Controlled Quantum Dense Coding in a Four-particle Non-maximally Entangled State via Local Measurements

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A controlled quantum dense coding scheme is investigated with a four-particle non-maximal quantum channel. The amount of classical information is shown to be capable of being controlled by the controllers through adjustments of the local measurement angles and to depend on the coefficients of the quantum channel; in addition, the four particles are distributed in two inverse ways in such an quantum channel. A restricted condition for distributing the particles to realize quantum dense coding in an arbitrary \((N+2)\)-particle quantum channel is proposed.

Keywords: Quantum dense coding, Four-particle entangled state, Local measurement, Unitary transformation

PACS number(s): 03.67.Hk, 03.65. Ud, 03.67. -a

Quantum entanglement plays a key role in quantum information theory and teleportation. Quantum dense coding (QDC) \cite{Bose2002,Lee2002,Bose2003} is one of the exhibitions of entanglement in quantum communication. Normally, the classical capacity of a transmission channel is 1 bit; however, in dense coding, with the help of entanglement, people can transmit two bits of classical information by sending only one qubit. Bose \textit{et al.} and Lee \textit{et al.} \cite{Bose2002,Lee2002} and Bose \cite{Bose2003} have generalized QDC between two parties to multiparticles and mixed state dense

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coding, respectively.

On the other hand, Hao et al. \[10\] have proposed a controlled dense coding scheme by using the three-particle Greenberger-Horne-Zeilinger (GHZ) state. In this scheme, one party (Alice) can transmit information to the second party (Bob) whereas the local measurement of the third party (Cliff) serves as quantum erasure. Cliff can control the quantum channel between Alice and Bob via a local measurement to realize controlled dense coding between Alice and Bob. Chen and Kuang \[11\] have generalized the controlled dense coding scheme of the three-particle GHZ quantum channel to the case of an \((N + 2)\)-particle GHZ quantum channel via a series of local measurements.

In this paper, we study controlled quantum dense coding in a four-particle non-maximal quantum channel via local measurements. Our goal consists of three aspects: (i) Study how the transmitted amount of classical information is controlled by the controllers through adjustments of the local measurement angles and how it depends on the coefficients of the entangled quantum channel. (ii) Discuss the distribution of the four particles in such a four-particle non-maximal quantum channel. (iii) Propose a restricted condition for how to distribute the particles to realize quantum dense coding in an arbitrary \((N + 2)\)-particle quantum channel.

Firstly, we review the QDC scheme. Let us assume that Alice and Bob initially share the Bell state \(|\phi^+\rangle\). Locally operating on her qubit, Alice obtains the four orthogonal Bell states \(\hat{I}|\phi^+\rangle = |\phi^+\rangle\), \(\hat{\sigma}_x|\phi^+\rangle = |\psi^+\rangle\), \(\hat{\sigma}_y|\phi^+\rangle = i|\psi^-\rangle\), and \(\hat{\sigma}_z|\phi^+\rangle = |\phi^-\rangle\). Alice then sends her qubit to Bob. By making a Bell measurement, Bob is able to obtain two bits of classical information. The four Bell states are defined by

\[
|\phi^\pm\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle), \tag{1}
\]

\[
|\psi^\pm\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle). \tag{2}
\]

Secondly, we now propose our scheme. Alice (party 2) and Bob (party 3) share a four-particle non-maximal quantum channel

\[
|\psi\rangle = (a|0000\rangle + b|1001\rangle + c|0110\rangle + d|1111\rangle)_{1234}, \tag{3}
\]

where the coefficients \(a, b, c, \) and \(d\) are real, and \(|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1\). We assume that qubit 2, qubit 3, qubit 1, and qubit 4 belong to Alice (party 2), Bob (party 3), party 1, and party 4, respectively.
We suppose that party 4 carries out a unitary operation on his qubit 4 in the following forms:

\[
|+\rangle_4 = \cos \theta_1 |0\rangle_4 + \sin \theta_1 |1\rangle_4, \\
|−\rangle_4 = \sin \theta_1 |0\rangle_4 - \cos \theta_1 |1\rangle_4;
\]
then, the four-particle non-maximal quantum channel can be rewritten as

\[
|\psi\rangle = |\varphi\rangle_{123} \otimes |+\rangle_4 + |\phi\rangle_{123} \otimes |−\rangle_4,
\]
where

\[
|\varphi\rangle_{123} = (a \cos \theta_1 |000\rangle + b \sin \theta_1 |100\rangle + c \cos \theta_1 |011\rangle + d \sin \theta_1 |111\rangle)_{123}, \\
|\phi\rangle_{123} = (a \sin \theta_1 |000\rangle - b \cos \theta_1 |100\rangle + c \sin \theta_1 |011\rangle - d \cos \theta_1 |111\rangle)_{123}.
\]

Party 4 can obtain two probable local measurement results from Eq. (6). One is \( |+\rangle_4 \), for which the state of qubits 1, 2, 3 collapses to \( |\varphi\rangle_{123} \); the other is \( |−\rangle_4 \), for which the state of qubits 1, 2, 3 collapses to \( |\phi\rangle_{123} \). We only consider the case in which the fourth party’s measurement result is \( |+\rangle_4 \), for which the state of qubits 1, 2, 3 collapses to \( |\varphi\rangle_{123} \) in Eq. (7).

Then, party 1 carries out a unitary operation on his qubit 1 in the following forms:

\[
|+\rangle_1 = \cos \theta_2 |0\rangle_1 + \sin \theta_2 |1\rangle_1, \\
|−\rangle_1 = \sin \theta_2 |0\rangle_1 - \cos \theta_2 |1\rangle_1.
\]

The three-particle state \( |\varphi\rangle_{123} \) can be rewritten as

\[
|\varphi\rangle_{123} = |\varphi\rangle_{23} \otimes |+\rangle_1 + |\phi\rangle_{23} \otimes |−\rangle_1,
\]
where

\[
|\varphi\rangle_{23} = (a \cos \theta_1 \cos \theta_2 + b \sin \theta_1 \sin \theta_2) |00\rangle_{23} + (c \cos \theta_1 \cos \theta_2 + d \sin \theta_1 \sin \theta_2) |11\rangle_{23}, \\
|\phi\rangle_{23} = (a \cos \theta_1 \sin \theta_2 - b \sin \theta_1 \cos \theta_2) |00\rangle_{23} + (c \cos \theta_1 \sin \theta_2 - d \sin \theta_1 \cos \theta_2) |11\rangle_{23}.
\]

Party 1 can also obtain two probable local measurement results from Eq. (11). If the measurement result is \( |+\rangle_1 \), the state of qubits 2, 3 collapses to \( |\varphi\rangle_{23} \). Otherwise, the state of qubits 2, 3 collapses to \( |\phi\rangle_{23} \). We only consider the case in which the first party’s measurement result is \( |+\rangle_1 \); then, the state of qubits 2, 3 collapses to \( |\varphi\rangle_{23} \) in Eq. (12).
After receiving the measurement results and information on the measurement angles from party 4 and party 1, Alice and Bob can obtain the two-particle maximally entangled state from the two-particle non-maximally entangled state in Eq. (12). If Alice induces an auxiliary qubit $|0\rangle_a$ and performs the unitary operation

$$\hat{U}_{2\ a} = \begin{pmatrix}
\tan \gamma & 0 & \sqrt{1 - \tan^2 \gamma} & 0 \\
0 & 1 & 0 & 0 \\
\sqrt{1 - \tan^2 \gamma} & 0 & -\tan \gamma & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$$

(14)
on her qubit 2 and on the auxiliary qubit, which are written under the basis $\{|0\rangle_2 |0\rangle_a, |1\rangle_2 |0\rangle_a, |0\rangle_2 |1\rangle_a, |1\rangle_2 |1\rangle_a\}$. In the unitary transformation of Eq. (14), $\tan \gamma$ is expressed by

$$\tan \gamma = \frac{c \cos \theta_1 \cos \theta_2 + d \sin \theta_1 \sin \theta_2}{a \cos \theta_1 \cos \theta_2 + b \sin \theta_1 \sin \theta_2}.$$  

(15)

The collective unitary operation $\hat{U}_{2\ a} \otimes \hat{I}_3$ transforms the direct product state $|\varphi\rangle_{23} \otimes |0\rangle_a$ to a three-particle entangled state:

$$|\varphi\rangle_{23a} = \sqrt{2} \sin \gamma |\varphi^+\rangle_{23} \otimes |0\rangle_a + \cos \gamma \sqrt{1 - \tan^2 \gamma} |00\rangle_{23} \otimes |1\rangle_a,$$

(16)

where $|\varphi^+\rangle_{23}$ is one of the Bell states of qubit 2 and qubit 3 as given by Eq. (11), $|00\rangle_{23}$ is the unentangled state of the two qubits, and the parameter angle $\gamma$ is defined by

$$\sin \gamma = \frac{1}{\sqrt{e}} (c \cos \theta_1 \cos \theta_2 + d \sin \theta_1 \sin \theta_2),$$

(17)

$$\cos \gamma = \frac{1}{\sqrt{e}} (a \cos \theta_1 \cos \theta_2 + b \sin \theta_1 \sin \theta_2),$$

(18)

with

$$e = (a \cos \theta_1 \cos \theta_2 + b \sin \theta_1 \sin \theta_2)^2 + (c \cos \theta_1 \cos \theta_2 + d \sin \theta_1 \sin \theta_2)^2.$$  

(19)

Alice and Bob can obtain a two-particle maximally entangled state when Alice measures the auxiliary qubit and obtains $|0\rangle_a$ from Eq. (16). From the above procedure, we obviously see that party 4 and party 1 control the entanglement between particle 2 and particle 3 with local measurements. The average classical amount of information transmitted from Alice to Bob adds up to

$$C = 1 + 2 |\sin \gamma|^2 = 1 + 2 [1 + \cot^2 \gamma]^{-1}.$$  

(20)
Thirdly, in order to expatiate on how the local measurement angles from party 1 and party 4 affect the amount of information, we discuss the expression of the transmitted classical amount of information in Eq. (20). (i) For the case of $|\tan \gamma| < 1$, the classical amount of information transmitted from Alice to Bob is less than two bits from Eq. (20) with Eq. (15). (ii) For the case of $\pi/4$ transformations and $|a| = |b| = |c| = |d| = 1/2$, party 1 and party 4 carry out $\theta_1 = \pi/4$ and $\theta_4 = \pi/4$, and the four-particle non-maximal quantum channel in Eq. (3) become a maximally entangled state. From Eq. (20) with Eqs. (17) – (19), we can see that the classical amount of information transmitted from Alice to Bob reaches a maximal value of two bits. Thus, we can conclude that the transmitted classical amount of information not only depends on the measurement angles, which are controlled by party 1 and party 2, but also depends on the coefficients of the four-particle non-maximal quantum channel.

Fourthly, we discuss the distribution of the four particles in such a four-particle non-maximal quantum channel. From Eq. (3) with Eq. (1) and Eq. (2), if quantum dense coding is to be realized, the four particles must be distributed in the following forms: (i) Particle 2 and particle 3 belong to Alice and Bob, respectively, and particle 1 and particle 4 as quantum erasure; then, Alice and Bob can obtain a two-particle non-maximally entangled state that is a linear combinations of states $\{|00\rangle_{23}, |11\rangle_{23}\}$. We have described such the case in much greater detail in this paper. (ii) If the particles are distributed in an opposite way, Alice and Bob can again obtain a two-particle non-maximally entangled state that is a linear combinations of states $\{|00\rangle_{23}, |11\rangle_{23}\}$. The remaining distribution of the four particles are unsuccessful. In a word, there are only two ways to distribute four particles to realize quantum dense coding in such a four-particle non-maximal quantum channel.

Finally, we propose a restricted condition on how to distribute the particles to realize quantum dense coding in an arbitrary $(N+2)$-particle quantum channel. Here, $N+2$ parties, possessing one particle each, share an arbitrary $(N+2)$-particle quantum channel. After $N$ particles are served as quantum erasure via a series of local measurements, to realize controlled quantum coding, the sender and the receiver must obtain a two-particle non-maximally entangled state that must be a linear combinations of states similar to one of states that are generated after the sender encodes her qubit.

It must be stressed that our scheme is valuable. Controlled quantum dense coding has been studied by others by employing a maximally entangled state, but depending on
the physical systems, a maximally entangled state can’t always be generated. We employ a partially entangled state instead of a maximally entangled state, which is convenient for physical systems. Our scheme in a four-particle non-maximal quantum channel is an extension of the controlled quantum dense coding scheme to other schemes employing a GHZ state as quantum channel. We first propose a restricted condition on how to distribute the particles to realize quantum dense coding in an arbitrary \((N+2)\)-particle quantum channel.

In summary, we have studied the QDC scheme between two fixed particles in a four-particle non-maximal quantum channel. We have found that the transmitted classical amount of information can be controlled by the controllers through adjusting the local measurement angles and that depends on the coefficients of the four-particle non-maximal quantum channel. We have shown that there are only two ways of distributing four particles in such a four-particle non-maximal quantum channel to realize quantum dense coding. We have proposed a restricted condition on how to distribute the particles to realize quantum dense coding in an arbitrary \((N+2)\)-particle quantum channel.

ACKNOWLEDGMENTS

This work was supported by the Korea Science and Engineering Foundation and by the National Natural Science Foundation of China under Grant No. 60261002.

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