On the four-term relation on Khovanov homology

Noboru Ito and Jun Yoshida

February 28, 2023

Contents

1 Introduction 1

2 Review on crossing-change on Khovanov homology 5
   2.1 Definitions and notations . . . . . . . . . . . . . . . . . . . . . . . . . . . 6
   2.2 The homotopy equivalence for $R_{III}$ . . . . . . . . . . . . . . . . . . . . . . 8
   2.3 The compatibility of $\hat{\Phi}$ with $R_{III}$ . . . . . . . . . . . . . . . . . . . . . . 10

3 Proof of Main Theorem A 14

4 Proof of Main Theorem B 17

1 Introduction

The goal of this paper is to categorify Kontsevich’s $4T$ relation on Vassiliev derivatives of Khovanov homology.

Every knot invariant is extended to singular knots by the Vassiliev skein relation ([3, 4, 1]):

$$v^{(r+1)}\begin{array}{c} \begin{array}{c} \vspace{-1mm} \hspace{-1mm}  \vspace{1mm} \hspace{-1mm} \end{array} \end{array} \begin{array}{c} \begin{array}{c} \vspace{-1mm} \hspace{-1mm}  \vspace{1mm} \hspace{-1mm} \end{array} \end{array} = v^{(r)}\begin{array}{c} \begin{array}{c} \vspace{-1mm} \hspace{-1mm}  \vspace{1mm} \hspace{-1mm} \end{array} \end{array} - v^{(r)}\begin{array}{c} \begin{array}{c} \vspace{-1mm} \hspace{-1mm}  \vspace{1mm} \hspace{-1mm} \end{array} \end{array} \right).$$

(1.1)

Shirokova and Webster [8] studied a categorified Vassiliev skein relation on Khovanov-Rozansky homology by computing Ext in a derived category. In the case of Khovanov homology, the authors [5, 6] found an explicit description as in the following theorem.

Theorem 1.1 ([6 Main Theorems A, B]). There is a degree-preserving morphism

$$\hat{\Phi} : \begin{array}{c} \begin{array}{c} \vspace{-1mm} \hspace{-1mm}  \vspace{1mm} \hspace{-1mm} \end{array} \end{array} \rightarrow \begin{array}{c} \begin{array}{c} \vspace{-1mm} \hspace{-1mm}  \vspace{1mm} \hspace{-1mm} \end{array} \end{array},$$

(1.2)
of complexes in the category of cobordisms $\text{Cob}_x^\ell$ in [2] which is compatible with Reidemeister moves in the following sense:

1. the diagrams below commute:

   \[ \Phi \circ \Phi^{-1} = \text{id} \]

2. there are degree $-1$ morphisms $\psi_O$ and $\psi_U$ which make the following diagrams chain homotopy commutative:

   \[ \Phi \circ \Phi^{-1} = \text{id} \]

where the morphisms $R_{\text{III}}^{\pm}$ are constructed using the “Kauffman trick” with respect to the crossings $c$.

Taking the mapping cone of the morphism $\widehat{\Phi}$, we extend Khovanov homology to singular links by

\[ \text{Cone}(\Phi) \cong \text{Cone}(\Phi^{-1}) \]

which is none other than a categorified version of Vassiliev skein relation (1.1). Indeed, the compatibility in Theorem 1.1 implies the invariance under the following moves:

\[ \Phi \circ \Phi^{-1} = \text{id} \]

In the case of polynomial knot invariants, say $v$, its Vassiliev derivatives satisfy the
Figure 1.1: The FI relation in Vassiliev theory

relations

\[ v^{(r)}(\begin{array}{c} \vdots \\ \vdots \end{array}) = 0, \quad (1.5) \]
\[ v^{(r)}(\begin{array}{c} \vdots \\ \vdots \end{array}) - v^{(r)}(\begin{array}{c} \vdots \\ \vdots \end{array}) = v^{(r)}(\begin{array}{c} \vdots \\ \vdots \end{array}) - v^{(r)}(\begin{array}{c} \vdots \\ \vdots \end{array}), \quad (1.6) \]

which are respectively called the FI and the 4T relations. We next aim at finding categorified analogues of them. We note that, besides the obvious proof of these relations, they are also understood in view of the homotopy theory of the space \( M \) of immersions \( S^1 \to \mathbb{R}^3 \). Vassiliev [9] introduced a stratification \( M = \bigcup_i M_i \) such that

- \( M_0 \) consists of smooth embeddings;
- \( M_1 \) consists of smooth immersions with exactly one double point;
- \( M_2 \) consists of
  (a) smooth injections with exactly one critical point and
  (b) smooth immersion with exactly two double points;
- \( M_3 \) contains immersions with triple points.

In this point of view, a crossing change is thought of as a “wall-crossing” passing through the codimension 1 stratum \( M_1 \). Hence, the FI relation (1.5) corresponds to the monodromy around the stratum \( M_2 \) as in Fig. 1.1. Similarly, the 4T relation (1.6) corresponds to the obstruction dual to the hexagon prism surrounding the triple point singularity in \( M_3 \) as in Fig. 1.2. In fact, the 4T relation (1.6) is proved by taking the alternating sum of the twelve diagrams in Fig. 1.2. Kontsevich [7] showed that the FI and the 4T relation are essential for finite type invariants: for the space \( V_m \) of the order \( m \) Vassiliev invariants, he gave an explicit description of the quotient \( V_m/V_{m-1} \) in terms of these relations.

Our goal is to categorify this framework. Theorem 1.1-(1) implies

\[ \begin{array}{c} \vdots \\ \vdots \end{array} \simeq 0, \quad (1.1) \]

which is none other than categorified FI relation. The next target is the 4T relation.
Main Theorem A. With regard to the morphism $\hat{\Phi}$ and the chain homotopies $\psi_O, \psi_U$ in Theorem 1.1, the following hold.

1. The following diagrams commute strictly:

$$
\begin{align*}
\xymatrix{
\begin{bmatrix}
 a \\
 b \\
 c 
\end{bmatrix} 
\ar[r]^\hat{\Phi}_a &
\begin{bmatrix}
 a \\
 b \\
 c 
\end{bmatrix} 
\ar[r]_{\Phi_b} &
\begin{bmatrix}
 a \\
 b \\
 c 
\end{bmatrix} \\
\begin{bmatrix}
 a \\
 b \\
 c 
\end{bmatrix} 
\ar[r]_{\Phi_b} &
\begin{bmatrix}
 a \\
 b \\
 c 
\end{bmatrix} 
\ar[r]^\hat{\Phi}_a &
\begin{bmatrix}
 a \\
 b \\
 c 
\end{bmatrix}
}
\end{align*}
$$

(1.7)

2. In the module of morphisms of degree $-1$, the following equation holds:

$$
\hat{\Phi}_a \psi_O \hat{\Phi}_a - \hat{\Phi}_b \psi_U \hat{\Phi}_b = 0
$$

(1.8)

The left hand side of the equation (1.8) is a $-1$-cocycle in a Hom complex associated with the boundary of the hexagon prism in Fig. 1.2 or the 2-monodromy around the triple point. Thus, the part (2) asserts that it is actually a trivial cycle. As a consequence, we obtain the following categorified analogue of 4T-relation:
Main Theorem B. There are morphisms

\[
\begin{bmatrix}
\begin{array}{cc}
\otimes & \otimes \\
\otimes & \otimes \\
\end{array}
\end{bmatrix} \rightarrow \begin{bmatrix}
\begin{array}{cc}
\otimes & \otimes \\
\otimes & \otimes \\
\end{array}
\end{bmatrix}, \quad \begin{bmatrix}
\begin{array}{cc}
\otimes & \otimes \\
\otimes & \otimes \\
\end{array}
\end{bmatrix} \rightarrow \begin{bmatrix}
\begin{array}{cc}
\otimes & \otimes \\
\otimes & \otimes \\
\end{array}
\end{bmatrix}
\]

whose mapping cones are chain homotopic to each other.

Indeed, by taking the mapping cones along the vertical edges of the hexagon prism diagram (1.2), we obtain the following (strictly) commutative diagram:

where \( R^O_{IV}, R^U_{IV} \) are the chain homotopy equivalences associated with the moves (1.4). The two morphisms in Main Theorem B are induced by taking mapping cones of (1.9) in two different directions.

We note that the equation (1.8) suggests studying an obstruction associated with a triple point in the stratum \( \mathcal{M}_3 \) for general knot homologies. Indeed, suppose \( H(K) \) is a knot homology equipped with a crossing-change morphism \( \hat{\Phi} \) which is compatible with Reidemeister moves in the sense of Theorem 1.1 (possibly up to homotopy). In this case, one can prove that the two hexagon diagrams (1.7) are commutative up to chain homotopies, say \( \alpha^+, \alpha^- \) with

\[
\partial(\alpha^-) = \hat{\Phi}_a R^O_{III} \hat{\Phi}_a - \hat{\Phi}_b R^U_{III} \hat{\Phi}_b, \quad \partial(\alpha^+) = \hat{\Phi}_a R^O_{III} \hat{\Phi}_a - \hat{\Phi}_b R^U_{III} \hat{\Phi}_b.
\]

For example, the part (1) in Main Theorem A asserts that we can take \( \alpha_\pm = 0 \). They together with the homotopies \( \psi_O \) and \( \psi_U \) in Theorem 1.1 yield a \(-1\)-cocycle

\[
\theta := \hat{\Phi}_a \psi_O \hat{\Phi}_a - \hat{\Phi}_b \psi_U \hat{\Phi}_b + \hat{\Phi}_c \alpha_- - \alpha_+ \hat{\Phi}_c
\]

in the Hom-complex, which corresponds to the 2-cycle generated by the faces of the hexagon prism Fig. 1.2. The part (2) in Main Theorem A then implies that we have \( \theta = 0 \) in the case of Khovanov homology.

2 Review on crossing-change on Khovanov homology

We begin with a review on Khovanov homology and the genus-one morphism \( \hat{\Phi} \).
2.1 Definitions and notations

We construct Khovanov homology following Bar-Natan’s formalism [2]. By cobordisms, we mean 2-dimensional cobordisms with corners. In this paper, we always use the “left-to-right” convention to depict them; e.g.

As an unfortunate consequence, the composition $W_1 \circ W_0$ of cobordisms $W_0$ and $W_1$ is, to the contrary to its notation, represented by gluing the right boundary of $W_0$ to the left boundary of $W_1$. Hence, we will not omit the composition operator “$\circ$” in composing “pictures” to clarify the composition order throughout the paper.

We define $\text{Cob}_2^\ell$ to be the additive closure of the category of formal sums of cobordisms modulo the three relations called $S_\cdot$, $T_\cdot$, and $4Tu$-relations (see [2]). Although a morphism of $\text{Cob}_2^\ell$ is hence a matrix of cobordisms, we often denote it by the formal sum of all its entries in the case where the domains and codomains are understood from the pictures. For example, the morphism

in $\text{Cob}_2^\ell$ is written as the following linear sum:

For each tangle diagram $D$, we construct $\llbracket D \rrbracket$ as a complex in $\text{Cob}_2^\ell$. We consider the following single saddle operations $\delta_-$ and $\delta_+$:

\[
\delta_- := \begin{align*}
&: \begin{array}{c}
\xrightarrow{\text{ }}
\end{array}
\rightarrow
\begin{array}{c}
\xrightarrow{\text{ }}
\end{array},
\
\delta_+ := \begin{align*}
&: \begin{array}{c}
\begin{array}{c}
\xrightarrow{\text{ }}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\text{ }}
\end{array}
\end{array}.
\end{align*}
\]
The complexes $\begin{bmatrix} \begin{array}{c|c|c}
\cdot & \cdot & \cdot \\
 & \delta & \\
 & & \\
\end{array} \end{bmatrix}$ and $\begin{bmatrix} \begin{array}{c|c|c}
\cdot & \cdot & \cdot \\
 & 0 & \\
 & & \\
\end{array} \end{bmatrix}$ are defined as follows:

$\begin{bmatrix} \begin{array}{c|c|c}
\cdot & \cdot & \cdot \\
 & \delta & \\
 & & \\
\end{array} \end{bmatrix}$: $\cdots \rightarrow 0 \rightarrow \begin{array}{c}
\cdot \\
\delta \\
\cdot \\
\end{array} \rightarrow \begin{array}{c}
0 \\
1 \\
0 \\
\end{array} \rightarrow \cdots$

$\begin{bmatrix} \begin{array}{c|c|c}
\cdot & \cdot & \cdot \\
 & 0 & \\
 & & \\
\end{array} \end{bmatrix}$: $\cdots \rightarrow 0 \rightarrow 0 \rightarrow \begin{array}{c}
\cdot \\
-\delta \\
\cdot \\
\end{array} \rightarrow \begin{array}{c}
0 \\
0 \\
\end{array} \rightarrow \cdots$

For general tangle diagram $D$, we define $\begin{bmatrix} \begin{array}{c|c|c}
\cdot & \cdot & \cdot \\
 & \delta & \\
 & & \\
\end{array} \end{bmatrix}$ by “stacking up” the complexes for all the crossings in $D$. Bar-Natan [2] proved that the resulting complex is invariant under Reidemeister moves and that it induces Khovanov homology for link diagrams. We call it the Khovanov complex of $D$.

In [6], the following degree-preserving crossing-change is discussed:

$\hat{\Phi} : \begin{bmatrix} \begin{array}{c|c|c}
\cdot & \cdot & \cdot \\
 & \delta & \\
 & & \\
\end{array} \end{bmatrix} \rightarrow \begin{bmatrix} \begin{array}{c|c|c}
\cdot & \cdot & \cdot \\
 & 0 & \\
 & & \\
\end{array} \end{bmatrix}$ .

(2.1)

We first consider the morphism $\Phi$ in the category $\mathcal{Cob}^2$ given by

$\Phi : \begin{bmatrix} \begin{array}{c|c|c}
\cdot & \cdot & \cdot \\
 & \delta & \\
 & & \\
\end{array} \end{bmatrix} \rightarrow \begin{bmatrix} \begin{array}{c|c|c}
\cdot & \cdot & \cdot \\
 & 0 & \\
 & & \\
\end{array} \end{bmatrix}$

(2.2)

where annuluses are attached to the identity cobordism in two different ways. The morphism $\hat{\Phi}$ is defined by the following diagram:

$\begin{bmatrix} \begin{array}{c|c|c}
\cdot & \cdot & \cdot \\
 & \delta & \\
 & & \\
\end{array} \end{bmatrix}$: $\cdots \rightarrow 0 \rightarrow \begin{array}{c}
\cdot \\
\delta \\
\cdot \\
\end{array} \rightarrow \begin{array}{c}
0 \\
1 \\
0 \\
\end{array} \rightarrow \cdots$

$\hat{\Phi}$

$\begin{bmatrix} \begin{array}{c|c|c}
\cdot & \cdot & \cdot \\
 & 0 & \\
 & & \\
\end{array} \end{bmatrix}$: $\cdots \rightarrow 0 \rightarrow 0 \rightarrow \begin{array}{c}
\cdot \\
-\delta \\
\cdot \\
\end{array} \rightarrow \begin{array}{c}
0 \\
0 \\
\end{array} \rightarrow \cdots$

In view of the Vassiliev skein relation, we extend Khovanov complex to singular tangle diagrams by setting $\begin{bmatrix} \begin{array}{c|c|c}
\cdot & \cdot & \cdot \\
 & \delta & \\
 & & \\
\end{array} \end{bmatrix}$ to be the mapping cone of $\hat{\Phi}$; more explicitly, it is the complex as follows:

$\begin{bmatrix} \begin{array}{c|c|c}
\cdot & \cdot & \cdot \\
 & \delta & \\
 & & \\
\end{array} \end{bmatrix}$: $\cdots \rightarrow 0 \rightarrow \begin{array}{c}
\cdot \\
\delta \\
\cdot \\
\end{array} \rightarrow \begin{array}{c}
0 \\
0 \\
1 \\
\end{array} \rightarrow \cdots$

Theorem 2.1 ([6]). The complex $\begin{bmatrix} \begin{array}{c|c|c}
\cdot & \cdot & \cdot \\
 & \delta & \\
 & & \\
\end{array} \end{bmatrix}$ is invariant under moves of singular tangles up to chain homotopy equivalences.
2.2 The homotopy equivalence for $R_{\text{III}}$

We next recall how the invariance under $R_{\text{III}}$ was proved in [2], where a technique so-called "Kauffman trick" was used. Indeed, by the definition of Khovanov complex, one has the following identifications:

\[
\begin{align*}
\text{Cone} & \xrightarrow{\delta_-} \text{Cone} \\
\text{Cone} & \xrightarrow{\delta_-} \text{Cone} \\
\text{Cone} & \xrightarrow{\delta_-} \text{Cone} \\
\text{Cone} & \xrightarrow{\delta_-} \text{Cone}
\end{align*}
\]

(2.4)

Notice that, in each pair of the moves, the domains and the codomains are identical or connected by two Reidemeister moves of type II. Unwinding the chain homotopy equivalences associated with those moves (see [2]), one sees that we have the commutative squares in the category of complexes in $\text{Cob}^2$:

\[
\begin{align*}
\text{Cone} & \xrightarrow{\delta_-} \text{Cone} \\
\text{Cone} & \xrightarrow{\delta_+} \text{Cone} \\
\text{Cone} & \xrightarrow{\delta_-} \text{Cone} \\
\text{Cone} & \xrightarrow{\delta_+} \text{Cone}
\end{align*}
\]

(2.5)

where $\gamma_0 = \{\gamma_{0i}\}_i$, and $\omega_0 = \{\omega_{0i}\}_i$, are morphisms of complexes which are zero except

\[
\begin{align*}
\gamma_0^{-1} & = - \\
\gamma_0^0 & = + \\
\gamma_0^1 & = + \\
\omega_0^0 & = + + +
\end{align*}
\]

(2.6)

(2.7)
while $F_{O-} = \{ F^i_{O-} \}$ and $F_{O+} = \{ F^i_{O+} \}$ are chain homotopies (i.e. degree $-1$ elements in the Hom complexes bounding appropriate elements) which are zero except

\[
F^0_{O-} := \begin{array}{c}
\vdots
\end{array}, \quad F^1_{O-} := \begin{array}{c}
\vdots
\end{array}, \quad F^0_{O+} := \begin{array}{c}
\vdots
\end{array}, \quad F^1_{O+} := \begin{array}{c}
\vdots
\end{array}. \tag{2.8}
\]

In fact, one can verify that $F_{O-}$ and $F_{O+}$ satisfy the equations in the Hom-complexes respectively:

\[
\partial(F_{O-}) = \delta_\gamma_0 - (-\omega_0)\delta_-, \quad \partial(F_{O+}) = \delta_+\omega_0 - \gamma_0\delta_+, \tag{2.9}
\]

so that they exhibit (2.5) as homotopy commutative squares. We then obtain a chain homotopy equivalence

\[
R_{III}^{-} : \begin{bmatrix}
\delta & \gamma \\
\omega & 0
\end{bmatrix} \Rightarrow \begin{bmatrix}
\delta & 0 \\
0 & \gamma
\end{bmatrix}, \quad R_{III}^{+} : \begin{bmatrix}
\gamma & 0 \\
0 & \omega
\end{bmatrix} \Rightarrow \begin{bmatrix}
\gamma & -\omega \\
0 & 0
\end{bmatrix}. \tag{2.10}
\]

More explicitly, under the identifications (2.4), the morphisms $R_{III}^{-}$ and $R_{III}^{+}$ in (2.10) are respectively represented by the matrices below:

\[
R_{III}^{-} = \begin{bmatrix}
-\omega_0 & -F_0^- \\
0 & \gamma_0
\end{bmatrix}, \quad R_{III}^{+} = \begin{bmatrix}
\gamma_0 & -F_0^+ \\
0 & \omega_0
\end{bmatrix}. \tag{2.11}
\]

Similarly, the chain homotopy equivalences

\[
R_{III}^{-} : \begin{bmatrix}
\delta & \gamma \\
\omega & 0
\end{bmatrix} \Rightarrow \begin{bmatrix}
\delta & 0 \\
0 & \gamma
\end{bmatrix}, \quad R_{III}^{+} : \begin{bmatrix}
\gamma & 0 \\
0 & \omega
\end{bmatrix} \Rightarrow \begin{bmatrix}
\gamma & -\omega \\
0 & 0
\end{bmatrix} \tag{2.12}
\]

are obtained by the following homotopy commutative squares:

\[
\begin{bmatrix}
\delta \\
\gamma
\end{bmatrix} \Rightarrow \begin{bmatrix}
\delta + \gamma
\end{bmatrix}, \quad \begin{bmatrix}
\delta \\
\gamma
\end{bmatrix} \Rightarrow \begin{bmatrix}
\delta + \gamma
\end{bmatrix}, \quad \begin{bmatrix}
\delta \\
\gamma
\end{bmatrix} \Rightarrow \begin{bmatrix}
\delta + \gamma
\end{bmatrix}, \quad \begin{bmatrix}
\delta \\
\gamma
\end{bmatrix} \Rightarrow \begin{bmatrix}
\delta + \gamma
\end{bmatrix}. \tag{2.13}
\]

9
where $\gamma_U$ and $\omega_U$ are morphisms of chain complexes in $\text{Cob}_2^L$ given by

$$
\gamma_{U_{-1}} := \begin{array}{c}
\text{Diagram 1} \\
\gamma_0 := \begin{array}{c}
\text{Diagram 2} \\
\gamma_1 := \begin{array}{c}
\text{Diagram 3}
\end{array}
\end{array}
\end{array},
$$

$$
\omega_U := \begin{array}{c}
\text{Diagram 4} \\
\omega_0 := \begin{array}{c}
\text{Diagram 5} \\
\omega_1 := \begin{array}{c}
\text{Diagram 6}
\end{array}
\end{array}
\end{array},
$$

(2.14)

(2.15)

while $F_{U_{-}}$ and $F_{U_{+}}$ are chain homotopies given by

$$
F_{U_{-0}} := \begin{array}{c}
\text{Diagram 7} \\
F_{U_{+0}} := \begin{array}{c}
\text{Diagram 8} \\
F_{U_{+1}} := \begin{array}{c}
\text{Diagram 9}
\end{array}
\end{array}
\end{array},
$$

$$
F_{U_{-1}} := \begin{array}{c}
\text{Diagram 10} \\
F_{U_{+1}} := \begin{array}{c}
\text{Diagram 11} \\
F_{U_{+1}} := \begin{array}{c}
\text{Diagram 12}
\end{array}
\end{array}
\end{array},
$$

(2.16)

so that they satisfy the equations

$$
\partial(F_{U_{-}}) = \delta_{-} \gamma_U - (-\omega_U)\delta_{-}, \quad \partial(F_{U_{+}}) = \delta_{+} \omega_U - \gamma_U\delta_{+}.
$$

(2.17)

The matrix representations of $R_{III}^{U_{-}}$ and $R_{III}^{U_{+}}$ are then given by

$$
R_{III}^{U_{-}} = \begin{bmatrix}
-\omega_U & -F_{U_{-}} \\
0 & \gamma_U
\end{bmatrix},
\quad
R_{III}^{U_{+}} = \begin{bmatrix}
\gamma_U & -F_{U_{+}} \\
0 & \omega_U
\end{bmatrix}.
$$

(2.18)

### 2.3 The compatibility of $\hat{\Phi}$ with $R_{III}$

We next discuss the compatibility of the morphism $\hat{\Phi}$ in (1.2) with the chain homotopy equivalences $R_{III}^{O_{\pm}}$ and $R_{III}^{U_{\pm}}$ in (2.10) and (2.12). More precisely, we prove the following result.

**Proposition 2.2** ([6, Theorem 4.1]). There are degree $-1$ morphisms $\psi_O$ and $\psi_U$ which make the following diagrams chain homotopy commutative:

$$
\begin{array}{ccc}
R_{III}^{O_{-}} & \xrightarrow{\hat{\Phi}_{\epsilon}} & R_{III}^{O_{+}} \\
\downarrow \psi_O \quad & \quad & \downarrow \psi_O \\
\hat{\Phi}_{\epsilon} \quad & \quad & \hat{\Phi}_{\epsilon}
\end{array}
\quad
\begin{array}{ccc}
R_{III}^{U_{-}} & \xrightarrow{\hat{\Phi}_{\epsilon}} & R_{III}^{U_{+}} \\
\downarrow \psi_U \quad & \quad & \downarrow \psi_U \\
\hat{\Phi}_{\epsilon} \quad & \quad & \hat{\Phi}_{\epsilon}
\end{array}.
$$

(2.19)
In view of “Kauffman trick”, we begin with the compatibility of the morphism $\Phi$ in (2.1). By direct computation, one can verify the following.

**Lemma 2.3.** There is a homotopy commutative square

\[
\begin{array}{ccc}
\text{Diagram 1} & \Phi & \text{Diagram 2} \\
\text{Diagram 3} & G_O & \text{Diagram 4} \\
\text{Diagram 5} & \Phi & \text{Diagram 6}
\end{array}
\]

where the chain homotopies $G_O = \{G_i^O\}_i$ and $G_U = \{G_i^U\}_i$ are given by

\[
G_0^O := \text{Diagram 7}, \quad G_1^O := \text{Diagram 8}, \quad G_0^U := \text{Diagram 9}, \quad G_1^U := \text{Diagram 10}
\]

and zero in the other degrees. More precisely, $G_O$ and $G_U$ respectively satisfy the following equations in the Hom-complexes:

\[
\partial(G_O) = \Phi(-\omega_O) - \omega_O \Phi, \quad \partial(G_U) = \Phi(-\omega_U) - \omega_U \Phi.
\]
Now, we have the following diagram of morphisms of complexes in $\text{Cob}_2^\ell$:

$$\begin{array}{cccccc}
\begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
0
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
& \xrightarrow{\delta_-} & \begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
0
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
& \xrightarrow{\Phi} & \begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
0
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
& \xrightarrow{-\delta_+} & 0 \\
0 & \downarrow{\gamma_0} & \begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
& \downarrow{-\omega_0} & \begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
& \downarrow{\gamma_0} & 0 \\
0 & \downarrow{\omega_0} & \begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
& \downarrow{\Phi} & \begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
& \downarrow{-\delta_+} & \begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{array}$$

where all square faces are strictly commutative or commutative up to chain homotopies $F_{O-}$, $G_O$, and $F_{O+}$. The last ingredients for the chain homotopy $\psi_O$ in Proposition 2.2 are the 2-homotopies bounding the homotopy commutative cubes in (2.24); indeed, since every complexes are supported in degrees $-1, 0, 1$, the cycles of degrees higher than 2 are trivial for degree reason. We define the elements

$$\begin{align*}
\Psi_{O-} & \in \text{Hom}^{-2}\left(\begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
, \begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}\right), \\
\Psi_{O+} & \in \text{Hom}^{-2}\left(\begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
, \begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}\right)
\end{align*}$$

by the following cobordisms (and zeros in the other degrees):

$$\begin{align*}
(P_{O-})^1 := \begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
, \\
(P_{O+})^1 := \begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{align*}$$

The direct computation shows that these elements satisfy the following equations in the Hom-complexes (e.g. see [6, Lemma 4.1]):

$$\partial(\Psi_{O-}) = -\Phi F_{O-} - G_O \delta_- , \quad \partial(\Psi_{O+}) = -\delta_+ G_O + F_{O+} \Phi .$$

$$\text{(2.26)}$$
Similarly, regarding the homotopy commutative diagram

![Homotopy commutative diagram](image)

we define the elements

\[ \Psi_{U-} \in \text{Hom}^{-2} \left( \begin{bmatrix} \begin{bmatrix} & \end{bmatrix} \\ \begin{bmatrix} & \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} & \end{bmatrix} \\ \begin{bmatrix} & \end{bmatrix} \end{bmatrix} \right) \], \quad \Psi_{U+} \in \text{Hom}^{-2} \left( \begin{bmatrix} \begin{bmatrix} & \end{bmatrix} \\ \begin{bmatrix} & \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} & \end{bmatrix} \\ \begin{bmatrix} & \end{bmatrix} \end{bmatrix} \right) \]

by the following cobordisms (and zeros in the other degrees):

\[ (\Psi_{U-})^1 := \begin{bmatrix} & \end{bmatrix}, \quad (\Psi_{U+})^1 := \begin{bmatrix} & \end{bmatrix} \].

As in the case of \( \Psi_{O\pm} \), they satisfy the following equations:

\[ \partial(\Psi_{U-}) = -\Phi F_{U-} - G_{U-} \delta_-, \quad \partial(\Psi_{U+}) = -\delta_+ G_{U} + F_{U+} \Phi \].

**Proof of Proposition 2.2.** In view of (2.4), we obtain isomorphisms

\[ \begin{bmatrix} \begin{bmatrix} & \end{bmatrix} \\ \begin{bmatrix} & \end{bmatrix} \end{bmatrix}^i \cong \begin{bmatrix} \begin{bmatrix} & \end{bmatrix} \\ \begin{bmatrix} & \end{bmatrix} \end{bmatrix}^i \oplus \begin{bmatrix} \begin{bmatrix} & \end{bmatrix} \\ \begin{bmatrix} & \end{bmatrix} \end{bmatrix}^{i+1} \], \quad \begin{bmatrix} \begin{bmatrix} & \end{bmatrix} \\ \begin{bmatrix} & \end{bmatrix} \end{bmatrix}^i \cong \begin{bmatrix} \begin{bmatrix} & \end{bmatrix} \\ \begin{bmatrix} & \end{bmatrix} \end{bmatrix}^i \oplus \begin{bmatrix} \begin{bmatrix} & \end{bmatrix} \\ \begin{bmatrix} & \end{bmatrix} \end{bmatrix}^{i-1} \] \]

with differentials represented respectively by the matrices

\[ D_1 := \begin{bmatrix} d & \delta_- \\ 0 & -d \end{bmatrix}, \quad D_2 := \begin{bmatrix} -d & -\delta_+ \\ 0 & d \end{bmatrix} \].

13
Similarly, we also have isomorphisms
\[
\begin{pmatrix}
\begin{pmatrix}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}
\end{pmatrix}
\cong
\begin{pmatrix}
\begin{pmatrix}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}
\end{pmatrix}
\oplus
\begin{pmatrix}
\begin{pmatrix}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}
\end{pmatrix}
\quad \text{with the differentials given in the same formulas as (2.31).}
\]

We then define the elements
\[
\psi_O \in \text{Hom}^{-1}
\begin{pmatrix}
\begin{pmatrix}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}
\end{pmatrix},
\psi_U \in \text{Hom}^{-1}
\begin{pmatrix}
\begin{pmatrix}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}
\end{pmatrix}
\]
which are represented by the following matrices with respect to the decompositions (2.30) and (2.32):
\[
\psi_O := \begin{pmatrix}
-\Psi_O^+ & 0 \\
G_O & -\Psi_O^-
\end{pmatrix}, \quad \psi_U := \begin{pmatrix}
-\Psi_U^+ & 0 \\
G_U & -\Psi_U^-
\end{pmatrix}.
\]

By virtue of the equations (2.11), (2.23), and (2.26), we have
\[
\partial(\psi_O) = D_2\psi_O + \psi_OD_1 = \begin{pmatrix}
\begin{pmatrix}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}
\end{pmatrix} = \begin{pmatrix}
\begin{pmatrix}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}
\end{pmatrix} + \begin{pmatrix}
\begin{pmatrix}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}
\end{pmatrix} + \begin{pmatrix}
\begin{pmatrix}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}
\end{pmatrix}.
\]

Observe that the last term in the right hand side of (2.34) vanishes; in fact, we have
\[
\delta^{-1}_+\Psi_O^- = -F_O + \Phi_O + \Phi_O + \delta^{-1}_+ - \Psi_O^- + \delta^{-1}_+ - \Psi_O^- = -F_O + \Phi_O + \Phi_O + \delta^{-1}_+.
\]

Thus, \(\psi_O\) makes the left square in (2.19) commute. The similar computation shows that \(\psi_U\) makes the right commute, so we obtain the result.

3 Proof of Main Theorem A

We now begin the proof of Main Theorem A. We directly compute the two compositions in the diagrams (1.7).
For this, we denote by $\Phi_L$ and $\Phi_R$ the morphisms

$$\Phi_L, \Phi_R : \begin{array}{|c|} \hline \hline \hline \end{array} \rightarrow \begin{array}{|c|} \hline \hline \end{array}$$

in the category $\text{Cob}_2$ which applies the morphism $\Phi$ given in (2.2) to the left and the right two strands respectively. Note that the tangle in the domain (and the codomain) appears as a common state in the homological degree 0 of the Khovanov complexes in Main Theorem A. Specifically, we have canonical inclusions

$$\begin{array}{|c|} \hline \hline \hline \end{array} \rightarrow \begin{array}{|c|} \hline \hline \end{array} = 0,$$

$$\begin{array}{|c|} \hline \hline \hline \end{array} \rightarrow \begin{array}{|c|} \hline \hline \end{array} = 0.$$

In what follows, we refer them as the states $s_0$ in these complexes.

**Lemma 3.1.** The compositions

$$\begin{array}{|c|} \hline \hline \hline \end{array} \rightarrow \begin{array}{|c|} \hline \hline \end{array} \rightarrow \begin{array}{|c|} \hline \hline \end{array} = 0,$$

$$\begin{array}{|c|} \hline \hline \hline \end{array} \rightarrow \begin{array}{|c|} \hline \hline \end{array} \rightarrow \begin{array}{|c|} \hline \hline \end{array} = 0,$$

both vanish except on the state $s_0$ where they equal to the composition

$$-\Phi_R \Phi_L : \begin{array}{|c|} \hline \hline \hline \end{array} \rightarrow \begin{array}{|c|} \hline \hline \end{array} \rightarrow \begin{array}{|c|} \hline \hline \end{array} = 0.$$

**Proof.** By (2.11), the composition (3.1) equals

$$\begin{array}{|c|} \hline \hline \hline \end{array} \rightarrow \begin{array}{|c|} \hline \hline \end{array} \rightarrow \begin{array}{|c|} \hline \hline \end{array} = 0.$$

Notice that the morphism $\Phi_a$ is non-zero only on the states where the crossing $a$ is smoothed vertically, and only such states appear in its image. By the definitions (2.6) and (2.8) of $\gamma_O$ and $F_O$, we hence have $\Phi_a \gamma_O \Phi_a = 0$ and $\Phi_a F_O = 0$. For the same reason, regarding (2.7), one sees that the composition $\Phi_{\omega_O} \Phi$ also vanishes except on the state $s_0$ where it is given by

$$\Phi_a^0 \omega_O \Phi_a^0 + \Phi_a^0 \Phi_a^0 = \Phi_a^0 \circ \Phi_a = \Phi_R \circ \Phi_a = \Phi_L = \Phi_R \Phi_L.$$
which ensures the assertion for (3.1).

As for the other composition, by (2.18), we have

\[
\hat{\Phi}_b R_{III} U_{III} \hat{\Phi}_b = \begin{bmatrix}
-\hat{\Phi}_b \omega_U \hat{\Phi}_b & -\hat{\Phi}_b F_{U-} \hat{\Phi}_b \\
0 & \hat{\Phi}_b \gamma_U \hat{\Phi}_b
\end{bmatrix}.
\]

By the same argument as above, we have

\[
\hat{\Phi}_b F_{U-} = 0, \quad \hat{\Phi}_b \gamma_U \hat{\Phi}_b = 0, \quad \text{and} \quad \hat{\Phi}_b \omega_U \hat{\Phi}_b = 0
\]

except the state \( s_0 \) where

\[
\hat{\Phi}_b \omega_U \hat{\Phi}_b = \begin{bmatrix}
\hat{\Phi}_b^0 \circ \left( \begin{array}{c}
\Phi_R \\
\Phi_L
\end{array} \right)
\end{bmatrix}.
\]

This completes the proof.

\[\square\]

**Lemma 3.2.** The compositions

\[
\begin{align*}
\left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \hat{\Phi}_a & \rightarrow \left[ \begin{array}{c} a \\ b \\ c \end{array} \right] R_{III} O \rightarrow \left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \hat{\Phi}_a, \quad (3.3) \\
\left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \hat{\Phi}_b & \rightarrow \left[ \begin{array}{c} a \\ b \\ c \end{array} \right] U_{III} \rightarrow \left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \hat{\Phi}_b, \quad (3.4)
\end{align*}
\]

both vanishes except at the state \( s_0 \) where they equal to the composition

\[
\Phi_R \Phi_L : \begin{array}{c}
\xrightarrow{=} \\
\xrightarrow{\circ} \\
\xleftarrow{=} \\
\xrightarrow{\circ}
\end{array}
\]

**Proof.** Since the proof is almost identical to Lemma 3.1, we omit it.

\[\square\]

**Proof of Main Theorem A.** The part (1) directly follows from Lemmas 3.1 and 3.2. As for the part (2), one can actually verify that the following two compositions are both trivial in a similar manner to Lemma 3.1

\[
\begin{align*}
\left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \hat{\Phi}_a & \rightarrow \left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \Psi O \rightarrow \left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \hat{\Phi}_a, \quad (3.5) \\
\left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \hat{\Phi}_b & \rightarrow \left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \Psi U \rightarrow \left[ \begin{array}{c} a \\ b \\ c \end{array} \right] \hat{\Phi}_b, \quad (3.6)
\end{align*}
\]

Indeed, by (2.33), we have

\[
\hat{\Phi}_a \Psi O \hat{\Phi}_a = \begin{bmatrix}
-\hat{\Phi}_a \Psi O \hat{\Phi}_a & 0 \\
\hat{\Phi}_a \Psi O \hat{\Phi}_a & -\hat{\Phi}_a \Psi O \hat{\Phi}_a
\end{bmatrix}.
\]

\[16\]
Notice that, seeing (2.25) and (2.21), \( \Psi_{O \pm} \) and \( G_O \) has no component both of whose domain and codomain are states such that the crossings \( \alpha \) are smoothed vertically. This implies that each entry of the matrix (3.7) and hence the composition (3.5) vanish. Similarly, the composition (3.6) vanishes. Consequently, we obtain

\[
\hat{\Phi}_a \psi O \hat{\Phi}_a - \hat{\Phi}_b \psi U \hat{\Phi}_b = 0 - 0 = 0 ,
\]

which is exactly the required equation.

\( \square \)

4 Proof of Main Theorem B

We finally prove the categorified 4\( T \)-relation Main Theorem B. Recall that the Khovanov complex on a singular point is defined as the mapping cone of the crossing-change morphism \( \hat{\Phi} \) as in (2.3). Thus, the homotopy commutative squares (2.19) induce chain homotopy equivalences

\[
R_O^{IV} : \begin{bmatrix} \alpha \beta \end{bmatrix} \to \begin{bmatrix} \alpha \gamma \beta \end{bmatrix} , \quad R_U^{IV} : \begin{bmatrix} \alpha \beta \end{bmatrix} \to \begin{bmatrix} \alpha \gamma \beta \end{bmatrix} \,. \tag{4.1}
\]

We assert that the diagram below commutes strictly:

\[
\begin{array}{c}
\begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \xrightarrow{\hat{\Phi}_a} \begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \xrightarrow{R_O^{IV}} \begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \xrightarrow{\hat{\Phi}_b} \begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \\
\begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \xrightarrow{R_U^{IV}} \begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \xrightarrow{\hat{\Phi}_a} \begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \xrightarrow{\hat{\Phi}_b} \begin{bmatrix} \alpha \beta \gamma \end{bmatrix}
\end{array} \tag{4.2}
\]

Notice that the counter-clockwise and clockwise compositions are respectively induced by the following homotopy commutative squares:

\[
\begin{array}{c}
\begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \xrightarrow{\hat{\Phi}_a R_O^{III} \hat{\Phi}_a} \begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \xrightarrow{\hat{\Phi}_b} \begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \\
\begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \xrightarrow{\hat{\Phi}_b R_U^{III} \hat{\Phi}_b} \begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \xrightarrow{\hat{\Phi}_a} \begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \\
\begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \xrightarrow{\hat{\Phi}_a R_O^{III} \hat{\Phi}_a} \begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \xrightarrow{\hat{\Phi}_b} \begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \\
\begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \xrightarrow{\hat{\Phi}_b R_U^{III} \hat{\Phi}_b} \begin{bmatrix} \alpha \beta \gamma \end{bmatrix} \xrightarrow{\hat{\Phi}_a} \begin{bmatrix} \alpha \beta \gamma \end{bmatrix}
\end{array} \tag{4.3}
\]
Main Theorem A asserts that they are actually identical, so the diagram (4.2) commutes.

We are now ready to prove Main Theorem B.

**Proof of Main Theorem B.** Since the diagram (4.2) is commutative, we have the following commutative diagram of complexes in $\mathcal{C}ob_2$:

This yields the following zigzag of chain homotopy equivalences between total complexes:

Note that, since the total complexes are isomorphic to the two-fold mapping cones, the first and the last complexes in (4.4) are respectively isomorphic to the mapping cones.
of the following forms:

\[
\text{Cone} \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{\rotatebox[origin=c]{90}{$\leftarrow$}}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{\rotatebox[origin=c]{90}{$\rightarrow$}}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right) \rightarrow \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{\rotatebox[origin=c]{90}{$\leftarrow$}}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{\rotatebox[origin=c]{90}{$\rightarrow$}}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right)
\end{array}
\right).
\]

By (4.4), they are chain homotopic to each other, and this completes the proof. □

Acknowledgments

The authors would like to thank Professor Keiichi Sakai for his comments. The work was partially supported by JSPS KAKENHI Grant Numbers JP20K03604, JP22K03603 and Toyohashi Tech Project of Collaboration with KOSEN.

References

[1] D. Bar-Natan. Vassiliev homotopy string link invariants. J. Knot Theory Ramifications, 4(1):13–32, 1995.

[2] D. Bar-Natan. Khovanov’s homology for tangles and cobordisms. Geom. Topol., 9:1443–1499, 2005.

[3] J. S. Birman. New points of view in knot theory. Bull. Amer. Math. Soc. (N.S.), 28(2):253–287, 1993.

[4] J. S. Birman and X.-S. Lin. Knot polynomials and Vassiliev’s invariants. Invent. Math., 111(2):225–270, 1993.

[5] N. Ito and J. Yoshida. Crossing change on Khovanov homology and a categorified Vassiliev skein relation. J. Knot Theory Ramifications, 29(7):2050051, 24, 2020. arXiv:1911.09308.

[6] N. Ito and J. Yoshida. A cobordism realizing crossing change on $\mathfrak{sl}_2$ tangle homology and a categorified Vassiliev skein relation. Topology Appl., 296:Paper No. 107646, 31, 2021.

[7] M. Kontsevich. Vassiliev’s knot invariants. In I. M. Gel’fand Seminar, volume 16 of Adv. Soviet Math., pages 137–150. Amer. Math. Soc., Providence, RI, 1993.

[8] N. Shirokova and B. Webster. Wall-crossing morphisms in Khovanov-Rozansky homology. arXiv:0706.1388, 2007.

[9] V. A. Vassiliev. Cohomology of knot spaces. In Theory of singularities and its applications, volume 1 of Adv. Soviet Math., pages 23–69. Amer. Math. Soc., Providence, RI, 1990.