Equation of state within gluon dominated QGP model in relativistic hydrodynamics approach

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Abstract. The dynamics of quark-gluon plasma (QGP) as a lump of deconfined free quarks and gluons is elaborated. Based on the first principal, we construct the Lagrangian that represents the dynamics of QGP. To induce a hydrodynamics approach, we substitute the gluon fields with flow fields. As a result, the derived equation of motion (E.O.M) for gluon dominated QGP shows a form similar to Euler equation; and the energy momentum tensor also represents explicitly the system of ideal fluid. Combining the E.O.M and energy momentum tensor, the pressure and energy density distribution are analytically derived.

1. Introduction
Recent experiments on heavy-ion collisions show a strong indication that hot dense deconfined phase of free quark and gluon, the so-called quark-gluon plasma (QGP), is conjectured to exist. The study of QGP itself has been carried out through a number of different approaches in previous works. Some results of these studies were obtained in the framework of quantum chromodynamics (QCD) theory by utilizing the lattice gauge calculation \cite{1,2}. Other calculations of QGP were based on the relativistic hydrodynamics approach \cite{3,4}. In the latter, the QGP could be either quark- \cite{4} or gluon- \cite{3} dominated matter. For the quark-dominated approach, a very small ratio of shear viscosity over entropy is required to get a good fit of the spectra of transverse momentum, energy density distribution and other physical observables that are obtained from experiments \cite{5–10}. On the other hand, the gluon-dominated plasma motivated by the discoveries of jet quenching in the heavy-ion collision at Brookhaven’s Relativistic Heavy Ion Collider (RHIC) indicates the shock waves in the form of Mach cone \cite{11,12}. The present paper adopts the so-called fluid QCD model \cite{13–15} to produce the equation of motion and energy-momentum tensor for quark and gluon in a lump of QGP, and subsequently to investigate the distributions of the pressure and energy density.

This paper is organized as follows. In Section 2 the fluid QCD model is briefly revisited, and the energy momentum tensor for ideal fluid is derived. Then, it is followed by the derivation for the equation of state and the explicit expression of gluon field in Section 3. Finally, the summary and discussion will be given in Section 4. Throughout this work, we use the natural units, i.e., $\hbar = c = 1$. 


2. Model

The Lagrangian of QGP that describes the unification of fermions and bosons from different gauge groups with preserving SU(3)$_F$ ⊗ U(1)$_G$ gauge symmetry can be written as [13, 14]

$$\mathcal{L} = i\bar{Q}\gamma^{\mu}\partial_{\mu}Q - m_Q\bar{Q}Q - \frac{1}{4}S_{\mu\nu}^{a}S^{a\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g_FJ_{\mu A}^{a}U^{a\mu} + g_GJ_{G\mu A}^{a}A^{\mu}_{\mu}. \quad (1)$$

SU(3)$_F$ ⊗ U(1)$_G$ represents a non-Abelian SU(3)$_F$ fluid fields containing quarks and antiquarks that interact with an electromagnetic field. $Q$ and $\bar{Q}$ represent the quark and antiquark triplets, $\gamma^{\mu}$ are Dirac gamma matrices, and $m_Q$ is the mass of quark. The term $\frac{1}{4}S_{\mu\nu}^{a}S^{a\mu\nu}$ is the gauge invariant kinetic term of gluon field. The gluon field strength tensor itself is expressed as $S_{\mu\nu}^{a} \equiv \partial_{\mu}U_{\nu}^{a} - \partial_{\nu}U_{\mu}^{a} + g_{F}f^{abc}U_{\mu}^{b}U_{\nu}^{c}$, where $U_{\mu}^{a}$ indicates gluon fields, $g_{F}$ is the strong coupling constant, and $f^{abc}$ is the structure constant of SU(3) gauge group. Similarly, the field strength tensor for $A_{\mu}$ is given by $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. The last two terms of (1), $J_{\mu A}^{a} = Q\gamma^{\mu}Q$ and $J_{G\mu} = \bar{Q}\gamma^{\mu}Q$, represent the quark currents from the SU(3) and U(1) gauge groups, respectively, whereas $g_{G}$ denotes the coupling constant of U(1) gauge group.

The QCD Lagrangian in (1) is constructed with a purpose of reproducing the energy-momentum tensor that has the same form as the energy momentum tensor of an ideal fluid [13]. In the succeeding step, we propose that each gluon field $U_{\mu}^{a}$, which is designed to act as the flow field in the system, be formulated in a configuration that inherently contains the relativistic flow. It is formulated as $U_{\mu}^{a} = (U_{0}^{a}, U^{a}) \equiv u_{\mu}\phi^{a}$ [14, 15]. Here, $\phi^{a}$ is the dimension one scalar field, and $u_{\mu} \equiv \gamma(1, v)$ is the relativistic velocity, with $\gamma = (1 - |v|^{2})^{-1/2}$.

This formulation provides a clue that a single gluonic field $U_{\mu}^{a}$ may behave as a fluid at certain scale beside its conventional point particle properties with a polarization vector $\epsilon_{\mu}$ in the form of $U_{\mu}^{a} = \epsilon_{\mu}\phi^{a}$. One can, then, consider that there is a kind of “phase transition”.

$$\text{hadronic state} \leftrightarrow \text{QGP state}.$$

As the gluon field behaves as a point particle, it is in a stable hadronic state and is characterized by its polarization vector. On the other hand in the pre-hadronic state (before hadronization) like hot QGP, the gluon field behaves as a highly energized flow particle, and the properties are dominated by its relativistic velocity.

The field $U_{\mu}^{a}$ is actually analogous to the gauge boson from U(1) gauge group in particle physics, where the polarization vector $\epsilon_{\mu}$ from the free photon solution $\sim \epsilon_{\mu}\exp(-ip_{\mu}x^{\mu})$ is replaced by the 4-velocity $u_{\mu}$. Recall that the wavefunction $U_{\mu}^{a}$ for a free particle satisfies $[g^{\mu\nu}(\partial^2 + m^2) - \partial^\mu \partial^\nu]U_{\mu}^{a} = 0$ with solution $U_{\mu}^{a} \sim \epsilon_{\mu}\phi^{a}\exp(-ip_{\mu}x^{\mu})$. For a massive vector particle, $m \neq 0$, so we have no choice but taking $\partial^\mu U_{\mu}^{a} = 0$. It is not a gauge condition like the case of massless particle. This then demands $p^{\mu}\epsilon_{\mu} = 0$. The number of independent polarization vectors is reduced from four to three in a covariant fashion. However, one can still perform another gauge transformation to the massless $U_{\mu}^{a}$, which makes finally only two degrees of freedom. Therefore, one should keep in mind that in the present model the spatial velocity has only two degrees of freedom, which means that one component must be described by another two vector components.

Furthermore, when Euler-Lagrange equation is applied to (1), one obtains

$$\frac{\partial}{\partial t}(\gamma v\phi^{a}) + \nabla(\gamma\phi^{a}) = -g_{F}\int d\mathbf{x}(J_{F\mu}^{a} + F_{\mu}^{a}), \quad (2)$$

where $J_{F0}^{a}$ denotes the covariant current originating from the quarks that are surrounded by and interact with the gluon “fluid”, while the term $F_{\mu}^{a}$ is induced by the fluid self-interaction and the interacting gauge fields $A_{\mu}^{a}$. Equation (2) is considered as the general relativistic fluid.
equation for single gluon field $U^a_\mu$ since in the nonrelativistic limit, i.e. $\phi^a \simeq 1$ and $\gamma \simeq 1 + \frac{1}{2}|v|^2$, such equation transforms to the classical equation of motion of fluid dynamics [14]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -g_F \int dx (J^a_\mu F^a_\mu + F^a_\mu) |_{\text{non-rel}} .$$

(3)

This shows that from a certain point of view the Lagrangian describes a general relativistic fluid system interacting with another gauge field and the matter inside. The flow characteristic that appears in the equation of motion comes from the contribution of gluon field $U^a_\mu$. This fact indicates that the system we are working with is a gluon-dominated QGP. In such system the terms that do not contain gluon field may be omitted. As a consequence, we have

$$\mathcal{L} = -\frac{1}{4} S^{a\mu\nu} S_{\mu\nu} + g_F J^a_\mu U^{a\mu} .$$

(4)

The Lagrangian given by (4) describes the kinetics of gluons, the self-interactions of gluons, and the interaction between quark with gluon “fluid”. Electromagnetic interactions also exist in the system, but they are suppressed due to the tiny value of electromagnetic coupling compared with that of the strong interaction. From the Lagrangian given in (4), we can derive the energy momentum tensor [16]

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta (\mathcal{L} \sqrt{-g})}{\delta g^{\mu\nu}}$$

$$= S^{a\mu\nu} S_{a\mu\nu} - g_{\mu\nu} \mathcal{L} + 2 g_F J^a_\mu U^{a\mu}$$

$$= [2 g_F g_{\mu\nu} J^a_\mu U^{a\mu} + \frac{g_F^2}{4} f^{abc} f_{abc} U^b_\mu U^c_\rho \epsilon^{\mu\nu\rho\sigma} U^\sigma_v]$$

$$- [g_F g_{\mu\nu} J^a_\mu U^{a\mu} - \frac{1}{4} g_{\mu\nu} \frac{g_F^2}{4} f^{abc} f_{abc} U^b_\mu U^c_\rho \epsilon^{\mu\nu\rho\sigma} U^\sigma_v] .$$

(5)

The factor $J^a_\mu U^{a\mu}$ in (5) might be expressed in a more elementary form. To this end, we can start with the assumption that the solution of Dirac equation for single color quark is in the form of $Q(p, x) = q(p) \exp(-ip \cdot x)$. By inserting this solution to the Dirac equation, i.e., $(i\gamma^\mu \partial_\mu - m)q(p) \exp(-ip \cdot x) = 0$, and multiplying with the antiquark solution, we obtain $\bar{q}_1 q_1 = 4p_\mu$. Since $J^a_\mu = \bar{Q}_a \gamma_\mu Q^a = \bar{q}_1 q_1 T^a = 4p_\mu T^a$, and $U^{a\mu} = u^\mu \phi^a$, we finally obtain $J^a_\mu U^{a\mu} = 4mq T^a \phi^a$. Note that we have utilized $u_\mu u^\mu = 1$ along with the assumption that all quarks and antiquarks have the same momenta $P^\mu$, and that the velocities of all gluons and quarks are homogeneous. Furthermore, due to the completely anticommutative property of the structure constant $f^{abc}$, the 2nd and 4th terms in (5) vanish. Then, the energy-momentum tensor in the function of field $\phi^a$ reads

$$T_{\mu\nu} = [8g_F m_q T^a \phi^a] u_\mu u_\nu - [4g_F m_q T^a \phi^a] g_{\mu\nu} .$$

(6)

With this form, $T_{\mu\nu}$ is obviously describing a system of perfect fluid. Furthermore, we assume that the gluon color states are homogeneous, i.e., $\phi^a = \phi$ for all $a = 1, \cdots, 8$. As the follow-on to this assumption, the generator $T^a$ can be compactly written as $T = \sum_a T^a$. The energy-momentum tensor then reduces to

$$T_{\mu\nu} = (8T g_F m_q \phi) u_\mu u_\nu - (4T g_F m_q \phi) g_{\mu\nu} .$$

(7)

It reveals the collective gluons flow in the system. Therefore, we can temporarily summarize that the system represents an isotropic homogeneous perfect fluid of gluon-dominated QGP. Furthermore, to discuss the dynamics of glue lumps, it is also plausible to take the gluon as a field that is independent of time, $\phi = \phi(x)$. If we compare (7) with energy-momentum tensor of ideal fluid $T_{\mu\nu} = (\mathcal{E} + \mathcal{P}) u_\mu u_\nu - \mathcal{P} g_{\mu\nu}$, it becomes obvious that

$$\mathcal{E} = \mathcal{P} = 4T g_F m_q \phi .$$

(8)
Here $\mathcal{P}$ and $\mathcal{E}$ denote the isotropic pressure and energy density for fluid field, respectively. Or, in currently discussed topic, it is the energy density and pressure distributions in the lump of gluon dominated QGP.

### 3. Expression for distribution field

The expression for the scalar field $\phi$ can be obtained by deriving the solution of (3). After applying the assumption of the gluon homogeneity, the equation can be expressed as follows.

$$\delta \partial_r \phi + \gamma \partial_r \phi + \xi + \beta \phi \partial_r \phi + \beta \phi \partial_r \phi - \lambda \partial_r \phi - \lambda \partial_r \phi = 0,$$

with $\delta = \gamma |\mathbf{v}|$, $\xi = g_F Q^2 T Q$, $\beta = i \gamma^2 |\mathbf{v}|^2$, $\lambda = (g_C / g_F) A \gamma$, and $\gamma = (1 - v^2)^{-\frac{1}{2}}$. To simplify but stay relevant, the pressure and energy density distribution of the lump of QGP is assumed to depend only on $r$, that is, $\phi = \phi(r)$ and $\partial_r \to d_r$. Therefore, the terms that involve the derivative of $t$ vanish,

$$\gamma d_r \phi + \beta \phi d_r \phi - \lambda d_r \phi + \xi = 0.$$  \hspace{1cm} (10)

The solution for this equation is

$$\phi = \Bigg( 2 \gamma \left( \frac{\exp[\frac{r A}{2\xi}] \Lambda_1[x]}{\exp[\frac{r A}{2\xi}] \beta \Lambda_2[x]} \right) - \gamma \left( \frac{\exp[\frac{r A}{2\xi}] \beta \Lambda_2[x]}{\exp[\frac{r A}{2\xi}] \beta \Lambda_2[x]} \right) \Bigg) \times \frac{1}{(\beta + \frac{r A}{2}\Lambda_2[x] - \frac{r A}{2}\Lambda_3[x] C_2)} \Bigg) \times \frac{1}{(\beta + \frac{r A}{2}\Lambda_2[x] - \frac{r A}{2}\Lambda_3[x] C_2)}.$$  \hspace{1cm} (11)

When we substitute $\delta, \xi, \beta, \lambda$ and $\gamma$ back to (11), then $\phi$ appears as

$$\phi = \left( \frac{2}{\sqrt{1 - v^2}} \left( \frac{\exp[\frac{r A}{2\xi}] \Lambda_1[x]}{\exp[\frac{r A}{2\xi}] (iv^2 \xi)^{1/3} \Lambda_2[x]} \right) - \frac{\exp[\frac{r A}{2\xi}] (iv^2 \xi)^{1/3} \Lambda_1[x]}{2^{1/3}} \right) \right) \right) \times \frac{1}{(\beta + \frac{r A}{2}\Lambda_2[x] - \frac{r A}{2}\Lambda_3[x] C_2)} \Bigg) \times \frac{1}{(\beta + \frac{r A}{2}\Lambda_2[x] - \frac{r A}{2}\Lambda_3[x] C_2)}.$$  \hspace{1cm} (12)

Here,

$$\Lambda_1[x] = \frac{1}{3^{1/6} \Gamma \left[ \frac{3}{2} \right]} + \frac{3^{1/6} x}{4 3^{1/6} \Gamma \left[ \frac{3}{2} \right]} + \frac{x^3}{6 3^{1/6} \Gamma \left[ \frac{3}{2} \right]} + \frac{x^4}{4 3^{1/6} \Gamma \left[ \frac{3}{2} \right]} + \mathcal{O}[x]^5,$$

$$\Lambda_2[x] = \frac{3^{1/6} x}{4 \Gamma \left[ \frac{1}{2} \right]} + \frac{x^2}{2 \Gamma \left[ \frac{1}{2} \right]} + \frac{x^3}{3^{1/3} \Gamma \left[ \frac{5}{6} \right]} + \frac{x^5}{30 \Gamma \left[ \frac{7}{6} \right]} + \mathcal{O}[x]^6,$$

$$\Lambda_3[x] = \frac{1}{3^{2/3} \Gamma \left[ \frac{5}{6} \right]} - \frac{3^{1/3} x}{3^{3/5} \Gamma \left[ \frac{5}{6} \right]} + \frac{x^3}{6 \Gamma \left[ \frac{3}{2} \right]} + \frac{x^4}{12 \Gamma \left[ \frac{3}{2} \right]} + \mathcal{O}[x]^5,$$

$$\Lambda_4[x] = -\frac{1}{3^{1/3} \Gamma \left[ \frac{3}{2} \right]} + \frac{2^{1/3} x}{2^{3/2} \Gamma \left[ \frac{3}{2} \right]} - \frac{x^3}{3 \Gamma \left[ \frac{3}{2} \right]} + \frac{x^5}{30 \Gamma \left[ \frac{7}{6} \right]} + \mathcal{O}[x]^6,$$  \hspace{1cm} (13)
and

\[ x = \frac{-i2v^2(\xi r + \frac{C_1}{\sqrt{1-v^2}}) + g^2 A^2}{2^{4/3}(-iv^2)^{2/3}}. \]  

(14)

So far, this nontrivial solution is well expressed. But to have a firm solution, some adjustments still need to be carried out in future works. For the rest, we will explore the expression for pressure and energy density in the system of gluon-dominated quark-gluon plasma.

4. Summary and discussion

We return to (8), \( \mathcal{E} = \mathcal{P} = 4T g_F m_Q \phi \), the energy and pressure can be expressed as follows [17]

\[ \rho = P = \int \mathcal{E} d^4x = \int \mathcal{P} d^4x = 4T g_F f_Q m_Q \int \int \phi dt dV, \]  

(15)

where \( \mathcal{E} \) and \( \mathcal{P} \) denote the isotropic pressure and density for single fluid field, respectively.

By adopting FRW geometry as the background, and doing integration on the spatial dimension, we get

\[ \rho = P = 4\pi \lambda \int R^3 \phi dt \int \frac{r^2 dr}{\sqrt{1 - kr^2}}. \]  

(16)

Here we have used \( dV = (R^3 r^2 \sin \theta) / \sqrt{1 - kr^2} d\theta \) and \( \lambda = 4T g_F f_Q m_Q \). Furthermore, if we assign \( \int_0^{2\pi} \int_0^\pi dV = 4\pi \int \frac{r^2 dr}{\sqrt{1 - kr^2}} \) as \( \zeta \), then the expression for \( P \) and \( \rho \) can be written as

\[ P = \rho = \lambda \zeta \int R^3 \phi dt. \]  

(17)

In a case when \( r \) is constant, \( p = \rho \propto \phi(t) dt \); and also if \( \phi(t) \propto 1/t \), then one arrives at \( P = \rho \propto \lambda \zeta R^3 / t \). It indicates that the pressure and energy in the system decrease asymptotically.

Figure 1. Pressure inside QGP-lump in the function of radius.
following the increases of time.

Figure 1 shows the total pressure inside the QGP-lump as a function of radius. It is drawn for the simplest condition, where $\phi$ is assumed as a constant, and the scale factor $R = 1$. $k = 1, 0, -1$ represent the space-time curvatures that describe the close, flat and open space-time. One of the obvious properties that appears here is the equation of state $P/\rho = 1$. Such a ratio indicates that in this model the QGP exists within radiation state. This estimate comes from the fact that in general the prerequisite condition for the radiation state is $P \sim \rho/3$, and $P/\rho$ is getting smaller along the transition from radiation state to matter state.

Finally, the study reveals observables that should be accessible through heavy ion collisions experiments at the RHIC and Large Hadron Collider (LHC). With currently proposed calculation results, such experiments are in future expected to become references for adjustments and verifications of the dynamics theory of macroscopic behavior of QGP.

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