Measures for orthogonal polynomials with unbounded recurrence coefficients

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August 28, 2014

Abstract

Systems of orthogonal polynomials whose recurrence coefficients tend to infinity are considered. A summability condition is imposed on the coefficients and the consequences for the measure of orthogonality are discussed. Also discussed are asymptotics for the polynomials.

Keywords: Orthogonal polynomials, unbounded recurrence coefficients, measures.

Mathematics Subject Classification Numbers: 42C05, 41A60.

1 Introduction

Let \( \{p_n(x)\}_{n=-1}^{\infty} \) be a system of polynomials satisfying the recurrence relations

\[
a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x) = x p_n(x), \quad p_0(x) = 1, \quad p_{-1}(x) = 0, \quad (1)
\]
with \( a_{n+1} > 0 \) and \( b_n \) real for \( n = 0, 1, \ldots \). By a theorem of Farvard these polynomials are orthonormal with respect to some positive probability measure supported on the real line. There has been much work done on the asymptotics or spectral properties of polynomials whose recurrence coefficients are unbounded \([5, 6, 11, 17]\) and here we are interested in the problem of constructing the orthogonality measure given the coefficients in the recurrence formula. This problem for the bounded case has been an area of ongoing intense investigation due to its connection to the discrete Schrödinger equation \([2, 4, 7, 10, 12, 13, 15, 16, 18, 19, 20]\) and to the connection between the continuum limits of the recurrence relations with varying recurrence coefficients and discrete integrable systems \([3, 13]\). However with regard to the construction of the measure of orthogonality almost all the results are for cases with bounded recurrence coefficients. Unfortunately many of the techniques developed for the bounded case cannot be applied to the unbounded case. Careful analysis of the work of Máté-Nevai \([15]\), Máté-Nevai-Totik \([16]\), and VanAssche-Geronimo \([20]\) for coefficients that are of bounded variation i.e.

\[
\sum_{n=1}^{\infty} |a_{n+1} - a_n| + |b_n - b_{n-1}| < \infty
\]

with limits \( a_n \to \frac{1}{2} \) and \( b_n \to 0 \) reveals that it is possible to modify these techniques so that they apply to certain cases when the recurrence coefficients tend to infinity. For bounded recurrence coefficients obeying the above criteria the absolutely continuous part of the orthogonality measure is given by

\[
du_{ac} = \frac{2}{\pi} \frac{\sqrt{1-x^2} dx}{|\xi(x)|^2 \prod_{n=1}^{\infty} |\zeta_n|^2(x)},
\]

where

\[
\zeta_n(x) := \frac{x - b_n + \sqrt{(x - b_n)^2 - 4a_n^2}}{2a_n}
\]

is the mapping function \( z = \zeta_n(x) \) of \( \mathbb{C}\backslash(\alpha_n, \beta_n) \) on \( \mathbb{C}\backslash\{|z| \leq 1\} \) normalized as \( \zeta(x)/x > 0 \) when \( x \to \infty \), and for \( \xi(x) \) we have the expressions

\[
\xi(x) = 1 + \sum_{k=1}^{\infty} \left\{ \frac{1}{\zeta_k(x)} - \frac{a_k/a_{k+1}}{\zeta_{k+1}(x)} \right\} \prod_{j=1}^{k} \frac{p_{k-1}(x)}{\zeta_j(x)},
\]

with \( a_{n+1} > 0 \) and \( b_n \) real for \( n = 0, 1, \ldots \).
or
\[
\xi(x) = \lim_{n \to \infty} \frac{p_n(x) - \frac{b_n}{b_{n+1}} p_{n-1}(x) / \zeta_{n+1}(x)}{\prod_{j=1}^{n} \zeta_j(x)},
\]

An extension of this formula was used in [1] to show the connection between the limit of varying recurrence coefficients and their corresponding orthogonality measure and an analog of this formula for the unbounded case is necessary in order to extend the above results to the unbounded case. This is done in the next section.

2 Unbounded Coefficients

We now consider the case when \(a_n > 0, n > 0\) and \(b_n\) real for \(n \geq 0\) tend in magnitude to infinity when \(n\) tends to infinity i.e.

\[
\lim_{n \to \infty} a_n = \infty \quad \text{and} \quad \lim_{n \to \infty} |b_n| = \infty.
\]

Let \(\rho(z) = z + \sqrt{z^2 - 1}\), \(t_{1,n} = \sqrt{\frac{a_n}{a_{n+1}}} \rho \left( \frac{x-b_n}{2\sqrt{a_n a_{n+1}}} \right)\), and \(t_{2,n} = \sqrt{\frac{a_n}{a_{n+1}}} \rho \left( \frac{x-b_n}{2\sqrt{a_n a_{n+1}}} \right)^{-1}\). We also set \(t_{i,j} = 1, i = 1, 2, j = -1, 0\). Since \(\rho\) maps the exterior of \([-1, 1]\) to the exterior of the unit circle these functions are nonzero. Let

\[
\phi_n(x) = p_n(x) - t_{2,n}p_{n-1}(x), \quad (3)
\]

and

\[
\phi^1_n(z) = p^1_n(x) - t_{2,n}p^1_{n-1}(x)
\]

where \(p^1_j, j = 1, 2, \ldots\) are the second kind polynomials of degree \(j - 1\) (we take \(p^1_0 = 0\)). Using the recurrence formula for the orthogonal polynomials and (3) yields

\[
\phi_{n+1} - t_{1,n} \phi_n = (t_{2,n} - t_{2,n+1}) p_n. \quad (4)
\]

We consider the systems of recurrence coefficients given by \(\{a_{n_0}^0, b_{n-1}^0\}_{n=1}^\infty, n_0 = 1, 2, \ldots\)

where

\[
a_{i}^{n_0} = \begin{cases} 
a_i & i < n_0 \\
a_{n_0} & i \geq n_0
\end{cases}
\]

and

\[
b_{i}^{n_0} = \begin{cases} 
b_i & i < n_0 \\
b_{n_0} & i \geq n_0
\end{cases}
\]
Let \( p_i^{n_0}(x) \) be the polynomials constructed satisfying
\[
a_{i+1}^{n_0} p_{i+1}^{n_0}(x) + b_i^{n_0} p_i^{n_0}(x) + a_i^{n_0} p_{i-1}^{n_0}(x) = x p_i^{n_0}(x)
\]  
\( p_{-1}^{n_0}(x) = 0, p_0^{n_0}(x) = 1. \) Set \( \phi_i^{n_0} = p_i^{n_0} - \rho(x-b_{n_0})^{-1} p_{i-1}^{n_0} \) and \( \phi_i^{n_0,1} = p_i^{n_0,1} - \rho(x-b_{n_0})^{-1} p_{i-1}^{n_0,1}. \) Then it is well known Geronimo and Case [7], Geronimo and Iliev [8], Germonius [10], Nevai [18] that the above polynomials are orthonormal with respect to a measure \( \mu^{n_0} \) where \( \mu^{n_0} \) is given by
\[
d\mu^{n_0}(x) = \begin{cases} \\
\frac{\sqrt{4a_n^{n_0}}}{2} \left| \phi_i^{n_0}(x) \right|^2 dx & x \in [b_{n_0} - 2a_{n_0}, b_{n_0} + 2a_{n_0}] \\
\sum_{i=1}^{m(n_i)} \mu_i^{n_0} \delta(x - x_i^{n_0})dx & x \text{ elsewhere},
\end{cases}
\]
where
\[
\mu_i^{n_0} = \frac{\phi_i^{n_0,1}(x_i^{n_0})}{\phi_i^{n_0}(x_i^{n_0})}.
\]
Here \( x_i \) is a zero of \( \phi_i^{n_0} \) for \( x \in \mathbb{R} \setminus \{a_{n_0} - 2b_{n_0}, a_{n_0} + 2b_{n_0}\} \) all of which are real and simple.
The following Lemma is found in [8] and has a simple proof.

**Lemma 1.** Suppose that the moment problem is determinate then \( \mu^{n_0}(x) \) converge weakly to \( \mu(x) \) where \( \mu(x) \) is the orthogonality measure associated with the polynomials \( \{p_i(x)\}_{i=-1}^{\infty}. \)

**Proof** It follows from equation (5) that the polynomials \( p_i^{n_0}(x) = p_i(x) \) for \( 0 \leq i \leq n_0 \) so that the moments of the original system \( s_i \) and the perturbed system \( s_i^{n_0} \) agree for \( 0 \leq i \leq 2n_0. \) The result now follows from the fact that the moment problem is determinate.

We now consider in more detail the properties of \( \mu. \) Set \( \mathbb{C}^+ = \{x \in \mathbb{C} : \Im x \geq 0\} \)

**Theorem 1.** Suppose \( \lim_{n \to \infty} a_n = \infty, \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 1, \lim_{n \to \infty} \left( \frac{b_n}{2a_{n+1}} \right) = d \neq \pm 1, \sum_{n=1}^{\infty} \frac{a_n}{a_{n+1}} - \frac{a_{n+1}}{a_{n+2}} < \infty, \sum_{n=1}^{\infty} \frac{1}{a_{n+1}} - \frac{1}{a_n} < \infty, \) and \( \sum_{n=0}^{\infty} \frac{b_n}{a_{n+1}} - \frac{b_{n+1}}{a_{n+2}} < \infty. \) Then
\[
\lim_{n \to \infty} \frac{\phi_n(x)}{\prod_{i=1}^{n-1} t_{1,i}} = g(x),
\]  
where the convergence is uniform on compact subsets of \( \mathbb{C}^+. \) Thus \( g \) is continuous for \( \Im x \geq 0 \) and analytic for \( \Im x > 0. \) The same holds for \( \lim_{n \to \infty} \frac{\phi^{1}_n(x)}{\prod_{i=1}^{n-1} t_{1,i}} = g^1(x) \).
Proof. Let $K$ be a compact subset of $\mathbb{C}^+$. It follows from the first three hypotheses that for every $x$ in $K$ there is an $k_0$ such that for $k \geq k_0$,

$$\frac{a_k}{a_{k+1}} \sqrt{\left(\frac{x - b_k}{2a_{k+1}}\right)^2} - \frac{a_k}{2a_{k+1}} \neq 0 = \frac{a_k}{a_{k+1}} \sqrt{\left(\frac{x - b_{k-1}}{2a_{k-1}}\right)^2} - \frac{a_k}{a_{k-1}} + \sqrt{\left(\frac{x - b_{k+1}}{2a_{k+2}}\right)^2} - \frac{a_{k+1}}{a_{k+2}} \quad (7)$$

Consequently

$$|t_{1,k+1} - \frac{a_k}{a_{k+1}} t_{2,k-1}^{-1}| \leq O(\epsilon_k + \epsilon_{k-1} + \epsilon_{k-2})$$

where

$$\epsilon_i = \left| \frac{1}{a_{i+1}} - \frac{1}{a_{i+2}} \right| + \left| \frac{a_i}{a_{i+1}} - \frac{a_{i+1}}{a_{i+2}} \right| + \left| \frac{b_i}{a_{i+1}} - \frac{b_{i+1}}{a_{i+2}} \right|,$$

for $i > 0$.

Let

$$G(k, m) = \begin{cases} \left( \prod_{i=m+1}^{k} t_{1,i} \right) \left( \prod_{j=k+1}^{m} t_{2,j} \right) & k \geq m \\ \sum_{i=m+1}^{k} t_{1,i+1} & k < m, \end{cases} \quad (8)$$

and $\hat{G}(k, m) = \prod_{i=k}^{m} t_{1,i}^{-1} G(k, m)$ for $k > 0$. Then from [20] p. 228,

$$\left( \frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} t_{2,k} \right) \hat{G}(k, m) = -1 + \frac{a_{k-1}}{a_k} t_{1,m-1} t_{1,m} R(k, m)$$

$$+ \sum_{i=k+1}^{m-1} t_{2,i}^{-1} \left\{ t_{1,i+1} - \frac{a_i}{a_{i+1}} t_{2,i}^{-1} \right\} R(k, i) \hat{G}(i, m) \quad (9)$$

where $R(k, m) = \prod_{j=k+1}^{m-1} t_{2,j-1}/t_{1,j}$. Since $\sqrt{\frac{a_{i+1}}{a_i}} |t_{2,j}| \leq 1 \leq \sqrt{\frac{a_{i+1}}{a_i}} |t_{1,j}|$ and

$$\lim_{k \to \infty} \left( \frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} t_{2,k+1} \right) = 1 - \rho(-d)^2$$

we see that for all $x \in K$ there is a constant $c$ and a $k_0$ such that $|R(k, m)| < c$ for $k \geq k_0$. Also for $k_0$ large enough there is an $r > 0$ such that $|\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} t_{1,k}| > r$ for all $k \geq k_0$. Thus Picard iteration gives $|\hat{G}(k, m)| \leq c \exp \{D \sum_{i=k}^{m} \epsilon_i\}$ for $k \geq k_0$.

Since $G(k, m)$ satisfies the relation

$$\frac{a_{k+1}}{a_{k+2}} G(k + 1, m) + G(k - 1, m) - (t_{1,k+1} + t_{2,k}) G(k, m) = \delta_{k,m}$$

[20 eq. (3.2)] after multiplication by $\prod_{i=k}^{m} t_{1,i}^{-1}$ this relation allows the computation of $\hat{G}(i, m), i = -1, 0$ so the above bound may be extended by induction to $-1 \leq k < k_0$. 

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The polynomials \( \{p_n(x)\}_{m=0}^{\infty} \) satisfy the relation [ eq. (3.5)]

\[
\hat{p}_m(x) = \hat{G}(-1, m) + \sum_{k=0}^{m-1} t_{1,k}(t_{1,k} - t_{1,k+1})\hat{p}_k(x)\hat{G}(k, m),
\]

where \( \hat{\phi}_n(z) = \left(\prod_{i=1}^{n-1} t_i^{-1}\right)\phi_n(z) \). Using the above bound on \( G(k, m) \) we can use Picard iteration yields ( [ eq. (3.10)])

\[
|\hat{p}_m(x)| \leq A \exp\left\{ B \sum_{k=1}^{m} |t_{1,k} - t_{1,k+1}| \right\} \leq A \exp\left\{ \tilde{B} \sum_{k=1}^{m} \epsilon_k \right\},
\]

where the constant \( A, B \) and \( \tilde{B} \) may be chosen uniformly for \( x \in K \). Finally from equation (4) we find

\[
\hat{\phi}_{n+1}(z) - \hat{\phi}_n(z) = (t_{2,n} - t_{2,n+1})\hat{p}_n(z),
\]

where \( \hat{\phi}_n(z) = \left(\prod_{i=1}^{n-1} t_i^{-1}\right)\phi_n(z) \). Thus \( \hat{\phi}_n(z) - \hat{\phi}_m(z) = \sum_{i=m}^{n-1} (t_{2,i} - t_{2,i+1})\hat{p}_j(z) \). The bound (11) can now be used to obtain

\[
|\hat{\phi}_n(z) - \hat{\phi}_m(z)| \leq O \left( \sum_{i=m}^{n} \epsilon_i \right).
\]

Thus \( \{\hat{\phi}_n(z)\} \) is a Cauchy sequence in every compact subset \( K \subset \mathbb{C}^+ \) which gives the uniform convergence. Since each \( t_{i,n}, \ i = 1, 2 \ n = 1, 2, \ldots \) is continuous for \( \Re x \geq 0 \) and analytic for \( \Re x > 0 \) the continuity and analyticity properties of \( g \) follow. An analogous argument gives the result for \( \hat{\phi}_n^1 \).

**Lemma 2.** Suppose the hypothesis of Theorem 1 hold with \(|d| > 1\). Then for each compact set \( K \subset \mathbb{C} \) there exits and \( N \) such that \( \prod_{i=1}^{N} t_{1,i}g(x) \) is analytic for \( x \in K \). The same is true for \( \prod_{i=2}^{N} t_{1,i}g^1(x) \).

**Proof.** If \( d > 1 \) then for each compact set \( K \subset \mathbb{C} \) there exists an \( N \) such for every \( x \in K, \ \Re(x) < b_N - 2a_N \). Thus for \( n \geq N, \ t_{i,n} \) are analytic for \( x \in K \) so the result follows from Theorem 1. A similar argument can be used for \( d < 1 \). The result for \( \prod_{i=2}^{N} t_{1,i}g^1(x) \) follows as above.
Theorem 2. Suppose the hypotheses of Theorem 1 hold and \( d \neq \pm 1 \). Suppose that the moment problem associated with the recurrence coefficients \( \{a_{n+1}, b_n\}_{n=0}^\infty \) is determinate. If \( |d| < 1 \) then \( \mu(x) \) is absolutely continuous and

\[
\mu'(x) = \frac{1}{a_1 \pi} \frac{\sqrt{1 - d^2}}{|g(x)|^2 \prod_{i=1}^\infty |\tilde{t}_{1,i}|^2}, \quad x \in \mathbb{R}
\]

If \( |d| > 1 \) then \( \mu \) is purely discrete with,

\[
d\mu = \sum_{i=0}^\infty \mu_i \delta(x - x_i) dx,
\]

where

\[
\mu_i = \frac{g^1(x_i)}{t_{1,1}(x_i)} \frac{d\phi(x_i)}{dx},
\]

and \( x_i \) are the real zeros of \( g(x) \). Here \( \tilde{t}_{1,n} = \rho \left( \frac{x - b_{2n}}{2a_{2n}} \right) \) and \( t_{1,1}(x_i) = \lim_{y \to 0} t_{1,1}(x_i + iy) \).

Proof. From the definition of \( \phi^{n_0}_{n_0} \) we see that \( \phi^{n_0}_{n_0} = \phi^{n_0}_{n_0} + (\rho(x - b_{2n})^{-1} - t_{2,n})p_{n-1} \). Thus Theorem 1 implies that \( \phi^{n_0}_{n_0} = \prod_{i=1}^{n_0-1} t_{1,i} \phi^{n_0}_{n_0} \) converges uniformly on compact subsets of \( \mathbb{C}^+ \) to \( g \). For \( |d| < 1 \) it follows that for each compact subset \( K \subset R \) there exists an \( N \), such that for all \( n > N, \tilde{t}_{1,n} = 1 \). Since for large enough \( n_0, K \subset [b(n_0) - 2a(n_0), b(n_0) + 2a(n_0)] \) we see that if \( n_0 \) is also greater than \( N \)

\[
d\mu_{ac}^{n_0}(x) = \frac{1}{\pi a_1 |\phi^{n_0}_{n_0}(x)|^2 \prod_{i=1}^{n_0-1} |\tilde{t}_{1,i}|^2 \sum_{i=1}^{n_0-1} |\tilde{t}_{1,i}|^2} \frac{\sqrt{1 - \left( \frac{x - b_{2n}}{2a_{2n}} \right)^2}}{dx}.
\]

Now from Theorem 1 and Lemma 1b

\[
\lim_{n_0 \to \infty} \frac{1}{\pi a_1 |\phi^{n_0}_{n_0}(x)|^2 \prod_{i=1}^{n_0-1} |\tilde{t}_{1,i}|^2} \frac{\sqrt{1 - \left( \frac{x - b_{2n}}{2a_{2n}} \right)^2}}{dx} = \frac{1}{\pi a_1 |g(x)| \prod_{i=1}^\infty |\tilde{t}_{1,i}|^2} \frac{\sqrt{1 - d^2}}{dx}
\]

where the convergence is uniform on compact subsets of \( R \). Note that \( \prod_{i=1}^\infty |\tilde{t}_{1,i}|^2 = \prod_{i=1}^{N} |\tilde{t}_{1,i}|^2 \) for \( x \in K \) which gives the result for \( |d| < 1 \). We now consider \( |d| > 1 \). Since the zeros of \( \phi^{n_0}_{n_0} \) are real and simple it follows from Theorem 1, Lemma 2, and Rouche’s theorem that the
zeros of $g$ are real and simple. The weak convergence of $\mu^{n_0}$ shows that $\mu$ is purely discrete and from Theorem 1 and Lemma 2 we find

$$\lim_{n_0 \to \infty} \mu^{n_0}_i = \lim_{n_0 \to \infty} \frac{\phi^{1,n_0}_n(x^{n_0}_i)}{dx} = \lim_{n_0 \to \infty} \frac{\hat{\phi}^{1,n_0}_n(x^{n_0}_i)}{t_{1,1}(x^{n_0}_i) \frac{d\phi^{n_0}_n(x^{n_0}_i)}{dx}} = \frac{g^1(x_i)}{t_{1,1}(x_i) \frac{dg(x_i)}{dx}},$$

which gives the result. \hfill \Box

The above results allow us to obtain asymptotics for the orthogonal polynomials in an analogous manner to some of those obtained in Máté-Nevai-Totik [16].

**Theorem 3.** Suppose the hypotheses of Theorem 1 hold. If $|d| < 1$ and $K$ is a compact subset of the real line then there exist an $N$ so that

$$\sqrt{a_{n+1}} \sqrt{1 - \left( \frac{x - b_{n+1}}{2\sqrt{a_{n+1}}a_{n+1}} \right)^2} \mu'(x)p_n(x) = \sqrt{\frac{1-d^2}{\pi}} \sin \left( \sum_{k=1}^{n} \arg t_{1,k} + \arg g(x) \right)$$

$$+ O\left( \sum_{k=n+1}^{\infty} \epsilon_k \right),$$

for all $n \geq N$ uniformly for $x \in K$. If $|d| > 1$ or $|d| < 1$ and $K$ is a compact subset of $\Re x > 0$ then

$$\frac{p_n(x) - t_{2,n}p_{n-1}(x)}{\prod_{i=1}^{n-1} t_{1,i}} = g(x) + O\left( \sum_{k=n}^{\infty} \epsilon_k \right),$$

uniformly for $x \in K$. For $|d| > 1$ this is also true for $K \subset \mathbb{C} \setminus \text{supp}(\mu)$.

**Proof.** Multiplication of equation (3) by $\frac{1}{\prod_{i=1}^{n-1} t_{1,i}}$ then using equation (13) and Theorem 1 yields

$$\frac{p_n(x) - t_{2,n}p_{n-1}(x)}{\prod_{i=1}^{n-1} t_{1,i}} = g(x) + O\left( \sum_{k=n}^{\infty} \epsilon_k \right).$$

This gives the result for $K \subset \Re x > 0$. For $|d| > 1$ and $K \subset \mathbb{C} \setminus \text{supp}(\mu)$ the result follows from Lemma 2. If $K$ is a compact subset of the real line and $|d| < 1$ then for large enough $n$, $K \subset [b_n - 2a_n, b_n + 2a_n]$ so taking the imaginary part of the above equation yields

$$\sqrt{\frac{a_n}{a_1}} \sqrt{1 - \left( \frac{x - b_n}{2\sqrt{a_{n+1}}a_n} \right)^2} p_{n-1}(x) = \left| g(x) \right| \prod_{i=1}^{n-1} |t_{1,i}| \sin \left( \sum_{k=1}^{n-1} \arg t_{1,k} + \arg g(x) \right)$$

$$+ O\left( \sum_{k=n}^{\infty} \epsilon_k \right).$$

The use of Theorem 2 implies equation (14). \hfill \Box
3 Acknowledgements

A.I. Aptekarev was supported by grant RScF-14-21-00025. He would also like to thank the School of Mathematics at GT for its hospitality (Fall 2004) when part of this work was completed. J.S. Geronimo was partially supported by Simons Foundation Grant 210169. He would also like to thank the JLU-GT Joint Institute for Theoretical Sciences for their hospitality where some of the work was carried out.

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