WELL-POSEDNESS AND LONG TERM BEHAVIOR OF A SIMPLIFIED ERICKSEN-LESLIE NON-AUTONOMOUS SYSTEM FOR NEMATIC LIQUID CRYSTAL FLOWS

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Abstract. We analyze a simplified Ericksen-Leslie model for nematic liquid crystal flows firstly introduced in [18] with non-autonomous forcing bulk term and boundary conditions on the order parameter field. We obtain existence of weak solutions in the two- and three-dimensional cases. We prove uniqueness, continuous dependence on initial conditions, forcing and boundary terms and also existence of strong solutions in the 2D case. Focusing on the 2D case, we then study the long term behavior of solutions by obtaining existence of global attractors for normal forcing terms (according to [21]). Finally, we prove the existence of exponential attractors for quasi-periodic forcing terms in the 2D model.

1. Introduction. In this paper we study the simplified Ericksen-Leslie model for nematic liquid crystal flows given by (see [18]):

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p &= -\nabla d^t \Delta d + g(t) \\
\nabla \cdot u &= 0 \\
\partial_t d + (u \cdot \nabla) d &= \Delta d - f(d) \\
|d| \leq 1 \quad &\text{in } \Omega \times (0, \infty); \\
u(x,0) = u_0, \quad d(x,0) = d_0 \quad &\text{for } x \in \Omega; \\
u(x,t) = 0, \quad d(x,t) = h(x,t) \quad &\text{on } \partial \Omega \times (0, \infty)
\end{aligned}
\]

where \( u \) is the eulerian velocity field, \( p \) is the pressure field and \( d \) is the order parameter field which describes the local orientation of the liquid crystal molecules. We recall that \( f(d) = \frac{1}{\epsilon} (|d|^2 - 1) d \) is obtained from the Ginzburg-Landau potential, which relaxes the physical constraint \( |d| = 1 \) (see [13], [17] or [25] for some interesting results without this relaxation). In this paper we consider two non-autonomous terms: one volume force on the linear momentum equation \( g(t) \) and the non-homogeneous Dirichlet boundary condition for the order parameter field \( h(t) \) (see [20] for some results with other boundary conditions).

Only some results are known on the well-posedness of the full classic model (see [5], [27] or [32] for a complete overview of the physical background and [16] for some analytical results) and much of the most recent results refer to the simplified...
system (1) which was first introduced in [18] by Lin e Liu (see, however, [34] or [28]
for a more complete model).

In section 2 we obtain well-posedness results for system (1). In particular, we
prove the existence of weak solutions both in the two- and three-dimensional cases
(theorem 2.2), uniqueness and continuous dependence of solutions on the initial
data, forcing and boundary terms (theorem 2.4) for the 2D case and existence of
strong solutions always in the two-dimensional case (theorem 2.6). See [11], [14]
and [15] for some additional regularity results.

We then focus on the long term behavior of solutions (see [34] and [33] for some
similar results). In section 3 we prove the existence of global attractors for the flow
under very general assumptions on the non-autonomous forcing terms. In particular,
we consider the newly class of normal functions introduced by Lu and Wu in 2005
(see [21] and [22]). We note that normal functions are obviously not necessary
translation compact whilst all translation compact functions are also normal. This
new class is a proper subset of the set of all translation bounded functions (i.e.
weakly translation compact functions, see [2]).

Finally in section 4 we prove the existence of exponential attractors for the flow
(see [23] for a review of this subject) considering quasi-periodic forcing terms.

2. Well-posedness. We start by studying the well-posedness of the just intro-
duced simplified Ericksen-Leslie model (1). We will essentially follow the proof
in [4] for the system without external driving force. We observe that, in [4], the
regularity assumptions on the time dependent lifting function (denoted here by \(d\),
see below equation (22)) seems in general not to be verified. We therefore give in
appendix A a fully revised proof of the existence of weak solutions in the 2D and 3D
cases. We will then complete the analysis in the 2D case by obtaining uniqueness
of weak solutions and continuous dependence on the data. Moreover, always in the
2D case, we will be able to prove existence of strong solutions.

2.1. Existence of weak solutions. We start by introducing the functional spaces
we will use in the analysis of (1). With \(L^2\) we will denote the standard function
space made up by vector valued \(L^2(\Omega)\) functions. Analogously, \(H^1\) will be the usual
vector Sobolev space constituted by componentwise \(H^1(\Omega)\) functions. Moreover, let
\[
H = \left\{ u \in C_0^\infty \mid \nabla \cdot u = 0 \right\}^{L^2} \quad \text{and} \quad V = \left\{ u \in C_0^\infty \mid \nabla \cdot u = 0 \right\}^{H^1_0}
\]
be the usual divergenceless spaces used in the analysis of Navier-Stokes equations
(see, e.g., [31]). The family \(\{w_n\}_n\) will be the Hilbert basis of \(V\) given by the
eigenfunctions of Stokes’ problem:
\[
\text{find } w_i : \quad (\nabla w_i, \nabla v) = \lambda_i (w_i, v) \quad \forall v \in V, \quad |w_i|_2 = 1 \quad (2)
\]
where, thanks to the spectral theorem, the sequence of \(\lambda_i\) is monotonically
increasing. From the spectral theory for compact operators we know that the functions \(w_i\)
form a complete orthonormal basis in \(H\) which is also orthogonal in \(H^1\). For
convenience we will write \(V^m = \langle w_1, w_2, \ldots, w_m \rangle\) for the finite dimensional subspace
of \(V\) spanned by the first \(m\) eigenfunction of Stokes’ problem. From the regularity
results for this problem (see [31]) and thanks to the finite-dimensionality of \(V^m\),
we know that the canonical embedding \(V^m \hookrightarrow H^2(\Omega)\) is compact. Finally, we will
denote by \(V^*\) the dual space of \(V\).
Notation. We will write $|w|_p$ to indicate the $L^p$ norm of $w$ and $|w|_{H^s}$ when referring to its $H^s$ norm. Sometimes, as in the definition of the nonlinear potential $f$ just after problem (1), we will write $|f(x,t)|$ or shortly $|f|$ when referring to the usual Euclidean norm of vectors in $\mathbb{R}^n$. Therefore, while at fixed time $|d|_2$ will be a real number, $|d|$ will be a real valued function on $\Omega$.

In all this work we will consider the following regularity assumptions:

1. the domain $\Omega$ will be an open bounded subset of $\mathbb{R}^n$, $n = 2, 3$, of class $C^{1,1}$;
2. the non-autonomous forcing term $g$ will belong to $L^2(0,T; V^*)$;
3. the non-autonomous boundary term $h$ will be bounded i.e. $|h| \leq 1$ a.e..
4. $h$ will be an element of $L^2(0,T; H^{3/2}(\partial \Omega))$ s.t. $\partial_t h \in L^2(0,T; H^{-1/2}(\partial \Omega))$.

We now give the definition of weak solutions for system (1).

**Definition 2.1.** Let $T > 0$. A pair $(u, d)$ is a weak solution to problem (1) if $(u, d) \in L^2(0,T; V \times H^s)$, $(\partial_t u, \partial_t d) \in L^p(0,T; V^* \times L^2)$ (with $p = 2$ when $n = 2$ and $p = 4/3$ when $n = 3$), $u(x,0) = u_0(x)$ in $L^2$ and $d(x,0) = d_0(x)$ in $H^s$, if $d(x,t) = h(x,t)$ on $\partial \Omega \times (0,T)$ in the sense of trace spaces and if

\[
\langle \partial_t u(t), v \rangle + \langle (u(t) \cdot \nabla) u(t), v \rangle + \nu (\nabla u(t), \nabla v) + (\Delta d(t), \nabla d(t) v) = \langle g(t), v \rangle
\]

holds for every $v \in V$, a.e. $t \in (0,T)$ and

\[
\partial_t d(t) + (u(t) \cdot \nabla) d = \Delta d - f(d(t)) \quad \text{and} \quad |d(x,t)| \leq 1
\]

hold almost everywhere in $\Omega \times (0,T)$.

**Remark 1.** We observe that this definition of weak solutions involves the following natural compatibility requirement on the initial and boundary conditions for the order parameter field: $d_0 = h$ on $\partial \Omega \times \{0\}$. This compatibility condition will involve some care in the definition of the phase space when studying the long term behavior of our system (see section 3.1 below).

The first result is the following:

**Theorem 2.2 (Weak existence).** Suppose that the hypothesis (1) to (4) are satisfied and let $u_0 \in H$, $d_0 \in H^1$ with $|d_0| \leq 1$ a.e., $d_0 = h(0)$ on $\partial \Omega$. Then there exists a weak solution $(u, d)$ to problem (1).

We give here only a brief sketch of the proof leaving the details to appendix A.

We start by studying the regularity of the solution of a lifting problem for the non-autonomous boundary conditions on the order parameter. We then give a semi-Galerkin formulation of problem (1) by considering the discretized problem for the velocity field leaving the other equations in the lifted form. Next we prove local existence of solutions for the approximating problem through a fixed point argument. However, the lifespan of these solutions depends in a critical way on the dimension of the approximating subspace $V^m$. Thus, before passing to the limit, we need to extend the approximating solutions. This is achieved by proving the following energy estimate which also holds for the approximating solutions (see page 439 below).
Lemma 2.3. Let the assumptions of theorem 2.2 hold. Then any weak solution of (1) satisfies for all $t > 0$ the estimate:

$$
|u(t)|^2 + |\nabla d(t)|^2 \\
\leq C_\Omega + |h(t)|^2_{H^{1/2}} + e^{-Ct}(|u_0|^2 + |\nabla d_0|^2) \\
+ C_\Omega \int_0^t |h(s)|^2_{H^{3/2}} ds + C_\Omega \int_0^t |\partial_t h(s)|^2_{H^{-1/2}} ds + \frac{1}{\nu} \int_0^t |g(s)|^2_{\mathcal{V}}. ds.
$$

(3)

Finally we end the proof by passing to the limit by means of standard arguments.

Remark 2. Lemma 2.3 justifies us in introducing global solutions defined for all $t > 0$. One actually only needs to observe that if $g \in L^2_{\text{loc}}([0, \infty); \mathcal{V}^*)$, $h \in L^2_{\text{loc}}([0, \infty); H^{3/2}(\partial\Omega)) \cap L^\infty(0, \infty; H^{1/2}(\partial\Omega))$ s.t. $\partial_t h \in L^2_{\text{loc}}([0, \infty); H^{-1/2}(\partial\Omega))$, then the size of the time interval $[t, t+T]$ on which the local solutions obtained by theorem 2.2 are defined is independent of $t$. Moreover, estimate (3) is uniform in $t$ and therefore any local solution can be extended by successive steps up to $\infty$.

Corollary 1 (Global weak existence). Suppose that the assumptions (1) to (4) are satisfied for any $T > 0$ and let $u_0 \in H$, $d_0 \in H^1$ with $|d_0| \leq 1$ a.e., $d_0 = h(0)$ on $\partial\Omega$. Then there exists a global weak solution $(u, d)$ satisfying (1) for all $T > 0$.

2.2. Uniqueness and continuous dependence on initial conditions in the 2D case. The following result holds:

Theorem 2.4 (Uniqueness and continuous dependence). Under the same assumptions of theorem 2.2, if $n = 2$, the weak solution of problem (1) is unique. Moreover, it continuously depends on the initial conditions $d_0$, $u_0$ and on the forcing terms $g$ and $h$, and the following estimate holds:

$$
|\delta u(t)|^2 + |\nabla \delta d(t)|^2 + \int_0^t (\nu|\nabla \delta u|^2 + |\Delta \delta d|^2) ds \leq \Phi(t) (1 + \Phi(t)) e^{\Psi(t)}
$$

(4)

where

$$
\Phi(t) = C \int_0^t \left( \frac{1}{\nu} |\nabla u|^2 + |\Delta d|^2 + |\nabla u|^2 + \frac{1}{e^2} \right) ds
$$

and

$$
\Psi(t) = |\delta u(t_0)|^2 + |\nabla \delta d(t_0)|^2
$$

$$
+ \int_{t_0}^t \left( \frac{1}{\nu} |\delta g|^2_{\mathcal{V}} + C(|\partial \delta h|^2_{H^{-1/2}} + C (1 + \frac{1}{e^2}) |\delta h|^2_{H^{3/2}}) ds.
$$

Remark 3. We observe also, as a simple corollary of estimate (4), that $(u, d) \in C(L^2 \times H^1)$.

Proof of theorem 2.4. As usual, let $(u_1, d_1)$ and $(u_2, d_2)$ be two solutions of system (1) respectively with forcing terms $g_1$ and $g_2$ and boundary conditions $h_1$ and $h_2$. We will use $\delta u = u_1 - u_2$ and $\delta d = d_1 - d_2$ to denote the difference between these two solutions and $\delta g = g_1 - g_2$, $\delta h = \partial_t h_1 - \partial_t h_2$ for the difference between the non-autonomous terms. By considering the difference of the equations solved by
Applying Gronwall’s inequality to this last estimate we eventually obtain (4).

We observe that, when uniqueness estimates are of concern, thanks to the Poincaré general case we are treating now, the regularity of strong solutions. In this section we will use the following assumptions:

\[ \text{strong solution in the 2D case.} \]

Sobolev’s inequalities we get:

\[ \frac{1}{2} \frac{d}{dt} (|\delta u|^2 + |\nabla \delta d|^2) + \nu |\nabla \delta u|^2 + |\Delta \delta d|^2 \]

\[ = (f(d_1) - f(d_2), \Delta \delta d) + H^{1/2}(\Omega) (\partial_t \delta d, \partial_t \delta h)_{H^{-1/2}(\Omega)}. \]

Summing up, recalling identity (34), using the estimate of lemma A.1, Hölder’s and Sobolev’s inequalities we get:

\[ \frac{1}{2} \frac{d}{dt} (|\delta u|^2 + |\nabla \delta d|^2) + \nu |\nabla \delta u|^2 + |\Delta \delta d|^2 \]

\[ \leq |\delta u|^2 |\nabla u|_2 + |\nabla \delta d|_4 |\Delta d|_2 |\delta u|_4 \]

\[ + |u|_4 |\nabla \delta u|_4 |\Delta \delta u|_2 + \frac{2}{c^2} |\delta u|^2 |\Delta \delta d|_2 \]

\[ + |\delta g|_2 |\nabla \delta u|_2 + |\partial_t \delta d|_{H^{1/2}(\Omega)} |\partial_t \delta h|_{H^{-1/2}(\Omega)} \]

\[ \leq C |\delta u|^2 |\nabla u|_2 + C |\nabla \delta u|_4 |\Delta d|_2 |\delta u|_4 \]

\[ + C |u|_4 |\nabla u|_4 |\nabla \delta u|_2 |\Delta \delta d|_2 + C |\delta u|^2 |\nabla \delta u|_2 + C |\delta d|_{H^2} |\partial_t \delta h|_{H^{-1/2}(\Omega)} \]

We observe that, when uniqueness estimates are of concern, thanks to the Poincaré inequality, the \( H^2 \) norm of \( \delta d \) can be replaced by the \( L^2 \) norm of \( \Delta \delta d \). In the general case we are treating now, the \( H^2 \) norm can be easily estimated as follows:

\[ |\delta d|_{H^2} \leq |\delta \bar{d}|_{H^2} + |\delta d|_{H^2} \]

\[ \leq C |\delta \Delta d|_2 + C |\delta h|_{H^{1/2}(\Omega)} \]

\[ \leq C |\delta d|_2 + C |\delta h|_{H^{1/2}(\Omega)}. \]

After repeatedly using Young’s inequality, reordering and neglecting the positive terms on the left hand side, we finally deduce:

\[ \frac{d}{dt} (|\delta u|^2 + |\nabla \delta d|^2) + \nu |\nabla \delta u|^2 + |\Delta \delta d|^2 \]

\[ \leq C \left( \frac{1}{\nu} |\nabla u|^2 + |\Delta d|^2 + |u|_2^2 |\nabla u|^2 + \frac{1}{c^2} \right) (|\delta u|^2 + |\nabla \delta d|^2) \]

\[ + \frac{3}{\nu} |\delta g|_2 |\nabla \delta u|_2 + C |\partial_t \delta h|^2_{H^{-1/2}(\Omega)} + C (1 + \frac{1}{\nu}) |\delta h|^2_{H^{1/2}(\Omega)}. \]

Applying Gronwall’s inequality to this last estimate we eventually obtain (4). \( \square \)

2.3. Strong solution in the 2D case. Having proved existence and uniqueness of weak solutions for system (1), we are now ready to investigate existence and regularity of strong solutions. In this section we will use the following assumptions:

(5) the domain \( \Omega \) will be an open bounded subset of \( \mathbb{R}^n \), with \( n = 2, 3 \) of class \( C^{2,1} \);

(6) the non-autonomous forcing term \( g \) will be an element of \( L^2(0, T; H) \);

(7) the non-autonomous boundary term \( h \) will belong to \( L^2(0, T; H^{5/2}(\partial \Omega)) \) and be such that \( \partial_t h \in L^2(0, T; H^{1/2}(\partial \Omega)) \).
We introduce the notion of strong solution for system (1).

**Definition 2.5.** A pair \((u, d)\) is a **strong solution** for problem (1) if it is a weak solution and moreover \((u, d) \in L^2(0, T; (H \cap H^2) \times H^1))\), \((\partial_t u, \partial_t d) \in L^2(0, T; H \times H^1))\), \(u(x, 0) = u_0(x)\) in \(H^1\) and \(d(x, 0) = d_0(x)\) in \(H^2\) and if

\[
\begin{align*}
\partial_t u(t) + (u(t) \cdot \nabla)u(t) - \nu \Delta u(t) + \nabla p(t) &= -(\nabla d(t))^T \Delta d(t) g(t) \\
\nabla \cdot u(t) &= 0 \\
\partial_t d(t) + (u(t) \cdot \nabla) d &= \Delta d - f(d(t))
\end{align*}
\]

holds almost everywhere in \(\Omega \times (0, T)\).

We now prove the following existence result.

**Theorem 2.6** (Strong existence). Suppose that the hypothesis (3), (5), (6) and (7) are satisfied and let \(u_0 \in V\), \(d_0 \in H^2\) with \(|d_0| \leq 1\) a.e., \(d_0 = h(0)\) on \(\partial \Omega\). Then there exists a strong solution \((u, d)\) of (1).

**Proof.** We start by considering the same lifting (22) and the same lifted problem (26) used in the proof of theorem 2.2. By using \(-\Delta u\) as a test function in the equation for the velocity field \(u\) and recalling that in the 2D case \((\langle u \cdot \nabla \rangle u, \Delta u) = 0\), we obtain:

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} |\nabla u|^2_2 + \nu |\Delta u|^2_2 &\leq |\nabla d|_\infty |\Delta d|_2 |\Delta u|_2 + |g|_2 |\Delta u|_2 \\
&\leq C |\nabla d|^2_2 |\Delta d|_2 |\Delta u|_2 + |g|_2 |\Delta u|_2 \\
&\leq C |\nabla u|^2_2 + \delta |d|^4_4 + \frac{C}{\delta^2} |\nabla d|^2_2 |\Delta d|^2_2 + \frac{2}{\nu} |g|^2_2
\end{align*}
\]

(5)

where \(\delta\) will be determined later.

To get regularity estimates for the order parameter \(d\) we can take the duality of the second equation in (26) with \(\Delta(\Delta \hat{d} - f(\hat{d}))\). We note that, since \(u|_{\partial \Omega} = 0\) and \(\hat{d}|_{\partial \Omega} = 0\), we have \((\Delta \hat{d} - f(\hat{d}))|_{\partial \Omega} = 0\) and therefore it satisfies Poincaré's inequality. Integrating by parts and observing that boundary terms vanish, after a few calculations we have:

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} |\Delta \hat{d} - f(\hat{d})|^2_2 + |\nabla(\Delta \hat{d} - f(\hat{d}))|^2_2 \\
= (\partial_t f(d), \Delta \hat{d} - f(\hat{d})) + \langle \langle \nabla u \cdot \nabla \rangle d, \nabla (\Delta \hat{d} - f(\hat{d})) \rangle \\
+ \langle \nabla \nabla d \cdot u, \nabla (\Delta \hat{d} - f(\hat{d})) \rangle
\end{align*}
\]

(6)

We now have to find bounds for every term on the right hand side of this last equation. We start by observing that \(\partial_t f(d) = \nabla d f(d) \cdot \partial_t d\). Recalling that \(|\nabla d f(d)|_\infty \leq C\) because \(|d| < 1\) by the maximum principle (see lemma A.1) and using the lifted equation, we obtain:

\[
\begin{align*}
- (\partial_t f(d), \Delta \hat{d} - f(\hat{d})) \\
= - (\nabla d f(d)(\Delta \hat{d} - f(\hat{d})), \Delta \hat{d} - f(\hat{d})) + (\nabla d f(d) \Delta \hat{d}, \Delta \hat{d} - f(\hat{d})) \\
+ (\nabla d f(d)(u \cdot \nabla) d, \Delta \hat{d} - f(\hat{d}))
\end{align*}
\]
Using the regularity assumptions of this section we have \( \tilde{d} \in L^\infty(0, T; \mathbb{H}^2) \). Moreover, from the weak regularity estimates of the previous sections, \( u \in L^\infty(0, T; L^2) \) and \( d \in L^\infty(0, T; \mathbb{H}^1) \) hold. Therefore, by proceeding as usual, we get:

\[
- \left( \partial_t f(d), \Delta \tilde{d} - f(d) \right) \leq C |\Delta \tilde{d} - f(d)|^2 + C |\Delta \tilde{d}| |\Delta \tilde{d} - f(d)|_2 + C |u|_\infty |\nabla d|_2 |\Delta \tilde{d} - f(d)|_2
\]

Likewise we can bound the second term in the right hand side of equation (6):

\[
\left\langle (\nabla u' \cdot \nabla) d, \nabla(\Delta \tilde{d} - f(d)) \right\rangle \leq |\nabla u|_4 |\nabla d|_4 |\nabla(\Delta \tilde{d} - f(d))|_2
\]

Arguing similarly for the third and last term, we have:

\[
\left\langle \nabla d \cdot u, \nabla(\Delta \tilde{d} - f(d)) \right\rangle \leq |d|_{W^{2,4}} |u|_4 |\nabla(\Delta \tilde{d} - f(d))|_2
\]

where \( \delta > 0 \) will be determined in a few passages.

We now gather all the results of this section. Summing up estimates (5) and (6) and using the last three inequalities, after reordering all terms, we get:

\[
\frac{1}{2} \frac{d}{dt} \left( |u|^2 + |\Delta \tilde{d} - f(d)|^2 \right) + \nu |\Delta u|^2 + \frac{1}{2} |\nabla(\Delta \tilde{d} - f(d))|^2 \leq 2 \delta |d|_{\mathbb{H}^3}^2 + \frac{C}{\delta \nu^2} |\Delta d|^2 + C |\Delta \tilde{d} - f(d)|_2^2
\]

(7)

where we have set \( K = C(1 + 1/\nu + 1/\delta) \) for the sake of simplicity.

Recalling the triangle inequality, we can easily bound the norms \( |d|_{\mathbb{H}^i} \), with \( i = 2, 3 \). In particular we have:

\[
|d|_{\mathbb{H}^2}^2 \leq 2 |\tilde{d}|_{\mathbb{H}^2}^2 + 2 |\tilde{d}|_{\mathbb{H}^2}^2 \leq C |\Delta \tilde{d}|^2 + 2 |\tilde{d}|_{\mathbb{H}^2}^2
\]

\[
\leq C |\Delta \tilde{d} - f(d)|^2 + C |f(d)|^2 + 2 |\tilde{d}|_{\mathbb{H}^2}^2
\]

\[
\leq C |\Delta \tilde{d} - f(d)|^2 + C
\]
and
\[ |d|_{L^3}^2 \leq 2|\Delta d|_{L^3}^2 + 2|d|_{H^3}^2 \leq C|\nabla \Delta \hat{d}|_{L^2}^2 + C|\Delta \hat{d}|_{L^2}^2 + 2|\Delta d|_{L^3}^2 \]
\[ \leq C|\nabla (\Delta \hat{d} - f(d))|_{L^2}^2 + C|\nabla f(d)|_{L^2}^2 + C|\Delta \hat{d}|_{L^2}^2 + 2|\Delta d|_{L^3}^2 \]
\[ \leq C|\nabla (\Delta \hat{d} - f(d))|_{L^2}^2 + 2|d|_{H^3}^2 + C \]
where in the last passage we have used the Poincaré inequality which holds for \( \Delta \hat{d} - f(d) \).

We now choose \( \delta < \frac{1}{4} \) so small that \( 2\delta |d|_{L^3}^2 \leq \frac{1}{4}|\nabla (\Delta \hat{d} - f(d))|_{L^2}^2 + O.T. \) and we observe that \( \delta \) depends only on the domain \( \Omega \) and on the boundary data \( h \). Using these bounds in (7) we finally obtain:

\[ \frac{d}{dt} \left( |\nabla u|^2_T + |\Delta \hat{d} - f(d)|^2_{T'} \right) + \nu \left( |\Delta u|^2_T + \frac{1}{4}|\nabla (\Delta \hat{d} - f(d))|^2_T \right) \leq \tilde{K}^2 |\Delta \hat{d} - f(d)|^2_{T'} + \tilde{K} |\nabla u|^2_{T'} + |d|^2_{H^3} + \frac{2}{\nu} |g|^2 + \tilde{K}. \]  

(8)

Setting \( A(t) = |\nabla u(t)|^2_T + |\Delta \hat{d}(t) - f(d(t))|^2_{T'} \), this estimate can be rewritten as

\[ \frac{d}{dt} A(t) \leq \tilde{K}^2 A^2(t) + |d|^2_{H^3} + \frac{2}{\nu} |g|^2 + \tilde{K}. \]  

(9)

Thanks to estimate (33) we have \( A \in L^1(0,T) \) for all \( T > 0 \). We can therefore apply Gronwall’s inequality and get:

\[ |\nabla u(t)|^2_T + |\Delta \hat{d}(t) - f(d(t))|^2_{T'} \leq \left( |\nabla u_0|^2_T + |f(d_0)|^2_{T'} + \tilde{K} t + \frac{1}{\nu} \int_0^t |d|^2_{H^3} ds + \frac{2}{\nu} \int_0^t |g|^2 ds \right) e^{\tilde{K}^2 \int_0^t |\nabla u(s)|^2 + |\Delta \hat{d}(s) - f(d(s))|^2_{T'} ds} \]

from which we can easily prove that \( u \in L^2(0,T;H^2) \cap L^\infty(0,T;H^1) \) and \( d \in L^2(0,T;H^1) \cap L^\infty(0,T;H^2) \) for all \( T > 0 \) as claimed. \( \square \)

3. Global attractors in the two-dimensional case. In this section we will study the existence of a global attractor for system (1). We will follow the approach of Chepyzhov and Vishik (see [2, Part 2]) as developed for less regular forcing terms by Lu et al. in [21] and [22]. We briefly recall the fundamental definitions we will use.

We will suppose that the time dependency can be completely described through a finite set of functions that we shall denote by \( \sigma(t) \). We will call \( \sigma(t) \) the time symbol or simply the symbol of the non-autonomous evolution equation. The set of all symbols of interest in a particular case will be called symbol space and will usually be denoted by \( \Sigma \).

We need some notions of dissipativeness for our evolution operator (see [2, Part 2]).

**Definition 3.1.** A set \( B_0 \subset X \) is said to be uniformly (with respect to \( \sigma \in \Sigma \)) absorbing for the family of processes \( \{U_\sigma(t,\tau)\} \), \( \sigma \in \Sigma \) if for any \( \tau \in \mathbb{R} \) and every \( B \in \mathcal{B}(X) \) there exists an absorption time \( t_0 = t_0(\tau, B) \geq \tau \) such that \( \cup_{\sigma \in \Sigma} U_\sigma(t,\tau)B \subset B_0 \) for all \( t \geq t_0 \).


Definition 3.2. A set \( K \subset X \) is uniformly (w.r.t. \( \sigma \in \Sigma \)) attracting for the family of processes \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma \) if it satisfies, for any fixed \( \tau \in \mathbb{R} \) and \( B \in \mathcal{B}(X) \), the following relation:

\[
\lim_{t \to \infty} \sup_{\sigma \in \Sigma} \text{dist}_X(U_\sigma(t, \tau)B, K) = 0
\]

where \( \text{dist}_X(A, B) \) is the usual Hausdorff semi-distance between subsets of a metric space \((X, d_X)\).

Definition 3.3. A closed set \( A_\Sigma \subset X \) is the uniform (w.r.t. \( \sigma \in \Sigma \)) attractor of the family of processes \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma \) if:

- \( A_\Sigma \) is uniformly (w.r.t. \( \sigma \in \Sigma \)) attracting (attracting property);
- \( A_\Sigma \) is contained in every other closed uniformly attracting set (minimality property).

Before stating the main result we will apply in this section, we still have to recall an additional definition (see [21]).

Definition 3.4. A family of processes \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma \) is uniformly (w.r.t. \( \sigma \in \Sigma \)) \( \omega \)-limit compact if for any \( \tau \in \mathbb{R} \) and any set \( B \in \mathcal{B}(X) \) the set

\[
B_t = \bigcup_{\sigma \in \Sigma} \bigcup_{s \geq t} U_\sigma(s, \tau)B
\]

is bounded for every \( t \) and \( \lim_{t \to \infty} \alpha(B_t) = 0 \) where \( \alpha \) is the Kuratowski measure of noncompactness defined by:

\[
\alpha(B) = \inf \{ \delta > 0 \mid B \text{ admits a finite cover by sets of diameter } \leq \delta \}.
\]

We can now report for the reader’s convenience the main result we will use in this section to prove the existence of a global attractor for system (1) (see [21, Section 2.3]).

Theorem 3.5. Let \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma \) be a uniformly (w.r.t. \( \sigma \in \Sigma \)) \( \omega \)-limit compact and \((X \times \Sigma, X)\)-weakly continuous family of processes acting in \( X \), let \( B_0 \) be a weakly compact (i.e., bounded) uniformly (w.r.t. \( \sigma \in \Sigma \)) weakly attracting set for \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma \) weakly attracting set for \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma \) be a weakly compact subset of some Banach space and let \( \{ T(t) \} \) be a weakly continuous invariant \( (T(t)\Sigma = \Sigma) \) semigroup on \( \Sigma \) satisfying the translation identity:

\[
U_\sigma(t + s, \tau + s) = U_{T(s)\sigma}(t, \tau) \quad \forall \sigma \in \Sigma, t, s, \tau \in \mathbb{R}, t \geq \tau, s \geq 0.
\]

Then the extended semigroup \( \{ S(t) \} \) defined on \( X \times \Sigma \) possesses the compact attractor \( \mathfrak{A} = \omega(B_0 \times \Sigma) \) (in the weak topology) which is strictly invariant with respect to \( \{ S(t) \} \): \( S(t)\mathfrak{A} = \mathfrak{A} \). Moreover:

- \( \Pi_X \mathfrak{A} = A_\Sigma \) is the uniform (w.r.t. \( \sigma \in \Sigma \)) attractor (in the strong topology) of the family of processes \( \{ U_\sigma(t, \tau) \}, \sigma \in \Sigma \);
- \( \Pi_{\Sigma} \mathfrak{A} = \Sigma \);
- the global attractor satisfies:

\[
\mathfrak{A} = \bigcup_{\sigma \in \Sigma} K_\sigma(0) \times \{ \sigma \};
\]

- the uniform attractor satisfies:

\[
A_\Sigma = \bigcup_{\sigma \in \Sigma} K_\sigma(0) = \omega_{0,\Sigma}(B_0)\]
where $K_\sigma(0)$ is the section at time $t = 0$ of the kernel $K_\sigma$ of the process $\{U_\sigma(t, \tau)\}$, that is:

$$K_\sigma(0) = \{u(0) \mid u \text{ is a bounded complete trajectory of } U_\sigma(t, \tau)\}.$$  

In addition we also recall a useful criterion to prove the uniform $\omega$-limit compactness for a given process (see [21, Theorem 2.3]).

**Proposition 1.** Let $X$ be a uniformly convex Banach space (any $L^p$ space with $p \neq 1, \infty$ will be suitable). Then the family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$ is uniformly (w.r.t. $\sigma \in \Sigma$) $\omega$-limit compact if and only if for any fixed $\tau \in \mathbb{R}$, $B \in B(X)$ and $\varepsilon > 0$ there exists $t_0 = t_0(\tau, B, \varepsilon) \geq \tau$ and a finite-dimensional subspace $X_1$ of $X$ such that:

- $\Pi (\cup_{\sigma \in \Sigma} \cup_{t \geq t_0} U_\sigma(t, \tau) B)$ is bounded;
- $|(\Pi - \Pi)(\cup_{\sigma \in \Sigma} \cup_{t \geq t_0} U_\sigma(t, \tau) u)|_X \leq \varepsilon$, $\forall u \in B$;

where $\Pi : X \to X_1$ is a bounded projector.

### 3.1. Bounded absorbing sets for system (1)

In order to apply the just introduced abstract theory to system (1), we need some preliminary estimates. Our first goal will be to find some absorbing sets for the trajectories of our system in various function spaces. Most of the results of this section are simple consequences of a suitable interpretation of the estimates obtained in the existence proofs.

In our case the symbol space will be generated by the two non-autonomous terms $g$ and $h$ by continuous time-shifts. Before introducing the symbol spaces we will use, we need to recall the following definition given first in [21].

**Definition 3.6.** Let $E$ be a Banach space. A function $f \in L^2_{loc}(\mathbb{R}; E)$ is normal if for every $\epsilon > 0$ there exists $\eta > 0$ such that:

$$\sup_{t \in \mathbb{R}} \int_{t-\eta}^{t+\eta} |\varphi(s)|^2_E ds \leq \epsilon.$$  

With $L^2_n(\mathbb{R}; E)$ we shall indicate the space of all normal functions taking values in $E$.

In particular we will use the following hypothesis:

- (8) if $g$ belongs to $L^2_n(0, \infty; V^*)$ and the boundary condition $h$ satisfies $h \in L^2_n(0, \infty; H^{1/2}(\partial \Omega))$ and $\partial h \in L^2_n(0, \infty; H^{-1/2}(\partial \Omega))$ and $|h| \leq 1$ a.e. we will have a “weaker” symbol space $\Sigma_0 = \mathcal{H}(g) \times \mathcal{H}(h)$;
- (9) if $g \in L^2_n(0, \infty; H)$, $h \in L^2_n(0, \infty; H^{5/2}(\partial \Omega))$, $\partial h \in L^2_n(0, \infty; H^{1/2}(\partial \Omega))$ and $|h| \leq 1$ a.e. we will consider the more regular symbol space $\Sigma_1 = \mathcal{H}(g) \times \mathcal{H}(h)$.

In both cases $\mathcal{H}(f)$ is the (weak) hull of $f$.

On account of the existence results of the previous section and, in particular, due to the compatibility condition $d_0 = h(0)$ on $\partial \Omega$, some care is needed in the definition of the phase space for our system. Indeed, we observe that we can decompose the state of our system as:

$$(u, d) = (u, \hat{d}) + (0, \bar{d})$$

where $\hat{d}$ and $\bar{d}$ are defined as in (31). Therefore we can consider the following natural phase space:

$$X = H \times H^1_0, \quad (u, \hat{d}) \in X.$$  

We also observe that a more geometric description of this phase space is possible. The situation for the velocity field being standard, we describe, for simplicity, only
the phase space framework for the order parameter. We introduce the map \( \Phi : H^1_0 \times \Sigma \to H^1 \), \( \Phi : (x, h(t)) \to x + d(0) \) which parametrizes a subset \( \tilde{X} \) of \( H^1 \). Since \( \Sigma \) is closed and regular, we observe that \( \tilde{X} \) is also closed. Moreover, from the translation invariance of \( \Sigma \), we deduce that \( \tilde{X} \) is itself invariant under the action of the family of processes \( U_\sigma(t, \tau), \sigma \in \Sigma \). The attraction properties, which we are going to prove in the present and subsequent sections, can therefore be interpreted as taking place on the parametrization of the phase space given by \( H^1_0 \times \Sigma \) in the metric locally induced by the map \( \Phi^{-1} \).

Thanks to theorem 2.2 and to the dissipation result which follows, we can also define the process associated with the solution operator of equation (1) acting on the phase space \( \tilde{X} \) indexed by a symbol \( \sigma \in \Sigma_0 \) (or \( \sigma \in \Sigma_1 \)).

**Theorem 3.7.** Under the regularity assumptions of theorem 2.2, system (1) admits a uniformly (w.r.t. \( \sigma \in \Sigma_0 \)) absorbing set \( B_0 \subset H \times H^1 \):

\[
B_0 = \{ (u, d) \in H \times H^1 \mid |u|^2 + |d|^2_{H^1} \leq \rho_0 \}
\]

where

\[
\rho_0 \Omega + \frac{16|\Omega|}{27C_0} + \frac{e^{C_0}}{e^{C_0} - 1} \left( \frac{1}{\nu} |g|^2_{L^2(\nu^*)} + C_0 |h|_{L^2(\nu^*)} \right) + C |h|_{L^2(\nu^*)}
\]

and the uniform (w.r.t. \( \sigma \in \Sigma_0 \)) absorbing time is given by:

\[
t_0(B) \frac{\ln(27C_0 e^4 \text{diam}(B)/8|\Omega|)}{C_0}
\]

Moreover, for \( t \geq t_0(B) \), we also have:

\[
\int_t^{t+1} |u(s)|^2_{H^1} ds + \int_t^{t+1} |d(s)|^2_{H^2} ds \leq \rho_1
\]

with

\[
\rho_1 = \max\{1, \nu\} \left( 2 \rho_0 + C_0 |h|_{L^2(\nu^*)} \right)
\]

**Proof.** Consider estimate (35). We only need to show that the two integrals are bounded if \( (g, h) \in \Sigma_0 \). Actually this can be easily shown under more general assumptions, namely it is enough that \( g \) and \( h \) are \( L^2 \) translation bounded (see [2, Section V.4]). Indeed we have:

\[
e^{-C_0 t} \int_0^t e^{C_0 s} |g(s)|^2_{\nu^*} ds = e^{-C_0 t} \sum_{i=0}^{n-1} \int_i^{i+1} e^{C_0 s} |g(s)|^2_{\nu^*} ds
\]

\[
\leq e^{-C_0 t} \sum_{i=0}^{n-1} e^{C_0 (i+1)} \int_i^{i+1} |g(s)|^2_{\nu^*} ds
\]

\[
\leq e^{-C_0 t} e^{C_0} \sum_{i=0}^{n-1} e^{C_0 i} \leq e^{-C_0 t} e^{C_0} \sum_{i=0}^{n-1} e^{C_0 i} \leq \frac{e^{C_0}}{e^{C_0} - 1} |g|^2_{L^2(\nu^*)}.
\]

Recalling a standard elliptic regularity estimate (cp. (31)) \( |\nabla d|_2 \leq |h|_{H^1/2} \) and that, under the present hypothesis, \( h \) is continuous and bounded with values in \( H^{1/2}(\partial \Omega) \), we obtain the absorbing set \( B_0 \) as claimed. We will denote by \( t_0(B) \) the absorption time of the bounded set \( B \) in \( B_0 \). In particular we observe that the absorption time \( t_0 \) can be obtained from the inequality:

\[
e^{-C_0 t} \left( |u_0|^2 + |d_0|^2_{H^1} \right) \leq \frac{8|\Omega|}{27C_0 e^4}
\]
which gives the claimed result.

In order to prove the second part of theorem 3.7, we only need to integrate equation (33) from \( t \) to \( t + 1 \) with \( t \) sufficiently large (it is enough to consider \( t \geq t_0 \)). This immediately gives the estimate we claimed. \( \Box \)

Analogously, starting from the result of section 2.3, we can prove the existence of absorbing sets bounded in the stronger topology of \( V \times H^2 \).

**Theorem 3.8.** Under the regularity assumptions of theorem 2.6, system (1) admits a uniformly (w.r.t. \( \sigma \)) absorbing set \( B_2 \subset V \times H^2 \):

\[
B_2 = \{ (u, d) \in V \times H^2 \mid |u|_{H^2}^2 + |d|_{H^2}^2 \leq \rho_2 \}
\]

and a uniform (w.r.t. \( \sigma \)) absorbing time given by \( t_2(B) = 1 + t_0(B) \). Moreover we have:

\[
\int_t^{t+1} |u(s)|_{H^2}^2 \, ds + \int_t^{t+1} |d(s)|_{H^2}^2 \, ds \leq \rho_3
\]

where \( \rho_2 \) and \( \rho_3 \) depend only on \( \nu, \varepsilon, \Omega, |h|_{L^2(R; H^{1/2})}, |\partial_t h|_{L^2}^{1/2}, |r|_{L^2}^{1/2}, |g|_{L^2}^{1/2}, |\partial_t g|_{L^2}^{1/2} \).

**Proof.** Consider estimate (9). If we use the uniform Gronwall’s inequality (see, for example, [30, Chap. 3, Sec. 1.1.3]), we can deduce for all \( t \geq 0 \):

\[
|\nabla u(t + \varepsilon)|_2^2 + |\Delta \tilde{d}(t + \varepsilon) - f(d(t + \varepsilon))|_2^2 \leq (\frac{1}{\varepsilon} \int_t^{t+\varepsilon} A(s) \, ds + K \varepsilon + \frac{2}{|\varepsilon|} \int_t^{t+\varepsilon} |\tilde{g}(s)|_2^2 \, ds) \cdot e^{C(\varepsilon) \int_t^{t+\varepsilon} A(s) \, ds}
\]

where \( A(t) \equiv |\nabla u(t)|_2^2 + |\Delta \tilde{d}(t) - f(d(t))|_2^2 \) satisfies \( A \in L^1(0, T) \) for all \( T > 0 \). By choosing \( \varepsilon = 1 \) and by recalling the regularity estimate (24) for the lifting solution \( \tilde{d} \), we then easily deduce the existence of the strong absorbing set \( B_2 \).

As in the proof of theorem 3.7, in order to obtain the second part of the theorem we only need to integrate estimate (8) from \( t \) to \( t + 1 \) with \( t \) sufficiently large (again it is enough to suppose \( t \) greater than the absorbing time in \( B_2 \)). \( \Box \)

We end this section introducing another absorbing set which will prove to be useful when dealing with exponential attractors in the next section.

**Corollary 2.** Under the regularity assumptions of theorem 2.6, system (1) admits the following uniform (w.r.t. \( \sigma \)) long-term bound:

\[
|\partial_t d|_2^2 \leq \rho_4 \quad \forall t \geq t_2(B)
\]

where

\[
\rho_4 = C \rho_0 \rho_2 + 3 \rho_2^2 + \frac{4}{9 \varepsilon^4} |\Omega|.
\]

**Proof.** This estimate can be obtained directly from the equation for the order parameter field in (1). Indeed we have:

\[
|\partial_t d|_2^2 \leq 3|\partial_t u|_2^2 |\nabla d|_2^2 + 3|\Delta d|_2^2 + 3|f(d)|_2^2
\]

\[
\leq 3|u|_2^2 |\nabla d|_2^2 + 3|\Delta d|_2^2 + 3|f(d)|_2^2
\]

\[
\leq C|u|_2^2 |\nabla u|_2^2 |\nabla d|_2^2 |H^2 + 3|\Delta d|_2^2 + 3\rho_2^2 |f(d)|_2^2
\]

\[
\leq C \rho_0 \rho_2 + 3 \rho_2^2 + \frac{4}{9 \varepsilon^4} |\Omega|
\]

which is the desired result. \( \Box \)
3.2. A smooth attractor. We can now apply theorem 3.5 to system (1) under strong regularity assumptions. More precisely we will prove the following result.

**Theorem 3.9.** Suppose that (5) and (9) hold, then the process \( \{U_{(g,h)}(t,\tau)\} \) generated by the solution operator of problem (1) possesses a compact uniform (w.r.t. \((g,h) \in \Sigma_1\)) attractor \( \mathcal{A}_{\Sigma_1} \) in \( V \times H^2 \) which uniformly (w.r.t. \((g,h) \in \Sigma_1\)) attracts the bounded sets in \( X \) in the product norm of \( X \). Moreover we have:

\[
\mathcal{A}_{\Sigma_1} = \bigcup_{(g,h) \in \Sigma_1} \mathcal{K}_{(g,h)}(0)
\]

where \( \mathcal{K}_{(g,h)} \) is the kernel of the process \( \{U_{(g,h)}(t,\tau)\} \) and where \( \mathcal{K}_{(g,h)} \) is nonempty for all \((g,h) \in \Sigma_1\).

**Remark 4.** From this result we deduce that all the solutions to (1) belonging to the kernel of the solution process are strong and globally bounded. We therefore deduce that system (1) holds a.e. on the kernel.

**Proof.** We begin our proof by observing that these regularity assumptions on the boundary term \( h \) imply that \( h \in L^\infty(0,\infty; H^{1/2}(\partial \Omega)) \).

Actually, with the above assumptions, we only have to prove \( \omega \)-limit compactness and weak continuity of the process defined by the solution operator in order to apply theorem 3.5 and prove our claim.

We start with \( \omega \)-limit compactness. We will consider again the lifted system (32) (forgetting all \( m \)'s). First of all, however, we prove that the lifted term \( \partial_t \nabla d \) is bounded and normal. From problem (31) tested against a function \( v \in H^1(\Omega) \), a simple integration by parts yields:

\[
\int_\Omega \nabla d : \nabla v = \int_{\partial \Omega} \partial_v d \cdot v.
\]

By choosing \( v = \hat{d} \) we eventually get:

\[
|\nabla \hat{d}|_2^2 \leq |\partial_v \hat{d}|_{H^{-1/2}} |h|_{H^{1/2}}
\leq C |\nabla \hat{d}|_2 |h|_{H^{1/2}} + C |\hat{d}|_2 |h|_{H^{1/2}}
\leq \frac{1}{2} |\nabla \hat{d}|_2^2 + C \left(1 + |h|_{H^{1/2}}^2\right).
\]

We then easily obtain:

\[
\sup_{t \in \mathbb{R}} \int_t^{t+\epsilon} |\nabla \hat{d}|_2^2 \leq C \sup_{t \in \mathbb{R}} \int_t^{t+\epsilon} |h|_{H^{1/2}}^2 + C \epsilon.
\]

If we apply this last estimate to \( \partial_t \hat{d} \) instead of \( \hat{d} \) with \( \partial_t h \) substituted to \( h \), we have then proved that \( \partial_t \nabla d \in L^2_\mathbb{R}(\mathbb{R}, L^2) \) since \( \partial_t h \in L^2_\mathbb{R}(\mathbb{R}, H^{1/2}) \).

We now recall proposition 1, which gives a straightforward way to check \( \omega \)-limit compactness for the process. Thanks to the Hilbert setting which provides a natural norm-reducing projection onto any linear subspace and on account of the absorbing sets previously identified, the first assumption of proposition 1 has already been verified. We still have to control for the “dissipativeness” of the higher modes. As subspaces we will consider \( V^n \) for the velocity field and the space \( D^m \) spanned by the first \( m \) eigenfunctions of the laplacian with Dirichlet homogeneous boundary conditions in \( \Omega \). Let \( \{\lambda_n\} \) and \( \{\mu_m\} \) be the ascending sequences of eigenvalues respectively for Stokes’s problem and Laplace’s problem on \( \Omega \) and let \( P_n \) and \( Q_m \)
be the projections on $V^n$ and $D^m$ respectively. In what follows, we set $u_1 = P_n u$, $d_1 = Q_n d$ and $u_2 = u - u_1$, $d_2 = d - d_1$.

Consider again the equation for the velocity field in (32) and take its scalar product in $L^2$ with $-\Delta u_2$. Using the orthogonality of the chosen base (notice, for example, that $(\Delta u, \Delta u_2) = |\Delta u_2|_2^2$), we obtain:

$$\frac{1}{2} \frac{d}{dt} |\nabla u_2|_2^2 + \nu |\Delta u_2|_2^2 = (u \cdot \nabla) u, \Delta u_2) + ((\nabla d)^t \Delta d, u_2) - (g(t), \Delta u_2). \quad (11)$$

As usual, we have to estimate all terms on the right hand side of this last expression. In order to obtain the desired estimates we recall a useful interpolation (see [1]).

**Lemma 3.10.** Let $f \in H^2(\Omega)$, let $\Omega \subset \mathbb{R}^2$ have a compact smooth boundary, then

$$|f|_{L^\infty} \leq C |f|_{H^1} \left( 1 + \ln \frac{|f|_{H^2}}{|f|_{H^1}} \right)^{1/2}$$

where the constant $C$ depends only on the domain $\Omega$.

We start by analyzing the well-known trilinear term of Navier-Stokes equations. We have:

$$|((u \cdot \nabla) u, \Delta u_2)| \leq |((u_1 \cdot \nabla) u, \Delta u_2)| + |((u_2 \cdot \nabla) (u_1 + u_2), \Delta u_2)|$$

$$\leq |u_1|_\infty |\nabla u_2|_2 |\Delta u_2|_2 + |u_2|_\infty |\nabla u_1|_2 |\Delta u_2|_2$$

$$\leq C |\nabla u_1|_2 \left( 1 + \ln \frac{|\Delta u_1|_2^2}{|\nabla u_2|_2^2} \right)^{1/2} |\Delta u_2|_2 |\nabla u_2|_2$$

$$+ C |u_2|_2^{1/2} |\Delta u_2|_2^{3/2} |\nabla u_2|_2.$$

Recalling the absorbing sets identified in the previous sections and noticing that $|\Delta u_2|_2^2 \leq \lambda_{n+1} |\nabla u_2|_2^2$, we finally obtain:

$$|((u \cdot \nabla) u, \Delta u_2)| \leq C \rho_2 (1 + \ln \lambda_{n+1})^{1/2} |\Delta u_2|_2 + C \rho_0^{1/4} \rho_2^{-1/2} |\Delta u_2|_2^{3/2}$$

$$\leq \frac{\nu}{12} |\Delta u_2|_2^2 + \frac{C \rho^2}{\nu} + \frac{C \rho_0 \rho_2^2}{\nu^3},$$

where $L \doteq 1 + \ln \lambda_{n+1}$.

The other nonlinear term can be estimated analogously as follows:

$$|((\nabla d)^t \Delta d, u_2)|$$

$$\leq |\nabla d_1|_\infty |\Delta d_2|_2 |\Delta u_2|_2 + |\nabla d_2|_4 (|\Delta d_1|_4 + |\Delta d_2|_4) |\Delta u_2|_2$$

$$\leq C |\nabla d_1|_{H^1} \left( 1 + \ln \frac{|\nabla d_1|_2^2}{|\nabla d_2|_2^2} \right)^{1/2} |\Delta u_2|_2 |\Delta d_2|_2$$

$$+ C |\nabla d_2|_2^{1/2} |\nabla d_2|_2^{1/2} \left( |\Delta d_1|_2^{1/2} |\Delta d_1|_{H^1} + |\Delta d_2|_2^{1/2} |\Delta d_2|_{H^1} \right) |\Delta u_2|_2$$

$$\leq C \rho M^{1/2} |\Delta u_2|_2$$

$$+ C \rho_0^{1/4} \rho_2^{1/2} \left( |\nabla \Delta d_1|_{H^1} + |\nabla \Delta d_2|_{H^1} \right) |\Delta u_2|_2 + C \rho_0^{1/4} \rho_2^{3/4} |\Delta u_2|_2$$

where $M \doteq 1 + \ln \mu_{m+1}$. By recalling that $|\nabla \Delta d_1|_2 \leq \mu_{m+1} |\Delta d_1|_2$, we get:

$$|((\nabla d)^t \Delta d, u_2)|$$

$$\leq \frac{\nu}{12} |\Delta u_2|_2^2 + \frac{1}{12} |\nabla \Delta d_2|_2^2 + C(\rho_0, \rho_2, M, \nu) + C \rho_0^{1/2} \rho_2^{3/2} \nu^{1/2} \rho_{m+1}.$$
The last term on the right hand side of equation (11) is easily dealt with:

\[ |(g, \Delta u_2)| \leq |g|_2|\Delta u_2|_2 \leq \frac{\nu}{12} |\Delta u_2|^2 + \frac{3}{\nu} |g|_2^2. \]

Putting everything together, we get the first half of the desired estimate:

\[ \frac{d}{dt}|\nabla u_2|^2 + \frac{3}{2} \nu |\Delta u_2|^2 \leq C(\rho_0, \rho_2, L, M, \nu) + \frac{6}{\nu} |g|_2^2 + \frac{1}{6} |\nabla \Delta \ddot{d}_2|^2 \frac{1}{\nu} + \frac{C^2 \rho_0 \rho_2^2 \nu^{1/2}}{\nu^{1/2} \mu_{m+1}}. \tag{12} \]

We now turn our attention to the equation for the order parameter. Testing the second equation in (32) with \( \Delta \Delta \ddot{d}_2 \), integrating by parts and using the orthogonality of the eigenbasis of the laplacian, we get:

\[
\frac{1}{2} \frac{d}{dt} |\Delta \ddot{d}_2|^2 + |\nabla \Delta \ddot{d}_2|^2 \\
= \left( |\nabla u^t \cdot \nabla d| \nabla \Delta \ddot{d}_2 + (\nabla \nabla \cdot u, \nabla \Delta \ddot{d}_2) \\
+ (\nabla f(d), \nabla \Delta \ddot{d}_2) + \left( \partial_t \nabla d, \nabla \Delta \ddot{d}_2 \right) - \right)_{H^{-1/2}} \left( \partial_t \ddot{d}, \partial_{\nu} \Delta \ddot{d}_2 \right)_{H^{-1/2}}.
\]

As with the equation for the velocity field, we now have to bound all terms on the right hand side of this last equality.

\[
|\left( |\nabla u^t \cdot \nabla d|, \nabla \Delta \ddot{d}_2 \right)| \\
\leq |\nabla u_2| \nabla \ddot{d}_2 \nabla \Delta \ddot{d}_2 + |\nabla u_2| |\nabla \ddot{d}_2| \nabla \Delta \ddot{d}_2 \\
\leq C \rho_2 |\nabla \ddot{d}_2|^2 + \frac{1}{4} \rho_2 \left( |\nabla u_1|^2 + |\nabla u_2|^2 \right) |\nabla \Delta \ddot{d}_2|^2 \\
\leq \frac{1}{12} |\nabla \Delta \ddot{d}_2|^2 + \frac{|\Delta u_2|^2}{10} + \frac{C}{\nu} \rho_0^2 + \frac{C}{\nu} \rho_0^2 + C \rho_0^{1/2} \rho_2^{3/2} \lambda_{n+1}^{1/2}.
\]

The second term is dealt with in a similar way. We only recall that usual elliptic regularity results for problem (31) give \( |d|_{H^3} \leq C(|\nabla \ddot{d}_2|^2 + |h|_{H^{1/2}}^2) \):

\[
|\left( |\nabla \nabla \cdot u|, \nabla \Delta \ddot{d}_2 \right)| \\
\leq |d|_{H^3} |u_1|_{H^1} \nabla \Delta \ddot{d}_2 + |d|_{H^3} |u_2|_{H^1} \nabla \Delta \ddot{d}_2 \\
\leq C \rho_2 |\nabla \ddot{d}_2|^2 L \nabla |u_1|^2 \nabla \Delta \ddot{d}_2 + C \rho_0^2 \rho_2^{1/2} \lambda_{n+1}^{1/2} |\nabla \Delta \ddot{d}_2|^2 \\
\leq \frac{1}{12} |\nabla \Delta \ddot{d}_2|^2 + \frac{|\Delta u_2|^2}{10} + \frac{C}{\nu} \rho_0 \rho_2.
\]

Finally the last two bulk terms can be estimated as follows:

\[
|\left( |\nabla f(d), \nabla \Delta \ddot{d}_2 \right)| \leq C |\nabla d| \nabla \Delta \ddot{d}_2 \leq \frac{1}{12} |\nabla \Delta \ddot{d}_2|^2 + C \rho_0
\]

and

\[
|\left( \partial_t \nabla d, \nabla \Delta \ddot{d} \right)| \leq \frac{1}{12} |\nabla \Delta \ddot{d}_2|^2 + C |\partial_t \nabla \ddot{d}_2|^2.
\]

In order to control the boundary term, we can write:

\[
\left| \left( \partial_{\nu} \ddot{d}, \partial_{\nu} \Delta \ddot{d}_2 \right) \right|_{H^{-1/2}} \leq C \left| \partial_{\nu} h \right|_{H^{1/2}(\partial \Omega)} |\Delta \ddot{d}_2|_{H^1} \\
\leq \frac{1}{12} |\nabla \ddot{d}_2|^2 + C \rho_2 + C \left| \partial_{\nu} h \right|_{H^{1/2}(\partial \Omega)}.
\]
Adding everything together, we eventually get:

\begin{align*}
  \frac{1}{2} \frac{d}{dt} |\Delta \tilde{d}_2|^2 + \frac{7}{12} |\nabla \Delta \tilde{d}_2|^2 \\
  \leq C(\rho_0, \rho_2, L, M, \nu) + C\rho_0^{1/2} \rho_2^{3/2} \lambda_{n+1}^{1/2} + C\rho_0^{1/2} \rho_2^{3/2} \mu_{m+1}^{1/2} \\
  + \frac{\nu}{4} |\Delta u_2|^2 + C|\partial_t \nabla u_2|^2 + C|\partial_t h|_{H^{1/2}(\partial \Omega)}^2.
\end{align*}

By recalling estimate (12) and adding the last inequality we have obtained, we find the desired bound on the higher modes of our solution:

\begin{align*}
  \frac{d}{dt} (|\nabla u_2|^2 + |\Delta d_2|^2) + \nu |\nabla u_2|^2 + |\Delta d_2|^2 \\
  \leq C(\rho_0, \rho_2, \nu) + C(\rho_0, \rho_2, \nu)(M + L) + C(\rho_0, \rho_2, \nu)(\lambda_{n+1}^{1/2} + \mu_{m+1}^{1/2}) \\
  + \frac{6}{\nu} |g|^2 + C|\partial_t \nabla d_2|^2 + C|\partial_t h|_{H^{1/2}(\partial \Omega)}^2.
\end{align*}

From Poincaré’s inequality in V and \( H_0^1 \) we have \( |\Delta u_2|^2 \geq \lambda_{n+1} |\nabla u_2|^2 \) and similarly \( |\nabla \Delta d_2|^2 \geq \mu_{m+1} |\Delta d_2|^2 \). By setting \( \kappa = \min \{ \nu \lambda_{n+1}, \mu_{m+1} \} \) and using Gronwall’s inequality we finally get:

\begin{align*}
  |\nabla u_2(t)|^2 + |\Delta d_2(t)|^2 \\
  \leq (|\nabla u_2(t_0)|^2 + |\Delta d_2(t_0)|^2) e^{-\kappa(t-t_0)} + C(\rho_0, \rho_2, \nu) \frac{M + L}{\kappa} \\
  + \frac{C(\rho_0, \rho_2, \nu)}{\kappa} (\lambda_{n+1}^{1/2} + \mu_{m+1}^{1/2}) + \frac{6}{\nu} \int_{t_0}^t e^{-\kappa(t-s)} |g(s)|^2 ds \\
  + C \int_{t_0}^t e^{-\kappa(t-s)} |\partial_t \nabla d(s)|^2 ds + C \int_{t_0}^t e^{-\kappa(t-s)} |\partial_t h(s)|_{H^{1/2}}^2 ds.
\end{align*}

All terms on the right hand side of last inequality can be made arbitrarily small by choosing \( n \) and \( m \) sufficiently large such that \( \nu \lambda_n \approx \mu_m \) and recalling estimate (10) for translation bounded functions and the normality assumption (see [21]). This proves \( \omega \)-limit compactness.

To complete the proof of the existence of the global attractor we still have to control weak continuity of the process with respect to initial data and to the symbol. Our argument follows [24, Lemma 2.1] with the obvious changes.

Consider again the lifted problem (32). Obviously the lifting problem is weakly continuous with respect to the initial boundary data so we have to care only of the lifted equation. Let \( \{u_{n_0}\} \subset V, \ u_{n_0} \to u_0, \ \{d_{n_0}\} \subset H^2, \ d_{n_0} \to d_0 \) and \( \{ (g_n, h_n) \} \subset \Sigma_1, (g_n, h_n) \rightharpoonup (g, h) \) be weakly convergent sequences of initial data and symbols. We want to prove \( U_{(g_0, h_0)}(t, \tau)(u_{n_0}, d_{n_0}) \to U_{(g, h)}(t, \tau)(u_0, d_0) \) in \( V \times H^2 \).

Let \( (u_{n}(t), d_{n}(t)) = U_{(g_n, h_n)}(t, \tau)(u_{0n}, d_{0n}) \). From the absorbing estimates of section 3.1 we know that \( \{u_{n}(t), d_{n}(t)\} \) is bounded in \( L^\infty(\tau, \infty; V \times H^2) \) and in \( L^2_{loc}(\tau, \infty; H^2 \times H^3) \). Moreover \( \{d_{n}\} \) is bounded in \( L^\infty(\tau, \infty; L^\infty) \). Directly from the equation we also get that \( \{\partial_t u_n, \partial_t d_n\} \) is bounded in \( L^2_{loc}(\tau, \infty; H \times H^1) \).
Next step will be proving the precompactness of the sequence \( \{(u_n(t), \dot{d}_n(t))\} \) in \( L^2_{\text{loc}}((\tau, \infty); \mathbf{V} \times \mathbf{H}^2) \). Actually we have:
\[
(u_n(t+a) - u_n(t), \mathbf{v}) = \int_{\tau}^{t+a} \left( \partial_t u_n(s), \mathbf{v} \right) \, ds
\]
\[
\leq a^{1/2} |\mathbf{v}|_2 \|\partial_t u_n\|_{L^2_{\text{loc}}(\mathbf{L}^2)} \leq Ca^{1/2} |\mathbf{v}|_2
\]
for all \( \mathbf{v} \in \mathbf{L}^2 \) and for a.e. \( t \in [\tau, T] \). By formally choosing \( \mathbf{v} = -\Delta (u_n(t+a) - u_n(t)) \) (actually we should use as usual the approximation 25) and integrating by parts, we obtain:
\[
\int_{\tau}^{T-a} |\nabla (u_n(t+a) - u_n(t))|^2 dt
\]
\[
\leq C_T a^{1/2} \int_{\tau}^{T-a} |\Delta (u_n(t+a) - u_n(t))|^2 dt
\]
\[
\leq C_T a^{1/2} \int_{\tau}^{T-a} |\Delta (u_n(t+a) - u_n(t))|^2 dt \leq C_T a^{1/2}.
\]
Since \( \{u_n\} \) is bounded in \( L^2(\tau, T; \mathbf{H}^2) \) it follows from [26, theorem 3] that \( \{u_n\} \) is precompact in \( L^2(\tau, T; \mathbf{V}) \) for all \( T > \tau \).

We can proceed analogously for the order parameter field by considering
\[
(\nabla (d_n(t+a) - \dot{d}_n(t)), \mathbf{w}) = \int_{t}^{t+a} \left( \partial_t \nabla \dot{d}_n(s), \mathbf{w} \right) \, ds
\]
\[
\leq a^{1/2} |\mathbf{w}|_2 \|\partial_t \nabla \dot{d}_n\|_{L^2_{\text{loc}}(\mathbf{L}^2)} \leq Ca^{1/2} |\mathbf{w}|_2
\]
for any \( \mathbf{w} \in \mathbf{L}^2 \) and a.e. \( t \in [\tau, T] \). If we take \( \mathbf{w} = -\nabla \Delta (d_n(t+a) - d_n) \in \mathbf{L}^2 \) a.e. \( t \geq \tau \), we get:
\[
\int_{\tau}^{T-a} |\Delta (\dot{d}_n(t+a) - \dot{d}_n(t))|^2 dt
\]
\[
\leq \int_{\tau}^{T-a} \|\partial_t \nabla \dot{d}_n(t+a) - \partial_t \nabla \dot{d}_n(t+a)\|_{\mathbf{H}^1/2} \, dt
\]
\[
+ C a^{1/2} \int_{\tau}^{T-a} |\nabla \Delta (\dot{d}_n(t+a) - \dot{d}_n)|_{\mathbf{H}^1} \, dt.
\]
However, noting that \( \Delta \dot{d} - f(h) - \partial_t h|_{\partial \Omega} = 0 \), the first term on the right hand side of this inequality can be estimated as follows:
\[
\int_{\tau}^{T-a} \|\partial_t \nabla \dot{d}_n(t+a) - \partial_t \nabla \dot{d}_n(t+a)\|_{\mathbf{H}^1/2} \, dt
\]
\[
\leq C \int_{\tau}^{T-a} |\dot{d}_n(t+a) - \dot{d}_n(t)|_{\mathbf{H}^1} \cdot |f(h_n(t+a)) - f(h_n(t)) + \partial_t h_n(t+a) - \partial_t h_n(t)|_{\mathbf{H}^{1/2}} \, dt.
\]
From estimate (13) we immediately have:
\[
|\nabla (d_n(t+a) - \dot{d}_n(t))|^2 \leq C_T a^{1/2}
\]
and therefore, on account of the regularity assumptions on \( h \), we deduce:
\[
\int_{\tau}^{T-a} \|\partial_t \nabla (d_n(t+a) - \dot{d}_n(t))\|_{\mathbf{H}^1/2} \, dt \leq C_T a^{1/4}.
\]
Since we already know that \( \{d_n\} \) is bounded in \( L^2(\tau, T - a; H^3) \), using the same lemma as before, we conclude that \( \{d_n\} \) is precompact in \( L^2(\tau, T - a; H^3) \).

From the boundedness and compactness of the sequences just proved, by means of a diagonal extraction process, we can find a subsequence of \( \{u_n, d_n\} \) that converges weakly* in \( L^\infty(\tau, \infty; V \times H^2) \), weakly in \( L^2_{loc}([\tau, \infty); H^2 \times H^3) \), strongly in \( L^2_{loc}(\tau, \infty); V \times H^2 \) to \( (u, d) \) and such that \( \{d_n\} \) converges weakly* in \( L^\infty(\tau, \infty; L^\infty) \). We note that \( (u, d) \) solves equation (1) (the passage to the limit in the equation is analogous to that treated in appendix A).

From the strong convergence we have \( (u_n(t), d_n(t)) \to (u(t), d(t)) \) strongly in \( V \times H^2 \) for a.e. \( t \geq \tau \). We therefore have:

\[
(\nabla u_n(t), v) \to (\nabla u(t), v) \quad \text{and} \quad (\Delta d_n(t), w) \to (\Delta d(t), w)
\]

for almost every \( t \geq \tau \) and any regular pair of functions \((v, w)\). We note that, from the previous estimates, \( (\nabla u_n(t), v) \) and \( (\Delta d_n(t), w) \) are equicontinuous and equicontinuous as functions of \( t \). Therefore the convergences hold for all \( t \geq \tau \), i.e. we have obtained weak continuity for the solution process we are studying. \( \square \)

3.3. A less regular attractor. We now want to investigate what happens when we consider less regular forcing terms of critical regularity. The main result we shall obtain is the following.

**Theorem 3.11.** Suppose that (1) and (8) hold, then the process \( \{U_{(g, h)}(t, \tau)\} \) associated to the solution of problem (1) possesses a compact uniform (w.r.t. \( (g, h) \in \Sigma_0 \)) attractor \( \mathcal{A}_{\Sigma_0} \) in \( X \) which uniformly (w.r.t. \( (g, h) \in \Sigma_0 \)) attracts the bounded sets in \( X \) in the norm of \( X \). Moreover we have:

\[
\mathcal{A}_{\Sigma_0} = \bigcup_{(g, h) \in \Sigma_0} K_{(g, h)}(0)
\]

where \( K_{(g, h)} \) is the kernel of the process \( \{U_{(g, h)}(t, \tau)\} \) and where \( K_{(g, h)} \) is nonempty for all \( (g, h) \in \Sigma_0 \).

**Proof.** As in the proof of theorem 3.9, we only have to check \( \omega \)-limit compactness and weak continuity in order to apply theorem 3.5. The argument leading to weak continuity can be carried over to the current setting with no significant changes. However, checking \( \omega \)-limit compactness involves a slightly more subtle estimate. Indeed, due to the structure of the nonlinear terms and in particular to the convective term \( u \cdot \nabla u \), the direct approach adopted in the last section does not lead to any useful estimate in this case.

We look for a bound on the time derivative of the solution fields in the natural weak norms. From the equation we have:

\[
|\partial_t u|_{H^s} \leq |(u \cdot \nabla) u|_{H^s} + \nu |\Delta u|_{H^s} + |\nabla d^t \Delta d|_{H^s} + |g|_{H^s},
\]

\[
|\partial_t d|_{2} \leq |(u \cdot \nabla) d|_{2} + |\Delta d|_{2} + |f(d)|_{2},
\]

where the nontrivial terms on the right hand side can be bounded as follows:

\[
|\Delta u|_{H^s} = |\nabla u|_{2}^2,
\]

\[
|\nabla d^t \Delta d|_{H^s} \leq C|d^t|_{2}^{1/2} |d|_{H^1}^{1/2} \leq C \rho_0^{1/2} |d|_{H^2}^{1/2},
\]

\[
|(u \cdot \nabla) d|_{2} \leq |u|_{4} |\nabla d|_{4} \leq C|u|_{4}^{1/2} |\nabla u|_{2}^{1/2} |\nabla d|_{2}^{1/2} |d|_{H^2}^{1/2}
\]

\[
\leq C \rho_0 |\nabla u|_{2} + C \rho_0 |d|_{H^2}.
\]
We observe that $|\mathbf{d}|_{H^2}^2 \leq C|\Delta \mathbf{d}|_{H^3}^2 + |\mathbf{h}|_{H^3/2}^2$ and therefore we have $(\partial_t u, \partial_t d) \in L^2_{\text{loc}}([\tau, \infty); \mathbf{V}^* \times \mathbf{L}^2)$. Moreover, the norm of $(\partial_t u, \partial_t d)$ is uniformly bounded in $[t, t + \delta t]$ w.r.t. $t \geq \tau$. Thanks to [26, Corollary 4], we deduce that

$$B_{[t, t + \delta t]} = \{(u(s), d(s)) = U(s, \tau)(u_\tau, d_\tau), (u_\tau, d_\tau) \in B_0, s \in [t, t + \delta t]\}$$

is precompact in $L^2(t, t + \delta t; H \times H^1)$.

From the precompactness of $B_{[t, t + \delta t]}$ we deduce that there exists a finite number of pairs $(u_1, d_1), \ldots, (u_N, d_N)$ such that for any $(u, d) \in B_{[t, t + \delta t]}$ there exists an $i$ that verifies

$$\int_t^{t + \delta t} (|u - u_i|^2 + |\nabla d - \nabla d_i|^2) \leq \epsilon \delta t.$$

Therefore there exists a time $\tilde{t} \in [t, t + \delta t]$ such that

$$|u(\tilde{t}) - u_i(\tilde{t})|^2 + |\nabla d(\tilde{t}) - \nabla d_i(\tilde{t})|^2 \leq \epsilon.$$

We now use the continuous dependence estimate (4) and get:

$$|u(t + \delta t) - u_i(t + \delta t)|^2 + |\nabla d(t + \delta t) - \nabla d_i(t + \delta t)|^2 \leq C \left(\left(\int_t^{t + \delta t} |u(\tilde{t}) - u_i(\tilde{t})|^2 + |\nabla d(\tilde{t}) - \nabla d_i(\tilde{t})|^2\right) + C \left(\sup_{t \geq \tau} \left(\frac{3}{2} |g|^2_{V^*} + C|\partial_t h|^2_{H^{-1/2}} + C |h|^2_{H^{1/2}}\right) ds\right)\right)$$

where all constants depend only on $\rho_0, \rho_1$ and are bounded w.r.t. $\delta t \leq 1$. Using the normality assumption (2) and the precompactness of the trajectories, we can bound the left hand side of the last inequality by a fixed constant times $\epsilon$. We have thus proven that $B_T = U_{T, \tau} B_0$ is compact for sufficiently big $T - \tau$, uniformly w.r.t. $\tau \in \mathbb{R}$. Since $B_0$ is absorbing, this also proves the $\omega$-limit compactness for the process. \qed

4. **Exponential Attractors.** Global attractors are not necessarily the only description of a dissipative dynamical system. A feature one usually wants to guarantee is an exponential attracting rate of trajectories to the attractor preserving the finite dimension. Moreover, some sort of continuous dependence of the attractor on the data is hoped for. This means, e.g., that small changes in the form of the forcing terms should not cause any relevant modification of the attracting sets (we refer to [23] for a complete overview of the subject).

One possible solution that guarantees these properties is given by exponential attractors which were proposed in the 90’s by Eden et al. (see [7]). Although some important drawbacks (notably the lack of uniqueness) still remain, this theory can easily be applied to the general setting of Banach spaces (see [8]) by using rather natural estimates on the solutions. We review briefly the standard theory recalling the results we shall use later.

**Definition 4.1.** Let $E$ be a Banach space. A compact set $\mathcal{M} \subset E$ is an exponential attractor for the semigroup $\{S(t)\}$ if it has finite fractal dimension, it is positively invariant and it attracts exponentially the bounded subsets of $E$.

In order to prove the existence of such absorbing sets we introduce the following dissipativity notion for a semigroup.
Definition 4.2. Let $E$ and $E_1$ be Banach spaces with $E_1$ compactly embedded in $E$, let $X$ be a bounded subset of $E_1$ and let $S : E \to E$. Then $S$ enjoys the smoothing property on $X$ if:

$$|Su - Sv|_{E_1} \leq C|u - v|_E \quad \forall u, v \in X.$$ 

Definition 4.3. Let $E$ and $E_1$ be Banach spaces with $E_1$ compactly embedded in $E$ and let $X$ be a bounded subset of $E_1$. Given positive constants $\delta$ and $K$, a (nonlinear) operator $S : E \to E$ is a smoothing operator, $S \in S_{\delta,K}(X)$, if

- $SO_\delta(X) \subset X$ where $O_\delta(X)$ is a neighborhood of $X$ of radius $\delta$ in the topology of $E_1$;
- $S$ enjoys the smoothing property on $O_\delta(X)$, that is:

$$|Su - Sv|_{E_1} \leq C|u - v|_E \quad \forall u, v \in O_\delta(X).$$

We can now state the main results we will use in this paper (see [8] for a proof).

Theorem 4.4. For every $S \in S_{\delta,K}(X)$, there exists an exponential attractor $M_S$ in the topology of $E_1$, that is:

1. $\dim_F(M_S) \leq C$;
2. $SM_S \subseteq M_S$;
3. $\text{dist}_{E_1}(S(n)X,M_S) \leq Ce^{-\alpha n}$, $n \in \mathbb{N}$.

Moreover, the map $S \mapsto M_S$ can be chosen such that it is Hölder continuous in the following sense:

$$\text{dist}_{M_{s_1}}(M_{s_2},M_{s_3}) \leq C|S_1 - S_2|^\beta.$$ 

Finally, $\alpha$, $\kappa$ and all other constants which appear in the preceding estimates depend only on $X$, $\delta$ and $K$, but they are otherwise independent of the particular semigroup $S \in S_{\delta,K}(X)$.

Lemma 4.5. Let $\{S(t)\}$, $S(t) : X \to X$ be a semigroup and let $t^*$ be a positive real number. If the discrete time semigroup generated by $S(t^*)$ has an exponential attractor $M^*$ and if the mapping $S(t)u$ is Lipschitz (or Hölder) on $[0,t^*] \times X$, then the set

$$\mathcal{M} = \bigcup_{t \in [0,t^*]} S(t)M^*$$

is an exponential attractor for the continuous-time semigroup $\{S(t)\}$ on $X$.

4.1. A discrete exponential attractor for system (1). This section and the next are devoted to prove the existence of an exponential attractor for system (1). We start by defining the symbol space we will use in this section.

Definition 4.6. Let $\Xi$ be a Banach space, let $(\alpha^1, \ldots, \alpha^k)$ be a $k$-tuple of rationally independent real numbers and let $\phi : \mathbb{R}^k \to \Xi$ be a Lipschitz continuous function which is $2\pi$-periodic in each argument, that is:

$$\phi(\omega^1, \ldots, \omega^i + 2\pi, \ldots, \omega^k) = \phi(\omega^1, \ldots, \omega^i, \ldots, \omega^k).$$

Then $\sigma(s) = \phi(\alpha^1 s, \alpha^2 s, \ldots, \alpha^k s)$ is said to be a (Lipschitz) quasi-periodic function with values in $\Xi$.

We recall that the translation hull of a (Lipschitz) quasi-periodic function can be identified with the $k$-dimensional real torus $\mathbb{T}^k$. In particular, setting

$$|\omega_1 - \omega_2|^2_{\mathbb{R}^k} = \sum_{i=1}^k (|\omega_i^1 - \omega_i^2| \mod 2\pi)^2$$
we have
\[ |\sigma_1 - \sigma_2|_E \leq C|\omega_1 - \omega_2|_T^k. \]  
(14)

We shall indicate with $QP(\Xi)$ the set of all (Lipschitz) quasi-periodic functions with values in $\Xi$. Our main assumption will be:

(10) $g \in QP(L^2)$, $h \in QP(H^{3/2}(\partial\Omega))$ and $\partial_t h \in QP(H^{1/2}(\partial\Omega))$.

The main result of this section is:

**Theorem 4.7.** Suppose that (3), (5) and (10) hold and let $\{S(t)\}$ be the extended semigroup associated to the solution operator of problem (1) acting on the extended phase space $X \times T^k$ (here $k$ is equal to the sum of the different irrationally independent periods of $h$ and $g$)\(^3\). Then there exists a finite time $t^*$ such that the discrete-time semigroup generated by $S(t^*)$ possesses a uniform (w.r.t. the initial phase $\theta \in \pi T^k$) exponential attractor.

**Proof.** Thanks to theorem 4.4, setting $X = H \times H^1$ and $X_1 = V \times H^2$, we will only need to prove that the extended semigroup $S : X \times T^k \to X \times T^k$ belongs to the class of operators $S_{\delta,K}(X)$ for suitable $\delta$, $K$ and $X$. It is therefore sufficient to show that there exists an absorbing bounded set $X \subset X_1 \times T^k$ and that the smoothing property holds (cf. definition 4.3).

Since $T^k$ is invariant under the action of the extended semigroup, the existence of a suitable absorbing set is a simple consequence of theorem 3.8. Therefore, in what follows, we choose $X = B_2 \times T^k$.

We now need to show the smoothing property. Since the proof of this result is quite lengthy, we give here a short overview of our (standard) argument. The main estimate we shall prove will be made up of three major contributions: the first arising from the difference equation for the velocity field obtained from (26) (without $ms$), the second coming from the difference lifted order parameter equation deduced again from (26) while the third and last derives from the difference time-dependent lifted problem got from (22). In any of the three cases, our aim will be to obtain inequalities of the form:

\[ \frac{d}{dt}|\delta|^2_{E_1} + |\delta|^2_{E_2} \leq C|\delta|^2_{E_1} + C|\delta|^2_{E_1} \]

where $E_2$ will be a Banach space (compactly) embedded in $E_1$ and where $\delta$ is any difference of solutions. Using then the uniform Gronwall inequality we get a time dependent bound on $|\delta(t)|_{E_1}$:

\[ |\delta(t)|^2_{E_1} \leq C(t) \int_0^t |\delta(s)|^2_{E_1} ds. \]

The smoothing property can then be obtained through the results of section 3.1.

We start by considering the equation for the velocity field in (26). Let $(u_1,d_1)$ and $(u_2,d_2)$ be two solutions to (26). Taking the difference of the equations for the velocity field, multiplying by $-\Delta w$ and integrating by parts, we obtain:

\[ \frac{1}{2} \frac{d}{dt} |\nabla w|^2_2 - ((\omega \cdot \nabla) u_1, \Delta w) - ((u_2 \cdot \nabla) w, \Delta w) + \nu |\Delta w|^2_2 \]

\[ = ((\nabla e)^{t} \Delta d_1, \Delta w) + ((\nabla d_2)^{t} \Delta e, \Delta w) + (g_1 - g_2, \Delta w) \]

\(^1\)We observe that here we have implicitly substituted the natural extension $T^l \oplus T^m$ of the phase space with the algebraically and geometrically equivalent space $T^{l+m}$. This equivalence can immediately be proven by considering the standard coordinate description of a $k$-dimensional torus through vectors of $\mathbb{R}^k$ with the proper identification.
for a.e. $t \in \mathbb{R}$, where we have set $w = u_1 - u_2$ and $e = d_1 - d_2$ (note that $w|_{\partial \Omega} = 0$).

As in the previous sections, we now bound all the non-linear terms. For the first part of the trilinear term coming from the Navier-Stokes equation we have:

$$\left| (w \cdot \nabla)u_1, \Delta w \right| \leq |w|_\infty |\nabla u_1|_2 |\Delta w|_2$$

$$\leq C |w|_2^{1/2} |\nabla u_1|_2 |\Delta w|_2^{3/2}$$

$$\leq \frac{\nu}{14} |\Delta w|_2^2 + \frac{C \beta^2}{\nu} |w|_2^2$$

where the last estimate follows from theorem 3.8. For the second term arising from the order parameter equation, we have:

$$\left| (u_2 \cdot \nabla)w, \Delta w \right| \leq |u_2|_4 |\nabla w|_4 |\Delta u_2|_2 |\Delta w|_2$$

$$\leq C |u_2|_2^{1/2} |\nabla u_2|_2^{1/2} |\nabla w|_2^{1/2} |\Delta w|_2^{3/2}$$

$$\leq \frac{\nu}{14} |\Delta w|_2^2 + \frac{C \beta^2}{\nu} |\nabla w|_2^2.$$

We now consider the two contributions coming from the nonlinear coupling with the order parameter equation. We have:

$$\left| (\nabla e)^t \Delta d_1, \Delta w \right| \leq |\nabla e|_\infty |\Delta d_1|_2 |\Delta w|_2$$

$$\leq C |\nabla e|_2^{1/2} |\Delta d_1|_2 |\Delta w|_2$$

$$\leq \frac{\nu}{14} |\Delta d_1|_2^2 + \beta |\Delta w|_2 + C \frac{\beta^2}{\nu^2 |\Delta w|_2^2} |\nabla e|_2^2$$

where $\beta$ is a positive real number that will be determined later. Similarly we get:

$$\left| (\nabla d_2)^t \Delta e, \Delta w \right| \leq |\nabla d_2|_4 |\Delta e|_4 |\Delta w|_2$$

$$\leq C |\nabla d_2|_2^{1/2} |\Delta d_2|_2^{1/2} |\Delta e|_2^{1/2} |\Delta w|_2$$

$$\leq \frac{\nu}{14} |\Delta d_2|_2^2 + \beta |\Delta e|_2 + C \frac{\beta^2}{\nu^2 |\Delta e|_2^2} |\nabla d_2|_2^2.$$

To obtain the first estimate we need to control the non-autonomous forcing term:

$$\left| (g_1 - g_2, \Delta w) \right| \leq |g_1 - g_2|_2 |\Delta w|_2 \leq \frac{\nu}{14} |\Delta w|_2^2 + \frac{7}{2\nu} |g_1 - g_2|_2^2.$$

By putting all these estimates together, we deduce the following inequality:

$$\frac{1}{2} \frac{d}{dt} |\nabla w|_2^2 + \nu \left( 1 - \frac{5}{14} \right) |\Delta w|_2^2$$

$$\leq 2 \beta |\Delta e|_2^2 + C \frac{\beta^2}{\nu} |\nabla w|_2^2 + C \frac{\beta^2}{\nu} |\Delta e|_2^2$$

$$+ C \frac{\beta^2}{\nu^2} |\nabla e|_2^2 + C \frac{\beta^2}{\nu^2} |\Delta e|_2^2 + \frac{7}{2\nu} |g_1 - g_2|_2^2. \tag{15}$$

We now consider the lifted equation for the order parameter field (see (26)). By taking the difference of the equations satisfied by the same two solutions $(u_1, d_1)$ and $(u_2, d_2)$ as above, multiplying by $\Delta (\Delta \tilde{e} - f(d_1) + f(d_2))$ and integrating by parts we have:

$$\langle \partial_t \Delta \tilde{e}, \Delta \tilde{e} - f(d_1) + f(d_2) \rangle + |\nabla (\Delta \tilde{e} - f(d_1)) + f(d_2)|_2^2$$

$$= \langle (\nabla u_1)^t \nabla e + \nabla \nabla e \cdot u_1, \nabla (\Delta \tilde{e} - f(d_1) + f(d_2)) \rangle$$

$$+ \langle (\nabla w)^t \nabla d_2 + \nabla \nabla d_2 \cdot w, \nabla (\Delta \tilde{e} - f(d_1) + f(d_2)) \rangle.$$
where we observe that \( \mathcal{C}_{\partial \Omega} = 0 \) and that \( \Delta \hat{e} - f(d_1) + f(d_2)|_{\partial \Omega} = 0 \). Completing the first term on the left hand side of this relation we obtain:
\[
\frac{1}{2} \frac{d}{dt} |\Delta \hat{e} - f(d_1) + f(d_2)|^2 + |\nabla (\Delta \hat{e} - f(d_1)) + f(d_2)|^2
\]
\[
= \langle (\nabla u_1)' \nabla w + \nabla \nabla e \cdot u_1, \nabla (\Delta \hat{e} - f(d_1)) + f(d_2) \rangle + \langle (\nabla w)' \nabla d_2 + \nabla \nabla d_2 \cdot w, \nabla (\Delta \hat{e} - f(d_1)) + f(d_2) \rangle + \langle \partial_t (f(d_1) - f(d_2)), \Delta \hat{e} - f(d_1) + f(d_2) \rangle .
\]

(16)

As usual, we have to estimate all the nonlinear terms appearing on the right hand side. We start with the four transport terms. We have:
\[
\left| \langle (\nabla u_1)' \nabla w, \nabla (\Delta \hat{e} - f(d_1)) + f(d_2) \rangle \right|
\leq |\nabla u_1| |\nabla \nabla e| |\nabla (\Delta \hat{e} - f(d_1)) + f(d_2)|
\leq C|\nabla u_1| |\nabla \nabla e|^{1/2} |\nabla (\Delta \hat{e} - f(d_1)) + f(d_2)|
\leq \frac{1}{16} |\nabla (\Delta \hat{e} - f(d_1)) + f(d_2)|^2 + \beta |\nabla \nabla e| + C \frac{\rho_0 \rho_2}{\beta} |\nabla \nabla e|.
\]

and, analogously, we get:
\[
\left| \langle (\nabla w)' \nabla d_2, \nabla (\Delta \hat{e} - f(d_1)) + f(d_2) \rangle \right|
\leq |\nabla w| |\nabla d_2| |\nabla (\Delta \hat{e} - f(d_1)) + f(d_2)|
\leq C|\nabla w|^{1/2} |\nabla \nabla d_2|^{1/2} |\nabla (\Delta \hat{e} - f(d_1)) + f(d_2)|
\leq \frac{1}{16} |\nabla (\Delta \hat{e} - f(d_1)) + f(d_2)|^2 + \beta |\nabla \nabla e| + C \frac{\rho_0 \rho_2}{\beta} |\nabla \nabla e|.
\]

We can also proceed in a similar way for the following two terms, obtaining:
\[
\left| \langle (\nabla d_2)' \nabla w, \nabla (\Delta \hat{e} - f(d_1)) + f(d_2) \rangle \right|
\leq |\nabla d_2| |\nabla w| |\nabla (\Delta \hat{e} - f(d_1)) + f(d_2)|
\leq C|\nabla d_2| |\nabla \nabla d_2|^{1/2} |\nabla (\Delta \hat{e} - f(d_1)) + f(d_2)|
\leq \frac{1}{16} |\nabla (\Delta \hat{e} - f(d_1)) + f(d_2)|^2 + \frac{\nu}{14} |\Delta w|^2 + C \frac{\rho_0 \rho_2}{\nu} |\nabla \nabla e|.
\]

and deducing:
\[
\left| \langle (\nabla d_2)' \nabla w, \nabla (\Delta \hat{e} - f(d_1)) + f(d_2) \rangle \right|
\leq |d_2| |\nabla w| |\nabla (\Delta \hat{e} - f(d_1)) + f(d_2)|
\leq C|d_2| |\nabla \nabla d_2|^{1/2} |\nabla (\Delta \hat{e} - f(d_1)) + f(d_2)|
\leq \frac{1}{16} |\nabla (\Delta \hat{e} - f(d_1)) + f(d_2)|^2 + \frac{\nu}{14} |\Delta w|^2 + C \frac{\rho_0 \rho_2}{\nu} |\nabla \nabla e|.
\]

We now have to consider the last term in (16). We start by observing that the following identity holds:
\[
\partial_t (f(d_1) - f(d_2)) = \nabla d f(d_1) \cdot \partial_t d_1 - \nabla d f(d_2) \cdot \partial_t d_2
= (\nabla d f(d_1) - \nabla d f(d_2)) \cdot \partial_t d_1 + \nabla d f(d_2) \cdot \partial_t e
\]
where with $\nabla_d$ we denote the gradient with respect to $d$. Before going on, we recall that the tensor norm we use throughout this work (Frobenius’ norm) is compatible with the standard Euclidean norm of vectors. Therefore we have:

$$|\nabla_d f(d_2) \cdot \partial_t e|^2_2 = \int_\Omega |\nabla_d f(d_2) \cdot \partial_t e|^2 \, d\Omega \leq \int_\Omega |\nabla_d f(d_2)|^2_2 |\partial_t e|^2 \, d\Omega \leq |\nabla_d f(d_2)|^2_\infty |\partial_t e|^2_2.$$  

By using these results we obtain:

$$\begin{align*}
&\left| \left\langle \partial_t (f(d_1) - f(d_2)), \Delta \tilde{e} - f(d_1) + f(d_2) \right\rangle \right| \\
&\leq |\nabla_d f(d_1) - \nabla_d f(d_2)|_\infty |\partial_t e|_2 |\Delta \tilde{e} - f(d_1) + f(d_2)|_2 \\
&\quad + |\nabla_d f(d_2)|_\infty |\partial_t e|_2 |\Delta \tilde{e} - f(d_1) + f(d_2)|_2.
\end{align*}$$

We observe that, thanks to lemma A.1, we have:

$$|\nabla_d f(d_1) - \nabla_d f(d_2)|_\infty \leq \frac{2\sqrt{10}}{\epsilon^2} |e|_\infty \quad \text{and} \quad |\nabla_d f(d_2)|_\infty \leq \frac{\sqrt{7}}{\epsilon^2}.$$  

In order to finish this part of our argument, we have to find an appropriate estimate for $|\partial_t e|_2$. From system 25, we obtain:

$$|\partial_t e|_2 \leq |(u_1 \cdot \nabla)e|_2 + |(w \cdot \nabla)d|_2 + |\Delta e|_2 + |f(d_1) - f(d_2)|_2 \leq |u_1|_4 |\nabla e|_4 + |w|_4 |\nabla d|_4 + |\Delta e|_2 + |f(d_1) - f(d_2)|_2 \leq C|u_1|^{1/2} |\nabla u_1|^{1/2} |\nabla e|^{1/2} |\nabla H^2| + C|w|^{1/2} |\nabla w|^{1/2} |\nabla d|^{1/2} |\nabla H^2| + |\Delta e|_2 + |f(d_1) - f(d_2)|_2 \leq C \rho_0^{1/4} \rho_2^{1/4} |e|_{H^2} + C \rho_0^{1/4} \rho_2^{1/4} |\nabla w|_2 + |e|_{H^2} + \frac{2}{\epsilon^2} |e|_2.$$

We can now obtain the desired estimate for the last term in (16). By using all the results above, recalling corollary 2 and $|e|_\infty \leq C|e|_{H^2}$, we get:

$$\begin{align*}
&\left| \left\langle \partial_t (f(d_1) - f(d_2)), \Delta \tilde{e} - f(d_1) + f(d_2) \right\rangle \right| \\
&\leq \frac{1}{2} |\Delta \tilde{e} - f(d_1) + f(d_2)|^2_2 + C \left( \frac{\rho_0^{1/2} \rho_2^{1/2}}{\epsilon^4} + 1 + \frac{1}{\epsilon^4} \right) |e|^2_{H^2} \\
&\quad + C \left( \frac{\rho_0^{1/2} \rho_2^{1/2}}{\epsilon^4} - |\nabla w|^2_2. \right.
\end{align*}$$

We can now substitute these estimates in (16) obtaining:

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} |\Delta \tilde{e} - f(d_1) + f(d_2)|^2_2 + \left( 1 - \frac{1}{4} \right) |\nabla (\Delta \tilde{e} - f(d_1) + f(d_2))|^2_2 \\
\leq 2\beta |e|_{H^3} + \frac{\nu}{2} |\Delta w|^2_2 + \frac{1}{2} |\Delta \tilde{e} - f(d_1) + f(d_2)|^2_2 \\
&\quad + C \left( \frac{\rho_0^{1/2} \rho_2^{1/2}}{\epsilon^4} + 1 + \frac{1}{\epsilon^4} \rho_0^{1/2} \rho_2^{1/2} + \frac{1}{\epsilon^8} \right) |e|^2_{H^2} \\
&\quad + C \left( \frac{\rho_0^{1/2} \rho_2^{1/2}}{\epsilon^4} + 1 + \frac{1}{\epsilon^4} \rho_0^{1/2} \rho_2^{1/2} \right) |\nabla w|^2_2. \quad (17)
\end{align*}$$
To deal with the term $|e|_{H^3}$ which appears on the right hand side of this last inequality, we observe that, thanks to $\tilde{e}|_{\partial \Omega} = 0$, we have:

$$|e|_{H^3}^2 \leq 2|\tilde{e}|_{H^3}^2 + 2|\tilde{e}|_{H^3}^2 \leq C|\nabla \Delta \tilde{e}|^2 + C|\Delta \tilde{e}|^2 + 2|\tilde{e}|_{H^3}^2 \leq C|\nabla \Delta \tilde{e} - f(d_1) + f(d_2)|^2 + C|\nabla (f(d_1) - f(d_2))|^2 + C|e|_{H^3}^2 + 2|\tilde{e}|_{H^3}^2.$$

Moreover $|\nabla (\Delta \tilde{e} - f(d_1) + f(d_2))|^2$ can easily be estimated by writing as above:

$$\nabla (\Delta \tilde{e} - f(d_1) + f(d_2)) = \nabla_{d_1} d_1 - \nabla_{d_2} f(d_2) \cdot \nabla_{d_2}$$

from which we obtain:

$$|\nabla (\Delta \tilde{e} - f(d_1) + f(d_2))|^2 \leq 2|\nabla_{d_1} d_1 - \nabla_{d_2} f(d_1)|^2 + 2|\nabla_{d_2} f(d_2)|^2 \leq C \frac{\rho_0}{c^2} |e|_{H^2}^2 + C \frac{1}{c^2} |\nabla e|^2 \leq C \frac{\rho_0 + 1}{c^2} |e|_{H^3}^2.$$

Summing up, we have:

$$|e|_{H^3}^2 \leq C|\nabla (\Delta \tilde{e} - f(d_1) + f(d_2))|^2 + C \left( 1 + \frac{1}{c^2} + \frac{\rho_0}{c^2} \right) |e|_{H^3}^2 + 2|\tilde{e}|_{H^3}^2. \quad (18)$$

We now only have to deal with $|e|_{H^3}$. This leads to the third and last preliminary estimate. Consider the difference of the equation satisfied by two solutions $\tilde{d}_1$ and $\tilde{d}_2$ of the lifting problem (22). Multiplying by $-\Delta \tilde{e}$ the resulting equation (where $\tilde{e} = \tilde{d}_1 - \tilde{d}_2$), standard computations give:

$$|\nabla \tilde{e}(t)|^2 + \int_0^t |\Delta \tilde{e}(s)|^2 ds \leq |\nabla \tilde{e}_0|^2 + \int_0^t |\delta h(s)|^2_{H^{3/2}(\partial \Omega)} ds + C \int_0^t |\partial \delta h(s)|^2_{H^{1/2}(\partial \Omega)} ds$$

where we have used the self explanatory notation $\delta h \equiv h_1 - h_2$. In order to obtain the desired estimate for $|\tilde{e}|_{H^3}$, we need also a higher regularity result. We start by considering again the difference of two solutions of problem (22), but this time we take the laplacian (in the sense of distributions) of the resulting equation and test it with $\Delta \tilde{e}$. After an integration by parts we get:

$$\frac{1}{2} \frac{d}{dt} |\Delta \tilde{e}|^2 + |\nabla \Delta \tilde{e}|^2 \leq H^{-1/2}(\partial \Omega) \langle \partial \delta \tilde{e}, \Delta \tilde{e} \rangle_{H^{1/2}(\partial \Omega)}.$$

Using directly equation (22) we then deduce:

$$\frac{1}{2} \frac{d}{dt} |\Delta \tilde{e}|^2 + |\nabla \Delta \tilde{e}|^2 \leq H^{-1/2}(\partial \Omega) \langle \partial \delta \tilde{e}, \Delta \tilde{e} \rangle_{H^{1/2}(\partial \Omega)} \leq C|\tilde{e}|_{H^3} |\partial \delta h|_{H^{1/2}(\partial \Omega)} \leq C \frac{1}{4} |\nabla \Delta \tilde{e}|^2 + |\delta h|^2_{H^{3/2}(\partial \Omega)} + C |\partial \delta h|^2_{H^{1/2}(\partial \Omega)}$$

whence we finally obtain:

$$\frac{1}{2} \frac{d}{dt} |\Delta \tilde{e}|^2 + \frac{3}{4} |\tilde{e}|_{H^3}^2 \leq C |\delta h|^2_{H^{3/2}(\partial \Omega)} + C |\partial \delta h|^2_{H^{1/2}(\partial \Omega)}. \quad (19)$$
We can now use the three main estimates we have obtained in the preceding pages. Summing together (15), (17) and (19) we have:

\[
\begin{align*}
&\frac{1}{2} \frac{d}{dt} |\nabla w|^2 + \frac{1}{2} \frac{d}{dt} |\Delta \tilde{e} - f(d_1) + f(d_2)|^2 + \frac{1}{2} \frac{d}{dt} |\Delta \tilde{e} - f(d_1) + f(d_2)|^2 \\
&\quad + \nu' |\Delta w|^2 + \frac{3}{4} |\nabla (\Delta \tilde{e} - f(d_1) + f(d_2))|^2 + \frac{3}{4} |\tilde{e} h|^2 \\
&\leq 4\beta |e|^2 H^1 + C \left( \frac{\rho_0^2}{\nu} + \frac{\rho_0^2}{\nu} + \frac{\rho_0^2}{\nu} + \frac{\rho_0^2}{\nu} + \frac{\rho_0^2}{\nu} + \frac{\rho_0^2}{\nu} + \frac{\rho_0^2}{\nu} + \frac{\rho_0^2}{\nu} \right) |\nabla w|^2 + \frac{1}{2} |\Delta \tilde{e} - f(d_1) + f(d_2)|^2 \\
&\quad + C \left( \frac{\rho_0^2}{\nu} + \frac{\rho_0^2}{\nu} + \frac{\rho_0^2}{\nu} + \frac{\rho_0^2}{\nu} + \frac{\rho_0^2}{\nu} + \frac{\rho_0^2}{\nu} + \frac{\rho_0^2}{\nu} + \frac{\rho_0^2}{\nu} \right) |\tilde{e} h|^2 \\
&\quad + \frac{C}{\nu} |\delta g|^2 + C |\delta h|^2 H^{1/2} + C |\delta h|^2 H^{1/2}.
\end{align*}
\]

Using (18) and choosing \( \beta \) sufficiently small (in particular, smaller than a constant which depends only on \( \Omega \)) we obtain:

\[
\begin{align*}
&\frac{d}{dt} |\nabla w|^2 + \frac{d}{dt} |\Delta \tilde{e} - f(d_1) + f(d_2)|^2 \\
&\leq C |\nabla w|^2 + |\Delta \tilde{e} - f(d_1) + f(d_2)|^2 + C |e|^2 H^1 \\
&\quad + C |\delta g|^2 + C |\delta h|^2 H^{1/2} + C |\delta h|^2 H^{1/2}.
\end{align*}
\]

where all the constants depend only on \( \rho_0, \rho_2, \rho_4, \Omega, \nu \) and \( \epsilon \). We recall inequality (14) introduced above:

\[
\begin{align*}
|\delta g|^2 &\leq C |\delta \theta|^2, \\
|\delta h|^2 H^{1/2} &\leq C |\delta \phi|^2, \\
|\delta h|^2 H^{1/2} &\leq C |\delta \phi|^2
\end{align*}
\]

where \( (\theta_i, \phi_i) \in T^k, i = 1, 2 \) and where \( \delta \theta = \theta_1 - \theta_2 \) and \( \delta \phi = \phi_1 - \phi_2 \). We also note that, under the action of the extended semigroup, the quantities \( |\delta \theta| \) and \( |\delta \phi| \) are conserved and therefore we have:

\[
\begin{align*}
&\frac{d}{dt} |\nabla w|^2 + \frac{d}{dt} |\Delta \tilde{e} - f(d_1) + f(d_2)|^2 \\
&\leq C |\nabla w|^2 + |\Delta \tilde{e} - f(d_1) + f(d_2)|^2 + C |e|^2 H^1 + C |\delta \theta|^2 + C |\delta \phi|^2.
\end{align*}
\]

On the other hand, we have:

\[
\begin{align*}
|e|^2 H^1 &\leq 2 |\tilde{e} h|^2 + 2 |\tilde{e} h|^2 \\
&\leq C |\Delta \tilde{e} h|^2 + C |\Delta \tilde{e} h|^2 + |\delta h|^2 H^{1/2} \\
&\leq C |\Delta \tilde{e} - f(d_1) + f(d_2)|^2 + C |f(d_1) - f(d_2)|^2 + C |\Delta \tilde{e}|^2 + C |\delta \phi|^2 \\
&\leq C |\Delta \tilde{e} - f(d_1) + f(d_2)|^2 + C \frac{C}{C_1} |e|^2 H^1 + C |\Delta \tilde{e}|^2 + C |\delta \phi|^2.
\end{align*}
\]

Before applying the uniform Gronwall inequality, we have to verify the integrability on the interval \([0, T]\) of all the arguments in the right hand side of (20). This
can be deduced from theorem 2.4 and from the following estimate:
\[
|\Delta \hat{e} - f(d_1) + f(d_2)|_2^2 \leq 2|\Delta \hat{e}|_2^2 + 2|f(d_1) - f(d_2)|_2^2
\]
\[
\leq 2|\hat{e}|_{H^2}^2 + 2|f(d_1) - f(d_2)|_2^2
\]
\[
\leq 4|e|_{H^2}^2 + 4|\hat{e}|_{H^2}^2 + \frac{C}{e^2}|e|_2^2.
\]
Using Gronwall’s inequality we get:
\[
\left|\nabla w(t)^2 + |\Delta \hat{e}(t) - f(d_1(t)) + f(d_2(t))|^2 + |\Delta \hat{e}(t)|^2 + |\delta \theta|^2 + |\delta \phi|^2
\right|
\]
\[
\leq \left(\frac{1}{t} \int_0^t \left(\left|\nabla w(s)^2 + |\Delta \hat{e}(s) - f(d_1(s)) + f(d_2(s))|^2 + |\Delta \hat{e}(s)|^2\right| ds
\right) e^{Ct}.
\]
Recalling estimate (4) and using again (21), we eventually obtain:
\[
\left|\nabla w(t)^2 + |e(t)|_{H^2}^2 + |\delta \theta|^2 + |\delta \phi|^2 \leq C(t) \left(|w_0|^2 + |e_0|^2_{H^2} + |\delta \theta|^2 + |\delta \phi|^2\right)
\]
where we observe that we have also used the following estimate:
\[
|\Delta \hat{e} - f(d_1) + f(d_2)|_2^2 \leq 4|\Delta e|_2^2 + 4|\Delta \hat{e}|_2^2 + C|e|_2^2.
\]
\[
\text{Remark 5.} \text{ Theorem 4.4 gives us also continuous dependence of the exponential attractor on the semigroup considered. When considering quasi-periodic symbols, it is easy to see that the extended semigroup continuously depends on the frequencies of the different periods characterizing the forcing terms. We therefore have continuity of the exponential attractor with respect to the frequencies of the symbols.}
\]
\[
\text{4.2. Exponential attractors.} \text{ We now prove that also an exponential attractor exists. To do this, we simply have to apply theorem 4.5 to system (1) (see \cite{7, Chapter 3} or \cite{9} for more details). Actually, the following result holds.}
\]
\[
\textbf{Theorem 4.8.} \text{ Let the same assumptions of theorem 4.7 be verified. Then there exists an exponential attractor } \mathcal{M} \text{ for the extended semigroup } \{S(t)\} \text{ on } X \times \mathbb{T}^k.
\]
\[
\text{Moreover, if } \Pi_1 \text{ and } \Pi_2 \text{ are the projections of the extended phase space on } X \text{ and } \mathbb{T}^k \text{ respectively, then } \Pi_1 \mathcal{M} \text{ is the uniform (w.r.t. } \theta \in \mathbb{T}^k) \text{ exponential attractor for the family of processes and } \Pi_2 \mathcal{M} = \mathbb{T}^k.
\]
\[
\text{Thanks to the study of global attractors in section 3, as an immediate consequence of this result, we have the following corollary.}
\]
\[
\textbf{Corollary 3.} \text{ Let the same assumptions of theorem 4.7 be verified. Then the global attractor of system (1) has finite fractal dimension.}
\]
\[
\text{Proof of theorem 4.8.} \text{ To prove our claim we only need to show that the extended semigroup } S(t) \text{ is Lipschitz continuous on the phase space and Hölder continuous in time. Indeed, the first statement follows easily from theorem 2.4, whereas for the second statement the following estimates hold:}
\]
\[
|u(t) - u(\tau)|_2 + |d(t) - d(\tau)|_{H^1} \leq \left|\int_\tau^t \frac{d}{ds} u(s) ds\right|_2 + \left|\int_\tau^t \frac{d}{ds} d(s) ds\right|_{H^1}
\]
\[
\leq \int_\tau^t |\partial_s u(s)|_2 ds + \int_\tau^t |\partial_s d(s)|_{H^1} ds
\]
\[
\leq (t - \tau)^{1/2} \left(|\partial_s u|_{L^2(\tau,t;H^1)}^2 + |\partial_s d|_{L^2(\tau,t;H^1)}^2\right)
\]
Lemma A.1. If \((u(s), d(s))\) is any solution of (1).

We begin with \(\partial_t u\). From the equation for the velocity field in (1) we get:

\[
|\partial_t u|_2 \leq |(u \cdot \nabla) u|_2 + \nu|\Delta u|_2 + |(\nabla d)^t \Delta d|_2 \\
\leq C|u|^{1/2}_{2} |\nabla u|^{1/2}_{2} |\Delta u|^{1/2}_{2} + \nu|\Delta u|_2 + C|\nabla d|^{1/2}_{2} |d|^{1/2}_{H^2} \\
\leq C\rho_0^{1/4} \rho_2^{1/2} |\Delta u|^{1/2}_{2} + \nu|\Delta u|_2 + C\rho_0^{1/4} \rho_2^{1/2} |d|^{1/2}_{H^3}.
\]

By squaring and integrating between \(\tau\) and \(t\), recalling the results of section 3.1, we easily find that on the absorbing set (that is on a neighborhood of the exponential attractor) \(|\partial_t u|_{L^2(\tau,t;H)}\) is bounded.

An analogous estimate can be obtained by taking the gradient of the equation for the order parameter. In particular we have:

\[
|\partial_t \nabla d|_2 \leq |\nabla u| |\nabla d|_4 + |u|_{W^{2,4}_s} + |d|_{H^3} + |\nabla f(d)|_2 \\
\leq C|\nabla u|^{1/2}_{2} |\Delta u|^{1/2}_{2} |\nabla d|^{1/2}_{2} |d|^{1/2}_{H^2} \\
+ C|u|^{1/2}_{2} |\nabla u|^{1/2}_{2} |\nabla d|^{1/2}_{2} |d|^{1/2}_{H^2} + |d|_{H^3} + C|\nabla d|_2 \\
\leq C\rho_0^{1/2} \rho_2^{1/2} |\Delta u|^{1/2}_{2} + C\rho_0^{1/4} \rho_2^{1/2} |d|^{1/2}_{H^3} + |d|_{H^3} + C|\nabla d|_2.
\]

Again simple calculations give a uniform bound on \(|\partial_t d|_{L^2(\tau,t;H^1)}\) on a neighborhood of the exponential attractor.

Remark 6. We observe that in [9], starting from the theory of pullback attractors, a slightly different notion of exponential attractor is introduced. Although here the setting of the problem is not exactly the same, the results now obtained (i.e. absorbing sets and smoothing property) are sufficient to prove the existence of a time dependent exponential attractor when the forcing terms are quasi-periodic. Moreover, thanks to the abstract results of [9], this attractor is continuous w.r.t. the forcing term considered. Since in the context of pullback attractors one considers a fixed non-autonomous forcing term, we also observe that the estimates of this section can easily be adapted to the more general case of arbitrary translation-bounded forcing terms. This leads to the existence of a positively invariant time-dependent exponential attractor even under these more general assumptions. It is interesting to notice that this approach does not require a somewhat involved definition of exponential attractor even under these more general assumptions. These features will however be studied in more detail in a forthcoming paper of which this remark represents only a preliminary account.

Appendix A. Proof of theorem 2.2 and of lemma 2.3. We give here a full proof of theorem 2.2 and lemma 2.3.

We start by summarizing some useful results about the nonlinear forcing term \(f(d)\) which appears in the equation for the order parameter field. We skip the elementary proof.

Notation. In analogy with the notation for vector norms introduced above, we will write \(|T|\) while referring to the usual Euclidean norm for tensors which is defined as \(|T|^2 = T_{ij} T_{ij}\) (where we leave out the sum over repeated indices).

Lemma A.1. If \(|d| \leq 1\) a.e. on \(\Omega \subset \mathbb{R}^n\), then:

\[
|f(d)| \leq \frac{2\sqrt{3}}{9c^2} \quad \text{and} \quad |\nabla_d f(d)|^2 \leq \frac{n+5}{c^4}.
\]
Moreover, if both \( \mathbf{d}_1 \) and \( \mathbf{d}_2 \) satisfy the above assumption, then:

\[
|f(\mathbf{d}_1) - f(\mathbf{d}_2)| \leq \frac{2}{\epsilon^2} |\mathbf{d}_1 - \mathbf{d}_2|
\]

and

\[
|\nabla_d(f(\mathbf{d}_1) - f(\mathbf{d}_2))| \leq \frac{2\sqrt{n + 8}}{\epsilon^2} |\mathbf{d}_1 - \mathbf{d}_2|.
\]

**Regularity of a time dependent lifting problem.** In order to deal with the non-autonomous boundary term, we consider a simple linear lifting problem:

\[
\begin{aligned}
\partial_t \tilde{\mathbf{d}} - \Delta \tilde{\mathbf{d}} &= 0 \quad \text{in } Q_T = \Omega \times (0, T); \\
\tilde{\mathbf{d}} &= \mathbf{h} \quad \text{on } \partial\Omega \times (0, T); \\
\tilde{\mathbf{d}}(0) &= \mathbf{d}_0 \quad \text{in } \Omega.
\end{aligned}
\] (22)

Existence and uniqueness of solutions for problem (22) follow easily from the usual theory for linear parabolic problems (see, e.g., [19]). In particular, the following useful lemmas hold.

**Lemma A.2.** Suppose that assumption (4) holds, let \( \mathbf{h} \in L^2(0, T; H^{1/2}(\partial\Omega)) \) and let \( \mathbf{d}_0 \in L^2 \). Then there exists a unique weak solution \( \tilde{\mathbf{d}} \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1) \) \( \forall T > 0 \) of problem (22). Moreover the following estimate holds:

\[
|\tilde{\mathbf{d}}(T)|^2 + \int_0^T |\nabla \tilde{\mathbf{d}}|^2 dt \leq |\mathbf{d}_0|^2 + \int_0^T C|h|_{H^{1/2}(\partial\Omega)}^2 dt.
\]

**Lemma A.3.** Let the same assumptions of lemma A.2 be verified.

- If assumption (4) holds and \( \mathbf{d}_0 \in H^1 \) is such that \( \mathbf{d}_0 = \mathbf{h}(0) \) on \( \partial\Omega \), then \( \tilde{\mathbf{d}} \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2) \) and the following estimate holds:

\[
|\tilde{\mathbf{d}}|_{H^2}^2 + \int_0^t |\tilde{\mathbf{d}}|^2_{H^2} dt \leq |\mathbf{d}_0|_{H^2}^2 + C \int_0^t \left( |\partial_t \mathbf{h}|_{H^{-1/2}(\partial\Omega)}^2 + |\mathbf{h}|_{H^{1/2}(\partial\Omega)}^2 \right) dt.
\] (23)

- If assumption (7) holds and \( \mathbf{d}_0 \in H^2 \) is such that \( \mathbf{d}_0 = \mathbf{h}(0) \) on \( \partial\Omega \), then \( \tilde{\mathbf{d}} \in L^\infty(0, T; H^2) \cap L^2(0, T; H^3) \). Moreover the following estimate holds:

\[
|\tilde{\mathbf{d}}|_{H^3}^2 + \int_0^t |\tilde{\mathbf{d}}|^2_{H^3} dt \leq |\mathbf{d}_0|_{H^3}^2 + C \int_0^t \left( |\partial_t \mathbf{h}|_{H^{1/2}(\partial\Omega)}^2 + |\mathbf{h}|_{H^{1/2}(\partial\Omega)}^2 \right) dt.
\] (24)

We also note that, under physically sound assumptions, the following maximum principle holds (see, e.g., [10]).

**Lemma A.4.** Let the same assumptions of lemma A.2 be verified. If in addition \( |\mathbf{h}| \leq 1 \ a.e. \ on \ \partial\Omega \times [0, T] \), then \( |\tilde{\mathbf{d}}| \leq 1 \ a.e. \ (x, t) \in \Omega \times [0, T] \).

**The semi-Galerkin approximation.** We now go back to the weak formulation of problem (1) and give its announced Galerkin approximation. We will use the usual Faedo-Galerkin method only for the velocity field. In particular, we will search a
solution $u^m \in C^1(0,T;V^m)$ and $d^m \in L^2(0,T;H^2) \cap L^\infty(0,T;H^1)$ such that:

$$\begin{aligned}
&\left( \partial_t u^m(t), v_m \right) + \left( (u^m(t) \cdot \nabla)v^m(t), v_m \right) + \nu \left( \nabla u^m(t), \nabla v_m \right) \forall v_m \in V^m; \\
&+ \left( \Delta d^m(t), \nabla d^m(t)v_m \right) = (g(t), v_m) \\
&\partial_t d^m(t) + (u^m(t) \cdot \nabla)d^m(t) = \Delta d^m(t) - f(d^m(t)) \quad \text{a.e. in } (0,T) \times \Omega;
\end{aligned}$$

$$|d^m| \leq 1$$

$$u^m(0) = u_{0m} \equiv P_m u_0$$

$$d^m(x,0) = d_0 \quad \text{for } x \in \Omega;$$

$$d^m(x,t) = h(x,t) \quad \text{on } \partial \Omega \times (0,T)$$

(25)

holds, where the linear operator $P_m : H \to V^m$ is the orthogonal (in $L^2$) projection on $V^m$. Actually also the following lifted problem will be important in the proof:

$$\begin{aligned}
&\left( \partial_t u^m(t), v_m \right) + \left( (u^m(t) \cdot \nabla)v^m(t), v_m \right) + \nu \left( \nabla u^m(t), \nabla v_m \right) \forall v_m \in V^m; \\
&+ \left( \Delta d^m(t), \nabla d^m(t)v_m \right) = (g(t), v_m) \\
&\partial_t \tilde{d}^m(t) + (u^m(t) \cdot \nabla)d^m(t) = \Delta d^m(t) - f(d^m(t)) \quad \text{a.e. in } (0,T) \times \Omega;
\end{aligned}$$

$$|d^m| \leq 1$$

$$u^m(0) = u_{0m} \equiv P_m u_0$$

$$\tilde{d}^m(x,0) = 0 \quad \text{for } x \in \Omega;$$

$$\tilde{d}^m(x,t) = 0 \quad \text{on } \partial \Omega \times (0,T)$$

(26)

where we write $\tilde{d}^m = d^m - \bar{d}$ and $\bar{d}$ is the solution of the lifting problem (22).

**Local time existence of solutions.** We will now apply a fixed point argument to prove existence of (at least) a solution on the time interval $[0,T_m]$ for the approximating problem (25). We start by introducing the following splitting.

1. Let $\bar{u}^m \in C(0,T;V^m)$ be a given velocity field. We look after the order parameter field $d^m \in L^2(0,T;H^2) \cap L^\infty(0,T;H^1)$ which solves:

$$\begin{aligned}
&\partial_t d^m + (\bar{u}^m(t) \cdot \nabla)d^m = \Delta d^m - f(d^m) \quad \text{in } \Omega \times (0,T); \\
d^m(x,0) = d_0 \quad \text{in } \Omega; \\
d^m(x,t) = h \quad \text{on } \partial \Omega \times (0,T).
\end{aligned}$$

(27a)

2. Let $d^m \in L^2(0,T;H^2) \cap L^\infty(0,T;H^1)$ be the order parameter field just determined. The second part of the splitting consists in finding a velocity field $u^m \in H^1(0,T;V^m)$ such that the following equation is satisfied:

$$\begin{aligned}
&\left( \partial_t u^m(t), v \right) + \left( (\bar{u}^m(t) \cdot \nabla)u^m(t), v \right) + \nu \left( \nabla u^m(t), \nabla v \right) \forall v \in V^m; \\
&u^m(0) = u^m_0 \quad \text{in } \Omega.
\end{aligned}$$

(27b)

We stress that in this problem the order parameter field $d^m$ is given.

**Remark 7.** The just introduced splitting and the fixed point argument of the following pages can be considered as possible starting points in the design of numerical scheme to solve problem (1). One possible advantage of this strategy is that one can use only existing programs that already efficiently solve Navier-Stokes equations and simple transport-diffusion equations without the need of implementing from scratch a whole new numerical algorithm.
Existence and uniqueness for problem (27a). The existence of a solution which satisfies equation (27a) can be obtained through a fixed point argument. We recall here only the main results which will be important in the sequel observing that here no assumptions on $\partial_t h$ are necessary.

Lemma A.5 (Weak maximum principle, [3]). Suppose that the hypotheses (1), (3) and (4) are satisfied, let $d_0 \in H^1$ such that $|d_0(x)| \leq 1$ a.e. $x \in \Omega$, $d_0 = h(0)$ on $\partial \Omega$ and let $u^m \in C(0,T; V^m)$. Then there exists a weak solution $d^m \in L^\infty(0,T; H^1) \cap L^2(0,T; H^2)$ to problem (27a). Moreover every such solution verifies $|d^m(x,t)| \leq 1$ a.e. $\Omega \times [0,T]$.

We now use the lifting function given by (22). Testing the lifted problem with $-\Delta \tilde{d}^m$, routine calculations give the following estimate.

Lemma A.6. Under the same assumptions of lemma A.5, let $M \in \mathbb{R}$ be a constant such that $\sup_{t \in [0,T]} |\nabla u^m|_2 \leq M$. Then the following estimate holds:

$$
|\nabla \tilde{d}^m(t)|^2 + \frac{1}{4} \int_0^t |\Delta \tilde{d}^m(s)|^2 ds \\
\leq T \frac{2}{3\sqrt{3}} |\Omega|^{1/2} + C(M^4 + M^2)|\nabla d|^2_{L^2(0,T; L^2)} + CM^2|\tilde{d}^m|^2_{L^2(0,T; H^2)}
$$

(28)

Moreover, it is easy to prove that the solution is unique and continuously depends on initial data so that the solution operator $S_d^m : C(0,T; V^m) \to L^\infty(0,T; H^1) \cap L^2(0,T; H^2)$, $S_d^m : u^m \to u^m$ is continuous.

Existence and uniqueness for problem (27b). The well-posedness of the problem for the velocity field (27b) can easily be obtained through the standard Galerkin approach. Before stating the main result and estimate, we only observe that since $\nabla d^m(t) \in L^2$, $\Delta d^m(t) \in L^2$ a.e. $t \in [0,T]$, the $d^m$-dependent forcing term can indeed be read as a scalar product in $L^2$ instead of being a duality. Moreover it is useful to recall the following vector identity:

$$
\nabla \cdot (\nabla d^m \otimes \nabla d^m) = (\nabla d^m)^t \Delta d^m + \frac{1}{2} \nabla |\nabla d^m|^2.
$$

(29)

Lemma A.7. Suppose that the hypotheses (1) and (2) are satisfied, let $u_0 \in H$ and let $d^m \in L^2(0,T; H^2) \cap L^\infty(0,T; H^1)$. Then there exists a time $T_m \leq T$ and a unique weak solution $u^m \in H^1(0,T; V^m)$ to problem (27b) for $t \in [0,T_m)$ which satisfies:

$$
|u^m(t)|^2_{L^2} + \nu |\nabla u^m|^2_{L^2(0,t; L^2)} \\
\leq |u_0|^2 + \frac{2}{\nu} |g|^2_{L^2(0,T_m; V^*)} + \frac{C_m}{\nu} T_m |\nabla d^m|^4_{L^\infty(0,T_m; L^2)}.
$$

(30)

We observe that the constant $C_m$, which explicitely depends on $m$, appears because we made use of the inverse inequality $|\nabla u^m|_{\infty} \leq C_m |\nabla u^m|_2$ which holds since $V^m$ is finite dimensional.

As before, we can also easily prove that the solution operator of problem (27b): $S_u^m : L^2(0,T; H^2) \cap L^\infty(0,T; H^1) \to H^1(0,T; V^m)$, $S_u^m(d^m) = u^m$ is continuous.
A fixed point result. We now prove existence and uniqueness of solutions for problem (25) for sufficiently short time intervals. Thanks to the estimates of the previous section, for all \( m \in \mathbb{N} \) we have \( \mathcal{S}_u^m \circ \mathcal{S}_d^m : C(0, T_m; V^m) \to H^1(0, T_m; V^m) \), \( \mathcal{S}_u^m \circ \mathcal{S}_d^m : \mathfrak{f}^m \mapsto u^m(d^m(\mathfrak{f}^m)) \) where the immersion \( H^1(0, T_m; V^m) \to C(0, T_m; V^m) \) is actually compact due to Rellich theorem and to the finite-dimensionality of \( V^m \).

Let now \( M > 0 \) such that \( |u_0|^2 + \frac{\alpha}{2} \|g^T\|^2_{L^2(0, T; V^c)} \leq \frac{M}{2} \). Thanks to estimates (28) and (30), if \( |\mathfrak{f}^m(t)|_2^2 \leq M \) for all \( t \in [0, T_m] \), we have:

\[
|u^m(t)|_2^2 \leq \frac{M}{2} + \frac{C_m}{\nu} T \| \nabla d^m \|^4_{L^\infty(0, T; L^2)}
\]

that is \( |u^m(t)|_2^2 \leq M \) for all \( t \in [0, T_m] \) where \( 0 < T_m \leq T \) is sufficiently small so that the norms of \( d^m \) are suitably bounded. We can therefore apply Schauder’s Theorem and obtain existence of a solution \( u^m \in H^1(0, T_m; V^m) \), \( d^m \in L^\infty(0, T_m; H^1) \cap L^2(0, T_m; H^2) \) to problem (25). Uniqueness can be proven in a standard way and we leave out the straightforward details.

Extending approximating solutions. Since \( T_m \to 0 \) when \( m \to \infty \), before passing to the limit on \( m \), we still need to extend these approximating solutions. The a priori estimates needed to accomplish this step can be obtained by considering a different lifting problem for system (1):

\[
\begin{aligned}
-\Delta d &= 0 & \text{in } \Omega; \\
\partial \Omega &= \mathbf{h} & \text{on } \partial \Omega.
\end{aligned}
\]  

(31)

From the standard theory for elliptic partial differential equations we know the following existence and regularity result.

**Lemma A.8.** Under assumptions (1) and (4) the lifting problem (31) has a unique solution \( d \) in \( H^1(0, T; L^2) \cap L^\infty(0, T; H^1) \cap L^2(0, T; H^2) \) and the following estimates hold for a.e. \( t > 0 \):

\[
\int_0^T |d|_{H^2}^2 \, dt \leq C \int_0^T |h|_{H^{3/2}(\partial \Omega)}^2 \, dt
\]

and

\[
\int_0^T |\partial_t d|_{L^2}^2 \, dt \leq C \int_0^T |\partial_t h|_{H^{-1/2}(\partial \Omega)}^2 \, dt.
\]

**Lemma A.9.** Under assumptions (5) and (7) the lifting problem (31) has a unique solution \( d \) in \( H^1(0, T; H^1) \cap L^\infty(0, T; H^2) \cap L^2(0, T; H^3) \). Moreover, the following estimates hold for a.e. \( t > 0 \):

\[
\int_0^T |d|_{H^3}^2 \, dt \leq C \int_0^T |h|_{H^{5/2}(\partial \Omega)}^2 \, dt
\]

and

\[
\int_0^T |\partial_t d|_{H^1}^2 \, dt \leq C \int_0^T |\partial_t h|_{H^{1/2}(\partial \Omega)}^2 \, dt.
\]
In the following, we will write \( \dot{d}^m \equiv d^m - \dot{d} \) to refer to this differently lifted function. We immediately observe that \( \dot{\varepsilon} \) and \( \dot{u} \) are uniformly bounded in time. We can now pass to the limit in (25). We recall that the lifting of the original equation (1) for \( \dot{d}^m \) and \( \dot{u} \) is bounded in \( \Omega \) and with \( \mu \) and \( \lambda \) first eigenvalues of the homogeneous Laplace-Dirichlet operator on \( \Omega \) we eventually have the following estimate:

\[
|u^m|^2 + |\nabla d^m|^2 \leq e^{-C_0 t} (|u_0|^2 + |\nabla d_0|^2) + \frac{8|\Omega|}{27 C_0 \epsilon^4} + C_\Omega |\partial_s h|_{H^{-1/2}}^2 + \frac{1}{\nu} |g|^2 |
\]

where \( C_\Omega \) is a constant depending on \( \Omega \) and \( \nu \). We have, indeed, shown that the \( L^2 \) norm of \( u^m \) and \( \nabla d^m \) are uniformly bounded in \( m \) and that we can extend all approximating solutions up to any fixed time \( T \).

**Passing to the limit.** We can now pass to the limit in (25). We recall that the sequence \( \{u^m, d^m\} \) is bounded in \( L^\infty(0, T; H \times H^1) \) and in \( L^2(0, T; V \times H^2) \) and is such that \( \{d^m\} \) is bounded in \( L^\infty(0, T; L^\infty) \).

Using identity (29), we can deduce directly from equation (35) that \( \partial_t u^m \) is bounded in \( L^p(0, T; V^*) \) and that \( \partial_t d^m \) is bounded in \( L^p(0, T; L^2) \), with \( p = 2 \) when \( n = 2 \) and \( p = 4/3 \) when \( n = 3 \). Actually, for the convective term, the following estimates hold (see [31]):

\[
|((u \cdot \nabla) v, w)| \leq C \left\{ \begin{array}{ll}
|u|^{1/2} |v|^{1/2} |w|^{1/2} & \text{for } n = 2 \\
|u|^{1/4} |v|^{3/4} |w|^{1/4} & \text{for } n = 3
\end{array} \right.
\]

and

\[
|((u \cdot \nabla) d, e)| \leq C \left\{ \begin{array}{ll}
|u|^{1/2} |d|^{1/2} |e|^{1/2} & \text{for } n = 2 \\
|u|^{1/4} |d|^{3/4} |e|^{1/4} & \text{for } n = 3
\end{array} \right.
\]
Moreover we can bound the nonlinear coupling term \((\Delta d, \nabla d \cdot v)\) by using the identity (29) and recalling that \(\nabla \cdot v = 0\), \(\forall v \in V\):

\[
|\langle \Delta d, \nabla d \cdot v \rangle| \leq |\langle \nabla d \otimes \nabla d, \nabla v \rangle| + \frac{1}{2} |\langle \nabla d^2, \nabla \cdot v \rangle|
\]

\[
\leq |\nabla d|^2 |v|_2 \leq |d|_{H^2}^{2-2\eta}|d|_{H^2}^{2\eta} |v|_H^2,
\]

where \(\eta = \frac{1}{2}\) for \(n = 2\) and \(\eta = \frac{3}{4}\) when \(n = 3\).

Using Banach-Alaoglu theorem it is thus possible to extract subsequences which converge weakly-\((\ast)\) in each one of the above functional spaces. The passage to the limit is then standard observing that, since from the weak maximum principle we have \(|d_m| \leq 1\), the potential term \(f(d_m)\) is dominated by a constant and we can use the dominated convergence theorem.

Before concluding we note that, passing to the limit in (35), we can also easily prove lemma 2.3.

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[18] F. Lin and C. Liu, *Nonparabolic dissipative systems modelling the flow of liquid crystals*, Comm. Pure Appl. Math., 48 (1995), 501–537.
[19] J.-L. Lions and E. Magenes, “Problèmes aux limites non homogènes et applications,” Volume 1, Dunod, Paris, 1968.
[20] C. Liu and J. Shen, *On liquid crystal flows with free-slip boundary conditions*, Discrete Contin. Dyn. Syst., 7 (2001), 307–318.
[21] S. Lu, H. Wu and C. Zhong, *Attractors for nonautonomous 2D Navier-Stokes equations with normal external forces*, Discrete Contin. Dyn. Syst., 13 (2005), 701–719.
[22] S. Lu, *Attractors for nonautonomous 2D Navier-Stokes equations with less regular normal forces*, J. Differential Equations, 230 (2006), 196–212.
[23] A. Miranville and S. Zelik, *Attractors for dissipative partial differential equations in bounded and unbounded domains*, in “Handbook of Differential Equations, Evolutionary Partial Differential Equations” Vol. 4 (eds. C. Dafermos and M. Pokorny), Elsevier, Amsterdam (2008), 103–200.
[24] R. Rosa, *The global attractor for the 2D Navier-Stokes flow on some unbounded domains*, Nonlinear Anal., 32 (1998), 71–85.
[25] S. Shkoller, *Well-posedness and global attractors for liquid crystals on Riemannian manifolds*, Comm. Partial Differential Equations, 27 (2002), 1103–1137.
[26] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl., 146 (1987), 65–96.
[27] I. Stewart, “The Static and Dynamic Continuum Theory of Liquid Crystals, A Mathematical Introduction,” Taylor & Francis, London and New York, 2004.
[28] H. Sun and C. Liu, *On energetic variational approaches in modelling the nematic liquid crystal flows*, Discrete Contin. Dyn. Syst., 23 (2009), 455–475.
[29] L. Tartar, “An Introduction to Sobolev Spaces and Interpolation Spaces,” Springer Verlag, Berlin Heidelberg, 2007.
[30] R. Temam, “Infinite-dimensional Dynamical Systems in Mechanics and Physics,” 2nd edition, Appl. Math. Sci., 68, Springer Verlag, New York Berlin Heidelberg, 1997.
[31] R. Temam, “Navier-Stokes Equations: Theory and Numerical Analysis,” Reprint of the 1984 edition, AMS, Chelsea Publishing, Providence RI, 2001.
[32] E. Virga, “Variational Theories for Liquid Crystals,” Applied Mathematics and Mathematical Computations, 8, Chapman & Hall, London, 1994.
[33] H. Wu, *Long-time behaviour for a nonlinear hydrodynamic system modelling the nematic liquid crystal flows*, Discrete Contin. Dyn. Syst., 20 (2010), 379–396.
[34] H. Wu, X. Xu and C. Liu, *Asymptotic behavior for a nematic liquid crystal model with different kinematic transport properties*, preprint, arXiv:0901.1751v2.

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