The semiclassical approximation for the Chern–Simons partition function

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Abstract

The semiclassical approximation for the partition function in Chern–Simons gauge theory is derived using the invariant integration method. Volume and scale factors which were undetermined and had to be fixed by hand in previous derivations are automatically taken account of in this framework. Agreement with Witten’s exact expressions for the partition function in the weak coupling (large $k$) limit is verified for gauge group $SU(2)$ and spacetimes $S^3$, $S^2 \times S^1$, $S^1 \times S^1 \times S^1$ and $L(p, q)$. 
There has been much interest in the pure Chern–Simons gauge theory with spacetime a general 3-dimensional manifold since E. Witten in 1989 gave a prescription for obtaining exact expressions for the partition function and expectation values of Wilson loops \([1]\). This prescription, which is based on a correspondence with 2D conformal field theory, leads in the case of the partition function to a new topological invariant of the 3-manifold\(^1\), and in the case of Wilson loops it leads to the Jones knot polynomial (and generalisations). However, it is far from clear that Witten’s exact prescription is compatible with standard approaches to quantum field theory, in particular with perturbation theory. It is of great theoretical interest to compare results obtained by Witten’s prescription with those obtained by standard approaches; this may lead to new insights into the scope or limitations of quantum field theory in general.

A basic prediction of perturbation theory is that the partition function should coincide with its semiclassical approximation in the weak coupling limit, corresponding to the asymptotic limit of large \(k\) in the case of Chern–Simons gauge theory. This has been investigated in a program initiated by D. Freed and R. Gompf \([2]\), and followed up by other authors \([3, 4, 5]\). In these works the large \(k\) asymptotics of the expressions for the partition function obtained from Witten’s prescription was evaluated for various classes of spacetime 3-manifolds with gauge group \(SU(2)\) and compared with expressions for the semiclassical approximations. Agreement was obtained, but only after fixing by hand the values of certain undetermined quantities (volume and scale factors) appearing in the expressions for the semiclassical approximation.

Our aim in this paper is to derive a complete, self-contained expression for the semiclassical approximation for the Chern–Simons partition function in which undetermined quantities do not appear. We do this using the invariant integration method introduced by A. Schwarz in \([6,\text{ App. II}]\). An important property of the resulting expression is that it is independent of the choice of invariant inner product in the Lie

\(^1\)This was subsequently shown by K. Walker \([7]\) to coincide with the 3-manifold invariant constructed in a rigorous framework by Reshetikhin and Turaev \([8]\).
algebra of $G$ which is required to evaluate it. We provide an explicit demonstration of this; it amounts to showing that the expression is independent of the scale parameter $\lambda$ determining the inner product $\langle a, b \rangle = -\frac{1}{\lambda} \text{Tr}(ab)$ in the Lie algebra.

The resulting expression for the semiclassical approximation is explicitly evaluated for gauge group $SU(2)$ and spacetime 3-manifolds $S^3$, $S^2 \times S^1$, $S^1 \times S^1 \times S^1$ and arbitrary lens space $L(p, q)$. The expression for the semiclassical approximation involves an integral over the moduli space of flat gauge fields, and the calculations involve cases where the moduli space is both discrete ($S^3$, $L(p, q)$) and continuous ($S^2 \times S^1$, $S^1 \times S^1 \times S^1$). After including the standard geometric counterterm in the phase factor we find complete agreement with the exact expressions for the partition function in the large $k$ limit. The techniques and mathematics involved in Witten’s exact prescription (conformal blocks/representation theory for Kac–Moody algebras, surgery techniques) are very different from those used to obtain the semiclassical approximation (gauge theory, Hodge theory, analytic continuation of zeta- and eta-functions). It is remarkable that not only the general features such as the asymptotic $k$-dependence, but also the precise numerical factors (including factors of $\pi$), are reproduced by the semiclassical approximation. Some details of our calculations, along with a more detailed description of the invariant integration method, will be provided in a forthcoming paper [9].

2 The invariant integration method

We briefly recall the invariant integration method of Schwarz [6, App. II]. Let $M$ be a closed manifold, $G$ a compact simple Lie group, $\mathfrak{g}$ the Lie algebra of $G$, $\Omega^q(M, \mathfrak{g})$ the space of $q$-forms on $M$ with values in $\mathfrak{g}$, $\mathcal{A} = \Omega^1(M, \mathfrak{g})$ the space of gauge fields (for simplicity we are assuming trivial bundle structure although this is not necessary), $\mathcal{G}$ the group of gauge transformations, i.e. the maps $\phi : M \to G$ acting on $\mathcal{A}$ by $\phi \cdot A = \phi A \phi^{-1} + \phi d\phi^{-1}$. A choice of metric on $M$ and $G$-invariant inner product in $\mathfrak{g}$ determine a $\mathcal{G}$-invariant inner product in each $\Omega^q(M, \mathfrak{g})$ and $\mathcal{G}$-invariant metrics on $\mathcal{A}$ and $\mathcal{G}$, which in turn determine a metric on $\mathcal{A}/\mathcal{G}$. Let $d^A_q : \Omega^q(M, \mathfrak{g}) \to \Omega^{q+1}(M, \mathfrak{g})$
denote the exterior derivative twisted by gauge field $A$ (then $-d_0^A = -\nabla^A$ is the generator of infinitesimal gauge transformations of $A$).

Consider the partition function of a gauge theory with action functional $S(A)$, formally given by

$$ Z(\alpha) = \frac{1}{V(G)} \int_A D[A] e^{-\frac{1}{\alpha^2} S(A)} $$

where $\alpha$ is the coupling parameter and $V(G)$ is the formal volume of $G$. Rewrite:

$$ Z(\alpha) = \frac{1}{V(G)} \int_{A/G} D[A] V([A]) e^{-\frac{1}{\alpha^2} S(A)} $$

where $[A] = G \cdot A$, the orbit of $G$ through $A$, and $V([A])$ is its formal volume. Let $C \subset A$ denote the subspace of absolute minima for $S$, and $M = C/G$ its moduli space. For $A_\theta \in C$ expand

$$ S(A_\theta + B) = S(A_\theta) + S^{(2)}_{A_\theta}(B) + S^{(3)}_{A_\theta}(B) + \ldots $$

where $S^{(p)}_{A_\theta}(B)$ is of order $p$ in $B \in \Omega^1(M, g)$. Then the asymptotics of $Z(\alpha)$ in the limit $\alpha \to 0$ is given by

$$ Z_{sc}(\alpha) = \frac{1}{V(G)} \int_M D[A_\theta] V([A_\theta]) e^{-\frac{1}{\alpha^2} S(A_\theta)} \int_{\tilde{T}_{[A_\theta]}} D[B] e^{-\frac{1}{\alpha^2} S^{(2)}_{A_\theta}(B)} $$

where $\tilde{T}_{[A_\theta]} = T_{[A_\theta]}(A/G) / T_{[A_\theta]} \cdot M$. Writing

$$ S^{(2)}_{[A_\theta]}(B) = \langle B, D_{A_\theta} B \rangle $$

where $D_{A_\theta}$ is a uniquely determined self-adjoint operator on $\Omega^1(M, g)$ with $\ker(D_{A_\theta}) = T_{A_\theta} C$ one gets

$$ \int_{\tilde{T}_{[A_\theta]}} D[B] e^{-\frac{1}{\alpha^2} S^{(2)}_{A_\theta}(B)} = \int_{\ker(D_{A_\theta})} D[B] e^{-\frac{1}{\alpha^2} \langle B, D_{A_\theta} B \rangle} = \det' \left( \frac{1}{\pi \alpha^2} D_{A_\theta} \right)^{-1/2} $$

Let $H_{A_\theta} \subset G$ denote the isotropy subgroup of $A_\theta$, i.e. the subgroup of gauge transformations which leave $A_\theta$ invariant. $H_{A_\theta}$ consists of constant gauge transformations,
corresponding to a subgroup of \( G \) which we also denote by \( H_{A_\theta} \) (see, e.g., [10, p.132]). Using the one-to-one map
\[
G/H_{A_\theta} \xrightarrow{\cong} \mathcal{G} \cdot A_\theta = [A_\theta] , \quad \phi \mapsto \phi \cdot A_\theta
\]
one gets
\[
V([A_\theta]) = |\det'(d_0^{A_\theta})|V(G/H_{A_\theta}) = V(\mathcal{G})V_G(H_{A_\theta})^{-1} \det'((d_0^{A_\theta})^* d_0^{A_\theta})^{1/2} \tag{2.7}
\]
where \( V_G(H_{A_\theta}) \) is the volume of \( H_{A_\theta} \) considered as a subspace of \( \mathcal{G} \). Substituting (2.6) and (2.7) in (2.4) leads to Schwarz’s expression for the semiclassical approximation [6, App.II, eq.(9)]:
\[
Z_{sc}(\alpha) = \int_\mathcal{M} D[A_\theta] V_G(H_{A_\theta})^{-1} e^{-\frac{i}{\alpha^2} S(A_\theta)} \det'((d_0^{A_\theta})^* d_0^{A_\theta})^{1/2} \det'\left(\frac{1}{\pi \alpha^2} D_{A_\theta}\right)^{-1/2} \tag{2.8}
\]
In the cases of interest \( \mathcal{M} \) is finite-dimensional (e.g. in the Yang–Mills theory it is an instanton moduli space), and the determinants in (2.8) can be zeta-regularised, leading to a finite expression for \( Z_{sc}(\alpha) \) (modulo any difficulties that may arise from \( \mathcal{M} \) not being a smooth compact manifold).

3 The semiclassical approximation in Chern–Simons gauge theory

In Chern–Simons gauge theory, with 3-dimensional \( M \) and gauge group \( G = SU(N) \), the negative number \(-\frac{1}{\alpha^2}\) in (2.1) is replaced by the purely imaginary number \( ik \) \((k \in \mathbb{Z})\). It is therefore natural to take \( \mathcal{C} \) to be the set of all critical points for the Chern–Simons action functional in this case. Then the elements \( A_\theta \) of \( \mathcal{C} \) are the flat gauge fields and \( \mathcal{M} \) is the moduli space of flat gauge fields. Expanding the Chern–Simons action functional
\[
S(A) = \frac{1}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{3.1}
\]
around a flat gauge field \( A_\theta \) one finds
\[
S^{(2)}_{A_\theta}(B) = \frac{1}{4\pi} \int_M \text{Tr}(B \wedge d_1^{A_\theta} B) \tag{3.2}
\]
To obtain $D_{A_\theta}$ from this we need a metric on $M$ and invariant inner product in $g$ to determine the inner product in $\Omega^1(M, g)$. The $G$-invariant inner products in $g$ are those of the form

$$\langle a, b \rangle_g = -\frac{1}{\lambda} \text{Tr}(ab)$$

(3.3)

specified by the scale parameter $\lambda \in \mathbb{R}_+$. Thus:

$$S_{A_\theta}^{(2)}(B) = \langle B, D_{A_\theta} B \rangle, \quad D_{A_\theta} = -\frac{\lambda}{4\pi} * d_1^{A_\theta}$$

(3.4)

where $*$ is the Hodge operator. Using the regularisation procedure of [5] we get

$$\det' \left( \frac{-ik}{\pi} D_{A_\theta} \right)^{-1/2} = \det' \left( \frac{ik \lambda}{4\pi^2} * d_1^{A_\theta} \right)^{-1/2}$$

$$= e^{\frac{2\pi}{\lambda} \eta(A_\theta)} \left( \frac{k \lambda}{4\pi^2} \right)^{-\zeta(A_\theta)/2} \det' \left( (d_1^{A_\theta})^* d_1^{A_\theta} \right)^{-1/4}$$

(3.5)

where $\eta(A_\theta)$ and $\zeta(A_\theta)$ are the analytic continuations to $s = 0$ of the eta function $\eta(s; * d_1^{A_\theta})$ and zeta function $\zeta(s; | * d_1^{A_\theta}|)$ respectively. In [5, eq. (23)] we showed that

$$\zeta(A_\theta) = \dim H^0(A_\theta) - \dim H^1(A_\theta)$$

(3.6)

where $H^q(A_\theta)$ is the $q$’th cohomology space of $d_1^{A_\theta}$. Substituting in (2.8) gives the following expression for the semiclassical approximation for the Chern–Simons partition function:

$$Z_{sc}(k) = \int_M D[A_\theta] V_g(H_{A_\theta})^{-1} e^{i\left( \frac{2\pi}{\lambda} \eta(A_\theta) + kS(A_\theta) \right)} \left( \frac{k \lambda}{4\pi^2} \right)^{-\zeta(A_\theta)/2} \tau'(A_\theta)^{1/2}$$

(3.7)

where $\tau'(A_\theta)$ is the Ray–Singer torsion of $d_1^{A_\theta}$ [11]. Note that this expression does not involve any undetermined quantities; all its ingredients are determined by the choice of metric on $M$ and scale parameter $\lambda$ in the invariant inner product for $g$. We will discuss below the metric dependence of (and its removal from) this expression. But first we derive the following:

**Theorem.** The semiclassical approximation for the Chern–Simons partition function given by (3.7) is independent of the scale parameter $\lambda$.

**Proof.** Since a scaling of the inner product in $g$ is equivalent to a scaling of the metric on $M$ in (3.7) the theorem can be obtained by the general metric-independence.
arguments of Schwarz in [12, §5]. But let us give an explicit derivation. We will show that the \( \lambda \)-dependence of \( V(H_{A\theta}) \) and \( D[A_{\theta}] \) factors out as

\[
V(H_{A\theta}) \sim \lambda^{-(\dim H_{A\theta})/2} \tag{3.8}
\]

\[
D[A_{\theta}] \sim \lambda^{-(\dim T_{[A_{\theta}]}M)/2} \tag{3.9}
\]

Then, since \( \dim H_{A\theta} = \dim H^0(A_{\theta}) \) and (in the generic case) \( \dim T_{[A_{\theta}]}M = \dim H^1(A_{\theta}) \), we have

\[
D[A_{\theta}] V(H_{A\theta})^{-1} \sim \lambda^{(\dim H^0(A_{\theta})-\dim H^1(A_{\theta}))/2}. \tag{3.10}
\]

This \( \lambda \)-dependence cancels against the \( \lambda \)-dependence of \((k\lambda/4\pi^2)^{-\zeta(A_{\theta})/2}\) in (3.7) due to (3.3). It is easy to see that all the other ingredients in (3.7) are independent of \( \lambda \) and it follows that (3.7) is \( \lambda \)-independent as claimed.

To derive (3.8)–(3.9) we begin with a general observation on the change in the volume element under a scaling of inner product in a vectorspace \( U \) of dimension \( d \). Let \( u_1, \ldots, u_d \) be an orthonormal basis for \( U \), then the volume element \( \text{vol} \in \Lambda^d U^* \) is the dual of \( u_1 \wedge \cdots \wedge u_d \in \Lambda^d U \), i.e. \( \text{vol}(u_1 \wedge \cdots \wedge u_d) = 1 \). If we scale the inner product in \( U \) by \( \langle \cdot, \cdot \rangle \rightarrow \langle \cdot, \cdot \rangle_{\lambda} = \frac{1}{\lambda} \langle \cdot, \cdot \rangle \) then an orthonormal basis for the new inner product is \( u^\lambda_1, \ldots, u^\lambda_d \) where \( u^\lambda_j = \sqrt{\lambda} u_j \). The new volume element \( \text{vol}_{\lambda} \), given by \( \text{vol}_{\lambda}(u^\lambda_1 \wedge \cdots \wedge u^\lambda_d) = 1 \), is \( \text{vol}_{\lambda} = \lambda^{-d/2} \text{vol} \). The relations (3.8) and (3.9) follow from this observation together with the fact that the metrics on \( H_{A\theta} \) and \( M \) depend on \( \lambda \) through a factor \( \frac{1}{\lambda} \) due to (3.3). This completes the proof.

Were it not for the metric-dependent phase factor \( e^{\frac{i\pi}{4} \eta(A_{\theta})} \) the semiclassical approximation (3.7) would be independent of the metric on \( M \) by the general arguments in [12, §5]. A related observation is that, as it stands, (3.7) cannot reproduce Witten’s exact formulae for the partition function at large \( k \) because the latter are not only metric-independent but also involve a choice of framing of \( M \). In [1] Witten resolved both of these problems by putting in by hand in the semiclassical approximation a

\footnote{In a previous preprint [13] we arrived at a \( \lambda \)-dependent expression for the semiclassical approximation for \( M = S^3 \). This was due to an error in our calculation equivalent to assuming \( \text{vol}_{\lambda} = \lambda^{d/2} \text{vol} \) instead of \( \lambda^{-d/2} \text{vol} \) in the argument above.}
phase factor (“geometric counterterm”) depending both on the metric and on the framing of $M$. It cancels the metric-dependence of $e^{\frac{i}{4\pi}\eta(A_\theta)}$ and transforms under a change of framing in the same way as the exact expression for the partition function.

We will also put in this factor here. We will carry out the calculations in the canonical framing of Atiyah [14]; then the inclusion of the geometric counterterm in the phase amounts to replacing $\eta(A_\theta) \rightarrow \eta(A_\theta) - \eta(0)$ in (3.7) [2].

To explicitly evaluate (3.7) we use the fact that the moduli space $\mathcal{M}$ can be identified with $\text{Hom}(\pi_1(M), G)/G$, the space of homomorphisms $\pi_1(M) \rightarrow G$ modulo the conjugation action of $G$. This leads, at least for the examples we consider below, to a one-to-one correspondence of the form

$$\tilde{\mathcal{M}} \equiv (G_1 \times \cdots \times G_s)/W \leftrightarrow \mathcal{M}, \quad \theta \leftrightarrow [A_\theta] \quad (3.11)$$

where each $G_i$ is a subspace of $G$, $W$ is a finite group acting on the $G_i$’s and $s$ is the number of generators of $\pi_1(M)$ which can be independently associated with elements of $G$ to determine a homomorphism $\pi_1(M) \rightarrow G$. The inner product (3.3) determines a measure $\mathcal{D}\theta$ on $\tilde{\mathcal{M}}$, and

$$\mathcal{D}[A_\theta] = |J_1(\theta)|\mathcal{D}\theta \quad (3.12)$$

where the Jacobi determinant $|J_1(\theta)|$ depends only on the metric on $M$ and $\mathcal{D}\theta$ depends only on $\lambda$. Similarly,

$$V_G(H_{A_\theta}) = |J_0(\theta)|V(H_{A_\theta}) \quad (3.13)$$

where $V(H_{A_\theta})$ is the volume of $H_{A_\theta}$ as a subspace of $G$, depending only on $\lambda$, and $|J_0(\theta)|$ depends only on the metric on $M$. (Explicitly, $|J_0(\theta)| = V(M)^{(\dim H_{A_\theta})/2}$.)

Putting all this into (3.7) gives

$$Z_{ac}(k) = \int_{\tilde{\mathcal{M}}} \mathcal{D}\theta V(H_{A_\theta})^{-1} e^{i\frac{\pi}{4}(\eta(A_\theta) - \eta(0)) + kS(A_\theta)} \left(\frac{k\lambda}{4\pi^2}\right)^{-\zeta(A_{\theta})/2} \tau(A_\theta)^{1/2} \quad (3.14)$$

where

$$\tau(A_\theta)^{1/2} = |J_0(\theta)|^{-1}|J_1(\theta)|\tau'(A_\theta)^{1/2}. \quad (3.15)$$
This quantity is the square root of the Ray–Singer torsion “as a function of the cohomology”, introduced and shown to be metric-independent in [15, §3]. Since \( \eta(A_\theta) - \eta(0) \) is known to be metric-independent [16] we see that the resulting expression (3.14) for \( Z_{sc}(k) \) is metric-independent as discussed above.

4 Explicit evaluations of the semiclassical approximation

We evaluate \( Z_{sc}(k) \) in the cases where \( G = SU(2) \) and \( M = S^3, S^2 \times S^1, S^1 \times S^1 \times S^1 \) and \( L(p, q) \), and compare with the expressions \( Z_W(k) \) for the partition function obtained from Witten’s exact prescription in the large \( k \) limit. To do the calculations we must choose a value for \( \lambda \); the answers are independent of the choice due to the theorem in §3. A basis for \( g = su(2) \) is

\[
a_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad a_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad a_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\]  

(4.1)

Since \( \text{Tr}(a_ia_j) = -\frac{1}{2} \delta_{ij} \) a convenient choice for \( \lambda \) is

\[
\lambda = 1/2,
\]

(4.2)

then \( \{a_1, a_2, a_3\} \) is an orthonormal basis for \( su(2) \), determining a left invariant metric on \( SU(2) \). The volume of \( SU(2) \) corresponding to this metric can be calculated to be

\[
V(SU(2)) = 16\pi^2.
\]

(4.3)

Define \( U(1) \subset SU(2) \) by

\[
U(1) = \left\{ e^{a_3\theta} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \mid \theta \in [0, 4\pi[ \right\}
\]  

(4.4)

Since \( a_3 \) is a unit vector in \( su(2) \), \( \frac{d}{ds} \bigg|_{s=0} e^{a_3(\theta+s)} = a_3 \) is a unit tangent vector to \( SU(2) \) at \( e^{a_3\theta} \) and it follows that the volume of \( U(1) \) in \( SU(2) \) is

\[
V(U(1)) = \int_0^{4\pi} d\theta = 4\pi.
\]

(4.5)

\( M = S^3 \).
$\pi_1(S^3)$ is trivial, so $\mathcal{M}$ consists of a single point corresponding to $A_\theta = 0$. Then $H_{A_\theta} = H_0 = G = SU(2)$, $\dim H^0(0) = 3$, $\dim H^1(0) = 0$, $\zeta(0) = 3 - 0 = 3$. In [9] we calculate the torsion \((3.15)\) to be

$$\tau(0) = 1$$ \hspace{1cm} (4.6)

Substituting in \((3.14)\) we get

$$Z_{sc}(k) = \frac{1}{V(G)} \left( \frac{k\lambda}{4\pi^2} \right)^{-\zeta(0)/2} \tau(0)^{1/2} = \frac{1}{16\pi^2} \left( \frac{k^2}{4\pi^2} \right)^{-3/2} = \sqrt{2}\pi k^{-3/2}. \hspace{1cm} (4.7)$$

This coincides with the exact formula [1, eq (2.26)] in the large $k$ limit:

$$Z_{W}(k) = \sqrt{\frac{2}{k + 2}} \sin \left( \frac{\pi}{k + 2} \right) \sim \sqrt{2}\pi k^{-3/2} \text{ for } k \to \infty. \hspace{1cm} (4.8)$$

$M = S^2 \times S^1$:

$$\pi_1(S^2 \times S^1) \cong \mathbb{Z},$$

so by standard arguments

$$\mathcal{M} \cong \text{Hom}(\mathbb{Z}, SU(2))/SU(2) \cong U(1)/\mathbb{Z}_2 \cong \tilde{\mathcal{M}} \hspace{1cm} (4.9)$$

where $U(1)$ is given by \((4.4)\) and the action of $\mathbb{Z}_2$ on $U(1)$ is generated by $e^{a_3\theta} \to e^{-a_3\theta}$. It follows that $\tilde{\mathcal{M}}$ can be identified with $[0, 2\pi]$ and $\int_{\tilde{\mathcal{M}}} D\theta(\cdots) = \int_{[0,2\pi]} d\theta(\cdots)$ where $\theta$ is the parameter in \((4.4)\). The isotropy group $H_{A_\theta}$ is the maximal subgroup of $SU(2)$ which commutes with $e^{a_3\theta}$, so $H_{A_\theta} = U(1)$ for $\theta \neq 0$. Hence $V(H_{A_\theta}) = 4\pi$, $\dim H^0(A_\theta) = \dim H^1(A_\theta) = 1$, $\zeta(A_\theta) = 1 - 1 = 0$. In [9] we show that $S(A_\theta) = 0$, $\eta(A_\theta) = 0$ for all $A_\theta$, and calculate

$$\tau(A_\theta) = (2 - 2\cos \theta)^2. \hspace{1cm} (4.10)$$

Putting all this into \((3.14)\) we get

$$Z_{sc}(k) = \int_{[0,2\pi]} d\theta \frac{1}{V(H_{A_\theta})} \left( \frac{k\lambda}{4\pi^2} \right)^{-\zeta(A_\theta)/2} \tau(A_\theta)^{1/2} = \int_{[0,2\pi]} d\theta \frac{1}{4\pi} \left( \frac{k^2}{4\pi^2} \right)^0 (2 - 2\cos \theta) = 1 \hspace{1cm} (4.11)$$

which coincides with the exact formula [9, eq. (4.31)]:

$$Z_{W}(k) = 1 \hspace{1cm} (4.12)$$
\[ M = S^1 \times S^1 \times S^1; \]
\[ \pi_1(S^1 \times S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}, \] so by standard arguments
\[ \mathcal{M} \cong \text{Hom}(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}, SU(2))/SU(2) \cong (U(1) \times U(1) \times U(1))/\mathbb{Z}_2 \equiv \tilde{\mathcal{M}} \] (4.13)

Set \( \theta = (\theta_1, \theta_2, \theta_3) \) where \( \theta_1, \theta_2, \theta_3 \) are 3 copies of the parameter for \( U(1) \) in (4.14); it follows from (4.13) that \( \tilde{\mathcal{M}} \) can be identified with \( [0, 2\pi] \times [0, 4\pi] \times [0, 4\pi] \) and
\[ \int_{\tilde{\mathcal{M}}} \mathcal{D}\theta(\cdots) = \int_{[0,2\pi] \times [0,4\pi] \times [0,4\pi]} d\theta_1 d\theta_2 d\theta_3 (\cdots) \]
We have \( \dim H^1(A_\theta) = \dim T_{\theta_1,\theta_2,\theta_3} \tilde{\mathcal{M}} = 3 \), \( H_{A_\theta} = U(1) \) (except for \( \theta = (0, 0, 0) \) and several other isolated points), so \( \dim H^0(A_\theta) = \dim(U(1)) = 1 \) and \( \zeta(A_\theta) = 1 - 3 = -2 \). In [9] we show that \( S(A_\theta) = 0, \eta(A_\theta) = 0 \) for all \( A_\theta \), and calculate
\[ \tau(A_\theta) = 1. \] (4.14)

Putting all this into (3.14) we get
\[ Z_{sc}(k) = \int_{[0,2\pi] \times [0,4\pi] \times [0,4\pi]} d\theta_1 d\theta_2 d\theta_3 \frac{1}{V(H_{A_\theta})} \left( \frac{k \lambda}{4\pi^2} \right)^{-\zeta(A_\theta)/2} \tau(A_\theta)^{1/2} \]
\[ = \int_{[0,2\pi] \times [0,4\pi] \times [0,4\pi]} d\theta_1 d\theta_2 d\theta_3 \frac{1}{4\pi} \left( \frac{k \lambda}{4\pi^2} \right)^{1/2} \cdot 1 \]
\[ = k \] (4.15)
which coincides in the large \( k \) limit with the exact formula [1, eq.(4.32)]:
\[ Z_W(k) = k + 1. \] (4.16)

\[ M = L(p,q); \]

In this case the quantities of interest have been calculated in [2, 3] and we quote the results. \( L(p,q) = S^3/\mathbb{Z}_p \) for a certain free action of \( \mathbb{Z}_p \) on \( S^3 \) specified by \( p \) and \( q \) (which must be relatively prime), so \( \pi_1(L(p,q)) = \mathbb{Z}_p \) and
\[ \mathcal{M} \cong \text{Hom}(\mathbb{Z}_p, SU(2))/SU(2) \cong \{ e^{i\kappa(4\pi n/p)} \mid 0 \leq n \leq p/2 \} \equiv \tilde{\mathcal{M}} \] (4.17)
(see [2]). Thus the moduli space is discrete, \( \theta \to n, A_\theta \to A_n, \int_{\tilde{\mathcal{M}}} \mathcal{D}\theta(\cdots) \to \sum_{0 \leq n \leq p/2} (\cdots) \) and \( \dim H^1(A_n) = \dim \mathcal{M} = 0 \). The isotropy group \( H_{A_n} \) is the maximal
subgroup of $SU(2)$ whose elements commute with $e^{a_3(4\pi n/p)}$, so for $0 < n < p/2$
$H_{A_n} = U(1)$, $V(H_{A_n}) = 4\pi$, $\dim H^0(A_n) = \dim H_{A_n} = 1$ and $\zeta(A_n) = 1 - 0 = 1$. For
$n = 0$, and for $n = p/2$ if $p/2$ is integer, $e^{a_3(4\pi n/p)} = \pm 1$, so in this case $H_{A_n} = SU(2)$,
$\dim H_{A_n} = 3$ and $\zeta(A_n) = 3 - 0 = 3$. By (3.14), the $k$-dependence of the summand is
$\sim k^{-\zeta(A_n)/2}$ and it follows that the terms corresponding to $n = 0$ and $n = p/2$ (if integer) do not contribute to the large $k$ asymptotics of $Z_{sc}(k)$, so we discard these in
the following; i.e. restrict to $0 < n < p/2$. In [3, eq.(5.3) and prop.5.2] it was shown
that

$$e^{i\pi/4(\eta(A_n) - \eta(0)) + kS(A_n)} = i e^{2\pi q^*(k+2)n^2/p}$$

(4.18)

where $q^*q = 1 \pmod p$. The torsion $\tau(A_n)$ is obtained from the calculations in [3] to be

$$\tau(A_n) = \frac{16}{p} \sin^2 \left( \frac{2\pi n}{p} \right) \sin^2 \left( \frac{2\pi q^*n}{p} \right)$$

(4.19)

It follows that the large $k$ asymptotics of (3.14) in this case is

$$Z_{sc}(k) \xrightarrow{k \to \infty} \frac{1}{V(H_{A_n})} \sum_{0 < n < p/2} \frac{1}{4\pi} e^{i\pi/4(\eta(A_n) - \eta(0)) + kS(A_n)} \left( \frac{k\lambda}{4\pi^2} \right)^{-\zeta(A_n)/2} \tau(A_n)^{1/2}$$

$$= \frac{\sqrt{2}}{\sqrt{p}} \sum_{n=1}^{[p/2]-1} \frac{1}{4\pi} i e^{2\pi q^*(k+2)n^2/p} \left( \frac{k\lambda}{4\pi^2} \right)^{-1/2} 4 \frac{1}{\sqrt{p}} \left| \sin \left( \frac{2\pi n}{p} \right) \sin \left( \frac{2\pi q^*n}{p} \right) \right|$$

(4.20)

This is precisely the formula for the large $k$ asymptotics of the exact partition function $Z_W(k)$ derived in [3, eq.(5.7)].

The calculation of $\tau(A_{\theta})$ in the preceding examples is carried out in [3] using the
equality between Ray–Singer torsion and the combinatorial R-torsion (both consid-
ered as functions of the cohomology) [17]. The R-torsion is calculated in each case
using a suitable cell decomposition of $M$ for which the combinatorial objects cor-
responding to the $|J_q(\theta)|$’s in (3.15) are equal to 1; then the combinatorial object corresponding to $\tau'(A_{\theta})$ is calculated to give (4.6), (4.10), (4.14) and (4.19) respectively. For the cases where $M$ is $S^2 \times S^1$ and $S^1 \times S^1 \times S^1$ the result $\eta(A_{\theta}) = 0$ is
obtained in [9] by decomposing $\Omega^1(M, g)$ into the direct sum of finite-dimensional subspaces invariant under $*d_1^A\theta$, and showing that $*d_1^A\theta$ has symmetric spectrum on each of these subspaces.

In future work we will be checking the agreement between the expression (3.14) for the semiclassical approximation and the exact formulae for the partition function in the large $k$ limit for other more complicated situations, e.g. when $M$ is a torus bundle [3, 18] or a Seifert manifold [4], and for gauge groups other than $SU(2)$. In certain cases, e.g. for certain Seifert manifolds, the $k$-dependence $\sim k^{\max\left\{-\zeta(A\theta)/2\right\}}$ predicted by the semiclassical approximation fails [4]. This is related to the moduli space $\mathcal{M}$ having certain singularities. It is an interesting problem to refine the derivation of the semiclassical approximation given here so that it also works for these cases.

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