Adaptive stabilization of two complex-conjugate dominant poles and quality indices of oscillating transient process in systems with interval parametric uncertainty

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Abstract. The paper is dedicated to a development of a controller synthesis method for control systems with an interval parametric uncertainty. The method combines two approaches to controller synthesis: adaptive one and interval one; also it is based on a pole domination principle. Therefore, the method allows providing desired control quality by placing two complex-conjugate dominant poles as necessary and reducing an influence of all other poles on a transient process in conditions of an interval parametric uncertainty. Application example of the synthesis method is provided; results of simulation modelling of a system, synthesized with the help of the method, are also provided.

1. Introduction
A vast variety of control objects or systems have uncertain parameters. Such parametric uncertainty must be considered during mathematical modelling or synthesizing control systems, which are to be reliable and robust, for example: industrial robots, aerial or underwater unmanned vehicles, etc.

There are several common approaches to synthesizing controllers for control systems with parametric uncertainty. The most widely used of them are adaptive control, fuzzy-logic control, neural-networks-based control and robust control, which considers uncertain parameters to vary in some known intervals. Each of these approaches has its own advantages and disadvantages. Adaptive control systems have stable control quality, but cannot adapt to rapid wide-range variations of parameters [1-3]. Neural-network-based controllers are able to provide desired control quality in such conditions, but require a vast amount of data to train a neural network [4]. Also, both adaptive controllers and neural-network based controllers have a complex structure. Fuzzy-logic controllers have a simpler structure, but the procedure of synthesizing a base of rules is not formal and requires a highly-qualified engineer to perform it [5]. Robust controllers have simple structure, though they are common linear controllers with constant parameters; procedures of their synthesis are formalized and require minimum data: only minimal and maximal estimates of uncertain parameters, but robust control systems often have insufficient dynamics and value of their quality indices may vary slightly due to parametric variations [6-14]. So, the problem of developing new methods of synthesizing a controller is highly relevant.

One of ways to improve existing synthesis methods is to combine them. This paper is dedicated to development of combined method of controller synthesis, which is based on adaptive and robust control simultaneously. The method will have advantages of both of these approaches: desired system dynamics, stable values of control quality indices, an ability to cope with rapid wide-range parametric variations, a minimum amount of data required, a simple structure of a controller and a strictly formalized synthesis algorithm. To develop such synthesis method is the main aim of the research. In order to reach the aim, a set of objectives must be accomplished: the problem must be set in terms of
mathematics, general expression for controller synthesis must be derived, an algorithm of an adaptive-
robust controller synthesis must be formulated and the algorithm must be applied to a synthesis 
problem. Let us now consider accomplishing each of these objectives.

2. Formulation of the problem 
The method is based on the robust approach to control, which considers all uncertain parameters of a 
system to vary in known limits or intervals. Each control system can be described by its characteristic
polynomial, which is a denominator of a transfer function of a closed-loop system. Coefficients of 
characteristic polynomial include all interval parameters of the system and, consequently, may also 
 vary in some intervals. Let us designate a characteristic polynomial with interval coefficients as an 
interval characteristic polynomial (ICP). General form of an ICP can be written as follows:

$$D(s, \overline{K}) = \sum_{i=0}^{n} [q_i(\overline{K})] \cdot s^i = \sum_{i=0}^{n} [q_i, q_i] \cdot s^i,$$

where $D(s, \overline{K})$ – is an ICP, $\overline{K}$ – is a vector of controller parameters, $q_i(\overline{K})$ – is an ICP coefficient 
of parameters of a controller, $q_i, q_i$ – are minimal and maximal values of each ICP 
coefficient, which include parameters of a controller.

Roots of (1), which are also poles of a system, migrate inside some regions of a complex plane due to 
parametric variations. This leads to instability of control quality indices: stability degree and 
oscillability degree. In order to reduce variations of control quality, a pole domination principle can be 
applied. According to this principle, in order to provide desired control quality in a system, it is 
 eno
gh to place one or two dominant poles according to desired values of stability degree and 
oscillability degree and then place all other non-dominant poles far away of dominant ones to reduce 
their influence on a transient process [15-17]. There are two practically significant ways to place 
dominant poles on a complex plane: one real dominant pole and two complex-conjugate dominant 
poles. The first variant allows providing an aperiodic transient process of desired setting time; the 
second variant allows determining a stability degree and an oscillability degree simultaneously. In the 
paper the second variant is considered: placing two complex-conjugate dominant poles. An example of 
such allocation is given in the figure 1.

![Figure 1. Example of poles allocation of an interval control system with two complex-conjugate dominant poles.](image)

In the figure 1 complex-conjugate dominant poles lie provide following values of quality indices: 
stability degree varies from $\eta_1 = 3$ to $\eta_2 = 3.8$; oscillability degree varies from $\mu_1 = 0.7$ to $\mu_2 = 1$. All 
other poles lie on the right of the line $\Re(x) = -8.2$ and their influence on a transient process is 
minimized. The aim of the paper is to develop a method, which will allow stabilizing dominant poles
in points of a complex plane, but not regions of it. This will make control quality of synthesized systems more stable. Let us now consider the main idea of the method.

The main idea of the method consists in decomposing an ICP in three parts: a dominant polynomial \( A(s) \); a non-dominant polynomial \( B(s, \vec{K}) \), which is a result of ICP division by dominant polynomial \( A(s) \), and a remainder \( R(s, \vec{K}) \) of such division. Decomposed ICP can be written as follows:

\[
D(s, \vec{K}) = A(s) \cdot B(s, \vec{K}) + R(s, \vec{K}).
\] (2)

In the paper a problem of placing two dominant complex-conjugate poles is considered, so, dominant polynomial \( A(s) \) can be written as follows:

\[
A(s) = (s - \lambda_1)(s - \lambda_2) = (s - \alpha - j\beta)(s - \alpha + j\beta),
\] (3)

where \( \lambda_1, \lambda_2 \) – are dominant complex-conjugate poles; \( \alpha, \beta \) – are their real and imaginary part.

Synthesis of a controller can be performed in three steps: defining dominant poles, placing roots of \( B(s, \vec{K}) \) far enough of dominant poles and providing \( R(s, \vec{K}) \) equality to zero. Now, we can formulate the problem of the research: to develop a method of placing roots of \( B(s, \vec{K}) \) and providing \( R(s, \vec{K}) \) equality to zero despite interval parametric uncertainty of both expressions.

3. Derivation of general expressions

Let us now derive expressions, which will allow calculating coefficients of a non-dominant polynomial and a remainder. As son, as we know the general form of an ICP (1) and a form of a dominant polynomial (3), we can divide (1) by (3) in general form and derive necessary expressions. They can be written as follows:

\[
\left[ b_i \right] = \left[ q_{i+2} \right] + x \cdot \left[ b_{i+1} \right] - y \cdot \left[ b_i \right], i \in n - 2...0; \tag{4}
\]

\[
R(s) = (q_1(z) + x \cdot b_0(z) - y \cdot b_i(z)) \cdot s + q_0(z) - y \cdot b_0(z); \tag{5}
\]

where \( \left[ b_i \right] \) – are coefficients of non-dominant polynomial; \( x \) – is a sum of dominant poles; \( y \) – is a production of dominant poles; \( n \) – is a degree of an ICP; \( R(s) \) – is a remainder; \( q_0(z), q_1(z), b_0(z), b_i(z) \) – are dependencies between coefficients of an ICP and a dominant polynomial and a vector of interval parameters \( z \). Let us now formulate the synthesis algorithm on a base of (4) and (5).

4. Development of the synthesis algorithm

In order to formulate the synthesis algorithm we will define required data, algorithm itself and resulting data.

First of all, before starting the synthesis procedure, the aim of the synthesis should be defined: desired values of stability degree and oscillability degree. The method is based on a pole domination principle, so, these parameters will be defined through desired allocation of dominant poles of a system. Also, parameters of a non-dominant poles allocation area must be define: maximal real part of a non-dominant pole and sector angle of non-dominant poles allocation. All these data will define a desired control quality of a system; an example of such definition is given in the figure 2.
In the figure 2 robust stability degree of the system $\eta$ and robust oscillability degree of the system $\mu_1$ is defined by dominant poles. Also in the figure 2 a maximal real part of a non-dominant pole $\delta$ and sector angle of non-dominant poles allocation $\mu_2$ are defined. It should be noticed, that $\mu_2$ must not exceed $\mu_1$.

Now, after the aim of synthesis is determined, a controller structure should be chosen. Transfer function of the controller must include three and more tunable parameters: two parameters are necessary to stabilize a pair of dominant poles; one parameter is necessary to place all non-dominant poles. In example of the method application a PID-controller will be used. Transfer function of a PID-controller can be written as follows:

$$W_{pl}(s, K_p, K_I, K_D) = K_p + \frac{K_I}{s} + K_D \cdot s,$$  \hspace{1cm} (6)

where $K_p, K_I, K_D$ – are proportional, integral and differential coefficients of PID-controller.

After choosing the controller, a transfer function of a whole system can be derived. Coefficients of denominator of such function, which is an ICP in a form (1), can be used in (4) and (5) to calculate coefficients of a non-dominant polynomial $B(s, K_p, K_I, K_D)$ and a remainder $R(s, K_p, K_I, K_D)$. After deriving a remainder, we should provide its equality to zero. To do this, we perform a substitution $s \to j\omega$, equate real and imaginary part of a remainder to zero and join these two equations in a system. In case of using a PID-controller (6) this system can be written as follows:

$$\begin{aligned}
&\text{Re}(R(j\omega, K_p, K_I, K_D, z)) = 0; \\
&\text{Im}(R(j\omega, K_p, K_I, K_D, z)) = 0.
\end{aligned}$$

By solving this system of equations functions, linking parameters of controller and interval parameters of a system with each other can be derived. General form of such function can be written as follows:

$$\begin{aligned}
K_p &= f(K_p, z); \\
K_I &= f(K_D, z).
\end{aligned}$$  \hspace{1cm} (7)

These functions are laws of controller adaptation, which allow stabilizing dominant poles in points of a complex plane. After deriving, these function must be substituted to expressions of $B(s, K_p, K_I, K_D)$ coefficients. This will allow deriving $B(s, K_D)$.

After expression for non-dominant polynomial $B(s, K_D)$, which depends from only one parameter of a controller, is derived, a D-partition method can be used to place its roots in desired area of a complex plane. An equation of a D-partition curve can be derived by substituting $s$ in $B(s, K_D)$ with
a border equation of a non-dominant poles allocation area. For example, in order to provide a minimal
real part of non-dominant poles, following substitution must be performed:
\[ s \rightarrow j \cdot \omega + \delta, \delta \leq 0, \]
(8)
In order to place non-dominant poles in a sector, a following substitution must be performed:
\[ s \rightarrow \frac{1}{\mu_2} \cdot \omega + j \cdot \omega, \mu \geq 0, \]
(9)
Performing a D-partition with both substitutions will allows placing non-dominant poles in a truncated sector, as shown in the figure 2. Constant value of the differential coefficient \( K_D \), chosen from a D-partition, will provide desired allocation of non-dominant poles.
Final form of the synthesized controller consists in laws of adaptation for proportional \( K_p \) and integral \( K_I \) coefficients (7) and a constant value of a differential coefficient \( K_D \), chosen from a D-partition.
To sum up all aforementioned information, the synthesis algorithm can be formulated:
1. Choose desired control quality and provide all required date as shown in the figure 2.
2. Choose an appropriate controller; derive a transfer function of the system and find the ICP in a form (1).
3. Derive coefficients of the non-dominant polynomial \( B(s, \vec{K}) \) and the remainder \( R(s, \vec{K}) \) with the help of (4) and (5).
4. Find adaptation laws (7) for two parameters of the controller.
5. Substitute (7) to \( B(s, \vec{K}) \) and derive equations of D-partition curves with the help of substitutions (8) and (9).
6. Choose values of other parameters of the controller from D-partition.
7. Controller is synthesized. It consists of adaptation laws for two of controller parameters and constant values of the rest of them.

Let us consider an application example of the algorithm.

5. Example of synthesis method application
Let us assume, that there is a problem of providing a desired control quality in a control system with interval parametric uncertainty. Transfer function of a control object of the system is given below:
\[ W(s) = \left\{ \frac{1}{[p_1] \cdot s^3 + [p_2] \cdot s^2 + [p_1] \cdot s + [p_0]} \right\}, \]
Where \([p_1]=[1:1.1], [p_2]=[28:32], [p_1]=[215:285] \) and \([p_0]=[350:370] \) – are interval parameters of the system.
According to the synthesis algorithm, let us now determine a desired control quality: \( \eta = 1 \), \( \delta = -10 \), \( \mu_1 = \mu_2 = 1 \). In order to reach the aim of synthesis a PID-controller (6) will be used. Considering this, an ICP of the system can be written as follows:
\[ D(s, K_p, K_I, K_D, \vec{p}) = \left\{ \left[ p_1 \right] \cdot s^4 + \left[ p_2 \right] \cdot s^3 + \left[ p_1 \right] + K_D \cdot s^2 + \left[ p_0 \right] \cdot K_p \cdot s + K_I \right\} \]
(10)
Let us now derive the non-dominant polynomial with the help of (4):
\[ B(s) = \left\{ [1:1.1] \cdot s^2 + [25.8:30] \cdot s + \right\} + \left\{ K_D + 152.8; K_D + 231.4 \right\} \]
(10)
After this, a remainder can be derived with the help of (5). Then, adaptation laws (7) can be found:
\[ K_p(K_D, p_1, p_2, p_3) = 2 \cdot p_1 - 4 \cdot p_2 + 4 \cdot p_3 + 2 \cdot K_D; \]
\[ K_I(K_D, p_0, p_1, p_2) = -p_0 + 2 \cdot p_1 - 2 \cdot p_2 + 2 \cdot K_D; \]
(11)
Now, let us choose the value of differential coefficient \( K_D \). To do this, a D-partition method will be applied. Equations of D-partition curves can be derived by substituting (8) and (9) to (10). First, let us find values of \( K_D \), which provide allocation of non-dominant poles inside the desired sector. D-partition curves in vertices of parametric polytope of the system are shown in the figure 3.
Figure 3. D-partition curves in a plane of $D_K$, providing allocation of non-dominant poles in the desired sector

From the figure 3 it is clear, that values $K_D = (0; 71.234)$ provide desired allocation of non-dominant poles.

Now let us find values of $K_D$, which provide allocation of non-dominant poles on the left from the line $\text{Re}(x) = -10$ on a complex plane. D-partition curves, in vertices of parametric polytope of the system are shown in the figure 4.

Figure 4. D-partition curves in a plane of $D_K$, providing allocation of non-dominant poles on the left of desired line

From the figure 4 it is clear, that all values $K_D > 47.3$ provide desired allocation of non-dominant poles. Intersection of intervals from these two D-partitions gives us a set of values $K_D = (47.3; 71.234)$, which provides desired allocation of non-dominant poles in a truncated sector. Let us choose $K_D = 50$.

Finally, the controller is synthesized. Expressions for its coefficients can be written as follows:

\[
K_p(p_1, p_2, p_3) = 2 \cdot p_1 - 4 \cdot p_2 + 4 \cdot p_3 + 100;
\]

\[
K_I(p_0, p_1, p_2) = -p_0 + 2 \cdot p_1 - 2 \cdot p_2 + 100;
\]

$K_D = 50$.

Poles allocation of the synthesized system is shown in the figure 5.
Figure 5. Poles allocation of the synthesized system

From the figure 5 it is clear, that synthesis was successful. Dominant poles are placed in desired points of a complex plane; allocation area of non-dominant poles is placed inside of the desired truncated sector.

In the figure 6 step responses of the synthesized system for different values of interval parameters are shown.

Figure 6. Step responses of the synthesized system

From the figure 6 it is clear, that control quality of the system is sufficient despite interval parametric uncertainty: setting time is equal to 2 seconds; overshoot does not exceed 5%.

6. Simulation modeling

Let us now test the behavior of the synthesized system, when uncertain parameters of the system vary faster, than setpoint signal. To do this, a SIMULINK model of the synthesized system was developed. Diagram of parametric variations during first 90 seconds of modeling is shown in the figure 7; input and output signals of the system are shown in the figure 8.
Figure 7. Variation of parameters during simulation modelling of the synthesized

Figure 8. Input and output signals of the synthesized system during simulation modelling

From figures 7 and 8 it is clear, that the system follows the sine input signal accurately. This means, that adaptation laws, synthesized with the help of the method, are effective.

7. Conclusion
The main aim of the research was reached: a new method of synthesis, which allows designing adaptive robust controllers, was developed. Controllers, synthesized with the help of the method, have advantages of interval and adaptive approaches to manipulating systems with uncertain parameters: a procedure of synthesis is simple; a controller structure is simple and strictly defined; a controller is able to damp rapid significant parametric disturbances and provides stable control quality. All these features were proved by simulation modelling in rigid conditions, when parameters of the system vary faster, than input signal does.

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9. References
[1] Ioannou P and Baldi P 2010 Robust Adaptive Control: The Control Systems Handbook (Boca Raton: CRC Press).
[2] Ioannou P and Sun J 2012 Robust Adaptive Control (United States: Courier Corporation).
[3] Feng G and Lozano R 1999 Adaptive Control Systems (Oxford: Newnes).
[4] Wang D, Huang J, Lan W and Li X 2009 Neural network-based robust adaptive control of nonlinear systems with unmodeled dynamics Math. and Comp. in Sim. 79(5) 1745-53.
[5] Lee H, Park J and Chen G 2001 Robust fuzzy control of nonlinear systems with parametric uncertainties IEEE Trans. on Fuzzy Syst. 9(2) 369-79.
[6] Polyak B and Scherbakov P 2002 Robust stability and control (Moscow: Science).
[7] Bhattacharyya S, Chapellat H and Keel L 1995 Robust control: The parametric approach (United States: Prentice-Hall).
[8] Nesenchuk A 2002 Parametric synthesis of qualitative robust control systems using root locus fields. Proc. of the 15th Triennial World Congress of The Int. Federation of Automatic Control (Barcelona) 387-387.
[9] Nesenchuk A and Fedorovich S 2008 Parametric synthesis method of integral systems on the basis of root locus curves of Kharitonov polynomials Autom. and Rem. Contr. 69(7) 1133-41.
[10] Nesenchuk A 2017 A method for synthesis of robust interval polynomials using the extended root locus. Proc. of the American Control Conf. (Seattle) 1715-25.
[11] Wang Y, Schinkel M and Hunt K 2002 PID and PID-like controller design by pole assignment within D-stable regions Asian J. of Contr. 4(4) 423-32.
[12] Zhmud V, Dimitrov L and Yadrishnikov O 2014 Calculation of regulators for the problems of mechatronics by means of the numerical optimization method Proc. of 12th Int. Conf. on Actual Problems of Electronic Instrument Engineering (Novosibirsk) 739-744.
[13] Zhmud V, Reva I and Dimitrov L 2017 Design of robust systems by means of the numerical optimization with harmonic changing of the model parameters J. of Phys.: Conf. Ser. 803(1).
[14] Tagami T and Ikeda K 2003 Design of robust pole assignment based on Pareto-optimal solutions Asian J. of Contr. 5(2) 195-205.
[15] Gayvoronskiy S and Ezangina T 2014 The synthesis of the robust stabilization system of cable tension for the test bench of weightlessness simulation Adv. Mat. Res. 1016 394-99.
[16] Gayvoronskiy S, Ezangina T and Khozhaev I 2015 The interval-parametric synthesis of a linear controller of speed control system of a descent submersible vehicle IOP Conf. Ser.: Mat. Sc. and Eng. 93.
[17] Gayvoronskiy S, Ezangina T and Khozhaev I 2017 Method of interval system poles allocation based on a domination principle Int. Conf. on Mechanical, System and Control Engineering (St. Petersburg) 245-249.