When are Stochastic Transition Systems Tameable?

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Abstract. A decade ago, Abdulla, Ben Henda and Mayr introduced the elegant concept of decisiveness for denumerable Markov chains \cite{Abdulla09}. Roughly speaking, decisiveness allows one to lift most good properties from finite Markov chains to denumerable ones, and therefore to adapt existing verification algorithms to infinite-state models. Decisive Markov chains however do not encompass stochastic real-time systems, and general stochastic transition systems (STSs for short) are needed. In this article, we provide a framework to perform both the qualitative and the quantitative analysis of STSs. Our first contribution is to define various notions of decisiveness (inherited from \cite{Abdulla09}), notions of fairness and of attractors for STSs, and explicit the relationships between them. As a second contribution, we define a notion of abstraction, together with natural concepts of soundness and completeness, and we give general transfer properties, which will be central to several verification algorithms on STSs. Our third contribution focuses on qualitative model-checking. Beyond (repeated) reachability properties for which our technics are strongly inspired by \cite{Abdulla09}, we use abstractions to design algorithms for the qualitative model-checking problem of arbitrary $\omega$-regular properties, when the STS admits a denumerable (sound and complete) abstraction with a finite attractor. Our fourth contribution is the design of generic approximation procedures for quantitative model-checking; in addition to extensions of \cite{Abdulla09} for general STSs, we design approximation algorithms for $\omega$-regular properties (once again by means of specific abstractions). Last, our fifth contribution consists in instantiating our framework with stochastic timed automata (STA) and generalized semi-Markov processes (GSMP), two models combining dense-time and probabilities. This allows us to derive decidability and approximability results for the verification of these two models. Some of these results were known from the literature, but our generic approach permits to view them in a unified framework, and to obtain them with less effort. We also derive interesting new approximability results for STA and GSMPs.

1 Introduction

Given its success for finite-state systems, the model checking approach to verification has been extended to various models based on automata, and including features such as time, probability and infinite data structures. These models allow one to represent software systems more faithfully, by representing timing constraints, randomization, and \textit{e.g.} unbounded call stacks. At the same time, they often offer the possibility to consider \textit{quantitative} verification questions, such as whether the best execution time meets a requirement, or whether the system is reliable with high probability. Quantitative verification is notably hard for infinite-state systems, and often requires the development of techniques dedicated to each class of models.

A decade ago, Abdulla, Ben Henda and Mayr introduced the concept of decisiveness for denumerable Markov chains \cite{Abdulla09}. Formally, a Markov chain is decisive \textit{w.r.t.} a set of states $F$ if runs almost-surely reach $F$ or a state from which $F$ can no longer be reached. The concept of decisiveness thus forbids some weird behaviours in denumerable Markov chains, and allows one to lift most good properties from finite Markov chains to denumerable ones, and therefore to adapt existing verification algorithms to infinite-state models. In particular, assuming

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decisiveness enables the quantitative model checking of (repeated) reachability properties, by providing an approximation scheme, which is guaranteed to terminate for any given precision for decisive Markov chains. Decisiveness also elegantly subsumes other concepts such as the existence of finite attractors, or coarseness [1].

Decisive Markov chains however are not general enough to represent stochastic real-time systems. Indeed, to faithfully model time in real-time systems, it is adequate to use dense time [4], that is, timestamps of events are taken from a dense domain (like the set of rational or of real numbers). This source of infinity for the state-space of the system is particularly difficult to handle: the state-space is non-denumerable (even continuous), the branching in the transition system is also non-denumerable, etc. For those reasons, stochastic real-time systems do not fit in the framework of decisive Markov chains of [1].

On the other hand, standard analysis techniques for non-stochastic real-time systems also cannot be easily adapted to stochastic real-time systems. Traditionally, these techniques rely on the design of appropriate finite abstractions, which preserve good properties of the original model. A prominent example of such an abstraction is that of the region automaton for timed automata [6]. However, these abstractions most often do not preserve quantitative properties and, in the context of stochastic systems they may be too coarse already for the evaluation of the probability of properties as simple as reachability properties.

A general framework to analyse a large class of stochastic real-time systems is thus lacking. In this article, we face this issue and provide a framework to perform the analysis of general stochastic transition systems (STSs for short). To do so, we generalize the main concepts of [1] (such as decisiveness, attractors), and standard notions for Markov chains (like fairness). STSs are purely stochastic Markov processes [29,30], that is, Markov chains with a continuous state-space. Note that, while this journal version builds on the conference paper [14], we choose here to phrase our results for time-homogeneous and Markovian models. As mentioned in [29], the Markovian assumption is not a severe restriction since many apparently non Markovian processes can be recast to Markovian models by changing the state space. In our opinion, this choice furthermore enabled the design of a richer and more elegant theory (compared to [14]).

Our first contribution is to define various notions of decisiveness (inherited from [1]), notions of fairness and of attractors in the general context of STSs. To complete the semantical picture, we explicit the relationships between these notions, in the general case of STSs, and also when restricting to denumerable Markov chains. Decisiveness or the existence of attractors will be later exploited to analyze properties for STSs.

As mentioned earlier, the analysis of real-time systems often requires the development of abstractions. As a second contribution, we define a notion of abstraction, which makes sense for STSs. Concepts of soundness and completeness are naturally defined for those abstractions, and general transfer properties are given, which will be central to several verification algorithms on STSs. The special case of denumerable abstractions is discussed, since it allows one to transfer more properties from the abstract system to the concrete one.

Our third contribution focuses on the qualitative model checking problem for various properties. In particular, we extend the results of [1] and show that, under some decisiveness assumptions, the almost-sure model checking of (repeated) reachability properties reduces to a simpler problem, namely to a reachability problem with probability 0. We advocate that this reduction simplifies the problem: in countable models, the 0-reachability amounts to the non existence of a path, in the underlying non-probabilistic system; beyond countable models, checking that a reachability property is satisfied with probability 0 amounts to exhibiting a
somehow regular set of executions with positive measure. Beyond repeated reachability properties, we use abstractions to design algorithms for the qualitative model-checking problem of arbitrary $\omega$-regular properties, in case the STS admits an abstraction with the finite attractor property. The latter contribution is completely new compared to the original results of [11] and our conference paper [14]. It is inspired by a procedure of [2] for probabilistic lossy channel systems, a special class of denumerable Markov chains with a finite attractor.

Our fourth contribution is the design of generic approximation procedures for the quantitative model-checking problem, inspired by the path enumeration algorithm of Purushothaman Iyer and Narashima [26]. Under some decisiveness assumptions, we prove that these approximation schemes are guaranteed to terminate. Assuming the STSs can be represented finitely and enjoy some smooth effectiveness properties, one derives approximation algorithms: one can approximate, up to a desired (arbitrary) precision, the probability of (repeated) reachability properties. Note that without these effectiveness properties, one cannot hope for algorithms, and this motivates our above formulation of “procedures”. Further, once again via the use of an abstraction with the finite attractor property, we design an approximation algorithm for $\omega$-regular properties; this algorithm makes use of the attractor of the abstract model to convert the quantitative analysis of an $\omega$-regular property into the quantitative verification of a reachability property in the concrete model. Up to our knowledge, this approach is completely new, and provides an interesting framework for quantitative verification of stochastic systems.

Our last contribution consists in instantiating our framework with high-level stochastic models, stochastic timed automata (STA) and generalized semi-Markov processes (GSMP), which are two models combining dense-time and probabilities. This allows us to derive decidability and approximability results for the verification of those models. Some of these results were known from the literature, e.g. the ones from [16], but our generic approach permits to view them in a unified framework, and to obtain them with less effort. We also derive interesting new approximability results for STA and GSMPs. In particular, the approximability results derived from this paper for STA are far more general than those obtained using an ad-hoc approach in [15].

The paper concludes with an overview of our main results, organized as a travel guide to STSs: it summarizes the relationships between all notions, and provides the reader recipes to analyze STSs.

The most technical proofs are postponed to the appendix. Pointers are given when relevant.

2 Preliminaries

In this section, we define the general model of stochastic transition systems, which are somehow Markov chains with a continuous state-space. This model corresponds to labelled Markov processes of [29] with a single action (hence removing non-determinism). We then define several probability measures, on infinite paths, but also on the state-space, which give different point-of-views over the behaviour of the systems. We continue by defining regular measurable events, and end up with defining deterministic Muller automata, and technical material for handling properties specified by these automata.
2.1 Stochastic transition systems

Given \((S, \Sigma)\) a measurable space (\(\Sigma\) is a \(\sigma\)-algebra over \(S\)), we write \(\text{Dist}(S, \Sigma)\) for the set of probability distributions over \((S, \Sigma)\). In the sequel, when the context is clear, we will omit the \(\sigma\)-algebra and simply write this set as \(\text{Dist}(S)\).

**Definition 1.** A stochastic transition system (STS) is a tuple \(T = (S, \Sigma, \kappa)\) consisting of a measurable analytic space \((S, \Sigma)\), and \(\kappa : S \times \Sigma \to [0,1]\) such that for every fixed \(s \in S\), \(\kappa(s, \cdot)\) is a probability measure and for each fixed \(A \in \Sigma\), \(\kappa(\cdot, A)\) is a measurable function. Function \(\kappa\) is the Markov kernel of \(T\).

Note that it is sufficient to define \(\kappa(s, \cdot)\) (for every \(s \in S\)) over a subset which generates the \(\sigma\)-algebra \(\Sigma\). The assumption that \((S, \Sigma)\) should be analytic is for the STSs to have smooth properties [30, Section 7.5].

Observe that if \(S\) is a denumerable set and \(\Sigma = 2^S\), then \(T\) is a denumerable Markov chain (DMC for short). If \(S\) is finite, the kernel \(\kappa\) then coincides with the standard probability matrix of the Markov chain. We now give two examples of STS.

**Example 1 (Denumerable Markov chain).** The first example is the DMC depicted in Figure 1.
We consider here \(T_1 = (S_1, \Sigma_1, \kappa_1)\) where

- \(S_1 = \mathbb{N}\),
- \(\Sigma_1 = 2^{S_1}\),
- for each \(i \geq 1\), \(\kappa_1(i, \{i+1\}) = p\) and \(\kappa_1(i, \{i-1\}) = 1-p\) with \(p \in ]0,1[\), and
- \(\kappa_1(0, \{1\}) = 1\).

This represents a random walk over the natural numbers.

Fig. 1. Random walk over \(\mathbb{N}\).

In the sequel, given a DMC \(T = (S, \Sigma, \kappa)\) and two states \(q, r \in S\), we will write \(\kappa(q, r)\) instead of \(\kappa(q, \{r\})\).

**Example 2 (Continuous-space Markov chain).** We now give a continuous variant of the previous random walk which models a simple queueing system. Precisely, we consider a queueing system with a single queue, a parameter \(\lambda\) for arrivals and \(\nu\) for serving times. Each state \(i \in \mathbb{N}\) is equipped with a non-negative real number that corresponds to the time that has elapsed since the beginning. Formally, we consider \(T_2 = (S_2, \Sigma_2, \kappa_2)\) with \(S_2 = \mathbb{N} \times \mathbb{R}_+\). We equip \(S_2\) with the \(\sigma\)-algebra generated by \(2^\mathbb{N} \times \mathcal{B}(\mathbb{R}_+)\) where \(\mathcal{B}(\mathbb{R}_+)\) is the Borel \(\sigma\)-algebra on \(\mathbb{R}_+\). Then intuitively, \(\kappa_2\) describes how the length of the queue evolves with time. Formally,
for each \( s, t \in \mathbb{R}_+ \),
\[
\kappa_2((0, t), (1, [0, s + t])) = \kappa_2((0, t), (1, [t, s + t])) = \int_0^s \lambda e^{-\lambda x} dx
\]
\[
\forall i \geq 1, \kappa_2((i, t), (i + 1, [0, s + t])) = \kappa_2((i, t), (i + 1, [t, s + t])) = \int_0^s \frac{\lambda}{\lambda + \nu} e^{-\lambda x} dx
\]
\[
\kappa_2((i, t), (i - 1, [0, s + t])) = \kappa_2((i, t), (i - 1, [t, s + t])) = \int_0^s \frac{\nu}{\lambda + \nu} e^{-\nu x} dx
\].

There will be more examples of STS with a continuous set of states in Section 7.

In the sequel, we fix an STS \( \mathcal{T} = (S, \Sigma, \kappa) \). We will give two semantical views on the behaviour of \( \mathcal{T} \): the first one is operational, in the sense that \( \mathcal{T} \) generates executions, with a measure over these executions; the second one observes how the state-space evolves over time.

The first point-of-view is the standard semantics of probabilistic systems and is widely used in the model-checking community, where the temporal aspects are important. From a state, a probabilistic transition is performed according to a fixed distribution, and the system resumes in one of the successor states. Among others, e.g. \cite{9}, uses this semantics for the standard model of continuous-time Markov chains. The second point-of-view bloomed more recently. It proposes to view probabilistic systems as transformers of probability distributions. Compared to the previous point-of-view, here one is interested not in states the system can be in, but rather in how the probability mass evolves along steps. This semantics was motivated by the ability to express different properties than the previous one \cite{11}. It has been considered for both discrete-time Markov chains \cite{24} and continuous-time models, e.g. those induced by stochastic Petri nets \cite{25}. These two point-of-views are two sides of the same coin, and we will use both in the following, though we are ultimately interested in properties related to the operational semantics.

### 2.2 A \( \sigma \)-algebra for measuring sets of infinite paths

The objective is now to interpret \( \mathcal{T} \) in an operational manner, and to define a probability measure over the set of infinite paths of \( \mathcal{T} \). We follow the lines of \cite{21}. A finite (resp. infinite) path of \( \mathcal{T} \) is a finite (resp. infinite) sequence of states. We write \( \text{Paths}(\mathcal{T}) \) for the set of infinite paths of \( \mathcal{T} \). In order to get a probability measure over \( \text{Paths}(\mathcal{T}) \), we need to equip this set with a \( \sigma \)-algebra. We therefore define for each finite sequence of measurable sets \( (A_i)_{0 \leq i \leq n} \in \Sigma^{n+1} \) the following set of infinite paths:
\[
\text{Cyl}(A_0, A_1, \ldots, A_n) = \{ \rho = s_0 \to s_1 \to \cdots \to s_n \to \cdots \mid \forall 0 \leq i \leq n, s_i \in A_i \}.
\]
This set is called a cylinder. We then equip \( \text{Paths}(\mathcal{T}) \) with the \( \sigma \)-algebra generated by the cylinders. We write it \( \mathcal{F}_T \).

Let \( \mu \) be an initial probability measure over \( \Sigma \), that is, \( \mu \in \text{Dist}(S) \). We define a probability measure \( \text{Prob}^T_\mu \) as follows. First we inductively define a probability measure over the cylinders. For every finite sequence of measurable subsets \( (A_i)_{0 \leq i \leq n} \in \Sigma^{n+1} \), we set:
\[
\text{Prob}^T_\mu(\text{Cyl}(A_0, A_1, \ldots, A_n)) = \int_{s_0 \in A_0} \text{Prob}^T_{\kappa(s_0)}(\text{Cyl}(A_1, \ldots, A_n)) d\mu(s_0),
\]
and we initialize with \( \text{Prob}^T_\mu(\text{Cyl}(A_0)) = \mu(A_0) \). From now on, we will use the classical notation \( \mu(ds_0) = d\mu(s_0) \). It should be noted that the value \( \text{Prob}^T_\mu(\text{Cyl}(A_0, A_1, \ldots, A_n)) \) is the result
of $n$ successive integrals and can be expressed as follows:

$$
\text{Prob}_T^T(\text{Cyl}(A_0, A_1, \ldots, A_n)) = \\
\int_{s_0 \in A_0} \int_{s_1 \in A_1} \cdots \int_{s_{n-1} \in A_{n-1}} \kappa(s_0, ds_1) \cdot \kappa(s_1, ds_2) \cdots \kappa(s_{n-2}, ds_{n-1}) \cdot \kappa(s_{n-1}, A_n) \cdot \mu(ds_0).
$$

Finally, using the classical Caratheodory’s theorem, $\text{Prob}_T^T$ can be extended in a unique way to the $\sigma$-algebra $\mathcal{F}_T$.

**Lemma 2.** $\text{Prob}_T^T$ defines a probability measure over $(\text{Paths}(T), \mathcal{F}_T)$.

The proof of Lemma 2 is classical and we omit it here. The interested reader may e.g. refer to the proof of [16, Proposition 3.2], which can easily be adapted to our context.

**Remark 1** Observe that if the initial distribution is a Dirac distribution $\delta_s$ over a single state $s \in S$, then we have that

$$
\text{Prob}^T_{\delta_s}(\text{Cyl}(A_0, \ldots, A_n)) = \begin{cases} 
0 & \text{if } s \notin A_0, \\
\text{Prob}^T_{\nu_{\{s\}}}(\text{Cyl}(A_1, \ldots, A_n)) & \text{otherwise}.
\end{cases}
$$

It follows that for every $\mu \in \text{Dist}(S)$, we can write

$$
\text{Prob}^T_{\mu}(\text{Cyl}(A_0, \ldots, A_n)) = \int_{s_0 \in A_0} \text{Prob}^T_{\delta_{s_0}}(\text{Cyl}(A_0, \ldots, A_n)) \mu(ds_0)
$$

and thus for every $\omega \in \mathcal{F}_T$,

$$
\text{Prob}^T_{\mu} (\omega) = \int_{s_0 \in S} \text{Prob}^T_{\delta_{s_0}}(\omega) \mu(ds_0).
$$

Remember that given two probability distributions $\mu$ and $\nu$ over some probability space $(S, \Sigma)$, $\mu$ and $\nu$ are equivalent if for each $A \in \Sigma$, $\mu(A) = 0 \Leftrightarrow \nu(A) = 0$. The proof of the following lemma is postponed to the technical appendix (page 43).

**Lemma 3.** Let $\mu$ and $\nu$ be two probability measures over $(S, \Sigma)$. If $\mu$ and $\nu$ are equivalent, then $\text{Prob}^T_{\mu}$ and $\text{Prob}^T_{\nu}$ are also equivalent.

### 2.3 STSs as transformers of probability measures

One can also interpret the dynamics of $T$ as a transformer of probability measures over $(S, \Sigma)$. For $\mu$ a probability measure over $\Sigma$, its transformation through $T$ can be defined as the measure $\Omega_T(\mu)$ defined for every $A \in \Sigma$ by:

$$
\Omega_T(\mu)(A) = \int_{s_0 \in S} \kappa(s_0, A) \cdot \mu(ds_0).
$$

It can be shown that $\Omega_T(\mu)$ is also a probability measure over $(S, \Sigma)$.

This interpretation offers a dual view on the STS $T$. Indeed, roughly speaking, $\Omega_T(\mu)(A)$ is the probability of being in $A$ after one step, when $\mu$ is the initial distribution on $T$. Given a distribution $\mu \in \text{Dist}(S)$ and given $A \in \Sigma$ such that $\mu(A) > 0$, we write $\mu_A$ for the conditional probability of $\mu$ given $A$, that is for each $B \in \Sigma$, $\mu_A(B) = \frac{\mu(A \cap B)}{\mu(A)}$. It should be observed that $\mu_A \in \text{Dist}(S)$. There is a strong relation between the transformer $\Omega_T(\mu)$ and the probability measure $\text{Prob}^T_{\mu}$ over paths defined previously, which we formalize below:
Lemma 4. Let $\mu \in \text{Dist}(S)$ be an initial distribution and let $(A_i)_{0 \leq i \leq n}$ be a sequence of measurable sets. Write $\nu_0 = \mu_{A_0}$, the conditional probability of $\mu$ given $A_0$, and for every $1 \leq j \leq n - 1$, write $\nu_j = (\Omega_T(\nu_{j-1}))_{A_j}$. Then, for every $0 \leq j \leq n$:

$$\text{Prob}^T_\mu(\text{Cyl}(A_0, A_1, \ldots, A_n)) =$$

$$\mu(A_0) \cdot \prod_{i=1}^{j}(\Omega_T(\nu_{i-1}))(A_i) \cdot \text{Prob}^T_{\mu_T(\nu_j)}(\text{Cyl}(A_{j+1}, \ldots, A_n)) .$$

The proof of this result is postponed to the technical appendix (page 50).

From this result, we can express the probability to reach $A$ in $n$ steps from the initial distribution $\mu$:

$$(\Omega_T^{(n)}(\mu))(A) = \text{Prob}^T_\mu(\text{Cyl}(S, \ldots, S, A)) .$$

2.4 Basic properties of paths in STSs

To define properties on the STS $T$, we use LTL-like notations, that will be interpreted as measurable subsets of $\text{Paths}(T)$. Let $\mathcal{L}_{S,\Sigma}$ be the set of formulas defined by the following grammar:

$$\varphi ::= B \mid \varphi_1 \text{U}_{\leq p} \varphi_2 \mid \varphi_1 \lor \varphi_2 \mid \varphi_1 \land \varphi_2 \mid \neg \varphi,$$

where $B \in \Sigma$, $\prec \in \{\geq, \leq, =\}$ is a comparison operator and $p \in \mathbb{N}$ is an integer. Given $\rho = (s_n)_{n \geq 0}$, we write $\rho_{\geq i} = (s_n)_{n \geq i} \in \text{Paths}(T)$ for each $i \geq 0$. Then the satisfaction relation of paths formulas is given as follows:

$$\rho \models B \iff s_0 \in B$$

$$\rho \models \varphi_1 \text{U}_{\leq p} \varphi_2 \iff \exists i \geq 0, i \succ p, \text{ s.t. } \rho_{\geq i} \models \varphi_2 \text{ and } \forall 0 \leq j < i, \rho_{\geq j} \models \varphi_1$$

$$\rho \models \varphi_1 \lor \varphi_2 \iff \rho \models \varphi_1 \text{ or } \rho \models \varphi_2$$

$$\rho \models \varphi_1 \land \varphi_2 \iff \rho \models \varphi_1 \text{ and } \rho \models \varphi_2$$

$$\rho \models \neg \varphi \iff \rho \not\models \varphi.$$

We write $\text{Ev}_T(\varphi)$ for the set of infinite paths $\rho$ in $T$ such that $\rho \models \varphi$. It is standard to show that the event $\text{Ev}_T(\varphi)$ is a measurable subset of $(\text{Paths}(T), \mathcal{F}_T)$ (see e.g. [35]). In particular, for every initial probability measure $\mu$, $\text{Prob}^T_\mu(\text{Ev}_T(\varphi))$ is well-defined. In the sequel, for simplicity, we often write $\text{Prob}^T_\mu(\varphi)$ instead of $\text{Prob}^T_\mu(\text{Ev}_T(\varphi))$.

We will also use classical notations like $T = S; \bot = \emptyset; \varphi_1 \text{U} \varphi_2 = \varphi_1 \text{U}_{\geq 0} \varphi_2; \mathbf{F} \varphi = \top \text{U} \varphi; \mathbf{F}_{\leq p}\varphi = \top \text{U}_{\leq p}\varphi; \mathbf{G} \varphi = \neg \mathbf{F}(\neg \varphi)$.

2.5 Labelled STSs and their properties

To ease the expression of rich properties over STSs, we extend the model with a labelling with atomic propositions.

Definition 5. A labelled STS (LSTS for short) is a tuple $T = (S, \Sigma, \kappa, \text{AP}, \mathcal{L})$, where $(S, \Sigma, \kappa)$ is an STS, $\text{AP}$ is a finite set of atomic propositions, and $\mathcal{L} : S \to 2^{\text{AP}}$ is a measurable labelling function.
Measures and other notions are extended in a straightforward way from STSs to LSTSs. We fix a finite set \( \text{AP} \) of atomic propositions and an LSTS \( \mathcal{T} = (S, \Sigma, \kappa, \text{AP}, \mathcal{L}) \).

A property over \( \text{AP} \) is a subset \( P \) of \( (2^\text{AP})^\omega \). An infinite path \( \rho = s_0s_1 \ldots \) of \( \mathcal{T} \) satisfies the property \( P \) whenever \( \mathcal{L}(s_0)\mathcal{L}(s_1)\mathcal{L}(s_2)\ldots \in P \), written \( \rho \models P \). \( \omega \)-regularity is a standard notion in computer science to characterise simple sets of infinite behaviours, and typical \( \omega \)-regular properties are Büchi and Muller properties. In order to express such properties, we introduce a new notation for the set of atomic propositions that are true infinitely often along a sequence of labels: for \( \omega = u_0u_1u_2 \ldots \in (2^\text{AP})^\omega \), we define \( \text{Inf}(\omega) = \{ a \in \text{AP} \mid \{ j \in \mathbb{N} \mid a \in u_j \} = \infty \} \). We extend this notation to paths in a natural way: if \( \rho = s_0s_1s_2 \ldots \in S^\omega \), writing \( \omega = \mathcal{L}(s_0)\mathcal{L}(s_1)\mathcal{L}(s_2) \ldots \), we define (with a slight abuse of notation) \( \text{Inf}(\rho) = \text{Inf}(\omega) \).

A Büchi property \( P \) over \( \text{AP} \) can be specified by a subset of atomic propositions \( F \subseteq \text{AP} \) as \( P = \{ \omega \in (2^\text{AP})^\omega \mid \text{Inf}(\omega) \cap F \neq \emptyset \} \). A Muller property over \( \text{AP} \) is a property \( P \) such that there exists \( F \subseteq 2^\text{AP} \) with \( P = \{ \omega \in (2^\text{AP})^\omega \mid \text{Inf}(\omega) \in F \} \).

**Remark 2** It should be noted that the set of infinite paths satisfying Büchi or Muller properties can be expressed using events as in Section 2.4. Indeed, for \( F \subseteq \text{AP} \) we write \( 2^\text{AP}_F = \{ u \in 2^\text{AP} \mid u \cap F \neq \emptyset \} \) and given \( a \in \text{AP} \), \( 2^\text{AP}_a = \{ u \in 2^\text{AP} \mid a \in u \} \). Then,

- the set of paths satisfying the Büchi property with acceptance condition \( F \) is \( \text{Ev}_T(\mathcal{G} \mathcal{F} (\bigvee_{u \in 2^\text{AP}_F} \mathcal{L}^{-1}(u))) \);

- the set of paths satisfying the Muller property with acceptance condition \( F \) is \( \text{Ev}_T(\bigvee_{F \in F} (\bigwedge_{a \in F} (\mathcal{G} \mathcal{F} (\bigvee_{u \in 2^\text{AP}_a} \mathcal{L}^{-1}(u)) \wedge \bigwedge_{a \notin F} \mathcal{F} (\mathcal{L}^{-1}(u))))) \).

It is well known that automata equipped with Büchi or Muller acceptance conditions capture all \( \omega \)-regular properties, and this also holds for deterministic Muller automata.

**Definition 6.** A deterministic Muller automaton (DMA) over \( \text{AP} \) is a tuple \( M = (Q, q_0, E, F) \) where:

- \( Q \) is a finite set of locations, and \( q_0 \in Q \) is the initial location;
- \( E \subseteq Q \times 2^\text{AP} \times Q \) is a finite set of edges;
- \( F \) is a Muller condition over \( Q \);

and such that

- \( M \) is deterministic: for all pair of edges \((q, u, q_1)\) and \((q, u, q_2)\) in \( E \), \( q_1 = q_2 \);
- \( M \) is complete: for every \( q \in Q \), for every \( u \in 2^\text{AP} \), there exists \((q, u, q') \in E \).

A deterministic Muller automaton \( M \) naturally gives rise to a property \( P_M \) defined by the language (over \( 2^\text{AP} \)) accepted by \( M \). Its semantics over infinite paths of \( \mathcal{T} \) is derived from that of property \( P_M \): if \( \rho \in \text{Paths}(\mathcal{T}) \), we write \( \rho \models M \) whenever \( \rho \models P_M \). Expanding Remark 2, one derives the standard fact that the set \( \mathcal{T}[M] \) is measurable, and we write \( \text{Prob}_\mu^\mathcal{T}(M) \) for \( \text{Prob}_\mu^\mathcal{T}(\mathcal{T}[M]) \).

**Remark 3** It is well known \([20]\) and \([24]\), Chapter 3] that for any LTL formula \( \varphi \) (the syntax given in the previous subsection, where we replace sets \( B \) by inverse images by \( \mathcal{L} \) of atomic propositions from \( \text{AP} \)), there is a deterministic Muller automaton \( M_\varphi \) that characterises \( \varphi \), that is: for every run \( \rho \), \( \rho \models \varphi \iff \rho \models M_\varphi \).
**Product STS.** To measure the probability of properties specified by a DMA \( M = (Q, q_0, E, F) \), it is convenient to build a new STS consisting of the product of \( T \) with \( M \). To this aim, we consider the discrete \( \sigma \)-algebra \( 2^Q \) on the finite set of locations \( Q \) of \( M \). The product \( S \times Q \) can then be equipped with the product \( \sigma \)-algebra \( \Sigma \times 2^Q \) defined as the smallest \( \sigma \)-algebra containing all rectangles, that is, all sets of the form \( A_1 \times A_2 \) with \( A_1 \in \Sigma \) and \( A_2 \in 2^Q \). Then, the product \( \sigma \)-algebra \( \Sigma \times 2^Q \) coincides with \( \Sigma' = \{ \bigcup_{q \in Q} C_q \times \{ q \} \mid \forall q \in Q, C_q \in \Sigma \} \) (see the proof in the appendix, page 51). Note that in the sequel, we will sometimes write \((C_q, q)\) instead of \( C_q \times \{ q \} \).

We now define the product of \( T \) with \( M \) as follows.

**Definition 7.** Given \( T = (S, \Sigma, \kappa, AP, \mathcal{L}) \) an LSTS and \( M = (Q, q_0, E, F) \) a DMA over \( AP \), we define the product of \( T \) with \( M \) as the LSTS \( T \times M = (S', \Sigma', \kappa', AP', \mathcal{L}') \) such that:

- \( S' = S \times Q \);
- \( \Sigma' \) is the product \( \sigma \)-algebra \( \Sigma \times 2^Q \);
- \( \kappa'((s, q), (A, q')) = \begin{cases} \kappa(s, A) & \text{if } (q, \mathcal{L}(s), q') \in E, \text{ and} \\ 0 & \text{otherwise}^4 \end{cases} \)
- \( AP' = Q \);
- \( \mathcal{L}'(s, q) = q \).

**Example 3.** We consider the random walk over \( \mathbb{N} \) of Example[1] We assume that it is equipped with the simple set of atomic propositions \( AP = \{ a \} \) and we assume that each state of the STS is labelled with \( a \). Let \( M \) be the DMA depicted on the left-hand side of Figure[2] The winning condition is given by \( F = \{ \{q_1, q_2\} \} \). The product \( T_1 \times M \) is then depicted on the right-hand side of Figure[2] It should be noted that we there assume that the system starts at \((0, q_0)\) however, there should be similar chains starting in \((i, q_0)\) for each \( i \geq 1 \). Note also that we did not specify the labels on the states: according to the definition each state is labelled with its current position in \( M \).

![Fig. 2. A Muller automaton M and the product T1 × M.](image)

We define on \( T \times M \) a Muller condition which is inherited from the one of \( M \) via the new labelling function \( \mathcal{L}' \): a run \( p \) satisfies the Muller condition \( F' \) whenever \( \mathcal{L}'(p') \) satisfies the Muller condition \( F \). We thus later simply use \( F \) instead of \( F' \).

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^4 Note that the above definition of \( \kappa' \) extends naturally to all elements of the \( \sigma \)-algebra \( \Sigma' \): each pair \((q, u)\) with \( q \in Q \) and \( u \in 2^Q \), there is a unique \( q' \in Q \) such that \((q, u, q') \in E \). Fix \((s, q) \in S \times Q \), write \( q' \) for the unique location such that \((q, \mathcal{L}(s), q') \in E \). Then for each \( A = \bigcup_{q \in Q} C_q \times \{ q \}, \kappa'((s, q), A) = \kappa'((s, q), (C_{q'}, q')) = \kappa(s, C_{q'}) \).
We now explain how initial distributions for $T$ are lifted to the product $T \ltimes M$. The idea is simple: the $T$-component is inherited from $T$, and the $M$-component is a Dirac measure over $q_0$, the initial state of $M$. In other words, when an initial distribution $\mu \in \text{Dist}(S)$ is fixed for $T$, the initial distribution of $T \ltimes M$ will be $\mu \times \delta_{q_0}$. We show that this allows to properly compute the probability of a property described by a DMA, with the following correspondence.

**Proposition 8.** Let $\mu \in \text{Dist}(S)$ be an initial distribution for $T$, and $M = (Q, q_0, E, F)$ be a DMA. Then:

$$\text{Prob}_\mu^T(T[M]) = \text{Prob}_{\mu \times \delta_{q_0}}^T(\{\rho \in \text{Paths}(T \ltimes M) \mid \rho \models F\}).$$

The formal proof of this statement is given in the appendix, page 51.

3 Nice properties of STSs

In [1], Abdulla et al. introduced the elegant concept of decisive Markov chain. Intuitively, decisiveness allows one to lift the good properties of finite Markov chains to infinite (but denumerable) Markov chains. We explain here how to extend and refine this concept and some related concepts to general STSs, and we establish relationships between these properties.

3.1 Several decisiveness notions

Decisiveness has been defined in [1] as a desirable property of denumerable Markov chains, since it implies that they behave essentially like finite Markov chains.

For $B \in \Sigma$ a measurable set, we define its **avoid-set** $\overline{B} = \{s \in S \mid \text{Prob}_\mu^T(FB) = 0\}$. It corresponds to the set of states from which the system will always avoid the set $B$ with probability 1. The set $\overline{B}$ enjoys the following properties, that obviously hold also in the context of denumerable Markov chains, but require proofs in our general context of STSs (those proofs are postponed to the technical appendix, page 53).

**Lemma 9.** Given $B \in \Sigma$, it holds that:

- $\overline{B}$ belongs to the $\sigma$-algebra $\Sigma$;
- for every $\mu \in \text{Dist}(\overline{B})$, $\text{Prob}_\mu^T(FB) = 0$;
- for every $\mu \in \text{Dist}(S)$, if $\mu((\overline{B})^c) > 0$, then $\text{Prob}_\mu^T(FB) > 0$;
- for every $\mu \in \text{Dist}(S)$, $\text{Prob}_\mu^T(FB) = \text{Prob}_\mu^T(FG\overline{B}) = \text{Prob}_\mu^T(GF\overline{B})$;
- for every $\mu \in \text{Dist}(S)$, $\text{Prob}_\mu^T(FB \vee F\overline{B}) = \text{Prob}_\mu^T(FB \vee (\neg B \cup \overline{B}))$.

Let us comment on the third and fourth properties stated in this lemma. The third item indicates that if we start from outside $\overline{B}$, then we will always have a positive probability to hit $B$. The fourth property says that $\overline{B}$ is some kind of sink: once we hit $\overline{B}$, we cannot escape. The other properties are rather straightforward to understand (even though proving the first property requires some technical developments).

We are now ready to define different decisiveness concepts. Two stem from [1] (though no initial distribution was fixed) while the third one was identified in [14] as a useful alternative.

**Definition 10.** Let $\mu$ be an initial probability distribution ($\mu \in \text{Dist}(S)$). Then:
\(- \mathcal{T} \) is said decisive w.r.t. \( B \) from \( \mu \) whenever \( \text{Prob}_{\mu}^T(\mathcal{F} B \lor \mathcal{F} \bar{B}) = 1 \); we then write that \( \mathcal{T} \) is \( \text{Dec}(\mu, B) \).

\(- \mathcal{T} \) is said strongly decisive w.r.t. \( B \) from \( \mu \) whenever \( \text{Prob}_{\mu}^T(\mathcal{G} \mathcal{F} B \lor \mathcal{F} \bar{B}) = 1 \); we then write that \( \mathcal{T} \) is \( \text{StrDec}(\mu, B) \).

\(- \mathcal{T} \) is said persistently decisive w.r.t. \( B \) from \( \mu \) whenever for every \( p \geq 0 \), \( \text{Prob}_{\mu}^T(\mathcal{F} \geq_p B \lor \mathcal{F} \geq_p \bar{B}) = 1 \); we then write that \( \mathcal{T} \) is \( \text{PersDec}(\mu, B) \).

Furthermore: \( \mathcal{T} \) is said (strongly, persistently) decisive w.r.t. \( B \) whenever it is (strongly, persistently) decisive w.r.t. \( B \) from every initial distribution \( \mu \); We then write that \( \mathcal{T} \) is \( \text{Dec}(B) \) (resp. \( \text{StrDec}(B), \text{PersDec}(B) \)). Also, given \( B \subseteq \Sigma \), we say that \( \mathcal{T} \) is (strongly, persistently) decisive w.r.t. \( B \) from \( \mu \) if it is \( \text{Dec}(\mu, B) \) (\( \text{StrDec}(\mu, B), \text{PersDec}(\mu, B) \)) for each \( B \in \mathcal{B} \). We write \( \mathcal{T} \) is \( \text{Dec}(\mu, B) \) (\( \text{StrDec}(\mu, B), \text{PersDec}(\mu, B) \)). We say that \( \mathcal{T} \) is (strongly, persistently) decisive w.r.t. \( B \) if it is \( \text{Dec}(B) \) (\( \text{StrDec}(B), \text{PersDec}(B) \)) for each \( B \in \mathcal{B} \). We write \( \mathcal{T} \) is \( \text{Dec}(B) \) (\( \text{StrDec}(B), \text{PersDec}(B) \)).

Intuitively, the (simple) decisiveness property says that, almost-surely, either \( B \) will eventually be visited, or states from which \( B \) can no more be reached will eventually be visited. It denotes a dichotomy between the behaviours of the STS \( \mathcal{T} \): there are those behaviours that visit \( B \), and those that do not visit \( B \), but then visit \( \bar{B} \); other behaviours have probability 0 to occur. Strong decisiveness imposes a similar dichotomy, but between behaviours that visit \( B \) infinitely often and behaviours that visit \( B \). Persistent decisiveness refines simple decisiveness, but by looking at an arbitrary horizon.

**Example 4.** Let us consider again the STS \( \mathcal{T}_1 \) of Example \( [\#] \) representing a discrete-time random walk. Since the chain is strongly connected, for each \( B \subseteq \mathbb{N}, \bar{B} = \emptyset \). Let us assume that \( p > 1/2 \) and that \( \mu = \delta_0 \), the Dirac distribution over state 0. Then it can be shown that for each set of states \( B \), \( \text{Prob}_{\mu}^T(\mathcal{F} B) = 1 \) and thus, \( \mathcal{T}_1 \) is \( \text{Dec}(\mu, B) \). However if \( \mu' = \delta_1 \) and \( B = \{0\} \) then \( \text{Prob}_{\mu'}^T(\mathcal{F} \{0\}) < 1 \); but since \( \bar{B} = \emptyset \), we derive that \( \mathcal{T}_1 \) is not \( \text{Dec}(\mu', B) \). Consider now for each \( i \geq 0 \), \( B_i = \{i\} \). Since \( p > 1/2 \), classical results on random walks imply that for each \( i \), \( \text{Prob}_{\mu}^T(\mathcal{G} \mathcal{F} B_i) = 0 \). And since \( \bar{B}_i = \emptyset \), we obtain that \( \mathcal{T}_1 \) is not \( \text{StrDec}(\mu, B_i) \).

Consider now the STS \( \mathcal{T}_2 \) of Example \( [\#] \). Assume that \( \lambda > \nu \) and that \( \mu = \delta_{(0,0)} \) and fix some \( T > 0 \). We consider \( B_1 = \{1\} \times [0,T] \). Then one can compute \( \bar{B} = \mathbb{N} \times [T,\infty] \). Note that here, as time almost-surely always progresses, \( \text{Prob}_{\mu}^T(\mathcal{F} \bar{B}) = 1 \). It thus follows that \( \mathcal{T}_2 \) is \( \text{Dec}(\mu, B) \) and \( \text{StrDec}(\mu, B) \).

### 3.2 Attractors

The notion of finite attractor has been used in several contexts like probabilistic lossy channel systems (see e.g. \( [\#],[\#] \)) and abstracted in \( [\#] \) in the context of denumerable Markov chains. A finite attractor is a finite set of states which is reached almost-surely from every state of the system. We lift this definition to our context, obviously relaxing the finiteness assumption, since it is very unlikely that systems with a continuous state-space will have finite attractors. Since the whole set of states is a trivial attractor, this general definition will prove useful once we are able to define attractors with some finiteness property, which will be done through abstractions in Section \( [\#] \).

**Definition 11.** Let \( \mu \in \text{Dist}(\Sigma) \) be an initial distribution. \( B \in \Sigma \) is a \( \mu \)-attractor for \( \mathcal{T} \) if \( \text{Prob}_{\mu}^T(\mathcal{F} B) = 1 \). Further, \( B \) is an attractor for \( \mathcal{T} \) if it is a \( \mu \)-attractor for every \( \mu \in \text{Dist}(\Sigma) \).
**Example 5.** Consider the random walk $T_1$ of Example 4 and assume again that $p > 1/2$. For $B = \{5\}$, it can be shown, as stated before, that $B$ is a $\mu$-attractor for $\mu = \delta_0$. However, for any distribution $\mu' \in \text{Dist}(\mathbb{N}_{\geq 6})$ over naturals greater than 6, $\text{Prob}^T_{\mu'}(F B) < 1$ and thus $B$ is not a $\mu'$-attractor.

On the other hand, if we assume $p \leq 1/2$, it is a well-known property of random walks that $\{0\}$ is reached almost-surely from every state, hence we can infer that any bounded subset $A$ of $\mathbb{N}$ is an attractor (for every initial distribution).

Attractors are very strong properties of STSs, and even in our general context, the following strong property is satisfied.

**Lemma 12.** If $B$ is an attractor for $T$ then for every initial distribution $\mu \in \text{Dist}(S)$,

$$\text{Prob}^T_\mu(GF B) = 1.$$

**Proof.** Let $B$ be an attractor for $T$, i.e. for each initial distribution $\mu \in \text{Dist}(S)$, $\text{Prob}^T_\mu(F B) = 1$. Towards a contradiction, assume that there is $\mu \in \text{Dist}(S)$ such that $\text{Prob}^T_\mu(F G B^c) < 1$. Then, $\text{Prob}^T_\mu(FG B^c) > 0$. Now remember that from the definitions, we have that

$$\text{Ev}_T(FG B^c) = \bigcup_{n \geq 0} \bigcap_{m \geq 0} \text{Cyl}(S, \ldots, S, B^c, \ldots, B^c).$$

It follows that there is $n \in \mathbb{N}$ such that

$$\lim_{m \to \infty} \text{Prob}^T_\mu(\text{Cyl}(S, \ldots, S, B^c, \ldots, B^c)) > 0.$$

From Lemma 4 if we write $\nu_0 = \mu$ and $\nu_j = \Omega_T(\nu_{j-1})$ for each $1 \leq j \leq n - 1$, we get that for each $m \geq 1$,

$$\text{Prob}^T_\mu(\text{Cyl}(S, \ldots, S, B^c, \ldots, B^c)) = \text{Prob}^T_{\Omega_T(\nu_{n-1})}(\text{Cyl}(B^c, \ldots, B^c))$$

since $\mu(S) = 1$ and for each $0 \leq j \leq n - 2$, $(\Omega_T(\nu_j))(S) = 1$. It can be seen that in this case, for each $0 \leq j \leq n - 1$, $\nu_j = \Omega_T^{j+1}(\mu)$. We write $\nu = \Omega_T(\nu_{n-1}) = \Omega_T^{(n)}(\mu) \in \text{Dist}(S)$. We thus get that

$$\lim_{m \to \infty} \text{Prob}^T_{\nu}(\text{Cyl}(B^c, \ldots, B^c)) = \text{Prob}^T_{\nu}(G B^c) > 0,$$

which contradicts the fact that $B$ is an attractor, hence a $\nu$-attractor, for $T$.

### 3.3 Fairness

Fairness is a standard notion in probabilistic systems [31,32,10], saying that something which is allowed infinitely often should happen infinitely often almost-surely. This can for instance be instantiated in denumerable Markov chains as follows: if a state $s$ is visited infinitely often, and the probability to move from $s$ to $s'$ is positive, then, almost-surely, infinitely often the state $s'$ is visited. It is well-known that not all Markov chains are fair, but finitely-branching Markov chains are fair. Fairness cannot be lifted directly to continuous state-space STSs (since
for two states $s$ and $s'$, the probability to move from $s$ to $s'$ is likely to be 0). A more careful definition of this notion must be provided for general STSs.

For $B \in \Sigma$, we define $\text{PreProb}^T(B) = \{B' \in \Sigma \mid \forall \mu' \in \text{Dist}(B'), \text{Prob}_\mu^T(\text{Cyl}(B', B)) > 0\}$, as the set of measurable sets $B'$ “from which” $B$ can be reached with positive probability. Note that, ideally we would like to define the maximal set that allows one to reach $B$, but the union of all such sets may not be measurable in our general context.

**Definition 13.** Let $\mu \in \text{Dist}(S)$ be some initial distribution, and $B \in \Sigma$. The STS $T$ is fair w.r.t. $B$ from $\mu$, written $T$ is fair($\mu, B$), if for every $B' \in \text{PreProb}^T(B)$, $\text{Prob}_\mu^T(\text{G F } B') > 0$ implies $\text{Prob}_\mu^T(\text{G F } B | \text{G F } B') = 1$.

We then write that $T$ is fair($\mu, B$).

As for decisiveness, we extend this definition to sets $B \subseteq \Sigma_2$, and use similar notations when we relax the fixed initial measure $\mu$. Finally, we say that $T$ is strongly fair whenever it is fair w.r.t. $B$ from $\mu$ for every $B \in \Sigma$ and every $\mu \in \text{Dist}(S)$.

**Example 6.** Consider again the random walk of Example 1. $T_1$ is strongly fair by observing that there is a positive lower bound on the non-zero probabilities to reach any set of states. Formally there exists $\epsilon > 0$ such that for each $B \subseteq S$, for each $B' \in \text{PreProb}^T(B)$ and for each $s \in B'$, $\kappa_1(s, B) \geq \epsilon$. It suffices to choose $\epsilon = \min(p, 1 - p) > 0$.

**Example 7 (Counter-example).** Consider now the DMC $T_3$ depicted in Figure 3. Consider $B = \{b\}$, $\mu = \delta_b$ and $B' = \{a_n \mid n \in \mathbb{N}\}$, $B' \in \text{PreProb}^T(B)$. It holds that $\text{Prob}_\mu^T(\text{G F } B') > 0$, however, $\text{Prob}_\mu^T(\text{G F } B | \text{G F } B') < 1$ and thus $T_3$ is not fair($\mu, B$).

![Fig. 3. A denumerable Markov chain $T_3$ that is not strongly fair.](image)

### 3.4 Relationships between the various properties

In this section, we compare all the notions, and give the precise links between all these notions. We first analyze the general case, and reinforce the results in the case of DMCs.

**General case** We can establish the following links between the notions of decisiveness and fairness. The first result is straightforward.

**Lemma 14.** For each $B \subseteq \Sigma$ and for each $\mu \in \text{Dist}(S)$, it holds that Dec($B$) (resp. StrDec($B$), PersDec($B$)) implies Dec($\mu, B$) (resp. StrDec($\mu, B$), PersDec($\mu, B$)), and fair($B$) implies fair($\mu, B$).
We also get straightforwardly from the definitions, the following implication.

**Lemma 15.** For each $B \subseteq \Sigma$ and for each $\mu \in \text{Dist}(S)$, it holds that $\text{StrDec}(\mu, B)$ implies $\text{Dec}(\mu, B)$, and $\text{PersDec}(\mu, B)$ implies $\text{Dec}(\mu, B)$.

It then turns out that strong decisiveness and persistent decisiveness are two equivalent notions.

**Lemma 16.** For each $B \subseteq \Sigma$ and for each $\mu \in \text{Dist}(S)$, it holds that $\text{StrDec}(\mu, B)$ is equivalent to $\text{PersDec}(\mu, B)$.

*Proof.* Fix $B \subseteq \Sigma$ and $\mu \in \text{Dist}(S)$. Fix $B \in \mathcal{B}$ and assume that $T$ is $\text{PersDec}(\mu, B)$, i.e. for each $p \geq 0$, $\text{Prob}_{\mu,T}^T(\mathbf{F} \geq p B \lor \mathbf{F} \geq p \mathbf{B}) = 1$. We want to show that $T$ is $\text{StrDec}(\mu, B)$, i.e. that $\text{Prob}_{\mu,T}^T(\mathbf{G} \mathbf{F} B \lor \mathbf{F} \mathbf{B}) = 1$, or equivalently that $\text{Prob}_{\mu,T}^T(\mathbf{F} \mathbf{G} B^c \land \mathbf{G} (\mathbf{B})^c) = 0$. We have that:

$$\text{Prob}_{\mu,T}^T(\mathbf{F} \mathbf{G} B^c \land \mathbf{G} (\mathbf{B})^c) \leq \sum_{p \geq 0} \text{Prob}_{\mu,T}^T(\mathbf{G} \geq p (B^c \land (\mathbf{B})^c))$$

$$= \sum_{p \geq 0} (1 - \text{Prob}_{\mu,T}^T(\mathbf{F} \geq p B \lor \mathbf{F} \geq p \mathbf{B}))$$

$$= 0$$

from the hypothesis.

Hence we get that $\text{Prob}_{\mu,T}^T(\mathbf{G} \mathbf{F} B \lor \mathbf{F} \mathbf{B}) = 1$ and thus $T$ is $\text{StrDec}(\mu, B)$ and $\text{StrDec}(\mu, B)$ as it holds true for each $B \in \mathcal{B}$.

Now fix again $B \in \mathcal{B}$ and assume that $T$ is $\text{StrDec}(\mu, B)$, i.e. $\text{Prob}_{\mu,T}^T(\mathbf{G} \mathbf{F} B \lor \mathbf{F} \mathbf{B}) = 1$. From Lemma 9 (fourth item), we get that $\text{Prob}_{\mu,T}^T(\mathbf{G} \mathbf{F} B \lor \mathbf{G} \mathbf{F} \mathbf{B}) = 1$ and it is then straightforward to establish that for each $p \geq 0$, $\text{Prob}_{\mu,T}^T(\mathbf{F} \geq p B \lor \mathbf{F} \geq p \mathbf{B}) = 1$. We hence deduce that $T$ is $\text{PersDec}(\mu, B)$ and thus $\text{PersDec}(\mu, B)$ as it holds true for each $B \in \mathcal{B}$. This concludes the proof.

Now, we have the following equivalences between the decisiveness notions.

**Lemma 17.** For each $B \subseteq \Sigma$, it holds that all three notions $\text{PersDec}(B)$, $\text{StrDec}(B)$ and $\text{Dec}(B)$ are equivalent.

*Proof.* Fix $B \subseteq \Sigma$. From Lemmas 13 and 16 it remains to prove that $\text{Dec}(B) \Rightarrow \text{StrDec}(B)$ or $\text{Dec}(B) \Rightarrow \text{PersDec}(B)$. We prove the last one. We pick $B \in \mathcal{B}$ and assume that $T$ is $\text{Dec}(B)$, i.e. for each $\mu \in \text{Dist}(S)$, $\text{Prob}_{\mu,T}^T(\mathbf{F} B \lor \mathbf{F} \mathbf{B}) = 1$. Pick $\mu \in \text{Dist}(S)$ and $i \geq 0$. We get that

$$\text{Prob}_{\mu,T}^T(\mathbf{G} \geq i B^c \land \mathbf{G} \geq i (\mathbf{B})^c) \leq \text{Prob}_{\mu,T}^T(\mathbf{G} (B^c \land (\mathbf{B})^c))$$

where $\mu_i = \Omega^{(i)}(\mu)$, from Lemma 4 and from a similar argument as in the proof of Lemma 12.

$$\leq 0$$

since $T$ is $\text{Dec}(B)$.

Hence for each $i \geq 0$, $\text{Prob}_{\mu,T}^T(\mathbf{F} \geq i B \lor \mathbf{F} \geq i \mathbf{B}) = 1$ and since it holds true for each $\mu \in \text{Dist}(S)$ and each $B \in \mathcal{B}$, we get that $T$ is $\text{PersDec}(B)$.

Finally, we show the following links between fairness and decisiveness.
Lemma 18. For each $B \subseteq \Sigma$ and for each $\mu \in \text{Dist}(S)$, it holds that $\text{StrDec}(\mu, B)$ implies $\text{fair}(\mu, B)$, and $\text{StrDec}(B)$ implies $\text{fair}(B)$.

Proof. Fix $B \subseteq \Sigma$ and $\mu \in \text{Dist}(S)$. Assume that $T$ is strongly decisive w.r.t. $B$ from $\mu$, that is for each $B \in \mathcal{B}$, $\Pr_T(\mu)(G F B \lor F \tilde{B}) = 1$. We want to prove that for each $B \in \mathcal{B}$, for each $B' \in \text{PreProb}(B)$ with $\Pr_T(\mu)(G F B') > 0$, we have that $\Pr_T(\mu)(G F B | G F B') = 1$.

Fix $B \in \mathcal{B}$ and $B' \in \text{PreProb}(B)$ such that $\Pr_T(\mu)(G F B') > 0$. We can notice that $\Pr_T(\mu)(G F B' \land F \tilde{B}) = 0$.

Indeed, towards a contradiction, assume that $\Pr_T(\mu)(G F B' \land F \tilde{B}) > 0$. Observe that

$$\text{Ev}_T(G F B' \land F \tilde{B}) = \bigcup_{n \geq 0} \bigcap_{m \geq 0} \bigcup_{l \geq m} \text{Cyl}(S, \ldots, S, \tilde{B}, S, \ldots, S, B').$$

Then, there are $n, m \in \mathbb{N}$ such that

$$\Pr_T(\mu)(\text{Cyl}(S, \ldots, S, \tilde{B}, S, \ldots, S, B')) > 0.$$

It follows, from Lemma 4 like seen previously, that there is $\nu \in \text{Dist}(S)$ ($\nu = \Omega_T^{(n)}(\mu)$), such that

$$\Pr_T(\nu)(\text{Cyl}(\tilde{B}, S, \ldots, S, B', B')) > 0.$$

And since $B' \in \text{PreProb}(B)$, we get that

$$\Pr_T(\nu)(\text{Cyl}(\tilde{B}, S, \ldots, S, B')) > 0.$$

Hence, $\nu(\tilde{B}) > 0$ and we can apply Lemma 9 (second item) to obtain a contradiction. Hence, equation (1) holds. We then write:

$$1 = \Pr_T(\mu)(G F B \lor F \tilde{B} \mid G F B') \quad \text{from strong decisiveness}$$

$$= \frac{\Pr_T(\mu)((G F B \lor F \tilde{B}) \land G F B'))}{\Pr_T(\mu)(G F B')}$$

$$= \frac{\Pr_T(\mu)((G F B \land G F B') \lor (F \tilde{B} \land G F B'))}{\Pr_T(\mu)(G F B')}$$

$$= \frac{\Pr_T(\mu)(G F B \land G F B')}{\Pr_T(\mu)(G F B')} \quad \text{from (1)}$$

$$= \Pr_T(\mu)(G F B \mid G F B')$$

which proves that $\text{StrDec}(\mu, B) \Rightarrow \text{fair}(\mu, B)$. Then, the implication $\text{StrDec}(B) \Rightarrow \text{fair}(B)$ is immediate since the previous implication holds for any initial distribution $\mu \in \text{Dist}(S)$.

We can summarize the previous implications as follows:
Proposition 19. For each $B \subseteq \Sigma$ and for each $\mu \in \text{Dist}(S)$, it holds that

\[
\begin{align*}
\mathcal{T} \text{ is Dec}(\mu, B) & \iff \mathcal{T} \text{ is StrDec}(\mu, B) \iff \mathcal{T} \text{ is PersDec}(\mu, B) \iff \mathcal{T} \text{ is fair}(\mu, B) \\
\mathcal{T} \text{ is Dec}(B) & \iff \mathcal{T} \text{ is StrDec}(B) \iff \mathcal{T} \text{ is PersDec}(B) \iff \mathcal{T} \text{ is fair}(B)
\end{align*}
\]

The three missing implications in the above proposition do actually not hold, as witnessed by the following example. We also illustrate the fact that $\text{Dec}(\mu, B)$ and $\text{fair}(\mu, B)$ are incomparable.

Example 8 (Counter-example). Consider the random walk $\mathcal{T}_1$ of Example I. We have shown in Example 6 that $\mathcal{T}_1$ is strongly fair. Now let us assume that $p > 1/2$ and let us consider the initial distribution $\mu = \delta_0$, the Dirac distribution over 0. Then from Example II $\mathcal{T}_1$ is decisive from $\mu$ w.r.t. any set of states. Again in this example, we have observed that it is not strongly decisive w.r.t. any set of the form $B = \{i\}$ with $i \geq 0$. This shows that we do not have $\text{Dec}(\mu, B) \Rightarrow \text{StrDec}(\mu, B)$, nor $\text{fair}(\mu, B) \Rightarrow \text{StrDec}(\mu, B)$ and $\text{fair}(B) \Rightarrow \text{StrDec}(B)$. And since $\mathcal{T}_1$ is not decisive from $\delta_1$ w.r.t. $\{0\}$, this also proves that $\text{fair}(\mu, B)$ does not imply $\text{Dec}(\mu, B)$.

In order to illustrate that $\text{Dec}(\mu, B)$ does not imply $\text{fair}(\mu, B)$ in general, we consider the denumerable Markov chain $\mathcal{T}_3$ of Example VII. We consider $B = \{b\}$ and $\mu = \delta_b$. It is easily observed that $\mathcal{T}_3$ is $\text{Dec}(\mu, B)$ as we start in $b$ with probability 1, but we have shown that $\mathcal{T}_3$ is not $\text{fair}(\mu, B)$.

The case of denumerable Markov chains If $\mathcal{T}$ is a DMC, i.e. if $S$ is at most denumerable and $\Sigma = 2^S$, we can complete the picture using the following result of [1].

Lemma 20. If $\mathcal{T}$ is a DMC that has a finite attractor, then $\mathcal{T}$ is decisive w.r.t. any set of states $B \subseteq S$.

We can sum up the previous implications as follows:

\[
\mathcal{T} \text{ DMC with a finite attractor} \quad \Rightarrow \quad \mathcal{T} \text{ is Dec}(2^S) \iff \mathcal{T} \text{ is StrDec}(2^S) \iff \mathcal{T} \text{ is PersDec}(2^S) \quad \Rightarrow \quad \mathcal{T} \text{ is strongly fair}
\]

4 Abstractions between STSs

While decisiveness is well-defined for general STSs, proving that a given STS $\mathcal{T}$ is decisive might be technical in general. A standard approach in model-checking to avoid such difficulties is to abstract the system into a simpler one, that can be analyzed and provides guarantees on the concrete system. We thus propose a notion of abstraction, which will help proving properties of general STSs. Also, through abstractions, we will be able to characterize meaningful attractors.
4.1 Abstraction

Let $\mathcal{T}_1 = (S_1, \Sigma_1, \kappa_1)$ and $\mathcal{T}_2 = (S_2, \Sigma_2, \kappa_2)$ be two STSs. Let $\alpha : (S_1, \Sigma_1) \to (S_2, \Sigma_2)$ be a measurable function. A set $B \subseteq \Sigma_1$ is said $\alpha$-closed whenever $B = \alpha^{-1}(\alpha(B))$: for every $s, s' \in S_1$, if $s \in B$ and $\alpha(s) = \alpha(s')$, then $s' \in B$. Following [22], we define the pushforward of $\alpha$ as $\alpha_# : \text{Dist}(S_1) \to \text{Dist}(S_2)$ by $\alpha_#(\mu)(M_2) = \mu(\alpha^{-1}(M_2))$ for every $\mu \in \text{Dist}(S_1)$ and for every $M_2 \subseteq \Sigma_2$. The role of the pushforward $\alpha_#$ is to transfer the measures from $(S_1, \Sigma_1)$ to $(S_2, \Sigma_2)$.

**Definition 21.** $\mathcal{T}_2$ is an $\alpha$-abstraction of $\mathcal{T}_1$ if

$$\forall \mu \in \text{Dist}(S_1), \quad \alpha_#(\Omega_{\mathcal{T}_1}(\mu)) \text{ is equivalent to } \Omega_{\mathcal{T}_2}(\alpha_#(\mu)).$$

From the definitions of $\Omega_{\mathcal{T}}$, $\alpha_#$ and equivalent measures, the notion of $\alpha$-abstraction equivalently requires that for every $\mu \in \text{Dist}(S_1)$ and every $A \subseteq S_2$,

$$\text{Prob}^{\mathcal{T}_1}_\mu(\text{Cyl}(S_1, \alpha^{-1}(A))) > 0 \iff \text{Prob}^{\mathcal{T}_2}_{\alpha_#(\mu)}(\text{Cyl}(S_2, A)) > 0.$$

Intuitively, the two STSs have the same “qualitative” steps.

The notion of $\alpha$-abstraction naturally extends to labelled STSs. $\mathcal{T}_2 = (S_2, \Sigma_2, \kappa_2, AP_2, \mathcal{L}_2)$ is an $\alpha$-abstraction of $\mathcal{T}_1 = (S_1, \Sigma_1, \kappa_1, AP_1, \mathcal{L}_1)$ whenever:

- $(S_2, \Sigma_2, \kappa_2)$ is an $\alpha$-abstraction of $(S_1, \Sigma_1, \kappa_1)$;
- $AP_1 = AP_2$;
- for every $s_1, s'_1 \in S_1$, $\alpha(s_1) = \alpha(s'_1) \Rightarrow \mathcal{L}_1(s_1) = \mathcal{L}_1(s'_1)$;
- for every $s \in S_1$, $\mathcal{L}_1(s) = a \Rightarrow \mathcal{L}_2(\alpha(s)) = a$.

The last two conditions imply that for each $a \in 2^{AP}$, $L_1^{-1}(\{a\})$ is $\alpha$-closed. Moreover, for each $a \in 2^{AP}$, $\alpha^{-1}(L_2^{-1}(\{a\})) = L_1^{-1}(\{a\})$.

**Basic properties** We now establish several technical results, which explicit how STSs are related through an $\alpha$-abstraction. The relationship is only qualitative, in the sense that it only relates positive reachability probabilities, but does not relate almost-sure or lower-bounded probabilities.

**Lemma 22.** Let $\alpha : (S_1, \Sigma_1) \to (S_2, \Sigma_2)$ be a measurable function. Then for every $s \in S_2$ and every $\mu \in \text{Dist}(\alpha^{-1}(\{s\}))$, $\alpha_#(\mu) = \delta_s$.

**Lemma 23.** Assume that $\mathcal{T}_2$ is an $\alpha$-abstraction of $\mathcal{T}_1$. Then, for every $i \in \mathbb{N}$, for every $\mu \in \text{Dist}(s_1)$, $\alpha_#(\Omega_T^{(i)}(\mu))$ is equivalent to $\Omega_{\mathcal{T}}^{(i)}(\alpha_#(\mu))$.

In other words, the above lemma states that for each $A \subseteq S_2$ and for each $i \in \mathbb{N}$,

$$\text{Prob}^{\mathcal{T}_1}_\mu(F =_{i} \alpha^{-1}(A)) > 0 \iff \text{Prob}^{\mathcal{T}_2}_{\alpha_#(\mu)}(F =_{i} A) > 0.$$

This can even be generalized to cylinders:

**Lemma 24.** Assume that $\mathcal{T}_2$ is an $\alpha$-abstraction of $\mathcal{T}_1$. Then for every $\mu \in \text{Dist}(S_1)$, for every $(A_i)_{0 \leq i \leq n} \in \Sigma_2^{n+1}$,

$$\text{Prob}^{\mathcal{T}_1}_\mu(\text{Cyl}(\alpha^{-1}(A_0), \ldots, \alpha^{-1}(A_n))) > 0 \iff \text{Prob}^{\mathcal{T}_2}_{\alpha_#(\mu)}(\text{Cyl}(A_0, \ldots, A_n)) > 0.$$
As an immediate consequence, the positivity of properties with bounded witnesses are preserved through $\alpha$-abstractions:

**Corollary 25.** Assume that $\mathcal{T}_2$ is an $\alpha$-abstraction of $\mathcal{T}_1$. Then for every $\mu \in \text{Dist}(S_1)$, for every $A, B \in \Sigma_2$:

$$\text{Prob}_{\mu}^{\mathcal{T}_1}(\text{Ev}_{\mathcal{T}_1}(\alpha^{-1}(A) \cup \alpha^{-1}(B))) > 0 \iff \text{Prob}_{\alpha \#(\mu)}^{\mathcal{T}_2}(\text{Ev}_{\mathcal{T}_2}(A \cup B)) > 0 .$$

Note that this however does not apply to liveness properties, such as $\text{Ev}_T(G F A)$ with $A \in \Sigma_2$. To ensure that these more involved properties are preserved via abstraction, we will strengthen the assumptions on the abstraction and on the STSs.

**Soundness and completeness of abstractions** We assume $\mathcal{T}_2$ is an $\alpha$-abstraction of $\mathcal{T}_1$.

**Definition 26.** Let $\mu \in \text{Dist}(S_1)$. The $\alpha$-abstraction $\mathcal{T}_2$ is $\mu$-sound whenever for every $B \in \Sigma_2$:

$$\text{Prob}_{\alpha \#(\mu)}^{\mathcal{T}_2}(F B) = 1 \implies \text{Prob}_{\mu}^{\mathcal{T}_1}(F \alpha^{-1}(B)) = 1 .$$

$\mathcal{T}_2$ is a sound $\alpha$-abstraction of $\mathcal{T}_1$ if it is $\mu$-sound for every $\mu \in \text{Dist}(S_1)$.

**Definition 27.** Fix $\mu \in \text{Dist}(S_1)$. The $\alpha$-abstraction $\mathcal{T}_2$ is $\mu$-complete whenever for every $B \in \Sigma_2$,

$$\text{Prob}_{\mu}^{\mathcal{T}_1}(F \alpha^{-1}(B)) = 1 \implies \text{Prob}_{\alpha \#(\mu)}^{\mathcal{T}_2}(F B) = 1 .$$

$\mathcal{T}_2$ is a complete $\alpha$-abstraction of $\mathcal{T}_1$ if it is $\mu$-complete for every $\mu \in \text{Dist}(S_1)$.

Sound and complete abstractions will guarantee that, up to $\alpha$, the same properties are satisfied almost-surely in $\mathcal{T}_1$ and $\mathcal{T}_2$ (provided some properties are satisfied by $\mathcal{T}_1$ and $\mathcal{T}_2$).

**Example 9.** Consider again the STSs $\mathcal{T}_1$ with parameter $p \in [0,1]$ and $\mathcal{T}_2$ with parameters $\lambda$ and $\nu > 0$ of Examples 1 and 2. Let $\alpha : S_2 \rightarrow S_1$ be the mapping defined as follows: for every $i \in \mathbb{N}$ and every $t \in \mathbb{R}_+$, $\alpha((i,t)) = i$. It can be shown that $\mathcal{T}_1$ is an $\alpha$-abstraction of $\mathcal{T}_2$. Moreover, $\mathcal{T}_1$ is sound and complete whenever $p > 1/2 \iff \lambda > \nu$.

When $\mathcal{T}_2$ is a DMC, soundness and completeness have a simpler characterization, which will be useful in the proofs.

**Lemma 28.** Assume $\mathcal{T}_2$ is a DMC. Then:

- $\mathcal{T}_2$ is an $\alpha$-abstraction of $\mathcal{T}_1$ iff for every $s, s' \in S_2$,

$$\kappa_2(s, \{s\}) > 0 \iff \forall \mu \in \text{Dist}(\alpha^{-1}(\{s\})), \text{Prob}_{\mu}^{\mathcal{T}_1}(\text{Cyl}(S_1, \alpha^{-1}(\{s\}))) > 0 .$$

- $\mathcal{T}_2$ is sound iff for every $s \in S_2$ and every $B \in \Sigma_2$,

$$\text{Prob}_{\delta_\alpha}(F B) = 1 \iff \forall \mu \in \text{Dist}(\alpha^{-1}(\{s\})), \text{Prob}_{\mu}^{\mathcal{T}_1}(F \alpha^{-1}(B)) = 1 .$$

- $\mathcal{T}_2$ is complete iff for every $s \in S_2$ and every $B \in \Sigma_2$,

$$\forall \mu \in \text{Dist}(\alpha^{-1}(\{s\})), \text{Prob}_{\mu}^{\mathcal{T}_1}(F \alpha^{-1}(B)) = 1 \implies \text{Prob}_{\delta_\alpha}(F B) = 1 .$$

The proof of Lemma 28 is postponed to the appendix, page 56.
4.2 Transfer of properties through abstractions

In this section, we explain how and under which conditions one can transfer interesting decisiveness, attractor and fairness properties of STSs through abstractions.

The case of sound abstractions

Proposition 29. If \( T_2 \) is a \( \mu \)-sound \( \alpha \)-abstraction of \( T_1 \), then for every \( B \in \Sigma_2 \):

\[
T_2 \text{ is } \text{Dec}(\alpha \#(\mu), B) \implies T_1 \text{ is } \text{Dec}(\mu, \alpha^{-1}(B)).
\]

In order to prove Proposition 29, we first show the following technical lemma, which relates avoid-sets in \( T_1 \) and in \( T_2 \).

Lemma 30. Let \( T_2 \) be an \( \alpha \)-abstraction of \( T_1 \). Then, for every \( B \in \Sigma_2 \):

\[
\alpha^{-1}(B) = \alpha^{-1}(-B).
\]

Proof. Fix \( B \in \Sigma_2 \). We have the series of equivalences:

\[
\begin{align*}
 s \in \alpha^{-1}(B) & \iff \text{Prob}_{\delta_s}(F \alpha^{-1}(B)) = 0 \\
 & \iff \text{Prob}_{\alpha \#(\delta_s)}(F B) = 0 \quad \text{(Corollary 20)}.
\end{align*}
\]

Now from Lemma 22 one can show that \( \alpha \#(\delta_s) = \delta_{\alpha(s)} \) by noticing that \( \delta_s \in \text{Dist}(\alpha^{-1}(\alpha(s))) \). Hence \( s \in \alpha^{-1}(B) \) iff \( \alpha(s) \in -B \) (i.e. \( s \in \alpha^{-1}(-B) \)), which concludes the proof.

We are now ready to prove Proposition 29.

Proof (Proof of Proposition 29). Fix \( B \in \Sigma_2 \) and assume that \( T_2 \) is \( \text{Dec}(\alpha \#(\mu), B) \), i.e.

\[
\text{Prob}_{\alpha \#(\mu)}(F B \lor F -B) = 1. \tag{2}
\]

To show that \( T_1 \) is \( \text{Dec}(\mu, \alpha^{-1}(B)) \), by Lemma 30 it suffices to prove that

\[
\text{Prob}_{\mu}(F \alpha^{-1}(B) \lor F \alpha^{-1}(-B)) = 1.
\]

The latter is immediate by (2) since \( T_2 \) is \( \mu \)-sound.

This result obviously extends to stronger decisiveness notions.

Corollary 31. If \( T_2 \) is a sound \( \alpha \)-abstraction of \( T_1 \), then for every \( B \in \Sigma_2 \):

\[
T_2 \text{ is } \text{Dec}(B) \text{ (or equiv. StrDec}(B), \text{PersDec}(B) \implies \\
T_1 \text{ is } \text{Dec}(\alpha^{-1}(B)) \text{ (or equiv. StrDec}(\alpha^{-1}(B)), \text{PersDec}(\alpha^{-1}(B))
\]

The definitions of attractor and of sound \( \alpha \)-abstraction yield a similar result:

Proposition 32. If \( T_2 \) is a sound \( \alpha \)-abstraction of \( T_1 \) and if \( A \in \Sigma_2 \) is an attractor for \( T_2 \), then \( \alpha^{-1}(A) \) is an attractor for \( T_1 \).

As a direct consequence of Lemma 20 and Corollary 31 we get the following result for denumerable abstractions, which will be crucial for designing approximation algorithms taking advantage of abstractions.
Proposition 33. Let $\mathcal{T}_2$ be a DMC with a finite attractor. If $\mathcal{T}_2$ is a sound $\alpha$-abstraction of $\mathcal{T}_1$, then $\mathcal{T}_1$ is decisive w.r.t. every $\alpha$-closed set.

Let us summarize the interesting results on denumerable abstractions. Assume $\mathcal{T}_2$ is an $\alpha$-abstraction of $\mathcal{T}_1$, and write $\mathcal{B} = \{\alpha^{-1}(B) \mid B \in \Sigma_2\}$, the set of $\alpha$-closed sets of $\Sigma_1$. The following implications hold true:

\[
\begin{align*}
\mathcal{T}_2 \text{ sound and DMC with finite attractor} & \quad \implies \quad \mathcal{T}_1 \text{ is Dec}(\mathcal{B}) \quad \iff \quad \mathcal{T}_1 \text{ is StrDec}(\mathcal{B}) \quad \iff \quad \mathcal{T}_1 \text{ is PersDec}(\mathcal{B})
\end{align*}
\]

$\mathcal{T}_1$ is fair($\mathcal{B}$)

Trickier transfers of properties We established that decisiveness properties could be transferred through sound abstractions. However in the next section, we will also see that soundness of an abstraction can be proved via decisiveness properties. It is therefore relevant to explore alternatives to prove decisiveness properties. We give here two frameworks where this can be done without any assumption on the abstraction.

First, we assume a denumerable abstraction, and lower bounds on probabilities of reachability properties.

Proposition 34. Let $\mathcal{T}_2$ be a DMC such that $\mathcal{T}_2$ is an $\alpha$-abstraction of $\mathcal{T}_1$. Assume that there is a finite set $A_2 = \{s_1, \ldots, s_n\} \subseteq S_2$ such that $A_2$ is an attractor for $\mathcal{T}_2$ and $A_1 = \bigcup_{i=1}^n \alpha^{-1}(s_i) = \alpha^{-1}(A_2)$ is an attractor for $\mathcal{T}_1$. Assume moreover that for every $1 \leq i \leq n$, for every $\alpha$-closed set $B$ in $\Sigma_1$, there exist $p > 0$ and $k \in \mathbb{N}$ such that:

- for every $\mu \in \text{Dist}(\alpha^{-1}(s_i))$, $\text{Prob}_{\mu}^{\mathcal{T}_1}(F^k B) \geq p$, or
- for every $\mu \in \text{Dist}(\alpha^{-1}(s_i))$, $\text{Prob}_{\mu}^{\mathcal{T}_1}(FB) = 0$.

Then $\mathcal{T}_1$ is decisive w.r.t. every $\alpha$-closed set.

We write $(\dagger)$ for the hypotheses over $\mathcal{T}_1$ in this proposition. The idea behind this result is that, with probability 1, the attractor of $\mathcal{T}_1$ will be visited infinitely often, and, if at each visit of the attractor, there is a positive probability to reach some ($\alpha$-closed) set $B$, since that probability is by assumption bounded from below, then $B$ will indeed be visited infinitely often with probability 1. This will allow to show the dichotomy between $B$ and $\overline{B}$ that is required for proving the decisiveness property. The full proof is given in the appendix, page 57, but we give here a sketch. Note that this kind of proofs appears quite often in the literature (see e.g. Lemma 3.4), but we have to do it carefully here, since the framework is rather general.

Proof (Sketch of proof). Fix $B \subseteq S_2$ and $\mu \in \text{Dist}(S_1)$. Towards a contradiction, assume that $\mathcal{T}_1$ is not $\mu$-decisive w.r.t. $B$: this means that $\text{Prob}_{\mu}^{\mathcal{T}_1}(G \alpha^{-1}(B^c) \land G \alpha^{-1}(\overline{B})^c) > 0$.

Since $A_1$ is an attractor, we deduce from Lemma 12 that

\[
\text{Prob}_{\mu}^{\mathcal{T}_1}(G \alpha^{-1}(B^c) \land G \alpha^{-1}(\overline{B})^c \land G F A_1) > 0.
\]

We write $A_2 \subseteq A_2$ for the non-empty set of states $s$ of $A_2$ such that

\[
\text{Prob}_{\mu}^{\mathcal{T}_1}(G \alpha^{-1}(B^c) \land G \alpha^{-1}(\overline{B})^c \land G F \alpha^{-1}(s)) > 0
\]

since $A_1 = \bigcup_{s \in A_2} \alpha^{-1}(s)$.
Then obviously $A'_2 \subseteq B^c \cap (\bar{B})^c$.

Since $A'_2 \subseteq (\bar{B})^c$, from Lemma 9 (third item), the hypothesis (†) and the finiteness of $A'_2$, we get that there is $p > 0$ and $k \in \mathbb{N}$ such that for every $s \in A'_2$ and every $\mu \in \text{Dist}(\alpha^{-1}(s))$,

$$\text{Prob}^\text{T}_\mu(F \leq k \bar{B}) \geq p.$$ 

Writing $A'_1$ for $\alpha^{-1}(A'_2)$, we can show that

$$0 < \text{Prob}^\text{T}_\mu(G \alpha^{-1}(B^c) \land G \alpha^{-1}((\bar{B})^c) \land G F A'_1)$$

$$\leq \text{Prob}^\text{T}_\mu(G \alpha^{-1}(B^c) \land G F A'_1)$$

$$\leq \lim_{n \to \infty} (1 - p)^n = 0$$

which is the required contradiction.

Second, we strengthen the hypothesis and assume a finite abstraction. The condition which applies in this case is the weakest property that we have seen, namely fairness!

**Proposition 35.** Let $\mathcal{T}_2$ be a finite Markov chain such that $\mathcal{T}_2$ is an $\alpha$-abstraction of $\mathcal{T}_1$. Fix $\mu \in \text{Dist}(S_1)$, and assume that $\mathcal{T}_1$ is $\mu$-fair w.r.t. every $\alpha$-closed set. Then $\mathcal{T}_1$ is $\mu$-decisive w.r.t. every $\alpha$-closed set.

**Proof (Sketch of proof).** We give here the main steps of the proof, the details are postponed to the appendix (page 58).

A key element of the proof relies on the fact that, since $\mathcal{T}_2$ is a finite MC, it can be viewed as a graph and we can talk of the bottom strongly connected components (BSCC) of $\mathcal{T}_2$. The first step of the proof aims at showing that, roughly speaking, the union of all BSCCs of $\mathcal{T}_2$ is a $\mu$-attractor for $\mathcal{T}_1$. More precisely, if $C = \{s \in S_2 \mid \exists C \in \text{BSCC}(\mathcal{T}_2), s \in C\}$, we prove that $\text{Prob}^\text{T}_\mu(\alpha^{-1}(C)) = 1$. This is shown thanks to the following arguments:

- for each $s \in S_2$, $\text{Prob}^\text{T}_\mu(G F \alpha^{-1}(s)) > 0$ implies that $s \in C$ – this uses the $\mu$-fairness assumption of $\mathcal{T}_1$ w.r.t. $\alpha$-closed sets, and the core property of BSCCs (we cannot escape from them);
- using Bayes formula, one can decompose the set of paths according to the states which are visited infinitely often (which corresponds to a decomposition according to the BSCC the path ultimately visit).

Once we have shown that $\alpha^{-1}(C)$ is a $\mu$-attractor for $\mathcal{T}_1$, it suffices to observe that for each $B \subseteq S_2$ and each BSCC $C$ of $\mathcal{T}_2$, either $B \cap C \neq \emptyset$, or $C \subseteq \bar{B}$. Transferring those observations to $\mathcal{T}_1$ and using Bayes formula to decompose $\text{Prob}^\text{T}_\mu(F \alpha^{-1}(B) \lor F \alpha^{-1}(\bar{B}))$ according to which BSCC is reached, it is easy to check that $\text{Prob}^\text{T}_\mu(F \alpha^{-1}(B) \lor F \alpha^{-1}(\bar{B})) = 1$.

### 4.3 Conditions for completeness and soundness

In our applications (Section 7), completeness will be for free. Indeed, a simple condition implies completeness as stated in the next lemma.

**Lemma 36.** If $\mathcal{T}_2$ is a finite Markov chain and an $\alpha$-abstraction of $\mathcal{T}_1$, then $\mathcal{T}_2$ is complete.
Proof. Pick \( s_0 \in S_2 \), and \( \mu \in \text{Dist}(\alpha^{-1}\{s_0\}) \) (in particular, \( \alpha_\#(\mu) = \delta_{s_0} \), the Dirac measure over \( \{s_0\} \)). Assume that \( \text{Prob}_{\mu}^{T_1}(F \alpha^{-1}(B)) = 1 \) but \( \text{Prob}_{\alpha_\#(\mu)}(F B) < 1 \).

Since \( T_2 \) is a finite Markov chain, there are \( s_1, \ldots, s_n \in S_2 \) such that
\[
\text{Prob}_{S_{s_0}}(\text{Cyl}(s_0, s_1, \ldots, s_n)) > 0
\]
and for each \( \rho = (s_i)_{i \geq 0} \in \text{Cyl}(s_0, \ldots, s_n) \) and for each \( i \geq 0, s_i \notin B \).

For each \( 0 \leq i \leq n \), we write \( A_i = \alpha^{-1}\{s_i\} \). Then, by Lemma [23] we get that \( \text{Prob}_{\mu}^{T_1}(\text{Cyl}(A_0, A_1, \ldots, A_n)) > 0 \). However, \( \text{Cyl}(A_0, A_1, \ldots, A_n) \cap \text{Ev}_T(F \alpha^{-1}(B)) = \emptyset \), yielding a contradiction.

Note that the above lemma does not hold for denumerable abstractions. To illustrate this, any two random walks over \( \mathbb{N} \) are abstractions of each others, and it is well-known that almost-sure reachability depends on the probability values.

In general, completeness can be guaranteed by some decisiveness condition. Note that, since finite Markov chains are always decisive, the next lemma actually subsumes the latter one, that we however found interesting to have as such.

**Lemma 37.** Let \( \mu \in \text{Dist}(S_1) \). Assume that \( T_2 \) is an \( \alpha \)-abstraction of \( T_1 \) and that \( T_2 \) is Dec(\( \alpha_\#(\mu) \)). Then, \( T_2 \) is a \( \mu \)-complete \( \alpha \)-abstraction.

**Proof.** Fix \( B \in \mathcal{B} \) and assume that \( \text{Prob}_{\mu}^{T_1}(F \alpha^{-1}(B)) = 1 \) but \( \text{Prob}_{\alpha_\#(\mu)}(F B) < 1 \).

Since \( T_2 \) is Dec(\( \alpha_\#(\mu) \)), we infer from Lemma [9](fifth item) that \( \text{Prob}_{\alpha_\#(\mu)}((\neg B) \cup \neg \overline{B}) > 0 \), and applying Corollary [25] we get that \( \text{Prob}_{\alpha_\#(\mu)}(\alpha^{-1}(\neg B) \cup \alpha^{-1}(\overline{B})) > 0 \). This contradicts the hypothesis that \( \text{Prob}_{\mu}^{T_1}(F \alpha^{-1}(B)) = 1 \).

Given the above results, we realize that soundness is often more critical than completeness, and showing it may require some effort. Below, we give a condition under which soundness holds.

**Proposition 38.** Let \( T_2 \) be an \( \alpha \)-abstraction of \( T_1 \). Assume \( T_1 \) is decisive w.r.t. every \( \alpha \)-closed sets. Then \( T_2 \) is a sound \( \alpha \)-abstraction of \( T_1 \).

**Proof.** Towards a contradiction assume that \( B \in \Sigma_2 \) is such that \( \text{Prob}_{\mu}^{T_1}(F \alpha^{-1}(B)) < 1 \). Since \( T_1 \) is decisive w.r.t. \( \alpha^{-1}(B) \) from \( \mu \), it holds from Lemma [9](fifth item) that \( \text{Prob}_{\alpha_\#(\mu)}(\neg \alpha^{-1}(B) \cup \neg \alpha^{-1}(\overline{B})) > 0 \). Applying Corollary [25] we get that \( \text{Prob}_{\alpha_\#(\mu)}((\neg B) \cup \neg \overline{B}) > 0 \), which contradicts the assumption that \( \text{Prob}_{\alpha_\#(\mu)}(F B) = 1 \).

**Remark 4** For \( T_2 \) an \( \alpha \)-abstraction of \( T_1 \), notice that completeness is ensured by a decisiveness assumption on \( T_2 \), whereas soundness requires \( T_1 \) being decisive w.r.t. every \( \alpha \)-closed set. While these conditions look very similar, the condition for soundness is actually harder to check since the abstract STS \( T_2 \) will be simpler than the original concrete STS \( T_1 \).

## 5 Qualitative analysis

In this section, we rely on the notions previously introduced and studied to design generic procedures for the qualitative analysis of properties of STSs, under some assumptions that
will be made precise. We emphasize that there are procedures rather than algorithms, since algorithms would require some effectiveness conditions on the STSs (numerical conditions, or decidability of some graph properties in the underlying non-stochastic model). Next, we will explicit necessary conditions to obtain algorithms from the generic procedures. For most natural STSs (and in particular for our applications – see Section 7), these conditions will be satisfied.

In the remainder of this section, we fix an STS $T = (S, \Sigma, \kappa)$.

5.1 Basic properties under decisiveness hypotheses

Our objective here is to describe generic procedures that capture the qualitative (almost-sure, positive) satisfaction of reachability and repeated reachability properties.

Given $B \in \Sigma$ a measurable set, recall that $\{s \in S \mid \text{Prob}_T(FB) = 0\}$ denotes its avoid-set. We start with two technical lemmas that will be useful to show various results thereafter. Their proofs are postponed to Appendix D, page 60.

**Lemma 39.** For every $\mu \in \text{Dist}(S)$

(i) $\text{Prob}_T(\neg B U \neg B) = 0$;
(ii) $\text{Prob}_T(GFB) = 0$.

**Lemma 40.** For every $\mu \in \text{Dist}(S)$, if $T$ is $\text{PersDec}(\mu, B)$, then $\text{Prob}_T(F\neg B U \neg B) = 0$.

Extending the approach of [1], we establish characterizations of the qualitative satisfaction of (repeated) reachability properties in terms of the positive satisfaction of reachability-like properties. We advocate that these are simpler to check on STSs: positive reachability amounts to guessing a “symbolic” path (or cylinder) leading to the target, and to showing that this path has a positive measure.

**Proposition 41 (Almost-sure reachability).** For every $\mu \in \text{Dist}(S)$

- if $\text{Prob}_\mu(FB) = 1$ then $\text{Prob}_\mu(\neg B U \neg B) = 0$;
- if $T$ is $\text{Dec}(\mu, B)$ and $\text{Prob}_\mu(\neg B U \neg B) = 0$, then $\text{Prob}_\mu(FB) = 1$.

**Proof.** We start with the first item. Since the event $\text{Ev}_T(FB)$ is almost-sure, we have

$$\text{Prob}_\mu(\neg B U \neg B) = \text{Prob}_\mu(\neg B U \neg B) \land FB$$

and then it is straightforward from point (i) of Lemma 39.

In order to prove the other implication, we need the assumption that $T$ is $\text{Dec}(\mu, B)$. We have that:

$$1 = \text{Prob}_\mu(FB U FB) = \text{Prob}_\mu(FB U (\neg B U \neg B)) \quad \text{from Lemma 9 (fifth item)}$$

$$= \text{Prob}_\mu(FB) + \text{Prob}_\mu(\neg B U \neg B) \quad \text{from Lemma 39 (point (i)).}$$

Then from $\text{Prob}_\mu(\neg B U \neg B) = 0$, we derive that $\text{Prob}_\mu(FB) = 1$.

This reduces the almost-sure model-checking of a reachability property to the 0-probability of an Until formula, a slight generalization of reachability properties.
Proposition 42 (Almost-sure repeated reachability). For every \( \mu \in \text{Dist}(S) \)
- if \( \text{Prob}_\mu^T(G F B) = 1 \) then \( \text{Prob}_\mu^T(F \widetilde{B} = 0) \);
- if \( T \) is \( \text{StrDec}(\mu, B) \) and \( \text{Prob}_\mu^T(F B) = 0 \), then \( \text{Prob}_\mu^T(G F B) = 1 \).

Proof. Since the event \( \text{Ev}_T(G F B) \) is almost-sure, we have
\[
\text{Prob}_\mu^T(F \widetilde{B}) = \text{Prob}_\mu^T(F B \land G F B)
\]
and then it is straightforward from point (ii) of Lemma 39.

In order to prove the second item, we assume that \( T \) is \( \text{StrDec}(\mu, B) \), i.e. \( \text{Prob}_\mu^T(G F B \lor F \widetilde{B}) = 1 \). By assumption, the event \( \text{Ev}_T(F B) \) has probability 0, and thus \( \text{Ev}_T(G F B) \) is almost-sure.

This reduces the almost-sure model-checking of a repeated reachability property to the 0-probability of a reachability property.

Proposition 43 (Positive repeated reachability). For every \( \mu \in \text{Dist}(S) \)
- if \( T \) is \( \text{Dec}(\mu, \widetilde{B}) \) and if \( \text{Prob}_\mu^T(G F B) > 0 \), then \( \text{Prob}_\mu^T(F \widetilde{B}) > 0 \);
- if \( T \) is \( \text{PersDec}(\mu, B) \) and if \( \text{Prob}_\mu^T(F \widetilde{B}) > 0 \), then \( \text{Prob}_\mu^T(G F B) > 0 \).

Proof. For the first item, we only require \( T \) to be \( \text{Dec}(\mu, \widetilde{B}) \), that is \( \text{Prob}_\mu^T(F \widetilde{B} \lor F \widetilde{B}) = 1 \). Since the event \( \text{Ev}_T(F \widetilde{B} \lor F \widetilde{B}) \) is almost-sure, we derive the equality:
\[
\text{Prob}_\mu^T(G F B) = \text{Prob}_\mu^T(G F B \land (F \widetilde{B} \lor F \widetilde{B}))
\]
Now from point (ii) of Lemma 39 we get that \( \text{Prob}_\mu^T(G F B \land (F \widetilde{B} \lor F \widetilde{B})) = \text{Prob}_\mu^T(G F B \land \widetilde{F} \widetilde{B}) \). Therefore \( \text{Prob}_\mu^T(G F B \land F \widetilde{B}) = \text{Prob}_\mu^T(G F B \land \widetilde{F} \widetilde{B}) = 0 \), and thus \( \text{Prob}_\mu^T(F \widetilde{B}) > 0 \).

Assume now that \( T \) is \( \text{PersDec}(\mu, B) \) and that \( \text{Prob}_\mu^T(F \widetilde{B}) > 0 \). Lemma 40 implies that \( \text{Prob}_\mu^T(F \widetilde{B}) < 1 \). Since \( \text{PersDec}(\mu, B) \) implies \( \text{StrDec}(\mu, B) \), it follows that \( \text{Prob}_\mu^T(G F B \lor \widetilde{F} \widetilde{B}) = 1 \) and thus, \( \text{Prob}_\mu^T(G F B) > 0 \).

This reduces the positive model-checking of a repeated reachability property to the positive model-checking of a reachability property.

Hence, in all cases, under some specific assumptions, the properties one wants to check are reduced to checking whether a specific reachability (or Until) property has positive probability. These are the simplest properties one can hope to be decidable in a class of models. Effectiveness hence relies here on the computation of avoid-sets, avoid-sets of avoid-sets, and on the decidability of the positive reachability (or Until) problem.

5.2 Basic properties through abstractions

Via abstractions, one can reduce the qualitative analysis of basic properties (reachability and repeated reachability) from the concrete model to the abstract model. Indeed, one can use the previous results, and show immediately:
**Proposition 44.** Assume $\mathcal{T}_2$ is an $\alpha$-abstraction of $\mathcal{T}_1$, and fix $B \in \Sigma_2$.

- Let $\mu \in \text{Dist}(S_1)$ be an initial distribution for $\mathcal{T}_1$. Assume that $\mathcal{T}_2$ is $\mu$-sound and $\mu$-complete. Then:
  \[ \text{Prob}_{\mu}^{\mathcal{T}_1}(\mathbf{F} \alpha^{-1}(B)) = 1 \iff \text{Prob}_{\alpha \#(\mu)}^{\mathcal{T}_2}(\mathbf{F} B) = 1. \]

- Assume that $\mathcal{T}_2$ is sound and complete, and that $\mathcal{T}_2$ is $\text{StrDec}(B)$. Then for every $\mu \in \text{Dist}(S_1)$:
  \[ \text{Prob}_{\mu}^{\mathcal{T}_1}(\mathbf{G} \mathbf{F} \alpha^{-1}(B)) = 1 \iff \text{Prob}_{\alpha \#(\mu)}^{\mathcal{T}_2}(\mathbf{G} \mathbf{F} B) = 1. \]

- Assume that $\mathcal{T}_2$ is sound and complete, and that $\mathcal{T}_2$ is $\text{PersDec}(B)$ and $\text{Dec}(\overline{B})$. Then for every $\mu \in \text{Dist}(S_1)$:
  \[ \text{Prob}_{\mu}^{\mathcal{T}_1}(\mathbf{G} \mathbf{F} \alpha^{-1}(B)) = 1 \iff \text{Prob}_{\alpha \#(\mu)}^{\mathcal{T}_2}(\mathbf{G} \mathbf{F} B) = 1. \]

This allows one to perform the qualitative analysis of (repeated) reachability properties in $\mathcal{T}_1$ on its abstraction $\mathcal{T}_2$, which is quite useful since $\mathcal{T}_2$ will usually be simpler than $\mathcal{T}_1$.

### 5.3 $\omega$-regular properties in DMCs with a finite attractor

In the case of DMCs with a finite attractor, the almost-sure satisfaction of properties given as a DMA can be characterized by finitely many reachability properties [2,13]. In many cases, this characterization yields an effective algorithm to decide the almost-sure satisfaction of $\omega$-regular properties. Note that, since DMA are closed under complement, the positive relationships between the BSCCs and attractors for $\mathcal{T}$.

We fix a finite set of atomic propositions $\text{AP}$, we let $\mathcal{T} = (S, \Sigma, \kappa, \text{AP}, \mathcal{L})$ be a labelled DMC and we also let $\mathcal{M} = (Q, q_0, E, \mathcal{F})$ be a DMA. The product $\mathcal{T} \times \mathcal{M}$ has been defined in Section 2.5.

First, since $\mathcal{M}$ has finitely many states, attractors transfer from $\mathcal{T}$ to the product $\mathcal{T} \times \mathcal{M}$. This is stated formally below, and proven in the technical appendix (page 61).

**Lemma 45.** Assume that $A$ is an attractor for $\mathcal{T}$. Then $A \times Q$ is an attractor for $\mathcal{T} \times \mathcal{M}$. Furthermore, if $A$ is finite, then so is $A \times Q$.

We assume that $\mathcal{T}$ has a finite attractor, hence applying the above result, the product $\mathcal{T} \times \mathcal{M}$ admits a finite attractor that we denote $B$. We write $\text{Graph}_{\mathcal{T} \times \mathcal{M}}(B)$ (or simply $\text{Graph}(B)$ when $\mathcal{T}$ and $\mathcal{M}$ are clear from the context) for the finite graph whose vertices are states of $B$, and in which there is an edge from $(s_1, q_1)$ to $(s_2, q_2)$ if $(s_1, q_1) \rightarrow^* (s_2, q_2)$ in $\mathcal{T} \times \mathcal{M}$. The bottom strongly connected components (BSCCs) of the graph $\text{Graph}_{\mathcal{T} \times \mathcal{M}}(B)$ play a central role in the model checking of $\omega$-regular properties of $\mathcal{T}$. Let us first discuss the relationships between the BSCCs and attractors for $\mathcal{T} \times \mathcal{M}$.

**Lemma 46.** The following properties are satisfied:

- The set $\{(s, q) \in C \mid \text{C BSCC of } \text{Graph}_{\mathcal{T} \times \mathcal{M}}(B)\}$ is an attractor of $\mathcal{T} \times \mathcal{M}$.
- If $C$ and $C'$ are two distinct BSCCs of $\text{Graph}_{\mathcal{T} \times \mathcal{M}}(B)$, for every $\mu \in \text{Dist}(S \times Q)$, $\text{Prob}_{\mu}^{\mathcal{T} \times \mathcal{M}}(\mathbf{F} C \land \mathbf{F} C') = 0$. 


If $C$ is a BSCC of $\text{Graph}_{\mathcal{T} \times \mathcal{M}}(B)$, for every $\mu \in \text{Dist}(C)$, $\text{Prob}_{\mu}^{\mathcal{T} \times \mathcal{M}}(G F C) = 1$.

**Proof.** The first property is obvious. The second property is a consequence of the fact that there is no path between two states of two different BSCCs. This second property implies that for each BSCC $C' \neq C$ of $\text{Graph}_{\mathcal{T} \times \mathcal{M}}(B)$ and for each $\mu \in \text{Dist}(C)$, $\text{Prob}_{\mu}^{\mathcal{T} \times \mathcal{M}}(F C') = 0$. From the first property and Lemma 12, we know that for each $\mu$ for each BSCC $C$, $\text{Prob}_{\mu}^{\mathcal{T} \times \mathcal{M}}(G F C) = 0$. This holds true in particular for each $\mu \in \text{Dist}(C)$ and thus, from the previous observation for such initial distributions, we get that $\text{Prob}_{\mu}^{\mathcal{T} \times \mathcal{M}}(G F C) = 1$ for each $\mu \in \text{Dist}(C)$.

From Lemma 16, the BSCCs of $\text{Graph}_{\mathcal{T} \times \mathcal{M}}(B)$ form an attractor, and once the system enters a BSCC $C$, only that BSCC will be visited again, and this will happen infinitely often. In particular, the satisfaction of the Muller condition in $\mathcal{T} \times \mathcal{M}$, inherited from $\mathcal{F}$, can be characterized by the BSCCs satisfying the Muller condition $\mathcal{F}$ (in a sense that we will make precise).

**Definition 47 (Good BSCC).** A BSCC $C$ of $\text{Graph}_{\mathcal{T} \times \mathcal{M}}(B)$ is good for $\mathcal{F}$, written $C \in \text{Good}_{\mathcal{T} \times \mathcal{M}}(\mathcal{F})$, if there exists $F \in \mathcal{F}$ such that

(a) for every state $(s, q)$, if there exists $(r, p) \in C$ with $(r, p) \rightarrow^* (s, q)$, then $q \in F$; and

(b) for every $q \in F$ there exists $s \in S$, there exists a state $(r, p) \in C$ such that $(r, p) \rightarrow^* (s, q)$.

Let $C$ be an arbitrary BSCC of $\text{Graph}_{\mathcal{T} \times \mathcal{M}}(B)$. We define the set $F_C = \{ q \in Q \mid \exists s \in S \exists (r, p) \in C \text{ s.t. } (r, p) \rightarrow^* (s, q) \}$ as the set of states of the Muller automaton that can be reached from $C$. Within a BSCC, all reachable states will actually be visited infinitely often almost-surely. More precisely, we state the following result:

**Lemma 48.** For every $(s, q) \in C$, $\text{Prob}_{\mathcal{B}(s, q)}^{\mathcal{T} \times \mathcal{M}}(\inf \subseteq F_C) = 1$.

**Proof.** Let $(s, q) \in C$, and $\rho = (s, q)(s_1, q_1)(s_2, q_2)\ldots$ a path in $\mathcal{T} \times \mathcal{M}$ starting at $(s, q)$. By definition of $F_C$, all $q_i$’s are in $F_C$, hence $\text{Prob}_{\mathcal{B}(s, q)}^{\mathcal{T} \times \mathcal{M}}(\inf \subseteq F_C) = 1$.

We now argue why all elements of $F_C$ are actually almost-surely visited infinitely often. Fix $p \in F_C$ and $(r, p)$ that is reachable from $C$. All two states of $C$ are reachable from each other; thus, from every state of $C$, $(r, p)$ is reachable through a finite path. Hence there is some $\nu > 0$ and $k \in \mathbb{N}$ such that for every state $(s', q') \in C$,

$$\text{Prob}_{\mathcal{B}(s', q')}^{\mathcal{T} \times \mathcal{M}}(F) \geq k (r, p)) \geq \nu .$$

Applying a reasoning similar to the proof of Proposition 34, we get that $\text{Prob}_{\mathcal{B}(s, q)}^{\mathcal{T} \times \mathcal{M}}(G F (r, p) \mid G F C) = 1$. Indeed, $\text{Prob}_{\mathcal{B}(s, q)}^{\mathcal{T} \times \mathcal{M}}(F G \neg (r, p) \land G F C) \leq \lim_{n \to \infty} (1 - \nu)^n = 0$. Thanks to the third item of Lemma 46, we obtain that

$$\text{Prob}_{\mathcal{B}(s, q)}^{\mathcal{T} \times \mathcal{M}}(G F (r, p)) = 1 .$$

We conclude that $\text{Prob}_{\mathcal{B}(s, q)}^{\mathcal{T} \times \mathcal{M}}(\inf \supseteq F_C) = 1$, which completes the proof.

As a consequence:

---

We recall that $\inf = F_C$ characterizes the set of runs $\rho'$ in $\mathcal{T} \times \mathcal{M}$ such that $\inf(\rho') = F_C$ ($\rho'$ is the labelling function of $\mathcal{T} \times \mathcal{M}$ such that $\rho'(s, q) = q$).
Corollary 49. For every initial distribution $\mu \in \text{Dist}(S)$ for $T$ and for every $q \in Q$, $\text{Prob}^{\omega}_{\mu \times \delta_{q_0}}(F \, C)$ implies $\text{Prob}^{\omega}_{\mu \times \delta_{q_0}}(\text{Inf} = F_C \mid F \, C) = 1$.

We can now completely characterize the probability of satisfying an $\omega$-regular property.

Theorem 50. Let $T$ be a labelled DMC with a finite attractor $B$, and $M = (Q, q_0, E, F)$ be a DMA. Then, for every initial distribution $\mu \in \text{Dist}(S)$ for $T$:

$$\text{Prob}^{\omega}_{\mu \times \delta_{q_0}}(\text{Inf} \in F) = \sum_{C \in \text{BSCC of Graph}(B)} \text{Prob}^{\omega}_{\mu \times \delta_{q_0}}(F \, C) \cdot \text{Prob}^{\omega}_{\mu \times \delta_{q_0}}(\text{Inf} \in F \mid F \, C).$$

Proof. Fix $\mu \in \text{Dist}(S)$ and $q \in Q$. As stated in Lemma 46, the BSCCs of Graph($B$) form an attractor for $T \times M$, and two BSCCs are probabilistically disjoint. Using Bayes formula with a disjunction over the BSCCs, we can write:

$$\text{Prob}^{\omega}_{\mu \times \delta_{q_0}}(\text{Inf} \in F) = \sum_{C \in \text{BSCC of Graph}(B)} \text{Prob}^{\omega}_{\mu \times \delta_{q_0}}(F \, C) \cdot \text{Prob}^{\omega}_{\mu \times \delta_{q_0}}(\text{Inf} \in F \mid F \, C)$$

where we say that $C$ is $\mu \times \delta_{q_0}$-reachable whenever $\text{Prob}^{\omega}_{\mu \times \delta_{q_0}}(F \, C) > 0$. Hence we deduce that:

$$\text{Prob}^{\omega}_{\mu \times \delta_{q_0}}(\text{Inf} \in F) = \sum_{C \in \text{BSCC of Graph}(B)} \text{Prob}^{\omega}_{\mu \times \delta_{q_0}}(F \, C) \cdot 1_F(F_C)$$

thanks to Corollary 49 where $1_F$ is the characteristic function of $F$ (that is, $1_F(F) = 1$ if $F \in F$, and $1_F(F) = 0$ otherwise). This concludes the proof of the Theorem.

As an immediate corollary of Theorem 50, we obtain a characterization for the almost-sure satisfaction of properties given as DMA.

Corollary 51 (Almost-sure $\omega$-regular property). Let $T$ be a labelled DMC with a finite attractor, and $M = (Q, q_0, E, F)$ be a DMA. Let $B$ be a finite attractor for $T \times M$. For every initial distribution $\mu \in \text{Dist}(S)$ for $T$:

$$\text{Prob}^{\omega}_{\mu \times \delta_{q_0}}(\text{Inf} \in F) = 1 \quad \text{if and only if}$$

every BSCC $C$ of Graph$_{T \times M}(B)$ such that $\text{Prob}^{\omega}_{\mu \times \delta_{q_0}}(F \, C) > 0$ is good for $F$.

In order to turn this characterization into a decision procedure, we need to be able to compute the attractor $B$ for $T \times M$, and to build the graph Graph$_{T \times M}(B)$; also one needs to be able to compute for every BSCC $C$ the set $F_C$.

5.4 $\omega$-regular properties of general STSs via abstraction and finite attractor

While the previous approach is adapted to DMCs, it does not apply directly to general STSs: indeed, it is unlikely that general STSs have finite attractors, and finiteness of the attractor is fundamental for the correctness of the approach. The idea will then be to rely on an abstraction, and to transfer properties through that abstraction.

Let $T_1 = (S_1, \Sigma_1, \kappa_1, \text{AP}, L_1)$ and $T_2 = (S_2, \Sigma_2, \kappa_2, \text{AP}, L_2)$ be two labelled STSs such that $T_2$ is a DMC, which is an $\alpha$-abstraction of $T_1$. Under certain conditions, we show how to
perform the qualitative model checking of \(\omega\)-regular properties on \(T_1\) by transferring the same analysis on \(T_2\). We assume the \(\omega\)-regular property is given by a DMA \(\mathcal{M} = (Q, q_0, E, F)\). We consider both the product \(T_1 \times \mathcal{M}\) and the product \(T_2 \times \mathcal{M}\).

First we justify why, within a slight abuse of terminology, \(T_2 \times \mathcal{M}\) can be viewed as an \(\alpha\)-abstraction of \(T_1 \times \mathcal{M}\). We also exhibit a sufficient condition under which it is sound.

**Lemma 52.** Let \(\alpha, \mathcal{M} : S_1 \times Q \rightarrow S_2 \times Q\) be the unique lifting of \(\alpha\) such that \(\alpha, \mathcal{M}(s, q) = (\alpha(s), q)\). If \(T_2\) is an \(\alpha\)-abstraction of \(T_1\), then \(T_2 \times \mathcal{M}\) is an \(\alpha, \mathcal{M}\)-abstraction of \(T_1 \times \mathcal{M}\). Furthermore, if \(T_1 \times \mathcal{M}\) is Dec \((B)\) where \(B = \{\alpha, \mathcal{M}^{-1}(B) \mid B \in \Sigma_2\}\), then \(T_2 \times \mathcal{M}\) is a sound \(\alpha, \mathcal{M}\)-abstraction of \(T_1 \times \mathcal{M}\).

While the proof of the first part of the lemma is technical hence postponed to the appendix (page 63), the second part of the lemma is a consequence of Proposition 68.

**Remark 5** In the sequel, our applications will be smooth enough to meet the hypothesis: \(T_1 \times \mathcal{M}\) is decisive w.r.t. \(\alpha, \mathcal{M}\)-closed sets. However we still have several open questions. The first one is the following: does soundness between \(T_2\) and \(T_1\) imply soundness between \(T_2 \times \mathcal{M}\) and \(T_1 \times \mathcal{M}\)? While this seems quite natural, it is surprisingly tricky. Although we did not manage to find a counter-example for this general question, we found one for a fixed initial distribution. It is described in Example 11 in the appendix (page 62) and highlights some difficulties we encounter when aiming at transferring analysis from the abstraction to the concrete model.

This justifies the fact that we assumed decisiveness. As we already know, if \(T_2\) is a sound \(\alpha\)-abstraction of \(T_1\) and \(T_2\) is decisive w.r.t. any set of states, then \(T_1\) is decisive w.r.t. any \(\alpha\)-closed sets. Then the second natural question is the following: does decisiveness w.r.t. \(\alpha\)-closed sets for \(T_1\) imply decisiveness w.r.t. \(\alpha, \mathcal{M}\)-closed sets for \(T_1 \times \mathcal{M}\)? Again, we do not have a general counter-example, but we have one for a fixed initial distribution. This is described in Example 11 in the appendix (page 62).

From now on, whenever \(T_1 \times \mathcal{M}\) is decisive w.r.t. \(\alpha, \mathcal{M}\)-closed sets and thus the previous result is applicable, we will abusively write \(\alpha\) for \(\alpha, \mathcal{M}\).

We focus now on the case where \(T_2\) has a finite attractor.\(^7\) Applying Lemma 45, \(T_2 \times \mathcal{M}\) has also a finite attractor, which we denote \(B_2\). We reuse notations of the previous subsection, in particular the graph of the attractor \(\text{Graph}_{T_2 \times \mathcal{M}}(B_2)\), and the set \(F_C\) of recurring states when \(C\) is a BSCC of that graph.

The following lemma is a counterpart to Lemma 46 for \(T_1\). Under the hypothesis that \(T_1 \times \mathcal{M}\) is decisive w.r.t. \(\alpha\)-closed sets, even though \(T_1 \times \mathcal{M}\) does not have a finite attractor, it has an attractor with an interesting structure inherited from \(T_2 \times \mathcal{M}\). In the sequel, we write \(B = \{\alpha^{-1}(B) \mid B \in \Sigma_2\}\).

**Lemma 53.** Assume \(T_2\) has a finite attractor, and assume that \(T_2 \times \mathcal{M}\) is a sound \(\alpha\)-abstraction of \(T_1 \times \mathcal{M}\). The following properties are satisfied:

- The set \(\alpha^{-1}\{((s, q) \in C \mid C \text{ BSCC of } \text{Graph}_{T_2 \times \mathcal{M}}(B_2))\}\) is an attractor of \(T_1 \times \mathcal{M}\).
- If \(C\) and \(C'\) are two distinct BSCCs of \(\text{Graph}_{T_2 \times \mathcal{M}}(B_2)\), for every \(\mu \in \text{Dist}(S_1 \times Q)\), \(\text{Prob}_{\mu^{T_1 \times \mathcal{M}}(F \alpha^{-1}(C) \land F \alpha^{-1}(C'))} = 0\).
- If \(C\) is a BSCC of \(\text{Graph}_{T_2 \times \mathcal{M}}(B_2)\), for every \(\mu \in \text{Dist}(\alpha^{-1}(C))\), \(\text{Prob}_{\mu^{T_1 \times \mathcal{M}}(G F \alpha^{-1}(C))} = 1\).

\(^7\) As \(T_2\) has a finite attractor it is decisive and thus \(T_2\) is a complete \(\alpha\)-abstraction of \(T_1\) by Lemma 47.
Proof. Since \( \mathcal{T}_2 \times \mathcal{M} \) is a sound \( \alpha \)-abstraction of \( \mathcal{T}_1 \times \mathcal{M} \), the first property derives from Proposition \( \text{32} \) and Lemma \( \text{46} \). The second property is a consequence of Lemma \( \text{46} \) and of the fact that \( \mathcal{T}_2 \times \mathcal{M} \) is an \( \alpha \)-abstraction of \( \mathcal{T}_1 \times \mathcal{M} \). Finally, the third property is, as in the proof of Lemma \( \text{46} \) a consequence of the second point and of Lemma \( \text{12} \).

We then prove a counterpart to Lemma \( \text{43} \) for \( \mathcal{T}_1 \), which shows that a BSCC is characterized by the set \( F_C \) of states that are visited infinitely often from \( C \).

**Lemma 54.** Assume \( \mathcal{T}_2 \) has a finite attractor, and assume that \( \mathcal{T}_2 \times \mathcal{M} \) is a sound \( \alpha \)-abstraction of \( \mathcal{T}_1 \times \mathcal{M} \). Let \( C \) be a BSCC of \( \text{Graph}_{\mathcal{T}_2 \times \mathcal{M}}(B_2) \), and \( \mu \in \text{Dist}(\alpha^{-1}(C)) \). Then:

\[
\text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(\text{Inf} = F_C) = 1.
\]

**Proof.** As already argued in the proof of Lemma \( \text{43} \) for every \( p \in F_C \), for every state \( s_2 \in C \), \( \text{Prob}_{\mathcal{S}_2}^{\mathcal{T}_2 \times \mathcal{M}}(F \ p) = 1 \) (we abusively write \( p \) for the measurable set \( S_2 \times \{ p \} \)). Since \( \mathcal{T}_2 \times \mathcal{M} \) is a sound \( \alpha \)-abstraction of \( \mathcal{T}_1 \times \mathcal{M} \), we derive for every \( \nu \in \text{Dist}(\alpha^{-1}(C)) \) that \( \text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(F \ p) = 1 \) (as before we abusively write \( p \) for \( S_1 \times \{ p \} = \alpha^{-1}(S_2 \times \{ p \}) \)). We can then show that for each \( \nu \in \text{Dist}(\alpha^{-1}(C)) \) and for each \( p \in F_C \),

\[
\text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(G \ F \ p) = 1.
\]

Indeed, towards a contradiction, assume that there is a distribution \( \nu \in \text{Dist}(\alpha^{-1}(C)) \) such that \( \text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(G \ F \ p) < 1 \), i.e. \( \text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(G \ F \ \neg p) > 0 \). From the third point of Lemma \( \text{43} \) we get that \( \text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(G \ F \ \alpha^{-1}(C) \wedge G \ \neg p) > 0 \). Now, observe that

\[
\text{Ev}_{\mathcal{T}_1 \times \mathcal{M}}(G \ F \ \alpha^{-1}(C) \wedge G \ \neg p) \subseteq \text{Ev}_{\mathcal{T}_1 \times \mathcal{M}}(\bigcup_{n \in \mathbb{N}} (F = n \alpha^{-1}(C) \wedge G \geq n \neg p)).
\]

It follows that there is \( n \in \mathbb{N} \) such that \( \text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(F = n \alpha^{-1}(C) \wedge G \geq n \neg p) > 0 \). From Lemma \( \text{4} \) we get that there is \( \nu' \in \text{Dist}(S'_1) \) (with \( S'_1 = S_1 \times Q \)) such that

\[
\text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(F = n \alpha^{-1}(C) \wedge G \geq n \neg p)
\]

\[
= \lim_{m \to \infty} \text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(\text{Cyl}(S'_1(1), \ldots, S'_1(n)) \wedge \neg p \wedge \neg p \wedge \cdots) \leq \lim_{m \to \infty} \text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(\text{Cyl}(\alpha^{-1}(C) \wedge \neg p, \neg p, \cdots) \wedge \cdots) \quad \text{from Lemma } \text{4}
\]

\[
= \lim_{m \to \infty} \nu'((\alpha^{-1}(C)) \wedge \nu'((\alpha^{-1}(C)) \cdot \text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(\text{Cyl}(\alpha^{-1}(C) \wedge \neg p, \neg p, \cdots))) \quad \text{from Lemma } \text{4}
\]

\[
= \nu'(\alpha^{-1}(C)) \cdot \text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(\text{Cyl}(\alpha^{-1}(C) \wedge \neg p)).
\]

From the assumption, we thus get that \( \text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(G \ p) \) which is the required contradiction. Hence, for each \( \nu \in \text{Dist}(\alpha^{-1}(C)) \) and for each \( p \in F_C \),

\[
\text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(G \ F \ p) = 1.
\]

It now suffices to show that, from any \( \nu \in \text{Dist}(\alpha^{-1}(C)) \), no other state is visited almost-surely infinitely often. Fix \( p' \notin F_C \). Then, by definition of \( F_C \), we have that \( \text{Prob}_{\mathcal{T}_2 \times \mathcal{M}}(\nu \ p') = 0 \) since \( \mathcal{T}_2 \times \mathcal{M} \) is an \( \alpha \)-abstraction of \( \mathcal{T}_1 \times \mathcal{M} \), we deduce that \( \text{Prob}_{\mathcal{T}_1 \times \mathcal{M}}(\text{Inf} = F_C) = 1 \), which is the claim of the lemma.
We are now in a position to decompose the probability to satisfy the Muller condition $\mathcal{F}$ in $\mathcal{T}_1 \times \mathcal{M}$ into the reachability probability of good BSCCs.

**Theorem 55.** Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two LSTSs such that $\mathcal{T}_2$ is a DMC with a finite attractor $B_2$, and $\mathcal{T}_2$ is an $\alpha$-abstraction of $\mathcal{T}_1$. Let $\mathcal{M} = (Q, q_0, E, \mathcal{F})$ be a DMA. Assume moreover that $\mathcal{T}_2 \times \mathcal{M}$ is an $\alpha$-sound abstraction of $\mathcal{T}_1 \times \mathcal{M}$. Then, for every initial distribution $\mu$ for $\mathcal{T}_1$ and for every $q \in Q$:

$$\text{Prob}_{\mu \times \delta_q}^{\mathcal{T}_1 \times \mathcal{M}}(\text{Inf} \in \mathcal{F}) = \sum_{C \in \text{Good}^{B_2}_{\mathcal{T}_2 \times \mathcal{M}}(\mathcal{F})} \text{Prob}_{\mu \times \delta_q}^{\mathcal{T}_1 \times \mathcal{M}}(\mathcal{F} \alpha^{-1}(C)) .$$

**Proof.** Applying Lemma 54 for every $\mu \in \text{Dist}(S_1)$, assuming $\text{Prob}_{\mu \times \delta_q}^{\mathcal{T}_1 \times \mathcal{M}}(\mathcal{F} \alpha^{-1}(C)) > 0$, then $\text{Prob}_{\mu \times \delta_q}^{\mathcal{T}_1 \times \mathcal{M}}(\text{Inf} = F_C \mid \mathcal{F} \alpha^{-1}(C)) = 1$. By the two first properties of Lemma 53 we can write the following Bayes formula, with a disjunction over the BSCCs of $\text{Graph}^{T_2 \times M}(B_2)$:

$$\text{Prob}_{\mu \times \delta_q}^{\mathcal{T}_1 \times \mathcal{M}}(\text{Inf} \in \mathcal{F}) = \sum_{C \in \text{BSCC of } \text{Graph}^{T_2 \times M}(B_2)} \text{Prob}_{\mu \times \delta_q}^{\mathcal{T}_1 \times \mathcal{M}}(\mathcal{F} \alpha^{-1}(C)) \cdot \text{Prob}_{\mu \times \delta_q}^{\mathcal{T}_1 \times \mathcal{M}}(\text{Inf} \in \mathcal{F} \mid \mathcal{F} \alpha^{-1}(C))$$

$$= \sum_{C \text{ BSCC of } \text{Graph}^{T_2 \times M}(B_2)} \text{Prob}_{\mu \times \delta_q}^{\mathcal{T}_1 \times \mathcal{M}}(\mathcal{F} \alpha^{-1}(C)) \cdot 1_{\mathcal{F}}(F_C)$$

$$= \sum_{C \in \text{Good}^{B_2}_{\mathcal{T}_2 \times \mathcal{M}}(\mathcal{F})} \text{Prob}_{\mu \times \delta_q}^{\mathcal{T}_2 \times \mathcal{M}}(\mathcal{F} \alpha^{-1}(C)) .$$

This concludes the proof of the theorem.

In particular, regarding the qualitative model checking of $\omega$-regular properties, we conclude with the following characterization of the almost-sure satisfaction of properties specified by a DMA.

**Corollary 56.** Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two LSTSs such that $\mathcal{T}_2$ is a DMC with a finite attractor, and $\mathcal{T}_2$ is an $\alpha$-abstraction of $\mathcal{T}_1$. Let $\mathcal{M} = (Q, q_0, E, \mathcal{F})$ be a DMA. Assume moreover that $\mathcal{T}_2 \times \mathcal{M}$ is an $\alpha$-sound abstraction of $\mathcal{T}_1 \times \mathcal{M}$. Then, for every initial distribution $\mu$ for $\mathcal{T}_1$ and for every $q \in Q$:

$$\text{Prob}_{\mu \times \delta_q}^{\mathcal{T}_1 \times \mathcal{M}}(\text{Inf} \in \mathcal{F}) = 1 \text{ if and only if } \text{Prob}_{\mu \times \delta_q}^{\mathcal{T}_2 \times \mathcal{M}}(\text{Inf} \in \mathcal{F}) = 1 .$$

Hence, this reduces the almost-sure model-checking of a property given by $\mathcal{M}$ in $\mathcal{T}_1$ to the almost-sure model-checking of a reachability property (applying Corollary 51). For the approach to be effective, it is sufficient that the analysis at the level of $\mathcal{T}_2 \times \mathcal{M}$ is effective.

As already quickly mentioned, under the hypotheses of Corollary 51, the abstraction $\mathcal{T}_2 \times \mathcal{M}$ is complete (since it has a finite attractor). Though it is not explicitly used, we could not have such an equivalence without some completeness of the abstraction.

**Remark 6 (Discussion on the approach of [16])** While the notion of abstraction was not precisely defined in [16] for stochastic timed automata, it was implicitly already there. Also, decidability of the almost-sure satisfaction was ensured thanks to a fairness condition. Using the terminology of the current paper, the framework was the following: $\mathcal{T}_1$ and $\mathcal{T}_2$ are two STSs such that $\mathcal{T}_2$ is a finite Markov chain which is an $\alpha$-abstraction of $\mathcal{T}_1$. Then the
condition for the abstraction to yield interesting results was that $T_1$ should be fair w.r.t. every $\alpha$-closed sets (the latter condition implying the fairness of $T_1 \times M$, for $M$ a DMA). Thanks to Proposition 35, this implies that $T_1 \times M$ is actually decisive w.r.t. $\alpha$-closed sets. Applying Proposition 35, we get that $T_2 \times M$ is sound abstraction of $T_1 \times M$. Given that $T_2$ is finite, it trivially has a finite attractor. Hence, the conditions of Theorem 55 are satisfied, and the approach of [10] was then a particular case of that theorem, when applied to specific subclasses of stochastic timed automata (details are provided in Subsection 7.1).

6 Approximate quantitative analysis

Beyond the qualitative analysis performed in the previous section, we will see that, under reasonable assumptions, one may derive approximation schemes to compute, within arbitrary precision, the probability of a given property. We consider first reachability, then repeated reachability and last $\omega$-regular properties.

For the next two subsections, fix an STS $T = (S, \Sigma, \kappa)$, and an initial distribution $\mu \in \text{Dist}(S)$.

6.1 Quantitative reachability under decisiveness hypotheses

In order to approximate the reachability probability of a set $B \in \Sigma$ in $T$, we define the two following sequences, similar to the ones given for decisive Markov chains [1]. For every $n \in \mathbb{N}$:

\[
\begin{align*}
    p_n^\text{Yes} &= \text{Prob}_\mu^T(F_{\leq n} B); \\
    p_n^\text{No} &= \text{Prob}_\mu^T(\neg B U_{\leq n} \tilde{B}).
\end{align*}
\]

Since the sequences of events $(F_{\leq n} B)_{n \in \mathbb{N}}$ and $(\neg B U_{\leq n} \tilde{B})_{n \in \mathbb{N}}$ are non-decreasing and converge respectively to $F B$ and $\neg B U \tilde{B}$, it is easy to determine the limit of the sequences $(p_n^\text{Yes})_n$ and $(p_n^\text{No})_n$, with no assumption on the model.

**Lemma 57.** The sequences $(p_n^\text{Yes})_n$ and $(p_n^\text{No})_n$ are non-decreasing and converge respectively to $\text{Prob}_\mu^T(F B)$ and $\text{Prob}_\mu^T(\neg B U \tilde{B})$.

Assuming now that $T$ is decisive w.r.t. $B$, the two limits are related.

**Corollary 58.** If $T$ is Dec($\mu$, $B$), then $\lim_{n \to +\infty} p_n^\text{Yes} + p_n^\text{No} = 1$.

**Proof.** From Lemma 57, it holds that

\[
\lim_{n \to +\infty} p_n^\text{Yes} + p_n^\text{No} = \text{Prob}_\mu^T(F B) + \text{Prob}_\mu^T(\neg B U \tilde{B})
= \text{Prob}_\mu^T(F B \lor (\neg B U \tilde{B})) \quad \text{from point (i) of Lemma 39}
= \text{Prob}_\mu^T(F B \lor \tilde{B}) \quad \text{from Lemma 9 (fifth item)}
= 1.
\]

The last equality comes from the decisiveness assumption.

Corollary 58 can be used to derive an approximation scheme for $\text{Prob}_\mu^T(F B)$. To obtain an $\varepsilon$-approximation, it suffices to evaluate $p_n^\text{Yes}$ and $p_n^\text{No}$ for larger and larger values of $n$, until $1 - p_n^\text{Yes} - p_n^\text{No} < \varepsilon$, and to return $p_n^\text{Yes}$. This scheme is effective as soon as one can compute $\tilde{B}$, and the probability (from $\mu$) of cylinders of the forms $\text{Cyl}(S, \ldots, S, B)$ and $\text{Cyl}(\neg B, \ldots, \neg B, B)$.

In case $p_n^\text{Yes}$ and $p_n^\text{No}$ cannot be computed exactly, but can only be approximated up to any desired error bound, this scheme can be refined to obtain a $2\varepsilon$-approximation for $\text{Prob}_\mu^T(F B)$.
6.2 Quantitative repeated reachability under decisiveness hypotheses

We now define two sequences that will yield an approximation scheme for a repeated reachability probability, under stronger assumptions on the model. For every $n \in \mathbb{N}$:

\[
\begin{align*}
q_n^{\text{Yes}} &= \operatorname{Prob}_\mu^T(\neg B \cup \neg n \ B); \\
q_n^{\text{No}} &= \operatorname{Prob}_\mu^T(\neg B \cup n \ B).
\end{align*}
\]

Here again, with no assumption on $\mathcal{T}$:

Lemma 59. The sequences $(q_n^{\text{Yes}})_n$ and $(q_n^{\text{No}})_n$ are non-decreasing and converge respectively to $\operatorname{Prob}_\mu^T(\neg B \cup \neg \ B)$ and $\operatorname{Prob}_\mu^T(\neg B \cup \ B)$.

Assuming now that $\mathcal{T}$ is persistently decisive w.r.t $B$ and decisive w.r.t $\neg B$, the two sequences are closely related.

Corollary 60. If $\mathcal{T}$ is $\text{PersDec}(\mu, B)$ and $\text{Dec}(\mu, \neg B)$, then the two sequences $(q_n^{\text{Yes}})_n$ and $(1 - q_n^{\text{No}})_n$ are adjacent and converge to $\operatorname{Prob}_\mu^T(\mathcal{G} \ \mathcal{F} \ B)$.

Proof. We first prove that

(a) $\operatorname{Prob}_\mu^T(\neg B \cup \neg B) = \operatorname{Prob}_\mu^T(\neg \mathcal{F} \ B)$ and
(b) $\operatorname{Prob}_\mu^T(\neg \ B \cup \ B) = \operatorname{Prob}_\mu^T(\mathcal{F} \ B)$.

Indeed from point (i) of Lemma 39, we get that

$\operatorname{Prob}_\mu^T(\neg B \cup \neg B) = \operatorname{Prob}_\mu^T(\neg (\mathcal{F} \ B) \land (\neg B \cup \neg B)) = \operatorname{Prob}_\mu^T(\mathcal{G} \neg B \land \mathcal{F} \ B)$ from the definition of $\neg B$.

where the last equality holds from Lemma 40 since $\mathcal{T}$ is $\text{PersDec}(\mu, B)$, yielding to point (a).

Now in order to establish point (b), notice first that $\operatorname{Prob}_\mu^T(\neg B \cup \neg B) \leq \operatorname{Prob}_\mu^T(\mathcal{F} \ B)$. To prove the other inequality, we have from Lemma 40 (still applicable) that:

$\operatorname{Prob}_\mu^T(\mathcal{F} \ B) = \operatorname{Prob}_\mu^T(\mathcal{F} \ B \land \neg (\mathcal{F} \ B)) = \operatorname{Prob}_\mu^T(\bigvee_{n \geq 0} (\mathcal{F} \ B \land \bigwedge_{p \geq 0} \mathcal{F} = p \neg \ B)) \leq \operatorname{Prob}_\mu^T(\bigvee_{n \geq 0} (\mathcal{F} \ B \land \bigwedge_{p < n} \mathcal{F} = p \neg \ B)) = \operatorname{Prob}_\mu^T(\neg B \cup \neg B)$

which gives us point (b). Now remember that $\text{PersDec}(\mu, B) \Rightarrow \text{StrDec}(\mu, B)$. Then, since $\mathcal{T}$ is $\text{PersDec}(\mu, B)$ and $\text{Dec}(\mu, B)$ and from Lemmas 39 (point (ii)) and 40 it follows that

- $\operatorname{Prob}_\mu^T(\mathcal{F} \ B) = 1 - \operatorname{Prob}_\mu^T(\neg \mathcal{F} \ B)$ and
- $\operatorname{Prob}_\mu^T(\mathcal{G} \ \mathcal{F} \ B) = 1 - \operatorname{Prob}_\mu^T(\mathcal{F} \ B)$.  

Using Lemma 59 it is now easy to show that \( \lim_{n \to +\infty} q_n^{Yes} + q_n^{No} = 1 \) and that \( \lim_{n \to +\infty} q_n^{Yes} = \lim_{n \to +\infty} 1 - q_n^{No} = \Prob^T_{\mu}(G F B) \).

Here also, under the assumptions of Corollary 60 we obtain an approximation scheme for the value \( \Prob^T_{\mu}(G F B) \). Effectiveness of the scheme relies on the computability of the avoid sets \( \hat{B} \) and \( B \), and on the effective computation of the probability of cylinders of the forms \( \text{Cyl}(\{\overline{B}, \ldots, \overline{B}, \overline{B}\}) \) and \( \text{Cyl}(\{\overline{B}, \ldots, \overline{B}, B\}) \). Similarly as before, in case \( q_n^{Yes} \) and \( q_n^{No} \) cannot be computed exactly, but can only be approximated up to any desired error bound, this scheme can be refined to obtain a \( 2\varepsilon \)-approximation for \( \Prob^T_{\mu}(G F B) \).

### 6.3 \( \omega \)-regular properties in DMC with a finite attractor

To go beyond reachability and repeated reachability, we now consider an \( \omega \)-regular property given by a DMA \( M = (Q, q_0, E, F) \). We assume that \( T = (S, \Sigma, \kappa, AP, L) \) is a labelled DMC.

In order to approximate the probability that the model satisfies this external specification, we assume that \( T \) has a finite attractor. Following Section 5.3, we consider the finite attractor \( B \) of \( T \times M \), and we apply Theorem 60 for each \( \mu \in \text{Dist}(S) \),

\[
\Prob_{\mu \times \delta_{q_0}}^{T \times M}(\inf \in F) = \sum_{C \in \text{Good}_{T \times M}(F)} \Prob_{\mu \times \delta_{q_0}}^{T \times M}(F C).
\]

Thus, the computation of the probability that a given model satisfies a given external specification is reduced to the computation of a reachability probability. Now, given that \( T \) and hence \( T \times M \) has a finite attractor, \( T \times M \) is \( \text{Dec}(\mu \times \delta_{q_0}, B) \) for any measurable set \( B \), so that we can apply the approximation scheme from Section 6.1 to obtain an approximation of the desired value.

The effectiveness of the approach relies on the effectiveness of the scheme for reachability, but also on the computability of an attractor for \( T \), and of the set of good BSCCs of the graph of the attractor.

### 6.4 \( \omega \)-regular properties of general STSs via abstraction and finite attractor

We assume the same framework as in Section 5.4, that is \( T_1 = (S_1, \Sigma_1, \kappa_1, AP, L_1) \) and \( T_2 = (S_2, \Sigma_2, \kappa_2, AP, L_2) \) are two LSTSs such that:

- \( T_2 \) is a sound \( \alpha \)-abstraction of \( T_1 \)
- \( T_2 \) is a DMC with a finite attractor \( B_2 \).

We consider again a DMA \( M = (Q, q_0, E, F) \), as well as the products \( T_1 \times M \) and \( T_2 \times M \).

Writing \( B = \{ \alpha^{-1}_M(B) \mid B \in \Sigma'_2 \} \), we assume that \( T_1 \times M \) is \( \text{Dec}(B) \). Remember that this implies, from Lemma 52, that \( T_2 \times M \) is a sound \( \alpha_M \)-abstraction of \( T_1 \times M \).

Fix an initial distribution \( \mu \) for \( T_1 \). Thanks to Theorem 59

\[
\Prob_{\mu \times \delta_{q_0}}^{T_1 \times M}(\inf \in F) = \sum_{C \in \text{Good}_{T_2 \times M}(F)} \Prob_{\mu \times \delta_{q_0}}^{T_1 \times M}(F \alpha^{-1}(C)).
\]

Thus, as previously, the computation of the probability that a given model satisfies a given external specification is reduced to the computation of a reachability probability. Since
we assumed \( T_1 \ltimes M \) to be \( \text{Dec}(B) \), we can use the approximation scheme from Section 6.1 to approximate the searched value.

Effectiveness of the approach requires effective numerical computations for the distributions, as well as good constructivity properties for various sets, like the BSCCs of the graph of the attractor, and avoid-sets of these, etc.

**Application to time-bounded verification of stochastic real-time systems.** Our initial motivation stems from real-time stochastic systems, that is, systems with both timing constraints and probabilistic choices. While we will not formally define a real-time stochastic system, let us give an informal description: a real-time system has in its state-space a component representing the time (via timestamps), which increases (almost-surely) while the system executes (this condition is ensured on the kernel). Then, under the assumption that the system is *almost-surely non-Zeno* (which is a desirable property in a real system), which means that almost-surely sequences of timestamps diverge, the system has natural attractors, which are all sets defined by an upper bound on the timestamp. Although these attractors may be infinite, they might be useful to analyze time-bounded properties. Fix a time bound \( \Delta \). Let \( T_1 \) be a real-time STS, and let \( T_2 \) be a denumerable \( \alpha \)-abstraction of \( T_1 \) which aggregates all states with timestamp larger than \( \Delta \) into a single absorbing state \( s_\Delta \). Then, if \( T_1 \) is almost-surely non-Zeno (which has to be proven “by hand”, or be structurally obvious), \( \alpha^{-1}(s_\Delta) \) is an attractor of \( T_1 \), and the abstraction \( T_2 \) will then be sound (by applying Propositions 33 and 35). This will therefore enable the quantitative analysis of time-bounded properties in \( T_1 \) (like time-bounded until or reachability properties).

We believe that some approximation results for time-bounded properties that can be found in the literature could be justified using that approach, for instance time-bounded verification of GSMPs [5]. It would be interesting to revisit results concerning the time-bounded analysis of stochastic hybrid systems (like [19,34]) using this new approach, and see whether they all fit in our framework.

### 7 Applications

The general approach to the qualitative and quantitative analysis of stochastic systems over a possibly continuous state-space can be instantiated in multiple frameworks. We do it here for two classes of models, namely stochastic timed automata and generalized semi-Markov processes.

#### 7.1 Stochastic timed automata

Stochastic timed automata (STA) [16] are stochastic real-time processes derived from timed automata [6] by randomizing both the delays and the edge choices. The semantics of a STA is naturally given via a STS as defined in this paper, although this had not been formulated this way originally.

Several decidability results have been proven for subclasses of STA, requiring the development of ad-hoc methods [7,15,17], and in [16], we proposed the first unifying method capturing all known decidability results for the qualitative model-checking problem: the so-called *thick graph* is a finite graph based on the standard region automaton construction for timed automata [6], which allows one to infer good transfer properties from this finite graph to the original STA when some *fairness* property is satisfied. The current work improves our
understanding of [14] and allows us both to unify all decidability and approximability results that were known, and to get new approximability results for the quantitative model-checking problem (of $\omega$-regular properties).

**Definition** To define the model properly, we first give some notations. Let $X$ be a finite set of clocks. We write $G(X)$ the set of guards defined as finite conjunctions of constraints of the form $x \leq c$, where $x \in X$, $\leq \in \{<,\leq,=,\geq,>\}$ and $c \in \mathbb{N}$. Guards are interpreted over clock valuations $\nu: X \rightarrow \mathbb{R}_{\geq 0}$ in a natural way – we then write $\nu \models g$. Also, for $\nu$ a valuation we define $[Y \leftarrow 0](\nu)$ the valuation assigning $0$ to every $x \in Y$ and $\nu(x)$ to each other clock, and if $d \in \mathbb{R}_{\geq 0}$, we write $\nu(x) + d$ for the valuation assigning $\nu(x) + d$ to every clock $x \in X$.

**Definition 61.** A stochastic timed automaton (STA) is a tuple

$$A = (L, \ell_0, X, E, (\mu_\gamma)_{\gamma \in L \times \mathbb{R}_{\geq 0}^X}, (w_e)_{e \in E})$$

where:

- $L$ is a finite set of states (or locations);
- $\ell_0 \in L$ is the initial state;
- $X$ is a finite set of clocks;
- $E \subseteq L \times G(X) \times 2^X \times L$ is a finite set of edges; and
- for every configuration $\gamma \in L \times \mathbb{R}_{\geq 0}^X$, $\mu_\gamma$ is a probability measure on the set of possible runs of the underlying timed automaton $\gamma$.

Originally, the semantics of an STA $A = (L, \ell_0, X, E, (\mu_\gamma)_{\gamma \in L \times \mathbb{R}_{\geq 0}^X}, (w_e)_{e \in E})$ was defined as a probability measure on the set of possible runs of the underlying timed automaton $(L, \ell_0, X, E)$: a run in such a timed automaton is an alternating sequence of delay transitions and of discrete transitions. A delay transition is of the form $(\ell, \nu) \xrightarrow{\gamma} (\ell, \nu + d)$, where $\mu_\gamma$ is a probability measure on the set of possible runs of the underlying timed automaton $\gamma$.

The probability measure was obtained by sampling delay transitions from a configuration $\gamma$ following distribution $\mu_\gamma$, and by sampling discrete transitions using the weights: the probability to take edge $e$ from configuration $\gamma$ is given by $p_\gamma(e) \overset{\text{def}}{=} \sum_{w_e \text{ enabled at } \gamma} w_e$ if $e$ is enabled at $\gamma$, and by $p_\gamma(e) \overset{\text{def}}{=} 0$ otherwise.

To have properly-defined measures we need some sanity assumptions on distributions $(\mu_\gamma)_{\gamma \in L \times \mathbb{R}_{\geq 0}^X}$: If we write $\lambda$ for the Lebesgue measure over $\mathbb{R}_+$, it must be the case that for each $\gamma \in L \times \mathbb{R}_+^X$, if $\lambda(I(\gamma)) > 0$ then $\mu_\gamma$ is equivalent to the restriction of $\lambda$ on $I(\gamma)$; Otherwise, it is the uniform distribution over the points of $I(\gamma)$.

We now give the semantics of an STA $A = (L, \ell_0, X, E, (\mu_\gamma)_{\gamma \in L \times \mathbb{R}_{\geq 0}^X}, (w_e)_{e \in E})$ as an STS $T_A = (S_A, \Sigma_A, \kappa_A)$ as follows. The set $S_A$ is the set of configurations $L \times \mathbb{R}_{\geq 0}^X$. $\Sigma_A$ is the

\[8\] Later we will also write $\gamma + d$ for the configuration $(\ell, \nu + d)$.
The thick graph abstraction The thick graph of [16] is an abstraction in our context. We write $M$ the thick graph of [16]. We say that whenever the

$$
\kappa_A(\gamma, B) = \sum_{e=(\ell, \nu, Y) \in E} \int_{d \in \mathbb{R}_{\geq 0}} 1_B(\ell', [Y \leftarrow 0](\nu + d)) \cdot p_{\gamma + d}(e) \, d\mu_\gamma(d)
$$

where $1_B$ is the characteristic function of $B$, and $p_{\gamma + d}(e)$ is equal to $\sum\{w_{\nu'} | e' \text{ enabled at } \gamma + d\}$ if $e$ is enabled at $\gamma + d$, and 0 otherwise. It gives the probability to hit set $B \subseteq S_A$ from configuration $\gamma$ in one step (composed of a delay transition followed by a discrete transition).

The probability measure on paths derived from $T_A$ in Section 2.2 coincides with the original definition of [16].

We fix for the rest of this section an STA $A = (L, \ell_0, X, E, (\mu_\gamma)_{\gamma \in L \times \mathbb{R}_{\geq 0}}, (w_e)_{e \in E})$, and $T_A = (S_A, \Sigma_A, \kappa_A)$ its corresponding STS.

The thick graph abstraction The thick graph of [16] is an abstraction in our context. To see this, we recall the concept of regions, that have been designed for standard timed automata [16]. We write $M_A$ the maximal integer appearing in a guard of $A$. Let $\nu, \nu' \in \mathbb{R}_{\geq 0}^X$ be two valuations over $X$. We say that $\nu$ and $\nu'$ are region-equivalent for $A$ whenever the following conditions hold:

1. for every $x \in X$, either both $\nu(x)$ and $\nu'(x)$ are strictly larger than $M_A$, or the integral parts of $\nu(x)$ and $\nu'(x)$ coincide;
2. for every $x, y \in X$ such that $\nu(x), \nu(y) \leq M_A$, writing $\{\}$ for the fractional part, $\{\nu(x)\} \leq \{\nu(y)\}$ if and only if $\{\nu'(x)\} \leq \{\nu'(y)\}$.

This region-equivalence has finite-index, and partitions the set of valuations $\mathbb{R}_{\geq 0}^X$ into classes which are called regions, and we write $R_A$ for the set of regions. If $\nu \in \mathbb{R}_{\geq 0}^X$, we write $[\nu]_A$ for the region to which $\nu$ belongs.

We define the abstraction $\alpha : L \times \mathbb{R}_{\geq 0}^X \rightarrow L \times R_A$ as the projection which associates $(\ell, \nu)$ onto $(\ell, [\nu]_A)$. Then we define the finite Markov chain $T_A^{5\alpha}$ as follows:

- its set of states is $L \times R_A$;
- there is an edge from $(\ell, r)$ to $(\ell', r')$ whenever there exists some $\nu \in r$ such that $\kappa_A((\ell, \nu), (\ell') \times r')) > 0$;
- from each state $(\ell, r) \in L \times R_A$, we associate the uniform distribution over $\{(\ell', r') \in L \times R_A | \text{there is an edge from } (\ell, r) \text{ to } (\ell', r')\}$.

By construction, we get:

**Lemma 62.** $T_A^{5\alpha}$ is a finite $\alpha$-abstraction of $T_A$.

Let us notice that finiteness of the abstraction implies completeness (Lemma 33).

As witnessed in [16] Appendix D.2, this abstraction may not give much information in general about the probability of linear-time properties in the original STA (see Example 10). However we will see that, in several cases, it helps to obtain decidability and approximability results (among which some are new).

Note that this is a local condition which is easy to check.
**Example 10 (Counter-example).** Consider the STA $A$ of Figure 4 with: $L = \{\ell_0, \ldots, \ell_4\}$, $X = \{x, y\}$ and the set of edges $E = \{e_0, \ldots, e_5\}$ is described on the figure. We assume that each edge has a weight of 1 and that each location is either equipped with a uniform distribution over delays (in $\ell_0$, $\ell_2$ and $\ell_4$) or a Dirac distribution over delays (in $\ell_1$ and $\ell_3$).

As said previously, it can be considered as a STS $T_A$ where the set of states is given by $L \times R^2_+ +$ and the Markov kernel is computed according to the distributions over the edges and the delays. We know that the thick graph viewed as a finite Markov chain, $T_A^{tg}$, is an $\alpha$-abstraction of $T_A$, but it can be shown that it is not sound. Indeed consider the Dirac distribution $\delta(\ell_0, (0,0))$ as the initial distribution, then the pushforward of $\alpha$ of this distribution is also a Dirac distribution on location $\ell_0$ in region $x = y = 0$. Consider the set of states $\{\ell_2\} \times R^2_+$ that we will abusively write $\ell_2$ and which is $\alpha$-closed. It has been shown in [16, Section 6.2.2] that $\operatorname{Prob}^{T_A^{tg}}_{\delta(\ell_0,0)}(F \ell_2) = 1$, but $\operatorname{Prob}^{T_A}_{\delta(\ell_0,x=y=0)}(F \ell_2) < 1$.

[Fig. 4. A two-clock STA $A$, for which $T_A^{tg}$ is not sound.]

**Remark 7** In [16], a condition for $T_A$ to be a useful abstraction was identified as fairness (the weakest notion defined in the current paper). Fairness is not sufficient for all the approximability results. Later, in [14], the condition was identified as (strong) decisiveness, but this was not sufficient for approximability of $\omega$-regular properties. Here we realize that we have a finite-attractor property (through an abstraction), and that conditions of Proposition 34 will actually always be fulfilled, allowing us to infer the whole class of decidability and approximability results through Proposition 38.

**Reactive STA** Following [17], the STA $A$ is said reactive whenever for every configuration $\gamma = (\ell, \nu) \in S_A$, $I(\gamma) = R_{\geq 0}$, and for every $\ell$, there exists a distribution $\mu_\ell$ with support $R_{\geq 0}$ such that for every $\nu \in R_+^X$, $\mu(\ell, \nu) = \mu_\ell$.

We take the notations used in the previous subsection for defining the thick-graph abstraction. A region $r$ is said memoryless whenever for every clock $x \in X$, either $\nu(x) = 0$ for every $\nu \in r$, or $\nu(x) > M_A$ for every $\nu \in r$. We write $R^m_A$ for the set of memoryless regions.

From [16, Lemma 13], which states that memoryless regions are visited infinitely often almost-surely from every configuration $\gamma \in S_A$, we get:

**Proposition 63.** The set $\alpha^{-1}(L \times R^m_A)$ is an attractor for $T_A$.

Using Propositions 34 and 38 we also get that:

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10 To give all arguments, this is easy to see that, in one step, one can ensure reaching a memoryless region by delaying at least $M_A + 1$ time units; since there is one single distribution which is applied at every configuration of a given location, the probability to do so is uniformly bounded from below from every configuration.
Proof. It can easily be shown that $L \times R^{\text{mem}}$ is a finite attractor of $\mathcal{T}_A^{\text{tg}}$. Thanks to Proposition 63, $\alpha^{-1}(L \times R^{\text{mem}})$ is an attractor for $\mathcal{T}_A^{\text{tg}}$. It remains to show the last condition of the hypotheses of Proposition 64. We therefore need to prove that for each $(\ell, r) \in L \times R^{\text{mem}}$, there are $p > 0$ and $k \in \mathbb{N}$ such that for each region $(\ell, r) \in L \times R^{\text{mem}}$:

- for each $\mu \in \text{Dist}(\alpha^{-1}(\ell, r))$, $\text{Prob}^\mathcal{T}_A(\mathcal{F} \leq k \alpha^{-1}(\ell, r), r) \geq p$, or
- for each $\mu \in \text{Dist}(\alpha^{-1}(\ell, r))$, $\text{Prob}_{\mu}^\mathcal{T}_A(\mathcal{F} \alpha^{-1}(\ell, r)) = 0$.

This is a consequence of [16] Lemma F.4 which says that from a memoryless region, the future (and its probability) is independent of the precise current configuration. This in particular implies that for two configurations $\gamma, \gamma' \in \alpha^{-1}(\ell, r)$, for every $\alpha$-closed set $B$, for every integer $k$, $\text{Prob}_{\gamma}^\mathcal{T}_A(\mathcal{F} = k B) = \text{Prob}_{\gamma'}^\mathcal{T}_A(\mathcal{F} = k B)$. By extension, for every $\mu \in \text{Dist}(\alpha^{-1}(\ell, r))$, $\text{Prob}_{\mu}^\mathcal{T}_A(\mathcal{F} = k B) = \text{Prob}_{\mu}^\mathcal{T}_A(\mathcal{F} = k B)$. This implies the expected bounds, by taking $B = \alpha^{-1}(\ell, r)$.

Similarly to labelled STS, we consider labelled STA, where each location is labelled by atomic propositions. As consequences of Sections 5 and 6, we get the following decidability and approximability results for reactive STA:

Corollary 65. Let $A$ be a reactive labelled STA, and $\mathcal{M}$ a DMA. Then:

1. we can decide whether $A$ satisfies almost-surely $\mathcal{M}$;
2. for every initial distribution $\mu$ which is numerically amenable w.r.t. $A^{11}$, we can compute arbitrary approximations of $\text{Prob}_\mu^\mathcal{T}_A(\mathcal{M})$.

Proof. This is an application of Theorem 54, Corollary 56 and of Sections 6.1 and 6.4. It should be noted that all the hypotheses are met:

- $\mathcal{T}_A^{\text{tg}} \times \mathcal{M}$ has a finite attractor: since $\mathcal{T}_A^{\text{tg}}$ is a finite MC then so is $\mathcal{T}_A^{\text{tg}} \times \mathcal{M}$ and we get a trivial finite attractor;
- $\mathcal{T}_A \times \mathcal{M}$ is decisive w.r.t. any $\alpha_{\mathcal{M}}$-closed sets.

This second point is a little more tricky. First one should realise that since $\mathcal{T}_A$ is a reactive, then $\mathcal{T}_A \times \mathcal{M}$ is also reactive since the condition to be reactive, concerns only the distributions over the delays on each location of the STA and those distributions are not modified from the product with $\mathcal{M}$. It should be noted that $\mathcal{T}_A^{\text{tg}} \times \mathcal{M}$ corresponds to the thick region graph abstraction of $\mathcal{T}_A \times \mathcal{M}$ since $\mathcal{M}$ does not influence the behaviour of $\mathcal{T}_A$. Then from Proposition 64 we know that $\mathcal{T}_A^{\text{tg}} \times \mathcal{M}$ is a sound $\alpha$-abstraction of $\mathcal{T}_A \times \mathcal{M}$. Since $\mathcal{T}_A^{\text{tg}} \times \mathcal{M}$ is a finite MC, we get that it is decisive w.r.t. any sets of states. We can thus conclude from Proposition 20.

Remark 8 We believe that the proposed approach through abstractions and finite attractors simplifies drastically the proof of decidability of almost-sure model-checking, and in particular avoids the ad-hoc but long and technical proof of [16, Lemma 7.14]. Furthermore, we obtain interesting approximability results, some of them being consequences of [14], but the general case of $\omega$-regular properties (in particular LTL properties) being new to this paper.

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11 We say that a distribution $\mu$ is numerically amenable w.r.t. $A$ if, given $k \in \mathbb{N}$, given $\varepsilon > 0$ and given a sequence of locations and regions $(\ell_0, r_0), (\ell_1, r_1), \ldots, (\ell_k, r_k)$, one can approximate $\text{Prob}_\mu^\mathcal{T}_A(\text{Cyl}((\ell_0, r_0), (\ell_1, r_1), \ldots, (\ell_k, r_k)))$ up to $\varepsilon$. 
Remark 9 Corollary 12 can be extended to properties expressed as deterministic and complete Muller timed automata (DCMTA), which are standard deterministic and complete timed automata [7] with a Muller accepting condition. Indeed, the product of a reactive STA with such a DCMTA is reactive. Hence, the whole theory that we have developed applies: the STS of the product has a finite sound abstraction. This allows to express rich properties with timing constraints and evaluate their likelihood in the STA.

Single-clock STA We will apply a similar reasoning to single-clock STA. We therefore assume that \( A \) is now a single-clock STA. As in [16] Section 7.1, we assume the following conditions:

(i) for all \( \ell \in L \), for all \([a, b] \subseteq \mathbb{R}_+\), the function \( \nu \rightarrow \mu_{(\ell, \nu)}([a, b]) \) is continuous;

(ii) if \( \gamma' = \gamma + t \) for some \( t \geq 0 \), and if \( 0 \notin I(\gamma + t', e) \) for each \( 0 \leq t' \leq t \), then \( \mu_{\gamma}(I(\gamma, e)) \leq \mu_{\gamma}(I(\gamma', e)) \);

(iii) there is \( 0 < \lambda_0 < 1 \) such that for every state \( \gamma \) with \( I(\gamma) \) unbounded, \( \mu_{\gamma}([0, 1]) \leq \lambda_0 \),

where for each \( \gamma = (\ell, \nu) \in L \times \mathbb{R}_+^{\infty} \) and for each \( e = (\ell, g, Y, \ell') \in E \), \( I(\gamma, e) = \{d \in \mathbb{R}_+ | \nu + d \geq g\} \). These requirements are technical, but they are rather natural and easily satisfiable. For instance, a timed automaton equipped with uniform (resp. exponential) distributions on bounded (resp. unbounded) intervals satisfy these conditions. If we assume exponential distributions on unbounded intervals, the very last requirement corresponds to the bounded transition rate condition in [21], required to have reasonable and realistic behaviours.

In [16] Section 7.1, there is no clear attractor property. From the details of the proofs we can nevertheless define \( A_{A}^{\text{max}} = \{(\ell, r_0) | \ell \in L\} \cup \{(\ell, r) \in L \times R_A | \forall (\ell, r) \rightarrow^* (\ell', r') \in T_{A}^{\text{ff}} \implies r' = r \} \), where \( r_0 \) is the region composed of the single null valuation.

Proposition 66. The set \( \alpha^{-1}(A_{A}^{\text{max}}) \) is an attractor for \( T_A \).

Proof. Let \( C = \{0\} \cup \{c \mid c \text{ constant appearing in a guard of } A\} \overset{\text{def}}{=} \{c_0 < c_1 < \cdots < c_h\} \).

The set of regions for \( A \) can be chosen as \( \{\{c_i\} | 0 \leq i \leq h\} \cup \{[c_{i-1}; c_i] | 1 \leq i \leq h\} \) (see [28]).

Following the proof of [16] Theorem 7.2], the set of infinite paths in \( A \) can be divided into (a) the set of paths that take resetting edges infinitely often, and (b) the set of paths that take resetting edges only finitely often.

We assume that the probability that (a) happens is positive, and we reason now in the \( \sigma \)-algebra which is conditioned by (a). Then under condition (a), \( \alpha^{-1}(\{(\ell, r_0) \mid \ell \in L\}) \) is reached almost-surely.

We assume that the probability that (b) happens is positive, and we reason now in the \( \sigma \)-algebra which is conditioned by (b). Under condition (b), almost-surely the value of the clock is non-decreasing along the path, and almost-surely a final region \( r \) is reached (that is, ultimately the value of the clock along the path belongs to \( r \) forever). We fix such a region \( r \), and we condition again with regard to that “final region” \( r \). We write \( E_r \) for the event (b) intersected with “the path ends up in \( r \)”. Let \( r' \) be a strict successor region of \( r \), with dimension at least as big as that of \( r \) (if \( r \) is an open interval, then \( r' \) has to be an open interval). There exists \( \alpha > 0 \) such that for every \( \nu \in r \), for every \( \ell \in L \), for every \( e = (\ell, g, Y, \ell') \) with \( \ell' \subseteq g \), \( \text{Prob}_{T_A_{\nu}}(\{(\ell, \nu) \sim r_0\} \geq \alpha \). Hence, using standard technics, we

\[ 12 \] In this context, complete means that from every configuration, for every subset of \( AP \), and every \( t \in \mathbb{R}_{\geq 0} \), there is an edge labelled by that subset which is enabled after \( t \) time units. So this is complete w.r.t time and actions.
show that with probability 1, if infinitely often such edges are enabled, infinitely often they will be taken; this contradicts hypothesis \( E_r \). Hence, under condition \( E_r \), with probability 1, one cannot visit infinitely often configurations enabling edges guarded by some strict time-successor \( r' \) of \( r \). Once this is assumed, we can then show that almost-surely, only finitely many resetting edges can be enabled. This means that, under condition \( E_r \), almost-surely, ultimately only states of \( \alpha^{-1}(\{(\ell, r) \in L \times R_A \mid \forall (\ell, r) \rightarrow^*(\ell', r') \text{ in } T_A^{ig} \text{ implies } r' = r\}) \) are visited. Hence, that set is an attractor, under condition (b).

Using some Bayes formula w.r.t. conditions (a) and (b), we conclude that \( \alpha^{-1}(A^\text{max}_A) \) is an attractor; this ends the proof.

As before, we get:

**Proposition 67.** \( T_A^{ig} \) is a sound \( \alpha \)-abstraction of \( T_A \).

**Proof.** We easily get that \( A^\text{max}_A \) is a finite attractor for \( T_A^{ig} \), whereas \( \alpha^{-1}(A^\text{max}_A) \) is an attractor for \( T_A \) (Proposition 66).

As for reactive STA, it remains to show the last property appearing in the hypotheses of Proposition 66. The required bounds obviously exist for the region \( r_0 \) (since only a single valuation belongs to \( r_0 \)). Furthermore, as argued in the proof of Proposition 66, when condition (b) is assumed, ultimately, the paths almost surely end up in \( \alpha^{-1}(\{(\ell, r) \in L \times R_A \mid \forall (\ell, r) \rightarrow^*(\ell', r') \text{ in } T_A^{ig} \text{ implies } r' = r\}) \), hence, ultimately, the STA behaves like a finite Markov chain. The required bounds can be inferred.

This allows to conclude that \( T_A^{ig} \) is a sound \( \alpha \)-abstraction of \( T_A \) (using Propositions 34 and 38).

As a consequence, we get the following decidability and approximability results for one-clock STA:

**Corollary 68.** Let \( A \) be a one-clock labelled STA, and \( M \) a DMA. Then:

1. we can decide whether \( A \) satisfies almost-surely \( M \);
2. for every initial distribution \( \mu \) which is numerically amenable w.r.t. \( A \), we can compute arbitrary approximations of \( \text{Prob}^T_A(\mu)(M) \).

**Proof.** Similarly to the proof of Corollary 65, this is an application of Theorem 55, Corollary 56 and of Sections 6.1 and 6.4. The facts that:

- \( T_A^{ig} \preceq M \) has a finite attractor, and
- \( T_A \preceq \alpha \)-closed sets.

can be deduced by similar arguments. We only observe that if \( T_A \) is a single-clock STA, then so is \( T_A \preceq M \) and that hypotheses (i), (ii) and (iii) are preserved through the product with \( M \) as those only concern distributions over the STA which are not altered from the product with \( M \).

**Remark 10** The proof of the existence of an attractor is very similar to the one we used for proving the fairness property in [16, Section 7.1]. However, for free, we get all the approximation results (as previously only few results could be inferred from [14])! It is worth noting that these results encompass the results of [17], where a strong assumption on cycles of the STA were made (but a closed-form for the probability could be computed). We remark here that the graph used in [17] is actually the graph of the attractor, as done in Section 5.5.
7.2 Generalized semi-Markov processes

A generalized semi-Markov process \[\text{[18][23]}\] is a stochastic process with a finite control, and built on a set of events. Each event is equipped with a random variable representing its duration: an event can either be a variable-delay event, defined by a density function, or be a fixed-delay event, modelled by a Dirac distribution. A transition is characterized by a set of events which expire, and schedules a set of new events. This model is known to generalize continuous-time Markov chains.

Definition

**Definition 69.** A generalized semi-Markov process (GSMP) is a tuple \(G = (Q, \mathcal{E}, \ell, u, f, E, \text{Succ})\) where

- \(Q\) is a finite set of states;
- \(\mathcal{E} = \{e_1, \ldots, e_p\}\) is a finite set of events;
- \(\ell : \mathcal{E} \to \mathbb{N}_{\geq 0}\) and \(u : \mathcal{E} \to \mathbb{N}_{\geq 0} \cup \{\infty\}\) are bounds such that for every \(e \in \mathcal{E}\), \(\ell(e) \leq u(e)\);
- \(f : \mathcal{E} \to \text{Dist}(\ell(e); u(e))\) assigns distributions to every event \(e \in \mathcal{E}\);
- \(E : Q \to 2^\mathcal{E}\) assigns to each state \(q\) a set of events enabled (or active) in \(q\);
- \(\text{Succ} : Q \times 2^\mathcal{E} \to \text{Dist}(Q)\) is the successor function defined for \((q, E)\) whenever \(E \subseteq E(q)\).

Each event \(e \in \mathcal{E}\) has an upper (resp. lower) bound \(u_e \overset{\text{def}}{=} u(e)\) (resp. \(\ell_e \overset{\text{def}}{=} \ell(e)\)) on its delay: the duration of event \(e\) is randomly chosen in the interval \([\ell_e, u_e]\) according to density \(f_e \overset{\text{def}}{=} f(e)\).

In contrast to fixed-delay events, \(e\) is called a variable-delay event, if \(\ell_e < u_e\). Events can alternatively be seen as random variables: with a variable-delay event is associated a density function and with a fixed-delay event is associated the corresponding Dirac distribution.

The semantics of a GSMP \(G\) is given as an STS \(T_G = (S_G, \Sigma_G, \kappa_G)\). There are two points-of-view to define the semantics of \(G\), one is through a residual-time semantics using races between events \[\text{[18]}\] (clocks behave like in timed automata), and the other is to sample the delay of an event once, when it is scheduled \[\text{[20]}\] (clocks are “countdown”). Though the results of \[\text{[18]}\] are stated using the first convention, we prefer the second option, since it is easier to understand the semantics. Note that the duality between the two allows obviously to interpret the results of \[\text{[18]}\] in our setting.

Configurations of \(G\) are elements of \(S_G = Q \times (\mathbb{R}_{\geq 0}^+)\) where \(\mathbb{R}_{\geq 0}^+ = \mathbb{R}_{\geq 0} \cup \{\bot\}\). Let \(q\) be a state and \(\nu \in (\mathbb{R}_{\geq 0}^+)\) a valuation for \(q\) whenever \(\nu(e) = \bot\) if \(e \notin E(q)\), and \(\nu(e) \in \mathbb{R}_{\geq 0}\) otherwise; in the latter case, \(\nu(e)\) is the remaining time for \(e\) before expiring.

Let \(\gamma = (q, \nu)\) be a configuration, and define \(E_0(\gamma) = \{e \in E(q) \mid \forall e' \in E(q), \nu(e) \leq \nu(e')\}\) and \(d(\gamma) = \nu(e)\) for \(e \in E_0(\gamma)\). From configuration \(\gamma\), there is a transition to any configuration \(\gamma' = (q', \nu')\) on occurrence of the set of events \(E_0(\gamma)\) after delay \(d(\gamma)\) whenever:

\[
\nu'(e) = \begin{cases} 
\bot & \text{if } e \notin E(q) \\
\nu(e) - d(\gamma) & \text{if } e \in (E(q) \cap E(q')) \setminus E_0(\gamma) \\
t & \text{otherwise, with } \ell_e \leq t \leq u_e.
\end{cases}
\]

Let \(\gamma = (q, \nu)\), and \(B = \{q'\} \times B'\). Then, assuming \(\text{Succ}(q, E_0(\gamma))(q') > 0\), we define:

\[
\kappa(\gamma, B) = \text{Succ}(q, E_0(\gamma))(q') \cdot \int_{(t_1, \ldots, t_p) \in B'} \left( \prod_{e \in E(q')} g_e(t_e) \right) dt_{e_1} \cdots dt_{e_p},
\]
where \( g_e(t) = f_e(t) \) if \( e \in E(q') \setminus (E(q) \setminus E_0(\gamma)) \); \( g_e(t) = \delta_{\nu(e) - d(\gamma)} \) if \( e \in (E(q) \cap E(q')) \setminus E_0(\gamma) \); \( g_e(t) = \delta_1 \) if \( e \notin E(q') \). In other words, for a newly activated event \( e \), its timestamp \( t_e \) is sampled (independently from the other events) according to density \( f_e \).

Before defining the abstraction, we will state two important results proven in [18].

**Two technical lemmas of [18]** In this part, we assume the GSMP \( G \) has no fixed-delay events, that is, for every event \( e \), \( \ell_e < u_e \). It is nevertheless worth to notice that the study of [18] is more precise, in that it allows some fixed-delay events, the restriction being that of single-ticking. However, our point is not the study of fixed-delay events, but an abstract view on results concerning GSMPs.

Let \( \varepsilon > 0 \). We define the notion of \( \varepsilon \)-separated configuration by \( \gamma = (q, \nu) \) is \( \varepsilon \)-*separated* if for every \( a, b \in \{0\} \cup \{\nu(e) \mid e \in E(q)\} \), either \( a = b \) or \( |a - b| > \varepsilon \). We write \( C^\varepsilon_G \) for the set of \( \varepsilon \)-separated configurations.

We recall here two technical lemmas (using our notations) which will be useful for our purpose:

**Lemma 70** (Lemma 1 of [18]). There exists \( \varepsilon > 0 \), \( m \in \mathbb{N} \) and \( p_1 > 0 \) such that for every \( \gamma \in S_G \), \( \text{Prob}_{\delta_r}(F_{\leq m} C^\varepsilon_G) \geq p_1 \).

**Lemma 71** (Lemma 2 of [18]). For every \( \varepsilon > 0 \) and \( k \in \mathbb{N} \), there is \( p_2 > 0 \) such that for all \( (q, r), (q', r') \in Q \times R^+_G \) such that there is a path \( (q, r) \rightarrow^k (q', r') \) in \( T^r_G \), for every \( \gamma = (q, \nu) \in C^\varepsilon_G \) with \( \nu \in r \), \( \text{Prob}_{\delta_r}(F = k \{(q', \nu') \mid \nu' \in r'\}) > p_2 \).

**The refined region graph abstraction** Due to the choice of the countdown-clock semantics (“clock values” decrease down to 0), the thick graph defined in subsection 7.1 has to be twisted a bit and refined with respect to the set of \( \varepsilon \)-separated configurations to be used in our framework.

Following Lemma 70, we select \( \varepsilon > 0 \), and w.l.o.g. we assume \( \varepsilon \) is of the form \( \frac{1}{d} \) with \( d \in \mathbb{N}_{>0} \). We let \( M_G \) be the maximal constant appearing in constants \( \{\ell_e \mid e \in \mathcal{E}\} \) and \( \{u_e \mid e \in \mathcal{E} \text{ and } u_e < \infty\} \). Each event \( e \in \mathcal{E} \) is virtually assigned a clock variable \( x_e \), and we consider a refinement of the region equivalence for clocks \( \{x_e \mid e \in \mathcal{E}\} \) w.r.t. maximal constant \( M_G \) and granularity \( \frac{1}{d} \) as follows. Two valuations \( \nu, \nu' \in (\mathbb{R}^{\geq 0})^\mathcal{E} \) are equivalent whenever the following conditions hold:

1. for every \( e \in \mathcal{E} \), either both \( \nu(e) \) and \( \nu'(e) \) are strictly larger than \( M_G \), or the integral parts of \( d \cdot \nu(e) \) and \( d \cdot \nu'(e) \) coincide;
2. for every \( e_1, e_2 \in \mathcal{E} \), for every \( e \in \frac{1}{d} \cdot \mathbb{N} \setminus [-M_G; M_G] \), for every \( \varepsilon \in \{<, \leq, =, \geq, >\} \), \( \nu(e_1) - \nu(e_2) \varepsilon c \) if and only if \( \nu'(e_1) - \nu'(e_2) \varepsilon c \).

Note that the above conditions refine the ones given in subsection 7.1 using diagonal constraints (6), and w.r.t. the granularity as well. We also realize that any region \( r \in R^\varepsilon_G \) has either only \( \varepsilon \)-separated configurations, or only non-\( \varepsilon \)-separated configurations. We write \( R^\varepsilon_G \) for the set of equivalence classes, also called regions.

We then define the abstraction \( \alpha : Q \times \mathbb{R}^{\geq 0} \rightarrow Q \times R^\varepsilon_G \) by projection, and the finite Markov chain \( T^r_G \) as follows:

- its set of states is \( Q \times R^\varepsilon_G \);
- there is an edge from \((q, r)\) to \((q', r')\) whenever there exists \( \nu \in r \) such that \( \kappa_G((q, \nu), \{q'\} \times r') > 0 \);
from each state \((q, r) \in Q \times R_G\), we associate the uniform distribution over \\{(q', r') \in Q \times R_G \mid \text{there is an edge from } (q, r) \text{ to } (q', r')\}.

Since \(T^{\text{rg}, \varepsilon}_G\) is just a rescaling of a standard region automaton, we immediately get:

**Lemma 72.** \(T^{\text{rg}, \varepsilon}_G\) is a finite \(\alpha\)-abstraction of \(T_G\).

As previously, we notice that the above abstraction is obviously complete (since it is finite).

It is argued in [18] that this abstraction is not always meaningful for having information about the almost-sure satisfaction of properties by \(G\).

**Analyzing GSMPs**

Let \(A^{\varepsilon}_G = \{(q, r) \in Q \times R_G \mid \alpha^{-1}(q, r) \subseteq C^{\varepsilon}_G\}\). As a direct consequence of Lemma 70 we get:

**Proposition 73.** The set \(\alpha^{-1}(A^{\varepsilon}_G)\) is an finite attractor for \(T_G\).

Finally, as for STAs and using Lemma 70 and 71, we also get:

**Proposition 74.** \(T^{\text{rg}, \varepsilon}_G\) is a sound \(\alpha\)-abstraction of \(T_G\).

As consequences, we get the following decidability and approximability results for GSMPs:

**Corollary 75.** Let \(G\) be a single-ticking labelled GSMP, and \(M\) be a DMA. Then:

1. we can decide whether \(G\) satisfies almost-surely \(M\);
2. for every initial distribution \(\mu\) which is numerically amenable w.r.t. \(G\), we can compute arbritary approximations of \(\text{Prob}^{T_G}_\mu(M)\).

**Proof.** Again, the proof is similar to the ones of Corollaries 65 and 68. We just notice that it is obvious that if \(T_G\) is a GSMP with no fixed-delay events, then so is \(T^{\varepsilon}\).

**Remark 11** We believe our approach gives new hints into the approximate quantitative model-checking of GSMPs, for which, up to our knowledge, only few results are known. For instance in [12], the authors approximate the probability of until formulas of the form “the system reaches a target before time \(T\) within \(k\) discrete events, while staying within a set of safe states” (resp. “the system reaches a target while staying within a set of safe states”) for GSMPs (resp. a restricted class of GSMPs which can be proved to be \textbf{PersDec}), and study numerical aspects. Our approach permits to do the same with any reachability or time-bounded until property on the whole class of single-ticking GSMPs. The numerical aspects in our computations can be dealt with as in [12].

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13 We say that a distribution \(\mu\) is numerically amenable w.r.t. \(G\) if, given \(k \in \mathbb{N}\), given \(\varepsilon > 0\) and given a sequence of states and refined regions \((q_0, r_0), (q_1, r_1), \ldots, (q_k, r_k)\), one can approximate \(\text{Prob}^{T_G}_\mu(\text{Cyl}(q_0, r_0)), (q_1, r_1), \ldots, (q_k, r_k))\) up to \(\varepsilon\).

14 We recall the discussion on non-Zeno real-time systems on page 34 and realize that single-ticking (with no 0-delay event) GSMPs are obviously almost-surely non-Zeno.
8 A small tour in the STSs

We now give an overview of the results presented in this paper. In the interest of space, not all precise statements are listed. For instance, we omit the results which assume a fixed initial distribution. Also, few notations are borrowed from the paper, yet the global picture is almost self-contained.

The idea is the following. Given an STS $T$ and a property $\varphi$, Figures 8 and 6 provide the assumptions $T$ should satisfy to be able to perform the qualitative or quantitative analysis of $\varphi$ on $T$. Note that when we consider abstractions $T_1 \xrightarrow{\alpha} T_2$, then we assume $T_1 = T$. Then, Figures 7, 8 and 9 summarize the relationships between the various notions. They should be used to know how to prove the properties that are expected of the model, either directly or via an abstraction (which needs to be designed).

![Diagram](image)

**Fig. 5.** Qualitative analysis: given $T$ an STS and $\varphi$ a property, decide whether $\text{Prob}^T(\varphi) = 0$ or $= 1$. $T$ is replaced with $T_1$ in case of an abstraction. The edge $\sim \sim \sim \sim \sim$ reads “amounts to”.

9 Conclusion

This paper deals with general stochastic transition systems (hence possibly continuous state-space Markov chains). We defined abstract properties of such stochastic processes, which allow one to design general procedures for their qualitative or quantitative analysis. Effectivity of the approach requires some effectiveness assumption on specific high-level formalisms that are used to describe the stochastic process. We have demonstrated the effectiveness of the approach on two classes of systems: stochastic timed automata on the one hand, and generalized semi-Markov processes on the other hand, can be instantiated in our framework. In both cases, we recover known results; but our approach yields further approximability results, which, up to our knowledge, are new.

We believe that, more importantly, we provide in this paper a methodology to understand stochastic models from a verification and algorithmics point-of-view. Section 8 gives a
T satisfying decisiveness properties \( \varphi \) (repeated) reach. property \( \approx \)

approx. scheme Sec. 6.1 and 6.2

T DMC with finite attractor \( \varphi \) given by (det.) automaton \( M \)

approx. scheme on \( T \times M \) applied to a reach. property given by the abstract graph of the attractor of \( T \times M \)

Sec. 6.3

\( T_1 \) STS and \( T_1 \xrightarrow{\alpha} T_2 \)

\( T_2 \) DMC with finite attractor \( \varphi \) given by (det.) automaton \( M \)

\( T_1 \times M \xrightarrow{\alpha, M} T_2 \times M \) sound abst.

approx. scheme on \( T_1 \times M \) applied to a reach. property given by the abstract graph of the attractor of \( T_2 \times M \)

Sec. 6.4

Fig. 6. Quantitative analysis (Section 6): given \( T \) an STS and \( \varphi \) a property, compute (or approximate) \( \text{Prob}_T(\varphi) \). In case of the abstraction, \( T_1 = T \). The edge \( \sim \) reads “amounts to”.

\(\begin{align*}
\mathcal{T} & \text{ is Dec}(\mathcal{B}) & \mathcal{T} \text{ is StrDec}(\mathcal{B}) \\
\mathcal{T} & \text{ is PersDec}(\mathcal{B}) & \mathcal{T} \text{ is fair}(\mathcal{B}) \\
\mathcal{T} \text{ is Dec}(\mathcal{B}) & \mathcal{T} \text{ is PersDec}(\mathcal{B}) & \mathcal{T} \text{ is fair}(\mathcal{B})
\end{align*}\)

\( \mathcal{B} = 2^S \)

\( T \) DMC with a finite attractor \( \mathcal{T} \times M \) DMC with a finite attractor

Lemma 45

\(\mathcal{T} \) finite MC

Fig. 7. Properties of STS (Section 3.4 and Lemma 45)

high-level description of our results, and of properties that should be satisfied by the stochastic model in order to apply our algorithms. In many cases, we showed that the hypotheses were really necessary to get the expected results, by providing counter-examples when the hypotheses are relaxed.

As future work, we plan to investigate new applications, such as for instance the real-time stochastic systems generated by stochastic Petri nets, or the infinite-state systems appearing in parameterized verification. Also, we would like to adopt a similar generic approach for processes with non-determinism like Markov decision processes, or even stochastic two-player games.

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\( T_2 \) DMC with finite attractor

\( T_1 \) satisfying (†) (cf page 20)

\( T_2 \) finite MC

\( T_1 \) fair w.r.t. \( \alpha \)-closed sets

\( T_1 \overset{\alpha}{\rightarrow} T_2 \) abstraction

\( \text{Prop. 34} \) \( T_1 \) decisive w.r.t. \( \alpha \)-closed sets

\( \text{Prop. 35} \) \( T_1 \) decisive w.r.t. \( \alpha \)-closed sets

\( \text{Coro. 31} \) \( T_1 \) decisive w.r.t. \( \alpha \)-closed sets

Fig. 8. Transfer of properties via abstractions

\( T_2 \) decisive

\( T_1 \overset{\alpha}{\rightarrow} T_2 \) sound abstraction

\( A_2 \) attractor of \( T_2 \)

\( \text{Prop. 32} \) \( \alpha^{-1}(A_2) \) attractor of \( T_1 \)

\( \text{Fig. 9.} \) Completeness and soundness of abstractions

\( T_1 \overset{\alpha}{\rightarrow} T_2 \) abstraction

\( \mathcal{T}_1 \times \mathcal{M} \overset{\alpha, \mathcal{M}}{\rightarrow} \mathcal{T}_2 \times \mathcal{M} \) abstraction

\( \triangle \) Soundness/completeness of \( \alpha \) does not imply soundness/completeness of \( \alpha, \mathcal{M} \)!

Condition for completeness:

\( T_2 \) DMC

\( T_1 \overset{\alpha}{\rightarrow} T_2 \) abstraction

\( \text{Lem. 34} \) \( T_1 \overset{\alpha}{\rightarrow} T_2 \) complete abstraction

Condition for soundness:

\( T_2 \) DMC and \( T_1 \) decisive w.r.t. \( \alpha \)-closed sets

\( T_1 \overset{\alpha}{\rightarrow} T_2 \) abstraction

\( \text{Prop. 38} \) \( T_1 \overset{\alpha}{\rightarrow} T_2 \) sound abstraction

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A Technical results of Section 2

**Lemma 3.** Let \( \mu \) and \( \nu \) be two probability measures over \((S, \Sigma)\). If \( \mu \) and \( \nu \) are equivalent, then \( \text{Prob}_\mu^T \) and \( \text{Prob}_\nu^T \) are also equivalent.

**Proof.** We have to show that for each \( \pi \in \mathcal{F}_T \), \( \text{Prob}_\mu^T (\pi) = 0 \Leftrightarrow \text{Prob}_\nu^T (\pi) = 0 \). Since the complement of each cylinder is a finite union of cylinders and since each denumerable unions of cylinders can be written as a denumerable disjoint union of cylinders, it suffices to show this for each cylinder \( \text{Cyl}(A_0, \ldots, A_n) \) with \( A_0, \ldots, A_n \in \Sigma \). We have to show that for each \( A_0, \ldots, A_n \in \Sigma \),

\[
\text{Prob}_\mu^T (\text{Cyl}(A_0, \ldots, A_n)) = 0 \Leftrightarrow \text{Prob}_\nu^T (\text{Cyl}(A_0, \ldots, A_n)) = 0.
\]

It should be observed that, by symmetry, it suffices to show one of the implications. First, assume \( n = 0 \) and fix \( A_0 \in \Sigma \). Then from the definition of \( \text{Prob}_\mu^T \) and \( \text{Prob}_\nu^T \) and from the hypothesis, we get that:

\[
\text{Prob}_\mu^T (\text{Cyl}(A_0)) = 0 \Leftrightarrow \mu(A_0) = 0 \Leftrightarrow \nu(A_0) = 0 \Leftrightarrow \text{Prob}_\nu^T (\text{Cyl}(A_0)) = 0.
\]

Now consider \( n = 1 \) and fix \( A_0, A_1 \in \Sigma \). Suppose that \( \text{Prob}_\mu^T (\text{Cyl}(A_0, A_1)) = 0 \), i.e. from the definition:

\[
\int_{s_0 \in A_0} \kappa(s_0, A_1) \mu(ds_0) = 0. \tag{3}
\]

Write \( B = \{ s_0 \in A_0 \mid \kappa(s_0, A_1) > 0 \} \). We can write \( B = \kappa(\cdot, A_1)^{-1}[0, 1] \cap A_0 \) which is in \( \Sigma \) from the hypotheses over \( \kappa \). From \( \text{(3)} \), we can easily check that \( \mu(B) = 0 \), which implies that \( \nu(B) = 0 \) and thus

\[
\int_{s_0 \in A_0} \kappa(s_0, A_1) \nu(ds_0) = 0.
\]

Using again the definition, it follows that \( \text{Prob}_\nu^T (\text{Cyl}(A_0, A_1)) = 0 \). Now, assume that \( n \geq 2 \), fix \( A_0, \ldots, A_n \in \Sigma \) and assume that \( \text{Prob}_\mu^T (\text{Cyl}(A_0, \ldots, A_n)) = 0 \). Remember that

\[
\text{Prob}_\mu^T (\text{Cyl}(A_0, \ldots, A_n)) = \int_{s_0 \in A_0} \left( \int_{s_1 \in A_1} \ldots \left( \int_{s_{n-1} \in A_{n-1}} \kappa(s_{n-1}, A_n) \kappa(s_{n-2}, ds_{n-1}) \right) \ldots \kappa(s_0, ds_1) \right) \mu(ds_0).
\]

We inductively define:

\[
\begin{align*}
B_{n-1} &= \kappa(\cdot, A_n)^{-1}[0, 1] \cap A_{n-1} \\
B_i &= \kappa(\cdot, B_{i+1})^{-1}[0, 1] \cap A_i \quad \forall 0 \leq i \leq n-2.
\end{align*}
\]

From the hypotheses over \( \kappa \), it is easily seen that for each \( 0 \leq i \leq n-1 \), \( B_i \in \Sigma \). Let us consider the value \( \int_{s_{n-1} \in A_{n-1}} \kappa(s_{n-1}, A_n) \kappa(s_{n-2}, ds_{n-1}) \). From the definition of \( B_{n-1} \), it holds that

\[
\int_{s_{n-1} \in A_{n-1}} \kappa(s_{n-1}, A_n) \kappa(s_{n-2}, ds_{n-1}) = \int_{s_{n-1} \in B_{n-1}} \kappa(s_{n-1}, A_n) \kappa(s_{n-2}, ds_{n-1}) = \text{Prob}_{\kappa(s_{n-2}, \cdot)}^T (\text{Cyl}(B_{n-1}, A_n)).
\]
We know that
\[
\text{Prob}_\kappa^T(\text{Cyl}(A_0, \ldots, A_n)) = 
\int_{s_0 \in A_0} \cdots \left( \int_{s_{n-2} \in A_{n-2}} \text{Prob}_\kappa^T(s_{n-3}, d\kappa(s_{n-2})) \right) \cdots \mu(ds_0).
\]

We prove the two following statements: for each \(0 \leq i \leq n-2\),
(a) \(\{s_i \in S \mid \text{Prob}_\kappa^T(s_{i+1}, \ldots, B_{n-1}, A_n) > 0\} \cap A_i = B_i\) and
(b) \(\int_{s_i \in A_i} \text{Prob}_\kappa^T(s_{i+1}, \ldots, B_{n-1}, A_n) \kappa(s_{i-1}, ds_i) = \text{Prob}_\kappa^T(s_{i-1}, Cyl(B_i, \ldots, B_{n-1}, A_n)),\)
where if \(i = 0\), \(\kappa(s_{i-1}, \cdot)\) will stand for the initial distribution \(\mu\). Point (a) is here in order to establish that the sets \(\{s_i \in S \mid \text{Prob}_\kappa^T(s_{i+1}, \ldots, B_{n-1}, A_n) > 0\} \cap A_i\) are measurable, and point (b) aims at reducing our integrals to sets whose images have positive values. It should be observed that the second point is an immediate consequence of the first point. We thus only need to prove point (a). We do this by induction over \(i\). First, if \(i = n-2\), we show that
\[
\{s_{n-2} \in S \mid \text{Prob}_\kappa^T(s_{n-2}, Cyl(B_{n-1}, A_n)) > 0\} = \{s_{n-2} \in S \mid \kappa(s_{n-2}, B_{n-1}) > 0\}
\]
which will ensure that (a) is satisfied. First assume that \(s_{n-2} \in S\) is such that
\[
\text{Prob}_\kappa^T(s_{n-2}, Cyl(B_{n-1}, A_n)) > 0.
\]
Towards a contradiction, assume that \(\kappa(s_{n-2}, B_{n-1}) = 0\). Then it holds that
\[
0 = \kappa(s_{n-2}, B_{n-1}) = \text{Prob}_\kappa^T(s_{n-2}, Cyl(B_{n-1})) \geq \text{Prob}_\kappa^T(s_{n-2}, Cyl(B_{n-1}, A_n)) > 0
\]
which is the needed contradiction. Now assume that \(\kappa(s_{n-2}, B_{n-1}) > 0\). Then from the definitions of \(B_{n-1}\) and of \(\text{Prob}_\kappa^T(s_{n-2}, \cdot)\), and from classical properties on integrals, it is straightforward to check that the second inclusion holds. Now suppose that point (a) holds for each \(i+1 \leq j \leq n-2\) for some \(i \geq 0\), and let us show that it is still true for \(i\). As before, it suffices to establish that
\[
\{s_i \in S \mid \text{Prob}_\kappa^T(s_{i+1}, Cyl(B_{i+1}, \ldots, B_{n-1}, A_n)) > 0\} = \{s_i \in S \mid \kappa(s_i, B_i) > 0\}.
\]
The first inclusion can be verified just like in the first case. Now assume that \(\kappa(s_i, B_{i+1}) > 0\). We know that
\[
\text{Prob}_\kappa^T(s_{i+1}, \ldots, B_{n-1}, A_n) = 
\int_{s_{i+1} \in B_{i+1}} \text{Prob}_\kappa^T(s_{i+1}, Cyl(B_{i+2}, \ldots, B_{n-1}, A_n)) \kappa(s_i, ds_{i+1}).
\]
Using the induction hypothesis over \(i+1\), we get that for each \(s_{i+1} \in B_{i+1}\),
\[
\text{Prob}_\kappa^T(s_{i+1}, Cyl(B_{i+2}, \ldots, B_{n-1}, A_n)) > 0.
\]
And since \(\kappa(s_i, B_{i+1}) > 0\), this induces that
\[
\text{Prob}_\kappa^T(s_i, Cyl(B_{i+1}, \ldots, B_{n-1}, A_n)) > 0.
\]
which concludes that point (a) is satisfied. Hence from points (a) and (b), we get that

\[ \text{Prob}^T_\mu(\text{Cyl}(A_0, \ldots, A_n)) = \text{Prob}^T_\mu(\text{Cyl}(B_0, \ldots, B_{n-1}, A_n)) = \int_{s_0 \in B_0} \text{Prob}^T_{\kappa(s_0, \cdot)}(\text{Cyl}(B_1, \ldots, B_{n-1}, A_n)) \mu(ds_0). \]

Since \( B_0 = \{ s_0 \in A_0 \mid \text{Prob}^T_{\kappa(s_0, \cdot)}(\text{Cyl}(B_1, \ldots, B_{n-1}, A_n)) > 0 \} \) and since \( \text{Prob}^T_\mu(\text{Cyl}(A_0, \ldots, A_n)) = 0 \), it follows that \( \mu(B_0) = 0 \). From the hypothesis, we thus get that \( \nu(B_0) \). Now observing that we can prove similarly that \( \text{Prob}^T_\nu(\text{Cyl}(A_0, \ldots, A_n)) = \text{Prob}^T_\nu(\text{Cyl}(B_0, \ldots, B_{n-1}, A_n)) \), we can establish that \( \text{Prob}^T_\nu(\text{Cyl}(A_0, \ldots, A_n)) = 0 \) which concludes the proof.

**Lemma 4.** Let \( \mu \in \text{Dist}(S) \) be an initial distribution and let \((A_i)_{0 \leq i \leq n} \) be a sequence of measurable sets. Write \( \nu_0 = \mu_{\text{A}_0} \), the conditional probability of \( \mu \) given \( \text{A}_0 \), and for every \( 1 \leq j \leq n-1 \), write \( \nu_j = (\Omega_T(\nu_{j-1}))_{\text{A}_j} \). Then, for every \( 0 \leq j \leq n \):

\[
\text{Prob}^T_\nu(\text{Cyl}(A_0, A_1, \ldots, A_n)) =
\mu(A_0) \cdot \prod_{i=1}^{j} (\Omega_T(\nu_{i-1}))(A_i) \cdot \text{Prob}^T_{\Omega_T(\nu_j)}(\text{Cyl}(A_{j+1}, \ldots, A_n)).
\]

**Proof.** The proof is by induction on \( j \). Assume that \( j = 0 \), we have to show: \( \text{Prob}^T_\mu(\text{Cyl}(A_0, A_1, \ldots, A_n)) = \mu(A_0) \cdot \text{Prob}^T_{\Omega_T(\nu_0)}(\text{Cyl}(A_1, \ldots, A_n)). \) First,

\[
\text{Prob}^T_\mu(\text{Cyl}(A_0, \ldots, A_n)) = \text{Prob}^T_\mu(\text{Cyl}(A_0) \cap \text{Cyl}(S, A_1, \ldots, A_n))
= \text{Prob}^T_\mu(\text{Cyl}(A_0)) \cdot \text{Prob}^T_\mu(\text{Cyl}(S, A_1, \ldots, A_n) \mid \text{Cyl}(A_0))
= \mu(A_0) \cdot \text{Prob}^T_{\mu_{\text{A}_0}}(\text{Cyl}(A_0, \ldots, A_n)).
\]

Now let us unfold \( \text{Prob}^T_{\Omega_T(\nu_0)}(\text{Cyl}(A_1, \ldots, A_n)) \):

\[
\text{Prob}^T_{\Omega_T(\nu_0)}(\text{Cyl}(A_1, \ldots, A_n)) = \int_{s_1 \in \text{A}_1} \text{Prob}^T_{\kappa(s_1, \cdot)}(\text{Cyl}(A_2, \ldots, A_n))(\Omega_T(\nu_0))(ds_1)
= \int_{s_1 \in \text{A}_1} \text{Prob}^T_{\kappa(s_1, \cdot)}(\text{Cyl}(A_2, \ldots, A_n)) \int_{s_0 \in S} \kappa(s_0, ds_1) \nu_0(ds_0)
= \int_{s_0 \in \text{A}_0} \left( \int_{s_1 \in \text{A}_1} \text{Prob}^T_{\kappa(s_1, \cdot)}(\text{Cyl}(A_2, \ldots, A_n)) \kappa(s_0, ds_1) \right) \mu_{\text{A}_0}(ds_0)
= \int_{s_0 \in \text{A}_0} \text{Prob}^T_{\kappa(s_0, \cdot)}(\text{Cyl}(A_1, \ldots, A_n)) \mu_{\text{A}_0}(ds_0)
= \text{Prob}^T_{\mu_{\text{A}_0}}(\text{Cyl}(A_0, \ldots, A_n)).
\]

Now fix \( 0 < j \leq \) and assume that for each for each \( 0 \leq i < j \) the equality above holds. We will prove that it is still the case for \( j \). First, observe that if \( j = n \) then the induction hypothesis
states that
\[
\text{Prob}_\mu^T(Cyl(A_0, A_1, \ldots, A_n)) = \mu(A_0) \cdot \prod_{i=1}^{n-1} (\Omega_T(\nu_{i-1}))(A_i) \cdot \text{Prob}_{\Omega_T(\nu_{n-1})}^T(Cyl(A_n)) \\
= \mu(A_0) \cdot \prod_{i=1}^{n-1} (\Omega_T(\nu_{i-1}))(A_i) \cdot \Omega_T(\nu_{n-1})(A_n) \\
= \mu(A_0) \cdot \prod_{i=1}^{n} (\Omega_T(\nu_{i-1}))(A_i)
\]

which is what we wanted. Otherwise, if \( j < n \), then the hypothesis induction states that
\[
\text{Prob}_\mu^T(Cyl(A_0, A_1, \ldots, A_n)) = \mu(A_0) \cdot \prod_{i=1}^{j-1} (\Omega_T(\nu_{i-1}))(A_i) \cdot \text{Prob}_{\Omega_T(\nu_{j-1})}^T(Cyl(A_j, \ldots, A_n)) \\
\]

Then using a similar argument as in the first case, we get that
\[
\text{Prob}_{\Omega_T(\nu_{j-1})}^T(Cyl(A_j, \ldots, A_n)) = \Omega_T(\nu_{j-1})(A_j) \cdot \text{Prob}_{\Omega_T(\nu_{j})}^T(Cyl(A_{j+1}, \ldots, A_n))
\]
since \( \Omega_T(\nu_{j}) = (\Omega_T(\nu_{j-1}))_{A_j} \). This concludes the proof.

Proof of the fact that the \( \sigma \)-algebra \( \Sigma \times 2^Q \) coincides with \( \Sigma' = \{ \bigcup_{q \in Q} C_q \times \{ q \} \mid \forall q \in Q, C_q \in \Sigma \} \) (stated page 9)

Proof. It suffices to show that

(i) \( \Sigma' \) contains all rectangles;
(ii) \( \Sigma' \subseteq \Sigma \times 2^Q \); and
(iii) \( \Sigma' \) is a \( \sigma \)-algebra.

Property (i) follows from the decomposition any rectangle \( A_1 \times A_2 \) into elements of \( \Sigma' \):
\[
A_1 \times A_2 = \bigcup_{q \in A_2} A_1 \times \{ q \} \cup \bigcup_{q \in A_2} \emptyset \times \{ q \}.
\]

Property (ii) is straightforward since for every \( q \in Q, C_q \times \{ q \} \in \Sigma \times 2^Q \) and thus, the union \( \bigcup_{q \in Q} C_q \times \{ q \} \) also belongs to the \( \sigma \)-algebra \( \Sigma \times 2^Q \).

We finally establish property (iii). First \( \Sigma' \) is non-empty as \( \emptyset \in \Sigma' \). Then, for \( A = \bigcup_{q \in Q} C_q \times \{ q \} \in \Sigma' \), the complement
\[
A^c = \bigcup_{q \in Q} C_q^c \times \{ q \}
\]
still belongs to \( \Sigma' \) since \( \Sigma \) is a \( \sigma \)-algebra and hence for each \( q, C_q^c \in \Sigma \).
Similarly, we get that \( \Sigma' \) is closed under denumerable unions.

Proposition 8. Let \( \mu \in \text{Dist}(S) \) be an initial distribution for \( T \), and \( \mathcal{M} = (Q, q_0, E, \mathcal{F}) \) be a DMA. Then:
\[
\text{Prob}_\mu^T(T[\mathcal{M}]) = \text{Prob}_\mu^{T \times \mathcal{M}}(\{ \rho \in \text{Paths}(T \times \mathcal{M}) \mid \rho \models \mathcal{F} \}).
\]
We will establish a link between distributions over Paths(\(T\)) and distributions over Paths(\(T \times M\)). In order to do so, we introduce some notations. Given \(A_0, A_1, \ldots, A_n \in \Sigma'\) we write for each \(i\), \(A_i = \bigcup_{q \in Q} A_{i,q} \times \{q\}\). Also given \(u_1, \ldots, u_n \in 2^{AP}\) and \(q \in Q\) we inductively define
\[
\begin{cases}
q_{u_1} = q' \in Q & \text{such that } (q, u_1, q') \in E \\
q_{u_1\ldots u_i} = q' \in Q & \text{such that } (q_{u_1\ldots u_i-1}, u_i, q') \in E, \ \forall 2 \leq i \leq n.
\end{cases}
\]
Observe that since \(M\) is deterministic, those states are uniquely defined. We then have the following result.

The proof of the above proposition will then be a direct consequence of the next lemma.

\textbf{Lemma 76.} For each initial distribution \(\mu \in \text{Dist}(S)\) for \(T\), for each state \(q \in Q\) of \(M\), for each \(n \in \mathbb{N}\) and for each \(A_0, \ldots, A_n \in \Sigma'\), it holds that
\[
\text{Prob}^{T \times M}_\mu(Cyl(A_0, A_1, \ldots, A_n)) = \sum_{u_1, \ldots, u_n \in 2^{AP}} \text{Prob}^{T}_\mu(Cyl(A_{0,q} \cap \mathcal{L}^{-1}(u_1), A_{1,q_1} \cap \mathcal{L}^{-1}(u_2), \ldots, A_{n-1,q_{u_1\ldots u_{n-1}} \cap \mathcal{L}^{-1}(u_n), A_{n,q_{u_1\ldots u_n}})).
\]

\textit{Proof.} We prove it by induction over \(n\). First if \(n = 0\), we have to show that for every \(\mu \in \text{Dist}(S)\), every \(q \in Q\) and every \(A_0 \in \Sigma'\),
\[
\text{Prob}^{T \times M}_\mu(Cyl(A_0)) = \text{Prob}^{T}_\mu(A_{0,q})
\]
which is trivial from the definition of \(\mu \times \delta_q\). Now fix \(n \geq 0\). Assume that for each \(0 \leq i \leq n\), the above property holds true and show that it is still the case for \(i = n + 1\). Let \(\mu \in \text{Dist}(S)\), \(q \in Q\) and \(A_0, \ldots, A_{n+1} \in \Sigma'\). We have that
\[
\text{Prob}^{T \times M}_\mu(Cyl(A_0, \ldots, A_{n+1}))
\]
\[=
\int_{(s_0, q') \in A_0} \text{Prob}^{T \times M}_\mu(Cyl(A_1, \ldots, A_{n+1}))d(\mu \times \delta_q)((s_0, q'))
\]
\[=
\int_{s_0 \in A_0,q} \text{Prob}^{T \times M}_{\kappa((s_0,q'),\cdot)}(Cyl(A_1, \ldots, A_{n+1}))d\mu(s_0)
\]
\[=
\sum_{u_1 \in 2^{AP}} \int_{s_0 \in A_0,q \cap \mathcal{L}^{-1}(u_1)} \text{Prob}^{T \times M}_{\kappa((s_0,q'),\cdot)}(Cyl(A_1, \ldots, A_{n+1}))d\mu(s_0)
\]
\[=
\sum_{u_1 \in 2^{AP}} \int_{s_0 \in A_0,q \cap \mathcal{L}^{-1}(u_1)} \text{Prob}^{T \times M}_{\kappa((s_0,q'),\cdot)}(Cyl(A_1, \ldots, A_{n+1}))d\mu(s_0)
\]
from unicity of \(q_{u_1}\). (4)

Using the induction hypothesis, we get that
\[
\text{Prob}^{T \times M}_{\kappa(s_0,\cdot)_{q_{u_1}}}(Cyl(A_1, \ldots, A_{n+1})) = \\
\sum_{u_2, \ldots, u_{n+1} \in 2^{AP}} \text{Prob}^T_{\kappa(s_0,\cdot)}(Cyl(A_{1,q_1} \cap \mathcal{L}^{-1}(u_2), \ldots, A_{n,q_{u_1\ldots u_n} \cap \mathcal{L}^{-1}(u_{n+1}), A_{n+1,q_{u_1\ldots u_{n+1}}}).
\]

Combining with (4), we thus obtain that
\[
\text{Prob}^{T \times M}_{\mu \times \delta_q}(Cyl(A_0, \ldots, A_{n+1})) = \\
\sum_{u_1, \ldots, u_{n+1} \in 2^{AP}} \text{Prob}^T_{\kappa(s_0,\cdot)}(Cyl(A_{0,q} \cap \mathcal{L}^{-1}(u_1), \ldots, A_{n,q_{u_1\ldots u_n} \cap \mathcal{L}^{-1}(u_{n+1}), A_{n+1,q_{u_1\ldots u_{n+1}}}))
\]
which concludes the proof.
B Technical results of Section 3

Lemma 9. Given $B \in \Sigma$, it holds that:

- $\widetilde{B}$ belongs to the $\sigma$-algebra $\Sigma$;
- for every $\mu \in \text{Dist}(B)$, $\text{Prob}^T_\mu(FB) = 0$;
- for every $\mu \in \text{Dist}(S)$, if $\mu((\widetilde{B})^c) > 0$, then $\text{Prob}^T_\mu(FB) > 0$;
- for every $\mu \in \text{Dist}(S)$, $\text{Prob}^T_\mu(F\widetilde{B}) = \text{Prob}^T_\mu(FG\widetilde{B}) = \text{Prob}^T_\mu(GF\widetilde{B})$;
- for every $\mu \in \text{Dist}(S)$, $\text{Prob}^T_\mu(FB \vee F\widetilde{B}) = \text{Prob}^T_\mu(FB \vee (\neg B \cup \widetilde{B}))$.

Proof. We first prove the first point. Remember that given $B \in \Sigma$, $\widetilde{B} = \{s \in S \mid \text{Prob}^T_{\delta_s}(FB) = 0\}$. Observe that we can write:

$$\widetilde{B} = \bigcap_{n \geq 0} \{s \in S \mid \text{Prob}^T_{\delta_s}(\text{Cyl}(\bar{S}, \ldots, S, B)) = 0\}.$$ 

It thus suffices to show that for each $n \geq 0$,

$$\{s \in S \mid \text{Prob}^T_{\delta_s}(\text{Cyl}(\bar{S}, \ldots, S, B)) = 0\} \in \Sigma.$$

We will use similar arguments as in the proof of Lemma 3. Remember that if $n \geq 1$, it holds that $\text{Prob}^T_{\delta_s}(\text{Cyl}(\bar{S}, \ldots, S, B)) = \text{Prob}^T_{\delta_s}(\text{Cyl}(\bar{S}, \ldots, S, B))$. First, if $n = 0$ then this set corresponds to the set $\{s \in S \mid \delta_s(B) = 0\} = B^c$ which is in $\Sigma$. Now if $n = 1$ then

$$\{s \in S \mid \text{Prob}_{\delta_s}(\text{Cyl}(B)) = 0\} = (\kappa(\cdot, B))^{-1}(\{0\})$$

which is in $\Sigma$ from the hypotheses over $\kappa$. Now assume that $n \geq 2$, it hold that

$$\text{Prob}^T_{\kappa(s)}(\text{Cyl}(\bar{S}, \ldots, S, B)) = \int_{s_1 \in S} \cdots \int_{s_{n-1} \in S} \kappa(s_{n-1}, B)\kappa(s_{n-2}, ds_{n-1}) \cdots \kappa(s_1, ds_2)\kappa(s, ds_1).$$

We inductively define:

$$\begin{cases} B_{n-1} = \kappa(\cdot, B)^{-1}([0, 1]) \\
B_i = \kappa(\cdot, B_{i+1})^{-1}([0, 1]) \quad \forall 0 \leq i \leq n-2. \end{cases}$$

From the hypotheses over $\kappa$, it holds that $B_i \in \Sigma$ for each $0 \leq i < n$. In the sequel, $s_0$ denotes $s$. As in the proof of Lemma 3 we can show that firstly, $\int_{s_{n-1} \in S} \kappa(s_{n-1}, B)\kappa(s_{n-2}, ds_{n-1}) = \text{Prob}^T_{\kappa(s_{n-2})}(\text{Cyl}(B_{n-1}, B))$ and that for each $1 \leq i \leq n-2$,

(a) $\{s_i \in S \mid \text{Prob}^T_{\kappa(s_i)}(\text{Cyl}(B_{i+1}, \ldots, B_{n-1}, B_i)) > 0\} = B_i$ and

(b) $\int_{s_i \in S} \text{Prob}^T_{\kappa(s_i)}(\text{Cyl}(B_{i+1}, \ldots, B_{n-1}, A_i))\kappa(s_{i-1}, ds_i) = \text{Prob}^T_{\kappa(s_{i-1})}(\text{Cyl}(B_i, \ldots, B_{n-1}, A_i)).$
It follows that
\[
\text{Prob}^T_{\kappa(s, \cdot)}(\text{Cyl}(S, \ldots, S, B)) = \text{Prob}^T_{\kappa(s, \cdot)}(\text{Cyl}(B_1, \ldots, B_n, B))
\]
\[
= \int_{s_1 \in B_1} \text{Prob}^T_{\kappa(s_1, \cdot)}(\text{Cyl}(B_2, \ldots, B_n, B))\kappa(s, ds_1)
\]
Now since for each \(s\in B_1\), \(\text{Prob}^T_{\kappa(s_1, \cdot)}(\text{Cyl}(B_2, \ldots, B_n, B)) > 0\), it holds that
\[
\text{Prob}^T_{\kappa(s_1, \cdot)}(\text{Cyl}(S, \ldots, S, B)) = 0
\]
if and only if \(\kappa(s, B_1) = 0\), i.e. if and only if \(s \notin B_0\). And since \(B_0 \in \Sigma\), it follows that \(B_0^c \in \Sigma\) and thus
\[
B_0^c = \{s \in S \mid \text{Prob}^T_{\delta}(\text{Cyl}(S, \ldots, S, B)) = 0\} \in \Sigma.
\]

The second property is a direct consequence of the definition of \(\bar{B}\).

We now focus on the third property. Towards a contradiction, assume that there is \(\mu \in \text{Dist}(S)\) such that \(\mu((\bar{B})^c) > 0\) but \(\text{Prob}^T_{\mu}(F\bar{B}) = 0\). It follows that there is \(s \in (\bar{B})^c\) such that \(\text{Prob}^T_{\delta}(F\bar{B}) = 0\) and thus \(s \in \bar{B}\) which is the wanted contradiction.

Let us show the fourth item. It should be observed that given \(\mu \in \text{Dist}(S)\), \(\text{Prob}^T_{\mu}(FG\bar{B}) \leq \text{Prob}^T_{\mu}(GF\bar{B}) \leq \text{Prob}^T_{\mu}(F\bar{B})\). It thus suffices to show that \(\text{Prob}^T_{\mu}(FG\bar{B}) = \text{Prob}^T_{\mu}(F\bar{B})\).

Since \(\text{Ev}_T(FG\bar{B}) \subseteq \text{Ev}_T(F\bar{B})\), towards a contradiction, we assume that \(\text{Prob}^T_{\mu}(F\bar{B} \land GF(\bar{B})^c) > 0\). Since
\[
\text{Ev}_T(F\bar{B} \land GF(\bar{B})^c) \subseteq \text{Ev}_T(\bigvee_{n \geq 0} (F = n\bar{B} \land F > n(\bar{B})^c))
\]
\[
= \bigcup_{n \geq 0} \bigcup_{m > 0} \text{Cyl}(S, \ldots, S, \bar{B}, S, \ldots, S, (\bar{B})^c)
\]
it follows that there is \(n \in \mathbb{N}\) and \(m > 0\) such that
\[
\text{Prob}^T_{\mu}(\text{Cyl}(S, \ldots, S, \bar{B}, S, \ldots, S, (\bar{B})^c)) > 0.
\]

From Lemma \(4\) writing \(\nu = \Omega_T^{(n)}(\mu)\), we get that
\[
\text{Prob}^T_{\nu}(\text{Cyl}(\bar{B}, S, \ldots, S, (\bar{B})^c)) > 0.
\]
And from the third property proven previously, we deduce that
\[
\text{Prob}^T_{\nu}(F\bar{B}) > 0
\]
with \(\nu \bar{B} \in \text{Dist}(\bar{B})\) which contradicts the second property of this lemma.

Finally, we prove the last property. It is straightforward by observing that the two events measured in this equality are exactly the same:
\[
\text{Ev}_T(FB \lor F\bar{B}) = \text{Ev}_T(FB \lor (\neg B \cup \bar{B})).
\]
C  Technical results of Section 4

Lemma 22. Let \( \alpha : (S_1, \Sigma_1) \to (S_2, \Sigma_2) \) be a measurable function. Then for every \( s \in S_2 \) and every \( \mu \in \text{Dist}(\alpha^{-1}(\{s\})) \), \( \alpha_\#(\mu) = \delta_s \).

Proof. Fix \( s \in S_2 \) and \( \mu \in \text{Dist}(\alpha^{-1}(\{s\})) \). For each \( A \in \Sigma_2 \), we have that \( (\alpha_\#(\mu))(A) = \mu(\alpha^{-1}(A)) \). If \( s \in A \), then \( \alpha^{-1}(\{s\}) \subseteq \alpha^{-1}(A) \) and thus \( \mu(\alpha^{-1}(A)) = 1 \). Otherwise, if \( s \notin A \), then \( \alpha^{-1}(\{s\}) \cap \alpha^{-1}(A) = \emptyset \) and thus \( \mu(\alpha^{-1}(A)) = 0 \). This directly implies that \( \alpha_\#(\mu) = \delta_s \).

Lemma 23. Assume that \( T_2 \) is an \( \alpha \)-abstraction of \( T_1 \). Then, for every \( i \in \mathbb{N} \), for every \( \mu \in \text{Dist}(s_1) \), \( \alpha_\#(\Omega_{T_1}^{(i)}(\mu)) \) is equivalent to \( \Omega_{T_2}^{(i)}(\alpha_\#(\mu)) \).

Proof. We show this by induction on \( i \). Case \( i = 1 \) is by definition. Fix some \( i \geq 1 \) and assume that the statement holds true for each \( 1 \leq j \leq i \). By induction hypothesis, we have that \( \alpha_\#(\Omega_{T_1}^{(j)}(\mu)) \) is equivalent to \( \Omega_{T_2}^{(j)}(\alpha_\#(\mu)) \). We want to show that \( \alpha_\#(\Omega_{T_1}^{(i)}(\mu)) \) is equivalent to \( \Omega_{T_2}^{(i)}(\alpha_\#(\mu)) \).

We first notice that \( \Omega_{T_2}(\alpha_\#(\Omega_{T_1}^{(i)}(\mu))) \) is equivalent to \( \Omega_{T_2}^{(i+1)}(\alpha_\#(\mu)) \). Indeed write \( \nu = \alpha_\#(\Omega_{T_1}^{(i)}(\mu)) \) and \( \nu' = \Omega_{T_2}^{(i)}(\alpha_\#(\mu)) \). From the induction hypothesis, we know that \( \nu \) and \( \nu' \). Following a similar argument as in the proof of Lemma 23 and from the definition of \( \Omega_{T_2} \), we can deduce that \( \Omega_{T_2}(\nu) \) is equivalent to \( \Omega_{T_2}(\nu') \). So it remains to show that \( \Omega_{T_2}(\alpha_\#(\mu')) \) is equivalent to \( \alpha_\#(\Omega_{T_1}(\mu')) \), when \( \mu' = \Omega_{T_1}^{(i)}(\mu) \). This is by definition of an \( \alpha \)-abstraction.

Lemma 24. Assume that \( T_2 \) is an \( \alpha \)-abstraction of \( T_1 \). Then for every \( \mu \in \text{Dist}(s_1) \), for every \( (A_i)_{0 \leq i \leq n} \in \Sigma_2^{n+1} \),

\[
\text{Prob}_{\mu}^{T_1}(\text{Cyl}(\alpha^{-1}(A_0), \ldots, \alpha^{-1}(A_n))) > 0 \iff \text{Prob}_{\alpha_\#(\mu)}^{T_2}(\text{Cyl}(A_0, \ldots, A_n)) > 0.
\]

Proof. We do the proof by induction on \( n \). The case \( n = 0 \) is obvious from the definition of \( \alpha_\# \). Now fix \( n \geq 1 \) and assume that for each \( 0 \leq k \leq n - 1 \), for each \( \mu \in \text{Dist}(S_1) \) and for each \( (A_i)_{0 \leq i \leq k} \in \Sigma_2^{k+1} \),

\[
\text{Prob}_{\mu}^{T_1}(\text{Cyl}(\alpha^{-1}(A_0), \ldots, \alpha^{-1}(A_k))) > 0 \iff \text{Prob}_{\alpha_\#(\mu)}^{T_2}(\text{Cyl}(A_0, \ldots, A_k)) > 0.
\]

We show that it is still the case for \( n \). Fix \( \mu \in \text{Dist}(S_1) \) and \( (A_i)_{i \geq n+1} \in \Sigma_2^{n+2} \). We let \( \nu_0 = \mu_{\alpha^{-1}(A_0)} \) and \( \nu'_0 = (\alpha_\#(\mu))_{A_0} \). Note that we hence assume that \( \mu(\alpha^{-1}(A_0)) > 0 \). We first realize that \( \nu'_0 = \alpha_\#(\nu_0) \). Indeed for each \( A \in \Sigma_2 \),

\[
(\alpha_\#(\nu_0))(A) = \nu_0(\alpha^{-1}(A)) = \frac{\mu(\alpha^{-1}(A \cap A_0))}{\mu(\alpha^{-1}(A_0))} = \frac{(\alpha_\#(\mu))(A \cap A_0)}{(\alpha_\#(\mu))(A_0)} = \nu'_0(A).
\]

Then, applying Lemma 23 we get:

\[
\text{Prob}_{\mu}^{T_1}(\text{Cyl}(\alpha^{-1}(A_0), \alpha^{-1}(A_1), \ldots, \alpha^{-1}(A_n))) = \mu(\alpha^{-1}(A_0)) \cdot \text{Prob}_{\nu_0}^{T_1}(\text{Cyl}(\alpha^{-1}(A_1), \ldots, \alpha^{-1}(A_n))).
\]
and
\[
\text{Prob}_{\alpha_\#(\mu)}^T_2(\text{Cyl}(A_0, A_1, \ldots, A_n)) = (\alpha_\#(\mu))(A_0) \cdot \text{Prob}_{\alpha_\#(\nu_0)}^T_1(\text{Cyl}(A_1, \ldots, A_n)).
\]

By definition of an \(\alpha\)-abstraction, the measures \(\Omega_{\text{Cyl}}(\nu_0)\) and \(\alpha_\#(\Omega_{\text{Cyl}}(\nu_0))\) are equivalent. Hence from Lemma 3
\[
\text{Prob}_{\alpha_\#(\nu_0)}^T_2(\text{Cyl}(A_1, \ldots, A_n)) > 0 \iff \text{Prob}_{\alpha_\#(\nu_0)}^T_2(\text{Cyl}(A_1, \ldots, A_n)) > 0.
\]

From the hypothesis of induction, we get that
\[
\text{Prob}_{\alpha_\#(\nu_0)}^T_2(\text{Cyl}(A_1, \ldots, A_n)) > 0 \implies \text{Prob}_{\alpha_\#(\nu_0)}^T_2(\text{Cyl}(\alpha^{-1}(A_1), \ldots, \alpha^{-1}(A_n))) > 0.
\]

Since \((\alpha_\#(\mu))(A_0) = \mu(\alpha^{-1}(A_0))\), we conclude:
\[
\text{Prob}_{\alpha_\#(\nu_0)}^T_2(\text{Cyl}(\alpha^{-1}(A_0), \alpha^{-1}(A_1), \ldots, \alpha^{-1}(A_n))) > 0 \iff \text{Prob}_{\alpha_\#(\mu)}^T_2(\text{Cyl}(A_0, A_1, \ldots, A_n)) > 0.
\]

We still have to consider the case where \(\mu(\alpha^{-1}(A_0)) = 0\). In that case, \((\alpha_\#(\mu))(A_0) = 0\) and thus
\[
\text{Prob}_{\alpha_\#(\nu_0)}^T_2(\text{Cyl}(\alpha^{-1}(A_0), \alpha^{-1}(A_1), \ldots, \alpha^{-1}(A_n))) = \text{Prob}_{\alpha_\#(\mu)}^T_2(\text{Cyl}(A_0, A_1, \ldots, A_n))
\]
which terminates the proof.

**Lemma 28.** Assume \(T_2\) is a DMC. Then:

- \(T_2\) is an \(\alpha\)-abstraction of \(T_1\) iff for every \(s, s' \in S_2\),
  \[
  \kappa_2(s, \{s'\}) > 0 \iff \forall \mu \in \text{Dist}(\alpha^{-1}(\{s\})), \text{Prob}_{\mu}^{T_1}(\text{Cyl}(S_1, \alpha^{-1}(\{s'\}))) > 0.
  \]

- \(T_2\) is sound iff for every \(s \in S_2\) and every \(B \in \Sigma_2\),
  \[
  \text{Prob}_{\kappa_2}^{T_2}(F B) = 1 \implies \forall \mu \in \text{Dist}(\alpha^{-1}(\{s\})), \text{Prob}_{\mu}^{T_1}(F \alpha^{-1}(B)) = 1.
  \]

- \(T_2\) is complete iff for every \(s \in S_2\) and every \(B \in \Sigma_2\),
  \[
  \forall \mu \in \text{Dist}(\alpha^{-1}(\{s\})), \text{Prob}_{\mu}^{T_1}(F \alpha^{-1}(B)) = 1 \implies \text{Prob}_{\kappa_2}^{T_2}(F B) = 1.
  \]

**Proof.** We handle the case of soundness. Indeed, assume that for each \(s \in S_2\) and for each \(B \in \Sigma_2\), the condition presented in the statement (second item) holds true. Then fix \(\mu \in \text{Dist}(S_1)\), \(B \in \Sigma_2\) and assume that \(\text{Prob}_{\kappa_2}^{T_2}(F B) = 1\) and show that \(\text{Prob}_{\mu}^{T_1}(F \alpha^{-1}(B)) = 1\). Towards a contradiction, assume that \(\text{Prob}_{\mu}^{T_1}(F \alpha^{-1}(B)) < 1\). Then, since \(T_2\) is a DMC, there is \(s \in S_2\) such that \(\mu(\alpha^{-1}(s)) > 0\) and
\[
\text{Prob}_{\mu_{\alpha^{-1}(s)}}^{T_1}(F \alpha^{-1}(B)) < 1.
\]

From the hypothesis, it follows that \(\text{Prob}_{\delta_s}^{T_2}(F B) < 1\). Observe that since \(\mu(\alpha^{-1}(s)) > 0\), we have that \((\alpha_\#(\mu))(s) > 0\). Hence we get a contradiction by noticing:
\[
\text{Prob}_{\alpha_\#(\mu)}^{T_2}(F B) \leq (\alpha_\#(\mu))(s) \cdot \text{Prob}_{\delta_s}^{T_2}(F B) < 1.
\]
Proposition 34. Let $\mathcal{T}_2$ be a DMC such that $\mathcal{T}_2$ is an $\alpha$-abstraction of $\mathcal{T}_1$. Assume that there is a finite set $A_2 = \{s_1, \ldots, s_n\} \subseteq S_2$ such that $A_2$ is an attractor for $\mathcal{T}_2$ and $A_1 = \bigcup_{i=1}^{n} \alpha^{-1}(s_i) = \alpha^{-1}(A_2)$ is an attractor for $\mathcal{T}_1$. Assume moreover that for every $1 \leq i \leq n$, for every $\alpha$-closed set $B$ in $\Sigma_1$, there exist $p > 0$ and $k \in \mathbb{N}$ such that:

- for every $\mu \in \text{Dist}(\alpha^{-1}(s_i))$, $\Pr^\mu_{\mathcal{T}_1}(F_{\leq k}B) \geq p$, or
- for every $\mu \in \text{Dist}(\alpha^{-1}(s_i))$, $\Pr^\mu_{\mathcal{T}_1}(B) = 0$.

Then $\mathcal{T}_1$ is decisive w.r.t. every $\alpha$-closed set.

Proof. Fix $B \subseteq S_2$ and $\mu \in \text{Dist}(S_1)$. We want to show that $\mathcal{T}_1$ is $\mu$-decisive w.r.t. $\alpha^{-1}(B)$. We therefore have to show that $\Pr^\mu_{\mathcal{T}_1}(F \alpha^{-1}(B) \lor \neg F \alpha^{-1}(\overline{B})) = 1$. Towards a contradiction we assume that $\Pr^\mu_{\mathcal{T}_1}(G (\neg \alpha^{-1}(B)) \land \neg \alpha^{-1}(\overline{B})) > 0$, i.e. $\Pr^\mu_{\mathcal{T}_1}(G \alpha^{-1}(B^c) \land \neg \alpha^{-1}(\overline{B})) > 0$.

Since $A_1 = \alpha^{-1}(A_2)$ is an attractor of $\mathcal{T}_1$, we deduce from Lemma 32 that $\Pr^\mu_{\mathcal{T}_1}(G F \alpha^{-1}(A_2)) = 1$, hence:

$$\Pr^\mu_{\mathcal{T}_1}(G \alpha^{-1}(B^c) \land \neg \alpha^{-1}(\overline{B})) > 0 \quad \text{(5)}$$

We let $A_2' \subseteq A_2$ be the subset of states $s$ of $A_2$ such that:

$$\Pr^\mu_{\mathcal{T}_1}(G \alpha^{-1}(B^c) \land \neg \alpha^{-1}(\overline{B})) > 0 \quad \text{(5')}$$

Due to equation (5), $A_2'$ is non-empty, and furthermore every such $s$ belongs to $B^c$ and $\overline{B}^c$.

We set $A'_1 = \alpha^{-1}(A_2')$.

In particular, $A'_1 \subseteq \alpha^{-1}(\overline{B}^c)$, hence from Lemma 39 (third item) we get that for every $\nu \in \text{Dist}(A'_1)$, $\Pr^\nu_{\mathcal{T}_1}(F \alpha^{-1}(B)) > 0$. According to hypothesis (i), for every $s \in A'_2$, we can find $p_s > 0$ and $k_s \in \mathbb{N}$ such that for every $\nu \in \text{Dist}(\alpha^{-1}(s))$,

$$\Pr^\nu_{\mathcal{T}_1}(F_{\leq k_s} \alpha^{-1}(B)) \geq p_s.$$

Then taking $p = \min\{p_s | s \in A'_1 \} > 0$ and $k = \max\{k_s | s \in A'_2\} \in \mathbb{N}$ (since $A'_2$ is finite), it holds that for every $\nu \in \text{Dist}(A'_1)$,

$$\Pr^\nu_{\mathcal{T}_1}(F_{\leq k} \alpha^{-1}(B)) \geq p \quad \text{hence} \quad \Pr^\nu_{\mathcal{T}_1}(G_{\leq k} \alpha^{-1}(B^c)) \leq 1 - p \quad \text{(6)}$$

From (6), we can deduce that:

$$0 < \Pr^\nu_{\mathcal{T}_1}(G \alpha^{-1}(B^c) \land \neg \alpha^{-1}(\overline{B})) \land G F A'_1 \leq \lim_{n \to \infty} (1 - p)^n = 0.$$

It remains to show the last inequality. We will prove it by induction as follows.

First we introduce some useful notations. We will write $B^n$ for the finite sequence $\alpha^{-1}(B^c), \ldots, \alpha^{-1}(B^c)$ where $\alpha^{-1}(B^c)$ occurs exactly $k$ times, and given $j \in \mathbb{N}$ we will write $S^n_j$ for the finite sequence $S_1, \ldots, S_1$ where $S_1$ occurs exactly $j$ times. Then observe that

$$\Pr_{\mathcal{T}_1}(G F (A'_1 \land G_{\leq k} \alpha^{-1}(B^c))) = \bigcap_{n \in \mathbb{N}} \bigcup_{(j_0, \ldots, j_n) \in \mathbb{N}^{n+1}} \text{Cyl}(S^{j_0}_{1^n}, A'_1, B^n, S^{j_1}_{1^n}, A'_1, B^n, \ldots, S^{j_n}_{1^n}, A'_1, B^n).$$
We will prove by induction over \( n \) that for each \( \nu \in \text{Dist}(S_1) \) and for each \( n \geq 0 \),

\[
\Pr_{\nu}^{T_1} \left( \bigcup_{(j_0, \ldots, j_n) \in \mathbb{N}^{n+1}} \text{Cyl}(S^{j_0}_1, A'_1, B^n_k, S^{j_1}_1, A'_1, B^n_k, \ldots, S^{j_n}_1, A'_1, B^n_k) \right) \leq (1 - p)^{n+1}.
\] (7)

First fix \( n = 0 \) and \( \nu \in \text{Dist}(S_1) \). It holds that for each \( j_0 \in \mathbb{N} \)

\[
\Pr_{\nu}^{T_1}(\text{Cyl}(S^0_1, A'_1, B^n_k)) \leq \Pr_{\nu}^{T_1}(\text{Cyl}(A'_1, B^n_k)) \quad \text{from Lemma 4.}
\]

\[
\leq 1 - p \quad \text{from (4).}
\]

Now fix \( m \geq 0 \) and assume that for each \( 0 \leq n \leq m \) and for each \( \nu \in \text{Dist}(S_1) \) the inequality (7) holds true. We want to show that it is still satisfied for \( n + 1 \). For each \( \nu \in \text{Dist}(S_1) \) and for each \( j_0 \in \mathbb{N} \) we have that

\[
\Pr_{\nu}^{T_1} \left( \bigcup_{(j_1, \ldots, j_{n+1}) \in \mathbb{N}^{n+1}} \text{Cyl}(S^{j_0}_1, A'_1, B^n_k, S^{j_1}_1, A'_1, B^n_k, \ldots, S^{j_{n+1}}_1, A'_1, B^n_k) \right)
\]

\[
\leq \Pr_{\nu}^{T_1} \left( \bigcup_{(j_1, \ldots, j_{n+1}) \in \mathbb{N}^{n+1}} \text{Cyl}(A'_1, B^n_k, S^{j_1}_1, A'_1, B^n_k, \ldots, S^{j_{n+1}}_1, A'_1, B^n_k) \right)
\]

\[
\quad \leq \nu'(A'_1) \prod_{i=1}^{k} (\Omega_{T_1}(\nu'_i))(\alpha^{-1}(B^n_c)) \cdot \Pr_{\nu'_{k+1}}^{T_1} \left( \bigcup_{(j_1, \ldots, j_{n+1}) \in \mathbb{N}^{n+1}} \text{Cyl}(S^{j_1}_1, A'_1, B^n_k, \ldots, S^{j_{n+1}}_1, A'_1, B^n_k) \right)
\]

\[
\leq \Pr_{\nu'}^{T_1}(\text{Cyl}(A'_1, B^n_k)) (1 - p)^{n+1} \quad \text{from Lemma 4 and hypothesis of induction}
\]

\[
\leq (1 - p)^{n+2} \quad \text{from the inequality (3).}
\]

Through the limits, we conclude that \( \Pr_{\nu}^{T_1}(\text{G F} (A'_1 \land G_{\leq k} \alpha^{-1}(B^n_c))) \leq \lim_{n \to \infty}(1 - p)^n = 0 \). This yields a contradiction and concludes the proof.

**Proposition 35.** Let \( T_2 \) be a finite Markov chain such that \( T_2 \) is an \( \alpha \)-abstraction of \( T_1 \). Fix \( \mu \in \text{Dist}(S_1) \), and assume that \( T_1 \) is \( \mu \)-fair w.r.t. every \( \alpha \)-closed set. Then \( T_1 \) is \( \mu \)-decisive w.r.t. every \( \alpha \)-closed set.

**Proof.** As \( T_2 \) is a finite Markov chain, it can be viewed as a graph. We can therefore speak of the bottom strongly connected components (BSCC) of \( T_2 \) (a BSCC is a subset \( C \subseteq S_2 \) such that for all \( s, s' \in C \), if \( s' \) is reachable from \( s \), then \( s \) is reachable from \( s' \) as well). We write \( \text{BSCC}(T_2) \) for the set of BSCCs of \( T_2 \). We define \( C = \{ s \in S_2 \mid \exists C \in \text{BSCC}(T_2), \ s \in C \} \). We first prove that \( \Pr_{\mu}^{T_1}(F \alpha^{-1}(C)) = 1 \). In order to establish this, we show that for each \( s \in S_2 \), \( \Pr_{\mu}^{T_1}(G F \alpha^{-1}(s)) > 0 \) implies that \( s \in C \). Indeed, pick \( s \in S_2 \) such that:

\[
\Pr_{\mu}^{T_1}(G F \alpha^{-1} \{s\}) > 0.
\]
We can state that for each \( k \geq 1 \) and for each \( s_0, s_1, \ldots, s_k \in S_2 \) with \( s_0 = s \) and such that for each \( 0 \leq i < k, \kappa_2(s_i, s_{i+1}) > 0 \), it holds that

\[
\text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(s_k) \mid \mathbb{P} \mathbf{F} \alpha^{-1}(s)) = 1.
\]

We prove this by induction over \( k \). First fix \( k = 1 \) and let \( s_1 \in S_2 \) such that \( \kappa_2(s, s_1) > 0 \). Then for every \( \nu \in \text{Dist}(\alpha^{-1}(s)) \), \( \text{Prob}^T_{\mu}(\text{Cyl}(\alpha^{-1}(s)), \alpha^{-1}(\{s_1\})) > 0 \). Hence \( \alpha^{-1}(s) \in \text{PreProb}^T(\{\alpha^{-1}(s_1)\}) \). And since \( \mathcal{T}_1 \) is fair w.r.t. \( \alpha \)-closed sets, we get that

\[
\text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(\{s_1\}) \mid \mathbb{P} \mathbf{F} \alpha^{-1}(\{s\})) = 1.
\]

Now fix \( k > 1 \) and assume that for each \( 1 \leq j < k \) and for each \( s_0, \ldots, s_j \in S_2 \) with \( s_0 = s \) and such that for each \( 0 \leq i < j, \kappa_2(s_i, s_{i+1}) > 0 \), it holds that

\[
\text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(s_j) \mid \mathbb{P} \mathbf{F} \alpha^{-1}(s)) = 1.
\]

We want to show that it is still the case for \( k \). Fix \( s_0, s_1, \ldots, s_k \in S_2 \) satisfying all the desired hypotheses. Using the induction hypothesis, we know that \( \text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(s_{k-1}) \mid \mathbb{P} \mathbf{F} \alpha^{-1}(s_0)) = 1 \) and \( \text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(s_k) \mid \mathbb{P} \mathbf{F} \alpha^{-1}(s_{k-1})) = 1 \). We can then compute:

\[
\text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(s_k) \mid \mathbb{P} \mathbf{F} \alpha^{-1}(s_0))
= \text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(s_k) \wedge \mathbb{P} \mathbf{F} \alpha^{-1}(s_{k-1}) \mid \mathbb{P} \mathbf{F} \alpha^{-1}(s_0))
= \text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(s_k) \mid \mathbb{P} \mathbf{F} \alpha^{-1}(s_{k-1})) \cdot \text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(s_{k-1}) \mid \mathbb{P} \mathbf{F} \alpha^{-1}(s_0))
= 1
\]

from the induction hypothesis. This shows that for every state \( s' \) which is reachable from \( s \) in \( \mathcal{T}_2 \),

\[
\text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(\{s'\}) \mid \mathbb{P} \mathbf{F} \alpha^{-1}(\{s\})) = 1.
\]

Then fix \( s' \) reachable from \( s \) in \( \mathcal{T}_2 \). We can show that \( s \) is also reachable from \( s' \). Towards a contradiction, assume that it is not the case. It follows that

\[
\text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(\{s\}) \wedge \mathbb{P} \mathbf{F} \alpha^{-1}(\{s'\})) = 0
\]

which is a contradiction with \( \text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(\{s\}) \mid \mathbb{P} \mathbf{F} \alpha^{-1}(\{s\})) = 1 \) and \( \text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(\{s\})) > 0 \). We deduce thus that \( s \) belongs to a BSCC of \( \mathcal{T}_2 \).

We can now prove that \( \text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F}^{-1}(C)) = 1 \). Indeed observe first that from the finiteness of \( \mathcal{T}_2 \), it holds that for every paths \( \rho = t_0t_1t_2 \ldots \in \text{Paths}(\mathcal{T}_1) \), there is \( s \in S_2 \) such that \( \{i \in \mathbb{N} \mid t_i \in \alpha^{-1}(s)\} \) is infinite. Keeping this in mind, we write \( S_2 = \{s_1, \ldots, s_k, s_{k+1}, \ldots, s_n\} \) where \( k \geq 1 \) and \( \{s_1, \ldots, s_k\} = C \). Then we can write

\[
\text{Paths}(\mathcal{T}_1) = \text{Ev}_{\mathcal{T}_1}(\mathbb{P} \mathbf{F} \alpha^{-1}(s_1)) \cup \text{Ev}_{\mathcal{T}_1}(\mathbb{P} \mathbf{F} \alpha^{-1}(s_2) \wedge \mathbb{P} \mathbf{G} \neg \alpha^{-1}(s_1)) \\
\cup \cdots \cup \text{Ev}_{\mathcal{T}_1}(\mathbb{P} \mathbf{F} \alpha^{-1}(s_n) \wedge \bigwedge_{i=1}^{n-1} \mathbb{F} \mathbf{G} \neg \alpha^{-1}(s_i)).
\]

From what we have shown previously, we now get that for each \( j \geq k + 1 \),

\[
0 = \text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(s_j)) \geq \text{Prob}^T_{\mu}(\mathbb{P} \mathbf{F} \alpha^{-1}(s_j) \wedge \bigwedge_{i=1}^{j-1} \mathbb{F} \mathbf{G} \neg \alpha^{-1}(s_i)).
\]
And we conclude that
\[ 1 = \Prob_{\mu}^{\tilde{\mathcal{T}}_1}(\text{Paths}(\mathcal{T}_1)) \]
\[ = \sum_{j=1}^{k} \Prob_{\mu}^{\tilde{\mathcal{T}}_1}(\mathbf{G} \mathbf{F} \alpha^{-1}(s_j) \land \bigwedge_{i=1}^{j-1} \mathbf{F} \mathbf{G} \neg \alpha^{-1}(s_i)) \]
\[ \leq \Prob_{\mu}^{\tilde{\mathcal{T}}_1}(\mathbf{F} \alpha^{-1}(\mathcal{C})). \]

We are now able to prove that $\mathcal{T}_1$ is $\text{Dec}(\mu, B)$. Fix $B \subseteq S_2$, we want to show that $\Prob_{\mu}^{\tilde{\mathcal{T}}_1}(\mathbf{F} \alpha^{-1}(B) \lor \mathbf{F} \alpha^{-1}(\tilde{B})) = 1$. We have that
\[ \Prob_{\mu}^{\tilde{\mathcal{T}}_1}(\mathbf{F} \alpha^{-1}(B) \lor \mathbf{F} \alpha^{-1}(\tilde{B})) = \sum_{C \in \text{BSCC}^{(T_2)}} \Prob_{\mu}^{\tilde{\mathcal{T}}_1}(\mathbf{F} \alpha^{-1}(C)) \cdot \Prob_{\mu}^{\tilde{\mathcal{T}}_1}(\mathbf{F} \alpha^{-1}(B) \lor \mathbf{F} \alpha^{-1}(\tilde{B}) \mid \mathbf{F} \alpha^{-1}(C)). \]

Now we fix some $C \in \text{BSCC}^{(T_2)}$ such that $\Prob_{\mu}^{\tilde{\mathcal{T}}_1}(\mathbf{F} \alpha^{-1}(C)) > 0$. There are two cases:

- first if there is $s \in C$ such that $s \in B$, then $\alpha^{-1}(s) \subseteq \alpha^{-1}(B)$ and thus $\Prob_{\mu}^{\tilde{\mathcal{T}}_1}(\mathbf{F} \alpha^{-1}(B) \lor \mathbf{F} \alpha^{-1}(\tilde{B}) \mid \mathbf{F} \alpha^{-1}(C)) = 1$;
- or for each $s \in C$, $s \in \tilde{B}$ which implies that $\alpha^{-1}(C) \subseteq \alpha^{-1}(\tilde{B})$ and in that case again $\Prob_{\mu}^{\tilde{\mathcal{T}}_1}(\mathbf{F} \alpha^{-1}(B) \lor \mathbf{F} \alpha^{-1}(\tilde{B}) \mid \mathbf{F} \alpha^{-1}(C)) = 1$.

We finally conclude that
\[ \Prob_{\mu}^{\tilde{\mathcal{T}}_1}(\mathbf{F} \alpha^{-1}(B) \lor \mathbf{F} \alpha^{-1}(\tilde{B})) = \sum_{C \in \text{BSCC}^{(T_2)}} \Prob_{\mu}^{\tilde{\mathcal{T}}_1}(\mathbf{F} \alpha^{-1}(C)) \]
\[ = \Prob_{\mu}^{\tilde{\mathcal{T}}_1}(\mathbf{F} \alpha^{-1}(\mathcal{C})) = 1. \]

D Technical results of Section 5

**Lemma 39.** For every $\mu \in \text{Dist}(S)$

(i) $\Prob_{\mu}^{\mathcal{T}}(\mathbf{F} B \land (\neg B \mathbf{U} \tilde{B})) = 0$;

(ii) $\Prob_{\mu}^{\mathcal{T}}(\mathbf{G} \mathbf{F} B \land \mathbf{F} \tilde{B}) = 0$.

**Proof.** We first prove point (i). Since $B$ cannot be reached while we are in $\neg B$, it holds that
\[ \Prob_{\mu}^{\mathcal{T}}(\mathbf{F} B \land (\neg B \mathbf{U} \tilde{B})) = \Prob_{\mu}^{\mathcal{T}}(\neg B \mathbf{U} (\tilde{B} \land \mathbf{F} B)). \]

Relaxing the constraint on the until, we get $\Prob_{\mu}^{\mathcal{T}}(\neg B \mathbf{U} (\tilde{B} \land \mathbf{F} B)) \leq \Prob_{\mu}^{\mathcal{T}}(\mathbf{F} (\tilde{B} \land \mathbf{F} B))$, and the latter is null by definition of $\tilde{B}$. This proves the first item.

Point (ii) is straightforward from the definition of $\tilde{B}$ by observing that $\Prob_{\mu}^{\mathcal{T}}(\mathbf{G} \mathbf{F} B \land \mathbf{F} \tilde{B}) \leq \Prob_{\mu}^{\mathcal{T}}(\mathbf{F} (\tilde{B} \land \mathbf{F} B)) = 0$.

**Lemma 40.** For every $\mu \in \text{Dist}(S)$, if $\mathcal{T}$ is $\text{PersDec}(\mu, B)$, then $\Prob_{\mu}^{\mathcal{T}}(\mathbf{F} \tilde{B} \land \mathbf{F} \tilde{B}) = 0$. 
Proof. Assume that $\mathcal{T}$ is $\text{PersDec}(\mu, B)$, i.e. for each $p \geq 0$, $\text{Prob}_\mu^T(F_{\geq p} \cap B) = 1$. Towards a contradiction, we suppose that $\text{Prob}_\mu^T(F_{m, n} \cap B) > 0$. Since

$$\text{Ev}_T(F_{m, n} \cap B) = \bigcup_{n \geq 0} \bigcup_{m \geq 0} \text{Ev}_T(F_{m, n} \cap B) \cap \text{Ev}_T(F_{m, n} \cap B),$$

we deduce that there are $n, m \geq 0$ such that $\text{Prob}_\mu^T(F_{m, n} \cap B) > 0$. We write $e$ for the event $e = \text{Ev}_T(F_{m, n} \cap B)$. We can show that $\text{Prob}_\mu^T(F_{m, n} \mid e) = 0$ and $\text{Prob}_\mu^T(F_{m, n} \cap B \mid e) = 0$. Indeed we get that:

$$\text{Prob}_\mu^T(F_{m, n} \mid e) = \frac{\text{Prob}_\mu^T((F_{m, n} \cap B) \mid e)}{\text{Prob}_\mu^T(e)} \leq \frac{\text{Prob}_\mu^T(F_{m, n} \cap B)}{\text{Prob}_\mu^T(e)} = 0$$

from the definition of $B$. The equality $\text{Prob}_\mu^T(F_{m, n} \cap B \mid e) = 0$ is proved similarly. Writing $q = \max(m, n)$, it follows that

$$\text{Prob}_\mu^T(F_{m, n} \cap B \mid e) = 0.$$ 

And since $\text{Prob}_\mu^T(e) > 0$, this contradicts the fact that $\mathcal{T}$ is $\text{PersDec}(\mu, B)$, which concludes the proof.

**Lemma 45.** Assume that $A$ is an attractor for $\mathcal{T}$. Then $A \times Q$ is an attractor for $\mathcal{T} \times M$. Furthermore, if $A$ is finite, then so is $A \times Q$.

We first prove the following lemma.

**Lemma 77.** Fix $\mu \in \text{Dist}(S)$ and assume that $A \in \Sigma$ is a $\mu$-attractor for $\mathcal{T}$. Then for each $q \in Q$, $A \times Q$ is a $(\mu \times \delta_q)$-attractor for $\mathcal{T} \times M$.

**Proof.** Fix $\mu \in \text{Dist}(S)$ and $A \in \Sigma$ such that $\text{Prob}_\mu^T(FA) = 1$. Fix $q \in Q$. We know that

$$\text{Ev}_{T \times M}(FA \times Q) = \text{Ev}_{T \times M}(\bigcup_{n \in \mathbb{N}} \text{Cyl}(S', S', A \times Q)).$$

Then from Lemma 76 we know that for each $n \in \mathbb{N}$

$$\text{Prob}_{\mu \times \delta_q}(\text{Cyl}(S', S', A \times Q)) = \sum_{u_1, \ldots, u_n \in \mathbb{A}^p} \text{Prob}_\mu^T(\text{Cyl}(L^{-1}(u_1), \ldots, L^{-1}(u_n), A))$$

$$= \text{Prob}_\mu^T(\text{Cyl}(S', S', A \times Q)).$$

As this holds true for each $n \geq 0$, we thus get that $\text{Prob}_{\mu \times \delta_q}(FA \times Q) = \text{Prob}_\mu^T(FA) = 1$ from the hypothesis. This concludes the proof.
Proof (Proof of Lemma 77). Fix $A \in \Sigma$ such that for each $\mu \in \text{Dist}(S)$, $\text{Prob}_\mu^T(F(A)) = 1$. We want to prove that for each $\nu \in \text{Dist}(S \times Q)$, $\text{Prob}_\nu^{T \times M}(F(A \times Q)) = 1$. Fix $\nu \in \text{Dist}(S \times Q)$ and compute:

$$\text{Prob}_\nu^{T \times M}(F(A \times Q)) = \sum_{q \in Q} \nu(S \times \{q\}) \cdot \text{Prob}_\nu^{T \times M}(F(A \times Q)).$$

Note that $\nu_{S \times \{q\}}$ induces a distribution $\nu_q \in \text{Dist}(S)$ as follows: for each $B \in \Sigma$, $\nu_q(B) = \nu_{S \times \{q\}}(B \times \{q\})$. Writing $\mu = \nu_q$ it then holds that $\nu_{S \times \{q\}} = \mu \cdot \delta_q$. We then get, from the hypothesis and Lemma 77, that $\text{Prob}_\nu^{T \times M}(F(A \times Q)) = 1$ for each $q \in Q$. Hence, $\text{Prob}_\nu^{T \times M}(F(A \times Q)) = \sum_{q \in Q} \nu(S \times \{q\}) = 1$ which concludes the proof.

Lemma 52. Let $\alpha_M : S_1 \times Q \to S_2 \times Q$ be the unique lifting of $\alpha$ such that $\alpha_M(s, q) = (\alpha(s), q)$. If $T_2$ is an $\alpha_M$-abstraction of $T_1$, then $T_2 \times M$ is an $\alpha_M$-abstraction of $T_1 \times M$. Furthermore, if $T_1 \times M$ is Dec$(B)$ where $B = \{\alpha_M^{-1}(B) \mid B \in \Sigma_q^2\}$, then $T_2 \times M$ is a sound $\alpha_M$-abstraction of $T_1 \times M$.

Proof. We first show that $T_2 \times M$ is an $\alpha_M$-abstraction of $T_1 \times M$. It suffices to show that for each $\mu \in \text{Dist}(S_1)$, for each $q, q' \in Q$ and for each $B_{q'} \in \Sigma_2$,

$$\text{Prob}_{\mu \times \delta_q}^{T_2 \times M}(\text{Cyl}(S_1 \times Q, \alpha_M^{-1}(B_{q'} \times \{q'\}))) > 0 \iff \text{Prob}_{\alpha_M(\mu \times \delta_q)}^{T_2 \times M}(\text{Cyl}(S_2 \times Q, B_{q'} \times \{q'\})) > 0. \quad (8)$$

Fix $\mu \in \text{Dist}(S_1)$, $q, q' \in Q$ and $B_{q'} \in \Sigma_2$. Write $u \in 2^{\text{AP}}$ for the unique label such that $(q, u, q') \in E$. In order to prove (8), we will use the fact that $T_2$ is an $\alpha$-abstraction of $T_1$. And in order to make the link with the wanted equivalence, we will use Lemma 77. We can establish that $(\alpha_M)(\#(\mu \times \delta_q)) = \alpha(\#(\mu) \times \delta_q)$. Indeed, given $p \in Q$ and $C_p \in \Sigma_2$, it holds that

$$(\alpha_M)(\#(\mu \times \delta_q))(C_p \times \{p\}) = (\mu \times \delta_q)(\alpha^{-1}(C_p) \{p\}) = \mu(\alpha^{-1}(C_p)) \cdot \delta_q(p) = \alpha(\#(\mu) \times \delta_q)(C_p \times \{p\}).$$

Hence we get that

$$\text{Prob}_{(\alpha_M)(\#(\mu \times \delta_q))}^{T_2 \times M}(\text{Cyl}(S_2 \times Q, B_{q'} \times \{q'\})) > 0 \iff \text{Prob}_{\alpha_M(\#(\mu \times \delta_q))}^{T_2 \times M}(\text{Cyl}(L_2^{-1}(u), B_{q'})) > 0$$

$$\iff \text{Prob}_{\alpha_M(\#(\mu \times \delta_q))}^{T_1 \times M}(\text{Cyl}(S_1 \times Q, \alpha_M^{-1}(B_{q'} \times \{q'\}))) > 0 \iff \text{Prob}_{\alpha_M(\#(\mu \times \delta_q))}^{T_1 \times M}(\text{Cyl}(S_1 \times Q, \alpha_M^{-1}(B_{q'} \times \{q'\}))) > 0$$

where the first and third equivalences hold from Lemma 77 and the second equivalence holds from the fact that $T_2$ is an $\alpha$-abstraction of $T_1$.

Finally, since $T_1 \times M$ is decisive w.r.t $\alpha_M^{-1}(B)$ for each $B \in \Sigma_q^2$ and since $T_2 \times M$ is an $\alpha_M$-abstraction of $T_1 \times M$, Proposition 35 allows us to conclude that $T_2 \times M$ is a sound $\alpha_M$-abstraction of $T_1 \times M$.

We give here the (partial) counter-example mentioned in Remark 5.
Example 11. We illustrate Remark 5 by exhibiting an example where soundness (w.r.t. a fixed distribution) as well as decisiveness properties do not transfer to the product with a deterministic Muller automaton.

Consider the DMC $T_1$ depicted on the left of Figure 10 which corresponds to the random walk over $\mathbb{N}$ from Example 1 when $p = 2/3$. Consider also the finite MC $T_2$ on the right of the same figure. Clearly enough, $T_2$ is an $\alpha$-abstraction of $T_1$ for the mapping $\alpha : \mathbb{N} \to \{s_0, s_1, s_2\}$ defined as follows: $\alpha(0) = s_0$, $\alpha(1) = s_1$ and $\alpha(i) = s_2$ for any $i \geq 2$.

Define $\mu = \delta_0$ as the initial distribution in $T_1$. For any $B \subseteq \mathbb{N}$, $\text{Prob}_\mu(T_1(B) \supseteq \emptyset) = 1$ and it follows that $T_2$ is a $\mu$-sound $\alpha$-abstraction of $T_1$. It should be noted that it is however not sound when considering $\mu' = \delta_1$ as initial distribution. Indeed, $\text{Prob}_{\mu'}(T_1(\emptyset)) < 1$ though $\text{Prob}_{\delta_1}(T_2(\emptyset)) = 1$ (and $\delta_0 = \alpha(\mu', s_0))$.

![Fig. 10. Left, $T_1$ a random walk over $\mathbb{N}$ and right, its sound finite abstraction $T_2$.](image)

Consider now the Muller automaton of Section 2 on the left of Figure 2. As stated in Example 11, when $p = 2/3$, $T_1$ is not $\mu$-sound for $T_1 \times \mathcal{M}$. This proves that $T_2 \times \mathcal{M}$ is not $(\mu \times \delta_0)$-sound for $T_1 \times \mathcal{M}$.

Now, observe that $T_1$ is decisive w.r.t. any set of states $B \subseteq \mathbb{N}$ from $\mu$ as we have seen that $\text{Prob}_\mu(T_1(B) \supseteq \emptyset) = 1$ for any set of states $B$. It should be noted that $T_2$ is not decisive by considering $\mu'$ as the initial distribution and $B = \{0\}$. In this case, $\{0\} = \emptyset$ and thus $\text{Prob}_{\mu'}(T_2 \times \mathcal{M}(\emptyset)) < 1$. Consider now $T_1 \times \mathcal{M}$, we have already shown that $\text{Prob}_{\mu \times \delta_0}(\emptyset) < 1$. It can be established that $\{0, q_2\} = (2N + 1) \times \{q_0, q_2\} \cup 2N \times \{q_1\}$ which are states not reachable from $(0, q_0)$. We deduce that $\text{Prob}_{\mu \times \delta_0}(\emptyset) < 1$. This shows that $T_1 \times \mathcal{M}$ is not decisive w.r.t. $\{0, q_2\}$ from $\mu \times \delta_0$. 
