Symmetric Orbifold Theories from Little String Residues

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Abstract

We study a class of Little String Theories (LSTs) of A type, described by \( N \) parallel M5-branes spread out on a circle and which in the low energy regime engineer supersymmetric gauge theories with \( U(N) \) gauge group. The BPS states in this setting correspond to M2-branes stretched between the M5-branes. Generalising an observation made in arXiv:1706.04425, we provide evidence that the BPS counting functions of special subsectors of the latter exhibit a Hecke structure in the Nekrasov-Shatashvili (NS) limit, i.e. the different orders in an instanton expansion of the supersymmetric gauge theory are related through the action of Hecke operators. We extract \( N \) distinct such reduced BPS counting functions from the full free energy of the LST with the help of contour integrals with respect to the gauge parameters of the \( U(N) \) gauge group. Physically, the states captured by these functions correspond to configurations where the same number of M2-branes is stretched between some of these neighbouring M5-branes, while the remaining M5-branes are collapsed on top of each other and a particular singular contribution is extracted. The Hecke structures suggest that these BPS states form the spectra of symmetric orbifold CFTs. We furthermore show that to leading instanton order (in the NS-limit) the reduced BPS counting functions factorise into simpler building blocks. These building blocks are the expansion coefficients of the free energy for \( N = 1 \) and the expansion of a particular function, which governs the counting of BPS states of a single M5-brane with single M2-branes ending on it on either side. To higher orders in the instanton expansion, we observe new elements appearing in this decomposition, whose coefficients are related through a holomorphic anomaly equation.

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1 Introduction

Little String Theories (LSTs) were first introduced in [1–7]. They are a type of quantum theories in 6 dimensions, which behave like ordinary quantum field theories (with point-like degrees of freedom) in the low energy regime, but whose UV completion requires the inclusion of string-like degrees of freedom. On the one hand side, LSTs serve in many aspects as toy models of string theory, with the only difference that the gravitational sector is absent: indeed, in practice, many examples of LSTs are obtained from (type II) string theory or M-theory through a particular decoupling limit, which sends the string coupling to zero, while leaving the string length finite. Thus studying properties of LSTs gives us an important window into string- or M-theory, which are intrinsically difficult to study by more direct means. On the other hand, conversely a better understanding of LSTs also provides us with more information about the (supersymmetric) gauge theories that are engineered in the low energy sector: due to their stringy origins, LSTs inherit numerous symmetries and dualities from string- and M-theory, remnants of which are still visible in the low energy gauge theories engineered by the LSTs.

In the same spirit, there exist many (geometric and computational) tools that have been developed in the framework of full-fletched string theory (or related applications), which allow us to perform many explicit computations for LSTs. For example, geometrical methods which have been used to classify conformal field theories in 6 dimensions or less [8–19], have recently also been deployed to attempt a classification of LSTs [20, 15]. Indeed, while an ADE-classification of LSTs is know since some time [1–7], recent efforts have focused on sharpening the list of all possible such theories.

Furthermore, a specific class of theories which have received a lot of attention recently are LSTs of type A. Such theories, compactified on a circle, have been studied using various dual approaches in string or M-theory: on the one hand side, they can be described by configurations of \( N \) parallel M5-branes that are separated along a circle \( S^1_\rho \) and compactified on a circle \( S^1_\tau \). BPS configurations in this setting correspond to M2-branes that are stretched between neighbouring M5-branes and wrapping \( S^1_\tau \). The partition function of LSTs compactified on \( S^1_\tau \), \( Z_{N,1} \), can be calculated by analysing the theory on the intersection of the M2- and M5-branes [21, 25, 26] using a 2-dimensional sigma-model description. In order to render \( Z_{N,1} \) well defined, the introduction of two regularisation parameters \( \epsilon_{1,2} \) is required. From the point of view of the low energy gauge theory description, the latter correspond to the parameters of the \( \Omega \)-background, which are needed to regularise the Nekrasov partition function. A dual approach is given by F-theory compactified on a class toric Calabi-Yau manifolds [27] called \( X_{N,1} \). The web diagram of the latter can directly be read off from the brane-web diagram

\(^1\)We refer the reader to [21,24] for more details on the brane setup.
representing the system of M2-M5-branes mentioned above \[21, 25, 26\]. Furthermore, \(Z_{N,1}\) in this approach is captured by the topological string partition function on \(X_{N,1}\), which in turn can be very efficiently calculated with the help of the topological vertex \[28, 29\]. The regularisation parameters \(\epsilon_{1,2}\) in this approach are intrinsic to the refined topological string \[30, 32\] (see also \[33-36\]).

Recent studies have exploited this efficient way to explicitly compute \(Z_{N,1}\) (or the corresponding free energy \(F_{N,1}\)) to study symmetries and other properties of the corresponding LSTs. In the process, numerous very interesting and unexpected structures have been discovered, among others:

(i) Dihedral and paramodular symmetries:
The Calabi-Yau manifold \(X_{N,1}\) engineers a supersymmetric gauge theory on \(\mathbb{R}^4 \times S^1 \times S^1\) with \(U(N)\) gauge group and matter in the adjoint representation. The Kähler moduli space of \(X_{N,1}\) can be understood as a subregion of a much larger so-called extended moduli space. Depending on the value of \(N\), there exist further regions in the latter which engineer supersymmetric gauge theories with different gauge structures and matter content. Many of these theories are dual to each other in the sense that they share the same partition function. The duality map, however, is intrinsically non-perturbative. More concretely, it was conjectured in \[37\] and proven in \[38, 39\] that the \(U(N)\) gauge theory above is dual to a circular quiver gauge theory with a gauge group made up of \(M'\) factors of \(U(N')\) and bifundamental matter, for any pair \((N', M')\) such that \(M'N' = N\) and \(\gcd(M', N') = 1\).\(^2\)

It was shown in \[41\], that this web of dualities implies additional symmetries for the partition function \(Z_{N,1}\) (as well as the free energy \(F_{N,1}\)). While it is clear (due to the structure of \(X_{N,1}\) as a double-elliptic fibration), that the latter are symmetric with respect to two modular groups called \(SL(2,\mathbb{Z})_\rho\) and \(SL(2,\mathbb{Z})_\tau\)\(^3\), it was shown in \[41\] that they also enjoy a dihedral symmetry, which (from the perspective of the gauge theories) acts in an intrinsically non-perturbative fashion. Moreover, it was argued in \[42\] that a particular subsector of the BPS-states (namely the sector of states which carry the same \(U(1)\) charges under all the generators of the Cartan subalgebra of the \(U(N)\) gauge group), is invariant under the level \(N\) paramodular group \(\Sigma_N \subset Sp(4,\mathbb{Q})\).

(ii) Hecke structures:
In \[42\] evidence was presented that in the Nekrasov-Shatashvili (NS) limit \[43, 44\] (which
\(^2\)In \[37\], the much stronger conjecture was put forth that the Calabi-Yau manifolds \(X_{N,M}\) and \(X_{N',M'}\) are dual to each other if \(NM = N'M'\) and \(\gcd(N,M) = \gcd(N',M')\). This implies a duality between gauge theories with gauge group \(U(N)^M\) and \(U(N')^{M'}\). Numerous examples have been successfully tested in \[37, 66, 39\]. Furthermore, the case \(\gcd(N,M) = 1\) has been proven in \[39\] for arbitrary values of \(\epsilon_{1,2}\) and a proof for generic \(N,M\) for \(\epsilon_{1,2} \to 0\) has been presented in \[40\].
\(^3\)The notation follows the Kähler parameters which act as modular parameters for the two groups.
in our notation essentially corresponds to the limit $\epsilon_2 \to 0$), the paramodular group $\Sigma_N$ that is present in the above mentioned subsector of BPS states, is further extended to $\Sigma^*_N$. The latter is obtained from $\Sigma_N$ through the inclusion of a further generator that exchanges the modular parameters $\rho$ and $\tau$ of the two modular groups mentioned above (see appendix D for details). This result corroborates the observation made in [45] that the states of the subsector of BPS states mentioned above (in the NS-limit) can be organised into a symmetric orbifold CFT. The latter in particular implies that the expansion coefficients of the reduced free energy (that only counts states in this particular BPS-subsector) are related through the action of Hecke operators. This relation was indeed observed in [45] in numerous examples.

(iii) Factorisation to leading instanton order and graph functions:

In [46] non-trivial evidence was provided that the free energy $F_{N,1}$ in the so-called unrefined limit (i.e. for $\epsilon_2 = -\epsilon_1$) factorises in a very intriguing fashion: for the examples $N = 2, 3, 4$, it was shown that the free energy to leading instanton order (from the perspective of the $U(N)$ gauge theory) can be decomposed into sums of products of the functions $H^{(r)}_{N=1}$ and $W^{(r)}_{\text{NS}}$. The former are the expansion coefficients of the instanton expansion of the free energy $F_{N=1,1}$, while the latter are the expansion coefficients of a function that governs the counting of BPS states of an M5-branes with a single M2-brane ending on either side. Furthermore, it was observed in [46] that the coefficients appearing in this expansion of $F_{N,1}$ resemble in many respects so-called modular graph functions, which have recently appeared in the study of scattering amplitudes in string theory [47–56]. Higher terms in the instanton expansion are more complicated: certain parts still allow to be factorised into simpler building blocks, however, on the one hand side the coefficient functions that appear in this way are more complicated and on the other hand the inclusion of Hecke transformations of $H^{(r)}_{N=1}$ and $W^{(r)}_{\text{NS}}$ is required. While primarily dealing with the unrefined free energy, preliminary results in [46] indicate similar decompositions (albeit more complicated) to be valid in the NS-limit.

The current paper is a continuation of the analysis of the symmetries and structures discovered in [42, 57, 46, 45]: focusing on the NS-limit of the free energy, we analyse the form of the free energy that was found in [42, 57, 46] and which is implied by the property (i) above. We observe new subsectors of the BPS-states that show a similar Hecke structure as discussed under (ii) above. Based on the examples of $N = 2$ and $N = 3$ (as well as partial results for $N = 4$), we observe for given $N$ and at each order in an expansion of $\epsilon_1$, $N$ distinct subsectors of the NS-limit of the BPS-free energy $F_{N,1}$ that exhibit such structures. We call the functions which count these BPS states at the $r$-th instanton order $C^{N,(r)}_{i,1,2s,0}$, where $i = 1, \ldots, N$ and $s \in \mathbb{N}$

\footnote{Explicit expressions for $H^{(r)}_{N=1}$ and $W^{(r)}_{\text{NS}}$ as well as more information can be found in appendix C}
indicates the order in an expansion in powers of $\epsilon_1$. The latter can abstractly be defined for general $N$ through contour integrals of (an expansion in powers of $\epsilon_1$ of the NS-limit of) $\mathcal{F}_{N,1}$ with respect to the gauge parameters $\hat{a}_1, \ldots, N$ of the $a_{N-1}$ gauge algebra (or their exponentials $Q_{\hat{a}_i} = e^{2\pi i \hat{a}_i}$ for $i = 1, \ldots, N$). These contours extract specific coefficients in a Fourier expansion of $\mathcal{F}_{N,1}$ in $Q_{\hat{a}_i}$ and/or particular poles in a limit where some of the $\hat{a}_i$ vanish (see eq. (3.1) for the abstract definition of the $\mathcal{C}_{i,(2s,0)}^{N,(r)}$). From a physical perspective the functions $\mathcal{C}_{i,(2s,0)}^{N,(r)}$ only receive contributions from M5-brane configurations where the same number of M2-branes is stretched between some of the adjacent M5-branes (see Fig. [4] for a schematic representation). From these configurations in turn specific poles are extracted in the limit where the remaining M5-branes coincide. The remaining functional dependence of $\mathcal{C}_{i,(2s,0)}^{N,(r)}$ is made up by two (remaining) Kähler moduli of $X_{N,1}$ (which we call $\rho$ and $S$). Finally, we can resum the $\mathcal{C}_{i,(2s,0)}^{N,(r)}$ into a Laurent series expansion $\mathcal{C}_{i}^{N,(r)}(\rho, S, \epsilon_1)$ in powers of $\epsilon_1$.

We observe that the functions $\mathcal{C}_{i,(2s,0)}^{N,(r)}$ obtained in this fashion show numerous interesting properties. First of all, they are quasi-Jacobi forms of index $rN$ and weight $2s - 2i$. Moreover, the functions for $r > 1$ can be obtained through the action of the $r$-th Hecke operator $\mathcal{H}_r$ (see (A.10) in appendix A for a definition) on $\mathcal{C}_{i,(2s,0)}^{N,(r=1)}$.

$$\mathcal{C}_{i,(2s,0)}^{N,(r)}(\rho, S) = \mathcal{H}_r \left[ \mathcal{C}_{i,(2s,0)}^{N,(1)}(\rho, S) \right], \quad \forall r \geq 1, \forall i = 1, \ldots, N. \quad (1.1)$$

Following the logic of [45], this suggests that the corresponding BPS states can be organised into a symmetric torus orbifold CFT. However, since the seed function (i.e. the initial function $\mathcal{C}_{i,(2s,0)}^{N,(r=1)}$) is different in each case, the corresponding target spaces are different for all $i = 1, \ldots, N$. Indeed, the functions $\mathcal{C}_{i}^{N,(r=1)}$ can be factorised in terms of $H_{N=1}^{(r=1)}$ and $W_{NS}^{(r=1)}$ in a very simple fashion (see eq. (3.9)). For $r > 1$, the $\mathcal{C}_{i}^{N,(r)}$ still can mostly be decomposed into $H_{N=1}^{(r=1)}(\rho, S)$ and $W_{NS}^{(r=1)}(\rho, S)$, up to remainder functions (see eq. (3.11)). The latter, however, are not arbitrary, but are connected by equations that strongly resemble holomorphic anomaly equations [58]. These results generalise the properties of the free energy discussed under (iii) above. Since the results in this paper are obtained by studying the examples of $N = 2$ and $N = 3$ (as well as partially $N = 4$) their generalisations to higher $N$ have to be considered conjectures. However, the large number of examples that all follow the same pattern, provides rather strong evidence in their favour.

This paper is organised as follows: In section 2 we review the LST partition function $Z_{N,1}$ and the associated free energy $\mathcal{F}_{N,1}$, as well as some of their properties discovered in recent publications. Due to the technical nature of some of the subsequent discussions, we provide a summary of the results of this paper in section 3. The sections 4, 5 and 6 provide a detailed discussion of the LST free energies for $N = 2$, $N = 3$ and $N = 4$ respectively.
Finally, section 7 contains our conclusions. Additional details on modular objects, explicit discussions of properties of the free energy, the discussion (and explicit expressions for some of their expansion coefficients) of the fundamental building blocks $H_{r N=1}^{(r)}$ and $W_{NS}^{(r)}$ as well as the definition of paramodular groups, which have been deemed too long for the main body of this paper have been relegated to four appendices.

2 Little String Free Energies and Their Properties

The Little String Theories (LSTs) of A-type that we are interested in can be studied by exploiting various dual descriptions in string or M-theory. On the one hand side, they can be described through configurations of parallel M5-branes compactified on a circle of circumference $\tau$ and spread out on a circle with circumference $\rho$, where the distances between the neighbouring M5-branes are denoted $(t_1, \ldots, t_N)$ such that,

$$\rho = t_1 + t_2 + \ldots + t_N .$$

BPS states in this setting are given by M2-branes. The latter are stretched between the neighbouring M5-branes and appear as strings in their worldvolumes, wrapping the circle $S^1_\tau$ on which the M5-branes are wrapped [21]. In this context, arbitrarily many M2-branes can be stretched between any of the neighbouring M5-branes (a schematic example is shown in Fig. 1). The space transverse to the M2-branes inside the M5-brane worldvolume is $\mathbb{R}^4_{\parallel}$ and the M2-branes appear as point particles in this space. The worldvolume theory of M2-branes has $\mathcal{N} = (4, 4)$ supersymmetry which is broken down to $\mathcal{N} = (0, 2)$ by a $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2} \times U(1)_m$ action on $\mathbb{R}^4_{\parallel} \times \mathbb{R}^4_{\perp}$ [21].

$$\begin{align*}
(z_1, z_2, w_1, w_2) \in \mathbb{C}^2_{\parallel} \times \mathbb{C}^2_{\perp} &\mapsto (z_1 e^{i\epsilon_1}, z_2 e^{i\epsilon_2}, w_1 e^{i(m+\epsilon_+)}, w_2 e^{i(m-\epsilon_+)}) . 
\end{align*}$$

The BPS degeneracies of the M2-branes is captured by the elliptic genus of the worldvolume theory which depends on the parameters $(\tau, t_1, \ldots, t_N, m, \epsilon_{1,2})$. This can be calculated by studying the gauge and matter content of the $(0, 2)$ worldvolume theory and using the techniques developed in [59, 61]. A different approach is to calculate the $(0, 2)$ elliptic genus of the sigma model to which the worldvolume theory flows in the infrared. The target space of the sigma model in this case is the product of Hilbert schemes of points on $\mathbb{C}^2_{\parallel}$ and the equivariant elliptic genus

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Figure 1: $N$ parallel M5-branes (orange) with $(n_1, \ldots, n_N)$ M2-branes (blue) stretched between them. The distances between the M5-branes are $t_1, \ldots, t_N$. We define $q = e^{i\epsilon_1}$ and $t = e^{-i\epsilon_2}$ so that the unrefined limit is $q = t$. 

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can be calculated using the details of the $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}$ action on the target space [62, 63, 21]. The theory on the worldvolume of the compactified M5-branes is the five dimensional $\mathcal{N} = 1^*$ quiver gauge theory. The partition function of this gauge theory captures the M2-brane BPS states as well and is given by the generating function of equivariant elliptic genera of the rank $N$ charge $K$ instanton moduli spaces $M(N, k)$ [21, 26].

A dual approach to describe the same LST is through F-theory compactified on a toric Calabi-Yau threefold [27] which in [22] was called $X_{N,1}$. The BPS string states are given by D3-branes wrapping various rational curves in the base of the Calabi-Yau threefold with Kähler parameters $t_1, \ldots, t_N$. The web diagram of the latter can be directly read off from the brane web configuration discussed before and is shown in Fig. 2. The latter figure also includes a definition (shown in blue) of a basis of the Kähler parameters of $X_{N,1}$: these include besides $t_1, \ldots, t_N$ also $\tau$ and $m$ which can be expressed in terms of the basis $(h_1, \ldots, h_N, m, v)$. From the perspective of F-theory compactified on $X_{N,1}$, the little string partition function $Z_{N,1}$ is captured by the topological string partition function on $X_{N,1}$ [20, 22]. The latter can be computed in an efficient manner using the refined topological vertex formalism [28, 29].

![Figure 2: Web diagram of $X_{N,1}$.](image)

In [21, 25, 26, 22, 64, 65] two different expansions of $Z_{N,1}$ and their interpretations were studied,

$$Z_{N,1} = \sum_k Q_k^k Z_k(t_1, \ldots, t_N, m, \epsilon_1, \epsilon_2), \quad (2.3)$$

where $Q_\tau = e^{2\pi i \tau}$ and $Q_{t_i} = e^{2\pi i t_i}$.

As discussed in Appendix B, $Z_k(t_1, \ldots, t_N, m, \epsilon_1, \epsilon_2)$ is the equivariant $(2, 2)$ elliptic genus of $M(N, k)$. This expansion of the partition function is natural when considering the theory on the M5-branes. The expansion in the second line in Eq. (2.3) gives the functions $Z_{k_1 \ldots k_N}$ which capture the degeneracy of configurations of M2-branes in which $k_i$ M2-branes are stretched between the $i$-th and $(i + 1)$-th M5-brane. The $Z_{k_1 \ldots k_N}$ is the equivariant $(0, 2)$ elliptic genus with target space $\bigotimes_{i=1}^N \text{Hilb}^{k_i}[\mathbb{C}^2]$ with right moving fermions coupled to a bundle $V_{k_1 \ldots k_N}$, the details of which are given in [21, 26].

In [26, 22] the following little string free energy $F_{N,1}^{\text{plet}}$ was discussed

$$F_{N,1}^{\text{plet}}(t_1, \ldots, N, m, \tau, \epsilon_1, \epsilon_2) = \text{Plog } Z_{N,1}(t_1, \ldots, N, m, \tau, \epsilon_1, \epsilon_2), \quad (2.4)$$

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where, Plog denotes the plethystic logarithm\textsuperscript{6} of $Z_{N,1}$. The exact form of $Z_{N,1}$ is given in Appendix B. From a physical perspective, $F_{N,1}^{\text{plet}}$ only counts single-particle BPS excitations of the LST, projecting out multi-particle states. Similar to the two equivalent expansions of the partition function in eq. (2.3), one can similarly consider two different ways of expanding $F_{N,1}^{\text{plet}}$

$$F_{N,1}^{\text{plet}}(t_1,...,N,m,\tau,\epsilon_{1,2}) = \sum_{k} Q_k^{\text{plet}}(t_1,...,N,m,\epsilon_{1,2}) = \sum_{k_1...k_N} Q_{k_1}^{\text{plet}} \cdots Q_{k_N}^{\text{plet}} F_{N,1}^{(k_1,...,k_N)}(\tau, m, \epsilon_{1,2}).$$

(2.5)

In previous work numerous properties of the free energy $F_{N,1}^{\text{plet}}$ (or some of the coefficients appearing in these two expansions) have been discovered. In the following we shall discuss some of them, which will turn out important for our current work:

- **Recursion relation:**
  In [23, 24] the counting functions of a particular class of single BPS states has been discussed: these states correspond to M-brane configurations of the type schematically shown in Fig. 1, however, they are special in the sense that they have one (or several neighbouring) M5-brane(s) with only a single M2-brane ending on them on either side. In the notation of Fig. 1 these are characterised by the fact that several adjacent $n_i$ are identical to 1, i.e.

$$\left(n_1, \ldots, n_k, 1, \ldots, 1, n_{k+m}, \ldots, n_N \right)_{m\text{-times}}.$$  

The BPS degeneracy of such states is captured by $F_{N,1}^{(k_1,...,k_N)}$ (defined in Eq. (2.5)) with $(k_1,\cdots,k_N) = (n_1,\ldots,n_k,1,\ldots,1,n_{k+m},\ldots,n_N)$. It was observed that in this case,

$$F_{N,1}^{(n_1,\ldots,n_k,1,\ldots,n_{k+m},\ldots,n_N)} = F_{N,1}^{(n_1,\ldots,n_k,1,n_{k+m},\ldots,n_N)} \cdot W(\tau, m, \epsilon_{1,2})^{m-1}.$$  

(2.7)

The relative factor $W$ appearing in this relation is a quasi-modular form and is given by,

$$W(\tau, m, \epsilon_{1,2}) = \frac{\theta_1(\tau, m + \epsilon_+) \theta_1(\tau, m - \epsilon_+) - \theta_1(\tau, m + \epsilon_-) \theta_1(\tau, m - \epsilon_-)}{\theta_1(\tau, \epsilon_1) \theta_2(\tau, \epsilon_2)},$$

(2.8)

with $\epsilon_\pm = \frac{\epsilon_1 \pm \epsilon_2}{2}$. Further information on this function and particular expansions that will be useful in the remainder of this paper can be found in Appendix C.2

- **Self-similarity:** In [24] it has been observed that in the NS-limit and in a certain region

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\textsuperscript{6}The plethystic logarithm of a function $g(x_1, x_2, \cdots, x_K)$ is given by $\text{Plog} g(x_1, x_2, \cdots, x_K) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log g(nx_1, nx_2, \cdots, nx_K)$ where $\mu(n)$ is the Möbius function.
of the Kähler moduli space of $X_{N,1}$, the part of the free energy that counts only single particle states, becomes directly related to the BPS counting function for the LST with $N = 1$ (and thus proportional to the free energy $F^\text{plet}_{N=1,1}$). With the notation introduced above, the particular region in the moduli space is defined as

$$t_1 = t_2 = \ldots = t_N = \frac{\rho}{N},$$

so that the M5-branes are all at equal distance from each other on the circle. In this region of the moduli space the free energy is a function of $(\tau, \rho, m)$ only and

$$F^\text{plet,NS}_{N,1} \left( \frac{\rho}{N}, \ldots, \frac{\rho}{N}, \tau, m, \epsilon \right) = F^\text{plet}_{N=1,1} \left( \frac{\rho}{N}, \tau, m, \epsilon \right),$$

where the Nekrasov-Shatashvili (NS) limit \cite{43, 44} is defined as,

$$F^\text{plet,NS}_{N,1} (t_1, \ldots, t_N, \tau, m, \epsilon) = \lim_{\epsilon_2 \to 0} \frac{\epsilon_2}{\epsilon_1} F^\text{plet}_{N,1} (t_1, \ldots, t_N, \tau, m, \epsilon_1).$$

- **Hecke structures and torus orbifold:** In \cite{15} contributions to the free energy coming from BPS states which carry the same charges under all the generators of the Cartan subalgebra of the gauge algebra $a_{N-1}$ were studied. In the language of the M-brane description, these correspond to configurations in which an equal number of M2-branes is stretched between any of the adjacent M5-branes. The degeneracy of such states are captured by,

$$F^\text{orb}_{N,1} (\rho, \tau, m, \epsilon_1, 2) = \sum_{k} Q^{\rho}_k \chi^k_{ell}(\text{Sym}^k M_{1 \ldots 1}) = \prod_{k,n,\ell,r} \left( 1 - Q^{\rho}_k Q^n_\tau Q^\ell_m q^r \right)^{-c(kn,\ell,r,s)}.$$  

Here $c(k, \ell, r)$ are the coefficients in the Fourier expansion of $\chi^k_{ell}(M_{1 \ldots 1})$ 

$$\chi^k_{ell}(M_{1 \ldots 1}) = \sum_{k,\ell,r} c(k, \ell, r) Q^{\rho}_k Q^{n}_\tau Q^{m}_m q^r.$$

\footnote{Here $F^\text{orb,NS}_{N,1}$ is the NS-limit of $F^\text{orb}_{N,1}$ in (2.12), i.e. $F^\text{orb,NS}_{N,1} (\rho, \tau, m, \epsilon_1) = \lim_{\epsilon_2 \to 0} \frac{\epsilon_2}{\epsilon_1} F^\text{orb}_{N,1} (\rho, \tau, m, \epsilon_1, 2)$.}
In this paper we shall discuss novel properties of the free energy, which (in a certain sense) generalise some of the points mentioned above. However, to render some of these properties more clearly visible, we shall choose to slightly modify two important points:

- Instead of the basis \( (t_1, \ldots, t_N, m, \tau) \), which is defined in Fig. 2 as certain Kähler parameters of \( X_{N,1} \), we prefer to work in a different basis given by the parameters \( (\hat{a}_1, \ldots, \hat{a}_N, S, R) \): this basis was first introduced in \cite{38,66,41} and allows for a more streamlined definition of some of the symmetries of the free energy. With respect to the web diagram of \( X_{N,1} \), this basis is shown in Fig. 3. Furthermore, as was discussed in \cite{41,42}, it can be obtained from the basis \( (t_1, \ldots, t_N, m, \tau) \), through the following linear transformation\(^8\):  

\[
R = \tau - 2N m + N \rho, \quad S = -m + \rho, \quad \hat{a}_i = t_{i+1}, \quad \forall \ i = 1, \ldots, N. \tag{2.15}
\]

This transformation is part of a symmetry group \( G_N \times \text{Dih}_N \), where \( \text{Dih}_N \) is a subgroup of the Weyl group of the \( U(N) \) gauge group and \( G_N \) is a (dihedral) symmetry group that is implied by a web of dualities of the little string theory (see \cite{41} for more details). Since \( G_N \times \text{Dih}_N \) leaves the free energy invariant, the results discussed above also hold when formulated in the new basis \( (\hat{a}_1, \ldots, \hat{a}_N, S, R) \). For further convenience we also introduce  

\[
Q_R = e^{2\pi i R}, \quad Q_S = e^{2\pi i S}, \quad Q_{\hat{a}_j} = e^{2\pi i \hat{a}_j}, \quad \forall j = 1, \ldots, N. \tag{2.16}
\]

- Instead of \( F_{N,1}^{\text{plet}} \) in (2.4) (which only counts single particle BPS states), we work with the full free energy  

\[
F_{N,1}(a_1, \ldots, N, S, R, \epsilon_{1,2}) = \ln Z_{N,1}(\hat{a}_1, \ldots, \hat{a}_N, S, R, \epsilon_{1,2}), \tag{2.17}
\]

\(^8\)We implicitly use \( t_{N+1} = t_1 \).
$\mathcal{F}_{N,1}$ can be expanded in powers of $Q_R$,

$$
\mathcal{F}_{N,1}(\hat{a}_{1,...,N}, S, R; \epsilon_{1,2}) = \sum_r Q_R^r P^{(r)}_{N}(\hat{a}_{1,...,N}, S; \epsilon_{1,2}), \tag{2.18}
$$

where we can also expand the coefficients $P^{(r)}(\hat{a}_{1,...,N}, S; \epsilon_{1,2})$ in powers of $\epsilon_{1,2}$

$$
P^{(r)}_{N}(\hat{a}_{1,...,N}, S; \epsilon_{1,2}) = \sum_{s_1, s_2} \epsilon_1^{s_1-1} \epsilon_2^{s_2-1} P^{(r)}_{N,(s_1,s_2)}(\hat{a}_{1,...,N}, S). \tag{2.19}
$$

We will mostly be interested in the NS-limit, i.e. $s_2 = 0$ and $s_1 \in \mathbb{N}_{\text{even}}$. Finally, we can expand the $P^{(r)}_{N,(s_1,s_2)}$ in powers of $Q_{\tilde{a}_i} = e^{2\pi i \tilde{a}_i}$

$$
P^{(r)}_{N,(s_1,s_2)}(\hat{a}_{1,...,N}, S) = \sum_{n_1,...,n_N} Q_{\tilde{a}_1}^{n_1} \cdots Q_{\tilde{a}_N}^{n_N} P^{(r)}_{(n_1,...,n_N)}(S). \tag{2.20}
$$

For later convenience, we will also use the notation $\underline{n} = \{n_1,...,n_N\}$. From the $P^{(r)}_{N,(s_1,s_2)}$ we construct the following (a priori formal) object

$$
H^{(r),(\{n_1,...,n_N\})}(\rho, S) = \sum_{k=0}^{\infty} Q_{\rho}^k P^{(r),\{n_1+k,n_2+k,...,n_N+k\}}_{N,(s_1,s_2)}(S), \tag{2.21}
$$

where $Q_{\rho} = e^{2\pi i \sum_{j=1}^{N} \tilde{a}_{ij}}$. The $P^{(r)}_{N,(s_1,s_2)}(\hat{a}_{1,...,N_2}, S)$ in (2.20) are resummed as

$$
P^{(r)}_{N,(s_1,s_2)}(\hat{a}_{1,...,N}, S) = H^{(r),(\{0,0,...,0\})}_{(s_1,s_2)}(\rho, S) + \sum_{\underline{n}} H^{(r),(\underline{n})}_{(s_1,s_2)}(\rho, S) Q_{\tilde{a}_1}^{n_1} \cdots Q_{\tilde{a}_N}^{n_N}. \tag{2.22}
$$

Here the summation is over all $\underline{n} = \{n_1,...,n_N\} \in (\mathbb{N} \cup \{0\})^N$ such that at least one of the $n_i = 0$. Furthermore, we implicitly assume that $\tilde{a}_N = \rho - \sum_{i=1}^{N-1} \tilde{a}_i$.

In the remainder of this paper we identify a limit in which the NS-limit of the free energy diverges but the residue of the second order pole counts BPS stats of a symmetric orbifold theory: For example, the partition function for the case $N = 2$ is discussed in Appendix B. In this case the partition function has a pole at $\tilde{a}_1 = \pm 2\epsilon_+$, while in the NS limit the free energy ${\mathcal{F}}_{N=2,1}$ has a pole at $\tilde{a}_1 = \pm \epsilon_1$. Terms of different order in $Q_R$ have different order poles at $\tilde{a}_1 = \pm \epsilon_1$ with different residues. If we expand the NS free energy in powers of $\epsilon_1$ then the coefficients have different order poles at $\tilde{a}_1 = 0$ with residues now shifted because of the $\epsilon_1$ expansion. The lowest order pole is of order two.

On a technical level, just as in previous work, we rely on studying series expansions of examples for small values of $N$, which reveal certain patterns. However, since the corresponding computations are rather technical, we will summarise our observations in the following section.
before presenting the computations for \( N = 2, N = 3 \) and \( N = 4 \) respectively.

## 3 Summary of Results

Due to the technical nature of some of the results of this paper, we provide a short overview of our main observations. For given \( N \), we start by extracting the following \( N \) functions from the (expansion coefficients of the) free energy \( P^{(r)}_{N,(2s,0)}(\hat{a}_1,\ldots,\hat{N},S) \) in (2.19)

\[
C_{i,(2s,0)}^{N,(r)}(\rho, S) = \frac{1}{(2\pi i)^{N_{P_{i-1}}} N} \sum_{\ell=0}^{\infty} \int_{0}^{\infty} \frac{dQ_{\hat{a}_1}}{Q_{\hat{a}_1}^{1+\ell}} \cdots \frac{dQ_{\hat{a}_N}}{Q_{\hat{a}_N}^{1+\ell}} \frac{dQ_{\hat{N}}}{Q_{\hat{N}}^{1+\ell}} \frac{dQ_{\hat{1}}}{Q_{\hat{1}}^{1+\ell}} \left( \frac{Q_{\hat{1}}^{2}}{Q_{\hat{1}}} \right)^{i-1} \frac{Q_{\hat{N}}^{2}}{Q_{\hat{N}}} \frac{Q_{\hat{1}}^{2}}{Q_{\hat{1}}} \cdots \frac{Q_{\hat{N}}^{2}}{Q_{\hat{N}}} \frac{d\hat{a}_i}{\hat{a}_i} \frac{d\hat{a}_2}{\hat{a}_2} \cdots \frac{d\hat{a}_{i-1}}{\hat{a}_{i-1}} d\hat{a}_i d\hat{a}_2 \cdots d\hat{a}_{i-1} d\hat{a}_1 \cdots d\hat{a}_{i-1} (\hat{a}_1 + \cdots + \hat{a}_{i-1})}
\]

\[
\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{d\hat{a}_{i+1}}{\hat{a}_{i+1}} \frac{d\hat{a}_{i+2}}{\hat{a}_{i+2}} \cdots \frac{d\hat{a}_{N-1}}{\hat{a}_{N-1}} \frac{d\hat{a}_N}{\hat{a}_N}
\]

The latter can be resummed into a (formal) series expansion in \( \epsilon_1 \)

\[
C_{i,(2s,0)}^{N,(r)}(\rho, S, \epsilon_1) = \sum_{s=0}^{\infty} \epsilon_1^{2s-2i} C_{i,(2s,0)}^{N,(r)}(\rho, S), \tag{3.2}
\]

which defines a Jacobi form of weight zero and index \( \epsilon N \). The contour integrals in (3.1) are understood to be small circles centred around the specified points.\(^9\) From a mathematical point of view, this extracts and combines specific coefficients in a mixed Fourier-Taylor expansion of \( P^{(r)}_{N} \); with respect to the variables \( \hat{a}_j \) for \( 1 \leq j \leq i-1 \) the prescription computes successively the coefficient of the second order pole at \( \hat{a}_j = -\hat{a}_1 - \cdots - a_{j-1} \). For the variables \( \hat{a}_j \) for \( i \leq j \leq N \) the prescription sums (weighted by \( Q_{\hat{a}_j}^\ell \)) the coefficients of the term \( Q_{\hat{a}_j}^\ell Q_{\hat{a}_{j+1}}^\ell \cdots Q_{\hat{a}_N}^\ell \) in a Fourier series expansion in terms of \( Q_{\hat{a}_j} = e^{2\pi i \hat{a}_j} \). From a physical perspective, the latter prescription combines contributions from M2-brane configurations (weighted by \( Q_{\hat{a}_j}^\ell \)), in which an equal number of \( \ell \) M2-branes is stretched between the M5-branes \( j \) and \( j+1 \) for \( i \leq j \leq N \).\(^10\) \(^11\)

\[\text{Figure 4: Brane configuration for extracting } C_{i,(2s,0)}^{N,(r)}(\rho, S, \epsilon_1) \text{ (see Fig. 4) for a schematic picture of a generic configuration, as well}\]

\(^9\)Although a priori it is a formal expansion but the \( i = 1 \) case given in eq. (3.5) and the \( i = 2 \) example discussed in Appendix B for \( N = 2 \) (see eq. (B.24)) shows that it is a Jacobi form involving \( \theta_1(\rho, z) \) and its derivatives.

\(^10\)We implicitly assume here that we are in a generic point in the moduli space, such that the free energy has isolated poles with respect to the various variables. In this case, the precise form of the contour is not crucial, and the prescription is simply designed to extract their residues.

\(^11\)In this notation it is understood that the M5 brane \( N + 1 \) is in fact the M5-brane 1.
as the figures Fig. 6, Fig. 8 and Fig. 10 for examples of the cases $N = 2, 3, 4$). Concerning the remaining M5-branes, we consider a region in the moduli space, in which they all form a stack on top of each other: from the resulting divergent expression, the contour integrals for $\hat{a}_j$ for $1 \leq j \leq i - 1$ in (3.1) extract successively the poles of the form $(\hat{a}_1 + \hat{a}_2 + \ldots + \hat{a}_{i-1})^{-2}, (\hat{a}_1 + \hat{a}_2 + \ldots + \hat{a}_{i-2})^{-2}, \ldots, \hat{a}_1^{-2}$. The total order of the divergence selected in this fashion is $2i - 2$: from all the examples we have explicitly calculated, this seems to be the highest singularity that appears in the prepotential at leading instanton order. This matches the analysis of the singularities of the partition function in appendix B.

The brane configuration of parallel M5-branes labelled $1, \ldots, i - 1$ stacked on top of each other (which is used to define the $C_{i,1}^{N,(r)}$) can also be understood geometrically: we recall that the Calabi-Yau threefold $X_{N,1}$ dual to the generic M5-brane configuration has a resolved $A_{N-1}$ singularity. The case in which $i - 1$ M5-branes are on top of each other corresponds to a partial resolution of the $A_{N-1}$ singularity so that it becomes a $A_{i-1}$ singularity,

$$A_{N-1} \mapsto A_{i-2} \times A_{N-i+1}^{\text{resolved}}. \quad (3.3)$$

The functions $C_{i,1}^{N,(r)}(\rho, S, \epsilon_1)$ were already studied before in [45]. As already briefly discussed in the previous section, there (based on the study of numerous explicit examples) the following relation was observed (we are using the notation introduced above)

$$C_{i,1}^{N,(r)}(\rho, S, \epsilon_1) = \mathcal{H}_r \left[ C_{i,1}^{N,(1)}(\rho, S, \epsilon_1) \right], \quad \forall r \geq 1 \quad (3.4)$$

where $\mathcal{H}_r$ is the $r$-th Hecke operator (see eq. (A.10) in appendix A for the definition) and

$$C_{i,1}^{N,(1)}(\rho, S, \epsilon_1) = \frac{\theta_+ \theta_- (\theta_+ \theta_1' - \theta_- \theta_1')}{\eta(\rho)^{3N} \theta_1(\rho, \epsilon_1)^N} \quad (3.5)$$

with $\theta_\pm = \theta_1(\rho, -S + \rho \pm \frac{\epsilon_1}{2})$. In this paper, based on a detailed study of $C_{i}^{N,(r)}(\rho, S, \epsilon_1)$ for $N = 2$ and $N = 3$ (and partially also for $N = 4$), we provide evidence that (3.4) can be generalised to all the functions defined in (3.1):

$$C_{i,(2s,0)}^{N,(r)}(\rho, S) = \mathcal{H}_r \left[ C_{i,(2s,0)}^{N,(1)}(\rho, S) \right], \quad \forall r \geq 1, \forall i = 1, \ldots, N, s \geq 0. \quad (3.6)$$

The $C_{i,(2s,0)}^{N,(1)}$ are index $N$ Jacobi forms of weight $2s - 2i$ and are coefficients in the $\epsilon_1$ expansion of $C_{i}^{N,(1)}(\rho, S, \epsilon_1)$ which is a Jacobi form in two complex variables $(S, \epsilon_1)$ [67]. The action of the Hecke operator $\mathcal{H}_r$ extends to the case of multiple complex variables as given in eq. (A.11) in
From eq. (3.6) it then follows that,

$$C_{i}^{N,(r)}(\rho, S, \epsilon_1) = \mathcal{H}_r \left[ C_{i}^{N,(1)}(\rho, S, \epsilon_1) \right], \quad \forall r \geq 1, \quad \forall i = 1, \ldots, N. \quad (3.7)$$

Assuming that this relation indeed holds for generic $r$, a generating function can be formed capturing the degeneracies BPS states in this subsector

$$Z_{i}^{(N)}(R, \rho, S, \epsilon_1) = \exp \left( \sum_{r=1}^{\infty} Q_{R}^{N_r} C_{i}^{N,(r)}(\rho, S, \epsilon_1) \right). \quad (3.8)$$

The eq. (3.7) together with the fact that the “seed function” $C_{i}^{N,(1)}(\rho, S, \epsilon_1)$ is a weight zero (index $N$) Jacobi form implies that $Z_{i}^{(N)}(R, \rho, S, \epsilon_1)$ can be interpreted as the partition functions of symmetric orbifold conformal field theories\footnote{The power $N_r$ of $Q_R$ in the summation in (3.8) is chosen such that $Z_{i}^{(N)}$ can be recognised more readily as a paramodular form with respect to the group $\Sigma_N$ (see appendix D).}. These symmetric orbifold theories arise from a special subsector of the full theory and are extracted using the contour integrals involving $a_1, \ldots, a_{i-1}$ in the NS limit. The fact that in this special subsector the moduli space of charge $r$ instantons can be realised as symmetric product of $r$ charge 1 instantons suggests that this special subsector is getting contributions from well separated instantons only \cite{68, 69}.

Furthermore, the study of the above mentioned examples has brought to light numerous interesting patterns, which suggest additional interesting properties of the functions $C_{i}^{N,(r)}(\rho, S, \epsilon_1)$: it was already remarked in \cite{46} that to order $\mathcal{O}(Q_R)$ (i.e. for $r = 1$), the free energy can be decomposed into simpler building blocks, which are given by the expansion coefficients of the free energy for $N = 1$ (see appendix \ref{app:C.1} for the definition) as well as the expansion of the function $W$ (see (C.4) in appendix \ref{app:C.2}), which governs the counting of BPS states of a single M5-brane with single M2-branes ending on it on either side (see sections \ref{sec:4.2} and \ref{sec:5.2} for the details in the cases $N = 2$ and $N = 3$ respectively). This decomposition is also reflected at the level of the functions $C_{i}^{N,(r=1)}(\rho, S, \epsilon_1)$ (and accordingly also for their expansion coefficients $C_{i,(2s,0)}^{N,(r=1)}(\rho, S)$), for which we find order by order in $\epsilon_1$

$$C_{i}^{N,(r=1)}(\rho, S, \epsilon_1) = \mathcal{Z}_{i,N}^{(1)}(H_{N=1}^{(1)}(\rho, S, \epsilon_1))^{i} (W_{NS}^{(1)}(\rho, S, \epsilon_1))^{N-i}, \quad \forall i = 1, \ldots, N, \quad (3.9)$$

where $\mathcal{Z}_{i,N}^{(1)}$ is a numerical factor: from the study of the examples $N = 2, 3, 4$ examples we conjecture that the factor $\mathcal{Z}_{i,N}^{(r)}$ only depends on $i$ and $N$. Modulo the factor $\mathcal{Z}_{i,N}^{(1)}$ the functions
$C_{i}^{N,(r=1)}(\rho, S, \epsilon_{1})$ satisfy the recursive relation:

$$
C_{i}^{N+1,(r=1)}(\rho, S, \epsilon_{1}) \sim C_{i}^{N,(r=1)}(\rho, S, \epsilon_{1}) W^{(1)}_{NS}(\rho, S, \epsilon_{1}), \\
C_{i+1}^{N+1,(r=1)}(\rho, S, \epsilon_{1}) \sim C_{i}^{N,(r=1)}(\rho, S, \epsilon_{1}) H^{(1)}_{N=1}(\rho, S, \epsilon_{1})
$$

(3.10)

Starting from a configuration of $(N+1)$ M5-branes with $i$ of them collapsed to form a stack, the first recursion relation suggests that the BPS states that contribute to the poles in $C_{i}^{N+1,(r=1)}$ can be counted from a similar configuration, where we remove one of the M5-branes that is not part of the stack and it is related to $C_{i}^{N,(r=1)}$ through multiplication with $W^{(1)}_{NS}(\rho, S, \epsilon_{1})$.

Similarly, the second recursion relation suggests that the effect of removing an M5-brane from the stack of collapsed branes on the counting function $C_{i}^{N+1,(r=1)}$ is described by multiplying $C_{i}^{N,(r=1)}$ with the function $H^{(1)}_{N=1}(\rho, S, \epsilon_{1})$.

To higher orders in $Q_{R}$ (i.e. for $r > 1$), the decomposition is more complicated. While we did not manage to identify all coefficients uniquely\(^{13}\), the examples we have studied suggest

$$
C_{i}^{N,(r)}(\rho, S, \epsilon_{1}) = z_{i,N}^{(r)} (H_{N=1}^{(r)}(\rho, S, \epsilon_{1}))^{i} (W^{(r)}_{NS}(\rho, S, \epsilon_{1}))^{N-i} + \mathcal{R}_{i}^{N,(r)}(\rho, S, \epsilon_{1}),
$$

(3.11)

where the rest-term $\mathcal{R}_{i}^{N,(r)}$ itself can be decomposed into combinations of $H_{N=1}^{(r)}(\rho, S, \epsilon_{1})$ where the coefficients are quasi-modular forms that only depend on $\rho$ and which can be written as harmonic polynomials in the Eisenstein series $(E_{2}(\rho), E_{4}(\rho), E_{6}(\rho))$ (see appendix A for the definitions). Moreover, different such polynomials are related through derivatives with respect to the Eisenstein series $E_{2}$, in the style of holomorphic anomaly equations. We refer the reader to sections 4.4 and 5.4 for the details in the cases $N = 2$ and $N = 3$ respectively.

In the following sections we shall present detailed computations for $N = 2$ and $N = 3$ (and partially also $N = 4$), which support the observations just outlined. After that we shall conclude in section 7.

### 4 Little String Theory with $N = 2$

The simplest non-trivial example is to consider a model of Little Strings, which is engineered by two M5-branes on a circle that probe a flat $\mathbb{R}^{4}$ transverse space\(^{14}\).

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\(^{13}\)They are, however, implicitly given through the relation (3.6).

\(^{14}\)The partition function function of the case when the transverse space is $\mathbb{C}^{2}/\mathbb{Z}_{M}$ is given in \[25, 26\]. It would be interesting to see if the Hecke structure we study in this paper is also present for $M > 1$. 

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4.1 Decomposition of the Free Energy

As explained above, the partition function and free energy of this particular LST is captured by the topological string on the toric Calabi-Yau threefold, $X_{2,1}$, whose web-diagram is shown in Fig. [5]. Here, we use a basis of Kähler parameters $(R, S, \rho, \hat{a}_1)$ where in addition to the parameters given in the figure, we have

$$\rho = \hat{a}_1 + \hat{a}_2, \quad R - 2S = v - m. \quad (4.1)$$

Starting from the partition function $Z_{2,1}$, we define the free energy

$$F_{2,1}(\tilde{a}_{1,2}, S, R, \epsilon_{1,2}) = \log Z_{2,1}(\tilde{a}_{1,2}, S, R, \epsilon_{1,2}). \quad (4.2)$$

We decompose the latter in terms of $H_{(s_1, s_2)}^{(r), \{n, 0\}}(\rho, S)$ (for $n \in \mathbb{N} \cup \{0\}$) as described in Section 2. Upon using the symmetries of the former, the summation in $(2.22)$ becomes

$$P_{2,(s_1, s_2)}^{(r)}(\tilde{a}_{1,2}, S) = H_{(s_1, s_2)}^{(r), \{0, 0\}}(\rho, S) + \sum_{n=1}^{\infty} H_{(s_1, s_2)}^{(r), \{n, 0\}}(\rho, S) \left( Q_{a_1}^n + \frac{Q_{\rho}^n}{Q_{a_1}^n} \right). \quad (4.3)$$

In the following we shall only discuss the so-called NS-limit [43, 44], i.e. we consider $s_2 = 0$. Only for $n = 0$ (which corresponds to the part of the free energy discussed in [45]), the $H_{(s, 0)}^{(r), \{n, 0\}}$ are (quasi) Jacobi forms. For $n > 0$, the $H_{(s, 0)}^{(r), \{n, 0\}}$ are no longer modular objects. However, following [57, 46], based on studying series expansions in $Q_{\rho}$ (and exploiting certain patterns arising in the expansion coefficients) we can conjecture the following generic form\footnote{We have verified that these expressions agree with an expansion of $P_{2,(s_1, s_2)}^{(r)}$ in $(4.3)$ following from the general definition $(2.17)$ and $(2.18)$ in terms of the partition function defined in $(B.1)$ and $(B.8)$, up to order $O(Q_{\rho}^{10})$ for $r = 1$, $O(Q_{\rho}^{16})$ for $r = 2$ and $O(Q_{\rho}^{12})$ for $r = 3$ and up to $2s = 8.$}

$$H_{(2s, 0)}^{(r), \{n, 0\}}(\rho, S) = \begin{cases} h_{0,(2s)}^{(r)}(\rho, S) & \text{for } n = 0 \\ \frac{1}{1 - Q_{\rho}^2} \sum_{k=1}^{r+s+1} n^{2k-1} b_{k,(2s)}^{(r)}(\rho, S) & \text{for } n > 0 \end{cases}, \quad (4.4)$$

where $b_{k,(2s)}^{(r)}$ is a (quasi-)Jacobi form of index $2r$ and weight $2s - 2 - 2k$. Using the standard Jacobi-forms $\phi_{-2,1}$ and $\phi_{0,1}$ (see appendix A for the definition), they can be cast into the form

$$b_{k,(2s)}^{(r)}(\rho, S) = \sum_{a=0}^{2r} h_{a,k,(2s)}^{(r)}(\rho, S) \left( \phi_{-2,1}(\rho, S) \right)^a \left( \phi_{0,1}(\rho, S) \right)^{2r-a}, \quad (4.5)$$

where $h_{a,k,(2s)}^{(r)}$ are (quasi-)modular forms of weight $2s - 2 - 2k + 2a$ and depth $sr + \delta_{k,0}$ which can be
expressed as homogeneous polynomials of the Eisenstein series \( \{ E_{2,4,6} \} \). For later convenience, the coefficients \( h_{a,k,(2s)}^{(r)} \) for \( r = 1, r = 2 \) and \( r = 3 \) are tabulated in Tables 1, 2 and 3 respectively.

4.2 Factorisation at Order \( O(Q_R) \)

As was already conjectured in [46], the coefficients \( H_{(2s,0)}^{(r=1),\{n,0\}}(\rho, S) \) can be factorised in terms of \( H_{(s,0)}^{(1),\{0\}}(\rho, S) \) and \( W_{(s,0)}(\rho, S) \), i.e. the coefficients that appear in the expansion of the free energy for \( N = 1 \) and the function \( W_{NS}^{(1)} \) defined in Eq. (C.8) (as reviewed in Appendix C.1), concretely

\[
H_{(2s,0)}^{(r=1),\{n,0\}}(\rho, S) = 2 \sum_{a+b=s} H_{(2a,0)}^{(1),\{0\}}(\rho, S) W_{(2b,0)}(\rho, S),
H_{(2s,0)}^{(r=1),\{n,0\}}(\rho, S) = \frac{1}{1 - Q^n} \sum_{a,b=0}^{s} H_{(2a,0)}^{(1),\{0\}}(\rho, S) H_{(2b,0)}^{(1),\{0\}}(\rho, S) M_{ab}^{(s)}(n), \quad \forall \ n \geq 1. \tag{4.6}
\]

Here \( M^{(s)} \) is a symmetric \((s + 1) \times (s + 1)\) dimensional matrix, that only depends on \( n \), which in [46] was conjectured to take the form

\[
M_{ab}^{(s)} = -2 \frac{(-1)^{a+b+n} n^{2s+1-2(a+b)}}{\Gamma(2s - 2(a + b - 1))}, \quad a, b \in \{0, \ldots, s\}, \tag{4.7}
\]

where it is understood that \( 1/\Gamma(-m) = 0 \) for \( m \in \mathbb{N} \cup \{0\} \). The first few instances of \( M^{(s)} \) are

\[
M^{(0)} = (-2n), \quad M^{(1)} = \begin{pmatrix} \frac{n^3}{3} & -2n & \frac{n^3}{3} & -2n \\ -2n & 0 & -2n & 0 \\ \frac{n^3}{3} & -2n & 0 & 0 \\ -2n & 0 & 0 & 0 \end{pmatrix}, \quad M^{(2)} = \begin{pmatrix} \frac{n^5}{60} & -2n & \frac{n^3}{3} & -2n \\ -\frac{n^3}{3} & -2n & 0 & 0 \\ -2n & 0 & 0 & 0 \\ -2n & 0 & 0 & 0 \end{pmatrix}. \tag{4.8}
\]

In [24] it was shown that the NS limit of the partition functions \( Z_{N,1} \) have a self-similar behavior\(^\dagger\) in a certain region of the Kähler moduli space i.e., for \( \hat{a}_1 = \hat{a}_2 = \cdots = \hat{a}_N = \hat{q} \). Relations such as (2.10) allow to infer non-trivial information about the free energy for generic \( N \), just based on the knowledge of the (much simpler) free energy for the configuration \( N = 1 \), albeit only at a specific point in the moduli space. From this perspective, (4.6) is similar in spirit to this self-similarity: they allow to obtain non-trivial information about the \( N = 2 \) free energy at leading instanton order just from the configuration \( N = 1 \). We shall see that relations of this type also exist for \( N > 2 \) and (to some extent) also generalise to higher orders in \( Q_R \).

4.3 Hecke Structures

The coefficients \( H_{(2s,0)}^{(r),\{n,0\}}(\rho, S) \) for \( r > 1 \) do not seem to exhibit simple factorisations of the type (4.6). We shall, however, in the following identify particular subsectors of the free energy

\(^\dagger\)The precise relation that was shown in [24] is eq. (2.10).
| s | k | \(h_{0,k,(2s)}^{(r=1)}\) | \(h_{1,k,(2s)}^{(r=1)}\) | \(h_{2,k,(2s)}^{(r=1)}\) |
|---|---|---|---|---|
| 0 | 0 | 0 | \(-\frac{1}{12}\) | \(-\frac{E_2}{6}\) |
| 1 | 0 | 0 | 0 | \(-2\) |
| 1 | 0 | \(\frac{1}{1152}\) | 0 | \(\frac{E_4-2E_2^2}{288}\) |
| 1 | 0 | 0 | \(\frac{1}{24}\) | \(-\frac{E_4}{12}\) |
| 2 | 0 | 0 | 0 | \(\frac{1}{3}\) |
| 2 | 0 | \(\frac{E_2}{55296}\) | \(\frac{5E_2^2-7E_4}{69120}\) | \(-\frac{10E_2^2-3E_2E_4+4E_6}{69120}\) |
| 1 | \(-\frac{1}{4608}\) | \(\frac{E_2}{576}\) | \(\frac{E_4}{576}\) | \(-\frac{10E_2^2+13E_4}{576}\) |
| 2 | 0 | \(\frac{1}{144}\) | 0 | \(\frac{E_2}{144}\) |
| 3 | 0 | 0 | 0 | \(-\frac{1}{60}\) |
| 3 | 0 | \(\frac{E_4}{2271840}\) | \(\frac{35E_2^2-21E_2E_4-29E_6}{17418240}\) | \(-\frac{70E_2^2-168E_2^2E_4-8E_2E_6+123E_4^2}{34836480}\) |
| 1 | \(-\frac{E_4}{110592}\) | \(\frac{E_2^2+4E_4}{27648}\) | \(\frac{-E_2}{3456}\) | \(-\frac{10E_2^2+13E_4}{3456}\) |
| 2 | \(-\frac{1}{27648}\) | \(-\frac{E_2}{3456}\) | \(\frac{1}{144}\) | \(-\frac{E_2}{3456}\) |
| 3 | 0 | \(\frac{1}{288}\) | 0 | \(-\frac{1}{252}\) |
| 4 | 0 | 0 | 0 | \(-\frac{1}{252}\) |
| 4 | 0 | \(\frac{118E_6+105E_2E_4-70E_4^3}{13377208320}\) | \(\frac{175E_2^3+210E_2^2E_4-130E_2E_6-381E_4^2}{5573896800}\) | \(\frac{1682E_{10}-350E_2^2-2030E_2E_4-1000E_2^2E_6+177E_2E_4^2}{10721510400}\) |
| 1 | \(-\frac{10E_2^2-7E_4}{53084160}\) | \(\frac{10E_2^2+30E_2E_4+11E_6}{19900560}\) | \(-\frac{350E_2^2-2730E_2E_4-1840E_2E_6-2283E_4^2}{1390459200}\) |
| 2 | \(\frac{E_2}{663552}\) | \(-\frac{E_2^2-E_4}{165888}\) | \(\frac{70E_2^2+273E_2E_4+92E_6}{17418240}\) |
| 3 | \(-\frac{1}{552960}\) | \(\frac{E_2}{69120}\) | \(-\frac{10E_2^2-13E_4}{69120}\) |
| 4 | 0 | \(\frac{1}{120960}\) | \(\frac{E_2}{60480}\) |
| 5 | 0 | 0 | \(-\frac{1}{181440}\) |

Table 1: Expansion coefficients \(h_{a,k,(2s)}^{(r=1)}\).
| $s$ | $k$ | $h_{0,k,(2s)}^{(r=2)}$ | $h_{1,k,(2s)}^{(r=2)}$ | $h_{2,k,(2s)}^{(r=2)}$ | $h_{3,k,(2s)}^{(r=2)}$ | $h_{4,k,(2s)}^{(r=2)}$ |
|-----|-----|------------------|------------------|------------------|------------------|------------------|
| 0   | 0   | $-\frac{1}{4608}$ | $-\frac{E_2}{1152}$ | $-\frac{E_4}{1152}$ | $E_6-E_2E_4$     |
| 1   | 0   | 0                | $-\frac{1}{96}$    | 0                | $-\frac{E_4}{12}$  |
| 2   | 0   | 0                | 0                | $-\frac{1}{12}$    | 0                |
| 3   | 0   | 0                | 0                | 0                | $-\frac{1}{24}$    |
| 1   | 0   | $\frac{1}{442368}$ | 0                | $\frac{5E_4-4E_2^2}{55296}$ | $\frac{4E_2E_4-3E_6}{6912}$ | $-\frac{16E_4^2E_4-16E_2E_6+37E_2^2}{27648}$ |
| 1   | 0   | $\frac{1}{4608}$  | $-\frac{E_2}{1152}$ | $\frac{E_4}{128}$  | $-\frac{E_2E_4+2E_6}{144}$ |
| 2   | 0   | 0                | $\frac{1}{128}$    | $-\frac{E_2}{144}$ | $\frac{53E_4}{1440}$ |
| 3   | 0   | 0                | 0                | $\frac{13}{576}$    | $-\frac{E_2}{288}$ |
| 4   | 0   | 0                | 0                | 0                | $\frac{1}{120}$    |
| 20  | $\frac{E_2}{3224^4}$ | $\frac{10E_2^2-21E_4}{13271040}$ | $\frac{87E_6-20E_2^2-59E_2E_4}{6635520}$ | $\frac{170E_2^2E_4+140E_2E_6}{3317760}$ | $\frac{E_4(746E_6-80E_2^2)}{3317760}$ | $-\frac{20E_2^2E_4-320E_2E_6-721E_2^2}{276480}$ |
| 1   | $-\frac{1}{884736}$ | $\frac{E_2}{55296}$ | $-\frac{20E_2^2-109E_4}{55296}$ | $\frac{9E_2E_4+13E_6}{13824}$ | $-\frac{80E_2^2E_4-320E_2E_6-721E_2^2}{276480}$ |
| 2   | 0   | $-\frac{5}{36864}$ | $\frac{E_2}{1536}$ | $-\frac{40E_2^2-593E_4}{138240}$ | $\frac{371E_2E_4+535E_6}{120960}$ |
| 3   | 0   | 0                | $-\frac{301}{183360}$ | $\frac{13E_2}{6012}$ | $-\frac{20E_2^2-901E_4}{138240}$ |
| 4   | 0   | 0                | 0                | $-\frac{7}{2880}$    | $\frac{E_2}{1140}$ |
| 5   | 0   | 0                | 0                | 0                | $-\frac{73}{120960}$ |

Table 2: Expansion coefficients $h_{a,k,(2s)}^{(r=2)}$. 
| s | k | $h_{0,k,(2s)}^{(r=3)}$ | $h_{1,k,(2s)}^{(r=3)}$ | $h_{2,k,(2s)}^{(r=3)}$ | $h_{3,k,(2s)}^{(r=3)}$ | $h_{4,k,(2s)}^{(r=3)}$ | $h_{5,k,(2s)}^{(r=3)}$ | $h_{6,k,(2s)}^{(r=3)}$ |
|---|---|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0 | 0 | 0 | $-\frac{1}{2985984}$ | $-\frac{E_k}{495664}$ | $-\frac{E_k}{124416}$ | $\frac{23E_k - 27E_k E_2}{186624}$ | $\frac{8E_2 E_4 - 9E_4^2}{20736}$ | $E_4 (20E_2 - 21E_4 E_2)$ |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 3: Expansion coefficients $h_{s,k,(2s)}^{(r=3)}$.
(as introduced in \([3.1]\)) for generic \(N\) that in fact do again factorise.

To this end, we define the following contour integrals

\[
C_{1,(2s,0)}^{\ell}(\rho, S) := \frac{1}{(2\pi i)^2} \sum_{\ell=0}^{\infty} Q_{\tilde{a}_1}^{\ell} \oint_{\mathbb{R}} \frac{d\hat{Q}_{\tilde{a}_1}}{Q_{\tilde{a}_1}^1} \oint_{\mathbb{R}} \frac{d\hat{Q}_{\tilde{a}_2}}{Q_{\tilde{a}_2}^1} P^{(r)}_{2,(2s,0)}(\hat{a}_1, \hat{a}_2, S), \tag{4.9}
\]

\[
C_{2,(2s,0)}^{\ell}(\rho, S) := \frac{1}{(2\pi i)^2} \frac{1}{r} \oint_{\mathbb{R}} \frac{d\hat{a}_1}{\hat{a}_1} \oint_{\mathbb{R}} \frac{d\hat{a}_2}{\hat{a}_2} P^{(r)}_{2,(2s,0)}(\rho, \hat{a}_1, S), \tag{4.10}
\]

where all contours are small circles around the origin\(^{17}\) and in \((4.10)\) we have implicitly used \(\hat{a}_2 = \rho - \hat{a}_1\). With these coefficient functions, we define the (a priori formal) series in \(\epsilon\)

\[
C_{a}^{N=2,(r)}(\rho, S, \epsilon) = \sum_{s=0}^{\infty} \epsilon^{2s-2a} C_{a,(2s,0)}^{N=2,(r)}(\rho, S), \quad \forall a = 1, 2. \tag{4.11}
\]

From the perspective of the M-brane-web, the functions \((4.9)\) and \((4.10)\) count certain BPS configurations of M2-branes stretched between two M5-branes on a circle. Due to the contour prescriptions, however, only certain configurations contribute, which are visualised in Fig. 6:

**Figure 6:** Brane web configurations made up from \(N = 2\) M5-branes (drawn in orange) spaced out on a circle, with various M2-branes (drawn in red and blue) stretched between them. (a) An equal number \(\ell\) of M2-branes is stretched between the M5-branes on either side of the circle. Configurations of this type are relevant for the computation of \(C_{1}^{N=2,(r)}\). (b) \(n_1\) M2-branes are stretched on one side of the circle and \(n_2 (\neq n_1)\) on the other side of the circle. Configurations of this type are relevant for the contributions \(C_{2}^{N=2,(r)}\).

- Combination \(C_{1,(2s,0)}^{N=2,(r)}\):
  
  Upon writing \(P^{(r)}_{2,(2s,0)}\) as a Fourier expansion in \(Q_{\tilde{a}_1,2}\) (similar to \(H^{(r),\{n_1,...,n_N\}}_{(2s,0)}\) in eq. \((2.20)\))

\[
P^{(r)}_{2,(2s,0)}(\hat{a}_1,2, S, \epsilon_{1,2}) = \sum_{n_1, n_2 = 0}^{\infty} Q_{\hat{a}_1}^{n_1} Q_{\hat{a}_2}^{n_2} P^{(r),\{n_1, n_2\}}_{(2s,0)}(S), \tag{4.12}
\]

\(^{17}\)The integrals are in fact designed to precisely extract the residues in a Laurent series expansion.
the contour prescriptions in (4.9) extract all terms with \( n_1 = n_2 \). \( C_{N=2,0}^{1,2} \) thus receives contributions only from those brane configurations, where an equal number of M2-branes is stretched between the two M5-branes on either side of the circle, as shown in Fig. 6(a). In fact, \( C_{1,2s,0}^{N=2,0} \) can equivalently be written as

\[
C_{1,2s,0}^{N=2,0}(\rho, S) = H^{(r),(n,0)}_{2s,0}(\rho, S), \tag{4.13}
\]

and \( C_{1,2s,0}^{N=2,0}(\rho, S, \epsilon_1) \) is in fact exactly the reduced free energy studied in \([45]\). Explicit expansions of \( C_{1,2s,0}^{N=2,0}(\rho, S, \epsilon_1) \) for \( r = 1, r = 2 \) and \( r = 3 \) can be recovered from Tables 1, 2 and 3, respectively from the coefficients with \( k = 0 \).

- **Combination \( C_{2,2s,0}^{N=2,0} \):**
  The function \( C_{2,2s,0}^{N=2,0} \) in (4.10) receives contributions from configurations in which \( n_1 \) M2-branes are stretched between the M5-branes on one side of the circle and \( n_2 \) (with \( n_2 \neq n_1 \)) on the other side, as schematically shown in Fig. 6(b). Furthermore, from each of these contributions, the contour integral extracts the pole of the type \( \hat{a}_1^{-2} \) (where it is important to write \( \hat{a}_2 = \rho - \hat{a}_1 \)).

In terms of the functions \( H^{(r),(n,0)}_{2s,0} \) in (4.4), the contour prescription in fact extracts the contributions of \( h^{(r)}_{k=1,2s} \)

\[
C_{2,2s,0}^{N=2,0}(\rho, S) = \frac{1}{r} h^{(r)}_{k=1,2s} . \tag{4.14}
\]

To intuitively understand this result, we introduce \([57]\)

\[
\mathcal{I}_\alpha(\rho, \hat{a}_1) = D_{\hat{a}_1}^{2\alpha} \mathcal{I}_0 = D_{\hat{a}_1}^{2\alpha} \sum_{n=1}^{\infty} \frac{n}{1 - Q^2} \left( Q^n + \frac{Q^n}{Q_{\hat{a}_1}^\alpha} \right), \quad \text{with} \quad D_{\hat{a}_1} = Q_{\hat{a}_1} \frac{\partial}{\partial Q_{\hat{a}_1}} . \tag{4.15}
\]

As argued in \([57]\), \( \mathcal{I}_0 \) can be written in terms of Weierstrass’s elliptic function \( \wp \) and the second Eisenstein series (see appendix A for the definitions)

\[
\mathcal{I}_0(\rho, \hat{a}_1) = \frac{1}{(2\pi i)^2} \left[ 2\zeta(2) E_2(\rho) + \wp(\hat{a}_1; \rho) \right] . \tag{4.16}
\]

Since Weierstrass’s elliptic function affords the following series expansion

\[
\wp(z; \rho) = \frac{1}{z^2} + \sum_{k=1}^{\infty} 2\zeta(2k + 2) (2k + 1) E_{2k+2}(\rho) z^{2k} , \tag{4.17}
\]
we have for the contour integral
\[ \oint d\hat{a}_1 \hat{a}_1 \mathcal{I}_\alpha(\rho, \hat{a}_1) = 2\pi i \delta_{\alpha 0}, \] (4.18)
such that with (4.3) and (4.4) we have (4.14). The factor $1/r$ in the latter relation is simply a convenient normalisation factor as will become apparent later on.

A more direct way to arrive at (4.14) is to start from the decomposition (4.3) and exchange the summations over $k$ and $n$
\[ P_{(2s,0)}^{(r)}(\hat{a}_{1,2}, S) = H_{(2s,0)}^{(r),\{0,0\}}(\rho, S) + \sum_{k=1}^{rs} h_{k,(2s)}^{(r)}(\rho, S) X_k(\hat{a}_{1,2}), \] (4.19)
where we introduced the shorthand notation
\[ X_k(\rho, \hat{a}_{1,2}) = \sum_{n=1}^{\infty} \sum_{b=0}^{\infty} n^{2k-1} \left( Q_{\hat{a}_1}^n + Q_{\hat{a}_2}^n \right) = \sum_{n=1}^{\infty} \sum_{b=0}^{\infty} \sum_{n=1}^{\infty} \sum_{b=0}^{\infty} n^{2k-1} Q_{\rho}^b \left( Q_{\hat{a}_1}^n + Q_{\hat{a}_2}^n \right). \] (4.20)
We can express $X_k$ in terms of the q-Polygamma function $\psi^{(m)}_q(z)$,
\[ \psi_q(z) = \frac{d \ln \Gamma_q(z)}{dz} = -\ln(1 - q) + \ln(q) \sum_{n=0}^{\infty} \frac{q^{n+z}}{1 - q^{n+z}}, \quad \psi_q^{(m)}(z) = \frac{d^m \psi_q(z)}{dz^m}, \] (4.21)
where $\Gamma_q$ is the q-Gamma function
\[ \Gamma_q(z) = (1 - q)^{1-z} \prod_{n=0}^{\infty} \frac{1 - q^{n+1}}{1 - q^{n+z}}. \] (4.22)
To this end, we interchange\(^{19}\) the sums in the last expression in (4.20) and find with (4.21)
\[ X_k(\rho, \hat{a}_{1,2}) = \frac{1}{\ln(Q_{\rho})^{k+1}} \left( \psi^{(2k-1)}_{Q_{\rho}}(\hat{a}_1/\rho) + \psi^{(2k-1)}_{Q_{\rho}}(\hat{a}_2/\rho) \right) \quad \text{for} \quad k \geq 1. \] (4.23)
The q-Gamma function $\Gamma_q(z)$ satisfies the identity $\Gamma_q(z + 1) = \frac{1}{1 - \frac{z}{q}} \Gamma_q(z)$ and, therefore,

\(^{18}\)This is possible since the sum over $k$ is finite.
\(^{19}\)This is possible for $|Q_{\rho}| < 1$ and $|Q_{\hat{a}_{1,2}}| < 1$. To see this, we consider for example
\[ \sum_{n=1}^{\infty} \sum_{b=0}^{\infty} n^{2k-1} |Q_{\rho}|^b |Q_{\hat{a}_1}|^n \leq \sum_{n=1}^{\infty} \sum_{b=0}^{\infty} n^{2k-1} |Q_{\rho}|^b |Q_{\hat{a}_2}|^n = \left( \sum_{n=1}^{\infty} n^{2k-1} |Q_{\hat{a}_1}|^n \right) \left( \sum_{b=0}^{\infty} |Q_{\rho}|^b \right) = \frac{\text{Li}_{1-2k}(|Q_{\hat{a}_1}|)}{1 - |Q_{\rho}|}, \]
where $\text{Li}_{1-2k}$ denotes the polylogarithm. Thus (4.20) is absolutely convergent, and therefore the summations can be interchanged.
for $z \to 0$ we obtain

$$\Gamma_q(z) = -\frac{1 - q}{z \ln(q)} + \mathcal{O}(z^0). \quad (4.24)$$

Thus the function $X_k(\rho, \hat{a}_{1,2})$ diverges for $\hat{a}_1 \to 0$ and in fact has a pole of order $k + 1$

$$X_k(\rho, \hat{a}_{1,2}) \sim -\frac{(2k - 1)!}{\hat{a}_1^{2k}} + O(\hat{a}_1^0) \quad (4.25)$$

Therefore the only contribution to the contour integral in (4.10) (which extracts the pole of order 2) stems from $X_1(\rho, \hat{a}_{1,2})$, thus yielding (4.14).

By comparing the explicit expressions for the contributions $C_{1,2,(2s,0)}^{N=2,(r)}$ and $C_{2,2,(2s,0)}^{N=2,(r)}$ to the free energy, we find that they satisfy the following recursion relation

$$C_{1,2,(2s,0)}^{N=2,(r)}(\rho, S) = \mathcal{H}_r \left[ C_{1,2,(2s,0)}^{N=2,(1)}(\rho, S) \right], \quad C_{2,2,(2s,0)}^{N=2,(r)}(\rho, S) = \mathcal{H}_r \left[ C_{2,2,(2s,0)}^{N=2,(1)}(\rho, S) \right]. \quad (4.26)$$

The normalisation factor $1/r$ appearing in the definition (4.10) was chosen to normalise the right hand side of the second equation above.

### 4.4 Decomposition of $C_{1,2,(2s,0)}^{N=2,(r)}$ and $C_{2,2,(2s,0)}^{N=2,(r)}$

In section 4.2 we have seen that the free energy in the NS limit factorises to order $\mathcal{O}(Q_R)$ as in eq. (4.6) with the basic building blocks given by the expansion coefficients of the free energy in the case $N = 1$. While the complete free energy at higher orders $\mathcal{O}(Q_R^r)$ (for $r > 1$) does not exhibit such a behaviour, the particular contributions $C_{1,2,(2s,0)}^{N=2,(r)}$ and $C_{2,2,(2s,0)}^{N=2,(r)}$ defined in (4.10) lend themselves to a generalisation of (4.6).

#### 4.4.1 Factorisation at Order $Q_R^1$

The first step is to establish the factorisation of $C_{1,2,(2s,0)}^{N=2,(r=1)}$ and $C_{2,2,(2s,0)}^{N=2,(r=1)}$, which are in fact induced by (4.6). Indeed, using (4.13) as well as (4.14), we have immediately

$$C_{1,2,(2s,0)}^{N=2,(r=1)}(\rho, S) = 2 \sum_{i,j=0}^{s} \delta_{s,i+j} H_{(2s,0)}^{(1),\{0\}}(\rho, S) W_{(2j,0)}(\rho, S), \quad (4.27)$$

$$C_{2,2,(2s,0)}^{N=2,(r=1)}(\rho, S) = -2 \sum_{i,j=0}^{s} \delta_{s,i+j} H_{(2s,0)}^{(1),\{0\}}(\rho, S) H_{(2j,0)}^{(1),\{0\}}(\rho, S). \quad (4.28)$$
Combining these expansion coefficients (in a series of $\epsilon_1$), we can equivalently write the following relations for the (a priori formal) series expansions
\begin{align*}
C_1^{N=2, (r=1)}(\rho, S, \epsilon_1) &= 2 H_{N=1}^{(1)}(\rho, S, \epsilon_1) W_{NS}^{(1)}(\rho, S, \epsilon_1), \\
C_2^{N=2, (r=1)}(\rho, S, \epsilon_1) &= -2 \left[H_{N=1}^{(1)}(\rho, S, \epsilon_1)\right]^2, \quad (4.29)
\end{align*}
where the coefficients $H_{N=1}^{(1)}$ and $W_{NS}^{(1)}$ are defined in (C.3) and (C.8) respectively.

### 4.4.2 Factorisation at Order $O(Q^2_R)$

Based on eq.(4.27) and (4.28), the first attempt to factorise the function $C_1^{N=2, (r=2)}$ to order $O(Q^2_R)$ would be to use a similar decomposition, except to replace $H_{N=1}^{(1)}$ and $W_{NS}^{(1)}$ by their order $O(Q^2_R)$ counterparts $H_{N=1}^{(2)}$ and $W_{NS}^{(1)}$ respectively. However, this does not fully reproduce the correct answer, instead we have
\begin{align*}
C_1^{N=2, (r=2)}(\rho, S, \epsilon_1) &= \frac{4}{3} H_{N=1}^{(2)}(\rho, S, \epsilon_1) W_{NS}^{(2)}(\rho, S, \epsilon_1) + \mathcal{R}_1^{(2)}(\rho, S, \epsilon_1), \\
C_2^{N=2, (r=2)}(\rho, S, \epsilon_1) &= -\frac{4}{3} \left[H_{N=1}^{(2)}(\rho, S, \epsilon_1)\right]^2 + \mathcal{R}_2^{(2)}(\rho, S, \epsilon_1), \quad (4.30)
\end{align*}
The additional contributions $\mathcal{R}_1^{(2)}$ are formal expansions in powers of $\epsilon_1$
\begin{equation}
\mathcal{R}_a^{(2)}(\rho, S, \epsilon_1) = \sum_{s=0}^{\infty} \epsilon_1^{2s-2a} \mathcal{R}_{a,(2s,0)}^{(2)}(\rho, S), \quad \forall a = 1, 2, \quad (4.31)
\end{equation}
where the $\mathcal{R}_{a,(2s,0)}^{(2)}(\rho, S)$ in turn can be decomposed as
\begin{equation}
\mathcal{R}_{a,(2s,0)}^{(2)}(\rho, S) = \sum_{i=0}^{4} \mathcal{R}_{a,i,(2s,0)}^{(2)}(\rho) (\phi_{-2,1}(\rho, S))^i \phi_{0,1}(\rho, S)^{4-i}, \quad (4.32)
\end{equation}
and the $\mathcal{R}_{a,i,(2s,0)}^{(2)}(\rho)$ are (quasi-)modular forms of weight $2s - 2 + 2i - 2a$ and the first few expressions are tabulated for $a = 1$ in Table 4 and for $a = 2$ in Table 5.

The functions $\mathcal{R}_a^{(2)}$ can themselves again be factorised where the basic building blocks are $H_{N=1}^{(1)}$
\begin{equation}
\mathcal{R}_a^{(2)}(\rho, S, \epsilon_1) = \mathcal{S}_a^{(2)}(\rho, \epsilon_1) \left[H_{N=1}^{(1)}(\rho, S, \epsilon_1)\right]^4. \quad (4.33)
\end{equation}
The only novel feature is the appearance of the functions $\mathcal{S}_a^{(2)}$, which are $S$-independent.
While we cannot write a closed form expression for the holomorphic anomaly in (quasi)Jacobi forms that are characterised through

\[
\frac{d \mathcal{G}_4^{(2)}(\rho, \epsilon_1)}{d E_2} = \frac{e_1^2}{6} \mathcal{G}_4^{(2)}(\rho, \epsilon_1), \quad \mathcal{G}_{2,4}(\rho, \epsilon_1) = \sum_{s=1}^{\infty} \epsilon_1^{2s-6} \left( \frac{(4s+1-1)(2s+1)}{3 \cdot 4^s \pi^{2s+1}} \right) \zeta(2s+2) E_{2s+2}(\rho). 
\]

(4.34)

While we cannot write a closed form expression for the holomorphic anomaly in \( \mathcal{R}_1^{(2)} \), we have

\[
\mathcal{G}_4 = \int dE_2 \mathcal{G}_{1,4}^{(2)} + \sum_{s=1}^{\infty} \epsilon_1^{2s-6} \left( \frac{(4s+1-1)(2s+1)}{18 \cdot 4^s \pi^{2s+1}} \right) \zeta(2s+2) e_{2s+4}(\rho),
\]

(4.35)

where \( e_{2s+4} \) is a polynomial in \( E_{4,6} \) of weight \( 2s+4 \), normalised such that \( e_{2s+4}(\rho) = 1 + O(Q_R) \).

### 4.4.3 Factorisation at Order \( Q_R^r \) for \( r > 2 \)

Following the decomposition (4.30) at order \( Q_R^2 \), we can consider similar expressions to higher orders. From the explicit examples we find to order \( Q_R^3 \)

\[
C_{1,4,2,(r=3)}^{N=2}(\rho, S, \epsilon_1) = \frac{3}{2} H_{N=1}^{(3)} W_{NS}^{(3)} + (H_{N=1}^{(1)})^6 \mathcal{S}_{1,(6,0)}(\rho, \epsilon_1)
\]

\[\text{Implicitly } e_{2s+4}(\rho) \text{ is of course fixed uniquely through the relation (4.26).}\]
and to order $Q^4_R$

\[
C_2^{N=2,(r=3)}(\rho, S, \epsilon_1) = \frac{3}{2} H_{N=1}^{(3)} H_{N=1}^{(3)} + (H_{N=1}^{(1)})^6 \mathcal{G}_{2,(6,0)}^{(r=3)}(\rho, \epsilon_1)
\]

\[
+ (H_{N=1}^{(1)})^4 H_{N=1}^{(2)} \mathcal{G}_{2,(4,1)}^{(r=3)}(\rho, \epsilon_1) + (H_{N=1}^{(1)})^2 (H_{N=1}^{(2)})^2 \mathcal{G}_{2,(2,2)}^{(r=3)}(\rho, \epsilon_1),
\]

(4.36)

Here $\mathcal{G}_{i,k}^{(r)}(\rho, \epsilon_1)$ are independent of $S$ and we find the following $\epsilon_1$-expansions for $r = 3$

\[
\frac{1}{\epsilon_1^1} \mathcal{G}_{1,(6,0)}^{(3)} = \frac{E_4(E_6 - E_2 E_4)}{2592} + \frac{e_1^2 (E_6^2 - 3E_2 E_4 E_6 + 2E_4^3)}{15552} + O(\epsilon_1^6)
\]

\[
\frac{1}{\epsilon_1^1} \mathcal{G}_{1,(4,1)}^{(3)} = \frac{E_4^2 - E_2 E_6}{162} - \frac{5e_1^2 E_4(E_2 E_4 - E_6)}{1944} + \frac{e_1^4 (-196E_2 E_4 E_6 + 123E_4^3 + 73E_6^2)}{23328} + O(\epsilon_1^6)
\]

\[
\frac{1}{\epsilon_1^1} \mathcal{G}_{1,(2,2)}^{(3)} = \frac{2(E_6 - E_2 E_4)}{81} + \frac{2e_1^2}{243} (E_2 - E_4 E_6) + \frac{e_1^4 E_4}{405} (E_6 - E_2 E_4) + O(\epsilon_1^6)
\]

\[
\frac{1}{\epsilon_1^1} \mathcal{G}_{2,(6,0)}^{(3)} = -\frac{E_2^3}{648} - \frac{E_4 E_6 e_1^2}{326} - \frac{e_1^4 (28E_4^3 + 15E_6^2)}{155520} + O(\epsilon_1^6)
\]

\[
\frac{1}{\epsilon_1^1} \mathcal{G}_{2,(4,1)}^{(3)} = -\frac{2E_6}{81} - \frac{5e_1^2 E_4 E_6 e_1^4}{486} + \frac{49E_4 E_6 e_1^4}{14580} + O(\epsilon_1^6), \quad \frac{1}{\epsilon_1^1} \mathcal{G}_{2,(2,2)}^{(3)} = -\frac{8E_4}{81} + \frac{8e_1^2 E_6}{243} - \frac{4E_2^4 e_1^4}{405} + O(\epsilon_1^6)
\]

and for $r = 4$

\[
\frac{1}{\epsilon_1^1} \mathcal{G}_{1,(8,0,0)}^{(4)} = \frac{-21E_2 E_4^3 - 31E_2 E_6^2 + 52E_4^2 E_6}{1741824} + \frac{13e_1^2 (E_4 E_6 (E_2 E_4 - E_6))}{497664}
\]

\[
+ e_1^4 (2E_2 (654E_4^3 + 4129E_4 E_6^2) + 6179E_4^2 E_6 + 3387E_6^3) + O(\epsilon_1^6)
\]

\[
\frac{1}{\epsilon_1^2} \mathcal{G}_{1,(6,1,0)}^{(4)} = \frac{-181E_2 E_4 E_6 + 129E_4^3 + 52E_6^3}{108864} + \frac{e_1^2 (-651E_4 E_6^3 - 313E_2 E_6^2 + 964E_4^2 E_6)}{653184}
\]

\[
+ \frac{e_1^4 E_4 (-12075E_2 E_4 E_6 + 5269E_4^3 + 6806E_6^2)}{13063680} + O(\epsilon_1^6)
\]
\[
\frac{1}{\varepsilon_1^4} S_{1,(4,2,0)}^{(4)} = -\frac{5}{176} (E_4(E_2E_4 - E_6) + \varepsilon_1^2 (-919E_2E_4E_6 + 540E_4^3 + 379E_6^2)
\]
\[
+ \frac{\varepsilon_1^4 (-2118E_2E_4^3 - 1303E_2E_6^2 + 3421E_2E_6)}{979776} + O (\varepsilon_1^5),
\]
\[
\frac{1}{\varepsilon_1^4} S_{1,(2,3,0)}^{(4)} = \frac{7}{18} (E_4^2 - E_2E_6) - \frac{103\varepsilon_1^2 (E_4E_2E_6 - E_6)}{243}
\]
\[
+ \frac{\varepsilon_1^4 (5386E_6^2 - 13887E_2E_4E_6 + 8501E_6^3)}{23160} + O (\varepsilon_1^5),
\]
\[
\frac{1}{\varepsilon_1^4} S_{1,(2,0,2)}^{(4)} = \frac{-15(E_2E_4 - E_6)}{128} + \frac{17\varepsilon_1^2 (E_2^2 - E_2E_6)}{256}
\]
\[
- \frac{1511\varepsilon_1^4 (E_4E_2E_4 - E_6)}{43008} + O (\varepsilon_1^5),
\]
\[
\frac{1}{\varepsilon_1^4} S_{2,(8,0,0)}^{(4)} = -\frac{(21E_4^3 + 31E_6^2)}{580608} - \frac{13E_4^2E_6\varepsilon_1}{165888} - \frac{\varepsilon_1^4 (654E_4^4 + 4129E_4E_6^2)}{104509440} + O (\varepsilon_1^5),
\]
\[
\frac{1}{\varepsilon_1^4} S_{2,(6,1,0)}^{(4)} = -\frac{181E_4E_6}{36288} - \frac{\varepsilon_1^2 (651E_4^3 + 313E_6^2)}{217728} - \frac{115E_4^2E_6\varepsilon_1}{41472} + O (\varepsilon_1^5),
\]
\[
\frac{1}{\varepsilon_1^4} S_{2,(4,2,0)}^{(4)} = -\frac{5E_4^2}{252} - \frac{919E_4E_6\varepsilon_1}{54432} - \frac{\varepsilon_1^4 (2118E_4^3 + 1303E_6^2)}{326592} + O (\varepsilon_1^5),
\]
\[
\frac{1}{\varepsilon_1^4} S_{2,(2,3,0)}^{(4)} = -\frac{7E_6}{81} - \frac{103E_4^2\varepsilon_1^2}{1701} - \frac{1543E_4E_6\varepsilon_1^4}{45360} + O (\varepsilon_1^5),
\]
\[
\frac{1}{\varepsilon_1^4} S_{2,(2,0,2)}^{(4)} = -\frac{45E_4}{128} - \frac{51E_4^2\varepsilon_1}{256} - \frac{1511E_4\varepsilon_1^4}{14336} + O (\varepsilon_1^5).
\]

Comparing these expressions suggests the following form

\[
\begin{aligned}
C_1^{N=2,(r)} &= \sum_{i_1,\ldots,i_r} \mathcal{S}_{a,(i_1,\ldots,i_r)}^{(r)} (H_{N=1}^{(1)})^{i_1} \cdots (H_{N=1}^{(1)})^{i_r} + \frac{2r}{\sigma_1(r)} \left\{ H_{N=1}^{(r)} W_{NS}^{(r)} \right\} \\
C_2^{N=2,(r)} &= \sum_{i_1,\ldots,i_r} \mathcal{S}_{a,(i_1,\ldots,i_r)}^{(r)} (H_{N=1}^{(1)})^{i_1} \cdots (H_{N=1}^{(1)})^{i_r} + \frac{2r}{\sigma_1(r)} \left\{ (-1) H_{N=1}^{(r)} H_{N=1}^{(r)} \right\}
\end{aligned}
\]  

(4.38)

Here the prime on the summation denotes the following conditions on \((i_1,\ldots, i_{r-1})\)

\[
\sum_{j=1}^{r} j i_j = 2r, \quad \text{and} \quad i_1 \in \mathbb{N}_{\text{even}}, \quad \text{and} \quad i_1 > 0,
\]

(4.39)

and \(\mathcal{S}_{a,(i_1,\ldots,i_{r-1})}^{(r)}\) are quasi-modular forms depending on \(\rho\) and \(\varepsilon_1\) which in particular satisfy

\[
\frac{\partial \mathcal{S}_{2,(i_1,\ldots,i_r)}^{(r)}}{\partial E_2(\rho)} (\rho, \varepsilon_1) = 0, \quad \mathcal{S}_{2,(i_1,\ldots,i_r)}^{(r)} (\rho, \varepsilon_1) = \frac{12}{r \varepsilon_1^2} \frac{\partial \mathcal{S}_{1,(i_1,\ldots,i_r)}^{(r)}}{\partial E_2(\rho)} (\rho, \varepsilon_1), \quad \forall r > 1.
\]

(4.40)

This generalises the first relation in (4.34) and also implies that \(\mathcal{S}_{2,(i_1,\ldots,i_r)}^{(r)}\) is a holomorphic Jacobi form. Notice also, for all examples we have computed so far \(\mathcal{S}_{2,(i_1,\ldots,i_r)}^{(r)} = 0\) for \(i_r > 0\).
5 Hecke Structure for $N = 3$

After discussing the free energy of the $N = 2$ LST, we continue with $N = 3$.

5.1 Decomposition of the Free Energy

The starting point is to compute the decomposition of the free energy. The web diagram representing $X_{3,1}$, which is relevant for the $N = 3$ free energy is shown in Fig. 7. In addition to the Kähler parameters shown in the figure, we also have

$$\rho = \hat{a}_1 + \hat{a}_2 + \hat{a}_3, \quad R - 3S = m - 2v, \quad (5.1)$$

From the partition function $Z_{3,1}$ we can compute the free energy

$$F_{3,1}(\hat{a}_{1,2,3}, S, R, \epsilon_{1,2}) = \log Z_{3,1}(\hat{a}_{1,2,3}, S, R, \epsilon_{1,2}),$$

Figure 7: Web diagram of $X_{3,1}$. As in the case $N = 2$, we focus exclusively on the NS-limit. In this case, following eq. (2.22), we can decompose the free energy in terms of $H^{(r),n}_{(2s,0)}$, where $n$ can be either of the following triples

$$\{0, 0, 0\}, \quad \{n, 0, 0\}, \quad \{n, n, 0\}, \quad \{n_1 + n_2, n_1, 0\}, \quad \text{with} \quad n, n_1, n_2 \in \mathbb{N}, \quad (5.2)$$

more concretely, we can write the following (a priori formal) decomposition

$$P^{(r)}_{3,(2s,0)}(\hat{a}_{1,2,3}, S) = H^{(r),\{0,0,0\}}_{(2s,0)}(\rho, S) + \sum_{n=1}^{\infty} H^{(r),\{n,0,0\}}_{(2s,0)}(\rho, S) \left( Q^n_{\hat{a}_1} Q^n_{\hat{a}_2} + \frac{Q^n_{\hat{a}_1}}{Q^n_{\hat{a}_2}} \right)$$

$$+ \sum_{n=1}^{\infty} H^{(r),\{n,n,0\}}_{(2s,0)}(\rho, S) \left( Q^n_{\hat{a}_1} Q^n_{\hat{a}_2} + \frac{Q^n_{\hat{a}_1}}{Q^n_{\hat{a}_2}} \right)$$

$$+ \sum_{n_1,n_2=1}^{\infty} H^{(r),\{n_1+n_2,n_1,0\}}_{(2s,0)}(\rho, S) \left( Q^{n_1+n_2}_{\hat{a}_1} Q^{n_1}_{\hat{a}_2} + \frac{Q^{n_1+n_2}_{\hat{a}_1} Q^{n_1}_{\hat{a}_2}}{Q^{n_1}_{\hat{a}_1} Q^{n_1}_{\hat{a}_2}} + (\hat{a}_1 \leftrightarrow \hat{a}_2) \right). \quad (5.3)$$

Comparing with an explicit expansion of the free energy (2.18) we observe that the coefficients $H^{(r),n}_{(2s,0)}$ can be written in the form

$$H^{(r),\{n,0,0\}}_{(2s,0)}(\rho, S) = \frac{1}{1 - Q^n_{\rho}} \sum_{k=1}^{r+1} n^{2k-1} f^{(r)}_{k,(2s)}(\rho, S) + \frac{Q^n_{\rho}}{(1 - Q^n_{\rho})^2} \sum_{k=1}^{r+1} n^{2k} g^{(r)}_{k,(2s)}(\rho, S),$$

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\begin{align}
H_{(2s,0)}^{(r),(n,n,0)}(\rho, S) &= \frac{1}{1 - Q_\rho^n} \sum_{k=1}^{r+1} n^{2k-1} f_{k,(2s)}^{(r)}(\rho, S) + \frac{1}{1 - Q_\rho^{n_2}} \sum_{k=1}^{r+1} n^{2k} g_{k,(2s)}^{(r)}(\rho, S), \\
H_{(2s,0)}^{(r),(n_1+n_2,n,0)}(\rho, S) &= \frac{n_2(n_2 + 2n_1)}{(1 - Q_\rho^{n_1})(1 - Q_\rho^{n_2})} \sum_{k=1}^{r+1} \sum_{\ell} p_{\ell,k,(2s)}^{(r)}(n_1, -n_1 - n_2) j_{\ell,k,(2s)}^{(r)}(\rho, S) + \frac{(n_2^2 - n_2^2)}{(1 - Q_\rho^{n_1})(1 - Q_\rho^{n_1+n_2})} \sum_{k=1}^{r+1} \sum_{\ell} p_{\ell,k,(2s)}^{(r)}(n_1, n_2) j_{\ell,k,(2s)}^{(r)}(\rho, S), \quad (5.4)
\end{align}

where \( f_{k,(2s)}^{(r)} \) are (quasi) Jacobi forms of index \( 3r \) and weight \( 2s - 2 - 2k \), \( g_{k,(2s)}^{(r)} \) are (quasi) Jacobi forms of index \( 3r \) and weight \( 2s - 4 - 2k \) and \( j_{a,k,(2s)}^{(r)} \) are (quasi) Jacobi forms of index \( 3r \) and weight \( 2s - 4 - 2k \). They can be written in the following fashion

\begin{align}
f_{a,k,(2s)}^{(r)}(\rho, S) &= - \sum_{a=0}^{3r} f_{a,k,(2s)}^{(r)}(\rho) (\phi_{-2,1}(\rho, S))^a (\phi_{0,1}(\rho, S))^{3r-a}, \\
g_{a,k,(2s)}^{(r)}(\rho, S) &= - \sum_{a=0}^{3r} g_{a,k,(2s)}^{(r)}(\rho) (\phi_{-2,1}(\rho, S))^a (\phi_{0,1}(\rho, S))^{3r-a}, \\
j_{a,\ell,k,(2s)}^{(r)}(\rho, S) &= - \sum_{a=0}^{3r} j_{a,\ell,k,(2s)}^{(r)}(\rho) (\phi_{-2,1}(\rho, S))^a (\phi_{0,1}(\rho, S))^{3r-a},
\end{align}

where \( f_{a,k,(2s)}^{(r)} \), \( g_{a,k,(2s)}^{(r)} \) and \( j_{a,\ell,k,(2s)}^{(r)} \) are quasi-modular forms of weight \( 2s - 2 - 2k \), \( 2s - 4 - 2k \) and \( 2s - 4 - 2k + 2a \) respectively. Similarly, we can expand

\begin{align}
H_{(2s,0)}^{(r),(0,0,0)}(\rho, S) &= - \sum_{a=0}^{3r} d_{a,(2s)}^{(r)}(\rho) (\phi_{0,1}(\rho, S))^a (\phi_{-2,1}(\rho, S))^{3r-a}. \quad (5.5)
\end{align}

where \( d_{a,(2s)}^{(r)} \) are quasi-Jacobi forms of weight \( 2s + 2k \). Furthermore, \( p_{\ell,k,(2s)}^{(r)}(n_1, n_2) \) in (5.4) are homogeneous polynomials in \( n_1, n_2 \) of order \( 2(k - 1) \), that are symmetric in the exchange of \( n_1 \leftrightarrow n_2 \). Explicit expressions for \( d_{a,k,(2s)}^{(r)} \), \( f_{a,k,(2s)}^{(r)} \), \( g_{a,k,(2s)}^{(r)} \) and \( j_{a,\ell,k,(2s)}^{(r)} \) as well as \( p_{k,(2s)}^{(r)}(n_1, n_2) \) for low values of \( s \) for \( r = 1 \) are tabulated in Table 6, Table 7, Table 8 and Table 9 and for \( r = 2 \) in Table 10, Table 11, Table 12 and Table 13 respectively.

5.2 Factorisation at Order \( \mathcal{O}(Q_R) \)

Following the results of \[46\] for \( N = 2 \), which have been reviewed in section 4.2, we expect that the free energy for the \( N = 3 \) LSTs in the NS-limit to order \( \mathcal{O}(Q_R) \) can be decomposed in terms of \( H_{(2s,0)}^{(1),(0)} \), similar to eq. (4.6). In the following we shall provide non-trivial evidence that such a decomposition indeed is possible.

From the Tables 8 and 6 we first notice that \( g_{a,k,(2s)}^{(r=1)}(\rho) = j_{a,k,(2s)}^{(r=1)}(\rho) \) and similarly,
Table 6: Expansion coefficients $a_{i}(2s)$.

| $s$ | $a_{0, (2s)}^{(r=1)}$ | $a_{1, (2s)}^{(r=1)}$ | $a_{2, (2s)}^{(r=1)}$ | $a_{3, (2s)}^{(r=1)}$ |
|-----|------------------|------------------|------------------|------------------|
| 0   | 0                | $\frac{1}{192}$ | $\frac{E_2}{48}$ | $\frac{E_2^2}{48}$ |
| 1   | $\frac{1}{18432}$ | $\frac{E_2}{9216}$ | $\frac{2E_4-3E_2^2}{4608}$ | $\frac{2E_2E_4-3E_2^3}{2304}$ |
| 2   | $\frac{E_2}{884736}$ | $\frac{45E_2^2-43E_4}{4423680}$ | $\frac{8E_6-21E_2E_4}{1105920}$ | $\frac{-45E_4^3+21E_2E_4^4+16E_2E_6-10E_4^2}{1105920}$ |
| 3   | $\frac{17E_4-5E_2^2}{424673280}$ | $\frac{315E_2^2-63E_4E_2-248E_6}{1486366480}$ | $\frac{315E_2^2-819E_2^2E_4-208E_2E_4+468E_4^2}{74378240}$ | $\frac{152E_2^2E_6-315E_2^2E_2-180E_2^4E_4+300E_2E_2E_4^2-112E_4E_6}{371589120}$ |

Table 7: Expansion coefficients $f_{a,k, (2s)}^{(r=1)}$.

| $s$ | $k$ | $f_{0,k, (2s)}^{(r=1)}$ | $f_{1,k, (2s)}^{(r=1)}$ | $f_{2,k, (2s)}^{(r=1)}$ | $f_{3,k, (2s)}^{(r=1)}$ |
|-----|-----|------------------|------------------|------------------|------------------|
| 0   | 1   | 0                | $\frac{1}{12}$ | $\frac{E_2}{6}$ | $\frac{E_4-3E_2^2}{288}$ |
| 1   | 1   | 0                | $\frac{1}{576}$ | 0 | $\frac{E_2}{36}$ |
| 2   | 0   | 0                | $\frac{1}{72}$ | $\frac{E_2}{18432}$ | $\frac{15E_2^2-17E_4}{92160}$ | $\frac{-45E_2^3+9E_2E_4+8E_6}{138240}$ |
| 2   | 2   | 0                | $\frac{1}{3456}$ | 0 | $\frac{3E_2^2-E_4}{1728}$ | $\frac{E_2}{720}$ |
| 3   | 3   | 0                | $\frac{1}{1440}$ | 0 | $\frac{E_2}{6}$ |
| 3   | 1   | $\frac{E_4}{2654208}$ | $\frac{E_2}{442368}$ | $\frac{315E_2^2-189E_2E_4-136E_6}{46448640}$ | $\frac{16E_6E_2-255E_2^3+504E_2E_4^2+315E_2^4}{46448640}$ |
| 2   | 2   | $\frac{1}{663552}$ | $\frac{E_4}{110592}$ | $\frac{17E_4-15E_2^2}{552960}$ | $\frac{-8E_6-9E_2E_4-45E_2^3}{829440}$ |
| 3   | 3   | $\frac{1}{69120}$ | 0 | $\frac{E_4-3E_2^2}{34560}$ | $\frac{E_2}{30240}$ |
| 4   | 4   | 0                | $\frac{1}{60480}$ | 0 | $\frac{E_2}{30240}$ |
\[
g_{0,k,(2s)}^{(r=1)} \quad g_{1,k,(2s)}^{(r=1)} \quad g_{2,k,(2s)}^{(r=1)} \quad g_{3,k,(2s)}^{(r=1)}
\]

| s | k | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 1/32 | 0 | 0 | 0 | 1/32 |
| 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | \frac{1}{884736} | \frac{1}{36864} | \frac{11E_{4}+15E_{2}^{2}}{245760} | \frac{184E_{6}+819E_{2}E_{4}+315E_{2}^{3}}{7744340} | \frac{13E_{4}+15E_{2}^{2}}{92160} | \frac{13E_{4}+15E_{2}^{2}}{92160} | \frac{1}{11520} |
| 2 | 0 | \frac{1}{3072} | \frac{1}{3072} | \frac{E_{2}}{512} | \frac{E_{2}}{512} | \frac{E_{2}}{512} | \frac{E_{2}}{512} | \frac{E_{2}}{512} |
| 3 | 0 | 0 | 0 | \frac{1}{384} | \frac{E_{2}}{192} | \frac{E_{2}}{192} | \frac{E_{2}}{192} | \frac{E_{2}}{192} |
| 4 | 0 | 0 | 0 | 0 | \frac{1}{384} | \frac{E_{2}}{192} | \frac{E_{2}}{192} | \frac{E_{2}}{192} |

Table 8: Expansion coefficients \(g_{n,k,(2s)}^{(r=1)}\).

\[
j_{0,\ell,k,(2s)}^{(r=1)} \quad j_{1,\ell,k,(2s)}^{(r=1)} \quad j_{2,\ell,k,(2s)}^{(r=1)} \quad j_{3,\ell,k,(2s)}^{(r=1)} \quad p_{\ell,k,(2s)}^{(r)}
\]

| s | k | \ell | 0 | 0 | 0 | 0 | 0 | 1 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 1/32 | 0 | 0 | 1/32 |
| 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | \frac{1}{884736} | \frac{1}{36864} | \frac{11E_{4}+15E_{2}^{2}}{245760} | \frac{184E_{6}+819E_{2}E_{4}+315E_{2}^{3}}{7744340} | \frac{13E_{4}+15E_{2}^{2}}{92160} | \frac{13E_{4}+15E_{2}^{2}}{92160} | \frac{1}{11520} |
| 2 | 0 | \frac{1}{3072} | \frac{1}{3072} | \frac{E_{2}}{512} | \frac{E_{2}}{512} | \frac{E_{2}}{512} | \frac{E_{2}}{512} | \frac{E_{2}}{512} |
| 3 | 0 | 0 | 0 | \frac{1}{384} | \frac{E_{2}}{192} | \frac{E_{2}}{192} | \frac{E_{2}}{192} | \frac{E_{2}}{192} |
| 4 | 0 | 0 | 0 | 0 | \frac{1}{384} | \frac{E_{2}}{192} | \frac{E_{2}}{192} | \frac{E_{2}}{192} |

Table 9: Expansion coefficients \(j_{n,k,(2s)}^{(r=1)}\).
| $s$ | $d_{0,(2s)}^{(r=2)}$ | $d_{1,(2s)}^{(r=2)}$ | $d_{2,(2s)}^{(r=2)}$ | $d_{3,(2s)}^{(r=2)}$ | $d_{4,(2s)}^{(r=2)}$ |
|-----|-------------------|-------------------|-------------------|-------------------|-------------------|
| 0   | 0                 | $\frac{1}{1769472}$ | $\frac{E_2}{221184}$ | $\frac{2E_4^2 + E_6}{221184}$ | $\frac{3E_2E_4 - 2E_6}{55296}$ |
| 1   | $\frac{1}{4112-24}$ | $\frac{1}{42467328}$ | $\frac{5E_4 - 4E_2^2}{14155776}$ | $\frac{7E_2E_4 - 2E_3^2 - 4E_6}{1769472}$ | $\frac{108E_2^2 - 208E_2E_4 + 12E_4^2}{10616832}$ |

Table 10: Expansion coefficients $d_{a,(2s)}^{(r=2)}$.

| $s$ | $d_{0,(2s)}^{(r=2)}$ | $d_{1,(2s)}^{(r=2)}$ | $d_{2,(2s)}^{(r=2)}$ | $d_{3,(2s)}^{(r=2)}$ | $d_{4,(2s)}^{(r=2)}$ |
|-----|-------------------|-------------------|-------------------|-------------------|-------------------|
| 0   | $\frac{24E_2E_4 - 32E_2E_6 + 9E_4^2}{110592}$ | $-E_6E_4^2 + 3E_2E_4^2 - 2E_4E_6$ | $\frac{1}{221184}$ | $\frac{E_4}{41568}$ | $\frac{3E_2E_4 - 2E_6}{1728}$ |
| 1   | $\frac{67E_2E_4 - 24E_2^2E_4 - 8(E_6E_2 + 4E_4E_4)}{884736}$ | $\frac{128E_2^2 + 78E_2E_4 + 72E_4^2E_2^4 - 384E_2E_4 + 14E_4^2}{2654208}$ | $\frac{1}{221184}$ | $\frac{E_4}{41568}$ | $\frac{3E_2E_4 - 2E_6}{1728}$ |

Table 11: Expansion coefficients $f_{a,(2s)}^{(r=2)}$.

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Furthermore, we can summarise Tables 6–9 through the following decompositions

\[
p_{\ell,k,(2s)}^{(r=2)}(n_1, n_2) = \sum_{a=0}^{k-1} n_1^{2k-2-2a} n_2^{2a}, \quad \text{and} \quad p_{\ell,k,(2s)}^{(r=1)}(n_1, n_2) = 0 \quad \forall \ell > 1. \quad (5.6)
\]

Furthermore, we can summarise Tables 6–9 through the following decompositions

\[
\begin{align*}
\sum_{k=1}^{s+1} n_1^{2k-1-1} n_1^{(r=1)}_{k,(2s)}(\rho, S) &= -\sum_{a=0}^{s} W_{(2s-2a)}(\rho, S) \sum_{i,j=0}^{a} A_{ij}^{(a)}(n) H_{(2i,0)}^{(1),\{0\}}(\rho, S) H_{(2j,0)}^{(1),\{0\}}(\rho, S), \\
\sum_{k=1}^{s+1} b_{k,(2s)}^{(r=1)}(\rho, S) &= -\sum_{a=0}^{s} W_{(2s-2a)}(\rho, S) \sum_{i,j=0}^{a} B_{ij}^{(a)}(n) H_{(2i,0)}^{(1),\{0\}}(\rho, S) H_{(2j,0)}^{(1),\{0\}}(\rho, S), \\
\sum_{k=1}^{s+1} \sum_{\ell} p_{\ell,k,(2s)}^{(r=1)}(n_1, n_2)_n^{(r)}(\rho, S) &= -\sum_{a=0}^{s} W_{(2s-2a)}(\rho, S) \sum_{i,j=0}^{a} C_{ij}^{(a)}(n_1, n_2) H_{(2i,0)}^{(1),\{0\}}(\rho, S) H_{(2j,0)}^{(1),\{0\}}(\rho, S),
\end{align*}
\]

which we conjecture to hold for generic values of \(s\) and where \(A_{ij}^{(a)}\), \(B_{ij}^{(a)}\) and \(C_{ij}^{(a)}\) are \((a + 1) \times (a + 1)\) matrices, whose components are given by

\[
\begin{align*}
A_{ij}^{(a)} &= \frac{2(-1)^{a+i+j} n_2^{2a+1-2(i+j)}}{\Gamma(2a - 2(i + j - 1))}, \\
B_{ij}^{(a)} &= \frac{2(-1)^{a+i+j} n_2^{2a+2-2(i+j)}}{\Gamma(2a + 2(i + j - 1))} \theta(a - i - j),
\end{align*}
\]

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\(s\) & \(k\) & \(g_{0,k,(2s)}^{(r=2)}\) & \(g_{1,k,(2s)}^{(r=2)}\) & \(g_{2,k,(2s)}^{(r=2)}\) & \(g_{3,k,(2s)}^{(r=2)}\) & \(g_{4,k,(2s)}^{(r=2)}\) & \(g_{5,k,(2s)}^{(r=2)}\) & \(g_{6,k,(2s)}^{(r=2)}\) \\
\hline
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\hline
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{tabular}
\caption{Expansion coefficients \(g_{a,k,(2s)}^{(r=2)}\).}
\end{table}
| $s$ | $k$ | $\ell$ | \( J_{0,\ell,k,2s} \) | \( J_{1,\ell,k,2s} \) | \( J_{2,\ell,k,2s} \) | \( J_{3,\ell,k,2s} \) | \( J_{4,\ell,k,2s} \) | \( J_{5,\ell,k,2s} \) | \( J_{6,\ell,k,2s} \) | \( P_{(r=2)_{\ell,k,2s}} \) |
|-----|-----|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0   | 1   | 1     | 0              | 0              | 0              | $\frac{1}{424^2}$ | 0              | 0              | 0              | 1              |
| 0   | 1   | 2     | 0              | 0              | 0              | 0              | $E_4$          | $\frac{96}{96}$ | 0              | 1              |
| 0   | 1   | 3     | 0              | 0              | 0              | 0              | 0              | 0              | $-\frac{E_6}{144}$ | 1              |
| 2   | 1   | 0     | 0              | 0              | 0              | $\frac{1}{1152}$ | 0              | 0              | 0              | 7$n_1^2 + 10m_1n_2 + 7n_2^2$ |
| 2   | 1   | 0     | 0              | 0              | 0              | 0              | 0              | $E_4$          | $\frac{480}{480}$ | 11$n_1^2 + 20m_1n_2 + 11n_2^2$ |
| 2   | 1   | 0     | 0              | 0              | 0              | 0              | 0              | $\frac{1}{576}$ | 0              | $(n_1^2 + n_1n_2 + n_2^2)(5n_1^2 + 9n_1n_2 + 5n_2^2)$ |
| 3   | 1   | 0     | 0              | 0              | 0              | 0              | 0              | 0              | $\frac{1}{144}$ | $(n_1 + n_2)^2(2n_1^4 + 4n_1^3n_2 + 9n_1^2n_2^2 + 4n_1n_2^3 + 2n_2^4)$ |

| $s$ | $k$ | $\ell$ | \( J_{0,\ell,k,2s} \) | \( J_{1,\ell,k,2s} \) | \( J_{2,\ell,k,2s} \) | \( J_{3,\ell,k,2s} \) | \( J_{4,\ell,k,2s} \) | \( J_{5,\ell,k,2s} \) | \( J_{6,\ell,k,2s} \) | \( P_{(r=2)_{\ell,k,2s}} \) |
|-----|-----|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1   | 1   | 1     | 0              | 0              | 0              | $\frac{1}{73728}$ | $-\frac{E_4}{18432}$ | $5E_4$ | $6144$ | $-\frac{3E_4E_2-5E_6}{2304}$ | $\frac{2E_4E_0+9E_6^2}{2304}$ | 1              |
| 2   | 1   | 0     | 0              | 0              | 0              | $\frac{5}{13824}$ | 0              | 0              | 0              | $n_1^2 + n_1n_2 + n_2^2$ |
| 2   | 2   | 0     | 0              | 0              | 0              | 0              | $-\frac{E_4}{9216}$ | 0              | 0              | $7n_1^2 + 10m_1n_2 + 7n_2^2$ |
| 2   | 3   | 0     | 0              | 0              | 0              | 0              | 0              | $\frac{E_4}{11520}$ | 0              | $83n_1^2 + 121m_1n_2 + 83n_2^2$ |
| 2   | 4   | 0     | 0              | 0              | 0              | 0              | 0              | $-\frac{E_4}{3840}$ | 0              | $11n_1^2 + 20m_1n_2 + 11n_2^2$ |
| 2   | 5   | 0     | 0              | 0              | 0              | 0              | 0              | $-\frac{E_4}{648}$ | 0              | $29n_1^2 + 48m_1n_2 + 29n_2^2$ |
| 3   | 1   | 0     | 0              | 0              | 0              | $\frac{1}{55296}$ | 0              | 0              | 0              | $(n_1^2 + n_1n_2 + n_2^2)(97n_1^2 + 141n_1n_2 + 97n_2^2)$ |
| 3   | 2   | 0     | 0              | 0              | 0              | 0              | 0              | $-\frac{E_4}{4608}$ | $\frac{3E_4}{2560}$ | $(n_1^2 + n_1n_2 + n_2^2)(5n_1^2 + 9n_1n_2 + 5n_2^2)$ |
| 4   | 1   | 0     | 0              | 0              | 0              | 0              | $\frac{1}{69120}$ | 0              | 0              | $92n_1^6 + 344n_1^5n_2 + 75n_1^4n_2^2 + 974n_1^3n_2^3 + 751n_1^2n_2^4 + 344n_1n_2^5 + 92n_2^6$ |
| 4   | 2   | 0     | 0              | 0              | 0              | 0              | 0              | $-\frac{E_4}{11520}$ | 0              | $(n_1 + n_2)^2(2n_1^4 + 4n_1^3n_2 + 9n_1^2n_2^2 + 4n_1n_2^3 + 2n_2^4)$ |
| 0   | 5   | 1     | 0              | 0              | 0              | 0              | 0              | 0              | $\frac{1}{120960}$ | $(n_1 + n_2)^2(20n_1^6 + 60n_1^5n_2 + 165n_1^4n_2^2 + 166n_1^3n_2^3 + 165n_1^2n_2^4 + 60n_1n_2^5 + 20n_2^6)$ |
Explicitly, for low values of \( a \) we have

\[
A^{(0)} = (-2n) , \quad A^{(1)} = \begin{pmatrix} \frac{n^3}{3} & -2n \\ -2n & 0 \end{pmatrix} , \quad A^{(2)} = \begin{pmatrix} -\frac{n^5}{60} & \frac{n^3}{3} & -2n \\ \frac{n^3}{3} & -2n & 0 \\ -2n & 0 & 0 \end{pmatrix} ,
\]

\[
B^{(0)} = (-n^2) , \quad B^{(1)} = \begin{pmatrix} \frac{n^4}{12} & -n^2 \\ -n^2 & 0 \end{pmatrix} , \quad B^{(2)} = \begin{pmatrix} -\frac{n^6}{360} & \frac{n^4}{12} & -n^2 \\ \frac{n^4}{12} & -n^2 & 0 \\ -n^2 & 0 & 0 \end{pmatrix} ,
\]

\[
C^{(0)} = (1) , \quad C^{(1)} = \begin{pmatrix} -\frac{n^2+n_2}{12} & 1 \\ 1 & 0 \end{pmatrix} , \quad C^{(2)} = \begin{pmatrix} -\frac{n_1+n_2^2}{12} & n_1+n_2^2 \\ n_1+n_2^2 & 1 \\ 1 & 0 \end{pmatrix} .
\] (5.9)

Notice that these matrices are very closely related and satisfy for example \( \partial_n B^{(a)}(n) = A^{(a)}(n) \).

Moreover, (5.7) yields a complete decomposition of the free energy for the \( N = 3 \) LSTs in the NS-limit, where the building blocks are only given by the free energy of the \( N = 1 \) LST \( H^{(1)}_{(2s,0)} \) and the expansion coefficients of the NS-limit of the function \( C_{1,2}(W_{(2s)}(\rho, S)) \).

### 5.3 Hecke Structures

Following the discussion in section 4.3 for the case \( N = 2 \), we will search for subsectors of the \( N = 3 \) free energy which in the NS-limit are related via Hecke transformations. We shall be able to identify three different contributions that are defined via certain contour integral prescriptions.

Generalising eq. (4.27) and eq. (4.28) to the case \( N = 3 \), we can define the following three subsectors of the \( N = 3 \) free energy

\[
C^{N=3,(r)}_{1,(2s,0)}(\rho, S) := \frac{1}{(2\pi i)^3} \sum_{\ell=0}^{\infty} Q_{\rho}^\ell \oint_0 dQ_{\tilde{a}_1} Q_{\tilde{a}_1}^{1+\ell} \oint_0 dQ_{\tilde{a}_2} Q_{\tilde{a}_2}^{1+\ell} \oint_0 dQ_{\tilde{a}_3} Q_{\tilde{a}_3}^{1+\ell} P_{2,(2s,0)}^{(r)}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, S) ,
\] (5.10)

\[
C^{N=3,(r)}_{2,(2s,0)}(\rho, S) := \frac{1}{(2\pi i)^3} \sum_{\ell=0}^{\infty} Q_{\rho}^\ell \oint_0 d\tilde{a}_1 \tilde{a}_1 Q_{\tilde{a}_1}^\ell \oint_0 dQ_{\tilde{a}_2} Q_{\tilde{a}_2}^{1+\ell} \oint_0 dQ_{\tilde{a}_3} Q_{\tilde{a}_3}^{1+\ell} P_{2,(2s,0)}^{(r)}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, S) ,
\] (5.11)

\[
C^{N=3,(r)}_{3,(2s,0)}(\rho, S, \epsilon_1) := \frac{1}{(2\pi i)^3} \sum_{\ell=0}^{\infty} Q_{\rho}^\ell \oint_0 d\tilde{a}_1 \tilde{a}_1 \oint_{\tilde{a}_1} d\tilde{a}_2 (\tilde{a}_1 + \tilde{a}_2) P_{2,(2s,0)}^{(r)}(\tilde{a}_1, \tilde{a}_2, \rho, S) .
\] (5.12)

Here the contour integral \( \oint_z \) is along a small circle around the point \( z \in \mathbb{Z} \), in such a way to extract the residue in a Laurent series. Furthermore, in the definition of \( C^{N=3,(r)}_{3,(2s,0)}(\rho, S) \) we have
implicitly used $\hat{a}_3 = \rho - \hat{a}_1 - \hat{a}_2$. Finally, as in the case of $N = 2$, we also introduce the following series in powers of $\epsilon_1$

$$C^{N=3,(r)}_a(\rho, S, \epsilon_1) = \sum_{s=0}^{\infty} \epsilon_1^{2s-2a} C^{N=3,(r)}_{a,(2s)}(\rho, S), \quad \forall a = 1, 2, 3, \quad (5.13)$$

In the case of $N = 3$, the free energy counts BPS states of 3 M5-branes separated on a circle with multiple M2-branes stretched between them. The functions $C^{N=3,(r)}_a$ only receive contributions from certain such configurations, as is schematically shown in Fig. 8:

- **Combination** $C^{N=3,(r)}_{1,(2s,0)}$:

  $C^{N=3,(r)}_{1,(2s,0)}$ can be described by extracting a particular class of terms in the Fourier expansion of $P(r)^{3,(2s,0)}$ in powers of $Q_{\hat{a}_1,1,3}$. Indeed, upon writing

  $$P(r)^{3,(2s,0)}(\hat{a}_1, \hat{a}_2, \hat{a}_3, S) = \sum_{n_1, n_2, n_3 = 0}^{\infty} Q_{\hat{a}_1}^{n_1} Q_{\hat{a}_2}^{n_2} Q_{\hat{a}_3}^{n_3} P_{(2s,0)}^{(r),\{n_1,n_2,n_3\}}(S), \quad (5.14)$$

  the contour prescriptions in $C^{N=3,(r)}_{1,(2s,0)}$ are designed to extract only those terms with $n_1 = n_2 = n_3$. Therefore, $C^{N=3,(r)}_{1,(2s,0)}$ receives contributions only from those brane configurations, in which an equal number of M2-branes is stretched between any two adjacent M5-branes, as is visualised in Fig. 8 (a). Following the definition of $H^{(r),0}_{(2s,0)}$ in (2.21), we find that
$\mathcal{C}_{1,(2s,0)}^{N=3,(r)}$ can equivalently be written as

$$
\mathcal{C}_{1,(2s,0)}^{N=3,(r)}(\rho, S) = H_{(2s,0)}^{(r),(0,0,0)}(\rho, S), \quad \mathcal{C}_{1,(2s,0)}^{N=3,(r)}(\rho, S, \epsilon_1) = \sum_{s=0}^{\infty} \epsilon_1^{2s-2} H_{(2s,0)}^{(r),(0,0,0)}(\rho, S).
$$

(5.15)

It is in fact the reduced free energy for $N = 3$ that was studied in [45]. Explicit expansions of $\mathcal{C}_{1,(2s,0)}^{N=3,(r)}$ for $r = 1$ and $r = 2$ can be recovered from Table 6 and Table 10.

- Combination $\mathcal{C}_{2,(2s,0)}^{N=3,(r)}$:
  The function $\mathcal{C}_{2,(2s,0)}^{N=3,(r)}$ in (5.11) extracts specific coefficients in a mixed Fourier- and Laurent series expansion of the free energy. Starting from the Fourier expansion (5.14), $\mathcal{C}_{2,(2s,0)}^{N=3,(r)}$ receives contributions only from coefficients with $n_1 \neq n_2 = n_3$. Physically, these correspond to configurations where an equal number $\ell$ of M2-branes is stretched between the second and third as well as third and first M5-branes, while a different number $n \neq \ell$ of M2-branes is stretched between the first and second M5-branes. Such configurations are schematically shown in Fig. 8 (b). Finally, the last contour integral in (5.11) over $\hat{a}_1$ extracts the second order pole in the Laurent expansion.

With respect to the decomposition (5.3), the coefficients $\mathcal{C}_{2,(2s,0)}^{N=3,(r)}$ can be written in the following form

$$
\mathcal{C}_{2,(2s,0)}^{N=3,(r)}(\rho, S) = \frac{1}{2\pi i} \oint_{\hat{a}_1} d\hat{a}_1 \sum_{n=1}^{\infty} \left[ H_{(2s,0)}^{(r),(n,0,0)}(\rho, S) Q^n_{\hat{a}_1} + H_{(2s,0)}^{(r),(n,n,0)}(\rho, S) \frac{Q^n_{\rho}}{Q^n_{\hat{a}_1}} \right].
$$

(5.16)

In order to perform the final contour integration over $\hat{a}_1$, we can use the conjectured form (5.4) of $H_{(2s,0)}^{(r),(n,0,0)}$ and $H_{(2s,0)}^{(r),(n,n,0)}$ to write for the integrand

$$
\mathcal{I}_{\alpha}^{N=3,(r)} = \sum_{n=1}^{\infty} \left[ H_{(2s,0)}^{(r),(n,0,0)}(\rho, S) Q^n_{\hat{a}_1} + H_{(2s,0)}^{(r),(n,n,0)}(\rho, S) \frac{Q^n_{\rho}}{Q^n_{\hat{a}_1}} \right] = \sum_{n=1}^{\infty} \frac{n^{2k-1} f_{k,(2s)}^{(r)}}{1 - Q^n_{\rho}} \left( Q^n_{\hat{a}_1} + Q^n_{\rho} \right) + \frac{n^{2k} Q^n_{\rho} g_{k,(2s)}^{(r)}}{Q^n_{\hat{a}_1}} \left( Q^n_{\hat{a}_1} + 1 \right) + \sum_{k=1}^{rs+1} f_{k,(2s)}^{(r)} \mathcal{I}_{k-1}(\rho, \hat{a}_1) + \sum_{k=1}^{rs+1} g_{k,(2s)}^{(r)} \sum_{n=1}^{\infty} \frac{n^{2k} Q^n_{\rho}}{Q^n_{\hat{a}_1}} \left( Q^n_{\hat{a}_1} + Q^{-n}_{\hat{a}_1} \right),
$$

(5.17)

where we have exchanged the order of summations and $\mathcal{I}_{\alpha}$ is defined in (4.15). With

---

22We remark in passing that contributions with $n = \ell$ in Fig. 8 (b) would give rise to terms with $H_{(2s,0)}^{(r),(0,0,0)}$ in (5.16). The latter, however, only depends on $\rho$ and not $\hat{a}_1$ and thus do not contribute to the contour integral over $\hat{a}_1$ in $\mathcal{C}_{2,(2s,0)}^{N=3,(r)}$. 

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39
\[
\frac{x}{(1-x)^2} = \sum_{\ell=1}^{\infty} \ell \cdot x^\ell, \text{ we can write for the sum over } n \text{ in the last term in (5.17)}
\]

\[
\sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} n^{2k} \ell Q^\ell \left( Q^{n}_{\tilde{a}_1} + Q^{-n}_{\tilde{a}_1} \right) = D_{\tilde{a}_1}^{2k} \sum_{n=1}^{\infty} Q^n \sum_{\ell | n} \frac{n}{\ell} \left( Q^\ell_{\tilde{a}_1} + Q^{-\ell}_{\tilde{a}_1} \right) := J_k(\rho, \tilde{a}_1) = D_{\tilde{a}_1}^{2k} J_0(\rho, \tilde{a}_1). 
\]

For \(0 < |Q| < 1\) the function \(J_0\) is in fact regular at \(\tilde{a}_1\), such that \(\int_0 \! d\tilde{a}_1 \tilde{a}_1 J_k(\rho, \tilde{a}_1) = 0\) for \(k \geq 0\). Using furthermore (4.18), we get

\[
\mathcal{C}^{N=3, (r)}_{2,(2s,0)} (\rho, S) = \frac{1}{r} \int_{k=1,(2s)}^{(r)} (\rho, S), \quad \mathcal{C}^{N=3, (r)}_{2, (2s,0)} (\rho, S, \epsilon_1) = \frac{1}{r} \sum_{s=0}^{\infty} \epsilon_1^{2s-1} \int_{k=1,(2s)}^{(r)} (\rho, S). \quad (5.18)
\]

- **Combination \(\mathcal{C}^{N=3, (r)}_{3,(2s,0)}\):**

  The function \(\mathcal{C}^{N=3, (r)}_{3,(2s,0)}\) in (5.12) receives contributions from M-brane configurations with \(n_i\) M2-branes stretched between the \(i\)th and \((i+1)\)th M5-brane (with \(n_i \neq n_j\) for \(i \neq j\)). The contour integrals, however, extract the second order poles for the successive limits \(\tilde{a}_2 \to -\tilde{a}_1\) and \(\tilde{a}_1 \to 0\). In the decomposition (5.3) only the terms with \(\mathcal{n} = \{n_1 + n_2, n_1, 0\}\) for \(n_{1,2} \geq 1\) contribute:

\[
\mathcal{C}^{N=3, (r)}_{3,(2s,0)} (\rho, S) = \frac{1}{(2\pi i)^2} \int_0 \! d\tilde{a}_1 \tilde{a}_1 \int_0 \! d\tilde{a}_2 (\tilde{a}_1 + \tilde{a}_2) \sum_{n_1, n_2=1}^{\infty} H_{(2s,0)}^{(r), \{n_1 + n_2, n_1, 0\}} (\rho, S) X_{n_1, n_2} (\tilde{a}_1, \tilde{a}_2, \rho), \quad (5.19)
\]

with

\[
X_{n_1, n_2} (\tilde{a}_1, \tilde{a}_2, \rho) = Q^{n_1 + n_2}_{\tilde{a}_1} Q^{n_1}_{\tilde{a}_2} + \frac{Q^n_{\tilde{a}_1} Q^n_{\tilde{a}_2}}{Q^{n_1}_{\tilde{a}_1} Q^{n_1}_{\tilde{a}_2}} + \frac{Q^{n_1 + n_2}_{\tilde{a}_1} Q^{n_1}_{\tilde{a}_2}}{Q^{n_1}_{\tilde{a}_1} Q^{n_1}_{\tilde{a}_2}} + (\tilde{a}_1 \leftrightarrow \tilde{a}_2) \quad (5.20)
\]

Using the conjectured form (5.4) of \(H_{(2s,0)}^{(r), \{n_1 + n_2, n_1, 0\}}\) we can write

\[
\mathcal{C}^{N=3, (r)}_{3,(2s,0)} (\rho, S) = \frac{1}{(2\pi i)^2} \int_0 \! d\tilde{a}_1 \tilde{a}_1 \int_0 \! d\tilde{a}_2 (\tilde{a}_1 + \tilde{a}_2) \sum_{n_1, n_2=1}^{r+1} \sum_{k=1}^{\infty} \sum_{k_1, k_2=0}^{\infty} X_{n_1, n_2} (\tilde{a}_1, \tilde{a}_2, \rho) 
\times \left[ Q^{k_1 n_1 + k_2 n_2}_{\rho} n_2 (n_2 + 2n_1) p^{(r)}_{\ell, k,(2s)} (n_1, -n_1 - n_2) \right]^{(r)}_{\ell, k,(2s)} (\rho, S) 
+ Q^{k_1 n_1 + k_2 (n_1 + n_2)} (n_1^2 - n_2^2) p^{(r)}_{\ell, k,(2s)} (n_1, n_2) \right]^{(r)}_{\ell, k,(2s)} (\rho, S). \quad (5.21)
\]
To further simplify this expression, let $\beta_{1,2} \in \mathbb{N}$ and consider

$$
\Upsilon = \frac{1}{(2\pi i)^2} \oint_0 d\hat{\alpha}_1 \hat{\alpha}_1 \oint d\hat{\alpha}_2 (\hat{\alpha}_1 + \hat{\alpha}_2) \sum_{n_1, n_2=1}^{\infty} n_1^{\beta_1} n_2^{\beta_2} \sum_{k_1, k_2=0}^{\infty} Q_{\rho}^{k_1 n_1 + k_2 n_2} X_{n_1, n_2}(\hat{\alpha}_1, \hat{\alpha}_2, \rho) .
$$

(5.22)

Assuming that $0 < |Q_{\rho}| < 1$, the factors $Q_{\rho}$ act as regulators for the sum over $n_{1,2}$ in the limit $Q_{\hat{\alpha}_{1,2}} \to 1$. The divergence that is relevant for the contour integrals therefore only stems from those terms where these factors are absent, namely for $k_1 = k_2 = 1$,

$$
\Upsilon = \frac{1}{(2\pi i)^2} \oint_0 d\hat{\alpha}_1 \hat{\alpha}_1 \oint d\hat{\alpha}_2 (\hat{\alpha}_1 + \hat{\alpha}_2) \sum_{n_1, n_2=1}^{\infty} n_1^{\beta_1} n_2^{\beta_2} \left[ (Q_{\hat{\alpha}_1} Q_{\hat{\alpha}_2})^{n_1} (Q_{\hat{\alpha}_1}^{n_2} + Q_{\hat{\alpha}_2}^{n_2}) \right]
$$

$$
= \frac{1}{(2\pi i)^2} \oint_0 d\hat{\alpha}_1 \hat{\alpha}_1 \oint d\hat{\alpha}_2 (\hat{\alpha}_1 + \hat{\alpha}_2) \left[ D_{\hat{\alpha}_1}^{\beta_1} (D_{\hat{\alpha}_1} - D_{\hat{\alpha}_2})^{\beta_2} \sum_{n_1, n_2=1}^{\infty} (Q_{\hat{\alpha}_1} Q_{\hat{\alpha}_2})^{n_1} Q_{\hat{\alpha}_1}^{n_2} Q_{\hat{\alpha}_2}^{n_2} \right] + D_{\hat{\alpha}_1}^{\beta_1} (D_{\hat{\alpha}_2} - D_{\hat{\alpha}_1})^{\beta_2} \sum_{n_1, n_2=1}^{\infty} (Q_{\hat{\alpha}_1} Q_{\hat{\alpha}_2})^{n_1} Q_{\hat{\alpha}_1}^{n_2} Q_{\hat{\alpha}_2}^{n_2} \right] \right) \right)
$$

(5.23)

Assuming that $0 < |Q_{\hat{\alpha}_{1,2}}| < 1$ we can perform the sum over $n_{1,2}$ to find

$$
\Upsilon = \frac{1}{(2\pi i)^2} \oint_0 d\hat{\alpha}_1 \hat{\alpha}_1 \oint d\hat{\alpha}_2 (\hat{\alpha}_1 + \hat{\alpha}_2) \left[ D_{\hat{\alpha}_1}^{\beta_1} (D_{\hat{\alpha}_1} - D_{\hat{\alpha}_2})^{\beta_2} \frac{Q_{\hat{\alpha}_1}^2 Q_{\hat{\alpha}_2}^2}{(1 - Q_{\hat{\alpha}_1} Q_{\hat{\alpha}_2})(1 - Q_{\hat{\alpha}_1})} \right] + D_{\hat{\alpha}_1}^{\beta_1} (D_{\hat{\alpha}_2} - D_{\hat{\alpha}_1})^{\beta_2} \frac{Q_{\hat{\alpha}_1}^2 Q_{\hat{\alpha}_2}^2}{(1 - Q_{\hat{\alpha}_1} Q_{\hat{\alpha}_2})(1 - Q_{\hat{\alpha}_2})} \right] \right)
$$

(5.24)

From the explicit series expansions

$$
\frac{Q_{\hat{\alpha}_1}^2 Q_{\hat{\alpha}_2}^2}{(1 - Q_{\hat{\alpha}_1} Q_{\hat{\alpha}_2})(1 - Q_{\hat{\alpha}_1})} = \frac{1}{(2\pi i)^2 \hat{\alpha}_1 (\hat{\alpha}_1 + \hat{\alpha}_2)} \left[ 1 + \pi i (2\hat{\alpha}_1 + \hat{\alpha}_2) + O(\hat{\alpha}_1^2) \right] \right), \\
\frac{Q_{\hat{\alpha}_1}^2 Q_{\hat{\alpha}_2}^2}{(1 - Q_{\hat{\alpha}_1} Q_{\hat{\alpha}_2})(1 - Q_{\hat{\alpha}_2})} = \frac{1}{(2\pi i)^2 \hat{\alpha}_2 (\hat{\alpha}_1 + \hat{\alpha}_2)} \left[ 1 + \pi i (\hat{\alpha}_1 + 2\hat{\alpha}_2) + O(\hat{\alpha}_2^2) \right] \right), \\
= - \frac{1 + \hat{\alpha}_1 \hat{\alpha}_2 + (\hat{\alpha}_1 + \hat{\alpha}_2)^2 + O((\hat{\alpha}_1 + \hat{\alpha}_2)^3)}{(2\pi i)^2 \hat{\alpha}_1 (\hat{\alpha}_1 + \hat{\alpha}_2)} \left[ 1 + \pi i (\hat{\alpha}_1 + 2\hat{\alpha}_2) + O(\hat{\alpha}_2^2) \right] \right) \right).
$$

(5.25)

it follows that $\Upsilon$ is only non-vanishing for $\beta_1 + \beta_2 = 2$. To understand why no higher derivatives may contribute, we define $\hat{\beta} = \hat{\alpha}_1 + \hat{\alpha}_2$ and consider respectively for the first

\[23\] Notice, in order to have a pole for $\hat{\alpha}_2 \to -\hat{\alpha}_1$ and $\hat{\alpha}_1 \to 0$, both $k_1$ and $k_2$ have to vanish simultaneously.
and second term in (5.24)

\[
\frac{Q_{a_{1,2}^t} Q_b}{1 - Q_{a_{1,2}^t}^t 1 - Q_b} = \left[ \frac{1}{2\pi i a_{1,2}^t} + \frac{i \pi a_{1,2}^t}{6} + \mathcal{O}(\bar{a}_1^2) \right] \left[ \frac{1}{2\pi i b} + \frac{i \pi b}{6} + \mathcal{O}(\bar{b}^2) \right]
\]

\[
= \left[ \frac{1}{2\pi i b} + \frac{i \pi b}{6} + \mathcal{O}(\bar{b}^2) \right] \times \left\{ \frac{1}{2\pi i a_1^t} + \frac{i \pi a_1^t}{6} + \mathcal{O}(\bar{a}_1^2) \right\} + \mathcal{O}(\bar{a}_1^2) + \mathcal{O}(\bar{b}^2)
\]

(5.26)

which only has poles of second order in \( \hat{a}_{1,2} \) and \( \hat{b} \) if hit with two derivatives. This implies that in (5.21) only terms with \( k = 1 \) (in which case \( \hat{p}_{\ell,k=1,(2s)} = \text{const.} \) are polynomials of order 0) contribute. Performing the explicit integrals, we obtain

\[
C_{3,(2s,0)}^{N=3,(r)}(\rho, S) = \frac{1}{r^2} \sum_{\ell} \hat{p}_{\ell,k=1,(2s)}^{(r)} \hat{h}_{\ell,k=1,(2s)}^{(r)}(\rho, S),
\]

\[
C_{3}^{N=3,(r)}(\rho, S, \epsilon_1) = \frac{1}{r^2} \sum_{s=0}^{\infty} \epsilon_1^{2s-4} \sum_{\ell} \hat{p}_{\ell,k=1,(2s)}^{(r)} \hat{h}_{\ell,k=1,(2s)}^{(r)}(\rho, S).
\]

(5.27)

By comparing the explicit expressions for the contributions \( C_{3,(2s,0)}^{N=3,(r)} \) for \( r = 1, 2 \) (and \( r = 3 \)) and \( s \) up to 4, we find that they are related through Hecke operators in the following fashion

\[
C_{3,(2s,0)}^{N=3,(r)}(\rho, S) = H_r \left[ C_{3,(2s,0)}^{N=3,(1)}(\rho, S) \right], \quad \forall a = 1, 2, 3.
\]

(5.28)

The normalisation factors \( 1/r \) and \( 1/r^2 \) appearing in the definition (5.11) and (5.12) were chosen to normalise the right hand side of (5.28).

### 5.4 Decomposition of \( C_{i}^{N=3,(r)} \)

#### 5.4.1 Factorisation at Order \( Q_{iR}^1 \)

Similar to the entire free energy \( P_{3,(2s,0)}^{(r)}(\hat{a}_{1,2,3}, S) \) (see eq. (5.7)) also the functions \( C_{a,(2s,0)}^{N=3,(r=1)} \) can be decomposed into small building blocks. Based on the examples provided in section 6.1 we find the following decomposition

\[
C_{1,(2s,0)}^{N=3,(r=1)} = 3 \sum_{a=0}^{s} H_{2s-2a}^{(1),\{0\}} \sum_{i,j=0}^{a} \delta_{a,i+j} W_{2i,0} W_{2j,0},
\]

\[
C_{2,(2s,0)}^{N=3,(r=1)} = -2 \sum_{a=0}^{s} H_{2s-2a}^{(1),\{0\}}(\rho, S) \sum_{i,j=0}^{a} \delta_{a,i+j} W_{2i,0}(\rho, S) H_{2j,0}^{(1),\{0\}}(\rho, S),
\]

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These expressions (as similar equations in the remainder of this subsection) are understood to hold order by order in an expansion of $\epsilon_1$. Combining these expansion coefficients (in a series of $\epsilon_1$), we can equivalently write

$$
C_{N=3,(r=1)}(\rho, S, \epsilon_1) = 3 H_{N=1}^{(1)}(\rho, S, \epsilon_1) W_{NS}^{(1)}(\rho, S, \epsilon_1) W_{NS}^{(1)}(\rho, S, \epsilon_1),
$$

$$
C_{N=3,(r=1)}(\rho, S, \epsilon_1) = -2 \epsilon_1 H_{N=1}^{(1)}(\rho, S, \epsilon_1) H_{N=1}^{(1)}(\rho, S, \epsilon_1) W_{NS}^{(1)}(\rho, S, \epsilon_1),
$$

$$
C_{N=3,(r=1)}(\rho, S, \epsilon_1) = \epsilon_1^2 H_{N=1}^{(1)}(\rho, S, \epsilon_1) H_{N=1}^{(1)}(\rho, S, \epsilon_1) H_{N=1}^{(1)}(\rho, S, \epsilon_1),
$$

where the coefficients $H_{N=1}^{(1)}$ and $W_{NS}^{(1)}$ are defined in (C.3) and (C.8) respectively.

### 5.4.2 Factorisation at Order $Q_R^2$

Following the example of $N = 2$ discussed in Section 4.4.3, we expect a decomposition of $C_{N=3,(r)}$ into more fundamental building blocks to hold also for $r > 1$. Indeed, for $r = 2$ we find

$$
C_{1}^{N=3,(2)} = \frac{4}{3} H_{N=1}^{(2)} W_{NS}^{(2)} W_{NS}^{(2)} + (H_{N=1}^{(1)})^6 T_{1,1,2},
$$

$$
C_{2}^{N=3,(2)} = -\frac{8}{9} H_{N=1}^{(2)} H_{N=1}^{(2)} W_{NS}^{(2)} + (H_{N=1}^{(1)})^4 T_{2,1,1},
$$

$$
C_{3}^{N=3,(2)} = \frac{4}{9} H_{N=1}^{(2)} H_{N=1}^{(2)} + (H_{N=1}^{(1)})^4 T_{3,1,1},
$$

and for $r = 3$

$$
C_{1}^{N=3,(3)} = \frac{27}{16} H_{N=1}^{(3)} W_{NS}^{(3)} W_{NS}^{(3)} + (H_{N=1}^{(1)})^9 T_{1,1,1,1},
$$

$$
C_{2}^{N=3,(3)} = -\frac{9}{8} H_{N=1}^{(3)} H_{N=1}^{(3)} W_{NS}^{(3)} + (H_{N=1}^{(1)})^7 T_{2,2,2},
$$

$$
C_{3}^{N=3,(3)} = \frac{9}{16} H_{N=1}^{(3)} H_{N=1}^{(3)} H_{N=1}^{(3)} + (H_{N=1}^{(1)})^5 T_{3,3,3}.
$$
Here $\mathcal{T}_{a,\mathbf{i}_1\mathbf{i}_2}^{(2)}$ are quasi modular forms that are independent of $S$, which satisfy

\[
\begin{align*}
\mathcal{T}_{3,(i_1,i_2)}^{(2)} &= \frac{3}{\epsilon_1^3} \frac{\partial \mathcal{T}_{2,(i_1,i_2)}^{(2)}}{\partial E_2(\rho)} = \frac{6}{\epsilon_1^2} \frac{\partial^2 \mathcal{T}_{1,(i_1,i_2)}^{(2)}}{(E_2(\rho))^2}, \\
\mathcal{T}_{3,(i_1,i_2,i_3)}^{(3)} &= \frac{2}{\epsilon_1^3} \frac{\partial \mathcal{T}_{2,(i_1,i_2,i_4)}^{(3)}}{\partial E_2(\rho)} = \frac{8}{3\epsilon_1^2} \frac{\partial^2 \mathcal{T}_{1,(i_1,i_2,i_3)}^{(3)}}{(E_2(\rho))^2}, \\
\mathcal{T}_{2,(i_1,i_2,i_3)}^{(3)} &= \frac{4}{\epsilon_1^2} \frac{\partial \mathcal{T}_{1,(i_1,i_2,i_3)}^{(3)}}{\partial E_2(\rho)},
\end{align*}
\]

and where $\mathcal{T}_{3,\mathbf{i}}^{(2)}$ can be expanded in $\epsilon_1$ as follows

\[
\begin{align*}
\frac{1}{\epsilon_1^6} \mathcal{T}_{3,(6,0)}^{(2)} &= \frac{E_6}{192} + \frac{E_2^4}{768} + \frac{11E_4E_6\epsilon_1^4}{46080} + \frac{\epsilon_1^6(7E_4^3 + 4E_6^2)}{290304} + O(\epsilon_1^8), \\
\frac{1}{\epsilon_1^6} \mathcal{T}_{3,(4,1)}^{(2)} &= \frac{E_4}{8} + \frac{E_6\epsilon_1^2}{48} + \frac{17E_2^2\epsilon_1^4}{5760} + \frac{31E_4E_6\epsilon_1^6}{80640} + O(\epsilon_1^8), \\
\mathcal{T}_{3,(2,2)}^{(2)} &= 0,
\end{align*}
\]

and similarly $\mathcal{T}_{3,\mathbf{i}}^{(2)}$ can be expanded as\footnote{To keep the length of this paper manageable, we refrain from explicitly writing $\mathcal{T}_{3,\mathbf{i}}^{(2,3)}$.}

\[
\begin{align*}
\frac{1}{3\epsilon_1^{12}} \mathcal{T}_{3,(9,0,0)}^{(3)} &= \frac{9E_4^3 + 4E_6^2}{62208} + \frac{19E_2^4E_6\epsilon_1^4}{124416} + \frac{\epsilon_1^6(493E_4^4 + 583E_6E_4^2)}{14929920} + O(\epsilon_1^8), \\
\frac{1}{3\epsilon_1^{10}} \mathcal{T}_{3,(7,1,0)}^{(3)} &= \frac{5E_4E_6}{1296} + \frac{\epsilon_1^2(13E_4^3 + 7E_6^2)}{7776} + \frac{215E_4^2E_6\epsilon_1^4}{186624} + O(\epsilon_1^6), \\
\frac{1}{3\epsilon_1^8} \mathcal{T}_{3,(5,2,0)}^{(3)} &= \frac{5E_4^2}{324} + \frac{19E_2E_6\epsilon_1^4}{1944} + \frac{\epsilon_1^4(2E_4^3 + 73E_6^2)}{46656} + O(\epsilon_1^6), \\
\frac{1}{3\epsilon_1^6} \mathcal{T}_{3,(3,3,0)}^{(3)} &= \frac{20E_6}{243} + \frac{\epsilon_1^2E_2^2}{27} + \frac{19E_4E_6\epsilon_1^4}{1458} + O(\epsilon_1^6), \\
\frac{1}{3\epsilon_1^4} \mathcal{T}_{3,(2,2,1)}^{(3)} &= \frac{E_4}{3} + \frac{\epsilon_1^2E_6}{9} + \frac{\epsilon_1^4E_4^2}{30} + O(\epsilon_1^6).
\end{align*}
\]

These examples suggest the following general form

\[
\begin{align*}
C_1^{N=3, (r)} &= \sum_{i_1, \ldots, i_r} \mathcal{T}_{a,(i_1,\ldots,i_r)}^{(r)} (H_{N=1}^{(1)})^i \ldots (H_{N=1}^{(r)})^i + \left( \frac{r}{\sigma_1(r)} \right)^2 \begin{cases} 
3H_{N=1}^{(r)}W_{NS}^{(r)}W_{NS}^{(r)} & \text{for } a = 1, \\
-2H_{N=1}^{(r)}H_{N=1}^{(r)}W_{NS}^{(r)} & \text{for } a = 2, \\
H_{N=1}^{(r)}W_{NS}^{(r)}W_{NS}^{(r)} & \text{for } a = 3,
\end{cases}
\end{align*}
\]

which generalises (4.38). Here the summation in (5.34) is restricted to

\[
\sum_{j=1}^{r} j i_j = 3r, \quad \text{and} \quad i_1 > 0,
\]

(5.35)
and the coefficients $\mathbf{Y}^{(r)}_{a(i_1,\ldots,i_r)}$ satisfy

$$\frac{\partial \mathbf{Y}^{(r)}_{a(i_1,\ldots,i_r)}}{\partial E_2(\rho)} = 0, \quad \mathbf{Y}^{(r)}_{a(i_1,\ldots,i_r)} = \frac{6}{r \epsilon_1^2} \frac{\partial \mathbf{Y}^{(r)}_{a(i_1,\ldots,i_r)}}{\partial E_2(\rho)}, \quad \mathbf{Y}^{(r)}_{a(i_1,\ldots,i_r)} = \frac{4}{r \epsilon_1^2} \frac{\partial \mathbf{Y}^{(r)}_{a(i_1,\ldots,i_r)}}{\partial E_2(\rho)}, \quad \forall r > 1.$$

(5.36)

The first equation in fact implies that $\mathbf{Y}^{(3)}_{a(i_1,\ldots,i_r)}$ are (holomorphic) Jacobi forms.

### 6 Hecke Structure for $N = 4$

In this section we present some partial results for the LST with $N = 4$. Since in this case the free energy is much more complicated than for $N = 2$ or $N = 3$, we shall not be able to achieve a full characterisation. However, the partial results we manage to extract fall in line with the patterns we have seen in the previous sections.

#### 6.1 Decomposition of the Free Energy

As in the previous cases, the starting point is to compute the decomposition of the free energy. The web diagram representing $X_{4,1}$, which is relevant for the $N = 4$ free energy is show in Fig. 9.

In addition to the Kähler parameters shown in the figure, we also have

$$\rho = \hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4, \quad R - 4S = v - 3m , \quad (6.1)$$

From the partition function $\mathcal{Z}_{4,1}$ we can compute the free energy

$$\mathcal{F}_{4,1}(\hat{a}_{1,2,3,4}, S, R, \epsilon_{1,2}) = \log \mathcal{Z}_{4,1}(\hat{a}_{1,2,3,4}, S, R, \epsilon_{1,2}) ,$$

As in the cases $N = 2$ and $N = 3$, we focus exclusively on the NS-limit. In this case, following eq. (2.22), we can decompose the free energy in terms of $H_{(2s,0)}^{(r),n}$, where $n$ can be either of the following combinations

$$\{0,0,0,0\}, \quad \{n,0,0,0\}, \quad \{n,n,0,0\}, \quad \{n_1+n_2,n_1,0,0\}, \quad \{n_1+n_2,n_1,0,n_1\}.$$
The first sum \( m = (m_1, m_2, m_3, m_4) \) is over all combinations appearing in (6.2), while the second sum is over distinct orbits of Dih_4 acting on \((\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4)\).

Conjectures for the \( H_{(2s,0)}^{(r=1), m} \) for all \( m \) and \( s = 0 \) have been presented in [57], which are of the form

\[
H_{(2s,0)}^{(r=1), m} = -\sum_{a=0}^{4} w_{a, (2s,0)}^{(r=1), m}(\rho) \left( \phi_{-1}(\rho, S) \right)^{a} \left( \phi_{0,1}(\rho, S) \right)^{4-a}
\]  

(6.4)

For the readers convenience, we recall some of the \( g_{(2s,0)}^{(r=1), m}(\rho) \) in Table 14 below, which turn out to be relevant for the further discussion.

| \( m \)     | \( a = 0 \) | \( a = 1 \) | \( a = 2 \) | \( a = 3 \) | \( a = 4 \) |
|------------|------------|------------|------------|------------|------------|
| (0, 0, 0, 0) | 0 | \( \frac{1}{2} \) | \( \frac{E_2}{256} \) | \( \frac{E_2}{256} \) | \( \frac{E_3}{432} \) |
| (n, 0, 0, 0) | 0 | 0 | \( \frac{n}{288(1-Q_p^n)} \) | \( \frac{nE_2}{256} + \frac{n^2Q_p^n}{12(1-Q_p^n)^2} \) | \( \frac{nE_2}{256} + \frac{n^2Q_p^n}{6(1-Q_p^n)^2} \) |
| (n, n, 0) | 0 | 0 | \( \frac{n}{288(1-Q_p^n)} \) | \( \frac{nE_2}{256} + \frac{n^2Q_p^n}{12(1-Q_p^n)^2} \) | \( \frac{nE_2}{256} + \frac{n^2Q_p^n}{6(1-Q_p^n)^2} \) |

Table 14: Expansion coefficients \( w_{a, (0,0)}^{(r=1), m} \).

While the structure of the \( H_{(2s,0)}^{(r=2), m} \) is in general more complicated, we have managed to identify particular patterns in some of them, which allow us to conjecture the following expressions

\[
H_{(0,0)}^{(r=0,0,0,0)} = \sum_{a=0}^{8} v_{a, (0,0)}^{(r=2), (0,0,0,0)}(\rho) \left( \phi_{-2,1}(\rho, S) \right)^{a} \left( \phi_{0,1}(\rho, S) \right)^{8-a},
\]

\[
H_{(0,0)}^{(r=0,n,0,0)} = \frac{1}{1-Q_p^n} \sum_{k=1}^{3} n^{2k-1} v_{1,k,0}^{(2), (n,0,0,0)}(\rho, S) + \frac{Q_p^n}{(1-Q_p^n)^2} \sum_{k=1}^{4} n^{2k-2} v_{2,k,0}^{(2), (n,0,0,0)}(\rho, S)
\]  

\[+ \frac{Q_p^n(1+Q_p^n)}{(1-Q_p^n)^3} \sum_{k=1}^{3} n^{2k+1} v_{3,k,0}^{(2), (n,0,0,0)}(\rho, S),
\]

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\[ H_{(0,0)}^{(r),(n,n,n,0)} = \frac{1}{1 - Q_{\rho}^n} \sum_{k=1}^{3} n^{2k-1} v_{1,k,(0)}^{(2),(n,0,0,0)}(\rho, S) + \frac{1}{(1 - Q_{\rho}^n)^2} \sum_{k=1}^{4} n^{2k-2} v_{2,k,(0)}^{(2),(n,0,0,0)}(\rho, S) \\
+ \frac{1 + Q_{\rho}^{n^2}}{(1 - Q_{\rho}^n)^3} \sum_{k=1}^{3} n^{2k+1} v_{3,k,(0)}^{(2),(n,0,0,0)}(\rho, S), \tag{6.5} \]

where

\[ v_{a,k,(0)}^{(2),(n,0,0,0)} = \sum_{i=2}^{8} v_{i,a,k,(0)}^{(r=2),(n,0,0,0)}(\rho) (\phi_{-2,1}(\rho, S))^{i} (\phi_{0,1}(\rho, S))^{8-i}, \tag{6.6} \]

where the coefficients \( v_{a,(0,0)}^{(r=2),(0,0,0,0)} \) are tabulated in Table 15 and \( v_{a,k,(0,0)}^{(r=2),(0,0,0,0)} \) in Table 16.

| \( a \) | \( v_{a,(0,0)}^{(r=2),(0,0,0,0)} \) |
|---|---|
| 0 | 0 |
| 1 | \(-\frac{1}{8E_{\frac{2}{3}}}\) |
| 2 | \(-\frac{E_{\frac{2}{3}}}{8E_{\frac{2}{3}}}\) |

| \( a \) | \( v_{a,(0,0)}^{(r=2),(0,0,0,0)} \) |
|---|---|
| 3 | \(-\frac{4E_{\frac{2}{3}}}{E_{4}} - E_{\frac{4}{3}}\) |
| 4 | \(\frac{3E_{a} - 2E_{\frac{3}{2}} - 6E_{2E_{4}}}{3E_{\frac{2}{3}}}\) |
| 5 | \(\frac{3E_{a} - 2E_{\frac{3}{2}} - 6E_{2E_{4}} - 9E_{2}}{2E_{\frac{2}{3}}}\) |

We have found evidence that other configurations in (6.2) afford similar expansions. However, due to the increased complexity, it is difficult to make conjectures based on the limited expansion of the free energy\(^{25}\).

### 6.2 Hecke Structures

Since the factorisation of the free energy for \( N = 4 \) at order \( Q_{R} \) and \( s = 0 \) in the fundamental building blocks \( H_{N=1}^{(1)} \) and \( W_{NS}^{(1)} \) was already commented on in \( [10] \), we directly turn to the extraction of contributions, that are related through Hecke transformations.

#### 6.2.1 Contour Prescription

Similar to eq. (4.27) and (4.28) for \( N = 2 \) and eq. (5.10), (5.11) and (5.12) for \( N = 3 \), we define the following three subsectors of the \( N = 4 \) free energy

\[ C_{i,(2s,0)}^{N=4,(r)}(\rho, S) = \frac{1}{(2\pi i)^{4r+1}} \sum_{n=0}^{\infty} Q_{\rho}^{n} \int_{\gamma} d\hat{a}_{1} \int_{\gamma} d\hat{a}_{2} (\hat{a}_{1} + \hat{a}_{2}) \ldots \int_{\gamma} d\hat{a}_{r-1} (\hat{a}_{1} + \ldots + \hat{a}_{r-1}) \]

\(^{25}\)At order \( Q_{R}^2 \) and for \( s = 0 \), we managed to compute coefficients up to \( O(Q_{R}^{14}) \).
| a | k | $v^{(2),(n,0,0,0)}$ | $u^{(2),(n,0,0,0)}$ | $v^{(2),(n,0,0,0)}$ | $u^{(2),(n,0,0,0)}$ | $v^{(2),(n,0,0,0)}$ | $u^{(2),(n,0,0,0)}$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | $-\frac{1}{4 \cdot 24^4}$ | $-E_2 \cdot 12 \cdot 24^4$ | $-E_2^2 - E_4$ | $\frac{4E_6 - 7E_2E_4}{3 \cdot 24^4}$ | $\frac{56E_2E_6 - 3E_4 \left( 16E_2^2 + 7E_4 \right)}{6 \cdot 24^4}$ |  |
| 2 | 0 | $-\frac{1}{12 \cdot 24^4}$ | $-E_2 \cdot 36 \cdot 24^4$ | $-2E_2^2 - E_4$ | $\frac{3E_6 - 13E_2E_4}{90 \cdot 24^4}$ |  |
| 3 | 0 | $-\frac{1}{3 \cdot 24^4}$ | $-E_2 \cdot 9 \cdot 24^4$ | $-2E_2^2 - E_4$ | $\frac{3E_6 - 13E_2E_4}{90 \cdot 24^4}$ |  |
| 2 | 1 | $-\frac{1}{2 \cdot 24^4}$ | $-E_2 \cdot 12 \cdot 24^4$ | $-E_2$ | $\frac{7E_4}{6 \cdot 24^4}$ | $\frac{27E_6 - 52E_2E_4}{15 \cdot 24^4}$ |  |
| 2 | 0 | $-\frac{7}{24^4}$ | $-7E_2$ | $\frac{7E_4}{6 \cdot 24^4}$ | $\frac{17E_6}{180 \cdot 24^4}$ |  |
| 3 | 0 | $-\frac{5}{12 \cdot 24^4}$ | $-\frac{5}{12 \cdot 24^4}$ | $-5E_2$ | $\frac{3 \cdot 24^4}{3 \cdot 24^4}$ |  |
| 3 | 0 | $-\frac{1}{15 \cdot 24^4}$ | $-\frac{1}{15 \cdot 24^4}$ | $-\frac{1}{15 \cdot 24^4}$ | $-\frac{1}{15 \cdot 24^4}$ |  |
| 3 | 1 | 0 | $0$ | $0$ | $0$ | $-\frac{E_4}{420}$ |  |
| 2 | 0 | 0 | $0$ | $0$ | $0$ | $-\frac{1}{13824}$ |  |
| 3 | 0 | 0 | $0$ | $0$ | $0$ | $-\frac{7}{34560}$ |  |
| 4 | 0 | 0 | $0$ | $0$ | $0$ | $0$ |  |
| 5 | 0 | 0 | $0$ | $0$ | $0$ | $0$ |  |

Table 16: Expansion coefficients $v^{(2),(n,0,0,0)}$.

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In the following, we shall exclusively focus on \( C_{1,(2s,0)}^{N=4,(r)} \) and \( C_{2,(2s,0)}^{N=4,(r)} \), for which the functions presented in (6.5) are relevant:

\[
\times \oint_0 dQ_{\hat{a}_1} \ldots \oint_0 dQ_{\hat{a}_4} P_{4,(2s,0)}^{(r)}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, S), \quad \forall i = 1, 2, 3, 4. \quad (6.7)
\]

Figure 10: Brane web configurations made up from \( N = 4 \) M5-branes (drawn in orange) spaced out on a circle, with various M2-branes (drawn in red and blue) stretched between them. (a) An equal number \( \ell \) of M2-branes is stretched between any two neighbouring M5-branes. Configurations of this type are relevant for the computation of \( C_{1,(2s,0)}^{N=4,(r)} \). (b) \( \ell \) M2-branes are stretched between M5-branes 2 and 3, 3 and 4 as well as 4 and 1, while \( n \neq \ell \) M2-branes between M5-branes 1 and 2. Configurations of this type are relevant for the computation of \( C_{2,(2s,0)}^{N=4,(r)} \).

- Combination \( C_{1,(2s,0)}^{N=4,(r)} \)
  As before, \( C_{1,(2s,0)}^{N=4,(r)} \) can be described by extracting a particular class of terms in the Fourier expansion of \( P_{4,(2s,0)}^{(r)} \) in powers of \( Q_{\hat{a}_1,2,3,4} \). Indeed, upon writing

\[
P_{4,(2s,0)}^{(r)}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, S) = \sum_{n_1,n_2,n_3,n_4=0}^{\infty} Q_{\hat{a}_1,2}^{n_1} Q_{\hat{a}_2,3}^{n_2} Q_{\hat{a}_3,4}^{n_3} Q_{\hat{a}_4}(6.8)
\]

the contour prescriptions for \( C_{1,(2s,0)}^{N=4,(r)} \) in (6.7) are designed to extract only those terms with \( n_1 = n_2 = n_3 = n_4 \). Therefore, \( C_{1,(2s,0)}^{N=3,(r)} \) receives contributions only from those brane configurations, in which an equal number of M2-branes is stretched between any two adjacent M5-branes, as is visualised in Fig. 10 (a). Following the definition of \( H_{(2s,0)}^{(r);a} \) in (2.21), we find that \( C_{1,(2s,0)}^{N=3,(r)} \) can equivalently be written as

\[
C_{1,(2s,0)}^{N=4,(r)}(\rho, S) = H_{(2s,0)}^{(r);\{0,0,0,0\}}(\rho, S), \quad C_{1,(2s,0)}^{N=4,(r)}(\rho, S, \epsilon_1) = \sum_{s=0}^{\infty} \epsilon_1^{2s-1} H_{(2s,0)}^{(r);\{0,0,0,0\}}(\rho, S). \quad (6.9)
\]
This is in fact the reduced free energy for \( N = 4 \) that was studied in [45]. Explicit expansions of \( C_1^{N=4,(r)} \) for \( r = 1 \) and \( r = 2 \) can be recovered from Table 14 and Table 15.

- Combination \( C_2^{N=4,(r)} \):

The function \( C_2^{N=4,(r)} \) in (6.7) extracts specific coefficients in a mixed Fourier- and Laurent series expansion of the free energy. Starting from the Fourier expansion (6.8) \( C_2^{N=4,(r)} \) receives contributions only from coefficients with \( n_1 \neq n_2 = n_3 = n_4 \). From the brane web picture, these correspond to configurations where an equal number \( \ell \) of M2-branes is stretched between the M5 branes 2 and 3, 3 and 4 as well as 4 and 1, while a different number \( n \neq \ell \) of M2-branes is stretched between the first and second M5-branes. Such configurations are schematically shown in Fig. 10 (b). Finally, the last contour integral in (5.11) over \( \hat{a}_1 \) extracts the second order pole in the Laurent expansion.

With respect to the decomposition (6.3), the coefficients \( C_2^{N=3,(r)} \) can be written in the following form

\[
C_2^{N=4,(r)}(\rho, S) = \frac{1}{2\pi i} \oint_0 d\hat{a}_1 \hat{a}_1 \sum_{n=1}^{\infty} \left[ H^{(r),\{n,0,0,0\}}(\rho, S) Q_\rho^n Q_{\hat{a}_1}^n + H^{(r),\{n,n,n,0\}}(\rho, S) Q_\rho^n Q_{\hat{a}_1}^{-n} \right].
\]

In order to perform the final contour integration over \( \hat{a}_1 \), we can use the conjectured form (6.5) of \( H^{(r),\{n,0,0,0\}} \) and \( H^{(r),\{n,n,n,0\}} \) for \( s = 0 \) and \( r = 1 \) and \( r = 2 \) to write for the integrand

\[
\mathcal{I}_c^{N=4,(r)} = \sum_{n=1}^{\infty} \left[ H^{(r),\{n,0,0,0\}}(\rho, S) Q_\rho^n Q_{\hat{a}_1}^n + H^{(r),\{n,n,n,0\}}(\rho, S) Q_\rho^n Q_{\hat{a}_1}^{-n} \right]
\]

\[
= \sum_{n=1}^{\infty} \sum_{k=1}^{3} \frac{n^{2k-1} v_{1,k,(0)}^{(r),(n,0,0,0)}}{1 - Q_\rho^n} \left( Q_\rho^n Q_{\hat{a}_1}^n + Q_\rho^n Q_{\hat{a}_1}^{-n} \right) + \sum_{n=1}^{\infty} \sum_{k=1}^{4} \frac{n^{2k-2} Q_\rho^n v_{2,k,0}^{(r),(n,0,0,0)}}{(1 - Q_\rho^n)^2} \left( Q_\rho^n Q_{\hat{a}_1}^n + Q_\rho^n Q_{\hat{a}_1}^{-n} \right)
\]

\[
= \sum_{k=1}^{3} v_{2,k,0}^{(r),(n,0,0,0)} \mathcal{I}_{k-1}(\rho, \hat{a}_1) + \sum_{k=1}^{4} v_{2,k,0}^{(r),(n,0,0,0)} \sum_{n=1}^{\infty} \frac{n^{2k-2} Q_\rho^n}{(1 - Q_\rho^n)^2} \left( Q_\rho^n Q_{\hat{a}_1}^n + Q_\rho^n Q_{\hat{a}_1}^{-n} \right)
\]

\[
+ \sum_{k=1}^{3} v_{3,k,0}^{(r),(n,0,0,0)} \sum_{n=1}^{\infty} \frac{n^{2k+1} Q_\rho^n (1 + Q_\rho^n)}{(1 - Q_\rho^n)^3} \left( Q_\rho^n Q_{\hat{a}_1}^n + Q_\rho^n Q_{\hat{a}_1}^{-n} \right),
\]

where we have exchanged the summation over \( k \) and \( n \) and \( \mathcal{I}_c \) is defined in (4.15). Using
the geometric series
\[
\frac{x}{(1-x)^2} = \sum_{\ell=1}^{\infty} \ell x^\ell, \quad \text{and} \quad \frac{x(1+x)}{(1-x)^3} = \sum_{\ell=1}^{\infty} \ell^2 x^\ell, \quad \text{for} \quad |x| < 1, \quad (6.12)
\]
we can write for the sum over \( n \) in the last two terms in (6.11)
\[
\sum_{n=1}^{\infty} \frac{n^{2k-2} Q_\rho^n}{(1 - Q_\rho^n)^2} (Q_{a_1}^{\ell} + Q_{a_1}^{-\ell}) = \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} n^{2k-2} Q_\rho^{\ell} (Q_{a_1}^{n} + Q_{a_1}^{-n}) \\
= D_{a_1}^{2k-2} \sum_{n=1}^{\infty} Q_\rho^n \sum_{\ell|n} n \ell (Q_{a_1}^{\ell} + Q_{a_1}^{-\ell}) = D_{a_1}^{2k-2} J_0^{(1)}(\rho, \hat{a}_1), \quad (6.13)
\]
\[
\sum_{n=1}^{\infty} \frac{n^{2k+1} Q_\rho^n (1 + Q_\rho^n)}{(1 - Q_\rho^n)^3} (Q_{a_1}^{n} + Q_{a_1}^{-n}) = \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} n^{2k+1} \ell^{2} Q_\rho^{\ell} (Q_{a_1}^{n} + Q_{a_1}^{-n}) \\
= D_{a_1}^{2k+1} \sum_{n=1}^{\infty} Q_\rho^n \sum_{\ell|n} (\frac{n}{\ell})^2 (Q_{a_1}^{\ell} + Q_{a_1}^{-\ell}) = D_{a_1}^{2k+1} J_0^{(2)}(\rho, \hat{a}_1). \quad (6.14)
\]
For \( 0 < |Q_\rho| < 1 \) the functions \( J_0^{(1)} \) and \( J_0^{(2)} \) are fact regular at \( \hat{a}_1 = 0 \), such that \( \oint_0 d\hat{a}_1 \hat{a}_1 D_{a_1}^{k} J_0^{(1,2)}(\rho, \hat{a}_1) = 0 \) for \( k \geq 0 \). Using furthermore (4.18), we get
\[
C_2^{N=4,(r)}(\rho, S) = \frac{1}{r} \mathcal{P}_{1,1,(0)}, \quad C_2^{N=4,(r)}(\rho, S, \epsilon_1) = \frac{1}{r} \sum_{s=0}^{\infty} \epsilon_1^{2s} \mathcal{P}_{1,1,(0)}^{(r)}(\rho, S), \quad (6.15)
\]
While we leave the study of the other functions to future work, we remark in passing that the \( \hat{a}_i \) of which we extract the pole \( Q_{a_i}^{-2} \) in (6.7) are consecutive. An interesting question is if it makes sense to define more general functions, as for example
\[
\widetilde{C}_{3,(2s=0)}^{N=4,(r)}(\rho, S) = \frac{1}{r^{s-1}} \sum_{\ell=0}^{\infty} Q_{a_1}^{\ell} \oint_0 \int \hat{a}_1 \hat{a}_3 (\hat{a}_1 + \hat{a}_3) \oint_{-\hat{a_1}}^{\hat{a_1}} \oint_{-\hat{a_3}}^{\hat{a_3}} \oint_{-\hat{a_1} + \hat{a_3}}^{\hat{a_1} + \hat{a_3}} dQ_{a_1}^{\ell} dQ_{a_3}^{\ell} + r^{(\rho)}_{4,(2s=0)}(\hat{a}_1,...,4, S), \quad (6.16)
\]
Both \( C_3^{N=4,(r)} \) and \( \widetilde{C}_{3,(2s=0)}^{N=4,(r)} \) receive contributions from slightly different M2-brane configurations (they are schematically shown in Fig. 11). With respect to the list in (6.2), the precise configurations are respectively
\[
C_3^{N=4,(r)}(\rho, S): \{n_1 + n_2, n_1, 0, 0\}, \quad \{n_1 + n_2, n_1, n_1, 0\}, \quad \{n_1 + n_2, n_1 + n_2, n_1, 0\}, \quad \{n, n, 0, 0\},
\]
\[
\widetilde{C}_3^{N=4,(r)}(\rho, S): \{n_1 + n_2, 0, n_1, 0\}, \quad \{n_1 + n_2, n_1, 0, n_1\}, \quad \{n_1 + n_2, n_1, n_1 + n_2, 0\}, \quad \{n, 0, n, 0\}.
\]
From [51], one can see that the corresponding \( H^{(r=1,2)}_{(0,0)} \) in the case of \( \widetilde{C}_3^{N=4,(r)} \) all involve poly-

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Figure 11: Brane web configurations made up from $N = 4$ M5-branes (drawn in orange) spaced out on a circle, with various M2-branes (drawn in red and blue) stretched between them. (a) An equal number $\ell$ of M2-branes is stretched between the neighbouring M5-branes 3 and 4 as well as 4 and 1, while different numbers of M2-branes $n_1 \neq n_2 \neq \ell$ are stretched between the M5-branes 1 and 2 as well as 2 and 3. Configurations of this type are relevant for the computation of $C_{3}^{N=4,(r)}$. (b) An equal number $\ell$ of M2-branes is stretched between the neighbouring M5-branes 2 and 3 as well as 4 and 1, while different numbers of M2-branes $n_1 \neq n_3 \neq \ell$ are stretched between the M5-branes 1 and 2 as well as 3 and 4. Configurations of this type are relevant for the computation of $\tilde{C}_{3,\{2s,0\}}^{N=4,(r)}$.

nomials of the $n$ or order 3 or higher (while there are contributions with polynomials of order 2 in the case of $C_{3}^{N=4,(r)}$). Following the experience of the previous sections, this suggests that (at least to leading order in $Q_{R}$) $\tilde{C}_{3,\{2s,0\}}^{N=4,(r)}$ in fact may be vanishing.

### 6.2.2 Factorisation and Hecke Relations

In the cases $N = 2$ and $N = 3$ we have observed that the functions $C_{a}^{N=2,(r=1)}$ and $C_{a}^{N=3,(r=1)}$ can be factorised as in eq. (4.28) and eq. (5.29) respectively. Analysing $C_{1,\{2s,0\}}^{N=4,(r=1)}$ for $s > 0$ is very complicated, however, based on the expansions presented above, we find

$$
C_{1,\{0,0\}}^{N=4,(r=1)}(\rho, S) = 4H_{\{0,0\}}^{(1)}(\rho, S) \left( W_{\{0,0\}}(\rho, S) \right)^3,
$$

$$
C_{2,\{0,0\}}^{N=4,(r=1)}(\rho, S) = \left( H_{\{0,0\}}^{(1)}(\rho, S) \right)^2 \left( W_{\{0,0\}}(\rho, S) \right)^2,
$$

which are indeed in agreement with the general conjectured form (3.9). Moreover, by comparing the explicit expressions for the contributions $C_{1,\{0,0\}}^{N=4,(r)}$ and $C_{2,\{0,0\}}^{N=4,(r)}$ for $r = 1$ and $r = 2$ to the free energy, we find that they satisfy the following recursion relation

$$
C_{1,\{0,0\}}^{N=4,(r=2)}(\rho, S) = \mathcal{H}_2 \left[ C_{1,\{0,0\}}^{N=4,(1)}(\rho, S) \right], \quad C_{2,\{0,0\}}^{N=4,(r=2)}(\rho, S) = \mathcal{H}_2 \left[ C_{2,\{0,0\}}^{N=4,(1)}(\rho, S) \right].
$$

(6.18)
which generalises the relations (4.26) and (5.28) to \( N = 4 \). In view of the results of the previous sections, we conjecture that this result in fact generalises not only for \( r > 2 \) and \( s > 0 \), but also to all functions \( C_i^{N=4, r} \) for \( i = 1, \ldots, 4 \).

7 Conclusions and Interpretation

Although the observations of the previous sections were only for the specific cases \( N = 2 \) and \( N = 3 \) (as well as partially for \( N = 4 \)) and for limited values of the order of \( Q_R \) (indicated by the index \( r \)) as well as \( \epsilon_1 \) (indicated by the index \( 2s \)), the fact that they exhibit a rather clear cut pattern leads us to believe that they hold in general (i.e. for generic \( N \) and generic values of \( r \) and \( s \)). To be concrete, we therefore conjecture that for given \( N \), to any instanton order \( 26r \) we can extract at every order \( \epsilon_2^{s-2} \) (for \( s \in \mathbb{N} \) \( N \) different functions \( C_i^{N, r} (\rho, S) \) (see eq. (3.1) for the definitions) for \( i = 1, \ldots, N \) from the NS-limit of the free energy \( P_N^r (\hat{a}_1, \ldots, N) \) that count very specific BPS states from the perspective of the M-brane webs. Indeed, focusing on configurations where the same number of M2-branes is stretched between \( N - i \) neighbouring M5-branes, they extract a particular polar part of the free energy when the remaining M5-branes are collapsed on top of each other. Viewed order by order in \( Q_R \), the formal series \( C_i^{N, r} (\rho, S, \epsilon_1) \) for different values of \( r \) are related through Hecke transformations (see eq. (3.6)). This generalises the observation made in [45], which in our language is the specific case \( C_i^{N, r} (\rho, S, \epsilon_1) \). Furthermore, following the logic put forward in [45, 42], the Hecke relation (3.6) suggests that the BPS-states counted by \( C_i^{N, r} (\rho, S, \epsilon_1) \) can be arranged into the form of a symmetric torus orbifold CFTs and we can define the corresponding CFT partition functions,

\[
Z_i^{(N)} (R, \rho, S, \epsilon_1) = \exp \left( \sum_{r \geq 1} Q_R^r C_i^{N, r} (\rho, S, \epsilon_1) \right). \tag{7.1}
\]

The relation (3.6) \( C_i^{N, r} (\rho, S, \epsilon_1) = H_r \left( C_i^{N, (1)} (\rho, S, \epsilon_1) \right) \) then implies [70],

\[
Z_i^{(N)} (R, \rho, S, \epsilon_1) = \exp \left( \sum_{r \geq 1} Q_R^r H_r \left( C_i^{N, (1)} (\rho, S, \epsilon_1) \right) \right) \tag{7.2}
\]

\[
= \sum_{r \geq 1} Q_R^r \chi_{\text{ell}} (\text{Sym}^r (\mathcal{M}_i)). \tag{7.3}
\]

Here \( Q_R \) keeps track of the symmetric products and we conjecture the existence of spaces \( \mathcal{M}_i \) with equivariant elliptic genus \( C_i^{N, (1)} (\rho, S, \epsilon_1) \). We only defined the terms with instanton

\footnote{Here we are taking the point of view of the \( U(N) \) gauge theory that is engineered from the Calabi-Yau threefold \( X_{N,1} \),}
in the language of the dual supersymmetric gauge theory, $O(Q_R^r)$ with $r > 0$, i.e. we have not included the terms coming from the little string partition function with $Q_R = 0$ which correspond to perturbative corrections in the dual gauge theory. Furthermore, to make a paramodular symmetry of the CFT partition function more manifest \cite{42}, we have used $Q_R^N$ as the generating function parameter rather than $Q_R$ in (7.3): indeed, the Hecke structure of eq.(4.26) implies that $Z_i^{(N)}(R, \rho, S, \epsilon_1)$ is the partition function of a symmetric orbifold conformal field theory on the torus that is invariant under the paramodular group $\Sigma_N \subset Sp(4, \mathbb{R})$ (see appendix D for the definition). To make invariance under $\Sigma_N$ more manifest, we remark that the Hecke structure of eq.(4.26) in $Z_i^{(N)}$ can be expressed in a product form \cite{70},

$$Z_i^{(N)} = \prod_{r, k, \ell} \left( 1 - Q_R^N Q^k Q^\ell q^p \right)^{-c_i(kr, \ell, p)} ,$$

(7.4)

where $c_i(k, \ell, p)$ are the Fourier coefficients of the 'seed function' $C_{i}^{N,(1)}(\rho, S, \epsilon_1)$,

$$C_{i}^{N,(1)}(\rho, S, \epsilon_1) = \sum_{k, \ell, p} c_i(k, \ell, p) Q^k Q^\ell q^p .$$

(7.5)

Thus, the partition function $Z_i^{(N)}(R, \rho, S, \epsilon_1)$ is an exponential lift of the Jacobi form $C_{i}^{N,(1)}(\rho, S, \epsilon_1)$ and related \cite{71} to a paramodular form of the group $\Sigma_N^*$ satisfying the property \cite{71},

$$Z_i^{(N)}(R, \rho, S, \epsilon_1) = Z_i^{(N)}(\frac{N}{N}, N R, S, \epsilon_1) .$$

(7.6)

Finally, we remark that $\Sigma_N^*$ acts on $\mathbb{H}_2$ the space of $2 \times 2$ matrices with positive imaginary part as in \cite{D.3}. The quotient $\Sigma_N^* \backslash \mathbb{H}_2$ is the moduli space of abelian surfaces with polarization $(1, N) \cite{72, 73}$. These abelian surfaces are precisely the ones appearing in the F-theory forming the fibers of the double elliptically fibered Calabi-Yau threefolds \cite{27}. It would be very interesting to have a clearer geometric interpretation of this result, for example understanding the target space of this CFT. We leave this question for future work.

The functions $C_{i}^{N,(r=1)}(\rho, S, \epsilon_1)$ at leading instanton order $O(Q_R)$ exhibit a factorisation into simpler building blocks which go beyond the known self-similarity and recursive structure (see section 2 for a review of both) of the free energy and extend preliminary results in \cite{46}: indeed $C_{i}^{N,(r=1)}(\rho, S, \epsilon_1)$ can be written as the product (3.9) where the building blocks $H_{N=1}^{(1)}(\rho, S, \epsilon_1)$ and $W_{NS}^{(1)}(\rho, S, \epsilon_1)$ stem either from the expansion of the free energy for $N = 1$ or govern the BPS-counting of a single M5-brane with single M2-branes attached to it on either side (for a review see appendix C). To higher order in $Q_R$, remnants of such a factorisation persist, but new elements appear as well (see eq. (3.11)). It is difficult to conjecture a closed form expression

\footnote{If the terms with $Q_R = 0$ are included in the definition (7.3) of the reduced partition function then $Z_i^{(N=2)}$ is precisely the paramodular form for $\Sigma_N^*$.}
of the latter, however, we have succeeded to show for \( N = 2 \) and \( N = 3 \) that they are governed by differential equations that are very similar to holomorphic anomaly equations.

The \( C_{i}^{N,(r=1)}(\rho, S, \epsilon_{1}) \) discussed in this work are specific contributions to the BPS-free energy of LSTs of type A. It would be very interesting to understand the geometric reason that makes these states special compared to others, such that they can be interpreted as part of the spectrum of a symmetric torus orbifold. This could give us the key to understanding if there are further sectors in the spectrum of the LSTs of A-type which exhibit similar properties. Furthermore, this may also give us a hint whether or not these various orbifold CFTs can be in any way connected via a duality transformation.

Another interesting observation is the fact that the \( C_{i}^{N,(r=1)}(\rho, S, \epsilon_{1}) \) (except for \( i = 1 \)) are obtained through contour integrals from the free energy \( P_{N}^{(r)}(\hat{a}_{1,...,N}) \) that select the coefficient of (a) pole(s) in \( \hat{a}_{1,...,N-1} \). In [74] the BPS counting of supersymmetric black holes has been discussed. It has been pointed out that the phenomenon of wall-crossing can be attributed to the polar part of a meromorphic Jacobi form that counts multi-centered black holes whose number can jump when crossing a wall. It would be interesting to analyse if a similar phenomenon takes place for the BPS counting functions discussed in this paper when we cross the loci \( \hat{a}_{i} = 0 \).

In the dual \( U(1)^{N} \) gauge theory \( \hat{a}_{i} \) are inverse coupling constants for each of the \( U(1) \) factors and crossing the \( \hat{a}_{i} = 0 \) locus corresponds to passing through infinite coupling region [75, 76]. It would be interesting to understand what happens in this case to the BPS-states that are counted by \( C_{i}^{N,(r=1)}(\rho, S, \epsilon_{1}) \). We leave this question for future work.

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A Modular Forms

Throughout this work we are using various different modular objects. This appendix compiles the definitions of all objects that are used in the main body of this article, as well as additional useful information and identities. For a more comprehensive review, we relegate the reader to the literature, e.g. [77–79].

A weak Jacobi form of the modular group \( \Gamma \cong SL(2, \mathbb{Z}) \) of index \( m \in \mathbb{Z} \) and weight \( w \in \mathbb{Z} \)
is a holomorphic function of the type

$$\phi: \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$$

$$(\rho, z) \mapsto \phi(\rho; z), \quad (A.1)$$

(where $\mathbb{H}$ is the upper complex plane) which behaves in the following manner under transformations of $\Gamma$

$$\phi \left( \frac{ap + b}{cp + d}, \frac{z}{cp + d} \right) = (c\rho + d)^w e^{\frac{2\pi imc^2}{c\rho + d}} \phi(\rho; z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

$$\phi(\rho; z + \ell_1 \rho + \ell_2) = e^{-2\pi i m (\ell_1^2 \rho + 2\ell_1 z)} \phi(\rho; z), \quad \forall \ell_1, \ell_2 \in \mathbb{N}, \quad (A.2)$$

Such functions allow a Fourier expansion of the form

$$\phi(z, \rho) = \sum_{n=0}^{\infty} \sum_{\ell \in \mathbb{Z}} c(n, \ell) Q^n_{\rho} e^{2\pi i \ell}, \quad \text{with} \quad Q_{\rho} = e^{2\pi i \rho}. \quad (A.3)$$

The Jacobi forms encountered throughout this work can be decomposed in terms of two basis functions, i.e. for index $m$ and weight $w$, we can write

$$\phi(\rho; z) = \sum_{a=0}^{m} f_a(\rho) (\phi_{0,1}(\rho, z))^a (\phi_{-2,1}(\rho, z))^{m-a}. \quad (A.4)$$

Here $\phi_{-2,1}$ and $\phi_{0,1}$ are Jacobi forms of index 1 and weight $-2$ and 0 respectively, which are defined as

$$\phi_{0,1}(\rho, z) = 8 \sum_{a=2}^{4} \frac{\theta^2_a(z; \rho)}{\theta^2_a(0, \rho)}, \quad \text{and} \quad \phi_{-2,1}(\rho, z) = \frac{\theta^2(z; \rho)}{\eta(\rho)}, \quad (A.5)$$

with $\theta_{a=1,2,3,4}(z; \rho)$ the Jacobi theta functions and $\eta(\rho)$ the Dedekind eta function. Furthermore, the $f_a(\rho)$ in (A.4) are modular forms of weight $w + 2a$. In practice, the $f_a(\rho)$ can be written as homogeneous polynomials in the Eisenstein series $E_{2n}$, which are modular forms of weight $2n$ and which are defined as

$$E_{2k}(\rho) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) Q^n_{\rho}, \quad \forall k \in \mathbb{N}, \quad (A.6)$$

where $B_{2k}$ are the Bernoulli numbers, while $\sigma_k(n)$ is the divisor function. We shall sometimes $\phi_{0,1}(\rho, z)$ defined below differs by a factor of 2 from its usual definition in the literature. As defined it is equal to the elliptic genus of $K3$.  

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also use the differently normalised functions
\[
G_{2k}(\rho) = 2\zeta(2k) + 2 \frac{(2\pi i)^{2k}}{(2k - 1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) Q_n^\rho = 2\zeta(2k)E_{2k}(\rho) .
\]  
(A.7)

The holomorphic Eisenstein series (i.e. the \(E_{2n}\) for \(n > 1\)) form a ring, which is generated by \(\{E_4, E_6\}\). Furthermore, most of the examples we encounter in this paper are in fact quasi-Jacobi forms, in the sense that the \(f_n(\rho)\) in their decomposition (A.4) also depend on the Eisenstein series \(E_2\): the latter is strictly speaking not a modular form. However, one can define the following non-holomorphic object
\[
\hat{E}_2(\rho, \bar{\rho}) = E_2(\rho) - \frac{6i}{\pi(\rho - \bar{\rho})} ,
\]  
(A.8)

which transforms with weight 2 under modular transformations.

Another object we will encounter in the main body of this paper is the Weierstrass elliptic function
\[
\wp(z; \rho) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k + 1)G_{2k+2}(\rho) z^{2k} .
\]  
(A.9)

which has a pole of order 2 in \(z\).

Finally, many of the results found in this paper use Hecke operators: these are maps from the space \(J_{w,m}(\Gamma)\) of Jacobi forms of index \(m\) and weight \(w\) into the space \(J_{w,km}(\Gamma)\) of Jacobi forms of index \(km\) and weight \(w\) for \(k \in \mathbb{N}\):
\[
\mathcal{H}_k : J_{w,m}(\Gamma) \longrightarrow J_{w,km}(\Gamma)
\]
\[
\phi(\rho; z) \longmapsto \mathcal{H}_k(\phi(\rho; z)) = k^{w-1} \sum_{d|k \mod d} d^{-w} \phi \left( \frac{k\rho + bd}{d^2}; \frac{kz}{d} \right) .
\]  
(A.10)

Hecke transformations of this type can also be extended to Jacobi forms that depend on more than one variable: let \(f_{w,\vec{m}}(\rho, \vec{z}) : \mathbb{H} \times \mathbb{C}^n \rightarrow \mathbb{C}\) be a Jacobi form with index-vector \(\vec{m}\). We then define
\[
\mathcal{H}_k : f_{w,\vec{m}}(\rho, \vec{z}) \longmapsto \mathcal{H}_k(f_{w,\vec{m}}(\rho, \vec{z})) = k^{w-1} \sum_{d|k \mod d} d^{-w} f_{w,\vec{m}} \left( \frac{k\rho + bd}{d^2}; \frac{k\vec{z}}{d} \right) .
\]  
(A.11)

For further use, we consider the case that \(f\) allows for a Laurent series expansion in one of the variables: let \((z_1, \dots, z_n) = (\vec{z}, z_n)\) (where \(\vec{z} \in \mathbb{C}^{n-1}\)) and let \((m_1, \dots, m_n) = (\vec{m}, m_n)\) be the
index vector of a Jacobi form that affords the following (convergent) Laurent series

\[ f_{w,(m_1,\ldots,m_n)}(\rho; \bar{z}, z_n) = \sum_{a} z_n^a f_{w+a,\vec{m}}(\rho, \bar{z}), \]  

(A.12)

where \( f_{w+a,\vec{m}}(\rho, \bar{z}) \) are Jacobi forms of weight \( w + a \) and index vector \( \vec{m} \). We then find for the action of the Hecke operator

\[ \mathcal{H}_k \left( f_{w,(m_1,\ldots,m_n)}(\rho; \bar{z}, z_n) \right) = k^{w-1} \sum \frac{d^w}{d \rho^w} f_{w,(m_1,\ldots,m_n)} \left( \frac{k\rho + bd}{d^2} ; \frac{k\bar{z}}{d} , \frac{kw}{d} \right) \]

\[ = \sum_{a} z_n^a k^{w+a-1} \sum \left. \frac{d^w}{d \rho^w} \right|_{d \equiv k \mod d} f_{w+a,\vec{m}} \left( \frac{k\rho + bd}{d^2} ; \frac{k\bar{z}}{d} \right) = \sum_{a} z_n^a \mathcal{H}_k \left( f_{w+a,\vec{m}}(\rho, \bar{z}) \right). \]  

(A.13)

B  \( (N, 1) \) Partition Functions

The topological string partition function of the Calabi-Yau threefold \( X_{N, 1} \) is given by \[ [21, 26, 22], \]

\[ Z_{N, 1}(\tau, \hat{a}, m, \epsilon_{1,2}) = \sum_{\lambda_1,\ldots,\lambda_N} Q_{\tau}^{\lambda_1 + \cdots + \lambda_N} Z_{\lambda_1,\ldots,\lambda_N}(\hat{a}, m, \epsilon_{1,2}), \quad \text{with} \quad \hat{a} = \{\hat{a}_1, \ldots, \hat{a}_N\}, \]  

(B.1)

and where the sum is over \( N \)-tuples of partitions of non-negative integers. The parts of the partition \( \lambda_\alpha \) are denoted by \( \lambda_{\alpha,i} \) with \( \lambda_{\alpha,1} \geq \lambda_{\alpha,2} \geq \lambda_{\alpha,3} \geq \cdots \). Each partition \( \lambda_\alpha \) corresponds to a Young diagram which is obtained by putting \( \lambda_{\alpha,i} \) boxes in the \( i \)-th column such that a box in the Young diagram can be assigned a coordinate \((i, j)\) as long as \( 1 \leq i \leq \ell(\lambda_\alpha), 1 \leq j \leq \lambda_{\alpha,i} \).

The transpose of a partition \( \lambda_\alpha \) is denoted by \( \lambda_\alpha^t \) and is defined as the partition corresponding to the Young diagram obtained by interchanging rows and columns of the Young diagram corresponding to \( \lambda_\alpha \). If we denote by \( \ell(\lambda_\alpha) \) the total number of non-zero parts of the partition \( \lambda_\alpha \) then we define,

\[ |\lambda_\alpha| = \sum_{i=1}^{\ell(\lambda_\alpha)} \lambda_{\alpha,i}, \quad ||\lambda_\alpha||^2 = \sum_{i=1}^{\ell(\lambda_\alpha)} \lambda_{\alpha,i}^2. \]  

(B.2)

As discussed in the main body of the paper, the topological string partition function (B.1) also captures the partition function of a supersymmetric gauge theories. Furthermore, from a geometric point of view, the instanton part of \( Z_{N, 1} \) is the generating function of equivariant elliptic genera of the instanton moduli space \( M(N, k) \),

\[ Z_{N, 1} = Z_0 \sum_k Q_k \chi_{\text{ell}} \left( M(N, k) \right), \]  

(B.3)
where $\chi_{\text{ell}}(X)$ denotes the equivariant elliptic genus of any manifold $X$,

$$
\chi_{\text{ell}}(X) = \text{Tr}_{H(X)}(-1)^{F_L+F_R} y^{F_L} q^{H} e^{2\pi i \hat{a} \cdot h}.
$$

(B.4)

Here the trace is over the RR sector, $F_{L,R}$ are the left and the right moving fermion numbers and $h_i$ are the Cartan generators of the symmetry group $G$ which acts on $X$ (and $\hat{a} \cdot h = \sum_{i=1}^{N} \hat{a}_i h_i$). The path integral representation of the above reduces to an index calculation,

$$
\chi_{\text{ell}}(X) = \int_{X} \text{ch}(E_{Q,y}) \ Td(X) = \int_{X} \dim(X) \ \prod_{i=1}^{\dim(X)} 2\pi i \xi_i \frac{\vartheta(\tau, m + \xi_i)}{\vartheta(\tau, \xi_i)}
$$

(B.5)

where $(y = e^{2\pi im})$,

$$
E_{Q,y} = y^{-\frac{d}{2}} \otimes_{\ell \geq 1} \left[ \wedge_{-yQ_{\ell}^{-1}} T_{X} \otimes \wedge_{-y^{-1}Q_{\ell}} T_{X} \otimes S_{Q_{\ell}} T_{X} \otimes S_{Q_{\ell}} T_{X} \right]
$$

(B.6)

and $x_i$ are the formal roots of the Chern polynomial. The relation between $Z_{\lambda_1 \cdots \lambda_N}$ and $\chi_{\text{ell}}(M(N, k))$ is given by,

$$
\chi_{\text{ell}}(M(N, k)) = \sum_{|\lambda_1| + \cdots + |\lambda_N| = k} Z_{\lambda_1 \cdots \lambda_N} / Z_0.
$$

(B.7)

The function $Z_{\lambda_1 \cdots \lambda_N}(\hat{a}, m, \epsilon_{1,2})$ in (B.1) is defined as,

$$
Z_{\lambda_1 \cdots \lambda_N}(\hat{a}, m, \epsilon_{1,2}) = Z_0 \prod_{\alpha=1}^{N} \frac{\vartheta_{\lambda_\alpha \lambda_\alpha}(Q_m)}{\vartheta_{\lambda_\alpha \lambda_\alpha}(Q_{\sqrt{\ell}})} \prod_{1 \leq \alpha < \beta \leq N} \frac{\vartheta_{\lambda_\alpha \lambda_\beta}(Q_{Q_{\epsilon}} Q_m) \vartheta_{\lambda_\alpha \lambda_\beta}(Q_{Q_{\epsilon}} Q_{m}^{-1})}{\vartheta_{\lambda_\alpha \lambda_\beta}(Q_{Q_{\epsilon}} \sqrt{\ell}) \vartheta_{\lambda_\alpha \lambda_\beta}(Q_{Q_{\epsilon}} \sqrt{\ell}^{-1})},
$$

(B.8)

where $Q_{Q_{\epsilon}} = e^{2\pi i (\hat{a}_{\alpha} - \hat{a}_{\beta})}$ and

$$
\vartheta_{\lambda \mu}(\rho, z) = \prod_{(i,j) \in \lambda} \theta_1\left(\rho; z^{-1} t^{-\frac{1}{2}} q^{-\lambda_i + j - \frac{1}{2}} \right) \prod_{(i,j) \in \mu} \theta_1\left(\rho; z^{-1} t^{\lambda_i - i + \frac{1}{2}} q^{\mu_j - j + \frac{1}{2}} \right)
$$

(B.9)

with $\theta_1(\rho, z)$ the Jacobi theta function and $\rho = \sum_{\alpha=1}^{N} \hat{a}_{\alpha}$. The factor $Z_0$ in eq. (B.3) and (B.8) is independent of $Q_{\epsilon}$ and is given by,

$$
Z_0 = \prod_{n=1}^{\infty} (1 - Q_{\rho}^{-n})^{-1} \left[ \prod_{1 \leq \alpha < \beta \leq N} F_{\alpha \beta} \right] \left[ \prod_{\alpha, \beta = 1}^{N} H_{\alpha \beta} \right]
$$

(B.10)

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where \( \tilde{Q}_{\alpha\beta} = Q_1 Q_2 \cdots Q_a Q_i^{-1} \cdots Q_b Q_m^{-b} \)

\[
F_{\alpha\beta} = \prod_{i,j=1}^{\infty} \frac{(1 - Q_{\alpha\beta} Q_m^{-1} t_i^{-1} q_j^{1/2})(1 - Q_{\alpha\beta} Q_n t_i^{1/2} q_j^{-1/2})}{(1 - Q_{\alpha\beta} t_i q_j^{-1})(1 - Q_{\alpha\beta} t_i^{-1} q_j)}
\]

\[
H_{\alpha\beta} = \prod_{n,i,j=1}^{\infty} \frac{(1 - Q^n_{\rho} \tilde{Q}_{\alpha\beta} Q_m^{-1} t_i^{-1} q_j^{1/2})(1 - Q^n_{\rho} \tilde{Q}_{\alpha\beta} Q_n t_i^{1/2} q_j^{-1/2})}{(1 - Q^n_{\rho} \tilde{Q}_{\alpha\beta} t_i q_j^{-1})(1 - Q^n_{\rho} \tilde{Q}_{\alpha\beta} t_i^{-1} q_j)}.
\]

B.1 Modular Transformation

To understand how the partition function \( Z_{N,1} \) transforms under the modular transformation,

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) : \quad (\rho, m, \epsilon_{1,2}, \tilde{a}_{\alpha\beta}) \mapsto \left( \frac{a \rho + b}{c \rho + d}, \frac{m}{c \rho + d}, \epsilon_{1,2}, \tilde{a}_{\alpha\beta} \right),
\]

which are generated by

\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

we need to determine the transformation properties of \( \vartheta_{\lambda\mu}(\rho, z) \). Although \( \vartheta_{\lambda\mu}(\rho, z) \) is not invariant under the \( T \) transformation, it is easy to see that the ratio \( \frac{\vartheta_{\lambda\mu}(\rho, z_1)}{\vartheta_{\lambda\mu}(\rho, z_2)} \) is invariant for any \( z_{1,2} \). In view of the structure of (B.8), this implies that \( Z_{N,1} \) is invariant under the \( T \) transformation. The ratio \( \frac{\vartheta_{\lambda\mu}(\rho, z_1)}{\vartheta_{\lambda\mu}(\rho, z_2)} \), however, is not invariant under the \( S \) transformation,

\[
\frac{\vartheta_{\lambda\mu}(\rho, z_1)}{\vartheta_{\lambda\mu}(\rho, z_2)} = e^{2 \pi i K_{\lambda\mu}} \frac{\vartheta_{\lambda\mu}(\rho, z_1)}{\vartheta_{\lambda\mu}(\rho, z_2)}
\]

\[
K_{\lambda\mu}(h_1, h_2) = \left( h_1^2 - h_2^2 \right) (|\lambda| + |\mu|) + \left( h_2 - h_1 \right) \left( \sum_{(i,j) \in \lambda} (\epsilon_2 (\lambda_i^j - i + \frac{1}{2}) - \epsilon_1 (\lambda - j + \frac{1}{2})) + \sum_{(i,j) \in \mu} (-\epsilon_2 (\lambda_i^j - i + \frac{1}{2}) + \epsilon_1 (\mu_i - j + \frac{1}{2})) \right),
\]

where \( z_{1,2} = e^{2\pi i h_{1/2}} \). Using the following identities,

\[
\sum_{(i,j) \in \lambda} \lambda_i^j = \sum_{(i,j) \in \mu} \lambda_i^j, \quad \text{and} \quad \sum_{(i,j) \in \lambda} (\lambda_i - j + \frac{1}{2}) = \frac{|\lambda|^2}{2},
\]

\[
K_{\lambda\mu}(h_1, h_2) \text{ appearing in eq. (B.14) can be simplified,}
\]

\[
K_{\lambda\mu}(h_1, h_2) = \left( h_1^2 - h_2^2 \right) (|\lambda| + |\mu|) + (h_1 - h_2) \left[ \epsilon_1 \frac{|\lambda|^2 - |\mu|^2}{2} + \epsilon_2 \frac{|\lambda'|^2 - |\mu'|^2}{2} \right].
\]
Notice that in the unrefined case \((\epsilon_2 = -\epsilon_1 = \epsilon)\) \(K_{\lambda\mu}(h_1, h_2)\) simplifies

\[
K_{\lambda\mu}(h_1, h_2) = \frac{1}{2}(h_1^2 - h_2^2)(|\lambda| + |\mu|) + (h_1 - h_2) \epsilon \left[ \kappa(\lambda) - \kappa(\mu) \right]
\]  

(B.17)

where \(\kappa(\lambda) = \frac{||\lambda||^2 - ||\lambda'||^2}{2}\).

Thus the function \(Z_{\lambda_1...\lambda_N}\) given in Eq.(B.8) transforms as,

\[
Z_{\lambda_1...\lambda_N} \mapsto e^{\frac{2\pi i K}{R}} Z_{\lambda_1...\lambda_N},
\]

(B.18)

with

\[
K_{\lambda_1...\lambda_N}(\hat{a}_{\alpha\beta}, m, \epsilon_+) = \sum_{\alpha=1}^{N} K_{\lambda_\alpha \lambda_\alpha}(m, -\epsilon_+) + \sum_{1 \leq \alpha < \beta \leq N} \left[ K_{\lambda_\alpha \lambda_\beta}(\hat{a}_{\alpha\beta} + m, \hat{a}_{\alpha\beta} - \epsilon_+) + K_{\lambda_\alpha \lambda_\beta}(\hat{a}_{\alpha\beta} - m, \hat{a}_{\alpha\beta} + \epsilon_+) \right]
\]

Here we have defined \(\hat{a}_{\alpha\beta} = \hat{a}_\alpha - \hat{a}_\beta\). Thus the partition function is not invariant under modular transformations (B.12) but can be made invariant at the expense of introducing a holomorphic anomaly [58].

B.2 Singularities

The function \(\vartheta_{\lambda\mu}(\rho, z)\) has some interesting properties. In the unrefined case it becomes proportional to a Kronecker delta function for \(z = 1\) and \(t = q\),

\[
\vartheta_{\lambda\mu}(\rho, 1) = \delta_{\lambda\mu} \prod_{(i,j) \in \lambda} \theta_1(\rho, q^{h(i,j)})\theta_1(\rho, q^{-h(i,j)}) = (-1)^{|\lambda|} \delta_{\lambda\mu} \prod_{(i,j) \in \lambda} \theta_1(\rho, q^{h(i,j)})^2.
\]

(B.19)

Since the partition function \(Z_{N,1}\) is a sum over all partitions, from Eq.(B.8) and Eq.(B.9) it follows that the partition function will have a pole whenever \(a_{\alpha\beta} \in S_{\lambda_\alpha \lambda_\beta}^1 \cup S_{\lambda_\alpha \lambda_\beta}^2\),

\[
S_{\lambda_\alpha \lambda_\beta}^1 = \{ \epsilon_1 \left( -\lambda_{\alpha,i} + j - \frac{1}{2} \right) + \epsilon_2 \left( \lambda_{\beta,j}^t - i + \frac{1}{2} \right) \pm \epsilon_+ \mid (i,j) \in \lambda_\alpha \}
\]

(B.20)

\[
S_{\lambda_\alpha \lambda_\beta}^2 = \{ \epsilon_1 \left( \lambda_{\beta,i} - j + \frac{1}{2} \right) + \epsilon_2 \left( -\lambda_{\alpha,j}^t + i - \frac{1}{2} \right) \pm \epsilon_+ \mid (i,j) \in \lambda_\beta \}
\]

Thus the total order of the poles in \(\hat{a}_{\alpha\beta}\) (counting with possible multiplicity) is \(2(|\lambda_\alpha| + |\lambda_\beta|)\).

The poles in the variable \(\hat{a}_{\alpha\beta}\) only depend on the shape of the pair of partitions \((\lambda_\alpha, \lambda_\beta)\) and therefore the pole structure for \(N > 2\) in the variables \(\hat{a}_{\alpha\beta}\) follows from the pole structure for the \(N = 2\) case in the variable \(\hat{a}_{12} = \hat{a}\).

The poles in the variable \(\hat{a}\) for the \(N = 2\) case form a nested sequence i.e., the set of poles at order \(Q_R^k\) are contained in the set of poles at order \(Q_R^{k+1}\). To see this, consider a pair of
partitions \((\lambda_1, \lambda_2)\), with \(|\lambda_1| + |\lambda_2| = k\), giving the set of poles \(\mathcal{S}_{\lambda_1, \lambda_2}\). For the case of \(N = 2\) consider the pair of partitions \((\lambda_1, \lambda_2) = ((k_1), 1^{k-k_1})\) which contribute to the coefficient of \(Q^k_\sigma\) for all \(k_1 = 0, 1, \ldots, k\). With this choice of the partitions the the set of possible poles in Eq.\((\text{B.20})\) becomes \((\sigma = 0, 1)\),

\[
\hat{a}_{1\sigma} = \hat{a} \in \{-(k_1 - 1) \epsilon_1 + k_2 \epsilon_2 - 2 \sigma \epsilon_+ | j = 0, \cdots, k_1 - 2\} \\
\{ (i - 2) \epsilon_2 + 2 \epsilon_+ \sigma | i = 1, \cdots, k - k_1\}.
\]

(B.21)

The free energy \(\ln(Z_{N,1})\) is a power series in \(\epsilon_{1,2}\) with coefficients which can are refined genus \(g\) amplitudes. Once the expansion in \(\epsilon_{1,2}\) has been carried out the coefficients, refined genus \(g\) amplitudes, now have poles at \(\hat{a}_{\alpha \beta} = 0\). In the paper we study the poles of the refined genus \(g\) amplitudes at \(\hat{a}_{\alpha \beta} = 0\) rather than the poles of the partition function which occur at various locations in the \((\epsilon_1, \epsilon_2)\) plane.

**Example:** Let us consider the case \(N = 2\) to first order in \(Q_\tau\). The free energy \((\text{B.17})\) is given by

\[
\mathcal{F}_{2,1} = \ln(Z_0) + Q_\tau \frac{\theta_{(1)(1)}(Q_m)}{\theta_{(1)(1)}(\sqrt{\frac{2}{q}})} \left[ \frac{\theta_{(1)(0)}(Q_{12} Q_m) \theta_{(1)(0)}(Q_{12} Q_m^{-1})}{\theta_{(0)(1)}(Q_{12} Q_m) \theta_{(0)(1)}(Q_{12} Q_m^{-1})} + \frac{\theta_{(0)(1)}(Q_{12} Q_m) \theta_{(0)(1)}(Q_{12} Q_m^{-1})}{\theta_{(0)(1)}(Q_{12} Q_m) \theta_{(0)(1)}(Q_{12} Q_m^{-1})} \right] + \cdots
\]

(B.22)

From Eq.\((\text{B.10})\) and Eq.\((\text{B.11})\) we see that as a function of \(\hat{a}_1\)

\[
\ln(Z_0) = A \ln(\hat{a}_1) + \cdots
\]

(B.23)

where \(A\) is independent of \(\hat{a}_1\). Thus the free energy diverges like,

\[
\mathcal{F}_{2,1} = A \ln(\hat{a}_1) - Q_\tau \frac{\theta_{1}(\rho, m + \epsilon_{-}) \theta_{1}(m - \epsilon_{-}) \theta_{1}(\rho, \hat{a}_1 + m + \epsilon_{+}) \theta_{1}(\rho, \hat{a}_1 - m + \epsilon_{+})}{\theta_{1}(\rho, \hat{a}_1) \theta_{1}(\rho, \hat{a}_1 + 2 \epsilon_{+})} \times
\]

\[
\left[ \frac{2}{(\hat{a}_1 + 2 \epsilon_{+})(\hat{a}_1 - 2 \epsilon_{+})} + \cdots \right].
\]

(B.24)

Thus we see that there is a pole at \(\hat{a}_1 = \pm 2 \epsilon_{+}\). However, if we first expand in \(\epsilon_{1,2}\) then we get a single pole \(\hat{a}_1 = 0\) of order two. This persists at higher order in \(Q_\tau\) and we see poles at \(\hat{a}_1 = 0\) of various even orders.
C Expansion Coefficients of the Basic Building Blocks

In this appendix we collect explicit expressions for the expansions of the free energy for \( N = 1 \), \( N = 2 \) and \( N = 3 \).

C.1 Coefficients of the \( N = 1 \) Free Energy

Due to their frequent use throughout the main body of this article, we tabulate the coefficients \( H^{(r),\{0\}}_{(2s,0)}(\rho, S) \) that appear in the expansion of the free energy to leading orders \( n \) \( r \) and \( s \). To this end, we decompose the former in the following fashion

\[
H^{(r),\{0\}}_{(2s,0)}(\rho, S) = \sum_{i=0}^{r} b^{(r)}_{i,(2s,0)}(\rho) \left( \phi_{-2,1}(\rho, S) \right)^{i} \left( \phi_{0,1}(\rho, S) \right)^{r-i},
\]

(C.1)

where \( b^{(r)}_{i,(2s,0)}(\rho) \) is a quasi-modular form of weight \( 2s + 2i - 2 \), which can be written as a polynomial in the Eisenstein series \( \{ E_{2}, E_{4}, E_{6} \} \). For \( r = 1 \), \( r = 2 \) and \( r = 3 \) the expansion coefficients are tabulated in 17, 18 and 19 respectively.

Following [45], the coefficients \( H^{(r),\{0\}}_{(2s,0)}(\rho, S) \) with \( r > 1 \) can be recovered from those with \( r = 1 \) through Hecke transformations, i.e.

\[
H^{(r),\{0\}}_{(s,0)}(\rho, S) = \mathcal{H}_{r} \left( H^{(1),\{0\}}_{(s,0)}(\rho, S) \right).
\]

(C.2)

The relations (4.26) for \( N = 2 \), (5.28) for \( N = 3 \) and (6.18) for \( N = 4 \) can be understood as generalisations of (C.2). Finally, for further use throughout the main body of this paper, we also introduce

\[
H^{(r)}_{N=1}(\rho, S, \epsilon_{1}) = \sum_{s=0}^{\infty} \epsilon_{1}^{2s-2} H^{(r),\{0\}}_{(s,0)}(\rho, S).
\]

(C.3)

C.2 Expansion of \( W(\rho, S, \epsilon_{1}, \epsilon_{2}) \)

In [22 45] the (quasi-)Jacobi form

\[
W(\rho, S, \epsilon_{1,2}) = \frac{\theta_{1}(\rho, S + \epsilon_{+})\theta_{1}(\rho, S - \epsilon_{+}) - \theta_{1}(\rho, S + \epsilon_{-})\theta_{1}(\rho, S - \epsilon_{-})}{\theta_{1}(\rho, \epsilon_{1})\theta_{2}(\rho, \epsilon_{2})}, \quad \text{with} \quad \epsilon_{\pm} = \frac{\epsilon_{1} \pm \epsilon_{2}}{2},
\]

(C.4)
Table 17: Coefficients in the expansion of $b_{i,(2s,0)}(\rho, S)$.

| $s$ | $b_{0,(2s,0)}^{(r=1)}$ | $b_{1,(2s,0)}^{(r=1)}$ |
|-----|------------------------|------------------------|
| 0   | 0                      | -1                     |
| 1   | 1/96                   | -E_4/48                |
| 2   | E_4/4608               | -5E_4^3+13E_4/23040    |
| 3   | 5E_4^2+7E_4/2211840    | -35E_4^3+273E_4E_4+184E_6/23222320 |
| 4   | 35E_4^3+147E_4E_4+124E_6/229534720 | -175E_4^3+2730E_4^2E_4+3680E_4E_6+5583E_4^2/2295347200 |

Table 18: Coefficients in the expansion of $b_{i,(2s,0)}^{(r=2)}(\rho, S)$.

| $s$ | $b_{0,(2s,0)}^{(r=2)}$ | $b_{1,(2s,0)}^{(r=2)}$ | $b_{2,(2s,0)}^{(r=2)}$ |
|-----|------------------------|------------------------|------------------------|
| 0   | 0                      | 1/16                   | 0                      |
| 1   | 1/1536                 | E_4/384                | 5E_4/384               |
| 2   | E_4/36864              | -5E_4^2+27E_4/92160    | 5(E_4E_4+2E_6)/92160   |
| 3   | 5E_4^2+13E_4/8847360   | -35E_4^3+567E_4E_4+1066E_6/4648640 | 5E_4^2E_4+20E_4E_6+53E_4^2/442368 |
| 4   | 70E_4^3+546E_4E_4+1067E_6/8918158880 | -175E_4^3+5670E_4^2E_4+21320E_4E_6+54303E_4^2/2295347200 | 70E_4^3E_4+420E_4^2E_4E_6+2226E_4^2E_4^2+53893E_4E_6E_6/445906944 |

Table 19: Coefficients in the expansion of $b_{i,(2s,0)}^{(r=3)}(\rho, S)$.

| $s$ | $b_{0,(2s,0)}^{(r=3)}$ | $b_{1,(2s,0)}^{(r=3)}$ | $b_{2,(2s,0)}^{(r=3)}$ | $b_{3,(2s,0)}^{(r=3)}$ |
|-----|------------------------|------------------------|------------------------|------------------------|
| 0   | 0                      | 1/432                  | 0                      | -E_4/36               |
| 1   | 1/41372                | -E_4/6912              | E_4/384                | -9E_4E_4+32E_6/5184   |
| 2   | E_4/663552             | -15E_4^2+151E_4/3317760 | 27E_4E_4+88E_6/165888 | -45E_4^2E_4+320E_4E_6+1333E_4^2/829440 |
| 3   | 5E_4^2+23E_4/108168320 | -105E_4^3-3171E_4E_4+10088E_6/1114767360 | 405E_4^2E_4+2640E_4E_6+10103E_4^2/79826240 | -315E_4^3E_4-3360E_4^2E_4E_6-27993E_4^2E_4^2-1034000E_4E_6E_6/278691840 |
was introduced, which governs the BPS-counting of a single M5-brane with on M2-brane ending on it on either side. In the NS-limit, expanding the latter in powers of $\epsilon_1$, we define

$$W_{NS}^{(1)}(\rho, S, \epsilon_1) = \lim_{\epsilon_2 \to 0} W(\rho, S, \epsilon_{1,2}) = \sum_{s=0}^{\infty} \epsilon_1^{2s} W_{(2s)}(\rho, S), \quad (C.5)$$

where to low orders in $s$, we find

$$W(0) = \frac{1}{24} (\phi_{0,1} + 2E_2 \phi_{-2,1}),$$

$$W(2) = -\frac{1}{576} (E_4 - E_2^2) \phi_{-2,1},$$

$$W(4) = \frac{5(E_4 - E_2^2)\phi_{0,1} + 2(5E_2^3 + 3E_2E_4 - 8E_6)\phi_{-2,1}}{552960},$$

$$W(6) = \frac{\phi_{-2,1}(35E_4^2 + 168E_2^2E_4 + 16E_2E_6 - 219E_4^2) - 7\phi_{0,1}(5E_2^3 + 3E_2E_4 - 8E_6)}{278691840}. \quad (C.6)$$

While not a function of $R$, following the free energy for $N = 1$ discussed in the previous appendix [C.1], we can define an extension of $W_{(2s)}$ to higher orders through

$$W_{(2s)}^{(r)}(\rho, S) = \mathcal{H}_r \left( W_{(2s)}(\rho, S) \right), \quad (C.7)$$

along with the building block

$$W_{NS}^{(r)}(\rho, S, \epsilon_1) = \sum_{s=0}^{\infty} \epsilon_1^{2s} W_{(2s)}^{(r)}(\rho, S). \quad (C.8)$$

For convenience we can give explicit expressions for the first few instances of $W_{(2s)}^{(r)}$. To this end, we introduce the decomposition

$$W_{(2s)}^{(r)}(\rho, S) = \sum_{i=0}^{r} t_{i,(2s)}^{(r)}(\rho) (\phi_{-2,1}(\rho, S))^i(\phi_{0,1}(\rho, S))^{r-i}, \quad (C.9)$$

where $t_{i,(2s)}^{(r)}$ is a quasi-modular form of weight $2s + 2i$, which can be written as a polynomial in the Eisenstein series. For $r = 1$, the expression [C.6] can be tabulated as

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Let \( N \in \mathbb{N} \) with \( N > 1 \). The degree two paramodular groups are subgroups of the symplectic group \( Sp(4, \mathbb{Q}) \) labelled by an integer \( N \) and defined as \([81, 82]\),

\[
\Sigma_N = \left\{ \begin{pmatrix}
* & N & * & * \\
* & * & * & * \\
* & N & * & * \\
N & N & N & * \\
\end{pmatrix} \in Sp(4, \mathbb{Q}), \quad * \in \mathbb{Z} \right\}.
\]  

(D.1)
\( \Sigma_N \) has the interesting property that \( \Sigma_N \Gamma_N \subset \Gamma_N \) where \( \Gamma_N \) is the lattice \( \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N \mathbb{Z} \) in \( Sp(4, \mathbb{Q}) \) and \( \Sigma_N \) acts through simple matrix multiplication. A very useful review of the degree \( n \) paramodular groups is given in [33].

In order to define the action of \( \Sigma_N \) on the free energies discussed in the main body of this paper, we furthermore introduce the period matrix

\[
\Omega = \begin{pmatrix} \rho & S \\ S & R \end{pmatrix} \in \mathbb{H}(2)
\]

where \( \mathbb{H}(2) \) is the space of \( 2 \times 2 \) matrices with positive imaginary part. We then define the action of \( \Sigma_N \) by

\[
g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Sigma_N : \quad \Omega \mapsto \Omega' = g \circ \Omega = (A \cdot \Omega + B) \cdot (C \cdot \Omega + D)^{-1},
\]

where \( A, B, C \) and \( D \) are \( 2 \times 2 \) matrices.

Following [82, 84, 85] one can define an extension of \( \Sigma_N \) to a subgroup of \( Sp(4, \mathbb{R}) \). To this end we introduce

\[
h_N = \begin{pmatrix} U_N & 0 \\ 0 & U_N^T \end{pmatrix} \subset Sp(4, \mathbb{R}), \quad \text{and} \quad U_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & N \\ 1 & 0 \end{pmatrix},
\]

and define

\( \Sigma_N^* = \Sigma_N \cup \Sigma_N h_N \subset Sp(4, \mathbb{R}) \).

Notice that \( h_N \) in (D.4) acts as

\[
h_N : \quad \Omega \mapsto \Omega' = h_N \circ \Omega = \begin{pmatrix} NR & S \\ S & \frac{\rho}{N} \end{pmatrix},
\]

which implies the symmetry \( f(R, \rho, S) = f\left(\frac{\rho}{N}, NR, S\right) \) for paramodular forms.

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