ON $L^\infty$ ESTIMATES FOR FULLY NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS ON HERMITIAN MANIFOLDS

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Abstract

Sharp $L^\infty$ estimates are obtained for general classes of fully non-linear PDE’s on Hermitian manifolds. The method builds on the method of comparison with auxiliary Monge-Ampère equations introduced earlier in the Kähler case by the authors in joint work with F. Tong, but uses this time auxiliary equations on open balls with a Dirichlet condition. The role of Yau’s theorem in the compact Kähler case is now played by the theorem of Caffarelli-Kohn-Nirenberg-Spruck for the Dirichlet problem. Global estimates are then deduced by combining arguments of Blocki with exponential estimates due to Kołodziej generalizing the classic inequality of Brezis-Merle. The method applies not just to compact Hermitian manifolds, but also to the Dirichlet problem, to open manifolds with a lower bound on their injectivity radii, and to $(n-1)$-form Monge-Ampère equations.

1 Introduction

A priori estimates are essential to the understanding of partial differential equations, and of these, $L^\infty$ estimates are of particular importance, as they are usually also required for other estimates. This is most evident in the case of Monge-Ampère equations in Kähler geometry, where $L^\infty$ estimates were first obtained by Yau [39] in his solution of the Calabi conjecture, and a sharp version was subsequently obtained by Kołodziej [27] using pluripotential theory. Very recently, an alternative proof of Kołodziej’s estimates using only PDE methods was provided in [20], which also gave new and sharp estimates for general classes of fully non-linear equations in Kähler geometry.

The goal of the present paper is to obtain sharp $L^\infty$ estimates for fully non-linear equations in the more general setting of Hermitian manifolds. There is a strong incentive to consider non-Kähler settings, as they appear increasingly frequently in complex geometry, symplectic geometry, and theoretical physics (see e.g. [15, 16, 17, 28, 30, 32] for surveys). In doing so, we shall still make use of the basic ideas in [20], namely comparisons with auxiliary complex Monge-Ampère equations and the resulting Trudinger inequalities. However, a significant new difficulty arises in the absence of the Kähler property: the auxiliary equations used in [20] are global equations which may not have a solution in this case, as complex Monge-Ampère equations on compact Hermitian manifolds are only solvable up to an undetermined constant [7, 35].

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The key to our approach for general Hermitian manifolds is still comparisons with auxiliary complex Monge-Ampère equations, but this time not on a compact complex manifold, but rather on an open ball with a Dirichlet boundary condition. The existence and regularity of the Monge-Ampère equation which we then need is a classic result of Caffarelli, Kohn, Nirenberg, and Spruck [3], which plays the same role for the Dirichlet problem as Yau’s existence theorem [39] did in [20] for the compact Kähler case. An important point is that the desired global estimates can still be obtained from the local estimates obtained in this manner near a minimum. This requires an adaptation of the original arguments of Blocki [1], combined with weak Harnack inequalities for supersolutions of linear elliptic equations [18], as well as an estimate of Kolodziej [27] which generalizes the one-dimensional estimate of Brezis-Merle [2].

The resulting method for \( L^\infty \) estimates, using only comparisons with a local equation, turns out to be both flexible and powerful. It bypasses the complicated integration by parts plaguing non-Kähler geometry and it applies to general classes of fully non-linear equations, not just on compact Hermitian manifolds, but also to the Dirichlet problem, to open manifolds under appropriate conditions, and to \( (n-1) \)-form Monge-Ampère equations.

We describe now more fully our results. Let \( (X, \omega) \) be a connected complex manifold of complex dimension \( n \), and \( \omega \) be a Hermitian metric. Let \( f : \Gamma \subset \mathbb{R}^n \to \mathbb{R}_+ \) be a nonlinear operator satisfying:

1. \( \Gamma \subset \mathbb{R}^n \) is a symmetric cone with
   \[
   \Gamma_n \subset \Gamma \subset \Gamma_1; \quad (1.1)
   \]
   Here \( \Gamma_k \) is the cone of vectors \( \lambda \) with \( \sigma_j(\lambda) > 0 \) for \( 1 \leq j \leq k \), where \( \sigma_k(\lambda) \) is the \( j \)-th symmetric polynomial in \( \lambda \). In particular, \( \Gamma_1 \) is the half-space defined by \( \lambda_1 + \cdots + \lambda_n > 0 \), and \( \Gamma_n \) is the first octant, defined by \( \lambda_j > 0 \) for \( 1 \leq j \leq n \).
2. \( f(\lambda) \) is symmetric in \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma \) and it is homogeneous of degree one;
3. \( \frac{\partial f}{\partial \lambda_j} > 0 \) for each \( j = 1, \ldots, n \) and \( \lambda \in \Gamma \);
4. There is a \( \gamma > 0 \) such that
   \[
   \prod_{j=1}^n \frac{\partial f}{\partial \lambda_j} \geq \gamma, \quad \forall \lambda \in \Gamma. \quad (1.2)
   \]

For any \( C^2 \) function \( \varphi : X \to \mathbb{R} \), we set \( \omega_\varphi = \omega + i\partial \bar{\partial} \varphi \) and let \( h_\varphi : TX \to TX \) be the endomorphism defined by \( \omega_\varphi \) relative to \( \omega \), i.e. in local coordinates, \( (h_\varphi)^j_i = g^{jk}(g_\varphi)_{ki} \), where \( g_{ji} \) denotes the components of \( \omega \) and \( (g^{jk}) \) its inverse. We denote by \( \lambda[h_\varphi] \) the (unordered) vector of eigenvalues of \( h_\varphi \).

Examples of such nonlinear operators \( f(\lambda) \) include:

- The complex Monge-Ampère operator: \( f(\lambda[h_\varphi]) = \left( \frac{\omega_\varphi^n}{\omega^n} \right)^{1/n}, \Gamma = \Gamma_n \)
- The complex Hessian operator [4, 13, 37]: \( f(\lambda[h_\varphi]) = \left( \frac{\omega_\varphi^k \wedge \omega^{n-k}}{\omega^n} \right)^{1/k}, \Gamma = \Gamma_k \).

2
The $p$-Monge-Ampère operator of Harvey and Lawson [25, 26]:

$$f(\lambda) = \left( \prod_{I} \lambda_{I} \right)^{\frac{n!}{(n-p)!p!}}$$

where $I$ runs over all distinct multi-indices $1 \leq i_{1} < \cdots < i_{p} \leq n$, $\lambda_{I} = \lambda_{i_{1}} + \cdots + \lambda_{i_{p}}$, and $\Gamma$ is the cone defined by $\lambda_{I} > 0$ for all $p$-indices $I$.

Clearly, any finite linear combination with positive coefficients of operators $f(\lambda)$ satisfying the above conditions (1-4) again satisfies the same conditions, on the cone defined as the intersection of the cones corresponding to each individual operator.

Given a smooth function $F$ on $X$, we consider the fully nonlinear partial differential equation

$$f(\lambda[h_{\varphi}]) = e^{F}, \text{ and } \lambda[h_{\varphi}] \in \Gamma.$$  \hspace{1cm} (1.3)

**Theorem 1** Assume that $(X, \omega)$ is a compact Hermitian manifold without boundary, and consider the equation (1.3) with the operator $f(\lambda)$ satisfying the conditions (1-4). Then for any $p > n$ and any $C^{2}$ solution $\varphi$ to (1.3), normalized to satisfy $\sup_{X} \varphi = 0$, we have

$$\sup_{X} |\varphi| \leq C,$$ \hspace{1cm} (1.4)

for some constant $C > 0$ depending only on $X, \omega, n, p, \gamma$, and $\|e^{nF}\|_{L^{1}(\log L)^{p}}$. Here the norm $\|e^{nF}\|_{L^{1}(\log L)^{p}}$ is defined by

$$\|e^{nF}\|_{L^{1}(\log L)^{p}} = \int_{X} (1 + |F|^{p})e^{nF}\omega^{n}$$

and has a natural interpretation as the $p$-th entropy of the function $e^{nF}$.

We survey briefly the previous known results related to this theorem. The case $(X, \omega)$ Kähler was established in [20], including even a stronger version allowing degenerations of the underlying Kähler class $\omega$ generalizing the case of the Monge-Ampère equation obtained in [14, 10]. So we concentrate on the case of $(X, \omega)$ non-Kähler. In this case, for the Monge-Ampère equation, $C^{0}$ estimates in the non-Kähler case were first obtained by Tosatti-Weinkove [35], building on earlier works of Cherrier [7], and a pointwise upper bound for $e^{nF}$ was needed. The stronger version requiring only weaker entropy bounds as described in Theorem 1 was first obtained by Dinew and Kolodziej [11, 12, 13, 27], using a non-trivial extension of pluripotential theory to the Hermitian setting. More recently a new approach using the theory of envelopes has been developed by Guedj and Lu [19]. The proof of Theorem 1 is arguably simpler than in all these approaches. More importantly, it is a PDE-based proof, and as such, it can apply to many more equations. The estimates in Theorem 1 appear to be the first $C^{0}$ estimates obtained in any generality for fully non-linear equations on compact Hermitian manifolds.
It may be worth noting an intriguing consequence of Theorem 1, or rather of its proof, as we shall see later in Section §3. Given a $C^2$ function $\varphi$ such that $\omega_\varphi = \omega + i\partial\bar{\partial}\varphi > 0$ defines a Hermitian metric on $X$, in general the following identity does not hold

$$\int_X (\omega + i\partial\bar{\partial}\varphi)^n = \int_X \omega^n,$$

unless extra conditions [11] are put on $\omega$. So it is interesting to estimate the lower/upper bound of the volume of $(X, \omega_\varphi)$, i.e. the integral $\int_X \omega_\varphi^n$. To this end, we define the relative volume $e^{nF}$ of $\omega_\varphi$ with respect to $\omega$ by $e^{nF} = \omega_\varphi^n/\omega^n$. Then $\varphi$ can be viewed as the solution of a complex Monge-Ampère equation:

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^{nF}\omega^n. \tag{1.5}$$

Normalizing $\varphi$ so that $\sup_X \varphi = 0$, we can apply Theorem 1 and its proof and obtain:

**Corollary 1** Given $p > n$ and $K > 0$, suppose $e^{nF}$ satisfies $\int_X e^{nF}|F|^p\omega^n \leq K$, then there is a constant $c > 0$ which depends on $n, \omega, p$ and $K$ such that

$$\int_X (\omega + i\partial\bar{\partial}\varphi)^n = \int_X e^{nF}\omega^n \geq c. \tag{1.6}$$

Thus an upper bound on the $p$-th entropy of relative volume $e^{nF}$ implies a positive lower bound of the $L^1$-norm of $e^{nF}$. This is trivial if $\omega$ is Kähler, but not in general. We remark that if $X$ admits a closed real $(1,1)$-form $\beta$ whose Bott-Chern class $[\beta] \in H^{1,1}_{BC}(X, \mathbb{C})$ is nef and big (i.e. $X$ is of Fujiki class $C$), by a weak transcendental Morse inequality of Demailly [9], equation (1.6) holds even without the assumption on $\int_X e^{nF}|F|^p\omega^n$. This is based on an observation of Tosatti [34]. For the convenience of readers, we provide a proof (see Lemma 8 in Section §3) of this fact using the arguments in [9, 34].

The strategy of the proof of Theorem 1 is to employ a local auxiliary complex Monge-Ampère equations as in [20] and a localized argument similar to that in [1]. It is now known that auxiliary Monge-Ampère equations on compact closed manifolds can be particularly effective, with recent major successes in the constant scalar curvature metric problem [5], the $L^\infty$ estimates in the Kähler case [20, 21, 22, 23], the corresponding treatment of parabolic equations [6], and applications to lower bounds for Green’s functions [24]. However, the use of a local auxiliary Monge-Ampère equation makes the method readily applicable to many other settings. In effect, the estimates are practically local, and we bypass integrations by parts, which lead to many complications when the torsion is not 0. We illustrate this by applying the local comparison method to the Dirichlet problem for manifolds with boundary, the case of open manifolds, and the $(n - 1)$-form Monge-Ampère equation.
We describe first the case of the Dirichlet problem. In this case, we let $X$ be an open subset with smooth boundary in a larger Hermitian manifold $(\tilde{X}, \omega)$. We consider the following Dirichlet boundary value problem on $X$:

$$f(\lambda[h_\varphi]) = e^F \quad \text{in } X, \quad \lambda[h_\varphi] \in \Gamma, \quad (1.7)$$

and $\varphi = \rho$ on $\partial X$ for some $\rho \in C^0(\partial X)$. Here $f$ is the nonlinear operator as in Theorem 1 and we require $\lambda[h_\varphi] \in \Gamma$ in $X$.

**Theorem 2** Given $p > n$, suppose $\varphi \in C^2(X) \cap C^0(\overline{X})$ is a solution to the equation (1.7). Then there is a constant $C > 0$ which depends only on $X, \omega, n, p, \gamma$ and, $\|e^{nF}\|_{L^1(\log L)^p}$, such that

$$\sup_X |\varphi| \leq \|\rho\|_{L^\infty(\partial X)} + C(1 + \|\varphi\|_{L^2}^2). \quad (1.8)$$

Furthermore, for complex Monge-Ampère and Hessian equations, if $\omega$ is Kähler, we can bound $\|\varphi\|_{L^1}$ in terms of other parameters.

Next, we consider the case of a noncompact complex manifold $(X, \omega)$, where $\omega$ is a complete Hermitian metric on $X$. We shall make the following assumption:

(A) There exists a constant $r_0 > 0$, so that, at each point $x \in X$, there exists a holomorphic coordinates chart $(U_x, z)$ centered at $x$ satisfying

$$\{|z| < 2r_0 \} \subset U_x, \quad \frac{1}{2} \omega \leq i\partial\bar{\partial}|z|^2 \leq 2\omega \quad \text{in } U_x. \quad (1.9)$$

With the assumption (1.9), we introduce two quantities associated to $\varphi$ and $e^F$, which we assume are bounded. Given a fixed $p > n$, we set

$$K_\varphi = \sup_{x \in X} \int_{U_x} |\varphi|^n \omega^n, \quad K_F = \sup_{x \in X} \int_{U_x} e^{nF} |F|^p \omega^n. \quad (1.10)$$

**Theorem 3** Let $(X, \omega)$ be a noncompact Hermitian manifold satisfying the assumption (A). Let $\varphi \in L^\infty(X) \cap C^2(X)$ be a solution to (1.7). Then there is a constant $C > 0$ which depends on $n, p, \omega, \gamma$ and $K_\varphi, K_F$ such that

$$\sup_X |\varphi| \leq C + \liminf_{z \to \infty} (-\varphi)_+$$

where $C$ is a constant depending on $X, \omega, n, p, \gamma, r_0$, and $K_\varphi, K_F$.

We remark that the assumption (1.9) is satisfied if the metric $\omega$ is Kähler, the Riemannian curvature is bounded and the injectivity radius of $(X, \omega)$ at each point is bounded below by some positive number (see [33]). In particular, it holds for the asymptotically locally Euclidean (ALE) Kähler manifolds. The assumption (1.9) also holds if we put some asymptotic behavior of the (Kähler) metric $\omega$ near infinity, for example, the asymptotically
conical (AC) and asymptotically cylindrical (ACyl) manifolds, which have been extensively studied in recent years to find canonical Kähler metrics.

Finally, we show that the local comparison method developed in this paper can apply to other second-order equations and not just scalar equations of the type (1.3). We illustrate this point, with the derivation of $C^0$ estimates for the $(n-1)$-form Monge-Ampère equation. Thus let $(X,\omega)$ be a compact Hermitian manifold with $\partial X = \emptyset$ and $\omega_h$ be another Hermitian metric on $X$. Suppose $\varphi$ is a $C^2$ function that satisfies the following so-called $(n-1)$-form Monge-Ampère equation

\begin{equation}
\left(\omega_h + \frac{1}{n-1}((\Delta_\omega \varphi)\omega - i\partial\bar{\partial}\varphi)\right)^n = e^F\omega^n, \tag{1.11}
\end{equation}

with $\sup_X \varphi = 0$ and $\omega_h + \frac{1}{n-1}((\Delta_\omega \varphi)\omega - i\partial\bar{\partial}\varphi) > 0$ on $X$. Here

\[\Delta_\omega \varphi = n \frac{i\partial\bar{\partial}\varphi \wedge \omega^{n-1}}{\omega^n}\]

is the complex Laplacian of $\varphi$ with respect to $\omega$. It has been shown by Tosatti-Weinkove [36] that, when $\omega$ is Kähler, the equation is solvable if $F$ is modified by a suitable additive constant. We refer the interested readers to [36] and references therein for more background on this equation. We are mainly interested in the $C^0$ a priori estimate of $\varphi$ satisfying (1.11).

**Theorem 4** Given $p > n$, there exists a constant $C > 0$ which depends on $n, \omega, p, \omega_h$ and $\|e^F\|_{L^1(\log L)^p(X,\omega)}$ such that

\[\sup_X |\varphi| \leq C.\]

As stressed in [36], the $C^0$ estimate is actually one of the main steps in the study of this equation. Our $C^0$ estimate improves on the one in [36] in at least two directions: it does not require that $\omega$ be Kähler, and it weakens the condition on $F$ from a pointwise to an $L^1(\log L)^p$ condition.

## 2 The key local estimate

Fix the Hermitian manifold $(X,\omega)$. We assume that there exists $r_0 > 0$ so that, at any point $x_0 \in X$, there exists a local holomorphic coordinate system $z$ centered at $x_0$, with $\omega(x_0) = i \sum_{j=1}^n dz^j \wedge d\bar{z}^j$ and

\[\frac{1}{2} |i\partial\bar{\partial}|z|^2 \leq \omega \leq 2i\partial\bar{\partial}|z|^2 \text{ in } B(x_0, 2r_0) \tag{2.1}\]

where $B(x_0, 2r_0)$ is the Euclidian ball of radius $2r_0$. When $X$ is compact, with or without boundary, then such an $r_0$ clearly exists, and can be viewed as determined by $X$ and $\omega$. Any dependence on $r_0$ can be absorbed into a dependence on $X$ and $\omega$. When $X$ is not compact, the existence of such an $r_0 > 0$ is an assumption that we always make. Without loss of generality we may assume $r_0 \leq 1/2$ throughout the paper.
Theorem 5 Let $\varphi$ be a $C^2$ solution on a Hermitian manifold $(X, \omega)$ of the equation
\[ f(\lambda[h, \varphi]) = e^F \quad (2.2) \]
where the operator $f(\lambda)$ satisfies the conditions (1-4) spelled out in Section §1. Let $x_0$ be any point in $X$, and assume that
\[ \varphi(z) \geq \varphi(x_0) \text{ for } z \in B(x_0, 2r_0). \quad (2.3) \]
Fix any $p > n$. Then we have
\[ -\varphi(x_0) \leq C \quad (2.4) \]
where $C$ is a constant depending only on $(X, \omega)$, $p$, $\gamma$, $\|e^n F\|_{L^1(\log L)^p}$, and $\|\varphi\|_{L^1}$. Here the $L^1$ and $L^1(\log L)^p$ norms are on the ball $B(x_0, 2r_0)$ with respect to the volume form $\omega^n$.

We will assume $-\varphi(x_0) \geq 2$, otherwise Theorem 5 already holds.

We break the proof into several steps. The first step is the most important, and requires a comparison of the solution $\varphi$ of (2.2) with the solution of an auxiliary Monge-Ampère equation. More precisely, set
\[ u_s(z) = \varphi(z) - \varphi(x_0) + \frac{1}{2}|z|^2 - s \quad (2.5) \]
for each $s$ with $0 < s < 2r_0^2$. It is convenient to set $\Omega = B(x_0, 2r_0)$. Then $u_s > 0$ on $\partial \Omega$, and thus the set $\Omega_s$ defined by
\[ \Omega_s = \{z \in \Omega; u_s(z) < 0\} \quad (2.6) \]
is an open and nonempty set with compact closure in $\Omega$. Set
\[ A_s = \int_{\Omega_s} (-u_s)e^n F \omega^n. \quad (2.7) \]
Formally, the auxiliary equation which we would like to consider is the following Dirichlet problem for the complex Monge-Ampère equation on the ball $\Omega$,
\[ (i\bar{\partial}\partial \psi_s)^n = \frac{(-u_s) \chi_{R^+}(x)}{A_s} e^n F \omega^n \text{ in } \Omega, \quad \psi_s = 0 \text{ on } \partial \Omega, \quad (2.8) \]
where $\chi_{R^+}(x)$ is the characteristic function of the positive real axis, and $\psi_s$ is required to be plurisubharmonic, $i\bar{\partial}\partial \psi_s \geq 0$. Since the right hand side would have singularities, we choose a sequence of smooth positive functions $\tau_k(x) : \mathbb{R} \rightarrow \mathbb{R}_+$ which converges to the function $x \cdot \chi_{R^+}(x)$ as $k \rightarrow \infty$. Let then the function $\psi_{s,k}$ be defined as the solution of the following Dirichlet problem
\[ (i\bar{\partial}\partial \psi_{s,k})^n = \frac{\tau_k(-u_s)}{A_{s,k}} e^n F \omega^n \text{ in } \Omega, \quad \psi_{s,k} = 0 \text{ on } \partial \Omega \quad (2.9) \]
with $i\partial\bar{\partial}\psi_{s,k} \geq 0$, and $A_{s,k}$ is defined by

$$A_{s,k} = \int_{\Omega} \tau_k(-u_s)e^{nF}\omega^n. \quad (2.10)$$

By the classic theorem of Caffarelli-Kohn-Nirenberg-Spruck [3], this Dirichlet problem admits a unique solution $\psi_{s,k}$ which is of class $C^\infty(\bar{\Omega})$. The maximum of $\psi_{s,k}$ can only be attained in $\partial\Omega$, and thus $\psi_{s,k} \leq 0$. By definition of the constants $A_{s,k}$, we also have $A_{s,k} \to A_s$ as $k \to \infty$, and

$$\int_{\Omega} (i\partial\bar{\partial}\psi_{s,k})^n = 1. \quad (2.11)$$

**Lemma 1** Let $u_s$ be a $C^2$ solution of the fully non-linear equation (2.2) and $\psi_{s,k}$ be the solutions of the complex Monge-Ampère equation (2.9) as defined above. Then we have

$$-u_s \leq \varepsilon(-\psi_{s,k})^{\frac{n}{n+1}} \text{ on } \bar{\Omega} \quad (2.12)$$

where $\varepsilon$ is the constant defined by $\varepsilon^{n+1} = A_{s,k}\gamma^{-1}\frac{(n+1)^n}{n^n}$. \rightarrow

**Proof.** We have to show that the function

$$\Phi = -\varepsilon(-\psi_{s,k})^{\frac{n}{n+1}} - u_s \quad (2.13)$$

is always $\leq 0$ on $\bar{\Omega}$. Let $x_{\text{max}} \in \bar{\Omega}$ be a maximum point of $\Phi$. If $x_{\text{max}} \in \bar{\Omega}\setminus\Omega_s$, clearly $\Phi(x_{\text{max}}) \leq 0$ by the definition of $\Omega_s$ and the fact that $\psi_{s,k} < 0$ in $\Omega$. If $x_{\text{max}} \in \Omega_s$, then we have $i\partial\bar{\partial}\Phi(x_{\text{max}}) \leq 0$ by the maximum principle.

Let $G^{\bar{j}} = \frac{\partial \log f(\lambda[h])}{\partial h_{ij}} = \frac{1}{f} \frac{\partial f(\lambda[h])}{\partial h_{ij}}$ be the linearization of the operator $\log f(\lambda[h])$. It follows from the structure conditions of $f$ that $G^{\bar{j}}$ is positive definite at $h_\varphi$ and

$$\det G^{\bar{j}} = \frac{1}{f^n} \det(\frac{\partial f(\lambda[h])}{\partial h_{ij}}) \geq \frac{\gamma}{f(\lambda)^n}. \rightarrow$$

The assumption that $f(\lambda)$ is homogeneous of degree 1 implies that

$$\text{tr}_G \omega_\varphi = \sum_{i,j} G^{\bar{j}}(\omega_\varphi)_{ji} = 1. \rightarrow$$

We calculate at the point $x_{\text{max}}$.

\begin{align*}
0 & \geq (\text{tr}_G i\partial\bar{\partial}\Phi)(x_{\text{max}}) = \sum_{i,j} G^{\bar{j}} \Phi_{ji} \\
& = \frac{n\varepsilon}{n+1} (-\psi_{s,k})^{\frac{1}{n+1}} (\text{tr}_G i\partial\bar{\partial}\psi_{s,k}) + \frac{\varepsilon n}{(n+1)^2} (-\psi_{s,k})^{\frac{n+2}{n+1}} |\nabla\psi_{s,k}|_G^2
\end{align*}
\[-\text{tr}_G \omega + \text{tr}_G (\omega - \frac{1}{2} i \partial \bar{\partial} |z|^2) \geq \frac{n \varepsilon}{n + 1} (-\psi_{s,k})^{- \frac{1}{n+1}} (\text{tr}_G i \partial \bar{\partial} \psi_{s,k}) - 1 \]
\[\geq \frac{n^2 \varepsilon}{n + 1} (-\psi_{s,k})^{- \frac{1}{n+1}} (\det G)^{1/n} [\det (\omega^{-1} \cdot i \partial \bar{\partial} \psi_{s,k})]^{1/n} - 1 \]
\[\geq \frac{n^2 \varepsilon}{n + 1} \gamma^{1/n} (-\psi_{s,k})^{- \frac{1}{n+1}} (-u_s)^{1/n} A_{s,k}^{1/n} - 1. \]

Here in the fourth line we use (2.1) and in the fifth line we apply the arithmetic geometric inequality. Hence at \(x_{\text{max}}\), we have
\[-u_s \leq A_{s,k} \gamma^{-1} \frac{(n + 1)^n}{n^{2n} \varepsilon^n} (-\psi_{s,k})^{\frac{n}{n+1}} = \varepsilon (-\psi_{s,k})^{\frac{n}{n+1}}, \]
by the choice of \(\varepsilon\). This implies \(\Phi(x_{\text{max}}) \leq 0\), and the claim is proved. Q.E.D.

Once we have a comparison between \(u_s\) and \(\psi_{s,k}\) as given in Lemma 1, we no longer need to know the differential equations satisfied by \(u_s\) and \(\psi_{s,k}\). It is most convenient to summarize then the remaining part of the proof of Theorem 5 in a lemma of general applicability:

**Lemma 2** Let \(0 < s < s_0 = 2r_0^2\). Assume that we have functions \(u_s, u_s > 0\) on \(\partial \Omega\), and let \(\Omega_s, A_s, A_{s,k}\) be the corresponding notions as defined by. Assume that the inequality (2.12) holds, that is,
\[-u_s \leq C(n, \gamma) A_{s,k}^{1/n} (-\psi_{s,k})^{\frac{n}{n+1}} \text{ on } \bar{\Omega} \]
for some constant \(C(n, \gamma)\), where \(\psi_{s,k}\) are plurisubharmonic functions on \(\Omega\), \(\psi_{s,k} = 0\) on \(\partial \Omega\), and \(f_\Omega(i \partial \bar{\partial} \psi_{s,k})^n = 1\). Then for any \(p > n\), we have
\[-\varphi(x_0) \leq C(n, \omega, \gamma, p, \|\varphi\|_{L^1(\Omega, \omega^n)}). \]

Clearly, Theorem 5 follows immediately from Lemma 1 and Lemma 2, so we devote the remaining part of this section to the proof of Lemma 2.

The main property of the functions \(\psi_{s,k}\) in Lemma 2 that we need is an exponential integral estimate, which can be stated as follows. Let \(\Omega \subset C^n\) be a bounded pseudoconvex domain. Then there are constants \(\alpha > 0\) and \(C > 0\) that depend only on \(n\) and \(\text{diam}(\Omega)\) such that
\[\int_{\Omega} e^{-\alpha \psi} dV \leq C, \]
for any \(\psi \in \text{PSH}(\Omega) \cap C^2(\Omega) \cap C^0(\bar{\Omega})\) with \(\psi|_{\partial \Omega} \equiv 0\), and \(f_\Omega(i \partial \bar{\partial} \psi)^n = 1\). This exponential estimate is due to Kolodziej [27], and it is a generalization of the inequality of Brezis-Merle
[2] in complex dimension one. While it has some similarity with the $\alpha$-invariant in Kähler geometry, it is a very different estimate as it requires information on the Monge-Ampère volume of $\psi$. It was first proved in [27] using pluripotential theory. Recently an elementary and elegant proof was given in [38] which in particular does not rely on pluripotential theory.

Returning to the proof of Lemma 2, we rewrite the comparison between $u_s$ and $\psi_{s,k}$ as

$$
\frac{(-u_s)^{(n+1)/n}}{A_{s,k}^{1/n}} \leq C(n, \gamma)(-\psi_{s,k}), \quad \text{in } \Omega_s.
$$

Take a small $\beta = \beta(n, \alpha, \gamma) > 0$ such that $\beta C(n, \gamma) \leq \alpha$ where $\alpha > 0$ is as in (2.16). We then get

$$
\int_{\Omega_s} \exp \left( \frac{(-u_s)^{(n+1)/n}}{A_{s,k}^{1/n}} \right) \omega^n \leq \int_{\Omega} \exp \left( \beta C(n, \gamma)(-\psi_{s,k}) \right) \omega^n \leq C,
$$

(2.17)

by the uniform integral estimate (2.16). We can now take the limit $k \to \infty$ and obtain the following exponential estimate for $u_s$,

**Lemma 3** For any $0 < s < s_0$, the functions $u_s$ satisfy the following inequality

$$
\int_{\Omega_s} \exp \left\{ \beta \frac{(-u_s)^{(n+1)/n}}{A_{s,k}^{1/n}} \right\} \omega^n \leq C
$$

(2.18)

where $\beta = \beta(n, \gamma, r_0)$ and $C = C(n, \gamma, r_0)$ are strictly positive constants depending only on $n, \gamma$ and $r_0$.

The next step is to show that this exponential inequality for $u_s$ implies the following estimate for $A_s$:

**Lemma 4** Fix $p > n$, and let $\delta_0 = \frac{p-n}{pm} > 0$. Then we have

$$
A_s \leq C_0(\int_{\Omega_s} e^{nF} \omega^n )^{1+\delta_0}
$$

(2.19)

where $C_0$ is a constant depending only on $n, p, \gamma, \omega$ and $\|e^{nF}\|_{L^1(\log L)^p}$.

**Proof.** As in [20], we apply Young’s inequality. For our purposes, it is convenient to state Young’s inequality as follows. Let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive strictly increasing function with $\lim_{x \to 0^+} \eta(x) = 0$. Let $\eta^{-1}$ be the inverse function of $\eta$, so that $\eta(\eta^{-1}(y)) = y$ and $\eta^{-1}(\eta(x)) = x$. Then for any $U, V \geq 0$, we have

$$
UV \leq \int_0^U \eta(x)dx + \int_0^V \eta^{-1}(y)dy.
$$

(2.20)
If we make a change of variables $y = \eta(x)$ in the second integral, this inequality can be rewritten as

$$UV \leq \int_0^U \eta(x) dx + \int_0^{\eta^{-1}(V)} x\eta'(x) dx \tag{2.21}$$

The first integral on the right hand side can be estimated by $U\eta(U)$, while the second integral can be estimated by

$$\int_0^{\eta^{-1}(V)} x\eta'(x) dx \leq \eta^{-1}(V) \int_0^{\eta^{-1}(V)} \eta'(x) dx = \eta^{-1}(V)\eta(\eta^{-1}(V)) = V\eta^{-1}(V). \tag{2.22}$$

Thus we arrive at the following inequality, which suffices for our purposes,

$$UV \leq U\eta(U) + V\eta^{-1}(V). \tag{2.23}$$

We return to the proof of Lemma 4. We apply the preceding Young’s inequality with $\eta(x) = (\log(1 + x))^p$, $U = e^{nF(z)}$, and a general positive function $V$. Then $\eta^{-1}(y) = \exp(y^{\frac{1}{p}}) - 1$, and we find

$$e^{nF}V \leq e^{nF}(\log(1 + e^{nF}))^p + V\{\exp(V^{\frac{1}{p}}) - 1\}. \tag{2.24}$$

Specializing further to $V = v(z)^p$ for a general non-negative function $v(z)$, we obtain

$$v(z)^pe^{nF} \leq e^{nF}(\log(1 + e^{nF}))^p + v^p(e^v - 1) \leq e^{nF}(1 + n|F|)^p + C_p e^{2v(z)}. \tag{2.25}$$

Let now

$$v(z) = \frac{\beta (u_s)^{(n+1)/n}}{2 A_s^{1/n}} \tag{2.26}$$

and integrate over $\Omega_s$ to obtain

$$\left(\frac{\beta}{2}\right)^p \int_{\Omega_s} \frac{(-u_s)^{(n+1)/p/n}}{A_s^{p/n}} e^{nF} \omega^n \leq \int_{\Omega_s} e^{nF}(1 + n|F|)^p \omega^n + \int_{\Omega_s} \exp\left(\frac{(-u_s)^{(n+1)/n}}{A_s^{1/n}}\right) \omega^n. \tag{2.27}$$

By Lemma 4, the right hand side is bounded by a constant $C(n, p, \beta, \|e^{nF}\|_{L^1(\log L)^p}, r_0)$. Thus we have

$$\int_{\Omega_s} (-u_s)^{\frac{p(n+1)}{n}} e^{nF} \omega^n \leq C(n, p, \beta, r_0) A_s^\frac{p}{n}. \tag{2.28}$$

This implies the following upper bound for $A_s$: by Hölder’s inequality, we can write

$$A_s = \int_{\Omega_s} (-u_s) e^{nF} \omega^n \leq \left( \int_{\Omega_s} (-u_s)^{p(n+1)/n} e^{nF} \omega^n \right)^{n/(p(n+1))} \cdot \left( \int_{\Omega_s} e^{nF} \omega^n \right)^{1/q} \leq A_s^{1/(n+1)} \cdot \left( \int_{\Omega_s} e^{nF} \omega^n \right)^{1/q}$$
where \( q > 1 \) is chosen such that \( \frac{1}{q} + \frac{n}{p(n+1)} = 1 \). This is the inequality to be proved in Lemma 4. Q.E.D.

We can now derive growth estimates for the function \( \phi(s) \) defined by

\[
\phi(s) = \int_{\Omega_s} e^{nF} \omega^n \tag{2.29}
\]

following the original strategy of De Giorgi [8]. We shall actually be interested in estimates as \( s \to 0^+ \), so the version relevant to our set-up is the one provided by [27].

**Lemma 5** The function \( \phi(s) \) is a monotone increasing function of \( s \in (0, s_0) \) which satisfies \( \lim_{s \to 0^+} \phi(s) = 0 \), \( \phi(s) > 0 \) for \( s \in (0, s_0) \), and

\[
t \phi(s - t) \leq C_0 \phi(s)^{1+\delta_0}, \quad \text{for all } 0 < t < s < s_0. \tag{2.30}
\]

**Proof.** The monotonicity of \( \phi \) is clear from the definition and the relation \( \Omega_s' \subset \Omega_s \) if \( 0 < s' < s < s_0 \). Note that \( x_0 \in \Omega_s \) for any \( s > 0 \). Being a nonempty open set, \( \Omega_s \) has positive measure. So \( \phi(s) > 0 \) for any \( s > 0 \). Furthermore, \( \varphi(z) - \varphi(x_0) + \frac{1}{2} |z|^2 \) is strictly positive in any small punctured neighborhood of \( x_0 \), so we have

\[
\cap_{s > 0} \overline{\Omega_s} = \{x_0\}.
\]

This shows that \( \phi(s) \to 0 \) as \( s \to 0^+ \).

For any \( t \in (0, s) \) and any \( z \in \Omega_{s-t} \subset \Omega_s \), we have

\[
0 > u_{s-t}(z) = u_s(z) + t, \quad \text{i.e. } -u_s(z) > t > 0.
\]

This observation shows that

\[
A_s = \int_{\Omega_s} (-u_s) e^{nF} \omega^n \geq \int_{\Omega_{s-t}} (-u_s) e^{nF} \omega^n \geq t \int_{\Omega_{s-t}} e^{nF} \omega^n. \tag{2.31}
\]

Note that \( \phi(s) = \int_{\Omega_s} e^{nF} \omega^n \) for \( s \in (0, s_0) \), then equations (2.19) and (2.31) imply that

\[
t \phi(s - t) \leq C_0 \phi(s)^{1+\delta_0}, \quad \text{for all } 0 < t < s < s_0. \tag{2.32}
\]

This finishes the proof of Lemma 5. Q.E.D.

**Lemma 6** Let \( \phi(s) \) be a function on \( (0, s_0) \) satisfying all the properties listed in the previous lemma. There is a constant \( c_0 > 0 \), depending only on \( s_0, C_0, \) and \( \delta_0 \) so that

\[
\phi(s_0) \geq c_0. \tag{2.33}
\]
Proof. We apply the iteration argument in [27].

Given the initial number \( s_0 \), we define inductively a decreasing sequence of positive numbers \( \{s_j\}_{j \geq 0} \) as follows: given \( s_j \) let

\[
s_{j+1} = \sup \{0 < s < s_j \mid \phi(s) \leq \frac{1}{2} \phi(s_j) \}.
\]

The continuity of \( \phi(s) \) implies that \( s_{j+1} \) is the largest \( s < s_j \) such that \( \phi(s) = \frac{1}{2} \phi(s_j) \). Thus we have \( \phi(s_{j+1}) = \frac{1}{2} \phi(s_j) \) and \( \frac{1}{2} \phi(s_j) < \phi(s) \leq \phi(s_j) \) for all \( s \in (s_{j+1}, s_j) \). Iterating this identity we infer that

\[
\phi(s_j) = 2^{-j} \phi(s_0), \quad \forall j.
\]

In particular, \( \lim_{j \to \infty} \phi(s_j) = 0 \) and Lemma 5 implies that \( \lim_{j \to \infty} s_j = 0 \), which exists since \( \{s_j\}_{j \geq 0} \) is a bounded decreasing sequence of positive numbers.

From (2.32), we get that for any \( s \in (s_{j+1}, s_j) \)

\[
\frac{1}{2} (s_j - s) \phi(s_j) \leq (s_j - s) \phi(s) \leq C_0 \phi(s_j)^{1 + \delta_0},
\]

so \( s_j - s \leq 2C_0 \phi(s_j)^{\delta_0} \). Letting \( s \to s_{j+1} \) we obtain

\[
-s_{j+1} + s_j \leq 2C_0 \phi(s_j)^{\delta_0} \leq 2C_0 2^{-\delta_0 j} \phi(s_0)^{\delta_0}.
\]

Taking summation over \( j \), we conclude that

\[
s_0 = \sum_{j=0}^{\infty} (s_j - s_{j+1}) \leq \frac{2C_0}{1 - 2^{-\delta_0}} \phi(s_0)^{\delta_0}.
\]

Thus we have

\[
\phi(s_0) \geq c_0
\]

for some constant \( c_0 > 0 \) depending on \( n, \omega, C_0, s_0 \). Q.E.D.

Proof of Theorem 5. Note that on \( \Omega_{s_0} \), \( u_{s_0} = \varphi + \frac{1}{2} |z|^2 - \varphi(x_0) - s_0 < 0 \), that is,

\[
-\varphi - \frac{1}{2} |z|^2 > -\varphi(x_0) - s_0 > 1 \quad \text{on} \quad \Omega_{s_0},
\]

and the last inequality holds from our assumption that \( s_0 < 1/2 \) and \( \varphi(x_0) < -2 \). From (2.35) we deduce that

\[
\log \left( \frac{- \varphi - 2^{-1} |z|^2}{(-\varphi(x_0) - s_0)^{1/2}} \right) > \log (\varphi(x_0) - s_0)^{1/2} > 0 \quad \text{on} \quad \Omega_{s_0}.
\]

Integrating (2.36) against the measure \( e^{nF} \omega^n \) over \( \Omega_{s_0} \), we deduce that

\[
\log (-\varphi(x_0) - s_0)^{1/2} \cdot \phi(s_0) = \log (-\varphi(x_0) - s_0)^{1/2} \int_{\Omega_{s_0}} e^{nF} \omega^n
\]

\[
< \int_{\Omega_{s_0}} \log \left( \frac{- \varphi - 2^{-1} |z|^2}{(-\varphi(x_0) - s_0)^{1/2}} \right) e^{nF} \omega^n
\]

13
We now apply the same Young’s inequality (2.24) as before, but this time with the following choice of the factor $V$,

$$ V = \log \frac{-\varphi - 2^{-1}|z|^2}{(-\varphi(x_0) - s_0)^{1/2}} =: \log W. \quad (2.37) $$

The inequality becomes

$$ (\log W) e^{nF} \leq e^{nF}(\log (1 + e^{nF}))^p + (\log W)\{\exp(\log W)^{\frac{1}{p}} - 1\} \quad (2.38) $$

Fix now $p > 1$. A rough upper bound of the two terms on the right hand side which suffices for our purposes is the following

$$ e^{nF}(\log (1 + e^{nF}))^p \leq e^{nF}(1 + |nF|)^p \quad (2.39) $$

and

$$ (\log W)\{\exp(\log W)^{\frac{1}{p}} - 1\} \leq (\log W)\{\exp(\frac{1}{p}\log W + C_p) - 1\} \leq C_p'(\log W)W^{\frac{1}{p}} \leq C_p'(1 + |W|). \quad (2.40) $$

Substituting back this inequality in the initial inequality (2.37), and estimating the integral over $\Omega_s$ by the full integral over $\Omega$, we find

$$ \log (-\varphi(x_0) - s_0)^{1/2} \cdot \phi(s_0) \leq \|e^{nF}\|_{L^1(\log L)^p} + C_p'' \int_{\Omega} (1 + \frac{-\varphi(z) - \frac{1}{2}|z|^2}{(-\varphi(x_0) - s_0)^{1/2}})\omega^n $$

$$ \leq \|e^{nF}\|_{L^1(\log L)^p} + C_p''(C_1(\omega) + \frac{\|\varphi\|_{L^1(\Omega)} + C_2(\omega)}{(-\varphi(x_0) - s_0)^{\frac{1}{2}}}) $$

In particular,

$$ \log (-\varphi(x_0) - s_0)^{1/2} \cdot \phi(s_0) \leq \frac{1}{2}\max\{\|e^{nF}\|_{L^1(\log L)^p} + C_p''C_1(\omega), C_p''\|\varphi\|_{L^1(\Omega)} + C_2(\omega)\}. $$

Since $\phi(s_0)$ is bounded from below by Lemma 6, it follows immediately that $-\varphi(x_0) - s_0$ is bounded above by a constant depending only on $X, \omega, n, p, \|e^{nF}\|_{L^1(\log L)^p}$ and $\|\varphi\|_{L^1}$. The proof of Theorem 5 is complete. Q.E.D.

We remark that compared to the compact Kähler case in [20], the constant $C > 0$ in Theorem 5 does not depend on the “energy” $E(\varphi) = \int_X (-\varphi)e^{nF}\omega^n$. This is because the function $\psi_{s,k}$ in the auxiliary equation (2.9) is local and plurisubharmonic in $\Omega_s$, while its counterpart in [20] is global and $\omega$-plurisubharmonic. Thus we do not need to shift the function $-\psi_{s,k}$ in (2.13) by a constant depending on $E(\varphi)$. The proof of Theorem 5 is in particular simpler since it does not require an argument for bounding the energy by the entropy as in [20].
3 The case of $X$ compact without boundary

Once we have Theorem 5, it is easy to establish all the other theorems listed in the Introduction. In this section, we begin with the case of $X$ compact without boundary, and give the proof of Theorem 1.

Proof of Theorem 1. Since $X$ is compact without boundary, the function $\varphi(z)$ must attain its minimum at some interior point $x_0$. By Theorem 5, $-\varphi(x_0)$ is bounded above by a constant depending only on $X, \omega, p, n, \gamma, \|e^{nF}\|_{L^1(\log L)^p}$, and $\|\varphi\|_{L^1(B(x_0,2r_0))}$. Thus the proof is complete once we have proved the following lemma giving a bound on the $L^1$-norm of $\varphi$ for which $\lambda[h_\varphi] \in \Gamma \subset \Gamma_1$.

Lemma 7 For any $\varphi \in C^2(X)$ such that $\lambda[h_\varphi] \in \Gamma \subset \Gamma_1$ and $\sup_X \varphi = 0$, there exists a uniform constant $C > 0$ depending only on $n, \omega$ such that

$$\int_X (-\varphi) \omega^n \leq C.$$ (3.1)

Proof. Note that $\lambda[h_\varphi] \in \Gamma_1$ means that $\text{tr}_\omega(\omega + i\partial\bar{\partial}\varphi) > 0$, that is, in local coordinates, the following holds

$$g^{ij} \frac{\partial^2 (-\varphi)}{\partial z_i \partial \bar{z}_j} < n.$$ (3.1)

If $\omega$ is Kähler, the lemma follows easily from the Green’s formula. In general, we will apply local elliptic estimates to $-\varphi$ over a finite cover of $(X, \omega)$. More precisely, we fix three sets of open covers $\{U_\ell\}_{\ell=1}^N$, $\{U'_\ell\}_{\ell=1}^N$ and $\{U''_\ell\}_{\ell=1}^N$, such that $U_\ell \subset U'_\ell \subset U''_\ell$ for each $\ell$. We can also view $U_\ell$ as a Euclidean ball of radius $r_\ell$ and $U'_\ell, U''_\ell$ as concentric balls with radii $2r_\ell, 3r_\ell$, respectively. The choice of these covers is fixed and depends only on $(X, \omega)$.

Assume $\sup_X \varphi = 0 = \varphi(p_1)$ is achieved at some point $p_1 \in U_1 \subset U'_1$. We can apply the standard elliptic estimate (see e.g. Theorem 8.18 in [18]) to the equation of $(-\varphi)$ as in (3.1) on $U'_1 \subset U''_1$, to conclude

$$\int_{U'_1} (-\varphi) \omega^n \leq C[(-\varphi)(p_1) + 1] = C_1,$$ (3.2)

for some constant $C_1 > 0$ depending on $U'_1 \subset U''_1$ and $\omega$ only. If $U'_1 \cap U'_2 \neq \emptyset$, then from (3.2) we get

$$\int_{U'_1 \cap U'_2} (-\varphi) \omega^n \leq C_1,$$

which by mean-value theorem implies that there is some point $p_2 \in U'_1 \cap U'_2$ such that $-\varphi(p_2) \leq C'_1$ for some $C'_1 > 0$. We can now apply again the elliptic estimate to $(-\varphi)$ which satisfies (3.1) in $U'_2 \subset U''_2$ to conclude that

$$\int_{U'_2} (-\varphi) \omega^n \leq C[(-\varphi)(p_2) + 1] \leq C_2,$$
for some uniform constant $C_2 > 0$. We can repeat this argument finitely many times, to deduce that $\int_{U_\ell} (-\varphi) \omega^n \leq C_F$ for each $\ell$, since $\{U_\ell\}_{\ell=1}^N$ is an open cover of $(X, \omega)$, which is assumed to be connected. From this, it is easy to see the bound of $L^1$-norm of $(-\varphi)$. Lemma 7 is proved.

The proof of Theorem 5 already contains a proof of Corollary 1. More precisely:

**Proof of Corollary 1.** By definition, $\varphi$ is a solution on $X$ of the complex Monge-Ampère equation

$$f(\lambda[h_\varphi]) = \left(\frac{\omega^n_\varphi}{\omega^n}\right)^{1/n} = e^F$$

which satisfies the conditions (1-4) described in the Introduction. Let $x_0$ be a point in $X$ where $\varphi$ attains its minimum. Then Lemmas 1-5 apply, implying that there exists a constant $c_0 > 0$ depending only on $n, p, \omega$, and $K$ such that

$$\phi(s_0) = \int_{\Omega_{s_0}} e^{nF} \omega^n \geq c_0.$$  

It is clear that $\phi(s_0) \leq \int_X (\omega + i\partial \bar{\partial} \varphi)^n$. Q.E.D.

We now provide a proof of the inequality (1.6), without the extra assumption on the $p$-th entropy of $e^{nF}$, but under an additional condition that $X$ admits a smooth closed real $(1,1)$-form $\beta$ whose Bott-Chern class is nef and big. The proof of Lemma 8 is motivated by an observation of Tosatti [34] using the Morse inequality of Demailly [9].

**Lemma 8** Under the assumption above, there exists a constant $c > 0$ that depends on $n, \omega$ and $\beta$ such that

$$\int_X (\omega + i\partial \bar{\partial} \varphi)^n \geq c,$$

for any $C^2$-function $\varphi \in PSH(X, \omega)$.

**Proof of Lemma 8.** By a weak transcendental Morse inequality of Demailly [9], one can show (see e.g. Theorem 3.1 in [34]) that for any $C^2$ function $u$, the following holds,

$$0 < \int_X \beta^n = \int_X \beta_u^n \leq \int_{X(\beta_u, 0)} \beta_u^n,$$ (3.3)

where $\beta_u = \beta + i\partial \bar{\partial} u$, and $X(\beta_u, 0)$ denotes the subset of $X$ where the $(1,1)$-form $\beta_u$ is nonnegative definite.

Since $\beta$ is a fixed $(1,1)$-form and $\omega$ is positive definite, there exists a constant $\Lambda > 0$ such that $\beta \leq \Lambda \omega$ on $X$. Apply (3.3) to the function $u = \Lambda \varphi$ for any $\varphi \in C^2 \cap PSH(X, \omega)$,
then we get

\[ 0 < \int_X \beta^n \leq \int_{X(\beta_u,0)} (\beta + i\partial\bar{\partial}\Lambda\varphi)^n \leq \int_{X(\beta_u,0)} (\Lambda\omega + \Lambda i\partial\bar{\partial}\varphi)^n \leq \Lambda^n \int_X (\omega + i\partial\bar{\partial}\varphi)^n, \]

where the inequality in the second line follows from \( 0 \leq \beta + i\partial\bar{\partial}\Lambda\varphi \leq \Lambda\omega + i\partial\bar{\partial}\Lambda\varphi \) over the set \( X(\beta_u,0) \). Q.E.D.

4 The case with boundary

We consider now the case of the Dirichlet problem for the equation (1.7) for an operator \( f(\lambda) \) satisfying the conditions (1-4), and give the proof of Theorem 2.

**Proof of Theorem 2.** In the case of the Dirichlet problem, we cannot normalize the solution to satisfy \( \sup_X \varphi = 0 \), and we need to establish an upper bound for \( \varphi \). However, this can be readily done by comparing \( \varphi \) to a solution of the Dirichlet problem for the Laplace-Beltrami equation on \( X \). More precisely, let \( h \) be the solution of the equation

\[ \Delta_\omega h = n + 1 \text{ in } X, \quad h = 0 \text{ on } \partial X. \quad (4.1) \]

where \( \Delta_\omega h = n(i\partial\bar{\partial}h \wedge \omega^{n-1})\omega^{-n} \) is the complex Laplacian defined by the Hermitian metric \( \omega \). In the Kähler case, \( \Delta_\omega \) coincides with the Laplace-Beltrami equation, and the unique solvability of the corresponding Dirichlet problem is well-known. It has been shown by Z. Lu [29] that the unique solvability of the Dirichlet problem still holds for general Hermitian metrics \( \omega \) on compact manifolds \( X \) with smooth boundary. Thus the function \( h \) exists and is unique, and a dependence on it can be absorbed into a dependence on \( (X, \omega) \).

Now \( \lambda[h,\varphi] \in \Gamma \subset \Gamma_1 \) implies that \( \Delta_\omega \varphi \geq -n \). Thus \( \Delta(\varphi + h) \geq 1 \) on \( X \), which implies that \( \varphi + h \) must attain its maximum on the boundary. Hence

\[ \sup_X (\varphi + h) \leq \sup_{\partial X}(\varphi + h) = \sup_{\partial X}\rho. \quad (4.2) \]

This gives an upper bound for \( \varphi \) by a constant depending only on \( X, \omega, \) and \( \rho \).

We can now replace \( \varphi \) by \( \varphi - \sup_X \varphi \), and thus \( \varphi \) is a non-positive solution to (1.7). Suppose \( \inf_X \varphi = \varphi(x_0) \) for some \( x_0 \in \overline{X} \). If \( x_0 \in \partial X \), we are done. Otherwise we assume \( x_0 \in X \) and

\[ \varphi(x_0) \leq \inf_{\partial X}\rho - 1. \quad (4.3) \]

Take a local coordinates system \((B(x_0, 2r_0), z)\) centered at \( x_0 \) and (2.1) holds.
If \( B(x_0, 2r_0) \subset X \), we can apply directly Theorem 5 and Lemma 5, which gives the desired bound for \(-\inf_X \varphi\), modulo an \( L^1(X, \omega)\)-bound of \((-\varphi)\).

If \( B(x_0, 2r_0) \cap \partial X \neq \emptyset \), we let \( s_0 = 2r_0^2 \), \( s \in (0, s_0) \), and consider the function

\[
u_s(z) = \varphi(z) - \varphi(x_0) + \frac{1}{2}|z|^2 - s, \quad \forall z \in \Omega = B(x_0, 2r_0) \cap X.
\] (4.4)

We observe that on \( \partial B(x_0, 2r_0) \cap X \), \( u_s > 2r_0^2 - s > 0 \); and on the other half of \( \partial \Omega \), \( \forall z \in B(x_0, 2r_0) \cap \partial X \), we have by (4.3)

\[
u_s(z) = \rho(z) - \varphi(x_0) + \frac{1}{2}|z|^2 - s \geq 1 - s > 0.
\]

Thus we have \( \nu_s|_{\partial \Omega} > 0 \). We can extend the function \( \nu_s \) to a smooth function on \( B(x_0, 2r_0) \) satisfying \( \nu_s > 0 \) on \( B(x_0, 2r_0) \setminus \Omega \). Since only the region when \( u_s < 0 \) is under consideration, the estimates in \S2 are independent of the extension of \( u_s \). We also extend \( F \) to be smooth function on \( B(x_0, 2r_0) \), and it will be clear from (4.5) that the extension is again irrelevant.

Denote the sub-level set of \( \nu_s \), \( \Omega_s = \{ z \in \Omega | \nu_s(z) < 0 \} \). By the discussions above, \( \Omega_s \subset X \). As in Section \S2, we consider the following (auxiliary) Dirichlet boundary problem (with the extended functions \( \nu_s \) and \( F \) above)

\[(i\ddbar \bar{\psi}_{s,k})^n = \frac{\tau_k(-\nu_s)}{A_{s,k}} e^{nF}\omega^n, \quad \text{in} \ B(x_0, 2r_0), \] (4.5)

and \( \bar{\psi}_{s,k} \equiv 0 \) on \( \partial B(x_0, 2r_0) \). Here \( \tau_k \) and \( A_{s,k} \) are the same as in Section \S2. We note that the calculations in Section \S2 are performed essentially in \( \Omega_s \), which is strictly contained in \( X \). Therefore, we still obtain Theorem 5 and Lemma 5, and hence a lower bound of \( \varphi(x_0) = \inf_X \varphi \), depending on an \( L^1 \) bound of \((-\varphi)\). We remark that because \( \partial X \neq \emptyset \), the a priori \( L^1 \)-norm of \( \varphi \) as in Lemma 7 in general does not hold.

We now focus on complex Monge-Ampère and Hessian equations with \( \omega \) being Kähler, i.e. \( d\omega = 0 \). We rewrite the equation (1.3) as \( (1 \leq k \leq n) \)

\[(\omega + i\ddbar \varphi)^k \wedge \omega^{n-k} = e^{\bar{F}} \omega^n \quad \text{in} \ X, \ \varphi = \rho \quad \text{on} \ \partial X.
\] (4.6)

Here we denote \( \bar{F} = kF \) to simplify notations. We will write \( \omega_{\varphi} = \omega + i\ddbar \varphi \in \Gamma_k \). When \( k = n \), (4.6) reduces to the Monge-Ampère equation. Since \( \varphi \) is always bounded above depending on \( \omega \) and \( \rho \), we may assume \( \sup_X \varphi \leq 0 \), and that \( \inf_X \varphi = \varphi(x_0) \leq \inf_{\partial X} \rho - 1 \) for some \( x_0 \in X^o \), otherwise we are done.

For simplicity of notations, we consider the function \( \bar{\varphi} = \min(0, \varphi - \inf_{\partial X} \rho) \), which is Lipschitz in \( X \) and \( \varphi|_{\partial X} \equiv 0 \). We calculate

\[
\int_X \log(-\bar{\varphi} + 1)(\omega_{\varphi}^k - \omega^k) \wedge \omega^{n-k}
\]
Combining the above, we obtain
\begin{equation}
\int_X \log (-\bar{\varphi} + 1) i\partial\bar{\partial}\varphi \wedge (\omega_{\varphi}^{k-1} + \omega_{\varphi}^{k-2} \wedge \omega + \cdots + \omega^{k-1}) \land \omega^{n-k} = \\
= \int_X \frac{1}{-\bar{\varphi} + 1} i\partial\bar{\varphi} \wedge \partial\bar{\varphi} \wedge (\omega_{\varphi}^{k-1} + \omega_{\varphi}^{k-2} \wedge \omega + \cdots + \omega^{k-1}) \land \omega^{n-k} \\
+ \int_{\partial X} \log (-\bar{\varphi} + 1) \partial\bar{\varphi} \wedge (\omega_{\varphi}^{k-1} + \omega_{\varphi}^{k-2} \wedge \omega + \cdots + \omega^{k-1}) \land \omega^{n-k}
\end{equation}
\begin{equation}
- \int_X \log (-\bar{\varphi} + 1) \bar{\partial}\bar{\varphi} \wedge \partial \left( (\omega_{\varphi}^{k-1} + \omega_{\varphi}^{k-2} \wedge \omega + \cdots + \omega^{k-1}) \land \omega^{n-k} \right)
\end{equation}
\begin{equation}
\geq \frac{4}{n} \int_X |\nabla \sqrt{-\bar{\varphi} + 1}|^2 \omega^n
\end{equation}
where the integral in (4.7) vanishes since \( \log (-\bar{\varphi} + 1)|_{\partial X} \equiv 0 \), and that in (4.8) is zero since we assume \( \omega \) is Kähler.

On the other hand, the left-hand-side of the equation above satisfies
\begin{equation}
\int_X \log (-\bar{\varphi} + 1) (\omega_{\varphi}^k - \omega^k) \land \omega^{n-k} = \int_X \log (-\bar{\varphi} + 1)(e^{\bar{\varphi}} - 1)\omega^n \leq \int_X \log (-\bar{\varphi} + 1)e^\bar{\varphi} \omega^n
\end{equation}
since \( \log (-\bar{\varphi} + 1) \geq 0 \). We can now apply Young’s inequality as in (2.24) with \( p = 1 \), \( v = \frac{1}{4} \log (-\bar{\varphi} + 1) \) to write

\begin{equation}
\log (-\bar{\varphi} + 1)e^{\bar{\varphi}} \leq 4 e^{\bar{\varphi}} (1 + |\bar{\varphi}|) + C \sqrt{-\bar{\varphi} + 1}
\end{equation}

and hence
\begin{equation}
\int_X \log (-\bar{\varphi} + 1)(\omega_{\varphi}^k - \omega^k) \land \omega^{n-k} \leq 4 \int_X e^{\bar{\varphi}} (1 + |\bar{\varphi}|)\omega^n + C \int_X \sqrt{-\bar{\varphi} + 1}\omega^n \\
\leq C + C \int_X \sqrt{-\bar{\varphi} + 1}\omega^n.
\end{equation}

Combining the above, we obtain

\begin{equation}
\int_X |\nabla(\sqrt{-\bar{\varphi} + 1} - 1)|^2 \omega^n \leq C + C \int_X (\sqrt{-\bar{\varphi} + 1} - 1)\omega^n.
\end{equation}

Observing that the function \( (\sqrt{-\bar{\varphi} + 1} - 1) \equiv 0 \) on \( \partial X \), we can apply the Poincaré inequality on \( (X, \omega) \) to get \( (C_p \text{ below is the Poincaré constant}) \)
\begin{equation}
\int_X (\sqrt{-\bar{\varphi} + 1} - 1)^2\omega^n \leq C_p \int_X |\nabla(\sqrt{-\bar{\varphi} + 1} - 1)|^2 \omega^n \leq C + C \int_X (\sqrt{-\bar{\varphi} + 1} - 1)\omega^n \\
\leq C + C \left( \int_X (\sqrt{-\bar{\varphi} + 1} - 1)^2\omega^n \right)^{1/2},
\end{equation}
from which we easily get \( \int_X (\sqrt{-\bar{\varphi} + 1} - 1)^2\omega^n \leq C \), hence
\begin{equation}
\int_X (-\bar{\varphi})\omega^n \leq C.
\end{equation}

Since \( \bar{\varphi} \) differs from \( \varphi \) by a uniform constant that depends only on \( \inf_{\partial X} \rho \), this proves the \( L^1(X, \omega^n) \)-bound of the function \( |\varphi| \). Q.E.D.
5 The case of non-compact manifolds

We now turn to the proof of Theorem 3, which is analogous to that of Theorem 2, so we only point out the very minor differences.

Proof of Theorem 3. First note we are allowing a dependence on the (local) $L^1$ norm of $\varphi$ in the constant $C$ in Theorem 3. With such an allowance, the upper bound of $\varphi$ is local since $\Delta \omega \varphi \geq -n$ and it follows from the standard elliptic estimates (e.g. Theorem 8.18 in [18]).

Next we derive the lower bound for $\varphi$. Let $\liminf_{z \to \infty} \varphi > -B$ for some positive constant $B$ which can be viewed as part of the data. Then there is a compact set $K \subset X$ with $\varphi \geq -B - 1$ for $z \in X \setminus K$. If $\varphi \geq -B$ on $K$, we are done. Otherwise, the infimum of $\varphi$ over $X$ must take place at some interior point $x_0$ of $K$. We can now apply Theorem 5, and obtain the desired bound for $-\varphi(x_0) = -\inf_X \varphi$. Q.E.D.

6 The case of $(n - 1)$-form Monge-Ampère equations

We come now to the $(n - 1)$-form Monge-Ampère equation and the proof of Theorem 4.

Proof of Theorem 4. Assume $\varphi$ is a $C^2$ solution to the $(n - 1)$-Monge-Ampère equation (1.11). We let $\tilde{\omega} = \omega_h + 1 \frac{1}{n-1}((\Delta \omega \varphi) \omega - i \partial \bar{\partial} \varphi) > 0$

be the Hermitian metric in the left-side of (1.11). Denote $\tilde{g}$ the metric associated to $\tilde{\omega}$. Similarly, $g$ denotes the metric associated to the background Hermitian metric $\omega$.

As in [36], we define a tensor $\Theta^{ij}$ by

$$
\Theta^{ij} = \frac{1}{n-1}((\text{tr}_\omega \omega) g^{ij} - \tilde{g}^{ij}).
$$

Lemma 9 The tensor $\Theta^{ij}$ is positive definite at any point, and satisfies

$$
det(\Theta^{ij}) \geq e^{-F} \det(g^{ij}), \text{ on } X. \quad (6.1)
$$

Proof. We can choose holomorphic coordinates at a given point $x_0 \in X$ such that $g^{ij}|_{x_0} = \delta_{ij}$ and $\tilde{g}^{ij}|_{x_0} = \lambda_i \delta_{ij}$. Note that $\lambda_i > 0$ for each $i = 1, \ldots, n$ since $\tilde{g}$ is positive. Then we have

$$
\Theta^{ij}|_{x_0} = \frac{1}{n-1}((\sum_{k \neq i} \frac{1}{\lambda_k}) \delta_{ij}
$$

which is clearly positive definite. Moreover,

$$
det(\Theta^{ij})|_{x_0} = \frac{1}{(n-1)^n} \prod_{i=1}^{n} \frac{1}{\lambda_k} \geq \prod_{i=1}^{n} \left( \prod_{k \neq i} \frac{1}{\lambda_k} \right)^{1/(n-1)} = \prod_{i=1}^{n} \frac{1}{\lambda_i} = e^{-F} \det(g^{ij}),
$$

20
where the inequality follows from the arithmetic-geometric (AG) inequality and the last equality follows from the equation (1.11), that is,
\[ \prod \lambda_i = \det \tilde{g}_{ij} = e^F \det g_{ij}, \text{ at } x_0. \]

This is the desired inequality. Q.E.D.

Next, we take a minimum point \( x_0 \) of \( \varphi \), i.e. \( \varphi(x_0) = \inf_X \varphi \). Let \((z_1, \cdots, z_n)\) be a local holomorphic coordinate system centered at \( x_0 \), such that on the Euclidean ball \( \Omega = B(x_0, 2r_0) \subset C^n \) defined by these coordinates, the following holds
\[ \frac{1}{2} i \partial \bar{\partial} |z|^2 \leq \omega \leq 2 i \partial \bar{\partial} |z|^2. \]  
(6.2)

We fix a small constant \( \epsilon' > 0 \) depending only on \( X, \omega, \omega_h \) such that
\[ \omega_h \geq 2 \epsilon' n - (\text{tr} \omega \omega_h) \omega. \]  
(6.3)

Let \( s_0 = 4 \epsilon' r_0^2 \). For any \( s \in (0, s_0) \), we define a function \( u_s \) similar to that in (2.5).
\[ u_s(z) := \varphi(z) - \varphi(x_0) + \epsilon'|z|^2 - s, \quad \forall z \in \Omega = B(x_0, 2r_0). \]  
(6.4)

Note that when \( z \in \partial \Omega \), \( u_s(z) \geq 4 \epsilon' r_0^2 - s > 0 \). Therefore the sub-level set of \( u_s \) \( \Omega_s := \{z | u_s(z) < 0\} \cap \Omega \) is relatively compact in \( \Omega \), and by definition \( \Omega_s \) is an open set.

With the same notations as in Section §2, we consider the following auxiliary Monge-Ampère equation with Dirichlet boundary condition on the Euclidean ball \( \Omega \).
\[ (i \partial \bar{\partial} \psi_{s,k})^n = \frac{\tau_k (-u_s)}{A_{s,k}} e^F \omega^n, \quad \text{in } \Omega, \]  
(6.5)
and \( \psi_{s,k} \equiv 0 \) on \( \partial \Omega \). Here \( A_{s,k} \) is a positive constant such that
\[ A_{s,k} = \int_{\Omega} \tau_k (-u_s) e^F \omega^n \to \int_{\Omega_s} (-u_s) e^F \omega^n =: A_s, \]  
as \( k \to \infty \). As noted before, the equation (6.5) admits a unique solution \( \psi_{s,k} \in PSH(\Omega) \cap C^\infty(\Omega) \) by the classic theorem of Caffarelli, Kohn, Nirenberg, and Spruck [3]. By definition \( \int_{\Omega} (i \partial \bar{\partial} \psi_{s,k})^n = 1 \), so by Kolodziej’s inequality [27, 38], \( \psi_{s,k} \) satisfies the uniform integral estimate as in (2.16) for some \( \alpha > 0 \) and \( C > 0 \).

The key is now to establish the analogue of Lemma 1, but where \( u_s \) is now constructed from the above form-Monge-Ampère equation, instead of solutions of the non-linear scalar equation \( f(\lambda) \):
Lemma 10 We have the comparison estimate on $\Omega$

$$-u_s \leq \varepsilon (-\psi_{s,k})^{\frac{n}{n+1}} \tag{6.6}$$

where $\varepsilon$ is defined by $\varepsilon^{1+n} = A_{s,k} \frac{(n+1)^n}{n^n}$.

Proof. We have to show that the test function

$$\Psi = -\varepsilon (-\psi_{s,k})^{\frac{n}{n+1}} - u_s$$

is $\leq 0$ on $\Omega$. This is clear if $\max \Omega \Psi = \Psi(x_0)$ for some $x_0 \in \Omega_s$. By the maximum principle we have $i\partial \bar{\partial} \Psi |_{x_0} \leq 0$. We define a linear operator $L \psi = \Theta^{i\bar{j}} v_{i\bar{j}}$. The fact that $\Theta^{i\bar{j}}$ is positive definite implies that $L \psi |_{x_0} \leq 0$. We calculate at $x_0$ as follows,

$$0 \geq L \psi |_{x_0} = \frac{n \varepsilon}{n+1} \Theta^{i\bar{j}}(\psi_{s,k})_{i\bar{j}} + \frac{\varepsilon n}{(n+1)^2} (-\psi_{s,k})^{-\frac{n+2}{n+1}} \Theta^{i\bar{j}}(\psi_{s,k})_{i}(\psi_{s,k})_{\bar{j}}$$

$$-n + \text{tr} \omega_h - \frac{e'}{n-1} (\text{tr} \omega \cdot \text{tr} \omega \omega h - \text{tr} \omega \omega C^n)$$

$$\geq \frac{n \varepsilon}{n+1} (-\psi_{s,k})^{-\frac{n+2}{n+1}} \Theta^{i\bar{j}}(\psi_{s,k})_{i\bar{j}} - n + \text{tr} \omega_h - \frac{2e'}{n-1} (\text{tr} \omega \omega h) \omega$$

$$\geq \frac{n^2 \varepsilon}{n+1} (-\psi_{s,k})^{-\frac{n+2}{n+1}} (\text{det} \Theta^{i\bar{j}}) [\text{det}(\psi_{s,k})]^{1/n} - n$$

$$\geq \frac{n^2 \varepsilon}{n+1} (-\psi_{s,k})^{-\frac{n+2}{n+1}} \left( \frac{(-u_s)^{1/n}}{A_{s,k}^{1/n}} - n. \right)$$

Here the first equality follows from the definition of $\Theta^{i\bar{j}}$ and the formula

$$\Theta^{i\bar{j}} \varphi_{i\bar{j}} = L \varphi = n - \text{tr} \omega_h$$

which is an easy consequence of the definition of $\tilde{\omega}$. The third line follows from the choice of $e'$ in (6.3). In the fourth line, we applied the standard arithmetic-geometric inequality. The last line follows from Lemma 9 and the definition of $\psi_{s,k}$. By the choice of $\varepsilon$, this implies that $\Psi(x_0) \leq 0$. Hence $\sup \psi \leq 0$. Q.E.D.

This means that we have now for the form Monge-Ampère equation the analogue of Lemma 1. We can then proceed in the same way as in Section §2 and apply Lemma 2 to obtain the lower bound of $\varphi(x_0) = \inf_X \varphi$. We remark that the uniform $L^1(X, \omega^n)$ bound of $\varphi$ still holds in this setting. This is because the assumption that $\tilde{\omega} > 0$ implies $\Delta \omega \varphi \geq -\text{tr} \omega \omega h \geq -C$. Lemma 7 then yields the desired $L^1$ bound of $(-\varphi)$ given our normalization $\sup_X \varphi = 0$. The proof of Theorem 4 is complete.

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