Remarks on the Liouville type problem in the stationary 3D Navier-Stokes equations

Dongho Chae
Department of Mathematics
Chung-Ang University
Seoul 156-756, Republic of Korea
email: dchae@cau.ac.kr

Abstract
We study the Liouville type problem for the stationary 3D Navier-Stokes equations on $\mathbb{R}^3$. Specifically, we prove that if $v$ is a smooth solution to (NS) satisfying $\omega = \text{curl } v \in L^q(\mathbb{R}^3)$ for some $\frac{3}{2} \leq q < 3$, and $|v(x)| \to 0$ as $|x| \to +\infty$, then either $v = 0$ on $\mathbb{R}^3$, or $\int_{\mathbb{R}^6} \Phi_+ dx dy = \int_{\mathbb{R}^6} \Phi_- dx dy = +\infty$, where $\Phi(x, y) := \frac{1}{4\pi} \omega(x) \cdot (x-y) \times (v(y) \times \omega(y)) \times \frac{|x-y|^3}{|x-y|^4}$, and $\Phi_{\pm} := \max\{0, \pm \Phi\}$. The proof uses crucially the structure of nonlinear term of the equations.

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1 Introduction
We consider the following stationary Navier-Stokes equations (NS) on $\mathbb{R}^3$.

\begin{align*}
(v \cdot \nabla)v &= -\nabla p + \Delta v, \\
\text{div } v &= 0,
\end{align*}

(1.1) (1.2)
where \( v(x) = (v_1(x), v_2(x), v_3(x)) \) and \( p = p(x) \) for all \( x \in \mathbb{R}^3 \). The system is equipped with the boundary condition:

\[
|v(x)| \to 0 \quad \text{uniformly as} \quad |x| \to +\infty.
\] (1.3)

In addition to (1.3) one usually also assume following finite enstrophy condition.

\[
\int_{\mathbb{R}^3} |
abla v|^2 \, dx < \infty,
\] (1.4)

which is physically natural. It is well-known that any weak solution of (NS) satisfying (1.4) is smooth. Actually, the regularity result for the \( L^\infty L^2_x \)-weak solution of the non-stationary Navier-Stokes equations proved in [2] implies immediately that \( v \in L^3(\mathbb{R}^3) \) is enough to guarantee the regularity. A long standing open question for solution of (NS) satisfying the conditions (1.3) and (1.4) is that if it is trivial (namely, \( v = 0 \) on \( \mathbb{R}^3 \)), or not. We refer the book by Galdi [3] for the details on the motivations and historical backgrounds on the problem and the related results. As a partial progress to the problem we mention that the condition \( v \in L^2_3(\mathbb{R}^3) \) implies that \( v = 0 \) (see Theorem X.9.5, pp. 729 [3]). Another condition, \( \Delta v \in L^\frac{6}{5}(\mathbb{R}^3) \) is also shown to imply \( v = 0 \) [1]. For studies on the Liouville type problem in the non-stationary Navier-Stokes equations, we refer [4]. Our aim in this paper is to prove the following:

**Theorem 1.1** Let \( v \) be a smooth solution to (NS) on \( \mathbb{R}^3 \) satisfying (1.3). Suppose there exists \( q \in [\frac{3}{2}, 3) \) such that \( \omega \in L^q(\mathbb{R}^3) \). We set

\[
\Phi(x, y) := \frac{1}{4\pi} \frac{\omega(x) \cdot (x - y) \times (v(y) \times \omega(y))}{|x - y|^3}
\] (1.5)

for all \((x, y) \in \mathbb{R}^3 \times \mathbb{R}^3\) with \( x \neq y \), and define

\[
\Phi_+(x, y) := \max\{0, \Phi(x, y)\}, \quad \Phi_-(x, y) := \max\{0, -\Phi(x, y)\}.
\]

Then, either

\[
v = 0 \quad \text{on} \quad \mathbb{R}^3,
\] (1.6)

or

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi_+(x, y) \, dx \, dy = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi_-(x, y) \, dx \, dy = +\infty.
\] (1.7)
Remark 1.1 One can show that if \( \omega \in L^{9/5}(\mathbb{R}^3) \) is satisfied together with (1.3), then (1.6) holds. In order to see this we first recall the estimate of the Riesz potential on \( \mathbb{R}^3 \),

\[
\| I_{\alpha}(f) \|_{L^q} \leq C \| f \|_{L^p}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{3}, \quad 1 \leq p < q < \infty,
\]

(1.8)

where

\[ I_{\alpha}(f) := C \int_{\mathbb{R}^3} \frac{f(y)}{|x - y|^{3-\alpha}} dy, \quad 0 < \alpha < 3 \]

for a positive constant \( C = C(\alpha) \). Applying (1.8) with \( \alpha = 1 \), we obtain by the Hölder inequality,

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\Phi(x, y)| dy dx \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\omega(x)| |\omega(y)||v(y)| \frac{dy}{|x - y|^2} dx
\]

\[
\leq \left( \int_{\mathbb{R}^3} |\omega(x)|^\frac{2}{3} dx \right)^\frac{3}{5} \left\{ \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{|\omega(y)||v(y)|}{|x - y|^2} dy \right)^\frac{2}{3} dx \right\}^\frac{3}{5}
\]

\[
\leq C \|\omega\|_{L^{9/5}} \left( \int_{\mathbb{R}^3} |\omega|^{\frac{2}{3}} |v|^{\frac{2}{3}} dx \right)^\frac{3}{5}
\]

\[
\leq C \|\omega\|_{L^{9/5}} \left( \int_{\mathbb{R}^3} |\omega|^\frac{2}{7} dx \right)^\frac{7}{9} \left( \int_{\mathbb{R}^3} |v|^\frac{2}{7} dx \right)^\frac{9}{7}
\]

\[
\leq C \|\omega\|_{L^{9/5}}^2 \|\nabla v\|_{L^\frac{7}{2}} \leq C \|\omega\|_{L^{9/5}}^3 < +\infty,
\]

where we used the Sobolev and the Calderon-Zygmund inequalities

\[
\|v\|_{L^2} \leq C \|\nabla v\|_{L^\frac{7}{2}} \leq C \|\omega\|_{L^{9/5}}
\]

(1.9)

in the last step. Thus, by the Fubini-Tonelli theorem, (1.7) cannot hold, and we are lead to (1.6) by application of the above theorem. We note that by (1.9) the condition \( \omega \in L^{9/5}(\mathbb{R}^3) \), on the other hand, implies the previously known sufficient condition \( v \in L^{9/2}(\mathbb{R}^3) \) of [3] mentioned above.

2 Proof of the main theorem

We first establish integrability conditions on the vector fields for the Biot-Savart’s formula in \( \mathbb{R}^3 \).
Proposition 2.1 Let $\xi = \xi(x) = (\xi_1(x), \xi_2(x), \xi_3(x))$ and $\eta = \eta(x) = (\eta_1(x), \eta_2(x), \eta_3(x))$ be smooth vector fields on $\mathbb{R}^3$. Suppose there exists $q \in [1, 3)$ such that $\eta \in L^q(\mathbb{R}^3)$. Let $\xi$ solve

$$\Delta \xi = -\nabla \times \eta, \quad (2.1)$$

under the boundary condition; either

$$|\xi(x)| \to 0 \quad \text{uniformly as} \quad |x| \to +\infty, \quad (2.2)$$

or

$$\xi \in L^s(\mathbb{R}^3) \quad \text{for some} \quad s \in [1, \infty). \quad (2.3)$$

Then, the solution of (2.1) is given by

$$\xi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x - y) \times \eta(y)}{|x - y|^3} \, dy \quad \forall x \in \mathbb{R}^3. \quad (2.4)$$

Proof We introduce a cut-off function $\sigma \in C^\infty_0(\mathbb{R}^N)$ such that

$$\sigma(|x|) = \begin{cases} 1 & \text{if} \ |x| < 1, \\ 0 & \text{if} \ |x| > 2, \end{cases}$$

and $0 \leq \sigma(x) \leq 1$ for $1 < |x| < 2$. For each $R > 0$ we define $\sigma_R(x) := \sigma\left(\frac{|x|}{R}\right)$. Given $\varepsilon > 0$ we denote $B_\varepsilon(y) = \{ x \in \mathbb{R}^3 \mid |x - y| < \varepsilon \}$. Let us fix $y \in \mathbb{R}^3$ and $\varepsilon \in (0, \frac{R}{2})$. We multiply (2.1) by $\frac{\sigma_R(|x - y|)}{|x - y|}$, and integrate it with respect to the variable $x$ over $\mathbb{R}^3 \setminus B_\varepsilon(y)$. Then,

$$\int_{\{|x - y| > \varepsilon\}} \frac{\Delta \xi \sigma_R}{|x - y|} \, dx = -\int_{\{|x - y| > \varepsilon\}} \frac{\sigma_R \nabla \times \eta(y)}{|x - y|} \, dx. \quad (2.5)$$

Since $\Delta \frac{1}{|x - y|} = 0$ on $\mathbb{R}^3 \setminus B_\varepsilon(y)$, one has

$$\frac{\Delta \xi \sigma_R}{|x - y|} = \sum_{i=1}^3 \partial_{x_i} \left( \frac{\partial_{x_i} \xi \sigma_R}{|x - y|} \right) - \sum_{i=1}^3 \partial_{x_i} \left( \frac{\xi \partial_{x_i} \sigma_R}{|x - y|} \right)$$

$$- \sum_{i=1}^3 \partial_{x_i} \left( \xi \sigma_R \partial_{x_i} \left( \frac{1}{|x - y|} \right) \right) + \frac{\xi \Delta \sigma_R}{|x - y|} + 2 \sum_{i=1}^3 \xi \partial_{x_i} \left( \frac{1}{|x - y|} \right) \partial_{x_i} \sigma_R.$$
Therefore, applying the divergence theorem, and observing \( \partial \nu \sigma_R = 0 \) on \( \partial B_\varepsilon(y) \), we have

\[
\int_{\{ |x-y| > \varepsilon \}} \frac{\Delta \xi \sigma_R}{|x-y|} dx = \int_{\{ |x-y| = \varepsilon \}} \frac{\partial \nu \xi}{|x-y|} dS \\
- \int_{\{ |x-y| = \varepsilon \}} \frac{\xi}{|x-y|^2} dS + \int_{\{ |x-y| > \varepsilon \}} \frac{\xi \Delta \sigma_R}{|x-y|} dx \\
- 2 \int_{\{ |x-y| > \varepsilon \}} \frac{(x-y) \cdot \nabla \sigma_R \xi}{|x-y|^3} dx
\]

(2.6)

where \( \partial \nu(\cdot) \) denotes the outward normal derivative on \( \partial B_\varepsilon(y) \). Passing \( \varepsilon \to 0 \), one can easily compute that

\[
\text{RHS of (2.6)} \to -4\pi \xi(y) + \int_{\mathbb{R}^3} \frac{\xi \Delta \sigma_R}{|x-y|} dx - 2 \int_{\mathbb{R}^3} \frac{(x-y) \cdot \nabla \sigma_R \xi}{|x-y|^3} dx \\
:= I_1 + I_2 + I_3.
\]

(2.7)

Next, using the formula

\[
\frac{\sigma_R \nabla \times \eta}{|x-y|} = \nabla \times \left( \frac{\sigma_R \eta}{|x-y|} \right) - \nabla \sigma_R \times \eta \frac{|x-y|}{|x-y|^3} + \frac{(x-y) \times \eta \sigma_R}{|x-y|^3},
\]

and using the divergence theorem, we obtain the following representation for the right hand side of (2.5).

\[
\int_{\{ |x-y| > \varepsilon \}} \frac{\sigma_R \nabla \times \eta}{|x-y|} dx = \int_{\{ |x-y| = \varepsilon \}} \nu \times \left( \frac{\eta}{|x-y|} \right) dS \\
- \int_{\{ |x-y| = \varepsilon \}} \nabla \sigma_R \times \eta \frac{|x-y|}{|x-y|^3} dx + \int_{\{ |x-y| > \varepsilon \}} \frac{(x-y) \times \eta \sigma_R}{|x-y|^3} dx,
\]

(2.8)

where we denoted \( \nu = \frac{y-x}{|y-x|} \), the outward unit normal vector on \( \partial B_\varepsilon(y) \). Passing \( \varepsilon \to 0 \), we easily deduce

\[
\text{RHS of (2.8)} \to - \int_{\mathbb{R}^3} \frac{\nabla \sigma_R \times \eta}{|x-y|} dx + \int_{\mathbb{R}^3} \frac{(x-y) \times \eta \sigma_R}{|x-y|^3} dx \\
:= J_1 + J_2 \quad \text{as} \quad \varepsilon \to 0.
\]

(2.9)
We now pass $R \to \infty$ for each term of (2.7) and (2.9) respectively below. Under the boundary condition (2.2) we estimate:

$$|I_2| \leq \int_{\{R \leq |x-y| \leq 2R\}} \frac{\xi(x)||\Delta\sigma_R(x-y)|}{|x-y|} \, dx$$

$$\leq \frac{\|\Delta\sigma\|_{L^\infty}}{R^2} \sup_{R \leq |x| \leq 2R} |\xi(x)| \left( \int_{\{R \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^3} \right)^{\frac{1}{3}} \left( \int_{\{R \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^3} \right)^{\frac{2}{3}}$$

$$\leq C\|\Delta\sigma\|_{L^\infty} \left( \int_{R}^{2R} \frac{dr}{r} \right)^{\frac{1}{3}} \sup_{R \leq |x-y| \leq 2R} |\xi(x)| \to 0$$

as $R \to \infty$ by the assumption (2.2), while under the condition (2.3) we have

$$|I_2| \leq \int_{\{R \leq |x-y| \leq 2R\}} \frac{\xi(x)||\Delta\sigma_R(x-y)|}{|x-y|} \, dx$$

$$\leq \frac{\|\Delta\sigma\|_{L^\infty}}{R^2} \|\xi\|_{L^\infty} \left( \int_{\{0 \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^s} \right)^{\frac{s-1}{s}} \left( \int_{\{0 \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^s} \right)^{\frac{1}{s}}$$

$$\leq C\|\Delta\sigma\|_{L^\infty} \left( \int_{R}^{2R} \frac{dr}{r} \right)^{\frac{1}{3}} \sup_{R \leq |x-y| \leq 2R} |\xi(x)| \to 0$$

as $R \to \infty$. Similarly, under (2.2)

$$|I_3| \leq 2 \int_{\{R \leq |x-y| \leq 2R\}} \frac{\xi(x)||\nabla\sigma_R(x-y)|}{|x-y|^2} \, dx$$

$$\leq \frac{C\|\nabla\sigma\|_{L^\infty}}{R} \sup_{R \leq |x| \leq 2R} |\xi(x)| \left( \int_{\{R \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^3} \right)^{\frac{1}{3}} \left( \int_{\{R \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^3} \right)^{\frac{2}{3}}$$

$$\leq C\|\nabla\sigma\|_{L^\infty} \left( \int_{R}^{2R} \frac{dr}{r} \right)^{\frac{1}{3}} \sup_{R \leq |x-y| \leq 2R} |\xi(x)| \to 0$$

as $R \to \infty$, while under the condition (2.3) we estimate

$$|I_3| \leq 2 \int_{\{R \leq |x-y| \leq 2R\}} \frac{\xi(x)||\nabla\sigma_R(x-y)|}{|x-y|^2} \, dx$$

$$\leq \frac{C\|\nabla\sigma\|_{L^\infty}}{R} \|\xi\|_{L^s(\{R \leq |x-y| \leq 2R\})} \left( \int_{\{0 \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^s} \right)^{\frac{s-1}{s}} \left( \int_{\{0 \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^s} \right)^{\frac{1}{s}}$$

$$\leq CR^{-\frac{3}{2}} \|\nabla\sigma\|_{L^\infty} \|\xi\|_{L^s} \to 0$$
as $R \to \infty$. Therefore, the right hand side of (2.6) converges to $-4\pi \xi(y)$ as $R \to \infty$. For $J_1, J_2$ we estimate

$$|J_1| \leq \int_{\{R \leq |x-y| \leq 2R\}} \frac{\|
abla \sigma_R\| \|\eta\|}{|x-y|} \, dx$$

$$\leq \frac{C \|
abla \sigma\|_{L^\infty}}{R} \|\eta\|_{L^q(R \leq |x-y| \leq 2R)} \left( \int_{\{0 \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^\frac{q-1}{q}} \right)^{\frac{q-1}{q}}$$

$$\leq C \|
abla \sigma\|_{L^\infty} \|\eta\|_{L^q(R \leq |x-y| \leq 2R)} R^{-\frac{2}{q}} \to 0$$

as $R \to \infty$. In passing $R \to \infty$ in $J_2$ of (2.9), in order to use the dominated convergence theorem, we estimate

$$\int_{\mathbb{R}^3} \frac{(x-y) \times \eta(y)}{|x-y|^3} \, dx \leq \int_{\{|x-y| < 1\}} \frac{\|\eta\|}{|x-y|^2} \, dx + \int_{\{|x-y| \geq 1\}} \frac{\|\eta\|}{|x-y|^2} \, dx$$

$$:= J_{21} + J_{22}. \quad (2.10)$$

$J_{21}$ is easy to handle as follows.

$$J_{21} \leq \|\eta\|_{L^\infty(B_1(y))} \int_{\{|x-y| < 1\}} \frac{dx}{|x-y|^2} = 4\pi \|\eta\|_{L^\infty(B_1(y))} < +\infty. \quad (2.11)$$

For $J_{22}$ we estimate

$$J_{22} \leq \left( \int_{\mathbb{R}^3} |\eta|^q \, dx \right)^{\frac{1}{q}} \left( \int_{\{|x-y| > 1\}} \frac{dx}{|x-y|^\frac{2q}{q-1}} \right)^{\frac{q-1}{q}}$$

$$\leq C \|\eta\|_{L^q} \left( \int_1^\infty \frac{r^{\frac{2}{q-1}} \, dr}{r^{\frac{q}{q-1}}} \right)^{\frac{q-1}{q}} < +\infty, \quad (2.12)$$

if $1 < q < 3$. In the case of $q = 1$ we estimate simply

$$J_{22} \leq \int_{\{|x-y| > 1\}} |\eta| \, dx \leq \|\eta\|_{L^1}. \quad (2.13)$$

Estimates of (2.10)-(2.13) imply

$$\int_{\mathbb{R}^3} \left| \frac{(x-y) \times \eta(y)}{|x-y|^3} \right| \, dx < +\infty.$$
Summarising the above computations, one can pass first $\varepsilon \to 0$, and then $R \to +\infty$ in (2.5), applying the dominated convergence theorem, to obtain finally (2.4). □

Corollary 2.1 Let $v$ be a smooth solution to (1.1)-(1.3) such that $\omega \in L^q(\mathbb{R}^3)$ for some $q \in \left[\frac{3}{2}, 3\right)$. Then, we have

$$v(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times \omega(y)}{|x-y|^3} \, dy,$$

(2.14)

and

$$\omega(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times (v(y) \times \omega(y))}{|x-y|^3} \, dy.$$

(2.15)

Proof Taking curl of the defining equation of the vorticity, $\nabla \times v = \omega$, using $\text{div } v = 0$, we have

$$\Delta v = -\nabla \times \omega,$$

which provides us with (2.14) immediately by application of Proposition 2.1. In order to show (2.15) we recall that, using the vector identity $\frac{1}{2} \nabla|v|^2 = (v \cdot \nabla)v + v \times (\nabla \times v)$, one can rewrite (1.1)-(1.2) as

$$-v \times \omega = -\nabla \left( p + \frac{1}{2} |v|^2 \right) + \Delta v.$$

Taking curl on this, we obtain

$$\Delta \omega = -\nabla \times (v \times \omega).$$

The formula (2.15) is deduced immediately from this equations by applying the proposition 2.1. For the allowed rage of $q$ we recall the Sobolev and the Calderon-Zygmund inequalities([5]),

$$\|v\|_{L^\frac{3q}{6-q}} \leq C \|\nabla v\|_{L^q} \leq C \|\omega\|_{L^q}, \quad 1 < q < 3,$$

(2.16)

which imply $v \times \omega \in L^{\frac{3q}{6-q}}(\mathbb{R}^3)$ if $\omega \in L^q(\mathbb{R}^3)$. We also note that $\frac{3}{2} \leq q < 3$ if and only if $1 \leq \frac{3q}{6-q} < 3$. □

Proof of Theorem 1.1 Under the hypothesis (1.3) and $\omega \in L^q(\mathbb{R}^3)$ with
$q \in [\frac{3}{2}, 3)$ both of the relations (2.14) and (2.15) are valid. We first prove the following.

**Claim:** For each $x, y \in \mathbb{R}^3$

$$0 \leq |\omega(x)|^2 = \int_{\mathbb{R}^3} \Phi(x, y) dy \leq \int_{\mathbb{R}^3} |\Phi(x, y)| dy < +\infty,$$  \hspace{1cm} (2.17)

and

$$0 = \int_{\mathbb{R}^3} \Phi(x, y) dx \leq \int_{\mathbb{R}^3} |\Phi(x, y)| dx < +\infty.$$  \hspace{1cm} (2.18)

**Proof of the claim:** We verify the following:

$$\int_{\mathbb{R}^3} |\Phi(x, y)| dy + \int_{\mathbb{R}^3} |\Phi(x, y)| dx < \infty \quad \forall (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3.$$  \hspace{1cm} (2.19)

Decomposing the integral, and using the Hölder inequality, we estimate

$$\int_{\mathbb{R}^3} |\Phi(x, y)| dy \leq |\omega(x)| \left( \int_{\{|x-y| \leq 1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} dy + \int_{\{|x-y| > 1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} dy \right) \leq \frac{4q-6}{3q} |\omega(x)||v||_{L^\infty(B_1(x))} ||\omega||_{L^\infty(B_1(x))} \int_{\{|x-y| \leq 1\}} \frac{dy}{|x-y|^{2q-6}} + C|\omega(x)||v||_{L^\infty(B_1(x))} ||\omega||_{L^\infty(B_1(x))} \int_{1}^{\infty} \frac{dr}{r^{\frac{6q}{3q-6}}} \leq C \frac{4q-6}{3q} < +\infty,$$  \hspace{1cm} (2.20)

where we used (2.16) and the fact that $\frac{4q-6}{3q-6} < -1$ if $\frac{3}{2} < q < 3$. In the case $q = \frac{3}{2}$ we estimate, instead,

$$\int_{\mathbb{R}^3} |\Phi(x, y)| dy \leq |\omega(x)| \left( \int_{\{|x-y| \leq 1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} dy + \int_{\{|x-y| > 1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} dy \right) \leq |\omega(x)||v||_{L^\infty(B_1(x))} ||\omega||_{L^\infty(B_1(x))} + |\omega(x)||v||_{L^1} ||\omega||_{L^{\frac{3}{2}}} < +\infty.$$  \hspace{1cm} (2.21)
We also have
\[
\int_{\mathbb{R}^3} |\Phi(x, y)|\, dx \leq |v(y)| |\omega(y)| \left( \int_{|x-y| \leq 1} \frac{|\omega(x)|}{|x-y|^2}\, dx + \int_{|x-y| > 1} \frac{|\omega(x)|}{|x-y|^2}\, dx \right)
\]
\[
\leq C|v(y)| |\omega(y)| |\omega|_{L^\infty(B_1(y))} + |v(y)| |\omega(y)| |\omega|_{L^q} \left( \int_{|x-y| > 1} \frac{dx}{|x-y|^{\frac{2}{q-1}}} \right)^{\frac{q-1}{q}}
\]
\[
\leq C|v(y)| |\omega(y)| |\omega|_{L^\infty(B_1(y))} + C|v(y)| |\omega(y)| |\omega|_{L^q} \left( \int_1^\infty r^{-\frac{2}{q-1}}\, dr \right)^{\frac{q-1}{q}} < +\infty,
\]
where we used the fact that \(-\frac{2}{q-1} < -1\) if \(\frac{3}{2} \leq q < 3\). From (2.15) we immediately obtain
\[
\int_{\mathbb{R}^3} \Phi(x, y)\, dy = \omega(x) \cdot \left( \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times (v(y) \times \omega(y))}{|x-y|^3}\, dy \right)
\]
\[
= |\omega(x)|^2 \geq 0, \quad \forall x \in \mathbb{R}^3
\]
and combining this with (2.20), we deduce (2.17). On the other hand, using (2.14), we find
\[
\int_{\mathbb{R}^3} \Phi(x, y)\, dx = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega(x) \cdot (x-y) \times (v(y) \times \omega(y))}{|x-y|^3}\, dx
\]
\[
= \left( \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega(x) \times (x-y)}{|x-y|^3}\, dx \right) \cdot v(y) \times \omega(y)
\]
\[
= v(y) \cdot v(y) \times \omega(y) = 0
\]
for all \(y \in \mathbb{R}^3\), and combining this with (2.22), we have proved (2.18). This completes the proof of the claim.

By the Fubini-Tonelli theorem we have
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi_+(x, y)\, dx\, dy = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi_+(x, y)\, dy\, dx := I_+,
\]
and
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi_-(x, y)\, dx\, dy = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi_-(x, y)\, dy\, dx := I_-.
\]
If (1.7) does not hold, then at least one of the two integrals $\mathcal{I}_+, \mathcal{I}_-$ is finite. In this case, using (2.25) and (2.26), we can interchange the order of integrations in repeated integral as follows.

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi(x, y) dx dy = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi_+(x, y) dx dy - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi_-(x, y) dx dy$$

$$= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi_+(x, y) dy dx - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi_-(x, y) dy dx$$

$$= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi(x, y) dy dx. \quad (2.27)$$

Therefore, from (2.23) and (2.24) combined with (2.27) provide us with

$$\int_{\mathbb{R}^3} |\omega(x)|^2 dx = \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \Phi(x, y) dy dx = \int_{\mathbb{R}^3} \Phi(x, y) dx dy = 0.$$

Hence,

$$\omega = 0 \quad \text{on} \quad \mathbb{R}^3. \quad (2.28)$$

We remark parenthetically that in deriving (2.28) it is not necessary to assume that $\int_{\mathbb{R}^3} |\omega(x)|^2 dx < +\infty$, and we do not need to restrict ourselves to $\omega \in L^2(\mathbb{R}^3)$. Hence, from (2.14) and (2.28), we conclude $v = 0$ on $\mathbb{R}^3$. □

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