ON A HITCHIN-THORPE INEQUALITY FOR MANIFOLDS WITH FOLIATED BOUNDARIES

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Abstract. We prove a Hitchin-Thorpe inequality for noncompact 4-manifolds with foliated geometry at infinity by extending on previous work by Dai and Wei. After introducing the objects at hand, we recall some preliminary results regarding the $G$-signature formula and the rho invariant, which are used to obtain expressions for the signature and Euler characteristic in our geometric context. We then derive our main result, and present examples.

Résumé: En se basant sur des travaux de Dai et Wei, on démontre une inégalité de Hitchin-Thorpe pour variété non-compactes de dimension 4, et munies d’une géométrie feuilletée à l’infini. Après avoir défini les notions pertinentes à cette étude, on rappelle quelques résultats concernant la formule de $G$-signature et l’invariant rho, qu’on utilise ici pour obtenir des expressions de la signature et de la caractéristique d’Euler dans notre cadre géométrique. On démontre ensuite nos résultats principaux avant de présenter quelques exemples.

1. Introduction

If $M$ is a closed compact and oriented 4-dimensional Einstein manifold, then its Euler characteristic $\chi(M)$ and its (Hirzebruch) signature $\tau(M)$ must satisfy $\chi(M) \geq 3|\tau(M)|/2$. This is the statement of the original Hitchin-Thorpe inequality, which was proved in [Hit74] and [ST69], and subsequently extended to various contexts, as in [Gro82], [Kot98], [Kot12] and [Sam98] to name a few. The most relevant generalization for the present work is that of Dai and Wei in [DW07], where the manifolds of interest are noncompact and have fibred geometries at infinity.

In the present note, we derive a Hitchin-Thorpe inequality for manifolds with a foliation structure (resolved by a fibration) at infinity. To be more precise, if $\overline{W}$ is the compactification at infinity of such a space $W$, the type of boundary $W = \partial M$, that we are considering is obtained as a quotient $\overline{W} = \overline{W}/\Gamma$, where $\overline{W}$ is the total space of a smooth fibration $F \to \overline{W} \to \Sigma$, and $\Gamma$ is a finite group acting smoothly and freely on $\overline{W}$, and also acting smoothly on the base $\Sigma$ in such a way that the projection $\phi : \overline{W} \to \Sigma$ is $\Gamma$-equivariant: $\phi(g \cdot w) = g \cdot \phi(w)$ for all $w \in \overline{W}$ and $g \in \Gamma$. If $\nu : \overline{W} \to \overline{W}/\Gamma$ is the quotient map with respect to the group action, then the fibration $\overline{W}$ induces a foliation atlas $F$ on the boundary $W = \overline{W}/\Gamma$ whose leaves are the images of the fibres $F$ under $\nu$.

In addition to the foliation on $W$, we assume that $\overline{M}$ is endowed with a $\mathcal{F}$-metric or a foliated cusp metric, which we now introduce. Let $c : I \times W \to M$ be a local diffeomorphism describing a tubular neighbourhood of the boundary (with $I = [0,1]$, $x \in C^\infty(M, \mathbb{R}_+)$ a boundary defining function for $W$, and $\tilde{\nu} := Id_I \times \nu$.

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be the covering $I_x \times \bar{W} \rightarrow I_x \times W$. A metric $g_F$ on $M$ is called a (product-type) $F$-metric (\cite{Roc12}, p.1314) if it satisfies
\[ \hat{\nu}^* c^* g_F = \frac{dx^2}{x^4} + \phi^* h + x^2 \kappa \in C^\infty \left( I \times \bar{W}, S^2 T^* (I \times \bar{W}) \right), \]
where $h$ is a $\Gamma$-invariant metric on $\Sigma$, and $\kappa \in C^\infty \left( \bar{W}, S^2 T^* \bar{W} \right)$ is a $\Gamma$-invariant tensor restricting to a metric on the fibres of $\bar{W} \rightarrow \Sigma$. With the same notations, a metric $g_{Fc}$ on $M$ is a (product-type) foliated cusp metric (\cite{Roc12}, (1.6) p.1314) if it is such that
\[ \hat{\nu}^* c^* g_{Fc} = \frac{dx^2}{x^2} + \phi^* h + x^2 \kappa \in C^\infty \left( I \times \bar{W}, S^2 T^* (I \times \bar{W}) \right). \]
Manifolds admitting such metrics are the main topic of \cite{Roc12}, where a pseudo-differential calculus adapted to this setting is constructed, before addressing the index theory of the associated Dirac-type operators. More recently, a study of their Hodge cohomology has been carried-out in \cite{GR15}.

We start our treatment by recalling some facts pertaining to the $G$-signature formula and the Atiyah-Bott-Lefschetz fixed-point formula in the setting of manifolds with boundaries, on which a finite group acts smoothly and has only isolated fixed-points in the interior. The results are then established in the third section, first by obtaining the signature and the Euler characteristic for a noncompact $4k$-dimensional manifold with a metric which is asymptotic to a foliated boundary or a foliated cusp metric, before specializing to the four dimensional case where $\bar{W}$ is a circle bundle over a compact Riemann surface, and finally deriving the Hitchin-Thorpe inequality. In the last section we illustrate the results with some examples.

2. Background Material

We give a brief exposition of the results needed for the upcoming sections.

2.1. The $G$-Signature Formula. The $G$-signature formula is a special case of the Lefschetz fixed point formula for elliptic complexes. In this section, $(X,g)$ is a $4k$-dimensional compact oriented Riemannian manifold with nonempty boundary $\bar{W}$, and a metric $g$ which is of product type near the boundary. Let $\Gamma \subset \text{Isom}(X)$ be a finite group of orientation-preserving isometries such that the action has only isolated fixed-points in the interior. We employ the following notations:

- $g\bar{B}$ is the (even) signature operator on $\bar{X}$, and $g\bar{A}$ is the odd signature operator on $\bar{W}$ (induced by $g|_{\bar{W}}$).
- For an eigenvalue $\lambda \in \text{Spec}^g \bar{A} \setminus \{0\}$ ($g\bar{A}$ is self-adjoint and elliptic), its associated eigenspace is $E_\lambda \subset L^2(\bar{W}, \Lambda^r \bar{W})$.
- For $r = 0, \cdots, m = 4k$, $\bar{H}^r$ denotes the image of the relative cohomology group $H^r(\bar{X}, \bar{W}; \mathbb{C})$ in the absolute cohomology $H^r(\bar{X}; \mathbb{C})$. Also, $\bar{H}_+^{2k}$ are the subspaces of $\bar{H}^{2k}$ on which the non-degenerate bilinear form induced by the cup product is positive or negative definite.
- For a fixed element $a \in \Gamma$, $a^*|_{\bar{W}}$ is the morphism induced in cohomology, $a^*_\lambda$ is the morphism induced on $E_\lambda$, and $a^*_x : T_x \bar{X} \rightarrow T_{a(x)} \bar{X}$ is the differential at $x \in \bar{X}$. 

Let $\phi : \tilde{X} \to \tilde{X}$ be an isometry that has only isolated fixed-points, and $dVol^g$ the volume form associated to $(\tilde{X}, g)$. For a fixed-point $x \in \tilde{X}$ of $\phi$, the differential $\phi_*|_x \in \text{Aut}(T_x \tilde{X})$ is also an isometry, and we have an orthogonal decomposition $T_x \tilde{X} = \oplus_{j=1}^{2k} V_j$ into $\phi$-invariant 2-planes. We choose an orthogonal basis $\{e_j, e'_j\}$ of $V_j$ for all $j = 1, \cdots, 2k$ such that

$$(dVol^g)_x(e_1, e'_1, \cdots, e_{2k}, e'_{2k}) = 1,$$

With respect to this basis, $\phi_*|_x$ is block-diagonal with components of the form

$$\left( \begin{array}{cc} \cos [\theta_{\phi,j}(x)] & -\sin [\theta_{\phi,j}(x)] \\ \sin [\theta_{\phi,j}(x)] & \cos [\theta_{\phi,j}(x)] \end{array} \right).$$

The numbers $\{\theta_{\phi,j}(x)\}_{j=1}^{2k}$ constitute a coherent system of angles for $\phi_*|_x$ ([AB68, p.473]).

Now, for a fixed isometry $a \in \Gamma$, we define the following topological quantities:

- The $G$-signature with respect to $a \in \Gamma$ is given by:
  $$\tau(a, \tilde{X}) = \text{Tr} \left( a^* |_{\tilde{H}_{2k}} \right) - \text{Tr} \left( a^* |_{\tilde{H}_{2k}} \right).$$

- The $G$-eta function with respect to $a$ is
  $$\eta_a(s, g\tilde{A}) = \sum_{\lambda \in \text{Spec} \lambda \{0\}} \text{sgn} \lambda |\lambda|^{-s} \text{Tr}(a_\lambda^*).$$

This function is holomorphic for $\text{Re}(s) \gg 0$ and admits a meromorphic continuation to the whole complex plane with no pole at $s = 0$ ([Don78]). The associated $G$-eta invariant is $\eta_a(g\tilde{A}) = \eta_a(0, g\tilde{A})$. For $a = \text{Id}$, this is the usual eta invariant.

- Let $\text{Fix}(a) \subset \tilde{X}$ be the fixed point set of $a \in \Gamma$. For $z \in \text{Fix}(a)$, the signature defect at that point is given by
  $$\text{def}(a, g\tilde{B})[z] = \prod_{j=1}^{2k} \left( \frac{\lambda_j + 1}{\lambda_j - 1} \right) = \prod_{j=1}^{2k} (-i) \cot \left( \theta_{a,j}(z)/2 \right),$$

where $\lambda_j = \exp(i\theta_{a,j}(z))$ is the $j$-th eigenvalue of the linear map $a_*|_z \in \text{Aut}(T_z \tilde{X})$ (Theorem 4.5.2 [Gil95], formula (7.2) [AB68]).

As a special case of a theorem proved by Donnelly ([Don78, Theorem 2.1; Gil95 Theorem 4.5.8]), we have the $G$-signature formula:

**Theorem 2.1.** Under the hypotheses of this subsection, the $G$-signature formula for $a \in \Gamma \setminus \{\text{Id}\}$ reads

$$\tau(a, \tilde{X}) = \sum_{z \in \text{Fix}(a)} \text{def}(a, g\tilde{B})[z] - \frac{1}{2} \eta_a(g\tilde{A}).$$

**Remark.** In [APS75a] and [Don78], the authors use the restriction of $g\tilde{A}$ to even forms rather than the odd signature operator itself, which is why they do not have this extra $(1/2)$ factor that appears here and in [DW07] next to the eta invariants.
2.2. The rho invariant of a finite covering. With the same notations as above, \( \Gamma \) acts smoothly and freely on \((\widetilde{W}, g_{\widetilde{W}})\), and \( \nu : \widetilde{W} \to W = \widetilde{W}/\Gamma \) is the induced covering projection. In this subsection, \( A_g \) denotes the odd signature operator associated to the metric induced by \( g \in C^\infty(\widehat{X}, S^2 T^* \widehat{X}) \) on \( (\gamma \tilde{A} = \nu^* A_g) \).

Atiyah, Patodi and Singer studied the signature with local coefficients in [APS75]. For a finite covering \( \widetilde{W} \xrightarrow{\nu} W \), and a one-dimensional unitary representation \( \alpha \) of \( \Gamma \) with associated flat bundle \( E_\alpha \to W \), they obtain the following expression ((I.5) in [Don78]):

\[
\eta(A_{g,\alpha}) = \frac{1}{|\Gamma|} \sum_{a \in \Gamma} \eta_a(\gamma \tilde{A}) \chi_\alpha(a),
\]

where \( A_{g,\alpha} \) is the operator induced by \( A_g \) on \( AT^* W \otimes E_\alpha \) and \( \chi_\alpha(a) \) is the character of \( a \in \Gamma \). If \( \alpha \) is the trivial representation, this formula reduces to ((I.6) in [Don78])

\[
\eta(A_g) = \frac{1}{|\Gamma|} \sum_{a \in \Gamma} \eta_a(\gamma \tilde{A}).
\]

The rho invariant of the finite covering \( \widetilde{W} \xrightarrow{\nu} W \) is then defined as:

\[
\rho(\widetilde{W}, W) = \frac{1}{2} \left[ \eta(\gamma \tilde{A}) - |\Gamma| \eta(A_g) \right] = \frac{1}{2} \sum_{a \neq Id} \eta_a(\gamma \tilde{A}).
\]

Eta invariants depend on the choice of a Riemannian metric, but this is not the case for the rho invariant of a finite covering:

Proposition 2.2. Let \( g_i, i = 0,1 \) be two Riemannian metrics on \( W \), \( \tilde{g}_i = \nu \circ g_i \) the pullback metrics on \( \widetilde{W} \), and \( \rho_i(\widetilde{W}, W) \) the corresponding rho invariants; then

\[
\rho_0(\widetilde{W}, W) = \rho_1(\widetilde{W}, W).
\]

Proof. Let \( \widetilde{M} = \widetilde{W} \times I \), \( M = W \times I \), \( \tilde{\nu} = \nu \times \text{Id}_I \) where \( I = [0,1] \subset \mathbb{R} \). For \( s \in I \), we have a metric on \( M \) given by \( h = (1 - s)g_0 + sg_1 + ds^2 \) with pullback \( \tilde{h} = \tilde{\nu}^* h \).

If \( \tilde{A}_i \) and \( A_i \) are the odd signature operators associated to \( \tilde{g}_i \) and \( g_i \), then applying the Atiyah-Patodi-Singer theorem to \( \widetilde{M} \) and \( M \) yields

\[
\tau(\widetilde{M}) = \int_{\widetilde{M}} L(\widetilde{M}) - \frac{1}{2} \left[ \eta(\tilde{A}_1) - \eta(\tilde{A}_0) \right],
\]

\[
\tau(M) = \int_M L(M) - \frac{1}{2} \left[ \eta(A_1) - \eta(A_0) \right].
\]

From the definition of the \( \rho_i \) and these formulas, we have

\[
\rho_1(\widetilde{W}, W) - \rho_0(\widetilde{W}, W) = \left[ \int_{\widetilde{M}} L(\widetilde{M}) - |\Gamma| \int_M L(M) \right] - \tau(\widetilde{M}) + |\Gamma| \tau(M).
\]

The first term in the RHS vanishes since \( \tilde{\nu} \) is a local isometry. To see that \( \tau(\widetilde{M}) \) vanishes, we consider the long exact sequence of relative cohomology

\[
\cdots \overset{\delta^*}{\to} H^{2k}(\widetilde{M}, \partial \widetilde{M}) \overset{i}{\to} H^{2k}(\widetilde{M}) \overset{i}{\to} H^{2k}(\partial \widetilde{M}) \overset{\delta^*}{\to} H^{2k+1}(\widetilde{M}, \partial \widetilde{M}) \overset{i}{\to} \cdots.
\]

Since \( \partial \widetilde{M} = \widetilde{W} \cup -\widetilde{W} \), \( H^{2k}(\partial \widetilde{M}) = H^{2k}(\widetilde{W}) \) and \( H^{2k+1}(\widetilde{M}, \partial \widetilde{M}) = H^{2k}(\widetilde{W}) \oplus H^{2k}(\widetilde{W}) \), the map \( i \) is injective, and the image of \( H^{2k}(\widetilde{M}, \partial \widetilde{M}) \) in \( H^{2k}(\widetilde{M}) \) is trivial, which means that \( \tau(\widetilde{M}) = 0 \). With the same argument we get \( \tau(M) = 0 \), from which \( \rho_1 = \rho_0 \) follows. \( \square \)
2.3. Manifolds with foliated boundaries. Let $M$ be a connected, oriented and noncompact manifold of dimension $4k$, $k \geq 1$.

**Definition 2.3.** We say that a complete Riemannian manifold $(M, g)$ has a foliated geometry at infinity resolved by a fibration if there exists a compactification at infinity $\overline{M} = M \cup W$ by a boundary $W = \partial \overline{M}$, and a field of symmetric bilinear forms $\overline{g} : \overline{M} \rightarrow S^2T^*\overline{M}$ satisfying the following conditions:

(i) The bilinear form $\overline{g}$ coincides with the initial metric on the interior: $\overline{g}_{|M} = g \in \Gamma(M, S^2T^*M)$;

(ii) The boundary $W$ is the base of a finite covering $\nu : \tilde{W} \rightarrow W = \overline{W}/\Gamma$, where $\Gamma$ is a finite group of orientation-preserving isometries acting freely on $\tilde{W}$;

(iii) The manifold $\tilde{W}$ is also the total space of a smooth fibration $F \rightarrow \tilde{W} \xrightarrow{\phi} \Sigma$, where $F$ and $\Sigma$ are closed compact oriented manifolds with $\dim F > 0$;

(iv) The group $\Gamma$ acts smoothly on the base $\Sigma$, and the projection $\phi : \tilde{W} \rightarrow \Sigma$ is $\Gamma$-equivariant.

Since we will only consider foliations resolved by a fibration in this paper, we will simply say that $M$ has a foliated geometry at infinity when it satisfies the definition above, and that the compactification $\overline{M}$ has a foliated boundary. Under these assumptions, we will use $F$ to denote the foliation of the boundary $W = \partial \overline{M}$ by the images of the fibres $F$ under the projection $\nu : \tilde{W} \rightarrow W$.

For the remaining of this section, we let $\overline{M}$ be a manifold with foliated boundary, and we fix a boundary defining function (bdf) $x \in C^\infty(M, \mathbb{R}^+)$ for $\overline{M}$, so that $W = \{x = 0\}$ and $x > 0$ on $M = \overline{M} \setminus W$. We will consider two classes of foliated cusp vector fields over $\overline{M}$, namely the $F$-vector fields:

$$\mathcal{V}_F(\overline{M}) := \left\{ \xi \in C^\infty(T\overline{M}) \mid \xi \cdot x \in C^\infty(\partial \overline{M}) \right\}$$

and the $F_c$-vector fields, defined as $\mathcal{V}_{F_c}(\overline{M}) := \frac{1}{x} \cdot \mathcal{V}_F(\overline{M})$. These are smooth sections of special vector bundles over $\overline{M}$ (Roc12, section 1):

**Definition 2.4.** The $F$-tangent bundle $\mathcal{X}T \overline{M}$ is the unique vector bundle over $\overline{M}$ such that $\mathcal{V}_F(\overline{M}) \simeq C^\infty(\overline{M}, \mathcal{X}T \overline{M})$, and the $F_c$-tangent bundle $\mathcal{X}_c T \overline{M}$ is the one such that $\mathcal{V}_{F_c}(\overline{M}) \simeq C^\infty(\overline{M}, \mathcal{X}_c T \overline{M})$.

Let $c : [0, 1] \times W \rightarrow \overline{M}$ be a local diffeomorphism onto a tubular neighbourhood of the boundary, and letting $\Gamma$ act trivially on $[0, 1]$, define the covering projection

$$\tilde{\nu} : [0, 1] \times \tilde{W} \rightarrow [0, 1] \times W, (x, p) \mapsto (x, \nu(p)).$$

Suppose $U_{\tilde{W}} \subset \tilde{W}$ is an open subset on which the covering projection $\nu$ restricts to a diffeomorphism $U_{\tilde{W}} \xrightarrow{\nu} U_W$. The bundles $\mathcal{X}T \overline{M}$ and $\mathcal{X}_c T \overline{M}$ are respectively related to the $\phi$- and the $d$-tangent bundles defined in section 3 of [DW07] by the following isomorphisms:

$$\mathcal{X}T([0, 1] \times U_W) \simeq \tilde{\nu}_* [\phi^* \mathcal{X}T([0, 1] \times U_{\tilde{W}})]$$

$$\mathcal{X}_c T([0, 1] \times U_W) \simeq \tilde{\nu}_* [d^* T([0, 1] \times U_{\tilde{W}})]$$

We will work with the following types of metrics on $\overline{M}$:

**Definition 2.5.** An exact foliated boundary metric $g_F$ ($F$-metric for short) on $\overline{M}$ is a smooth Riemannian metric on $\mathcal{X}T \overline{M}$ that admits a decomposition of the following form on $c([0, 1] \times W) \subset \overline{M}$:

$$\tilde{\nu}^* c^* g_F = \frac{dx^2}{x^2} + \frac{\phi^* h}{x^2} + \kappa + x^2 B,$$
where $h \in \Gamma(\Sigma, S^2T^*\Sigma)$ is a $\Gamma$-invariant Riemannian metric on the base space of $\overline{W}$, $\kappa \in \Gamma(\overline{W}, S^2T^*\overline{W})$ is a $\Gamma$-invariant tensor restricting to a metric on the fibres of $\overline{W}$ and possibly depending smoothly on $x$, and $B$ is the pullback of a smooth section of $S^2[\overline{F}T^*\overline{M}]$. If $B \equiv 0$, we will say that $g_F$ is an asymptotic $\mathcal{F}$-metric. If we have a decomposition of the form:

$$\hat{\nu}^* c^*(\hat{g})_F = \frac{dx^2}{x^4} + \phi^* h + g_F,$$

where this time $\{g_F(y)\}_{y \in \Sigma}$ is a family of $\Gamma$-invariant Riemannian metrics on the fibres $F$ of $\overline{W}$ smoothly parametrized by the base $\Sigma$, we say that $g_F$ is a product-type $\mathcal{F}$-metric.

A foliated cusp metric $g_{Fc}$ on $\overline{M}$ is a Riemannian metric on $\overline{F} \times \overline{M}$ that is obtained from an $\mathcal{F}$-metric by a conformal rescaling $g_{Fc} = x^2 g_F$.

Remark.

1) The meaning of the term “product-type” metric differs from one author to another. In this paper, we are following the convention used in [Vai01] and [DW07], but in [MM98] and [Roc12], “product-type” is what we call “asymptotic” here.

2) Having a globally defined product-type metric requires the choice of a splitting $T \overline{W} = T^V \overline{W} \oplus T^H \overline{W}$ into horizontal and vertical subbundles. This is an implicit assumption in [DW07].

3) Following the terminology of [DW07], an asymptotic metric is defined as a metric of the form $g_F = \hat{g}_F + xA$, where $\hat{g}_F$ is product-type and $A \in \Gamma(M, S^2[\overline{F}T^*\overline{M}])$ is a tensor such that $A(x^2 \partial_x, \cdot) \equiv 0$. This is compatible with our definition in the following sense: in a coordinate chart near the boundary, we may take the Taylor expansion of the restriction $(g_F)|_{TF \times TF}$ w.r.t. the boundary defining function $x$ to obtain

$$\hat{\nu}^* c^*(\hat{g})_F|_{TF \times TF} = \kappa|_{TF \times TF} = g_F + x \cdot \hat{\kappa},$$

where $\{g_F\}$ is a family of metrics on $F$ as in the last definition (not depending on $x$). We thus have a local decomposition $\kappa = g_F + x \cdot \hat{\nu}^* c^* A$, with $A \in \Gamma(M, S^2[\overline{F}T^*\overline{M}])$ satisfying $A(x^2 \partial_x, \cdot) \equiv 0$.

Going back to definition 2.3, we required the existence of a symmetric bilinear form $\hat{g} : \overline{M} \to S^2T^*\overline{M}$ that extends the complete metric $g$ on the interior $M$. The purpose of $\hat{g}$ is to encode the singular behaviour of $g$ at infinity, which is why we did not assume that $\hat{g}$ is a smooth section of $S^2T^*\overline{M}$. In the next section, we will use the following definition:

**Definition 2.6.** A manifold with foliated boundary $(M, g)$ will be called an $\mathcal{F}$-manifold if the extension $\hat{g}$ coincides with an exact $\mathcal{F}$-metric on the compactification $\overline{M}$. Similarly, if $\hat{g} \equiv g_{Fc}$ for some exact $\mathcal{F}_c$-metric, we will call $(M, g)$ an $\mathcal{F}_c$-manifold. Moreover, the prefixes “asymptotic” and “product-type” will be employed to designate the particular $\mathcal{F}/\mathcal{F}_c$-metrics considered.

3. Results

Throughout this section, $(M, g)$ is a $4k$-dimensional manifold with foliated geometry at infinity, and for brevity, we use the same notations as in definitions 2.3 and 2.5 without specifying all the properties of the objects $\hat{g}$, $\kappa$, $\{g_F(y)\}_{y \in \Sigma}$, $A$ and $B$. 
3.1. Index formulæ. Our first fundamental result is a generalisation of theorems 4.3 and 3.6 of [DW07]. As before, \( \tilde{\Gamma} : \overline{M} \to \mathbb{R}_+ \) is a fixed boundary defining function, the map \( e : [0, 1]_x \times W \to \overline{M} \) is a local diffeomorphism onto a tubular neighborhood of the boundary \( W = \partial M \), and \( \tilde{\nu} : [0, 1]_x \times \tilde{W} \to [0, 1]_x \times W \) is the covering projection induced by the \( \Gamma \)-action on the fibre bundle \( F \to \tilde{W} \to \phi : (\Sigma, h) \).

**Theorem 3.1.** Let \( (M, (g_F)|_M) \) be an \( F \)-manifold, where \( g_F = \tilde{g}_F + x^2 B \) is an exact metric on \( S^2[F^* TM] \) such that \( \tilde{g}_F \) is asymptotic and \( B \in \Gamma(S^2[F^* TM]) \).

Let \( \tilde{g}_F \) be the pullback of \( (\tilde{g}_F)|_{c[0, 1] \times W} \) to \( [0, 1]_x \times \tilde{W} \) and \( F \tilde{\nabla} \) be the Levi-Civita connection of \( g_F \). The signature and Euler characteristic are given by:

\[
\tau(M) = \int_M L(M, F \nabla) + L_{CS}(\partial M, F \nabla) + \frac{1}{1!} \left\{ \rho(\tilde{W}, W) - \frac{1}{2} \lim_{\epsilon \to 0} \eta(\tilde{W}, (\tilde{g}_F)|_{x=\epsilon}) \right\},
\]

\[
\chi(M) = \int_M e(M, F \nabla) + e_{CS}(\partial M, F \nabla),
\]

where the terms \( L_{CS}(\partial M, F \nabla) \) and \( e_{CS}(\partial M, F \nabla) \) denote the Chern-Simons boundary corrections, \( \rho(\tilde{W}, W) \) is the rho invariant introduced in subsection 2.3 and \( \eta(\tilde{W}, (\tilde{g}_F)|_{x=\epsilon}) \) is the eta invariant of the odd signature operator associated to the metric \((\tilde{g}_F)|_{x=\epsilon}\) on the hypersurface \( \{ x = \epsilon \} \subset [0, 1]_x \times \tilde{W} \).

If \( (M, (g_F)|_M) \) is an \( F \)-manifold with \( g_F = x^2 \tilde{g}_F \), one has similar index formulæ as above, with \( F \tilde{\nabla} \) instead of \( F \tilde{\nabla} \) and \( L_{CS}(\partial M, F \nabla) = e_{CS}(\partial M, F \nabla) = 0 \).

**Remark.** The signature formula above does not follow from the index formula of [Roc12] since the signature operator is not Fredholm.

**Proof.** Let \( 0 < \epsilon \ll 1 \), and set \( M_\epsilon = \overline{M} \setminus c[0, \epsilon[ \times W \). In this case \( \partial M_\epsilon \approx W \) is the hypersurface \( \{ x = \epsilon \} \) in \( \overline{M} \), and topologically, one has \( \tau(M_\epsilon) = \tau(M) \) and \( \chi(M_\epsilon) = \chi(M) \). We prove the result for asymptotic metrics \( \tilde{g}_F \) and \( \tilde{g}_F = x^2 \tilde{g}_F \) first, assuming that they decompose as

\[
\tilde{\nabla} c^* \tilde{g}_F = \frac{dx^2}{x^2} + \frac{\phi^* h}{x^2} + \kappa \quad \text{and} \quad \tilde{\nabla} c^* \tilde{g}_F = \frac{dx^2}{x^2} + \phi^* h + x^2 \kappa
\]
on \( c(0, 1] \times W \subset \overline{M} \). We recall the following facts:

(a) If \( \tilde{g}_c \) is an auxiliary metric on \( \overline{M} \) such that

\[
\tilde{\nabla} c^* \tilde{g}_c = \frac{dx^2}{\epsilon^2} + \frac{\phi^* h}{\epsilon^2} + \kappa \quad \text{and} \quad (\tilde{g}_c)|_{\partial M_\epsilon} \equiv (\tilde{g}_c)|_{\partial M},
\]

then it induces the same odd signature operator as \( \tilde{g}_F \) on \( \partial M_\epsilon \).

(b) If \( P(M_\epsilon, \tilde{\nabla}) \) is a characteristic form computed with the curvature of some Levi-Civita connection \( \tilde{\nabla} \), we have in our case a transgression form \( TP(M_\epsilon, c^* \tilde{\nabla}, F \tilde{\nabla}) \) satisfying

\[
dTP(M_\epsilon, c^* \tilde{\nabla}, F \tilde{\nabla}) = P(M_\epsilon, F \tilde{\nabla}) - P(M_\epsilon, c^* \tilde{\nabla}),
\]

where \( F \tilde{\nabla} \) and \( c^* \tilde{\nabla} \) are the Levi-Civita connections of \( \tilde{g}_F \) and \( \tilde{g}_c \) respectively.

(c) Going back to the hypotheses on \( W \approx \partial M \) (def. 2.3), we can introduce a finite covering \( \Gamma \to [\epsilon, 1] \times \tilde{W} \to [\epsilon, 1] \times \partial M_\epsilon \), where \( \Gamma \) acts trivially on \([\epsilon, 1] \). From subsection 2.3, we know that the eta invariants of the odd signature operators of \((W, (\tilde{g}_F)|_{x=\epsilon}\) and \((\tilde{W}, (\tilde{g}_F)|_{x=\epsilon}\) are related by the equation:

\[
\frac{1}{2} \eta(W, (\tilde{g}_F)|_{x=\epsilon}) = \frac{1}{1!} \left[ \frac{1}{2} \eta(\tilde{W}, (\tilde{g}_F)|_{x=\epsilon}) - \rho(\tilde{W}, W) \right].
\]

(d) If we look at the metrics \( \tilde{g}_F = x^2 \tilde{g}_F \) and \( \epsilon^2 \tilde{g}_c \) on \( M_\epsilon \), we see that

\[
\epsilon^2 (\tilde{g}_c)|_{\partial M_\epsilon} = \epsilon^2 (\tilde{g}_F)|_{\partial M_\epsilon} = (\tilde{g}_F)|_{\partial M_\epsilon},
\]
and since the eta invariant is not modified by rescaling the associated metric, we moreover have
\[
\eta(\tilde{W}, (\tilde{g}_e)|_{x=\varepsilon}) = \eta(W, (\tilde{g}_\phi)|_{x=\varepsilon}) = \eta(W, (x^2\tilde{g}_\phi)|_{x=\varepsilon}).
\]
Applying the Atiyah-Patodi-Singer theorem to \((M, \tilde{g}_e)\) and using (a) and (b) above, we obtain for any \(\varepsilon > 0\) sufficiently small that
\[
\tau(M) = \int_{M,} L(M_\varepsilon, \hat{\nabla}) - \frac{1}{2} \phi(W, (\tilde{g}_e)|_{x=\varepsilon})
\]
\[
= \int_{M,} L(M_\varepsilon, \hat{\nabla}) - \int_{\partial M_\varepsilon} TL(M_\varepsilon, \hat{\nabla}, \hat{F}\hat{\nabla}) - \frac{1}{2} \phi(W, (\tilde{g}_e)|_{x=\varepsilon})
\]
\[
= \left[ \int_{M,} L(M_\varepsilon, \hat{\nabla}) \right]_{(A)} - \left[ \int_{\partial M_\varepsilon} TL(M_\varepsilon, \hat{\nabla}, \hat{F}\hat{\nabla}) \right]_{(B)}
\]
\[
- \frac{1}{|1|} \left[ \frac{1}{2} \phi(W, (\tilde{g}_e)|_{x=\varepsilon}) \right]_{(C)} + \frac{1}{|1|} \rho(\tilde{W}, W)
\]
and similarly
\[
\chi(M) = \left[ \int_{M,} e(M_\varepsilon, \hat{\nabla}) \right]_{(D)} - \left[ \int_{\partial M_\varepsilon} Te(M_\varepsilon, \hat{\nabla}, \hat{F}\hat{\nabla}) \right]_{(E)}
\]
It then remains to take the limits of the terms (A)-(E) as \(\varepsilon \to 0\). The terms (A) and (D) tend to the invariant integrals in the statement of the theorem, the correction \(\rho(\tilde{W}, W)\) is invariant under changes of metrics, while for (B) and (E), we have by definition that
\[
L_{CS}(\partial M, F\hat{\nabla}) = - \lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} TL(M_\varepsilon, \hat{\nabla}, \hat{F}\hat{\nabla}),
\]
\[
e_{CS}(\partial M, F\hat{\nabla}) = - \lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} Te(M_\varepsilon, \hat{\nabla}, \hat{F}\hat{\nabla}).
\]
In the case where we use a foliated cusp metric \(\tilde{g}_{F_\varepsilon}\), we obtain the same expressions as above for \(\tau(M)\) and \(\chi(M)\), with \(\hat{\nabla}\) replaced by \(\hat{F}\hat{\nabla}\), and \(\eta(\tilde{W}, (\tilde{g}_\phi)|_{x=\varepsilon})\) replaced by \(\eta(W, (x^2\tilde{g}_\phi)|_{x=\varepsilon})\). By facts (c) and (d), these eta invariants coincide. Regarding the Chern-Simons corrections, since \(\nu : \tilde{W} \to W\) is a local isometry, we have:
\[
\int_{\partial M_\varepsilon} TL(M_\varepsilon, \hat{\nabla}, \hat{F}\hat{\nabla}) = \frac{1}{|1|} \int_{\{\varepsilon\} \times \tilde{W}} TL([\varepsilon, 1] \times \tilde{W}, \nu^*(\hat{\nabla}), \hat{d}\hat{\nabla})|_{x=\varepsilon},
\]
\[
\int_{\partial M_\varepsilon} Te(M_\varepsilon, \hat{\nabla}, \hat{F}\hat{\nabla}) = \frac{1}{|1|} \int_{\{\varepsilon\} \times \tilde{W}} Te([\varepsilon, 1] \times \tilde{W}, \nu^*(\hat{\nabla}), \hat{d}\hat{\nabla})|_{x=\varepsilon},
\]
where \(\hat{d}\hat{\nabla}\) the Levi-Civita connection of \(x^2\tilde{g}_\phi\). Taking \(\varepsilon \to 0\) then yields \(L_{CS}(\partial M, F\hat{\nabla}) = e_{CS}(\partial M, F\hat{\nabla}) = 0\) by proposition A.4.
If we now consider an exact metric \(g_{\tilde{F}} = \tilde{g}_{\tilde{F}} + x^2B\) with Levi-Civita connection \(\tilde{F}\hat{\nabla}\), the transgression integrals in equations (3.1) and (3.2) are replaced by \(\int_{\partial M_\varepsilon} TL(M_\varepsilon, \hat{\nabla}, \hat{F}\hat{\nabla})\) and \(\int_{\partial M_\varepsilon} Te(M_\varepsilon, \hat{\nabla}, \hat{F}\hat{\nabla})\) respectively. Again, by expressing these as integrals on \(\{\varepsilon\} \times \tilde{W}\) and applying proposition A.4 we have:
\[
L_{CS}(\partial M, F\hat{\nabla}) = - \lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} TL(M_\varepsilon, \hat{\nabla}, \hat{F}\hat{\nabla}) = - \lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} TL(M_\varepsilon, \hat{\nabla}, \hat{F}\hat{\nabla}),
\]
\[
e_{CS}(\partial M, F\hat{\nabla}) = - \lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} Te(M_\varepsilon, \hat{\nabla}, \hat{F}\hat{\nabla}) = - \lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} Te(M_\varepsilon, \hat{\nabla}, \hat{F}\hat{\nabla}),
\]
and likewise, \( L_{CS}(\partial \overline{M}, F, \nabla) = e_{CS}(\partial \overline{M}, F, \nabla) = 0 \) for exact foliated cusp metrics.

\( \square \)

\textbf{Remark.} The reader should be warned that the corresponding result in \cite{DW07} contains a mistake. In theorems 3.6 and 4.3 of \cite{DW07}, it is stated that the Chern-Simons terms vanish for fibres \( F \) of arbitrary (positive) dimension, but this is false in general. A counter example for the Euler characteristic for even dimensional fibres is given by \( M = \mathbb{R}^{2k} \times S^{2p} \) equipped with the product of the standard metrics \( g \), which is a fibred-boundary metric for the boundary defining function \( 1/r \), where \( r \) is the radial coordinate on \( \mathbb{R}^{2k} \). The boundary at infinity is \( S^{2k-1} \times S^{2p} \), viewed as a trivial \( S^{2p} \)-bundle over \( S^{2k-1} \). The Pfaffian of the curvature of \( g \) vanishes in this case since one of the factors is flat, but we still have \( \chi(M) \neq 0 \), which means that the Chern-Simons term does not vanish. Dai and Wei have been made aware of this mistake, and they intend to publish an erratum that rectifies this issue (see last subsection of appendix A for more details).

As seen in the previous proof, the fact that \( \nu : \tilde{W} \to W \) is a local isometry allows us to compute transgression integrals on \([0, 1] \times \tilde{W} \) endowed with a fibred boundary metric. As an obvious consequence of proposition A.6 we have the following sufficient conditions for the vanishing of the Chern-Simons corrections with an \( F \)-metric:

\textbf{Proposition 3.2.} Let \( \overline{M} \) be a 4k-dimensional manifold with foliated boundary. Suppose there’s a splitting \( T\overline{M} = TV \oplus TH \overline{W} \) into horizontal and vertical sub-bundles for the cover \( \tilde{W} \to \Sigma \) of \( W = \partial \overline{M} \), and consider an exact \( F \)-metric \( g_F \) on \( \tilde{F}T\overline{M} \) of the form

\[ g_F = \tilde{g}_F + x \cdot A + x^2 \cdot B, \quad \text{with} \ A, B \in \Gamma(\overline{M}, S^2[\tilde{F}T^*\overline{M}]), \]

where \( \tilde{g}_F \) is a product metric, and where the pullback \( \tilde{A} = \nu^*e^*(A|_{\tilde{W} \times [0, 1]}) \) of \( A \) satisfies the following:

\begin{align*}
(i) \quad & \tilde{A}(x^2 \partial_x, \cdot) = 0 \quad \text{and} \quad \tilde{A}(xY_1, xY_2) = O(x), \\
(ii) \quad & \left( V_1 \cdot \tilde{A}(V_2, xY_1) - V_2 \cdot \tilde{A}(V_1, xY_1) - \tilde{A}([V_1, V_2], xY_1) \right)_{|x=0} = 0,
\end{align*}

for all \( V_1, V_2 \in \mathfrak{x}(\Sigma) \) and all \( V_1, V_2 \in \Gamma(\tilde{W}, TV \tilde{W}) \). Then:

(a) If the dimension of the fibres \( F \) of \( \tilde{W} \) is odd, then the boundary correction to the Euler characteristic vanishes:

\[ e_{CS}(\partial \overline{M}, F, \nabla) = 0. \]

(b) If \( \dim F = 1 \), then the correction to the Hirzebruch signature vanishes:

\[ L_{CS}(\partial \overline{M}, F, \nabla) = 0. \]

An interesting case for applications to asymptotically locally flat (ALF) manifolds, is when \( \dim(M) = 4 \) and \( \phi : \tilde{W} \to \Sigma \) is a circle bundle. We now list the assumptions made for the remaining of this section:

- **The circle bundle \( \tilde{W} \to \Sigma \):** Suppose \( \Sigma \) is a compact Riemann surface, and let \( (E, g_E) \to (\Sigma, h) \) be an oriented rank 2 Euclidian vector bundle. We denote by \( \tilde{W} = S(E) \) the circle subbundle of \( E \), and by \( \tilde{X} = D(E) \) the disc bundle over \( \Sigma \) such that \( \partial \tilde{X} = \tilde{W} \).

- **Action of \( \Gamma \):** The finite group \( \Gamma \) acts as follows on \( \tilde{W} = S(E) \): On the surface \( \Sigma \), the action is smooth and there are only isolated fixed points. We assume that we have a faithful representation of \( \Gamma \) on \( \mathbb{C} \), and that the action on \( S^1 \) is smooth and
free. This $\Gamma$-action is extended to $\mathcal{X} = D(E)$ as follows: the action remains the same on the base $\Sigma$, and if $a \in \Gamma$ acts on a fibre of $\tilde{W}$ via $\theta \mapsto a(\theta)$, then the same isometry sends a point $(r, \theta)$ in a fibre of $\mathcal{X}$ to $(r, a(\theta))$. With this requirement, the fixed points are isolated, and strictly included in $\tilde{X} \setminus \tilde{W}$. 

- **Foliated boundary metrics on $\overline{M}$**: The Levi-Civita connection of $g_{F}$ induces a splitting $TW = T^{V}W \oplus T^{H}W$, so we may consider product-type foliated boundary metrics on $\overline{M}$. We assume that $\mathcal{F}$-metrics on $\overline{M}$ decompose as

\begin{equation}
\hat{\nu}^{*}c^{*}g_{F} = \left( \frac{dx^{2}}{\varepsilon^{4}} + \frac{\phi^{*} h}{\varepsilon^{2}} + g_{F} \right) + \hat{\nu}^{*}c^{*} (xA + x^{2}B)
\end{equation}

near $W = \partial \overline{M}$, where $g_{F} \in \Gamma(\tilde{W}, S^{2}[T^{V}\tilde{W}]^{*})$ is a family of Riemannian metrics on $S^{1}$ smoothly parametrized by $\Sigma$, and $A, B \in \Gamma(\overline{M}, S^{2}[\mathcal{F}^{*}T^{*}\overline{M}])$ are symmetric bilinear forms such that the pullback $\hat{A}$ of $A_{\epsilon \in [0,1] \times W}$ to $[0,1]_{x} \times W$ satisfies condition (i) in proposition 3.2. (ii) is automatically satisfied since the fibre is 1-dimensional. The only difference with the exact metric definition in 3.2 is the decomposition $\kappa = g_{F} + x\hat{A}$. 

Under these assumptions, and using the notations of subsection 2.1, one has:

**Corollary 3.3.** Let $M$ be a 4-dimensional manifold with foliated geometry at infinity satisfying the hypotheses above, and let $g$ be an exact $\mathcal{F}$- or $\mathcal{F}_{\kappa}$-metric on $\overline{M}$. The topological invariants of $M$ are given by:

\begin{equation}
\tau(M) = \int_{\overline{M}} L(M, \eta^{\nabla}) - \frac{1}{|\Gamma|} \left\{ \sum_{a \neq \text{Id}} \sum_{z \in \text{Fix}(a)} \text{def}(a, \eta^{\nabla} \hat{B})[z] + \frac{\chi(E)}{3} \right\} + \epsilon(E),
\end{equation}

\begin{equation}
\chi(M) = \int_{\overline{M}} \epsilon(M, \eta^{\nabla}),
\end{equation}

where $\chi(E)$ is the Euler characteristic of the vector bundle $E$, and $\epsilon(E)$ is defined as

\begin{equation}
\epsilon(E) = \begin{cases} 
-1; & \chi(E) < 0 \\
0; & \chi(E) = 0 \\
1; & \chi(E) > 0 
\end{cases}
\end{equation}

This is an immediate consequence of the following lemma:

**Lemma 3.4.** For an $\mathcal{F}$-metric of the form 3.3 satisfying equations 3.3, the Chern-Simons terms vanish:

\begin{equation}
L_{CS}(\partial M, \mathcal{F}^{*}\nabla) = c_{CS}(\partial M, \mathcal{F}^{*}\nabla) = 0,
\end{equation}

the limit of the eta invariant in theorem 3.1 coincides with the following adiabatic limit:

\begin{equation}
\lim_{\varepsilon \to 0} \frac{1}{2} \eta(\tilde{W}, (g_{\epsilon})_{|z = \varepsilon}) = \frac{1}{2} \lim_{\varepsilon \to 0} \eta(S(E), \varepsilon^{-2}\phi^{*}h + g_{F}) = \frac{\chi(E)}{3} - \epsilon(E),
\end{equation}

and the rho invariant is given by:

\begin{equation}
\rho(\tilde{W}, W) = (|\Gamma| - 1) \epsilon(E) - \sum_{a \neq \text{Id}} \sum_{z \in \text{Fix}(a)} \text{def}(a, \eta^{\nabla} \hat{B})[z].
\end{equation}

**Proof.** By proposition 3.2 it is sufficient to consider that the $\mathcal{F}$-metric $g_{F}$ and the auxiliary metric $g_{\epsilon}$ are of product-type:

\begin{equation}
\hat{\nu}^{*}c^{*}g_{F} = \left( \frac{dx^{2}}{\varepsilon^{4}} + \frac{\phi^{*} h}{\varepsilon^{2}} + g_{F} \right) \quad \text{and} \quad \hat{\nu}^{*}c^{*}g_{\epsilon} = \left( \frac{dx^{2}}{\varepsilon^{4}} + \frac{\phi^{*} h}{\varepsilon^{2}} + g_{F} \right).
\end{equation}
This is because the perturbations $x A + x^2 B \in \Gamma(S^2|\mathcal{T}^* M|)$ to $g_{\mathcal{F}}$ that we are allowing yield vanishing transgression integrals when $\epsilon \to 0$. Also, since $(g_{\mathcal{F}})|_{x= \epsilon} \equiv (\epsilon^{-2}\phi^* h + g_{\mathcal{F}})$ on $[0,1]\times \tilde{W}$, theorem 3.1 and proposition 5.2 imply that:

$$\chi(M) = \int_M e(M, \nabla^F) \quad \text{and} \quad \tau(M) = \int_M L(M, \nabla^F) - \frac{1}{2} \lim_{\epsilon \to 0} \eta(\tilde{W}, \epsilon^{-2}\phi^* h + g_{\mathcal{F}}).$$

The last limit coincides with the adiabatic limit of the eta invariant of the odd signature operator associated to the metric $(\epsilon^{-2}\phi^* h + g_{\mathcal{F}})$ on $\tilde{W} = S(E)$, so as a special case of theorem 3.2 in [DZ95] (also formula (5.4) in [DW07]), we have:

$$\frac{1}{2} \lim_{\epsilon \to 0} \eta(S(E), \epsilon^{-2}\phi^* h + g_{\mathcal{F}}) = \frac{\chi(E)}{3} - \epsilon(E).$$

To determine the expression of the rho invariant, we use the results (and notations) of subsection 2.1 on the spaces $\tilde{X} = D(E)$ and $\tilde{W} = S(E)$. From the G-signature formula and the definition of $\rho$ in terms of the G-eta invariants, we have:

$$\rho(\tilde{W}, W) = \sum_{a \neq Id} \left[ \tau(a, \tilde{X}) - \sum_{z \in \text{Fix}(a)} \text{def}(a, \tilde{g} \tilde{B}) | z \right].$$

It remains to specify the value of $\tau(a, \tilde{X}) = \text{Tr} \left( a^* | \tilde{H}_z^2 \right) - \text{Tr} \left( a^* | \tilde{H}_z^2 \right)$ under our hypotheses. We have the following (complex) cohomology group isomorphisms

$$H^2(\tilde{X}) \overset{\text{homotopy}}{\cong} H^2(\Sigma) \cong \mathbb{C},$$

$$H^2(\tilde{X}, \tilde{W}) \overset{\text{def}}{=} H^2(\tilde{X} \setminus \tilde{W}) \overset{\text{Poincaré}}{\cong} H^2(\tilde{X} \setminus \tilde{W}) \overset{\text{homotopy}}{\cong} H^2(\Sigma) \cong \mathbb{C}.$$

Since the $a^*$ are induced by orientation preserving isometries, we have that $a^* |_{H^2(\tilde{X})} = Id_{H^2(\tilde{X})}$. Moreover, the image $\tilde{H}_z^2 = \tilde{H}_z^2 \oplus \tilde{H}_z^2$ is at most 1-dimensional, so one of the subspaces must be trivial, hence $\tau(a, \tilde{X}) \in \{\pm 1, 0\}$. On the other hand, the spaces $\tilde{H}_z^2$ are the subspaces on which the quadratic form given by

$$Q : H^0(\Sigma) \otimes H^0(\Sigma) \to \mathbb{R} ; \alpha \otimes \beta \mapsto \langle \alpha \circ \beta \sim [e_E], [\Sigma] \rangle,$$

is either positive or negative definite ($[e_E]$ is the Euler class of $E$ here, c.f. section 5 of [DW07]). This results from the Thom isomorphism $\cdot \circ \Phi : H^0(\Sigma) \to H^2(\tilde{X} \setminus \tilde{W})$ and the fact that $\langle \Phi^2, [\Sigma] \rangle = \langle [e_E], [\Sigma] \rangle$, where $\Phi$ denotes the Thom class. The sign of $\tau(a, \tilde{X})$ is hence that of $Q(1, 1) = \chi(E)$, i.e. $\tau(a, \tilde{X}) = \epsilon(E)$, and the result follows.

**Remark:** For practical computations, having explicit expressions for the action of $\Gamma$ should allow one to determine coherent systems of angles $\{\theta_{a,j}(z)\}_{j=1}^2$ for the fixed points $z \in \text{Fix}(a)$ of $a \in \Gamma$. In this case, the boundary correction to the signature is

$$\frac{1}{2} \lim_{\epsilon \to 0} \eta(W, (g_{\mathcal{F}})|_{x= \epsilon}) = \frac{1}{|\Gamma|} \left[ \frac{\chi(E)}{3} - \sum_{a \neq Id} \sum_{z \in \text{Fix}(a)} \prod_{j=1}^2 \cot \left( \theta_{a,j}(z)/2 \right) \right] - \epsilon(E).$$

### 3.2. Hitchin-Thorpe inequality

Here, we are interested in an obstruction to the existence of Einstein metrics on the noncompact 4-manifolds that we are considering.
Theorem 3.5. Let \( M \) be a 4-dimensional manifold with foliated geometry at infinity satisfying the assumptions listed before corollary 3.3. If \( M \) admits an exact Einstein \( F \)- or \( F_c \)-metric, then

\[
\chi(M) \geq \frac{3}{2} \left( \tau(M) - \epsilon(E) + \frac{1}{|T|} \left\{ \sum_{a \neq \text{Id}} \sum_{z \in \text{Fix}(a)} \text{def}(a, g) \hat{B}(z) + \frac{\chi(E)}{3} \right\} \right).
\]

If equality occurs, then the universal cover of \( M \) is a complete Ricci-flat (anti-)self-dual manifold.

Proof. On a 4-dimensional Riemannian manifold \((M, g)\) (with \( g \) an exact foliated boundary or foliated cusp metric), the Riemann tensor decomposes as (1.128, [Bes08])

\[
\hat{g}R = \begin{pmatrix} \hat{W}^+ + \frac{S}{12} \text{Id} & \hat{Z} \\ \hat{W}^- + \frac{S}{12} \text{Id} & 0 \end{pmatrix},
\]

where \( \hat{W}^+ \) and \( \hat{W}^- \) are respectively the self-dual and anti-self-dual parts of the Weyl tensor, \( S \) is the scalar curvature and \( \hat{Z} = \text{Ric} - \frac{S}{4} \text{Id} \) is the traceless part of the Ricci tensor. This decomposition allows us to re-express the Euler form and the Hirzebruch \( L \)-polynomial in terms of the components of \( \hat{g}R \), so that our invariant integrals become (6.31 and 6.34, [Bes08]):

\[
\frac{3}{2} \int_M L(M, \hat{g} \nabla) = \frac{1}{8\pi^2} \int_M \left( |\hat{W}^+|^2 - |\hat{W}^-|^2 \right) dVol^g
\]

\[
= \frac{3}{2} \left[ \tau(M) + \frac{1}{|T|} \left\{ \frac{1}{2} \lim_{\varepsilon \to 0} \eta(\hat{A}_\varepsilon) - \rho(\hat{W}, W) \right\} \right],
\]

\[
\int_M \epsilon(M, \hat{g} \nabla) = \frac{1}{8\pi^2} \int_M \left( |\hat{W}^+|^2 + |\hat{W}^-|^2 - |\hat{Z}|^2 + \frac{S^2}{24} \right) dVol^g = \chi(M).
\]

Using the fact that \( \hat{Z} = 0 \) if \( g \) is an Einstein metric, adding and subtracting the integrals above leads to

\[
\chi(M) \geq \frac{3}{2} \left[ \tau(M) + \frac{1}{|T|} \left\{ \frac{1}{2} \lim_{\varepsilon \to 0} \eta(\hat{A}_\varepsilon) - \rho(\hat{W}, W) \right\} \right],
\]

and the inequality of the statement follows from lemma 3.4. When equality occurs, the equations above yield

\[
\int_M \left( |\hat{W}^\pm|^2 + \frac{S^2}{24} \right) dVol^g = 0 \implies S = |\hat{W}^\pm| = 0.
\]

The vanishing of the scalar curvature means that \( M \) is Ricci-flat, and the vanishing of the (anti-) self-dual part of the Weyl tensor implies that the holonomy group of the pullback of \( g \) to the universal cover of \( M \) is contained in \( SU(2) \) (by section 3 of [Hit74]), which proves the second claim of the theorem. \( \square \)

4. Examples

Our examples are built on quotients of the Gibbons-Hawking ansatz by cyclic groups, which are studied extensively in [GH79]. In appendix B we give some details and references on multi-Taub-NUT metrics. Let \( k \geq 1 \) be an integer, and suppose that \( \{p_j\}_{j=1}^k \) are points in \( \mathbb{R}^3 \) of coordinates \( p_j = (\cos(2\pi j/k), \sin(2\pi j/k), 0) \) with respect to the origin. We consider the principal circle bundle

\[
S^1 \longrightarrow M \longrightarrow \mathbb{R}^3 \setminus \{p_j\}_{j=1}^k,
\]
whose first Chern class yields \((-1)\) when paired with the homology class associated to a sphere centred at one of the monopoles \(p_j\). We equip \(M\) with the multi-Taub-NUT metric
\[
g = \pi^* [V \cdot ((dx^1)^2 + (dx^2)^2 + (dx^3)^2)] + \pi^*(V^{-1}) \cdot (d\theta + \pi^* \omega)^2,
\]
where the function \(V : \mathbb{R}^3 \setminus \{p_j\} \to \mathbb{R}\) is defined as:
\[
V(x) = 1 + \frac{1}{2} \sum_{j=1}^k \frac{1}{|x - p_j|},
\]
and determines \(g\) uniquely. \(\theta\) is a coordinate on the fibres, and \(\omega\) is a connection 1-form on \(\mathbb{R}^3 \setminus \{p_j\}\) (unique up to “gauge transformations”) of curvature \(d\omega = \ast dV\). By suitably adding points \(\{q_i\}\) to \(M\) above the monopoles, we obtain a smooth Ricci-flat hyper-Kähler completion \((M_0, g_0)\) of \((M, g)\) [AKL89], with only one asymptotically locally flat end at infinity. If \((\overline{M}_0, g_0)\) is the compactification at infinity of \((M_0, g_0)\), the boundary \(\partial\overline{M}_0\) is diffeomorphic to a quotient of a Hopf fibration by a \(\mathbb{Z}_k\)-action on the fibres [Hit79], i.e. it is the total space of a fibration of the form
\[
S^1 \to S^3/\mathbb{Z}_k \xrightarrow{\varphi} S^2.
\]
Let \((r, \varphi, \psi)\) be the spherical coordinates on \(\mathbb{R}^3\) \((r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2})\), and consider the metric \(h = (d\varphi)^2 + \sin^2 \varphi (dv)^2\) on the base \(S^2\) of \(\partial M_0\). If we introduce the boundary defining function \(\rho := 1/\pi(r)\) on \(\pi^{-1}(\{r > 1\}) \subset \overline{M}_0\), then the multi-Taub-NUT metric can be rewritten as:
\[
g_0 = \frac{d\rho^2}{\rho^2} + \frac{\phi^* h}{\rho^2} + \kappa,
\]
where \(\kappa = V^{-1}(d\theta + \omega)^2 \in \pi(\partial \overline{M}_0, S^2 I^* \partial \overline{M}_0)\) restricts to the metric \(V^{-1}(d\theta)^2\) on the fibres, and depends smoothly on \(\rho\), which means that \(g_0\) is an asymptotic \(\phi\)-metric.

In the examples below, we consider quotients of \(\partial \overline{M}_0\) by the action of a cyclic group, in which case \(\overline{M}_0\) gives rise to an asymptotic \(F\)-metric (after averaging with respect to the group action).

Finally, the invariants of interest for our discussion are the following ([GH79], (5.4) in [DW07]):
\[
\chi(M_0) = k, \quad \tau(M_0) = 1 - k \quad \text{and} \quad \frac{1}{2} \lim_{\eta} (\partial \overline{M}_0, \varphi, g_0, \partial \overline{M}_0) = \frac{k}{3} - 1.
\]
To simplify the notations, we will write \(\rho\) instead of \(\rho(\partial \overline{M}_0, \partial \overline{M}_0, \Gamma)\), and \(\eta A\) instead of \(\eta(\partial \overline{M}_0, \varphi, g_0, \partial \overline{M}_0)\).

**Example 4.1.** Let \(\Gamma = \mathbb{Z}_k\), and suppose that the generator \(1 \in \Gamma\) acts as follows on \(\overline{M}_0\). On the base \(\mathbb{R}^3\), the group \(\Gamma\) acts via a rotation of \(\frac{2\pi}{k}\) about the axis \(Oz\) (w.r.t the canonical orientation). As explained on p.106 of [Wri12], this action can be lifted to an action on \(\overline{M}_0\) by isometries, but not in a unique fashion. The action on \(\overline{M}_0\) is uniquely determined once we specify how \(\Gamma\) acts on the fibre above the origin in \(\mathbb{R}^3\), and to ensure that it is free, we require that \(1 \in \Gamma\) acts as multiplication by \(e^{\frac{2\pi}{k}}\) on \(\pi^{-1}(\{0\}) \approx S^1\) (as it is done in section 6 of [Gri13]). We consider the manifold \(M_1 := \overline{M}_0/\Gamma\) with boundary \(\partial M_1 = \partial \overline{M}_0/\Gamma\), and induced metric \(g_1\), which is a Ricci-flat foliated boundary metric (thus Einstein in particular). Since
the quotient map \( \overline{M}_0 \to M_1 \) is a local isometry, we have:

\[
\chi(M_1) - \frac{3}{2} \tau(M_1) + \frac{1}{k} (\eta^A - \rho) = \int_{M_1} e(M_1, g^i \nabla) - \frac{3}{2} \int_{M_1} L(M_1, g^i \nabla) \\
= \frac{1}{k} \left\{ \int_{\overline{M}_0} e(M_0, g^0 \nabla) - \frac{3}{2} \int_{\overline{M}_0} L(M_0, g^0 \nabla) \right\} \\
= \frac{1}{k} \left\{ \chi(M_0) - \frac{3}{2} \tau(M_0) + \eta \right\} \\
= \frac{1}{k} \left\{ k - \frac{3}{2} (1 - k + \frac{k}{3} - 1) \right\} = \frac{k - k}{k} = 0.
\]

We note that our inequality reduces to that of theorem 1.1 (and cor. 1.2) of [DW07] since \( M_1 \) is a global quotient and \( \overline{M}_0 \) is a manifold with fibred boundary.

**Example 4.2.** We consider \( M_1 \) of the previous example, and we make \( l \geq 1 \) blow-ups in its interior. We obtain the space \( X_1 = M_1 \# \mathbb{CP}^2 \), where \( \mathbb{CP}^2 \) is the complex projective plane with reverse orientation, for which we have \( \chi(\mathbb{CP}^2) = 3 \) and \( \tau(\mathbb{CP}^2) = -1 \). By the identities

\[
\tau(A\#B) = \tau(A) + \tau(B),
\]

\[
\chi(A\#B) = \begin{cases} 
\chi(A) + \chi(B) & \text{for } \dim A = \dim B = 2m + 1 \\
\chi(A) + \chi(B) - 2 & \text{for } \dim A = \dim B = 2m 
\end{cases}
\]

we have

\[
\tau(X_1) + \frac{1}{k} (\eta^A - \rho) = \frac{1}{k} \left[ \int_{\overline{M}_0} e(\overline{M}_0, g^0 \nabla) \right] + l \cdot \tau(\mathbb{CP}^2) = -\left( \frac{2}{3} + l \right),
\]

\[
\chi(X_1) = \frac{1}{k} \left[ \int_{\overline{M}_0} e(\overline{M}_0, g^0 \nabla) \right] + l \cdot (\chi(\mathbb{CP}^2) - 2) = 1 + l.
\]

Thus, for any \( l \geq 1 \), we get

\[
\chi(X_1) - \frac{3}{2} \tau(X_1) + \frac{1}{k} (\eta^A - \rho) = (l + 1) - \left| \left( 1 + \frac{3}{2} l \right) \right| = -\frac{l}{2} < 0.
\]

Using the expressions of the invariant integrals in terms of the tensors \( W^\pm \) and \( Z \) (as in the proof of 3.6), this inequality implies that \( |Z|^2 > 2|W^\pm|^2 + (S^2/24) \geq 0 \), which means that \( X_1 \) doesn’t admit an Einstein exact \( F/F_e \)-metric since the component \( Z = Ric - \frac{3}{2} \mathcal{L} \) can’t vanish.

**Example 4.3.** We start by modifying \( M_1 \) of the previous example to construct a space which isn’t a global quotient of a manifold with fibred boundary. The presentation here is largely based upon example 26 of [GR15]. Let \( B\Gamma \) be the classifying space of \( \Gamma \)-bundles, and \( \theta : \partial M_1 \to B\Gamma \) the map corresponding to the covering \( \partial M_0 \to \partial M_1 = \partial M_0/\Gamma \). Let \( \gamma : S^1 \to \partial M_1 \) be a loop such that \( [\theta \circ \gamma] \in \pi_1(B\Gamma) \) is nonzero, and \( \tilde{\gamma} : S^1 \to M_1 \cap \partial M_1 \) a smooth translation of \( \gamma \) into the interior of \( M_1 \). The loop \( \tilde{\gamma}(S^1) \) has a trivialized tubular neighborhood in \( M_1 \) since the tangent bundle of \( \partial M_1 \) is trivial (the boundary is 3-dimensional and orientable). We now perform a codimension 3 surgery along \( \tilde{\gamma}(S^1) \subset M_1 \) ( [LM89], p.299): let \( V \approx S^1 \times B^3 \) be a closed subset of \( M_1 \) such that \( \tilde{\gamma}(S^1) \subset \partial V \), and consider \( N \approx B^3 \times S^2 \) such that \( \partial V \approx \partial N \approx S^1 \times S^2 \), then define

\[
M_2 := (M_1 \setminus V^\circ) \cup_{\partial V} N,
\]

with \( \partial M_2 = \partial M_1 \). In the manifold \( M_2 \), the loop \( \tilde{\gamma}(S^1) \) is contractible, which implies that the map \( \theta : \partial M_2 \to B\Gamma \) cannot be extended to the interior \( M_2 \setminus \partial M_2 \), and hence that \( M_2 \) is not a global quotient.
ON A HITCHIN-THORPE INEQUALITY FOR MANIFOLDS WITH FOLIATED BOUNDARIES

We need to determine the topological invariants of $M_2$. Since $\partial M_2 = \partial M_0 / \Gamma$, we have the same boundary correction as before. For the Euler characteristic, we use the identity
\[ \chi(A) = \chi(A) + \chi(B) - \chi(A \cap B) \]
on $M_2 = (M_2 \setminus N^0) \cup N$ and $M_1 = (M_1 \setminus V^0) \cup V$ to obtain
\[ \chi(M_2) = \chi(M_1) + 2 + \frac{1}{k} \left[ \int_{M_0} \epsilon(M_0, \partial_0 \nabla) \right] = 3, \]

since $\chi(M_2 \setminus N) = \chi(M_1 \setminus V)$ by construction, $\chi(N) = 2$ and $\chi(V) = 0$ by homotopy equivalences, and $\chi(\partial N) = 0$ since $\dim \partial N = 3$. For the Hirzebruch signature of $M_2$, we have the following equations:
\[ \tau(M_1) = \tau(M_1 \setminus V) + \tau(V), \]
\[ \tau(M_2) = \tau(M_2 \setminus N) + \tau(N). \]

We have $H^2(V) = H^2(S^1) = 0$, so the restriction of the cup product to the image of the map $H^2(V, \partial V) \to H^2(V)$ is zero, and by the definition of the signature for a manifold with boundary, $\tau(V) = 0$. To determine $\tau(N)$, we consider the trivial bundle $E = \mathbb{R}^2 \times S^2$ over $S^2$, and set $N = D(E)$ and $\partial N = S(E)$ (resp. the disk and circle subbundles of $E \to S^2$). Using the Künneth formula and the Thom isomorphism, we obtain the following nonzero cohomology groups:
\[ H^r(N, \partial N) \simeq H^r(D(E), S(E)) \simeq H^{r-2}(S^2) = \mathbb{R} \text{ for } r = 2, 4, \]
\[ H^r(\partial N) \simeq \mathbb{R} \text{ for } r = 0, \ldots, 3. \]

Since $H^*(N) \simeq H^*(S^2)$, the long exact sequence for relative cohomology then reads
\[ 0 \to H^1(\partial N) \xrightarrow{\delta} H^2(N, \partial N) \xrightarrow{j} H^2(N) \to H^2(\partial N) \to 0, \]
and we see that $\delta$ is injective, and that $j$ is zero, which means that $\tau(N) = 0$. Finally:
\[ \tau(M_2) = \tau(M_2 \setminus N) = \tau(M_1 \setminus V) = \tau(M_1), \]
so that
\[ \tau(M_2) + \frac{1}{k} (\eta^A - \rho) = -\frac{2}{3}. \]

To illustrate theorem 3.5, consider the space $X_2 = M_2 \#_l \mathbb{C}P^2$ with $l \geq 1$ and the blow-ups in the interior. Proceeding as in the second example to determine $\chi(X_2)$ and $\tau(X_2)$, we have:
\[ \tau(X_2) + \frac{1}{k} (\eta^A - \rho) = -\left( \frac{2}{3} + l \right), \]
\[ \chi(X_2) = l + 3, \]
and therefore
\[ \chi(X_2) - \frac{3}{2} \left| \tau(X_2) + \frac{1}{k} (\eta^A - \rho) \right| = \frac{4 - l}{2}. \]

By theorem 3.5, the space $X_2$ cannot admit an Einstein exact $\mathcal{F}/\mathcal{F}_e$-metric if $l > 4$. 

Appendix A. Chern-Simons corrections for $\phi$- and $d$-metrics

Let $N$ and $F$ be closed compact oriented manifolds such that $\dim N + \dim F = 4k - 1$ for some $k \geq 1$ and $\dim F > 0$. We consider a smooth fibration $F \to W \overset{\phi}{\to} N$, where the total space $W$ is the boundary of a compact manifold $M$ with fixed boundary defining function $x : M \to \mathbb{R}_+$. In this appendix, we are interested in the behaviour of the Chern-Simons terms of the Euler characteristic and the signature of $M$ as $x \to 0$, assuming that this space is equipped with a fibred boundary or a fibred cusp metric.

A.1. Riemannian metrics on the boundary: In addition to the fibration structure of $W$, we assume that we have the following geometric objects:

- A splitting $T_W = T^V W \oplus T^H W$ into vertical and horizontal subbundles, where $T^H W$ is identified with $\phi^* TN$ and $T^V W = \ker(d\phi)$ (i.e. a connection on $W$);
- A Riemannian metric $h \in \Gamma(N, S^2 T^* N)$ on $N$;
- A family of Riemannian metrics $\{g_F(y)\}_{y \in N}$ on the fibre space $F$ that is smoothly parametrized by the base $N$.

We thus have a metric $\phi^* h$ on the subbundle $T^H W$, and by interpreting the family $\{g_F\}$ as a field of symmetric bilinear forms $\tau \in \Gamma(W, S^2 (T^* W)^*)$, we may then define $g_W = \phi^* h + \tau$, that gives a Riemannian submersion $\phi : (W, g_W) \to (N, h)$ for which the splitting $T^V W \oplus T^H W$ is orthogonal.

A.2. Riemannian metrics on $M$: In the upcoming discussion, we work on a collar neighborhood of the boundary in $M$ that is diffeomorphic to $[0, 1]_x \times W$. The local coordinates on $N$ will be denoted by $\{y^i\}_{i=1}^{\dim N}$, and we will write $\{z^a\}_{a=1}^{\dim F}$ for those on the fibre $F$. We will use the following notational convention for tensors on $M$ near the boundary:

- The index 0 is reserved for the boundary defining function: $\partial_0 = \frac{\partial}{\partial x}$;
- The indices $i, j, k, l \in \{1, \cdots, \dim N\}$ designate variables on the base of $W$: $\partial_i = \frac{\partial}{\partial y^i}$;
- The indices $a, b, c, d \in \{1, \cdots, \dim F\}$ refer to coordinates on the fiber $F$: $\partial_a = \frac{\partial}{\partial z^a}$;
- Greek indices will be used to designate arbitrary indices in the summation convention: $\{\partial_\alpha\} = \{\partial_0, \partial_i, \partial_a\}$.

We now set-up the notations for the metrics that we will be dealing with, which are smooth metrics on the $\phi$- and $d$-tangent bundles, $\phi TM$ and $dTM$ (\cite{DW07}, section 3).

Let $\tilde{g}_{\phi} = (dx/x^2)^2 + (\phi^* h/x^2) + \tau$ be a product-type fibred boundary metric, and $\tilde{g}_d = x^2 \tilde{g}_{\phi}$ the associated product-type fibred cusp metric ($\phi$- and $d$-metrics for short), where $g_W = \phi^* h + \tau$ is the submersion metric discussed above. Near $W = \partial M$, we have the following local coordinate expressions:

$$
\tilde{g}_{\phi} = \frac{dx^2}{x^4} + \frac{h_{ij}(y)}{x^2} dy^i \otimes dy^j + \tau_{ab}(y, z) dz^a \otimes dz^b,
$$

$$
\tilde{g}_d = \frac{dx^2}{x^2} + h_{ij}(y) dy^i \otimes dy^j + x^2 \tau_{ab}(y, z) dz^a \otimes dz^b.
$$
Let $A, B \in \Gamma(M, S^2[\phi T^* M])$ be symmetric bilinear forms, with $A$ such that $A(x^2 \partial_\nu, \cdot) \equiv 0$ and $A(x \partial_\nu, x \partial_\nu) = O(x)$. Asymptotic metrics are first order perturbations (in $x$) of product-type metrics, and will be denoted by:

$$\hat{g}_\phi = \tilde{g}_\phi + x \cdot A, \quad \hat{g}_d = x^2 \tilde{g}_d.$$  

**Exact metrics** are second order perturbations of product-type metrics, and will be denoted by

$$g_\phi = \tilde{g}_\phi + x \cdot A + x^2 \cdot B = \hat{g}_\phi + x^2 \cdot B, \quad g_d = x^2 \tilde{g}_d.$$  

For asymptotic metrics, we still have an orthogonal decomposition $\phi T([0,1]_x \times W) = \langle x^2 \partial_x \rangle \oplus TW$ near the boundary. From now onward, asymptotic $\phi$-metrics will be expressed as:

$$\hat{g}_\phi = \frac{dx^2}{x^4} + \phi^* h + \kappa \quad \text{and} \quad \hat{g}_d = \frac{dx^2}{x^2} + \phi^* h + x^2 \kappa,$$

where $\kappa = \tau + xA \in \Gamma(W, S^2 T^* W)$ is a bilinear form on the boundary, depending smoothly on the bdf $x$, and restricting to a metric on the fibres $F$. As for exact metrics, these are the most general smooth Riemannian metrics on $\phi TM$ and $\phi^* TM$ that we consider here (for a fixed bdf $x$), but all the properties we are interested in are coming from their “asymptotic part”.

For exact $\phi$- and $d$-metrics, the Levi-Civita covariant derivatives will be denoted by $\phi \nabla$ and $d\nabla$, the connection 1-forms by $\phi \omega$ and $d\omega$, and the curvature 2-forms by $\phi \Omega$ and $d\Omega$. To designate the corresponding objects associated to the product-type and asymptotic metrics, we will use the same superscripts on the left and add a “tilde” (product) or a “hat” (asymptotic) above.

**Remark.** 1) The factor $x^2$ in exact metrics is the smallest exponent for $x$ that gives a well-defined covariant derivative $\phi \nabla : \Gamma(\phi TM) \to \Gamma(\phi^* TM \otimes T^* M)$. Indeed, taking $g_\phi = \hat{g}_\phi + xB$ with $B \in \Gamma(M, S^2[\phi^* T^* M])$ would give $\langle d\omega, \phi \nabla_{\partial_x} \partial_x \rangle = 0(x^{-1})$, which blows-up as $x \to 0$.

2) In the local frame given by $\{\partial_\alpha \otimes dx^\alpha\}$ on $\text{End} TM$, an element $B \in \Gamma(M, S^2[\phi^* T^* M])$ decomposes as

$$B = \left( \begin{array}{ccc}
\frac{1}{x^2} B_{00} & \frac{1}{x} B_{0i} & \frac{1}{x} B_{0\alpha} \\
\frac{1}{x} B_{i0} & \frac{1}{x} B_{ij} & \frac{1}{x} B_{i\alpha} \\
\frac{1}{x} B_{\alpha 0} & \frac{1}{x^2} B_{\alpha i} & \frac{1}{x^2} B_{\alpha\beta} 
\end{array} \right),$$

where the $B_{\alpha\beta}$ are all smooth on $M$.

Finally, we introduce the auxiliary metrics $\tilde{g}_\varepsilon$ and $g_\varepsilon$ on $TM$ with $\varepsilon \in [0,1]$. The first one is of asymptotic type:

$$\tilde{g}_\varepsilon := \frac{dx^2}{\varepsilon^2} + \frac{\phi^* h}{\varepsilon^2} + \kappa,$$

while the second is a product metric near the boundary:

$$g_\varepsilon := \frac{dx^2}{\varepsilon} + \frac{\phi^* h}{\varepsilon^2} + \tau.$$

Their restrictions to $W = \partial M$ blow-up the metrics $(\phi^* h + \kappa)$ and $g_W = (\phi^* h + \tau)$ resp. in the direction of the base, and they coincide with $\phi$-metrics on hypersurfaces $\{ x = \varepsilon \} \subset M$:

$$\tilde{g}_\varepsilon|_{\{ x = \varepsilon \}} = (\tilde{g}_\phi)|_{\{ x = \varepsilon \}}; \quad g_\varepsilon|_{\{ x = \varepsilon \}} = (\tilde{g}_\phi)|_{\{ x = \varepsilon \}}.$$

Auxiliary metrics are introduced to apply the Atiyah-Patodi-Singer index theorem. The symbols $\varepsilon \nabla$, $\varepsilon \omega$ and $\varepsilon \Omega$ will respectively denote the Levi-Civita connection, the connection 1-form and the curvature 2-form of $g_\varepsilon$ and $\varepsilon^2 \cdot g_\varepsilon$, while $\varepsilon \nabla$, $\varepsilon \omega$ and $\varepsilon \Omega$ will designate the same objects for $\tilde{g}_\varepsilon$, and $\varepsilon^2 \cdot \tilde{g}_\varepsilon$. 
We now have an important technical result, that relates the connection 1-forms of the metrics defined above:

**Lemma A.1.** Let \( M_\varepsilon = \{ x \geq \varepsilon \} \subset M \) with boundary \( \partial M_\varepsilon = \{ x = \varepsilon \} \) for \( 0 < \varepsilon \ll 1 \). Then:

\[
\begin{align*}
(\phi \omega - \phi \bar{\omega})_{|\partial M_\varepsilon} &\in \varepsilon \cdot \Omega^1 (\partial M_\varepsilon, \text{End}(\phi TM)_{|\partial M_\varepsilon}), \\
(\phi \omega - d\phi \bar{\omega})_{|\partial M_\varepsilon} &\in \varepsilon \cdot \Omega^1 (\partial M_\varepsilon, \text{End}(d\phi TM)_{|\partial M_\varepsilon}).
\end{align*}
\]

**Remark.** In all the proofs of this appendix, we will focus on the dependence on \( x \) of the objects involved rather than giving precise expressions. To obtain the entries of the connection 1-forms, one first needs to determine the Christoffel symbols of the metrics at hand. The general procedure for computing these is as follows: Since \( \phi TM \) is isomorphic to \( TM \) for \( x \neq 0 \), the Christoffel symbols \( \Gamma^a_{\alpha\beta} \) of \( g_\phi \) with respect to the basis \( \{ \partial_a \} \) are computed by means of the usual formula

\[
\Gamma^a_{\beta\mu} = \frac{(g_{\phi}^{-1})_{\alpha\nu}}{2} [-\partial_{\nu}(g_{\phi})_{\beta\mu} + \partial_{\beta}(g_{\phi})_{\mu\nu} + \partial_{\mu}(g_{\phi})_{\nu\beta}],
\]

and then re-expressed in the basis \( \{ x^2 \partial_x, x \partial_i, \partial_a \} \) or in an orthonormal frame \( \{ x^2 \partial_x, x e_i, e_a \} \) using the appropriate transformation rule \( (\Gamma^a_{\alpha\beta}) \) are not components of a tensor. To obtain the Christoffel symbols of a \( d \)-metric, one uses the fact that \( g_\phi \) and \( g_\phi \) are related by a conformal rescaling. For instance, the covariant derivatives of product-type \( \phi \) - and \( d \)-metrics are related by:

\[
d\bar{\nabla}_X Y = \frac{1}{x} \left[ \phi \bar{\nabla}_X (x \cdot Y) \right] + \left( \frac{dx}{x} Y \right) X - \bar{g}_\phi (X, xY) \cdot \bar{g}_\phi^{-1} \left( \frac{dx}{x} \right).
\]

for all \( X \in \mathfrak{X}(M) \) and \( Y \in \Gamma^{dTM} = x^{-1} \cdot \Gamma(\phi TM) \), and one has analogous equations for perturbed metrics.

**Proof.** Let \( \Gamma^a_{\alpha\beta} \) and \( \tilde{\Gamma}^a_{\alpha\beta} \) be the Christoffel symbols of \( g_\phi \) and \( \tilde{g}_\phi \) in the basis \( \{ x^2 \partial_x, x \partial_i, \partial_a \} \). A direct computation yields:

\[
(A.1) \quad (\phi \omega - \phi \bar{\omega})_{\mu} = \left( \Gamma^a_{\alpha\beta} - \tilde{\Gamma}^a_{\alpha\beta} \right) dx^\mu = \begin{pmatrix} 0 & f^0_{\alpha} dx & 0 \\ f^0_{\alpha} dx & f^0_{\alpha} dx & f^0_{\alpha} dx \end{pmatrix} + x \cdot E,
\]

where \( E \in \Omega^1(M, \text{End}(\phi TM)) \) and \( f^0_{\alpha} \in \mathcal{C}^\infty(M) \) are of order \( 0 \) in \( x \), and where we have used the following convention for the entries of the connection 1-forms:

\[
\phi \omega^\beta_{\alpha} = \left( \frac{dx^a}{x^\beta_{\alpha}}, \phi \nabla_{\partial_a} (x^\alpha \partial_\beta) \right) \cdot dx^\mu = \Gamma^a_{\beta\mu} \cdot dx^\mu,
\]

with \( x^\beta_{\alpha} \partial_a \) and \( (dx^a/x^\beta_{\alpha}) \) being shorthands that designate the fields \( x^2 \partial_x, x \partial_i, \partial_a \in \Gamma(\phi TM) \) and \( (dx/x^2), (dy/x), dz^a \in \Gamma(\phi T^*M) \). By restricting to \( \partial M_\varepsilon = \{ x = \varepsilon \} \) in equation \( (A.1) \), we omit the terms in \( dx \) to find that indeed

\[
(\phi \omega - \phi \bar{\omega})_{|\partial M_\varepsilon} \in \varepsilon \cdot \Omega^1 (\partial M_\varepsilon, \text{End}(\phi TM)_{|\partial M_\varepsilon}).
\]

The same computation applied to a \( d \)-metric proves the second claim, namely that

\[
(d \omega - d \bar{\omega})_{|\partial M_\varepsilon} \in \varepsilon \cdot \Omega^1 (\partial M_\varepsilon, \text{End}(d\phi TM)_{|\partial M_\varepsilon}).
\]

Finally, in a \( \tilde{g}_d \)-orthonormal frame \( \{ x \partial_x, e_i, \frac{1}{x} e_a \} \) with transition matrix

\[
\Lambda = \begin{pmatrix} x & 0 & 0 \\ 0 & e^i_j & e^i_a \\ 0 & \frac{1}{x} e^i_j & \frac{1}{x} e^i_a \end{pmatrix},
\]
we find that the only nonvanishing entries of \((\tilde{d}\tilde{\omega} - \tilde{\omega})|_{x=\varepsilon}\) are:
\[
(\tilde{d}\tilde{\omega} - \tilde{\omega})^a_b|_{x=\varepsilon} = -(\tilde{d}\tilde{\omega} - \tilde{\omega})^a_b|_{x=\varepsilon} = -\varepsilon \cdot \varepsilon^a_b \kappa_{ab}\big|_{x=\varepsilon} \cdot dx^3 + O(\varepsilon^2), \forall \alpha, \beta \neq 0
\]
and the third claim follows. \(\square\)

The next proposition is used to prove lemma 3.4.

**Proposition A.2.** Considering a product metric \(\tilde{g}_\phi = (dx/x^2)^2 + (\phi^* h/x^2) + \tau\) and the asymptotic metric,
\[
\tilde{g}_\phi = \frac{dx^2}{x^4} + \frac{\phi^* h}{x^2} + \kappa = \tilde{g}_\phi + xA,
\]
suppose that \(\forall Y \in \mathfrak{X}(N)\) and \(\forall Z, V \in \Gamma(M, T^*W)\), the tensor \(\kappa = \tau + xA \in \Gamma(W, S^2T^*W)\) satisfies the identity:
\[
(\ref{A.2}) \quad (Z \cdot \kappa(V, Y^H) - V \cdot \kappa(Z, Y^H) - \kappa([Z, V], Y^H))|_{x=0} = 0,
\]
where \(Y^H \in \Gamma(M, T^*W)\) is the lift of \(Y\). Then the difference of connection 1-forms \((\tilde{d}\tilde{\omega} - \tilde{\omega})\) is such that:
\[
(\tilde{d}\tilde{\omega} - \tilde{\omega})|_{\partial M_\varepsilon} \in \varepsilon \cdot \Omega^1 \big( \partial M_\varepsilon, \text{End}(\phi^*TM)|_{\partial M_\varepsilon} \big).
\]

**Proof.** Define \(\tilde{\theta} := (\tilde{d}\tilde{\omega} - \tilde{\omega})\), and write
\[
\kappa_{ab} = \tau_{ab} + x \cdot A_{ab}.
\]
Now consider a local orthonormal frame \(\{x^2\partial_x, xe_i, e_a\} \subset \Gamma(\phi^*TM)\) for \(\tilde{g}_\phi\) with transition matrix
\[
\Lambda = \begin{pmatrix}
x^2 & 0 & 0 \\
0 & xe_i & xe_a \\
0 & e_i & e_a
\end{pmatrix}.
\]
A direct computation of the entries of \(\tilde{\theta}\) yields that:
\[
\tilde{\theta}|_{x=\varepsilon} = \begin{pmatrix}
0 & 0 & 0 \\
\tilde{\theta}_{ij}|_{x=\varepsilon} & 0 \\
0 & -\tilde{\theta}_{ij}|_{x=\varepsilon}
\end{pmatrix} + \varepsilon \cdot E,
\]
with \(E \in \Omega^1 \big( \partial M_\varepsilon, \text{End}(\phi^*TM)|_{\partial M_\varepsilon} \big)\), and such that for some \(C \in \Omega^1(\partial M_\varepsilon)\) and any \(\alpha \neq 0:\)
\[
(\ref{A.4}) \quad \tilde{\theta}_{ij}|_{x=\varepsilon} = \frac{1}{2} \Lambda^a \Lambda^b \left(-\partial_a \kappa_{bj} + \partial_b \kappa_{aj} + \varepsilon \cdot \partial_j A_{ab}\right) dy^j + \varepsilon \cdot C.
\]
In local coordinates, equation \((\ref{A.2})\) becomes:
\[
(\kappa([Z, V], Y^H) - [Z \cdot \kappa(V, Y^H) - V \cdot \kappa(Z, Y^H)])|_{x=0} = Z^a V^b Y^j (-\partial_a \kappa_{bj} + \partial_b \kappa_{aj})|_{x=0} = 0,
\]
for any \(Z^a, V^b \in C^\infty(F)\) and \(Y^j \in C^\infty(N)\), which simply means that \(\forall j \in 1, \cdots, \dim N\) and \(\forall a, b \in \{1, \cdots, \dim F\}\), we have \((\partial_a \kappa_{bj} - \partial_b \kappa_{aj}) = O(x)\), and we get \(\tilde{\theta}|_{x=\varepsilon} \in \varepsilon \cdot \Omega^1 \big( \partial M_\varepsilon, \text{End}(\phi^*TM)|_{\partial M_\varepsilon} \big)\). \(\square\)

**Remark.** The condition \(A(x\partial_i, x\partial_j) = O(x)\) on the tensor \(A \in \Gamma(M, S^2[\phi^*TM])\) is necessary for the proof above to work. If \(A(x\partial_i, x\partial_j) = O(x^b)\), on the one hand \(\tilde{\theta}|_{x=\varepsilon} \neq O(\varepsilon)\), but more importantly, we can’t apply the Atiyah-Patodi-Singer theorem to \((M_\varepsilon, g_\varepsilon)\) anymore.

A central object in the upcoming computations is the restriction to \(\partial M_\varepsilon\) of the curvature \(\tilde{\omega}^\varepsilon\) associated to the auxiliary metric \(g_\varepsilon\). We have the following fact:
Lemma A.3. Let \( h^\Omega \in \Omega^2(N, \text{End} \, TN) \) be the connection 2-form of the metric \( h \) on the base space \( N \), and let \( ^\kappa \Omega(y) \) be the curvature 2-form of the metric \( \kappa_{TF_0} \) on the fibre \( F_y = \phi^{-1}(\{y\}) \subset W \). Then:
\[
^\varepsilon \Omega|_{\partial M_\varepsilon} = \begin{pmatrix}
0 & 0 & 0 \\
0 & \phi^* \left( h^\Omega \right) & 0 \\
0 & 0 & \kappa_\varepsilon(y) + \alpha(y)
\end{pmatrix} + \varepsilon \cdot \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & E
\end{pmatrix},
\]
where \( E \in \Omega^2(\partial M_\varepsilon, \text{End}(T\partial M_\varepsilon)) \) and \( \alpha(y) = \alpha_a dy^a \wedge dz^a \) locally.

Proof. First, regarding \( \varepsilon \nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM) \), we have:
\[
\langle dx^\alpha, \varepsilon \nabla_{\partial_\alpha} \rangle = \langle dx^\mu, \varepsilon \nabla_{\partial_\mu} \rangle = 0,
\]
where:
\[
\langle dx^\mu, \varepsilon \nabla_{\partial_\mu} \rangle = \left( \Gamma_{\alpha \beta}^{\mu} \right)_{\varepsilon \nabla_{\partial_\mu}} \quad \forall \mu, \alpha, \beta \neq 0.
\]
Once we express \( \varepsilon \omega \) in a \( \varepsilon \)-orthonormal frame \( \{\varepsilon^2 \partial_\varepsilon, \varepsilon c_1, c_a\} \) near the boundary \( W \), we find:
\[
\varepsilon \omega_j^i = \Gamma_{jk}^i dy^k + \varepsilon \gamma_{ja}^i dz^a, \quad \varepsilon \omega_a = \varepsilon \gamma_{ja}^i dy^k + \Gamma_{jk}^i(y) dz^c,
\]
\[
\varepsilon \omega_a = -\varepsilon \omega_a = \varepsilon \gamma_{ja}^i dz^a, \quad \varepsilon \omega_0 = -\varepsilon \omega_0 = 0,
\]
with \( \gamma_{ja}^i \in C^\infty(M) \) of order 0 in \( \varepsilon \). We observe that \( \Gamma_{jk}^i dy^k \) give the entries of the connection 1-form associated to the metric \( \partial \) on \( N \), while \( \Gamma_{jk}^i(y) dz^c \) are those of the Levi-Civita connection form of the metric \( \kappa_{TF_0} \) on the fibre \( F_y = \phi^{-1}(\{y\}) \). Using the Maurer-Cartan equation
\[
^\varepsilon \Omega = d^\varepsilon \omega + \varepsilon \omega \wedge \varepsilon \omega,
\]
one obtains the stated result:
\[
^\varepsilon \Omega_j = h^\Omega_j + O(\varepsilon), \quad ^\varepsilon \Omega_a = -^\varepsilon \Omega_a = O(\varepsilon),
\]
\[
^\varepsilon \Omega_a = ^\kappa \Omega_a(y) + \partial_i \Gamma_{ja}^i(y) dy^i \wedge dz^c + O(\varepsilon),
\]
where:
\[
^h \Omega_j = \left( \partial_j \Gamma_{jk} + \Gamma_{ja}^i \Gamma_{jk}^i \right) dy^j \wedge dy^k,
\]
\[
^\kappa \Omega_a(y) = \left( \partial_i \Gamma_{ja}^i + \Gamma_{ja}^i \Gamma_{jk}^i \right) dy^j \wedge dz^c \wedge dz^d.
\]

A.3. Vanishing of Chern-Simons terms: The first result here is valid for arbitrary even dimensions \( \dim M = 2m \). With the same notations as in lemma A.1, we have:

Proposition A.4. Let \( \tilde{g}_\phi \) and \( \tilde{g}_d \) be asymptotic metrics, and for some \( B \in \Gamma(M, S^2[\phi^* TM]) \), consider the exact metrics
\[
g_\phi = \tilde{g}_\phi + x^2 B \quad \text{and} \quad g_d = \tilde{g}_d + x^d B,
\]
and an asymptotic auxiliary metric \( \tilde{g}_c \) on \( TM \) such that \( (\tilde{g}_c)|_{\partial M_\varepsilon} \equiv (\tilde{g}_\phi)|_{\partial M_\varepsilon} \). Then for a given invariant polynomial \( P \in S^n(\mathfrak{so}_{2m}^\ast(\mathbb{R})) \), the Chern-Simons boundary correction to \( P \) obtained from an exact or an asymptotic \( \phi \)-metric are the same:
\[
\lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} T \left( M, \varepsilon \tilde{\nabla}^\phi, \phi \tilde{\nabla} \right) = \lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} T \left( M, \varepsilon \tilde{\nabla}, ^\phi \tilde{\nabla} \right),
\]
and these corrections vanish in the case of \( d \)-metrics:
\[
\lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} T \left( M, \varepsilon \tilde{\nabla}, d \tilde{\nabla} \right) = \lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} T \left( M, \varepsilon \tilde{\nabla}, d \tilde{\nabla} \right) = 0.
Proof. Recall that:

$$P(M, \phi \nabla) \equiv P(\phi \Omega, \ldots, \phi \Omega), \quad TP(M, \phi \nabla, \phi \nabla) \equiv m \int_0^1 dt P(\phi \omega - \varepsilon \hat{\omega}, \phi \Omega, \ldots, \phi \Omega),$$

where $\phi \Omega_i$ is the curvature 2-form of the interpolation connection $\phi \nabla_i = \varepsilon \hat{\nabla} + t(\phi \nabla - \varepsilon \hat{\nabla}).$ As in the proof of theorem 3.1 we have

$$\int_{\partial M_\varepsilon} \phi \nabla = \int_{M_\varepsilon} P(M, \phi \nabla) - \int_{M_\varepsilon} \phi \Omega,$$

which leads to

$$\int_{\partial M_\varepsilon} TP(M, \phi \nabla) = \int_{\partial M_\varepsilon} TP(M, \phi \nabla) + \int_{\partial M_\varepsilon} TP(M, \phi \nabla, \phi \nabla)$$

$$= \int_{\partial M_\varepsilon} TP(M, \phi \nabla)$$

+ $$\int_{\partial M_\varepsilon} m \left[ \int_0^1 P(\phi \omega - \phi \hat{\omega}, \phi \Omega, \ldots, \phi \Omega) dt \right]_{\partial M_\varepsilon},$$

with $\phi \Omega$ the curvature of the interpolation connection $t \phi \nabla + (1 - t) \phi \hat{\nabla}.$ We note that for some $E(t) \in \Omega^{2n-2}(M_c):$

$$P(\phi \omega - \phi \hat{\omega}, \phi \Omega, \ldots, \phi \Omega) \equiv P(\phi \omega - \phi \hat{\omega}, \phi \Omega) = P(\phi \Omega)_{|x=\varepsilon} + dx \wedge E(t),$$

so that

$$\left[ \int_0^1 P(\phi \omega - \phi \hat{\omega}, \phi \Omega, \ldots, \phi \Omega) dt \right]_{\partial M_\varepsilon} = \int_0^1 P(\phi \Omega)_{|x=\varepsilon} dt,$$

and by lemma A.1

$$\int_{\partial M_\varepsilon} TP(M, \phi \nabla) = \int_{\partial M_\varepsilon} TP(M, \phi \nabla) + \varepsilon \cdot \int_{\partial M_\varepsilon} Q.$$ 

Since this is also valid for $d \hat{\nabla}$ and $d \nabla,$ we get:

$$\lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} TP(M, \phi \nabla, d \nabla) = \lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} TP(M, \phi \nabla, d \hat{\nabla}),$$

and

$$\lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} TP(M, \phi \nabla, d \nabla) = \lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} TP(M, \phi \nabla, d \hat{\nabla}).$$

On the other hand, we also have $(d \hat{\omega} - \varepsilon \hat{\omega})_{|x=\varepsilon} = O(\varepsilon)$ by lemma A.1 so that $TP(M, \phi \nabla, d \hat{\nabla}) = \varepsilon Q$ for some $Q \in \Omega^{n-1}(\partial M),$ and the vanishing of the last two limits above follows. \qed

The next proposition links the Chern-Simons corrections of exact and product-type $\phi$-metrics:

**Proposition A.5.** Let $M$ be a $2m$-dimensional manifold with fibred boundary and $P \in S^n(\mathfrak{so}_{2m}(\mathbb{R}))$ an invariant polynomial. Let $\tilde{g}_\phi$ and $g_\varepsilon$ be product metrics on $\phi TM$ and $TM$ resp., and consider the metrics

$$g_\varepsilon = \tilde{g}_\phi + x A + x^2 B \quad \text{and} \quad g_\varepsilon = g_\varepsilon + \varepsilon \cdot A_{|x=\varepsilon},$$

with $B \in \Gamma(S^2[\phi^* M]),$ and $A \in \Gamma(S^2[\phi^* M])$ a symmetric bilinear form such that

(i) $A(x^2 \partial_\varepsilon, \cdot) \equiv 0$ and $A(x Y_1, x Y_2) = O(x),$

(A.5) (ii) $(V_1 \cdot A(V_2, x Y_1) - V_2 \cdot A(V_1, x Y_1) - A([V_1, V_2], x Y_1))_{|x=0} = 0,$
for all $Y_1, Y_2 \in \mathfrak{X}(N)$ and all $V_1, V_2 \in \Gamma(W, T^2 W)$. One then has:

$$\lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} TP(M, \varepsilon \nabla, \phi \nabla) = \lim_{\varepsilon \to 0} \int_{\partial M_\varepsilon} TP(M, \varepsilon \nabla, \phi \nabla).$$

Remark. Let $\kappa = x \cdot A$. Condition (i) in this statement means that $\kappa \in \Gamma(W, S^2 T^* W)$, while condition (ii) is equivalent to equation (A.2). Also, we identified the vectors $Y \in \mathfrak{X}(N)$ with their horizontal lifts to $W$.

Proof. With the usual notations for the Levi-Civita connections, we start by writing:

$$[P(M, \phi \nabla) - P(M, \varepsilon \nabla)] = \left[ P(M, \phi \nabla) - P(M, \varepsilon \nabla) \right] + \left[ P(M, \varepsilon \nabla) - P(M, \phi \nabla) \right] - \left[ P(M, \varepsilon \nabla) - P(M, \varepsilon \nabla) \right] + \left[ P(M, \varepsilon \nabla) - P(M, \varepsilon \nabla) \right],$$

then, integrating on $M_\varepsilon$ and applying Stokes theorem, we have for some $E(t) \in \Omega^{2m-2}(\partial M_\varepsilon)$ and curvature forms $\phi \hat{\nabla} t$, $\varepsilon \hat{\nabla} t$ and $\varepsilon \Omega t$ of appropriate interpolation connections that:

$$\int_{\partial M_\varepsilon} TP(M, \varepsilon \nabla, \phi \nabla) - \int_{\partial M_\varepsilon} TP(M, \varepsilon \nabla, \phi \nabla) = \int_{\partial M_\varepsilon} m \left[ \int_0^1 dt \cdot \left( dx \wedge E(t) + P(\phi \omega - \phi \hat{\nabla} t) \right) \right],$$

by lemma (A.1) we have $(\phi \omega - \phi \hat{\nabla} t)|_{x=\varepsilon} \in \varepsilon \cdot \Omega_\varepsilon(\partial M_\varepsilon, \text{End}(\phi TM)|_{\partial M_\varepsilon})$, and by proposition (A.2)

$$(\phi \hat{\nabla} t - \phi \nabla)|_{x=\varepsilon}, (\varepsilon \hat{\nabla} t - \varepsilon \omega)|_{x=\varepsilon} \in \varepsilon \cdot \Omega_\varepsilon(\partial M_\varepsilon, \text{End}(\phi TM)|_{\partial M_\varepsilon}),$$

which means that for some $Q \in \Omega^{2m-1}(\partial M_\varepsilon)$, we get:

$$\int_{\partial M_\varepsilon} TP(M, \varepsilon \nabla, \phi \nabla) - \int_{\partial M_\varepsilon} TP(M, \varepsilon \nabla, \phi \nabla) = \varepsilon \cdot \int_{\partial M_\varepsilon} Q,$$

and the result follows once we take the limit as $\varepsilon \to 0$. \hfill \Box

The last result focuses on the cases needed for our Hitchin-Thorpe inequality, namely when $P \in S^m(\mathfrak{so}_{2m}(\mathbb{R}))$ is the Hirzebruch $L$-polynomial or the Pfaffian.

**Proposition A.6.** Let $M$ be compact a 4k-manifold with fibred boundary $F \to \partial M \xrightarrow{\phi} N$ and consider a splitting $TW = T^2 W \oplus T^H W$ for $W = \partial M$. Let $g_\phi$ be an exact metric on $M$ of the form

$$g_\phi = \tilde{g}_\phi + x \cdot A + x^2 \cdot B,$$

where $\tilde{g}_\phi$ is a product metric, and $A$ satisfies the following:

(i) $A(x^2 \partial_x, \cdot) \equiv 0$ and $A(xY_1, xY_2) = O(x)$,

(ii) $(V_1 \cdot A(V_2, xY_1) - V_2 \cdot A(V_1, xY_1) - A([V_1, V_2], xY_1))|_{x=0} = 0$,

for all $Y_1, Y_2 \in \mathfrak{X}(N)$ and all $V_1, V_2 \in \Gamma(W, T^2 W)$. 


If the dimension of the fibre $F$ is odd, and if $\hat{g}_x$ is an asymptotic auxiliary metric such that $(\hat{g}_x)|_{\partial M_x} \equiv (\hat{g}_\phi + xA)|_{\partial M_x}$, then the Chern-Simons correction associated to the Euler form vanishes:

$$e_{CS}(\partial M, \phi \nabla) = -\lim_{\varepsilon \to 0} \int_{\partial M_x} Te(M, \varepsilon \nabla, \phi \nabla) = 0.$$  

Furthermore, when $F$ is one-dimensional, the Chern-Simons correction associated to the Hirzebruch $L$-polynomial also vanishes:

$$L_{CS}(\partial M, \phi \nabla) = -\lim_{\varepsilon \to 0} \int_{\partial M_x} TL(M, \varepsilon \nabla, \phi \nabla) = 0.$$  

**Proof.** It is sufficient to consider $g_\phi = \hat{g}_\phi$ and $g_\varepsilon = \hat{g}_\varepsilon$ by proposition \(A.3\), and we thus carry-out the computations in a $\hat{g}_\phi$-orthonormal frame $\{x^2 \partial_{x^2}, x^i, e_a\}$ of $\hat{T}M$ near the boundary (using the same notations as above for indices, decompositions of matrices, etc.). Defining $\hat{\theta} := (\hat{\omega} - \varepsilon \omega)$, we employ the interpolation connection $\hat{\nabla}_\varepsilon = \varepsilon \nabla + t\hat{\theta}$, and write $\hat{\omega}_\varepsilon$ and $\hat{\Omega}_\varepsilon$ for its connection and curvature forms. On $\partial M_x = \{x = \varepsilon\}$, the only nonvanishing entries of $\hat{\theta}|_{x=\varepsilon}$ are found to be

$$\hat{\theta}^0|_{x=\varepsilon} = -\hat{\theta}^0|_{x=\varepsilon} = dy^i$$

and since $\hat{\omega}_\varepsilon = \varepsilon \omega + t\hat{\theta}$, the Maurer-Cartan equation yields:

$$\hat{\Omega}_\varepsilon = d(\varepsilon \omega + t\hat{\theta}) + (\varepsilon \omega + t\hat{\theta}) \wedge (\varepsilon \omega + t\hat{\theta})$$

or a $\varepsilon$-dimensional manifold, the Euler form is given by

$$e(M, \phi \nabla) = Pf \left( \frac{\phi \hat{\Omega}}{2\pi} \right) = \frac{c_m}{(2m)!} \sum_{A \in S_{2m}} (-1)^{|A|} \phi_\sigma(2) \wedge \cdots \wedge \phi_\sigma(2m),$$

for a constant $c_m \in \mathbb{R} \setminus \{0\}$. If we write

$$\int_{\partial M_x} Te(M, \varepsilon \nabla, \phi \nabla) = \int_{\partial M_x} \left[ \int_0^1 \left( Pf(\varepsilon \omega - \varepsilon \omega, \phi \hat{\Omega}_t)|_{\partial M_x} + d\varepsilon \wedge E(t) \right) dt \right]_{\partial M_x}$$

$$= \int_{\partial M_x} \left[ Pf(\varepsilon \omega - \varepsilon \omega, \phi \hat{\Omega}_t)|_{\partial M_x} + d\varepsilon \wedge E(t) \right]_{\partial M_x} dt,$$

where $E(t) \in \Omega^{2m-2}(M_x)$, and

$$P(\tilde{\theta}, \phi \hat{\Omega}_t)|_{\partial M_x} = \frac{c_m}{(2m)!} \sum_{A \in S_{2m}} (-1)^{|A|} \left[ \tilde{\theta}_\sigma(2) \wedge (\phi \hat{\Omega}_t)_{\sigma(3)} \wedge \cdots \wedge (\phi \hat{\Omega}_t)_{\sigma(2m)} \right]_{x=\varepsilon}.$$

Now for $\dim F = 2f + 1$ and $\dim N = 2n$, the last expression combined with equations \(A.8\), \(A.9\) and \(A.10\) yield that the nonvanishing summands which are proportional to the volume form on $\partial M_x$ necessarily come from products such as:

$$\phi_\sigma(2) \wedge \cdots \wedge \phi_\sigma(2m) \wedge (\phi \hat{\Omega}_t)^{a_1} \wedge \cdots \wedge (\phi \hat{\Omega}_t)^{a_{2f+1}} \wedge (\phi \hat{\Omega}_t)^{a_{2f+2} \cdots a_{2f+1}} \wedge (\phi \hat{\Omega}_t)^{a_{2f+2} \cdots a_{2f+1}} \wedge \cdots \wedge (\phi \hat{\Omega}_t)^{a_{2f+1} \cdots a_{2f+1}} \wedge \phi_\sigma(2) \wedge \cdots \wedge \phi_\sigma(2m) \wedge \phi_\sigma(2) \wedge \cdots \wedge \phi_\sigma(2m).$$
where the \(i_j\)’s are indices coming from the base \(N\) and the \(a_j\)’s are associated to the fibre \(F\). Again by equation (A.7), the presence of factors \(\hat{\phi}_{\Omega}^{i_1} \in \varepsilon \cdot \Omega^2(\partial M)\) for odd dimensional fibres leads to:

\[
e_{CS}(\partial M, \varepsilon, \hat{\phi}) = - \lim_{\varepsilon \to 0} \int_{\partial M} T_{e}(M, \varepsilon, \phi) = \lim_{\varepsilon \to 0} \left[ \varepsilon \cdot \int_{\partial M} Q \right] = 0,
\]

for some \(Q \in \Omega^{2m-1}(\partial M)\).

**L-polynomial:** Let \(M\) be 4k-dimensional with one-dimensional fibres of the boundary. If we write

\[
\int_{\partial M} TL(M, \varepsilon, \phi) = \int_{\partial M} \int_{0}^{1} P(\hat{\phi}, \phi_{\Omega})|_{x=\varepsilon} dt,
\]

then the summands of \(P(\hat{\phi}, \phi_{\Omega})|_{x=\varepsilon}\) that are proportional to a volume form on \(\partial M\) are obtained from products of the form:

\[
dy^{i_1} \& \left[ \left( \hat{\phi}_{\Omega}^{i_2} \right)_{a_2} \oplus \left( \hat{\phi}_{\Omega}^{i_4} \right)_{a_4} \oplus \cdots \oplus \left( \hat{\phi}_{\Omega}^{i_{2n-1}} \right)_{a_{2n-1}} \right] \& \left( \hat{\phi}_{\Omega}^{a_n} \right)_{a_n},
\]

\[
\text{and}
\]

\[
dy^{i_1} \& \left[ \left( \hat{\phi}_{\Omega}^{i_2} \right)_{a_2} \oplus \left( \hat{\phi}_{\Omega}^{i_4} \right)_{a_4} \oplus \cdots \oplus \left( \hat{\phi}_{\Omega}^{i_{2n-1}} \right)_{a_{2n-1}} \right] \& \left( \hat{\phi}_{\Omega}^{a_n} \right)_{a_n},
\]

which means that if \(\dim F = 1\) then \(TL(M, \varepsilon, \phi) = \varepsilon Q\) for some \(Q \in \Omega^{4k-1}(\partial M)\), and the claim follows. \(\square\)

### A.4 Counter-examples:

We discuss some cases in which propositions \(\text{A.4}\) and \(\text{A.6}\) do not hold anymore, as well as the mistakes in [DWO7]. We use the same notations as above.

1. As shown in propositions \(\text{A.3}\) and \(\text{A.2}\), the conditions \(\text{A.3}\) on the perturbation term \(A \in \Gamma(S^2(\hat{\phi} T^*M))\) for an asymptotic metric are necessary for the vanishing of the Chern-Simons corrections, and for the equality

\[
\lim_{\varepsilon \to 0} \int_{\partial M} TP(M, \varepsilon, \phi) = \lim_{\varepsilon \to 0} \int_{\partial M} TP(M, \varepsilon, \phi),
\]

to hold. The analog of proposition \(\text{A.5}\) in [DWO7] is Lemma 3.7, and claims the same equality, but the statement doesn’t hold without the conditions \(\text{A.3}\) and this causes theorem 3.6 of [DWO7] to be erroneous too. At the end of p.563, the equation

\[
S = xS_I + dx \otimes S'
\]

is false (in our notations, \(S \equiv \hat{\phi} = \phi - \phi\)). The correct expressions are equations \(\text{A.3}\) and \(\text{A.4}\) in the proof of proposition \(\text{A.2}\).

2. In lemma 4.2 of [DWO7], the claim is that for an asymptotic metric \(g_{\phi} = \tilde{g}_{\phi} + x A\) with \(A \in \Gamma(S^2(\hat{\phi} T^*M))\) such that \(A(x^2 \partial_{x}, \cdot) \equiv 0\), one has.

\[
\lim_{\varepsilon \to 0} \int_{\partial M} TP(M, \varepsilon, \phi) = 0.
\]

Here, the conditions \(\text{A.3}\) of proposition \(\text{A.3}\) along with the restrictions on \(\dim F\) in proposition \(\text{A.3}\) are necessary for this statement to be true, as well as for theorem 4.3 in [DWO7]. The mistake in the proof of lemma 4.2 of [DWO7] is the formula

\[
P(\theta, \Omega_1, \cdots, \Omega_{\ell}) = \pi^*(\alpha) + O(\varepsilon)
\]
on page 566, which comes from computational mistakes in the decomposition of \( \Omega_t \). The correct decomposition of \( \Omega_t \) is given in equation (A.7) of the proof of (A.6).

3) If the fibre \( F \) is even-dimensional, then \( \lim_{\varepsilon \to 0} \int_{\partial M} Te(\varepsilon \nabla, \tilde{\phi}) \) does not vanish in general. Going back to the proof of proposition (A.6) if we take \( \dim F = 2f \) and \( \dim N = 2n + 1 \) and employ the same notations, we find that the summands in the integrand of \( Te(\varepsilon \nabla, \tilde{\phi})|_{\partial M_f} \) that are multiples of a volume form on \( \partial M_f \) come from products such as

\[
\tilde{\phi}_{0}^j \land \left[ \left( \Phi_{a}^{i} \right)_{1}^{i} \land \cdots \land \left( \Phi_{a}^{i} \right)_{n}^{n+1} \right] \land \left[ \left( \Phi_{a}^{i} \right)_{a1}^{a2} \land \left( \Phi_{a}^{i} \right)_{a3}^{a4} \land \cdots \land \left( \Phi_{a}^{i} \right)_{a2j-1}^{a2j} \right],
\]

which implies that in all generality, \( Te(\varepsilon \nabla, \tilde{\phi})|_{\partial M_f} \) is not of order at least one in \( \varepsilon \) (c.f. equation (A.9)).

4) If \( \dim F > 1 \), then \( \lim_{\varepsilon \to 0} \int_{\partial M} TL(M, \varepsilon \nabla, \tilde{\phi}) \) does not vanish in general. Recall that for a 4k-dimensional manifold \((M, \tilde{g}_0)\), the Hirzebruch \( L \)-polynomial is given by

\[
L(M, \tilde{\phi}) \equiv L_k[p_1, \ldots, p_k] = \sum_{\alpha} a_\alpha \left( \frac{\phi}{2\pi} \right)^{\alpha},
\]

where the multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k \) are such that

\[
\alpha_1 + 2\alpha_2 + \cdots + j\alpha_j + \cdots + k\alpha_k = k,
\]

and where the coefficients \( a_\alpha \) are rational, the \( p_j(\phi/2\pi) \) denotes the \( j \)-th Pontryagin form, defined as the form of degree 4\( j \) in the expansion

\[
\det \left( \text{Id}_{4k} + \frac{\phi}{2\pi} \right) = 1 + p_1 \left( \frac{\phi}{2\pi} \right) + \cdots + p_j \left( \frac{\phi}{2\pi} \right)^{j} + \cdots + p_{4k} \left( \frac{\phi}{2\pi} \right)^{4k},
\]

and \( L_k[p_1, \ldots, p_k] \) is homogeneous of degree 2\( k \) in the entries of \( (\phi/2\pi) \). We look at the case where \( k = 2 \) (dim \( \partial M = 7 \)) and dim \( F = 3 \), in which one has:

\[
p_1(\Omega/2\pi) = \frac{1}{2(2\pi)^2} \text{Tr}(\Omega^2),
\]

\[
p_2(\Omega/2\pi) = \frac{1}{8(2\pi)^4} \left( \left( \text{Tr}(\Omega^2) \right)^2 - \text{Tr}(\Omega^4) \right),
\]

and

\[
L_2[p_1, p_2](\Omega/2\pi) = \frac{1}{45} \left( 7p_2 - (p_2)^2 \right) (\Omega/2\pi) = \frac{(2\pi)^{-4}}{360} \left[ 5 \left( \text{Tr}(\Omega^2) \right)^2 - 7 \text{Tr}(\Omega^4) \right].
\]

Using the notations in the proof of proposition (A.6) analyzing the integrand of \( TL(M, \varepsilon \nabla, \tilde{\phi})|_{\partial M_f} \) yields that for some constant \( c \neq 0 \):

\[
P(\tilde{\phi}, \tilde{\phi}_t) = c \cdot \sum_{i=1}^{\text{dim } N=4} \tilde{\phi}_i^0 \land \left( \Phi_{a}^{i} \right)^{0}_{1} \land \cdots \land \sum_{b=1}^{\text{dim } F=3} \left( \Phi_{a}^{i} \right)^{b}_{a} \land \left( \Phi_{a}^{i} \right)^{b}_{a} + O(\varepsilon),
\]

and by equations (A.6) and (A.7) above, \( TL(M, \varepsilon \nabla, \tilde{\phi})|_{\partial M_f} \) is not of order 1 in \( \varepsilon \), and may not necessarily vanish when \( \varepsilon \to 0 \).
Appendix B. Multi Taub-NUT metrics

As above, we have \( k \) points \( \{ p_j \}_{j=1}^k \) in \( \mathbb{R}^3 \), a principal circle bundle \( M \to \mathbb{R}^3 \setminus \{ p_j \} \) of degree -1 near each \( p_j \), and we are considering the following metric on \( M \):

\[
g = \pi^*(V - ((dx^1)^2 + (dx^2)^2 + (dx^3)^2)) + \pi^*(V^{-1}) \cdot (d\theta + \pi^* \omega)^2,
\]

where \( \theta \) is a coordinate on the fibres, \( \omega \) is a connection 1-form on \( \mathbb{R}^3 \setminus \{ p_j \} \) with curvature \( d\omega = *_{\mathbb{R}^3} dV \), and the function \( V : \mathbb{R}^3 \setminus \{ p_j \}_{j=1}^k \to \mathbb{R} \) is defined as:

\[
V(x) = 1 + \frac{1}{2} \sum_{j=1}^{k} \frac{1}{|x - p_j|}.
\]

The harmonic function \( V \) determines \( g_0 \) uniquely, since for any gauge transformation \( \omega \to \omega + df \) with \( f \in C^\infty(\mathbb{R}^3 \setminus \{ p_j \}) \), we can make the change of variable \( \theta \to \theta + f \) to keep the same metric (WW90, section 6.2).

Now we briefly recall the construction of the smooth completion \( (M_0, g_0) \), following section 3 of EST1 (see also AKLS9 and Lab94). For \( 0 < \delta < \min_{1 \leq j \leq k} |p_i - p_j| \), we consider the punctured open balls \( B_i = \mathbb{B}^3_\delta(p_i) \setminus \{ p_i \} \subset \mathbb{R}^3 \) \( (i = 1, \cdots, k) \), and note that \( \pi^{-1}(B_i) \) are diffeomorphic to \( \mathbb{B}^3_\delta(0) \setminus \{ 0 \} \) in such a way that the \( S^1 \) action coincides with scalar multiplication on \( \mathbb{R}^3 \simeq \mathbb{C}^2 \). We then define

\[
M_0 := (M \cup \bigcup_{j=1}^k \mathbb{B}_\delta^3(q_j))/\sim,
\]

where \( \sim \) stands for the identification of the \( \pi^{-1}(B_i) \) with punctured 4-balls \( \mathbb{B}_\delta^4(q_j) \setminus \{ q_j \} \). The map \( \pi : M \to \mathbb{R}^3 \setminus \{ p_j \} \) is then smoothly extended to \( \pi : M_0 \to \mathbb{R}^3 \) with \( q_i = \pi^{-1}(p_i), \) and \( \pi \) acts as the projection to the space of \( S^1 \)-orbits (under scalar multiplication) when restricted to the balls \( \mathbb{B}_\delta^3(q_i) \). To see that we can also extend \( g \) to a smooth metric \( g_0 \) on \( M_0 \), we note that in the vicinity of the points \( q_i \), we can write \( g = g^F_i + \alpha_i \) where \( g^F_i \) is isometric to the flat Euclidean metric on \( \mathbb{B}_\delta^3(q_i) \), and \( \alpha_i \) is a symmetric bilinear form which is smooth on the same neighbourhood.

Indeed, let \( (r, \varphi, \psi) \) be the spherical coordinates centred at \( p_i \in \mathbb{R}^3 \), and write \( V = V_i + f_i \) with \( V_i(x) = (2|x - p_i|)^{-1} = 1/(2r) \) and \( f_i \) smooth on \( \mathbb{B}^3_\delta(p_i) \). Since

\[
\ast_{\mathbb{R}^3} dV_i = -\frac{1}{2} \sin \varphi d\varphi \wedge d\psi = d\left( \frac{\cos \varphi d\psi}{2} \right),
\]

we may take \( \omega_i(x) = (\cos \varphi d\psi)/2 \) as a local connection 1-form for the curvature \( \ast dV_i \), and the restriction \( g_{\mathbb{B}_\delta^3(q_i)} = g^F_i + \alpha_i \) is then given by:

\[
g^F_i = \pi^*[V_i \cdot ((dx^1)^2 + (dx^2)^2 + (dx^3)^2)] + \pi^*(V_i^{-1}) \cdot (d\theta + \pi^* \omega_i)^2
\]

\[
= \frac{(dr)^2}{2r} \left[ (d\varphi)^2 + (d\psi)^2 + (\cos 2\theta) \right] + 2 \cos \varphi d\varphi \wedge d\psi,
\]

\[
\alpha_i = f_i \pi^*((dx^1)^2 + (dx^2)^2 + (dx^3)^2) + V_i^{-1} \cdot \left[ -(f_i/V_i)(d\theta + \pi^* \omega_i)^2 + \pi^*(\omega - \omega_i)^2 \right].
\]

On \( \mathbb{B}_\delta^3(q_i) \), we introduce the following coordinates (EH79):

\[
y^1 + iy^2 = \sqrt{2r} \cos \left( \frac{\varphi}{2} \right) \exp \left[ i \left( \frac{\theta + \psi}{2} \right) \right]
\]

\[
y^3 + iy^4 = \sqrt{2r} \sin \left( \frac{\varphi}{2} \right) \exp \left[ i \left( \frac{\theta - \psi}{2} \right) \right],
\]

and we obtain that \( g^F_i = (dy^1)^2 + (dy^2)^2 + (dy^3)^2 + (dy^4)^2 \). Letting \( y \to q_i \) in \( M_0 \), we have \( \alpha_i \to f_i \pi^*((dx^1)^2 + (dx^2)^2 + (dx^3)^2)|_{p_i} \), and it is then obvious that \( g = g^F_i + \alpha_i \) can be smoothly extended to \( q_i \).
Finally, the condition $d\omega = \ast R^3 dV$ implies that $g$ is Ricci-flat, and we can define 3 compatible parallel complex structures $\{J_i\}_{i=1}^3$ on $M$ by taking

$$J_i : \pi^* dx^i \mapsto -V^{-1}(d\theta + \pi^* \omega), \pi^* dx^j \mapsto \pi^* dx^3,$$

and permuting the action on these forms for $J_2$ and $J_3$. Since $M_0$ is simply connected, we obtain that $g_0$ is a complete Ricci-flat hyper-Kähler metric for this space.

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