DYNAMICS OF A CONTINUOUS PIECEWISE AFFINE MAP OF THE SQUARE

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ABSTRACT. We present a one-parameter family of continuous, piecewise affine, area preserving maps of the square, which are inspired by a dynamical system in game theory. Interested in the coexistence of stochastic and (quasi-)periodic behaviour, we investigate invariant annuli separated by invariant circles. For certain parameter values, we explicitly construct invariant circles both of rational and irrational rotation numbers, and present numerical experiments of the dynamics on the annuli bounded by these circles.

1. Introduction

Piecewise affine maps and piecewise isometries have received a lot of attention, either as simple, computationally accessible models for complicated dynamical behaviour, or as a class of systems with their own unique range of dynamical phenomena. For a list of examples, see [1, 2, 3, 4, 6, 7, 9, 10, 13, 15, 16, 17, 22, 23, 24] and references therein.

In this paper, we investigate a family of piecewise affine homeomorphisms of the unit square, motivated by a particular class of dynamical systems modeling learning behaviour in game theory, the so-called Fictitious Play dynamics (see [19, 20, 21]). It was shown in [21] that this dynamics can be represented by a flow on $S^3$ with a topological disk as a global first return section, whose first return map is continuous, piecewise affine, area preserving and fixes the boundary of the disk pointwise. For an investigation of the itinerary structure and some numerical simulations of such first return maps, see [15].

The maps considered in this paper are chosen to satisfy these properties, and the nine-piece construction considered here seems to be the simplest possible (nontrivial) example satisfying all of them. Its qualitative behaviour resembles that seen in [15] for the first return maps of Fictitious Play; in particular, the ways in which stochastic and (quasi-)periodic behaviour coexist seem to be of similar type, giving rise to similar phenomena.

In Sections 2 and 3 we present a geometric construction of our family of maps and describe its basic formal properties. Then, interested in the long-term behaviour of iterates of the maps, our next goal is to establish the existence of certain invariant regions. For that, in Section 4 we develop some technical results about periodic orbits and invariant curves, and in Section 5 prove their existence for certain parameter values. Finally, in Section 6 we discuss the dynamics for more general parameter values, present numerical observations and discuss open questions.

2. Construction of the map

Let us denote the unit square by $S = [0,1] \times [0,1]$. We construct a one-parameter family of continuous, piecewise affine maps $F_\theta : S \to S$, $\theta \in (0, \pi/4)$, as follows (see Fig. 1 for an illustration).

Denote the four vertices of $S$ by $E_1 = (0,0)$, $E_2 = (1,0)$, $E_3 = (1,1)$ and $E_4 = (0,1)$. In the following we will use indices $i \in I = \{1,2,3,4\}$ with cyclic order, i.e., with the understanding that index $i + 1$ is 1 for $i = 4$ and index $i - 1$ is 4 for $i = 1$.

Let $\theta \in (0, \pi/4)$, and for $i \in I$ let $L_i$ be the ray through $E_i$, such that the angle between the segment $E_i E_{i+1}$ and $L_i$ is $\theta$. Let $P_i \in \text{int}(S)$ be the point $L_{i-1} \cap L_i$, then the $P_i$, $i \in I$, form a smaller square inside $S$. Now we divide $S$ into the following nine regions (see Fig 1(left)):
Since square which has the same y-coordinate as \( \Delta \) square by is continuous, its derivative we call these the points \((i, E)\). Now, we repeat the same construction 'in reverse orientation', to obtain a second, very similar partition of \( S \), as shown in Fig.1(right). Here we denote the vertex of the inner square which has the same y-coordinate as \( P_1 \) by \( P'_1 \), and the other vertices of the inner square by \( P'_2, P'_3, P'_4 \), in counterclockwise order. For \( i \in I \) we denote the triangles \( \mathcal{A}_i' = \Delta(E_i, E_{i+1}, P'_i), \mathcal{B}_i' = \Delta(E_i, P'_i, P'_{i-1}) \) and the square \( C' = \square(P'_1, P'_2, P'_3, P'_4) \).

Finally, the map \( F = F_0: S \to S \) is uniquely defined by the data

- \( F(E_i) = E_i, i \in I \);
- \( F(P_i) = P'_i, i \in I \);
- \( F \) affine on each of the pieces \( \mathcal{A}_i, \mathcal{B}_i, \) and \( C \).

3. Properties of \( F \)

It is easy to see that \( F \) is a piecewise affine homeomorphism of \( S \), with \( F(\mathcal{A}_i) = \mathcal{A}_i' \) and \( F(\mathcal{B}_i) = \mathcal{B}_i' \), for each \( i \in I \), and \( F(C) = C' \). Moreover, \( F \) is area and orientation preserving, since \( \det dF = 1 \) everywhere. Note that while \( F \) is continuous, its derivative \( dF \) has discontinuity lines along the boundaries of the pieces; we call these the break lines. Note also that \( F_{|\partial S} = \text{id} \).

We denote \( P_1 = (s, t) \), where \( t \in (0, \frac{1}{2}) \) and \( s \in (0, \frac{1}{2}) \). The coordinates of the other points \( P_i \) and \( P'_i \) are then given by symmetry. Simple geometry gives that \( s, t, \) and \( \theta \) satisfy

\[
    t - t' = s^2, \quad t = \sin^2 \theta, \quad s = \sin \theta \cos \theta. \tag{3.1}
\]

The map \( F \) is given by three types of affine maps \( A, B, \) and \( C \):

- \( A: \mathcal{A}_1 \to \mathcal{A}_1' \) is a shear fixing \( E_1E_2 = [0, 1] \times [0] \) and mapping \( P_1 \) to \( P'_1 \):

\[
    A(x, y) = \begin{pmatrix} 1 & \frac{1 - 2s}{2} \\ 0 & 1 \end{pmatrix} \left( \begin{array}{c} x \\ y \end{array} \right).
\]

It leaves invariant horizontal lines and moves points in \( \text{int}(\mathcal{A}_1) \) to the right (since \((1 - 2s)/t > 0\)).
Let $S$ denote the rotation about $Z$ by the angle $\alpha = \frac{3\pi}{2} - \theta$, where $\theta$ is the parameter angle in the construction of the map $F$. All other pieces of $F$ are analogous, by symmetry of the construction. To capture this high degree of symmetry, we make the following observations which follow straight from the definition of $F$.

**Lemma 3.1.** Let $R$ denote the rotation about $Z = \left(\frac{1}{2}, \frac{1}{2}\right)$, mapping $P_i$ to $P'_i$, $i \in I$:

$$C(x, y) = \left(\frac{2s}{1 - 2t}, \frac{2t - 1}{2s}\right) \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} 1 - t - s \\ t - s \end{array}\right).$$

The rotation angle is $\alpha = \frac{3\pi}{2} - \theta$, where $\theta$ is the parameter angle in the construction of the map $F$.

**Lemma 3.2.** Let $S(x, y) = (y, x)$. Then $F$ is $S$-reversible, i.e., $S$ conjugates $F$ and $F^{-1}$:

$$S \circ F \circ S = F^{-1}.$$

Further, $F$ is $T_1$- and $T_2$-reversible for the reflections $T_1(x, y) = (1 - y, x)$ and $T_2(x, y) = (x, 1 - y)$.

Heuristically, $F$ acts similarly to a twist map: The iterates $F^n(X)$ of any point $X \in \text{int}(S)$ rotate counterclockwise about $Z$ as $n \to \infty$. The 'rotation angle' (the angle between $XX$ and $ZF(X)$) is not constant, but it is bounded away from zero as long as $X$ is bounded away from $\partial S$; in particular, every point whose orbit stays bounded away from the boundary runs infinitely many times around the centre.

Note also that the rotation angle is monotonically decreasing along any ray emanating from the centre $Z$. However, it is not strictly decreasing, as all points in $C$ rotate by the same angle; this sets the map $F$ apart from a classical twist map, for which strict monotonicity (the "twist condition") is usually required.

### 4. Invariant Circles

Clearly, the circle inscribed to the inner square $\mathcal{C}$ and all concentric circles in it centred at $Z$ are invariant under $F$, which acts as a rotation on these circles. When $\theta$ is a rational multiple of $\pi$, the rotation $C$ is a rational (periodic) rotation, and a whole regular $n$-gon inscribed to $\mathcal{C}$ is $F$-invariant, see Fig. [4].

We are interested in other invariant circles encircling $Z$, as these form barriers to the motion of points under $F$ and provide a partitioning of $\mathcal{S}$ into $F$-invariant annuli. Numerical simulations indicate that such curves exist for many parameter values $\theta$ and create invariant annuli, on which the motion is predominantly stochastic.

This section follows closely the arguments of Bullett [6], where in a similar way, invariant circles are studied for a piecewise linear version of the standard map. The idea is to study the orbits of the points where the invariant circles intersect break lines and to prove that these follow a strict symmetry pattern, forming so-called cancellation orbits.

We consider invariant circles $\Gamma$ on which $F$ preserves the $S^1$-order of points. This, for example, is the case if all rays from $Z$ intersect the circle $\Gamma$ in precisely one point. For an
Lemma 4.1. Let $\Gamma_1, \Gamma_2$ be two invariant circles for $F$ encircling $Z$. If $\Gamma_1$ is contained in the component of $S \setminus \Gamma_2$ containing $Z$, then $\rho_{\Gamma_1} \geq \rho_{\Gamma_2}$.

In other words, if there is a family of such nested invariant circles, their rotation number is monotonically decreasing as the circles approach $\partial S$. It also follows that the rotation number $\rho$ of any orbit is bounded above by the rotation number on the centre piece $C$, i.e.,

$$0 \leq \rho \leq \frac{\alpha}{2\pi} = \frac{1 - \theta}{4\pi}.$$

We now consider $F$-invariant circles near the boundary $\partial S$, which do not intersect the centre piece $C$. Any such curve $\Gamma$ intersects exactly two types of break line segments: the segments $B_i \cap A_i = E_i F_i$ and $A_i \cap B_{i+1} = L_{i+1} F_{i+1}$, $i \in \mathcal{I}$. Let us call these intersection points $U_i = \Gamma \cap (B_i \cap A_i)$ and $V_i = \Gamma \cap (A_i \cap B_{i+1})$.

We say that an invariant curve which encircles $Z$ and on which $F$ preserves the $S^1$-order is rotationally symmetric, if it is invariant under the rotation $R$ (cf. Lemma 3.1): $R(\Gamma) = \Gamma$.

In the remainder of this section we will use the fact that any invariant circle $\Gamma$ with rational rotation number is of one of the following two types [12]:

- pointwise periodic, that is, $F|_{\Gamma}$ is conjugate to a rotation of the circle;
- non-periodic, that is, $F|_{\Gamma}$ is not conjugate to a rotation; however, in this case, $F|_{\Gamma}$ still has at least one periodic orbit.

Following the ideas in [13], we now prove a number of results illustrating the importance of the orbits of $U_i$ and $V_i$ for the invariant circle containing them.

Lemma 4.2. Let $\Gamma$ be a rotationally symmetric invariant circle disjoint from $C$. Assume that $F|_{\Gamma}$ has rational rotation number $\rho_{\Gamma} = \frac{\alpha}{\theta} \in \mathbb{Q}$, and that $F|_{\Gamma}$ is periodic. Then for $i \in \mathcal{I}$ the orbit $\text{orb}(U_i) = \{ F^k(U_i) : n \in \mathbb{Z}\}$ contains some $V_j$, $j \in \mathcal{I}$, and vice versa.

Proof. By symmetry it is sufficient to show the result for $U_1$. Suppose for a contradiction that $V_j \notin \text{orb}(U_1)$ for all $j \in \mathcal{I}$. Let $O = (\bigcup_j \text{orb}(U_j)) \cup (\bigcup_j \text{orb}(V_j))$, which by periodicity of $F$ is finite. Let $X, Y \in O$ be the two points closest to $U_1$ on either side along $\Gamma$. Consider the line segment $XY$ which crosses the break line going through $U_1$. Then $F(XY)$ is a ‘bent’ line, consisting of two straight line segments, so that $F$ maps the triangle $\Delta(X, U_1, Y)$ to a quadrangle.

Now, by assumption and periodicity of $F$, there exists $k > 1$ such that $F^k(U_1) = U_j$ for some $j \in \mathcal{I}$ and $F^l(U_1) \notin \{ U_i : i \in \mathcal{I} \} \cup \{ V_j : j \in \mathcal{I} \}$ for $1 \leq l < k$. This implies that the triangle $\Delta(F(X), F(U_1), F(Y))$ is mapped by $F^{k-1}$ to the triangle $\Delta(F^k(X), F(U_1), F(Y))$ without bending any of its sides (since $X$ and $Y$ are the points in $O$ closest to $U_1$).

By symmetry we have that $\Delta(F^k(X), F^k(U_1), F^k(Y)) = \tilde{R}(\Delta(X, U_1, Y))$, where $\tilde{R}$ is the rotation about $Z$ by one of the angles $0, \frac{\pi}{2}, \frac{3\pi}{2}$, and hence

$$\text{area}(\Delta(F^k(X), F^k(U_1), F^k(Y))) = \text{area}(\Delta(X, U_1, Y)).$$

But $F$ is area preserving, so

$$\text{area}(\Delta(F^k(X), F^k(U_1), F(Y))) = \text{area}(\Delta(F(X), F(U_1), F(Y))),$$

which is a contradiction because $F$ maps $\Delta(X, U_1, Y)$ to a quadrangle which either properly contains or is properly contained in $\Delta(F(X), F(U_1), F(Y))$. $\square$

With a slightly bigger effort, we can extend the result to the case of non-periodic $F|_{\Gamma}$.

Lemma 4.3. Let $\Gamma$ be a rotationally symmetric invariant circle disjoint from $C$. Assume that $F|_{\Gamma}$ has rational rotation number $\rho_{\Gamma} = \frac{\alpha}{\theta} \in \mathbb{Q}$, and that $F|_{\Gamma}$ is not periodic. Then for $i \in \mathcal{I}$, the orbit of $U_i$ contains some $V_j$, $j \in \mathcal{I}$, and vice versa.
Proof. As in the previous lemma, we give a proof for \( U_1 \), the other cases following by symmetry. We distinguish two cases:

**Case 1: \( U_1 \) non-periodic for \( F \).**

Let us write \( z_k = F^k(U_1) \) for \( k \in \mathbb{Z} \). Since \( \rho_F = \frac{4}{9} \in \mathbb{Q} \), there exist points \( Q \) and \( Q' \) in \( \Gamma \), each periodic of period \( q \), such that \( z_{nq} \to Q \) and \( z_{-nq} \to Q' \) as \( n \to \infty \). Note that \( F^4 \) is affine in a sufficiently small neighbourhood on either side of \( Q \). Then for sufficiently large \( N \), the points \( z_{nq}, n > N \), lie on the straight line segment \( z_{Nq}Q \) (the contracting direction at \( Q \)). Hence for large \( n \), \( \Gamma \) contains the straight line segment \( z_{Nq}Q \). Analogously, for large \( n \), the straight line segment \( Q'z_{-nq} \) is contained in \( \Gamma \).

In particular, \( \ell = \sum_{-(N+1)q < m < nq} z_m \) is contained in \( \Gamma \) for large \( m \). But \( U_1 \in F^{(n+1)q}(\ell) \), so \( F^{nq}(\ell) \) has a kink for large \( n \) unless \( z_{Nq} = V_j \) for some \( j \) (note that since \( U_1 \) is non-periodic, \( U_1 \notin \text{orb}(U_1), i \in I \)). Since \( F^{nq}(\ell) \) is near \( Q \) for large \( n \), it has to be straight, and it follows that \( V_j \in \text{orb}(U_1) \) for some \( j \).

**Case 2: \( U_1 \) periodic for \( F \).**

In this case \( F^4(U_1) = U_1 \) and the argument is similar to the proof of Lemma 4.3. Assume for a contradiction that \( V_j \notin \text{orb}(U_1) \) for all \( j \). By symmetry, this implies \( V_j \notin \bigcup \text{orb}(U_1) \). Pick \( X, Y \in \bigcup \text{orb}(U_1) \) nearest to \( V_j \) from each side and denote by \( S \) the segment of \( \Gamma \) between \( X \) and \( Y \). Since the straight line segment \( XY \) crosses the break line which contains \( V_j \), its image \( F(XY) \) has a kink. Therefore, since \( F \) is area preserving, the area between \( F(S) \) and \( F(X)F(Y) \) is either greater or less than the area between \( S \) and \( XY \). For \( 0 \leq k \leq q \), the area between \( F^k(S) \) and \( F^k(X)F^k(Y) \) is equal to the area between \( F^{k+1}(S) \) and \( F^{k+1}(X)F^{k+1}(Y) \), unless \( F^k(S) \) contains one of the \( V_j \). Whenever \( V_j \in F^k(S) \) for some \( j \) and \( k \) (which can happen at most four times for \( 0 \leq k < q \)), this area decreases or increases. By symmetry, these up to four changes have the same form, so the area either always decreases or always increases. Note on the other hand that \( F^q(S) = S \), \( F^q(X) = X \) and \( F^q(Y) = Y \) (since \( X \) and \( Y \) are in the \( q \)-periodic orbit of \( U_1 \)). So if we denote the region between \( S \) and \( XY \) by \( \Omega \), \( \text{area}(F^q(\Omega)) \neq \text{area}(\Omega) \), which contradicts the fact that \( F \) is area preserving. This finishes the proof.

Combing the above lemmas, we get the following result.

**Proposition 4.4.** Let \( \Gamma \) be a rotationally symmetric invariant circle disjoint from \( C \) with rotation number \( \rho_F = \frac{4}{9} \in \mathbb{Q} \). Then for every \( i \in I \), the \( F \)-orbit of \( U_i \) contains some \( V_j \), \( j \in I \), and vice versa. Moreover, every such orbit contains an equal number \( n \) of the \( U_i \) and \( V_j \), which are traversed in alternating order. If \( n \geq 2 \), then any such orbit is periodic.

**Remark 4.5.** In [6], Bullet coined the term ‘cancellation orbits’ for these orbits of break points on an invariant circle, reflecting the insight that each ‘kink’ introduced by the discontinuity of \( dF \) at one such point needs to be ‘cancelled out’ by an appropriate ‘reverse kink’ at another \( dF \)-discontinuity point, if the invariant circle has rational rotation number.

However, cancellation orbits can also occur on invariant circles with \( \rho \notin \mathbb{Q} \). In that case these orbits are not periodic, and each cancellation orbit would only contain one of the \( U_i \) and one of the \( V_j \). We will see examples for this in Section 5.

We now show that cancellation orbits determine the behaviour of the whole map \( F|_\Gamma \).

**Proposition 4.6.** Let \( \Gamma \) be a rotationally symmetric invariant circle disjoint from \( C \). Then \( F|_\Gamma \) is periodic if and only if the \( U_i \) and \( V_j \)-orbits are periodic.

**Proof.** Of course, if the rotation number \( \rho^* \) is irrational, neither \( F|_\Gamma \) nor the break point orbits can be periodic, so we only need to consider rational rotation number.

Further, if \( F|_\Gamma \) is periodic, so are all cancellation orbits. For the converse, suppose for a contradiction that \( U_i \) and \( V_j \), \( i \in I \), are periodic, but \( F|_\Gamma \) is not. We repeat an argument already familiar from the proof of Lemma 4.3.
Pick any non-periodic point \( P \in \Gamma \), then there exists \( Q \in \Gamma \), such that \( F^n(Q) \to Q \) as \( n \to \infty \). Note that \( F^n \) is affine in a sufficiently small neighbourhood on either side of \( Q \). Then for \( n > N \) sufficiently large, the points \( F^n(P) \) lie on a straight line segment from \( F^n(Q) \) to \( Q \) (the contracting direction at \( Q \)). So \( \Gamma \) contains a straight line segment \( S \) which expands under \( F^{-q} \). This expansion cannot continue indefinitely, so \( F^{-m}(I) \) must meet some \( U_i \) (or \( V_i \)) for some \( m \). But then this \( U_i \) (or \( V_i \)) cannot be periodic, which contradicts the assumption. \( \square \)

5. Special parameter values

In this section, we will show that for a certain countable subset of parameter values \( \theta \in (0, \frac{\pi}{4}) \), the map \( F = F_\theta \) has invariant circles of the form described in the previous section.

**Theorem 5.1.** There exists a sequence of parameter angles \( \theta_1, \theta_2, \ldots \in (0, \frac{\pi}{4}) \), \( \theta_k \to \frac{\pi}{4} \) as \( k \to \infty \), such that for each \( K, F_{\theta_k} \) has a countable collection of invariant circles \( \{ \Gamma^N_K : N \geq 0 \} \), each of rational rotation number \( \rho(\Gamma^N_K) = 1/(4(K+N)) \). The curves \( \Gamma^N_K \) consist of straight line segments, are rotationally symmetric and converge to the boundary \( \partial S \) as \( N \to \infty \).

**Proof.** We will prove the result by explicitly finding periodic orbits in \( \bigcup_i (A_i \cup B_i) \), which hit the break lines whenever passing from one of the pieces to another. More precisely, we will show that for \( K \geq 3 \) and \( N \geq 0 \), there is a parameter value \( \theta = \theta_K \) and a point \( X_N \in E_1P_1 \), such that \( F^n(X_N) \in B_1 \) for \( 0 \leq n < K \), \( F^K(X_N) \in E_1P_1 \), \( F^n(X_N) \in A_1 \) for \( K \leq n < K+N \), and \( F^{N+K}(X_N) = R(X_N) \in E_2P_1 \), where \( R \) is the counterclockwise rotation by \( \frac{\pi}{4} \) about the centre of the square, see Fig 2. By symmetry this clearly gives a periodic cancellation orbit, and an invariant circle is then given by

\[
\Gamma^K_N = \bigcup_{i=0}^{4(K+N)} F^i(X_N)F^{i+1}(X_N).
\]

We use the following two lemmas, whose proofs we leave to the appendix.

**Lemma 5.2.** For every \( K \in \mathbb{N} \), \( K \geq 3 \), there exists a parameter \( \theta_K \in (0, \frac{\pi}{4}) \), such that for \( F = F_{\theta_K} \), \( F^K(E_1P_1) \subset B_1 \) for \( 0 \leq k < K \), and \( F^K(E_1P_1) = E_1P_1 = B_1 \cap A_1 \). For \( K \to \infty \), the angle \( \theta_K \) tends to \( \frac{\pi}{4} \).
Lemma 5.3. For every $\theta \in (0, \frac{\pi}{4})$ and $N \geq 0$ there exists a point $Y_N \in \overrightarrow{P_1P_2} = B_1 \cap A_1$ such that $F^n(Y_N) \in A_1$ for $0 \leq n < N$ and $F^N(Y_N) \in \overrightarrow{P_1P_2} = A_1 \cap B_2$. For $N \to \infty$, the points $Y_N$ converge to $E_1$.

By Lemma 5.2 for every $K \geq 3$ we can find $\theta_K$, such that $F^K$ maps $\overrightarrow{P_1P_2}$ to $\overrightarrow{P_1P_2}$ (in $B_1$). Then by Lemma 5.3 for any $N \geq 0$ there exists $X_N \in \overrightarrow{P_1P_2}$ such that $F^K(X_N) = Y_N \in \overrightarrow{P_1P_2}$ and $F^{K+N}(X_N) \in \overrightarrow{P_1P_2}$, and it can be seen that each open line segment $(F^K(X_N), F^{K+1}(X_N)), k = 0, \ldots, K + N - 1$, lies in the interior of either $B_1$ ($0 \leq k < K$) or $A_1$ ($K \leq k < K + N$), see Fig. 2.

Further, $B$ preserves the quadratic form $Q_B(x, y) = n(x^2 + y^2) - xy$. With $X_N = (x_1, y_1) \in \overrightarrow{P_1P_2}$ and $F^K(X_N) = (x_3, y_3) \in \overrightarrow{P_1P_2}$ as above, a simple calculation shows that $Q_B(X_N) = Q_B(F^K(X_N))$ implies that $x_1 = y_2$. Then, with $F^{K+N}(X_N) = (x_3, y_3) \in \overrightarrow{P_1P_2}$, one has $y_3 = y_2$, as $A$ preserves the $y$-coordinate. We have that $R(\overrightarrow{P_1P_2}) = \overrightarrow{P_1P_2}$, and since $x_1 = y_3$, it follows that $F^{K+N}(X_N) = R(X_N)$. Rotational symmetry (Lemma 5.4) then implies that $X_N$ is a periodic point of period $4(K + N)$ for $F$, and the line segments connecting the successive $F$-iterates of $X_N$ form a rotationally symmetric invariant circle for $F$.

Hence we obtain, for each $K \geq 3$, a parameter $\theta_K$ such that $F_{\theta_K}$ has a sequence of rotationally invariant circles $\Gamma_K^N$, $N \geq 0$, each consisting of straight line segments. Moreover, Lemma 5.3 implies that $\Gamma_K^N \to \partial S$ (in the Hausdorff metric) and $\rho_{\Gamma_K} = 1/(4(K + N)) \to 0$ as $N \to \infty$.

In the proof of the theorem, for a sequence of special parameter values $\theta_K$ we constructed periodic orbits hitting the break lines and invariant circles made up of line segments connecting the points of the periodic orbits. A closer look at the behaviour of $F$ between any two consecutive invariant circles in this construction reveals that in fact the dynamics of $F$ in these regions is very simple for these parameter values.

To see this, let $\theta = \theta_K$, $K \geq 3$, $N \geq 0$, and take $X \in \Gamma_K^N \cap \overrightarrow{P_1P_2}$, $Y \in \Gamma_K^{N+1} \cap \overrightarrow{P_1P_2}$. Then $X$ and $Y$ are periodic cancellation orbits of periods $4(K + N)$ and $4(K + N + 1)$ on the respective invariant circles. Let $R$ be the quadrangle with vertices $X, Y, F(Y), F(X)$. Then $F^{K+N}(X) = R(X)$ and $F^{K+N+1}(Y) = R(Y)$ lie in $\overrightarrow{P_1P_2} = R(\overrightarrow{E_1P_2})$, and it is easy to see that $F^{K+N}$ maps the triangle $D_1 = \Delta(X, F(Y), F(X))$ affinely to the triangle $R(\Delta(X, Y, F(X)))$ and $F^{K+N+1}$ maps $D_1 = \Delta(X, Y, F(Y))$ affinely to $R(\Delta(Y, F(Y), F(X)))$. This gives a piecewise affine map $\Phi: D_1 \cup D_2 = R \to R(\partial S)$ of the simple form shown in Fig. 3. By symmetry, the first return map of $F$ to the quadrangle $R$ is then the fourth iterate of such map, and is easily seen to preserve the $y$-coordinate.

It follows immediately that in fact all points in the annulus between $\Gamma_K^N$ and $\Gamma_K^{N+1}$ lie on invariant circles. These invariant circles are of the same form as the $\Gamma_K^N$, that is, they consist of line segments parallel to those explicitly constructed in the proof of Theorem 5.1. The rotation number of the invariant circle through the point $W \in \overrightarrow{E_1P_1}$ changes continuously (in fact, linearly) from $1/(4(K + N))$ to $1/(4(K + N + 1))$, as $W$ goes from $X$ to $Y$.

![Figure 3](image-url)
These invariant circles take on both rational and irrational rotation numbers, and their intersections with the break lines are not necessarily periodic as those of the $\Gamma^0_{\frac{3}{K}}$, but their orbits still form cancellation orbits, since $F^K(\overline{E_1P_4}) = \overline{E_1P_3}$ (see Remark 4.5). We get the following corollary.

**Corollary 5.4.** For $\theta = \theta_K$, $K \geq 3$, as in Theorem 5.1, the annulus between $\partial S$ and $\Gamma^0_{\frac{3}{K}}$ (the invariant circle containing $P_i$ and $P'_i$, $i \in I$) is completely foliated by rotationally symmetric invariant circles with rotation numbers continuously and monotonically varying from 0 on $\partial S$ to $1/(4K)$ on $\Gamma^0_{\frac{3}{K}}$.

**Remark 5.5.** One can check that $K = 3$ in the proof of Theorem 5.1 is obtained by setting $\theta = \frac{\pi}{8}$, corresponding to $t = (2 - \sqrt{2})/4$ and $s = \sqrt{2}/4$. This is the case when $C$ is the rotation by $\frac{\pi}{4}$ on $C$. For $K \geq 4$, exact values for $\theta$ are less easy to determine explicitly.

In this special case $\theta = \frac{\pi}{8}$, the map $F$ turns out to be of a very simple form, allowing a complete description of the dynamics on all of $S$. By a similar argument applied to the region inside $\Gamma^0_{\frac{3}{3}}$ (containing the rotational part $C$), the statement of Corollary 5.4 can then be strengthened, stating that in this case the whole space $S$ is foliated by invariant circles, with rotation numbers varying continuously and monotonically from 0 on $\partial S$ to $\frac{1}{8}$ on the invariant octagon $O$ inscribed in $C$. The invariant circles between $\Gamma^0_{\frac{3}{3}}$ and $O$ each consist of twelve straight line segments, parallel to the twelve segments of $\Gamma^0_{\frac{3}{3}}$, see Fig. 4.

6. **General parameter values and discussion**

For other parameter values than the ones considered in the previous section, we generally cannot prove the existence of any invariant circles. Let us briefly mention more general ‘higher order’ cancellation orbits and invariant circles.

Recall that in Theorem 5.1 we constructed a family of invariant circles consisting of line segments connecting successive orbit points of periodic cancellation orbits. The chosen cancellation orbits were of the simplest possible kind, where a point on any break line is mapped by a certain number of iterations to the next possible break line.
It is possible to construct invariant circles from one or several more complicated periodic cancellation orbits (for other values of $\theta$ than the ones in Theorem 5.1). On such periodic cancellation orbit, the iterates of a point on a break line would cross several break lines before hitting one. The resulting invariant circle would still consist of straight line segments, but in such a way that a given segment and its image under $F$ are not adjacent on the circle.

Doing this ‘higher order’ construction is conceptionally not much more difficult, but certainly more tedious than the ‘first order’ construction of Theorem 5.1. The effort to construct even just a single higher order periodic cancellation orbit seems futile, unless a more general scheme to construct all or many of them at once can be found.

It is unclear whether invariant circles of this type exist for all $\theta$, and whether for typical $\theta$ there exist piecewise line segment invariant circles (of rational or irrational rotation number) containing non-periodic cancellation orbits, as the ones seen in Corollary 5.4.

**Question 6.1.** For which parameter values $\theta \in (0, \pi/4)$ does $F_\theta$ have invariant circles with periodic cancellation orbits (and, therefore, rational rotation number)? For which $\theta$ are there piecewise line segment invariant circles with non-periodic cancellation orbits?

Moreover, we can at this point not rule out the existence of invariant circles (outside of $C$) of an entirely different kind, not consisting of straight line segments. Indeed, some numerical experiments seem to indicate the occurrence of invariant regions with smooth boundaries, but it is unclear whether this is due to the limited resolution (see, for example, left picture in Fig. 5 and Fig. 9).

By Proposition 4.4, the intersections of any invariant circle of rational rotation number with the break lines form cancellation orbits. For irrational rotation number, this need not be the case. Note, however, that under irrational rotation, the ‘kink’ introduced to an invariant circle at its intersection with a break line propagates densely to the entire circle, unless it is cancelled by eventually being mapped to another break line intersection. Hence, an invariant circle of irrational rotation number would either contain a cancellation orbit for each pair of break lines, or otherwise would be geometrically complicated, namely nowhere differentiable.

**Question 6.2.** Are there invariant circles for $F$ (outside of $C$) which are not comprised of a finite number of straight line segments? Are there invariant circles whose intersections with the break lines do not form cancellation orbits?

As for the dynamics of $F$ between invariant circles, we can only point to numerical evidence that annuli between consecutive invariant circles form ergodic components interspersed with ‘elliptic islands’. An elliptic island consists of a periodic point of, say, period $p$, surrounded by a family of ellipses which are invariant under $F^p$, such that $F^p$ acts as an irrational rotation on each of these ellipses; this is referred to as ‘quasi-periodic’ behaviour. In the case when $F^p$ is a rational rotation, these quasi-periodic invariant circles would take on the shape of polygons, consisting entirely of $p$-periodic points. The rest of the annuli seems to be filled with what is often referred to as ‘stochastic sea’, that is, the dynamics seems to be ergodic and typical orbits seem to fill these regions densely.

As in many similar systems (e.g., perturbations of the standard map), numerical observations seem to indicate that these ‘ergodic regions’ have positive Lebesgue measure (see Figures 5-9). This is related to questions surrounding the famous ‘quasi-ergodic hypothesis’, going back to Ehrenfest [8] and Birkhoff [5], conjecturing that typical Hamiltonian dynamical systems have dense orbits on typical energy surfaces (see also [11]).

1Due to the rotational symmetry of the system, a periodic cancellation orbit could contain one, two, or four pairs of break points. In the first two cases, the union of all rotated copies of the cancellation orbit would need to be considered, to form an invariant circle by adding straight line segments between successive points in this union.
Figure 5. The first $5 \cdot 10^5$ iterates of three initial points for $\theta = \frac{\pi}{11}$. Each of the orbits seems to be confined to an invariant annulus and fill this annulus densely, except for a number of elliptic islands.

Figure 6. Zoom-in of the first two orbits from Fig.5, showing that the invariant annuli contain large numbers of smaller and smaller elliptic island chains. Further zoom-in (and longer orbits) reveal increasingly intricate patterns of such quasi-periodic elliptic regions.

For a piecewise linear version of the standard map, Bullet [6] established a number of results on cancellation orbits and invariant circles of both rational and irrational rotation numbers. Wojtkowski [23, 24] showed that the map is almost hyperbolic and deduced that it has ergodic components of positive Lebesgue measure. Almost hyperbolicity here means the almost everywhere existence of invariant foliations of the space (or an invariant subset) by transversal local contracting and expanding fibres. Equivalently, this can be expressed through the existence of invariant contracting and expanding cone fields. By classical theory (see, for example, [18]) almost hyperbolicity implies certain mixing properties of the map, and, in particular, the existence of an at most countable collection of ergodic components of positive Lebesgue measure.

A similar kind of cone construction as in [23, 24] seems to be more difficult for the map studied in this paper. One important difference is that the piecewise linear standard map in these papers is a twist map, which is not strictly the case for the map $F$ studied here (see [14] and references therein for an overview over the numerous classical results for twist maps, mostly based on Birkhoff and Aubry-Mather theory). The additional property that $F$ equals the identity on the boundary of the square also sets it apart. In particular, the motion of points under $F$ close to the boundary can be arbitrarily slow, that is, take arbitrarily many iterations to pass through the piece $\mathcal{A}_i$ (while the number of iterations for a passage through $\mathcal{B}_i$ remains bounded). This seems to make it more difficult to explicitly
construct an invariant contracting or expanding cone field, as was done for the piecewise linear standard map. Moreover, such an invariant cone field construction cannot be carried out uniformly for all $\theta$. In fact, as can be seen from Corollary 5.4 there are parameter values $\theta_K$, $K = 3, 4, \ldots$, for which almost hyperbolicity cannot hold on large parts of $S$, as the dynamics is completely integrable on the annulus between the invariant circle $\Gamma^0_K$ and $\partial S$ (and even on all of $S$ for $\theta = \theta_3 = \frac{\pi}{8}$, see Remark 5.5).

We are led to leave the following as a question.

**Question 6.3.** Are there parameter values $\theta$ for which the map $F_\theta$ is almost hyperbolic on some invariant subset of $S$? How large is the set of parameters $\theta$ for which this is the case?

While it does not seem likely that almost hyperbolicity can be shown for almost all $\theta$, numerical evidence suggests that for typical $\theta$, the map $F_\theta$ has a finite number of ergodic components of positive Lebesgue measure, in which typical points have dense orbits.

**Conjecture 6.4.** For Lebesgue almost all $\theta \in (0, \frac{\pi}{4})$, there is a finite number of $F$-invariant sets $A_1, \ldots, A_m$, each of positive Lebesgue measure, such that $F_{|A_i} : A_i \to A_i$ is ergodic for every $i = 1, \ldots, m$. Each $A_i$ is a topological annulus with a certain number of elliptic islands removed from it, and, together with the elliptic islands and the invariant disk inscribed in $C$, the $A_i$ form a partition of $S$. 

Figure 7. The first $10^8$ iterates of four initial points for $\theta = \frac{\pi}{20}$. Each seems to be a dense orbit in an invariant annulus. The four annuli (without a number of elliptic islands and the invariant periodic regular 20-gon inscribed in $C$) seem to be the ergodic components of $F$. 

Figure 8. The first $10^5$ iterates of two initial points for $\theta = \frac{\pi}{5}$. Each of the orbits seems to densely fill a thin invariant annulus. The rectangle seems to be partitioned into finitely many such invariant annuli (which get thinner and more numerous as $\theta \to 0$).

Figure 9. Zoom-in of the orbits from Fig.8. The thin invariant annuli contain periodic island chains, which under even stronger magnification could be seen to be surrounded by further, finer, islands of quasi-periodic motion.

Numerical experiments also seem to indicate that for many parameter values, the way in which chaotic and (quasi-)periodic behaviour coexist, that is, the structure of invariant annuli containing families of quasi-periodic elliptic islands, can be quite rich, see Fig.5 and Fig.6. Besides the total measure of such quasi-periodic elliptic islands, it would be also interesting to know whether a general scheme for their itineraries, periods and rotation numbers can be found.

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we have that quadratic form (3.2). Then, using \( P_4 = (1-s,t) \) and \( P_1 = (s,t) \), we calculate
\[
Q_B(P_4) = Q_B(P_1) = \left(1 - s^2 + t^2 - \frac{(1-s)t}{t}\right) - \left(s^2 + t^2 - \frac{st}{t}\right) = 0,
\]
and hence \( Q_B(P_4) = Q_B(P_1) = c \). So the line \( \{x \in \mathcal{B}_1 : Q_B(x) = c\} \) is a segment of a hyperbola connecting \( P_4 \) and \( P_1 \). Therefore, with \( V = \{x \in \mathcal{B}_1 : Q_B(x) \leq c\} \), we get
\[
F(V) = B(V) \subset (V \cup \mathcal{A}_1).
\]
Further, note that \( B \) maps rays through \( E_1 \) to other rays through \( E_1 \). This implies that all points on the straight line segment \( E_1P_4 = \mathcal{A}_1 \cap \mathcal{B}_1 \) remain in the piece \( \mathcal{B}_1 \) for equally many iterations of the map \( F \), before being mapped into \( \mathcal{A}_1 \). In particular, if for \( X \in E_1P_4 \) we have that
\[
F^k(X) \in \begin{cases} \mathcal{B}_1 & \text{if } 0 \leq k < K, \\ E_1P_1 & \text{if } k = K, \end{cases}
\]
then the same holds for every other point \( X' \in E_1P_4 \), and \( F^K(E_1P_4) = E_1P_1 \).

We will now show that there exists a sequence of parameter values \( \theta_k, K \geq 3, \) such that for \( F = F_{\theta_k}, F^k(P_4) = B^k(P_4) \in \mathcal{B}_1 \) for \( 0 \leq k \leq K \) and \( F^K(P_4) = P_1 \). For this, we need a few elementary facts about the map \( B \), which follow from straightforward (but rather tedious) calculations:

- Let \( f(t) = \sqrt{1-4t^2} \). Then the hyperbolic map \( B \) has two eigendirections
\[
\nu_1 = \left(1 + f(t)\right) 2t, \quad \nu_2 = \left(1 + f(t)\right)
\]
with corresponding eigenvalues
\[
\lambda_1 = \lambda = \frac{4t^2 - 2t + (2s-1)(1+f(t))}{2(t-s)} > 1, \quad \lambda_2 = \lambda^{-1} < 1.
\]
- By a linear change of coordinates \( \Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) mapping \( \nu_1 \) and \( \nu_2 \) to \((1,0)\) and \((0,1)\), respectively, one gets a conjugate linear map
\[
\hat{B} = \Phi \circ B \circ \Phi^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.
\]

Setting \( \Phi(P_4) = \Phi(t, 1-s) =: (x_1, y_1) \) and \( \Phi(P_1) = \Phi(s, t) =: (x_2, y_2) \), a somewhat tedious calculation gives
\[
\frac{x_2}{x_1} = \frac{2s + t(f(t) - 1)}{2t^2 + (1-s)(f(t) - 1)}.
\]

Now, since \( \hat{B}^K = \Phi \circ \hat{B}^K \circ \Phi^{-1} \), it follows that \( B^K(P_4) = P_1 \) is equivalent to \( \hat{B}^K \Phi(P_4) = \Phi(P_1) \). By the simple form of \( \hat{B}^K \), this is equivalent to \( x_2 = \lambda^k x_1 \), that is,
\[
K = \frac{\log(x_2/x_1)}{\log(\lambda)}.
\]

Substituting (6.2) and (6.3) into (6.4), and using \( s = \sqrt{1-t^2} \) from (3.1), we get an expression \( K = K(t) \) for \( 0 < t < \frac{1}{2} \). Then \( K \) is differentiable and strictly monotonically decreasing as a function of \( t \), and (by application of L'Hôpital's rule)
\[
K \rightarrow \begin{cases} 2 & \text{as } t \rightarrow 0, \\ \infty & \text{as } t \rightarrow \frac{1}{2}. \end{cases}
\]

Since \( \sin^2 \theta = t \) with \( t \in (0, \frac{1}{2}) \), \( \theta \in (0, \frac{\pi}{2}) \), we get that for each \( K \geq 3 \) there exists \( \theta_K \in (0, \frac{\pi}{2}) \) such that \( B^k(P_4) = P_1 \), hence \( B^k(E_1P_4) = E_1P_1 \), as claimed. \( \square \)
Proof of Lemma 5.3. First, the shear map \( A : \mathcal{A}_1 \to \mathcal{A}_1 \cup \mathcal{B}_2 \) is such that any point \((x, y) \in \mathcal{A}_1\) is mapped to \( F(x, y) = A(x, y) = (x + \tilde{c}y, y), \tilde{c} > 0\). By continuity, it follows that for any \( N \geq 0 \) there exists a point \( Y_N \in \mathcal{E}_1 \cap \mathcal{B}_1 \) such that \( F^n(Y_N) \in \mathcal{A}_1 \) for \( 0 \leq n < N \) and \( F^N(Y_N) \in \mathcal{E}_2 \cap \mathcal{B}_2 \). Clearly, \( Y_0 = P_1 = (s, t) \), and one can calculate that

\[
y_N = \frac{1}{1 + N(1 - 2s)}(s, t), \quad N \geq 0,
\]

and \( Y_N \to (0, 0) = E_1 \) as \( N \to \infty \). \( \square \)

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