CP Violation in the Decay $\eta \to \pi^+\pi^-\gamma$

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Abstract

We study the CP violating effects in the decay of $\eta \to \pi^+\pi^-\gamma$. We show that to have CP violation in the decay, one has to consider both linear and circular photon polarizations. In the standard model, the polarizations are vanishingly small. However, model-independently, i.e. using only experimental constraint imposed by the limit on $Br(\eta \to \pi^+\pi^-)$ it can be up to $O(10\%)$. We also explore various possible operators and we find that the tensor type operator, possibly arising from a nonzero CP violating electric dipole moment of the strange quark, can induce a sizable linear photon polarization.
Both CP violation (CPV) and time-reversal (T) violation (TV) have been measured experimentally in the $K^0$ system \[1\]. More recently, CPV has also been seen in another flavor-changing $B$-meson decays \[2\]. However, the origin of the violations remains unclear. In the standard model, CPV or TV arises from a unique physical phase in the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix \[3\]. It is also intimately connected to flavor changing processes. To ensure that this phase is indeed the source of CP or T violation, and to gain a deeper understanding of the phenomenon one needs to look for new CPV interactions; especially those outside the $K^0$ system. It would be particularly interesting if CP or T symmetry is also violated in flavor-conserving processes such as $\eta$ decays and the neutron electric dipole moment ($d_n$) \[4, 5\]. The information here is very limited. Thus far the efforts have been concentrated mainly in the search for the electric dipole moments (EDM) of the neutron and the electron and a few nuclear reactions (for a review see \[6\]). For EDM’s, the experimental limit of $d_n < 10^{-25} e \text{ cm}$, which is far from the standard model prediction of $O(10^{-32}) e \text{ cm}$, provides stringent constraints on various CP violating models beyond the CKM paradigm. However, it is known that it will be very hard to improve the limit for another few orders due to limitations of current technology.

In this paper we wish to investigate the use of $\eta$ to decays as probes of CPV in the flavor conserving sector. This is prompted by the possibility of producing a large number of $\eta$ mesons in high statistics experiments. This new possibility can provide us with tools to gain more knowledge on rare $\eta$ decay processes and offer unique probes of new physics. As we shall see the $\eta$ meson can yield complementary information on strangeness conserving CPV which can only be indirectly gleaned from neutron EDM studies.

To begin we will study CPV effects in

$$\eta(p) \rightarrow \pi^+(p_+)\pi^-(p_-)\gamma(k, \epsilon),$$

where $\epsilon$ is the photon polarization and obvious kinematics notation. This decay has a branching ratio of 4.75 percent and relatively simple final states. Under the Lorentz and gauge invariances, the general amplitude for the decay in Eq. (1) is given by

$$\mathcal{M} = \frac{i}{m_{\eta}^3} \{-M \varepsilon_{\mu\rho\lambda} p_+^\mu p_-^\rho \gamma^\lambda + E[\epsilon \cdot p_+)(k \cdot p_-) - (\epsilon \cdot p_-(k \cdot p_+)]\},$$

where the terms corresponding to $M$ and $E$ stand for magnetic and electric transitions, which are P-conserving and violating, respectively, since $C(\eta) = +$ and $P(\eta) = -$. We note that since $C(\pi^+\pi^-\gamma) = (-1)^{l+1}$ with $l$ being the angular momentum of the dipion, the $\pi^+\pi^-$ state has to be in an odd angular momentum state if $C$ is conserved. The diagrams that contribute to the decay in Eq. (1) are shown in Figure 1. We note that the terms associated with $M$ and $E$ in Eq. (2) are given only up to third order in momenta and $M$ and $E$ are functions of Lorentz scalars.
In the $\eta$ rest frame, without loss of generality we can choose the decay to be in the $x-z$ plane. In this frame, the photon momentum can be directed along the $z$-axis and we denote its energy by $E_\gamma$. The amplitude in Eq. (2) then becomes

$$\mathcal{M} = i m_\eta^{-2} E_\gamma [ \hat{M} \cdot (\hat{e} \times \hat{p}_+) - E\hat{e} \cdot \hat{p}_+].$$

(3)

If both $M$ and $E$ are $C$-even states, i.e., $M = M(s, (k \cdot q)^2)$ and $E = E(s, (k \cdot q)^2)$ with $s = (p_+ + p_-)^2 = m_\eta^2 - 2m_\eta E_\gamma$ and $q = p_+ - p_-$, the terms with $M$ and $E$ in Eq. (3) are $(+,+,+)$ and $(+,+,+).$ In the absence of final state interactions, $M$ and $E$ are purely real due to the $CPT$ theorem. In this case, the existence of the $E$ term is clearly a indication of direct CP violation since $CP(\eta) = -$. We note that $M$ and $E$ are $C$-odd if $M = m_\eta^{-2} k \cdot q M'(s, (k \cdot q)^2)$ and $E = m_\eta^{-2} k \cdot q E'(s, (k \cdot q)^2)$ [7, 8]. The $C$ violating interaction with $C(M) = -$ can be induced in models such as noncommutative QED associated with studies in string theory [9] through $\eta \rightarrow 3\gamma$ [10]. However, the effect is negligibly small and experimentally it is consistent with zero [11].

In general, to observe a CP violating effect, one needs to study the interference between the $M$ and $E$ terms with explicit photon polarization. To show this, we write the squared amplitude from Eq. (3) as

$$|\mathcal{M}|^2 = m_\eta^{-4} E_\gamma^2 \left\{ |M|^2 |\hat{k} \cdot (\hat{e} \times \hat{p}_+)|^2 + |E|^2 |\hat{e} \cdot \hat{p}_+|^2 + E^* M [\hat{k} \cdot (\hat{p}_+ \times \hat{e})] (\hat{e} \cdot \hat{p}_+)^* + M^* E [\hat{k} \cdot (\hat{p}_+ \times \hat{e})]^* (\hat{e} \cdot \hat{p}_+) \right\}. \quad (4)$$

It is easily seen that the interference terms in Eq. (4) are related to the triple momentum correlation of $\hat{k} \cdot (\hat{p}_+ \times \hat{e})$, which is odd under the time-reversal transformation. Observing this correlation would be a sign of direct T violation. By summing over photon polarizations, from Eq. (4) the partial decay rate of the Dalitz plot density is

$$\frac{d\Gamma}{dE_+ dE_-} = \frac{1}{64\pi^3 m_\eta} \sum_{spin} |\mathcal{M}|^2 \propto |E|^2 + |M|^2, \quad (5)$$

where $E_\pm$ are the $\pi^\pm$ energies. In terms of $E_\gamma$, it can be also written as

$$\frac{d\Gamma}{dE_\gamma d\cos \theta} = \frac{1}{512\pi^3} \left( \frac{E_\gamma}{m_\eta} \right)^3 \beta^3 \left( 1 - \frac{2E_\gamma}{m_\eta} \right) \sin^2 \theta \left[ |E|^2 + |M|^2 \right], \quad (6)$$

where $\beta = (1 - 4m_\eta^2/s)^{1/2}$ and $\theta$ is the angle between $\pi^+$ and $\gamma$ in the $\pi^+\pi^-$ rest frame. As seen from Eqs. (4) and (5), there is no interference term between $M$ and $E$, and therefore, no CP or T violation can be detected without the explicit measurement of the photon polarization.

From Eq. (4), we can define the photon polarization in terms of the well known
density matrix
\[
\rho = \begin{pmatrix}
|E|^2 & E^* M \\
EM^* & |M|^2
\end{pmatrix} = \frac{1}{2} \left(|E|^2 + |M|^2\right) \left[1 + \vec{S}(E, \theta) \cdot \vec{\tau} \right]
\] (7)

where \(\vec{\tau} = (\tau_1, \tau_2, \tau_3)\) denotes the Pauli matrices, and \(\vec{S}\) is the Stokes vector of the photon with components

\[
S_1(E, \theta) = 2 \text{Re} \left( E^* M \right) / \left(|E|^2 + |M|^2\right),
\]

\[
S_2(E, \theta) = 2 \text{Im} \left( E^* M \right) / \left(|E|^2 + |M|^2\right),
\]

\[
S_3(E, \theta) = \left(|E|^2 - |M|^2\right) / \left(|E|^2 + |M|^2\right),
\] (8)

respectively. Here, we have used

\[
\vec{e}_1 = \frac{\vec{\rho}_+ \times \vec{k}}{|(\vec{\rho}_+ \times \vec{k}) \times \vec{k}|},
\]

\[
\vec{e}_2 = \frac{\vec{\rho}_+ \times \vec{k}}{|\vec{\rho}_+ \times \vec{k}|},
\] (9)

as the two independent polarization vectors. The Stokes parameters \(S_{1,2}\) in Eq. (8) can be expressed as

\[
S_{1(2)}(E, \theta) = \frac{d\Gamma_{+(L)} - d\Gamma_{-(R)}}{d\Gamma_{+(L)} + d\Gamma_{-(R)}},
\] (10)

where \(d\Gamma_{\pm}\) and \(d\Gamma_{L,R}\) refer to photons polarized in the directions

\[
\vec{e}_\pm = \frac{1}{\sqrt{2}} (\vec{e}_2 \pm \vec{e}_1),
\] (11)

which is 45° with respect to the decay plane, and

\[
\vec{e}_{L,R} = \frac{1}{\sqrt{2}} (1, \mp i, 0),
\] (12)

respectively. We can also define the integrated Stokes parameters \(S_i(E, \theta)\), for example, by integrating \(\theta\) in Eq. (10) for each \(d\Gamma_{\alpha}\) (\(\alpha = \pm, L, R\)). The two parameters of \(S_{1,2}\) are sometimes called as the linear and circular polarizations. It is clear that \(S_1\) and \(S_2\) are related to CPV. If \(E = 0\), one has that \(S_1 = S_2 = 0\) and \(S_3 = -1\) and there is no CP violation.

\(^{a}\)The analysis for the photon polarization in the case of \(K_L \to \pi^+ \pi^- \gamma\) can be found in Ref. \(^{[12]}\).
Phenomenologically, the decay rate of $\eta \to \pi^+\pi^-\gamma$ is described by the magnetic term from the box-anomaly and resonance contributions $[13]$. Explicitly, for example, in a chiral model it was found that $[14, 15, 16]$

$$M = M^+(s) \simeq -\frac{e m^3_\eta}{4 \pi^2 f^3_\pi} \times \frac{\sqrt{3}}{6} \left(1 - \frac{3m^2_\rho}{m^2_\rho - s}\right),$$

(13)

where $C(M^+) = +$, $f_\pi = 93$ MeV and $m_\rho = 770$ MeV. From Eq. (13), we see that the dominant contribution to the decay is from a C-even state. If we include terms that are higher order in momentum and thus nonleading we have

$$M = M^+ + M^-, \quad E = E^+ + E^-,$$

(14)

where $C(M^\pm, E^\pm) = \pm$. With final state interactions, in Table 1, we show the possible violations of symmetries for the photon polarizations $S_i$ with $M$ and $E$ given in Eq. (14).

Table 1: Possible violations of symmetries for the photon polarizations $S_i$ due to the interferences between $M^+$ and $M^−$, $E^+$ with final state interactions.

| Interference  | $S_1$ | $S_2$ | $S_3$ |
|---------------|-------|-------|-------|
| $M^+M^−$      | $-$   | $-$   | $C, CP$ |
| $M^+E^+$      | $P, CP$ | $P, CP$ | $-$ |
| $M^+E^−$      | $C, P$ | $C, P$ | $-$ |

Table 2: Same as Table 1 but without final state phases.

| Interference  | $S_1$ | $S_2$ | $S_3$ |
|---------------|-------|-------|-------|
| $M^+M^−$      | $-$   | $-$   | $-$   |
| $M^+E^+$      | $P, CP$ | $-$   | $-$   |
| $M^+E^−$      | $-$   | $C, P$ | $-$   |

We now study the possible interactions which would yield the electric transitions in Eq. (2). First of all, it is easy to see that the $E$ term can be induced first going through the $\pi^+\pi^-$ intermediate state which violates CP symmetry because $CP(\pi^+\pi^-) = +$, and then radiating the photon from the pions. These bremsstrahlung terms are shown in Figures 1b and 1c. Explicitly, we have that

$$E^+(\eta \to (\pi^+\pi^-)^* \to \pi^+\pi^-\gamma) = \frac{e m^3_\eta g_{\gamma\pi\pi}}{(p_+ \cdot k)(p_- \cdot k)},$$

(15)
where $g_{\eta\pi\pi}$ is the effective coupling for $\eta \rightarrow \pi^+\pi^-$. From the experimental limit of $Br(\eta \rightarrow \pi^+\pi^-) < 3.3 \times 10^{-4}$, we find that

$$|g_{\eta\pi\pi}|^{\text{exp}} < 1.2 \times 10^{-4} \text{ GeV}.$$  

(16)

To illustrate the order of limits on the CP violating effects, in Figure 2, by using Eqs. (10), (13) and (15), we show $S_1(E_\gamma)$ with a real upper value of $g_{\eta\pi\pi}$ in Eq. (16). However, if we assume that the relative strong phase $\theta$ between the terms of $M^-$ and $E^+$ is $\delta$, i.e., $g_\eta = |g_\eta|e^{i\theta}$, from Figure 2, we get that

$$|S_{1,2}(E_\gamma)| < 0.2 \cos \delta, \ 0.2 \sin \delta, \ \text{and} \ S_3 \simeq -1,$$  

(17)

for $E_\gamma > 20$ MeV. It is interesting to see from Eq. (17) that one can get rid of the strong phase by measuring $S_1^2 + S_2^2$ to give the pure CP violating effect.

Within the Standard model the sources for the decay of $\eta \rightarrow \pi^+\pi^-$ can arise from the CKM phase, and/or the strong $\theta$ term in QCD [17, 18]. New physics such as spontaneous CP violating models can also lead to this decay [19]. From the experimental data on CP violation, it is found that $|g_{\eta\pi\pi}|$ are less than $2.6 \times 10^{-16}, 2 \times 10^{-10},$ and $5 \times 10^{-11}$ GeV, for the above three sources, respectively. In these cases, the CP violating effect such as the parameters of $|S_{1,2}|$ are expected to be less than $2.2 \times 10^{-12}, 1.7 \times 10^{-6}$ and $4.2 \times 10^{-7}$, respectively, which are vanishingly small and thus undetectable. This is not surprising at all since the constraints from the CP violating parameters from $\epsilon$ and $\epsilon'$ in the $K^0$-system as well as $d_\alpha$ are very strong in these models. To evade these constraints, one has to search for some unconventional sources of CPV which do not contribute directly to $\epsilon$, $\epsilon'$ and $d_\alpha$ and yet has a contribution in $\eta \rightarrow \pi^+\pi^-$, and we use a factorization approximation

$$|\eta_\gamma|$$  

between the terms of $M^-$ and $E^+$.

Hence, we construct flavor-conserving CP violating four-fermion operators involving two strange quarks together with combinations of other light quarks.

Explicitly, we study the four-fermion operator, given by

$$\mathcal{O} = \frac{1}{m_\eta^3} G \bar{s}s \sigma_{\mu\nu} \gamma_5 (p - k)^\nu s \bar{u}u,$$  

(18)

where $u(s)$ stands for the up (strange) quark and $G$ is a dimensionless parameter originating from short distance physics and it can be taken real due to the CPT theorem and taking $C(G) = +$. We note that there is no charge asymmetry in the decay due to the $C$ invariance and we also note that one may discuss similar operators with appropriate color generators and indices included in Eq. (13).

To calculate its contribution to $\eta \rightarrow \pi^+\pi^-\gamma$, we use a factorization approximation that the $\pi^+\pi^-$ part is from $p\bar{q}^\mu q$ and the $\eta\gamma$ transition involves only part containing $\bar{\eta}\gamma$. The dipion state for the $M^+$ transition in Eq. (13) is in a state of angular momentum $l_{\pi\pi} = 1$ and isospin $I = 1$, while that in Eq. (15) is $l_{\pi\pi} = 0$ and isospin $I = 0$, with the final state phase shifts being $\delta_1$ and $\delta_0$, respectively, and $\delta = \delta_1 - \delta_0$.

We note that $S_1^2 + S_2^2 = 1 - S_3^2$ which is zero if there is no CP violation.
strangeness, \( i.e. \),

\[
< \eta | \mathcal{O} | \pi^+ \pi^- \gamma > \sim \frac{1}{m_\eta^3} G < \eta | i \bar{s} \sigma_{\mu \nu} \gamma_5 (p - k)^\nu s | \gamma > < 0 | \bar{u} \gamma^\mu u | \pi^+ \pi^- > . \tag{19}
\]

This can be viewed as the photon being emitted directly from the structure part shown in Figure 3. We define the form factors for the \( \eta \to \gamma \) transition by

\[
< \eta | \bar{s} \sigma_{\mu \nu} \gamma_5 (p - k)^\nu s | \gamma > = \text{i} e [\epsilon_\mu (k \cdot p) - (\epsilon \cdot p) k_\mu] \frac{F(s)}{m_\eta} . \tag{20}
\]

The form factor of \( F(s) \) can be estimated in various QCD models such as the light front quark model (LFQM) \cite{20} and the relativistic quark model (QM) \cite{21}. In the LFQM \cite{20}, we find that, for example, \( F(0) \sim 0.19 \) which is consistent with the result from the QM \cite{21}. We shall assume that \( F(s) \sim F(0) \) to illustrate our numerical values for CP violation.

From Eqs. (18) and (20), we obtain that

\[
E \sim 2 e F(s) G . \tag{21}
\]

For \( G < O(1) \), we have that \(|M| > > |E|\). From Eqs. (10), (13) and (21), we find that

\[
S_1(E_\gamma, \theta) = S_1(E_\gamma) \sim \frac{96 \pi^2 F(s) G}{\sqrt{3}} \left( \frac{f_\pi}{m_\eta} \right)^3 \left( \frac{m_\eta^2 - s}{2m_\rho^2 + s} \right) , \\
S_2 \sim 0 , \text{ and } S_3 \sim -1 , \tag{22}
\]

where we have assumed that \( C(G) = +, i.e. \), \( E \sim E^+ \). The results in Eq. (22) indicate that the only interesting CPV observable is the linear polarization \( S_1 \) unless \( G \) are C-odd, such as when they contain higher order momentum terms, i.e. \( G \propto k \cdot q \).

But in this case the circular polarization of \( S_2 \) which can be non-zero are both CP and T conserving as shown in Table 2 and therefore we shall not discuss it in the remaining of the paper. In Figure 4, we show the linear polarization of the Stokes parameter \( S_1 \) as a function of \( E_\gamma = (m_\eta^2 - s)/2m_\eta \) with \( G \sim 1 \). For the maximal energy of the photon, i.e., \( E_\gamma \simeq 0.2 \text{ GeV}, S_1(E_\gamma) \) is about 21%.

We now discuss possible constraints on \( G \) and models that may lead to the operator in Eq. (18). As we mentioned previously, the operator in Eq. (18) cannot directly generate the decay of \( \eta \to \pi^+ \pi^- \). It is also important to note that it cannot directly induce \( d_n \) either; unlike the scalar-type operators studied in Ref. \cite{22}. There are basically no direct constraints for \( G \) from both low and high energy experiments. In principle, \( G \sim O(1) \) is allowed. A search for the photon polarization in \( \eta \to \pi^+ \pi^- \gamma \) will therefore provide interesting limits on \( G \) in the event of a negative search.

Our study have focused on the general aspects of CPV in Eq.(1) and appeal to experimental bounds as much as possible without committing to models explicit of
$CP$ violation. However, one would be wondering how the operator in Eq. (18) can be realized from some theoretical models.

In the standard model, there is no contribution to the operator in Eq. (18) at the lowest order and also to 1-loop. However, the operator can be induced in models in which the strange quark has an EDM. But, it has to be generated from a weak loop in a CP violating theory as well. Details are highly model dependent; however, we can expect that $G$ will not be larger than $G_F m_s^2 \simeq 0.35 \times 10^{-5}$. This leads to the expectation that $S_1$ is less than $10^{-6}$.

Finally, we remark that the operator given by the replacement of $s$ by $u$ or $d$ in Eq. (18) can also induce $S_1$ but it is very small due to the constraint from the neutron EDM.

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Figure 1: Diagrams which contribute to $\eta \to \pi^+\pi^-\gamma$.

Figure 2: The Stokes parameter $S_1(E_\gamma)$ as a function of $E_\gamma$ due to $\eta \to \pi^+\pi^-$. 
Figure 3: Structure dependent contribution to $\eta \rightarrow \pi^+\pi^-\gamma$.

Figure 4: The Stokes parameter $S_1(E_\gamma)$ as a function of $E_\gamma$ due to the operator in Eq. (18).