SYMMETRIC POSITIVE SOLUTIONS TO NONLINEAR CHOQUARD EQUATIONS WITH POTENTIALS

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ABSTRACT. Existence results for a class of Choquard equations with potentials are established. The potential has a limit at infinity and it is taken invariant under the action of a closed subgroup of linear isometries of \( \mathbb{R}^N \). As a consequence, the positive solution found will be invariant under the same action. Power nonlinearities with exponent greater or equal than two or less than two will be handled. Our results include the physical case.

1. INTRODUCTION

This paper is devoted to the study of existence results for the following Choquard equation

\[
\begin{aligned}
-\Delta u + V(x)u &= (I_{\alpha} * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \\
\end{aligned}
\]

with \( N \geq 2, \alpha < N \), and \( I_{\alpha} \) represents the Riesz potential of order \( \alpha \), defined for every point \( x \in \mathbb{R}^N \setminus \{0\} \) by

\[
I_{\alpha}(x) = \frac{A_{\alpha}}{|x|^{N-\alpha}}, \quad \text{where} \quad A_{\alpha} = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma(\alpha/2)2^\alpha \pi^{N/2}},
\]

where \( \Gamma \) denotes the Gamma function (see [25]). The exponent \( p \) lies in the range

\[
\frac{N-2}{N+\alpha} \leq \frac{1}{p} < \frac{N}{N+\alpha},
\]

and the potential \( V \) satisfies

\[
V \in C^0(\mathbb{R}^N), \quad \inf_{x \in \mathbb{R}^N} V(x) > 0, \quad \text{and} \quad \lim_{|x| \to \infty} V(x) = V_\infty > 0.
\]

This partial differential equation arises in several physical models; it has been introduced in [24] (see also [16]) in the context of quantum mechanics, and it also corresponds to the stationary case associated to the nonlinear Hartree equation (see e.g. [17] and for further references see [20]).

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The condition on the exponent $p$ and the Hardy-Littlewood-Sobolev inequality implies that the right hand side on $(P_V)$ is well defined for every $u \in H^1(\mathbb{R}^N)$, so that under (1.2) any solution turns out to be a critical point of the action functional

$$I_V(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p. \quad (1.3)$$

When $V(x) \leq V_\infty$, $V \not\equiv V_\infty$, the existence of a least action solution, corresponding to a minimum point of $I_V$ on

$$\mathcal{N}_V : \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : \langle I'_V(u), u \rangle = 0 \right\}$$

has been first proved in [17] by means of the well-known concentration compactness method (see also [20, 21, 28]).

But, when $V(x)$ approaches $V_\infty$ from above, or oscillating, the search of a minimum point on $\mathcal{N}_V$ is useless and higher action level solutions have to be sought; this immediately requires a deep study of the possible lack of compactness of a bounded Palais-Smale sequence due to the unboundedness of the domain. In this path, versions of the well-known Splitting Lemma, firstly introduced by [26], are of great help as compactness results, as this tool furnishes compactness at quantized energy intervals whenever the so-called problem at infinity

$$\begin{cases} -\Delta u + V_\infty u = (I_\alpha * |u|^p) |u|^{p-2} u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \quad (P_\infty)$$

has a unique positive solution. As a consequence, a good knowledge concerning existence and uniqueness properties of $(P_\infty)$ turns out to be a cornerstone of the research of bound states of $(P_V)$. The existence of positive solutions to this autonomous problem goes back to [16] and [17], and these results have been extended in [22], where it is shown that the problem $(P_\infty)$ has a positive radially symmetric least action solution $\omega \in C^2(\mathbb{R}^N)$ for any $p$ satisfying (1.1).

Besides, the uniqueness property is known for $(P_\infty)$ if $p = 2$, $\alpha = 2$ and $N = 3, 4, 5$, (see [16, 18, 30] and [31] for a generalization). Consequently, in this range of the parameters one can exploit the above approach to get solutions for $(P_V)$ and we have followed this strategy in [19] (see also [29] where a non-autonomous case has been studied for $N = 3$ and $\alpha = p = 2$).

On the other hand, one can exploit symmetric properties of the potentials $V$ in order to look for critical points of $I_V$ enjoying the same kind of symmetry, by minimizing $I_V$ on a symmetric $\mathcal{N}_V$.

The introduction of symmetry into play naturally increases the least action level and, at the same time, allows to construct a Palais-Smale sequence which is also a minimizing one. Then, the key point becomes again proving that a “symmetric” minimizing level lies in a range where compactness properties hold. In order to do this, one can build a suitable competitor making use of $\omega$, a least action solution of $(P_\infty)$. More precisely, consider the action of $G$ a closed subgroup of linear isometries of $\mathbb{R}^N$ and set

$$\ell(G) = \min \left\{ \# Gx : x \in \mathbb{R}^N \setminus \{0\} \right\}$$
where \( \#Gx \) is the cardinality of the \( G \)-orbit of \( x \), (see (2.1) in Section 2). Then, a good competitor is made of a sum of suitable translation of \( \omega \) centred in points that are far away from each others. So that, in order to estimate the functional, decay properties of \( \omega \) are required.

When studying the nonlinear Schrödinger equation following this strategy, the exponential decay of the least action solution of the corresponding problem at infinity plays a crucial role (see e.g. [4]); since, due to this fast decay one can split the action level of this competitor into a suitable multiple of the least action level of \((P_\infty)\).

In the case of \((P_\infty)\) the decay of \( \omega \) is of exponential type when \( p \geq 2 \), and existence results of symmetric solutions have been proved in [10] under suitable hypotheses on the group of symmetries \( G \).

But, when \( p < 2 \) one sees the real non-locality feature of \((P_\infty)\) as the nonlinearity in this case is not, roughly speaking, “focusing enough” and the decay of \( \omega \) is not exponential any more but it is actually of polynomial type (see [20,21] for a comprehensive discussion on this topic).

Our contributions in this paper are twofold. First of all, we succeed in proving existence results in the case \( p < 2 \) by performing the above mentioned asymptotic analysis even when \( \omega \) decays polynomially and under assumptions on the group of symmetries \( G \) weaker than the one in [10].

In particular, our existence result for \( p < 2 \) is the following one.

**Theorem 1.1.** Let \( G \) be a closed subgroup of the linear isometries of \( \mathbb{R}^N \), with \( \ell(G) \geq 2 \), and finite. Assume that the exponent \( p \) is such that

\[
\frac{N + \alpha}{N} < p < 2
\tag{1.4}
\]

and that the potential \( V(x) \) satisfies (1.2) and

\[
V(x) \leq V_\infty + A_0 (1 + |x|)^{-\beta}, \quad \forall x \in \mathbb{R}^N, \text{ with } A_0 > 0, \text{ and } \beta > \frac{N - \alpha}{2 - p}. \tag{1.5}
\]

Then, if \( V \) is \( G \)-invariant, Problem \( (P_V) \) has a positive \( G \)-invariant solution.

In proving this result different phenomena, compared to the Schrödinger equation, appear due to the fact that \( p < 2 \). First of all, \( \mathcal{N}_V \) is not of class \( C^1 \); however, the approach introduced in [27] can be exploited to deal with this situation. More relevant are the difficulties arising when studying the term involving the nonlinearity. Indeed, classical algebraic inequalities as in [4,10] do not apply and the effects of the nonlinearity are spread in a less concentrated region, so that interactions between two different translations of \( \omega \) become relevant even if the centres of translation are far away from each other (see Lemma 3.5). This phenomenon, due to the polynomial decay of \( \omega \), resembles to what occur in other context involving nonlocal operators.

Once one has dealt with the nonlinearity, then the integral term involving the potentials has to be compared with the analogous term in \((P_\infty)\); in this comparison, decay estimates such as (1.5) are useful and, coherently with the decay of \( \omega \), we assume here a polynomial type decay on the potential \( V \); this appears to be another novelty compared with the nonlinear Schrödinger equation and it is closer to what happen in the zero mass case (see [8,9]).
Existence results of symmetric solutions for \((P_\nu)\), also in the presence of a magnetic potential, have been obtained in \([6]\) for \(p\) satisfying (1.1) when \(\ell(G) = \infty\), or for \(p \geq 2\) and satisfying (1.1) and for potentials \(V(x)\) going to \(V_\infty\) from below. While, the case of \(\ell(G)\) finite, and \(V(x)\) approaching to \(V_\infty\) from above or oscillating has been tackled in \([10]\) for \(p \geq 2\) and \(\ell(G) \geq 3\).

So this is, to our knowledge, the first existence result of a positive solution enjoying a finite number of symmetries when \(p < 2\).

In addition, we succeed in obtaining a refined asymptotic analysis also for \(p \geq 2\). This enables us to weaken the request on the decay of \(V\) and on the group of symmetry allowing \(\ell(G) \geq 2\).

Our results depending on the range where the exponent \(\alpha\) lies are the following.

**Theorem 1.2.** Let \(G\) be a closed subgroup of the linear isometries of \(\mathbb{R}^N\), with \(\ell(G) \geq 2\) and finite. Assume (1.2) and

\[
\frac{N - 2}{N + \alpha} < \frac{1}{p} < \frac{1}{2}, \quad \text{or} \quad p = 2, \quad \alpha \leq N - 1. \tag{1.6}
\]

Let \(\mu_G\) be defined in (2.4) and suppose that, for every \(x \in \mathbb{R}^N\) it holds

\[
V(x) \leq V_\infty + A_0(1 + |x|)^\nu e^{-\beta|x|}, \quad \text{with} \quad A_0 > 0, \quad \beta \geq \mu_G \sqrt{V_\infty} \tag{1.7}
\]

with the exponent \(\sigma\) satisfying the following condition depending on the constant \(\mu_G\)

\[
\begin{cases}
\sigma \in \mathbb{R} & \text{if } \beta > \mu_G \sqrt{V_\infty} \\
\sigma < \min\{-1, -\frac{N-1}{2} + 2\tau_1\} & \text{if } \beta = \mu_G \sqrt{V_\infty},
\end{cases} \tag{1.8}
\]

and

\[
\tau_1 = \begin{cases}
0 & \text{if } p > 2 \text{ or } p = 2, \ \alpha < N - 1 \\
\nu \sqrt{V_\infty} & \text{if } p = 2, \ \alpha = N - 1,
\end{cases}
\]

with \(\nu > 0\) given in (2.11).

Then, if \(V\) is \(G\)-invariant, Problem \((P_\nu)\) has a positive \(G\)-invariant solution.

**Theorem 1.2** does not cover the case \(p = 2\) and \(\alpha \in (N - 1, N)\). This is because for \(\alpha\) lying in this range the decay of least action solutions of \((P_\infty)\) has a perturbation in the exponential term (see (2.9)) so that, hypothesis (1.7) is not suitable any more, and we will prove the following result.

**Theorem 1.3.** Let \(G\) be a closed subgroup of linear isometries of \(\mathbb{R}^N\), with \(\ell(G) \geq 2\) and finite. Assume (1.2) and

\[
p = 2, \quad \alpha \in \left(\frac{N - 1, N - \frac{1}{2}}{2}\right), \quad \text{and let} \quad \gamma = 1 - N + \alpha \in \left(0, \frac{1}{2}\right). \tag{1.9}
\]

Let \(\mu_G\) be defined in (2.4) and suppose that, for every \(x \in \mathbb{R}^N\) it holds

\[
V(x) \leq V_\infty + A_0(1 + |x|)^\nu e^{-\beta|x|+c'|x|}, \quad \text{with} \quad A_0 > 0, \quad \beta \geq \mu_G \sqrt{V_\infty} \tag{1.10}
\]

where (recalling (2.9), and (2.11)) the constants \(\gamma' \in [0, 1), c' \geq 0, \sigma \in \mathbb{R}\) are such that

(i) If \(\beta > \mu_G \sqrt{V_\infty}\), then \(\gamma' \in [0, 1), c' \geq 0, \sigma \in \mathbb{R}\).
(ii) If \( \beta = \mu_G \sqrt{V_\infty} \), we assume \( \gamma' \leq \gamma \) and \( c' \geq 0, \quad \sigma \in \mathbb{R}, \quad \text{if } \gamma' < \gamma \).

(iii) If \( \beta = \mu_G \sqrt{V_\infty} \) and \( \gamma' = \gamma \) we assume that \( \mu_G < 2, \quad c' \leq 2^{1-\gamma} c_G \mu^\gamma_G \) and \( \sigma \) is such that
\[
\begin{cases}
\sigma \in \mathbb{R}, & \text{if } c' < 2^{1-\gamma} c_G \mu^\gamma_G, \\
\sigma < \frac{N-1}{2} + \frac{\gamma}{2} + 2 \tau_2 & \text{if } c' = 2^{1-\gamma} c_G \mu^\gamma_G, \quad \text{with } \tau_2 = \begin{cases}
0 & \text{if } \alpha < N - \frac{1}{2} \\
\frac{\sqrt{\gamma}}{8} & \text{if } \alpha = N - \frac{1}{2}.
\end{cases}
\end{cases}
\]

Then if \( V \) is \( G \)-invariant, Problem (P\( \sigma \)) has a positive \( G \)-invariant solution.

Theorems 1.2 and 1.3 extend Theorem 1.3 in [10] under various aspects. First of all, as already mentioned, we also include the case \( \ell(G) = 2 \). Moreover, even in the case \( \ell(G) = 3 \), we improve the decay assumptions on \( V \), and we also study the threshold case \( \beta = \mu_G \sqrt{V_\infty} \).

As mentioned above, when \( p > 2 \) the decay of \( \omega \) is analogous to the one of the unique positive solution of the nonlinear autonomous Schrödinger equation (see [5]), so that, we are naturally lead to assume that the potential \( V(x) \) approach its limit at infinity decaying in an exponential way too. Moreover, differently to the case of Theorem 1.1 here we see the effect of the symmetry in the decay due to the presence of the constant \( \mu_G \), which takes into account the least distance between two elements of the \( G \)-orbit of a point \( x \) in the unit sphere. This marks another relevant difference with the case \( p < 2 \) as \( \mu_G \) does not play any role in this latter case.

When the exponent \( \beta \) in the decay of \( V(x) \) reaches the threshold \( \mu_G \sqrt{V_\infty} \), the corrections become important. In particular, in Theorem 1.2 this role is played by the polynomial part and we have to take into account that when \( p = 2 \) and \( \alpha = N - 1 \) a polynomial perturbation with exponent \( \tau_1 \) (see (2.8)) appears in the decay of \( \omega \), so that we are naturally lead to assume (1.8).

In the setting of Theorem 1.3 we first have to consider the fact that when \( \alpha \) overcomes \( N - 1 \), the decay of \( \omega \) changes again and an exponential perturbation arises (see (2.9)); in this situation one has to face new difficulties, which can be overcome by means of some new technical lemma (see Lemma 4.1, 4.2) which we believe that may be of independent interest. In addition, in this case the threshold is achieved when both \( \beta = \mu_G \sqrt{V_\infty} \) and \( \gamma' = \gamma \); at this point we need the condition \( \mu_G < 2 \) and we see that the constant \( c' \) becomes relevant: if \( c' < 2^{1-\gamma} c_G \mu^\gamma_G \) (\( c_G \) given in (2.9)) then the exponential term still guide the asymptotic, while if \( c' = 2^{1-\gamma} c_G \mu^\gamma_G \) again the polynomial part starts being the leading term and we arrive at the condition on \( \sigma \).

When \( p = 2, \alpha > N - \frac{1}{2} \), new perturbations appear in the decay of \( \omega \) (see (22)) and we expect that similar results could be obtained, by slightly modifying our arguments and at the price of some heavy technicalities. The first step would be to prove an extension of Lemma 4.1 and 4.2 to the case of functions with more involved exponential corrections.

Concluding, let us point out that our conditions on the potential \( V \) are somewhat sharp, meaning that, if they are not satisfied the decay of \( V \) may be not comparable with the asymptotic decay of the solutions of the limit problem, see also Remark 4.7.
This paper is organized as follows: in Section 2 we give the variational setting of the problem and some preliminary results, whereas in Section 3 we study the case $p < 2$, and prove Theorem 1.1. Theorem 1.2 and Theorem 1.3 will be proved in Section 4 through a unified approach.

2. Setting of the problem and preliminaries

In this section we introduce the symmetric framework in which we settle our problem. Let us observe that the use of symmetry turns out to be a useful and largely exploited tool when looking for existence results to \((P_0)\) (see [1, 11, 12, 15]).

In what follows, $G$ will represent a closed subgroup of linear isometries of $\mathbb{R}^N$.

Define the $G$-orbit of $x$ as $Gx = \{gx : g \in G\}$, and $\#Gx$ its cardinality. We set
\[
\ell(G) = \min \left\{ \#Gx : x \in \mathbb{R}^N \setminus \{0\} \right\}. \tag{2.1}
\]

As mentioned in the Introduction, the case $\ell(G) = +\infty$ has been treated in [6, Theorem 1.1]. Here, we will assume $\ell(G) < +\infty$.

**Remark 2.1.** In general, there may exist points such that $\#Gx > \ell(G)$. For instance, take in $\mathbb{R}^4 \cong \mathbb{C} \times \mathbb{C}$ the group $G = \mathbb{Z}_2 \times \mathbb{Z}_3$, where $\mathbb{Z}_l$ is the cyclic group generated by the $l$-th roots of the unit. Then the point $x = (0,0,0,1) \cong (0,i)$ has $\#Gx = 3$, and $\ell(G) = 2$ as it possible to see taking $y = (1,0,0,0) \cong (1,0)$.

In Section 3 we will just use the notion of $\ell(G)$, while in Section 4 the minimum distance between two different orbit points will play a role in the exponential decay estimates.

More precisely, let us consider the set $\Sigma$ given by
\[
\Sigma = \left\{ x \in S^{N-1} : \#Gx = \ell(G) \right\}. \tag{2.2}
\]

Let us define
\[
\mu(Gz) = \begin{cases} 
\inf \{|gz - hz| : g, h \in G, gz \neq hz\}, & \text{if } \#Gz \geq 2 \\
2|z| & \text{if } \#Gz = 1,
\end{cases} \tag{2.3}
\]
for every $z \in \Sigma$, and the extremum
\[
\mu_G = \inf_{z \in \Sigma} \mu(Gz). \tag{2.4}
\]

The following properties of $\Sigma$ and $\mu_G$ will be useful in Section 4.

**Lemma 2.2.** The set $\Sigma \neq \emptyset$ is a compact, $G$-invariant subset of $\mathbb{R}^N$ and $\mu_G$ is achieved.

**Proof.** The set $\Sigma$ is nonempty, because $\ell(G)$ is attained, and for every $x \in \mathbb{R}^N \setminus \{0\}$ such that $\#Gx = \ell(G)$, then $x/|x| \in S^{N-1}$ and $\#Gx = \#G(x/|x|)$, since elements in $G$ are linear transformations, so that $x/|x| \in \Sigma$. In addition, the $G$-invariance property is a direct consequence of the definition.

In order to prove that $\Sigma$ is closed, let $(x_n) \in \Sigma$ be such that $x_n \to x$. Arguing by contradiction, assume that $x \notin \Sigma$. Then the orbit of $x$ contains a number of points greater than $\ell(G)$, so that there exist $g_1, \ldots, g_{\ell+1} \in G$ with $g_ix \neq g_jx$ for every $i \neq j$, for
i, j = 1, . . . , ℓ + 1. As \( g_i x_n \rightarrow g_i x \) for every \( i \), we get that \( g_i x_n \neq g_j x_n \) if \( i \neq j \), for \( n \) large enough, i.e., \#Gx_n \geq \ell + 1 \), which cannot be as \( x_n \in \Sigma \). This shows that \( \Sigma \) is closed, and as it is contained in \( S^{N-1} \), it turns out to be a compact set.

Let us now define the function \( f : \Sigma \rightarrow \mathbb{R} \) as

\[
f(x) := \mu(Gx), \quad \text{where } \mu_G \text{ is defined in (2.3)},
\]

and prove that \( f \) is continuous. Let \( x_n \rightarrow x \) in \( \Sigma \). Choose \( g_1, \ldots, g_\ell \in G \) such that \( Gx = \{g_1 x, \ldots, g_\ell x\} \). Arguing as before, one obtains that \( g_i x_n \neq g_j x_n \) for \( i, j = 1, \ldots, \ell(G) \) and for \( n \) sufficiently large, so that \( G(x_n) = \{g_1 x_n, \ldots, g_\ell x_n\} \), because \( x_n \in \Sigma \). Then, the continuity immediately follows if \#Gx = 1, otherwise, it results

\[
f(x_n) = \min_{i \neq j} |g_i x_n - g_j x_n| \rightarrow \min_{i \neq j} |g_i x - g_j x| = f(x) \quad \text{as } n \rightarrow \infty.
\]

So \( f \) is continuous, as claimed. As a consequence, \( \mu_G \) is achieved. \( \square \)

The influence of symmetries will appear in the decay estimates for \( p \geq 2 \) through the constant \( \mu_G \). In the following remarks we give some information on \( \mu_G \) that will be useful in Section 4 and that illustrate some hypotheses of Theorem 1.3.

**Remark 2.3.** Notice that \( 0 < \mu_G \leq 2 \). Moreover, if \( \mu_G = 2 \) then \( \ell(G) = 1 \) or \( \ell(G) = 2 \).

Indeed, the first inequality is a direct consequence of the fact that \( \mu_G \) is attained. On the other hand, the second inequality follows by observing that the distance between two distinct points on the unit sphere is less or equal than two.

Furthermore, suppose by contradiction that \( \mu_G = 2 \) and \( \ell(G) \geq 3 \). Then, there exists \( x \in \Sigma \) such that \( \mu(Gx) = 2 \) and there exist \( g_1, g_2, g_3 \in G \) such that \( g_i x \neq g_j x \). Without loss of generality, we can assume that \( |g_1 x - g_2 x| = \mu(Gx) = \mu_G = 2 \), but then \( |g_1 x - g_3 x| < 2 \) as \( |g_i x| = 1 \) for every \( i = 1, 2, 3 \), which contradicts the fact that the minimum is \( \mu_G = 2 \).

**Remark 2.4.** In conclusion (iii) of Theorem 1.3 we assume \( \mu_G < 2 \). Notice that one can find groups such that \( \mu_G < 2 \) and \( \ell(G) = 2 \). For instance, let \( g \) the linear isometry in \( \mathbb{R}^3 \) which corresponds to a clockwise rotation of angle \( \pi/2 \) around the \( y \) axis, followed by a clockwise rotation of angle \( \pi \) around the \( z \) axis. Take the closed group acting on \( \mathbb{R}^3 \) generated by \( g \). Then \( \ell(G) \geq 2 \), as every point on the sphere is mapped by \( g \) in a point different from itself.

Moreover, consider the north pole \( N = (0, 0, 1) \). This point is mapped into \((1, 0, 0)\) by \( g \) and \( g^{-1} \), and \( g^2(N) = N = (g^{-1})^2(N) \), thus \#G\( N = 2 \), \( \ell(G) = 2 \) and \( N \in \Sigma \). However, the distance between \( N \) and \((1, 0, 0)\) is less than 2, thus \( \mu_G < 2 \).

As observed in the introduction, our results cover the case \( \ell(G) = 2 \). Note that there are many groups satisfying \( \ell(G) \geq 3 \) when \( N = 2n \) is even, but this is not the case if \( N \) is odd. For further remarks concerning \( \ell(G) \) see [10], pg.4.

We will work in the functional space

\[
H^1_G = \{u \in H^1(\mathbb{R}^N) : u(gx) = u(x) \quad \text{for any } g \in G, x \in \mathbb{R}^N\}
\]

dowered, thanks to (1.2), with the scalar product and norm

\[
(u, v)_V = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x) uv), \quad \|u\|^2_V = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2).
\]
Every symmetric solution to \( P \) is a critical point of the action functional \( I_V : H^1_G \to \mathbb{R} \) defined in (1.3). Indeed, \( I_V(u) \) is \( G \)-invariant as \( V \) is, so that the principle of symmetric criticality (23) applies.

Hypothesis (1.1) and Hardy-Littlewood-Sobolev inequality imply that \( I_V \) is a \( C^1 \) functional on \( H^1_G \), (see [20], Proposition 3.1), so that we can define
\[
\langle I_V(u), u \rangle = \| u \|^2_V - \int_{\mathbb{R}^N} (I_a * |u|^p) |u|^p
\]
and
\[
\mathcal{N}^G_V = \left\{ u \in H^1_G(\mathbb{R}^N) \setminus \{0\} : \langle I_V(u), u \rangle = 0 \right\}, \quad c^G_V = \inf_{u \in \mathcal{N}^G_V} I_V(u) \tag{2.5}
\]
Notice that twice differentiability of \( I_V \) holds only for \( p \geq 2 \), (see for instance [22]). As a consequence, \( \mathcal{N}^G_V \) is not, in general, a differentiable manifold. In order to overcome these difficulties we will use the approach in [27] (see Section 3.2).

In an analogous way, let us define \( I_\infty : H^1(\mathbb{R}^N) \to \mathbb{R} \) by
\[
I_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty u^2) - \frac{1}{2p} \int_{\mathbb{R}^N} (I_a * |u|^p) |u|^p
\]
where \( H^1(\mathbb{R}^N) \) is endowed with the scalar product and the norm
\[
(u, v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + V_\infty uv), \quad \| u \|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty u^2) \tag{2.6}
\]
and accordingly \( \mathcal{N}^G_\infty \) and \( c^G_\infty \) are defined for \( P_\infty \).

The existence of a least action solution to \( P_\infty \) is proved, under assumption (1.1), in Theorem 3.2 in [20]. Moreover, weak solutions are classical, and, up to translation and inversion of the sign, positive and radially symmetric, see [16, 22]. Precise decay asymptotic for solutions to \( P_\infty \) are given in Propositions 6.3, 6.5 and Remark 6.1 in [22], (see also [21]), depending on the value of \( p \).

Summarizing the following result holds.

**Theorem 2.5** (Theorem 4 pg.157 in [22]). Assume \( \alpha \in (0, N) \) and that \( p \) satisfies (1.1). Let \( \omega \) be a least action solution to \( P_\infty \). Then the following asymptotic estimates hold.

1. If \( p < 2 \), there exists a positive constant \( c \) such that
\[
\omega(x) = (c + o(1)) |x|^{-\frac{N+\gamma}{2}} \quad \text{as } |x| \to \infty. \tag{2.7}
\]
2. Under (1.6), it results
\[
\omega(x) = (c + o(1)) |x|^{-\frac{N+\gamma}{2}} e^{-\sqrt{V_\infty}|x|} \quad \text{as } |x| \to \infty. \tag{2.8}
\]
where \( \tau_1 = 0 \) if \( p > 2 \) or \( p = 2 \) and \( \alpha < N - 1 \); while \( \tau_1 = \frac{\sqrt{V_\infty}}{2} \) when \( p = 2 \) and \( \alpha = N - 1 \) and where \( \nu \) is a positive constant depending on the \( L^2(\mathbb{R}^N) \) norm of \( \omega \) (see (2.11) below).
3. If \( p = 2 \) and \( N - 1 < \alpha \leq N - \frac{1}{2} \), then \( \omega \) decays as follows
\[
\omega(x) = (c + o(1)) |x|^{-\frac{N+\gamma}{2}} e^{-\sqrt{V_\infty}|x|+c|x|^\gamma}, \quad \text{with } \gamma = 1 - N + \alpha, \tag{2.9}
\]
and where \( c_\gamma = \frac{1}{2} \nu^{1-\gamma} \sqrt{V_\infty}; \tau_2 = 0 \) if \( \alpha < N - \frac{1}{2} \) and \( \tau_2 = \frac{\sqrt{V_\infty}}{8} \) when \( \alpha = N - 1/2 \), and \( \nu \) is as in (2.11) below.
The above result shows that the decay of the least action solutions strongly depends on the interaction of the Riesz potential and on the nonlinearity. First of all, when \( p > 2 \) we see the same decay behavior as in the local case (see [5]); while for \( p < 2 \) the presence of the convolution term forces the decay to be of polynomial type, more resembling the case of nonlocal fractional operators (see [14] and [22] p. 157-158).

The threshold is \( p = 2 \). As observed in [19], in this range we see different perturbations on the decay of \( \omega \) depending on \( \alpha \). In general, it holds

\[
\omega(x) = (c + o(1)) \frac{e^{-\sqrt{V_\infty}Q(|x|)}}{|x|^{\frac{N-1}{2}}} \quad \text{as } |x| \to \infty, \tag{2.10}
\]

where

\[
Q(t) = \int_0^t \sqrt{1 - \frac{t^{N-\alpha}}{s^{N-\alpha}}} \, ds, \quad v^{N-\alpha} = \frac{1}{V_\infty} \frac{\Gamma \left( \frac{N-\alpha}{2} \right)}{\Gamma \left( \frac{\alpha}{2} \right) \pi^{(N-\alpha)/2}} \int_{\mathbb{R}^N} |\omega|^2, \tag{2.11}
\]

and \( v \) only depends on \( \|\omega\|_2^2 \) (see [22]). Then, taking into account the Taylor expansion of the square root, one can see that (2.8) still holds when \( \alpha < N - 1 \); while a perturbation in the polynomial part occurs if \( \alpha = N - 1 \) (which includes the physical case \( N = 3, \alpha = 2, p = 2 \)), and more and more perturbations appear as \( \alpha \) increases. In particular, if \( N - 1 < \alpha \leq N - \frac{1}{2} \), the decay becomes as stated in (2.9). As a last information, when \( \alpha > N - \frac{1}{2} \) the decay will include more and more terms in the Taylor expansion of the function \( Q \) (for more details see also Remark 6.1 in [21]).

In order to obtain analogous decay estimates on the convolution term the following lemma will be crucial

**Lemma 2.6.** Let \( h \geq 0, h \in L^\infty \) such that

\[
\sup_{\mathbb{R}^N} h(x) (1 + |x|)^s < +\infty, \quad \text{for some } s > N. \tag{2.12}
\]

Then

\[
I_\alpha * h(x) = I_\alpha(x) \|h\|_1 (1 + o(1)). \tag{2.13}
\]

Moreover, let \( f \in L^p_{\text{loc}}(\mathbb{R}^N), f \geq 0, \) be such that

\[
\sup_{\mathbb{R}^N} f(x) (1 + |x|)^\eta < +\infty, \quad \text{with } p\eta > N.
\]

For every \( z_1, z_2 \in \mathbb{R}^N \), it results

\[
\limsup_{|x| \to \infty} |x - z_1|^{(N-\alpha)\frac{p-1}{p}} |x - z_2|^{(N-\alpha)\frac{1}{p}} \int_{\mathbb{R}^N} \frac{f(y - z_1)^{p-1} f(y - z_2)}{|y - x|^{N-\alpha}} \, dy < +\infty. \tag{2.14}
\]

**Proof.** The first conclusion follows immediately from Lemma 6.2 in [22]. In order to prove the second one, we observe that

\[
|y - x|^{N-\alpha} = |y - x|^{\frac{p-1}{p}(N-\alpha)} |y - x|^{(N-\alpha)\frac{1}{p}}.
\]
Therefore, by applying Hölder’s inequality, one has
\[
\int_{\mathbb{R}^N} \frac{f(y - z_1)^p - f(y - z_2)}{|y - x|^{N-a}} dy \leq \left( \int \frac{f(y - z_1)^p}{|y - x|^{N-a}} \right)^{\frac{p-1}{p}} \left( \int \frac{f(y - z_2)^p}{|y - x|^{N-a}} \right)^{\frac{1}{p}} \]
\[
= \left[ \int \frac{f(y)^p dy}{|y + z_1 - x|^{N-a}} \right]^{\frac{p-1}{p}} \left[ \int \frac{f(y)^p dy}{|y + z_2 - x|^{N-a}} \right]^{\frac{1}{p}} \]
\[
= \left[ (I_h * f^p)(x - z_1) \right]^{\frac{p-1}{p}} \left[ (I_h * f^p)(x - z_2) \right]^{\frac{1}{p}}.
\]
The conclusion follows by applying (2.13) with \(h = f^p\), and \(s = p\eta > N\). \(\square\)

As an immediate consequence of Lemma 2.6, we get the following asymptotic decay of the convolution term
\[
I_h * \omega^p(x) = I_h(x) \|\omega\|^p (1 + o(1)).
\] (2.15)

Indeed, if \(p < 2\), (2.7) yields that (2.12) is satisfied by \(h = \omega^p\) with \(s = p\frac{N-a}{2-N}\). Note that \(s > N\) as \(p > \frac{2N}{N-a}\), which is always true in our setting, thanks to (1.1). Moreover, in the case \(p \geq 2\), then (2.12) is satisfied by \(h = \omega^p\) for any \(s\).

Let us conclude this section by introducing the threshold that will guide our study. Let \(\Sigma\) be defined in (2.2). Then for every \(z \in \Sigma\) there are \(g_1, \ldots, g_{\ell(G)} \in G\) such that \(g_i \neq g_j\) whenever \(g_i \neq g_j\). We denote with \(\omega_{i,R}(x)\) a solution of the limit problem translated in \(Rg_i z\), namely
\[
\omega_{i,R}(x) = \omega(x - Rg_i z), \quad \text{for } i = 1, \ldots, \ell(G).
\] (2.16)

Then we define
\[
\varepsilon_R^i = \int_{\mathbb{R}^N} (I_h * \omega_{i,R}^p) \omega_{i,R}^{p-1} \omega_{j,R} = \int_{\mathbb{R}^N} \left[ \nabla \omega_{i,R} \cdot \nabla \omega_{j,R} + V \omega_{i,R} \omega_{j,R} \right],
\]
\[
\varepsilon_R = \sum_{i \neq j} \varepsilon_R^i.
\] (2.17)

In the following sections we will see that \(\varepsilon_R\) has different asymptotic decays depending on \(p\). This will lead us to assume different decay assumptions on the potentials \(V\) in order to get our existence results. Moreover, in order to prove that \(c_{\ell(G)}^G\) given in (2.5) is an action level where the Palais-Smale condition holds, we will evaluate \(I_V\) on the competitor
\[
\chi_{R,z} = \sum_{i=1}^{\ell(G)} \omega_{i,R}, \quad \text{where } \omega_{i,R} \text{ is defined in (2.16)}.
\] (2.18)

3. Case \(p < 2\)

This section is devoted to the proof of Theorem 1.1, which will be carried on by minimizing \(I_V\) on \(N_{\ell(G)}^G\). Since it is known that the Palais-Smale condition is satisfied for any level below a suitable value which depends on \(\ell(G)\) and on \(c_{\ell(G)}^G\), (see Proposition 3.1 in [6]), the main point consists in finding a competitor in \(N_{\ell(G)}^G\) showing that the minimum value belongs to the range where compactness holds.
In order to do this evaluation, we first analyze the decay \( \epsilon_R \) as \( R \to \infty \) (see Lemma 3.1), and in Lemma 3.4 we show that the part involving the potential is actually small with respect to \( \epsilon_R \). Then, we will analyze the behavior of integrals involving the nonlinearities; as explained in the Introduction, the presence of a power \( p < 2 \) prevents one from exploiting algebraic results for power-like nonlinearities, (see [4, 10]), but we will take care of the non-locality feature of the problem by performing a very careful analysis of the integrals involved, and on the behavior of \( \epsilon_R \) (see Lemma 3.5 and Proposition 3.6). Then in subsection 3.2 we will conclude the proof also exploiting the approach in [27] as for \( p < 2 \), \( N^C_{\Psi} \) is not of class \( C^1 \).

### 3.1. Asymptotical analysis

#### Lemma 3.1.

Let \( p \) satisfy \( \text{(1.4)} \). Then, for \( R \) large enough

\[
\epsilon_R^{ij} \sim R^{-\frac{N-a}{2-p}}, \quad \text{where } \epsilon_R^{ij} \text{ is introduced in (2.17)}.
\]

#### Remark 3.2.

Lemma 3.1 shows that for \( p \geq 2 \), the distance between any two points in the orbit of \( z \) does not play any role in determining the decay of \( \epsilon_R \). This marks a relevant difference with the local case and with the case \( p \geq 2 \) (see Lemma 4.4).

#### Remark 3.3.

Let \( a_{ij} := |g_iz - g_jz| \). Notice that Lemma 2.2 and Remark 2.3 imply that \( 0 < \mu_G \leq \mu(Gz) \leq d_{ij} = |g_iz - g_jz| \leq 2 \). Indeed, as \( g_i \) is an isometry, \( |g_iz| = |z| = 1 \), and distinct points on the sphere have distance \( \leq 2 \).

#### Proof.

Let us first observe that, exploiting (2.17), (2.7) and (2.15), one has

\[
\epsilon_R^{ij} \leq C \int_{\mathbb{R}^N} (1 + |x-Rg_iz|)^{-\frac{N-a}{2-p}} (1 + |x-Rg_jz|)^{-\frac{N-a}{2-p}} dx.
\]

We now apply Lemma A.1 with \( a = a' = -\frac{N-a}{2-p} \) and \( \xi = Rg_i z - Rg_j z \), and take into account (1.1) to get \( \epsilon_R^{ij} \leq CR^{-\frac{N-a}{2-p}} \).

In order to get the estimates from below, one takes into consideration that \( \omega \) is positive, radially symmetric and decreasing to obtain that

\[
\inf_{x \in B_1(0)} I_a * \omega^p(x) \geq \inf_{x \in B_1(0)} \frac{A_a}{R_0^{N-a}} \int_{B_{R_0}(x)} \omega^p(y) dy \geq A_a \frac{|B_{R_0}(0)|}{R_0^{N-a}} \min_{y \in B_{R_0+1}(0)} \omega^p(y) \geq C > 0.
\]

Hence, again exploiting (2.7), one has (denoting with \( C \) possibly different constants)

\[
\epsilon_R^{ij} \geq \int_{B_1(Rg_iz)} (I_a \ast \omega^p R_0 \omega_{R_0}^{p-1} \omega_{R_0} R_0 = \int_{B_1(0)} (I_a \ast \omega^p(x)) \omega_{R_0}^{p-1}(x) \omega(x-R(g_iz - g_jz)) dx
\]

\[
\geq \inf_{x \in B_1(0)} (I_a \ast \omega^p(x) \omega_{R_0}^{p-1}(x)) \int_{B_1(0)} \omega(x-R(g_iz - g_jz)) dx
\]

\[
\geq C \int_{B_1(0)} (1 + |x-R(g_iz - g_jz)|)^{-\frac{N-a}{2-p}} \geq CR^{-\frac{N-a}{2-p}},
\]

where the last inequality can be deduced observing that, as pointed out in Remark 3.3 \( d_{ij} \leq 2 \) so that, if \( |x| < 1 \), the following inequality holds for every \( R \geq 1 \)

\[
1 + |x-R(g_iz - g_jz)| < 1 + |x| + R |g_iz - g_jz| < 4R.
\]
Lemma 3.4. If \( p \) satisfies (1.4) and \( V \) satisfies (1.5) then it holds
\[
\mathcal{A}_V := \int_{\mathbb{R}^N} (V(x) - V_\infty) (\chi_{R \bar{z}})^2 \leq o(\varepsilon_R).
\]

**Proof.** The conclusion is a direct consequence of (2.7) and Lemma A.1 as
\[
\int_{\mathbb{R}^N} (V(x) - V_\infty) \omega^2_{L,R} \leq C \int_{\mathbb{R}^N} (1 + |x|)^{-\beta} (1 + |x - Rg_i z|)^{-2 \frac{N-a}{2-p}} \leq CR^{-\tau}
\]
where \( \tau = \min\{\beta, 2 \frac{N-a}{2-p} - 2\} > 0 \).

In order to compare the asymptotic behavior of the nonlinearity term with respect to \( \varepsilon_R \), we first need to deepen our knowledge of the behavior of the threshold \( \varepsilon_R \).

Having in mind (2.17), let us define, for \( i, j = 1, \ldots, \ell(G) \)
\[
\varepsilon_{ij}^{ij} = \int_{\mathcal{R}_{kl}} \omega^p_{l,R}(\zeta) \omega^{p-1}_{L,R}(\theta) \omega_{L,R}(\theta) \frac{d\zeta d\theta}{|\theta - \zeta|^{N-a}}
\]  
(3.2)

where the set \( \mathcal{R}_{kl} \) is defined, for \( k, l = 1, \ldots, \ell(G) \) and for \( \rho \in (0, \min_{i,j} d_{ij}/2) \) fixed, as
\[
\mathcal{R}_{kl} := \{ (\theta, \zeta) : |\theta - Rg_k z| < \rho R, |\zeta - Rg_l z| < \rho R \}.
\]
(3.3)

In the following lemma we detect all the contribution terms in \( \varepsilon_R \) that actually play a relevant role. We will see that in this study the presence of the convolution term will be important.

**Lemma 3.5.** The following expansion holds
\[
\varepsilon_R = \sum_{i \neq j} (\varepsilon_{ij}^{ij} + \varepsilon_{ji}^{ji} + \varepsilon_{ii}^{ii}) + o(\varepsilon_R).
\]

**Proof.** Taking into account (2.17), we need to show that, for every \( (i, j) \), \( \varepsilon_{ij}^{ij} \) restricted to the set \( (\mathcal{R}_{ij} \cup \mathcal{R}_{ji} \cup \mathcal{R}_{ii})^c \) is \( o(\varepsilon_R) \). First notice that
\[
(\mathcal{R}_{ij} \cup \mathcal{R}_{ji} \cup \mathcal{R}_{ii})^c = \Omega_1 \cup \Omega_2 \cup \mathcal{R}_{jj},
\]
where
\[
\Omega_1 = (B_i \cup B_j)^c \times \mathbb{R}^N \quad \text{where} \quad B_k = B_{\rho R}(Rg_k z),
\]
\[
\Omega_2 = (B_i \cup B_j) \times (B_i \cup B_j)^c.
\]

In Figure 1 we draw an example for \( i = 1, j = 2, \ell(G) = 2 \). Let us start estimating the integral on \( \Omega_1 \) and define the following subsets of \( \mathbb{R}^N \)
\[
E^+ = \{ \theta \in \mathbb{R}^N : |\theta - Rg_i z| > |\theta - Rg_j z| \},
\]
\[
E^- = \{ \theta \in \mathbb{R}^N : |\theta - Rg_i z| < |\theta - Rg_j z| \}.
\]
(3.4)
Then, exploiting (2.7), and (2.15) we have

$$\iint_{\Omega} \omega_{i,R}^p(\zeta) \omega_{i,R}^{p-1}(\theta) \omega_{j,R}(\theta) \frac{d\zeta d\theta}{|\theta - \zeta|^{N-\alpha}} \leq \int_{(B_i \cup B_j)^c} (I_{\alpha} * \omega_{i,R}^p(\theta)) \omega_{i,R}^{p-1}(\theta) \omega_{j,R}(\theta) \ d\theta$$

$$\leq C \int_{(B_i \cup B_j)^c} |\theta - R g_i z|^{-\frac{N\alpha}{2-p}} |\theta - R g_j z|^{-\frac{N\alpha}{2-p}} \ d\theta$$

$$\leq C \int_{(B_i \cup B_j)^c \cap E^+} |\theta - R g_i z|^{-2} d\theta$$

$$+ C \int_{(B_i \cup B_j)^c \cap E^-} |\theta - R g_j z|^{-2} d\theta$$

$$\leq C \int_{\rho R}^{+\infty} r^{-2} \frac{\rho}{2-p+N-1} \ d\rho = CR^{N-2} = o(\epsilon R),$$

where the last inequality can be deduced noting that $(B_i \cup B_j)^c \cap E^+ \subset (B_i)^c$ (similarly $(B_i \cup B_j)^c \cap E^- \subset (B_j)^c$), and applying Lemma 3.1.
Now we estimate the integral on $\Omega_2$. We have, exchanging integrals, and exploiting (2.17) and (2.14), with $f = \omega$, $z_1 = Rg_i z$ and $z_2 = Rg_j z$

$$\int\int_{\Omega_2} \frac{\omega^p_{i,R}(\zeta) \, \omega^{p-1}_{i,R}(\theta) \omega_{j,R}(\theta)}{|\theta - \zeta|^{N-a}} \, d\zeta d\theta \leq \int (B_i \cup B_j)^c \int_{\mathbb{R}^N} \frac{\omega^p_{i,R}(\zeta) \omega^{p-1}_{i,R}(\theta) \omega_{j,R}(\theta)}{|\theta - \zeta|^{N-a}} \, d\zeta d\theta \leq C \int_{(B_i \cup B_j)^c} \frac{\omega^p_{i,R}(\zeta) \omega^{p-1}_{i,R}(\theta) \omega_{j,R}(\theta)}{|\zeta - Rg_i z|^{(N-a)\frac{p}{p-2}} + |\zeta - Rg_j z|^{\frac{N-a}{p}}} \, d\zeta \leq C \int_{(B_i \cup B_j)^c} \frac{\omega^p_{i,R}(\zeta) \omega^{p-1}_{i,R}(\theta) \omega_{j,R}(\theta)}{|\zeta - Rg_i z|^{(N-a)(\frac{p}{p-2})} + |\zeta - Rg_j z|^{\frac{N-a}{p}}} \, d\zeta.

Then splitting in $E^+$ and in $E^-$ (see (3.4)) and applying Lemma 3.1 one has

$$\int\int_{\Omega_2} \frac{\omega^p_{i,R}(\zeta) \, \omega^{p-1}_{i,R}(\theta) \omega_{j,R}(\theta)}{|\theta - \zeta|^{N-a}} \, d\zeta d\theta \leq C \int_{(B_i \cup B_j)^c \cap E^+} |\zeta - Rg_i z|^{-\frac{2N}{p-2}} \, d\zeta + C \int_{(B_i \cup B_j)^c \cap E^-} |\zeta - Rg_i z|^{-\frac{2N}{p-2}} \, d\zeta \leq CR^{-\frac{2N}{p-2} + N} = o(\varepsilon_R).$$

To complete the proof we have to study $\varepsilon_{jj}$. Recall that $\mathcal{R}_{jj} = B_i \times B_j$; moreover, (2.7) yields

$$\int\int_{\mathcal{R}_{jj}} \frac{\omega^p_{i,R}(\zeta) \, \omega^{p-1}_{i,R}(\theta) \omega_{j,R}(\theta)}{|\theta - \zeta|^{N-a}} \, d\zeta d\theta \leq C \int B_i \int_{B_j} \frac{\omega_{j,R}(\theta) \, d\theta}{(1 + |\zeta - Rg_i z|)^{\frac{p}{p-2}} \int_{B_j} (1 + |\theta - Rg_i z|)^{(p-1)\frac{N-a}{p-2}} \, d\theta \, d\theta} \int B_j \int_{B_i} \frac{\omega_{j,R}(\theta) \, d\theta}{(1 + |\zeta - Rg_i z|)^{\frac{p}{p-2}} \int_{B_i} (1 + |\theta - Rg_i z|)^{(p-1)\frac{N-a}{p-2}} \, d\theta \, d\theta}.

In addition, let us observe that for every $j \neq i$, and for every $\zeta \in \mathbb{R}^N$ it holds

$$|\zeta - Rg_j z| < \rho R \Rightarrow |\zeta - Rg_i z| \geq |Rg_j z - Rg_i z| - |\zeta - Rg_j z| > \rho R.

Then, recalling (2.15), one gets

$$\int\int_{\mathcal{R}_{jj}} \frac{\omega^p_{i,R}(\zeta) \, \omega^{p-1}_{i,R}(\theta) \omega_{j,R}(\theta)}{|\theta - \zeta|^{N-a}} \, d\zeta d\theta \leq \frac{C}{R^{2(p-1)\frac{N-a}{p-2}}} \int_{B_j} (I_{a} * \omega_{j,R}(\zeta)) \, d\zeta \leq \frac{C}{R^{2(p-1)\frac{N-a}{p-2}}} \int B_j \frac{d\zeta}{(1 + |\zeta - Rg_j z|)^{N-a}}.

Computing the last integral by using polar coordinates yields (denoting with C possibly different constants)

$$\int\int_{\mathcal{R}_{jj}} \frac{\omega^p_{i,R}(\zeta) \, \omega^{p-1}_{i,R}(\theta) \omega_{j,R}(\theta)}{|\theta - \zeta|^{N-a}} \, d\zeta d\theta \leq CR^{1-p+1}\frac{N-a}{p} R^a = o(\varepsilon_R),$$

where the last equality follows from direct computations, taking into account Lemma 3.1 and (1.1).

We are now in the position to deal with the nonlinearity term. \qed
Proposition 3.6. Let \( \chi_{R,z}^p \) be defined in (2.18), \( \varepsilon_R \) in (2.17) and \( \varepsilon_{ki}^p \) in (3.2). It results
\[
\int_{\mathbb{R}^N} (I_a \ast \chi_{R,z}^p) \chi_{R,z}^p \geq \sum_{k=1}^{\ell(G)} \int_{\mathbb{R}^N} (I_a \ast \omega_{k,R}^p) \omega_{k,R}^p + p\sum_{k=1}^{\ell(G)} \sum_{i \neq k} \varepsilon_{ki}^p + o(\varepsilon_R).
\]

Proof. Recalling (3.3), let us set
\[
\Omega = (\mathbb{R}^N \times \mathbb{R}^N) \setminus \left( \bigcup_{k,l=1}^{\ell(G)} \mathcal{R}_{kl} \right),
\]
and observe that from our choice of \( \rho \) it follows that the sets \( \mathcal{R}_{kl} \) are disjoint, so that
\[
\int_{\mathbb{R}^N} (I_a \ast \chi_{R,z}^p) \chi_{R,z}^p \geq \sum_{k,l=1}^{\ell(G)} \int_{\mathcal{R}_{kl}} \chi_{R,z}^p(\zeta) \chi_{R,z}^p(\theta) \, d\zeta d\theta + \int_{\Omega} \int_{\mathbb{R}^N} \chi_{R,z}^p(\zeta) \chi_{R,z}^p(\theta) \, d\zeta d\theta
\]
\[
\geq \sum_{k,l=1}^{\ell(G)} \int_{\mathcal{R}_{kl}} \chi_{R,z}^p(\zeta) \chi_{R,z}^p(\theta) \, d\zeta d\theta
\]
\[
= \sum_{k=1}^{\ell(G)} \int_{\mathcal{R}_{kk}} \chi_{R,z}^p(\zeta) \chi_{R,z}^p(\theta) \, d\zeta d\theta
\]
\[
+ \sum_{k=1}^{\ell(G)} \sum_{i \neq k} \int_{\mathcal{R}_{kl}} \chi_{R,z}^p(\zeta) \chi_{R,z}^p(\theta) \, d\zeta d\theta
\]
where the inequality follows from the positivity of the function \( \omega \). Let us consider the first integral term on the right side. Taking into account (2.18) and applying Bernoulli’s inequality
\[
\chi_{R,z}^p = \left( \sum_{i=1}^{\ell(G)} \omega_{i,R}^p \right)^p \geq \omega_{k,R}^p + p \sum_{i \neq k} \omega_{k,R}^{p-1} \omega_{i,R}^p,
\]
we get
\[
\int_{\mathcal{R}_{kk}} \chi_{R,z}^p(\zeta) \chi_{R,z}^p(\theta) \frac{d\zeta d\theta}{|\theta - \zeta|^{N-a}} \geq \int_{\mathcal{R}_{kk}} \omega_{k,R}^p(\zeta) \omega_{k,R}^{p-1}(\theta) \, d\zeta d\theta
\]
\[
+ p \sum_{i \neq k} \int_{\mathcal{R}_{kk}} \omega_{k,R}(\zeta) \omega_{k,R}^{p-1}(\theta) \omega_{i,R}(\zeta) \, d\zeta d\theta
\]
\[
+ p \sum_{i \neq k} \int_{\mathcal{R}_{kk}} \omega_{k,R}^{p-1}(\zeta) \omega_{i,R}(\zeta) \omega_{k,R}(\theta) \, d\zeta d\theta
\]
\[
+ p^2 \sum_{i \neq k} \int_{\mathcal{R}_{kk}} \omega_{k,R}^{p-1}(\zeta) \omega_{i,R}(\zeta) \omega_{k,R}^{p-1}(\theta) \, d\zeta d\theta.
\]
Note that the last term can be neglected, as it is positive, and that the following equality holds (see (3.2))

$$\int_{\mathcal{R}_{ik}} \omega_{k,R}^p(\theta) \omega_{k,R}^{p-1}(\zeta) \omega_{i,R}(\zeta) \frac{d\zeta}{|\theta - \zeta|^{N-\alpha}} \ d\theta = \int_{\mathcal{R}_{ik}} \omega_{k,R}^p(\zeta) \omega_{k,R}^{p-1}(\theta) \omega_{i,R}(\theta) \frac{d\theta}{|\zeta - \theta|^{N-\alpha}} \ d\zeta = \epsilon^i_{kk}.$$ 

Exploiting these facts into (3.6) one obtains

$$\int_{\mathcal{R}_{ik}} \chi_{R_{ik}}^p(\zeta) \chi_{R_{ik}}^p(\theta) \frac{d\zeta}{|\theta - \zeta|^{N-\alpha}} \ d\theta \geq \int_{\mathcal{R}_{ik}} \omega_{k,R}^p(\zeta) \omega_{k,R}^p(\theta) \frac{d\zeta}{|\theta - \zeta|^{N-\alpha}} \ d\zeta + 2p \sum_{i \neq k} \epsilon^i_{kk}. \quad (3.7)$$

On the other hand

$$\int_{\mathcal{R}_{ik}} \omega_{k,R}^p(\zeta) \omega_{k,R}^p(\theta) \frac{d\zeta}{|\theta - \zeta|^{N-\alpha}} \ d\theta = \int_{\mathbb{R}^N} (I_a * \omega^p_{k,R}) \omega^p_{k,R} - \int_{(\mathcal{R}_{ik})^c} \omega_{k,R}^p(\zeta) \omega_{k,R}^p(\theta) \frac{d\zeta}{|\theta - \zeta|^{N-\alpha}} \ d\zeta,$$

so that, (3.7) becomes

$$\int_{\mathcal{R}_{ik}} \chi_{R_{ik}}^p(\zeta) \chi_{R_{ik}}^p(\theta) \frac{d\zeta}{|\theta - \zeta|^{N-\alpha}} \ d\theta \geq 2p \sum_{i \neq k} \epsilon^i_{kk} + \int_{\mathbb{R}^N} (I_a * \omega^p_{k,R}) \omega^p_{k,R}(x) \ dx \ \tag{3.8}$$

$$- \int_{(\mathcal{R}_{ik})^c} \omega_{k,R}^p(\zeta) \omega_{k,R}^p(\theta) \frac{d\zeta}{|\theta - \zeta|^{N-\alpha}} \ d\zeta.$$

By using (2.7), (2.15) and recalling (3.4), we obtain

$$\int_{(\mathcal{R}_{ik})^c} \omega_{k,R}^p(\zeta) \omega_{k,R}^p(\theta) \frac{d\zeta}{|\theta - \zeta|^{N-\alpha}} \ d\theta \leq \int_{(B_k)^c} \omega_{k,R}^p(\zeta) d\zeta \int_{\mathbb{R}^N} \omega_{k,R}^p(\theta) \frac{d\theta}{|\theta - \zeta|^{N-\alpha}} \ d\zeta$$

$$+ \int_{(B_k)^c} \omega_{k,R}^p(\theta) d\theta \int_{\mathbb{R}^N} \omega_{k,R}^p(\zeta) \frac{d\zeta}{|\theta - \zeta|^{N-\alpha}} \ d\zeta$$

$$\leq C \int_{(B_k)^c} |\theta - R_{G_{ik}}|^{-2\frac{\alpha}{N-\alpha}} d\theta$$

$$= C \int_{r \leq r_R} r^{-2\frac{\alpha}{N-\alpha}} r^{N-1} \ dr = CR^{N-2\frac{\alpha}{N-\alpha}} = o(\varepsilon_R), \quad (3.9)$$

where the last equality follows from Lemma 3.1 and hypothesis (1.1).

Using (3.9) and (3.8) one deduces the following information concerning the contributions on $\mathcal{R}_{ik}$

$$\sum_{k=1}^{\ell(G)} \int_{\mathcal{R}_{ik}} \chi_{R_{ik}}^p(\zeta) \chi_{R_{ik}}^p(\theta) \frac{d\zeta}{|\theta - \zeta|^{N-\alpha}} \ d\theta \geq \sum_{k=1}^{\ell(G)} \int_{\mathbb{R}^N} (I_a * \omega^p_{k,R}) \omega^p_{k,R} + 2p \sum_{i \neq k} \epsilon^i_{kk} + o(\varepsilon_R).$$

Exploiting this in (3.5) we have

$$\int_{\mathbb{R}^N} (I_a * \chi_{R_{ik}}^p) \chi_{R_{ik}}^p \geq \sum_{k=1}^{\ell(G)} \int_{\mathbb{R}^N} (I_a * \omega^p_{k,R}) \omega^p_{k,R} + 2p \sum_{i \neq k} \epsilon^i_{kk} + o(\varepsilon_R) \tag{3.10}$$

$$+ \sum_{k=1}^{\ell(G)} \sum_{i \neq k} \int_{\mathcal{R}_{ik}} \chi_{R_{ik}}^p(\zeta) \chi_{R_{ik}}^p(\theta) \frac{d\zeta}{|\theta - \zeta|^{N-\alpha}} \ d\zeta.$$
Let us now study the integral terms on \( R_{kl} \) with \( k \neq l \). By applying Bernoulli’s inequality with respect to \( \omega_{j,R} \) one has

\[
\chi^p_{R,z} \geq \omega^p_{j,R} + p \sum_{i \neq j} \omega^{p-1}_{j,R} \omega_{i,R} \geq \omega^p_{j,R} + p \omega^{p-1}_{j,R} \omega_{i,R}.
\]

Computing the product and dropping some terms by positivity, one gets

\[
\chi^p_{R,z}(\zeta) \chi^p_{R,z}(\theta) \geq p \omega^p_{j,R}(\zeta) \omega^{p-1}_{j,R}(\theta) \omega_{i,R}(\theta) + p \omega^p_{j,R}(\theta) \omega^{p-1}_{j,R}(\zeta) \omega_{i,R}(\zeta).
\]

Since it results

\[
\iint_{R_{kl}} \frac{\chi^p_{R,z}(\zeta) \chi^p_{R,z}(\theta)}{|\theta - \zeta|^{N-a}} \, d\zeta d\theta = \frac{1}{2} \iint_{R_{kl}} \frac{\chi^p_{R,z}(\zeta) \chi^p_{R,z}(\theta)}{|\theta - \zeta|^{N-a}} \, d\zeta d\theta + \frac{1}{2} \iint_{R_{kl}} \frac{\chi^p_{R,z}(\zeta) \chi^p_{R,z}(\theta)}{|\theta - \zeta|^{N-a}} \, d\zeta d\theta
\]

we can apply (3.11) with \( j = k \) and \( i = l \) in the first integral and with \( j = l \) and \( i = k \) in the second one, to obtain

\[
\iint_{R_{kl}} \frac{\chi^p_{R,z}(\zeta) \chi^p_{R,z}(\theta)}{|\theta - \zeta|^{N-a}} \, d\zeta d\theta \geq \frac{p}{2} (\epsilon^l_{ik} + \epsilon^l_{kl} + \epsilon^k_{ik} + \epsilon^k_{kl}).
\]

(3.12)

Then, recalling (3.10), and (3.12), one has

\[
\int_{\mathbb{R}^N} (I_\alpha \ast \chi^p_{R,z}) \chi^p_{R,z} \geq \sum_{k=1}^{\ell(G)} \int_{\mathbb{R}^N} (I_\alpha \ast \omega^p_{k,R}) \omega^p_{k,R} \, dx + 2p \sum_{k=1}^{\ell(G)} \sum_{i \neq k} \epsilon^k_{ik}
\]

\[
+ p \sum_{k=1}^{\ell(G)} \sum_{i \neq k} (\epsilon_{ik} + \epsilon^l_{ik} + \epsilon^k_{ik} + \epsilon^l_{kl}) + o(\varepsilon_R)
\]

\[
= \sum_{k=1}^{\ell(G)} \int_{\mathbb{R}^N} (I_\alpha \ast \omega^p_{k,R}) \omega^p_{k,R} + 2p \sum_{k=1}^{\ell(G)} \sum_{i \neq k} \epsilon^k_{ik}
\]

\[
+ p \sum_{k=1}^{\ell(G)} \sum_{i \neq k} (\epsilon^l_{ik} + \epsilon^l_{kl}) + o(\varepsilon_R).
\]

So that, Lemma 3.5 implies

\[
2p \sum_{k=1}^{\ell(G)} \sum_{i \neq k} \epsilon^k_{ik} + p \sum_{k=1}^{\ell(G)} \sum_{i \neq k} \epsilon^l_{ik} = p \sum_{k=1}^{\ell(G)} \sum_{i \neq k} \epsilon^k_{ik} + p \sum_{k=1}^{\ell(G)} \sum_{i \neq k} \epsilon^l_{ik} + \epsilon^l_{kl}
\]

\[
= p \sum_{k=1}^{\ell(G)} \sum_{i \neq k} \epsilon^k_{ik} + p \sum_{l \neq k=1}^{\ell(G)} (\epsilon^l_{kl} + \epsilon^l_{kl} + \epsilon^k_{kk})
\]

\[
= p \sum_{k=1}^{\ell(G)} \sum_{i \neq k} \epsilon^k_{ik} + p \varepsilon_R + o(\varepsilon_R),
\]

yielding the conclusion. \( \square \)
3.2. Proof of Theorem 1.1. In this subsection we will complete the proof of Theorem 1.1. Let us start, recalling the useful properties concerning $N_V^G$.

**Lemma 3.7.** The following conclusions hold.

1. For each $u \in H^1_G \setminus \{0\}$ there exists a unique $T(u) > 0$ such that $T(u)u \in N_V^G$ (see (2.5)). Moreover, $T(u)u$ is the unique global maximum of $I_V(tu)$, $t \in [0, +\infty)$.
2. $c_V^G$ defined in (2.5) is strictly positive.
3. The set $N_V^G$ is a closed topological manifold of $H^1(\mathbb{R}^N)$ homeomorphic to the unit sphere.

Lemma 3.7 is a straightforward adaptation of Lemma 2.8, Proposition 2.9 and Corollary 2.10 in [27], as in our case $E^+ = H^1_G$ and $E^- = \emptyset$. Then, we just sketch the argument.

**Proof.** For any $u \in H^1_G \setminus \{0\}$

$$\langle I_V'(ru), ru \rangle = \|ru\|_V^2 - r^{2p-2} \int_{\mathbb{R}^N} (I_u * |u|^p) |u|^p$$

which is positive for $r > 0$ sufficiently small, it goes to $-\infty$ for $r \to +\infty$ and it is strictly decreasing in $r \in (0, \infty)$. Then, there exists a unique $T = T(u) > 0$ such that $T(u)u$ is the unique global maximum of $I_V(tu)$ and $T(u)u \in N_V^G$.

The Hardy-Littlewood-Sobolev inequality immediately implies $c_V^G > 0$.

In addition, the map $\hat{m} : H^1_G \setminus \{0\} \to N_V^G$ defined as $\hat{m}(u) = T(u)u$ is continuous, and its restriction to the unit sphere is a homeomorphism between $S^1$ and $N_V^G$ because $\hat{m}(u)$ is the unique global maximum of $I_V$ restricted to the set $\mathbb{R}^+u$ and $I_V$ is coercive on $N_V^G$ as

$$I_V(u) = \frac{1}{2} \left( 1 - \frac{1}{p} \right) \|u\|_V^2, \quad \forall u \in N_V^G. \quad \square$$

**Remark 3.8.** Having defined $\Psi : S^1 \hookrightarrow \mathbb{R}$ by $\Psi(u) = I_V(\hat{m}(u))$, and following the same arguments as in Proposition 2.9 and Corollary 2.10 in [27], it turns out that

$$\inf_{S^+} \Psi = \inf_{N_V^G} I_V = c_V^G, \quad \text{where } S^+ = \{w \in H^1_G : \|w\| = 1\}.$$ 

Moreover, $\Psi$ is $C^1$ and $\Psi'(u)v = T(u)I_V'(\hat{m}(u))v$. From this, we deduce that $u$ is a critical point of $\Psi$ on $S^+$ if and only if $\hat{m}(u)$ is a critical point of $I_V$ on $N_V^G$.

We are now in the position to detect the suitable action level where it is possible to recover a compactness property.

**Proposition 3.9.** Let $T_R := T(\chi_{R,\varepsilon})$ be defined in Lemma 3.7 and assume (1.4), (1.5). Then, the following inequality holds

$$I_V(T_R\chi_{R,\varepsilon}) \leq \ell(G)c_\infty - \frac{1}{2} \sum_{i=1}^{\ell(G)} \sum_{k \neq i} \epsilon_{\tilde{\mu}} + o(\varepsilon_R), \quad \text{as } R \to +\infty.$$
Proof. Let us first notice that, following Conclusion (1) of Lemma 3.7, it is easy to obtain that $T_R := T(\chi_{R,z})$ is given by

$$T_R^{2p-2} = \frac{||\chi_{R,z}||^2_V}{\int_{\mathbb{R}^N} (I_\alpha * \chi_{R,z}^p) \chi_{R,z}^p}.$$  

(3.13)

On the other hand, Lemma 3.4 and (2.17) yield

$$||\chi_{R,z}||^2_V \leq \sum_{i=1}^{\ell(G)} ||\omega_{i,R}||^2 + \sum_{i \neq j} \int_{\mathbb{R}^N} [\nabla \omega_{i,R} \cdot \nabla \omega_{j,R} + V_\alpha \omega_{i,R} \omega_{j,R}] + o(\varepsilon_R)$$

$$= \sum_{i=1}^{\ell(G)} ||\omega_{i,R}||^2 + \varepsilon_R + o(\varepsilon_R).$$

This, together with (3.13) and Proposition 3.6 implies

$$T_V(T_R \chi_{R,z}) = T_R \left[ \frac{1}{2} ||\chi_{R,z}||^2_V - \frac{T_R^{2p-2}}{2p} \int_{\mathbb{R}^N} (I_\alpha * \chi_{R,z}^p) \chi_{R,z}^p \right]$$

$$\leq \left( \frac{1}{2} - \frac{1}{2p} \right) \frac{\left[ \sum_{i=1}^{\ell(G)} ||\omega_{i,R}||^2 + \varepsilon_R + o(\varepsilon_R) \right]^{\frac{p}{p-1}}}{\left[ \int_{\mathbb{R}^N} (I_\alpha * \chi_{R,z}^p) \chi_{R,z}^p \right]^{\frac{1}{p-1}}}.$$ 

Using the expansion $(a + \varepsilon_t)^a = a^a + a^{a-1}t + o(t)$ and the notation

$$a := \sum ||\omega_{i,R}||^2 = \ell(G) ||\omega||^2,$$

we get

$$T_V(T_R \chi_{R,z}) \leq \left( \frac{1}{2} - \frac{1}{2p} \right) \left[ a + \varepsilon_R + o(\varepsilon_R) \right]^{\frac{p}{p-1}} \left[ a + p \varepsilon_R + p \sum_{i=1}^{\ell(G)} \sum_{k \neq i} e_{ik}^j + o(\varepsilon_R) \right]^{-\frac{1}{p-1}}$$

$$= \left( \frac{1}{2} - \frac{1}{2p} \right) \left[ a^{\frac{p}{p-1}} + \frac{p}{p-1} a^{\frac{1}{p-1}} \varepsilon_R + o(\varepsilon_R) \right]$$

$$\cdot \left[ a^{-\frac{p}{p-1}} - \frac{1}{p-1} a^{-\frac{p}{p-1}} (p \varepsilon_R + p \sum_{i=1}^{\ell(G)} \sum_{k \neq i} e_{ik}^j) + o(\varepsilon_R) \right]$$

$$= \left( \frac{1}{2} - \frac{1}{2p} \right) \left[ a - \frac{p}{p-1} \sum_{i=1}^{\ell(G)} \sum_{k \neq i} e_{ik}^j + o(\varepsilon_R) \right]$$

$$= \ell(G) c_\infty^G - \frac{1}{2} \sum_{i=1}^{\ell(G)} e_{ii}^{ij} + o(\varepsilon_R).$$

We are now in the position to prove Theorem 1.1.
Proof of Theorem 1.1. Let us first prove that
\[ c_G^V < \ell(G)c_G^\infty. \] (3.14)

This inequality can be obtained arguing as in estimate (3.1); indeed from (3.2) we infer
\[ \varepsilon_{ik}^{ii} = \int \int_{\{ |\zeta| \leq \rho R, |\theta| \leq \rho R \} \omega_p(\zeta)\omega_p^{p-1}(\theta) |\theta - \zeta|^{N-\alpha} \omega(\theta - R(g_iz - g_kz)) d\zeta d\theta \]
\[ \geq \int \int_{B(0) \times B(0)} \omega_p(\zeta)\omega_p^{p-1}(\theta) |\theta - \zeta|^{N-\alpha} \omega(\theta - R(g_iz - g_kz)) d\zeta d\theta \]
\[ \geq C \inf_{B(0) \times B(0)} \omega_p(\zeta)\omega_p^{p-1}(\theta) \int_{B(0)} \omega(\theta - R(g_iz - g_kz)) d\theta \]
\[ \geq CR \frac{\varepsilon_{ik}^{ii}}{C} \]

where \( C > 0 \) denotes possibly different constants and the last inequality comes from (2.7). This, together with Lemma 3.1 and Proposition 3.9 yield (3.14).

We can now reach the conclusion arguing as in the proof of Theorem 1.1 in [27]: we construct a minimizing Palais-Smale sequence for \( I_V \), then, taking into account (3.14), we can apply Proposition 3.1 in [6] to deduce that \( u_n \) is compact. Therefore, there exists \( u \in \mathcal{N}_G^V \) such that \( I_V(u) = c_G^V \). As \( |u| \in \mathcal{N}_G^V \) too, and \( c_G^V = I_V(u) = I_V(|u|) \) we can choose \( u \) positive. Hence by Lemma 3.7 we have a \( G \)-invariant positive solution. \( \square \)

4. Case \( p \geq 2 \)

This section is devoted to the proof of Theorem 1.2 and Theorem 1.3. The theoretical strategy of the proof is analogous to the previous section. In addition, in this case, the nonlinearities can be treated as in [10] and the main point is to deal with the potential term. For this range of exponents, the solutions of the limit problem \( (P_\infty) \) have an exponential decay, so instead of Lemma A.1 we will apply a result proved in [3] (see Lemma A.2) when \( p > 2 \) or \( p = 2 \) and \( \alpha < N - 1 \).

While, if \( p = 2 \) and \( \alpha \in (N - 1, N - 1/2] \), the solutions of the limit problem \( (P_\infty) \) have an exponential correction, see (2.9), and we will need to extend Lemma A.2 in order to treat these different decays, (Lemma 4.1 and 4.2). These results will allow us to prove that also in this situation the integral involving the potential decays faster. Then, the proofs of Theorem 1.2 and Theorem 1.3 will be given in Subsection 4.2.

4.1. Asymptotic Analysis. In this Subsection we first prove two extensions of Lemma A.2 to functions with an exponential correction in the decays. The proof, which is partly inspired by [3], requires a very careful analysis, and we will need to split it into two different Lemma, proved arguing in different ways depending on the coefficients. Thanks to these Lemma we will be able to perform the asymptotic study as in subsection 3.1.

**Lemma 4.1.** Let \( u, v \) be two continuous, positive radial functions such that
\[ u \sim |x|^\alpha e^{-b|x| + c|x|^\gamma}, \quad v \sim |x|^\alpha e^{-b'|x| + c'|x|^\gamma'}, \text{ as } |x| \to \infty, \] (4.1)
where \( b, b', c, c' > 0, a, a' \in \mathbb{R}, \gamma \in (0, 1), \) and \( \gamma' \in [0, 1) \). Then the following estimate holds
\[
\int_{\mathbb{R}^N} u_\xi v \sim |\xi|^a e^{-b|\xi|+c|x-\xi|^\gamma} \quad \text{if } b < b', \text{ or if } b = b' \text{ and } \gamma > \gamma',
\]
where \( u_\xi(x) = u(x - \xi) \).

**Proof.** The proof is quite lengthy, so that it will be divided into steps.

**Step 1.** We preliminarily give a bound from below. By the positivity and the continuity of \( v \) one has that \( v \geq C > 0 \) on \( B_1(0) \), the ball centred at zero with radius one, this together with the fact that the function \( f(t) = t^a e^{-bt+ct^\gamma} \) is decreasing if \( t \) is sufficiently large, yields
\[
\int_{\mathbb{R}^N} u_\xi v \geq \int_{B_1(0)} u_\xi v \geq C \int_{B_1(0)} |x-\xi|^a e^{-b|x-\xi|+c|x-\xi|^\gamma} \, dx \\
\geq C(|\xi| + 1)^a e^{-b|\xi|+c|\xi|+1} \sim C |\xi|^a e^{-b|\xi|+c|\xi|^\gamma}.
\]  

**Step 2.** In this step we will show that
\[
\int_{\mathbb{R}^N} u_\xi v \leq \int_0^{\xi_0 - r_0} dr \int_{\mathbb{R}^{N-1}} u_\xi v dy + C \left[ 2^{a_0} \xi_0^{a_0-\frac{b_0 c_0'}{2}} + \xi_0^a e^{-b_0 c_0' + c_0' r_0} + \xi_0^a e^{-b_0 c_0' + c_0' r_0} \right],
\]  
where \( x = (r, y) \in \mathbb{R} \times \mathbb{R}^{N-1}, \quad \xi = (\xi_0, 0, \ldots, 0), \quad \text{and } r_0 \in (1, \xi_0/2). \)

Let us observe that \( r_0 \) will be fixed sufficiently large and the notation on \( \xi \) can be taken up to rotations. In order to prove (4.3), we split the integral as follows
\[
\int_{\mathbb{R}^N} u_\xi v = \int_{-\infty}^{r_0} dr \int_{\mathbb{R}^{N-1}} u_\xi v dy + \int_{r_0}^{\xi_0-r_0} dr \int_{\mathbb{R}^{N-1}} u_\xi v dy + \int_{\xi_0-r_0}^{+\infty} dr \int_{\mathbb{R}^{N-1}} u_\xi v dy.
\]  
As \( u, v \) are radial functions, by performing the change of variables \( r' = \xi_0 - r \) one gets
\[
\int_{-\infty}^{r_0} dr \int_{\mathbb{R}^{N-1}} u_\xi v dy + \int_{r_0}^{+\infty} dr \int_{\mathbb{R}^{N-1}} u_\xi v dy = \int_{-\infty}^{r_0} dr \int_{\mathbb{R}^{N-1}} (u_\xi v + uv_\xi) dy.
\]  

Now, note that for every \( r < r_0 \) it results
\[
|x - \xi| = \sqrt{(\xi_0 - r)^2 + |y|^2} \geq |\xi_0 - r| > \xi_0 - r_0,
\]
then thanks to the monotonicity properties of the function \( f(t) = t^a e^{-bt+ct^\gamma} \) already observed, one deduces that, for \( \xi_0 \) sufficiently large there exists a positive constant \( C \) such that
\[
|x - \xi|^a e^{-b|x-\xi|+c|x-\xi|^\gamma} \leq C \xi_0^a e^{-b_0 c_0' + c_0' r_0} , \quad |x - \xi|^a e^{-b'|x-\xi|+c'|x-\xi|^\gamma'} \leq C \xi_0^a e^{-b_0' c_0' + c_0' r_0}'.
\]  
These facts and (4.1) yield (with \( C \) possibly different constants)
\[
\int_{-\infty}^{r_0} dr \int_{\mathbb{R}^{N-1}} (u_\xi v + uv_\xi) dy \leq C \xi_0^a e^{-b_0 c_0' + c_0' r_0} \int_{-\infty}^{r_0} dr \int_{\mathbb{R}^{N-1}} v dy \\
+ C \xi_0^a e^{-b_0' c_0' + c_0' r_0'} \int_{-\infty}^{r_0} dr \int_{\mathbb{R}^{N-1}} u dy
\]  
(4.6)
where the last inequality is deduced observing that \( u, v \in L^1(\mathbb{R}^N) \), so that (4.3) holds.

**Step 3.** In this step we are going to show that

\[
\int_{r_0}^{\tilde{r}_0-r_0} dr \int_{\mathbb{R}^{N-1}} u \xi v \, dy \leq C \tilde{\xi}_0^a e^{-b\tilde{\xi}_0+c\tilde{\xi}_0^a}, \quad \text{if } b < b'.
\]  

(4.7)

First of all, we can find \( \tilde{b}, C \) positive constants such that

\[
e^{-b'|x|-b|\xi-x|+c'|x|'+c|\xi-x|'} \leq e^{-b\tilde{\xi}_0-(b'-b)|x|+c'|x|'+c|\xi-x|'} \leq C e^{-b\tilde{\xi}_0-b|x|+c\tilde{\xi}_0^a}.
\]  

(4.8)

In addition, taking into account (4.4) it results

\[|x|^{a'} |\xi - x|^{a} \leq r^{a'} |\xi_0 - r|^{a}, \quad \text{when } a', a < 0,
\]

while, if both \( a \) and \( a' \) are positive, by direct computations, we can find a positive constant \( C \) such that

\[|x|^{a'} |\xi - x|^{a} \leq e^{a' \gamma} |\xi_0 - r|^{a} (1 + |y|)^{a'} (1 + |y|)^a.
\]

Then, noting that \( |x| \geq r + |y|/2 \), and using (4.8), we obtain (denoting with \( C \) possibly different constants)

\[
\int_{r_0}^{\tilde{r}_0-r_0} dr \int_{\mathbb{R}^{N-1}} u \xi v \, dy \leq C e^{-b\tilde{\xi}_0+c\tilde{\xi}_0^a} \int_{r_0}^{\tilde{r}_0-r_0} dr \int_{\mathbb{R}^{N-1}} r^{a'} |\xi_0 - r|^{a} e^{-b\tilde{\xi}_0} e^{-\tilde{b}\tilde{\xi}_0} h(y) \, dy
\]

\[
\leq C e^{-b\tilde{\xi}_0} \int_{r_0}^{\tilde{r}_0-r_0} dr \int_{\mathbb{R}^{N-1}} r^{a} e^{-b\tilde{\xi}_0} \, dy,
\]

where we have used that \( e^{-b\tilde{\xi}_0} h(y) \in L^1(\mathbb{R}^{N-1}) \). When \( a \) and \( a' \) have opposite sign an analogous argument leads to the same conclusion. This last integral can be now estimated exactly as in [3] (Lemma 3.7 pp. 108-109), and we get (4.7).

**Step 4.** In this step we will consider the case \( b = b' \) and we will show that

\[
\int_{r_0}^{\tilde{r}_0-r_0} dr \int_{\mathbb{R}^{N-1}} u \xi v \, dy \leq \int_{r_0}^{\tilde{r}_0} dr \int_{\{|y|<r\}} (u \xi v + u v) \, dy
\]

\[
+ C e^{-b\tilde{\xi}_0} \left[ \tilde{\xi}_0^a e^{c\tilde{\xi}_0'} + \tilde{\xi}_0^a e^{c\tilde{\xi}_0} \right].
\]

(4.9)

Performing the change of variables \( r' = r - \tilde{\xi}_0 \) and taking into account the symmetry properties of \( u \) and \( v \), one gets

\[
\int_{\tilde{r}_0/2}^{\tilde{r}_0-r_0} dr \int_{\mathbb{R}^{N-1}} u \xi v \, dy = \int_{r_0}^{\tilde{r}_0/2} dr \int_{\mathbb{R}^{N-1}} u v \, dy
\]

so that

\[
\int_{r_0}^{\tilde{r}_0-r_0} dr \int_{\mathbb{R}^{N-1}} u \xi v \, dy = \int_{r_0}^{\tilde{r}_0} dr \int_{\{|y|>r\}} (u \xi v + u v) \, dy
\]

\[
+ \int_{r_0}^{\tilde{r}_0} dr \int_{\{|y|<r\}} (u \xi v + u v) \, dy.
\]

(4.10)

Then, in order to show (4.9) we have to study the first integral on the right hand side.
Notice that $|x| \leq r + |y|$, and $|\xi - x| \leq \xi_0 + |y|$, so that as $\gamma, \gamma' \in [0, 1)$, $|x|^\gamma \leq r^\gamma + |y|^\gamma \leq 2|y|^\gamma$, and $|\xi - x|^\gamma \leq \xi_0^\gamma + |y|^\gamma$. Moreover, as $|y| > r$ in the integral under study, it holds that $|x| \geq \sqrt{2}r$ and $(1 - 1/\sqrt{2})|x| \geq (1 - 1/\sqrt{2})|y|$, summing up, one gets $|x| \geq r + (1 - 1/\sqrt{2})|y|$. Thus

$$|x| + |\xi - x| > \xi_0 + \lambda |y|, \quad \lambda = 1 - 1/\sqrt{2},$$

yielding

$$e^{-b(|x-\xi|+|x|)}+c|x-\xi|^\gamma+c'|x|^\gamma' \leq e^{-b\xi_0+c\xi_0^\gamma}e^{-\lambda b|y|+c|y|^\gamma+2c'|y|^\gamma'}. \quad (4.12)$$

Moreover, $|x| \geq r$, and $|\xi - x| \geq \xi_0$, so that

$$|x - \xi|^a |x|^d' \leq C |r|^d' \xi_0^a \leq C \xi_0^a, \quad \text{if } a, d' < 0,$$

$$|x - \xi|^a |x|^d' \leq C |y|^d' (\xi_0^a + |y|^a) \leq C \xi_0^a |y|^{a+d'}, \quad \text{if } a, d' > 0,$$

where the last inequality follows from the fact that $|y| \geq r$. In the case in which $a$ and $d'$ have opposite sign we will obtain a combination of the previous estimates. This, together with (4.11) and (4.12), implies

$$\int_{r_0}^{\xi_0} \int_{|y| > r} u \nabla v \leq C_{\xi_0}^a e^{-b\xi_0+c\xi_0^\gamma} \int_{r_0}^{\xi_0} \int_{|y| > r} |y|^d' |y|^d' e^{-\lambda b|y|+c|y|^\gamma+2c'|y|^\gamma'} dr dp$$

$$\leq C_{\xi_0}^a e^{-b\xi_0+c\xi_0^\gamma} \int_{r_0}^{\xi_0} \int_{r}^{\xi_0} |y|^d' |y|^d' e^{-\lambda b|y|+c|y|^\gamma+2c'|y|^\gamma'} dr dp$$

$$= C_{\xi_0}^a e^{-b\xi_0+c\xi_0^\gamma} \int_{r_0}^{\xi_0} \int_{r_0}^{\xi_0} \rho |y|^d' |y|^d' e^{-\lambda b\rho+c\rho^\gamma+2c'\rho^\gamma'} dr dp$$

$$\leq C_{\xi_0}^a e^{-b\xi_0+c\xi_0^\gamma} \int_{r_0}^{\xi_0} \rho^{N-1} |y|^d' |y|^d' e^{-\lambda b\rho+c\rho^\gamma+2c'\rho^\gamma'} dr dp$$

$$\leq C_{\xi_0}^a e^{-b\xi_0+c\xi_0^\gamma},$$

where we recall that $\lambda > 0$ is introduced in (4.11). Arguing analogously, one obtains (4.9).

**Step 5.** In this step we will prove

$$\int_{r_0}^{\xi_0} dr \int_{|y| < r} (u \nabla v + uv \xi) dy \sim C_{\xi_0}^a e^{-b\xi_0} \int_{r_0}^{\xi_0} \frac{r^{N-1}}{2} + \frac{a'}{2} e^{c r^\gamma} + c (\xi_0 - r)^\gamma' dr$$

$$+ \xi_0^d' e^{-b\xi_0} \int_{r_0}^{\xi_0} \frac{r^{N-1}}{2} + \frac{a'}{2} e^{c r^\gamma} + c (\xi_0 - r)^\gamma' dr. \quad (4.13)$$

Recalling (4.4) and taking into consideration that $|y| \leq r$, one has

$$r < |x| = \sqrt{r^2 + |y|^2} < \sqrt{2}r, \quad \frac{\xi_0}{2} < |\xi - x| < |\xi_0 - r| + r = \xi_0. \quad (4.14)$$

Moreover, let us take $h = |y|^2$ and consider, for every $s, t \in (0, 2)$, and $d, d' > 0$, the function

$$f(h) := d' |x|^t + d |\xi - x|^t = d' (r^2 + h)^{\frac{t}{2}} + d ( (\xi_0 - r)^2 + h)^{\frac{t}{2}}.$$
Then, (4.14) yields for $h \in [0, r^2]$ and $\zeta_0 > 2r$
\[
d' \frac{s}{2} 2^{\frac{1}{5}} r^{2-\frac{2}{5}} \leq f'(h) \leq \left( d' \frac{t}{2} r^{2-\frac{2}{5}} + d' \frac{s}{2} r^{2-\frac{2}{5}} \right),
\]
so that
\[
d' \frac{s}{2} 2^{\frac{1}{5}} r^{2-\frac{2}{5}} h + f(0) \leq f(h) \leq f(0) + \left( d' \frac{t}{2} r^{2-\frac{2}{5}} + d' \frac{s}{2} r^{2-\frac{2}{5}} \right) h, \quad \forall h \in [0, r^2].
\]
As a consequence, the following inequality holds for every $s, t \in (0, 2)$
\[
d'r^s + d(\zeta_0 - r)^t + d' \frac{s}{2} 2^{\frac{1}{5}} r^{2-\frac{2}{5}} |y|^2 < d' |x|^t + d |\zeta - x|^t
\]
\[
< d'r^s + d(\zeta_0 - r)^t + \left( d' \frac{t}{2} r^{2-\frac{2}{5}} + d' \frac{s}{2} r^{2-\frac{2}{5}} \right) |y|^2.
\]
Using these information (both for $s = \gamma'$, $t = \gamma$, $d = c$, $d' = c'$, and for $s = t = 1$, $d = d' = b$), together with (4.1) and (4.14), we obtain
\[
\int_{|y| < r} u^{e^v} dy \sim \zeta_0^\alpha e^{-b_0 \gamma^0 \rho^\frac{\gamma}{\gamma - 1} (\zeta_0 - r)^\gamma} \int_{|y| < r} e^{-\frac{b_1}{2} |y|^{\gamma}} dy\]
where $b_1, b_2, b_3$ are positive constants which depend on the parameters and whether we are considering estimates from above or below. Notice that, for every $r > r_0$ and for $r_0$ fixed sufficiently large, (depending on the parameters but not on $\zeta_0$),
\[
\frac{b_1 |y|^2}{2} r \leq \frac{|y|^2}{r} (b_1 - \frac{b_2}{r_0^{\gamma}} - \frac{b_3}{r_0^{\gamma - \gamma}}) \leq \frac{b_1 |y|^2}{r}.
\]
Therefore, choosing $b = b_1$ in the estimate from below and $b = b_1/2$ in the one from above, it follows
\[
\int_{r_0}^{\zeta_0} \frac{\xi}{\xi} dr \int_{|y| < r} u^{e^v} dy \sim \zeta_0^\alpha e^{-b_0 \gamma^0 \rho^\frac{\gamma}{\gamma - 1} (\zeta_0 - r)^\gamma} \int_{r_0}^{\zeta_0} \frac{\xi}{\xi} dr \int_{|y| < r} e^{-\frac{b_1}{2} |y|^{\gamma}} dy \]
\[
= \zeta_0^\alpha e^{-b_0 \gamma^0 \rho^\frac{\gamma}{\gamma - 1} (\zeta_0 - r)^\gamma} \int_{r_0}^{\zeta_0} \frac{\xi}{\xi} dr \int_{|y| < \sqrt{r}} e^{-b_1 |y|^{\gamma}} dy'.
\]
The last integral is bounded from above by the integral in the whole $\mathbb{R}^N$ and, since $r > 1$, it is bounded from below by the integral on $B_1(0)$, both finite; so that it results
\[
\int_{r_0}^{\zeta_0} dr \int_{|y| < r} u^{e^v} dy \sim \zeta_0^\alpha e^{-b_0 \gamma^0 \rho^\frac{\gamma}{\gamma - 1} (\zeta_0 - r)^\gamma} \int_{r_0}^{\zeta_0} \frac{\xi}{\xi} dr \int_{|y| < \sqrt{r}} e^{-b_1 |y|^{\gamma}} dy'.
\]
By similar computations, exchanging the role of the coefficients, one has (4.13).
\textbf{Step 6.} In this step we will conclude the proof.

Let us start dealing with the first integral on the right hand side of (4.13), and notice that on the interval $[r_0, \zeta_0/2]$ one has $r' < (\zeta_0/2)^\gamma$ and $(\zeta_0 - r)^\gamma \leq (\zeta_0 - r_0)^\gamma$. Hence
for any \( \gamma > \gamma'' > \gamma' \geq 0 \) one has
\[
\int_{r_0}^{\xi_0} r^{\frac{N-1}{2} + a' e^{r'} + c(\xi_0-r)^\gamma} dr \leq e^{c'(\xi_0/2)^\gamma + c(\xi_0-r_0)^\gamma + (\xi_0/2)^\gamma''} \int_{r_0}^{\xi_0} r^{\frac{N-1}{2} + a' e^{-r''}} dr \\
\leq C e^{c'(\xi_0/2)^\gamma + (\xi_0/2)^\gamma'' + c(\xi_0-r_0)^\gamma}.
\]

As \( \gamma > \gamma'' > \gamma' \geq 0 \), one gets \( e^{c'(\xi_0/2)^\gamma + (\xi_0/2)^\gamma'' + c(\xi_0-r_0)^\gamma} \sim e^{c_0^\gamma} \) if \( \xi_0 \) is sufficiently large, so that
\[
\int_{r_0}^{\xi_0} r^{\frac{N-1}{2} + a' e^{r'} + c(\xi_0-r)^\gamma} dr \leq C e^{c_0^\gamma}.
\]

(4.16)

When arguing on the second integral in (4.13), we take into account that the role of \( \gamma \) and \( \gamma' \) are exchanged, and as \( \gamma > \gamma' \) we obtain
\[
\int_{r_0}^{\xi_0} r^{\frac{N-1}{2} + a' e^{r'} + c(\xi_0-r)^\gamma} dr \leq C e^{c(\xi_0/2)^\gamma}, \quad \text{if } \xi_0 \text{ is big enough.} \quad (4.17)
\]

Finally, exploiting (4.2), (4.3), (4.7) one gets the conclusion if \( b < b' \). When \( b = b' \) and \( \gamma > \gamma' \), one takes into account (4.2), (4.3), (4.9), (4.13), (4.16) and (4.17) to conclude the proof.

In the next lemma we analyze the case \( b = b' \) and \( \gamma = \gamma' \), concluding the extension of Lemma A.2 useful in our context.

**Lemma 4.2.** Let \( u, v \) be two continuous, positive radial functions such that (4.1) is satisfied with \( b' = b > 0, \gamma' = \gamma \in (0,1), c, c' > 0 \) and \( a, a' \in \mathbb{R} \). Then the following estimate holds
\[
\int_{\mathbb{R}^N} u \tilde{u} v \sim |\xi|^{\frac{N-1}{2} + a + a' - \frac{1}{\gamma}} e^{-b |\xi| + \tilde{c} |\xi|^\gamma}, \quad \text{with } \tilde{c} = \left( (c')^{1-\gamma} + c^{1-\gamma} \right)^{1-\gamma},
\]
where \( u_\xi(x) = u(x - \xi) \).

**Remark 4.3.** Let us observe that in Lemma 4.2, we assume \( c, c' > 0 \) or \( \gamma, \gamma' > 0 \) as the cases \( c, c' \leq 0 \) or \( \gamma, \gamma' \leq 0 \) are already contained in Lemma A.2.

Moreover, the case \( c' \geq 0, \gamma' > 0 \) is equivalent to our assumption \( c' > 0, \gamma' \geq 0 \), and if \( b > b' \) the result can be proved as well, by exchanging the role of \( b, b' \) and \( \gamma, \gamma' \).

**Proof.** Let us start proving estimates from above and assuming, without loss of generality, that \( c \geq c' > 0 \). As in the proof of Lemma 4.1, we use the notation in (4.4) with \( r_0 \) such that \( 1 < r_0 < (1 + (c'/c)^{1/(1-\gamma)})^{-1} \xi_0/2 < \xi_0/2 \). Steps 1, 2, 4, 5 in the proof of Lemma 4.1 are still valid, hence we can take into account (4.3), (4.9), (4.13), and obtain
\[
\int_{\mathbb{R}^N} u \tilde{u} v \leq C \xi_0^{a} e^{-b_0 \xi} \int_{r_0}^{\xi_0} r^{\frac{N-1}{2} + a' e^{r'} + c(\xi_0-r)^\gamma} dr \\
+ C \xi_0^{a'} e^{-b_0 \xi} \int_{r_0}^{\xi_0} r^{\frac{N-1}{2} + a e^{r'} + c(\xi_0-r)^\gamma} dr + C \xi_0^{a} e^{-b_0 \xi} + C \xi_0^{a'} e^{-b_0 \xi}.
\]

(4.18)

However, the estimate of the integrals in the right hand side of (4.18), which in case \( \gamma > \gamma' \) corresponds to Step 6 in the proof of Lemma 4.1, requires a more accurate analysis when considering the case \( \gamma = \gamma' \).
Let us define
\[
g(r) := c' r^{\gamma} + c(\xi_0 - r)^{\gamma},
\]
and observe that it has a maximum in the interval \([r_0, \xi_0/2]\) in the point \(\hat{r}\), where
\[
\hat{r} = \frac{\xi_0}{1 + (c/c')^{1-\gamma}}, \quad \text{and} \quad g(\hat{r}) = \xi_0^{\gamma} \left( (c')^{1-\gamma} + c^{1-\gamma} \right)^{1-\gamma}.
\]  
As \(c \geq c'\), \(\hat{r} \leq \xi_0/2\); then we can write
\[
\int_{r_0}^{\xi_0/2} r^{N-1+a'} e^{\gamma r^{\gamma} + c(\xi_0 - r)^{\gamma}} dr = \int_{r_0}^{\hat{r}/2} r^{N-1+a'} e^{\gamma r^{\gamma} + c(\xi_0 - r)^{\gamma}} dr
\]
and, noting that \(g(r) \leq g(\hat{r}/2)\) in \((r_0, \hat{r}/2)\), one obtains
\[
\int_{r_0}^{\hat{r}/2} r^{N-1+a'} e^{\gamma r^{\gamma} + c(\xi_0 - r)^{\gamma}} dr \leq g(\hat{r}/2) \int_{r_0}^{\hat{r}/2} r^{N-1+a'} dr \leq C \hat{r}^{\frac{N-1}{2} + a'} e^{g(\hat{r}/2)}. \quad \text{(4.22)}
\]
In order to estimate the second integral in (4.21) we need to study the behavior of \(g\) near \(\hat{r}\). By Taylor expansion at the maximum point \(\hat{r}\), one has
\[
g(r) = g(\hat{r}) + \frac{1}{2} g''(r_1)(\hat{r} - r)^2 \quad \text{(4.23)}
\]
where \(r_1\) belongs to the interval of extrema \(r\) and \(\hat{r}\) so that \(r_1 \in (\hat{r}/2, \xi_0/2)\). In addition,
\[
g''(r) = \gamma(\gamma - 1) h(r) \quad \text{where} \quad h(t) = \frac{c'}{t^{2-\gamma}} + \frac{c}{(\xi_0 - t)^{2-\gamma}} \quad \text{(4.24)}
\]
and \(h(t)\) has a global minimum point at \(\hat{t}\) such that
\[
\frac{\xi_0}{2} \geq \hat{t} = \frac{\xi_0}{1 + (c/c')^{1-\gamma}} \geq \hat{r}, \quad \text{as} \quad c \geq c'.
\]
Hence, taking into account that \(\gamma < 1\), (4.24) together with (4.23) yields
\[
g(r) \leq g(\hat{r}) - \frac{1}{2} c_1 \gamma (1 - \gamma) \xi_0^{\gamma - 2}(\hat{r} - r)^2. \quad \text{(4.25)}
\]
Exploiting (4.25) into the second integral on the right hand side of (4.21), one has
\[
\int_{\hat{r}/2}^{\xi_0/2} r^{N-1+a'} e^{\gamma r^{\gamma} + c(\xi_0 - r)^{\gamma}} dr \leq e^{g(\hat{r})} \int_{\hat{r}/2}^{\xi_0/2} r^{N-1+a'} e^{-c_1 \xi_0^{\gamma - 2}}(\hat{r} - r)^2 dr
\]
\[
= e^{g(\hat{r})} \int_{\hat{r}/2}^{\hat{r}} r^{N-1+a'} e^{-c_1 \xi_0^{\gamma - 2}}(\hat{r} - r)^2 dr + e^{g(\hat{r})} \int_{\hat{r}}^{\xi_0/2} r^{N-1+a'} e^{-c_1 \xi_0^{\gamma - 2}}(\hat{r} - r)^2 dr \quad \text{(4.26)}
\]
where \(c_1 = \frac{1}{2} c_1 \gamma (1 - \gamma)\). Let us study the first integral on the right hand side and note that (4.20) implies that there exist two positive constants \(C_1 < C_2\) such that
\[
C_1 \xi_0^{N-1+a'} \leq r^{N-1+a'} \leq C_2 \xi_0^{N-1+a'}, \quad \forall r \in [\hat{r}/2, \xi_0/2]. \quad \text{(4.27)}
\]
Performing the change of variables \( w = (\xi_0^{\gamma/2}-2)^{1/2}(\hat{r} - r) \), one obtains

\[
\int_{\hat{r}/2}^{\hat{r}} r^{\frac{N+1}{2}+a'} e^{-\xi_0^{\gamma/2}(\hat{r}-r)^2} \, dr \leq C \xi_0^{\frac{N+1}{2}+a'-\frac{1}{2}} \int_0^{2\xi_0^{\gamma/2}} e^{-w^2} \, dw \leq C \xi_0^{\frac{N+1}{2}+a'-\frac{1}{2}}.
\]

Using again (4.27) in the second integral and performing the change of variables \( w = (\xi_0^{\gamma/2}-2)^{1/2}(\hat{r} - r) \), one deduces that

\[
\int_{\hat{r}/2}^{\hat{r}} r^{\frac{N+1}{2}+a'} e^{-\xi_0^{\gamma/2}(\hat{r}-r)^2} \, dr \leq C \xi_0^{\frac{N+1}{2}+a'-\frac{1}{2}},
\]

so that, (4.26) becomes

\[
\int_{\hat{r}/2}^{\hat{r}} r^{\frac{N+1}{2}+a'} e^{c r^\gamma + c(\xi_0 - r)^\gamma} \, dr \leq C \xi_0^{\frac{N+1}{2}+a'-\frac{1}{2}} e^{\delta(\hat{r})}.
\]

Exploiting this last information and (4.22) together with the fact that \( g(\hat{r}/2) < g(\hat{r}) \) into (4.21) one obtains

\[
\int_{\hat{r}/2}^{\hat{r}} r^{\frac{N+1}{2}+a'} e^{c r^\gamma + c(\xi_0 - r)^\gamma} \leq C \xi_0^{\frac{N+1}{2}+a'-\frac{1}{2}} e^{\delta(\hat{r})}.
\]

As a consequence, from (4.18), and taking into account that \( g(\hat{r}) > c \xi_0^{\gamma} \) (thanks to (4.20)), we deduce that

\[
\int_{\mathbb{R}^N} u \xi^\nu \leq C \xi_0^{\frac{N+1}{2}+a'+a'-\frac{1}{2}} e^{-b \xi_0 + g(\hat{r})} + C \xi_0^{\gamma} e^{-b \xi_0} \int_{\hat{r}/2}^{\hat{r}} r^{\frac{N+1}{2}+a'} e^{c r^\gamma + c(\xi_0 - r)^\gamma} \, dr,
\]

for \( c \geq c' \). Notice that, if \( c = c' \), in order to study the integral on the right hand side we can repeat the argument above exchanging the roles of \( a \) and \( a' \), and we obtain

\[
\int_{\mathbb{R}^N} u \xi^\nu \leq \xi_0^{\frac{N+1}{2}+a'+a'-\frac{1}{2}} e^{-b \xi_0 + g(\hat{r})},
\]

and the proof of estimates from above in this case is complete, recalling (4.20).

On the other hand, if \( c > c' \), then we need to estimate differently the integral appearing on the right hand side of (4.28). We consider the function

\[
g(r) := g(\xi_0 - r) = cr^\gamma + c'(\xi_0 - r)^\gamma
\]

and notice that it is increasing in the interval \([\hat{r}/2, \xi_0/2]\), so that

\[
\int_{\hat{r}/2}^{\xi_0/2} r^{\frac{N+1}{2}+a} e^{c r^\gamma + c'(\xi_0 - r)^\gamma} \, dr \leq C e^{\delta(\xi_0/2)} \int_{\hat{r}/2}^{\xi_0/2} r^{\frac{N+1}{2}+a} \, dr.
\]

In addition, \( \hat{r} < \xi_0/2 \) as \( c > c' \) (see (4.20)), so that \( g(\xi_0/2) = g(\xi_0/2) < g(\hat{r}) \), as \( \hat{r} \) is the maximum point of \( g \) in \([\hat{r}/2, \xi_0/2]\). Hence, integrating one has

\[
\int_{\hat{r}/2}^{\xi_0/2} r^{\frac{N+1}{2}+a} e^{c r^\gamma + c'(\xi_0 - r)^\gamma} \, dr \leq C \xi_0^{\frac{N+1}{2}+a'-\frac{1}{2}} e^{\delta(\hat{r})}.
\]

Therefore, taking into account (4.28), one has that (4.29) holds in this case too.
To conclude the proof, we need to prove the estimate from below. We notice that on the interval \([\frac{r}{2}, r]\) the function \(h\) given in (4.24) attains its maximum on the boundary. In any case, one has

\[ h(r_1) \leq c_2 \delta_0^{\gamma - 2}, \quad \text{where } r_1 \in (\frac{r}{2}, r), \]

with a suitable constant \(c_2 > 0\). Exploiting a second order Taylor expansion of the function \(g\) as in (4.23) one deduces that

\[
\int_{\frac{r}{2}}^{r} r^{\frac{N-1}{2} + a' \varepsilon' r^\gamma + c(\hat{\delta}_0 - r)^\gamma} \, dr \geq \int_{\frac{r}{2}}^{r} r^{\frac{N-1}{2} + a' \varepsilon(\hat{\delta}_0 - r)^\gamma} \, dr
\]

\[ = e^{\varepsilon(\hat{\delta}_0 - r)^\gamma} \int_{\frac{r}{2}}^{r} r^{\frac{N-1}{2} + a' \varepsilon(\hat{\delta}_0 - r)^\gamma} \, dr, \quad (4.30) \]

where \(\hat{\varepsilon} = \frac{1}{2} c_2 \gamma (1 - \gamma)\). Taking into account (4.27) and performing the change of variable \(w = (c \hat{\delta}_0^{\gamma - 2})^{1/2} (\hat{\delta}_0 - r)\), yields

\[
\int_{\frac{r}{2}}^{r} r^{\frac{N-1}{2} + a' \varepsilon(\hat{\delta}_0 - r)^\gamma} \, dr \geq C \hat{\delta}_0^{\frac{N-1}{2} + a' \varepsilon(\hat{\delta}_0 - r)^\gamma} \int_{\frac{r}{2}}^{r} e^{-c \hat{\delta}_0^{\gamma - 2} (\hat{\delta}_0 - r)^2} \, dr
\]

\[ \geq C \hat{\delta}_0^{\frac{N-1}{2} + a' \varepsilon(\hat{\delta}_0 - r)^\gamma} \int_{0}^{\hat{\delta}_0^{\gamma - 2} (\hat{\delta}_0 - r)^2} e^{-w^2} \, dw \]

\[ \geq C \hat{\delta}_0^{\frac{N-1}{2} + a' \varepsilon(\hat{\delta}_0 - r)^\gamma} \int_{0}^{1} e^{-w^2} \, dw \geq C \hat{\delta}_0^{\frac{N-1}{2} + a' \varepsilon(\hat{\delta}_0 - r)^\gamma}, \quad (4.31) \]

for \(\hat{\delta}_0\) sufficiently large. Finally, as \(u, v\) are positive functions and recalling (4.15), (4.20),

\[
\int_{R^N} u^{\sigma} v^{\sigma} \geq \int_{r_0}^{\hat{\delta}_0/2} dr \int_{|y| < r} u^{\sigma} v^{\sigma} \, dy \geq C \hat{\delta}_0 e^{-b_0} \int_{\frac{r}{2}}^{r} r^{\frac{N-1}{2} + a' \varepsilon' r^\gamma + c(\hat{\delta}_0 - r)^\gamma} \, dr,
\]

thanks to the fact that \(1 < r_0 < (1 + (c' \hat{\delta}_0^{\gamma - 2})^{1/(1 - \gamma)})^{-1} \hat{\delta}_0/2 < \hat{\delta}_0/2\). This together with (4.30) and (4.31), gives the desired estimate from below and completes the proof. \(\square\)

We can now give the asymptotic decay of \(\varepsilon_R\) introduced in (2.17).

**Lemma 4.4.** For every \(z \in \Sigma\), let \(\mu(Gz)\) be defined in (2.3). The following conclusions hold.

(i) If \(p\) satisfies (1.6), then

\[ \varepsilon_R \sim R^{-\frac{N-1}{2} + 2\tau_1 e^{-\mu(Gz)\sqrt{\varepsilon_R}}, \]

where \(\tau_1\) is introduced in (2.8).

(ii) If \(p = 2\) and \(\alpha \in (N - 1, N - \frac{1}{2})\), then

\[ \varepsilon_R \sim R^{-\frac{N-1}{2} + \frac{\gamma}{2} + 2\tau_2 e^{-\mu(Gz)\sqrt{\varepsilon_R} \gamma + 2^{1-\gamma} c_\gamma (\mu(Gz) R)}, \]

where \(\gamma = \alpha - (N - 1), c_\gamma\) and \(\tau_2\) are given in (2.9).

**Remark 4.5.** Notice that, for \(p = 2\) and any \(\alpha \in (0, N)\), one can easily give a bound from below on \(\varepsilon_R\), which however in general is far from being sharp. One has

\[ \varepsilon_R^{ij} \geq CR^{-\frac{N-1}{2} e^{-d_{ij} \sqrt{\varepsilon_R}}, \]

see Remark 3.3 in [19] for the the case \(d_{ij} = 2\): exactly the same proof also works for the more general case we are considering here.
This estimate turns out to be enough in order to consider the case \( \ell(G) \geq 3 \), and \( \beta > \mu_G \sqrt{\nu_\infty} \), see also [10], as the leading term in the asymptotic analysis is the linear part in the exponential. On the other hand, in other cases, and in particular if \( p = 2 \), \( \alpha \geq N - 1 \), exponential and polynomial corrections turn out to be relevant as well, and a more careful analysis is needed. Lemmas A.2, 4.1, 4.2 will be crucial.

Proof. Recalling (2.17) and performing a change of variable one obtains
\[
\epsilon_R^j = \int_{\mathbb{R}^N} (I_a \ast \omega^p)(x) \omega^{p-1}(x) \omega(x-R(g_j z - g_j z)) dx.
\]
We are going to apply Lemma A.2 with
\[
v = I_a \ast \omega^p \omega^{p-1}, \quad u = \omega, \quad \xi_{ij} = R(g_j z - g_j z), \quad |\xi_{ij}| = R d_{ij},
\]
where \( d_{ij} \) is introduced in Remark 3.3. If \( p > 2 \), one takes into account (2.8) and (2.15) to deduce that \( u \) and \( v \) satisfy the assumptions of Lemma A.2 with \( a = -\frac{N-1}{2} \), \( b = \sqrt{\nu_\infty} \), \( a' = -(p-1) \frac{N-1}{2} - N + \alpha \) and \( b' = (p-1) \sqrt{\nu_\infty} \). Since \( b < b' \), it results
\[
\epsilon_R^j \sim e^{-\sqrt{\nu_\infty} d_{ij} R} R^{-\frac{N-1}{2}}, \quad as \ R \to \infty.
\]
Then, observing that, by definition, \( \mu(Gz) \leq d_{ij} \) and it is achieved (see Lemma 2.2 and Remarks 2.3, 3.3, 2.17) yields the conclusion.

When \( p = 2 \), it follows that \( b = b' \). Furthermore, if \( \alpha < N - 1 \), (2.15) and (2.8) still hold so that the conclusion follows as in the case \( p > 2 \).

When \( \alpha = N - 1 \), one takes into account (2.8) and obtains \( a = \frac{p}{2} \sqrt{\nu_\infty} - \frac{N-1}{2} \), and \( a' = a - 1 \), so that, \( a' < a \) and \( a' > -\frac{N+1}{2} \) as \( \nu > 0 \). Then, the proof of the first conclusion is completed observing that \( a + a' + \frac{N+1}{2} = \nu \sqrt{\nu_\infty} - \frac{N-1}{2} \) and applying Lemma A.2.

In order to prove the second conclusion, we perform the same choice as (4.32). As before \( b = b' \), but Lemma A.2 cannot be applied, as it does not include decay such as (2.9). We can instead exploit Lemma 4.2 with \( a = -\frac{N-1}{2} + \tau_2, \quad a' = a - N + \alpha, \quad \gamma' = \gamma, \quad c = c' = c_\gamma \). \( \square \)

All the estimates above hold for any \( z \in \Sigma \). In order to compare the asymptotic decay of the potential integral term with \( \epsilon_R \), we need to choose a suitable \( z \). From now on, taking into account Lemma 2.2 we fix \( z \in \Sigma \) such that
\[
\mu_G = \mu(Gz),
\]
where \( \mu_G \) and \( \mu(Gz) \) are defined in (2.3), 2.4).

Lemma 4.6. Let \( \epsilon_R \) be defined in (2.17) and \( \mu_G \) be introduced in (2.4). Moreover, let \( z \) be fixed such that (4.33) holds. Assume (1.6) and (1.7) or (1.9) and (1.10). Then it results
\[
A_V := \int_{\mathbb{R}^N} (V(x) - V_\infty)(\chi_{Rz})^2 \leq o(\epsilon_R), \quad as \ R \to +\infty.
\]

Proof. Let us first assume that \( p \) satisfies (1.6), and \( V \) satisfies (1.7). As in the proof of Lemma 3.4 we first observe that
\[
\int_{\mathbb{R}^N} (V(x) - V_\infty) \omega_R^2 \leq C \int_{\mathbb{R}^N} |x|^\alpha e^{-\beta |x|} \omega_R^2.
\]
Take
\[ u = \omega^2, \quad v = |x|^{\sigma} e^{-\beta|x|} \quad \xi_i = Rg_i z, \text{ with } |\xi_i| = R \text{ for every } i = 1, \ldots, \ell(G). \]

Observe that (2.8) together with the fact that \( \mu(G) \leq 2 \) implies that \( u \) satisfies the following upper bound
\[ u \leq Ce^{-\mu(G)\sqrt{\nabla u}|x|} |x|^{-N+1}. \]

Let us first assume that \( \beta > \mu_G\sqrt{\nabla u} = \mu(G)\sqrt{\nabla u}, \) due to (4.33). We apply Lemma A.2 with
\[ a = -N + 1 + 2\tau_1, \quad b = \mu(G)\sqrt{\nabla u}, \quad a' = \sigma, \quad b' = \beta, \] (4.34)

where \( \tau_1 \) is given in (2.8). Hence, \( \mathcal{A}_V \) satisfies
\[ \mathcal{A}_V \leq e^{-\mu(G)\sqrt{\nabla u} R} R^{-N+1+2\tau_1} \]

and Lemma 4.4 implies that this is \( o(\varepsilon_R) \).

If \( \beta = \mu_G\sqrt{\nabla u} = \mu(G)\sqrt{\nabla u}, \) then we have as before (4.34) with \( b' = b \) and we apply conclusion (ii) in Lemma A.2. Thus, if \( \sigma < a = -N + 1 + 2\tau_1, \) it holds
\[ \mathcal{A}_V \leq \left\{ \begin{array}{ll}
    e^{-\mu(G)\sqrt{\nabla u} R} R^{-N+1+2\tau_1+\sigma+N+1} & \text{if } \sigma > -\frac{N+1}{2} \\
    e^{-\mu(G)\sqrt{\nabla u} R} R^{-N+1+2\tau_1} & \text{if } \sigma = -\frac{N+1}{2} \\
    e^{-\mu(G)\sqrt{\nabla u} R} R^{-N+1+2\tau_1} & \text{if } \sigma < -\frac{N+1}{2}.
\end{array} \right. \]

These estimates and Lemma 4.4 show that \( \mathcal{A}_V = o(\varepsilon_R) \) when \( \sigma < a. \) An analogous argument can be performed when \( \sigma > a, \) yielding the first conclusion.

Let us now assume that \((1.9)\) and \((1.10)\) hold. In this case we take
\[ u = \omega^2, \quad v = |x|^{\sigma} e^{-\beta|x|+\gamma'|x|'}, \quad \xi_i = Rg_i z, \text{ with } |\xi_i| = R. \]

Note that, (2.9) implies
\[ u \sim C |x|^{-N+1+2\tau_2} e^{-2\sqrt{\nabla u}|x|+2c_1|x|}. \] (4.35)

If \( \beta > \mu_G\sqrt{\nabla u} = \mu(G)\sqrt{\nabla u}, \) we apply Lemma 4.1 with
\[ a = -N + 1 + 2\tau_2, \quad b = \mu(G)\sqrt{\nabla u}, \quad c = 2^{1-\gamma} c_G \mu(G)^{\frac{\gamma'}{2}} \quad \sigma = \sigma, \quad b' = \beta. \]

Thus, \( \mathcal{A}_V = o(\varepsilon_R), \) taking into account Lemma 4.4 and recalling that \( \gamma > 0. \)

Let now \( \beta = \mu_G\sqrt{\nabla u} = \mu(G)\sqrt{\nabla u}. \) The case \( \mu_G < 2, \) is taken into account both in conclusions (ii) and (iii) of Theorem 1.3 and we will handle it at the same time: we take into consideration (4.35) and we apply Lemma 4.1 with
\[ a = -N + 1 + 2\tau_2, \quad b = 2\sqrt{\nabla u}, \quad c = 2c_\gamma, \quad a' = \sigma, \quad b' = \beta, \]

yielding \( \mathcal{A}_V \leq C R^2 e^{-\mu_G\sqrt{\nabla u} R+c' R'}. \) Then Lemma 4.4 and (1.10) yield the conclusion if either \( \gamma' < \gamma \) or \( \gamma' = \gamma \) and \( c' < 2^{1-\gamma} c_G \mu_G^{\frac{\gamma'}{2}} \) or \( \gamma' = \gamma, \quad c' = 2^{1-\gamma} c_G \mu_G^{\frac{\gamma'}{2}} \) and \( \sigma \) satisfies the hypotheses in conclusion (iii) in Theorem 1.3. In the last case \( \beta = \mu_G\sqrt{\nabla u} \) and \( \mu_G = 2, \)

(1.10) lead us to assume the hypotheses in conclusion (ii) namely \( \gamma' < \gamma, \) then Lemma 4.1 implies
\[ \mathcal{A}_V \leq C R^{-N+1+2\tau_2} e^{-2\sqrt{\nabla u} R+2c_\gamma R'}. \]
Then, one deduces that \( A_V = o(\varepsilon_R) \) exploiting Lemma 4.4, using that \( 2^{1-\gamma}c_\gamma (\mu_G)^\gamma = 2c_\gamma \) as \( \mu_G = 2 \) and recalling that \( \gamma > 0 \).

**Remark 4.7.** An inspection of the proofs above provides examples of potentials not satisfying our assumptions and for which the associated integral term \( A_V \) is not \( o(\varepsilon_R) \).

In particular, take \( V(x) = V_\infty + |x|^\sigma e^{-\beta |x| + c|x|^\gamma} \), \( \mu_G = 2 \), \( \gamma' = \gamma > 0 \) and \( c' > 0 \). In this case, by Lemma 4.2 one gets

\[
\int_{\mathbb{R}^N} (V(x) - V_\infty) \omega^2_{t,R} \sim C R^\frac{2-N}{\frac{\sigma}{2} + \frac{\gamma}{2}} e^{-2\sqrt{\gamma}c_\gamma (\mu_G)^\gamma},
\]

where \( \tilde{c} = \left( (c')^{\frac{1}{\gamma}} + (2c_\gamma)^{\frac{1}{\gamma}} \right)^{1-\gamma} \). On the other hand, from Lemma 4.4 we deduce that \( \varepsilon_R \) decays as follows

\[
\varepsilon_R \sim R^{-\frac{N-1}{2} + \frac{\gamma}{2} + 2\gamma c_\gamma (\mu_G)^\gamma} e^{-\mu (\Sigma) \sqrt{\gamma} + 2^{1-\gamma}c_\gamma (\mu_G)^\gamma}.
\]

Notice that \( \mu_G = 2 \) implies \( \mu (\Sigma) = 2 \) for any \( z \in \Sigma \). Hence we need to take into account the exponential correction and as it holds \( \tilde{c} > 2c_\gamma \) for any choice of \( c' > 0 \), we deduce that \( A_V \) is not \( o(\varepsilon_R) \).

### 4.2. Proof of Theorem 1.2 and 1.3

We will follow the same strategy of Theorem 1.1. Here, the Nehari manifold is \( C^* \), as \( \mathcal{I}_V \) is \( C^2 \) if \( p \geq 2 \). Moreover, we point out that conclusions (1) and (2) of Lemma 3.7 are still true in the setting \( p \geq 2 \).

The analog of Proposition 3.6 will be the following

**Proposition 4.8.** If \( p \geq 2 \), then

\[
\int_{\mathbb{R}^N} \left( I_{2-R} \chi_{R,z}^p \right) \chi_{R,z}^p \geq \sum_{i=1}^{\ell(G)} \int_{\mathbb{R}^N} \left( I_{2-R} \omega_i^p \right) \omega_i^p + 2(p-1)\varepsilon_R. \tag{4.37}
\]

For \( p = 2 \) a sharper estimate holds:

\[
\int_{\mathbb{R}^N} \left( I_{2-R} \chi_{R,z}^2 \right) \chi_{R,z}^2 \geq \sum_{i=1}^{\ell(G)} \int_{\mathbb{R}^N} \left( I_{2-R} \omega_i^2 \right) \omega_i^2 + 4\varepsilon_R. \tag{4.38}
\]

**Proof.** The first statement is an immediate consequence of [10, Lemma 5.3], whereas the second one follows by direct computations, see also [19].

**Remark 4.9.** The inequality proved in Proposition 4.8 for \( p > 2 \) is not consistent with the case \( p = 2 \). This is because (4.37) lies on an algebraic inequality of Bernoulli’s type (see formula (5.2) in [10]), while (4.38) is obtained by direct computations. We believe that it would be possible to improve (4.37) following the argument of [2, Lemma 2.2].

In order to prove our existence results the following estimate will be crucial.

**Proposition 4.10.** Let \( z \) be fixed in (4.33). Assume (1.6) and (1.7) or (1.9) and (1.10). Then, the following inequality holds

\[
\mathcal{I}_V (T(\chi_{R,z}) \chi_{R,z}) \leq \begin{cases} 
\ell(G) c^G_{\infty} - \frac{p-2}{2p} \varepsilon_R + o(\varepsilon_R), & \text{if } p > 2, \\
\ell(G) c^G_{\infty} - \varepsilon_R + o(\varepsilon_R), & \text{if } p = 2,
\end{cases}
\]

as \( R \to +\infty \) and where \( T(\chi_{R,z}) \) is defined in Lemma 3.7.
Proof. Let us first notice that, following Conclusion (1) of Lemma 3.7, it is easy to obtain that $T_R := T(x_R,z)$ is given by
\[T_R^{2p-2} = \frac{||x_{R,z}||_V^2}{\int_{\mathbb{R}^N}{(I_{\alpha} * x_{R,z})}^p_{x_{R,z}}}.\]
Therefore, taking into account (2.6), (2.17), Proposition 4.8, Lemma 4.6 and that $\omega_i$ is a solution of Problem (P_∞) it results
\[\mathcal{I}_V(T_R x_{R,z}) = \left(\frac{1}{2} - \frac{1}{2p}\right) T_R^2 ||x_{R,z}||_V^2 = \left(\frac{1}{2} - \frac{1}{2p}\right) \frac{(||x_{R,z}||_V^2)^{\rho}}{\left[\int_{\mathbb{R}^N}{(I_{\alpha} * x_{R,z})}^p_{x_{R,z}}\right]^{\rho}} \sum_{i=1}^{\ell(G)} ||\omega_{i,R}||^2 + b_p \varepsilon_R \leq \left(\frac{1}{2} - \frac{1}{2p}\right) \frac{\sum_{i=1}^{\ell(G)} ||\omega_{i,R}||^2 + b_p \varepsilon_R}{\left[\sum_{i=1}^{\ell(G)} ||\omega_{i,R}||^2 + b_p \varepsilon_R\right]^{\rho}} \right],\]
where
\[b_p = \begin{cases} 2(p - 1) & \text{if } p > 2 \\ 4 & \text{if } p = 2. \end{cases}\]
Notice that $b_p > p$ for any $p \geq 2$. Using the expansion $(a + t)^a = a^a + a^{a-1}t + o(t)$ and the notation $a := \sum ||\omega_{i,R}||^2$, we get
\[\mathcal{I}_V(T_R x_{R,z}) \leq \left(\frac{1}{2} - \frac{1}{2p}\right) \left[a + \varepsilon_R + o(\varepsilon_R)\right]^{\rho} \left[a + b_p \varepsilon_R\right]^{-\rho} \leq \left(\frac{1}{2} - \frac{1}{2p}\right) \left[a + \varepsilon_R + o(\varepsilon_R)\right]^{\rho} \left[a + b_p \varepsilon_R\right]^{-\rho} \leq \left(\frac{1}{2} - \frac{1}{2p}\right) \left[a - \varepsilon_R + o(\varepsilon_R)\right]^{\rho} = \ell(G)c_\infty^G - \frac{b_p - p}{2p} \varepsilon_R + o(\varepsilon_R),\]
where the last equality comes from the fact that $c_\infty^G = \left(\frac{1}{2} - \frac{1}{2p}\right) ||\omega_{i,R}||^2$. \hfill \square

We now prove our main results in case $p \geq 2$.

Proof of Theorems 1.2 and 1.3. Let us take a minimizing sequence for $c_V^G$ and exploit Ekeland’s Variational Principle [13] to construct a minimizing sequence which is also a Palais-Smale for $\mathcal{I}_V$ restricted on $\mathcal{N}_V^G$, then arguing as in Corollary 3.2 in [11] (see also Lemma 2.2 and Lemma 2.5 in [7]) we obtain a subsequence $u_n$ which is a Palais-Smale sequence in the whole $H_1^G$.

Take $z$ satisfying (4.33), exploit Proposition 4.10 to apply Proposition 3.1 in [6] and deduce that $u_n$ is compact. Then, there exists $u \in \mathcal{N}_V^G$ such that $\mathcal{I}_V(u) = c_V^G$. Also $|u| \in \mathcal{N}_V^G$ and $c_V = \mathcal{I}_V(u) = \mathcal{I}_V(|u|)$ so that we can take $u$ positive.

Hence, we have a $G$-invariant positive solution.
Appendix A. Technical Lemma

Lemma A.1 (Lemma 4.1 in [8]). Let $u, v : \mathbb{R}^N \to \mathbb{R}$ be two continuous functions such that
\[
    u(x) \leq C(1 + |x|)^a, \quad v(x) \leq C(1 + |x|)^{a'}
\]
as $|x| \to \infty$, where $a, a' < 0$ such that $a + a' < -N$. Let $\xi \in \mathbb{R}^N$ such that $|\xi| \to \infty$. We denote $u_\xi(x) = u(x - \xi)$. Then the following asymptotic estimate holds:
\[
    \int_{\mathbb{R}^N} u_\xi v \leq C |\xi|^\tau
\]
where $\tau = \max\{a, a', a + a' + N\} < 0$.

Lemma A.2 (Lemma 3.7 in [3]). Let $u, v : \mathbb{R}^N \to \mathbb{R}$ be two positive continuous radial functions such that
\[
    u(x) \sim |x|^a e^{-b|x|}, \quad v(x) \sim |x|^{a'} e^{-b'|x|}
\]
as $|x| \to \infty$, where $a, a' \in \mathbb{R}$, and $b, b' > 0$. Let $\xi \in \mathbb{R}^N$ such that $|\xi| \to \infty$. We denote $u_\xi(x) = u(x - \xi)$. Then the following asymptotic estimates hold:
(i) If $b < b'$,
\[
    \int_{\mathbb{R}^N} u_\xi v \sim e^{-b|\xi|} |\xi|^a.
\]
A similar expression holds if $b > b'$, by replacing $a$ and $b$ with $a'$ and $b'$.
(ii) If $b = b'$, suppose that $a \geq a'$. Then:
\[
    \int_{\mathbb{R}^N} u_\xi v \sim \begin{cases} 
    e^{-b|\xi|} |\xi|^{a + a' + \frac{N+1}{2}} & \text{if } a' > -\frac{N+1}{2}, \\
    e^{-b|\xi|} |\xi|^a \log |\xi| & \text{if } a' = -\frac{N+1}{2}, \\
    e^{-b|\xi|} |\xi|^a & \text{if } a' < -\frac{N+1}{2}.
    \end{cases}
\]

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