The quantum modes of the (1+1)-dimensional oscillators in general relativity

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Abstract

The quantum modes of a new family of relativistic oscillators are studied by using the supersymmetry and shape invariance in a version suitable for (1+1) dimensional relativistic systems. In this way one obtains the Rodrigues formulas of the normalized energy eigenfunctions of the discrete spectra and the corresponding rising and lowering operators.

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1 Introduction

In general relativity, the geometric models play the role of kinetics, helping us to understand the characteristics of the classical or quantum free motion on a given background. One of the simplest geometric models in (1+1) dimensions is that of the quantum relativistic oscillator (RO) defined as a free massive scalar particle on the anti-de Sitter static background [1, 2, 3]. Recently, we have generalized this model to a family of quantum models of RO whose metrics are one-parameter deformations (i.e. conformal transformations) of the anti-de Sitter or de Sitter ones [4]. As it is shown in Refs. [5], the deformed anti-de Sitter metrics give the relativistic correspondents the usual nonrelativistic Pöschl-Teller (PT) problems while the deformed de Sitter metrics generate relativistic Rosen-Morse (RM) problems [6]. A remarkable property of these RO is that all of them have as nonrelativistic limit just the usual nonrelativistic harmonic oscillator (NRHO) [4].

In these relativistic models the Klein-Gordon equation is analytically solvable in the same manner as the Schrödinger equation of the mentioned well-studied nonrelativistic problems. This allows one to study the RO by using the successful methods of supersymmetry and shape invariance [7] with minimal changes requested by the specific form the Klein-Gordon equation [8]. In this way one can derive the normalized energy eigenfunctions of the discrete energy spectrum and the form of the shift operators of the energy basis that are involved in the structure of the dynamical algebras [9].

Here we would like to present a systematic study of our family of RO based on the supersymmetry and shape invariance of the relativistic potentials, pointing out the main specific features of the PT and RM relativistic problems. We believe that this first example of a family of metrics generating analytically solvable quantum problems could be of interest for further investigations concerning the supersymmetry of other solvable relativistic quantum models in (3+1) dimensions [10, 11] or more [12].

We start in Sec.2 with a short review of the (1+1) relativistic scalar quantum mechanics constructed as the one-particle restriction of the theory of the scalar free field on curved space-time. We define the state space in the coordinate representation and we introduce the coordinate and momentum operators. In Sec.3 we present the relativistic PT and RM oscillators giving their energy spectra and the energy eigenfunctions up to normalization.
factors. The relativistic supersymmetry and the shape invariance of the relativistic PT and RM potentials are used in the next section for deriving the definitive form of the normalized energy eigenfunctions of the discrete energy spectra. The Sec. 5. is devoted to the properties of the shift operators of the energy bases of our RO. Therein we recover the known shift operators of the PT models [13] and we derive those of the RM models.

2 Relativistic quantum mechanics

It is well-known that the one-particle relativistic quantum mechanics cannot be constructed as an independent consistent theory because of some difficulties related to the probabilistic interpretation of the relativistic wave functions. The good theory of the relativistic quantum systems is in fact the quantum field theory where the second quantization guarantees a coherent probabilistic interpretation. In these conditions, the relativistic quantum mechanics, in the sense of general relativity, can be seen as the one-particle restriction of the quantum field theory on curved backgrounds. Herein the quantum modes of the scalar particle in external gravitational field are given by the particular solutions of the free Klein-Gordon equation. These may form a basis in the space of wave functions organized as a Hilbert space with respect to the scalar product derived from the expression of the conserved electric charge [14].

2.1 The Klein-Gordon equation in special frames

Let us consider a (1+1)-dimensional background with a static local chart (i.e., natural frame) of holonomic coordinates \((u^0, u^1) \equiv (t, u)\) where the metric tensor defined on the space domain \(D_u\) is \(g_{\mu\nu}(u)\), \(\mu, \nu = 0, 1\), and \(g = \det(g_{\mu\nu})\). The one-particle quantum modes of a scalar field \(\phi\) of the mass \(m\), minimally coupled with the gravitational field, are given by the Klein-Gordon equation

\[
\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi \right) + m^2 \phi = 0, \quad (1)
\]
written in natural units with $\hbar = c = 1$. Since in the static charts the energy, $E$, is conserved, the Klein-Gordon equation has a set of fundamental solutions (of positive and negative frequencies),

$$\phi_E^{(+)}(t, u) = \frac{1}{\sqrt{2E}} e^{-iEt} U_E(u), \quad \phi^{(-)} = (\phi^{(+)})^*, \quad (2)$$

which depend on the static energy eigenfunctions $U_E$. These may be orthonormal (in usual or generalized sense) with respect to the relativistic scalar product \[14\]

$$\langle U, U' \rangle = \int_{D_u} du \mu(u) U(u)^* U'(u) \quad (3)$$

where

$$\mu = \sqrt{-\tilde{g}_{00}} \quad (4)$$

is the specific relativistic weight function of the scalar field. The Hilbert space of the square integrable functions with respect to this scalar product is denoted by $L^2(D_u, \mu)$. Obviously, the set of the wave functions $U_E$ represents an usual or generalized basis in this space. This will be called the energy basis.

In the case of the static backgrounds, a change of the space coordinates does not change the quantum modes. On the other hand, it is known that in $(1+1)$ dimensions any static background has a special natural frame where the metric is the conformal transformation of the Minkowski flat one. Starting with any natural frame $(t, u)$, the space coordinate of the special frame $(t, x)$ reads

$$x = \chi(u) = \int du \mu(u) + \text{const.,} \quad (5)$$

where the constant assures the condition $\chi(0) = 0$. In the special frame we have $\tilde{g}_{00}(x) = -\tilde{g}_{11}(x)$ and $\tilde{\mu}(x) = 1$. Therefore, the scalar product \[14\] becomes just the usual one of the Hilbert space $L^2(D)$ where $D$ is the domain of the coordinate $x$ corresponding to $D_u$.

If we denote $\tilde{g}_{00} = 1 + v$, then we can write the line element of the special frame as

$$ds^2 = [1 + v(x)](dt^2 - dx^2) \quad (6)$$

while the Klein-Gordon equation takes the form

$$\left[ -\frac{d^2}{dx^2} + V_R(x) \right] U_E(x) = (E^2 - m^2) U_E(x) \quad (7)$$
where $V_R = m^2v$. We say that this is the relativistic potential since in the nonrelativistic limit $V_R/2m$ becomes just the usual potential of the corresponding Schrödinger equation.

### 2.2 Observables

The linear operators on $\mathcal{L}^2$ (denoted here using boldface) can be defined either by giving their matrix elements in a countable basis of $\mathcal{L}^2$ or as differential operators in the coordinate representation. The most general differential operator we use in an arbitrary frame $(t, u)$ has the form

$$ (DU)(u) = i \left[ f(u) \frac{d}{du} + h(u) \right] U(u), \quad (8) $$

depending on two real functions $f$ and $h$. Its adjoint with respect to the scalar product (3) is

$$ D^\dagger = D + i \left[ \frac{1}{\mu} \frac{d(\mu f)}{du} - 2h \right] 1 \quad (9) $$

where $1$ is the identity operator. Hereby, we see that for $h = \partial_u(\mu f)/2\mu$ the operator $D$ is self-adjoint [13].

The consequence is that we can introduce a unique pair of self-adjoint coordinate and momentum operators defining their action in any natural frame $(t, u)$. One can easily verify that the following definitions,

$$ (XU)(u) = \chi(u)U(u), \quad (PU)(u) = i \frac{1}{\mu(u)} \frac{dU(u)}{du}, \quad (10) $$

are satisfactory since the commutation relation

$$ [P, X] = i1 \quad (11) $$

is just the desired one. Obviously, in the special frame $(t, x)$ these operators have the same action as in the case of the Minkowski flat space-time, namely

$$ (XU)(x) = xU(x), \quad (PU)(x) = i \frac{dU(x)}{dx}. \quad (12) $$
Furthermore, one can put the Klein-Gordon equation (7) in operator form,
\[ H^2 = m^2 \mathbf{1} + \Delta[V_R], \]
where \( H \) is the Hamiltonian operator defined by \( HU_E = EU_E \) and
\[ \Delta[V] = P^2 + V(X). \]

Thus we have obtained the main operators on \( L^2 \). The whole algebra of observables is that freely generated by the operators \( X \) and \( P \), like in the Schrödinger picture of the nonrelativistic one-dimensional quantum mechanics.

### 3 Relativistic Oscillators

The new geometric models of RO we discuss here are simple systems of free test particles that move on static backgrounds simulating oscillations. This means that there are local charts of coordinates \((t, u)\) where an observer at \( u = 0 \) moving along the direction \( \partial_t \) observes an oscillatory geodesic motion. These charts called proper natural frames have line elements [4],
\[ ds^2 = g_{00} dt^2 + g_{11} du^2 = \frac{1 + (1 + \lambda) \omega^2 u^2}{1 + \lambda \omega^2 u^2} dt^2 - \frac{1 + (1 + \lambda) \omega^2 u^2}{(1 + \lambda \omega^2 u^2)^2} du^2, \]
depending on a real parameter \( \lambda \). Thus one obtains a family of metrics which are conformal transformations either of the anti-de Sitter metric (as given in Ref.[1]) or of the de Sitter one. The anti-de Sitter metric with \( \lambda = -1 \) is also included in this family. A special case is that of \( \lambda = 0 \) when we say that the line element
\[ ds^2 = (1 + \omega^2 u^2)(dt^2 - du^2) \]
defines the normal RO. In Ref [4] it is shown that the quantum models with \( \lambda \leq 0 \) have countable energy spectra while for \( \lambda > 0 \) the energy spectra are mixed, with a finite discrete sequence and a continuous part. All these models will be presented here in the special frames \((t, x)\) associated with the proper frames \((t, u)\) defined above. The advantage is that in the special frames our RO appear either as PT or as RM relativistic systems [5] which can be analytically solved like those known from the nonrelativistic quantum mechanics.
3.1 The relativistic Pöschl-Teller models

Let us consider first the models with \( \lambda < 0 \) when the metrics are conformal transformations of the anti-de Sitter one. We denote
\[
\lambda = -\epsilon^2, \quad \hat{\omega} = \epsilon \omega, \quad \epsilon \geq 0
\]
and calculate the space coordinate of the special frame. According to Eq.(13) we obtain
\[
x = \frac{1}{\hat{\omega}} \arcsin \hat{\omega} u
\]
while from Eq.(15) we write the line element in this frame,
\[
ds^2 = \left(1 + \frac{1}{\epsilon^2} \tan^2 \hat{\omega} x\right) (dt^2 - dx^2),
\]
where the space domain is \( D = (-\pi/2\hat{\omega}, \pi/2\hat{\omega}) \) because of the event horizon at \( \pm \pi/2\hat{\omega} \). The relativistic potential,
\[
V_{PT}(k, x) = \frac{m^2}{\epsilon^2} \tan^2 \hat{\omega} x = \hat{\omega}^2 k(k - 1) \tan^2 \hat{\omega} x,
\]
is a PT one depending on the new parameter
\[
k = \sqrt{\frac{m^2}{\epsilon^2 \hat{\omega}^2} + \frac{1}{4} + \frac{1}{2}}.
\]
In the following we use \( k \) instead of \( m \), as the main parameter of the PT models that will be denoted from now by \( (k) \).

The Klein-Gordon equation (7) of the model \((k)\) with the potential (20) can be written as
\[
\left[-\frac{1}{\hat{\omega}^2} \frac{d^2}{dx^2} + \frac{k(k - 1)}{\cos^2 \hat{\omega} x}\right] U(x) = \nu^2 U(x)
\]
where
\[
\nu^2 = \frac{E^2}{\hat{\omega}^2} - \left(1 - \frac{1}{\epsilon^2}\right) \frac{m^2}{\hat{\omega}^2} = \frac{E^2}{\hat{\omega}^2} + (1 - \epsilon^2)k(k - 1).
\]
Its solutions
\[
U(x) \sim \sin^{2s} \hat{\omega} x \cos^{2p} \hat{\omega} x F\left(s + p - \frac{\nu}{2}, s + p + \frac{\nu}{2}, 2s + \frac{1}{2}, \sin^2 \hat{\omega} x\right),
\]
are expressed in terms of Gauss hypergeometric functions \([16]\) whose parameters \(s\) and \(p\) are solutions of the equations \(2s(2s-1) = 0\) and \(2p(2p-1) = k(k-1)\). The wave functions \([24]\) have good physical meaning only when \(F\) is a polynomial selected by a suitable quantization condition (since otherwise \(F\) is strongly divergent for \(x \to \pm \pi/2\hat{\omega}\)). Therefore, we introduce the quantum number \(n_s\) and impose

\[
\nu = 2(n_s + s + p), \quad n_s = 0, 1, 2, \ldots.
\]  

(25)

In addition, we choose the boundary conditions of the regular modes \([10]\) given by \(2s = 0, 1\) and \(2p = k\). Then the energy levels

\[
E_{k,n}^2 = \hat{\omega}^2[(k+n)^2 + (\epsilon^2 - 1)k(k-1)]
\]  

(26)

depend only on the main quantum number, \(n = 2n_s + 2s\), which takes even values if \(s = 0\) and odd values for \(s = 1/2\). Particularly for the anti-de Sitter model with \(\epsilon = 1\) we recover the well-known result \(E_{k,n} = \omega(k+n)\) \([3]\).

The next step is to derive the concrete form of the normalized energy eigenfunctions corresponding to these energy levels. According to Eqs.\((24)\) and \((25)\), these are

\[
U_{k,n}(x) = N_{k,n} \sin^{2s} \hat{\omega}x \cos^{k} \hat{\omega}xF \left( -n_s, n_s + k + 2s, 2s + \frac{1}{2}, \sin^2 \hat{\omega}x \right),
\]  

(27)

where the normalization constants \(N_{k,n}\) might be calculated with the help of the scalar product of \(L^2(D)\). However, there is another efficient method based on supersymmetry and shape invariance \([7]\) giving directly the Rodrigues formula of the normalized eigenfunctions. This will be presented in the next section.

We specify that our PT models are well-defined for any \(k \in [1, \infty)\) since the limit \(k \to 1\) (when \(m \to 0\)) has a good physical meaning. Indeed, in this case the massless particle remains confined to the rectangular infinite well of width \(\pi/\hat{\omega}\) having the equidistant energy levels

\[
E_{1,n} = \hat{\omega}(n + 1),
\]  

(28)

corresponding to the normalized eigenfunctions

\[
U_{1,n}(x) = \sqrt{\frac{2\hat{\omega}}{\pi}} \sin(n + 1) \left( \frac{\pi}{2} - \hat{\omega}x \right), \quad n = 0, 1, 2, \ldots.
\]  

(29)

Notice that this is a pure relativistic model since its nonrelativistic limit does not make sense.
3.2 The relativistic Rosen-Morse models

For \( \lambda > 0 \) the metrics of RO are conformal transformations of the de Sitter metric. Now we change the significance of \( \epsilon \) and put

\[
\lambda = \epsilon^2, \quad \hat{\omega} = \epsilon \omega, \quad \epsilon \geq 0. \tag{30}
\]

Furthermore, from Eq.(3) we find

\[
x = \frac{1}{\hat{\omega}} \arcsinh \hat{\omega}u \tag{31}
\]

and from Eq.(5) we obtain the line element

\[
ds^2 = \left( 1 + \frac{1}{\epsilon^2} \tanh^2 \hat{\omega}x \right) (dt^2 - dx^2) \tag{32}
\]

in the special frame where the space domain is \( D = (-\infty, \infty) \). These metrics define relativistic RM models whose potentials,

\[
V_{RM}(j, x) = \frac{m^2}{\epsilon^2} \tanh^2 \hat{\omega}x = \hat{\omega}^2 j(j+1) \tanh^2 \hat{\omega}x, \tag{33}
\]

depend on the parameter

\[
j = \sqrt{\frac{m^2}{\epsilon^2 \hat{\omega}^2} + \frac{1}{4} - \frac{1}{2}}. \tag{34}
\]

Like in the case of PT models, we consider that \( j \) is the main parameter of the RM models, denoted by \( (j) \).

Now the Klein-Gordon equation is

\[
\left[ \frac{1}{\hat{\omega}^2} \frac{d^2}{dx^2} + \frac{j(j+1)}{\cosh^2 \hat{\omega}x} \right] U(x) = \hat{\nu}^2 U(x) \tag{35}
\]

where

\[
\hat{\nu}^2 = -\frac{E^2}{\hat{\omega}^2} + \left( 1 + \frac{1}{\epsilon^2} \right) \frac{m^2}{\hat{\omega}^2} = -\frac{E^2}{\hat{\omega}^2} + (1 + \epsilon^2)j(j+1). \tag{36}
\]

The solutions

\[
U(x) \sim \sinh^{2s} \hat{\omega}x \cosh^{2p} \hat{\omega}xF \left( s + p - \frac{\hat{\nu}}{2}, s + p + \frac{\hat{\nu}}{2}, 2s + 1, -\sinh^2 \hat{\omega}x \right), \tag{37}
\]

9
depend on the parameters \( s \) and \( p \) which satisfy \( 2s = 0, 1 \) and \( 2p(2p + 1) = j(j + 1) \). Like in the nonrelativistic case, the relativistic RM models have mixed energy spectra with a finite discrete sequence and a continuous part [4].

The discrete levels arise from the quantization condition

\[
\hat{\nu} = 2(n_s + s + p) , \quad n_s = 0, 1, 2, ...
\]

One can show that the corresponding energy eigenfunctions are square integrable only when \( 2p = -j \) and the quantum main number \( n = 2n_s + 2s \) takes the values \( n = 0, 1, ..., n_{\text{max}} < j \). Therefore, these are

\[
U_{j,n}(x) = N_{j,n} \sinh^{2s} \omega x \cosh^{-j} \omega x F \left( -n_s, n_s - j + 2s, 2s + \frac{1}{2}, -\sinh^2 \omega x \right).
\]

Hence it results that the discrete energy spectrum is finite having \( n_{\text{max}} + 1 \) levels where \( n_{\text{max}} \) is the highest integer smaller than \( j \). This spectrum is included in the domain \([m, m\sqrt{1 + 1/\epsilon^2})\) since the energy levels are

\[
E_{j,n}^2 = \omega^2[-(n - j)^2 + (\epsilon^2 + 1)j(j + 1)] , \quad n = 0, 1, 2...n_{\text{max}}.
\]

The definitive form of the normalized energy eigenfunctions of the discrete spectrum will be calculated in the next section by using the shape invariance of the RM potentials.

The continuous spectrum cover the domain \([m\sqrt{1 + 1/\epsilon^2}, \infty)\) where we have \( \hat{\nu} = i|\hat{\nu}| \). The generalized energy eigenfunctions of this spectrum are tempered distributions of the form (37) with \( 2s = 0, 1 \) and \( 2p = -j \). We note that for \( m \to 0 \) (when \( j \to 0 \)) the discrete spectrum disappears while the continuous one becomes \([0, \infty)\). In this model the massless test particle move like in flat space-time and, therefore, it does not have nonrelativistic limit.

### 3.3 The normal RO and the nonrelativistic limit

Our family of RO is continuous in \( \lambda = 0 \) [4]. This means that the limits for \( \epsilon \to 0 \) of the PT and RM models must coincide. Indeed, according to
Eqs. (18) and (31) we find that in this limit \( x \to u \) while from Eqs. (21) and (34) it results that for any model with \( m \neq 0 \) we have \( k \to \infty \), \( j \to \infty \) but

\[
\lim_{\epsilon \to 0} \epsilon^2 k = \lim_{\epsilon \to 0} \epsilon^2 j = \frac{m}{\omega}. \tag{41}
\]

Furthermore, we can verify that the finite discrete spectra of the models with \( \lambda > 0 \) become countable while the continuous spectra disappear in a such a manner that the RM models and the PT ones have the same limit which is just the normal RO (with \( \lambda = 0 \)). The special frame of this model coincides with the proper one where the metric is defined by Eq. (16). Therefore, the relativistic potential is

\[
V_0(u) = \lim_{\epsilon \to 0} V_{PT}(x) = \lim_{\epsilon \to 0} V_{RM}(x) = m^2 \omega^2 u^2 \tag{42}
\]

and the Klein-Gordon equation

\[
\left[ -\frac{d^2}{du^2} + m^2 \omega^2 u^2 \right] U_n^{(0)}(u) = (E_n^2 - m^2) U_n^{(0)}(u) \tag{43}
\]

gives the familiar energy eigenfunctions of the NRHO,

\[
U_n^{(0)} = \left( \frac{m \omega}{\pi} \right)^{1/4} \frac{1}{\sqrt{n!2^n}} e^{-m \omega^2 u^2/2} H_n(\sqrt{m \omega} u), \tag{44}
\]

(where \( H_n \) are Hermite polynomials), but relativistic energy levels,

\[
E_n^{(0)} = m^2 + 2m \omega (n + \frac{1}{2}). \tag{45}
\]

In the nonrelativistic limit (defined by \( m/\omega \to \infty \)), the normal RO becomes the NRHO with the potential \( V_0/2m \) and usual energy levels. The nonrelativistic limit of the other models, with \( m \neq 0 \) and \( \lambda \neq 0 \), can be easily calculated if we observe that, according to Eqs. (21) and (34), this is equivalent with the limit \( \lambda \to 0 \) and, in addition, \( m \gg \omega \). Hereby it results that all the RO with \( m > 0 \) have the same nonrelativistic limit like that of the normal RO of the mass \( m \), namely the usual NRHO. On the other hand, it is interesting that in this way we can show that the parameter \( \lambda \), or the parameter \( \epsilon \) related to it, does not have a direct nonrelativistic equivalent, since the terms involving \( \lambda \) vanish in this limit.
4 Supersymmetry and Shape invariance

A relativistic supersymmetric quantum mechanics can be constructed in the same way as the nonrelativistic one. The main problem here is to find the operator which should play the role of Hamiltonian. We shall show that this is the operator \([14]\) with a suitable translated potential.

4.1 Supersymmetry

Let us start with a (1+1)-dimensional relativistic model with the potential \(V_R\) giving finite or countable energy spectrum. First we denote the energy levels by \(E_n^{(-)}\) and the corresponding energy eigenfunctions by \(U_n^{(-)}\). Then Eq.(7) in the special frame can be written as

\[
\Delta[V]U_n^{(-)} = d_n^{(-)}U_n^{(-)}, \quad n = 0, 1, 2, ...
\]  

(46)

where

\[
V_{-} = V_R - (E_0^{(-)} - m^2)
\]  

(47)

and

\[
d_n^{(-)} = E_n^{(-)2} - E_0^{(-)2}.
\]  

(48)

Herein we have translated the spectrum of \(\Delta\) in a such a manner to accomplish the condition \(d_0^{(-)} = 0\) we need for defining the superpotential \([7]\)

\[
W(x) = -\frac{1}{U_0^{(-)}} \frac{dU_0^{(-)}(x)}{dx}.
\]  

(49)

Then we have \(V_{-} = W^2 - W'\) (with the notation \(\prime = \partial_x\)) and the supersymmetric partner (superpartner) potential of \(V_{-}\) reads \(V_{+} = W^2 + W' = -V_{-} + 2W^2\). Furthermore, we introduce the operator

\[
A = -iP + W(X)
\]  

(50)

which satisfies

\[
[A, A^{\dagger}] = 2W'(X)
\]  

(51)
and help us to write

\[ \Delta[V_-] = A^\dagger A, \quad \Delta[V_+] = AA^\dagger. \] (52)

Now, like in the nonrelativistic case [7], we can convince ourselves that the spectrum of the eigenvalue problem

\[ \Delta[V_+]U_n^{(+)} = d_n^{(+)}U_n^{(+)} \] (53)

coincides with that of Eq.(46), apart of the eigenvalue \( d_0^{(-)} = 0 \). Thus we have \( d_n^{(+)} = d_{n+1}^{(-)}, n = 0, 1, 2, \ldots \), while the normalized eigenfunctions of \( \Delta[V_-] \) and \( \Delta[V_+] \) satisfy

\[ AU_n^{(-)} = \eta \sqrt{d_n^{(-)}} U_{n-1}^{(+)} , \quad A^\dagger U_{n-1}^{(+)} = \eta^* \sqrt{d_n^{(-)}} U_n^{(-)} \] (54)

where \( \eta \) is an arbitrary phase factor.

Hence, we can say that the (1+1) relativistic supersymmetric quantum mechanics has the same main features as the nonrelativistic one. It remains us to study the shape invariance of the relativistic potentials of the RO related through supersymmetry.

4.2 Shape invariance

Let us consider the PT model (k) and identify \( U_n^{(-)} \equiv U_{k,n} \) and \( E_n^{(-)} \equiv E_{k,n} \). Then the differences (48) are

\[ d_n^{(-)} \equiv d_{k,n} = E_{k,n}^2 - E_{k,0}^2 = \omega^2 n(n + 2k), \] (55)

and from Eqs.(47) and (20) we obtain

\[ V_-(k, x) = V_{PT}(k, x) + m^2 - E_{k,0}^2 = \omega^2 [k(k - 1) \tan^2 \omega x - k], \] (56)

On the other hand, the normalized ground-state eigenfunction calculated from Eq.(27),

\[ U_{k,0}(x) = \left( \frac{\omega^2}{\pi} \right)^{\frac{1}{4}} \left[ \frac{\Gamma(k + 1)}{\Gamma(k + \frac{3}{2})} \right]^{\frac{1}{2}} \cos^k \omega x , \] (57)
gives the superpotential \( W(k, x) = \hat{\omega} k \tan \hat{\omega} x \) which allows us to find the superpartner of \( V_- \),
\[
V_+(k, x) = -V_-(k, x) + 2W(k, x)^2 = \hat{\omega}^2 [k(k + 1) \tan^2 \hat{\omega} x + k].
\] (58)

Moreover, with this superpotential the operator (50) reads
\[
A_k = -i\mathbf{P} + \hat{\omega} k \tan \hat{\omega} \mathbf{X} = -\cos^k \hat{\omega} \mathbf{X} (i\mathbf{P}) \cos^{-k} \hat{\omega} \mathbf{X}
\] (59)
while from Eq.(51) we obtain
\[
[A_k, A_k^\dagger] = 2k \hat{\omega}^2 \frac{1}{k+1} \left( \frac{1}{2} \right) ^2.
\] (60)

Now we observe that the potentials \( V_-(k) \) and \( V_+(k) \) are shape invariant since
\[
V_+(k, x) = V_-(k + 1, x) + \hat{\omega}^2 (2k + 1).
\] (61)
Consequently, we can identify \( U_n^{(+)} \equiv U_{k+1,n} \) which means that the normalized energy eigenfunctions satisfy
\[
A_k U_{k,n} = \sqrt{d_{k,n}} U_{k+1,n-1}, \quad A_k^\dagger U_{k+1,n-1} = \sqrt{d_{k,n}} U_{k,n}.
\] (62)
as it results from Eqs.(52) with \( \eta = 1 \). Thus we have related the energy eigenfunctions of the model \( (k) \) with those of its superpartner model, \( (k+1) \). In general, we can write any normalized energy eigenfunction of the model \( (k) \) as
\[
U_{k,n} = \frac{1}{\hat{\omega}^n \sqrt{n!}} \left[ \frac{\Gamma(n + 2k)}{\Gamma(2n + 2k)} \right] ^{1/2} A_k^\dagger A_{k+1}^\dagger \ldots A_{k+n-1}^\dagger U_{k+n,0}\right). \] (63)
where \( U_{k+n,0} \) is the normalized ground-state eigenfunction of the model \( (k+n) \) given by Eq.(57).

For the relativistic RM models we use the same method starting with the model \( (j) \) and denoting \( U_n^{(-)} \equiv U_{j,n} \) and \( E_n^{(-)} \equiv E_{j,n} \). Then the differences (18) are
\[
d_n^{(-)} \equiv d_{j,n} = E_{j,n}^2 - E_{j,0}^2 = \hat{\omega}^2 n(2j - n),
\] (64)
and, according to (33), we have
\[
V_-(j, x) = V_{RM}(j, x) + m^2 - E_{j,0}^2 = \hat{\omega}^2 [j(j + 1) \tanh^2 \hat{\omega} x - j].
\] (65)
From Eq.(39) we find the normalized ground-state eigenfunction

$$U_{j,0}(x) = \left(\frac{\hat{\omega}^2}{\pi}\right)^{\frac{1}{4}} \left[\frac{\Gamma(j + \frac{1}{2})}{\Gamma(j)}\right]^{\frac{1}{2}} \cosh^{-j} \hat{\omega} x,$$  \hspace{1cm} (66)

giving the superpotential

$$W(j, x) = \hat{\omega} j \tanh \hat{\omega} x.$$

Now the operator (50) reads

$$A_j = -i P + \hat{\omega} j \tanh \hat{\omega} X = -\cosh^{-j} \hat{\omega} X (i P) \cosh^{j} \hat{\omega} X$$  \hspace{1cm} (68)

while Eq.(51) gives

$$[A_j, A_j^\dagger] = 2 j \hat{\omega}^2 \mathbf{1} - \frac{1}{2j} (A_j + A_j^\dagger)^2.$$  \hspace{1cm} (69)

The potentials $V_+(j)$ and $V_-(j)$ are shape invariant since

$$V_+(j, x) = V_-(j - 1, x) + \hat{\omega}^2 (2j - 1).$$  \hspace{1cm} (70)

Consequently, as in the previous case, we find that the normalized energy eigenfunctions satisfy

$$A_j U_{j,n} = \sqrt{d_{j,n}} U_{j-1,n-1}, \quad A_j^\dagger U_{j-1,n-1} = \sqrt{d_{j,n}} U_{j,n}$$  \hspace{1cm} (71)

if we take $\eta = 1$ in Eqs.(52). Thus we have obtained the relation between the sets of energy eigenfunctions of the superpartner models $(j)$ and $(j - 1)$. Moreover, we can also express the normalized eigenfunctions as

$$U_{j,n} = \frac{1}{\hat{\omega}^n \sqrt{n!}} \left[\frac{\Gamma(2j - 2n + 1)}{\Gamma(2j - n + 1)}\right]^{\frac{1}{2}} A_j^\dagger A_{j-1}^\dagger \ldots A_{j-n+1}^\dagger U_{j-n,0}.$$  \hspace{1cm} (72)

where now $U_{j-n,0}$ is the normalized ground-state eigenfunction of the model $(j - n)$ given by Eq.(66).
4.3 The normalized energy eigenfunctions

The normalization of the energy eigenfunctions of the PT models may be easily done in usual way but for the RM models there are some technical difficulties that can be avoided by using the previous results. Indeed, we observe that the Eqs. (63) and (72) are nothing else than the operator form of the Rodrigues formulas of the normalized eigenfunctions (in our phase convention with $\eta = 1$). Therefore, it remains only to rewrite their expressions in usual form.

For the PT models we replace the operator (59) in Eq.(63) which takes the form

$$U_{k,n}(x) = \frac{(-1)^n}{\hat{\omega}^n \sqrt{n!}} \frac{\Gamma(n + 2k)}{\Gamma(2n + 2k)} \cos^{-k} \hat{\omega} x \frac{d}{dx} \frac{1}{\cos \hat{\omega} x} d \cdots \frac{1}{\cos \hat{\omega} x} \frac{d}{dx} \cos \hat{\omega} x U_{k,n,0}(x).$$

Then, according to Eqs. (18) and (57), we obtain the final Rodrigues formula of the normalized energy eigenfunctions of the PT models in proper frames,

$$U_{k,n}(u) = \left(\frac{\hat{\omega}^2}{\pi}\right)^{\frac{1}{4}} \frac{(-1)^n}{\hat{\omega}^n \sqrt{n!}} \frac{\Gamma(2k + n) \Gamma(k + n + 1)}{\Gamma(2k + 2n) \Gamma(k + n + \frac{1}{2})} \frac{d^n}{du^n}(1 - \hat{\omega}^2 u^2)^{k + n - \frac{1}{2}}.$$ \hspace{1cm} (73)

In the same way we can derive the Rodrigues formula for the normalized energy eigenfunctions of the RM models. By using the Eqs.(68) and (72) we find the normalized energy eigenfunctions of the discrete spectrum in the proper frame,

$$U_{j,n}(u) = \left(\frac{\hat{\omega}^2}{\pi}\right)^{\frac{1}{4}} \frac{(-1)^n}{\hat{\omega}^n \sqrt{n!}} \frac{\Gamma(2j - 2n + 1) \Gamma(j - n + \frac{1}{2})}{\Gamma(2j - n + 1) \Gamma(j - n)} \frac{d^n}{du^n}(1 + \hat{\omega}^2 u^2)^{-j + n - \frac{1}{2}}.$$ \hspace{1cm} (74)

Of course, as it was expected, this formula gives square integrable functions only for $n \leq n_{\text{max}}$. On the other hand, here the problem of ”normalization”
of the generalized energy eigenfunctions of the continuous spectrum remains open since there are not yet efficient procedures for doing this.

Hence we have obtained the definitive formulas of the normalized energy eigenfunctions of our RO corresponding to the discrete energy levels. These can be written now in terms of Jacobi or Gegenbauer polynomials [16] and even as associated Legendre functions [13, 17] but only when \( k \) and \( j \) are integer numbers. In the particular case of the PT model with \( k = 1 \) the trigonometric form (29) can be derived from Eq.(74) by using the properties of the Tchebyshev polynomials.

In Sec.3.3 we have seen that the normal RO has the same energy eigenfunctions as the NRHO. Now, by taking into account that for large arguments, \( z \), we have \( \Gamma(z + a)/\Gamma(z + b) \sim z^{(a-b)} \) and by using Eqs.(41) we verify directly that

\[
\lim_{\epsilon \to 0} U_{k,n}(u) = \lim_{\epsilon \to 0} U_{j,n}(u) = U^{(0)}(u).
\]

Because of these properties we can say that the eigenfunctions (74) and (75) represent relativistic generalizations of the NRHO eigenfunctions, different from that of the algebraic method [2].

5 Shift operators

In our models only one main quantum number is involved and, therefore, in each model we must have a pair of shift operators, i.e. the rising and the lowering operators of the energy basis. In general, the shift operators are different from those of the supersymmetry apart the shift operators of the normal RO that are up to factors just those of the supersymmetry since this model is its own superpartner.

Let us start with this simplest case since here the energy eigenfunctions are similar to those of the NRHO. Consequently, we can take over the well-known results from the nonrelativistic theory defining the differential operators

\[
(aU)(u) = \frac{1}{\sqrt{2m\omega}} \left( \frac{d}{du} + m\omega u \right) U(u), \tag{77}
\]

\[
(a^\dagger U)(u) = \frac{1}{\sqrt{2m\omega}} \left( -\frac{d}{du} + m\omega u \right) U(u). \tag{78}
\]
of the Heisenberg-Weyl algebra. Obviously, they are the desired shift operators which obey \([a, a^\dagger] = 1\) giving us the operator of number of quanta \(N = a^\dagger a\) and

\[
X = \frac{1}{\sqrt{2 m \omega}} (a^\dagger + a), \quad P = -i \sqrt{\frac{m \omega}{2}} (a^\dagger - a). \tag{79}
\]

Moreover, it is natural to find that

\[
\lim_{\epsilon \to 0} A_k = \lim_{\epsilon \to 0} A_j = \sqrt{2 m \omega} a. \tag{80}
\]

For the models with \(\lambda \neq 0\) the shift operators differ from those of supersymmetry. They can be calculated directly by using the action of the supersymmetry operators and the form of the normalized energy eigenfunctions derived above. In the case of \(\lambda = -\epsilon^2\), after a few manipulation, we find that the shift operators of the PT model \((k)\) can be defined in the proper frame as

\[
(A_{k, (+)} U_{k,n})(u) = \frac{1}{\hat{\omega} \sqrt{2k}} \left[ - (1 + \hat{\omega}^2 u^2) \frac{d}{du} + \hat{\omega}^2 u(k + n) \right] U_{k,n}(u), \tag{81}
\]

\[
(A_{k, (-)} U_{k,n})(u) = \frac{1}{\hat{\omega} \sqrt{2k}} \left[ (1 - \hat{\omega}^2 u^2) \frac{d}{du} + \hat{\omega}^2 u(k + n) \right] U_{k,n}(u). \tag{82}
\]

Their shifting action is

\[
A_{k, (+)} U_{k,n} = C_{k,n}^{(+)} U_{k,n+1}, \quad A_{k, (-)} U_{k,n} = C_{k,n}^{(-)} U_{k,n-1}, \tag{83}
\]

where

\[
C_{k,n}^{(+)} = \frac{1}{\sqrt{2k}} \left[ \frac{(2k + n)(k + n)}{k + n + 1} \right]^{\frac{1}{2}} \sqrt{n+1}, \tag{84}
\]

\[
C_{k,n}^{(-)} = \frac{1}{\sqrt{2k}} \left[ \frac{(2k + n - 1)(k + n)}{k + n - 1} \right]^{\frac{1}{2}} \sqrt{n}. \tag{85}
\]

If we rewrite the action of the operators (81) and (82) in the special frame \((t, x)\) then we recover the result of Ref.[13]. Furthermore, we can verify the commutation relation

\[
[A_{k, (-)}, A_{k, (+)}] U_{k,n} = \left( 1 + \frac{n}{k} \right) U_{k,n} \tag{86}
\]
and the identity
\[ 2k \mathbf{A}_{k,(+)} \mathbf{A}_{k,(−)} U_{k,n} = n(2k + n - 1)U_{k,n} \]
which is just the Klein-Gordon equation in operator form [7]. In the limit \( \epsilon \to 0 \) we have [8]
\[ \lim_{\epsilon \to 0} \mathbf{A}_{k,(+)} = a\dagger, \quad \lim_{\epsilon \to 0} \mathbf{A}_{k,(−)} = a. \]  (88)

With the same procedure we find the shift operators of the RM model \((j)\) in the proper frame,
\[ (\mathbf{A}_{j,(+)} U_{j,n})(u) = \frac{1}{\sqrt{2j}} \left[ -(1 + \omega^2 u^2) \frac{d}{du} + \omega^2 u(j - n) \right] U_{j,n}(u), \quad (89) \]
\[ (\mathbf{A}_{j,(−)} U_{j,n})(u) = \frac{1}{\sqrt{2j}} \left[ (1 + \omega^2 u^2) \frac{d}{du} + \omega^2 u(j - n) \right] U_{j,n}(u), \quad (90) \]
which have the action
\[ \mathbf{A}_{j,(+)} U_{j,n} = C_{j,n}^{(+)} U_{j,n+1}, \quad \mathbf{A}_{j,(−)} U_{j,n} = C_{j,n}^{(−)} U_{j,n-1}, \quad (91) \]
where
\[ C_{j,n}^{(+)} = \frac{1}{\sqrt{2j}} \left[ \frac{(2j - n)(j - n)}{j - n - 1} \right]^{\frac{1}{2}} \sqrt{n + 1}, \quad (92) \]
\[ C_{j,n}^{(−)} = \frac{1}{\sqrt{2j}} \left[ \frac{(2j - n + 1)(j - n)}{j - n + 1} \right]^{\frac{1}{2}} \sqrt{n}. \quad (93) \]
They obey the commutation rule
\[ [\mathbf{A}_{j,(−)}, \mathbf{A}_{j,(+)}] U_{j,n} = \left( 1 - \frac{n}{j} \right) U_{j,n} \quad (94) \]
and give us the operator form of the Klein-Gordon equation for discrete levels,
\[ 2j \mathbf{A}_{j,(+)} \mathbf{A}_{j,(−)} U_{j,n} = n(2j - n + 1)U_{j,n}. \quad (95) \]
Finally we find that
\[ \lim_{\epsilon \to 0} \mathbf{A}_{j,(+)} = a\dagger, \quad \lim_{\epsilon \to 0} \mathbf{A}_{j,(−)} = a. \quad (96) \]

We must specify that the shift operators of the models with \( \lambda \neq 0 \) have two important properties, namely: they are not pure differential operators on \( L^2(D_{u,\mu}) \) and, moreover, the raising and lowering operators are not adjoint with each other, i.e. \( \mathbf{A}_{k,(±)} \neq (\mathbf{A}_{k,(±)})\dagger \) and similarly for the RM models.
6 Comments

In this article we have studied the quantum modes of a family of (1+1) RO by using the methods of a supersymmetric relativistic quantum mechanics similar with the well-known nonrelativistic one. This was possible since the form of the Klein-Gordon equation in the special frames is very close to that of the Schrödinger equation, allowing us to introduce the relativistic potentials involved in supersymmetry and to exploit their shape invariance.

However, our relativistic theory has some new interesting features due to the fact that the mass is not only involved in the formula of the energy levels but also plays here the role of a coupling constant. For this reason there are some kind of regularities leading to a very simple parametrization in a such a manner that for any pair of superpartner models we have either $\Delta k = \pm 1$ or $\Delta j = \mp 1$. Thus $k$ and $j$ simulate the behavior of quantum numbers even though they can not be considered as eigenvalues of self-adjoint operators [8]. On the other hand, the models with superpartner potentials can be seen as having particles of different masses moving on the same background. The consequence is that the masses of the sets of superpartner PT or RM models appear as being quantized according to the formulas $m_k^2 = \varepsilon^2 \tilde{\omega}^2 k(k - 1)$ and $m_j^2 = \varepsilon^2 \tilde{\omega}^2 j(j + 1)$ respectively. These remarkable properties helped us to easily write down the Rodrigues formulas of the normalized energy eigenfunctions of the discrete spectra and to find the corresponding shift operators.

Concluding we can say that our family of models brings together the main solvable problems with parity-symmetric potentials of the one-dimensional quantum mechanics, interpreted as relativistic oscillators in the sense that all these models (apart those with $k = 1$ and $j = 0$) lead to the NRHO in the nonrelativistic limit.

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