Growth of generalized Weyl algebras over polynomial algebras and Laurent polynomial algebras

Dedicated to Professor Yuqun Chen on the Occasion of His 65th Birthday

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Received March 17, 2022; accepted June 6, 2022; published online September 21, 2022

Abstract We study the growth and the Gelfand-Kirillov dimension (GK-dimension) of the generalized Weyl algebra (GWA) \( A = D(\sigma, a) \), where \( D \) is a polynomial algebra or a Laurent polynomial algebra. Several necessary and sufficient conditions for \( \text{GKdim}(A) = \text{GKdim}(D) + 1 \) are given. In particular, we prove a dichotomy of the GK-dimension of GWAs over the polynomial algebra in two indeterminates, i.e., \( \text{GKdim}(A) \) is either 3 or \( \infty \) in this case. Our results generalize several existing results in the literature and can be applied to determine the growth, GK-dimension, simplicity and cancellation properties of some GWAs.

Keywords Gelfand-Kirillov dimension, generalized Weyl algebra, polynomial automorphism

MSC(2020) 16P90, 16S36, 16P40, 16S32

1 Introduction and main results

Let \( k \) be a field. All the algebras under consideration are associative unital \( k \)-algebras and all the automorphisms of algebras are \( k \)-automorphisms. Let \( D \) be an algebra, \( a \) be a central element of \( D \) and \( \sigma \) be an automorphism of \( D \). The \textit{generalized Weyl algebra} (GWA, for short) \( A = D(\sigma, a) \) over \( D \) is defined as the algebra generated by \( D \) and two indeterminates \( x \) and \( y \) subject to the relations

\[
xd = \sigma(d)x, \quad yd = \sigma^{-1}(d)y, \quad yx = a, \quad xy = \sigma(a) \quad \text{for all } d \in D.
\]

GWAs are natural generalizations of Weyl algebras and form an important class of noncommutative algebras. Many algebras of common interest fall into this class, for example, Weyl algebras, certain quantum enveloping algebras, some iterated Ore extensions, Noetherian (generalized) down-up algebras, and Heisenberg algebras (see [6,11,55] for more examples). GWAs were introduced by Bavula [6] in the early 1990s. Since then their structures and representations have been studied intensively, for example, their dimension and growth [8,10,12,55], their derivations, isomorphisms and automorphisms [1,9,21,25,47], and their homological and geometric properties [16,19,30,36,37,51].

The notion of growth, introduced by Gelfand and Kirillov [22] for algebras and by Milnor [41] for groups, is a fundamental object of study in theories of algebras (not necessarily associative) and groups.
The Gelfand-Kirillov dimension (GK-dimension, for short) is a main tool to quantify growth of algebras and groups. The GK-dimension of an algebra \( A \) is defined as

\[
\text{GKdim}(A) := \sup_{V} \limsup_{n \to \infty} \log n \dim_k \left( \sum_{i=0}^{n} V^i \right),
\]

where the supremum is taken over all the finite-dimensional subspaces \( V \) of \( A \). The GK-dimension basically measures the growth of an algebra (rather than the growth of a ring [52]). For a finitely generated commutative algebra, its Krull dimension and GK-dimension coincide. Thus the GK-dimension can be viewed as a noncommutative analogue of the Krull dimension. It turns out that the GK-dimension is a very useful and powerful tool for investigating noncommutative algebras, for example, in recent studies of classification problems (see, e.g., [2, 23, 50]) and cancellation problems (see, e.g., [14, 18, 35, 48]). For basic properties and applications of the GK-dimension of algebras and groups, we refer the readers to [31]. For the GK-dimension of other algebraic structures, see, e.g., [3, 4, 29, 43].

There have been in the literature a number of results concerning the GK-dimension of skew polynomial extensions related to derivations and/or automorphisms, for example, GK-dimensions of Ore extensions [26, 38, 53], of PBW-extensions [39], and of differential difference algebras/modules [54]. Recently, Zhao et al. [55] investigated general properties of GK-dimensions of GWAs.

In this paper, we mainly study the growth and the GK-dimension of GWAs over polynomial algebras \( P_n := k[z_1, z_2, \ldots, z_n] \) (especially \( P_2 \)) and that over Laurent polynomial algebras \( L_n := k[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_n^{\pm 1}] \).

We have two motivations for this paper. The first motivation is the fact that lots of GWAs studied in the literature are ones over \( P_n \) or \( L_n \) and thus this subclass of GWAs is of fundamental importance in practice. For example, (quantum) Weyl algebras, quantum planes and primitive quotients of the quantum enveloping algebra \( U_q(sl_2) \) of the special linear Lie algebra \( sl_2 \); Noetherian generalized down-up algebras, the quantum Heisenberg algebra and the universal enveloping algebra \( U(sl_2) \) are GWAs over \( P_2 \); the group algebra of the (discrete) Heisenberg group (see Example 4.11) and the homogenized enveloping algebra of some Lie algebra (see [27, Example 2.4(ii)]) are GWAs over \( L_2 \). Our second motivation is that since the automorphism groups \( \text{Aut}(P_2) \) and \( \text{Aut}(L_n) \) are well understood (see [28, 49] for \( \text{Aut}(P_2) \) and Theorem 5.4 for \( \text{Aut}(L_n) \)), sharper theorems can be expected when one investigates the GK-dimension of GWAs over \( P_2 \) and \( L_n \).

Let \( A = D(\sigma, a) \) be a GWA. It was proved in [55] that the difference \( \text{GKdim}(A) - \text{GKdim}(D) \) can be any positive integer or the infinity. If \( \sigma \) is locally algebraic (see Definition 2.10) meaning that every finite-dimensional subspace of \( D \) is contained in a finite-dimensional \( \sigma \)-stable subspace of \( D \), then

\[
\text{GKdim}(A) - \text{GKdim}(D) = 1.
\]

Although \( \text{GKdim}(A) - \text{GKdim}(D) = 1 \) does not imply in general that \( \sigma \) is locally algebraic (see Example 3.8), we prove that for GWAs over \( P_n \) and \( L_n \), this implication does hold. More generally, we have the following theorem.

**Theorem 1.1.** Let \( A = D(\sigma, a) \) be a GWA. Let \( P_n := k[z_1, z_2, \ldots, z_n] \) be the polynomial algebra in \( n \) indeterminates and \( Q_n \) be the fraction field of \( P_n \). If \( P_n \subseteq D \subseteq Q_n \), then the following statements are equivalent:

(a) \( \text{GKdim}(A) = \text{GKdim}(D) + 1 \).

(b) \( \sigma \) is locally algebraic.

If \( k \) is algebraically closed, \( D \) is a finitely generated field and \( a \neq 0 \), then (a) and (b) are equivalent to each of the following statements:
(c) $\sigma$ has a finite order.
(d) $A$ is not simple.

Theorem 1.1 is an analogue of [53, Theorem 1.1] but note that we do not assume that $k$ is algebraically closed to get (a) $\iff$ (b) (see Proposition 3.7 for more results for the algebraically closed case).

When considering GWAs over $P_2$, we have more elaborate results. Our main results on the GK-dimension of GWAs over $P_2$ can be summarized as follows.

**Theorem 1.2.** Let $A = k[z_1, z_2](\sigma, a)$ be a GWA. Then the following statements are equivalent:

(a) $\text{GKdim}(A) = 3$.
(b) $\text{GKdim}(A) < \infty$.
(c) $A$ has polynomial growth.
(d) $A$ does not have exponential growth.
(e) $\sigma$ is locally algebraic.
(f) $\sigma$ is conjugate to a triangular automorphism.
(g) $\sigma^m$ is conjugate to a triangular automorphism for some $m \geq 1$.

The growth of algebras (see Definition 2.6) can be prescribed within a rather wide range of functions. Recently, Bell and Zelmanov [13] gave a complete characterization of the functions that can occur as the growth functions of algebras. Theorem 1.2 indicates that GWAs over $L_n$ are summarized in the following theorem.

**Theorem 1.3.** Let $A = k[z_1^{k_1}, z_2^{k_2}, \ldots, z_n^{k_n}](\sigma, a)$ be a GWA. Then the following statements are equivalent:

(a) $\text{GKdim}(A) = n + 1$.
(b) $\sigma$ is locally algebraic.
(c) $M$ has a finite order, i.e., $M^m$ equals the identity matrix for some $m \geq 1$.
(d) $M$ is power-bounded, i.e., there exists an $N > 0$ such that the absolute values of all the entries of $M^m$ are less than $N$ for all $m \geq 1$.

Theorem 1.3 enables us to determine the GK-dimension by computation of integer matrices under certain conditions. Given $n \geq 1$, Levitt and Nicolas [34] showed that the maximum order $G(n)$ of a matrix in $GL(n, \mathbb{Z})$ satisfies a Landau-type estimate $\ln G(n) \sim \sqrt{n} \ln n$ and all the possible orders of matrices in $GL(n, \mathbb{Z})$ can be determined by factorization of natural numbers less than $G(n)$ (see Remark 5.7).

Our results can be applied to determine the GK-dimension, the simplicity (see Corollary 3.9) and cancellation properties (see Corollary 5.11) of some GWAs.
The rest of this paper is organized as follows. In Section 2, we recall basic definitions and list some results from the literature. In Section 3, we study the sensitive multiplicity condition and prove Theorem 1.1. In Section 4, we focus on the growth and the GK-dimension of GWAs over polynomial algebras, where Theorem 1.2 is proved. In Section 5, we first investigate automorphisms of Laurent polynomial algebras, and then study the GK-dimension of GWAs over Laurent polynomial algebras, and finally prove Theorem 1.3.

2 Preliminaries

In this section, we recall some terminology related to generalized Weyl algebras and the Gelfand-Kirillov dimension. Our main references for the GK-dimension are [31] and [40, Chapter 8].

Let $k$ be a field and $k^* = k \setminus \{0\}$. Unless otherwise stated, all the spaces in this paper are $k$-spaces, $\dim(V)$ denotes the dimension of a $k$-space $V$, all the algebras are associative unital $k$-algebras, and all the automorphisms are $k$-algebra automorphisms. Denote the space spanned by a set $S$ by $kS$, particularly by $ka$ if $S = \{a\}$. Given the subspaces $V$ and $W$ of an algebra $D$, define $VW = \sum_{v \in V, w \in W} k(vw)$, $V^n = k$ and $V^0 = V^{n-1}V$ for $n \geq 1$, and write $VW = Va$ if $W = ka$. We use $Z(D)$ (resp. $\text{Aut}(D)$, $\text{id} = \text{id}_D$) to denote the center (resp. the automorphism group, the identity automorphism) of $D$.

**Definition 2.1** (See [6]). Let $D$ be an algebra, $\sigma \in \text{Aut}(D)$ and $a \in Z(D)$. The *generalized Weyl algebra* (GWA, for short) $A = D(\sigma, a) = D(x, y; \sigma, a)$ with the *base algebra* $D$ is defined as the algebra generated by $D$ and indeterminates $x$ and $y$, subject to the defining relations

$$xd = \sigma(d)x, \quad yd = \sigma^{-1}(d)y, \quad yx = a, \quad xy = \sigma(a) \quad \text{for all } d \in D.$$

The automorphism $\sigma$ and the element $a$ are called the *defining automorphism* and the *defining element* of the GWA $A$, respectively.

The following example indicates why the algebra $D(\sigma, a)$ is called a GWA.

**Example 2.2.** The first Weyl algebra $A_1$ over $k$ is the ring of polynomials in indeterminates $x$ and $y$ with coefficients in $k$, subject to the relation $yx - xy = 1$. Then

$$A_1 \cong D(\sigma, a), \quad x \mapsto x, \quad y \mapsto y,$$

where $D = k[h]$, $a = h = xy$ and $\sigma(h) = h - 1$. The $n$-th Weyl algebra can be presented as an iterated GWA (see [55] for more detailed examples of GWAs).

A $\mathbb{Z}$-*filtration* of the algebra $A$ is a sequence of subspaces

$$\cdots \subseteq F_{i-1} \subseteq F_i \subseteq F_{i+1} \subseteq \cdots, \quad i \in \mathbb{Z}$$

such that $1 \in F_0$, $F_iF_j \subseteq F_{i+j}$ for all $i, j \in \mathbb{Z}$ and $A = \bigcup_{i \in \mathbb{Z}} F_i$. An algebra with a $\mathbb{Z}$-filtration is called a $\mathbb{Z}$-filtered algebra. The vector space

$$\text{gr}(A) = \bigoplus_{i \in \mathbb{Z}} F_i/F_{i-1}$$

equipped with the linear multiplication defined by the rule

$$(a + A_{i-1})(b + F_{j-1}) = ab + F_{i+j-1}$$

is called the *associated graded algebra* of $A$. The linear mapping $\text{gr} : A \to \text{gr}(A)$ is defined by $\text{gr}(a) = a + F_{i-1}$ for all $a \in F_i/F_{i-1}$. For a subspace $V \subseteq A$, define $\text{gr}(V) = \sum_{v \in V} k\text{gr}(v)$. The following lemma lists some basic properties of the map $\text{gr}$. For more properties related to filtered algebras and graded algebras, see [31, Chapter 6].

**Lemma 2.3** (See [53, Lemma 2.1]). Let $A$ be a $\mathbb{Z}$-filtered algebra, $W$ be a finite-dimensional subspace
of $A$ and $a \in A$. Then

(i) $\text{gr}(W(a)) \supseteq \text{gr}(W) \text{gr}(a)$;
(ii) $\dim(W) = \dim(\text{gr}(W))$;
(iii) $\dim(W^m) \geq \dim((\text{gr}(W))^m)$.

The GWA $A = D(x, y; \sigma, a)$ is a free left $D$-module with a basis $\{x^i, y^j : i \geq 0, j > 0\}$. There is a natural $\mathbb{Z}$-graded structure for a GWA, i.e.,

$$D(\sigma, a) = \bigoplus_{i \in \mathbb{Z}} A_i,$$

where $A_i = Dx^i$ if $i > 0$, $A_i = Dy^{-1}$ if $i < 0$ and $A_i = D$ if $i = 0$. Let $F_i = \bigcup_{j \leq i} A_j$. Then

$$\cdots \subseteq F_{i-1} \subseteq F_i \subseteq F_{i+1} \subseteq \cdots, \quad i \in \mathbb{Z}$$

is a $\mathbb{Z}$-filtration for $A$. Recall that given $\sigma \in \text{Aut}(D)$, the Ore extension $D[x; \sigma]$ over an algebra $D$ is the algebra generated by $D$ and $x$ subject to the relation $xd = \sigma(d)x$ for all $d \in D$. The following lemma follows immediately from the $\mathbb{Z}$-graded structure of GWAs.

**Lemma 2.4.** The Ore extension $D[x; \sigma]$ is a subalgebra of the GWA $D(\sigma, a)$.

It is routine to check that GWAs have the following universal property (compare it with [40, Chapter 1.2.5]).

**Proposition 2.5.** Let $D(x, y; \sigma, a)$ be a GWA and $B$ be an algebra. Suppose that $\phi : D \rightarrow B$ is an algebra homomorphism, and $x', y' \in B$ satisfy

$$x' \phi(d) = \phi(\sigma(d))x', \quad y' \phi(d) = \phi(\sigma^{-1}(d))y', \quad y'x' = \phi(a), \quad x'y' = \phi(\sigma(a)) \quad \text{for all } d \in D.$$

Then there exists a unique algebra homomorphism $\psi : D(x, y; \sigma, a) \rightarrow B$ such that $\psi(x) = x'$, $\psi(y) = y'$ and the following diagram:

$$\begin{array}{ccc}
D & \rightarrow & D(x, y; \sigma, a) \\
\phi \downarrow & & \downarrow \psi \\
B & \rightarrow & B
\end{array}$$

is commutative.

Let $\Phi$ denote the set of all the eventually monotone increasing functions $f : \mathbb{N} \rightarrow \mathbb{R}^+$. For $f, g \in \Phi$, set $f \preceq^* g$ if and only if there exist $c, m, N \in \mathbb{N}$ such that

$$f(n) \leq cg(mn) \quad \text{for all } n > N$$

and set $f \sim g$ if $f \preceq^* g$ and $g \preceq^* f$. With a slight abuse of notation, we often write $f(n) \sim g(n)$ to mean that $f \sim g$. One can check that $\sim$ is an equivalent relation on $\Phi$. For $f \in \Phi$, the equivalence class $\mathcal{G}(f) \in \Phi/\sim$ is called the growth of the function $f$. At the risk of abusing the notation, we also write the growth of $f$ as $\mathcal{G}(f(n))$. The partial order on $\Phi/\sim$ induced by $\preceq^*$ is denoted by $\leq$.

If an algebra $A$ is generated (as an algebra) by a subspace $V$, then $V$ is called a generating subspace of $A$. If $V$ and $W$ are two finite-dimensional generating subspaces of $A$, then $\mathcal{G}(d_V) = \mathcal{G}(d_W)$ (see [31, Lemma 1.1]), where $d_V(n) = \dim(\sum_{i=0}^n V^i)$ for all $n \in \mathbb{N}$. Thus we can define the notion of the growth of algebras, which is independent of the choice of generating subspaces.

**Definition 2.6.** Let $A$ be a finitely generated algebra.

(i) Suppose that $A$ is generated by a finite-dimensional subspace $V$. Then $d_V$ is called the growth function of $A$ with respect to $V$ and $\mathcal{G}(A) := \mathcal{G}(d_V)$ is called the growth of $A$.

(ii) A finitely generated algebra $A$ is said to have
• polynomial growth if \( G(A) = G(n^d) \) for some \( d \in \mathbb{N} \);
• exponential growth if \( G(A) = G(e^n) \);
• intermediate growth if \( G(A) < G(e^n) \) yet \( G(A) \not\in G(n^m) \) for all \( m \in \mathbb{N} \);
• alternative growth if \( A \) has either polynomial growth or exponential growth.

Functions within a wide range can be realized as growth functions of finitely generated algebras. Particularly, exponential growth is the largest that an algebra can have. Recently, Bell and Zelmanov [13] gave a complete characterization of the functions that can occur as the growth functions of algebras.

The Gelfand-Kirillov dimension is a numerical characteristic of growth, which works for not only finitely generated algebras but also infinitely generated algebras.

**Definition 2.7.** Let \( A \) be an algebra. The Gelfand-Kirillov dimension (GK-dimension, for short) of \( A \) is defined as

\[
\text{GKdim}(A) := \sup_{V} \limsup_{n \to \infty} \log_n \dim V(n),
\]

where the supremum is taken over all the finite-dimensional subspaces \( V \) of \( A \).

A finite-dimensional subspace \( V \subseteq A \) is called a subframe of \( A \) if \( 1 \in V \). If \( A \) is generated by a subframe \( V \) (and thus \( A \) is finitely generated), then

\[
\text{GKdim}(A) = \limsup_{n \to \infty} \log_n \dim(V^n).
\]

The following well-known fact is useful in the computation with the growth functions and the GK-dimension.

**Lemma 2.8.** If \( f(n) \in \Phi \) is a polynomial of degree \( d \in \mathbb{N}^* \), then

\[
g(n) := f(0) + f(1) + \cdots + f(n)
\]

is a polynomial of degree \( d + 1 \).

The following lemma provides useful properties of GWAs, which will be used later.

**Lemma 2.9** (See [9, Lemma 2.7]). Let \( A = D(\sigma, a) \) be a GWA and let \( \tau \in \text{Aut}(D) \). Then

(i) \( A \cong D(\tau^{-1}\sigma\tau, \tau^{-1}(a)) \);
(ii) \( A \cong D(\sigma^{-1}, \sigma(a)) \);
(iii) if \( \lambda \) is a central unit in \( D \), then \( A \cong D(\sigma; a\lambda) \). In particular, if \( a \) is a unit, then

\[
A \cong D(x^{\pm 1}; \sigma, 1) = D[x^{\pm 1}; \sigma],
\]

the twisted Laurent polynomial algebra over \( D \).

**Proof.** The isomorphisms are given, respectively, by (i) \( x \mapsto x, y \mapsto y \) and \( d \mapsto \tau^{-1}(d) \) for all \( d \in D \); (ii) \( x \mapsto y, y \mapsto x \) and \( d \mapsto d \) for all \( d \in D \); (iii) \( x \mapsto x\lambda^{-1}, y \mapsto y \) and \( d \mapsto d \) for all \( d \in D \).

**Definition 2.10.** Suppose that \( D \) is an algebra and \( \sigma, \tau \in \text{Aut}(D) \).

(i) \( \tau \) is conjugate to \( \sigma \) if \( \tau = \eta^{-1}\sigma\eta \) for some \( \eta \in \text{Aut}(D) \).
(ii) A subspace \( V \) of \( D \) is said to be \( \sigma \)-stable if \( \sigma(V) = V \).
(iii) \( \sigma \) has a finite order if \( \sigma^m = \text{id} \) for some \( m \geq 1 \).
(iv) \( \sigma \) is inner if there exists a unit \( u \in D \) such that \( \sigma(d) = ud\sigma^{-1} \) for all \( d \in D \).
(v) \( \sigma \) is locally algebraic if every finite-dimensional subspace of \( D \) is contained in a finite-dimensional \( \sigma \)-stable subspace of \( D \).

**Lemma 2.11.** An automorphism \( \sigma \in \text{Aut}(D) \) is locally algebraic if and only if for every \( a \in D \) the set \( \{\sigma^i(a) : i \in \mathbb{N}\} \) is contained in a finite-dimensional subspace of \( D \).

**Proof.** (\( \Rightarrow \)) Suppose that \( \sigma \) is locally algebraic. Then for every \( a \in D \), the subspace \( k[a] \) is finite-
dimensional and thus contained in a finite-dimensional \( \sigma \)-stable subspace \( U \). As a result,
\[
\{ \sigma^i(a) : i \in \mathbb{N} \} \subseteq U.
\]

(\( \Leftarrow \)) Suppose that \( V = k\{a_1, a_2, \ldots, a_m\} \) is a finite-dimensional subspace of \( D \). By the assumption, each space \( k\{a_i\} \) is contained in a finite-dimensional \( \sigma \)-stable subspace \( U_i \subseteq D \), \( 1 \leq i \leq m \). Then
\[
V \subseteq U := U_1 + U_2 + \cdots + U_m.
\]

Clearly, \( U \) is finite-dimensional and \( \sigma \)-stable.

Local algebraicity has a close relation to the GK-dimension of algebras extended by automorphisms.

**Lemma 2.12** (See [55, Lemma 3.1 and Theorem 3.4]). Let \( A = D(\sigma, a) \) be a GWA. Then
(i) \( \text{GKdim}(A) \geq \text{GKdim}(D) + 1 \);
(ii) \( \text{GKdim}(A) = \text{GKdim}(D) + 1 \) if moreover \( \sigma \) is locally algebraic.

**Lemma 2.13** (See [53, Theorem 1.1]). Let \( D \) be a commutative domain over an algebraically closed field \( k \) such that the fraction field of \( D \) is finitely generated as a field, and let \( \sigma \in \text{Aut}(D) \). Then the following statements are equivalent:
(i) \( \text{GKdim}(D[x, \sigma]) = \text{GKdim}(D) + 1 \).
(ii) \( \sigma \) is locally algebraic.

If moreover \( D \) is a field, then (i) and (ii) are equivalent to the following statement:
(iii) \( \sigma \) has a finite order.

We conclude this section by two lemmas of GWAs due to Bavula [6, 7].

**Lemma 2.14** (See [6]). Let \( A = D(\sigma, a) \) be a GWA. Suppose \( a \neq 0 \). If \( D \) is Noetherian (resp. without zero divisors), then \( A \) is Noetherian (resp. without zero divisors).

**Lemma 2.15** (See [7, Theorem 4.2]). Let \( A = D(\sigma, a) \) be a GWA. Then \( A \) is simple if and only if the following conditions hold:
(i) \( D \) has no proper ideals that are \( \sigma \)-stable.
(ii) \( \sigma^m \) is not inner for all \( m \geq 1 \).
(iii) \( Da + Da^i(a) = D \) for all \( i \in \mathbb{N} \).
(iv) \( a \) is regular in \( D \), i.e., \( ad \neq 0 \) and \( da \neq 0 \) for all \( 0 \neq d \in D \).

### 3 The sensitive multiplicity condition

Let \( P_n = k[z_1, z_2, \ldots, z_n] \) and \( Q_n \) be the fraction field of \( P_n \). In this section, we first prove that all the algebras \( D \) such that \( P_n \subseteq D \subseteq Q_n \) satisfy the sensitive multiplicity condition (see Definition 3.1) without assuming that \( k \) is algebraically closed and then we prove Theorem 1.1.

The notion of the sensitive multiplicity condition was introduced by Zhang [53], and it plays an important role in the GK-dimension of skew polynomial extensions. Recall that \( a \in A \) is called a regular element in the algebra \( A \) if \( ab \neq 0 \) and \( ba \neq 0 \) for all \( 0 \neq b \in A \).

**Definition 3.1.** Let \( A \) be an algebra of \( \text{GKdim}(A) = d \). We say that \( A \) satisfies the sensitive multiplicity condition (SMD for short) \( \text{SM}(V_0, c, d) \) if there exist a finite-dimensional subspace \( V_0 \) of \( A \) and a constant \( c > 0 \) such that if \( W \) is a finite-dimensional subspace of \( A \) containing \( V_0 a \) for some regular element \( a \in A \), then \( \dim(W^n) \geq c \dim(W)m^d \) for all \( m \in \mathbb{N} \).

**Lemma 3.2** (See [53, Lemma 3.1(2)]). Let \( A \) be a commutative domain and \( Q(A) \) be the fraction field of \( A \). Then \( Q(A) \) satisfies \( \text{SM}(V_0, c, d) \) if and only if \( A \) satisfies \( \text{SM}(V_0 a, c, d) \) for some \( 0 \neq a \in A \) such that \( V_0 a \subseteq A \).

**Proof.** Suppose that \( Q(A) \) satisfies \( \text{SM}(V_0, c, d) \). Since \( V_0 \) is finite-dimensional, we can write \( V_0 = V a^{-1} \) for some finite-dimensional subspace \( V \subseteq A \) and \( 0 \neq a \in A \). Suppose that \( W \) is a finite-dimensional
subspace of $A$ such that $W \supseteq Vb$ for some regular element $b \in A$. Then $W \supseteq V_0(ab)$ and $ab \in Q(A)$ is regular. By the SMC for $Q(A)$,

$$\dim(W^m) \geq c\dim(W)m^d, \quad m \in \mathbb{N}.$$ 

Thus $A$ satisfies SM($V_0a, c, d$) for $a \in A$ regular and $V_0a = V \subseteq A$.

Suppose that $A$ satisfies SM($V_0a, c, d$) for some $V_0 \subseteq Q(A)$ and $0 \neq a \in A$ such that $V_0a \subseteq A$. Let $W$ be a finite-dimensional subspace of $Q(A)$ and $V_0q \subseteq W$ for some $0 \neq q \in Q(A)$. We can write $W = W'b^{-1}$ for some finite-dimensional subspace $W' \subseteq A$ and $0 \neq b \in A$ such that $a^{-1}qb \in A$. Then

$$(V_0a)(a^{-1}qb) = V_0qb \subseteq Wb = W',$$

and thus by the SMC for $A$,

$$\dim(W^m) = \dim(W'm^m) \geq c\dim(W)m^d = c\dim(W)m^d, \quad m \in \mathbb{N}.$$ 

Thus $Q(A)$ satisfies SM($V_0, c, d$).

**Lemma 3.3.** If $D$ is a commutative algebra satisfying the SMC and $\sigma \in \text{Aut}(D)$ is not locally algebraic, then $\text{GKdim}(D, \sigma(a)) \geq \text{GKdim}(D) + 2$.

**Proof.** By Lemma 2.4, $D[x, \sigma]$ is a subalgebra of $D(\sigma, a)$. Thus, by [31, Lemma 3.1] and [53, Proposition 3.3(1)], we have

$$\text{GKdim}(D(\sigma, a)) \geq \text{GKdim}(D[x, \sigma]) \geq \text{GKdim}(D) + 2,$$

which completes the proof.

Following the proof of [53, Theorem 3.2], we have the following lemma without the assumption that $k$ is algebraically closed.

**Lemma 3.4.** For $n \geq 0$, the polynomial algebra $P_n$ satisfies SM($V_n, c_n, n$), where

$$V_n = k + kz_1 + \cdots + kz_n, \quad c_n = \frac{1}{2^n\sqrt{5^{n(n+1)}}}.$$ 

**Proof.** The proof of [53, Theorem 3.2] with small modification still works for our lemma.

**Corollary 3.5.** Suppose that $D$ is an algebra such that $P_n \subseteq D \subseteq Q_n$. Then $D$ satisfies the SMC.

**Proof.** Let $V = k + kz_1 + \cdots + kz_n \subseteq D$. Suppose that $W$ is a finite-dimensional subspace of $D$ such that $Va \subseteq W$ for some $0 \neq a \in D$. Note that $\text{GKdim}(D) = n$ since $\text{GKdim}(P_n) = \text{GKdim}(Q_n) = n$. By Lemmas 3.4 and 3.2, $Q_n$ satisfies SM($V_n, c_n, n$), where $V_n$ and $c_n$ are as in Lemma 3.4. Hence by the definition of the SMC,

$$\dim(W^m) \geq c\dim(W)m^n \quad \text{for some} \ c > 0 \ \text{and} \ m \in \mathbb{N},$$

which means that $D$ satisfies the SMC.

Now we have the following result without the assumption that $k$ is algebraically closed.

**Proposition 3.6.** Let $A = D(\sigma, a)$ be a GWA, where $P_n \subseteq D \subseteq Q_n$. Then the following statements are equivalent:

(i) $\text{GKdim}(A) = \text{GKdim}(D) + 1$.

(ii) $\sigma$ is locally algebraic.

**Proof.** By Corollary 3.5, $D$ satisfies the SMC. Then the statement follows from Lemmas 2.12 and 3.3.
Proposition 3.6 is a partial converse of Lemma 2.12(ii). If \( k \) is algebraically closed, we have the following proposition, which is an analogue of Lemma 2.13.

**Proposition 3.7.** Let \( A = D(\sigma, a) \) be a GWA, where \( D \) is a commutative domain over an algebraically closed field \( k \) such that the fraction field of \( D \) is finitely generated as a field. Then the following statements are equivalent:

(i) \( \text{GKdim}(A) = \text{GKdim}(D) + 1 \).

(ii) \( \sigma \) is locally algebraic.

If moreover \( D \) is a field, then (i) and (ii) are equivalent to the following statement:

(iii) \( \sigma \) has a finite order.

**Proof.** (ii) \( \Rightarrow \) (i) follows from Lemma 2.12. (i) \( \Rightarrow \) (ii) follows from Lemma 3.3.

(ii) \( \Rightarrow \) (iii). The proof of Lemma 2.13 (see [53, p.369]) still works for our statements.

Note that (i) \( \Rightarrow \) (ii) in Proposition 3.7 fails if \( D \) is not commutative (see Example 3.8).

**Example 3.8.** Let \( D = k[z^{\pm 1}, s][d; \sigma] \) be an Ore extension over the polynomial ring \( k[z^{\pm 1}, s] := k[z^{\pm 1}][s] \), where the automorphism \( \sigma \) is defined by \( \sigma(z) = z \) and \( \sigma(s) = zs \). Let \( A = D(x, y; \tau, 1) \) be a GWA, where \( \tau \in \text{Aut}(D) \) satisfies \( \tau(z) = z, \tau(s) = s \) and \( \tau(d) = z^{-1}d \). Then \( \tau \) is not locally algebraic but \( \text{GKdim}(A) = \text{GKdim}(D) + 1 = 5 \).

**Proof.** First note that \( D \) is the differential difference algebra defined in [54, Example 3.12] and \( \text{GKdim}(D) = 4 \) (see [54, p.494]). By Lemma 2.12, \( \text{GKdim}(A) \geq \text{GKdim}(D) + 1 = 5 \). It remains to show that \( \text{GKdim}(A) \leq 5 \). Rewrite \( A \) as an Ore extension over \( C := k[x^{\pm 1}, z^{\pm 1}, s] \), i.e., \( A = C[d; \delta] \), where \( \delta \in \text{Aut}(C) \) is defined by \( \delta(z) = z, \delta(x) = zx \) and \( \delta(s) = zs \). Let \( V = k[x^{\pm 1}, z^{\pm 1}, s, 1] \). Then \( W = V + kd \) is a finite-dimensional generating subspace of \( A \). By induction, we can prove that for \( m \geq 1 \),

\[
W^m = \sum_{i=0}^{m} W^{m-i} d^i, \quad W_i := V + \delta(V) + \cdots + \delta^i(V)
\]

(see also [53, (3.3.1) and (3.3.2)]) and for \( 0 \leq i \leq m \),

\[
W_i = k\{1, z^{\pm 1}, z^j x, z^{-j} x^{-1}, z^j s : 0 \leq j \leq i\}.
\]

Thus

\[
W_i^{m-i} \subseteq k\{z^{p_1} x^{p_2} s^{p_3} : -i(m - i) \leq p_1 \leq i(m - i), -(m - i) \leq p_2 \leq m - i, 0 \leq p_3 \leq m - i\}.
\]

As a result,

\[
\dim(W^m) = \sum_{i=0}^{m} \dim(W_i^{m-i}) \leq \sum_{i=0}^{m} (2i(m - i) + 1)(2(m - i) + 1)(m - i + 1) \sim m^5,
\]

where \( \sim \), meaning the equivalence of two functions in \( \Phi \) (see Section 2), follows from Lemma 2.8. Therefore, \( \text{GKdim}(A) = \limsup_{m \to \infty} \log_m \dim(W^m) \leq 5 \) as desired.

Proposition 3.7 together with Lemma 2.15 gives a criterion of the simplicity for certain GWAs.

**Corollary 3.9.** Suppose that \( k \) is algebraically closed and \( A = D(\sigma, a) \) is a GWA, where \( D \) is a finitely generated field and \( a \neq 0 \). Then the following statements are equivalent:

(i) \( A \) is simple.

(ii) \( \text{GKdim}(A) \neq \text{GKdim}(D) + 1 \).

**Proof.** Note that if \( D \) is a field and \( a \neq 0 \), then the conditions (i)–(iii) in Lemma 2.15 are satisfied automatically. As a result, under the assumption, \( A \) is simple if and only if \( \sigma^m \) is not inner for all \( m \geq 1 \), if and only if \( \sigma^m \neq \text{id} \) for all \( m \geq 1 \), or if and only if \( \text{GKdim}(A) \neq \text{GKdim}(D) + 1 \) (by Proposition 3.7).
Proof of Theorem 1.1. It follows from Propositions 3.6 and 3.7 and Corollary 3.9.

4 The GK-dimension of GWAs over polynomial algebras

In this section, we study the GK-dimension of GWAs over \( P_n := k[z_1, z_2, \ldots, z_n] \), in particular over \( P_2 \). We first prove a lemma, which will be used later.

Lemma 4.1. Let \( A = D(x, y; \sigma, a) \) be a GWA, \( m \in \mathbb{N}^* \) and \( B \) be the subalgebra of \( A \) generated by \( D \cup \{x^m, y^m\} \). Then \( B = D(x^m, y^m; \sigma^m, b) \) is a GWA and \( \text{GKdim}(A) = \text{GKdim}(B) \), where

\[
b = b_m = \sigma^{-(m-1)}(a)\sigma^{-(m-2)}(a)\cdots\sigma^{-1}(a)a.
\]

Proof. First note that \( x^m d = \sigma^m(d)x^m \) and \( y^m d = \sigma^m(d)y^m \) for all \( d \in D \). Let

\[
b = \sigma^{-(m-1)}(a)\sigma^{-(m-2)}(a)\cdots\sigma^{-1}(a)a \in D.
\]

By using the defining relations of \( A \), we see that \( b = y^m x^m \). We claim that \( b \) is central in \( D \). For each \( d \in D \) and \( i \in \mathbb{Z} \),

\[
\sigma^i(a)d = \sigma^i(\sigma^{-i}(d)) = \sigma^i(\sigma^{-i}(d)a) = d\sigma^i(a),
\]

which implies \( bd = db \) and the claim holds.

Similarly, \( \sigma^i(b) \) is central in \( D \) for all \( i \in \mathbb{N} \). Thus

\[
\sigma^m(b) = \sigma(a)\sigma^2(a)\cdots\sigma^m(a) = \sigma^m(a)\sigma^{m-1}(a)\cdots\sigma(a) = x^m y^m.
\]

Therefore, \( B = D(x^m, y^m; \sigma^m, b) \) is a GWA.

Since \( A \) is finitely generated as a right \( B \)-module, we obtain by [31, Proposition 5.5] that \( \text{GKdim}(A) = \text{GKdim}(B) \).

Now we can prove the following proposition.

Proposition 4.2. Suppose that \( A = D(\sigma, a) \) is a GWA.

(i) If \( \sigma^m \) is an inner automorphism for some \( m \geq 1 \), then \( \text{GKdim}(A) = \text{GKdim}(D) + 1 \).

(ii) If \( \sigma \) has a finite order, then \( \text{GKdim}(A) = \text{GKdim}(D) + 1 \).

Proof. (i) Let \( A = D(x, y; \sigma, a) \) and \( B \) be the subalgebra of \( A \) generated by \( D \cup \{x^m, y^m\} \). Suppose that \( u \in D \) is an invertible element such that \( \sigma^m(d) = udu^{-1} \) for all \( d \in D \). By Lemma 4.1, \( B = D(x^m, y^m; \sigma^m, b) \) where \( b \) is a central element in \( D \). We further claim that \( B = D(u^{-1}x^m, uy^m, id, b) \).

For each \( d \in D \),

\[
u^{-1}x^m d = u^{-1}\sigma^m(d)x^m = u^{-1}ud^{-1}x^m = du^{-1}x^m
\]

and \( uy^m d = duy^m \) similarly. Note that \( \sigma^{\pm m}(u) = u \) and \( \sigma^{\pm m}(u^{-1}) = u^{-1} \). Thus

\[
uy^m u^{-1}x^m = u\sigma^{\pm m}(u^{-1})y^m x^m = y^m x^m = b
\]

and

\[
u^{-1}x^m uy^m = x^my^m = \sigma^m(b) = ubu^{-1} = b,
\]

where the last equality holds since \( b \) is central in \( D \). Thus, \( B = D(u^{-1}x^m, uy^m, id, b) \).

Now it follows from Lemmas 4.1 and 2.12 that \( \text{GKdim}(A) = \text{GKdim}(B) = \text{GKdim}(D) + 1 \).

(ii) follows from (i).

Proposition 3.7 implies that the converse of Proposition 4.2(ii) also holds under certain conditions, while it does not hold in general (see Example 5.8).

Corollary 4.3. Let \( D \) be an algebra and \( D[x, y] \) be the polynomial algebra in two variables over \( D \). Suppose that \( I \) is the ideal of \( D[x, y] \) generated by \( xy - a \) for some \( a \in Z(D) \). Then \( \text{GKdim}(D[x, y]/I) \)
Proof. By Proposition 2.5, there exists a homomorphism $\psi : D(x, y; \text{id}, a) \to D(x, y)/I$ such that $\psi(x) = x$, $\psi(y) = y$ and $\psi(d) = d$ for all $d \in D$. Note that the homomorphism

$$\psi' : D[x, y] \to D(x, y; \text{id}, a), \quad x \mapsto x, \quad y \mapsto y, \quad d \mapsto d \text{ for all } d \in D$$

maps $I$ to 0 since $\psi'(xy-a) = xy-a = 0$. Hence $\psi'$ induces a homomorphism

$$\psi'' : D[x, y]/I \to D(x, y; \text{id}, a).$$

It is easy to check that $\psi''\psi = \text{id}_{D[x, y]}$ and $\psi\psi'' = \text{id}_{D[x, y]/I}$. Thus $D[x, y]/I \cong D(x, y; \text{id}, a)$. It follows from Proposition 4.2 that \(\text{GKdim}(D[x, y]) = \text{GKdim}(D(x, y; \text{id}, a)) = \text{GKdim}(D) + 1\).

Now we consider GWAs $P_n(\sigma, a)$ over $P_n$. Denote by $\deg(f)$ the total degree of $f \in P_n$. If $n = 0$, then $\sigma = \text{id}_k$ and $\text{GKdim}(k(\sigma, a)) = 1$. If $n = 1$, then $\sigma \in \text{Aut}(k[z])$ is determined by $\sigma(z) = bz + c$ for some $b \in k^*$ and $c \in k$. Hence every $\sigma \in \text{Aut}(k[z])$ is algebraically and thus $\text{GKdim}(k[z](\sigma, a)) = \text{GKdim}(k[z]) + 1 = 2$. For $n \geq 2$, we have the following lemma.

**Lemma 4.4.** Let $D = P_n, n \geq 2$, and $A = D(x, y; \sigma, a)$ be a GWA over $D$. Suppose that there exist $z \in D$ and a real number $r \geq 2$ such that $\deg(\sigma^{m+1}(z)) \geq r \deg(\sigma^m(z))$ for all $m \in \mathbb{N}$. Then $A$ has exponential growth. In particular, $\text{GKdim}(A) = \infty$.

**Proof.** Suppose that there exist $z \in D$ and $r \geq 2$ such that $\deg(\sigma^{m+1}(z)) \geq r \deg(\sigma^m(z))$ for all $m \in \mathbb{N}$. Consider elements of $A$ of the form

$$z^{q_0}x^{q_1}z^{q_2} \cdots z^{q_{p-1}}xz^p,$$

where $p \in \mathbb{N}$ and each $q_i$ is either 0 or 1. There are $2^{p+1}$ such elements (denoted by $a_1, \ldots, a_{2^{p+1}}$) for each $p \in \mathbb{N}$. It is easy to see that $a_1 \in V^{2^{p+1}}$, where $V = k + kz_1 + \cdots + kz_n + kx$.

**Claim.** $a_1, \ldots, a_{2^{p+1}}$ are linearly independent for all $p \in \mathbb{N}$.

**Proof.** Suppose that there exists a $q \in \mathbb{N}$ such that $a_1, \ldots, a_{2^{q+1}}$ are linearly dependent. Note that

$$z^{q_0}x^{q_1}z^{q_2} \cdots z^{q_{p-1}}xz^p = z^{q_0}(z^{q_1})\sigma(z^{q_2}) \cdots \sigma^p(z^q)x^p$$

and thus the degree with respect to $x$ of each $a_i$ is $q$. Write $a_i = t_i x^q$, $i = 1, 2, \ldots, 2^{q+1}$, $t_i \in D$. Since $a_1, \ldots, a_{2^{q+1}}$ are linearly dependent, so are $t_1, \ldots, t_{2^{q+1}}$. Thus there exist $t_1$ and $t_2$ such that $\deg(t_1) = \deg(t_2)$. Namely, there are two sequences $(\epsilon_0, \ldots, \epsilon_q) \neq (\epsilon_0', \ldots, \epsilon_q')$, each $\epsilon_i, \epsilon_i' \in \{0, 1\}$, such that

$$\epsilon_0d_0 + \cdots + \epsilon_qd_q = \epsilon_0'd_0 + \cdots + \epsilon_q'd_q,$$  \hspace{1cm} (4.1)

where each $d_i := \deg(\sigma^i(z))$. Let $l$ be the maximal index such that $\epsilon_l \neq \epsilon_l'$. Clearly $l \geq 1$. Without loss of generality, assume $\epsilon_l = 1$ and $\epsilon_l' = 0$. Then (4.1) becomes

$$\epsilon_0d_0 + \cdots + \epsilon_{l-1}d_{l-1} + d_l = \epsilon_0'd_0 + \cdots + \epsilon_{l-1}'d_{l-1}.$$  

By the hypothesis, $d_{m+1} \geq rd_m$ for all $m \in \mathbb{N}$, which implies that $d_l \leq (1/r)^{l-1}d_l$. Thus

$$d_l = (\epsilon_0' - \epsilon_0)d_0 + \cdots + (\epsilon_{l-1}' - \epsilon_{l-1})d_{l-1}$$

$$\leq d_0 + \cdots + d_{l-1} \leq \sum_{i=0}^{l-1} \left(\frac{1}{r}\right)^{l-i} d_l = \frac{1 - (\frac{1}{r})^l}{r - 1} d_l < d_l,$$

which leads to a contradiction. Hence the claim holds.

It follows from the claim that $\dim(V^{2^{p+1}}) \geq 2^{p+1}$ for all $p \in \mathbb{N}$ and thus $A$ has exponential growth.
Convention 4.5. For an automorphism \( \sigma : P_n \to P_n \) defined by \( \sigma(z_i) = f_i \in P_n, 1 \leq i \leq n \), we define \( \sigma = (f_1, f_2, \ldots, f_n) \).

Recall that an automorphism \( \sigma = (f_1, f_2, \ldots, f_n) \in \text{Aut}(P_n) \) is said to be triangular if \( f_i = \lambda_i z_i + g_i \), where \( \lambda_i \in k^* \) for \( 1 \leq i \leq n \), \( g_i \in k[z_{i+1}, \ldots, z_n] \) for \( 1 \leq i < n \) and \( g_n \in k \).

**Lemma 4.6.** Suppose that \( \sigma \in \text{Aut}(P_n) \) is conjugate to a triangular automorphism. Then \( \sigma \) is locally algebraic.\(^*\)

**Proof.** Let \( \delta, \tau \in \text{Aut}(P_n) \) and \( \tau = \delta^{-1} \sigma \delta \) be triangular. Note that for every finite-dimensional subspace \( U \subseteq P_n \), the subset \( \delta^{-1}(U) \subseteq P_n \) is also finite-dimensional. Since \( \tau \) is triangular and a triangular automorphism of \( P_n \) is locally algebraic (see [33, Proposition 2]), we have that \( \tau \) is locally algebraic. Hence, there exists a finite-dimensional subspace \( V_1 \subseteq P_n \) such that \( \delta^{-1}(U) \subseteq V_1 \) and \( \tau(V_1) = V_1 \). Thus, \( \delta(V_1) \) is a finite-dimensional subspace of \( P_n \) and \( U \subseteq \delta(V_1) \). Note that \( \sigma(\delta(V_1)) = \delta \tau \delta^{-1}(\delta(V_1)) = \delta(V_1) \), i.e., \( \delta(V_1) \) is \( \sigma \)-stable. Therefore \( \sigma \) is locally algebraic. \( \square \)

The converse of Lemma 4.6 is true in the case \( n = 2 \) (see Theorem 4.9). However, it does not hold in general (see the following example).

**Example 4.7.** The Nagata automorphism (see [42])

\[
(x - 2y(xz + y^2) - z(xz + y^2)^2, y + z(xz + y^2), z) \in \text{Aut}(k[x, y, z])
\]

is locally algebraic but not conjugate to a triangular automorphism (see [5, 20]).

**Corollary 4.8.** Let \( A = P_n(\sigma, a) \) be a GWA. If \( \sigma^m \) is conjugate to a triangular automorphism for some \( m \geq 1 \), then \( \text{GKdim}(A) = n + 1 \).

**Proof.** Let \( D = P_n \), and \( B \) and \( b \) be as in Lemma 4.1. Then \( B = D(\sigma^m, b) \) and \( \text{GKdim}(A) = \text{GKdim}(B) \) by Lemma 4.1. By the assumption and Lemma 4.6, \( \sigma^m \) is locally algebraic and thus \( \text{GKdim}(B) = n + 1 \) by Lemma 2.12. Hence \( \text{GKdim}(A) = n + 1 \). \( \square \)

Now we consider the case \( n = 2 \).

**Theorem 4.9.** Let \( A = k[z_1, z_2](\sigma, a) \) be a GWA. Then \( \text{GKdim}(A) = 3 \) if \( \sigma \) is conjugate to a triangular automorphism, and \( \text{GKdim}(A) = \infty \) otherwise.

**Proof.** Let \( D = k[z_1, z_2] \). Suppose that \( \sigma \) is conjugate to a triangular automorphism. By Corollary 4.8, \( \text{GKdim}(A) = 3 \).

Suppose that \( \sigma \) is not conjugate to a triangular automorphism. Let \( \pi = (z_2, z_1) \in \text{Aut}(D) \) (see Convention 4.5). By [32, Lemma 3], \( \sigma \) is conjugate to an automorphism \( \sigma' \) of the form

\[
\sigma' = \tau_1 \tau_2 \cdots \tau_s \pi, \quad s \geq 1,
\]

where for each positive integer \( i (i \leq s) \),

\[
\tau_i = (\alpha_i, z_1 + \beta_i(z_2), u_i z_2 + v_i), \quad \alpha_i, u_i \in k^*, \quad v_i \in k, \quad \beta_i(z_2) \in k[z_2], \quad d_i := \text{deg}(\beta_i) > 1.
\]

We claim that

\[
\sigma'(z_2) = \lambda z_2^{d_1 \cdots d_s} + \text{(lower total degree terms)}, \quad \lambda \in k^*.
\]

We prove the claim by induction on \( s \). If \( s = 1 \), then \( \sigma'(z_2) = \tau_1(z_2) = \alpha_1 z_1 + \beta_1(z_2) = \lambda z_2^{d_1} + \alpha_1 z_1 \).

Thus the claim holds for \( s = 1 \). Suppose that the claim holds for automorphisms with \( s \geq 1 \). For \( \sigma' = \tau_1 \tau_2 \cdots \tau_{s+1} \pi \), we have

\[
\sigma'(z_2) = \tau_1 \pi(\lambda z_2^{d_1 \cdots d_{s-1}} + T) = \tau_1(\lambda z_2^{d_1 \cdots d_{s-1}}) + \tau_1(\pi(T)) = \lambda (\alpha_1 z_1 + \beta_1(z_2))^{d_1 \cdots d_{s-1}} + \tau_1(\pi(T)) = \lambda z_2^{d_1 \cdots d_{s-1}} + T' + \tau_1(\pi(T)),
\]
where $\lambda, \lambda' \in k^*$, $\deg(T) < d_2d_3 \cdots d_{s+1}$ and $\deg(T') < d_1d_2 \cdots d_{s+1}$. Since
\[ \deg(\tau_1\pi(T)) = \deg(\tau_1(T)) < d_1d_2 \cdots d_{s+1}, \]
we have
\[ \sigma'(z_2) = \lambda'z_2^{d_1d_2 \cdots d_{s+1}} + (\text{lower total degree terms}) \]
as desired.

It follows from the claim that
\[ \deg(\sigma'^{m+1}(z_2)) = \deg((\tau_1\pi\tau_2 \cdots \pi\tau_s\pi)^{m+1}(z_2)) = (d_1 \cdots d_s)^{m+1} \geq (d_1 \cdots d_s)^{m} = 2\deg(\sigma^m(z_2)). \]
Now, by Lemma 4.4, $D(\sigma', b)$ has exponential growth and GKdim($D(\sigma', b)$) = $\infty$ for all $b \in k[z_1, z_2]$. Since $\sigma$ is conjugate to $\sigma'$, we may suppose $\sigma = \delta^{-1}\sigma'\delta$ for some $\delta \in \text{Aut}(k[z_1, z_2])$. Then by Lemma 2.9, we have
\[ k[z_1, z_2][\sigma, a] \cong k[z_1, z_2][\sigma', \delta^{-1}(a)]. \]
Hence GKdim($D(\sigma, a)$) = GKdim($D(\sigma', \delta^{-1}(a))$) = $\infty$. \hfill $\square$

**Remark 4.10.** Note that a key element in the proof of Theorem 4.9 is [32, Lemma 3], which is based on the famous theorem of Jung [28] and van der Kulk [49] that asserts that $\text{Aut}((P_m)$ is generated by triangular automorphisms and the permutation $\pi = (z_2, z_1)$, i.e., all $\sigma \in \text{Aut}(P_2)$ can be written as a product of triangular automorphisms (and their inverses) and $\pi$. Shestakov and Umirbaev [45] proved that there exist automorphisms $\tau \in \text{Aut}(P_2)$ that are not tame. As a result, such $\tau$ cannot be presented as a product of triangular automorphisms (and their inverses) and affine automorphisms (i.e., automorphisms $\delta = (f_1, f_2, f_3) \in \text{Aut}(P_3)$ with $\deg(f_i) = 1$ for $i = 1, 2, 3$) of $P_3$.

We can now put everything together to prove Theorem 1.2.

**Proof of Theorem 1.2.** (c) $\Rightarrow$ (d) and (f) $\Rightarrow$ (g) are trivial. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (e) follow from Theorem 4.9 and Proposition 3.6, respectively. (d) $\Rightarrow$ (f) follows from the proof of Theorem 4.9. (g) $\Rightarrow$ (a) follows from Corollary 4.8.

(c) $\Rightarrow$ (c). Suppose that $\sigma$ is locally algebraic. Let $V$ be a $\sigma$-stable generating subframe of $P_2$ such that $a \in V$ (and thus $\sigma(a) \in V$). Then $W := V + xV + ky$ is a generating subframe of $A$. Since
\[ W^{2m} = (V + kx + ky)^{2m} \supseteq V^m + V^mx + \cdots + V^mx^m, \quad m \geq 1, \]
we have
\[ \dim(W^{2m}) > (m + 1)\dim(V^m), \quad m \geq 1. \] (4.2)

On the other hand, we claim that
\[ W^m \subseteq \sum_{i=1}^{m} V^mx^i + \sum_{i=1}^{m} V^my^i + V^m, \quad m \geq 1. \] (4.3)

The claim is obvious when $m = 1$. Suppose that the inclusion (4.3) holds for some $m \geq 1$. Then
\[ W^{m+1} \subseteq W\left(\sum_{i=1}^{m} V^mx^i + \sum_{i=1}^{m} V^my^i + V^m\right) \quad \text{(induction hypothesis)} \]
\[ = kx \sum_{i=1}^{m} V^mx^i + kx \sum_{i=1}^{m} V^my^i + kxV^m \]
Hence the inclusion (4.3) holds for any m ≥ 1. By the \(\mathbb{Z}\)-graded structure of GWAs, the sum in (4.3) is direct and thus
\[
\dim(W^m) \leq (2m + 1) \dim(V^m), \quad m \geq 1.
\]
(4.4)
Combining (4.2) and (4.4) gives
\[
(m + 1) \dim(V^m) \leq \dim(W^{2m}) \leq (4m + 1) \dim(V^{2m}), \quad m \geq 1.
\]
Since the growth of \(P_2\) is independent of the choice of generating subspaces,
\[
\dim(V^m) \sim \dim(kz_1 + kz_2)^m \sim m^2.
\]
We obtain that \(\dim(W^m) \sim m^3\) and thus \(A\) has polynomial growth. Figure 1 summarizes the relations we have proved among the items in Theorem 1.2, which completes the proof.

We conclude this section with an example, which shows that the dichotomy of the GK-dimension appearing in Theorem 4.9 (i.e., GKdim(\(P_2(\sigma, a)\)) is either 3 or \(\infty\)) cannot be extended to that of GWAs over a general base algebra of GK-dimension two. Note that it is proved by Rogalski [44] that if \(k\) is algebraically closed and \(K/k\) is a finitely generated field extension with GKdim(\(K\)) = 2, then every “big subalgebra” [44, Definition 6.1] of the GWA \(K(x, y; \alpha, 1) = K[x, x^{-1}; \alpha]\) has the same GK-dimension \(d \in \{3, 4, 5, \infty\}\).

**Example 4.11.** Let \(H_m = H_m(\alpha_1, \alpha_2) := k[z_1^{\pm 1}, z_2^{\pm 1}]/\{x, y; \sigma_m, a\}\) be a GWA, where
\[
\sigma_m(z_1) = \alpha_1 z_1, \quad \sigma_m(z_2) = \alpha_2 z_1^{\pm 1} z_2, \quad \alpha_1, \alpha_2 \in k^*, \quad m \in \mathbb{N}.
\]
Then GKdim(\(H_m\)) = 4 for all \(m \neq 0\) and GKdim(\(H_0\)) = 3.

![Figure 1](image-url) Implication relations among the items in Theorem 1.2
Proof. Without loss of generality, we assume $\alpha_1 = \alpha_2 = 1$. In fact, since $V = k[z_1^{\pm 1}, z_2^{\pm 1}, x, y]$ is a generating subspace of both $H_m(\alpha_1, \alpha_2)$ and $H_m(1, 1)$, we have $\text{GKdim}(H_m(\alpha_1, \alpha_2)) = \text{GKdim}(H_m(1, 1))$. We may also suppose $m \geq 0$, since $H_m \cong H_{-m}$ via the isomorphism determined by $z_1 \mapsto z_1^{-1}$, $z_2 \mapsto z_2$, $x \mapsto x$ and $y \mapsto y$.

It is clear that $\sigma_0$ is locally algebraic and thus $\text{GKdim}(H_0) = 3$. Now assume $m \geq 1$. Note that $H_1$ is the group algebra of the first (discrete) Heisenberg group (also known as the generic quantum torus [27, Example 2.5(ii)]). It is well known that $\text{GKdim}(H_1) = 4$ (see, e.g., [31, Example 11.10]). By Lemma 4.1, the subalgebra $B = k[z_1^{\pm 1}, z_2^{\pm 1}][x^m, y^m; \sigma_1^m, 1] \subseteq H_1$ is a GWA and $\text{GKdim}(B) = \text{GKdim}(H_1) = 4$. By the universal property of GWAs (see Proposition 2.5), the mappings

$$
\psi : H_m \to B, \quad z_1 \mapsto z_1, \quad z_2 \mapsto z_2, \quad x \mapsto x^m, \quad y \mapsto y^m
$$

and

$$
\psi' : B \to H_m, \quad z_1 \mapsto z_1, \quad z_2 \mapsto z_2, \quad x^m \mapsto x, \quad y^m \mapsto y
$$

determine two algebra homomorphisms such that $\psi \psi' = \text{id}_B$ and $\psi' \psi = \text{id}_{H_m}$. Hence $H_m \cong B$ and thus $\text{GKdim}(H_m) = \text{GKdim}(B) = 4$.

5 The GK-dimension of GWAs over Laurent polynomial algebras

In this section, we first characterize the automorphisms of the Laurent polynomial algebra

$$
L_n = k[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_n^{\pm 1}]
$$

and then investigate the GK-dimension of GWAs over $L_n$. Finally, we prove Theorem 1.3 and give an application on cancellation problem.

Denote by $\mathbb{Z}^{m \times n}$ (resp. $\text{GL}(n, \mathbb{Z})$) the set of all the $m \times n$ integer matrices (resp. the general linear group of degree $n$ over $\mathbb{Z}$). For $M = (a_{ij}) \in \mathbb{Z}^{m \times n}$, let $M[i, j] := a_{ij}$ be the $(i, j)$-th element of $M$, $M[i, -] := (a_{i1}, a_{i2}, \ldots, a_{in})$ be the $i$-th row of $M$, and $M[-, j] := (a_{1j}, a_{2j}, \ldots, a_{nj})^T$ be the $j$-th column of $M$, where $T$ stands for the transpose operation.

It is well known that the unit group of $L_n$ is

$$
U(L_n) = \{a z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n} : \alpha \in k^*, i_1, i_2, \ldots, i_n \in \mathbb{Z}\}.
$$

Since an automorphism $\sigma \in \text{Aut}(L_n)$ sends a unit to a unit, we have

$$
\sigma(z_i) = a_i z_1^{a_{i1}} z_2^{a_{i2}} \cdots z_n^{a_{in}}, \quad a_i \in k^*, \quad a_{i1}, a_{i2}, \ldots, a_{in} \in \mathbb{Z}, \quad i = 1, 2, \ldots, n. \tag{5.1}
$$

The matrix $M = M_\sigma = (a_{ij}) \in \mathbb{Z}^{m \times n}$ (resp. the scalar $\sigma = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (k^*)^n$) is uniquely determined by the automorphism $\sigma$ and we write $\sigma = (M, \alpha)$. We introduce the following convention.

Conjecture 5.1. Define $z := (z_1, z_2, \ldots, z_n)$. Given rows $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in L_n^\ast$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in L_n^\ast$, a column $b = (b_1, b_2, \ldots, b_n)^T \in \mathbb{Z}^{n \times 1}$, and a matrix $M \in \mathbb{Z}^{n \times n}$, we use the following notation:

$$
\alpha b := a_1^b z_1^{a_{11}} z_2^{a_{12}} \cdots z_n^{a_{1n}} \in L_n,
\alpha^M := (\alpha^{M[-1]}, \alpha^{M[-2]}, \ldots, \alpha^{M[-n]}) \in L_n^\ast,
\alpha \beta := (\alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_n \beta_n) \in L_n^\ast.
$$

In particular, the element-wise multiplication

$$
\alpha \beta = (\alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_n \beta_n), \quad \forall \alpha, \beta \in (k^*)^n
$$
Proof. (i) and (iii) are obvious. (ii) can be checked directly.

Theorem 5.4. Let \( L_n = k[z_1^{±1}, z_2^{±1}, \ldots, z_n^{±1}] \). Then \( \text{Aut}(L_n) \cong \text{GL}(n, \mathbb{Z}) \times (k^*)^n \) as groups, where the group operation of the semiproduct is given by Lemma 5.3(i).

Proof. It follows from Lemma 5.3(ii) that the mapping

\[
f : \text{Aut}(L_n) \rightarrow \text{GL}(n, \mathbb{Z}) \times (k^*)^n, \quad \sigma \mapsto (M_\sigma, \alpha_\sigma)
\]

is well defined and bijective. It is routine to check that \( \text{GL}(n, \mathbb{Z}) \times (k^*)^n \) has a semidirect structure with the product given by Lemma 5.3(i).

Definition 5.5. Suppose \( M \in \mathbb{Z}^{n \times n} \) and

\[
\|M\| := \max_{1 \leq i,j \leq n} |M[i,j]|.
\]

(i) \( M \) is called power-bounded (by \( N \)) if there exists an \( N \in \mathbb{N}^* \) such that \( \|M^m\| < N \) for all \( m \in \mathbb{N} \).

(ii) \( M \) has a finite order if \( M^m = I \) for some \( m \in \mathbb{N}^* \), and the smallest such \( m \) is called the order of \( M \), denoted by \( o(M) \).

Note that if \( \sigma = (\alpha, M) \in \text{Aut}(L_n) \) has a finite order, then so does \( M \). But, as demonstrated in Example 5.8, the reverse is not true in general.

Lemma 5.6. Let \( \sigma = (\alpha, M) \in \text{Aut}(L_n) \), \( a \in L_n \) and \( A = L_n(x, y; \sigma, a) \). Then the following statements are equivalent:

(i) \( M \) is power-bounded.

(ii) \( M \) has a finite order.

(iii) \( \sigma \) is locally algebraic.

Proof. (i) \( \Rightarrow \) (ii). Suppose that \( \|M^m\| < N \) for some \( N \geq 1 \) and all \( m \in \mathbb{N} \). Then the entries of matrix \( M^m \) must be integers between \(-N\) and \( N \) for all \( m \in \mathbb{N} \). Hence, \( \{M^m : m \in \mathbb{N}\} \) is a finite subgroup of \( \text{GL}(n, \mathbb{Z}) \). Thus, \( M \) is of finite order.
(ii) \Rightarrow (iii). Suppose that \( M \) has a finite order. Then \( \{M^i : i \geq 0\} \) is a finite set. Let \( N \) be the maximal number that appears as an entry of the matrices \( \{M^i : i \geq 0\} \).

**Claim.** \( \sigma^m(z) = \beta_mz^m \) for \( m \geq 1 \), where
\[
\beta_m = \alpha^{1+M+M^2+\cdots+M^{m-1}}, \quad m \geq 1.
\]

**Proof.** The claim holds for \( m = 1 \) by (5.3). Assume that the claim holds for some \( m \geq 1 \). Then by (5.3) and Lemma 5.2,
\[
\sigma^{m+1}(z) = \sigma(\beta_m z^m) = \beta_m \sigma(z)^m = \beta_m(\alpha z ^M )^m = \beta_m \alpha^M z^m = \beta_{m+1} z^{m+1}.
\]

Thus the claim holds for all \( m \geq 1 \).

By the claim, we have that
\[
\sigma^n(z_i) \in V := k\{z_1^i z_2^j \cdots z_n^l : -N \leq i_j \leq N, 1 \leq j \leq n\}, \quad m \geq 1,
\]
where \( V \) is a finite-dimensional subspace with \( \dim(V) \leq (2N + 1)^n \). Suppose \( T = z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} \in L_n \), and each \( a_i \in \mathbb{Z} \). Then
\[
\sigma^n(T) = (\sigma(z_1))^{a_1} (\sigma(z_2))^{a_2} \cdots (\sigma(z_n))^{a_n},
\]
which is contained in the finite-dimensional subspace
\[
V_T := k\{z_1^i z_2^j \cdots z_n^l : -N' \leq i_j \leq N', 1 \leq j \leq n\}
\]
for all \( m \geq 1 \), where \( N' = N(|a_1| + |a_2| + \cdots + |a_n|) \). As a result, for each Laurent polynomial \( p \in L_n \), i.e.,
\[
p = c_1 T_1 + c_2 T_2 + \cdots + c_s T_s, \quad c_i \in k^*, \quad T_i \text{ is Laurent monomial,} \quad i = 1, 2, \ldots, n,
\]
we have that \( \sigma^m(p) \) is contained in the finite-dimensional space \( V_{T_1} + V_{T_2} + \cdots + V_{T_s} \) for all \( m \geq 1 \). Therefore, by Lemma 2.11, \( \sigma \) is locally algebraic.

(iii) \Rightarrow (i). Suppose that \( M \) is not power-bounded, i.e., for every \( l > 1 \), there exists an \( m = m(l) > 1 \) such that \( \|M^m\| > l \). Without loss of generality, assume \( \|M^m\| = \|M^m[1,1]\| \) for infinitely many \( m \geq 1 \).

Let
\[
V = k z_1 + k z_2 + \cdots + k z_n + k z_1^{-1} + k z_2^{-1} + \cdots + k z_n^{-1}.
\]
It follows from the above claim that
\[
\sigma^m(z_i) = c z_1^{M^m[1,1]} z_2^{M^m[2,1]} \cdots z_n^{M^m[n,1]} \notin V.
\]

Thus the subspace \( k\{\sigma^m(z_1) : m \geq 1\} \) is not finite-dimensional. Therefore, \( \sigma \) is not locally algebraic.

**Proof of Theorem 1.3.** It follows from Proposition 3.6 and Lemma 5.6.

**Remark 5.7.** It follows from Lemma 5.6 that the elements of finite order in \( \text{GL}(n, \mathbb{Z}) \) play an important role. Given \( n \geq 1 \), there are only finitely many integers that can be the orders of matrices in \( \text{GL}(n, \mathbb{Z}) \). Levitt and Nicolas [34] showed that the maximum order \( G(n) \) of elements of finite order in \( \text{GL}(n, \mathbb{Z}) \) satisfies a Landau-type estimate \( G(n) \sim \sqrt{n} \ln n \) and there is a method of dynamical programming to compute \( G(n) \). Following [34], Table 1 gives the values of \( G(n) \) for \( n \leq 20 \), where \( G(n) \) is omitted for odd \( n = 2p + 1 \), since \( G(2p + 1) = G(2p) \).

| \( n \) | 1 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( G(n) \) | 2 | 6 | 12 | 30 | 60 | 120 | 210 | 420 | 840 | 1,260 | 2,520 |

*Table 1* Maximum orders of matrices in \( \text{GL}(n, \mathbb{Z}) \)
By using [34, Proposition 1.1], we can determine whether an integer is the order of an element of $\text{GL}(n,\mathbb{Z})$. Note that for a given order $o$, there may be many matrices of order $o$ (see Example 5.8).

**Example 5.8.** Suppose that $c \in k^*$ and $\sigma = \sigma_q \in \text{Aut}(L_2)$ is defined as

$$\sigma(z_1) = cz_1^{1-q}z_2^{-q}, \quad \sigma(z_2) = z_1^{-q}z_2^{-1}, \quad q \in \mathbb{Z}.$$  

Then $\sigma^m \neq \text{id}$ for all $m \geq 1$. But

$$M_o = \begin{pmatrix} 1-q & q \\ 2-q & q-1 \end{pmatrix}$$

has a finite order $o(M) = 2$ for all $q \in \mathbb{Z}$ and thus

$$\text{GKdim}(L_2(\sigma, a)) = \text{GKdim}(L_2) + 1 = 3$$

by Theorem 1.3.

We conclude this section by an application of Theorem 1.3 to the cancellation problem.

**Definition 5.9** (See [48, Definitions 0.1 and 0.8]). Let $A$ be an algebra.

(i) We say that $A$ is cancellative (resp. strongly cancellative) if the isomorphism

$$A[x] \cong B[x] \text{ (resp. } A[x_1, \ldots, x_m] \cong B[x_1, \ldots, x_m])$$

implies $A \cong B$ for all the algebras $B$ (resp. for all the algebras $B$ and all the integers $m \geq 1$).

(ii) We say that $A$ is $\sigma$-algebraically cancellative (resp. strongly $\sigma$-algebraically cancellative) if the isomorphism between Ore extensions (resp. iterated Ore extensions)

$$A[x; \sigma'] \cong B[x; \sigma''] \text{ (resp. } A[x_1; \sigma'_1]\cdots[x_m; \sigma'_m] \cong B[x_1; \sigma''_1]\cdots[x_m; \sigma''_m])$$

implies $A \cong B$ for all the algebras $B$ and the locally algebraic automorphisms $\sigma'$ and $\sigma''$ (resp. for all the algebras $B$, the locally algebraic automorphisms $\sigma'_i$ and $\sigma''_i$, $1 \leq i \leq m$ and all the integers $m \geq 1$).

The following lemma follows directly from the definitions and its proof is omitted.

**Lemma 5.10.** Let $A$ be an algebra.

(i) If $A$ is strongly cancellative (resp. strongly $\sigma$-algebraically cancellative), then $A$ is cancellative (resp. $\sigma$-algebraically cancellative).

(ii) If $A$ is $\sigma$-algebraically cancellative (resp. strongly $\sigma$-algebraically cancellative), then $A$ is cancellative (resp. strongly cancellative).

Theorem 1.3 together with the results from [48] can be applied to get some cancellation properties of GWAs.

**Corollary 5.11.** Let $A = L_n(\sigma, a)$ be a GWA with $a \neq 0$ such that one of the equivalent conditions in Theorem 1.3 is satisfied. Then $A$ is strongly $\sigma$-algebraically cancellative. As a consequence, $A$ is strongly cancellative and cancellative.

**Proof.** Since $L_n$ is a finitely generated domain and $a \neq 0$, it follows from Lemma 2.14 that $A$ is a finitely generated domain. It is easy to check that the identity element 1 is a controlling element (see [48, Definition 4.2]) of $A$, i.e., $D(1) = A$. By the assumption and Theorem 1.3,

$$\text{GKdim}(A) = n + 1 < \infty,$$

and thus $A$ is strongly $\sigma$-algebraically cancellative by [48, Theorem 0.9].

**Acknowledgements.** This work was supported by Huizhou University (Grant Nos. hzu202001 and 2021JB022) and the Guangdong Provincial Department of Education (Grant Nos. 2020KTSCX145 and 2021ZDJS080). The author thanks the referees for their careful reading, helpful suggestions and useful references. Part of this work...
was done during a visit to James J. Zhang at the University of Washington. The author expresses deep gratitude to James J. Zhang for many stimulating conversations on the subject, and to the department of Mathematics at the University of Washington for the hospitality during his visit. The author wishes to thank Yu Li, Wenchao Zhang and Xiaotong Sun for helpful discussions.

References
1. Almulhem M, Brzeziński T. Skew derivations on generalized Weyl algebras. J Algebra, 2018, 493: 194–235
2. Andruchewitsch N, Angiono I, Heckenberger I. On Nichols algebras of infinite rank with finite Gelfand-Kirillov dimension. Atti Accad Naz Lincei Rend Lincei Mat Appl, 2020, 31: 81–101
3. Bai Y X, Chen Y Q, Zhang Z R. Gelfand-Kirillov dimension of bicommutative algebras. Linear Multilinear Algebra, 2022, in press
4. Bao Y H, Ye Y, Zhang J J. Truncation of unitary operads. Adv Math, 2020, 372: 107290
5. Bass H. A non-triangular action of $G_a$ on $A^3$. J Pure Appl Algebra, 1984, 33: 1–5
6. Bavula V V. Generalized Weyl algebras and their representations. St Petersburg Math J, 1993, 4: 71–92
7. Bavula V V. Filter dimension of algebras and modules, a simplicity criterion of generalized Weyl algebras. Comm Algebra, 1996, 24: 1971–1992
8. Bavula V V. Global dimension of generalized Weyl algebras. In: Representation Theory of Algebras, vol. 18. Providence: Amer Math Soc, 1996, 81–107
9. Bavula V V, Jordan D A. Isomorphism problems and groups of automorphisms for generalized Weyl algebras. Trans Amer Math Soc, 2001, 353: 769–794
10. Bavula V V, Lenagan T H. Krull dimension of generalized Weyl algebras with noncommutative coefficients. J Algebra, 2001, 235: 315–358
11. Bavula V V, Lu T. The quantum Euclidean algebra and its prime spectrum. Israel J Math, 2017, 219: 929–958
12. Bavula V V, van Oystaeyen F. Krull dimension of generalized Weyl algebras and iterated skew polynomial rings: Commutative coefficients. J Algebra, 1998, 208: 1–34
13. Bell J P, Zelmanov E. On the growth of algebras, semigroups, and hereditary languages. Invent Math, 2021, 224: 683–697
14. Bell J P, Zhang J J. Zariski cancellation problem for noncommutative algebras. Selecta Math (NS), 2017, 23: 1709–1737
15. Bruns W, Gubeladze J. Polytopes, Rings, and K-Theory. New York: Springer, 2009
16. Brzeziński T. Noncommutative differential geometry of generalized Weyl algebras. SIGMA Symmetry Integrability Geom Methods Appl, 2016, 12: 050
17. Ceken S, Palmieri J H, Wang Y H, et al. The discriminant controls automorphism groups of noncommutative algebras. Adv Math, 2015, 269: 551–584
18. Chan K, Gaddis J, Won R, et al. Reflexive hull discriminants and applications. Selecta Math (NS), 2022, 28: 40
19. Ferraro L, Gaddis J, Won R. Simple $Z$-graded domains of Gelfand-Kirillov dimension two. J Algebra, 2020, 562: 433–465
20. Furter J-P. Quasi-locally finite polynomial endomorphisms. Math Z, 2009, 263: 473–479
21. Gaddis J, Won R. Fixed rings of generalized Weyl algebras. J Algebra, 2019, 536: 149–169
22. Gelfand I M, Kirillov A A. Sur les corps liés aux algèbres enveloppantes des algèbres de Lie. Publ Math Inst Hautes Études Sci, 1966, 31: 5–19
23. Goodearl K R, Zhang J J. Non-affine Hopf algebra domains of Gelfand-Kirillov dimension two. Glasg Math J, 2017, 59: 563–593
24. Grigorchuk R, Pak I. Groups of intermediate growth: An introduction. Enseign Math, 2008, 54: 251–272
25. Gutierrez J, Valqui C. Bivariant K-theory of generalized Weyl algebras. J Noncommut Geom, 2020, 14: 639–666
26. Hub C, Kim C O. Gelfand-Kirillov dimension of skew polynomial rings of automorphism type. Comm Algebra, 1996, 24: 2317–2323
27. Jordan D A, Sason N. Reversible skew Laurent polynomial rings and deformations of Poisson automorphisms. J Algebra Appl, 2009, 8: 733–757
28. Jung H W E. Über ganze birationale transformationen der Ebene. J Reine Angew Math, 1942, 184: 161–174
29. Khoroshkin A, Piontkovski D. On generating series of finitely presented operads. J Algebra, 2015, 426: 377–429
30. Klyuev D. Twisted traces and positive forms on generalized $q$-Weyl algebras. SIGMA Symmetry Integrability Geom Methods Appl, 2022, 18: 009
31. Krause G R, Lenagan T H. Growth of Algebras and Gelfand-Kirillov Dimension. Providence: Amer Math Soc, 2000
32. Lane D R. Fixed points of affine Cremona transformations of the plane over an algebraically closed field. Amer J Math, 1975, 97: 707–732
33. Leroy A, Matczuk J, Okninski J. On the Gelfand-Kirillov dimension of normal localizations and twisted polynomial rings. In: Perspectives in Ring Theory. NATO ASI Series, vol. 233. Dordrecht: Springer, 1988, 205–214
34 Levitt G, Nicolas J-L. On the maximum order of torsion elements in GL(n, Z) and Aut(F_n). J Algebra, 1998, 208: 630–642
35 Lezama O, Wang Y H, Zhang J J. Zariski cancellation problem for non-domain noncommutative algebras. Math Z, 2019, 292: 1269–1290
36 Liu L Y. On homological smoothness of generalized Weyl algebras over polynomial algebras in two variables. J Algebra, 2018, 498: 228–253
37 Liu L Y, Ma W. Batalin-Vilkovisky algebra structures on Hochschild cohomology of generalized Weyl algebras. Front Math China, 2022, in press
38 Lorenz M. On the Gelfand-Kirillov dimension of skew polynomial rings. J Algebra, 1982, 77: 186–188
39 Matczuk J. The Gelfand-Kirillov dimension of Poincaré-Birkhoff-Witt extensions. In: Perspectives in Ring Theory. NATO ASI Series, vol. 233. Dordrecht: Springer, 1988, 221–226
40 McConnell J C, Robson J C. Noncommutative Noetherian Rings. Graduate Studies in Mathematics, vol. 30. Providence: Amer Math Soc, 2001
41 Milnor J. A note on curvature and fundamental group. J Differential Geom, 1968, 2: 1–7
42 Nagata M. On Automorphism Group of k[x,y]. Tokyo: Kinokuniya, 1972
43 Qi Z H, Xu Y J, Zhang J J, et al. Growth of nonsymmetric operads. Indiana Univ Math J, 2023, in press
44 Rogalski D. GK-dimension of birationally commutative surfaces. Trans Amer Math Soc, 2009, 361: 5921–5945
45 Shestakov I P, Umirbaev U U. Poisson brackets and two-generated subalgebras of rings of polynomials. J Amer Math Soc, 2003, 17: 181–196
46 Smith M K. Universal enveloping algebras with subexponential but not polynomially bounded growth. Proc Amer Math Soc, 1976, 60: 22–24
47 Suárez-Alvarez M, Vivas Q. Automorphisms and isomorphisms of quantum generalized Weyl algebras. J Algebra, 2019, 424: 540–552
48 Tang X, Zhang J J, Zhao X G. Cancellation of Morita and skew types. Israel J Math, 2021, 244: 467–500
49 van der Kulk W. On polynomial rings in two variables. Nieuw Arch Wiskd (5), 1953, 3: 33–41
50 Wang D G, Zhang J J, Zhuang G. Connected Hopf algebras of Gelfand-Kirillov dimension four. Trans Amer Math Soc, 2015, 367: 5597–5632
51 Won R. The noncommutative schemes of generalized Weyl algebras. J Algebra, 2018, 506: 322–349
52 Wu Q S. Gelfand-Kirillov dimension under base field extension. Israel J Math, 1991, 73: 289–296
53 Zhang J J. A note on GK dimension of skew polynomial extensions. Proc Amer Math Soc, 1997, 125: 363–373
54 Zhang Y, Zhao X G. Gelfand-Kirillov dimension of differential difference algebras. LMS J Comput Math, 2014, 17: 485–495
55 Zhao X G, Mo Q H, Zhang Y. Gelfand-Kirillov dimension of generalized Weyl algebras. Comm Algebra, 2018, 46: 4403–4413