Fluid-Structure Coupling Vibration of an Elastically Restrained Circular Plate

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Abstract. In this paper, an analytical method is presented to calculate the fluid-structure coupling vibration of an elastically restrained circular plate. The displacement of the plate is expanded in series of dry mode shapes and the motion of the fluid is described by velocity potential function. Considering the equilibrium differential equation of the plate and the velocity continuous conditions at the interface of fluid and structure, the Galerkin method and expansion of Fourier-Bessel series are used to establish the governing equations of the system. The free vibration characteristics of the circular plate in contact with fluid are studied and the present method is verified by comparison with numerical results. Furthermore, the influence of spring stiffness and fluid depth on the free vibration characteristics of the circular plate is investigated.

Keywords: elastic boundary conditions, circular plate, fluid-structure interaction, Galerkin method, Fourier-Bessel series.

1. Introduction
Circular plates are widely applied in engineering. In many cases, circular plates are in contact with fluid, such as micro pumps, the nuclear reactor internal components, disk brakes, etc. The existence of fluid decreases the free vibration frequencies of the circular plate, which has a significant influence on the dynamic characteristics of the circular plate. Therefore, it is meaningful to study the fluid-structure coupling vibration of circular plates.

Kwak [1] studied the vibration characteristics of a circular plate floating on a infinite fluid region. By using Nondimensionalized Added Virtual Mass Incremental (NAVMI) to evaluate the influence of the fluid, the coupled natural frequencies of the plate are derived approximately. Kwak [2] applied Hankel transform to calculating the hydrodynamic characteristics of a circular plate clamped in a rigid baffle. Kwak and Han [3] calculated the vibration characteristic of a circular plate floating on a fluid with finite depth. References [1]–[3] ignored the influence of fluid on the mode shapes of the circular plate, thus the results obtained are not accurate. By using Rayleigh-Ritz method to establish the governing equation, Amabili and Kwak [4] gained more accurate results. It was also found that the calculation deviation by using NAVMI method is less for lower order frequencies. Bauer [5] studied the coupling vibration of an elastic cover of a cylindrical container and the fluid inside the container. On basis of [5], Bauer and Chiba [6] considered the influence of fluid viscosity. Amabili [7] investigated the vibration charastics of a circular plate floating on the fluid in a container, and the influence of free surface wave is studied.
Yousefzadeh [8] analysed the vibration characteristics of a functionally graded circular plate floating on fluid. Cheung and Zhou [9] studied the free vibration of a flexible thin plate placed into a circular hole and elastically connected to the rigid bottom slab of a circular cylindrical container. Eftekhar [10] introduced differential quadrature method (DQM) to solve the fluid-structure vibration of the circular plate. Tariverdilo et al. [11] solved the coupled natural frequencies of the plate in contact with fluid by employing Fourier-Bessel series and variational formulation, respectively. But the method was not applicable to axisymmetric modes of the circular plate. Jeong [12] calculated the fluid-structure coupling vibration of two identical circular plates, and the elements of the added mass matrix was adjusted for axisymmetric modes. However, the result error of the out of phase mode was relatively large.

On basis of [11] and [12], this paper investigates the fluid-structure coupling vibration of an elastically restrained circular plate in contact with fluid. The expansion of Fourier Bessel series and Galerkin method are combined to establish the governing equations. Considering the properties of Bessel functions of index 0, the added constraint equation is introduced to calculate the coupled frequencies of axisymmetric modes. Furthermore, the effects of spring stiffness and fluid depth on the vibration characteristics of the plate are studied.

2. Theoretical Background

2.1. Formulations for Circular Plates

Figure 1 depicts an elastically restrained circular plate coupled with an incompressible and inviscid fluid. The radius and thickness of the plate are \( a \) and \( h \), respectively. The fluid depth of is \( d \). The side walls at \( r=a \) are considered to be rigid walls. The distributed stiffness of translational springs and rotational springs are \( K \) and \( K_\phi \), respectively.

Assuming the circular plate satisfies Kirchhoff’s thin plate theory, the dynamic equilibrium equation of the system can be written as

\[
D \nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = P
\]

where \( w \), \( D \), \( \rho \) are the deflection, bending stiffness and density of the circular plate, respectively. \( P \) is the fluid pressure and \( \nabla^2 \) is the Laplace operator in polar coordinates.

According to modal orthogonality theory, the plate deflection can be expanded in series of mode shapes in vacuum:

\[
 w(r, \theta, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_{nm} W_{nm}(r, \theta)e^{i\omega t}
\]

where \( W_{nm} \) is the mode shapes of the circular plate in vacuum, \( q_{nm} \) is a generalized coordinate, \( n \) and \( m \) represent numbers of nodal diameters and nodal circles, respectively.

The mode shape of the circular plate in vacuum can be written as [13]:

\[
 W_{nm}(r, \theta) = [J_n(\lambda_{nm}r) + C_{nm} I_n(\lambda_{nm}r)] \cos(n\theta)
\]
where \( J_n \) and \( I_n \) are the Bessel function and the modified Bessel function of the first kind, \( \lambda_{nm} \) is the frequency parameter defined by
\[
\lambda_{nm} = \sqrt{\omega_{nm}^2 \rho h / D} \quad (4)
\]
Where \( \omega_{nm} \) represents the natural frequency of the circular plate in vacuum. When the size and material of the plate are certain, the parameters \( \lambda_{nm} \) and \( C_{nm} \) are determined by \( K \) and \( K \phi \).

### 2.2. Velocity potential function of the fluid

Assuming the movement of the fluid is irrotational, the velocity potential function \( \Phi(r, \theta, x, t) \) can be introduced to describe the fluid movement. Considering the fluid is incompressible and inviscid, \( \Phi(r, \theta, x, t) \) satisfies the Laplace equation:
\[
\nabla^2 \Phi = 0 \quad (5)
\]
Separate the function \( \Phi \) into time and space terms:
\[
\Phi(r, \theta, x, t) = \omega \Phi_{\theta}(r, \theta, x)e^{i\omega t} \quad (6)
\]
where \( \phi \) is the spatial velocity potential, which also satisfies the Laplace equation. Using method of separation of variables, the general solution of the function \( \phi \) can be derived:
\[
\phi(r, \theta, x) = (a_n x + b_n) \delta_{n0} \cos \theta + \sum_{n=1}^{\infty} J_n(\beta_n r) [E_n \sinh(\beta_n x) + F_n \cosh(\beta_n x)] \cos n\theta \quad (7)
\]
Where \( \delta_{n0} \) is Kronecker delta function, \( a_n, b_n, E_n, F_n, \beta_n \) are undetermined coefficients and \( \beta_n > 0 \).

The boundary condition at the rigid walls is:
\[
\left. \frac{\partial \phi}{\partial r} \right|_{r=a} = 0 \quad (8)
\]
where \( J_n(\beta_n a) = 0 \) \( (9) \)

The boundary condition at the free surface is
\[
P_{r=0} = \rho_0 \frac{\partial \phi}{\partial r} \bigg|_{r=0} = \rho_0 \omega \Phi_{\theta, r} = \rho_0 \omega \Phi_{\theta, \phi} = 0 \quad (10)
\]
Substitution of Eq. (3) into Eq. (10) gives
\[
b_0 = 0 \quad (11)
\]
\[
F_n = 0 \quad (12)
\]
Hence the velocity potential function \( \phi \) can be reduced to
\[
\phi(r, \theta, x) = \cos n\theta \left[ a_n x \delta_{n0} + \sum_{n=1}^{\infty} J_n(\beta_n r) E_n \sinh(\beta_n x) \right] \quad (13)
\]

### 2.3. Velocity Continuity at the Fluid-Structure Interface

At the fluid-structure interface, the velocity continuity condition should be satisfied:
\[
\left. \frac{\partial \Phi}{\partial t} \right|_{t=0} = \left. \frac{\partial \Phi}{\partial x} \right|_{x=d} \quad (14)
\]
yields
\[
\sum_{n=1}^{\infty} q_n R_m(r) = a_n \delta_{n0} + \sum_{n=1}^{\infty} E_n \beta_n J_n(\beta_n r) \cosh(\beta_n d) \quad (15)
\]
where \( R_n(r) = J_n(\lambda_n r) + C_n I_n(\lambda_n r) \) is the radial component of the \( W_n \). According to the orthogonality of the eigenfunctions of Bessel equations, multiplying both sides of Eq. (15) by \( r J_n(\beta_n r) \), and integrating on domain \([0, a]\) gives

\[
E_n = \frac{\sum_{n=0}^{M} q_{nm} \int_{0}^{a} r R_n(\beta_n r) J_n(\beta_n r) dr}{\beta_n \cosh(\beta_n a) \int_{0}^{a} r [J_n(\beta_n r)]^2 dr}
\]

(16)

When \( n=0 \), constant function is also eigenfunctions of the Bessel equation of order 0. Multiplying both sides of Eq. (15) by \( r \), and integrating on domain \([0, a]\) yields

\[
\sum_{n=0}^{\infty} q_{nm} \int_{0}^{a} r R_{nm}(r) dr - \frac{a^2}{2} a_n = 0
\]

(17)

Substitution of Eq. (16) into Eq. (13), the velocity potential function can be written as

\[
\phi(r, \theta, x) = \cos n\theta \left[a_n x \delta_{n0} + \sum_{m=1}^{M} T_{mn} \int_{0}^{a} r R_{nm}(r) J_m(\beta_m r) \sinh(\beta_m x) dr \right]
\]

(18)

where

\[
T_{mn} = \frac{\int_{0}^{a} r R_{nm}(r) J_m(\beta_m r) dr}{\beta_m \int_{0}^{a} r [J_m(\beta_m r)]^2 dr}
\]

(19)

2.4. Establishment of the Governing Equations

Employing linearized Bernoulli’s equation, the fluid pressure exerted on the circular plate is

\[
P = \rho_0 \frac{\partial \Phi}{\partial t} = \rho_0 \omega^2 e^{i\omega t} \phi|_{t=-d}
\]

(20)

where \( \rho_0 \) is the density of fluid. And the equilibrium equation can be expressed as

\[
D \sum_{n=0}^{M} q_{nm} \nabla^2 W_{nm} - \rho h \omega^2 \sum_{n=0}^{M} q_{nm} W_{nm} - \rho_0 \omega^2 \left[a_n d \delta_{n0} + \sum_{m=1}^{S} T_{mn} \int_{0}^{a} r R_{nm}(r) J_m(\beta_m r) \sinh(\beta_m x) dr \right] \cos n\theta = 0
\]

(21)

Multiplying both sides of Eq. (21) by \( W_{nm}(r, \theta) \) and integrating over the circular plate gives

\[
\rho h \left(\omega^2_0 - \omega^2\right) \sum_{n=0}^{M} q_{nm} \int_{0}^{a} r R_{nm}^2 dr - \rho_0 \omega^2 \left[a_n d \delta_{n0} + \sum_{m=1}^{S} T_{mn} \int_{0}^{a} r J_m(\beta_m r) R_{nm}^2 dr \right] = 0
\]

(22)

Truncating \( m \) and \( s \) to \( M \) and \( S \), Eq. (22) can be expressed in matrix form. When \( n > 0 \), Eq. (22) can be written as

\[
\left[\rho h P_n - \omega^2 (\rho h Z_n + \rho_0 G_n)\right] q_n = 0, \quad n > 0
\]

(23)

where \( q_n = [q_{n0}, q_{n1}, \ldots, q_{nM}]^T \) is the coefficient vector and \( P_n, Z_n, G_n \) are \((M+1) \times (M+1)\) matrices whose elements are given by

\[
P_{nm} = \omega_0^2 \sum_{k=0}^{M} \int_{0}^{a} r R_{nk}^2 (r) dr \delta_{km}, \quad k, m = 0, 1, 2, \ldots, M
\]

(24)
When \( n = 0 \), to derive the coefficients \( a_0 \) and \( q_0 \), Eq. (17) and Eq. (22) should be solved simultaneously. And the governing equation for \( n = 0 \) can be written as:

\[
\left[\begin{array}{c}
\rho h p_0 - \omega^2 \left( \rho h Z_n + \rho h G_0 \right) \\
\frac{1}{l_0} \frac{1}{-a^2/2}
\end{array}\right]
\left[\begin{array}{c}
q_0 \\
a_0
\end{array}\right] = 0, \quad n = 0
\]

where \( c_0 \) and \( l_0 \) are column vector and row vector of dimension \( M+1 \), respectively. The elements of the vectors are defined by

\[
c_{0,k} = \int_0^a r R_{mk} (r) dr
\]

\[
l_{0,m} = \int_0^a r R_{mk} (r) dr
\]

Eqs. (23) and (27) give the governing equations in matrix form for any fixed number \( n \). By solving the generalized eigenvalue problem, the natural frequencies and the mode shapes of the circular plate in contact with fluid can be derived.

3. Numerical Results and Discussion

3.1. Verification of Results

To check the validity of the present method, the finite element analysis is carried out using FEM software ANSYS. The values of the parameters used in computation are listed in Table 1.

| Parameter                      | Value           |
|--------------------------------|-----------------|
| plate radius                   | \( a = 0.12m \) |
| plate thickness                | \( h = 0.002m \) |
| plate density                  | \( \rho = 2700kg/m^3 \) |
| plate Young modulus            | \( E = 69Gpa \) |
| plate Poisson’s ratio          | \( \nu = 0.3 \) |
| fluid density                  | \( \rho_0 = 1000kg/m^3 \) |
| fluid depth                    | \( d = 0.02m \) |
| distributed stiffness of translational springs | \( K = 10^5N/m^2 \) |
| distributed stiffness of rotational springs | \( K_\phi = 10^5N/\text{rad} \) |

It can be found that, the results of the present method agree well with the FEM results, and the relative error is less than 1%.

| \( n \) | \( m \) | FEM (Hz) | Theory (Hz) | Discrepancy (%) |
|--------|--------|----------|-------------|-----------------|
| 0      | 0      | 4.1      | 4.1         | 0.0             |
| 0      | 1      | 192.0    | 192.2       | -0.2            |
| 0      | 2      | 755.6    | 748.9       | -0.9            |
| 1      | 0      | 34.9     | 34.9        | 0.0             |
| 1      | 1      | 401.9    | 401.9       | 0.0             |
| 1      | 2      | 1192.3   | 1181.6      | -0.9            |
| 1      | 3      | 2516.8   | 2477.2      | -1.6            |
| 2      | 0      | 112.3    | 112.3       | 0.0             |
| 2      | 1      | 684.6    | 686.7       | -0.3            |
| 2      | 2      | 1702.8   | 1702.8      | 0.0             |
| 2      | 3      | 3329.7   | 3279.3      | -1.7            |
| 3      | 0      | 235.3    | 235.7       | -0.3            |
| 3      | 1      | 1033.8   | 1031.8      | -0.8            |
| 3      | 2      | 2347.0   | 2312.9      | -1.5            |
| 3      | 3      | 4262.2   | 4167.2      | -2.2            |
3.2. Effect of Spring Stiffness and Fluid Depth

Table 3 lists the coupled natural frequencies of the circular plate with different spring stiffness. The table shows that for range of $K$ and $K_\phi$ between 10 and $10^9$, the natural frequencies of the circular plate increase significantly as the spring stiffness increases. When the spring stiffness range from $10^9$ to $10^{10}$, the coupled natural frequencies remain nearly the same, which means the boundary conditions can be seemed to be clamped boundary conditions when the spring stiffness is higher than $10^9$.

| $K$  | $K_\phi$ | (0,0)  | (0,1)  | (0,2)  | (1,0)  | (1,1)  | (1,2)  |
|------|---------|--------|--------|--------|--------|--------|--------|
| 10   | 10      | 0.4    | 148.9  | 687.6  | 4.9    | 350.6  | 1114.5 |
| $10^3$ | $10^3$  | 4.1    | 192.2  | 748.9  | 34.9   | 401.9  | 1181.6 |
| $10^5$ | $10^5$  | 40.0   | 243.2  | 873.6  | 68.9   | 482.7  | 1348.1 |
| $10^7$ | $10^7$  | 155.6  | 510.6  | 1021.1 | 305.5  | 707.3  | 1451.9 |
| $10^9$ | $10^9$  | 162.8  | 675.7  | 1630.8 | 348.0  | 1071.5 | 2273.7 |
| $10^{10}$ | $10^{10}$ | 162.8  | 675.7  | 1630.8 | 348.4  | 1075.9 | 2296.0 |

To evaluate the fluid effects on the free vibration of the circular plate, the normalized natural frequency, defined as the ratio of the coupled natural frequencies to the corresponding natural frequencies in vacuo, is introduced. As can be seen in 0, the values of normalized natural frequencies are always smaller than 1. It means that, due to the contribution of added mass, the existence of fluid reduces the natural frequencies of the circular plate. As can be also inferred from the figure, with the increase of $n$ or $m$, the normalized frequencies increase. That is to say, for the mode with higher order, the added fluid mass acting on the plate is less. And this conclusion applies to circular plates with different spring stiffness.

Figure 2. Normalized natural frequencies of the circular plate with different spring stiffness

It can be inferred that, for plates with lower spring stiffness, the difference between dry mode shapes and wet mode shapes is smaller. When the spring stiffness $K=K_\phi=10^3$, the influence of fluid on the mode shapes of the plate can be ignored, and it is feasible to replace wet mode shapes with corresponding dry mode shapes to simplify the calculation.
Figure 3. Comparison of the mode shapes of the circular plates in vacuum and in fluid

Figure 4. Evolution of normalized natural frequency with fluid depth

0 depicts the evolution of normalized natural frequencies with fluid depth. As shown in the figure, when the normalized fluid depth \(d/a\) ranges from 0.01 to 1, the normalized natural frequencies reduce significantly, which means increase of added fluid mass.

With increase of the fluid depth, the decrease rate of the frequencies reduces. It can be seen from the figure that for each frequency of the plate there is a threshold fluid depth. And when the fluid depth reaches the threshold, the natural frequency remains unchanged with growth of \(d/a\). This threshold is smaller for the mode with bigger \(n\) or \(m\). When \(d/a > 1.8\), the coupled frequency of arbitrary mode of the circular plate remains unchanged with variation of the fluid depth.
4. Conclusion

In this paper, the free vibration characteristics of an elastically restrained circular plate in contact with fluid is obtained by using Fourier Bessel series and Galerkin method. The present method is proved to be accurate by comparing with FEM.

Then the vibration characteristics of circular plates with several spring stiffness is studied. The results show that the existence of fluid reduces the natural frequencies of the plate significantly, and it has more influence on the lower order frequencies.

Furthermore, the influence of fluid depth is investigated. Results show that in low depth range, the coupled natural frequencies reduce rapidly with fluid depth. When the fluid depth is greater than 1.8 times the radius of the circular plate, the influence of the increase of fluid depth can be ignored.

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