A 4/3-approximation for TSP on cubic 3-edge-connected graphs

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1 Introduction

We consider the travelling salesman problem on metrics which can be viewed as the shortest path metric of an undirected graph with unit edge-lengths. Finding a TSP tour in such a metric is then equivalent to finding a connected Eulerian subgraph in the underlying graph. Since the length of the tour is the number of edges in this Eulerian subgraph our problem can equivalently be stated as follows: Given an undirected, unweighted graph \( G = (V, E) \) find a connected Eulerian subgraph, \( H = (V, E') \) with the fewest edges. Note that \( H \) could be a multigraph.

In this paper we consider the special case of the problem when \( G \) is 3-regular (also called cubic) and 3-edge-connected. Note that the smallest Eulerian subgraph contains at least \( n = |V| \) edges. In fact, in the shortest path metric arising out of such a graph the Held-Karp bound for the length of the TSP tour would also be \( n \). This is because we can obtain a fractional solution to the sub-tour elimination LP (which is equivalent to the Held-Karp bound) of value \( n \) by assigning \( 2/3 \) to every edge in \( G \).

Improving the approximation ratio for metric-TSP beyond 3/2 is a long standing open problem. For the metric completion of cubic 3-edge connected graphs Gamarnik et.al. obtained an algorithm with an approximation guarantee slightly better than 3/2. The main result of this paper is to improve this approximation guarantee to 4/3 by giving a polynomial time algorithm to find a connected Eulerian subgraph with at most \( 4n/3 \) edges. This matches the conjectured integrality gap for the sub-tour elimination LP for the special case of these metrics.

2 Preliminaries

Let \( n \) be the number of vertices of the given graph \( G \). Let \( d(x) \) denote the degree of \( x \). A 2-factor in \( G \) is a subset of edges \( X \) such that every vertex has degree 2 in \( X \). Let \( \sigma(X) \) denote the minimum size of components of \( X \). Given two distinct edges \( e_1 = x_1v \) and \( e_2 = x_2v \) incident on a vertex \( v \), let \( G_v^{e_1,e_2} \) denote the graph obtained by replacing \( e_1, e_2 \) by the edge \( x_1x_2 \). The vertex \( v \) is said to be split off. We call a cut \((S, \overline{S})\) essential when both \( S \) and \( \overline{S} \) contain at least one edge each.

We will need the following results for our discussion

**Lemma 1** (Peterson). Every bridgeless cubic graph has a 2-factor.

**Lemma 2** (Mader). Let \( G = (V, E) \) be a \( k \)-edge-connected graph, \( v \in V \) with \( d(v) \geq k+2 \). Then there exists edges \( e_1, e_2 \in E \) such that \( G_{v}^{e_1,e_2} \) is homeomorphic to a \( k \)-edge-connected graph.

**Lemma 3** (Jackson, Yoshimoto). Let \( G \) be a 3-edge-connected graph with \( n \) vertices. Then \( G \) has a spanning even subgraph in which each component has at least \( \min(n,5) \) vertices.
3 Algorithm

Our algorithm can be broadly split into three parts. We first find a 2-factor of the cubic graph that has no 3-cycles and 4-cycles. Next, we compress the 5-cycles into ‘super-vertices’ and split them using Lemma 2 to get a cubic 3-edge-connected graph $G'$ again. Repeatedly applying the first part on $G'$ and compressing the five cycles gives a 2-factor with no 5-cycle on the vertices of the original graph. We ‘expand’ back the super-vertices to form $X$ that is a subgraph of $G$. We finally argue that $X$ can be modified to get a connected spanning even multi-graph using at most $4/3(n)$ edges.

The starting point of our algorithm is Theorem 3 [2]. In fact [2] proves the following stronger theorem.

**Theorem 1.** Let $G$ be a 3-edge-connected graph with $n$ vertices, $u_2$ be a vertex of $G$ with $d(u_2) = 3$, and $e_1 = u_1u_2; e_2 = u_2u_3$ be edges of $G$. (it may be the case that $u_1 = u_3$). Then $G$ has a spanning even subgraph $X$ with $\{e_1,e_2\} \subset E(X)$ and $\sigma(X) \geq \min(n,5)$.

The proof of this theorem is non-constructive. We refer to the edges $e_1, e_2$ in the statement of the theorem as “required edges”. We now discuss the changes required in the proof given in [2] to obtain a polynomial time algorithm which gives the subgraph $X$ with the properties as specified in Theorem 1. Note that we will be working with a 3-regular graph (as against an arbitrary graph of min degree 3 in [2]) and hence the even subgraph $X$ we obtain will be a 2-factor.

1. If $G$ contains a non-essential 3-edge cut then we proceed as in the proof of Claim 2 in [2]. This involves splitting $G$ into 2 graphs $G_1, G_2$ and suitably defining the required edges for these 2 instances so that the even subgraphs computed in these 2 graphs can be combined. This step is to be performed whenever the graph under consideration has an essential 3-edge cut.

2. Since $G$ is 3-regular we do not require the argument of Claim 6.

3. Since $G$ has no essential 3-edge cut and is 3-regular, a 3-cycle in $G$ implies that $G$ is $K_4$. In this case we can find a spanning even subgraph containing any 2 required edges.

4. The process of eliminating 4-cycles in the graph involves a sequence of graph transformations. The transformations are as specified in [2] but the order in which the 4-cycles are considered depends on the number of required edges in the cycle. We first consider all such cycles which do not have any required edges, then cycles with 2 required edges and finally cycles which have one required edge.

Since with each transformation the number of edges and vertices in the graph reduces we would eventually terminate with a graph, say $G'$, with girth 5. We find a 2-factor in $G'$, say $X'$ and undo the transformations (as specified in [2]) in the reverse order in which they were done to obtain a 2-factor $X$ in the original graph $G$ which has the properties of Theorem 1.

Suppose the 2-factor obtained $X$ contains a 5-cycle $C$. We compress the vertices of $C$ into a single vertex, say $v_C$, and remove self loops. $v_C$ has degree 5 and we call this vertex a super-vertex. We now use Lemma 2 to replace two edges $x_1v_C$ and $x_2v_C$ incident at $v_C$ with the edge $x_1x_2$ while preserving 3-edge connectivity. The edge $x_1x_2$ is called a super-edge. Since the graph obtained is cubic and 3-edge connected we can once again find a 2-factor, each of whose cycles has length at least 5. If there is a 5-cycle which does not contain any super-vertex or super-edge we compress it and repeat the above process. We continue doing this till we obtain a 2-factor, say $X$, each of whose cycles is either of length at least 6 or contains a super-vertex or a super-edge.
In the 2-factor $X$ we replace every super-edge with the corresponding edges. For instance the super-edge $x_1x_2$ would get replaced by edges $x_1v_C$ and $x_2v_C$ where $v_C$ is a super-vertex obtained by collapsing the vertices of a cycle $C$. After this process $X$ is no more a 2-factor but an even subgraph. However, the only vertices which have degree more than 2 are the super-vertices and they can have a maximum degree 4. Let $X$ denote this even subgraph.

Consider some connected component $W$ of $X$. We will show how to expand the super-vertices in $W$ into 5-cycles to form an Eulerian subgraph with at most $\lfloor 4|W'|/3 \rfloor - 2$ edges, where $|W'|$ is number of vertices in the expanded component. For each component we will use 2 more edges to connect this component to the other components to obtain a connected Eulerian subgraph with at most $\lfloor 4n/3 \rfloor - 2$ edges. Note that the subgraph we obtain may use an edge of the original graph at most twice.

We now consider two cases depending on whether $W$ contains a super-vertex.

1. $W$ has no super-vertices. Then, $W$ is a cycle with at least 6 vertices and hence Eulerian. Since $|W|/3 \geq 2$ the claim follows.

2. $W$ has at least one super-vertex, say $s$. We will discuss the transformations for a single super-vertex and this will be repeated for the other super-vertices. Note that $s$ has degree 2 or 4.

![Figure 1: On Expanding a super-vertex with degree 2](image)

If $s$ has degree 2, then the 2 edges incident on the 5-cycle corresponding to $s$ would be as in Figure 1. In both cases we obtain an Eulerian subgraph. By this transformation we have added 4 vertices and at most 5 edges to the subgraph $W$.

Suppose the super-vertex $s$ has degree 4 in the component $W$. $W$ may not necessarily be a component of the subgraph $X$ as it might have been obtained after expanding a few super-vertices, but that will not effect our argument. Let $C$ be the 5-cycle corresponding to this super-vertex and let $v_1, v_2, v_3, v_4, v_5$ be the vertices on $C$ (in order). Further let $v'_1$ be the vertex not in $C$ adjacent to $v_1$. Let $v_5v'_5$ be the edge incident on $C'$ that is not in the subgraph $W$.

We replace the vertex $s$ in $W$ with the cycle $C$ and let $W'$ be the resulting subgraph. Note that by dropping edges $v_1v_2$ and $v_3v_4$ from $W'$ we obtain an Eulerian subgraph which includes all vertices of $C$. However, this subgraph may not be connected as it could be the case that edges $v_1v_2$ and $v_3v_4$
Figure 2: Expanding a super-vertex with degree 4 when $v_1v_2$ and $v_3v_4$ do not form a 2-edge-cut of the sub-graph constructed till now.

Figure 3: Expanding a super-vertex with degree 4 when $v_1v_2$ and $v_3v_4$ form a 2-edge-cut.

form an edge-cut in $W'$. If this is the case then we apply the transformation as shown in Figure 3.

This ensures that $W'$ remains connected and is Eulerian. Note that as a result of this step we have added 4 vertices and at most 4 edges to the subgraph $W$.

Let $W'$ be the component obtained by expanding all the super-vertices in $W$. Suppose initially, component $W$ had $k_1$ super-vertices of degree 2, $k_2$ super-vertices of degree 4 and $k_3$ vertices of degree 2. This implies $W$ had $k_1 + 2k_2 + k_3$ edges. On expanding a super-vertex of degree 2, we add 5 edges in the worst case. On expanding a super-vertex of degree 4, we add 4 edges in the worst case. So, the total number of edges in $W'$ is at most $6k_1 + 6k_2 + k_3$ while the number of vertices in $W'$ is exactly $5k_1 + 5k_2 + k_3$. Note that $k_1 + k_2 + k_3 \geq 5$ and if $k_1 + k_2 + k_3 = 5$ then $k_1 + k_2 \geq 1$. Hence, $2k_1 + 2k_2 + k_3 \geq 6$ and this implies that the number of edges in $W'$ is at most $\lceil 4|V(W')|/3 \rceil - 2$.

4 Conclusions

We show that any cubic 3-edge connected graph contains a connected Eulerian subgraph with at most $4n/3$ edges. It is tempting to conjecture the same for non-cubic graphs especially since the result in [2] holds for all 3-edge connected graphs. The example of a $K_{3,n}$ demonstrates that this conjecture would be false. A $K_{3,n}$ is 3-edge connected and any connected Eulerian subgraph contains at least $2n$ edges.
References

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