A quantum-geometrical description of fracton statistics

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Abstract

We consider the fractal characteristic of the quantum mechanical paths and we obtain for any universal class of fractons labeled by the Hausdorff dimension defined within the interval $1 < h < 2$, a fractal distribution function associated with a fractal von Neumann entropy. Fractons are charge-flux systems defined in two-dimensional multiply connected space and they carry rational or irrational values of spin. This formulation can be considered in the context of the fractional quantum Hall effect-FQHE and number theory.

keywords: Fractal distribution function; fractal von Neumann entropy; fractional quantum Hall effect.

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I. INTRODUCTION

We make out a review of some concepts introduced by us in the literature, such as [1–6]: fractons, universal classes $h$ of particles, fractal spectrum, duality symmetry between classes $h$ of particles, fractal supersymmetry, fractal distribution function, fractal von Neumann entropy, fractal index etc. We apply these ideas in the context of the FQHE and number theory.

Fractons are charge-flux systems which carry rational or irrational values of spin. These objects are defined in two-dimensional multiply connected space and are classified in universal classes $h$ of particles or quasiparticles, with the fractal parameter or Hausdorff dimension $h$, defined in the interval $1 < h < 2$. It is related to the quantum paths and can be extracted from the propagators of the particles in the momentum space [7]. The particles are collected in each class take into account the fractal spectrum

$$
\frac{1}{2} = 1 - \nu, \quad 0 < \nu < 1; \quad h - 1 = \nu - 1, \quad 1 < \nu < 2;
$$

$$
\frac{3}{2} = 3 - \nu, \quad 2 < \nu < 3; \quad h - 1 = \nu - 3, \quad 3 < \nu < 4; \text{etc.}
$$

(1)

and the spin-statistics relation $\nu = 2s$, valid for such fractons. The fractal spectrum establishes a connection between $h$ and the spin $s$ of the particles: $h = 2 - 2s$, $0 \leq s \leq \frac{1}{2}$. Thus, there exists a mirror symmetry behind this notion of fractal spectrum. Given the statistical weight for these classes of fractons

$$
\mathcal{W}[h, n] = \frac{[G + (nG - 1)(h - 1)]!}{[nG]![G + (nG - 1)(h - 1) - nG]!}
$$

(2)

and from the condition of the entropy be a maximum, we obtain the fractal distribution function [2]

$$
n[h] = \frac{1}{\mathcal{Y}[\xi] - h}
$$

(3)

The function $\mathcal{Y}[\xi]$ satisfies the equation

$$
\xi = \left\{ \mathcal{Y}[\xi] - 1 \right\}^{h-1} \left\{ \mathcal{Y}[\xi] - 2 \right\}^{2-h},
$$

(4)

with $\xi = \exp \{(\epsilon - \mu)/KT\}$. We understand the fractal distribution function as a quantum-geometrical description of the statistical laws of nature, since the quantum path is a fractal curve and this reflects the Heisenberg uncertainty principle.

We can obtain for any class its distribution function considering Eq.(3) and Eq.(4). For example, the universal class $h = \frac{3}{2}$ with distinct values of spin $\left\{ \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \cdots \right\}_{h=\frac{3}{2}}$ has a specific fractal distribution

$$
n\left[ \frac{3}{2} \right] = \frac{1}{\sqrt{\frac{1}{4} + \xi^2}}.
$$

(5)

We also have
\[\xi^{-1} = \left\{ \Theta[\mathcal{Y}] \right\}^{h-2} - \left\{ \Theta[\mathcal{Y}] \right\}^{h-1}\]  
(6)

where

\[\Theta[\mathcal{Y}] = \frac{\mathcal{Y}^{\xi} - 2}{\mathcal{Y}^{\xi} - 1}\]  
(7)

is the single-particle partition function. We verify that the classes \(h\) satisfy a duality symmetry defined by \(\tilde{h} = 3 - h\). So, fermions and bosons come as dual particles. As a consequence, we extract a fractal supersymmetry which defines pairs of particles \((s, s + \frac{1}{2})\). In this way, the fractal distribution function appears as a natural generalization of the fermionic and bosonic distributions for particles with braiding properties. Therefore, our approach is a unified formulation in terms of the statistics which each universal class of particles satisfies: from a unique expression we can take out any distribution function. In some sense, we can say that fermions are fractons of the class \(h = 1\) and bosons are fractons of the class \(h = 2\).

The free energy for particles in a given quantum state is expressed as

\[\mathcal{F}[h] = K T \ln \Theta[\mathcal{Y}].\]  
(8)

Hence, we find the average occupation number

\[n[h] = \xi \frac{\partial}{\partial \xi} \ln \Theta[\mathcal{Y}].\]  
(9)

The fractal von Neumann entropy per state in terms of the average occupation number is given as \([1,2]\)

\[S_G[h, n] = K \left[ [1 + (h-1)n] \ln \left\{ \frac{1 + (h-1)n}{n} \right\} - [1 + (h-2)n] \ln \left\{ \frac{1 + (h-2)n}{n} \right\} \right].\]  
(10)

and it is associated with the fractal distribution function (Eq.3).

The entropies for fermions \(\left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots \right\}_{h=1}\) and bosons \(\left\{ 0, 1, 2, \cdots \right\}_{h=2}\), can be recovered promptly

\[S_G[1] = -K \left\{ n \ln n + (1 - n) \ln(1 - n) \right\}\]  
(11)

and

\[S_G[2] = K \left\{ (1 + n) \ln(1 + n) - n \ln n \right\} .\]  
(12)

Now, as we can check, each universal class \(h\) of particles, within the interval of definition has its entropy defined by the Eq.(10). Thus, for fractons of the self-dual class \(\left\{ \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \cdots \right\}_{h=\frac{3}{2}}\), we have

\[S_G\left[\frac{3}{2}\right] = K \left\{ (2 + n) \ln \sqrt{\frac{2 + n}{2n}} - (2 - n) \ln \sqrt{\frac{2 - n}{2n}} \right\}.\]  
(13)
and for two more examples, the dual classes \( \left\{ \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \ldots \right\}_{h=\frac{4}{3}} \) and \( \left\{ \frac{1}{6}, \frac{5}{6}, \frac{7}{6}, \ldots \right\}_{h=\frac{5}{6}} \),

the entropies read as

\[
S_{G} \left[ \frac{4}{3} \right] = K \left\{ (3 + n) \ln \sqrt[3]{3 + n} - (3 - 2n) \ln \sqrt[3]{3 - 2n} \right\} 
\]

(14)

and

\[
S_{G} \left[ \frac{5}{3} \right] = K \left\{ (3 + 2n) \ln \sqrt[3]{3 + 2n} - (3 - n) \ln \sqrt[3]{3 - n} \right\}. 
\]

(15)

We have also introduced the topological concept of fractal index, which is associated with each class. As we saw, \( h \) is a geometrical parameter related to the quantum paths of the particles and so, we define [3]

\[
i_{f} [h] = \frac{6}{\pi^{2}} \int_{0}^{1} \frac{d\xi}{\xi} \ln \left\{ \Theta \left[ Y (\xi) \right] \right\}. 
\]

(16)

We obtain for the bosonic class \( i_{f}[2] = 1 \), for the fermionic class \( i_{f}[1] = 0.5 \) and for some classes of fractons, we have \( i_{f}[\frac{3}{2}] = 0.6, i_{f}[\frac{4}{3}] = 0.56, i_{f}[\frac{5}{3}] = 0.656 \). For the interval of the definition \( 1 \leq h \leq 2 \), there exists the correspondence \( 0.5 \leq i_{f}[h] \leq 1 \), which signalizes the connection between fractons and quasiparticles of the conformal field theories, in accordance with the unitary \( c < 1 \) representations of the central charge. For \( \nu \) even it is defined by

\[
c[\nu] = i_{f} [h, \nu] - i_{f} \left[ h, \frac{1}{\nu} \right] 
\]

(17)

and for \( \nu \) odd it is defined by

\[
c[\nu] = 2 \times i_{f} [h, \nu] - i_{f} \left[ h, \frac{1}{\nu} \right], 
\]

(18)

where \( i_{f} [h, \nu] \) means the fractal index of the universal class \( h \) which contains the statistical parameter \( \nu = 2s \) or the particles with distinct values of spin, which obey specific fractal distribution function. For example, we obtain the results

\[
c[0] = i_{f} [2, 0] - i_{f} [h, \infty] = 1; 
\]

\[
c[1] = 2 \times i_{f} [1, 1] - i_{f} [1, 1] = 0.5; etc. 
\]

(19)

We have noted in [3], for the first time, an unsuspected connection between fractal geometry and conformal field theories, which second our considerations is expressed by Eqs.(16,17,18).

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In another way, the central charge \( c[\nu] \) can be obtained using the Rogers dilogarithm function, i.e.

\[
c[\nu] = \frac{L[x^{\nu}]}{L[1]}, 
\]

(20)

with \( x^{\nu} = 1 - x, \ \nu = 0, 1, 2, 3, etc. \) and

\[
L[x] = -\frac{1}{2} \int_{0}^{x} \left\{ \frac{\ln(1 - y)}{y} + \frac{\ln y}{1 - y} \right\} dy, \ 0 < x < 1. 
\]

(21)

Thus, we have established a connection between fractal geometry and number theory, given that the dilogarithm function appears in this context, besides another branches of mathematics [8].
II. FRACTIONAL QUANTUM HALL EFFECT

Such ideas can be applied in the context of the FQHE. This phenomenon is characterized by the filling factor parameter $f$, and for each value of $f$ we have the quantization of Hall resistance and a superconducting state along the longitudinal direction of a planar system of electrons, which are manifested by semiconductor doped materials, i.e. heterojunctions, under intense perpendicular magnetic fields and lower temperatures [9].

The parameter $f$ is defined by $f = N\frac{\phi}{\phi_0}$, where $N$ is the electron number, $\phi_0$ is the quantum unit of flux and $\phi$ is the flux of the external magnetic field throughout the sample. The spin-statistics relation is given by $\nu = 2s = 2\frac{\phi}{\phi_0}$, where $\phi$ is the flux associated with the charge-flux system which defines the fracton $(h, \nu)$. According to our approach there is a correspondence between $f$ and $\nu$, numerically $f = \nu$. This way, we verify that the filling factors observed experimentally appear into the classes $h$ and from the definition of duality between the equivalence classes, we note that the FQHE occurs in pairs of these dual topological quantum numbers $(f, \tilde{f}) = (\frac{1}{3}, \frac{2}{3}), (\frac{5}{3}, \frac{4}{3}), (\frac{4}{3}, \frac{5}{3}), (\frac{2}{3}, \frac{7}{7}), (\frac{2}{5}, \frac{3}{5}), (\frac{3}{7}, \frac{4}{7}), (\frac{4}{7}, \frac{5}{7}) \text{ etc.}$

All the experimental data satisfy this symmetry principle. In this way, our formulation can predicting FQHE, that is, consider the duality symmetry discovered by us [2]. Thus, each Hall state is described by a system of quasiparticles(fractons) such that for a given value of filling factor $f$, the spin of the objects which constitute the physical system is $s = f/2$. We understand here fractons as modelling collective excitations of a two-dimensional electron gas under special conditions like FQHE.

We also observe that our approach, in terms of equivalence classes for the filling factors, embodies the structure of the modular group as discussed in the literature [2,10]. We have that the transitions allowed are those generated by the condition $|p_2q_1 - p_1q_2| = 1$, with $h_1 = \frac{p_1}{q_1}$ and $h_2 = \frac{p_2}{q_2}$. For example, we have the transitions between the classes

$$\left\{ \frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \cdots \right\}_{h=\frac{5}{7}}; \left\{ \frac{2}{5}, \frac{8}{5}, \frac{12}{5}, \cdots \right\}_{h=\frac{2}{5}}; \left\{ \frac{3}{7}, \frac{11}{7}, \frac{17}{7}, \cdots \right\}_{h=\frac{3}{7}}; \left\{ \frac{4}{9}, \frac{14}{9}, \frac{22}{9}, \cdots \right\}_{h=\frac{4}{9}}; \left\{ \frac{5}{11}, \frac{17}{11}, \frac{27}{11}, \cdots \right\}_{h=\frac{5}{11}}; \left\{ \frac{6}{13}, \frac{20}{13}, \frac{32}{13}, \cdots \right\}_{h=\frac{6}{13}} \text{ etc.}$$

This way, we define the universality classes of the quantum Hall transitions, which are labeled by the fractal parameter $h$. The topological character of these quantum numbers comes from the relation between $h$ and $f$, by the fractal spectrum.

III. NUMBER THEORY

We observe again that our formulation to the universal class $h$ of particles with any values of spin $s$ establishes a connection between Hausdorff dimension $h$ and the central charge $c[\nu]$. Besides this, we have obtained a relation between the fractal parameter and the Rogers dilogarithm function, through the concept of fractal index, which is defined in terms...
of the partition function associated with each universal class of particles. As a result, we have a connection between fractal geometry and number theory. Thus,

\[ c[\nu] = \frac{L[x^\nu]}{L[1]} = i_f[h, \nu] - i_f[h, \frac{1}{\nu}], \quad \nu = 0, 2, 4, \text{etc.} \quad (22) \]

Also we have established a connection between the fractal parameter \( h \) and the Farey sequences of rational numbers. Now, the fractal curve is continuous and nowhere differentiable, it is self-similar, it does not depend on the scale and has fractal dimension just in the interval \( 1 < h < 2 \). Given a closed path with length \( L \) and resolution \( R \), the fractal properties of this curve can be determined by \( h - 1 = \lim_{R \to 0} \frac{\ln L/l}{\ln R} \), where \( l \) is the usual length for the resolution \( R \) and the curve is covering with \( l/R \) spheres of diameter \( R \).

Farey series \( F_n \) of order \( n \) is the increasing sequence of irreducible fractions in the range \( 0 - 1 \) whose denominators do not exceed \( n \). They satisfy the properties

P1. If \( h_1 = \frac{p_1}{q_1} \) and \( h_2 = \frac{p_2}{q_2} \) are two consecutive fractions \( \frac{p_1}{q_1} > \frac{p_2}{q_2} \), then \( |p_2q_1 - q_2p_1| = 1 \).

P2. If \( \frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3} \) are three consecutive fractions \( \frac{p_1}{q_1} > \frac{p_2}{q_2} > \frac{p_3}{q_3} \), then \( \frac{p_2}{q_2} = \frac{p_1}{q_1} + \frac{p_3}{q_3} \).

P3. If \( \frac{p_1}{q_1}, \frac{p_2}{q_2} \) are consecutive fractions in the same sequence, then among all fractions between the two, \( \frac{p_1 + p_2}{q_1 + q_2} \) is the unique reduced fraction with the smallest denominator.

We have the following

**Theorem** [6]: The elements of the Farey series \( F_n \) of the order \( n \), belong to the fractal sets, whose Hausdorff dimensions are the second fractions of the fractal sets. The Hausdorff dimension has values within the interval \( 1 < h < 2 \), which are associated with fractal curves.

We observe that the sets obtained are dual sets and, in particular, we have a fractal selfdual set, with Hausdorff dimension \( h = \frac{3}{2} \). In this way, taking into account the fractal spectrum and the duality symmetry between sets, we can extract for any Farey series of rational numbers, fractal sets whose Hausdorff dimension is the second fraction of the set.

## IV. CONCLUSIONS

We have introduced a unified description of particles with distinct values of spin in terms of their statistics. From a unique expression, the fractal distribution function, we can take out distribution functions for any universal class \( h \) of particles. The Hausdorff dimension of the fractal quantum paths of the fractons are determined by the fractal spectrum. We have here a quantum-geometrical description of the statistical laws of nature.

We verify along these ideas that the FQHE occurs in pairs of dual filling factors. These quantum numbers get their topological character from the Hausdorff dimension, a geometrical parameter associated with the fractal curves of the particles. We can check that all experimental results satisfy the symmetry principle discovered by us, the duality symmetry between universal classes \( h \) of particles. The idea of supersymmetry, in some sense, appears in this context of the condensed matter and the universality classes of the quantum Hall transitions are established.

We emphasize that our formulation is supported by symmetry principles: mirror symmetry behind the fractal spectrum, duality symmetry between classes \( h \) of particles, fractal supersymmetry, modular group behind the quantum Hall transitions.
In another direction, we have established a connection between Number Theory and Physics relating fractal geometry and dilogarithm function through the concept of fractal index. Also we have determined an algorithm for computation of the Hausdorff dimension of any fractal set related to the Farey sequences of rational numbers.

Finally, we are thinking about fracton quantum computing from the possible perspective of fractons qubits.
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