The four-way intersection problem for latin squares

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Abstract

For \( \mu \) given latin squares of order \( n \), they have \( k \) intersection when they have \( k \) identical cells and \( n^2 - k \) cells with mutually different entries. For each \( n \geq 1 \) the set of integers \( k \) such that there exist \( \mu \) latin squares of order \( n \) with \( k \) intersection is denoted by \( I\mu[n] \). In a paper by P. Adams et al. (2002), \( I^3[n] \) is determined completely. In this paper we completely determine \( I^4[n] \) for \( n \geq 16 \). For \( n \leq 16 \), we find out most of the elements of \( I^4[n] \).

Keywords. Latin square, intersection of latin squares, 4-way intersection.

1 Introduction and preliminaries

A partial latin rectangle is an \( r \times n \) (\( r \leq n \)) array such that each cell is either empty or consists of a symbol from a set of \( n \) distinct symbols (e.g. \{1, 2, \ldots, n\}), and that each symbol appears at most once in each row and in each column. A latin rectangle is a partial latin rectangle when all cells are non-empty. A (partial) latin square of order \( n \) is an \( n \times n \) (partial) latin rectangle. We assume the set of symbols \{1, 2, \ldots, n\} are used in latin squares of order \( n \).

A \( \mu \)-way latin square (\( \mu \geq 2 \)) of order \( n \) is a set of \( \mu \) latin squares of order \( n \) with the following property: the \( \mu \) entries in cells with the same

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coordinate are either all the same, or all different. The cells with the same entries are often called fixed cells or identical cells interchangeably. A \( \mu \)-way latin square has \( k \) intersection when the number of fixed cells is exactly \( k \). Similarly, \( \mu \)-way latin rectangle of order \( r \times n \) is defined.

For each \( n \geq 1 \), by \( I^\mu[n] \) we denote the set of all integers \( k \) such that there exist \( \mu \)-way latin squares of order \( n \) with \( k \) intersection. Determining \( I^\mu[n] \) is called \( \mu \)-way latin intersection problem.

The intersection problem has arisen in many combinatorial areas such as latin squares \cite{2, 7, 8}, Steiner triple systems \cite{9}, \( m \)-cycle systems \cite{1}, and design theory \cite{4}. For an old survey on intersection problem see \cite{4}. Intersection problem in latin squares was introduced at first by Fu \cite{8} for two latin squares. Later, Fu and Fu \cite{6} proposed an intersection problem for 3 latin squares which is a relaxation of the problem we have defined above for \( \mu = 3 \). Instead of having entries of \( n^2 - k \) cells mutually distinct in all three latin squares, they only require not all of these entries to be equal. Adams et al. \cite{2} have completely determined \( I^3[n] \) for \( n \geq 1 \). There was an error in \cite{2}, in showing that \( 35 \in I^3[8] \). We have found it by a computer program and shown in Figure 1.

For \( \mu \geq 2 \), a \( \mu \)-way latin trade of volume \( s \) is defined as follows: A group of \( \mu \) partial latin rectangles such that each of them have precisely the same \( s \) filled cells. If cell \((i, j)\) is filled, then its entry is different in all \( \mu \) partial latin rectangles. Moreover, for any relevant \( i \), the set of entries of \( i^{th} \) row is the same for all \( \mu \) partial latin rectangles, and similarly for relevant columns \( j \). In \cite{3}, there are some useful results on \( \mu \)-way latin trades which we have used for our work.

Note that from each \( \mu \)-way latin square, by considering all cells which have identical fixed elements as empty cells, we obtain a \( \mu \)-way latin trade. We will refer to the set of cells which have fixed elements as intersection part and to its complement as trade part.

Given a \( \mu \)-way latin square (latin trade) \( L \), its skeleton is a binary matrix \( S \) where \( S_{i,j} \) is 1 if and only if cell \((i, j)\) is in the intersection part (is an empty cell, respectively). Let \( r_i \) and \( c_j \) be the number of ones in the \( i^{th} \) row and \( j^{th} \) column of \( S \) respectively. We call \((r_1, r_2, \ldots, r_n)\) and \((c_1, c_2, \ldots, c_n)\) the row sequence and column sequence of \( L \), respectively. An example of a 3-way latin square and its skeleton is given in Figure 1.

Next, we present some old (which are referenced) and new results which are used to determine \( I^4[n] \) for \( n \geq 1 \).

**Lemma 1.1** \cite{3} If \( \min\{m, n\} \geq \mu \) then there exists a \( \mu \)-way latin trade of order \( m \times n \) of volume \( mn \).
Let $T$ be a $\mu$-way latin trade of order $n$. In the following results, $R_i$ and $C_j$ denote the set of elements of row $i$ and column $j$ of $T$, respectively.

Next lemma is an immediate result from the definition of $\mu$-way latin trades.

**LEMMA 1.2** \[^3\] Let $T$ have a nonempty cell $(i, j)$. Then $|R_i \cap C_j| \geq \mu$.

**COROLLARY 1.1** \[^3\] Let $T$ have a nonempty cell $(i, j)$. If $|R_i| = |C_j| = \mu$ then $R_i = C_j$.

**COROLLARY 1.2** Assume $i_1 \neq i_2$ and cells $(i_1, j), (i_2, j)$ are nonempty. If $|C_j| = \mu + 1$ and $|R_{i_1}| = |R_{i_2}| = \mu$ then

$$R_{i_1} \cup R_{i_2} = C_j, \quad |R_{i_1} \cap R_{i_2}| = \mu - 1.$$ 

Similarly, assume that $j_1 \neq j_2$ and cells $(i, j_1), (i, j_2)$ are nonempty. If $|R_i| = \mu + 1$ and $|C_{j_1}| = |C_{j_2}| = \mu$, then

$$C_{j_1} \cup C_{j_2} = R_i, \quad |C_{j_1} \cap C_{j_2}| = \mu - 1.$$ 

**Proof.** By symmetry, it suffices to show only the first part. According to Lemma [1.2] we have $R_{i_1}, R_{i_2} \subseteq C_j$. Therefore, there are two possibilities: $|R_{i_1} \cap R_{i_2}| = \mu - 1$, or $|R_{i_1} \cap R_{i_2}| = \mu$ (in which case $R_{i_1} = R_{i_2}$). We show the latter case is not possible. If the latter case happens then there exists an element $x \in C_j \setminus (R_{i_1} \cup R_{i_2})$. This means that $x$ can appear in at most $\mu - 1$ cells of column $j$, which is a contradiction as each element should appear in exactly $\mu$ cells of a row or column. \[\square\]
**LEMMA 1.3** If an element appears at least \( n - \mu + 1 \) times in the intersection part of a \( \mu \)-way latin square \( L \), then it appears only in the intersection part of \( L \).

**Proof.** By Lemma 1.2 if an element appears in the trade, then it appears in at least \( \mu \) rows (and \( \mu \) columns) of each partial latin square in the trade. Hence, there are at most \( n - \mu \) available positions for that element in the intersection part.

In [2] it is shown that \( I^3[2n] \supseteq I^3[n] + I^3[n] + I^3[n] + I^3[n] \) for \( n \geq 1 \). This can be simply generalized to \( I^\mu[2n] \) for any \( \mu \geq 4 \). Using this generalization, together with the fact that any latin square of order \( i \) can be embedded in a latin square of order \( 2n \) when \( i \leq n \), we obtain the following proposition.

**PROPOSITION 1.1** For any \( n \geq 1 \) and \( \mu \geq 2 \) we have

\[
I^\mu[2n] \supseteq \left( I^\mu[n] + I^\mu[n] + I^\mu[n] + I^\mu[n] \right) \cup \left( \bigcup_{i=1}^{n} (I^\mu[i] + \{(2n)^2 - i^2\}) \right).
\]

**LEMMA 1.4** Let \( L \) be a \( \mu \)-way latin square of order \( n \) with \( k \) fixed cells. If \( p \) is the number of elements of \( L \) which appear only in the intersection part then

\[
p \geq \left\lceil n - \frac{n^2 - k}{\mu} \right\rceil.
\]

**Proof.** From the definition of \( \mu \)-way latin square, each of the other \( n - p \) elements appears at least \( \mu \) times in the cells of the trade part of \( L \). So, each of these \( n - p \) elements can appear at most \( n - \mu \) times in the cells of the intersection part. Hence, \( k \leq pn + (n - p)(n - \mu) \), or equivalently, \( \mu p \geq \mu n - n^2 + k \).

**LEMMA 1.5** Let \( L \) be a 4-way latin square of order 7 and \( \{a_i\} \) be its row (or column) sequence. None of the sequences \( \{7, 3\} \), \( \{7, 7, 2\} \) and \( \{7, 7, 7, 1\} \) can be a subsequence of \( \{a_i\} \).

**Proof.** We show that \( \{7, 3\} \) can not be a subsequence of the row sequence of any 4-way latin square of order 7. Other statements are similar. Suppose, on the contrary, that there is such an \( L \). By a permutation on row and columns of \( L \), we may assume that the first and second rows have 7 and 3 fixed cells, respectively, and the fixed cells of the second row are located at the first 3 columns. By a permutation on the elements, we may assume that
Consider the cell (2, 4) in the trade part. Since there should appear four different elements distinct from \(x, y, z\) and 4, we should have \(4 \in \{x, y, z\}\). By a similar argument for the rest of cells in the second row, we have \(5, 6, 7 \in \{x, y, z\}\) which is impossible.

With the same approach as in the above, we can show the following two lemmata as well.

**LEMMA 1.6** Let \(L\) be a 4-way latin square of order 6 and \(\{a_i\}\) be its row (or column) sequence. None of the sequences \(\{6, 2\}\) and \(\{6, 6, 1\}\) can be a subsequence of \(\{a_i\}\).

**LEMMA 1.7** Let \(L\) be a 4-way latin square of order 5 and \(\{a_i\}\) be its row (or column) sequence. The sequence \(\{5, 1\}\) cannot be a subsequence of \(\{a_i\}\).

## 2 Constructions

In this section, we introduce four techniques which contribute to the generation of the majority of 4-way intersections. The first technique is inspired by [2] and the rest are new. We start with illustrating the first technique by an example, then elaborating the technique in the sequel.

**EXAMPLE 2.1** Consider the following partial 4-way latin squares \(A, B,\) and \(C\).
\[ A = \begin{bmatrix} 2345 & 3254 & 1453 & 5432 \\ 3254 & 1523 & 5321 & 2345 \\ 1523 & 5321 & 2345 & 3254 \\ 5321 & 2345 & 3254 & 1523 \end{bmatrix} \]

\[ B = \begin{bmatrix} 9678 \\ 8967 \\ 7896 \\ 6789 \end{bmatrix} \]

\[ C = \begin{bmatrix} 6789 \\ 8769 \\ 9876 \\ 7986 \end{bmatrix} \]

By combining these partial 4-way latin squares, we obtain a 4-way latin square of order 9 with 17 fixed cells.

**TECHNIQUE 2.1** \([n \rightarrow 2n + 1\) technique\] This technique constructs a \(\mu\)-way latin square of order \(2n + 1\) by generating and combining three partial \(\mu\)-way latin squares \(A, B,\) and \(C\). Partial latin squares \(A, B,\) and \(C\) are generated as follows. Let \(A'\) be a \(\mu\)-way latin square of order \(n + 1\) with
elements from \(\{1, \ldots, n+1\}\). \(A\) is constructed by embedding, symmetrically, the first \(n\) rows of \(A'\), at the up-right and down-left corners of a square of order \(2n + 1\) and laying the \((n + 1)\text{th}\) row of \(A'\) at the down-right corner, diagonally.

\(B\) is constructed by embedding a \(\mu\)-way latin square \(B'\) of order \(n\) with elements from \(\{n + 2, \ldots, 2n + 1\}\) at the top-left corner of a square of order \(2n + 1\).

\(C\) is made by embedding a partial \(\mu\)-way latin square of order \(n + 1\), say \(C'\), at the down-right corner of a square of order \(2n + 1\). Note that the elements of \(C'\) are from \(\{n + 2, \ldots, 2n + 1\}\) and diagonal cells of \(C'\) are empty.

The following lemma is a generalization of Lemma 2.3 in [2]. As in Proposition 1.1 by the fact that any latin square of order \(i\) can be embedded in a latin square of order \(2n + 1\) when \(i \leq n\) we have

\[
\bigcup_{i=1}^{n} (I^4[i] + \{(2n + 1)^2 - i^2\}) \subseteq I^4[2n + 1].
\]

**LEMMA 2.1** If \(n \geq 4\) then

\[
I^4[2n + 1] \supseteq \{I^4[n] + (n + 1)\{[0, n - 4] \cup \{n\}\} + C\} \cup X,
\]

where \(C = \bigcup_{i=1}^{n-3} \{2tn, 2tn-t, 2tn-n\} \cup \{0, 1, 2\} \cup (2n+1)\{0, n-3\} \cup \{n+1\} \cup (n+1)\{1, 2n-7\} \cup [n + 1, 2n - 3]\) and \(X = \bigcup_{i=1}^{n} (I^4[i] + \{(2n + 1)^2 - i^2\})\).

**TECHNIQUE 2.2** [Trade-into-Trade technique] In this technique we consider a \(\mu'\)-way latin square of order \(n\). Then for each \(i, 1 \leq i \leq \mu'\), we substitute each entry of unfixed cells in the \(i\text{th}\) latin square with a proper \(\mu_i\)-way latin trade of order \(m\). In this way we obtain a \((\sum_{i=1}^{\mu'} \mu_i)\)-way latin square of order \(mn\).

Let’s illustrate a simple case of this method with the following example.

**EXAMPLE 2.2** Consider the following 2-way latin square of order 4 with 9 fixed cells.

|    | \(A_1\) | \(A_2\) | \(A_3\) | \(A_4\) |
|----|--------|--------|--------|--------|
| \(A_1\) |   |   |   |   |
| \(A_2\) |   |   |   |   |
| \(A_3\) |   |   |   |   |
| \(A_4\) |   |   |   |   |
Now we replace each $A_i$, $1 \leq i \leq 4$, with a 2-way latin trade. Note that elements of any two trades corresponding to two different $A_i$ are disjoint. In this way, we obtain the following 4-way latin square of order 8 with 36 fixed cells.

\[
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 &
\end{array}
\begin{array}{ccccccccc}
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 &
\end{array}
\begin{array}{ccccccccc}
5 & 6 & 7 & 8 & 12 & 21 & 34 & 43 &
\end{array}
\begin{array}{ccccccccc}
6 & 5 & 8 & 7 & 21 & 12 & 43 & 34 &
\end{array}
\begin{array}{ccccccccc}
3 & 4 & 56 & 65 & 7 & 8 & 12 & 21 &
\end{array}
\begin{array}{ccccccccc}
4 & 3 & 65 & 56 & 8 & 7 & 21 & 12 &
\end{array}
\begin{array}{ccccccccc}
7 & 8 & 12 & 21 & 34 & 43 & 56 & 65 &
\end{array}
\begin{array}{ccccccccc}
8 & 7 & 21 & 12 & 43 & 34 & 65 & 56 &
\end{array}
\]

More generally, this technique can be used to construct fine structures which are defined in [5].

Let’s consider an example before explaining next technique.

**EXAMPLE 2.3** Since the following three partial latin squares are completable to a latin square of order 9 we have $46 \in I^3[9]$.

\[
A_1 = \\
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
4 & 1 & 2 & 3 & 5 & 6 & 7 & 9 \\
5 & 6 & 7 & 8 & 9 & 3 & 4 & 12 \\
9 & 5 & 6 & 7 & 8 & 2 & 3 & 41 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 89
\end{array} \\
A_2 = \\
\begin{array}{cccccccc}
4 & 1 & 2 & 3 & 5 & 6 & 7 & 9 \\
3 & 4 & 1 & 2 & 5 & 6 & 7 & 89 \\
9 & 5 & 6 & 7 & 8 & 2 & 3 & 41 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 89 \\
5 & 6 & 7 & 8 & 9 & 1 & 2 & 34
\end{array} \\
A_3 = \\
\begin{array}{cccccccc}
3 & 4 & 1 & 2 & 5 & 6 & 7 & 8 \\
2 & 3 & 4 & 1 & 5 & 6 & 7 & 89 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 89 \\
5 & 6 & 7 & 8 & 9 & 1 & 2 & 34 \\
9 & 5 & 6 & 7 & 8 & 4 & 1 & 23
\end{array}
\]
**TECHNIQUE 2.3 [Gear technique 1]** Consider a completable partial latin square of order $n$, as in the following figure, where $A = \{1, \ldots, a\}$, $B = \{a+1, \ldots, n\}$ and $A \cup B$ are the sets of entries at the specified portion. Then by cyclically permuting the rows of the two subrectangles of order $y \times b$ and $(y + x) \times a$, we obtain a $\mu$-way latin square of order $n$ (after filling the empty cells identically, in all latin squares).

![Diagram of a partial latin square](image)

Sufficient conditions for having such a completable partial latin square are

1. $a \geq x + \mu - 1$,
2. $b \geq x + \mu - 1$,
3. $a + b = n$,
4. $x \geq 1$ and
5. $y \geq \mu$.

Briefly, conditions (1), (2), and (3) ensure that latin subrectangles using elements of $A$, $B$ and $A \cup B$ can be constructed, conditions (2) and (3) guarantee that the partial latin square is completable, and condition (4) is intrinsic in the technique, condition (5) is needed when permuting the rows,

Clearly, we can obtain $\mu$-way latin trades using Technique 2.3 and since we don’t require the completability of the partial latin square for latin trades, we can relax the condition $b \geq x + \mu - 1$ to $b \geq \mu - 1$ and obtain the following Proposition which is used for confining possible members of $I^4[n]$ in the next section.
PROPOSITION 2.1 For any \( i \in \{0, \ldots, \mu\} \), there exists a \( \mu \)-way latin trade of volume \( s \in \{\mu(3\mu - i), \mu(3\mu - i) + 1, \ldots, \mu(3\mu - i) + (\mu - i)\} \).

Proof. (sketch) For each \( i \in \{0, \ldots, \mu\} \), consider Technique 2.3 for generating latin trades, with the following parameters:

\[
x = 1, n = 3\mu - i - 1, y = \mu \quad \text{and} \quad a \in \{\mu, \mu + 1, \ldots, 2\mu - i\}
\]

As proved in [3], there exists a \( \mu \)-way latin trade of volume \( s \) for any \( s \geq 3\mu^2 + \mu - 1 \). Hence we get the following corollary.

COROLLARY 2.1 For any \( s \geq 3\mu^2 - \mu \), there exists a \( \mu \)-way latin trade of volume \( s \).

The next technique is similar to Technique 2.3. As for Technique 2.3 we start with an example to explain the technique.

EXAMPLE 2.4 Since the following partial latin squares are completable to a latin square of order 13 we have 128 \( \in I^4[13] \).

\[
\begin{array}{cccccc|cccccc}
B_1 & & & & & & & & & & & \\
6 & 8 & 9 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 \\
7 & 9 & 6 & 3 & 1 & 5 & 4 & 2 & 8 & 6 & 9 & 1 \\
8 & 1 & 2 & 3 & 9 & 6 & 7 & 8 & 5 & 4 & 9 & 2 \\
9 & 2 & 3 & 4 & 7 & 8 & 5 & 6 & 1 & 3 & 4 & 5 \\
\end{array}
\hspace{1cm}
\begin{array}{cccccc|cccccc}
B_2 & & & & & & & & & & & \\
9 & 6 & 7 & 8 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 \\
8 & 9 & 6 & 7 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\
7 & 8 & 2 & 3 & 9 & 6 & 4 & 5 & 1 & 2 & 3 & 4 \\
9 & 0 & 7 & 8 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 \\
\end{array}
\hspace{1cm}
\begin{array}{cccccc|cccccc}
B_3 & & & & & & & & & & & \\
5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 \\
3 & 8 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 \\
1 & 2 & 3 & 5 & 6 & 4 & 9 & 0 & 7 & 8 & 9 & 0 \\
6 & 7 & 8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 & 1 & 2 \\
\end{array}
\]

TECHNIQUE 2.4 [Gear technique 2] Similar to Technique 2.3 we take a completable partial latin square of order \( n \) (the following figure) where \( A = \{1, \ldots, a\} \) and \( B = \{a + 1, \ldots, a + b\} \). Then by permuting the rows of the two subrectangles of order \( a \times (y + x) \) and \( b \times y \), we obtain a \( \mu \)-way latin square of order \( n \) with \( n^2 - (a(x + y) + by) \) fixed cells.
Similar to Technique 2.3, sufficient conditions for having a completable partial latin square suitable for this technique are the following.

1. \( a, b \geq x + \mu - 1 \),
2. \( a + b + c = n \),
3. \( x \geq 1 \),
4. \( c \geq y \geq \mu \),
5. \( a + b \geq x + y \) and

3 Main results

In this section first we introduce some notations. Then we give some proofs of the existence and non-existence of our results on 4-way intersection problem. These will lead to the proof of our main theorem stated at the end of this section.

For each \( n \geq 1 \), define

\[
J^4[n] := [0, n^2 - 27] \cup \{n^2 - 25, n^2 - 24, n^2 - 23, n^2 - 20, n^2 - 16, n^2\}
\]

\[
:= [0, n^2] \setminus ([1, 15] \cup \{17, 18, 19, 21, 22, 26\}).
\]

Since the possible volumes for 4-way latin trades are \( N \setminus ([1, 15] \cup \{17, 18, 19, 21, 22, 26\}) \) (see [3]) we have \( I^4[n] \subseteq J^4[n] \).

The following lemmata are generalization of Corollary 2.1 and Corollary 2.2 of [2]:
LEMMA 3.1 If $I^4[n] \supseteq [0, \lceil n^2/2 \rceil]$, then $I^4[2n] \supseteq [0, 3n^2] \cup \{I^4[n] + \{3n^2\}\}$.

LEMMA 3.2 If $n \geq 4$ and $I^4[n] \supseteq [0, 7n + 4]$, then

$I^4[2n + 1] \supseteq [0, (2n + 1)^2 - n^2] \cup \{I^4[n] + \{(2n + 1)^2 - n^2\}\}$.

Now, these two corollaries are immediate:

COROLLARY 3.1 If $I^4[n] = J^4[n]$ and $I^4[n] \supseteq [0, \lceil n^2/2 \rceil]$, then $I^4[2n] = J^4[2n]$.

COROLLARY 3.2 If $n \geq 4$, $I^4[n] = J^4[n]$ and $I^4[n] \supseteq [0, 7n + 4]$, then $I^4[2n + 1] = J^4[2n + 1]$.

3.1 Proofs of existence

Here, we mention the method of obtaining non-trivial members of $I^4[n]$, for each $n \geq 4$.

• $n = 4$:
  ◦ $\{0\}$: by Lemma 1.1

• $n = 5$:
  ◦ $\{0, 5\}$: by Lemma 1.1
  ◦ $\{1\}$: computer search.

• $n = 6$:
  ◦ $\{0, 6, 12\}$: by Lemma 1.1
  ◦ $\{1, 2, 3, 4, 5, 7, 8, 11\}$: computer search.

• $n = 7$:
  ◦ $\{0, 7, 14, 21\}$: by Lemma 1.1
  ◦ $\{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 16, 17, 19\}$: computer search.

• $n = 8$:
  ◦ $\{0, 8, 16, 24, 32\}$: by Lemma 1.1
  ◦ $\{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 25, 28, 33\}$: computer search
• $n = 9$:
  - $\{0, 9, 18, 27, 36, 45\}$: by Lemma 1.1
  - $\{2, 3, 4, 6, 7, 8, 10, 11, 12, 13, 14, 15\}$: computer search
  - $\{1, 5, 16, 17, 20, 21, 25, 29, 37, 41, 61, 65\}$: by Technique 2.1
  - $\{22, 23, 31, 32, 40\}$: by Technique 2.3

• $n = 10$:
  - $\{0, 10, 20, 30, 40, 50, 60\}$: by Lemma 1.1
  - $\{1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 15, 16, 25, 26, 27, 28, 31, 32, 35\} \cup \{36, 51, 52, 55, 56, 57, 61, 75, 76, 80, 84\}$: by Proposition 1.1
  - $\{14, 24, 34, 44, 45, 46, 54\}$: by Technique 2.3
  - $\{77\}$: computer search

• $n = 11$:
  - $\{4, 21, 38, 45, 50, 51, 54, 59, 60, 62, 65, 70\}$: by Technique 2.3
  - remaining elements of $I^4[11] \setminus R^4[11]$ : by Technique 2.1

• $n = 12$:
  - $\{117, 121\}$: computer search
  - remaining elements of $I^4[12] \setminus R^4[12]$ : by Proposition 1.1

• $n = 13$:
  - $\{146\}$: computer search
  - $\{128, 129\}$: by Technique 2.4
  - remaining elements of $I^4[13] \setminus R^4[13]$ : by Technique 2.1

• $n = 14$:
  - $\{169, 173\}$: computer search
  - $\{135\}$: by Technique 2.3
  - $\{146\}$: by Technique 2.4
  - remaining elements of $I^4[14] \setminus R^4[14]$ : by Proposition 1.1
• $n = 15$:
  - $\{202\}$: by adding fixed cells, one can obtain a 4-way latin rectangle of $5 \times 15$ from the 4-way latin rectangle found for the case $77 \in I_4[10]$ in the Appendix. Then, obviously, adding fixed rows to this 4-way rectangle result in $202 \in I_4[15]$.
  - $\{198\}$: computer search
  - $\{160\}$: by Technique 2.3
  - $\{170, 171, 173, 174, 175\}$: by Technique 2.4
  - remaining elements of $I_4[15] \setminus R_4[15]$ : by Technique 2.1.

• $n = 16$:
  - $\{233\}$: similar to $202 \in I_4[15]$ argument
  - $\{229\}$: computer search
  - $I_4[16]$ : by Proposition 1.1

• $n = 17$:
  - $\{266\}$: similar to $202 \in I_4[15]$ argument
  - $\{262\}$: by adding fixed cells, one can obtain a 4-way latin rectangle of $5 \times 17$ from the 4-way latin rectangle found for the case $117 \in I_4[12]$ in the Appendix. Then, obviously, adding fixed rows to this 4-way rectangle result in $262 \in I_4[17]$.
  - $\{215\}$: by Technique 2.3
  - $\{218, 223, 224\}$: by Technique 2.4
  - $\{220\}$: Using the latin rectangle $(17, 220)$ shown in Appendix as input, we can make this intersection with a technique similar to Technique 2.4. This latin rectangle has three row permuting sub-rectangles. Note that the first row of one of these subrectangles is separate from other rows of it.
  - remaining elements of $I_4[17]$ : by Technique 2.1

• $n = 18$:
  - $\{301\}$: similar to $202 \in I_4[15]$ argument
  - $\{297\}$: similar to $262 \in I_4[17]$ argument
  - remaining elements of $I_4[18]$ : by Proposition 1.1
• $n = 19$:
  ◦ $\{338\}$: similar to $202 \in I^4[15]$ argument
  ◦ $\{334\}$: similar to $262 \in I^4[17]$ argument
  ◦ $\{264, 273, 274, 278, 279\}$: by Technique \ref{2.3}
  ◦ $\{268\}$: Using the latin rectangle $(19, 268)$ shown in Appendix as input, we can make this intersection with a technique similar to Technique \ref{2.4}. This latin rectangle has three row permuting subrectangles.
  ◦ remaining elements of $I^4[17]$ by Technique \ref{2.1}

• $n = 20$:
  ◦ $\{373\}$: similar to $262 \in I^4[17]$ argument
  ◦ remaining elements of $I^4[20]$ : by Proposition \ref{1.1}

• $n = 21$:
  ◦ $\{414\}$: similar to $262 \in I^4[17]$ argument
  ◦ $\{354, 358\}$: computer search
  ◦ remaining elements of $I^4[21]$ : by Technique \ref{2.1}

• $n = 22$:
  ◦ $\{457\}$: similar to $262 \in I^4[17]$ argument
  ◦ remaining elements of $I^4[22]$ : by Proposition \ref{1.1}

• $n = 23$:
  ◦ $\{502\}$: similar to $262 \in I^4[17]$ argument
  ◦ remaining elements of $I^4[23]$ : by Technique \ref{2.1}

• $24 \leq n \leq 31$:
  ◦ for odd $n$, $I^4[n]$ : by Technique \ref{2.1}
  ◦ for even $n$, $I^4[n]$ : by Proposition \ref{1.1}

• $n \geq 32$:
  ◦ for even $n$, $I^4[n]$ : by Corollary \ref{3.1}
  ◦ for odd $n$, $I^4[n]$ : by Corollary \ref{5.2}
3.2 Proofs of non-existence

Here, we prove that certain intersection sizes are not possible for small values of \( n \). In most of the proofs, we assume that a 4-way latin square of corresponding intersection size exists. Then by argument on the extendibility and completability of 'its trade' to a 4-way latin square of needed order, we reach a contradiction.

**Proof of** \( 2 \notin I^4[5] \): Suppose \( 2 \in I^4[5] \). By Lemma 1.2, we know that each row (and column) has at most 1 fixed cell. By a permutation on rows and columns, we may assume that \((1,1)\) and \((2,2)\) are the fixed cells. If these fixed cells have different elements, then there are only 3 elements left for the trade at \((1,2)\). Hence, they must have the same element, say 1. Now, it is easy to verify that there is not enough room for 1 in the third row as 1 cannot appear in \((3,1)\) and \((3,2)\).

**Proof of** \( 9 \notin I^4[5] \text{ and } 20 \notin I^4[6] \): In each case, the volume of the trade is 16 and since there are at least four filled cells in each row and in each column of the trade, and also since each element appears at least 4 times in the trade, that is indeed a 4-way latin square of order 4. But, this trade cannot be extended to a 4-way latin square of order 5 or 6, as necessary condition for embedding a latin square of order \( m \) in a latin square of order \( n \) is that \( n \geq 2m \) or \( m = 2n \).

**Proof of** \( 13, 16 \notin I^4[6] \text{ and } 22, 24, 25, 26, 29, 33 \notin I^4[7] \): For these intersections, we produced all possible decreasing row sequences. Then, according to Lemma 1.5 and 1.6, we ruled out all of them. (Note that we can similarly prove \( 9 \notin I^4[5] \text{ and } 20 \notin I^4[6] \) by means of Lemma 1.7 and 1.6).

**Proof of** \( 9 \notin I^4[6] \): Suppose \( 9 \in I^4[6] \). There are at least four filled cells in each row and column of its trade. So, its trade is \( 5 \times 6 \) or \( 6 \times 6 \). \( 5 \times 6 \): There are two possible skeletons (cells containing a dot correspond to empty cells of trade) for this trade. The first skeleton is as below.

```
  1  2  3  4  5  6
  7  8  9 10 11 12
 13 14 15 16 17 18
 19 20 21 22 23 24
 25 26 27 28 29 30
 31 32 33 34 35 36
```
As \( |X| \leq 6 \) (since there is a row of six nonempty cells \( |X| = 6 \)) and each element of \( X \) should appear in at least four rows we have \( |R_i \cap R_j| = 4 \) for \( i \neq j \in \{1, 2, 3\} \). Now consider \( x \in X \setminus R_3 \). Clearly \( x \in R_1 \) and \( x \in R_2 \) (hence \( x \in R_1 \cap R_2 \)). By Lemma 1.2 we have \( C_3 \subset R_1 \) and \( C_3 \subset R_2 \) (hence \( C_3 \subset R_1 \cap R_2 \)). Since \( |C_3| = 4 \) we have \( x \in C_3 = R_1 \cap R_2 \). This yields that cell \((3, 3)\) of this trade cannot be filled to get a 4-way latin square of needed order.

The second possible skeleton is

As we have a row of six nonempty cells we have \( X = \{1, 2, \ldots, 6\} \). Suppose \( 6 \notin X \setminus R_2 \). To be able to fill the cell \((2, 3)\) in order to obtain the needed 4-way latin square, we should have \( 6 \notin C_1, C_2, C_3 \). But this is a contradiction as \( 6 \) should appear in at least 4 columns of this trade.

6 × 6: In this trade, there are at least three rows and three columns with two empty cells. We can assume the first three rows and columns are so. There is one row and one column with only one empty cell. By permutation, assume that the last row and column are like this and hence cell \((6, 6)\) is empty. Suppose \( R_6 = \{1, 2, \ldots, 5\} \). We can assume, by Corollary 1.2 that \( C_1 = \{1, 2, 4, 5\} \), \( C_2 = \{1, 2, 3, 4\} \) and \( C_3 = \{1, 2, 3, 5\} \). According to Corollary 1.1 we can assume that \( R_i = C_i \) for \( i = 1, 2, 3 \) and hence by Corollary 1.2 we have \( C_6 = \{1, 2, \ldots, 5\} \). Each element should appear in at least four columns, so \( X = \{1, 2, \ldots, 5\} \). Now one can simply check that empty cells of three first rows and columns cannot be filled in a latin way to obtain the 4-way latin square of needed order.

• **Proof of** \( 20 \notin I^4[7] \): Let \( L \) be a 4-way latin square with intersection size 20. The row (or column) sequence of \( L \) can only contain \( \{0, 1, 2, 3, 7\} \) (Lemma 1.3). The row (or column) sequence cannot contain more than two 7s as there are only 20 cells in the intersection part. If it contains only one 7, it cannot contain 3 and if it contains exactly two 7s, it cannot contain 2 or 3 (Lemma 1.5). In both cases, the maximum intersection size would be 19 \((7 + 6 \times 2)\) or 19 \((7 + 7 + 5 \times 1)\).
Hence, the row sequence of $L$ does not contain 7. Similarly, there can be no 1s and there is exactly one 2, and (up to symmetry) the only valid row (and column) sequence is $\{a_i\} = \{3,3,3,3,3,3,2\}$. Without loss of generality, suppose that the first column has intersection size 2 and these intersections are the two top cells of this column. At least one of the two first rows has intersection size 3. Suppose this row is the first one. Furthermore, suppose that these 3 intersections are located in the first 3 cells of this row. Applying Corollary 1.1, we deduce that the same set of numbers (Suppose $\{1,2,3\}$) appears in the intersection part in the last four columns. So, there are 3 numbers that appear at least four times in $L$. By Lemma 1.3 any of these numbers appears exactly 7 times in the intersection part. Hence, $L$ has intersection size at least $7 \times 3 = 21 > 20$.

- **Proof of $37 \notin I^4[8], 54 \notin I^4[9]$ and $73 \notin I^4[10]$:** We prove $54 \notin I^4[9]$ and the other ones are similar. Suppose $54 \in I^4[9]$. There are at least four filled cells in each row and column of its trade. So, its trade is $5 \times 6$ or $6 \times 6$.

  5 × 6: Each element of this trade should appear in at least four cells of it. So at most six elements are in this trade. Since there is a row of six nonempty cells, we have $X = \{1,2,\ldots,6\}$. To extend this trade to a 4-way latin square of order 9, we should place other three elements, \{7,8,9\}, in the added rows and empty cells. But this is not feasible.

  6 × 6: The same as 6 × 6 case of 9 \notin I^4[6] we can prove that $X = \{1,2,\ldots,5\}$. But this trade cannot be extended to the desired 4-way latin square using other four elements, \{6,7,8,9\}.

- **Proof of $39 \notin I^4[8]$ and $56 \notin I^4[9]$:** We prove $39 \notin I^4[8]$ and the other one is similar. Suppose $39 \in I^4[8]$. There are at least four filled cells in each row and column of its trade. So, its trade is $5 \times 5$ or $5 \times 6$ or $6 \times 6$.

  5 × 5: In this trade we have $|X| \leq 6$. So it cannot be extended to a 4-way latin square of order 8.

  5 × 6: In this trade we have $|X| \leq 6$ (one can show $|X| = 5$). So it cannot be extended to a 4-way latin square of order 8.

  6 × 6: In this trade, there are five rows and five columns containing two empty cells. There are one row and one column with a single empty cell. We can assume the last row and column contain a single empty cell. If we show that $R_1 = R_2 = \cdots = R_5 = C_1 = C_2 = \cdots = C_5$ then we have a contradiction since $|X| \geq 5$ (last row has five nonempty
Let $H = (A, B)$ be a bipartite graph constructed from this trade, as follows. Corresponding to each first five rows we have a vertex in $A$ and corresponding to each first five columns we have a vertex in $B$. Vertex $a \in A$ is adjacent to vertex $b \in B$ if and only if their correspondent rows and columns have a nonempty cell in common. As $|R_i| = |C_i| = 4$ for $i \in \{1, 2, \ldots, 5\}$, to prove $R_1 = R_2 = \cdots = R_5 = C_1 = C_2 = \cdots = C_5$ it suffices to show the graph $H$ is connected. To show this, consider that $H$ is a graph obtained from a $K_{5,5}$ by deleting two perfect matchings from it. So, $H$ is a 3-regular bipartite graph with five vertices in each partition and hence connected. A possible skeleton of this trade is as below

![Skeleton of the trade](image)

- **Proof of $40 \notin I^4[8]$ and $57 \notin I^4[9]$**: We prove $40 \notin I^4[8]$ and the other one is similar. Suppose $40 \in I^4[8]$. There are at least four filled cells in each row and column of its trade. So, its trade is $5 \times 5$ or $4 \times 6$ or $5 \times 6$ or $6 \times 6$.

  5 × 5: Each element of this trade should appear in at least four cells of it. So at most six elements are in this trade. But this trade cannot be extended to the desired 4-way latin square of order 8.

  4 × 6: Since each element should appear in at least four cells of this trade we have $X = \{1, 2, \ldots, 6\}$. So this trade cannot be extended to the desired 4-way latin square of order 8 using the remaining two elements.

  5 × 6: This trade has at least one row with two empty cells. Suppose the first row has two empty cells. Each column has an empty cell. We can assume the last two cells of first row are empty. So by Corollary 1.1 we have $R_1 = C_1 = C_2 = C_3 = C_4$. This yields that $X = R_1$. Therefore, this trade cannot be extended to the desired 4-way latin square of order 8 using the remaining four elements.

  6 × 6: Each row and column has two nonempty cells. Similar to 6 × 6 case of $39 \notin I^4[8]$ we can show that $X = \{1, 2, 3, 4\}$. And hence this trade cannot be extended to the desired 4-way latin square of order 8 using the remaining four elements.
• **Proof of** $41 \notin I^4[8]$ and $58 \notin I^4[9]$: We prove $41 \notin I^4[8]$ and the other one is similar. Suppose $41 \in I^4[8]$. There are at least four filled cells in each row and column of its trade. So, its trade is $5 \times 5$. It has two empty cells which are not in a common row or column. As $|X| = 5$, this trade is not extendible to the desired 4-way latin square.

• **Proof of** $44 \notin I^4[8]$: There are at least four filled cells in each row and column of its trade. So, its trade is $5 \times 5$ or $4 \times 5$.

4×5: One can simply see that $|X| = 5$. So, this trade is not extendible to the desired 4-way latin square.

5×5: Each row and column contains an empty cell. Similar to $6 \times 6$ case of $39 \notin I^4[8]$ we can show that $X = \{1, 2, 3, 4\}$. Hence this trade is not extendible to the desired 4-way latin square.

The results given above follow the following (main) theorem. Note that $R^4[n]$ denotes undecided values.

**THEOREM 3.1 (MAIN)** We have,

• $I^4[n] = J^4[n]$ for $1 \leq n \leq 6$,
• $I^4[7] \supseteq [0, 17] \cup \{19, 21, 49\}$, $R^4[7] = \{18\}$,
• $I^4[8] \supseteq J^4[8] \setminus (R^4[8] \cup \{35, 37, 39, 40, 41, 44\})$, $R^4[8] = \{26, 27, 29, 30, 31, 34\}$
• $I^4[9] \supseteq J^4[9] \setminus (R^4[9] \cup \{54, 56, 57, 58\})$, $R^4[9] = \{19, 24, 28, 33, 34, 35, 39, 42, 43, 44, 47, 48, 49, 50, 51, 52, 53\}$,
• $I^4[10] \supseteq J^4[10] \setminus (R^4[10] \cup \{73\})$, $R^4[10] = \{9, 13, 17, 18, 19, 21, 22, 23, 29, 33, 37, 38, 39, 41, 42, 43, 47, 48, 49, 53, 57, 58, 59, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72\}$,
• $I^4[11] \supseteq J^4[11] \setminus R^4[11]$, $R^4[11] = \{74, 75, 78, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 92, 93, 94, 98\}$,
• $I^4[12] \supseteq J^4[12] \setminus R^4[12]$, $R^4[12] = \{93, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107\}$,
• $I^4[13] \supseteq J^4[13] \setminus R^4[13]$, $R^4[13] = \{118, 119, 121, 122, 123, 124, 125, 126, 130, 131, 132, 142\}$,
• $I^4[14] \supseteq J^4[14] \setminus R^4[14]$, $R^4[14] = \{137, 139, 141, 142, 143, 144, 145\}$,
• $I^4[15] \supseteq J^4[15] \setminus R^4[15]$, $R^4[15] = \{162, 164, 166, 167, 168, 172\}$, and
• $I^4[n] = J^4[n]$ for any $n \geq 16$. 

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4 Conclusion

The so-called intersection problem has been considered for many different combinatorial structures, including latin squares. This intersection problem basically takes a pair of structures, with the same parameters and based on the same underlying set, and determines the possible number of common sub-objects which they may have (such as blocks, entries, etc.). The intersection problem has also been extended from consideration of pairs of combinatorial structures to sets of three, or even sets of $\mu$, where $\mu$ may be larger than 3. In this paper, we studied the problem of determining, for all orders $n$, the set of integers $k$ for which there exists 4 latin squares of order $n$ having precisely $k$ identical cells, with their remaining $n^2 - k$ cells different in all four latin squares, denoted by $I^4_n$.

We have completely determined $I^4_n$ for $n \geq 16$ but there are still undecided values for $n \leq 15$. The smallest undecided question is whether $18 \in I^4_7$. If the answer to this question is yes, it can be concluded that $137 \in I^4_{14}$ and $162 \in I^4_{15}$ using Proposition 1.1 and Technique 2.1. All other undecided values ($R^4[x], 8 \leq x \leq 15$, section 3) should be answered directly and can not be solved (at least with) techniques discussed in this paper.

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Appendix

In each of the following, the top two numbers in each row indicate the order of latin square and the achieved intersection number, respectively. Elements of the set \{1, 2, \ldots, 9, a, b, c, \ldots, z\} are used as entries. Latin rectangles containing underlined entries, are the input needed for Techniques 2.3 and 2.4 where the underlined entries show fixed cells. Fixed or permuting parts are separated by double lines.
|    | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|----|---|---|---|---|---|---|---|
| 1  |   |   |   |   |   |   |   |
| 2  | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3  | 4 | 5 | 6 | 7 | 1 | 2 | 3 |
| 4  | 8 | 9 | 10| 11| 12| 13| 14|
| 5  | 15| 16| 17| 18| 19| 20| 21|
| 6  | 22| 23| 24| 25| 26| 27| 28|
| 7  | 29| 30| 31| 32| 33| 34| 35|

|    | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|----|---|---|---|---|---|---|---|
| 1  |   |   |   |   |   |   |   |
| 2  | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3  | 8 | 9 | 10| 11| 12| 13| 14|
| 4  | 15| 16| 17| 18| 19| 20| 21|
| 5  | 22| 23| 24| 25| 26| 27| 28|
| 6  | 29| 30| 31| 32| 33| 34| 35|
| 7  | 36| 37| 38| 39| 40| 41| 42|

|    | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|----|---|---|---|---|---|---|---|
| 1  |   |   |   |   |   |   |   |
| 2  | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3  | 8 | 9 | 10| 11| 12| 13| 14|
| 4  | 15| 16| 17| 18| 19| 20| 21|
| 5  | 22| 23| 24| 25| 26| 27| 28|
| 6  | 29| 30| 31| 32| 33| 34| 35|
| 7  | 36| 37| 38| 39| 40| 41| 42|

|    | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|----|---|---|---|---|---|---|---|
| 1  |   |   |   |   |   |   |   |
| 2  | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3  | 8 | 9 | 10| 11| 12| 13| 14|
| 4  | 15| 16| 17| 18| 19| 20| 21|
| 5  | 22| 23| 24| 25| 26| 27| 28|
| 6  | 29| 30| 31| 32| 33| 34| 35|
| 7  | 36| 37| 38| 39| 40| 41| 42|

|    | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|----|---|---|---|---|---|---|---|
| 1  |   |   |   |   |   |   |   |
| 2  | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3  | 8 | 9 | 10| 11| 12| 13| 14|
| 4  | 15| 16| 17| 18| 19| 20| 21|
| 5  | 22| 23| 24| 25| 26| 27| 28|
| 6  | 29| 30| 31| 32| 33| 34| 35|
| 7  | 36| 37| 38| 39| 40| 41| 42|

|    | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|----|---|---|---|---|---|---|---|
| 1  |   |   |   |   |   |   |   |
| 2  | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3  | 8 | 9 | 10| 11| 12| 13| 14|
| 4  | 15| 16| 17| 18| 19| 20| 21|
| 5  | 22| 23| 24| 25| 26| 27| 28|
| 6  | 29| 30| 31| 32| 33| 34| 35|
| 7  | 36| 37| 38| 39| 40| 41| 42|

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## 8.6

| 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|----|----|----|----|----|----|----|----|
| 2  | 1  | 45 | 36 | 65 | 98 | 34 | 57 |
| 3  | 4  | 56 | 34 | 78 | 35 | 67 | 48 |
| 4  | 5  | 67 | 56 | 87 | 78 | 45 | 32 |
| 5  | 6  | 78 | 67 | 56 | 45 | 32 | 12 |

## 8.7

| 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|----|----|----|----|----|----|----|----|
| 2  | 1  | 45 | 36 | 65 | 98 | 34 | 57 |
| 3  | 4  | 56 | 34 | 78 | 35 | 67 | 48 |
| 4  | 5  | 67 | 56 | 87 | 78 | 45 | 32 |
| 5  | 6  | 78 | 67 | 56 | 45 | 32 | 12 |

## 8.9

| 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|----|----|----|----|----|----|----|----|
| 2  | 1  | 45 | 36 | 65 | 98 | 34 | 57 |
| 3  | 4  | 56 | 34 | 78 | 35 | 67 | 48 |
| 4  | 5  | 67 | 56 | 87 | 78 | 45 | 32 |
| 5  | 6  | 78 | 67 | 56 | 45 | 32 | 12 |

## 8.10

| 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|----|----|----|----|----|----|----|----|
| 2  | 1  | 45 | 36 | 65 | 98 | 34 | 57 |
| 3  | 4  | 56 | 34 | 78 | 35 | 67 | 48 |
| 4  | 5  | 67 | 56 | 87 | 78 | 45 | 32 |
| 5  | 6  | 78 | 67 | 56 | 45 | 32 | 12 |

## 8.11

| 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|----|----|----|----|----|----|----|----|
| 2  | 1  | 45 | 36 | 65 | 98 | 34 | 57 |
| 3  | 4  | 56 | 34 | 78 | 35 | 67 | 48 |
| 4  | 5  | 67 | 56 | 87 | 78 | 45 | 32 |
| 5  | 6  | 78 | 67 | 56 | 45 | 32 | 12 |

## 8.12

| 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|----|----|----|----|----|----|----|----|
| 2  | 1  | 45 | 36 | 65 | 98 | 34 | 57 |
| 3  | 4  | 56 | 34 | 78 | 35 | 67 | 48 |
| 4  | 5  | 67 | 56 | 87 | 78 | 45 | 32 |
| 5  | 6  | 78 | 67 | 56 | 45 | 32 | 12 |

## 8.13

| 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|----|----|----|----|----|----|----|----|
| 2  | 1  | 45 | 36 | 65 | 98 | 34 | 57 |
| 3  | 4  | 56 | 34 | 78 | 35 | 67 | 48 |
| 4  | 5  | 67 | 56 | 87 | 78 | 45 | 32 |
| 5  | 6  | 78 | 67 | 56 | 45 | 32 | 12 |

## 8.14

| 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|----|----|----|----|----|----|----|----|
| 2  | 1  | 45 | 36 | 65 | 98 | 34 | 57 |
| 3  | 4  | 56 | 34 | 78 | 35 | 67 | 48 |
| 4  | 5  | 67 | 56 | 87 | 78 | 45 | 32 |
| 5  | 6  | 78 | 67 | 56 | 45 | 32 | 12 |
| 8,15 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|---|---|
| 1 |   |   |   |   |   |   |   |   |
| 2 |   |   |   |   |   |   |   |   |
| 3 |   |   |   |   |   |   |   |   |
| 4 |   |   |   |   |   |   |   |   |
| 5 |   |   |   |   |   |   |   |   |
| 6 |   |   |   |   |   |   |   |   |
| 7 |   |   |   |   |   |   |   |   |
| 8 |   |   |   |   |   |   |   |   |

| 8,17 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|---|---|
| 1 |   |   |   |   |   |   |   |   |
| 2 |   |   |   |   |   |   |   |   |
| 3 |   |   |   |   |   |   |   |   |
| 4 |   |   |   |   |   |   |   |   |
| 5 |   |   |   |   |   |   |   |   |
| 6 |   |   |   |   |   |   |   |   |
| 7 |   |   |   |   |   |   |   |   |
| 8 |   |   |   |   |   |   |   |   |

| 8,18 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|---|---|
| 1 |   |   |   |   |   |   |   |   |
| 2 |   |   |   |   |   |   |   |   |
| 3 |   |   |   |   |   |   |   |   |
| 4 |   |   |   |   |   |   |   |   |
| 5 |   |   |   |   |   |   |   |   |
| 6 |   |   |   |   |   |   |   |   |
| 7 |   |   |   |   |   |   |   |   |
| 8 |   |   |   |   |   |   |   |   |

| 8,19 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|---|---|
| 1 |   |   |   |   |   |   |   |   |
| 2 |   |   |   |   |   |   |   |   |
| 3 |   |   |   |   |   |   |   |   |
| 4 |   |   |   |   |   |   |   |   |
| 5 |   |   |   |   |   |   |   |   |
| 6 |   |   |   |   |   |   |   |   |
| 7 |   |   |   |   |   |   |   |   |
| 8 |   |   |   |   |   |   |   |   |

| 8,20 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|---|---|
| 1 |   |   |   |   |   |   |   |   |
| 2 |   |   |   |   |   |   |   |   |
| 3 |   |   |   |   |   |   |   |   |
| 4 |   |   |   |   |   |   |   |   |
| 5 |   |   |   |   |   |   |   |   |
| 6 |   |   |   |   |   |   |   |   |
| 7 |   |   |   |   |   |   |   |   |
| 8 |   |   |   |   |   |   |   |   |

| 8,21 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|---|---|
| 1 |   |   |   |   |   |   |   |   |
| 2 |   |   |   |   |   |   |   |   |
| 3 |   |   |   |   |   |   |   |   |
| 4 |   |   |   |   |   |   |   |   |
| 5 |   |   |   |   |   |   |   |   |
| 6 |   |   |   |   |   |   |   |   |
| 7 |   |   |   |   |   |   |   |   |
| 8 |   |   |   |   |   |   |   |   |
|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 |
| 5 | 6 | 7 | 8 | 1 | 3 | 4 | 2 |
| 3 | 4 | 5 | 7 | 2 | 6 | 3 | 1 |
| 8 | 2 | 1 | 4 | 3 | 6 | 5 | 8 |
| 7 | 4 | 5 | 6 | 7 | 8 | 1 | 3 |

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 |
| 5 | 6 | 7 | 8 | 1 | 3 | 4 | 2 |
| 3 | 4 | 5 | 7 | 2 | 6 | 3 | 1 |
| 8 | 2 | 1 | 4 | 3 | 6 | 5 | 8 |
| 7 | 4 | 5 | 6 | 7 | 8 | 1 | 3 |

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 |
| 5 | 6 | 7 | 8 | 1 | 3 | 4 | 2 |
| 3 | 4 | 5 | 7 | 2 | 6 | 3 | 1 |
| 8 | 2 | 1 | 4 | 3 | 6 | 5 | 8 |
| 7 | 4 | 5 | 6 | 7 | 8 | 1 | 3 |
| 9, 22 | 9, 23 | 9, 31 |
|-------|-------|-------|
| 7 8 9 6 | 7 8 9 5 | 7 8 9 6 |
| 6 7 8 9 5 1 2 3 4 5 | 5 6 7 8 9 4 1 2 3 | 6 7 8 9 5 1 2 3 4 5 |
| 9 6 7 8 4 5 1 2 3 | 9 5 6 7 8 3 4 1 2 | 9 6 7 8 4 5 1 2 3 |
| 8 9 6 7 3 4 5 1 2 | 8 9 5 6 7 2 3 4 1 | 8 9 6 7 3 4 5 1 2 |
| 3 5 2 1 9 6 4 8 7 | 4 3 9 1 2 7 5 6 8 | 3 5 2 1 9 6 4 8 7 |
| 4 3 5 2 8 9 7 6 1 | 3 8 1 2 4 9 6 7 5 | 4 3 5 2 8 9 7 6 1 |
| 5 4 1 3 2 8 9 7 6 | 7 4 2 3 1 8 9 5 6 | 5 4 1 3 2 8 9 7 6 |

| 10, 14 |
|--------|
| 8 9 a 7 | 7 8 9 a 6 1 2 3 4 5 6 |
| 7 8 9 a 6 1 2 3 4 5 6 | 8 9 a 7 1 2 3 4 5 6 |
| 9 a 7 8 9 5 6 1 2 3 4 5 | 7 8 9 a 6 1 2 3 4 5 6 |
| a 7 8 9 5 6 1 2 3 4 5 | a 7 8 9 5 6 1 2 3 4 5 |
| 9 a 7 8 9 5 6 1 2 3 4 5 | 9 a 7 8 9 5 6 1 2 3 4 5 |
| 6 5 4 2 3 9 7 a 1 8 | 6 5 4 2 3 9 7 a 1 8 |
| 5 1 2 6 a 3 4 8 7 9 | 5 1 2 6 a 3 4 8 7 9 |
| 4 6 1 3 2 a 9 5 8 7 | 4 6 1 3 2 a 9 5 8 7 |
| 2 3 6 5 9 4 8 7 a 1 | 2 3 6 5 9 4 8 7 a 1 |
| 3 4 5 1 8 7 a 9 6 2 | 3 4 5 1 8 7 a 9 6 2 |

| 10, 24 | 10, 34 | 10, 44 |
|--------|--------|--------|
| 8 9 7 6 | 3 4 5 6 | 8 9 a 7 |
| 7 8 9 a 6 1 2 3 4 5 6 | 2 9 7 8 a 1 | 7 8 9 a 6 1 2 3 4 5 6 |
| 7 8 9 a 6 1 2 3 4 5 6 | 8 9 a 7 1 2 3 4 5 6 | 7 8 9 a 6 1 2 3 4 5 6 |
| a 7 8 9 5 6 1 2 3 4 5 | a 7 8 9 5 6 1 2 3 4 5 | a 7 8 9 5 6 1 2 3 4 5 |
| 9 a 7 8 9 5 6 1 2 3 4 5 | 9 a 7 8 9 5 6 1 2 3 4 5 | 9 a 7 8 9 5 6 1 2 3 4 5 |
| 4 6 2 5 a 3 8 9 1 7 | 4 6 2 5 a 3 8 9 1 7 | 4 6 2 5 a 3 8 9 1 7 |
| 6 5 1 2 3 4 9 a 7 8 | 6 5 1 2 3 4 9 a 7 8 | 6 5 1 2 3 4 9 a 7 8 |
| 5 3 6 1 9 a 4 7 8 2 | 5 3 6 1 9 a 4 7 8 2 | 5 3 6 1 9 a 4 7 8 2 |

| 10, 46 | 10, 54 |
|--------|--------|
| 6 7 8 9 a 5 1 2 3 4 | 5 6 7 8 9 a 4 1 2 3 |
| 6 7 8 9 a 5 1 2 3 4 | 5 6 7 8 9 a 4 1 2 3 |
| 9 a 5 6 7 8 9 3 4 1 2 | 9 a 5 6 7 8 9 3 4 1 2 |
| 9 a 5 6 7 8 9 3 4 1 2 | 9 a 5 6 7 8 9 3 4 1 2 |
| 8 9 2 1 3 4 a 7 6 5 | 8 9 2 1 3 4 a 7 6 5 |
| 7 3 a 2 4 1 8 6 5 9 | 7 3 a 2 4 1 8 6 5 9 |
| 8 5 a 1 3 2 9 4 6 7 | 8 5 a 1 3 2 9 4 6 7 |
| 11,4 | 11,21 | 11,38 |
|------|-------|-------|
| 9 a  | 9 a   | 9 a   |
| 8 a  | 7 a   | 7 a   |
| b 8  | b 7   | b 7   |
| a 6  | a b   | a b   |
| b 5  | b a   | b a   |
| 5 4  | 3 2   | 3 2   |
| 2 3  | 1 7   | 1 7   |
| 7 5  | 2 3   | 2 3   |

| 11,45 | 11,50 | 11,51 |
|-------|-------|-------|
| 8 a  | 7 a   | 6 7   |
| 9 a  | 8 a   | 8 9   |
| b 6  | b 7   | b 6   |
| a 5  | a 4   | a 6   |
| 3 2  | 1 7   | 1 4   |
| 5 1  | 2 3   | 3 5   |
| b 7  | b a   | b a   |
| 4 3  | 6 2   | 6 1   |

| 11,54 | 11,59 | 11,60 |
|-------|-------|-------|
| 9 a  | 9 a   | 9 a   |
| 8 a  | 7 a   | 7 a   |
| b 6  | b 7   | b 6   |
| a 5  | a 4   | a 5   |
| 3 2  | 1 7   | 1 4   |
| 6 2  | 1 3   | 2 1   |
| b 3  | b a   | b a   |
| 5 6  | 4 8   | 4 3   |

| 11,62 | 11,65 | 11,70 |
|-------|-------|-------|
| 6 6  | 6 9   | 6 5   |
| 8 9  | 8 7   | 8 9   |
| b 5  | b 5   | b 8   |
| a 5  | a 6   | a 6   |
| 3 4  | 1 3   | 2 1   |
| 7 5  | 6 7   | 5 1   |

| 11,62 | 11,65 | 11,70 |
|-------|-------|-------|
| 9 a  | 9 a   | 9 a   |
| 8 a  | 7 a   | 7 a   |
| b 6  | b 7   | b 6   |
| a 5  | a 4   | a 5   |
| 3 2  | 1 7   | 1 4   |
| 6 2  | 1 3   | 2 1   |
| b 3  | b a   | b a   |
| 5 6  | 4 8   | 4 3   |

| 11,62 | 11,65 | 11,70 |
|-------|-------|-------|
| 9 a  | 9 a   | 9 a   |
| 8 a  | 7 a   | 7 a   |
| b 6  | b 7   | b 6   |
| a 5  | a 4   | a 5   |
| 3 2  | 1 7   | 1 4   |
| 6 2  | 1 3   | 2 1   |
| b 3  | b a   | b a   |
| 5 6  | 4 8   | 4 3   |
### 17, 215

|   | d | e | f | g | h | 8 | 9 | a | b | c | 6 | 7 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| d | h | d | e | f | g | h | c | 8 | 9 | a | b | 5 | 6 | 7 | 1 | 2 | 3 |
| g | h | d | e | f | g | b | c | 8 | 9 | a | 4 | 5 | 6 | 7 | 1 | 2 |
| f | g | h | d | e | f | d | e | f | g | h | 1 | 2 | 3 | 4 | 5 | 8 | 9 |
| e | f | g | h | d | 2 | 1 | 4 | 3 | 6 | 9 | 8 | 5 | a | c | 7 | b |
| 17, 224 |
|---------|
| 6 7 8 9 | a d b c e f g h | 1 2 3 4 5 |
| d e f g h | 6 7 8 9 | a b c | 5 1 2 3 4 |
| h d e f g | c 6 7 8 9 | a b | 4 5 1 2 3 |
| g h d e f | b c 6 7 8 9 | a | 3 4 5 1 2 |
| f g h d e | 1 2 3 4 5 6 7 | 8 9 | a b c |
| g f e h d | 2 1 4 3 6 5 8 | 7 | a 9 c b |

| 19, 264 |
|---------|
| b c d e f g h i j | 8 9 a | 1 2 3 4 5 6 7 |
| a a b c d e f g h i j | 8 9 | 7 1 2 3 4 5 6 |
| 9 a b c d e f g h i j | 8 6 7 1 2 3 4 5 |
| 8 9 a b c d e f g h i j | 5 6 7 1 2 3 4 |
| j 8 9 a b c d e f g h i j | 4 5 6 7 1 2 3 |
| i j 8 9 a b c d e f g h i j | 3 4 5 6 7 1 2 |
| h i j f g 4 a c 6 7 1 5 2 3 | d 9 8 b c |

| 19, 268 |
|---------|
| g h i j c d e f | 8 9 a b | 1 2 3 4 5 6 |
| g h i j c d e f | 7 8 9 | a b 2 3 4 5 6 1 |
| c d e f g h i j b | 7 8 9 3 4 5 6 1 2 |
| j c d e f g h i a b | 7 8 9 4 5 6 1 2 3 |
| i j c d e f g h a b c | 1 2 3 4 5 9 7 8 6 |
| h i j c d e f g 5 1 2 3 4 6 9 7 8 6 a |

| 19, 273 |
|---------|
| g h i j d | 1 2 3 4 5 6 7 8 9 a b c |
| d e f g h i j c | 1 2 3 4 5 6 7 8 9 a b |
| j d e f g h i b c | 1 2 3 4 5 6 7 8 9 a |
| i j d e f g h a b c | 1 2 3 4 5 6 7 8 9 |
| h i j 9 c b a e 8 6 f g 7 1 4 d 5 2 3 |

| 19, 274 |
|---------|
| d e f g h i j c | 1 2 3 4 5 6 7 8 9 a b |
| c d e f g h i j b | 1 2 3 4 5 6 7 8 9 a |
| j c d e f g h i a b | 1 2 3 4 5 6 7 8 9 |
| i j c d e f g h a b c | 1 2 3 4 5 6 7 8 |
| h i j 9 a b f g 8 4 6 7 e d 3 c 1 2 5 |
### 19.278

| 9 | a | b | c | d | e | f | g | h | i | j | 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 9 | a | b | c | d | e | f | g | h | i | j | 7 | 1 | 2 | 3 | 4 | 5 | 6 |
| j | 8 | 9 | a | b | c | d | e | f | g | h | i | j | 6 | 7 | 1 | 2 | 3 | 4 |
| i | j | 8 | 9 | a | b | c | d | e | f | g | h | i | 6 | 7 | 1 | 2 | 3 | 4 |
| h | i | j | d | f | a | g | c | 7 | 8 | 6 | 2 | b | e | 5 | 1 | 9 | 3 |

### 19.279

| 8 | 2 | a | b | c | d | e | f | g | h | i | j | 7 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 7 | 8 | 9 | a | b | c | d | e | f | g | h | i | j | 6 | 1 | 2 | 3 | 4 | 5 |
| j | 7 | 8 | 9 | a | b | c | d | e | f | g | h | i | 5 | 6 | 1 | 2 | 3 | 4 |
| i | j | 7 | 8 | 9 | a | b | c | d | e | f | g | h | 4 | 5 | 6 | 1 | 2 | 3 |
| h | i | j | g | e | f | a | c | 7 | 8 | 6 | 2 | b | 4 | e | 5 | 1 | 9 |

### 21.354

| a | b | c | d | e | f | g | h | i | j | k | l | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i | j | k | l | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| b | a | d | c | e | f | g | h | i | j | l | k | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| c | d | a | b | g | h | i | e | j | l | k | i | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| g | h | i | j | k | l | a | b | c | d | e | f | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 |
| i | g | h | i | j | k | l | a | b | c | d | e | f | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 |
| k | i | g | h | i | j | e | f | a | b | c | d | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 |
| j | k | l | g | h | i | 1 | 2 | 3 | 4 | 5 | 6 | a | b | c | 7 | 8 | 9 | d | e | f |

### 21.358

| 6 | 7 | 8 | 9 | a | b | c | d | e | f | g | h | i | j | k | l | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 7 | 8 | 9 | a | b | c | d | e | f | g | h | i | j | 5 | 1 | 2 | 3 | 4 |
| l | e | f | g | h | i | j | k | d | 6 | 7 | 8 | 9 | a | b | c | 4 | 5 | 1 | 2 |
| k | i | e | f | g | h | i | j | c | d | 6 | 7 | 8 | 9 | a | b | 3 | 4 | 5 | 1 | 2 |
| i | k | i | e | f | g | h | i | j | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | a | b | c | d |
| i | j | k | l | e | f | g | h | 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 | a | 9 | c | d | b |
| h | i | j | k | l | e | f | g | 3 | 4 | 1 | 2 | 7 | 8 | 5 | 6 | b | c | d | 9 | a |

### 10.77

| 6 | 7 | 8 | 9 | 5 | a | 3_{12} | 13_{42} | 14_{31} | 5_{21} |
|---|---|---|---|---|---|---|---|---|---|
| 9 | a | 6 | 7 | 4 | 9 | 3_{23} | 8 | 2_{15} | 5_{12} |
| 8 | 9 | a | 6 | 7 | 2_{51} | 1_{32} | 4_{25} | 2_{35} | 1_{34} |
| 7 | 8 | 9 | a | 6 | 3_{25} | 2_{43} | 5_{14} | 4_{25} | 3_{14} |
|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 6 | 7 | 8 | 9 | a | b |
| c | 6 | 7 | 8 | 9 | 4 | b |
| b | c | 6 | 7 | 8 | 9 | a |
| a | b | c | 6 | 7 | 8 | 9 |
| 9 | a | b | c | 6 | 7 | 8 |

| 6 | 7 | 8 | 9 | a | b | c | 5 |
| d | 6 | 7 | 8 | 9 | a | b | 4 |
| c | d | 6 | 7 | 8 | 9 | a | b |
| a | b | c | 6 | 7 | 8 | 9 |

| 6 | 7 | 8 | 9 | a | b | c | 5 |
| e | 6 | 7 | 8 | 9 | a | b | 4 |
| d | c | 6 | 7 | 8 | 9 | a | b |
| a | b | c | 6 | 7 | 8 | 9 |

| 6 | 7 | 8 | 9 | a | b | c | 5 |
| f | 6 | 7 | 8 | 9 | a | b | 4 |
| e | d | 6 | 7 | 8 | 9 | a | b |
| a | b | c | 6 | 7 | 8 | 9 |

| 7 | 8 | 9 | a | b | c | 5 |
| e | 7 | 8 | 9 | a | b | 6 |
| d | c | 7 | 8 | 9 | 4 | b |
| a | b | c | 7 | 8 | 9 |
| 9 | a | b | c | 7 | 8 | 9 |

| 7 | 8 | 9 | a | b | c | 5 |
| f | 7 | 8 | 9 | a | b | 6 |
| e | f | 7 | 8 | 9 | a | b |
| a | b | c | 7 | 8 | 9 |

| 7 | 8 | 9 | a | b | c | d | e | 5 |
| f | 7 | 8 | 9 | a | b | 6 | e |
| e | f | 7 | 8 | 9 | a | b | 4 |
| d | e | f | 7 | 8 | 9 | a | b |
| c | d | e | 7 | 8 | 9 | a | b |

| 7 | 8 | 9 | a | b | c | d | e | 5 |
| f | 7 | 8 | 9 | a | b | 6 | e |
| e | f | 7 | 8 | 9 | a | b | 4 | c |
| d | e | f | 7 | 8 | 9 | a | b |
| c | d | e | 7 | 8 | 9 | a | b |
|   | 7 | 8 | 9 | a | b | c | d | e | f | 5 | g | 3126 | 1234 | 6341 | 4612 | 2463 |
|---|---|---|---|---|---|---|---|---|---|---|---|-----|-----|-----|-----|-----|
| g | 7 | 8 | 9 | a | b | c | d | e | f | 5 | g | 3213 | e | 2341 | 154 | 425 | 4532 |
| f | g | 7 | 8 | 9 | a | b | 4 | d | e | 3152 | 6213 | c | 1635 | 2561 | 3261 |
| e | f | g | 7 | 8 | 9 | a | b | c | d | 2531 | 1362 | 4123 | 416 | 254 | 3645 |
| d | e | f | g | 7 | 8 | 9 | a | b | c | 1325 | 2631 | 3412 | 4563 | 8146 | 9254 |