

Scaling properties of the cluster distribution of a critical nonequilibrium model

Marta Chaves and Maria Augusta Santos

Departamento de Física and Centro de Física do Porto, Faculdade de Ciências, Universidade do Porto
Rua do Campo Alegre 687 – 4150 Porto – Portugal

(January 10, 2022)

A geometric approach to critical fluctuations of a nonequilibrium model is reported. The two-dimensional majority vote model was investigated by Monte Carlo simulations on square lattices of various sizes and a detailed scaling analysis of cluster statistical and geometric properties was performed. The cluster distribution exponents and fractal dimension were found to be the same as those of the (two-dimensional) Ising model. This result, which cannot be derived purely from the known bulk critical behaviour, widens our knowledge about the range of validity of the Ising universality class for nonequilibrium systems.

PACS numbers: 05.70.Ln, 05.50.+q, 64.60.Ht, 64.60.Ak, 02.70.Lq

KEY WORDS: nonequilibrium systems, scaling, cluster properties

I. INTRODUCTION

Nonequilibrium models, defined by a stochastic evolution rule, often develop long range correlations in space and time. As a consequence of the absence of the microscopic reversibility constraint, critical phenomena in those systems exhibit a wider range of universality classes than in the usual thermal phase transitions. A lot of research activity in nonequilibrium systems in recent years was concerned with the characterization of steady states critical behaviour and also with some time-dependent properties [1].

The relation between the singularities in macroscopic quantities and the corresponding internal cluster structure in critical equilibrium systems received considerable attention in the seventies and eighties [2-8]. A scaling theory for the cluster distribution was presented [3], involving the standard critical indices plus a new exponent describing the compactness of the clusters. Geometric clusters were however found to be too large to describe correlations between spins. The concept of critical clusters (also called physical clusters or droplets) was introduced [5] and is the basis of fast algorithms of thermal equilibration [9].

A number of nonequilibrium models have critical behaviour characterized by equilibrium Ising exponents [10-20]; it is however not known if their structure on the ‘microscopic’ level is also Ising-like - as Alonso et al [21] have found for a nonequilibrium random field model - or rather a distinct one. The influence of the dynamics on the system steady state geometric properties is yet to be clarified.

Cluster-flipping rules have also been used to propose new nonequilibrium models and universality classes [22]. Nonequilibrium models are defined by a particular, usually local, dynamic rule - modifying this rule often leads asymptotically to a distinct steady state. Since local dynamics suffer from critical slowing down, it is desirable to devise alternative (faster!) dynamics preserving the statistical properties of a particular steady state. This raises the interesting question of how to relate geometric to physical clusters for a particular nonequilibrium model. As a first step to answer these questions, we have analysed the cluster structure of a nonequilibrium model that belongs to the two-dimensional kinetic Ising universality class, the 2d majority vote model [23][1][20]. The results are compared with those obtained in the Ising case.

II. MODEL AND FORMALISM

'Spin' variables (voters) \( \{\sigma_i\} \) = ±1 are defined on the sites of a square lattice and evolve in time by single-spin flips with a probability \( W_i \) given by

\[
W_i = \frac{1}{2} \left[ 1 - \sigma_i (1 - 2q) S \left( \sum_\delta \sigma_{i+\delta} \right) \right]
\]

(1)

where \( S(x) = \text{sign}(x) \) if \( x \neq 0 \), \( S(0) = 0 \) and the sum is over nearest neighbours of \( \sigma_i \); \( q \) is the control parameter and we will restrict to \( 0 \leq q \leq 0.5 \) (alignment with neighbours is favoured). At \( q = q_c = 0.075 \) (for a square lattice),
the system undergoes a nonequilibrium phase transition from a ferromagnetic (low $q$) to a paramagnetic (higher $q$) state; the (static and dynamic) critical behaviour is Ising-like [10,11,20].

In what follows, a cluster is a connected set of spins such that each spin has at least one nearest neighbour in the same state. The number of spins in the cluster is the cluster size (or mass) $s$; spins with at least one neighbour with the opposite orientation belong to the cluster perimeter. By analogy with the Ising case, one expects that at criticality there are non-spanning clusters of all sizes, i.e. the cluster size distribution $n_s$ (average number of clusters of size $s$ per lattice site) has power-law form

$$n_s \sim s^{-\tau} \quad (2)$$

The average perimeter of $s$-size clusters (per spin) $P_s$ scales as

$$P_s \sim s^\sigma \quad (3)$$

where $\sigma$ measures the degree of compactness of the clusters and varies between 1 (ramified clusters) and $1 - \frac{1}{d}$ (compact clusters in $d$ dimensions). Corrections to scaling are expected for $q$ slightly away from $q_c$ and take the form

$$n_s(s, \epsilon) \sim s^{-\tau} f(s^q \epsilon) \quad (4)$$

$$P_s(s, \epsilon) \sim s^\sigma g(s^q \epsilon) \quad (5)$$

with $\epsilon = \frac{q - q_c}{q_c}$. Formulas above are valid for $1 \ll s \ll L^d$, where $L$ is the system linear size. Assuming expressions (2)-(5) hold for the present model, it is straightforward to see that, as in the equilibrium case [3,4], the distribution exponents $\tau$, $\sigma$ and $y$ are related to the usual critical indices by

$$\beta = \frac{\tau - 2}{y} \quad (6)$$

$$\alpha = 1 - \frac{\tau - \sigma - 1}{y} \quad (7)$$

so one additional exponent is sufficient to describe the scaling properties of the non-spanning clusters. For an Ising system the spanning cluster is compact in the ferromagnetic phase but becomes a fractal (fractal dimension $D$) at the critical point. In contrast to percolation, where $D$ is related to the euclidean dimension $d$ through critical exponents of the same model [24], for Ising or Potts models it was shown [18,25,26] that $D$ is independent from the corresponding bulk exponents. It can be obtained from the second moment of the (finite) cluster distribution (the percolative susceptibility) [8]

$$\chi_p = \sum_s s^2 n_s \sim \epsilon^{d-2D} \quad (8)$$

or, for a finite system at criticality,

$$\chi_p(q_c, L) \sim L^{-d+2D} \quad (9)$$

A direct measurement of the fractal dimension $D$ of the 'infinite' cluster is made by counting the mass within a distance $r$ from its center of mass

$$s(r) \sim r^D \quad (10)$$

or, alternatively, by using the relation between average size of the 'infinite' cluster $s_\infty$ and lattice side $L$

$$s_\infty \sim L^D \quad (11)$$

Assuming that the finite clusters have the same fractal dimension [3,21,22,27], the dependence of cluster size on mean radius of gyration $R_s$ also yields $D$

$$s \sim R_s^D \quad (12)$$
III. MONTE CARLO SIMULATIONS AND RESULTS

We have performed simulations of the model described in the previous section on square lattices of $32^2$, $40^2$, $64^2$, $128^2$ and $200^2$ sites with periodic boundary conditions for several values of $q \leq q_c$. After reaching a stationary state for each value of $q$, the system evolution was monitored for up to $1.2 \times 10^6$ MCS; configurations were analysed every 200 MCS. Clusters of either ‘aligned’ or ‘reversed’ spins (compared to the dominant sample orientation) were identified and counted separately, together with the respective perimeters. Double logarithmic plots of $n_s(s)$ and $P_s(s)$ - for reversed spins - are displayed in figures 1 and 2, respectively, for several $q$ values; the slopes of the corresponding linear regions are consistent with the Ising exponents $\tau = 2.05$ and $\sigma = 0.68$. From fig.2 we notice that the perimeter scaling law (3) is only valid for intermediate size clusters, larger clusters being more ramified than small ones, as found in the Ising case [21]. A better estimate of $\tau$, together with the correction to scaling exponent $\gamma$, are obtained from the best scaling fit to the data of the function [6]

$$F(tS^y) = t^{1/y} \sum_{s=1}^{S} s^\gamma n_s(s, \epsilon)$$

as shown in fig.3. This data collapse yields $\tau = 2.05 \pm 0.005$ and $\gamma = 0.445 \pm 0.01$.

The study of the fractal dimension of finite (inverted) clusters at $q_c$ is presented in fig.4 for the largest system size ($L = 200$). In order to reduce statistical scatter, the results were collected in bins of growing size (bin width increases by a factor of 1.5 for each successive interval). The best fit to the data yields $D = 1.938 \pm 0.003$; smaller systems produced slightly higher values, so we estimate $D = 1.945 \pm 0.01$. This is in agreement with the value $D = 1.95 \pm 0.02$ obtained from the finite size relation (9) for $L = 32, 40, 64, 128$ and coincides, within error, with the exact Ising value $D = 1.947$.

Fig.5 shows a direct evaluation of the fractal dimension of the percolating cluster at $q_c$. The best fit for the $L = 200$ system gives $D = 1.970 \pm 0.003$ to be compared with $D = 1.968$ obtained from relation (11) for $L = 64, 128$ and 200. This slightly higher value of $D$ obtained from the infinite clusters data is interpreted as a finite size effect: even for the largest lattice, the magnetization is finite at $q_c$ and the percolating cluster was always found to have the dominant orientation of the sample. Hence, we were probably not probing the incipient infinite cluster but rather a more compact object.

IV. CONCLUSION

We have performed a detailed numerical study of the statistical properties of geometric clusters close to the critical point of a two-dimensional majority vote model. Scaling was confirmed with distribution exponents $\tau = 2.05 \pm 0.005$, $\sigma = 0.68 \pm 0.01$ and $\gamma = 0.445 \pm 0.01$, indistinguishable from their equilibrium Ising values, as reported in reference [21] for another model in the same universality class. The fractal dimension associated with the cluster distribution at criticality $D = 1.945 \pm 0.01$ was also found to coincide with its Ising value $D = \frac{13}{10}$. A direct determination of $D$ from the geometric properties of the 'infinite' cluster produced a slightly higher value, which we believe is due to the finite system size. The percolative susceptibility diverges with an exponent $\gamma_p \equiv -2 + 2D = 1.90$ larger than the Ising value $\gamma = 1.75$, indicating that, as expected, purely geometric clusters do not coincide with critical ones. Coniglio and Klein’s rule [31] to reduce geometric to physical clusters - introducing ‘active bonds’ with a temperature-dependent probability - is not easily adaptable to this system. The difficulty to relate the probability for an ‘active bond’ with the noise parameter $q$ is due to the non-hamiltonian nature of the model. The study of the majority vote phase transition as a percolative transition of (suitably defined) physical clusters is a challenge for future work.

ACKNOWLEDGMENTS

We thank J M B Oliveira for his help in the implementation of the cluster counting algorithm and J F Mendes for useful discussions. This work was partially financed by Praxix XXI (Portugal) within project PRAXIS/2/2.1/Fis/299/94; M C has benefitted from a junior research grant (BIC) from Praxix XXI.
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V. FIGURE CAPTIONS

Fig.1 - Double logarithmic plot of the cluster size distribution. From bottom to top $q = 0.070, 0.071, 0.072, 0.074$ and $0.0747$ (successive curves have been shifted one unit up for clarity). The straight lines have slope $-2.05$. Lattice size $L = 200$; averages over 5000 or 6000 ($q = 0.0747$) samples.

Fig.2 - Log-log plot of the perimeter length $P_s$ as a function of size $s$. Same conditions as in fig.1. The dotted lines have slope 0.68. The inset shows a zoom of the linear region for the five $q$ values; the straight line is a least-square fit and has slope 0.683.

Fig.3 - Scaling function $F$ versus $\epsilon S^\nu$ for $\tau = 2.05$ and $\nu = 0.445$ (see text). The arrows indicate end of data for (left to right) $q = 0.0747, 0.074, 0.072, 0.071$ and 0.070. Lattice size $L = 200$.

Fig.4 - Cluster size v.s. mean radius of gyration in double logarithmic scale for $L = 200$ and $q = q_c$; data binned as described in the text. The best fit line has slope 1.938. Averages over 5000 samples.

Fig.5 - Mass within distance $r$ from center of mass, $s_r \equiv s(r)$, for percolating clusters at $q_c$. Averages over 5000 samples. System sizes $L = 64(\square), 128(\bigcirc)$ and 200($+$). The straight line has slope $D = 1.970$. 
