Branching diffusions in random environment

Christian Böinghoff* and Martin Hutzenthaler†

University of Frankfurt and University of Munich (LMU)

October 8, 2018

Abstract

We consider the diffusion approximation of branching processes in random environment (BPREs). This diffusion approximation is similar to and mathematically more tractable than BPREs. We obtain the exact asymptotic behavior of the survival probability. As in the case of BPREs, there is a phase transition in the subcritical regime due to different survival opportunities. In addition, we characterize the process conditioned to never go extinct and establish a backbone construction. In the strongly subcritical regime, mean offspring numbers are increased but still subcritical in the process conditioned to never go extinct. Here survival is solely due to an immortal individual, whose offspring are the ancestors of additional families. In the weakly subcritical regime, the mean offspring number is supercritical in the process conditioned to never go extinct. Thus this process survives with positive probability even if there was no immortal individual.

1 Introduction and main results

Branching processes in random environment (BPREs) have been introduced by Smith and Wilkinson (1969) (see also Smith (1968)) and have attracted considerable interest in the last decade (e.g. [3–6,8,15,20,21,30,41,42]). On the one hand this is due to the more realistic model compared with classical branching processes. On the other hand this is due to interesting properties such as phase transitions in the subcritical regime. Here we consider the diffusion approximation of BPREs which can be viewed as continuous mass branching process in random environment. Our results are qualitatively analogous to discrete mass BPREs. The main observation of this article is that the diffusion approximation of BPREs is a simple model (3 parameters) having explicit formulas for various expressions. In particular, the contributions of the branching process and of the environment are explicit in terms of the parameters. These properties make the diffusion approximation interesting for applications.

The diffusion approximation of BPREs has been conjectured by Keiding (1975) and has been established by Kurtz (1978). This diffusion approximation is the strong solution \((Z_t, S_t)_{t \geq 0}\) of the stochastic differential equations (SDEs)

\[
\begin{align*}
dZ_t &= \frac{1}{2} \sigma_e^2 Z_t dt + Z_t dS_t + \sqrt{\sigma^2 Z_t} dW_t^{(b)} \\
dS_t &= \alpha dt + \sqrt{\sigma^2} dW_t^{(e)}
\end{align*}
\]

for \(t \geq 0\) where \(S_0 = 0\). The parameters satisfy \(\alpha \in \mathbb{R}, \sigma_e \in [0, \infty)\) and \(\sigma_b \in (0, \infty)\). The processes \((W_t^{(b)})_{t \geq 0}\) and \((W_t^{(e)})_{t \geq 0}\) are independent standard Brownian motions. We denote the process \((Z_t, S_t)_{t \geq 0}\) as branching diffusion

*Research supported by the German Research Foundation (DFG) and the Russian Foundation of Basic Research (Grant DFG-RFBR 08-01-91954)
†Research supported by the Institute for Mathematical Sciences of the National University of Singapore
AMS 2010 subject classification: 60J80; 60K37, 60J60

Key words and phrases: Branching process, random environment, diffusion approximation, Laplace transform, survival probability, backbone construction, immortal individual, ultimate survival
in random environment (BDRE). Moreover, we will refer to \((S_t)_{t \geq 0}\) as the associated Brownian motion, which is a non-standard Brownian motion. To be accurate, the conjecture of Keiding (1975) did not include the term \(\frac{1}{2}\sigma^2 Z_t \, dt\) which is a characteristic part for random environment. Moreover, Helland (1981) gave an inaccurate “proof” of N. Keiding’s conjecture. So we attribute the correct statement of the diffusion approximation of BPRES to Kurtz (1978). The BDRE has not been studied since Kurtz (1978). For this reason, we first discuss properties of the BDRE obtained by Kurtz (1978) beginning with the diffusion approximation.

First we introduce BPRES in order to state the diffusion approximation. Our formulation follows the notation of Afanasyev et al. \cite{[4]}. Let \(\Delta\) be the Polish space of probability measures on \(\mathbb{N}_0 := \{0, 1, 2, \ldots\}\) equipped with the metric of total variation. Fix \(n \in \mathbb{N} := \{1, 2, \ldots\}\) for the moment. Let \(\Pi^{(n)} = (Q_0^{(n)}, Q_1^{(n)}, \ldots)\) be a sequence of independent and identically distributed random variables taking values in \(\Delta\). Conditioned on \(\Pi^{(n)}\) the BPRE \((Z_i^{(n)})_{i \in \mathbb{N}_0}\) is defined recursively through

\[
Z_{i+1}^{(n)} := \sum_{j=1}^{Z_i^{(n)}} s_{j,i}, \quad i \in \mathbb{N}_0,
\]

where \(Z_0^{(n)}\) is independent of \(\Pi^{(n)}\) and where \((s_{j,i})_{j,i \in \mathbb{N}_0}\) conditioned on \(\Pi^{(n)}\) are independent random variables with distribution

\[
P(s_{j,i}^{(n)} = k | \Pi^{(n)}) = Q_i^{(n)}(k), \quad \forall \, j,i,k \in \mathbb{N}_0.
\]

Let the mean of the environment at time \(i \in \mathbb{N}_0\) be defined through

\[
m(Q_i^{(n)}) := \sum_{k=0}^{\infty} k Q_i^{(n)}(k).
\]

Define a continuous time version of the BPRE through \(Z_t^{(n)} = Z_{\lfloor t \rfloor}^{(n)}\) where \(\lfloor t \rfloor := \max\{m \in \mathbb{N}_0 : m \leq t\}\) for every \(t \in [0, \infty)\). The associated random walk \((S_t^{(n)})_{t \geq 0}\) is defined through

\[
S_t^{(n)} := \sqrt{n} \sum_{i=0}^{\lfloor t \rfloor - 1} \log \left( m(Q_i^{(n)}) \right), \quad t \in [0, \infty).
\]

This random walk is central for the BPRE as it determines the mean of the BPRE:

\[
E[Z_t^{(n)} | \Pi^{(n)}] = E[Z_0^{(n)}] \prod_{i=0}^{\lfloor t \rfloor - 1} m(Q_i^{(n)}) = E[Z_0^{(n)}] \exp \left( \frac{S_t^{(n)}}{\sqrt{n}} \right), \quad t \in [0, \infty).
\]

We included the factor \(\sqrt{n}\) in the definition of the associated random walk to have the usual scaling in the limit as \(n \to \infty\).

Next we let \(n \to \infty\) to obtain the diffusion approximation. The following assumptions mainly ensure that the associated random walk converges to a Brownian motion with infinitesimal drift \(\alpha \in \mathbb{R}\) and infinitesimal standard deviation \(\sigma_c \in [0, \infty)\):

\[
\lim_{n \to \infty} n \cdot E \left[ m(Q_0^{(n)}) - 1 \right] = \alpha \in \mathbb{R}
\]

\[
\lim_{n \to \infty} n \cdot E \left[ (m(Q_0^{(n)}) - 1)^2 \right] = \sigma_c^2 \in [0, \infty)
\]

\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \sum_{k=0}^{\infty} \left( \frac{k}{m(Q_0^{(n)})} - 1 \right)^3 \cdot Q_i^{(n)}(k) \right] < \infty.
\]
So the branching process is near-critical as \( m(Q_0^{(n)}) \to 1 \) in distribution as \( n \to \infty \). If \( \sigma_e > 0 \), then the environment comprises both supercritical and subcritical phases. In addition, we suppose that

\[
\lim_{n \to \infty} \mathbb{E} \left[ \sum_{k=0}^{\infty} \left( \frac{k}{m(Q_0^{(n)})} - 1 \right)^2 Q_0^{(n)}(k) \right] = \sigma_e^2 \in (0, \infty). \tag{10}
\]

So \( \alpha \) is a parameter of expected super-/subcriticality, \( \sigma_e \) is a parameter for the standard deviation of the offspring mean around the critical value 1 and \( \sigma_e^2 \) is the mean offspring variance per individual per generation. Under the above assumptions, Corollary 2.18 of Kurtz (1978) implies that the suitably rescaled BPRE converges in distribution to a diffusion.

**Proposition 1.** Assume that \( Z_0^{(n)}/n \to Z_0 \) in distribution as \( n \to \infty \). Under the assumptions \( \text{[9]}, \text{[10]}, \text{[11]} \) and \( \text{[12]} \) we have that

\[
\left( \frac{Z_t^{(n)}}{n}, \frac{S_t^{(n)}}{\sqrt{n}} \right)_{t \geq 0} \xrightarrow{w} (Z_t, S_t)_{t \geq 0} \tag{11}
\]

in the Skorohod topology (see e.g. [17]) where the limiting diffusion is the strong solution of the SDEs \( \text{[12]} \).

The assumptions of Corollary 2.18 of Kurtz (1978) are checked in Section \( \text{[2]} \).

Inserting the random environment \( (S_t)_{t \geq 0} \) into the diffusion equation \( \text{[11]} \) of the BDRE, we see that \( (Z_t)_{t \geq 0} \) solves the stochastic differential equation

\[
dZ_t = \left( \alpha + \frac{1}{2}\sigma_e^2 \right) Z_t \, dt + \sqrt{\sigma_e^2 Z_t} \, dW_t^{(e)} + \sqrt{\sigma_b^2 Z_t} \, dW_t^{(b)} \tag{12}
\]

for \( t \in [0, \infty) \). Comparing with Feller’s branching diffusion (i.e. \( \text{[12]} \) with \( \sigma_e = 0 \)) there are two differences. First there is an additional drift term \( \frac{1}{2}\sigma_e^2 Z_t dt \). Second there is an additional diffusion term \( \sigma_e Z_t dW_t^{(e)} \). Both terms originate in the conditional expectation

\[
\mathbb{E}[Z_t|S_t] = \mathbb{E}[Z_0] \exp(S_t), \quad t \in [0, \infty), \tag{13}
\]

almost surely, which is a geometric Brownian motion and solves the SDE

\[
dY_t = \left( \alpha + \frac{1}{2}\sigma_e^2 \right) Y_t \, dt + \sigma_e Y_t \, dW_t^{(e)}, \quad Y_0 = \mathbb{E}[Z_0], \tag{14}
\]

for \( t \geq 0 \).

Now we come to properties of the BDRE which embody the branching property conditioned on the environment. Theorem 2.10 of Kurtz (1978) implies that the BDRE is in fact a reweighted and time-changed branching diffusion (see Lemma \( \text{[10]} \) below for a different proof):

**Proposition 2.** Assume \( \alpha \in \mathbb{R} \), \( \sigma_b \in (0, \infty) \) and \( \sigma_e \in [0, \infty) \). Let \( (W_t^{(b)})_{t \geq 0} \) and \( (W_t^{(e)})_{t \geq 0} \) be independent standard Brownian motions. Let \( (F_t)_{t \geq 0} \) be the strong solution of

\[
dF_t = \sqrt{F_t} \, dW_t^{(b)} \tag{15}
\]

for \( t \in [0, \infty) \) and let \( S_t := \alpha t + \sigma_e W_t^{(e)} \) for \( t \in [0, \infty) \). Moreover define \( (\tau(t))_{t \geq 0} \) through

\[
\tau(t) := \int_0^t e^{-S_s} \sigma_b^2 \, ds \tag{16}
\]

for \( t \in [0, \infty) \). Then

\[
(F_{\tau(t)} e^{S_t}, S_t)_{t \geq 0} \tag{17}
\]

is a weak solution of \( \text{[11]} \), that is, is a version of the BDRE \( \text{[11]} \).
Due to this property, many results on Feller’s branching diffusion carry over to the BDRE (1). For example, 
$(Z_t e^{-S_t})_{t \geq 0}$ is a time-changed Feller branching diffusion and is therefore infinitely divisible. Another simple
implication of Proposition 2 is an explicit formula for the Laplace transform of the BDRE (1) conditioned on
the environment, which has not been reported yet. We agree on the convention that

$$
\frac{c}{0} := \begin{cases} 
\infty & \text{if } c \in [0, \infty] \\
0 & \text{if } c = 0
\end{cases}, \quad \frac{c}{\infty} := 0 \text{ for } c \in [0, \infty) \quad \text{and that } 0 \cdot \infty = 0.
$$

(18)

Throughout the paper, the notation $\mathbb{P}$ and $\mathbb{E}$ refers to the starting point of the involved process, e.g. $\mathbb{P}(Z_t \in \cdot) := \mathbb{P}(Z_t \in \cdot | Z_0 = z)$ for $z \in [0, \infty)$ or $\mathbb{P}((Z_t, S_t) \in \cdot) := \mathbb{P}((Z_t, S_t) \in \cdot | (Z_0, S_0) = (z, s))$ for $z \in [0, \infty)$ and
$s \in \mathbb{R}$.

**Corollary 3.** Assume $\alpha \in \mathbb{R}$, $\sigma_b \in (0, \infty)$ and $\sigma_c \in [0, \infty)$. Let $(Z_t, S_t)_{t \geq 0}$ be the strong solution of (1). Then we have that

$$
\mathbb{E}^{(z,0)} \left[ \exp \left( -\lambda Z_t \right) \right] = \exp \left( -\int_0^t \frac{\sigma}{2} \exp \left( \Sigmab \right) ds + \frac{1}{2} \exp \left( -\Sigmab \right) \right)
$$

for all $t, z, \lambda \in [0, \infty)$ almost surely.

The proof is deferred to Section 3. If $\sigma_c = 0$, then (19) is just the Laplace transform of Feller’s branching
diffusion with criticality parameter $\alpha$ and branching rate $\sigma_b^2$.

The simplicity of the right-hand side of (19) derives from the fact that the distribution of the integral of the
squared geometric Brownian motion with drift $\beta \in \mathbb{R}$,

$$
A_t^{(\beta)} := \int_0^t \exp \left( 2(\beta s + W_s^{(\sigma)}) \right) ds, \quad t \in [0, \infty)
$$

(20)
is well understood (see e.g. [10, 14, 34, 43]). Even more, the density of the joint distribution of $(A_t^{(\beta)}, W_t^{(\sigma)} + \beta t)$
is known rather explicitly for every $t \in [0, \infty)$. Define

$$
a_t(x, u) du := \mathbb{P}(A_t^{(\beta)} \in du | W_t^{(\sigma)} + \beta t = x)
$$

for $t, u \in (0, \infty)$ and $x \in \mathbb{R}$. Then the density of $(A_t^{(\beta)}, W_t^{(\sigma)} + \beta t)$ satisfies that

$$
\frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{x^2}{2t} \right) a_t(x, u) = \frac{1}{u} \exp \left( -\frac{1}{2u} \left( 1 + e^{2x} \right) \right) \theta_{c/r} \tau, u)(t)
$$

(22)

where

$$
\theta_{r}(t) = \frac{r}{\sqrt{2\pi r t}} \exp \left( \frac{\pi^2}{2t} \right) \int_0^\infty \exp \left( -\frac{y^2}{2t} \right) \exp \left( -r \cosh(y) \right) \sinh(y) \sin \left( \frac{\pi y}{t} \right) dy
$$

(23)

for all $t, u, r \in (0, \infty)$ and $x \in \mathbb{R}$, see Proposition 2 of Yor (1992). Using the explicit formula (22) allows to
answer rather fine questions by elementary (but sometimes nontrivial) calculations. Thereby the BDRE (1)
becomes one of the most tractable processes in the class of BPREs.

The following corollary of Corollary 3 provides an explicit expression for the survival probability. Define the parameter $\beta \in [-\infty, \infty]$ and the function $f : [0, \infty] \rightarrow [0, 1]$ through

$$
\beta := -\frac{2\alpha}{\sigma_c^2} \quad \text{and} \quad f(x) := 1 - \exp \left( -\frac{\sigma_c^2}{\sigma_b^2} x \right), \quad x \in [0, \infty],
$$

(24)

if $\sigma_c \neq 0$ and through $\beta = 0$ and $f \equiv 0$ if $\sigma_c = 0$. 

4
Corollary 4. Assume $\alpha \in \mathbb{R}$, $\sigma_\beta \in (0, \infty)$ and $\sigma_c \in [0, \infty)$. Let $(Z_t, S_t)_{t \geq 0}$ be the strong solution of (1). Then

$$p_{c, \beta}(a) := \mathbb{P}\left( \frac{1}{2A_v^{(\beta)}} \in da \right) = \frac{e^{-\beta^2 v^2/2e^{2\pi^2 v^2}}}{\sqrt{2\pi v}} \Gamma\left(\beta + 2/2\right)^{-\alpha a - (\beta + 1)/2} \int_0^\infty e^{-\xi^2/2v} \sinh(\xi) \cosh(\xi) \propto (s + (\cosh(\xi))^2)^{1/4} \, d\xi \, ds \, da$$

(27)

for every $t \in (0, \infty)$ and every $z \in [0, \infty]$ where the density function of $1/2A_v^{(\beta)}$ satisfies

$$\mathbb{P}(t > 0 \mid (S_s)_{s \leq t}) = 1 - \exp\left( -\frac{z}{\int_0^t \sigma_s^2 \exp(-S_s) \, ds} \right)$$

(25)

for every $t \in (0, \infty)$ and every $z \in [0, \infty)$ almost surely. If $\beta > -1$ and $\sigma_c > 0$, then

$$\mathbb{P}(t > 0) = \mathbb{E}\left[ f\left( \frac{z}{\sqrt{2A_v^{(\beta)}}} \right) \right] = \int_0^\infty \int_0^\infty f(z_\alpha) p_{c, \beta}(z_\alpha) \, da$$

(26)

for every $t \in (0, \infty)$ and every $z \in [0, \infty)$ where the density function of $1/(2A_v^{(\beta)})$ satisfies

$$\mathbb{P}(t > 0 \mid (S_s)_{s \leq t}) = 1 - \exp\left( -\frac{z}{\int_0^t \sigma_s^2 \exp(-S_s) \, ds} \right)$$

The proof is deferred to Section 3.

The asymptotic behavior of the survival probability strongly depends on $\alpha$. As in the case of classical branching processes, the survival probability stays positive, converges to zero polynomially fast or converges to zero exponentially fast according to whether the process is supercritical ($\alpha > 0$), critical ($\alpha = 0$) or subcritical ($\alpha < 0$), respectively. Now in case of a random environment it is known for BPREs that there is another phase transition in the subcritical regime. For BDREs this phase transition turns out to occur at $\alpha = -\sigma_c^2$. We adopt the standard notation of the literature on BPREs for the different regimes and say that the BDRE is weakly subcritical if $-\sigma_c^2 < \alpha < 0$, intermediate subcritical if $\alpha = -\sigma_c^2$ and strongly subcritical if $\alpha < -\sigma_c^2$. The following theorem establishes the asymptotic behavior of the survival probability of BDREs including explicit expressions for the limiting constants. For the rest of this article, we concentrate on the subcritical regime; the supercritical regime is then subject of the forthcoming paper [23].

Theorem 5. Assume $\alpha \in \mathbb{R}$ and $\sigma_\beta, \sigma_c \in (0, \infty)$. Let $(Z_t, S_t)_{t \geq 0}$ be the strong solution of (1). Then we have that

$$\lim_{t \to \infty} \mathbb{P}(t > 0) = 1 - \left( 1 + \frac{\sigma_c^2}{\sigma_\beta^2} \cdot z \right)^{-\frac{\alpha}{2\sigma_c^2}} > 0 \quad \text{if } \alpha > 0$$

(28)

$$\lim_{t \to \infty} \sqrt{t} \mathbb{P}(t > 0) = \sqrt{\frac{\alpha}{\pi \sigma_c}} \log \left( 1 + \frac{\sigma_c^2}{\sigma_\beta^2} \cdot z \right) > 0 \quad \text{if } \alpha = 0$$

(29)

$$\lim_{t \to \infty} \sqrt{t} e^{\frac{z^2}{2\sigma_\beta^2}} \mathbb{P}(t > 0) = \frac{\sigma_\beta^2}{\sigma_c^2} \int_0^\infty f(z_\alpha) \phi_\beta(a) \, da > 0 \quad \text{if } \frac{\alpha}{\sigma_c^2} \in (-1, 0)$$

(30)

$$\lim_{t \to \infty} \sqrt{t} e^{\frac{z^2}{2\sigma_\beta^2}} \mathbb{P}(t > 0) = \frac{z \sqrt{2\sigma_c}}{\sqrt{\pi} \sigma_\beta} > 0 \quad \text{if } \frac{\alpha}{\sigma_c^2} = -1$$

(31)

$$\lim_{t \to \infty} \sqrt{t} e^{\frac{z^2}{2\sigma_\beta^2}} \mathbb{P}(t > 0) = \frac{2 - \alpha - \sigma_c^2}{\sigma_\beta^2} > 0 \quad \text{if } \frac{\alpha}{\sigma_c^2} < -1$$

(32)

for every $z \in (0, \infty)$, where $\phi_\beta : (0, \infty) \to (0, \infty)$ is defined as

$$\phi_\beta(a) = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{2\pi}} \Gamma\left(\beta + 2/2\right) e^{-a - \beta/2u^{(\beta - 1)/2}e^{-u}} \sinh(\xi) \cosh(\xi) \propto (u + a(\cosh(\xi))^2)^{1/4} \, d\xi \, du$$

(33)

for every $a \in (0, \infty)$. 

The proof is deferred to Section 4.

Let us compare the convergence rate of the survival probability with the classical case \( \sigma_c = 0 \) of Feller’s branching diffusion. Pars pro toto we discuss the critical regime \( \alpha = 0 \). In that case, the survival probability of Feller’s branching diffusion is of order \( O\left(\frac{1}{t}\right) \) whereas it is of order \( O\left(\frac{1}{\sqrt{t}}\right) \) if \( \sigma_c > 0 \) as \( t \to \infty \). So a branching process in random environment has a higher probability to survive. The reason for this is that there is a positive probability of experiencing a long supercritical phase. More precisely, for every \( \varepsilon > 0 \), the event that the critical Brownian motion \( \{S_t\}_{t \geq 0} \) stays above \( \varepsilon \) from time \( \varepsilon \) until time \( t \) is of order \( O\left(\frac{1}{\sqrt{t}}\right) \) as \( t \to \infty \). On this event the branching process is supercritical and survives with positive probability. This explains the slower convergence order \( O\left(\frac{1}{\sqrt{t}}\right) \) as \( t \to \infty \).

Note that the expectation \( \mathbb{E}^z[Z_t] = z \exp\left((\alpha + \frac{\sigma^2}{2})t\right), z \in (0, \infty) \), changes its qualitative behavior as \( t \to \infty \) at \( \alpha = -\frac{\sigma^2}{2} \). The phase transition for the survival probability, however, is at \( \alpha = -\sigma^2 \). Here is an heuristic. If the associated Brownian motion (drift \( \alpha < 0 \)) is negative for almost all of the time, then we expect the BDRE to behave like Feller’s branching diffusion. In that case we expect that \( \mathbb{P}^z(Z_t > 0) \sim \text{const} \cdot \mathbb{E}^z[Z_t] = \text{const} \cdot z \exp\left((\alpha + \frac{\sigma^2}{2})t\right), z \in (0, \infty) \), as \( t \to \infty \). This gives indeed the exponential decay rate in the strongly subcritical regime. However, the associated Brownian motion might be positive until time \( t > 0 \). On this event the BDRE is supercritical and survives with positive probability. The probability of this event decreases like \( \exp\left(-\frac{\sigma^2}{2\sqrt{t}}\right) \) (times polynomial terms) as \( t \to \infty \). This exponential decay rate follows from an application of the Cameron-Martin-Girsanov theorem (e.g. Theorem IV.38.5 in [32]). This gives the exponential decay rate in the weakly subcritical regime. Now the phase transition for the survival probability occurs when these two exponential decay rates \( \alpha + \frac{\sigma^2}{2} \) and \( -\frac{\sigma^2}{2\sqrt{t}} \) coincide, namely at \( \alpha = -\sigma^2 \).

For BPREs Afanasyev (1979) was the first to observe different regimes for the survival probability in the subcritical regime. Independently hereof Dekking (1987) rediscovered this dichotomy. For more recent results on the speed of decay of the survival probability, see Corollary 1.2 of [3] for the critical case, Corollary 1.2 of [3] for the weakly subcritical case, Theorem 1 of [11] for the intermediate subcritical case and Theorem 1.1 of [5] for the strongly subcritical case. Its derivation, however, is sometimes involved and, in general, there are no simple expressions for the limiting constants. Only the case of linear-fractional offspring distributions is known to admit explicit limiting constants, see [2].

Next we investigate the event of survival in more detail and condition the BDRE on the event of ultimate survival. The method of conditioning a Markov process to stay nonnegative has been applied in various situations (e.g. [7],[9],[32]). However, we have not found a suitable formulation for the case of multi-dimensional diffusions. As such a formulation is of independent interest, we include it in the following lemma. For this, define a set \( \mathcal{Q} \) of functions as

\[
\mathcal{Q} := \left\{ q : [0, \infty) \to [0, \infty) \mid \lim_{t \to \infty} \frac{q(t + s)}{q(t)} = 1 \text{ for all } s \in [0, \infty) \right\}.
\]  

(34)

Note that \( q \in \mathcal{Q} \) if and only if \( q \circ \log \) is slowly varying at infinity; see Galambos and Seneta (1973) for this notion.

**Lemma 6.** Let \( d, m \in \mathbb{N} \) and let \( I \subset \mathbb{R}^d \) and \( A \subset I \) be Borel measurable sets. In addition let the drift vector \( \mu : I \to \mathbb{R}^d \) and the diffusion matrix \( \sigma : I \to \mathbb{R}^{d \times m} \) be Borel measurable functions. Moreover, let \( (X_t)_{t \geq 0} \) be a Markov process and a weak solution of the stochastic differential equation

\[
dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dW_t
\]  

(35)

for \( t \in [0, \infty) \) with initial value \( X_0 \in I \) where \( (W_t)_{t \geq 0} \) is an \( m \)-dimensional standard Brownian motion. Assume that there exist a twice continuously differentiable function \( \eta : I \to [0, \infty) \), a function \( q \in \mathcal{Q} \) and values \( \lambda, p \in [0, \infty) \) with the following properties:

- \( \lim_{t \to \infty} q(t) e^{At} \mathbb{P}^x = \left( X_t \notin A \right) = \eta(x) \) for all \( x \in I \),
The proof is deferred to Section 5. Define $\bar{I} := \{ x \in I : \eta(x) > 0 \}$. Then there exists a process $(\bar{X}_t)_{t \geq 0}$ with state space $\bar{I}$ such that
\begin{equation}
\mathbb{P}^x \left( (X_s)_{s \in [0,t]} \in \bullet \mid X_T \notin A \right) \xrightarrow{T \to \infty} \mathbb{P}^x \left( (\bar{X}_s)_{s \in [0,t]} \in \bullet \right)
\end{equation}
for all $x \in \bar{I}$ and all $t \in [0, \infty)$, such that $(\bar{X}_t)_{t \geq 0}$ is a weak solution of the SDE
\begin{equation}
d\bar{X}_t = \left( \sigma \frac{\nabla \eta}{\eta} \sigma \right) (\bar{X}_t) dt + \mu(\bar{X}_t) dt + \sigma(\bar{X}_t) dW_t
\end{equation}
for $t \in [0, \infty)$ where $\nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right)$ and such that
\begin{equation}
\mathbb{E}^x [g(\bar{X}_t)] = \frac{\mathbb{E}^x [\eta(\bar{X}_t)g(\bar{X}_t)]}{e^{-\lambda t} \eta(x)}
\end{equation}
for all $x \in \bar{I}$, $t \in [0, \infty)$ and all Borel measurable functions $g : I \to [0, \infty)$.

The proof is deferred to Section 5.

We will apply Lemma \ref{lemma5} to our bivariate process $(Z_t, S_t)_{t \geq 0}$ and to the set $A = \{ 0 \} \times \mathbb{R}$. Lemma \ref{lemma5} shows that the limiting constants of Theorem \ref{theorem5} play an important role for conditioning on ultimate survival. Theorem \ref{theorem5} shows that $\eta(z, s) = \vartheta(z)$, $(z, s) \in [0, \infty) \times \mathbb{R}$, where the function $\vartheta : [0, \infty) \to [0, \infty)$ is defined through
\begin{equation}
\vartheta(z) = \begin{cases}
1 - \left( 1 + \frac{\sigma^2}{2} \cdot z \right)^{-\frac{\alpha}{\sigma^2}} & \text{if } \alpha > 0 \\
\frac{\sqrt{\sigma}}{\sqrt{2}} \log \left( 1 + \frac{\sigma^2}{2} \cdot z \right) & \text{if } \alpha = 0 \\
z \frac{\sqrt{\sigma}}{\sqrt{2}} e^{-\frac{\alpha - \sigma^2}{\sigma^2}} & \text{if } \frac{\alpha}{\sigma^2} = -1 \\
z \frac{\sqrt{\sigma}}{\sqrt{2}} & \text{if } \frac{\alpha}{\sigma^2} < -1
\end{cases}
\end{equation}
for every $z \in [0, \infty)$ where $\phi_\beta$ is defined in Theorem \ref{theorem5}. Moreover, define $\lambda := 0$ for $\alpha \geq 0$, $\lambda := \frac{\sigma^2}{2\sigma^2} - \alpha$ for $\alpha \in (-\sigma^2, 0)$ and $\lambda := -\left( \alpha + \frac{\sigma^2}{2} \right)$ for $\alpha \leq -\sigma^2$. The following theorem characterizes the BDRE conditioned to never go extinct.

**Theorem 7.** Assume $\alpha \in \mathbb{R}$ and $\sigma_0, \sigma_e \in (0, \infty)$. Let $(Z_t, S_t)_{t \geq 0}$ be the strong solution of \ref{SDE}. Let $(\bar{Z}_t, \bar{S}_t)_{t \geq 0}$ denote the process $(Z_t, S_t)_{t \geq 0}$ conditioned to never go extinct. Then this process is a weak solution of the SDEs
\begin{align}
\quad_dZ_t &= \left( \sigma_0^2 \frac{\partial}{\partial (Z_t)} \vartheta(Z_t) \right) \frac{\partial}{\partial (Z_t)} Z_t + \frac{1}{2} \frac{\sigma^2}{\sigma^2} \frac{\partial}{\partial (Z_t)} Z_t dt + Z_t d\bar{S}_t + \sqrt{\sigma_0^2} \frac{\partial}{\partial (Z_t)} Z_t dW_t^{(b)} \\
\quad_d\bar{S}_t &= \left( \alpha + \sigma_e^2 \frac{\partial}{\partial (Z_t)} \vartheta(Z_t) \right) \frac{\partial}{\partial (Z_t)} Z_t dt + \sigma_e dW_t^{(c)}
\end{align}
for $t \in [0, \infty)$ and satisfies
\begin{equation}
\mathbb{E}^{(z, s)} [g(Z_t, \bar{S}_t)] = e^{\lambda t} \frac{\mathbb{E}^{(z, s)} [\vartheta(Z_t)g(Z_t, \bar{S}_t)]]}{\mathbb{E}^{(z, s)} [\vartheta(Z_t)]}
\end{equation}
for all \((z, s) \in (0, \infty) \times \mathbb{R}, t \in [0, \infty)\) and all Borel measurable functions \(g: [0, \infty) \times \mathbb{R} \rightarrow [0, \infty)\). If \(\alpha \in (-\sigma^2, \infty)\), then the function \((0, \infty) \ni z \mapsto \sigma^2 z^2 \vartheta'(z)/\vartheta(z) \in \mathbb{R}\) is strictly monotonic decreasing and satisfies \(\lim_{z \to 0} \sigma^2 z^2 \vartheta'(z)/\vartheta(z) = \sigma^2\) and \(\lim_{z \to \infty} \sigma^2 z^2 \vartheta'(z)/\vartheta(z) = \max(-\alpha, 0)\). If \(\alpha \in (-\infty, -\sigma^2]\), then \(\sigma^2 z^2 \vartheta'(z)/\vartheta(z) = \sigma^2\) for all \(z \in (0, \infty)\). If \(\alpha > -\sigma^2\), then \(\lim_{t \to \infty} \bar{Z}_t = \infty\) in distribution. If \(\alpha < -\sigma^2\), then

\[
P^z(\bar{Z}_t \in dy) \xrightarrow{w_{t \to \infty}} c y (\sigma^2 + \sigma^2 y) \xrightarrow{w_{t \to \infty}} dy
\]

(weak convergence of measures on \((0, \infty)\)) for every \(z \in (0, \infty)\) where \(c = c(\alpha, \sigma^2, \sigma^2) \in (0, \infty)\) is a normalizing constant such that the right-hand side is a probability distribution.

The proof is deferred to Section 6.

Theorem 7 exhibits a difference in the survival opportunities between the weakly subcritical and strongly subcritical regimes. In the strongly subcritical regime, \(\vartheta\) is a linear function and the SDEs of the conditioned process \((\bar{Z}_t, \bar{S}_t)_{t \geq 0}\) simplify to

\[
d\bar{Z}_t = \left(\sigma^2 + \frac{1}{2} \sigma^2 \bar{Z}_t\right) dt + \bar{Z}_t d\bar{S}_t + \sqrt{\sigma^2 \bar{Z}_t} dW^{(b)}
\]

\[
d\bar{S}_t = (\alpha + \sigma^2) dt + \sigma c dW^{(c)}
\]

for \(t \in [0, \infty)\). The drift of the associated Brownian motion \((\bar{S}_t)_{t \geq 0}\) is increased by \(\sigma^2\), but is still negative. Thus the conditioned process survives solely due to the immigration term \(\sigma^2 dt\). In the weakly subcritical regime, the drift of the associated Brownian motion \((\bar{S}_t)_{t \geq 0}\) is strictly positive. Thus the conditioned process in the weakly subcritical regime survives due to a supercritical environment with positive probability. The immigration term \(\sigma^2 dz\vartheta'(z)/\vartheta(z) dt\) is not needed for this. The effect of this immigration term is to ensure survival with full probability. Another observation in the weakly subcritical regime is that the environment in the conditioned process depends on the population size. The reason for this is that the survival probability of a supercritical BDRE depends on the initial mass. In addition it is intuitive that the environment in the conditioned process needs to be less beneficial if the population size is large. Formally this means that the additional drift term \(\sigma^2 z^2 \vartheta'(z)/\vartheta(z)\) is decreasing in \(z \in (0, \infty)\). More precisely, this function decreases from \(\sigma^2\) to \(\max(-\alpha, 0)\) as the population size increases. Moreover, Theorem 7 provides an explicit quantification of the dependence of the additional drift term on the conditioned process on the population size.

In the strongly subcritical and in the intermediately subcritical regimes, conditioning on ultimate survival affects each individual in the same way so that the conditioned process is again a BDRE except for an additional immigration term. In the case of a constant environment, branching processes with immigration may be represented as a branching process with an additional immortal individual. The trajectory of the immortal individual is referred to as spine or backbone. This backbone construction goes back to Kallenberg (1977) for branching processes in discrete time and has later been established e.g. for branching processes in continuous time (Gorostiza and Wakolbinger 1991), for the Dawson-Watanabe superprocess (Evans 1993), for the infinite-variance \((1 + \beta)\)-superprocess (Etheridge and Williams 2003) or for general continuous-state branching processes (Lambert 2007). In all of these backbone constructions, families evolve independently of each other. The next theorem establishes the backbone construction for BDREs conditioned on ultimate survival in the strongly subcritical and in the intermediately subcritical regimes. Here the families are correlated through the environment. Due to Proposition 2 however, this correlation is rather explicit.

We understand a family to be a single ancestor together with the progeny of that individual. The total mass hereof as a function of time is an excursion from 0 as a single individual has mass 0 in the diffusion approximation. We denote the space of continuous excursions from 0 as

\[
U := \{\chi \in C((-\infty, \infty), [0, \infty)) : T_0(\chi) \in (0, \infty), \chi_t = 0 \ \forall t \in (-\infty, 0] \cup [T_0(\chi), \infty)\}
\]

where \(T_0(\chi) := \inf\{t > 0: \chi_t = 0\} \in [0, \infty]\) is the first hitting time of 0 (\(\inf \emptyset := \infty\)). Let \((F_t)_{t \geq 0}\) be the strong solution of the SDE

\[
dF_t = \sqrt{F_t} dW_t
\]
for \( t \in [0, \infty) \). The law of families of the process \((F_t)_{t \geq 0}\) is the excursion measure \( Q \) which is a \( \sigma \)-finite measure on the excursion space \( U \) and which is uniquely determined by

\[
\int g(\chi) \, Q(d\chi) = \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{E}^\delta \left[ g((F_t)_{t \geq 0}) \right]
\]

for all bounded, continuous functions \( g: C([0, \infty), [0, \infty)) \to \mathbb{R} \) depending on a finite time interval and such that there is an \( \varepsilon > 0 \) such that \( g(\chi) = 0 \) for all \( \chi \in C([0, \infty), (0, \infty)) \) with \( \sup_{t \geq 0} \chi_t \leq \varepsilon \). Such a measure \( Q_F \) exists according to Theorem 1 in [24].

**Theorem 8.** Assume \( \sigma_\theta, \sigma_\varepsilon \in (0, \infty) \) and \( \alpha \in (-\infty, -\sigma_\varepsilon^2] \). Let \((W_i^{(c)})_{i \geq 0}\) be a standard Brownian motion. Define

- \( \tilde{S}_t := (\alpha + \sigma_\varepsilon^2) t + \sigma_\varepsilon W_i^{(c)} \) for \( t \in [0, \infty) \),
- \((\tilde{\tau}(t))_{t \geq 0}\) through \( \tilde{\tau}(t) := \int_0^t e^{-\tilde{S}_t^{-1/2} \alpha} \, dt \) for \( t \in [0, \infty) \),
- a Poisson point process \( \mathcal{P} \) on \([0, \infty) \times U\) with intensity measure \( dy \times Q \) and
- a Poisson point process \( \tilde{\mathcal{P}} \) on \([0, \infty) \times U\) with intensity measure \( dt \times Q \).

Assume the ingredients \((W_i^{(c)})_{i \geq 0}, \mathcal{P} \) and \( \tilde{\mathcal{P}} \) to be independent. Let \((\tilde{Z}_t, \tilde{S}_t)_{t \geq 0}\) denote the BDRE \((Z_t, S_t)_{t \geq 0}\) started in \( Z_0 = z \in (0, \infty) \) and conditioned to never go extinct and define a process \((\tilde{Z}_t)_{t \geq 0}\) through \( \tilde{Z}_0 := z \) and

\[
\tilde{Z}_t := \sum_{(y, \chi) \in \mathcal{P}} \mathbb{1}_{y \leq z} \chi_{\tilde{\tau}(t)} e^{\tilde{S}_t} + \sum_{(u, \chi) \in \tilde{\mathcal{P}}} \chi_{\tilde{\tau}(t) - u} e^{\tilde{S}_t}
\]

for \( t \in (0, \infty) \). Then \((\tilde{Z}_t, \tilde{S}_t)_{t \geq 0}\) and \((\tilde{Z}_t, \tilde{S}_t)_{t \geq 0}\) are equal in distribution.

The proof is deferred to Section 7.

The excursions \((y, \chi) \in \mathcal{P}\) are the families whose ancestor lived before time 0. Due to conditioning on ultimate survival, there is an immortal individual. Offspring of this individual are the ancestors of families \((s, \chi) \in \tilde{\mathcal{P}}\). Conditioned on the environment, all of these families evolve independently of each other. Note that the environment appears in only through the time-change and through the reweighting of the critical excursion paths. In the critical and weakly subcritical regimes, the conditioned process \((\tilde{Z}_t, \tilde{S}_t)_{t \geq 0}\) is not a BDRE with immigration. A representation with independent families is therefore not possible. In view of Theorem 8 the SDEs can still be interpreted as follows: Birth events of the immortal individual are accepted only with probability \( \sigma^2 \) (at \( 0, 1 \)) if the current population size is \( z \in (0, \infty) \). Moreover the additional drift of \((S_t)_{t \geq 0}\) is not \( \sigma^2 \) as in the strongly subcritical regime but \( \sigma_z^2 \) if the current population size is \( z \in (0, \infty) \).

## 2 Diffusion approximation

**Proof of Proposition 3.** We derive Proposition 3 from Corollary 2.18 of Kurtz (1978). To check the assumptions thereof we need more notation. Define

\[
\alpha_i^{(n)} := \sum_{k=0}^{\infty} \left( \frac{k}{m(Q_i^{(n)})} - 1 \right)^2 Q_i^{(n)}(k)
\]

for all \( i \in \mathbb{N}, n \in \mathbb{N} \) and note that \( \alpha_i^{(n)}, i \in \mathbb{N} \), are independent and identically distributed for every \( n \in \mathbb{N} \). According to assumption (10), the expectation of \( \alpha_i^{(n)} \) converges to \( \sigma_i^2 \) as \( n \to \infty \) for every \( i \in \mathbb{N} \). Therefore the law of large numbers for triangular independent sequences implies that

\[
A_n(t) := \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \alpha_i^{(n)} \to t \sigma_i^2 \quad \text{as} \quad n \to \infty
\]

(49)
almost surely for every $t \in [0, \infty)$. The rescaled associated random walk converges (see [35], Theorem 3) to a Brownian motion $(S_t)_{t \geq 0}$, that is,

$$\left( \frac{S_{tn}}{\sqrt{n}} \right)_{t \geq 0} \xrightarrow{w} (S_t)_{t \geq 0} \quad \text{as } n \to \infty. \quad (50)$$

The Brownian motion $(S_t)_{t \geq 0}$ has drift $\alpha$ due to assumption (11) and to $\log(x) \approx x - 1$ for all $x$ in a neighbourhood of 1. Furthermore, $(S_t)_{t \geq 0}$ has infinitesimal variance $\sigma^2_t$ due to assumption (5). Moreover, Corollary 2.18 of Kurtz (1978) requires a third moment condition which follows from (9) and from

$$E \left[ \frac{1}{n^{3/2}} \sum_{i=0}^{\lfloor nt \rfloor - 1} \sum_{k=0}^{\infty} \frac{k}{m(Q_t^{(n)})} - 1 \right]^{3} \lesssim \sup_{n \in N} E \left[ \sum_{k=0}^{\infty} \frac{k}{m(Q_t^{(n)})} - 1 \right]^{3} \cdot Q_0^{(n)}(k) \xrightarrow{n \to \infty} 0.$$

Having checked all assumptions, Proposition 1 follows from Corollary 2.18 of Kurtz (1978).

\[ \square \]

### 3 The Laplace transform and the extinction probability

**Proof of Corollary 3** It suffices to prove (19) for the version (17) of the BDRE due to Proposition 2. The Laplace transform of Feller’s branching diffusion $(F_t)_{t \geq 0}$ satisfies

$$E^z \left[ \exp \left( -\lambda F_t \right) \right] = \exp \left( -\frac{z}{2} t + \frac{z}{\lambda} \right) \quad \text{for } t, \lambda \in [0, \infty), \quad (51)$$

(e.g., Example 26.11 of [29]). Thus we get for the Laplace transform of $F_{\tau(t)}e^{S_t}$ that

$$E^{(z,0)} \left[ \exp \left( -\lambda F_{\tau(t)}e^{S_t} \right) \right] = \exp \left( -\frac{z}{2} \tau(t) + \frac{z}{\lambda \tau(t)} \right)
= \exp \left( -\frac{z}{\int_0^t \sigma^2_u du} \exp \left( -S_s \right) ds + \frac{1}{\lambda} \exp(-S_t) \right) \quad (52)$$

for all $t, z, \lambda \in [0, \infty)$ almost surely. This completes the proof.

\[ \square \]

**Proof of Corollary 4** Fix $t \in [0, \infty)$ and $z \in [0, \infty)$. Letting $\lambda \to \infty$ in formula (19) for the Laplace transform and applying the dominated convergence theorem yields that

$$E^{(z,0)} \left[ Z_t > 0 \mid (S_s)_{s \leq t} \right] = \lim_{\lambda \to \infty} E^{(z,0)} \left[ 1 - \exp \left( -\lambda Z_t \right) \mid (S_s)_{s \leq t} \right]
= \lim_{\lambda \to \infty} \left[ 1 - \exp \left( -\frac{z}{\int_0^t \sigma^2_u du} \exp \left( -S_s \right) ds + \frac{1}{\lambda} \exp(-S_t) \right) \right]
= 1 - \exp \left( -\frac{z}{\int_0^t \sigma^2_u du} \exp(-S_s) ds \right) \quad (53)$$

almost surely. This proves (25). Now assume that $\sigma_e > 0$ and recall that $\beta = -\frac{2\alpha}{\sigma_e^2}$. Note that

$$\begin{align*}
-\alpha s - \sigma_e W_s \quad &\quad 0 \leq s \leq t \\
\frac{d}{ds} \left( -\frac{2\alpha}{\sigma_e^2} s + 2W_s \right) \quad &\quad 0 \leq s \leq t.
\end{align*} \quad (54)$$

10
\textbf{Proof.} Let for every \( \lambda \) we have that \( t \) intermediately and strongly subcritical regimes. \( c \) which is the lower bound in (57).

Inserting \( S_s = \alpha s + \sigma_s W_s^{(c)} \), \( s \leq t \), into (53), applying (54) and the time substitution \( u := \frac{z^2}{\lambda^2} \), we get that
\[
\mathbb{P}^z\left( Z_t > 0 \mid (S_s)_{s \leq t} \right) = 1 - \exp\left( -\frac{z^t}{\int_0^t \sigma_s^2 \exp(-\alpha s - \sigma_s W_s^{(c)}) \, ds} \right)
\]
\[
= 1 - \exp\left( -\frac{z}{\int_0^t \sigma_s^2 \exp\left( -2\lambda^2 \frac{\sigma_s^2}{\lambda^2} s + 2W_s^{(c)} \right) \, ds} \right)
\]
\[
= 1 - \exp\left( -\frac{z}{\int_0^t \sigma_s^2 \exp\left( 2\beta u + 2W_u^{(c)} \right) \, du} \right)
\]
\[
= 1 - \exp\left( -\frac{za^2}{2\sigma_s^2 A_s^{(\beta)}} \right) = f\left( \frac{z}{2A_s^{(\beta)}} \right)
\]
almost surely. Taking expectations, we arrive at
\[
\mathbb{P}^z\left( Z_t > 0 \right) = \mathbb{E}\left[ f\left( \frac{z}{2A_s^{(\beta)}} \right) \right] = \int_0^\infty f(z)\rho(z/2,\beta(a) \, da.
\]
The last step is equation (2.5) in Matsumoto and Yor (2003) which requires \( \beta > -1 \).

\section{4 Asymptotic behavior of the survival probability}

Throughout this section, let \( (Z_t, S_t)_{t \geq 0} \) be the strong solution of (1) and \( f \) be defined as in (24).

\subsection{4.1 General results}

Recall \( A_t^{(\gamma)} \), \( t \in [0, \infty) \), \( \gamma \in \mathbb{R} \), from (29). First we provide sufficient conditions under which \( \mathbb{E}\left[ z/A_t^{(\gamma)} \right] \) and \( \mathbb{E}\left[ (1 - \exp(-z/A_t^{(\gamma)})) \right] \), \( z \geq 0 \), have the same asymptotics as \( t \to \infty \) for \( \gamma \in \mathbb{R} \). We will use this for the intermediate and strongly subcritical regimes.

\textbf{Lemma 9.} Let \( (Y_t)_{t \geq 1} \) be a family of non-negative random variables. Assume that there exist a function \( c: [1, \infty) \to [0, \infty) \) and a constant \( a \in [0, \infty) \) such that \( \lim_{t \to \infty} ct\mathbb{E}[Y_t] = a \) and \( \limsup_{t \to \infty} ct\mathbb{E}[Y_t^2] = 0 \). Then we have that
\[
\lim_{t \to \infty} ct\mathbb{E}\left[ 1 - \exp\left( -\lambda Y_t \right) \right] = \lambda a
\]
for every \( \lambda \in [0, \infty) \).

\textbf{Proof.} Let \( \lambda \in [0, \infty) \) be fixed. The upper bound follows from \( 1 - e^{-\lambda x} \leq \lambda x \), \( x \geq 0 \) and from \( \lim_{t \to \infty} ct\mathbb{E}[Y_t] = a \). The lower bound results from \( 1 - e^{-\lambda x} \geq \lambda x - \frac{\lambda^2 x^2}{2} \) for every \( x \geq 0 \). Applying this, we get that
\[
ct\mathbb{E}\left[ 1 - \exp(-\lambda Y_t) \right] \geq ct\mathbb{E}\left[ \lambda Y_t - \frac{\lambda^2 Y_t^2}{2} \right] = ct\mathbb{E}[Y_t] - \frac{\lambda^2 ct\mathbb{E}[Y_t^2]}{2}
\]
for every \( t \geq 0 \). The assumptions of the lemma then yield that
\[
\liminf_{t \to \infty} ct\mathbb{E}\left[ 1 - \exp( -\lambda Y_t) \right] \geq \liminf_{t \to \infty} \left( ct\mathbb{E}[Y_t] - \frac{\lambda^2 ct\mathbb{E}[Y_t^2]}{2} \right) = \lambda a,
\]
which is the lower bound in (57). \( \square \)
The Cameron-Martin-Girsanov theorem (e.g., Theorem IV.38.5 in [36]) for a standard Brownian motion $(W_s)_{s \geq 0}$ asserts that

$$E \left[ h\left((W_s + \theta s)_{0 \leq s \leq t}\right) \right] = E \left[ \exp(\theta W_t - \theta^2 t/2) h((W_s)_{0 \leq s \leq t}) \right]$$  \hspace{1cm} (60)

for every $\theta \in \mathbb{R}$, every measurable function $h: C([0, t], \mathbb{R}) \to [0, \infty)$ and every $t \in [0, \infty)$. The next lemma will allow us to deduce the intermediate and the strongly subcritical case from the critical and supercritical case, respectively, by changing the drift through (60).

**Lemma 10.** Let $\gamma \in \mathbb{R}$ and let $(A_t^{(\gamma)})_{t \geq 0}$ be defined as in (20). Then we have that

$$E \left[ \frac{1}{2A_t^{(\gamma)}} \right] = e^{-(2\gamma - 2)t} E \left[ \frac{1}{2A_t^{(-\gamma - 2)}} \right]$$  \hspace{1cm} (61)

$$E \left[ \frac{1}{(2A_t^{(\gamma)})^2} \right] \leq e^{-(2\gamma - 2)t} E \left[ \frac{1}{2A_{t/2}^{(-\gamma - 2)}} \right]$$  \hspace{1cm} (62)

for every $t \in (0, \infty)$.

**Proof.** Let $t \in (0, \infty)$ and $\gamma \in \mathbb{R}$ be fixed. Applying (60) with $\theta = 2$, we obtain that

$$E \left[ \frac{1}{2A_t^{(\gamma)}} \right] = E \left[ \frac{1}{2 \int_0^t \exp \left( 2(\gamma s + W_s^{(\gamma)}) \right) ds} \right] = E \left[ \frac{\exp(2W_t^{(\gamma)} - 2t/2)}{2 \int_0^t \exp \left( 2((\gamma - 2)s + W_s^{(\gamma)}) \right) ds} \right]$$

$$= e^{-2t} E \left[ \frac{1}{2 \int_0^t \exp \left( 2((\gamma - 2)s + W_s^{(\gamma)} - W_t^{(\gamma)}) \right) ds} \right] = e^{-2t} E \left[ \frac{1}{2 \int_0^t \exp \left( 2((-\gamma - 2)s + W_s^{(\gamma)}) \right) ds} \right]$$

$$= e^{-2(\gamma - 2)t} E \left[ \frac{1}{(2A_t^{(\gamma)})^2} \right],$$

where we used the substitution $u := t - s$. This is the first claim of the lemma.

For the second moment of $1/(2A_t^{(\gamma)})$, we use analogous arguments to obtain that

$$E \left[ \frac{1}{(2A_t^{(\gamma)})^2} \right] = E \left[ \frac{1}{\left( \int_0^t \exp \left( 2(\gamma s + W_s^{(\gamma)}) \right) ds \right)^2} \right] = E \left[ \frac{\exp(2W_t^{(\gamma)} - 2t/2)}{(\int_0^t \exp \left( 2((\gamma - 2)s + W_s^{(\gamma)}) \right) ds)^2} \right]$$

A rough estimate allows us to split the integral into two independent parts:

$$E \left[ \frac{1}{(2A_t^{(\gamma)})^2} \right] = e^{-2t} E \left[ \frac{\exp(2W_t^{(\gamma)})}{(\int_0^t \exp \left( 2((\gamma - 2)s + W_s^{(\gamma)}) \right) ds)^2} \right]$$

$$\leq e^{-2t} \left[ \frac{1}{2 \int_{t/2}^t \exp \left( 2((\gamma - 2)s + W_s^{(\gamma)}) \right) ds} \cdot \frac{\exp(2W_t^{(\gamma)})}{\int_0^{t/2} \exp \left( 2((\gamma - 2)s + W_s^{(\gamma)}) \right) ds} \right]$$

$$= e^{-2t} E \left[ \frac{1}{2A_{t/2}^{(\gamma - 2)}} \right] \cdot E \left[ \frac{1}{2 \int_{t/2}^t \exp \left( 2((\gamma - 2)s + W_s^{(\gamma)} - W_t^{(\gamma)}) \right) ds} \right]$$

(63)

Substituting $u := t - s$ yields that

$$E \left[ \frac{1}{(2A_t^{(\gamma)})^2} \right] \leq e^{-2t(\gamma - 2)t} E \left[ \frac{1}{2A_{t/2}^{(\gamma - 2)}} \right] \cdot E \left[ \frac{\exp(-2(\gamma - 2)t)}{2 \int_{t/2}^t \exp \left( 2((-\gamma - 2)(s - t) + W_s^{(\gamma)}) \right) ds} \right]$$

$$= e^{-2t(\gamma - 2)t} E \left[ \frac{1}{2A_{t/2}^{(\gamma - 2)}} \right] \cdot E \left[ \frac{1}{2 \int_{t/2}^t \exp \left( 2((-\gamma - 2)u + W_u^{(\gamma)}) \right) ds} \right]$$

(64)

$$= e^{-(2\gamma - 2)t} E \left[ \frac{1}{2A_{t/2}^{(\gamma - 2)}} \right] \cdot E \left[ \frac{1}{2A_{t/2}^{(-\gamma - 2)}} \right],$$

12
which is the second claim of the lemma.

4.2 The supercritical regime

In this subsection, we will prove the result of Theorem 5 in the case of \((S_t)_{t \geq 0}\) having positive drift \(\alpha > 0\). Here we will prove for \(\alpha > 0\) that, starting from \(z > 0\), the probability of survival is strictly positive in the limit as \(t \to \infty\) and for every \(z \in [0, \infty)\) is given by

\[
\lim_{t \to \infty} \mathbb{P}(Z_t > 0) = \mathbb{E} \left[ 1 - \exp \left( - \frac{\sigma^2}{\sigma_0^2} z G \frac{2z}{\sigma_0^2} \right) \right] = 1 - \left( 1 + \frac{\sigma^2}{\sigma_0^2} z \right)^{-\frac{2z}{\sigma_0^2}},
\]

where

\[
\mathbb{P}(G \in dx) = \frac{1}{\Gamma(\nu)} x^{\nu-1} e^{-x} dx, \quad x \in (0, \infty).
\]

The last equality in (65) follows from the explicit formula for the Laplace transform of the gamma-distribution (e.g. [38]), that is, for every \(\lambda \geq 0\) and \(\nu \in (0, \infty)\), \(\mathbb{E}[\exp(-\lambda G)] = \frac{1}{1+\lambda}\). We use the following result of Dufresne (1990) (see also [44]).

Lemma 11. Let \((B_t)_{t \geq 0}\) be a standard Brownian motion. For all \(a \neq 0\) and \(b > 0\),

\[
\left( \int_0^\infty \exp(aB_s - bs) ds \right)^{-1/2} \frac{a^2}{2} G \frac{2z}{\sigma_0^2}
\]

where \(G\) has distribution (66).

Proof of the supercritical case of Theorem 5. Let \(z \geq 0\) be fixed. As \(1 - e^{-x} \leq 1, x \geq 0\), we may apply the dominated convergence theorem. Using (25) in Corollary 4 and continuity of \(f\), we get that

\[
\lim_{t \to \infty} \mathbb{P}(Z_t > 0) = \lim_{t \to \infty} \mathbb{E} \left[ 1 - \exp \left( - \frac{\sigma^2}{\sigma_0^2} z G \frac{2z}{\sigma_0^2} \right) \right] = \mathbb{E} \left[ 1 - \exp \left( - \frac{\sigma^2}{\sigma_0^2} z G \frac{2z}{\sigma_0^2} \right) \right].
\]

\[\square\]

4.3 The critical regime

Next we study the case \(\alpha = 0\) and prove [29]. For simplicity, denote \(A_t := A_t^{(0)}\) for \(t \geq 0\). The next lemma yields the convergence.

Lemma 12. Let \(g: [0, \infty) \to [0, \infty)\) be a Borel measurable function. Assume for some constants \(c, p \in (0, \infty)\) that \(g(x) \leq cx^p\) for every \(x \geq 0\). Then we get that

\[
\lim_{t \to \infty} \sqrt{t} \cdot \mathbb{E} \left[ g \left( \frac{1}{2A_t} \right) \right] = \int_0^\infty g(a) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}} da < \infty.
\]

Proof. Let \(g\) be a function fulfilling the conditions of the lemma. Instead of (25), we will use a simpler expression for the density of \(A_t\) for \(t \in (0, \infty)\). According to Theorem 4.1 of Dufresne (2001), the density function of \(1/(2A_t)\) is given by

\[
\mathbb{P} \left( \frac{1}{2A_t} \in da \right) = \frac{\sqrt{2\pi}}{\sqrt{2\pi}^2} \frac{1}{\sqrt{\pi}} \int_0^\infty \exp \left( -a \left( \cosh(y) \right)^2 - \frac{y^2}{2t} \right) \cosh(y) \cos \left( \frac{\pi y}{2t} \right) dy da =: p_t(a) da
\]

(70)
on \((0, \infty)\) for every \(t \in (0, \infty)\). As for every \(x \geq 0\), \(\frac{1}{2} e^x \leq \cosh(x) \leq e^x\), \(|\cos(x)| \leq 1\) and \(g(x) \leq cx^2\), we obtain that

\[
\int_0^\infty \int_0^\infty \sup_{t \geq 1} |\frac{e^{\sqrt{\pi t}} g(a)}{\sqrt{\pi t} \sqrt{a}} \exp \left( -a \left( \cosh(y) \right)^2 - \frac{y^2}{2t} \right) \cosh(y) \cos \left( \frac{\pi y}{2t} \right) | \, dy \, da \\
\leq \int_0^\infty \int_0^\infty e^{\frac{a}{4} x^2} \frac{1}{x^2} \exp \left( -\frac{ae^{2y}}{4} \right) (\exp(y)) \, dy \, da.
\]

(71)

Using the substitution \(y := ax\), we see that

\[
\int_0^\infty a^{p+\frac{1}{2}} e^{-ax} \, da = \int_0^\infty g^{p+\frac{1}{2}} x^{\frac{1}{2} x^2 - p} e^{-y} \frac{1}{x} \, dy = \frac{1}{x^{1+p}} \Gamma \left( p + \frac{1}{2} \right)
\]

(72)

for every \(x > 0\). Now applying Fubini’s theorem and (72) yields that

\[
\int_0^\infty \int_0^\infty a^{p+\frac{1}{2}} \exp \left( -\frac{ae^{2y}}{4} \right) (\exp(y)) \, dy \, da = \int_0^\infty \int_0^\infty a^{p+\frac{1}{2}} \exp \left( -\frac{ae^{2y}}{4} \right) (\exp(y)) \, dy \, da
\]

\[
= \Gamma \left( p + \frac{1}{2} \right) \left( \frac{4}{e^{2y}} \right)^{p+\frac{1}{2}} e^{y} \, dy = \Gamma \left( p + \frac{1}{2} \right) 2^{2p+1} \int_0^\infty e^{-2py} \, dy
\]

\[
= \Gamma \left( p + \frac{1}{2} \right) 2^{2p+1} \frac{1}{2p} < \infty.
\]

(73)

Thus the integrals in (71) are finite. Applying the dominated convergence theorem, we get that

\[
\lim_{t \to \infty} \sqrt{t} \cdot \mathbb{E} \left[ g \left( \frac{1}{2 A_t} \right) \right] = \lim_{t \to \infty} \sqrt{t} \int_0^\infty g(a) p_\pi(a) \, da
\]

\[
= \int_0^\infty g(a) \int_0^\infty \frac{\sqrt{2\pi}}{\sqrt{\pi} \sqrt{a}} \frac{1}{\sqrt{\pi}} \exp \left( -a \left( \cosh(y) \right)^2 - \frac{y^2}{2t} \right) \cosh(y) \cos \left( \frac{\pi y}{2t} \right) \, dy \, da
\]

\[
= \int_0^\infty g(a) \int_0^\infty \frac{\sqrt{2\pi}}{\sqrt{a}} \frac{1}{\sqrt{\pi}} \exp \left( -a \left( \cosh(y) \right)^2 \right) \cosh(y) \, dy \, da.
\]

(74)

To complete the lemma, we will simplify the inner integral in the above equation. Noting that \((\cosh(y))^2 = 1 + (\sinh(y))^2\), \(y \in \mathbb{R}\), and using the substitution \(x := \sqrt{2a} \sinh(y)\) yields that

\[
\int_0^\infty \frac{\sqrt{2\pi}}{\sqrt{a}} \frac{1}{\sqrt{\pi}} \exp \left( -a \left( \cosh(y) \right)^2 \right) \cosh(y) \, dy = \int_0^\infty \frac{\sqrt{2\pi}}{\sqrt{a}} \frac{1}{\sqrt{\pi}} \exp \left( -a \left( 1 + (\sinh(y))^2 \right) \right) \cosh(y) \, dy
\]

\[
= \int_0^\infty \frac{\sqrt{2\pi}}{\sqrt{a}} \frac{1}{\sqrt{\pi}} \exp \left( -a - \frac{x^2}{2} \right) \frac{1}{\sqrt{2a}} e^{-x} \, dx = \frac{1}{\sqrt{2\pi}} e^{-a}
\]

for all \(a \in (0, \infty)\).

\[\square\]

**Proof of the critical case of Theorem**

Fix \(z \geq 0\). As \(f(zx) \leq \frac{z^2 x^2}{2}\) for every \(x \geq 0\), the assumptions of Lemma 12 are met with \(g = f\). Applying Corollary 11 and Lemma 12 we get that

\[
\lim_{t \to \infty} \sqrt{t} \cdot \mathbb{P}^z (Z_t > 0) = \frac{2}{\sigma_c} \lim_{t \to \infty} \sqrt{\frac{t \sigma_c^2}{4}} \cdot \mathbb{E} \left[ f \left( \frac{z}{2 A_t^{\sigma_c^2/4}} \right) \right] = \frac{2}{\sigma_c} \int_0^\infty f(za) \frac{1}{\sqrt{2\pi}} e^{-a} \, da.
\]

(75)

Next we have that

\[
\int_0^\infty (1 - e^{-ax}) \frac{e^{-x}}{x} \, dx = \log(1 + c)
\]

(76)
for every $c \in [0, \infty)$ which can be checked by differentiating both sides. Using this result with $c = \frac{\sigma_t^2}{\sigma_b^2}z$ and recalling the definition (24) of $f$, we get that

$$
\lim_{t \to \infty} \sqrt{t} \cdot \mathbb{P}^z (Z_t > 0) = \frac{2}{\sigma_e} \int_0^\infty \left(1 - \exp \left(-\frac{\sigma_t^2}{\sigma_b^2}a \right) \right) \frac{1}{\sqrt{2\pi}} \frac{e^{-a}}{a} \, da = \sqrt{2} \sqrt{\frac{1}{2\pi \sigma_e}} \log \left(1 + \frac{\sigma_t^2}{\sigma_b^2} \cdot z \right),
$$

which proves Theorem 5 in the case $\alpha = 0$.

\[ \square \]

### 4.4 The weakly subcritical regime

Next, we turn to the case $-\sigma_e^2 < \alpha < 0$ and prove (30).

**Lemma 13.** Let $\gamma > 0$ and let $g : [0, \infty) \to [0, \infty)$ be a Borel measurable function. Assume for some $b > \gamma/2$ and $c > 0$ that $g(x) \leq cx^b$ for every $x \geq 0$. Then we get that

$$
\lim_{t \to \infty} E^{\gamma^2 t/2} \left[ g \left( \frac{1}{2\lambda_t^{(\gamma)}} \right) \right] = \int_0^\infty g(a) \phi_\gamma (a) \, da < \infty.
$$

**Proof.** Let $\gamma > 0$ be fixed and let $g$ be a function fulfilling the conditions of the lemma with constants $b > \gamma/2$ and $c > 0$. Recall the density of $\frac{1}{2\lambda_t^{(\gamma)}}$, $t > 0$, from (27). With the substitution $a = \gamma^2 t$, we get for $t \in (0, \infty)$ that

$$
\int_0^\infty \int_0^\infty \int_0^\infty \sup_{n \geq 1} \frac{t e^{\pi^2/2t}}{\sqrt{2\pi}^2} \Gamma \left( \frac{\gamma + 2}{2} \right) g(a) e^{-a} \gamma/2 e^{-\xi/2t} \left( \frac{u}{a} \right)^{(\gamma-1)/2} e^{-u} \sinh(\xi) \cosh(\xi) \sin(\pi \xi / t) \left( \frac{u}{a} + (\cosh(\xi))^2 \right)^{(\gamma+2)/2} \frac{d\xi}{a} \, du \, da
$$

In order to apply the dominated convergence theorem, we will show that the integrand on the right-hand side is dominated for $t \in [1, \infty)$ by an integrable function. As for $x \geq 0$, $|\sin(x)| \leq x$, $\sinh(x) \leq \cosh(x) \leq e^x$, $\cosh(x) \geq e^x/2$ and by assumption, $g(x) \leq cx^b$, there is a constant $d = d_4 > 0$ such that

$$
\int_0^\infty \int_0^\infty \int_0^\infty \sup_{n \geq 1} \frac{t e^{\pi^2/2t}}{\sqrt{2\pi}^2} \Gamma \left( \frac{\gamma + 2}{2} \right) g(a) e^{-a} \gamma/2 e^{-\xi/2t} \left( \frac{u}{a} \right)^{(\gamma-1)/2} e^{-u} \sinh(\xi) \cosh(\xi) \sin(\pi \xi / t) \left( \frac{u}{a} + (\cosh(\xi))^2 \right)^{(\gamma+2)/2} \frac{d\xi}{a} \, du \, da
$$

\[ \square \]
The first summand on the right-hand side is finite as $\frac{2}{1 + \varphi} > -1$ and $\gamma > 0$. Recall $b - \gamma/2 > 0$. Choose $0 < \varepsilon < \min(b - \gamma/2, \gamma/2)$. Using $a^x \leq a^y$ for every $a \in [0,1]$ and $0 \leq y \leq x$, and applying Fubini's theorem, we estimate the second summand on the right-hand side of (80) as follows:

$$\int_0^1 \int_0^\infty \int_0^\infty a^{b - \gamma/2} u^{(\gamma - 1)/2} e^{-u} \frac{e^{-\xi/\gamma}}{4ue^{-2\xi + a}((\gamma + 2)/2)} d\xi du da \\ \leq \int_0^1 \int_0^\infty \int_0^1 a^{\epsilon} u^{(\gamma - 1)/2} e^{-u} \frac{e^{-\xi/\gamma}}{4ue^{-2\xi + a}((\gamma + 2)/2)} d\xi du da \\ \leq \int_0^\infty \int_0^\infty \int_0^1 (4ue^{-2\xi + a} + a)^{\epsilon} u^{(\gamma - 1)/2} e^{-u} \frac{e^{-\xi/\gamma}}{(4ue^{-2\xi + a}((\gamma + 2)/2))} da d\xi du \\ = \int_0^\infty \int_0^\infty \int_0^1 \frac{1}{4ue^{-2\xi + a}((\gamma + 2)/2)^{\epsilon}} da d\xi du \\ \leq \frac{4\gamma - \epsilon}{4\gamma - \epsilon} \int_0^\infty \int_0^\infty u^{-1/2 + \varepsilon} e^{-u} e^{-2\xi/\gamma} du d\xi < \infty. \quad (81)$$

Thus, applying dominated convergence in (79) and using $t \sin(\pi\xi/t) \xrightarrow{t \to \infty} \pi\xi$ for every $\xi \in [0,\infty)$, we get that

$$\lim_{t \to \infty} t^{3/2} e^{2\gamma t/2} \mathbb{E} \left[ g \left( \frac{1}{2A_t} \right) \right] = \Gamma \left( \frac{\gamma + 2}{2} \right) \int_0^\infty \int_0^\infty \int_0^\infty \lim_{t \to \infty} t e^{\pi^2/2t} \frac{1}{\sqrt{2\pi}} g(a)e^{-a\gamma/2 e^{-\xi/\gamma} u^{(\gamma - 1)/2} e^{-u}} \sinh(\xi) \cosh(\xi) \sin(\pi\xi/t) \left( u + a(\cosh(\xi))^2 \right)^{(\gamma + 2)/2} d\xi du da \\ = \frac{1}{\sqrt{2\pi}} \Gamma \left( \frac{\gamma + 2}{2} \right) \int_0^\infty \int_0^\infty \int_0^\infty g(a)e^{-a\gamma/2 e^{-\xi/\gamma} u^{(\gamma - 1)/2} e^{-u}} \sinh(\xi) \cosh(\xi) \left( u + a(\cosh(\xi))^2 \right)^{(\gamma + 2)/2} d\xi du da \\ = \int_0^\infty g(a) \phi_1(\gamma) da, \quad (82)$$

which is the claim of the lemma.

Proof of Theorem 5 in the weakly subcritical case. Assume $-\sigma^2 > \alpha < 0$. Let $z \geq 0$ be fixed and $\beta = -2\alpha/\sigma^2$. By Corollary 4, we obtain that

$$\lim_{t \to \infty} t^{3/2} e^{2\gamma t/2} \cdot \mathbb{P}(Z_1 > 0) = \frac{8}{\sigma^2} \lim_{t \to \infty} \left( \frac{\sigma^2 t}{4} \right)^{3/2} e^{\left( \frac{z^2}{4} \right)^{1/2} \frac{z^2}{4} \left( 2A_{t/4}^{1/2} \right)^{1/4}} \mathbb{E} \left[ f \left( \frac{z}{2A_{t/4}^{1/2}} \right) \right]. \quad (83)$$

As for every $x \geq 0$, $f(zx) \leq \frac{5\sigma^2}{2\pi} x$ and $0 < \beta < 2$, the assumptions of Lemma 13 with $g = f$, $\gamma = \beta$ and $b = 1 > \frac{3}{2}$ are fulfilled. Applying Lemma 13 to (83) proves Theorem 5 in the weakly subcritical case.

The following lemma for the weakly subcritical case will be needed later. Recall $\beta = -2\alpha/\sigma^2$ and $\vartheta$ from (99).
Lemma 14. Assume $\sigma_n, \sigma_c \in (0, \infty)$ and $\alpha \in (-\sigma_c^2, 0)$. Then $\vartheta \in C^\infty ([0, \infty), \mathbb{R})$ and

$$\lim_{z \to \infty} \frac{\vartheta(z)}{z^{\frac{\sigma_c^2}{\sigma_c^2}} \log(z)} = \frac{2}{\beta} \frac{\sqrt{2\pi}}{\beta \sigma_c^2 \sin \left( \frac{\pi \sigma_c^2}{2\beta} \right)} \left( \frac{\sigma_c^2}{\sigma_c^2} \right)^{\frac{\sigma_c^2}{2\beta}}$$

(84)

$$\lim_{z \to \infty} \frac{z^{\vartheta'}(z)}{z^\frac{\sigma_c^2}{\sigma_c^2} \log(z)} = \frac{2\sqrt{2\pi}}{\sigma_c^2 \sin \left( \frac{\pi \sigma_c^2}{2\beta} \right)} \left( \frac{\sigma_c^2}{\sigma_c^2} \right)^{\frac{\sigma_c^2}{2\beta}}.$$

(85)

Proof. In order to prove $\vartheta \in C^\infty ([0, \infty), \mathbb{R})$, note that

$$\int_0^\infty \left| \frac{\partial^n f(za)}{\partial a^n} \right| \varphi_\beta(a) da = \int_0^\infty \frac{2^{\frac{2n}{\sigma_c^2}} a^n}{\sigma_c^2} \exp \left( -\frac{\sigma_c^2}{\sigma_c^2} za \right) \varphi_\beta(a) da \leq \frac{2^{\frac{2n}{\sigma_c^2}}}{\sigma_c^2} \int_0^\infty a^n \varphi_\beta(a) da$$

(86)

for all $z \in [0, \infty)$ and all $n \in \mathbb{N}$. The right-hand side of (86) is finite according to Lemma 13 (note $1 > \beta/2$). The dominated convergence theorem thus implies that $\vartheta \in C^\infty ([0, \infty), \mathbb{R})$. Moreover, we get that

$$\vartheta^{(n)}(z) = \frac{8}{\sigma_c^2} \int_0^\infty f^{(n)}(za)a^n \varphi_\beta(a) da$$

(87)

for all $z \in [0, \infty)$, $n \in \mathbb{N}_0$. For proving (84), we substitute $b := za$ and get that

$$\frac{\vartheta(z)}{z^{\frac{\sigma_c^2}{\sigma_c^2}} \log(z)} = \frac{1}{z^{\frac{\sigma_c^2}{\sigma_c^2}} \log(z)} \frac{8}{\sigma_c^2} \int_0^\infty f(za)\varphi_\beta(a) da = \frac{8}{\sigma_c^2} \int_0^\infty \frac{1}{z^{\frac{\sigma_c^2}{\sigma_c^2} + 1} \log(z)} \varphi_\beta \left( \frac{b}{z} \right) db$$

(88)

for all $z \in (0, \infty)$. Next we rewrite the definition (84) of $\varphi_\beta$ and see that

$$\frac{1}{\sqrt{2\pi}} \Gamma \left( \frac{\beta + 2}{2} \right) e^{-\frac{b}{2} - \beta/2} \int_0^\infty u^{(\beta - 1)/2} e^{-u} \int_0^\infty \frac{\sinh(\xi) \cosh(\xi) \xi}{\log(z) \left( u + \frac{b}{2}(\cosh(\xi))^2 \right)^{\frac{3+\beta}{2}}} d\xi du$$

(89)

for all $b \in (0, \infty)$ and all $z \in (0, \infty)$. Using the substitution $x := \cosh(\xi) \sqrt{b/(zu)}$ and noting that $dx/d\xi = \sinh(\xi) \sqrt{b/(zu)}$, we obtain that

$$\int_0^\infty \frac{\sinh(\xi) \cosh(\xi) \xi}{\log(z) \left( u + \frac{b}{2}(\cosh(\xi))^2 \right)^{\frac{3+\beta}{2}}} d\xi = \frac{u}{bu^{\frac{3+\beta}{2}}} \int_0^\infty \cosh(\xi) \sqrt{\frac{u}{2\pi}} \cdot \sinh(\xi) \sqrt{\frac{u}{2\pi}} \log(z) \left( 1 + \frac{u}{bu}(\cosh(\xi))^2 \right)^{\frac{3+\beta}{2}} d\xi$$

$$= \frac{1}{bu^{\frac{3+\beta}{2}}} \int_0^\infty x \arccosh \left( \frac{\sqrt{u}}{x} \right) \log(z) \left( 1 + x^2 \right)^{\frac{3+\beta}{2}} dx$$

(90)

for all $b \in (0, \infty)$, $u \in (0, \infty)$ and all $z \in (0, \infty)$. Note that $\arccosh(y) = \log(y + \sqrt{y^2 - 1})$ for all $y \in [1, \infty)$ and therefore $\arccosh(y) \leq \log(2y)$ for all $y \in [1, \infty)$. Consequently, we have that $\arccosh(x \sqrt{\frac{u}{b}})/\log(z) \leq (\log(x \sqrt{\frac{u}{b}}) \vee 1) + \log(2) + \frac{\log(u)}{\log(2)} \leq 3(\log(x \sqrt{\frac{u}{b}}) \vee 1)$ for all $x \in (\sqrt{b/(zu)}, \infty)$ and all $z \in [3, \infty)$. Let us prove that we may apply the dominated convergence theorem. From (SS), (SS) and (SS) we see that with $c := \frac{\sigma_c^2}{\sigma_c^2} \sqrt{2\pi}$, we get that

$$\frac{\vartheta(z)}{z^{\frac{\sigma_c^2}{\sigma_c^2}} \log(z)} = c \int_0^\infty f(b) e^{-\frac{b}{2} - \frac{\beta}{2} - 1} \int_0^\infty u^{\frac{1}{2}} e^{-u} \int_0^\infty \frac{x \arccosh \left( \frac{\sqrt{u}}{x} \right) \log(z) \left( 1 + x^2 \right)^{\frac{3+\beta}{2}}} d\xi du db$$

$$\leq c \int_0^\infty f(b) b^{-\frac{\beta}{2} - 1} \int_0^\infty u^{\frac{1}{2}} e^{-u} \int_0^\infty \frac{x \arccosh \left( \frac{\sqrt{u}}{x} \right) \log(z) \left( 1 + x^2 \right)^{\frac{3+\beta}{2}}}{(1 + x^2)^{\frac{3+\beta}{2}}} 3(\log(x \sqrt{\frac{u}{b}}) \vee 1) dx du db$$
for all $z \in [3, \infty)$. The right-hand side is finite as $\beta/2 \in (0, 1)$ and as $f(b) \leq (b \sigma_b^2 / \sigma_c^2) \wedge 1$ for $b \in [0, \infty)$. Thus we may apply the dominated convergence theorem and get that

$$
\lim_{z \to \infty} \frac{\vartheta(z)}{z^2 \log(z)} = e \int_0^\infty f(b) b^{-\beta/2-1} \int_0^\infty u^{-3/4-1} e^{-u} \int_0^\infty \frac{\arcsinh \left( \frac{x}{z} \sqrt{\frac{\beta}{\beta}} \right)}{\log(z)} \frac{x}{(1+x^2)^{\frac{\beta}{2}}} dx \, du \, db
$$

(91)

For the last step we used that $\arcsinh(z)/\log(z) \to 1$ as $z \to \infty$. We will simplify the integrals in (91). Using integration by parts, we obtain that

$$
\int_0^\infty f(b) b^{-\beta/2-1} db = \int_0^\infty b^{-\beta/2-1} \left( 1 - \exp \left( - \frac{\sigma_b^2}{\sigma_c^2} b \right) \right) db = \frac{2 \sigma_b^2}{\beta \sigma_c^2} \left( \frac{\sigma_b^2}{\sigma_c^2} \right)^{1-\frac{\beta}{2}} \Gamma \left( 1 - \frac{\beta}{2} \right). \tag{92}
$$

Inserting (92) into (91) and using $\Gamma \left( \frac{1}{2} \right) = \sqrt{\pi}$, $\Gamma (1+\frac{1}{2}) = \frac{4}{2} \Gamma \left( \frac{1}{2} \right)$ and Euler’s reflection principle $\Gamma (x) \Gamma (1-x) = \frac{\pi}{\sin(\pi x)}$ for $x \in (0, 1)$, we get that

$$
\lim_{z \to \infty} \frac{\vartheta(z)}{z^2 \log(z)} = \frac{8}{\sigma_c^2} \frac{1}{\sqrt{2\pi}} \beta \left( \frac{\beta}{2} \right) \frac{2 \sigma_b^2}{\beta \sigma_c^2} \left( \frac{\sigma_b^2}{\sigma_c^2} \right)^{1-\frac{\beta}{2}} \Gamma \left( 1 - \frac{\beta}{2} \right) \cdot \Gamma \left( \frac{1}{2} \right) \cdot \frac{1}{2 \beta} = \frac{2 \sqrt{2\pi}}{\sigma_c^2 \beta \sin \left( \frac{\pi \beta}{2} \right)} \left( \frac{\sigma_b^2}{\sigma_c^2} \right)^{-\frac{\beta}{2}}
$$

This proves (S3). Paralleling the above arguments yields (S5).

The following lemma implies that the additional drift term in (10) is strictly decreasing in the weakly subcritical regime.

**Lemma 15.** Assume $\sigma_b, \sigma_c \in (0, \infty)$ and $\alpha \in (-\sigma_c^2, \infty)$. Then the function $(0, \infty) \ni z \mapsto \sigma_c^2 z \vartheta’(z) / \vartheta(z) \in \mathbb{R}$ is strictly monotonic decreasing and satisfies $\lim_{z \to 0} \sigma_c^2 z \vartheta’(z) / \vartheta(z) = \sigma_c^2$ and $\lim_{z \to \infty} \sigma_c^2 z \vartheta’(z) / \vartheta(z) = \max (-\alpha, 0)$.

**Proof.** In all regimes $\alpha \in \mathbb{R}$, we have that $\vartheta’(0) \in (0, \infty)$ and $\vartheta(0) = 0$ and, therefore,

$$
\lim_{z \to 0} \sigma_c^2 z \vartheta’(z) / \vartheta(z) = \sigma_c^2 \frac{\lim_{z \to 0} \vartheta’(z) / \vartheta(z)}{\lim_{z \to 0} \vartheta’(z) / \vartheta(z)} = \sigma_c^2 \frac{\vartheta’(0)}{\vartheta(0)} = \sigma_c^2. \tag{93}
$$

In the supercritical regime $\alpha > 0$, we see that

$$
\frac{z \vartheta’(z)}{\vartheta(z)} = \frac{z^2 \vartheta’(z)}{\vartheta(z)} \frac{1}{1 - \left( 1 + \frac{\sigma_b^2}{\sigma_c^2} \cdot z \right) \frac{1}{\alpha \sigma_c^2 z}} = \frac{2 \alpha z}{\sigma_b^2 + \sigma_c^2 z} \left( 1 + \frac{\sigma_b^2}{\sigma_c^2} \cdot z \right) \frac{1}{\alpha \sigma_c^2 z} - 1
$$
for all \( z \in (0, \infty) \). The derivative hereof is strictly negative

\[
\frac{d}{dz} \frac{z \vartheta'(z)}{\vartheta(z)} = \frac{2\alpha \sigma_y^2 \left( \left( 1 + \frac{\sigma_x^2}{\sigma_y^2} \right) \frac{2\vartheta'}{\vartheta} - 1 \right) - 4\alpha^2 z \left( 1 + \frac{\sigma_x^2}{\sigma_y^2} \right) \frac{2\vartheta'}{\vartheta}}{(\sigma_x^2 + \sigma_y^2)^2 \left( \left( 1 + \frac{\sigma_x^2}{\sigma_y^2} \right) \frac{2\vartheta'}{\vartheta} - 1 \right)^2}
\]

\[
= \int_0^z 4\alpha^2 \left( 1 + \frac{\sigma_x^2}{\sigma_y^2} \cdot y \right) \frac{2\vartheta'}{\vartheta} - 1 dy - 4\alpha^2 \int_0^z \left( 1 + \frac{\sigma_x^2}{\sigma_y^2} \cdot y \right) \frac{2\vartheta'}{\vartheta} + y \frac{2\alpha^2 \left( 1 + \frac{\sigma_x^2}{\sigma_y^2} \right)}{(\sigma_x^2 + \sigma_y^2)^2} \left( \left( 1 + \frac{\sigma_x^2}{\sigma_y^2} \right) \frac{2\vartheta'}{\vartheta} - 1 \right)^2 dy
\]

\[
\leq -4\alpha^2 \int_0^z y \frac{2\alpha^2 \left( 1 + \frac{\sigma_x^2}{\sigma_y^2} \cdot y \right)}{(\sigma_x^2 + \sigma_y^2)^2} \left( \left( 1 + \frac{\sigma_x^2}{\sigma_y^2} \right) \frac{2\vartheta'}{\vartheta} - 1 \right)^2 < 0
\]

for all \( z \in (0, \infty) \). Moreover, it is clear that \( \lim_{z \to \infty} z \vartheta'(z)/\vartheta(z) = 0 \).

In the critical regime \( \alpha = 0 \), we get that

\[
\frac{z \vartheta'(z)}{\vartheta(z)} = \frac{\sigma_y^2}{\sigma_x^2 + \sigma_y^2 \log \left( 1 + \frac{\sigma_x^2}{\sigma_y^2} \right)}
\]

for all \( z \in (0, \infty) \). The derivative hereof is strictly negative

\[
\frac{d}{dz} \frac{z \vartheta'(z)}{\vartheta(z)} = \frac{\sigma_y^2 \sigma_x^2 \log \left( 1 + \frac{\sigma_y^2}{\sigma_x^2} \right) - \sigma_x^2 \sigma_y^2}{(\sigma_x^2 + \sigma_y^2)^2 \left( \log \left( 1 + \frac{\sigma_x^2}{\sigma_y^2} \right) \right)^2} < 0
\]

for all \( z \in (0, \infty) \) where we used the inequality \( \log(1 + x) < x \) for all \( x \in (0, \infty) \). Moreover, it is clear that \( \lim_{z \to \infty} z \vartheta'(z)/\vartheta(z) = 0 \).

For the rest of the proof, we assume that \( \alpha \in (-\sigma_x^2, 0) \). Recall \( \beta = -2\alpha/\sigma_y^2 \). Lemma 14 implies that

\[
\lim_{z \to \infty} \frac{z \vartheta'(z)}{\vartheta(z)} = \sigma_y^2 \left[ \lim_{z \to \infty} \frac{z \vartheta'(z)}{z \vartheta(z)} \right] = \sigma_y^2 \left[ \lim_{z \to \infty} \frac{\sqrt{\pi} \vartheta (z)}{\vartheta(z)} \right] = \sigma_y^2 \frac{\sqrt{\pi}}{\sqrt{\pi}} = \sigma_y^2 = \alpha.
\]

It remains to prove monotonicity of \( (0, \infty) \ni z \mapsto z \vartheta'(z)/\vartheta(z) \). The first derivative hereof is

\[
\frac{d}{dz} \frac{z \vartheta'(z)}{\vartheta(z)} = \frac{\left( \frac{\beta}{2} z \vartheta(z) - z \vartheta'(z) \right) \vartheta'(z) + \vartheta(z) \left( \left( 1 - \frac{z}{2} \right) \vartheta'(z) + z \vartheta''(z) \right)}{(\vartheta(z))^2}
\]

for all \( z \in (0, \infty) \). We will show that the right-hand side is negative for all \( z \in (0, \infty) \). Define a function \( \tilde{\phi}_\beta : (0, \infty) \to [0, \infty) \) through \( \tilde{\phi}_\beta(a) := \frac{\beta}{2} \psi(\vartheta(a)) a^{\frac{\beta}{2} + 1} \) for all \( a \in (0, \infty) \). Using the substitution \( \eta = \sqrt{\alpha} \cosh(\xi) \),
we rewrite this function as
\[
\tilde{\phi}_\beta(a) = \frac{8}{\sigma_c^2} \int_0^\infty \int_0^\infty \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{\beta + 2}{2}\right) e^{-u} u^{(\beta-1)/2} e^{-u} \frac{\sinh(\xi) \cosh(\xi)}{(u + a(\cosh(\xi))^2)^{\frac{\beta}{2}}} d\xi du \cdot a
\]
\[
= \frac{8}{\sigma_c^2} \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{\beta + 2}{2}\right) e^{-a} \int_0^\infty u^{(\beta-1)/2} e^{-u} \int_0^\infty \frac{\sinh(\xi) \cdot \text{arccosh}\left(\frac{u}{\sqrt{a}}\right)}{(u + \eta^2)^{\frac{\beta}{2}}} \frac{d\eta}{\sqrt{a} \sinh(\xi)} du \cdot a
\]
\[
= \frac{8}{\sigma_c^2} \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{\beta + 2}{2}\right) e^{-a} \int_0^\infty u^{(\beta-1)/2} e^{-u} \int_0^\infty \mathbb{1}_{\eta > \sqrt{a}} \text{arccosh}\left(\frac{u}{\sqrt{a}}\right) - \frac{\eta}{(u + \eta^2)^{\frac{\beta}{2}}} d\eta du
\]
for all \(a \in (0, \infty)\). As \((0, \infty) \ni a \mapsto e^{-a} \text{arccosh}\left(\frac{u}{\sqrt{a}}\right) \mathbb{1}_{\eta > \sqrt{a}}\) is strictly monotonic decreasing for every \(\eta \in (0, \infty)\), we conclude that \(\tilde{\phi}_\beta\) is strictly monotonic decreasing and that \(\tilde{\phi}'_\beta(a) < 0\) for all \(a \in (0, \infty)\). Moreover \(\tilde{\phi}_\beta(a)\) decays at least exponentially fast as \(a \to \infty\) so that \(f_a^\infty b^\beta \tilde{\phi}_\beta(b) \, db < \infty\) for all \(a, p \in (0, \infty)\). Applying (A7) and integration by parts twice, it follows that
\[
\tilde{\phi}^{(n)}(z) = \frac{8}{\sigma_c^2} \int_0^\infty f^{(n)}(za)a^n \tilde{\phi}_\beta(a) \, da = \int_0^\infty f^{(n)}(za)a^{n-\frac{\beta}{2}-1} \tilde{\phi}'_\beta(a) \, da
\]
\[
= -\int_0^\infty f^{(n)}(za) \int_0^\infty b^{n-\frac{\beta}{2}-1} \tilde{\phi}'_\beta(b) \, db \cdot \int_0^\infty f^{(n+1)}(za)z \frac{b^{n-\frac{\beta}{2}}}{n-\frac{\beta}{2}} \tilde{\phi}'_\beta(b) \, db \, da
\]
\[
= -\int_0^\infty f^{(n+1)}(za)z a^{n-\frac{\beta}{2}} \tilde{\phi}_\beta(a) \, da - z \int_0^\infty f^{(n+1)}(za) \int_0^\infty \frac{b^{n-\frac{\beta}{2}}}{n-\frac{\beta}{2}} \tilde{\phi}'_\beta(b) \, db \, da
\]
for all \(z \in (0, \infty)\) and all \(n \in \mathbb{N}_0\). Recall that \(\tilde{\phi}'_\beta(b) < 0\) for all \(b \in (0, \infty)\) and note that \(f_0^{(1)}(za) a^{1-\frac{\beta}{2}} \tilde{\phi}'_\beta(a) > 0\) for all \(a, b, z \in (0, \infty)\). Thus (96) with \(n = 0\) implies that
\[
\left(1 - \frac{\beta}{2}\right) \tilde{\phi}(z) < -\frac{\beta}{2} \frac{1}{2-\frac{\beta}{2}} \int_0^\infty f^{(1)}(za) z a^{1-\frac{\beta}{2}} \tilde{\phi}_\beta(a) \, da = -\frac{8}{\sigma_c^2} \int_0^\infty f^{(1)}(za) a^{1-\frac{\beta}{2}} \tilde{\phi}_\beta(a) \, da = z \tilde{\phi}'(z)
\]
for all \(z \in (0, \infty)\). Moreover \(f^{(2)}(za) a^{1-\frac{\beta}{2}} \tilde{\phi}'_\beta(b) > 0\) for all \(a, b, z \in (0, \infty)\). So (96) with \(n = 1\) yields that
\[
\left(1 - \frac{\beta}{2}\right) \tilde{\phi}'(z) < -\frac{\beta}{2} \frac{1}{2-\frac{\beta}{2}} \int_0^\infty f^{(2)}(za) z a^{1-\frac{\beta}{2}} \tilde{\phi}_\beta(a) \, da = -\frac{8}{\sigma_c^2} \int_0^\infty f^{(2)}(za) a^{1-\frac{\beta}{2}} \tilde{\phi}_\beta(a) \, da = -z \tilde{\phi}''(z)
\]
for all \(z \in (0, \infty)\). Applying the inequalities (A7) and (A8) to the right-hand side of (99) and using \(\tilde{\phi}'(z), \tilde{\phi}(z) > 0\) for all \(z \in (0, \infty)\), we conclude that \(\frac{1}{1-\frac{\beta}{2}} (\tilde{\phi}'(z)/\tilde{\phi}(z)) < 0\) for all \(z \in (0, \infty)\) which implies that \((0, \infty) \ni z \mapsto z \tilde{\phi}''(z)/\tilde{\phi}(z)\) is strictly monotonic decreasing.

4.5 The intermediately subcritical regime

In this subsection, we will prove (11).

Proof of Theorem 2 in the intermediately subcritical case. Let \(z \geq 0\) be fixed for the moment. Lemma 11 asserts that \(\mathbb{E}\left[\frac{1}{\sigma_2^a}\right] = e^{-2t}\mathbb{E}\left[\frac{1}{\sigma_2^a}\right] \leq e^{-2t}\mathbb{E}\left[\frac{1}{\sigma_2^a}\right]^2\) for \(t \in (0, \infty)\). According to Lemma 12 we have that \(\lim_{t \to \infty} \sqrt{t} \mathbb{E}\left[\frac{1}{\sigma_2^a}\right] = f_0^{\infty} \frac{a}{\sqrt{2\pi}} e^{-a^2/2} \, da < \infty\) and, therefore, \((\mathbb{E}\left[\frac{1}{\sigma_2^a}\right])^2\) decays like \(t^{-1}\) as \(t \to \infty\). Thus the
Lemma 10, we obtain for the first and second moment of $\alpha$ which proves Theorem 5 in the case $t$ for every $X$ Markov property of $(\beta$ Note that (2 $\alpha = 1/2A_t^{(2)}$, $t \geq 1$. Applying Corollary 4 and Lemma 9, we get that

$$\lim_{t \to \infty} \sqrt{t} e^{2t} \cdot \mathbb{E}[Z_t > 0] = \sqrt{\frac{4}{\sigma^2}} \lim_{t \to \infty} \sqrt{\frac{\sigma^2 t}{4}} e^{-z^2/4} \mathbb{E}[f\left(\frac{z}{2A_t^{(2)}}\right)] = \frac{2}{\sigma e} - \frac{z^2}{\sigma^2} \int_0^\infty \frac{1}{\sqrt{2\pi} a} da = \frac{\sqrt{2\sigma^2}}{\sqrt{\pi \sigma^2}}$$

which proves Theorem 5 in the case $\alpha = -\sigma^2.

4.6 The strongly subcritical regime

Finally, we will prove (12).

Proof of Theorem 5 in the strongly subcritical case. Let $\beta = -2\alpha/\sigma^2$ and assume $\alpha < -\sigma^2$. Let $t > 0$ and $z \in [0, \infty)$ be fixed. The main tool for the proof is Lemma 9. Let us check the conditions of that lemma. Using Lemma 10 we obtain for the first and second moment of $1/(2A_t^{(2)})$ that

$$\mathbb{E}\left[\frac{1}{2A_t^{(2)}}\right] = e^{-(2\beta-2)t} \mathbb{E}\left[\frac{1}{2A_t^{(2)-(\beta-2)}}\right]$$

$$\mathbb{E}\left[\frac{1}{(2A_t^{(2)})^2}\right] = e^{-(2\beta-2)t} \mathbb{E}\left[\frac{1}{2A_t^{(2)-(\beta-2)}}\right] \cdot \mathbb{E}\left[\frac{1}{2A_t^{(2)-(\beta-2)}}\right]$$

for every $t \in (0, \infty)$. As $\beta > 2$, the monotone convergence theorem and Lemma 11 yield that

$$\lim_{t \to \infty} \mathbb{E}\left[\frac{1}{2A_t^{(2)-(\beta-2)}}\right] = \mathbb{E}\left[\frac{1}{2A_t^{(2)-(\beta-2)}}\right] = \mathbb{E}[G_{\beta-2}] = \beta - 2 < \infty. \quad (101)$$

Using relation (1.1) from [34] and monotone convergence, we get that

$$\lim_{t \to \infty} \mathbb{E}\left[\frac{1}{2A_t^{(2)-(\beta-2)}}\right] = \lim_{t \to \infty} \mathbb{E}\left[\frac{1}{2A_t^{(2)-(\beta-2)}}\right] - \mathbb{E}[G_{\beta-2}] = 0. \quad (102)$$

Thus, by (99), (101) and (102), the assumptions of Lemma 9 are met with $c_t := e^{(2\beta-2)t}$ and $Y_t = 1/(2A_t^{(2)})$, $t \geq 1$. By Corollary 4, Lemma 9 and 11, we obtain that

$$\lim_{t \to \infty} e^{(2\beta-2)\sigma^2 t/4} \cdot \mathbb{P}[Z_t > 0] = \lim_{t \to \infty} e^{(2\beta-2)\sigma^2 t/4} \cdot \mathbb{E}\left[f\left(\frac{z}{2A_t^{(2)}}\right)\right] = \frac{z\sigma^2}{\sigma^2} \lim_{t \to \infty} \mathbb{E}\left[\frac{1}{2A_t^{(2)-(\beta-2)}}\right] = \frac{z\sigma^2}{\sigma^2} (\beta - 2) = \frac{z\sigma^2}{\sigma^2} (\beta - 2)$$

Note that $(2\beta-2)\sigma^2/4 = -\alpha + \frac{\sigma^2}{2}$. Thus, (103) is the claim of Theorem 5 for the strongly subcritical case.

5 Proof of Lemma 6

Proof of Lemma 6. Fix $t \in [0, \infty)$ and $x \in I$ for the moment. As $\eta(x) > 0$, there exists a $T_0 \in [0, \infty)$ such that $\mathbb{P}^x(X_T \notin A) > 0$ for all $T \in [T_0, \infty)$. Let $\phi: C([0, t], I) \to \mathbb{R}$ be a bounded and Borel measurable function. The Markov property of $(X_s)_{s \geq 0}$ implies that

$$\mathbb{E}\phi((X_s)_{s \leq t}) \mathbb{I}_{X_T \notin A} = \mathbb{E}\phi((X_s)_{s \leq t}) \mathbb{E}\mathbb{I}_{X_T \notin A} \mathbb{E}(X_s)_{s \leq t}$$

$$= \mathbb{E}\phi((X_s)_{s \leq t}) \mathbb{E}X_t (X_T \notin A) \quad (104)$$
for all $T \in [0, \infty)$. Consequently, we get for the conditional expectation that

$$\mathbb{E}^x \left[ \phi \left( (X_s)_{s \leq t} \right) | X_{T+t} \notin A \right] = \frac{\mathbb{E}^x \left[ \phi \left( (X_s)_{s \leq t} \right) \mathbb{P}^{X_t} (X_T \notin A) \right]}{\mathbb{P}^x (X_{T+t} \notin A)} = \frac{\mathbb{E}^x \left[ \phi \left( (X_s)_{s \leq t} \right) q(T)e^{\lambda(T+t)} \mathbb{P}^{X_t} (X_T \notin A) \right]}{q(T) e^{\lambda(T+t)}} q(T) e^{\lambda(T+t)}$$

for all $T \in [T_0, \infty)$. Due to $\sup_{T \in [1, \infty)} q(T) e^{\lambda(T+t)} (X_T \notin A) \leq c (1 + \|X_t\|^p)$ for some constant $c \in [0, \infty)$ and due to $\mathbb{E}[\|X_t\|^p] < \infty$, we may apply the dominated convergence theorem and obtain that

$$\lim_{T \to \infty} \mathbb{E}^x \left[ \phi \left( (X_s)_{s \leq t} \right) | X_{T+t} \notin A \right] = \frac{\mathbb{E}^x \left[ \phi \left( (X_s)_{s \leq t} \right) \lim_{T \to \infty} q(T)e^{\lambda(T+t)} \mathbb{P}^{X_t} (X_T \notin A) \right]}{\lim_{T \to \infty} q(T) e^{\lambda(T+t)}} q(T) e^{\lambda(T+t)}$$

$$= \frac{\mathbb{E}^x \left[ \phi \left( (X_s)_{s \leq t} \right) \eta(X_t) \right]}{e^{-\lambda \eta(x)}}$$

If $\phi \equiv 1$, then the left-hand side is equal to 1 and, therefore, the right-hand side is equal to 1. Consequently the right-hand side defines a probability distribution on $C([0, t], \mathbb{R}^d)$ for every $t \in [0, \infty)$. These probability distributions are consistent for $t \in [0, \infty)$. So Kolmogorov’s extension theorem (e.g. [29]) implies existence of a stochastic process $(X_t)_{t \geq 0}$ having continuous sample paths and being uniquely determined by

$$\lim_{T \to \infty} \mathbb{E}^x \left[ \phi \left( (X_s)_{s \leq t} \right) | X_{T+t} \notin A \right] = \mathbb{E}^x \left[ \phi \left( (X_s)_{s \leq t} \right) \right] = \mathbb{E}^x \left[ \phi \left( (X_s)_{s \leq t} \right) \eta(X_t) \right] e^{-\lambda \eta(x)}$$

for all $t \in [0, \infty)$. This proves ([29]). Moreover if $t \in [0, \infty)$ and if $g: I \to [0, \infty)$ is a Borel measurable function, then (105) follows from (106) with $\phi ((X_s)_{s \leq t}) := \min (g(X_t), n)$, $n \in \mathbb{N}$, and from the monotone convergence theorem as $n \to \infty$.

Next we identify the linear operator of the martingale problem solved by $(X_t)_{t \geq 0}$. Similar arguments as above imply that

$$\eta(x) = 0 = \lim_{T \to \infty} q(T + t) e^{\lambda(T + t)} \mathbb{P}^x (X_{T+t} \notin A)$$

$$= \lim_{T \to \infty} q(T + t) e^{\lambda T} \mathbb{E}^x \left[ \lim_{T \to \infty} q(T)e^{\lambda(T+t)} \mathbb{P}^{X_t} (X_T \notin A) \right] = e^{\lambda T} \mathbb{E}^x [\eta(X_t)]$$

for all $x \in I \setminus \overline{I}$ and for all $t \in [0, \infty)$. Moreover, the Markov property of $(X_t)_{t \geq 0}$, the relation

$$\mathbb{E}^x [\eta(X_t)] - \eta(x) = \eta(x) (e^{-\lambda t} - 1) = - \int_0^t \eta(x) \lambda e^{-\lambda s} ds = \int_0^t \mathbb{E}^x [-\lambda \eta(X_s)] ds$$

for $x \in I$ and $t \in [0, \infty)$ and Proposition 4.1.7 of [17] imply that

$$d \eta(X_t) = -\lambda \eta(X_t) dt + dM_t$$

for $t \in [0, \infty)$ where $(M_t)_{t \geq 0}$ is a suitable martingale. Let $g: I \to \mathbb{R}$ be bounded and twice continuously differentiable. Itō’s lemma shows that

$$dg(X_t) = (\mathcal{G}g)(X_t) dt + (\nabla g \sigma)(X_t) dW_t$$

for all $t \in [0, \infty)$ where $\mathcal{G}g := \nabla g \mu + \frac{1}{2} \text{tr}(\sigma^T (\nabla^T \nabla g) \sigma)$. Thus Itō’s lemma and symmetry of $\sigma \sigma^T$ result in

$$d e^{\lambda t} \eta(X_t) g(X_t)$$

$$= \lambda e^{\lambda t} \eta(X_t) g(X_t) dt + e^{\lambda t} d\eta(X_t) g(X_t)$$

$$= \lambda e^{\lambda t} \eta(X_t) g(X_t) dt + e^{\lambda t} \eta(X_t) g(X_t) dt + e^{\lambda t} g(X_t) dt + e^{\lambda t} \left( \nabla \eta \sigma \sigma^T g \right)(X_t) dt$$

$$= e^{\lambda t} \eta(X_t) (\mathcal{G}g)(X_t) dt + e^{\lambda t} (\eta \nabla g \sigma)(X_t) dW_t + e^{\lambda t} g(X_t) dM_t + e^{\lambda t} \left( \nabla \eta \sigma \sigma^T g \right)(X_t) dt$$

22
for all \( t \in [0, \infty) \). Taking expectations, we infer that
\[
\mathbb{E}^x \left[ e^{\lambda t} \eta(X_t) g(X_t) \right] - \eta(x) g(x) = \int_0^t \mathbb{E}^x \left[ e^{\lambda u} \eta(X_u) (\mathcal{G} g) (X_u) \right] du
\] (111)
for all \( t \in [0, \infty) \) and all \( x \in I \) where \( \mathcal{G} g := g + \frac{1}{2} \nabla \eta \sigma^2 \nabla^2 g \). This implies for the Markov process \((\bar{X}_t)_{t \geq 0}\) that
\[
\mathbb{E}^x \left[ g(\bar{X}_t) \right] - g(x) = \int_0^t \mathbb{E}^x \left[ (\mathcal{G} g) (\bar{X}_u) \right] du
\] (112)
for all \( t \in [0, \infty) \) and all \( x \in \tilde{I} \). Now Proposition 4.1.7 in [17] implies that \((\bar{X}_t)_{t \geq 0}\) is a solution of the martingale problem for \( \mathcal{G} \). Finally, Theorem V.20.1 of [37] shows that \((\bar{X}_t)_{t \geq 0}\) is a weak solution of the SDE [37]. This completes the proof. \( \square \)

### 6 The BDRE conditioned to never go extinct

**Proof of Theorem 7**. Fix \( \alpha \in \mathbb{R} \) and \( \sigma_0, \sigma_e \in (0, \infty) \). We will prove Theorem 7 by applying Lemma 6 to the process \((X_t)_{t \geq 0} = (Z_t, S_t)_{t \geq 0}\) which has state space \( I := [0, \infty) \times \mathbb{R} \). Define \( \mu : I \to \mathbb{R}^2 \) and \( \sigma : I \to \mathbb{R}^{2 \times 2} \) by
\[
\mu(z, s) = \left( \frac{\alpha + \frac{1}{2} \sigma^2 z}{\alpha} \right) \quad \text{and} \quad \sigma(z, s) = \begin{pmatrix} \sqrt{\sigma_0^2} & \sigma e^z \\ 0 & \sigma_e \end{pmatrix}
\] (113)
for all \((z, s) \in [0, \infty) \times \mathbb{R}\). We set \( A = \{0\} \times \mathbb{R} \). Note that \( \{(Z_t, S_t) \notin A\} = \{Z_t > 0\} \) for all \( t \in [0, \infty) \). Moreover, define \( \eta : I \to [0, \infty) \) through \( \eta(z, s) := \vartheta(z) \) for all \((z, s) \in [0, \infty) \times \mathbb{R}\). We will check the assumptions of Lemma 6 for the different regimes separately. In all cases we have that
\[
\mathbb{E}^x[|Z_t|^2] + \mathbb{E}^x[|S_t|^2] \leq z^2 \mathbb{E}^0[e^{2S_t}] + 2s^2 + 2\mathbb{E}^0[S_t^2] = z^2 e^{2\alpha t + \sigma_0^2 t} + 2s^2 + 2\sigma_0^2 t + 2\sigma_e^2 t < \infty
\] (114)
for all \((z, s) \in [0, \infty) \times \mathbb{R}\) and all \( t \in [0, \infty) \).

**The supercritical regime** Let \( q \equiv 1 \in \mathcal{Q} \), \( \lambda = 0 \) and \( p = 0 \). Theorem 6 implies that
\[
\lim_{t \to \infty} q(t) e^{\lambda t} \mathbb{P}^{(z,s)}(Z_t > 0) = \eta(z, s) = \vartheta(z) = 1 - \left( 1 + \frac{\sigma_0^2}{\sigma_e} \right)^{-\frac{2z}{\sigma_e}}
\] (115)
for all \((z, s) \in [0, \infty) \times \mathbb{R}\). The function \( \eta \) is twice continuously differentiable and satisfies \( \eta(z, s) > 0 \) if and only if \((z, s) \in (0, \infty) \times \mathbb{R}\). Moreover, it is clear that \( q(t) e^{\lambda t} \mathbb{P}^{(z,s)}(Z_t > 0) \leq 1 \) for all \((z, s) \in I\) and for all \( t \in [0, \infty) \).

**The critical regime** Define \( q \in \mathcal{Q} \) through \( q(t) = \sqrt{t} \) for \( t \in [0, \infty) \) and let \( \lambda = 0 \) and \( p = 1 \). Theorem 5 implies that
\[
\lim_{t \to \infty} q(t) e^{\lambda t} \mathbb{P}^{(z,s)}(Z_t > 0) = \eta(z, s) = \vartheta(z) = \frac{\sqrt{2}}{\sqrt{\pi \sigma_e}} \log \left( 1 + \frac{\sigma_0^2}{\sigma_e} \cdot z \right)
\] (116)
for all \((z, s) \in [0, \infty) \times \mathbb{R}\) and thus \( \eta \) is twice continuously differentiable. Note that \( \eta(z, s) > 0 \) if and only if \((z, s) \in (0, \infty) \times \mathbb{R}\). Moreover, Corollary 6 implies that
\[
\frac{1}{1 + \|(z, s)\|} \sup_{t \in [1, \infty)} q(t) e^{\lambda t} \mathbb{P}^{(z,s)}(Z_t > 0) = \frac{1}{1 + \|(z, s)\|} \sup_{t \in [1, \infty)} \sqrt{t} \mathbb{E} \left[ f \left( \frac{z}{2A \sigma_e^2/4} \right) \right] \leq \frac{z}{1 + z} \sup_{t \in [1, \infty)} \sqrt{t} \frac{\sigma_0^2}{\sigma_e^2} \mathbb{E} \left[ \frac{1}{2A \sigma_e^2/4} \right]
\] (117)
for all \((z, s) \in [0, \infty) \times \mathbb{R}\). The right-hand side is finite according to Lemma 12 and is uniformly bounded in \((z, s) \in [0, \infty) \times \mathbb{R}\).
The weakly subcritical regime Define \( q \in Q \) through \( q(t) = \sqrt{t} \) for \( t \in [0, \infty) \) and let \( \lambda = \frac{\sigma^2}{2\pi^2} \) and \( p = 1 \). Theorem 3 implies that

\[
\lim_{t \to \infty} \frac{1}{1 + \|(z, s)\|} \sup_{t \in [1, \infty)} q(t) e^{\lambda t \mathbb{P}^{(z, s)}(Z_t > 0)} = \frac{1}{1 + \|(z, s)\|} \sup_{t \in [1, \infty)} \sqrt{t} e^{\frac{\sigma^2}{2\pi^2} t^2} E \left[ f \left( \frac{z}{2A^{(2)}_{t\sigma^2/4}} \right) \right]
\]

for all \((z, s) \in [0, \infty) \times \mathbb{R}\). The function \( \eta \) is twice continuously differentiable according to Lemma 13. Note that \( \eta(z, s) > 0 \) if and only if \((z, s) \in (0, \infty) \times \mathbb{R}\). Moreover, Corollary 4 implies that

\[
\leq \frac{z}{1 + z \sigma^2} \left( \frac{4}{\sigma^2} \right) \sup_{t \in [1, \infty)} \sqrt{t} e^{\frac{\sigma^2}{2\pi^2} t^2} E \left[ \frac{1}{2A^{(2)}_{t\sigma^2/4}} \right]
\]

for all \((z, s) \in [0, \infty) \times \mathbb{R}\). The right-hand side is finite according to Lemma 13 and is uniformly bounded in \((z, s) \in [0, \infty) \times \mathbb{R}\).

The intermediately subcritical regime Define \( q \in Q \) through \( q(t) = \sqrt{t} \) for \( t \in [0, \infty) \) and let \( \lambda = \frac{\sigma^2}{2\pi^2} \) and \( p = 1 \). Theorem 3 implies that

\[
\lim_{t \to \infty} \frac{1}{1 + \|(z, s)\|} \sup_{t \in [1, \infty)} q(t) e^{\lambda t \mathbb{P}^{(z, s)}(Z_t > 0)} = \frac{1}{1 + \|(z, s)\|} \sup_{t \in [1, \infty)} \sqrt{t} e^{\frac{\sigma^2}{2\pi^2} t^2} E \left[ f \left( \frac{z}{2A^{(2)}_{t\sigma^2/4}} \right) \right]
\]

for all \((z, s) \in [0, \infty) \times \mathbb{R}\). The function \( \eta \) is twice continuously differentiable and satisfies \( \eta(z, s) > 0 \) if and only if \((z, s) \in (0, \infty) \times \mathbb{R}\). Corollary 4 implies that

\[
\leq \frac{z}{1 + z \sigma^2} \left( \frac{4}{\sigma^2} \right) \sup_{t \in [1, \infty)} \sqrt{t} e^{\frac{\sigma^2}{2\pi^2} t^2} E \left[ \frac{1}{2A^{(2)}_{t\sigma^2/4}} \right]
\]

for all \((z, s) \in [0, \infty) \times \mathbb{R}\). The last step follows from Lemma 10. The right-hand side of (121) is finite due to \(-(\beta - 2) < 0\) and due to Lemma 11 and is uniformly bounded in \((z, s) \in [0, \infty) \times \mathbb{R}\).
Application of Lemma \[6\] After having checked all assumptions, we apply Lemma \[6\]. The additional drift term is

\[
\frac{1}{\eta(z,s)} (\sigma^2 \nabla^T \eta)(z,s) = \frac{1}{\eta(z)} \left( \begin{array}{cc} \sqrt{\sigma_e^2} z & \sigma_e z \\ \sigma_e z & \sigma_e \end{array} \right) \left( \begin{array}{c} \dot{\varphi}(z) \\ 0 \end{array} \right) \\
= \frac{1}{\eta(z)} \left( \sigma_e^2 z + \sigma_e^2 \dot{z}^2 \right) \dot{\varphi}(z)
\]

(122)

for \((z,s) \in \tilde{I} = (0,\infty) \times \mathbb{R}\). Inserting this into (37), we get for \((\hat{Z}_t, \hat{S}_t)_{t \geq 0}\) that

\[
d\hat{Z}_t = \frac{\dot{\varphi}(\hat{Z}_t)}{\partial (\hat{Z}_t)} \left( \sigma_e^2 \hat{Z}_t + \sigma_e^2 \hat{S}_t^2 \right) dt + \left( \frac{1}{2} \sigma_e^2 \hat{Z}_t + \alpha \hat{Z}_t \right) dt + \int (\sigma_e^2 \hat{Z}_t) dW_t^{(b)} + \int \sigma_e dW_t^{(c)}
\]

\[
d\hat{S}_t = \frac{\dot{\varphi}(\hat{Z}_t)}{\partial (\hat{Z}_t)} \sigma_e^2 \hat{Z}_t dt + \alpha dt + \sigma_e dW_t^{(c)}.
\]

Therefore \((\hat{Z}_t, \hat{S}_t)_{t \geq 0}\) solves the SDEs (10). Moreover, Lemma \[6\] implies that the conditioned process satisfies (11). In addition Lemma \[14\] establishes the properties of the function \((0,\infty) \ni z \mapsto \sigma_e^2 \dot{z}(z)/\varphi(z)\).

It remains to establish the limit of \(\hat{Z}_t\) as \(t \to \infty\). Note that \((\hat{Z}_t)_{t \geq 0}\) is a one-dimensional diffusion with drift term \(\mu(z) := \frac{\dot{\varphi}(z)}{\varphi(z)} (\sigma_e^2 z + \sigma_e^2 \hat{x}^2) + \frac{1}{2} \sigma_e^2 z + \alpha z\), \(z \in (0,\infty)\) and diffusion term \(\sigma^2(z) := \sigma_e^2 z + \sigma_e^2 z^2\), \(z \in (0,\infty)\). Define a scale function \(R: [0,\infty) \to [0,\infty, \infty]\) through

\[
R(z) := \int_1^z \exp \left( - \int_1^y \frac{2\mu(x)}{\sigma^2(x)} dx \right) dy
\]

(123)

for all \(z \in [0,\infty]\). Standard results (e.g. [27]) show that \(\hat{Z}_t \to \infty\) in distribution as \(t \to \infty\) if \(R(0) = -\infty\) and \(R(\infty) < \infty\). Let \(\alpha \in \mathbb{R}\). We rewrite the integral in the exponent on the right-hand side of (123) as

\[
\int_1^y \frac{\dot{\varphi}(x)}{\varphi(x)} (\sigma_e^2 z + \sigma_e^2 \hat{x}^2 + \sigma_e^2 z + 2\alpha z) = 2 \int_1^y \dot{\varphi}(x) dx + \int_1^y \sigma_e^2 + 2\alpha = 2 \int_1^y \dot{\varphi}(x) dx + \int_1^y \sigma_e^2 + 2\alpha dx
\]

(124)

for all \(y \in (0,\infty)\). By (124), there exists a constant \(c \in (0,\infty)\) such that

\[
R(z) = c \int_1^z \frac{1}{(\dot{\varphi}(y))^2} (\sigma_e^2 z + \sigma_e^2 \hat{x}^2 - \frac{\sigma^2}{\sigma_e^2} + \frac{\sigma^2}{\sigma_e^2}) dy
\]

(125)

for all \(z \in [0,\infty)\). As \(\dot{\varphi}(z)/z \to c\) as \(z \to 0\) for a constant \(c = \hat{c}(\alpha, \sigma_e, \sigma_b) > 0\), we have that \(\lim_{z \to 0} R(z) = \int_1^z \frac{1}{y^2} dy = -\infty\). Next we show that \(R(\infty) < \infty\) whenever \(\alpha > -\sigma_e^2\). If \(\alpha > 0\), then \(\dot{\varphi}(z) = 1 - \left( 1 + \frac{\sigma^2}{\sigma_e^2} \right) \frac{\sigma^2}{\sigma^2} \log(1 + \frac{\sigma^2}{\sigma_e^2}) \).

\[
and thus \(R(\infty) < \infty\). In the case \(\alpha = 0\) (and thus \(\beta = 0\)), we have that \(\dot{\varphi}(z) = \frac{\sigma^2}{\sigma^2} \log(1 + \frac{\sigma^2}{\sigma_e^2})\). From \(\int_1^z \frac{1}{y^2 \log(y)} dy < \infty\), we deduce that \(R(\infty) < \infty\). Next let \(\alpha \in (-\sigma_e^2, 0)\). Lemma \[14\] implies that there is a constant \(c \in (0,\infty)\) such that

\[
R(z) \sim \hat{c} \int_1^z \frac{1}{(y^{\beta/2} \log(y))^2} (\sigma_e^2 + \sigma_e^2 y) \beta^{-1} dy
\]

(126)

as \(z \to \infty\). As \(\int_1^\infty \frac{1}{y^{\beta/2} \log(y)} dy < \infty\), this implies that \(R(\infty) < \infty\). Finally assume that \(\alpha < -\sigma_e^2\). Theorem V.54.5 in [36] implies that

\[
P^x (Z_t \in dy) \overset{w}{\to} \frac{2}{\sigma_e^2(x)} \exp \left( \int_0^y \frac{2 \mu(u)}{\sigma^2(u)} du \right) dy
\]

(127)
for every $z \in (0, \infty)$ if there exists a normalizing constant $\tilde{c} \in (0, \infty)$ such that the right-hand side is a probability distribution. Due to (124) we need to show that
\[
\frac{1}{\sigma_{\tilde{c}}^2 y + \sigma_{\tilde{c}}^2 y^2} (\theta(y))^2 \left( \sigma_\theta^2 + \sigma_{\tilde{c}}^2 y^2 \right)^{2 + \alpha} = \left( 2 - \frac{e}{\sigma_{\tilde{c}}^2} \right)^2 y \left( \sigma_\theta^2 + \sigma_{\tilde{c}}^2 y^2 \right)^{2 + \alpha}
\] (128)
is integrable over $y \in (0, \infty)$. This function is bounded over $(0, 1]$ and is of order $O(y^{1 + \frac{\alpha}{\sigma_{\tilde{c}}^2}})$ as $y \to \infty$. As $\alpha < -\sigma_{\tilde{c}}^2$, there exists a normalizing constant $\tilde{c}$ such that the right-hand side of (127) is a probability distribution.

7 Family decomposition of BDREs with immigration

Let $\alpha, \theta \in \mathbb{R}$, $\sigma_b \in (0, \infty)$ and $\sigma_e \in [0, \infty)$. In this section we consider the BDRE with immigration/emigration which is the solution of the SDEs
\[
dZ_t = \theta dt + \frac{1}{2} \sigma_b^2 dZ_t dt + Z_t dS_t + \sqrt{\sigma_b^2} Z_t dW^{(b)}_t
\]
\[
dS_t = \alpha dt + \sqrt{\sigma_e^2} dW^{(e)}_t
\] (129)
for $t \geq 0$ where $S_0 = 0$. The family decomposition of the BDRE with immigration will be a corollary of the family decomposition of Feller’s branching diffusion with immigration. For this, we first need to generalize Proposition 2 to include immigration.

**Lemma 16.** Assume $\alpha, \theta \in \mathbb{R}$, $\sigma_b \in (0, \infty)$ and $\sigma_e \in [0, \infty)$. Let $(F_t)_{t \geq 0}$ be a weak solution of
\[
dF_t = \frac{\theta}{\sigma_b^2} dt + \sqrt{F_t} dW^{(b)}_t
\] (130)
for $t \in [0, \infty)$ and let $S_t := \alpha t + \sigma_e W^{(e)}_t$ for $t \in [0, \infty)$ be independent of $(F_t)_{t \geq 0}$. Moreover, define $(\tau(t))_{t \geq 0}$ through $\tau(t) := \int_0^t e^{-S_u^2} dS_u$ for $t \in [0, \infty)$. Then
\[
(F_{\tau(t)} e^{S_t}, S_t)_{t \geq 0}
\] (131)
is a weak solution of (129).

**Proof.** Fix $\alpha, \theta \in \mathbb{R}$, $\sigma_b \in (0, \infty)$ and $\sigma_e \in [0, \infty)$. Define $Z_t := F_{\tau(t)} e^{S_t}$ for $t \in [0, \infty)$. Itô’s lemma together with independence of $(F_t)_{t \geq 0}$ and of $(S_t)_{t \geq 0}$ imply that
\[
dZ_t = dF_{\tau(t)} e^{S_t}
\]
\[
= e^{S_t} dF_{\tau(t)} + F_{\tau(t)} e^{S_t} dS_t + \frac{1}{2} F_{\tau(t)} e^{S_t} \sigma_e^2 dt
\]
\[
= e^{S_t} \frac{\theta}{\sigma_b^2} dt + e^{S_t} \sqrt{F_{\tau(t)}} dW^{(b)}_{\tau(t)} + Z_t dS_t + \frac{1}{2} Z_t \sigma_e^2 dt
\]
\[
= e^{S_t} \frac{\theta}{\sigma_b^2} \tau'(t) dt + \frac{\sigma_b^2}{\sqrt{\sigma_b^2}} F_{\tau(t)} e^{S_t} \sigma_e \frac{1}{\sqrt{\sigma_b^2}} dW^{(b)}_{\tau(t)} + Z_t dS_t + \frac{1}{2} Z_t \sigma_e^2 dt
\]
\[
= \theta dt + \frac{1}{2} \sigma_e^2 Z_t dt + Z_t dS_t + \sqrt{\sigma_b^2} Z_t \frac{1}{\sqrt{\tau(t)}} dW^{(b)}_{\tau(t)}
\] (132)
for \( t \in [0, \infty) \). As \((\tau(t))_{t \geq 0}\) and \((W_t^{(b)})_{t \geq 0}\) are independent, the process \((W_t)_{t \geq 0}\) defined through \(W_t := \int_0^t \frac{1}{\sqrt{\tau(s)}} dW_t^{(b)}, \quad t \in [0, \infty)\), is a continuous martingale and a Markov process satisfying
\[
E [W_t^2] = E \left[ \left( \int_0^t \frac{1}{\sqrt{\tau(s)}} dW_t^{(b)}(s) \right)^2 \right] = E \left[ \int_0^t \frac{1}{\tau(s)} d\tau(s) \right] = t
\]
for all \( t \in [0, \infty) \). Thus \((W_t)_{t \geq 0}\) is a standard Brownian motion according to Lévy’s characterization (e.g. Theorem IV.33.1 of [37]). Moreover \((W_t)_{t \geq 0}\) and \((W_t^{(c)})_{t \geq 0}\) are independent. Therefore (132) implies that \((Z_t, S_t)_{t \geq 0}\) is a weak solution of (129).

Let \( \sigma_b \in (0, \infty) \) and let \((F_t)_{t \geq 0}\) be the solution of the SDE
\[
dF_t = \sqrt{\sigma_b^2 F_t} dW_t
\]
for \( t \in [0, \infty) \). Recall the associated excursion measure \(Q \) on \(U\) from (16). The following family decomposition of Feller’s branching diffusion is a special case of the family decomposition of the Dawson-Watanabe superprocess with immigration (see [33] and [11]). Recall the excursion space \(U\) from (14).

**Lemma 17.** Let \( \theta \in [0, \infty), \alpha \in \mathbb{R} \) and \( \sigma_b \in (0, \infty) \). Let \( \mathbb{P}_0 \) be a Poisson point process on \([0, \infty) \times U\) with intensity measure \(dy \times Q\) and let \( \mathbb{P}_0\) be an independent Poisson point process on \([0, \infty) \times U\) with intensity measure \(\theta dt \times Q\). Then the process \((\tilde{F}_t)_{t \geq 0}\) defined through \(\tilde{F}_0 = x\) and
\[
\tilde{F}_t := \sum_{(y, \chi) \in \mathbb{P}_0} \mathbb{1}_{y \leq x} \chi_t + \sum_{(s, \chi) \in \mathbb{P}_0} \chi_{t-s}
\]
for \( t \in (0, \infty) \) is a weak solution of the SDE
\[
d\tilde{F}_t = \theta dt + \sqrt{\sigma_b^2 \tilde{F}_t} dW_t, \quad \tilde{F}_0 = x
\]
for \( t \in [0, \infty) \) and for each \( x \in [0, \infty) \).

**Proof of Theorem 3** Fix \( \sigma_b, \sigma_e \in (0, \infty) \) and \( \alpha \in (-\infty, -\sigma_e^2] \). Define a process \((\tilde{F}_t)_{t \geq 0}\) through \(\tilde{F}_0 = x\) and through
\[
\tilde{F}_t := \sum_{(y, \chi) \in \mathbb{P}} \mathbb{1}_{y \leq x} \chi_t + \sum_{(s, \chi) \in \mathbb{P}} \chi_{t-s}
\]
for \( t \in (0, \infty) \). Then Lemma 17 shows that \((\tilde{F}_t)_{t \geq 0}\) is a weak solution of
\[
d\tilde{F}_t = dt + \sqrt{\tilde{F}_t} dW_t
\]
for \( t \in [0, \infty) \). Lemma 16 implies that \((\tilde{Z}_t, \tilde{S}_t)_{t \geq 0}\) is a solution of (133) due to Theorem 7. As the solution of (133) is unique in law, we conclude that \((\tilde{Z}_t, \tilde{S}_t)_{t \geq 0}\) and \((\tilde{Z}_t, \tilde{S}_t)_{t \geq 0}\) have the same distribution.

**References**

[1] **Afanasiev, V. I.** On the survival probability of a subcritical branching process in a random environment. Dep. VINITI (1979), No. M1794–79 (in Russian).
[2] Afanasyev, V. I. Limit theorems for a conditional random walk and some applications. Diss. cand. sci., MSU, Moscow, 1980.

[3] Afanasyev, V. I., Böinghoff, C., Kersting, G., and Vatutin, V. A. Limit theorems for a weakly subcritical branching process in a random environment. to appear in J. Theoret. Probab., DOI: 10.1007/s10959-010-0331-6 (2011).

[4] Afanasyev, V. I., Geiger, J., Kersting, G., and Vatutin, V. A. Criticality for branching processes in random environment. Ann. Probab. 33, 2 (2005), 645–673.

[5] Afanasyev, V. I., Geiger, J., Kersting, G., and Vatutin, V. A. Functional limit theorems for strongly subcritical branching processes in random environment. Stochastic Process. Appl. 115, 10 (2005), 1658–1676.

[6] Bansaye, V., and Berestycki, J. Large deviations for branching processes in random environment. Markov Process. Related Fields 15, 4 (2009), 493–524.

[7] Bertoin, J., and Doney, R. A. On conditioning a random walk to stay nonnegative. Ann. Probab. 22, 4 (1994), 2152–2167.

[8] Böinghoff, C., Dyakonova, E. E., Kersting, G., and Vatutin, V. A. Branching processes in random environment which extinct at a given moment. Markov Process. Related Fields 16, 2 (2010), 329–350.

[9] Cattiaux, P., Collet, P., Lambert, A., Martínez, S., Méleard, S., and San Martín, J. Quasi-stationary distributions and diffusion models in population dynamics. Ann. Probab. 37, 5 (2009), 1926–1969.

[10] Comtet, A., Monthus, C., and Yor, M. Exponential functionals of Brownian motion and disordered systems. J. Appl. Probab. 35, 2 (1998), 255–271.

[11] Dawson, D. A. Measure-valued Markov processes. In École d’Été de Probabilités de Saint-Flour XXI—1991, vol. 1541 of Lecture Notes in Math. Springer, Berlin, 1993, pp. 1–260.

[12] Dekking, F. M. On the survival probability of a branching process in a finite state i.i.d. environment. Stochastic Process. Appl. 27, 1 (1987), 151–157.

[13] Dufresne, D. The distribution of a perpetuity, with applications to risk theory and pension funding. Scand. Acturial. J. 1990, 1 (1990), 39–79.

[14] Dufresne, D. The integral of geometric Brownian motion. Adv. in Appl. Probab. 33, 1 (2001), 223–241.

[15] Dyakonova, E. E. On subcritical multi-type branching process in random environment. In Fifth Colloquium on Mathematics and Computer Science, Discrete Math. Theor. Comput. Sci. Proc., AI. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2008, pp. 397–404.

[16] Etheridge, A. M., and Williams, D. R. E. A decomposition of the $(1 + \beta)$-superprocess conditioned on survival. Proc. Roy. Soc. Edinburgh Sect. A 133, 4 (2003), 829–847.

[17] Ethier, S. N., and Kurtz, T. G. Markov processes: Characterization and convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986.

[18] Evans, S. N. Two representations of a conditioned superprocess. Proc. Roy. Soc. Edinburgh Sect. A 123, 5 (1993), 959–971.

[19] Galambos, J., and Seneta, E. Regularly varying sequences. Proc. Amer. Math. Soc. 41 (1973), 110–116.
[20] Geiger, J., and Kersting, G. The survival probability of a critical branching process in random environment. *Theory Probab. Appl.* 45, 3 (2002), 518–526.

[21] Geiger, J., Kersting, G., and Vatutin, V. A. Limit theorems for subcritical branching processes in random environment. *Ann. Inst. H. Poincaré Probab. Statist.* 39, 4 (2003), 593–620.

[22] Gorostiza, L. G., and Wakolbinger, A. Persistence criteria for a class of critical branching particle systems in continuous time. *Ann. Probab.* 19, 1 (1991), 266–288.

[23] Helland, I. S. Minimal conditions for weak convergence to a diffusion process on the line. *Ann. Probab.* 9, 3 (1981), 429–452.

[24] Hutzenthaler, M. The Virgin Island Model. *Electron. J. Probab.* 14 (2009), no. 39, 1117–1161 (electronic).

[25] Hutzenthaler, M. Supercritical branching diffusions in random environment. *Electron. Commun. Probab.* 16, 2 (2011), no. 69, 781–791 (electronic).

[26] Kaltenberg, O. Stability of critical cluster fields. *Math. Nachr.* 77 (1977), 7–43.

[27] Karatzas, I., and Shreve, S. E. *Brownian motion and stochastic calculus*, second ed., vol. 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[28] Keiding, N. Extinction and exponential growth in random environments. *Theor. Population Biology* 8 (1975), 49–63.

[29] Klenke, A. *Probability theory*. Universitext. Springer-Verlag London Ltd., London, 2008. A comprehensive course, Translated from the 2006 German original.

[30] Kozlov, M. V. On large deviations of branching processes in a random environment: a geometric distribution of the number of descendants. *Discrete Math. Appl.* 16, 2 (2006), 155–174.

[31] Kurtz, T. G. Diffusion approximations for branching processes. In *Branching processes (Conf., Saint Hippolyte, Que., 1976)*, vol. 5 of *Adv. Probab. Related Topics*. Dekker, New York, 1978, pp. 269–292.

[32] Lambert, A. Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct. *Electron. J. Probab.* 12 (2007), no. 14, 420–446.

[33] Li, Z., and Shiga, T. Measure-valued branching diffusions: immigrations, excursions and limit theorems. *J. Math. Kyoto Univ.* 35, 2 (1995), 233–274.

[34] Matsumoto, H., and Yor, M. On Dufresne's relation between the probability laws of exponential functionals of Brownian motions with different drifts. *Adv. Appl. Prob.* 35 (2003), 184–206.

[35] Rackauskas, A., and Suquet, C. Hölderian invariance principle for Hilbertian linear processes. *ESAIM Probab. Stat.* 13 (2009), 261–275.

[36] Rogers, L. C. G., and Williams, D. *Diffusions, Markov processes and martingales. Vol. 2*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.

[37] Rogers, L. C. G., and Williams, D. *Diffusions, Markov processes and martingales. Vol. 2*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.
[38] Severini, T. A. *Elements of distribution theory*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2005.

[39] Smith, W. L. Necessary conditions for almost sure extinction of a branching process with random environment. *Ann. Math. Statist* 39 (1968), 2136–2140.

[40] Smith, W. L., and Wilkinson, W. E. On branching processes in random environments. *Ann. Math. Statist.* 40 (1969), 814–827.

[41] Vatutin, V. A. A limit theorem for an intermediate subcritical branching process in a random environment. *Theory Probab. Appl.* 48, 3 (2004), 481–492.

[42] Wang, H.-X., and Fang, D. Asymptotic behaviour of population-size-dependent branching processes in Markovian random environments. *J. Appl. Probab.* 36, 2 (1999), 611–619.

[43] Yor, M. On some exponential functionals of Brownian motion. *Adv. in Appl. Probab.* 24, 3 (1992), 509–531.

[44] Yor, M. Sur certaines fonctionnelles exponentielles du mouvement brownien réel. *J. Appl. Probab.* 29, 1 (1992), 202–208.