{\Lambda}^{(l,m)}(\lambda, \{\lambda_1^{(1)}\}) \right] = 0 \), in this case gives

\[
X_2^{(l,m)} \bar{\Lambda}^{(l,m)}(\lambda, \{\lambda_1^{(1)}\})^2 + X_1^{(l,m)} \bar{\Lambda}^{(l,m)}(\lambda, \{\lambda_1^{(1)}\}) + X_0^{(l,m)} = 0,
\]

leaving us with the following expression for the eigenvalues \( \bar{\Lambda}^{(l,m)}(\lambda, \{\lambda_1^{(1)}\}) \),

\[
\bar{\Lambda}^{(l,m)}(\lambda, \{\lambda_1^{(1)}\}) = \frac{-X_1^{(l,m)} \pm \sqrt{(X_1^{(l,m)})^2 - 4X_2^{(l,m)}X_0^{(l,m)}}}{2X_2^{(l,m)}}.
\]

In order to avoid an overcrowded section we have collected the functions \( X_i^{(l,m)} \) in the appendix A. The relation (46) together with (42) and (43) determines the eigenvalues of \( T^{(l,m)}(\lambda) \) in the sector \( n_1 = 1 \) without any constraint for the boundary parameters.

For general \( n_1 \) the nested problem consists in the diagonalization of the auxiliary transfer matrix \( \bar{T}^{(l,m)}(\lambda, \{\lambda_j^{(1)}\}) \) whose dimension is \( 2^{n_1} \times 2^{n_1} \). This problem can be tackled by a second Bethe ansatz in the lines of [23, 24]. Except for the case \( \bar{T}^{(1,1)}(\lambda, \{\lambda_j^{(1)}\}) \), no constraint is required for the boundary parameters.

In order to do that we first note that the associated boundary matrices \( \bar{K}^{(l)}(\lambda) \) also satisfy \( \left[ \bar{K}^{(l)}(\lambda), \bar{K}^{(l)}(\mu) \right] = 0 \). Thus they can be written as

\[
\bar{K}^{(l)}(\lambda) = \mathcal{G}^{(l)}(\lambda) \bar{D}^{(l)}(\lambda) \left( \mathcal{G}^{(l)} \right)^{-1}
\]

where \( \bar{D}^{(l)}(\lambda) \) is a diagonal matrix and \( \mathcal{G}^{(l)}(\lambda) \) is independent of the spectral parameter \( \lambda \). The matrices \( \mathcal{G}^{(1)}(\lambda) \) are then given by

\[
\mathcal{G}^{(1)}(\lambda) = \begin{pmatrix} h_1^\pm & -h_3^\pm \\ -h_2^\pm & h_4^\pm \end{pmatrix},
\]

while the diagonal matrices \( \bar{D}^{(1)}(\lambda) \) turn out to be

\[
\bar{D}^{(1)}(\lambda) = \begin{pmatrix} 1 & 0 \\ \frac{h_0^\pm - \lambda - \frac{1}{2}}{h_0^\pm + \lambda + \frac{1}{2}} & \lambda - \frac{1}{2} \end{pmatrix},
\]

\[
\bar{D}^{-(1)}(\lambda) = \begin{pmatrix} \lambda & 0 \\ \frac{h_0^- + \lambda - 1}{h_0^- - \lambda} & 1 \end{pmatrix}.
\]

By way of contrast, the matrices \( \bar{K}^{(2)}(\lambda) \) are proportional to the identity and the relation (47) becomes trivial.
Next we proceed by inserting terms $\mathcal{G}_A^{(l)} \left( \mathcal{G}_A^{(l)} \right)^{-1}$ in between all the elements of the double-row operator (10). In this way the transfer matrix $\tilde{T}^{(l,m)}(\lambda, \{\lambda_j^{(1)}\})$ can be rewritten as

$$
\tilde{T}^{(l,m)}(\lambda, \{\lambda_j^{(1)}\}) = \text{Tr}_A \left[ D_A^{(l)}(\lambda) \tilde{r}_A^{(l)}(\lambda + \lambda_1^{(1)} - 1) \ldots \tilde{r}_{A_{n_1}}^{(l)}(\lambda + \lambda_{n_1}^{(1)} - 1) \tilde{K}_A^{-(l,m)}(\lambda) \tilde{r}_{A_{n_1}}^{(l)}(\lambda - \lambda_{n_1}^{(1)}) \ldots \tilde{r}_A^{(l)}(\lambda - \lambda_1^{(1)}) \right]
$$

(50)

where $\tilde{r}_A^{(l)}(\lambda) = \left( \mathcal{G}_A^{(l)} \right)^{-1} r_A^{(l)}(\lambda) \mathcal{G}_A^{(l)}$ and $\tilde{K}_A^{-(l,m)}(\lambda) = \left( \mathcal{G}_A^{(l)} \right)^{-1} \tilde{K}_A^{-(m)}(\lambda) \mathcal{G}_A^{(l)}$. Fortunately in this case the gauge transformation on the operator $r_A^{(l)}(\lambda)$ can be reversed through the transformation $r_A^{(l)}(\lambda) = \left( \mathcal{G}_A^{(l)} \right)^{-1} \tilde{r}_A^{(l)}(\lambda) \mathcal{G}_A^{(l)}$. Then the transfer matrix $\tilde{T}^{(l,m)}(\lambda, \{\lambda_j^{(1)}\})$ defined as

$$
\tilde{T}^{(l,m)}(\lambda, \{\lambda_j^{(1)}\}) = \prod_{j=1}^{n_1} \left( \mathcal{G}_j^{(l)} \right)^{-1} \tilde{T}^{(l,m)}(\lambda, \{\lambda_j^{(1)}\}) \prod_{j=1}^{n_1} \mathcal{G}_j^{(l)}
$$

(51)

is given by

$$
\tilde{T}^{(l,m)}(\lambda, \{\lambda_j^{(1)}\}) = \text{Tr}_A \left[ D_A^{(l)}(\lambda) r_A^{(l)}(\lambda + \lambda_1^{(1)} - 1) \ldots r_{A_{n_1}}^{(l)}(\lambda + \lambda_{n_1}^{(1)} - 1) \tilde{K}_A^{-(l,m)}(\lambda) r_{A_{n_1}}^{(l)}(\lambda - \lambda_{n_1}^{(1)}) \ldots r_A^{(l)}(\lambda - \lambda_1^{(1)}) \right].
$$

(52)

Clearly, the operators $\tilde{T}^{(l,m)}(\lambda, \{\lambda_j^{(1)}\})$ and $\tilde{T}^{(l,m)}(\lambda, \{\lambda_j^{(1)}\})$ share the same eigenvalues and the eigenvectors $|\tilde{F}\rangle$ of $\tilde{T}^{(l,m)}(\lambda, \{\lambda_j^{(1)}\})$ are related to $|F\rangle$ by

$$
|\tilde{F}\rangle = \prod_{j=1}^{n_1} \left( \mathcal{G}_j^{(l)} \right)^{-1} |F\rangle.
$$

(53)

A careful examination of the matrix $\tilde{K}^{-(1,1)}(\lambda)$ reveals that it consists of a non-diagonal matrix which still makes difficult to find an appropriate reference state to perform a Bethe ansatz analysis. However, the requirement that $\tilde{K}^{-(1,1)}(\lambda)$ is upper or lower triangular allows the standard $sl(2)$ highest weight states to be used as pseudovacuum state in the algebraic Bethe ansatz framework. By requiring that $\tilde{K}^{-(1,1)}(\lambda)$ is upper triangular we find two possible classes of restrictions for the boundary parameters, namely

$$
C(1,1,a) : \quad h_3^{-} h_4^{+} = h_4^{-} h_3^{+}
$$

(54)

$$
C(1,1,b) : \quad h_1^{-} h_2^{+} = h_2^{-} h_1^{+}.
$$

(55)

On the other hand the matrix $\tilde{K}^{-(1,2)}(\lambda)$ is already diagonal and the diagonalization of the double-row operator $\tilde{T}^{(1,2)}(\lambda, \{\lambda_j^{(1)}\})$ can be performed without any restriction for the boundary parameters.
For the case $\bar{T}^{(2,1)}(\lambda, \{\lambda_j^{(1)}\})$ we shall adopt a different strategy. In that case we insert terms $\mathcal{G}_A^{-1}(\mathcal{G}_A^{-1})^{-1}$ in between all the elements of the respective double-row operator \cite{10}. In this way the transfer matrix $\bar{T}^{(2,1)}(\lambda, \{\lambda_j^{(1)}\})$ can be rewritten as

$$
\bar{T}^{(2,1)}(\lambda, \{\lambda_j^{(1)}\}) = \text{Tr}_A \left[ \tilde{K}_A^{+(2,1)}(\lambda) \tilde{r}_{A1}^{(1)}(\lambda + \lambda_1^{(1)} - 1) \ldots \tilde{r}_{A_{n1}}^{(1)}(\lambda + \lambda_{n1}^{(1)} - 1) D_A^{-1}(\lambda) \tilde{r}_{A_{n1}}^{(1)}(\lambda - \lambda_{n1}^{(1)}) \ldots \tilde{r}_{A1}^{(1)}(\lambda - \lambda_1^{(1)}) \right]
$$

(56)

where $\tilde{r}_{Aj}^{(1)}(\lambda) = (\mathcal{G}_A^{-1})^{-1} r_{Aj}^{(1)}(\lambda) \mathcal{G}_A^{-1}$ and $\tilde{K}_A^{+(2,1)}(\lambda) = (\mathcal{G}_A^{-1})^{-1} \bar{K}_A^{+(2,1)}(\lambda) \mathcal{G}_A^{-1}$.

In the same way as before \cite{51} we can define a new transfer matrix $T^{(2,1)}(\lambda, \{\lambda_j^{(1)}\})$ sharing the same eigenvalues with $\bar{T}^{(2,1)}(\lambda, \{\lambda_j^{(1)}\})$. The new double-row operator $\tilde{T}^{(2,1)}(\lambda, \{\lambda_j^{(1)}\})$ is given by

$$
\tilde{T}^{(2,1)}(\lambda, \{\lambda_j^{(1)}\}) = \text{Tr}_A \left[ \tilde{K}_A^{+(2,1)}(\lambda) r_{A1}^{(1)}(\lambda + \lambda_1^{(1)} - 1) \ldots r_{A_{n1}}^{(1)}(\lambda + \lambda_{n1}^{(1)} - 1) D_A^{-1}(\lambda) r_{A_{n1}}^{(1)}(\lambda - \lambda_{n1}^{(1)}) \ldots r_{A1}^{(1)}(\lambda - \lambda_1^{(1)}) \right]
$$

(58)

due to the relation

$$
r_{Aj}^{(1)}(\lambda) = (\mathcal{G}_A^{-1})^{-1} \tilde{r}_{Aj}^{(1)}(\lambda) \mathcal{G}_A^{-1}.
$$

(59)

For instance, the matrix $\tilde{K}_A^{+(2,1)}(\lambda)$ is also diagonal and the eigenvalues $\tilde{\Lambda}^{(2,1)}(\lambda, \{\lambda_j^{(1)}\})$ can be obtained for general values of the boundary parameters. The remaining case $\tilde{T}^{(2,2)}(\lambda, \{\lambda_j^{(1)}\})$ contains only diagonal $K$-matrices and its diagonalization is straightforward through the algebraic Bethe ansatz. Considering that the diagonalization of double-row transfer matrices with diagonal $K$-matrices has been well explained in the literature, see for instance \cite{20, 37}, we restrict ourselves in presenting only the final solution for $\Lambda^{(l,m)}(\lambda)$.

$$
\Lambda^{(l,m)}(\lambda) = Q_1^{(l,m)}(\lambda) a_1^{2l}(\lambda) \prod_{i=1}^{n_1} \frac{(\lambda - \lambda_i^{(1)} + \frac{1}{2}) (\lambda + \lambda_i^{(1)} + \frac{1}{2})}{(\lambda - \lambda_i^{(1)} - \frac{1}{2}) (\lambda + \lambda_i^{(1)} - \frac{1}{2})} + Q_2^{(l,m)}(\lambda) b^{2L}(\lambda) \prod_{i=1}^{n_1} \frac{(\lambda - \lambda_i^{(1)} + \frac{1}{2}) (\lambda + \lambda_i^{(1)} + \frac{1}{2})}{(\lambda - \lambda_i^{(1)} - \frac{1}{2}) (\lambda + \lambda_i^{(1)} - \frac{1}{2})} \prod_{i=1}^{n_2} \frac{(\lambda - \lambda_i^{(2)} - 1) (\lambda + \lambda_i^{(2)} - 1)}{(\lambda - \lambda_i^{(2)}) (\lambda + \lambda_i^{(2)})} + Q_3^{(l,m)}(\lambda) b^{2L}(\lambda) \prod_{i=1}^{n_2} \frac{(\lambda - \lambda_i^{(2)} + 1) (\lambda + \lambda_i^{(2)} + 1)}{(\lambda - \lambda_i^{(2)}) (\lambda + \lambda_i^{(2)})},
$$

(60)

12
where the functions $Q_{i}^{(l,m)}(\lambda)$ are given by

\[
Q_1^{(1,1,a)}(\lambda) = -\frac{(\lambda + \frac{1}{2})(\lambda + h_0^+ - \frac{1}{2})}{(\lambda - \frac{1}{2})(\lambda + h_0^- + \frac{1}{2})} \quad Q_1^{(1,1,b)}(\lambda) = -\frac{(\lambda + \frac{1}{2})(\lambda + h_0^+ - \frac{1}{2})}{(\lambda - \frac{1}{2})(\lambda + h_0^- + \frac{1}{2})} \\
Q_2^{(1,1,a)}(\lambda) = \frac{(\lambda + \frac{1}{2})(\lambda + h_0^- - \frac{1}{2})}{(\lambda - \frac{1}{2})(\lambda + h_0^- + \frac{1}{2})} \quad Q_2^{(1,1,b)}(\lambda) = \frac{(\lambda + \frac{1}{2})(\lambda + h_0^- + \frac{1}{2})}{(\lambda - \frac{1}{2})(\lambda + h_0^- + \frac{1}{2})} \\
Q_3^{(1,1,a)}(\lambda) = \frac{(\lambda + h_0^-)(\lambda - h_0^+ + \frac{1}{2})}{(\lambda - h_0^-)(\lambda + h_0^+ + \frac{1}{2})} \quad Q_3^{(1,1,b)}(\lambda) = \frac{(\lambda - h_0^- + 1)(h_0^+ - \lambda - \frac{1}{2})}{(\lambda - h_0^-)(h_0^+ - \lambda + \frac{1}{2})}
\]

In order to capture the two classes of restrictions (54, 55), in which we are able to present the eigenvalue $\Lambda^{(1,1)}(\lambda)$, we have used $Q_{i}^{(1,1,a)}(\lambda)$ and $Q_{i}^{(1,1,b)}(\lambda)$ to denote the functions $Q_{i}^{(1,1)}(\lambda)$ obtained under the constraints $C(1,1,a)$ and $C(1,1,b)$ respectively. The set of rapidities $\{\lambda_{1}^{(2)}, \ldots, \lambda_{n_2}^{(2)}\}$, introduced in the diagonalization of $T^{(l,m)}(\lambda, \{\lambda_{j}^{(1)}\})$, together with the set $\{\lambda_{1}^{(1)}, \ldots, \lambda_{n_1}^{(1)}\}$ are required to satisfy the following Bethe ansatz equations,

\[
\begin{align*}
\left(\frac{\lambda_{j}^{(1)} - \frac{1}{2}}{\lambda_{j}^{(1)} + \frac{1}{2}}\right)^{2L} \Theta_1^{(l,m)}(\lambda_{j}^{(1)}) &= \prod_{i=1}^{n_1} \frac{(\lambda_{j}^{(1)} - \lambda_{i}^{(2)} - \frac{1}{2})(\lambda_{j}^{(1)} + \lambda_{i}^{(2)} - \frac{1}{2})}{(\lambda_{j}^{(1)} - \lambda_{i}^{(2)} + \frac{1}{2})(\lambda_{j}^{(1)} + \lambda_{i}^{(2)} + \frac{1}{2})} \\
\prod_{i=1}^{n_1} \frac{(\lambda_{j}^{(2)} - \lambda_{i}^{(1)} + \frac{1}{2})(\lambda_{j}^{(2)} + \lambda_{i}^{(1)} + \frac{1}{2})}{(\lambda_{j}^{(2)} - \lambda_{i}^{(1)} - \frac{1}{2})(\lambda_{j}^{(2)} + \lambda_{i}^{(1)} - \frac{1}{2})} \Theta_2^{(l,m)}(\lambda_{j}^{(2)}) &= \prod_{i \neq j}^{n_2} \frac{(\lambda_{j}^{(1)} - \lambda_{i}^{(1)} - 1)(\lambda_{j}^{(1)} + \lambda_{i}^{(1)} + 1)}{(\lambda_{j}^{(1)} - \lambda_{i}^{(1)} + 1)(\lambda_{j}^{(1)} + \lambda_{i}^{(1)} - 1)}
\end{align*}
\]

The functions $\Theta_1^{(l,m)}(\lambda)$ are given by

\[
\Theta_1^{(1,1,a)}(\lambda) = 1 \quad \Theta_1^{(1,1,a)}(\lambda) = \frac{(\lambda + h_0^+ - \frac{1}{2})(\lambda - h_0^-)}{(\lambda - h_0^+ + \frac{1}{2})(\lambda + h_0^-)}
\]
\[ \Theta_{1}^{(1,1,a)}(\lambda) = \frac{(h_0^- - \lambda - \frac{1}{2})}{(h_0^- + \lambda - \frac{1}{2})} \quad \Theta_{2}^{(1,1,b)}(\lambda) = \frac{(\lambda + h_0^+ - \frac{1}{2}) (\lambda + h_0^- - 1)}{(\lambda - h_0^+ + \frac{1}{2}) (\lambda - h_0^- + 1)} \] (72)

\[ \Theta_{1}^{(1,2)}(\lambda) = \frac{(h_0^+ + \lambda + \frac{1}{2})}{(h_0^- - \lambda - \frac{1}{2})} \quad \Theta_{2}^{(1,2)}(\lambda) = \frac{(h_0^+ + \lambda - \frac{1}{2})}{(h_0^- - \lambda - \frac{1}{2})} \] (73)

\[ \Theta_{1}^{(2,1)}(\lambda) = \frac{(h_0^- - \lambda + 1)}{(h_0^+ + \lambda + 1)} \quad \Theta_{2}^{(2,1)}(\lambda) = \frac{(h_0^- - \lambda)}{(h_0^+ + \lambda)} \] (74)

\[ \Theta_{1}^{(2,2)}(\lambda) = \frac{(\lambda - h_0^+ - 1) (\lambda + h_0^- + \frac{1}{2})}{(\lambda + h_0^- + 1) (\lambda - h_0^- - \frac{1}{2})} \quad \Theta_{2}^{(2,2)}(\lambda) = 1 \] (75)

where \( \Theta_{i}^{(1,1,a)}(\lambda) \) and \( \Theta_{i}^{(1,1,b)}(\lambda) \) denote the functions \( \Theta_{i}^{(1,1)}(\lambda) \) under the constraints \( C(1,1,a) \) and \( C(1,1,b) \) respectively.

Considering our results so far, the eigenvalues \( E^{(l,m)} \) of the hamiltonian \( H^{(l,m)} \) are given by

\[ E^{(l,m)} = -\sum_{j=1}^{n_1} \frac{1}{(\lambda_j^{(1)})^2 - \frac{1}{4}}, \] (76)

where the rapidities \( \lambda_j^{(1)} \) satisfy the Bethe ansatz equations (70) with the corresponding index \( (l,m) \).

## 4 Concluding Remarks

In this work we have constructed supersymmetric t-J models with integrable open boundaries through the Quantum Inverse Scattering Method. Four different kinds of open boundaries are obtained: one having only diagonal elements, two with one diagonal boundary and the other non-diagonal, and one with two non-diagonal boundaries. The exact solution of the corresponding models is obtained by means of the algebraic Bethe ansatz. We also showed that the covering transfer matrix of the supersymmetric t-J model built from a general non-diagonal solution of the reflection equation possess a trivial reference state. Furthermore, we also presented the one-particle eigenvalue of the corresponding transfer matrix without any restriction for the boundary parameters.

One interesting possibility that deserves attention is the generalization of the above results for the \( q \)-deformed t-J model based on the \( U_q[sl(2[1])] \) symmetry. Here we remark that progresses have been reported for the XXZ model with non-diagonal open boundaries [26–28] and the extension of these results for the \( q \)-deformed t-J model deserves to be investigated.

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Appendix A: Auxiliary functions

In this appendix we present the functions $X_{i}^{(l,m)}$ required in section 3 for the diagonalization of the $n_1 = 1$ nested problem.

\begin{align*}
X^{(l,m)}_2 & = \ (1 - 2\lambda)^2 \\
X^{(l,m)}_1 & = (1 - 2\lambda) \left\{ \left( 4Y_3^{(l,m)}\lambda + 2Y_2^{(l,m)} \right) \left[ \lambda^2 + (1 - \lambda_1^{(1)})\lambda_1^{(1)} \right] + 2Y_1^{(l,m)}\lambda + Y_0^{(l,m)} \right\} \\
X^{(l,m)}_0 & = \lambda \left( 4Y_3^{(l,m)}\lambda + 4Y_2^{(l,m)}Y_3^{(l,m)} \right) \left[ \lambda^4 + 2\lambda_1^{(1)}(1 - \lambda_1^{(1)})\lambda^2 + (\lambda_1^{(1)})^3(\lambda_1^{(1)} - 2) \right] \\
& + \ (\lambda_1^{(1)})^2 \left( -2Z_{10}^{(l,m)}\lambda^2 + 2Z_1^{(l,m)}\lambda + Z_0^{(l,m)} \right) + \lambda_1^{(1)} \left( 2Z_7^{(l,m)}\lambda^2 + 2Z_6^{(l,m)}\lambda + Z_5^{(l,m)} \right) \\
& + \ Z_4^{(l,m)}\lambda^4 - 2Z_3^{(l,m)}\lambda^3 + Z_2^{(l,m)}\lambda^2 + Z_1^{(l,m)}\lambda + Z_0^{(l,m)} \\
\end{align*}

The functions $Y_{i}^{(l,m)}$ and $Z_{i}^{(l,m)}$ contain the dependence on the elements of the $K$-matrices and they are explicitly given by

\begin{align*}
Y_3^{(l,m)} & = k_{22}^{-(m)}(\lambda)k_{22}^{+(l)}(\lambda) + k_{32}^{-(m)}(\lambda)k_{23}^{+(l)}(\lambda) + k_{23}^{-(m)}(\lambda)k_{32}^{+(l)}(\lambda) + k_{33}^{-(m)}(\lambda)k_{33}^{+(l)}(\lambda) \\
Y_2^{(l,m)} & = k_{11}^{-(m)}(\lambda)k_{22}^{+(l)}(\lambda) - k_{22}^{-(m)}(\lambda)k_{22}^{+(l)}(\lambda) - k_{32}^{-(m)}(\lambda)k_{23}^{+(l)}(\lambda) - k_{23}^{-(m)}(\lambda)k_{32}^{+(l)}(\lambda) \\
& + \ k_{11}^{-(m)}(\lambda)k_{33}^{+(l)}(\lambda) - k_{33}^{-(m)}(\lambda)k_{33}^{+(l)}(\lambda) \\
Y_1^{(l,m)} & = k_{33}^{-(m)}(\lambda)k_{22}^{+(l)}(\lambda) - k_{32}^{-(m)}(\lambda)k_{23}^{+(l)}(\lambda) - k_{23}^{-(m)}(\lambda)k_{32}^{+(l)}(\lambda) + k_{22}^{-(m)}(\lambda)k_{33}^{+(l)}(\lambda) \\
& + \ k_{11}^{-(m)}(\lambda)k_{33}^{+(l)}(\lambda) - k_{22}^{-(m)}(\lambda)k_{33}^{+(l)}(\lambda) \\
Y_0^{(l,m)} & = k_{11}^{-(m)}(\lambda)k_{22}^{+(l)}(\lambda) - k_{22}^{-(m)}(\lambda)k_{22}^{+(l)}(\lambda) + k_{32}^{-(m)}(\lambda)k_{23}^{+(l)}(\lambda) + k_{23}^{-(m)}(\lambda)k_{32}^{+(l)}(\lambda) \\
& + \ k_{11}^{-(m)}(\lambda)k_{33}^{+(l)}(\lambda) - k_{22}^{-(m)}(\lambda)k_{33}^{+(l)}(\lambda) \\
Z_{10}^{(l,m)} & = k_{11}^{-(m)}(\lambda)^2(k_{22}^{+(l)}(\lambda))^2 - k_{22}^{-(m)}(\lambda)k_{22}^{+(l)}(\lambda)(k_{22}^{+(l)}(\lambda))^2 - (k_{22}^{-(m)}(\lambda))^2(k_{22}^{+(l)}(\lambda))^2 \\
& - \ 2k_{23}^{-(m)}(\lambda)k_{32}^{-(m)}(\lambda)(k_{22}^{+(l)}(\lambda))^2 + 2k_{22}^{-(m)}(\lambda)k_{33}^{-(m)}(\lambda)(k_{22}^{+(l)}(\lambda))^2 \\
& - \ 2k_{22}^{-(m)}(\lambda)k_{22}^{+(l)}(\lambda)(k_{22}^{+(l)}(\lambda)) - k_{32}^{-(m)}(\lambda)^2k_{22}^{+(l)}(\lambda)^2 \\
& - \ 2k_{22}^{-(m)}(\lambda)k_{22}^{+(l)}(\lambda)k_{22}^{+(l)}(\lambda) - 2k_{23}^{-(m)}(\lambda)k_{23}^{-(m)}(\lambda)k_{22}^{+(l)}(\lambda)k_{32}^{+(l)}(\lambda) \\
& - \ 2k_{22}^{-(m)}(\lambda)k_{22}^{+(l)}(\lambda)k_{33}^{+(l)}(\lambda) - 10k_{23}^{-(m)}(\lambda)k_{32}^{-(m)}(\lambda)k_{23}^{+(l)}(\lambda)k_{32}^{+(l)}(\lambda) \\
& + \ 4k_{22}^{-(m)}(\lambda)k_{32}^{-(m)}(\lambda)k_{23}^{+(l)}(\lambda)k_{32}^{+(l)}(\lambda) - 2k_{11}^{-(m)}(\lambda)k_{32}^{-(m)}(\lambda)k_{22}^{+(l)}(\lambda)k_{23}^{+(l)}(\lambda) \\
& - \ 2(k_{33}^{-(m)}(\lambda))^2k_{32}^{+(l)}(\lambda)k_{32}^{+(l)}(\lambda) - (k_{22}^{-(m)}(\lambda))^2(k_{23}^{+(l)}(\lambda))^2 \\
& + \ 2(k_{11}^{-(m)}(\lambda))^2k_{22}^{+(l)}(\lambda)k_{33}^{+(l)}(\lambda) - 2k_{22}^{-(m)}(\lambda)k_{22}^{+(l)}(\lambda)k_{33}^{+(l)}(\lambda) \\
& + \ 2(k_{22}^{-(m)}(\lambda))^2k_{22}^{+(l)}(\lambda)k_{33}^{+(l)}(\lambda) + 4k_{23}^{-(m)}(\lambda)k_{32}^{-(m)}(\lambda)k_{22}^{+(l)}(\lambda)k_{33}^{+(l)}(\lambda) \end{align*}
\[ Z_{9}^{(l,m)} = k_{11}^{(m)}(\lambda) k_{33}^{(m)}(\lambda) (k_{22}^{(l)}(\lambda))^{2} - 2(k_{22}^{(m)}(\lambda))^{2}(k_{22}^{(l)}(\lambda))^{2} - 2k_{23}^{(m)}(\lambda) k_{32}^{(m)}(\lambda)(k_{22}^{(l)}(\lambda))^{2} \]

\[ - k_{11}^{(m)}(\lambda) k_{33}^{(m)}(\lambda) (k_{22}^{(l)}(\lambda))^{2} + 2k_{22}^{(m)}(\lambda) k_{32}^{(m)}(\lambda)(k_{22}^{(l)}(\lambda))^{2} \]

\[ + 2k_{23}^{(m)}(\lambda) k_{32}^{(m)}(\lambda)(k_{22}^{(l)}(\lambda)) + (k_{11}^{(m)}(\lambda))^{2}(k_{33}^{(l)}(\lambda))^{2} \]

\[ - 2k_{23}^{(m)}(\lambda) k_{32}^{(m)}(\lambda)(k_{33}^{(l)}(\lambda))^{2} - 2k_{11}^{(m)}(\lambda) k_{32}^{(m)}(\lambda)(k_{33}^{(l)}(\lambda))^{2} \]

\[ + 2k_{22}^{(m)}(\lambda) k_{33}^{(m)}(\lambda)(k_{33}^{(l)}(\lambda))^{2} - (k_{33}^{(m)}(\lambda))^{2}(k_{33}^{(l)}(\lambda))^{2} \quad (A.8) \]

\[ Z_{8}^{(l,m)} = -k_{11}^{(m)}(\lambda) k_{33}^{(m)}(\lambda) (k_{22}^{(l)}(\lambda))^{2} + (k_{22}^{(m)}(\lambda))^{2}(k_{22}^{(l)}(\lambda))^{2} + k_{23}^{(m)}(\lambda) k_{32}^{(m)}(\lambda)(k_{22}^{(l)}(\lambda))^{2} \]

\[ - k_{11}^{(m)}(\lambda) k_{33}^{(m)}(\lambda) (k_{22}^{(l)}(\lambda))^{2} - 2k_{22}^{(m)}(\lambda) k_{33}^{(m)}(\lambda)(k_{22}^{(l)}(\lambda))^{2} \]

\[ - 2k_{11}^{(m)}(\lambda) k_{32}^{(m)}(\lambda)(k_{22}^{(l)}(\lambda)) + 2k_{22}^{(m)}(\lambda) k_{33}^{(m)}(\lambda)(k_{22}^{(l)}(\lambda))^{2} \]

\[ + (k_{32}^{(m)}(\lambda))^{2}(k_{23}^{(l)}(\lambda))^{2} - 2k_{11}^{(m)}(\lambda) k_{25}^{(m)}(\lambda)(k_{22}^{(l)}(\lambda))^{2} \]

\[ + 2k_{22}^{(m)}(\lambda) k_{25}^{(m)}(\lambda)(k_{22}^{(l)}(\lambda)) + (k_{32}^{(m)}(\lambda))^{2}(k_{23}^{(l)}(\lambda))^{2} \]

\[ + 6k_{23}^{(m)}(\lambda) k_{32}^{(m)}(\lambda)(k_{23}^{(l)}(\lambda))^{2} - 2k_{22}^{(m)}(\lambda) k_{33}^{(m)}(\lambda)(k_{32}^{(l)}(\lambda))^{2} \]

\[ + (k_{32}^{(m)}(\lambda))^{2}(k_{23}^{(l)}(\lambda))^{2} - 2k_{23}^{(m)}(\lambda)(k_{32}^{(l)}(\lambda))^{2} + 2k_{22}^{(m)}(\lambda) k_{33}^{(m)}(\lambda)(k_{33}^{(l)}(\lambda))^{2} \]

\[ - 2k_{23}^{(m)}(\lambda) k_{33}^{(m)}(\lambda)(k_{33}^{(l)}(\lambda))^{2} - 2k_{11}^{(m)}(\lambda) k_{23}^{(m)}(\lambda)(k_{33}^{(l)}(\lambda))^{2} \]

\[ + 2k_{23}^{(m)}(\lambda) k_{33}^{(m)}(\lambda)(k_{33}^{(l)}(\lambda))^{2} - (k_{33}^{(m)}(\lambda))^{2}(k_{33}^{(l)}(\lambda))^{2} \quad (A.9) \]
\[ Z^{(l,m)}_7 = (k^{(l,m)}_{11}(\lambda))^2(k^{(l)}_{22}(\lambda))^2 - 2k^{(m)}_{11}(\lambda)k^{(m)}_{22}(\lambda)(k^{(l)}_{22}(\lambda))^2 + (k^{(m)}_{22}(\lambda))^2(k^{(l)}_{22}(\lambda))^2 \]

\[ Z^{(l,m)}_6 = k^{(m)}_{11}(\lambda)k^{(m)}_{22}(\lambda)(k^{(l)}_{22}(\lambda))^2 + 2k^{(m)}_{23}(\lambda)k^{(m)}_{32}(\lambda)(k^{(l)}_{22}(\lambda))^2 \]

\[ Z^{(l,m)}_5 = (k^{(l,m)}_{11}(\lambda))^2(k^{(l)}_{22}(\lambda))^2 - k^{(m)}_{11}(\lambda)k^{(m)}_{22}(\lambda)(k^{(l)}_{22}(\lambda))^2 - k^{(m)}_{23}(\lambda)k^{(m)}_{32}(\lambda)(k^{(l)}_{22}(\lambda))^2 \]
\[ Z_4^{(l,m)} = (k_{11}^{-m}(\lambda))^2(k_{22}^{+(l)}(\lambda))^2 - 2k_{11}^{-m}(\lambda)k_{22}^{-(m)}(\lambda)(k_{22}^{+(l)}(\lambda))^2 - 3(k_{22}^{-(m)}(\lambda))^2(k_{22}^{+(l)}(\lambda))^2 \]

\[ Z_3^{(l,m)} = k_{11}^{-m}(\lambda)k_{22}^{-(m)}(\lambda)(k_{22}^{+(l)}(\lambda))^2 - 2(k_{22}^{-(m)}(\lambda))^2(k_{22}^{+(l)}(\lambda))^2 - 3k_{11}^{-m}(\lambda)k_{33}^{-(m)}(\lambda)(k_{22}^{+(l)}(\lambda))^2 \]
\[
\begin{align*}
Z_2^{(l,m)} &= 3k_{11}^{(m)}(\lambda)k_{22}^{(m)}(\lambda)k_{33}^{(l)}(\lambda)2^2 - (k_{22}^{(m)}(\lambda))^2(k_{22}^{(l)}(\lambda))^2 + 3k_{23}^{(m)}(\lambda)k_{32}^{(m)}(\lambda)(k_{22}^{(l)}(\lambda))^2 \\
+ k_{11}^{(m)}(\lambda)k_{22}^{(m)}(\lambda)k_{33}^{(l)}(\lambda))^2 - 2(k_{33}^{(m)}(\lambda))^2(k_{33}^{(l)}(\lambda))^2 \quad (A.15)
\end{align*}
\]

\[
\begin{align*}
Z_1^{(l,m)} &= (k_{11}^{(m)}(\lambda))^2(k_{22}^{(l)}(\lambda))^2 - k_{11}^{(m)}(\lambda)k_{22}^{(m)}(\lambda)k_{22}^{(l)}(\lambda))^2 - k_{29}^{(m)}(\lambda)k_{32}^{(m)}(\lambda)(k_{22}^{(l)}(\lambda))^2 \\
- k_{11}^{(m)}(\lambda)k_{22}^{(m)}(\lambda)k_{33}^{(l)}(\lambda))^2 + k_{11}^{(m)}(\lambda)k_{32}^{(m)}(\lambda)(k_{22}^{(l)}(\lambda))^2 \\
+ 4(k_{11}^{(m)}(\lambda))^2k_{32}^{(l)}(\lambda)k_{32}^{(l)}(\lambda) - 6k_{11}^{(m)}(\lambda)k_{22}^{(m)}(\lambda)k_{32}^{(l)}(\lambda)k_{33}^{(l)}(\lambda) \\
+ (k_{22}^{(m)}(\lambda))^2k_{22}^{(l)}(\lambda)k_{32}^{(l)}(\lambda) - 4k_{22}^{(m)}(\lambda)k_{22}^{(m)}(\lambda)k_{32}^{(l)}(\lambda)k_{33}^{(l)}(\lambda) \\
- 6k_{11}^{(m)}(\lambda)k_{32}^{(m)}(\lambda)k_{32}^{(l)}(\lambda)k_{33}^{(l)}(\lambda) + 6k_{22}^{(m)}(\lambda)k_{32}^{(m)}(\lambda)k_{22}^{(l)}(\lambda)k_{33}^{(l)}(\lambda) \\
+ (k_{33}^{(m)}(\lambda))^2k_{22}^{(l)}(\lambda)k_{33}^{(l)}(\lambda) - 2(k_{11}^{(m)}(\lambda))^2k_{22}^{(l)}(\lambda)k_{33}^{(l)}(\lambda) \\
+ 4k_{11}^{(m)}(\lambda)k_{22}^{(m)}(\lambda)k_{33}^{(l)}(\lambda)k_{33}^{(l)}(\lambda) - (k_{22}^{(m)}(\lambda))^2k_{33}^{(l)}(\lambda)k_{33}^{(l)}(\lambda) \\
+ 2k_{22}^{(m)}(\lambda)k_{33}^{(m)}(\lambda)k_{22}^{(l)}(\lambda)k_{33}^{(l)}(\lambda) + 4k_{11}^{(m)}(\lambda)k_{33}^{(m)}(\lambda)k_{22}^{(l)}(\lambda)k_{33}^{(l)}(\lambda) \\
- 4k_{22}^{(m)}(\lambda)k_{33}^{(m)}(\lambda)k_{33}^{(l)}(\lambda) - (k_{33}^{(m)}(\lambda))^2k_{33}^{(l)}(\lambda)k_{33}^{(l)}(\lambda) \\
+ (k_{11}^{(m)}(\lambda))^2k_{33}^{(l)}(\lambda)k_{33}^{(l)}(\lambda) - k_{11}^{(m)}(\lambda)k_{22}^{(m)}(\lambda)(k_{33}^{(l)}(\lambda))^2 - k_{23}^{(m)}(\lambda)k_{32}^{(m)}(\lambda)(k_{33}^{(l)}(\lambda))^2 \\
- k_{11}^{(m)}(\lambda)k_{33}^{(m)}(\lambda)(k_{33}^{(l)}(\lambda))^2 + k_{22}^{(m)}(\lambda)k_{33}^{(m)}(\lambda)(k_{33}^{(l)}(\lambda))^2 \quad (A.17)
\end{align*}
\]
\[ Z_0^{(l,m)} = \left( (k_{11}^{-(m)}(\lambda))^2 - k_{11}^{-(m)}(\lambda)k_{22}^{-(m)}(\lambda) - k_{23}^{-(m)}(\lambda)k_{32}^{-(m)}(\lambda) - k_{11}^{-(m)}(\lambda)k_{33}^{-(m)}(\lambda) \right) 
+ \left( k_{22}^{-(m)}(\lambda)k_{33}^{-(m)}(\lambda) \right) \left( k_{22}^{+(l)}(\lambda)k_{33}^{+(l)}(\lambda) - k_{23}^{+(l)}(\lambda)k_{32}^{+(l)}(\lambda) \right) \] (A.18)