STABILITY OF CRANK-NICOLSON COMPACT SCHEME FOR CONVECTION-DIFFUSION EQUATIONS USING MATRIX METHOD

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Abstract. The fully discrete problem for convection-diffusion equation comprising of compact approximations for spatial discretization and Crank-Nicolson scheme for temporal discretization is considered. Gerschgorin circle theorem is applied to estimate the eigen values of amplification matrix. The stability of the fully discrete problem is proved using matrix method.

Keywords: Gerschgorin circle theorem, Matrix method for stability, Compact schemes, Crank-Nicolson method, Convection-diffusion equation.

1. Introduction

The convection–diffusion equation is ubiquitous in several phenomenons, for example, option pricing problems in stock market [1, 2], computational fluid dynamics [3, 4], and in various other physical systems [5, 6, 7, 8]. The analytical solution of the convection-diffusion equations is only obtained in a few cases and it is not available in general. Therefore, rich theory of numerical methods is essential to solve such problems. In literature, various numerical methods have been developed for solving convection-diffusion equation [9, 10, 11]. Second order accurate finite difference method (FDM) is one of the most established numerical approach because of its effortless implementation and desirable stability properties [12, 13, 14]. Here, the stability of an FDM implies that the errors made at one time step do not magnify as the computations are carried forward in time.

In fact, high-order accurate FDM can be developed by increasing the number of grid points in computational stencil. However, the implementation of boundary conditions becomes tedious in those cases. Moreover, the corresponding coefficient matrices in fully discrete problem have more non-zero entries. Further, stability condition may become more restrictive. Therefore, high-order accurate FDMs have been developed using smaller (compact) stencils where the non-zero entries of coefficient matrix are cumbersome but tractable. Such high-order accurate FDMs, as studied in [15, 16, 17], are commonly known as compact schemes. Due to the above mentioned parsimony and high-order accuracy of compact scheme as compared to FDM, the former is even more crucial for solving multi-dimensional PDEs [18, 19].

Since compact approximations to compute the derivatives of functions have been derived in many ways, let us briefly discuss the available literature in this direction. A MATLAB suit for compact approximations of various order derivatives is presented in [20]. A fundamental work about compact approximations and its characteristics is presented in [21]. In [22] and [23], the method of undetermined coefficients and the polynomial interpolation were adopted to derive the compact approximations on non-uniform grids respectively.

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It has always been a challenge to obtain a stable numerical scheme to solve the PDEs numerically. The stable compact schemes have already been developed for convection-diffusion equation and other partial differential equations (PDEs) \[24, 25\]. However, the stability of these schemes is studied via von-Neumann (Fourier stability) analysis. In principle, the applicability of von-Neumann stability analysis is restricted to the linear constant coefficient finite difference formulas on uniform grids, see Theorem 4.8, pp. 166 \[26\]. In this manuscript, the compact scheme developed in \[27\] is considered to discretize the convection-diffusion equation. Its stability is proved via matrix method using Gerschgorin circle theorem (GCT) under certain assumptions. To the best of our knowledge, this is the first work to establish the matrix method stability using GCT for any PDE using Crank-Nicolson compact scheme.

The present manuscript is structured as follows: The model problem is given in Sec 2. The fully discrete problem for convection-diffusion equation is presented in Sec. 3. The stability of fully discrete problem is proved in Sec. 4. Sec. 5 includes the concluding remarks with some future research directions.

2. The Model Problem

Let us consider the following convection-diffusion equation on a bounded domain

\[
\frac{\partial \phi}{\partial v} + A \frac{\partial \phi}{\partial x} = B \frac{\partial^2 \phi}{\partial x^2}, \tag{1}
\]

where \(A\) and \(B\) are constant. After taking the transformation \(u = \frac{B}{A} \phi, z = \frac{A}{B} x\) in the above equation (1), we get

\[
\frac{\partial u}{\partial v} + \epsilon \frac{\partial u}{\partial z} = \epsilon \frac{\partial^2 u}{\partial z^2}, \tag{2}
\]

for \(z\) in a finite open interval \((z_l, z_r)\), \(v > 0\), and \(\epsilon = \frac{A^2}{B} > 0\). The Eq. (2) is equipped with the following initial and boundary conditions

\[
u(0, z) = f(z), z \in (z_l, z_r) \tag{3}
\]

\[
u(v, z_l) = g_1(v), \quad \nu(v, z_r) = g_2(v), v > 0. \tag{4}
\]

We assume that \(g_1\) and \(g_2\) are smooth functions such that \(g_1(0) = f(z_l)\) and \(g_2(0) = f(z_r)\).

3. The Fully Discrete Problem

In this section, the fully discrete problem for Eq. (2) is presented. For the sake of simplicity, a uniformly spaced mesh is considered in both temporal and spatial domain. For fixed \(N\), we define \(z_q = z_l + q \delta z, 0 \leq q \leq N\) for a fixed space step size \(\delta z = \frac{z_r - z_l}{N}\). Also for fixed \(M\), consider the \(m^{th}\) time step as \(m \delta v\) for constant time step size \(\delta v\) and \(m = 0, 1, ..., M\). Let \(u^m_q\) denote the solution of (2) at \(m^{th}\) time level and at space grid point \(z_q\). Following Eq. (6) in \[27\], the fully discrete problem for (2) using Crank-Nicolson compact scheme at space grid point \(z_q\) and time level \(m\) is

\[
\Delta_t u^m_q - \frac{\delta z^2}{12} (\Delta_x^2 u^m_q - \Delta_{xx}^2 u^m_q) + \frac{c}{2} \left( \Delta_x u^m_q - \left(1 + \frac{\delta z^2}{12}\right) \Delta_{xx} u^m_q \right) + \frac{c}{2} \left( \Delta_x u^{m+1}_q - \left(1 + \frac{\delta z^2}{12}\right) \Delta_{xx} u^{m+1}_q \right) = O(\delta v^2, \delta z^4). \tag{5}
\]
Here $\Delta^+_v u_q^m$, $\Delta_x u_q^m$, and $\Delta_{xz} u_q^m$ represent finite difference approximations for first order time derivative, first order space derivative, and second order space derivative of $u$ respectively at $m^{th}$ time level and space grid point $z_q$. The expressions for the same are as follows:

\[
\Delta^+_v u_q^m = \frac{u_{q+1}^m - u_q^m}{\delta v}, \quad \Delta_x u_q^m = \frac{u_{q+1}^m - u_{q-1}^m}{2\delta z}, \quad \Delta_{xz} u_q^m = \frac{u_{q+1}^m - 2u_q^m + u_{q-1}^m}{\delta z^2}. \tag{6}
\]

If $U_q^m$ denotes the approximate value of $u_q^m$, then using relation (6) in (5) and rearranging the terms, we get the following fully discrete problem for all $1 \leq q \leq N - 1$

\[
U_{q-1}^{m+1} \left( \frac{2 + \delta z}{24\delta v} - \frac{c}{4\delta z} - \frac{c + \frac{5c^2}{12}}{2\delta z^2} \right) + U_q^{m+1} \left( \frac{5}{6\delta v} + \frac{c + \frac{5c^2}{12}}{\delta z^2} \right) +
\]

\[
U_{q+1}^{m+1} \left( \frac{2 - \delta z}{24\delta v} + \frac{c}{4\delta z} - \frac{c + \frac{5c^2}{12}}{2\delta z^2} \right) = U_q^{m+1} \left( \frac{2 + \delta z}{24\delta v} + \frac{c}{4\delta z} + \frac{c + \frac{5c^2}{12}}{2\delta z^2} \right),
\]

with $U_0^m = g_1(v_m)$, and $U_N^m = g_2(v_m)$, for all $m > 0$. Now we will prove the stability of the fully discrete problem (7) in the following section.

4. Stability

In this section, we prove the stability of fully discrete problem (7). Suppose $U^m$ denotes the vector $[U_1^m, U_2^m, \cdots, U_{N-1}^m]$. We also introduce the following constants, depending on $\delta z$ and $\delta v$:

\[
c_1 = \frac{(2 + \delta z)}{24\delta v}, \quad c_2 = \frac{5}{6\delta v}, \quad c_3 = \frac{2 - \delta z}{24\delta v},
\]

\[
y_1 = -\frac{c}{4\delta z} - \frac{c + \frac{5c^2}{12}}{2\delta z^2}, \quad y_2 = \frac{c + \frac{5c^2}{12}}{2\delta z^2}, \quad y_3 = \frac{c}{4\delta z} - \frac{c + \frac{5c^2}{12}}{2\delta z^2}.
\]

Set $b := \frac{\delta v}{\delta z}$, to simplify

\[
y_1 = -\frac{c}{2\delta v} \left[ 1 + \frac{\delta z}{2} \right] b + \frac{\delta v}{12}, \quad y_2 = \frac{c}{2\delta v} \left[ 2b + \frac{\delta v}{6} \right], \quad y_3 = -\frac{c}{2\delta v} \left[ 1 - \frac{\delta z}{2} \right] b + \frac{\delta v}{12}.
\]

The fully discrete problem (7) can be written as

\[
(X + Y)U^{m+1} = (X - Y)U^m + F^m, \tag{8}
\]

where

\[
X := \begin{bmatrix}
c_2 & c_3 & \cdots \\
c_1 & c_2 & c_3 & \cdots \\
& \vdots \end{bmatrix}, \quad Y := \begin{bmatrix}
y_2 & y_3 & \cdots \\
y_1 & y_2 & y_3 & \cdots \\
& \vdots
\end{bmatrix},
\]

and

\[
F^m = [(c_1 - y_1)g_1(v_m) - (c_1 + y_1)g_1(v_{m+1}), 0, \ldots, 0, (c_3 - y_3)g_2(v_m) - (c_3 + y_3)g_2(v_{m+1})].
\]
We assume that $0 < \delta z < 2$ now onwards to ensure positivity of the constant $c_3$. Evidently, $X$ is diagonally dominant, and hence invertible. Therefore, using $W := X^{-1}Y$, Eq. (8) can be rewritten as

$$(I + W)U^{m+1} = (I - W)U^m + X^{-1}F^m.$$  \hspace{1cm} (10)

The stability of fully discrete problem (7) will be proved by estimating the upper bound of the eigen values of matrix $W$. We will prove that $\|W\|_2 < 1$ and real part of the eigen values of matrix $W$ is positive. It is shown in Theorem 3 that it suffices to prove these two conditions for stability of fully discrete problem (7). The element of matrix $X^{-1}$ are required to prove these conditions. In order to get the elements of $X^{-1}$, the following result is borrowed from pp. 137 in [28]:

**Remark 1** (Inverse of a tridiagonal Toeplitz matrix). *Since $X$ is Toeplitz, for all $1 \leq q, q' \leq N - 1$ the entries of $X^{-1}$ can be obtained using the following formula*

$$(X^{-1})_{q,q'} = \frac{(-1)^{q-q'} \sqrt{c_1 c_3}}{c_3} \left( \frac{c_2}{2 \sqrt{c_1 c_3}} \right)^{q-q'} \frac{p_{(q\wedge q')-1} \times p_{N-1-(q\vee q')}}{p_{N-1}}.$$  \hspace{1cm} (11)

*where $q \wedge q'$ and $q \vee q'$ are minimum and maximum of $\{q, q'\}$ respectively and*

$$p_n(x) = (2x)^n \left[ 1 + \sum_{n'=1}^{\lfloor n/2 \rfloor} (-1)^{n'} \left( \frac{n - n'}{n'} \right) \left( \frac{1}{4x^2} \right)^{n'} \right].$$

Note that, due to the assumption $\delta z < 2$, $c_3$ is positive and hence the expression in (11) is real.

Furthermore, in order to locate the eigen values of amplification matrix, the following result is borrowed from pp. 61 in [29]:

**Remark 2** (Gerschgorin Circle Theorem). *Let us consider a matrix $B = [b_{i,j}]$. Suppose $M_q$ denotes the sum of the modulus of the elements of $q^{th}$ row of matrix $B$ while excluding the diagonal element $b_{i,i}$. Then each eigen value of the matrix $B$ lies inside or on the boundary of at least one of the circles*

$$|\lambda - b_{i,i}| = M_q,$$

*where $\lambda$ denotes any eigen value of matrix $B$.*

Now the main results will be proved in the following Lemma 3 and Theorem 4.

**Lemma 3.** If $\delta z < 2$ and

$$\delta v < \sqrt{\frac{5}{12}} \left( \frac{2c}{\delta z^2} + \frac{c}{6} \right),$$  \hspace{1cm} (12)

we have $\|W\|_2 < 1$.

**Proof.** As $W = X^{-1}Y$, it is enough to prove that $\|X^{-1}\|_2 \|Y\|_2 \leq 1$. To this end, we first obtain upper bounds for $\|X^{-1}\|_2$ and $\|Y\|_2$ below. Let $X^*$ denote the transpose of $X$. Since $(X^{-1})^*X^{-1} = (XX^*)^{-1}$, $s$ is a singular value of $X^{-1}$ iff $s^2$ is an eigenvalue of $(XX^*)^{-1}$. 


Equivalently, \( s^{-2} \) is an eigenvalue of \( Z = XX^* \), whose entries are as follows

\[
Z_{q,q'} = \begin{cases} 
\sum_{i=1}^{3} c_i^2 & \text{if } 1 < q' = q < N - 1 \\
\sum_{i=2}^{3} c_i^2 & \text{if } q' = q = 1 \\
\sum_{i=1}^{2} c_i^2 & \text{if } q' = q = N - 1 \\
c_2(c_1 + c_3) & \text{if } |q' - q| = 1 \\
c_1c_3 & \text{if } |q' - q| = 2 \\
0 & \text{else},
\end{cases}
\tag{13}
\]

for all \( 1 \leq q, q' \leq N - 1 \). Thus

\[
\|X^{-1}\|_2 = \max \{ s \mid s \text{ is a singular value of } X^{-1} \} = \frac{1}{\sqrt{\rho_{\min}(XX^*)}},
\]

where \( \rho_{\min}(XX^*) \) is the minimum of the modulus of the eigenvalue of \( XX^* \). A lower bound of spectrum of \( X^*X \) will be obtained using Gerschgorin’s Circle Theorem (GCT). To facilitate the application the theorem, the centers and radius of Gerschgorin’s disks are calculated below. From (13), it is clear that we need to consider four different discs, namely

\[
D(\sum_{i=2}^{3} c_i^2, c_2(c_1 + c_3) + c_1c_3), \quad D(\sum_{i=1}^{3} c_i^2, 2c_2(c_1 + c_3) + c_1c_3),
\]

\[
D(\sum_{i=1}^{3} c_i^2, 2c_2(c_1 + c_3) + 2c_1c_3), \quad D(\sum_{i=1}^{2} c_i^2, c_2(c_1 + c_3) + c_1c_3),
\]

where \( D(a, r) \) denotes the disc with center \( a \) and radius \( r \). The values of \( (a - r) \) for all four discs are

\[
\frac{1}{\delta v^2} \left( \frac{5}{9} - \frac{(2 - \delta z)\delta z}{288} \right), \quad \frac{1}{\delta v^2} \left( \frac{4}{9} - \frac{(4 - \delta z^2)}{192} \right),
\]

\[
\frac{1}{\delta v^2} \left( \frac{4}{9} - \frac{(4 - \delta z^2)}{144} \right), \quad \frac{1}{\delta v^2} \left( \frac{5}{9} + \frac{(2 + \delta z)\delta z}{288} \right)
\]

respectively. Since the third member is the least, by applying GCT we get

\[
\rho_{\min}(XX^*) \geq \frac{1}{\delta v^2} \left( \frac{4}{9} - \frac{(4 - \delta z^2)}{144} \right) = \frac{60 + \delta z^2}{(12\delta v)^2} > \frac{5}{12(\delta v)^2}.
\]
Therefore, \( \| X^{-1} \|_2 < \sqrt{\frac{12}{5}} \delta v \). Further, we have

\[
\| Y \|_\infty = \max_{1 \leq q' \leq N-1} \sum_{q=1}^{N-1} |Y_{q,q'}|
\]

\[
= |y_1| + |y_2| + |y_3|
\]

\[
= \frac{1}{\delta v} \left[ \frac{bc}{2} + \frac{c\delta v}{24} + \frac{bc}{12} + \frac{bc}{2} + \frac{c\delta v}{24} \right]
\]

\[
= \frac{2bc}{\delta v} + \frac{c}{6},
\]

since \( \delta z < 2 \). Similarly, we have

\[
\| Y \|_1 = \max_{1 \leq q' \leq N-1} \sum_{q=1}^{N-1} |Y_{q,q'}|
\]

\[
= |y_1| + |y_2| + |y_3|
\]

\[
= \frac{2bc}{\delta v} + \frac{c}{6}.
\]

Using the relation between matrix norms, \( \| Y \|_2 \leq \sqrt{\| Y \|_1 \| Y \|_\infty} \), we have

\[
\| Y \|_2 \leq \left( \frac{2bc}{\delta v} + \frac{c}{6} \right).
\]

Given \( W = X^{-1}Y \) and \( b = \frac{\delta v}{\delta z} \), we have

\[
\| W \|_2 \leq \| X^{-1} \|_2 \| Y \|_2
\]

\[
\leq \sqrt{\frac{12}{5}} \left( \frac{2c}{\delta z} + \frac{c}{6} \right) \delta v.
\]

The result follows from \( \{12\} \). \( \square \)

**Theorem 4.** Assume that the real parts of the eigenvalues of \( W \) are positive. If \( \delta z \), and \( \delta v \) are as in Lemma \( \{3\} \), then fully discrete problem \( \{7\} \) is stable.

**Proof.** From Lemma \( \{3\} \) we know that \( (I + W)^{-1} \) exists and can be written as \( \sum_{n=0}^{\infty} (-W)^n \). Therefore, we can rewrite equation \( \{10\} \) as

\[
U^{m+1} = HU^m + (X + Y)^{-1} F^m,
\]

where

\[
H = \left( \sum_{n=0}^{\infty} (-W)^n \right) (I - W).
\]

Note that if \( \rho \) and \( \beta \) is a pair of eigenvalue and eigenvector of \( W \), we get

\[
H \beta = \left( \sum_{n=0}^{\infty} (-\rho)^n \right) (1 - \rho) \beta = \frac{1 - \rho}{1 + \rho} \beta.
\]
Consequently, $\frac{1-\rho}{1+\rho}$ is an eigenvalue of $H$, the amplification matrix. Hence the fully discrete problem (7) is stable provided $\frac{1-\rho}{1+\rho} < 1$ for each eigenvalue $\rho$ of $W$. Since we have assumed that the real part of $\rho$ is positive, the modulus of $1 - \rho$ is smaller than that of $1 + \rho$. Thus

$$|1 - \rho| < 1.$$  

Hence the fully discrete problem (7) is stable. \[ \Box \]

**Remark 5.** Theorem 4 holds under the assumption that the real parts of the eigenvalues of $W$ are positive. That assumption is shown to be true theoretically in Remark 6 and Lemma 7 with no additional assumption on the model parameters. Hence, this is not an unrealistic assumption.

**Remark 6.** For $N = 2$, the matrices $X$, $Y$, and $W$ are just scalars, and their values are

$$X = c_2, \quad Y = y_2, \quad \text{and} \quad W = \frac{y_2}{c_2} = \frac{3c}{5} \left( 2b + \frac{\delta v}{6} \right) > 0.$$  

Thus, Theorem 4 holds true for $N = 2$ case.

**Lemma 7.** Assume $N = 3$ and $\delta z < 2$. If $\rho$ is an eigenvalue of $W$, then the real part of $\rho$ is positive for sufficiently small $\delta z > 0$.

**Proof.** Using the fact that $Y$ is tri-diagonal, we can write for $1 \leq q, q' \leq N - 1$

$$W_{q,q'} = (X^{-1})_{q,q'} - 1 Y_{q'-1, q'} + (X^{-1})_{q,q'} Y_{q,q'} + (X^{-1})_{q,q'+1} Y_{q'+1, q'}$$

$$= (X^{-1})_{q,q'} - 1 y_3 + (X^{-1})_{q,q'} y_2 + (X^{-1})_{q,q'+1} y_1. \quad (14)$$

In the above expression we mean

$$(X^{-1})_{q,q'} = 0,$$  

if either of $q$ or $q'$ is not in $\{1, \ldots, N - 1\}$.

The fraction $\frac{c}{2Nc_2c_3}$ appearing in (11) has a value $\frac{10}{\sqrt{4 - \delta z^2}}$. Thus from (11), we have

$$(X^{-1})_{q,q'} = (-1)^{q-q'} \frac{24\delta v}{\sqrt{4 - \delta z^2}} \left( \sqrt{\frac{2 + \delta z}{2 - \delta z}} \right)^{q-q'} \frac{p_{(q',q')} - 1 \times p_{N-1 - (q',q')}}{p_{N-1}} \left( \frac{10}{\sqrt{4 - \delta z^2}} \right). \quad (15)$$

Since $\delta z < 2$, $y_1$ and $y_3$ have an identical sign which is opposite of $y_2$. Moreover from Remark 6 (14) and (X^{-1})_{q,q'-1} and (X^{-1})_{q,q'+1} are having identical signs which is opposite of $(X^{-1})_{q,q'}$. Hence $(X^{-1})_{q,q'-1} y_1$, $(X^{-1})_{q,q'} y_2$, and $(X^{-1})_{q,q'+1} y_3$ all have an identical sign. Therefore the absolute value of their sum is equal to the sum of their absolute values. Thus for $1 \leq q, q' \leq N - 1$, we write

$$|W_{q,q'}| = |(X^{-1})_{q,q'} - 1 y_3| + |(X^{-1})_{q,q'} y_2| + |(X^{-1})_{q,q'} + 1 y_1|. \quad (16)$$

Again, we apply GCT to locate the eigenvalues of $W$ by finding the Gershgorin disks $D(a_q, r_q)$ corresponding to the $q^{th}$ row. From (15), and (16), the center $a_q$ and radius $r_q$ for all $1 \leq q \leq N - 1$ are given by

$$a_q = W_{q,q} = (X^{-1})_{q,q} - 1 y_3 + (X^{-1})_{q,q} y_2 + (X^{-1})_{q,q} + 1 y_1 \quad (17)$$

$$r_q = \sum_{q \neq q} |W_{q,q'}|. \quad (18)$$
In order to show positivity of the real part of eigenvalues, it is sufficient to show that \( a_q > r_q \) for all \( 1 \leq q \leq 2 \). To start with \( q = 1 \), we get

\[
a_1 = (X^{-1})_{1,1} y_2 + (X^{-1})_{1,2} y_1
\]

\[
= \frac{24\delta v}{\sqrt{4 - \delta^2}} \left[ \frac{p_0 p_1}{p_2} \left( \frac{10}{\sqrt{4 - \delta^2}} \right) y_2 - \left( \frac{2 + \delta z}{2 - \delta} \right)^{-1} \frac{p_0 p_0}{p_2} \left( \frac{10}{\sqrt{4 - \delta^2}} \right) y_1 \right]
\]

\[
= \frac{24\delta v}{\sqrt{4 - \delta^2}} \left[ \frac{p_0 p_1}{p_2} - \left( \frac{2 + \delta z}{2 - \delta} \right) \frac{y_1}{p_2} \left( \frac{10}{\sqrt{4 - \delta^2}} \right) \right].
\]

Using \( p_0 = 1 \), and

\[
r_1 = |W_{1,2}| = |(X^{-1})_{1,1} y_3| + |(X^{-1})_{1,2} y_2|
\]

\[
= \frac{24\delta v}{\sqrt{4 - \delta^2}} \left[ \frac{p_0 p_1}{p_2} \left( \frac{10}{\sqrt{4 - \delta^2}} \right) |y_3| + \left( \frac{2 + \delta z}{2 - \delta} \right)^{-1} \frac{p_0 p_0}{p_2} \left( \frac{10}{\sqrt{4 - \delta^2}} \right) |y_2| \right]
\]

\[
= \frac{24\delta v}{\sqrt{4 - \delta^2}} \left[ \frac{-p_0 p_1}{p_2} - \left( \frac{2 + \delta z}{2 - \delta} \right) \frac{y_1}{p_2} \left( \frac{10}{\sqrt{4 - \delta^2}} \right) \right],
\]

From above we have

\[
(a_1 - r_1) = \left( \frac{24\delta v}{\sqrt{4 - \delta^2}} \right) \left[ \frac{y_2 p_1}{p_2} - \left( \frac{2 + \delta z}{2 - \delta} \right) \frac{y_1}{p_2} + \frac{y_3 p_1}{p_2} \right] \left( \frac{10}{\sqrt{4 - \delta^2}} \right).
\]

We wish to show the positivity of the above expression as \( \delta z \to 0 \). Since \( \lim_{\delta z \to 0} b = \infty \), we take the repeated limit approach by fixing the value of \( b \) first and letting \( \delta z \to 0 \), and then allowing \( b \to \infty \). As \( \delta z \to 0 \), (19) becomes

\[
\left[ (y_2 + y_3) \frac{p_1}{p_2} - (y_1 + y_2) \frac{1}{p_2} \right] (5) = \left[ \left( \frac{c}{2\delta v} \left( \frac{b + \delta v}{12} \right) \right) \frac{p_1 - 1}{p_2} \right] (5) \to \infty
\]

as \( c > 0 \), \( p_1(5) - 1 = 9 > 0 \), \( p_2(5) = 99 > 0 \). Now, using the continuity of (19) w.r.t. \( \delta z \), we conclude that there is a sufficiently small \( \delta z > 0 \) such that (19) is positive. Thus, for sufficiently small \( \delta z \)

\[
a_1 > r_1.
\]

Similarly, for \( q = 2 \), we get

\[
a_2 = (X^{-1})_{2,1} y_3 + (X^{-1})_{2,2} y_2
\]

\[
= \frac{24\delta v}{\sqrt{4 - \delta^2}} \left[ - \left( \frac{2 + \delta z}{2 - \delta} \right) \left( \frac{p_0 p_0}{p_2} \left( \frac{10}{\sqrt{4 - \delta^2}} \right) \right) y_3 + \left( \frac{p_1 p_0}{p_2} \left( \frac{10}{\sqrt{4 - \delta^2}} \right) \right) y_2 \right]
\]

\[
= \frac{24\delta v}{\sqrt{4 - \delta^2}} \left[ \frac{y_2 p_1}{p_2} - \left( \frac{2 + \delta z}{2 - \delta} \right) \frac{y_3}{p_2} \left( \frac{10}{\sqrt{4 - \delta^2}} \right) \right],
\]

\[
\text{for sufficiently small } \delta z.
\]
using \( p_0 = 1 \), and

\[
r_2 = |W_{2,1}| = |(X^{-1})_{2,1}y_2| + |(X^{-1})_{2,2}y_1|
\]

\[
= \frac{24\delta v}{\sqrt{4 - \delta z^2}} \left[ \left( \frac{2 + \delta z}{2 - \delta z} \right) \frac{p_0 p_0}{p_2} \left( \frac{10}{\sqrt{4 - \delta z^2}} \right) |y_2| + \frac{p_1 p_0}{p_2} \left( \frac{10}{\sqrt{4 - \delta z^2}} \right) |y_1| \right]
\]

\[
= \frac{24\delta v}{\sqrt{4 - \delta z^2}} \left[ \left( \frac{2 + \delta z}{2 - \delta z} \right) \frac{y_2}{p_2} - \frac{y_1 p_1}{p_2} \right] \left( \frac{10}{\sqrt{4 - \delta z^2}} \right).
\]

From above we have,

\[
(a_2 - r_2) = \left( \frac{24\delta v}{\sqrt{4 - \delta z^2}} \right) \left[ \frac{y_2 p_1}{p_2} - \left( \sqrt{\frac{2 + \delta z}{2 - \delta z}} \right) \frac{y_2}{p_2} - \left( \sqrt{\frac{2 + \delta z}{2 - \delta z}} \right) \frac{y_1 p_1}{p_2} \right] \left( \frac{10}{\sqrt{4 - \delta z^2}} \right).
\]

(21)

As \( \delta z \to 0 \), (21) becomes

\[
\left[ (y_2 + y_1) \frac{p_1}{p_2} - (y_3 + y_2) \frac{1}{p_2} \right] (5) = \left[ \left( \frac{c}{2\delta v} \left( b + \frac{\delta v}{12} \right) \right) \frac{p_1 - 1}{p_2} \right] (5) \xrightarrow{b \to \infty} \infty
\]

as \( c, p_1(5) - 1, p_2(5) \) are all positive. On similar lines of \( q = 1 \) case, we have

\[
a_2 > r_2.
\]

(22)

for sufficiently small \( \delta z \). The result follows from (20) and (22). \( \square \)

**Remark 8.** In the above Lemma \[4\] it is observed for \( N = 3 \) that only two terms are present in the expressions of the center and the radius of the Gerschgorin’s disks corresponding to the first and the last rows. However, the above fact is true for any general \( N \). Therefore, if all the eigenvalues of \( W \) lie in the Gerschgorin’s disks corresponding to the first and the last rows, the assertion of Lemma \[7\] holds true for general \( N \).

5. Conclusions and future directions

The application of Gerschgorin circle theorem was facilitated to estimate the eigen values of amplification matrix. A matrix method approach was developed to establish the sta-
bility of Crank-Nicolson compact schemes for convection-diffusion equations. As a future work, the proposed methodology may be investigated for variable coefficient problems, multi-
dimensional problems, and for system of PDEs etc.

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