Formation of fractal structure in many-body systems with attractive power-law potentials

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We study the formation of fractal structure in one-dimensional many-body systems with attractive power-law potentials. Numerical analysis shows that the range of the index of the power for which fractal structure emerges is limited. Dependence of the growth rate on wavenumber and power-index is obtained by linear analysis of the collisionless Boltzmann equation, which supports the numerical results.

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I. INTRODUCTION

Formation of spatial structures is an interesting and important phenomenon in nature. It is seen over a wide range, from protein folding in biological systems [1, 2] to large-scale structure in the universe [3]. The theoretical origins of such structures are quite important and will be classified into several classes. One of the most interesting areas within the field of dynamical systems is that some remarkable structure and organization is created dynamically by the mutual interaction among the elements [4, 5].

Recently we have discovered that spatial structure with fractal distribution emerges spontaneously from uniformly random initial conditions in a one-dimensional self-gravitating system, that is the sheet model [6]. What is noteworthy in this phenomenon is that the
spatial structure is not given at the initial condition, but dynamically created from a state without spatial correlation. Succeeding research clarified that the structure is created first in small spatial scale then grows up to large scale through hierarchical clustering \cite{7}, and the structure is transient \cite{8}. It is quite interesting that some remarkable spatial structures are emerged instead of monotonous thermal relaxation in Hamiltonian system.

The emergence of fractal structure is a typical example that systems of many degrees of freedom are self-organized by dynamics themselves. Hence to clarify its dynamical mechanism is very important subject toward understanding physics of self-organization of matter.

A way to clarify the dynamical mechanism is to know which class of pair potential can form the fractal structure. Here we note an important fact that fractal structure does not have characteristic spatial scale, nor does the potential of the sheet model, since the pair potential is power of the distance. Hence the scale free property of potential may be a keystone to understand the dynamical mechanism. The question is if the fractal structure can be formed in not only the sheet model, but also other systems with power-law potentials.

In this paper we study the possibility that the fractal structure can be formed in more general systems without characteristic spatial scale which is extended from sheet model. Here we adopt the model as the system with attractive power-law potentials. At first we examine the formation of the fractal structure by numerical simulation for various values of the power index of the potential. Next we perform linear analysis to consider the numerical results.

In Sec.\textbf{II} we introduce many-body systems with power-law potentials. In Sec.\textbf{III} we review the formation of power-law correlation in the sheet model. In Sec.\textbf{IV} we carry out numerical simulation. In Sec.\textbf{V} we analyze linear perturbation of the collision-less Boltzmann equation. The final section is devoted to summary and discussions.
II. MODEL

We consider the model where many particles with an uniform mass interact with purely attractive pair potential of power-law, which is described by the Hamiltonian \[9\]

\[ H = K + U = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i=1}^{N-1} \sum_{j>i}^{N} |x_i - x_j|^\alpha, \]  

(1)

where \(x_i\) and \(p_i\) are the position and momentum a particle, respectively. The first term is kinetic energy and the second term is potential energy. For simplicity in this paper we consider the system where motion of particles is bounded to one-dimensional direction.

For the special case \(\alpha = 1\), the Hamiltonian (1) applies to a system of \(N\) infinite parallel mass sheets, where each sheet extends over a plane parallel to the \(yz\) plane and moves along the \(x\) axis under the mutual gravitational attraction of all the other sheets. The Hamiltonian of the sheet model \([10, 11, 12, 13]\) is usually written in the form

\[ H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i=1}^{N-1} \sum_{j>i}^{N} |x_i - x_j|. \]  

(2)

In the previous letters \([6, 7, 8]\) we investigated the time evolution of the sheet model (2) to show that fractal structure emerges from non-fractal initial conditions. In the Secs. IV and V we will clarify the relation between the power index of the pair potential \(\alpha\) and the formation of the fractal structure by investigating time evolution of more general model (1) numerically and analytically.

III. FORMATION OF FRACTAL STRUCTURE IN THE SHEET MODEL \((\alpha = 1)\)

Before we investigate the time evolution of the model (1), we briefly review our previous works \([6, 7, 8]\) on the model (2). We found that fractal structure is formed in the sheet model (2) from uniformly-random initial conditions \([6, 7]\). In Figure 1 we show a typical example of such fractal structure and a snapshot of the \(\mu\)-space configuration.
IV. NUMERICAL SIMULATION

In this section we examine by computer simulations if the fractal structure is formed in the systems \([\Pi]\). Here we carry out the case \(\alpha \geq 1\) to avoid divergence when two particles come close.

The non-fractal initial conditions where fractal structure is emerged are characterized typically by those of virial ratio \(2E_{\text{kin}}/E_{\text{pot}} = 0\) in the sheet model \([6]\). (Spatial distribution is set to be random.) These state of zero velocity dispersion corresponds to the limiting case of zero thermal fluctuation, and is called cold-random condition. Therefore we adopt this cold-random initial condition to investigate the time evolution for various value of \(\alpha\).

In our simulation we use 4-th order of the symplectic integrator with a fixed time step \(\Delta t = 2\pi/10^4\) and \(N = 65536\) particles. In what follows we show numerical results for two typical examples: \(\alpha = 1.125\) and \(\alpha = 1.5\). For other values of \(\alpha\) behavior of the systems varies gradually in accordance with the change of \(\alpha\).

A. The case \(\alpha = 1.125\)

At first, we consider the case the interaction force deviates slightly from sheet model. Here we show the numerical results in the case \(\alpha = 1.125\) in Figs.2 and 3. We display particle distribution in \((x, v)\) space (\(\mu\)-space) in Fig.2. In the course of time evolution we see that many whirlpools nest in the \(\mu\)-space to form the hierarchical structure. In Fig.3 we show a box counting dimension of the spatial distribution. We can see that the dimension is \(D \simeq 0.83\) (Fig.3). Therefore our numerical results suggest that the fractal structure is formed. These behaviors are similar qualitatively to the sheet model, \(\alpha = 1\) \([6]\). We find that fractal structure can be formed as well as the sheet model.
B. The case $\alpha = 1.5$

Next, we consider the interaction deviate further from sheet model. Time evolution change qualitatively as $\alpha$ increases. Here we show the numerical results in the case $\alpha = 1.5$. We display particle distribution in $\mu$-space in Fig.4 and in Fig.5 we show a box counting dimension for spatial distribution. Differently from the case $\alpha = 1.125$, a single spiral is rolled up in $\mu$-space. Therefore fractal structure is not formed.

We can summarize these numerical results in this section that the fractal structure can be constructed as well as the sheet model, but range of $\alpha$ for which the fractal structure is created is limited; it can not be constructed for large value of $\alpha$.

V. ANALYSIS OF LINEAR PERTURBATION

In this section we clarify the physical reason analytically why fractal structure can not be constructed for the large value of the power index of the potential in numerical simulation in the previous section. The formation of the fractal structure occurs at the relative early stage in the whole-evolution history[8]. Then it is instructive for understanding the mechanism by which the structure is formed to know the qualitative properties of the short-term behaviors by linear analysis.

In this section we derive the dispersion relation from the collisionless Boltzmann equation (CBE), which describes the growing rate of the linear perturbation [14, 15].

A. Collisionless Boltzmann equation

CBE is defined by

$$\left\{ \frac{\partial}{\partial t} + p \frac{\partial}{\partial x} + \int_{-\infty}^{\infty} dx' F(x-x') \left( \int_{-\infty}^{\infty} dp' f(x', p', t) \right) \frac{\partial}{\partial p} \right\} f(x, p, t) = 0,$$

where $F$ is 2-particle force which is related with a pair-potential $U$ by

$$F(x) = -\frac{dU}{dx},$$

(3)
and $f$ is the one-particle distribution function. For simplicity we consider the system is extended in infinite region $-\infty < x < \infty$. It is clear that the state of uniform spatial density with an arbitrary velocity distribution,

$$ f(x, p, t) = f_0(p), $$

is a stationary state. We impose the following small perturbation over the stationary state

$$ f(x, p, t) = f_0(p) + \delta f(x, p, t). $$

The linearized equation for $\delta f$ is

$$ \left( \frac{\partial}{\partial t} + p \frac{\partial}{\partial x} \right) \delta f(x, p, t) = - \int_{-\infty}^{\infty} dx' F(x - x') \int_{-\infty}^{\infty} dp' \delta f(x', p', t) \frac{\partial}{\partial p} f_0(p). $$

Now we define the Fourier-Laplace transform by that $x$ is Fourier transformed and $t$ is Laplace transformed, that is

$$ \tilde{\delta f}(k, p, \omega) \equiv \int_0^{\infty} dte^{-i\omega t} \int_{-\infty}^{\infty} dx e^{ikx} \delta f(x, p, t). $$

Fourier transform is

$$ \hat{\delta f}(k, p, t) \equiv \int_{-\infty}^{\infty} dx e^{ikx} \delta f(x, p, t), $$

and

$$ \hat{F}(k) \equiv \int_{-\infty}^{\infty} dx e^{ikx} F(x). $$

Then the Fourier-Laplace transformed equation of Eq.(7) is

$$ \varepsilon_k(\omega) \tilde{\delta f}_k(\omega) = \frac{-1}{i(\omega + kp)} \hat{\delta f}(k, p, 0), $$

where $\varepsilon_k(\omega)$ is

$$ \varepsilon_k(\omega) \equiv 1 + \int_{-\infty}^{\infty} dp \frac{\hat{F}(k)}{i(\omega + kp)} \frac{\partial f_0}{\partial p}(p). $$
B. Dispersion relation

From the inverse-Laplace transform of Eq. (11),

$$\delta f_k(t) = \int_{-\infty}^{\infty} e^{-i\omega t} \tilde{f}_k(\omega) \frac{d\omega}{2\pi}$$  \hspace{1cm} (13)

$$= \int_{-\infty}^{\infty} e^{i\omega t} \frac{1}{\varepsilon_k(\omega) i(-\omega + kp)} \delta f(k, p, 0) \frac{d\omega}{2\pi}.$$ \hspace{1cm} (14)

Now we continuously move the integration contour to upper half of complex $\omega$ plane while avoiding the singular points. Then the contributions from except of the pole can be neglected, because of the factor $\exp(i\omega t)$ ($Im(\omega) > 0$).

Then the growth rate of each mode of the fluctuation is obtained by the solution of the equation

$$\varepsilon_k(\omega) = 0.$$ \hspace{1cm} (15)

Eq. (15) is the dispersion relation. If Eq. (15) has the solution where the inequality $Im(\omega) < 0$ is satisfied, the fluctuation is unstable.

C. Dynamical stability of systems with power-law potentials

Now we consider the case that the potential is power-law, the pair-potential $U$ is

$$U(x) = A|x|^\alpha.$$ \hspace{1cm} (16)

Assuming the interaction is attractive, $A > 0$. $\alpha = 1$ for “sheet model”. The Fourier-transformed potential is

$$\hat{U}(k) \equiv \int_{-\infty}^{\infty} dx e^{ikx} A|x|^\alpha$$ \hspace{1cm} (17)

$$= 2A \left\{ - \left( \sin \frac{\alpha \pi}{2} \right) \frac{\Gamma(\alpha + 1)}{|k|^{\alpha+1}} \right\} \quad (\alpha \neq 0, 2, 4, \cdots, -1, -3, \cdots).$$ \hspace{1cm} (18)

For simplicity, we choose the stationary state as

$$f_0 = n_0 \delta(p),$$ \hspace{1cm} (19)

where $n_0$ is the number density of particles.
The dispersion relation is

\[ \varepsilon_k(\omega) = 1 + 2n_0A \left\{ \left( \frac{\sin \frac{\alpha \pi}{2}}{k^{\alpha-1}} \right) \frac{\Gamma(\alpha + 1)}{|k|^{\alpha-1}} \right\} \frac{1}{\omega^2} = 0, \]  

(20)

and \( \omega \) which satisfy the dispersion relation is

\[ \omega^2 = -2n_0A \left\{ \left( \frac{\sin \frac{\alpha \pi}{2}}{k^{\alpha-1}} \right) \frac{\Gamma(\alpha + 1)}{|k|^{\alpha-1}} \right\}. \]  

(21)

When \( \omega^2 < 0 \), the system is unstable. Eq. (21) can be reduced to

\[ \omega = \pm i \sqrt{2n_0A \left\{ \left( \frac{\sin \frac{\alpha \pi}{2}}{k^{\alpha-1}} \right) \frac{\Gamma(\alpha + 1)}{|k|^{\alpha-1}} \right\} \cdot |k|^{(1-\alpha)/2}}. \]  

(22)

From Eq. (22) we can classify the evolution of the perturbation into three types; (i) when \( 0 < \alpha < 1 \), the growing rate increase monotonously for \(|k|\). (ii) when \( \alpha = 1 \), the fluctuations for all scale grows at same rate. (iii) when \( 1 < \alpha < 2 \), the growing rate decrease monotonously for \(|k|\).

The cold-random condition corresponds to the mixed states of fluctuation with all wavelength, so called “white noise”. The larger the value of \( \alpha \) is, the smaller the growth rate of the fluctuation with large wavevector is. This is consistent with the numerical results in Sec. IV that a large whirlpool is formed in \( \mu \) space in the case for the large value of \( \alpha \).

VI. CONCLUDING REMARK

In this paper we have studied structure formation of many-body systems with power-law potentials. Firstly we have investigated structure formation in this model by numerical simulation. As results, we have found that behaviors of time evolution are different depending on the power index of the potential. When \( \alpha \) is slightly above 1, fractal structure is formed similar to the sheet model \[ 6 \]. On the other hand, when \( \alpha \gtrsim 1.5 \), fractal structure is not formed.

In order to explain these numerical results, we have also analyzed linear perturbation of collisionless Boltzmann equation to derive the dispersion relation which represent the growth rate of each mode of the fluctuation. As results, we have found that for large value
of $\alpha$ the growth rate of small scale is suppressed. In addition we can explain the sheet model ($\alpha = 1$) is marginal in the sense all scale fluctuations grow at same rate. There is a slight difference between the initial condition used in the numerical simulation and the stationary state employed in the perturbation, that is, we set $N$ particles in finite region for simulation, whereas the unperturbed state $f_0$ is infinitely spread. Nonetheless we think the linear perturbation grabs the core of the instability, especially for short time scale.

In our simulation the fractal structure is formed through hierarchical clustering. That is, clusters are created first in small spatial scale, then grow to large scale. The spatial randomness in the initial condition and finiteness of the number of particles $N$ imply that relative fluctuation in mass density is large for small spatial scale. This enhancement of initial fluctuation in small spatial scale is probably the “seed” of spatial structure formation.

Fractal properties of the structures can differ from place to place of the system. Most fractal structures are not exactly self-similar but can contain various inner structures. Using generalized dimension $D_q$ one can unveil details of fractal structures. Here we show the generalized dimension of the spatial structure formed in the power-law potential model with $\alpha = 1.125$ in Fig.6. We find that the generalized dimension is almost constant ($D_q \simeq 0.83$) for some range of $q$ at $t = 5.0$ (Fig.6 top). Therefore we conclude that the fractal structure formed is mono-fractal at $t = 5.0$ as far as we observed.

As discussed in our previous paper [8], however, the fractal structure is a transient state and relaxes finally. We find that the multifractal nature emerges due to the relaxation at late times (the bottom of Fig.6). The boxcounting dimension $D_0$ has relaxed to almost 1, while the correlation dimension $D_2$ has not yet (the bottom of Fig.6). In other words, the relaxation of the boxcounting dimension $D_0$ is much faster than the correlation dimension $D_2$. The detailed analysis will be a future work.

We noted that scale free property of the potential may be a key-stone to understand the dynamical mechanism of the emergence of fractal spatial structure. In this paper we have clarified that scale free property does not immediately imply fractal structure. Potentials with different power-index makes different temporal behaviors, and the gravitational system
(the sheet model) is a special case in all the 1-dimensional systems with scale-free potential.

As relevant works, structure formation in other one-dimensional self-gravitating systems in an expanding universe has been studied \[13\, 18\]. They claimed that their system is not a simple fractal or even a regular multi-fractal, but bifractal \[13\]. Indications of this behavior have also been found for the “quintic” model \[19\]. It is a future work to clarify the relevance between the models comprehensibly. In addition, the dependence of the spatial dimension on the structure formation will be also interesting and important subject of future.

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APPENDIX A: PROOF OF EQ.(11)

In this appendix we lead Eq.(11) by the Fourier-Laplace transform of Eq.(7). In terms of the transform \[11\], Eq. (7) become following: The first term of the left hand side of Eq. (7) is transformed to

$$\int_0^\infty dt e^{-i\omega t} \int_{-\infty}^\infty dx e^{ikx} \delta f(x, p, t) = \left[ e^{-i\omega t} \tilde{\delta f}(k, p, t) \right]_{t=0}^{t=\infty} - (i\omega) \int_0^\infty dt e^{-i\omega t} \tilde{\delta f}(k, p, t)$$

(A1)

$$= -\tilde{\delta f}(k, p, t = 0) + i\omega \tilde{\delta f}(k, p, \omega),$$

(A2)

here \(\omega\) is a complex number and \(Im\omega < 0\). The second term of the left hand side of Eq. (7) is transformed to

$$\int_0^\infty dt e^{-i\omega t} \int_{-\infty}^\infty dx e^{ikx} p \frac{\partial}{\partial x} \delta f(x, p, t) = -kp \int_0^\infty dt e^{-i\omega t} \tilde{\delta f}(k, p, t) = -ikp \tilde{\delta f}(k, p, \omega).$$

(A3)

Then the left hand side of Eq. (7) is transformed to

$$\text{lhs} = -i(-\omega + kp) \tilde{\delta f}(k, p, \omega) - \tilde{\delta f}(k, p, 0).$$

(A4)
On the other hand, the right hand side of Eq. (7) is transformed to

\[(\text{rhs}) = - \int_0^\infty dt e^{-i\omega t} \int_0^\infty dx e^{ikx} \left\{ \int_0^\infty dx' F(x - x') \int_{-\infty}^{\infty} dp' \delta f(x', p', t) \frac{\partial}{\partial p} f_0(p) \right\} = \int_{-\infty}^\infty dy e^{iky} F(y) \int_{-\infty}^{\infty} dp' \tilde{\delta} f(k, p', \omega) \frac{\partial f_0}{\partial p}(p),\]

where

\[\delta f(x', p', \omega) \equiv \int_0^\infty dt e^{-i\omega t} \delta f(x', p', t). \quad (A5)\]

Eq. (7) is transformed to

\[\tilde{\delta} f(k, p, \omega) = -\tilde{F}(k) \frac{\partial f_0}{\partial p}(p) \cdot \left( \int_{-\infty}^{\infty} dp' \tilde{\delta} f(k, p', \omega) \right) + \frac{-1}{i(-\omega + kp)} \tilde{\delta} f(k, p, 0), \quad (A6)\]

where

\[\tilde{\delta} f_k(\omega) \equiv \int_{-\infty}^\infty dp \tilde{\delta} f(k, p, \omega).\]

Integrating both side of Eq. (A6) by \(p\), we obtain

\[\left(1 + \int_{-\infty}^{\infty} dp \frac{\tilde{F}(k)}{i(-\omega + kp)} \frac{\partial f_0}{\partial p}(p)\right) \tilde{\delta} f_k(\omega) = \frac{-1}{i(-\omega + kp)} \tilde{\delta} f(k, p, 0). \quad (A7)\]

Using Eq. (12), we obtain Eq. (11).

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FIG. 1: An example of fractal structure is formed of the model (2) from non-fractal structure. The number of particles is \( N = 2^{15} \). The left figure represents the \( \mu \) space at \( t = 9.375 \). The right figure represents the box-counting dimension.

FIG. 2: The snap-shots of the \( \mu \) space for \( \alpha = 1.125 \). \( N = 65536 \). Time are \( t = 4.4, 5.0, 9.4 \) from the top to the bottom.

FIG. 3: Box counting dimension \( D \) of the spatial distribution for \( \alpha = 1.125 \) and \( t = 5.0 \). The plus symbols represent our data, and solid and dashed lines correspond to \( D = 0.83 \) and \( D = 1 \), respectively.
FIG. 4: The snap-shots of the $\mu$ space for $\alpha = 1.5$. $N = 65536$. Time are $t = 11.0, 15.7$ and $23.6$ from the top to the bottom.

FIG. 5: Box counting dimension $D$ of the spatial distribution for $\alpha = 1.5$ and $t = 23.6$.

FIG. 6: The snap-shot of the $\mu$ space for $\alpha = 1.125$ (left). The generalized dimension $D_q$ (right). Time are $t = 5.0, 9.4$ and $37.7$ from the top to the bottom.