Topological Defects in Ferromagnetic, Antiferromagnetic and Cyclic Spinor Condensates – A Homotopy Theory

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Abstract

We apply the homotopy group theory in classifying the topological defects in atomic spin-1 and spin-2 Bose-Einstein condensates. The nature of the defects depends crucially on the spin-spin interaction between the atoms. We find the topologically stable defects both for spin-1 ferromagnetic and anti-ferromagnetic states, and for spin-2 ferromagnetic and cyclic states. With this rigorous approach we clarify the previously controversial identification of symmetry groups and order parameter spaces for the spin-1 anti-ferromagnetic state, and show that the spin-2 cyclic case provides a rare example of a physical system with non-Abelian line defects, like those observed in biaxial nematics. We also show the possibility to produce vortices with fractional winding numbers of $\frac{1}{2}$, $\frac{1}{3}$ and their multiples in spinor condensates.

1 Introduction

The all-optical trapping of Bose-Einstein condensates (BEC)\cite{1,2} has opened up a new direction in the study of dilute atomic gases, i.e., the spinor condensates with degenerate internal degrees of freedom of the hyperfine spin $F$. For alkali atoms with $F = 1$, both experiments and theories have shown two possible kinds of spin correlations in the atom species, namely ferromagnetic (e.g. $^{87}$Rb \cite{3,4,5}) or antiferromagnetic (e.g. $^{23}$Na \cite{3,4,6}). With the experimental success of condensing alkali bosons with $F > 1$ such as $^{85}$Rb \cite{7} and $^{133}$Cs \cite{8}, and the unusual stability of the $F = 2$ state (against spin-exchange) in $^{87}$Rb \cite{9}, one expects that defects with much richer structure can be created in the future. A remarkable feature here is that both the gauge symmetry $U(1)$ and the spin symmetry $SO(3)$ are involved, a situation similar to superfluid $^3$He where three different continuous
symmetries (orbital, spin and gauge) are broken either independently or in a connected fashion [10, 11].

Topological defects and excitations in the spinor BECs have been studied theoretically by several groups [3, 4, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Stoof and Khawaja [12] showed that ferromagnetic condensates have long-lived Skyrmion excitations, which are nonsingular but topologically nontrivial pointlike spin textures. Moreover, they also found that spin-1 Bose-Einstein antiferromagnets have singular pointlike topological spin textures [13], which are analogous to the ’t Hooft-Polyakov magnetic monopoles in particle physics. Coreless vortices were demonstrated to be thermodynamically stable in ferromagnetic $F = 1$ spinor condensates under rotation [14, 16] and were phase imprinted in a $F = 1$ sodium condensate experimentally [21]. Yip [17] has performed a systematic study on vortex structures and presented several axisymmetric and non-axisymmetric vortices for $F = 1$ antiferromagnetic BEC. Martikainen et al. [15] proposed and demonstrated numerically a method to create monopoles in three dimensional two-component condensates. Linear defects were studied by Leonhardt and Volovik [18], who pointed out the existence of Alice strings in the condensate of $^{23}$Na.

Most of the work on this subject is based on the original identification of the order parameter spaces by Ho [3]. After the original studies it was also claimed by Zhou that a discrete symmetry of $Z_2$ type was missed in the case of antiferromagnetic spin-1 condensate [19, 20] and therefore the topological defects would manifest totally different structures. In this article we present a rigorous topological study that both solves this spin-1 controversy, and reveals interesting aspects of spin-2 systems. The phases of spin-2 spinor condensates are characterized by a pair of parameters $|\langle F \rangle|$ and $|\Theta|$ describing the ferromagnetic order and the formation of singlet pairs, respectively [22, 23, 24, 25]. It turns out that for the so called cyclic phase the fundamental group that determines the nature of possible stable topological defects is non-Abelian. The only known physical example of such a system so far has been the biaxial nematic liquid crystal.

The organization of this paper is as follows: In the following section, we shall review the basic physics of the spinor condensate and discuss the possible ground states for hyperfine spin $F = 1$ and $F = 2$. In Section 3, we give a brief introduction of the homotopy theory of the defect classification, taking the nematic liquid crystal and superfluid $^3$He as examples. We present our calculation of the homotopy groups for spinor condensates in Sections 4-7. The non-Abelian fundamental group for the cyclic phase and its indications are discussed in detail and the order parameter spaces are easily identified in a correct way following our procedure of symmetry breaking. We summarize our results in Section 8.

2 Spinor Condensate

Neutral atomic gases can be confined in conventional magnetic traps with the availability of hyperfine states being restricted by the requirement that the trapped atoms remain in weak-field seeking states. Alkali atoms with a nuclear spin of $I = 3/2$, such as $^{87}$Rb and $^{23}$Na, have three weak-field seeking states at small field. A far-off-resonant optical trap, however,
confines atoms regardless of their hyperfine state. Thus, the atomic spin is liberated from the requirements of magnetic trapping and becomes a new degree of freedom. In particular, all atoms in the lower hyperfine manifold, for example the \( F = 1 \) hyperfine manifold of sodium, can be stably trapped simultaneously. Such multi-component optically trapped condensates are represented by an order parameter which is a vector in hyperfine spin space, and are thus called spinor Bose-Einstein condensates. The spin relaxation collisions in spinor condensates allow for population exchange among hyperfine states without trap loss. Theoretical studies started with the determination of the ground state structure in mean field theory for both spin-1 [3] [4] and spin-2 [22] [23] [24] [25] [26] [27] cases. Law et al. [28] [29] investigated the spin correlation beyond mean-field limit and the spin-mixing dynamics due to the nonlinear interaction in the spinor condensate. The dynamics is sensitive to the relative phase and particle number distribution among the individual components of the condensate. Ho and Yip [30] later found that the ground state of a spin-1 Bose gas with an antiferromagnetic interaction was a fragmented condensate in uniform magnetic fields. Zhou [19] [20] showed that the low energy spin dynamics in the system can be mapped into an \( o(n) \) nonlinear sigma model. The formation of ground state spin domains, metastable states and quantum tunneling were observed in a series experiments at MIT [6] [31] [32] [33].

The discussions in this paper, however, mainly concern the possible ground states in mean-field theory.

### 2.1 Spin-1 case

The ground states of the spinor condensate are determined through the minimization of the energy functional with the constraint of the conservation of the atom number and magnetization [33]. An \( F = 1 \) spinor Bose-Einstein condensate is described by a three-component order parameter \( \psi(r) = \left( \psi_{+1}, \psi_0, \psi_{-1} \right)^T \). In second quantized notation, the Hamiltonian describing a weakly-interacting Bose gas can be obtained from the Gross-Pitaevskii theory [3]

\[
H = \int d^3 r \left\{ \hat{\Psi}_i^\dagger(r) \left( -\frac{\hbar^2 \nabla^2}{2m} + U(r) \right) \hat{\Psi}_j(r) \delta_{ij} + \frac{1}{2} g_0 \hat{\Psi}_i^\dagger(r) \hat{\Psi}_j^\dagger(r) \hat{\Psi}_i(r) \hat{\Psi}_j(r) + \frac{1}{2} g_2 \hat{\Psi}_i^\dagger(r) \hat{\Psi}_j^\dagger(r) (F_a)_{ik} (F_a)_{jl} \hat{\Psi}_k(r) \hat{\Psi}_l(r) \right\} \tag{1}
\]

where \( \hat{\Psi}_i(r) \) is the field annihilation operator for an atom with mass \( m \) in hyperfine state \( |1, i\rangle \) at position \( r \) with \( i = +1, 0, -1 \) and \( U(r) \) is the trapping potential. Here the repeated indices are summed. The scattering lengths \( a_0 \) and \( a_2 \) characterize collisions between atoms through the total spin 0 and 2 channels, respectively. \( g_0 = \frac{4 \pi \hbar^2 a_0 + 2 a_2}{3} \) is interaction strength through the “density” channel, and \( g_2 = \frac{4 \pi \hbar^2 a_2 - a_0}{3} \) is that through the “spin” channel.
It is convenient to express the order parameter as $\psi(r) = \sqrt{n(r)} \zeta(r)$ where $n(r)$ is the atomic density and $\zeta(r)$ is a three-component spinor $\zeta(r) = (\zeta_1, \zeta_0, \zeta_{-1})^T = (x_+ e^{i\theta_+}, x_0 e^{i\theta_0}, x_- e^{i\theta_-})^T$ of normalization $|\zeta|^2 = 1$. Here $x$ and $\theta$ are the amplitudes and phases of the components. The spinor determines the average local spin by means of $\langle F \rangle = \zeta^\dagger(r) F \zeta(r)$, and $F$ are the usual spin-1 matrices with

$$F_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which obey the commutation relations $[F_a, F_b] = i\epsilon_{abc} F_c$. We thus obtain the energy functional

$$K = \int d^3 \mathbf{r} \left\{ \psi^\dagger \left( -\frac{\hbar^2 \nabla^2}{2m} \right) \psi + (U(\mathbf{r}) - \mu) n + \frac{n^2}{2} \left( g_0 + g_2 \langle F \rangle^2 \right) \right\}$$

$$= \int d^3 \mathbf{r} \left( K_0 + n^2 g_2 \langle F \rangle^2 / 2 \right)$$

(2)

where $K_0$ is the density-dependent part and the chemical potential $\mu$ determines the number of atoms in the condensate. It is obvious that all spinors related to each other by gauge transformation $e^{i\theta} e^{iF_x \alpha} e^{iF_y \beta} e^{iF_z \gamma}$ are energetically degenerate in zero external magnetic field, where $(\alpha, \beta, \gamma)$ are the Euler angles. The ground-state spinor is determined by minimizing the spin-dependent mean-field interaction energy, $n^2 g_2 \langle F \rangle^2 / 2$.

There are two distinct states depending on the sign of the interaction parameter $g_2$:

- $g_2 > 0$ (i.e. $a_2 > a_0$, e.g. $^{23}\text{Na}$): anti-ferromagnetic or polar state as the condensate lowers its energy by minimizing its average spin, i.e. by making $\langle F \rangle = 0$. The ground state spinor is then one of a degenerate set of spinors, the “polar” states, corresponding to all possible rotations of the hyperfine state $m_F = 0$, i.e.

$$\zeta(\mathbf{r}) = e^{i\theta} U \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e^{i\theta} \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} e^{-i\alpha} \sin \beta \\ \frac{1}{\sqrt{2}} \cos \beta \end{pmatrix}$$

(3)

- $g_2 < 0$ (i.e. $a_2 < a_0$, e.g. $^{87}\text{Rb}$): ferromagnetic as the condensate lowers its energy by maximizing its average spin, i.e. by making $\langle F \rangle = 1$. In this case the ground state spinors correspond to all rotations of the hyperfine state $m_F = 1$, i.e.

$$\zeta(\mathbf{r}) = e^{i\theta} U \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e^{i(\theta-\gamma)} \begin{pmatrix} e^{-i\alpha} \cos^2 \frac{\beta}{2} \\ \sqrt{2} \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \\ e^{i\alpha} \sin^2 \frac{\beta}{2} \end{pmatrix}$$

(4)
2.2 Magnetic Field

One can tailor the ground state structure with an external magnetic field and the effects of field inhomogeneities and quadratic Zeeman shifts modify the spin-dependent interaction energy into \[6\]

\[ K_{\text{spin}} = \left( c (\langle F \rangle^2 - p \langle F_z \rangle + q \langle F_z^2 \rangle) \right) n \]  

where \( c = g_2 n/2 \). The linear Zeeman shift \( p = g \mu_B B_z + p_0 \), where \( g \) is the Landé \( g \)-factor and \( \mu_B \) is the Bohr magneton, comes from the field gradient \( B \) along the long axis \( z \) of the condensate, while the last term gives the quadratic Zeeman shift from homogeneous field which is always positive for spin-1 condensate in a weak field. Assuming conservation of total spin, we have included a Lagrange multiplier \( p_0 \) into \( p \). For a system with zero total spin, \( p_0 \) cancels the linear Zeeman shift due to a homogeneous bias \( B_0 \), yielding \( p = 0 \). Positive (negative) values of \( p \) are achieved for condensates with a positive (negative) overall spin. The parameters \( p \) and \( q \) can be related to the individual level shifts by (energies in units of the hyperfine splitting \( E_{\text{HFS}} \))

\[ 2p = E_- - E_+ \]
\[ 2q = E_- + E_+ - 2E_0 \]  

where the Zeeman energies \( E_+, E_0 \) and \( E_- \) of the \( m_F = +1, 0, -1 \) can be expressed according to the Breit-Rabi formula \[34\] as

\[ E_+ = -\frac{1}{8} - \frac{1}{2} \sqrt{1 + x + x^2} \]
\[ E_0 = -\frac{1}{8} - \frac{1}{2} \sqrt{1 + x^2} \]
\[ E_- = -\frac{1}{8} - \frac{1}{2} \sqrt{1 - x + x^2} \]  

with \( x = g \mu_B B / E_{\text{HFS}} \).

Including the non-diagonal terms of the mean field interaction, we may minimize the energy functional

\[ K_{\text{spin}} / n = c \left( x_+^2 - x_-^2 \right)^2 + 2c x_0^2 \left( x_+^2 + x_-^2 + 2x_+ x_- \cos \phi \right) \]
\[ -p \left( x_+^2 - x_-^2 \right) + q \left( x_+^2 + x_-^2 \right) \]  

by means of the Lagrange multiplier method subjected to the constraint of normalization

\[ g = x_+^2 + x_0^2 + x_-^2 - 1 = 0 \]  

where \( \phi = \theta_+ + \theta_- - 2\theta_0 \). The solutions to the first derivatives of the Lagrange multiplier function \( \mathcal{X} = K_{\text{spin}} / n - \lambda g \) can be classified into the following table of spinors with their corresponding energies
We notice that spinors 4-7 are only well-defined in some specific regions in the \( p \)-\( q \) plane, i.e., the quantities under the square root must be non-negative. For example, spinor 7 may only exist for \( q^2 + 4cq - p^2 < 0 \) and \( q^2 > p^2 \), and in addition we must have \( \phi = 0 \) or \( \pi \). The ground state spinors obtained by minimizing the energy functional can be indicated in the so-called spin-domain phase diagrams (Figure 1 in ref. [6]). For \( c = 0 \), the Zeeman energy causes the cloud to separate into three pure domains with \( m_F = +1, 0, -1 \) and with boundaries at \( |p| = q \). For \( c > 0 \), a spin domain with mixed \( m_F = \pm 1 \) components, i.e., spinor 4, appears in the anti-ferromagnetic phase diagram. For \( c < 0 \), all three components are generally miscible and have no sharp boundaries, which corresponds to spinor 7.

### 2.3 Conservation of Magnetization

Although conservation of the magnetization was included in the above section, it was not separately discussed. Consequently the results do not easily apply to systems with fixed values of the magnetization \( \mathcal{M} \). The ground state structures as given in [6] correspond to the actual ground state as realized through an \( \mathcal{M} \) non-conserving evaporation process (e.g. in the presence of a non-zero \( B \)-field) that serves as a reservoir for condensate magnetization. On the other hand, the phase diagram for fixed values of \( \mathcal{M} \) was also explicitly discussed [35], which could physically correspond to experimental ground states (with/without a \( B \)-field) due to an \( \mathcal{M} \) conserving evaporation process. This requires the introduction of two Lagrange multipliers during the minimization subjected to conservation constraints for both the atomic number \( N \) and magnetization \( \mathcal{M} \), which in the mean-field approximation are given by

\[
N = \int d^3 \mathbf{r} n(\mathbf{r}) \left( x_+^2(\mathbf{r}) + x_0^2(\mathbf{r}) + x_-^2(\mathbf{r}) \right),
\]

\[
\mathcal{M} = \int d^3 \mathbf{r} n(\mathbf{r}) \left( x_+^2(\mathbf{r}) - x_-^2(\mathbf{r}) \right).
\]
We restrict the discussion here to the situation that equation (9) and
\[ h = x_+^2 - x_-^2 - m = 0 \]  
are satisfied where \( m = M/N \). With the definition \( x = x_+^2 + x_-^2 \), we can assort the possible spinors minimizing the Lagrange multiplier function \( X = K_{\text{spin}}/n - \lambda g - \delta h \) into the following classes (where the energy zero point has been moved to \( pm \)):

| spinors                                                                 | energies                        |
|-------------------------------------------------------------------------|---------------------------------|
| \( e^{i\theta_1} \sqrt{\frac{1+m}{2}}, 0, e^{i\theta_0} \sqrt{\frac{1-m}{2}} \) | \( cm^2 + q \)                  |
| \( e^{i\theta_1} \sqrt{\frac{1-m}{2}}, 0, e^{i\theta_0} \sqrt{\frac{1+m}{2}} \) | \( -cm^2 + (2c + q) m \)       |
| \( 0, e^{i\theta_0} \sqrt{1 + m}, e^{i\theta_1} \sqrt{1 - m} \)          | \( -cm^2 - (2c + q) m \)       |
| \( e^{i\theta_1} \sqrt{\frac{1+m}{2}}x_{\pm}, e^{i\theta_0} \sqrt{1 - x_{\pm}}x_{\pm}, e^{i\theta_1} \sqrt{\frac{1-m}{2}}x_{\pm} \) | \( cm^2 + g_{\pm}(x_{\pm}) + qx_{\pm} \) |

We still have the spinors 1-3 which are the same as in above section, however, they only exist for special values \( m = +1, -1, 0 \), respectively. While spinor 5(6) is confined to the positive (negative) values of \( m \), 4 and 7 may exist for the whole region \(-1 \leq m \leq 1\). In spinor 7 with three nonzero components, the phase convention remains \( \phi = 0 \) or \( \pi \) and the minimum is reached when \( x = x_m \) where \( x_m \) is determined by
\[ g_{\pm}'(x_m) + q = 0 \]  
with \( g_{\pm}(x) = 2c(1 - x) \left( x \pm \sqrt{x^2 - m^2} \right) \) for ferromagnetic(+) or anti-ferromagnetic(−) interaction, respectively. The ground state spinor phase diagram for a homogeneous condensate may be determined in the \( m-q \) plane, as indicated for positive \( m \) case in Figure 4 of ref. [35]. For \( c = 0 \), spinor 5 will always dominate except that on the boundary \( q = 0 \) we have spinor 7. For \( c < 0 \), spinor 7 will dominate. For \( c > 0 \), a curve \( q = 2c(1 - \sqrt{1 - m^2}) \) divides spinors 4 and 7.

### 2.4 Spin-2 case

For \(^{23}\text{Na}\) and \(^{87}\text{Rb}\) with regular hyperfine multiplets, the lower hyperfine state \( F = 1 \) has lower energy than the upper state \( F = 2 \). Experimentally only atoms in the lower hyperfine states can be confined in the optical trap. Those in the upper hyperfine states will leave the trap by spin-flip scattering. Since spin-flip scattering is strong in \(^{23}\text{Na}\), only the high-field seeking stretched state \(|2, -2\rangle\) exhibits reasonable stability, experiments with more complex spinor condensate do not seem to be possible [36]. On the other hand, optically trapped \(^{87}\text{Rb}\) has proved to be a candidate for spin-2 Bose gas [37] with rich spin dynamics and magnetization conservation was also observed during the mixing [5]. In the case of \(^{85}\text{Rb}\), the lowest multiplet has spin \( F = 2 \) and a negative s-wave scattering length in zero field. With the success to Bose condense \(^{85}\text{Rb}\) in magnetic traps [7], it is conceivable that an \( F = 2 \) spinor condensate might be trapped optically in lower hyperfine states, provided
that the three particle losses when the field is reduced through the Feshbach resonance are not too large.

Bose systems require that the total angular momentum of two colliding spin-2 particles is restricted to 0, 2, and 4. The effective low-energy Hamiltonian including the interaction energy describing binary collisions via the $s$-wave scattering can be generally expressed as 

$$H = \int d^3r \left\{ \hat{\Psi}^\dagger_i(r) \left( -\frac{\hbar^2 \nabla^2}{2m} + U(r) \right) \hat{\Psi}_j(r) \delta_{ij} 
+ \frac{1}{2} c_0 \hat{\Psi}^\dagger_i(r) \hat{\Psi}^\dagger_j(r) \hat{\Psi}_i(r) \hat{\Psi}_j(r) 
+ \frac{1}{2} c_1 \hat{\Psi}^\dagger_i(r) \hat{\Psi}^\dagger_j(r) (F_a)_{ik} (F_a)_{jl} \hat{\Psi}_k(r) \hat{\Psi}_l(r) 
+ \frac{1}{2} 5c_2 \hat{\Psi}^\dagger_i(r) \hat{\Psi}^\dagger_j(r) \langle 2i; 2j|00 \rangle \langle 00|2k; 2l \rangle \hat{\Psi}_k(r) \hat{\Psi}_l(r) \right\}$$

(13)

where $\langle 00|2k; 2l \rangle$ is the Clebsch-Gordan coefficient for combining two spin-2 particles with $m_F = k$ and $l$ into a spin singlet $|0, 0 \rangle$. The parameters

$$c_0 = \frac{4\pi \hbar^2 4a_2 + 3a_4}{m},$$
$$c_1 = \frac{4\pi \hbar^2 a_4 - a_2}{m},$$
$$5c_2 = \frac{4\pi \hbar^2 3a_4 - 10a_2 + 7a_0}{7}$$

(14)

describe the density-density interaction, spin-spin interaction, and formation of the singlet pair, respectively. The spinor $\zeta(r)$ with five components $\zeta(r) = (\zeta_{+2}, \zeta_{+1}, \zeta_{0}, \zeta_{-1}, \zeta_{-2})$ normalized to unity, determines the average local spin as $\langle \mathbf{F} \rangle = \zeta^\dagger(r) \mathbf{F} \zeta(r)$, and $\mathbf{F}$ are the $5 \times 5$ spin-2 matrices which obey the same commutation relations $[F_a, F_b] = i\epsilon_{abc} F_c$.

In the mean-field approach the properties of a spinor condensate are determined by the spin-dependent energy functional

$$K_{\text{spin}} = \left( c_1 \langle \mathbf{F} \rangle^2 + c_2 |\Theta|^2 - p \langle F_z \rangle + q \langle F_z^2 \rangle \right) n$$

(15)

where $\Theta = 2\zeta_{+2}\zeta_{-2} - \zeta_{+1}\zeta_{-1} + \zeta_0^2$ represents a singlet pair of identical spin-2 particles and is invariant under any rotation. The parameters $p$ and $q$ are related to the individual level shifts by

$$p = \frac{1}{12} (E_{+2} - E_{-2}) + \frac{2}{3} (E_{-1} - E_{+1})$$
$$q = -\frac{1}{24} (E_{+2} + E_{-2}) + \frac{2}{3} (E_{-1} + E_{+1}) - \frac{5}{4} E_0$$

(16)
The Breit-Rabi formula [34] in the case of $^{23}\text{Na}$ or $^{87}\text{Rb}$ ($F = 2$ is the upper hyperfine state with higher energy) gives

$$
E_{+2} = -\frac{1}{8} + \frac{1}{2} (1 + x) \\
E_{+1} = -\frac{1}{8} + \frac{1}{2} \sqrt{1 + x + x^2} \\
E_0 = -\frac{1}{8} + \frac{1}{2} \sqrt{1 + x^2} \\
E_{-1} = -\frac{1}{8} + \frac{1}{2} \sqrt{1 - x + x^2} \\
E_{-2} = -\frac{1}{8} + \frac{1}{2} (1 - x)
$$

In weak field, the quadratic Zeeman splitting is always negative, i.e., $q = -\frac{1}{16} x^2 + O(x^4)$. In the case of $^{85}\text{Rb}$ ($I = 5/2$) the lowest multiplet has spin $F = 2$. From the individual level shift

$$
E_{+2} = -\frac{1}{12} - \frac{1}{2} \sqrt{1 + \frac{4}{3} x + x^2} \\
E_{+1} = -\frac{1}{12} - \frac{1}{2} \sqrt{1 + \frac{2}{3} x + x^2} \\
E_0 = -\frac{1}{12} - \frac{1}{2} \sqrt{1 + x^2} \\
E_{-1} = -\frac{1}{12} - \frac{1}{2} \sqrt{1 - \frac{2}{3} x + x^2} \\
E_{-2} = -\frac{1}{12} - \frac{1}{2} \sqrt{1 - \frac{4}{3} x + x^2}
$$

we easily see $q$ is always positive for $^{85}\text{Rb}$ at small field, $q = \frac{1}{36} x^2 + O(x^4)$. Unlike in the case of spin-1, the whole $p$–$q$ plane is accessible experimentally for a spin-2 condensate.

The ground state magnetization must be aligned with the external field, i.e., along $z$-axis, implying $\langle F^2 \rangle = \langle F_z \rangle^2$ in eq. (15). Minimization of the spin dependent energy functional using the similar Lagrange multiplier method leads to three possible phases, one more compared to the spin-1 case. These phases are characterized by a pair of parameters $|\langle F \rangle|$ and $|\Theta|$ describing the ferromagnetic order and the formation of singlet pairs, respectively. For convenience, we only consider the linear Zeeman shift $p$ due to the magnetic field:

- **Polar/Anti-ferromagnetic phases**

  $$
P : \sqrt{\frac{1}{2}} \left( e^{i\theta_{+2}} \sqrt{1 + \frac{p}{4c_1 - c_2}}, 0, 0, 0, e^{i\theta_{-2}} \sqrt{1 - \frac{p}{4c_1 - c_2}} \right) \\
P_1 : \sqrt{\frac{1}{2}} \left( 0, e^{i\theta_{+1}} \sqrt{1 + \frac{p}{2(c_1 - c_2)}}, 0, e^{i\theta_{-1}} \sqrt{1 - \frac{p}{2(c_1 - c_2)}}, 0 \right) \\
P_0 : e^{i\theta_0} (0, 0, 1, 0, 0)
$$
with energies $c_2 - p^2 / (4c_1 - c_2)$, $c_2 - p^2 / 4 (c_1 - c_2)$ and $c_2$ respectively. Here $\theta_i$ are arbitrary phases for the corresponding components. These states are energetically degenerate in the absence of the external field with energy $c_2$ and parameters $\langle F \rangle = 0$ and $|\Theta| = 1$.

- **Ferromagnetic phases**

  $F : e^{i\theta} (1, 0, 0, 0, 0)$

  $F' : e^{i\theta+1} (0, 1, 0, 0, 0)$

  with energies $4c_1 - 2p$ and $c_1 - p$ respectively. This phase has a non-vanishing parameter $|\langle F \rangle| = 1$ indicating the ferromagnetic order and $|\Theta| = 0$.

- **Cyclic phase**

  $C : \frac{1}{2} \left( e^{i\phi} \left( 1 + \frac{p}{4c_1} \right), 0, \sqrt{2 - \frac{p^2}{8c_1^2}}, 0, e^{-i\phi} \left( -1 + \frac{p}{4c_1} \right) \right)$

  with energy $-p^2 / 4c_1$ and $\phi$ an arbitrary phase. This is a nonmagnetic phase which has no spin-1 analog and was referred to as the cyclic state because of its close analog to the $d$-wave BCS superfluids. Both parameters are zero, $\langle F \rangle = 0$ and $|\Theta| = 0$.

Recent experiments observed clear evidence of polar behaviour for $F = 2$ spinor condensate of $^{87}\text{Rb}$, and the slow dynamics of prepared cyclic ground states showed the $F = 2$ state to be close to the cyclic phase [26, 37]. However, the nature of the spinor condensate which depends on the $s$-wave scattering lengths for the total spins 0, 2, and 4, may be changed into other phases by an offset magnetic field.

### 3 Outline of the homotopy theory of defects

We sketch out the procedure which has been widely used in the study of topological defects in ordered media such as liquid crystals, superfluid $^3\text{He}$ and heavy-fermion superconductors. The explicit use of homotopy for topological classification of defects was made by some French [38, 39, 40, 41] and Russian authors [42, 43]. The results were well summarized in two review articles [39, 44]. The central feature of the classification scheme of the defects emerges from examining the mappings of closed curves in physical space into the order-parameter space (OPS).

The order parameter of a system has associated with it a group of transformations $G$. The set of all transformations in $G$ that leave the reference order parameter $f$ (chosen arbitrarily but thereafter fixed) unchanged is known as the isotropy group $H = \{ g \in G | gf = f \}$. The OPS can then be taken to be the space of cosets of $H$ in $G$: $M = G/H$. In terms of broken symmetry, the fact that the ordering breaks the underlying symmetry is expressed in the fact that $H$ is only a subgroup of $G$. The description that follows will be valid for
any group $G$ that acts transitively on $M$, i.e., if $f_1$ and $f_2$ are possible values of the order parameter, then there is a transformation $g$ in $G$ which takes $f_1$ into $f_2$: $f_2 = gf_1$.

Homotopy groups of the order-parameter space describe physical defects\[44\]. The $n$-th homotopy group $\pi_n(M)$ of the space $M$ consists of the equivalence classes of continuous maps from $n$-dimensional sphere $S_n$ to the space $M$. Two maps are equivalent if they are homotopic to one another. In three dimensional space, the first homotopy group, also called the fundamental group, $\pi_1(M)$ describes singular line defects and domain walls, which are non-singular defects. The second homotopy group $\pi_2(M)$ describes singular point defects and non-singular line defects. These can be calculated with the help of the fundamental theorem: Let $G$ be a connected, simply connected continuous group and $H_0$ be the set of points in $H$ that can be connected to the identity by a continuous path lying entirely in $H$. Then we have the isomorphisms

$$\pi_1(M) = H/H_0, \quad \pi_2(M) = \pi_1(H_0).$$ (17)

For the theorem to hold, it is necessary that $\pi_0(G) = \pi_1(G) = \pi_2(G) = 0$, meaning that $G$ has only one connected piece, any loop in $G$ can be shrunk continuously to a point, and $G$ has a vanishing second homotopy group. While the second homotopy groups are always Abelian, the fundamental groups can either be Abelian (each element constitutes a conjugacy class), or non-Abelian (the line defects are characterized by the conjugacy classes instead of the elements). In Figure 1 we give a schematic description of the procedure for calculation of homotopy groups.

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Figure 1: A schematic description of the procedure for calculation of homotopy groups.
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A ready example for illustrating the above procedure is the biaxial nematics, whose symmetry is that of a rectangular box (proper point group $D_2$). If $G$ is taken to be $SO(3)$ then the isotropy subgroup $H$ is the four-element group consisting of the identity and $180^\circ$ rotations about three mutually perpendicular axes ($D_2$). Order parameter space is thus identified as $M = SO(3)/D_2$. If, however, we take $G$ to be $SU(2)$, the universal covering group of $SO(3)$, then $H$ is expanded to the non-Abelian quaternion group $Q$ (known as the lift or double group) with eight elements

$$Q = \{\pm 1, \pm i\sigma_x, \pm i\sigma_y, \pm i\sigma_z\}.$$ (18)

The natural representation for the order parameter space of a biaxial nematic turns out to be $M = SU(2)/Q$. Since it is a discrete subgroup of $SU(2)$, $H/H_0 = H$. Thus $\pi_1(M) = Q$.  


and \( \pi_2(M) = 0 \). There are no stable point defects in biaxial nematics and the line defects are characterized by five conjugacy classes of group \( Q \)

\[
C_0 = \{1\}, \quad \overline{C_0} = \{-1\}, \quad C_x = \{\pm i\sigma_x\}, \quad C_y = \{\pm i\sigma_y\}, \quad C_z = \{\pm i\sigma_z\}.
\]

(19)

The class \( C_0 \) contains removable trivial defects; \( \overline{C_0} \) contains defects in which the object rotates about 360\(^\circ\) as the defect line is encircled; the other three classes contain defects in which the rotation is through 180\(^\circ\) about each of the three distinct symmetry axes. The defects here are non-commutative, providing an example with non-Abelian fundamental group.

Another illustrative example is the dipole-free \( A \)-phase of \( ^3\text{He} \), which affords an unusual example of a case where \( G \) must be bigger than \( \text{SO}(3) \). The order parameter is the product of an arbitrary unit 3-vector \( \vec{n} \) and a complex 3-vector of the form \( \hat{u} + i\hat{v} \), where \( \hat{u} \) and \( \hat{v} \) are an orthonormal pair. The orientations of \( \hat{u} \) and \( \hat{v} \) are uncoupled. Take the reference order parameter to be \( \hat{A}_{ij} = z_i x_j + y_j \), the group \( G \) can be taken to be the direct product of \( \text{SO}(3) \) with itself: \( G = \text{SO}(3) \times \text{SO}(3) \), elements of \( G \) consisting of pairs \((R, R')\) of distinct rotations. The isotropy group \( H \) consists of elements of the form \((R(z, \theta), 1)\) and \((R(\hat{u}, \pi), R(\hat{z}, \pi))\) for any axis \( \hat{u} \) in the \( x-y \) plane. To construct a simply connected \( G \), we must replace each \( \text{SO}(3) \) by \( \text{SU}(2) \). Determining the lift of \( H \) from \( \text{SO}(3) \times \text{SO}(3) \) to its covering group \( \text{SU}(2) \times \text{SU}(2) \), we find the isotropy group consists of 4 pieces \( \{H_0, gH_0, g^2H_0, g^3H_0\} \) with the connected component of the identity \( H_0 = \{(u(z, \theta), 1)\} \) and \( g = (u(\hat{x}, \pi), u(\hat{z}, \pi)) \). In this article, the notations \( R \) and \( u \) represent the rotations in \( \text{SO}(3) \) and \( \text{SU}(2) \), respectively. The fundamental group is thus isomorphic to the cyclic group of order 4, \( \pi_1(M) = Z_4 \) and \( \pi_2(M) = Z \).

For the spinor condensate it seems natural to identify the underlying symmetry group as \( \text{U}(1) \times \text{SO}(3) \), the groups in the direct product representing the gauge and spin degrees of freedom respectively. This group is not simply connected, i.e., \( \pi_1(\text{U}(1) \times \text{SO}(3)) \neq 0 \). To apply the theorem, however, it is again essential that one chooses the group \( G \) to be simply connected. We proceed by specifying the symmetry group as its universal covering group \( R \times \text{SU}(2) \), with the group of real numbers \( R \) representing any translation \( \theta \in (-\infty, +\infty) \) in the phase of the condensate. For \( F = 1 \), we use the 3D representation of the group \( \text{SU}(2) \) in order to obtain the isotropy group, e.g., a rotation \( u(z, \alpha) \) around axis \( z \) by angle \( \alpha \) takes the form of a diagonal matrix \( \text{Diag}(e^{-i\alpha}, 1, e^{i\alpha}) \), a rotation \( u(y, \beta) \) around axis \( y \) by angle \( \beta \) takes the form of

\[
\begin{pmatrix}
\frac{1}{2} (1 + \cos \beta) & -\sin \beta & \frac{1}{2} (1 - \cos \beta) \\
\sin \beta / \sqrt{2} & \cos \beta & -\sin \beta / \sqrt{2} \\
\frac{1}{2} (1 - \cos \beta) & \sin \beta & \frac{1}{2} (1 + \cos \beta)
\end{pmatrix}.
\]

The two elements \( \pm u(z, \alpha) \) are represented by the same matrix \( \text{Diag}(e^{-i\alpha}, 1, e^{i\alpha}) \) in this even representation of \( \text{SU}(2) \), though we know (and should always bear in mind) that \( u(z, \alpha + 2\pi) = -u(z, \alpha) \) while \( u(z, \alpha + 4\pi) = u(z, \alpha) \).
4 Calculation of the homotopy groups

There are two possible ground states in $F = 1$ case. For the ferromagnetic state, the isotropy group $H$ is constructed by the set of transformations which leave the reference order parameter $(1, 0, 0)^T$ invariant. From the degenerate family of the ground state spinor eq. we know immediately that the angles should satisfy

$$\beta = 0, \theta - \alpha - \gamma = 2n\pi$$

with $n$ an integer. The elements in group $H$ are the combination of a translational part and a rotational part $H = \{(\theta, u(z, \theta)), (\theta, u(z, \theta + 2\pi))\} = \{(\theta, \pm u(z, \theta))\}$. Evidently this group includes two disconnected components—the connected component of the identity $H_0 = \{\theta, u(z, \theta)\}$ is isomorphic to $R$. The group $H/H_0$ is isomorphic to the integers modulo 2, i.e., $Z_2$. The second homotopy group $\pi_2$ is trivial and we arrive at the same result as that in Ref. [12]

$$\pi_1(M) = Z_2, \quad \pi_2(M) = 0, \text{ (spin-1 FM state).} \quad (21)$$

A ferromagnetic spin-1 condensate may have therefore only singular vortices with winding number one while the point-like defects are topologically unstable. Alternatively we may take the symmetry group as SU(2) because we can produce all possible gauge transformations by absorbing $\theta$ into the Euler angle $\gamma$. The isotropy group is discrete and isomorphic to $Z_2$, which gives exactly the same result.

The polar state emerges if the atoms in the condensate interact anti-ferromagnetically. In the ground state eq. 3, the reference parameter $(0, 1, 0)^T$ is left invariant for just those elements with

$$\beta = 0, \quad \theta = 2n\pi \quad \text{or} \quad \beta = \pi, \theta = (2n + 1)\pi. \quad (22)$$

Thus the isotropy group $H$ includes now the transformations in which both the rotation and the translation leave the spinor unchanged, and those in which the rotation takes the reference spinor $(0, 1, 0)^T$ to $(0, -1, 0)^T$ and the translation takes it back, i.e., a $\pi$ rotation about arbitrary axis perpendicular to $\hat{z}$ combined with a $\pi$ translation in $\theta$ (or any odd multiples of $\pi$). The latter invariance is identical to the Ising gauge symmetry emphasized in eq. (14) of Ref. [20]. The full isotropy group is the union of these two sets, $H = \{(2n\pi, u(z, \alpha)), ((2n + 1)\pi, gu(z, \alpha))\}$ where $g = u(y, \pi)$. There are infinitely many discrete components in $H$, while the connected component of the identity $H_0 = \{(0, u(z, \alpha))\}$ is isomorphic to $U(1)$. The elements with an even translational parity are of the form $(2n\pi, I)H_0$, and those with an odd parity are of the form $((2n + 1)\pi, g)H_0$. The group $H/H_0$ is therefore isomorphic to the group of integers $Z$ through the isomorphism $((2n + j)\pi, g^j)H_0 \mapsto 2n + j$ for $j = 0, 1$. We recover the conclusion that line and point defects in spin-1 polar state can be classified by integer winding numbers,

$$\pi_1(M) = Z, \quad \pi_2(M) = Z, \text{ (spin-1 Polar state).} \quad (23)$$

Thus the $Z_2$ term does not appear in the homotopy group. We argue that the identification of the OPS in Ref. [19, 20] is also incorrect (see below). Physically there are indeed infinite
number of line defects corresponding to integer and half-integer vortices (eq. (27) in Ref. [20]). On the other hand, it is the Ising symmetry that leads to half-vortices ($j = 1$), which have been shown to be the unique linear defects in polar condensate in addition to the usual integer vortices ($j = 0$) [18]. If we move around a closed path in the condensate we note that when we return to the starting point the angle $\theta$ has changed by some amount. If we define the change in this angle divided by $2\pi$ to be the winding number, we see from the elements of $H/H_0$ that the winding number can be either an integer $n$ or a half-integer $n + 1/2$.

5 Spin-2 Bose condensate

We next apply the same approach to the BEC of spin-2 bosons. The defects which may be created in spin-2 condensate exhibit even more elaborate structures due to quantum correlations among bosons. For $F = 2$ we have to use the 5D representation of $SU(2)$, e.g., the rotation $u(z, \alpha)$ is represented by matrix $\text{Diag}(e^{-2i\alpha}, e^{-i\alpha}, 1, e^{i\alpha}, e^{2i\alpha})$ and $u(y, \beta)$ takes the form of

\[
\begin{pmatrix}
\cos^4 \frac{\beta}{2} & -\sin \beta \cos^2 \frac{\beta}{2} & \frac{\sqrt{6}}{4} \sin^2 \beta & -\sin \beta \sin^2 \frac{\beta}{2} & \sin^4 \frac{\beta}{2} \\
\sin \beta \cos^2 \frac{\beta}{2} & \cos \beta + \cos 2\beta & -\frac{\sqrt{6}}{4} \sin 2\beta & \cos \beta - \cos 2\beta & -\sin \beta \sin^2 \frac{\beta}{2} \\
\frac{\sqrt{6}}{4} \sin^2 \beta & \frac{\sqrt{6}}{4} \sin 2\beta & \cos \beta - \cos 2\beta & -\sqrt{6} \sin 2\beta & \sqrt{6} \sin^2 \beta \\
\sin \beta \sin^2 \frac{\beta}{2} & \cos \beta - \cos 2\beta & \frac{\sqrt{6}}{4} \sin 2\beta & \cos \beta + \cos 2\beta & -\sin \beta \cos^2 \frac{\beta}{2} \\
\sin^4 \frac{\beta}{2} & \sin \beta \sin^2 \frac{\beta}{2} & \frac{\sqrt{6}}{4} \sin 2\beta & \cos \beta - \cos 2\beta & \cos^4 \frac{\beta}{2}
\end{pmatrix}
\]

The calculations of the degenerate family of the ground state spinors and the corresponding homotopy groups are straightforward and some results have been reported in [47]. Here we pick up some interesting features in our results, focusing on the symmetry properties of the defects in comparison with those in other ordered media. We first consider the defects in the absence of an external field and the effect of magnetic field will be discussed later.

We start with the case of the ferromagnetic state $F$. Equating the general expression for the ground state spinor

\[
\zeta = e^{i(\theta - 2\gamma)} \begin{pmatrix}
e^{-2i\alpha} \cos^4 \frac{\beta}{2} \\
e^{-i\alpha} \sin \beta \cos^2 \frac{\beta}{2} \\
\frac{\sqrt{6}}{4} \sin^2 \beta \\
e^{i\alpha} \sin \beta \sin^2 \frac{\beta}{2} \\
e^{2i\alpha} \sin^4 \frac{\beta}{2}
\end{pmatrix}
\]

with the reference spinor $(1, 0, 0, 0, 0)^T$ leads to the requirement for the isotropy group $H$

\[
\beta = 0, \theta - 2\alpha - 2\gamma = 2n\pi.
\]
by $\theta/2, \theta/2 + \pi, \theta/2 + 2\pi, \theta/2 + 3\pi$ respectively. Hence the group $H$ is composed of 
four pieces $H = \{(\theta, u(z, n\pi + \theta/2))\}$. Here it is important to show that the four 
components are not connected: there does not exist a continuous path in $H$ which connects one 
component to another, though the rotational parts themselves are connected. The connected 
component of the identity $H_0 = \{(\theta, u(z, \theta/2))\}$ is again isomorphic to $R$. If we define an 
element $g$ of the group $R \times SU(2)$ by $(0, u(z, \pi))$, we see that the quotient group $H/H_0$ has 
the same structure as the cyclic group of order 4, i.e., $\{e, g, g^2, g^3\}$ and we conclude that 

$$\pi_1(M) = Z_4, \quad \pi_2(M) = 0, \text{ (spin-}2\text{ }F\text{ state).} \quad (26)$$

It is interesting to check how the group $Z_4$ characterizes vortices for state $F$. In spin-1 
case there is only one topologically stable line defect, that is, a vortex with winding number 
one. Equation (26) shows that there are three stable vortices for spin-2 condensates. We 
can set $\theta - 2\gamma = 2m\varphi, -\alpha = m\varphi, \beta = \pi t$ in the ground state for $F$ state, Eq. (24), 
which leads to a family of spinor states parametrized by a parameter $t$ between 0 and 1. 
Here $m > 0$ is an integer, $\varphi$ is the azimuthal angle. When $t$ evolves from 0 to 1, the 
$4m\varphi$ vortex state $\zeta(t = 0) = (e^{i4m\varphi}, 0, 0, 0)^T$ evolves continuously to the vortex free 
state $\zeta(t = 1) = (0, 0, 0, 1)^T$. This shows that vortices with winding number $4m$ are 
topologically unstable. Similarly, by multiplying factors $e^{ik\varphi}(k = 1, 2, 3)$ one obtains the 
following correspondences 

$$e^{i(4m+k)\varphi}(1,0,0,0,0)^T \rightarrow e^{ik\varphi}(0,0,0,0,1)^T \quad (27)$$

i.e., the vortices with winding numbers $4m + k$ may evolve into vortices with winding 
numbers $k$, respectively. There are thus three classes of topologically stable line defects. 
Together with the uniform state, they form the fundamental group $Z_4$. Non-trivial vortices 
are those in which the reference spinor rotates through 180°, 360° or 540° about the $z$-axis 
when the defect line is circulated. Straightforwardly for ferromagnetic condensates with 
spin $F$, the fundamental group $\pi_1(M) = Z_{2F}$ characterizes $(2F - 1)$ classes of stable line 
defects.

Spin variations in the ferromagnetic states in general lead to superflows [3, 25]. To 
illustrate the coreless (or Skyrmion) vortices in spin-2 case, we set $\theta - 2\tau = 2\varphi, \alpha = \varphi$ in 
the spinor degenerate family (24) and consider the condensate 

$$\zeta(\mathbf{r}) = \begin{pmatrix} 
\cos^4 \frac{\beta}{2} \\
\cos^2 \frac{\beta}{2} \\
\sin^2 \frac{\beta}{2} \\
\sin \frac{\beta}{2} \\
\cos \frac{\beta}{2} 
\end{pmatrix} \quad (28)$$

where $\beta = \beta(\mathbf{r})$ is an increasing function of $\mathbf{r}$ starting from $\beta = 0$ at $\mathbf{r} = 0$. The superfluid 
velocity does not depend on $z$ and it is cylindrically symmetric 

$$v_s = \frac{\hbar}{M} [2\nabla \varphi - 2\cos \beta \nabla \varphi] = \frac{2\hbar}{Mr} (1 - \cos \beta) \hat{\varphi} \quad (29)$$
i.e., the coreless vortex may exist in the spin-2 case, with only the velocity doubling its value compared to the spin-1 case \[3\]. The velocity vanishes instead of diverging at \(r = 0\) because \(\beta(0) = 0\). This is called a coreless vortex. For a Mermin-Ho vortex \[45\], the bending angle \(\beta\) must be \(\pi/2\) at the boundary of the condensate, while for an Anderson-Toulouse \[46\] vortex \(\beta\) must be \(\pi\), i.e.

\[
\beta(R) = \pi/2, \text{ for Mermin-Ho} \\
\beta(R) = \pi, \text{ for Anderson-Toulouse.}
\]  

(30)

6 Non-Abelian homotopy groups

Media with non-Abelian fundamental groups are especially interesting from the topological point of view. The only illustrative example in ordered media so far have been biaxial nematic liquid crystals \[48\]. Their multiplication table has been verified experimentally \[49\].

We have found that the cyclic state \(C\) provides another physically realistic example in which the fundamental group is non-commutative. A rotation and a gauge transformation of the reference spinor \(\frac{1}{2}\left(e^{i\phi}, 0, \sqrt{2}, 0, -e^{-i\phi}\right)^T\) in zero field produce the following degenerate family

\[
\zeta = \frac{1}{2}e^{i\theta} \begin{pmatrix}
e^{-2i\alpha} \left(\cos^4 \frac{\beta}{2} e^{i\phi-2i\tau} + \frac{\sqrt{2}}{2} \sin^2 \beta - \sin^4 \frac{\beta}{2} e^{-i\phi+2i\tau}\right) \\
e^{-i\alpha} \sin \beta \left(\cos^2 \frac{\beta}{2} e^{i\phi-2i\tau} - \sqrt{3} \cos \beta + \sin^2 \frac{\beta}{2} e^{-i\phi+2i\tau}\right) \\
e^{i\alpha} \sin^2 \beta e^{i\phi-2i\tau} + \sqrt{2} \sin^2 \beta e^{-i\phi+2i\tau} \\
e^{2i\alpha} \left(\sin^4 \frac{\beta}{2} e^{i\phi-2i\tau} + \sqrt{3} \cos \beta + \cos^2 \frac{\beta}{2} e^{-i\phi+2i\tau}\right)
\end{pmatrix}
\]  

The reference spinor is left invariant by the elements of three sets characterized by the translations in the phase of the condensate \(\theta\):

- For \(\theta = 2n\pi\), one must have \(\beta = 0, \alpha + \tau = m\pi, \text{ or } \beta = \pi, \alpha - \tau = -\phi + \frac{\pi}{2} + m\pi\);
- For \(\theta = \frac{2\pi}{3} + 2n\pi\), one must have \(\beta = \frac{\pi}{2}, \alpha + \tau = -\frac{\pi}{2} + m\pi, \alpha - \tau = -\phi + m'\pi\);
- For \(\theta = \frac{4\pi}{3} + 2n\pi\), one must have \(\beta = \frac{\pi}{2}, \alpha + \tau = \frac{\pi}{2} + m\pi, \alpha - \tau = -\phi + m'\pi\).

Here \(m\) and \(m'\) are integers satisfying \(m + m' = \text{odd}\). For all possible transformations we need take the values \(m, m' = 0, 1, 2, 3\) so that there are eight possibilities

\[
\begin{align*}
m &= 0, m' = 1, 3 \\
m &= 1, m' = 0, 2 \\
m &= 2, m' = 1, 3 \\
m &= 3, m' = 0, 2
\end{align*}
\]
This gives the isotropy group
\[
H = \{ \pm I, \pm a, \pm b, \pm c, \\
\pm d, \pm e, \pm f, \pm g, \\
\pm d^2, \pm e^2, \pm f^2, \pm g^2 \}.
\] (31)

The spin rotations \( a = u(z, \pi), b = u(y, \pi)u(z, \phi + \pi/2) \) and \( c = ba \) satisfy \( a^2 = b^2 = e^2 = -I \), while \( d = u(z, \pi/4 + \phi/2)u(y, \pi/2)u(z, \pi/4 - \phi/2) \), \( e = -da, f = -ad \) and \( g = -ada \) satisfy \( d^3 = e^3 = f^3 = g^3 = -I \). Each element in the first, second, third row is associated with an additional phase change \( 2n\pi, 2\pi/3 + 2n\pi, 4\pi/3 + 2n\pi \) respectively. It is a discrete group, and \( H_0 \) consists of the identity \((0, I)\) only. The fundamental theorems identify that
\[
\pi_1(M) = H, \quad \pi_2(M) = 0, \text{(spin-2 C state)}.
\] (32)

The elements in the fundamental group are non-commutative, for example \( ab = -c \neq ba \).

The criterion for the topological equivalence of defects applies in the most general case in terms of conjugacy classes of the fundamental group. Two line defects are topologically equivalent if and only if they are characterized by the same conjugacy class. Defects can still be labelled by the elements of the first homotopy group, but if these elements belong to the same conjugacy class, corresponding defects can be continuously transformed to one another. However, if they belong to different conjugacy classes this is not possible. It is thus necessary to classify the group into the following conjugacy classes:

\[
C_0(n) = \{ I \}_n, \quad \overline{C_0(n)} = \{ -I \}_n, \quad C_2(n) = \{ \pm a, \pm b, \pm c \}_n, \\
C_3(n + 1/3) = \{ d, e, f, g \}_{n+1/3}, \quad \overline{C_3}(n + 1/3) = \{ -d, -e, -f, -g \}_{n+1/3}, \\
C_3^2(n + 2/3) = \{ d^2, e^2, f^2, g^2 \}_{n+2/3}, \quad \overline{C_3^2}(n + 2/3) = \{ -d^2, -e^2, -f^2, -g^2 \}_{n+2/3}
\]

with the subscripts standing for the winding numbers of the defects. Physically this indicates the feasibility of creating not only vortices with any integer winding number but also fractional quantum vortices. The class \( C_0(n) \) describes defects in which the phase of the spinor is changed by \( 2\pi n \) as the defect line is encircled. Note that only \( C_0(0) \) corresponds to trivial defects. In the case of \( \overline{C_0}(n) \) phase change of \( 2\pi n \) is accompanied by a 360\(^\circ\) rotation about z-axis. The element \( a \) with winding number \( n \) in the class \( C_2(n) \) depicts a defect in which the spinor rotates through 180\(^\circ\) about the z-axis and changes phase by \( 2\pi n \) as the line is encircled. The multiplication table of conjugacy classes is shown in table 1. Only half of table is shown because the class multiplication is commutative. Winding numbers have been omitted for clarity. When two classes are multiplied the winding number of the resulting class is the sum of the individual winding numbers. It shows that, for example, when we combine defect \( C_2(n) \) with \( C_2(-n) \) they can either annihilate each other \((C_0(0))\) or form defect \( \overline{C_0}(0) \) or \( C_2(0) \), the result depending on how they are brought together. Interesting features of this non-Abelian fundamental group include the topological instability of the defects and their interaction, i.e., entanglement when two of them are brought to cross with each other [44].
Table 1: The multiplication table of the conjugacy classes of the cyclic phase.

|        | $C_0$ | $C_2$ | $C_3$ | $C_2^3$ | $C_3^2$ | $C_3^3$ |
|--------|-------|-------|-------|---------|---------|---------|
| $C_0$  | $C_0$ |       |       |         |         |         |
| $C_2$  | $C_2$ | $6C_0 + 6C_2 + 4C_2$ |       |         |         |         |
| $C_3$  | $C_2^3$ | $3(C_3 + C_2^3)$ | $3C_2^3 + C_3^2$ |       |         |         |
| $C_3^2$ | $C_3$ | $3(C_3 + C_2^3)$ | $C_3^2 + 3C_3^2$ | $3C_2^3 + C_3^2$ |         |         |
| $C_3^3$ | $C_3^2$ | $3(C_3^2 + C_2^2)$ | $4C_0 + 2C_2$ | $4C_0 + 2C_2$ | $3C_3 + C_3$ |         |

The defects can be further grouped into classes, which form an Abelian group isomorphic to the first homology group of the order parameter space $[50]$. This coarser classification is more general than the homotopic one because two defects are considered equivalent also, if they can be transformed into each other via a catalyzation process consisting of splitting a line singularity into two and recombining them beyond a third one. All elements labelled by elements of the commutator subgroup $D$ of $\pi_1(M)$ can be catalyzed away by this procedure. $D$ is generated by the commutators $\delta \tau \delta^{-1} \tau^{-1}$ of all pairs of elements $\delta, \tau \in \pi_1(M)$. The elements of $\pi_1(M)/D$ are unions of conjugacy classes. In our case $D$ is the union of the conjugacy classes with winding number zero, $D = C_0(0) \cup C_0(0) \cup C_2(0)$ and the first homology group is

$$\pi_1(M)/D = \left\{ C_0 \cup C_0 \cup C_2, C_3 \cup C_3, C_2^3 \cup C_3^3 \right\}. \quad (33)$$

The homology theory assembles the conjugacy classes further into three sets for each $n$, in which the defects are labeled by the winding numbers $n, n+1/3, n+2/3$ respectively $[47]$. Two defects in the same conjugacy classes can be continuously converted into one another by local surgery, while two defects in the same homology class can be deformed into one another by the catalyzation procedure.

### 7 Order Parameter Spaces

Like the quaternion group $Q$ for biaxial nematics, the fundamental group $[31]$ is the lift of a point group in $R \times SU(2)$. To find the remaining discrete symmetry group for the cyclic state, and, in addition, to clarify the controversial identification of the OPS for spin-1 case, in the remaining of this paper we turn to describe the system in terms of rotations in $SO(3)$, e.g., two elements $\pm u(z, \alpha)$ in $SU(2)$ are mapped into one $R(z, \alpha)$ in $SO(3)$ with $R(z, \alpha + 2\pi) = R(z, \alpha)$.

The OPS for $F = 1$ polar state was identified as $U(1) \times S^2$ in Refs. $[3, 12]$. An extra $Z_2$ symmetry was claimed in Ref. $[19]$ so the author concluded the OPS as $U(1) \times S^2/Z_2$. Here we show that previous studies are incorrect. Taking the group $G$ as $U(1)_G \times SO(3)_S$ where the subscripts stand for the gauge and spin symmetries respectively, we see that the isotropy group $H$ consists of two separate parts, $\left\{ (e^{i\theta}, R(z, \alpha)) \right\}$ and $\left\{ (e^{i\pi}, R(y, \pi)R(z, \alpha)) \right\}$.
The rotations in the first part constitute the group SO(2), while the elements in the second part are just those in the group O(2) but not in SO(2) with determinants −1. The combination of these two parts gives the full isotropy group as O(2) where both gauge and spin symmetries are involved. The OPS is the quotient \( G/H = (U(1)_G \times SO(3)_S) / O(2)_{G+S} \) and here it is not possible to apply the fundamental theorem for \( G \) is not any more simply connected. One may wonder if we can factorize the OPS further as

\[
G/H = (U(1)_G \times SO(3)_S) / O(2)_{G+S} \\
= (U(1)_G \times SO(3)_S) / (SO(2) \times Z_2)_{G+S} \\
= U(1) \times S^2/Z_2
\]

However it is incorrect because though in 3 dimensional space we have \( O(3) = SO(3) \times Z_2 \) but it is not true in 2 dimensional case, i.e. \( O(2) \neq SO(2) \times Z_2 \). The spin and gauge symmetries are broken in a connected fashion just as in the system of \(^3\)He \[10, 11\]. Table 2 summarizes our result in comparison with the previous studies.

### Table 2: Comparison of the OPS and fundamental groups for spin-1 polar condensate

|                | OPS                  | \( \pi_1(M) \) |
|----------------|----------------------|---------------|
| Ho, Stoof, etc.| \( U(1) \times S^2 \) | \( Z \)       |
| Zhou           | \( U(1) \times S^2/Z_2 \) | \( Z \times Z_2 \) |
| This paper     | \( (U(1) \times SO(3)) / O(2) \) | \( Z \)       |

Figure 2: Symmetries of the defects in biaxial nematics \((D_2)\) and cyclic state \(C\) in spin-2 condensate \((T)\). The dot at the center of the rectangle stands for axis \(z\). The dashed lines represent 2-fold axes, except that with a triangle for 3-fold axis.

For the ferromagnetic state the group \( H \) may be obtained if one notices that the \( 2\pi \) difference in the rotational angle does not give another component as it did in the case of
SU(2). We have $H = \{ (e^{i\theta}, R(z, \theta)) \}$ which is isomorphic to $U(1)_{G+S}$. This means that there is a remaining symmetry $U(1)$ in the symmetry broken system. The OPS is thus factorized as $(U(1)_{G} \times SO(3)_S) / U(1)_{G+S} = SO(3)_{S+G}$.

The discrete symmetry group of defects in the spin-2 cyclic state $C$ can be shown to be isomorphic to the tetrahedral group $T$. We continue to represent $G$ as $U(1)_G \times SO(3)_S$. The isotropy group (31) is shrunk to a group of 12 elements if one understands the rotation in the sense of SO(3) (i.e., $a = R(z, \pi)$),

$$H = \{ I, a, b, c, \epsilon d, \epsilon e, \epsilon f, \epsilon g, \epsilon^2 d^2, \epsilon^2 e^2, \epsilon^2 f^2, \epsilon^2 g^2 \},$$ (34)

where $\epsilon = \exp(2\pi i/3)$ comes from the gauge transformation and $\epsilon d$, for instance, is an abbreviation for the element $(\epsilon, d)$. Three 2-fold rotational axes are $z$ and 2 lines in $xy$ plane perpendicular to each other (which lie on axes $x$ and $y$ if we choose the arbitrary phase $\phi = \pi/2$). The elements $\epsilon d, \epsilon e, \epsilon f, \epsilon g$ are four 3-fold axes. The symmetries remaining in the symmetry broken states for biaxial nematics and spin-2 cyclic state are shown in Figure 2. The OPS for state $C$ can be identified as $(U(1)_{G} \times SO(3)_S) / T_{G+S}$.

It should be noted that an applied magnetic field $B$ changes the defect structure severely by reducing the degenerate family of the spinor. We take again the cyclic state as an example. The symmetry group in this case is $U(1) \times SO(2)$ because the magnetic field chooses its direction automatically as the quantization axis. From the spinor

$$\frac{1}{2} \begin{pmatrix} (1 + p)e^{i\phi_1} \\ 0 \\ \sqrt{2 - 2p^2}e^{i(\phi_1 + \phi_2)/2} \\ 0 \\ (-1 + p)e^{i\phi_2} \end{pmatrix},$$

where $\phi_{1,2}$ are two arbitrary phases and $p \sim B$, we easily see the possibility to create vortices in any of the three nonzero components with winding number for $i$–th component $n_i$ confined by $n_1 + n_5 = 2n_3$.

8 Summary

Our main findings are summarized in Table 3. We have determined the nature of the topological defects in spin-1 and spin-2 condensates. The order parameter spaces are identified as the spaces of the coset of the isotropy group $H$ in the transformation group $G$. Topologically stable vortices with winding numbers larger than unity may be created in the ferromagnetic state for condensates with $F > 1$, up to the value $(2F - 1)$. The line defects in the spin-2 cyclic state $C$ exhibit non-commutative features, resulting e.g. in line defects with winding numbers of $1/3$ and its multiples. It also turns out that in the zero field $U(1) \times SO(3)$ does not act transitively on the order-parameter space of the polar phase and thus the defect structure remains unsolved.
Table 3: Main results on calculation of the OPS and homotopy groups

|        | OPS                      | $\pi_1$ | $\pi_2$ |
|--------|--------------------------|---------|---------|
| Spin-1 FM | SO(3)                    | $Z_2$   | 0       |
| Spin-1 AFM | (U(1) $\times$ SO(3)) / O(2) | $Z$    | $Z$    |
| Spin-2 $F$ | SO(3)/$Z_2$              | $Z_4$   | 0       |
| Spin-2 $F'$ | SO(3)                   | $Z_2$   | 0       |
| Spin-2 $C$ | (U(1) $\times$ SO(3)) / $T$ | $H$ eq. [31] | 0       |

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