MULTIDIMENSIONAL CONTINUED FRACTION AND RATIONAL APPROXIMATION

ZONGDUO DAI, KUNPENG WANG, AND DINGFENG YE

Abstract. The classical continued fraction is generalized for studying the rational approximation problem on multi-formal Laurent series in this paper, the construction is called \( m \)-continued fraction. It is proved that the approximants of an \( m \)-continued fraction converge to a multi-formal Laurent series, and are best rational approximations to it; conversely for any multi-formal Laurent series an algorithm called \( m \)-CF transform is introduced to obtain its \( m \)-continued fraction expansions; moreover, strict \( m \)-continued fractions, which are \( m \)-continued fractions imposed with some additional conditions, and multi-formal Laurent series are in 1-1 correspondence. It is shown that \( m \)-continued fractions can be used to study the multi-sequence synthesis problem.

1. Introduction

Continued fraction [14, 21, 22, 28, 29] is a useful tool in dealing with many number theoretic problems and in numerical computing. It is well-known that the simple continued fraction expansion of a single real number gives the best solution to its rational approximation problem. Many people have contrived to construct multidimensional continued fractions in dealing with the rational approximation problem for multi-reals. The archetypal example of a multidimensional generalization of simple continued fraction is the Jacobi-Perron algorithm (JPA), see [2]. This algorithm and its modification are extensively studied [12, 13, 16, 19]. These algorithms are borrowed to study the same problem for multi-formal Laurent series [8, 11]. But none of these algorithms guarantee best rational approximation in general. In this paper, we deal with the multi-rational approximation problem over the formal Laurent series field \( F((z^{-1})) \): given an element \( r \in F((z^{-1}))^m \), find \( p \in F[z]^m, q \in F[z] \), such that \( p/q \) approximates \( r \) as close as possible while \( \deg(q) \) is bounded. In this setting, we give a natural generalization of the simple 1-dimensional continued fraction to the multidimensional case, and demonstrate that approximants of the continued fraction expansions of a
multi-formal Laurent series \( r \) we define indeed give the best rational estimates of \( r \).

The rational approximation problem of the multi-formal Laurent series is motivated by sequence synthesis problem, which has applications in the field of communication and cryptography. It is known that the single-sequence synthesis problem can be solved by the iterative Berlekamp-Massey (BM) algorithm \([1, 15]\), the Mills algorithm using continued fractions \([17]\), and the Euclidean algorithm as presented by Sugiyama Y. et al. \([23]\). Some consequential work shows that the continued fraction algorithm is a powerful tool \([3, 18, 20, 27]\). Though for solving sequence synthesis problem the continued fraction algorithm is equivalent to the BM algorithm and Euclidean algorithm, some mysterious data structures of the latter have natural interpretation in view of the former \([4, 5]\). The BM algorithm and the Euclidean algorithm can be generalized to solve the multi-sequence synthesis problem \([6, 7]\), which can also be solved by the lattice basis reduction algorithm as presented by Wang L.P. et al. \([24, 25, 26]\). It is natural to think of the generalization of continued fractions to multidimensional case.

Our main contributions in this paper are: 1, we give a definition of continued fraction in the multidimensional case; 2, set up the transformation from continued fractions to elements of \( F(((z^{-1}))^m) \); 3, prove that the approximants of a continued fraction are best rational approximants of its corresponding element; 4, give the procedures of computing the continued fraction expansions of a given element of \( F(((z^{-1}))^m) \); 5, give a natural 1-1 correspondence between strict \( m \)-continued fractions, which are \( m \)-continued fractions imposed with some additional conditions, and elements of \( F(((z^{-1}))^m) \); 6. apply the theory to solve the multi-sequence synthesis problem.

The rest of this paper is arranged as follows: Section 2 lists the notations used in the paper. Section 3 deals with the indexed valuation of \( F(((z^{-1}))^m) \). Section 4 is the detailed definition of the problem of multidimensional rational approximation. Section 5 defines \( m \)-pre-continued fractions and \( m \)-continued fractions and gives some of their basic properties. Section 6 states the main results of this paper. Section 7 completes all proofs.

2. Notations

In this paper we always keep the following notations.

\( F \) denotes a field, \( F((z^{-1})) = \left\{ \sum_{t \geq i} a_t z^{-t} \mid i \in Z, a_t \in F \right\} \) is the formal Laurent series field \([22]\), where \( Z \) denotes the integer ring. \( F(z) \) denotes the rational fraction field over \( F \), \( F[z] \) denotes the polynomial ring over \( F \). The discrete valuation on \( F((z^{-1})) \) is denoted by \( v(\cdot) \), i.e., \( v(\sum_{t \geq i} a_t z^{-t}) = i \) if \( a_i \neq 0 \). For \( \alpha = \sum_{d \geq -t \geq 0} a_t z^{-t} + \sum_{1 \leq t < \infty} a_t z^{-t} \in F_2((z^{-1})) \), the polynomial part \( \sum_{d \geq -t \geq 0} a_t z^{-t} \) is denoted by \([\alpha]\); and the remaining part \( \sum_{1 \leq t < \infty} a_t z^{-t} \) is denoted by \( \{\alpha\} \).

\( m \) denotes a positive integer, \( Z_m \) denotes the finite set \( \{1, 2, \ldots, m\} \). \( F((z^{-1}))^m \) denotes the set of all \( m \)-tuples over \( F((z^{-1})) \), similar for \( F(z)^m \).
and $F[z]^m$, and we always write those $m$-tuples in column form. For an element $\underline{r} = (r_1(z), r_2(z), \ldots, r_m(z))^\tau$ in $F((z^{-1}))^m$, where $\tau$ means transpose, we denote $([r_1(z)], [r_2(z)], \ldots, [r_m(z)])^\tau$ by $[\underline{r}]$, and 

$((r_1(z)), (r_2(z)), \ldots, (r_m(z)))^\tau$ by $\{\underline{r}\}$.

$I_n$ denotes the identity matrix of order $n$ for any positive integer $n$, and $e_j$ denotes the $j$-th column of the matrix $I_n$ for $1 \leq j \leq n$. Denote by $M_{a,b}(F((z^{-1})))$ the set of all possible matrices of order $a \times b$ over $F((z^{-1}))$ for any integers $a$ and $b$, and similar for $M_{a,b}(F[z])$.

### 3. Indexed valuation on $F((z^{-1}))^m$

In this section the indexed valuation on $F((z^{-1}))^m$ is introduced, which will be in use in studying the rational approximation problem for multidimensional Laurent series.

**Definition 3.1. (Linear ordering on $Z_m \times Z$)**

For any two elements $(h, v)$ and $(h', v')$ in $Z_m \times Z$, we say $(h, v) < (h', v')$ if $v < v'$, or $v = v'$ and $h < h'$.

It is clear that the ordering defined above on $Z_m \times Z$ is linear [10], i.e., any two elements can be compared.

**Lemma 3.2.** Let $(i, x)$ and $(j, y)$ be any two elements in $Z_m \times Z$, then $(i, x) < (j, y)$ if and only if $x + L_{i,j} \leq y$, or equivalently if and only if $x - l_{j,i} < y$, where $l_{i,j} = 1$ if $i > j$, $l_{i,j} = 0$ if $i \leq j$; $L_{i,j} = 1$ if $i \geq j$, $L_{i,j} = 0$ if $i < j$.

**Proof.** It is easy to check. \quad $\square$

For any $(j, x) \in Z_m \times Z$, we call the non-zero element $z^x e_j$ in $F((z^{-1}))^m$ a monomial, and denote

$$Iv(z^x e_j) = (j, -x) \in Z_m \times Z.$$ (3.1)

**Notation 3.3.** Denote by $\mathcal{M}$ the set of all monomials in $F((z^{-1}))^m$. Any element $\underline{r}$ in $F((z^{-1}))^m$ can be expressed uniquely as a sum of the following form:

$$\underline{r} = \sum_{\underline{m} \in \mathcal{M}} c_{\underline{m}} \underline{m}, \ c_{\underline{m}} \in F, \ \underline{m} \in \mathcal{M}. $$ (3.2)

We will say $\underline{m}$ is a monomial of $\underline{r}$ and denote $\underline{m} \in \underline{r}$ if $c_{\underline{m}} \neq 0$. The total number (finite or infinite) of the nonzero coefficients $c_{\underline{m}}$ will be called the *Hamming weight* of $\underline{r}$, denoted as $w_H(\underline{r}) = \tau$.

**Definition 3.4. (Indexed valuation)**

For an nonzero element $\underline{r} = (r_1(z), r_2(z), \ldots, r_m(z))^\tau$ in $F((z^{-1}))^m$, we call

$$Iv(\underline{r}) = \min\{Iv(\underline{m}) \mid \underline{m} \in \mathcal{M}, \ \underline{m} \in \underline{r} \} \in Z_m \times Z$$
the indexed valuation of \( r \), where \( Iv(m) \) is defined as in (3.1). If \( Iv(r) = (h,v) \), then \( v \) will be called the valuation of \( r \) and denoted by \( v(r) \), and \( h \) will be called the index of \( r \) and denoted by \( I(r) \). It is clear that

\[
\begin{align*}
   v(r) &= \min\{v(r_i(z)), 1 \leq i \leq m\}, \\
   I(r) &= \min\{i | 1 \leq i \leq m, v(r_i(z)) = v(r)\},
\end{align*}
\]

where \( v(r_i(z)) \) is the discrete valuation on \( F((z^{-1})) \). By convention \( v(0) = \infty \), \( Iv(0) = (1, \infty) \), and \( Iv(r) < Iv(0) \forall r \neq 0 \).

For \( r \in F((z^{-1}))^m \) such that \( w_H(r) = r < \infty \), define

\[
\text{Supp}(r) = \{Iv(m) | m \in \mathcal{M}, m \in r\},
\]

which will be called the support of \( r \), and denote by \( \text{Supp}^+(r) \) the largest element in \( \text{Supp}(r) \) if \( w_H(r) < \infty \). The following lemma is clear.

**Lemma 3.5.** The indexed valuation is a surjective mapping \( Iv: F((z^{-1}))^m \rightarrow (\mathbb{Z}_m \times \mathbb{Z}) \cup (1, \infty) \) which satisfies

1. \( Iv(r) = (1, \infty) \iff r = 0 \).
2. \( Iv(ru(z)) = (h, v + Iv(u(z))) \) for all \( r \in F((z^{-1}))^m \) and \( u(z) \in F((z^{-1})) \), where \( (h, v) = Iv(r) \).
3. \( Iv(r + s) \geq \min\{Iv(r), Iv(s)\} \) for all \( r, s \in F((z^{-1}))^m \). The equality holds true if \( Iv(r) \neq Iv(s) \).

4. **Rational Approximation**

In this section we give a detailed definition for the rational approximation problem on multi-formal Laurent series. Let \( r \in F((z^{-1}))^m, 0 \neq q(z) \in F[z], p(z) \in F[z]^m \), we call \( \frac{p(z)}{q(z)} \) a rational approximant to \( r \) if \( v(r - \frac{p(z)}{q(z)}) > \deg(q(z)) \), and call \( Iv(r - \frac{p(z)}{q(z)}) \) the precision of \( \frac{p(z)}{q(z)} \) to \( r \). Here it is worth pointing out that the components of \( \frac{p(z)}{q(z)} \) have a common denominator \( q(z) \). It is easy to see that \( \frac{p(z)}{q(z)} \) is a rational approximant to \( r \) if and only if \( p(z) = \lfloor r q(z) \rfloor \). So any nonzero polynomial \( q(z) \) can be the denominator of a rational approximant to \( r \), and such a rational approximant is unique. As a matter of convenience, any nonzero polynomial \( q(z) \) will be called a denominator of \( r \) with precision \( Iv(r - \frac{\lfloor r(z)q(z) \rfloor}{q(z)}) \).

**Definition 4.1.** (Best Rational approximant) Let \( r \in F((z^{-1}))^m \). We call \( \frac{p(z)}{q(z)} \in F(z)^m \) a best rational approximant of \( r \) if denominators of \( r \) with degree lower than \( \deg(q(z)) \) have precision lower than \( Iv(r - \frac{p(z)}{q(z)}) \), and denominators of \( r \) with degree same as \( q(z) \) have precision no greater than \( Iv(r - \frac{p(z)}{q(z)}) \). In this case, \( q(z) \) is also called a best denominator of \( r \).
Let $S$ be the set of all elements $\underline{r}$ of $F((z^{-1}))^m$ with $v(\underline{r}) > 0$. For the rational approximation problem on $F((z^{-1}))^m$, we need only consider elements in $S$. In fact, if we write $\underline{r} = [\underline{q}] + \{\underline{q}\}$, then, $\{\underline{q}\} \in S$, and $[\underline{q}] + \frac{\underline{p}(z)}{\underline{q}(z)}$ is a rational approximant to $\underline{r}$ of precision $(h, n)$, if and only if $\frac{\underline{b}(z)}{\underline{q}(z)} \in F(z)^m$ is a rational approximant to $\{\underline{q}\}$ of the same precision $(h, n)$; and $q(z)$ is a best denominator of $\underline{r}$ if and only if it is a best denominator of $\{\underline{q}\}$.

In studying the best rational approximation problem for $\underline{r} \in S$, motivated by the simple continued fractions in the case $m = 1$, we will introduce the multidimensional continued fraction in the next section.

5. Multidimensional continued fractions

In this section we introduce the $m$-pre-continued fractions and the $m$-continued fractions, and give some of their basic properties.

5.1. $m$-pre-continued fractions.

Definition 5.1. ($m$-pre-continued fraction) Let

\begin{equation}
C = [a_0, h_1, a_1, h_2, a_2, \ldots, h_k, a_k, \ldots], \quad 0 \leq k < \omega,
\end{equation}

where $h_k \in Z_n$ and $a_k \in F[z]^m$ for all $k : 0 \leq k < \omega$, and $\omega$ is a positive integer or $\infty$, then $C$ will be called an multidimensional-pre-continued fraction, or simply $m$-pre-continued fraction, and $\omega$ the length of $C$. We always assume $a_0(z) = 0$, since we will see $a_0$ is irrelevant to our concern.

Notation 5.2. Denote by $E_h$ the matrix of order $(m + 1)$ which comes by exchanging the $h$-th column and the $(m + 1)$-th column of the identity matrix $I_{m+1}$:

\begin{equation}
E_h = (e_1 \ e_2 \ e_{h-1} \ e_{m+1} \ e_{h+1} \cdot \cdot \cdot \ e_m \ e_h) \in M_{m+1,m+1}(F[z]).
\end{equation}

From $C$ defined as (5.1) we define iteratively the square matrices $B_k$ of order $(m + 1)$ over $F[z]$: \begin{equation}
B_0 = I_{m+1}, \quad B_k = B_{k-1}E_{h_k}A(a_k) \in M_{m+1,m+1}(F[z]), \quad k \geq 1,
\end{equation}
where

\begin{equation}
A(a_k) = \begin{pmatrix}
I_m & a_k \\
0 & 1
\end{pmatrix} \in M_{m+1,m+1}(F[z]),
\end{equation}

and denote by $b_k$ the last column of $B_k$, and let $p_k = p_k(z) \in F[z]^m$ and $q_k = q_k(z) \in F[z]$ be components of $b_k$, that is

\begin{equation}
b_k = (p_k, q_k)^T.
\end{equation}

Some properties of the sequence $\{(p_k, q_k)\}_{k \geq 0}$ are given in the following proposition.

Proposition 5.3. \hspace{2cm} (1) $\gcd(q_k(z), p_{k,1}(z), p_{k,2}(z), \ldots, p_{k,m}(z)) = 1$ for $k \geq 0$, where $p_{k,j}(z)$ is the $j$-th component of $p_k$. 
(2) For $k \geq 0$, let $P_{k-1}$ be the matrix of size $m \times m$ and $Q_{k-1}$ the matrix of size $1 \times m$ such that
\begin{equation}
B_k = \begin{pmatrix}
P_{k-1} & P_k \\
Q_{k-1} & q_k
\end{pmatrix},
\end{equation}
and denote by $P_{k-1,j}(\in F[z]^m)$ the $j$th column of $P_{k-1}$, and $Q_{k-1,j}(\in F[z])$ the $j$th component of $Q_{k-1}$, $1 \leq j \leq m$. Then for $k \geq 1$,
\begin{equation}
\begin{pmatrix}
p_k \\
q_k
\end{pmatrix} = \begin{pmatrix}
P_{k-1} \\
Q_{k-1}
\end{pmatrix} a_k(z) + \begin{pmatrix}
P_{k-2,h_k} \\
Q_{k-2,h_k}
\end{pmatrix},
\end{equation}
or explicitly
\begin{align*}
p_k &= p_{k-1} a_{k,h_k}(z) + \sum_{j \neq h_k, 1 \leq j \leq m} P_{k-2,j} a_{k,j}(z) + P_{k-2,h_k}, \\
q_k &= q_{k-1} a_{k,h_k}(z) + \sum_{j \neq h_k, 1 \leq j \leq m} Q_{k-2,j} a_{k,j}(z) + Q_{k-2,h_k}.
\end{align*}

(3) For $1 \leq k < \omega$ and $j \in \mathbb{Z}_m$, denote by $l(k,j)$ the largest positive integer $k' \leq k$ such that $h_{k'} = j$, and let $l(k,j) = 0$ if no such $k'$ exists. Then for $k \geq 1$,
\begin{align*}
(P_{k-1,j}, Q_{k-1,j}) &= \begin{cases}
  (p_{l(k,j)-1}, q_{l(k,j)-1}) & \text{if } l(k,j) \geq 1, \\
  (c_j, 0) & \text{if } l(k,j) = 0.
\end{cases}
\end{align*}

Proof. It is easy to prove. \qed

5.2. Conditions 1-4 and $m$-continued fractions.

Definition 5.4. (Conditions 1-4) For the $m$-pre-continued fraction $C$ defined as [1], we define condition 1 as below:
- **Condition 1**: $\deg(a_{k,h_k}(z)) \geq 1$, $\forall 1 \leq k < \omega$, where $a_{k,h_k}(z)$ denotes the $h_k$-component of $a_k(z)$.

In the sequel, we always assume $C$ satisfies the condition 1, and associate it with the following quantities (for each $k : 1 \leq k < \omega$):
\begin{align}
\begin{cases}
t_k = \deg(a_{k,h_k}(z)), & t_0 = 0, \\
v_{k,j} = \sum_{i=j, i \leq k} t_i, & v_{0,j} = 0, \\
v_k = v_{k,h_k}, & v_0 = 0,
\end{cases}
\end{align}
and the diagonal matrix $D_k$ (for each $k : 1 \leq k < \omega$):
\begin{align}
D_k &= \text{Diag.}(z^{-v_{k,1}}, z^{-v_{k,2}}, \ldots, z^{-v_{k,m}}) \\
&= \begin{pmatrix}
z^{-v_{k,1}} & & \\
& \ddots & \\
& & z^{-v_{k,m}}
\end{pmatrix}.
\end{align}

We define conditions 2-4 on $C$ as below (by convention $\infty - 1 = \infty$):
- **Condition 2**: $(h_k, v_{k-1,h_k}) < (h_{k+1}, v_{k+1}), 1 \leq k < \omega - 1$.
- **Condition 3**: $1v(D_k a_k) = (h_k, v_{k-1,h_k}), 1 \leq k < \omega$. 

• **Condition 4:** $\text{Supp}^+(D_k a_k) < (h_{k+1}, v_{k+1})$, $1 \leq k < \omega - 1$.

**Definition 5.5.** An $m$-pre-continued fraction $C$ is called an $m$-continued fraction if $C$ satisfies the conditions 1-3. An $m$-continued fraction $C$ is said to be strict if it satisfies the condition 4.

It is clear that Conditions 1, 3 and 4 imply Condition 2. These conditions can be stated in some equivalent forms as shown in the following proposition.

**Proposition 5.6.** Let $C$, which is defined as (5.1), satisfies the condition 1, keep all notations made for it, and denote

$$a_k = (a_{k,1}(z), \cdots , a_{k,j}(z), \cdots , a_{k,m}(z))^\tau, 1 \leq k < \omega,$$

where $a_{k,j}(z) \in F[z]$. Moreover, we associate $C$ with some quantities as shown in the following proposition.

Let $l_{i,j}$ and $L_{i,j}$ be defined as Lemma 3.2. Then

1. For $k : 1 \leq k < \omega - 1$, the following conditions are equivalent:
   a. $(h_k, v_{k-1}, h_k) < (h_{k+1}, v_{k+1})$.
   b. $(h_k, n_k) < (h_{k+1}, n_{k+1})$.
   c. $v_{k-1}, h_k - v_{k,j-1} + L_{h_k,h_{k+1}} \leq t_{k+1}$.

2. For $k : 1 \leq k < \omega$, the following two conditions are equivalent:
   a. $Iv(D_k a_k) = (h_k, v_{k-1}, h_k)$.
   b. $\deg(a_{k,j}(z)) \leq v_{k,j} - v_{k,j-1} - h_{k,j}, \forall 1 \leq j \leq m, j \neq h_k$.

3. Assume the condition 3 holds true. For $k : 1 \leq k < \omega - 1$, the following two conditions are equivalent:
   a. $\text{Supp}^+(D_k a_k) < (h_{k+1}, v_{k+1})$.
   b. For every $j : 1 \leq j \leq m$, $a_{k,j}(z)$ is of the form
      $$a_{k,j}(z) = \sum_{0 \leq x \leq X_{k,j}} a_{k,j,x} z^{t_{k,j}-x}, a_{k,j,x} \in F,$$
      where $X_{k,j} = \min\{t_{k,j}, x_{k,j}\}$, $t_{k,j} = v_{k,j} - v_{k,j-1} - h_{k,j}, x_{k,j} = v_{k+1} - v_{k,j-1} - h_{k,j} - L_{h_{k,j},h_{k+1}}$.

4. Assume both of the condition 2 and 3 hold true. For $k : 1 \leq k < \omega$ and $j \neq h_k$,
   $$\begin{cases} \deg(a_{k,j}(z)) < d_k - d_{l(k,j)-1} & \text{if } l(k,j) \geq 1, \\ \deg(a_{k,j}(z)) \leq 0 & \text{if } l(k,j) = 0. \end{cases}$$

**Proof.** It is easy to prove. □

6. **Main Results**

In this section we state the main results of this paper.
6.1. **Approximants of m-continued fraction-I.** In this subsection, we always let $C$, which is defined as (5.1), be an $m$-continued fraction. We keep all notations made for it in section 5.

**Theorem 6.1.** \( \deg(q_k(z)) = d_k \) for all \( k : 0 \leq k < \omega \).

Based on the above theorem, each pair \((p_k, q_k)\) provides a rational fraction \( \frac{p_k(z)}{q_k(z)} \), which will be called the \( k \)th approximant (or, convergent) of the \( m \)-continued fraction \( C \). The properties of these approximants are summarized in the following theorems.

**Theorem 6.2.**

\[
I v \left( \frac{p_{k-1}(z)}{q_{k-1}(z)} - \frac{p_k(z)}{q_k(z)} \right) = (h_k, n_k), \quad 1 \leq k < \omega.
\]

As a consequence, the sequence \( \{ \frac{p_k}{q_k} \}_{k \geq 1} \) is convergent in the case \( \omega = \infty \).

Denote

\[
\varphi(C) = \begin{cases} \frac{p_{\omega-1}(z)}{q_{\omega-1}(z)} & \text{if } \omega < \infty, \\
\lim_{k \to \infty} \frac{p_k(z)}{q_k(z)} & \omega = \infty.
\end{cases}
\]

We see \( \varphi(C) \in F((z^{-1}))^m \), and call \( C \) an \( m \)-continued fraction expansion of \( \varphi(C) \). Denote by \( C(r) \) the set of all possible \( m \)-continued fraction expansions of \( r \).

**Corollary 6.3.** Let \( r = \varphi(C) \). For \( k : 1 \leq k < \omega \), we have

1. \( I v(r - \frac{p_{k-1}(z)}{q_{k-1}(z)}) = (h_k, n_k) \). As a consequence, \( r q_{k-1}(z) - p_{k-1}(z) = \{r q_{k-1}(z)\} \) and \( I v(\{r q_{k-1}(z)\}) = (h_k, v_k) \).
2. \( I v(r) = (h_1, t_1) \).
3. \( I v(p_k(z)) = (h_1, -d_k + t_1) \). As a consequence, \( \deg(p_k(z)) = d_k - t_1 \), where \( \deg(p_k(z)) \) denotes the largest one among \( \deg(p_{k,j}(z)) \), \( 1 \leq j \leq m \).

**Proof.** It is easy to prove. \( \square \)

**Theorem 6.4.** Let \( r = \varphi(C) \). Assume \( q(z) \in F[z] \), \( d_k \leq \deg(q(z)) < d_{k+1} \) for some \( 0 \leq k < \omega \) (\( d_\omega = \infty \) if \( \omega < \infty \)). Then

\[
I v(r - \frac{r q(z)}{q(z)} \leq I v(r - \frac{p_k(z)}{q_k(z)}).
\]

As a consequence, each \( \frac{p_k(z)}{q_k(z)} \), \( 0 \leq k < \omega \), is a best rational approximant to \( r \), and the degree of any best denominator of \( r \) must be \( d_k \) for some \( k : 0 \leq k < \omega \).
6.2. \textit{m-CF Transform and m-continued fractions-I.} Given an element \( \underline{r} \in F((z^{-1}))^m \) with \( \underline{r} \neq \underline{0} \) and \( v(\underline{r}) > 0 \). In this subsection we introduce a transform, which we call \textit{m-CF Transform}, which may produce the \textit{m}-continued fraction expansions of \( \underline{r} \).

\textbf{Definition 6.5.} (\textit{D-matrix}) We call a diagonal matrix over \( F((z^{-1})) \) a \textit{D-matrix} if each of its diagonal elements is a power of \( z \).

\textbf{m-CF Transform:} Given \( \underline{r} \in F((z^{-1}))^m \), \( \underline{r} \neq \underline{0} \) and \( v(\underline{r}) > 0 \). Initially, take \( \underline{a}_0 = \underline{0} \in F[z^{-1}]^m \), \( \Delta_{-1} = I_m \) (the identity matrix of order \( m \)), and \( \beta_0 = \underline{r} \). Suppose \( [\underline{a}_0, h_1, \underline{a}_1, h_2, \underline{a}_2, \ldots, h_{k-1}, \underline{a}_{k-1}] \), \( \Delta_{k-2} = \text{Diag.}(\ldots, z^{-c_{k-1}}, \ldots) \) which is a \( D \)-matrix of order \( m \), and \( \beta_{k-1} = (\ldots, \beta_{k-1-j}, \ldots)^r \in F((z^{-1}))^m \) have been defined for an integer \( k \geq 1 \). If \( \beta_{k-1} = \underline{0} \), the algorithm terminates and denote \( \omega = k \). If \( \beta_{k-1} \neq \underline{0} \), then do the following steps:

1. Take \( h_k = I(\Delta_{k-2}\beta_{k-1}) \in Z_m \).
2. Take \( \Delta_{k-1} = \text{Diag.}(\ldots, z^{-c_{k-j}}, \ldots) \) where \( c_{k,j} = c_{k-1,j} \) if \( j \neq h_k \), and \( c_{k,h_k} = v(\Delta_{k-2}\beta_{k-1}) \in Z \).
3. Take \( \underline{a}_k = [\rho_k] - \underline{g}_k \) and \( \beta_k = \{\rho_k\} + \underline{g}_k \), where \( \rho_k = (\ldots, \rho_{k,j}, \ldots)^r \in F((z^{-1}))^m \), \( \rho_{k,j} \in F((z^{-1})) \), \( \rho_{k,j} = \frac{\beta_{k-1,j}}{\beta_{k-1,h_k}} \) if \( j \neq h_k \), \( \rho_{k,h_k} = \frac{1}{\beta_{k-1,h_k}} \).

Denote \( \omega = \infty \) if the above procedure never terminates.

An \( m \)-CF transform on \( \underline{r} \) will result in an \( m \)-pre-continued fraction: \( C = [\underline{0}, h_1, \underline{a}_1, h_2, \underline{a}_2, \ldots, h_k, \underline{a}_k, \ldots] \). Note that \( \underline{g}_k \) may not be unique at each step \( k \geq 1 \), different choices of \( \underline{g}_k \) will give different \( C \).

\textbf{Definition 6.6.} We denote by \( T(\underline{r}) \) the set of all possible \( m \)-pre-continued fractions obtained from \( \underline{r} \) by \( m \)-CF transforms.

\textbf{Definition 6.7.} (\textit{\( \Delta \)-polynomial part of \( \underline{r} \)}) Let \( \underline{r} \in F((z^{-1}))^m \), \( \Delta \) be a \( D \)-matrix. Write \( A = [\underline{r}] = \sum_{m \in \mathcal{M}} c_m \underline{m} \), \( c_m \in F \), \( \alpha = [\underline{r}] \), where \( \mathcal{M} \) is defined in Notation \( \text{(3.3)} \). We call

\[ |\underline{r}|_{\Delta} = \sum_{m \in \text{Supp}_{\Delta,\omega,-}(A)} c_m \underline{m} \]

the \textit{\( \Delta \)-polynomial part} of \( \underline{r} \), where

\[ \text{Supp}_{\Delta,\omega,-}(A) = \{ m \in \mathcal{M} \mid m \in A, \text{Iv}(\Delta m) < \text{Iv}(\Delta \alpha) \}. \]

Denote \( |\underline{r}|_{\Delta}^+ = \sum_{m \in \text{Supp}_{\Delta,\alpha,+}(A)} c_m \underline{m} \), where

\[ \text{Supp}_{\Delta,\alpha,+}(A) = \{ m \in \mathcal{M} \mid m \in A, \text{Iv}(\Delta \alpha) < \text{Iv}(\Delta m) \}. \]

We see \( |\underline{r}| = |\underline{r}|_{\Delta} + |\underline{r}|_{\Delta}^+ \), by noting that the mapping \( m \mapsto \Delta m \) is a bijection on \( \mathcal{M} \).
Lemma 6.8. Let \( r \in F((z^{-1}))^m \), \( A = |r| \), \( \alpha = \{r\} \); and let \( \Delta \) be a D-matrix. Assume \( \epsilon \in F[z]^m \), and \( a = A - \epsilon \). Then the following conditions are equivalent:

\[
\begin{align*}
(1) & \quad \overline{a} = |r|_\overline{\Delta}. \\
(2) & \quad Iv(\Delta \alpha) \leq Iv(\Delta \epsilon) \text{ if } a = 0, \text{ and } Supp^+(\Delta \overline{a}) < Iv(\Delta \alpha) \leq Iv(\Delta \epsilon) \text{ if } a \neq 0, \\
(3) & \quad \begin{cases} 
\text{Supp}(a) \cup \text{Supp}(\epsilon) = \text{Supp}(A), \\
\text{Supp}(a) \subseteq \text{Supp}_{\Delta,\alpha,-}(A), \\
\text{Supp}(\epsilon) \subseteq \text{Supp}_{\Delta,\alpha,+}(A),
\end{cases}
\end{align*}
\]

where \( \text{Supp}_{\Delta,\alpha,-}(A) \) and \( \text{Supp}_{\Delta,\alpha,+}(A) \) are defined as (6.1) and (6.2). In particular, \( \overline{|r|_\Delta} = r \) for any \( r \in F[z]^m \).

Proof. It is easy to prove. \( \square \)

Definition 6.9. (Mapping \( \psi \)) Define \( \psi(r) \) to be the element in \( T(r) \), obtained from \( r \) by the m-CF transform, where at each step we choose \( a_k = \lfloor R_k^{-1} \Delta_k \rfloor \Delta_{k-1} \), i.e., we choose \( a_k = \lfloor R_k^{-1} \Delta_k \rfloor \Delta_{k-1} \).

The properties of m-CF transform are summarized in the following theorem.

Theorem 6.10. Denote by \( S \) the set of all possible \( r \in F((z^{-1}))^m \) with \( v(r) > 0 \), and by \( C^* \) the set of all possible strict m-continued fractions. Then

\[
\begin{align*}
(1) & \quad T(r) = C(r) \quad \forall r \in S. \\
(2) & \quad The \ mapping \ \psi \ is \ a \ bijection \ from \ S \ onto \ C^*, \ and \ \varphi \ is \ its \ inverse.
\end{align*}
\]

6.3. An application to multi-sequence synthesis problem. In this subsection we show m-continued fractions can be used to solve the multi-sequence synthesis problem. The multi-sequence synthesis problem was stated in terms of linear feedback shift registers [6, 7]. For convenience, we restate it by means of the indexed valuation on \( F((z^{-1}))^m \). For any given sequence \( r = \{c_t\}_{t \geq 0} \) over \( F \), where \( c_t \in F \), we identify it with the formal Laurent series \( r(z) = \sum_{t \geq 0} c_t z^{-1-t} \) with valuation larger than 0, and let \( r^{(n)} = (c_0, c_1, \ldots, c_{n-1}) \) be the length \( n \) prefix of the sequence \( r \). For any given multi-sequences \( \underline{r} = (r_1, \ldots, r_j, \ldots, r_m) \), where each \( r_j = \{c_{j,t}\}_{t \geq 0} \) is a sequence over \( F \), we identify it with the element \( \underline{r} = (r_1(z), r_2(z), \ldots, r_m(z))^\tau \in F((z^{-1}))^m \) with valuation larger than 0, where \( r_j(z) = \sum_{t \geq 0} c_{j,t} z^{-1-t} \) is the formal Laurent series identified with the \( j \)th sequence \( r_j \), and let \( \underline{r}^{(n)} = (r_1^{(n)}, r_2^{(n)}, \ldots, r_m^{(n)})^\tau \) be the length \( n \) prefix of the multi-sequences \( \underline{r} \). Given a polynomial \( q(z) \) over \( F \), we call it a characteristic polynomial of \( r^{(n)} \) if \( Iv\{\underline{r} - \frac{\underline{r}^{(n)}}{q(z)}\} > (m, n) \), or equivalently \( Iv\{\underline{r} q(z)\} > (m, n - \deg(q(z))) \); and call it a minimal polynomial of \( \underline{r}^{(n)} \) if it is a characteristic polynomial of \( r^{(n)} \) of the smallest degree; and call \( \deg(q(z)) \) the linear complexity of \( r^{(n)} \), denoted by \( L_n(r) \), if \( q(z) \) is a minimal polynomial of \( r^{(n)} \). The multi-sequence synthesis problem is: Given a
multi-sequences \( r \in F((z^{-1}))^m \), find a minimal polynomial and the linear complexity of \( r^{(n)} \) for each \( n \geq 1 \).

This problem is solved by using \( m \)-continued fractions as shown in the following corollary.

**Corollary 6.11.** Given a multi-sequences \( r \in F((z^{-1}))^m \), let \( C \in T(r) \). Write \( C \) in the form of (5.1). Then \( q_k(z) \) is a minimal polynomial of \( r^{(n)} \) and \( L_n(r) = d_k \) for \( n_k \leq n < n_k+1 \), \( 0 \leq k < \omega \), where we let \( n_\omega = \infty \) in the case \( \omega < \infty \), and \( n_0 = 1 \).

**Proof.** It is easy to prove. \( \square \)

7. **Proofs of Main Results**

In this section we complete the proofs for all results of this paper.

7.1. **Approximants of \( m \)-continued fraction-II.**

**Proof of Theorem 6.1.** Keep all the notations made for \( C \). We prove it by induction on \( k \). For \( k = 1 \), we have \( q_1(z) = a_{1,h_1}(z) \) by Proposition 5.3 and \( \deg(q_1(z)) = t_1 = d_1 \). Assume we are done for \( k < k \), where \( k \geq 2 \). From Proposition 5.6 we have

\[
q_k(z) = q_{k-1}(z)a_{k,h_k}(z) + Q_{k-2,h_k} + \sum_{j \neq h_k, 1 \leq j \leq m} Q_{k-2,j}a_{k,j}(z).
\]

The wanted result \( \deg(q_k(z)) = d_k \) follows by observing the following facts:

- \( \deg(q_{k-1}(z)a_{k,h_k}(z)) = d_{k-1} + t_k = d_k \) (induction assumption).
- If \( Q_{k-2,j} \neq 0 \), from Proposition 5.3 we see \( l(k-1,j) \geq 1 \), then \( \deg(Q_{k-2,j}) = \deg(q_{l(k-1,j)-1}) = d_{l(k-1,j)-1} \) (by induction assumption). In particular, \( \deg(Q_{k-2,h_k}) < d_k \).
- For \( j \neq h_k \) and \( Q_{k-2,j} \neq 0 \), from the above and Proposition 5.6 we get \( \deg(Q_{k-2,j}a_{k,j}(z)) < d_{l(k-1,j)-1} + d_k - d_{l(k,j)-1} \), then \( \deg(Q_{k-2,j}a_{k,j}(z)) < d_k \), since \( l(k-1,j) = l(k,j) \) in this case. \( \square \)

**Lemma 7.1.** Let \( C \), which is defined as (5.1), be an \( m \)-continued fraction, and \( \varphi = \varphi(C) \), and keep all notations made for it. Let

\[
y_{r,t} = \frac{p_r(z)}{q_r(z)} - \frac{p_t(z)}{q_t(z)}, \forall 0 \leq r < t,
\]

\[
z_{r,t} = \frac{p_r(z)q_t(z) - p_t(z)q_r(z)}{q_r(z)}, \forall 0 \leq r < t.
\]

Then

1. \( Iv(z_{r,t}) = (h,v) \), if and only if \( Iv(y_{r,t}) = (h,v + d_r + d_t) \).
2. The following statements are equivalent to each other:
   - \( Iv(y_{r-1,t}) = (h_t, n_t), \) if \( 0 < t < k \),
   - \( Iv(z_{r-1,t}) = (h_t, v_t - d_t), \) if \( 0 < t < k \),
   - \( Iv(y_{r,t}) = (h_{r+1}, n_{r+1}), \) if \( 0 \leq r < t < k \),
   - \( Iv(z_{r,t}) = (h_{r+1}, v_{r+1} - d_t), \) if \( 0 \leq r < t < k \).
Proof. The item (1) can be verified easily by Theorem 6.4. The item (2) can be obtained by a routing proof.

Proof of Theorem 6.2: Keep all the notations made for $C$. It is enough to prove that $Iv(y_{-1,t}) = (h_t,n_t)$ for $t \geq 1$, which is equivalent to

\begin{equation}
Iv(z_{-1,t}) = (h_t,v_t - d_t), \quad 1 \leq t
\end{equation}

from Lemma 7.1. For $t = 1$, $z_{0,1} = p_1(z)$, then $Iv(z_{0,1}) = Iv(p_1(z)) = (h_1,d_1 - t_1) = (h_1,v_1 - d_1)$ by Corollary 6.3. Suppose we are done for $t < k$, where $k \geq 2$, which together with Lemma 7.1 implies

\begin{equation}
Iv(z_{r,t}) = (h_{r+1},v_{r+1} - d_t), \quad 0 \leq r < t < k.
\end{equation}

By Proposition 5.3 we have

\begin{equation}
\begin{aligned}
&z_{k-1,k} \\
&= p_{k-1}q_k - p_{k}q_{k-1} \\
&= p_{k-1}(q_{k-1}a_{k,h_k}(z) + \sum_{j \neq h_k} Q_{k-2,j}a_{k,j}(z) + Q_{k-2,h_k}) - (p_{k-1}a_{k,h_k}(z) + \sum_{j \neq h_k} P_{k-2,j}a_{k,j}(z) + P_{k-2,h_k}q_{k-1}) \\
&= -(p_{k-2,h_k}q_{k-1} - p_{k-1}Q_{k-2,h_k}) - \sum_{j \neq h_k} (P_{k-2,j}q_{k-1} - p_{k-1}Q_{k-2,j})a_{k,j}(z) \\
&= -Z_{k,h_k} - \sum_{j \neq h_k} Z_{k,j}a_{k,j}(z),
\end{aligned}
\end{equation}

where

\begin{equation}
Z_{k,j} = p_{k-2,j}q_{k-1} - p_{k-1}Q_{k-2,j} \quad \forall 1 \leq j \leq m.
\end{equation}

By Proposition 5.3 we can get

\begin{equation}
Z_{k,j} = \begin{cases}
Z_{l(k-1,j)}^{-1,-1,k-1} & \text{if } l(k-1,j) \geq 1, \\
Z_{j}q_{k-1}(z) & \text{if } l(k-1,j) = 0.
\end{cases}
\end{equation}

Hence by (7.2) we get

\begin{equation}
Iv(Z_{k,j}) = (j,v_{l(k-1,j)} - d_{k-1}).
\end{equation}

In particular, we have

\begin{equation}
Iv(Z_{k,h_k}) = (h_k,v_{l(k-1,h_k)} - d_{k-1}) = (h_k,v_k - d_k).
\end{equation}

For $j \neq h_k$, we have

\begin{equation}
\begin{aligned}
Iv(Z_{k,j}a_{k,j}(z)) \geq \ & (j,v_{l(k-1,j)} - d_{k-1} - (v_{l(k,j)} - v_{l(k-1,h_k)} - l_{h_k,j})) \\
&= (j,v_{l(k-1,h_k)} - d_{k-1} + l_{h_k,j}) \\
&> (h_k,v_{l(k-1,h_k)} - d_{k-1}) \\
&= (h_k,v_k - d_k),
\end{aligned}
\end{equation}

\end{proof}
where the first line comes from \(v(k,j) = v_{k,j}\), the second line comes from \(l(k,j) = l(k-1,j)\) since \(j \neq h_k\), the last two lines come from definitions. This together with Lemma 7.2 and Lemma 7.3 leads to equation (7.1) when \(t = k\).

**Lemma 7.2.** Let \(C\), which is defined as (5.7), be an \(m\)-continued fraction, and \(r = \varphi(C)\), and keep all notations made for it. Assume \(c_i(z) \in F[z]\), \(\deg(c_i(z)) < t_{i+1}\), \(0 \leq i < \omega - 1\). Then

1. \(\text{Iv}(\{rq_i(z)c_i(z)\}) = (h_{i+1}, v_{i+1} - \deg(c_i(z)))\).
2. \(\text{Iv}(\{zq_i(z)c_i(z)\}) \neq \text{Iv}(\{rq_j(z)c_j(z)\})\) for all \(0 \leq j \neq i < \omega\) and \(c_i(z)c_j(z) \neq 0\).

**Proof.** It is easy to prove. □

**Proof of Theorem 6.4.** Keep all the notations made for \(C\). This theorem is true in the case \(k+1 = \omega < \infty\), since \(r = \frac{p_i(z)}{q_i(z)}\) in this case. We need only consider the case \(k+1 < \omega\). Denote \(d = \deg(q(z))\). We have \(d_k \leq d < d_{k+1}\) by the assumption. It is enough to prove \(\text{Iv}(\{rq(z)\}) \leq (h_{k+1}, n_{k+1} - d)\).

We can write \(q(z) = \sum_{0 \leq i \leq k} c_i(z)q_i(z)\), where \(\deg(c_i(z)) < \deg(q_{i+1}(z)) - \deg(q_i(z)) = t_{i+1}\) for \(i \geq 0\) (note that \(q_0(z) = 1\) and \(\deg(c_k(z)) = d - d_k \geq 0\). Note that \(\{rq(z)\} = \sum_{0 \leq i \leq k} \{rq_i(z)c_i(z)\}\), from Lemma 7.2 and Lemma 8.3 we have

\[
\text{Iv}(\{rq(z)\}) = \min \{ \text{Iv}(\{rq_i(z)c_i(z)\}) \mid c_i(z) \neq 0, 0 \leq i \leq k \} \\
\leq \text{Iv}(\{rq_k(z)c_k(z)\}) \\
= (h_{k+1}, v_{k+1} - \deg(c_k(z))) = (h_{k+1}, n_{k+1} - d). □
\]

### 7.2. \(m\)-CF Transform and \(m\)-continued fractions-II

In this subsection, at first we restate the \(m\)-CF Transform in terms of matrices, then we prove Theorem 6.10. Before giving the matrix version of \(m\)-CF Transform we give some necessary definitions.

**Definition 7.3.** (Base matrix and its \(D\)-component)

Let \(R\) be a matrix of order \(m\) over \(F((z^{-1}))\), we call it a base matrix if each of the columns of \(R\) is nonzero, and the index of the \(j\)-th column of \(R\) is \(j\) for each \(1 \leq j \leq m\). For a base matrix \(R\), the \(D\)-matrix \(\Delta = \text{Diag.}(z^{-c_1}, z^{-c_2}, \ldots, z^{-c_m})\) will be called the \(D\)-component of \(R\), where \(c_j\) denotes the valuation of the \(j\)-th column of \(R\).

**Proposition 7.4.** (Matrix version of \(m\)-CF Transform) The \(m\)-CF Transform which is introduced in subsection 6.2. can be restated in terms of matrices as follows:

Given \(r \in F((z^{-1}))^m\), \(r \neq 0\) and \(v(r) > 0\). Initially let \(R_{-1} = I_m\), \(r_0 = r\). Suppose \((-R_{k-2} \text{mod}_{k-1}) \in M_{m,m+1}(F((z^{-1}))\), \(k \geq 1\), is defined, where \(R_{k-2} \in M_{m,m}(F((z^{-1}))\) and \(r_{k-1} \in F((z^{-1}))^m\). If \(r_{k-1} = 0\) the algorithm terminates and denote \(\omega = k\). If \(r_{k-1} \neq 0\) then do the following steps:

1. Let \(h_k = I(r_{k-1}) \in Z_m\).
(2) Let $R_{k-1}$ be the matrix of order $m$ and $s_{k-2} \in F((z^{-1}))^m$ such that $(-R_{k-1} s_{k-2}) = (-R_{k-2} \tau_{k-1})E_{h_k} \in M_{m,m+1}(F((z^{-1})))$, where $E_{h_k}$ is defined in Notation 7.2.

(3) Let $R_{k-1}^{-1}s_{k-2} = A_k + \alpha_k \in F((z^{-1}))^m$, where $A_k = [R_{k-1}^{-1}s_{k-2}]$, and $\alpha_k = \{R_{k-1}^{-1}s_{k-2}\}$.

(4) Let $u_k$ be any one in $F[z]^m$ such that $u_k = A_k - \xi_k$, where $\xi_k \in F[z]^m$, $Iv(\Delta_{k-1}\alpha_k) \leq Iv(\Delta_{k-1}\xi_k)$, and $\Delta_{k-1}$ is the $D$-component of $R_{k-1}$.

(5) Let $\tau_k \in F((z^{-1}))^n$ such that $(-R_{k-1} \tau_k) = (-R_{k-1} s_{k-2})A(u_k) \in M_{m,m+1}(F((z^{-1})))$.

Denote $\omega = \infty$ if the above procedure never terminates.

Proof. It follows directly from the following two lemmas (Lemma 7.6 and Lemma 7.3).

**Lemma 7.5.** Keep notations made in Proposition 7.2. Denote $\beta_{k-1}^* = \alpha_{k-1} + \epsilon_{k-1}$. For $k \geq 1$, we have

1. $\beta_{k-1}^* = 0$ if and only if $\tau_{k-1} = 0$.
2. $R_{k-1}$ is a base matrix.
3. $\tau_{k-1} = R_{k-2}\beta_{k-1}^*$.
4. $v(\Delta_{k-j}) = \begin{cases} v(\Delta_{k-j}) & \text{if } j \neq h_k, \\ v(\Delta_{k-2}\beta_{k-1}^*) & \text{if } j = h_k. \end{cases}$
5. Denote $R_{k-1}^{-1}\Delta_{k-1} = (\cdots, \rho_{k,j}, \cdots)^{\tau} \in F((z^{-1}))^m$, $\rho_{k,j}^{*} \in F((z^{-1}))$. Then $\rho_{k,j}^{*} = \frac{\beta_{k-1,j}^{*}}{\beta_{k-1,h_k}^{*}}$ if $j \neq h_k$, $\rho_{k,h_k}^{*} = \frac{1}{\beta_{k-1,h_k}^{*}}$, where $\beta_{k-1,j}^{*}$ is the $j$-th component of $\beta_{k-1}^{*}$.

Proof. It is easy to check.

**Lemma 7.6.** Any base matrix $R$ is invertible, and $Iv(Rr) = Iv(\Delta r)$ for all $r \in F((z^{-1}))^n$, where $\Delta$ is the $D$-component of $R$.

Proof. It is easy to prove.

We divide Theorem 6.10 in the following two parts, then prove them separately.

**Part 1 of Theorem 6.10** $T(\tau) \subseteq C(\tau)$, and $\psi(\tau)$ is a strict $m$-continued fraction.

**Part 2 of Theorem 6.10** $C(\tau) \subseteq T(\tau)$, Moreover, if $C \in C(\tau)$ is strict, then $C = \psi(\tau)$.

To start the proof, we make a notation which will be used later.

**Notation 7.7.** Let $R$ be a base matrix, define

\[ v(R) = (c_1, c_2, \cdots, c_j, \cdots, c_m), \]

where $c_j$ denotes the valuation of the $j$-th column of $R$. We use the notation $(h, v) > v(R)$ to denote the fact that $v > c_h$. Notations $(h, v) \leq v(R)$, $v(R) > (h, v)$, etc. should be interpreted similarly.
Lemma 7.8. Let $\Delta$ be a $D$-matrix, $0 \neq \underline{r} \in F((z^{-1}))^m$. Then $\nu(\Delta \underline{r}) > \nu(\Delta)$ if $\nu(\underline{r}) > 0$ (including $\underline{r} = \underline{0}$); and $\nu(\Delta \underline{r}) \leq \nu(\Delta)$ if $\underline{0} \neq \underline{r} \in F[z]^m$.

Proof. It is easy to prove. □

Lemma 7.9. Let $R$ be a base matrix, $\Delta$ be the $D$-component of $R$, $\underline{s} \in F((z^{-1}))^m$, $A = [R^{-1}\underline{s}]$, and $\alpha = \{R^{-1}\underline{s}\}$. Let $\epsilon \in F[z]^m$, and $\underline{a} = A - \epsilon$. Then the following conditions are equivalent:

1. $\nu(\Delta \epsilon) \leq \nu(\Delta \underline{a})$.
2. $\epsilon = \underline{0}$, or $\nu(\Delta \epsilon) < \nu(\Delta \underline{a})$.
3. $\nu(-Ra + \underline{s}) = \nu(\Delta \epsilon)$.
4. $\nu(-Ra + \underline{s}) > \nu(\Delta)$.

Proof. It is easy to prove. □

Lemma 7.10. Let $R$ be a base matrix, $\Delta$ be the $D$-component of $R$, $\underline{s} \in F((z^{-1}))^m$, $A = [R^{-1}\underline{s}]$, and $\alpha = \{R^{-1}\underline{s}\}$. Let $\epsilon \in F[z]^m$, and $\underline{a} = A - \epsilon$. If $\nu(R) > \nu(\underline{s})$ and $\nu(\Delta \epsilon) \leq \nu(\Delta \underline{a})$, then

1. $\nu(\underline{s}) = \nu(\Delta \underline{a}) = \nu(\Delta \epsilon)$, and $\underline{a} \neq \underline{0}$.
2. $\nu(\Delta \underline{a}) < \nu(-Ra + \underline{s}) > \nu(\Delta)$.
3. $\underline{a} = [R^{-1}\underline{s}]^{-2}$ if and only if $\text{Supp}^+(\Delta \underline{a}) < \nu(-Ra + \underline{s}) > \nu(\Delta)$.

Proof. It is easy to prove. □

Lemma 7.11. Let $\underline{r} \in F((z^{-1}))^m$, $\nu(\underline{r}) > 0$, and keep the quantities obtained by acting the $m$-CF transform on $\underline{r}$, say, $h_k, R_{k-1}, s_k, \alpha_k, \Delta_{k-1}, \epsilon_k, a_k, r_k$, and $\omega$, etc. Let

$$C = [a_0, h_1, a_2, h_2, a_3, \ldots, h_k, a_k, \ldots] \in T(\underline{r}).$$

Then, for $k : 1 \leq k < \omega$ we have

1. $R_{k-1}$ is a base matrix, so the $m$-CF transform is well defined.
2. $\nu(R_{k-1}) > \nu(s_k) = \nu(\Delta_{k-1}a_k) < \nu(r_k) > \nu(\Delta_{k-1})$.
3. $\text{deg}(a_k, \text{ht}(z)) \geq 1$, i.e., $C$ satisfies the condition 1.
4. Based on the item (3), we may keep the quantities associated with $C$, which are defined as in section 5, say, $t_k, D_k, v_{k,j}, v_k$, etc. Then

$$\begin{cases}
\nu(\Delta_{k-1,j}) = \nu(R_{k-1,j}) = (j, v_{k,j}). \\
\Delta_{k-1} = D_k. \\
\nu(r_k) = (h_k, v_k). \\
\nu(s_{k-2}) = (h_k, v_{k-1, h_k}).
\end{cases}$$

Proof. It is easy to prove. □

Proof of Part 1 of Theorem 6.10

From Lemma 7.11 whose notations are kept, we see $C$ satisfies the conditions 1, so, in order to prove $C$ is an $m$-continued fraction, it is enough to prove $C$ satisfies the conditions 2 and 3. From Lemma 7.11 we can see:

- For $k : 1 \leq k < \omega - 1$, we have $(h_k, v_{k-1, h_k}) = \nu(s_{k-2}) < \nu(r_k) = (h_{k+1}, v_{k+1})$, which means condition 2 is satisfied.
• For $k: 1 \leq k < \omega$, we have $(h_k, v_{k-1, h_k}) = Iv(s_{k-2}) = Iv(\Delta_{k-1} s_k) = Iv(D_k s_k)$, which shows that condition 3 is satisfied.

In order to prove $C$ is strict if $C = \psi(\rho)$, it is enough to prove $C$ satisfies the condition 4. From $C = \psi(\rho)$ we have $a_k = |R_{k-1}^{-1} s_{k-2}|_{\Delta_{k-1}}$.

From Lemma 7.11 we see $R_{k-1}$ is a base matrix, and $v(R_{k-1}) > Iv(s_{k-2})$; from definition of $m$-CF transform we see $Iv(\Delta_{k-1}(R_{k-1}^{-1} s_{k-2})) \leq Iv(\Delta_{k-1} s_{k})$, so we can apply Lemma 7.10 (taking $R = R_{k-1}, s = s_{k-2}, \epsilon = \epsilon_k$) to get $\text{Supp}^+(\Delta_{k-1} a_k) < Iv(-R_{k-1} a_k + s_{k-2}) = Iv(r_k)$. Then, by Lemma 7.11 we get $\text{Supp}^+(\Delta_{k-1} a_k) < Iv(r_k) = (h_{k+1}, v_{k+1})$ for $1 \leq k < \omega - 1$, i.e., the condition 4 is satisfied.

Finally we prove $\varphi(C) = r$. Keep all the notations made for $C$ in section 5. By definition of $m$-CF transform, we can see $(-R_{k-1}, r_k) = (-R_{k-2}, r_{k-1})E_{h_k} A(a_k) = \cdots = (-I_m, \rho)B_k$. This leads to $r_k = -p_k + q_k$ (the last column of $B_k$ is $(p_k, q_k)^T$), so $r - p_k/q_k = r_k/q_k$. By Lemma 7.11 we have $Iv(r_k) = (h_{k+1}, v_{k+1})$, then $Iv(r_k/q_k) = (h_{k+1}, v_{k+1} + \deg q_k) = (h_{k+1}, v_{k+1} + d_k)$. Therefore $\lim_{k \to \infty} p_k/q_k = r$ since $\lim_{k \to \infty} d_k = \infty$, in other words, $\varphi(C) = r$.

Lemma 7.12. Let $C = [a_0, h_1, a_1, h_2, a_2, \ldots, h_k, a_k, \ldots]$, which is defined as in Lemma 6.4, be an $m$-continued fraction, and keep all notations made for it, say, $v_k$, $D_k$, etc. Let $\rho = \varphi(C)$. Define $R_{k-1} \in M_{m,m}(F((z^{-1})))$, $r_k \in F((z^{-1}))^m$ and $s_{k-2} \in F((z^{-1}))^m$ iteratively as below ($1 \leq k < \omega$):

\[
\begin{cases}
R_{k-1} = I_m, & r_0 = \rho \\
(-R_{k-1}, s_{k-2}) = (-R_{k-2}, r_{k-1})E_{h_k} A(a_k) = \cdots = (-I_m, \rho)B_k \\
(-R_{k-1}, r_k) = (-R_{k-1}, s_{k-2})A(a_k) \in M_{m,m+1}(F((z^{-1})))
\end{cases}
\]

Denote by $R_{k-1,j}$ the $j$-th column of $R_{k-1}$, denote by $\Delta_{k-1}$ the $D$-component of $R_{k-1}$, and denote $s_k = [R_{k-1}^{-1} s_{k-2}] - a_k$. Then, for $1 \leq k < \omega$, we have

\begin{enumerate}
\item $Iv(r_k) = (h_k, v_k)$.
\item $R_{k-1}$ is a base matrix, and $\Delta_{k-1} = D_k$.
\item $Iv(r_k) = Iv(-R_{k-1} a_k + s_{k-2}) > v(\Delta_{k-1})$.
\item $v(R_{k-1}) > Iv(s_{k-2})$.
\item $Iv(\Delta_{k-1}(R_{k-1}^{-1} s_{k-2})) \leq Iv(\Delta_{k-1} s_k)$.
\end{enumerate}

Proof. It is easy to prove.

\begin{proof}
\end{proof}

Proof of Part 2 of Theorem 6.10
Keep notations made in Lemma 6.12. From Lemma 7.12 we have $Iv(r_k) = h_k$ and $Iv(\Delta_{k-1}(R_{k-1}^{-1} s_{k-2})) \leq Iv(\Delta_{k-1} s_k)$ for $1 \leq k < \omega$, which show that $r_k$ is an $m$-CF transform, so $C \in T(r)$. To prove $C = \psi(\rho)$ when $C$ is strict, it is enough to show $a_k = [R_{k-1}^{-1} s_{k-2}]_{\Delta_{k-1}}$ for $k: 1 \leq k < \omega$. In fact, if $C$ is strict, we have $\text{Supp}^+(D_k a_k) < (h_{k+1}, v_{k+1})$ for $1 \leq k < \omega - 1$, then $\text{Supp}^+(\Delta_{k-1} a_k) < (h_{k+1}, v_{k+1}) = Iv(r_k) = Iv(-R_{k-1} a_k + s_{k-2}) > v(\Delta_{k-1})$
by Lemma 7.12, thus $a_k = \lfloor R_{k-1}^{-1} s_{k-2} \delta_{k-1} \rfloor$ according to Lemma 7.10 (taking $R = R_{k-1}, s = s_{k-2}, \delta = \delta_k$), since all the conditions for applying Lemma 7.10 are satisfied by Lemma 7.12.

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E-mail address: yangdai@public.bta.net.cn
E-mail address: kunpengwang@263.net
E-mail address: ydf@is.ac.cn

State Key Laboratory of Information Security (Graduate School of Chinese Academy of Science), Beijing, 100039