Optimal solutions to matrix-valued Nehari problems and related limit theorems

A.E. Frazho, S. ter Horst and M.A. Kaashoek

Abstract

In a 1990 paper Helton and Young showed that under certain conditions the optimal solution of the Nehari problem corresponding to a finite rank Hankel operator with scalar entries can be efficiently approximated by certain functions defined in terms of finite dimensional restrictions of the Hankel operator. In this paper it is shown that these approximants appear as optimal solutions to restricted Nehari problems. The latter problems can be solved using relaxed commutant lifting theory. This observation is used to extend the Helton and Young approximation result to a matrix-valued setting. As in the Helton and Young paper the rate of convergence depends on the choice of the initial space in the approximation scheme.

1 Introduction

Since the 1980s, the Nehari problem played an important role in system and control theory, in particular, in the \( H^\infty \)-control solutions to sensitivity minimization and robust stabilization, cf., [9]. In system and control theory the Nehari problem appears mostly as a distance problem: Given \( G \) in \( L^\infty \), determine the distance of \( G \) to \( H^\infty \), that is, find the quantity \( d := \inf \{ \| G - F \|_\infty \mid F \in H^\infty \} \) and, if possible, find an \( F \in H^\infty \) for which this infimum is attained. Here all functions are complex-valued functions on the unit circle \( T \). It is well-known that the solution to this problem is determined by the Hankel operator \( H \) which maps \( H^2 \) into \( K^2 = L^2 \ominus H^2 \) according to the rule \( Hf = P_- (Gf) \), where \( P_- \) is the orthogonal projection of \( L^2 \) onto \( K^2 \). Note that \( H \) is uniquely determined by the Fourier coefficients of \( G \) with negative index. Its operator norm determines the minimal distance. In fact, \( d = \| H \| \) and the infimum is attained. Furthermore, if \( H \) has a maximizing vector \( \varphi \), that is, \( \varphi \) is a non-zero function in \( H^2 \) such that \( \| H \varphi \| = \| H \| \| \varphi \| \), then the AAK theory [1, 2] (see also [18]) tells us that the best approximation \( \hat{G} \) of \( G \) in \( H^\infty \) is unique and is given by

\[
\hat{G}(e^{it}) = G(e^{it}) - \frac{(H \varphi)(e^{it})}{\varphi(e^{it})} \quad \text{a.e.}
\]

By now the connection between the Nehari problem and Hankel operators is well established, also for matrix-valued and operator-valued functions, and has
been put into the larger setting of metric constrained interpolation problems, see, for example, the books [6, Chapter IX], [13, Chapter XXXV], [7, Chapter I], [17, Chapter 5] and [3, Chapter 7], and the references therein.

The present paper is inspired by Helton-Young [14]. Note that formula (1.1) and the maximizing vector \( \varphi \), may be hard to compute, especially if \( H \) has large or infinite rank. Therefore, to approximate the optimal solution (1.1), Helton-Young [14] replaces \( H \) by the restriction \( \hat{H} = H|_{H^2 \ominus z^n q H^2} \) to arrive at

\[
\tilde{G}(e^{it}) = G(e^{it}) - \frac{(\hat{H} \tilde{\varphi})(e^{it})}{\tilde{\varphi}(e^{it})}, \quad \text{a.e.} 
\]  

(1.2)
as an approximant of \( \hat{G} \). Here \( n \) is a positive integer, \( q \) is a polynomial and \( \tilde{\varphi} \) is a maximizing vector of \( \hat{H} \). Note that a maximizing vector \( \tilde{\varphi} \) of \( \hat{H} \) always exists, since \( \text{rank} \hat{H} \leq n + \text{deg} q \), irrespectively of the rank of \( H \) being finite, or not.

In [14] it is shown that \( \tilde{G} \) is a computationally efficient approximation of the optimal solution \( \hat{G} \) when the zeros of the polynomial \( q \) are close to the poles of \( G \) in the open unit disk \( D \) that are close to the unit circle \( T \). To be more precise, it is shown that if \( G \) is rational, i.e., \( \text{rank} H < \infty \), and \( \|H\| \) is a simple singular value of \( H \), then \( \|\hat{G} - \tilde{G}\|_{\infty} \) converges to 0 as \( n \to \infty \). This convergence is proportional to \( r^n \) if the poles of \( G \) in \( D \) are within the disc \( D_r = \{ z \in \mathbb{C} \mid |z| < r \} \), and the rate of convergence can be improved by an appropriate choice of the polynomial \( q \).

It is well-known that the Nehari problem fits in the commutant lifting framework, and that the solution formula (1.1) follows as a corollary of the commutant lifting theorem. We shall see that the same holds true for formula (1.2) provided one uses the relaxed commutant lifting framework of [8]; cf., Corollary 2.5 in [8].

To make the connection with relaxed commutant lifting more precise, define \( R_n \) to be the orthogonal projection of \( H^2 \) onto \( H^2 \ominus z^n q H^2 \), and put \( Q_n = SR_n \), where \( S \) is the forward shift on \( H^2 \). Then the operators \( R_n \) and \( Q_n \) both map \( H^2 \) into \( H^2 \ominus z^n q H^2 \), and the restriction operator \( H_n := H|_{H^2 \ominus z^n q H^2} \) satisfies the intertwining relation \( V \circ H_n R_n = H_n Q_n \). Here \( V \) is the compression of the forward shift \( V \) on \( L^2 \) to \( K^2 \). Given this intertwining relation, the relaxed commutant lifting theorem [8, Theorem 1.1] tells us that there exists an operator \( B_n \) from \( H^2 \ominus z^n q H^2 \) into \( L^2 \) such that

\[
P \cdot B_n = H_n, \quad VB_n R_n = B Q_n, \quad \|B_n\| = \|H_n\|. \tag{1.3}
\]

The second identity in (1.3) implies (see Lemma 2.2 below) that for a solution \( B_n \) to (1.3) there exists a unique function \( \Phi_n \in L^2 \) such that the action of \( B_n \) is given by

\[
(B_n h)(e^{it}) = \Phi_n(e^{it}) h(e^{it}) \quad \text{a.e.} \quad (h \in H^2 \ominus z^n q H^2). \tag{1.4}
\]

Furthermore, since \( H_n \) has finite rank, there exists only one solution \( B_n \) to (1.3) (see Proposition 2.3 below), and if \( \psi_n = \tilde{\varphi} \) is a maximizing vector of \( H_n \), then
this unique solution is given by (1.4) with $\Phi_n$ equal to

$$\Phi_n(e^{it}) = \frac{(H_n \psi_n)(e^{it})}{\psi_n(e^{it})} = \frac{(H \varphi)(e^{it})}{\varphi(e^{it})}, \quad \text{a.e.} \tag{1.5}$$

Thus $G - \widetilde{G}$ appears as an optimal solution to a relaxed commutant lifting problem.

This observation together with the relaxed commutant lifting theory developed in the last decade, enabled us to extend the Helton-Young convergence result for optimal solutions in [14] to a matrix-valued setting, that is, to derive an analogous convergence result for optimal solutions to matrix-valued Nehari problems; see Theorem 3.1 below. A complication in this endeavor is that formula (1.1) generalizes to the vector-valued case, but not to the matrix-valued case. Furthermore, in the matrix-valued case there is in general no unique solution. We overcome the latter complication by only considering the central solutions which satisfy an additional maximum entropy-like condition. On the way we also derive explicit state space formulas for optimal solutions to the classical and restricted Nehari problem assuming that the Hankel operator is of finite rank and satisfies an appropriate condition on the space spanned by its maximizing vectors. These state space formulas play an essential role in the proof of the convergence theorem.

This paper consists of 6 sections including the present introduction. In Section 2 which has a preliminary character, we introduce a restricted version of the matrix-valued Nehari problem, and use relaxed commutant lifting theory to show that it always has an optimal solution. Furthermore, again using relaxed commutant lifting theory, we derive a formula for the (unique) central optimal solution. In Section 3 we state our main convergence result. In Section 4 the formula for the (unique) central optimal solution derived in Section 2 is developed further, and in Section 5 this formula is specified for the classical Nehari problem. Using these formulas Section 6 presents the proof of the main convergence theorem.

Notation and terminology. We conclude this introduction with a few words about notation and terminology. Given $p, q$ in $\mathbb{N}$, the set of positive integers, we write $L^2_{q \times p}$ for the space of all $q \times p$-matrices with entries in $L^2$, the Lebesgue space of square integrable functions on the unit circle. Analogously, we write $H^2_{q \times p}$ for the space of all $q \times p$-matrices with entries in the classical Hardy space $H^2$, and $K^2_{q \times p}$ stands for the space of all $q \times p$-matrices with entries in the space $K^2 = L^2 \ominus H^2$, the orthogonal compliment of $H^2$ in $L^2$. Note that each $F \in L^2_{q \times p}$ can be written uniquely as a sum $F = F_+ + F_-$ with $F_+ \in H^2_{q \times p}$ and $F_- \in K^2_{q \times p}$. We shall refer to $F_+$ as the analytic part of $F$ and to $F_-$ as its co-analytic part. When there is only one column we simply write $L^2_p$, $H^2_p$ and $K^2_p$ instead of $L^2_{p \times 1}$, $H^2_{p \times 1}$ and $K^2_{p \times 1}$. Note that $L^2_p$, $H^2_p$ and $K^2_p$ are Hilbert spaces and $K^2_p = L^2_p \ominus H^2_p$. Finally, $L^\infty_{q \times p}$ stands for the space of all $q \times p$-matrices whose entries are essentially bounded on the unit circle with respect to the Lebesgue measure, and $H^\infty_{q \times p}$ stands for the space of all $q \times p$-matrices whose entries are analytic and uniformly bounded on the open unit disc $\mathbb{D}$. Note that
each $F \in L^\infty_{q \times p}$ belongs to $L^2_{q \times p}$ and hence the analytic part $F_+$ and the co-analytic part $F_-$ of $F$ are well defined. These functions belong to $L^2_{q \times p}$ and it may happen that neither $F_+$ nor $F_-$ belong to $L^\infty_{q \times p}$. In the sequel we shall need the following embedding and projection operators:

$$E : \mathbb{C}^p \to H^2_p, \quad Eu(\lambda) = u \quad (z \in \mathbb{D});$$

$$\Pi : K^2_q \to \mathbb{C}^q, \quad \Pi f = \frac{1}{2\pi} \int_0^{2\pi} e^{-it} f(e^{it}) dt.$$ 

Throughout $G \in L^\infty_{q \times p}$, and $H : H^2_p \to K^2_q$ is the Hankel operator defined by the co-analytic part of $G$, that is, $Hf = P_-(Gf)$ for each $f \in H^2_p$. Here $P_-$ is the orthogonal projection of $L^2_q$ onto $K^2_q$. Note that $V_-H = HS$, where $S$ is the forward shift on $H^2_p$ and $V_-$ is the compression to $K^2_q$ of the forward shift $V$ on $L^2_q$.

Finally, we associate with the Hankel operator $H$ two auxiliary operators involving the closure of its range, i.e., the space $\mathcal{X} = \text{Im} H$, as follows:

$$Z : \mathcal{X} \to \mathcal{X}, \quad Z = V_-|_{\mathcal{X}},$$

$$W : H^2_p \to \mathcal{X}, \quad Wf = Hf \quad (f \in H^2_p).$$

Note that $\mathcal{X} := \text{Im} H$ is a $V_-$-invariant subspace of $K^2_q$. Hence $Z$ is a well-defined contraction. Furthermore, if rank $H$ is finite, then the spectral radius $r_{\text{spec}}(Z)$ is strictly less than one and the co-analytic part $G_-$ of $G$ is the rational matrix function given by

$$G_-(\lambda) = (\Pi|_{\mathcal{X}})(\lambda I - Z)^{-1}WE.$$

In system theory the right hand side of the above identity is known as the restricted backward shift realization of $G_-$; see, for example, [5, Section 7.1]. This realization is minimal, and hence the eigenvalues of $Z$ coincide with the poles of $G_-$ in $\mathbb{D}$. In particular, $r_{\text{spec}}(Z) < 1$. Since $V_-H = HS$, we have $ZW = WS$. Furthermore, Ker $H^* = K^2_q \odot \mathcal{X}$.

### 2 Restricted Nehari problems and relaxed commutant lifting

In this section we introduce a restricted version of the Nehari problem, and we prove that it is equivalent to a certain relaxed commutant lifting problem. Throughout $\mathcal{M}$ is a subspace of $H^2_p$ such that

$$S^*\mathcal{M} \subset \mathcal{M}, \quad \text{Ker } S^* \subset \mathcal{M}.$$ 

With $\mathcal{M}$ we associate operators $R_{\mathcal{M}}$ and $Q_{\mathcal{M}}$ acting on $H^2_p$, both mapping $H^2_p$ into $\mathcal{M}$. By definition $R_{\mathcal{M}}$ is the orthogonal projection of $H^2_p$ onto $S^*\mathcal{M}$ and $Q_{\mathcal{M}} = SR_{\mathcal{M}}$. 


We begin by introducing the notion of an $\mathcal{M}$-norm. We say that $\Phi \in L_{q\times p}^2$ has a finite $\mathcal{M}$-norm if $\Phi h \in L_q^2$ for each $h \in \mathcal{M}$ and the map $h \mapsto \Phi h$ is a bounded linear operator, and in that case we define

$$\|\Phi\|_{\mathcal{M}} = \sup\{\|\Phi h\|_{L_q^2} \mid h \in \mathcal{M}, \|h\|_{H_p^2} \leq 1\}.$$ 

If $\mathcal{M}$ is finite dimensional, then each $\Phi \in L_{q\times p}^2$ has a finite $\mathcal{M}$-norm. Furthermore, $\Phi \in L_{q\times p}^\infty$ has a finite $\mathcal{M}$-norm for every choice of $\mathcal{M}$, and in this case $\|\Phi\|_{\mathcal{M}} \leq \|\Phi\|_\infty$, with equality if $\mathcal{M} = H_p^2$. Note that $\Phi \in L_{q\times p}^2$ has a finite $\mathcal{M}$-norm and $G \in L_{q\times p}^\infty$ imply $G - \Phi$ has a finite $\mathcal{M}$-norm.

We are now ready to formulate the $\mathcal{M}$-restricted Nehari problem. Given $G \in L_{q\times p}^\infty$ and a subspace $\mathcal{M}$ of $H_p^2$, we define the optimal $\mathcal{M}$-restricted Nehari problem to be the problem of determining the quantity

$$d_\mathcal{M} := \inf\{\|G - F\|_{\mathcal{M}} \mid F \in H_{q\times p}^2 \text{ and } F \text{ has a finite } \mathcal{M}\text{-norm}\},$$

and, if possible, to find a function $F \in H_{q\times p}^2$ of finite $\mathcal{M}$-norm at which the infimum is attained. In this case, a function $F$ attaining the infimum is called an optimal solution. The suboptimal variant of the problem allows the norm $\|G - F\|_{\mathcal{M}}$ to be larger than the infimum. When $\mathcal{M} = H_p^2$, the problem coincides with the classical matrix-valued Nehari problem in $L_{q\times p}^\infty$. In [15, 10] the case where $\mathcal{M} = H_p^2 \otimes S^k H_p^2$, with $k \in \mathbb{N}$, was considered.

**Proposition 2.1.** Let $G \in L_{q\times p}^\infty$, and let $\mathcal{M}$ be a subspace of $H_p^2$ satisfying the conditions in (2.1). Then the $\mathcal{M}$-restricted Nehari problem has an optimal solution and the quantity $d_\mathcal{M}$ in (2.2) is equal to $\gamma_\mathcal{M} := \|H\|_\mathcal{M}$, where $H : H_p^2 \rightarrow K_q^2$ is the Hankel operator defined by the co-analytic part of $G$.

We shall derive the above result as a corollary to the relaxed commutant lifting theorem [8, Theorem 1.1], in a way similar to the way one proves the Nehari theorem using the classical commutant lifting theorem (see, for example, [6, Section II.3]). For this purpose we need the following notion. We say that an operator $B$ from $\mathcal{M}$ into $L_q^2$ is defined by a $\Phi \in L_{q\times p}^2$ if the action of $B$ is given by

$$(Bh)(e^{it}) = \Phi(e^{it})h(e^{it}) \ a.e. \ (h \in \mathcal{M}).$$

In that case, $\Phi$ has a finite $\mathcal{M}$-norm, and $\|\Phi\|_{\mathcal{M}} = \|B\|$. When (2.3) holds we refer to $\Phi$ as the defining function of $B$. The following lemma characterizes operators $B$ from $\mathcal{M}$ into $L_q^2$ defined by a function $\Phi \in L_{q\times p}^2$ in terms of an intertwining relation.

**Lemma 2.2.** Let $\mathcal{M}$ be a subspace of $H_p^2$ satisfying (2.1), and let $B$ be a bounded operator from $\mathcal{M}$ into $L_q^2$. Then $B$ is defined by a $\Phi \in L_{q\times p}^2$ if and only if $B$ satisfies the intertwining relation $VBR_M = BQ_M$. In that case, $\Phi(\cdot)u = BEu(\cdot)$ for any $u \in \mathbb{C}^p$ and $\|B\| = \|\Phi\|_{\mathcal{M}}$.

**Proof.** This result follows by a modification of the proof of Lemma 3.2 in [11]. We omit the details. \hfill \Box
Proof of Proposition 2.1. Put $\gamma_M = \|H|_M\|$. Recall that the Hankel operator $H$ satisfies the intertwining relation $V_-H = HS$. This implies $V_-H|_M R_M = H|_M Q_M$. Here $R_M$ and $Q_M$ are the operators defined in the first paragraph of the present section. Since $Q_M^* Q_M = R_M^* R_M$ and $V$ is an isometric lifting of $V_-$, the quintet
\[\{H|_M, V_-, V, R_M, Q_M, \gamma_M\}\] (2.4)
is a lifting data set in the sense of Section 1 in [8]. Thus Theorem 1.1 in [8] guarantees the existence of an operator $B$ from $M$ into $L_2^q$ with the properties
\[P_-B = H|_M, \quad VBR_M = BQ_M, \quad \|B\| = \gamma_M.\] (2.5)
By Lemma 2.2 the second equality in (2.5) tells us there exists a $\Phi \in L_2^q \times p$ defining $B$, that is, the action of $B$ is given by (2.3). As $\Phi(\cdot)u = BEu(\cdot)$, the first identity in (2.5) shows that $G_- = \Phi_-$, and hence $F := G - \Phi \in H_2^q \times p$. Furthermore,
\[\|G - F\|_M = \|\Phi\|_M = \|B\| = \gamma_M,
\]
because of the third identity in (2.5). Thus the quantity $d_M$ in (2.2) is less than or equal to $\gamma_M$.

It remains to prove that $d_M \geq \gamma_M$. In order to do this, let $\tilde{F} \in H_2^q \times p$ and have a finite $M$-norm. Put $\tilde{\Phi} = G - \tilde{F}$. Then $\tilde{\Phi}$ has a finite $M$-norm. Let $\tilde{B}$ be the operator from $M$ into $L_2^q$ defined by $\tilde{\Phi}$. Since $\tilde{F} \in H_2^q \times p$, we have $G_- = \tilde{\Phi}_-$, and hence the first identity in (2.5) holds with $\tilde{B}$ in place of $B$. It follows that
\[\|G - \tilde{F}\|_M = \|\tilde{\Phi}\|_M = \|\tilde{B}\| \geq \|H|_M\| = \gamma_M.
\]
This completes the proof.

In the scalar case, or more generally in the case when $p = 1$, the optimal solution is unique. Moreover this unique solution is given by a formula analogous to (1.2); cf., [1]. This is the contents of the next proposition which is proved in much the same way as the corresponding result for the Nehari problem. We omit the details.

Proposition 2.3. Assume $p = 1$, that is, $G \in L_\infty^q$ and $M$ a subspace of $H^2$ satisfying (2.1). Assume that $H|_M$ has a maximizing vector $\psi \in M$. Then there exists only one optimal solution $F$ to the $M$-restricted Nehari problem (2.5), and this solution is given by
\[F(e^{it}) = G(e^{it}) - \frac{(H\psi)(e^{it})}{\psi(e^{it})} \quad \text{a.e.}\] (2.6)
In general, if $p > 1$ the optimal solution is not unique. To deal with this non-uniqueness, we shall single out a particular optimal solution.

First note that the proof of Proposition 2.1 shows that there is a one-to-one correspondence between the optimal solutions of the $M$-restricted Nehari problem of $G$ and all interpolants for $H|_M$ with respect to the lifting data set (2.4),
that is, all operators $B$ from $\mathcal{M}$ into $L^2_q$ satisfying (2.5). This correspondence is given by

$$B \mapsto F = G - \Phi,$$

where $\Phi$ is the defining function of $B$. (2.7)

Next we use that the relaxed commutant lifting theory tells us that among all interpolants for $H|_{\mathcal{M}}$ with respect to the lifting data set (2.4) there is a particular one, which is called the central interpolant for $H|_{\mathcal{M}}$ with respect to the lifting data set (2.4); see [8, Section 4]. This central interpolant is uniquely determined by a maximum entropy principle (see [8, Section 8]) and given by an explicit formula using the operators appearing in the lifting data set.

Using the correspondence (2.7) we say that an optimal solution $F$ of the $\mathcal{M}$-restricted Nehari problem of $G$ is the central optimal solution whenever $\Phi := G - F$ is the defining function of the central interpolant $B$ for $H|_{\mathcal{M}}$ with respect to the lifting data set (2.4). Furthermore, using the formula given in [8, Section 4] for the central interpolant the correspondence (2.7) allows us to derive a formula for the central optimal solution. To state this formula we need to make some preparations.

As before $\gamma_{\mathcal{M}} = \|H|_{\mathcal{M}}\|$. Note that $\|HP_{\mathcal{M}}S\| \leq \|HP_{\mathcal{M}}\| = \|H|_{\mathcal{M}}\|$, where $P_{\mathcal{M}}$ is the orthogonal projection of $H^2(\mathbb{C}^p)$ on $\mathcal{M}$. This allows us to define the following defect operators acting on $H^2(\mathbb{C}^p)$

$$D_{\mathcal{M}} = (\gamma_{\mathcal{M}}^2 I - P_{\mathcal{M}}H^*HP_{\mathcal{M}})^{1/2} \text{ on } H^2(\mathbb{C}^p),$$

$$D^2_{\mathcal{M}} = (\gamma_{\mathcal{M}}^2 I - S^*P_{\mathcal{M}}H^*HP_{\mathcal{M}}S)^{1/2} \text{ on } H^2(\mathbb{C}^p).$$ (2.8)

(2.9)

For later purposes we note that $S^*D^2_{\mathcal{M}}S = D^2_{\mathcal{M}}$. Next define

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} : H^2_p \rightarrow \begin{bmatrix} \mathbb{C}^q \\ H^2_p \end{bmatrix},$$

$$\omega(D_{\mathcal{M}}Q_{\mathcal{M}}) = \begin{bmatrix} \Pi H_{\mathcal{M}}R_{\mathcal{M}} \\ D_{\mathcal{M}}R_{\mathcal{M}} \end{bmatrix} \text{ and } \omega|_{\text{ker } Q_{\mathcal{M}}D_{\mathcal{M}}} = 0.$$ (2.10)

(2.11)

From the relaxed commutant lifting theory we know that $\omega$ is a well defined partial isometry with initial space $F = \text{Im } D_{\mathcal{M}}Q_{\mathcal{M}}$. Furthermore, the forward shift operator $V$ on $L^2_q$ is the Sz.-Nagy-Schäffer isometric lifting of $V_-$. Then as a consequence of [8, Theorem 4.3] and the above analysis we obtain the following result.

**Proposition 2.4.** Let $G \in L^\infty_{q\times p}$, and let $\mathcal{M}$ be a subspace of $H^2_q$ satisfying the conditions in (2.4). Then the central optimal solution $F_{\mathcal{M}}$ to the $\mathcal{M}$-restricted Nehari problem is given by $F_{\mathcal{M}} = G - \Phi_{\mathcal{M}}$, where $\Phi_{\mathcal{M}} \in L^2_{q\times p}$ has finite $\mathcal{M}$-norm, the co-analytic part of $\Phi_{\mathcal{M}}$ is equal to $G_-$, and the analytic part $\Phi_{\mathcal{M},+}$ of $\Phi_{\mathcal{M}}$ is given by

$$\Phi_{\mathcal{M},+}(\lambda) = \omega_1(I - \lambda\omega_2)^{-1}D_{\mathcal{M}}E.$$ (2.12)

Here $E$ is defined by (1.6), and $\omega_1$ and $\omega_2$ are defined by (2.10) and (2.11).
It is this central optimal solution \( F_M \) we shall be working with. From Corollary 4.4 in \([8]\) (see also \([10, \text{Theorem 1.1}]\)) we know that \( F = \text{Im} D_M Q_M = D_M \) implies that the central solution of \((2.5)\) is the only optimal solution to the \(M\)-restricted Nehari problem. The latter fact will play a role in Section \(4\).

3 Statement of the main convergence result

Let \( G \in L^\infty_{q \times p} \), and let \( H \) be the Hankel operator defined by the co-analytic part of \( G \). In our main approximation result we shall assume that the following two conditions are satisfied:

(C1) \( H \) has finite rank,

(C2) none of the maximizing vectors of \( H \) belongs \( SH^2_p \), and the space spanned by the maximizing vectors of \( H \) has dimension \( p \).

Note that (C1) is equivalent to \( G \) being the sum of a rational matrix function with all its poles in \( D \) and a matrix-valued \( H^\infty \) function.

In the scalar case the second part of (C2) implies the first part. To see this let \( p = q = 1 \), and assume that the space spanned by the maximizing vectors of \( H \) is one dimensional. Let \( Sv \) be a maximizing vector of \( H \). Since \( S \) is an isometry and \( V - H = HS \), we have \( v \neq 0 \) and

\[
\|H\|v = \|H\|Sv = \|HSv\| = \|V - Hv\| \leq \|H\|v \leq \|H\|v.
\]

Thus the inequalities are equalities, and \( v \) is a maximizing vector of \( H \). As the space spanned by the maximizing vectors of \( H \) is assumed to be one dimensional, \( v \) must be a scalar multiple of \( Sv \), which can only happen when \( v = 0 \), which contradicts \( v \neq 0 \). Thus the first part of (C2) is fulfilled. Next observe that for \( p = q = 1 \) the statement “the space spanned by the maximizing vectors of \( H \) has dimension one” is just equivalent to the requirement that \( \|H\| \) is a simple singular value of \( H \), which is precisely the condition used in Theorem 2 of the Helton-Young paper \([14]\).

As we shall see in Section \(5\) the two conditions (C1) and (C2) guarantee that the solution to the optimal Nehari problem is unique.

For our approximation scheme we fix a finite dimensional subspace \( M_0 \) of \( H^2_p \) invariant under \( S^* \), and we define recursively

\[
M_k = \text{Ker} S^* \oplus S M_{k-1}, \quad k \in \mathbb{N}.
\] (3.1)

Since \( M_0 \) is invariant under \( S^* \), the space \( M^0_1 \) is invariant under \( S \), and the Beurling-Lax theorem tells us that \( M^0_1 = \Theta H^2_p \), where \( \Theta \in H^\infty_{p \times \ell} \) and can be taken to be inner. Using this representation one checks that \( M_k = H^2_p \ominus z^k \Theta H^2_\ell \) for each \( k \in \mathbb{N} \). It follows that \( M_0 \subset M_1 \subset M_2 \subset \cdots \) and \( \bigcup_{k \geq 0} M_k = H^2_p \). Furthermore,

\[
S^* M_k \subset M_k \quad \text{and} \quad \text{Ker} S^* \subset M_k, \quad k \in \mathbb{N}.
\] (3.2)

Note that the spaces \( M_k = H^2 \ominus z^k q H^2 \), \( k = 1, 2, \ldots \), appearing in \([14]\) satisfy \(4\) with \( M_0 = H^2 \ominus q H^2 \).
Theorem 3.1. Let \( G \in L_{q\times p}^\infty \). Assume that conditions (C1) and (C2) are satisfied, and let the sequence of subspaces \( \{ \mathcal{M}_k \}_{k \in \mathbb{N}} \) be defined by (3.1) with \( \mathcal{M}_0 \) a finite dimensional \( S^* \)-invariant subspace of \( H^2 \). Let \( F \) be the unique optimal solution to the Nehari problem for \( G \), and for each \( k \in \mathbb{N} \) let \( F_k \) be the central optimal solution to the \( \mathcal{M}_k \)-restricted Nehari problem. Then \( G - F \) is a rational function in \( H^\infty_{q\times p} \), and for \( k \in \mathbb{N}_+ \) sufficiently large, the same holds true for \( G - F_k \). Furthermore, \( \| F_k - F \|_\infty \to 0 \) for \( k \to \infty \). More precisely, if all the poles of \( G \) inside \( \mathbb{D} \) are within the disk \( \mathbb{D}_r = \{ \lambda \mid |\lambda| < r \} \), for \( r < 1 \), then there exists a number \( L > 0 \) such that \( \| F_k - F \|_\infty < Lr^k \) for \( k \) large enough.

Improving the rate of convergence is one of the main issues in [13], where it is shown that for the case when the poles of \( G \) inside \( \mathbb{D} \) are close to the unit circle, that is, \( r \) close to 1, convergence with \( \mathcal{M}_0 = \{0\} \) may occur at a slow rate. In [13] it is also shown how to choose (in the scalar case) a scalar polynomial \( q \) so that the choice \( \mathcal{M}_0 = H^2 \ominus qH^2 \) increases the rate of convergence. In fact, if the roots of \( q \) coincide with the poles of \( G \) in \( \mathbb{D}_r \setminus \mathbb{D}_0 \), then starting with \( \mathcal{M}_0 = H^2 \ominus qH^2 \) the convergence is of order \( \mathcal{O}(r_0^k) \) rather than \( \mathcal{O}(r^k) \). In Section 6 we shall see that Theorem 3.1 remains true if \( r < 1 \) is larger than the spectral radius of the operator \( V_{-|H\mathcal{M}_k^\bot} \), and thus again the convergence rate can be improved by an appropriate choice of \( \mathcal{M}_0 \). To give a trivial example: when \( \mathcal{M}_0 \) is chosen in such a way that it includes \( \text{Im} H^* \), all the central optimal solutions \( F_k \) in Theorem 3.1 coincide with the unique optimal solution solution \( F \) to the Nehari problem.

4 The central optimal solution revisited

As before \( G \in L_{q\times p}^\infty \) and \( H \) is the Hankel operator defined by the co-analytic part of \( G \). Furthermore, \( \mathcal{M} \) is a subspace of \( H^2_q \) satisfying (2.1). In this section we assume that \( \| HP_M S \| < \gamma_M = \| HP_M \| \). In other words, we assume that the defect operator \( D_M^0 \) defined by (2.3) is invertible. This additional condition allows us to simplify the formula for the central optimal solution to the \( \mathcal{M} \)-restricted Nehari problem presented in Proposition 2.4. We shall prove the following theorem.

Theorem 4.1. Let \( G \in L_{q\times p}^\infty \), and let \( \mathcal{M} \) be a subspace of \( H^2_q \) satisfying (2.1). Assume the defect operator \( D_M^0 \) defined by (2.3) is invertible, and put

\[
\Lambda_M = D_M^{0,-2} S^* D_M^2.
\]

Then \( r_{\text{spec}}(\Lambda_M) \leq 1 \), and the central optimal solution \( F_M \) to the \( \mathcal{M} \)-restricted Nehari problem is given by \( F_M = G - \Phi_M \), where \( \Phi_M \in L_{q\times p}^2 \) has finite \( \mathcal{M} \)-norm, the co-analytic part of \( \Phi_M \) is equal to \( G_- \), and the analytic part of \( \Phi_M \) is given by

\[
\Phi_M^+(\lambda) = \Pi H (I - \lambda \Lambda_M)^{-1} \Lambda_M E = N_M(\lambda) M_M(\lambda)^{-1} \quad (\lambda \in \mathbb{D}),
\]

where

\[
N_M(\lambda) = \Pi H (I - \lambda S^*)^{-1} \Lambda_M E, \quad M_M(\lambda) = I - \lambda E^* (I - \lambda S^*)^{-1} \Lambda_M E.
\]
In particular, $M(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$.

The formulas in the above theorem for the central optimal solution are inspired by the formulas for the central suboptimal solution in Sections IV.3 and IV.4 of [7].

We first prove two lemmas. In what follows $P_M$ and $R_M$ are the orthogonal projections of $H^2_p$ onto $M$ and $S^*M$, respectively, and $Q_M = SR_M$.

Lemma 4.2. Let $M$ be a subspace of $H^2_p$ satisfying (2.1). Then

$$R_M = S^*P_MS, \quad R_MS^* = S^*P_M, \quad Q_M = P_MS.$$

(4.4)

Proof. Note that

$$(S^*P_MS)^2 = S^*P_MSS^*P_MS = S^*P_MS - S^*P_M(I - SS^*)P_MS.$$

Since $I - SS^*$ is the orthogonal projection onto Ker $S^*$, the second part of (2.1) implies that $P_M(I - SS^*) = I - SS^*$. Thus $(S^*P_MS)^2 = S^*P_MS$, and hence $S^*P_MS$ is an orthogonal projection. The range of this orthogonal projection is $S^*M$, and therefore the first identity in (4.4) is proved.

Using this first identity and $P_M(I - SS^*) = I - SS^*$ we see that

$$R_MS^* = S^*P_MSS^* = S^*P_M - S^*P_M(I - SS^*) = S^*P_M.$$

Thus the second identity in (4.4) also holds. Finally,

$$Q_M = SR_M = (R_MS^*)^* = (S^*P_M)^* = P_MS.$$

Thus (4.4) is proved.

Lemma 4.3. Let $G \in L_{q,p}^\infty$ and let $M$ be a subspace of $H^2_p$ satisfying (2.1). Assume the defect operator $D_M$ defined by (2.9) is invertible. Then the range $F$ of the operator $D_MQ_M$ is closed and the orthogonal projection of $H^2_p$ onto $F$ is given by

$$P_F = D_MQ_MD_M^{-1}Q_MD_M.$$

(4.5)

Proof. We begin with two identities:

$$D_MP_M = P_MD_M, \quad D_M^2 = P_MD_M.$$  

(4.6)

Since $P_M$ is an orthogonal projection, the first equality in (4.6) follows directly from the definition of $D_M$ in (2.8). To prove the second, we use the second identity in (4.4). Taking adjoints and using the fact that $R_M$ and $P_M$ are orthogonal projections, we see that $P_MS = SR_M$. It follows that $D_M^2$ is also given by

$$D_M^2 = (\gamma^2M I - R_MS^*H^*HSR_M)^{1/2}.$$  

(4.7)

From this formula for $D_M^2$ the second identity in (4.6) is clear.
Now assume that \( D_M^2 \) is invertible, and let \( P \) be the operator defined by the right hand side of \((4.5)\). Clearly, \( P \) is selfadjoint. Let us prove that \( P \) is a projection. Using the second equality in \((4.6)\) we have

\[
P^2 = D_M Q_M D_M^{\circ -2} Q_M^* D_M = D_M Q_M D_M^{\circ -2} (R_M S^* D_M S R_M) D_M^{\circ -2} Q_M D_M
\]

Observe that \( Q_M R_M = S R_M = Q_M \). Since \( D_M^{\circ 2} = S^* D_M S \), it follows that

\[
P^2 = D_M Q_M D_M^{\circ -2} Q_M^* D_M = P.
\]

Thus \( P \) is an orthogonal projection. This implies that \( D_M Q_M \) has a closed range, and \( P_F = P \).

**Proof of Theorem 4.1** Our starting point is formula \((2.12)\). Recall that \( \omega_1 \) and \( \omega_2 \) are zero on \( \text{Ker} Q_M^* D_M \). From Lemma \((4.3)\) we know that \( D_M Q_M \) has a closed range. It follows that \( \omega_1 = \omega_1 P_F \) and \( \omega_2 = \omega_2 P_F \), where \( P_F \) is the orthogonal projection of \( H_F^2 \) onto \( F = \text{Im} D_M Q_M \). Using the formula for \( P_F \) given by \((4.5)\), the second intertwining relation in \((4.6)\), the identities in \((4.4)\) and the definition of \( \omega \) in \((4.10)\), \((4.11)\) we compute

\[
\omega_1 D_M = \omega_1 P_F D_M = \omega_1 D_M Q_M D_M^{\circ -2} Q_M^* D_M^2 = \Pi H R_M D_M^{\circ -2} R_M S^* D_M^2 = \Pi H D_M^{\circ 2} R_M S^* D_M^2
\]

and

\[
\omega_2 D_M = \omega_2 P_F D_M = \omega_2 D_M Q_M D_M^{\circ -2} Q_M^* D_M^2 = D_M R_M D_M^{\circ -2} Q_M^* D_M^2 = D_M \Lambda M P_M.
\]

Furthermore, using the intertwining relations in \((4.6)\) and the second identity in \((4.11)\) we see that \( R_M \Lambda M = \Lambda M P_M \). In particular, \( \Lambda M \) leaves \( \mathcal{M} \) invariant.

Let us now prove that \( r_{\text{spec}}(\Lambda M) \leq 1 \). Note that

\[
r_{\text{spec}}(\omega_2) = r_{\text{spec}}(\omega_2 P_F) = r_{\text{spec}}(D_M R_M D_M^{\circ -2} S^* D_M) = r_{\text{spec}}(R_M D_M^{\circ -2} S^* D_M^2) = r_{\text{spec}}(R_M \Lambda M) = r_{\text{spec}}(\Lambda M P_M).
\]

Thus \( r_{\text{spec}}(\Lambda M P_M) \leq 1 \), because \( \omega_2 \) is contractive. Since \( \Lambda M \) leaves \( \mathcal{M} \) invariant, we see that relative to the orthogonal decomposition \( H_F^2 = \mathcal{M} \oplus \mathcal{M}^\perp \) the operator \( \Lambda M \) decomposes as

\[
\Lambda M = \begin{bmatrix} P_M \Lambda M P_M & * \\ 0 & (I - P_M) \Lambda M (I - P_M) \end{bmatrix}.
\]

(4.8)
Note that \((I - P_M)(I - R_M) = (I - P_M)\). Using the latter identity, the formulas \(2.8\) and \(4.7\), and the intertwining relations in \(4.6\), we obtain
\[
(I - P_M)\Lambda_M(I - P_M) = (I - P_M)(I - R_M)\Lambda_M(I - P_M)
= (I - P_M)(I - R_M)S^*(I - P_M)
= (I - P_M)S^*(I - P_M).
\]

Thus \((I - P_M)\Lambda_M(I - P_M)\) is a contraction. Hence \(r_{\text{spec}}((I - P_M)\Lambda_M(I - P_M) \leq 1\). But then \(4.8\) shows that \(r_{\text{spec}}(\Lambda_M) \leq 1\).

Next, using that \(\Lambda_M M \subset S^* M \subset M\) and \(\text{Im } E = \text{Ker } S^* \subset M\), we obtain for each \(\lambda \in \mathbb{D}\) that
\[
\Phi_{M,+}(\lambda) = \omega_1(I - \lambda \omega_2)^{-1}D_M E = \omega_1D_M(I - \lambda \Lambda_M P_M)^{-1}E
= \Pi\Lambda_M P_M(I - \lambda \Lambda_M P_M)^{-1}E = \Pi(1 - \lambda \Lambda_M P_M)^{-1}\Lambda_M P_M E
= \Pi(1 - \lambda \Lambda_M)^{-1}\Lambda_M E,
\]
which gives formula \(1.2\).

Finally, to see that \(1.3\) holds, note that \(\Lambda_M S = I\). Hence \(\Lambda_M\) is a left inverse of \(S\). Since \(E\) is an isometry with \(\text{Im } E = \text{Ker } S^*\), we have \(\Lambda_M = S^* + \Lambda_M E E^*\). Therefore, for each \(\lambda \in \mathbb{D}\),
\[
\Phi_{M,+}(\lambda) = \Pi(1 - \lambda \Lambda_M)^{-1}\Lambda_M E = \Pi(1 - \lambda S^* - \lambda \Lambda_M E E^*)^{-1}\Lambda_M E
= \Pi(1 - \lambda S^* - \lambda \Lambda_M E E^*)^{-1}(1 - \lambda S^*)^{-1}\Lambda_M E
= \Pi(1 - \lambda S^*)^{-1}\Lambda_M E(1 - \lambda E^* (1 - \lambda S^*)^{-1}\Lambda_M E)^{-1}
= N(\lambda)M(\lambda)^{-1}.
\]

In particular, \(M(\lambda)\) is invertible.

\(\square\)

Remark. From \(R_M \Lambda_M = \Lambda_M P_M\) we see that \(\Lambda_M\) leaves \(M\) invariant. Thus, if \(M\) in Theorem \(4.1\) is finite dimensional, then \(\Phi_{M,+}\) in \(1.2\) is a rational function in \(H^2_{p \times q}\), and hence \(\Phi_{M,+}\) is a rational \(p \times q\) matrix function which has no pole in the closed unit disk.

Next we present a criterion in terms of maximizing vectors under which Theorem \(4.1\) applies.

Proposition 4.4. Assume \(\text{rank } HP_M\) is finite. Then \(D^\circ_M\) is invertible if and only if none of the maximizing vectors of \(HP_M\) belongs to \(SH^2_p\).

Proof. A vector \(h \in H^2_p\) is a maximizing vector of \(HP_M\) if and only if \(0 \neq h \in D^\perp_M\). Thus we have to show that invertibility of \(D^\circ_M\) is equivalent to \(D^\perp_M \cap SH^2_p = \{0\}\).

Assume \(D^\perp_M \cap SH^2_p \neq \{0\}\). Thus, using the definition of a maximizing vector, there exists \(Sv\) with \(v \neq 0\) such that \(\|HP_M Sv\| = \gamma_M \|Sv\|\). Since \(S\) is an isometry we see that \(\|HP_M Sv\| = \gamma_M \|v\|\). It follows that \(v\) is in the kernel of \(D^\circ_M\), and hence \(D^\circ_M\) is not invertible.
Conversely, assume that \( D_M^1 \cap \mathcal{SH}_p^2 = \{0\} \). Note that \( \text{rank} (HP_M S) \) is also finite. Hence \( HP_M S \) has a maximizing vector, say \( v \). We may assume that \( \|v\| = 1 \). By our assumption the vector \( Sv \) is not a maximizing vector of \( HP_M \). Hence

\[
\|HP_M S\| = \|HP_M S\|\|v\| = \|HP_M Sv\| < \|HP_M\|\|Sv\| = \gamma_M \|Sv\| = \gamma_M.
\]

Therefore \( D_M^2 = \gamma_M^2 I - S^* P_M H^* H P_M S \) is positive definite, and thus invertible. Consequently, \( D_M^2 \) is invertible. \( \square \)

For later purposes we mention the following. It is straightforward to prove that \( D_M^2 \) is invertible if and only if the operator \( \gamma_M^2 I - HP_M SS^* P_M H^* \) is invertible, and in that case we have

\[
\begin{align*}
\Lambda_M P_M H^* &= R_M H^* V_1^* (\gamma_M^2 I - HP_M SS^* P_M H^*)^{-1} \times \\
&\quad (\gamma_M^2 I - HP_M H^*), \quad (4.9) \\
\Lambda_M E &= -R_M H^* V_1^* (\gamma_M^2 I - HP_M SS^* P_M H^*)^{-1} H E. \quad (4.10)
\end{align*}
\]

These formulas can be simplified further using the operators \( Z \) and \( W \) associated to the Hankel operator \( H \) which have been introduced at the end of Section II, see (1.8) and (1.9). Recall that \( \mathcal{X} = \text{Im} H \). Since \( K_4^2 \cap \mathcal{X} = \text{Ker} H^* \), the space \( \mathcal{X} \) is a reducing subspace for the operators \( \gamma_M^2 I - HP_M SS^* P_M H^* \) and \( \gamma_M^2 I - HP_M H^* \). Furthermore,

\[
\begin{align*}
\Delta_M := (\gamma_M^2 I - HP_M SS^* P_M H^*)|_{\mathcal{X}} &= \gamma_M^2 I_{\mathcal{X}} - Z W R_M W^* Z^*, \quad (4.11) \\
\Xi_M := (\gamma_M^2 I - HP_M H^*)|_{\mathcal{X}} &= \gamma_M^2 I_{\mathcal{X}} - W P_M W^*. \quad (4.12)
\end{align*}
\]

Note that \( \Delta_M \) is invertible if and only if \( D_M^2 \) is invertible. Using the above operators, (4.9) and (4.10) can be written as

\[
\begin{align*}
\Lambda_M P_M W^* &= R_M W^* Z^* \Delta_M^{-1}, \quad \Lambda_M E = -R_M W^* Z^* \Delta_M^{-1} W E. \quad (4.13)
\end{align*}
\]

**Corollary 4.5.** Let \( G \in L_p^\infty \), and let \( M \) be a subspace of \( H_p^2 \) satisfying (2.1). Assume the operator \( \Delta_M \) defined by (4.11) is invertible. Then the defect operator \( D_M^2 \) defined by (2.9) is invertible, and the functions \( N_M \) and \( M_M \) appearing in (4.3) are also given by

\[
\begin{align*}
N_M(\lambda) &= N_{M,1}(\lambda) + N_{M,2}(\lambda), \quad (4.14) \\
N_{M,1}(\lambda) &= -\Pi HW^* (I - \lambda Z^*)^{-1} Z^* \Delta_M^{-1} W E \quad (4.15) \\
N_{M,2}(\lambda) &= \Pi H (I - \lambda S^*)^{-1} (I - R_M) W^* Z^* \Delta_M^{-1} W E. \quad (4.16)
\end{align*}
\]

and

\[
\begin{align*}
M_M(\lambda) &= M_{M,1}(\lambda) + M_{M,2}(\lambda), \quad (4.17) \\
M_{M,1}(\lambda) &= I + \lambda E^* W^* (I - \lambda Z^*)^{-1} Z^* \Delta_M^{-1} W E \quad (4.18) \\
M_{M,2}(\lambda) &= -\lambda E^* (I - \lambda S^*)^{-1} (I - R_M) W^* Z^* \Delta_M^{-1} W E. \quad (4.19)
\end{align*}
\]
Furthermore, if \( r_{\text{spec}}(Z^*\Delta_{\mathcal{M}}^{-1}\Xi_{\mathcal{M}}) < 1 \), then \( M_{\mathcal{M},1}(\lambda) \) is invertible for \( |\lambda| \leq 1 \) and
\[
M_{\mathcal{M},1}(\lambda)^{-1} = I - \lambda E*W*(I - \lambda Z^*\Delta_{\mathcal{M}}^{-1}\Xi_{\mathcal{M}})^{-1}Z^*\Delta_{\mathcal{M}}^{-1}WE, \quad |\lambda| \leq 1. \quad (4.20)
\]

**Proof.** For operators \( A \) and \( B \) the invertibility of \( I + AB \) is equivalent to the invertibility of \( I + BA \). Using this fact it is clear that the invertibility of \( D_{\mathcal{M}}^* \) follows from the invertibility of \( \Delta_{\mathcal{M}} \). Hence we can apply Theorem 4.1. Writing \( R_M \) as \( I - (I - R_M) \) and using (4.13), we see that (4.14) holds with \( N_{\mathcal{M},2} \) being given by (4.16) and with
\[
N_{\mathcal{M},1}(\lambda) = -\Pi H(I - \lambda S^*)^{-1}W*Z^*\Delta_{\mathcal{M}}^{-1}WE. \quad (4.21)
\]

The intertwining relation \( WS = ZW \) yields \( (I - \lambda S^*)^{-1}W* = W*(I - \lambda Z^*)^{-1} \). Using the latter identity in (4.21) yields (4.15). In a similar way one proves the identities (4.17)-(4.19).

To complete the proof assume \( r_{\text{spec}}(Z^*\Delta_{\mathcal{M}}^{-1}\Xi_{\mathcal{M}}) < 1 \). Then the inversion formula for \( M_{\mathcal{M},1}(\lambda) \) follows from the standard inversion formula from [4, Theorem 2.2.1], where we note that the state operator in the inversion formula equals
\[
Z^* - Z^*\Delta_{\mathcal{M}}^{-1}WE*W^* = Z^*\Delta_{\mathcal{M}}^{-1}(\gamma^2_{\mathcal{M}} I - ZWR_MW*Z - WEE*W^*)
\]
\[
= Z^*\Delta_{\mathcal{M}}^{-1}(\gamma^2_{\mathcal{M}} I - W(\lambda S_{\mathcal{M}}S^* + EE^*)W^*)
\]
\[
= Z^*\Delta_{\mathcal{M}}^{-1}(\gamma^2_{\mathcal{M}} I - W(S^*P_M + EE^*P_M)W^*)
\]
\[
= Z^*\Delta_{\mathcal{M}}^{-1}(\gamma^2_{\mathcal{M}} I - WP_MW^*) = Z^*\Delta_{\mathcal{M}}^{-1}\Xi_{\mathcal{M}},
\]
as claimed. Here we used the second identity in (4.14), and the fact that \( P_M E = E \), because \( \text{Im } E = \text{Ker } S^* \subset \mathcal{M} \).

**5 The special case where \( \mathcal{M} = H^2_p \)**

Throughout this section \( \mathcal{M} = H^2_p \), that is, we are dealing with the \( H^2_p \)-restricted Nehari problem, which is just the usual Nehari problem. Since \( \mathcal{M} = H^2_p \), we will suppress the index \( \mathcal{M} \) in our notation, and just write \( D, D^p, D, D^c, \Lambda, \) etc. instead of \( D_{\mathcal{M}}, D_{\mathcal{M}}^*, D_{\mathcal{M}}, D_{\mathcal{M}}^c, \Lambda_{\mathcal{M}}, \) etc. In particular,
\[
\gamma = \|H\|, \quad D = (\gamma^2 I - H^*H)^{1/2}, \quad D^c = (\gamma^2 I - S^*H^*HS)^{1/2}. \quad (5.1)
\]

We shall assume (cf., the first paragraph of Section 3) that the following two conditions are satisfied
\begin{enumerate}
  \item [(C1)] \( H \) has finite rank,
  \item [(C2)] none of the maximizing vectors of \( H \) belongs \( SH^2_p \), and the space spanned by the maximizing vectors of \( H \) has dimension \( p \).
\end{enumerate}
Note that the space spanned by the maximizing vectors of $H$ is equal to $\text{Ker } D = D^\perp$, where $D$ is the closure of the range of $D$. As $H^2_p = \text{Ker } S^* \oplus SH^2$, we see that

\[(C2) \iff H^2_p = \text{Ker } D^+ \iff H^2_p = \text{Ker } S^* + D.\] (5.2)

Here $\oplus$ means direct sum, not necessarily orthogonal direct sum.

Let $Z$ and $W$ be the operators defined by (1.8) and (1.9), respectively, and 
\[
\Delta = \gamma^2 I_X - ZWW^*Z^* , \quad \Xi = \gamma^2 I_X - WW^*.\] (5.3)

We shall prove the following theorem.

**Theorem 5.1.** Let $G \in L^2_{q \times p}$, and assume that the Hankel operator $H$ associated with the co-analytic part of $G$ satisfies conditions (C1) and (C2). Then the operator $\Delta$ defined by the first identity in (5.3) is invertible and the Nehari problem associated with $G$ has a unique optimal solution $F \in H^\infty_{q \times p}$. Moreover, this unique solution is given by $F = G^+ - \Phi^+$, where $G^+$ is the analytic part of $G$ and $\Phi^+$ is the rational $q \times p$ matrix-valued $H^\infty$ function given by

\[
\Phi^+(\lambda) = N(\lambda)M(\lambda)^{-1}, \quad \text{where}
\]
\[
N(\lambda) = -\Pi HW^*(I_X - \lambda Z^*)^{-1}Z^*\Delta^{-1}WE,
\]
\[
M(\lambda) = I + \lambda E^*W^*(I_X - \lambda Z^*)^{-1}Z^*\Delta^{-1}WE.
\]

Furthermore, $r_{\text{spec}}(Z^*\Delta^{-1}\Xi) < 1$, and the inverse of $M(\lambda)$ is given by
\[
M(\lambda)^{-1} = I - \lambda E^*W^*(I_X - \lambda Z^*\Delta^{-1}\Xi)^{-1}Z^*\Delta^{-1}WE.
\]

Here $\Xi$ is the operator defined by the second identity in (5.3).

The fact that condition (C2) implies uniqueness of the optimal solution follows from [2]: cf., Theorem 7.5 (2) in [3]. It will be convenient first to prove the following lemma.

**Lemma 5.2.** Assume $H$ is compact and (C2) is satisfied. Then the following holds.

(i) The operator $D^\circ$ is invertible, and the range of $DS$ is closed and equal to $D$. In particular, the optimal solution to the Nehari problem is unique.

(ii) The subspace $\text{Ker } D = D^\perp$ of $H^2_p$ is cyclic for $S$.

(iii) The operators $\omega_2 = DD^\circ - S^*D$ and $\Lambda = D^\circ - S^*D$ are well-defined and strongly stable.

**Proof.** We split the proof into three parts according to the three items.

**Part 1.** We prove (i). Since $H$ is compact, the selfadjoint operator $D$ has closed range and a finite dimensional null space. Thus $D$ is a Fredholm operator of index zero. See [12, Section XI.1] for the definitions of these notions. Note
$S$ is a Fredholm operator of index $p$. Thus $DS$ is also a Fredholm operator. In particular, the range of $DS$ is closed, and hence $\mathcal{F} := DS\mathcal{H}_p^2 = DS\mathcal{H}_p^2$. Moreover,

$$\text{ind}(DS) = \text{ind}(D) + \text{ind}(S) = -p.$$ 

Here $\text{ind}$ denotes the index of a Fredholm operator, and we used the fact ([12, Theorem XI.3.2.]) that the index of a product of two Fredholm operators is the sum of the indices of the factors. On the other hand, since $\text{Ker} D \cap S\mathcal{H}_p^2$ consists of the zero vector only, we see that $\text{Ker} DS = \{0\}$, and hence, using the definition of the index, we have $p = \text{codim} DS\mathcal{H}_p^2$. But $DS\mathcal{H}_p^2 \subset D\mathcal{H}_p^2 = \mathcal{D}$ and, by the third part of (5.2), we have $\text{codim} \mathcal{D} = p$ Thus $\mathcal{F} = \mathcal{D}$. The latter implies that the central solution of (2.5) is the only optimal solution of the Nehari problem; see the remark made at the end of Section 2.

Finally, $\text{Ker} DS = \{0\}$ and $DS$ has closed range, yields $D^2 = S^*D^2S$ is invertible. This completes the proof of (i).

Part 2. We prove (ii). We begin with a remark. From (i) we know that that $D^0$ is invertible. Thus the operators $\omega_2 = DD^{o-2}S^*D$ and $\Lambda = D^{o-2}S^*D^2$ are well defined. Clearly, $\omega_2D = DA$, and hence $\omega^k_2D = DA^k$ for $k = 0, 1, 2, \ldots$. It follows that

$$\Lambda^{k+1} = D^{o-2}S^*D^2\Lambda^k = D^{o-2}S^*DA_2^k, \quad k = 0, 1, 2, \ldots \quad (5.4)$$

Since $\omega_2$ is a contraction, we conclude that $\sup_{k>0} \|\Lambda^k\| < \infty$.

Our aim is to prove that $H^2_p = \bigcap_{k=0}^{\infty} S^k\mathcal{D}^\perp$. Take $h \in H^2_p$ perpendicular to $\bigcap_{k=0}^{\infty} S^k\mathcal{D}^\perp$. The latter is equivalent to $S^kh$ being perpendicular to $\mathcal{D}^\perp$ for $k = 0, 1, 2, \ldots$, that is, $S^kh \in \mathcal{D}$ for $k = 0, 1, 2, \ldots$. Recall that the range of $D$ is closed, because $H$ is compact. Thus for each $k = 0, 1, 2, \ldots$ the vector $S^kh = Dh_k$ for some $h_k \in \mathcal{D}$. Thus $S^{k+1}h = S^kDh_k$. Since $D^0$ is invertible, Lemma 4.3 specified for the case $\mathcal{M} = H^2_p$ tells us that $P := DS^2D^2S^*D$ is the orthogonal projection of $H^2_p$ onto $\mathcal{F} = D = \text{Im} D$. Thus for $k = 0, 1, 2, \ldots$ we have

$$S^kh = Dh_k = DPH_k = D^2SD^{o-2}S^*Dh_k = D^2SD^{o-2}S^{k+1}h = \Lambda^kS^{k+1}h,$$

and by induction $h = \Lambda^kS^*h$. Since $\lim_{k \to 0} \|S^{k+1}h\| = 0$, and $\sup_{k>0} \|\Lambda^k\| < \infty$, it follows that $\|h\| = 0$. Hence $h = 0$, and we can conclude that $\bigcap_{k=0}^{\infty} S^k\mathcal{D}^\perp = H^2_p$. This proves (ii).

Part 3. We prove (iii). We already know that $\omega_2$ and $\Lambda$ are well defined. We first prove that $\omega_2$ is strongly stable, that is, $\lim_{k \to \infty} \omega^k_2v = 0$ for any $v \in H^2_p$. Note that $\omega_2DS = D$. Hence $\omega^k_2DS^k = D$ for $k = 0, 1, 2, \ldots$. Since $\mathcal{D}^\perp = \text{Ker} D$, we have for any nonnegative integers $k, l$ that $\omega^{k+l}DS^k\mathcal{D}^\perp = \omega^k_2SD^\perp = 0$. In other words, the kernel of $\omega^k_2$ includes $\mathcal{K}_k := \bigcap_{\nu=0}^{\infty} S^{\nu}\mathcal{D}^\perp$. Let $v \in H^2_p$. According to (ii), we have $\bigcap_{\nu=0}^{\infty} S^{\nu}\mathcal{D}^\perp = H^2_p$. Thus $P_{\mathcal{K}_k}v \to 0$, with $\mathcal{K}_k = H^2_p \cap \mathcal{K}_k$, and since $\omega_2$ is contractive, we find that

$$\|\omega^k_2v\| = \|\omega^k_2P_{\mathcal{K}_k}v\| \leq \|P_{\mathcal{K}_k}v\| \to 0.$$
Thus $\omega_2$ is strongly stable, as claimed, and the fact that $\Lambda$ is strongly stable follows immediately from (6.4).

**Proof of Theorem 5.1** From Lemma 5.2 (i) we know that $D^o$ is invertible, and the optimal solution is unique. Since the invertibility of $D^o$ implies the invertibility of $\Delta$, we can apply Theorem 4.1 and Corollary 4.5 with $\mathcal{M} = H^2_p$ to get the desired formula for $\Phi_+$. Note that $R_{H^2_p} = I$, and hence in this case the functions appearing in (4.16) and (4.19) are identically zero. Thus

$$\omega(T) = \omega_1(T) = \omega_2(T),$$

and the formula for its inverse follow by specifying the final part of Corollary 4.5 for the case when $\mathcal{M} = H^2_p$. 

**6 Convergence of central optimal solutions**

Throughout $G \in L^\infty_{q,p}$ and $H$ is the Hankel operator defined by the co-analytic part of $G$. We assume that conditions (C1) and (C2) formulated in the first paragraph of Section 3 are satisfied. Furthermore, $\mathcal{M}_0$ is a finite dimensional $S^*$-invariant subspace of $H^2_p$, and $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \ldots$ is a sequence of subspaces of $H^2_p$ defined recursively by (3.3). We set $P_k = P_{\mathcal{M}_k}$. From the remarks made in the paragraph preceding Theorem 3.1 one sees that

$$I - P_k = S^k(I - P_0)S^k, \quad S^*P_k = P_{k-1}S^* \quad P_k E = E. \quad (k \in \mathbb{N}). \quad (6.1)$$

Here $E$ is the embedding operator defined by (1.9).

In this section we will proof Theorem 3.1. In fact we will show that with an appropriate choice of the initial space $\mathcal{M}_0$ convergence occurs at an ever faster rate than stated in Theorem 3.1. We start with a lemma that will be of help when proving the increased rate of convergence.

**Lemma 6.1.** Let $Z$ and $W$ be the operators defined by (1.8) and (1.9), respectively, and put $\mathcal{X}_0 = WM_0^2 \subset \mathcal{X}$. Then $\mathcal{X}_0$ is $Z$-invariant of $\mathcal{X} = \text{Im} W$, and $r_{\text{spec}}(Z_0) \leq r_{\text{spec}}(Z)$. Furthermore, let the operators $Z_0 : \mathcal{X}_0 \rightarrow \mathcal{X}_0$ and $W_0 : H^2_p \rightarrow \mathcal{X}_0$ be defined by $Z_0 = Z|_{\mathcal{X}_0}$ and $W_0 = \Pi_{\mathcal{X}_0} W$, where $\Pi_{\mathcal{X}_0}$ is the orthogonal projection of $\mathcal{X}$ onto $\mathcal{X}_0$. Then

$$Z^k W(I - P_0) = \Pi_{\mathcal{X}_0} Z^k_0 W_0(I - P_0), \quad k = 0, 1, 2, \ldots \quad (6.2)$$
\textbf{Proof.} Since \( ZW = WS \) and \( \mathcal{M}_0^{\perp} \) is invariant under \( S \), we see that \( X_0 \) is invariant under \( Z \), and thus \( r_{\text{spec}}(Z_0) \leq \gamma(X_0) \). From the definition of \( Z_0 \) and \( W_0 \) we see that \( Z_\Pi X_0 = \Pi X_0 Z_0 \) and \( \Pi W_0 (I - P_0) = W(I - P_0) \). Thus
\[
Z^kW(I - P_0) = Z^k\Pi X_0 W_0 (I - P_0) = \Pi X_0 Z_0^k W_0 (I - P_0), \quad k = 0, 1, 2, \ldots
\]
This proves (6.2).

Assume \( 0 < r < 1 \) such that the poles of \( G \) inside \( \mathbb{D} \) are in the open disc \( \mathbb{D}_r \). As mentioned in the introduction, the poles of \( G \) inside \( \mathbb{D} \) coincide with the eigenvalues of \( Z \). Thus \( r_{\text{spec}}(Z) < r \). By Lemma 6.1 \( r_{\text{spec}}(Z_0) \leq r_{\text{spec}}(Z) < r \). In what follows we fix \( 0 < r_0 < 1 \) such that \( r_{\text{spec}}(Z_0) < r_0 < r \). We will show that the convergence of the central optimal solutions \( F_k \) in Theorem 3.1 is proportional to \( r_k \).

For simplicity, we will adapt the notation of Section 5, and write \( \gamma, \Delta, N \) and \( M \) instead of \( \gamma_{H^2}, \Delta_{H^2}, N_{H^2} \) and \( M_{H^2} \). Furthermore, we use the abbreviated notation \( P_k, \gamma_k, \Lambda_k, \Xi_k \) and \( \Delta_k \) for the operators \( P_{\mathcal{M}_k}, \gamma_{\mathcal{M}_k}, \Lambda_{\mathcal{M}_k}, \Xi_{\mathcal{M}_k} \), and \( \Delta_{\mathcal{M}_k} \) appearing in Section 3 for \( \mathcal{M} = \mathcal{M}_k \).

As a first step towards the proof of our convergence result we prove the following lemma.

\textbf{Lemma 6.2.} Assume conditions (C1) and (C2) are satisfied. Then \( \Delta_k \to \gamma_0 \), and for \( k \in \mathbb{N} \) large enough \( \Delta_k \) is invertible, and \( \Delta_k^{-1} \to \gamma_0^{-1} \).

\textbf{Proof.} We begin with a few remarks. Recall that for \( \mathcal{M} \) in (2.1) the operator \( R_{\mathcal{M}} \) is defined to be the orthogonal projection of \( H_\mu^2 \) onto \( S^* \mathcal{M} \); see the first paragraph of Section 2. For \( \mathcal{M} = \mathcal{M}_k \) we have \( S^* \mathcal{M}_k = \mathcal{M}_{k-1} \) by (3.1), and thus \( \mathcal{M} = \mathcal{M}_k \) implies \( R_{\mathcal{M}_k} = P_{k-1} \). It follows that the operator \( \Delta_k \) is given by \( \Delta_k = \gamma_k^2 I - ZW P_{k-1} W^* Z^* \); c.f., the second part of (4.11). From the invertibility of \( D_{\mathcal{M}_k}^* \) we obtain that \( \Delta_k \) is invertible as well; see the first paragraph of the proof Corollary 4.5. The identities in (4.13) for \( \mathcal{M} = \mathcal{M}_k \) now take the form
\[
\Lambda_k P_k W^* = P_{k-1} W^* Z^* \Delta_k^{-1} \Xi_k, \quad \Lambda_k E = -P_{k-1} W^* Z^* \Delta_k^{-1} WE. \quad (6.3)
\]
Observe that \( \gamma_k^2 = \| H P_k \|^2 = \| P_k H^* \|^2 = r_{\text{spec}}(H P_k H^*) = \| H P_k H^* \|. \) By a similar computation \( \gamma^2 = \| H H^* \|. \) Thus, using (6.1) and (6.2),
\[
|\gamma^2 - \gamma_k^2| = \| H H^* \| - \| H P_k H^* \| \leq \| H H^* - H P_k H^* \| \\
= \| H S_k (I - P_0) S_k^* H^* \| = \| Z_0^k W_0 (I - P_0) W_0^* Z_0^k \| \\
\leq \| Z_0^k \| \| H \| \| (I - P_0) \| \| H^* \| \| Z_0^k \| = \| H \| \| Z_0^k \|.
\]
It follows that \( \gamma_k^2 \to \gamma_0^2 \).

Next, we obtain
\[
\Delta_k = \gamma_k^2 I - ZW P_{k-1} W^* Z^* \\
= \gamma_k^2 I - ZW W^* Z + ZW S_k^k (I - P_0) S_k^{k-1} W^* Z^* \\
= \Delta + (\gamma_k^2 - \gamma^2) I + P_{\mathcal{M}_0} Z_0^{k-1} W_0 (I - P_0) W_0 Z_0^k P_{\mathcal{M}_0}.
\]
Clearly the second and third summand converge to zero proportional to \( r \), and thus we may conclude that \( \Delta \rightarrow r \Delta \).

Since \( \Delta \) is invertible by Theorem 6.2, the result of the previous paragraph implies that for \( k \) large enough \( \Delta_k \) is invertible and \( \| \Delta_k^{-1} \| < L \) for some \( L > 0 \) independent of \( k \). Consequently \( \Delta_k^{-1} \rightarrow r \Delta^{-1} \).

**Proof of Theorem 5.1 (with \( r \)-convergence).** We split the proof into four parts. Throughout \( k \in \mathbb{N} \) is assumed to be large enough so that \( \Delta_k \) is invertible; see Lemma 5.2.

**Part 1.** Let \( N \) and \( M \) be as in Theorem 5.1. Put

\[
N_{k,1}(\lambda) = -\Pi HW^*(I - \lambda Z^*)^{-1}Z^*\Delta_k^{-1}WE, \quad M_{k,1}(\lambda) = I + \lambda E^*W^*(I - \lambda Z^*)^{-1}Z^*\Delta_k^{-1}WE. \tag{6.4}
\]

Since the only dependence on \( k \) in \( N_{k,1} \) and \( M_{k,1} \) occurs in the form of \( \Delta_k \), it follows from Lemma 6.2 that

\[
M_{k,1} \rightarrow r \bar{\partial} M \quad \text{and} \quad N_{k,1} \rightarrow r_0 N. \tag{6.5}
\]

**Part 2.** From Corollary 4.5 we know that

\[
N_k(\lambda) = N_{k,1}(\lambda) + N_{k,2}(\lambda), \quad N_{k,2}(\lambda) = \Pi H\Gamma_k(\lambda)Z^*\Delta_k^{-1}WE, \quad M_k(\lambda) = M_{k,1}(\lambda) + M_{k,2}(\lambda), \quad M_{k,2}(\lambda) = -\lambda E^*\Gamma_k(\lambda)Z^*\Delta_k^{-1}WE. \tag{6.6}
\]

Here \( \Gamma_k(\lambda) = (I - \lambda S^*)^{-1}(I - P_{k-1})W^* \). In this part we show that \( M_{k,2} \rightarrow r_0 \).

Using the first identity in (6.1), the intertwining relation \( ZW = WS \), and (6.2) we see that

\[
\Gamma_k(\lambda) = (I - \lambda S^*)^{-1}S^{k-1}(I - P_0)S^{*k-1}W^* = (I - \lambda S^*)^{-1}S^{k-1}(I - P_0)W_0^*Z_0^{*k-1}\Pi X_0.
\]

Next we use that

\[
(I - \lambda S^*)^{-1}S^{k-1} = \sum_{j=0}^{k-2} \lambda^j S^{k-1-j} + \lambda^{k-1}(I - \lambda S^*)^{-1}.
\]

Thus \( \Gamma_k(\lambda) = \Gamma_{k,1}(\lambda) + \Gamma_{k,2}(\lambda) \), where

\[
\Gamma_{k,1}(\lambda) = \sum_{j=0}^{k-2} \lambda^j S^{k-1-j} \Pi X_0, \quad \Gamma_{k,2}(\lambda) = \lambda^{k-1}(I - \lambda S^*)^{-1}(I - P_0)W_0^*Z_0^{*k-1}\Pi X_0.
\]

Now recall that \( M_0 \) is \( S^* \)-invariant, and write \( S_0 = P_0SP_0 = P_0S \). The fact that \( M_0 \) is finite dimensional implies \( \text{r}_\text{spec}(S_0) < 1 \). The computation

\[
(I - \lambda S^*)^{-1}(I - P_0)W_0^* = (I - \lambda S^*)^{-1}W_0^* - (I - \lambda S^*)^{-1}P_0W_0^* = W_0^*(I - \lambda Z_0^*)^{-1} - (I - \lambda S_0^*)^{-1}P_0W_0^*,
\]

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shows that \((I−λS^*)^{-1}(I−P_0)W_0^*\) is uniformly bounded on \(\mathbb{D}\). Since \(r_{\text{spec}}(Z_0) < r_0 < 1\), we conclude that \(Γ_{k,2} \to r_0\).

Next observe that \(E^*(\sum_{j=0}^{k-2} \lambda^j S^{k-1-j}) = 0\), and thus \(E^*Γ_{k,1}(\lambda) = 0\) for each \(k \in \mathbb{N}\). We conclude that

\[
M_{k,2}(\lambda) = −λE^*Γ_{k,2}(\lambda)Z^*Δ_k^{-1}WE.
\]

But then \(Γ_{k,2} \to r_0\) 0 implies that the same holds true for \(M_{k,2}\), that is, \(M_{k,2} \to r_0\). Indeed, this follows from the above identity and the fact that the sequence \(Δ_k^{-1}\) is uniformly bounded.

**Part 3.** In this part we show that \(N_{k,2} \to r_0\). To do this we first observe that

\[
ΠHS^{k-1-j} = ΠV^{k-1-j}H = ΠV^{k-1-j}P_XW = ΠP_XZ^{k-1-j}W.
\]

Post-multiplying this identity with \(I−P_0\) and using (6.2) yields

\[
ΠHS^{k-1-j}(I−P_0) = ΠP_XZ_0^{k-1-j}W_0(I−P_0).
\]

It follows that

\[
N_{k,2}(\lambda) = \left(\sum_{j=0}^{k-2} \lambda^j ΠP_XZ_0^{k-1-j}\right)W_0(I−P_0)W_0^*Z_0^{k-1} + ΠHΠΓ_{k,2}(\lambda)Z^*Δ_k^{-1}WE.
\]  

(6.9)

From the previous part of the proof we know that \(Γ_{k,2} \to r_0\), and by Lemma 6.2 the sequence \(Δ_k^{-1}\) is uniformly bounded. It follows that the second term in the right hand side of (6.9) converges to zero with a rate proportional to \(r_0^k\). Note that for \(λ \in \mathbb{D}\) we have

\[
\|\sum_{j=0}^{k-2} \lambda^j ΠP_XZ_0^{k-1-j}\| \leq \sum_{j=0}^{k-2} \|Z_0\|^{k-1-j} \leq \sum_{j=1}^{∞} \|Z_0\| \leq \frac{L_0r_0}{1−r_0}.
\]

Since \(r_{\text{spec}}(Z_0) < r_0 < 1\), we also have \(\|Z_0^{k-1}\| \to r_0\). It follows that the first term in the right hand side of (6.9) converges to zero with a rate proportional to \(r_0^k\). We conclude that \(N_{k,2} \to r_0\).

**Part 4.** To complete the proof, it remains to show that \(M_{k,1}^{-1}(\lambda) \to r_0\) \(M_{k,1}^{-1}(\lambda)\) uniformly on \(\overline{\mathbb{D}}\). By similar computations as in the proof of Lemma 6.2 it follows that \(ξ_k \to r_0\) \(ξ\). Hence \(Z^*Δ_k^{-1}ξ_k \to r_0^k Z^*Δ^{-1}ξ\). By Theorem 5.1 we have \(r_{\text{spec}}(Z^*Δ_k^{-1}ξ_k) < 1\). Thus for \(k\) large enough also \(r_{\text{spec}}(Z^*Δ_k^{-1}ξ_k) < 1\), and \(M_{k,1}(\lambda)\) is invertible on \(\mathbb{D}\). From the fact that \(M_{k,1} \to r_0^k M\), we see that \(M_{k,1}^{-1} \to r_0^{-k} M^{-1}\), with \(M_{k,1}^{-1}\) and \(M^{-1}\) indicating here the functions on \(\overline{\mathbb{D}}\) with values \(M_{k,1}(\lambda)^{-1}\) and \(M(\lambda)^{-1}\) for each \(λ \in \overline{\mathbb{D}}\). In particular, the functions \(M_{k,1}^{-1}\) are uniformly bounded on \(\mathbb{D}\) by a constant independent of \(k\), which implies

\[
I + M_{k,1}^{-1}M_{k,2} \to r_0 I, \quad (I + M_{k,1}^{-1}M_{k,2})^{-1} \to r_0 I.
\]

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As a consequence

\[ M_k^{-1} = (M_{k,1} + M_{k,2})^{-1} = (I + M_{k,1}^{-1} M_{k,2})^{-1} M_{k,1}^{-1} \to_{r_0} I \cdot M^{-1} = M^{-1}, \]

which completes the proof. \(\blacksquare\)

**Concluding remarks**

Note that the functions \(M_{k,1}\) and \(N_{k,1}\) given by (6.4) and (6.5) converge with a rate proportional to \(r_0^{2k}\) rather than \(r_0^k\); cf., (6.6). Consequently the same holds true for \(M_{k,1}^{-1}\). Thus a much faster convergence may be achieved when \(N_{k,1} M_{k,1}^{-1}\) are used instead of \(N_{k} M_{k}^{-1}\). However, for the inverse of \(M_{k,1}\) to exist on \(\mathbb{D}\) we need \(k\) to be large enough to guarantee \(r_{\text{spec}}(Z^* \Delta_k \Xi_k) < 1\), and it is at present not clear how large \(k\) should be.

For the scalar case condition (C2) is rather natural. Indeed (see the second paragraph of Section 3) for the scalar case condition (C2) is equivalent to the requirement that the largest singular value of the Hankel operator is simple. The latter condition also appears in model reduction problems. In the matrix-valued case (C2) seems rather special. We expect that a version of Theorem 3.1 can be proved by only using the first part of (C2), that is, by assuming that none of the maximizing vectors of the Hankel operator belongs to \(\mathcal{SH}_2\); cf., Proposition 4.4. However, note that in that case the optimal solution of the Nehari problem may not be unique.

Computational examples show that it may happen that the approximations of the optimal solution to the Nehari problem considered in this paper oscillate to the optimal solution when the initial space \(\mathcal{M}_0 = \{0\}\). Although the rate of convergence can be improved considerably by choosing a different initial space \(\mathcal{M}_0\), the same examples show that the approximations still oscillate in much the same way as before to the optimal solution. This suggests that approximating the optimal solution may not be practical in some problems. In this case, one may have to adjust these approximating optimal solutions. We plan to return to this phenomenon in a later paper.

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