Distributed Order Calculus: an Operator-Theoretic Interpretation

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Abstract

Within the functional calculi of Bochner-Phillips and Hirsch, we describe the operators of distributed order differentiation and integration as functions of the classical differentiation and integration operators respectively.

1. Introduction and Preliminaries. In the distributed order calculus [11], used in physics for modeling ultraslow diffusion and relaxation phenomena, we consider derivatives and integrals of distributed order. The definitions are as follows.

Let \( \mu \) be a continuous non-negative function on \([0, 1]\). The distributed order derivative \( D^{(\mu)} \) of weight \( \mu \) for a function \( \varphi \) on \([0, T]\) is

\[
(D^{(\mu)} \varphi)(t) = \int_0^1 (D^{(\alpha)} \varphi)(t) \mu(\alpha) \, d\alpha
\]

where \( D^{(\alpha)} \) is the Caputo-Dzhrbashyan regularized fractional derivative of order \( \alpha \), that is

\[
(D^{(\alpha)} \varphi)(t) = \frac{1}{\Gamma(1-\alpha)} \left[ \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} \varphi(\tau) \, d\tau - t^{-\alpha} \varphi(0) \right], \quad 0 < t \leq T.
\]

Denote

\[
k(s) = \int_0^1 \frac{s^{-\alpha}}{\Gamma(1-\alpha)} \mu(\alpha) \, d\alpha, \quad s > 0.
\]

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It is obvious that $k$ is a positive decreasing function. The definition (1)-(2) can be rewritten as

$$
(D^{(\mu)} \varphi)(t) = \frac{d}{dt} \int_0^t k(t - \tau) \varphi(\tau) d\tau - k(t)\varphi(0).
$$

(4)

The right-hand side of (4) makes sense for a continuous function $\varphi$, for which the derivative $\frac{d}{dt} \int_0^t k(t - \tau) \varphi(\tau) d\tau$ exists.

If a function $\varphi$ is absolutely continuous, then

$$
(D^{(\mu)} \varphi)(t) = \int_0^t k(t - \tau) \varphi'(\tau) d\tau.
$$

(5)

Below we always assume that $\mu \in C^3[0, 1]$, $\mu(1) \neq 0$, and either $\mu(0) \neq 0$, or $\mu(\alpha) \sim a\alpha^\nu$, $a, \nu > 0$, as $\alpha \to 0$. Under these assumptions (see [11]),

$$
k(s) \sim s^{-1}(\log s)^{-2}\mu(1), \quad s \to 0,
$$

$$
k'(s) \sim -s^{-2}(\log s)^{-2}\mu(1), \quad s \to 0,
$$

so that $k \in L_1(0, T)$ and $k$ does not belong to any $L_p$, $p > 1$. We cannot differentiate under the integral in (4), since $k'$ has a non-integrable singularity.

It is instructive to give also the asymptotics of the Laplace transform

$$
\mathcal{K}(z) = \int_0^\infty k(s)e^{-zs}ds.
$$

Using (4) we find that

$$
\mathcal{K}(z) = \int_0^1 z^{\alpha-1}\mu(\alpha) d\alpha,
$$

so that $\mathcal{K}(z)$ can be extended analytically to an analytic function on $\mathbb{C} \setminus \mathbb{R}_-$, $\mathbb{R}_- = \{ z \in \mathbb{C} : \text{Im } z = 0, \text{Re } z \leq 0 \}$. If $z \in \mathbb{C} \setminus \mathbb{R}_-\, |z| \to \infty$, then [11]

$$
\mathcal{K}(z) = \frac{\mu(1)}{\log z} + O\left((\log |z|)^{-2}\right);
$$

(6)

see [11] for further properties of $\mathcal{K}$.

The distributed order integral $II^{(\mu)}$ is defined as the convolution operator

$$
(II^{(\mu)} f)(t) = \int_0^t \kappa(t - s)f(s)ds, \quad 0 \leq t \leq T,
$$

(7)
where \( \kappa(t) \) is the inverse Laplace transform of the function \( z \mapsto \frac{1}{zK(z)} \),

\[
\kappa(t) = \frac{d}{dt} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt} \frac{1}{zK(z)} \, dz, \quad \gamma > 0.
\] (8)

It was proved in [11] that \( \kappa \in C^\infty(0, \infty) \), and \( \kappa \) is completely monotone; for small values of \( t \),

\[
\kappa(t) \leq C \log \frac{1}{t}, \quad |\kappa'(t)| \leq Ct^{-1} \log \frac{1}{t},
\] (9)

If \( f \in L^1(0, T) \), then \( D(\mu)I(\mu)f = f \).

The aim of this paper is to clarify the operator-theoretic meaning of the above constructions. It is well known that fractional derivatives and integrals can be interpreted as fractional powers of the differentiation and integration operators in various Banach spaces; see, for example, [1, 4, 9, 15].

Let \( A \) be the differential operator \( Au = -\frac{du}{dx} \) in \( L^p(0, T) \), \( 1 \leq p < \infty \), with the boundary condition \( u(0) = 0 \). Its domain \( D(A) \) consists of absolutely continuous functions \( u \in L^p(0, T) \), such that \( u(0) = 0 \) and \( u' \in L^p(0, T) \). We show that on \( D(A) \) the distributed order differentiation coincides with the function \( L(-A) \) of the operator \( -A \), where \( L(z) = zK(z) \), and the function of an operator is understood in the sense of the Bochner-Phillips functional calculus (see [13, 2, 16]).

Moreover, if \( p = 2 \), then the distributed order integration operator \( \mathbb{I}(\mu) \) equals \( \mathcal{N}(J) \), where \( \mathcal{N}(x) = \frac{1}{L(x)} \), \( J \) is the integration operator, \( (Ju)(t) = \int_0^t u(\tau) \, d\tau \). This result is obtained within Hirsch’s functional calculus [7, 8] giving more detailed results for a more narrow class of functions. As by-products, we obtain an estimate of the semigroup generated by \( -L(-A) \), and an expression for the resolvent of the operator \( \mathbb{I}(\mu) \).

2. Functions of the differentiation operator. The semigroup \( U_t \) of operators on the Banach space \( X = L^p(0, T) \) generated by the operator \( A \) has the form

\[
(U_t f)(x) = \begin{cases} 
  f(x-t), & \text{if } 0 \leq t \leq x < T; \\
  0, & \text{if } 0 < x < t,
\end{cases}
\]

\( x \in (0, T), \; t \geq 0 \). This follows from the easily verified formula for the resolvent \( R(\lambda, A) = (A - \lambda I)^{-1} \) of the operator \( A \):

\[
(R(\lambda, A)u)(x) = -\int_0^x e^{-\lambda(x-y)}u(y) \, dy;
\] (10)

see [10] for a similar reasoning for operators on \( L^p(0, \infty) \). The semigroup \( U_t \) is nilpotent, \( U_t = 0 \) for \( t > T \); compare Sect. 19.4 in [6]. It follows from the expression (10) and the Young inequality that \( \|R(\lambda, A)\| \leq \lambda^{-1}, \; \lambda > 0 \), so that \( U_t \) is a \( C_0 \)-semigroup of contractions.
In the Bochner-Phillips functional calculus, for the operator $A$, as a generator of a contraction semigroup, and any function $f$ of the form

$$f(x) = \int_0^\infty (1 - e^{-tx}) \sigma(dt) + a + bx, \quad a, b \geq 0,$$

where $\sigma$ is a measure on $(0, \infty)$, such that

$$\int_0^\infty \frac{t}{1+t} \sigma(dt) < \infty,$$

the subordinate $C_0$-semigroup $U^f_t$ is defined by the Bochner integral

$$U^f_t = \int_0^\infty (U_s u) \sigma_t(ds)$$

where the measures $\sigma_t$ are defined by their Laplace transforms,

$$\int_0^\infty e^{-sx} \sigma_t(ds) = e^{-tf(x)}.$$  

The class $\mathcal{B}$ of functions (11) coincides with the class of Bernstein functions, that is functions $f \in C([0, \infty) \cap C^\infty(0, \infty)$, for which $f'$ is completely monotone. Below we show that $\mathcal{L} \in \mathcal{B}$.

The generator $A^f$ of the semigroup $U^f_t$ is identified with $-f(-A)$. On the domain $D(A)$,

$$A^f u = -au + bAu + \int_0^\infty (U_t u - u) \sigma(dt), \quad u \in D(A).$$

**Theorem 1.** (i) If $u \in D(A)$, then $A^\mathcal{L} u = -\mathcal{L}^{(u)} u$.

(ii) The semigroup $U^\mathcal{L}_t$ decays at infinity faster than any exponential function:

$$\|U^\mathcal{L}_t\| \leq C e^{-rt} \quad \text{for any } r > 0.$$  

(iii) The resolvent $R(\lambda, -A^\mathcal{L})$ of the operator $-A^\mathcal{L}$ has the form

$$(R(\lambda, -A^\mathcal{L})) u(x) = \int_0^x r_{\lambda}(x-s) u(s) \, ds, \quad u \in X,$$

where

$$r_{\lambda}(s) = \frac{1}{\lambda} \frac{d}{ds} u_{\lambda}(s),$$
and \(u_\lambda\) is the solution of the Cauchy problem
\[
\mathbb{D}^{(\mu)} u_\lambda = \lambda u_\lambda, \quad u_\lambda(0) = 1. \tag{16}
\]

(iv) The inverse \((-A^\mathcal{L})^{-1}\) coincides with the distributed order integration operator \(\mathbb{I}^{(\mu)}\).

(v) The resolvent of \(\mathbb{I}^{(\mu)}\) has the form
\[
(\mathbb{I}^{(\mu)} - \lambda I)^{-1} u = -\frac{1}{\lambda} u - \frac{1}{\lambda^2} r_{1/\lambda} * u, \quad \lambda \neq 0. \tag{17}
\]

Proof. Let \(\sigma(dt) = -k'(t) dt\). By (3),
\[
k'(t) = -\int_0^1 \frac{\alpha t^{-\alpha - 1}}{\Gamma(1 - \alpha)} \mu(\alpha) d\alpha,
\]
so that
\[
\int_0^\infty \frac{t}{1 + t} \sigma(dt) = \int_0^1 \frac{\alpha \mu(\alpha)}{\Gamma(1 - \alpha)} d\alpha \int_0^\infty \frac{t^{-\alpha}}{1 + t} dt.
\]

Using the integral formula 2.2.5.25 from [14] we find that
\[
\int_0^\infty \frac{t}{1 + t} \sigma(dt) = \pi \int_0^1 \frac{\alpha \mu(\alpha)}{(\sin \alpha \pi) \Gamma(1 - \alpha)} d\alpha < \infty.
\]

Let us compute the function (11) with \(a = b = 0\). We have
\[
f(x) = - \int_0^\infty (1 - e^{-tx}) k'(t) dt = x \int_0^\infty e^{-tx} k(t) dt = x K(x) = \mathcal{L}(x).
\]

The corresponding expression (12) for \(A^\mathcal{L} u, \ u \in D(A)\), is as follows:
\[
(A^\mathcal{L} u) (x) = - \int_0^\infty [(U_t u)(x) - u(x)] k'(t) dt = - \int_0^x [u(x - t) - u(x)] k'(t) dt + u(x) \int_x^\infty k'(t) dt
\]
\[
= -k(x) u(x) - \int_0^x [u(x - t) - u(x)] k'(t) dt.
\]

By (4), we find that \(A^\mathcal{L} u = -\mathbb{D}^{(\mu)} u, \ u \in D(A)\).

The function \(\mathcal{L}(z)\) is holomorphic for \(\text{Re } z > 0\). We will need a detailed information (refining (6)) on the behavior of \(\text{Re } \mathcal{L}(\sigma + i\tau), \ \sigma, \tau \in \mathbb{R}, \ \sigma > 0\), when \(|\tau| \to \infty\). We have
\[
\text{Re } \mathcal{L}(\sigma + i\tau) = \int_0^1 \varphi(\alpha, \sigma, \tau) \mu(\alpha) d\alpha
\]
where

\[ \varphi(\alpha, \sigma, \tau) = (\sigma^2 + \tau^2)^{\alpha/2} \cos \left( \alpha \arctan \frac{\tau}{\sigma} \right). \]

We check directly that \( \varphi(\alpha, \sigma, 0) = \sigma^\alpha \),

\[ \frac{\partial \varphi(\alpha, \sigma, \tau)}{\partial \tau} = \alpha(\sigma^2 + \tau^2)^{\alpha/2 - 1} \cos \left( \alpha \arctan \frac{\tau}{\sigma} \right) \left[ \tau - \sigma \tan \left( \alpha \arctan \frac{\tau}{\sigma} \right) \right] \geq 0, \]

and \( \frac{\partial \varphi(\alpha, \sigma, \tau)}{\partial \tau} > 0 \) for \( \alpha < 1 \). This means that the function \( g_\sigma(\tau) = \text{Re} L(\sigma + i\tau) \) (which is even in \( \tau \)) is strictly monotone increasing in \( \tau \) for \( \tau > 0 \). Its minimal value is

\[ g_\sigma(0) = \int_0^1 \sigma^\alpha \mu(\alpha) \, d\alpha. \]

On the other hand,

\[ \text{Re} L(\sigma + i\tau) \geq \int_0^1 (\sigma^2 + \tau^2)^{\alpha/2} \cos \frac{\alpha \pi}{2} \mu(\alpha) \, d\alpha = \frac{\alpha}{\pi} \int_0^{\pi/2} (\sigma^2 + \tau^2)^{t/\pi} \mu \left( \frac{2t}{\pi} \right) \cos t \, dt \]

\[ = \frac{2}{\pi} \int_0^{\pi/2} e^{qt} \mu \left( \frac{2t}{\pi} \right) \cos t \, dt = \frac{2}{\pi} e^{q\pi/2} \int_0^{\pi/2} e^{-qs} \mu \left( 1 - \frac{2}{\pi} s \right) \sin s \, ds \]

where \( q = \frac{1}{\pi} \log(\sigma^2 + \tau^2) \). By Watson’s asymptotic lemma (see [12]), since \( \mu(1) \neq 0 \), we have

\[ \int_0^{\pi/2} e^{-qs} \mu \left( 1 - \frac{2}{\pi} s \right) \sin s \, ds \sim C q^{-2} \]

where \( C \) does not depend on \( \sigma, \tau \). Roughening the estimate a little we find that

\[ e^{-t \text{Re} L(\sigma + i\tau)} \leq C e^{-t q |\tau|^{\frac{1}{2} - \varepsilon}} \] (18)

where \( 0 < \varepsilon < \frac{1}{2} \) can be taken arbitrarily, and the positive constants \( C \) and \( \rho \) do not depend on \( \sigma \) and \( \tau \).

It follows from (18) (see [3]) that for each \( t > 0 \) the function \( x \mapsto e^{-t L(x)} \) is represented by an absolutely convergent Laplace integral. This means that the measure \( \sigma_t(ds) \) has a density \( m(t, s) \) with respect to the Lebesgue measure. Moreover,

\[ m(t, s) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{zs} \cdot e^{-t L(z)} \, dz, \quad \gamma > 0. \] (19)

Since \( U_t = 0 \) for \( t > T \), we have

\[ U_t^f u = \int_0^T (U_s u) m(t, s) \, ds. \] (20)
The representation (19) yields the expression

\[ m(t, s) = \frac{e^{\gamma s}}{\pi} \int_{0}^{\infty} e^{i\tau} e^{-tL(\gamma + i\tau)} \, d\tau, \]

\[ L(\gamma + i\tau) = \int_{0}^{1} (\gamma + i\tau)^{\alpha} \mu(\alpha) \, d\alpha, \quad 0 \leq \tau < \infty. \]

We have

\[ |m(t, s)| \leq \frac{e^{\gamma s}}{\pi} \int_{0}^{\infty} e^{-t g(\tau)} \, d\tau. \]

The above monotonicity property of \( g \) makes it possible to apply to the last integral the Laplace asymptotic method [12]. We obtain that, for large values of \( t \),

\[ |m(t, s)| \leq Ct^{-1} e^{\gamma s} e^{-t g(0)}. \]

Changing \( \gamma \) and \( C \) we can make the coefficient \( g(0) \) arbitrarily big. By (20), this leads to the estimate (13).

Due to (13), the resolvent

\[ R(\lambda, A^C) = -\int_{0}^{\infty} e^{-\lambda t} U^C_t \, dt, \]  

(21)

is an entire function, so that \( A^C \) has no spectrum.

It follows from (21) that

\[ R(\lambda, -A^C) = \int_{0}^{\infty} e^{\lambda t} U^C_t \, dt, \]

and if \( u \in X, \text{Re} \lambda \leq 0 \), then

\[ (R(\lambda, -A^C)u)(x) = \int_{0}^{\infty} e^{\lambda t} dt \int_{0}^{x} u(x - s) m(t, s) \, ds = \int_{0}^{x} r(\lambda)(x - s) u(s) \, ds \]

where

\[ r(\lambda) = \int_{0}^{\infty} e^{\lambda t} m(t, s) \, dt. \]  

(22)

For a fixed \( \omega \in (\frac{1}{2}, 1) \), let us deform the contour of integration in (19) from the vertical line to the contour \( S_{\gamma, \omega} \) consisting of the arc

\[ T_{\gamma, \omega} = \{ z \in \mathbb{C} : |z| = \gamma, |\arg z| \leq \omega \pi \}, \]

and two rays

\[ \Gamma_{\gamma, \omega}^{\pm} = \{ z \in \mathbb{C} : |\arg z| = \pm \omega \pi, |z| \geq \gamma \}. \]
The contour $S_{\gamma, \omega}$ is oriented in the direction of growth of $\arg z$. By Jordan’s lemma,

$$m(t, s) = \frac{1}{2\pi i} \int_{S_{\gamma, \omega}} e^{zs} \cdot e^{-tL(z)} \, dz.$$  

Under this integral, we may integrate in $t$, as required in (22). We find that

$$r_\lambda(s) = \frac{1}{2\pi i} \int_{S_{\gamma, \omega}} \frac{e^{zs}}{L(z) - \lambda} \, dz, \quad s > 0$$  

(for $\Re \lambda > 0$, $\gamma$ should be taken big enough).

If $\lambda = 0$, the right-hand side of (23) coincides with that of (8) (see also the formula (3.4) in [11]), and we prove that $(-A^c)^{-1} = \Pi_\nu$.

For $\lambda \neq 0$, we rewrite (23) as

$$r_\lambda(s) = \frac{1}{2\pi i \lambda} \int_{S_{\gamma, \omega}} \frac{e^{zs} L(z)}{L(z) - \lambda} \, dz - \frac{1}{2\pi i \lambda} \int_{S_{\gamma, \omega}} e^{zs} \, dz. \quad (24)$$

For $0 < s < T$, we have

$$\int_{S_{\gamma, \omega}} e^{zs} \, dz = - \lim_{R \to \infty} \int_{|z|=R, \omega \pi < |\arg z| < \pi} e^{zs} \, dz,$$

$$\leq 2R \int_{\omega \pi}^\pi e^{R s \cos \varphi} d\varphi \leq 2R \pi (1 - \omega) e^{R s \cos \omega \pi} \to 0,$$

as $R \to \infty$.

Thus, the second integral in (24) equals zero, and it remains to compare (24) with the formula (2.15) of [11] giving an integral representation of the function $u_\lambda$.

The formula (17) follows from (15) and the general connection between the resolvents of an operator and its inverse ([10], Chapter 3, formula (6.18)). The theorem is proved.

Note that the expression (17) for the resolvent of a distributed order integration operator is quite similar to the Hille-Tamarkin formula for the resolvent of a fractional integration operator (see [6], Sect. 23.16). In our case, the function $u_\lambda$ is a counterpart of the function $z \mapsto E_\alpha(\lambda z^\alpha)$ (for the order $\alpha$ case). However, in our situation no analog of the entire function $E_\alpha$ (the Mittag-Leffler function) has been identified so far. Accordingly, our proof of (17) is different from the reasoning in [6].

3. Functions of the integration operator. In this section we assume that $p = 2$.

Hirsch’s functional calculus deals with the class $\mathcal{R}$ of functions which are continuous on $\mathbb{C} \setminus (-\infty, 0)$, holomorphic on $\mathbb{C} \setminus (-\infty, 0]$, transform the upper half-plane into itself, and transform the semi-axis $(0, \infty)$ into itself. The class $\mathcal{R}$ is a subclass of $\mathcal{B}$.  

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Another important class of functions is the class $S$ of Stieltjes functions

$$f(z) = a + \int_0^\infty \frac{d\rho(\lambda)}{z + \lambda}, \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

where $a \geq 0$, $\rho$ is a non-decreasing right-continuous function, such that $\int_0^\infty \frac{d\rho(t)}{1 + t} < \infty$. If $f$ is a nonzero function from $S$, then the function

$$\tilde{f}(z) = \frac{1}{f(z-1)}$$

also belongs to $S$.

If $f \in S$, then the function $H_f(z) = f(z^{-1})$ belongs to $R$. It has the form

$$H_f(z) = a + \int_0^\infty \frac{z}{1 + \lambda z} d\rho(\lambda), \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

For some classes of linear operators $V$, the function $H_f(V)$ is defined as a closure of the operator

$$W x = ax + \int_0^\infty V(I + \lambda V)^{-1} x d\rho(\lambda), \quad x \in D(V).$$

In particular, this definition makes sense if $-V$ is a generator of a contraction $C_0$-semigroup, and in this case the above construction is equivalent to the Bochner-Phillips functional calculus [2, 4]. In addition, by Theorem 2 of [7], if $(-V)^{-1}$ is also a generator of a contraction $C_0$-semigroup, then

$$[H_f(V)]^{-1} = H_{\tilde{f}}(V^{-1}). \quad (25)$$

In order to apply the above theory to our situation, note that [7]

$$z^\alpha = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \frac{z}{1 + \lambda z} \lambda^{-\alpha} d\lambda, \quad 0 < \alpha < 1,$$

whence

$$L(z) = \int_0^\infty \frac{z}{1 + \lambda z} \beta(\lambda) d\lambda$$

where

$$\beta(\lambda) = \int_0^1 \frac{\lambda^{-\alpha} \mu(\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} d\alpha.$$

Thus $L(z) = H_f(z)$, with

$$f(z) = \int_0^\infty \frac{1}{z + \lambda} \beta(\lambda) d\lambda.$$
It follows from Watson’s lemma [12] that \( \beta(\lambda) \leq C(\log \lambda)^{-2} \) for large values of \( \lambda \). Therefore
\[
\int_{0}^{\infty} \frac{\beta(\lambda)}{1 + \lambda} d\lambda < \infty.
\]

Denote \( N(z) = H_f(z) = \frac{1}{\mathcal{L}(z)} \).

If \( V = -A \), then \((-V)^{-1} = -J\), where \( J \) is the integration operator. It is easy to check that \( \langle (J + J^*)u, u \rangle \geq 0 \) (\( \langle \cdot, \cdot \rangle \) is the inner product in \( L_2(0, T) \)). Therefore \(-J\) is a generator of a contraction semigroup.

After these preparations, the equality (25) implies the following result.

**Theorem 2.** The operator \( \mathbb{I}^{(\mu)} \) of distributed order integration and the integration operator \( J \) are connected by the relation
\[
\mathbb{I}^{(\mu)} = N(J).
\]

**References**

[1] Bakaev N. Yu., Tarasov R. P. Semigroups and a method for stably solving the Abel equation. // Siberian Math. J. – 1978. – 19. – P. 1–5.

[2] Berg C., Boyadzhiev Kh., deLaubenfels R. Generation of generators of holomorphic semigroups. // J. Austral. Math. Soc. A. – 1993. – 55. – P. 246–269.

[3] Ditkin V. A., Prudnikov A. P. Integral transforms and operational calculus. – Oxford: Pergamon Press. 1965. – XI, 529 p.

[4] Gohberg I. C., Krein M. G. Theory and applications of Volterra operators in Hilbert space. – Providence: American Mathematical Society. 1970. – X, 430 p.

[5] Gorbachuk V. I., Knyazuk A. V. Boundary values of solutions of differential-operator equations. // Russian Math. Surveys. – 1989. – 44, No. 3. – P. 67–111.

[6] Hille E., Phillips R. S. Functional analysis and semigroups. – Providence: American Mathematical Society. 1957. – XII, 808 p.

[7] Hirsch F. Int´egrales de r´esolvantes et calcul symbolique. // Ann. Inst. Fourier. – 1972. – 22, No. 4. – P. 239–264.

[8] Hirsch F. Domaines d’opérateurs représentés comme intégrales de résolvantes. // J. Funct. Anal. – 1976. – 23. – P. 199–217.

[9] Jacob N., Krägeloh A. M. The Caputo derivative, Feller semigroups, and the fractional power of the first order derivative on \( C_\infty(\mathbb{R}_0^+) \). // Fract. Calc. Appl. Anal. – 2002. – 5. – P. 395–410.
[10] Kato T. Perturbation theory for linear operators. – Berlin: Springer. – 1966. – XX, 592 p.

[11] Kochubei A. N. Distributed order calculus and equations of ultraslow diffusion. // J. Math. Anal. Appl. (to appear); math-ph/0703046.

[12] Olver F. W. J. Asymptotics and special functions. – New York: Academic Press. – 1974. – XVI, 572 p.

[13] Phillips R. S. On the generation of semigroups of linear operators. // Pacif. J. Math. – 1952. – 2. – P. 343–369.

[14] Prudnikov A. P., Brychkov Yu. A., Marichev O. I. Integrals and series. Vol. 1: Elementary functions, New York: Gordon and Breach. – 1986. – 798 p.

[15] Samko S. G., Kilbas A. A., Marichev O. I. Fractional integrals and derivatives: theory and applications. – New York: Gordon and Breach. 1993. – xxxvi, 976 p.

[16] Schilling R. L. Subordination in the sense of Bochner and a related functional calculus. // J. Austral. Math. Soc. A. – 1998. – 64. – P. 368–396.