Interaction of electromagnetic perturbations with infalling observers inside spherical charged black holes

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The electromagnetic radiation that falls into a Reissner-Nordström black hole is known to develop a “blue sheet”, namely, an infinite concentration of energy density at the Cauchy horizon. The interaction of these divergent electromagnetic fields with infalling matter was recently analyzed (L. M. burko and A. Ori, Phys. Rev. Lett. 74, 1064 (1995)). Here, we give a more detailed description of that analysis: We consider classical electromagnetic fields (that were produced during the collapse and then backscattered into the black hole), and investigate the blue-sheet effects of these fields on infalling objects within two simplified models of a classical and a quantum absorber. These effects are found to be finite and even negligible for typical parameters of a supermassive black hole.

I. INTRODUCTION

The Kerr-Newman black hole (BH), which in view of the no-hair theorems is expected to be the stationary outcome of gravitational collapse, is an exact (electro-) vacuum solution of the Einstein field equations. However, one expects a generic collapse process to be accompanied by perturbations, which may exist before the onset of the collapse, or develop during it. Consequently, one would not expect to find an astrophysical BH to be an exact Kerr-Newman solution, but rather a perturbed one.

The possibility to fall into a black hole and re-emerge in another universe is one of the most intriguing open questions of General Relativity. The spacetime of un perturbed BHs, such as Kerr, seems to allow this possibility. However, the Kerr geometry is highly symmetrical; It is not a priori clear whether more realistic solutions to the Einstein equations, which are not unperturbed, still allow for the possibility to traverse the inner horizon safely and re-emerge in a different universe. Consequently, perturbed BHs have been under investigation during the last three decades.

Realistic BHs are the outcome of gravitational collapse. In such a realistic (a-symmetric) collapse, non-vanishing multipole moments of various fields develop in the star, and consequently electromagnetic and gravitational waves are emitted from the surface of the collapsing star. As these waves propagate outwards, some fraction of them is backscattered off the spacetime curvature and captured by the BH. This process leads to a “tail” of radiation, which at late times decays according to an inverse-power law both for the spherical case and for the spinning case.

To re-emerge in another universe it is necessary to traverse a certain null hypersurface known as the inner horizon or the Cauchy horizon (CH). In order to cross the CH safely, it is necessary that the spacetime be either nonsingular there or have at the most only a weak singularity. Therefore, it is troubling that the CH is a surface of infinite blue-shift, i.e., infalling radiation (electromagnetic or gravitational), even very mild and well-behaved in the external universe, is infinitely blue-shifted at the CH.

There are two types of problems which may cause difficulties for an observer who wishes to cross the CH. First, a divergent spacetime curvature develops on the CH. This curvature singularity results from two sources: the infinite blue shift of infalling gravitational waves (which lead to the divergence of the gradients of the metric perturbations) and the infinite energy-momentum tensor (which can be taken as a dynamic source term for the Einstein equations) of the infinitely blue-shifted infalling electromagnetic radiation. Second, the same infinitely blue-shifted electromagnetic waves cause an infinite flux of radiation, which might heat any infalling observer unlimitedly.

Penrose [9] argued that the CH was unstable against small perturbations. His arguments for the infinite blue shift at the CH relied on a geometric-optics approximation. Penrose argued that perturbations originated in an infinitely long (external) time are concentrated in a finite (proper) time of the infalling object. Thus, unless these perturbations decay at least exponentially fast in external time, the infinite concentration will lead to the blue sheet. Later, other works considered the wave equation for the evolving perturbations. Simpson and Penrose [10] re-affirmed numerically (for linear electromagnetic perturbations) the qualitative arguments of Penrose. Gürel et al. [11] and Chandrasekhar and Hartle [12] calculated the projection of the energy-momentum tensor on the world-line of an infalling observer, and inferred from its divergence on the CH that the radiation absorbed by the observer was also divergent. Nevertheless, the fundamental fields (i.e., the scalar field, the electromagnetic four-potential, and the metric perturbations) were found to be regular. It is the gradients of the fundamental fields which diverge on the CH. Therefore, it was suggested by Ori [13,14] that although there is a true curvature singularity at the CH, this singularity is rather weak. Namely, despite the divergence of the tidal force, the actual tidal distortion experienced by infalling observers (as they hit the singular CH) is finite, and for typical pa-
rameters – even negligible. To cross the CH safely there still remains, though, the other aforementioned potential problem. Namely, the possible complete burning up of any physical body at the CH, due to the divergent electromagnetic field there (and the associated energy flux). This subject was considered recently by Burko and Ori [13]. In this Paper we give a more detailed account of that work.

Physical BHs are expected to spin very fast. However, it turns out that the mathematical analysis involved with the evolution of the multipole moments is very complicated, due to the axial symmetry of the Kerr background. For this reason, it is often common to work with a toy model, within which the mathematical analysis is much simpler. Of course, the toy model should preserve the most essential and relevant properties of the realistic BH. Thus, most work (including the present one) is done in the framework of the Reissner-Nordström (RN) BH. (The RN solution is involved, however, with a certain complication, which arises from the coupling of the gravitational and the electromagnetic fields.) The RN spacetime is the unique electrically charged, spherically symmetric, static vacuum solution of the coupled Einstein-Maxwell equations. For obvious reasons physical BHs are not expected to be significantly charged. The vanishing angular momentum of the RN BH is also an unrealistic feature. Nevertheless, it turns out that the internal causal structure of the RN BH is very similar to the internal structure of the Kerr BH: In both spacetimes the singularities are timelike, and are located beyond a CH; Both spacetimes have a wormhole-like topology, which may allow for a travel to other asymptotically-flat universes. Consequently, it is believed that the RN BH is a physically justifiable model for realistic BHs.

In this paper we analyze the interaction of an infalling object with the divergent electromagnetic field which we expect to find at the CH. Throughout this paper we shall assume that the infalling object is much smaller than the typical radius of curvature near the CH. However, electromagnetic perturbations in the RN geometry are always accompanied by gravitational perturbations. Typically, the latter produce a diverging curvature at the CH (due to the infinite blue shift), which means that the radius of curvature vanishes there. Our assumption is valid only if—for the sake of evaluating the radiative electromagnetic effects—we ignore the gravitational perturbations. The modification of the radiative interaction by the metric perturbations is obviously a non-linear effect, as it is quadratic in the perturbation’s amplitude. This non-linear effect remains the subject of future research.

We do not expect, however, this non-linear effect to significantly alter the linear-order interaction. The gravitational analogue of this problem—the object’s interaction with the divergent tidal forces—demonstrates this reasoning: As implied from the analysis of Ref. [14] on the strength of the CH singularity in spinning BHs—where it has been demonstrated that the non-linear gravitational interaction with an object may be negligible—we do not expect higher-order contributions to change the general picture significantly. Thus, in this work we restrict ourselves to linear effects only, and take the background to be RN (and not a gravitationally-perturbed background). In addition, recent fully-nonlinear numerical simulations have shown that at the asymptotic past of the CH the metric perturbations vanish, in accord with perturbation analyses [16].

The organization of this Paper is as follows: In section II we summarize the definitions and the notation we use. In section III we briefly review the formalism given by Chandrasekhar [17] for the determination of the tetrad components of the electromagnetic field tensor $F_{αβ}$ at the CH for given perturbations. In section IV we obtain the tetrad components of the electric and magnetic fields for initial perturbations which decay according to an inverse-power law. In section V we transform these tetrad components to their tensorial counterparts and write them in the rest-frame of a freely-falling observer. In section VI and section VII we use these fields to calculate their interaction with (very simplified) classical and quantum absorbers, respectively. We then discuss (section VIII) the results, and argue that although the electromagnetic fields diverge at the CH, the interaction of the field does not necessarily cause ultimate destruction of the infalling observer.

It should be noted that we only treat here classical radiation, and ignore quantum processes, especially pair-production effects. When these effects are taken into consideration [18], it should be expected that they may change our results here considerably. Yet, we believe that our main conclusion, namely, that the singular CH is not the edge of spacetime, will still be relevant even after consideration of the quantum effects.

II. DEFINITIONS AND NOTATION

We write the line element of an unperturbed RN BH in the form

$$ds^2 = e^2ν( dx^0)^2 - e^{2μ2}( dx^2)^2 - r^2 dΩ^2,$$  \hspace{1cm} (1)

where the coordinates are $(x^0, x^1, x^2, x^3) = (t, r, θ)$, $dΩ^2$ is the unit two-sphere line element, and the metric coefficients are $e^{2ν} = e^{-2μ2} = (r^2 - 2Mr + Q^2)/r^2 ≡ Δ/r^2$, $M, Q$ being the mass and electric charge, respectively, of the RN BH, and where $r$ is the radial Schwarzschild coordinate, defined such that circles of radius $r$ have circumference $2πr$. The general form of the line element (1) is preserved under polar perturbations (sometimes called even-parity perturbations); On the other hand, axial perturbations (called also odd-parity perturbations), will lead in general to non-vanishing off-diagonal metric. [Axial perturbations are characterized by the non-vanishing
of the metric functions \( \omega, q_2, q_3 \) (the non-vanishing of these metric coefficients induce a dragging of the inertial frame and impart a rotation to the BH), while polar perturbations are those which alter the values of the metric functions \( \nu, \mu_2, \mu_3 \) and \( \psi \) (which are in general non-zero for the unperturbed BH). Therefore, the form of the metric of a generally-perturbed RN BH will be more complicated than the line element (1). It has been shown [17], that a metric of sufficient generality is of the form

\[
\begin{aligned}
\frac{ds^2}{e^{2\nu} (dx^0)^2} - e^{2\psi} (dx^1 - \omega dx^0 - q_2 dx^2 - q_3 dx^3)^2 = e^{2\mu_2} (dx^2)^2 - e^{2\mu_3} (dx^3)^2. \\
(2)
\end{aligned}
\]

Since the unperturbed RN background is spherically symmetric, we may consider only axisymmetric perturbation modes without any loss of generality. (This is because all non-axisymmetric modes can be received from the axisymmetric modes, if the unperturbed spacetime is spherically symmetric [17].) The metric (2) involves seven functions, namely, \( \nu, \psi, \mu_2, \mu_3, \omega, q_2, \) and \( q_3 \). Because the Einstein equations involve only six independent functions, not all of the seven functions can be determined independently, and there is a gauge-fixing freedom on the metric coefficients. However, this gauge freedom involves only the metric coefficients \( \omega, q_2, \) and \( q_3 \); There is no gauge freedom in the determination of the polar perturbations.

The horizons of the RN BH are the event horizon \( r_+ \) and the inner horizon \( r_- \), which are located at the roots of \( \Delta \), namely, at \( r_{\pm} = M \pm (M^2 - Q^2)^{1/2} \). We denote the surface gravity of the event horizon and the CH by \( \kappa_{\pm} \equiv (r_+ - r_-)/r_{\pm}^2 \), respectively. We define the Eddington-Finkelstein null coordinates \( u = r_+ - t \) and \( v = r_+ + t \), where \( r_+ \) is the Regge-Wheeler ‘tortoise’ coordinate defined by \( d/ dr_* = (\Delta/r^2) d/d r \). The coordinate \( t \) is spacelike between the event and the Cauchy horizons, and we take \( t = +\infty \) at the event horizon. In this paper we are interested in the sections of the event horizon represented by \( u = -\infty \) and of the inner horizon represented by \( v = +\infty \). (These are the sections which intersect in the standard Penrose diagram at future timelike infinity of the external universe.) We assume that the object moves along a typical radial world line that intersects the event horizon and the CH at some finite values \( v = v_0 \) and \( u = u_0 \), respectively. Accordingly, the trajectory of the object can be described by the function \( r(\tau) \) and by \( u_0 \), where \( \tau \) is the proper time of the infalling object. We set \( \tau (r = r_-) = 0 \).

III. THE CHANDRASEKHAR FORMALISM

In this section we briefly review the Chandrasekhar formalism for the evolution of the polar perturbations [17], and describe the algorithm which can be constructed from it for the determination of the Maxwell tensor.
marize the algorithm we can construct from the Chandrasekhar formalism (or from its generalization to dipole modes) for the calculation of the perturbed metric functions and the components of the Maxwell field strength tensor, given the initial perturbing fields \( Z_{1}^{(\pm)} \), \((i = 1, 2)\). When we give reference to a specific equation in Chapter 5 of Ref. \[17\], a letter "C" will precede the equation number. We first find the functions \( H_{1}^{(+)} \) by the algebraic equations (C186)-(C187). Now, we calculate the function \( \Phi \) by Eq. (C196). The next step is to obtain the functions defined by Eqs. (C190), and complete the solution with Eqs. (C191)-(C195). The perturbations of the metric (1) will be given then by Eqs. (C158) and the perturbations in the tetrad components of the Maxwell field strength tensor will be given by Eqs. (C159-C160). In the following we shall transform from the tetrad components of the Maxwell tensor to their tensorial counterparts using the tetrad

\[
\begin{align*}
\epsilon(0)_{\mu} &= e^{\nu}(1 \ 0 \ 0 \ 0) \\
\epsilon(1)_{\mu} &= r \sin \theta(0 \ 1 \ 0 \ 0) \\
\epsilon(2)_{\mu} &= e^{-\nu}(0 \ 0 \ 1 \ 0) \\
\epsilon(3)_{\mu} &= r(0 \ 0 \ 0 \ 1).
\end{align*}
\]

IV. DERIVATION OF THE MAXWELL FIELD NEAR THE CH – TETRAD COMPONENTS

In what follows, we consider an isolated charged BH, surrounded by electromagnetic waves, which we treat as a linear perturbation. (In fact, because of the non-vanishing electric field of the background, this linear perturbation consists of both electromagnetic and gravitational waves \[11\,12\].) We calculate the asymptotic behavior of the electromagnetic perturbation near the CH. The class of perturbations that we consider here is the one which is inherent to any non-spherical gravitational-collapse; These are the electromagnetic perturbations which result from the evolution of non-vanishing electromagnetic multipole moments (in the star) during the collapse. When these perturbations propagate outwards, some fraction of them is backscattered off the spacetime curvature and captured by the BH. This process leads to a “tail” of infalling radiation at the event horizon which at late times \((v \gg M)\) decays like \((v/M)^{-(2l+2)}\), where \(l\) is the multipole order of the mode \[8\]. We treat this electromagnetic field according to the formalism by Chandrasekhar [13] for \(l \geq 2\) polar modes and the extension of this formalism by Burko [14] for dipole \((l = 1)\) polar modes. (The extension of the formalism is needed as we \textit{a posteriori} find that the effects we study are dominated by the dipole mode. This mode is not treated properly by Ref. \[17\].)

The Components of \( B_{\mu \nu} \)

Our goal now is to find an approximate expression [to the leading order in \((r - r_{-})\) and \((\kappa - \kappa_{-})^{-1}\)] for the electromagnetic field an infalling observer measures on arrival at the CH. We shall find that some components of the Maxwell field strength tensor diverge there, while other components remain finite. Therefore, we shall restrict ourselves here to the evaluation of the divergent components only. (The other components can be found analogously.) We shall consider here an \(l \geq 2\) polar mode of infalling radiation. It should be stated that the most dominant effect is \textit{not} caused by the \(l \geq 2\) modes but by the \(l = 1\) mode. The reason we choose here to discuss the \(l \geq 2\) modes rather than the \(l = 1\) mode is that the formalism of Ref. \[17\] is inapplicable for the treatment of dipole modes, as mentioned in section III (see Ref. \[19\] for details). It turns out, that when one calculates the dipole perturbations according to the generalized formalism of Ref. \[19\] the results remain qualitatively unchanged. Therefore, we may consider here only the \(l \geq 2\) modes. When we conclude the perturbation analysis we shall give the final result for the dipole perturbations too.

Let us take, then, the electromagnetic and gravitational perturbations near the CH to be \[11\]:

\[
Z_{1}^{(+)} = av^{-(2l+2)} + bu^{-(2l+2)}
\]

\[
Z_{2}^{(+)} = cv^{-(2l+2)} + du^{-(2l+2)}.
\]

We now use the algorithm described in section III to obtain the electromagnetic field near the CH. From \(Z_{1,2}^{(+)}\) we can find the functions \(H_{1,2}^{(+)}\) by Eqs. (C186-C187). The function \(\Phi\) is given by Eq. (C196). Substituting Eq. (C196) in Eqs. (C186-C187), we obtain

\[
\Phi(r, t) = \frac{1}{q_{1}^{2} + |q_{1}|q_{2}} \int_{r}^{\infty} \left\{ \frac{Z_{1}^{(+)} [nr|q_{1}|q_{2}]^{1/2} + \sqrt{2n}Q_{s}q_{1}}{nrq_{1} - Q_{s}\sqrt{2n|q_{1}|q_{2}}} \right\} e^{-\nu} \frac{dr}{\omega r}.
\]

As we are interested in the development of the perturbations near the CH, we write an approximate expression for \(\Phi(r, t)\) to the leading term in \(\kappa - r_{-}\). To do this we expand in \(r - r_{-}\), and obtain

\[
\frac{e^{-\nu}}{\omega r} dr \approx \frac{e^{-\kappa - r_{-}/2}}{nr_{-}^{2} + 3Mr_{-} - 2Q_{s}^{2}} + O(e^{-\frac{1}{2}\kappa - r_{-}}) dr_{s}.
\]

Substituting in \(\Phi\) we get:

\[
\Phi(r_{s}, t) \approx a_{1} \int_{r_{s}}^{\infty} e^{-\frac{1}{2}\kappa - r_{-}} \left[ (a_{2}a_{2} + a_{3}c)u^{-(2l+2)} \right. \\
+ \left. (a_{2}b_{2} + a_{3}d)u^{-(2l+2)} \right] dr_{s},
\]

where
\[ a_1 = (q_1^2 + |q_1q_2|)^{-1} \left( nr^2 + 3Mr - 2Q^2 \right)^{-1} \]
\[ a_2 = nr\sqrt{(-q_1q_2) + \sqrt{2n}}Q_*q_1 \]
\[ a_3 = nrq_1 - \sqrt{2n} \sqrt{(-q_1q_2)Q_*}. \]

It can be verified (for a formal proof see Ref. [22]) that the integral \( \int_x^\infty e^{-gz}z^{-(2l+2)} \, dz \) can be represented by the asymptotic series
\[ \frac{1}{g} e^{-gz} z^{-(2l+2)} \sum_{p=0}^{\infty} (-1)^p \frac{(p + 2l + 1)!}{(2l + 1)! g^p} x^{-p}. \]

Since we are interested primarily in very large values of \( x \), i.e., we are interested in the regions of spacetime closest to the CH, the zeroth order approximation of the asymptotic series suffices for our needs. Therefore, we shall take
\[ \int_x^\infty e^{-gz}z^{-(2l+2)} \, dz \approx \frac{1}{g} e^{-gz} z^{-(2l+2)}. \]

To use this for the evaluation of \( \Phi \) we first change the variables in Eq. (6), and integrate the two terms separately. Thus, we obtain:
\[ \Phi(r_0, t) \approx -\frac{2a_1}{\kappa} e^{-2\kappa - r} \left[ (a_2a + a_3c)v^{-(2l+2)} + (a_2b + a_3d)u^{-(2l+2)} \right]. \]  

In order to calculate \( B_{03} \), we first need to have \( H^1_{1,r} \).
[See Eqs. (C190) and (C195).] We readily find that
\[ H^1_{1,r} = \frac{dr_0}{dr} \frac{d}{dr_0} H^1_{1,r} \]
\[ = -(2l + 2) \frac{e^{-2\nu}}{q_1^2 + |q_1q_2|} \left[ \alpha v^{-(2l+3)} + \beta u^{-(2l+3)} \right], \]
where \( \alpha = q_1a - (-q_1q_2)^{1/2}c \) and \( \beta = q_1b - (-q_1q_2)^{1/2}d \).

We can obtain the following approximate expression to the dominant term in \( e^{-\nu} \) (as \( e^{-\nu} \) diverges on the CH it is clear that the stronger the dependence of the exponent in \( \nu \) the faster the divergence):
\[ B_{03}(r, t) \approx -\frac{Q_4^2}{r^2} H^1_{1,r}(r, t) \]
\[ = \left( \frac{2l + 2}{\Delta} \frac{Q_4^2}{q_1^2 + |q_1q_2|} \right) \left[ \alpha v^{-(2l+3)} + \beta u^{-(2l+3)} \right]. \]  

In Eq. \[ \text{(3)} \] we kept only the leading term in \( e^{-\nu} \). Similarly, we also find the dominant term in \( e^{-\nu} \) of \( B_{23}(r, t) \) near the CH. Namely, in view of Eq. (C190) and Eq. (C194) we get
\[ B_{23}(r, t) \approx -\frac{Q_4^2}{r^2} H^1_{1,r}(r, t) \]
\[ = \left( \frac{Q_4^2}{r^2} \frac{1}{q_1^2 + |q_1q_2|} \right) \left[ \alpha v^{-(2l+2)} + \beta u^{-(2l+2)} \right]. \]

We thus see that both \( B_{03}(r, t) \) and \( B_{23}(r, t) \) are the results of linear differential operators acting on the perturbing fields \( Z_i^{(+)} \). As shown in section III [see Eqs. (C159)-(C160)] the formalism can be now used to obtain frequency-dependent tetrad components of the Maxwell tensor. As the expressions for the functions \( B_{\mu\nu} \) are independent of the frequency, it is clear that there is a need to adapt the two representations of the fields (the temporal representation and the frequency representation).

It turns out that the tetrad components of the Maxwell tensor which lead to divergencies are \( F_{0(3)}(r, \sigma) \) and \( F_{2(3)}(r, \sigma) \).

The Components of \( F_{(\mu)(\nu)}(r, t, \theta) \)

Our goal now is to obtain the frequency-independent expression for the electromagnetic field. As \( F_{(\mu)(\nu)}(r, \sigma, \theta) \) does not involve the frequency [in view of Eq. (C159)], it is possible to compute \( F_{(0)(3)}(r, t, \theta) \) directly, without considering the subtleties of the Fourier transform. It can be shown that for the correct performance of the transformation from the frequency representation to the temporal representation one should only replace the frequency \( \sigma \) of Eq. (C160) with \(-id/\,dt\). (For a rigorous proof of this scheme—which is not as trivial as it may seem—see Ref. [22].) We thus obtain:
\[ F_{(0)(3)}(r, t, \theta) \approx \left( \frac{1 + l}{r} + \frac{1}{q_1^2 + |q_1q_2|} \right) \left[ \alpha v^{-(2l+3)} \right. \]
\[ + \left. \beta u^{-(2l+3)} \right] P_{2,\theta} \]  
\[ F_{(2)(3)}(r, t, \theta) \approx \left( \frac{1 + l}{r} + \frac{1}{q_1^2 + |q_1q_2|} \right) \left[ \alpha v^{-(2l+3)} + \beta u^{-(2l+3)} \right] P_{2,\theta}. \]

Here, \( P_l(\cos \theta) \) denotes the Legendre polynomial of order \( l \).

V. Derivation of the Maxwell Field Near the CH – Tensorial Components

This section is built in the following way: We first write down the tensorial components of the Maxwell tensor in the Schwarzschild coordinates, which are known (see, e.g., Ref. [23]) to be singular at the CH. We then transform from the Schwarzschild coordinates to Kruskal-Szekeres coordinates, which are regular at the CH. Finally, we transform from the Kruskal-Szekeres coordinates to the rest frame of the infalling object. (The motivation behind this last transformation is given below.)

The Schwarzschild Coordinates

After finding the tetrad components of the electromagnetic field we shall now find the tensorial compo-
ments. From now on we omit the explicit angular dependence of the Legendre polynomials, to make the expressions simpler and more compact. We may do this because none of the transformations to be performed below changes the angular coordinates. To allow for this omission we re-define the components of the Maxwell tensor. Schematically, we separate the variables by \( F_{(\mu)(\nu)}(r, t, \theta) = F_{(\mu)(\nu)}(r, t) \Theta_{(\mu)(\nu)}(\theta) \). Hence, from now on \( F_{(\mu)(\nu)}(r, t) \) should be understood accordingly. The tensorial components can be obtained from the tetrads \( F_{\mu\nu} = F_{(\alpha)(\beta)} e^{(\alpha)}_{\mu} e^{(\beta)}_{\nu} \), namely,

\[
F_{03}(r, t) = -r e^{\alpha} F_{(0)(3)}(r, t) \\
\approx \frac{(l+1)\mu}{q_1^2 + |q_1 q_2|} \left[ \alpha v^{-(2l+3)} + \beta u^{-(2l+3)} \right]
\]

\[
F_{23}(r, t) = r e^{-\nu} F_{(2)(3)}(r, t) \\
\approx \frac{e^{-2\nu}(l+1)\mu}{q_1^2 + |q_1 q_2|} \left[ \alpha v^{-(2l+3)} - \beta u^{-(2l+3)} \right].
\]

**Transforming to Regular Coordinates**

**Transforming to Kruskal-Szekeres Coordinates**

Now, we wish to transform the Maxwell strength field tensor to a co-moving reference frame, i.e., a frame in which the infalling observer is at rest. We do that in order that we could relate the results for different observers and because of the fact that to use a (classical or quantum) local theory for the matter-radiation interaction we need to express the electromagnetic field as measured by the physical system in question, and as a function of its local (proper) time. First, we transform from the coordinates \((t, \phi, r, \theta)\) to the coordinates \((t, \phi, r_*, \theta)\). It is clear, that the only component (out of the two relevant components) of \( F_{\mu\nu} \) which is changed by this transformation is \( F_{23} \). In fact, we get that

\[
F_{r\theta} = e^{2\nu} F_{\phi\theta} = \frac{(l+1)\mu}{q_1^2 + |q_1 q_2|} \left[ \alpha v^{-(2l+3)} - \beta u^{-(2l+3)} \right].
\]

Now, we transform to the coordinates \((u, \phi, v, \theta)\). (The coordinates \(u\) and \(v\) are defined in section II.) The line element for the unperturbed RN background in the Eddington-Finkelstein coordinates is \( ds^2 = \frac{\kappa^2}{\kappa^2 - 1} du dv - r^2 (u, v) d\Omega^2 \). We thus find that

\[
F_{u\theta} = F_{r\phi} - F_{r\theta} = -\frac{2(l+1)\mu}{|q_1 q_2|} \beta u^{-(2l+3)}
\]

\[
F_{v\theta} = F_{r\phi} + F_{r\theta} = \frac{2(l+1)\mu}{|q_1 q_2|} \alpha v^{-(2l+3)}.
\]

Now, we define the Kruskal-Szekeres future directed null coordinates \( U, V \) by:

\[
\ln \left( -\frac{1}{2} \kappa_- V \right) = -\frac{1}{2} \kappa_- u,
\]

\[
\ln \left( -\frac{1}{2} \kappa_- U \right) = -\frac{1}{2} \kappa_- v.
\]

To obtain the form of the metric in these coordinates we first need to have an explicit expression for \( r_*(r) \), because in transforming from the the Eddington-Finkelstein coordinates to the Kruskal-Szekeres coordinates we find for the line element

\[
ds^2 = -\frac{(r_+ - r)(r - r_-)}{r^2} \kappa_- r_* (r_* U, V) d\Omega^2.
\]

As we defined \( r_* \) in section II only through its differential, we realize that when we integrate to obtain \( r_*(r) \) we may add an arbitrary integration constant. This integration constant can be chosen in such a way, that the line element near the CH assumes a quasi-Minkowskian form\(^{[1]}\), i.e., \( ds^2 = [-1 + O(UV)] dU dV - r^2 d\Omega^2 \). Thus, we integrate \( dr_* / dr \) and obtain\(^{[1]}\)

\[
r_*(r) = (r - r_-) + \frac{1}{\kappa_+} \ln \frac{r_+ - r}{r_+ - r_-} - \frac{1}{\kappa_-} \ln \frac{r - r_-}{r - r_-} + \frac{1}{\kappa_-} \ln \frac{r_+ - r_- / \kappa_+}{(r_+ - r_-) \kappa_- / \kappa_+}.
\]

Inserting this in the line element, we find

\[
ds^2 = -\left( \frac{r_+ - r}{r} \right)^2 \left( \frac{r_+ - r_-}{r_+ - r_-} \right)^{1 + \kappa_- / \kappa_+} e^{\kappa_- (r - r_-)} dU dV - r^2(U, V) d\Omega^2.
\]

Here, \( r \) is the implicit function of the coordinates \( UV = 4 \exp[-\kappa_- r_*(r(U, V))] / \kappa^2 \). This metric is of course regular in the domain between the two horizons, and at the CH, as should be expected from Ref. \([24, 25]\). Due to the regularity of the metric (17) at the CH, we can work on a sufficiently small neighborhood where spacetime is as close to Minkowski spacetime as we wish (which is, in fact, trivial, as any non-singular spacetime possesses this property). In these coordinates we obtain:

\(^{[1]}\)This is the line element not only exactly on the CH but also very close to it. (Of course, this line element is regular at the CH.) We shall use this quasi-Minkowskian form when we construct interaction models for the radiation with matter (see sections VI and VII).

\(^{[2]}\)Notice the difference between this form of \( r_*(r) \) and the forms of Ref. \([1, 2]\). Also notice that in Ref. \([1]\) the definition for the surface gravity is different from ours.
we demand that at the intersection of the orbit with the infalling object is in motion relative to this reference frame of the object’s rest frame, as generally we should find that like coordinates, yet not appropriate for the description coupled equations

\[
F_{V\theta} = -\frac{2}{\kappa V} F_{v\theta} \tag{18}
\]

\[
F_{U\theta} = -\frac{2}{\kappa U} F_{u\theta} = \frac{2(2l + 2)}{\kappa U} \frac{\mu}{q_1^2 + |q_1 q_2|} \beta u^{-(2l+3)}. \tag{19}
\]

Transformation to the infalling object’s rest-frame

Eqs. (18,19) are not enough for our needs. The reason for this is two-fold: First, we should like to calculate the interaction of the electromagnetic field near the CH with the matter comprising the infalling object. A description in a flat spacetime of course simplifies the intricate radiative processes considerably. A flat-spacetime description also allows us to use the standard notions of electric and magnetic fields. Second, the region where the electromagnetic field assumes exceptionally high values is a very ‘narrow’ region near the CH. In this narrow region, because spacetime is approximately Minkowskian, spacetime curvature is negligible in the background. Additionally, it will turn out (see sections VI and VII) that we shall need to describe the interaction of the infalling object with the radiation field along the entire trajectory, and, in addition, we shall also wish to compare different observers arriving at totally different points on the CH. (The reasons for this will become clear in sections VI and VII.) To do that we cannot be satisfied with a completely local description of the CH, and therefore we transform now to co-moving coordinates, which allow us to compare different observers, as for all we set the proper time equal to zero on arrival to the CH. Let us now define the coordinates \( Z, T \) by: \( U = Z - T \) and \( V = Z + T \). (Note, that \( Z \) is a timelike coordinate and \( T \) is a spacelike coordinate.) The coordinates \( Z, T \) are Minkowski-like coordinates, yet not appropriate for the description of the object’s rest frame, as generally we should find that the infalling object is in motion relative to this reference frame.

We now transform to co-moving Minkowski coordinates \( \bar{z}, \bar{t} \), adapted to the trajectory in question. Namely, we demand that at the intersection of the orbit with \( V = 0 \), the newly defined coordinates be such, that \( \ddot{\bar{z}} = 0 \) (and \( \dot{\bar{z}} = 1 \)). In addition, we set \( \bar{z} = \bar{t} = 0 \) at that intersection point. This transformation is thus defined by the coupled equations

\[
\begin{align*}
(1 0 0 0) & \equiv \dot{x}^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \dot{x}^\beta \\
\eta^{\alpha'\beta'} & \equiv g^{\alpha'\beta'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta'}}{\partial x^{\beta}},
\end{align*}
\]

where \( \eta^{\alpha\beta} \) is the metric tensor of a Minkowski spacetime and a dot denotes differentiation with respect to the proper time of the infalling object. It is clear, that these transformation equations define the special-relativistic Lorentz boost generating the co-moving coordinates uniquely. The solution of these equations is

\[
\begin{pmatrix}
\frac{\partial \bar{z}}{\partial z} & \frac{\partial \bar{t}}{\partial z} \\
\frac{\partial \bar{z}}{\partial \tau} & \frac{\partial \bar{t}}{\partial \tau}
\end{pmatrix} = \begin{pmatrix}
\dot{\bar{z}} & \dot{\bar{t}} \\
\dot{\bar{z}} & \dot{\bar{t}}
\end{pmatrix}.
\]

We now define the null coordinates \( \bar{u} = \bar{z} - \bar{t} \) and \( \bar{v} = \bar{z} + \bar{t} \). (Note that both \( \bar{u} \) and \( \bar{v} \) vanish at the intersection of the trajectory with \( V = 0 \).) Transforming to these coordinates we find

\[
F_{V\theta} = \frac{\partial \bar{v}}{\partial V} F_{v\theta} + \frac{\partial \bar{u}}{\partial V} F_{u\theta},
\]

\[
F_{U\theta} = \frac{\partial \bar{v}}{\partial U} F_{v\theta} + \frac{\partial \bar{u}}{\partial U} F_{u\theta}.
\]

It can be shown that

\[
\frac{\partial \bar{u}}{\partial V} = 0, \quad \frac{\partial \bar{u}}{\partial U} = \dot{V}, \quad \frac{\partial \bar{v}}{\partial V} = \dot{U}, \quad \frac{\partial \bar{v}}{\partial U} = 0.
\]

Hence, we get that

\[
F_{V\theta} = \frac{1}{U} F_{V\theta}, \quad F_{u\theta} = \frac{1}{V} F_{U\theta}.
\]

It can be readily shown [22] that on the CH \( \dot{U}(r_-) = 1/(2\kappa) e^{-\kappa u_0/\kappa} \) and \( \dot{V}(r_-) = -2\kappa e^{\kappa u_0/\kappa} \). Using this, it can be shown that on the CH \( \bar{v} = \dot{U} \bar{V} \) and \( \bar{u} = \dot{V} U - 4\kappa \kappa u_0/\kappa \). After performing all the transformations, we end up with the required expression for the divergent component of the Maxwell tensor as measured by an infalling observer in his rest frame. We substitute

\[
v = -\frac{2}{\kappa} \ln \left( -\frac{1}{2} \kappa - \frac{\kappa}{\bar{U}} \right)
\]

and

\[
u = -\frac{2}{\kappa} \ln \left[ -\frac{1}{2} \kappa - \frac{1}{\bar{V}} \left( \bar{u} + 4 \kappa \kappa u_0/\kappa \right) \right],
\]

and obtain

\[
F_{V\theta} = \frac{4(l + 1)}{\kappa - U V} \frac{\mu}{q_1^2 + |q_1 q_2|} \alpha u^{-(2l+3)} = C'_{r_-} \left( \frac{\ln |\kappa - \bar{v}| + \frac{1}{2} \kappa - u_0 + \ln |\bar{r}|}{\kappa - \bar{v}} \right)^{-(2l+3)}, \tag{20}
\]

\[
F_{u\theta} = \frac{4(l + 1)}{\kappa - U V} \frac{\mu}{q_1^2 + |q_1 q_2|} \beta u^{-(2l+3)} = -C'_{r_-} \left( \frac{\ln |\kappa - \bar{u}| + 4 \kappa \kappa u_0/\kappa}{\kappa - \bar{u}} \right)^{-(2l+3)} \times \left( \frac{\ln |\kappa - \bar{u}| + 4 \kappa \kappa u_0/\kappa - \ln |4\kappa|}{\kappa - \bar{u}} \right)^{-(2l+3)}, \tag{21}
\]
where

\[
C' = \frac{4(l+1)\mu}{q_{\tau}^2 + |q_1 q_2|} \frac{(\kappa_-/2)^{2l+3}}{r} \frac{1}{r},
\]
\[
C'' = \frac{4(l+1)\mu}{q_{\tau}^2 + |q_1 q_2|} \beta \frac{(\kappa_-/2)^{2l+3}}{r} \frac{1}{r}.
\]

The form in which we represented \( F_{\theta\theta} \) might obscure the simplicity of its meaning. In fact, all we need is the value on the inner horizon, where \( \bar{u} = 0 \). We readily find that on the CH

\[
F_{\theta\theta} = \frac{l+1}{r} \frac{r-\mu}{q_{\tau}^2 + |q_1 q_2|} \beta u_0^{(2l+3)}.
\]

From this expression the regularity of \( F_{\theta\theta} \) on the CH is self evident.

The final form for the electromagnetic field near the CH that an infalling observer measures

We may construe each point of the matter comprising the infalling observer as being located in the center of its own spatial coordinate system, i.e., we may take \( \bar{z} = 0 \) or, equivalently, \( \bar{t} = \bar{u} = \bar{t} \). Identifying the coordinate \( \bar{t} \) with the observer’s proper time \( \tau \), we realize that \( \bar{v}, \bar{u} \) vanish on the CH. We get that \( F_{\theta\theta} \) remains finite on the CH, while \( F_{\bar{\theta}\bar{\theta}} \) diverges.

We can write the required divergent components of the electric and magnetic fields near the CH as measured by the infalling observer in an orthonormal Cartesian tetrad frame in which the observer is at rest. As the tetrad introduced in section III is orthonormal, it is only natural to take here the same tetrad base. Thus, we get:

\[
F_{(\bar{z})(\bar{\gamma})} = F_{\bar{\theta}\bar{\theta}} \epsilon^{\bar{\gamma}}_{(\bar{z})} \epsilon^\theta_{(\bar{\gamma})} = F_{\bar{\theta}\bar{\theta}} \epsilon^\theta_{(\bar{\gamma})} \mu \gamma^0 = \frac{1}{r} F_{\bar{\theta}\bar{\theta}},
\]
\[
F_{(\bar{t})(\bar{\gamma})} = F_{\bar{\theta}\bar{\theta}} \epsilon^{\bar{\gamma}}_{(\bar{t})} \epsilon^\theta_{(\bar{\gamma})} = F_{\bar{\theta}\bar{\theta}} \epsilon^\theta_{(\bar{\gamma})} \gamma^0 = \frac{1}{r} F_{\bar{\theta}\bar{\theta}}.
\]

As the tetrad base is Cartesian and orthonormal, it is easily shown that both \( \epsilon^\theta_{(\bar{t})} \) and \( \epsilon^\theta_{(\bar{z})} \) identically equal unity. We now denote \( E_{\bar{\theta}} \equiv F_{\bar{\theta}\bar{\theta}} \) and \( B_{\bar{\theta}} \equiv -F_{\bar{\theta}\bar{\theta}} \), where \( F_{\bar{\theta}\bar{\theta}} \approx F_{\bar{\theta}\bar{\theta}} \approx F_{\bar{\theta}\bar{\theta}} \).

\[
E_\bar{\gamma} = -B_\bar{\gamma} \approx \frac{C}{\kappa_-} \ln |\kappa_- \tau| + \frac{1}{\kappa_-} u_0 + \ln |\bar{r}|^{-2(2l+3)}. \]

From these expressions for the divergent components of the electromagnetic field we may neglect the term dependent on \( \bar{r} \), as for typical observers \( \bar{r} \) is neither vanishing nor divergent, and is typically of order unity. (On the other hand, \( \ln |\kappa_- \tau| \) diverges, and \( u_0 \) is typically very large too.) From now on, for the sake of brevity we shall call these electromagnetic components simply \( E \) and \( B \), respectively. Thus, we conclude that

\[
E = -B \approx \frac{C'}{\kappa_- \tau} \left( \ln |\kappa_- \tau| + \frac{1}{2} \kappa_- u_0 \right)^{-(2l+3)}.
\]

It is clear from Eq. (22) that the electromagnetic field— as measured by the infalling observer—diverges. This conclusion agrees with the results of Refs. [9,11,12,26,27].

The above analysis was done for \( l \geq 2 \) polar modes. The analysis can be repeated for \( l = 1 \) polar modes analogously using the formalism of Ref. [19]. When this is done, it turns out that the electric and magnetic fields can still be expressed by Eq. (22). However, the correct expression for \( C' \) is now \( C'(l=1) = -2a (\kappa_-/2)^5 / (q_1 r_-) \).

VI. INTERACTION OF THE RADIATION WITH MATTER: CLASSICAL ABSORBER

We now consider the interaction of the electromagnetic field (23) with the matter comprising the infalling object. Here, we consider a classical object, and in the next section we consider a quantum object. In both the present and the consecutive sections we assume that the object is much smaller than the typical radius of curvature between the event and the inner horizons, and hence the effects of curvature are negligible. This arises from our assumption that non-linear effects will not have important effects on the radiative interaction of the field with the infalling object. (See the discussion in section I.) Consequently, we can imagine the object as being at rest in its locally co-moving Minkowski frame when an electromagnetic impulse of the shape (23) comes from null infinity and interacts with it. In what follows we shall use the co-moving Cartesian coordinates defined in section V, but for simplicity we shall omit here the ‘bars.’

Namely, we shall use the coordinates \( \tau \ x \ y \ z \).

Despite the flatness of spacetime in the observer’s rest-frame, the interaction of the electromagnetic field with the matter from which the observer is made is extremely complicated. Therefore, we take a very simplified (toy-) model for the matter-field interaction, which still embodies the most essential properties of more realistic interactions. We take the matter to be composed of classical “atoms.” Each “atom” is composed of two point-like oppositely charged particles, with charges \( \pm e \) and masses \( \mu_\pm \), correspondingly. We denote the reduced mass by \( \mu \). (Do not confuse \( \mu \) here, which is the reduced mass, with \( \mu \) in the previous two sections.) The system interacts with an external force, which in our case is the Lorentz
force induced by the blue-shifted incoming electromagnetic field. This external force acts to change the separation distance between the two particles of the “atom.” We presume small deviations from equilibrium (to be justified \textit{a posteriori}), and hence take a linear restoring force, i.e., \( F = -\mu \omega^2 X \), where \( X \) is the deviation (of the particles’ separation) from equilibrium, and \( \omega \) is the resonance frequency. The dipole is chosen to be aligned in the \( \partial/\partial \theta \) direction (to allow for a maximum interaction with the field). We take the initial conditions to be \( X = 0 \) and \( \dot{X} = 0 \). The excitation of the system by the field may be characterized by \( X, \dot{X} \), and also by the total absorbed mechanical energy \( \mathcal{E}_c \). We shall show that all three variables are finite, small, and for typical parameters even negligible.

The equation of motion is
\[
\mu \ddot{X} + \mu \omega^2 X = e E(\tau), \tag{23}
\]
where \( E(\tau) \) is the divergent component of the electric field \([22]\). (The contribution of the magnetic field is neglected, as the ratio of the electric term to the magnetic term in the expression for the Lorentz force is proportional to the system’s internal velocity \( \dot{X} \), which is taken to be small—a presumption which is justified \textit{a posteriori}.)

The solution of this equation (with our initial conditions) is
\[
X(\tau) = -\frac{1}{2i\omega \mu} e^{-i\omega \tau} \int_{-T}^{\tau} E(\tau') e^{i\omega \tau'} d\tau' + \text{c.c.} \tag{24}
\]
\[
\dot{X}(\tau) = \frac{1}{2} \frac{e}{\mu} e^{-i\omega \tau} \int_{-T}^{\tau} E(\tau') e^{i\omega \tau'} d\tau' + \text{c.c.,} \tag{25}
\]
where \( T \) is the time of infall from the event horizon to the CH. Calculating the sum of the kinetic and potential energies, we find for the total absorbed mechanical energy
\[
\mathcal{E}_c(\tau) = \frac{1}{2} \mu \omega^2 X^2 + \frac{1}{2} \mu \dot{X}^2
= \frac{1}{2} \mu \left( \frac{e}{\mu} \right)^2 \left| \int_{-T}^{\tau} E(\tau') e^{i\omega \tau'} d\tau' \right|^2. \tag{26}
\]
In all of the three expressions \([24],[25]\), and \([26]\) we need to calculate the integral
\[
I = \int_{-T}^{\tau} E(\tau') e^{i\omega \tau'} d\tau'. \tag{27}
\]
The evaluation of this integral for typical parameters is done explicitly in Appendix A. The evaluation yields \([\text{Eq. (A27)}]\), to the leading orders in \((\kappa- u_0)^{-1}\)
\[
I(\tau = 0) \approx -\frac{C'}{(2l + 2)\kappa} \left( \frac{1}{2} \kappa - u_0 \right)^{-2(l+2)}
+ \frac{C'}{\kappa} \frac{1}{\pi i} \left( \frac{1}{2} \kappa - u_0 \right)^{-(2l+3)}, \tag{28}
\]
Substituting in the explicit expressions for \( X, \dot{X} \), and \( \mathcal{E}_c \) we obtain, to the leading order in \((\kappa- u_0)^{-1}\)
\[
\mathcal{E}_c(0) \approx \frac{1}{2(2l+2)\kappa}\left( \frac{e}{\mu} \right)^2 \left( \frac{1}{2} \kappa - u_0 \right)^{-2(l+2)} \tag{29}
\]
\[
\dot{X}(0) \approx \frac{1}{2l + 2} \frac{C'}{\kappa} \left( \frac{e}{\mu} \right) \left( \frac{1}{2} \kappa - u_0 \right)^{-2(l+2)} \tag{30}
\]
\[
X(0) \approx \frac{1}{2} \frac{C'}{\kappa \omega} \left( \frac{e}{\mu} \right) \left( \frac{1}{2} \kappa - u_0 \right)^{-(2l+3)} \tag{31}
\]
We find that the strength of the excitation depends on \( u_0 \). However, \( u_0 \) cannot be directly evaluated by an outside observer, who wishes to predict the excitation strength should he jump into the BH. Thus, it would be worthwhile to express the excitation strength in terms of the \textit{external} parameters of the problem. This can be done once we realize that \( du_0/ dv_0 = -1 \). (The proof is given in Ref. \([22]\).) We find that the infalling observer can increase \( |u_0| \) by simply waiting outside the BH before jumping in and thus increasing the value of \( v_0 \). For a sufficiently large \( |u_0| \) we get that \( \mathcal{E}_c(0), X(0) \) and \( \dot{X}(0) \) are finite and arbitrarily small.

We conclude that despite the divergence of the electromagnetic field, the excitation of the classical system (and the energy absorbed) is finite, and becomes arbitrarily small, for late-time observers (large \( v_0 \)). Therefore, the behavior of the charged classical system obtained by the above analysis is regular, and despite the divergence on the CH of the external force acting on the system, the energy absorbed by it is finite and negligible for a sufficiently large \( v_0 \).

VII. INTERACTION OF THE RADIATION WITH MATTER: QUANTUM ABSORBER

As a quantum analogue to the preceding classical system we take (again) a very simplified model, which, we believe, captures the essential properties for the interaction we study and neglects all irrelevant details. Let us deal then with a non-degenerate quantum system obeying the Schrödinger equation. We shall evaluate the excitation of the system by considering the transition of the system from its ground state to an excited state. When there are many (or even an infinite number of) excited states, our considerations can be generalized for the analysis of the excitation. In the following we shall discuss, then, only the excitation between two quantum states. The unperturbed eigenstates of the system are the ground state \( |\psi_0\rangle e^{-i\omega_0 t} \) and the excited state \( |\psi_f\rangle e^{-i\omega_f t} \). (That is, the eigenstates of the quantum system when there is no electromagnetic field due to the blue sheet. This could be, for instance, the eigenstates of the system in its original orbit around the BH before the jump in, or its eigenstates when the system crosses the event...
horizon.) Namely, the obvious evolution in time is given by the standard oscillatory dependence, and then $|\psi_n\rangle$ is independent of the time, where $n = i, f$. The perturbed wave function would be given then by

$$|\psi\rangle = \sum_n a_n |\psi_n\rangle e^{-i\omega_n t},$$

where $a_n$ are the expansion coefficients. The system is taken initially at its ground state $|\psi_i\rangle$. Therefore, we take $a_{i,\text{initial}}^{\text{initial}} = 1$ and $a_{f,\text{initial}}^{\text{initial}} = 0$. Assuming small transitions (an assumption which will be justified a posteriori), we take $a_{f,\text{final}}^{\text{final}} \approx 1$ too. The system is perturbed by the pulse of the electromagnetic field. In the Coulomb gauge, which is consistent with our treatment, the interaction Hamiltonian is $\mathcal{H} = (e/m)\mathbf{A} \cdot \mathbf{p}$, where we keep only linear terms in the perturbation, which is assumed to be small. Here, $\mathbf{A}$ is the electromagnetic vector-potential written in the Coulomb gauge, and $\mathbf{p}$ is the 3-momentum. (The explicit form of $\mathbf{A}$ is not important for our needs here, although it can be found from $\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla \phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$, where $\phi$ is the temporal component of the four-potential.) It can be easily verified that any vector potential of the form $\mathbf{A} = \mathbf{A}(\tilde{t})\mathbf{e}_y$, where $\mathbf{e}_y$ is a unit vector in the $y$ direction, is consistent with these expressions and with Eq. (2). It can be further shown that this form for the vector potential is consistent also with the Coulomb gauge condition, namely, with $\nabla \cdot \mathbf{A} = 0$. In what follows, we take the temporal component of the four-potential to vanish. This can be done consistently with the Coulomb gauge. The vector potential can thus be written explicitly as $\mathbf{A} = -\int \mathbf{E}(\tau') d\tau'$.

We are interested in the effects of the blue sheet, namely, with the effects of the pulse of the electromagnetic field on our system. Therefore, we shall assume, for the sake of brevity and simplicity, that the electromagnetic pulse ends at the CH, or, in other words, that for each point of the system the electric and the magnetic fields vanish for positive proper time. More accurately, if the system’s spatial extension in the $z$ direction is $\delta z$ from the center of the system, we shall examine the excitation of the system at proper time (of the system’s central point) $\tau > \delta z > 0$. More conveniently, we shall look at the system’s state at $\tilde{t} > 0$, when there is no electromagnetic field associated with the blue sheet. [Of course, as physics beyond the CH is as yet unknown, we do not suggest here that there are no perturbing fields on the other side of the CH. Our point here, is that for the sake of the calculation of the blue-sheet effects, the specific form of the fields beyond the CH is irrelevant. Moreover, our choice here (to set the electromagnetic field equal to zero for positive $\tilde{t}$) is no worse than any other choice (in view of the present knowledge of the physics beyond the CH).] We shall calculate the energy absorbed by the system, as a measure for the strength of its excitation due to the divergent electromagnetic field. By first order time-dependent perturbation theory $a_f^{\text{final}}$ is given by

$$a_f^{\text{final}} = -\frac{i}{\hbar} \frac{e}{m} \int_{-\infty}^{\tau} \langle \psi_f | \mathbf{A} \cdot \mathbf{p} | \psi_i \rangle e^{i\omega_{f,i} \tau'} d\tau',$$

where $\Omega_{f,i} = \omega_f - \omega_i$. Integrating by parts, we find that

$$a_f^{\text{final}} = \frac{e}{m \hbar \Omega_{f,i}} \int_{-\infty}^{\tau} \langle \psi_f | \mathbf{A} \cdot \mathbf{p} | \psi_i \rangle (\partial \mathbf{A}/\partial \mathbf{r}) \cdot \mathbf{v} e^{i\omega_{f,i} \tau'} d\tau'
+ \frac{e}{m \hbar \Omega_{f,i}} \int_{-\infty}^{\tau} \langle \psi_f | \mathbf{E} \cdot \mathbf{v} | \psi_i \rangle e^{i\omega_{f,i} \tau'} d\tau'
- \frac{e}{m \hbar \Omega_{f,i}} \int_{-\infty}^{\tau} \langle \psi_f | \mathbf{E} \cdot \mathbf{v} | \psi_i \rangle e^{i\omega_{f,i} \tau'} d\tau'
\equiv a_{f,1}^{(1)} + a_{f,2}^{(2)}.$$

The first term does not contribute to the physical excitation. The problem with $a_{f,1}^{(1)}$ is that apparently one could (incorrectly) infer that even after the perturbation vanishes, $a_{f,1}^{\text{final}}$ continues to evolve. To be more specific, it looks as though the system is being perturbed (and excited) even when there is no perturbation at all. This is, of course, an erroneous result. We shall resolve this “paradox” in Appendix B, and conclude that $a_{f,1}^{\text{final}}$ describes a pure gauge distortion of the wave-function, which can be set equal to zero by a proper gauge transformation. That is, when we adjust the gauge such that $\mathbf{A}$ vanishes for $\tau > 0$, the above problematic term simply disappears. Therefore, we are thus left only with the gauge-independent energy absorption $\mathcal{E}_q(\tau)$ which is given by

$$\mathcal{E}_q(\tau) = \left| a_{f,2}^{(2)} \right|^2 \hbar \Omega_{f,i},$$

$$= \frac{1}{\hbar \Omega_{f,i}} \frac{e^2}{m^2} \left| \int_{-\infty}^{\tau} \langle \psi_f | \mathbf{E}(\tau', \mathbf{r}) \cdot \mathbf{v} | \psi_i \rangle e^{i\omega_{f,i} \tau'} d\tau' \right|^2$$

$$= \frac{1}{\hbar \Omega_{f,i}} \frac{e^2}{m^2} \left| \left( \int_{-\infty}^{\tau} e^{i\omega_{f,i} \tau'} \mathbf{E}(\tau) d\tau \right) \cdot \mathbf{p} | \psi_i \rangle \right|^2.$$  

The physical energy-absorption we are interested in is the energy absorption just after the system has fully crossed the CH, in the meaning described previously. As we are interested here in the effects of the blue sheet, let us suppose then that after the CH the electric field vanishes, and therefore the quantum system does not undergo any further excitation. The integrand in Eq. (33) vanishes for $\tilde{t} > 0$, and therefore we may change the variables in the integration, and get

$$\mathcal{E}_q(\tilde{v} = 0) = \frac{1}{\hbar \Omega_{f,i}} \frac{e^2}{m^2} \left| \langle \psi_f | e^{-i\Omega_{f,i} \tilde{v}} \right| \left( \int_{-\infty}^{0} e^{i\Omega_{f,i} \tau} \mathbf{E}(|\tilde{v}| d\tilde{v}) \right) \mathbf{p} | \psi_i \rangle \right|^2$$

$$= \frac{1}{\hbar \Omega_{f,i}} \frac{e^2}{m^2} \left| \langle \psi_f | e^{-i\Omega_{f,i} \tilde{v}} \mathbf{p} | \psi_i \rangle \right|^2.$$
The integral in Eq. (34) is the same integral as in the expression for the total mechanical energy of the classical oscillator at the CH, namely, the integral in Eq. (A1). Therefore, using Eq. (A27) we get that to the leading order in \((\kappa - u_0)^{-1}\)

\[
\mathcal{E}_q(0) = \frac{1}{(2l + 2)^2} \frac{1}{\hbar \Omega_f} \left( \frac{e}{m} \right)^2 \frac{C^2}{\kappa} C^2 \left| \langle \psi_f | e^{-i\Omega_f \hat{\mathbf{p}}_g} | \psi_i \rangle \right|^2 \times \left( \frac{1}{2\kappa - u_0} \right)^{-2(2l+2)}.
\] (37)

The matrix element in Eq. (37) is regular and does not diverge on the CH any more than anywhere else. We also see the close correspondence between the classical and the quantum systems. As the matrix element in Eq. (37) has dimensions of momentum squared, we see that the energy gap between the states times the probability amplitude for the quantum system indeed has the dimensions of mass. In the classical system the absorbed energy is equal to the product of half the reduced mass of the system and the square of the internal velocity of the system. We see, that the two expressions for the absorbed energy in the two systems indeed correspond.

The advantage of the classical treatment is that it does not involve perturbation theory. It suffers, though, from the fact that it is a classical model, while actual physical matter is intrinsically quantum. Thus, the two models contribute to the understanding of each other, and augment our understanding of the interaction of the blue sheet inside a RN BH with infalling objects.

\section{VIII. DISCUSSION}

In this Paper we investigated the following question: Are physical objects necessarily burnt up by the blue sheet inside a BH? This is a key question for a more complete understanding of the internal structure of BHs and for an understanding of the possibility to fall into a BH and re-emerge in another universe. To address this question, we analyzed the interaction of the blue sheet with an infalling physical object. We first derived an explicit expression for the electromagnetic field as measured in the rest frame of the infalling observer, and then we modeled the interaction of the blue sheet with the observer in two ways: a classical model and a quantum model. In both models we calculated a measure for the strength of the interaction of the blue sheet with the infalling object.

We have shown, that the divergence of the energy density of the radiation (or, even, the divergence of the integral of this energy density over proper time) at the CH does not mean that any physical object falling into a BH will necessarily be completely burnt up by the radiation.

Even though we toy-modeled the matter comprising the infalling object in a very simplified way, we believe that our models capture the essence of the interaction of radiation with matter. Therefore, we conclude that the classical Maxwell radiation created during the collapse of the star will not necessarily destroy objects at the CH. We find, that the interaction of the blue sheet with physical objects is finite. Moreover, for typical parameters of astrophysical supermassive BHs this interaction is even arbitrarily small. In fact, one can diminish the extent of the interaction by simply waiting outside the BH before jumping into it. This means that if a spaceship is in orbit around the BH, and an astronaut wishes to fall into it (hoping to re-emerge in another universe), he should just wait in the spaceship, and postpone the beginning of his unusual odyssey. According to our analysis, the longer he waits, the smaller the interaction, and the safer the voyage.

We should remember though, that throughout this Paper we have ignored other inherent radiation sources, such as the cosmic background radiation. In addition, a more realistic treatment of the interaction of the blue sheet with infalling objects will have to consider QED effects, which we have ignored here completely. Such QED effects, and in particular effects caused by pair-production, are expected to change the general picture portrayed by our analysis considerably. These quantum effects might be crucial for a more complete understanding of the blue sheet and its interaction with infalling objects. We showed elsewhere, that these QED effects could be fatal for a human-being observer (due to his high vulnerability to \(\gamma\) rays), but typical physical objects of similar or smaller size might survive it. These results do not provide support to the idea that no continuation of the geometry beyond the CH is physically reasonable.

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\section*{APPENDIX A: EVALUATION OF THE INTEGRAL}

In order to evaluate the integral

\[
I = \int_{-\infty}^{\tau} e^{i\omega \tau'} E(\tau') \, d\tau'
\] (A1)

let us divide the integration region into three qualitatively different regions, denoted by \(a, b\) and \(c\), respectively. In region \(a\) we assume that \(-\infty < \tau < -\omega^{-1}\). Since the variation of \(E\) with \(\tau\) is dominated by \(1/\tau\), in
region a the electric field changes very slowly compared with the exponential in the integrand. Hence, we shall approximate the evaluation of the integral by taking the electric field outside the integration. We thus obtain that

\[ I_a \approx e^{i\omega \tau} E(\tau)/(i\omega). \]

We now evaluate the error associated with this approximation: Integrating by parts (in region a), we obtain

\[ I_a = \frac{1}{i\omega} \left[ e^{i\omega \tau} E(\tau) - \int_{-\infty}^{\tau} e^{i\omega \tau'} dE/d\tau' \, d\tau' \right], \tag{A2} \]

or

\[ I_a \approx \frac{1}{i\omega} \left[ e^{i\omega \tau} E(\tau) - \frac{1}{i\omega} \frac{dE}{d\tau} \int_{-\infty}^{\tau} e^{i\omega \tau'} \, d\tau' \right]. \tag{A3} \]

Hence, the error in the evaluation of \( I_a \) is:

\[ |\Delta I_a| \approx \left| \frac{1}{i\omega} e^{i\omega \tau} \left[ E(\tau) - \frac{1}{i\omega} \frac{dE}{d\tau} - E(\tau) \right] \right| = \frac{1}{\omega^2} \left| \frac{dE}{d\tau} \right|. \tag{A4} \]

As we can take (for region a) \( E(\tau) \approx C'/(\kappa_+ \tau) \), we get that

\[ |\Delta I_a| \approx \frac{C'}{\kappa_+ (\omega \tau)^2}. \tag{A5} \]

The relative error is, thus,

\[ \frac{|\Delta I_a|}{I_a} \approx \left| \frac{\frac{1}{(\omega \tau)^2}}{\frac{1}{\tau}} \right| = \left| \frac{1}{\omega \tau} \right|. \tag{A6} \]

We choose now for the proper time \( \tau \) at the boundary between regions a and b the following value:

\[ \tau_a \approx (\kappa_- \omega)^{-1/2}, \tag{A7} \]

and therefore we obtain

\[ \left| \frac{\Delta I_a}{I_a} \right| \approx \sqrt{\frac{\kappa_-}{\omega}} \ll 1. \tag{A8} \]

[For typical parameters (see below) \( \kappa_-/\omega \) is of order \( 10^{-29} \).] Hence, we find that the approximation we take for region a is valid.

Region c is defined by \( -\omega^{-1} \ll \tau < 0 \), i.e., by the requirement that the electric field varies very fast compared with the exponential term, and therefore we can take the latter outside the integration. Before we perform this explicitly let us deal with region b, where most of the problems lie.

Region b is in between regions a and c. The main difficulty in the evaluation of the contribution of region b to the integral (A1) comes from the neighborhood of \( \tau \approx -\omega^{-1} \). In that region neither of the two approximations (the one for region a and the one for region c) is valid. It turns out, however, that in that region there is a different approximation we can use: the logarithm-dependent term in the electric field \( (22) \) does not change much in comparison with the \( 1/\tau \) term in region b. It turns out, that if we assume that \( \omega^{-1} \ll M \ll -u_0 \) (which is a very plausible assumption for physical BHs), region b overlaps with both regions a and c. Thus, using the three different approximations (for regions a, b, and c), we can calculate the integral (A1) for the entire interval \(-\infty < \tau < 0 \). In fact, the matching between regions a and b is done automatically because of the combination of two facts: first, in region a the integral follows the electric field adiabatically, and does not have a ‘memory’ of its values in former times; second, region b overlaps with region a. Therefore, the integral (A1) assumes the form

\[ I = \frac{E_0}{\kappa_-} \int_{-\infty}^{\tau} \frac{1}{\tau} e^{i\omega \tau'} \, d\tau' \equiv \frac{E_0}{\kappa_-} I_b, \tag{A9} \]

where \( E_0 = C' \left( \ln |\kappa_- \omega^{-1}| + \frac{1}{2} \kappa_- u_0 \right)^{-(2l+3)} \). We justify the protraction of the lower limit of the integration interval from \(-T \) to \(-\infty \) by the vanishing value of this expression in the limit of \( \tau \to -\infty \). Region b extends up to \( \tau = \tau_b \). We take \( \tau_b \) such that \( \vartheta = \omega/\tau_b \), where \( \vartheta \ll 1 \) is a dimensionless constant, to be evaluated. (In Ref. [22] we calculate an optimal value of the boundary between regions b and c, i.e., we calculate \( \vartheta \). However, this optimal value for \( \vartheta \) is of no crucial importance. Yet, it turns out that this optimal value is \( \vartheta \approx -(2l + 3)(\kappa_- u_0/2)^{-1} \ln |(2l + 3)(\kappa_- u_0/2)^{-1}| \). This means that both approximations a and b are valid near \( \tau = \tau_b \). We write the integral in Eq. (A9) as

\[ \int_{-\infty}^{\tau_b} \frac{1}{\tau} e^{i\omega \tau} \, d\tau = \int_{-\infty}^{\vartheta} \frac{1}{\omega \tau} e^{i\omega \tau} \, d(\omega \tau). \tag{A10} \]

[In the approximation for region b we suppose that \( \omega^{-1} \ll M \ll -u_0 \): we take the BH mass to be \( 10^5 M_\odot \), where \( M_\odot \) denotes the solar mass. (Even though a typical mass for a supermassive galactic BH may be “only” about \( 10^7-10^8 M_\odot \); this does not affect our analysis.)] This mass is equivalent to a time period of \( 5 \cdot 10^3 \) seconds (as \( 1 M_\odot \) is equivalent to \( 5 \mu \text{sec} \)). We also take the time of the jump into the BH to be of the order of magnitude of a typical cosmological time scale, e.g., we take \( u_0 = -10^9 \) years. This means that \( \kappa_- u_0 \) is of the order of \( 6 \cdot 10^{12} \). If we take \( \omega \) to be of the order of \( 10^{16} \text{sec}^{-1} \), which is a typical angular frequency for atomic processes, we get that \( \ln |(\kappa_- \omega^{-1})| \approx -40 \). The typical infall time for a BH \( T \approx M \), and therefore \( \ln |(\kappa_- T)| \approx 1 \). This means that the variation in the logarithmic dependent term throughout the protracted region b is negligible in comparison with the magnitude of \( \kappa_- u_0 \), which justifies the approximation we made for region b.] Hence, we get that in region b the integral becomes:

\[ I_b = \int_{-\infty}^{\tau_b} \cos \frac{\omega \tau}{\omega \tau} \, d(\omega \tau) + i \int_{-\infty}^{\tau_b} \sin \frac{\omega \tau}{\omega \tau} \, d(\omega \tau), \tag{A11} \]
or, more conveniently,

$$I_b \equiv (I_1 + iI_2).$$  \hfill (A12)

We now calculate each term separately:

$$I_2 = \int_0^\infty \frac{\sin y'}{y'} \, dy' + \int_0^\nu \frac{\sin y'}{y'} \, dy' = \frac{1}{2}\pi + \text{Si}(y),$$  \hfill (A13)

where \(\text{Si}(\tau)\) is the sine integral defined by \(\text{Si}(z) = \int_0^z \frac{\sin t}{t} \, dt\) and \(y = \omega \tau\). As in the overlap region between regions \(b\) and \(c\) we have \(-\omega \tau \ll 1\), the sine integral is much smaller than unity, and we thus obtain that

$$I_2 \approx \frac{1}{2}\pi.$$  \hfill (A14)

Turn now to \(I_1\):

$$I_1 = \int_{-\infty}^\nu \frac{\cos y'}{y'} \, dy' = \int_{-\infty}^{-1} \frac{\cos y'}{y'} \, dy' + \int_{-1}^y \frac{\cos y'}{y'} \, dy',$$

or,

$$I_1 \equiv I_{11} + I_{12}.$$  \hfill (A15)

It is straightforward to show that the integral \(I_{11}\) is bounded and of order unity. An explicit calculation yields\footnote{In fact, \(I_{11} = \text{Ci}(1)\), where \(\text{Ci}(z)\) is the cosine integral defined by \(\text{Ci}(z) = \gamma + \ln z + \int_0^z (-1 + \cos t)/t \, dt\), where \(\gamma\) is Euler’s constant.} \(I_{11} = \int_{-\infty}^{-1} \frac{\cos y'}{y'} \, dy' \approx 0.3374\). \hfill (A16)

We now write the integral \(I_{12}\) as:

$$I_{12} = \int_{-1}^y \frac{-1 + \cos y'}{y'} \, dy' + \int_{-1}^y \frac{1}{y} \, dy',$$

or,

$$I_{12} = \int_{-1}^y \frac{-1 + \cos y'}{y'} \, dy' + \ln |y|.$$  \hfill (A17)

The integral on the right-hand side of (A17) can be written as:

$$\int_{-1}^y \frac{-1 + \cos y'}{y'} \, dy' = A + O(y^2),$$  \hfill (A18)

where \(A = \int_1^0 (-1 + \cos y'/y') \, dy'\). Again, it is straightforward to show that the integral \(A\) is bounded and of order unity, and numerical calculation yields\footnote{In fact, \(A = \sum_{p=1}^{\infty} (-1)^{p+1} \frac{1}{2p(2p)}\).} \(A \approx 0.2398\). Therefore, we get that

$$I_b \approx K + \ln |y| + O(y^2) + i \left[ \frac{\pi}{2} + O(y) \right],$$  \hfill (A19)

where \(K \equiv \text{Ci}(1) + A \approx 0.5772\), and where we kept only the leading terms. We see from Eq. (A19) that \(I_b\) contributes to both the real and the imaginary parts of \(I\). However, we now explicitly see the reason for which region \(b\) cannot be protracted all the way to \(y = 0\): the logarithmic term in \(\text{Re}(I_b)\) diverges as \(y \to 0\). In Ref.\footnote{see Ref. \[22\], we obtain \(\omega \tau_b\). It is shown there, that when the value of \(y\) at the boundary between regions \(b\) and \(c\) is negligible in comparison with the contribution of \(I_e\) to the real part of \(I\). Therefore, we obtain that on the boundary of regions \(b\) and \(c\), \(I\) is proportional to \((\kappa - u_0)^{-2(2+3)}\) due to the proportionality to \(E_0\). Let us now evaluate the error in our approximation, and thus show that the approximation is valid. The exact value of the integral (A1) in region \(b\) is

$$I_b^{\text{exact}} = \int_{\tau_a}^{\tau_b} \frac{1}{\tau} \left( \ln |\kappa - \tau| + \frac{1}{2} \ln |\kappa - u_0| \right)^{-(2l+3)} e^{i\omega \tau} \, d\tau$$

$$= \left( \ln |\kappa - \tau_b| + \frac{1}{2} \ln |\kappa - u_0| \right)^{-(2l+3)} \int_{\tau_a}^{\tau_b} \frac{1}{\tau} e^{i\omega \tau} \, d\tau$$

$$- (2l + 3) \int_{\tau_a}^{\tau_b} \left( \int_{\tau}^{2\tau} \frac{1}{\tau} e^{i\omega \tau} \, d\tau \right)$$

$$\times \left( \ln |\kappa - \tau| + \frac{1}{2} \ln |\kappa - u_0| \right)^{-(2l+4)} \, d\tau.$$  \hfill (A19)

Our approximation is

$$I_b^{\approx} = \left( \ln |\kappa - \tau_b| + \frac{1}{2} \ln |\kappa - u_0| \right)^{-(2l+3)} \int_{\tau_a}^{\tau_b} \frac{1}{\tau} e^{i\omega \tau} \, d\tau.$$  \hfill (A19)

Therefore, the relative error is

$$\frac{|I_b^{\approx} - I_b^{\text{exact}}|}{I_b^{\approx}} \approx (2l + 3) \left| \frac{\int_{\tau_a}^{\tau_b} \left( \int_{\tau}^{2\tau} \frac{1}{\tau} e^{i\omega \tau} \, d\tau \right) \frac{1}{\tau} \, d\tau}{\ln |\kappa - \tau_b| + \frac{1}{2} \ln |\kappa - u_0|} \right|$$

$$\approx (2l + 3) \left| \frac{\ln |\omega \tau_b|}{\ln |\kappa - \tau_b| + \frac{1}{2} \ln |\kappa - u_0|} \right|.$$  \hfill (A19)

Taking now the optimal value for the boundary between regions \(b\) and \(c\) (see Ref. \[22\]), we obtain

$$\frac{\Delta I_b}{I_b} \approx (2l + 3) \left| \frac{\ln |\frac{1}{2} \kappa - u_0|}{\frac{1}{2} \ln |\kappa - u_0|} \right| \ll 1.$$  \hfill (A20)

Thus, our approximation for region \(b\) is valid.

Region \(c\) is defined by the requirement that \(\theta < \omega \tau < 0\). This means that the electric field varies very fast compared with the angular frequency of the oscillator. Therefore, we may take the \(e^{i\omega \tau}\) term out of the integration in
Eq. (A1). The remaining integral is easily solvable, and we get

$$I_c = \int_{\theta \omega^{-1}}^{\tau} \frac{1}{K} \frac{1}{(|K - \tau| + \frac{1}{2} K - u_0)^{2 l + 3}} d\tau'. \quad (A21)$$

An explicit integration yields, then

$$I_c = -\frac{1}{(2 l + 2) K} \left\{ \frac{1}{(\ln|K - \tau| + \frac{1}{2} K - u_0)^{2 l + 2}} - \frac{1}{(\ln|K - \theta \omega^{-1}| + \frac{1}{2} K - u_0)^{2 l + 2}} \right\},$$

which for \( \tau = 0 \) becomes

$$I_c(\tau = 0) = \frac{1}{(2 l + 2) K} \left( \frac{1}{2} K - u_0 \right)^{-(2 l + 2)} \quad (A22).$$

Let us now evaluate the error of our calculation for region \( c \). The exact integral for \( \tau = 0 \) is

$$I_c^{\text{exact}} = \int_{\tau_b}^{\tau} e^{i \omega \tau'} \frac{1}{K} \left( \ln|K - \tau'| + \frac{1}{2} K - u_0 \right)^{-(2 l + 3)} d\tau'$$

$$\approx \int_{\tau_b}^{\tau} (1 + i \omega \tau') \frac{1}{K} \left( \ln|K - \tau'| + \frac{1}{2} K - u_0 \right)^{-(2 l + 3)} d\tau'$$

$$= \int_{\tau_b}^{\tau=0} \frac{1}{K} \left( \ln|K - \tau'| + \frac{1}{2} K - u_0 \right)^{-(2 l + 3)} d\tau'$$

$$+ i \frac{\omega}{K} \int_{\tau_b}^{\tau=0} \left( \ln|K - \tau'| + \frac{1}{2} K - u_0 \right)^{-(2 l + 3)} d\tau'. \quad (A23)$$

For the typical parameters we chose we thus obtain

$$\left| \frac{I_c^{\text{exact}} - I_c^{\text{approx.}}}{I_c^{\text{approx.}}} \right| (\tau = 0) \ll 1. \quad (A23)$$

For the evaluation of Eq. (A23) we again used the results of Ref. [24]. From the evaluation of the error associated with our approximation for the contribution of region \( c \) to \( I \), it is clear that the most dominant error in \( I_c \) is imaginary. Therefore, we should also verify, that this imaginary error is negligible in comparison with the imaginary part of the contribution of region \( b \) to \( I \). Repeating the relative error evaluation, we now obtain,

$$\left| \frac{I_c^{\text{exact}} - I_c^{\text{approx.}}}{(E_0/\kappa_+)} \right| (\tau = 0)$$

$$\lesssim \frac{\omega_{r_b}}{K} \int_{\tau_b}^{\tau=0} \left( \ln|K - \tau'| + \frac{1}{2} K - u_0 \right)^{-(2 l + 3)} d\tau'$$

$$\approx \frac{\omega_{r_b}}{2 l + 2} \left( \frac{1}{2} K - u_0 \right)^{-(2 l + 3)} \lesssim \frac{2 l + 3}{2 l + 2} \left( \frac{1}{2} K - u_0 \right)^2.$$

Collecting the contributions to \( I \) from all three regions, we find that the dominant contribution (at \( \tau = 0 \)) to \( \text{Re}(I) \) comes from region \( c \). This contribution is proportional to \((\kappa - u_0)^{-(2 l + 3)}\) (Eq. (A24)). The dominant contribution (at \( \tau = 0 \)) to \( \text{Im}(I) \), however, comes from region \( b \), and is proportional to \((\kappa - u_0)^{-(2 l + 3)}\). Hence, we find for the required integral

$$I(\tau = 0) = \text{Re}[I(0)] + i \text{Im}[I(0)], \quad (A24)$$

where

$$\text{Re}[I(0)] = -\frac{C'}{(2 l + 2) K} \left( \frac{1}{2} K - u_0 \right)^{-(2 l + 2)}$$

$$+ O \left( \frac{\kappa - u_0}{(\kappa - u_0)^{-(2 l + 3)}} \right) \quad (A25)$$

and

$$\text{Im}[I(0)] = \frac{C'}{2 l + 2 K} \left( \frac{1}{2} K - u_0 \right)^{-(2 l + 3)} + O \left( \frac{\kappa - u_0}{(\kappa - u_0)^{-(2 l + 4)}} \right). \quad (A26)$$

When \( \tau = 0 \) we thus obtain, to the leading order in \((\kappa - u_0)^{-1}\),

$$I(0) \approx -\frac{C'}{(2 l + 2) K} \left( \frac{1}{2} K - u_0 \right)^{-(2 l + 2)}$$

$$+ \frac{C'}{K - 2} \left( \frac{1}{2} K - u_0 \right)^{-(2 l + 3)} \quad (A27).$$

Note, that \( X(\tau = 0) \) is dominated by region \( b \) [Eq. (11)], and \( \hat{X}(\tau = 0) \) and \( \hat{E}_c(\tau = 0) \) are dominated by region \( c \) [Eqs. (23) and (33)], as well as \( \hat{E}_b(0) \) [Eq. (37)].

APPENDIX B: THE QUANTUM FORMALISM

Consider an electric field \( E_x(\bar{v}) \) in flat spacetime, which vanishes for \( \bar{v} < \bar{v}_0 \). After \( \bar{v} = \bar{v}_0 \) the field \( E_x(\bar{v}) \) rises,
and vanishes again for $\bar{v} \geq \bar{v}_1$. This electric field can be obtained from a potential, whose only non-vanishing component is $A_x(\bar{v})$, which vanishes for $\bar{v} < \bar{v}_0$, starts changing at $\bar{v} = \bar{v}_0$, and assumes a final constant value $A_0$ for $\bar{v} \geq \bar{v}_1$. (We note that any such field is consistent with the Maxwell field equations.) We notice that

$$A_0 = \int_{\bar{v}_0}^{\bar{v}_1} E_x(\bar{v}) \, d\bar{v}. \quad (B1)$$

Let us take a quantum system, which initially is in the eigenstate $|i\rangle$. We calculate $a_f$ according to first-order time-dependent perturbation theory. In the Coulomb gauge, in which our vector potential is written, the interaction term in the Hamiltonian is $-\vec{A} \cdot \vec{p}$. We calculate $a_f$ for $\bar{v} > \bar{v}_1$. More exactly, we assume that the system is centered about $\bar{z} = 0$, and that its typical spatial extension is $\delta \bar{z}$. We calculate $a_f$ at the time $\bar{v}_2 > \bar{v}_1 + \delta \bar{z}$, such that for all points of the system $\bar{v} > \bar{v}_1$. This means that the system has fully crossed the region of non-vanishing electromagnetic field. Hence, the electromagnetic field vanishes everywhere at the instant we calculate $a_f$. We now notice that at that instant $A_x = A_0 \neq 0$ in general. Therefore, the interaction term does not vanish formally, and from first-order-perturbation theory we get

$$a_f = -\frac{i}{\hbar m} \langle f | A_0 p_x | i \rangle e^{i \Omega_f \tau}, \quad (B2)$$

which generally is non-vanishing. Integration of (B2) yields

$$a_f = -\frac{i}{\hbar m} \left( \text{const} + \frac{A_0}{i \Omega_f} \langle f | p_x | i \rangle e^{i \Omega_f \tau} \right). \quad (B3)$$

($A_0$ is constant, and therefore we could take it out of the matrix element.) This may lead us to a weird (and wrong) conclusion: even thought there is no electromagnetic field, $a_f$ apparently continues to evolve. The resolution of this apparent “paradox” is as follows. If we write the state of the system for $\bar{v} > \bar{v}_2$ in the standard gauge appropriate for that state (namely, $A_x = 0$ instead of $A_x = A_0$), we find that $a_f$ is constant. [Initially, since we started with $E = 0$, i.e., with a vanishing perturbing field, we chose the standard gauge for the description of the system, namely, we chose $A_x = 0$. (This is, of course, the gauge in which the system’s wave-functions take their standard form.) Now, when we come to interpret the final state (in which again $E = 0$), we again use the gauge which is most natural for this situation, namely, we again take $A_x = 0$. This requires us to transform our original results [Eq. (B6)] to the new gauge.]

Let us show this now: The full wave-function is given by

$$\psi(\tau) = a_i(\tau) |i\rangle e^{-i \omega_i \tau} + \sum_{\{f\}} a_f(\tau) |f\rangle e^{-i \omega_f \tau}. \quad (B4)$$

We take $a_i(\tau) \approx 1$, which is consistent with small transition amplitudes. Substituting (B3) in (B4) we find (for $\bar{v} > \bar{v}_2$):

$$\psi(\tau) = \left( |i\rangle - \frac{i}{\hbar m} \frac{A_0}{i \Omega_f} \langle f | p_x | i \rangle |f\rangle \right) e^{-i \omega_i \tau}$$

$$- \sum_{\{f\}} c_f \frac{i}{\hbar m} \langle f | e^{-i \omega_f \tau}. \quad (B5)$$

The constant term in Eq. (B3) may depend on $f$, and therefore was given in Eq. (B5) an appropriate index. The gauge transformation

$$A_x^{\text{new}} = A_x^{\text{old}} - A_0 \quad (B6)$$

of the potential leads to the following gauge transformation of the wave-function:

$$\psi^{\text{new}} = \exp \left( \frac{ie}{\hbar} \int A_0 dx \right) \psi^{\text{old}}. \quad (B7)$$

Since we work to first order in the perturbation, we may use

$$\psi^{\text{new}} \approx \left( 1 + \frac{ie}{\hbar} A_0 x \right) \psi^{\text{old}}. \quad (B8)$$

Substituting Eq. (B5) for $\psi^{\text{old}}$ we get that

$$\psi^{\text{new}} \approx \left( 1 + \frac{ie}{\hbar} A_0 x \right) \left[ \left( |i\rangle - \sum_{\{f\}} \frac{i}{\hbar m} \frac{A_0}{i \Omega_f} \langle f | p_x | i \rangle |f\rangle \right) \right.$$

$$\left. \times e^{-i \omega_i \tau} - \sum_{\{f\}} c_f \frac{i}{\hbar m} \langle f | e^{-i \omega_f \tau} \right]$$

$$\approx \left( |i\rangle + \frac{ie}{\hbar} A_0 x |i\rangle - \sum_{\{f\}} \frac{1}{\hbar m} \frac{A_0}{\Omega_f} \langle f | p_x | i \rangle |f\rangle \right)$$

$$\times e^{-i \omega_i \tau} - \sum_{\{f\}} c_f \frac{i}{\hbar m} \langle f | e^{-i \omega_f \tau}$$

and therefore we obtain

$$\psi^{\text{new}} \approx |i\rangle e^{-i \omega_i \tau} - \sum_{\{f\}} c_f \frac{i}{\hbar m} \langle f | e^{-i \omega_f \tau}, \quad (B9)$$

after keeping linear terms in the interaction only, and using the relation

$$\sum_{\{f\}} \frac{1}{\Omega_f} \langle f | p_x | i \rangle |f\rangle = imx |i\rangle. \quad (B10)$$

Hence, we find that the transformation to the natural gauge ($A = 0$) removes the second term in the brackets in Eq. (B5). We now get the following conclusion: The
correct way to understand the result of the perturbative calculation is in the gauge where the vector potential vanishes. In this gauge, the second term in Eq. (B3) vanishes, and we obtain that
\[ a_f^{\text{new}} = \text{const}, \]  
(B11)
as should be expected (because the electromagnetic field vanishes).

Finally, we find the specific form for the constant term of Eqs. (B3) and (B11). To obtain this, we compare \( a_f^{(2)} \) [see Eq. (B4)] to the right-hand side of Eq. (B3) and obtain that
\[ a_f^{\text{final, new}} = -\frac{e}{m} \frac{1}{\hbar \bar{f}_i} \int_{-\infty}^{\tau} \langle \psi_f | \mathbf{E} \cdot \mathbf{p} | \psi_i \rangle e^{i \Omega_f i t} dt. \]  
(B12)
After the perturbation vanishes, this expression is constant.

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