A naive parametrization for the vortex-sheet problem.

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Abstract

We consider the dynamics of a vortex sheet that evolves by the Birkhoff-Rott equations. We understand the fluid as a weak solution of the incompressible Euler equations where the vorticity is given by a delta function on a curve multiplied by an amplitude. We show that as long as the weak solution is of this type we obtain the continuity of the pressure. We study solutions with finite energy which implies zero mean amplitude. In this context we choose a term in the tangential direction for the motion of the vortex sheet for which we prove ill-posedness for non analytic initial data.

1 Introduction

We consider a velocity field \( v = (v_1, v_2) \) satisfying the incompressible 2D- Euler equations

\[
\begin{align*}
v_t + (v \cdot \nabla)v &= -\nabla p, \\
\nabla \cdot v &= 0.
\end{align*}
\]

We study weak solutions of the system with a vorticity \( \omega = \nabla \times v \) supported on the curve \( z(\alpha, t) \) given by

\[
\omega(x, t) = \varpi(\alpha, t)\delta(x - z(\alpha, t)),
\]

i.e. a vortex sheet, where \( \delta \) is the delta function. Here we shall assume for the curve the following scenarios:

- Periodicity in the horizontal space variable: \( z(\alpha + 2k\pi, t) = z(\alpha, t) + (2k\pi, 0) \).
- A closed contour: \( z(\alpha + 2k\pi, t) = z(\alpha, t) \).
- A near planar vortex sheet: \( \lim_{\alpha \to \infty} (z(\alpha, t) - (\alpha, 0)) = 0 \).

The vortex sheet \( z(\alpha, t) \) evolves satisfying the equation,

\[
z_t(\alpha, t) = BR(z, \varpi)(\alpha, t) + c(\alpha, t)\partial_\alpha z(\alpha, t),
\]

where the Birkhoff-Rott integral on the curve, which comes from Biot-Savart law, is given by

\[
BR(z, \varpi)(\alpha, t) = \frac{1}{2\pi} PV \int \frac{(z(\alpha, t) - z(\beta, t))^\perp}{|z(\alpha, t) - z(\beta, t)|^2} \varpi(\beta, t) d\beta,
\]
and $c(\alpha, t)$ represents the re-parametrization freedom. Then we can close the system using Bernoulli’s law with the equation:

$$\varpi_t = \partial_\alpha(c \varpi).\quad (1.6)$$

We study an initial data for the amplitude of the vorticity with mean zero which is preserved by equation (1.6). From Biot-Savart law, at first expansion, the expression at infinity is of the order of $1/|x|^2$ if $\varpi$ for a closed curve $\partial$ near planar at infinity. To obtain a velocity field in $L^2$ it is necessary to have $\int \varpi = 0$ (for more details see [11]). In the periodic case, $z(\alpha + 2\pi k, t) = z(\alpha, t) + (2\pi k, 0)$, the following classical identity for complex numbers

$$\frac{1}{\pi} \left(\frac{1}{z} + \sum_{k \geq 1} \frac{2z}{z^2 - (2\pi k)^2}\right) = \frac{1}{2\pi \tan(z/2)},$$

yields (ignoring the variable $t$)

$$v(x) = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \varpi(\beta) \left(\frac{\tanh(\frac{x_2 - z_2(\beta)}{2})(1 + \tan^2(\frac{x_1 - z_1(\beta)}{2}))}{2\tanh(\frac{x_1 - z_1(\beta)}{2}) + \tan^2(\frac{x_2 - z_2(\beta)}{2})} + \frac{\tan(\frac{x - z(\beta)}{2})(\tan(\frac{2\pi + z(\beta)}{2}) - 1)}{\tan^2(\frac{x_1 - z_1(\beta)}{2}) + \tan^2(\frac{x_2 - z_2(\beta)}{2})}\right) d\beta,$$

for $x \neq z(\alpha, t)$. Then

$$\lim_{x_2 \to \pm \infty} v(x, t) = \mp \frac{1}{4\pi} \int_{-\pi}^{\pi} \varpi(\beta) d\beta(1, 0),$$

and to have the same value at infinity it is necessary again mean zero.

The problem of existence of weak solutions of the Euler equations for a general initial velocity in $L^2$ is open [11]. There is solution for this problem but the velocity field becomes a Laplace-Young measure (see [2]). Constantin, E and Titi [6] prove a condition of regularity in 3D within the chain of Besov spaces, $v \in L^3([0, T], B_3^{1,\infty}) \cap C([0, T], L^2)$ with $\alpha > 1/3$, for weak solutions conserving energy (Onsager’s conjecture). Nevertheless there are results of non-uniqueness for weaker solutions with zero initial data that becomes nontrivial (see [15] and [16]) even for velocity fields in $L^2$, i.e. $v(x, t) \in L^\infty_c([0, T] \times L^2)$ (see [9]). There is also a result of uniqueness for a vorticity in $L^1 \cap L^\infty$ due to Yudovich [20].

For the particular case of a vortex sheet there are many papers which consider the case of $\varpi$ with a distinguished sign. We can point out the work of Delort [1] where he prove global existence of weak solutions for initial velocity in $L^3_{loc}$ and vorticity a positive Radon measure. A simpler proof can be found in [10] due to Majda. Existence for a particular case of a Radon measure with non distinguished sign is shown in [13]. Also, in the case of an analytic initial data, a local existence result for the vortex sheet is given by Sulem, Sulem, Bardos and Frisch in [18]. Duchon and Robert [3] prove global-existence for a peculiar initial data. They consider a particular $c(\alpha, t)$ which gives $z_0(\alpha, t) = 0$ and therefore if one parametrizes initially $z_0(\alpha) = (\alpha, y_0(\alpha))$ the free boundary is given in terms of a function and the equations (1.4) become

$$y_t(\alpha, t) = \frac{1}{2\pi} PV \int \frac{(\alpha - \beta) + (y(\alpha, t) - y(\beta, t))\partial_\alpha y(\alpha, t)}{(|\alpha - \beta|^2 + (y(\alpha, t) - y(\beta, t))^2)} \varpi(\beta, t) d\beta.$$
and \( c(\alpha, t) \) in equation (1.6) is given by
\[
c(\alpha, t) = \frac{1}{2\pi} PV \int \frac{(y(\alpha, t) - y(\beta, t))}{(\alpha - \beta)^2 + (y(\alpha, t) - y(\beta, t))^2} \varpi(\beta, t) d\beta.
\]
A similar approach is done by Caflish and Orellana [4] to show also global-existence for particular initial data and moreover they give an argument to prove ill-posedness in \( H^s \) for \( s > 3/2 \). They choose \( c(\alpha, t) = 0 \) which implies \( \varpi(\alpha, t) = \varpi_0(\alpha) \). If \( \varpi_0(\alpha) \) has a distinguish sign, the following change of variable is legitimate
\[
d\sigma = \varpi_0(\beta)d\beta
\]
and equations (1.4) can be written as
\[
z_t(\alpha, t) = \frac{1}{2\pi} PV \int \frac{(z(\alpha, t) - z(\beta, t))^1}{|z(\alpha, t) - z(\beta, t)|^2} d\beta,
\]
which is the Birkhoff-Rott equation. By taking \( z(\alpha, t) = (\alpha + \varepsilon_1(\alpha, t), \varepsilon_2(\alpha, t)) \) (or \( \varpi(\alpha, t) = 1 + \varepsilon_1(\alpha, t) \) and \( y(\alpha, t) = \varepsilon_2(\alpha, t) \)) in the parametrization of Duchon and Robert and linearizing in (1.7) yields
\[
\partial_t \varepsilon_1 = -\frac{1}{2} \Lambda(\varepsilon_2), \quad \partial_t \varepsilon_2 = -\frac{1}{2} \Lambda(\varepsilon_1).
\]
Therefore
\[
\tilde{\varepsilon}_1(\xi, t) = \frac{\tilde{\varepsilon}_1(\xi, 0) + \tilde{\varepsilon}_2(\xi, 0)}{2} e^{-\pi|\xi| t} + \frac{\tilde{\varepsilon}_1(\xi, 0) - \tilde{\varepsilon}_2(\xi, 0)}{2} e^{\pi|\xi| t},
\]
\[
\tilde{\varepsilon}_2(\xi, t) = \frac{\tilde{\varepsilon}_1(\xi, 0) + \tilde{\varepsilon}_2(\xi, 0)}{2} e^{-\pi|\xi| t} - \frac{\tilde{\varepsilon}_1(\xi, 0) - \tilde{\varepsilon}_2(\xi, 0)}{2} e^{\pi|\xi| t},
\]
where \( \Lambda \) is the operator \( \Lambda = (-\Delta)^{\frac{1}{2}} \).

Since the initial data \( \varepsilon_1(\xi, 0) = \varepsilon_2(\xi, 0) \) only oscillate the dissipative waves, it follows global-existence even for non-regular initial data. Applying Fourier techniques to the nonlinear case
\[
\partial_t \varepsilon_1 = -\frac{1}{2} \Lambda(\varepsilon_2) + T(\varepsilon_1, \varepsilon_2), \quad \partial_t \varepsilon_2 = -\frac{1}{2} \Lambda(\varepsilon_1) + S(\varepsilon_1, \varepsilon_2),
\]
yields that these particular initial data, small enough, activate only the dissipative waves and control the nonlinear operators \( T \) and \( S \) obtaining global in time solutions.

The main idea to show ill-posedness of Caflisch and Orellana is to consider the following function
\[
s_0(\gamma, t) = \varepsilon(1 - i)[(1 - e^{-t/2 - i\gamma})^{1+\nu} - (1 - e^{-t/2 + i\gamma})^{1+\nu}]
\]
which is a solution of the linearization of equation (1.7). For \( 0 < \nu < 1 \), \( s_0 \) has a infinite curvature at \( \gamma = t = 0 \). Then they prove that a function \( r(\gamma, t) \) exists such that \( z(\gamma, t) = \gamma + s_0 + r \) is an analytic solution of equation (1.7) with infinite curvature at \( \gamma = t = 0 \). Then they obtain ill-posedness in Sobolev spaces in the Hadamard sense using the following symmetry properties:

If \( z(\gamma, t) \) is a solution of (1.7) then so are \( z_0(\gamma, t) = \pi(\gamma, -t), z_s(\gamma, t) = z(\gamma, t - t_0) \) and \( z_n(\gamma, t) = n^{-1}z(n\gamma, nt) \).
Ill-posedness was also proved by Ebin [8] using different techniques. A study of the existence of solutions of equation (1.7) in less regular spaces than $H^s$ can be found in [19]. We also quote that the first evidence of singularities with analytic initial data was given by Moore in [12].

In this paper our first step will be to deduce the equation of motion of the vortex sheet from the weak formulation of the Euler equations. In order to do that we show the equality of pressure when we take the limits at each side of the curve for weak solutions satisfying (1.3) (see proposition 2.1). In section 3 we shall study the case in which the term in the tangential direction is given by $c(\alpha, t) = \frac{1}{2} H(\varpi)(\alpha, t)$, where $H \varpi$ is the Hilbert transform of the function $\varpi$ (see [17]) given by

$$H \varpi(\alpha) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{\varpi(\beta)}{\alpha - \beta} d\beta,$$

and also

$$H \varpi(\alpha) = \frac{1}{\pi} PV \int_{-\pi}^{\pi} \frac{\varpi(\beta)}{2 \tan((\alpha - \beta)/2)} d\beta,$$

in the periodic domains. This term is just of the same order that the Birkhoff-Rott integral over $\varpi$ for a regular curve [7]. In fact, for $z(\alpha, t) = (\alpha, 0)$, we have exactly

$$BR(z, \varpi)(\alpha, t) = \frac{1}{2} H(\varpi)(\alpha, t)(0, 1).$$

Then we show ill-posedness for the equation of the amplitude (1.6) for a non-analytic initial data with mean zero.

2 The evolution equation

In this section we shall obtain the classic Birkhoff-Rott equations from the weak formulation of the Euler equations for the velocity. For this purpose we must prove the continuity of the pressure over the vortex sheet which is a property that has been taken in the literature, as far as we know, as an assumption. One alternative deduction, which does not need to prove the continuity of the pressure, can be found in [14] where the authors use the weak formulation for the Euler equations for the vorticity in 2D. We consider weak solutions of the system (1.1, 1.2): for any smooth functions $\eta$ and $\zeta$ compactly supported on $[0, T) \times \mathbb{R}^2$, i.e. in the space $C_c^\infty([0, T) \times \mathbb{R}^2)$, we have

$$\int_0^T \int_{\mathbb{R}^2} \left( v \cdot (\eta_t + v \cdot \nabla \eta) + p \nabla \cdot \eta \right) dx dt + \int_{\mathbb{R}^2} v_0(x) \cdot \eta(x, 0) dx = 0$$

and

$$\int_0^T \int_{\mathbb{R}^2} v \cdot \nabla \zeta dx dt = 0,$$

where $v_0(x) = v(x, 0)$ is the initial data.
Let us assume that the vorticity is given by a delta function on the curve \( z(\alpha, t) \) multiplied by an amplitude, i.e.
\[
\omega(x, t) = \varpi(\alpha, t) \delta(x - z(\alpha, t)),
\]
where \( z(\alpha, t) \) splits the plane in two domains \( \Omega^j(t) \) \((j = 1, 2)\).

Then by Biot-Savart law we get
\[
v(x, t) = \frac{1}{2\pi} \text{PV} \int \frac{(x - z(\beta, t))^\perp}{|x - z(\beta, t)|^2} \varpi(\beta, t) d\beta
\]
for \( x \neq z(\alpha, t) \). We have
\[
\begin{align*}
v^2(z(\alpha, t), t) &= BR(z, \varpi)(\alpha, t) + \frac{1}{2} \frac{\varpi(\alpha, t)}{\partial_\alpha z(\alpha, t)^2} \partial_\alpha z(\alpha, t), \\
v^1(z(\alpha, t), t) &= BR(z, \varpi)(\alpha, t) - \frac{1}{2} \frac{\varpi(\alpha, t)}{\partial_\alpha z(\alpha, t)^2} \partial_\alpha z(\alpha, t),
\end{align*}
\]
where \( v^j(z(\alpha, t), t) \) denotes the limit velocity field obtained approaching the boundary in the normal direction inside \( \Omega^j \) and \( BR(z, \varpi)(\alpha, t) \) is given by (1.5). It is easy to check that the velocity \( v \) (2.4) satisfies (2.2).

**Proposition 2.1** Let us consider a weak solution \((v, p)\) satisfying (2.1–2.2) where \( \nabla \times v = \omega \) is given by (2.3). Then we have the following identity
\[
p^1(z(\alpha, t), t) = p^2(z(\alpha, t), t),
\]
where \( p^j(z(\alpha, t), t) \) denotes the limit pressure obtained approaching the boundary in the normal direction inside \( \Omega^j \).

**Proof:** We shall show that the Laplacian of the pressure is as follows
\[
\Delta p(x, t) = F(x, t) + f(\alpha, t) \delta(x - z(\alpha, t)),
\]
where \( F \) is regular in \( \Omega^j(t) \) and discontinuous on \( z(\alpha, t) \). The amplitude of the delta function \( f \) is regular. The inverse of the Laplacian by means of the Newtonian potential gives the continuity of the pressure on the free boundary (see [7]). Here we shall give the argument for a close curve; the proof for the other cases being analogous.

The expression for the conjugate of the velocity in complex variables
\[
\overline{v}(z, t) = \frac{1}{2\pi i} \text{PV} \int \frac{1}{z - z(\alpha, t)} \overline{\varpi}(\alpha, t) d\alpha,
\]
for \( z \neq z(\alpha, t) \) allows us to accomplish the fact that
\[
\partial_z \overline{v}(z, t) = \frac{1}{2\pi i} \text{PV} \int \frac{-\overline{\varpi}(\alpha, t)}{(z - z(\alpha, t))^2} d\alpha = \frac{1}{2\pi i} \text{PV} \int \frac{-\partial_\alpha z(\alpha, t)}{(z - z(\alpha, t))^2} \overline{\varpi}(\alpha, t) d\alpha.
\]
Therefore
\[
\partial_z \overline{v}(z, t) = \frac{1}{2\pi i} \text{PV} \int \frac{1}{z - z(\alpha, t)} \partial_\alpha(\overline{\varpi}(\alpha, t)) d\alpha
\]
(2.6)
for a regular parametrization with $\partial_\alpha z(\alpha, t) \neq 0$. In a similar way

$$
\partial^2_\alpha \pi(z, t) = \frac{1}{2\pi} PV \int \frac{1}{z - z(\alpha, t)} \partial_\alpha (\frac{1}{\partial_\alpha z(\alpha, t)}) (\alpha, t) d\alpha. 
$$

These identities allow us to get the values of $\nabla v^j(z(\alpha, t), t)$ and $\nabla^2 v^j(z(\alpha, t), t)$. As for the velocity, the limits are different, but we can compute the values.

To get the above formula for the pressure we take the weak type identity (2.1) with $\eta(x, t) = \nabla \lambda(x, t)$. We can compute then the Laplacian of the pressure in a weak sense due to

$$
\int_0^T \int_{\mathbb{R}^2} p \Delta \lambda dx dt = - \int_0^T \int_{\mathbb{R}^2} v \cdot \nabla \lambda_t dx dt - \int_0^T \int_{\mathbb{R}^2} v \cdot (v \cdot \nabla^2 \lambda) dx dt - \int_{\mathbb{R}^2} v_0(x) \cdot \nabla \lambda(x, 0) dx
$$

Then

$$
I_1 = I_3 = 0
$$

by the incompressible condition. We define

$$
\Omega^1_\varepsilon(t) = \{x \in \Omega^1(t) : \text{dist} (x, \partial \Omega^1(t)) \geq \varepsilon\}
$$

$$
\Omega^2_\varepsilon(t) = \{x \in \Omega^2(t) : \text{dist} (x, \partial \Omega^2(t)) \geq \varepsilon\}.
$$

We decompose as follows $I_2 = J_3 + J_4 + J_5 + J_6$ where

$$
J_3 = - \int_0^T \int_{\mathbb{R}^2} (v_1)^2 \partial^2_{x_1} \lambda dx dt, \quad J_4 = - \int_0^T \int_{\mathbb{R}^2} v_1 v_2 \partial_{x_2} \partial_{x_1} \lambda dx dt,
$$

$$
J_5 = - \int_0^T \int_{\mathbb{R}^2} v_1 v_2 \partial_{x_1} \partial_{x_2} \lambda dx dt, \quad J_6 = - \int_0^T \int_{\mathbb{R}^2} (v_2)^2 \partial^2_{x_2} \lambda dx dt.
$$

Using the sets $\Omega^j_\varepsilon(t)$ and the identity (2.6) we get

$$
J_3 = - \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega^1_\varepsilon(t)} (v_1)^2 \partial^2_{x_1} \lambda dx dt + \int_0^T \int_{\Omega^2_\varepsilon(t)} (v_1)^2 \partial^2_{x_1} \lambda dx dt
$$

$$
= \int_0^T \int_{\mathbb{R}^2} 2 v_1 \partial_{x_1} v_1 \partial_{x_1} \lambda dx dt
$$

$$
+ \int_0^T \int_{-\pi}^{\pi} ((v_1^2(z(\alpha, t), t))^2 - (v_1^1(z(\alpha, t), t))^2) \partial_{x_1} \lambda(z(\alpha, t), t) \partial_\alpha z_2(\alpha, t) d\alpha dt
$$

$$
= K_1 + K_2.
$$

The term $K_1$ trivializes because the subtle integration by parts and the identity (2.7) give

$$
K_1 = - \int_0^T \int_{\mathbb{R}^2} 2(v_1 \partial^2_{x_1} v_1 + (\partial_{x_1} v_1)^2) \lambda dx dt - \int_0^T \int_{-\pi}^{\pi} \tilde{f}(\alpha, t) \lambda(z(\alpha, t), t) d\alpha dt
$$
for $\bar{f}(\alpha, t) = 2(v_2^2(z(\alpha, t), t)\partial_{x_2} v_1(z(\alpha, t), t) - v_1^1(z(\alpha, t), t)\partial_{x_1} v_1(z(\alpha, t), t))\partial_\alpha z_2(\alpha, t)$. The first term in $K_1$ is part of $F(x, t)$ and the second of $f(\alpha, t)$.

We can rewrite $K_2$ as follows

$$K_2 = -2\int_0^T \int_{-\pi}^{\pi} \omega BR_1 \frac{\partial_\alpha z_1}{|\partial_\alpha z|^2} \partial_{x_1} \lambda(z) \partial_\alpha z_2 d\alpha dt.$$  \hspace{1cm} (2.8)

We continue with $J_4$

$$J_4 = \int_0^T \int_{\mathbb{R}^2} (v_2 \partial_{x_2} v_1 + v_1 \partial_{x_2} v_2) \partial_{x_2} \lambda dxdt$$

$$- \int_0^T \int_{-\pi}^{\pi} \left( (v_1^2 v_2^1)(z(\alpha, t), t) - (v_1^1 v_2^1)(z(\alpha, t), t) \right) \partial_{x_1} \lambda(z(\alpha, t), t) \partial_\alpha z_1(\alpha, t) d\alpha dt$$

$$= K_3 + K_4.$$

We deal with the term $K_3$ in a similar way as with $K_1$.

We can rewrite $K_4$ as follows

$$K_4 = -\int_0^T \int_{-\pi}^{\pi} \omega BR_1 \frac{\partial_\alpha z_2}{|\partial_\alpha z|^2} + \omega BR_2 \frac{\partial_\alpha z_1}{|\partial_\alpha z|^2} \partial_{x_1} \lambda(z) \partial_\alpha z_1 d\alpha dt.$$  \hspace{1cm} (2.9)

For $J_5$ we split

$$J_5 = \int_0^T \int_{\mathbb{R}^2} (v_2 \partial_{x_1} v_1 + v_1 \partial_{x_1} v_2) \partial_{x_2} \lambda dxdt$$

$$+ \int_0^T \int_{-\pi}^{\pi} \left( (v_1^2 v_2^1)(z(\alpha, t), t) - (v_1^1 v_2^1)(z(\alpha, t), t) \right) \partial_{x_2} \lambda(z(\alpha, t), t) \partial_\alpha z_2(\alpha, t) d\alpha dt$$

$$= K_5 + K_6.$$

We proceed for $K_5$ in a similar manner as with $K_1$.

We obtain for $K_6$ the following expression

$$K_6 = \int_0^T \int_{-\pi}^{\pi} [\omega BR_1 \frac{\partial_\alpha z_2}{|\partial_\alpha z|^2} + \omega BR_2 \frac{\partial_\alpha z_1}{|\partial_\alpha z|^2}] \partial_{x_2} \lambda(z) \partial_\alpha z_2 d\alpha dt.$$  \hspace{1cm} (2.10)

With $J_6$ one finds

$$J_6 = \int_0^T \int_{\mathbb{R}^2} 2v_2 \partial_{x_2} v_2 \partial_{x_2} \lambda dxdt$$

$$- \int_0^T \int_{-\pi}^{\pi} \left( (v_2^2(z(\alpha, t), t))^2 - (v_2^1(z(\alpha, t), t))^2 \right) \partial_{x_2} \lambda(z(\alpha, t), t) \partial_\alpha z_2(\alpha, t) d\alpha dt$$

$$= K_7 + K_8.$$

For $K_7$ we proceed as before. We obtain for $K_8$ the following expression

$$K_8 = -2\int_0^T \int_{-\pi}^{\pi} \omega BR_2 \frac{\partial_\alpha z_2}{|\partial_\alpha z|^2} \partial_{x_2} \lambda(z) \partial_\alpha z_1 d\alpha dt.$$  \hspace{1cm} (2.11)
We now sum as follows $K_2 + K_4 + K_6 + K_8 = L_2$, then

$$L_2 = - \int_0^T \int_{-\pi}^\pi \frac{\varpi(\alpha, t)}{\partial_{\alpha}z(\alpha, t)}^2 BR(z, \varpi)(\alpha, t) \cdot \partial_{\alpha}z(\alpha, t) \cdot \nabla \lambda(z(\alpha, t), t) d\alpha d\tau.$$

An integration by parts in the variable $\alpha$ in $L_2$ gives the last term of $f(\alpha, t)$. The formula for the Laplacian of $p$ is found.

Next we shall obtain the equation for the curve $z(\alpha, t)$. We start from equation (2.1) with $\eta(x, 0) = 0$, which is

$$\int_0^T \int_{\Omega(t)} [v \cdot (\eta_t + v \cdot \nabla \eta) + p \nabla \cdot \eta] dx dt = 0. \quad (2.12)$$

Again we can split the equation (2.12) in the following way

$$\lim_{\varepsilon \to 0} \left( \int_0^T \int_{\Omega_1^\varepsilon(t)} [v \cdot (\eta_t + v \cdot \nabla \eta) + p \nabla \cdot \eta] dx dt + \int_0^T \int_{\Omega_2^\varepsilon(t)} [v \cdot (\eta_t + v \cdot \nabla \eta) + p \nabla \cdot \eta] dx dt \right) = 0,$$

where $\Omega_1^\varepsilon(t)$ and $\Omega_2^\varepsilon(t)$ have been defined previously. We will study the first terms in detail. Integrating by parts we obtain

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_1^\varepsilon(t)} v \cdot \eta_t dx dt = \int_0^T \int_{\Omega_1^\varepsilon(t)} (v^1 \cdot \eta)(z_t \cdot \partial_{\alpha}z) dx dt - \int_0^T \int_{\Omega_1^\varepsilon(t)} v_t \cdot \eta dx dt.$$

Similarly for the other terms we have

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_2^\varepsilon(t)} v \cdot (v \cdot \nabla) \eta dx dt = - \int_0^T \int_{\Omega_1^\varepsilon(t)} (v^1 \cdot \eta)(v^1 \cdot \partial_{\alpha}z) dx dt - \int_0^T \int_{\Omega_2^\varepsilon(t)} \eta \cdot (v \cdot \nabla) v dx dt.$$

Operating in a similar way with the integral over $\Omega_2^\varepsilon(t)$ yields the following equations

$$\varpi \cdot (\partial_t z - BR(z, \varpi)) \cdot \partial_{\alpha}z = 0, \quad (2.13)$$
$$v_t + (v \cdot \nabla) v = -\nabla p \quad \text{over } \Omega^1 \text{ and } \Omega^2, \quad (2.14)$$

where the derivatives of $v$ on $\partial \Omega^1$ and $\partial \Omega^2$ have to be understood like the limits in the normal direction to the curve $z(\alpha, t)$ and we have used the continuity of the pressure.

Next we close the system giving the evolution equation for the amplitude of the vorticity $\varpi(\alpha, t)$ by means of Bernoulli’s law. Using (2.1) for $x \neq z(\alpha, t)$ we get $v(x, t) = \nabla \phi(x, t)$ where

$$\phi(x, t) = \frac{1}{2\pi} PV \int \arctan \left( \frac{x_2 - z_2(\beta, t)}{x_1 - z_1(\beta, t)} \right) \varpi(\beta, t) d\beta.$$
We define
\[ \Pi(\alpha, t) = \phi^2(z(\alpha, t), t) - \phi^1(z(\alpha, t), t), \]
where again \( \phi^j(z(\alpha, t), t) \) denotes the limit obtained approaching the boundary in the normal direction inside \( \Omega^j \). It is clear
\[ \partial_\alpha \Pi(\alpha, t) = (\nabla \phi^2(z(\alpha, t), t) - \nabla \phi^1(z(\alpha, t), t)) \cdot \partial_\alpha z(\alpha, t) \]
\[ = (v^2(z(\alpha, t), t) - v^1(z(\alpha, t), t)) \cdot \partial_\alpha z(\alpha, t) \]
\[ = \varpi(\alpha, t), \]
therefore
\[ \int \varpi(\alpha, t)d\alpha = 0. \]
Now we can check that
\[ \phi^2(z(\alpha, t), t) = IT(z, \varpi)(\alpha, t) + \frac{1}{2}\Pi(\alpha, t) \]
\[ \phi^1(z(\alpha, t), t) = IT(z, \varpi)(\alpha, t) - \frac{1}{2}\Pi(\alpha, t), \]
where
\[ IT(z, \varpi)(\alpha, t) = \frac{1}{2\pi}PV \int \arctan \left( \frac{z_2(\alpha, t) - z_2(\beta, t)}{z_1(\alpha, t) - z_1(\beta, t)} \right) \varpi(\beta, t)d\beta. \]
Using the Bernoulli’s law in (1.1), inside each domain, we have
\[ \phi_t(x, t) + \frac{1}{2}|v(x, t)|^2 + p(x, t) = 0. \]
Taking the limit it follows
\[ \phi^2_t(z(\alpha, t), t) + \frac{1}{2}|v^2(z(\alpha, t), t)|^2 + p^2(z(\alpha, t), t) = 0, \]
and since \( p^1(z(\alpha, t), t) = p^2(z(\alpha, t), t) \) we get
\[ \phi^2_t(z(\alpha, t), t) - \phi^1_t(z(\alpha, t), t) + \frac{1}{2}|v^2(z(\alpha, t), t)|^2 - \frac{1}{2}|v^1(z(\alpha, t), t)|^2 = 0. \]
Then it is clear that
\[ \phi^2_t(z(\alpha, t), t) = \partial_t(\phi^2(z(\alpha, t), t)) - z_t(\alpha, t) \cdot \nabla \phi^2(z(\alpha, t), t) \]
and using (2.5) together (2.15) in (2.16) we obtain
\[ \Pi_t(\alpha, t) = \varpi(\alpha, t)(z_t(\alpha, t) - BR(z, \varpi)(\alpha, t)) \cdot \frac{\partial_\alpha z(\alpha, t)}{|\partial_\alpha z(\alpha, t)|^2}. \]
Taking one derivative with respect to \( \alpha \) on (2.17) yields equation (1.6).
Finally it is easy to show that the solutions of the system (1.4) and (1.6) provide weak solutions of the Euler’s equation.
Given a curve $z(\alpha, t) \in C^1, \delta$ and a function $\varpi(\alpha, t) \in C^1, \delta$ such that the equations (1.4) and (1.6) are satisfied, we define the velocity $v(x, t)$ by the expression (2.4) and the pressure by
$$p(x, t) = -\phi_t(x, t) - \frac{1}{2} |v(x, t)|^2$$
over $\Omega^1$ and $\Omega^2$, where the potential $\phi(x, t)$ is given by $v = \nabla \phi$. From equation (1.6) we have that the pressure is continuous over the vortex sheet. In order to check that $v(x, t)$ and $p(x, t)$ are weak solutions of Euler’s equations we just have to introduce them in the first member of (2.1) and (2.2) and integrate by parts.

3 Ill-posedness for the amplitude equation.

In this section we choose the tangential term $c(\alpha, t) = \frac{1}{2} H \varpi(\alpha, t)$ which gives the following closed equation for the amplitude of the vorticity
$$\varpi_t - \frac{1}{2} (\varpi H \varpi)_\sigma = 0,$$
(3.1)
$$\varpi(\sigma, 0) = \varpi_0(\sigma).$$
(3.2)

We shall prove the following theorem:

**Theorem 3.1** Let $\varpi_0 \in H^s(\mathbb{T})$ with $s > \frac{3}{2}$ and
$$\int_T \varpi_0 = 0.$$

Then if there exist a point $\sigma_0$ where $\varpi_0(\sigma_0) > 0$ and $\varpi_0$ is not $C^\infty$ in $\sigma_0$, there is no solution of equation (3.1) in the class $C([0, T); H^s(\mathbb{T}))$ with $s > \frac{3}{2}$ and $T > 0$. In addition, $\varpi_0 \in C^\infty$ is not sufficient to obtain existence.

**Remark 3.2** In the case of the real line $\mathbb{R}$ equation (3.1) is also ill-posed, in $H^s$ with $s > 3/2$, for a non analytic initial. For more details see [5].

**Proof:** We will proceed by a contradiction argument.

Let us assume that there exist a solution of equation (3.1) in the class $C([0, T), H^s(\mathbb{T}))$ with $\varpi(\sigma, 0) = \varpi_0(\sigma)$.

First we have to note that if the initial data, $\varpi_0$, is of mean zero the solution $\varpi$ will remain of mean zero.

Now, taking the Hilbert transform on equation (3.1) yields
$$\partial_t H \varpi - \frac{1}{2} (H \varpi H \varpi_\sigma - \varpi \varpi_\sigma) = 0,$$
where we have used the following properties of the Hilbert transform for a periodic function with mean zero:
- $H(H \varpi) = -\varpi$. 

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We denote the complex valued function $z(\sigma, t) = H \varpi(\sigma, t) - i \varpi(\sigma, t)$ which satisfies
\[ \partial_t z - \frac{1}{2} zz_{\sigma} = 0. \] (3.3)

Take $P_\sigma(u)$ to be the Green's function of the Laplacian for the Dirichlet problem in the unit ball
\[ P_\sigma(u) \equiv \frac{1}{2\pi} \frac{1 - |u|^2}{|u - \sigma|^2}, \]
and $P \varpi(u)$ will be
\[ P \varpi(u) \equiv \int_{\partial B(0,1)} P_\sigma(u) \varpi(\sigma) d\sigma. \]

Therefore
\[ Z(u) = P(H \varpi - i \varpi)(u) \quad \text{with} \quad u = re^{i\sigma}, \]
is an analytic function on the unit ball. Applying $P$ to the equation (3.3) yields
\[ \partial_t Pz = \frac{1}{2} P(zz_{\sigma}), \]
where we can write the second term in the following way
\[ P(zz_{\sigma}) = Pz(Pz)_\sigma, \]
since both terms have the same restriction to the boundary of the unit ball and both are harmonic.

Thus, we have for $Z(u, t)$ the equation
\[ Z_t - \frac{1}{2} ZZ_{\sigma} = 0 \quad \text{on} \quad u \in B(0,1), \]
hence
\[ Z_t - \frac{1}{2} iu ZZ_u = 0 \quad \text{on} \quad u \in B(0,1), \] (3.4)
\[ Z(u, 0) = Z_0(u) = P(H \varpi_0 - i \varpi_0)(x). \] (3.5)

We will define the complex trajectories $X(u, t)$ by
\[ \frac{dX(u, t)}{dt} = -\frac{1}{2} iX(u, t)Z(X(u, t), t), \]
\[ X(u, 0) = u, \quad u \in B(0,1). \]

For sufficiently small $t$, by Picard's Theorem, these trajectories exist and $X(u, t) \in B(0,1)$. Therefore
\[ \frac{dZ(X(u, t), t)}{dt} = \partial_t Z(X(u, t), t) - \frac{1}{2} iX(u, t)Z(X(u, t), t)Z_u(X(u, t), t) = 0. \]

\[ \bullet \quad H(\varpi H \varpi) = \frac{1}{2}((H \varpi)^2 - \varpi^2). \]
Thus, we have
\[ Z(X(u, t), t) = Z_0(u), \]
and
\[ \frac{dX(u, t)}{dt} = -\frac{1}{2} iX(u, t)Z_0(u). \]
Moreover
\[ X(u, t) = u e^{-\frac{1}{2} iZ_0(u)t}. \]
Taking modules in the last expression we obtain
\[ R(u, t) = |X(u, t)| = re^{-\frac{1}{2} P\overline{w}_0(re^{i\sigma})t}. \]
If we consider a point \[ e^{i\sigma_0} = u_0 \in \partial B(0, 1) \text{ with } \overline{w}_0(\sigma_0) > 0, \text{ then} \]
\[ R(u_0, t) = e^{-\frac{1}{2} u_0(\sigma_0)t} < 1. \]
Hence \[ X(u_0, t) \in B(0, 1) \text{ for all } t > 0, \text{ and a continuity argument yields} \]
\[ Z(X(\sigma_0, t), t) = z_0(\sigma_0) = H \overline{w}_0(\sigma_0) - i\overline{w}_0(\sigma_0), \]
where to simplify we denote \[ X(u_0, t) = X(\sigma_0, t). \] Then we have
\[ X(\sigma_0, t) = e^{i(\sigma_0 - \frac{1}{2} z_0(\sigma_0)t)}. \]
Taking a derivative with respect to \[ \sigma_0 \] on this equation we find that
\[ \frac{dX(\sigma_0, t)}{d\sigma_0} = i(1 - \frac{1}{2} z_0(\sigma_0)t)X(\sigma_0, t). \]
With the chain’s rule we obtain
\[ \frac{dZ}{dX}(X(\sigma_0, t), t) iX(\sigma_0, t) = \frac{dZ}{d\Theta}(X(\sigma_0, t), t) = \frac{z_0(\sigma_0)}{(1 - \frac{1}{2} z_0(\sigma_0)t)}, \]
where
\[ X(\sigma_0, t) = R(\sigma_0, t)e^{i\Theta(\sigma_0, t)}. \]
Taking two derivatives
\[ \frac{d^2 Z}{d\Theta^2}(X(\sigma_0, t), t) = \frac{z_0(\sigma_0)}{(1 - \frac{1}{2} z_0(\sigma_0)t)^3}. \]
For the n-th derivative we have
\[ \frac{d^n Z}{d\Theta^n}(X(\sigma_0, t), t) = \frac{d^n z_0}{d\sigma^n}(\sigma_0)}{(1 - \frac{1}{2} z_0(\sigma_0)t)^{n+1}} + \text{“lower terms”}. \]
We observe that \((1 - \frac{1}{2} z_0(\sigma_0)t) \neq 0 \text{ for } t \text{ small enough.} \)
Then if \(w_0\) is not \(C^\infty\) in \(\sigma_0\) this is a contradiction since \(Z(u, t)\) is analytic on \(X(\sigma_0, t)\) for all \(t > 0.\)
In addition, if \( \pi_0(\sigma) > 0 \) and \( \frac{d^n \pi_0}{d\sigma^n}(\sigma_0) = 0 \) \( \forall n \) but \( \pi_0 \) is not constant on any neighborhood of \( \sigma_0 \), we can conclude
\[
\frac{d^3 \Im Z}{d\Theta}(X^1(\sigma_0, t), X^2(\sigma_0, t)) = 0.
\]
Continuing this process we obtain that all derivatives satisfy
\[
\frac{d^n \Im Z}{d\Theta^n}(X^1(\sigma_0, t), X^2(\sigma_0, t)) = 0.
\]
The imaginary part \( \Im Z(x_1, x_2, t) \) is analytic on \((x_1, x_2) = (X^1(\sigma_0, t), X^2(\sigma_0, t))\) for all \( t > 0 \), thus \( \Im Z(x_1, x_2) \) is constant over the circumference, \( R = R(\sigma_0, t) \), and this is a contradiction if \( \pi_0 \) is not constant.

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