The Rigged Hilbert Space Formulation of Quantum Mechanics and its Implications for Irreversibility

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Abstract

Quantum mechanics in the Rigged Hilbert Space formulation describes quasistationary phenomena mathematically rigorously in terms of Gamow vectors. We show that these vectors exhibit microphysical irreversibility, related to an intrinsic quantum mechanical arrow of time, which states that preparation of a state has to precede the registration of an observable in this state. Moreover, the Rigged Hilbert Space formalism allows the derivation of an exact golden rule describing the transition of a pure Gamow state into a mixture of interaction-free decay products.
Introduction

The idea of using the Rigged Hilbert Space formulation to describe quantum mechanics already occurred in 1966 in [1] and was later elaborated in [2] and [3]. This formalism recaptures all standard results of quantum mechanics and makes the mathematical theory of Dirac’s bras and kets rigorous. Unexpectedly it was found [3, 4] that it also allows the description of resonances and decaying states. Resonances in scattering theory are usually defined as the poles of the analytically continued $S$-matrix. The Rigged Hilbert Space provides a means to describe these states by means of vectors. We call these vectors Gamow vectors. Their time evolution is given by a semigroup instead of the usual unitary one-parameter group [4, 5]. This particular feature made people aware of a connection between Gamow vectors and irreversibility. For example Antoniou, Prigogine et. al. who tried to obtain intrinsic irreversibility on the quantum level [6] advocated the use of this method to explain irreversibility on the microphysical level [7]. It finally could be shown in [8] that the occurrence of the two semigroups is linked to a quantum mechanical arrow of time which stems from the fact, that states must be prepared before observables can be measured in them.

In section 1 we briefly describe the Rigged Hilbert Space formalism of quantum mechanics. A more detailed presentation is provided by the above references, e. g. [5].

Section 2 is devoted to the derivation of a quantum mechanical arrow of time from the time shift between the preparation and registration procedure. This quantum mechanical arrow of time is based on an earlier arrow of time formulated by Ludwig and based on the Bohr-Ludwig interpretation of quantum mechanics [9] which however was only formulated for the experimental preparation and registration apparatuses and not incorporated into quantum mechanics. The reason for this was, that the standard quantum mechanics in Hilbert space is reversible and cannot accommodate an arrow of time.

In section 3.1 a mathematical formulation of the quantum mechanical arrow of time for general scattering experiments is given. We will cast it in a form which reappears in section 3.2, where the quantum mechanical arrow of time in connection with Gamow vectors is discussed. Finally, we consider decaying states and show how these results lead to an exact golden rule.

1 The Rigged Hilbert Space Formalism of Quantum Mechanics

By considering different topologies on a given linear space, we can produce various complete topological spaces. This idea is behind the Rigged Hilbert Space (R.H.S.)
formalism, which uses Gel’fand triplets of the form
\[ \Phi \subset \mathcal{H} \subset \Phi^\times, \]  
(1)
where \( \mathcal{H} \) represents a Hilbert space with the well-known Hilbert space topology \( \tau_\mathcal{H} \) and \( \Phi \) a subspace with a nuclear topology \( \tau_\Phi \). This topology is chosen to be stronger than the Hilbert space topology.

Furthermore, \( \Phi^\times \) is the conjugate space of \( \Phi \), i.e., the space of all \( \tau_\Phi \)-continuous antilinear functionals \( F(\phi), \phi \in \Phi \), in the following denoted by \( F(\phi) = \langle \phi|F \rangle \).

Notice, that \( \langle \phi|F \rangle \) with \( F \in \Phi^\times \) is an extension of the ordinary Hilbert space scalar product \( (\phi,f) \) for \( f \in \mathcal{H} \cong \mathcal{H}^\times \).

The construction of the above triplet is such that \( \Phi \) is \( \tau_\mathcal{H} \)-dense in \( \mathcal{H} \). The space \( \Phi \) will be used later to define the physical states.

In order to describe observables, we need the following triplet of linear operators
\[ A^\dagger|_\Phi \subset A^\dagger \subset A^\times, \]
(2)
where \( A \) has to be a \( \tau_\Phi \)-continuous operator in \( \Phi \) but is not a closed operator in \( \mathcal{H} \). The operator \( A^\dagger \) is the Hilbert space adjoint and \( A^\times \) the closure of \( A \). In general \( A^\dagger \) and \( A^\times \) are not \( \tau_\mathcal{H} \)-continuous. \( A^\times \), defined by
\[ \langle A^\times|F \rangle = \langle A^\dagger|F \rangle \quad \text{for all } \phi \in \Phi \text{ and } |F \rangle \in \Phi^\times \]
(3)
is a continuous operator in \( \Phi^\times \), called the conjugate operator of \( A \). Thus \( A^\times \) is defined as the extension of \( A^\dagger \) to \( \Phi^\times \). In this formalism, observables are represented by elements of an algebra of \( \tau_\Phi \)-continuous operators whereas in the usual Hilbert space formulation they are given by linear (unbounded) operators defined on \( \mathcal{H} \). In the Hilbert space formulation, of quantum mechanics (pure) states are described by elements of \( \mathcal{H} \); in the Rigged Hilbert Space formulation only elements of \( \Phi \) represent physically preparable states.

Generalized eigenvectors \( F_\omega \) of the operator \( A \) with eigenvalue \( \omega \) are defined by
\[ \langle A^\times|F_\omega \rangle = \omega \langle \phi|F_\omega \rangle \quad \text{for all } \phi \in \Phi, \, F \in \Phi^\times. \]
(4)
The Dirac kets are generalized eigenvectors in this sense and the expansion of a physical state vector \( \phi \) for an observable \( H \) in terms of a basis system of eigenkets, is given by the generalized basis vector expansion which Dirac already used but which was proven only by the Nuclear Spectral Theorem [10]
\[ \phi = \int_0^\infty dE|E > \langle E|\phi > + \sum_{E_n} |E_n)(E_n|\phi) \quad \text{for every } \phi \in \Phi^\times, \]
(5)
where
\[ H|E_n) = E_n|E_n), \quad H^\times|E > = E|E > \]
(6)
with $E_n = -|E_n|$ being the discrete eigenvalues and $E$ elements of the absolutely continuous spectrum of $H$.

The R.H.S.-formalism of quantum mechanics has several advantages in comparison with the ordinary Hilbert space formalism:

1. The topology of the space $\Phi$ excludes states with infinite energy which are elements of $\mathcal{H}$.

2. Wave functions $\phi(E)$ — omitting in this notation the dependence on additional quantum numbers — are always smooth functions. In the Hilbert space formulation, the wave function is any function $h(E)$ in the class of Lebesgue square integrable functions. As $|\phi(E)|$ describes apparatus resolution, the Rigged Hilbert Space formulation is closer to the experimental situation.

3. In contrast to the Hilbert space formulation, the algebra of observables is in the R.H.S.formalism represented by continuous operators and every essentially self-adjoint observable has a complete set of generalized eigenvectors in $\Phi^\times$.

4. In addition to the Dirac kets, the R.H.S. contains generalized eigenvectors with complex eigenvalues called Gamow vectors, which are appropriate for the description of resonances and decaying states.

5. The Gamow vectors which are generalized eigenvectors of the self-adjoint energy operator with complex eigenvalues have a Breit-Wigner energy distribution as necessary for the analysis of resonance scattering experiments and are associated to the resonance poles of the $S$-matrix.

6. The time-evolution of the Gamow vectors is exactly exponential which is the basis of microphysical irreversibility.

It is shown in [5, p. 48], that

$$L^2(\mathbb{R}) = \mathcal{H}_+^2 \oplus \mathcal{H}_-^2$$

(7)

where $\mathcal{H}_+^2$ (respectively $\mathcal{H}_-^2$) represents the space of Hardy class functions from above (respectively below).

Defining $\Delta := \mathcal{H}_\pm \cap S$, where $S$ is the Schwartz space, we obtain two Gel’fand triplets

$$\Delta_\pm \subset \mathcal{H}_\pm \subset (\Delta_\pm)^\times.$$  

(8)

By a suitable (non-unitary) [3, p. 69] map these triplets define a pair of R.H.S.s

$$\Phi_\pm \subset \mathcal{H} \subset (\Phi_\pm)^\times.$$  

(9)

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\(^1\mathcal{H}_+^p\) consists of all complex analytic functions $G_+(E+iy)$ defined on the upper half of the complex plane which are $p$-integrable for any fixed $y > 0$ and for which \(\sup_{y>0} \int_{-\infty}^{\infty} |G(E+iy)|^p < \infty\). Their boundary values on the real line exist for almost all $E$ and is $p$-integrable. $\mathcal{H}_-^p$ is analogously defined on the lower half plane.
The physical interpretation of these two triplets will be the following: $\Phi_-$ is the space of the preparations, i.e. the physical in-states, $\Phi_+$ is the space of the registrations, i.e. the detected out-states of the scattering experiment.

It can be shown, that the restriction of the unitary time evolution operator $U(t) = e^{iHt}$ to $\Phi_\pm$ is a $\tau_\Phi$-continuous operator with $U(t)\Phi_+ \subset \Phi_+$ only for $t \geq 0$ and analogously a $\tau_\Phi$-continuous operator with $U(t)\Phi_- \subset \Phi_-$ only for $t \leq 0$. This results in a pair of evolution semigroups instead of the ordinary unitary evolution group $U(t)$ which is the mathematical manifestation of irreversibility.

2 Quantum Mechanics and the Arrow of Time

In the conventional description of quantum mechanics irreversibility is not accounted for intrinsically and the time evolution is given by the unitary one-parameter group of operators $U(t) = e^{iHt}$ which is generated by the Hamiltonian $H$ and well-defined for $-\infty < t < \infty$. Instead, an arrow of time is considered to be a result of an irreversible act of measurement and was postulated by von Neumann by a formalism which is known as the “collapse of the wave function” [11, 12]:

In usual quantum mechanics a state $W$ changes according to

$$W(t) = U^\dagger(t)W(0)U(t)$$

where $U(t) = e^{iHt}$ is a unitary one-parameter group of operators generated by the Hamiltonian $H$. As the time evolution is unitary, there is no arrow of time and we have

$$S[W(t)] \equiv -k\text{Tr}[W(t)\ln(W(t))] = -k\text{Tr}[W(0)\ln(W(0))] = S[W(0)].$$

Now, according to the postulate of the collapse of the wave function, the state changes as a result of an “idealized measurement of the first kind” such that the state $W$ in which the observable $B = \sum_i b_i E_{b_i}$ is measured collapses into $W'$ according to

$$W \overset{\text{collapse}}{\rightarrow} W' = \sum_{b_i} E_{b_i} W E_{b_i} \overset{\text{reading of result } b_f}{\rightarrow} E_{b_f} W E_{b_f}.$$  

Only $S[W'] \geq S[W]$ can be shown, and the measurement is considered to be the cause of irreversibility. It is important to notice that the “collapse of the wave function” is a mathematical idealization expressing the fact that measurement affects the states and that an immediate repetition of a measurement should lead to the same result.

The arrow of time we will present in the following is of a different nature and is located on the microphysical level. It manifests itself as exponential decay and can be formulated in terms of states and observables. The same idea has already been used by Ludwig [9] who however formulated this arrow of time in terms of an experimental apparatus only and extrapolated the time evolution into the past when he transcribed
the observational facts into the mathematical theory. This was necessary because his Hilbert space formulation of quantum mechanics does not allow for an intrinsic quantum mechanical arrow of time according to the Bohr-Ludwig interpretation of quantum mechanics. The division of an experiment into a preparation apparatus, which prepares physical states for the experiment, and a registration apparatus, which measures properties of the microsystem, is crucial. The microsystem itself is the agent by which the preparation apparatus acts on the registration apparatus. Whereas the state was prepared by the preparation apparatus the measurements of the observables $F$ is done by the registration apparatus and the expectation value represents the average value of an ensemble of measurements:

$$\text{Tr}(WF) = \sum_{\phi} (\phi, F\phi) = \sum_{\phi} <\phi|\psi><\psi|\phi>.$$  \hspace{1cm} (13)

Here, we choose the special case that the statistical operator $W$ is given by the vector $\phi$ describing a pure state and the self-adjoint operator $F$ is the projection operator $|\psi><\psi|$ describing the special observable called decision observable or property.

Whereas the apparatuses are not described by quantum mechanics, the measured value obtained by the measuring apparatus is predicted by quantum theory. We observe the dynamics of the microsystem as the time translation of the registration procedure relative to the preparation procedure. Thus, if a time direction is distinguished for the time translation of the registration apparatus relative to the preparation apparatus then this is a distinguished direction for the time evolution of the microphysical system described by quantum mechanics. In terms of the apparatus it is possible to describe the arrow of time independently of a mathematical theory. We call this arrow the preparation $\rightarrow$ registration arrow and characterize it by the following statement [9]:

Time translation of the registration apparatus relative to the preparation apparatus makes sense only by an amount $\tau \geq 0$.

Translated into the quantum mechanical notions of the mathematical theory, this arrow is formulated as:

An observable $|\psi(\tau)><\psi(\tau)|$ — in particular a projection operator — can be measured in a state $\phi = \phi(0)$ only after the state has been prepared, i.e. for $\tau \geq 0$.

This is the formulation in the Heisenberg picture, in the Schrödinger picture it is given as:

The state $\phi(t)$ must be prepared before an observable $|\psi><\psi| = |\psi(0)><\psi(0)|$ can be measured in that state, i.e. at $t \leq 0$. 
3 Mathematical Formulation of the Quantum Mechanical Arrow of Time in a Resonance Scattering Experiment

3.1 Resonances of the $S$-Matrix and Gamow Vectors

The design of a scattering experiment is the following: A mixture of initial states $\phi^{in}$ is prepared and evolves according to the free Hamiltonian $K$

$$\phi^{in}(t) = e^{-iKt/\hbar}\phi^{in}. \tag{14}$$

The evolution through the interaction region is given by the Hamiltonian $H = V + K$. The result is a state $\phi^{out}$,

$$\phi^{out}(t) = S\phi^{in}(t), \quad S = \Omega^{-\dagger}\Omega^{+} \tag{15}$$

where

$$\Omega^{+}\phi^{in}(t) \equiv \phi^{+}(t) = e^{-iHt/\hbar}\phi^{+} = \Omega^{-}\phi^{out}(t) \tag{16}$$

$$\Omega^{-}\psi^{out}(t) \equiv \psi^{-}(t) = e^{-iHt/\hbar}\psi^{-}, \tag{17}$$

and $\Omega^{+}$ and $\Omega^{-}$ are the Møller wave operators. The state vector $\phi^{in}$ is determined by the preparation apparatus. The state vector $\phi^{out}$ is determined by the preparation apparatus and the dynamics. The detector does not register $\phi^{out}$, but a property $|\psi^{out}><\psi^{out}|$, where $\psi^{out}$ is determined by the registration apparatus. The probability to measure the observable $|\psi^{out}><\psi^{out}|$ in the state $\phi^{in}$ is then given by $|<\phi^{in}|\psi^{out}>|^2$ which is the modulus square of the $S$-matrix.

The entries of the $S$-matrix are given by the elements

$$\left(\psi^{out}(t), \phi^{out}(t)\right) = \left(\psi^{out}(t), S\phi^{in}(t)\right), \tag{18}$$
which, for all $t$, can be given by

$$
(\psi^{\text{out}}(t), S\phi^{\text{in}}(t)) = (\Omega^{-}\psi^{\text{out}}(t), \Omega^{+}\phi^{\text{in}}(t))
= (\psi^{-}(t), \phi^{+}(t))
= \int_{\sigma(H)} dE <\psi^{-}|E^{-}> S_{I}(E + i0) < +E|\phi^{+} > .
$$

Figure 1: Paths of integration

Here $S_{I}(E + i0)$ indicates integration along the upper rim of the cut along the real axis in the first sheet. Since $S_{I}(E + i0) = S_{II}(E - i0)$ this integration along that sheet can be realized by the integration over the lower rim of the cut in the second sheet. The + preceding $E$ in $< +E|$ and the − following $E$ in $|E^{-} >$ underline this fact.

In order to define now the Gamow vector in terms of the $S$-matrix we consider the simplest model and we make the following assumptions (if there are more resonance
poles and also bound states the arguments are easily generalized):

- There are no bound or virtual states.

- The spectrum $\sigma(H)$ of $H = K + V$ is given by the positive real axis which implies that there is a cut from 0 to $\infty$ on the Riemann surface for the analytic continuation $S(z)$ of the $S(E)$-matrix.

- There is a pair of poles on the second sheet of the complex energy surface of the $S$-matrix $S(z)$ at the pole position:

$$z_R = E_R - i \frac{\Gamma}{2}, \quad z^*_R = E_R + i \frac{\Gamma}{2}. \quad (20)$$

Physically, the poles in the second sheet of the $S$-matrix are associated with resonances. We will associate to these pole positions autonomous resonance states and we call the corresponding generalized state vectors the Gamow vectors. To accomplish this we deform the contour of integration in (19) into the second sheet and obtain

$$\left< \psi^- | \phi^+ \right> = \int_{C_-} dz < \psi^- | \omega^- > S_{II}(z) < +z | \phi^+ > + \oint dz < \psi^- | z^- > \frac{s_{-1}}{z - z_R} < +z | \phi^+ > \quad (21)$$

where $s_{-1} = i \Gamma$ is the residuum of $S_{II}(z)$ at $z_R$ and $C_-$ is the curve in the lower second sheet which is shown in figure 1. We also have assumed here, that $< \psi^- | z^- > = < -z | \psi^- >^*$ and $< +z | \phi^+ >$ are analytic functions in the lower half plane of the second sheet of the Riemann surface. As the first integral is not related to the resonance we will ignore this term in the following.

The second integral in (21) can now be evaluated by means of the Cauchy Theorem, which yields

$$< \psi^- | z_R^+ > (2\pi \Gamma) < +z_R | \phi^+ > = \oint dz < \psi^- | z^- > < +z | \phi^+ > \frac{i \Gamma}{z - z_R} \quad (22)$$

We now demand further, that the second integral of (21) is given by a Breit-Wigner integral, i.e. that (22) takes the form

$$< \psi^- | z_R^- > < +z_R | \phi^+ > 2\pi \Gamma = \int_{-\infty}^{+\infty} dE < \psi^- | E^- > < +E | \phi^+ > \frac{i \Gamma}{E - (E_R - i \frac{\Gamma}{2})} \quad (23)$$

The reason for this requirement is that we want to associate the resonance “state” vector with the pole of the $S$-matrix and the resonance state vector should have a Breit-Wigner energy distribution.
In order to deform the contour of integration from the one in (22) into an integration from \(-\infty\) to \(+\infty\) as in (23), we need to make the following requirement of the energy wave functions:

\[
< E|\psi^{out} > = < E|\psi^- > \in S \cap \mathcal{H}^2_+ \quad \text{or} \quad < \psi^-|E^- > \in S \cap \mathcal{H}^2_- \quad (24)
\]

\[
< E|\phi^{in} > = < +E|\phi^+ > \in S \cap \mathcal{H}^2_- \quad \text{or} \quad < \phi^+|E^+ > \in S \cap \mathcal{H}^2_+ \quad (25)
\]

because then \(< \psi^-|E^- > < +E|\phi^+ > \in S \cap \mathcal{H}^2_+ \) and (23) is a special case of the Titchmarsh theorem (cf. Appendix). In here \(\mathcal{H}^2_+\) means Hardy class from above and \(\mathcal{H}^2_-\) means Hardy class from below.

If the conditions (24) and (25) are fulfilled, we say that \(\psi^-\) is a “very well-behaved vector from above” and \(\phi^+\) a “very well-behaved vector from below” and write this as

\[
\psi^- \in \Phi_+, \quad \phi^+ \in \Phi_- \quad (26)
\]

This means the space of very well behaved vectors from below, \(\Phi_-\), describes the state vectors prepared by the preparation apparatus, and the space of very well behaved vectors from above, \(\Phi_+\), describes the observables detected by the registration apparatus. We know that a more careful investigation would show that \(\Phi_+ \cap \Phi_- \neq \{0\}\) (zero vector) \([4]\). We shall see below, that conditions (24) and (25) are exactly the mathematical statement describing the quantum mechanical arrow of time. It will thus come as no surprise that the time development of the Gamow vectors is given by two semigroups instead of a unitary evolution group.

It may seem curious that we label the vectors \(\psi^-\), \(\phi^+\) by superscripts that are opposite to the subscripts by which we label their spaces \(\Phi_+\), \(\Phi_-\). The reason for this is that the labelling of the vectors comes from the convention of scattering theory developed by physicists and the labelling of the spaces comes from the convention for the Hardy class spaces \(\mathcal{H}^2_+\) and \(\mathcal{H}^2_-\) used by mathematicians. Both subjects were developed independently and without knowing of each other. The agreement between \(\{\psi^-\}\) and \(\Phi_+\) as well as \(\{\phi^+\}\) and \(\Phi_-\) is an example of “the unreasonable effectiveness of mathematics in the natural sciences” (E. P. Wigner).

We are now able to define the Gamow vectors \(|z_R^-\rangle\) and \(|z_R^+\rangle\) associated with the poles at \(z_R\) and \(z_R^*\). In analogy to (23) we apply the Titchmarsh Theorem to the function \(< \psi^-|E^- \rangle \in \mathcal{H}^2_-\) with value \(< \psi^-|z_R^-\rangle\) at \(z_R\) to obtain:

\[
< \psi^-|z_R^-\rangle = -\frac{1}{2\pi i} \int_{\infty, I}^{-\infty, I} dE < \psi^-|E^- \rangle \frac{1}{E-z_R^-} \quad \text{for all} \quad < -E|\psi^- > \in S \cap \mathcal{H}^2_-(27)
\]

and to the function \(< \phi^+|E^+ \rangle \in \mathcal{H}^2_+\) with value \(< \phi^+|z_R^+\rangle\) at \(z_R^*\) to get:

\[
< \phi^+|z_R^+\rangle = \frac{1}{2\pi i} \int_{-\infty, I}^{\infty, I} dE < \phi^+|E^+ \rangle \frac{1}{E-z_R^+} \quad \text{for all} \quad < +E|\phi^+ > \in S \cap \mathcal{H}^2_-(28)
\]
Notice, that in equation (27), integration is taken along the lower rim on the second sheet whereas in equation (28) along the upper rim on the second sheet.

It is clear that \(|z_{R}^{−}\rangle\) defined by (27) is an element of \(\Phi_{+}^{}\) and \(|z_{R}^{+}\rangle\) defined by (28) is an element of \(\Phi_{−}^{}\). These generalized vectors (functionals) are the two kinds of Gamow vectors. It is crucial that the integration in (27) and (28) extends from \(-\infty\) to \(+\infty\) though the spectrum of \(H\) is bounded from below.

Replacing \(\psi^{-}\) by \(H\psi^{-}\) (respectively \(\phi^{+}\) by \(H\phi^{+}\)) in equation (27) (respectively equation (28)) and using \(<H\psi^{-}|E^{−}>=E<\psi^{-}|E^{−}>(\text{respectively}<H\phi^{+}|E^{+}>=E<\phi^{+}|E^{+}>)\) we can infer that the Gamow vector \(|z_{R}^{−}\rangle\) (respectively \(|z_{R}^{+}\rangle\)) is a generalized eigenvector of \(H\) with the eigenvalue \(z_{R}\) (respectively \(z_{R}^{*}\)).

\[
\begin{align*}
< H\psi^{-} | z_{R}^{−} ⟩ &= \psi^{-} | H^{−} | z_{R}^{−} ⟩ = z_{R} < \psi^{-} | z_{R}^{−} ⟩ & (29) \\
\text{or } H^{−} | z_{R}^{−} ⟩ &= z_{R} | z_{R}^{−} ⟩; | z_{R}^{−} ⟩ ∈ \Phi_{+}^{}. \\
< H\phi^{+} | z_{R}^{+} ⟩ &= \phi^{+} | H^{−} | z_{R}^{+} ⟩ = z_{R}^{*} < \phi^{+} | z_{R}^{+} ⟩ & (30) \\
\text{or } H^{−} | z_{R}^{+} ⟩ &= z_{R}^{*} | z_{R}^{+} ⟩; | z_{R}^{+} ⟩ ∈ \Phi_{−}^{.}
\end{align*}
\]

### 3.2 Time Evolution of the Gamow Vectors

In the usual Hilbert space \(\mathcal{H}\), the time evolution, given by \(e^{iHt}\), is a continuous operator for all \(t \in \mathbb{R}\). The time evolution operators of the Gamow vectors \(|z_{R}^{−}\rangle \in Φ^{+}\) and \(|z_{R}^{+}\rangle \in Φ^{−}\) are the conjugate operators of \(U(t)|_{Φ^{+}} = e^{iHt}|_{Φ^{+}}\) and \(U(t)|_{Φ^{−}} = e^{iHt}|_{Φ^{−}}\), respectively. The conjugate operators can only be defined if the operators themselves are continuous operators with respect to the topology in \(Φ\). In the case of \(U(t)|_{Φ^{+}}\) this is only the case for \(t ≥ 0\) and we denote the conjugate by \(\overline{e_{−}^{−iHt}}\).

On the other hand the operator \(U(t)|_{Φ^{−}}\) is a continuous operator in \(Φ\) only for \(t ≤ 0\) and here, the conjugate is denoted by \(\overline{e_{−}^{−iHt}}\). Thus, \(\overline{e_{−}^{−iHt}}\) exists only for \(t ≥ 0\) and \(\overline{e_{+}^{−iHt}}\) exists only for \(t ≤ 0\).

Now, the time evolution of the Gamow vectors \(|z_{R}^{−}\rangle \in Φ^{+}\) can be calculated to be

\[
\begin{align*}
\langle e^{iHt}\psi^{-} | z_{R}^{−} ⟩ &≡ \langle \psi^{-} | \overline{e_{+}^{−iHt}} | z_{R}^{−} ⟩ \\
&= e^{-iE_{R}t}e^{\Sigma t} \langle \psi^{-} | z_{R}^{−} ⟩ \text{ for all } \psi^{-} \in Φ^{+}, \text{and for } t ≥ 0 \quad (31)
\end{align*}
\]

and the time evolution of \(|z_{R}^{+}\rangle\) is calculated to be:

\[
\begin{align*}
\langle e^{iHt}\phi^{+} | z_{R}^{+} ⟩ &≡ \langle \phi^{+} | \overline{e_{−}^{−iHt}} | z_{R}^{+} ⟩ \\
&= e^{-iE_{R}t}e^{\Sigma t} \langle \phi^{+} | z_{R}^{+} ⟩ \text{ for all } \phi^{+} \in Φ^{−}, \text{and for } t ≤ 0 \quad (32)
\end{align*}
\]

Thus the vector \(|z_{R}^{−}\rangle\) is an exponentially decaying state vector. If \(\langle \psi^{-} | z_{R}^{−} ⟩^{2}\) is the probability of finding the Gamow state \(|z_{R}^{−}\rangle\) with the detector that registers the
property $|\psi^-><\psi^-|$ at $t = 0$ then

$$|<\psi^-| e^{-iHt}|z_R^->|^2 = e^{-\Gamma t} |<\psi^-|z_R^->|^2$$

(33)

is the probability to find this Gamow state at the time $t$. Thus the probability decreases exponentially towards 0 as $t$ grows. Analogously, the increasing state $|z_R^{*+}>$ is defined for $t \leq 0$ with probability increasing exponentially from 0 in the distant past to $|<\phi^+|z_R^{*+}>|^2$ at $t = 0$.

This behaviour can be summarized by saying that $|z_R^->$ and $|z_R^{*+}>$ have an intrinsic arrow of time which evolves only from 0 to $\infty$ or from $-\infty$ to 0, i.e. $|z_R^->$ does not grow and $|z_R^{*+}>$ does not decay.

Thus a microphysical system, described by such a state vector, is equipped with an arrow of time.

### 3.3 A New Generalized Basis Vector Expansion

The basis vector expansions, of which the Nuclear Spectral Theorem is an example, provide very important tools for quantum mechanics. They are generalizations of the well known basis vector expansion $\vec{X} = \sum_{i=1}^{3} x^i \vec{e}_i$ in the three dimensional Euclidean space $\mathbb{R}^3$. In the $N$-dimensional complex space this expansion in terms of eigenvectors $e_i, i = 1, \ldots, N$ of any self-adjoint linear operator $H$ is called the fundamental theorem of the linear algebra. In the infinite dimensional Hilbert space $\mathcal{H}$ this expansion

$$\mathcal{H} \ni h = \sum_{n=1}^{\infty} \sum_b |E_n, b)(b, E_n|h); \ |E_n, b) = e_i \in \mathcal{H}$$

(34)

is only correct for a subset of self-adjoint operators $H$ (compact). Here we have called $b$ the degeneracy index: $H|E_{n_j}, b) = E_n|E_{n_j}, b)$. For any arbitrary self-adjoint operator a generalization of holds, which is given by the fundamental Nuclear Spectral Theorem of the generalized Dirac basis vector expansion (5). (To take degeneracy into account the label $b$ should be added: $|E, b>$ and a sum should be taken over all values of the quantum numbers $b$.) The set of generalized eigenvalues $\{E_n, E\}$ is called the spectrum (we have assumed that the spectrum of $H$ was absolutely continuous and discrete).

With the help of the Gamow vectors we can generalize the Dirac basis vector expansion further. For a self-adjoint Hamiltonian $H = K + V$ for which there are only $N$ simple poles of the $S$-matrix at $z_{R_i} = E_i - i\frac{\Gamma_i}{2}, i = 1, \ldots, N$ one obtains (from a generalization of equation (21) from 1 to $N$ poles) a new generalized basis vector expansion

$$\phi^+ = \sum_{i=1}^{N} \sum_b |z_{R_i}^-, b > (2\pi \Gamma_i) < +b, z_{R_i}|\phi^+ >$$

$$+ \int_{-\infty}^{0} \sum_b dE |E, b^+>< +b, E|\phi^+ > \text{ for every } \phi^+ \in \Phi_-.$$ 

(35)
We have ignored the sum over the discrete spectrum \( \sum_{n=1}^{\infty} |E_n)(E_n| \phi^+ \), corresponding to bound states.

In contrast to Dirac's basis vector expansion (3) which holds for every \( \phi \in \Phi \), the new basis vector expansion (35) holds only for \( \phi^+ \in \Phi_+ \subset \Phi = \Phi_+ + \Phi_- \) and contains generalized eigenvectors \( |z_R^- \rangle \in \Phi^- \). It means that a vector \( \phi^+ \) representing a state (ensemble) prepared by a preparation apparatus of a scattering experiment can be expanded into a superposition of Gamow vectors representing exponentially decaying resonance "states" and a background term

\[
\phi^+_{bg} = \int_{0}^{-\infty} \sum_b dE |E, b^+ > \langle +b, E| \phi^+ >; < +b, E| \phi^+ > \in \mathcal{H}_2^2 \cap \mathcal{S}.
\]  

(36)

The integration in the background term is taken in the second sheet along the negative real axis (and the values of \( < +E| \phi^+ > \) can be calculated from the physical values \( < +E| \phi^+ > = < E| \phi^{in} > \) with \( 0 \leq E < +\infty \) given by the energy distribution of the incoming beam using the von Winter Theorem (3) and could have been replaced by integration along other equivalent contours. The result (35) shows that resonance "states" can take a very similar position as bound states in the basis vector expansion of states \( \phi^+ \in \Phi_- \) prepared by a scattering experiment. These state vectors \( \phi^+ \) can indeed be given by a superposition of exponentially decaying resonance states

\[ \psi_i^D = |z_R^- > \sqrt{2\pi \Gamma_i} \]  

(38)

3.4 The Quantum Mechanical Arrow of Time and Resonance Scattering

We now want to discuss the connection between the Gamow vector’s intrinsic arrow of time and the more general quantum mechanical arrow of time that we had introduced in section 2 and which was nothing else but a general expression of causality. For this purpose, we apply that general arrow of section 2 to the resonance scattering experiment.

We choose as \( t = 0 \) the point in time at which the preparation of the state \( \phi^{in}(t) \) is completed and after which the registration of \( |\psi^{out} > < \psi^{out}| \) begins. Since no preparations take place for \( t > 0 \), the energy distribution given by the preparation apparatus must vanish and we require therefore that \( < E| \phi^{in}(t) > = 0 \) for \( t > 0 \) and for all physical values of \( E \). And since as a general result of scattering theory

\[
< E| \phi^{in} > = < +E| \phi^+ > \quad \text{and} \quad < E| \psi^{out} > = < -E| \psi^- >,
\]  

(37)

where \( |E > \) and \( |E^\pm > \) are defined by the Lippman-Schwinger equation

\[
|E^\pm > = |E > \pm \frac{1}{E - H \pm i\Omega} V|E > = \Omega^+|E >
\]  

(38)
and by $K|E > = E|E >$ and $H|E^\pm = E|E^\pm >$, we formulate this requirement of the quantum mechanical arrow of time as:

$$0 = \int_{-\infty}^{\infty} dE <^+ E|\phi^+ (t) > = \int_{-\infty}^{\infty} dE <^+ E|e^{-iEt/\hbar}|\phi^+ >$$

$$= \int_{-\infty}^{\infty} dE <^+ E|\phi^+ > e^{-iEt/\hbar} \equiv \mathcal{F}(t) \quad \text{for } t > 0.$$  \hspace{1cm} (39)

Notice that $\mathcal{F}$ is the Fourier transform of the energy wave function $<^+ E|\phi^+ > = < E|\phi^{in} >$.

Similarly, since all registrations take place after $t = 0$, we have

$$0 = \int_{-\infty}^{\infty} dE <E|\psi_{out} (t) >$$

$$= \int_{-\infty}^{\infty} dE <E|\psi^- > e^{-iEt/\hbar} \equiv \mathcal{G}(t) \quad \text{for } t < 0,$$  \hspace{1cm} (40)

where $\mathcal{G}(t)$ is the Fourier transform of $<- E|\psi^- >$ (here, $-$ in $<- E|$ refers to integration along the lower rim of the first sheet).

The equations (39) and (40) are the mathematical formulations of the general quantum mechanical arrow of time, namely of “no preparations for $t > 0$” and of “no registrations for $t < 0$”, respectively. From the mathematical statements (39) and (40) one can derive the mathematical properties of the spaces $\Phi_-$ and $\Phi_+$ in the following way: As $<^+ E|\phi^+ >$ as well as $<- E|\psi^- >$ are the energy distributions of the experimental apparatuses, they are supposed to be smooth, well-behaved functions of $E$. Thus, the Theorem of Paley-Wiener is applicable, which says that a square integrable function $G_+(E)$ (respectively $G_-(E)$) belongs to $\mathcal{H}_2^+$ (respectively $\mathcal{H}_2^-$) if and only if it is the Fourier transform of a square integrable function which vanishes on the interval $(0, \infty)$ (respectively $(-\infty, 0)$). In our situation this yields

$$<- E|\psi^- > \in \mathcal{H}_2^+ \cap S \quad \text{(or } \psi^- \in \Phi_+)$$

$$<^+ E|\phi^+ > \in \mathcal{H}_2^+ \cap S \quad \text{(or } \phi^+ \in \Phi_-).$$  \hspace{1cm} (41) (42)

These are precisely the conditions (24) and (25) which we obtained above from the requirement that the pole term of the $S$-matrix has a Breit-Wigner energy distribution. This means that we could have taken the quantum mechanical arrow of time in its mathematical formulation (39) and (40) as the starting point and derived from it the Rigged Hilbert Spaces $\Phi_+ \subset \mathcal{H} \subset \Phi_+^\times$ of observables (or out-“states” $\psi^-$) and $\Phi_- \subset \mathcal{H} \subset \Phi_-^\times$ of states (or in-states $\phi^+$) and their Gamow vectors $|z_R^- >$ and $z_R^+ >$ respectively.

### 4 Exact Golden Rule

Decaying states can be considered as resonances for which the production process is ignored. As an example one can think of the radiative transitions of excited atoms
to lower states $A$ by $A^* \rightarrow A + \gamma$. Whereas stationary states are rare in physics, decaying states are numerous and thus constitute an important part of physics.

If the decaying system is isolated, its state $W(t)$ evolves in time according to the exact Hamiltonian $H$ and is given by

$$W(t) = e^{-iHt}W(0)e^{iHt}, \quad \text{with } H = K + V \geq 0.$$  \hspace{1cm} (43)

As registration apparatus we choose detectors which surround $A^*$. We suppose that the decay products are far enough from each other so that they do not interact after the decay. Then the observables are projection operators $\Lambda$ (or positive operators) on the space of physical states of the decay products and are given by

$$\Lambda = \int_0^\infty dE \sum_b |E, b><E, b|$$  \hspace{1cm} (44)

where $|E, b>$ are eigenvectors of $K$ with eigenvalue $E$ and not eigenvectors of $H$.

Denote by $\mathcal{P}(t)$ the transition probability, i. e. the probability to find the decay product $\Lambda$ in the state $W(t)$. Then $\mathcal{P}(t)$ is given by

$$\mathcal{P}(t) = \text{Tr}(\Lambda W(t)).$$  \hspace{1cm} (45)

Inserting for $W(t) = |\psi_t><\psi_t|$ with $\psi_t \in \mathcal{H}$ one can show that $\mathcal{P}(t)$ is identically zero for all $t$ if it was so for some time interval in the past \[\text{[13]}\]. For the derivative of $\mathcal{P}(t)$ at $t = 0$, i. e. for the initial decay rate $\dot{\mathcal{P}}(t = 0)$ however, the standard treatment \[\text{[14]}\] leads to an approximate formula given by the Golden Rule, which shows that $\mathcal{P}(t = 0) \neq 0$ as it should be. Since nothing more than the most fundamental assumptions of Hilbert space quantum mechanics enter in this derivation we have to conclude that the transition probability cannot be derived in the framework of the Hilbert space formulation in a consistent way.

In contrast to this, in the R.H.S.-formulation one can derive an exact Golden Rule for $\mathcal{P}(t)$ and obtain $\dot{\mathcal{P}}(t)$ as a derivative of it if one chooses for the decaying state $W(t)$ the Gamow state given by the vector $\psi^G = |z_R^- > f$ (with $f$ being a normalization constant):

$$W(t) = |\psi^G(t)><\psi^G(t)| = e^{-\Gamma t}|\psi^G><\psi^G| \text{ for } t \geq 0 \text{ only.}$$  \hspace{1cm} (46)

Then one obtains

$$\mathcal{P}(t) = 1 - e^{-\Gamma t} \int_0^\infty dE \sum_{b \neq b_G} \frac{|<E, b|V|\psi^G>|^2}{(E - E_R)^2 + \left(\frac{\Gamma}{2}\right)^2}, \text{ for } t \geq 0,$$  \hspace{1cm} (47)

and from this by differentiation:

$$\dot{\mathcal{P}}(t) = e^{-\Gamma t}2\pi \int_0^\infty dE \sum_{b \neq b_G} \frac{\Gamma}{2\pi} \frac{|<E, b|V|\psi^G>|^2}{(E - E_R)^2 + \left(\frac{\Gamma}{2}\right)^2}, \text{ for } t \geq 0.$$

$$\dot{\mathcal{P}}(t) = e^{-\Gamma t}2\pi \int_0^\infty dE \sum_{b \neq b_G} \frac{\Gamma}{2\pi} \frac{|<E, b|V|\psi^G>|^2}{(E - E_R)^2 + \left(\frac{\Gamma}{2}\right)^2}, \text{ for } t \geq 0.$$

$$\dot{\mathcal{P}}(t) = e^{-\Gamma t}2\pi \int_0^\infty dE \sum_{b \neq b_G} \frac{\Gamma}{2\pi} \frac{|<E, b|V|\psi^G>|^2}{(E - E_R)^2 + \left(\frac{\Gamma}{2}\right)^2}, \text{ for } t \geq 0.$$

$$\dot{\mathcal{P}}(t) = e^{-\Gamma t}2\pi \int_0^\infty dE \sum_{b \neq b_G} \frac{\Gamma}{2\pi} \frac{|<E, b|V|\psi^G>|^2}{(E - E_R)^2 + \left(\frac{\Gamma}{2}\right)^2}, \text{ for } t \geq 0.$$
From this one finds — using the fact that the probability to find the decay product at time $t \leq 0$ must be 0, $\mathcal{P}(0) = 0$ — that $\mathcal{P}(t) = 1 - e^{-\Gamma t}$ and $\dot{\mathcal{P}}(0) = \Gamma$ (i.e., the initial decay rate is equal to the imaginary part of the resonance pole position of the $S$-matrix, which had already been shown above in section 3.1 to be equal to the width of the Breit-Wigner energy distribution).

In the Born approximation, which is defined by

$$\psi^G \rightarrow f^d, \quad E_R \rightarrow E_d, \quad \frac{\Gamma}{2E_R} \rightarrow 0,$$

(49)

where $Kf^d = E_d f^d$ is the eigenvector of the free Hamiltonian approximating the decaying state $\psi^G$, one obtains

$$\dot{\mathcal{P}}(0) = \Gamma = 2\pi \int_0^{\infty} dE \sum_{b \neq b_G} |<E,b|V|f^d>|^2 \delta(E - E_d) .$$

(50)

This is the standard Golden Rule for the transition from the excited but non-interacting state $f^d$ into the mixture of non-interacting decay products.

5 Summary and Conclusions

Quasistationary microphysical systems and their resonances can be described by Gamow vectors which are generalized eigenvectors in a suitably chosen space of self-adjoint Hamiltonians with complex eigenvalues. The description of Gamow vectors is not possible in the Hilbert space, but the Rigged Hilbert Space allows to define Gamow vectors in $\Phi^\times$. These vectors are associated to the resonance poles of the $S$-matrix.

The time evolution of the Gamow vectors is governed by semigroups of operators in the spaces $\Phi^\pm$ and thus displays irreversibility. This is a microphysical arrow of time which is built in the microphysical systems described by the Gamow vectors. The conjugate semigroups of operators in $\Phi^\pm$ describe a general quantum mechanical arrow of time which is the theoretical description of the “preparation $\rightarrow$ registration” arrow of time of the experimental apparatuses.

This “preparation $\rightarrow$ registration” arrow of time has been known for some time, but could not be transcribed faithfully from the experimental observation into the mathematical theory of quantum mechanics in Hilbert space. Finally, with the Gamow vector to describe a decaying state, one can derive a sensible result for the transition probability into non-interacting decay products and obtain the standard Golden Rule for the transition rate as the Born approximation. The Gamow vector provides the link that was missing from the Hilbert space formulation to connect the theoretically and experimentally defined quantities for the decay phenomena.
Appendix

Theorem 1 (Titchmarsh) Let $G_{\pm}(z) \in H_{\mathbb{R}}^0$. Then, for any $z = E_0 + iy$ with $y \geq 0$, one has:

$$G_{\pm}(z) = \pm \frac{1}{2\pi i} \int_0^\infty G_{\pm}(E) \frac{1}{E - z} \text{Im}z \geq 0$$

(51)

and

$$\int_0^\infty \frac{G_{\pm}(E)}{E - z^*} \text{d}E = 0.$$  

(52)

In our case the function under consideration is $G_-(E) = <\psi^-|E^- > < E|\phi^+ >$. Similarly, we obtain

$$< \phi^+|z^*_R^+ > < z^*_R|\psi^- > =$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} dE < \phi^+|E^+ > < -E|\psi^- > \frac{1}{E-(E_R-i\frac{\chi}{2})},$$

(53)

for $z^*_R = E_R + i\frac{\chi}{2}$ and the function $G_+(E) = < \phi^+|E^+ > < -E|\psi^- >$. Notice, that here the integration takes place along the upper edge of the real axis in the second sheet.

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