Study of nearly invariant subspaces with finite defect in Hilbert spaces

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Abstract. In this article, we briefly describe nearly $T^{-1}$ invariant subspaces with finite defect for a shift operator $T$ having finite multiplicity acting on a separable Hilbert space $\mathcal{H}$ as a generalization of nearly $T^{-1}$ invariant subspaces introduced by Liang and Partington in Complex Anal. Oper. Theory 15(1) (2021) 17 pp. In other words, we characterize nearly $T^{-1}$ invariant subspaces with finite defect in terms of backward shift invariant subspaces in vector-valued Hardy spaces by using Theorem 3.5 in Int. Equations Oper. Theory 92 (2020) 1–15. Furthermore, we also provide a concrete representation of the nearly $T_B^{-1}$ invariant subspaces with finite defect in a scale of Dirichlet-type spaces $D_\alpha$ for $\alpha \in [-1, 1]$ corresponding to any finite Blaschke product $B$, as was done recently by Liang and Partington for defect zero case (see Section 3 of Complex Anal. Oper. Theory 15(1) (2021) 17 pp).

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1. Introduction

The structure of the invariant subspaces of an operator $T$ plays an important role to study the action of $T$ on the full space in a better way. To that aim, the study of (almost) invariant subspaces were initiated and a suitable investigation of these brings the concept such as near invariance. The study of nearly invariant subspaces for the backward shift in the scalar-valued Hardy space $H^2(\mathbb{D}, \mathbb{C})$ were introduced by Hayashi [13], Hitt [14] and then Sarason [23] in the context of kernels of Toeplitz operators. Going further, Chalendar et al. [2] gave a complete characterization of nearly invariant subspaces under the backward shift operator acting on the vector-valued Hardy space, providing a vectorial generalization of a result of Hitt. In 2004, Erard [8] investigated the nearly invariant subspaces related to multiplication operators in Hilbert spaces of analytic functions. The concept of nearly invariant subspaces of finite defect for the backward shift in the scalar-valued Hardy space...
was introduced by Chalendar et al. [4] and provided a complete characterization of these spaces in terms of backward shift invariant subspaces. A recent work by Chattopadhyay et al. [5] characterizes nearly invariant subspace of finite defect for the backward shift operator acting on the vector-valued Hardy space and provides a vectorial generalization of Chalendar–Gallardo–Gutiérrez–Partington algorithm. In this connection, we also mention that a similar type of connection was obtained independently by O’Loughlin [19]. Recently, Liang and Partington introduced the notion of nearly $T^{-1}$ invariant subspaces in general, the Hilbert space setting [16] and provided a representation of nearly $T^{-1}$ invariant subspaces for the shift operator $T$ with finite multiplicity acting on a separable infinite dimensional Hilbert space $\mathcal{H}$ in terms of backward shift invariant subspaces on the vector-valued Hardy spaces as an application of Corollary 4.5 given in [2]. Moreover, they also gave a description of the nearly $T_B^{-1}$ invariant subspaces for the operator $T_B$ of multiplication by $B$ in a scale of Dirichlet-type spaces [16], where $B$ is any finite Blaschke product.

Motivated by the work of Liang and Partington [16], we also introduce the notion of nearly $T^{-1}$ invariant subspaces with finite defect (see Definition 2.1) for an left invertible operator $T$ acting on a separable infinite dimensional Hilbert space as a generalization of nearly $T^{-1}$ invariant subspaces. The purpose of this article is to study nearly $T^{-1}$ invariant subspaces with finite defect for a shift operator $T$ with finite multiplicity acting on a separable Hilbert space. In other words, we provide a characterization of nearly $T^{-1}$ invariant subspaces with finite defect in terms of backward shift invariant subspaces in vector-valued Hardy spaces by using Theorem 3.5 in [5]. Moreover, we also give a concrete representation of the nearly $T_B^{-1}$ invariant subspaces with finite defect in a scale of Dirichlet-type spaces $D_\alpha$ for $\alpha \in [-1, 1]$ corresponding to any finite Blaschke product $B$ by using similar ideas mentioned in [8, 16] with an appropriate modification, providing a generalization of results of Liang and Partington in a sense that they already proved the result for defect zero setting (Section 3, [16]). There are also many other contributions related with this topic and the interested reader can also refer to [1, 7] and the references therein. In order to state the precise contribution of this paper, we need to recapitulate some useful notations and definitions.

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and $B(\mathcal{H})$ denote the set of all bounded linear operators acting on $\mathcal{H}$. The $\mathbb{C}^m$-valued Hardy space [20] over the unit disc $\mathbb{D}$ is denoted by $H^2(\mathbb{D}, \mathbb{C}^m)$ and defined by

$$H^2(\mathbb{D}, \mathbb{C}^m) := \left\{ F(z) = \sum_{n \geq 0} A_n z^n : \| F \|^2 = \sum_{n \geq 0} \| A_n \|^2_{\mathbb{C}^m} < \infty, \ A_n \in \mathbb{C}^m \right\}.$$ 

We can also view the above Hilbert space as the direct sum of $m$-copies of $H^2(\mathbb{D}, \mathbb{C})$ or sometimes it is useful to see the above space as a tensor product of two Hilbert spaces $H^2(\mathbb{D}, \mathbb{C})$ and $\mathbb{C}^m$, that is,

$$H^2(\mathbb{D}, \mathbb{C}^m) \equiv \bigoplus_{m} H^2(\mathbb{D}, \mathbb{C}) = H^2(\mathbb{D}, \mathbb{C}) \otimes \mathbb{C}^m.$$ 

On the other hand, the space $H^2(\mathbb{D}, \mathbb{C}^m)$ can also be defined as the collection of all $\mathbb{C}^m$-valued analytic functions $F$ on $\mathbb{D}$ such that

$$\| F \| = \left[ \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \| F(re^{i\theta}) \|^2_{\mathbb{C}^m} d\theta \right]^{\frac{1}{2}} < \infty.$$ 

Moreover, the nontangential boundary limit (or radial limit)

$$F(e^{i\theta}) := \lim_{r \to 1^-} F(re^{i\theta})$$
exists almost everywhere on the unit circle $\mathbb{T}$ (for more details, see [18], I.3.11). Therefore, $H^2(\mathbb{D}, \mathbb{C}^m)$ can be embedded isometrically as a closed subspace of $L^2(\mathbb{T}, \mathbb{C}^m)$ by identifying $H^2(\mathbb{D}, \mathbb{C}^m)$ through the nontangential boundary limits of the $H^2(\mathbb{D}, \mathbb{C}^m)$ functions. Let $S$ denote the forward shift operator (multiplication by the independent variable) acting on $H^2(\mathbb{D}, \mathbb{C}^m)$, that is, $SF(z) = zF(z), z \in \mathbb{D}$. The adjoint of $S$ is denoted by $S^*$ and defined in $H^2(\mathbb{D}, \mathbb{C}^m)$ as the operator

$$S^*(F)(z) = \frac{F(z) - F(0)}{z}, \quad F \in H^2(\mathbb{D}, \mathbb{C}^m)$$

which is known as backward shift operator. The Banach space of all $L(\mathbb{C}^r, \mathbb{C}^m)$ (set of all bounded linear operators from $\mathbb{C}^r$ to $\mathbb{C}^m$)-valued bounded analytic functions on $\mathbb{D}$ is denoted by $H^\infty(\mathbb{D}, L(\mathbb{C}^r, \mathbb{C}^m))$ and the associated norm is

$$\|F\|_\infty = \sup_{z \in \mathbb{D}} \|F(z)\|.$$

Moreover, the space $H^\infty(\mathbb{D}, L(\mathbb{C}^r, \mathbb{C}^m))$ can be embedded isometrically as a closed subspace of $L^\infty(\mathbb{T}, L(\mathbb{C}^r, \mathbb{C}^m))$. Note that each $\Theta \in H^\infty(\mathbb{D}, L(\mathbb{C}^r, \mathbb{C}^m))$ induces a bounded linear map $T_\Theta \in H^\infty(\mathbb{D}, L(\mathbb{C}^r, \mathbb{C}^m))$ defined by

$$T_\Theta F(z) = \Theta(z)F(z) \cdot (F \in H^2(\mathbb{D}, \mathbb{C}^r))$$

The elements of $H^\infty(\mathbb{D}, L(\mathbb{C}^r, \mathbb{C}^m))$ are called the multipliers and are determined by $\Theta \in H^\infty(\mathbb{D}, L(\mathbb{C}^r, \mathbb{C}^m))$ if and only if $ST_\Theta = T_\Theta S$,

where the shift $S$ on the left-hand side and the right-hand side act on $H^2(\mathbb{D}, \mathbb{C}^m)$ and $H^2(\mathbb{D}, \mathbb{C}^r)$ respectively. A multiplier $\Theta \in H^\infty(\mathbb{D}, L(\mathbb{C}^r, \mathbb{C}^m))$ is said to be inner if $T_\Theta$ is an isometry, or equivalently, $\Theta(e^{it}) \in L(\mathbb{C}^r, \mathbb{C}^m)$ is an isometry almost everywhere with respect to the Lebesgue measure on $\mathbb{T}$. Inner multipliers are among the most important tools for classifying invariant subspaces of reproducing kernel Hilbert spaces. For instance, see Theorem 1.1.

**Theorem 1.1** [24]. A non-zero closed subspace $\mathcal{M} \subseteq H^2(\mathbb{D}, \mathbb{C}^m)$ is shift invariant if and only if there exists an inner multiplier $\Theta \in H^\infty(\mathbb{D}, L(\mathbb{C}^r, \mathbb{C}^m))$ such that

$$\mathcal{M} = \Theta H^2(\mathbb{D}, \mathbb{C}^r),$$

for some $r \ (1 \leq r \leq m)$.

Consequently, the space $\mathcal{M}^\perp$ of $H^2(\mathbb{D}, \mathbb{C}^m)$ which is invariant under $S^*$ (backward shift) can be represented as

$$\mathcal{K}_\Theta := \mathcal{M}^\perp = H^2(\mathbb{D}, \mathbb{C}^m) \ominus \Theta H^2(\mathbb{D}, \mathbb{C}^r),$$

which is also known as model spaces ([9,10,17,18]). Let $P_m : L^2(\mathbb{T}, \mathbb{C}^m) \rightarrow H^2(\mathbb{D}, \mathbb{C}^m)$ be an orthogonal projection onto $H^2(\mathbb{D}, \mathbb{C}^m)$ defined by

$$\sum_{n=-\infty}^{\infty} A_n e^{int} \mapsto \sum_{n=0}^{\infty} A_n e^{int}.$$

Therefore, $P_m(F) = (Pf_1, Pf_2, \ldots, Pf_m)$, where $P$ is the Riesz projection on $H^2(\mathbb{D}, \mathbb{C})$ [9] and $F = (f_1, f_2, \ldots, f_m) \in L^2(\mathbb{T}, \mathbb{C}^m)$. Also note that for any $\Phi \in L^\infty(\mathbb{T}, L(\mathbb{C}^m, \mathbb{C}^m))$, the Toeplitz operator $T_\Phi : H^2(\mathbb{D}, \mathbb{C}^m) \rightarrow H^2(\mathbb{D}, \mathbb{C}^m)$ is defined by $T_\Phi(F) = P_m(\Phi F)$.
for any $F \in H^2(D, \mathbb{C}^m)$. Next, we introduce a special family of Hilbert spaces of analytic functions. Let $\alpha$ be any real number. Then the Dirichlet-type spaces are denoted by $D_\alpha \equiv D_\alpha(D)$ and defined by

$$D_\alpha \equiv D_\alpha(D) := \left\{ f : D \to \mathbb{C} : f(z) = \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} (n+1)^\alpha |a_n|^2 < \infty \right\}.$$ 

Then each $D_\alpha$ is a Hilbert space with respect to the norm

$$\| f \|_\alpha := \left( \sum_{n=0}^{\infty} (n+1)^\alpha |a_n|^2 \right)^{\frac{1}{2}}.$$

Note that the particular instances of $\alpha$ yield the well-known Hilbert spaces of analytic functions on $D$. More precisely, when $\alpha = 0$, we get the Hardy space $H^2(D, \mathbb{C})$ and we have the classical Bergman space $A_2$, and $\alpha = 1$ correspond to the Dirichlet space $D$. Since $\| f \|_\gamma < \| f \|_\beta$ for $\gamma < \beta$, the continuous inclusion $D_\beta \subset D_\gamma$ holds for any $\gamma < \beta$. For more information about Dirichlet-type spaces, we refer to [1] and references therein.

Recall that an analytic function $u$ is said to be a multiplier of $D_\alpha$ if for any $f \in D_\alpha$, $uf \in D_\alpha$ that is, the analytic Toeplitz operator $T_u : f \mapsto uf$ is defined everywhere on $D_\alpha$ (hence bounded by closed graph theorem). Furthermore, one can easily check that any finite Blaschke product $B$ is a multiplier for each $D_\alpha$ spaces. Note that a finite Blaschke product is given by

$$B(z) = e^{i\theta} \prod_{k=1}^{N} \frac{z - z_k}{1 - \overline{z_k}z}, \quad (z \in D)$$

where $\alpha_i \in \mathbb{D}$ and the degree of $B$ is just the number of zeros $\{z_1, \ldots, z_N\}$, counted with multiplicity. Moreover, finite Blaschke products play an important role in mathematics. We refer [12,25] for more on the subject of multipliers of $D_\alpha$ and the qualitative study of finite Blaschke product respectively. The famous Wold Decomposition Theorem [6] implies that for any Blaschke product $B$, each element $f \in H^2(D, \mathbb{C})$ has the following decomposition:

$$f(z) = \sum_{n=0}^{\infty} B^n(z)h_n(z),$$

where $h_n$ belongs to the model space $K_B = H^2(D, \mathbb{C}) \ominus BH^2(D, \mathbb{C})$. An analogous theorem for Dirichlet-type spaces $D_\alpha(D)$ is the following.

**Theorem 1.2** [11, Theorem 3.1], [3, Theorem 2.1]. Suppose $\alpha \in [-1, 1]$ and $B$ is a finite Blaschke product. Then $f \in D_\alpha(D)$ if and only if $f = \sum_{n=0}^{\infty} B^n h_n$ (convergence in $D_\alpha(D)$ norm) with $h_n \in K_B = H^2(D, \mathbb{C}) \ominus BH^2(D, \mathbb{C})$ and

$$\sum_{n=0}^{\infty} (n+1)^\alpha \| h_n \|_{H^2}^2 < \infty. \quad \text{(1.1)}$$

Moreover, since $B$ is a finite Blaschke product, then $K_B$ is finite dimensional and hence we can consider other (equivalent) norms here, such as $\| h \|_{D_\alpha}$. 
The nearly invariant subspaces related to the multiplication operator $M_u$ in the Hilbert space of analytic functions has been studied by Erard [8]. In fact, Erard gave the definition of “nearly invariant under division by $u$”, which is same as “nearly $M_u^{-1}$ invariant”, a special case of the notion of nearly $T^{-1}$ invariant subspaces for any left invertible operator $T \in B(\mathcal{H})$ recently introduced by Liang and Partington [16] and the definition is the following.

**Definition 1.3** [16, Definition 1.2]

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and $T \in B(\mathcal{H})$ be left invertible. Then a closed subspace $\mathcal{M} \subset \mathcal{H}$ is said to be nearly $T^{-1}$ invariant if for every $g \in \mathcal{H}$ such that $Tg \in \mathcal{M}$, it holds that $g \in \mathcal{M}$.

It is well known that the shift operator acting on a separable Hilbert space is a generalization of the unilateral shift $S$ and the operator $T_B$ on $H^2(\mathbb{D}, \mathbb{C}^n)$. Recall that an operator $T \in B(\mathcal{H})$ is said to be a shift operator if it is an isometry and $T^*$ converges strongly to zero, that is, $\|T^n h\| \to 0$ as $n \to \infty$ for all $h \in \mathcal{H}$ [22]. Equivalently, an isometry $T \in B(\mathcal{H})$ is a shift operator if and only if $T$ is pure, that is, $\cap_{n=0}^{\infty} T^n \mathcal{H} = \{0\}$. Therefore, it is easy to observe that shift operator is an isometry and left invertible. Moreover, the multiplicity of a shift operator $T \in B(\mathcal{H})$ is defined to be the dimension of $\text{Ker} \ T^* = \mathcal{H} \ominus T \mathcal{H}$. As we have discussed earlier, Liang and Partington [16] have characterized nearly $T^{-1}$ invariant subspaces for a shift operator $T \in B(\mathcal{H})$ with finite multiplicity and furthermore they also studied the nearly $T_B^{-1}$ invariant subspaces corresponding to a finite Blaschke product $B$ in a scale of Dirichlet-type spaces $D_\alpha$ for $\alpha \in [-1, 1]$. The main aim of this article is to first introduce the notion of nearly $T^{-1}$ invariant subspaces with finite defect for a shift operator $T \in B(\mathcal{H})$ with finite multiplicity and then characterize those subspaces in terms of backward shift invariant subspaces in vector-valued Hardy spaces. Furthermore, we also study the nearly $T_B^{-1}$ invariant subspaces in a scale of Dirichlet-type spaces $D_\alpha$ for $\alpha \in [-1, 1]$ corresponding to a finite Blaschke product $B$ and provide a concrete representation of it by generalizing some results of Erard [8] and using the concept of the equivalent norm introduced by Liang and Partington (see Section 3, [16]) in our context.

The rest of the paper is organized as follows: In Section 2, we introduce the notion of nearly $T^{-1}$ invariant subspaces with finite defect for a left invertible operator $T \in B(\mathcal{H})$ and give a characterization of nearly $T^{-1}$ invariant subspaces with finite defect for the shift operator $T \in B(\mathcal{H})$ with finite multiplicity. In Section 3, we deal with the study of nearly $T_B^{-1}$ invariant subspaces with finite defect corresponding to a finite Blaschke product $B$ in a scale of Dirichlet-type spaces $D_\alpha$ for $\alpha \in [-1, 1]$.

2. Characterization of nearly invariant subspaces with finite defect for the shift operator

In this section, we study nearly $T^{-1}$ invariant subspaces with finite defect for a shift operator $T \in B(\mathcal{H})$ having finite multiplicity. Now we introduce the notion of nearly $T^{-1}$ invariant subspaces with finite defect for any left invertible operator $T \in B(\mathcal{H})$ as a generalization of nearly $T^{-1}$ invariant subspaces.
DEFINITION 2.1

Let \( T \in \mathcal{B}(\mathcal{H}) \) be left invertible. Then a closed subspace \( \mathcal{M} \) of \( \mathcal{H} \) is said to be nearly \( T^{-1} \) invariant with finite defect \( p \) if there exists a \( p \) dimensional subspace \( \mathcal{F} \) (which may be taken to be orthogonal to \( \mathcal{M} \)) such that for any \( f \in \mathcal{H} \) with \( Tf \in \mathcal{M} \), it holds that \( f \in \mathcal{M} \oplus \mathcal{F} \).

The following lemma which is almost similar to Lemma 2.2 of [16] gives a connection of nearly invariant subspaces with same defect between similar operators.

Lemma 2.2. Let \( T_1 \in \mathcal{B}(\mathcal{H}_1) \) and \( T_2 \in \mathcal{B}(\mathcal{H}_2) \) be two left invertible operators such that they are similar by some invertible operator \( V : \mathcal{H}_1 \to \mathcal{H}_2 \), so that \( T_2 = VT_1 V^{-1} \). Let \( \mathcal{M} \) be a nearly \( T_1^{-1} \) invariant subspace with defect \( p \) in \( \mathcal{H}_1 \); then \( V(\mathcal{M}) \) is also a nearly \( T_2^{-1} \) invariant subspace with the same defect \( p \) in \( \mathcal{H}_2 \).

Proof. Suppose \( g \in \mathcal{H}_2 \) such that \( T_2 g \in V \mathcal{M} \). Then we want to show that \( g \in V \mathcal{M} \oplus VF \), where \( F \) is the \( p \) dimensional defect space for \( \mathcal{M} \) in \( \mathcal{H}_1 \). Since \( T_2 g = VT_1 V^{-1} g \in V \mathcal{M} \), it implies that \( T_1 V^{-1} g \in V \mathcal{M} \). Moreover, since \( \mathcal{M} \) is nearly \( T_1^{-1} \) invariant with defect space \( F \), then we must have \( V^{-1} g \in \mathcal{M} \oplus F \). Thus \( g \in V(\mathcal{M} \oplus F) = V \mathcal{M} \oplus VF \), proving that \( V(\mathcal{M}) \) is a nearly \( T_2^{-1} \) invariant subspace with defect \( p \) in \( \mathcal{H}_2 \). \( \square \)

Now onwards we always assume \( T \in \mathcal{B}(\mathcal{H}) \) is a shift operator with multiplicity \( m \), throughout this section. Let \( \{e_1, e_2, \ldots, e_m\} \) be an orthonormal basis of \( \mathcal{K} = \mathcal{H} \ominus T \mathcal{H} \) and let \( \delta_j^m = (0, 0, \ldots, 1, \ldots, 0) \) with 1 in the \( j \)-th place be an orthonormal basis of \( \mathcal{K}_z = H^2(\mathbb{D}, \mathbb{C}^m) \ominus zH^2(\mathbb{D}, \mathbb{C}^m) \) for \( j = 1, 2, \ldots, m \). By considering the following two orthogonal decompositions

\[
\mathcal{H} = \bigoplus_{i=0}^{\infty} T^i \mathcal{K} \quad \text{and} \quad H^2(\mathbb{D}, \mathbb{C}^m) = \bigoplus_{i=0}^{\infty} z^i \mathcal{K}_z,
\]

we have an unitary mapping \( U : \mathcal{H} \to H^2(\mathbb{D}, \mathbb{C}^m) \) defined by

\[
U(T^i e_j) = z^i \delta_j^m. \quad (2.1)
\]

Therefore the following diagram (2.2) corresponding to the shift operator \( T : \mathcal{H} \to \mathcal{H} \) with multiplicity \( m \) and the unilateral shift \( S : H^2(\mathbb{D}, \mathbb{C}^m) \to H^2(\mathbb{D}, \mathbb{C}^m) \) is commutative:

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{T} & \mathcal{H} \\
U \downarrow & & \downarrow U \\
H^2(\mathbb{D}, \mathbb{C}^m) & \xrightarrow{S} & H^2(\mathbb{D}, \mathbb{C}^m)
\end{array}
\quad (2.2)
\]

Therefore, from the above commutative diagram (2.2), we get

\[
S^n U = U T^n, \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (2.3)
\]
Now onwards, we denote $P_M$ as the orthogonal projection of $\mathcal{H}$ onto a closed subspace $M$ of $\mathcal{H}$. The following lemma gives an upper bound concerning the dimension of the subspace $M \ominus (M \cap T\mathcal{H})$.

**Lemma 2.3.** Let $T \in \mathcal{B}(\mathcal{H})$ be a shift operator with multiplicity $m$ and let $M$ be a non-trivial closed subspace of $\mathcal{H}$ such that $M \nsubseteq T\mathcal{H}$ (that means $M$ is not properly contained in $T\mathcal{H}$). Then

\[ 1 \leq r := \dim(M \ominus (M \cap T\mathcal{H})) \leq m. \quad (2.4) \]

**Proof.** Since $T$ is a shift operator with multiplicity $m$, then $\dim(\mathcal{H} \ominus T\mathcal{H}) = m$. Moreover, since $M \nsubseteq T\mathcal{H}$, then $M \ominus (M \cap T\mathcal{H}) \neq \{0\}$. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $\mathcal{H} \ominus T\mathcal{H}$. Our claim is that $\{P_M e_1, \ldots, P_M e_m\}$ generates $M \ominus (M \cap T\mathcal{H})$. Indeed, for any $g \in M \ominus (M \cap T\mathcal{H})$ with $\langle g, P_M e_i \rangle = 0$ for all $i \in \{1, \ldots, m\}$ implies $g = 0$ and hence $1 \leq r := \dim(M \ominus (M \cap T\mathcal{H})) \leq m$. $\square$

Next by using condition (2.3) and the above Lemma 2.3, we have the following result.

**Lemma 2.4.** Let $M$ be a non-trivial nearly $T^{-1}$ invariant subspace with finite defect $p$ and let $G_0 = [g_1, g_2, \ldots, g_r]^t$ be a $r \times 1$ matrix with $\{g_1, g_2, \ldots, g_r\}$ an orthonormal basis of $M \ominus (M \cap T\mathcal{H})$ (note that the superscript $t$ denotes the transpose of a matrix). Then $F_0 = [Ug_1, Ug_2, \ldots, Ug_r]^t$ is an $r \times m$ matrix with $\{Ug_1, Ug_2, \ldots, Ug_r\}$ an orthonormal basis for $U_M \ominus (U_M \cap zH^2(\mathbb{D}, \mathbb{C}^m))$.

**Proof.** The proof is straightforward and we leave it to the reader. $\square$

Going further, we need the following useful lemma, similar to Liang and Partington [16].

**Lemma 2.5.** Suppose $T : \mathcal{H} \to \mathcal{H}$ is a shift operator, and let $U$ be as in (2.3). Let $g_1, g_2, \ldots, g_n \in \mathcal{H}$ and $h_1, h_2, \ldots, h_n \in H^2(\mathbb{D}, \mathbb{C})$ be such that

\[ (Ug_1)h_1 + (Ug_2)h_2 + \cdots + (Ug_n)h_n \in H^2(\mathbb{D}, \mathbb{C}^m), \]

and suppose there exist sequences of polynomials $\{\{p_{li}\}_{1 \leq i \leq n, l \in \mathbb{N}}\}$ with $p_{li} \to h_i$ in $H^2(\mathbb{D}, \mathbb{C})$ as $l \to \infty$ for $1 \leq i \leq n$ such that $(Ug_1)p^1_{li} + (Ug_2)p^2_{li} + \cdots + (Ug_n)p^n_{li} \to (Ug_1)h_1 + (Ug_2)h_2 + \cdots + (Ug_n)h_n$ in $H^2(\mathbb{D}, \mathbb{C}^m)$ as $l \to \infty$. Then

\[ U^*[(Ug_1)h_1 + (Ug_2)h_2 + \cdots + (Ug_n)h_n] = h(T)g, \quad (2.5) \]

where

\[ h(T)g = \lim_{l \to \infty} [p^1_{li}(T)g_1 + p^2_{li}(T)g_2 + \cdots + p^n_{li}(T)g_n]. \]

and $h(T) = [h_1(T), h_2(T), \ldots, h_n(T)], g = [g_1, g_2, \ldots, g_n]^t$.

Now we are in a position to state and prove our main result in this section which provides an isometric relation between nearly $T^{-1}$ invariant subspaces with defect $p$ and the backward shift invariant subspaces of $H^2(\mathbb{D}, \mathbb{C}^{r+p}) = H^2(\mathbb{D}, \mathbb{C}^r) \times H^2(\mathbb{D}, \mathbb{C}^p)$. 


Theorem 2.6. Suppose $T$ is a shift operator with multiplicity $m$ and $\mathcal{M} \subset \mathcal{H}$ is a non trivial nearly $T^{-1}$ invariant subspace with defect $p$ and let $\mathcal{F}$ be the corresponding $p$ dimensional defect space. Let $F_1 = [f_1, f_2, \ldots, f_p]^t$ be a $p \times 1$ matrix containing an orthonormal basis $\{f_1, f_2, \ldots, f_p\}$ of $\mathcal{F}$. Then

(i) In the case when $\mathcal{M} \not\subset T\mathcal{H}$, there exist a non negative integer $r' \leq r + p$ and an inner multiplier $\Phi \in H^\infty (\mathbb{D}, \mathcal{L}(\mathbb{C}^{r'}, \mathbb{C}^{r'+p}))$, unique up to an unitary equivalence such that

$$\mathcal{M} = \{f \in \mathcal{H} : f = K_0(T)G_0 + T K_1(T)F_1 : (K_0, K_1) \in H^2(\mathbb{D}, \mathbb{C}^{r'+p}) \oplus \Phi H^2(\mathbb{D}, \mathbb{C}^{r'})\},$$

(2.6)

where $G_0 = [g_1, g_2, \ldots, g_r]^t$ is an $r \times 1$ matrix with $\{g_1, g_2, \ldots, g_r\}$ an orthonormal basis of $\mathcal{M} \oplus (\mathcal{M} \cap T\mathcal{H})$ and also there exists an isometry

$$Q : \mathcal{M} \rightarrow H^2(\mathbb{D}, \mathbb{C}^{r'+p}) \text{ defined by } Q(f) = (K_0, K_1).$$

(ii) In the case when $\mathcal{M} \subset T\mathcal{H}$, there exists a non negative integer $p' \leq p$ and an inner multiplier $\Theta \in H^\infty (\mathbb{D}, \mathcal{L}(\mathbb{C}^{p'}, \mathbb{C}^{p'}))$ which is unique up to unitary constant such that

$$\mathcal{M} = \{f \in \mathcal{H} : f = TK_1(T)F_1 : K_1 \in H^2(\mathbb{D}, \mathbb{C}^p) \oplus H^2(\mathbb{D}, \mathbb{C}^{p'}), \}$$

(2.7)

and also there exists an isometry

$$R : \mathcal{M} \rightarrow H^2(\mathbb{D}, \mathbb{C}^p) \text{ defined by } R(f) = K_1.$$

Proof. From Lemma 2.2 and using (2.1), we say that $U\mathcal{M}$ is a nearly $S^*$ invariant subspace of $H^2(\mathbb{D}, \mathbb{C}^m)$ with defect $p$ and the corresponding defect space is $U\mathcal{F} \subset H^2(\mathbb{D}, \mathbb{C}^m)$. Therefore, by applying our recent Theorem 3.5 in [5, Theorem 3.5, Case (i)] (see also Theorem 3.4, [19]) corresponding to nearly $S^*$ invariant subspace with finite defect in vector-valued Hardy space $H^2(\mathbb{D}, \mathbb{C}^m)$, we have

$$U\mathcal{M} = \left\{ F \in H^2(\mathbb{D}, \mathbb{C}^m) : F(z) = F_0(z)^t K_0(z) + \sum_{j=1}^p z k_j(z)Uf_j(z) : (K_0, k_1, \ldots, k_p) \in \mathcal{K} \right\},$$

where $\mathcal{K} \subset H^2(\mathbb{D}, \mathbb{C}^r) \times H^2(\mathbb{D}, \mathbb{C}) \times \cdots \times H^2(\mathbb{D}, \mathbb{C})$ is a closed $S^* \oplus \cdots \oplus S^*$-invariant subspace of the vector-valued Hardy space $H^2(\mathbb{D}, \mathbb{C}^{r+p})$,

$$\|F\|^2 = \|K_0\|^2 + \sum_{j=1}^p \|k_j\|^2,$$

(2.8)

and $F_0$ given in Lemma 2.4. Therefore, by Beurling–Lax–Halmos theorem on $H^2(\mathbb{D}, \mathbb{C}^{r+p})$, there exists a non negative integer $r' \leq r + p$ and an inner multiplier $\Phi \in H^\infty (\mathbb{D}, \mathcal{L}(\mathbb{C}^{r'}, \mathbb{C}^{r'+p}))$ unique up to unitary equivalence such that $\mathcal{K} = H^2(\mathbb{D}, \mathbb{C}^{r'+p}) \oplus$...
Thus, if we consider \( f \in \mathcal{M} \), then there exists \((K_0, k_1, k_2, \ldots, k_p) \in \mathcal{K}\) such that

\[
Uf = [Ug_1, Ug_2, \ldots, Ug_r]K_0 + \sum_{j=1}^{p} S k_j Uf_j
\]

and

\[
\|f\|^2 = \|Uf\|^2 = \|K_0\|^2 + \sum_{j=1}^{p} \|k_j\|^2. \tag{2.9}
\]

Let \( K_0 = (k_0^0, k_0^0, \ldots, k_0^0) \in H^2(\mathbb{D}, \mathbb{C}^r) \), then

\[
Uf = [Ug_1, Ug_2, \ldots, Ug_r]K_0 + \sum_{j=1}^{p} S k_j Uf_j
\]

\[
= \sum_{i=1}^{r}(Ug_i)k_i^0 + \sum_{j=1}^{p} S(Uf_j)k_j
\]

and therefore, by using Lemma 2.5, we get

\[
U^*(Uf) = U^* \left[ \sum_{i=1}^{r}(Ug_i)k_i^0 + \sum_{j=1}^{p} S(Uf_j)k_j \right]
\]

\[
= U^* \left[ \sum_{i=1}^{r}(Ug_i)k_i^0 + \sum_{j=1}^{p} U(Tf_j)k_j \right]
\]

\[
= K_0(T)G_0 + TK_1(T)F_1,
\]

and hence

\[
f = K_0(T)G_0 + TK_1(T)F_1,
\]

where \( K_1 = (k_1, k_2, \ldots, k_p) \in H^2(\mathbb{D}, \mathbb{C}^r) \). Therefore,

\[
\mathcal{M} = \{f \in \mathcal{H} : f = K_0(T)G_0 + TK_1(T)F_1 : (K_0, K_1) \in H^2(\mathbb{D}, \mathbb{C}^{r+p}) \ominus \Phi H^2(\mathbb{D}, \mathbb{C}^r) \}.
\]

Moreover, the relation (2.8) gives the existence of an isometry \( V : U\mathcal{M} \rightarrow H^2(\mathbb{D}, \mathbb{C}^{r+p}) \ominus \Phi H^2(\mathbb{D}, \mathbb{C}^r) \). Now if we define \( Q = Vu \), then \( Q : \mathcal{M} \rightarrow H^2(\mathbb{D}, \mathbb{C}^{r+p}) \ominus \Phi H^2(\mathbb{D}, \mathbb{C}^r) \) is an isometry and the isometric relation is given by (2.9). This completes the proof of (i).

For Case (ii), we assume \( \mathcal{M} \subset T\mathcal{H} \) and hence \( \mathcal{M} \ominus (\mathcal{M} \cap T\mathcal{H}) = \{0\} \). Therefore, again by applying [5, Theorem 3.5, Case (ii)], we have

\[
U\mathcal{M} = \left\{ F \in H^2(\mathbb{D}, \mathbb{C}^m) : F(z) = \sum_{j=1}^{p} z k_j(z) Uf_j(z) : (k_1, \ldots, k_p) \in \mathcal{K} \right\},
\]
where $\mathcal{K} \subset H^2(\mathbb{D}, \mathbb{C}) \times \cdots \times H^2(\mathbb{D}, \mathbb{C})$ is a closed $S^* \oplus \cdots \oplus S^*$-invariant subspace of the vector-valued Hardy space $H^2(\mathbb{D}, \mathbb{C}^p)$ and
\[
\|F\|^2 = \sum_{j=1}^{p} \|k_j\|^2. \tag{2.10}
\]

Similarly as in Case (i), there exists a non-negative integer $p' \leq p$ and an inner multiplier $\Theta \in H^\infty(\mathbb{D}, \mathcal{L}(C^{p'}, C^{p'}))$ unique up to an unitary equivalence such that $\mathcal{K} = H^2(\mathbb{D}, \mathbb{C}^p) \ominus \Phi H^2(\mathbb{D}, \mathbb{C}^{p'})$. Moreover, if $K_1 = (k_1, k_2, \ldots, k_p) \in H^2(\mathbb{D}, \mathbb{C}^p)$, then
\[
\mathcal{M} = \{ f \in \mathcal{H} : f = TK_1(T)F_1 : K_1 \in H^2(\mathbb{D}, \mathbb{C}^p) \ominus \Theta H^2(\mathbb{D}, \mathbb{C}^{p'}) \}.
\]

Furthermore, equation (2.10) gives an existence of an isometry $W : UM \to H^2(\mathbb{D}, \mathbb{C}^p) \ominus \Phi H^2(\mathbb{D}, \mathbb{C}^{p'})$ and therefore, if we define $R = WU$, then $R : \mathcal{M} \to H^2(\mathbb{D}, \mathbb{C}^p) \ominus \Theta H^2(\mathbb{D}, \mathbb{C}^{p'})$ is an isometry. This completes the proof of (ii).

Motivated by Corollary 2.6 in [16], we have the following corollary which characterize the nearly $T_B^{-1}$ invariant subspace with finite defect $p$ in $H^2(\mathbb{D}, \mathbb{C})$, as a consequence of the above Theorem 2.6. Note that for any finite Blaschke $B$ with degree $m$, the operator $T_B : H^2(\mathbb{D}, \mathbb{C}) \to H^2(\mathbb{D}, \mathbb{C})$ is a shift operator with multiplicity $m$.

**COROLLARY 2.7**

*Let $\mathcal{M} \subset H^2(\mathbb{D}, \mathbb{C})$ be a non trivial nearly $T_B^{-1}$ invariant subspace with defect $p$, where $B$ is a finite Blaschke of degree $m$ having at least one zero in $\mathbb{D} \setminus \{0\}$. Let $G_0 = [g_1, g_2, \ldots, g_r]'$ be an $r \times 1$ matrix with $\{g_1, g_2, \ldots, g_r\}$ an orthonormal basis of $\mathcal{M} \ominus (\mathcal{M} \cap T_B \mathcal{H})$ and let $F_1 = [f_1, f_2, \ldots, f_p]'$ be a $p \times 1$ matrix containing an orthonormal basis $\{f_1, f_2, \ldots, f_p\}$ of the defect space $\mathcal{F}$. Then there exists a non-negative integer $r' \leq r + p$ and an inner multiplier $\Phi \in H^\infty(\mathbb{D}, \mathcal{L}(C^{r'}, C^{r'+p}))$, unique up to unitary equivalence such that*

\[
\mathcal{M} = \{ f \in \mathcal{H} : f = K_0(T_B)G_0 + T_B K_1(T_B)F_1 : (K_0, K_1)
\in H^2(\mathbb{D}, C^{r'+p}) \ominus \Phi H^2(\mathbb{D}, C^{r'}) \}. \tag{2.11}
\]

The following example gives a better understanding of the above corollary which is same as Example 2.7 in [16] with a small variation.

**Example 2.8.** Let us define $B_a(z) = \frac{a - z}{1 - \bar{a}z}$ for any $a \in \mathbb{D} \setminus \{0\}$. Now consider the subspace
\[
\mathcal{M} = B_a(z) \cdot \left\{ \sqrt{1, z^2, z^6, z^8, z^{10}, \ldots} \oplus \sqrt{z, z^3, z^5, \ldots, z^{2m+1}} \right\},
\]
for some $m \in \mathbb{N} \cup \{0\}$. Then $\mathcal{M}$ is a nearly $T_B^*$ invariant subspace of $H^2(\mathbb{D}, \mathbb{C})$ with defect 1. It is easy to observe that $\dim(\mathcal{M} \ominus (\mathcal{M} \cap T_{z^2} H^2(\mathbb{D}, \mathbb{C}))) = 2$, $G_0 = B_a(z) \cdot [1, z]'$ and
the defect space is $\mathcal{F} = \langle z^d \phi_\alpha(z) \rangle$ with $F_1 = [z^d \phi_\alpha(z)]$. Therefore, for any $f \in \mathcal{M}$, we have

$$f(z) = \left[ \sum_{k=0}^{\infty} a_k z^{2k}, \sum_{k=0}^{\infty} a_k 2^{2k} \right] G_0(z) + T_{z^2} \left[ \sum_{k=0}^{\infty} b_k z^{2k} \right] F_1,$$

where the constants $a_k, a_k$ and $b_k$ satisfy the following:

$$\begin{align*}
    a_k &\in \mathbb{C} \text{ for } k \in \{0, 1\} \text{ and } a_k = 0 \text{ for } k \geq 2, \\
    a_k &\in \mathbb{C} \text{ for } k \in \{0, 1, \ldots, m\} \text{ and } a_k = 0 \text{ for } k \geq m + 1, \\
    b_k &\in \mathbb{C} \text{ for } k \geq 0.
\end{align*}$$

Moreover, Equation (2.11) along with the above discussions conclude

$$\mathcal{M} = \{ f \in \mathcal{H} : f = K_0(T_{z^2}) G_0 + T_{z^2} K_1(r_{z^2}) F_1 : (K_0, K_1) \\
\in H^2(\mathbb{D}, \mathbb{C}^{2+1}) \otimes \Phi H^2(\mathbb{D}, \mathbb{C}) \},$$

where $\Phi \in H^\infty(\mathbb{D}, L(\mathbb{C}, \mathbb{C}^3))$ is an inner multiplier such that $\Phi(z) = (z^2, z^{m+1}, 0) \in \mathbb{C}^3$.

3. Description of nearly $T_B^{-1}$ invariant subspaces with defect for finite Blaschke $B$ in $D_\alpha$ spaces

In this section, we discuss about nearly $T_B^{-1}$ invariant subspaces with finite defect corresponding to any finite Blaschke product $B$ in a scale of $D_\alpha$ spaces for $\alpha \in [-1, 1]$ by combining the ideas of Erard [8] and Liang and Partington [16] with appropriate changes. Recall that any finite Blaschke product $B$ is a multiplier of each $D_\alpha$, that is, the multiplication operator $T_B : D_\alpha \rightarrow D_\alpha$ is defined everywhere and bounded. Moreover, the operator $T_B$ is bounded below but not an isometry. We refer to the reader concerning the work of Lance and Steffin [15] in connection with the study of multiplication invariant subspaces of Hardy spaces. In [8], Erard studied the nearly invariant subspaces corresponding to lower bounded multiplication operator $M_u$ on the Hilbert space of analytic functions $\mathcal{H}$ and there are four conditions concerning the pairs $(\mathcal{H}, u)$ which are as follows:

(i) $\mathcal{H}$ is a Hilbert space and a linear subspace of $\mathcal{O}(\mathcal{W}) := \{ f : \mathcal{W} \rightarrow \mathbb{C} | f \text{ is analytic} \}$, where $\mathcal{W}$ is an open subset of $\mathbb{C}^d (d \in \mathbb{N})$,

(ii) $u \in \mathcal{O}(\mathcal{W})$ satisfies $uh \in \mathcal{H}$ for all $h \in \mathcal{H}$,

(iii) for all $w \in \mathcal{W}$, the evaluation $\mathcal{H} \rightarrow \mathbb{C}, h \rightarrow h(w)$ is continuous,

(iv) there exists $c > 0$ such that for all $h \in \mathcal{H}$, $c\|h\|_{\mathcal{H}} \leq \|uh\|_{\mathcal{H}}$.

Corresponding to the above pair $(\mathcal{H}, u)$, the lower bound of the multiplication operator $M_u$ relative to the norm $\|.,\|_{\mathcal{H}}$ is defined by

$$\gamma_{\mathcal{H}, M_u} = \sup \{ c > 0 : \forall h \in \mathcal{H}, c\|h\|_{\mathcal{H}} \leq \|uh\|_{\mathcal{H}} \} \in (0, \infty).$$

(3.1)

For simplicity, we denote $\gamma_{\mathcal{H}, M_u}$ by $\gamma$. In particular, for the pair $(\mathcal{H}, u(z) = z)$, Erard gives a connection between nearly backward shift invariant subspaces in $\mathcal{H}$ and a backward shift invariant subspaces in $H^2(\mathbb{D}, \mathbb{C})$ (see Theorem 5.1 in [8]). Note that the operator $T_B :
$D_\alpha \to D_\alpha$ is more general than $M_\alpha : H^2(\mathbb{D}, \mathbb{C}) \to H^2(\mathbb{D}, \mathbb{C})$ and the characterizations for nearly $T_B^{-1}$ invariant subspaces in $D_\alpha$ for $\alpha \in [-1, 1]$ corresponding to the finite Blaschke product $B$ is due to Liang and Partington (see Theorem 3.4 and Theorem 3.7 in [16]) by applying some results of Erard [8]. Here our main aim is to characterize nearly $T_B^{-1}$ invariant subspaces with finite defect in $D_\alpha$ for $\alpha \in [-1, 1]$ corresponding to the finite Blaschke product $B$. To achieve our goal, we need to first extend two important results (namely Approximation lemma and Factorization theorem) due to Erard [8]. Before we proceed, note that if $T : H \to H$ is a bounded operator that is bounded from below, then $T$ has closed range and $T^*T$ is invertible. Now using the ideas in [8] with suitable modification, we have the following lemma which provides a generalization of Lemma 2.1 in [8].

**Lemma 3.1 (Approximation lemma).** Let $H$ be a Hilbert space and let $T : H \to H$ be a bounded operator such that for all $h \in H$, $\|h\|_H \leq \|Th\|_H$. Suppose $M$ is a nearly $T^{-1}$ invariant subspace of $H$ with defect $p$ (i.e. the dimension of the defect space $F$ is $p$). We set $R = (T^*T)^{-1}T^*P_{M \cap TH}$, $Q = P_{M \oplus (M \cap TH)}$, $S = P_F$. Then $\|R\| \leq 1$, and for all $h \in M$ and $m \in \mathbb{N}$, we have

$$h = \sum_{k=0}^{m} T^k QR^k h + T^{m+1} R^{m+1} + T \sum_{k=1}^{m} T^{k-1} SR^k h$$

(3.2)

and

$$\|h\|^2_H \geq \sum_{k=0}^{\infty} \|QR^k h\|^2_H + \sum_{k=1}^{\infty} \|SR^k h\|^2_H.$$  

(3.3)

**Proof.** Consider $h \in H$ and write $P_{M \cap TH}(h) = Th_0$. Then we have

$$TRh = T(T^*T)^{-1}T^*Th_0 = Th_0 = P_{M \cap TH}(h).$$

(3.4)

Thus for any $h \in H$, we have $\|Rh\| \leq \|TRh\| = \|P_{M \cap TH}(h)\| \leq \|h\|$ and hence $\|R\| \leq 1$. Suppose $h \in M$ and hence by using (3.4), we conclude that $TRh \in M$. Since $M$ is a nearly $T^{-1}$ invariant subspace with defect $p$, then we have

$$Rh \in M \oplus F.$$  

(3.5)

Moreover, by using (3.4) and since $T$ is bounded below, we have for any $h \in M$,

$$h = Qh + TRh$$

(3.6)

and

$$\|h\|^2 \geq \|Qh\|^2 + \|TRh\|^2 \geq \|Qh\|^2 + \|Rh\|^2.$$  

(3.7)

Since $Rh \in M \oplus F$ (by (3.5)), then we have

$$Rh = P_M Rh + SRh$$
which implies that $Rh - SRh \in \mathcal{M}$. Note that since (3.6) is true for any $h \in \mathcal{M}$, therefore, if we replace $h$ by $Rh - SRh$ in (3.6), we get

$$Rh = QRh + TR^2h + SRh. \quad (3.8)$$

Now it is easy to observe that $R(\mathcal{M} \oplus \mathcal{F}) \subseteq \mathcal{M} \oplus \mathcal{F}$ and hence $R^mh \in \mathcal{M} \oplus \mathcal{F}$, $\forall m \in \mathbb{N}$. Therefore, by induction from (3.8), we get for any $m \in \mathbb{N}$,

$$R^mh = Q R^mh + TR^{m+1}h + SR^mh \quad (3.9)$$

and since $T$ is bounded below, we have

$$\|R^mh\|^2 \geq \|QR^mh\|^2 + \|R^{m+1}h\|^2 + \|SR^mh\|^2. \quad (3.10)$$

Finally by combining (3.6) and (3.9), we have

$$h = \sum_{k=0}^{m} T^k QR^kh + TR^{m+1}h + T \sum_{k=1}^{m} T^{k-1} SR^kh, \quad m \in \mathbb{N}$$

and moreover equations (3.7) and (3.10) yield that

$$\|h\|^2_{\mathcal{H}} \geq \sum_{k=0}^{\infty} \|QR^kh\|^2_{\mathcal{H}} + \sum_{k=1}^{\infty} \|SR^kh\|^2_{\mathcal{H}}.$$ 

This completes the proof.

□

**Remark 3.2.** Under the same assumption as in Lemma 3.1, let $\mathcal{M}$ be a nearly $T^{-1}$ invariant subspace of $\mathcal{H}$ with defect $p$ such that $\mathcal{M} \subseteq T\mathcal{H}$ and let $\mathcal{F}$ be the corresponding $p$ dimensional defect space having an orthonormal basis $\{e_j\}_{j=1}^p$. Then for any $h \in \mathcal{M}$ and $m \in \mathbb{N}$, we have

$$h = T^{m+1}R^mh + T \sum_{k=1}^{m} T^{k-1} SR^kh \quad \text{and} \quad \|h\|^2_{\mathcal{H}} \geq \sum_{k=1}^{\infty} \|SR^kh\|^2_{\mathcal{H}}. \quad (3.11)$$

Next we denote $D(0, a) := \{z \in \mathbb{C} : |z| < a\}$. As an application of the above Approximation lemma and mimicking the ideas given in [8] with appropriate changes, we have the following theorem which gives a generalization of Theorem 3.2 in [8].

**Theorem 3.3 (Factorization theorem).** Assume that the pair $(\mathcal{H}, u)$ satisfies the four conditions (i)–(iv) given above. Let $\mathcal{M}$ be a nearly $M_u^{-1}$ invariant subspace of $\mathcal{H}$ with defect $p$ and let $\mathcal{F}$ be the corresponding defect space. Let $\{g_i\}_{i \in I}$ be an orthonormal basis of $\mathcal{M} \ominus (\mathcal{M} \cap M_u \mathcal{H})$ and let $\{e_j\}_{j=1}^p$ be an orthonormal basis of $\mathcal{F}$. Moreover, we also assume that

$$\bigcap_{n \in \mathbb{N}} u^n|_{u^{-1}(D(0, \gamma))} \mathcal{H}|_{u^{-1}(D(0, \gamma))} = \{0\}, \quad (3.12)$$
where $\mathcal{H}|_{u^{-1}(D(0,\gamma))}$ consists of the restrictions to $u^{-1}(D(0,\gamma))$ of the functions of $\mathcal{H}$. Then

(i) In the case when $\mathcal{M} \nsubseteq M_u\mathcal{H}$, for all $h \in \mathcal{M}$, there exist $(q_i)_{i \in I}$ and $(h_j)_{j=1}^p$ in $\mathcal{O}(u^{-1}(D(0,\gamma)))$ such that

$$h = \sum_{i \in I} g_i q_i + \gamma^{-1} M_u \sum_{j=1}^p e_j h_j$$

on $u^{-1}(D(0,\gamma))$ for all $i \in I$ and $j \in \{1, \ldots, p\}$, and also there exist $(c_{ki})_{k \in \mathbb{N}_0} \in \mathbb{C}^I$ and $(b_{kj})_{k \in \mathbb{N}} \in \mathbb{C}^p$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with

$$q_i = \sum_{k=0}^{\infty} c_{ki} \left( \frac{u}{\gamma} \right)^k, \quad h_j = \sum_{k=1}^{\infty} b_{kj} \left( \frac{u}{\gamma} \right)^{k-1}$$

(3.13)

and

$$\sum_{i \in I} \sum_{k=0}^{\infty} |c_{ki}|^2 + \sum_{j=1}^{p} \sum_{k=1}^{\infty} |b_{kj}|^2 \leq \|h\|_{\mathcal{H}}^2. \quad (3.14)$$

(ii) In the case when $\mathcal{M} \subseteq M_u\mathcal{H}$, for all $h \in \mathcal{M}$, there exists $(h_j)_{j=1}^p$ in $\mathcal{O}(u^{-1}(D(0,\gamma)))$ such that

$$h = \gamma^{-1} M_u \sum_{j=1}^p e_j h_j \quad \text{on } u^{-1}(D(0,\gamma))$$

for all $j \in \{1, 2, \ldots, p\}$ and also there exists $(b_{kj})_{k \in \mathbb{N}} \in \mathbb{C}^p$ such that

$$h_j = \sum_{k=1}^{\infty} b_{kj} \left( \frac{u}{\gamma} \right)^{k-1} \quad \text{and} \quad \sum_{j=1}^{p} \sum_{k=1}^{\infty} |b_{kj}|^2 \leq \|h\|^2.$$ 

Proof.

(i) First we consider $T = \gamma^{-1} M_u$. Then $T$ satisfies the hypothesis of Lemma 3.1. Now we define $R$, $Q$, $S$ as in Lemma 3.1 and let $h \in \mathcal{M}$. Then we define a family of sequences $\{(c_{ki})_{k \in \mathbb{N}_0}\}_{i \in I}$, $\{(b_{kj})_{k \in \mathbb{N}}\}_{j=1}^p$ of complex numbers by the following equations:

$$QR^k h = \sum_{i \in I} c_{ki} g_i, \quad k \in \mathbb{N}_0 \quad \text{and} \quad SR^k h = \sum_{j=1}^{p} b_{kj} e_j, \quad k \in \mathbb{N}.$$

Therefore by using (3.2) and (3.3), we get

$$h = \sum_{k=0}^{m} T^k QR^k h + T^{m+1} R^{m+1} h + \sum_{k=1}^{m} T^k SR^k h$$

$$= \sum_{k=0}^{m} \sum_{i \in I} c_{ki} T^k g_i + T^{m+1} R^{m+1} h + T \sum_{k=1}^{m} \sum_{j=1}^{p} b_{kj} T^{k-1} e_j, \quad \text{for all } h \in \mathcal{M},$$
and hence

$$h = \sum_{k=0}^{m} \sum_{i \in I} c_{ki} \left( \frac{u}{\gamma} \right)^{k} g_{i} + \left( \frac{u}{\gamma} \right)^{m+1} R^{m+1} h + \gamma^{-1} u \sum_{k=1}^{m} \sum_{j=1}^{p} b_{kj} \left( \frac{u}{\gamma} \right)^{k-1} e_{j}$$  \hspace{1cm} (3.15)$$

and

$$\sum_{i \in I} \sum_{k=0}^{\infty} |c_{ki}|^2 + \sum_{j=1}^{p} \sum_{k=1}^{\infty} |b_{kj}|^2 \leq \|h\|^2,$$  \hspace{1cm} (3.16)

so that for all $i \in I$ and $j \in \{1, 2, \ldots, p\}$,

$$\sum_{k=0}^{\infty} |c_{ki}|^2 < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} |b_{kj}|^2 < \infty.$$

Therefore it follows that for all $i \in I$, the series $\sum_{k=0}^{\infty} c_{ki} \left( \frac{u}{\gamma} \right)^{k}$ converges uniformly on compact subsets of $u^{-1}(D(0, \gamma))$, so that its sum, which we denote by $q_{i}$, belongs to $O(u^{-1}(D(0, \gamma)))$. Similarly the series $\sum_{k=1}^{\infty} b_{kj} \left( \frac{u}{\gamma} \right)^{k-1}$ also converges uniformly on compact subsets of $u^{-1}(D(0, \gamma))$ and hence the sum of the series denoted by $h_{j}$ also belongs to $O(u^{-1}(D(0, \gamma)))$. Let $w \in u^{-1}(D(0, \gamma))$, then by using Cauchy–Schwarz inequality and (3.16), we obtain

$$\sum_{i \in I} |(g_{i} q_{i})(w)| \leq \left( \sum_{i \in I} |g_{i}(w)|^2 \right)^{\frac{1}{2}} \left( \sum_{i \in I} |q_{i}(w)|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i \in I} \langle g_{i}, k_{w} \rangle^2 \right)^{\frac{1}{2}} \left( \sum_{i \in I} \sum_{k=0}^{\infty} |c_{ki}|^2 \right) \left( \sum_{k=0}^{\infty} |u(w)|^{2k} / \gamma^{2k} \right)^{\frac{1}{2}} \leq \|Q_{k_{w}}\|_{\mathcal{H}} \|h\|_{\mathcal{H}} \frac{1}{\sqrt{1 - |u(w)|^2 / \gamma^2}}$$

and

$$\sum_{j=1}^{p} |(e_{j} h_{j})(w)| \leq \|S_{k_{w}}\|_{\mathcal{H}} \|h\|_{\mathcal{H}} \frac{1}{\sqrt{1 - |u(w)|^2 / \gamma^2}},$$

and hence both the series $\sum_{i \in I} g_{i} q_{i}$ and $\sum_{j=1}^{p} e_{j} h_{j}$ converges at each point of $u^{-1}(D(0, \gamma))$. Now from equation (3.15), we obtain

$$(h - \sum_{i \in I} g_{i} q_{i} - \sum_{j=1}^{p} e_{j} h_{j})|_{u^{-1}(D(0, \gamma))} \in \bigcap_{m \in \mathbb{N}} u^{m}|_{u^{-1}(D(0, \gamma))} \mathcal{H}|_{u^{-1}(D(0, \gamma))},$$
which along with the hypothesis (3.12) implies that

\[
h = \sum_{i \in I} g_i q_i + \gamma^{-1} M_u \sum_{j=1}^{p} e_j h_j \text{ on } u^{-1}(D(0, \gamma)) \quad \text{and} \quad \sum_{i \in I} \sum_{k=0}^{\infty} |c_{ki}|^2
\]

\[+ \sum_{j=1}^{p} \sum_{k=1}^{\infty} |b_{kj}|^2 \leq \|h\|^2.
\]

(ii) Again we consider \( T = \gamma^{-1} M_u \). Therefore, by using Remark 3.2 and proceeding as in Case (i), we obtain

\[
h = \gamma^{-1} M_u \sum_{j=1}^{p} e_j h_j \text{ on } u^{-1}(D(0, \gamma)) \quad \text{and} \quad \sum_{j=1}^{p} \sum_{k=1}^{\infty} |b_{kj}|^2 \leq \|h\|^2.
\]

\[\square\]

Now we are in a position to describe the nearly \( T^{-1}_B \) invariant subspaces with defect \( p \) corresponding to a finite Blaschke \( B \) in Dirichlet type spaces \( D_{\alpha} \) for \( \alpha \in [-1, 1] \) by applying a similar type of mechanism used by Liang and Partington [16]. From now onwards, we assume that \( B \) is Blaschke product of degree \( m \) and therefore for any non trivial nearly \( T^{-1}_B \) invariant subspace \( \mathcal{M} \) in \( D_{\alpha} \) with defect \( p \) and \( \mathcal{M} \not\subset T_B D_{\alpha} \), we have

\[1 \leq r := \dim(\mathcal{M} \ominus (\mathcal{M} \cap T_B D_{\alpha})) \leq m
\]

which follows by similar argument as in Lemma 2.3. In the sequel, we now endow the space \( D_{\alpha} \) with two different equivalent norms introduced by Liang and Partington (see Section 3, [16]) according to the cases \( \alpha \in [-1, 0) \) and \( \alpha \in [0, 1] \) and hence we divide the analysis into two subsections. Note that we do rest of the analysis below by following a similar type of idea used by Liang and Partington [16] with an appropriate modification.

3.1 \( \alpha \in [-1, 0) \)

Note that we need to endow the space \( D_{\alpha} \) with a norm in such a way so that we can get a nice lower bound of the operator \( T_B \). Keeping this information in mind, we endow the space \( D_{\alpha} \) for \( \alpha \in [-1, 0) \) with the modified equivalent norm introduced by Liang and Partington [16] which we denote by \( \| \cdot \|_1 \) as follows: for any \( f = \sum_{n=0}^{\infty} f_n B^n \) with \( f_n \in K_B \),

\[
\| f \|_1^2 := \sum_{n=0}^{G-1} G^\alpha \| f_n \|_{H^2(\mathbb{D}, \mathbb{C})}^2 + \sum_{n=G}^{\infty} (n+1)^\alpha \| f_n \|_{H^2(\mathbb{D}, \mathbb{C})}^2,
\]

where \( G \) is a fixed and sufficiently large positive number to be specified below and moreover we have the same lower bound of \( T_B \) as obtained in [Section 3, [16]] which is as follows:

\[
\gamma_1 := \left( 1 - \frac{1}{G+1} \right)^{-\alpha/2}.
\]
Thus from the definition of lower bound, it follows that for any $f \in D_{\alpha}$,

$$\|T_B f\|_1^2 = \|B f\|_1^2 \geq \gamma_1^2 \|f\|_1^2$$

and hence the operator $T := \gamma_1^{-1} T_B : D_{\alpha} \to D_{\alpha}$ satisfies

$$\|T f\|_1^2 = \|\gamma_1^{-1} T_B f\|_1^2 \geq \|f\|_1^2$$

for any $f \in D_{\alpha}$.

Note that the pair $(D_{\alpha}, T_B)$ also satisfies conditions (i)–(iv) with lower bound $\gamma_1$ given in (3.18). Furthermore, as in [16], we choose $G$ large enough so that $\gamma_1$ satisfies $B^{-1}(D(0, \gamma_1)) \supset s \mathbb{D}$ with $s \mathbb{D}$ a disc containing all the zeros of $B$ which ensures that

$$\|\gamma_1^{-1} B\|_{H^\infty(s \mathbb{D})} < 1.$$ (3.19)

Moreover, the operator $T := \gamma_1^{-1} T_B$ satisfies all the assumptions in Lemma 3.1 together with the fact that

$$\bigcap_{m \in \mathbb{N}} B^m D_{\alpha}|_{s \mathbb{D}} = \bigcap_{m \in \mathbb{N}} T^m D_{\alpha}|_{s \mathbb{D}} = \{0\}.$$\\\\

Combining the above facts together with Theorem 3.3 implies the following lemma, providing a generalization of Lemma 3.6 in [16].

**Lemma 3.4.** Let $M$ be a non trivial nearly $T_B^{-1}$ invariant subspace of $D_{\alpha}$ with defect $p$ for $\alpha \in [-1, 0)$ and let $F$ be the corresponding $p$ dimensional defect space. Let $\{f_i\}_{i=1}^r$ and $\{e_j\}_{j=1}^p$ be an orthonormal basis of $M \ominus (M \cap T_B D_{\alpha})$ and $F$ respectively. Then for all $f \in M$, there exist $\{q_i\}_{i=1}^r$ and $\{h_j\}_{j=1}^p$ in $O(s \mathbb{D})$ such that

$$f = \sum_{i=1}^r f_i q_i + \gamma_1^{-1} T_B \sum_{j=1}^p e_j h_j \quad \text{on } s \mathbb{D},$$ (3.20)

for all $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, p\}$, and also there exist $(a_{ki})_{k \in \mathbb{N}_0} \in \mathbb{C}^r$ and $(b_{kj})_{k \in \mathbb{N}} \in \mathbb{C}^p$ with

$$q_i = \sum_{k=0}^\infty a_{ki} (\gamma_1^{-1} B)^k \quad \text{on } s \mathbb{D}, \quad h_j = \sum_{k=1}^\infty b_{kj} (\gamma_1^{-1} B)^{k-1} \quad \text{on } s \mathbb{D}$$ (3.21)

and

$$\sum_{i=1}^r \sum_{k=0}^\infty |a_{ki}|^2 + \sum_{j=1}^p \sum_{k=1}^\infty |b_{kj}|^2 \leq \|f\|_D^2.$$ (3.22)
Remark 3.5. If the subspace $\mathcal{M} \subseteq T_B \mathcal{D}_\alpha$, then using the same notation as in Lemma 3.4, for all $f \in \mathcal{M}$, there exists $\{h_j\}_{j=1}^p$ in $\mathcal{O}(sD)$ such that

$$f = \gamma_1^{-1} T_B \sum_{j=1}^p e_j h_j \text{ on } sD,$$

and also there exists $(b_{kj})_{k \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ with

$$h_j = \sum_{k=1}^\infty b_{kj} \left( \gamma_1^{-1} B \right)^{k-1} \text{ and } \sum_{j=1}^p \sum_{k=1}^\infty |b_{kj}|^2 \leq \|f\|^2_{\mathcal{D}_\alpha}.$$

Here our main aim is to describe the nearly $T_B^{-1}$ invariant subspaces of $\mathcal{D}_\alpha$ with finite defect for $\alpha \in [-1, 0)$ in terms of $T_B^{-1}$ invariant subspaces of $\mathcal{H}^2(sD, \mathbb{C}^{r+p})$. In order to get a connection with invariant subspaces of $\mathcal{H}^2(sD, \mathbb{C}^{r+p})$, we introduce the same unitary mapping $U_s : \mathcal{H}^2(sD, \mathbb{C}^{r+p}) \to \mathcal{H}^2(D, \mathbb{C}^{r+p})$ as mentioned in [Section 3, [16]] and the map is defined by

$$(U_s f)(z) = f(sz).$$

If we denote $T_s^* := U_s T_B^{-1} U_s^*$, then we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{H}^2(sD, \mathbb{C}^{r+p}) & \xrightarrow{T_B^{-1}} & \mathcal{H}^2(sD, \mathbb{C}^{r+p}) \\
U_s \downarrow & & \downarrow U_s \\
\mathcal{H}^2(D, \mathbb{C}^{r+p}) & \xrightarrow{T_s^*} & \mathcal{H}^2(D, \mathbb{C}^{r+p})
\end{array} \quad (3.23)$$

Since the disc $sD$ contains all the zeros of $B$, the symbol $B^{-1}$ lies in $L^\infty(s\mathbb{T})$ and therefore by using the fact $B^{-1}(sz) = B(s^{-1}z)$ on $\mathbb{T}$ and by repeating the identical calculations as done in [16], we conclude

$$(T_s^* f)(z) = T_{B(s^{-1}z)} f(z). \quad (3.24)$$

For more details about (3.24), see (3.18), Section 3 in [16]. Now we state our main theorem in this subsection concerning nearly $T_B^{-1}$ invariant subspaces with defect $p$ in $\mathcal{D}_\alpha$ spaces with $\alpha \in [-1, 0)$ based on the above notations which gives a generalization of Theorem 3.7 in [16].

**Theorem 3.6.** Let $\mathcal{M}$ be a nearly $T_B^{-1}$ invariant subspace of $\mathcal{D}_\alpha$ with finite defect $p$ for $\alpha \in [-1, 0)$ and let $\mathcal{F}$ be the corresponding $p$ dimensional defect space. Let $E_0 := \{e_1, e_2, \ldots, e_p\}$, where $\{e_j\}_{j=1}^p$ is an orthonormal basis of $\mathcal{F}$ using norm $\|\cdot\|_1$. Then

(i) In the case when $\mathcal{M} \not\subseteq T_B \mathcal{D}_\alpha$, if $F_0 := \{f_1, f_2, \ldots, f_r\}$ is a matrix containing an orthonormal basis $\{f_i\}_{i=1}^r$ of $\mathcal{M} \cap (\mathcal{M} \cap T_B \mathcal{D}_\alpha)$, there exists a linear subspace $\mathcal{N} \subset \mathcal{H}^2(sD, \mathbb{C}^{r+p})$ such that

$$\mathcal{M} = \{f \in \mathcal{D}_\alpha : f = F_0 q + \gamma_1^{-1} T_B E_0 h \text{ on } sD : (q, h) \in \mathcal{N}\} \text{ on } sD,$$
together with

\[
(1 - \|\gamma_1^{-1}B\|_{H^\infty(D)}^2)^{1/2} (\|q\|_{H^2(D,C')}^2 + \|h\|_{H^2(D,C')}^2)^{1/2} \leq \|f\|_{D_\alpha}.
\]

Moreover, \( N \) is invariant under \( T_B^{-1} \) and hence \( U_s(N) \) is invariant under \( T_s^* = U_sT_{B^{-1}}U_s^* \) in \( H^2(D, C^{\infty}) \).

(ii) In the case when \( M \subset T_B D_\alpha \), then there exists a linear subspace \( N \subset H^2(D, C^p) \) such that

\[
M = \{ f \in D_\alpha : f = \gamma_1^{-1}T_B E_0 h : h \in N \} \text{ on } sD,
\]

together with

\[
(1 - \|\gamma_1^{-1}B\|_{H^\infty(D)}^2)^{1/2} \|h\|_{H^2(D,C^p)} \leq \|f\|_{D_\alpha}.
\]

Moreover, \( N \) is invariant under \( T_B^{-1} \) and hence \( U_s(N) \) is invariant under \( T_s^* = U_sT_{B^{-1}}U_s^* \) in \( H^2(D, C^p) \) (Note that here \( U_s : H^2(D, C^p) \to H^2(D, C^p) \)).

**Proof.**

(i) For \( f \in M \subset D_\alpha \) with \( \alpha \in [-1,0) \), Equation (3.20) in Lemma 3.4 implies

\[
f = \sum_{i=1}^{r} f_i q_i + \gamma_1^{-1}T_B \sum_{j=1}^{p} e_j h_j = F_0 q + \gamma_1^{-1}T_B E_0 h, \text{ on } sD
\]

(3.25)

where \( q = [q_1, q_2, \ldots, q_r]^T \) and \( h = [h_1, h_2, \ldots, h_p]^T \). Using the facts (3.19) and (3.21), we obtain the following for all \( i \in \{1,2,\ldots,r\} \) and \( j \in \{1,2,\ldots,p\} \):

\[
\|q_i\|_{H^2(D)} = \|\sum_{k=0}^{\infty} a_{ki} (\gamma_1^{-1}B)^k\|_{H^2(D)} \leq \sum_{k=0}^{\infty} |a_{ki}| \|\gamma_1^{-1}B\|_{H^\infty(D)}^k \leq \left( \sum_{k=0}^{\infty} \|\gamma_1^{-1}B\|_{H^\infty(D)}^{2k} \right)^{1/2} \left( \sum_{k=0}^{\infty} |a_{ki}|^2 \right)^{1/2} = (1 - \|\gamma_1^{-1}B\|_{H^\infty(D)}^2)^{-1/2} \left( \sum_{k=0}^{\infty} |a_{ki}|^2 \right)^{1/2}
\]

and

\[
\|h_j\|_{H^2(D)} \leq (1 - \|\gamma_1^{-1}B\|_{H^\infty(D)}^2)^{-1/2} \left( \sum_{k=0}^{\infty} |b_{kj}|^2 \right)^{1/2}.
\]

Therefore, the above estimates along with the inequality in (3.22) yields

\[
\|q\|_{H^2(D,C')}^2 = \sum_{i=1}^{r} \|q_i\|_{H^2(D)}^2 \leq (1 - \|\gamma_1^{-1}B\|_{H^\infty(D)}^2)^{-1} \left( \sum_{i=1}^{r} \sum_{k=0}^{\infty} |a_{ki}|^2 \right)^{1/2} \leq (1 - \|\gamma_1^{-1}B\|_{H^\infty(D)}^2)^{-1} \|f\|_{D_\alpha}^2 < +\infty
\]
and
\[ \|h\|_{H^2(sD, C^p)}^2 \leq (1 - \|\gamma_1^{-1} B\|_{H^\infty(sD)})^{-1}\|f\|_{D^\alpha}^2 < +\infty. \]

Thus the above implies
\[ q = \sum_{k=0}^{\infty} A_k (\gamma_1^{-1} B)^k \in H^2(sD, C^p), \quad \text{where } A_k = [a_{k1}, a_{k2}, \ldots, a_{kr}]^t \]

and
\[ h = \sum_{k=1}^{\infty} B_k (\gamma_1^{-1} B)^{k-1} \in H^2(sD, C^p), \quad \text{where } B_k = [b_{k1}, b_{k2}, \ldots, b_{kp}]^t. \]

Moreover, Equation (3.22) implies for all \( f \in \mathcal{M} \),
\[ \|q\|_{H^2(sD, C^p)}^2 + \|h\|_{H^2(sD, C^p)}^2 \leq (1 - \|\gamma_1^{-1} B\|_{H^\infty(sD)})^{-1}\|f\|_{D^\alpha}^2. \quad (3.26) \]

Now we define a linear subspace as follows:
\[ \mathcal{N} := \{(q, h) \in H^2(sD, C^p) \times H^2(sD, C^p) : \exists f \in \mathcal{M}, f = F_0q + \gamma_1^{-1} T_B E_0h \text{ on } sD \}, \]

satisfying for any \( f \in \mathcal{M}, \exists (q, h) \in \mathcal{N} \) such that \( f = F_0q + \gamma_1^{-1} T_B E_0h \) on \( sD \). Next, we show that \( \mathcal{N} \) is invariant under \( T_B^{-1} \). By considering \( T = \gamma_1^{-1} T_B \) in Lemma 3.1, Equation (3.2) with \( m = 0 \) implies
\[ f = Qf + TRf = Qf + \gamma_1^{-1} T_B Rf. \]

Moreover, on \( sD \), the above equation together with (3.25) yields
\[
F_0q + \gamma_1^{-1} T_B E_0h \\
= Q(F_0q + \gamma_1^{-1} T_B E_0h) + \gamma_1^{-1} T_B R(F_0q + \gamma_1^{-1} T_B E_0h) \\
= F_0A_0 + \gamma_1^{-1} B R(F_0q + \gamma_1^{-1} T_B E_0h),
\]

which further satisfies
\[ F_0(q - A_0) + \gamma_1^{-1} T_B E_0h = \gamma_1^{-1} B R(F_0q + \gamma_1^{-1} T_B E_0h). \]

Next, by using the fact that \( T_B \) is injective, we conclude from the above that
\[ \gamma_1^{-1} R(F_0q + \gamma_1^{-1} T_B E_0h) \\
= F_0\left( \sum_{k=1}^{\infty} A_k \gamma_1^{-k} B^{k-1} \right) + \gamma_1^{-1} E_0h = F_0(T_B^{-1} q) + \gamma_1^{-1} E_0h. \quad (3.27) \]
Moreover, by using the fact that \( R(F_0q + \gamma_1^{-1}TB E_0h) \in \mathcal{M} \oplus \mathcal{F} \), we obtain

\[
R(F_0q + \gamma_1^{-1}TB E_0h) = P^\mathcal{M} R(F_0q + \gamma_1^{-1}TB E_0h) + E_0B_1.
\]

Thus by combining equations (3.27) and (3.28), we get

\[
\gamma_1^{-1} P^\mathcal{M} R(F_0q + \gamma_1^{-1}TB E_0h)
= F_0(T_{B^{-1}}q) + \gamma_1^{-1} E_0 \left( \sum_{k=2}^{\infty} B_k (\gamma_1^{-1} B)^{k-1} \right)
= F_0(T_{B^{-1}}q) + \gamma_1^{-1} TB E_0(T_{B^{-1}}h).
\]

Note that \( \gamma_1^{-1} P^\mathcal{M} R(F_0q + \gamma_1^{-1}TB E_0h) \in \mathcal{M} \) and hence from the definition of \( \mathcal{N} \), we conclude \((T_{B^{-1}}q, T_{B^{-1}}h) \in \mathcal{N}\). Thus \( \mathcal{N} \) is \( T_{B^{-1}} \) invariant in \( H^2(s\mathbb{D}, \mathbb{C}^p) \). Finally, by using the diagram (3.23), we have \( T^*_s(U_s(\mathcal{N})) \subset U_s(\mathcal{N}) \), that is, \( U_s(\mathcal{N}) \) is invariant under \( T^*_s \).

(ii) If \( \mathcal{M} \subset T_B \mathcal{D}_{\alpha}, \) then by using Remark 3.5 and proceeding as in Case (i), we obtain a linear subspace \( \mathcal{N} \subset H^2(s\mathbb{D}, \mathbb{C}^p) \) such that

\[
\mathcal{M} = \{ f \in \mathcal{D}_{\alpha} : f = \gamma_1^{-1} TB E_0h : h \in \mathcal{N} \} \text{ on } s\mathbb{D},
\]

together with

\[
(1 - \|\gamma_1^{-1} B\|^2_{H^\infty(s\mathbb{D})})^{1/2}\|h\|_{H^2(s\mathbb{D}, \mathbb{C}^p)} \leq \|f\|_{\mathcal{D}_{\alpha}}.
\]

Moreover, \( \mathcal{N} \) is invariant under \( T_{B^{-1}} \) and \( U_s(\mathcal{N}) \) is invariant under \( T_s = U_s T_{B^{-1}} U^*_s \) in \( H^2(\mathbb{D}, \mathbb{C}^p) \). (Note that here \( U_s : H^2(s\mathbb{D}, \mathbb{C}^p) \to H^2(\mathbb{D}, \mathbb{C}^p) \). This completes the proof. \( \square \)

### 3.2 \( \alpha \in [0, 1] \)

Here we consider \( \mathcal{D}_{\alpha} \) spaces with \( \alpha \in [0, 1] \) and \( B \) is a finite Blaschke product of degree \( m \). We now endow \( \mathcal{D}_{\alpha} \) with the following equivalent norm introduced by Liang and Partington [16] which we denote by \( \|\cdot\|_2 \) and is defined by

\[
\|f\|_2 := \sum_{n=0}^{\infty} (n + 1)^\alpha \|g_n\|^2_{H^2(\mathbb{D}, \mathbb{C})}
\]

for any \( f = \sum_{n=0}^{\infty} g_n B^n \) with \( g_n \in \mathcal{K}_B \) (see Theorem 1.2). Therefore, we have

\[
\|T_B f\|_2^2 = \|B f\|_2^2 = \sum_{n=0}^{\infty} (n + 2)^\alpha \|g_n\|^2_{H^2(\mathbb{D}, \mathbb{C})} \geq \|f\|_2^2
\]

which implies that the operator \( T_B : (\mathcal{D}_{\alpha}, \|\cdot\|_2) \to (\mathcal{D}_{\alpha}, \|\cdot\|_2) \) is lower bounded and the lower bound \( (3.1) \) of \( T_B \) relative to the norm \( \|\cdot\|_2 \) is \( \gamma_2 := 1 \). Moreover, the pair \((\mathcal{D}_{\alpha}, B)\)
also satisfies the conditions (i)–(iv). Furthermore, it is easy to check that $B^{-1}(D(0, 1)) = B^{-1}(D) = D$ and $\bigcap_{m \in \mathbb{N}} B^m \mathcal{D}_\alpha = \{0\}$ on $D$. These facts along with Theorem 3.3 (with $\mathcal{H} = \mathcal{D}_\alpha$, $u = B$, $\gamma = \gamma_2 = 1$ and $I = \{1, 2, \ldots, r\}$) gives the following lemma which is a generalization of Lemma 3.3 in [16].

**Lemma 3.7.** Let $\mathcal{M}$ be a non trivial nearly $T^{-1}$ invariant subspace of $\mathcal{D}_\alpha$ for $\alpha \in [0, 1]$ such that $\mathcal{M} \not\subseteq T_B \mathcal{D}_\alpha$ and let $\{f_i\}_{i=1}^r$ and $\{e_j\}_{j=1}^p$ be an orthonormal basis of $\mathcal{M} \ominus (\mathcal{M} \cap T_B \mathcal{D}_\alpha)$ and the defect space $\mathcal{F}$ respectively. Then for any $f \in \mathcal{M}$, there exist $\{q_i\}_{i=1}^r$ and $\{h_j\}_{j=1}^p$ in $O(D)$ such that

$$f = \sum_{i=1}^r f_i q_i + T_B \sum_{j=1}^p e_j h_j$$

for any $i \in \{1, 2, \ldots, r\}$; $j \in \{1, 2, \ldots, p\}$ and also there exist $(c_k)_{k \in \mathbb{N}_0} \in \mathbb{C}^N$ and $(d_{kj})_{k \in \mathbb{N}} \in \mathbb{C}^N$ with

$$q_i = \sum_{k=0}^\infty c_k B^k, \quad h_j = \sum_{k=1}^\infty d_{kj} B^{k-1}$$

(3.30)

and

$$\sum_{i=1}^r \sum_{k=0}^\infty |c_k|^2 + \sum_{j=1}^p \sum_{k=1}^\infty |d_{kj}|^2 \leq \|f\|_{\mathcal{D}_\alpha}^2.$$  
(3.31)

**Remark 3.8.** If $\mathcal{M} \subseteq T_B \mathcal{D}_\alpha$, then using the same notation as in Lemma 3.7 for any $f \in \mathcal{M}$, there exists $\{h_j\}_{j=1}^p$ in $O(D)$ such that

$$f = T_B \sum_{j=1}^p e_j h_j$$

and also there exists $(b_{kj})_{k \in \mathbb{N}} \in \mathbb{C}^N$ with

$$h_j = \sum_{k=1}^\infty b_{kj} B^{k-1} \quad \text{and} \quad \sum_{j=1}^p \sum_{k=1}^\infty |b_{kj}|^2 \leq \|f\|_{\mathcal{D}_\alpha}^2.$$  
(3.32)

Now we are in a position to describe the nearly $T^{-1}$ invariant subspace with defect $p$ in $\mathcal{D}_\alpha$ for $\alpha \in [0, 1]$, providing a generalization of Theorem 3.4 in [16]. Due to Lemma 2.2, without loss of generality, we assume $B(0) = 0$.

**Theorem 3.9.** Let $\mathcal{M}$ be a nearly $T^{-1}$ invariant subspace of $\mathcal{D}_\alpha$ with finite defect $p$ for $\alpha \in [0, 1]$ and let $\mathcal{F}$ be the $p$ dimensional defect space. Let $E_0 := \{e_1, e_2, \ldots, e_p\}$, where $\{e_j\}_{j=1}^p$ is an orthonormal basis of $\mathcal{F}$ using norm $\|\cdot\|_2$. Then we have as follows:
(i) In the case when $\mathcal{M} \not\subset T_{B}D_{\alpha}$, if $F_{0} := [f_{1}, f_{2}, \ldots, f_{r}]$ is a matrix containing an orthonormal basis $\{f_{i}\}_{i=1}^{r}$ of $\mathcal{M} \ominus (\mathcal{M} \cap T_{B}D_{\alpha})$, there exists a linear subspace $\mathcal{N} \subset H^{2}(\mathbb{D}, \mathbb{C}^{r})$ such that

$$\mathcal{M} = \{f \in D_{\alpha} : f = F_{0}q + T_{B}E_{0}h : (q, h) \in \mathcal{N}\}$$

together with

$$\|q\|^{2}_{H^{2}(\mathbb{D}, \mathbb{C}^{r})} + \|h\|^{2}_{H^{2}(\mathbb{D}, \mathbb{C}^{p})} \leq \|f\|^{2}_{D_{\alpha}}.$$

Moreover, $\mathcal{N}$ is $T_{\overline{B}}$ invariant.

(ii) In the case $\mathcal{M} \subset T_{B}D_{\alpha}$, there exists a linear subspace $\mathcal{N} \subset H^{2}(\mathbb{D}, \mathbb{C}^{p})$ such that

$$\mathcal{M} = \{f \in D_{\alpha} : f = T_{B}E_{0}h : h \in \mathcal{N}\}$$

together with

$$\|h\|^{2}_{H^{2}(\mathbb{D}, \mathbb{C}^{p})} \leq \|f\|^{2}_{D_{\alpha}},$$

and $\mathcal{N}$ is $T_{\overline{B}}$ invariant.

**Proof.**

(i) For $f \in \mathcal{M} \subset D_{\alpha}$ with $\alpha \in [0, 1]$. By applying Lemma 3.7, we get

$$f = \sum_{i=1}^{r} f_{i}q_{i} + T_{B} \sum_{j=1}^{p} e_{j}h_{j} = F_{0}q + T_{B}E_{0}h,$$

(3.33)

where $q = [q_{1}, q_{2}, \ldots, q_{r}]^{t}$ and $h = [h_{1}, h_{2}, \ldots, h_{p}]^{t}$. Next, by using the facts (3.30) and (3.31), we obtain the following norm equalities and norm estimates for any $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, p\}$:

$$\|q_{i}\|^{2}_{H^{2}(\mathbb{D}, \mathbb{C})} = \sum_{k=0}^{\infty} |c_{ki}|^{2}, \quad \|h_{j}\|^{2}_{H^{2}(\mathbb{D}, \mathbb{C})} = \sum_{k=1}^{\infty} |d_{kj}|^{2},$$

and hence

$$\|q\|^{2}_{H^{2}(\mathbb{D}, \mathbb{C}^{r})} + \|h\|^{2}_{H^{2}(\mathbb{D}, \mathbb{C}^{p})} = \sum_{i=1}^{r} \|q_{i}\|^{2}_{H^{2}(\mathbb{D}, \mathbb{C})} + \sum_{j=1}^{p} \|h_{j}\|^{2}_{H^{2}(\mathbb{D}, \mathbb{C})}$$

$$= \sum_{i=1}^{r} \sum_{k=0}^{\infty} |c_{ki}|^{2} + \sum_{j=1}^{p} \sum_{k=1}^{\infty} |d_{kj}|^{2} \leq \|f\|^{2}_{D_{\alpha}}.$$

Thus it follows that

$$q = \sum_{k=0}^{\infty} C_{k}B^{k} \in H^{2}(\mathbb{D}, \mathbb{C}^{r}), \quad \text{where} \ C_{k} = [c_{k1}, c_{k2}, \ldots, c_{kr}]^{t}$$

and

$$h = \sum_{k=1}^{\infty} D_{k}B^{k-1} \in H^{2}(\mathbb{D}, \mathbb{C}^{p}), \quad \text{where} \ D_{k} = [d_{k1}, d_{k2}, \ldots, d_{kp}]^{t}.$$
Now we define a linear subspace as follows:

\[ \mathcal{N} := \{(q, h) \in H^2(\mathbb{D}, \mathbb{C}^r) \times H^2(\mathbb{D}, \mathbb{C}^p) : \exists f \in \mathcal{M} \text{ such that } f = F_0q + T_B E_0h\} \]

satisfying for any \( f \in \mathcal{M}, \exists (q, h) \in \mathcal{N} \) such that

\[ f = F_0q + T_B E_0h \quad \text{with } \| f \|_{\mathcal{D}_\alpha}^2 \geq \| q \|^2_{H^2(\mathbb{D}, \mathbb{C}^r)} + \| h \|^2_{H^2(\mathbb{D}, \mathbb{C}^p)}. \]

Next we show that \( \mathcal{N} \) is invariant under \( T_B \). Consider \( T = T_B \) and \( \mathcal{H} = \mathcal{D}_\alpha \) for \( \alpha \in [0, 1] \) in Lemma 3.1 and therefore the corresponding operator \( R, Q \) and \( S \) in Lemma 3.1 becomes

\[ R = (T_B^*T_B)^{-1}T_B^*P_{\mathcal{M} \cap T_B \mathcal{D}_\alpha}, \quad Q = P_{\mathcal{M} \oplus (\mathcal{M} \cap T_B \mathcal{D}_\alpha)} \quad \text{and} \quad S = P_F \quad \text{and hence Equation (3.2)} \]

with \( m = 0 \) implies that for any \( f \in \mathcal{M} \),

\[ f = Qf + TRf = Qf + T_B Rf, \]

which together with (3.33) yields

\[ F_0q + T_B E_0h = Q(F_0q + T_B E_0h) + T_B R(F_0q + T_B E_0h) = F_0C_0 + BR(F_0q + T_B E_0h), \]

which further satisfies

\[ F_0(q - C_0) + T_B E_0h = BR(F_0q + T_B E_0h). \]

Since \( T_B \) is injective, from the above, we conclude that

\[ R(F_0q + T_B E_0h) = F_0 \left( \sum_{k=1}^{\infty} C_k B^{k-1} \right) + E_0h = F_0(T_B q) + E_0h. \]  \hspace{1cm} (3.34)

On the other hand, note that \( R(F_0q + T_B E_0h) \in \mathcal{M} \oplus \mathcal{F} \) and hence

\[ R(F_0q + T_B E_0h) = P_{\mathcal{M}} R(F_0q + T_B E_0h) + E_0D_1. \]  \hspace{1cm} (3.35)

Thus by combining (3.34) and (3.35), we get

\[ P_{\mathcal{M}} R(F_0q + T_B E_0h) = F_0(T_B q) + E_0 \left( \sum_{k=2}^{\infty} D_k B^{k-1} \right) = F_0(T_B q) + T_B E_0(T_B h). \]

Since \( P_{\mathcal{M}} R(F_0q + T_B E_0h) \in \mathcal{M} \), from the definition of \( \mathcal{N} \), it follows that \( (T_B q, T_B h) \in \mathcal{N} \). Thus \( \mathcal{N} \) is \( T_B \) invariant in \( H^2(\mathbb{D}, \mathbb{C}^{r+p}) \).

(ii) If \( \mathcal{M} \subset T_B \mathcal{D}_\alpha \), then by using Remark 3.8 and proceeding similarly as in Case (i), we obtain a linear subspace \( \mathcal{N} \subset H^2(\mathbb{D}, \mathbb{C}^p) \) such that

\[ \mathcal{M} = \{ f \in \mathcal{D}_\alpha : f = T_B E_0h : h \in \mathcal{N} \} \quad \text{together with } \| h \|_{H^2(\mathbb{D}, \mathbb{C}^p)} \leq \| f \|_{\mathcal{D}_\alpha}. \]
and \( \mathcal{N} \) is \( T_B \) invariant in \( H^2(\mathbb{D}, \mathbb{C}^p) \). This completes the proof.

Next, we consider a special case of (2.2) as discussed in [Section 3, [16]]:

\[
\begin{array}{ccc}
H^2(\mathbb{D}, \mathbb{C}^{r+p}) & \xrightarrow{T} & H^2(\mathbb{D}, \mathbb{C}^{r+p}) \\
\downarrow U & & \downarrow U \\
H^2(\mathbb{D}, \mathbb{C}^{m(r+p)}) & \xrightarrow{S} & H^2(\mathbb{D}, \mathbb{C}^{m(r+p)})
\end{array}
\] (3.36)

Then \( SU = UT_B \) holds for the unilateral shift \( S \) : \( H^2(\mathbb{D}, \mathbb{C}^{m(r+p)}) \rightarrow H^2(\mathbb{D}, \mathbb{C}^{m(r+p)}) \) and \( T_B : H^2(\mathbb{D}, \mathbb{C}^{r+p}) \rightarrow H^2(\mathbb{D}, \mathbb{C}^{r+p}) \) having multiplicity \( m(r+p) \). Using this fact, we end the section with the following remark concerning finite dimensional nearly \( T_B^{-1} \) invariant subspaces of \( D_\alpha \) for \( \alpha \in [0, 1] \) which is almost identical to Remark 3.5 in [16].

**Remark 3.10.** Note that the subspace \( \mathcal{N} \) is not closed in general. In the above Theorem 3.9, if we consider \( \mathcal{M} \) is finite dimensional, then \( \mathcal{N} \subset H^2(\mathbb{D}, \mathbb{C}^{r+p}) \) is also finite dimensional and hence closed. Then from Beurling–Lax–Halmos theorem and using diagram (3.36), we obtain that there exists a non negative integer \( l \) with \( l \leq m(r+p) \) and an inner multiplier \( \Phi \in H^\infty(\mathbb{D}, L(C^l, C^{m(r+p)})) \) such that

\[
\mathcal{N} = U^*(H^2(\mathbb{D}, C^{m(r+p)}) \ominus \Phi H^2(\mathbb{D}, C^l))
\]

and hence

\[
\mathcal{M} = \{ f \in D_\alpha : f = F_0q + T_BE_0h : (q, h) \in U^*(H^2(\mathbb{D}, C^{m(r+p)}) \ominus \Phi H^2(\mathbb{D}, C^l)) \}.
\]

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