Graph Fourier transform based on singular value decomposition of the directed Laplacian

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Abstract
The Graph Fourier transform (GFT) is a fundamental tool in graph signal processing. In this paper, based on singular value decomposition of the Laplacian, we introduce a novel definition of GFT on directed graphs, and use the singular values of the Laplacian to carry the notion of graph frequencies. We show that the proposed GFT has its frequencies and frequency components evaluated by solving some constrained minimization problems with low computational cost, and it could represent graph signals with different modes of variation efficiently. Moreover, the proposed GFT is consistent with the conventional GFT in the undirected graph setting, and on directed circulant graphs, it is the classical discrete Fourier transform, up to some rotation, permutation and phase adjustment.

Keywords
Graph signal processing · Graph Fourier transform · Directed graphs · Singular value decomposition

Mathematics Subject Classification 94C15 · 94A12 · 94A15 · 42C05

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1 Introduction

Graph signal processing provides an innovative framework to represent, analyze and process data sets residing on networks, and its mathematical foundation is closely related to applied and computational harmonic analysis and spectral graph theory [1, 3, 20, 21, 23, 25, 29, 31]. The graph Fourier transform (GFT) is one of fundamental tools in graph signal processing that decomposes graph signals into different frequency components and represents them by different modes of variation [1, 20, 23, 25, 29].

Let $G = (V, E)$ be a weighted (un)directed graph of order $N$ containing no loops or multiple edges, and denote the associated adjacency matrix, in-degree matrix and (in-degree) Laplacian by $A, D$ and $L := D - A$ respectively. The GFT has been well-studied in the undirected graph setting [4, 16, 18, 21, 31, 34]. A conventional definition of GFT of a graph signal $x$ in the undirected graph setting is given by

$$\mathcal{F}x := V^T x, \quad (1.1)$$

where the signal $x = [x_i]_{i \in V} \in \mathbb{R}^N$ takes value $x_i$ at the vertex $i \in V$, $V = [v_0, \ldots, v_{N-1}]$ is the orthogonal matrix in the eigen-decomposition

$$L = VA^T = \sum_{k=0}^{N-1} \lambda_k v_k v_k^T \quad (1.2)$$

of the positive semi-definite Laplacian $L$, and $A = \text{diag}(\lambda_0, \ldots, \lambda_{N-1})$ is a diagonal matrix of nonnegative eigenvalues with diagonal entries being in a nondecreasing order, i.e., $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{N-1}$. The eigenvalues $\lambda_k$ and the associated eigenvectors $v_k, 0 \leq k \leq N - 1$, of the Laplacian $L$ are considered as frequencies and frequency components of the GFT in (1.1), see [22] and references therein for natural ordering of frequencies and frequency components. It is known that the GFT in (1.1) is an orthogonal transform (hence the Parseval’s identity holds) and on a cycle graph it becomes essentially the classical discrete Fourier transform.

Directed graphs have been widely used to describe the interaction structures of networks in which pairwise relations between agents are not mutual and equitable, such as individuals and organizations in a social network and leaders and followers in citation graphs [13, 19, 27, 32]. The GFT on directed graphs is an important tool to identify the patterns and quantify the influence of members and communities of a social network, and to understand the dynamic of a network. Several approaches have been proposed to define the GFT on directed graphs, however there is no well-accepted definition yet [4–7, 11, 12, 18, 22, 24–26, 28, 30, 31, 34, 36]. In this paper, we introduce a novel definition of GFT on directed graphs, which is based on singular value decomposition of the associated Laplacian, see Definition 2.1.

The GFT in (1.1) does not easily generalize to directed graphs, since in the directed graph setting the associated Laplacian $L$ is not symmetric and it does not have the eigen-decomposition (1.2). A natural approach is to replace the eigen-decomposition (1.2) of the Laplacian $L$ by the Jordan decomposition $L = VJV^{-1}$, and then to define
the GFT of a signal \( x \) on a directed graph by

\[
\mathcal{F} x = V^{-1} x
\]  

(1.3)

and interpret the eigenvalues and corresponding eigenvectors of the Laplacian \( L \) as frequencies and frequency components [6, 7, 24, 25, 30]. The GFT in (1.3) could have complex frequencies and it is not always unitary. More critically, Jordan decomposition of the Laplacian \( L \) on directed graphs could be numerically unstable and computationally expensive, and hence it could be difficult to be applied for graph spectral analysis and decomposition on a large network, see [7] for a modified Jordan decomposition with some numerical stability. Our GFT in Definition 2.1 is based on the singular value decomposition (SVD)

\[
L = U \Sigma V^T = \sum_{k=0}^{N-1} \sigma_k u_k v_k^T
\]  

(1.4)

of the Laplacian \( L \) and has its nonnegative singular values \( \sigma_k, 0 \leq k \leq N - 1 \), as frequencies and \( u_k, v_k, 0 \leq k \leq N - 1 \), as the associated left/right frequency components, where

\[
U = [u_0, \ldots, u_{N-1}] \text{ and } V = [v_0, \ldots, v_{N-1}]
\]  

(1.5)

are orthogonal matrices, and the diagonal matrix \( \Sigma = \text{diag}(\sigma_0, \ldots, \sigma_{N-1}) \) has singular values deployed on the diagonal in a nondecreasing order, i.e., \( 0 = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{N-1} \). Compared with the GFT in (1.3), a significant advantage of the proposed SVD-based GFT is the numerical stability and low computational cost.

Denote the standard inner product on \( \mathbb{R}^N \) by \( \langle \cdot, \cdot \rangle \) and define \( \| x \|_2 = \sqrt{\langle x, x \rangle} \) for \( x \in \mathbb{R}^N \) and a graph signal \( x \) on a graph of order \( N \). Given a signal \( x \) on a directed graph, define its quadratic variation by

\[
QV(x) = x^T L x = \frac{1}{2} x^T (L + L^T) x
\]  

(1.6)

[4, 12, 21, 31]. In the undirected graph setting, frequencies \( \lambda_k \) and their corresponding frequency components \( v_k, 0 \leq k \leq N - 1 \), of the GFT in (1.1) can be obtained via solving the following constrained minimization problems

\[
\left\{
\begin{array}{l}
\lambda_k = \min_{x \in W_k^\perp} \text{ with } \| x \|_2 = 1 \ QV(x) \\
v_k = \arg \min_{x \in W_k^\perp} \text{ with } \| x \|_2 = 1 \ QV(x)
\end{array}
\right.
\]  

(1.7)

inductively for \( 1 \leq k \leq N - 1 \), where \( \lambda_0 = 0 \), the initial \( v_0 \) is usually selected by \( N^{-1/2} \mathbf{1} = N^{-1/2}[1, \ldots, 1]^T \), and \( W_k^\perp, 1 \leq k \leq N - 1 \), are the orthogonal complements of the space spanned by \( v_k, 0 \leq k' \leq k - 1 \). We remark that the quadratic variation \( QV \) in (1.6) overlooks the edge directivity in the directed graph setting. To define the GFT on directed graphs, several directed variations to measure
the change of signals along the graph structure have been proposed [12, 26, 28]. The authors in [12, 26, 28] define frequencies and frequency components of GFT on directed graphs via solving some constrained optimization problems with directed variations as their objective functions, see Remark 3.1 for detailed explanations. In Sect. 3, we show that right frequency components \( v_k, 0 \leq k \leq N - 1 \), of the proposed GFT can be obtained via solving constrained minimization problems (1.7) with the objective function \( QV(x) = \sqrt{x^T L^T L x} \), see (3.8). In Sect. 3, we also observe that bandlimiting in the low frequencies of the proposed GFT provides good approximations to graph signals with regularity, see Theorem 3.2. Compared with the GFTs based on constrained optimization of directed variations in [26, 28], major differences are that the GFT proposed in this paper coincides with the conventional GFT (1.1) in the undirected graph setting, see (2.10), and that on directed circulant graphs, it is essentially the classical discrete Fourier transform, up to certain rotation, permutation and phase adjustment, see Theorem 2.3.

We say that a directed graph \( E \) is an Eulerian graph if the in-degree and out-degree are the same at each vertex, and that \( E^T \) is the transpose of the graph \( E \) if they have the same vertex set and the adjacency matrix is the transpose \( A^T \) of the adjacency matrix \( A \) of the original graph \( E \). As a consequence, we see that for an Eulerian graph \( E \), the Laplacian of its transpose \( E^T \) is the transpose of the original Laplacian \( L \) on the graph \( E \). A familiar measurement to qualify the “symmetry” of a directed Eulerian graph \( E \) is the largest singular value of the difference \( L - L^T \) between the Laplacian \( L \) of the directed Eulerian graph \( E \) and the Laplacian \( L^T \) of the transpose graph \( E^T \), i.e.,

\[
\sigma_{asym} = \max_{\|x\|_2 = 1} \| (L - L^T)x \|_2
\]

[17]. To further understand the “symmetry” of a directed Eulerian graph \( E \), we introduce a family of directed Eulerian graphs \( E_t, 0 \leq t \leq 1 \), to connect an Eulerian graph \( E \) to its transpose graph \( E^T \), where \( E_t, 0 \leq t \leq 1 \), share the same vertex set \( V \) with the graph \( E \) and have adjacency matrices

\[
A_t = (1 - t)A + tA^T
\]

being linear combinations of the adjacency matrices of the graph \( E \) and its transpose graph \( E^T \). In Sect. 4 we consider GFTs \( F_t \) on the directed Eulerian graphs \( E_t, 0 \leq t \leq 1 \), and study algebraic and analytic properties of the corresponding frequencies and frequency components of the GFT \( F_t, 0 \leq t \leq 1 \), see (4.8), (4.10), and Theorems 4.1, 4.2 and 4.3.

GFTs should be designed to have energy of smooth graph signals concentrated mainly at low frequencies, see Theorem 3.2. In Sect. 5, bandlimiting procedure to the low frequencies of the proposed SVD-based GFT is demonstrated to denoise the hourly temperature data set collected at 218 locations in the United States effectively (and hence represent the data set approximately with few frequency components), and it outperforms the ones associated with the SOC and PAMAL-based GFTs in [26] and the magnetic-Laplacian-based GFT in [11, 36]. In Sect. 6, we collect all proofs.
Notation  Bold lower cases and capitals are used to represent the column vectors and matrices respectively. Denote the Hermitian transpose and transpose of a matrix $A$ by $A^H$ and $A^T$ respectively, and use $1$, $0$, $I$ and $O$ to represent a vector with all 1 s, a row/column vector with all 0 s, an identity matrix, and a zero matrix of appropriate size.

2 Graph Fourier transforms on directed graphs

Let $\mathcal{G} = (V, E)$ be a weighted directed graph of order $N$ containing no loops or multiple edges, and denote the associated Laplacian by $L = D - A$, where the adjacency matrix $A = (a_{ij})_{i,j \in V}$ has nonzero weights $a_{ij} \neq 0$ only when there is a directed edge from node $j$ to node $i$, and the in-degree matrix $D = \text{diag}(d_i)_{i \in V}$ has the in-degree $d_i = \sum_{j \in V} a_{ij}$ of node $i$, $i \in V$ as its diagonal entries. The Laplacian $L$ has eigenvalue zero and the constant signal $1$ as an associated eigenvector, i.e.,

$$L1 = 0. \quad (2.1)$$

In this section, based on the eigen-decomposition (2.3) of the self-adjoint dilation

$$S(L) := \begin{pmatrix} O & L \\ L^T & O \end{pmatrix} \in \mathbb{R}^{2N \times 2N} \quad (2.2)$$

of the Laplacian $L$, we propose a novel definition of GFT on directed graphs, see Definition 2.1. The proposed GFT preserves the Parseval’s identity, see (2.7), and in the undirected graph setting, it coincides with the conventional GFT in (1.1), see (2.10). Circulant graphs have been widely used in image processing [8, 9, 14, 15]. In Theorem 2.3, we show that the proposed SVD-based GFT on a directed circulant graph is essentially the classical discrete Fourier transform, up to certain rotation, permutation and phase adjustment.

Let orthogonal matrices $U = [u_0, \ldots, u_{N-1}]$, $V = [v_0, \ldots, v_{N-1}]$ and diagonal matrix $\Sigma = \text{diag}(\sigma_0, \ldots, \sigma_{N-1})$ be as in the SVD (1.4) of the Laplacian $L$. Then the self-adjoint dilation $S(L)$ in (2.2) has the following eigendecomposition

$$S(L) = F \begin{pmatrix} \Sigma & O \\ O & -\Sigma \end{pmatrix} F^T, \quad (2.3)$$

where

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} U & U \\ V & -V \end{pmatrix} \in \mathbb{R}^{2N \times 2N} \quad (2.4)$$

is an orthogonal matrix. Using the above orthogonal matrix $F$, we define the GFT and inverse GFT on the directed graph $\mathcal{G}$ as follows.
Definition 2.1  Let $F$ be the orthogonal matrix in (2.4). We define graph Fourier transform $\mathcal{F} : \mathbb{R}^N \rightarrow \mathbb{R}^{2N}$ and inverse graph Fourier transform $\mathcal{F}^{-1} : \mathbb{R}^{2N} \rightarrow \mathbb{R}^N$ on the directed graph $\mathcal{G}$ by

$$\mathcal{F}x := F^T \left( \frac{x}{\sqrt{2}} \right) = \left( \frac{(U^T + V^T)x}{2} \right)$$

and

$$\mathcal{F}^{-1} \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) = \left( \begin{array}{cc} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{array} \right) F \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) = \frac{1}{2} (U(z_1 + z_2) + V(z_1 - z_2)), \quad (2.6)$$

where $x$ is a graph signal on $\mathcal{G}$ and $z_1, z_2 \in \mathbb{R}^N$.

The GFT in (2.5) provides a tool to analyze and represent signals in the spectral domain. Shown in Fig. 1 is a piecewise constant signal on the weighted Minnesota traffic graph (top left), the frequencies $\sigma_k, 0 \leq k \leq N - 1$, of the proposed GFT (solid line on the top right), and the first and next $N$ components of the GFT of a piecewise constant signal on the graph (bottom left and right). We observe that the piecewise constant signal on the weighted Minnesota graph has its energy concentrated mainly at low frequencies of the proposed GFT in (2.5).

By the orthogonality of the matrix $F$, one may verify that the Parseval’s identity holds, i.e.,

$$\|x\|_2 = \|\mathcal{F}x\|_2, \quad x \in \mathbb{R}^N, \quad (2.7)$$

and the inverse GFT $\mathcal{F}^{-1}$ is the pseudo-inverse of the GFT $\mathcal{F}$, i.e.,

$$\mathcal{F}^{-1} \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) = \arg \min_{z \in \mathbb{R}^N} \left\| \mathcal{F}z - \frac{1}{2} (U(z_1 + z_2) + V(z_1 - z_2)) \right\|_2, \quad z_1, z_2 \in \mathbb{R}^N. \quad (2.8)$$

Therefore the original graph signal $x$ can be reconstructed from its GFT $\mathcal{F}x$,

$$\mathcal{F}^{-1} \mathcal{F}x = x, \quad x \in \mathbb{R}^N. \quad (2.9)$$

For the case that the graph $\mathcal{G}$ is undirected, orthogonal matrices $U$ and $V$ in (1.4) can be selected to be the same, i.e., $U = V$. Then the corresponding GFT of a graph signal $x$ becomes

$$\mathcal{F}x = \left( \begin{array}{c} V^T x \\ 0 \end{array} \right). \quad (2.10)$$

This shows that, in the undirected graph setting, the proposed SVD-based GFT is essentially the same as the well-accepted GFT in (1.1) on undirected graphs.

Remark 2.2  Let $\mathcal{G} = (V, E)$ be a directed graph of order $N$ with its adjacency matrix denoted by $A = (a_{ij})_{i,j \in V}$. Define the magnetic Laplacians $L_q, q \geq 0$, on $\mathcal{G}$ by

$$L_q := D - H_q, \quad (2.11)$$
Fig. 1 Plotted on the top left is a piecewise constant signal $x_0$ on a weighted Minnesota traffic graph of order $N = 2640$ with the weights $a_{ij}$ on adjacent edges $(i, j)$ being randomly chosen in the interval $[0, 2]$. On the top right in solid and dashed lines are the frequencies $\sigma_k$, $0 \leq k \leq N - 1$, in (3.2) and $\lambda_{k; q}$, $0 \leq k \leq N - 1$, in (2.12) with $q = 1/4$ respectively. On the bottom left and right are the first $N$ components $(U^T + V^T)x_0/2$ and the second $N$ components $(U^T - V^T)x_0/2$ of the GFT $\mathcal{F}x_0$ of the signal $x_0$ respectively. The relative percentage of signal energy $(\sum_{k=0}^{M-1} |(u_k + v_k)^T x_0/2|^2 + |(u_k - v_k)^T x_0/2|^2)^{1/2}/\|x_0\|_2$ for the first $M = 20, 50$ and 100 frequencies (about 0.76%, 1.89% and 3.79% of the total 2640 frequencies) are 0.7005, 0.7655, 0.8166, where $\|x\|_2 = \sqrt{x^T x}$ denotes the energy of the signal $x$.

where $D_s = \text{diag}(\sum_{j \in V} \frac{a_{ij} + a_{ji}}{2} \text{sgn}(a_{ij}))_{i \in V}$ is the average of the in-degree and out-degree matrices and $H_q = (\frac{a_{ij} + a_{ji}}{2} e^{2\pi \sqrt{-1} q (a_{ij} - a_{ji})})_{i, j \in V}$ is the Hadamard product of the symmetrized adjacency matrix $A_s = (A + A^T)/2$ and the Hermitian phase matrix with the complex entries $\gamma_{q; ij} = e^{2\pi \sqrt{-1} q (a_{ij} - a_{ji})}$, $i, j \in V$, which encode the edge direction into the phase in the complex plane. The magnetic Laplacians $L_q$, $q \geq 0$, are used in quantum physics to describe the phenomenology of a free charged particle on a graph, where $q$ is called electric charge, see [11] and references therein. As the magnetic Laplacians $L_q$, $q \geq 0$, are positive semi-definite Hermitian matrices, they have the following eigendecomposition

$$L_q = \sum_{k=0}^{N-1} \lambda_{k; q} v_k; q v_k^H; q,$$

(2.12)
Plot on the left and right are the real part $\Re(V^H_{q}x_0)$ and imaginary part $\Im(V^H_{q}x_0)$ of the GFT in (2.13) by $q$-ML with $q = 1/4$, where $x_0$ is the piecewise constant signal on a weighted Minnesota traffic graph plotted in top left of Fig. 1. The relative percentage of signal energy $\left(\sum_{k=0}^{M-1}|v^H_{k,q}x_0|^2\right)^{1/2}/\|x_0\|_2$ for the first $M = 20, 50$ and $100$ frequencies (about $0.76\%$, $1.89\%$ and $3.79\%$ of the total $2640$ frequencies) are $0.1289, 0.2385, 0.3289$ respectively.

where $0 \leq \lambda_{0,q} \leq \cdots \leq \lambda_{N-1,q}$ and $V_q = [v_{0,q}, \ldots, v_{N-1,q}]$ is a unitary matrix. Based on the eigendecomposition (2.12) of the magnetic Laplacians $L_q, q \geq 0$, we can define the GFT of a signal $x$ on the directed graph $G$ as follows,

$$\mathcal{F}_q x = V^H_q x,$$  

(2.13)

and interpret $\lambda_{k,q}$ and $v_{k,q}, 0 \leq k \leq N - 1$, as the corresponding frequencies and frequency components [11, 36]. We use the abbreviation $q$-ML to denote the above GFT $\mathcal{F}_q$ based on magnetic Laplacian $L_q, q \geq 0$. Shown on the top right of Fig. 1 are the frequencies $\lambda_{k,1/4}, 0 \leq k \leq N - 1$, (dashed line) on the weighted Minnesota traffic graph of order $N = 2640$, where the lowest frequency $\lambda_{0,1/4}$ is $0.0174$. It is observed that frequencies of the GFT in (2.5) and (2.13) have the similar patterns, except that the lowest frequency of the GFT in (2.5) is zero by (3.3) and the lowest frequency $\lambda_{0,q}$ of the GFT in (2.13) is not always zero. Plotted in Fig. 2 are the real and imagery part of the GFT $\mathcal{F}_q x_0$ of a piecewise constant signal on the Minnesota traffic graph in Fig. 1. Unlike the concentration at low frequencies of the GFT in (2.5), it is noticed from Fig. 2 that the piecewise constant signal on the weighted Minnesota graph may have its energy distributed at almost all frequencies of the GFT in (2.13) by $q$-ML, cf. Fig. 1.

For $N \geq 1$ and a set $Q = \{q_1, \ldots, q_L\}$ of positive integers ordered with $1 \leq q_1 < \cdots < q_L \leq N - 1$, let the directed circulant graph $C_d := C_d(N, Q)$ generated by $Q$ be the unweighted graph with the vertex set $V_N = \{0, 1, \ldots, N - 1\}$ and the edge set $E_N(Q) = \{(i, i + q \text{ mod } N), i \in V_N, q \in Q\}$, where we say that $a = b \text{ mod } N$ if $(a - b)/N$ is an integer [8, 9, 14, 15]. Set

$$P(z) = L - \sum_{l=1}^{L} z^{q_l},$$  

(2.14)
and denote the discrete Fourier transform matrix by

\[ W := \left( N^{-1/2} \omega_{N}^{ij} \right)_{0 \leq i,j \leq N-1}, \tag{2.15} \]

where \( \omega_{N} = \exp(2\pi \sqrt{-1}/N) \) is the \( N \)-th root of the unit. One may verify that the Laplacian \( L_{C_d} \) on the directed circulant graph \( C_d \) is a circulant matrix with eigenvalues \( P(\omega_{N}^{k}), 0 \leq k \leq N-1 \), and that the \( k \)-th column of the discrete Fourier transform matrix \( W \) as a unit eigenvector associated with the eigenvalue \( P(\omega_{N}^{k}), 0 \leq k \leq N-1 \).

Define the phases \( \theta_{k} \in [−\pi, \pi) \) of \( P(\omega_{N}^{k}), 0 \leq k \leq N-1 \), by

\[ \exp(\sqrt{-1} \theta_{k}) = \begin{cases} 1 & \text{if } P(\omega_{N}^{k}) = 0 \\ \frac{P(\omega_{N}^{k})}{|P(\omega_{N}^{k})|} & \text{if } P(\omega_{N}^{k}) \neq 0. \end{cases} \]

Then

\[ P(\omega_{N}^{k}) = |P(\omega_{N}^{k})| \exp(\sqrt{-1} \theta_{k}), 0 \leq k \leq N-1. \tag{2.16} \]

Let

\[ R = \begin{cases} \text{diag}(1, R_2, \ldots, R_2) & \text{if } N \text{ is odd} \\ \text{diag}(1, R_2, \ldots, R_2, 1) & \text{if } N \text{ is even} \end{cases} \tag{2.17} \]

be the block diagonal matrix with number one and the \( 2 \times 2 \) unitary matrix \( R_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\sqrt{-1} \\ 1 & \sqrt{-1} \end{pmatrix} \) as its diagonal blocks, and define the diagonal matrix

\[ \Theta = \text{diag}(\exp(\sqrt{-1} \theta_{0}), \ldots, \exp(\sqrt{-1} \theta_{N-1})) \tag{2.18} \]

with phases \( \exp(\sqrt{-1} \theta_{k}) \) in (2.16) as its diagonal entries. In Proposition 6.1 of Sect. 6.1, we show that the Laplacian matrix \( L_{C_d} \) on the directed circulant graph \( C_d \) has the following SVD,

\[ L_{C_d} = U \Sigma V^T, \tag{2.19} \]

where \( P_0 \) and \( P_1 \) are permutation matrices (see (6.5) and (6.6) for explicit expressions),

\[ U = W \Theta P_0^T R P_1 \text{ and } V = WP_0^T R P_1 \tag{2.20} \]

are orthogonal matrices with real entries, and

\[ \Sigma = \text{diag}(\sigma_0, \ldots, \sigma_{N-1}) \tag{2.21} \]

has diagonal entries being a nondecreasing rearrangement of the magnitudes \( |P(\omega_{N}^{k})|, 0 \leq k \leq N-1 \), in (2.16). Based on the above SVD of the Laplacian matrix \( L_{C_d} \), we observe that the GFT in Definition 2.1 is essentially the classical discrete Fourier transform

\[ \text{DFT}(x) := WHx, \ x \in \mathbb{R}^N, \tag{2.22} \]

up to certain rotation \( R \), phase adjustment \( \Theta \) and permutations \( P_0 \) and \( P_1 \).
Theorem 2.3 Let $N \geq 1$, $Q = \{q_1, \ldots, q_L\}$ be a set of positive integers with $1 \leq q_1 < \cdots < q_L \leq N - 1$, $C_d(N, Q)$ be the directed circulant graph generated by $Q$, and take the SVD (2.19) of the Laplacian matrix $L_{C_d}$ on $C_d(N, Q)$. Then the corresponding GFT in (2.5) is given by

$$F_x = \frac{1}{2} \left( \begin{array}{cc} P_1 & O \\ O & P_1 \end{array} \right)^T \left( \begin{array}{cc} R & O \\ O & R \end{array} \right)^H \left( \begin{array}{cc} P_0 & O \\ O & P_0 \end{array} \right) \left( \begin{array}{cc} \Theta & \Theta \\ \Theta & \Theta \end{array} \right)^H \left( \begin{array}{cc} \text{DFT}(x) \\ \text{DFT}(x) \end{array} \right),$$

(2.23)

where $x$ is a signal on $C_d(N, Q)$, and the rotation $R$, the phase adjustment matrix $\Theta$, and the permutations $P_0$ and $P_1$, are given in (2.17), (2.18), (6.5) and (6.6) respectively.

Proof By (2.5), (2.19) and (2.20), we have

$$F_x = \frac{1}{2} \left( \begin{array}{cc} (U^H + V^H)x \\ (U^H - V^H)x \end{array} \right) = \frac{1}{2} \left( \begin{array}{cc} P_1^T R^H P_0 (\Theta^H + I) W^H x \\ P_1^T R^H P_0 (\Theta^H - I) W^H x \end{array} \right)$$

$$= \frac{1}{2} \left( \begin{array}{cc} P_1^T O \\ O P_1^T \end{array} \right) \left( \begin{array}{cc} R^H & O \\ O & R^H \end{array} \right) \left( \begin{array}{cc} P_0 & O \\ O & P_0 \end{array} \right) \left( \begin{array}{cc} \Theta^H & I \\ \Theta^H & -I \end{array} \right) \left( \begin{array}{cc} \text{DFT}(x) \\ \text{DFT}(x) \end{array} \right).$$

Then the conclusion (2.23) about the graph Fourier transform $F_x$ of a graph signal $x$ on the circulant graph $C_d(N, Q)$ follows. 

3 Frequencies and frequency components

Let $\mathcal{G} = (V, E)$ be a directed graph of order $N$ containing no loops or multiple edges, and $U = [u_0, \ldots, u_{N-1}]$, $V = [v_0, \ldots, v_{N-1}]$ and $\Sigma = \text{diag}(\sigma_0, \ldots, \sigma_{N-1})$ be the orthogonal matrices and diagonal matrix in the SVD (1.4) of the associated Laplacian $L$. In this section, we propose to use singular values $\sigma_k$, $0 \leq k \leq N - 1$, of the Laplacian $L$ to carry the graph frequencies of the GFT in Definition 2.1, and to take the columns $u_k$ and $v_k$, $0 \leq k \leq N - 1$, of orthogonal matrices $U$ and $V$ as its left/right frequency components. In Remarks 2.2 and 3.1, we compare the proposed SVD-based GFT with the Magnetic-Laplacian-based GFT in [11, 36] and the GFTs in [26, 28]. We observe that frequencies of the proposed GFT have similar pattern to the ones in [11, 26, 30, 36], see Fig. 3. Based on the SVD (1.4) of the Laplacian, we propose an effective algorithm (3.8) to evaluate frequencies and left/right frequency components. We finish this section by showing that graph signals with regularity may have their energy mainly concentrated on the low frequencies of the proposed GFT $\mathcal{F}$, see Theorem 3.2.

Let the self-adjoint dilation $S(L)$ be as in (2.3) and $F$ be as in (2.4). By (2.3) and (2.4), we have the following “commutativity property” for the proposed GFT,

$$F^T S(L) = \left( \begin{array}{cc} \Sigma & O \\ O & -\Sigma \end{array} \right) F^T.$$

(3.1)

Recall that in the undirected graph setting, the SVD (1.4) of the Laplacian $L$ can be chosen to be the same as its eigen-decomposition (1.2) and the proposed SVD-based
GFT is essentially the GFT in (1.1) by (2.10). So following the terminology of GFT on undirected graphs, we use

$$0 \leq \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{N-1}$$

(3.2)

to carry the notion of frequencies of the GFT in Definition 2.1, see Fig. 1 for frequencies of a weighted Minnesota traffic graph of size $N = 2640$.

By (2.1), the GFT in Definition 2.1 has zero as its lowest frequency, i.e.,

$$\sigma_0 = 0.$$  

(3.3)

Shown in Figs. 3 and 4 are a directed unweighted graph of size 15 containing three clusters connected with a directed cycle [26, Fig. 1(c)], and its frequencies and frequency components obtained by the splitting orthogonality constraint method (SOC for abbreviation) [26, Algorithm 1], the proximal alternating minimized augmented Lagrangian methods (PAMAL for abbreviation) [26, Algorithms 2 and 3], the Jordan decomposition method (Jordan) in (1.3) [6, 7, 24], the magnetic-Laplacian based method ($q$-ML) in (2.13) [11, 36], and the SVD-based approach proposed in this paper. It is observed that frequencies (3.2) obtained by our approach have similar pattern to the ones in SOC, PAMAL, Jordan and $q$-ML, and the lowest frequency obtained by our approach and the ones in SOC and PAMAL are always zero, while the $q$-ML may have positive lowest frequency.

For $0 \leq k \leq N - 1$, we obtain from the SVD (1.4) that

$$u_k^T L = \sigma_k v_k^T \quad \text{and} \quad L v_k = \sigma_k u_k,$$

(3.4)

[10, 16]. Then we call $u_k$ and $v_k$, $0 \leq k \leq N - 1$, as the left and right frequency components associated with frequency $\sigma_k$, or $k$-th left (right) frequency components in

\[ 
\begin{array}{c}
\text{Frequencies obtained by different approaches} \\
\hline
\text{SVD} & \text{SOC} & \text{PAMAL} & \text{Jordan} & \text{$1/4$-ML} & \text{$1/2$-ML} \\
\hline
\end{array}
\]
Fig. 4  Plot from top to bottom are left frequency components, right frequency components of our proposed GFT, the SOC and PAMAL frequency components associated with $k$-th frequency, where $k = 0, 1, 2$ from left to right and the directed unweighted graph has three clusters of 5 knots connected with a directed cycle [26, Fig. 1(c)], see Fig. 3

short, respectively. By (2.1), the right frequency component associated with frequency zero can be selected as

$$v_0 = N^{-1/2} 1.$$  \hfill (3.5)
The left frequency component $u_0$ associated with frequency zero is not always a multiple of the constant signal $1$. One may verify that it can be so chosen that

$$u_0 = N^{-1/2}1$$  \hspace{1cm} (3.6)

if and only if $G$ is an Eulerian graph, in which the in-degree and out-degree are the same at each vertex.

In the undirected graph setting, the left and right frequency components can be selected as the same, and they can be obtained by solving a family of constrained optimization problems inductively,

$$u_k = v_k = \arg\min_{x \in W_k^\perp} \|x\|_2 = 1 QV(x)$$

$$= \arg\min_{x \in W_k^\perp} \|x\|_2 = 1 \|Lx\|_2,$$  \hspace{1cm} (3.7)

with the initials $u_0 = v_0 = N^{-1/2}1$, where the quadratic variation $QV(x)$ of a graph signal $x$ is given in (1.6), and for $1 \leq k \leq N - 1$, $W_k^\perp$ is the orthogonal complement of the space spanned by $v_k'$, $0 \leq k' \leq k - 1$. Denote the average and standard deviation of a vector $x \in \mathbb{R}^N$ by

$$m(x) = N^{-1}1^T x \quad \text{and} \quad \text{SD}(x) = N^{-1/2} \|x - m(x)1\|_2,$$

the null space of $L^T$ by $\ker(L^T)$, and the dimension of a linear space $W$ by $\dim W$. Based on the standard algorithm to evaluate the singular value decomposition and Courant-Fischer-Weyl min-max principle, we can apply the following approach to construct frequencies $\sigma_k$ and frequency components $v_k$ and $u_k$, $0 \leq k \leq N - 1$, of the proposed GFT:

$$\sigma_0 = 0 \quad \text{and} \quad v_0 = N^{-1/2}1$$  \hspace{1cm} (3.8a)

for $k = 0$, and

$$\begin{cases} 
\sigma_k = \min_{x \in W_k^\perp} \|x\|_2 = 1 \|Lx\|_2 \\
= \min_{\dim W = k+1} \max_{x \in W, \|x\|_2 = 1} \|Lx\|_2 \\
= \max_{\dim W = N-k} \min_{x \in W, \|x\|_2 = 1} \|Lx\|_2 \\
v_k = \arg\min_{x \in W_k^\perp} \|x\|_2 = 1 \|Lx\|_2 
\end{cases}$$  \hspace{1cm} (3.8b)

inductively for $1 \leq k \leq N - 1$, and let $u_k$, $0 \leq k \leq k_0$, be an orthonormal basis of the null space $\ker(L^T)$ with

$$u_0 = \arg\min_{x \in \ker(L^T)} \|x\|_2 = 1 \text{ and } m(x) \geq 0 \text{ SD}(x),$$  \hspace{1cm} (3.8c)

and define

$$u_k = \sigma_k^{-1}Lv_k, \quad k_0 < k \leq N - 1,$$  \hspace{1cm} (3.8d)

where $k_0$ is the largest index such that $\sigma_{k_0} = 0$. We remark that the left frequency component $u_0$ associated with zero frequency in the above construction satisfies (3.6) if $G$ is an Eulerian graph, and that $k_0 = 0$ if the Laplacian $L$ has rank $N - 1$, or equivalently if the graph $G$ is strongly connected. Shown in Fig. 4 are frequencies and
frequency components of a directed unweighted graph of size 15 containing three clusters connected by a directed cycle [26, Fig. 1(c)]. We observe that frequency components with low frequencies may have certain clustering property and oscillation pattern related to the graph topology.

In addition to the quadratic variation $QV(x)$ in (1.6) and $\|Lx\|_2$ in (3.8), several directed variations have been proposed to measure the variation of a graph signal $x = [x_i]_{i \in V}$ along the directed graph structure, including graph directed variation

$$GDV(x) = \sum_{i,j \in V} a_{ji}(x_i - x_j)^+ \quad (3.9)$$

and directed variation

$$DV(x) = \sum_{i,j \in V} a_{ji}(x_i - x_j)^+)^2, \quad (3.10)$$

where weight $a_{ij}$ is the $(i, j)$-th entry of the adjacency matrix $A$ and $t^+ = \max(t, 0)$ for any real number $t \in \mathbb{R}$ [26, 28]. The directed variations (3.9) and (3.10) are used in [26, 28] to define GFT on directed graphs, see Remark 3.1 below for the comparisons with the GFT in Definition 2.1.

**Remark 3.1** In [26], the authors use the graph directed variation $GDV(x)$ in (3.9) as Lovász extension of the cut size function, and define the GFT with frequency components $\mathbf{v}_k$ and frequencies $\lambda_k = GDV(\mathbf{v}_k), 0 \leq k \leq N - 1$, being ordered so that $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{N-1}$, where $\mathbf{V} = [\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}]$ is the solution of the following constrained minimization problem

$$\min_{\mathbf{V}} \sum_{k=0}^{N-1} GDV(\mathbf{v}_k) \quad (3.11)$$

subject to $\mathbf{V}^T \mathbf{V} = \mathbf{I}$ and $\mathbf{v}_0 = N^{-1/2} \mathbf{1}$. To deal with the nonsmooth objective function and non-convex orthogonality constraints in (3.11), the authors present two iterative algorithms, splitting orthogonality constraints (SOC) and proximal alternating minimization augmented Lagrange (PAMAL), to solve relaxed versions of the constrained minimization problem (3.11), see [26, Algorithms 1, 2, 3]. The above two implementations are more numerically stable than the GFT in (1.3) based on the Jordan decomposition of Laplacian, however they may fail to describe different modes of variation over the directed graph, see [28, Fig. 1]. Compared with the GFT proposed in this paper where only the SVD of the Laplacian matrix of size $N \times N$ is required, it needs to perform SVD of a matrix of size $N \times N$ at each iteration of the iterative SOC and PAMAL algorithms.

In [28], the authors use the directed variation $DV(x)$ in (3.10) to measure the signal variation along the graph structure, and define the GFT with frequency components $\mathbf{v}_k$ and frequencies $\lambda_k = DV(\mathbf{v}_k), 0 \leq k \leq N - 1$, being ordered so that $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{N-1}$, where $\mathbf{V} = [\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}]$ is the solution of the following constrained
problem,
\[
\min_{\mathbf{v}} \sum_{k=1}^{N-1} |\mathbf{D}(\mathbf{v}_k) - \mathbf{D}(\mathbf{v}_{k-1})|^2 \tag{3.12}
\]
subject to \( \mathbf{V}^T \mathbf{V} = \mathbf{I} \), \( \mathbf{v}_0 = N^{-1/2} \mathbf{1} \) and \( \mathbf{v}_{N-1} = \arg \max_{\|\mathbf{v}\|_2 = 1} \mathbf{D}(\mathbf{v}) \). Based on the feasible method for optimization over the Stiefel manifold in [33], the authors develop an iterative algorithm to solve the constrained problem (3.12), see [28, Algorithms 1 and 2]. At each iteration, the proposed algorithm involves a matrix inversion and the computational complexity is about \( O(N^3) \). Also as mentioned in [28, Remark 1], for the directed cycle graph (the circulant graph \( C_d(N, Q) \) generated by \( Q = \{1\} \)), the proposed GFT in [28] fails to obtain the discrete Fourier transform in (2.22), cf. Theorem 2.3 for our SVD-based GFT in the directed circulant graph setting.

In the undirected graph setting, the energy \( \|\mathbf{Lx}\|_2 = (\|\mathbf{Lx}\|_2 + \|\mathbf{L}^T \mathbf{x}\|_2)/2 \) of a graph signal \( \mathbf{x} \) is widely used to measure its smoothness [4, 20, 21, 29, 31]. For a directed Eulerian graph \( \mathcal{G} \), it is known that the transpose \( \mathbf{L}^T \) of the Laplacian on the graph \( \mathcal{G} \) coincides with the Laplacian of the transpose graph \( \mathcal{G}^T \). Then we may use \( (\|\mathbf{Lx}\|_2 + \|\mathbf{L}^T \mathbf{x}\|_2)/2 \) to measure regularity of a graph signal \( \mathbf{x} \) on a Eulerian graph. In this section, we finish with the observation that bandlimiting in the low frequencies of the proposed GFT provides good approximations to graph signals \( \mathbf{x} \) with regularity measured by \( (\|\mathbf{Lx}\|_2 + \|\mathbf{L}^T \mathbf{x}\|_2)/2 \) for arbitrary directed graphs, see Sect. 6.2 for the proof.

**Theorem 3.2** Let \( \mathbf{L} \) be the Laplacian matrix on \( \mathcal{G} \), and \( \mathbf{u}_k, \mathbf{v}_k, \sigma_k, 0 \leq k \leq N - 1 \), be as in (1.4). For a frequency bandwidth \( M \in \{1, 2, \ldots, N\} \), define the low frequency components of a graph signal \( \mathbf{x} \) on \( \mathcal{G} \) with bandwidth \( M \) by
\[
\mathbf{x}_M = \frac{1}{2} \sum_{k=0}^{M-1} (z_{1,k} + z_{2,k}) \mathbf{u}_k + (z_{1,k} - z_{2,k}) \mathbf{v}_k = \frac{1}{2} \sum_{k=0}^{M-1} (\mathbf{u}_k \mathbf{u}_k^T + \mathbf{v}_k \mathbf{v}_k^T) \mathbf{x}, \tag{3.13}
\]
where \( z_{1,k} = (\mathbf{u}_k + \mathbf{v}_k)^T \mathbf{x}/2 \) and \( z_{2,k} = (\mathbf{u}_k - \mathbf{v}_k)^T \mathbf{x}/2, 0 \leq k \leq M - 1 \). Then
\[
\|\mathbf{x} - \mathbf{x}_M\|_2 \leq \frac{1}{2\sigma_M} (\|\mathbf{Lx}\|_2 + \|\mathbf{L}^T \mathbf{x}\|_2). \tag{3.14}
\]

**4 Graph Fourier transform on directed Eulerian graphs**

Let \( \mathcal{E} = (V, E) \) be an Eulerian graph of order \( N \) containing no loops or multiple edges, and \( \mathcal{E}_t, 0 \leq t \leq 1 \), be a family of directed Eulerian graphs that share the same vertex set \( V \) with the graph \( \mathcal{E} \) and have adjacency matrices \( \mathbf{A}_t \) given in (1.9). In this section, we consider frequencies, frequency components and graph Fourier transforms on Eulerian graphs \( \mathcal{E}_t, 0 \leq t \leq 1 \), to connect the graph \( \mathcal{E} \) and its transpose graph \( \mathcal{E}^T \). It is observed that frequencies and frequency components on the Eulerian graphs \( \mathcal{E}_t, 0 \leq t \leq 1 \), have certain symmetric properties, see (4.8) and Theorem 4.2.
We say that frequencies \(\sigma_k(t), 0 \leq k \leq N - 1\), of the Eulerian graphs \(E_t, 0 \leq t \leq 1\), are *simple* if
\[
0 = \sigma_0(t) < \sigma_1(t) < \cdots < \sigma_{N-1}(t), \quad 0 \leq t \leq 1.
\]

In Theorem 4.1, we show that frequencies and frequency components are differentiable about \(0 \leq t \leq 1\), if frequencies of the Eulerian graphs \(E_t, 0 \leq t \leq 1\), are simple.

From the estimation in Theorem 4.1, we conclude that frequencies and frequency components have slow variation on \(0 \leq t \leq 1\) when the measurement \(\sigma_{\text{asym}}\) in (1.8) to qualify the symmetry of the directed Eulerian graph \(\mathcal{G}\) is small, see (4.15) and (4.16).

Recall that an Eulerian graph \(E\) has the same in-degree and out-degree at each vertex, the Laplacians \(L_t\) of the graphs \(E_t\), \(0 \leq t \leq 1\), are given by
\[
L_t = (1 - t)L + tL^T,
\]
and satisfy
\[
L_t1 = 0, \quad 0 \leq t \leq 1.
\]

By the continuity of the Laplacian \(L_t, 0 \leq t \leq 1\), we can find an SVD
\[
L_t = U_t\Sigma_tV_t^T
\]
with initials \((U_0, V_0, \Sigma_0) = (U, V, \Sigma)\) such that orthogonal matrices \(U_t, V_t\) and diagonal matrices
\[
\Sigma_t = \text{diag}(\sigma_0(t), \ldots, \sigma_{N-1}(t))
\]
of singular values of Laplacians \(L_t\) in a nondecreasing order are continuous about \(0 \leq t \leq 1\), where \(L = U\Sigma V^T\) is the SVD (1.4) of the Laplacian \(L\). Using the SVD (4.4) of the Laplacian \(L_t\), we can define GFT of a signal \(x\) on the graph \(E_t\) (and also on \(E = E_0\) as they have the same vertex set) by
\[
\mathcal{F}_t x = \frac{1}{2} \left( U_t^T x + V_t^T x \right), \quad 0 \leq t \leq 1.
\]

By (4.3), we have
\[
\sigma_0(t) = 0, \quad 0 \leq t \leq 1.
\]

By the SVD (4.4), \((\sigma_k(t))^2, 0 \leq k \leq N - 1\), are eigenvalues of matrices \(L_t^T L_t\) and \(L_t L_t^T\). This together with the nonnegative nondecreasing order of singular values \(\sigma_k(t), 0 \leq k \leq N - 1\), and the observation that \(L_{1-t}L_t^T = L_t^T L_t, \quad 0 \leq t \leq 1\), proves that
\[
\sigma_k(1 - t) = \sigma_k(t), \quad 0 \leq t \leq 1.
\]

Shown in Fig. 5 are the graph frequencies \(\sigma_k(t), 0 \leq k \leq N - 1\), of Eulerian graphs \(E_t, 0 \leq t \leq 1\), of order \(N = 64\).

Observe from (4.2) that \(L_t, 0 \leq t \leq 1\), satisfy
\[
L_t - L_s = (t - s)(L^T - L), \quad 0 \leq t, s \leq 1.
\]
This together with the Courant-Fischer-Weyl min-max principle

\[
\sigma_k(t) = \min_{\dim W = k+1} \max_{x \in W, \|x\|_2 = 1} \|L_t x\|_2 = \max_{\dim W = N-k} \min_{x \in W, \|x\|_2 = 1} \|L_t x\|_2
\]

implies that \(\sigma_k(t), 0 \leq k \leq N - 1\), are Lipschitz functions,

\[
|\sigma_k(t) - \sigma_k(s)| \leq \sigma_{asym} |t - s|, \quad 0 \leq t, s \leq 1.
\] (4.10)

In the following theorem, we consider the differentiability of frequencies and left/right frequency components with respect to \(0 \leq t \leq 1\), when \(\sigma_k(t), 0 \leq k \leq N - 1\), are simple. By (4.10), we see that the simple requirement (4.1) is met if all eigenvalues of Laplacian on \(E_{1/2}\) are simple, and the directed Eulerian graph \(E\) is close to its undirected counterpart \(E_{1/2}\) in the sense that

\[
0 < \sigma_{asym} \leq \alpha \min_{1 \leq k \leq N-1} \sigma_k(1/2) - \sigma_{k-1}(1/2)
\]

for some \(0 < \alpha < 1\).

**Theorem 4.1** Let \(E_t, 0 \leq t \leq 1\), be the family of directed Eulerian graphs to connect a directed Eulerian graph \(E\) and its transpose graph \(E^T\), and the associated Laplacians \(L_t\) in (4.2) have the SVDs (4.4) such that orthogonal matrices \(U_t = [u_0(t), \ldots, u_{N-1}(t)], V_t = [v_0(t), \ldots, v_{N-1}(t)]\) and diagonal matrices \(\Sigma_t = \text{diag}(\sigma_0(t), \ldots, \sigma_{N-1}(t))\) of singular values of Laplacians \(L_t\) in a nondecreasing order are continuous about \(t\) and satisfy

\[
u_0(t) = v_0(t) = N^{-1/2} I.
\] (4.11)
Then for any $1 \leq k \leq N - 1$, the $k$-th frequency $\sigma_k(t)$ and frequency components $\mathbf{u}_k(t)$, $\mathbf{v}_k(t)$ of the graph Fourier transform $\mathcal{F}_t$, $0 \leq t \leq 1$, are differentiable about $t$ if $\sigma_k(t)$ is a simple singular value, i.e., $\sigma_k(t) \neq \sigma_{k'}(t)$ for all $k' \neq k$. Moreover, for all $1 \leq k \leq N - 1$,

$$
\frac{d\sigma_k(t)}{dt} = \mathbf{v}_k^T(t)(\mathbf{L}^T - \mathbf{L})\mathbf{u}_k(t),
$$

(4.12)

$$
\frac{d\mathbf{u}_k(t)}{dt} = \sum_{k' = 1}^{N-1} \left( a_{k,k'}(t)\mathbf{u}_{k'}^T(t)(\mathbf{L}^T - \mathbf{L})\mathbf{v}_k(t) - b_{k,k'}(t)\mathbf{v}_{k'}^T(t)(\mathbf{L}^T - \mathbf{L})\mathbf{u}_k(t) \right)\mathbf{u}_{k'}(t),
$$

(4.13)

and

$$
\frac{d\mathbf{v}_k(t)}{dt} = \sum_{k' = 1}^{N-1} \left( -a_{k,k'}(t)\mathbf{v}_{k'}^T(t)(\mathbf{L}^T - \mathbf{L})\mathbf{u}_k(t) + b_{k,k'}(t)\mathbf{u}_{k'}^T(t)(\mathbf{L}^T - \mathbf{L})\mathbf{v}_k(t) \right)\mathbf{v}_{k'}(t),
$$

(4.14)

where

$$
a_{k,k'}(t) = \begin{cases} 
    \sigma_k(t)((\sigma_k(t))^2 - (\sigma_{k'}(t))^2)^{-1} & \text{if } k' \neq k \\
    4\sigma_k(t)^{-1} & \text{if } k' = k,
\end{cases}
$$

and

$$
b_{k,k'}(t) = \begin{cases} 
    \sigma_{k'}(t)((\sigma_k(t))^2 - (\sigma_{k'}(t))^2)^{-1} & \text{if } k' \neq k \\
    -4\sigma_k(t)^{-1} & \text{if } k' = k.
\end{cases}
$$

The detailed proof of Theorem 4.1 will be given in Sect. 6.3.

Let the degree $\sigma_{asym}$ of asymmetry of the directed Eulerian graph $\mathcal{E}$ be as in (1.8). By Theorem 4.1, we have

$$
\left| \frac{d\sigma_k(t)}{dt} \right| \leq \sigma_{asym}.
$$

(4.15)

Set

$$
C(t) = \max_{1 \leq k,k' \leq N-1} |a_{k,k'}(t)| + \max_{1 \leq k,k' \leq N-1} |b_{k,k'}(t)|.
$$

By (4.13) and the orthogonality of the matrices $\mathbf{U}$ and $\mathbf{V}$, we obtain

$$
\left\| \frac{d\mathbf{u}_k(t)}{dt} \right\|_2 \leq \left( \max_{1 \leq k_1,k_2 \leq N-1} |a_{k_1,k_2}(t)| \right) \left\| (\mathbf{L}^T - \mathbf{L})\mathbf{v}_k(t) \right\|_2
$$

$$
+ \left( \max_{1 \leq k_1,k_2 \leq N-1} |b_{k_1,k_2}(t)| \right) \left\| (\mathbf{L}^T - \mathbf{L})\mathbf{u}_k(t) \right\|_2
$$

$$
\leq C(t)\sigma_{asym}, \quad 1 \leq k \leq N - 1.
$$
Following a similar argument to \( \| d\mathbf{v}_k(t) / dt \|_2 \), then we have

\[
\max \left( \left\| d\mathbf{u}_k(t) / dt \right\|_2, \left\| d\mathbf{v}_k(t) / dt \right\|_2 \right) \leq C(t)\sigma_{\text{asym}}, \quad 1 \leq k \leq N - 1.
\] (4.16)

This concludes that frequencies and frequency components have small variations about \( 0 \leq t \leq 1 \) when the degree \( \sigma_{\text{asym}} \) of asymmetry of the Eulerian graph \( \mathcal{E} \) is small.

Under the simplicity assumption (4.1) for all singular values \( \sigma_k(t), 0 \leq k \leq N - 1 \), in addition to the symmetry (4.8) for the graph frequencies, we have certain symmetric property for the orthogonal matrices \( \mathbf{U}_t \) and \( \mathbf{V}_t \), \( 0 \leq t \leq 1 \), see Sect. 6.4 for the detailed proof.

**Theorem 4.2** Let the family of directed Eulerian graphs \( \mathcal{E}_t, 0 \leq t \leq 1 \), the associated Laplacians \( \mathbf{L}_t \), the singular value decompositions \( \mathbf{L}_t = \mathbf{U}_t \mathbf{\Sigma}_t \mathbf{V}_t \) be as in Theorem 4.1. If the singular values \( \sigma_k(t), 0 \leq k \leq N - 1 \), satisfy (4.1), then for all \( 0 \leq t \leq 1 \),

\[
\mathbf{U}_t = \mathbf{V}_{1-t}
\] (4.17)

and

\[
\mathcal{F}_{1-t}\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathcal{F}_t\mathbf{x},
\] (4.18)

where \( \mathbf{x} \) is a graph signal on the Eulerian graph \( \mathcal{E} \).

For the case that \( \mathcal{E} \) is undirected (hence an Eulerian graph), the orthogonal matrices \( \mathbf{U}_t \) and \( \mathbf{V}_t \), \( 0 \leq t \leq 1 \), in the singular value decompositions (4.4) can be chosen to be independent on \( t \). In the following theorem, we show that \( (\mathbf{L}_t^T)^2 = \mathbf{L}^2 \) is a necessary condition for some pair of orthogonal matrices \( \mathbf{U}_t \) and \( \mathbf{V}_t \), \( 0 \leq t \leq 1 \), is identical, see Sect. 6.5 for the proof.

**Theorem 4.3** Let \( \mathbf{U}_t, \mathbf{V}_t \) be the orthogonal matrices in the singular value decompositions (4.4) of the Laplacians \( \mathbf{L}_t, 0 \leq t \leq 1 \), in (4.2). If there exist \( t_0 \neq t_1 \in [0, 1] \) such that

\[
(\mathbf{U}_{t_0}, \mathbf{V}_{t_0}) = (\mathbf{U}_{t_1}, \mathbf{V}_{t_1}),
\] (4.19)

then \( (\mathbf{L}_t^T)^2 = \mathbf{L}^2 \).

**5 Numerical simulations**

Graph Fourier transforms (GFTs) should be designed to decompose graph signals into different frequency components, In this section, we consider denoising the hourly temperature data set collected at 218 locations in the United States on August 1st, 2010 [2, 9, 35] and demonstrate the denoising performances via the bandlimiting \( \mathbf{P}_M \) at the first \( M \)-frequencies of GFTs proposed in this paper and [11, 26, 36]. Here for the SVD-based GFT proposed in this paper, the bandlimiting \( \mathbf{P}_M \) of a graph signal \( \mathbf{x} \)
Fig. 6 Plotted are the hourly temperature data $x(t_9)$ and $x(t_{21})$, on August 1st, 2012 at the first $M$-frequencies is given by

$$P_M x = \mathcal{F}^{-1} \left( \mathbf{O}_M \mathbf{O} \right) \mathcal{F} x = \frac{1}{2} \sum_{k=0}^{M-1} \langle x, u_k \rangle u_k + \langle x, v_k \rangle v_k,$$

where $\chi_M$ is a diagonal matrix with the first $M$ diagonal entries taking value one and all others taking value zero, and for $0 \leq k \leq M - 1$, the $k$-th left/right frequency components $u_k, v_k$ are $k$-th columns of orthogonal matrices $U$ and $V$ in the SVD (1.4) respectively. For the GFTs given by splitting orthogonality constraints (SOC) and proximal alternating minimized augmented Lagrangian (PAMAL), the bandlimiting $P_M$ of a graph signal $x$ at the first $M$-frequencies is given by

$$P_M x = \sum_{k=0}^{M-1} \langle x, v_k \rangle v_k,$$

where $v_k, 0 \leq k \leq M - 1$, is the $k$-th frequency component in [26]. Similarly for the GFTs based on the magnetic Laplacian $L_q, q \geq 0$, the bandlimiting $P_M$ of a graph signal $x$ at the first $M$-frequencies is determined by

$$P_M x = \sum_{k=0}^{M-1} v_{k; q} v_{k; q}^H x,$$

where $v_{k; q}, 0 \leq k \leq M - 1$, is the $k$-th frequency component in (2.12) [11, 36].

Let the underlying graph $G$ of the U.S. temperature data set have 218 vertices representing locations of weather stations, edges given by 5-nearest neighboring stations in physical distances, and weights on the edges are randomly chosen in $[0.8, 1.2]$, and denote the U.S. temperature measured in Fahrenheit on August 1st, 2010 by $x(t_m), 1 \leq m \leq 24$, see Fig. 6 for two snapshots of the data set at 9:00 AM and 9:00 PM, and [2, Fig. 6] for snapshots at 0:00 AM and 12:00 PM. Shown in Fig. 7 are the
Fig. 7 Plotted are the averages of ISNRs and SNRs of denoising the hourly temperature data set $y(t_m), 1 \leq m \leq 24$, on August 1st, 2012 over $1000 \times 24$ trials, via bandlimiting $P_M$ at the first $M$-frequencies of the SVD, SOC, PAMAL and $q$-ML with $q = \frac{1}{2}$ and $q = \frac{1}{4}$, where $M = 25$ (left) and $M = 50$ (right).

denoising performances in ISNR and SNR to apply the bandlimiting projection $P_M$ to the noisy observations

$$y(t_m) = x(t_m) + \eta(t_m), \ 1 \leq m \leq 24,$$  \hspace{1cm} (5.1)

corrupted with additive random noises $\eta(t_m)$ with entries being i.i.d. and having mean zero and variance $c \in [4, 16]$. Here the input signal-to-noise ratio (ISNR) and the output signal-to-noise ratio (SNR) are defined by

$$\text{ISNR} = -20 \log_{10} \frac{\|y - x\|_2}{\|x\|_2} \quad \text{and} \quad \text{SNR} = -20 \log_{10} \frac{\|\hat{x} - x\|_2}{\|x\|_2},$$

where the original signal $x$, the noisy measurement $y$ and the denoised signal $\hat{x}$ are given by the hourly temperature data $x(t_m)$, the noisy temperature data $y(t_m)$ in (5.1), and the denoised signal $P_My(t_m)$ by bandlimiting $P_M$ the noisy temperature data $y(t_m)$ to the first $M$-frequencies respectively. We observe from Fig. 7 that the SVD-based GFT proposed in this paper outperforms the SOC and PAMAL-based GFTs in Remark 3.1 and the $q$-ML-based GFTs in Remark 2.2 with $q = \frac{1}{2}$ and $q = \frac{1}{4}$ on denoising the U.S. hourly temperature data set on August 1st, 2010 by bandlimiting $P_M$ at the first $M$-frequencies. It is also noticed that the SOC and PAMAL-based GFTs in [26] have very similar performance on denoising the temperature data set. The possible reason is that they are based on different relaxations of the same constrained minimization problem (3.11).

6 Proofs

In the section, we collect the proofs of Theorems 2.3, 3.2, 4.2 and 4.3.
6.1 The graph Fourier transform on circulant graphs

In this subsection, we discuss the GFT on circulant graphs.

Write \( Q = \{q_1, \ldots, q_L\} \) with \( 1 \leq q_1 < q_2 < \cdots < q_L \leq N - 1 \), and \( W = [w_0, \ldots, w_{N-1}] \). Observe that the Laplacian matrix \( L_{C_d} = (c_{ij})_{0 \leq i, j \leq N-1} \) on the circulant graph \( C_d := C_d(N, Q) \) is a circulant matrix with \( ij \)-th entries \( c_{ij} \), given by

\[
c_{ij} = \begin{cases} 
L & \text{if } j = i \\
-1 & \text{if } j - i \in Q \mod N \\
0 & \text{otherwise}
\end{cases}
\]

Then one may verify that

\[
L_{C_d} w_k = P(\omega_N^k)w_k, \quad 0 \leq k \leq N - 1, \tag{6.1}
\]

where \( P \) is the polynomial symbol of the circulant matrix \( L_{C_d} \) defined by (2.14).

Let

\[
M = \text{diag}(|P(1)|, |P(\omega_N)|, \ldots, |P(\omega_N^{N-1})|), \tag{6.2}
\]

be the diagonal matrix with magnitudes \( |P(\omega_N^k)|, 0 \leq k \leq N - 1 \), of the symbol \( P \) on all \( N \)-th unit roots. Then we can reformulate (6.1) in the following matrix form,

\[
L_{C_d} W = W \Theta M, \tag{6.3}
\]

where \( \Theta \) is the diagonal matrix in (2.18).

Let \( Q \) be a permutation matrix to rearrange \( |P(\omega_N^k)|, 0 \leq k \leq N - 1 \), in a nondecreasing order, with 0 as the first index, and indices \( k \) and \( N - k \), \( 1 \leq k < N/2 \), next each other. This together with \( |P(\omega_N^k)| = |P(\omega_N^{N-k})|, 1 \leq k \leq N - 1 \), implies that the diagonal matrix \( \Sigma \) in (2.21) satisfies

\[
\Sigma = Q^T M Q. \tag{6.4}
\]

Let \( e_k, 0 \leq k \leq N - 1 \), be the unit vectors with zero entries except the \( k \)-th entry taking value 1, and define the permutation matrices \( P_0 \) and \( P_1 \) by

\[
P_0 = \begin{cases} 
[e_0, e_1, e_{N-1}, e_2, e_{N-2}, \ldots, e_{(N-1)/2}, e_{(N+1)/2}] & \text{if } N \text{ is odd} \\
[e_0, e_1, e_{N-1}, e_2, e_{N-2}, \ldots, e_{N/2-1}, e_{N/2+1}, e_{N/2}] & \text{if } N \text{ is even}
\end{cases} \tag{6.5}
\]

and

\[
P_1 = P_0 Q. \tag{6.6}
\]

Therefore the conclusion in Theorem 2.3 about the GFT on the circulant graph \( C_d(N, Q) \) reduces to establishing the SVD (2.19) for the Laplacian matrix \( L_{C_d} \).

**Proposition 6.1** Let \( L_{C_d} \) be the Laplacian matrix on the circulant graph \( C_d := C_d(N, Q) \), and \( W, R, \Theta, \Sigma, P_0 \) and \( P_1 \) be as in (2.15), (2.17), (2.18), (2.21), (6.5)
and (6.6) respectively. Then the matrices \( U \) and \( V \) in (2.20) are orthogonal matrices with real entries, and the SVD (2.19) holds for the Laplacian matrix \( L_{\mathcal{C}_d} \).

**Proof** The conclusions are trivial for \( N = 1 \) and \( N = 2 \). So we assume that \( N \geq 3 \) from now on. First we divide two cases, \( N \geq 3 \) is odd and even, to prove that matrices \( U \) and \( V \) in (2.20) are orthogonal matrices with real entries. Define

\[
\tilde{U} = \mathbf{W}\Theta_0^T\mathbf{R} = [\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_{N-1}] \tag{6.7}
\]

and

\[
\tilde{V} = \mathbf{W}\mathbf{P}_0^T\mathbf{R} = [\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_{N-1}] \tag{6.8}
\]

As \( \mathbf{P}_1 \) is a permutation matrix, \( U = \tilde{U}\mathbf{P}_1 \) and \( V = \tilde{V}\mathbf{P}_1 \), it suffices to show that \( \tilde{U} \) and \( \tilde{V} \) are orthogonal matrices with real entries.

**Case 1**: \( N = 2K + 1 \) for some integer \( K \geq 1 \).

By (2.17), (6.7) and (6.8), we have

\[
\begin{align*}
\tilde{u}_0 &= \tilde{v}_0 = w_0 = N^{-1/2}1 \in \mathbb{R}^N, \\
\tilde{v}_{2k-1} &= \frac{w_k + w_{N-k}}{\sqrt{2}} = \frac{w_k + w_{N-k}}{\sqrt{2}}, \\
\tilde{v}_{2k} &= \frac{w_k - w_{N-k}}{\sqrt{2}} = \frac{w_k - w_{N-k}}{\sqrt{2}},
\end{align*}
\tag{6.9}
\]

and

\[
\begin{align*}
\tilde{u}_{2k-1} &= \exp(\sqrt{-1}0_k)w_k + \exp(-\sqrt{-1}0_k)w_{N-k} \in \mathbb{R}^N, \\
\tilde{u}_{2k} &= \exp(\sqrt{-1}0_k)w_k - \exp(-\sqrt{-1}0_k)w_{N-k} \in \mathbb{R}^N,
\end{align*}
\tag{6.10}
\]

for \( 1 \leq k \leq K \). Therefore \( \tilde{U} \) and \( \tilde{V} \) are square matrices with real entries. This together with the unitary property for the discrete Fourier transform matrix \( \mathbf{W} \), the phase matrix \( \Theta \) and the rotation matrix \( \mathbf{R} \), and the orthogonality of the permutation matrix \( \mathbf{P}_1 \) implies that

\[
\tilde{U}^T\tilde{U} = \tilde{U}^H\tilde{U} = R^H\mathbf{P}_0\Theta^H\mathbf{W}^H\mathbf{W}\Theta_0^T\mathbf{R} = I
\]

and

\[
\tilde{V}^T\tilde{V} = \tilde{V}^H\tilde{V} = R^H\mathbf{P}_0\mathbf{W}^H\mathbf{W}\mathbf{P}_0^T\mathbf{R} = I.
\]

This proves that \( \tilde{U} \) and \( \tilde{V} \) (and hence \( U \) and \( V \) in (2.20)) are orthogonal matrices with real entries for the case that \( N \) is odd.

**Case 2**: \( N = 2K + 2 \) for some integer \( K \geq 1 \).

Using the similar argument used in Case 1, we can show that (6.9) and (6.10) hold. In addition, we have

\[
\tilde{u}_{2K+1} = \tilde{v}_{2K+1} = N^{-1/2}(1, -1, \ldots, 1, -1)^T \in \mathbb{R}^N \tag{6.12}
\]

Therefore \( \tilde{U} \) and \( \tilde{V} \) are square matrices with real entries. The orthogonal property for the matrices \( \tilde{U} \) and \( \tilde{V} \) can be established in a similar way used in Case 1. This proves that \( \tilde{U} \) and \( \tilde{V} \) (and hence \( U \) and \( V \) in (2.20)) are orthogonal matrices with real entries for the case that \( N \) is even.
Next we establish the SVD (2.19) for the Laplacian matrix \( L_{C_d} \). By \( |P(\omega_N^i)|^2 = |P(\omega_N^{N-i})|^2 \), \( 1 \leq i \leq N - 1 \), one may verify that

\[
MP_0^T R P_0 = P_0^T R P_0 M. \tag{6.13}
\]

By (2.20), (6.3), (6.4), (6.6), (6.13), and the permutation property \( Q^T Q = I \), we obtain

\[
L_{C_d} V = L_{C_d} W P_0^T R P_1 = W \Theta M P_0^T R P_0 Q = W \Theta P_0^T R P_0 M Q = U \Sigma.
\]

This together with the real orthogonal property for the matrices \( U \) and \( V \) proves the SVD in (2.19) for the Laplacian \( L_{C_d} \), and hence completes the proof. \( \square \)

### 6.2 Proof of Theorem 3.2

By (1.4), we have

\[
\|L x\|_2^2 = x^T V 6^2 V^T x = \sum_{k=0}^{N-1} \sigma_k^2 (v_k^T x)^2 \geq \sigma_M^2 \sum_{k=M}^{N-1} (v_k^T x)^2 \tag{6.14}
\]

and

\[
\|L^T x\|_2^2 = x^T U 6^2 U^T x \geq \sigma_M^2 \sum_{k=M}^{N-1} (u_k^T x)^2. \tag{6.15}
\]

From (2.9) and (3.13), it follows that

\[
\|x - x_M\|_2 = \frac{1}{2} \left\| \sum_{k=M}^{N-1} (u_k u_k^T + v_k v_k^T) x \right\|_2 \\
\leq \frac{1}{2} \left( \sum_{k=M}^{N-1} (u_k^T x)^2 \right)^{1/2} + \frac{1}{2} \left( \sum_{k=M}^{N-1} (v_k^T x)^2 \right)^{1/2}.
\]

This together with (6.14) and (6.15) completes the proof.

### 6.3 Proof of Theorem 4.1

Let \( S(L_t) \) be the self-adjoint dilation of the Laplacian \( L_t \), \( 0 \leq t \leq 1 \). By (4.4), we have

\[
S(L_t) F_t = \frac{1}{\sqrt{2}} \begin{pmatrix} L_t V_t & -L_t V_t \\ L_t^T U_t & L_t^T U_t \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} U_t \Sigma_t & -U_t \Sigma_t \\ V_t \Sigma_t & V_t \Sigma_t \end{pmatrix} \\
= \frac{1}{\sqrt{2}} \begin{pmatrix} U_t & U_t \\ V_t & -V_t \end{pmatrix} \begin{pmatrix} \Sigma_t & 0 \\ 0 & -\Sigma_t \end{pmatrix} = F_t \Lambda_t, \tag{6.16}
\]

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where
\[
F_t := \frac{1}{\sqrt{2}} \begin{pmatrix} U_t & U_t \\ V_t & -V_t \end{pmatrix} = [z_0(t), z_1(t), \ldots, z_{2N-1}(t)]
\]  
(6.17)
and
\[
\Lambda_t := \begin{pmatrix} \Sigma_t & 0 \\ 0 & -\Sigma_t \end{pmatrix} = \text{diag}(\lambda_0(t), \lambda_2(t), \ldots, \lambda_{2N-1}(t)).
\]  
(6.18)

By (4.10) and the assumption on \(k\)-th frequency \(\sigma_k(t), 1 \leq k \leq N - 1\), we can find \(\delta > 0\) such that for all \(0 \leq s \leq 1\) with \(|s - t| < \delta\),
\[
\lambda_k(s) = \sigma_k(s)
\]  
(6.19)
is a simple eigenvalue of self-adjoint dilation \(S(L_\sigma)\) of the Laplacian \(L_\sigma, 0 \leq s \leq 1\), and \(z_k(s)\) is an associated eigenvector with norm one. This together with (6.16) and (6.17) implies that
\[
\begin{pmatrix} \sigma_k(t)I - S(L_\sigma) z_k(t) \\ z_k(t)^T \end{pmatrix} = \begin{pmatrix} F_t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_k(t)I - \Lambda_t & e_k \\ e_k^T & 0 \end{pmatrix} \begin{pmatrix} F_t & 0 \\ 0 & 1 \end{pmatrix}^T,
\]  
(6.20)
is nonsingular, where \(e_k, 0 \leq k \leq 2N - 1\), are unit vectors of size \(2N\) with all entries taking value zero except value one at \(k\)-th entry.

Define a map \(H : \mathbb{R}^{2N} \times \mathbb{R} \times [0, 1] \to \mathbb{R}^{2N+1}\) by
\[
H(z, \lambda, t) = \left( \frac{\lambda z - S(L_\sigma)z}{z^T}, \lambda, t \right).
\]  
(6.21)

Then
\[
\nabla_{z, \lambda} H(z, \lambda, t) = \begin{pmatrix} \lambda I - S(L_\sigma) & z \\ z^T & 0 \end{pmatrix} \quad \text{and} \quad \nabla_t H(z, \lambda, t) = \begin{pmatrix} S(L^T - L)z \\ 0 \end{pmatrix}.
\]  
(6.22)

By (6.16), (6.19), (6.20), (6.22), and the implicit function theorem, there exists \(0 < \tilde{\delta} < \delta\) such that for all \(s\) with \(|s - t| < \tilde{\delta}\), \((z_k(s)^T, \sigma_k(s))\) is the unique solution of
\[
H(z, \lambda, s) = 0
\]
in the neighborhood of \((z_k(t)^T, \sigma_k(t))\). Applying the implicit function theorem again and using (6.20), (6.22), we obtain
\[
\frac{d(z_k(t))}{dt} = -\left(\nabla_{z, \lambda} H(z, \lambda, t)\right)^{-1} \nabla_t H(z, \lambda, t) = -\begin{pmatrix} F_t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\sigma_k(t)I - \Lambda_t)^T e_k \\ e_k^T \\ 0 \end{pmatrix} \begin{pmatrix} F_t & 0 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} S(L^T - L)z_k(t) \\ 0 \end{pmatrix},
\]  
(6.23)
where the diagonal matrix $\sigma_k(t)I - \Lambda_t$ is the pseudo-inverse of $\sigma_k(t)I - \Lambda_t$ with $k'$-th diagonal entries being $(\sigma_k(t) - \sigma_k'(t))^{-1}$ for $k \neq k' \leq N - 1$, 0 for $k' = k$ and $(\sigma_k(t) + \sigma_k'(t))^{-1}$ for $N \leq k' \leq 2N - 1$. Substituting $z_k(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} u_k(t) \\ v_k(t) \end{pmatrix}$ into (6.23), we prove (4.12).

Let

$$A_{k,t} = \frac{1}{2} \left( (\sigma_k(t)I - \Sigma_t)^\dagger + (\sigma_k(t)I + \Sigma_t)^{-1} \right) = \text{diag}(a_{k,0}(t), \ldots, a_{k,N-1}(t)) \quad (6.24)$$

and

$$B_{k,t} = \frac{1}{2} \left( (\sigma_k(t)I - \Sigma_t)^\dagger - (\sigma_k(t)I + \Sigma_t)^{-1} \right) = \text{diag}(b_{k,0}(t), \ldots, b_{k,N-1}(t)). \quad (6.25)$$

By (6.17) and (6.23), we obtain

$$\frac{du_k(t)}{dt} = U_t A_{k,t} U_t^T (L^T - L)v_k(t) - U_t B_{k,t} V_t^T (L^T - L)u_k(t) \quad (6.26)$$

and

$$\frac{dv_k(t)}{dt} = V_t B_{k,t} U_t^T (L^T - L)v_k(t) - V_t A_{k,t} V_t^T (L^T - L)u_k(t). \quad (6.27)$$

By (4.11), we have

$$u_0^T (L^T - L) = v_0^T (L^T - L) = N^{-1/2} I^T (L^T - L) = 0.$$ 

This together with (6.24)-(6.27) completes the proof of (4.13) and (4.14).

### 6.4 Proof of Theorem 4.2

By (4.1), the SVD (4.4) for the Laplacian $L_t$, $0 \leq t \leq 1$ is unique, up to a sign for each eigenvector, i.e.,

$$\tilde{U}_t = U_t S_t \quad \text{and} \quad \tilde{V}_t = V_t S_t \quad (6.28)$$

for any orthogonal pairs $(U_t, V_t)$ and $(\tilde{U}_t, \tilde{V}_t)$ in the singular value decomposition (4.4), where $S_t$ is a diagonal matrix with $\pm 1$ as its diagonal entries. In our setting, we observe from the SVD (4.4) that

$$L_t = U_t \Sigma_t V_t^T = V_{1-t} \Sigma_t U_{1-t}^T = L_{1-t}^T, \quad 0 \leq t \leq 1. \quad (6.29)$$

By (6.28) and (6.29), and orthogonality property for $U_t$ and $V_t$, $0 \leq t \leq 1$, there exists diagonal matrices $S_t$, $0 \leq t \leq 1$, with diagonal entries $\pm 1$ such that

$$V_{1-t}^T U_t = U_{1-t}^T V_t = S_t. \quad (6.30)$$

Recall that $U_t$ and $V_t$ are continuous about $t \in [0, 1]$. This, together with (6.30) and the observation that $S_t$, $0 \leq t \leq 1$, have entries taking values $\pm 1$, implies that $S_t$, $0 \leq t \leq 1$, is independent on $t$, i.e.,

\[ \gcd \]
\[ S_t = S_{1/2}, \quad 0 \leq t \leq 1. \] (6.31)

For \( t = 1/2 \), \( L_t \) is a symmetric matrix with all eigenvalues being simple by (4.1), which implies that \( U_{1/2} = V_{1/2} \). Hence

\[ S_{1/2} = I. \]

This together with (6.30) and (6.31) proves (4.17). The relationship (4.18) between GFTs \( \mathcal{F}_{1-t} \) and \( \mathcal{F}_t \), \( 0 \leq t \leq 1 \), follows directly from (2.5) and (4.17).

### 6.5 Proof of Theorem 4.3

Set \( U = U_{t_0} = U_{t_1} \). By (4.4) and (4.19), we have

\[
((1 - t_0)L + t_0L^T)((1 - t_1)L + t_1L^T)^T = L_{t_0}(L_{t_1})^T = U \Sigma_{t_0} \Sigma_{t_1} U^T = U \Sigma_{t_1} \Sigma_{t_0} U^T = L_{t_1}(L_{t_0})^T = ((1 - t_1)L + t_1L^T)((1 - t_0)L + t_0L^T)^T.
\]

Simplifying the above equality and using \( t_0 \neq t_1 \) proves the conclusion that \( (L^T)^2 = L^2 \).

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