Heavy Traffic Limits for GI/H/n Queues: Theory and Application

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Abstract

We consider a GI/H/n queueing system. In this system, there are multiple servers in the queue. The inter-arrival time is general and independent, and the service time follows hyper-exponential distribution. Instead of stochastic differential equations, we propose two heavy traffic limits for this system, which can be easily applied in practical systems. In applications, we show how to use these heavy traffic limits to design a power efficient cloud computing environment based on different QoS requirements.

1 Introduction

Many large queueing systems, like call centers and data centers, contain thousands of servers. For call centers, it is common to have 500 servers in one call center [1]. For data centers, Google has more than 45 data centers
as of 2009, and each of them contains more than 1000 machines [2]. When
the number of servers goes to infinity, many queueing systems should be
stable as long as the traffic intensity $\rho_n < 1$ (i.e., the arrival rate is smaller
than the service capacity). The traffic intensity for a queueing system with
$n$ servers can be thought of as the rate of job arrivals divided by the rate
at which jobs are serviced. At the same time, the queueing systems should
work efficiently, which means that $\rho_n$ should approach 1, i.e., $\lim_{n \to \infty} \rho_n = 1$.
This regime of operation is called the heavy traffic regime. Our paper focuses
on establishing heavy traffic limits, and using these limits to design a power
efficient cloud based on different QoS requirements.

Some classical results on heavy traffic limits are given by Iglehart in [3],
Halfin and Whitt in [4], and summarized by Whitt in Chapter 5 of his recent
book [5]. This heavy traffic limit ($(1 - \rho_n)\sqrt{n}$ goes to a constant as $n$ goes to
infinity) is now called the Halfin-Whitt regime. Recently, the behavior of the
normalized queue length in this regime has been studied by A. A. Puhalskii
and M. I. Reiman [6], J. Reed [7], D. Gamarnik and P. Momeilovic [8], and
Ward Whitt [9][10]. Based on these studies, some design and control policies
are proposed in [11][14].

Our work differs from prior work in three key aspects. First, literature
on heavy traffic limits that is based on analysis of call center systems does
not capture various unique features of large queueing systems today, such as
the cloud computing environment. Many of those works assume a Poisson
arrival process and exponential service time [11][14]. Perhaps appropriate
for smaller systems, these models need to be generalized for today’s larger
systems such as increasingly complex call-centers and cloud computing en-
vironments. The arrival process in such complex and large systems may
be independent, but more general. More importantly, the service times of
jobs are quite varied and unlikely to be accurately modeled by an exponential service time distribution. In [9], Whitt also considers the hyper-exponential distributed service time, but only with two stages and where one of them always has zero mean. Second, although some QoS metrics (especially Quality-Efficiency-Driven (QED)) have been extensively studied in some call center scenarios [11][12][13], the QoS requests can be more complex, because of the wide variety of application needs, especially in the cloud computing environment [16][18]. And third, while there are studies that give heavy traffic solutions for more general scenarios [6][7], these solutions can only be described by complex stochastic differential equations, which are quite cumbersome to use and provide little insight.

In this paper, we build a system model for general and independent inter-arrival process and hyper-exponentially distributed service times. As mentioned earlier, the general arrival process can be used to characterize a variety of arrival distributions for the queueing system. The main motivation for studying the hyper-exponential distribution is that it can capture the high degree of variability in the service time. For example, the hyper-exponential distribution can characterize any coefficient of variation (standard deviation divided by the mean) greater than 1. Since the service time of jobs is expected to be highly variable from job to job, the hyper-exponential distribution is well suited to model the service times for today’s queueing systems.

To satisfy the QoS and save operation cost at the same time, we characterize the performance of the queueing system for four different types of QoS requirements: Zero-Waiting-Time (ZWT), Minimal-Waiting-Time (MWT), Bounded-Waiting-Time (BWT) and Probabilistic-Waiting-Time (PWT) (the precise definitions are given in Section 2). Since the heavy
traffic limits for the ZWT and PWT classes can be directly derived from the current literature (details in our technical report [19]), we simply list their results, and focus instead on the MWT and BWT classes for which we develop new heavy traffic limits. We use the heavy traffic limits to characterize the relationship between the traffic intensity and the number of servers in the queueing systems.

In applications, we show how to use these heavy traffic limit results to determine the number of active machines in a cloud to ensure that the QoS requirements are met and the cloud operates in a stable and cost efficient manner. Cloud computing environments are rapidly deployed by the industry as a means to provide efficient computing resources. A significant fraction of the overall cost of operating a cloud is the amount of power it consumes, which is related to the number of machines in operation. In order to efficiently manage the power cost associated with cloud computing, we develop the foundations for designing a cloud computing environment. In particular, we aim to determine how many machines a cloud should have to sustain a specific system load and a certain level of QoS, or equivalently how many machines should be kept awake at any given time. Finally, using simulations we show that depending on the QoS requirements of the cloud, the cloud needs substantially different number of machines. We also show that the number of operational machines in simulations are consistent with the proposed design based on the new set of heavy traffic limit results. Although the number of operational machines is derived from heavy traffic limits, simulation results indicate that it is a good methodology, even when the number of machines is finite, but large.

The main contributions of this paper are:

- This paper makes new contributions to heavy traffic analysis, in that it
derives new heavy traffic limits for two important QoS classes (MWT and BWT) for queueing systems when the arrival process is general and the service times are hyper-exponentially distributed.

• Using the heavy traffic limits results, this paper answers the important question for enabling a power efficient cloud computing environment as an application: How many machines should a cloud have to sustain a specific system load and a certain level of QoS, or equivalently how many machines should be kept awake at any given time?

The paper is organized as follows. In Section 2, we present the system model of the queueing system, and describe the four different classes of QoS requirements. Based on this model, we develop heavy traffic limits results in Section 3 and Section 4 for the MWT and BWT classes correspondingly. Using these heavy traffic limits results and the results in our technical report [19], in Section 5 we consider cloud computing environment as an application and compute the operational number of machines needed for different classes of clouds. Simulation results are also provided in Section 6. Finally, we conclude this paper in Section 7.

2 System Model and QoS Classes

2.1 System Model and Preliminaries

We assume that the queueing system consists of a large number of servers, out of which \( n \) are active/operational at any given time. A larger \( n \) will result in better QoS at the expense of higher operational cost.

We assume that the job arrivals to the system are independent with rate \( \lambda_n \) and coefficient of variation \( c \).
We also assume that the service time \( v \) of the system satisfies the hyper-exponential distribution as given below.

\[
P(v > t) = \sum_{i=1}^{k} P_i e^{-\mu_i t}
\]  

(1)

Without loss of generality, we assume that

\[
0 < \mu_1 < \mu_2 < ... < \mu_k < \infty;
\]

(2)

\[
P_i > 0, \forall i \in 1,...k; \sum_{i=1}^{k} P_i = 1.
\]

The maximum buffer size that holds the jobs that are yet to be scheduled is assumed to be unbounded. The service priority obeys a first-come-first-serve (FCFS) rule. In this paper we consider a service model where each job is serviced by one server. All servers are considered to have similar capability.

### 2.2 Definition of QoS Classes

Before we give the definition of different QoS classes, we first provide some notations that will be used throughout this section. Here, we let \( n \) denote the total number of servers. For a given \( n \), we let \( T_n \) denote the time that a job is in the system before departure, \( Q_n \) denote the total number of jobs in the system, \( W_n \) denote the time that the job waits in the system before being processed. For two functions \( f(n) \) and \( g(n) \) of \( n \), \( g(n) = o(f(n)) \) if and only if \( \lim_{n \to \infty} g(n)/f(n) = 0 \). Also, we use \( \sim \) as equivalent asymptotics, i.e., \( f(n) \sim g(n) \) means that \( \lim_{n \to \infty} f(n)/g(n) = 1 \). We also use \( \phi(\cdot) \) and \( \Phi(\cdot) \) as probability density function and cumulative distribution function of normal distribution, and use \( \varphi_X(\cdot) \) as the characteristic function of the random variable \( X \).

We now provide precise definitions of the various QoS classes described
in the introduction. Since we are interested in studying the performance of the system in the heavy traffic limit, we let the traffic intensity $\rho \to 1$ as $n \to \infty$ in the case of each QoS class we study.

### 2.2.1 Zero-Waiting-Time (ZWT) Class

A system of the ZWT class is one for which

$$\lim_{n \to \infty} P\{Q_n \geq n\} = 0$$

The ZWT class corresponds to the class that provides the strictest of the QoS requirements we consider here. For such systems, the requirement is that an arriving job needs to wait in the queue is zero. Loosely speaking, a system of the ZWT class corresponds to having a QoS requirement that the jobs need to be served as soon as they arrive into the system.

### 2.2.2 Minimal-Waiting-Time (MWT) Class

For this class, the QoS requirement is

$$\lim_{n \to \infty} P\{Q_n \geq n\} = \alpha,$$

where $\alpha$ is a constant such that $0 < \alpha < 1$.

This requirement is less strict than the ZWT class. There is a nonvanishing probability that the jobs queue of the system is not empty. Roughly speaking, a system of the MWT class corresponds to the situation when jobs are served with some probability as soon as they arrive into the system.
2.2.3 Bounded-Waiting-Time (BWT) Class

For this class,

\[ \lim_{n \to \infty} P\{Q_n \geq n\} = 1 \]
\[ P\{W_n > t_1\} \sim \delta_n, \]

where

\[ \lim_{n \to \infty} \delta_n = 0. \]

The BWT class corresponds to the class for which the probability of waiting time \( W_n \) to exceed a constant threshold \( t_1 \) decreases to 0 as \( n \) goes to infinity. The decreasing rate has equivalent asymptotics with \( \delta_n \). This means that the waiting time \( W_n \) is between 0 and \( t_1 \) with probability 1, as \( n \) goes to infinity.

2.2.4 Probabilistic-Waiting-Time (PWT) Class

For this class,

\[ \lim_{n \to \infty} P\{Q_n \geq n\} = 1 \]
\[ \lim_{n \to \infty} P\{W_n > t_2\} = \delta, \]

where \( \delta \) is a given constant and satisfies \( 0 < \delta < 1 \).

The PWT class corresponds to the class that provides the least strict QoS requirements of the four types of systems considered here. Hence, the probability that the waiting time \( W_n \) is greater than some constant threshold \( t_2 \) is non-zero, for large enough \( n \). This means that the QoS requirement for this system is such that the waiting time \( W_n \) is between 0 and \( t_2 \) with probability \( 1 - \delta \), as \( n \) goes to infinity.

Further discussions and details on the four classes is given in Section 6.
and our technical report [19]. For the rest of the paper, we will mainly focus on developing new heavy traffic limits for the MWT and BWT classes.

## 3 Heavy Traffic Limit Analysis for the MWT class

The following result tells us how the number of servers must scale in the heavy traffic limit for the MWT class.

### Proposition 1. Assume

\[
\lim_{n \to \infty} \rho_n = 1, \quad (3)
\]

\[
\lim_{n \to \infty} P\{Q_n \geq n\} = \alpha, \quad (4)
\]

then

\[
L \leq \lim_{n \to \infty} (1 - \rho_n) \sqrt{n} \leq U, \quad (5)
\]

where

\[
U = \left( \sum_{i=1}^{k} \beta_U^{(i)} \frac{P_i}{\mu_i} \right) \sqrt{\mu}, \quad (6)
\]

\[
L = \max_{i \in \{1, \ldots, k\}} \left\{ \beta_L^{(i)} \frac{P_i}{\mu_i} \right\} \sqrt{\mu}, \quad (7)
\]

\[
\mu = \left( \sum_{i=0}^{k} \frac{P_i}{\mu_i} \right)^{-1}, \quad \rho_n = \frac{\lambda_n}{n\mu}, \quad (8)
\]

\[
\beta_U^{(i)} = (1 + c^2 P_i) \psi_U, \quad (9)
\]

\[
\frac{\alpha}{k} = [1 + \sqrt{2\pi} \psi_U \Phi(\psi_U) \exp(\psi_U^2/2)]^{-1},
\]

\[
\beta_L^{(i)} = (1 + c^2 P_i) \psi_L, \quad (10)
\]

\[
\alpha = [1 + \sqrt{2\pi} \psi_L \Phi(\psi_L) \exp(\psi_L^2/2)]^{-1},
\]
\begin{equation}
0 \leq \alpha \leq 1, \quad 0 \leq \beta_L \leq \infty, \quad 0 \leq \beta_U \leq \infty.
\end{equation}

In Proposition 1, \( \psi_U \) is the solution of Eq. (9), and \( \beta_U^{(i)} \) can be computed using \( \psi_U \). Similarly, \( \psi_L \) is the solution of Eq. (10), and \( \beta_L^{(i)} \) can be computed using \( \psi_L \). Thus, upper bound \( U \) in Eq. (11) and lower bound \( L \) in Eq. (7) can be achieved using \( \beta_U^{(i)}, \beta_L^{(i)} \), and other parameters.

To prove Proposition 1, we construct an artificial system structure. The arrival process and the capacity of a single server are same as the original system. In the artificial system, we assume that there are \( k \) types of jobs. For each arrival, we know the probability of \( i \)th type job is \( P_i \), and the service time of each \( i \)th type job is exponentially distributed with mean \( 1/\mu_i \).

Thus, the service time \( v \) of the system can be viewed as a hyper-exponential distribution which satisfies Eq. (1). We also assume that there is an omniscient scheduler for the artificial system. This scheduler can recognize the type of arriving jobs, and send them to the corresponding queue. For arrivals of type \( i \), the scheduler sends them to the \( i \)th queue, which contains \( n_i \) servers. Then the arrival rate of the \( i \)th queue is \( P_i \lambda_n \). Also, the priority of each separated queue obeys the FCFS rule. The artificial system is shown in Fig. 1.

**Lemma 2.** For the \( i \)th separated queue, the inter-arrival time \( \{Y_j^{(i)}, \quad j = 1, 2, \ldots\} \) is i.i.d., and the coefficient of variance \( c^{(i)} = \sqrt{1 + (c^2 - 1)P_i} \).

**Proof.** For the \( i \)th separated queue in Fig. 1, the inter-arrival time \( Y^{(i)} \) is a summation of inter-arrival times of a certain number of consecutive arrivals in the original queue. The number of the summands is a random variable \( k_j^{(i)} \). \( k_j^{(i)} \) is equal to the number of original arrivals between \( (j - 1) \)th and \( j \)th arrivals in the \( i \)th separated queue.

Based on the structure of the artificial system, \( k_j^{(i)} \) is an independent
random variable with geometric distribution with parameter $P_i$. Assume
$\{X_1, X_2, \ldots\}$ are the inter-arrival times in the original queueing system. Note
that $\{X_1, X_2, \ldots\}$ are also independent of $k_j^{(i)}$, because $k_j^{(i)}$ is only dependent
on the distribution of the service time. Then, for each $i$, the inter-arrival
time $Y_j^{(i)}$, $j = 1, 2, \ldots$ is i.i.d..

Let $t$ be the index of the first inter-arrival time within the $j^{th}$ inter-
arrival time in separated queue $i$. Then, $Y_j^{(i)} = X_t + X_{t+1} + \ldots + X_{t+k_j^{(i)}-1}$.
So,

$$E(Y_j^{(i)}) = E(X_t + X_{t+1} + \ldots + X_{t+k_j^{(i)}-1})$$
$$= E(E(X_t + X_{t+1} + \ldots + X_{t+k_j^{(i)}-1} | k_j^{(i)}))$$
$$= E(k_j^{(i)} E(X_t)) = E(k_j^{(i)}) E(X_t), \quad (12)$$
and

\[
\text{Var}(Y_{ij}^{(i)}) = E \left( (Y_{ij}^{(i)})^2 \right) - \left( E(Y_{ij}^{(i)}) \right)^2
\]

\[
= E \left( (X_t + X_{t+1} + \ldots + X_{t+k_j(i)-1})^2 \right) - \left( E(Y_{ij}^{(i)}) \right)^2
\]

\[
= E \left( E \left( (X_t + X_{t+1} + \ldots + X_{t+k_j(i)-1})^2 \mid k_j(i) \right) \right) - \left( E(Y_{ij}^{(i)}) \right)^2
\]

\[
= E \left( E \left( X_t^2 + X_{t+1}^2 + \ldots + X_{t+k_j(i)-1}^2 \mid k_j(i) \right) \right) - \left( E(Y_{ij}^{(i)}) \right)^2
\]

\[
= E \left( k_j(i)^2 \text{Var}(X_t) \right) - \left( E(Y_{ij}^{(i)}) \right)^2
\]

\[
= E(k_j(i)^2)(E(X_t)^2 + k_j(i)^2 \text{Var}(X_t)) - E(k_j(i))^2 E(X_t)^2
\]

\[
= \text{Var}(k_j(i)) E(X_t)^2 + E(k_j(i)) \text{Var}(X_t).
\]

Thus, we can achieve the coefficient of variation \(c^{(i)}\) for all the separated queues as below.

\[
c^{(i)} = \sqrt{\frac{\text{Var}(Y_{ij}^{(i)})}{\left( E(Y_{ij}^{(i)}) \right)^2}}
\]

\[
= \sqrt{\frac{\text{Var}(k_j) (E(X_t)^2 + E(k_j)^2 \text{Var}(X_t))}{E(k_j)^2 E(X_t)^2}}
\]

\[
= \sqrt{\frac{1 - P_i}{\sqrt{\frac{1}{P_i} E(X_t)^2 + \frac{1}{P_i} \text{Var}(X_t)}}} = \sqrt{1 + (c^2 - 1) P_i}.
\]

Remark 3. If the arrival process is Poisson, \(c = 1\), then \(c^{(i)} = 1\), \(\forall i = 1, 2, \ldots k\). If the arrival process is deterministic, \(c = 0\), then the interarrival time of each separated queue has a geometric distribution, and \(c^{(i)} = \sqrt{1 - P_i}, \forall i = 1, 2, \ldots k\).

Proof of Proposition 1. To prove this proposition, we must prove both the upper and the lower bounds of the limit. For the upper bound, we consider
the Artificial System I, which satisfies the following condition:

$$\lim_{n_i \to \infty} (1 - \rho_{n_i})\sqrt{n_i} = \beta_{U}^{(i)},$$  \hspace{1cm} (15)$$

where

$$\rho_{n_i} = \frac{P_i \lambda_i}{n_i \mu_i},$$

$$\beta_{U}^{(i)} = \frac{(1 + (c(i))^2)\psi_U}{2} = \frac{c^2 - 1}{2}P_i\psi_U,$$  \hspace{1cm} (16)$$

and

$$\alpha = \frac{1 + \sqrt{2\pi\psi_U\Phi(\psi)} \exp(\psi^2/2)}{1 + \sqrt{2\pi\psi_U\Phi(\psi)} \exp(\psi^2/2)} = \frac{1 + \sqrt{2\pi\psi_U\Phi(\psi)} \exp(\psi^2/2)}{1 + \sqrt{2\pi\psi_U\Phi(\psi)} \exp(\psi^2/2)}.$$  \hspace{1cm} (17)$$

The result of Theorem 4 in [4] shows that

$$\lim_{n \to \infty} P\{Q_n \geq n\} = \alpha_{c}$$  \hspace{1cm} (18)$$

if and only if

$$\lim_{n \to \infty} (1 - \rho_n)\sqrt{n} = \beta, \hspace{1cm} (19)$$

under the following conditions:

$$\beta = \frac{(1+c^2)\psi}{2},$$

$$\alpha_{c} = \frac{1 + \sqrt{2\pi\psi_U\Phi(\psi)} \exp(\psi^2/2)}{1 + \sqrt{2\pi\psi_U\Phi(\psi)} \exp(\psi^2/2)}.$$  \hspace{1cm} (20)$$

By applying this result into Artificial System I, for each individual queue, we have

$$\lim_{n_i \to \infty} P\{Q_{n_i}^{(i)} \geq n_i\} = \frac{1}{k}, \hspace{1cm} \forall i \in \{1, \ldots k\},$$  \hspace{1cm} (21)$$

where $Q_{n_i}^{(i)}$ is the length of the $i^{th}$ separated queue.
Let \( n_U = \sum_{i=1}^{k} n_i \), \( Q_{n_U} = \sum_{i=1}^{k} Q_{n_i}^{(i)} \). Then, for Artificial System I, we have

\[
P\{Q_{n_U} \geq n_U\} = P\left( \sum_{i=1}^{k} Q_{n_U}^{(i)} \geq \sum_{i=1}^{k} n_i \right)
\leq P\left( \bigcup_{i=1}^{k} \{Q_{n_i}^{(i)} \geq n_i\} \right) \leq \sum_{i=1}^{k} P\{Q_{n_i}^{(i)} \geq n_i\}.
\]  

(22)

By taking the limit on both sides,

\[
\lim_{n_i \to \infty} P\{Q_{n_U} \geq n_U\} 
\leq \lim_{n_i \to \infty} \left( \sum_{i=1}^{k} P\{Q_{n_i}^{(i)} \geq n_i\} \right)
= \left( \sum_{i=1}^{k} \lim_{n_i \to \infty} P\{Q_{n_i}^{(i)} \geq n_i\} \right) = \alpha
\]  

(23)

From Eq. (23), we know that when Artificial System I has \( n_U \) servers, the probability that queue length \( Q_{n_U} \) is greater than or equal to \( n_U \) is asymptotically less than or equal to \( \alpha \). Observe that the original system needs no more servers than Artificial System I since there may be some idle servers in Artificial System I, even when the other job queues are not empty. Based on the asymptotic optimality of FCFS in our system [20–23], to satisfy the same requirement, the original system does not need more servers than Artificial System I. By using Eqs. (15) and (16), we can solve for \( n_i \). That is,

\[
n \leq n_U = \sum_{i=1}^{k} n_i = \sum_{i=1}^{k} \left( \frac{P_i \lambda_n}{\mu_i} + \beta_{U}^{(i)} \sqrt{\frac{P_i \lambda_n}{\mu_i}} \right)
= \frac{\lambda_n}{\mu} + \sqrt{\frac{\lambda_n}{\mu} \left( \sum_{i=1}^{k} \beta_{U}^{(i)} \sqrt{\frac{P_i}{\mu_i}} \right)} \sqrt{\mu}.
\]  

(24)

Since \( \lim_{n_i \to \infty} \rho_i = 1 \), we ignore the factor \( \sqrt{\frac{1}{\rho_i}} \) and achieve Eq. (24). By
taking Eq. (24) into the definition of $\rho_n$ in Eq. (8), we can directly achieve the upper bound Eq. (6) of Eq. (5).

For the lower bound, we consider Artificial System II, which has similar structure as Artificial System I and Fig. 1 but $n_i$ satisfies the following conditions.

\[
n_i = \begin{cases} 
\frac{p_i \lambda_n}{\mu_i}, & i \in \{1, \ldots, k\}, i \neq m \\
\frac{p_m \lambda_n}{\mu_m} + \beta_L^{(m)} \sqrt{\frac{p_m \lambda_n}{\mu_m}}, & i = m
\end{cases}
\]

where

\[
\beta_L^{(i)} = \frac{(1 + (c^{(i)})^2)\psi}{2} = (1 + \frac{c^2 - 1}{2}P_i)\psi_L,
\]

\[
m = \inf \arg\max_{i \in \{1, \ldots, k\}} \left( \beta_L^{(i)} \sqrt{\frac{P_i}{\mu_i}} \right),
\]

and

\[
\alpha = [1 + \sqrt{2\pi} \psi_L \Phi(\psi_L) \exp(\psi_L^2/2)]^{-1}.
\]

Then,

\[
\lim_{n_m \to \infty} (1 - \rho_{nm}) \sqrt{n_m} = \beta_L^{(m)},
\]

where

\[
\rho_{nm} = \frac{p_m \lambda_n}{n_m \mu}.
\]

By substituting Eqs. (18-20) into Eqs. (25-27), the reader can verify the following result for Artificial System II.

\[
\lim_{n_i \to \infty} P\{Q_n^{(i)} \geq n_i\} = \begin{cases} 
1, & i \in \{1, \ldots, k\}, i \neq m \\
\alpha, & i = m
\end{cases}
\]

Define $n_L = \sum_{i=1}^k n_i$. If the original system has $n_L$ servers, then we can construct a scheduler based on Artificial System II. This scheduler can
make QoS of the arrivals satisfy Eq. (30). By the effect of the scheduler, this queueing discipline is neither FCFS nor work conserving. The original system, needs more servers than Artificial System II to satisfy Eq. (4) (see details in our technical report [19]). Therefore, \( n \) should be greater than or equal to \( n_L \), i.e.,

\[
 n \geq n_L = \sum_{i=1}^{k} n_i = \sum_{i=1}^{k} \left( \frac{P_i \lambda_n}{\mu_i} \right) + \beta^{(m)}_L \sqrt{\frac{P_m \lambda_n}{\mu_m}}
\]

\[
= \frac{\lambda_n}{\mu} + \sqrt{\frac{\lambda_n}{\mu} \max_{i \in \{1,...,k\}} \left\{ \beta^{(i)}_L \sqrt{\frac{P_i}{\mu_i}} \right\}} \sqrt{\mu}.
\]

By taking Eq. (31) into the definition of \( \rho_n \) in Eq. (8), we can directly achieve the lower bound Eq. (7) of Eq. (5).

**Corollary 4.** If the arrival process is Poisson process, we have a tighter upper bound \( \hat{U} \), which satisfies the following equation.

\[
\hat{U} = \left( \sum_{i=1}^{k} \sqrt{\frac{P_i}{\mu_i}} \right) \sqrt{\mu \hat{\psi}_U},
\]

where

\[
\mu = \left( \sum_{i=0}^{k} \frac{P_i}{\mu_i} \right)^{-1}, \quad \rho_n = \frac{\lambda_n}{n \mu},
\]

\[
1 - (1 - \alpha)^{\frac{1}{2}} = [1 + \sqrt{2\pi \hat{\psi}_U} \Phi(\hat{\psi}_U) \exp (\hat{\psi}_U^2 / 2)]^{-1},
\]

\[
0 \leq \alpha \leq 1, \quad 0 \leq \hat{\psi}_U \leq \infty.
\]

**Proof.** For Poisson arrival process, we can easily achieve that \( c = 1 \) and \( c^{(i)} = 1, \forall i \in \{1, 2, ..., k\} \). We consider a similar Artificial System III, which has same structure as Artificial System II. Let Artificial System III satisfy the following conditions.
\[
\lim_{n_i \to \infty} (1 - \rho_{n_i}) \sqrt{n_i} = \hat{\psi}_U,
\]

where
\[
\rho_{n_i} = \frac{P_i \lambda_n}{n_i \mu_i},
\]

and
\[
1 - (1 - \alpha)^{\frac{1}{k}} = \left[1 + \sqrt{2\pi \hat{\psi}_U \Phi(\hat{\psi}_U)} \exp \left(\frac{\hat{\psi}_U^2}{2}\right)\right]^{-1}.
\]

Similarly to Artificial System II, for each individual queue, we have
\[
\lim_{n_i \to \infty} P\{Q_{n_i}^{(i)} \geq n_i\} = 1 - (1 - \alpha)^{\frac{1}{k}}, \quad \forall i \in \{1, \ldots, k\},
\]

where \(Q_{n_i}^{(i)}\) is the length of the \(i^{th}\) separated queue.

Let \(n_U = \sum_{i=1}^{k} n_i\). Since arrival process is Poisson process, by the Colouring Theorem \cite{24}, the arrival process in each separated queue is independent Poisson process. Then, for Artificial System III, we have
\[
P\{Q_{n_U} \geq n_U\} = 1 - P\{Q_{n_U} < n_U\}
\]
\[
\leq 1 - \prod_{i=1}^{k} \left(1 - P\{Q_{n_i}^{(i)} \geq n_i\}\right)
\]

where
\[
Q_{n_U} = \sum_{i=1}^{k} Q_{n_i}^{(i)}.
\]

By taking the limits on each sides, we can achieve that
\[
\lim_{n_i \to \infty, \forall i \in \{1, \ldots, k\}} P\{Q_{n_U} \geq n_U\}
\]
\[
\leq \lim_{n_i \to \infty, \forall i \in \{1, \ldots, k\}} \left(1 - \prod_{i=1}^{k} \left(1 - P\{Q_{n_i}^{(i)} \geq n_i\}\right)\right)
\]
\[
= 1 - \prod_{i=1}^{k} \left(1 - \lim_{n_i \to \infty} P\{Q_{n_i}^{(i)} \geq n_i\}\right) = \alpha
\]
From Eq. (42), we know that when artificial system I has $n_U$ servers, the probability that queue length $Q_{n_U}$ is greater than or equal to $n_U$ is asymptotically less than or equal to $\alpha$. To satisfy the same requirement, the original system does not need more servers than Artificial System III. By using Eqs. (36) and (37), we can get the expression of $n_i$. That is,

$$n \leq n_U = \sum_{i=1}^{k} n_i = \sum_{i=1}^{k} \left( \frac{P_i \lambda_n}{\mu_i} + \psi_U \sqrt{\frac{P_i \lambda_n}{\mu_i}} \right) = \frac{\lambda_n}{\mu} + \sqrt{\frac{\lambda_n}{\mu}} \left( \sum_{i=1}^{k} \sqrt{\frac{P_i}{\mu_i}} \right) \sqrt{\mu \psi_U}. \quad (43)$$

By taking Eq. (43) into the definition of $\rho_n$ in Eq. (33), we can directly achieve the upper bound Eq. (32).

Since for Poisson arrival process, $c = 1$ and $c(i) = 1$, for $i \in \{1, 2, ..., k\}$, then $\beta^{(i)}_U = \psi_U$ in Eq. (9). Since $(1 - \frac{\alpha}{k})^k$ is an increasing function, then $(1 - \frac{\alpha}{k})^k \geq 1 - \alpha$. Thus, $1 - (1 - \alpha)^\frac{1}{k} \geq \frac{\alpha}{k}$. We can directly achieve that $\psi_U \geq \hat{\psi}_U$, i.e., Eq. (42) is a tighter upper bound than Eq. (6) for Poisson arrival process.

Remark 5. When $k = 1$, the service time reduces to an exponential distribution. Based on the Proposition 4, we can see that $U = L = \beta_U = \beta_L \triangleq \beta$ in this scenario, i.e., $\lim_{n \to \infty} (1 - \rho_n) \sqrt{n} = \beta$. Thus, Proposition 4 in our paper is consistent with Proposition 1 and Theorem 4 in [4].

Corollary 6. The solution $\tilde{U}$ of the following optimization problem results in a tighter upper bound for the Eq. (5).

$$\min_{\alpha_1, ..., \alpha_k} \frac{\sum_{j=1}^{k} \beta_j \sqrt{\frac{P_j}{\mu_j}}}{\sqrt{\sum_{j=1}^{k} \frac{P_j}{\mu_j}}}, \quad (44)$$
\[
\text{s.t. } \sum_{j=1}^{k} \alpha_j \leq \alpha, \quad (45)
\]

where

\[
\beta_j = (1 + \frac{c^2}{2} P_j) \psi_j,
\]

\[
\alpha_j = \left[ 1 + \sqrt{2\pi \psi_j^* \Phi(\psi_j^*) \exp \left( \frac{\psi_j^2}{2} \right)} \right]^{-1},
\]

\[
0 \leq \alpha_j \leq 1, \quad 0 \leq \beta_j \leq \infty, \quad \forall j.
\]

**Proof.** It is not necessary to choose all the \(\alpha_j\) equally. Once Eq. 22 is satisfied, it is sufficient to find an upper bound. Thus, the minimum of all the upper bounds are a new tighter upper bound for Proposition 1.

**Remark 7.** Since the corresponding objective value of every \(\{\alpha_j, \ j = 1, ..., k\}\) in the feasible set of the optimization problem (44-45) is an upper bound of the limit in 6. If we choose \(\tilde{\alpha}_j = \frac{\alpha}{k}, \forall j = 1, ..., k\), it is easy to check that the value of \(\{\tilde{\alpha}_j, \ j = 1, ..., k\}\) is in the feasible set, and the objective value is same as the upper bound in Eq. 6.

**Corollary 8.** The solution \(\tilde{U}\) of the following optimization problem results in a tighter upper bound for Poisson arrival process.

\[
\min_{\alpha_1, ..., \alpha_k} \frac{\sum_{j=1}^{k} \beta_j \sqrt{\frac{P_j}{\mu_j}}}{\sqrt{\sum_{j=1}^{k} \frac{P_j}{\mu_j}}}, \quad (47)
\]

\[
\text{s.t. } \lim_{s \to \infty} \int_{-\infty}^{\infty} \left[ \prod_{j=1}^{k} \phi(Q_j^*) \left( \sqrt{\frac{P_j}{\mu_j}} \right) \frac{1 - \exp(-its)}{it} \right] dt \leq 2\pi \alpha, \quad (48)
\]

where

\[
\beta_j = \frac{(1+c^2)\psi_j}{2},
\]

\[
\alpha_j = \left[ 1 + \sqrt{2\pi \psi_j^* \Phi(\psi_j^*) \exp \left( \frac{\psi_j^2}{2} \right)} \right]^{-1},
\]

\[
0 \leq \alpha_j \leq 1, \quad 0 \leq \beta_j \leq \infty, \quad \forall j,
\]

and the probability density function of \(Q_j^*\) is
$$f_j(x) = \begin{cases} 
\alpha_j \beta_j \exp(-\beta_j x), & \text{when } x > 0 \\
(1 - \alpha_j) \frac{\phi(x + \beta_j)}{\Phi(\beta_j)}, & \text{when } x < 0 
\end{cases}.$$  \tag{50}

**Proof.** We construct a new comparable system with similar structure as Fig. 1. For sub-queue \( j \), let the probability that queue length \( Q_j \) is greater than or equal to \( n_j \) be \( \alpha_j \). Then, the total number of servers \( n \) is

\[
n = \sum_{j=1}^{k} n_j = \left( \sum_{j=1}^{k} \frac{P_j}{\mu_j} \right) \lambda + \left( \sum_{j=1}^{k} \beta_j \sqrt{\frac{P_j}{\mu_j}} \right) \sqrt{\lambda} 
\tag{51}
\]

where \( \mu \) is same as Eq. (8).

For each arrival, the end-to-end time \( D \) of the original system is less than or equal to the end-to-end time \( \tilde{D} \) of the compared separated system in stochastic ordering [20,25,27]. Then, there exists a sample space \( \Omega \), such that \( D(\omega) \leq \tilde{D}(\omega) \) [28,29]. In this sample space \( \Omega \), the queue length \( Q(\omega) \) of the original system is less than or equal to the total queue length \( \tilde{Q}(\omega) \) of the compared artificial system for all \( \omega \in \Omega \). Thus, \( Q \leq \tilde{Q} \) in the stochastic ordering. We represent this stochastic ordering as \( Q \leq_{st} \tilde{Q} \).

By the definition of the stochastic ordering [29], for the same number \( n \), \( P(\tilde{Q} \geq n) \geq P(Q \geq n) \). In other words, if we assume that the QoS of the artificial system can satisfy \( P(\tilde{Q} \geq n) \leq \alpha \), then, to achieve the same QoS, the original system needs no more than \( n \) servers. For this reason, we can achieve a tighter upper bound for Eq. (5).

Now, consider the artificial system with the same QoS. We define \( \hat{Q}_j \) as
\[ \frac{Q_j - n_j}{\sqrt{n_j}}. \] Then,

\[
\alpha \geq P \left( \sum_{j=1}^{k} \tilde{Q}_j \geq n \right) = P \left( \sum_{j=1}^{k} (n_j + \sqrt{n_j} \tilde{Q}_j) \geq n \right) = P \left( \sum_{j=1}^{k} \sqrt{n_j} \tilde{Q}_j \geq 0 \right) = P \left( \sum_{j=1}^{k} \sqrt{P_j} \mu_j \tilde{Q}_j \geq 0 \right)
\] (52)

From Theorems 1 and 4 in [4], we can achieve the probability of normalized queue length as Eq. (50). Then, the characteristic function of \( \sum_{j=1}^{k} \sqrt{P_j} \mu_j \tilde{Q}_j \) in Eq. (52) is

\[
\varphi_{\sum_{j=1}^{k} \sqrt{P_j} \mu_j \tilde{Q}_j} (t) = \prod_{j=1}^{k} \varphi_{\sqrt{P_j} \mu_j \tilde{Q}_j} \left( \sqrt{P_j} \mu_j t \right)
\] (53)

By Levy’s inversion theorem [30], the Eq. (52) can be written as

\[
\alpha \geq P \left( \sum_{j=1}^{k} \sqrt{P_j} \mu_j \tilde{Q}_j \geq 0 \right) \geq \frac{1}{2\pi} \lim_{s \to \infty} \int_{-\infty}^{\infty} \left[ \prod_{j=1}^{k} \varphi_{\sqrt{P_j} \mu_j \tilde{Q}_j} \left( \sqrt{P_j} \mu_j \right) \frac{1 - \exp(-its)}{it} \right] dt
\] (54)

Thus, from Eq. (51) and (54), the solution of optimization problem (47-48) is an upper bound of the limit in Eq. (5) for the artificial system. Then, for the original system, no more servers are needed under the same value of traffic intensity, i.e., the upper bound of the artificial system is also an upper bound for the original system.

Remark 9. If we choose any \( \{\alpha_j, j = 1, \ldots, k\} \) in the feasible set of the optimization problem (47-48), then the corresponding objective value is an upper bound for Poisson arrivals. If we choose \( \tilde{\alpha}_j = 1 - (1 - \alpha)^{\frac{1}{k}}, \forall j = 1, \ldots, k \), it is easy to check that the value of \( \{\tilde{\alpha}_j, j = 1, \ldots, k\} \) is in the feasible set, and the objective value is same as the upper bound in Eq. (52).
4 Heavy Traffic Limit Analysis for the BWT Class

The following result provides conditions under which the waiting time of a job is bounded by a constant $t_1$ but the probability that new arrivals need to wait approaches one in the heavy traffic scenario.

**Proposition 10.** Assume

\[
\lim_{n \to \infty} \delta_n = 0,
\]

then

\[
\lim_{n \to \infty} \rho_n = 1 \quad (56)
\]

\[
\lim_{n \to \infty} P\{Q_n \geq n\} = 1 \quad (57)
\]

\[
P\{W_n > t_1\} \sim \delta_n \quad (58)
\]

if and only if

\[
\lim_{n \to \infty} \frac{(1 - \rho_n)n}{-\ln \delta_n} = \tau
\]

\[
\lim_{n \to \infty} \delta_n \exp (k\sqrt{n}) = \infty, \quad \forall k > 0
\]

where

\[
\tau = \frac{\mu^2 \sigma^2 + c^2}{2\mu t_1}, \quad \rho_n = \frac{\lambda_n}{n\mu},
\]

\[
\mu = \left( \sum_{i=1}^{k} \frac{P_i}{\mu_i} \right)^{-1}, \quad \sigma^2 = 2 \sum_{i=1}^{k} \left( \frac{P_i}{\mu_i^2} \right) - \left( \sum_{i=1}^{k} \frac{P_i}{\mu_i} \right)^2
\]

**Remark 11.** The main reason why Proposition 10 can be derived from Proposition 4 is due to the asymptotic rate of $\rho_n$. Although $\lim_{n \to \infty} (1 - \rho_n)\sqrt{n}$ is no longer a constant, it still has a constant lower and upper bound, i.e., it is still on a constant “level”.
Proof of Proposition 10. To prove Proposition 10 we must prove both necessary and sufficient conditions.

Necessary Condition: From the heavy traffic results given by Kingman [31] and Kollerstrom [32,33], the equilibrium waiting time in our system can be shown to asymptotically follow an exponential distribution with parameter

$$\frac{2(E(v_n) - E(s_n))}{\text{Var}(\frac{s_n}{n}) + \text{Var}(v_n)}.$$ \hspace{1cm} (63)

In Eq. (63), \(s_n\) is the service time, and \(v_n\) is the inter-arrival time. Assume the mean and variance of service time is \(\mu^{-1}\) and \(\sigma^2\). Then, we get

$$P(W_n \geq t_1) \sim \exp \left( -\frac{2(1 - \rho_n)nt_1}{\sigma^2 + \frac{c^2}{n^2}} \right).$$ \hspace{1cm} (64)

Since \(c_n = c\) and for this class the equilibrium waiting time satisfies that \(P(W_n \geq t_1) \sim \delta_n\), it implies that

$$\lim_{n \to \infty} \frac{(1 - \rho_n)n}{- \ln \delta_n} = \tau,$$ \hspace{1cm} (65)

where

$$\tau \triangleq \frac{\mu^2 \sigma^2 + c^2}{2\mu t_1},$$

$$\mu = \left( \sum_{i=1}^{k} \frac{P_i}{\mu_i} \right)^{-1}, \quad \sigma^2 = 2 \sum_{i=1}^{k} \left( P_i \lambda_i \right) - \left( \sum_{i=1}^{k} P_i \right)^2.$$ 

Based on Proposition 11 from \(\lim_{n \to \infty} P(Q_n \geq n) = 1\), we can achieve that \(\lim_{n \to \infty} (1 - \rho_n)\sqrt{n} = 0\), i.e., \(\lim_{n \to \infty} \frac{\ln \delta_n}{\sqrt{n}} = 0\). This means that \(\ln \frac{1}{\delta_n} = o(\sqrt{n})\). Hence, \(\lim_{n \to \infty} \delta_n \exp(k\sqrt{n}) = \infty, \quad \forall k > 0\). Thus, Eq. (60) is achieved.

Sufficient Condition: When Eq. (60) is satisfied, we get \(\ln \frac{1}{\delta_n} = o(n)\), i.e.,

$$\lim_{n \to \infty} \frac{\ln \delta_n}{n} = 0,$$ 

which is equivalent to \(\lim_{n \to \infty} \rho_n = 1\) based on Eq. (59). Hence,
Eq. (56) is achieved.

Now, based on Eqs. (60) and (56), and using the heavy traffic limit result Eqs. (9)-(11), the lower bound in Proposition I should satisfy that

\[ L = 0. \] 

(66)

By applying Eq. (66) in Eq. (71), we can directly obtain \( \lim_{n \to \infty} P\{Q_n \geq n\} = 1 \). Hence, Eq. (57) is satisfied.

Based on Eq. (59), it can be shown that

\[
\lim_{n \to \infty} \exp \left[ -n(1 - \rho_n)/\tau \right] = 1.
\] 

(67)

Based on Eq. (64), we get \( \lim_{n \to \infty} \frac{P\{W_n > t_1\}}{\delta_n} = 1 \). That is \( P\{W_n > t_1\} \sim \delta_n \).

Eq. (58) is achieved.

Remark 12. Let \( k = 1 \), then \( \mu_1 = \mu \) and \( P_1 = 1 \). We can directly achieve the scenario with exponential distributed service time from Proposition 10.

In the case of exponential distributed service time, the Proposition 10 still holds, and \( \tau \) can be simplified to \( \frac{1 + c^2}{2\mu t_1} \).

Corollary 13. Comparing the two cases in Proposition 10 and Remark 12, assume that they have the same parameters (\( t_1 \) and \( \mu \)) and functions (\( \rho_n \) and \( \delta_n \)), which satisfies Eqs. (75-78). Then, the hyper-exponential distributed service time needs a larger number of servers than the case of exponential distributed service time.

Proof. Using Eq. (62) and Eq. (61), we obtain

\[
\tau = \frac{\mu^2\sigma^2 + c^2}{2\mu t_1} = \frac{2 \sum_{i=1}^{k} \left( \frac{P_i}{\mu_i} \right) + (c^2 - 1) \left( \sum_{i=1}^{k} \frac{P_i}{\mu_i} \right)^2}{2 \left( \sum_{i=1}^{k} \frac{P_i}{\mu_i} \right) t_1}.
\] 

(68)
Based on Jensen’s Inequality, we can get that

$$\sum_{i=1}^{k} \left( \frac{P_i}{\mu_i^2} \right) \geq \left( \sum_{i=1}^{k} \frac{P_i}{\mu_i} \right)^2.$$  \hspace{1cm} (69)

Then,

$$\tau \geq \frac{(c^2 + 1) \left( \sum_{i=1}^{k} \frac{P_i}{\mu_i} \right)}{2\mu_1} = \frac{c^2 + 1}{2\mu_1}. \hspace{1cm} (70)$$

Then, in Eq. (69), the limit ($\tau$) for hyper-exponential distributed service time is greater than the limit ($\frac{c^2 + 1}{2\mu_1}$) for exponential distributed service time. Thus, for same $\rho_n$ and $\delta_n$, hyper-exponential distributed service time needs more servers than exponential distributed service time.

Consider Eq. (2) which defines the hyper-exponential service time, we can also get that Eq. (70) achieves equality if and only if $k = 1$.

Next, we will use the results obtained in Propositions 11 and 10 to compute heavy traffic limits when the cloud has different QoS requirements. These results will then provide guidelines on how many machines to keep active to meet the QoS requirements of the cloud.

5 Applications in Cloud Computing

The concept of cloud computing can be traced back to the 1960s, when John McCarthy claimed that “computation may someday be organized as a public utility” [34]. In recent years, cloud computing has received increased attention from the industry [16]. Many applications of cloud computing, such as utility computing [35], Web 2.0 [36], Google app engine [37], Amazon web services [38-39] and Microsoft’s Azure services platform [40], are widely used today. Some future application opportunities are also discussed by Michael Armbrust et al. in [16]. With the rapid growth of cloud based
applications, many definitions, concepts, and properties of cloud computing have emerged [16,18,41–43]. Cloud computing is an attractive alternative to the traditional dedicated computing model, since it makes such services available at a lower cost to the end users [16,17]. In order to provide services at a low cost, the cost of operating the cloud itself, needs to be kept low. In [44], based on detailed cost analysis of the cloud, 30% of the ongoing cost is electrical utility costs, and more than 70% of the ongoing cost is power-related cost which also includes power distribution and cooling costs. Some typical companies, like Google, have already claimed that their annual energy costs exceed their server costs [45]. And the power consumption of Google is 260 million watts [46]. So, power related cost, which is directly dependent on the number of operational machines in the cloud, is a significant fraction of the total cost of operating a cloud.
In [43], P. McFedries points out that clouds are typically housed in massive buildings and may contain thousands of machines. This claim is consistent with the fact that large data centers today often have thousands of machines [16]. The service system of a cloud can be viewed as a queueing system. Based on the stability and efficiency discussions in Section 1, we focus on the behavior of a cloud in the heavy traffic scenarios. Figure 2 shows the basic architecture. Using the new set of heavy traffic limit results developed in Section 3 and 4, we can achieve the design criteria of power efficient cloud computing environment, which allows for general and independent arrival processes and hyper-exponential distributed service times.

5.1 Heavy Traffic Limits for Different Classes of Clouds

As discussed earlier, it is important that the cloud operates stably, which means that the traffic intensity $\rho_n$ should be less than 1. Further, the cloud also needs to work efficiently, which means that the traffic intensity $\rho_n$ should be as close to 1 as possible and should approach 1 as $n \to \infty$. The different classes of clouds will result in different heavy traffic limits, and will thus be governed by different design rules for the number of operational machines $n$ and traffic intensity $\rho_n$. From the known literature [11, 12, 31–33, 47, 48], one can easily derive the heavy traffic limits for the ZWT and PWT classes. The derivation is also explicitly shown in our technical report [19], and so, here, to save space, we simply state how $n$ and $\rho_n$ should scale to satisfy the QoS requirements of various clouds.
5.1.1 ZWT Class

For a cloud of ZWT Class, using Proposition 1, we observe that

\[(1 - \rho_n)\sqrt{n} \to \infty, \quad (71)\]

from Eqs. (18)-(20). If we define \(f(n)\) as \(1 - \rho_n\), then

\[\lim_{n \to \infty} f(n) = 0,\]
\[\lim_{n \to \infty} f(n)\sqrt{n} = \infty. \quad (72)\]

5.1.2 MWT Class

Applying the result of Proposition 1, we can show that the QoS of a cloud of MWT Class can be satisfied if

\[L \leq \lim_{n \to \infty} (1 - \rho_n)\sqrt{n} \leq U. \quad (73)\]

\(U\) and \(L\) can be computed from Eq. (6)–Eq. (11) in Proposition 1.

5.1.3 BWT Class

We can satisfy the QoS requirement of this class by applying Proposition 10 to obtain

\[\lim_{n \to \infty} \frac{(1 - \rho_n)n}{-\ln \delta_n} = \tau, \quad (74)\]

where \(\tau\) can be computed by Eq. (61) and Eq. (62).

For a cloud of BWT Class, not all functions \(\delta_n\), which decrease to 0, as \(n\) goes to infinity, can satisfy the condition. An appropriate \(\delta_n\) that can be used to satisfy the QoS of BWT Class should satisfy the condition Eq. (60) given in Proposition 10. Then, the waiting time of jobs for BWT Class is
between 0 and $t$ almost surely as $n \to \infty$.

### 5.1.4 PWT Class

The QoS requirement of a cloud of PWT Class cloud based on Eq. (64) satisfies

$$P\{W_n \geq t_2\} \sim e^{\frac{-2n\mu(1-\rho)n}{\mu^2\sigma^2+c^2}}.$$

For a cloud of PWT Class, to satisfy its QoS requirement, the traffic intensity must scale as

$$\lim_{n \to \infty} (1 - \rho_n)n = \gamma,$$

where

$$\gamma = \frac{-(\mu^2\sigma^2 + c^2) \ln \delta}{2\mu t_2}.$$  

Here, $\mu$ and $\sigma$ are same as Eq. (62).

### 5.2 Number of Operational Machines for Different Classes

As discussed in Section 1, an important motivation of cloud computing is to maximize the workload that the cloud can support and at the same time satisfy the QoS requirements of the users. Based on the heavy traffic limits shown in Sections 3 and 4, we have different heavy traffic limits for different cloud classes (The details of the ZWT and PWT classes are shown in our technical report [19]). Thus, in order for the cloud to work efficiently and economically, we need to compute the least number of machines that the cloud needs to continue operating for a given QoS requirement.

When $\rho$ is closed to 1 and $n$ is large, the heavy traffic limit is a good methodology to approximate the relationship between $\rho$ and $n$. Based on the heavy traffic limits, we list the minimum number of machines that the cloud needs to provide under four classes of clouds, as below.
• The ZWT class: The $\rho_n$ and $n$ satisfy that $1 - \rho_n \sim f(n)$. Then, the number of operational machines $n$ is $\lceil f^{-1}(1 - \rho) \rceil$.

• The MWT class: The $\rho_n$ and $n$ satisfy that $L \leq (1 - \rho_n)\sqrt{n} \leq U$. Then, for the number of optimal machines $n$, the lower bound is $\lceil (\frac{L}{1 - \rho})^2 \rceil$, and the upper bound is $\lceil (\frac{U}{1 - \rho})^2 \rceil$.

• The BWT class: The $\rho_n$ and $n$ satisfy that $(1 - \rho_n)n = \tau$. Then, the number of operational machines $n$ is $\lceil \frac{\tau \ln \delta_n}{\rho - 1} \rceil$.

• The PWT class: The $\rho_n$ and $n$ satisfy that $(1 - \rho_n)n = \gamma$. Then, the number of operational machines $n$ is $\lceil \frac{\gamma}{1 - \rho} \rceil$.

Since there are many advanced techniques that can be used to estimate the parameter $\rho$ and this is not the main focus of this paper, we assume that the parameter $\rho$ can be estimated from the data. The number of machines can then be determined by the QoS requirements and the estimated $\rho$, as shown above.

6 Numerical Analysis

6.1 Evaluation Setup

We assume that the cloud can accommodate at most $N$ machines. Clearly, to reduce power consumption, we want to keep the number of powered servers to a minimum while at the same time satisfying the corresponding QoS requirements. The parameters for the four classes are as follows:

1. For the ZWT class, we choose $f(n) = n^{-k_1}$, where $k_1 = 0.25$.

2. For the MWT class, we choose the waiting probability $\alpha = 0.005$. 
3. For the BWT class, we choose $\delta_n = \exp \left( -n^{\frac{1}{4}} \right)$, which satisfies Eq. (60), and $t_1 = 0.5$.

4. For the PWT class, we choose the probability threshold $\delta = 0.1$ and $t_2 = 1$.

6.2 Necessity of Class-based Design

We first choose a simple process–Poison process–for arrivals, and choose an exponential service time distribution (i.e., $\mu = 0.3$, which is the simplest case of the hyper-exponential distribution).

The results characterizing the relationship between the number $n$ of requested machines and the traffic intensity $\rho$ are shown in Fig. 3 for $N = 10000$. The figure shows that with a larger pool of machines, not only a large number of jobs, but also a higher intensity of the offered load can be sustained, especially for clouds with more stringent QoS requirements.

From Fig. 3, we can also see that the number of machines needed for a given value of $\rho$ is quite different for different QoS classes. Classes with
higher QoS require several times more machines than classes with lower QoS under the same traffic intensity $\rho$, which implies that different number of operational machines are necessary for different QoS classes, even for the simplest case in our scenarios.

In Fig. 4, we now choose a hyper-exponential distributed service time, with $\mu = [1\ 8\ 20]$ and $P = [0.6\ 0.25\ 0.15]$. The results characterizing the relationship between the number $n$ of requested machines and the traffic intensity $\rho$ are shown in Fig. 4 for $N = 10000$. The figure is similar to the exponential distributed service time case shown in Fig. 3. The difference is that there are only upper and lower bounds for the MWT class in this scenario. However, even though there is a certain gap between the upper and lower bounds for the MWT class, the number of requested operational machines is still different from other classes when $n$ is large enough.

Note that Figs. 3 and 4 can also be used to find the maximal traffic intensity a cloud can support while satisfying a given QoS requirement for a given number of machines in the cloud.
Figure 5: Additional Operational Machines for Exponential Distributed Service Time

Figure 6: Additional Operational Machines for Hyper-exponential Distributed Service Time
Figure 7: Traffic Intensity for Exponential Distributed Service Time

Given an arrival rate $\lambda$, the basic request number of machines is equal to $\frac{\lambda}{\mu}$. However, it is not enough to satisfy the different QoS requirements. For different QoS requirements, the corresponding number of machines are shown in Figs. 5 and 6. Figs. 5 and 6 are under the same scenarios as Figs. 3 and 4 correspondingly. From these two figures, we can see that, for the same arrival rate, different classes need different additional number of machines to satisfy different QoS requirements. Similarly, given an arrival rate $\lambda$, the heaviest traffic intensity the system can support under a given QoS requirement is shown in Figs. 7 and 8.

6.3 Evaluation for the MWT and BWT Classes

For the MWT class, we also choose the same distribution of service time as above (i.e., $\mu = [1 \ 8 \ 20]$ and $P = [0.6 \ 0.25 \ 0.15]$) as an example. The performance of the MWT class is shown in Fig. 9.

We define a ratio to evaluate the tightness of the upper and lower bounds
Figure 8: Traffic Intensity for Hyper-exponential Distributed Service Time

Figure 9: Simulation results for the queueing systems of the MWT Class (Log Y-Axis)
for the clouds of the MWT class as below.

\[ r \triangleq \frac{U}{L} = r_1 r_2, \quad (76) \]

where

\[ r_1 = \frac{\psi_U}{\psi_L}, \]
\[ r_2 = \frac{\sum_{i=1}^{k} \left( 1 + \frac{c^2 - 1}{2} P_i \right) \sqrt{\frac{P_i}{\mu_i}}}{\max_{i \in \{1, \ldots, k\}} \left\{ (1 + \frac{c^2 - 1}{2} P_i) \sqrt{\frac{P_i}{\mu_i}} \right\}}. \quad (77) \]

For a given \( k \), \( r_1 \) and \( r_2 \) are independent. \( r_2 \) is determined by how the sum \( \sum_{i=1}^{k} \left( 1 + \frac{c^2 - 1}{2} P_i \right) \sqrt{\frac{P_i}{\mu_i}} \) dominates the largest item \( \max_{i \in \{1, \ldots, k\}} \left\{ (1 + \frac{c^2 - 1}{2} P_i) \sqrt{\frac{P_i}{\mu_i}} \right\} \). Its domain is interval \([1, k]\). \( r_1 \) is determined by parameter \( k \) and \( \alpha \), and is independent of \( P \) and \( \mu \). For different values of \( k \) and \( \alpha \), the corresponding ratio \( r_1 \) is shown in Fig. 10. From Fig. 10 we can see that \( r_1 \) is typically a small constant, even when \( \alpha \) and \( k \) are large (e.g. if \( \alpha = 0.15 \) and \( k = 20 \), then \( r_1 \) is less than 2).

The performance of the BWT class is shown in Fig. 11.

For non-Poisson arrival processes, we also select 2-state Erlang distribu-
Figure 11: Simulation results for the queueing systems of the BWT Class

tion and deterministic distribution as examples. The simulation results for
the MWT and BWT classes are shown in Figs. 12 and 13.

We have used the heavy traffic limit results to design the cloud for finite
values of \( n \) in Figs. 9, 11, 12 and 13. From these figures, we observe that the
simulation results closely follow the result obtained from the heavy traffic
analysis even when the number of machines is not very large (e.g., only 100)
and traffic is not very heavy (e.g., \( \rho = 0.85 \)).

7 Conclusion

In this paper, we study the heavy traffic limits of GI/H/n queues. First,
we classify the queueing systems into four classes based on the QoS require-
ments. Then, we develop heavy traffic limits that characterize the perform-
ance of the queueing systems for different types of QoS requirements. For
the MWT and BWT classes, new heavy traffic limits are derived. Based the
analysis of heavy traffic limits for different classes in this paper and existing
results, we show the relationship between heavy traffic limits and QoS re-
(a) The MWT Class with Erlang Arrivals ($E_{r_2}$)  
(b) The MWT class with Deterministic Arrivals

Figure 12: Simulation Results for the MWT class with Other Arrival Processes (Log Y-Axis)

(a) The BWT class with Erlang Arrivals ($E_{r_2}$)  
(b) The BWT Class with Deterministic Arrivals

Figure 13: Simulation Results for the BWT class with Other Arrival Processes
quirements to obtain design rules in the cloud computing environment as an application. The numerical results show that different rules are necessary for computing the number of operational machines for different cloud classes, and show the performance of the heavy traffic limits when the number of operational machines is finite. In the future, we plan to extend our work to jobs that need multiple servers or multiple stages, and apply it to improve widely-used frameworks, such as MapReduce.

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