A technical remark on the Donaldson-Futaki invariant for Fano reductive group compactifications

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Abstract

We present an elementary way of computing the Donaldson-Futaki invariant associated to a test-configuration of an anti-canonically polarized Fano reductive group compactification. In addition, we explain how to obtain examples of real Einstein manifolds from K-stable Fano manifolds.

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1 Introduction

A metric on a manifold is said to be canonical if its curvature is optimal in some variational sense. In Kähler geometry, the best studied canonical metrics are the extremal ones, which are critical points of the Calabi functional. Extremal metrics include the constant scalar curvature Kähler (cscK) metrics, among which are the Kähler-Einstein (KE) metrics.

The Yau-Tian-Donaldson (YTD) conjecture is an umbrella term for the problem of finding necessary and sufficient conditions for the existence of an extremal Kähler metric. This article is concerned with the YTD conjecture for positive KE metrics on certain group compactifications that generalize toric manifolds, and applications to real Einstein geometry.
Before stating our results, let us review some basic knowledge on Kähler metrics. For a thorough treatment of many of the topics that we present in sections 1.1 and 1.2 below, see [13, 26].

1.1 The Mabuchi functional

Let $X$ be a compact Kähler manifold of dimension $n$. Fix a Kähler class $\Omega \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$, and a basepoint $\omega_0 \in \Omega$. Let

$$M_\Omega = \{ \phi \in C^\infty(X, \mathbb{R}) \mid \omega_\phi := \omega_0 + i \partial \bar{\partial} \phi > 0 \}$$

be the space of Kähler potentials of Kähler metrics in $\Omega$, $S := \frac{2n \pi c_1(X) \cup \Omega^{n-1}}{\Omega^n}$ be the average scalar curvature, and $S_\phi$ be the scalar curvature of $\omega_\phi$. The scalar curvature 1-form $\sigma$ on $M_\Omega$ is given as

$$\sigma_\phi(\psi) := \langle \psi, \overline{S} - S_\phi \rangle_{L^2(\omega_\phi)} = \int_X \psi (\overline{S} - S_\phi) \frac{\omega_\phi^n}{n!},$$

for any $\phi \in M_\Omega$, $\psi \in T_{M_\Omega, \phi} = C^\infty(X, \mathbb{R})$. Consider the 4th order, self-adjoint, Lichnerowicz operator, acting on functions on $X$ by $L(f) = D^* D(f)$, where $D(f) = \nabla_a \nabla_b f$, $\nabla$ is the Chern connection, and $D^*$ is the formal adjoint of $D$. Any $f, g \in C^\infty(X, \mathbb{R})$ can be viewed as commuting vector fields on $M_\Omega$. The scalar curvature 1-form is closed because $f \cdot \sigma_\phi(g) = \langle -2L(f), g \rangle_{L^2(\omega_\phi)}$ so that

$$d \sigma_\phi(f, g) = f \cdot \sigma_\phi(g) - g \cdot \sigma_\phi(f) - \sigma_\phi([f, g]) = -2\langle L(f), g \rangle_{L^2(\omega_\phi)} + 2\langle g, L(f) \rangle_{L^2(\omega_\phi)} = 0.$$

By the Poincaré Lemma, $\sigma$ must be exact as $M_\Omega$ is contractible.

The Mabuchi functional (or K-energy) is the uniquely defined functional $K : M_\Omega \to \mathbb{R}_{\geq 0}$, satisfying $dK = \sigma$ and $K(0) = 0$. In fact, $K$ descends to a functional $\overline{K}$ on $\Omega$ whose Euler-Lagrange equation is the cscK equation, $S(\omega) = \overline{S}$, $\omega \in \Omega$ [21]. The K-energy has some interesting properties. For example, it is geodesically convex, and geodesics here are defined relative to the Riemannian metric on $M_\Omega$, which at any $\phi \in M_\Omega$ is given by

$$\langle f_1, f_2 \rangle_\phi = \int_X f_1 f_2 \frac{\omega_\phi^n}{n!},$$

for any $f_1, f_2 \in T_{M_\Omega, \phi}$. In some cases, cscK metrics can be seen to be energy minimizers.
1.2 K-stability

Let \( L \to X \) be an ample line bundle. Then for some \( r > 0 \), there is an embedding \( \epsilon : X \hookrightarrow \mathbb{CP}^{N_r} \), \( \epsilon(p) = [s_0(p) : \cdots : s_{N_r}(p)] \), where \( (s_i)_{i=0}^{N_r} \) is a basis of the space of global holomorphic sections \( H^0(X, L^r) \). A test-configuration of the polarization \( (X, L) \) is a 1-parameter deformation \( (X_t, L_t) \) such that \( X_t \simeq X \) unless possibly when \( t = 0 \). Precisely, it is the data of (1) a choice of embedding \( \epsilon : X \hookrightarrow \mathbb{CP}^{N_r} \) and (2) a \( \mathbb{C}^* \)-action on \( \mathbb{CP}^{N_r} \), or equivalently, a 1-parameter subgroup \( \lambda : \mathbb{C}^* \to \text{GL}_{N_r+1}(\mathbb{C}) \).

Now, let \( \omega_{FS} \) be the Fubini-Study metric on \( \mathbb{CP}^{N_r} \). For all \( t \neq 0 \), \( \omega_t := \frac{1}{t} \epsilon^* \omega_{FS} \) is a Kähler metric on \( X_t \). Up to normalization, the Donaldson-Futaki (DF) invariant of the test configuration \( (\epsilon, \lambda) \) is the rational number \( \text{DF}(\epsilon, \lambda) = \lim_{t \to \infty} \frac{dK(\omega_t)}{dt} \). So the DF-invariant describes the asymptotics of Mabuchi’s functional along special metric degenerations. The polarization \( (X, L) \) is K-stable if the DF-invariant is positive for all test-configurations \( (X_t, L_t) \) with \( X_0 \simeq X \).

In practice, the main issue one encounters when trying to check the K-stability condition is classifying all inequivalent test-configurations. However, this issue can be overcome in highly symmetric cases, such as polarized reductive group compactifications, which will be the focus of our work here.

1.3 Results and organization

In this note, we study a special case of the below stated partial solution of the YTD conjecture, and some of its ramifications in the real setting. Theorem 1 was proved in [5]; see also [27].

**Theorem 1.** Let \( X \) be a compact Kähler Fano manifold \( (c_1(X) > 0) \). Then, there exists a KE form \( \omega \in 2\pi c_1(X) \) iff \( (X, K_X^{-1}) \) is K-stable, where \( K_X^{-1} \) is the anti-canonical line bundle of \( X \). Moreover, whenever such a form exists, it is unique.

Polarized compactifications \( (X, L) \) of a connected complex reductive group \( G \) have a combinatorial description in terms of polytopes. This description leads to a classification of all relevant test-configurations and to a simplification of the DF-invariant, and hence of the K-stability condition. The latter is encoded in the combinatorics of the polytope. Let us be more precise. If \( W \) is the Weyl group of \( G \) w.r.t. to a maximal torus \( T \),
the combinatorial counterpart of \((X,L)\) is a \(W\)-invariant convex polytope \(P\). Choose a set of positive roots \(\Phi^+\) in the root system of \((G,T)\). Define \(P^+\) to be the part of \(P\) that meets the positive Weyl chamber defined relative to \(\Phi^+\). A test-configuration of the polarization is then given by a piecewise linear, \(W\)-invariant, convex function \(f\) on \(P\) that is affine linear on \(P^+\). Denote by \(2\rho\) the sum of the positive roots in \(\Phi^+\). At the polytope level, Fano is the condition that the distance between \(2\rho\) and any codimension one face of \(P^+\) that does not meet the boundary of the positive Weyl chamber is equal to one (cf. section 2.3). Any integral over \(X\) transforms into a Riemann integral over \(P\), thereby allowing us to compute the DF invariant in an elementary way.

**Theorem 2.** Assume that \(P\) satisfies the Fano condition. Let \(\text{Vol}_{DH}(P^+) := \int_{P^+} H_d d\mu\) and \(\text{bar}_{DH}(P^+) := \frac{1}{\text{Vol}_{DH}(P^+)} \int_{P^+} xH_d d\mu\) be the volume, respectively the barycenter of \(P^+\) w.r.t. the Duistermaat-Heckman (DH) measure. The DF-invariant of the test-configuration represented by \(f\) is

\[
-F_1(f) = \frac{1}{2\text{Vol}_{DH}(P^+)} \int_{P^+} \langle \nabla f, x - 2\rho \rangle H_d d\mu
= \frac{1}{2} \langle \text{bar}_{DH}(P^+) - 2\rho, \nabla f \rangle.
\]

Theorem 2 can be interpreted as a reductive analog of Theorem C in [11]. Indeed, Theorem 2 recovers Delcroix’s barycentric criterion for K-stability (cf. Theorem 5).

The second observation made in this note is about an application of Theorem 1 to real Einstein geometry. Let \((X,J)\) be a complex manifold, and \(\sigma : (X,J) \to (X,J)\) be an anti-holomorphic involution with non-empty fixed point set. By the real part of \(X\) w.r.t. to \(\sigma\), we mean the fixed point set of \(\sigma\). The uniqueness of Fano KE metrics in a fixed Kähler class is linked to the existence of real Einstein submanifolds.

**Theorem 3.** Let \((X,\omega)\) be a Fano KE manifold, and \(f : X \to X\) be a bi-anti-holomorphic map. Then, \(f\) is an anti-isometry of \((X,\omega)\), and an isometry of the underlying Riemannian manifold. As a result, the real part of a Fano KE manifold w.r.t. any anti-holomorphic involution is Einstein.

In section 2 we recall the basics of the theory of reductive groups and their compactifications. In section 3 we prove Theorem 2. In section 4 we
explain how to obtain Delcroix’s barycentric criterion by means of Theorem 2. In section 5 we prove Theorem 3.

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2 Reductive group compactifications

Good references for the theory of reductive groups and the representation theory that we survey here are [3, 4, 14, 25].

Let $G$ be a connected complex linear algebraic group. There is a maximal connected normal solvable subgroup of $G$, which is denoted by $R(G)$, and called the radical of $G$. There is also a maximal connected normal unipotent subgroup $R_u(G) \subseteq R(G)$ of $G$, called the unipotent radical of $G$. If $R_u(G) = \{e\}$, then $G$ is called a complex reductive group. Indeed, any semisimple $G$ is reductive since $R(G) = \{e\}$. Equivalently, $G$ is reductive iff all of its rational representations are direct sums of irreducible representations or if $G$ is isomorphic to the complexification of a maximal compact subgroup. From now on, the letter $G$ will always be used to denote a reductive group.

A reductive group compactification (for $G$) is a normal irreducible projective $G \times G$-variety $X$ that has an open dense orbit that is equivariantly isomorphic to $G$, where the $G \times G$-action on $G \subseteq X$ is $(g_1, g_2)x = g_1xg_2^{-1}$. In this context, a polarization is an ample $G \times G$-linearized line bundle on $X$. Reductive group compactifications are a special instance of the stable reductive varieties from [1].

Example 1. The toric variety of a lattice polytope $P \subset \mathbb{R}^n$ is a reductive group compactification for a torus $T \simeq (\mathbb{C}^*)^n$. For example, consider the toric variety of the standard 3-simplex, which is $\mathbb{C} \mathbb{P}^3$. However $\mathbb{C} \mathbb{P}^3$ is, at the same time, a compactification of the reductive group $\text{PGL}_2(\mathbb{C})$. Let $M_2(\mathbb{C})$ stand for the
space of $2 \times 2$ complex matrices. Indeed, $\text{PGL}_2(\mathbb{C}) \times \text{PGL}_2(\mathbb{C})$ acts on the projectivization $\mathbb{P}(M_2(\mathbb{C}))$, which can be identified with $\mathbb{C}P^3$, and the action has an open and dense orbit $\{[a : b : c : d]|ad - bc \neq 0\} \subset \mathbb{C}P^3$ that is isomorphic to $\text{PGL}_2(\mathbb{C})$. Projective 3-space viewed in this light is the wonderful compactification of $\text{PGL}_2(\mathbb{C})$. See Example 2.3 in [10] for a brief discussion on wonderful compactifications, as well as [9] and references therein, for a more detailed introduction to the topic.

A non-toric reductive group compactification is the wonderful compactification of $\text{PGL}_3(\mathbb{C})$.

2.1 Reductive Lie algebras

Choose a maximal torus $T \cong (\mathbb{C}^*)^n$ in $G$. The Lie algebra of $G$ is $g = Z(g) \oplus [g, g]$, where $Z(g)$ is the center of $g$ and where $[g, g]$ is the Lie algebra of the semisimple Lie group $[G, G]$. Let $t$ be the Lie algebra of $T$, and $t_{ss} = t \cap [g, g]$ be its semisimple part. The adjoint action of $[g, g]$ on itself, when restricted to $t_{ss}$, is simultaneously diagonalizable and thus has an eigenspace decomposition

$$[g, g] = t_{ss} \oplus \bigoplus_{\alpha \in \Phi} [g, g]_\alpha,$$

where

$$[g, g]_\alpha = \{X \in [g, g] | ad(H)(X) = \alpha(H)X \text{ for all } H \in t_{ss}\},$$

and

$$\Phi = \{\alpha \in t_{ss}^* \setminus \{0\} | [g, g]_\alpha \neq 0\}.$$

We define the root system of $(G, T)$ to be $\Phi$. Thus, we are associating to $G$ the root system of the derived Lie algebra $[g, g]$. The Weyl group of $(G, T)$ is $W = N_G(T)/T$, where

$$N_G(T) = \{g \in G | gTg^{-1} = T\}$$

is the normalizer of $T$.

Suppose that $\Delta \subset \Phi$ is a set of simple roots. This means that $\Delta$ is a basis of $\text{Span}_\mathbb{C}(\Phi)$ and that for each $b \in \Phi$, $b = \sum_{\delta \in \Delta} a_\delta \delta$ satisfies either $a_\delta > 0$ or $a_\delta < 0$ for all $\delta \in \Delta$. The set of positive roots w.r.t. $\Delta$ is

$$\Phi^+ = \{\alpha \in \Phi | a = \sum_{\delta \in \Delta} a_\delta \delta, a_\delta > 0\}.$$
We define the weight lattice $M$ of $G$ to be the character lattice of $T$ and the coweight lattice $N$ of $G$ to be the lattice of 1-parameter subgroups of $T$. Both $M$ and $N$ have rank equal to $\dim T = n$. The Lie algebra $t$ can be identified with $M_\mathbb{R} := M \otimes \mathbb{R}$ and its dual with $N_\mathbb{R} := N \otimes \mathbb{R}$.

Relative to a set of positive roots $\Phi^+ \subset N_\mathbb{R}$, the positive Weyl chamber is

$$M^+_\mathbb{R} := \{ x \in M_\mathbb{R} | \langle \alpha, x \rangle \geq 0 \text{ for all } \alpha \in \Phi^+ \},$$

where $\langle \cdot, \cdot \rangle$ is the natural bilinear pairing between $t$ and its dual. The expected $W$-action on $T$ induces a $W$-action on $M_\mathbb{R}$, whose fundamental domain is $M^+_\mathbb{R}$.

### 2.2 Polytopes

Now let us fix a choice of maximal torus $T \subset G$ with character lattice $M$, and Lie algebra $t$. Let $W$ be the Weyl group of $(G, T)$, and let $\Phi$ denote the root system of $(G, T)$ with selected positive roots $\Phi^+$. We declare $2\rho$ to be the sum of these positive roots.

There is a one-to-one correspondence between lattice points $\lambda \in M^+_\mathbb{R}$ and irreducible $G$–representations $E_\lambda$, and to a lattice point $\lambda \in M^+_\mathbb{R}$ corresponds a $G \times G$–representation $\text{End}(E_\lambda)$. The dimension of $\text{End}(E_\lambda)$ is a polynomial

$$\dim(\text{End}(E_\lambda)) = (\dim(E_\lambda))^2 = H_d(\lambda) + H_{d-1}(\lambda) + \ldots$$

in $\lambda$, and here $H_d$ stands for the degree $d$ homogeneous part of the polynomial $\dim(\text{End}(E_\lambda))$, $H_{d-1}$ stands for the degree $d - 1$ part, and so on.

Let $P^+ := P \cap M^+_\mathbb{R}$ and $C(P^+) \subseteq M_\mathbb{R} \times \mathbb{R}$ be the cone over $(P^+, 1)$. Consider the finitely generated algebra

$$R_P = \bigoplus_{\lambda \in C(P^+) \cap (M \times \mathbb{Z})} \text{End}(E_\lambda).$$

Let $(X, L)$ be a polarized compactification of $G$. Denote the Zariski closure of $T$ in $X$ by $V$. Then, $V$ is a toric subvariety of $X$ so that $(V, L|_V)$ is a polarized toric variety. To $(X, L)$ we can associate the $W$–invariant lattice polytope that corresponds to $(V, L|_V)$ by the combinatorial dictionary of toric varieties. Conversely, to any $W$–invariant lattice polytope $P \subseteq M_\mathbb{R}$, we can associate a polarized reductive group compactification $(X_P, L_P)$, where $X_P = \text{Proj}(R_P)$ and $L_P = \mathcal{O}(1)$.
Example 2. The interval $P = [-2, 2]$ is the polytope of $\mathbb{CP}^3$ viewed as the $\text{PGL}_2(\mathbb{C})$–compactification that we described in Example 1.

2.3 The Fano condition

This section is based on the article [23]. Denote the Zariski closure of $T$ in $X_P$ by $Z$, which is a toric subvariety. When $X_P$ is Fano, the support function $v$ of $P$ is of the form $v = v_K + v_Z$, where $v_K(x) = \langle 2\rho, x \rangle$ for all $x$ in the positive Weyl chamber, $v_K(wx) = v_K(x)$ for all $w \in W$, and $v_Z(x) = -g_{-K_Z}(-x)$, where $-g_{-K_Z}$ is the support function of the anticanonical line bundle of the subvariety $Z \subset X_P$. Since $P$ is also the polytope of $Z$, the associated fan $\Sigma_P$ determines the toric subvariety $Z$. From the theory of toric varieties, $-K_Z = \sum_{\rho \in \Sigma(1)} D_\rho$, where $\Sigma(1)$ is the set of 1-dimensional cones of $\Sigma_P$ and $D_\rho$ is a prime torus invariant divisor on $Z$. The support function $g_{-K_Z}$ has the property that $g_{-K_Z}(u_\rho) = -1$ for all $\rho \in \Sigma(1)$, where $u_\rho$ is the minimal generator of the ray $\rho$. In particular, if $a_i$ is the inward pointing normal to the $i$-th codimension one face of $P$, $g_{-K_Z}(a_i) = -g_{-K_Z}(-a_i) = -1$. Then, $v(a_i) = \langle a_i, 2\rho \rangle - 1$, and so the facet presentation of the polytope is

$$P = \{ x \in M_{\mathbb{R}} | \langle a_i, x \rangle \geq \langle a_i, 2\rho \rangle - 1 \}.$$ 

Consequently, the equation that defines the $i$-th boundary face of $P$ is $f_i(x) = \langle a_i, x - 2\rho \rangle + 1$ so that $f_i(2\rho) = 1$.

2.4 The Alexeev-Katzarkov formula

Following Donaldson’s construction of toric test-configurations [12], Alexeev and Katzarkov construct reductive test families, which correspond to convex rational $W$–invariant piecewise linear (PL) functions on $P$ [2]. Moreover, they compute the DF-invariant associated to such test families.

**Theorem 4.** (Theorem 3.3 [2]) Let $f$ be a convex rational $W$–invariant PL function on $P$. Then the DF-invariant of the corresponding test-configuration is given by the formula

$$-F_1(f) = \frac{1}{2} \int_{\partial P} H_d d\mu \left( \int_{\partial P} f H_d d\sigma + 2 \int_{P} f H_{d-1} d\mu - a \int_{\partial P} f H_d d\mu \right).$$


where
\[ a = \frac{\int_{\partial P} H_d d\sigma + 2 \int_{\partial P} H_{d-1} d\mu}{\int_{\partial P} H_d d\mu}. \]

Here \( d\mu \) is the Lebesgue measure restricted to \( P \), and the boundary measure \( d\sigma \) is a positive measure on \( \partial P \) that is normalized so that on each codimension one face, which is defined by an equation \( l(x) := \langle a, x \rangle = c \), \( d\sigma \wedge dl = \pm d\mu \) holds.

In the sequel, we obtain a number of identities that together with the Fano condition will allow us to simplify Alexeev’s and Katzarkov’s DF-invariant from Theorem 4.

3 Computation of the Donaldson-Futaki invariant

Choose once and for all an isomorphism \( M_\mathbb{R} \cong \mathbb{R}^n \) so that \( P \) can be viewed as though contained in \( \mathbb{R}^n \).

Lemma 1. Let \( \Phi^+ = \{\alpha_1, \ldots, \alpha_r\} \), \( c = \prod_{i=1}^r \langle \alpha_i, \rho \rangle^2 \), where \( \rho = \frac{1}{2} \sum_{i=1}^r \alpha_i \), and let \( \{e_j\}_{j=1}^n \) be the standard basis of \( \mathbb{R}^n \). Then,
1. \( H_d(x) = \frac{1}{c} \prod_{i=1}^r \langle \alpha_i, x \rangle^2 \),
2. \( H_{d-1}(x) = \frac{1}{c} \sum_{j=1}^r 2 \langle \alpha_j, x \rangle \langle \alpha_j, \rho \rangle \langle \alpha_1, x \rangle^2 \ldots \langle \alpha_j, x \rangle^2 \ldots \langle \alpha_r, x \rangle^2 \),
3. \( \nabla H_d(x) = \frac{1}{c} \sum_{j=1}^n \left( \sum_{i=1}^n 2 \langle \alpha_i, x \rangle \langle \alpha_i, e_j \rangle \langle \alpha_1, x \rangle^2 \ldots \langle \alpha_j, x \rangle^2 \ldots \langle \alpha_r, x \rangle^2 \right) e_j \),
4. \( \langle \nabla H_d(x), \rho \rangle = H_{d-1}(x) \),
5. \( \langle \nabla H_d(x), x \rangle = 2r H_d(x) \), and
6. for any smooth function \( f : P \to \mathbb{R} \),
   \[ \text{div}((x - 2\rho)f H_d) = \langle \nabla f, x - 2\rho \rangle H_d + (2r + n)f H_d - 2f H_{d-1}. \]
Proof. Let $E_x$ be an irreducible representation with highest weight $x$. To prove 1. and 2., we make use of the Weyl dimension formula

$$\dim(E_x) = \frac{\prod_{i=1}^{r} \langle \alpha_i, x + \rho \rangle}{\prod_{i=1}^{r} \langle \alpha_i, \rho \rangle}.$$ 

From the expression

$$\dim(E_x)^2 = \frac{1}{c} \prod_{i=1}^{r} (\langle \alpha_i, x \rangle^2 + 2 \langle \alpha_i, x \rangle \langle \alpha_i, \rho \rangle + \langle \alpha_i, \rho \rangle^2),$$

it follows that if $d$ is the highest degree homogeneous part of the polynomial $\dim(E_x)^2$, then

$$H_d(x) = \frac{1}{c} \prod_{i=1}^{r} \langle \alpha_i, x \rangle^2,$$

and the $(d - 1)$–degree homogeneous part of $\dim(E_x)^2$ is

$$H_{d-1}(x) = \frac{1}{c} \sum_{j=1}^{r} 2 \langle \alpha_j, x \rangle \langle \alpha_j, \rho \rangle \langle \alpha_1, x \rangle^2 \ldots \langle \alpha_j, x \rangle^2 \ldots \langle \alpha_r, x \rangle^2.$$

For 3., note that $\frac{\partial}{\partial x_j} \langle \alpha_i, x \rangle = \langle \alpha_i, e_j \rangle$ so that

$$\frac{\partial}{\partial x_j} H_d(x) = \frac{1}{c} \sum_{i=1}^{r} 2 \langle \alpha_i, x \rangle \langle \alpha_i, e_j \rangle \langle \alpha_1, x \rangle^2 \ldots \langle \alpha_j, x \rangle^2 \ldots \langle \alpha_r, x \rangle^2,$$

and hence

$$\nabla H_d(x) = \sum_{j=1}^{r} \frac{\partial}{\partial x_j} H_d(x)e_j = \frac{1}{c} \sum_{j=1}^{n} \left( \sum_{i=1}^{r} 2 \langle \alpha_i, x \rangle \langle \alpha_i, e_j \rangle \langle \alpha_1, x \rangle^2 \ldots \langle \alpha_j, x \rangle^2 \ldots \langle \alpha_r, x \rangle^2 \right)e_j.$$ 

For 4., notice that
\[ \nabla H_d(x) = \frac{1}{c} \sum_{j=1}^{n} \left( \frac{1}{c} \sum_{i=1}^{n} 2 \langle \alpha_i, x \rangle \langle \alpha_i, e_j \rangle \langle \alpha_1, x \rangle^2 \ldots \langle \alpha_j, x \rangle^2 \ldots \langle \alpha_r, x \rangle^2 e_j \right) \]

\[ = \sum_{i=1}^{r} \left( \frac{1}{c} 2 \langle \alpha_i, x \rangle \langle \alpha_1, x \rangle^2 \ldots \langle \alpha_i, x \rangle^2 \ldots \langle \alpha_r, x \rangle^2 \right) \sum_{j=1}^{n} \langle \alpha_i, e_j \rangle e_j \]

\[ = \sum_{i=1}^{r} \left( \frac{1}{c} 2 \langle \alpha_i, x \rangle \langle \alpha_1, x \rangle^2 \ldots \langle \alpha_i, x \rangle^2 \ldots \langle \alpha_r, x \rangle^2 \right) \alpha_i \]

and then

\[ \langle \nabla H_d(x), \rho \rangle = \sum_{i=1}^{r} \frac{1}{c} 2 \langle \alpha_i, x \rangle \langle \alpha_i, \rho \rangle \langle \alpha_1, x \rangle^2 \ldots \langle \alpha_i, x \rangle^2 \ldots \langle \alpha_r, x \rangle^2 = H_{d-1}(x). \]

For 5., observe that since

\[ \nabla H_d(x) = \sum_{i=1}^{r} \left( \frac{1}{c} 2 \langle \alpha_i, x \rangle \langle \alpha_1, x \rangle^2 \ldots \langle \alpha_i, x \rangle^2 \ldots \langle \alpha_r, x \rangle^2 \right) \alpha_i, \]

and since for each \( i, \)

\[ \left\langle \frac{1}{c} 2 \langle \alpha_i, x \rangle \langle \alpha_1, x \rangle^2 \ldots \langle \alpha_i, x \rangle^2 \ldots \langle \alpha_r, x \rangle^2 \alpha_i, x \right\rangle = 2 \left( \frac{1}{c} \langle \alpha_1, x \rangle^2 \ldots \ldots \langle \alpha_r, x \rangle^2 \right), \]

indeed we have that

\[ \langle \nabla H_d(x), x \rangle = 2r \left( \frac{1}{c} \prod_{i=1}^{r} \langle \alpha_i, x \rangle^2 \right) = 2r H_d(x). \]

The above identities now imply the last point. Namely,

\[ \text{div}((x - 2\rho)f H_d) = \langle \nabla (f H_d), x - 2\rho \rangle + \text{div}(x - 2\rho)f H_d \]

\[ = \langle \nabla f, x - 2\rho \rangle H_d + \langle \nabla H_d, x - 2\rho \rangle f + nf H_d \]

\[ = \langle \nabla f, x - 2\rho \rangle H_d + \langle \nabla H_d, x \rangle f - 2\langle \nabla H_d, \rho \rangle f + nf H_d \]

\[ = \langle \nabla f, x - 2\rho \rangle H_d + (2r + n)f H_d - 2f H_{d-1}. \]
Proof of Theorem [2]. Suppose that $\partial P^+$ has $k$ codimension one faces $\partial P_i^+$. Let $\{\partial P_i^+ : i = 1, \ldots, m\}$ be the set of all codimension one faces of $P^+$ that do not intersect the boundary of the positive Weyl chamber. Suppose that $\partial P_i^+$ is defined by $\langle a_i, x \rangle - c_i = 0$ and set $f_i(x) := \langle a_i, x \rangle - c_i$. The (inward) unit normal vector field to $\partial P_i^+$ is $-\nabla f_i \parallel \nabla f_i \parallel = -\frac{a_i}{\parallel a_i \parallel}$. Since $P$ satisfies the Fano condition, for $x \in \partial P_i^+$, we have that

$$\langle (x - 2\rho) f_H d, -\frac{a_i}{\parallel a_i \parallel} \rangle = -H_d f \left( \langle x - 2\rho, \frac{a_i}{\parallel a_i \parallel} \rangle \right)$$

$$= -\frac{H_d(x) f}{\parallel a_i \parallel} \left( \langle x, a_i \rangle - \langle 2\rho, a_i \rangle \right)$$

$$= \frac{H_d(x) f}{\parallel a_i \parallel} \left( 2\rho, a_i \rangle - c_i \right)$$

$$= \frac{H_d(x) f}{\parallel a_i \parallel}.$$

The divergence theorem implies that

$$\int_{P^+} \text{div}(\langle (x - 2\rho) f_H d, -\frac{a_i}{\parallel a_i \parallel} \rangle) d\mu = \sum_{i=1}^{m} \int_{\partial P_i^+} H_d f \frac{f}{\parallel a_i \parallel} d\sigma_i + \sum_{i=m+1}^{k} \int_{\partial P_i^+} \langle (x - 2\rho) f_H d, -\frac{a_i}{\parallel a_i \parallel} \rangle d\sigma_i,$$

where $d\sigma_i$ is the standard Lebesgue measure on $\partial P$ with domain restricted to $\partial P_i$. When $i = m + 1, \ldots, k$, $\partial P_i^+$ is in the boundary of the positive Weyl chamber, and

$$\int_{\partial P_i^+} \langle (x - 2\rho) f_H d, \frac{a_i}{\parallel a_i \parallel} \rangle d\sigma_i = 0.$$

Then

$$\int_{P^+} \text{div}(\langle (x - 2\rho) f_H d \rangle) d\mu = \sum_{i=1}^{m} \int_{\partial P_i^+} H_d(x) f \frac{d\sigma_i}{\parallel a_i \parallel}$$

and the right hand side is the definition of

$$\int_{\partial P^+} f H_d d\sigma.$$

By 6. of Lemma [11] taking $f = 1$, we obtain that

$$\text{div}(\langle (x - 2\rho) H_d \rangle) = (2r + n)H_d - 2H_{d-1}.$$
Then, by the divergence theorem,
\[
\int_{\partial P^+} H_d d\sigma = (2r + n) \int_{P^+} H_d d\mu - 2 \int_{P^+} H_{d-1} d\mu.
\]
Hence,
\[
a = \frac{\int_{\partial P^+} H_d d\sigma + 2 \int_{P^+} H_{d-1} d\mu}{\int_{P^+} H_d d\mu} = 2r + n.
\]

Upon substituting the above calculations into Alexeev's and Katzarkov's DF-invariant (cf. Theorem 4), again using 6. of Lemma 1 to rewrite the first integral, we find that
\[
-F_1(f) = \frac{1}{2} \int_{P^+} H_d d\mu \int_{P^+} \langle \nabla f, x - 2\rho \rangle H_d d\mu.
\]
Suppose that \( f \) on \( P^+ \) is given as \( f(x) = \sum_{j=1}^n b_j x_j + k \). Put \( b = (b_1, \ldots, b_n) \), \( x = (x_1, \ldots, x_n) \) and \( 2\rho = (2\rho_1, \ldots, 2\rho_n) \), and let \( e_j \) be the \( j \)-th standard basis vector of \( \mathbb{R}^n \). Then, \( \langle \nabla f, x - 2\rho \rangle = \sum_{j=1}^n b_j (x_j - 2\rho_j) \) and it follows that
\[
-F_1(f) = \frac{1}{2 Vol_{DH}(P^+)} \int_{P^+} \langle \nabla f, x - 2\rho \rangle H_d d\mu
\]
\[
= \frac{1}{2 Vol_{DH}(P^+)} \left( \sum_{j=1}^n b_j \int_{P^+} x_j H_d d\mu - \sum_{j=1}^n b_j (2\rho_j) Vol_{DH}(P^+) \right)
\]
\[
= \frac{1}{2} \left( \langle bar_{DH}(P^+), \sum_{j=1}^n b_j e_j \rangle - \sum_{j=1}^n b_j (2\rho_j) \right)
\]
\[
= \frac{1}{2} \left( \langle bar_{DH}(P^+), b \rangle - \langle b, 2\rho \rangle \right)
\]
\[
= \frac{1}{2} (\langle bar_{DH}(P^+), 2\rho \rangle, \nabla f).
\]

4 **K-stability as a combinatorial criterion**

Next we show how to use Theorem 1 alongside Theorem 2 to verify Del-Croix's barycentric criterion (Theorem 5).
From the works [7] or [19], it follows that all relevant test-configurations are defined by affine linear functions on $P^+ \subset \mathbb{R}^n$. So when $(X_P, K_P^{-1})$ is $K$-stable, $-F_1(f) \geq 0$ for any linear function $f : P^+ \to \mathbb{R}$. In particular, for the coordinate function $x^i(x) := \langle x, e_i \rangle$, where $e_i$ is the $i$-th standard basis vector of $\mathbb{R}^n$, $-F_1(x^i) \geq 0$. Then since $\langle \nabla x^i, x^i - 2\rho \rangle = x^i(x) - x^i(2\rho)$, it follows that

$$-F_1(x^i) = \frac{1}{2Vol_{DH}(P^+)} \int_{P^+} (x^i(x) - x^i(2\rho))H_d d\mu.$$

So $-F_1(x^i) \geq 0$ implies

$$\frac{1}{Vol_{DH}(P^+)} \int_{P^+} x^i(x)H_d d\mu \geq \frac{x^i(2\rho)}{Vol_{DH}(P^+)} \int_{P^+} H_d d\mu$$

or more succinctly, $x^i(b) \geq x^i(2\rho)$, where $b$ is the barycenter of $P^+$ w.r.t. the DH measure.

Conversely, suppose that $x^i(b) \geq x^i(2\rho)$ so that $-F_1(x^i) \geq 0$. Take any affine linear function $f(x) = \sum_{i=1}^n a_ix^i + b$ on $P^+$, and note that in order to extend $f$ in a $W$-invariant way so that it is a convex function on $P$, we need that $a_i \geq 0$ for all $i$. Then since

$$\langle \nabla f, x^i(2\rho) \rangle = \sum_{i=1}^n a_i(x^i(x) - x^i(2\rho)),$$

$$-F_1(f) = \sum_{i=1}^n a_i(-F_1(x^i)) \geq 0,$$

and so the anti-canonical polarization of $X_P$ is K-stable.

By Theorem 1, the condition $b \in 2\rho + \Theta$, where $\Theta$ is the relative interior of the cone $C(\Phi^+)$, is equivalent to the existence of a KE metric on $X_P$.

**Theorem 5.** (Theorem 1.5 [9], Theorem A [10]) Let $X$ be a smooth Fano compactification of a reductive group $G$ with associated polytope $P$. Let $\Phi$ be the root system of a pair $(G, T)$. The barycenter of $P^+$ w.r.t. the DH measure is $b := \frac{1}{\int_{P^+} H_d d\mu} \int_{P^+} xH_d d\mu$. There exists a KE metric on $X$ iff $b \in 2\rho + \Theta$.

Theorem 5 has been generalized to smooth Fano spherical varieties (cf. Theorem A [11]).

Let us now briefly revisit Fano KE toric varieties as a special case of reductive ones. As pointed out in [2], in the toric case, $H_d = 1, H_{d-1} = 0,
$P^+ = P$, and $2\rho = 0$. The defining functions of each codimension 1 face $\partial P_i$ of $P$ are of the form $f_i(x) = \langle a_i, x \rangle + 1$. Assuming that the polytope has $m$ boundary faces,

$$\sigma(\partial P) := \sum_{i=1}^{m} \int_{\partial P_i} \langle -a_i, x \rangle \frac{d\sigma_i}{\|a_i\|} = \int_P \text{div}(x) d\mu = n\mu(P).$$

Hence $n = \frac{\sigma(\partial P)}{\mu(P)}$, and since $\text{div}(xf) = \langle x, \nabla f \rangle + nf$, we obtain

$$-F_1(f) = \frac{1}{2\mu(P)} \int_P \langle \nabla f, x \rangle d\mu - n \int_P f d\mu = \frac{1}{2\mu(P)} \int_P \langle x, \nabla f \rangle d\mu.$$

Observe that for $X_P$ a Fano toric manifold, since $P = P^+$, the $K$–stability of $(X_P, K^{-1}_X)$ is equivalent to the non-negativity of the DF-invariant of affine linear functions on $P$. But then we must have that for any affine linear function $f(x) = \sum_{i=1}^{n} -a_i x^i + b$ on $P$, both $-F_1(f) \geq 0$ and $-F_1(-f) \geq 0$, which holds iff $-F_1(f) = 0$. Denote the barycenter of $P$ by $b$. Then,

$$-F_1(f) = \frac{1}{2\mu(P)} \int_P \langle \nabla f, x \rangle d\mu = \frac{1}{2} \sum_{i=1}^{n} \frac{a_i}{\mu(P)} \int_P x^i d\mu = 0$$

iff $x^i(b) = \frac{1}{\mu(P)} \int_P x^i d\mu = 0$ for all $i$. Thus, we have recovered Wang’s and Zhu’s barycenter characterization of Fano KE toric varieties [28] (see also [20]).

**Corollary 1.** Let $X$ be a smooth Fano toric variety associated to a polytope $P$, and let $f$ be an affine linear function on $P$. The DF-invariant of the associated test-configuration is given by

$$-F_1(f) = \frac{1}{2\mu(P)} \int_P \langle \nabla f, x \rangle d\mu.$$

Moreover, $X$ is KE iff the barycenter of $P$ is the origin.
5 An application: real Einstein submanifolds

In this section, we explain one way to obtain examples of real Einstein manifolds from K-stable anti-canonically polarized Fano manifolds. We begin with a brief review of the real geometry that will be needed here.

5.1 Totally geodesic submanifolds and real structures

Let \((M,g)\) be a Riemannian manifold and \((V,g|_V)\) be a connected submanifold. Recall that \((V,g|_V)\) is totally geodesic if any geodesic in \((V,g|_V)\) is a geodesic in \((M,g)\). For example, geodesics are always 1-dimensional totally geodesic submanifolds.

The second fundamental form of \(V\) is the difference
\[
II(X,Y) := \nabla^M_X Y - \nabla^V_X Y,
\]
where \(\nabla^M\) and \(\nabla^V\) are the Levi-Civita connections of \(g\), respectively \(g|_V\). In the sequel, we will use the following fact, a proof of which can be found in [16] or [17].

**Proposition 1.** The submanifold \((V,g|_V)\) of \((M,g)\) is totally geodesic iff \(II = 0\).

Suppose that \((V,g|_V)\) is totally geodesic in \((M,g)\). Thanks to the Gauss equation [18], we have that for any \(X,Y,Z,W \in T_{p,V}\),
\[
R^M_{\nabla^M}(X,Y,Z,W) = R^V_{\nabla^V}(X,Y,Z,W),
\]
where \(R^M\) is the Riemann curvature tensor of \(M\) and \(R^V\) is that of \(V\). So in particular, if \((M,g)\) is Einstein, \((V,g|_V)\) is Einstein as well. For the following theorem, see for instance, [16].

**Theorem 6.** Each connected component of the fixed point set of an isometry of \((M,g)\) is a totally geodesic submanifold.

Recall that if \((X,J)\) is a complex manifold and \(\sigma : X \to X\) is a smooth map, then \(\sigma\) is anti-holomorphic iff \(\sigma J = -J\sigma\). Moreover, a bi-anti-holomorphic map \(\sigma\) satisfies \(-J = \sigma^* J := \sigma^{-1} J\sigma\).

**Definition 1.** Let \((X,J)\) be a complex manifold and \(\sigma : X \to X\) be an anti-holomorphic map that is involutive \((\sigma^2 = Id_X)\). The real part of \(X\) w.r.t. \(\sigma\) is the fixed point set of \(\sigma\). Any such pair \(((X,J),\sigma)\) of complex manifold and anti-holomorphic involution is often called a real structure.

Note that the fixed point set of a map \(\sigma\) as in Definition 1 could be empty, however as soon as it is non-empty, we know that it will be a real submanifold of dimension equaling that of \(X\). It should be noted that the
real part could have more than one connected component. More information on real structures can be found in [6].

Remark 1. Let $\omega$ be a Kähler metric, which has the local form $\omega = i g_{jk} dz^j \wedge d\bar{z}^k$. By the Riemannian metric associated to $\omega$, we mean the hermitian metric $g = g_{jk}(dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j)$. We will sometimes denote this metric by $g_\omega$. Also, the phrase any real part is to be understood as the real part w.r.t. any anti-holomorphic involution.

5.1.1 Basic properties of anti-holomorphic maps

For the terminology, notation and complex geometry concepts that we make use of here, see [15].

Lemma 2. Let $X$ be an $n$-dimensional complex manifold, and $f : X \to X$ be an anti-holomorphic map. Then, $\partial f^* = f^* \bar{\partial}$ and $\bar{\partial} f^* = f^* \partial$.

Proof. Let $p \in X$, $(U_i, \phi_i)$ be a chart containing $p$, $(U_j, \phi_j)$ be a chart containing $f(p)$, and suppose that $f(U_i) \subseteq U_j$. Write $f_k$ for the $k$-th component of $\phi_j \circ f \circ \phi_i^{-1}$, which is a function of $(z^1, \ldots, z^n) \in \phi_i(U_i) \subset \mathbb{C}^n$. Interpreting partial derivatives in the adequate sense, we have that

$$f^*(dz^k) = \frac{\partial f_k}{\partial z_m} dz^m + \frac{\partial f_k}{\partial \bar{z}_m} d\bar{z}^m = \frac{\partial f_k}{\partial \bar{z}_m} d\bar{z}^m,$$

since $f_k$, as the composition of the holomorphic coordinate function $z^k$ and the anti-holomorphic function $f \circ \phi_i^{-1}$, is anti-holomorphic. Similarly,

$$f^*(d\bar{z}^k) = \frac{\partial \bar{f}_k}{\partial z_m} dz^m,$$

since $\bar{f}_k$ is anti-holomorphic and hence $\bar{f}_k$ is holomorphic. These calculations imply that the pull-back $f^*$ induces type-reversing maps $A^{p,q}(X) \to A^{q,p}(X)$.

Let $\Pi^{p,q} : A^k(X) = \bigoplus_{r+s=k} A^{r,s}(X) \to A^{p,q}(X)$ be the projection operator so that $\partial = \Pi^{p+1,q} \circ d$ and $\bar{\partial} = \Pi^{p,q+1} \circ d$, where $d : A^{p,q}(X) \to A^{p+1,q}(X) \oplus A^{p,q+1}(X)$ is the $\mathbb{C}$-linear extension of the exterior derivative. Let $\beta = \sum_{r+s=k} b_{i_1, \ldots, i_r} dz^{i_1} \wedge \cdots \wedge dz^{i_r} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_s} \in A^k(X)$. Then,
\[ f^*(\Sigma^{pq}) = f^*(b_{i_p,j_q} dz^{j_1} \wedge \cdots \wedge dz^{j_q} \wedge \bar{d}z^{i_1} \wedge \cdots \wedge \bar{d}z^{i_q}) \]
\[ = (b_{i_p,j_q} \circ f)(f^* dz^{j_1}) \wedge \cdots (f^* dz^{j_q}) \wedge (f^* d\bar{z}^{i_1}) \wedge \cdots (f^* d\bar{z}^{i_q}) \]
\[ = \Sigma^{pq} \left( \sum_{r+s=k} (b_{i_r,j_s} \circ f)(f^* dz^{j_1}) \wedge \cdots (f^* dz^{j_r}) \wedge (f^* d\bar{z}^{i_1}) \wedge \cdots (f^* d\bar{z}^{i_s}) \right) \]
\[ = \Sigma^{pq}(f^* \beta). \]

Therefore, for any \( \gamma \in A^{pq}(X) \),
\[ \partial(f^* \gamma) = (\Sigma^{p+1,q} \circ d)(f^* \gamma) \]
\[ = (\Sigma^{p+1,q} \circ f^* d\gamma \]
\[ = f^*(\Pi^{p,q+1} \circ d\gamma \]
\[ = f^*(\bar{\partial} \gamma). \]

A similar computation shows that also \( \bar{\partial}(f^* \gamma) = f^*(\partial \gamma). \)

**Lemma 3.** Let \((X, \omega)\) be a Kähler manifold, and \(V \subseteq X\) be a complex submanifold. Suppose that induced Kähler form \(\omega|_V\) is given in terms of a Kähler potential \(\psi\), and that there is an anti-holomorphic map \(f : X \to X\) that satisfies \(f^* \psi = \psi\). Then, \(f\) is an anti-isometry of \((V, \omega|_V)\) and an isometry of \((V, g|_V)\), where \(g\) is the Riemannian metric associated to \(\omega\).

**Proof.** By Lemma 2 it follows that
\[ f^* \omega = i f^* (\partial \bar{\partial} \psi) \]
\[ = i \bar{\partial} \partial f^* \psi \]
\[ = -i \partial \bar{\partial} \psi \]
\[ = -\omega \]
on \(V\). Let \(Y\) and \(Z\) be vector fields on \(V\). The second claim follows from the calculation.
\[ f^*g(Y,Z) = g(f,Y,f,Z) \]
\[ = \omega(f,Y,Jf,Z) \]
\[ = -\omega(f,Y,f,JZ) \]
\[ = -f^*\omega(Y,JZ) \]
\[ = \omega(Y,JZ) \]
\[ = g(Y,Z). \]

5.1.2 Examples of totally geodesic real parts of Kähler manifolds

Example 3. Consider the Fubini-Study form \( \omega_{FS} \) on \( \mathbb{C}P^n \). Recall that to any point \([z_0 : \cdots : z_n]\) in the open subset \( U_i = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n|z_i \neq 0\} \) corresponds the coordinate \((\frac{z_0}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_n}{z_i}) \) in \( \mathbb{C}^n \). Put \( w_1 = \frac{z_0}{z_i}, \ldots, w_n = \frac{z_n}{z_i} \).

Then, \( \omega_{FS} = i\partial\bar{\partial}\log(1 + \sum_{j=1}^n |w_j|^2) \). The associated Riemannian metric is
\[
g = \left(\frac{1 + \sum_{j=1}^n |w_j|^2}{1 + \sum_{j=1}^n |w_j|^2}\right)(dz^i \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^i).
\]

Consider the standard complex conjugation \( \sigma : \mathbb{C}P^n \to \mathbb{C}P^n \), defined by \( \sigma([z_0 : \cdots : z_n]) = [\bar{z}_0 : \cdots : \bar{z}_n] \). The map \( \sigma \) is anti-holomorphic, and in fact, \((\mathbb{C}P^n, \sigma)\) is a real structure. Moreover,
\[
\sigma^*\left( \log(1 + \sum_{j=1}^n |w_j|^2) \right) = \log(1 + \sum_{j=1}^n |\bar{w}_j|^2) = \log(1 + \sum_{j=1}^n |w_j|^2).
\]
So by Lemma 3, \( \sigma^*g = g \) on each Kähler submanifold \( U_i \subseteq \mathbb{C}P^n \), and this is precisely what it means for \( \sigma \) to be an isometry of \((\mathbb{C}P^n, g)\).

The real part of \( \mathbb{C}P^n \) w.r.t. \( \sigma \) is \( \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n|\text{Im}(z_i) = 0 \text{ for all } i\} = \mathbb{R}P^n \). By Theorem 6, it follows that \((\mathbb{R}P^n, g|_{\mathbb{R}P^n})\) is totally geodesic.

It is worth mentioning that \( \mathbb{R}P^n \) is one of the simplest examples of a real toric variety, as defined in [24]. Moreover, \( \mathbb{R}P^n \) can be turned into a small cover of the standard \( n \)-simplex by taking the real torus action to be the induced subgroup action of \((Z_2)^n \subseteq (\mathbb{R}^*)^n \subseteq (\mathbb{C}^*)^n \). Small covers were introduced in [8].
Example 4. Let $f_1,\ldots,f_k \in \mathbb{C}[z_0,\ldots,z_n]$ be homogeneous polynomials with real coefficients; i.e. $f_i = \sum_{j=1}^{n+1} q_j z_0^{a_{ij}^0} \cdots z_n^{a_{ij}^n}$, where $\sum_{k=0}^{n} \alpha^{ij}_k$ is a constant and $q_j \in \mathbb{R}$. If $V := \mathbb{V}(f_1,\ldots,f_k) \subseteq \mathbb{C}\mathbb{P}^n$ is smooth, then $(V,\omega_{\mathbb{F}_S}|_V)$ is a Kähler manifold, and the restricted complex conjugation map $\sigma|_V$ is an isometry of $(V,g|_V)$ (cf. Example 3). Note that the fixed point set of $\sigma|_V$ is the real part $V \cap \mathbb{R}\mathbb{P}^n$.

Here is a specific instance of this construction: take $f = \sum_{j=1}^{n+1} z_j^2 - z_0^2$, and consider the variety $V := \mathbb{V}(f) \subset \mathbb{C}\mathbb{P}^{n+1}$. The fixed point set of $\sigma|_V$ is $V \cap \mathbb{R}\mathbb{P}^{n+1} \simeq S^n$, which must then be a totally geodesic submanifold of $(V,g|_V)$.

We conclude this section with an observation about toric varieties, and their totally geodesic subvarieties (Proposition 2). Note that we are using the terminology from Remark 1 in the statement of the proposition.

Lemma 4. Let $T$ and $T'$ be topological spaces and assume that $T'$ is Hausdorff. Let $f,g : T \to T'$ be continuous maps and $U \subseteq T$ be a dense subset. If $f|_U = g|_U$, then $f = g$ on $T$.

Proof. Let $C = \{x \in T | f(x) = g(x)\} \subseteq T$. Define a function $F : T \to T' \times T'$ by $F(x) = (f(x),g(x))$, which is continuous. Since $T'$ is Hausdorff, $T' \times T'$ and the subspace $F(T)$ are Hausdorff as well. Hence, the diagonal subspace $\Delta = \{(a,a) | a \in \mathbb{R}\} \subset F(T)$ is closed. But since $C = F^{-1}(\Delta)$, $C$ is a closed subspace of $T$. By definition, the closure $cl(U)$ is the smallest closed subset of $T$ that contains $U$, and clearly $U \subseteq C$. Therefore, $T = cl(U) \subseteq C$. And then, $T = C$ and $f = g$ on $T$. \qed

Proposition 2. Any real part of a smooth projective toric variety $(X,\omega)$ is a totally geodesic submanifold of $(X,\omega|_V)$.

Proof. Let $X$ be a smooth projective toric variety of dimension $n$. Let $\omega$ be a $(S^1)^n$–invariant Kähler form on $X$. Indeed, if $T^n_C \subseteq X$ denotes the open dense $(\mathbb{C}^*)^n$–orbit, then $\omega|_{T_C^n} = i\partial \bar{\partial} f$, for a convex real-valued function $f$ of the real variables $x_i = \log|z_i|^2$, where the $z_i$ are the standard holomorphic coordinates on $(\mathbb{C}^*)^n$. \qed
Consider an anti-holomorphic involution $\sigma : X \to X$ such that the restriction $\sigma|_{T^n_X}$ is complex conjugation. Then, $\sigma|_{T^n_X}^* f = f$, and so by Lemma 3 $\sigma|_{T^n_X}$ is an anti-isometry of $(T^n_X, \omega|_{T^n_X})$ and an isometry of $(T^n_X, g|_{T^n_X})$.

We can think of both $\sigma^* g_\omega$ and $g_\omega$ as continuous functions from the toric manifold $X$ into the space $B$ of all symmetric bilinear forms on each tangent space of $X$, which is a manifold, hence is Hausdorff. Applying Lemma 4 to $f = \sigma^* g_\omega$, $g = g_\omega$, $T = X$, $T' = B$, and $U = T^n_X$, which recall is open and dense in $X$, we conclude that $\sigma$ is an isometry of $(X, g_\omega)$. The claim now follows from Theorem 6.

5.2 Real Einstein manifolds from K-stable Kähler manifolds

Let us make some preliminary observations that will lead to the proof of Theorem 3. The assumption that $\sigma : (X, J) \to (X, J)$ is bi-anti-holomorphic is there to ensure that the pullback of the complex structure $J$ is defined, and also to prevent $\sigma$ from pulling back Kähler forms to degenerate forms.

**Lemma 5.** Let $(X, J)$ be a complex manifold and $\sigma : X \to X$ be a bi-anti-holomorphic map. If $\omega \in 2\pi c_1(X)$, then $-\sigma^* \omega \in 2\pi c_1(X)$.

**Proof.** Since we are assuming that our choice of manifold $X$ is fixed, the first Chern class of $X$ depends on the complex structure only. Thus, we write $c_1(J)$ in place of $c_1(X) := c_1(T_X)$. Now, simply note that

$$[-\sigma^* \omega] = -\sigma^* [\omega]$$
$$= -\sigma^* (2\pi c_1(J))$$
$$= -2\pi c_1(\sigma^* J)$$
$$= -2\pi c_1(-J)$$
$$= [\omega].$$

Here we have used the naturality of Chern classes to go from the second to the third equality, and the identity $c_k(-J) = (-1)^k c_1(J)$ (see Lemma 14.9 in [22]) to deduce the last line.

**Lemma 6.** Let $(X, J)$ be a complex manifold, $\omega$ be a KE form on $X$, and $\sigma : X \to X$ be a bi-anti-holomorphic map. Then $-\sigma^* \omega$ is a KE form on $X$. 

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Proof. Let us write $g$ for the Riemannian metric associated to $\omega$. Observe that

\begin{align*}
\sigma^* g(Y, Z) &= g(\sigma Y, \sigma Z) \\
&= \omega(\sigma Y, J\sigma Z) \\
&= -\omega(\sigma Y, \sigma JZ) \\
&= -\sigma^* \omega(Y, JZ)
\end{align*}

and that since locally we have that $\omega = ig_{ij}dz^i \wedge d\bar{z}^j$, that $\sigma$ is anti-holomorphic implies that $\sigma^* \omega = i(\sigma^* g_{ij})dz^i \wedge d\bar{z}^j$. Using Lemma 2, we compute

\[ \sigma^* \text{Ric}(\omega) = \sigma^*(-i\partial \bar{\partial} \log \det(g)) = -i\partial \bar{\partial} \log \det(-\sigma^* g) = \text{Ric}(\sigma^* \omega). \]

It follows that $-\sigma^* \omega = -\sigma^* \text{Ric}(\omega) = \text{Ric}(-\sigma^* \omega).$ \hfill \square

Proof of Theorem 3. Lemmas 5 and 6 imply that $-f^* \omega \in 2\pi c_1(X)$ is another KE form on $X$. But Theorem 1 forces $-f^* \omega = \omega$, proving that $f$ is an anti-isometry of $(X, \omega)$. That $f$ is an isometry of $(X, g_\omega)$ practically follows from the second calculation in the proof of Lemma 3.

By Theorem 6, the fixed point set of $f$ must be Einstein. Specifically when $f$ is an involution, the real part of $X$ w.r.t. $f$ is Einstein. \hfill \square

Concrete examples of real Einstein manifolds that are real parts of K-stable smooth Fano varieties can therefore be obtained by means of Theorem 3 and the polytope classification from section 4.

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