Essential norms of weighted composition operators between Hardy spaces $H^p$ and $H^q$
for $1 \leq p, q \leq \infty$

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Abstract
We complete the different cases remaining in the estimation of the essential norm of a weighted composition operator acting between the Hardy spaces $H^p$ and $H^q$ for $1 \leq p, q \leq \infty$. In particular we give some estimates for the cases $1 = p \leq q \leq \infty$ and $1 \leq q < p \leq \infty$.

1 Introduction
Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ denote the open unit disk in the complex plane. Given two analytic functions $u$ and $\varphi$ defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, one can define the weighted composition operator $uC_{\varphi}$ that maps any analytic function $f$ defined on $\mathbb{D}$ into the function $uC_{\varphi}(f) = u(f \circ \varphi)$. In [10], de Leeuw showed that the isometries in the Hardy space $H^1$ are weighted composition operators, while Forelli [8] obtained this result for the Hardy space $H^p$ when $1 < p < \infty$, $p \neq 2$. Another example is the study of composition operators on the half-plane. A composition operator in a Hardy space of the half-plane is bounded if and only if a certain weighted composition operator is bounded on the Hardy space of the unit disk (see [13] and [14]).

2010 Mathematics Subject Classification: Primary 47B33; Secondary 30H10, 46E15.
Key words and phrases: weighted composition operator, essential norm, Carleson measure, Hardy space.
When \( u \equiv 1 \), we just have the composition operator \( C_\varphi \). The continuity of these operators on the Hardy space \( H^p \) is ensured by the Littlewood’s subordination principle, which says that \( C_\varphi(f) \) belongs to \( H^p \) whenever \( f \in H^p \) (see [1], Corollary 2.24). As a consequence, the condition \( u \in H^\infty \) suffices for the boundedness of \( uC_\varphi \) on \( H^p \). Considering the image of the constant functions, a necessary condition is that \( u \) belongs to \( H^p \). Nevertheless a weighted composition operator needs not to be continuous on \( H^p \), and it is easy to find examples where \( uC_\varphi(H^p) \nsubseteq H^p \) (see Lemma 2.1 of [3] for instance).

In this note we deal with weighted composition operators between \( H^p \) and \( H^q \) for \( 1 \leq p, q \leq \infty \). Boundedness and compactness are characterized in [2] for \( 1 \leq p \leq q < \infty \) by means of Carleson measures, while essential norms of weighted composition operators are estimated in [5] for \( 1 < p \leq q < \infty \) by means of an integral operator. For the case \( 1 \leq q < p < \infty \), boundedness and compactness of \( uC_\varphi \) are studied in [5], and Gorkin and MacCluer in [9] gave an estimate of the essential norm of a composition operator acting between \( H^p \) and \( H^q \).

The aim of this paper is to complete the different cases remaining in the estimation of the essential norm of a weighted composition operator. In section 2 and 3, we give an estimate of the essential norm of \( uC_\varphi \) acting between \( H^p \) and \( H^q \) when \( p = 1 \) and \( 1 \leq q < \infty \) and when \( 1 \leq p < \infty \) and \( q = \infty \). Sections 4 and 5 are devoted to the case where \( \infty \geq p > q \geq 1 \).

Let \( \mathbb{D} \) be the closure of the unit disk \( \mathbb{D} \) and \( \mathbb{T} = \partial \mathbb{D} \) its boundary. We denote by \( dm = dt/2\pi \) the normalised Haar measure on \( \mathbb{T} \). If \( A \) is a Borel subset of \( \mathbb{T} \), the notation \( m(A) \) as well as \( |A| \) will design the Haar measure of \( A \). For \( 1 \leq p < \infty \), the Hardy space \( H^p(\mathbb{D}) \) is the space of analytic functions \( f : \mathbb{D} \to \mathbb{C} \) satisfying the following condition

\[
\|f\|_p = \sup_{0<r<1} \left( \int_{\mathbb{T}} |f(re^{i\theta})|^p \, dm(\zeta) \right)^{1/p} < \infty.
\]

Endowed with this norm, \( H^p(\mathbb{D}) \) is a Banach space. The space \( H^\infty(\mathbb{D}) \) is consisting of every bounded analytic function on \( \mathbb{D} \), and its norm is given by the supremum norm on \( \mathbb{D} \).

We recall that any function \( f \in H^p(\mathbb{D}) \) can be extended on \( \mathbb{T} \) to a function \( f^* \) by the following formula: \( f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \). The limit exists almost everywhere by Fatou’s theorem, and \( f^* \in L^p(\mathbb{T}) \). Moreover, \( f \mapsto f^* \) is an into isometry from \( H^p(\mathbb{D}) \) to \( L^p(\mathbb{T}) \) whose image, denoted by \( H^p(\mathbb{T}) \) is the closure (weak-star closure for \( p = \infty \)) of the set of polynomials in \( L^p(\mathbb{T}) \). So
we can identify $H^p(D)$ and $H^p(T)$, and we will use the notation $H^p$ for both of these spaces. More on Hardy spaces can be found in [11] for instance.

The essential norm of an operator $T : X \to Y$, denoted $\|T\|_e$, is given by

$$\|T\|_e = \inf \{ \|T - K\| \mid K \text{ is a compact operator from } X \text{ to } Y \}.$$ 

Observe that $\|T\|_e \leq \|T\|$, and $\|T\|_e$ is the norm of $T$ seen as an element of $B(X,Y)/K(X,Y)$ where $B(X,Y)$ is the space of all bounded operators from $X$ to $Y$ and $K(X,Y)$ is the subspace consisting of all compact operators.

Notation: we will write $a \approx b$ whenever there exists two positive universal constants $c$ and $C$ such that $cb \leq a \leq Cb$. In the sequel, $u$ will be a non-zero analytic function on $D$ and $\varphi$ will be a non-constant analytic function defined on $D$ satisfying $\varphi(D) \subset D$.

### 2 $uC\varphi \in B(H^1, H^q)$ for $1 \leq q < \infty$

Let us first start with a characterization of the boundedness of $uC\varphi$ acting between $H^p$ and $H^q$:

**Theorem 2.1** (see [5, Theorem 4]). Let $u$ be an analytic function on $D$ and $\varphi$ an analytic self-map of $D$. Let $0 < p \leq q < \infty$. Then the weighted composition operator $uC\varphi$ is bounded from $H^p$ to $H^q$ if and only if

$$\sup_{a \in D} \int_T |u(\zeta)|^q \left( \frac{1 - |a|^2}{|1 - \bar{a}\varphi(\zeta)|^2} \right)^{q/p} \, dm(\zeta) < \infty.$$ 

As a consequence $uC\varphi$ is a bounded operator as soon as $uC\varphi$ is uniformly bounded on the set $\{ k_a^{1/p} \mid a \in D \}$ where $k_a$ is the normalized kernel defined by $k_a(z) = (1 - |a|^2)/(1 - \bar{a}z)^2$, $a \in D$. Note that $k_a^{1/p} \in H^p$ and $\|k_a^{1/p}\|_p = 1$. These kernels play a crucial role in the estimation of the essential norm of a weighted composition operator:

**Theorem 2.2** (see [5, Theorem 5]). Let $u$ be an analytic function on $D$ and $\varphi$ an analytic self-map of $D$. Assume that the weighted composition operator $uC\varphi$ is bounded from $H^p$ to $H^q$ with $1 < p \leq q < \infty$. Then

$$\|uC\varphi\|_e \approx \limsup_{|a| \to 1} \left( \int_T |u(\zeta)|^q \left( \frac{1 - |a|^2}{|1 - \bar{a}\varphi(\zeta)|^2} \right)^{q/p} \, dm(\zeta) \right)^{\frac{1}{q}}.$$
The aim of this section is to give the corresponding estimate for the case \( p = 1 \). We shall prove that the previous theorem is still valid for \( p = 1 \):

**Theorem 2.3.** Let \( u \) be an analytic function on \( \mathbb{D} \) and \( \varphi \) an analytic self-map of \( \mathbb{D} \). Suppose that the weighted composition operator \( uC_\varphi \) is bounded from \( H^1 \) to \( H^q \) for a certain \( 1 \leq q < \infty \). Then we have

\[
\|uC_\varphi\|_e \approx \limsup_{|a| \to 1^-} \left( \int_\mathbb{T} |u(\zeta)|^q \left( \frac{1 - |a|^2}{|1 - a\varphi(\zeta)|^2} \right)^q \, dm(\zeta) \right)^{\frac{1}{q}}.
\]

Let us start with the upper estimate:

**Proposition 2.4.** Let \( uC_\varphi \in B(H^1, H^q) \) with \( 1 \leq q < \infty \). Then there exists a positive constant \( \gamma \) such that

\[
\|uC_\varphi\|_e \leq \gamma \limsup_{|a| \to 1^-} \left( \int_\mathbb{T} |u(\zeta)|^q \left( \frac{1 - |a|^2}{|1 - a\varphi(\zeta)|^2} \right)^q \, dm(\zeta) \right)^{\frac{1}{q}}.
\]

The main tool of the proof is the use of Carleson measures. Assume that \( \mu \) is a finite positive Borel measure on \( \mathbb{D} \) and let \( 1 \leq p, q < \infty \). We say that \( \mu \) is a \((p, q)\)-Carleson measure if the embedding \( J_\mu : f \in H^p \mapsto f \in L^q(\mu) \) is well defined. In this case, the closed graph theorem ensures that \( J_\mu \) is continuous. In other words, \( \mu \) is a \((p, q)\)-Carleson measure if there exists a constant \( \gamma_1 > 0 \) such that for every \( f \in H^p \),

\[
(2.1) \quad \int_{\mathbb{D}} |f(z)|^q \, d\mu(z) \leq \gamma_1 \|f\|_p^q.
\]

Let \( I \) be an arc in \( \mathbb{T} \). By \( S(I) \) we denote the Carleson window given by

\[
S(I) = \{ z \in \mathbb{D} \mid 1 - |I| \leq |z| < 1, \ z/|z| \in I \}.
\]

Let us denote by \( \mu_\mathbb{D} \) and \( \mu_\mathbb{T} \) the restrictions of \( \mu \) to \( \mathbb{D} \) and \( \mathbb{T} \) respectively. The following result is a version of a theorem of Duren (see [7], p.163) for measures on \( \mathbb{D} \):

**Theorem 2.5** (see [11, Theorem 2.5]). Let \( 1 \leq p < q < \infty \). A finite positive Borel measure \( \mu \) on \( \mathbb{D} \) is a \((p, q)\)-Carleson measure if and only if \( \mu_\mathbb{T} = 0 \) and there exists a constant \( \gamma_2 > 0 \) such that

\[
(2.2) \quad \mu_\mathbb{D}(S(I)) \leq \gamma_2 |I|^q/p \quad \text{for any arc } I \subset \mathbb{T}.
\]
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Notice that the best constants $\gamma_1$ and $\gamma_2$ in (2.1) and (2.2) are comparable, meaning that there is a positive constant $\beta$ independent of the measure $\mu$ such that $(1/\beta)\gamma_2 \leq \gamma_1 \leq \beta \gamma_2$.

The notion of Carleson measure was introduced by Carleson in [2] as a part of his work on the corona problem. He gave a characterization of measures $\mu$ on $\mathbb{D}$ such that $H^p$ embeds continuously in $L^p(\mu)$.

Examples of such Carleson measures are provided by composition operators. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map and let $1 \leq p, q < \infty$. The boundedness of the composition operator $C_\varphi : f \mapsto f \circ \varphi$ between $H^p$ and $H^q$ can be rephrased in terms of $(p,q)$-Carleson measures. Indeed, denote by $m_\varphi$ the pullback measure of $m$ by $\varphi$, which is the image of the Haar measure $m$ of $\mathbb{T}$ under the map $\varphi^*$, defined by

$$m_\varphi(A) = m(\varphi^{-1}(A))$$

for every Borel subset $A$ of $\overline{\mathbb{D}}$. Then

$$\|C_\varphi(f)\|_q^q = \int_{\mathbb{T}} |f \circ \varphi|^q \, dm = \int_{\mathbb{D}} |f|^q \, dm_\varphi = \|J_{m_\varphi}(f)\|_q^q$$

for all $f \in H^p$. Thus $C_\varphi$ maps $H^p$ boundedly into $H^q$ if and only if $m_\varphi$ is a $(p,q)$-Carleson measure.

In the sequel we will denote by $r\mathbb{D}$ the open disk of radius $r$, in other words $r\mathbb{D} = \{z \in \mathbb{D} \mid |z| < r\}$ for $0 < r < 1$. We will need the following lemma concerning $(p,q)$-Carleson measures:

**Lemma 2.6.** Take $0 < r < 1$ and let $\mu$ be a finite positive Borel measure on $\mathbb{D}$. Let

$$N_r^* := \sup_{|a| \geq r} \int_{\mathbb{D}} |k_a(w)|^{\frac{q}{p}} \, d\mu(w).$$

If $\mu$ is a $(p,q)$-Carleson measure for $1 \leq p \leq q < \infty$ then so is $\mu_r := \mu_{\mathbb{D} \setminus r\mathbb{D}}$. Moreover one can find an absolute constant $M > 0$ satisfying $\|\mu_r\| \leq MN_r^*$ where $\|\mu_r\| := \sup_{I \subset \mathbb{T}} \frac{\mu_r(S(I))}{|I|^{1/p}}$.

We omit the proof of Lemma 2.6 here, which is a slight modification of the proof of Lemma 1 and Lemma 2 in [3] using Theorem 2.5.

In the proof of the upper estimate of Theorem 2.2 in [3], the authors use a decomposition of the identity on $H^p$ of the form $I = K_N + R_N$ where $K_N$ is the partial sum operator defined by $K_N(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{N} a_n z^n$, and they use the fact that $(K_N)$ is a sequence of compact operators that is uniformly bounded in $B(H^p)$ and that $R_N$ converges pointwise to zero on
Nevertheless the sequence \((K_N)\) is not uniformly bounded in \(B(H^1)\). In fact, \((K_N)\) is uniformly bounded in \(B(H^p)\) if and only if the Riesz projection \(P : L^p \to H^p\) is bounded [15, Theorem 2], which occurs if and only if \(1 < p < \infty\). Therefore we need to use a different decomposition for the case \(p = 1\). Since \(K_N\) is the convolution operator by the Dirichlet kernel on \(H^p\), we shall consider the Fejér kernel \(F_N\) of order \(N\).

Let us define 
\[
K_N : H^1 \to H^1
\]
\[
to be the convolution operator associated to \(F_N\) that maps \(f \in H^1\) to 
\[
K_N f = F_N \ast f \in H^1
\]
and \(R_N = I - K_N\).

Then \(\|K_N\| \leq 1\), \(K_N\) is compact and for every \(f \in H^1\), \(\|f - K_N f\|_1 \to 0\) following Fejér’s theorem. If \(f(z) = \sum_{n \geq 0} \hat{f}(n) z^n \in H^1\), then 
\[
K_N f(z) = \sum_{n=0}^{N-1} \left(1 - \frac{n}{N}\right) \hat{f}(n) z^n.
\]

**Lemma 2.7.** Let \(1 \leq q < \infty\) and suppose that \(uC_\phi \in B(H^1, H^q)\). Then 
\[
\|uC_\phi\|_e \leq \liminf_N \|uC_\phi R_N\|.
\]

**Proof.**
\[
\|uC_\phi\|_e = \|uC_\phi K_N + uC_\phi R_N\|_e = \|uC_\phi R_N\|_e \quad \text{since } K_N \text{ is compact}
\]
and the result follows taking the lower limit.

We will need the following lemma for an estimation of the remainder \(R_N\):

**Lemma 2.8.** Let \(\varepsilon > 0\) and \(0 < r < 1\). Then \(\exists N_0 = N_0(r) \in \mathbb{N}, \forall N \geq N_0,\)
\[
|R_N f(w)|^q < \varepsilon \|f\|_1^q,
\]
for every \(|w| < r\) and for every \(f \in H^1\).

**Proof.** Let \(K_w(z) = 1/(1 - \bar{w}z), w \in \mathbb{D}, z \in \mathbb{D}\). \(K_w\) is a bounded analytic function on \(\mathbb{D}\). It is easy to see that for every \(f \in H^1,\)
\[
\langle R_N f, K_w \rangle = \langle f, R_N K_w \rangle
\]
where \(|w| < r, N \geq 1\) and
\[
\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta
\]
for $f \in H^1$ and $g \in H^\infty$. Then we have $|R_N f(w)| = |\langle R_N f, K_w \rangle| = |\langle f, R_N K_w \rangle| \leq \|f\|_1 \|R_N K_w\|_\infty$. Take $|w| < r$ and choose $N_0 \in \mathbb{N}$ so that for every $N \geq N_0$ one has $r^N \leq \varepsilon^{1/q}(1 - r)/2$ and $1/N \sum_{n=1}^{N-1} nr^n \leq (1/2)\varepsilon^{1/q}$.

Since

$$R_N K_w(z) = R_N \left( \sum_{n=0}^{\infty} \bar{w}^n z^n \right) = \sum_{n=0}^{N-1} \frac{n}{N} \bar{w}^n z^n + \sum_{n=N}^{\infty} \bar{w}^n z^n,$$

one has

$$\|R_N K_w\|_\infty < \frac{1}{N} \sum_{n=0}^{N-1} nr^n + \sum_{n=N}^{\infty} r^n \leq \varepsilon^{1/q}.$$

Thus $|R_N f(w)|^q \leq \varepsilon \|f\|^q_q$ for every $f$ in $H^1$.

Proof of Proposition 2.4. Denote by $\mu$ the measure which is absolutely continuous with respect to $m$ and whose density is $|u|^q$, and let $\mu_\varphi = \mu \circ \varphi^{-1}$ be the pullback measure of $\mu$ by $\varphi$. Fix $0 < r < 1$. For every $f \in H^1$, we have

$$\|(uC_\varphi R_N f)w\|^q_q = \int_T |u(\zeta)|^q \left| \left( (R_N f) \circ \varphi \right)(\zeta) \right|^q \, dm(\zeta)$$

$$= \int_T \left| \left( (R_N f) \circ \varphi \right)(\zeta) \right|^q \, d\mu(\zeta)$$

$$= \int_{\mathbb{D} \setminus \bar{D}} |R_N f(w)|^q \, d\mu_\varphi(w)$$

$$+ \int_{\mathbb{D} \cap \bar{D}} |R_N f(w)|^q \, d\mu_\varphi(w)$$

$$= I_1(N, r, f) + I_2(N, r, f).$$

Let us first show that $\lim_{N \uparrow \infty} \sup_{\|f\|_1 = 1} I_2(N, r, f) = 0$. For $\varepsilon > 0$, Lemma 2.8 gives us an integer $N_0(r)$ such that for every $N \geq N_0(r)$,

$$I_2(N, r, f) = \int_{r \bar{D}} |R_N f(w)|^q \, d\mu_\varphi(w)$$

$$\leq \varepsilon \|f\|_q \|\mu_\varphi\|_{r \bar{D}}$$

$$\leq \varepsilon \|f\|_q \|\mu_\varphi\|_{\bar{D}}$$

$$\leq \varepsilon \|f\|_q \|u\|_{q_q}.$$

So, $r$ being fixed, we have $\lim_{N \uparrow \infty} \sup_{\|f\|_1 = 1} I_2(N, r, f) = 0$.

Now we need an estimate of $I_1(N, r, f)$. The continuity of $uC_\varphi : H^1 \to H^q$ ensures that $\mu_\varphi$ is a $(1, q)$-Carleson measure, and therefore $\mu_{\varphi, r} := \mu_{\varphi, \mathbb{D} \cap r \bar{D}}$
is also a $(1, q)$-Carleson measure by using Lemma 2.6 for $p = 1$. It follows that
\[ \int_{\mathbb{D} \setminus r \mathbb{D}} |R_N f(w)|^q \, d\mu_{\varphi, r}(w) \leq \gamma_1 \|R_N f\|_1^q \]
\[ \leq \beta \|\mu_{\varphi, r}\| \|R_N f\|_1^q \]
\[ \leq 2^q \beta MN_r^* \|f\|_q^q \]
using Lemma 2.6 and the fact that $\|R_N\| \leq 1 + \|K_N\| \leq 2$ for every $N \in \mathbb{N}$. We take the supremum over $B_{H^1}$ and take the lower limit as $N$ tends to infinity in (2.3) to obtain
\[ \liminf_{N \to \infty} \|uC_\varphi R_N\|^q \leq 2^q \beta MN_r^*. \]
Now as $r$ goes to 1 we have:
\[ \lim_{r \to 1} N_r^* = \limsup_{|a| \to 1^-} \int_{\mathbb{T}} |k_a(w)|^q \, d\mu_{\varphi}(w) \]
\[ = \limsup_{|a| \to 1^-} \int_{\mathbb{T}} |u(\zeta)|^q \left( \frac{1 - |a|^2}{|1 - a\varphi(\zeta)|^2} \right)^q \, dm(\zeta) \]
and we obtain the estimate announced using Lemma 2.7. □

Now let us turn to the lower estimate in Theorem 2.2. Let $1 \leq q < \infty$. Consider $F_N$ the Fejér kernel of order $N$, and define $K_N : H^q \to H^q$ the convolution operator associated to $F_N$ and $R_N = I - K_N$. Then $(K_N)_N$ is a sequence of uniformly bounded compact operators in $\mathcal{B}(H^q)$, and $\|R_N f\|_q \to 0$ for all $f \in H^q$.

**Lemma 2.9.** There exists $0 < \gamma \leq 2$ such that whenever $uC_{\varphi}$ is a bounded operator from $H^1$ to $H^q$ with $1 \leq q < \infty$, one has
\[ \frac{1}{\gamma} \limsup_{N} \|R_N uC_{\varphi}\| \leq \|uC_{\varphi}\|_{e}. \]

**Proof.** Take $K \in \mathcal{B}(H^1, H^q)$ a compact operator. Since $(K_N)$ is uniformly bounded, one can find $\gamma > 0$ satisfying $\|R_N\| \leq 1 + \|K_N\| \leq \gamma$ for all $N > 0$, and we have:
\[ \|uC_{\varphi} + K\| \geq \frac{1}{\gamma} \|R_N (uC_{\varphi} + K)\| \]
\[ \geq \frac{1}{\gamma} \|R_N uC_{\varphi}\| - \frac{1}{\gamma} \|R_N K\|. \]
Now use the fact that $(R_N)$ goes pointwise to zero in $H^q$, and consequently $(R_N)$ converges strongly to zero over the compact set $K(B_{H^1})$ as $N$ goes to infinity. It follows that $\|R_NK\| \longrightarrow 0$, and

$$\|uC_\varphi + K\| \geq \frac{1}{\gamma} \limsup_N \|R_NuC_\varphi\|$$

for every compact operator $K : H^1 \rightarrow H^q$.

**Proposition 2.10.** Let $u$ be an analytic function on $\mathbb{D}$ and $\varphi$ an analytic self-map of $\mathbb{D}$. Assume that $uC_\varphi \in B(H^1, H^q)$ with $1 \leq q < \infty$. Then

$$\|uC_\varphi\|_e \geq \frac{1}{\gamma} \limsup_{|a| \to 1^-} \left( \int_T |u(\zeta)|^q \left( \frac{1 - |a|^2}{|1 - \bar{a}\varphi(\zeta)|^2} \right)^q \, dm(\zeta) \right)^{\frac{1}{q}}.$$ 

**Proof.** Since $k_a$ is a unit vector in $H^1$,

$$\|R_NuC_\varphi\| = \|uC_\varphi - K_NuC_\varphi\| \geq \|uC_\varphi k_a\|_q - \|K_NuC_\varphi k_a\|_q.$$ 

**First case:** $q > 1$

Since $(k_a)$ converges to zero for the topology of uniform convergence on compact sets in $\mathbb{D}$ as $|a|$ goes to 1, so does $uC_\varphi(k_a)$. The topology of uniform convergence on compact sets in $\mathbb{D}$ and the weak topology agree on $H^q$, therefore it follows that $uC_\varphi(k_a)$ goes to zero for the weak topology in $H^q$ as $|a|$ goes to 1. Since $K_N$ is a compact operator, it is completely continuous and carries weak-null sequences to norm-null sequences. So $\|K_N(uC_\varphi(k_a))\|_q \to 0$ when $|a| \to 1$, and

$$\|R_NuC_\varphi\| \geq \limsup_{|a| \to 1^-} \|uC_\varphi(k_a)\|_q.$$ 

Taking the upper limit as $N \to \infty$, we obtain the result using Lemma 2.9.

For the second case we will need the following computational lemma:

**Lemma 2.11.** Let $\varphi$ be an analytic self-map of $\mathbb{D}$. Take $a \in \mathbb{D}$ and $N \geq 1$ an integer. Denote by $\alpha_p(a)$ the $p$-th Fourier coefficient of $C_\varphi(k_a/(1 - |a|^2))$, so that for every $z \in \mathbb{D}$ we have

$$k_a(\varphi(z)) = (1 - |a|^2) \sum_{p=0}^{\infty} \alpha_p(a) z^p.$$ 

Then there exists a positive constant $M = M(N) > 0$ depending on $N$ such that $|\alpha_p(a)| \leq M$ for every $p \leq N$ and every $a \in \mathbb{D}$.
Proof. Write \( \varphi(z) = a_0 + \psi(z) \) with \( a_0 = \varphi(0) \in \mathbb{D} \) and \( \psi(0) = 0 \). If we develop \( k_a(z) \) as a Taylor series and replace \( z \) by \( \varphi(z) \) we obtain:

\[
k_a(\varphi(z)) = (1 - |a|^2) \sum_{n=0}^{\infty} (n + 1)(\bar{a})^n \varphi(z)^n.
\]

Then

\[
\alpha_p(a) = \left\langle \sum_{n=0}^{\infty} (n + 1)(\bar{a})^n \varphi(z)^n, z^p \right\rangle
= \sum_{n=0}^{\infty} (n + 1)(\bar{a})^n \sum_{j=0}^{n} \binom{n}{j} a_0^{-j} \left\langle \psi(z)^j, z^p \right\rangle.
\]

where \( \left\langle f, g \right\rangle = \int_{\mathbb{T}} f \bar{g} \, dm \). Note that \( \left\langle \psi(z)^j, z^p \right\rangle = 0 \) if \( j > p \) since \( \psi(0) = 0 \), and consequently

\[
\alpha_p(a) = \sum_{n=0}^{\infty} (n + 1)(\bar{a})^n \sum_{j=0}^{\min(n,p)} \binom{n}{j} a_0^{-j} \left\langle \psi(z)^j, z^p \right\rangle
= \sum_{j=0}^{p} \sum_{n=j}^{\infty} (n + 1)(\bar{a})^n \binom{n}{j} a_0^{-j} \left\langle \psi(z)^j, z^p \right\rangle
= \sum_{j=0}^{p} \left\langle \psi(z)^j, z^p \right\rangle \sum_{n=j}^{\infty} (n + 1)(\bar{a})^n \binom{n}{j} a_0^{-j}.
\]

In the case where \( a_0 \neq 0 \) we obtain

\[
\alpha_p(a) = \sum_{j=0}^{p} \left\langle \psi(z)^j, z^p \right\rangle a_0^{-j} \sum_{n=j}^{\infty} (n + 1) \binom{n}{j} (\bar{a}a_0)^n
= \sum_{j=0}^{p} \left\langle \psi(z)^j, z^p \right\rangle a_0^{-j} \frac{(j + 1)(\bar{a}a_0)^j}{(1 - \bar{a}a_0)^{j+2}}
= \sum_{j=0}^{p} \left\langle \psi(z)^j, z^p \right\rangle \frac{(j + 1)(\bar{a})^j}{(1 - \bar{a})^{j+2}}
\]

using the following equalities for \( x = \bar{a}a_0 \in \mathbb{D} \):

\[
\sum_{n=0}^{\infty} (n + 1) \binom{n}{j} x^n = \frac{\left( \sum_{n=j}^{\infty} \binom{n}{j} x^{n+1} \right)'}{x^j} = \frac{(x^{j+1} - 1)}{(1 - x)^{j+2}}
\]

Note that the last expression obtained for \( \alpha_p(a) \) is also valid for \( a_0 = 0 \).
Thus, for $0 \leq p \leq N$ we have the following estimates:

\[
|\alpha_p(a)| \leq \sum_{j=0}^{p} \left| \langle \psi(z)^j, z^p \rangle \right| \frac{j+1}{(1 - |a_0|)^{j+2}} \\
\leq \sum_{j=0}^{p} \|\psi^j\|_{\infty} \frac{N+1}{(1 - |a_0|)^{N+2}} \\
\leq \frac{(N+1)^2}{(1 - |a_0|)^{N+2}} \max_{0 \leq j \leq N} \|\psi^j\|_{\infty} \\
\leq M,
\]

where $M$ is a constant independent from $a$. \hfill \Box

**Second case: $q = 1$**

In this case, it is no longer for the weak topology but for the weak-star topology of $H^1$ that $uC_{\varphi}(k_{a})$ tends to zero when $|a| \to 1$. Nevertheless, it is still true that $\|K_N uC_{\varphi}(k_{a})\|_1 \to 0$ as $|a| \to 1$. Indeed if $f(z) = \sum_{n \geq 0} \hat{f}(n)z^n \in H^1$, then

\[
K_N f(z) = \sum_{n=0}^{N-1} \left( 1 - \frac{n}{N} \right) \hat{f}(n)z^n.
\]

We have the following development:

\[
k_{a}(\varphi(z)) = (1 - |a|^2) \sum_{n=0}^{\infty} \alpha_n(a) z^n.
\]

Denote by $u_n$ the $n$-th Fourier coefficient of $u$, so that

\[
uC_{\varphi}(k_{a})(z) = (1 - |a|^2) \sum_{n=0}^{\infty} \left( \sum_{p=0}^{n} \alpha_p(a)u_{n-p} \right) z^n, \ \forall z \in \mathbb{D}.
\]

It follows that

\[
\|K_N uC_{\varphi}(k_{a})\|_1 \leq (1 - |a|^2) \sum_{n=0}^{N-1} \left( 1 - \frac{n}{N} \right) \left| \sum_{p=0}^{n} \alpha_p(a)u_{n-p} \right| \|z^n\|_1.
\]

Now using estimates from Lemma 2.11 one can find a constant $M > 0$ independent from $a$ such that $|\alpha_p(a)| \leq M$ for every $a \in \mathbb{D}$ and $0 \leq p \leq N - 1$. Use the fact that $\|z^n\|_1 = 1$ and $|u_p| \leq \|u\|_1$ to deduce that there is a constant $M' > 0$ independent from $a$ such that

\[
\|K_N uC_{\varphi}(k_{a})\|_1 \leq M'(1 - |a|^2)\|u\|_1
\]
for all \( a \in \mathbb{D} \). Thus \( K_N u C_\varphi(k_a) \) converges to zero in \( H^1 \) when \( |a| \to 1 \), and take the upper limit in \( 2.4 \) when \( a \) tends to \( 1^- \) to obtain

\[
\| R_N u C_\varphi \| \geq \limsup_{|a| \to 1} \| u C_\varphi(k_a) \|_1, \quad \forall N \geq 0.
\]

We conclude with Lemma 2.9 and observe that \( \gamma = \sup \| R_N \| \leq 2 \) since \( \| R_N \| \leq 1 + \| K_N \| \leq 2 \).

3 \( u C_\varphi \in B(H^p, H^\infty) \) for \( 1 \leq p < \infty \)

Let \( u \) be a bounded analytic function. Characterizations of boundedness and compactness of \( u C_\varphi \) as a linear map between \( H^p \) and \( H^\infty \) have been studied in [3] for \( p \geq 1 \). Indeed,

\[
u C_\varphi \in B(H^p, H^\infty) \text{ if and only if } \sup_{z \in \mathbb{D}} \frac{|u(z)|^p}{1 - |\varphi(z)|^2} < \infty
\]

and

\[
u C_\varphi \text{ is compact if and only if } \| \varphi \|_\infty < 1 \text{ or } \lim_{|\varphi(z)| \to 1} \frac{|u(z)|^p}{1 - |\varphi(z)|^2} = 0.
\]

In the case where \( \| \varphi \|_\infty = 1 \) we let

\[
M_{\varphi}(u) = \limsup_{|\varphi(z)| \to 1} \frac{|u(z)|}{(1 - |\varphi(z)|^2)^{\frac{p}{2}}}
\]

As regarding Theorem 1.7 in [12], it seems reasonable to think that the essential norm of \( u C_\varphi \) is equivalent to the quantity \( M_{\varphi}(u) \). We first have a majorization:

**Proposition 3.1.** Let \( u \) be an analytic function on \( \mathbb{D} \) and \( \varphi \) an analytic self-map of \( \mathbb{D} \). Suppose that \( u C_\varphi \) is a bounded operator from \( H^p \) to \( H^\infty \), where \( 1 \leq p < \infty \) and that \( \| \varphi \|_\infty = 1 \). Then

\[
\| u C_\varphi \|_e \leq 2 M_{\varphi}(u).
\]

**Proof.** Let \( \varepsilon \) be a real positive number, and pick \( r < 1 \) satisfying

\[
\sup_{|\varphi(z)| \geq r} \frac{|u(z)|}{(1 - |\varphi(z)|^2)^{\frac{p}{2}}} \leq M_{\varphi}(u) + \varepsilon.
\]

We approximate \( u C_\varphi \) by \( u C_\varphi K_N \) where \( K_N : H^p \to H^p \) is the convolution operator by the Fejér kernel of order \( N \), where \( N \) is chosen so that \( |R_N f(w)| < \varepsilon \| f \|_1 \) for every \( f \in H^1 \) and every \( |w| < r \) (Lemma 2.8 for
Let $q = 1$. We want to show that $\|uC_\varphi - uC_\varphi K_N\| = \|uC_\varphi R_N\| \leq \max(2M_\varphi(u) + 2\varepsilon, \varepsilon\|u\|_\infty)$, which will prove our assertion. If $f$ is a unit vector in $H^p$, then the norm of $uC_\varphi R_N(f)$ is equal to

$$\max \left( \sup_{|\varphi(z)| \geq r} |u(z)(R_Nf) \circ \varphi(z)|, \sup_{|\varphi(z)| < r} |u(z)(R_Nf) \circ \varphi(z)| \right).$$

We want to estimate the first term. If $\omega \in \mathbb{D}$, we denote by $\delta_\omega$ the linear functional on $H^p$ defined by $\delta_\omega(f) = f(\omega)$. Then $\delta_\omega \in (H^p)^*$ and $\|\delta_\omega\|_{(H^p)^*} = 1/(1 - |w|^2)^{1/p}$ for every $w \in \mathbb{D}$. Therefore

$$\sup_{|\varphi(z)| \geq r} |u(z)(R_Nf) \circ \varphi(z)| \leq 2 \sup_{|\varphi(z)| \geq r} \frac{|u(z)|}{(1 - |\varphi(z)|^2)^{1/p}},$$

using the fact that $\|R_Nf\|_p \leq 2$.

For the second term, since $|\varphi(z)| < r$ we have

$$|u(z)R_Nf(\varphi(z))| \leq \|u\|_{\infty}|R_Nf(\varphi(z))| \leq \varepsilon\|u\|_{\infty}\|f\|_1 \leq \varepsilon\|u\|_{\infty},$$

which ends the proof.

On the other hand, we have the lower estimate:

**Proposition 3.2.** Let $u$ be an analytic function on $\mathbb{D}$ and $\varphi$ an analytic self-map of $\mathbb{D}$ satisfying $\|\varphi\|_{\infty} = 1$. Suppose that $uC_\varphi$ is a bounded operator from $H^p$ to $H^\infty$, where $1 \leq p < \infty$. Then

$$\frac{1}{2}M_\varphi(u) \leq \|uC_\varphi\|_e.$$

**Proof.** Assume that $uC_\varphi$ is not compact, implying $M_\varphi(u) > 0$. Let $(z_n)$ be a sequence in $\mathbb{D}$ satisfying

$$\lim_n |\varphi(z_n)| = 1 \quad \text{and} \quad \lim_n \frac{|u(z_n)|}{(1 - |\varphi(z_n)|^2)^{1/p}} = M_\varphi(u).$$

Consider the sequence $(f_n)$ defined by

$$f_n(z) = k_{\varphi(z_n)}(z)^{1/p} = \left(\frac{1 - |\varphi(z_n)|^2}{1 - \varphi(z_n)z}\right)^{1/p}.$$

Each $f_n$ is a unit vector of $H^p$. Let $K : H^p \to H^\infty$ be a compact operator.
First case: $p > 1$

Since the sequence $(f_n)$ converges to zero for the weak topology of $H^p$ and $K$ is completely continuous, the sequence $(Kf_n)$ converges to zero for the norm topology in $H^\infty$. Use that $\|uC\varphi + K\| \geq \|uC\varphi(f_n)\|_\infty - \|Kf_n\|_\infty$ and take the upper limit when $n$ tends to infinity to obtain

$$\|uC\varphi + K\| \geq \limsup_n \|uC\varphi(f_n)\|_\infty$$

$$\geq \limsup_n \|u(z_n)\| |f_n(\varphi(z_n))|$$

$$\geq \limsup_n \frac{|u(z_n)|}{(1 - |\varphi(z_n)|^2)^{\frac{1}{p}}}$$

$$\geq M\varphi(u).$$

Second case: $p = 1$

Let $\varepsilon > 0$. Since the sequence $(f_n)$ is no longer weakly convergent to zero in $H^1$, we cannot assert that $(Kf_n)_n$ goes to zero in $H^\infty$. Nevertheless, passing to subsequences, one can assume that $(Kf_{n_k})_k$ converges in $H^\infty$, and hence is a Cauchy sequence. So we can find an integer $N > 0$ such that for every $k$ and $m$ greater than $N$ we have $\|Kf_{n_k} - Kf_{n_m}\| < \varepsilon$. We deduce that

$$\|uC\varphi + K\| \geq \left\| (uC\varphi + K) \left( \frac{f_{n_k} - f_{n_m}}{2} \right) \right\|_\infty$$

$$\geq \frac{1}{2} \|uC\varphi(f_{n_k} - f_{n_m})\|_\infty - \frac{\varepsilon}{2}$$

$$\geq \frac{1}{2} |u(z_{n_k})| |f_{n_k}(\varphi(z_{n_k})) - f_{n_m}(\varphi(z_{n_m}))| - \frac{\varepsilon}{2}$$

$$\geq \frac{|u(z_{n_k})|}{2 (1 - |\varphi(z_{n_k})|^2)} - \frac{|u(z_{n_k})| (1 - |\varphi(z_{n_k})|^2)}{2 (1 - |\varphi(z_{n_k})|^2)^2} - \frac{\varepsilon}{2}$$

Now take the upper limit as $m$ goes to infinity ($k$ being fixed) and recall that $\lim_m |\varphi(z_{n_m})| = 1$ and $|\varphi(z_{n_k})| < 1$ to obtain

$$\|uC\varphi + K\| \geq \frac{|u(z_{n_k})|}{2 (1 - |\varphi(z_{n_k})|^2)} - \frac{\varepsilon}{2}$$

for every $k \geq N$. It remains to make $k$ tend to infinity to have

$$\|uC\varphi + K\| \geq \frac{1}{2} M\varphi(u) - \frac{\varepsilon}{2}.$$

Combining Proposition 3.1 and Proposition 3.2 we obtain the following estimate:
Theorem 3.3. Let \( u \) be an analytic function on \( \mathbb{D} \) and \( \varphi \) an analytic self-map of \( \mathbb{D} \) satisfying \( \| \varphi \|_\infty = 1 \). Suppose that \( uC_\varphi \) is a bounded operator from \( H^p \) to \( H^\infty \), where \( 1 \leq p < \infty \). Then \( \| uC_\varphi \|_e \approx M_\varphi(u) \). More precisely, we have the following inequalities:

\[
\frac{1}{2} M_\varphi(u) \leq \| uC_\varphi \|_e \leq 2M_\varphi(u).
\]

Note that if \( p > 1 \) one can replace the constant 1/2 by 1.

4 \( uC_\varphi \in B(H^\infty, H^q) \) for \( \infty > q \geq 1 \)

In this setting, boundedness of the weighted composition operator \( uC_\varphi \) is equivalent to saying that \( u \) belongs to \( H^q \), and \( uC_\varphi \) is compact if and only if \( u = 0 \) or \( |E_\varphi| = 0 \) where \( E_\varphi = \{ \zeta \in \mathbb{T} \mid \varphi^*(\zeta) \in \mathbb{T} \} \) is the extremal set of \( \varphi \) (see [3]). We give here some estimates of the essential norm of \( uC_\varphi \) that appear in [9] for the special case of composition operators:

Theorem 4.1. Let \( u \in H^q \), with \( \infty > q \geq 1 \) and \( \varphi \) be an analytic self-map of \( \mathbb{D} \). Then \( \| uC_\varphi \|_e \approx \left( \int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta) \right)^{\frac{1}{q}} \). More precisely,

\[
\frac{1}{2} \left( \int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta) \right)^{\frac{1}{q}} \leq \| uC_\varphi \|_e \leq 2 \left( \int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta) \right)^{\frac{1}{q}}.
\]

We start with the upper estimate:

Proposition 4.2. Let \( u \in H^q \), with \( \infty > q \geq 1 \) and \( \varphi \) be an analytic self-map of \( \mathbb{D} \). Then

\[
\| uC_\varphi \|_e \leq 2 \left( \int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta) \right)^{\frac{1}{q}}.
\]

Proof. Take \( 0 < r < 1 \). Since \( \| r\varphi \|_\infty \leq r < 1 \), the set \( E_{r\varphi} \) is empty and therefore the operator \( uC_{r\varphi} \) is compact. Thus \( \| uC_\varphi \|_e \leq \| uC_\varphi - uC_{r\varphi} \| \). But

\[
(4.1) \quad \| uC_\varphi - uC_{r\varphi} \|^q = \sup_{\|f\|_\infty \leq 1} \int_{\mathbb{T}} |u(\zeta)|^q |f(\varphi(\zeta)) - f(r\varphi(\zeta))|^q \, dm(\zeta).
\]

If \( |E_\varphi| = 1 \) then the integral in (4.1) coincides with

\[
\int_{E_\varphi} |u(\zeta)|^q |f(\varphi(\zeta)) - f(r\varphi(\zeta))|^q \, dm(\zeta)
\]

which is less than \( 2^q \int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta) \). If \( |E_\varphi| < 1 \) we let \( F_\varepsilon = \{ \zeta \in \mathbb{T} \mid |\varphi^*(\zeta)| < 1 - \varepsilon \} \) for \( \varepsilon > 0 \), which is a nonempty set for \( \varepsilon \) sufficiently small.
(Let us mention here that an element $\zeta \in \mathbb{T}$ needs not to satisfy neither $\zeta \in E$ nor $\zeta \in \bigcup_{\varepsilon > 0} F_\varepsilon$. It can happen that the radial limit $\varphi^*(\zeta)$ does not exist, but this happens only for $\zeta$ belonging to a set of measure zero). We will use the pseudohyperbolic distance $\rho$ defined for $z$ and $w$ in the unit disk by $\rho(z, w) = |z - w|/|1 - \bar{w}z|$. The Pick-Schwarz’s theorem ensures that $\rho(f(z), f(w)) \leq \rho(z, w)$ for every function $f \in B_{H^\infty}$. As a consequence the inequality $|f(z) - f(w)| \leq 2\rho(z, w)$ holds for every $w$ and $z$ in $\mathbb{D}$.

If $\zeta$ is an element of $F_\varepsilon$ then

$$\rho(\varphi(\zeta), r\varphi(\zeta)) = \frac{(1 - r)|\varphi(\zeta)|}{1 - r|\varphi(\zeta)|^2} \leq \frac{1 - r}{1 - r(1 - \varepsilon)^2}.$$

One can choose $0 < r < 1$ satisfying $\sup_{F_\varepsilon} \rho(\varphi(\zeta), r\varphi(\zeta)) < \varepsilon/2$, and therefore

$$|f(\varphi(\zeta)) - f(r\varphi(\zeta))| \leq 2\sup_{F_\varepsilon} \rho(\varphi(\zeta), r\varphi(\zeta)) \leq \varepsilon$$

for all $\zeta \in F_\varepsilon$ and for every function $f$ in the closed unit ball of $H^\infty$. It follows from these estimates and (4.1) that

$$\|uC_{\varphi} - uC_{r\varphi}\|_q \leq \sup_{\|f\|_\infty \leq 1} \left( \int_{F_\varepsilon} |u(\zeta)|^q \varepsilon^q \, dm(\zeta) + \int_{\mathbb{T}\setminus F_\varepsilon} 2^q |u(\zeta)|^q \, dm(\zeta) \right)$$

$$\leq \varepsilon^q \|u\|_q^q + 2^q \int_{\mathbb{T}\setminus F_\varepsilon} |u(\zeta)|^q \, dm(\zeta).$$

Make $\varepsilon$ tend to zero to deduce the upper estimate.

Let us turn to the lower estimate:

**Proposition 4.3.** Suppose that $\varphi$ is an analytic self-map of $\mathbb{D}$ and $u \in H^q$ with $\infty > q \geq 1$. Then

$$\|uC_{\varphi}\|_e \geq \frac{1}{2} \left( \int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta) \right)^{\frac{1}{q}}.$$

**Proof.** Take a compact operator $K \in B(H^\infty, H^q)$. Since the sequence $(z^n)_{n \in \mathbb{N}}$ is bounded in $H^\infty$, there exists an increasing sequence of integers $(n_k)_{k \geq 0}$ such that $(K(z^{n_k}))_{k \geq 0}$ converges in $H^q$. For any $\varepsilon > 0$ one can find $N \in \mathbb{N}$ such that for every $k, m \geq N$ we have $\|Kz^{n_k} - Kz^{n_m}\|_q < \varepsilon$. If $0 < r < 1$, we let $g_r(z) = g(rz)$ for a function $g$ defined on $\mathbb{D}$. Take $k \geq N$. Then there exists $0 < r < 1$ such that

$$\|u\varphi^{n_k}\|_r \geq \|u\varphi^{n_k}\|_q - \varepsilon.$$
For all \( m \geq N \) we have
\[
\| uC\varphi + K \| \geq \left\| (uC\varphi + K) \left( \frac{z^{n_k} - z^{n_m}}{2} \right) \right\|_q \\
\geq \frac{1}{2} \| u(\varphi^{n_k} - \varphi^{n_m}) \|_q - \frac{\varepsilon}{2} \\
\geq \frac{1}{2} \| (u\varphi^{n_k})_r - (u\varphi^{n_m})_r \|_q - \frac{\varepsilon}{2} \\
\geq \frac{1}{2} \left( \| (u\varphi^{n_k})_r \|_q - \| (u\varphi^{n_m})_r \|_q \right) - \frac{\varepsilon}{2} \\
\geq \frac{1}{2} \left( \| u\varphi^{n_k} \|_q - \| (u\varphi^{n_m})_r \|_q \right) - \varepsilon.
\]

Let us make \( m \) tend to infinity, keeping in mind that \( 0 < r < 1 \) and \( \| \varphi_r \|_\infty < 1 \):
\[
\| (u\varphi^{n_m})_r \|_q \leq \| u\|_q \| (\varphi_r)^{n_m} \|_\infty \leq \| u\|_q \| \varphi_r \|_\infty^m \xrightarrow{m \to \infty} 0.
\]
Thus \( \| uC\varphi + K \| \geq (1/2)\| u\varphi^{n_k} \|_q - \varepsilon \) for all \( k \geq N \). We conclude noticing that
\[
\| u\varphi^{n_k} \|_q = \left( \int_{E\varphi} |u(\zeta)\varphi(\zeta)^{n_k}|^q \ dm(\zeta) \right)^{\frac{1}{q}} \xrightarrow{k \to \infty} \left( \int_{E\varphi} |u(\zeta)|^q \ dm(\zeta) \right)^{\frac{1}{q}}.
\]

\[\square\]

5 \( uC\varphi \in B(H^p, H^q) \) for \( \infty > p > q \geq 1 \)

In [9], the authors give an estimate of the essential norm of a composition operator between \( H^p \) and \( H^q \) for \( 1 < q < p < \infty \). The proof makes use of the Riesz projection from \( L^q \) onto \( H^q \), which is a bounded operator for \( 1 < q < \infty \). Since it is not bounded from \( L^1 \) to \( H^1 \) (\( H^1 \) is not even complemented in \( L^1 \)) there is no way to use a similar argument. So we need a different approach to get some estimates for \( q = 1 \). A solution is to make use of Carleson measures. First, we give a characterization of the boundedness of \( uC\varphi \) in terms of a Carleson measure. In the case where \( p > q \), Carleson measures on \( \overline{\mathbb{D}} \) are characterized in [11]. Denote by \( \Gamma(\zeta) \) the Stolz domain generated by \( \zeta \in \mathbb{T} \), i.e. the interior of the convex hull of the set \( \{\zeta\} \cup (\alpha \mathbb{D}) \), where \( 0 < \alpha < 1 \) is arbitrary but fixed.

Theorem 5.1 (see [11] Theorem 2.2). Let \( \mu \) be a measure on \( \overline{\mathbb{D}} \), \( 1 \leq q < p < \infty \) and \( s = p/(p-q) \). Then \( \mu \) is a \( (p,q) \)-Carleson measure on \( \overline{\mathbb{D}} \)
if and only if \( \zeta \mapsto \int_{\Gamma(\zeta)} \frac{d\mu(z)}{1-|z|^2} \) belongs to \( L^s(\mathbb{T}) \) and \( \mu_T = Fdm \) for a function \( F \in L^s(\mathbb{T}) \).

This leads to a characterization of the continuity of a weighted composition operator between \( H^p \) and \( H^q \):

**Corollary 5.2.** Let \( u \) be an analytic function on \( \mathbb{D} \) and \( \varphi \) an analytic self-map of \( \mathbb{D} \). For \( 1 \leq q < p < \infty \), the weighted composition operator \( uC_{\varphi} : H^p \rightarrow H^q \) is bounded if and only if \( G : \zeta \in \mathbb{T} \mapsto G(\zeta) = \int_{\Gamma(\zeta)} \frac{d\mu(z)}{1-|z|^2} \) belongs to \( L^s(\mathbb{T}) \) for \( s = p/(p-q) \) and \( \mu_{\varphi|_T} = Fdm \) for a certain \( F \in L^s(\mathbb{T}) \), where \( d\mu = |u|^qdm \) and \( \mu_{\varphi} = \mu \circ \varphi^{-1} \) is the pullback measure of \( \mu \) by \( \varphi \).

**Proof.** \( uC_{\varphi} \) is a bounded operator if and only if there exists \( \gamma > 0 \) such that for any \( f \in H^p \), \( \int_{\mathbb{T}} |u(\zeta)|^q |f \circ \varphi(\zeta)|^q \, dm(\zeta) \leq \gamma \| f \|_p^q \), which is equivalent (via a change of variables) to \( \int_{\mathbb{T}} |f(z)|^q \, d\mu_{\varphi}(z) \leq \gamma \| f \|_p^q \) for every \( f \in H^p \). This exactly means that \( \mu_{\varphi} \) is a \((p,q)\)-Carleson measure. This is equivalent by Theorem 5.1 to the condition announced.

If \( f \in H^p \), the Hardy-Littlewood maximal nontangential function \( Mf \) is defined by \( Mf(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)| \) for \( \zeta \in \mathbb{T} \). For \( 1 < p < \infty \), \( M \) is a bounded operator from \( H^p \) to \( L^p \) and we will denote its norm by \( \| M \|_p \). The following lemma is the analogue version of Lemma 2.6 for the case \( p > q \).

**Lemma 5.3.** Let \( \mu \) be a positive Borel measure on \( \overline{\mathbb{D}} \). Assume that \( \mu \) is a \((p,q)\)-Carleson measure for \( 1 \leq q < p < \infty \). Let \( 0 < r < 1 \) and \( \mu_r := \mu_{|D_rD|} \). Then \( \mu_r \) is a \((p,q)\)-Carleson measure, and there exists a positive constant \( \gamma \) such that for every \( f \in H^p \),

\[
\int_{\mathbb{D}} |f(z)|^q \, d\mu_r(z) \leq (\| F \|_s + \gamma \| M \|_p \| \widetilde{G}_r \|_s) \| f \|_p^q
\]

where \( d\mu_T = Fdm \) and \( \widetilde{G}_r(\zeta) = \int_{\Gamma(\zeta)} \frac{d\mu(z)}{1-|z|^2} \). In addition, \( \| \widetilde{G}_r \|_s \rightarrow 0 \) as \( r \rightarrow 1 \).

We use the notation \( \widetilde{G}_r \) to avoid any confusion with the notation introduced before for \( \varphi \) and its radial function \( \varphi_r \).

**Proof.** Being a \((p,q)\)-Carleson measure only depends on the ratio \( p/q \) (see [I Lemma 2.1]), so we have to show that \( \mu_r \) is a \((p/q,1)\)-Carleson measure. From the definition it is clear that \( \widetilde{G}_r \leq G \in L^s(\mathbb{T}) \). Moreover \( d\mu_{r|_T} =
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\[ d\mu_T = F dm \in L^s(\mathbb{T}). \]

Corollary 5.2 ensures the fact that \( \mu_r \) is a \((p, q)\)-Carleson measure.

Let \( f \) be in \( H^p \). Then

\[
\int_{\mathbb{T}} |f(\zeta)|^q \ d\mu_r(\zeta) = \int_{\mathbb{T}} |f(\zeta)|^q \ d\mu(\zeta) = \int_{\mathbb{T}} |f(\zeta)|^q F(\zeta) \ dm(\zeta)
\]

\[
\leq \left( \int_{\mathbb{T}} |f(\zeta)|^p \ dm(\zeta) \right)^{\frac{q}{p}} \|F\|
\]

(5.1)

\[
\leq \|f\|_p^q \|F\|
\]

using Hölder’s inequality with conjugate exponents \( p/q \) and \( s \).

For \( z \neq 0, z \in \mathbb{D} \), let \( \tilde{I}(z) = \{ \zeta \in \mathbb{T} \mid z \in \Gamma(\zeta) \} \). In other words \( \zeta \in \tilde{I}(z) \leftrightarrow z \in \Gamma(\zeta) \). Then

\[
m\left( \tilde{I}(z) \right) \approx 1 - |z|
\]

and

\[
\int_{\mathbb{D}} |f(z)|^q \ d\mu_r(z) \approx \int_{\mathbb{D}} |f(z)|^q \left( \int_{\tilde{I}(z)} \ dm(\zeta) \right) \frac{d\mu_r(z)}{1 - |z|^2}
\]

\[
= \int_{\mathbb{T}} \int_{\Gamma(\zeta)} |f(z)|^q \ \frac{d\mu_r(z)}{1 - |z|^2} \ dm(\zeta)
\]

\[
\leq \int_{\mathbb{T}} M f(\zeta)^q \ \frac{d\mu_r(z)}{1 - |z|^2} \ dm(\zeta)
\]

where \( M f(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)| \) is the Hardy-Littlewood maximal nontangential function. We apply Hölder’s inequality to obtain

\[
\int_{\mathbb{D}} |f(z)|^q \ d\mu_r(z) \leq \gamma \|M f\|_p^q \|\tilde{G}_r\|_s \leq \gamma \|M\|_p^q \|\tilde{G}_r\|_s \|f\|_p^q,
\]

(5.3)

where \( r \) is a positive constant that appears in (5.2). Combining (5.1) and (5.3) it follows that

\[
\int_{\mathbb{D}} |f(z)|^q \ d\mu_r(z) \leq (\|F\|_s + \gamma \|M\|_p^q \|\tilde{G}_r\|_s) \|f\|_p^q.
\]

It remains to show that \( \|\tilde{G}_r\|_s \to 0 \) when \( r \to 1 \). We will make use of Lebesgue’s dominated convergence theorem. Clearly we have \( 0 \leq \tilde{G}_r \leq G \in L^s(\mathbb{T}) \), so we need to show that \( \tilde{G}_r(\zeta) \to 0 \) as \( r \to 1 \) for \( m \)-almost every \( \zeta \in \mathbb{T} \). Let \( A = \{ \zeta \in \mathbb{T} \mid G(\zeta) < \infty \} \). It is a set of full measure \( (m(A) = 1) \) since \( G \in L^s(\mathbb{T}) \). Write \( \tilde{G}_r(\zeta) = \int_{\Gamma(\zeta)} \tilde{f}_r(z) \ d\mu(z) \) with \( \tilde{f}_r(z) = \mathbb{1}_{\mathbb{D}\setminus r\mathbb{D}}(z)(1 - |z|^2)^{-1} \), \( z \in \Gamma(\zeta) \). For every \( \zeta \in A \) one has

\[
|\tilde{f}_r(z) | \leq \frac{1}{1 - |z|^2} \in L^1(\Gamma(\zeta), \mu) \text{ since } \zeta \in A,
\]

\[
\tilde{f}_r(z) \to 0 \text{ for all } z \in \Gamma(\zeta) \subset \mathbb{D}.
\]
Lebesgue’s dominated convergence theorem in $L^1(\Gamma(\zeta), \mu)$ ensures that $\tilde{G}_r(\zeta) = \|\tilde{f}_r\|_{L^1(\Gamma(\zeta), \mu)}$ tends to zero as $r$ tends to 1 for $m$—almost every $\zeta \in \mathbb{T}$, which ends the proof.

\textbf{Theorem 5.4.} Let $u$ be an analytic function on $\mathbb{D}$ and $\varphi$ an analytic self-map of $\mathbb{D}$. Assume that $uC_\varphi$ is a bounded operator from $H^p$ to $H^q$, with $\infty > p > q \geq 1$. Then

$$\|uC_\varphi\|_e \leq 2 \|C_\varphi\|_{p/q}^{1/q} \left( \int_{E_\varphi} |u(\zeta)|^{\frac{pq}{p-q}} \, dm(\zeta) \right)^{\frac{p-q}{pq}},$$

where $\|C_\varphi\|_{p/q}$ denotes the norm of $C_\varphi$ acting on $H^{p/q}$.

\textbf{Proof.} We follow the same lines as in the proof of the upper estimate in Proposition 2.4: we have the decomposition $I = K_N + R_N$ in $B(H^p)$, where $K_N$ is the convolution operator by the Fejér kernel, and

$$\|uC_\varphi\|_e \leq \liminf_N \|uC_\varphi R_N\|.$$ 

We also have, for every $0 < r < 1$,

$$\|(uC_\varphi R_N)f\|_q = \int_{\mathbb{D} \setminus r\mathbb{D}} |R_Nf(w)|^q \, d\mu_\varphi(w) + \int_{r\mathbb{D}} |R_Nf(w)|^q \, d\mu_\varphi(w)
= I_1(N, r, f) + I_2(N, r, f).$$

As in the $p \leq q$ case, we show that

$$\limsup_N I_2(N, r, f) = 0.$$ 

The measure $\mu_\varphi$ being a $(p, q)$-Carleson measure, we use Lemma 5.3 to have the following inequality

$$I_1(N, r, f) \leq (\|F\|_s + \gamma \|M\|_p \|G_r\|_s) \|R_Nf\|_p$$

for every $f \in H^p$. As a consequence

$$\|uC_\varphi\|_e \leq \liminf_N \left( \sup_{\|f\|_p \leq 1} I_1(N, r, f) \right)^{\frac{1}{q}} \leq 2(\|F\|_s + \gamma \|M\|_p \|G_r\|_s)^{\frac{1}{q}}$$

using the fact that $\sup_N \|R_N\| \leq 2$. Now we make $r$ tend to 1, keeping in mind that $\|G_r\|_s \to 0$. We obtain

$$\|uC_\varphi\|_e \leq 2 \|F\|_s^{1/q}.$$
Essential norms of weighted composition operators on Hardy spaces

It remains to see that we can choose $F$ in such a way that

$$\|F\|_s \leq \|C_\varphi\|_{p/q} \left( \int_{E_\varphi} |u(\zeta)|^{\frac{pq}{p-q}} \, dm(\zeta) \right)^{1/s}.$$ 

Indeed, if $f \in C(T) \cap H^{p/q}$, we apply Hölder’s inequality with conjugates exponents $p/q$ and $s$ to have

$$\left| \int_T f \, d\mu_{\varphi,T} \right| = \left| \int_{E_\varphi} |u|^q f \circ \varphi \, dm \right| \leq \int_{E_\varphi} |u|^q |f \circ \varphi| \, dm \leq \|C_\varphi(f)\|_{p/q} \left( \int_{E_\varphi} |u|^q \, dm \right)^{1/s},$$

meaning that $\mu_{\varphi,T} \in (H^{p/q})^*$, which is isometrically isomorphic to $L^s(T)/H_0^s$, where $H_0^s$ is the subspace of $H^s$ consisting of functions vanishing at zero. If we denote by $N(\mu_{\varphi,T})$ the norm of $\mu_{\varphi,T}$ viewed as an element of $(H^{p/q})^*$, then one can choose $F \in L^s(T)$ satisfying

$$\|F\|_s = N(\mu_{\varphi,T}) \leq \|C_\varphi\|_{p/q} \left( \int_{E_\varphi} |u|^{pq/(p-q)} \, dm \right)^{1/s}$$

and $\mu_{\varphi,T} = F \, dm$ (see for instance [11], p. 194). Finally we have

$$\|uC_\varphi\|_s \leq 2\|C_\varphi\|_{p/q} \left( \int_{E_\varphi} |u(\zeta)|^{\frac{pq}{p-q}} \, dm(\zeta) \right)^{\frac{p-q}{pq}}.$$ 

Although we have not be able to give a corresponding lower bound of this form for the essential norm of $uC_\varphi$, we have the following result:

**Proposition 5.5.** Let $1 \leq q < p < \infty$, and assume that $uC_\varphi \in B(H^p, H^q)$. Then

$$\|uC_\varphi\|_s \geq \left( \int_{E_\varphi} |u(\zeta)|^q \, dm(\zeta) \right)^{\frac{1}{q}}.$$ 

**Proof.** Take a compact operator $K$ from $H^p$ to $H^q$. Since it is completely continuous, and the sequence $(z^n)$ converges weakly to zero in $H^p$, $(K(z^n))_n$ converges to zero in $H^q$. Hence

$$\|uC_\varphi + K\| \geq \|(uC_\varphi + K)z^n\|_q \geq \|uC_\varphi(z^n)\|_q - \|K(z^n)\|_q.$$
for every \( n \geq 0 \). Taking the limit as \( n \) tends to infinity, we have

\[
\|uC_\phi\|_e \geq \left( \int_{E_\phi} |u(\zeta)|^q \, dm(\zeta) \right)^{\frac{1}{q}}.
\]

\( \square \)

Acknowledgements

The author is grateful to the referee for his careful reading and for the several suggestions made for improvement.

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