Euler–Maruyama Approximations for Stochastic McKean–Vlasov Equations with Non-Lipschitz Coefficients

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Abstract
In this paper, we study a type of stochastic McKean–Vlasov equations with non-Lipschitz coefficients. Firstly, by an Euler–Maruyama approximation the existence of its weak solutions is proved. Then we observe the pathwise uniqueness of its weak solutions. Finally, it is shown that the Euler–Maruyama approximation has an optimal strong convergence rate.

Keywords Euler–Maruyama approximations · Stochastic McKean–Vlasov equations · Non-Lipschitz conditions · Convergence rates

Mathematics Subject Classification 60H10

1 Introduction
Given $T > 0$, suppose that a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$ and a standard $d$-dimensional Brownian motion $W_t$ on the probability space are given. Consider the following stochastic McKean–Vlasov equation (SMVE) on $\mathbb{R}^d$:

$$
\begin{cases}
X_t = \xi + \int_0^t b(X_s, \mu_s) \, ds + \int_0^t \sigma(X_s, \mu_s) \, dW_s, \\
\mu_s = \text{probability distribution of } X_s,
\end{cases}
$$

(1)
where $\xi$ is a $\mathcal{F}_0$-measurable random variable and the coefficients $b : \mathbb{R}^d \times \mathcal{M}_{\lambda^2}(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathcal{M}_{\lambda^2}(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^d$ are Borel measurable ($\mathcal{M}_{\lambda^2}(\mathbb{R}^d)$ is defined in Sect. 2.1).

If $b$ and $\sigma$ in Eq. (1) are independent of the probability distribution $\mu_t$ of the process at time $t$, Eq. (1) is a standard Markov process and has been well studied in the literature (c.f. [3,5]). Besides, there are stochastic differential equations whose coefficients depend not only on the process but also on the probability distribution of the process at time $t$ as indicated in Eq. (1). The study on SMVEs was initiated by McKean [6] who was inspired by Kac’s program in Kinetic Theory. And then there have been numerous results (c.f. [10]). Let us mention some works. Recently, Huang and Wang [4] studied the existence and the uniqueness of strong solutions for Eq. (1) under some integrable conditions. Besides, if the diffusion coefficient is independent of $\mu_t$, the second named author [7] showed that under Lipschitz and linear growth conditions, Eq. (1) has a unique mild solution in a real separated Hilbert space, and the Euler approximation of the mild solution converges to itself. Later, under more general conditions than that in [7], Govindan and Ahmed [2] proved Eq. (1) has a unique mild solution, and the Yosida approximation of the mild solution converges to itself. If $b$ and $\sigma$ depend on $\mu_t$ as follows:

$$
\int_0^t b[X_s, \mu_s] \, ds = \int_0^t \int_{\mathbb{R}^d} b(X_s, y) \mu_s(dy) \, ds,
$$

$$
\int_0^t \sigma[X_s, \mu_s] \, ds = \int_0^t \int_{\mathbb{R}^d} \sigma(X_s, y) \mu_s(dy) \, ds,
$$

where $b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ are Borel measurable, Sznitman [10] investigated the existence and the uniqueness of strong solutions for Eq. (1) with a fixed point argument if the coefficients are globally Lipschitz continuous. Recently, Chi [1] proved that if the coefficients are continuous and satisfy linear growth condition, a weak solution of the multivalued SMVE exists by the Euler approximation.

In this paper, we study Eq. (1) under non-Lipschitz conditions. Firstly, we establish the weak existence of Eq. (1) under a linear growth condition. Next, the pathwise uniqueness is obtained under two non-Lipschitz conditions. Thus, by the weak existence and the pathwise uniqueness, we prove the existence and the uniqueness of a strong solution for Eq. (1). Finally, the convergence rate of the Euler–Maruyama approximation is discussed.

It is worthwhile to mention our conditions and methods. We give two non-Lipschitz conditions which cannot be covered by the conditions in [4]. Moreover, our conditions are more straight than that in [4]. Besides, we prove the existence of martingale solutions of Eq. (1) by an Euler–Maruyama approximation, which implies its weak existence. Thus, a number of complex calculation, as that in [8,11], is avoided.

The rest of the paper is organized as follows. In Sect. 2, we recall some basic notations and give some necessary concepts and assumptions. And then we prove the existence and the uniqueness of a strong solution of Eq. (1) in Sect. 3. In Sect. 4, the convergence rate of the Euler–Maruyama approximation is investigated.
The following convention will be used throughout the paper: $C$ with or without indices will denote different positive constants whose values may change from one place to another.

2 The Framework

In the section, we recall some basic notations and give some necessary concepts and assumptions.

2.1 Notations

In the subsection, we introduce notations used in the sequel.

Let $C(R^d)$ be the space of continuous functions on $R^d$. And let $C^k_0(R^d)$ be the collection of all continuous functions which have bounded, continuous partial derivatives of every order up to $k$ where $k$ is a positive integer. Let $\partial_{ij}$ denote the differentiation with respect to the coordinates with corresponding numbers (e.g., $\partial_{ij}(f) := \frac{\partial^2 f(x)}{\partial x^i \partial x^j}$).

Let $B(R^d)$ be the Borel $\sigma$-algebra on $R^d$ and $M(R^d)$ be the space of all probability measures defined on $B(R^d)$ carrying the usual topology of weak convergence.

For convenience, we shall use $|\cdot|$ and $\|\cdot\|$ for norms of vectors and matrices, respectively. Furthermore, let $\langle \cdot, \cdot \rangle$ denote the scalar product in $R^d$. Let $A^*$ denote the transpose of the matrix $A$.

Define the Banach space

$$C_\rho(R^d) := \left\{ \varphi \in C(R^d), \| \varphi \|_{C_\rho(R^d)} = \sup_{x \in R^d} \frac{|\varphi(x)|}{(1 + |x|)^2} + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|} < \infty \right\}.$$ 

Let $M^s_{\lambda^2}(R^d)$ be the Banach space of signed measures $m$ on $B(R^d)$ satisfying

$$\|m\|_{\lambda^2}^2 := \int_{R^d} (1 + |x|)^2 |m|(dx) < \infty,$$

where $|m| = m^+ + m^-$ and $m = m^+ - m^-$ is the Jordan decomposition of $m$. Let $M_{\lambda^2}(R^d) = M^s_{\lambda^2}(R^d) \cap M(R^d)$ be the set of probability measures on $B(R^d)$. We put on $M_{\lambda^2}(R^d)$ a topology induced by the following metric:

$$\rho(\mu, \nu) := \sup_{\|\varphi\|_{C_\rho(R^d)} \leq 1} \left| \int_{R^d} \varphi(x) \mu(dx) - \int_{R^d} \varphi(x) \nu(dx) \right|.$$ 

Then, $(M_{\lambda^2}(R^d), \rho)$ is a complete metric space.
2.2 Some Concepts

In the subsection, we introduce the concepts of strong solutions, weak solutions and pathwise uniqueness. Consider Eq. (1), i.e.,

\[
\begin{aligned}
X_t &= \xi + \int_0^t b(X_s, \mu_s) \, ds + \int_0^t \sigma(X_s, \mu_s) \, dW_s, \\
\mu_s &= \text{probability distribution of } X_s.
\end{aligned}
\]

**Definition 2.1** We say that Eq. (1) admits a strong solution with the initial value \( \xi \) if there exists a continuous process \( X = \{X_t; 0 \leq t \leq T\} \) on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})\) such that

(i) \( \mathbb{P}(X_0 = \xi) = 1 \),
(ii) \( X_t \in \mathcal{F}_W^t \), where \( \{\mathcal{F}_W^t\}_{t \in [0,T]} \) stands for the \( \sigma \)-field filter generated by \( W \),
(iii) it holds that

\[
\int_0^t \left( |b(X_s, \mu_s)| + \|\sigma(X_s, \mu_s)\|^2 \right) \, ds < +\infty, \quad \text{a.s.} \mathbb{P},
\]

and

\[
X_t = \xi + \int_0^t b(X_s, \mu_s) \, ds + \int_0^t \sigma(X_s, \mu_s) \, dW_s, \quad 0 \leq t \leq T.
\]

From the above definition, we know that \( \mu_0 = \mathbb{P} \circ \xi^{-1} \).

**Definition 2.2** We say that Eq. (1) admits a weak solution with the initial law \( \mu_0 \) if there exists a stochastic space \( \hat{S} := (\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \in [0,T]}, \hat{\mathbb{P}}) \), a \( d \)-dimensional standard Brownian motion \( \hat{W} \) as well as a \( \{\hat{\mathcal{F}}_t\}_{t \in [0,T]} \)-adapted process \( \hat{X} \) defined on \( \hat{S} \) such that

(i) \( \hat{\mathbb{P}} \circ \hat{X}_0^{-1} = \mu_0 \),
(ii) it holds that

\[
\int_0^t \left( |b(\hat{X}_s, \hat{\mu}_s)| + \|\sigma(\hat{X}_s, \hat{\mu}_s)\|^2 \right) \, ds < +\infty, \quad \text{a.s.} \hat{\mathbb{P}},
\]

and

\[
\hat{X}_t = \hat{X}_0 + \int_0^t b(\hat{X}_s, \hat{\mu}_s) \, ds + \int_0^t \sigma(\hat{X}_s, \hat{\mu}_s) \, d\hat{W}_s, \quad 0 \leq t \leq T.
\]

Such a weak solution is denoted by \((\hat{S}; \hat{W}, \hat{X})\).

**Definition 2.3 (Pathwise Uniqueness)** Suppose \((\hat{S}; \hat{W}, \hat{X}^1)\) and \((\hat{S}; \hat{W}, \hat{X}^2)\) are two weak solutions with \( \hat{X}_0^1 = \hat{X}_0^2 \). If \( \hat{\mathbb{P}}(\hat{X}_t^1 = \hat{X}_t^2, t \geq 0) = 1 \), then we say that the pathwise uniqueness holds for Eq. (1).
2.3 Some Assumptions

In the subsection, we give out some assumptions:

**\( (H_1) \)** The functions \( b, \sigma \) are continuous in \((x, \mu)\) and satisfy for \((x, \mu) \in \mathbb{R}^d \times \mathcal{M}_{2,2}(\mathbb{R}^d)\)

\[
|b(x, \mu)|^2 + \|\sigma(x, \mu)\|^2 \leq L_1 (1 + |x|^2 + \|\mu\|^2_2),
\]

where \(L_1 > 0\) is a constant.

**\( (H_2) \)** The functions \( b, \sigma \) satisfy for \((x_1, \mu_1), (x_2, \mu_2) \in \mathbb{R}^d \times \mathcal{M}_{2,2}(\mathbb{R}^d)\)

\[
2(x_1 - x_2, b(x_1, \mu_1) - b(x_2, \mu_2)) + \|\sigma(x_1, \mu_1) - \sigma(x_2, \mu_2)\|^2 \leq L_2 \left( \kappa_1 |x_1 - x_2|^2 + \kappa_2 \left( \rho^2(\mu_1, \mu_2) \right) \right),
\]

where \(L_2 > 0\) is a constant and \(\kappa_i(x), i = 1, 2\) are two positive, strictly increasing, continuous concave function and satisfies \(\kappa_i(0) = 0, \int_0^1 \frac{1}{\kappa_1(x) + \kappa_2(x)} \, dx = \infty\).

**\( (H'_2) \)** The functions \( b \) and \( \sigma \) satisfy for \((x_1, \mu_1), (x_2, \mu_2) \in \mathbb{R}^d \times \mathcal{M}_{2,2}(\mathbb{R}^d)\)

\[
|b(x_1, \mu_1) - b(x_2, \mu_2)| \leq \lambda_1 \left( |x_1 - x_2|^2 \gamma_1(|x_1 - x_2|) + \rho(\mu_1, \mu_2) \right),
\]

\[
\|\sigma(x_1, \mu_1) - \sigma(x_2, \mu_2)\|^2 \leq \lambda_2 \left( |x_1 - x_2|^2 \gamma_2(|x_1 - x_2|) + \rho^2(\mu_1, \mu_2) \right),
\]

where \(\lambda_1 > 0\) is a constant and \(\gamma_i(x)\) is a positive continuous function, bounded on \([1, \infty)\) and satisfying

\[
\lim_{x \downarrow 0} \frac{\gamma_i(x)}{\log(x^{-1})} = \delta_i < \infty, \quad i = 1, 2.
\]

**Remark 2.4** If \(b(x, \mu)\) satisfies \(H'_2\), it holds that for \((x_1, \mu_1), (x_2, \mu_2) \in \mathbb{R}^d \times \mathcal{M}_{2,2}(\mathbb{R}^d)\)

\[
\langle x_1 - x_2, b(x_1, \mu_1) - b(x_2, \mu_2) \rangle \\
\leq |x_1 - x_2||b(x_1, \mu_1) - b(x_2, \mu_2)| \\
\leq \lambda_1 \left( |x_1 - x_2|^2 \gamma_1(|x_1 - x_2|) + |x_1 - x_2| \rho(\mu_1, \mu_2) \right) \\
\leq \lambda_1 \left( |x_1 - x_2|^2 \gamma_1(|x_1 - x_2|) + \frac{|x_1 - x_2|^2}{2} + \frac{\rho^2(\mu_1, \mu_2)}{2} \right) \\
\leq \lambda_1 \left( |x_1 - x_2|^2 \gamma_1(|x_1 - x_2|) + |x_1 - x_2|^2 + \rho^2(\mu_1, \mu_2) \right).
\]

Besides, by the proof of Theorem 2.3 in [11], we know that there exists a \(0 < \eta < \frac{1}{e}\) such that

\[
x^2 \gamma_i(x) \leq \kappa_\eta(x^2), \quad i = 1, 2,
\]
where
\[
\kappa_\eta(x) := \begin{cases} 
0, & x = 0, \\
x \log x^{-1}, & 0 < x \leq \eta, \\
(\log \eta^{-1} - 1)x + \eta, & x > \eta,
\end{cases}
\]
is a positive, strictly increasing, continuous concave function and satisfies \(\kappa_\eta(0) = 0,\)
\[
\int_0^\infty \frac{1}{\kappa_\eta(x)+x} \, dx = \infty.
\]
Thus,
\[
\langle x_1 - x_2, b(x_1, \mu_1) - b(x_2, \mu_2) \rangle \leq \lambda_1 \left( \kappa_\eta(|x_1 - x_2|^2) + |x_1 - x_2|^2 + \rho^2(\mu_1, \mu_2) \right).
\]
If \(\sigma(x, \mu)\) satisfies \((H_2)\), by the similar deduction to above it holds that
\[
\| \sigma(x_1, \mu_1) - \sigma(x_2, \mu_2) \|_2 \leq \lambda_2 \left( |x_1 - x_2|^2 \gamma_2(|x_1 - x_2|) + \rho^2(\mu_1, \mu_2) \right)
\leq \lambda_2 \left( \kappa_\eta(|x_1 - x_2|^2) + \rho^2(\mu_1, \mu_2) \right).
\]
That is, \((H_2')\) implies \((H_2)\).

3 The Existence and the Uniqueness of Strong Solutions

In the section, we study the existence and the uniqueness of strong solutions for Eq. (1). The main result is the following theorem.

**Theorem 3.1** Suppose that \((H_1)-(H_2)\) hold and \(\mathbb{E} |\xi|^2p < \infty\) for any \(p > 1\). Then, Eq. (1) has a unique strong solution.

The proof of the above theorem is made up of two parts—the existence and the path-wise uniqueness of weak solutions. Firstly, we prove the existence of weak solutions for Eq. (1). To do that, we introduce martingale solutions for Eq. (1). Set
\[
\mathcal{W} := C([0, T], \mathbb{R}^d), \quad \mathcal{W} = \mathcal{B}(\mathcal{W}),
\]
\[
\mathcal{W}_t := C([0, t], \mathbb{R}^d), \quad \mathcal{W}_t = \cap_{s>t} \mathcal{B}(\mathcal{W}_s), \quad t \in [0, T].
\]

**Definition 3.2** A probability measure \(P\) on \((\mathcal{W}, \mathcal{W})\) is called a martingale solution of Eq. (1) with the initial law \(\mu_0\), if
\[
M_t^f := f(w_t) - f(w_0) - \int_0^t (\mathcal{A}(\mu_s) f)(w_s) \, ds, \quad f \in C_0^2(\mathbb{R}^d),
\]
is a continuous \(\mathcal{W}_t\)-adapted martingale, where \(\mu_s := P \circ w_s^{-1}\) denotes the law of \(w_s\) under \(P\) and
\[
(\mathcal{A}(\mu) f)(x) := \frac{1}{2} (\sigma(x, \mu) \sigma^*(x, \mu))^{ij} \partial_{ij}^2 f + b^i(x, \mu) \partial_i f.
\]
Here and hereafter we use the convention that the repeated indices stand for the summation. We have the relationship between martingale solutions and weak solutions as follows.

**Proposition 3.3** The existence of martingale solutions implies the existence of weak solutions and vice versa.

Since its proof is similar to that of [1, Proposition 2.10], we omit it. Next, we give a lemma which will take an important part in the sequel.

**Lemma 3.4** Suppose \( b(x, \mu) \) and \( \sigma(x, \mu) \) satisfy (H1). If \((\hat{S}; \hat{W}, \hat{X})\) is a weak solution to Eq. (1), where \( \hat{\mathbb{E}}(\cdot) := \mathbb{E}_{\hat{\mathbb{P}}}(\cdot) \) denotes the expectation under \( \hat{\mathbb{P}} \), it follows that for \( p \geq 1 \)

\[
\mathbb{E}(|\hat{X}_t|^{2p}) \leq C(1 + \mathbb{E}|\hat{X}_0|^{2p})e^{Ct}, \quad 0 \leq t \leq T, \quad (4)
\]

\[
\mathbb{E}(|\hat{X}_t - \hat{X}_s|^{2p}) \leq C(1 + \mathbb{E}|\hat{X}_0|^{2p})(t - s)^p, \quad 0 \leq s < t \leq T, \quad (5)
\]

where \( C > 0 \) is a constant depending on \( T, p, L_1 \).

**Proof** Set \( \tau_k := \inf \{t \geq 0, |\hat{X}_t| \geq k\}, k \in \mathbb{N} \). If these inequalities (4) and (5) hold for the process \( \hat{X}_{\tau_k} \), let \( k \to +\infty \), by Fatou’s Lemma it follows that these inequalities (4) and (5) also hold for \( \hat{X}_t \). So we might as well suppose that \( \hat{X}_t \) is bounded.

For Eq. (1), by the Hölder inequality and BDG inequality, it holds that

\[
\mathbb{E}|\hat{X}_t|^{2p} \leq 3^{2p-1}\left( \mathbb{E}|\hat{X}_0|^{2p} + \mathbb{E}|\int_0^t b(\hat{X}_s, \hat{\mu}_s) \, ds|^{2p} \right)
\]

\[
+ \mathbb{E}\left| \int_0^t \sigma(\hat{X}_s, \hat{\mu}_s) \, d\hat{W}_s \right|^{2p} \right)
\]

\[
\leq 3^{2p-1}\left( \mathbb{E}|\hat{X}_0|^{2p} + \mathbb{E}|\int_0^t b(\hat{X}_s, \hat{\mu}_s) \, ds|^{2p} \right)
\]

\[
+ [p(2p - 1)]^{p-1}\mathbb{E}\left( \int_0^t \|\sigma(\hat{X}_s, \hat{\mu}_s)\|^{2p} \, ds \right)^{p-1} \right)
\]

\[
\leq 3^{2p-1}\left( \mathbb{E}|\hat{X}_0|^{2p} + t^{2p-1}\left( \int_0^t \mathbb{E}|b(\hat{X}_s, \hat{\mu}_s)|^{2p} \, ds \right) \right)
\]

\[
+ [p(2p - 1)]^{p-1}\left( \int_0^t \mathbb{E}\|\sigma(\hat{X}_s, \hat{\mu}_s)\|^{2p} \, ds \right) \right)
\]

\[
\leq C\left( \mathbb{E}|\hat{X}_0|^{2p} + \int_0^t \mathbb{E}|b(\hat{X}_s, \hat{\mu}_s)|^{2p} + \|\sigma(\hat{X}_s, \hat{\mu}_s)\|^{2p} \, ds \right) \right)
\]

\[
\leq C\left( \mathbb{E}|\hat{X}_0|^{2p} + \int_0^t \mathbb{E}(1 + |\hat{X}_s|^{2p} + \|\hat{\mu}_s\|^{2p}) \, ds \right) \right)
\]

\[
\leq C\left( \mathbb{E}|\hat{X}_0|^{2p} + \int_0^t \mathbb{E}(1 + |\hat{X}_s|^{2p} + \hat{\mathbb{E}}(1 + |\hat{X}_s|^{2p})) \, ds \right) \right)
\]

\[
\leq C\left( 1 + \mathbb{E}|\hat{X}_0|^{2p} + \int_0^t \mathbb{E}|\hat{X}_s|^{2p} \, ds \right), \quad 0 \leq t \leq T,
\]
where $C > 0$ is a constant depending on $T$, $p$, $L_1$. By Gronwall’s inequality, one can get (4).

By the similar deduction to above, we obtain

$$
\hat{E} | \hat{X}_t - \hat{X}_s |^{2p} \leq C_p \hat{E} \left( | \int_{s}^{t} b(\hat{X}_u, \hat{\mu}_u) \, du |^{2p} + | \int_{s}^{t} \sigma(\hat{X}_u, \hat{\mu}_u) \, d\hat{W}_u |^{2p} \right)
$$

$$
\leq C_p T (t-s)^{p-1} \int_{s}^{t} \hat{E} \left( | b(\hat{X}_u, \hat{\mu}_u) |^{2p} + \| \sigma(\hat{X}_u, \hat{\mu}_u) \|^{2p} \right) \, du
$$

$$
\leq C_p T (t-s)^{p-1} \int_{s}^{t} (1 + \hat{E} | \hat{X}_u |^{2p}) \, du
$$

$$
\leq C (1 + \hat{E} | \hat{X}_0 |^{2p})(t-s)^p, \quad 0 \leq s < t \leq T.
$$

The proof is completed. □

**Proposition 3.5** Suppose that $(H_1)$ holds and $\mathbb{E} | \xi |^{2p} < \infty$ for any $p > 1$. Then, there exists a martingale solution to Eq. (1).

**Proof** Firstly, for fixed $n \in \mathbb{N}$, consider the following Euler–Maruyama approximation equation

$$
\text{d}X^n_t = b(X^n_t, \mu^n_t) \, dt + \sigma(X^n_t, \mu^n_t) \, dW_t, \quad (6)
$$

where $X^n_0 = \xi$, $t_n = \left[ \frac{t}{2^n} \right]$ and $[a]$ denotes the integer part of $a$. By solving a deterministic problem, this equation can be solved step by step. That is, there exists a solution $X^n$ to Eq. (6). By (2) and Lemma 3.4, we have

$$
\mathbb{E}(|X^n_t|^2) \leq (1 + \mathbb{E} | \xi |^{2p})e^{Ct}, \quad 0 \leq t \leq T,
$$

$$
\mathbb{E}(|X^n_t - X^n_s|^2) \leq (1 + \mathbb{E} | \xi |^{2p}) (t-s)^p, \quad 0 \leq s < t \leq T, \quad (7)
$$

where $C$ is independent of $n$. Since $\mathbb{E} | \xi |^{2p} < +\infty$, we further have

$$
\sup_{n \geq 1} \mathbb{E} | X^n_0 |^{2p} = \mathbb{E} | \xi |^{2p} < +\infty,
$$

$$
\sup_{n \geq 1} \mathbb{E}(|X^n_t - X^n_s|^2) \leq C (1 + \mathbb{E} | \xi |^{2p}) (t-s)^p \leq C(t-s)^p.
$$

Set $P^n = \mathbb{P} \circ (X^n)^{-1}$, and then by Lemma 20.3 in [3, P. 185] we derive that $\{P^n\}$ is tight. So there exist a subsequence still denoted by $\{P^n\}$ and $P^0$ such that $P^n$ weakly converges to $P^0$ as $n \to +\infty$.

Now set

$$
M^{n,f}_t := f(w_t) - f(w_0) - \frac{1}{2} \int_0^t (\sigma(w_s, \mu^n_s) \sigma^*(w_s, \mu^n_s))^{ij} \partial_{ij}^2 f(w_s) \, ds
$$

$$
- \int_0^t b^i(w_s, \mu^n_s) \partial_i f(w_s) \, ds, \quad f \in C^2_0(\mathbb{R}^d).
$$
Since Eq. (6) has a weak solution $X^n$, by Proposition 3.3, we know that there exists a martingale solution $P^n$ on $(\mathcal{W}, \mathcal{F}')$ of Eq. (6), which yields that $M^{n,f}$ is a continuous $\mathcal{F}_t$-adapted martingale under $P^n$. So for any continuous, bounded and $\mathcal{F}_s$-measurable functional $G_s$,

$$E^{P^n}((M^{n,f}_t - M^{n,f}_s)G_s) = 0, \quad 0 \leq s < t \leq T.$$ 

To prove that $P^0$ on $(\mathcal{W}, \mathcal{F}')$ is a martingale solution to Eq. (1), we just need to prove that $M^f_t$ defined by (3) is a continuous $\bar{\mathcal{W}}_t$-adapted martingale under $P^0$. That is,

$$E^{P^0}((M^f_t - M^f_s)G_s(w)) = \int_{\mathcal{W}} \left( f(w_t) - f(w_s) - \int_s^t \mathcal{A}(\mu_u) f(w_u) \, du \right) G_s(w) P^0(dw) = 0.$$ 

Note that $P^n$ weakly converges to $P^0$. Thus, it is clear that

$$\lim_{n \to \infty} \int_{\mathcal{W}} (f(w_t) - f(w_s)) G_s(w) P^n(dw) = \int_{\mathcal{W}} (f(w_t) - f(w_s)) G_s(w) P^0(dw).$$ 

We now prove that

$$\lim_{n \to \infty} \int_{\mathcal{W}} \left( \int_s^t b^i(\mu_{u_n}, \mu_{u_n}) \partial_i f(w_u) \, du \right) G_s(w) P^n(dw) = \int_{\mathcal{W}} \left( \int_s^t b^i(\mu_u, \mu_u) \partial_i f(w_u) \, du \right) G_s(w) P^0(dw), \quad (8)$$

and

$$\lim_{n \to \infty} \int_{\mathcal{W}} \left( \int_s^t (\sigma(\mu_{u_n}) \sigma^*(\mu_{u_n}) \partial_j f(w_u) \, du \right) G_s(w) P^n(dw) = \int_{\mathcal{W}} \left( \int_s^t (\sigma(\mu_u) \sigma^*(\mu_u) \partial_j f(w_u) \, du \right) G_s(w) P^0(dw). \quad (9)$$

With the help of Theorem c.6 [3, P. 324] and the weak convergence of $P^n$ to $P^0$, we know that there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and $\mathcal{W}$-valued processes $\tilde{X}^n$, $\tilde{X}$ on it satisfying

(i) The law of $\tilde{X}^n$ and $\tilde{X}$ are $P^n$ and $P^0$, respectively,

(ii) $\tilde{X}^n \xrightarrow{a.s.} \tilde{X}$ as $n \to \infty$.

Based on (i), (8) (9) become

$$\lim_{n \to \infty} E^{\tilde{P}} \left( \int_s^t b^i(\tilde{X}^n_u, \mu_{u_n}) \partial_i f(\tilde{X}^n_u) \, du \right) G_s(\tilde{X}^n) = E^{\tilde{P}} \left( \int_s^t b^i(\tilde{X}_u, \mu_u) \partial_i f(\tilde{X}_u) \, du \right) G_s(\tilde{X}), \quad (10)$$

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and
\[
\lim_{n \to \infty} \mathbb{E}_{\tilde{\mathbb{P}}} \left( \left( \int_s^t \left( \sigma(\tilde{X}_{un}^n, \mu_{un}^n) \sigma^*(\tilde{X}_{un}^n, \mu_{un}^n) \right)^{ij} \delta_{ij} f(\tilde{X}_{un}^n) d\tilde{u} \right) G_s(\tilde{X}_u^n) \right) = \mathbb{E}_{\tilde{\mathbb{P}}} \left( \left( \int_s^t \left( \sigma(\tilde{X}_u, \mu_u) \sigma^*(\tilde{X}_u, \mu_u) \right)^{ij} \delta_{ij} f(\tilde{X}_u) d\tilde{u} \right) G_s(\tilde{X}_u) \right). \tag{11}
\]

In the following, we are devoted to proving (10). On one side, by (ii), it holds that \( \tilde{X}_{un}^n \xrightarrow{a.s.} \tilde{X}_u \) for \( u \in [s, t] \) as \( n \to \infty \). Next, we observe \( \rho(\mu_{un}^n, \mu_u) \). By the definition of \( \rho \), it holds that
\[
\rho(\mu_{un}^n, \mu_u) = \sup_{\|\varphi\|_{C^p(\mathbb{R}^d)} \leq 1} \left| \mathbb{E}_{\tilde{\mathbb{P}}} \varphi(\tilde{X}_{un}^n) - \mathbb{E}_{\tilde{\mathbb{P}}} \varphi(\tilde{X}_u) \right| \leq \sup_{\|\varphi\|_{C^p(\mathbb{R}^d)} \leq 1} \left| \mathbb{E}_{\tilde{\mathbb{P}}} \varphi(\tilde{X}_{un}^n) - \varphi(\tilde{X}_u) \right| \leq \mathbb{E}_{\tilde{\mathbb{P}}} \left| \tilde{X}_{un}^n - \tilde{X}_u \right|.
\]

Note that for any \( \lambda > 0 \),
\[
\int_{|\tilde{X}_{un}^n| > \lambda} |\tilde{X}_{un}^n| d\tilde{\mathbb{P}} = \int_{|\tilde{X}_{un}^n| > \lambda} \frac{|\tilde{X}_{un}^n|^p}{\lambda^p} \lambda^p d\tilde{\mathbb{P}} \leq \int_{|\tilde{X}_{un}^n| > \lambda} \frac{|\tilde{X}_{un}^n|^{2p}}{\lambda^p} d\tilde{\mathbb{P}} \leq \frac{1}{\lambda^p} \mathbb{E}_{\tilde{\mathbb{P}}} \left| \tilde{X}_{un}^n \right|^{2p} = \frac{1}{\lambda^p} \mathbb{E} |X_{un}^n|^{2p}.
\]

Thus, by (7) we have that
\[
\lim_{\lambda \to \infty} \sup_{n \geq 1} \int_{|\tilde{X}_{un}^n| > \lambda} |\tilde{X}_{un}^n| d\tilde{\mathbb{P}} = 0,
\]
and then \( \{\tilde{X}_{un}^n, n \geq 1\} \) is uniformly integrable. Based on [3, Theorem 4.5], one can know that uniform integrability of \( \{\tilde{X}_{un}^n, n \geq 1\} \) and almost sure convergence of \( \tilde{X}_{un} \) to \( \tilde{X}_u \) imply that \( \lim_{n \to \infty} \mathbb{E}_{\tilde{\mathbb{P}}} \left| \tilde{X}_{un}^n - \tilde{X}_u \right| = 0 \) and furthermore \( \lim_{n \to \infty} \rho(\mu_{un}^n, \mu_u) = 0 \).

On the other side, by (i) (2) and (7), it holds that
\[
\mathbb{E}_{\tilde{\mathbb{P}}} \left| b(\tilde{X}_{un}^n, \mu_{un}^n) \right| = \mathbb{E} \left| b(X_{un}^n, \mu_{un}^n) \right| \leq \mathbb{E} \left( \mathbb{E} \left| b(X_{un}^n, \mu_{un}^n) \right|^{2p} \right)^{1/2p} \leq C \left( \mathbb{E} \left( 1 + |X_{un}^n|^{2p} + \|\mu_{un}^n\|_{\mathbb{L}^2}^{2p} \right) \right)^{1/2p}.
\]
\[
\leq C \left( \mathbb{E} \left( 1 + |X^n_{u_n}|^{2p} + \mathbb{E}(1 + |X^n_{u_n}|^{2p}) \right) \right)^{1/2p} \\
\leq C \left( \mathbb{E}(1 + |X^n_{u_n}|^{2p}) \right)^{1/2p} < \infty.
\]

By the continuity of \(b\) and the dominated convergence theorem, we obtain (10). By the similar means, one can prove (11). The proof is now completed. \(\Box\)

So, by Proposition 3.3 and Theorem 3.5, we know that Eq. (1) has a weak solution. Next, we prove that pathwise uniqueness holds for Eq. (1) under certain conditions. The following lemma is known (c.f. [9, Lemma 116, P.79 and Lemma 144, P.113]). For the readers’ convenience, we give a short proof.

Lemma 3.6 For any \(t \geq 0\), if \(y_t\) satisfies \(0 \leq y_t \leq \int_0^t (\kappa_1(y_s) + \kappa_2(y_s)) \, ds < \infty\), where \(\kappa(u)\) satisfies the conditions in (H\(_2\)), then \(y_t \equiv 0, \forall t \geq 0\).

Proof Set \(z_t := \int_0^t (\kappa_1(y_s) + \kappa_2(y_s)) \, ds\), and then we just need to prove \(z_t = 0\). Note that \(z_t\) is absolutely continuous and nondecreasing. Thus, it holds that

\[
\frac{dz_t}{dt} = \kappa_1(y_t) + \kappa_2(y_t) \leq \kappa_1(z_t) + \kappa_2(z_t).
\]

Let \(t_0 := \sup\{t \geq 0; z_s = 0, \forall s \in [0, t]\}\). If \(t_0 < \infty\), then \(z_t > 0, t > t_0\). Therefore, we have

\[
\infty = \int_0^{z_0 + \varepsilon} \frac{du}{\kappa_1(u) + \kappa_2(u)} = \int_{t_0}^{t_0 + \varepsilon} \frac{dz_t}{\kappa_1(z_t) + \kappa_2(z_t)} \leq \int_{t_0}^{t_0 + \varepsilon} dt \leq \varepsilon, \quad \forall \varepsilon > 0,
\]

which is a contradiction. So \(t_0 = \infty\) and \(z_t = 0\). \(\Box\)

Proposition 3.7 Suppose that \((H_2)\) holds. Then, the pathwise uniqueness holds for Eq. (1).

Proof Suppose that \((\hat{S}; \hat{W}, \hat{X}^1)\) and \((\hat{S}; \hat{W}, \hat{X}^2)\) are two weak solutions to Eq. (1) with \(\hat{X}^1_0 = \hat{X}^2_0\). Set

\[
Z_t := \hat{X}^1_t - \hat{X}^2_t,
\]

and then \(Z_t\) satisfies

\[
Z_t = \int_0^t \left( b(\hat{X}^1_s, \hat{\mu}^1_s) - b(\hat{X}^2_s, \hat{\mu}^2_s) \right) \, ds + \int_0^t \left( \sigma(\hat{X}^1_s, \hat{\mu}^1_s) - \sigma(\hat{X}^2_s, \hat{\mu}^2_s) \right) \, d\hat{W}_s.
\]
Applying the Itô formula to $|Z_t|^2$, we obtain that

$$|Z_t|^2 = \int_0^t 2\langle Z_s, b(\hat{X}_s^1, \mu_s^1) - b(\hat{X}_s^2, \mu_s^2) \rangle \, ds + \int_0^t \| \sigma(\hat{X}_s^1, \mu_s^1) - \sigma(\hat{X}_s^2, \mu_s^2) \|^2 \, ds$$

$$+ \int_0^t 2\langle Z_s, (\sigma(\hat{X}_s^1, \mu_s^1) - \sigma(\hat{X}_s^2, \mu_s^2)) \rangle \, d\hat{W}_s.$$ 

By taking the expectation on both sides, one can have

$$\hat{E} |Z_t|^2 = \hat{E} \int_0^t 2\langle Z_s, b(\hat{X}_s^1, \mu_s^1) - b(\hat{X}_s^2, \mu_s^2) \rangle \, ds$$

$$+ \hat{E} \int_0^t \| \sigma(\hat{X}_s^1, \mu_s^1) - \sigma(\hat{X}_s^2, \mu_s^2) \|^2 \, ds.$$

Put $G_t := \hat{E} |Z_t|^2$, and by (H2), it holds that

$$G_t = \hat{E} \int_0^t \left( 2\langle Z_s, b(\hat{X}_s^1, \mu_s^1) - b(\hat{X}_s^2, \mu_s^2) \rangle + \| \sigma(\hat{X}_s^1, \mu_s^1) - \sigma(\hat{X}_s^2, \mu_s^2) \|^2 \right) \, ds$$

$$\leq L_2 \hat{E} \int_0^t \left( \kappa_1(|Z_s|^2) + \kappa_2 \left( \rho^2(\mu_s^1, \mu_s^2) \right) \right) \, ds.$$

Note that

$$\rho(\mu_s^1, \mu_s^2) = \sup_{\|\varphi\|_{C_p(\mathbb{R}^d)} \leq 1} \left| \int_{\mathbb{R}^d} \varphi(x) \mu_s^1(dx) - \int_{\mathbb{R}^d} \varphi(x) \mu_s^2(dx) \right|$$

$$= \sup_{\|\varphi\|_{C_p(\mathbb{R}^d)} \leq 1} \left| \hat{E} \varphi(\hat{X}_s^1) - \hat{E} \varphi(\hat{X}_s^2) \right|$$

$$\leq \sup_{\|\varphi\|_{C_p(\mathbb{R}^d)} \leq 1} \hat{E} \left| \varphi(\hat{X}_s^1) - \varphi(\hat{X}_s^2) \right|$$

$$\leq \hat{E} \left| \hat{X}_s^1 - \hat{X}_s^2 \right|,$$  \hspace{1cm} (12)

and

$$\rho^2(\mu_s^1, \mu_s^2) \leq \left( \hat{E} \left| \hat{X}_s^1 - \hat{X}_s^2 \right| \right)^2 \leq \hat{E} \left| \hat{X}_s^1 - \hat{X}_s^2 \right|^2 = \hat{E} |Z_s|^2 = G_s.$$

Thus, by the Jensen inequality, we get that

$$G_t \leq L_2 \hat{E} \int_0^t \left( \kappa_1(|Z_s|^2) + \kappa_2(G_s) \right) \, ds \leq L_2 \int_0^t \left( \kappa_1(\hat{E}|Z_s|^2) + \kappa_2(G_s) \right) \, ds$$

$$= L_2 \int_0^t \left( \kappa_1(G_s) + \kappa_2(G_s) \right) \, ds.$$
By Lemma 3.6 we have that $G_t = 0$ and then $Z_t = 0, \forall t \geq 0, \text{ a.s.}$. Therefore, the pathwise uniqueness is right. \hfill \square

Finally, Theorem 3.1 can be proved by Theorems 3.5, 3.7 and [5, Proposition 3.20, P.309].

4 The Convergence Rate for the Euler–Maruyama Approximation

In the section, we consider the convergence rate for the Euler–Maruyama approximation $\{X^n_t\}$ defined in (6), i.e.,

$$X^n_t = \xi + \int_0^t b(X^n_{s_n}, \mu^n_{s_n}) ds + \int_0^t \sigma(X^n_{s_n}, \mu^n_{s_n}) dW_s,$$

where $s_n = \lfloor \frac{t}{2^n} \rfloor$ and $[a]$ denotes the integer part of $a$.

Theorem 4.1 Suppose $b$ and $\sigma$ satisfy $(\mathbf{H}_1)$ and $(\mathbf{H}'_2)$ and $\mathbb{E}|\xi|^{2p} < \infty$ for any $p > 1$. Then, there exists a $T_0 > 0$ such that

$$\mathbb{E}\left( \sup_{t \in [0,T]} |X^n_t - X_t|^2 \right) = O(2^{-n}T_0),$$

where $O(2^{-n}T_0)$ means that $\frac{O(2^{-n}T_0)}{2^{-n}T_0}$ is bounded.

Proof Set $H_t := X^n_t - X_t$, and then $H_t$ satisfies

$$H_t = \int_0^t \left( b(X^n_{s_n}, \mu^n_{s_n}) - b(X_s, \mu_s) \right) ds + \int_0^t \left( \sigma(X^n_{s_n}, \mu^n_{s_n}) - \sigma(X_s, \mu_s) \right) dW_s.$$

It follows from the Itô formula that

$$|H_t|^2 = J_1 + J_2 + J_3,$$

where

$$J_1 := \int_0^t 2\langle H_s, b(X^n_{s_n}, \mu^n_{s_n}) - b(X_s, \mu_s) \rangle ds,$$

$$J_2 := \int_0^t 2\langle H_s, (\sigma(X^n_{s_n}, \mu^n_{s_n}) - \sigma(X_s, \mu_s)) dW_s \rangle,$$

$$J_3 := \int_0^t \| \sigma(X^n_{s_n}, \mu^n_{s_n}) - \sigma(X_s, \mu_s) \|^2 ds.$$
For $J_1$, by $(H^2_\eta)$ and (12) it holds that

$$
\mathbb{E}|J_1| \leq 2\mathbb{E}\int_0^t |H_s| |b(X^n_{s_n}, \mu^n_{s_n}) - b(X_s, \mu_s)| \, ds
$$

$$
\leq 2\mathbb{E}\int_0^t \left( |H_s| |b(X^n_{s_n}, \mu^n_{s_n}) - b(X^n_s, \mu^n_s)| + |H_s| |b(X^n_s, \mu^n_s) - b(X_s, \mu_s)| \right) \, ds
$$

$$
\leq \mathbb{E}\int_0^t |H_s|^2 + |b(X^n_{s_n}, \mu^n_{s_n}) - b(X^n_s, \mu^n_s)|^2 \, ds
$$

$$
+ 2\mathbb{E}\int_0^t \lambda_1 (|H_s|^2 + |b(X^n_s, \mu^n_s)|) \, ds
$$

$$
\leq \mathbb{E}\int_0^t |H_s|^2 \, ds + 2\mathbb{E}\int_0^t \lambda_1 \left( |X^n_{s_n} - X^n_s|^2 \gamma_1(|X^n_{s_n} - X^n_s|) \right) \, ds
$$

$$
+ 2\mathbb{E}\int_0^t \lambda_1 |H_s|^2 \gamma_1(|H_s|) \, ds + \lambda_1 \mathbb{E}\int_0^t |H_s|^2 \, ds + \lambda_1 \mathbb{E}\int_0^t \rho^2(\mu^n_s, \mu_s) \, ds
$$

$$
\leq C\mathbb{E}\int_0^t |H_s|^2 \, ds + C_{\lambda_1} \mathbb{E}\int_0^t \kappa_2(\rho(\mu^n_s, \mu_s)) \, ds + C_{\lambda_1} \int_0^t \mathbb{E}(|X^n_{s_n} - X^n_s|^2) \, ds
$$

$$
+ C_{\lambda_1} \int_0^t \kappa_2(|H_s|^2) \, ds,
$$

where in the last inequality the following result is used that

$$
x \gamma_1(x) \leq \kappa_\eta(x), \quad x > 0,
$$

$$
x^2 \gamma_1(x) \leq \kappa_\eta(x^2),
$$

and for $0 < \eta < \frac{1}{e}$

$$
\kappa_\eta(x) = \begin{cases} 
0, & x = 0, \\
x \log x^{-1}, & 0 < x \leq \eta, \\
(\log \eta^{-1} - 1)x + \eta, & x > \eta.
\end{cases}
$$

Here the properties of $\kappa_\eta$ can be referred to in Remark 2.4. And then, the Jensen inequality gives that

$$
\mathbb{E}|J_1| \leq C \int_0^t \mathbb{E}|H_s|^2 \, ds + C_{\lambda_1} \int_0^t \kappa_2((\mathbb{E}|X^n_{s_n} - X^n_s|^2)^{1/2}) \, ds
$$

$$
+ C_{\lambda_1} \int_0^t \mathbb{E}(|X^n_{s_n} - X^n_s|^2) \, ds + C_{\lambda_1} \int_0^t \kappa_2(\mathbb{E}|H_s|^2) \, ds,
$$

and furthermore,

$$
\mathbb{E}\left( \sup_{r \in [0,T]} |J_1| \right) \leq C \int_0^T \mathbb{E}\left( \sup_{r \in [0,s]} |H_r|^2 \right) \, ds
$$

$$
+ C_{\lambda_1} \int_0^T \kappa_2\left(\left(\mathbb{E}\left( \sup_{r \in [0,s]} |X^n_{s_n} - X^n_s|^2 \right) \right)^{1/2}\right) \, ds.
$$
\begin{align}
&+ C_{\lambda_1} \int_0^T \mathbb{E} \left( \sup_{r \in [0,s]} \left| X_n^{r_n} - X_r^n \right|^2 \right) ds \\
&+ C_{\lambda_1} \int_0^T \kappa_\eta \left( \mathbb{E} \left( \sup_{r \in [0,s]} \left| H_r \right|^2 \right) \right) ds. 
\end{align}

(13)

Similarly, we obtain

\begin{align}
\mathbb{E} \left( \sup_{t \in [0,T]} |J_3| \right) &\leq C \int_0^T \mathbb{E} \left( \sup_{r \in [0,s]} \left| H_r \right|^2 \right) ds \\
&+ C_{\lambda_2} \int_0^T \kappa_\eta \left( \mathbb{E} \left( \sup_{r \in [0,s]} \left| X_n^{r_n} - X_r^n \right|^2 \right) \right) ds \\
&+ C_{\lambda_2} \int_0^T \mathbb{E} \left( \sup_{r \in [0,s]} \left| X_n^{r_n} - X_r^n \right|^2 \right) ds \\
&+ C_{\lambda_2} \int_0^T \kappa_\eta \left( \mathbb{E} \left( \sup_{r \in [0,s]} \left| H_r \right|^2 \right) \right) ds.
\end{align}

(14)

For \( J_2 \), by (H2), (14), the BDG inequality and the Young inequality, one can get that

\begin{align}
\mathbb{E} \left( \sup_{t \in [0,T]} |J_2| \right) &\leq C \mathbb{E} \left( \int_0^T \left| H_t \right|^2 \right) \left\| \sigma (X_{s_n}^n, \mu_{s_n}^n) - \sigma (X_s, \mu_s) \right\|^2 ds \right)^{1/2} \\
&\leq C \mathbb{E} \left( \sup_{t \in [0,T]} \left| H_t \right|^2 \right) \int_0^T \left\| \sigma (X_{s_n}^n, \mu_{s_n}^n) - \sigma (X_s, \mu_s) \right\|^2 ds \right)^{1/2} \\
&\leq \frac{1}{4} \mathbb{E} \left( \sup_{t \in [0,T]} \left| H_t \right|^2 \right) + C \mathbb{E} \int_0^T \left\| \sigma (X_{s_n}^n, \mu_{s_n}^n) - \sigma (X_s, \mu_s) \right\|^2 ds \\
&\leq \frac{1}{4} \mathbb{E} \left( \sup_{t \in [0,T]} \left| H_t \right|^2 \right) + C \int_0^T \mathbb{E} \left( \sup_{r \in [0,s]} \left| H_r \right|^2 \right) ds \\
&+ C_{\lambda_2} \int_0^T \kappa_\eta \left( \mathbb{E} \left( \sup_{r \in [0,s]} \left| X_n^{r_n} - X_r^n \right|^2 \right) \right) ds \\
&+ C_{\lambda_2} \int_0^T \mathbb{E} \left( \sup_{r \in [0,s]} \left| X_n^{r_n} - X_r^n \right|^2 \right) ds \\
&+ C_{\lambda_2} \int_0^T \kappa_\eta \left( \mathbb{E} \left( \sup_{r \in [0,s]} \left| H_r \right|^2 \right) \right) ds.
\end{align}

(15)
Combining (13)–(15), we know that

\[
\mathbb{E}\left( \sup_{t \in [0,T]} |H_t|^2 \right) \leq C \int_0^T \mathbb{E}\left( \sup_{r \in [0,s]} |H_r|^2 \right) \, ds \\
+ C \int_0^T \kappa^2 \eta \left( \left( \mathbb{E}\left( \sup_{r \in [0,s]} |X_{r_n}^n - X_r^n|^2 \right) \right)^{\frac{1}{2}} \right) \, ds \\
+ C \int_0^T \kappa \eta \left( \mathbb{E}\left( \sup_{r \in [0,s]} |X_{r_n}^n - X_r^n|^2 \right) \right) \, ds \\
+ C \int_0^T \mathbb{E}\left( \sup_{r \in [0,s]} |X_{r_n}^n - X_r^n|^2 \right) \, ds \\
+ C \int_0^T \kappa \eta \left( \mathbb{E}\left( \sup_{r \in [0,s]} |H_r|^2 \right) \right) \, ds,
\]

where \( C > 0 \) is a constant depending on \( \lambda_1, \lambda_2 \). Next, we estimate

\[
\mathbb{E}\left( \sup_{r \in [0,T]} |X_{r_n}^n - X_r^n|^2 \right).
\]

Note that for \( r_n = \frac{i}{2^n} T \leq r < \frac{i+1}{2^n} T, i = 0, 1, 2, \ldots, 2^n - 1 \)

\[
X_r^n = X_{r_n}^n + \int_{r_n}^r b(X_s^n, \mu_{s_n}) \, ds + \int_{r_n}^r \sigma(X_s^n, \mu_{s_n}) \, dW_s \\
= X_{r_n}^n + b(X_{r_n}^n, \mu_{r_n}^n)(r - r_n) + \sigma(X_{r_n}^n, \mu_{r_n}^n)(W_r - W_{r_n}).
\]

By (2), (7), the Hölder inequality and the BDG inequality, it holds that

\[
\mathbb{E}\left( \sup_{\frac{i}{2^n} T \leq r < \frac{i+1}{2^n} T} |X_r^n - X_{r_n}^n|^2 \right) \\
\leq 2 \mathbb{E}\left( |b(X_{r_n}^n, \mu_{r_n}^n)|^2 \left| \frac{i+1}{2^n} T - \frac{i}{2^n} T \right|^2 \right) \\
+ 2 \mathbb{E}\left( \|\sigma(X_{r_n}^n, \mu_{r_n}^n)\|^2 \sup_{\frac{i}{2^n} T \leq r < \frac{i+1}{2^n} T} |W_r - W_{r_n}|^2 \right) \\
\leq C 2^{-n} T + C \left( \mathbb{E}\left\|\sigma(X_{r_n}^n, \mu_{r_n}^n)\right\|^4 \right)^{1/2} \left( \mathbb{E}\sup_{\frac{i}{2^n} T \leq r < \frac{i+1}{2^n} T} |W_r - W_{r_n}|^4 \right)^{1/2} \\
\leq C 2^{-n} T,
\]
where the constant \( C > 0 \) is independent of \( n \). Thus, we obtain

\[
\mathbb{E}\left( \sup_{t \in [0,T]} |H_t|^2 \right) \leq C \int_0^T \mathbb{E}\left( \sup_{r \in [0,t]} |H_r|^2 \right) ds + CT \kappa_\eta \left( \mathbb{E}\left( \sup_{r \in [0,t]} |H_r|^2 \right) \right) ds \\
+ CT \kappa_\eta^2 (2^{-n}T)^{1/2} + CT \kappa_\eta (C2^{-n}T) + CT (2^{-n}T)
\]

\[
\leq C \int_0^T \left( \mathbb{E}\left( \sup_{r \in [0,t]} |H_r|^2 \right) + \kappa_\eta \left( \mathbb{E}\left( \sup_{r \in [0,t]} |H_r|^2 \right) \right) \right) ds \\
+ CT \kappa_\eta^2 (2^{-n}T)^{1/2} + CT \kappa_\eta (C2^{-n}T) + CT (2^{-n}T).
\]

By Lemma 144 in [9, P. 113] and Lemma 2.1 in [11], we have

\[
\mathbb{E}\left( \sup_{t \in [0,T]} |H_t|^2 \right) \leq A \exp(-CT),
\]

where \( A := CT \kappa_\eta^2 (C(2^{-n}T)^{1/2}) + CT \kappa_\eta (C2^{-n}T) + CT (2^{-n}T) \). Thus, there exists a \( T_0 > 0 \) such that

\[
\mathbb{E}\left( \sup_{t \in [0,T_0]} |X^n_t - X_t|^2 \right) = O(2^{-n}T_0).
\]

If \( T_0 \geq T \), the proof is over; if \( T_0 < T \), on \([T_0, 2T_0], [2T_0, 3T_0], \ldots, [[T_0/T_0]T_0, T] \), by the same way to the above we deduce and conclude that

\[
\mathbb{E}\left( \sup_{t \in [0,T]} |X^n_t - X_t|^2 \right) = O(2^{-n}T_0).
\]

The proof is completed. \( \square \)

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