ELEMENTS OF TOPOLOGICAL ALGEBRA

III. THE CLOSED CATEGORY OF FILTERS

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Abstract. We explore the structure of Fil, the category of filters and germs of admissible partial functions. In particular, we show that Fil is a nonsymmetric closed category, as defined in [5].

Introduction

In this paper, we study the category of filters, defined almost exactly as defined in [2], the only difference being that we admit as objects in the category of filters, Fil, filters that contain the empty set. This necessitates that we define germs of functions using explicit partial functions.

The point of the paper is not this minor change, but the definition we give of nonsymmetric closed category structure ([5] - see [3, VII.1 and VII.7] for the canonical treatment of the symmetric case) on the category Fil. We feel the need for a self-contained definition and exploration of the properties of this category, to facilitate forthcoming more detailed explorations of applications, such as [6] and [4], briefly mentioned in Section 10.

1. Filters

In this section, we will begin to define Fil, the category of filters and germs, with a discussion of filters. Recall

Definition 1.1. A filter F on a set S is a set of subsets of S such that

1) S ∈ F;
2) if F ∈ F and F ⊆ F′ ⊆ S, then F′ ∈ F; and
3) if F, F′ ∈ F then F ∩ F′ ∈ F.

We will denote the set of filters on S by Fil S. Note that some definitions include another condition: that F be proper, i.e., that F not contain the empty set. However, when we assume this, we shall explicitly call the filter a proper filter.

Remark 1.2. A filter F ∈ Fil S uniquely determines S, as S = ∪ F.
Ordering the set of filters. Fil $S$ admits a partial ordering, which we (unlike some authors) take to be reverse inclusion:

**Proposition 1.3.** Let $\mathcal{F}, \mathcal{G} \in \text{Fil } S$. The following are equivalent (and, if they hold, we will say $\mathcal{F} \leq \mathcal{G}$):

(1) $\mathcal{G} \subseteq \mathcal{F}$, and
(2) For any $G \in \mathcal{G}$, there is an $F \in \mathcal{F}$ with $F \subseteq G$.

Filter bases.

**Definition 1.4.** We say that a set $B$ of subsets of $S$ is a filter base or base for a filter if $F, F' \in B$ imply there is a $\bar{F} \in B$ such that $\bar{F} \subseteq F \cap F'$.

**Definition 1.5.** If $B$ is a filter base (of subsets of $S$), then the set of subsets $F = \{F \subseteq S \mid \exists B \in B \text{ such that } B \subseteq F\}$ is a filter $F$, which is the least (in the above ordering) such that $B \subseteq F$, and which we denote by $\text{Fg}_S B$ or simply $\text{Fg} B$. In this case, we also say that $B$ is a base for $F$. If $B$ is not a filter base, then the least filter $F$ such that $B \subseteq F$, and which we still denote by $\text{Fg} B$, is $\text{Fg} B'$, where $B'$ is the set of finite intersections of elements of $B$ (and is a filter base). In either case, we say that $F = \text{Fg} B$ is the filter generated by $B$.

Subfilters. If $F \in \text{Fil } S$, then we say that a filter $F' \in \text{Fil } S$ is a subfilter of $F$ if $F' \leq F$.

We will denote by $\text{Fil } F$ the set of subfilters $F' \leq F$, i.e., the interval sublattice $I_{\text{Fil } S} [\bot, F]$.

Fil $S$ is coalgebraic. Recall that a lattice is coalgebraic if its dual is algebraic [4, Definition 15.1].

**Proposition 1.6.** Let $\mathcal{F}$ be a filter. Then $\text{Fil } \mathcal{F}$ is a coalgebraic complete lattice, where

1. $\bigvee_i \mathcal{F}_i = \bigcap_i \mathcal{F}_i$ and
2. $\bigwedge_i \mathcal{F}_i = \text{Fg} \{ \bigcup_i \mathcal{F}_i \}$.

$f(\mathcal{F})$ and $f^{-1}(\mathcal{G})$.

**Definition 1.7.** Let $S$ and $T$ be sets, and $f : S \to T$ a function. If $\mathcal{F} \in \text{Fil } S$, then we define

$$f(\mathcal{F}) = \text{Fg} \{ f(F) \mid F \in \mathcal{F} \};$$

if $\mathcal{G} \in \text{Fil } T$, then we define

$$f^{-1}(\mathcal{G}) = \text{Fg} \{ f^{-1}(G) \mid G \in \mathcal{G} \}.$$

**Proposition 1.8.** We have

1. If $f : S \to T$ is a function, $\mathcal{F} \in \text{Fil } S$, and $\mathcal{G} \in \text{Fil } T$, then

$$f(\mathcal{F}) \leq \mathcal{G} \iff \mathcal{F} \leq f^{-1}(\mathcal{G}).$$

2. If in addition $g : T \to W$ is a function and $\mathcal{H} \in \text{Fil } W$, then

$$g(f(\mathcal{F})) = (gf)(\mathcal{F})$$

and

$$f^{-1}(g^{-1}(\mathcal{H})) = (gf)^{-1}(\mathcal{H}).$$
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2. Partial Functions; Restriction

If $S, T$ are sets, a partial function $f : S \to T$ is a method or rule $f$ which somehow assigns an element $f(s) \in T$ to $s$, for some, but not necessarily all, elements $s \in S$.

**Definition 2.1.** We denote the domain of definition of $f$, the subset of $s \in S$ such that $f(s)$ is defined, by $\text{dd}(f)$. We denote the range of $f$, the set of elements of the form $f(s)$ for some $s \in S$, by $\text{r}(f)$.

**Definition 2.2.** If $f : S \to T$, $g : T \to W$ are partial functions, then the composite partial function of $f$ and $g$, denoted $g \circ f$, is the partial function that assigns an element $s \in S$ to $g(f(s))$, if both $s \in \text{dd}(f)$ and $f(s) \in \text{dd}(g)$.

**Definition 2.3.** If $f$ and $g$ are partial functions from $S$ to $T$, then we say that $f$ is a restriction of $g$ if $\text{dd}(f) \subseteq \text{dd}(g)$ and $f(s) = g(s)$ for $s \in \text{dd}(f)$. If $f$ is a partial function on $S$, and $D \subseteq S$, then we denote by $f|_D$ the restriction of $f$ to $D$, i.e., the rule which assigns $f(s)$ to $s$ for $s \in D \cap \text{dd}(f)$ and does not assign anything to elements not in $D \cap \text{dd}(f)$.

**f(D) and $f^{-1}(D)$ when $f$ is a Partial Function.** If $f : S \to T$ is a partial function, and $D \subseteq S$, we define

$$f(D) = \{ f(s) \mid s \in D \cap \text{dd}(f) \}$$

and if $D' \subseteq T$,

$$f^{-1}(D') = \{ s \in S \mid s \in \text{dd}(f) \text{ and } f(s) \in D' \}.$$

**Lemma 2.4.** Let $f : S \to T$, $g : T \to W$ be partial functions. Then $\text{dd}(g \circ f) = f^{-1}(\text{dd}(g))$.

**Proof.** Referring to the definitions, we have

$$s \in \text{dd}(g \circ f) \iff s \in \text{dd}(f) \text{ and } f(s) \in \text{dd}(g) \iff s \in f^{-1}(\text{dd}(g)).$$

**Lemma 2.5.** Let $f$ and $g$ be partial functions from $S$ to $T$. If $f = g|_{\hat{D}}$ for some $\hat{D} \subseteq S$ then

1. If $D \subseteq S$, then $f(D) \subseteq g(D)$, with $f(D) = g(D)$ when $D \subseteq \hat{D}$, and
2. if $D' \subseteq T$, then $f^{-1}(D') \subseteq g^{-1}(D')$.

**Proof.** (1): We have

$$t \in f(D) \iff t \in g|_{\hat{D}}(D) \iff \exists s \in \hat{D} \cap D \cap \text{dd}(g) \text{ such that } t = g(s) \iff \exists s \in D \cap \text{dd}(g) \text{ such that } t = g(s) \iff s \in g(D),$$

with equivalence if $D \subseteq \hat{D}$. 

We have
\[ s \in f^{-1}(D') \iff s \in \text{dd}(f) \text{ and } f(s) \in D' \]
\[ \iff s \in \text{dd}(g|_D) \text{ and } g|_{D'}(s) \in D' \]
\[ \iff s \in \hat{D} \cap \text{dd}(g) \text{ and } g(s) \in D' \]
\[ \implies s \in \text{dd}(g) \text{ and } g(s) \in D' \]
\[ \iff s \in g^{-1}(D'). \]

\[ \square \]

**Proposition 2.6.** We have

1. If \( f : S \to T \) is a partial function, \( D \subseteq S \), and \( D' \subseteq T \), then
\[ f(D) \subseteq D' \iff D \subseteq (S - \text{dd}(f)) \cup f^{-1}(D'); \]

2. if in addition, there is another partial function \( g : T \to W \), and \( D'' \subseteq W \), then
\[ g(f(D)) = (g \circ f)(D) \]
and
\[ f^{-1}(g^{-1}(D'')) = (g \circ f)^{-1}(D''). \]

**Proof.** (1) We have
\[ f(D) \subseteq D' \iff (s \in D \text{ and } s \in \text{dd}(f) \implies f(s) \in D') \]
\[ \iff (s \notin D \text{ or } s \notin \text{dd}(f) \text{ or } f(s) \in D') \]
\[ \iff (s \in D \implies (s \notin \text{dd}(f) \text{ or } f(s) \in D')) \]
\[ \iff D \subseteq (S - \text{dd}(f)) \cup f^{-1}(D'); \text{ and} \]

(2): we have
\[ w \in g(f(D)) \iff \exists s \text{ such that } s \in \text{dd}(f) \text{ and } f(s) \in \text{dd}(g) \text{ and } w = g(f(s)) \]
\[ \iff \exists s \text{ such that } s \in \text{dd}(g \circ f) \text{ and } (g \circ f)(s) = w \]
\[ \iff w \in (g \circ f)(D); \text{ and} \]

\[ s \in f^{-1}(g^{-1}(D'')) \iff s \in \text{dd}(f) \text{ and } f(s) \in g^{-1}(D'') \]
\[ \iff s \in \text{dd}(f) \text{ and } f(s) \in \text{dd}(g) \text{ and } g(f(s)) \in D'' \]
\[ \iff s \in \text{dd}(g \circ f) \text{ and } (g \circ f)(s) \in D'' \]
\[ \iff s \in (g \circ f)^{-1}(D''). \]

\[ \square \]
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If \( f \) and \( f^{-1}(G) \) are \textbf{Partial Functions}. If \( S \) and \( T \) are sets, \( f : S \to T \) is a partial function, and \( F \) is a filter of subsets of \( S \), then we define

\[
f(F) = \{ f(F) | F \in F \}.
\]

On the other hand, given \( f \) and a filter \( G \) of subsets of \( T \), then we define

\[
f^{-1}(G) = \{ f^{-1}(G) | G \in G \}.
\]

\textbf{Remark 2.7.} Note that for a total function \( f \) (i.e. if \( \text{dd}(f) = S \)), this definition coincides with the definition (Definition 1.7) given previously. Also, note that the mappings \( D \mapsto f(D) \) and \( D' \mapsto g^{-1}(D') \) are monotone, and, consequently, take filter bases to filter bases.

\textbf{Theorem 2.8.} We have

(1) If \( f \) is a partial function from \( S \) to \( T \), \( F \) is a filter of subsets of \( S \) such that \( \text{dd}(f) \in F \), and \( G \) is a filter of subsets of \( T \), then

\[
f(F) \leq G \iff F \leq f^{-1}(G);
\]

(2) and if in addition, we have a partial function \( g : T \to W \), and \( H \) is a filter of subsets of \( W \), then

\[
g(f(F)) = (g \circ f)(F)
\]

and

\[
f^{-1}(g^{-1}(H)) = (g \circ f)^{-1}(H).
\]

\textbf{Proof.} First, we note that by Remark 2.7, if we have \( f : S \to T \) and \( F \in \text{Fil} S \), then

\[
f(F) = \{ f(F) | F \in F \}
\]

where \( \text{Up} B \), for a subset \( B \) of a lattice (in this case, the lattice of subsets of \( S \)), denotes the set of elements of the lattice greater than or equal to an element in \( B \). Similarly, if \( G \in \text{Fil} T \), then

\[
f^{-1}(G) = \{ f^{-1}(G) | G \in G \}
\]

Then, to prove the statements of the Theorem, we consider that
(1): by Proposition 2.6, and since \( \text{dd}(f) \in \mathcal{F} \),

\[
\begin{align*}
f(\mathcal{F}) \leq \mathcal{G} & \iff Fg\{ f(F) \mid F \in \mathcal{F} \} \leq \mathcal{G} \\
& \iff \forall G \in \mathcal{G}, \exists F \in \mathcal{F} \text{ such that } f(F) \subseteq G \\
& \iff \forall G \in \mathcal{G}, \exists F \in \mathcal{F} \text{ such that } F \subseteq (S - \text{dd}(f)) \cup f^{-1}(G) \\
& \iff \forall G \in \mathcal{G}, \exists F \in \mathcal{F} \text{ such that } F \cap \text{dd}(f) \subseteq f^{-1}(G) \\
& \iff \forall G \in \mathcal{G}, \exists F \in \mathcal{F} \text{ such that } F \subseteq f^{-1}(G) \\
& \iff \mathcal{F} \leq Fg\{ f^{-1}(G) \mid G \in \mathcal{G} \} \\
& \iff \mathcal{F} \leq f^{-1}(\mathcal{G});
\end{align*}
\]

(2): we also have

\[
H \in g(f(\mathcal{F})) \iff H \in Fg\{ g(G) \mid G \in Fg\{ f(F) \mid F \in \mathcal{F} \} \} \\
\iff H \in Fg\{ g(f(F)) \mid F \in \mathcal{F} \} \\
\iff H \in (g \circ f)(\mathcal{F}), \text{ and}
\]

\[
F \in f^{-1}(g^{-1}(\mathcal{H})) \iff F \in Fg\{ f^{-1}(G) \mid G \in Fg\{ g^{-1}(H) \mid H \in \mathcal{H} \} \} \\
\iff F \in Fg\{ f^{-1}(g^{-1}(H)) \mid H \in \mathcal{H} \} \\
\iff F \in (g \circ f)^{-1}(\mathcal{H}).
\]

\[\square\]

3. The Category \( \text{LPartial} \)

**Admissible domains of definition.** We would like to consider as equivalent, functions (or partial functions) on a set \( S \) which have a common restriction, and to work with the resulting equivalence classes of partial functions that we will call germs in Section 4. In this plain form, the equivalence relation is uninteresting, because any two partial functions on \( S \) have a common restriction to the empty set. For this reason, we will limit this relation of having a common restriction to considering two partial functions on \( S \) as equivalent, if and only if they have a common restriction to a subset of \( S \) that belongs to a specified set of admissible domains of definition. In order that this result in an equivalence relation, we will require the specified set of admissible domains of definition be a filter of subsets of \( S \). We call partial functions from \( S \) to \( T \), defined on some set in the filter \( \mathcal{F} \in \text{Fil} S \), *admissible partial functions from \( \mathcal{F} \) to \( T \).* We already saw, in Theorem 2.8(1), a use of the condition that a partial function be admissible, although we didn’t yet call it that.

**Notation 3.1.** If \( \mathcal{F} \) is a filter on some set \( S \), \( \mathcal{G} \) is a filter on some set \( T \), and \( G \in \mathcal{G} \), we denote by \( \text{Partial}(\mathcal{F}, \mathcal{G}) \) (\( \text{LPartial}(\mathcal{F}, \mathcal{G}) \)), the set of admissible partial functions from \( \mathcal{F} \) to \( \mathcal{G} \), (respectively the set of local admissible partial functions from \( \mathcal{F} \) to \( \mathcal{G} \)) and by \( \text{Partial}(\mathcal{F}, \mathcal{G}, G) \) (\( \text{LPartial}(\mathcal{F}, \mathcal{G}, G) \)) the set of admissible partial functions from \( S \) to \( T \).
(respectively the set of local admissible partial functions from $\mathcal{F}$ to $\mathcal{G}$), such that for some $F \in \mathcal{F}$, $f(F) \subseteq G$.

**Remark 3.1.** These notations will be useful not only for defining (in this section and the next) the categories $\text{LPartial}$ and $\text{Fil}$, but also for defining a nonsymmetric closed structure on $\text{Fil}$ in Sections 8 and 9. Note that elements of $\text{LPartial}(\mathcal{F}, \mathcal{G}, \mathcal{G})$ are not necessarily arrows in any category, although they play a role in the definitions of categories in the later sections we mentioned.

**Locality.** In order for admissible partial functions, and germs of admissible partial functions, to be the arrows of categories, we will need to impose another condition, locality, that the partial functions must satisfy. The problem is that if $f$ and $g$ are partial functions, $\text{dd}(g \circ f)$ may be so small that $g \circ f$ is not admissible. Thus, suppose $f$ is an admissible partial function from $\mathcal{F}$ to $T$, where $\mathcal{F} \in \text{Fil} \, S$, and $g$ is an admissible partial function from $\mathcal{G}$ to $W$, where $\mathcal{G} \in \text{Fil} \, T$. As $\text{dd}(g \circ f) = f^{-1}(\text{dd}(g))$ and $\text{dd}(g)$ could be any element of $\mathcal{G}$, what we want to require of $f$ is that for any $G \in \mathcal{G}$, $f^{-1}(G) \in \mathcal{F}$. For, if that is true, then because $\mathcal{F}$ is a filter, $g \circ f$ will defined on $\text{dd}(f) \cap f^{-1}(G) \in \mathcal{F}$ and will be admissible.

**Definition 3.2.** We say that $f$ is local (with respect to $\mathcal{G}$) if for each $G \in \mathcal{G}$, $f^{-1}(G) \in \mathcal{F}$, or in other words, there is an $F \in \mathcal{F}$ such that $F \subseteq \text{dd}(f)$ and $f(F) \subseteq G$.

**Proposition 3.3.** If $f : S \to T$ is a partial function, admissible with respect to $\mathcal{F}$ and local with respect to $\mathcal{G}$, and $g : T \to W$ is a partial function, admissible with respect to $\mathcal{G}$ and local with respect to $\mathcal{H}$, then $g \circ f : S \to W$ is a partial function, admissible with respect to $\mathcal{F}$ and local with respect to $\mathcal{H}$.

**Definition 3.4.** We denote the set of partial functions from $S = \bigcup \mathcal{F}$ to $T = \bigcup \mathcal{G}$, admissible with respect to $\mathcal{F}$ and local with respect to $\mathcal{G}$, by $\text{LPartial}(\mathcal{F}, \mathcal{G})$. This defines a category $\text{LPartial}$, where the identity arrow from $\mathcal{F}$ to itself is just the identity function on $S = \bigcup \mathcal{F}$.

**Proof that $\text{LPartial}$ is a category.** If $f : \mathcal{F} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{H}$, then $g \circ f$ is defined as the partial function with domain of definition $\text{dd}(f) \cap f^{-1}(g)$, sending $s \in \text{dd}(f)$ to $g(f(s))$. That is, it is the partial function corresponding to the relational product of $f$ and $g$, seen as relations. The axioms of a category are immediate. 

## 4. Germs and the Category $\text{Fil}$

**Germs of partial functions.**

**Definition 4.1.** If $\mathcal{F}$ is a filter of subsets of a set $S$, then a germ of admissible partial functions from $\mathcal{F}$ to a set $T$ is an $\equiv_\mathcal{F}$-equivalence class of such partial functions, where $f \equiv_\mathcal{F} g$ iff for some $F \in \mathcal{F}$, $\text{dd}(f) \cap F = \text{dd}(g) \cap F$ and $f(s) = g(s)$ for all $s \in \text{dd}(f) \cap \text{dd}(g) \cap F$.

If $f$ is a partial function, we will denote its germ (the $\equiv_\mathcal{F}$-equivalence class containing $f$) by $\Gamma f$, or by $f/\mathcal{F}$. We will also use $\Gamma$ as a set-function, so that if $Y$ is a set of admissible partial functions from $\mathcal{F}$ to $T$, $\Gamma(Y)$ will denote the set of germs of the partial functions in $Y$. We will use $\Gamma$ in this way particularly in two cases: we will shortly define the hom-set $\text{Fil}(\mathcal{G}, \mathcal{H}) =$
\( \Gamma(\text{Partial}(\mathcal{H}, \mathcal{G})) \) of the category \( \text{Fil} \), and we will later define an internal hom-object \( \mathcal{G}^{\mathcal{H}} = Fg\{ \Gamma(\text{Partial}(\mathcal{H}, \mathcal{G}, G) \} \) for the category \( \text{Fil} \) using the base of sets \( \Gamma(\text{Partial}(\mathcal{H}, \mathcal{G}, G)). \)

**Theorem 4.2.** Let \( f, g \) be partial functions from \( S \) to \( T \), and \( \mathcal{F} \) a filter of subsets of \( S \) such that \( f \equiv \mathcal{F} g \). We have

1. If \( \mathcal{F}' \) is a subfilter of \( \mathcal{F} \), then \( f(\mathcal{F}') = g(\mathcal{F}') \), and
2. if \( \mathcal{G} \) is a filter of subsets of \( T \), then \( f^{-1}(\mathcal{G}) \cap \mathcal{F} = g^{-1}(\mathcal{G}) \cap \mathcal{F} \)

**Proof.** Let \( \mathcal{F} \in \mathcal{F} \) be such that \( \text{dd}(f) \cap \mathcal{F} = \text{dd}(g) \cap \mathcal{F} \) and \( f = g \) on \( \text{dd}(f) \cap \text{dd}(g) \cap \mathcal{F} \). Then

1. Since \( \mathcal{F} \in \mathcal{F} \), and \( \mathcal{F}' \leq \mathcal{F} \), there is an \( \mathcal{F}' \in \mathcal{F}' \) such that \( \mathcal{F}' \subseteq \mathcal{F} \). We have \( \text{dd}(f) \cap \mathcal{F}' = \text{dd}(g) \cap \mathcal{F}' \) and \( f = g \) on \( \text{dd}(f) \cap \text{dd}(g) \cap \mathcal{F}' \). Thus,

   \[
   f(\mathcal{F}') = Fg\{ f(\mathcal{F}') | \mathcal{F}' \in \mathcal{F}' \}
   = Fg\{ f(\mathcal{F}') | \mathcal{F}' \in \mathcal{F}' \text{ and } \mathcal{F}' \subseteq \mathcal{F} \}
   = Fg\{ g(\mathcal{F}') | \mathcal{F}' \in \mathcal{F}' \text{ and } \mathcal{F}' \subseteq \mathcal{F} \}
   = Fg\{ g(\mathcal{F}') | \mathcal{F}' \in \mathcal{F}' \}
   = g(\mathcal{F}').
   \]

2. If \( \mathcal{F} \in \mathcal{F} \) is such that \( \text{dd}(f) \cap \mathcal{F} = \text{dd}(g) \cap \mathcal{F} \) and \( f(s) = g(s) \) for all \( s \in \text{dd}(f) \cap \text{dd}(g) \cap \mathcal{F} \), then the same statement is true for any smaller \( \mathcal{F} \). Consequently, we have

   \[
   f^{-1}(\mathcal{G}) \cap \mathcal{F} = Fg\{ f^{-1}(\mathcal{G}) \cap \mathcal{F}'' | \mathcal{G} \in \mathcal{G}, \mathcal{F}'' \in \mathcal{F} \}
   = Fg\{ g^{-1}(\mathcal{G}) \cap \mathcal{F}'' | \mathcal{G} \in \mathcal{G}, \mathcal{F}'' \in \mathcal{F} \}
   = g^{-1}(\mathcal{G}) \cap \mathcal{F};
   \]

**Notation 4.1.** We continue to use roman letters \( f, g \), etc. to denote partial functions, and will use greek letters \( \varphi, \gamma \), etc. for germs.

**\( \varphi(\mathcal{F}) \) and \( \varphi^{-1}(\mathcal{G}) \).**

**Notation 4.2.** Let \( \mathcal{F}, \mathcal{F}' \) be filters of subsets of \( S \) such that \( \mathcal{F}' \leq \mathcal{F} \), and let \( \mathcal{G} \) be a filter of subsets of \( T \). If \( \varphi \) is a germ of partial functions from \( S \) to \( T \) and admissible wrt \( \mathcal{F} \), then we define

\[
\varphi(\mathcal{F}') = f(\mathcal{F}')
\]

and

\[
\varphi^{-1}(\mathcal{G}) = f^{-1}(\mathcal{G}) \cap \mathcal{F},
\]

where \( f \) is any admissible partial function representing \( \varphi \). By Theorem 4.2, these formulae are independent of the choice of \( f \).

**Proposition 4.3.** Let \( \mathcal{F} \in \text{Fil} S \), and \( \mathcal{G} \in \text{Fil} T \). If \( \varphi \) is a germ of partial functions admissible wrt \( \mathcal{F} \), then the following are equivalent:
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(1) For some admissible partial function \( f : \mathcal{G} \to T \) representing \( \varphi \), \( f \) is local with respect to \( \mathcal{G} \);
(2) For every admissible partial function \( f : \mathcal{G} \to T \) representing \( \varphi \), \( f \) is local with respect to \( \mathcal{G} \);
(3) \( \varphi(\mathcal{F}) \leq \mathcal{G} \).

Proof. Certainly (2) \( \implies \) (1).

\( 1 \implies 3 \): Let \( f : \mathcal{G} \to T \) be an admissible partial function representing \( \varphi \), local with respect to \( \mathcal{G} \), and suppose that we are given \( G \in \mathcal{G} \). Since \( f \) is local wrt \( \mathcal{G} \), there is an \( F \in \mathcal{F} \) such that \( f(F) \subseteq G \). This shows that \( \varphi(\mathcal{F}) = f(\mathcal{F}) \leq \mathcal{G} \).

\( 3 \implies 2 \): Assume that \( \varphi(\mathcal{F}) \leq \mathcal{G} \), and let \( f : \mathcal{G} \to T \) be an admissible partial function representing \( \varphi \). Let \( G \) be any element of \( \mathcal{G} \). Since \( \varphi(\mathcal{F}) \leq \mathcal{G} \), there is an \( f \in \mathcal{F} \) such that \( f(F) \subseteq G \). Since \( G \) was any element of \( \mathcal{G} \), this shows that \( f \) is local wrt \( \mathcal{G} \).

\( \square \)

Theorem 4.4. Let \( \mathcal{F} \) be a filter of subsets of a set \( S \), and let \( \mathcal{G} \) be a filter of subsets of another set \( T \). We have

(1) \( \equiv_{\mathcal{F}} \) is an equivalence relation on the set of partial functions from \( S \) to \( T \);
(2) if \( f \equiv_{\mathcal{F}} g \), then \( f \) is admissible wrt \( \mathcal{F} \) iff \( g \) is admissible wrt \( \mathcal{F} \);
(3) if \( f \equiv_{\mathcal{F}} g \), then \( f \) is local wrt \( \mathcal{G} \) iff \( g \) is local wrt \( \mathcal{G} \);
(4) \( \equiv_{\mathcal{F}} \) is an equivalence relation on the set of partial functions from \( S \) to \( T \) local wrt \( \mathcal{G} \);
(5) If \( f \) and \( g \) are admissible (wrt \( \mathcal{F} \)) functions from \( S \) to \( T \), and \( f \equiv_{\mathcal{F}} g \), then there is an \( F \in \mathcal{F} \) such that \( F \subseteq dd(f) \cap dd(g) \) and \( f|_F = g|_F \).
(6) If \( f, f' : S \to T \) are partial functions, admissible with respect to \( \mathcal{F} \) and local with respect to \( \mathcal{G} \), with \( f \equiv_{\mathcal{F}} f' \), and \( g, g' \in \mathbf{Fil}(\mathcal{G}, \bigcup \mathcal{H}) \) are admissible with respect to \( \mathcal{H} \), with \( g \equiv g' \), then \( (g \circ f) \equiv_{\mathcal{F}} (g' \circ f') \).

Proof. (1): Let \( f \equiv_{\mathcal{F}} g \equiv_{\mathcal{F}} h \). Then there is an \( F \in \mathcal{F} \) such that \( dd(f) \cap F = dd(g) \cap F \) and \( f \equiv g \) on \( dd(f) \cap dd(g) \cap F \), and an \( F' \in \mathcal{F} \) such that \( dd(g) \cap F' = dd(h) \cap F' \) and \( g \equiv h \) on \( dd(g) \cap dd(h) \cap F' \). Then \( F \cap F' \in \mathcal{F} \), \( dd(f) \cap F \cap F' = dd(g) \cap F \cap F' \), \( dd(g) \cap F \cap F' = dd(h) \cap F' \), and \( f \equiv h \) on \( dd(f) \cap dd(g) \cap dd(h) \cap F \cap F' = dd(f) \cap dd(g) \cap dd(h) \cap F \cap F' \). Thus, \( \equiv_{\mathcal{F}} \) is transitive. Reflexivity and symmetricity are obvious.

(2): \( f \) is admissible wrt \( \mathcal{F} \) iff \( dd(f) \in \mathcal{F} \), and likewise, \( g \) is admissible iff \( dd(f) \in \mathcal{F} \). If \( f \equiv_{\mathcal{F}} g \), then there is an \( F \in \mathcal{F} \) such that \( dd(f) \cap dd(g) \cap F \); if \( f \) is admissible so that \( dd(f) \in \mathcal{F} \), then \( dd(g) \cap F \notin F \), which implies that \( dd(g) \notin \mathcal{F} \). Thus, \( f \) admissible implies \( g \) admissible. The converse follows by symmetry.

(3): If \( f \equiv_{\mathcal{F}} g \), then there is an \( F \in \mathcal{F} \) such that \( f = g \) on \( dd(f) \cap dd(g) \cap F \), and if \( f \) is local wrt \( \mathcal{G} \), then for any \( G \in \mathcal{G} \) there is an \( f \in \mathcal{F} \) such that \( f(F) \subseteq G \). Then \( f(F \cap F) \subseteq G \), which implies that \( g(F \cap F) \subseteq G \). Thus, \( g \) is local wrt \( \mathcal{G} \). The converse follows by symmetry.

(4): Follows from (1) and (3).

(5): We have \( dd(f) \cap dd(g) \in \mathcal{F} \), and there is an \( F \in \mathcal{F} \) such that \( f \) and \( g \) are equal on \( dd(f) \cap F \) and \( dd(g) \cap F \). We let \( F = dd(f) \cap dd(g) \cap F \).
(6): \(dd(g \circ f) = dd(f) \cap f^{-1}(dd(g))\). \(dd(g' \circ f') = dd(f') \cap (f')^{-1}(dd(g'))\). Let \(F \in \mathcal{F}\) be such that \(dd(f) \cap F = dd(f') \cap F\) and \(f = f'\) on \(dd(f) \cap dd(f') \cap F\), and let \(G \in \mathcal{G}\) be such that \(dd(g) \cap G = dd(g') \cap G\) and \(g = g'\) on \(dd(g) \cap dd(g') \cap G\). We have [perhaps we need to show \(f^{-1}(dd(g') \cap G) \cap F = (f')^{-1}(dd(g') \cap G) \cap F\)]

\[
dd(g \circ f) \cap F \cap f^{-1}(G) = dd(f) \cap f^{-1}(dd(g)) \cap F \cap f^{-1}(G) \\
= [dd(f) \cap F] \cap [f^{-1}(dd(g) \cap G) \cap F] \\
= [dd(f') \cap F] \cap [(f')^{-1}(dd(g') \cap G) \cap F] \\
= dd(f') \cap (f')^{-1}(dd(g')) \cap F \cap (f')^{-1}(G) \\
= dd(g' \circ f') \cap F \cap (f')^{-1}(G)
\]

and if \(s \in F \cap f^{-1}(G)\), then \(g(f(s)) = g'(f(s)) = g'(f'(s))\). \(\square\)

**Remark 4.5.** Looking at the statements of the Theorem, it makes sense to call the germ \(f/\mathcal{F}\) of a partial function \(f : S \to T\) admissible (wrt \(\mathcal{F} \in \text{Fil}\,S\)) if \(f\) is admissible wrt \(\mathcal{F}\), and local (wrt \(\mathcal{G} \in \text{Fil}\,T\)) if \(f\) is local wrt \(\mathcal{G}\).

**Galois Connection.** Now we want to show that like the mappings \(\mathcal{F}' \mapsto f(\mathcal{F}')\) and \(\mathcal{G} \mapsto f^{-1}(\mathcal{G}) \wedge \mathcal{F}\), the mappings \(\mathcal{F}' \mapsto \varphi(\mathcal{F}')\) and \(\mathcal{G} \mapsto \varphi^{-1}(\mathcal{G})\) constitute a Galois connection:

**Theorem 4.6.** Let \(\mathcal{F}\) and \(\mathcal{G}\) be filters, on sets \(S\) and \(T\), respectively. Let \(\mathcal{F}'\) be a filter such that \(\mathcal{F}' \leq \mathcal{F}\). If \(\varphi\) is a germ of partial functions from \(S\) to \(T\) admissible with respect to \(\mathcal{F}\), then

\[
\varphi(\mathcal{F}') \leq \mathcal{G} \iff \mathcal{F}' \leq \varphi^{-1}(\mathcal{G}).
\]

**Proof.** Let \(f\) represent \(\varphi\). Then by Theorem 4.2 and the notation that follows it,

\[
\varphi(\mathcal{F}') \leq \mathcal{G} \iff f(\mathcal{F}') \leq \mathcal{G} \\
\iff \mathcal{F}' \leq f^{-1}(\mathcal{G}) \wedge \mathcal{F} \\
\iff \mathcal{F}' \leq f^{-1}(\mathcal{G}) \\
\iff \mathcal{F}' \leq \varphi^{-1}(\mathcal{G}).
\]

\(\square\)

**Notation.** Just as we denote by \(f|_F\) the restriction to \(F\) of an admissible partial function \(f\) we can restrict a germ \(\varphi\) to a smaller subdomain filter. Thus if \(\varphi = f/\mathcal{F}\), and \(\mathcal{F} \leq \mathcal{F}\), we can form \(\varphi/\mathcal{F} = (f/\mathcal{F})/\mathcal{F}\). because since \(\mathcal{F} \leq \mathcal{F}\), there is an \(\mathcal{F}' \in \mathcal{F}\) such that \(\mathcal{F}' \subseteq dd\,f\).

**Proposition 4.7.** In this situation, \((f/\mathcal{F})/\mathcal{F} = f/\mathcal{F}\).

**The category Fil.** If \(S, T\) are sets, \(\mathcal{F} \in \text{Fil}\,S\), and \(\mathcal{G} \in \text{Fil}\,T\), then we denote the set of germs of partial functions from \(S\) to \(T\), admissible with respect to \(\mathcal{F}\) and local with respect to \(\mathcal{G}\), by \(\text{Fil}(\mathcal{F}, \mathcal{G})\). This defines \(\text{Fil}\), the category of filters, where the identity arrow from \(\mathcal{F}\) to \(\mathcal{F}\) is the germ \(1_S/\mathcal{F}\). We denote by \(\Gamma\) the functor from \(\text{LPartial}\) to \(\text{Fil}\) that takes an admissible, local partial function \(f \in \text{LPartial}(\mathcal{F}, \mathcal{G})\) to its germ \(f/\mathcal{F} \in \text{Fil}(\mathcal{F}, \mathcal{G})\).
Remark 4.8. Although we have needed to handle the boundary case that we mentioned previously, of filters $F$ such that $\{\} \in F$, sometimes we know that $\{\} \notin F$. In this case, proofs can sometimes be simplified by avoiding the need to work with partial functions, for, if $\gamma \in \text{Fil}(F, \mathcal{G})$, and $\mathcal{G}$ is such that $\{\} \notin \mathcal{G}$, then $\gamma$ has a representative $g$ which is total, i.e. such that $\text{dd}(g) = \bigcup \mathcal{G}$.

5. Factorization of Arrows in $\text{Fil}$

In this section, we will define a factorization system $\langle E^{\text{Fil}}, M^{\text{Fil}} \rangle$ for the category $\text{Fil}$. See [4, Section 2] for a discussion of this concept. We will show that this factorization is a so-called epi, monic factorization system. Finally, we look at the the $M^{\text{Fil}}$-subobject lattice of an object $F \in \text{Fil}$, and show that it can be identified with the lattice of filters $F'$ such that $F' \leq F$.

The factorization system $\langle E^{\text{Fil}}, M^{\text{Fil}} \rangle$. We define the subcategory $E^{\text{Fil}}$ of $\text{Fil}$ to contain all germs $\varphi : F \to \mathcal{G}$ such that $\varphi(F) = \mathcal{G}$. We define the subcategory $M^{\text{Fil}}$ to contain all germs $\varphi : F \to \mathcal{G}$ having the form $f/F$, where $f$ is an admissible partial function one-one on its domain of definition.

Theorem 5.1. $\langle E^{\text{Fil}}, M^{\text{Fil}} \rangle$ is a factorization system in $\text{Fil}$, such that $M^{\text{Fil}}$ consists of monic, and $E^{\text{Fil}}$ of epi, arrows.

Proof. Let $\varphi : F \to \mathcal{G}$, where $F \in \text{Fil}S$ and $\mathcal{G} \in \text{Fil}T$. If $\varphi = f/F$, then $f$ is a local partial function from $F$ to $f(F)$, and if we define $\epsilon : F \to f(F)$ by $\epsilon = f/F$ and $\mu : f(F) \to \mathcal{G}$ by $\mu = 1_T/f(f(F))$, we have a suitable factorization $\varphi = \mu \circ \epsilon$. For, $\epsilon \in E^{\text{Fil}}(F, f(F))$ and $\mu \in M^{\text{Fil}}(f(F), \mathcal{G})$.

If $\varphi \in E^{\text{Fil}}(F, \mathcal{G})$, then let $\alpha, \beta : \mathcal{G} \to \mathcal{H}$, and suppose that $\alpha \circ \varphi = \beta \circ \varphi$. Let $a, b$, and $f$ be partial functions representing $\alpha, \beta$, and $\varphi$, where $a$ and $b$ can be taken to have $\text{dd}(a) = \text{dd}(b) = G \in G$. Let $F \in \mathcal{F}$ be smaller than $\text{dd}(f)$, such that $f(F) \subseteq G$, and such that $(a \circ f)|_{F} = (b \circ f)|_{F}$. Since $(a \circ f)|_{F} = (b \circ f)|_{F}$, $a|_{f(F)} = b|_{f(F)}$, showing that $\alpha = \beta$ because (remembering that $\varphi(F) = \mathcal{G}$) $f(F) \in \mathcal{G}$. Thus, $\varphi$ is epi.

If $\varphi \in M^{\text{Fil}}(F, \mathcal{G})$, let $f$ be an admissible partial function representing $\varphi$, and such that $f$ is one-one on $\text{dd}(f) = F \in F$. Let $\alpha, \beta : \mathcal{H} \to \mathcal{F}$ be such that $\varphi \circ \alpha = \varphi \circ \beta$. Let $a$ and $b$ be representatives of $\alpha$ and $\beta$ having the same domain of definition $H \in \mathcal{H}$, which is such that $a(H) \subseteq F$ and $b(H) \subseteq F$, and such that $f \circ a = f \circ b$. (Such representatives can always be constructed by restriction, since $\alpha$ and $\beta$ are local and $\varphi \circ \alpha = \varphi \circ \beta$.) However, $f$ is one-one, implying $a = b$, which implies that $\alpha = \beta$. We have proved that $\varphi$ is monic.
Suppose now that germs $\epsilon$, $\alpha$, $\beta$, and $\mu$ are given, such $\epsilon \in E^\Fil$, and $\mu \in M^\Fil$, and forming a diagram

\[
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\epsilon} & \mathcal{F} \\
\downarrow{\alpha} & & \downarrow{\beta} \\
\mathcal{H} & \xleftarrow{\mu} & \mathcal{G}
\end{array}
\]

which commutes. Let $\epsilon = e/\mathcal{F}$, $\alpha = a/\mathcal{F}$, $\beta = b/\mathcal{W}$, and $\mu = m/\mathcal{G}$, where $\text{dd}(e) = \text{dd}(a)$, $r(e) \subseteq \text{dd}(b)$, $r(a) \subseteq \text{dd}(m)$, $m$ is one-one, and furthermore $b \circ e = m \circ a$.

Let $e$ and $a$, considered as functions from $\text{dd}(e)$ to $\text{dd}(b)$ and from $\text{dd}(a)$ to $\text{dd}(m)$ respectively, be factored as $e = m[e] \circ \tilde{e}[e]$ and $a = m[a] \circ \tilde{e}[a]$ in the category $\textbf{Set}$. (The notations $m[f]$, and $\tilde{e}[f]$ for an arrow $f$, which we employ only in this proof, are defined in [4, Section 2]; $e = m[e] \circ \tilde{e}[e]$ is the canonical chosen factorization of the function $e$ in the usual factorization system of the category $\textbf{Set}$.) We have a commutative diagram

\[
\begin{array}{ccc}
\text{dd}(e) & = & \text{dd}(a) \\
\downarrow{\text{d}} & & \downarrow{\text{d}} \\
r(e) & \xrightarrow{\text{d}} & r(a) \\
\downarrow{b \circ m[e]} & & \downarrow{m \circ m[a]} \\
r(b \circ m[e]) & = & r(m \circ m[a])
\end{array}
\]

in the category $\textbf{Set}$ which is uniquely diagonalized as shown by a function we denote by $\text{d}$. For, $\tilde{e}[e]$ is an onto function, and $m \circ m[a]$ is a one-one function.

The function $\text{d}$ is an admissible partial function from $\mathcal{W}$ to $\mathcal{T}$ because, $\epsilon$ being in $E^\Fil$, $r(e) \in \mathcal{W}$. $\text{d}$ is local, because if $G \in \mathcal{G}$, there is an $F \in \mathcal{F}$ such that $a(F) \subseteq G$, and then $d(\tilde{e}[e](F)) \subseteq G$; however, $\tilde{e}[e](F) \in \mathcal{W}$ because $e \in E^\Fil$.

Let $\delta \in \text{Fil}(\mathcal{W}, \mathcal{G})$ be defined by $\delta = d/\mathcal{W}$. Since $e$, $a$, $b$, and $m$, and $\text{d}$ are all admissible, local partial functions, the diagram of germs commutes. The uniqueness of the diagonal arrow $\delta$ follows by invoking either the fact that $E^\Fil$ consists of epi, or the fact that $M^\Fil$ consists of monic arrows of $\Fil$.

Epi and monic arrows in $\Fil$.

**Theorem 5.2.** Let $\varphi \in \Fil(\mathcal{F}, \mathcal{G})$, where $\mathcal{F} \in \Fil S$ and $\mathcal{G} \in \Fil T$. We have

1. If $\varphi$ is epi, then $\varphi \in E^\Fil$; and
2. if $\varphi$ is monic, then $\varphi \in M^\Fil$.

**Proof.** (1): Suppose that $\varphi \in \Fil(\mathcal{F}, \mathcal{G})$ but $\varphi \notin E^\Fil(\mathcal{F}, \mathcal{G})$, i.e. that $\varphi(\mathcal{F}) < \mathcal{G}$. This means that if $\varphi = f/\mathcal{F}$, there is an $F \in \mathcal{F}$ such that $F \subseteq \text{dd}(f)$ and $f(F) \notin \mathcal{G}$, thus, for all $G \in \mathcal{G}$, $G \not\subseteq f(F)$, so there is an element $g_G \in G$ such that $g_G \notin f(F)$. Let $W = \{0, 1\}$. Let
a : T → W send all elements to 0. Let b : T → W be the same, except for elements of the form \( g_G \) (for any \( G \)), which it should send to 1. \( a \) and \( b \) are total functions, so admissible.

By construction, \( a/G \neq b/G \). But, we have \( a \circ f = b \circ f \), because if \( x \in F \), we cannot have \( f(x) = g_G \) for any \( G \). Thus, \( \varphi \) is not epi.

(2): Suppose \( \varphi \in \text{Fil}(\mathcal{F}, \mathcal{G}) \), but \( \varphi \not\in \mathcal{M}^\text{Fil}(\mathcal{F}, \mathcal{G}) \). That is, we assume that if \( \varphi = f/\mathcal{F} \), for an admissible partial function \( f : \mathcal{F} \rightarrow T \), then \( f \) is not one-one. For every \( F \in \mathcal{F} \), such that \( F \subseteq \text{dd}(f) \), there are \( a_F, b_F \in F \) such that \( a_F \neq b_F \) but \( f(a_F) = f(b_F) \). Let \( W \) be the set of \( F \in \mathcal{F} \) such that \( F \subseteq \text{dd}(f) \), and define \( a : W \rightarrow S, b : W \rightarrow S \) by \( a : F \mapsto a_F \) and \( b : F \mapsto b_F \). Let \( \mathcal{H} \in \text{Fil}W \) be defined as \( \mathcal{H} = a^{-1}(\mathcal{F}) \cap b^{-1}(\mathcal{F}) \). The functions \( a \) and \( b \) are total functions, hence admissible. By monotonicity, we have

\[
\begin{align*}
    a(\mathcal{H}) &= a(a^{-1}(\mathcal{F}) \cap b^{-1}(\mathcal{F})) \\
    &\leq a(a^{-1}(\mathcal{F})) \\
    &\leq \mathcal{F}
\end{align*}
\]

and similarly, \( b(\mathcal{H}) \leq \mathcal{F} \). Thus, \( a/\mathcal{H} = \alpha \) and \( b/\mathcal{H} = \beta \) are local germs, and we have \( \varphi \circ \alpha = \varphi \circ \beta \) because \( f \circ a = f \circ b \). However, \( \alpha \neq \beta \), for, if \( H \in \mathcal{H} \), then there exist \( F_a, F_b \in \mathcal{F} \) such that \( a^{-1}(F_a) \cap b^{-1}(F_b) \subseteq H \), and letting \( F = F_a \cap F_b \), we have \( a^{-1}(F) \cap b^{-1}(F) \subseteq H \). Then \( a_F \) and \( b_F \in H \), so \( a|_H \neq b|_H \). This proves \( \alpha \neq \beta \), and the contrapositive, that \( \varphi \) is not monic.

**Isomorphisms.** As usual with factorization systems, the isomorphisms in \( \text{Fil} \) are precisely those arrows contained both in \( \text{E}^{\text{Fil}} \) and in \( \text{M}^{\text{Fil}} \). (This follows from axioms [4, Section 2, (F3) and (F4)].) This allows us to characterize them:

**Proposition 5.3.** An arrow \( \varphi \in \text{Fil}(\mathcal{F}, \mathcal{G}) \) is an isomorphism in \( \text{Fil} \) iff there is a partial function \( f \) representing \( \varphi \) such that \( f \) is one-one and \( f(\mathcal{F}) = \mathcal{G} \).

**The partially-ordered sets \( \mathcal{F}/\mathcal{M}^{\text{Fil}} \).** Recall [4, Section 2.2] that if we have a factorization system \( (\text{E}, \text{M}) \) in a category \( \mathcal{C} \), \( c \in \mathcal{C} \), and arrows \( m \) and \( m' \) with common codomain \( c \), then we say \( m \leq m' \) when there is a diagram

\[
\begin{tikzcd}
\text{dom} m & \text{dom} m' \\
& c \\
m \arrow[ru] & m' \arrow[lu]
\end{tikzcd}
\]

where \( f \) is an arrow making the diagram commutative. If \( f \) is an isomorphism, so that \( m \leq m' \) and \( m' \leq m \), then we say that \( m \) and \( m' \) are equivalent, or \( m \sim m' \). The \( \leq \) relation defines a preorder and the \( \sim \) relation defines an equivalence relation; we denote the corresponding partially-ordered set of equivalence classes (wrt \( \sim \)) by \( c/\text{M} \).

**Theorem 5.4.** Let \( \mathcal{F} \in \text{Fil} \). Then \( \mathcal{F}/\mathcal{M}^{\text{Fil}} \) is isomorphic to the complete lattice \( \text{Fil} \mathcal{F} \).
Proof. The elements of $\mathcal{F}/\mathcal{M}^{\text{Fil}}$ are equivalence classes of arrows of $\text{Fil}$ with codomain $\mathcal{F}$. Given a germ $\mu : \mathcal{G} \to \mathcal{F}$, we map $\mu$ to $\mu(\mathcal{G}) \in \text{Fil} \mathcal{F}$. If we have a diagram

$$
\begin{array}{c}
\mathcal{G} \\
\downarrow \varphi \\
\mathcal{F}
\end{array} 
\begin{array}{cc}
\rightarrow
\\
\downarrow \\
\mathcal{G}' \quad \mu \\
\downarrow \\
\mathcal{F} \quad \mu'.
\end{array}
$$

(5.1)

where $\varphi$ is an isomorphism, then because $\mu = \mu' \circ \varphi$, $\mu(\mathcal{G}) = \mu'(\varphi(\mathcal{G}))$. Now, $\varphi$, being an isomorphism, is an arrow of $\mathcal{E}^{\text{Fil}}$, and by definition, this means that $\varphi(\mathcal{G}) = \mathcal{G}'$. Thus, $\mu(\mathcal{G}) = \mu'(\mathcal{G}')$. In other words, $\mu \sim \mu'$ implies $\mu(\mathcal{G}) = \mu'(\mathcal{G}')$. Thus we have defined a mapping $Z : \mathcal{F}/\mathcal{M}^{\text{Fil}} \to \text{Fil} \mathcal{F}$, which takes the equivalence class $[\mu]$ of an arrow $\mu : \mathcal{G} \to \mathcal{F}$ to $\mu(\mathcal{G})$.

Suppose now that we have the diagram 5.1, absent $\varphi$, but knowing that $\mu(\mathcal{G}) \leq \mu'(\mathcal{G}')$ (in the lattice $\text{Fil} \mathcal{F}$), and we will construct an arrow $\varphi$ witnessing $\mu \leq \mu'$. Let $m : \mathcal{G} \to S$, $m' : \mathcal{G}' \to S$ be admissible one-one partial functions representing the germs $\mu$ and $\mu'$, respectively.

Let $f$ be the partial function $(m')^{-1} \circ m$. We will show that $f$ is a local, admissible partial function, such that $\varphi = f/\mathcal{G}$ completes Diagram 5.1.

Let $K' \in \mathcal{G}'$. We have $K' \cap \mathcal{G}' \subseteq \mathcal{G}'$ since $\mathcal{G}' \subseteq \mathcal{G}'$. Then $m'(K' \cap \mathcal{G}') \subseteq m'(\mathcal{G}')$. Since $m(\mathcal{G}) \leq m'(\mathcal{G}')$ by assumption, there is an $L \in m(\mathcal{G})$ such that $L \subseteq m'(K' \cap \mathcal{G}')$. There is a $K \in \mathcal{G}$ such that $K \subseteq G = \text{dd}(m)$ and $K \subseteq L$. Then, $K \subseteq \text{dd}(f)$ and $f(K) \subseteq K'$. So, $f$ is local.

Certainly, $m|_K = m' \circ f|_K$, implying that $\mu = \mu' \circ \varphi$. □

6. Properties of the Category $\text{Fil}$

In this section, we will verify that the category $\text{Fil}$ satisfies the basic properties in the list [4, 3.1]. (A number of subsequent theorems in [4] will follow.)

Theorem 6.1. $\mathcal{M}^{\text{Fil}}$ is well-powered.

Proof. If $\mathcal{F} \in \text{Fil} S$, then by Theorem 5.4, $\mathcal{F}/\mathcal{M}^{\text{Fil}} \cong \text{Fil} \mathcal{F}$, which is a small set because $S$ is a small set. □

Theorem 6.2. $\text{Fil}$ has limits of all finite diagrams.

Proof. It suffices to show that there are equalizers, and products of finite tuples of objects.

(Equalizers): Suppose $\alpha, \beta : \mathcal{F} \to \mathcal{G}$, where $\mathcal{F} \in \text{Fil} S$ and $\mathcal{G} \in \text{Fil} T$. Let $\langle \mathcal{H}, \mu \rangle$ be the equalizer and its arrow to $\mathcal{F}$, if they exist, in the diagram

$$
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\mu} & \mathcal{F} \\
\alpha \downarrow & & \downarrow \beta \\
\mathcal{G},
\end{array}
$$
where we know that $\mu$ needs to be monic because that is always the case for an equalizer, and from our analysis of the factorization system $(\mathbf{E}^\text{Fil}, \mathbf{M}^\text{Fil})$, that if the equalizer exists, we can choose $\mathcal{H} \in \text{Fil}\mathcal{F}$ and $\mu$ the germ with respect to $\mathcal{H}$ of the identity function $1_\mathcal{S}$. Also, we know that if $\{\} \in \mathcal{G}$, then $S = \{\}$ and $\mu$ is an isomorphism, but if not, then by Remark 4.8, there are total functions $m, a, \text{and } b$ representing $\mu, \alpha, \text{and } \beta$ respectively.

We choose

$$\mathcal{H} = Fg\{ [a = b] \mid a/\mathcal{F} = \alpha \text{ and } b/\mathcal{F} = \beta \},$$

where $[a = b]$ stands for the set of $x \in S$ on which the partial functions $a$ and $b$ are defined and $a(x) = b(x)$. We have $\mathcal{H} \leq \mathcal{F}$, because if $F \in \mathcal{F}$, $a/\mathcal{F} = \alpha$, and $b/\mathcal{F} = \beta$, then $[a|_F = b|_F] \subseteq F$.

Suppose now that $\mathcal{K}$ is a filter, and $\gamma \in \text{Fil}(\mathcal{K}, \mathcal{F})$ is such that $\alpha \circ \gamma = \beta \circ \gamma$. Let $a, b,$ and $g$ be admissible partial functions representing $\alpha, \beta,$ and $\gamma$ respectively. We have $[a \circ g = b \circ g] = K \in \mathcal{K}$. Let $h : K \to S$ be defined as $h : x \mapsto g(x)$. Clearly $\alpha \circ (h/\mathcal{K}) = \beta \circ (h/\mathcal{K})$. We must show that $h/\mathcal{K}$ is local. If $F \in \mathcal{F}$, then $F \supseteq [a|_F = b|_F] \in \mathcal{H}$. Let $K' = [(a|_F)g = (b|_F)g]$; we have $K' \in \mathcal{K}$ and $g(K') \subseteq F$.

The germ $h/\mathcal{K}$ is unique, because, $\mu$ being one-one on its domain of definition, $\mu \in \mathbf{M}^\text{Fil}$, and by Theorem 5.1, then, $\mu$ is monic.

(Products of finite tuples of filters): Given a tuple of filters $\mathcal{F}_i$, on sets $S_i$, let $S = \Pi_i S_i$, and let $\mathcal{P} = \bigwedge_i \pi_i^{-1}(\mathcal{F}_i)$, where the $\pi_i : S \to S_i$ are the projections. We claim that if the tuple $\mathcal{F}_i$ is finite, then $\mathcal{P}$ is the product, with product cone the germs $\pi_i/\mathcal{P}$.

Given a filter $\mathcal{G}$ (on a set $\mathcal{G}$), and germs $\varphi_i : \mathcal{G} \to \mathcal{F}_i$, let $f_i : X_i \to S_i$ be a partial function representing $\varphi_i$, for each $i$, where $X_i \in \mathcal{G}$. We define $X = \bigcap_i X_i \in \mathcal{G}$ (using the fact that the index set is finite) and we can define $f : X \to S$, using the universal property of the product. It is clear that for each $i$, $(\pi_i/\mathcal{P}) \circ (f/\mathcal{G}) = \varphi_i$.  

**Theorem 6.3.** $\text{Fil}$ has pullbacks of small tuples of arrows in $\mathbf{M}^\text{Fil}$.

**Proof.** Let $\mathcal{F} \in \text{Fil}\mathcal{S}$, and for each $i$ in some small index set which we take to be an ordinal without loss of generality (as long as we assume the Axiom of Choice), let $\mathcal{G}_i \in \text{Fil}$ and $\gamma_i \in \mathbf{M}^\text{Fil}(\mathcal{G}_i, \mathcal{F})$. We will prove that the diagram

$$
\begin{array}{c}
\mathcal{G}_0 \\
\downarrow \gamma_0 \\
\mathcal{F} \\
\downarrow \gamma_1 \\
\mathcal{G}_1 \\
\downarrow \gamma_2 \\
\mathcal{G}_2 \\
\vdots
\end{array}
$$

has a pullback.

By Theorem 5.4 and because the lattice $\text{Fil}\mathcal{F}$ is complete, the $\mathbf{M}^\text{Fil}$-subobjects $\gamma_i/\mathbf{M}^\text{Fil}$ have a meet, $\mathcal{H}$. Let $\eta : \mathcal{H} \to \mathcal{F}$ be the germ $1_{\mathcal{S}/\mathcal{H}}$. By Theorem 5.4, we can then draw the
pullback diagram

\[
\begin{array}{c}
\mathcal{H} \\
\downarrow \varphi_0 \downarrow \varphi_1 \downarrow \varphi_2 \downarrow \cdots \\
\mathcal{G}_0 \downarrow \gamma_0 \downarrow \mathcal{G}_1 \downarrow \gamma_1 \downarrow \mathcal{G}_2 \downarrow \gamma_2 \downarrow \cdots \\
\bigcup \mathcal{F}
\end{array}
\]

where for all \( i \), \( \gamma_i \circ \varphi_i = 1_{S/H} \), with the \( \varphi_i \) coming from Theorem 5.4 and the fact that for each \( i \), \( \mathcal{H} \leq \mathcal{G}_i \). \( \square \)

**Theorem 6.4.** Fil has coproducts of all small tuples of arrows.

*Proof.* Let \( \mathcal{F}_i \in \text{Fil} S_i \), indexed by a small set which, without loss of generality, we take to be an ordinal number. Let \( Z \) be the disjoint union of the sets \( S_i \), and for each \( i \), let \( j_i \) be the insertion of \( S_i \) into \( Z \). Then, let \( \bigvee j_i(\mathcal{F}_i) \) be the join over \( i \) of the filters \( j_i(\mathcal{F}_i) \). The join \( \mathcal{C} = \bigvee j_i(\mathcal{F}) \) is the filter of subsets \( \mathcal{C} \subseteq Z \) such that for all \( i \), there is an \( F_i \in \mathcal{F}_i \) with \( j_i(\mathcal{F}_i) \subseteq \mathcal{C} \). Let \( \iota_i = j_i/\mathcal{F}_i \) for each \( i \).

Thus we have the diagram

\[
\begin{array}{c}
\mathcal{F}_0 \downarrow \iota_0 \downarrow \mathcal{F}_1 \downarrow \iota_1 \downarrow \mathcal{F}_2 \downarrow \iota_2 \downarrow \cdots \\
\mathcal{C}
\end{array}
\]

in the category Fil, and we will show that \( \mathcal{C} \) is a coproduct of the tuple of \( \mathcal{F}_i \).
Suppose we are given a filter $X \in \text{Fil}_W$, and a cocone of germs $\xi_i : F_i \to X$, as shown in the following diagram, and we will construct and prove uniqueness of the dotted arrow $\lambda$ such that for all $i$, $\xi_i = \lambda \circ \iota_i$, proving the universal property.

For each $i$, let $x_i : F_i \to W$ be a partial function representing $\xi_i$. Let $C = \bigcup_i F_i \subseteq Z$. Let $\ell : C \to W$ be the arrow given by the universal property of the disjoint union (in $\text{Set}$). Then let $\lambda = \ell/C$. $\ell$ is an admissible partial function; we must show that it is local, that its germ $\lambda$ satisfies $\xi_i = \lambda \circ \iota_i$ for all $i$, and that if $\lambda'$ is any germ satisfying those equations, $\lambda' = \lambda$.

Each partial function $x_i$ is local, which means that if $X \in \mathcal{X}$, there is an $F'_i \in F_i$ such that $x_i(F'_i) \subseteq X$. It follows that $\ell'$, defined as the restriction of $\ell$ to $C' = \bigcup_i j_i(F'_i)$, is local.

Now suppose that we have any arrow $\lambda' : \mathcal{C} \to \mathcal{X}$ such that for all $i$, $\xi_i = \lambda' \circ \iota_i$. Let $\ell' : C' \to W$ be an admissible, local partial function representing $\lambda'$. For each $i$, let $F''_i$ be the subset of $S_i$ such that $\ell_i \circ j_i = \ell' \circ j_i$ on $F''_i$. We know that $F''_i \in F_i$ because $\lambda \circ \iota_i = \xi_i = \lambda' \circ \iota_i$. Then $C'' \subseteq C$ where $C'' = \bigcup_i j_i(F''_i)$, and $\ell = \ell'$ on $C'$. This shows that $\lambda = \ell/C = \ell'/C = \lambda'$.

**Theorem 6.5.** $M^{\text{Fil}}$ consists of monic arrows of $\text{Fil}$.

**Proof.** More than that, by Theorems 5.1 and 5.2, arrows in $\text{Fil}$ are monic iff they belong to $M^{\text{Fil}}$. □

**Theorem 6.6.** $E^{\text{Fil}}$ is stable under pullbacks along arrows of $\text{Fil}$.

**Proof.** Let $\varepsilon \in E^{\text{Fil}}(F, G)$, and let us pull it back along $\varphi : \mathcal{H} \to \mathcal{G}$, giving $\mathcal{P}$ and $\varepsilon' \in \text{Fil}(\mathcal{P}, \mathcal{H})$ which we want to prove is in $E^{\text{Fil}}$.

First, let us deal with the boundary case in which $\emptyset \in \mathcal{G}$. Then also $\emptyset \in \mathcal{F}$ and $\varepsilon$ is an isomorphism, whence the pullback $\varepsilon'$ is too, so that $\varepsilon' \in E^{\text{Fil}}$. 

\[\begin{array}{ccc}
\mathcal{F}_0 & \xrightarrow{\iota_0} & \mathcal{F}_1 \\
\downarrow{\xi_0} & & \downarrow{\xi_1} \\
\mathcal{F}_2 & & \mathcal{F}_2 \\
\downarrow{\iota_2} & & \downarrow{\xi_2} \\
\mathcal{X} & & \mathcal{X} \\
\end{array}\]
Assuming, on the contrary, that \( \{ \} \notin \mathcal{G} \), we can use Remark 4.8, and drawing the detailed diagram

\[
\begin{array}{c}
\mathcal{P} \\
\downarrow \psi \\
\mathcal{H} \\
\downarrow \varphi \\
\mathcal{G}
\end{array}
\]

we can assume that there are total functions \( f \) and \( e \) representing \( \varphi \) and \( \varepsilon \), respectively. We can construct our pullback \( \varepsilon' \) by first constructing the product

\[
\begin{array}{c}
\mathcal{H} \times \mathcal{F} \\
\downarrow \pi \\
\mathcal{H} \\
\downarrow \pi' \\
\mathcal{F}
\end{array}
\]

and then the equalizer of the arrows \( \varphi \circ \pi, \varepsilon \circ \pi' : \mathcal{H} \times \mathcal{F} \to \mathcal{G} \):

\[
\begin{array}{c}
\mathcal{P} \\
\downarrow \lambda \\
\mathcal{H} \times \mathcal{F} \\
\downarrow \varepsilon \circ \pi' \\
\mathcal{G}
\end{array}
\]

after which we can set \( \varepsilon' = \pi \circ \lambda \) and \( \psi = \pi' \circ \lambda \).

To show \( \varepsilon' \in E^{\text{Fil}} \), we need to show that \( \mathcal{H} = \varepsilon'(\mathcal{P}) \), or in other words, since we have \( \varepsilon'(\mathcal{P}) \subseteq \mathcal{H} \) just because \( \varepsilon' \) is an arrow, that \( \mathcal{H} \leq \varepsilon'(\mathcal{P}) \). To show this, it will suffice to show that for some admissible partial function \( e' \) representing \( \varepsilon' \), and any \( P \in \mathcal{P} \), there is an \( H \in \mathcal{H} \) such that \( H \subseteq e'(P) \).

Examining the proof of the existence of equalizers, we see that under our current assumption that \( \{ \} \notin \mathcal{G} \), not only \( \varphi \) and \( \varepsilon \), but also \( \lambda \) can be represented by a total function, and so can \( \pi \) and \( \pi' \) from the proof of the existence of finite products. Thus, the partial function \( e' \) representing \( \varepsilon' \) can be assumed to be a total function.

Let \( P \in \mathcal{P} \). Let \( F \in \mathcal{F} \), and \( H \in \mathcal{H} \), be such that \( f(H) \subseteq e(F) \). (This is possible because \( \varepsilon \in E^{\text{Fil}} \) and \( f \) is local.) At the same time, let \( H \) and \( F \) be such that \( \pi^{-1}(H) \cap (\pi')^{-1}(F) \subseteq P \). Consider the diagram in \textbf{Set}, which is a pullback diagram:

\[
\begin{array}{c}
P' = H \times_{e(F)} F \\
\downarrow \pi' \downarrow \pi' \\
H \\
\downarrow \pi' \downarrow \pi' \\
F
\end{array}
\]

\[
\begin{array}{c}
\downarrow f |_{H} \\
\downarrow \pi' |_{P'} \\
\downarrow e(F)
\end{array}
\]

\[
\begin{array}{c}
\downarrow \varepsilon |_{F} \\
\downarrow \varepsilon |_{F}
\end{array}
\]
III. THE CLOSED CATEGORY OF FILTERS

where we know very well that in \( \textbf{Set} \), since \( e \) maps \( F \) onto \( e(F) \), the function \( \pi \) also maps \( P' \) onto \( H \). But \( P' \subseteq P \). Since \( H \) can be made as small as desired in the filter \( \mathcal{H} \), this proves that \( \varepsilon' = \pi/P \in \mathcal{E}^{\text{Fil}} \).

\[ \square \]

7. THE CORE OF A FILTER

**Definition 7.1.** Let \( F \) be a filter of subsets of \( S \). The subset \( \bigcap_{F \in \mathcal{F}} F \subseteq S \) is called the core of \( F \) and denoted by \( \text{core} F \). If \( \varphi \in \text{Fil}(F, \mathcal{G}) \), then \( \text{core} \varphi \) will denote the restriction of \( f \) to the filter \( \text{Fil}(\text{core} F) \).

**Proposition 7.2.** The mapping \( S \mapsto \{ S \} \) and the functor \( \Gamma \) sending \( f : S \to S' \) to \( f/\{ S \} \), define a functor \( L : \textbf{Set} \to \text{LPartial} \). There are adjunctions \( \langle L, \text{core}, \eta, \varepsilon \rangle : \textbf{Set} \to \text{LPartial} \), where \( \eta_S = 1_S \) and \( \varepsilon_{(S,F)} \) is the inclusion of \( \text{core} F \) into \( S \), and \( \langle \Gamma \circ L, \text{core}, \alpha \rangle : \textbf{Set} \to \text{Fil} \), where \( \Gamma : \text{LPartial} \to \text{Fil} \) is the functor defined in Subsection 4.

**Proof.** To have these adjunctions, we must have isomorphisms

\[ \chi_{S,G} : \text{LPartial}(L(S), \mathcal{G}) \cong \text{Set}(S, \text{core} \mathcal{G}) \]

and

\[ \chi'_{S,G} : \text{Fil}(\Gamma(L(S)), \mathcal{G}) \cong \text{Set}(S, \text{core} \mathcal{G}) \]

natural in \( S \) and \( \mathcal{G} \).

Since \( L(S) = \{ S \} \), for \( L(S) \) be admissible, a partial function between values of the functor \( L \) must be a total function, and the functor \( \Gamma \) does nothing. Thus, the isomorphisms \( \chi \) and \( \chi' \) simply relate total functions to total functions. Naturality in \( S \) and in \( \mathcal{G} \) is straightforward. \( \square \)

**Remark 7.3.** Thus, if we decide to study objects in \( \text{Fil} \) that have additional structure, such as groups or other algebra structures, we have available right adjoint *forgetful* functors that will yield groups or other algebras. However, note that the forgetting that the core functor does can be very extensive. For example, a filter can have an empty core, or a core with just one element. Certainly we should expect to have more of interest to study in many such cases than the trivial group.

8. MONOIDAL PRODUCTS

\( \mathcal{F} \boxtimes \mathcal{G} \). Suppose that \( \mathcal{F} \) and \( \mathcal{G} \) are filters of subsets of sets \( S \) and \( T \), respectively. If \( F \in \mathcal{F} \) and \( g : F \to \mathcal{G} \) (i.e., \( g \) is any function assigning a subset in the filter \( \mathcal{G} \) to each element of \( F \)), then we define \( F \boxtimes g = \{ (s, t) \mid s \in F, t \in g(s) \} \). More generally, if \( g : F \to \text{Sub} \ S \) is any function such that \( \{ s \mid g(s) \subseteq \mathcal{G} \} \in \mathcal{F} \), then we define \( F \boxtimes g \) just the same, as \( \{ (s, t) \mid s \in F, t \in g(s) \} \).
Theorem 8.1. We have

(1) The set
\[ \{ F \circ g \mid F \in \mathcal{F}, g : F \to \mathcal{G} \} \]
is a base for a filter \( \mathcal{F} \circ \mathcal{G} \) of subsets of \( S \times T \);

(2) the filter \( \mathcal{F} \circ \mathcal{G} \) consists of those subsets \( H \subseteq S \times T \) such that
\[ \{ s \mid \{ t \mid \langle s, t \rangle \in H \} \in \mathcal{G} \} \in \mathcal{F} ; \]

(3) every subset in \( \mathcal{F} \circ \mathcal{G} \) has the form \( S \circ g \) for some \( g : S \to \text{Sub} T \).

Proof. (1): Given \( F_1 \circ g_1 \) and \( F_2 \circ g_2 \), we have \( F \circ g \subseteq (F_1 \circ g_1) \cap (F_2 \circ g_2) \), where \( F = F_1 \cap F_2 \) and for \( s \in F \), \( g(s) = g_1(s) \cap g_2(s) \).

(2): Given \( H \) satisfying the condition, let \( F = \{ s \mid \{ t \mid \langle s, t \rangle \in H \} \in \mathcal{G} \} \in \mathcal{F} \). For each \( s \in F \), let \( g(s) = \{ t \mid \langle s, t \rangle \in H \} \in \mathcal{G} \). Then \( F \circ g \subseteq H \), so \( H \in \mathcal{F} \circ \mathcal{G} \).

On the other hand, if \( D \circ f \) is an element of the base of \( \mathcal{F} \circ \mathcal{G} \), then it clearly satisfies the condition. Then we need only see that the set of subsets satisfying the condition is closed upward.

(3): Given \( H \in \mathcal{F} \circ \mathcal{G} \), let \( g(s) = \{ t \in T \mid \langle s, t \rangle \in H \} \). Then \( H = S \circ g \).

Part (2) of the Theorem suggests some notation we will use later:

Notation 8.1. If \( \mathcal{F} \) is a filter of subsets of a set \( S \), \( \mathcal{G} \) is filter of subsets of a set \( T \), and we have a subset \( X \in \mathcal{F} \circ \mathcal{G} \), then we define

1. \( F_{\mathcal{F}, \mathcal{G}, X} = \{ s \in S \mid \{ t \in T \mid \langle s, t \rangle \in X \} \in \mathcal{G} \} \) and
2. \( h_{\mathcal{F}, \mathcal{G}, X} = \{ s \in F_{\mathcal{F}, \mathcal{G}, X} \mapsto \{ t \in T \mid \langle s, t \rangle \in X \} \} \).

Monoidal products in \( \text{LPartial} \) and \( \text{Fil} \).

Definition 8.2. If \( f \in \text{LPartial}(\mathcal{F}, \mathcal{F'}) \) and \( g \in \text{LPartial}(\mathcal{G}, \mathcal{G'}) \), where \( dd(f) = F \) and \( dd(g) = G \), then we define \( f \circ g : \mathcal{F} \circ \mathcal{G} \to \mathcal{F'} \circ \mathcal{G'} \) to be the partial function with domain of definition \( F \times G \), sending a pair \( \langle s, t \rangle \) to \( \langle f(s), g(t) \rangle \).

Theorem 8.3. We have

(1) The foregoing defines a functor \( \square_p : \text{LPartial} \times \text{LPartial} \to \text{LPartial} \).

(2) If \( f \equiv_F f' \) and \( g \equiv_G g' \), then \( (f \circ_p g) \equiv_{F \circ_p G} (f' \circ_p g') \).

(3) Setting \( \square_g = \square_p \) on objects, and \( (f/F) \circ_p (g/G) = (f \circ_p g)/\mathcal{F} \circ \mathcal{G} \) on arrows, defines a functor \( \square_p : \text{Fil} \times \text{Fil} \to \text{Fil} \).

(4) \( \Gamma(-1) \circ_p -2 = \Gamma(-1) \circ_p \Gamma(-2) : \text{LPartial} \times \text{LPartial} \to \text{Fil} \).

(5) \( \text{core}(-1) \circ_p -2 = \text{core}(-1) \times \text{core}(-2) : \text{LPartial} \times \text{LPartial} \to \text{Set} \).

(6) \( \text{core}(-1) \circ_p -2 = \text{core}(-1) \times \text{core}(-2) : \text{Fil} \times \text{Fil} \to \text{Set} \).

Proof. (1): Let \( f \in \text{LPartial}(\mathcal{F}, \mathcal{F'}) \), and \( g \in \text{LPartial}(\mathcal{G}, \mathcal{G'}) \). \( F \times G \in \mathcal{F} \circ \mathcal{G} \), because \( F \times G = F \circ [s \in F \to G] \). Thus, \( f \circ g \) is admissible. To show \( f \circ g \) is local, consider a base element \( F' \circ h' \in \mathcal{F'} \circ \mathcal{G'} \), where \( F' \in \mathcal{F} \) and \( h' : F' \to \mathcal{G'} \). We have \( f^{-1}(F') \in \mathcal{F} \). Consider
now $f^{-1}(F') \Box h$, where $h : f^{-1}(F') \to G$ is defined by setting $h(s) = g^{-1}(h'(f(s)))$. Then if $(s,t) \in f^{-1}(F') \Box h$, we have

$$(f \Box g)(s,t) = \langle f(s), g(t) \rangle \in F' \Box h',$$

because $f(s) \in F'$ and $g(t) \in h'(f(s))$.

(2): Suppose $f|_F = f'|_F$ and $g|_G = g'|_G$, where $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Then if $(s,t) \in F \times G$, we have $(f \Box g)(s,t) = (f' \Box g')\langle s,t \rangle$. But $F \times G \in F \Box G$, so $(f \Box g) \equiv_{F \Box G} (f' \Box g')$.

(3): Follows from (2).

(4): Follows by the definition of $\Box_g$ in (3).

(5), (6): The core functor is a right adjoint functor, after all.

Unit object, unit and associativity natural isomorphisms, and coherence. If $S$, $T$, and $W$ are sets, let $\alpha_{S,T,W} : S \times (T \times W) \to (S \times T) \times W$ be the function defined by $\langle s, (t,w) \rangle \mapsto \langle (s,t), w \rangle$. If $\mathcal{D}$, $\mathcal{D}'$, and $\mathcal{D}''$ are filters on $S$, $T$, and $W$ respectively, we set $\alpha^p_{\mathcal{D},\mathcal{D}',\mathcal{D}''} = \alpha_{S,T,W}$, considered as a partial function. We set $\alpha^q_{\mathcal{D},\mathcal{D}',\mathcal{D}''} = \alpha_{S,T,W}/(\mathcal{D} \Box (\mathcal{D}' \Box \mathcal{D}''))$.

**Theorem 8.4.** $\alpha^p_{\mathcal{D},\mathcal{D}',\mathcal{D}''} \in \text{LPartial}(\mathcal{D} \Box (\mathcal{D}' \Box \mathcal{D}''), (\mathcal{D} \Box \mathcal{D}') \Box \mathcal{D}'')$ and is an isomorphism. Similarly, its germ in $\mathcal{F}$ is an isomorphism.

**Proof.** $\alpha^p_{\mathcal{D},\mathcal{D}',\mathcal{D}''}$ is admissible because $\alpha_{S,T,W}$ is a total function. To show it is an isomorphism, it suffices to show that both it and its inverse (in $\text{Set}$) are local.

If $X \in \mathcal{D} \Box (\mathcal{D}' \Box \mathcal{D}'')$, then $X = S \Box f$ where $f : S \to \text{Sub}(T \times W)$. In this proof, we will denote the subset $X = S \Box f$ of $S \times (T \times W)$ by $X[f]$.

On the other hand, if $Y \in (\mathcal{D} \Box \mathcal{D}') \Box \mathcal{D}'$, then $Y = Y[h,k] = (S \Box h) \Box k$ where $h : S \to \text{Sub}T$ and $k : S \times T \to \text{Sub}W$.

To see that $\alpha^{-1}_{S,T,W}(\mathcal{D} \Box (\mathcal{D}' \Box \mathcal{D}'')) = \mathcal{D} \Box (\mathcal{D}' \Box \mathcal{D}'')$, consider $X[f]$, and we will find $h$ and $k$ such that $\alpha^{-1}_{S,T,W}(Y[h,k]) = X[f]$. Let $h : S \to \text{Sub}T$ be defined by $h : s \mapsto p_1(f(s))$, where $p_1 : T \Box g \to T$ is the projection to the first component of a pair. Let $k : S \Box h \to \text{Sub}W$ be defined by $k : (s,t) \mapsto p_2(f(s))$, where $p_2 : T \Box g \to \text{Sub}W$ projects a pair to its second component. We have

$$(s,(t,w)) \in Y[h,k] \iff (\langle s,t \rangle, w) \in Y[h,k] 
\iff s \in S \text{ and } t \in h(s) \text{ and } w \in k(s,t) 
\iff s \in S \text{ and } (t,w) \in h(s) \Box k(s,t) 
\iff (s,(t,w)) \in X[f];$$

it follows that both $\alpha_{\mathcal{D},\mathcal{D}',\mathcal{D}''}$ and $\alpha^{-1}_{\mathcal{D},\mathcal{D}',\mathcal{D}''}$ are local.

Applying the functor $\Gamma$, we obtain the corresponding statements about $\alpha^q_{\mathcal{D},\mathcal{D}',\mathcal{D}''}$.

**Notation 8.2.** Let $u$ denote the filter $\{1\} = \{0\}$ on the one-element set $1 = \{0\}$.  

\[\boxed{\text{III. THE CLOSED CATEGORY OF FILTERS 21}}\]
Proposition 8.5. These definitions yield natural isomorphisms

$$\alpha^p : -1\Box_p(-2\Box_p-3) \cong (-1\Box_p-2)\Box_p-3,$$

$$\alpha^g : -1\Box_g(-2\Box_g-3) \cong (-1\Box_g-2)\Box_g-3,$$

$$\lambda^p : u\Box_p- \cong -,$$

$$\lambda^g : u\Box_g- \cong -,$$

$$\varrho^p : -\Box_pu \cong -,$$ and

$$\varrho^g : -\Box_gu \cong -$$

making \(\text{LPartial, } \Box_p, u, \alpha^p, \lambda^p, \varrho^p\) and \(\text{Fil, } \Box_g, u, \alpha^g, \lambda^g, \varrho^g\) into monoidal categories [3, p. VII.1], [5, Definition 1.1], and the functors \(\Gamma : \text{LPartial} \to \text{Fil, core}^p : \text{LPartial} \to \text{Set, and core} : \text{Fil} \to \text{Set}\) are strict morphisms of monoidal categories [3, p. VII.1]/[5, Definition 1.7].

9. The Closed Category Structure of \text{Fil}

Let \(\mathcal{G}\) be a filter of subsets of a set \(T\), \(\mathcal{H}\) be a filter of subsets of another set \(W\), \(q \in \text{LPartial}(\mathcal{G}, \mathcal{G}')\), and \(r \in \text{LPartial}(\mathcal{H}', \mathcal{H}).\) Since \text{Fil} is a category, composition with \(\gamma = q/\mathcal{G}\) on the left induces a function

$$\text{Fil}(\mathcal{H}, \gamma) : \text{Fil}(\mathcal{H}, \mathcal{G}) \to \text{Fil}(\mathcal{H}', \mathcal{G})$$

and composition with \(\gamma' = r/\mathcal{H}\) on the right produces a function

$$\text{Fil}(\gamma', \mathcal{G}) : \text{Fil}(\mathcal{H}, \mathcal{G}) \to \text{Fil}(\mathcal{H}', \mathcal{G});$$

and we have also

$$\text{Fil}(\gamma', \gamma) = \text{Fil}(\gamma', \mathcal{G}) \circ \text{Fil}(\mathcal{H}, \gamma) = \text{Fil}(\mathcal{H}, \gamma) \circ \text{Fil}(\gamma', \mathcal{G}) : \text{Fil}(\mathcal{H}, \mathcal{G}) \to \text{Fil}(\mathcal{H}', \mathcal{G}'),$$

where hopefully, the reader will recognize easily that these functions are simply forms of the \text{Hom} functor for the category \text{Fil}. We mention them to clarify our notation in what follows.

The internal \text{Hom} functor of \text{Fil}. We will be defining the internal \text{Hom} functor for \text{Fil} using, among other things, the mapping that takes an admissible (but not necessarily local) partial function to its germ. That is, if we have filters \(\mathcal{H}\) (of subsets of \(W\)) and \(\mathcal{G}\) (of subsets of \(T\)) then we can take the germ of an element of \(\text{Partial}(\mathcal{H}, \mathcal{G})\) (See Notation 3.1, and Section 4 where germs are defined), giving an arrow in \(\text{Fil}(\mathcal{H}, \mathcal{G}).\) However, formation of germs is useful more generally: If \(G \in \mathcal{G},\) then recall that we denote the set of admissible partial functions \(f : W \to T,\) such that there is an \(H \in \mathcal{H}\) such that \(f(H) \subseteq G,\) by \(\text{Partial}(\mathcal{H}, \mathcal{G}, G).\) If \(f\) is such a function, we can form the germ \(f/\mathcal{H}\) and get an element of the set of germs of
Proof.

(1): clear.

Proposition 9.1. We have

(1) The subsets $\text{Fil}(\mathcal{H}, \mathcal{G}, G)$, for $G \in \mathcal{G}$, form a base for a filter $G^\mathcal{H}$ of subsets of $\text{Fil}(\mathcal{H}, \mathcal{G}, T)$, and core $G^\mathcal{H} = \text{Fil}(\mathcal{H}, \mathcal{G})$.

(2) $(-1)^{-2}$ is a functor from $\text{Fil} \times \text{Fil}^{op}$ to $\text{Fil}$.

Proof. (1): clear.

(2): Given $\mathcal{G}$ and $\mathcal{H}$, other filters $\mathcal{G}'$ and $\mathcal{H}'$, and germs $\gamma \in \text{Fil}(\mathcal{G}, \mathcal{G}')$, $\rho \in \text{Fil}(\mathcal{H}', \mathcal{H})$, we must show that $\gamma$ gives rise to an arrow $\gamma^\mathcal{H} \in \text{Fil}(\mathcal{G}^\mathcal{H}, \mathcal{G}'^{\mathcal{H}'})$, and $\rho$ gives rise to an arrow $\rho^\mathcal{G} : \text{Fil}(\mathcal{G}^\mathcal{H}, \mathcal{G}'^{\mathcal{H}'})$.

Let $g$ be a partial function, admissible with respect to $\mathcal{G}$ and local with respect to $\mathcal{G}'$. (Every $\gamma$ has such a representative by definition of $\text{Fil}(\mathcal{G}, \mathcal{G}')$, so let us say that $g$ is a representative of $\gamma$.) Composition with $g$ on the left is a (total) function from $\text{Partial}(\mathcal{H}, \mathcal{G}, T)$ to $\text{Partial}(\mathcal{H}, \mathcal{G}', T')$ (where $\mathcal{G}'$ is a filter of subsets of the set $T'$). The germ of this total function is admissible (the germ of a total function always is); to show it is local, consider a basic set $\text{Partial}(\mathcal{H}, \mathcal{G}', \mathcal{G}')$ where $\mathcal{G}' \in \mathcal{G}'$. If $G \in \mathcal{G}$ is such that $g(G) \subseteq \mathcal{G}'$ (such a $G$ exists because $g$ is local with respect to $\mathcal{G}'$), then composition with $g$ on the left maps the basic set $\text{Partial}(\mathcal{H}, \mathcal{G}, G)$ into $\text{Partial}(\mathcal{H}, \mathcal{G}', G')$, as needed to show composition with $g$ is a local function, and so, the germ of the composition function is local. Thus the arrow $\gamma^\mathcal{H} : \mathcal{G}^\mathcal{H} \to \mathcal{G}'^{\mathcal{H}'}$.

Now, let $r : \bigcup \mathcal{H}' \to W$ be a partial function, admissible with respect to $\mathcal{H}'$, local with respect to $\mathcal{H}$, and representing $\rho$. Composition with $r$ on the right is once again a total function from $\text{Partial}(\mathcal{H}, \mathcal{G}, T)$ to $\text{Partial}(\mathcal{H}', \mathcal{G}, T)$, because if $f \in \text{Partial}(\mathcal{H}, \mathcal{G}, T)$, then it is admissible, and we have $\text{dd}(f) \in \mathcal{H}$. Then since $r$ is local, we have $\text{dd}(f \circ r) \in \mathcal{H}$, so that $f \circ r \in \text{Partial}(\mathcal{H}', \mathcal{G}, T)$ – i.e., it is an admissible partial function and we conclude that composition with the germ, $\rho$, is total and admissible. For locality, suppose now that we have $G \in \mathcal{G}$; we want to show that there is a basic set in the filter $\mathcal{G}^\mathcal{H}$ that will map into $\text{Partial}(\mathcal{H}', \mathcal{G}, G)$, and we will show that $\text{Partial}(\mathcal{H}, \mathcal{G}, G)$ will serve. Given $f \in \text{Partial}(\mathcal{H}, \mathcal{G}, G)$, we have $f^{-1}(G) \in \mathcal{H}$. If we form $f \circ r$, then since $r$ is local, we see that $(f \circ r)^{-1}(G) = r^{-1}(\text{dd}(r) \cap f^{-1}(G))) \in \mathcal{H}'$, so $f \circ r \in \text{Partial}(\mathcal{H}', \mathcal{G}, G)$. Since we have now shown that composition with $r$ on the right is admissible and local, so is its germ; thus, the arrow $\rho^\mathcal{G} : \mathcal{G}^\mathcal{H} \to \mathcal{G}'^{\mathcal{H}'}$.

Clearly we have produced a functor $(-1)^{-2} : \text{Fil} \times \text{Fil}^{op} \to \text{Fil}$.

For this next definition, we make use of Notation 8.1.

Definition 9.2. Let $\mathcal{H}$ be a filter. For every pair of filters $\langle \mathcal{F}, \mathcal{G} \rangle$, let

$$\chi^\mathcal{H}_{\mathcal{F}, \mathcal{G}} : \text{Fil}(\mathcal{F} \square \mathcal{H}, \mathcal{G}) \to \text{Fil}(\mathcal{F}, \mathcal{G}^\mathcal{H})$$
be the total function mapping a germ $\kappa$

to

$$
\chi_{F,G}^H(\kappa) = \{ s \in F_{F,G,\text{dd}(q)} \mapsto [w \in h_{F,G,\text{dd}(q)}(s) \mapsto q(s,w)]/H \}/F \in \text{Fil}(F,G^H)
$$

where $q \in \text{LPartial}(F\Box H,G)$ is a representative of the arrow (admissible, local germ) $\kappa$.

Some propositions about this mapping, and Notation 8.1, that we will need later:

**Lemma 9.3.** Let $F$, $\tilde{F}$, $G$, $\tilde{G}$, $H$ be filters of subsets of sets $S$, $\tilde{S}$, $T$, $\tilde{T}$, and $W$ respectively, and let $q \in \text{LPartial}(F\Box H,G)$, $\tilde{q} \in \text{LPartial}(F,\tilde{F})$, $\bar{q} \in \text{LPartial}(G,\tilde{G})$, and $\bar{q} = q \circ (\tilde{q}\Box_p H)$. We have

1. $F_{\tilde{F},H,\text{dd}(\tilde{q})} = \bar{q}^{-1}(F_{F,H,\text{dd}(q)})$;
2. if $s \in F_{\tilde{F},H,\text{dd}(\tilde{q})}$, then $h_{\tilde{F},H,\text{dd}(\tilde{q})}(s) = h_{F,H,\text{dd}(q)}(\bar{q}(s))$;
3. $F_{\tilde{F},H,\text{dd}(\tilde{q}\circ q)} \subseteq F_{\tilde{F},H,\text{dd}(\tilde{q})}$; and
4. if $s \in F_{\tilde{F},H,\text{dd}(\tilde{q}\circ q)}$, then $h_{\tilde{F},H,\text{dd}(\tilde{q}\circ q)}(s) \subseteq h_{\tilde{F},H,\text{dd}(\tilde{q})}(s)$.

**Proof.** (1):

$$
F_{\tilde{F},H,\text{dd}(\tilde{q})} = \{ s \in \tilde{S} \mid \{ w \in W \mid \langle s, w \rangle \in \text{dd}(\tilde{q}) \} \in H \}
$$

$$
= \{ s \in \tilde{S} \mid \{ w \in W \mid \langle s, w \rangle \in (\text{dd}(\tilde{q}) \times W) \cap (\bar{q} \Box_p H)^{-1}(\text{dd}(q)) \} \in H \}
$$

$$
= \{ s \in \text{dd}(\tilde{q}) \mid \{ w \in W \mid \langle s, w \rangle \in (\bar{q} \Box_p H)^{-1}(\text{dd}(q)) \} \in H \}
$$

$$
= \tilde{q}^{-1}(\{ s \in S \mid \{ w \in W \mid \langle s, w \rangle \in \text{dd}(q) \} \in H \})
$$

$$
= \tilde{q}^{-1}(F_{F,H,\text{dd}(q)}).
$$

(2):

$$
h_{\tilde{F},H,\text{dd}(\tilde{q})}(s) = \{ w \in W \mid \langle s, w \rangle \in \text{dd}(\tilde{q}) \}
$$

$$
= \{ w \in W \mid \langle \bar{q}(s), w \rangle \in \text{dd}(q) \}
$$

$$
= h_{F,H,\text{dd}(q)}(\bar{q}(s)).
$$

(3):

$$
F_{\tilde{F},H,\text{dd}(\tilde{q}\circ q)} = \{ s \in S \mid \{ w \in W \mid \langle s, w \rangle \in \text{dd}(\tilde{q} \circ q) \} \in H \}
$$

$$
= \{ s \in S \mid \{ w \in W \mid \langle s, w \rangle \in \text{dd}(q) \cap q^{-1}(\text{dd}(q)) \} \in H \}
$$

$$
\subseteq \{ s \in S \mid \{ w \in W \mid \langle s, w \rangle \in \text{dd}(q) \} \in H \}
$$

$$
= F_{\tilde{F},H,\text{dd}(q)}.
$$

(4):

$$
h_{\tilde{F},H,\text{dd}(\tilde{q}\circ q)}(s) = \{ w \in W \mid \langle s, w \rangle \in \text{dd}(\tilde{q} \circ q) \}
$$

$$
= \{ w \in W \mid \langle s, w \rangle \in \text{dd}(q) \cap q^{-1}(\text{dd}(q)) \}
$$

$$
\subseteq \{ w \in W \mid \langle s, w \rangle \in \text{dd}(q) \}
$$

$$
= h_{F,H,\text{dd}(q)}(s).
$$
Remark 9.4. Note that just as in our discussion of the natural transformation \( \alpha \), there is not much mystery about where the partial functions we define send the elements in their domains.

**Theorem 9.5.** We have

1. \( \chi^{\mathcal{H}}_{F,G} \) is a well-defined, one-one, and onto function;
2. \( \chi^{\mathcal{H}} \) is a natural isomorphism from the functor \( \text{Fil}(-,\mathcal{H}) : \text{Fil}^{\text{op}} \times \text{Fil} \to \text{Set} \) to the functor \( \text{Fil}(-,\mathcal{H}) : \text{Fil}^{\text{op}} \times \text{Fil} \to \text{Set} \), resulting in an adjunction
\[
\langle(-\mathcal{H},\mathcal{H}),\chi^{\mathcal{H}}\rangle : \text{Fil} \to \text{Fil};
\]
3. we have a nonsymmetric closed structure
\[
\langle\text{Fil},\varnothing,u,\alpha,\lambda,\varrho,\{\chi^{\mathcal{H}}\}_{\mathcal{H} \in \text{Fil}}\rangle
\]
on the category \( \text{Fil} \), where \( u,\alpha = \alpha^{\mathcal{G}},\lambda = \lambda^{\mathcal{G}},\varrho = \varrho^{\mathcal{G}} \) are defined as in Section 8.

**Proof.** (1): If \( q : F\mathcal{H} \to T \) and \( q' : F'\mathcal{H}' \) are representatives of \( \kappa \), then \( q \) and \( q' \) agree on \( F\mathcal{H} \), for some \( \hat{F} \in \mathcal{F} \) and \( \hat{h} : \hat{F} \to \mathcal{H} \) such that \( F\mathcal{H} \subseteq F\mathcal{H} \cap F'\mathcal{H}' \). Then for every \( s \in \hat{F} \), \( \hat{h}(s) \cap h'(s) \subseteq h(s) \), and we have
\[
[w \in h(s) \mapsto q(s,w)]/\mathcal{H} = [w \in \hat{h}(s) \mapsto q(s,w)]/\mathcal{H} = [w \in h'(s) \mapsto q'(s,w)]/\mathcal{H},
\]
proving that \( \chi^{\mathcal{H}}_{F,G}(\kappa) \) does not depend on the choice of \( q \).

On the other hand, if we have the same \( q \) and \( q' \), except that \( q \neq q' \), then for any \( \hat{F}\mathcal{H} \in \mathcal{F}\mathcal{H} \) with \( \hat{F} \subseteq F \cap F' \) and \( \hat{h} : \hat{F} \to \mathcal{H} \) with \( \hat{h}(s) \subseteq h(s) \cap h'(s) \) for \( s \in \hat{F} \), there is an \( \hat{s} \in \hat{F} \) such that \([w \in \hat{h}(s) \mapsto q(s,w)] \neq [w \in \hat{h}(s) \mapsto q(s,w)]\), which means that there is some \( \langle\hat{s},w\rangle \in \hat{F}\mathcal{H} \) such that \( q(\hat{s},w) \neq q'(\hat{s},w) \), and it follows that \([w \in \hat{h}(\hat{s}) \mapsto q(\hat{s},w)] \neq [w \in \hat{h}(\hat{s}) \mapsto q'(\hat{s},w)]\) and \( \chi^{\mathcal{H}}_{F,G}(q/\mathcal{F}) \neq \chi^{\mathcal{H}}_{F,G}(q'/\mathcal{F}) \). Thus, \( \chi^{\mathcal{H}}_{F,G} \) is one-one.

\( \chi^{\mathcal{H}}_{F,G}(\kappa) \) is admissible because it is the germ of a partial function with domain \( F \). It is local, because if \( X \in \mathcal{G}^{\mathcal{H}} \), there is a \( G \in \mathcal{G} \) such that \( \Gamma(\text{Partial}(\mathcal{H},\mathcal{G},G)) \subseteq X \), and we will have
\[
\chi^{\mathcal{H}}_{F,G}(q|_{F \times G}) \in X;
\]
thus, \( \chi^{\mathcal{H}}_{F,G} : \text{Fil}(\mathcal{F}\mathcal{H},\mathcal{G}) \to \text{Fil}(\mathcal{F},\mathcal{G}^{\mathcal{H}}) \).

To show \( \chi^{\mathcal{H}}_{F,G} \) is onto, let us be given \( \rho \in \text{Fil}(\mathcal{F},\mathcal{G}^{\mathcal{H}}) \), and an admissible, local partial function \( r : \mathcal{F} \to \mathcal{G}^{\mathcal{H}} \) representing \( \rho \), and define
\[
\tilde{\chi}^{\mathcal{H}}_{F,G}(\rho) = \langle s,w \in \text{dd}(r)\mathcal{H}[s \mapsto \text{dd}(y(s))] \mapsto y(s)(w)\rangle/(\mathcal{F}\mathcal{H})
\]
where for every \( s \in \text{dd}(r) \), \( y(s) \) is some representative of the germ of admissible partial functions \( r(s) \in \mathcal{G}^{\mathcal{H}} \). Since we made choices here, this is a one-to-many relation. Ignoring
for the moment that this is not a function, and just fixing our choices in defining \( \chi_{\mathcal{F},G}^H(\rho) \), we see that

\[
\chi_{\mathcal{F},G}^H \left( \chi_{\mathcal{F},F}^H(\rho) \right) = \chi_{\mathcal{F},G}^H \left( ([s, w] \in \text{dd}(r) \implies [s \implies \text{dd}(y(s))] \implies y(s)(w)] \cap (\mathcal{F} \square \mathcal{H}) \right)
\]
\[
= [s \in \text{dd}(r) \implies [w \in \text{dd}(y(s))] \implies y(s)(w)] \cap \mathcal{H} / \mathcal{F}
\]
\[
= [s \in \text{dd}(r) \implies y(s) / \mathcal{H}] / \mathcal{F}
\]
\[
= [s \in \text{dd}(r) \implies r(s)] / \mathcal{F}
\]
\[
= r / \mathcal{F} = \rho,
\]
showing that \( \chi_{\mathcal{F},G}^H \) is onto.

(2) In order to show that \( \chi_{\mathcal{F},G}^H \) is natural in \( \mathcal{F} \), it suffices to show that if in addition to having \( \kappa \) as we have assumed, we have \( \bar{\kappa} \in \text{Fil}(\mathcal{F}, \mathcal{F}) \), where \( \bar{\mathcal{F}} \) is a filter of subsets of a set \( \bar{S} \), then

\[
\chi_{\mathcal{F},G}^H(\kappa \circ (\bar{\kappa} \square \mathcal{H})) = (\chi_{\mathcal{F},G}^H(\kappa)) \circ \bar{\kappa} : \bar{\mathcal{F}} \to \mathcal{G}^H;
\]

indeed, if we set \( \hat{q} = q \cap (\hat{q} \square \mathcal{H}) : \bar{\mathcal{F}} \square \mathcal{H} \to \mathcal{G} \), we have

\[
\chi_{\mathcal{F},G}^H(\kappa \circ (\bar{\kappa} \square \mathcal{H})) = \chi_{\mathcal{F},G}^H \left( q / (\bar{\mathcal{F}} \square \mathcal{H}) \circ (\hat{q} \square \mathcal{H}) / (\bar{\mathcal{F}} \square \mathcal{H}) \right)
\]
\[
= \chi_{\mathcal{F},G}^H \left( \hat{q} / \bar{\mathcal{F}} \square \mathcal{H} \right)
\]
\[
= [s \in F_{\mathcal{F},H,\text{dd}(\hat{q})} \implies [w \in h_{\mathcal{F},H,\text{dd}(\hat{q})}(s) \implies \hat{q}(s, w)] / \mathcal{H} / \bar{\mathcal{F}}
\]
\[
= [s \in F_{\mathcal{F},H,\text{dd}(\hat{q})} \implies [w \in h_{\mathcal{F},H,\text{dd}(\hat{q})}(s) \implies q(\hat{q}(s), w)] / \mathcal{H} / \bar{\mathcal{F}}
\]
\[
= [s \in \bar{q}^{-1}(F_{\mathcal{F},H,\text{dd}(\hat{q})}) \implies [w \in h_{\mathcal{F},H,\text{dd}(\hat{q})}(\bar{q}(s)) \implies q(\bar{q}(s), w)] / \mathcal{H} / \bar{\mathcal{F}}
\]
\[
= ([s \in F_{\mathcal{F},H,\text{dd}(\hat{q})} \implies [w \in h_{\mathcal{F},H,\text{dd}(\hat{q})}(s) \implies q(s, w)] / \mathcal{H} \circ \bar{\hat{q}}) / \bar{\mathcal{F}}
\]
\[
= ([s \in F_{\mathcal{F},H,\text{dd}(\hat{q})} \implies [w \in h_{\mathcal{F},H,\text{dd}(\hat{q})}(s) \implies q(s, w)] / \mathcal{H} / \bar{\mathcal{F}}) \circ (\bar{\hat{q}} / \bar{\mathcal{F}})
\]
\[
= (\chi_{\mathcal{F},G}^H(\kappa)) \circ \bar{\kappa}
\]
where we use Lemma 9.3(1) and (2).

To show that \( \chi_{\mathcal{F},G}^H \) is natural in \( \mathcal{G} \), we must show that

\[
\chi_{\mathcal{F},G}^H(\bar{\kappa} \circ \kappa) = \bar{\kappa}^H \circ (\chi_{\mathcal{F},G}^H(\kappa)) : \mathcal{F} \to \bar{\mathcal{G}}^H,
\]
for any \( \bar{q} / \mathcal{G} = \bar{\kappa} : \mathcal{G} \to \bar{\mathcal{G}} \), where \( \bar{\kappa}^H \) is the usual shorthand for \( \bar{\kappa}^{1_H} \). Considering the two sides of Equation 9.2, we have
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\[
\chi^H_{F,G}(\tilde{\kappa} \circ \kappa) = \chi^H_{F,G}(\tilde{q}/G \circ q/(F \square H)) = \chi^H_{F,G}((\tilde{q} \circ q)/(F \square H)) = \chi^H_{F,G}((\tilde{q}/G \circ q/(F \square H)) = \chi^H_{F,G}(\tilde{q}/G \circ q/(F \square H)) = (\tilde{q}/G)^H \circ \chi^H_{F,G}(q/(F \square H)) = (\tilde{q}/G)^H \circ (\chi^H_{F,G}(q/(F \square H))) \]

using Lemma 9.3(3) and (4), which force germs with respect to \( F \) and \( H \) to be the same.

□

Remark 9.6. Note that each component \( \eta^H_F : F \to (F \square H)^H \) of the unit natural transformation \( \eta^H \) is the germ of the function sending \( s \in S \) to the germ of the function sending \( w \in W \) to \( \langle s, w \rangle \). Each component \( \varepsilon^H_G : G^H \square H \to G \) of the counit natural transformation \( \varepsilon^H \) is the germ of the function sending \( \langle q/(G^H \square H), w \rangle \) to \( q(w) \) for \( w \in \text{dd}(q) \).

10. APPLICATIONS

Uniform spaces on filters. In [6], we discuss in detail the theory of uniform spaces with an underlying filter, instead of an underlying set. It should not come as a shock that there is a close relationship between \( \text{Fil} \) and uniform spaces, when we consider that a uniformity on a set is defined as a filter of entourages. \( \text{Fil} \) has a factorization system (Section 5), and we show in [4] that a certain list of properties of a category with factorization system support a theory of generalized equivalences, of which uniformities on a set are an example. These are the properties of \( \text{Fil} \) (and its factorization system) that we proved in Section 6.

Thus, we consider in [6] the category of uniform spaces on filters, which we continue to denote by \( \text{Unif} \) because it is so natural to do so. A uniformity on a filter is a small generalization from a uniformity on a set, but one which manifests interesting new phenomena, especially when completion is considered. If we have a uniformity on a filter, the points of the filter that are not in the core play an interesting role. They are not closed points, and can never be limits of a cauchy filter, but, there can be cauchy filters consisting entirely of non-core points. These cauchy filters give rise to new elements when we apply the functor of (hausdorff) completion, \( C : \text{Unif} \to \text{Unif} \).

This becomes especially important when we try to define a function space and make the category of uniform spaces on filters into a closed category. Indeed this is possible, if we admit the possibility of a nonsymmetric closed category. The elements of the function space are germs of admissible partial functions, without regard to being local or uniformly continuous,
which properties hold only for germs in the core of the function space object\(^1\). However, these non-core germs can give rise to new arrows when we complete the hom-objects and, using the fact\([5,\text{Section 1}]\) that completion is a \textit{monoidal functor}, form the category \(\mathcal{C}(\text{Unif})\), as defined in \([5,\text{Section 3}]\). The new arrows include an inverse to the unit natural arrow from any uniform space into its completion, so that it becomes possible to work with the assumption that all spaces are complete.

\textbf{Ind[Fil], a base category for Topological Algebra.} In \([4]\), we introduce the theory of \textbf{Ind[Fil]}, a cartesian-closed category suitable for the study of Topological Algebra. In particular, for varieties of algebras that are congruence-modular, \textbf{Ind[Fil]} has properties that allow us to generalize Day’s Theorem\([1]\), something that is not possible\([7]\) in \textbf{Unif}. Any hausdorff, compactly-generated topological algebra can be made into an object of \textbf{Ind[Fil]}, and a factorization system (Section 5) in that category allows us to associate with that object, a structure lattice analogous to the congruence lattice or lattice of compatible uniformities on that algebra, which will be modular if the algebra belongs to a congruence-modular variety.

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