I. INTRODUCTION

There has been known to exist various ways to measure the velocity of a classical wave, such as the phase velocity \( v_p = \omega/k \), the group velocity \( v_g = d\omega/dk \), etc., with \( \omega \) and \( k \) respectively denoting the angular frequency and the wavenumber. In a different context, for a matter wave described by Schrödinger’s or Dirac’s equation, an effective velocity, defined as

\[
v = \frac{\nabla S}{m} = \frac{j}{\rho}, \tag{1}
\]

where \( S \) denotes the quantum Hamilton-Jacobi function [1], and density \( \rho \) and current \( j \) are governed by the probability conservation equation, is often to serve as an important factor to evaluate many dynamical properties of a quantum system [2]. It is notable that with the quantum dispersion relations the \( v_p \) and the \( v_g \) for free particles should, if nonvanishing, have the same sign [3].

Nevertheless this is not true of Eq. (1) with all free-particle states, particularly with coherent ones that have no well-defined momentum. For instance, given the simplest superposed state

\[
|\psi\rangle = |k_1\rangle + a|k_2\rangle, \tag{2}
\]

with \( k_1 = 0 \), \( k_2 = k > 0 \) for simplicity, it was found that a peculiar phenomenon, termed quantum backflow, could happen. That is, Eq. (1), which is equal, in the position representation, to

\[
v = \frac{\hbar}{m} \text{Im}[\frac{\partial_x \psi}{\psi}] = \frac{\hbar k}{m} \frac{a + \cos kx}{1 + a^2 + 2a \cos kx}, \tag{3}
\]

may take negative values with certain \( a \) and \( k \) [4], while the expectation of momentum \( \langle \hat{p} \rangle \) remains positive.

This phenomenon can be well understood in, e.g., the pilot-wave theory [5–7] — an ontological interpretation of quantum mechanics. Accordingly, the \( v \) in (1) should be understood as the velocity of a particle that is moving along a certain trajectory. In this case, the particle in state (2), while not subject to any classical potential, is indeed subject to a nonzero quantum potential [1]

\[
Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} = \frac{\hbar^2 k^2 (1 + a \cos kx)(a + \cos kx)}{2m (1 + a^2 + 2a \cos kx)^2}, \tag{4}
\]

with \( R = \sqrt{\psi^* \psi} \). This potential, when positive, may reflect the particle backward, therefore leading to the backflows (see also Fig. 1). (It is also related to the arrival-time problems [8, 9].) The standard momentum \( \langle \hat{p} \rangle \) remains unchanged since it does not take account of such potentials.

Let us stress that the phenomenon, that a superposition of components with some positive properties yield an opposite collective movement, is not unique in quantum theory; similar phenomena — e.g., the phase velocity versus the group velocity — do exist in the field of classical optics. But there are distinctions: (i) in classical optics, \( \omega \) and \( k \) in each component are independent quantities, but in quantum mechanics they are connected by the dispersion relations [10]; (ii) the occurrence of backflows not only depends on the position — it could be that it may happen in some region but may not elsewhere — but is also time changing (see, e.g., [4]); (iii) most importantly, quantum mechanical wavefunctions, according to Born’s rule, have a particle-wave duality, while classical waves do not.

By far, there have been many results on the backflow [4, 11] and its related phenomena [8, 9], such as the weak value [4] and the superoscillation [12, 13]. Yet,
less is known about its relativistic extensions, except for just a few [14–16]. Particularly, in [16], it was shown that for Klein-Gordon and Dirac waves, which consist of superpositions of many plane waves, the local group velocity can exceed the speed of light $c$, yielding potentially observable phenomena outside the spectrum of the plane wave. This type of superluminality involves very similar concepts as quantum backflow. Results in the present paper share similarities with those in [16]:

- Objects discussed are massive quantum particles.
- All constituent plane waves are positive and subluminal.
- One-dimensional waves are considered.

On the other hand, there are a few differences:

- Rather than weak values, we are focused on quantum backflow, which is also related to the Zitterbewegung.
- We use four-component spinors with more degrees of freedom.
- We consider only two waves in a superposition.
- Dynamical behaviors were also investigated in [16].

Thus, the present paper can fairly be regarded as a timely extension of [16].

It is very meaningful to investigate the relativistic effects upon quantum backflow. For one thing, relativistic quantum mechanics provides more degrees of freedom in the system than that with only orbital part of wavefunctions considered. For another, concepts like negative energies, spins, helicities and negative-positive energy transition [17, 18] have been widely believed to be responsible for a few unique quantum phenomena, such as the beam split in the Stern-Gerlach experiment, the spin-orbital angular momentum conservation, the Zitterbewegung, etc. Thus, more studies on how relativistic features affect quantum backflow are inevitable.

It is notable that in measuring, for instance, the momentum $k_i$, one needs to have the state $|\psi\rangle = \sum_i a_i |k_i\rangle$ coupled with a meter initially in $|0\rangle$, then, according to time evolution, the combined system becomes entangled, i.e., $|\psi\rangle \otimes |0\rangle \rightarrow \sum_i a_i |k_i\rangle \otimes |i\rangle$, such that the $i$th state of the meter, $|i\rangle$, with a probability $|a_i|^2$, indicates $|k_i\rangle$ of the system. Hence, for different components $|k_i\rangle \otimes |i\rangle$’s, one must require the combined momenta to be equal to one another, though $|k_i\rangle$’s alone are clearly not. The same argument for concerns of conservation applies to other quantities like energy, angular momentum, etc.

II. QUANTUM BACKFLOW OF THE FREE DIRAC PARTICLE FOR SPIN $\frac{1}{2}$

Since $v$ defined in (1) is proportional to $j$ just up to a positive quantity $\rho$, in studying backflows below, we

\begin{equation}
\Psi = \frac{1}{\sqrt{A}} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \quad \chi = \frac{c \sigma \cdot p}{E + mc^2} \varphi, \quad A = \Psi^\dagger \Psi, \quad (9)
\end{equation}

and taking $p = (0, 0, \hbar k)$ for simplicity, we then get $\sigma \cdot p = \lambda \hbar k$ with $\lambda = \pm 1$, so that for a certain $k$ the solutions
read

\[ \Psi(E_+, k, \lambda) = \left( \frac{2|E|}{|E| + mc^2} \right)^{-1/2} \left[ \frac{\phi_\lambda}{\sqrt{|E| + mc^2}} \varphi_\lambda \right] e^{ikz}, \]

(10)

\[ \Psi(E_-, k, \lambda) = \left( \frac{2|E|}{|E| + mc^2} \right)^{-1/2} \left[ -\frac{\lambda \chi\hbar k}{|E| + mc^2} \varphi_\lambda \right] e^{ikz}, \]

(11)

where \( \varphi_{\pm 1} = [1,0]^T, \varphi_{-1} = [0,1]^T, \) and \( E = E_{\pm} = \pm \sqrt{c^2\hbar^2k^2 + m^2c^4}. \)

Given \( k > 0, \) we now list a summary of our results with a variety of free-particle states [19]:

1. \( \Psi = \Psi(E_+, k, \lambda): \)

\[ j_x = j_y = 0, \quad j_z = \frac{c^2\hbar k}{|E|} \times \text{Sgn}(E_+), \]

(12)

where \( \text{Sgn}(E) \) denotes the sign function of support \( E. \)

2. \( \Psi = [\Psi(E_+, k, \lambda) + ae^{i\phi} \Psi(E_-, k, \lambda)]/\sqrt{1 + a^2}: \)

\[ j_x = j_y = 0, \]

\[ j_z = \frac{c^2}{(1 + a^2)|E|} \left[ \hbar k(1 - a^2) + 2\lambda mc \cos \phi \right] \]

\[ = \frac{mc^2}{(1 + a^2)|E|} \left[ \gamma v(1 - a^2) + 2\lambda \cos \phi \right], \]

where \( \gamma = 1/\sqrt{1 - (v/c)^2}, \ v \in [0, 1]. \) For instance, we obtain at \( \phi = \pi, \lambda = 1 \) a negative \( j_z \) in the domain

\[ a \in (a_c, 1), \quad a_c = \sqrt{\frac{\gamma - 1}{\gamma + 1}}. \]

(14)

3. \( \Psi = [\Psi(E_+, k, \lambda) + ae^{i\phi} \Psi(E_-, -k, \lambda)]/\sqrt{1 + a^2}: \)

\[ j_x = j_y = 0, \]

\[ j_z = \frac{c^2\hbar k}{|E|} \left[ \frac{2\lambda ac}{1 + a^2} \cos(2kz - \phi) \right] \]

\[ = v + \frac{2\lambda ac}{1 + a^2} \cos(2kz - \phi). \]

Similarly to case 2, at \( \phi = \pi, \lambda = 1, \) we get a negative \( j_z \) around \( z = n\pi/k, \ n = 0, \pm 1, \pm 2, \cdots, \) by taking, for instance, \( v = ac, \ m = 1. \)

4. \( \Psi = [\Psi(E_+, k, \lambda) + ae^{i\phi} \Psi(E_-, k, \lambda)]/\sqrt{1 + a^2}: \)

\[ j_x = \frac{2ac}{1 + a^2} \cos \phi, \quad j_y = \frac{2\lambda ac}{1 + a^2} \sin \phi, \]

\[ j_z = \frac{c^2\hbar k 1 - a^2}{|E|} \left[ \frac{1}{1 + a^2} > 0. \right] \]

Most intriguingly, here \( j_x \) and \( j_y \) can be nonvanishing despite that \( p_x = p_y = 0. \)

5. \( \Psi = [\Psi(E_+, k, \lambda) + ae^{i\phi} \Psi(E_-, -k, \lambda)]/\sqrt{1 + a^2}: \)

\[ j_x = \frac{2amc^3}{(1 + a^2)|E|} \cos(2kz - \phi), \]

\[ j_y = -\frac{2\lambda amc^3}{(1 + a^2)|E|} \sin(2kz - \phi), \]

\[ j_z = \frac{c^2\hbar k}{|E|} > 0. \]

Similarly to case 4, \( j_x \) and \( j_y \) can be nonvanishing despite that \( p_x = p_y = 0. \)

6. \( \Psi = [\Psi(E_+, 0, \lambda) + ae^{i\phi} |E|/(|E| + m)]^{1/2} \Psi(E_+, k, \lambda): \)

\[ j_x = j_y = 0, \]

\[ j_z = \pm \frac{ac^2\hbar k}{|E| + mc^2} \left[ a + \cos(kz + \phi) \right]. \]

\( j_z \) can be negative around \( z = 2n\pi/k, \ n = 0, \pm 1, \pm 2, \cdots, \) for both the positive-energy case, for instance at \( \phi = \pi, \lambda = 1, \ a = 1/\sqrt{2}, \ m = 1, \) and the negative-energy case, for instance at \( \phi = 0, \lambda = 1, \ a = 1/\sqrt{2}, \ m = 1. \)

7. \( \Psi = \Psi(E_-, 0, \lambda) + ae^{i\phi} |E|/(|E| + m)]^{1/2} \Psi(E_-, k, \lambda): \)

\[ j_x = j_y = 0, \]

\[ j_z = \pm \frac{ac^2\hbar k}{|E| + mc^2} \left[ a + \cos(kz - \phi) \right]. \]

Similarly to case 6, \( j_z \) can be negative around \( z = 2n\pi/k, \ n = 0, \pm 1, \pm 2, \cdots, \) for instance at \( \phi = \pi, \lambda = 1, \ a = 1/\sqrt{2}, \ m = 1. \)

Some remarks are in order. First, it is notable that in case 1 backflow always occurs with the negative-energy state. This makes sense because the negative-energy state indicates an antiparticle, which, when traveling forward, can be understood as a particle in the positive-energy state traveling backward. It also makes sense to case 2, in which the state can effectively be seen as a superposition of two components that have opposite momenta. Then, case 3 is remarkable, since each component has a positive momentum (effectively), and backflow still occurs. Cases 4 and 5 show that different helicities in superposition may result in nonzero current, positive or negative, in the \( x \) or \( y \) direction, despite that the corresponding momenta both equal zero [20]. Cases 6 and 7 resemble ones in the nonrelativistic limit (c.f. Eqs. (2) and (3)) but, here, negative energies are also considered (see also [14, 15]).

III. FIELD-THEORETIC FORMALISM

In quantum field theory the wave function \( \Psi \) is promoted as the field operator of fermions, which can be fur-
there expanded in terms of annihilation and creation operators satisfying fermionic anticommutation relations [21]:

$$\Psi(x) = (2\pi)^{-3/2} \sum_\sigma \int dp \left[ a_{p,\sigma} u(p, \sigma)e^{ip\cdot x} + b_{p,\sigma}^\dagger v(p, \sigma)e^{-ip\cdot x} \right],$$

(20)

where \( \{a_p, a_p^\dagger\} = \{b_p, b_p^\dagger\} = \delta_{pp'} \), \( \{a_p, a_{p'}\} = \{b_p, b_{p'}\} = [a_p, b_p^\dagger] = 0 \), and \( u(p, \sigma) \) and \( v(p, \sigma) \) are determined by the representation of the Lorentz group that describe the fermions with a certain spin \( \sigma \) (which should not be confused with the Pauli matrices in previous sections).

Let us consider a model of spin-1/2 with the field operator distributed only around \( p = 0 \). We fix the spin to be upward \( (\sigma = 1/2) \), many labels being omitted without confusion. The field operator at \( t = 0 \) becomes then

$$\Psi(x) \bigg|_{t=0} \rightarrow (2\pi)^{-3/2} \int_{|p| \leq \epsilon} dp \left[ a_{p,\frac{1}{2}} u(0, \frac{1}{2})e^{ip\cdot x} + b_{p,\frac{1}{2}}^\dagger v(0, \frac{1}{2})e^{-ip\cdot x} \right]$$

$$\approx (2\pi)^{-3/2} \int_{|p| \leq \epsilon} dp \left[ a_{0,\frac{1}{2}} u(0, \frac{1}{2}) + b_{0,\frac{1}{2}}^\dagger v(0, \frac{1}{2}) \right]$$

(21)

$$= (2\pi)^{-3/2} \int_{|p| \leq \epsilon} dp \left[ a_{0,\frac{1}{2}} u(0, \frac{1}{2}) + b_{0,\frac{1}{2}}^\dagger v(0, \frac{1}{2}) \right]$$

$$\approx (2\pi M)^{-3/2} \int_{|p| \leq \epsilon} dp \left[ a_{p,\frac{1}{2}} u(p, \frac{1}{2}) + b_{p,\frac{1}{2}}^\dagger v(p, \frac{1}{2}) \right],$$

where \( M \), possibly depending on the small quantity \( \epsilon \), equals a finite, positive constant due to the way of how the field operator is expanded around \( p = 0 \), \( u \) and \( v \) are short for \( u(0, \frac{1}{2}) \) and \( v(0, \frac{1}{2}) \), respectively, and

$$u(0, \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v(0, -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

(22)

$$u(0, -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v(0, \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$}

The current operator, in analogy with (5), is then computed in the \( z \)-axis

$$j_z \bigg|_{t=0} = (2\pi M)^{-3/2} \int_{|p| \leq \epsilon} dp \left[ u_{p,\frac{1}{2}}^\dagger + v_{p,\frac{1}{2}} \right]$$

$$\times \left[ \begin{array}{c} \sigma_z \\ (2\pi M)^{-3/2} \int_{|p| \leq \epsilon} dp \left[ a_{p',\frac{1}{2}} u_{p',\frac{1}{2}} + b_{p',\frac{1}{2}}^\dagger v_{p',\frac{1}{2}} \right] \end{array} \right]$$

$$= \left( 2\pi M \right)^{-3} \int_{|p| \leq \epsilon} dp \left[ a_{p,\frac{1}{2}} u_{p,\frac{1}{2}} + b_{p,\frac{1}{2}}^\dagger v_{p,\frac{1}{2}} \right]$$

$$+ \left( 2\pi M \right)^{-3} \int_{|p| \leq \epsilon} dp \cdot 1.$$ (23)

Here, the first term after integration is proportional to the normal ordered charge operator \( N \); the second term, which is positive and finite, depends on the way we make approximations in (21), but might become infinite if we perform a more general calculation with less approximations — e.g., when \( \epsilon \) is large. This is undesired, and so the physical current operator should subtract the second term [22]:

$$j_z \bigg|_{t=0} = \left[ j_z - (2\pi M)^{-3} \int_{|p| \leq \epsilon} dp \cdot 1 \right]_{t=0} = (2\pi M)^{-3} N,$$ (24)

This now has both negative and positive eigenvalues, even around \( p = 0 \). The result differs from either relativistic or nonrelativistic quantum mechanical results. (Obviously, \( j_{x,y} \), as well as other cases with \( \sigma = -1/2 \), can be computed in similar ways.) The reason for the nonzero probability current with almost vanishing momenta is that the current (5), when expressed in terms of field operators, has nothing to do with probability flows, but is a Hermitian operator which is proportional to the normal ordered charge operator.

IV. SUMMARY

We have investigated the quantum backflows of a free Dirac particle in superposed states with various energies, momenta, and helicities. In nonrelativistic quantum mechanics, if backflows occur, it requires necessarily that the state must involve at least two different momenta; however, in this paper, we have shown that energies and helicities, which are featured in relativistic quantum mechanics, also play an indispensable role in the occurrence of backflows. Some conditions, under which backflows occur, have no counterpart in the nonrelativistic quantum mechanics.

In the end, we have used the quantized wavefunctions (i.e., the field operators) to study the odd behaviors of the probability current at \( t = 0 \) around \( p = 0 \). We have shown that the current operator is proportional to the
normal ordered charge operator, and so it has nothing to do with any sort of probabilities. In this paper, we do not try to give an ontological interpretation of our results (except for what has been done by using the pilot-wave theory in the nonrelativistic case, Eq. (4)), because, to the best of our knowledge, most of ontological interpretations are in need of considering some presumed configuration spaces, which often suffer the localization problem when it comes to the context of relativity [23]. Nevertheless, dynamical behaviors, as well as considering more kinds of superposed states and more general systems, may deserve subsequent investigations in the future. We suggest that the results in this paper may shed new light into relativistic electron vortices [24].

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