REVISITING RIESZ TRANSFORMS FOR HERMITE AND SPECIAL HERMITE OPERATORS

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Abstract. In this paper we prove weighted mixed norm estimates for Riesz transforms associated to Hermite and special Hermite operators. The estimates are shown to be equivalent to vectorvalued estimates for a sequence of operators defined in terms of Laguerre functions of different type.

1. Introduction

The boundedness of Riesz transforms associated to Hermite and special Hermite operators have been studied by several authors, see [18] and [17]. Here in this article we are interested in mixed norm estimates. To fix the notation let $H = -\Delta + |x|^2$ be the Hermite operator on $\mathbb{R}^d$ which can be written as $H = \frac{1}{2} \sum_{j=1}^{d} (A_j A_j^* + A_j^* A_j)$, where $A_j = \frac{\partial}{\partial x_j} + x_j$ and $A_j^* = -\frac{\partial}{\partial x_j} + x_j$ are the annihilation and creation operators. The Riesz transforms associated to the Hermite operator are defined by $R_j f = A_j H^{-\frac{1}{2}}$ and $R_j^* = A_j^* H^{-\frac{1}{2}}$, $j = 1, 2, \ldots, d$. It is well known that they are bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$ and weak type $(1, 1)$.

Given a weight function $w(r)$ defined on $\mathbb{R}^+ = (0, \infty)$ we consider the mixed norm space $L^{p,2}(\mathbb{R}^d, w)$ which consists of all measurable functions $f$ on $\mathbb{R}^d = \mathbb{R}^+ \times S^{d-1}$ for which

$$\|f\|_{L^{p,2}(\mathbb{R}^d, w)} = \int_0^\infty \left( \int_{S^{d-1}} |f(r\omega)|^2 d\omega \right)^{\frac{p}{2}} w(r)r^{d-1} dr$$

are finite. One of our main results in this paper is the following.

Theorem 1.1. For $j = 1, 2, \ldots, d$ we have

$$\|R_j f\|_{L^{p,2}(\mathbb{R}^d, w)} \leq C\|f\|_{L^{p,2}(\mathbb{R}^d, w)}$$

for all $1 < p < \infty$, and $w \in A^{d-1}_p(\mathbb{R}^+)$. Similar estimates are valid for $R_j^*$ also.

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In the above theorem $A_p^\alpha(\mathbb{R}^+)$ is the Muckenhoupt’s $A_p^\alpha$ class defined on $\mathbb{R}^+$ for $\alpha \geq -\frac{1}{2}$ with respect to the measure $d\mu(r) = r^{2\alpha+1}dr$. The above result leads to a vector valued inequality for a sequence of Laguerre Riesz transforms, see Theorem 2.6.

We also consider the Riesz transforms $S_j$ and $\overline{S}_j$ associated to the special Hermite operator $L = -2 \sum_{j=1}^{d}(Z_j \overline{Z}_j + \overline{Z}_j Z_j)$ where $Z_j = \frac{\partial}{\partial z_j} + \frac{1}{4} z_j$ and $\overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - \frac{1}{4} \overline{z}_j$. Again using spectral theorem we define $S_j = Z_j L^{-\frac{1}{2}}$ and $\overline{S}_j = \overline{Z}_j L^{-\frac{1}{2}}$. For these operators we prove

**Theorem 1.2.** For $j = 1, 2, \ldots, d$ we have

$$\|S_j f\|_{L^p,2(C^d,w)} \leq C \|f\|_{L^p,2(C^d,w)}$$

for all $1 < p < \infty$, and $w \in A^d_{p-1}(\mathbb{R}^+)$. Similar estimates are valid for $\overline{S}_j$ also.

In this article we give three different proofs of Theorem 1.1. In a forthcoming paper [1] we prove mixed norm estimates for Riesz transforms associated to Dunkl harmonic oscillator. Of the three proofs mentioned above, only one proof can be modified to treat the case of Dunkl harmonic oscillator.

2. **Riesz transforms for the Hermite Operator**

2.1. **Mixed norm estimates for Hermite Riesz transforms.** We first recall some results on Hermite expansions required for the proof of Theorem 1.1. If $e^{-tH}$ is the Hermite semigroup generated by $H$, then it is well known that

$$e^{-tH} f(x) = \int_{\mathbb{R}^d} K_t(x,y) f(y) dy$$

where the kernel is explicitly given by

$$K_t(x,y) = (2\pi)^{-d} (\sinh 2t)^{-\frac{d}{2}} e^{-\frac{1}{4} t^2} e^{-\frac{1}{2} (\coth t)|x-y|^2} e^{-\frac{1}{4} (\tanh t)|x+y|^2}.$$

The operator $H^{-\frac{1}{2}}$ is then defined by

$$H^{-\frac{1}{2}} f(x) = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-tH} f(x) t^{-\frac{1}{2}} dt$$

and consequently the Riesz transforms $R_j$ are given by

$$R_j f(x) = \int_{\mathbb{R}^d} R_j(x,y) f(y) dy$$

where

$$(2.1) \quad R_j(x,y) = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \left( \frac{\partial}{\partial x_j} + x_j \right) K_t(x,y) t^{-\frac{1}{2}} dt.$$

The following estimates on the kernel $R_j$ are known (see Theorem 3.3 in [17]).
Proposition 2.1. For each \( j = 1, 2, \ldots, d; \ x, y \in \mathbb{R}^d \) and \( x \neq y \), there exist constants \( C_j \) and \( C'_j \) such that

\[
\begin{align*}
(1) \ |R_j(x, y)| & \leq C_j |x - y|^{-d} \\
(2) \ |\nabla_x R_j(x, y)| & \leq C'_j |x - y|^{-d-1}.
\end{align*}
\]

As \( R_j \) are clearly bounded on \( L^2(\mathbb{R}^d) \), the above estimates lead to the boundedness of \( R_j \) on \( L^p(\mathbb{R}^d), \ 1 < p < \infty \) and also the weak type \((1, 1)\) estimate. As we are interested in mixed norm estimates, we consider \( L^{p, 2}(\mathbb{R}^d) \) as \( L^p(\mathbb{R}^+, L^2(\mathbb{S}^{d-1})) \) i.e, as the \( L^p \) space on \( \mathbb{R}^+ \) taken with respect to the measure \( d\mu_{\frac{d}{2} - 1} \) of functions taking values in the Hilbert space \( \mathcal{H} = L^2(\mathbb{S}^{d-1}) \).

Thus given \( f \in L^{p, 2}(\mathbb{R}^d) \) and \( s \in \mathbb{R}^+ \) we consider \( f(s) \) defined on \( \mathbb{S}^{d-1} \) by \( f(s)(\omega) = f(s\omega) \) as an element of \( \mathcal{H} \). Note that

\[
\|f\|_{L^{p, 2}(\mathbb{R}^d)} = \left( \int_0^\infty \|f(s)\|_{L^2(\mathbb{S}^{d-1})}^p d\mu_{\frac{d}{2} - 1}(s) \right)^{\frac{1}{p}}.
\]

For \( r, s \in \mathbb{R}^+, \ r \neq s \) consider the linear operator \( R_j(r, s) : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1}) \) defined by

\[
R_j(r, s)\varphi(\omega) = \int_{\mathbb{S}^{d-1}} R_j(r\omega, s\omega')\varphi(\omega')d\omega'.
\]

It is then clear that \( R_j(r, s) \) is a bounded operator on \( L^p(\mathbb{S}^{d-1}) \) and

\[
\|R_j(r, s)\|_{Op} \leq \sup_{\omega} \int_{\mathbb{S}^{d-1}} |R_j(r\omega, s\omega')|d\omega'.
\]

We can now view of \( R_j \) as an integral operator defined on \( L^p(\mathbb{R}^+, \mathcal{H}) \). Indeed, we see that

\[
(2.2) \quad R_j f(r\omega) = \int_0^\infty (R_j(r, s)f(s))(\omega)s^{d-1}ds
\]

and hence we will study \( R_j \) by means of singular integral operators on \( L^p(\mathbb{R}^+, \mathcal{H}) \).

For any \( \alpha \geq -\frac{1}{2} \) we consider \( \mathbb{R}^+ \) equipped with the measure \( d\mu_\alpha(r) = r^{2\alpha + 1}dr \) as a homogeneous space. Let \( B(a, b) \) stand for the ball of radius \( b > 0 \) centered at \( a \in \mathbb{R}^+ \). We say that a non negative and locally integrable function \( w \) is in \( A^p_\alpha(\mathbb{R}^+) \), \( 1 < p < \infty \) if \( w \) satisfies

\[
\left( \frac{1}{\mu_\alpha(Q)} \int_Q w(r)d\mu_\alpha \right) \left( \frac{1}{\mu_\alpha(Q)} \int_Q w(r)^{-\frac{p}{p'}} d\mu_\alpha \right)^{p-1} \leq C
\]

for all intervals \( Q \subset \mathbb{R}^+ \). We consider a singular integral operator \( T \) acting on vector valued (\( \mathcal{H}\)-valued ) functions on \( \mathbb{R}^+ \) associated to an operator
valued kernel $K$ which is defined on $\mathbb{R}^+ \times \mathbb{R}^+$ and taking values in $\mathcal{L}(\mathcal{H}, \mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$. That is to say

$$Tf(r) = \int_0^\infty K(r,s)f(s)d\mu_\alpha(s)$$

for all bounded and compactly supported vector valued functions $f$ and $r \notin \text{suppf}$. The following theorem tells us the boundedness of $T$ under some conditions on $K$.

**Theorem 2.2.** Suppose that $T$ defined as above is bounded on $L^2(\mathbb{R}^+, \mathcal{H})$.

Assume that $K$ satisfies the following conditions

1. $\|K(r,s)\|_{op} \leq C_1 \left( \mu_\alpha(B(r,|r-s|)) \right)^{-1}$ and
2. $\|\partial_r K(r,s)\|_{op} \leq C_2 |r-s|^{-1} \left( \mu_\alpha(B(r,|r-s|)) \right)^{-1}$ for $r \neq s$.

Then $T$ is bounded on $L^p(\mathbb{R}^+, \mathcal{H})$, $1 < p < \infty$ and weak type $(1, 1)$. More generally, if $w \in A^p_\alpha(\mathbb{R}^+)$, then we have

$$\|Tf\|_{L^p(\mathbb{R}^+, \mathcal{H}, w d\mu_\alpha)} \leq C \|f\|_{L^p(\mathbb{R}^+, \mathcal{H}, w d\mu_\alpha)}$$

for $1 < p < \infty$.

**Proof.** By an easy calculation using the hypothesis on $K$, we can show that there exists a positive constant $C$ such that for all $s, t \in \mathbb{R}^+$

$$\int_{|r-s| > 2|s-t|} \|K(r,s) - K(r,t)\|_{op} d\mu_\alpha(r) \leq C$$

and

$$\int_{|r-s| > 2|s-t|} \|K(s,r) - K(t,r)\|_{op} d\mu_\alpha(r) \leq C.$$ 

For the un-weighted case, the boundedness of $T$ is proved in Theorem 1.1 in [9] and for the weighted case, the boundedness of $T$ can be proved by imitating the proof of Theorem 7.11, page no. 144 in [6] by changing the notations and notions (definitions) into our setting. \hfill \Box

As we have observed, the Riesz transforms can be considered as operators defined on $L^2(\mathbb{R}^+, \mathcal{H}; \mu_{\frac{d}{2}-1})$, $\mathcal{H} = L^2(S^{d-1})$:

$$R_jf(r) = \int_0^\infty R_j(r,s)f(s)d\mu_{\frac{d}{2}-1}(s).$$

Then $R_j$ is clearly bounded on $L^2(\mathbb{R}^+, \mathcal{H}; \mu_{\frac{d}{2}-1})$. So, in order to prove Theorem 1.1, all we need is the following proposition giving estimates on the operator valued kernels $R_j(r,s)$.

**Proposition 2.3.** For any $r \neq s$ we have

1. $\|R_j(r,s)\|_{op} \leq C_1 \left( \mu_{\frac{d}{2}-1}(B(r,|r-s|)) \right)^{-1}$
2. $\|\partial_r R_j(r,s)\|_{op} \leq C_2 |r-s|^{-1} \left( \mu_{\frac{d}{2}-1}(B(r,|r-s|)) \right)^{-1}$


Proof. As we have already observed
\[ \|R_j(r, s)\|_{Op} \leq \sup_{\omega} \int_{S^{d-1}} |R_j(r\omega, s\omega')|d\omega'. \]
Using the estimate (1) of Proposition 2.1, we get
\[ \|R_j(r, s)\|_{Op} \leq C \sup_{\omega} \int_{S^{d-1}} |r\omega - s\omega'|^{-d}d\omega'. \]
(2.3)
Integrating in polar coordinates, the last integral is a constant multiple of
\[ \int_0^\pi (r^2 + s^2 - 2rs\cos \theta)^{-\frac{d}{2}}(\sin \theta)^{d-2}d\theta \]
which is bounded by
\[ c \int_0^1 (r^2 + s^2 - 2rsu)^{-\frac{d}{2}}(1 - u)^{\frac{d-3}{2}}du. \]
In order to estimate this we make use of the following lemma (see Lemma 5.3 in [3])

**Lemma 2.4.** Let \( c \geq \frac{1}{2}, 0 < B < A \) and \( \lambda > 0 \). Then
\[ \int_0^1 \frac{(1 - u)^{c-\frac{1}{2}}}{(A - Bu)^{c + \lambda + \frac{d}{2}}}du \leq \frac{C}{A^{c + \frac{d}{2}}(A - B)^{\lambda}}. \]

Appealing to this lemma with \( A = r^2 + s^2, B = 2rs, c = \frac{d}{2} - 1 \) and \( \lambda = \frac{1}{2} \) we obtain
\[ \|R_j(r, s)\|_{Op} \leq C|r - s|^{-1}(r^2 + s^2)^{-\left(\frac{d+1}{2}\right)}. \]
The desired estimate follows as \( |r - s|(r^2 + s^2)^{-\frac{d-1}{2}} \) is comparable to \( \mu_{\frac{d}{2} - 1}(B(r, |r - s|)) \). In order to get the estimate on the derivative we note that
\[ \frac{\partial}{\partial r} R_j(r, s)\varphi(\omega) = \frac{\partial}{\partial r} \int_{S^{d-1}} R_j(r\omega, s\omega')\varphi(\omega')d\omega' \]
\[ = \sum_{i=1}^d \int_{S^{d-1}} \frac{\partial}{\partial x_i} R_j(r\omega, s\omega')\varphi(\omega')\omega_i d\omega' \]
The estimate (2) of Proposition 2.1 can be used to bound the operator norm of \( \frac{\partial}{\partial r} R_j(r, s) \). This leads to
\[ \|\frac{\partial}{\partial r} R_j(r, s)\|_{Op} \leq C \sup_{\omega} \int_{S^{d-1}} |r\omega - s\omega'|^{-d-1}d\omega'. \]
As before, using Lemma 2.4 we get the desired estimate. \( \square \)

This completes the proof of Theorem 1.1.
2.2. Another proof of Theorem 1.1: In this subsection we give another proof of Theorem 1.1 following an idea of Rubio de Francia. This method described briefly in [15] is based on an extension of a theorem of Marcinkiewicz and Zygmund as expounded in Herz and Riviere [10]. Indeed, we make use of the following Lemma which can be found in [10].

Lemma 2.5. Let \((G, \mu)\) and \((H, \nu)\) be arbitrary measure spaces and \(T : L^p(G) \to L^p(G)\) a bounded linear operator. Then if \(p \leq q \leq 2\) or \(p \geq q \geq 2\), there exists a bounded linear operator \(\tilde{T} : L^p(G; L^q(H)) \to L^p(G; L^q(H))\) with \(\|\tilde{T}\| \leq \|T\|\) such that for \(g \in L^p(G; L^q(H))\) of the form \(g(x, \xi) = f(\xi)u(x)\) where \(f \in L^p(G)\) and \(u \in L^q(H)\) we have

\[
(\tilde{T}g)(\xi, x) = (Tf)(\xi)u(x).
\]

The idea of Rubio de Francia is as follows (we are indebted to Gustavo Garrigos for bringing this to our attention). Suppose \(T : L^p(\mathbb{R}^d, dx) \to L^p(\mathbb{R}^d, dx)\) is a bounded linear operator. Then by the lemma of Herz and Riviere, it has an extension \(\tilde{T}\) to \(\mathcal{H}\) valued functions on \(\mathbb{R}^d\) where \(\mathcal{H}\) is the Hilbert space \(L^2(K)\), \(K = SO(d)\). Moreover, the extension satisfies

\[
(\tilde{T}f)(x, k) = Tg(x)h(k) \text{ if } f(x, k) = g(x)h(k), \ x \in \mathbb{R}^d, \ k \in SO(d).
\]

Given \(f \in L^p(\mathbb{R}^d, dx)\) consider \(\hat{f}(x, k) = f(k.x)\). Then

\[
\int_{\mathbb{R}^d} (\int_{K} f(k, x, k) |dk|)^{\frac{1}{2}} dx \text{ can be calculated as follows. If } x = r\omega, \ \omega \in S^{d-1}, \ \hat{f}(x, k) = f(rk \cdot \omega) \text{ and hence}
\]

\[
\int_{K} \hat{f}(x, k)^2 dk = \int_{K/\omega} \left( \int_{K/\omega} |f(rk \cdot \omega)|^2 d\mu \right) d\nu
\]

where \(K_\omega = \{k \in K : k \cdot \omega = \omega\}\) is the isotropy subgroup of \(K\), \(d\nu\) is the Haar measure on \(K_\omega\) and \(d\mu\) is the \(K_\omega\) invariant measure on \(K/K_\omega\) which can be identified with \(S^{d-1}\). Hence

\[
\int_{K} \hat{f}(x, k)^2 dk = c \int_{S^{d-1}} |f(r\omega)|^2 d\sigma(\omega).
\]

Therefore,

\[
\int_{\mathbb{R}^d} \left( \int_{K} \hat{f}(x, k)^2 dk \right)^{\frac{1}{2}} dx = c' \int_{0}^{\infty} \left( \int_{S^{d-1}} |f(r\omega)|^2 r^{d-1} d\sigma(\omega) \right)^{\frac{1}{2}} dr.
\]

Let us define \(\rho(k)f(x) = f(k \cdot x)\) so that \(\hat{f}(x, k) = \rho(k)f(x)\). If \(T\) commutes with rotation i.e. \(T\rho(k) = \rho(k)T\) then

\[
\tilde{T}\hat{f}(x, k) = T(\rho(k)f)(x) = \rho(k)(Tf)(x) = (Tf)(k \cdot x).
\]

The boundedness of \(\tilde{T}\) on \(L^p(\mathbb{R}^d, \mathcal{H})\) gives

\[
\int_{\mathbb{R}^d} \left( \int_{K} |Tf(k \cdot x)|^2 dk \right)^{\frac{1}{2}} dx \leq C \int_{\mathbb{R}^d} \left( \int_{K} |f(k \cdot x)|^2 dk \right)^{\frac{1}{2}} dx
\]

which translates into the mixed norm estimate for \(T\).

Given a unit vector \(u \in S^{d-1}\) let us consider the operator \(T_uf = \sum_{j=1}^{d} u_j R_j f(x)\) where \(R_j = A_j H^{-\frac{d}{2}}\) are the Hermite Riesz transforms. This operator \(T_u\) is
not rotation invariant but has a nice transformation property under the action of $SO(d)$. Indeed,

$$T_u f(x) = (x \cdot u + u \cdot \nabla) H^{-\frac{1}{2}} f(x)$$

and as $H^{-\frac{1}{2}}$ commutes with $\rho(k)$ it follows that

$$T_u \rho(k) f = \rho(k) T_k u f$$
or

$$T_{k^{-1}u} \rho(k) f = \rho(k) T_u f.$$

This leads us to

$$T_u f(k \cdot x) = \sum_{j=1}^{d} (k^{-1} \cdot u)_j R_j (\rho(k) f)(x).$$

We make use of this in proving Theorem 1.1.

The operator $R_j$ are singular integral operators and hence bounded on $L^p(\mathbb{R}^d, wdx)$ for any weight function $w \in A_p(\mathbb{R}^d)$, $1 < p < \infty$. By the lemma of Herz and Riviere, $R_j$ extends as a bounded operator on $L^p(\mathbb{R}^d, H; wdx)$ where $H = L^2(S^{d-1})$. When $w$ is radial, it can be easily checked that

$$\int_{\mathbb{R}^d} \left( \int_{\mathcal{K}} |\rho(k) f(x)|^2 dk \right)^{\frac{p}{2}} w(x) dx \leq c \int_{0}^{\infty} \left( \int_{S^{d-1}} |f(r\omega)|^2 d\sigma(\omega) \right)^{\frac{p}{2}} w(r) r^{d-1} dr$$

Moreover, by the result of Duoandikoetxea (Theorem 3.2 in [7]), any radial $w \in A_p(\mathbb{R}^d)$ if and only if $w(r) \in A^{\frac{d}{p}-1}(\mathbb{R}^+)$. From the identity

$$T_u f(k \cdot x) = \sum_{j=1}^{d} (k^{-1} \cdot u)_j R_j (\rho(k) f)(x)$$

we obtain

$$\left( \int_{\mathbb{R}^d} \left( \int_{\mathcal{K}} |T_u f(k \cdot x)|^2 dk \right)^{\frac{p}{2}} w(x) dx \right)^{\frac{1}{p}} \leq c \sum_{j=1}^{d} \left( \int_{\mathbb{R}^d} \left( \int_{\mathcal{K}} |\rho(k) f(x)|^2 dk \right)^{\frac{p}{2}} w(x) dx \right)^{\frac{1}{p}}$$

which translates into the required inequality of Theorem 1.1 by taking $u$ to be coordinate vectors.

2.3. Laguerre Riesz transforms. For each $\alpha \geq -\frac{1}{2}$ we consider the Laguerre differential operator

$$L_\alpha = -\frac{d^2}{dr^2} + r^2 - \frac{2\alpha + 1}{r} \frac{d}{dr}$$
whose normalized eigenfunctions are given by
\[ \psi_\alpha^k(r) = \left( \frac{2\Gamma(k + 1)}{\Gamma(k + \alpha + 1)} \right)^{1/2} L_\alpha^k(r^2) e^{-\frac{1}{2}r^2} \]
where \( L_\alpha^k(r) \) are Laguerre polynomials of type \( \alpha \). These functions form an orthonormal basis for \( L^2(\mathbb{R}^+, d\mu_\alpha) \). The operator \( L_\alpha \) generates the semi-group \( T_\alpha^t = e^{-tL_\alpha} \) whose kernel is given by
\[ K_\alpha^t(r, s) = \sum_{k=0}^{\infty} e^{-(4k+2\alpha+2)t} \psi_\alpha^k(r) \psi_\alpha^k(s). \]
(2.9)

The generating function identity (1.1.47 in [18]) for Laguerre functions gives the explicit expression
\[ K_\alpha^t(r, s) = (\sinh 2t)^{-1} e^{-\frac{1}{2}(\coth 2t)(r^2+s^2)} (rs)^{-\alpha} I_\alpha \left( \frac{rs}{\sinh 2t} \right) \]
where \( I_\alpha = e^{-\frac{\pi}{2}i\alpha} J_\alpha(iz) \) is the modified Bessel function.

The Laguerre Riesz transforms \( R_\alpha = (\frac{\partial}{\partial r} + r)^{-\frac{1}{2}} \) have been studied in [13] and it is known that they are bounded on \( L^p(\mathbb{R}^+, d\mu_\alpha) \), \( 1 < p < \infty \). Here we are interested in a vector valued inequality for the sequence of Riesz transforms \( R_{\alpha+m} \) where \( \alpha = \frac{d}{2} - 1 \).

Theorem 2.6. Let \( d \geq 2 \) and \( \alpha = \frac{d}{2} - 1 \). Then for any \( 1 < p < \infty \), \( w \in A_p(\mathbb{R}^+) \) we have
\[ \int_0^\infty \left( \sum_{m=0}^{\infty} r^{2m} |R_{\alpha+m} f_m(r)|^2 \right)^{\frac{p}{2}} w(r) d\mu_{\frac{d}{2}-1}(r) \leq C \int_0^\infty \left( \sum_{m=0}^{\infty} |f_m(r)|^2 \right)^{\frac{p}{2}} w(r) d\mu_{\frac{d}{2}-1}(r) \]
where \( f_m \in L^p(\mathbb{R}^+, d\mu_\alpha) \), \( \tilde{f}_m(r) = r^{-m} f_m(r) \).

Before proving this theorem we remark that in [2] where the authors have studied the boundedness of Hermite Riesz transforms in terms of polar coordinates, the above theorem has been proved first from which our main theorem can be deduced. Here we have already proved Theorem 1.1 and now we will show how the above result can be deduced.

Coming to the proof of the above theorem, consider the vector of the Riesz transforms
\[ Rf(x) = (R_1 f(x), R_2 f(x), \cdots, R_d f(x)). \]
If \( \nabla \) stand for the gradient then it follows that \( Rf(x) = (x + \nabla) H^{-\frac{1}{2}} f(x) \). Let \( \mathcal{H}_m \) stand for the space of spherical harmonics of degree \( m \) of dimension
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d(m). Fix an orthonormal basis \( Y_{m,j}, j = 1, 2, \ldots, d(m) \) for \( \mathcal{H}_m \) consisting of real valued spherical harmonics. Any \( f \in L^{p,2}(\mathbb{R}^d) \) has an expansion

\[
f(r\omega) = \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} f_{m,j}(r) Y_{m,j}(\omega).
\]

In view of the Hecke - Bochner formula (Theorem 3.4.1 in [18]) for the Hermite projections, the operator \( H^{-\frac{1}{2}} \) preserves each \( \mathcal{H}_m \) and consequently

\[
H^{-\frac{1}{2}} f(r\omega) = \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} F_{m,j}(r) Y_{m,j}(\omega).
\]

Here \( F_{m,j}(r) \) are given by

\[
F_{m,j}(r) = \int_{S^{d-1}} H^{-\frac{1}{2}} f(r\omega) Y_{m,j}(\omega) d\omega.
\]

The gradient \( \nabla \) can be split into radial and angular parts as follows:

\[
\nabla = \frac{1}{r} \nabla_0 + \nabla_0 \frac{\partial}{\partial r} \text{ if } x = r\omega, \omega \in S^{d-1} \text{ is the polar decomposition. Here } \nabla_0 \text{ acts in the } \omega - \text{variable, see [4]. In view this formula we see that}
\]

\[
(x + \nabla)(F_{m,j}(r) Y_{m,j}(\omega)) = \left(r + \frac{\partial}{\partial r}\right) F_{m,j}(r) Y_{m,j}(\omega) \omega + \frac{1}{r} F_{m,j}(r) \nabla_0 Y_{m,j}(\omega).
\]

Thus we see that

\[
Rf(r\omega) = \left(\sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} \left(r + \frac{\partial}{\partial r}\right) F_{m,j}(r) Y_{m,j}(\omega)\right) \omega + \frac{1}{r} \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} F_{m,j}(r) U_{m,j}(\omega)
\]

where \( U_{m,j}(\omega) = (U_{m,j}^1(\omega), \ldots, U_{m,j}^d(\omega)) = \nabla_0 Y_{m,j}(\omega) \). We make use of the following facts: (i) \( \omega.\nabla_0 Y_{m,j} = 0 \) and (ii) \( \int_{S^{d-1}} \nabla_0 Y_{m,j} \cdot \nabla_0 Y_{m',j'} d\omega = m(m + d - 2) \delta_{m,m'} \delta_{j,j'} \). For a proof of these facts see Lemma 2.2 in [14]. Therefore, \(|Rf(r\omega)|^2 = \sum_{j=1}^{d} |R_j f(r\omega)|^2 \) is given by the sum of

\[
\left| \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} \left(r + \frac{\partial}{\partial r}\right) F_{m,j}(r) Y_{m,j}(\omega) \right|^2
\]

and

\[
\frac{1}{r^2} \sum_{k=0}^{d} \left| \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} F_{m,j}(r) U_{m,j}^k(\omega) \right|^2
\]

where we have used the facts (i). Integrating over \( S^{d-1} \) and using the fact (ii) we see that

\[
\int_{S^{d-1}} \left( \sum_{j=1}^{d} |R_j f(r\omega)|^2 \right) d\omega
\]
is given by the sum of
\[ \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} \left| \left( r + \frac{\partial}{\partial r} \right) F_{m,j}(r) \right|^2 \]
and
\[ \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} \frac{1}{r^2} m(m + d - 2) |F_{m,j}(r)|^2. \]

Therefore, our main theorem gives the following two inequalities:

\[
\int_0^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} \left| \left( r + \frac{\partial}{\partial r} \right) F_{m,j}(r) \right|^2 \right)^{\frac{p}{2}} w(r) r^{d-1} dr \\
\leq C \int_0^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |f_{m,j}(r)|^2 \right)^{\frac{p}{2}} w(r) r^{d-1} dr \tag{2.11}
\]

and

\[
\int_0^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} \frac{1}{r^2} m(m + d - 2) |F_{m,j}(r)|^2 \right)^{\frac{p}{2}} w(r) r^{d-1} dr \\
\leq C \int_0^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |f_{m,j}(r)|^2 \right)^{\frac{p}{2}} w(r) r^{d-1} dr. \tag{2.12}
\]

We will now restate the first of the above inequality in terms of Laguerre-Riesz transforms which will immediately prove Theorem 2.6.

Appealing to Hecke - Bochner formula for the Hermite projections it is easy to see that

\[
e^{-tH}(f_{m,j}Y_{m,j})(r\omega) = c_d r^{m} Y_{m,j}(\omega) T_t^{\alpha+m} \tilde{f}_{m,j}(r) \tag{2.13}
\]

where \( \tilde{f}_{m,j}(r) = r^{-m} f_{m,j}(r) \), \( \alpha = \frac{d}{2} - 1 \) and \( T_t^{\alpha+m} = e^{-tL_{\alpha+m}} \). For a different proof of this see Proposition 3.2 in [11]. Consequently,

\[
F_{m,j}(r) = c_d r^{m} \frac{1}{\sqrt{\pi}} \int_0^{\infty} T_t^{\alpha+m} \tilde{f}_{m,j}(r)t^{-\frac{1}{2}} dt \\
= c_d r^{m} L_{\alpha+m}^{-\frac{1}{2}} \tilde{f}_{m,j}(r).
\]

This leads to the conclusion that

\[
\left( r + \frac{\partial}{\partial r} \right) F_{m,j}(r) = c_d r^{m} R^{\alpha+m} \tilde{f}_{m,j}(r) + c_d mr^{m-1} L_{\alpha+m}^{\frac{1}{2}} \tilde{f}_{m,j}(r).
\]
As \( mr^{m-1}L_{\alpha+m}^{-\frac{1}{2}} \tilde{f}_{m,j}(r) = \frac{m}{r} F_{m,j}(r) \), from the main theorem we get the inequality
\[
\int_0^\infty \left( \sum_{m=0}^\infty \sum_{j=1}^d r^{2m} |L_{\alpha+m}^{-\frac{1}{2}} \tilde{f}_{m,j}(r)|^2 \right)^{\frac{p}{2}} w(r)r^{d-1}dr \\
\leq C \int_0^\infty \left( \sum_{m=0}^\infty \sum_{j=1}^d |f_{m,j}(r)|^2 \right)^{\frac{p}{2}} w(r)r^{d-1}dr.
\]
This completes the proof of Theorem 2.6.

Remark 2.7. We also have the inequality
\[
\int_0^\infty \left( \sum_{m=0}^\infty \sum_{j=1}^d m^2 r^{2m-2} |L_{\alpha+m}^{-\frac{1}{2}} \tilde{f}_{m,j}(r)|^2 \right)^{\frac{p}{2}} w(r)r^{d-1}dr \\
\leq C \int_0^\infty \left( \sum_{m=0}^\infty \sum_{j=1}^d |f_{m,j}(r)|^2 \right)^{\frac{p}{2}} w(r)r^{d-1}dr.
\]
which has been proved in [2] by first establishing uniform estimates for the kernels of the operators taking \( f_{m,j} \) into \( \frac{m}{r}L_{\alpha+m}^{-\frac{1}{2}} \tilde{f}_{m,j}(r) \).

2.4. Another proof of the inequality (2.11). In this subsection we give a different proof of the inequality
\[
\int_0^\infty \left( \sum_{m=0}^\infty \sum_{j=1}^d \left| \left( \frac{\partial}{\partial r} + r \right) F_{m,j}(r) \right|^2 \right)^{\frac{p}{2}} w(r)r^{d-1}dr \\
\leq C \int_0^\infty \left( \sum_{m=0}^\infty \sum_{j=1}^d |f_{m,j}(r)|^2 \right)^{\frac{p}{2}} w(r)r^{d-1}dr.
\]
Let us set \( T_t = e^{-tH} \) which is an integral operator with kernel \( K_t(x,y) \). Note that \( K_t \) depends only on \( |x|, |y| \) and \( x \cdot y \). We write
\[
K_t(r, s, u) = c_d (\sinh 2t)^{-\frac{1}{4}} e^{-\frac{1}{4} (\coth 2t)(r^2 + s^2) + \frac{1}{2} (\csc 2t) rs u}
\]
so that \( K_t(x, y) = K_t(r, s, u) \) with \( r = |x|, s = |y| \) and \( u = x' \cdot y' \). With this notation
\[
F_{m,j}(r) = \int_{S^{d-1}} \int_0^\infty T_t f(r\omega) Y_{m,j}(\omega) t^{-\frac{1}{2}} dt d\omega.
\]
Let \( P_m^{\frac{d}{2} - 1} (u) \) stand for the ultraspherical polynomial normalised so that \( P_m^{\frac{d}{2} - 1}(1) = 1 \). Recall that Funk - Hecke formula gives (See [5])
\[
\int_{S^{d-1}} \varphi(x' \cdot y') Y_{m,j}(y') dy' = Y_{m,j}(x') \int_{-1}^1 \varphi(u) P_m^{\frac{d}{2} - 1}(u)(1 - u^2)^{\frac{d-3}{2}} du.
\]
Applying this formula we see that

\[
\int_{S^{d-1}} T_t f(rx')Y_{m,j}(x')dx' \\
= \int_0^\infty \int_{S^{d-1}} f(sy')(\int_{S^{d-1}} K_t(rx', sy')Y_{m,j}(x')dx')s^{d-1}dy's\,ds \\
= \int_0^\infty \left( \int_{-1}^1 K_t(r, s, u)P_m^\frac{d}{2} - 1(u)(1 - u^2)\frac{d-3}{2} \,du \right) f_{m,j}(s)s^{d-1}ds.
\]

On the other hand we also have

\[
\int_{S^{d-1}} K_t(x,y)P_m^\frac{d}{2} - 1(x' \cdot y')dy' = \int_{-1}^1 K_t(r, s, u)P_m^\frac{d}{2} - 1(u)(1 - u^2)^{\frac{d-3}{2}} \,du.
\]

Consequently,

\[
\int_{S^{d-1}} T_t f(rx')Y_{m,j}(x')dx' \\
= \int_0^\infty \left( \int_{S^{d-1}} K_t(x,y)P_m^\frac{d}{2} - 1(x' \cdot y')f_{m,j}(s) \right)dy's^{d-1}ds.
\]

If we define \(K_{t,m}(x,y) = K_t(x,y)P_m^\frac{d}{2} - 1(x' \cdot y')\) then we have

\[
F_{m,j}(r) = \int_0^\infty K_{m}(r,s)f_{m,j}(s)s^{d-1}ds
\]

where

\[
K_{m}(r,s) = \int_0^\infty \left( \int_{S^{d-1}} K_{t,m}(x,y)dy' \right) t^{-\frac{1}{2}} \,dt.
\]

We will use this expression to estimate the kernel \(K_m\) and their derivatives.

**Proposition 2.8.**

1. \(\left| \left( \frac{\partial}{\partial r} + r \right) K_m(r,s) \right| \leq C \left( \mu_d \frac{d}{2} - 1(B(r, |r - s|)) \right)^{-1}
2. \(\left| \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} + r \right) K_m(r,s) \right| \leq C |r - s|^{-1} \left( \mu_d \frac{d}{2} - 1(B(r, |r - s|)) \right)^{-1}

**Proof.** \(\left( \frac{\partial}{\partial r} + r \right) K_m(r,s) = \int_0^\infty \int_{S^{d-1}} \left( \frac{\partial}{\partial r} + r \right) K_t(x,y)P_m^\frac{d}{2} - 1(x' \cdot y')dy't^{-\frac{1}{2}} \,dt.\)

Since \(\frac{\partial}{\partial r} K_t(x,y) = \sum_{j=1}^d x_j \frac{\partial}{\partial x_j} K_t(x,y)\) and \(r = \sum_{j=1}^d x_j x_j\) we need to estimate

\[
\sum_{j=1}^d x_j \int_0^\infty \int_{S^{d-1}} \left( \frac{\partial}{\partial x_j} + x_j \right) K_t(x,y)P_m^\frac{d}{2} - 1(x' \cdot y')dy't^{-\frac{1}{2}} \,dt
\]

which is given by

\[
\sum_{j=1}^d x_j \int_{S^{d-1}} R_j(x,y)P_m^\frac{d}{2} - 1(x' \cdot y')dy'
\]
where $R_j(x,y)$ are the kernels of the Riesz transforms. The estimates on $R_j(x,y)$ can be used along with the fact that $|\mathcal{P}_m^d(u)| \leq 1$ to get the estimate

$$\int_{S^{d-1}} |x' - sy'|^{-d} dy'$$

which gives the required estimate. The estimate on the gradient is proved in a similar way using the estimate for $\frac{\partial}{\partial x_i} R_j(x,y)$. □

Remark 2.9. It is also possible to prove uniform estimates for the kernels of the operators taking $f_{m,j}$ into $F_{m,j}$. These estimates lead to a different proof of Theorem 1.1 as in [2].

3. Riesz transforms for the special Hermite operator

3.1. Special Hermite operator and Riesz transforms. In this section we recall preliminaries about special Hermite expansion (which are discussed in Chapter 1 and Chapter 2. of [15]) required for the proof of Theorem 1.2. For $f \in L^2(\mathbb{C}^d)$, the special Hermite expansion of $f$ is given by

$$f(z) = (2\pi)^{-d} \sum_{k=0}^{\infty} f \times \varphi_k(z)$$

where $\varphi_k = L_k^{d-1}(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2}$ are the Laguerre functions, $L_k^{d-1}$ being Laguerre polynomials of type $(d - 1)$. And the convolution in the above is called the twisted convolution defined as follows: for given $f, g \in L^2(\mathbb{C}^d)$,

$$f \times g(z) = \int_{\mathbb{C}^d} f(z-w)g(w)e^{\frac{i}{2}z \cdot \overline{w}} dw.$$ 

Now consider the the special Hermite operator $L$ given by

$$L = -2 \sum_{j=1}^{d} (Z_j \overline{Z}_j + Z_j \overline{Z}_j) = -\Delta_z + \frac{1}{4} |z|^2 - iN$$

where $\Delta_z$ is the Laplacian on $\mathbb{C}^d$, $Z_j = \left( \frac{\partial}{\partial x_j} + \frac{1}{4} y_j \right)$, $Z_j = \left( \frac{\partial}{\partial x_j} - \frac{1}{4} z_j \right)$ and $N = \sum_{j=1}^{d} (x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j})$ with $z = x + iy \in \mathbb{C}^d$. It is known that $L$ is a positive, symmetric and elliptic operator on $\mathbb{C}^d$ (for more details see Chapter 1 and Chapter 2. in [15]). Special Hermite expansion is the spectral decomposition of $L$ and $(2\pi)^{-d}(f \times \varphi_k)$ is the projection of $f$ onto the eigenspace corresponding to the eigenvalue $(2k + d)$. If $e^{-tL}$ is the special Hermite semigroup generated by $L$ then

$$e^{-tL} f(z) = (2\pi)^{-d} \sum_{k=0}^{\infty} e^{-(2k+d)t} f \times \varphi_k(z)$$

(3.1)
for functions \( f \in L^2(\mathbb{C}^d) \). For Schwartz class functions \( f \), it is clear that 
\[
e^{-tL}f(z) = f \times p_t(z),
\]
where
\[
p_t(z) = (2\pi)^{-d} \sum_{k=0}^{\infty} e^{-(2k+d)t} \phi_k(z).
\]
The generating function for Laguerre functions of type \((d-1)\) leads to the explicit formula
\[
p_t(z) = (2\pi)^{-d} (\cosh t)^{-d-e^{-\frac{1}{2}|z|^2} \coth t}.
\]
(3.2)
The operator \( L^{-\frac{1}{2}} \) can be defined using spectral theorem, which is also given by
\[
L^{-\frac{1}{2}}f(z) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-tL}f(z)t^{-\frac{1}{2}} dt.
\]
Consequently the Riesz transforms \( S_j \) for the special Hermite operator are defined by
\[
S_j f(z) = Z_j L^{-\frac{1}{2}} f(z) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} Z_j e^{-tL}f(z)t^{-\frac{1}{2}} dt
\]
\[
= f \times s_j(z)
\]
where
\[
s_j(z) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} Z_j p_t(z)t^{-\frac{1}{2}} dt.
\]
It is known that \( S_j \) are bounded on \( L^p(\mathbb{C}^d), \ 1 < p < \infty \) and weak type \((1,1)\) (see Theorem 2.2.2 in [18]). The mixed norm estimates for the Riesz transforms \( S_j \) will be proved in subsection 3.3.

3.2. Bigraded spherical harmonics. In order to study mixed norm estimates for the Riesz transforms associated to special Hermite expansions, we need to make use of several properties of bigraded spherical harmonics. If \( \mathcal{H}_N \) stands for the space of (ordinary) spherical harmonics of degree \( N \) on \( S^{2d-1} \) then we have an action of the unitary group \( U(d) \) on \( \mathcal{H}_N \). Under this action \( \mathcal{H}_N \) decomposes into irreducible pieces \( \mathcal{H}_{m,n} \) where \( m \) and \( n \) are non-negative integers with \( m + n = N \). The members of \( \mathcal{H}_{m,n} \) are called bigraded spherical harmonics.

Let \( \Delta_z = \Delta_x + \Delta_y \) stand for the Laplacian on \( \mathbb{C}^d \) identified with \( \mathbb{R}^{2d} \). Then bigraded solid harmonics of bidegree \((m,n)\) are harmonic functions on \( \mathbb{C}^d \) of the form
\[
P(z) = \sum_{|\alpha|=m} \sum_{|\beta|=n} c_{\alpha,\beta} z^\alpha \overline{z}^\beta.
\]
Note that \( P(z) \) satisfies the homogeneity condition \( P(\lambda z) = \lambda^m \overline{\lambda}^n P(z) \) for any \( \lambda \in \mathbb{C} \). The elements of \( \mathcal{H}_{m,n} \) are just restrictions to \( S^{2d-1} \) of bigraded
solid harmonics of bidegree \((m,n)\). As in the case of spherical harmonics we have
\[
L^2(S^{2d-1}) = \bigoplus_{m,n \geq 0} \mathcal{H}_{m,n}.
\]
Here we use the standard inner product
\[
(f,g)_{L^2(S^{2d-1})} = \int_{S^{2d-1}} f(z') \overline{g(z')} dz'
\]
on the unit sphere. Let \(d(m,n)\) stand for the dimension of \(\mathcal{H}_{m,n}\). We fix an orthonormal basis \(\{Y_{j}^{m,n} : j = 1, 2, \ldots, d(m,n)\}\) for \(\mathcal{H}_{m,n}\).

We write \(z = x + iy\) for elements of \(\mathbb{C}^d\) and identify \(z\) with \((x,y) \in \mathbb{R}^{2d}\). We use the notation \(\nabla^x\) and \(\nabla^y\) for the gradient on \(\mathbb{R}^d\) in the \(x\) and \(y\) variables respectively. Let \(\nabla = (\nabla^x, \nabla^y)\) stand for the gradient on \(\mathbb{R}^d\). Define the complex gradient \(\nabla^z = \frac{1}{2}(\nabla^x - i \nabla^y)\). Then it is clear that
\[
\nabla^z = \frac{1}{r} \nabla^0 + \frac{1}{2r} \frac{1}{2} \nabla^0 + \eta \frac{\partial}{\partial r}.
\]
Here \(\nabla^0\) is the spherical part of the gradient \(\nabla\). By writing \(\nabla^0 = (\nabla^0_x, \nabla^0_y)\) we obtain the decomposition
\[
\nabla^x = \frac{1}{r} \nabla^0_x + \frac{1}{2} \nabla^0_x + \frac{\partial}{\partial r}, \quad \nabla^y = \frac{1}{r} \nabla^0_y + \frac{\partial}{\partial r}.
\]
This leads us to the decomposition
\[
\nabla^z = \frac{1}{r} \nabla^0 + \frac{1}{2r} \frac{1}{2} \nabla^0 + \frac{\partial}{\partial r}
\]
where \(\nabla^0 = \frac{1}{2}(\nabla^0_x - i \nabla^0_y)\). With these notations we can now prove the following result.

**Proposition 3.1.** Let \(P_{m,n}\) be any bigraded solid harmonic of bidegree \((m,n)\). With the notation \(\langle z, w \rangle = z \cdot \overline{w} = \sum_{j=1}^{d} z_j \overline{w_j}\) we have the following:

1. (a) \(\langle z, \nabla^z P_{m,n}(z) \rangle = m \overline{P_{m,n}(z)}\)
   
   (b) \(\langle z, \nabla^0 P_{m,n}(z) \rangle = \frac{m}{2} - n \overline{P_{m,n}(z)}\)
   
   (c) \(\langle \zeta, \nabla^0 P_{m,n}(\zeta) \rangle = \frac{1}{2} (\zeta \cdot \nabla^0 P_{m,n}(\zeta) - \eta \cdot \nabla^0 P_{m,n}(\zeta))\) where \(\zeta = \xi + i \eta\).

2. \(\langle \nabla^z P_{m,n}, \nabla^z P_{m',n'}(z) \rangle = \frac{1}{r} \langle \nabla^0 P_{m,n}, \nabla^0 P_{m',n'}(z) \rangle + \frac{1}{4r}((3m+n)m' + (m-n)n')P_{m,n}P_{m',n'}\).
(3) For any two functions \( f \) and \( g \) on \( S^{2d-1} \) then
\[
\langle \nabla_0^z f(\zeta), \nabla_0^z g(\zeta) \rangle = \frac{1}{4} \nabla_0^z f(\zeta) \cdot \nabla_0^z g(\zeta) + \frac{i}{4} (\nabla_0^z f(\zeta) \cdot \nabla_0^y g(\zeta) - \nabla_0^y f(\zeta) \cdot \nabla_0^y g(\zeta)).
\]

(4) \( \int_{S^{2d-1}} \nabla_0^z f(\zeta) \cdot \nabla_0^y g(\zeta) d\zeta = - \int_{S^{2d-1}} (\nabla_0^y \cdot \nabla_0^y f(\zeta)) g(\zeta) d\zeta + (2d-1) \int_{S^{2d-1}} (\eta \cdot \nabla_0^x f(\zeta)) g(\zeta) d\zeta. \)

(5) \( \int_{S^{2d-1}} \langle \nabla_0^z P_{m,n}(\zeta), \nabla_0^y P_{m',n'}(\zeta) \rangle d\zeta = \lambda_d(m,n) \langle P_{m,n}, P_{m',n'} \rangle \) where \( \lambda_d(m,n) = \frac{1}{4} (m+n)^2 + (4d-3)m - n \).

**Proof.** 1(a) follows by simple calculation:
\[
\langle z, \nabla^z P_{m,n} \rangle = \sum_{j=1}^{d} \overline{z_j} \frac{\partial}{\partial z_j} P_{m,n}(z).
\]

If \( P_{m,n} = \sum_{|\alpha|=m} \sum_{|\beta|=n} c_{\alpha \beta} z^\alpha \overline{z}^\beta \) then
\[
\overline{z_j} \frac{\partial}{\partial z_j} P_{m,n}(z) = \sum_{|\alpha|=m} \sum_{|\beta|=n} \overline{c_{\alpha \beta}} \alpha_j \overline{z}^\alpha z^\beta.
\]

Summing over \( j \) and noting that \( |\alpha| = m \), we get the result. To prove 1(b) we use (3.6) which gives \( \nabla_0^z = r \nabla^z - \frac{1}{r} \partial \overline{z} \). Hence
\[
\nabla_0^z P_{m,n}(z) = r \nabla^z P_{m,n}(z) - \frac{1}{2} \overline{z} \frac{\partial}{\partial r} P_{m,n}(z).
\]

By virtue of homogeneity \( \frac{\partial}{\partial r} P_{m,n}(z) = \frac{1}{2} (m+n) P_{m,n}(z) \) and therefore
\[
\nabla_0^z P_{m,n}(z) = r \nabla^z P_{m,n}(z) - \frac{(m+n)^2}{2r} \overline{z} P_{m,n}(z).
\]

Taking inner product with \( z \) and using 1(a) we get
\[
\langle z, \nabla_0^z P_{m,n}(z) \rangle = r p_{m,n}(z) - \frac{r}{2} (m+n) P_{m,n}(z) = \frac{r}{2} (m-n) P_{m,n}(z).
\]

To prove 1(c) we make use of the fact that \( \langle \xi, \eta \rangle \cdot \nabla P_{m,n}(\zeta) = 0 \) (see Lemma 2.2 in [14]).
\[
\langle \zeta, \nabla_0^z P_{m,n}(z) \rangle = \frac{1}{2} ((\xi - i\eta) \cdot (\nabla_0^z - i
abla_0^y) P_{m,n}(\zeta)).
\]

Simplifying and using \( \langle \xi, \eta \rangle \cdot \nabla P_{m,n}(\zeta) = 0 \) we obtain
\[
\langle \zeta, \nabla_0^z P_{m,n}(z) \rangle = \frac{i}{2} (\xi \cdot \nabla_0^y P_{m,n}(z) - \eta \cdot \nabla_0^y P_{m,n}(z)).
\]

Coming to the proof of (2) we see that
\[
\nabla^z P_{m,n}(z) = \frac{1}{r} \left( \nabla_0^z + \frac{1}{2} \overline{z} \frac{\partial}{\partial r} \right) P_{m,n}(z)
\]
\[
= \frac{1}{r} \left( \nabla_0^z P_{m,n}(z) + \frac{(m+n)}{2r} \overline{z} P_{m,n}(z) \right).
\]
We have a similar expression for $\nabla^z P_{m',n'}(z)$. Consequently, $\langle \nabla^z P_{m,n}(z), \nabla^z P_{m',n'}(z) \rangle$ is given by
\[
\frac{1}{r^2} \left( (\nabla_0^z P_{m,n}(z), \nabla_0^z P_{m',n'}(z)) + \frac{(m+n)}{2r} P_{m,n}(z) \langle \nabla_0^z P_{m',n'}(z) \rangle \right)
\]
\[
+ \frac{(m'+n')}{2r} \mathcal{P}_{m',n'}(z) \langle \nabla_0^z P_{m,n}(z), \nabla_0^z P_{m',n'}(z) \rangle + \frac{(m+n)(m'+n')}{4r^2} r^2 P_{m,n}(z) \mathcal{P}_{m',n'}(z).
\]

Using 1(b) and simplifying we get
\[
\langle \nabla^z P_{m,n}(z), \nabla^z P_{m',n'}(z) \rangle = \frac{1}{r^2} \langle \nabla_0^z P_{m,n}(z), \nabla_0^z P_{m',n'}(z) \rangle + \frac{1}{4r} ((3m+n)m' + (m-n)n') P_{m,n}(z) \mathcal{P}_{m',n'}(z).
\]

Recalling the definition of $\nabla_0^z$ we see that
\[
\langle \nabla_0^z f(\zeta), \nabla_0^z g(\zeta) \rangle = \frac{1}{4} (\langle \nabla_0^z \iota \nabla_0^z f(\zeta), \nabla_0^z \iota \nabla_0^z g(\zeta) \rangle).
\]

The right hand side simplifies into
\[
\nabla_0^z f(\zeta) \cdot \nabla_0 \mathcal{F}(\zeta) + i \langle \nabla_0^z \iota \nabla_0^z f(\zeta), \nabla_0^z \iota \nabla_0^z \mathcal{F}(\zeta) - \nabla_0^y \iota f(\zeta) \cdot \nabla_0^y \mathcal{F}(\zeta) \rangle
\]
which is the required expression for proving (3). Considering now the integral in (4) which can be evaluated using Proposition 8.7, Chapter 1 in [4]. Writing $\nabla_0 = (D_1, \cdots, D_d, D_{d+1}, \cdots, D_{2d})$ so that $\nabla_0^z = (D_1, D_2, \cdots, D_d)$, $\nabla_0^y = (D_{d+1}, \cdots, D_{2d})$ we have
\[
\int_{S^{2d-1}} \nabla_0^z f(\zeta) \cdot \nabla_0^y \mathcal{F}(\zeta) d\zeta = \sum_{j=1}^d \int_{S^{2d-1}} D_j f(\zeta) D_{d+j} \mathcal{F}(\zeta) d\zeta
\]
\[
= - \sum_{j=1}^d \int_{S^{2d-1}} D_{d+j} D_j f(\zeta) \mathcal{F}(\zeta) d\zeta + (2d-1) \sum_{j=1}^d \eta_j D_j f(\zeta) \mathcal{F}(\zeta) d\zeta
\]
\[
= - \int_{S^{2d-1}} \langle \nabla_0^y \nabla_0^z f(\zeta), \mathcal{F}(\zeta) \rangle d\zeta + (2d-1) \int_{S^{2d-1}} (\eta \cdot \nabla_0^y f(\zeta)) \mathcal{F}(\zeta) d\zeta.
\]

This proves (4). Finally in order to prove (5) we integrate (3) over $S^{2d-1}$ and make use of (4). The result is , with $P = P_{m,n}$, $Q = P_{m',n'}$
\[
\int_{S^{2d-1}} \langle \nabla_0^z P(\zeta), \nabla_0^z Q(\zeta) \rangle d\zeta = \frac{1}{4} \int_{S^{2d-1}} \nabla_0 P(\zeta) \cdot \nabla_0 \mathcal{Q}(\zeta)(\zeta) d\zeta
\]
\[
+ \frac{i}{4} (2d-1) \int_{S^{2d-1}} (\eta \cdot \nabla_0^z P(\zeta) - \xi \cdot \nabla_0^y P(\zeta)) \mathcal{Q}(\zeta) d\zeta
\]
since $\nabla_0^z \cdot \nabla_0^y f = \nabla_0^y \cdot \nabla_0^z f$. Using 1(c) we convert the second integral in above into $-\frac{2d-1}{2} \int_{S^{2d-1}} \langle \zeta, \nabla_0^z P(\zeta) \rangle \mathcal{Q}(\zeta) d\zeta$. Since $P = P_{m,n}$ it follows that $\mathcal{P}$ is of bigraded $(n, m)$ and hence by 1(b) we have $\langle \zeta, \nabla_0^z \mathcal{P}(\zeta) \rangle = \frac{(n-m)}{2} P(\zeta)$.
Using this we have obtained
\[
\int_{S^{2d-1}} \langle \nabla_0^2 P, \nabla_0^2 Q \rangle \, d\zeta = \frac{1}{4} \int_{S^{2d-1}} \nabla_0 P(\zeta) \cdot \nabla_0 Q(\zeta) \, d\zeta + \frac{(2d-1)}{4} (m-n) \int_{S^{2d-1}} P(\zeta) \overline{Q}(\zeta) \, d\zeta.
\]

The first integral simplifies to
\[
- \int_{S^{2d-1}} \Delta_0 P(\zeta) \overline{Q}(\zeta) \, d\zeta = (m+n)(m+n+2d-2) \int_{S^{2d-1}} P(\zeta) \overline{Q}(\zeta) \, d\zeta
\]
where \(\Delta_0\) is the spherical Laplacian for which \(P\) is an eigenfunction. Thus
\[
\int_{S^{2d-1}} \langle \nabla_0^2 P, \nabla_0^2 Q \rangle \, d\zeta = \frac{1}{4} ((m+n)(m+n+2d-2) + (2d-1)(m-n)) \int_{S^{2d-1}} (\nabla_0 P \cdot \nabla_0 Q)(\zeta) \, d\zeta
\]
which can be simplified to prove (5). \(\square\)

3.3. Mixed norm estimates for \(S_j\). In this subsection we prove the mixed norms estimates for \(S_j\) stated in Theorem 1.2. As in Section 2.2 we use the idea of Rubio de Francia along with some weighted norm inequalities satisfied by \(S_j\) and the transform properties of the vector \(S = (S_1, S_2, \ldots, S_d)\) under the action of the unitary group. As the ideas can be applied even to higher order Riesz transforms we start with the following definition.

Given a bigraded solid harmonic \(P \in \mathcal{H}_{m,n}\) we define \(R_P\) the higher order Riesz transform associated to \(P\) by the prescription
\[
W(R_P f) = W(f) G(P) H^{-\frac{1}{2}} (m+n).
\]
Here \(W(f)\) is the Weyl transform of \(f\) and \(G(P)\) is the Weyl correspondence of the polynomial \(P\). We refer to Geller [8], Thangavelu [19] and Sanjay-Thangavelu [16], for various facts about these objects. We let \(\rho(k)f(z) = f(k \cdot z)\) stand for the action of \(U(d)\) on functions. We also make use of the metaplectic representation \(\mu(k)\) which is defined by the property
\[
W(\rho(k)f) = \mu(k)^* W(f) \mu(k).
\]
The operator \(\mu(k)\), \(k \in U(d)\) are unitary on \(L^2(\mathbb{R}^d)\). The following proposition is easy to prove.

**Proposition 3.2.** For any \(k \in U(d)\) we have
\[
\rho(k) R_P f = R_{\rho(k)P} (\rho(k)f).
\]

**Proof.** Indeed, we notice that
\[
W(R_{\rho(k^{-1})P} f) = W(f) G(\rho(k^{-1})P) H^{-\frac{1}{2}} (m+n) = W(f) \mu(k) G(P) \mu(k)^* H^{-\frac{1}{2}} (m+n).
\]
Since \(\mu(k)^*\) commutes with \(H^{-\frac{1}{2}} (m+n)\) we can rewrite the above as
\[ W(R_{\rho(k^{-1})P}f) = \mu(k)W(\rho(k)f)G(P)H^{-\frac{1}{2}(m+n)}\mu(k)^*. \]

This simply means that
\[ W(\rho(k)R_{\rho(k^{-1})P}f) = W(\rho(k)f)G(P)H^{-\frac{1}{2}(m+n)} \]

which proves the proposition. \(\square\)

In [16] it has been shown that the operators \( R_P \) are Oscillatory singular integral operators. Appealing to the theorem of [12] we obtain

**Theorem 3.3.** For any \( P \in \mathcal{H}_{m,n} \) the operators \( R_P \) satisfy the weighted norm inequality
\[
\left( \int_{\mathbb{C}^d} |R_P f(z)|^p w(z) dz \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{C}^d} |f(z)|^p w(z) dz \right)^{\frac{1}{p}}
\]

for \( 1 < p < \infty \) whenever \( w \in A_p(\mathbb{C}^d) \).

Let us specialize to the the case where \( P_w(z) = \sum_{j=1}^{d} z_j \overline{w}_j, w \in S^{2d-1} \) so that \( R_{P_w} = \sum_{j=1}^{d} \overline{w}_j S_j \). In view of the proposition we get the identity
\[ R_{P_w} f(k \cdot z) = R_{P_{w}}(\rho(k)f)(z), \]

which is the same as saying
\[
\sum_{j=1}^{d} \overline{w}_j S_j f(k \cdot z) = \sum_{j=1}^{d} (k \cdot w)_j S_j(\rho(k)f)(z)
\]

We can now use this identity along with the above weighted norm inequality to use the idea of Rubio de Francia to get the required inequality. We need to remember that radial weight function \( w \in A_p(\mathbb{C}^d) \) if and only if \( w(r) \in A_p^{d-1}(\mathbb{R}^+) \). By taking \( w = e_j \), the coordinate vectors we get Theorem 1.2.

In the next subsection we apply Theorem 1.2 to get some vector valued inequality for Laguerre Riesz transforms which are more refined than those proved in Theorem 2.6.

**3.4. The vector of Riesz transforms.** Recall that the Riesz transforms associated to the special Hermite operator \( L \) are given by \( S_j = Z_j L^{-\frac{3}{2}} \) and \( \overline{S}_j = \overline{Z}_j L^{-\frac{3}{2}}, j = 1, 2, \ldots, d \) with \( Z_j = \frac{\partial}{\partial z_j} + \frac{1}{4} z_j \) and \( \overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - \frac{1}{4} \overline{z}_j \). Let \( Sf = (S_1, \cdots, S_d) f \) and \( \overline{S}f = (\overline{S}_1, \cdots, \overline{S}_d) f \) stand for the vectors of Riesz transforms; note that \( Sf = (\nabla^{\overline{z}} + \frac{1}{4} \overline{z}) L^{-\frac{3}{2}} f \) and \( \overline{S}f = (\nabla^z - \frac{1}{4} z) L^{-\frac{3}{2}} f \).

We first calculate the spherical harmonic coefficients of \( Sf \) and \( \overline{S}f \).

Let \( e^{-tL} \) stand for the special Hermite semigroup which is given by twisted convolution with
\[ p_t(z) = c_d(\sinh t)^{-d} e^{-\frac{1}{2}(\coth t)|z|^2} \]
That is to say,
\[ e^{-tL}f(z) = f \times p_t(z) = \int_{\mathbb{C}^d} f(z-w)p_t(w)e^{\frac{t}{2}d(z,\overline{w})}dw. \]

We can express \( L^{-\frac{1}{2}}f(z) \) in terms of \( e^{-tL}f \) in the usual way
\[ L^{-\frac{1}{2}}f(z) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-tL}f(z)t^{-\frac{1}{2}}dt. \]

In order to find the spherical harmonic expansion of \( L^{-\frac{1}{2}}f(z) \) we calculate the coefficients of \( f \times p_t(z) \). For \( \delta > -\frac{1}{2} \) let us define
\[ \varphi^\delta_k(r) = \left( \frac{\Gamma(k+1)2^{-\delta}}{\Gamma(k+\delta+1)} \right)^{\frac{1}{2}}L_k^\delta\left(\frac{1}{2}r^2\right)e^{-\frac{1}{2}r^2} \]
so that \( \{\varphi^\delta_k : k = 0, 1, 2, \ldots\} \) forms an orthonormal basis for \( L^2(\mathbb{R}^+, r^{2\delta+1}dr) \).

We set
\[ R^\delta_k(g) = \frac{\Gamma(k+1)2^{-\delta}}{\Gamma(k+\delta+1)} \int_0^\infty g(r)L_k^\delta\left(\frac{1}{2}r^2\right)e^{-\frac{1}{2}r^2}r^{2\delta+1}dr. \]

We fix an orthonormal basis \( \{Y^{i,j}_{m,n} : j = 1, 2, \ldots, d(m,n), \ m, n \in \mathbb{N} \} \) for \( L^2(S^{2d-1}) \) consisting of bigraded spherical harmonics. Let
\[ k_t^i(r, s) = \sum_{k=0}^{\infty} e^{-2(k+\delta+1)t} \langle \varphi^\delta_k(r) \varphi^\delta_k(s) \rangle, \]
which can be expressed in terms of the kernel \( K_t^i(r, s) \) introduced in Subsection 2.2. Indeed, \( k_t^i(r, s) = 2^{-\delta}K_t^\delta\left(\frac{r}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right) \).
Let \( f_{m,n} \) stand for the spherical harmonic coefficients of \( f \).

**Proposition 3.4.** For each \( m, n \in \mathbb{N}, \ j = 1, 2, \ldots, d(m,n) \) we have
\[ \int_{S^{2d-1}} f \times p_t(r\zeta)Y^{i,j}_{m,n}(\zeta)d\zeta = \mathcal{T}^{d+m+n-1}_t f_{m,n}^i(r)Y^{i,j}_{m,n}(\zeta) \]
where
\[ \mathcal{T}^{d+m+n-1}_t g(r) = e^{-t(m-n)} \int_0^\infty g(s)(rs)^{m+n}k_t^{d+m+n-1}(r, s)s^{2d-1}ds. \]

The proposition follows immediately from the Hecke - Bochner formula for the special Hermite projections (see Theorem 2.6.1 or Equation 2.6.10 in [19]). According to this formula, when \( f(r\zeta) = g(r)Y(\zeta) \) with \( Y \in \mathcal{H}_{m,n} \) one has
\[ f \times \varphi_k(r\zeta) = c_dR^{d+m+n-1}(\overline{g})\varphi_k^{d+m+n-1}(r)Y(r)^{m+n} \]
where \( \overline{g}(r) = r^{-(m+n)}g(r) \). This shows that, for \( f \) as above,
\[ e^{-tL}f(r\zeta) = \sum_{k=m}^{\infty} e^{-2(k+\delta)t}R^{d+m+n-1}(\overline{g})\varphi_k^{d+m+n-1}(r)Y(r)^{m+n} \]
In order to find the spherical harmonic coefficients of \( Sf \), calculate the action of \( \mathcal{L} \) on \( f \) and consequently \( L^{-\frac{d}{2}}F \). Using this we can easily prove the following proposition on the vectors \( (F, Y) \) of Riesz transforms.

**Remark 3.5.** As a by-product of the above calculation we get the interesting formula

\[
\int_{\mathbb{S}^{2d-1}} p_t(rz' - sw') e^{-\frac{1}{2}t(r^2 + s^2)} Y^j_{m,n}(w') \, dw' = Y^j_{m,n}(z') (r^2 + s^2) Y^j_{m,n}(r) e^{t(m-n)}. 
\]

**Corollary 3.6.** If \( F^j_{m,n}(r) \) stand for the spherical harmonic coefficients of \( L^{-\frac{d}{2}}F \), then we have

\[
F^j_{m,n}(r) = \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty e^{-t(m-n)} (r^2 + s^2)^{-\frac{1}{2}} (r, s) F^j_{m,n}(s) s^{2d-1} t^{-\frac{1}{2}} ds dt.
\]

Thus we have obtained the following expansion,

\[
L^{-\frac{d}{2}}f(r \zeta) = \sum_{m,n=0}^{\infty} \sum_{j=1}^{d(m,n)} F^j_{m,n}(r) Y^j_{m,n}(\zeta).
\]

In order to find the spherical harmonic coefficients of \( Sf \) and \( \nabla f \), let us calculate the action of \( \nabla^2 + \frac{1}{4} \zeta \) and \( \nabla^2 - \frac{1}{4} \zeta \) on functions of the form \( F(r)Y(\zeta) \) where \( Y \in \mathcal{H}_{m,n} \). In view of (3.6) we have

\[
\nabla^2(FY)(z) = \frac{1}{r} F(r) \nabla^2 Y(\zeta) + \frac{1}{2r} \frac{\partial F}{\partial r}(r) \zeta,
\]

\[
\nabla^2(FY)(z) = \frac{1}{r} F(r) \nabla^2 Y(\zeta) + \frac{1}{2r} \frac{\partial F}{\partial r}(r) \zeta.
\]

and consequently

\[
\left( \nabla^2 + \frac{1}{4} \zeta \right)(FY)(r \zeta) = \frac{1}{2} \left( \frac{\partial}{\partial r} + \frac{1}{2r} \right) F(r)Y(\zeta) \zeta + \frac{1}{r} F(r) \nabla^2 Y(\zeta),
\]

\[
\left( \nabla^2 - \frac{1}{4} \zeta \right)(FY)(r \zeta) = \frac{1}{2} \left( \frac{\partial}{\partial r} - \frac{1}{2r} \right) F(r)Y(\zeta) \zeta + \frac{1}{r} F(r) \nabla^2 Y(\zeta).
\]

Using this we can easily prove the following proposition on the vectors \( (Sf, \nabla f) \) of Riesz transforms.

**Proposition 3.7.** The square of the \( L^2(\mathbb{S}^{2d-1}) \) norm of \( \langle Sf, \nabla f \rangle + \langle \nabla f, \nabla f \rangle \) is the sum of the following five terms:

\[
(1) \sum_{m,n=0}^{\infty} \sum_{j=1}^{d(m,n)} \left| \frac{1}{2} \left( \frac{\partial}{\partial r} + \frac{1}{2r} \right) F^j_{m,n}(r) \right|^2 = A_1(r)^2,
\]
\(\sum_{m,n=0}^{\infty} \sum_{j=1}^{d(m,n)} \left| \frac{1}{2} \left( \frac{\partial}{\partial r} + \frac{1}{2} \right) F_{m,n}^j(r) \right|^2 = A_2(r)^2,\)

\(\sum_{m,n=0}^{\infty} \sum_{j=1}^{d(m,n)} \frac{1}{r^2} \lambda_d(m, n) |F_{m,n}^j(r)|^2 = A_3(r)^2,\)

\(\sum_{m,n=0}^{\infty} \sum_{j=1}^{d(m,n)} \frac{1}{r^2} \lambda_d(n, m) |F_{m,n}^j(r)|^2 = A_4(r)^2,\)

\(\sum_{m,n=0}^{\infty} \sum_{j=1}^{d(m,n)} \frac{1}{2} \left( \frac{m-n}{2r} \right)^2 |F_{m,n}^j(r)|^2 = A_5(r).\)

**Proof.** In view of the above expression for \(\left( \nabla^z + \frac{1}{4} \right)(FY)\) we see that

\[\langle \left( \nabla + \frac{1}{4} \right)(F_{m,n}^j r^j Y_{m,n}(r\zeta)), \left( \nabla + \frac{1}{4} \right)(F_{m',n'}^j r^j Y_{m',n'}^j(r\zeta)) \rangle\]

is the sum of the three terms

\[\frac{1}{2} \left( \frac{\partial}{\partial r} + \frac{1}{2} \right) F_{m,n}^j(r) \frac{1}{2} \left( \frac{\partial}{\partial r} + \frac{1}{2} \right) F_{m',n'}^j(r) Y_{m,n}(r) Y_{m',n'}^j(r\zeta),\]

\[\frac{1}{r^2} F_{m,n}^j(r) F_{m',n'}^j(r) \langle \nabla\hat{\zeta} Y_{m,n}, \nabla\hat{\zeta} Y_{m',n'}^j(r\zeta) \rangle,\]

and

\[2\Re \frac{1}{2} \left( \frac{\partial}{\partial r} + \frac{1}{2} \right) F_{m,n}^j(r) Y_{m,n}(r) \langle \nabla\hat{\zeta}, \nabla\hat{\zeta} Y_{m',n'}^j(r\zeta) \rangle \frac{1}{r} F_{m',n'}^j(r).\]

Integrating over \(S^{2d-1}\) and making use of Proposition 3.1 we can see that \(\langle Sf, Sf \rangle\) is the sum of the terms (1), (3) and

\[(3.7) \quad 2\Re \sum_{m,n=0}^{\infty} \sum_{j=1}^{d(m,n)} \frac{1}{2} \left( \frac{\partial}{\partial r} + \frac{1}{2} r^j \right) F_{m,n}^j(r) \left( \frac{m-n}{2r} \right) F_{m,n}^j(r).\]

And similarly \(\langle \overline{Sf}, \overline{Sf} \rangle\) is the sum of (2), (4) and

\[(3.8) \quad 2\Re \sum_{m,n=0}^{\infty} \sum_{j=1}^{d(m,n)} \frac{1}{2} \left( \frac{\partial}{\partial r} - \frac{1}{2} r^j \right) F_{m,n}^j(r) \left( \frac{n-m}{2r} \right) F_{m,n}^j(r).\]

By adding \(\langle Sf, Sf \rangle\) with \(\langle \overline{Sf}, \overline{Sf} \rangle\), we will get the required result.

\(\square\)

3.5. **Revisiting Laguerre Riesz transforms:** In the course of the proof of Proposition 3.7 we have shown that \(\langle Sf, Sf \rangle\) is the sum of the terms (1), (3) and (3.7) of Proposition 3.7. By Cauchy-Schwarz inequality the third term \(\langle \overline{Sf}, \overline{Sf} \rangle\) is dominated by the sum of first two terms and hence the mixed norm estimates

\[\|S_j f\|_{L^p(C^d, w)} \leq C \|f\|_{L^p(C^d, w)}\]

will follow once we prove the estimate

\[\left( \int_0^\infty A_j(r)^p w(r) r^{2d-1} dr \right)^{\frac{1}{p}} \leq C \|f\|_{L^p(C^d, w)}\]

for \(j = 1, 2\). On the other hand from Theorem 1.2 and Proposition 3.7 we should be able to deduce the above inequality for all \(j = 1, 2, \ldots, 5\). Since
the term $A_5$ is not nonnegative, this expectation may be false. What we can
deduce from Theorem 1.2 is the following.

Let $L^p_h(C^d)$ stand for the subspace of $L^p(C^d)$ consisting of $f$ for which
\[
\int_{S^{2d-1}} f(r\zeta)Y^j_{m,n}(\zeta)d\zeta = 0
\]
for all $m < n$. Similarly we can define $L^p_{ah}(C^d)$ with the condition
\[
\int_{S^{2d-1}} f(r\zeta)Y^j_{m,n}(\zeta)d\zeta = 0
\]
for all $m \geq n$. Clearly, $L^p_h(C^d) \oplus L^p_{ah}(C^d) = L^p(C^d)$. For $f \in L^p_h(C^d)$, all the
terms appearing in Proposition 3.7, including $A_5$ are nonnegative. Hence
we get the following result.

**Theorem 3.8.** For $f \in L^p_h(C^d)$, $1 < p < \infty$ and $w \in A^{d-1}_p(\mathbb{R}^+)$ we have the inequalities
\[
\left( \int_0^\infty A_j(r)^p w(r)r^{2d-1}dr \right)^{\frac{1}{p}} \leq C \|f\|_{L^p(C^d,w)}
\]
for $j = 1, 2, \ldots, 5$.

It is also possible to deduce the above result directly as a consequence
of the result for the Hermite Riesz transforms. For example, consider the
inequality
\[
\int_0^\infty \left( \sum_{m \geq n} \sum_{j=1}^{d(m,n)} \frac{(m+n)^2}{r^2} |F^j_{m,n}(r)|^2 \right)^{\frac{p}{2}} w(r)r^{2d-1}dr
\]
\[
\leq C \int_0^\infty \left( \sum_{m \geq n} \sum_{j=1}^{d(m,n)} |f^j_{m,n}(r)|^2 \right)^{\frac{p}{2}} w(r)r^{2d-1}dr.
\]
for $f \in L^p_h(C^d)$. Recall that
\[
F^j_{m,n}(r) = \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty e^{-t(m-n)}(rs)^{m+n+k^d+m+n-k^{d+N-1}}(r,s)f^j_{m,n}(s)s^{2d-1}t^{-\frac{1}{2}}dtds.
\]
When $m \geq n$ the above is dominated by
\[
|F^j_{m,n}(r)| \leq \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty (rs)^N k^{d+N-1}(r,s)|f^j_{m,n}(s)|s^{2d-1}t^{-\frac{1}{2}}dtds
\]
where $m + n = N$. The terms on the right hand side are precisely the
the corresponding terms in the study of Riesz transforms on $\mathbb{R}^{2d}$. Then we can
appeal to (2.12) to get the desired inequality.

Similarly, we can prove the inequality
\[
\int_0^\infty \left( \sum_{m \geq n} \sum_{j=1}^{d(m,n)} \frac{1}{2} \left( \frac{\partial}{\partial r} \pm \frac{1}{2} r \right)^2 F^j_{m,n}(r) \right)^{\frac{p}{2}} w(r)r^{2d-1}dr
\]
This result will be proved once we show that the operators taking $f^j_{m,n}$ into
\[
\left(\frac{\partial}{\partial r} \pm \frac{1}{2} r\right) F^j_{m,n}(r)
\]
are singular integral operators on the homogeneous space $(\mathbb{R}^+, d\mu_{d-1})$ whose kernels satisfy uniform estimates. But this is easy to see: the kernels of these operators are given by
\[
\frac{1}{2} \left( \frac{\partial}{\partial r} \pm \frac{1}{2} r \right) \int_0^\infty e^{-t(m-n)} (rs)^{m+n} k_t^{d+m+n-1}(r,s) t^{-\frac{d}{2}} dt
\]
Since $k_t^{d+m+n-1}(r,s) = 2^{d-m-n} K_t^{d+m+n-1} \left( \sqrt{r}, \sqrt{s} \right)$, it is enough to estimate the kernels
\[
\left( \frac{\partial}{\partial r} \pm r \right) \int_0^\infty e^{-t(m-n)} (rs)^{m+n} K_t^{d+m+n-1}(r,s) t^{-\frac{d}{2}} dt
\]
Since we are assuming $m \geq n$, the estimation of this is similar to that of
\[
\left( \frac{\partial}{\partial r} \pm r \right) \int_0^\infty (rs)^N K_t^{N+d-1}(r,s) t^{-\frac{d}{2}} dt
\]
which has been already done in Section 2. This proves the required estimates and hence the result.

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