Two monads on the category of graphs

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Submitted: May 30, 2017; Accepted: mmm dd, yyyy; Published: XX
Mathematics Subject Classifications: 05C70, 51E10, 18C15

Abstract

We introduce two monads on the category of graphs and prove that their Eilenberg-Moore categories are isomorphic to the category of perfect matchings and the category of partial Steiner triple systems, respectively. As a simple application of these results, we describe the product in the categories of perfect matchings and partial Steiner triple systems.

1 Introduction

Despite of the fact that there is a considerable amount of literature about graph homomorphisms and their properties, the attempts to apply category theory in graph theory appear to be rather rare. A quick Google Scholar search in the content of the 872 papers citing the monograph Graphs and Homomorphisms by Hell and Nešetřil [4] results in 66 hits for the word “functor”, 32 hits for the word “adjoint” and only 2 hits for the word “monad”.

In the present note we prove that two classical notions of graph theory arise as instances of a category-theoretic notion of an algebra for a monad: the notion of a perfect matching on a graph and the notion of a partial Steiner triple system. As a simple application of these results, we describe the product in the categories of perfect matchings and partial Steiner triple systems.

2 Preliminaries

For here undefined notions of category theory we refer to introductory books [1, 10, 6]. For the more brave among readers, see the classic monograph [5].

2.1 The category Graph

In this note we shall deal solely with simple, loopless and undirected graphs. If $G$ is a graph, we write $V(G)$ for the (possibly empty) set of vertices of $G$ and $E(G)$ for the set of edges of $G$. The edges of a graph $G$ are identified with two-element sets of vertices of $G$. The adjacency relation on the set of vertices of a graph $G$ is denoted by $\sim_G$. We shall mostly drop the index $G$ on $\sim_G$, whenever there is no danger of confusion.

∗This research is supported by grants VEGA 2/0069/16, 1/0420/15, Slovakia and by the Slovak Research and Development Agency under the contract APVV-14-0013.

1 One could say that one of the aims of this paper is to increment these numbers by one.
For two graphs $G$ and $H$, a *morphism of graphs* $f : G \to H$ is a mapping of sets $f : V(G) \to V(H)$ such that, for all $u, v \in V(G)$, $u \sim v$ implies that $f(u) \sim f(v)$. The morphism of graphs is usually called a *homomorphism* \[8].

Clearly, the composition of two morphisms is a morphism and an identity map $\text{id}_G$ on $V(G)$ is a morphism. So the class of all graphs equipped with morphisms of graphs forms a category, which we will call $\text{Graph}$.

Let us state without proof some facts about $\text{Graph}$.

- The empty graph is the initial object of $\text{Graph}$.
- There is no terminal object, hence $\text{Graph}$ is not complete.
- $\text{Graph}$ has all equalizers.
- $\text{Graph}$ lacks some coequalizers, so it is not a cocomplete category.
- Disjoint union of graphs is the coproduct of graphs.
- $\text{Graph}$ has binary products.

From the point of view of category theory, the lack of certain limits and colimits appears to be somewhat problematic. This is why most of the categorically-minded authors prefer to deal with graphs that admit loops, see for example \[2\]. Perhaps surprisingly, lack of loops in $\text{Graph}$ will be necessary prove the main results of the present note.

### 2.2 Monads and their algebras

Recall, that a monad on a category $\mathcal{C}$ is a triple $(\mathcal{T}, \eta, \mu)$, where $\mathcal{T}$ is an endofunctor of $\mathcal{C}$ and $\eta : \text{id}_\mathcal{C} \to \mathcal{T}$, $\mu : \mathcal{T} \circ \mathcal{T} \to \mathcal{T}$ are natural transformations of endofunctors of $\mathcal{C}$ such that, $\mu \circ (T\eta) = \mu \circ (\eta T) = \text{id}_\mathcal{T}$ and $\mu \circ (T\mu) = \mu \circ (\mu T)$. That means, for every object $A$ of $\mathcal{C}$, the diagrams

\[
\begin{align*}
T(A) & \xrightarrow{T(\eta_A)} T^2(A) & \xrightarrow{\eta_{T(A)}} T(A) \\
T(A) & \xrightarrow{\mu_A} T^2(A) & \xrightarrow{\mu_T(A)} T(A)
\end{align*}
\]

\[1\] commute.

**Example 1.** Consider an endofunctor $\mathcal{P}$ of the category of sets $\text{Set}$ that takes a set to its powerset $\mathcal{P}(X)$ and a mapping $f : X \to Y$ of sets to a mapping $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ that takes a subset of $X$ to its image under $f$. For a set $X$, $\eta_X : X \to \mathcal{P}(X)$ is given by the rule $\eta_X(x) = \{x\}$ and $\mu_X : \mathcal{P}^2(X) \to X$ is given by $\mu_X(\Omega) = \bigcup \Omega$, where $\Omega \in \mathcal{P}^2(X)$ is a system of subsets of $X$.

Then $(\mathcal{P}, \eta, \mu)$ is a monad, usually called the **powerset monad**.

Although the monads are interesting and useful structures per se (see for example the seminal paper \[9\] for applications in the theory of programming), the main use of monads is in their deep connection with the notion of an adjoint pair of functors.

Every monad induces an adjunction between the underlying category $\mathcal{C}$ and another category $\mathcal{C}^\mathcal{T}$. The objects of $\mathcal{C}$ are called *algebras for the monad* $\mathcal{T}$ and (in many cases) can be described as “an object of $\mathcal{C}$ equipped with an additional structure”. The morphisms of $\mathcal{C}^\mathcal{T}$ can be then thought of as “the $\mathcal{C}$-morphisms preserving the additional structure”.

Let $(\mathcal{T}, \eta, \mu)$ be a monad on a category $\mathcal{C}$. The category of algebras for $(\mathcal{T}, \eta, \mu)$, also known as the *Eilenberg-Moore category* for $(\mathcal{T}, \eta, \mu)$ is a category (denoted by $\mathcal{C}^\mathcal{T}$), such that objects (called *algebras*...
for the monad $T$ of $C^T$ are pairs $(A, \alpha)$, where $\alpha : T(A) \to A$, such that the diagrams

\[
\begin{array}{c}
A \xrightarrow{\eta_A} T(A) \\
\downarrow{1_A} \downarrow{\alpha} \\
A
\end{array}
\]  

(2)

\[
\begin{array}{c}
T^2(A) \xrightarrow{T(\alpha)} T(A) \\
\downarrow{\mu_A} \downarrow{\alpha} \\
T(A) \xrightarrow{\alpha} A
\end{array}
\]  

(3)

commute. A morphism of algebras $h : (A_1, \alpha_1) \to (A_2, \alpha_2)$ is a $C$-morphism $h : A_1 \to A_2$ such that the diagram

\[
\begin{array}{c}
T(A_1) \xrightarrow{T(h)} T(A_2) \\
\downarrow{\alpha_1} \downarrow{\alpha_2} \\
A_1 \xrightarrow{h} A_2
\end{array}
\]  

(4)

commutes.

Example 2. The category of algebras $\text{Set}^P$ for the powerset monad $(P, \eta, \mu)$ is isomorphic to the category $\text{Sup}$ in which the objects are all complete lattices and morphisms are mappings that preserve all (even infinite) joins.

The aforementioned adjunction between $C$ and $C^T$ is given by a pair of functors $F \dashv U$. The forgetful right adjoint functor $U : C^T \to C$ maps an algebra $(A, \alpha)$ to its underlying object, $U(A, \alpha) = A$ and a morphism of algebras to its underlying $C$-morphism. The left adjoint $F : C \to C^T$ maps an object $A$ of $C$ to the pair $(T(A), \mu_A)$. This pair is always an algebra for the monad $T$. Such adjunctions (and adjunctions equivalent to them) are called monadic.

On the other hand, every adjunction $F \dashv U$ induces, in a canonical way, a monad on the domain of the left adjoint $F$. So every adjunction can be “upgraded” to a monadic one.

3 The perfect matching monad and its algebras

In this section, we will introduce a monad $T$ on $\text{Graph}$ and prove that the category of algebras for $T$ is isomorphic to the category of graphs equipped with a perfect matching. We call this monad a perfect matching monad.

Recall, that a perfect matching \cite{7} on a graph $A$ is a set $M$ of edges of $A$ such that no two edges in $M$ have a vertex in common and $\bigcup M = V(A)$.

In the present note it will be of advantage if we use an alternative definition, that is clearly equivalent to the usual one.

Definition 1. Let $A$ be a graph. A perfect matching on $A$ is a mapping $m : V(A) \to V(A)$ such that, for all $x \in V(A)$, $\{x, m(x)\}$ is an edge of $A$ and $m \circ m = id_{V(A)}$

The category of perfect matchings is a category, where

- the objects are all pairs $(A, m)$, where $m$ is a perfect matching on a graph $A$ and
- the morphisms $f : (A, m) \to (Y, m)$ are graph homomorphisms $f : A \to Y$, that preserve the $m$, meaning that for all vertices $x$ of $A$, $f(m(x)) = m(f(x))$. 
The category of perfect matchings is denoted by $\text{Perf}$.

For a graph $A$, $T(A)$ is a graph that extends the graph $A$ by new leaf vertices, attaching a new leaf (or a new pendant edge) vertex at every vertex of $A$.

Let us introduce a notation that will turn out to be useful in our description of the perfect matching monad. The set of vertices of $T(A)$ is the set $V(A) \times \{0, 1\}$; we denote the vertices of $T(A)$ by $x_i$, where $x \in V(A)$, $i \in \{0, 1\}$. The vertices of $T(A)$ with $i = 0$ mirror the original vertices, the vertices with $i = 1$ are the new leaves. The edges $T(A)$ are of one of two types: either the edge is of the form $\{x_0, y_0\}$, where $\{x, y\} \in E(A)$, or it is of the form $\{x_0, x_1\}$, where $x \in V(A)$.

It is easy to see that this construction is functorial, so $T$ is an endofunctor of the category of graphs. Explicitly, if $f : A \to B$ is a graph homomorphism, then $T(f) : T(A) \to T(B)$ is given by the rule $T(f)(x_i) = f(x_i)$.

Moreover, for every graph $A$ we clearly have a morphism $\eta_A : A \to T(A)$ given by the rule $\eta_A(x) = x_0$. There is another morphism $\mu_A : T(T(A)) \to T(A)$, given by the rule $\mu_A(v_{ij}) = v_{i\oplus j}$, where $\oplus$ denotes the exclusive or operation on the set $\{0, 1\}$, also known as the addition in the 2-element cyclic group $\mathbb{Z}_2$.

Now note how this operation folds, in a natural way, the new pendant edges $\{v_{00}, v_{01}\}$ and $\{v_{10}, v_{11}\}$ that were added in the second iteration of $T$ onto the edge $\{v_0, v_1\}$. These families of maps determine natural transformations $\eta : \text{id}_{\text{Graph}} \to T$, $\mu : T^2 \to T$.

**Theorem 1.** The triple $(T, \eta, \mu)$ is a monad on the category of graphs.

**Proof.** Let us prove that, for every graph $A$, the monad axioms are satisfied. This is a consequence of the fact that $(\{0, 1\}, \oplus, 0)$ is a monoid. Indeed, let us check the validity of the associativity axiom. Let $x_{ijk} \in V(T^3(A))$. Then

$$(\mu_A \circ \mu_A)(v_{ijk}) = \mu_A(v_{ij\oplus k}) = v_{i\oplus (j\oplus k)} \quad \text{and} \quad (\mu_A \circ T(\mu_A))(v_{ijk}) = \mu_A(v_{i(j\oplus k)}) = v_{(i\oplus j)\oplus k}.$$ 

Similarly, the unit axioms are satisfied because $0$ is a unit for the operation $\oplus$. \hfill $\square$

**Theorem 2.** The category $\text{Perf}$ is isomorphic to the category of algebras for the monad $(T, \eta, \mu)$.

**Proof.** Let $\alpha : T(A) \to A$ be an algebra for $T$. The triangle diagram $\square$ means that, for all $x \in V(A)$, $\alpha(x_0) = x$. So every algebra $\alpha$ is completely determined by its values on the new leaves $x_1$ of $T(A)$. We claim that the mapping $m : V(A) \to V(A)$ given by the rule $m(x) = \alpha(x_1)$ is a perfect matching on $A$. Indeed, since $\alpha$ is a homomorphism of graphs, it must take the edge $\{x_0, x_1\}$ of $T(A)$ to an edge of $A$, so $\{\alpha(x_0), \alpha(x_1)\} = \{x, m(x)\}$ is an edge of $A$. To prove that $m \circ m = \text{id}_V(A)$, consider the
commutes, that means, to prove the equality a graph homomorphism is easy to see. Clearly, the triangle diagram (2) commutes. Let us prove that (3)
is a perfect matching. W e claim that (A, α) is an algebra for the monad T. The fact that α is a graph homomorphism is easy to see. Clearly, the triangle diagram (2) commutes. Let us prove that (3) commutes, that means, to prove the equality α(x_i j) = α(x_{i ⊕ j}). For i = 0,

\[ α(x_i j) = α(x_{0 j}) = α(x_j) = α(x_{0 ⊕ j}). \]

For j = 0,

\[ α(x_i j) = α(x_{i 0}) = α(x_i) = α(x_{i ⊕ 0}). \]

For i = j = 1,

\[ α(x_{1 1}) = α(m(x)1) = m(m(x)) = x = α(x_0) = α(x_{1 ⊕ 1}), \]

since m is a perfect matching.

It remains to prove the there constructions are functorial and, as functors, inverse to each other. The proof of this is completely straightforward and is thus omitted. □

**Example 3.** Let G be the graph from Figure 1. It has two perfect matchings, see Figure 2. Under the isomorphism described in the proof of Theorem 2, these two perfect matchings correspond to two algebras (G, α_1), (G, α_2) for the monad T that are characterized by

| v     | a | b | c | d | a_1 | a_2 |
|-------|---|---|---|---|-----|-----|
| a_1(v)| a | b | c | d | a_1 | a_2 |
| a_2(v)| a | b | c | d | a_1 | a_2 |

Recall that for two graphs A, B, their product A × B in the category Graph is a graph with vertex set V(A × B) = V(A) × V(B) such that for (a_1, b_1), (a_2, b_2) ∈ V(A × B) we have (a_1, b_1) ∼_{A,B} (a_2, b_2) if and only if a_1 ∼_A a_2 and b_1 ∼_B b_2. We write p_A and p_B for the projections from A × B onto A and B, respectively: p_A(a, b) = a, p_B(a, b) = b.

**Corollary 1.** Let (A, m), (B, m) be perfect matchings. Then their product in Perf has the A × B as the underlying graph and the perfect matching on A × B is given by the rule m(a, b) = (m(a), m(b)).

**Proof.** Since the functor U : Graph^T → Graph is a right adjoint in a monadic adjunction, it creates all limits that exist in Graph, see [3] Proposition IV.4.1. For products in Graph^T this means that
Figure 3: The endofunctor $S$ adds a new triangle over every edge.

$(A, \alpha) \times (B, \beta)$ is the algebra $(A \times B, \gamma)$, where $\gamma$ is the unique arrow that makes the diagram

\[
\begin{array}{ccc}
T(A) & \xrightarrow{T(p_A)} & T(A \times B) \\
\alpha & \downarrow \gamma & \downarrow \beta \\
A & \xrightarrow{p_A} & A \times B \\
\end{array}
\]

commute. Explicitly, this means that $\gamma : T(A \times B) \to A \times B$ is given by the rule $\gamma((a, b)_1) = (\alpha(a_1), \beta(b_1))$. If we translate this fact into the language of perfect matching via the isomorphism $\text{Graph}^T \cong \text{Perf}$ that we constructed in the proof of Theorem 2 the product $(A, m) \times (B, m)$ is a matching on $A \times B$ is given by $m(a, b) = (m(a), m(b))$. Indeed,

\[m(a, b) = \gamma((a, b)_1) = (\alpha(a_1), \beta(b_1)) = (m(a), m(b)).\]

Similarly, one can prove that an equalizer of two parallel morphisms $f, g : (A, m) \to (B, m)$ in the category $\text{Perf}$ is given by the restriction of $m$ to the induced subgraph $E \hookrightarrow A$, where $V(E) = \{v \in V(A) : f(v) = g(v)\}$.

4 The Steiner triple monad and its algebras

In this section, we will introduce a monad $S$ on $\text{Graph}$ and prove that the category of algebras for $S$ is isomorphic to the category of partial Steiner triple systems. We call this monad a Steiner triple monad.

Recall, that a partial Steiner triple system is a finite set $A$ equipped with a system of 3-element subsets $\Omega_A$ such that every 2-element subset $\{u, v\}$ of $A$ is in at most one set in $\Omega_A$. A partial Steiner triple system is complete if every 2-element subset of $A$ occurs in exactly one set in $\Omega_A$. A complete partial Steiner triple system is called simply a Steiner triple system.

The category of partial Steiner triple systems has pairs $(A, \Omega_A)$ as objects. For a pair of objects $(A, \Omega_A)$ and $(B, \Omega_B)$ a morphism is a mapping $f : A \to B$ such that for every triple $\{u, v, w\} \in \Omega_A$, $\{f(u), f(v), f(w)\} \in \Omega_B$. We denote this category by $\text{PSTS}$.

Let us describe the monad $S$. For a graph $G$, the graph $S(G)$ is the graph that can be described as “a copy of $G$ with a new triangle over every edge”. Note the similarity with the perfect matching monad: one could say that $S$ does something similar to $T$, but one dimension higher.

Formally, it will be of advantage to define $S(G)$ as a graph with the set of vertices $V(S(G)) = \{\{u\} : u \in V(G)\} \cup E(G)$ and, for all $X, Y \in V(S(G))$, $X \sim_{S(G)} Y$ if and only if one of the following is true:
• \( X = \{u\}, Y = \{v\} \) and \( u \sim v \).
• \( X = \{u, v\}, Y = \{v\} \) and \( u \sim v \).
• \( X = \{u\}, Y = \{u, v\} \) and \( u \sim v \).

Clearly, \( S \) is the object part of an endofunctor on the category \( \mathbf{Graph} \); for a morphism of graphs \( f : G \to H, S(f) : S(G) \to S(H) \) is given by \( S(f(u)) = f(\{u\}) \) for \( u \in V(G) \) and \( S(f(\{u, v\})) = S(\{f(u), f(v)\}) \) for \( \{u, v\} \in E(G) \).

For every graph \( G \), there is a morphism \( \eta_G : G \to S(G) \) given by the rule \( \eta_G(v) = \{v\} \). It is obvious that this family of morphisms gives us a natural transformation \( \eta : \text{id}_{\mathbf{Graph}} \to S \).

The last piece of data we need is a natural transformation \( \mu : S^2 \to S \). For every edge \( \{u, v\} \) of a graph \( G \), \( S(G) \) contains (essentially) the original edge and two new edges, forming a triangle with vertices \( \{u\}, \{v\}, \{u, v\} \). Repeating this construction once again, we see that \( S^2(G) \) consists of “triangles with an inscribed triangle”, one for every edge of the original graph \( G \), see Figure 3. There is a clear candidate for the desired mapping \( \mu_G : S^2(G) \to S(G) \); \( \mu_G \) just folds the three outer triangles onto the inner one. Note that a formal description of \( \mu_G \) is very simple: the vertex set \( V(S^2(G)) \) consists of certain systems of sets of vertices of \( G \). Then \( \mu_G(X) \) is simply the symmetric difference of all sets in the system \( X \); in symbols \( \mu_G(X) = \Delta X \). For example \( \mu_G(\{\{u\}, \{u, v\}\}) = \{u\} \Delta \{u, v\} = \{v\} \) and \( \mu_G(\{\{u, v\}\}) = \{u, v\} \).

It is obvious that \( \mu_G \) is a morphism in \( \mathbf{Graph} \) and that the family of morphisms \( (\mu_G)_{G \in \mathbf{Graph}} \) is a natural transformation from \( S^2 \) to \( S \).

**Theorem 3.** \((S, \mu, \eta)\) is a monad on the category of graphs.

**Proof.** We need to prove that, putting \( T = S \) in (11), both triangles and the square commute.

Let \( \{u, v\} \) be a vertex of \( S(G) \), that means \( u \sim v \) in \( G \). Then, in particular, \( u \neq v \) and we may chase \( \{u, v\} \) around the triangles:

\[
\mu_G(S(\eta_G(\{u, v\}))) = \mu_G(\{\eta_G(u), \eta_G(v)\}) = \mu_G(\{\{u\}, \{v\}\}) = \{u\} \Delta \{v\} = \{u, v\}
\]

\[
\mu_G(\eta_G(\{u, v\})) = \mu_G(\{\{u, v\}\}) = \{u, v\}
\]

The case of a vertex of the form \( \{u\} \) is trivial.

The fact that the square in (11) commutes follows from the fact that the symmetric difference \( \Delta \) is a commutative and associative operation with a neutral element \( \emptyset \) and that \( Y \Delta Y = \emptyset \).

In detail, every element of \( V(S^3(G)) \) is some set of sets of sets of the form

\[
\{(X_i^1, X_i^2, \ldots, X_i^{k_i}), (X_j^1, X_j^2, \ldots, X_j^{k_j}), \ldots, (X_n^1, X_n^2, \ldots, X_n^{k_n})\}
\]

where each \( X_i^j \) is a set of vertices of \( G \). (In fact, \( n \in \{1, 2\} \), each \( k_i \in \{1, 2\} \) and every \( X_i^j \) is either a singleton or a pair, but we shall not need any of these facts.) The morphism \( S(\mu_G) \) maps this element to the system of sets

\[
\{X_1^1 \Delta X_2^1 \Delta \ldots \Delta X_{k_1}^1, X_1^2 \Delta X_2^2 \Delta \ldots \Delta X_{k_2}^2, \ldots, X_1^n \Delta X_2^n \Delta \ldots \Delta X_{k_n}^n\}
\]

in \( V(S^2(G)) \) and this is mapped by \( \mu_G \) to the set

\[
(X_1^1 \Delta X_2^1 \Delta \ldots \Delta X_{k_1}^1)(X_1^2 \Delta X_2^2 \Delta \ldots \Delta X_{k_2}^2) \Delta \ldots \Delta (X_1^n \Delta X_2^n \Delta \ldots \Delta X_{k_n}^n)
\]

Chasing the element (6) the other way around the square (11), \( \mu_{S(G)} \) maps it to

\[
\{X_1^1, X_2^1, \ldots, X_{k_1}^1 \} \Delta \{X_1^2, X_2^2, \ldots, X_{k_2}^2 \} \Delta \ldots \Delta \{X_1^n, X_2^n, \ldots, X_{k_n}^n \}
\]

this amounts to keeping just those sets \( X_i^j \) that occur odd number of times. We then apply \( \mu_G \) to the resulting system of sets, so we get an expression exactly like (7), but with those \( X_i^j \) that occur even number of times removed. However, removing those sets does not change the value of the expression and we have proved that the square commutes.

\[\square\]
Theorem 4. The category of algebras $\text{Graph}^S$ for the Steiner triple monad is isomorphic to the category $\text{PSTS}$.

Proof. Let $(G, \alpha)$ be an algebra for the Steiner triple monad. There are two types of vertices in $S(G)$: singletons and pairs. The triangle axiom (2) tells us that $\alpha(\{u\}) = u$ for all $u \in V(G)$. Thus, every algebra $\alpha$ is completely determined by its value on pairs, that means, edges of $G$. Since $\alpha$ is a morphism of graphs, the image of every triangle in $S(G)$ under $\alpha$ is a triangle in $G$. Hence for every $\{u, v\} \in E(G)$, $\{(u, v), \{u, v\}\}$ is a triangle in $S(G)$ and its image is the triangle

$$\{\alpha(\{u\}), \alpha(\{v\}), \alpha(\{u, v\})\} = \{u, v, \alpha(\{u, v\})\}.$$ 

Thus, for every edge $\{u, v\}$ of $G$, $\alpha$ selects a triangle in $G$ with vertices $u, v$ and $\alpha(\{u, v\})$.

We claim that the set $V(G)$ equipped with the system of triples

$$\Omega_\alpha = \{\{u, v, \alpha(\{u, v\})\} : u \sim_G v\}$$

is a partial Steiner triple system.

It is clear that a pair of vertices $\{u, v\}$ of $G$ occurs as a subset in at least one of the triples in $\Omega_\alpha$ if and only if $\{u, v\}$ is an edge of $G$. It remains to prove that every edge occurs in exactly one triple in $\Omega_\alpha$ or, in other words, that the triple that arises from an edge $\{u, v\}$ is the same as the triple that arises from the edge $\{u, \alpha(\{u, v\})\}$.

In terms of properties of the $\alpha$ mapping, this amounts to

$$\alpha(\{u, \alpha(\{u, v\})\}) = v.$$ 

However, this follows by the commutativity of the square (3), because

$$(\alpha \circ S(\alpha))(\{\{u\}, \{u, v\}\}) = \alpha((\alpha(\{u\}), \alpha(\{u, v\}))) = \alpha(\{u, \alpha(\{u, v\})\})$$

and

$$(\alpha \circ \mu_G)(\{\{u\}, \{u, v\}\}) = \alpha(\{u\} \Delta \{u, v\}) = \alpha(\{\{u\}\}) = v.$$ 

To prove that this construction is functorial, let us write $F(G, \alpha)$ for the partial Steiner triple system $(V(G), \Omega_\alpha)$ constructed above. Every $f : (G, \alpha) \rightarrow (G', \alpha')$ induces a $\text{PSTS}$-morphism $F(f) : F(G, \alpha) \rightarrow F(G', \alpha')$. Indeed, for all $\{u, v, \alpha(\{u, v\})\} \in \Omega_\alpha$,

$$\{f(u), f(v), f(\alpha(\{u, v\}))\} = \{f(u), f(v), \alpha(S(f)(\{u, v\}))\} = \{f(u), f(v), \alpha(\{f(u), f(v)\})\},$$

because $f$ is a morphism of algebras. $F$ is then a functor from $\text{Graph}^T$ to $\text{PSTS}$.

On the other hand, let $(A, \Omega_A)$ be a partial Steiner triple system. Let $G$ be a graph with $V(G) = A$ and $u \sim_G v$ if and only if $u \neq v$ and $u, v \in X$ for some $X \in \Omega_A$. Let us define a morphism of graphs $\alpha : S(G) \rightarrow G$ by the rules $\alpha(\{u\}) = u$ and $\alpha(\{u, v\}) = w$, where $w$ is the unique element of $A$ such that $\{u, v, w\} \in \Omega_A$.

Clearly, $\alpha$ is a morphism of graphs. We need to prove that $(G, \alpha)$ is an algebra for the Steiner triple monad. The triangle axiom (2) is clearly satisfied by $\alpha$, so let us check the square axiom (3). Let $K$ be a vertex of $S^2(G)$. We need to prove that $\alpha(\mu_G(K)) = \alpha(S(\alpha)(K))$. Let $u, v \in X$ for some $X \in \Omega_A$, so that $\{u, v\}$ is an edge of $G$.

If $K = \{\{u, v\}, \{u, v\}\}$, then $\alpha(\mu_G(K)) = \alpha(\{u, v\} \Delta \{v\}) = \alpha(\{v\}) = v$ and $\alpha(S(\alpha)(\{u, v\}, \{u\})) = \alpha(\alpha(\{u, v\}), \{u\}) = v$, because $X = \{u, v, \alpha(\{u, v\})\}$.

The cases of $K = \{\{u, v\}, K = \{\{u\}\}$ and $K = \{\{u\}, \{v\}\}$ are trivial and thus omitted.

To prove the functoriality of this construction, let us write $U(A, \Omega_A) = (G, \alpha)$. We claim that every morphism in $\text{PSTS}$ $f : (A_1, \Omega_{A_1}) \rightarrow (A_2, \Omega_{A_2})$ induces a morphism $U(f) : U(A_1, \Omega_{A_1}) \rightarrow U(A_2, \Omega_{A_2})$. This amounts to proving that (3) commutes.
If \( \{u, v\} \) is an edge of \( U(A_1, \Omega_{A_1}) \), then \( \{u, v, \alpha_1(\{u, v\})\} \in \Omega_{A_1} \) is the unique triple that contains \( u \) and \( v \). Since \( f \) is a morphism of partial Steiner triple systems, \( \{f(u), f(v), f(\alpha_1(\{u, v\}))\} \in \Omega_{A_2} \). Further, \( \alpha_2(S(f)(\{u, v\})) = \alpha_2(\{f(u), f(v)\}) \) and \( \{f(u), f(v), \alpha_2(S(f)(\{u, v\}))\} \) is the unique triple that contains \( f(u) \) and \( f(v) \). Therefore \( f(\alpha_1(\{u, v\})) = \alpha_2(S(f)(\{u, v\})) \), meaning that (3) commutes.

For a singleton vertex \( \{u\} \) of \( U(A_1, \Omega_{A_1}) \),
\[
f(\alpha_1(\{u\})) = f(u) = \alpha_2(\{f(u)\}) = \alpha_2(S(f)(\{u\})).
\]

Therefore \( U \) is a functor and it is easy to check that \( UF = id_{\text{Graph}^S} \) and \( FU = id_{\text{PSTS}} \).

**Corollary 2.** Let \( (A, \Omega_A), (B, \Omega_B) \) be partial Steiner triple systems. Then their product in \( \text{PSTS} \) is \( (A \times B, \Omega_{A \times B}) \), where \( X \in \Omega_{A \times B} \) if and only if \( X \) is a 3-element subset of \( A \times B \) such that \( p_A(X) \in \Omega_A \) and \( p_B(X) \in \Omega_B \).

**Proof.** We can describe the product in the category \( \text{Graph}^S \) using the diagram (3): the product of two algebras \( (A, \alpha) \) and \( (B, \gamma) \) for the Steiner triple monad is the algebra \( (A \times B, \gamma) \), where \( \gamma \) is given by the rules
\[
\gamma(\{(a, b)\}) = (\alpha(\{a\}), \beta(\{b\})) = (a, b)
\]
\[
\gamma(\{(a_1, b_1), (a_2, b_2)\}) = (\alpha(\{a_1, a_2\}), \beta(\{b_1, b_2\}))
\]
for every vertex \( (a, b) \) and edge \( \{(a_1, b_1), (a_2, b_2)\} \) of the graph \( A \times B \). That means that the product of partial Steiner triple systems \( (A, \Omega_A) \) and \( (B, \Omega_B) \) in \( \text{PSTS} \) is \( (A \times B, \Omega_{A \times B}) \) with
\[
\Omega_{A \times B} = \{(a_1, b_1), (a_2, b_2), (\alpha(\{a_1, a_2\}), \beta(\{b_1, b_2\})) : a_1 \sim_A a_2 \text{ and } b_1 \sim_B b_2\},
\]
where \( (A, \alpha) \) and \( (B, \beta) \) are the algebras associated with \( (A, \Omega_A) \) and \( (B, \Omega_B) \), respectively. The statement then easily follows. \( \square \)

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