Variable separated ODE method–A powerful tool for testing traveling wave solutions of nonlinear equations

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Abstract: The variable separated ODE method is extended by choosing the additional variable separated equation as the general elliptic equation. More exact traveling wave solutions of nonlinear equations are obtained by using the method of comparison of coefficients and the known solutions of the auxiliary equation.

Keywords: Variable separated ODE method; General elliptic equation; Auxiliary equation.

1 Introduction

In 2002, we established a direct algebraic method for solving sine–Gordon type and sinh–Gordon type equations [1], which was named as the variable separated ODE method by Wazwaz [2]. Further applications and generalizations were considered by Wazwaz [2, 3, 4, 5, 6, 7, 8, 9, 10] and others [11, 12, 13], and they successfully solved many generalized sine–Gordon type, sinh-Gordon type, cosine–Gordon, cosh–Gordon type, sine–cosine–Gordon type, sinh–cosh–Gordon type and Liouville type equations, etc. These studies showed that the variable separated ODE method works effectively if the equation involves sine, cosine, hyperbolic sine, and hyperbolic cosine functions. If it is not the case, can we use the variable separated ODE method to solve the nonlinear equations? In other words, can we solve the commonly used nonlinear equations by means of the variable separated ODE method? The purpose of this paper is to give a positive answer to this question. The key to solve the problem is how to select the additional variable separated equation

\[ u'(\xi) = \frac{du}{d\xi} = G(u), \] (1)

which satisfied by the wave transformation \( u(x, t) = u(\xi) , \xi = x - \omega t \). In general, the selection of Eq. (1) is can be performed by many choices, such as the Riccati equation, the Bernoulli equation and the general elliptic equation, etc. In particular, more exact traveling wave solutions for the given nonlinear equations are obtained by choosing the general elliptic
equation as our additional variable separated equation. The advantage of our method is that it does not need to solve the nonlinear equations and the exact traveling wave solutions of the nonlinear equations can be obtained by comparing the coefficients of the related polynomials and using the known solutions of the additional variable separated Eq. (1). Therefore, the method is simpler than the existing direct algebraic methods.

The paper is organized as follows. In section 2 we shall give the description of the method. The introduction of the auxiliary equations and the list of their solutions are also presented. Some examples are presented in the section (3). Section (4) will give some discussions.

2 Description of the method

As described in [1], we first make the wave transformation \( u(x, t) = u(\xi), \xi = x - \omega t \) to carry out a given nonlinear PDE into an equivalent ODE. Substituting Eq. (1) into the given ODE yields a system of algebraic equations that can be solved to determine the unknown parameters. In this paper, the additional variable separated Eq. (1) will be taken as the general elliptic equation [14, 15, 16, 17, 18] of the form

\[
F^{r2}(\xi) = c_0 + c_1 F(\xi) + c_2 F^2(\xi) + c_3 F^3(\xi) + c_4 F^4(\xi),
\]

where \( c_i \) (\( i = 0, 1, 2, 3, 4 \)) are constants. Here we shall use a new classification of solutions for Eq. (2) as the following five cases.

Case 1 \( c_0 = c_1 = 0 \).

\[
\begin{align*}
F_1(\xi) & = \frac{2c_2}{\varepsilon \sqrt{\Delta} \cosh (\sqrt{c_2} \xi) - c_3}, \quad \Delta > 0, c_2 > 0, \\
F_2(\xi) & = \frac{2c_2}{\varepsilon \sqrt{-\Delta} \sinh (\sqrt{c_2} \xi) - c_3}, \quad \Delta < 0, c_2 > 0, \\
F_{3a}(\xi) & = \frac{2c_2}{\varepsilon \sqrt{\Delta} \cos (\sqrt{-c_2} \xi) - c_3}, \\
F_{3b}(\xi) & = \frac{2c_2}{\varepsilon \sqrt{\Delta} \sin (\sqrt{-c_2} \xi) - c_3}, \quad \Delta > 0, c_2 < 0, \\
F_4(\xi) & = -\frac{c_2}{c_3} \left[ 1 + \varepsilon \tanh \left( \frac{\sqrt{c_2}}{2} \xi \right) \right], \quad \Delta = 0, c_2 > 0, \\
F_5(\xi) & = -\frac{c_2}{c_3} \left[ 1 + \varepsilon \coth \left( \frac{\sqrt{c_2}}{2} \xi \right) \right], \quad \Delta = 0, c_2 > 0, \\
F_6(\xi) & = \frac{\varepsilon}{\sqrt{c_4} \xi}, \quad c_2 = c_3 = 0, c_4 > 0, \\
F_7(\xi) & = \frac{4c_3}{c_5^2 \xi - 4c_4}, \quad c_2 = 0,
\end{align*}
\]

where \( \Delta = c_3^2 - 4c_2c_4, \varepsilon = \pm 1. \)
Case 2 \( c_3 = c_4 = 0. \)

\[
F_8(\xi) = -\frac{c_1}{2c_2} + \frac{\varepsilon\sqrt{\delta}}{2c_2} \cosh(\sqrt{c_2}\xi), \quad \delta > 0, c_2 > 0,
\]

\[
F_9(\xi) = -\frac{c_1}{2c_2} + \frac{\varepsilon\sqrt{-\delta}}{2c_2} \sinh(\sqrt{c_2}\xi), \quad \delta < 0, c_2 > 0,
\]

\[
F_{10a}(\xi) = -\frac{c_1}{2c_2} + \frac{\varepsilon\sqrt{\delta}}{2c_2} \cos(\sqrt{c_2}\xi),
\]

\[
F_{10b}(\xi) = -\frac{c_1}{2c_2} + \frac{\varepsilon\sqrt{-\delta}}{2c_2} \sin(\sqrt{c_2}\xi), \quad \delta > 0, c_2 < 0,
\]

\[
F_{11}(\xi) = -\frac{c_1}{2c_2} + e^{\varepsilon\sqrt{c_2}\xi}, \quad \delta = 0, c_2 > 0,
\]

\[
F_{12}(\xi) = \varepsilon\sqrt{c_0}\xi, \quad c_1 = c_2 = 0,
\]

\[
F_{13}(\xi) = -\frac{c_0}{c_1} + \frac{c_1}{4} \xi^2, \quad c_2 = 0,
\]

where \( \delta = c_1^2 - 4c_0c_2, \varepsilon = \pm 1. \)

Case 3 \( c_1 = c_3 = 0. \)

\[
F_{14}(\xi) = \varepsilon \sqrt{-\frac{c_2}{2c_4}} \tanh \left( \sqrt{-\frac{c_2}{2}} \xi \right), \quad \Delta_1 = 0, c_2 < 0, c_4 > 0,
\]

\[
F_{15}(\xi) = \varepsilon \sqrt{-\frac{c_2}{2c_4}} \coth \left( \sqrt{-\frac{c_2}{2}} \xi \right), \quad \Delta_1 = 0, c_2 < 0, c_4 > 0,
\]

\[
F_{16a}(\xi) = \varepsilon \sqrt{\frac{c_2}{2c_4}} \tan \left( \sqrt{\frac{c_2}{2}} \xi \right),
\]

\[
F_{16b}(\xi) = \varepsilon \sqrt{\frac{c_2}{2c_4}} \cot \left( \sqrt{\frac{c_2}{2}} \xi \right), \quad \Delta_1 = 0, c_2 > 0, c_4 > 0;
\]

\[
F_{17}(\xi) = \sqrt{-\frac{c_2m^2}{c_4(m^2 + 1)}} \mathrm{sn} \left( \sqrt{-\frac{c_2}{m^2 + 1}} \xi \right), \quad c_0 = \frac{c_2^2m^2}{c_4(m^2 + 1)^2}, c_2 < 0, c_4 > 0,
\]

\[
F_{18}(\xi) = \sqrt{-\frac{c_2m^2}{c_4(2m^2 - 1)}} \mathrm{cn} \left( \sqrt{-\frac{c_2}{2m^2 - 1}} \xi \right), \quad c_0 = \frac{c_2^2m^2(m^2 - 1)}{c_4(2m^2 - 1)^2},
\]

\( c_2 > 0, c_4 < 0, \)

\[
F_{19}(\xi) = \sqrt{-\frac{c_2}{c_4(2 - m^2)}} \mathrm{dn} \left( \sqrt{-\frac{c_2}{2 - m^2}} \xi \right), c_0 = \frac{c_2^2(1 - m^2)}{c_4(2 - m^2)^2}, c_2 > 0, c_4 < 0,
\]

\[
F_{20}(\xi) = \varepsilon \left( -\frac{4c_0}{c_4} \right)^{\frac{1}{3}} \mathrm{ds} \left( -\frac{4c_0c_4}{\xi}, \sqrt{\frac{2}{2}} \right), \quad c_2 = 0, c_0c_4 < 0,
\]

\[
F_{21}(\xi) = \varepsilon \left( \frac{c_0}{c_4} \right)^{\frac{1}{4}} \left[ \mathrm{ns} \left( 2(c_0c_4)^{\frac{1}{2}} \xi, \sqrt{\frac{2}{2}} \right) + \mathrm{cs} \left( 2(c_0c_4)^{\frac{1}{2}} \xi, \sqrt{\frac{2}{2}} \right) \right],
\]

\( c_2 = 0, c_0c_4 > 0, \)
where $\Delta_1 = c_2^2 - 4c_0c_4, \varepsilon = \pm 1$.

Case 4 $c_2 = c_4 = 0$.

$$F_{22}(\xi) = \varphi \left( \frac{\sqrt{c_3}}{2}, g_2, g_3 \right), \quad g_2 = -\frac{4c_1}{c_3}, g_3 = -\frac{4c_0}{c_3}, c_3 > 0.$$  

Case 5 $c_0 = 0$.

$$F_{23}(\xi) = -\frac{8c_2 \tanh^2 \left( \sqrt{\frac{-c_4}{12}} \xi \right)}{3c_3 \left( 3 + \tanh^2 \left( \sqrt{\frac{-c_4}{12}} \xi \right) \right)},$$

$$F_{24}(\xi) = -\frac{8c_2 \coth^2 \left( \sqrt{\frac{-c_4}{12}} \xi \right)}{3c_3 \left( 3 + \coth^2 \left( \sqrt{\frac{-c_4}{12}} \xi \right) \right)}, \quad c_2 < 0, \quad c_1 = \frac{8c_2^2}{27c_3}, \quad c_4 = \frac{c_3^2}{4c_2};$$

$$F_{25}(\xi) = \frac{8c_2 \tan^2 \left( \sqrt{\frac{-c_4}{12}} \xi \right)}{3c_3 \left( 3 - \tan^2 \left( \sqrt{\frac{-c_4}{12}} \xi \right) \right)},$$

$$F_{26}(\xi) = \frac{8c_2 \cot^2 \left( \sqrt{\frac{-c_4}{12}} \xi \right)}{3c_3 \left( 3 - \cot^2 \left( \sqrt{\frac{-c_4}{12}} \xi \right) \right)}, \quad c_2 > 0, \quad c_1 = \frac{8c_2^2}{27c_3}, \quad c_4 = \frac{c_3^2}{4c_2};$$

$$F_{27}(\xi) = -\frac{c_3}{4c_4} \left[ 1 + \varepsilon \sqrt{\left( \frac{c_3}{4m\sqrt{c_4}} \xi \right)} \right],$$

$$F_{28}(\xi) = -\frac{c_3}{4c_4} \left[ 1 + \varepsilon \sqrt{\left( \frac{c_3}{4m\sqrt{c_4}} \xi \right)} \right],$$

$$c_4 > 0, \quad c_1 = \frac{c_3^2(1 - m^2)}{32m^2c_4^2}, \quad c_2 = \frac{c_3^2(5m^2 - 1)}{16m^2c_4};$$

$$F_{29}(\xi) = -\frac{c_3}{4c_4} \left[ 1 + \varepsilon \sqrt{\left( \frac{c_3}{4m\sqrt{c_4}} \xi \right)} \right],$$

$$F_{30}(\xi) = -\frac{c_3}{4c_4} \left[ 1 + \varepsilon \sqrt{\left( \frac{c_3}{4m\sqrt{c_4}} \xi \right)} \right],$$

$$c_4 > 0, \quad c_1 = \frac{c_3^2(1 - m^2)}{32c_4^2}, \quad c_2 = \frac{c_3^2(5 - m^2)}{16c_4};$$

$$F_{31}(\xi) = -\frac{c_3}{4c_4} \left[ 1 + \varepsilon \sqrt{\left( \frac{c_3}{4m\sqrt{c_4}} \xi \right)} \right],$$

$$F_{32}(\xi) = -\frac{c_3}{4c_4} \left[ 1 + \varepsilon \sqrt{\left( \frac{c_3}{4m\sqrt{c_4}} \xi \right)} \right],$$

$$c_4 < 0, \quad c_1 = \frac{c_3^2}{32m^2c_4^2}, \quad c_2 = \frac{c_3^2(4m^2 + 1)}{16m^2c_4};$$
$F_{33}(\xi) = -\frac{c_3}{4c_4} \left[ 1 + \frac{\varepsilon}{\sqrt{1 - m^2}} \text{dn} \left( \frac{c_3}{4\sqrt{c_4(m^2 - 1)}} \xi \right) \right],$

$F_{34}(\xi) = -\frac{c_3}{4c_4} \left[ 1 + \frac{\varepsilon}{\text{dn} \left( \frac{c_3}{4\sqrt{c_4(m^2 - 1)}} \xi \right)} \right],$

$c_4 < 0, c_1 = \frac{c_3^3m^2}{32c_4^4(m^2 - 1)}, c_2 = \frac{c_3^2(5m^2 - 4)}{16c_4(m^2 - 1)};$

$F_{35}(\xi) = -\frac{c_3}{4c_4} \left[ 1 + \frac{\varepsilon}{\text{cn} \left( \frac{c_3}{4\sqrt{c_4(1-m^2)}} \xi \right)} \right],$

$F_{36}(\xi) = -\frac{c_3}{4c_4} \left[ 1 + \frac{\varepsilon \text{dn} \left( \frac{c_3}{4\sqrt{c_4(1-m^2)}} \xi \right)}{\sqrt{1 - m^2} \text{cn} \left( \frac{c_3}{4\sqrt{c_4(1-m^2)}} \xi \right)} \right],$

$c_4 > 0, c_1 = \frac{c_3^3}{32c_4^4(1-m^2)}, c_2 = \frac{c_3^2(4m^2 - 5)}{16c_4(m^2 - 1)};$

$F_{37}(\xi) = -\frac{c_3}{4c_4} \left[ 1 + \varepsilon \text{dn} \left( \frac{-c_3}{4\sqrt{-c_4}} \xi \right) \right],$

$F_{38}(\xi) = -\frac{c_3}{4c_4} \left[ 1 + \frac{\varepsilon \sqrt{1 - m^2}}{\text{dn} \left( \frac{-c_3}{4\sqrt{-c_4}} \xi \right)} \right], c_4 < 0, c_1 = \frac{c_3^3m^2}{32c_4^4}, c_2 = \frac{c_3^2(m^2 + 4)}{16c_4},$

where $\varepsilon = \pm 1.$

### 3 Illustrative examples

In this section we shall give three illustrative examples for using our method to solve nonlinear equations.

**Example 1** The modified Benjamin–Bona–Mahony (mBBM) equation

$$u_t + u_x + u^2 u_x + u_{xxt} = 0.$$  \hspace{1cm} (3)

Substituting the wave transformation $u(x, t) = u(\xi), \xi = x - \omega t$ into (3) and integrating once with respect to $\xi$ leads the following ODE

$$(1 - \omega) u(\xi) + \frac{1}{3} u^3(\xi) - \omega u''(\xi) + B = 0,$$

which can be rewritten as

$$u''(\xi) = \frac{B}{\omega} + \frac{1 - \omega}{\omega} u(\xi) + \frac{1}{3\omega} u^3(\xi),$$  \hspace{1cm} (4)
where \( B \) is the integration constant.

In accordance with the form of Eq. (4), we can take the Eq. (1) as the general elliptic equation for unknown function \( u(\xi) \) of the form

\[
u'^2(\xi) = G(u) = c_0 + c_1 u(\xi) + c_2 u^2(\xi) + c_3 u^3(\xi) + c_4 u^4(\xi),
\]

which has the second form

\[
u''(\xi) = \frac{c_1}{2} + c_2 u(\xi) + \frac{3c_3}{2} u^2(\xi) + 2c_4 u^3(\xi), \quad (5)
\]

where \( c_i (i = 0, 1, 2, 3, 4) \) are undetermined parameters.

Comparing the coefficients of \( u^j \ (j = 0, 1, 2, 3) \) in Eq. (4) and Eq. (5) yields

\[
c_0 = c_0, \quad c_1 = \frac{2B}{\omega}, \quad c_2 = \frac{1 - \omega}{\omega}, \quad c_3 = 0, \quad c_4 = \frac{1}{6\omega}. \quad (6)
\]

When \( c_0 = B = 0 \), if we take (6) into the solutions of the general elliptic equation (2) given by Case 1, then the exact traveling wave solutions of the mBBM equation are obtained as follows

\[
u_1(x, t) = \varepsilon \sqrt{6(1 - \omega)} \text{csch}\left(\frac{1 - \omega}{\omega} (x - \omega t)\right), \quad 0 < \omega < 1,
\]

\[
u_2(x, t) = \varepsilon \sqrt{6(\omega - 1)} \sec\left(\frac{\omega - 1}{\omega} (x - \omega t)\right), \quad \omega > 1,
\]

\[
u_3(x, t) = \varepsilon \sqrt{6(\omega - 1)} \csc\left(\frac{\omega - 1}{\omega} (x - \omega t)\right), \quad \omega > 1,
\]

\[
u_4(x, t) = \frac{\sqrt{6\varepsilon}}{x - t + \xi_0}, \quad \omega = 1.
\]

When \( c_0 \neq 0, B = 0 \), taking (6) into the solutions of the general elliptic equation (2) given by Case 3, we get the following exact traveling wave solutions of the mBBM equation

\[
u_5(x, t) = \varepsilon \sqrt{3(\omega - 1)} \tanh\left(\frac{\omega - 1}{2\omega} (x - \omega t)\right), \quad \omega > 1,
\]

\[
u_6(x, t) = \varepsilon \sqrt{3(\omega - 1)} \coth\left(\frac{\omega - 1}{2\omega} (x - \omega t)\right), \quad \omega > 1,
\]

\[
u_7(x, t) = \varepsilon \sqrt{3(1 - \omega)} \tan\left(\frac{1 - \omega}{2\omega} (x - \omega t)\right), \quad 0 < \omega < 1,
\]

\[
u_8(x, t) = \varepsilon \sqrt{3(1 - \omega)} \cot\left(\frac{1 - \omega}{2\omega} (x - \omega t)\right), \quad 0 < \omega < 1,
\]
\[ u_9(x,t) = \sqrt{\frac{6(\omega - 1)m^2}{m^2 + 1}} \text{sn} \left( \sqrt{\frac{\omega - 1}{\omega(m^2 + 1)}} (x - \omega t), m \right), \omega > 1, \]

\[ u_{10}(x,t) = \varepsilon (-24c_0)^{\frac{1}{4}} \text{ds} \left( \left( -\frac{2}{3} c_0 \right)^{\frac{1}{4}} (x - t), \frac{\sqrt{2}}{2} \right), \omega = 1, c_0 < 0, \]

\[ u_{11}(x,t) = \varepsilon (6c_0)^{\frac{1}{4}} \left[ \text{ns} \left( 2 \left( \frac{c_0}{6} \right)^{\frac{1}{4}} (x - t), \frac{\sqrt{2}}{2} \right) + \text{cs} \left( 2 \left( \frac{c_0}{6} \right)^{\frac{1}{4}} (x - t), \frac{\sqrt{2}}{2} \right) \right], \omega = 1, c_0 > 0. \]

For solutions \( u_j (j = 5, 6, 7, 8) \), the condition \( \Delta_1 = c_2^2 - 4c_0c_4 = \left( \frac{1-\omega}{\omega} \right)^2 - \frac{2\omega}{\omega} = 0 \) leads \( c_0 = \frac{3(1-\omega)^2}{2\omega} \). For solution \( u_9 \), we have \( c_0 = \frac{6(1-\omega)^2m^2(m^2-1)}{\omega(2m^2-1)^2} \). But the solutions \( u_j (j = 5, 6, 7, 8, 9) \) do not require any condition and they are true for any arbitrary constant \( c_0 \neq 0 \). Therefore, it does not need to calculate the value of \( c_0 \).

**Example 2** Consider the nonlinear Schrödinger (NLS) equation

\[ iu_t + \alpha u_{xx} + \beta |u|^2 u = 0, \quad \text{(7)} \]

where \( \alpha \) and \( \beta \) are constants.

Taking the transformation

\[ u(x,t) = v(\xi)e^{i\eta}, \xi = x - \omega t, \eta = kx + ct, \quad \text{(8)} \]

with undetermined constants \( c, k \) and \( \omega \) into Eq. (7) and separating the real part and the imaginary part yields

\[
\begin{cases}
-\omega v'\xi) + 2akv'(\xi) = 0, \\
- cv\xi) + \alpha \left( v''\xi) - k^2 v\xi) \right) + \beta v^3\xi) = 0.
\end{cases}
\]

The first equation leads \( k = \frac{\omega}{2\alpha} \) and the second equation becomes

\[-cv\xi) + \alpha \left( v''\xi) - \frac{\omega^2}{4\alpha^2} v\xi) \right) + \beta v^3\xi) = 0.
\]

This equation can be rewritten as

\[ v''\xi) = \frac{\omega^2 + 4\alpha c}{4\alpha^2} v\xi) - \frac{\beta}{\alpha} v^3\xi). \quad \text{(9)} \]

If we choose the Eq. (11) as the general elliptic equation

\[ v'^2\xi) = G^2(v) = c_0 + c_1 v\xi) + c_2 v^2\xi) + c_3 v^3\xi) + c_4 v^4\xi), \]

then by differentiating the above equation with respect to \(\xi\), we obtain

\[
v''(\xi) = \frac{c_1}{2} + c_2 v(\xi) + \frac{3c_3}{2} v^2(\xi) + 2c_4 v^3(\xi).
\] (10)

By comparing the coefficients of \(v^j\) \((j = 0, 1, 2, 3)\) in Eq. (9) and Eq. (10), the unknown parameters are now determined to be

\[
c_0 = c_0, c_1 = 0, c_2 = \frac{\omega^2 + 4\alpha c}{4\alpha^2}, c_3 = 0, c_4 = -\frac{\beta}{2\alpha}.
\] (11)

When \(c_0 = 0\), inserting (11) into the solutions of the general elliptic equation given by Case 1 and using (8) we get the following exact traveling wave solutions of NLS equation

\[
u_1(x, t) = \varepsilon \sqrt{\frac{\omega^2 + 4\alpha c}{2\alpha}} \text{sech} \left[ \frac{1}{2} \sqrt{\frac{\omega^2 + 4\alpha c}{\alpha^2}} (x - \omega t) \right] e^{i\left(\frac{\omega}{2\alpha} x + ct\right)}, \omega^2 + 4\alpha c > 0, \alpha \beta > 0,
\]

\[
u_2(x, t) = \varepsilon \sqrt{-\frac{\omega^2 + 4\alpha c}{2\alpha}} \text{csch} \left[ \frac{1}{2} \sqrt{\frac{\omega^2 + 4\alpha c}{\alpha^2}} (x - \omega t) \right] e^{i\left(\frac{\omega}{2\alpha} x + ct\right)}, \omega^2 + 4\alpha c > 0, \alpha \beta < 0,
\]

\[
u_3(x, t) = \varepsilon \sqrt{\frac{\omega^2 + 4\alpha c}{2\alpha}} \sec \left[ \frac{1}{2} \sqrt{-\frac{\omega^2 + 4\alpha c}{\alpha^2}} (x - \omega t) \right] e^{i\left(\frac{\omega}{2\alpha} x + ct\right)}, \omega^2 + 4\alpha c < 0, \alpha \beta < 0,
\]

\[
u_4(x, t) = \varepsilon \sqrt{\frac{\omega^2 + 4\alpha c}{2\alpha}} \csc \left[ \frac{1}{2} \sqrt{-\frac{\omega^2 + 4\alpha c}{\alpha^2}} (x - \omega t) \right] e^{i\left(\frac{\omega}{2\alpha} x + ct\right)}, \omega^2 + 4\alpha c < 0, \alpha \beta > 0,
\]

where \(\omega, \xi_0\) are arbitrary constants.

When \(c_0 \neq 0\), substituting (11) into the solutions of general elliptic equation given by Case 3, we obtain the exact traveling wave solutions of NLS equation as follows

\[
u_5(x, t) = \varepsilon \sqrt{-\frac{\beta}{2\alpha}} (x - \omega t + \xi_0) e^{i\left(\frac{\omega}{2\alpha} x + ct\right)}, \omega^2 + 4\alpha c = 0, \alpha \beta > 0,
\]

\[
u_6(x, t) = \varepsilon \sqrt{-\frac{2(\omega^2 + 4\alpha c)}{\alpha^2}} \tanh \left[ \frac{1}{4} \sqrt{-\frac{2(\omega^2 + 4\alpha c)}{\alpha^2}} (x - \omega t) \right] e^{i\left(\frac{\omega}{2\alpha} x + ct\right)}, \omega^2 + 4\alpha c < 0, \alpha \beta < 0,
\]

\[
u_7(x, t) = \varepsilon \sqrt{-\frac{2(\omega^2 + 4\alpha c)}{\alpha^2}} \coth \left[ \frac{1}{4} \sqrt{-\frac{2(\omega^2 + 4\alpha c)}{\alpha^2}} (x - \omega t) \right] e^{i\left(\frac{\omega}{2\alpha} x + ct\right)}, \omega^2 + 4\alpha c < 0, \alpha \beta < 0,
\]
\[ u_8(x,t) = \frac{\varepsilon}{2} \sqrt{-\frac{\omega^2 + 4\alpha c}{\alpha \beta}} \tan \left[ \frac{1}{4} \sqrt{\frac{2(\omega^2 + 4\alpha c)}{\alpha^2}} (x - \omega t) \right] e^{i \left( \frac{\omega}{\alpha} x + ct \right)}, \]

\[ \omega^2 + 4\alpha c > 0, \alpha \beta < 0, \]

\[ u_9(x,t) = \frac{\varepsilon}{2} \sqrt{-\frac{\omega^2 + 4\alpha c}{\alpha \beta}} \cot \left[ \frac{1}{4} \sqrt{\frac{2(\omega^2 + 4\alpha c)}{\alpha^2}} (x - \omega t) \right] e^{i \left( \frac{\omega}{\alpha} x + ct \right)}, \]

\[ \omega^2 + 4\alpha c > 0, \alpha \beta < 0, \]

\[ u_{10}(x,t) = \sqrt{\frac{m^2(\omega^2 + 4\alpha c)}{2\alpha \beta(2m^2 - 1)}} \cn \left( \frac{1}{2} \sqrt{\frac{\omega^2 + 4\alpha c}{\alpha^2(2m^2 - 1)}} (x - \omega t), m \right) e^{i \left( \frac{\omega}{\alpha} x + ct \right)}, \]

\[ \omega^2 + 4\alpha c > 0, \alpha \beta > 0, \frac{1}{2} < m^2 < 1, \]

\[ u_{11}(x,t) = \sqrt{\frac{m^2(\omega^2 + 4\alpha c)}{2\alpha \beta(m^2 + 1)}} \sn \left( \frac{1}{2} \sqrt{\frac{\omega^2 + 4\alpha c}{\alpha^2(m^2 + 1)}} (x - \omega t), m \right) e^{i \left( \frac{\omega}{\alpha} x + ct \right)}, \]

\[ \omega^2 + 4\alpha c > 0, \alpha \beta > 0, \]

\[ u_{12}(x,t) = \sqrt{\frac{\omega^2 + 4\alpha c}{2\alpha \beta(2 - m^2)}} \dn \left( \frac{1}{2} \sqrt{\frac{\omega^2 + 4\alpha c}{\alpha^2(2 - m^2)}} (x - \omega t), m \right) e^{i \left( \frac{\omega}{\alpha} x + ct \right)}, \]

\[ \omega^2 + 4\alpha c > 0, \alpha \beta < 0, \]

\[ u_{13}(x,t) = \varepsilon \left( \frac{-2\omega^2 c_0}{\beta c} \right)^{\frac{1}{4}} \ds \left( \left( \frac{-8\beta c_0}{\omega^2} \right)^{\frac{1}{4}} (x - \omega t), \frac{\sqrt{2}}{2} \right) e^{-i \left( \frac{\omega}{\alpha} x - ct \right)}, \]

\[ \omega^2 + 4\alpha c = 0, \beta c_0 < 0, \]

\[ u_{14}(x,t) = \varepsilon \left( \frac{\omega^2 c_0}{2\beta c} \right)^{\frac{1}{4}} \left[ \ns \left( 2 \left( \frac{2\beta c_0}{\omega^2} \right)^{\frac{1}{4}} (x - \omega t), \frac{\sqrt{2}}{2} \right) \right] e^{-i \left( \frac{\omega}{\alpha} x - ct \right)}, \]

\[ \omega^2 + 4\alpha c = 0, \beta c_0 c > 0. \]

It is noted that the solutions \( u_j \) (\( j = 10, 11, 12 \)) do not require any condition, so it does not need to calculate \( c_0 \) from the conditions of solutions.

**Example 3** The combined KdV–mKdV equation

\[
\dot{u}_t + 6 \left( \alpha u + \beta u^2 \right) u_x + \gamma u_{xxx} = 0, \tag{12}
\]

in which \( \alpha, \beta, \gamma \) are constants.

Substituting the wave transformation \( u(x,t) = u(\xi), \xi = x - \omega t \) into Eq. (12) and integrating once with respect to \( \xi \), we obtain

\[-\omega u + 3\alpha u^2 + 2\beta u^3 + \gamma u'' - C = 0, \]

where \( C \) is the constant of integration. Now this equation can be rewritten as

\[
u'' = \frac{C}{\gamma} + \frac{\omega}{\gamma} u - \frac{3\alpha}{\gamma} u^2 - \frac{2\beta}{\gamma} u^3. \tag{13}
\]
Suppose that the \( u(\xi) \) satisfies the general elliptic equation, then we have

\[
u'' = \frac{c_1}{2} + c_2 u + \frac{3c_3}{2} u^2 + 2c_4 u^3. \tag{14}\]

Equating the coefficients of \( u^j \) \((j = 0, 1, 2, 3)\) in Eq. \( (13) \) and Eq. \( (14) \) yields

\[
c_0 = c_0, c_1 = \frac{2C}{\gamma}, c_2 = \frac{\omega}{\gamma}, c_3 = \frac{2\alpha}{\gamma}, c_4 = -\frac{\beta}{\gamma}. \tag{15}\]

When \( c_0 = c_1 = 0 \), inserting \( (15) \) into the solutions in Case 1 will give the following exact traveling wave solutions of the combined KdV–mKdV equation

\[
u_1(x,t) = \frac{\omega}{\varepsilon \sqrt{\alpha^2 + \beta \omega}} \cosh \sqrt{\frac{\omega}{\gamma}} (x - \omega t) + \alpha, \quad \alpha^2 + \beta \omega > 0, \omega \gamma > 0,
\]

\[
u_2(x,t) = \frac{\omega}{\varepsilon \sqrt{-(\alpha^2 + \beta \omega)}} \sinh \sqrt{\frac{\omega}{\gamma}} (x - \omega t) + \alpha, \quad \alpha^2 + \beta \omega < 0, \omega \gamma > 0,
\]

\[
u_3(x,t) = \frac{\omega}{\varepsilon \sqrt{\alpha^2 + \beta \omega}} \cos \sqrt{\frac{\omega}{\gamma}} (x - \omega t) + \alpha, \quad \alpha^2 + \beta \omega > 0, \omega \gamma < 0,
\]

\[
u_4(x,t) = \frac{\omega}{\varepsilon \sqrt{\alpha^2 + \beta \omega}} \sin \sqrt{\frac{\omega}{\gamma}} (x - \omega t) + \alpha, \quad \alpha^2 + \beta \omega > 0, \omega \gamma < 0,
\]

\[
u_5(x,t) = -\frac{\alpha}{2\beta} \left[ 1 + \varepsilon \tanh \frac{1}{2} \sqrt{-\frac{\alpha^2}{\beta \gamma}} \left( x + \frac{\alpha^2}{\beta} t \right) \right], \quad \beta \gamma < 0,
\]

\[
u_6(x,t) = -\frac{\alpha}{2\beta} \left[ 1 + \varepsilon \coth \frac{1}{2} \sqrt{-\frac{\alpha^2}{\beta \gamma}} \left( x + \frac{\alpha^2}{\beta} t \right) \right], \quad \beta \gamma < 0,
\]

and the stationary solution

\[
u_7(x,t) = -\frac{2\alpha \gamma}{\alpha^2 x^2 + \beta \gamma}, \omega = 0.
\]

When \( c_0 = 0 \), we need to consider the following seven cases.

(1). If \( c_1 = \frac{8c_2^2}{27c_3}, c_4 = \frac{c_2^3}{4c_2} \), then we obtain from \( (15) \) that

\[
c_1 = -\frac{4\alpha^3}{27\beta^2 \gamma}, c_2 = -\frac{\alpha^2}{\beta \gamma}, c_3 = -\frac{2\alpha}{\gamma}, \omega = -\frac{\alpha}{\gamma}, C = -\frac{2\alpha^3}{27\beta^2}. \tag{16}\]

To take \( (16) \) into \( F_j \) \((j = 23, 24, 25, 26)\), we obtain the following exact traveling solutions of the combined KdV–mKdV equation

\[
u_8(x,t) = -\frac{4\alpha \tanh^2 \frac{1}{6} \sqrt{\frac{3\alpha^2}{\beta \gamma}} \left( x + \frac{\alpha^2}{\beta} t \right)}{3\beta \left( 1 + \tanh^2 \frac{1}{6} \sqrt{\frac{3\alpha^2}{\beta \gamma}} \left( x + \frac{\alpha^2}{\beta} t \right) \right)^2}, \beta \gamma > 0,
\]
equation as follows

\[ u_9(x, t) = -\frac{4\alpha \coth^2 \frac{1}{6} \sqrt{\frac{3\alpha^2}{\beta \gamma}} (x + \frac{\alpha^2}{\beta} t)}{3\beta \left( 3 + \coth^2 \frac{1}{6} \sqrt{\frac{3\alpha^2}{\beta \gamma}} (x + \frac{\alpha^2}{\beta} t) \right)^2}, \beta \gamma > 0, \]

\[ u_{10}(x, t) = \frac{4\alpha \tan^2 \frac{1}{6} \sqrt{-\frac{3\alpha^2}{\beta \gamma}} (x + \frac{\alpha^2}{\beta} t)}{3\beta \left( 3 - \tan^2 \frac{1}{6} \sqrt{-\frac{3\alpha^2}{\beta \gamma}} (x + \frac{\alpha^2}{\beta} t) \right)^2}, \beta \gamma < 0, \]

\[ u_{11}(x, t) = \frac{4\alpha \cot^2 \frac{1}{6} \sqrt{-\frac{3\alpha^2}{\beta \gamma}} (x + \frac{\alpha^2}{\beta} t)}{3\beta \left( 3 - \cot^2 \frac{1}{6} \sqrt{-\frac{3\alpha^2}{\beta \gamma}} (x + \frac{\alpha^2}{\beta} t) \right)^2}, \beta \gamma < 0. \]

(2). If \( c_1 = \frac{c_3(5m^2-1)}{10m_c^2c_4}, c_2 = \frac{c_3(5m^2-1)}{10m_c^2c_4}, \) then (15) gives the following parameters

\[ c_1 = -\frac{\alpha^3(m^2-1)}{4m^2\beta^2\gamma}, c_2 = -\frac{\alpha^2(5m^2-1)}{4m^2\beta \gamma}, c_3 = -\frac{2\alpha}{\gamma}, \]

\[ c_4 = -\frac{\beta}{\gamma}, \omega = -\frac{\alpha^2(5m^2-1)}{4m^2\beta}, C = -\frac{\alpha^3(m^2-1)}{8m^2\beta^2}. \]  

(17)

Inserting (18) into \( F_j (j = 27, 28) \) yields the exact solutions of the combined KdV–mKdV equation as follows

\[ u_{12}(x, t) = -\frac{\alpha}{2\beta} \left[ 1 + \varepsilon \frac{\alpha}{2m^2\beta} \sqrt{\frac{\beta}{\gamma}} \left( x + \frac{\alpha^2(5m^2-1)}{4m^2\beta} t \right) \right], \beta \gamma < 0, \]

\[ u_{13}(x, t) = -\frac{\alpha}{2\beta} \left[ 1 + \frac{\varepsilon}{\frac{\alpha}{2m^2\beta}} \sqrt{-\frac{\beta}{\gamma}} \left( x + \frac{\alpha^2(5m^2-1)}{4m^2\beta} t \right) \right], \beta \gamma < 0. \]

(3). If \( c_1 = \frac{c_3(1-m^2)}{2c_4^2}, c_2 = \frac{c_3(5-m^2)}{10c_4^2}, \) then it solves from (15) that

\[ c_1 = -\frac{\alpha^3(1-m^2)}{4\beta^2\gamma}, c_2 = -\frac{\alpha^2(5-m^2)}{4\beta \gamma}, c_3 = -\frac{2\alpha}{\gamma}, \]

\[ c_4 = -\frac{\beta}{\gamma}, \omega = -\frac{\alpha^2(m^2-5)}{\beta}, C = -\frac{\alpha^3(m^2-1)}{8m^2\beta^2}. \]  

(18)

Substituting (18) into \( F_j (j = 29, 30) \), we obtain the exact traveling wave solutions of Eq. (12) as following

\[ u_{14}(x, t) = -\frac{\alpha}{2\beta} \left[ 1 - \varepsilon \frac{\alpha}{2\gamma} \sqrt{-\frac{\gamma}{\beta}} \left( x - \frac{\alpha^2(m^2-5)}{\beta} t \right) \right], \beta \gamma < 0, \]

\[ u_{15}(x, t) = -\frac{\alpha}{2\beta} \left[ 1 - \frac{\varepsilon}{\frac{\alpha}{2\gamma}} \sqrt{\frac{\gamma}{\beta}} \left( x - \frac{\alpha^2(m^2-5)}{\beta} t \right) \right], \beta \gamma < 0. \]
(4). If \(c_1 = \frac{c_1^3}{32c_4^2c_4^4}, c_2 = \frac{c_2^2(4m^2+1)}{16m^4c_4^2}, \) then we obtain from (19) that

\[
\begin{align*}
c_1 &= -\frac{\alpha^3}{4m^2\beta^2\gamma}, \quad c_2 = -\frac{\alpha^2(4m^2+1)}{4m^2\beta\gamma}, \quad c_3 = -\frac{2\alpha}{\gamma}, \\
c_4 &= -\frac{\beta}{\gamma}, \quad \omega = -\frac{\alpha^2(4m^2+1)}{4m^2\beta}, \quad C = -\frac{\alpha^3}{8m^2\beta^2}.
\end{align*}
\]

(19)

To take (19) into \(F_j\) \((j = 31, 32)\) we obtain the following two types of exact traveling wave solutions for Eq. (12)

\[
\begin{align*}
u_{16}(x, t) &= -\frac{\alpha}{2\beta} \left[ 1 + \varepsilon \text{cn} \frac{\alpha}{2m\gamma} \sqrt{\beta^2} \left( x + \frac{\alpha^2(4m^2+1)}{4m^2\beta^2} t \right) \right], \beta\gamma > 0, \\
u_{17}(x, t) &= -\frac{\alpha}{2\beta} \left[ 1 + \frac{\varepsilon}{\sqrt{1-m^2}} \text{sn} \frac{\alpha}{2m\gamma} \sqrt{\beta^2} \left( x + \frac{\alpha^2(4m^2+1)}{4m^2\beta^2} t \right) \right], \beta\gamma > 0.
\end{align*}
\]

\[(5). \text{ If } c_1 = \frac{c_1^3}{32c_4^2(m^2-1)}, c_2 = \frac{c_2^2(5m^2-4)}{16c_4(m^2-1)}, \text{ then (15) is solved as}

\[
\begin{align*}
c_1 &= -\frac{\alpha^3 m^2}{4\beta^2\gamma(m^2-1)}, \quad c_2 = -\frac{\alpha^2(4m^2+1)}{4m^2\beta\gamma}, \quad c_3 = -\frac{2\alpha}{\gamma}, \\
c_4 &= -\frac{\beta}{\gamma}, \quad \omega = -\frac{\alpha^2(5m^2-4)}{4\beta(m^2-1)}, \quad C = -\frac{\alpha^3 m^2}{8\beta^2(m^2-1)}.
\end{align*}
\]

(20)

Inserting (20) into \(F_j\) \((j = 33, 34)\) leads the exact traveling wave solutions of Eq. (12)

\[
\begin{align*}
u_{18}(x, t) &= -\frac{\alpha}{2\beta} \left[ 1 + \frac{\varepsilon}{\sqrt{1-m^2}} \text{dn} \frac{\alpha}{2m\gamma} \sqrt{\beta^2} \left( x + \frac{\alpha^2(5m^2-4)}{4\beta(m^2-1)} t \right) \right], \\
u_{19}(x, t) &= -\frac{\alpha}{2\beta} \left[ 1 + \frac{\varepsilon}{\sqrt{1-m^2}} \text{sn} \frac{\alpha}{2m\gamma} \sqrt{\beta^2} \left( x + \frac{\alpha^2(5m^2-4)}{4\beta(m^2-1)} t \right) \right],
\end{align*}
\]

where \(\beta\gamma > 0.\)

(6). If \(c_1 = \frac{c_1^3}{32c_4^2(1-m^2)}, c_2 = \frac{c_2^2(4m^2-5)}{16c_4(1-m^2)},\) then (15) gives the following parameters

\[
\begin{align*}
c_1 &= -\frac{\alpha^3}{4\beta^2\gamma(m^2-1)}, \quad c_2 = -\frac{\alpha^2(4m^2+1)}{4m^2\beta\gamma}, \quad c_3 = -\frac{2\alpha}{\gamma}, \\
c_4 &= -\frac{\beta}{\gamma}, \quad \omega = -\frac{\alpha^2(4m^2-5)}{4\beta(m^2-1)}, \quad C = -\frac{\alpha^3}{8\beta^2(m^2-1)}.
\end{align*}
\]

(21)

Substituting (21) into \(F_j\) \((j = 35, 36)\) we obtain the exact traveling wave solutions of Eq.
as following

\[ u_{20}(x, t) = -\frac{\alpha}{2\beta} \left[ 1 + \frac{\varepsilon}{\sqrt{\gamma} \beta(1-m^2)} \left( x + \frac{\alpha^2(4m^2+5)}{4\beta(m^2-1)} t \right) \right], \beta\gamma < 0, \]

\[ u_{21}(x, t) = -\frac{\alpha}{2\beta} \left[ 1 - \frac{\varepsilon\sqrt{1-m^2} \left( x + \frac{\alpha^2(4m^2+5)}{4\beta(m^2-1)} t \right)}{\sqrt{1-m^2} \left( x + \frac{\alpha^2(4m^2-5)}{4\beta(m^2-1)} t \right)} \right], \beta\gamma < 0, \]

(7). If \( c_1 = \frac{c_3m^2}{32c_4}, c_2 = \frac{c_3(m^2+4)}{16c_4} \), then (15) leads the following parameters

\[ c_1 = -\frac{\alpha^3m^2}{4\beta^2\gamma}, c_2 = -\alpha^2(4m^2+1) \frac{4m^2\beta\gamma}{4\gamma}, c_3 = -\frac{2\alpha}{\gamma}, \]

\[ c_4 = -\frac{\beta}{\gamma}, \omega = -\frac{\alpha^2(m^2+4)}{4\beta}, C = -\frac{\alpha^3m^2}{8\beta^2}. \]

(22)

Taking (22) into \( F_j (j = 37, 38) \) we get the following exact traveling wave solutions to Eq. (12)

\[ u_{22}(x, t) = -\frac{\alpha}{2\beta} \left[ 1 + \varepsilon\sqrt{\frac{\gamma}{\beta}} \left( x + \frac{\alpha^2(m^2+4)}{4\beta} t \right) \right], \beta\gamma > 0, \]

\[ u_{23}(x, t) = -\frac{\alpha}{2\beta} \left[ 1 + \frac{\varepsilon\sqrt{1-m^2} \left( x + \frac{\alpha^2(m^2+4)}{4\beta} t \right)}{\sqrt{1-m^2} \left( x + \frac{\alpha^2(m^2+4)}{4\beta} t \right)} \right], \beta\gamma > 0. \]

4 Discussions

By our method one can use the known solutions of the additional variable separated equation Eq. (1) to test the traveling wave solutions of a given nonlinear equation. This is convenient for judging the solutions of a given nonlinear equation without solving the nonlinear equation. The method also does not require the expansion expression of the traveling wave solution, so it does not need to determine the order of the traveling wave solutions from the homogeneous balancing principle. In addition, we can neglect some conditions of Eq. (1) and determine the integration constants by using the elementary comparison method. Therefore, the method can be used to find more exact solutions of wide class of nonlinear equations.

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