SEMI–NORMS OF THE BERGMAN PROJECTION

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Abstract. It is known that the Bergman projection operator maps the space of essentially bounded functions in the unit ball in the $d$–dimensional complex vector space onto the Bloch space of the unit ball. This paper deals with the various semi–norms of the Bergman projection. We improve some recent results.

1. Introduction and the Main Result

Throughout the whole paper $d$ will be a positive integer. Let $\langle \cdot , \cdot \rangle$ stands for the inner product in the complex $d$–dimensional space $\mathbb{C}^d$ given by

$$\langle z, w \rangle = z_1 \overline{w}_1 + \cdots + z_d \overline{w}_d,$$

where $z = (z_1, \ldots, z_d)$ and $w = (w_1, \ldots, w_d)$ are coordinate representations of $z, w \in \mathbb{C}^n$ in the standard base $\{e_1, \ldots, e_d\}$ of $\mathbb{C}^d$. The standard norm in $\mathbb{C}^d$, induced by the inner product, is denoted by $| \cdot |$.

Denote by $B$ the unit ball $\{z \in \mathbb{C}^d : |z| < 1\}$ in $\mathbb{C}^d$ and let $S = \partial B$ be the unit sphere. Normalized Lebesgue measure on the unit ball (sphere) is denoted by $dv$ ($d\tau$).

Let $L^p(B)$, $1 \leq p < \infty$ stands for the Lebesgue space of all measurable functions in the unit ball of $\mathbb{C}^d$ which modulus with the exponent $p$ is integrable. For $p = \infty$ let it be the space of all essentially bounded measurable functions. Denote by $\| \cdot \|_p$ the usual norm on $L^p(B)$ ($1 \leq p \leq \infty$). Recall that

$$\|f\|_p^p = \int_B |f(z)|^p dv(z)$$

for $f \in L^p (1 \leq p < \infty)$.

Following the Rudin monograph [2] as well as the Forelli and Rudin work [2], associate with each complex number $s = \sigma + it$, $\sigma > -1$ the integral kernel

$$K_s(z, w) = \frac{(1 - |w|^2)^s}{(1 - \langle z, w \rangle)^{d+1+s}},$$

and let

$$T_s f(z) = c_s \int_B K_s(z, w) f(w) dv(w) \quad (z \in B).$$

We understand $f$ is a such one function that the preceding integral is defined. The coefficient $c_s$ in the previous relation is chosen in the way that for the weighted measure in the unit ball

$$dv_s(w) = c_s (1 - |w|^2)^s dv(w)$$

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holds \( \nu_s(B) = 1 \) (then \( T_s 1 = 1 \)). Using the polar coordinates

\[
\int_B h(z) \, dv(z) = 2d \int_0^1 r^{2d-1} \, dr \int_S h(r \zeta) \, d\tau(\zeta),
\]

one can show that

\[
c_s^{-1} = n \, B(s+1,d) = \frac{\Gamma(s+1) \, \Gamma(d+1)}{\Gamma(d+s+1)},
\]

where \( \Gamma \) and \( B \) are Euler functions.

The operator \( T_s \) is the Bergman projection operator. Bergman type projections are central operators when dealing with questions related to analytic function spaces. One usually wants to prove that the Bergman projections are bounded. However, the exact operator norm is often difficult to obtain.

Forelli and Rudin [23] proved that \( T_s : L^p(B) \to L^p(B) = L^p(B) \cap H(B) \), where \( H(B) \) is the space of all analytic functions in the unit ball, is a bounded (and surjective) operator if and only if \( \sigma > \frac{1}{p} - 1 \), where \( 1 \leq p < \infty \). Moreover, they find \( \| T_s : L^1(B) \to L^1(B) \| \) for \( \sigma > 0 \) and \( \| T_s : L^2(B) \to L^2(B) \| \) for \( \sigma > -\frac{1}{2} \). It seems that the calculation of \( \| T_s : L^p(B) \to L^p(B) \| \) in other cases is not an easy problem.

Mateljević and Pavlović [6] extended some of Forelli and Rudin results considering the case \( p \in (0, 1) \).

On the other hand, it is known that the operator \( T_s \) (for every \( \sigma > -1 \)) project \( L^\infty(B) \) continuously onto the Bloch space \( B \) of the unit ball in \( \mathbb{C}^d \), what can be seen from Theorem 3.4 in [12], or the Choe paper [1]. Recall that the Bloch space \( B \) contains all functions \( f(z) \) analytic in \( B \) for which \( \sup_{z \in B} (1 - |z|^2) \left| \nabla f(z) \right| \) is finite. Here \( \nabla f(z) = \left( \frac{\partial f(z)}{\partial z_1}, \ldots, \frac{\partial f(z)}{\partial z_n} \right) \) is the complex gradient of \( f \) at \( z \). For the Bloch space in several dimensions we refer to [10][11][12].

In order to state our main result here, let us first introduce some notation.

Let \( n \) be any positive integer and denote

\[
\tilde{d} = \left( \frac{n + d - 1}{d - 1} \right).
\]

Let \( | \cdot |_{\mathbb{C}^d} \) be a norm on \( \mathbb{C}^d \) which satisfies

\[
|Z|_{\mathbb{C}^d} = |Z|, \quad Z \in \mathbb{C}^d.
\]

For a multi–index \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \) let \( D_\alpha^z \) stands for the differential operator \( D_{z_1}^{\alpha_1} \cdots D_{z_n}^{\alpha_n} \). Denote by \( D_z : H(B) \to H(B)^d \) the following operator

\[
D_z f(z) = (\ldots, D_\alpha^z f(z), \ldots).\]

The vector \( (\ldots, D_\alpha^z f(z), \ldots) \in \mathbb{C}^d \) contains all partial derivatives \( D_\alpha^z f(z) \) such that for the multi–index \( \alpha \in \mathbb{N}^d \) holds \( |\alpha| = n \).

In the Bloch space \( B \) we introduce the following semi–norm

\[
\| f \|_B = \sup_{z \in B} (1 - |z|^2)^n |D_z f(z)|_{\mathbb{C}^d}.
\]

Recall that \( f \in B \) if and only if \( \sup_{z \in B} (1 - |z|^2)^n |D_\alpha^z f(z)| \) is finite for all multi–indexes \( \alpha \in \mathbb{N}^d, |\alpha| = n \), i.e., if and only if \( \sup_{z \in B} \max_{|\alpha| = n} (1 - |z|^2)^n |D_\alpha^z f(z)| \) is finite. It follows from this characterization of the Bloch space that \( f \in B \) if and only if \( \| f \|_B < \infty \).

For \( \zeta \in \mathbb{C}^d \) denote \( Z(\zeta) = (\ldots, \zeta^\alpha, \ldots) \in \mathbb{C}^d \) (this vector contains all \( \zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_d^{\alpha_d} \), where \( |\alpha| = n \)).
In what follows we consider the semi–norm of $T_\sigma$ given by

$$\|T_\sigma : L^\infty(B) \to B\| = \sup_{\|G\|_\infty \leq 1} \|T_\sigma G\|_B.$$ 

Our aim in this paper is to prove the following result.

**Theorem 1.1.** For the Bergman projection operator $T_\sigma$ holds

$$\|T_\sigma : L^\infty(B) \to B\| = C \frac{\Gamma(\lambda + n) \Gamma(n)}{\Gamma^2(\frac{d}{2} + \frac{\lambda}{2})}$$

for every $\sigma > -1$, where we have denoted

$$\lambda = d + 1 + \sigma$$

and

$$C = \max_{|\zeta| = 1} |Z(\zeta)|_{C^d}.$$ 

Note that $|Z(w)|_{C^d} \leq C$, $w \in \mathbb{C}^d$, $|w| < 1$.

In the next section we give a proof of this result. After that we compare our result with some known.

2. Proof of the result

In order to prove our main result, we need some auxiliary results. These will be collected in lemmas which follows.

It is well known that bi–holomorphic mappings of $B$ onto itself have the form

$$\varphi_z(\omega) = \frac{z - \frac{\langle \omega, z \rangle}{|z|^2} z - (1 - |z|^2)^{1/2} (\omega - \frac{\langle \omega, z \rangle}{|z|^2} z)}{1 - \langle \omega, z \rangle} \quad (z \in B),$$

up to unitary transformations.

The known identities

(1) \quad $1 - |\varphi_z(\omega)|^2 = \frac{(1 - |z|^2)(1 - |\omega|^2)}{|1 - \langle \omega, z \rangle|^2}$

and

(2) \quad $(1 - \langle \omega, z \rangle)(1 - \langle \varphi_z(\omega), z \rangle) = 1 - |z|^2$

for $z, \omega \in B$, will be useful in the proof of the next

**Lemma 2.1.** For every $z \in B$ we have

$$(1 - |z|^2)^n \int_B \frac{\Phi(w)}{|1 - \langle z, w \rangle|^{\lambda+n}} dv_\sigma(w) = \int_B \frac{\Phi(\varphi_z(\omega))}{|1 - \langle z, \omega \rangle|^{\lambda-n}} dv_\alpha(\omega),$$

where $\Phi(w)$ is a function in the unit ball $B$ such that the integral on the left side exists.

**Proof.** The real Jacobian of $\varphi_z(\omega)$ is given by the expression

$$(J_{R\varphi_z})(\omega) = \left\{ \frac{1 - |z|^2}{|1 - \langle \omega, z \rangle|^2} \right\}^{d+1} \quad (\omega \in B).$$
Using (1) we obtain the next relation for the pull–back measure
\[ dv_\sigma(\varphi_z(\omega)) = c_\sigma (1 - |\varphi_z(w)|^2)^\sigma (J_{\varphi_z})(\omega) dv(\omega) \]
\[ = c_\sigma \left( \frac{(1 - |\omega|^2)(1 - |z|^2)}{|1 - \langle \omega, z \rangle|^2} \right)^\sigma \frac{1 - |z|^2}{|1 - \langle \omega, z \rangle|^2} dv(\omega) \]
\[ = \left\{ \frac{1 - |z|^2}{|1 - \langle \omega, z \rangle|^2} \right\}^\lambda dv_\sigma(\omega). \]

Denote the integral on the left side of our lemma by \( J \). Introducing the change of variables \( w = \varphi_z(\omega) \) and using the preceding result for the pull-back measure, we obtain
\[ J = \int_B \frac{(1 - |z|^2)^{\alpha} \Phi(\varphi_z(\omega))}{|1 - \langle z, \varphi_z(\omega) \rangle|^{\lambda + n} |1 - \langle z, \omega \rangle|^{2\lambda}} dv_\sigma(\omega) \]
\[ = \int_B \frac{(1 - |z|^2)^{\lambda + n} \Phi(\varphi_z(\omega))}{|1 - \langle z, \varphi_z(\omega) \rangle|^{\lambda + n} |1 - \langle z, \omega \rangle|^{2\lambda}} dv_\sigma(\omega) \]
\[ = \int_B \frac{(1 - |z|^2)^{\lambda + n} \Phi(\varphi_z(\omega))}{|1 - \langle z, \varphi_z(\omega) \rangle|^{\lambda + n} |1 - \langle z, \omega \rangle|^{2\lambda}} dv_\sigma(\omega) \]
\[ = \int_B \frac{\Phi(\varphi_z(\omega))}{|1 - \langle z, \omega \rangle|^{\lambda - n}} dv_\sigma(\omega). \]

In third equality we have used the identity (2) \( \square \)

For the next lemma see [9] Proposition 1.4.10 as well as [4].

**Lemma 2.2.** For \( z \in B \), \( c \) real, \( t > -1 \) define
\[ J_{c,t}(z) = \int_B \frac{(1 - |w|^2)^t}{|1 - \langle z, w \rangle|^{2+1+t+c}} dv(w). \]
Then \( J_{c,t}(z) \) is bounded in \( B \) for \( c < 0 \).

Moreover, \( J_{c,t} \) depends only on \( |z| \), it is increasing in this value, and
\[ \sup_{z \in B} J_{c,t}(z) = J_{c,t}(e_1) = \frac{\Gamma(d + 1) \Gamma(t + 1) \Gamma(-c)}{\Gamma^2\left(\frac{d}{2} + 1 + \frac{t}{2} - \frac{c}{2}\right)}. \]

The following lemma is known as Vitali theorem. It is a content of Theorem 26.2 in [3].

**Lemma 2.3 (Vitali).** Let \( X \) be a measure space with finite measure \( \mu \) and let \( h_m : X \to \mathbb{C} \) be a sequence of functions that is uniformly integrable, i.e., such that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \), independent of \( m \), satisfying
\[ \mu(E) < \delta \implies \int_E |h_m| d\mu < \varepsilon. \]

Now: if \( \lim_{m \to \infty} h_m(x) = h(x) \) a.e., then
\[ \lim_{m \to \infty} \int_X h_m d\mu = \int_X h d\mu. \]

In particular, if
\[ \sup_m \int_X |h_m|^s d\mu < \infty \text{ for some } s > 1, \]
then (†) and (‡) hold.
We start now with a proof of our theorem. We divide it into two parts.

**Part I**

The aim of this part is to establish that

\[
\|T_\sigma : L^\infty \to B\| \leq C \frac{\Gamma(\lambda + n) \Gamma(n)}{\Gamma^2(\frac{\lambda}{2} + \frac{n}{2})}
\]

We have to estimate from above the following expression

\[
\|T_\sigma : L^\infty \to B\| = \sup \left\{ (1 - \left|z\right|^2)^n |D_z(T_\sigma G)(z)| : z \in B, \|G\|_\infty \leq 1 \right\}
\]

Let \(z \in B\) be fixed for a moment. For every \(Z^* \in C^d\) we have

\[
\langle D_z(T_\sigma G)(z), Z^* \rangle = c_\sigma \int_B \langle D_z K_\sigma(z, w), Z^* \rangle G(w) dw(w).
\]

Therefore

\[
\left| \langle D_z(T_\sigma G)(z), Z^* \rangle \right| \leq c_\sigma \int_B |\langle D_z K_\sigma(z, w), Z^* \rangle| |G(w)| dw(w)
\]

\[
\leq c_\sigma \int_B |\langle D_z K_\sigma(z, w), Z^* \rangle| dw(w)
\]

for every \(G \in L^\infty(B)\) which satisfies \(\|G\|_\infty \leq 1\).

Since for a multi-index \(\alpha \in \mathbb{N}^d, |\alpha| = n\) we have

\[
D^\alpha K_\sigma(z, w) = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \frac{(1 - \left|z\right|^2)^n}{(1 - \langle z, w \rangle)^{\lambda+n}} w^\alpha,
\]

we obtain (by using Lemma 2.1 in the last equality)

\[
(1 - \left|z\right|^2)^n |\langle D_z(T_\sigma G)(z), Z^* \rangle| \leq \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} (1 - \left|z\right|^2)^n \int_B \frac{|\langle Z(w), Z^* \rangle|}{\left|1 - \langle z, w \rangle\right|^{\lambda+n}} dv_\sigma(w)
\]

\[
= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \int_B \frac{|\langle Z(\varphi_z(\omega)), Z^* \rangle|}{\left|1 - \langle z, \omega \rangle\right|^{\lambda-n}} dv_\sigma(\omega).
\]

Since the space \(C^d\) with the norm \(|\cdot|_{C^d}\) is (as a finite dimensional space) reflexive, it follows

\[
(1 - \left|z\right|^2)^n |D_z(T_\sigma G)(z)|_{C^d} = \sup_{|Z^*|_{C^d} = 1} (1 - \left|z\right|^2)^n |\langle D_z(T_\sigma G)(z), Z^* \rangle|
\]

\[
\leq \sup_{|Z^*|_{C^d} = 1} \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \int_B \frac{|\langle Z(\varphi_z(\omega)), Z^* \rangle|}{\left|1 - \langle z, \omega \rangle\right|^{\lambda-n}} dv_\sigma(\omega)
\]

(in the last two expression we mean the conjugate norm of \(Z^* \in C^d\)).

Now, since for \(|Z^*|_{C^d} = 1\) (then also \(|Z^*|_{C^d} = 1\), by our assumption) we have

\[
|\langle Z(\varphi_z(\omega)), Z^* \rangle| \leq |Z(\varphi_z(\omega))|_{C^d} |Z^*|_{C^d} \leq C
\]

(recall that \(C\) stands for the maximum of \(|Z(\zeta)|_{C^d}\) for \(\zeta \in S\), we infer

\[
(1 - \left|z\right|^2)^n |D_z(T_\sigma G)(z)|_{C^d} \leq \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} c_\sigma C \int_B \frac{(1 - |\omega|^2)^\sigma}{\left|1 - \langle z, \omega \rangle\right|^{\lambda-n}} dv(\omega)
\]

\[
= C \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} c_\sigma J_{-n,\sigma}(z).
\]
By Lemma 2.2 we have
\[
\frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} c_\sigma J_{-n,\sigma}(z) \leq \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} c_\sigma J_{-n,\sigma}(e_1)
\]
\[
= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} - \frac{\Gamma(\lambda + 1) \Gamma(n)}{\Gamma(\lambda) \Gamma(d + 1) \Gamma(d + 1) \Gamma(\sigma + 1) \Gamma(n)}
\]
\[
= \frac{\Gamma(\lambda + n) \Gamma(n)}{\Gamma^2(\frac{\lambda}{2} + \frac{n}{2})}
\]
for all \( z \in B \).
Altogether, we obtain
\[
\|T_\sigma : L^\infty \to B\| \leq C \frac{\Gamma(\lambda + n) \Gamma(n)}{\Gamma^2(\frac{\lambda}{2} + \frac{n}{2})},
\]
what we wanted to prove.

**Part II**

We are going to prove now the reverse inequality
\[
\|T_\sigma : L^\infty \to B\| \geq C \frac{\Gamma(\lambda + n) \Gamma(n)}{\Gamma^2(\frac{\lambda}{2} + \frac{n}{2})}.
\]

It is enough to find a sequence \( \{z_m \in B\} \) and a sequence of functions \( \{G_m(w) \in L^\infty(B) : \|G_m\| \leq 1\} \) such that
\[
\liminf_{m \to \infty} \left(1 - |z_m|^2\right)^n \|D_z(T_\sigma G_m)(z_m)\| \geq C \frac{\Gamma(\lambda + n) \Gamma(n)}{\Gamma^2(\frac{\lambda}{2} + \frac{n}{2})}.
\]

We have denoted by \( C \) the value \( \sup_{|\zeta|=1} |Z(\zeta)|_{C^\lambda} \). Assume that the maximum is attained for \( \zeta_0 \in C^n \). Thus we have \( |\zeta_0| = 1 \) and
\[
|Z(\zeta_0)|_{C^\lambda} = C.
\]
Since
\[
|Z(\zeta_0)|_{C^\lambda} = \sup_{|\zeta^*|_{C^\lambda} = 1} |\langle Z(\zeta_0), \zeta^* \rangle|,
\]
we have
\[
|Z(\zeta_0)|_{C^\lambda} = |\langle Z(\zeta_0), Z_0^\lambda \rangle| = C
\]
for some concrete \( |Z_0^\lambda|_{C^\lambda} = 1 \) (we mean conjugate norm of \( Z_0^\lambda \)).

Take \( z_m = \varepsilon_m \cdot \zeta_0 \), where \( \{\varepsilon_m\} \subseteq (0, 1) \) is a sequence of increasing numbers such that \( \lim_{m \to \infty} \varepsilon_m = 1 \).
Denote
\[
G_m(w) = \frac{\langle Z(w), Z_0^\lambda \rangle \left(1 - \langle z_m, w\rangle\right)^{\lambda + n}}{\langle Z(w), Z_0^\lambda \rangle \left(1 - \langle z_m, w\rangle\right)^{\lambda + n}} (w \in B).
\]
Then \( G_m \in L^\infty(B) \) and \( \|G_m\| = 1 \).
Since
\[
D^2_w K_\sigma(z, w) = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \frac{(1 - |w|^2)^{\sigma}}{(1 - \langle z, w\rangle)^{\lambda + n}} w^\sigma,
\]
it follows
\[
\langle D_z K_\sigma(z, w), Z_0^\lambda \rangle = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \frac{(1 - |w|^2)^{\sigma}}{(1 - \langle z, w\rangle)^{\lambda + n}} \langle Z(w), Z_0^\lambda \rangle.
\]
Therefore,
\[
\left| \langle \mathcal{D}_z(T_\sigma G_m)(z_m), \overline{Z_0} \rangle \right|
= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \int_B \frac{|\langle Z(w), Z_0^* \rangle|}{|\langle Z(w), Z_0 \rangle|} \frac{|\langle Z(w), Z_0^* \rangle|}{|\langle Z(w), \overline{Z_0} \rangle|} \frac{1 - \overline{\langle z_m, w \rangle}^{\lambda+n}}{|1 - \overline{\langle z_m, w \rangle}^{\lambda+n}} \, dv_\sigma(w)
= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \int_B \frac{|\langle Z(w), Z_0^* \rangle|}{|1 - \overline{\langle z_m, w \rangle}^{\lambda+n}} \, dv_\sigma(w).
\]

It follows
\[
(1 - |z_m|^2)^n \left| \langle \mathcal{D}_z(T_\sigma G_m)(z_m), \overline{Z_0} \rangle \right|
= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} (1 - |z_m|^2)^n \int_B \frac{|\langle Z(w), Z_0^* \rangle|}{|1 - \overline{\langle z_m, w \rangle}^{\lambda+n}} \, dv_\sigma(w)
= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \int_B \frac{|\langle Z(\varphi_{z_m}(\omega)), Z_0^* \rangle|}{|1 - \overline{\langle z_m, \omega \rangle}^{\lambda-n}} \, dv_\sigma(\omega),
\]
where we have used Lemma 2.1 in the last equality.

Since
\[
\left| \mathcal{D}_z(T_\sigma G_m)(z_m) \right|_{C^d} = \sup_{|W^*|_{C^d} = 1} \left| \langle \mathcal{D}(T_\sigma G_m)(z_m), W^* \rangle \right|,
\]
we mean the conjugate norm of \( W \in C^d \) we have
\[
\liminf_{m \to \infty} (1 - |z_m|^2)^n \left| \mathcal{D}_z(T_\sigma G_m)(z_m) \right|_{C^d}
\geq \liminf_{m \to \infty} (1 - |z_m|^2)^n \left| \langle \mathcal{D}_z(T_\sigma G_m)(z_m), \overline{Z_0} \rangle \right|
= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \liminf_{m \to \infty} \int_B \frac{|\langle Z(\varphi_{z_m}(\omega)), Z_0^* \rangle|}{|1 - \overline{\langle z_m, \omega \rangle}^{\lambda-n}} \, dv_\sigma(\omega).
\]

We will prove that there exist and will calculate
\[
\lim_{m \to \infty} \int_B \frac{|\langle Z(\varphi_{z_m}(\omega)), Z_0^* \rangle|}{|1 - \overline{\langle z_m, \omega \rangle}^{\lambda-n}} \, dv_\sigma(\omega).
\]
We use the Vitali theorem.

For fixed \( \omega \in B \) we have
\[
\lim_{m \to \infty} \frac{|\langle Z(\varphi_{z_m}(\omega)), Z_0^* \rangle|}{|1 - \overline{\langle z_m, \omega \rangle}^{\lambda-n}} = \frac{|\langle Z(\zeta_0), Z_0^* \rangle|}{|1 - \overline{\langle \zeta_0, \omega \rangle}^{\lambda-n}} = \frac{C}{|1 - \overline{\langle \zeta_0, \omega \rangle}^{\lambda-n}}.
\]

Since for \( s = \frac{\lambda - n}{\lambda-n} > 1 \) (the parameter in Lemma 2.3), according to Lemma 2.2 (take \( c = -\frac{1}{2} \text{ and } t = \sigma \))
\[
\sup_m \int_B \left( \frac{|\langle Z(\varphi_{z_m}(\omega)), Z_0^* \rangle|}{|1 - \overline{\langle z_m, \omega \rangle}^{\lambda-n}} \right)^s \, dv_\sigma(\omega) \leq C^s \sup_m \frac{\int_B \frac{dv_\sigma(\omega)}{|1 - \overline{\langle z_m, \omega \rangle}^{\lambda-n}}}{|1 - \overline{\langle \zeta_0, \omega \rangle}^{\lambda-n}|^s} < \infty
\]
(note that
\[
|\langle Z(\varphi_{z_m}(\omega)), Z_0^* \rangle| \leq |Z(\varphi_{z_m}(\omega))|_{C^d} |Z_0^*|_{C^d} \leq C,
\]
by the Vitali theorem we conclude
\[
\lim_{m \to \infty} \int_B \frac{|\langle Z(\varphi z_m(\omega)), Z_0^* \rangle|}{|1 - \langle z_m, \omega \rangle|^{\lambda - n}} dv_\sigma(\omega) = \int_B \lim_{m \to \infty} \frac{|\langle Z(\varphi z_m(\omega)), Z_0^* \rangle|}{|1 - \langle z_m, \omega \rangle|^{\lambda - n}} dv_\sigma(\omega)
\]
\[
= C \int_B \frac{dv_\sigma(\omega)}{|1 - \langle \zeta_0, \omega \rangle|^{\lambda - n}} = C c_\sigma \lambda J_{-n,\sigma}(\zeta_0).
\]
Finally, again by Lemma 2.2 we have
\[
\lim \inf_{m \to \infty} \left(1 - |z_m|^2\right)^n |D_\sigma(T_\sigma G_m)(z_m)| \geq \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} c_\sigma J_{-n,\sigma}(e_1)
\]
\[
= C \Gamma(\lambda + n) \Gamma(n) \Gamma(2 + \frac{\lambda}{2}).
\]
This finishes the proof of the second part.

3. SOME COROLLARIES

We consider here some corollaries of our main result and we mention some earlier results concerning the various norms of the Bergman projection operator.

The following one is just a reformulation of our main theorem in the case \(d = 1\).

**Corollary 3.1.** Let \(|\cdot|_C\) be a norm on \(C\). Let a semi–norm on the Bloch space \(B\) of the unit disc \(U\) in the complex plane be given by
\[
\|f\|_B = \sup_{z \in U} (1 - |z|^2)^n |f^{(n)}(z)|_C.
\]
Then, for the Bergman projection operator \(T_\sigma : L^\infty(U) \to B\) holds
\[
\|T_\sigma : L^\infty(U) \to B\| = C \frac{\Gamma(2 + \sigma + n)}{\Gamma^2(1 + \frac{\lambda}{2} + \frac{\sigma}{2})}
\]
for every \(\sigma > -1\), where
\[
C = \max_{|\zeta| = 1} |\zeta|_C.
\]
In [4], we find the semi–norm of \(T_\sigma\) w.r.t. the semi–norm on \(B\) given by
\[
\|f\|_B = \sup_{z \in B} (1 - |z|^2) |\nabla f(z)|, \quad f \in B.
\]
In this case it was obtained
\[
\|T_\sigma : L^\infty(B) \to B\| = \frac{\Gamma(\lambda + 1)}{\Gamma^2(\frac{\lambda}{2} + \frac{\sigma}{2})}.
\]
Particularly, for \(\sigma = 0\) and \(d = 1\) we put \(P = T_0\) (then we have the original Bergman projection). Perälä proved that
\[
\|P : L^\infty(U) \to B\| = \frac{8}{\pi},
\]
which is the main results of [7].

If the norm in above corollary is given in the standard way, then \(C = 1\), and therefore, for the ordinary Bergman projection we have
\[
\|P : L^\infty(U) \to B\| = 4 \frac{(n + 1) \Gamma^2(n)}{n \Gamma^2(\frac{n}{2})}.
\]
For $Z = (Z_1, \ldots, Z_d) \in \mathbb{C}^d$ denote
$$|Z|_p = \left\{ |Z_1|^p + \cdots + |Z_d|^p \right\}^{1/p} \quad (1 \leq p < \infty)$$
and
$$|Z|_\infty = \max_{1 \leq j \leq d} |Z_j|.$$  

If a semi–norm on $B$ is given by
$$\|f\|_B = \sup_{z \in B} (1 - |z|^2)^n |D_z f(z)|_\infty$$
$$= \sup_{z \in B} (1 - |z|^2)^n \max_{|\alpha|=n} |D_\alpha z f(z)|, \quad f \in B,$$
then it is not hard to see that $C = 1$. Therefore, by our theorem, we have
$$\|T_\sigma \colon L^\infty(B) \to B\| = \frac{\Gamma(\lambda + n) \Gamma(n)}{\Gamma^2(\frac{\lambda}{\sigma} + \frac{n}{2})}.$$  

This result is also obtained in [5], which serves as a motivation for the present paper. We extend this result in the following

**COROLLARY 3.2.** For $1 \leq p < \infty$ let a semi–norm on $B$ be given by
$$\|f\|_B = \sup_{z \in B} (1 - |z|^2)^n |D_z f(z)|_p$$
$$= \sup_{z \in B} (1 - |z|^2)^n \left\{ \sum_{|\alpha|=n} |D_\alpha z f(z)|^p \right\}^{1/p}, \quad f \in B.$$  

Then the semi–norm of $T_\sigma$ is
$$\|T_\sigma \colon L^\infty(B) \to B\| = C_p \frac{\Gamma(\lambda + n) \Gamma(n)}{\Gamma^2(\frac{\lambda}{\sigma} + \frac{n}{2})},$$
where
$$d^n \sigma^{-\frac{n}{2}} \leq C_p \leq d^n \sigma^{-\frac{n}{2}}, \quad 1 \leq p < 2,$$
and
$$C_p = 1, \quad 2 \leq p < \infty.$$  

Particularly, for $n = 1$ (then $\tilde{d} = d$), we obtain
$$C_p = \begin{cases} 
 d^{\frac{1}{p}} - \frac{1}{2}, & \text{if } 1 \leq p < 2, \\
 1, & \text{if } 2 \leq p < \infty.
\end{cases}$$

This corollary follows from our main result and the next simple

**LEMMA 3.3.** We have
$$d^n \sigma^{-\frac{n}{2}} \leq \max_{|\zeta|=1} |Z(\zeta)|_p \leq d^n \sigma^{-\frac{n}{2}}, \quad 1 \leq p < 2,$$
and
$$\max_{|\zeta|=1} |Z(\zeta)|_p = 1, \quad 2 \leq p < \infty.$$
PROOF. It is known that for every $W \in \mathbb{C}^d$ holds the sharp inequalities

$$|W|_p \leq \tilde{d}^{\frac{p}{p-1}} |W|, \quad 1 \leq p < 2,$$

and

$$|W|_p \leq |W|, \quad 2 \leq p < \infty.$$  

If $|\zeta| = 1$, then $|Z(\zeta)| \leq 1$;

$$|Z(\zeta)|^2 = \sum_{|\alpha|=n} |\zeta|^2 = \sum_{|\alpha|=n} \{|\zeta_1|^2\}^{\alpha_1} \cdots \{|\zeta_d|^2\}^{\alpha_d}$$

$$\leq \prod_{j=1}^{n} (|\zeta_1|^2 + \cdots + |\zeta_d|^2) = 1.$$  

We infer $|Z(\zeta)|_p \leq \tilde{d}^{\frac{p}{p-2}}$ if $1 \leq p < 2$, and $|Z(\zeta)|_p \leq 1$ if $2 \leq p < \infty$. On the other hand, if we take $\zeta = (d^{-\frac{1}{2}}, \ldots, d^{-\frac{1}{2}})$ for $1 \leq p < 2$, and $\zeta = e_1$ in the case $2 \leq p < \infty$, we find $|Z(\zeta)|_p = \tilde{d}^{\frac{p}{p-2}} (1 \leq p < 2)$, and $|Z(\zeta)|_p = 1 (2 \leq p < \infty)$. This proves the lemma. □

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