On the Renormalization of Hamiltonians

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Abstract

We introduce a novel method for the renormalization of the Hamiltonian operator in Quantum Field Theory in the spirit of the Wilson renormalization group. By a series of unitary transformations that successively decouples the high-frequency degrees of freedom and partially diagonalizes the high-energy part, we obtain the effective Hamiltonian for the low energy degrees of freedom. We successfully apply this technique to compute the 2-loop renormalized Hamiltonian in scalar $\lambda \phi^4$ theory.

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1 Introduction

The subject of renormalization for the Hamiltonian operator has been looked into extensively for the past decade [1]. Yet it is hardly being used in any practical computation in field theory, since usually all the intermediate steps are non-covariant and, generally, rather complicated compared to the standard approach. Nevertheless the concept of “integrating out” of the high frequency degrees of freedom has much more physical meaning precisely in the Hamiltonian framework. One way to present it would be to set up the whole procedure as a Born-Oppenheimer approximation used in atomic physics [2]. Another way to look at it would be to introduce a unitary transformation in order to decouple “high” and “low” modes and then look at the low-energy part of the spectrum [3]. In this letter we want to introduce a renormalization technique in the spirit of the Wilsonian renormalization [4], i.e., integration of fast degrees of freedom, appropriate for the Hamiltonian (Schrödinger) formalism [5]. It is similar in essence to both [2] and [3], but appears to be very different in practice. Basically what we want to see is how the high frequency degrees of freedom modify the low energy Hamiltonian operator, or equivalently, what Hamiltonian in term of low frequency modes produces the same low energy physics as the original Hamiltonian. This technique was successfully applied in a previous paper by the authors to the 1-loop renormalization of Abelian and non-Abelian gauges theories [6].

For the sake of explicitness let us consider a system described by a Hamiltonian $H(\Lambda)$ defined up to the momentum scale (cut-off) $\Lambda$; that is, $H(\Lambda)$ produces finite results. Now if $\mu$ is a lower momentum scale, we want to find a new Hamiltonian operator $H(\mu)$ that generates the same results for all the physical processes which do not involve momenta greater than $\mu$. The definition of $H(\mu)$ is simple: $H(\mu)$ is the projection of the original Hamiltonian $H(\Lambda)$ onto the low frequency subspace, i.e., the high frequency vacuum,

$$H(\mu) = \mathcal{P}_{\text{low}} H(\Lambda) \mathcal{P}_{\text{low}}.$$  \hspace{1cm} (1)

The claim is that, at least perturbatively, it is possible to decouple the low frequency modes from the high frequency modes and thus give sense to the notion of “low frequency subspace”. In essence we will show that after a suitable unitary transformation we can partially diagonalize the Hamiltonian and construct the vacuum for high momentum modes. Therefore we can rewrite eq. (1) as

$$H(\mu) = \langle 0_{\text{high}} | U^\dagger(\Lambda, \mu) \ H(\Lambda) \ U(\Lambda, \mu) | 0_{\text{high}} \rangle.$$  \hspace{1cm} (2)

Now let us explain how to construct the unitary operator $U(\Lambda, \mu)$. As we have explained before, the main goal is to identify the ground state of the high frequency modes and then project the whole Hamiltonian onto this state. So if we can find a unitary transformation that: i) separates the low energy modes from the high energy modes and, ii) diagonalizes the high energy part of the Hamiltonian (for example in terms of the creation and annihilation operators of the high frequencies), we are done: the ground state will be then the state
annihilated by all the high frequency annihilation operators. This task is of course difficult, but in fact we need less than that. Since we only need to identify the vacuum state of the high momenta Hamiltonian and not the whole spectrum, only a partial diagonalization is enough. In fact, only those non-diagonal terms containing purely creation operators or purely annihilation operators have to be removed from the Hamiltonian. Other non-diagonal terms containing both creation and annihilation operators do not change the vacuum state of the theory (once the aforementioned terms have been removed).

We now show how this work in more detail. Suppose that after a unitary transformation we bring a Hamiltonian to the form $H_{\text{diag}} + V$ where $V$ contains only terms with at least one creation operator and one annihilation operator. That is, $V$ can be written as $V = a_i^\dagger M_{ij} a_j$ where $M$ is an arbitrary operator and $i, j$ are generic indices. Then, standard perturbation theory tell us that the correction to an arbitrary state $|n\rangle$ is given by

$$|\delta n\rangle = \sum_{l \neq n} \frac{\langle l|V|n\rangle}{(E_l - E_n)} |l\rangle + \sum_{l, n \neq m} \frac{\langle l|V|m\rangle\langle m|V|n\rangle}{(E_l - E_m)(E_m - E_n)} |l\rangle + \cdots$$  (3)

Therefore, if $|n\rangle \equiv |0\rangle$, the vacuum state, it is annihilated by $V$ and there is no correction to it at any order in perturbation theory. The ground state is unaffected by $V$.

Now we can proceed to find the unitary transformation of eq. (2). First we split the original Hamiltonian into four pieces,

$$H = H_1 + H_2 + V_A + V_B.$$  (4)

Here $H_1$ contains only the modes with momenta less than $\mu$, $H_2$ is the free part for all the modes with momenta greater than $\mu$ and $V_A + V_B$ contains mixing terms and all non-diagonal high-momentum operators; $V_A$ contains the “pure” terms that have only high frequency creation operators or high frequency annihilation operators, but not both, and $V_B$ the “impure” remaining terms (we assume here that $V_A$ and $V_B$ are normal-ordered with respect to the free perturbative vacuum). Then we break the unitary operator $U$ in a product series, $U = U_0 U_1 \cdots U_n \cdots$ and we compute all the terms successively. The objective of each individual $U_n$ is to partially diagonalize the Hamiltonian, at a given order in $\lambda$ (a generic coupling of the theory) and $\mu/\Lambda$. Let us show how to achieve this.

We proceed iteratively by first diagonalizing the Hamiltonian at leading order in $\lambda$, up to the desired accuracy in $\mu/\Lambda$. Consider first a unitary operator $U_0$ written as $e^{i\Omega_0}$. Now we perform a unitary transformation on equation (4), expanding in powers of $\Omega$ (in the general case $\Omega$ is at least of order $\lambda$ so at a given order only a finite number of terms are needed):

$$e^{-i\Omega_0}(H_1 + H_2 + V_A + V_B)e^{i\Omega_0} = H_1 + H_2 + V_A + V_B + i[H_1, \Omega_0] + i[H_2, \Omega_0] + i[V_A, \Omega_0] + i[V_B, \Omega_0] \cdots$$  (5)

As we have explained above, we want to eliminate the “pure” mixing terms $V_A$, while we generate an expansion in $\mu/\Lambda$. Thus, we impose the following condition
on $\Omega_0$,
\begin{equation}
  i [H_2, \Omega_0] + V_A = 0.
\end{equation}

This equation can be solved perturbatively and since commutators with
$H_2$ generate time derivatives of the high frequency fields we have the desired
expansion. Notice that equation (6) defines $\Omega_0$ up to terms that commute
with $H_2$. This ambiguity corresponds to the redefinition of the low energy
Hamiltonian $H_1$ by a unitary transformation. We will assume some kind of a
“minimal” scheme - namely that $\Omega_0$ does not have a part that commutes
with $H_2$.

After $\Omega_0$ is chosen to cancel $V_A$ in the effective Hamiltonian, a new mixing
term of order $\lambda$ appears from $i [H_1, \Omega_0]$. However this new term is of a higher
order in $\mu/\Lambda$ and will be eliminated by the next unitary transformation
$U_1 = e^{i \Omega_1}$. Explicitly,
\begin{equation}
  e^{-i \Omega_1} e^{-i \Omega_0} (H_1 + H_2 + V_A + V_B) e^{i \Omega_0} e^{i \Omega_1} = H_1 + H_2 + V_B +
  i [H_1, \Omega_0] + i [H_2, \Omega_1] + i [H_1, \Omega_1] + \text{second order terms} \ldots
\end{equation}

and we now choose $\Omega_1$ so that
\begin{equation}
  i [H_1, \Omega_0] + i [H_2, \Omega_1] = 0.
\end{equation}

Using equations (6) and (8) we obtain:
\begin{equation}
  e^{-i \Omega_1} e^{-i \Omega_0} (H_1 + H_2 + V_A + V_B) e^{i \Omega_0} e^{i \Omega_1} = H_1 + H_2 + V_B +
  i [V_B, \Omega_0] + i [V_B, \Omega_1] + i [H_1, \Omega_1] - \frac{1}{2} [H_1, \Omega_0] \cdot \Omega_0 -
  \frac{1}{2} [H_1, \Omega_0] \cdot \Omega_1 + i [H_1, \Omega_1] + \text{higher order terms} \ldots
\end{equation}

Now it is easy to deduce the logic of the procedure: the next step is to
introduce $\Omega_2$ in order to cancel $i [H_1, \Omega_1]$, which is of higher order in $\mu/\Lambda$, and
continue with the same process until we have attained the desired accuracy.
Notice that by construction
\begin{equation}
  [H_2, \Omega_{n+1}] = - [H_1, \Omega_n]
\end{equation}

and $H_2 \approx \Lambda, H_1 \approx \mu$, then
\begin{equation}
  \Omega_{n+1} \approx \frac{\mu}{\Lambda} \Omega_n,
\end{equation}

and any new $\Omega$ is smaller than the previous one by a factor of $\mu/\Lambda$.

Of course so far we have only eliminated the high momenta degrees of freedom
up to the first order in the coupling constant. Requiring the absence of the $\lambda^2$ - order mixing terms will lead to the introduction of a whole new series
of unitary transformations, and the same arguments can be applied to them.

As a final remark, notice that the “partial” diagonalization at first order in
$\lambda$ defines a correct low-energy effective Hamiltonian (after projecting onto the
high frequency vacuum) which is valid up to the order $\lambda^3$. The reason is that
since all the relevant terms of order $\lambda$ have been cancelled, the introduction of
the new series of $\Omega$’s of order $\lambda^2$ will only modify the Hamiltonian at order $\lambda^4$. 
2 A quantum-mechanical example

In this section we illustrate the ideas of the previous section by studying in some detail a quantum mechanical “toy model” where the main ingredients of our technique can be presented in a simpler context.

Consider the following Hamiltonian of two coupled anharmonic oscillators,

\[ H = \frac{\omega_1}{2} (P^2 + Q^2) + \frac{\omega_2}{2} (p^2 + q^2) + \lambda (Q + q)^4 \]  

where \( \omega_1 \ll \omega_2 \). Now we want to find the effective Hamiltonian for \( P \) and \( Q \) resulting from “integrating out” the high frequency modes \( p \) and \( q \). We choose to compute the effective Hamiltonian up to order \( \lambda^3 \) and \( (\omega_1/\omega_2)^3 \).

According to the discussion of the previous section we divide the Hamiltonian into four pieces as in eq. (4),

\[ H = H_1 + H_2 + V_A + V_B \]  

where the various terms have the form:

\[ H_1 = \frac{\omega_1}{2} (P^2 + Q^2) + \lambda \left( \frac{3}{4} + 3 Q^2 + Q^4 \right), \]

\[ H_2 = \omega_2 a^\dagger a, \]

\[ V_A = \lambda \left\{ \frac{3}{2} a^2 + \frac{3}{2} a^4 + \frac{1}{4} a^4 + \frac{1}{4} a^{14} + \sqrt{2} Q (3a + 3 a^\dagger + a^3 + a^{13}) + 3 Q^2 (a^2 + a^{12}) + 2 \sqrt{2} Q^3 a + a^\dagger \right\}, \]

\[ V_B = \lambda \left\{ 3 a^\dagger a + a^\dagger a^3 + 6 a^\dagger a^2 + a^{13} + 3 \sqrt{2} Q (a^\dagger a^2 + a^{12} a) + 6 Q^2 a^\dagger a \right\}. \]

In eq. (13), \( a \) and \( a^\dagger \) are the annihilation and the creation operators respectively of the high frequency states, defined through \( a \equiv (p + iq)/\sqrt{2} \) and \( a^\dagger \equiv (p - iq)/\sqrt{2} \).

Now we are ready to perform the first unitary transformation \( U_0 = e^{i\Omega_0} \). As explained in Section 1, \( \Omega_0 \) is chosen in order to satisfy equation (3). Using the expressions given in eq. (13) we obtain:

\[ \Omega_0 = i \lambda \frac{1}{\omega_2} \left\{ \frac{3}{4} (a^\dagger a - a^4) + \sqrt{2} Q \left( 3(a^\dagger a - a) + \frac{1}{3} (a^{13} - a^3) \right) + \frac{3}{2} Q^2 (a^\dagger a - a^2) + 2 \sqrt{2} Q^3 (a^\dagger a - a) + \frac{1}{16} (a^{14} - a^4) \right\}. \]

After performing the unitary transformation with the above expression for \( \Omega_0 \), we obtain the leading order approximation of the diagonalization of the whole Hamiltonian: we have eliminated the non-diagonal “pure” terms \( V_A \) at the expense of creating new ones in \([H_1, \Omega_0]\). However those new terms are of order of \( \omega_1/\omega_2 \) as can be seen clearly from eqs. (14) and (15).
The following step is then to cancel this newly created non-diagonal terms with the next unitary transformation $U_1 = e^{\Omega_1}$ where $\Omega_1$ satisfies equation (8). A straightforward calculation gives

$$
\Omega_1 = i\lambda \frac{\omega_1}{\omega_2} \left\{ \frac{3}{4}(a^{\dagger 2} + a^2) + i\sqrt{2} P \left( 3(a^{\dagger} + a) + \frac{1}{9}(a^{\dagger 3} + a^3) \right) + i \frac{3}{2} QP (a^{\dagger 2} + a^2) - i6\sqrt{2}Q^2 P (a^{\dagger} + a) \right\}. \tag{16}
$$

Finally, to reach the desired accuracy in $\frac{\Omega_2}{\omega_2}$ we need also $\Omega_2$, defined through the relation $[H_1, \Omega_1] + [H_2, \Omega_2] = 0$. We find

$$
\Omega_2 = -\lambda \frac{\omega^2_1}{\omega_2} \left\{ 12(a^{\dagger} - a) - i\sqrt{2} Q \left( 3(a^{\dagger} - a) - \frac{1}{27}(a^{\dagger 3} - a^3) \right) - i \frac{3}{4} (P^2 + Q^2) (a^{\dagger 2} - a^2) + i12\sqrt{2} QP^2 (a^{\dagger} - a) i6\sqrt{2} Q^3 (a^{\dagger} - a) \right\}. \tag{17}
$$

Now we are ready to compute the “low energy” effective Hamiltonian. Recalling the discussion of the previous section, we have the following expression for the effective Hamiltonian, up to the order $\lambda^3$ and $1/\omega_2^3$:

$$
H_{eff} = \langle 0 | \left\{ H_1 + H_2 + V_B + \frac{i}{2} [V_A, \Omega_0] + i[V_B, \Omega_0] + i[V_B, \Omega_1] + i[V_B, \Omega_2] - \frac{1}{2}[[H_1, \Omega_0], \Omega_0] - \frac{1}{2}[[H_1, \Omega_0], \Omega_1] - \frac{1}{3}[[V_A, \Omega_0], \Omega_0] - \frac{1}{2}[[V_A, \Omega_0], \Omega_1] - \frac{1}{2}[[V_B, \Omega_0], \Omega_0] - [[V_B, \Omega_0], \Omega_1] - \frac{i}{6}[[H_1, \Omega_0], \Omega_0], \Omega_0] \right\} | 0 \rangle. \tag{18}
$$

Here $| 0 \rangle$ is the vacuum associated to the operators $a$ and $a^{\dagger}$ leaving the $P$ and $Q$ operators untouched. For the sake of completeness we have written the whole expression derived from the rules described in Section 1. However, some of the terms vanish upon projection.

Finally after a long but straightforward computation we have the desired low frequency Hamiltonian:

$$
H_{eff} = \frac{\omega_1}{2} (P^2 + Q^2) + \lambda \left( \frac{3}{4} + 3 Q^2 + Q^4 \right) + \lambda^2 \left( -\frac{21}{8} \frac{1}{\omega_2} + \frac{29}{3} \frac{\omega_1}{\omega_2^2} + \frac{153}{4} \frac{\omega_1^2}{\omega_2^4} + \frac{31}{2} \frac{\omega_1}{\omega_2} + \frac{45}{2} \frac{\omega_1^2}{\omega_2^2} + 144 \frac{\omega_1^2}{\omega_2^6} \right) Q^2 - \frac{166}{9} \frac{\omega_1^2}{\omega_2^4} P^2 + \frac{162}{2} \frac{\omega_1^2}{\omega_2^6} QP - 81 \frac{\omega_1^2}{\omega_2^4} Q^2 P^2 + 288 \frac{i}{\omega_2} \frac{\omega_1^2}{\omega_2^4} Q^3 P + \left( \frac{36}{\omega_1^2} \frac{1}{\omega_2^2} - \frac{33}{\omega_2} \right) Q^4 + \frac{72}{\omega_1^2} \frac{\omega_1^2}{\omega_2^6} Q^4 P^2 - \frac{8}{\omega_2} \frac{\omega_1^2}{\omega_2^4} \frac{Q^6}{\omega_2^8} + \lambda^3 \left( \frac{333}{16} \frac{1}{\omega_2^2} - 178 \frac{\omega_1}{\omega_2} + \frac{168}{4} \frac{1}{\omega_2^2} - \frac{3653}{2} \frac{\omega_1}{\omega_2^2} \right) Q^2 + \left( \frac{888}{\omega_1^2} - 3132 \frac{\omega_1}{\omega_2^2} \right) \frac{Q^4}{\omega_2^4} + \left( \frac{534}{\omega_1^2} - 1224 \frac{\omega_1}{\omega_2^2} \right) Q^6 + \frac{96}{\omega_2} \frac{Q^8}{\omega_2^8} \right) + O(\lambda^4, 1/\omega_2^4). \tag{19}
$$
The skeptical reader can verify, using standard Rayleigh-Schrödinger perturbation theory, that the spectrum of the effective Hamiltonian (19) is the same as the low energy spectrum (i.e., the part of the spectrum that remains finite if $\omega_2 \to \infty$) of the original Hamiltonian (12).

### 3 Scalar Field Theory

In this section we apply the same formalism to the case of a scalar field with the quartic self-interaction, $\lambda \phi^4$. We will determine the effective Hamiltonian $H$ up to the two-loop order.

Before proceeding to the loop calculation we have to explain how renormalization is performed in our formalism. As was shown in Section 1, by projecting to the vacuum state for the “high” momentum modes we obtain the effective Hamiltonian $H_{\text{eff}}$. Since our system is supposed to have had an ultraviolet cut-off $\Lambda$ from the very beginning, $H_{\text{eff}}$ will explicitly depend on this UV scale.

The renormalization procedure consists of modifying the original Hamiltonian $H$ by introducing renormalization $Z$-factors that depend on the UV cut-off $\Lambda$ and some arbitrary “renormalization scale” $M$. Each of the $Z$’s depends on $\Lambda$ and $M$ in such a way that the effective Hamiltonian obtained from it does not depend on $\Lambda$; in fact it has to look exactly like the original one except that all $\Lambda$’s should be replaced by $\mu$ - the scale down to which we are integrating out.

Let us start from the “bare” Hamiltonian

$$H = \int d^3x \left( \frac{1}{2} \Pi_\phi^2(x) + \frac{1}{2} \Phi(x) \left( -\vec{\nabla}^2 + m^2 \right) \Phi(x) + \lambda \Phi^4(x) \right), \quad (20)$$

and introduce a $Z$ factor for each composite operator,

$$H = \int \left( \frac{Z}{2} \Pi_\phi^2(x) + \frac{Z}{2} \Phi(x) \left( -\vec{\nabla}^2 \right) \Phi(x) + Z_mZ_\phi m^2 \Phi^2(x) + \lambda Z_\lambda Z_\phi^2 \Phi^4(x) \right) \quad (21)$$

Each of the $Z$-factors has a perturbative expansion in $\lambda$, where we are assuming that $\lambda$ has been defined by some renormalization prescription at the renormalization scale $M$ ($\lambda$ is the “renormalized” coupling in the language of the standard renormalization group). Generically:

$$Z = 1 + f_1(\Lambda)\lambda + f_2(\Lambda)\lambda^2 + f_3(\Lambda)\lambda^3 + \cdots \quad (22)$$

The functions $f_n$ will be chosen order-by-order from the requirement that after integration of the modes from $\mu$ to $\Lambda$ all the corrections sum up in such a way that $Z(\Lambda) \to Z(\mu)$. When doing the one-loop corrections one can therefore assume that all the $Z$’s are initially 1 and choose the corresponding $f$’s from the condition that high-cutoff dependence be cancelled after computing the $H_{\text{eff}}$ to one loop. For the second loop we use the one-loop $Z$-corrected Hamiltonian and determine $f_n$ to the next order in $\lambda$ by the same procedure.

According to our philosophy we have to identify the purely “low” part $H_1$, the free “high” momentum part $H_2$ and the interaction terms $V_A$ and $V_B$. First
By virtue of the equations (23), \( H(20) \) can be rewritten as sum on the four pieces \( V \) the interaction part forms the result with respect to “high” modes we can identify all the terms that contain substituting this definition in the expressions (25) and (26) and normal-ordering 

Hamiltonian due to the normal-ordering of the terms of type \( \phi \)

Notice that already at this stage we have some contributions to the effective 
a_\omega

Here \( \omega_k = \sqrt{k^2 + m^2} \) and can be taken \( \omega_k \approx |k| \) for \( \mu >> m \). The operators \( a_k \) and \( a_k^\dagger \) satisfy the standard commutation relations \( [a_k, a_k^\dagger] = \delta_{kp} \). Upon substituting this definition in the expressions (25) and (26) and normal-ordering the result with respect to “high” modes we can identify all the terms that contain only creation or only annihilation operators; these terms form \( V_A \). The rest of the interaction part forms \( V_B \).

\[
H_2 = \sum_{\mu<k<\Lambda} \omega_k a_k^\dagger a_k,
\]

\[
V_A = \lambda \sum \left\{ \frac{e^{i(k+p+q+r)x}a_k a_p a_q a_r}{4\sqrt{\omega_k \omega_p \omega_q \omega_r}} + \frac{2 \phi(x) e^{i(k+p+q)x} a_k a_p a_q}{\sqrt{2} \omega_k \omega_p \omega_q} + \frac{3 \phi^2(x) e^{ikx} a_k}{\sqrt{2} \omega_k a_k^\dagger a_k} \right\} + \frac{3 \phi^2(x) e^{i(k-p+q)x} a_k a_p}{\sqrt{\omega_k \omega_p \omega_q}} + \frac{3 \phi(x) e^{i(k-p+q)x} a_k a_p a_q a_r}{\sqrt{\omega_k \omega_p \omega_q \omega_r}} + \frac{6 \phi(x) e^{i(p-q-k)x} a_k^\dagger a_p a_q}{\sqrt{2} \omega_k \omega_p \omega_q} + \text{h. c.} \right\}.
\]

\[
V_B = \lambda \sum \left\{ \frac{e^{i(p+q+r-k)x} a_k^\dagger a_p a_q a_r}{\sqrt{\omega_k \omega_p \omega_q \omega_r}} + \frac{3 \phi(x) e^{i(p+q-k)x} a_k^\dagger a_p a_q}{\sqrt{2} \omega_k \omega_p \omega_q} + \text{h. c.} \right\}.
\]

Notice that already at this stage we have some contributions to the effective Hamiltonian due to the normal-ordering of the terms of type \( \phi^2(x)a_k a_k^\dagger \); this term explicitly depends on the UV cut-off \( \Lambda \) and can be included into \( H_1 \),

\[
\delta H_1 = 6\lambda \sum_{\mu<k<\Lambda} \frac{1}{2\omega_k} \int \phi^2(x) \approx 3\lambda \frac{\Lambda^2 - \mu^2}{4\pi^2} \int \phi^2(x),
\]
where we have made the standard replacement \( \sum_k \rightarrow \int \frac{d^4k}{(2\pi)^4} \). Equation (11) is, of course, the standard “tadpole”, one-loop mass renormalization.

In order to pick up the only other one-loop contribution, the coupling constant renormalization, we have to follow the general procedure of Section I and determine \( \Omega_0 \) for the first unitary transformation. Using equation (8) we deduce

\[
\Omega_0 = (-i)\lambda \left\{ \frac{4 \phi^2(x)e^{i k_x x} a_k}{\sqrt{2\omega_k \omega_k}} + \frac{3 \phi^2(x)e^{i (k+p)x} a_k a_p}{\sqrt{\omega_k \omega_p (\omega_k + \omega_p)}} + \frac{3}{4} \frac{1}{\omega_p} \frac{a_k a_{-k}}{\omega_k} + \frac{2\phi(x)e^{i(k+p+q)x} a_k a_p a_q}{\sqrt{2\omega_k \omega_p \omega_q (\omega_k + \omega_p + \omega_q)}} + \frac{e^{i(k+p+q+r)x} a_k a_p a_q a_r}{4 \sqrt{\omega_k \omega_p \omega_q \omega_r (\omega_k + \omega_p + \omega_q + \omega_r)}} - \text{h.c.} \right\} \tag{32}
\]

To find the correction to the Hamiltonian we expand the unitary transformations and project onto the high energy vacuum. The contributions to \( H_{\text{eff}} \) are exactly the same as in equation (18).

Now we have to determine the potentially divergent contributions (if \( \Lambda \rightarrow \infty \)) that will emerge after projection to the vacuum state. In doing so the following naive power-counting rule is useful:

a) Each \( \Omega_n \) brings \( k^{(n+1)} \) (momentum) to the denominator.

b) Each contraction contributes with \( k^{-1} \).

c) For each loop integration we have a contribution of \( k^3 \).

Therefore, each term for which the total degree of divergence of a term is greater than or equal to zero, has to be taken into account. Of course, there may be overlapping divergences in computing the proper expressions, but those have to be studied individually, term by term.

By inspection of equation (18), it is easy to see that at this order there are no contributions to the terms \( H^2 \) and \( \Phi(-\vec{\nabla})\Phi \) (i.e., \( Z_\pi = Z_\phi = 1 \)) and the only other one-loop term is the coupling constant renormalization. This contribution comes from the commutator \([V_A, \Omega_0]\) when contracting twice the high energy fields. The rest of the terms are not important at this stage: the commutators \([V_A, \Omega_0]\) and \([H_1, \Omega_0]\) are zero when projected onto the vacuum. After projecting onto the high frequency vacuum we have

\[
\delta H_1 |_{\phi^4} = -18\lambda^2 \int \phi(x)\phi(y) \frac{dp dq}{(2\pi)^6} \frac{1}{\omega_k \omega_p (\omega_k + \omega_p)} e^{i(p-q)(x-y)}
\]

\[
= -9\lambda^2 \frac{2\pi^2}{2\pi^2} \ln \left( \frac{\Lambda}{\mu} \right) \left( \int \phi^4(x) \, dx \right).
\tag{33}
\]

Expressions (31) and (33) give the two contributions to \( H_{\text{eff}} \) at one-loop. From them we can deduce the renormalization factors \( Z_m \) and \( Z_\lambda \):

\[
Z_m = 1 - 3\lambda \frac{(\Lambda^2 - M^2)}{2\pi^2}, \quad Z_\lambda = 1 + \frac{9\lambda^2}{2\pi^2} \ln \left( \frac{\Lambda}{M} \right). \tag{34}
\]

This finishes the one-loop renormalization procedure. (In refs. 4, 7 there is an extra one-loop counterterm of order \( \lambda \) due to the definition of the “effective” wave functional in the Schrödinger representation). Before going to the
we determine the operator $\Omega$ only divergent contribution comes from the first term. Following equation (8) however using the power counting rules described above we can see that the subleading divergent part that gives rise to the two-loop wave-function renormalization, 

$$
\delta H_{1,\phi^2} = -\frac{12\lambda^2}{(2\pi)^9} \left\{ \int d^3r \phi(r) \phi(-r) \right\} \int \frac{d^3p d^3q}{\omega_p \omega_q \omega_{p-q}(\omega_p + \omega_q + \omega_{p-q})}
$$

$$
= -\frac{3\lambda^2}{\pi^4} (2 \ln 2 - 1) \Lambda^2 \int \frac{1}{2} \phi^2(x) d^4x + \frac{3\lambda^2}{8\pi^4} \ln \left( \frac{\Lambda}{\mu} \right) \int \frac{1}{2} \left( \nabla \phi(x) \right)^2 \quad (35)
$$

Extra contributions to $H_{\text{eff}}$ at two-loops come from the terms

$$
\delta H = \langle 0 \rangle \left( -\frac{1}{2} [[H_1, \Omega_0], \Omega_0] + i \left[ V_B, \Omega_1 \right] - \frac{1}{2} [[H_1, \Omega_1], \Omega_1] \right) |0\rangle. \quad (36)
$$

However using the power counting rules described above we can see that the only divergent contribution comes from the first term. Following equation (8) we determine the operator $\Omega_1$:

$$
\Omega_1 = -(i) \sum \left\{ \frac{4}{\sqrt{2\omega_k\omega_k^2}} [H_1, \phi^3(x)] e^{ikx} a_k + \frac{3}{\sqrt{2\omega_k\omega_p}} [H_1, \phi^2(x)] e^{i(k+p)x} a_k a_p + \frac{2}{\sqrt{2\omega_k\omega_p\omega_p^2}} [H_1, \phi(x)] e^{i(k+p+q)x} a_k a_p a_q + \frac{e^{i(k+p+q+r)x} a_k a_p a_q a_r}{4\omega_k\omega_p\omega_q\omega_r(\omega_k + \omega_p + \omega_q + \omega_r)^2} - \text{h.c.} \right\} \quad (37)
$$

and after evaluating a momentum integral similar to the one in eq. (35), we finally get

$$
\delta H_{1,\phi^2} = -\frac{3\lambda^2}{8\pi^4} \ln \left( \frac{\Lambda}{\mu} \right) \left\{ \int \frac{1}{2} \pi_\phi^2 d^4x \right\} \quad (38)
$$

It is important to point out that the corrections to $(\nabla \phi)^2$, eq. (35), and $\pi^2$ eq. (38), have equal magnitude and opposite sign as it must be, due to the Lorentz covariance of the theory. In fact, since $\pi_\phi$ is represented by $\frac{d\delta}{d\phi}$, then if $\phi^2$ gets corrected by $Z$ then $\pi_\phi^2$ should change by $\frac{1}{2}$, so that the equal time commutator is preserved. Now we can define the two-loop wave-function renormalization factors $Z_\phi$ and $Z_\pi$ as

$$
Z_\phi = 1 - 3 \frac{\lambda^2}{8\pi^4} \ln \left( \frac{\Lambda}{\mu} \right), \quad (39)
$$

$$
Z_\pi = \frac{1}{Z_\phi} = 1 + 3 \frac{\lambda^2}{8\pi^4} \ln \left( \frac{\Lambda}{\mu} \right). \quad (40)
$$

Finally there is one more two-loop contribution that comes at the order $\lambda^3$ and renormalizes the $\phi^4(x)$ term. It comes from the terms

$$
\delta H = \langle 0 \rangle \left( -\frac{1}{3} [[V_A, \Omega_0], \Omega_0] - \frac{1}{2} [[V_B, \Omega_0], \Omega_0] \right) |0\rangle \quad (41)
$$
After a somewhat tedious computation we get

$$
\delta H^{(2-\text{loop})}_{\phi^4} = \frac{27}{2} \frac{1}{\pi^4} \lambda^3 \left\{ \ln \left( \frac{\Lambda}{\mu} \right) + \left[ \ln \left( \frac{\Lambda}{\mu} \right) \right]^2 \right\} \int \phi^4(x) \, d^3x. \quad (42)
$$

To compute the entire two-loop correction we have to add to this result the contribution of the one-loop counterterms, i.e., the one-loop terms with the order $\lambda$ contributions to the $Z$-factors. Thus, we can finally deduce the value of the $Z_\lambda$ factor at two-loops,

$$
Z_\lambda = 1 + \frac{9}{2\pi^2} \log(\Lambda/M) \, \lambda^2 + \left( \frac{81}{4\pi^4} (\log(\Lambda/M))^2 - \frac{51}{4\pi^4} \log(\Lambda/M) \right) \, \lambda^3. \quad (43)
$$

From equations (39) and (43) we get the correct two-loop $\beta$-function of the theory,

$$
\beta(\lambda) = \frac{\partial \lambda}{\partial \log M} \bigg|_{\Lambda} = -( (\partial_\lambda (\lambda Z_\lambda)|_{\Lambda,M})^{-1} \lambda \frac{\partial Z_\lambda}{\log M} \bigg|_{\lambda,\Lambda}
= \frac{9}{2\pi^2} \lambda^2 - \frac{51}{4\pi^4} \lambda^3, \quad (44)
$$

and the correct two-loop anomalous dimension $\gamma$:

$$
\gamma(\lambda) = \frac{1}{2} \frac{\partial \log Z_\phi}{\partial \log M} \bigg|_{\Lambda} = \frac{3}{16\pi^2} \lambda^2. \quad (45)
$$

In summary, we have described in this letter a novel perturbative technique of renormalization in the Hamiltonian (Schrödinger) formalism. We have showed that this method successfully gives the two-loop renormalized Hamiltonian for a scalar Field Theory with a quartic potential. Moreover, in a previous work we have shown that this technique also gives a consistent one-loop renormalization for the Quantum Electrodynamics and the Yang-Mills Hamiltonians where results similar to those from Coulomb gauge covariant calculations were found.

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