PRANDTL-BATCHelor FLOWS ON AN ANNULUS

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Abstract. For steady two-dimensional Navier-Stokes flows with a single eddy (i.e. nested closed streamlines) in a simply connected domain, Prandtl (1905) and Batchelor (1956) found that in the inviscid limit, the vorticity is constant inside the eddy. In this paper, we consider the generalized Prandtl-Batchelor theory for the forced steady Navier-Stokes equations on an annulus. First, we observe that in the limit of infinite Reynolds number, if forced steady Navier-Stokes solutions has nested closed streamlines on an annulus, then the inviscid limit is a rotating shear flow uniquely determined by the external force and boundary conditions. We call solutions of steady Navier-Stokes equations with the above property Prandtl-Batchelor flows. Then, by constructing higher order approximate solutions of the forced steady Navier-Stokes equations and establishing the validity of Prandtl boundary layer expansion, we give a rigorous proof of the existence of Prandtl-Batchelor flows on an annulus with the wall velocities slightly different from the rigid-rotations along the same direction.

1. Introduction

The study of steady Navier-Stokes equations has a long history, starting with the classical paper of J.Leary [30], in which Leray proved that the nonhomogeneous boundary problem of steady Navier-Stokes equations in a two-dimensional bounded domain $\Omega$ with $C^2$ smooth boundary $\partial \Omega = \bigcup_{j=1}^{N} \Gamma_j$ has a solution under the assumption $\int_{\Gamma_j} u \cdot n dl = 0, j = 1, \cdots, N$. V.Korobkov, K.Pileckas and R.Russo in [29] obtained the existence theorem under the assumption $\sum_{j=1}^{N} \int_{\Gamma_j} u \cdot n dl = 0$. However, the problem of inviscid limit of steady Navier-Stokes equations is still largely open. One of the main difficulties is that there are infinitely many solutions for the steady Euler equations in a closed region, which is very different from the time-dependent inviscid limit where the leading Euler flow is uniquely determined by the initial and boundary conditions. Thus, a selection principle must be found to obtain the physical solution. It is unknown how to choose the Euler flow except for some special case, for example, Prandtl in [38] considered the steady motion of slightly viscous incompressible fluid in a simply-connected region. By integrating the steady Navier-Stokes equations along a closed streamline and letting the Reynolds number converge to infinity, Prandtl formally found that the vorticity of steady Euler flow must be constant in a region of nested closed streamlines (i.e. a single eddy). However, Prandtl didn’t give an approach to determine this constant. This important property was rediscovered later by Batchelor (1956) in [1]. This class of result is now usually referred to as Prandtl-Batchelor theory and such laminar flows are called Prandtl-Batchelor flows in the literature. Furthermore, for Prandtl-Batchelor flows on a circular disk, the Euler flow must be Couette flows $(ar, 0)$ and the constant $a$ was given in [1, 47] by exactly solving steady the nonlinear Prandtl equations in the Von Mises transformation. The formula to determine $a$ is usually referred to as the Batchelor-Wood formula. Similar results were given by Feymann and Lagerstrom in [5]. However, for Prandtl-Batchelor flows on a more general domain, it’s difficult to determine the limiting vorticity constant and only approximate results were obtained in [1, 5, 28, 42, 43, 47]. In addition, the Prandtl-Batchelor theory for forced steady Navier-Stokes...
equations has been considered by H. Okamoto in [34] on torus when the external force is proportional to viscous force, and derived a necessary condition for the limiting Euler flow. H. Okamoto in [35] studied the stationary Kolmogolov flows on two-dimensional torus by numerical experiments and found that under a certain condition, the Navier-Stokes flows are “nearly singular” (nonsmooth) for large Reynolds numbers.

The Prandtl-Batchelor theory has important applications in many studies involving laminar flows with small viscosity, for example, the nonlinear critical layer theory near shear flows in [31]. The Prandtl-Batchelor theory can also be applied to general two-dimensional advection diffusion equation of a passive scalar field $\theta(x,y)$

$$u \cdot \nabla \theta - R^{-1} \Delta \theta = 0,$$

where $u(x,y)$ is a given steady incompressible flow and $R^{-1}$ is the diffusion coefficient. For example, homogenization of potential vorticity in the ocean circulation theory [37, 40, 41] and flux expulsion in the self-excited dynamo theory [32, 33, 46].

Despite the importance of Prandtl-Batchelor theory and its wide applications, there is relatively few mathematical works on this problem. Kim initiated a mathematical study of Prandtl-Batchelor flows on a disk in a series of works [22, 23, 24, 25, 26, 27, 28]. In particular, when the boundary velocity is slightly different from a constant, the well-posedness of the Prandtl equations under the Batchelor-Wood condition was shown in [23] and some asymptotic study of the boundary layer expansion was given in [22]. However, he didn’t prove the convergence of the boundary layer expansion in the limit of infinite Reynolds number. Recently, by constructing higher order approximate solutions of the Navier-Stokes equations and establishing the validity of Prandtl boundary layer expansion, we give a rigorous proof of the existence of Prandtl-Batchelor flows on a disk with the wall velocity slightly different from the rigid-rotation in [6].

In this paper, we extend the Prandtl-Batchelor theory to the forced steady Navier-Stokes equations on an annulus. The limiting Euler flow in this case is selected in the following way. Under some suitable assumptions, if the solution of steady Navier-Stokes equations with nested closed streamlines converges to steady Euler flows, then the limiting Euler flow must be a rotating shear flow $(u_e(r), 0)$ satisfying (1.7). In fact, F. Hamel and N. Nadirashvili in [15] proved that if steady Euler flow with rigid wall boundary conditions in two-dimensional annulus domain has no stagnation point, then the flow is circular in the sense that the streamlines are concentric. By integrating the steady Navier-Stokes equations along a circle and letting the Reynolds number converge to infinity, we can show that the velocity $u_e(r)$ and the external force $F_u$ must satisfy the ODE in (1.7). And the Batchelor-Wood formula determines the boundary conditions for the velocity $u_e(r)$ in (1.7). Combining the ODE and the boundary conditions, we obtain the system (1.7) for the Euler flow. We call the above selection the Prandtl-Batchelor theory since it describes which Euler flow is selected in the limit $\varepsilon \to 0$. In the Appendix, we will give a proof of the above property under the following assumptions: for sufficiently small viscosity i) the steady flows of Navier-Stokes equations have nested closed streamlines; ii) any interior domain of the annulus is separated from the boundary layer uniformly for vanishing viscosity; iii) inside the interior domain, steady Navier-Stokes solutions converge to a steady Euler solution in $C^2$. However, given a domain, boundary data and external force, it is hard to understand the streamline structures of the steady solutions of Navier-Stokes equations and control the boundary layer in a domain, hence it seems difficult to verify the above assumptions rigorously. Following the argument of [6], by constructing higher order approximate solutions and establishing the convergence of Prandtl boundary layer expansion, we construct a class of Prandtl-Batchelor flows to the steady Navier-Stokes equations on an annulus with both the wall velocities slightly different from the rigid-rotations along the same direction.
More precisely, we consider the forced steady Navier-Stokes equations in a two-dimensional annulus centered on zero $A := B_1(0) \setminus B_{r_0}(0)$

\[
\begin{align*}
&\begin{cases}
  \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \nabla p^\varepsilon - \varepsilon^2 \Delta \mathbf{u}^\varepsilon = \varepsilon^2 \mathbf{F}, \\
  \nabla \cdot \mathbf{u}^\varepsilon = 0,
\end{cases} \\
&\text{with rotating boundary conditions}
\end{align*}
\]  

(1.1)

with rotating boundary conditions
\[
\mathbf{u}^\varepsilon|_{\partial B_1} = (\alpha + \eta f(\theta))\mathbf{t}, \quad \mathbf{u}^\varepsilon|_{\partial B_{r_0}} = (\beta + \eta g(\theta))\mathbf{t},
\]

(1.2)

where $\varepsilon^2 > 0$ is reciprocal to Reynolds number, $0 < r_0 < 1$, $\alpha > 0$, $\beta > 0$, $\mathbf{u}^\varepsilon$ is the velocity, $p^\varepsilon$ is the pressure, $\mathbf{F}$ is the external force, $\eta$ is a small number, $\mathbf{t}$ is the unit tangential vector to $\partial B_1$ or $\partial B_{r_0}$, $f(\theta), g(\theta)$ are $2\pi$-periodic smooth functions.

Let $\mathbf{n}$ be the unit normal to $\partial B_1$ and
\[
\mathbf{u}^\varepsilon = u^\varepsilon(\theta, r)\mathbf{t} + v^\varepsilon(\theta, r)\mathbf{n}, \quad \mathbf{F} = F_u(\theta, r)\mathbf{t} + F_v(\theta, r)\mathbf{n},
\]

then (1.1) reads
\[
\begin{align*}
&\begin{cases}
  u^\varepsilon u^\varepsilon_\theta + rv^\varepsilon u^\varepsilon_r + u^\varepsilon v^\varepsilon_r + p^\varepsilon_\theta - \varepsilon^2 \left( \frac{u^\varepsilon}{r} + ru^\varepsilon_r + u^\varepsilon + \frac{2}{r}v^\varepsilon_\theta - \frac{u^\varepsilon_\theta}{r} \right) = \varepsilon^2 r F_u, \\
  u^\varepsilon v^\varepsilon_\theta + rv^\varepsilon v^\varepsilon_r - (u^\varepsilon)^2 + r p^\varepsilon_r - \varepsilon^2 \left( \frac{v^\varepsilon}{r} + rv^\varepsilon_r + v^\varepsilon + \frac{2}{r}u^\varepsilon_\theta - \frac{v^\varepsilon_\theta}{r} \right) = \varepsilon^2 r F_v, \\
  u^\varepsilon_\theta + (rv^\varepsilon)_r = 0, \\
  u^\varepsilon(\theta, 1) = \alpha + \eta f(\theta), \quad v^\varepsilon(\theta, 1) = 0, \\
  u^\varepsilon(\theta, r_0) = \beta + \eta g(\theta), \quad v^\varepsilon(\theta, r_0) = 0,
\end{cases}
\end{align*}

(1.3)

where $(\theta, r) \in \Omega := [0, 2\pi] \times [r_0, 1]$. Formally, as $\varepsilon \to 0$, we obtain the steady Euler equations for $(u_e, v_e)$

\[
\begin{align*}
&\begin{cases}
  u_e \partial_\theta u_e + v_e r \partial_r u_e + u_e v_e + \partial_\theta p_e = 0, \\
  u_e \partial_\theta v_e + v_e r \partial_r v_e - (u_e)^2 + r \partial_r p_e = 0, \\
  \partial_\theta u_e + \partial_r (rv_e) = 0.
\end{cases}
\end{align*}

(1.4)

We will show the existence of solution $(u^\varepsilon, v^\varepsilon)$ to (1.3) which converges to a solution of steady Euler equations (1.4) satisfying (1.7). Following the argument of [6], we first choose a steady Euler flow $(u_e(r), 0)$ with $u_e(r)$ satisfying (1.7), then construct a class of Prandtl-Batchelor flows $(u^\varepsilon, v^\varepsilon)$ to (1.3) by perturbing this steady Euler flow. Generally, there is a mismatch between the tangential velocities of the Euler flow $u_e$ and the prescribed Navier-Stokes flows $u^\varepsilon$. Due to the mismatch on the boundary, Prandtl in 1904 formally introduced the boundary layer theory to correct this mismatch. However, the justification of this formal boundary expansion is a challenging problem.

The studies on steady Prandtl equations and linearized steady Prandtl equations can be found in [3] [13] [36] [41] [45] and the validity of Prandtl boundary layer expansion has also some important progresses, see [7] [8] [10] [11] [16] [17] [18] [20]. Moreover, the stability in Sobolev space for some class shear flow of Prandtl type has been studied in [2] [9].

In the current work, we assume that
\[
\int_0^{2\pi} f(\theta)d\theta = \int_0^{2\pi} g(\theta)d\theta = 0,
\]

(1.5)

and take the leading order Euler flow $(u_e(\theta, r), v_e(\theta, r))$ to be the following shear flow
\[
u_e(\theta, r) = u_e(r), \quad v_e(\theta, r) = 0,
\]

(1.6)
where \( u_e(r) \) satisfies the second order ordinary differential equations

\[
\begin{align*}
\dot{u}_e''(r) + \frac{u_e'(r)}{r} - \frac{u_e(r)}{r^2} &= -F_u(r), \\
\dot{u}_e(1) &= \left( \alpha^2 + \frac{\eta^2}{2\pi} \int_0^{2\pi} f^2(\theta) d\theta \right)^{1/2}, \\
\dot{u}_e(r_0) &= \left( \beta^2 + \frac{\eta^2}{2\pi} \int_0^{2\pi} g^2(\theta) d\theta \right)^{1/2},
\end{align*}
\]

(1.7)

where \( F_u(r) = \frac{1}{2\pi} \int_0^{2\pi} F_u(\theta, r) d\theta, r \in [r_0, 1] \). There is an important basis for this choice—the generalized Prandtl-Batchelor theory, see Appendix C. This theory shows that if the Euler flow is rotating shear \( u_e(r)e_\theta \) in the vanishing viscosity limit, then \( u_e(r) \) must satisfy the second order ordinary differential equation in (1.7), and the Batchelor-Wood formula in Lemma 2.1 gives the boundary conditions for \( u_e(r) \) in (1.7). Moreover, the solvability of linearized second order Euler equations in the matched asymptotic expansion also makes \( u_e(r) \) necessary to be this form, see Remark 2.1.

**Remark 1.1.** If the external force \( F = 0 \), then \( u_e(r) = ar + \frac{b}{r}(a, b \text{ can be uniquely determined by the boundary condition in (1.7)}) \) is the Couette-Taylor flow and the associated vorticity is constant 2a. If the external force \( F = \frac{2}{r}e_\theta \), then \( u_e(r) = ar + \frac{b}{r} + cr \ln r \) and the associated vorticity is \( 2a + c + 2c \ln r(a, b \text{ can be uniquely determined by the boundary condition in (1.7)}).

Next we introduce the steady Prandtl equations near \( r = 1 \)

\[
\begin{align*}
\dot{u}_p(1) + u_p(0) \partial_\theta u_p(0) + (v_p^{(1)}(\theta, 0)) \partial_Y u_p(0) - \partial_Y \partial_Y u_p(0) &= 0, \\
\dot{u}_p(0) + \partial_\theta v_p(1) &= 0, \\
v_p(0) = (\theta + 2\pi, Y), \quad v_p(0) = v_p(\theta + 2\pi, Y), \\
v_p(0)|_{Y=0} = \alpha + \eta f(\theta) - u_e(1), \quad \lim_{Y \to +\infty} u_p(0) &= \lim_{Y \to +\infty} v_p(1) = 0
\end{align*}
\]

(1.8)

and the steady Prandtl equations near \( r = r_0 \)

\[
\begin{align*}
\dot{u}_p(0) + \dot{u}_p(0) \partial_\theta \dot{u}_p(0) + (\dot{v}_p^{(1)}(\theta, 0)) r_0 \partial_Z \dot{u}_p(0) - r_0 \partial_Z \partial_Z \ddot{u}_p(0) &= 0, \\
\dot{u}_p(0) + r_0 \partial_\theta \dot{v}_p(1) &= 0, \\
\dot{u}_p(0) = \beta + \eta g(\theta) - u_e(r_0), \quad \lim_{Z \to +\infty} \dot{u}_p(0) &= \lim_{Z \to +\infty} \dot{v}_p(1) = 0,
\end{align*}
\]

(1.9)

where \( Y = \frac{r-1}{\varepsilon} \) and \( Z = \frac{r-r_0}{\varepsilon} \). The above steady Prandtl equations will be derived by matched asymptotic expansions and their solvability will be studied in the next section.

Now our main theorem is stated as follows.

**Theorem 1.2.** Assume that \( f(\theta), g(\theta), F_u(\theta, r), F_v(\theta, r) \) are smooth functions satisfying (1.5) and the solution \( u_e(r) \) of (1.7) has a positive lower bound, then there exist \( \varepsilon_0 > 0, \eta_0 > 0, C > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0) \), the Navier-Stokes equations (1.3) have a solution \((u^\varepsilon(\theta, r), v^\varepsilon(\theta, r))\) which satisfies

\[
\begin{align*}
\left\| u^\varepsilon(\theta, r) - u_e(r) - u_p^{(0)}(\theta, \frac{r-1}{\varepsilon}) - \hat{u}_p^{(1)}(\theta, \frac{r-r_0}{\varepsilon}) \right\|_{L^\infty(\Omega)} &\leq C \varepsilon, \\
\| v^\varepsilon \|_{L^\infty(\Omega)} &\leq C \varepsilon,
\end{align*}
\]

where \((u_e(r), 0)\) is the shear flow satisfying (1.7), \((u_p^{(0)}, v_p^{(1)})\) is the solution of steady Prandtl equations (1.8) and \((\hat{u}_p^{(0)}, \hat{v}_p^{(1)})\) is the solution of steady Prandtl equations (1.9).
Moreover, for any $r_0 < r_1 < r_2 < 1$, there holds
\[ \lim_{\varepsilon \to 0} \| w^\varepsilon - w_e(r) \|_{L^\infty(A_{r_1,r_2})} = 0, \]
where $w^\varepsilon(\theta,r)$ is the vorticity of $(u^\varepsilon(\theta,r), v^\varepsilon(\theta,r))$, $w_e(r)$ is the vorticity of $(u_e(r),0)$ and $A_{r_1,r_2} = B_{r_2}(0) \setminus B_{r_1}(0)$.

**Remark 1.3.** If $\bar{F}_u(r) \geq 0$, $\forall r \in [r_0,1]$, then the solution $u_e(r)$ of (1.7) has a positive lower bound by maximum principle.

**Remark 1.4.** We note that the positivity assumption of $u_e(r)$ in Theorem 1.2 is a sharp condition for structural stability of shear flow $(u_e(r),0)$. The structural stability of rotating shear flow $(u_e(r),0)$ means that

- The rotating shear flow $(u_e(r),0)$ is a solution to (1.7) with external force $(F_u,F_v) = (u''_e(r) + u'_e(r) - u_e(r)/r^2,0)$ and constant boundary conditions.
- Under the small perturbation on external force and boundary conditions, the solution to (1.7) is still nearly rotating shear flow (“rotating shear flow” + “small error term”).

On the one hand, Theorem 1.2 shows that if $u_e(r)$ has a positive lower bound, then the shear flow $(u_e(r),0)$ is structurally stable. On the other hand, if $u_e(r)$ has a degenerate point, small perturbations on boundary and force will cause hyperbolic singularities for the Euler flow, then the “cat eye” structure may appear, thus we can’t expect structural stability. Hence, in this sense, the positivity of $u_e(r)$ is a sharp condition for structural stability of shear flow $(u_e(r),0)$ and also for the existence of PB flow.

**Remark 1.5.** We write $\Omega(r) = u_e(r)/r$ as the angular velocity of the Euler flow. From the Rayleigh equation for rotating shear flow
\[ (\Omega - c)(D_s^2 - \frac{\alpha^2}{r^2}) \phi - \frac{rD^2\Omega + 3D\Omega}{r} \phi = 0, \quad D = \frac{d}{dr}, \quad D_s = \frac{d}{dr} + \frac{1}{r}, \]
it is easy to deduce that if the shear flow $(u_e(r),0)$ is dynamically unstable for the Euler equations, $(u_e(r) + Cr,0)$ is also dynamically unstable (Lyapunov unstable) for any constant $C$. Furthermore, it is expected that if the shear flow $(u_e(r),0)$ is dynamically unstable for the Euler equations, it is also dynamically unstable for the Navier-Stokes equations with small viscosity. For any dynamically unstable shear flow $(u_e(r),0)$, we choose $C$ such that $u_e(r) + Cr$ has positive below bound. Thus the shear flow $(u_e(r) + Cr,0)$ is dynamically unstable, but structurally stable from Theorem 1.2. Hence, dynamical stability and structural stability are totally separate.

**Remark 1.6.** The condition $\int_0^{2\pi} f(\theta)d\theta = \int_0^{2\pi} g(\theta)d\theta = 0$ can be dropped due to the fact
\[ \alpha + \eta f(\theta) = \alpha + \frac{\eta}{2\pi} \int_0^{2\pi} f(\theta)d\theta + \eta \bar{f}(\theta), \]
where $\int_0^{2\pi} \bar{f}(\theta)d\theta = 0$. Moreover, the smoothness of $f(\theta), g(\theta), F_u(\theta,r), F_v(\theta,r)$ can be relaxed, but we don’t pursue this issue in this manuscript.

**Remark 1.7.** We believe that our current method can be applicable to the channel flow. More precisely, we consider the two-dimensional forced steady Navier-Stokes equations
\[
\begin{align*}
\mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \nabla p^\varepsilon - \varepsilon^2 \Delta \mathbf{u}^\varepsilon &= \varepsilon^2 \mathbf{F}, \quad (x,y) \in [0,2\pi] \times [0,1] \\
\nabla \cdot \mathbf{u}^\varepsilon &= 0, \quad (x,y) \in [0,2\pi] \times [0,1] \\
\mathbf{u}^\varepsilon(x + 2\pi,y) &= \mathbf{u}^\varepsilon(x,y), \quad y \in [0,1] \\
\mathbf{u}^\varepsilon(x,0) &= (\alpha + \eta f(x))\mathbf{e}_1, \quad \mathbf{u}^\varepsilon(x,0) &= (\beta + \eta g(x))\mathbf{e}_1,
\end{align*}
\]
where $\alpha > 0, \beta > 0, \mathbf{e}_1 = (1,0)^T$, $f(x),g(x)$ are $2\pi$-periodic smooth functions and $\eta$ is a small number. In Appendix C, we proved that under some assumption the inviscid limit of the solutions
\(u^\varepsilon\) to (1.10) must satisfy \(u^\varepsilon_r(y) = \frac{1}{2\pi} \int_0^{2\pi} F_1(x, y) dx, \ \forall y \in [0,1]\), where \(F_1\) is the first component of \(F\). We believe that following the argument of the current paper, when the boundary conditions are slightly different with uniform motion, we can construct solutions \(u^\varepsilon\) to (1.10) which convergence to the shear flow \((u_\varepsilon(y), 0)^T\) when \(\varepsilon \to 0\).

Now we present a sketch of the proof and some key ideas.

**Step 1: Construction of the approximate solution.** We construct an approximate solution \((u^a, v^a)\) by matched asymptotic expansion. The leading order of \((u^a, v^a)\) is

\[
\left( u_\varepsilon(r) + u_p^{(0)}(\theta, \frac{r-1}{\varepsilon}) + u_\theta^{(0)}(\theta, \frac{r-1}{\varepsilon}), 0 \right).
\]

The details of constructing the approximate solution will be given in Section 2. After the construction of the approximate solution, we derive the equations (3.1) for the error \((u, v) := (u^\varepsilon - u^a, v^\varepsilon - v^a)\). Notice that the nonlinear term can be easily handled by higher order approximation, hence we only need to consider the linearized error equations

\[
\begin{aligned}
&u^a u_\theta + u^a v_\theta + v^a r u_r + v^a u_r + v^a u + v^a u + p^a - \varepsilon^2(r u_{rr} + \frac{u^a r}{r} + 2 \frac{u^a}{r} + u_r - \frac{u}{r}) = F_1,
&u^a v_\theta + u^a r v_r + v^a r v_r + v^a r v_r - 2 u^a u + r p_r - \varepsilon^2(r v_{rr} + \frac{v^a r}{r} - 2 \frac{v^a}{r} + v_r - \frac{v}{r}) = F_2,
&u_\theta + (r v)_r = 0,
&u(\theta + 2\pi, r) = u(\theta, r), \ v(\theta + 2\pi, r) = v(\theta, r),
&u(\theta, 1) = 0, \ v(\theta, 1) = 0,
&u(\theta, 0) = 0, \ v(\theta, 0) = 0.
\end{aligned}
\]

**Step 2: Linear stability estimate for (1.11).** Equations (1.11) are the linearized Navier-Stokes equations around the approximate solution \((u^a, v^a)\). For simply, we assume \((u^a, v^a) \approx (u_\varepsilon(r) + u_p^{(0)}(\theta, \frac{r-1}{\varepsilon}), 0)\) because we can deal with the boundary layer profile near \(r = r_0\) in the similar way. Since \(|u_\theta^a| = |\partial_\theta u_p^{(0)}| \lesssim \eta\), the leading order of the above system can be simplified as

\[
\begin{aligned}
&u^a u_\theta + v^a r u_r + v^a u_r + v^a u + p^a - \varepsilon^2(r u_{rr} + \frac{u^a r}{r} + 2 \frac{u^a}{r} + u_r - \frac{u}{r}) = F_1,
&u^a v_\theta - 2 u^a u_a + r p_r - \varepsilon^2(r v_{rr} + \frac{v^a r}{r} - 2 \frac{v^a}{r} + v_r - \frac{v}{r}) = F_2,
&u_\theta + (r v)_r = 0,
&u(\theta + 2\pi, r) = u(\theta, r), \ v(\theta + 2\pi, r) = v(\theta, r),
&u(\theta, 1) = 0, \ v(\theta, 1) = 0,
&u(\theta, 0) = 0, \ v(\theta, 0) = 0,
\end{aligned}
\]

where \(u^a\) can be regarded as \(u_\varepsilon(r) + u_p^{(0)}(\theta, \frac{r-1}{\varepsilon})\).

The linear stability estimate consists of a basic energy estimate and positivity estimate. In fact, it is easy to know that the basic energy estimate is not good enough for closing the estimate because \(\varepsilon\) is small. The key point is the following observation which gives the important positivity estimate: \(u^a\) is strictly positive, we should make use of the terms \(u^a u_\theta\) and \(u^a v_\theta\) to obtain a positive quantity from the convective term. To do this, we choose \((- \frac{r v^a}{u^a})_r, (\frac{r v^a}{u^a})_\theta\) as the multiplier, the pressure is eliminated due to the divergence-free condition and the diffusion term is dominated easily due to the small viscosity \(\varepsilon^2\). Here we only give a sketch of deriving the key positive quantity as follows. Firstly one has

\[
- \int^{1}_{r_0} \int_0^{2\pi} (u^a u_\theta + v^a r u_r + v^a u_r) \left( \frac{r v^a}{u^a} \right)_r \, d\theta dr
+ \int^{1}_{r_0} \int_0^{2\pi} (u^a v_\theta - 2 u^a u_a) \left( \frac{r v^a}{u^a} \right)_\theta \, d\theta dr
\]
Here are the details of obtaining the last step will be given in subsection 3.2.

Since \( u^a \approx u_\epsilon(r) + u_p^{(0)}(\theta, \frac{r-1}{\epsilon}) \), the boundary layer profile \( u_p^{(0)}(\theta, Y) \) is small in the Prandtl variable \( Y = \frac{r-1}{\epsilon} \), we deduce that

\[
\int_{r_0}^1 \int_0^{2\pi} \frac{1}{u^a} \left( r \partial_r^2 u_p^{(0)} + \partial_r u_p^{(0)} - \frac{u_p^{(0)}}{r} \right) (rv)^2 d\theta dr \\
= \int_{r_0}^1 \int_0^{2\pi} \frac{1}{u^a} \left( rY^2 \partial_r^2 u_p^{(0)} + \varepsilon Y^2 \partial_r u_p^{(0)} - \frac{(\varepsilon Y)^2 u_p^{(0)}}{r} \right) (rv)^2 d\theta dr \\
\lesssim \eta \int_{r_0}^1 \int_0^{2\pi} \left( \frac{rv}{r-1} \right)^2 d\theta dr \lesssim \eta \int_{r_0}^1 \int_0^{2\pi} (r(r_v))^2 d\theta dr \lesssim \eta \int_{r_0}^1 \int_0^{2\pi} u^2 d\theta dr,
\]

where we have used the Hardy inequality.

Moreover, if the Euler flow is Taylor-Couette flow, that is \( u_\epsilon(r) = ar + \frac{b}{r} \), then \( ru''_\epsilon + u'_\epsilon - \frac{u_\epsilon}{r} = 0 \).

So

\[
\int_{r_0}^1 \int_0^{2\pi} (ru^2_\theta + rv^2_\theta) d\theta dr + \int_{r_0}^1 \int_0^{2\pi} \frac{1}{u^a} \left( ru^2_{\theta\theta} + u^2_r - \frac{u^a}{r} \right) (rv)^2 d\theta dr \\
\geq (1 - C\eta) \int_{r_0}^1 \int_0^{2\pi} (ru^2_\theta + rv^2_\theta) d\theta dr,
\]

where the constant \( C \) is independent of \( \eta \). So we obtain a positive quantity \( \|(u_\theta, v_\theta)\|_2 \). When \( u_\epsilon \) is strictly positive, we can also obtain the same estimate which will be given in the text.

However, the positive quantity \( \|(u_\theta, v_\theta)\|_2 \) don’t contain zero-frequency of \( (u, v) \) which is needed to obtain the \( L^\infty \) estimate for the error \( (u, v) \). Notice that \( \int_0^{2\pi} v(\theta, r) d\theta = 0 \) because of the divergence-free condition and the boundary conditions, we can dominate \( \|v\|_2 \) by \( \|v_\theta\|_2 \) using the Poincaré inequality. However, \( u_0(r) := \frac{1}{2\pi} \int_0^{2\pi} u(\theta, r) d\theta \neq 0 \), we need to obtain the estimate of \( |u_0(r)| \) from the basic energy estimate.

Now we give a sketch of the basic energy estimate. Choose \( u_0 \) as a multiplier to the first equation in \( (1.12) \). The diffusion term is

\[
\int_{r_0}^1 \int_0^{2\pi} -\varepsilon^2 \left( ru_{\theta\theta} + \frac{u_{\theta\theta}}{r} + 2 \frac{v_\theta}{r} + u_r - \frac{u}{r} \right) u_0(r) d\theta dr = \varepsilon^2 \int_{r_0}^1 \int_0^{2\pi} \left( r|u_\theta|^2 + \frac{u^2_\theta}{r} \right) d\theta dr.
\]

We decompose the approximate solution as \( u^a \approx u_\epsilon(r) + u_p^{(0)}(\theta, \frac{r-1}{\epsilon}) \). Due to the Euler flow is radial, we deduce that the following quantity vanishes

\[
\int_{r_0}^1 \int_0^{2\pi} (u_\epsilon u_\theta + vr u_{\theta\theta} + vu_\theta + p_\theta) u_0 d\theta dr = 0,
\]
where we used $\int_0^{2\pi} v(\theta, r)d\theta = 0$. Moreover, the Prandtl part can be handled by the Hardy inequality as above

$$\int_{r_0}^1 \int_0^{2\pi} (u_p^{(0)} u_\theta + vr\partial_r u_p^{(0)} + vu_p^{(0)}) u_\theta d\theta dr$$

$$= \int_{r_0}^1 \int_0^{2\pi} \varepsilon(Y u_p^{(0)} u_\theta + \frac{vr}{r-1} Y^2 \partial_r u_p^{(0)} + v Y u_p^{(0)}) \frac{u_\theta}{r-1} d\theta dr \lesssim \eta \varepsilon \|(u_\theta, v_\theta)\|_2 \|u_\theta\|_2.$$

Thus, we obtain that $\varepsilon^2 \|u_\theta\|_2 \leq \|(u_\theta, v_\theta)\|_2$. One can see Lemma 3.2 for the basic energy estimate.

Combining the positivity estimate and basic energy estimate, we obtain the linear stability of equations (1.12).

**Step 3: $L^\infty$ estimate for (1.11).** To close the nonlinearity, we need the $L^\infty$ estimate. Using the anisotropic Sobolev embedding, we only need to estimate $\|(u_\theta, v_\theta)\|_2$ which can be obtained by studying the Stokes equations.

Finally, combining the linear stability estimate and $L^\infty$ estimate, we establish the well-posedness of the error equations by contraction mapping theorem.

The paper is organized as follows. In Section 2, we construct the approximate solution by the matched asymptotic expansion method. In Section 3, we derive the error equations and establish the linear stability estimate which consists of the basic energy estimate and positivity estimate. In Section 4, we firstly establish the “$H^2$ estimate” for a Stokes system, then obtain $L^\infty$ estimate by the anisotropic Sobolev embedding, finally complete the proof of Theorem 1.2 by combining the linear stability estimate and $L^\infty$ estimate.

2. Construction of approximate solutions

In this section, we construct an approximate solution of the Navier-Stokes equations (1.3) by the matched asymptotic expansion.

2.1. Euler expansions.

Away from the boundary, we make the following formal expansions

$$u^\varepsilon(\theta, r) = u_e^{(0)}(\theta, r) + \varepsilon u_e^{(1)}(\theta, r) + \text{h.o.t.},$$

$$v^\varepsilon(\theta, r) = v_e^{(0)}(\theta, r) + \varepsilon v_e^{(1)}(\theta, r) + \text{h.o.t.},$$

$$p^\varepsilon(\theta, r) = p_e^{(0)}(\theta, r) + \varepsilon p_e^{(1)}(\theta, r) + \text{h.o.t.},$$

here and in what follows, “h.o.t.” means higher order terms.

2.1.1. Equations for $(u_e^{(0)}, v_e^{(0)}, p_e^{(0)})$.

By substituting the above expansions into (1.3) and collecting the $\varepsilon$-zeroth order terms, we deduce that $(u_e^{(0)}, v_e^{(0)}, p_e^{(0)})$ satisfies the following steady nonlinear Euler equations

$$\begin{align*}
    u_e^{(0)} \partial_\theta u_e^{(0)} + rv_e^{(0)} \partial_r u_e^{(0)} + u_e^{(0)} v_e^{(0)} + \partial_\theta p_e^{(0)} &= 0, \\
    u_e^{(0)} \partial_\theta v_e^{(0)} + rv_e^{(0)} \partial_r v_e^{(0)} - (u_e^{(0)})^2 + r \partial_r p_e^{(0)} &= 0, \\
    \partial_\theta u_e^{(0)} + \partial_r (rv_e^{(0)}) &= 0.
\end{align*}$$

(2.1)

In our current work, we assume that the leading order Euler flow $(u_e^{(0)}, v_e^{(0)})$ is the shear flow in (1.6). Then equations (2.1) lead to

$$\partial_\theta p_e^{(0)}(\theta, r) = 0, \quad \partial_r p_e^{(0)}(\theta, r) = \frac{1}{r} u_e^{(0)}.$$


Thus we can write
\[
p_e(0)(\theta, r) = p_e(r), \quad p_e'(r) = \frac{1}{r} u_e^2(r). \tag{2.2}
\]

2.2.1. Equations for \((u_e^{(1)}, v_e^{(1)}, p_e^{(1)})\).

By collecting the \(\varepsilon\)-order terms, we deduce that \((u_e^{(1)}, v_e^{(1)}, p_e^{(1)})\) satisfies the following linearized Euler equations in \(\Omega\)

\[
\begin{aligned}
&u_e(r)\partial_\theta u_e^{(1)} + r v_e^{(1)} u_e' + u_e(r) v_e^{(1)} + \partial_\theta p_e^{(1)} = 0, \\
u_e(r)\partial_\theta v_e^{(1)} - 2u_e(r) u_e^{(1)} + r \partial_r p_e^{(1)} = 0, \\
\partial_\theta u_e^{(1)} + \partial_r (r v_e^{(1)}) = 0,
\end{aligned}
\tag{2.3}
\]

which are equipped with the boundary conditions

\[
v_e^{(1)}|_{r=1} = -v_p^{(1)}|_{Y=0}, \quad v_e^{(1)}|_{r=r_0} = -v_p^{(1)}|_{Y=0}, \quad v_e^{(1)}(\theta, r) = v_e^{(1)}(\theta + 2\pi, r), \tag{2.4}
\]

where \(v_p^{(1)}\) is defined in (2.7) and (2.10) respectively in next subsection.

2.2. Prandtl expansions near \(\partial B_1\).

We introduce the scaled variable \(Y = \frac{r - 1}{\varepsilon} \in (-\infty, 0]\) and make the following Prandtl expansions near \(\partial B_1\)

\[
u_e = u_e(0)(\theta, r) + u_e(0)(\theta, Y) + \varepsilon [u_e(1)(\theta, r) + u_e(1)(\theta, Y)] + \varepsilon^2 [u_e(2)(\theta, r) + u_e(2)(\theta, Y)] + \text{h.o.t.},
\]

\[
v_e = v_e(0)(\theta, r) + v_e(0)(\theta, Y) + \varepsilon [v_e(1)(\theta, r) + v_e(1)(\theta, Y)] + \varepsilon^2 [v_e(2)(\theta, r) + v_e(2)(\theta, Y)] + \text{h.o.t.},
\tag{2.5}
\]

\[
p_e = p_e(0)(\theta, r) + p_e(0)(\theta, Y) + \varepsilon [p_e(1)(\theta, r) + p_e(1)(\theta, Y)] + \varepsilon^2 [p_e(2)(\theta, r) + p_e(2)(\theta, Y)] + \text{h.o.t.},
\]

where as \(Y \to -\infty\)

\[
\partial_\theta ^m \partial_Y ^n v_\nu(\theta, Y) \to 0, \quad \partial_\theta ^m \partial_Y ^n p_\nu(\theta, Y) \to 0, \tag{2.6}
\]

here \(l, m \geq 0, i = 0, 1, \cdots\), and satisfy the following boundary conditions

\[
u_e(0)(\theta, 1) + \nu_e(0)(\theta, 0) = \alpha + \eta f(\theta), \quad u_e(1)(\theta, 1) + u_e(1)(\theta, 0) = 0, \quad i \geq 1,
\]

\[
v_e(1)(\theta, 1) + v_p(1)(\theta, 0) = 0, i \geq 0, \quad \lim_{Y \to -\infty} (u_p(0)(\theta, Y), \partial_\theta Y u_p(1)(\theta, Y)) = (0, 0).
\]

2.2.1. Equations for \((v_p^{(0)}, p_p^{(0)})\).

By substituting the above expansions into (2.3) and collecting the \(\frac{1}{\varepsilon}\) order terms, we get

\[
\partial_\theta Y v_\nu(0)(\theta, Y) = 0, \quad \partial_\theta Y p_\nu(0)(\theta, Y) = 0,
\]

which together with (2.6) implies

\[
v_p^{(0)} = 0, \quad p_p^{(0)} = 0.
\]

2.2.2. Equations for \((u_p^{(0)}, v_p^{(1)}, p_p^{(1)})\).

By substituting the above expansions into (2.3) and collecting the \(\varepsilon\)-zeroth order terms, we obtain the following steady Prandtl equations for \((u_p^{(0)}, v_p^{(1)})\)

\[
\begin{aligned}
(u_e(1) + u_p^{(0)}) \partial_\theta u_p^{(0)} + (v_p^{(1)} - v_p^{(1)}(\theta, 0)) \partial_\theta Y u_p^{(0)} + \partial_\theta Y u_p^{(0)} = 0, \\
\partial_\theta Y u_p^{(0)} + \partial_\theta Y v_p^{(1)} = 0, \\
u_p^{(0)}(\theta, Y) = u_p^{(0)}(\theta + 2\pi, Y), \quad v_p^{(0)}(\theta, Y) = v_p^{(0)}(\theta + 2\pi, Y), \\
u_p^{(0)}|_{Y=0} = \alpha + \eta f(\theta) - u_e(1), \quad \lim_{Y \to -\infty} u_p^{(0)} = \lim_{Y \to -\infty} v_p^{(1)} = 0
\end{aligned}
\tag{2.7}
\]
and the pressure $p_p^{(1)}$ satisfies
\[
\partial_Y p_p^{(1)}(\theta, Y) = (u_e^{(0)})^2(\theta, Y) + 2u_e(1)u_p^{(0)}(\theta, Y), \quad \lim_{Y \to -\infty} p_p^{(1)}(\theta, Y) = 0. \quad (2.8)
\]

### 2.2.3. Equations for $(u_p^{(1)}, v_p^{(2)})$

By substituting the above expansions into the first and third equation of (1.3) and collecting the $\varepsilon$-order terms, we obtain the following linearized steady Prandtl equations for $(u_p^{(1)}, v_p^{(2)})$

\[
\begin{cases}
(u_e(1) + u_p^{(0)}) \partial_Y u_p^{(1)} + (v_e(1) + v_p^{(1)}) \partial_Y u_p^{(1)} + (v_p^{(2)} - v_p^{(2)}(\theta, 0)) \partial_Y u_p^{(0)} \\
+ u_p^{(1)} \partial_Y u_p^{(0)} - \partial_Y Y u_p^{(1)} = f_1(\theta, Y),
\end{cases}
\]

\[
\begin{aligned}
\partial_Y u_p^{(1)} + \partial_Y v_p^{(2)} + \partial_Y (Y v_p^{(1)}) &= 0, \\
u_p^{(1)}(\theta, Y) &= u_p^{(1)}(\theta + 2\pi, Y), \quad v_p^{(2)}(\theta, Y) = v_p^{(2)}(\theta + 2\pi, Y), \\
\lim_{Y \to -\infty} \partial_Y u_p^{(1)} &= \lim_{Y \to -\infty} v_p^{(2)} = 0
\end{aligned}
\]

and

\[
\begin{aligned}
f_1(\theta, Y) &= - \partial_Y p_p^{(1)} + Y \partial_Y Y u_p^{(0)} + \partial_Y u_p^{(0)} \\
&\quad - u_p^{(0)} \left( \partial_Y u_e^{(1)}(\theta, 1) + v_e^{(1)}(\theta, 1) + v_p^{(1)} \right) - (u_e(1)Y + u_e^{(1)}(\theta, 1)) \partial_Y u_p^{(0)} \\
&\quad - (\partial_Y v_e^{(1)}(\theta, 1) + v_e^{(1)}(\theta, 1))Y \partial_Y u_p^{(0)} - (u_e(1) + Y \partial_Y u_p^{(0)} + u_e^{(1)})v_p^{(1)}.
\end{aligned}
\]

### 2.3. Prandtl expansions near $\partial B_{r_0}$

Similarly as above, we introduce the scaled variable $Z = \frac{r - r_0}{\varepsilon} \in [0, +\infty)$ and make the following Prandtl expansions near $\partial B_{r_0}$

\[
\begin{align*}
u^\varepsilon &= u_e^{(0)}(\theta, r) + \tilde{u}_p^{(0)}(\theta, Z) + \varepsilon [u_e^{(1)}(\theta, r) + \tilde{u}_p^{(1)}(\theta, Z)] + \varepsilon^2 [u_e^{(2)}(\theta, r) + \tilde{u}_p^{(2)}(\theta, Z)] + \text{h.o.t.}, \\
v^\varepsilon &= v_e^{(0)}(\theta, r) + \tilde{v}_p^{(0)}(\theta, Z) + \varepsilon [v_e^{(1)}(\theta, r) + \tilde{v}_p^{(1)}(\theta, Z)] + \varepsilon^2 [v_e^{(2)}(\theta, r) + \tilde{v}_p^{(2)}(\theta, Z)] + \text{h.o.t.}, \\
p^\varepsilon &= p_e^{(0)}(\theta, r) + \tilde{p}_p^{(0)}(\theta, Z) + \varepsilon [p_e^{(1)}(\theta, r) + \tilde{p}_p^{(1)}(\theta, Z)] + \varepsilon^2 [p_e^{(2)}(\theta, r) + \tilde{p}_p^{(2)}(\theta, Z)] + \text{h.o.t.,}
\end{align*}
\]

where as $Z \to +\infty$

\[
\partial_Y^l \partial_Y^m \tilde{u}_p^{(i)}(\theta, Z) \to 0, \quad \partial_Y^l \partial_Y^m \tilde{p}_p^{(i)}(\theta, Z) \to 0,
\]

here $l, m \geq 0, i = 0, 1, \cdots$, and there hold the following boundary conditions

\[
\begin{align*}
u_e^{(0)}(\theta, r_0) + \tilde{u}_p^{(0)}(\theta, 0) &= \beta + \eta g(\theta), \quad u_e^{(i)}(\theta, r_0) + \tilde{u}_p^{(i)}(\theta, 0) = 0, \quad i \geq 1, \\
v_e^{(i)}(\theta, r_0) + \tilde{v}_p^{(i)}(\theta, 0) = 0, \quad i \geq 0, \quad \lim_{Z \to +\infty} (\tilde{u}_p^{(0)}(\theta, Z), \partial_Z \tilde{u}_p^{(1)}(\theta, Z)) = (0, 0).
\end{align*}
\]

#### 2.3.1. Equations for $(\hat{v}_p^{(0)}, \hat{p}_p^{(0)})$

By substituting the above expansions into (1.3) and collecting the $\frac{1}{\varepsilon}$ order terms, we get

\[
\partial_Z \hat{v}_p^{(0)}(\theta, Z) = 0, \quad \partial_Z \hat{p}_p^{(0)}(\theta, Z) = 0,
\]

which together with (2.6) implies

\[
\hat{v}_p^{(0)} = 0, \quad \hat{p}_p^{(0)} = 0.
\]
2.3.2. Equations for \((u_p^{(0)}, v_p^{(1)}, p_p^{(1)})\).

By substituting the above expansions into (2.13) and collecting the \(\varepsilon\)-zeroth order terms, we obtain the following steady Prandtl equations for \((\hat{u}_p^{(0)}, \hat{v}_p^{(1)})\)

\[
\begin{aligned}
&\left\{ \begin{array}{l}
(u_e(r_0) + \hat{u}_p^{(0)})\partial_\theta \hat{u}_p^{(0)} + (\hat{v}_p^{(1)} - \hat{v}_p^{(0)}(\theta, 0))r_0\partial_Z \hat{u}_p^{(0)} - r_0\partial_ZZ \hat{u}_p^{(0)} = 0, \\
\partial_\theta \hat{u}_p^{(0)} + r_0\partial_Z \hat{v}_p^{(1)} = 0, \\
\hat{u}_p^{(0)}(\theta, Z) = \hat{u}_p^{(0)}(\theta + 2\pi, Z), \quad \hat{v}_p^{(0)}(\theta, Z) = \hat{v}_p^{(0)}(\theta + 2\pi, Z), \\
\hat{u}_p^{(0)}|_{Z=0} = \beta + \eta g(\theta) - u_e(r_0), \quad \lim_{Z \to +\infty} \hat{u}_p^{(0)} = \lim_{Z \to +\infty} \hat{v}_p^{(1)} = 0
\end{array} \right.
\end{aligned}
\] (2.10)

and the pressure \(p_p^{(1)}\) satisfies

\[
r_0\partial_Z p_p^{(1)}(\theta, Z) = (\hat{u}_p^{(0)})^2(\theta, Z) + 2u_e(r_0)\hat{u}_p^{(0)}(\theta, Z), \quad \lim_{Z \to +\infty} p_p^{(1)}(\theta, Z) = 0. \tag{2.11}
\]

2.3.3. Equations for \((\hat{u}_p^{(1)}, \hat{v}_p^{(2)})\).

By substituting the above expansions into the first and third equation of (1.3) and collecting the \(\varepsilon\)-order terms, we obtain the following linearized steady Prandtl equations for \((\hat{u}_p^{(1)}, \hat{v}_p^{(2)})\)

\[
\begin{aligned}
&\left\{ \begin{array}{l}
(u_e(r_0) + \hat{u}_p^{(0)})\partial_\theta \hat{u}_p^{(1)} + (\hat{v}_p^{(1)}(\theta, r_0) + \hat{v}_p^{(0)}(\theta, 0))r_0\partial_Z \hat{u}_p^{(1)} + \hat{u}_p^{(1)}\partial_\theta \hat{u}_p^{(0)} \\
\partial_\theta \hat{v}_p^{(1)} + r_0\partial_Z \hat{v}_p^{(2)} + \partial_Z(Z\hat{v}_p^{(1)}) = 0,
\end{array} \right.
\end{aligned}
\] (2.12)

where

\[
\hat{f}_1(\theta, Y) = -\partial_\theta \hat{p}_p^{(1)} + Z\partial_ZZ \hat{u}_p^{(0)} + \partial_Z \hat{u}_p^{(0)} - \hat{u}_p^{(0)}(\partial_\theta u_e^{(1)}(\theta, r_0) + \hat{v}_p^{(1)}(\theta, r_0) + \hat{v}_p^{(0)}(\theta, 0)) - (u_e'(r_0)Z + u_e^{(1)}(\theta, r_0))\partial_\theta \hat{u}_p^{(0)} - (r_0\partial_\theta v_e^{(1)}(\theta, r_0) + v_e^{(1)}(\theta, r_0) + \hat{v}_p^{(1)}(\theta, 0))Z\partial_Z \hat{u}_p^{(0)} - [u_e(r_0) + r_0\hat{u}_e'(r_0)]\hat{v}_p^{(1)}.
\]

Before performing the higher order expansions we then aim to solve the above Euler equations and Prandtl equations.

2.4. Solvabilities of Euler equations and Prandtl equations.

The order in which we solve the equation is as follows

\[
(u_e(r), 0) \to (\hat{u}_p^{(0)}, \hat{v}_p^{(1)})/(\hat{u}_p^{(0)}, \hat{v}_p^{(1)}) \to (u_e^{(1)}, v_e^{(1)}) \to (\hat{u}_p^{(1)}, \hat{v}_p^{(2)})/(\hat{u}_p^{(1)}, \hat{v}_p^{(2)}).
\]

2.4.1. Prandtl system and its solvability.

We first derive some necessary conditions for the solvability of Prandtl equations, one can also refer to Lemma 2.1 in [6].

**Lemma 2.1.** (Batchelor-Wood formula [22] [23]) Let \(\int_0^{2\pi} f(\theta)d\theta = \int_0^{2\pi} g(\theta)d\theta = 0\). If the nonlinear Prandtl equations (2.7) has a solution \((u_p^{(0)}, v_p^{(1)})\) and (2.10) have a solution \((\hat{u}_p^{(0)}, \hat{v}_p^{(1)})\) which satisfy

\[
u_e(1) + u_p^{(0)}(\theta, Y) > 0, \quad \forall Y \leq 0; \quad u_e(r_0) + \hat{u}_p^{(0)}(\theta, Z) > 0, \quad \forall Z \geq 0,
\]

\[
\|u_p^{(0)}\| + \|\hat{u}_p^{(0)}\| \leq M,
\]
where $M > 0$ is a constant, then there hold

$$u_ε^2(1) = \alpha^2 + \frac{\eta^2}{2\pi} \int_0^{2\pi} f^2(\theta) d\theta,$$

(2.13)

and

$$u_ε^2(r_0) = \beta^2 + \frac{\eta^2}{2\pi} \int_0^{2\pi} g^2(\theta) d\theta.$$

(2.14)

Proof. We only need to prove (2.13) since (2.14) can be derived similarly. We introduce the von Mises variable

$$\psi = \int_0^Y \left( u_e(1) + u_p(0)(\theta, z) \right) dz, \quad U^{(0)}(\theta, \psi) = u_e(1) + u_p(0)(\theta, Y).$$

Then from (2.7), we deduce that $U^{(0)}$ satisfies

$$\begin{cases}
2U_\theta^{(0)} = (U^{(0)})^2 \psi, \\
U^{(0)}(\theta, \psi) = U^{(0)}(\theta + 2\pi, \psi), \\
U^{(0)}(\theta, \psi = 0) = \alpha + \eta f(\theta), \quad \lim_{\psi \to -\infty} U^{(0)} = u_e(1).
\end{cases}$$

(2.15)

Here we have used the facts:

$$\partial_\theta u_p^{(0)} = U_\theta^{(0)} + U_\psi^{(0)} \int_0^Y \partial_\theta u_p^{(0)}(\theta, z) dz$$

$$= U_\theta^{(0)} + U_\psi^{(0)} \left( v_p^{(1)}(\theta, 0) - v_p^{(1)}(\theta, Y) \right)$$

$$= U_\theta^{(0)} - U_\psi^{(0)} \left( v_p^{(1)}(\theta, 1) + v_p^{(1)}(\theta, Y) \right),$$

$$\partial_Y u_p^{(0)} = U_\theta^{(0)} U_\psi^{(0)}.$$

Integrating the first equation in (2.15) from 0 to $2\pi$ about $\theta$ leads to

$$\frac{\partial^2}{\partial \psi^2} \int_0^{2\pi} (U^{(0)})^2(\theta, \psi) d\theta = 0.$$

Notice that $U^{(0)}$ is bounded at $\psi \to -\infty$, we deduce that

$$\frac{\partial}{\partial \psi} \int_0^{2\pi} (U^{(0)})^2(\theta, \psi) d\theta = 0.$$

Therefore combining the boundary condition in (2.15), we deduce that

$$u_e^2(1) = \frac{1}{2\pi} \int_0^{2\pi} \left( \alpha + \eta f(\theta) \right)^2 d\theta = \alpha^2 + \frac{\alpha \eta}{\pi} \int_0^{2\pi} f(\theta) d\theta + \frac{\eta^2}{2\pi} \int_0^{2\pi} f^2(\theta) d\theta$$

$$= \alpha^2 + \frac{\eta^2}{2\pi} \int_0^{2\pi} f^2(\theta) d\theta.$$

Thus, we complete the proof of this lemma. \qed
Next we aim to solve the steady Prandtl equations (2.17), one can refer to Corollary 2.2 in [6].

**Proposition 2.2.** There exists \( \eta_0 > 0 \) such that for any \( \eta \in (0, \eta_0) \) and any \( j, k, l \in \mathbb{N} \cup \{0\} \), the equations (2.15) have a unique solution \( U \), then (2.16) is equivalent to

\[
\int_{-\infty}^{0} \int_{0}^{2\pi} \left| \partial_\theta \partial_\psi^k (U^{(0)} - u_e(1)) \right|^2 \langle \psi \rangle^{2l} d\theta d\psi \leq C(j, k, l) \eta^2,
\]

here \( \langle \psi \rangle = \sqrt{1 + \psi^2} \).

**Proof.** We will use the contraction mapping theorem to prove the desired conclusions and divide the proof into five steps.

**Step 1: Derivation of equivalent equations.**

Let \( Q(\theta, \psi) := (U^{(0)})^2(\theta, \psi) - u_e^2(1) \) and we rewrite (2.15) as

\[
\begin{align*}
Q_\theta &= U^{(0)} Q_{\psi \psi}, \\
Q(\theta, \psi) &= Q(\theta + 2\pi, \psi), \\
Q|_{\psi = 0} &= \alpha^2 + 2\alpha \eta f(\theta) + \eta^2 f(\theta) - u_e^2(1), \quad Q|_{\psi \to -\infty} = 0.
\end{align*}
\]

Defining

\[
G(Q) = Q - 2u_e(1) \sqrt{Q + u_e^2(1) + 2u_e^2(1)},
\]

then (2.16) is equivalent to

\[
\begin{align*}
Q_\theta - u_e(1) Q_{\psi \psi} &= (G(Q))_\theta, \\
Q(\theta, \psi) &= Q(\theta + 2\pi, \psi), \\
Q|_{\psi = 0} &= \alpha^2 + 2\alpha \eta f(\theta) + \eta^2 f(\theta) - u_e^2(1), \quad Q|_{\psi \to -\infty} = 0.
\end{align*}
\]

Let \( Q_0 \) be the solution to

\[
\begin{align*}
(Q_0)_\theta &= u_e(1)(Q_0)_{\psi \psi}, \\
Q_0(\theta, \psi) &= Q_0(\theta + 2\pi, \psi), \\
Q_0|_{\psi = 0} &= \alpha^2 + 2\alpha \eta f(\theta) + \eta^2 f(\theta) - u_e^2(1), \quad Q_0|_{\psi \to -\infty} = 0,
\end{align*}
\]

which will be solved in Appendix A, then (2.17) is equivalent to

\[
\begin{align*}
Q_\theta - u_e(1) Q_{\psi \psi} &= (H(Q))_\theta, \\
Q(\theta, \psi) &= Q(\theta + 2\pi, \psi), \\
Q|_{\psi = 0} = 0, \quad Q|_{\psi \to -\infty} = 0,
\end{align*}
\]

where

\[
H(Q) = Q + Q_0 - 2u_e(1) \sqrt{Q + Q_0 + u_e^2(1) + 2u_e^2(1)}.
\]

Defining the linear operator \( L : \Lambda \to \Phi \) such that

\[
\Phi = L \Lambda \iff \Phi(\theta, \psi) = \Phi(\theta + 2\pi, \psi),
\]

\[
\Phi|_{\psi = 0} = 0, \quad \Phi|_{\psi \to -\infty} = 0,
\]

then (2.19) is equivalent to

\[
\begin{align*}
Q &= L(H(Q)), \\
Q(\theta, \psi) &= Q(\theta + 2\pi, \psi), \\
Q|_{\psi = 0} = 0, \quad Q|_{\psi \to -\infty} = 0.
\end{align*}
\]
Defining the function space $X$ as follows

$$X = \left\{ Q : Q(\theta, \psi) = Q(\theta + 2\pi, \psi), \, Q|_{\psi=0} = Q|_{\psi=-\infty} = 0, \right. $$

$$\left. \|Q\|_X^2 = \sum_{j+k\leq m} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \left| \partial_{\theta}^j \partial_{\psi}^k Q(\psi)^\dagger d\theta d\psi < +\infty \right \} \right.$$ 

and a ball $B_0$ in $X$

$$B_0 = \{ Q \in X : \|Q\|_X \leq \nu \},$$

here $m$ is a positive integer and $\nu$ is a small number which will be determined later.

In the next three steps we aim to verify that $\mathcal{L} \circ \mathcal{H}$ is a contraction map from $B_0$ to $B_0$ with suitable small $\nu$.

**Step 2:** Boundedness of $\mathcal{L}$ in $X$. In this step, we prove that for any $m \geq 0, l \geq 0$, there holds

$$\sum_{j+k\leq m} \| \partial_{\theta}^j \partial_{\psi}^k \Phi(\psi)^\dagger \|_2 \leq C(m, l) \sum_{j+k\leq m,q\leq l} \| \partial_{\theta}^j \partial_{\psi}^k \Lambda(\psi)^q \|_2. \quad (2.22)$$

First, we prove (2.22) for $l = 0$. Multiplying the equation in (2.20) by $\Phi$ and integrating with respect to $(\theta, \psi) \in (0, 2\pi) \times (-\infty, 0)$, we obtain for any $\lambda > 0$

$$u(1)\|\Phi\|_2 \leq C(\lambda)\|\Lambda\|_2 + \lambda\|\Phi\|_2. \quad (2.23)$$

Then, multiplying the equation in (2.20) by $\Phi_{\theta}$ and integrating with respect to $(\theta, \psi) \in (0, 2\pi) \times (-\infty, 0)$, one has

$$\|\Phi_{\theta}\|_2 \leq C(\lambda)\|\Lambda_{\theta}\|_2 + \lambda\|\Phi_{\theta}\|_2. \quad (2.24)$$

Combining (2.23)-(2.24) and choosing small $\lambda > 0$ we get

$$\|(\Phi, \Phi_{\theta})\|_2 \leq C((\Lambda, \Lambda_{\theta}))_2. \quad (2.25)$$

Integrating (2.20) with respect to $\theta \in (0, 2\pi)$ gives

$$\frac{d^2}{d\psi^2} \int_{0}^{2\pi} \Phi(\theta, \psi) d\theta = 0,$$

which and $\Phi|_{\psi=0} = \Phi|_{\psi=-\infty} = 0$ imply

$$\int_{0}^{2\pi} \Phi(\theta, \psi) d\theta = 0. \quad (2.26)$$

Due to (2.26) and the Poincaré inequality we have

$$\|\Phi\|_2 \leq C\|\Phi_{\theta}\|_2 \leq C((\Lambda, \Lambda_{\theta}))_2. \quad (2.27)$$

For any $j \geq 0, k \geq 2$, from the equation (2.20), we deduce that

$$\| \partial_{\theta}^j \partial_{\psi}^k \Phi \|_2 \leq C(\| \partial_{\theta}^j \partial_{\psi}^{k-2} \Phi_{\theta}\|_2 + \| \partial_{\theta}^j \partial_{\psi}^{k-2} \Lambda_{\theta}\|_2). \quad (2.28)$$

For any $j \geq 0$, applying $\partial_{\theta}^j$ to the equation in (2.20), multiplying the resultant equation by $\partial_{\theta}^j \Phi$, integrating with respect to $(\theta, \psi) \in (0, 2\pi) \times (-\infty, 0)$ and using the Young inequality one has

$$\| \partial_{\theta}^j \partial_{\psi} \Phi \|_2 \leq C\| \partial_{\theta}^{j+1} \Lambda\|_2 + C\| \partial_{\theta}^j \Phi\|_2. \quad (2.29)$$

For any $j \geq 0$, multiplying the equation in (2.20) by $\partial_{\theta}^{2j-1} \Phi$, integrating with respect to $(\theta, \psi) \in (0, 2\pi) \times (-\infty, 0)$, one can get

$$\| \partial_{\theta}^j \Phi \|_2^2 = \int_{-\infty}^{\infty} \int_{0}^{2\pi} \partial_{\theta}^{2j-1} \Phi \Lambda_{\theta} d\theta d\psi \leq C\| \partial_{\theta}^j \Phi\|_2\| \partial_{\theta}^j \Lambda\|_2. \quad (2.30)$$
Applying $\partial_\theta^{m-1}$ to the equation in (2.20), multiplying the resultant equation by $\partial_\theta^{m-1}\Phi^{2l}$, integrating with respect to $(\theta, \psi) \in (0, 2\pi) \times (-\infty, 0)$ and using the Young inequality one has

$$
\left| \partial_\theta^{m-1}\Phi^{l}\right|_2 \leq C \left| \partial_\theta^{m-1}\Lambda^{l}\right|_2 + C \left| \partial_\theta^{m-1}\Phi^{l-1}\right|_2.
$$

(2.36)

Multiplying the equation in (2.20) by $\partial_\theta^{m-1}\Phi^{2l}$, integrating with respect to $(\theta, \psi) \in (0, 2\pi) \times (-\infty, 0)$, we can get

$$
\left| \partial_\theta^{m-1}\Phi^{l}\right|_2 \leq C \left| \partial_\theta^{m-1}\Lambda^{l}\right|_2 + C \left| \partial_\theta^{m-1}\Phi^{l-1}\right|_2.
$$

(2.37)

Thanks to (2.35), (2.36) and (2.37), we obtain that for any $m \geq 2$, $l \geq 1$, there holds

$$
\sum_{j+k=m} \left| \partial_\theta^{j}\partial_\psi^{k}\Phi^{l}\right|_2 \leq C(m, l) \sum_{j+k \leq m-1, q \leq l} \left| \partial_\theta^{j}\partial_\psi^{k}\Phi^{q}\right|_2 + C(m, l) \sum_{j+k \leq m} \left| \partial_\theta^{j}\partial_\psi^{k}\Lambda^{q}\right|_2.
$$

(2.38)
This complete the proof of (2.22). Thus, we obtain
\[ \| \mathcal{L} \Lambda \|_X \leq C \| \Lambda \|_X, \quad \forall \Lambda \in X. \] (2.39)

**Step 3:** $\mathcal{L} \circ \mathcal{H}$ is a continuous map from $B_0$ to $B_0$. In this section, we first prove that for any $m \geq 2, l \geq 0$, there holds
\[ \sum_{j+k \leq m} \left\| \partial^j_\theta \partial^k_\psi \mathcal{H}(Q) \langle \psi \rangle^l \right\|_2 \leq C(m, l) \left( \sum_{j+k \leq m} \left\| \partial^j_\theta \partial^k_\psi Q \langle \psi \rangle^l \right\|_2 + \sum_{j+k \leq m} \left\| \partial^j_\theta \partial^k_\psi Q^0 \langle \psi \rangle^l \right\|_2 \right)^2. \] (2.40)

Set
\[ \tilde{H}(x) = x - 2u_\epsilon(1) \sqrt{x + u_\epsilon^2(1)} + 2u_\epsilon^2(1), \quad |x| \ll u_\epsilon^2(1), \]
then $\mathcal{H}(Q) = \tilde{H}(Q + Q_0)$. Direct computation gives
\[ |\tilde{H}'(x)| \leq C|x|, \quad |\tilde{H}^{(k)}(x)| \leq C, \quad k \geq 2. \] (2.41)

Since
\[ \mathcal{H}(Q) = \left( \sqrt{Q + Q_0 + u_\epsilon^2(1)} - u_\epsilon(1) \right)^2 = \left( \frac{Q + Q_0}{\sqrt{Q + Q_0 + u_\epsilon^2(1)} + u_\epsilon(1)} \right)^2, \]
it’s easy to get
\[ \mathcal{H}^2(Q) \langle \psi \rangle^l \leq C \| Q + Q_0 \|_{L^\infty}^2 \left( Q^2 \langle \psi \rangle^l + Q_0^2 \langle \psi \rangle^l \right), \quad l \geq 0. \]

Using (2.41), we deduce that
\[ (\partial_\psi \mathcal{H}(Q))^2 \langle \psi \rangle^l \leq C \| Q + Q_0 \|_{L^\infty}^2 \left( Q_\psi^2 \langle \psi \rangle^l + (Q_0)^2 \langle \psi \rangle^l \right), \quad l \geq 0, \]
\[ (\partial_\theta \mathcal{H}(Q))^2 \langle \psi \rangle^l \leq C \| Q + Q_0 \|_{L^\infty}^2 \left( Q_\theta^2 \langle \psi \rangle^l + (Q_0)^2 \langle \psi \rangle^l \right), \quad l \geq 0. \]

Thus, we obtain
\[ \sum_{j+k \leq 1} \left\| \partial^j_\theta \partial^k_\psi \mathcal{H}(Q) \langle \psi \rangle^l \right\|_2 \leq C \| Q + Q_0 \|_{L^\infty} \left( \sum_{j+k \leq 1} \left\| \partial^j_\theta \partial^k_\psi Q \langle \psi \rangle^l \right\|_2 + \sum_{j+k \leq 1} \left\| \partial^j_\theta \partial^k_\psi Q^0 \langle \psi \rangle^l \right\|_2 \right). \] (2.42)

Since
\[ \partial_\psi \mathcal{H}(Q) = \tilde{H}'(Q + Q_0)(\partial_\psi Q + \partial_\psi Q_0) + \tilde{H}''(Q + Q_0)(\partial_\theta Q + \partial_\theta Q_0)(\partial_\psi Q + \partial_\psi Q_0), \]
thus, using (2.41), we obtain
\[ \left\| \partial_\psi \mathcal{H}(Q) \langle \psi \rangle^l \right\|_{L^2} \leq C \| Q + Q_0 \|_{L^\infty} \left( \left\| Q_\psi \langle \psi \rangle^l \right\|_{L^2} + \left\| (Q_0)^\psi \langle \psi \rangle^l \right\|_{L^2} \right) + C \| Q + Q_0 \|_{L^4} \left\| (Q_\theta + (Q_0)^\theta) \langle \psi \rangle^l \right\|_{L^4} \leq C \| Q + Q_0 \|_{L^\infty} \left( \left\| Q_\psi \langle \psi \rangle^l \right\|_{L^2} + \left\| (Q_0)^\psi \langle \psi \rangle^l \right\|_{L^2} \right) + C \| Q + Q_0 \|_{H^1} \left( \left\| Q_\theta + (Q_0)^\theta \right\|_{H^1} \left\| (Q_0)^\psi \langle \psi \rangle^l \right\|_{H^1}. \right. \]

Hence by Sobolev imbedding, we deduce that
\[ \left\| \partial_\psi \mathcal{H}(Q) \langle \psi \rangle^l \right\|_{L^2} \leq C \left( \sum_{j+k \leq 2} \left\| \partial^j_\theta \partial^k_\psi Q \langle \psi \rangle^l \right\|_{L^2} + \sum_{j+k \leq 2} \left\| \partial^j_\theta \partial^k_\psi Q^0 \langle \psi \rangle^l \right\|_{L^2} \right)^2. \]
Same estimates hold for $\|\partial_{\theta\theta} \mathcal{H}(Q) \langle \psi \rangle^1 \|_2$ and $\|\partial_{\psi\psi} \mathcal{H}(Q) \langle \psi \rangle^1 \|_2$. Thus, combing the estimate \[2.32\], we arrive at
\[
\sum_{j+k\leq 2} \|\partial_{\theta} \partial_{\psi}^k \mathcal{H}(Q) \langle \psi \rangle^1 \|_2 \leq C \left( \sum_{j+k\leq 2} \|\partial_{\theta} \partial_{\psi}^k Q \langle \psi \rangle^1 \|_2 + \sum_{j+k\leq 2} \|\partial_{\theta}^j \partial_{\psi}^k Q_0 \langle \psi \rangle^1 \|_2 \right)^2.
\]
Using \[2.41\] and repeating the above arguments, we obtain \[2.40\].

Consequently, if we take $\nu = \|Q_0\|_X$ and $\eta$ small enough, then
\[
\|\mathcal{L} \mathcal{H}(Q)\|_X \leq C\|\mathcal{H}(Q)\|_X \leq 4C\nu^2 \leq 4C\eta\nu \leq \nu, \forall Q \in B_0,
\]
here we have used \[5.2\]. Thus, $\mathcal{L} \mathcal{H}$ is a continuous map from $B_0$ to $B_0$ for small $\eta$.

**Step 4:** $\mathcal{L} \circ \mathcal{H}$ is a contraction map in $B_0$.

Noting firstly that
\[
\mathcal{H}(Q_1) - \mathcal{H}(Q_2) = (Q_1 - Q_2) \left( 1 - \frac{2u_e(1)}{\sqrt{Q_1 + Q_0 + u_e^2(1)} + \sqrt{Q_2 + Q_0 + u_e^2(1)}} \right)
\]
and
\[
1 - \frac{2u_e(1)}{\sqrt{Q_1 + Q_0 + u_e^2(1)} + \sqrt{Q_2 + Q_0 + u_e^2(1)}} = \frac{1}{Q_1 + Q_0} \left( \sqrt{Q_1 + Q_0 + u_e^2(1)} + \sqrt{Q_2 + Q_0 + u_e^2(1)} \right)
\]
\[
+ \frac{1}{Q_2 + Q_0} \left( \sqrt{Q_1 + Q_0 + u_e^2(1)} + \sqrt{Q_2 + Q_0 + u_e^2(1)} \right) \left( \sqrt{Q_1 + Q_0 + u_e^2(1)} + u_e(1) \right)
\]
With the help of the similar arguments in Step 3, we can obtain that there exist $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$, there holds
\[
\|\mathcal{L} \mathcal{H}(Q_1) - \mathcal{L} \mathcal{H}(Q_2)\|_X \leq C\|\mathcal{H}(Q_1) - \mathcal{H}(Q_2)\|_X \leq C\|Q_1 - Q_2\|_X (\|Q_0\|_X + \|Q_1\|_X + \|Q_2\|_X) \leq C\|Q_0\|_X \|Q_1 - Q_2\|_X \leq \frac{1}{2}\|Q_1 - Q_2\|_X,
\]
that is, $\mathcal{L} \mathcal{H}$ is a contraction map in $B_0$.

**Step 5:** Existence and uniqueness of the system \[2.15\]. By the standard contraction mapping principle, we know that there exists $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$ and any $j, k, l \in \mathbb{N} \cup \{0\}$, the equation \[2.15\] has a unique solution $\mathcal{U}^{(0)}$ which satisfies
\[
\int_{-\infty}^{0} \int_{0}^{2\pi} \left| \partial_{\theta} \partial_{\psi}^k (\mathcal{U}^{(0)} - u_e(1)) \right|^2 \langle \psi \rangle \, d\theta d\psi \leq C(j, k, l)\eta^2,
\]
this completes the proof of this proposition. □

Returning to the equations \[2.7\], we have the following result.

**Proposition 2.3.** There exists $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$, the equations \[2.7\] have a unique solution $(u_p^{(0)}, v_p^{(1)})$ which satisfies
\[
\sum_{j+k\leq m} \int_{-\infty}^{0} \int_{0}^{2\pi} \left| \partial_{\theta} \partial_{\psi}^k (u_p^{(0)}, v_p^{(1)}) \right|^2 \langle Y \rangle \, d\theta dY \leq C(m, l)\eta^2, \forall m, l \geq 0,
\]
\[
\int_{0}^{2\pi} v_p^{(1)}(\theta, Y) d\theta = 0, \forall Y \leq 0.
\]
Finally, solving (2.3), we obtain $p_p^{(1)}(\theta, Y)$ which decay very fast as $Y \to -\infty$.

By the same argument, we can obtain the well-posedness of equations (2.10).

**Proposition 2.4.** There exists $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$, the equations (2.10) have a unique solution $(\tilde{u}_p^{(1)}, \tilde{v}_p^{(1)})$ which satisfies

$$\sum_{j+k \leq m} \int_{-\infty}^{0} \int_{0}^{2\pi} \left| \frac{\partial}{\partial \theta} \frac{\partial}{\partial r} (\tilde{u}_p^{(0)}, \tilde{v}_p^{(1)}) \right|^2 (Y)^{2l} d\theta dY \leq C(m, l)\eta^2, \quad m, l \geq 0,$$

$$\int_{0}^{2\pi} \tilde{v}_p^{(1)}(\theta, Z) d\theta = 0, \quad \forall Z \geq 0. \quad (2.44)$$

Similarly, we can obtain $\hat{p}_p^{(1)}$ by solving (2.11) and $\hat{p}_p^{(1)}$ decay very fast as $Z \to +\infty$.

Here we give a general strategy for constructing high order approximate solution. We first construct $(u_e^{(1)}, v_e^{(1)}, p_e^{(1)})$ by solving the linearized Euler equations (2.3) and $(u_p^{(1)}, v_p^{(2)})$ by solving the linearized Prandtl equations (2.9). Then, because $A_{1\infty} := \lim_{Y \to -\infty} u_p^{(1)}$ is a nonzero constant, we need to change it to $\tilde{u}_p^{(1)} = u_p^{(1)} - A_{1\infty}$. Finally, we modify $u_e^{(1)}$ into $\tilde{u}_e^{(1)}$ by adding a radial function, see (2.69) for the details. Notice the structure of the linearized Euler equations (2.3) and the fact that only the value of $u_e^{(1)}$ at $r = 1$ appear in the equation (2.9), we can easily deduce that the modified $(\tilde{u}_e^{(1)}, \tilde{v}_e^{(1)}, \tilde{p}_e^{(1)})$ and $(\tilde{u}_p^{(1)}, \tilde{v}_p^{(2)})$ still satisfies the equation (2.3) and (2.9). The higher order approximate solutions $(\tilde{u}_e^{(i)}, \tilde{v}_e^{(i)}, \tilde{p}_e^{(i)})(i \geq 2)$ and $(\tilde{u}_p^{(i)}, \tilde{v}_p^{(i+1)})(i \geq 2)$ can be constructed by the same approach.

### 2.4.2. Linearized Euler equations for $(u_e^{(1)}, v_e^{(1)}, p_e^{(1)})$ and their solvability.

**Proposition 2.5.** There exists $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$, the linearized Euler equations (2.3) have a solution $(u_e^{(1)}, v_e^{(1)}, p_e^{(1)})$ which satisfies

$$\| \frac{\partial}{\partial \theta} \frac{\partial}{\partial r} (u_e^{(1)}, v_e^{(1)}) \|_2 \leq C(k, j)\eta, \quad \forall j, k \geq 0. \quad (2.45)$$

**Proof.** Eliminating pressure $p_e^{(1)}$ in the equation (2.3), we obtain

$$-u_e(r)\left( \frac{\partial^2 v_e^{(1)}}{\partial r^2} + \frac{3\partial v_e^{(1)}}{r^2} \right) + v_e^{(1)} \left( u_e''(r) + \frac{u_e'(r)}{r} - \frac{2u_e(r)}{r^2} \right) = 0.$$

Notice that $\triangle = \partial^2 r + \frac{\partial}{\partial r}$, we deduce that

$$-u_e(r)\triangle (rv_e^{(1)}) + \left( u_e''(r) + \frac{u_e'(r)}{r} - \frac{2u_e(r)}{r^2} \right) (rv_e^{(1)}) = 0,$$

hence we obtain the following equations for $rv_e^{(1)}$ in $\Omega$

$$\begin{cases} -\triangle (rv_e^{(1)}) + \left( u_e''(r) + \frac{u_e'(r)}{r} - \frac{2u_e(r)}{r^2} \right) (rv_e^{(1)}) = 0, \\ rv_e^{(1)}|_{r=1} = -v_p^{(1)}|_{Y=0}, \quad rv_e^{(1)}|_{r=r_0} = -r_0v_p^{(1)}|_{Z=0}. \quad (2.46) \end{cases}$$

Firstly, we deduce that equations (2.46) a unique solution $rv_e^{(1)}$. In fact, we assume that $f(\theta, r)$ solving the equations

$$\begin{cases} -\triangle f + \left( u_e''(r) + \frac{u_e'(r)}{r} - \frac{2u_e(r)}{r^2} \right) f = 0, \\ f|_{r=1} = 0, \quad f|_{r=r_0} = 0, \end{cases}$$

then multiple $rf$ and integrating in $\Omega$, we obtain

$$\int_{0}^{2\pi} \int_{r_0}^{1} \left( r(\partial_r f)^2 + (\partial_\theta f)^2 + \frac{1}{u_e(r)} (ru_e''(r) + u_e'(r) - \frac{2u_e(r)}{r^2}) f^2 \right) d\theta dr = 0.$$
Let $g = \frac{f}{u_e}$, we arrive at
\[
\int_0^{2\pi} \int_{r_0}^1 \left( u_e^2(r)(\partial_r g)^2 + \frac{u_e^2(r)g^2}{r} - \frac{u_e^2(r)g^2}{r} \right) d\theta dr = 0.
\]
Notice that \( \int_0^{2\pi} g(\theta, r) d\theta = 0, \quad \forall r \in [r_0, 1] \), by Poincaré inequality, we deduce that $g = 0$, hence $f = 0$. Thus, equation \( (2.46) \) has a unique solution.

Then, notice \( (2.43) \) and \( (2.44) \), we deduce that
\[
\int_0^{2\pi} v_e^{(1)}(\theta, r, t) d\theta = 0, \quad \forall r \in [r_0, 1].
\]
Moreover, due to \( \int_0^{2\pi} v_p^{(1)}(\theta, 0) d\theta = \int_0^{2\pi} v_p^{(1)}(\theta, 0) d\theta = 0 \) and the divergence-free condition, we deduce that
\[
\int_0^{2\pi} v_e^{(1)}(\theta, r, t) d\theta = 0, \quad \forall r \in [r_0, 1].
\]

Then we can construct $u_e^{(1)}$ by solving the following equation
\[
\begin{aligned}
\partial_\theta u_e^{(1)} + \partial_r (r v_e^{(1)}(\theta, t)) &= 0, \\
u_e^{(1)}(\theta, r) &= u_e^{(1)}(\theta + 2\pi, r).
\end{aligned}
\]

After obtaining \( (u_e^{(1)}, v_e^{(1)}) \), we construct $p_e^{(1)}$ as following
\[
p_e^{(1)}(\theta, r) := \phi(r) - \int_0^\theta [u_e(r)\partial_\theta u_e^{(1)} + ru_e v_e^{(1)} + u_e v_e^{(1)}](\theta', r) d\theta',
\]
where $\phi(r)$ is a function which satisfies
\[
r\partial_r \phi(r) + u_e(r)\partial_\theta v_e^{(1)}(0, r) - 2u_e(r)u_e^{(1)}(0, r) = 0.
\]
Combining the equations of \( (u_e^{(1)}, v_e^{(1)}) \), it’s direct to obtain
\[
u_e \partial_\theta v_e^{(1)} - 2u_e u_e^{(1)} + r \partial_r p_e^{(1)} = 0.
\]
Hence, \( (u_e^{(1)}, v_e^{(1)}, p_e^{(1)}) \) solves the equation \( (2.3) \) and satisfies \( (2.45) \).

\[
\begin{aligned}
2.4.3. \quad \text{Linearized Prandtl equations for } (u_p^{(1)}, v_p^{(2)}), (\bar{u}_p^{(1)}, \bar{v}_p^{(2)}) \text{ and their solvabilities.}
\end{aligned}
\]

In this subsection, we consider the solvabilities of \( (2.3) \) and \( (2.12) \). One can also refer to Proposition 2.5 in \[6\].

**Proposition 2.6.** There exists $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$, the equations \( (2.3) \) have a unique solution \( (u_p^{(1)}, v_p^{(2)}) \) which satisfies
\[
\sum_{j+k \leq m} \int_0^{2\pi} \int_0^1 \left| \partial_\theta^j \partial_Y^k (u_p^{(1)} - A_{1\infty}) - A_{1\infty} \right|^2 (Y)^{2l} d\theta dY \leq C(m, l)\eta^2, \quad \forall m, l \geq 0,
\]
\[
\int_0^{2\pi} v_p^{(2)}(\theta, Y) d\theta = 0, \quad \forall Y \leq 0,
\]
where $A_{1\infty} := \lim_{Y \to -\infty} u_p^{(1)}(\theta, Y)$ is a constant which satisfies $|A_{1\infty}| \leq C\eta$.

**Proof.** Let $\eta \in C_c^\infty((-\infty, 0])$ satisfy
\[
\eta(0) = 1, \quad \int_0^{+\infty} \eta(y) dy = 0.
\]
For simplicity, we set
\[
\bar{u} := u_e(1) + u_p^{(0)}, \quad \bar{v} := v_e^{(1)}(\theta, 1) + v_p^{(1)},
\]
\[
\bar{u}_p := u_e(1) + u_p^{(0)}, \quad \bar{v}_p := v_e^{(1)}(\theta, 1) + v_p^{(1)},
\]
\[
\bar{p} := p_e^{(1)}(\theta, 1) + p_p^{(1)}.
\]
\[
\bar{u}_p := u_e^{(1)} + u_p^{(0)} + \int_0^1 \int_0^{2\pi} \left( \partial_\theta^j \partial_Y^k (u_p^{(1)} - A_{1\infty}) - A_{1\infty} \right) (Y)^{2l} d\theta dY,
\]
\[
\bar{v}_p := v_e^{(1)}(\theta, 1) + v_p^{(1)} + \int_0^1 \int_0^{2\pi} \left( \partial_\theta^j \partial_Y^k (v_p^{(1)} - A_{1\infty}) - A_{1\infty} \right) (Y)^{2l} d\theta dY,
\]
\[
\bar{p} := p_e^{(1)}(\theta, 1) + p_p^{(1)} + \int_0^1 \int_0^{2\pi} \left( \partial_\theta^j \partial_Y^k (p_p^{(1)} - A_{1\infty}) - A_{1\infty} \right) (Y)^{2l} d\theta dY.
\]
\[ u := v_p^{(1)} + v_e^{(1)}(\theta, 1)\eta(Y), \quad v := v_p^{(2)} - v_p^{(2)}(\theta, 0) + Y v_p^{(1)} - \partial_\theta u_e^{(1)}(\theta, 1) \int_0^Y \eta(z) \, dz. \]

Then, the equations (2.49) reduce to
\[
\begin{align*}
\bar{u} \partial_\theta u &+ \bar{v} \partial_Y u + u \partial_\theta \bar{u} + v \partial_Y \bar{u} - \partial_Y \gamma u = \bar{f}, \\
\partial_\theta u + \partial_Y v &= 0, \\
Y u(\theta, Y) &= u(\theta + 2\pi, Y), \quad v(\theta, Y) = v(\theta + 2\pi, Y) \\
u|_{Y=0} = v|_{Y=0} = 0, \quad \lim_{Y \to -\infty} \partial_Y u = 0 \tag{2.49}
\end{align*}
\]

where \( \bar{f}(\theta, Y) \) is 2\( \pi \)-periodic function and decays fast as \( Y \to -\infty \).

We can solve the equations (2.49) by considering the following approximate system. Let \( \gamma > 0 \) be a constant, we consider the following elliptic equation
\[
\begin{align*}
\bar{u} \partial_\theta u &+ \bar{v} \partial_Y u + \left[ \int_0^\gamma \partial_\gamma u \gamma(\theta, z) \, dz \right] \partial_Y \bar{u} + u \gamma \partial_\gamma \bar{u} - \partial_Y u \gamma - \gamma \partial_\theta u \gamma = \bar{f}, \\
u\gamma(\theta, Y) &= u\gamma(\theta + 2\pi, Y), \\
u\gamma|_{Y=0} &= 0. \tag{2.50}
\end{align*}
\]

We expect the solution of this equation is in \( \tilde{H}_0^1 = \{ u | \partial_\theta u \in L^2, \partial_Y u \in L^2, u|_{Y=0} = 0 \} \) rather than \( H_0^1 = \{ u | u \in L^2, \partial_\theta u \in L^2, \partial_Y u \in L^2, u|_{Y=0} = 0 \} \). Now we establish apriori estimate of equation (2.50). Multiplying the first equation in (2.50) by \( u \gamma \) and integrating in \( (\theta, r) \in (0, 2\pi) \times (-\infty, 0) \), we obtain that
\[
\int_{-\infty}^0 \int_0^{2\pi} \left[ - \partial_Y u \gamma - \gamma \partial_\theta u \gamma \right] u \gamma \, d\theta dY = \| \partial_\gamma u \gamma \|_2^2 + \| \partial_\theta u \gamma \|_2^2.
\]

Recall the estimates (2.43) and (2.45), we have
\[
| \partial_\theta^j \partial_Y^k (\bar{u} - u_e(1))(Y) | \leq C(j, k, l) \eta, \\
| \partial_\theta^j \partial_Y^k (\bar{v} - v_e(1)(\theta, 1))(Y) | \leq C(j, k, l) \eta, \quad | \partial_\theta^j v_e^{(2)}(\theta, 1) | \leq C(j) \eta,
\]

thus we can deduce that
\[
- \int_{-\infty}^0 \int_0^{2\pi} \left[ \bar{u} \partial_\theta u \gamma + \bar{v} \partial_Y u \gamma + \left( \int_0^\gamma \partial_\gamma u \gamma(\theta, z) \, dz \right) \partial_Y \bar{u} + u \gamma \partial_\gamma \bar{u} \right] u \gamma \, d\theta dY + \int_{-\infty}^0 \int_0^{2\pi} \bar{f} u \gamma \, d\theta dY
\]
\[
\leq \int_{-\infty}^0 \int_0^{2\pi} \frac{1}{2} \left[ \partial_\theta \bar{u} + \partial_Y \bar{v} \right] (u \gamma)^2 \, d\theta dr + \| Y^2 \bar{u} Y \|_\infty \left\| \frac{1}{Y} \int_0^\gamma \partial_\gamma u \gamma \, dz \right\|_2 \left\| \frac{1}{Y} \bar{u} \gamma \right\|_2
\]
\[
+ \| Y^2 \bar{u} \theta \|_\infty \left\| \frac{1}{Y} \bar{u} \gamma \right\|_2^2 + \| Y \bar{f} \|_2 \left\| \frac{1}{Y} \bar{u} \gamma \right\|_2
\]
\[
\leq C \eta \left( \| \partial_\theta u \gamma \|_2^2 + \| \partial_Y u \gamma \|_2^2 \right) + C \| Y \bar{f} \|_2 \| \partial_Y u \gamma \|_2,
\]

where we use \( \bar{u}_\theta + \bar{v}_Y = 0 \) and the Hardy inequality
\[
\left\| \int_0^\gamma \partial_\gamma u \gamma \, dz \right\|_2 \leq C \| \partial_\theta u \gamma \|_2, \quad \left\| \frac{1}{Y} \bar{u} \gamma \right\|_2 \leq C \| \partial_Y u \gamma \|_2.
\]
By collecting the above estimates, we obtain
\[ \|\partial_Y u^\gamma\|^2 + \gamma \|\partial_\theta u^\gamma\|^2 \leq C\eta [\|\partial_\theta u^\gamma\|^2 + \|\partial_Y u^\gamma\|^2] + C\|Y\bar{f}\|_2 \|\partial_Y u^\gamma\|_2, \]
where \( C \) is independent on \( \eta \) and \( \gamma \). If \( \eta \) is small enough, we deduce that
\[ \|\partial_Y u^\gamma\|^2 + \gamma \|\partial_\theta u^\gamma\|^2 \leq C\eta [\|\partial_\theta u^\gamma\|^2 + C\|Y\bar{f}\|_2^2]. \quad (2.51) \]

Next we multiply the first equation in (2.50) by \( \partial_\theta u^\gamma \) and integrate in \((\theta, r) \in (0, 2\pi) \times (-\infty, 0)\), we arrive at
\[
\int_{-\infty}^{0} \int_{0}^{2\pi} \left[ \bar{u} \partial_\theta u^\gamma + \bar{v} \partial_Y u^\gamma + \left( \int_{Y} \partial_\theta u^\gamma(\theta, z)dz \right) \partial_Y \bar{u} + u^\gamma \partial_\theta \bar{u} \right] \partial_\theta u^\gamma d\theta dY = 0.
\]

Moreover, there holds
\[
\int_{-\infty}^{0} \int_{0}^{2\pi} \left[ \bar{u} \partial_\theta u^\gamma + \bar{v} \partial_Y u^\gamma + \left( \int_{Y} \partial_\theta u^\gamma(\theta, z)dz \right) \partial_Y \bar{u} + u^\gamma \partial_\theta \bar{u} \right] \partial_\theta u^\gamma d\theta dY = 0.
\]

Moreover, there holds
\[
\int_{-\infty}^{0} \int_{0}^{2\pi} \left[ \bar{u} \partial_\theta u^\gamma + \bar{v} \partial_Y u^\gamma + \left( \int_{Y} \partial_\theta u^\gamma(\theta, z)dz \right) \partial_Y \bar{u} + u^\gamma \partial_\theta \bar{u} \right] \partial_\theta u^\gamma d\theta dY \leq \|\bar{v}\|_\infty \|\partial_Y u^\gamma\|_2 \|\partial_\theta u^\gamma\|_2 + \|u^\gamma Y\infty\| \left( \|\partial_\theta u^\gamma\|_2 \right)^2.
\]

It’s direct to obtain
\[
\int_{-\infty}^{0} \int_{0}^{2\pi} \left[ \bar{u} \partial_\theta u^\gamma + \bar{v} \partial_Y u^\gamma + \left( \int_{Y} \partial_\theta u^\gamma(\theta, z)dz \right) \partial_Y \bar{u} + u^\gamma \partial_\theta \bar{u} \right] \partial_\theta u^\gamma d\theta dY \leq \|\bar{v}\|_\infty \|\partial_Y u^\gamma\|_2 \|\partial_\theta u^\gamma\|_2 + \|u^\gamma Y\infty\| \left( \|\partial_\theta u^\gamma\|_2 \right)^2.
\]

Thus, we obtain
\[
\alpha \|\partial_\theta u^\gamma\|_2^2 \leq C\eta [\|\partial_Y u^\gamma\|^2 + \|\partial_\theta u^\gamma\|^2] + C\|\bar{f}\|_2 \|\partial_\theta u^\gamma\|_2.
\]

We then choose \( \eta_0 \) small enough such that for any \( \eta \in (0, \eta_0) \), there holds
\[ \alpha \|\partial_\theta u^\gamma\|_2^2 \leq C\eta [\|\partial_Y u^\gamma\|^2 + \|\partial_\theta u^\gamma\|^2] + C\|\bar{f}\|_2^2. \quad (2.52) \]

It follows from (2.51) and (2.52), we have
\[ \alpha \|\partial_\theta u^\gamma\|_2^2 + \|\partial_Y u^\gamma\|^2 + \gamma \|\partial_\theta u^\gamma\|^2 \leq C\|Y\bar{f}\|_2^2 + C\|\bar{f}\|_2. \quad (2.53) \]

According to the first equation in (2.50), we deduce
\[
\|\partial_Y u^\gamma\|^2 + \gamma \|\partial_\theta u^\gamma\|^2 \leq \|\partial_Y u^\gamma + \gamma \partial_\theta u^\gamma\|^2 \leq \left( \|\partial_Y u^\gamma\|^2 + \|\partial_\theta u^\gamma\|^2 \right)^2 \leq C\|Y\bar{f}\|_2^2 + C\|\bar{f}\|_2^2. \quad (2.54)
\]

Collecting the estimates (2.53) and (2.54), we obtain
\[ \alpha \|\partial_\theta u^\gamma\|_2^2 + \|\partial_Y u^\gamma\|^2 + \|\partial_Y u^\gamma\|^2 + \gamma \|\partial_\theta u^\gamma\|^2 + \gamma \|\partial_\theta u^\gamma\|^2 \leq C\|Y\|_2^2 + C\|\bar{f}\|_2^2. \]
which shows the existence and uniqueness of solution for system (2.50) for any $\gamma > 0$ in $\dot{H}_0^1$. Moreover the solution is smooth if $f$ is smooth. Set

$$u := \lim_{\gamma \to 0} u^\gamma, \quad v := \int_0^y \partial_\theta u(\theta, \gamma) d\gamma,$$

then

$$\alpha \| \partial_\theta u \|^2_2 + \| \partial_Y u \|^2_2 + \| \partial_Y^2 u \|^2_2 \leq C \langle Y \rangle \| \tilde{f} \|^2_2$$

(2.55)

and $(u, v)$ solves the system (2.49) except the boundary condition $\lim_{Y \to -\infty} \partial_Y u = 0$.

Finally, we show that the derivatives of $u, v$ decay fast as $Y \to -\infty$. Let $\psi = \int_0^y \tilde{u}(\theta, z) dz$ and

$$\tilde{u}(\theta, \psi) = u(\theta, Y(\theta, \psi)), \quad \tilde{v}(\theta, \psi) = v(\theta, Y(\theta, \psi)), \quad F(\theta, \psi) = \tilde{f}(\theta, Y(\theta, \psi)),$$

then there holds

$$\begin{cases}
\partial_\theta \tilde{u} - \partial_\psi (a(\theta, \psi) \partial_\psi \tilde{u}) + b(\theta, \psi) \tilde{u} + c(\theta, \psi) \tilde{v} = F(\theta, \psi), \\
\tilde{u}(\theta + 2\pi, \psi) = \tilde{u}(\theta, \psi), \\
\tilde{u}(\theta, 0) = 0, \quad \lim_{\psi \to -\infty} \partial_\psi \tilde{u}(\theta, \psi) = 0,
\end{cases}$$

(2.56)

where

$$a(\theta, \psi) = \tilde{u}(\theta, Y(\theta, \psi)), \quad b(\theta, \psi) = \frac{\partial_\theta \tilde{u}(\theta, Y(\theta, \psi))}{\tilde{u}(\theta, Y(\theta, \psi))}, \quad c(\theta, \psi) = \frac{\partial_Y \tilde{u}(\theta, Y(\theta, \psi))}{\tilde{u}(\theta, Y(\theta, \psi))}.$$  

Notice that there exist $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$, there holds

$$\frac{\alpha}{2} \leq \tilde{u}(\theta, Y) \leq \alpha, \quad \forall (\theta, Y) \in [0, 2\pi] \times (-\infty, 0].$$

Thus, we deduce that $\frac{\alpha}{2} \leq \frac{|\psi|}{|Y|} \leq \alpha$.

$$\| \partial_\theta \tilde{u} \|_2 + \| \partial_\psi \tilde{u} \|_2 \leq C(l) \| F(\psi) \|^2_2,$$

(2.57)

where

$$\tilde{u}_\neq = \tilde{u} - u_0(\psi), \quad u_0(\psi) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}(\theta, \psi) d\theta.$$  

From (2.55), we deduce that

$$\| \partial_\theta \tilde{u} \|_2 + \| \partial_\psi \tilde{u} \|_2 \leq C \| F(\psi) \|_2,$$

thus (2.57) holds for $l = 0$.

For any $l \geq 1$, multiplying $\tilde{u}_\neq \psi^{2l}$ in (2.56) and integrating in $[0, 2\pi] \times (-\infty, 0]$, we obtain

$$\int_0^{2\pi} \int_{-\infty}^0 \partial_\theta \tilde{u}_\neq \psi^{2l} d\psi d\theta = \int_0^{2\pi} \int_{-\infty}^0 \partial_\psi (a(\theta, \psi) \partial_\psi \tilde{u}_\neq \psi^{2l}) d\psi d\theta$$

$$= \int_0^{2\pi} \int_{-\infty}^0 [F(\theta, \psi) - b(\theta, \psi) \tilde{u} - c(\theta, \psi) \tilde{v}] \tilde{u}_\neq \psi^{2l} d\psi d\theta.$$  

Obviously, $I_1 = 0$. Due to the fast decay of $b(\theta, \psi), c(\theta, \psi)$ as $\psi \to -\infty$, we deduce that

$$|I_3| \leq C \| F(\psi) \|^2_2 \| \partial_\theta \tilde{u}_\neq \psi^{l-1} \|_2 + C \| \partial_\theta \tilde{u} \|_2^2 \leq C \| F(\psi) \|^2_2 + C \| \partial_\theta \tilde{u}_\neq \psi^{l-1} \|_2.$$
Moreover, there holds
\[ I_2 = \int_0^{2\pi} \int_{-\infty}^0 a(\theta, \psi) \partial_\psi \bar{u} \partial_\psi \tilde{u} \bar{u} \psi^2 d\psi d\theta + 2l \int_0^{2\pi} \int_{-\infty}^0 a(\theta, \psi) \partial_\psi \bar{u} \partial_\psi \tilde{u} \psi^{2l-1} d\psi d\theta. \]

Notice that \(a(\theta, \psi) = u_e(1) + u_p^{(0)}(\theta, Y(\theta, \psi))\), we deduce that
\[ I_{21} = u_e(1) \int_0^{2\pi} \int_{-\infty}^0 \partial_\psi \bar{u} \partial_\psi \tilde{u} \bar{u} \psi^2 d\psi d\theta + \int_0^{2\pi} \int_{-\infty}^0 u_p^{(0)}(\theta, Y(\theta, \psi)) \partial_\psi \bar{u} \partial_\psi \tilde{u} \bar{u} \psi^2 d\psi d\theta \]
\[ = u_e(1) \int_0^{2\pi} \int_{-\infty}^0 \partial_\psi \bar{u} \partial_\psi \tilde{u} \bar{u} \psi^2 d\psi d\theta + \int_0^{2\pi} \int_{-\infty}^0 u_p^{(0)}(\theta, Y(\theta, \psi)) \partial_\psi \bar{u} \partial_\psi \tilde{u} \bar{u} \psi^2 d\psi d\theta \]
\[ \geq \frac{\alpha}{2} \| \partial_\psi \bar{u} \psi \|_2^2 - C \| \partial_\psi \bar{u} \|_2^2. \]

Moreover, by the Hölder inequality and the Poincaré inequality, there holds
\[ |I_{22}| \leq C(l) \| \partial_\psi \bar{u} \psi \|_2 \| \partial_\theta \bar{u} \psi \|_2. \]

Thus, we obtain
\[ \| \partial_\psi \bar{u} \psi \|_2 \leq C(l) \| \partial_\theta \bar{u} \psi \|_2 + C \| F(\psi) \|_2. \]  

(2.58)

Noticing
\[ \partial_\theta \bar{u} - a(\theta, \psi) \partial_\psi \bar{u} = F(\theta, \psi) + \partial_\psi a(\theta, \psi) \partial_\psi \bar{u} - b(\theta, \psi) \bar{u} - c(\theta, \psi) \tilde{v}, \]
we deduce that
\[ \| [\partial_\theta \bar{u} - a(\theta, \psi) \partial_\psi \bar{u}] \psi^j \|_2^2 \leq \| [F(\theta, \psi) + \partial_\psi a(\theta, \psi) \partial_\psi \bar{u} - b(\theta, \psi) \bar{u} - c(\theta, \psi) \tilde{v}] \psi^j \|_2^2. \]

The right side can be controlled by
\[ C \| F(\psi) \|_2^2 + C \| \| \partial_\theta \bar{u} \|_2^2 + \| \partial_\psi \bar{u} \|_2^2 \|_2. \]

Moreover, there holds
\[ \| [\partial_\theta \bar{u} - a(\theta, \psi) \partial_\psi \bar{u}] \psi^j \|_2^2 \]
\[ = \| \partial_\theta \bar{u} \psi^j \|_2^2 + \| a(\theta, \psi) \partial_\psi \bar{u} \psi^j \|_2^2 - 2 \int_0^{2\pi} \int_{-\infty}^0 a(\theta, \psi) \partial_\theta \bar{u} \partial_\psi \bar{u} \psi^{2l} d\psi d\theta \]
\[ \geq \| \partial_\theta \bar{u} \psi^j \|_2^2 + \frac{\alpha}{2} \| \partial_\psi \bar{u} \psi^j \|_2^2 - 2 \int_0^{2\pi} \int_{-\infty}^0 a(\theta, \psi) \partial_\theta \bar{u} \partial_\psi \bar{u} \psi^{2l} d\psi d\theta. \]

Integrating by parts, we deduce that
\[ \mathcal{I} = 2 \int_0^{2\pi} \int_{-\infty}^0 \partial_\psi \partial_\psi a(\theta, \psi) \partial_\theta \bar{u} \partial_\psi \bar{u} \psi^{2l} d\psi d\theta + 4l \int_0^{2\pi} \int_{-\infty}^0 a(\theta, \psi) \partial_\theta \bar{u} \partial_\psi \bar{u} \psi^{2l-1} d\psi d\theta \]
\[ + 2 \int_0^{2\pi} \int_{-\infty}^0 a(\theta, \psi) \partial_\theta \bar{u} \partial_\psi \bar{u} \psi^{2l} d\psi d\theta. \]

Obviously, there holds
\[ |I_1| + |I_3| \leq C(\| \partial_\theta \bar{u} \|_2^2 + \| \partial_\psi \bar{u} \|_2^2). \]
Moreover,
\[
|I_2| = \left| 4 u_\epsilon(1) \int_0^{2\pi} \int_{-\infty}^{2\pi} \partial_\theta \tilde{u}^{2l-1} \partial_\psi \tilde{u} \psi d\theta d\psi + 4 l \int_0^{2\pi} \int_{-\infty}^{2\pi} u_p^{(0)}(\theta, Y(\theta, \psi)) \partial_\theta \tilde{u}^{2l-1} \partial_\psi \tilde{u} \psi d\theta d\psi \right|
\]
\[
= \left| 4 u_\epsilon(1) \int_0^{2\pi} \int_{-\infty}^{2\pi} \partial_\theta \tilde{u}^{2l-1} \partial_\psi \tilde{u} \psi d\theta d\psi + 4 l \int_0^{2\pi} \int_{-\infty}^{2\pi} u_p^{(0)}(\theta, Y(\theta, \psi)) \partial_\theta \tilde{u}^{2l-1} \partial_\psi \tilde{u} \psi d\theta d\psi \right|
\]
\[
\leq C(l) \|\partial_\theta \tilde{u}^{l-1}\|_2 \|\partial_\psi \tilde{u}^{l-1}\|_2 + C(l)(\|\partial_\theta \tilde{u}\|^2 + \|\partial_\psi \tilde{u}\|^2).
\]
Thus, we obtain
\[
\|\partial_\theta \tilde{u}^{l-1}\|_2 + \|\partial_\psi \tilde{u}^{l-1}\|_2 \leq C(l) \|\partial_\psi \tilde{u}^{l-1}\|_2 + C(l)(F(\psi)^{l+1})_2.
\]
Combining the estimates (2.58) and (2.59), we obtain that for any \(l \geq 1\), there holds
\[
\|\partial_\theta \tilde{u}^{l-1}\|_2 + \|\partial_\psi \tilde{u}^{l-1}\|_2 \leq C(l)(\|\partial_\psi \tilde{u}^{l-1}\|_2 + \|\partial_\theta \tilde{u}^{l-1}\|_2) + C(l)(F(\psi)^{l})_2.
\]
Thus, by induction, we obtain (2.57).
Furthermore, by the Hardy inequality, we have
\[
\|\partial_\psi \tilde{u}^{l-1}\|_2 \leq C \|\partial_\psi \tilde{u}^{l-1}\|_2,
\]
hence there holds
\[
\|\partial_\theta \tilde{u}^{l-1}\|_2 + \|\partial_\psi \tilde{u}^{l-1}\|_2 \leq C \|\partial_\psi \tilde{u}^{l-1}\|_2.
\]
Returning to the original variable, we obtain that for any \(l \geq 1\), there holds
\[
\|\langle Y \rangle^l \partial_\theta Y \tilde{u}\|_2 + \|\langle Y \rangle^{l-1} \partial_\theta u\|_2 \leq C(l)(\|\langle Y \rangle^{l+1} \tilde{f}\|_2^2.
\]
Furthermore, by induction, we obtain that for any \(m \in \mathbb{N}_+, l \in \mathbb{N}\), there holds
\[
\sum_{j+k \leq m} \left( \|\langle Y \rangle^l \partial_\theta^j \partial_\psi^k \partial_\theta \tilde{u}\|_2^2 + \|\langle Y \rangle^{l-1} \partial_\theta^j \partial_\psi^k \tilde{f}\|_2^2 \right)
\]
\[
\leq C(m, l) \sum_{j+k \leq m} \left( \|\langle Y \rangle^{l+1} \partial_\theta^j \partial_\psi^k \tilde{f}\|_2^2 \leq C(m, l)\eta^2.
\]
Noticing that \(\lim_{Y \to -\infty} \langle u_\theta, u_Y \rangle = 0\) and \(A_{1\infty} := \lim_{Y \to -\infty} u(\theta, Y)\) is a constant independent on \(\theta\),
then by the Hardy inequality we have for any \(l \geq 2\)
\[
\|\langle Y \rangle^{l-2} (u - A_{1\infty})\|_2 \leq C(l)(\|Y^{l-1} \partial_\theta u\|_2 \leq C(l)\eta^2.
\]
This completes the proof of this proposition.

Next, we construct the pressure \(p_p^{(2)}(\theta, Y)\). Consider the equation
\[
\partial_Y p_p^{(2)}(\theta, Y) = g_1(\theta, Y), \quad \lim_{Y \to -\infty} p_p^{(2)}(\theta, Y) = 0,
\]
where
\[
g_1(\theta, Y) = -Y \partial_Y p_p^{(1)} + \partial_\psi v_p^{(1)} - u_\epsilon(1) \partial_\theta v_p^{(1)} - u_p^{(0)}(\partial_\theta v_p^{(1)}(\theta, 1)) + \partial_\psi v_p^{(1)}
\]
\[
- \partial_Y v_p^{(1)}(v_p^{(1)}(\theta, 1) + v_p^{(1)} - 2Y u_\epsilon(1)u_p^{(0)} + u_\epsilon(1)u_p^{(1)} + [u_\epsilon^{(1)}(\theta, 1) + A_1]u_p^{(0)} + u_p^{(0)}u_p^{(1)}),
\]
here and below,
\[
u_p^{(1)} = u_p^{(1)} - A_{1\infty}.
\]
g_1(\theta, Y) can be obtained by replacing \(u_p^{(1)}\) by \(\tilde{u}_p^{(1)}\) in the expansion (2.3) and putting the new expansion into the second equation of (1.3), then collecting the \(\varepsilon^1\)-order terms together. Notice that \(g_1(\theta, Y)\) decay fast as \(Y \to -\infty\), we can get \(p_p^{(2)}(\theta, Y)\) by solving (2.5) and deduce that \(p_p^{(2)}(\theta, Y)\) decay fast as \(Y \to -\infty\).

By the same argument as above, we can obtain the well-posedness of equations (2.12).
Proposition 2.7. There exists $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$, the equations (2.62) have a unique solution $(\mathring{u}_p^{(1)}, \mathring{v}_p^{(1)})$ which satisfies

$$
\sum_{j+k \leq m} \int_{0}^{2\pi} \left| \frac{\partial^j \theta^k}{\partial Z^{m-l}} (\hat{u}_p^{(1)} - \hat{A}_{1\infty}, \hat{v}_p^{(2)}) \right|^2 \langle Z \rangle^{2l} d\theta dZ \leq C(m,l)\eta^2, \ \forall m,l \geq 0,
$$

(2.61)

where $\hat{A}_{1\infty} := \lim_{Z \to +\infty} \hat{u}_p^{(1)}(\theta, Z)$ is a constant which satisfies $|\hat{A}_{1\infty}| \leq C\eta$.

Similarly, we can also construct $\hat{p}_p^{(2)}(\theta, Z)$ which decays fast as $Z \to +\infty$.

2.4.4. Linearized Euler system for $(u_e^{(2)}, v_e^{(2)}, p_e^{(2)})$ and its solvability.

Let $r_1 = \frac{1 + 2n}{3}, r_2 = \frac{3 + 2n}{3}$ and $\chi(r) \in C^\infty([r_0, 1])$ be an increasing smooth function such that

$$
\chi(r) = \begin{cases} 
0, & r \in [r_0, r_1], \\
1, & r \in [r_2, 1]
\end{cases}
$$

and

$$
\bar{u}_e^{(1)}(\theta, r) := u_e^{(1)}(\theta, r) + \chi(r) \hat{A}_{1\infty} + (1 - \chi(r))\hat{A}_{1\infty}, \ \bar{v}_e^{(1)}(\theta, r) = v_e^{(1)}(\theta, r),
$$

$$
\bar{p}_e^{(1)}(\theta, r) := p_e^{(1)}(\theta, r) + \int_{r_0}^{r} \frac{2u_e(s)}{s} [\chi(s)A_{1\infty} + (1 - \chi(s))\hat{A}_{1\infty}] ds,
$$

then $(\bar{u}_e^{(1)}, \bar{v}_e^{(1)}, \bar{p}_e^{(1)})$ also satisfies the linearized Euler equations (2.3) with the boundary condition (2.4).

Putting

$$
u^\varepsilon(\theta, r) = u_e(r) + \varepsilon \bar{u}_e^{(1)}(\theta, r) + \varepsilon^2 u_e^{(2)}(\theta, r) + h.o.t.,$$

$$v^\varepsilon(\theta, r) = \varepsilon \bar{v}_e^{(1)}(\theta, r) + \varepsilon^2 v_e^{(2)}(\theta, r) + h.o.t.,$$

$$p^\varepsilon(\theta, r) = p_e(r) + \varepsilon \bar{p}_e^{(1)}(\theta, r) + \varepsilon^2 p_e^{(2)}(\theta, r) + h.o.t.$$

into the Navier-Stokes equations (1.3), we obtain the following linearized Euler equations for $(u_e^{(2)}, v_e^{(2)}, p_e^{(2)})$

$$
\begin{cases}
\begin{array}{l}
\nu_e(r) \partial_r u_e^{(2)} + r v_e^{(2)}(r) u_e^{(2)}(r) + \nu_e(r) v_e^{(2)}(r) + \nu_e^{(1)} \partial_r \bar{u}_e^{(1)} \\
+ \bar{v}_e^{(1)} \partial_r \bar{u}_e^{(1)} + \bar{v}_e^{(2)} = r u''_e(r) + u'_e(r) - \frac{u_e(r)}{r} + r F_u,
\end{array}
\end{cases}
\begin{cases}
\nu_e(r) \partial_r v_e^{(2)} - 2 \nu_e(r) u_e^{(2)} + r \partial_r p_e^{(2)} + \nu_e^{(1)} \partial_r \bar{v}_e^{(1)} + \bar{v}_e^{(1)} r \partial_r \bar{v}_e^{(1)} - (\bar{v}_e^{(1)})^2 = r F_v,
\end{cases}
\begin{cases}
\partial_r u_e^{(2)} + \partial_r (r v_e^{(2)}) = 0,
\end{cases}
\tag{2.62}
$$

with the boundary conditions

$$
v_e^{(2)}(r=1) = -v_p^{(2)}|_{y=0}, \ \nu_e^{(2)}(r=r_0) = -v_p^{(2)}|_{Z=0}, \ \nu_e^{(2)}(\theta, r) = v_e^{(2)}(\theta + 2\pi, r).
$$

(2.63)

Proposition 2.8. The linearized Euler system (2.62) has a solution $(u_e^{(2)}, v_e^{(2)}, p_e^{(2)})$ which satisfies

$$
\|{\partial^j \theta^k}(u_e^{(2)}, v_e^{(2)})\|_2 \leq C(j,k), \ \forall j,k \geq 0.
$$

(2.64)

Proof. First, we show that there holds

$$
\int_0^{2\pi} \left[ \bar{u}_e^{(1)} \partial_\theta \bar{u}_e^{(1)} + \bar{v}_e^{(1)} \partial_r \bar{v}_e^{(1)} + \bar{v}_e^{(1)} \bar{v}_e^{(1)} \right] d\theta = 0, \ \forall r \in [r_0, 1].
$$

(2.65)
We set
\[ \tilde{v}_e^{(1)}(\theta, r) = \sum_{n \neq 0} V_n(r)e^{in\theta}, \quad \tilde{u}_e^{(1)}(\theta, r) = a(r) - \sum_{n \neq 0} \frac{\langle r V_n \rangle'(r)}{in} e^{in\theta}, \]
here we have used (2.47) and the third equation of (2.3).

Then, it’s easy to obtain that
\[ \tilde{v}_e^{(1)} r \partial_r \tilde{u}_e^{(1)}(\theta, r) + \tilde{u}_e^{(1)} \tilde{v}_e^{(1)}(\theta, r) \]
\[ = (ra'(r) + a(r))\tilde{v}_e^{(1)}(\theta, r) - \tilde{v}_e^{(1)}(\theta, r) \sum_{n \neq 0} \frac{e^{in\theta}}{in} [r \langle r V_n \rangle' + (r V_n)'] \]  
(2.66)
Moreover, due to \(-\Delta (r \tilde{v}_e^{(1)}) + U_e(r)(r \tilde{v}_e^{(1)}) = 0\), where \(U_e(r) = \frac{1}{u_e(r)} \left( u_e''(r) + \frac{u_e'(r)}{r} - \frac{u_e(r)}{r^2} \right)\), we deduce that
\[ \sum_{n \neq 0} e^{in\theta} [r^2 \langle r V_n \rangle'' + r \langle r V_n \rangle' - (n^2 + r^2 U_e(r)) r V_n] = 0, \]
hence there holds \(r \langle r V_n \rangle'' + (r V_n)' = (n^2 + r^2 U_e(r)) V_n\).

Thus, by (2.66) we deduce that
\[ \tilde{v}_e^{(1)} r \partial_r \tilde{u}_e^{(1)}(\theta, r) + \tilde{u}_e^{(1)} \tilde{v}_e^{(1)}(\theta, r) \]
\[ = (ra'(r) + a(r))\tilde{v}_e^{(1)}(\theta, r) + \tilde{v}_e^{(1)}(\theta, r) \partial_\theta \tilde{v}_e^{(1)}(\theta, r) - \frac{r^2 U_e(r)}{2} \partial_\theta \left( \sum_{n \neq 0} \frac{e^{in\theta}}{in} V_n \right)^2, \]
(2.67)
hence (2.65) can be obtained.

Then, we can follow the line of the construction of \((u_e^{(1)}, v_e^{(1)}, p_e^{(1)})\) to construct \((u_e^{(2)}, v_e^{(2)}, p_e^{(2)})\): eliminating pressure \(p_e^{(2)}\) in the equation (2.62) and obtaining the elliptic equation for \(rv_e^{(2)}\):
\[ - u_e(r) \Delta (rv_e^{(2)}) + \left( u_e''(r) + \frac{u_e'(r)}{r} - \frac{u_e(r)}{r^2} \right) (rv_e^{(2)}) \]
\[ = - r \partial_r \left( \tilde{u}_e^{(1)} \partial_\theta \tilde{u}_e^{(1)} + \tilde{v}_e^{(1)} r \partial_r \tilde{u}_e^{(1)} + \tilde{v}_e^{(1)} \tilde{v}_e^{(1)} \right) + \partial_\theta \left( \tilde{u}_e^{(1)} \partial_\theta \tilde{v}_e^{(1)} + \tilde{v}_e^{(1)} r \partial_r \tilde{v}_e^{(1)} - (\tilde{v}_e^{(1)})^2 \right) \]
\[ + r \partial_r \left( ru_e''(r) + u_e'(r) - \frac{u_e(r)}{r} + F_u \right) - \partial_\theta (r F_v). \]
This elliptic equation with boundary condition (2.64) is solvable by noticing (1.7) and (2.65). Then \(u_e^{(2)}\) can be obtained from the divergence-free condition; finally, construct pressure \(p_e^{(2)}\) from the first equation in (2.62). Thus, \((u_e^{(2)}, v_e^{(2)}, p_e^{(2)})\) satisfies the equation (2.62) and (2.64).

Remark 2.1. In this remark, we show that the solvability of (2.62) also determine the form of \(u_e(r)\). In fact, integrating the first equation of (2.62) with respect to \(\theta\) in \([0, 2\pi]\), notice \(\int_0^{2\pi} v_e^{(2)}(\theta, r) d\theta = 0\) and (2.65), we deduce that
\[ u_e''(r) + \frac{u_e'(r)}{r} - \frac{u_e(r)}{r^2} + F_u = 0, \]
this is the equation in (1.7).
Let $A_1(r)$ be a smooth function such that
\[
\begin{align*}
A_1(r) &= A_1(1) = 0, \\
\frac{1}{2\pi} \int_0^{2\pi} \left( v_{e}^{(1)} \partial_r (ru_{e}^{(2)}) + v_{e}^{(2)} \partial_r (ru_{e}^{(1)}) - \left(r \partial_{rr} u_{e}^{(1)} + \partial_r u_{e}^{(1)} - \frac{u_{e}^{(1)}}{r}\right) \right) d\theta,
\end{align*}
\]
then $\|\partial^k_{\theta} A_1(r)\|_\infty \leq C(k)\eta$.

Set
\[
\begin{align*}
\tilde{\bar{u}}_{e}^{(1)} &= \bar{u}_{e}^{(1)} + A_1(r), \quad \tilde{\bar{\nu}}_{e}^{(1)} = \tilde{\nu}_{e}^{(1)}, \quad \tilde{\bar{p}}_{e}^{(1)} = \tilde{p}_{e}^{(1)} + \int_{r_0}^{r} \frac{2u_{e}(s)}{s} A_1(s) ds,
\end{align*}
\]
then $(\tilde{\bar{u}}_{e}^{(1)}, \tilde{\bar{\nu}}_{e}^{(1)}, \tilde{\bar{p}}_{e}^{(1)})$ is also a solution of (2.3) with the boundary condition (2.4), and there holds
\[
\|\partial^j_{\theta} \partial^k_{\tau} (\tilde{\bar{u}}_{e}^{(1)}, \tilde{\bar{\nu}}_{e}^{(1)})\|_\infty \leq C(j, k)\eta, \quad \forall k, j \geq 0.
\]

Thus, by the same argument as Proposition 2.8, we deduce that the following linearized Euler system
\[
\begin{align*}
\begin{cases}
u_{e}(r)\partial_{\theta}\bar{u}_{e}^{(2)} + r v_{e}^{(2)} \bar{u}_{e}^{(1)} + u_{e}(r) v_{e}^{(2)} + \partial_{\theta} p_{e}^{(2)} + \bar{u}_{e}^{(1)} \partial_{\theta} \bar{u}_{e}^{(1)} \\
+ \bar{v}_{e}^{(1)} r \partial_{\theta} \bar{v}_{e}^{(1)} + \bar{u}_{e}^{(1)} \bar{v}_{e}^{(1)} = r u_{e}^{(r)}(r) + \bar{u}_{e}(r) - \frac{u_{e}(r)}{r} + r F_{\theta}, \\
u_{e}(r)\partial_{\theta} v_{e}^{(2)} - 2 u_{e}(r) u_{e}^{(2)} + r \partial_{\tau} p_{e}^{(2)} + \bar{u}_{e}^{(1)} \partial_{\theta} \bar{v}_{e}^{(1)} + \bar{u}_{e}^{(1)} r \partial_{\theta} \bar{v}_{e}^{(1)} - (\tilde{\bar{u}}_{e}^{(1)})^2 = r F_{\tau}, \\
\partial_{\theta} u_{e}^{(2)} + r \partial_{\tau} v_{e}^{(2)} + v_{e}^{(2)} = 0
\end{cases}
\end{align*}
\]
with the boundary conditions
\[
\begin{align*}
v_{e}^{(2)}|_{r=1} = -v_{p}^{(2)}(0) Y=0, \quad v_{e}^{(2)}|_{r=r_0} = -\bar{v}_{e}^{(2)}(0) Z=0, \quad v_{e}^{(2)}(\theta, r) = v_{e}^{(2)}(\theta + 2\pi, r),
\end{align*}
\]
has a solution $(\tilde{\bar{u}}_{e}^{(2)}, \tilde{\bar{\nu}}_{e}^{(2)}, \tilde{\bar{p}}_{e}^{(2)})$ which satisfies
\[
\|\partial^j_{\theta} \partial^k_{\tau} (\tilde{\bar{u}}_{e}^{(2)}, \tilde{\bar{\nu}}_{e}^{(2)})\|_\infty \leq C(j, k).
\]

2.4.5. Linearized Prandtl equations for $(u_{p}^{(2)}, v_{p}^{(3)}), (\tilde{\bar{u}}_{p}^{(2)}, \tilde{\bar{\nu}}_{p}^{(3)})$ and their solvabilities.

Let
\[
\begin{align*}
u_{p}^{(2)}(\theta, r) &= u_{p}^{(2)}(\theta, Y) + \varepsilon [\tilde{\bar{u}}_{e}^{(1)}(\theta, r) + \tilde{\bar{v}}_{e}^{(1)}(\theta, Y)] + \varepsilon^2 [\tilde{\bar{u}}_{e}^{(2)}(\theta, r) + u_{e}^{(2)}(\theta, Y)] + \text{h.o.t.}, \\
u_{p}^{(3)}(\theta, r) &= \varepsilon [\tilde{\bar{v}}_{e}^{(1)}(\theta, r) + \tilde{\bar{p}}_{e}^{(1)}(\theta, Y)] + \varepsilon^2 [\tilde{\bar{v}}_{e}^{(2)}(\theta, r) + v_{e}^{(2)}(\theta, Y)] + \varepsilon^3 [\tilde{\bar{p}}_{e}^{(3)}(\theta, r) + p_{e}^{(3)}(\theta, Y)] + \text{h.o.t.}, \\
p_{p}^{(2)}(\theta, r) &= p_{e}(r) + \varepsilon [\tilde{\bar{p}}_{e}^{(1)}(\theta, r) + p_{e}^{(1)}(\theta, Y)] + \varepsilon^2 [\tilde{\bar{p}}_{e}^{(2)}(\theta, r) + p_{e}^{(2)}(\theta, Y)] + \varepsilon^3 p_{e}^{(3)}(\theta, r) + \text{h.o.t.},
\end{align*}
\]
with the boundary conditions
\[
\begin{align*}
\tilde{\bar{u}}_{p}^{(2)}(\theta, 1) + u_{e}^{(2)}(\theta, 0) = 0, \quad v_{p}^{(3)}(\theta, 1) + v_{e}^{(3)}(\theta, 0) = 0, \quad \lim_{Y \to -\infty} (\partial_{\tau} u_{p}^{(2)}, v_{p}^{(3)})(\theta, Y) = (0, 0),
\end{align*}
\]
we obtain the following linearized steady Prandtl equations for $(u_{p}^{(2)}, v_{p}^{(3)})$
\[
\begin{align*}
\begin{cases}
\begin{aligned}
(\bar{u}_{e}(1) + u_{p}^{(0)}(\theta, Y) + v_{p}^{(1)}(\theta, Y) + \varepsilon [\tilde{\bar{u}}_{e}^{(1)}(\theta, 1) + \tilde{\bar{v}}_{e}^{(1)}(\theta, Y)] + \varepsilon^2 [\tilde{\bar{u}}_{e}^{(2)}(\theta, 1) + u_{e}^{(2)}(\theta, Y)] + \text{h.o.t.}
\end{aligned}
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
\begin{aligned}
\partial_{\theta} u_{p}^{(2)} + \partial_{\tau} v_{p}^{(3)} + \partial_{\tau} (Y v_{p}^{(2)}) = 0, \\
u_{p}^{(2)}(\theta, Y) = u_{p}^{(2)}(\theta + 2\pi, Y), \quad v_{p}^{(3)}(\theta, Y) = v_{p}^{(3)}(\theta + 2\pi, Y), \\
u_{p}^{(2)}|_{Y=0} = -\bar{u}_{e}^{(2)}|_{r=1}, \quad \lim_{Y \to -\infty} \partial_{\tau} u_{p}^{(2)}(\theta, Y) = \lim_{Y \to -\infty} v_{p}^{(3)}(\theta, Y) = 0.
\end{aligned}
\end{cases}
\end{align*}
\]
here
\[
f_2(\theta, Y) = -\partial_\theta p_\theta^2 + Y \partial_Y u_\theta^{(1)} + \partial_Y u_\theta^{(1)} + \partial_\theta u_\theta^{(0)} - u_\theta^{(0)} - \tilde{u}_\theta^{(1)} \partial_\theta u_\theta^{(1)} - v_\theta^{(2)} \partial_Y u_\theta^{(1)} - \sum_{i+j=2} v_\theta^{(i)} Y \partial_Y u_\theta^{(j)}
\]
\[
- \sum_{k=0}^2 \sum_{i+j=2-k, (k,j) 
\neq (0,2)} \frac{\partial^k \tilde{v}_\theta^{(i)}(\theta,1)}{k!} Y^k \partial_Y \tilde{v}_\theta^{(j)} + \tilde{v}_\theta^{(j)} \partial^k \partial_\theta \tilde{v}_\theta^{(i)}(\theta,1) Y^k + u_\theta^{(2)}(\theta,1) \partial_\theta u_\theta^{(0)}
\]
\[
- \sum_{k=0}^2 \sum_{i+j=2-k} \frac{\partial^k \tilde{v}_\theta^{(i)}(\theta,1)}{k!} Y^k \partial_Y \tilde{v}_\theta^{(j)} + (v_\theta^{(2)}(\theta,1) \partial_Y \tilde{v}_\theta^{(j)}) Y^k
\]
\[
- \sum_{k=0}^2 \sum_{i+j=3-k, (k,j) 
\neq (0,2),(0,0)} \frac{\partial^k \tilde{v}_\theta^{(i)}(\theta,1)}{k!} Y^k \partial_Y \tilde{v}_\theta^{(j)}
\]
where \( \tilde{u}_\theta^{(0)} = u_\theta^{(0)}, \tilde{u}_\theta^{(2)} = u_\theta^{(2)}, \tilde{v}_\theta^{(0)} = v_\theta^{(0)}, \tilde{v}_\theta^{(2)} = v_\theta^{(2)} \).

Similarly, let
\[
u^e(\theta, r) = u_e(r) + \tilde{u}_e^{(0)}(\theta, Z) + \varepsilon [\tilde{v}_e^{(1)}(\theta, r) + (\tilde{u}_e^{(1)}(\theta, Z) - \tilde{A}_1)] + \varepsilon^2 [\tilde{u}_e^{(2)}(\theta, r) + \tilde{v}_e^{(2)}(\theta, Z)] + \text{h.o.t.}, \]
\[
v^e(\theta, r) = \varepsilon [\tilde{v}_e^{(1)}(\theta, r) + \tilde{v}_e^{(2)}(\theta, Z)] + \varepsilon^2 [\tilde{v}_e^{(3)}(\theta, r) + \tilde{v}_e^{(3)}(\theta, Z)] + \varepsilon^3 [\tilde{v}_e^{(3)}(\theta, r) + \tilde{v}_e^{(3)}(\theta, Z)] + \text{h.o.t.}, \]
\[
p^e(\theta, r) = p_e(r) + \varepsilon [\tilde{p}_e^{(1)}(\theta, r) + \tilde{p}_e^{(2)}(\theta, Z)] + \varepsilon^2 [\tilde{p}_e^{(2)}(\theta, r) + \tilde{p}_e^{(2)}(\theta, Z)] + \varepsilon^3 [\tilde{p}_e^{(3)}(\theta, Z) + \text{h.o.t.}, \]
with the following boundary conditions
\[
\tilde{u}_e^{(2)}(\theta, r_0) + \tilde{v}_e^{(2)}(\theta, 0) = 0, \quad \tilde{v}_e^{(3)}(\theta, r_0) + \tilde{v}_e^{(3)}(\theta, 0) = 0, \quad \lim_{Z \to +\infty} (\partial_Z \tilde{v}_e^{(2)}(\theta, Z), \tilde{v}_e^{(3)}(\theta, Z)) = (0, 0), \]
we obtain the following linearized Prandtl equations for \( (\tilde{u}_e^{(2)}, \tilde{v}_e^{(3)}) \)
\[
\begin{cases}
(u_e(r_0) + \tilde{u}_e^{(0)}) \partial_\theta \tilde{u}_e^{(2)} + (v_e^{(1)}(\theta, r_0) + \tilde{v}_e^{(1)}(\theta, r_0)) r_0 \partial_Z \tilde{u}_e^{(2)} + (\tilde{u}_e^{(2)}(\theta, r_0)) \partial_\theta \tilde{u}_e^{(0)} + (\tilde{v}_e^{(3)} - \tilde{v}_e^{(3)}(\theta, 0)) r_0 \partial_Z \tilde{u}_e^{(0)} - \tilde{v}_e^{(3)}(\theta, r_0) \partial_Z \tilde{u}_e^{(2)} = \tilde{f}_2(\theta, Z), \\
\partial_\theta \tilde{v}_e^{(2)} + r_0 \partial_Z \tilde{v}_e^{(3)} + \partial_Z (Z \tilde{v}_e^{(3)}) = 0, \\
\tilde{u}_e^{(2)}(\theta, Z) = \tilde{u}_e^{(2)}(\theta + 2\pi, Z), \quad \tilde{v}_e^{(3)}(\theta, Z) = \tilde{v}_e^{(3)}(\theta + 2\pi, Z), \\
\tilde{u}_e^{(2)}|_{Z=0} = -u_e^{(2)}|_{r=r_0}, \quad \lim_{Z \to +\infty} (\partial_Z \tilde{u}_e^{(2)}, \tilde{v}_e^{(3)}(\theta, Z)) = (0, 0).
\end{cases}
\] (2.75)

The expressions of functions \( \tilde{f}_2(\theta, Z) \) are similar to \( f_2(\theta, Y) \), we omit the details here.

**Proposition 2.9.** There exists \( \eta_0 > 0 \) such that for any \( \eta \in (0, \eta_0) \), the equations (2.74) have a unique solution \((\tilde{u}_e^{(2)}, \tilde{v}_e^{(3)})\) and the equations (2.75) have a unique solution \((\tilde{u}_p^{(2)}, \tilde{v}_p^{(3)})\) which satisfies
\[
\sum_{j+k \leq m} \int_0^{2\pi} \int_0^{2\pi} \left| \frac{\partial_\theta \partial_Y^k (u_p^{(2)} - A_{2\infty}, v_p^{(3)})}{Y} \right|^2 Y^{2l} d\theta dY \leq C(m, l), \quad \forall m, l \geq 0,
\]
\[
\sum_{j+k \leq m} \int_0^{+\infty} \int_0^{2\pi} \left| \frac{\partial_\theta \partial_Z^k (u_p^{(2)} - \tilde{A}_{2\infty}, v_p^{(3)})}{Z} \right|^2 Z^{2l} d\theta dZ \leq C(m, l), \quad \forall m, l \geq 0,
\]
(2.76)
\[
\int_0^{2\pi} v_p^{(3)}(\theta, Y) d\theta = \int_0^{2\pi} \tilde{v}_p^{(3)}(\theta, Z) d\theta = 0, \quad \forall Y \leq 0, \quad Z \geq 0
\]
where
\[A_{2\infty} := \lim_{Y \to -\infty} u_p^{(2)}(\theta, Y), \quad \tilde{A}_{2\infty} := \lim_{Z \to +\infty} \tilde{u}_p^{(2)}(\theta, Z).\]
and satisfies $|A_{2\infty}| + |\hat{A}_{2\infty}| \leq C$.

The proof is same with Proposition 2.6 by noticing that $f_2(\theta, Y)$ decays very fast as $Y \to -\infty$, we omit the details.

We construct the pressure $p^{(3)}_p(\theta, Y)$ by considering the equation

$$
\partial_\theta p^{(3)}_p(\theta, Y) = g_2(\theta, Y), \quad \lim_{Y \to -\infty} p^{(3)}_p(\theta, Y) = 0, \tag{2.77}
$$

where

$$
g_2(\theta, Y) = \partial_\theta v_p^{(2)} + Y \partial_\theta v_p^{(1)} + \partial_\theta v_p^{(1)} - 2 \partial_\theta u_p^{(0)} - Y \partial_\theta p_p^{(2)} - \sum_{i+j=3} v_p^{(i)} \partial_\theta v_p^{(j)} - \sum_{i+j=2} \left( \tilde{u}_p^{(i)} \partial_\theta v_p^{(j)} + v_p^{(i)} Y \partial_\theta v_p^{(j)} - \tilde{u}_p^{(i)} \tilde{v}_p^{(j)} + \tilde{v}_p^{(i)} \tilde{v}_c^{(j)}(\theta, 1) \right)
$$

$$
- \frac{1}{k!} \sum_{i+j=2-k} \left( \frac{\partial^k \tilde{u}_c^{(i)}(\theta, 1)}{k!} Y^k \partial_\theta v_p^{(j)} + \frac{\partial^k \partial_\theta \tilde{v}_c^{(j)}(\theta, 1)}{k!} Y^k \tilde{u}_p^{(i)} \right)
$$

$$
- \frac{2}{k!} \sum_{i+j=2-k} \left( \frac{\partial^k \tilde{u}_c^{(i)}(\theta, 1)}{k!} Y^k \tilde{v}_p^{(j)} + \frac{\partial^k \partial_\theta \tilde{v}_c^{(j)}(\theta, 1)}{k!} Y^k \tilde{u}_p^{(i)} \right),
$$

where $u^{(0)}_p = u^{(0)}_p$, $u^{(1)}_c = u^{(1)}_c(r)$, $\tilde{u}^{(2)}_c = \tilde{u}^{(2)}_c + A_{2\infty}$, $\tilde{v}^{(2)}_c = \tilde{v}^{(2)}_c$, and here and below $\tilde{u}^{(2)}_p = u^{(2)}_p - A_{2\infty}$. $g_2(\theta, Y)$ can be derived by the same argument as $g_1(\theta, Y)$. Moreover, notice that $g_2(\theta, Y)$ decay fast as $Y \to -\infty$, we can obtain $p^{(3)}_p$ by solving (2.77) and $p^{(3)}_p$ also decay as $Y \to -\infty$.

Similarly, we can construct $p^{(3)}_p(\theta, Z)$ and $\tilde{v}^{(3)}_p(\theta, Z)$ which also decay fast as $Z \to +\infty$.

### 2.4.6. Linearized Euler system for $(u^{(3)}_c, v^{(3)}_c, p^{(3)}_c)$ and its solvability.

Let

$$
\tilde{u}^{(2)}_c(\theta, r) := \tilde{u}^{(2)}_c(\theta, r) + \chi(r)A_{2\infty} + (1 - \chi(r))\tilde{A}_{2\infty}, \quad \tilde{v}^{(2)}_c(\theta, r) = \tilde{v}^{(2)}_c(\theta, r),
$$

$$
\tilde{p}^{(2)}_c(\theta, r) := \tilde{p}^{(2)}_c(\theta, r) + \int_{r_0}^r \frac{2u_c(s)}{s} [\chi(s)A_{2\infty} + (1 - \chi(s))\tilde{A}_{2\infty}] ds,
$$

then $(\tilde{u}^{(2)}_c, \tilde{v}^{(2)}_c, \tilde{p}^{(2)}_c)$ also satisfies the linearized Euler equations (2.71) with the boundary conditions (2.72), and there holds

$$
\|\partial_\theta^j \partial_r^k (\tilde{u}^{(2)}_c, \tilde{v}^{(2)}_c)\|_\infty \leq C(j, k), \quad \forall k, j \geq 0. \tag{2.78}
$$

Putting

$$
u^{(3)}(\theta, r) = u^{(3)}_c(r) + \sum_{i=1}^3 \varepsilon^i \tilde{u}^{(i)}_c(\theta, r) + \text{h.o.t.},
$$

$$
\nu^{(3)}(\theta, r) = \sum_{i=1}^3 \varepsilon^i \tilde{v}^{(i)}_c(\theta, r) + \text{h.o.t.},
$$

$$
p^{(3)}(\theta, r) = p^{(3)}_c(r) + \sum_{i=1}^3 \varepsilon^i \tilde{p}^{(i)}_c(\theta, r) + \text{h.o.t.},
$$

$$
\|\partial_\theta^j \partial_r^k (\tilde{u}^{(2)}_c, \tilde{v}^{(2)}_c)\|_\infty \leq C(j, k), \quad \forall k, j \geq 0. \tag{2.78}
$$

Putting

$$
\|\partial_\theta^j \partial_r^k (\tilde{u}^{(2)}_c, \tilde{v}^{(2)}_c)\|_\infty \leq C(j, k), \quad \forall k, j \geq 0. \tag{2.78}
$$
into the Navier-Stokes equations \[13\], we find that \((\tilde{u}_e^{(3)}, \tilde{v}_e^{(3)}, \tilde{p}_e^{(3)})\) satisfies the following linearized Euler equations in \(\Omega\)
\[
\begin{aligned}
&\begin{cases}
  u_e(r)\partial_\theta \tilde{u}_e^{(3)} + r \tilde{v}_e^{(3)} u'_e(r) + u_e(r) \tilde{v}_e^{(3)} + \partial_\theta \tilde{p}_e^{(3)} + f_e(\theta, r) = 0, \\
  u_e(r)\partial_\theta \tilde{v}_e^{(3)} - 2u_e(r)\tilde{u}_e^{(3)} + r\partial_r \tilde{p}_e^{(3)} + g_e(\theta, r) = 0, \\
  \partial_\theta \tilde{u}_e^{(3)} + \partial_r(\tilde{r} \tilde{v}_e^{(3)}) = 0,
\end{cases}
\end{aligned}
\]
equipped with the following boundary conditions
\[
\tilde{v}_e^{(3)}|_{r=1} = -\tilde{v}_p^{(3)}|_{\gamma=0}, \quad \tilde{v}_e^{(3)}|_{r=r_0} = -\tilde{v}_e^{(3)}|_{\theta=0}, \quad \tilde{v}_e^{(3)}(\theta, r) = \tilde{v}_e^{(3)}(\theta + 2\pi, r),
\]
where
\[
\begin{aligned}
f_e(\theta, r) &= \tilde{u}_e^{(1)} \partial_\theta \tilde{u}_e^{(2)} + \tilde{u}_e^{(2)} \partial_\theta \tilde{u}_e^{(1)} + \tilde{v}_e^{(1)} r\partial_r \tilde{u}_e^{(2)} + \tilde{v}_e^{(2)} r\partial_r \tilde{v}_e^{(1)} + \tilde{u}_e^{(1)} \tilde{v}_e^{(2)} + \tilde{u}_e^{(2)} \tilde{v}_e^{(1)} \\
&\quad - \left(\frac{\partial_\theta \tilde{u}_e^{(1)}}{r} + r\partial_r \tilde{v}_e^{(1)} + \partial_r \tilde{u}_e^{(1)} - \frac{2}{r} \partial_\theta \tilde{v}_e^{(1)} \right), \\
g_e(\theta, r) &= \tilde{u}_e^{(1)} \partial_\theta \tilde{v}_e^{(2)} + \tilde{u}_e^{(2)} \partial_\theta \tilde{v}_e^{(1)} + \tilde{v}_e^{(1)} r\partial_r \tilde{v}_e^{(2)} + \tilde{v}_e^{(2)} r\partial_r \tilde{v}_e^{(1)} - 2\tilde{u}_e^{(1)} \tilde{v}_e^{(2)} \\
&\quad - \left(\frac{\partial_\theta \tilde{v}_e^{(1)}}{r} + r\partial_r \tilde{v}_e^{(1)} + \partial_r \tilde{v}_e^{(1)} - \frac{2}{r} \partial_\theta \tilde{v}_e^{(1)} \right).
\end{aligned}
\]

**Proposition 2.10.** The linearized Euler equations \[2.79\] have a solution \((\tilde{u}_e^{(3)}, \tilde{v}_e^{(3)}, \tilde{p}_e^{(3)})\) which satisfies
\[
\|\partial_\theta^k \partial_r^j (\tilde{u}_e^{(3)}, \tilde{v}_e^{(3)})\|_2 \leq C(j, k), \quad \forall j, k \geq 0. \tag{2.80}
\]

**Proof.** We only need to check \(\int_0^{2\pi} f_e(\theta, r)d\theta = 0\), that is,
\[
\int_0^{2\pi} \left(\tilde{v}_e^{(1)} \partial_r (r\tilde{u}_e^{(2)}) + \tilde{v}_e^{(2)} \partial_r (r\tilde{u}_e^{(1)}) - \left(r\partial_r \tilde{u}_e^{(1)} + \partial_r \tilde{u}_e^{(1)} - \frac{\tilde{u}_e^{(1)}}{r}\right)\right)d\theta = 0, \quad \forall r \in [r_0, 1]. \tag{2.81}
\]
By \[2.68\], we only need to prove
\[
\int_0^{2\pi} \left(\frac{v_e^{(1)} \partial_r (ru_e^{(2)}) + v_e^{(2)} \partial_r (ru_e^{(1)})}{r}\right)(\theta, r)d\theta = 0, \quad \forall r \in [r_0, 1].
\]
Notice that \(v_e^{(1)} = \tilde{v}_e^{(1)}, \int_0^{2\pi} v_e^{(2)} d\theta = 0\) and \(u_e^{(1)} = \tilde{u}_e^{(1)} + A_1(r)\), thus we only need to prove
\[
\int_0^{2\pi} \left(\frac{v_e^{(1)} \partial_r (ru_e^{(2)}) - r\tilde{u}_e^{(2)}}{r}\right) + (v_e^{(2)} - \tilde{v}_e^{(2)})\partial_r (ru_e^{(1)})\right)(\theta, r)d\theta = 0, \quad \forall r \in [r_0, 1].
\]
For simplicity, we write \([v^{(1)}, u^{(1)}] = [v_e^{(1)}, u_e^{(1)}]\), and \([v^{(2)}, u^{(2)}] = [v_e^{(2)} - \tilde{v}_e^{(2)}, u_e^{(2)} - \tilde{u}_e^{(2)}]\). Notice that \(v^{(1)}\) satisfies equations \[2.46\], i.e.
\[-r \Delta (rv^{(1)}) + Brv^{(1)} = 0,
\]
where \(B = \frac{r}{u_e(r)}(u_e''(r) + u_e'(r) - \frac{u_e(r)}{r})\) is a radial function. By equations \[2.62\] and \[2.71\], \(v^{(2)}\) satisfies the following equation:
\[-r \Delta (rv^{(2)}) + Brv^{(2)} = f^{(2)},
\]
where
\[
\begin{align*}
f^{(2)} &= \frac{r}{u_e} \left[ r^2 v^{(1)} \Delta A_1 - r A_1 \Delta (r v^{(1)}) - \frac{A_1}{r} v^{(1)} \right] \\
&= \frac{r}{u_e} \left[ r^2 v^{(1)} \Delta A_1 - r A_1 B v^{(1)} - \frac{A_1}{r} v^{(1)} \right].
\end{align*}
\]
We set \( I[h] = \sum_{n \neq 0} \frac{\epsilon^{inb}}{in} h_n(r) \) for \( h \) satisfying \( \int_0^{2\pi} h d\theta = 0 \). Since \( u^{(2)}_\theta = -(r v^{(2)})_r \), there holds
\[
\begin{align*}
(r u^{(2)} - \int_0^{2\pi} r u^{(2)} d\theta)_r &= -I[(r (r v^{(2)})_r)] \\
&= -I[r \Delta (r v^{(2)}) - \frac{v^{(2)}_\theta}{r}] = I[f^{(2)}] + \frac{v^{(2)}_\theta}{r} - r B I[v^{(2)}].
\end{align*}
\]
Similarly,
\[
\begin{align*}
(r u^{(1)} - \int_0^{2\pi} r u^{(1)} d\theta)_r &= \frac{v^{(1)}_\theta}{r} - r B I[v^{(1)}].
\end{align*}
\]
Thus we have
\[
\begin{align*}
\int_0^{2\pi} \left( v^{(1)}_\theta (r u^{(2)}_e - r \tilde{u}^{(2)}_e) + (v^{(2)}_e - \tilde{v}^{(2)}_e) \partial_r (r \tilde{u}^{(1)}_e) \right) (\theta, r) d\theta \\
&= \int_0^{2\pi} \left( v^{(1)}_\theta (r u^{(2)} - v^{(2)}_e) \partial_r (ru^{(1)}_e) \right) d\theta \\
&= \int_0^{2\pi} \left( v^{(1)}_\theta I[f^{(2)}] + \frac{v^{(1)}_\theta v^{(2)}_\theta}{r} + \frac{v^{(2)}_e v^{(1)}_\theta}{r} - r B v^{(1)}_e I[v^{(2)}] - r B v^{(2)}_e I[v^{(1)}] \right) d\theta \\
&= \int_0^{2\pi} v^{(1)}_\theta I[f^{(2)}] d\theta.
\end{align*}
\]
Moreover, notice that \( f^{(2)} \) is a radial function times \( v^{(1)} \), we deduce that
\[
\begin{align*}
\int_0^{2\pi} v^{(1)}_\theta I[f^{(2)}] d\theta &= 0,
\end{align*}
\]
this complete the proof.

2.4.7. Linearized Prandtl equations for \((u_p^{(3)}, v_p^{(4)}), (\tilde{u}_p^{(3)}, \tilde{v}_p^{(4)})\) and their solvabilities.

Let
\[
\begin{align*}
u^{\varepsilon}(\theta, r) &= u_e(r) + u_p^{(0)}(\theta, Y) + \sum_{i=1}^2 \kappa^{i} \left[ \tilde{u}^{(i)}_e (\theta, r) + \tilde{u}^{(i)}_p (\theta, Y) \right] + \kappa^{3} [\tilde{u}^{(3)}_e (\theta, r) + u_p^{(3)}(\theta, Y)] + \text{h.o.t.}, \\
v^{\varepsilon}(\theta, r) &= \sum_{i=1}^3 \kappa^{i} \left[ \tilde{v}^{(i)}_e (\theta, r) + v^{(i)}_p (\theta, Y) \right] + \kappa^{4} v^{(4)}_p (\theta, Z) + \text{h.o.t.}, \\
p^{\varepsilon}(\theta, r) &= p_e(r) + \sum_{i=1}^3 \kappa^{i} \left[ \tilde{p}^{(i)}_e (\theta, r) + p^{(i)}_p (\theta, Y) \right] + \kappa^{4} p^{(4)}_p (\theta, Y) + \text{h.o.t.},
\end{align*}
\]
with the boundary conditions
\[
\begin{align*}
\tilde{v}^{(3)}_e (\theta, 1) + v^{(3)}_p (\theta, 0) &= 0, \\
\lim_{Y \to \infty} \partial_Y v^{(3)}_p (\theta, Y) &= v^{(4)}_p (\theta, 0) = 0,
\end{align*}
\]
we obtain the following linearized Prandtl problem for \((u_p^{(3)}, v_p^{(4)})\)

\[
\begin{aligned}
&\left\{
\begin{array}{l}
(u_e^{(1)} + v_p^{(0)}) \partial_\theta u_p^{(0)} + (v_e^{(1)}(\theta, 1) + v_p^{(1)}) \partial_Y u_p^{(3)} + (u_p^{(3)} + \hat{u}_e^{(3)}(\theta, 1)) \partial_\theta u_p^{(0)} \\
v_p^{(4)} \partial_Y u_p^{(0)} - \partial_Y Y u_p^{(3)} = f_3(\theta, Y)
\end{array}
\right.
\end{aligned}
\]

\[
\begin{aligned}
&\partial_Y u_p^{(3)} + \partial_Y v_p^{(4)} + \partial_Y (Y v_p^{(3)}) = 0,
\end{aligned}
\]

\[
\begin{aligned}
&u_p^{(3)}(\theta, Y) = u_p^{(3)}(\theta + 2\pi, Y), \quad v_p^{(4)}(\theta, Y) = v_p^{(4)}(\theta + 2\pi, Y),
\end{aligned}
\]

\[
\begin{aligned}
&u_p^{(3)}|_{Y=0} = -\hat{u}_e^{(3)}(\theta, 1), \quad \lim_{Y \to \infty} \partial_Y u_p^{(3)}(\theta, Y) = v_p^{(4)}(\theta, 0) = 0
\end{aligned}
\]

and

\[
\begin{aligned}
&\partial_Y v_p^{(4)}(\theta, Y) = g_3(\theta, Y), \quad p_p^{(4)}(\theta, 0) = 0,
\end{aligned}
\]

where

\[
\begin{aligned}
f_3(\theta, Y) &= -\partial_\theta p_p^{(3)} + Y \partial_Y Y u_p^{(3)} + \partial_Y u_p^{(3)} + \sum_{k=0}^{1} (-1)^k \frac{Y^k \partial_{\theta \theta} u_p^{(1-k)}}{k!} + 2\partial_\theta v_p^{(2)} - 2Y \partial_Y v_p^{(1)}
\end{aligned}
\]

\[
\begin{aligned}
&- \sum_{k=0}^{1} (-1)^k \frac{Y^k \partial_{\theta \theta} u_p^{(1-k)}}{k!} - \sum_{i+j=3, i \leq 2} \hat{u}_e^{(i)}(\theta, 1) \partial_Y \hat{u}_p^{(j)} + \sum_{i+j=3} \left[ v_p^{(i)} Y \partial_Y \hat{u}_p^{(j)} + \hat{u}_e^{(i)} v_p^{(j)} \right]
\end{aligned}
\]

\[
\begin{aligned}
g_3(\theta, Y) &= \partial_Y (Y v_p^{(3)}) + Y \partial_Y Y v_p^{(2)} + \partial_Y v_p^{(2)} + \partial_Y v_p^{(2)} + \sum_{k=0}^{1} (-1)^k \frac{Y^k \partial_{\theta \theta} u_p^{(2-k)}}{k!} - \sum_{i+j=4} \left[ v_p^{(i)} \partial_Y v_p^{(j)} - \sum_{k=0}^{2} \frac{\partial_{\theta} v_p^{(i)}(\theta, 1) Y \partial_Y v_p^{(j)}}{k!} \right]
\end{aligned}
\]

with \(u_p^{(0)} = u_p^{(0)}, \hat{u}_e^{(0)} = u_e(r), \hat{u}_p^{(3)} = u_p^{(3)}\).
Similarly, let
\[ u^\varepsilon(\theta, r) = u^\varepsilon_0(\theta, Z) + \sum_{i=1}^{2} \varepsilon^i \left[ \hat{u}_e^{(i)}(\theta, r) + (\hat{v}_p^{(i)}(\theta, Z) - \hat{A}_{1\infty}) \right] + \varepsilon^3 \left[ \hat{u}_e^{(3)}(\theta, r) + \hat{v}_p^{(3)}(\theta, Z) \right] + \text{h.o.t.,} \]
\[ v^\varepsilon(\theta, r) = \sum_{i=1}^{3} \varepsilon^i \left[ \hat{v}_e^{(i)}(\theta, r) + \hat{v}_p^{(i)}(\theta, Z) \right] + \varepsilon^4 \hat{v}_p^{(4)}(\theta, Z) + \text{h.o.t.,} \]
\[ p^\varepsilon(\theta, r) = p^\varepsilon_0(r) + \sum_{i=1}^{3} \varepsilon \left[ \hat{p}_e^{(i)}(\theta, r) + \hat{p}_p^{(i)}(\theta, Z) \right] + \varepsilon^4 \hat{p}_p^{(4)}(\theta, Z) + \text{h.o.t.,} \]
with the following boundary conditions
\[ \hat{u}_e^{(3)}(\theta, r_0) + \hat{v}_p^{(3)}(\theta, 0) = 0, \quad \lim_{Z \to +\infty} \partial_Z \hat{u}_p^{(3)}(\theta, Z) = \hat{v}_p^{(4)}(\theta, 0) = 0, \]
we obtain the following linearized Prandtl equations for \((\hat{u}_p^{(3)}, \hat{v}_p^{(4)})\)
\[
\begin{cases}
(u_e(r_0) + \hat{u}_p^{(0)}) \partial_\theta \hat{u}_p^{(3)} + (v_e^{(1)}(\theta, r_0) + \hat{v}_p^{(1)}) r_0 \partial_Z \hat{u}_p^{(3)} + \hat{u}_p^{(3)} \partial_\theta \hat{u}_p^{(0)} \\
\partial_\theta \hat{v}_p^{(3)} + r_0 \partial_Z \hat{v}_p^{(4)} + \partial_Z (Z \hat{v}_p^{(3)}) = 0,
\end{cases}
\tag{2.84}
\]
and the pressure \(\hat{p}_p^{(4)}\) satisfies
\[
\partial_Z \hat{p}_p^{(4)}(\theta, Z) = \hat{g}_3(\theta, Z), \quad \hat{p}_p^{(4)}(\theta, 0) = 0.
\tag{2.85}
\]
The functions \((\hat{f}_3(\theta, Z), \hat{g}_3(\theta, Z))\) are similar to \((f_3(\theta, Y), g_3(\theta, Y))\), we omit the details here.

**Proposition 2.11.** There exists \(\eta_0 > 0\) such that for any \(\eta \in (0, \eta_0)\), the equations \(2.82\) have a unique solution \((u_p^{(3)}, v_p^{(4)})\) and the equations \(2.84\) have a unique solution \((\hat{u}_p^{(3)}, \hat{v}_p^{(4)})\) which satisfy
\[
\sum_{0 < j + k \leq m} \int_{-\infty}^{0} \int_{0}^{2\pi} |\partial_\theta^j \partial_\sigma^k (u_p^{(3)}, v_p^{(4)})|^2 |Y|^{2l} d\theta dY \leq C(m, l), \quad \forall m, l \geq 0,
\]
\[
\sum_{0 < j + k \leq m} \int_{0}^{+\infty} \int_{0}^{2\pi} |\partial_\theta^j \partial_\sigma^k (u_p^{(3)}, v_p^{(4)})|^2 |Z|^{2l} d\theta dZ \leq C(m, l), \quad \forall m, l \geq 0,
\tag{2.86}
\]
\[
\int_{0}^{2\pi} v_p^{(4)}(\theta, Y) d\theta = \int_{0}^{2\pi} \hat{v}_p^{(4)}(\theta, Z) d\theta = 0, \quad \forall Y \leq 0, \quad Z \geq 0
\]
and
\[
\|(u_p^{(3)}, v_p^{(4)})\|_{\infty} + \|\hat{(u}_p^{(3)}, \hat{v}_p^{(4)})\|_{\infty} \leq C.
\]

The proof is same as with Proposition 2.6, we omit the details. Moreover, we can construct \((p_p^{(4)}, \hat{p}_p^{(4)})\) by solving the equations \(2.83\) and \(2.85\).

2.5 Approximate solutions.

In this subsection, we construct an approximate solution of Navier-Stokes equations \(1.3\). Set
\[
\tilde{u}_p^{\varepsilon}(\theta, r) := \chi(r) \left( u_p^{(0)}(\theta, Y) + \varepsilon \tilde{u}_p^{(1)}(\theta, Y) + \varepsilon^2 \tilde{u}_p^{(2)}(\theta, Y) + \varepsilon^3 u_p^{(3)}(\theta, Y) \right) + (1 - \chi(r)) \left( u_p^{(0)}(\theta, Z) + \varepsilon \left[ \tilde{u}_p^{(1)}(\theta, Z) - \hat{A}_{1\infty} \right] + \varepsilon^2 \left[ \tilde{u}_p^{(2)}(\theta, Z) - \hat{A}_{2\infty} \right] + \varepsilon^3 \tilde{u}_p^{(3)}(\theta, Z) \right)
\]
\begin{align*}
\tilde{v}_p^a(\theta, r) &:= \chi(r) \left( \sum_{i=1}^{4} \varepsilon_i \tilde{v}_p^{(i)}(\theta, Y) \right) + (1 - \chi(r)) \left( \sum_{i=1}^{4} \varepsilon_i \tilde{v}_p^{(i)}(\theta, Z) \right) \\
:= &\chi(r) v_p^a + (1 - \chi(r)) \tilde{v}_p^a, \\
\tilde{p}_p^a(\theta, r) &:= \chi^2(r) \left( \sum_{i=1}^{4} \varepsilon_i \tilde{p}_p^{(i)}(\theta, Y) \right) + (1 - \chi(r))^2 \left( \sum_{i=1}^{4} \varepsilon_i \tilde{p}_p^{(i)}(\theta, Z) \right) \\
:= &\chi^2(r) p_p^a + (1 - \chi(r))^2 \tilde{p}_p^a,
\end{align*}
and
\begin{align*}
\tilde{v}_c^a(\theta, r) &:= u_c^a(\theta, r) + \sum_{i=1}^{3} \varepsilon_i \tilde{u}_c^{(i)}(\theta, r), \\
\tilde{p}_c^a(\theta, r) &:= p_c^a(\theta, r) + \sum_{i=1}^{3} \varepsilon_i \tilde{p}_c^{(i)}(\theta, r).
\end{align*}

We construct an approximate solution \((u^a, v^a, p^a)\)
\begin{align*}
u^a(\theta, r) &:= u_c^a(\theta, r) + \tilde{u}_p^a(\theta, r) + \varepsilon^4 h(\theta, r), \\
v^a(\theta, r) &:= v_c^a(\theta, r) + \tilde{v}_p^a(\theta, r), \\
p^a(\theta, r) &:= p_c^a(\theta, r) + \tilde{p}_p^a(\theta, r).
\end{align*}
where the corrector \(h(\theta, r)\) will be given in Appendix B which satisfies
\[
h(\theta, 1) = 0, \ h(\theta, r_0) = 0, \ ||\partial^2 \partial^2_k h||_2 \leq C(j, k) \varepsilon^{-k}
\]
and makes \((u^a, v^a)\) be divergence-free
\[
u^a_{\theta} + (r v^a)_r = 0.
\]
Moreover, \((u^a, v^a)\) satisfies the following boundary conditions
\[
\begin{align*}
u^a(\theta + 2\pi, r) &= \nu^a(\theta, r), \ v^a(\theta + 2\pi, r) = v^a(\theta, r), \\
\nu^a(\theta, 1) &= \alpha + \eta f(\theta), \ v^a(\theta, 1) = 0, \\
\nu^a(\theta, r_0) &= \beta + \eta g(\theta), \ v^a(\theta, r_0) = 0.
\end{align*}
\]
By collecting the estimates \((2.70), (2.77), (2.80)\), we deduce that
\[
||u^a - u_c(\theta, r)||_\infty + ||(\partial_r u^a - u_c(\theta, r), \partial_\theta u^a)||_\infty \leq C\varepsilon(\eta + \varepsilon),
\]
\[
||v^a||_\infty + ||(\partial_r v^a, \partial_\theta v^a)||_\infty \leq C\varepsilon(\eta + \varepsilon),
\]
and collecting the estimates \((2.43), (2.45), (2.76), (2.86)\) one has
\[
||(Y^j \partial^k Y^a u_p^a_{\theta}^a, Y^j \partial^k Y^a u_p^a_{r}^a)||_\infty \leq C(\eta + \varepsilon), \ \||(Y^j \partial^k Z^a_{\theta}^a, Y^j \partial^k Z^a_{r}^a)||_\infty \leq C\varepsilon(\eta + \varepsilon), \ \forall j \leq 2, k \leq 2. \ \ (2.91)
\]
Similarly, there holds
\[
||(Z^j \partial^k Z^a_{\theta}^a, Z^j \partial^k Z^a_{r}^a)||_\infty \leq C(\eta + \varepsilon), \ \||(Z^j \partial^k Z^a_{\theta}^a, Z^j \partial^k Z^a_{r}^a)||_\infty \leq C\varepsilon(\eta + \varepsilon), \ \forall j \leq 2, k \leq 2.
\]
By (1.6), (2.87), (2.90) and (2.91), we deduce that there exists $\varepsilon_0 > 0, \eta_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0)$, there holds

$$u^a(\theta, r) \geq d_0 > 0, \ \forall (\theta, r) \in \Omega,$$

(2.92)

where $d_0$ is a fixed number.

Finally, set

$$R_u^a := u^a u_\theta^a + v^a rv_r + u^a v^a + p_\theta^a - \varepsilon^2 \left( r u_\theta^a + \frac{u_\theta^a}{r} + 2 \frac{v_\theta}{r} + u_r^a - \frac{u^a}{r} \right) - \varepsilon^2 r F_u,$$

$$R_v^a := u^a v_\theta^a + v^a rv_r - (u^a)^2 + r p_r^a - \varepsilon^2 \left( r v_\theta^a + \frac{v_\theta}{r} - 2 \frac{u_\theta}{r} + v_r^a - \frac{v^a}{r} \right) - \varepsilon^2 r F_v,$$

then there holds

$$\|R_u^a\|_2 + \|\partial_\theta R_u^a\|_2 \leq C \varepsilon^4, \ \|R_v^a\|_2 + \|\partial_\theta R_v^a\|_2 \leq C \varepsilon^4$$

and $(u^a, v^a, p^\alpha)$ satisfies

$$\begin{cases}
\begin{align*}
& u^a u_\theta^a + v^a rv_r + u^a v^a + p_\theta^a - \varepsilon^2 \left( r u_\theta^a + \frac{u_\theta^a}{r} + 2 \frac{v_\theta}{r} + u_r^a - \frac{u^a}{r} \right) = \varepsilon^2 r F_u + R_u^a, \quad (\theta, r) \in \Omega, \\
& u^a v_\theta^a + v^a rv_r - (u^a)^2 + r p_r^a - \varepsilon^2 \left( r v_\theta^a + \frac{v_\theta}{r} - 2 \frac{u_\theta}{r} + v_r^a - \frac{v^a}{r} \right) = \varepsilon^2 r F_v + R_v^a, \quad (\theta, r) \in \Omega, \\
& u_\theta^a + (rv^\alpha)_r = 0, \quad (\theta, r) \in \Omega, \\
& u^a(\theta + 2\pi, r) = u^a(\theta, r), \quad v^a(\theta + 2\pi, r) = v^a(\theta, r), \quad (\theta, r) \in \Omega, \\
& u^a(\theta, 1) = \alpha + f(\theta)\eta, \quad v^a(\theta, 1) = 0, \quad \theta \in [0, 2\pi], \\
& u^a(\theta, r_0) = \beta + g(\theta)\eta, \quad v^a(\theta, r_0) = 0, \quad \theta \in [0, 2\pi].
\end{align*}
\end{cases}$$

(2.93)

3. LINEAR STABILITY ESTIMATES OF ERROR EQUATIONS

In this section, we derive the error equations and establish the linear stability estimate of the error equations.

3.1. Error equations.

Set the error

$$u := u^\varepsilon - u^a, \quad v := v^\varepsilon - v^a, \quad p := p^\varepsilon - p^\alpha,$$

then there hold

$$\begin{cases}
\begin{align*}
& u^a u_\theta + uu_\theta + uu_\theta + v^a rv_r + v^a rv_r + v^a u + vu + u^a + vv + p_\theta - \varepsilon^2 \left( r u_\theta + \frac{u_\theta}{r} + 2 \frac{v_\theta}{r} + u_r + \frac{u^a}{r} \right) = R_u^a, \\
& v^a v_\theta + uv_\theta + v^a rv_r + v^a rv_r + v^a u + vu + u^a + vv + p_\theta - \varepsilon^2 \left( r v_\theta + \frac{v_\theta}{r} - 2 \frac{u_\theta}{r} + v_r + \frac{v^a}{r} \right) = R_v^a, \\
& u_\theta + (rv^\alpha)_r = 0, \\
& u(\theta + 2\pi, r) = u(\theta, r), \quad v(\theta + 2\pi, r) = v(\theta, r), \\
& u(\theta, 1) = 0, \quad v(\theta, 1) = 0, \\
& u(\theta, r_0) = 0, \quad v(\theta, r_0) = 0.
\end{align*}
\end{cases}$$

(3.1)

We set

$$S_u := u^a u_\theta + v^a rv_r + uu_\theta + v^a rv_r + v^a u + uu^a,$$

$$S_v := v^a v_\theta + v^a rv_r + uu_\theta + v^a rv_r - 2uu^a,$$
then the error equations become

\[
\begin{cases}
-\varepsilon^2\left(ru_{rr} + \frac{r u_r}{r} + 2\frac{u_\theta}{r} + u_r - \frac{u}{r}\right) + p_\theta + S_u = R_u, \\
-\varepsilon^2\left(rv_{rr} + \frac{r v_r}{r} - 2\frac{u_\theta}{r} + v_r - \frac{v}{r}\right) + r p_r + S_v = R_v, \\
u_\theta + (rv)_r = 0, \\
u(\theta, r) = u(\theta + 2\pi, r), \quad v(\theta, r) = v(\theta + 2\pi, r), \\
u(\theta, 1) = 0, \quad v(\theta, 1) = 0, \\
u(\theta, r_0) = 0, \quad v(\theta, r_0) = 0,
\end{cases}
\]

(3.2)

where

\[R_u := R^a_u - u_\theta v_r - vr u_r - vu, \quad R_v := R^a_v - v_\theta u_r - vu u_r + u^2.\]

3.2. Linear stability estimate.

In this part, we consider the following linear system in \(\Omega\)

\[
\begin{cases}
-\varepsilon^2\left(ru_{rr} + \frac{r u_r}{r} + 2\frac{u_\theta}{r} + u_r - \frac{u}{r}\right) + p_\theta + S_u = F_1, \\
-\varepsilon^2\left(rv_{rr} + \frac{r v_r}{r} - 2\frac{u_\theta}{r} + v_r - \frac{v}{r}\right) + r p_r + S_v = F_2, \\
u_\theta + (rv)_r = 0, \\
u(\theta, r) = u(\theta + 2\pi, r), \quad v(\theta, r) = v(\theta + 2\pi, r), \\
u(\theta, 1) = 0, \quad v(\theta, 1) = 0, \\
u(\theta, r_0) = 0, \quad v(\theta, r_0) = 0,
\end{cases}
\]

(3.3)

and establish its linear stability estimate (3.4).

Firstly, due to the third equation in (3.3) and the boundary condition of \(v\), we easily deduce that

\[
\int_0^{2\pi} v(\theta, r)d\theta = 0, \quad \forall \ r \in [r_0, 1].
\]

Hence, there holds

\[
\int_0^{2\pi} u^2(\theta, r)d\theta \leq \int_0^{2\pi} v^2(\theta, r)d\theta, \quad \forall \ r \in [r_0, 1].
\]

The following stability estimate is a direct result of the basic energy estimate (3.5) and positivity estimate (3.10).

**Proposition 3.1.** Let \((u, v)\) be a smooth solution of (3.3), then there exist \(\varepsilon_0 > 0, \eta_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0), \) there holds

\[
\varepsilon^2 \|u_r\|_2^2 + \|(u_\theta, v_\theta)\|_2^2 \leq C\|(F_1, F_2)\|_2^2 + C\left|\int_{r_0}^{1} \int_0^{2\pi} (uF_1 + vF_2) d\theta dr\right|.
\]

(3.4)

3.2.1. Basic energy estimate.

In this subsection, we establish the following basic energy estimate.

**Lemma 3.2.** Let \((u, v)\) be a smooth solution of (3.3), then there exist \(\varepsilon_0 > 0, \eta_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0), \) there holds

\[
\varepsilon^2 \|u_r\|_2^2 \leq C\|(u_\theta, v_\theta)\|_2^2 + C\left|\int_{r_0}^{1} \int_0^{2\pi} [uF_1 + vF_2] d\theta dr\right|.
\]

(3.5)
Proof. Multiplying the first equation in (3.2) by $u$, the second equation in (3.2) by $v$, adding them together and integrating in $\Omega$, we obtain that

\[
-\varepsilon^2 \int_{r_0}^{1} \int_{0}^{2\pi} \left[ (ru_{rr} + \frac{u \theta \theta}{r} + 2v \frac{\theta}{r} + u_r - \frac{u}{r}) u + (rv_{rr} + \frac{v \theta \theta}{r} - 2\frac{u \theta}{r} + v_r - \frac{v}{r}) v \right] d\theta dr
\]

\[
+ \int_{r_0}^{1} \int_{0}^{2\pi} (p \theta u + p_r v) d\theta dr + \int_{r_0}^{1} \int_{0}^{2\pi} (u S_u + v S_v) d\theta dr
\]

\[
= \int_{r_0}^{1} \int_{0}^{2\pi} [u F_u + v F_v] d\theta dr.
\]

Next we deal with them term by term.

**The diffusion term:** Introducing by parts, we obtain

\[
-\varepsilon^2 \int_{r_0}^{1} \int_{0}^{2\pi} \left[ (ru_{rr} + \frac{u \theta \theta}{r} + 2v \frac{\theta}{r} + u_r - \frac{u}{r}) u + (rv_{rr} + \frac{v \theta \theta}{r} - 2\frac{u \theta}{r} + v_r - \frac{v}{r}) v \right] d\theta dr
\]

\[
= \varepsilon^2 \int_{r_0}^{1} \int_{0}^{2\pi} \left( \frac{u^2}{r} + \frac{v^2}{r} + \frac{u^2 + v^2}{r} + \frac{2\theta \theta}{r} \right) d\theta dr + \varepsilon^2 \int_{r_0}^{1} \int_{0}^{2\pi} r(u_r^2 + v_r^2) d\theta dr
\]

\[
= \varepsilon^2 \int_{r_0}^{1} \int_{0}^{2\pi} \left( \frac{u^2 \theta}{r} + \frac{v^2}{r} + \frac{v - u}{r} \right)^2 d\theta dr + \varepsilon^2 \int_{r_0}^{1} \int_{0}^{2\pi} r(u_r^2 + v_r^2) d\theta dr \geq r_0 \varepsilon^2 \|u_r\|^2. \tag{3.6}
\]

**Pressure term:** Integrating by parts and using the divergence-free condition, we deduce that

\[
\int_{r_0}^{1} \int_{0}^{2\pi} (p \theta u + p_r(v)) d\theta dr = 0. \tag{3.7}
\]

**Linear term:** Integrating by parts and using the divergence-free condition, we obtain

\[
\int_{r_0}^{1} \int_{0}^{2\pi} (u S_u + v S_v) d\theta dr
\]

\[
= \int_{r_0}^{1} \int_{0}^{2\pi} [u^2 u \theta u + ru^2 u_r u + uu \theta^2 u + vu \theta^2 r u + (v^2 u + vu^2) u] d\theta dr
\]

\[
+ \int_{r_0}^{1} \int_{0}^{2\pi} [u^2 v \theta v + rv^2 v_r v + uv \theta^2 v + v^2 v_r v - 2vu^2 v] d\theta dr
\]

\[
= \int_{r_0}^{1} \int_{0}^{2\pi} [u^2 u \theta^2 + v^2 r v_r^2 + v^2 u^2 + uv^2 \theta + uv (r u_r^2 - u^2)] d\theta dr, \tag{3.8}
\]

where we used

\[
\int_{r_0}^{1} \int_{0}^{2\pi} (u^2 u \theta u + rv^2 u_r u) d\theta dr = \int_{r_0}^{1} \int_{0}^{2\pi} (u^2 v \theta v + rv^2 v_r v) d\theta dr = 0.
\]

Next, we divide into four steps to estimate (3.8) term by term.

1) **The term** $u^2 u \theta^2$: We divide it into the Euler part and Prandtl part. Notice that $\partial \theta u = 0$, integrating by parts and using (2.9), we obtain

\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} u^2 \partial \theta u \partial u \theta d\theta dr \right| = \left| \int_{r_0}^{1} \int_{0}^{2\pi} u^2 \partial \theta (u^2 - u_e) d\theta dr \right|
\]

\[
= \left| \int_{r_0}^{1} \int_{0}^{2\pi} 2uu \theta (u^2 - u_e) d\theta dr \right| \leq C\varepsilon(\eta + \varepsilon)\|u_r\|_2\|u\|_2.
\]
Moreover, integrating by parts, using hardy inequality and (2.91), we have
\[
| \int_{r_0}^{1} \int_{0}^{2\pi} u^2 \partial_\theta u_\theta^a d\theta dr | = | \int_{r_0}^{1} \int_{0}^{2\pi} 2uu_\theta u_\theta^a d\theta dr |
\]
\[
= \varepsilon | \int_{r_0}^{1} \int_{0}^{2\pi} \frac{2u}{r - 1} u_\theta Y u_\theta^a d\theta dr | \leq C\varepsilon (\eta + \varepsilon) \| u_r \|_2 \| u_\theta \|_2.
\]
Similarly, there holds
\[
| \int_{r_0}^{1} \int_{0}^{2\pi} u^2 \partial_\theta \hat{u}_\theta^a d\theta dr | = | \int_{r_0}^{1} \int_{0}^{2\pi} 2uu_\theta \hat{u}_\theta^a d\theta dr |
\]
\[
= \varepsilon | \int_{r_0}^{1} \int_{0}^{2\pi} \frac{2u}{r - r_0} u_\theta Z u_\theta^a d\theta dr | \leq C\varepsilon (\eta + \varepsilon) \| u_r \|_2 \| u_\theta \|_2.
\]
Thus, we obtain
\[
| \int_{r_0}^{1} \int_{0}^{2\pi} u^2 u_\theta^a d\theta dr | \leq C\varepsilon (\eta + \varepsilon) \| u_r \|_2 \| u_\theta \|_2.
\]

2) The term \( v^2 r v_\theta^a \) and \( uv v_\theta^a \): Combining (2.90) and (2.91), we get
\[
\| v_\theta^a \|_\infty \leq C(\eta + \varepsilon).
\]
Thus
\[
| \int_{r_0}^{1} \int_{0}^{2\pi} v^2 r v_\theta^a d\theta dr | = | \int_{r_0}^{1} \int_{0}^{2\pi} (rv)^2 \frac{v_\theta^a}{r} d\theta dr | \leq C(\eta + \varepsilon) \int_{r_0}^{1} \int_{0}^{2\pi} u_\theta^a d\theta dr.
\]
The same argument gives that
\[
| \int_{r_0}^{1} \int_{0}^{2\pi} v_\theta^a u v d\theta dr | \leq C\varepsilon (\eta + \varepsilon) \| v_\theta \|_2 \| u_r \|_2.
\]

3) The term \( v^a u^2 \): We decompose \( u(\theta, r) = u_0(r) + \tilde{u}(\theta, r) \), where \( u_0(r) = \frac{1}{2\pi} \int_{0}^{2\pi} u(r, \theta)d\theta \), and notice that \( \int_{0}^{2\pi} v^a(r, \theta)d\theta = 0 \), \( \forall r \in [r_0, 1] \), we obtain
\[
| \int_{r_0}^{1} \int_{0}^{2\pi} v^a u^2 d\theta dr | = | \int_{r_0}^{1} \int_{0}^{2\pi} v^a(2u_0 \tilde{u} + \tilde{u}^2) d\theta dr | \leq C\varepsilon (\eta + \varepsilon)(\| u_\theta \|_2 \| u_r \|_2 + \| u_\theta \|_2^2),
\]
where we used \( \| \tilde{u} \|_2 \leq C\| u_\theta \|_2 \).

4) The term \( uv(ru_\theta^a - u^a) \): Finally, we deal with the remainder two terms which involve \( u, v \) together
\[
\int_{r_0}^{1} \int_{0}^{2\pi} [uv ru_\theta^a - uv u^a] d\theta dr.
\]
We first consider the leading Euler flow \( u_\alpha(r) \). We decompose \( u(\theta, r) = u_0(r) + \tilde{u}(\theta, r) \) as above, then
\[
| \int_{r_0}^{1} \int_{0}^{2\pi} uv ru_\theta' d\theta dr | = | \int_{r_0}^{1} \int_{0}^{2\pi} (u_0 + \tilde{u}) v ru_\theta' d\theta dr |
\]
\[
= | \int_{r_0}^{1} \int_{0}^{2\pi} \tilde{u} v ru_\theta' d\theta dr | \leq C \| u_\theta \|_2 \| v \|_2,
\]
where we used
\[
\int_{0}^{2\pi} v(\theta, r)d\theta = 0, \forall r \in [r_0, 1].
\]
Moreover, by (2.90), it’s easy to obtain that
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} u v r \partial_r (u^a_r - u_e) d\theta dr \right| \leq C\varepsilon (\eta + \varepsilon) \|v_\theta\|_2 \|u_r\|_2.
\]
Thus, we obtain that
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} u v r \partial_r u^a_r d\theta dr \right| \leq C\varepsilon (\eta + \varepsilon) \|v_\theta\|_2 \|u_r\|_2 + C \|u_\theta\|_2 \|v_\theta\|_2.
\]
The same argument gives
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} v u v u^a_r d\theta dr \right| \leq C\varepsilon (\eta + \varepsilon) \|v_\theta\|_2 \|u_r\|_2 + C \|u_\theta\|_2 \|v_\theta\|_2.
\]
For the Prandtl part, using the Hardy inequality, divergence-free condition and (2.91), we deduce that
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} u v r \partial_r u^a_p d\theta dr \right| = \varepsilon \left| \int_{r_0}^{1} \int_{0}^{2\pi} \frac{r v u}{(r - 1)^2} \partial_r u^a_p d\theta dr \right|
\leq C\varepsilon (\eta + \varepsilon) \left( \int_{r_0}^{1} \int_{0}^{2\pi} \frac{u^2}{(r - 1)^2} d\theta dr \right)^{\frac{1}{2}} \left( \int_{r_0}^{1} \int_{0}^{2\pi} \frac{(r v)^2}{(r - 1)^2} d\theta dr \right)^{\frac{1}{2}}
\leq C\varepsilon (\eta + \varepsilon) \|u_\theta\|_2 \|u_r\|_2.
\]
The same argument gives
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} u v r \partial_r \hat{u}^a_p d\theta dr \right| \leq C\varepsilon (\eta + \varepsilon) \|u_\theta\|_2 \|u_r\|_2.
\]
Recalling that \( \hat{u}^a_p = \chi(r) u^a_p + (1 - \chi(r)) \hat{u}^a_p \) and the fast decay of \( (u^a_p, \hat{u}^a_p) \) away from the boundary, we deduce that
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} u v r \partial_r \hat{u}^a_p d\theta dr \right| \leq \varepsilon (\eta + \varepsilon) \|u_\theta\|_2 \|u_r\|_2.
\]
Moreover, by the same argument, we can obtain
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} u v u^a_p d\theta dr \right| \leq C\varepsilon (\eta + \varepsilon) \|v_\theta\|_2 \|u_r\|_2,
\]
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} u v \hat{u}^a_p d\theta dr \right| \leq C\varepsilon (\eta + \varepsilon) \|v_\theta\|_2 \|u_r\|_2.
\]
Summing these estimates together, we obtain
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} [u v u^a_r - v u v^a_r] d\theta dr \right| \leq C\varepsilon (\eta + \varepsilon) (\|u_\theta\|_2 + \|v_\theta\|_2) \|u_r\|_2 + C \|u_\theta\|_2 \|v_\theta\|_2.
\]
Finally, summing the above four steps we obtain
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} (u S_u + v S_v) d\theta dr \right| \leq C\varepsilon (\eta + \varepsilon) \|(u_\theta, v_\theta)\|_2 \|u_r\|_2 + C \|u_\theta\|_2 \|v_\theta\|_2 + C\varepsilon (\eta + \varepsilon) \|u_\theta\|_2^2
\leq \frac{\eta \varepsilon^2}{4} \|u_r\|_2 + C \|(u_\theta, v_\theta)\|_2.
\]
(3.9)
Based on the estimates (3.6), (3.7) and (3.9), we obtain (3.5). \( \square \)
3.2.2. Positivity estimate.

In this subsection, we establish the following positivity estimate.

**Lemma 3.3.** Let \((u, v)\) be a smooth solution of (3.3), then there exist \(\varepsilon_0 > 0, \eta_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0)\), there holds

\[
\| (u_\eta, v_\eta) \|_2^2 \leq C \varepsilon^2 (\eta + \varepsilon) \| u_r \|_2^2 + C \| (F_1, F_2) \|_2^2. \tag{3.10}
\]

**Proof.** Multiplying the first equation by \(-\left(\frac{r^2}{u^3}\right)_r\), and the second equation by \(\left(\frac{r}{u^3}\right)_\theta\), integrating in \(\Omega\) and summing two terms together, we arrive at

\[
\begin{align*}
- \int_{r_0}^{1} \int_{0}^{2\pi} \left[ - \varepsilon^2 (ru_{rr} + \frac{u_{\theta \theta} r}{r} + 2 \frac{v}{r} + u_r - \frac{u}{r}) + p_{\theta} + S_u \right] \left( \frac{r^2 v}{u^3} \right)_r d\theta dr \\
+ \int_{r_0}^{1} \int_{0}^{2\pi} \left[ - \varepsilon^2 (rv_{rr} + \frac{v_{\theta \theta} r}{r} - 2 \frac{u_{\theta} r}{r} + v_r - \frac{v}{r}) + r p_r + S_v \right] \left( \frac{r v}{u^3} \right)_\theta d\theta dr \\
= \int_{r_0}^{1} \int_{0}^{2\pi} \left[ - F_1 \left( \frac{r^2 v}{u^3} \right)_r + F_2 \left( \frac{r v}{u^3} \right)_\theta \right] d\theta dr. \tag{3.11}
\end{align*}
\]

Firstly, we choose \(\varepsilon_0 > 0, \eta_0 > 0\) such that (2.92) holds.

**Positivity term:** We first deal with these terms which are related to \(S_u, S_v\). Recall that

\[
\begin{align*}
S_u : &= u^a u_\theta + v^a r u_r + uu^a_\theta + vru^a_\theta + v^a u + uu^a, \\
S_v : &= u^a v_\theta + v^a rv_r + uu^a_\theta + vr^a_{\theta} - 2uu^a,
\end{align*}
\]

we first handle the following terms

\[
\begin{align*}
- \int_{r_0}^{1} \int_{0}^{2\pi} (u^a u_\theta + v^a rv_r + uu^a_\theta + vru^a_\theta + v^a u + uu^a) \left( \frac{r^2 v}{u^3} \right)_r d\theta dr \\
+ \int_{r_0}^{1} \int_{0}^{2\pi} (u^a v_\theta - 2uu^a) \left( \frac{rv}{u^3} \right)_\theta d\theta dr := I_1 + I_2.
\end{align*}
\]

By (2.90), (2.91) and (2.92), we deduce that

\[
I_2 := \int_{r_0}^{1} \int_{0}^{2\pi} \left( vr^2_{\theta} - 2ruv_r - \frac{vvgr^a_{\theta}}{u^a} + \frac{2uvru^a_{\theta}}{u^a} \right) d\theta dr
\]

\[
= \int_{r_0}^{1} \int_{0}^{2\pi} \left( vr^2_{\theta} - \frac{vvgr^a_{\theta}}{u^a} + \frac{2uvru^a_{\theta}}{u^a} \right) d\theta dr
\]

\[
\geq (1 - C \eta - C \varepsilon) \int_{r_0}^{1} \int_{0}^{2\pi} rv^2_{\theta} d\theta d\theta + \int_{r_0}^{1} \int_{0}^{2\pi} \frac{2uvru^a_{\theta}}{u^a} d\theta dr,
\]

where we used

\[
\int_{r_0}^{1} \int_{0}^{2\pi} -2ruv_r d\theta dr = \int_{r_0}^{1} \int_{0}^{2\pi} 2u_\theta rv d\theta dr = - \int_{r_0}^{1} \int_{0}^{2\pi} 2(rv)_r v dr d\theta = 0.
\]

Moreover, by the Hardy inequality, (2.90), (2.91) and (2.92), there holds

\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} \frac{2uvru^a_{\theta}}{u^a} d\theta dr \right|
\]

\[
\leq \varepsilon \left| \int_{r_0}^{1} \int_{0}^{2\pi} \frac{2uvru^a_{\theta}}{(r-1)u^a} d\theta dr \right| + \varepsilon \left| \int_{r_0}^{1} \int_{0}^{2\pi} \frac{2uvru^a_{\theta}}{(r-r_0)u^a} d\theta dr \right| + \left| \int_{r_0}^{1} \int_{0}^{2\pi} \frac{2uvru^a_{\theta}}{u^a} d\theta dr \right|
\]

\[
\leq C \varepsilon (\eta + \varepsilon) \| u_r \|_2 \| v_r \|_2 + \int_{r_0}^{1} \int_{0}^{2\pi} \frac{2uvru^a_{\theta}}{u^a} d\theta dr.
\]
\[ \leq C(\eta + \varepsilon) \int_{r_0}^{1} \int_{0}^{2\pi} r v_0^2 d\theta dr + C \varepsilon^2 (\eta + \varepsilon) \|u_r\|^2_2 + \bigg| \int_{r_0}^{1} \int_{0}^{2\pi} \frac{2uv\partial_\theta u_\theta^a}{u^a} d\theta dr \bigg|. \]

By (2.90), it is easy to get
\[ \bigg| \int_{r_0}^{1} \int_{0}^{2\pi} \frac{2uv\partial_\theta u_\theta^a}{u^a} d\theta dr \bigg| \leq C\varepsilon(\eta + \varepsilon) \|u_r\|_2 \|v_\theta\|_2. \]

Thus,
\[ I_2 \geq (1 - C\eta - C\varepsilon) \int_{r_0}^{1} \int_{0}^{2\pi} r v_0^2 d\theta dr - C\varepsilon^2 (\eta + \varepsilon) \|u_r\|^2_2. \]

For \( I_1 \), we first decompose it
\[
I_1 = - \int_{r_0}^{1} \int_{0}^{2\pi} \left[ u^a u_\theta + ru v_\theta^a + vu^a \right] \left( - \frac{r}{u^a} u_\theta + \frac{r}{u^a} \right) d\theta dr \\
= \int_{r_0}^{1} \int_{0}^{2\pi} r u_\theta^2 d\theta dr + \int_{r_0}^{1} \int_{0}^{2\pi} r^2 v^2 u_\theta^a u_\theta^a d\theta dr + \int_{r_0}^{1} \int_{0}^{2\pi} r u v_\theta d\theta dr \\
- \int_{r_0}^{1} \int_{0}^{2\pi} \frac{r}{u^a} u_\theta^2 u_\theta d\theta dr - \int_{r_0}^{1} \int_{0}^{2\pi} \frac{r}{u^a} u_\theta^2 (ru v_\theta^a) d\theta dr \\
- \int_{r_0}^{1} \int_{0}^{2\pi} \frac{ru v_\theta^a}{r} (ru v_\theta^a) d\theta dr := \sum_{i=1}^{6} I_{1i}.
\]

Then, integrating by parts and using the divergence-free condition, we deduce that
\[
I_{12} = \int_{r_0}^{1} \int_{0}^{2\pi} r^2 v^a u_\theta^a u_\theta d\theta dr \\
= - \int_{r_0}^{1} \int_{0}^{2\pi} r^2 v^a u_\theta^a (ru v_\theta^a) d\theta dr = \frac{1}{2} \int_{r_0}^{1} \int_{0}^{2\pi} \left( \frac{ru v_\theta^a}{u^a} \right) (ru v_\theta^a) d\theta dr,
\]
\[
I_{13} = \int_{r_0}^{1} \int_{0}^{2\pi} r u v_\theta d\theta dr = - \int_{r_0}^{1} \int_{0}^{2\pi} r (ru v_\theta^a) d\theta dr = 0,
\]
\[
I_{14} = - \int_{r_0}^{1} \int_{0}^{2\pi} \frac{ru v_\theta^a}{r} u_\theta^2 u_\theta d\theta dr \\
= \int_{r_0}^{1} \int_{0}^{2\pi} \frac{ru v_\theta^a}{r} (ru v_\theta^a) d\theta dr = - \frac{1}{2} \int_{r_0}^{1} \int_{0}^{2\pi} \left[ u^a \left( \frac{r v_\theta^a}{u^a} \right) r \right] (ru v_\theta^a) d\theta dr.
\]

Finally, summing the terms \( I_{11}, \ldots, I_{16} \), we obtain
\[
I_1 = \int_{r_0}^{1} \int_{0}^{2\pi} r u_\theta^2 d\theta dr \\
+ \int_{r_0}^{1} \int_{0}^{2\pi} \left[ \frac{1}{2} \left( \frac{ru v_\theta^a}{u^a} \right) - \frac{1}{2} \left[ u^a \left( \frac{r v_\theta^a}{u^a} \right) r - u^a \left( \frac{r}{u^a} \right) r - u^a \left( \frac{r}{u^a} \right) r \right] (ru v_\theta^a) d\theta dr.
\]

Direct computation gives
\[
\frac{1}{2} \left( \frac{ru v_\theta^a}{u^a} \right) r - \frac{1}{2} \left[ u^a \left( \frac{r v_\theta^a}{u^a} \right) r - u^a \left( \frac{r}{u^a} \right) r - u^a \left( \frac{r}{u^a} \right) r \right]
\]
\[
= \left( \frac{ru v_\theta^a}{u^a} \right) r - \left( \frac{u^a}{u^a} \right) \left( \frac{r}{u^a} \right) r
\]
\[
= u^a (u^a + ru v_\theta^a) - r (u^a)^2 - r \left( \frac{u^a}{u^a} \right)^2 - \frac{r}{(u^a)^2} \left( \frac{u^a}{u^a} \right)^2.
\]
Notice that $u^a = u_e^a + \tilde{u}_p^a$, there holds

$$I_1 = \int_{r_0}^{1} \int_{0}^{2\pi} ru_\theta^a d\theta dr + \int_{r_0}^{1} \int_{0}^{2\pi} \frac{1}{u^a} \left( \partial_r u_e^a + r \partial r u_e^a - \frac{u_e^a}{r} \right) (rv)^2 d\theta dr$$

$$= \int_{r_0}^{1} \int_{0}^{2\pi} ru_\theta^a d\theta dr + \int_{r_0}^{1} \int_{0}^{2\pi} \frac{1}{u^a} \left( \partial_r u_e^a + r \partial r u_e^a - \frac{u_e^a}{r} \right) (rv)^2 d\theta dr$$

$$+ \int_{r_0}^{1} \int_{0}^{2\pi} \frac{1}{u^a} \left( \partial_r \tilde{u}_p^a + r \partial r \tilde{u}_p^a - \frac{\tilde{u}_p^a}{r} \right) (rv)^2 d\theta dr$$

$$= \int_{r_0}^{1} \int_{0}^{2\pi} ru_\theta^a d\theta dr + \int_{r_0}^{1} \int_{0}^{2\pi} \frac{1}{u^a} \left( \partial_r u_e^a + r \partial r u_e^a - \frac{u_e^a}{r} \right) (rv)^2 d\theta dr$$

$$+ \int_{r_0}^{1} \int_{0}^{2\pi} \frac{1}{u^a} \left( \partial_r \tilde{u}_p^a + r \partial r \tilde{u}_p^a - \frac{\tilde{u}_p^a}{r} \right) (rv)^2 d\theta dr.$$

By (2.90), (2.91) and (2.92), we deduce that $\frac{1}{u^a} - \frac{1}{u_e} \leq C(\eta + \varepsilon)$, hence

$$\left| \int_{r_0}^{1} \int_{0}^{2\pi} \left( \frac{1}{u^a} - \frac{1}{u_e} \right) \left( \partial_r u_e^a + r \partial r u_e^a - \frac{u_e^a}{r} \right) (rv)^2 d\theta dr \right| \leq C(\eta + \varepsilon) \| rv \|_2^2 \leq C(\eta + \varepsilon) \| u_\theta \|_2^2.$$

Moreover, by the Hardy inequality, (2.91) and (2.92), we deduce that

$$\left| \int_{r_0}^{1} \int_{0}^{2\pi} \frac{1}{u^a} \left( \partial_r u_e^a + r \partial r u_e^a - \frac{u_e^a}{r} \right) (rv)^2 d\theta dr \right|$$

$$= \int_{r_0}^{1} \int_{0}^{2\pi} \frac{(r - 1)^2}{u^a} \left( \partial_r u_e^a + r \partial r u_e^a - \frac{u_e^a}{r} \right) (rv)^2 d\theta dr$$

$$= \int_{r_0}^{1} \int_{0}^{2\pi} \frac{Y^2}{u^a} \left( \varepsilon \theta \partial_r u_e^a + r \partial r \theta u_e^a - \varepsilon^2 \frac{u_e^a}{r} \right) (rv)^2 d\theta dr \leq C(\eta + \varepsilon) \| u_\theta \|_2^2.$$

Similarly, there holds

$$\left| \int_{r_0}^{1} \int_{0}^{2\pi} \frac{1}{u_e^a} \left( \partial_r \tilde{u}_p^a + r \partial r \tilde{u}_p^a - \frac{\tilde{u}_p^a}{r} \right) (rv)^2 d\theta dr \right|$$

$$= \int_{r_0}^{1} \int_{0}^{2\pi} \frac{(r - r_0)^2}{u_e^a} \left( \partial_r \tilde{u}_p^a + r \partial r \tilde{u}_p^a - \frac{\tilde{u}_p^a}{r} \right) (rv)^2 d\theta dr \leq C(\eta + \varepsilon) \| u_\theta \|_2^2.$$

Recall that $\tilde{u}_p^a = \chi(r) u_e^a + (1 - \chi(r)) \tilde{u}_p^a$ and the fast decay of $(u_e^a, \tilde{u}_p^a)$ away from the boundary, we deduce that

$$\left| \int_{r_0}^{1} \int_{0}^{2\pi} \frac{1}{u^a} \left( \partial_r \tilde{u}_p^a + r \partial r \tilde{u}_p^a - \frac{\tilde{u}_p^a}{r} \right) (rv)^2 d\theta dr \right| \leq C(\eta + \varepsilon) \| u_\theta \|_2^2.$$

Thus, we deduce that

$$I_1 \geq \int_{r_0}^{1} \int_{0}^{2\pi} ru_\theta^a d\theta dr + \int_{r_0}^{1} \int_{0}^{2\pi} \frac{1}{u_e^a} \left( \partial_r u_e^a + r \partial r u_e^a - \frac{u_e^a}{r} \right) (rv)^2 d\theta dr - C(\eta + \varepsilon) \| u_\theta \|_2^2$$

$$\geq \int_{r_0}^{1} \int_{0}^{2\pi} ru_\theta^a d\theta dr + \int_{r_0}^{1} \int_{0}^{2\pi} \frac{1}{u_e^a} \left( \partial_r u_e^a + r \partial r u_e^a - \frac{u_e^a}{r} \right) (rv)^2 d\theta dr - C(\eta + \varepsilon) \| u_\theta \|_2^2,$$

where we used the fact $u_e^a(\theta, r) := u_e(r) + \sum_{i=1}^{3} \varepsilon^i \tilde{u}_e^{(i)}(\theta, r)$. 
Finally, summing the estimates for $I_1$ and $I_2$, we obtain

$$\frac{1}{r_0} \int_{r_0}^{2\pi} \left( u^a u_\theta + vr u^a_r + vu^a \right) \left( \frac{r^2 v}{u^a} \right) d\theta dr + \int_{r_0}^{1} \int_{0}^{2\pi} \left( u^a v_\theta - 2uu^a \right) \left( \frac{ru}{u^a} \right) d\theta dr \geq \left( 1 - C \eta - C \varepsilon \right) \int_{r_0}^{1} \int_{0}^{2\pi} r(u^2_\theta + v^2_\theta) d\theta dr$$

$$+ \int_{r_0}^{1} \int_{0}^{2\pi} \frac{1}{r} u \left( \partial_r u_e + r \partial_r v_e - \frac{u_e}{r} \right) (rv)^2 d\theta dr - C \varepsilon^2 (\eta + \varepsilon) \|u_r\|_2.$$ 

Next, we deal with the remainder terms in $S_u$

$$- \int_{r_0}^{1} \int_{0}^{2\pi} \left[ v^a r u_r + uu_\theta + v^a u \right] \left( \frac{r^2 v}{u^a} \right) \frac{r}{u^a} \frac{d\theta dr}{\theta}$$

$$= - \int_{r_0}^{1} \int_{0}^{2\pi} \left[ rv^a u_r + uu_\theta + v^a u \right] \left( \frac{r}{u^a} \right) d\theta dr - \int_{r_0}^{1} \int_{0}^{2\pi} uu_\theta \left( \frac{r}{u^a} \right) d\theta dr$$

and the remainder terms in $S_v$

$$\int_{r_0}^{1} \int_{0}^{2\pi} \left[ v^a r u_r + uu_\theta + v^a u \right] \left( \frac{r^2 v}{u^a} \right) \frac{d\theta dr}{\theta}$$

$$= \int_{r_0}^{1} \int_{0}^{2\pi} \left[ rv^a u_r + uu_\theta + v^a u \right] \left( \frac{r}{u^a} \right) d\theta dr - \int_{r_0}^{1} \int_{0}^{2\pi} uu_\theta \left( \frac{r}{u^a} \right) d\theta dr$$

The remainder terms in $S_u$: By (2.90), (2.91) and (2.92), we have

$$\int_{r_0}^{1} \int_{0}^{2\pi} \left[ v^a r u_r + uu_\theta + v^a u \right] \frac{d\theta dr}{\theta} \leq C \varepsilon (\eta + \varepsilon) \|u_r\|_2 \|u_\theta\|_2,$$

Moreover, by the Hardy inequality, (2.90), (2.91) and (2.92), we deduce

$$\int_{r_0}^{1} \int_{0}^{2\pi} \left[ v^a r u_r + uu_\theta + v^a u \right] \frac{d\theta dr}{\theta} \leq C \varepsilon (\eta + \varepsilon) \|u_r\|_2 \|u_\theta\|_2,$$
The same argument gives thus, there holds
\[ \int_{r_0}^{1} \int_{0}^{2\pi} \frac{r^2 v u_a r \partial_r u_a^a}{(u^a)^2} d\theta dr \leq C \varepsilon (\eta + \varepsilon) \| u_r \|_2 \| u_\theta \|_2, \]

Finally, we obtain
\[ |I_3^u| \leq (\eta + \varepsilon) \int_{r_0}^{1} \int_{0}^{2\pi} r u_\theta^2 d\theta dr + C \varepsilon^2 (\eta + \varepsilon) \int_{r_0}^{1} \int_{0}^{2\pi} u_r^2 d\theta dr. \]

The same argument gives
\[ |I_4^u| \leq (\eta + \varepsilon) \int_{r_0}^{1} \int_{0}^{2\pi} r u_\theta^2 d\theta dr + C \varepsilon^2 (\eta + \varepsilon) \int_{r_0}^{1} \int_{0}^{2\pi} u_r^2 d\theta dr. \]

For \( I_2^u \), the same argument as above gives
\[ \int_{r_0}^{1} \int_{0}^{2\pi} \frac{u u_\theta r u_\theta^a}{(u^a)^2} d\theta dr \leq C \varepsilon (\eta + \varepsilon) \| u_r \|_2 \| u_\theta \|_2. \]

Moreover, direct computation gives
\[ \int_{r_0}^{1} \int_{0}^{2\pi} u_\theta^2 (\frac{r}{u^a}) r v d\theta dr = \int_{r_0}^{1} \int_{0}^{2\pi} \left( \frac{u u_\theta^2 r v}{u^a} - \frac{u u_\theta^2 r v}{(u^a)^2} \right) d\theta dr. \]

The same argument as above gives
\[ \int_{r_0}^{1} \int_{0}^{2\pi} \frac{u u_\theta^2 r v}{(u^a)^2} d\theta dr \leq C \varepsilon (\eta + \varepsilon) \| u_r \|_2 \| u_\theta \|_2. \]

By the Hardy inequality, (2.40), (2.91) and (2.92), we deduce that
\[ \int_{r_0}^{1} \int_{0}^{2\pi} \frac{u u_\theta^2 r v \cdot r \partial_r u_e^a}{(u^a)^2} d\theta dr \leq C \varepsilon (\eta + \varepsilon) \| u_r \|_2 \| u_\theta \|_2. \]

Furthermore, using the above argument and noting the fact \( \| \partial_r u_e^a \|_2 \leq C \varepsilon (\eta + \varepsilon) \), there holds
\[ \int_{r_0}^{1} \int_{0}^{2\pi} \frac{u u_\theta^2 r v \cdot r \partial_r u_e^a}{(u^a)^2} d\theta dr \leq C \varepsilon (\eta + \varepsilon) \| u_r \|_2 \| u_\theta \|_2. \]

Finally, we obtain
\[ |I_2^h| \leq (\eta + \varepsilon) \int_{r_0}^{1} \int_{0}^{2\pi} r (u_\theta^2 + v_\theta^2) d\theta dr + C \varepsilon^2 (\eta + \varepsilon) \int_{r_0}^{1} \int_{0}^{2\pi} u_r^2 d\theta dr. \]

Thus, there holds
\[ \int_{r_0}^{1} \int_{0}^{2\pi} \left[ v^a r u_r + uu_\theta^a + v^a u \right] \frac{(r^2 v^a)}{u^a} d\theta dr \leq C (\eta + \varepsilon) \int_{r_0}^{1} \int_{0}^{2\pi} r (u_\theta^2 + v_\theta^2) d\theta dr + C \varepsilon^2 (\eta + \varepsilon) \int_{r_0}^{1} \int_{0}^{2\pi} u_r^2 d\theta dr. \]

(3.12)
The remainder terms in $S_c$: Using $v^a = O(\varepsilon (\eta + \varepsilon))$, we deduce that

$$|I_1^v| \leq C \varepsilon (\eta + \varepsilon) \|u_\theta\|_2 \|v_\theta\|_2,$$

For $I_3^v$, using the Hardy inequality, (2.90), (2.91) and (2.92), we deduce

$$\left| \int_{r_0}^{1} \int_{0}^{2\pi} r \partial_r v^a_e v^a_{\theta} \overline{v_\theta} \overline{r} d\theta dr \right| \leq C \varepsilon (\eta + \varepsilon) \|u_\theta\|_2 \|v_\theta\|_2,$$

$$\left| \int_{r_0}^{1} \int_{0}^{2\pi} r \partial_r v^a_e v^a_{\theta} \overline{v_\theta} \overline{r} d\theta dr \right| = \left| \int_{r_0}^{1} \int_{0}^{2\pi} (r - 1) \partial_r v^a_e \overline{v_\theta} \overline{r} - 1 \overline{u^a}_d d\theta dr \right| \leq C \varepsilon (\eta + \varepsilon) \|u_\theta\|_2 \|v_\theta\|_2,$$

$$\left| \int_{r_0}^{1} \int_{0}^{2\pi} r \partial_r v^a_e v^a_{\theta} \overline{v_\theta} \overline{r} d\theta dr \right| = \left| \int_{r_0}^{1} \int_{0}^{2\pi} Y \partial_r v^a_e \overline{v_\theta} \overline{r} - 1 \overline{u^a}_d d\theta dr \right| \leq C \varepsilon (\eta + \varepsilon) \|u_\theta\|_2 \|v_\theta\|_2,$$

hence there holds

$$\left| \int_{r_0}^{1} \int_{0}^{2\pi} r \partial_r v^a_e v^a_{\theta} \overline{v_\theta} \overline{r} d\theta dr \right| \leq C \varepsilon (\eta + \varepsilon) \|u_\theta\|_2 \|v_\theta\|_2.$$

The same argument gives

$$\left| \int_{r_0}^{1} \int_{0}^{2\pi} r v^a_e v^a_{\theta} \overline{v_\theta} \overline{r} \overline{u^a_d} d\theta dr \right| \leq C \varepsilon (\eta + \varepsilon) \|u_\theta\|_2^2.$$

Collecting these estimates, we get

$$|I_3^v| \leq C \varepsilon (\eta + \varepsilon) \|u_\theta\|_2 (\|u_\theta\|_2 + \|v_\theta\|_2).$$

Thus, we obtain

$$\left| \int_{r_0}^{1} \int_{0}^{2\pi} \left[ v^a rv_r + u v^a_\theta + v r v_r \right] \left( \frac{r v}{u^a} \right)_\theta \overline{v_\theta} \overline{r} d\theta dr \right| \leq C \varepsilon (\eta + \varepsilon) \int_{r_0}^{1} \int_{0}^{2\pi} r (u^2_\theta + v^2_\theta) d\theta + C \varepsilon^2 (\eta + \varepsilon) \int_{r_0}^{1} \int_{0}^{2\pi} u^2_r d\theta d\theta. \tag{3.13}$$

Finally, collecting (3.12) and (3.13), we can choose $\varepsilon_0 > 0$, $\eta_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\eta \in (0, \eta_0)$, there holds

$$\left| \int_{r_0}^{1} \int_{0}^{2\pi} S_u \left( \frac{r^2 v}{u^a} \right)_r d\theta dr \right| + \int_{r_0}^{1} \int_{0}^{2\pi} S_v \left( \frac{r v}{u^a} \right)_\theta d\theta dr \geq \left( 1 - C \eta - C \varepsilon \right) \int_{r_0}^{1} \int_{0}^{2\pi} r (u^2_\theta + v^2_\theta) d\theta d\theta \tag{3.14}$$

Pressure estimate: Integrating by parts, we deduce that

$$- \int_{r_0}^{1} \int_{0}^{2\pi} p_\theta \left( \frac{r^2 v}{u^a} \right)_r d\theta dr + \int_{r_0}^{1} \int_{0}^{2\pi} r p_r \left( \frac{r v}{u^a} \right)_\theta d\theta dr = \int_{r_0}^{1} \int_{0}^{2\pi} p_r \left( \frac{r v}{u^a} \right)_r d\theta dr - \int_{r_0}^{1} \int_{0}^{2\pi} p_{\theta r} \left( \frac{r^2 v}{u^a} \right)_\theta d\theta dr = 0. \tag{3.15}$$

Diffusion term: Finally, we deal with the diffusion term:

$$- \int_{r_0}^{1} \int_{0}^{2\pi} \left[ - \varepsilon^2 (ru_{rr} + \frac{u_\theta \theta}{r} + 2 \frac{u_\theta}{r} + u_r - \frac{u}{r}) \right] \left( \frac{r^2 v}{u^a} \right)_r d\theta dr.$$
and

\[ \int_0^{2\pi} \int_{r_0}^{r_1} \frac{1}{r} \frac{r^2(v^2)_{rr}}{u^2} \, dr \, d\theta = \int_0^{2\pi} \int_{r_0}^{r_1} r(r^2v_{rr})_{r} + rv \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_{r_0}^{r_1} \frac{1}{r} \frac{r^2v_{rr}}{u^2} \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_{r_0}^{r_1} \frac{1}{r} \frac{v^2}{u^2} \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_{r_0}^{r_1} \left( \frac{1}{2u^2} \right) \, dr \, d\theta \]

\[ \leq C \int_0^{2\pi} \int_{r_0}^{r_1} (\frac{u^2}{2} + v^2) \, dr \, d\theta \]

and

\[ \int_0^{2\pi} \int_{r_0}^{r_1} \frac{1}{r} \frac{r^2}{u^2} \, dr \, d\theta = \int_0^{2\pi} \int_{r_0}^{r_1} \frac{r^2v_{rr}}{u^2} \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_{r_0}^{r_1} \frac{1}{r} \frac{r^2v_{rr}}{u^2} \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_{r_0}^{r_1} \frac{1}{r} \frac{r^2v_{rr}}{u^2} \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_{r_0}^{r_1} \left( \frac{1}{2u^2} \right) \, dr \, d\theta \]

\[ \leq \frac{\eta}{\varepsilon^2} \|ru\|_2^2 + C\eta \|u_r\|_2^2 + \left( \int_0^{2\pi} \int_{r_0}^{r_1} \frac{1}{r} \frac{r^2v_{rr}}{u^2} \, dr \, d\theta \right) \]

\[ \leq \frac{\eta}{\varepsilon^2} \|ru\|_2^2 + C\eta \|u_r\|_2^2. \]

The other low order terms can be easily estimated as follows

\[ \left| \int_0^{2\pi} \int_{r_0}^{r_1} (u_r + \frac{2}{r} v_\theta - \frac{1}{r} u) \frac{(r^2v)_r}{u^2} \, dr \, d\theta \right| \]

\[ \leq \frac{1}{\varepsilon} \int_0^{2\pi} \int_{r_0}^{r_1} r(u^2_\theta + v^2_\theta) \, dr \, d\theta + C\varepsilon \int_0^{2\pi} \int_{r_0}^{r_1} u^2_r \, dr \, d\theta. \]
Thus,

\[ |I_1| \leq C(\eta + \varepsilon) \int_0^{2\pi} \int_{r_0}^1 r(u_0^2 + v_0^2)drd\theta + C\varepsilon^2(\eta + \varepsilon) \int_0^{2\pi} \int_{r_0}^1 u_r^2 drd\theta. \]

By the same argument as above, we can deduce the following estimate for \( I_2 \):

\[ |I_2| = \varepsilon^2 \int_0^{2\pi} \int_{r_0}^1 \left| \frac{1}{r} u_{r\theta} + ru_{rr} + u_r + \frac{2}{r} v_{\theta} - \frac{1}{r} u \right| r^2 v_{\theta} \frac{u_r}{u^2} drd\theta \]
\[ \leq C(\eta + \varepsilon) \int_0^{2\pi} \int_{r_0}^1 (u_0^2 + v_0^2)drd\theta + C\varepsilon^2 \int_0^{2\pi} \int_{r_0}^1 u_r^2 drd\theta. \]

Summing the estimates of \( I_1 \) and \( I_2 \), we obtain that

\[ |I_1| + |I_2| \leq C(\eta + \varepsilon)\|\sqrt{r}(u_0, v_0)\|_2^2 + C\varepsilon^2(\eta + \varepsilon)\|u_r\|_2^2. \]

For \( I_3 \), the high order terms can be estimated by integrating by parts:

\[ \left| \int_0^{2\pi} \int_{r_0}^1 \left( \frac{1}{r} v_{r\theta} + rv_{r\theta} \right) \frac{ru_{\theta}}{u^2} drd\theta \right| \leq \frac{C}{\varepsilon}\|\sqrt{r}(u_\theta, v_\theta)\|_2^2. \]

The lower order terms can be estimated by the divergence-free condition and (2.92)

\[ \left| \int_0^{2\pi} \int_{r_0}^1 (v_r - \frac{2}{r} u_\theta - \frac{1}{r} v) \frac{r u_{\theta}}{u^2} drd\theta \right| \leq C \int_0^{2\pi} \int_{r_0}^1 (u_0^2 + v_0^2)drd\theta. \]

Thus, we obtain

\[ |I_3| \leq C\varepsilon\|\sqrt{r}(u_\theta, v_\theta)\|_2^2. \]

For \( I_4 \), the same argument as above gives that

\[ |I_4| \leq C\varepsilon^2\|\sqrt{r}(u_\theta, v_\theta)\|_2^2. \]

Finally, we can choose \( \varepsilon_0 > 0, \eta_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0) \), there holds

\[ |I_3| + |I_4| \leq C\varepsilon\|\sqrt{r}(u_\theta, v_\theta)\|_2^2. \]

Hence, we deduce that there exist \( \varepsilon_0 > 0, \eta_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0) \), the diffusive term (3.16) can be controlled by

\[ C(\eta + \varepsilon) \int_0^{2\pi} \int_{r_0}^1 (u_0^2 + v_0^2)drd\theta + C\varepsilon^2(\eta + \varepsilon)\|u_r\|_2^2 \]

which combining with the estimates (3.14) and (3.15) implies that there exist \( \varepsilon_0 > 0, \eta_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0) \), the lift hand side of (3.11) has a lower bound

\[ \left( 1 - C\eta - C\varepsilon \right) \int_0^{2\pi} \int_{r_0}^1 (u_0^2 + v_0^2)drd\theta + \int_0^{2\pi} \int_{r_0}^1 \left( \frac{1}{u_e} \left( \partial_r u_e + ru \partial r u_e - \frac{u_e}{r} \right) \right) (rv)^2 d\theta dr - C\varepsilon^2(\eta + \varepsilon)\|u_r\|_2^2. \]

Let \( g = \frac{ru}{u_e} \), then

\[ \int_0^{2\pi} \int_{r_0}^1 (u_0^2 + v_0^2)drd\theta = \int_0^{2\pi} \int_{r_0}^1 \left( \frac{ru}{u_e} \right)^2 (u_e g^2 + \frac{u_e^2}{g^2}) drd\theta \]

\[ = \int_0^{2\pi} \int_{r_0}^1 \left( ru_e g^2 + \frac{u_e^2}{g^2} \right) drd\theta - \int_0^{2\pi} \int_{r_0}^1 u_e \left( \partial_r u_e + ru \partial r u_e \right) g^2 drd\theta. \]

Thus,

\[ \left( 1 - C\eta - C\varepsilon \right) \int_0^{2\pi} \int_{r_0}^1 (u_0^2 + v_0^2)drd\theta + \int_0^{2\pi} \int_{r_0}^1 \left( \frac{1}{u_e} \left( \partial_r u_e + ru \partial r u_e - \frac{u_e}{r} \right) \right) (rv)^2 d\theta dr \]
By the Poincaré inequality,
\[
\int_0^{2\pi} \int_{r_0}^1 \left( r u_e g_r^2 + \frac{u_e^2}{r} g_\theta^2 - \frac{u_e^2}{r} g_r^2 \right) dr d\theta \geq C(\eta + \varepsilon) \int_0^{2\pi} \int_{r_0}^1 r(u_\theta^2 + v_\theta^2) dr d\theta.
\]

Thus, we directly estimated as follows:
\[
\int_0^{2\pi} \int_{r_0}^1 \left( r u_e g_r^2 + \frac{u_e^2}{r} g_\theta^2 - \frac{u_e^2}{r} g_r^2 \right) dr d\theta \geq C \int_0^{2\pi} \int_{r_0}^1 \frac{u_e^2}{r} g_r^2 dr d\theta,
\]
where we used the fact that \( u_e \) has a positive lower bound and upper bound in the second inequality.

Thus
\[
\int_0^{2\pi} \int_{r_0}^1 \left( \frac{1}{2} r u_e g_r^2 + \frac{u_e^2}{2C} g_r^2 + \frac{u_e^2}{2C} g_\theta^2 + \left( 1 - \frac{1}{2C} \right) \frac{u_e^2}{r} g - \frac{u_e^2}{r} g_r^2 \right) dr d\theta
\]
\[
= \int_0^{2\pi} \int_{r_0}^1 \left( \frac{1}{2} r u_e g_r^2 + \frac{u_e^2}{2C} g_\theta^2 \right) dr d\theta
\]
\[
\geq \tilde{C} \int_0^{2\pi} \int_{r_0}^1 r(u_\theta^2 + v_\theta^2) dr d\theta,
\]
where we used
\[
|u_\theta|^2 = |(rv)_r|^2 = |(gu_e)_r|^2 \leq C(g^2 + g_e^2).
\]

So (3.17) has a lower bound
\[
\left( \tilde{C} - C\eta - C\varepsilon \right) \int_0^{2\pi} \int_{r_0}^1 r(u_\theta^2 + v_\theta^2) dr d\theta - C\varepsilon^2(\eta + \varepsilon)\|u_r\|^2_2.
\]

(3.18)

Thanks to the Young inequality and Poincaré inequality, the right hand side of (3.11) can be directly estimated as follows:
\[
\left| \int_0^{2\pi} \int_{r_0}^1 \left[ F_1 \left( \frac{r^2 v}{u^a} \right)_r + F_2 \left( \frac{r^2 v}{u^a} \right)_g \right] d\theta dr \right|
\]
\[
= \left| \int_0^{2\pi} \int_{r_0}^1 \left[ F_1 \left( \frac{r(v)_r + rv}{u^a} \right) - \frac{r^2 v u_r^a}{(u^a)^2} \right] + F_2 \left( \frac{rv_\theta}{u^a} - \frac{rv u_\theta^a}{(u^a)^2} \right) \right] d\theta dr \right|
\]
\[
\leq \frac{\tilde{C}}{4} \int_0^{2\pi} \int_{r_0}^1 r(u_\theta^2 + v_\theta^2) dr d\theta + C\|F_1, F_2\|^2_2.
\]

(3.19)

Finally, collecting the estimate (3.18) and (3.19) together, we obtain (3.10), this complete the proof of this lemma.

\[\Box\]

4. Existence of error equations

To deal with the nonlinear term and close the estimate, we need to estimate \( \|(u, v)\|_\infty \). To this end, we consider the following \( \mathcal{H}^2 \) estimate for a Stokes system.
4.1. “$H^2$ estimate” for Stokes system.

We consider the Stoke equations

$$
\begin{align*}
\begin{cases}
-\varepsilon^2 \left( r u_{rr} + \frac{u_{\theta \theta}}{r} + 2 \frac{v_{\theta}}{r} + u_r - \frac{u}{r} \right) + p_\theta = f_u,
-\varepsilon^2 \left( r v_{rr} + \frac{v_{\theta \theta}}{r} - 2 \frac{u_{\theta}}{r} + v_r - \frac{v}{r} \right) + r p_r = g_v,
\end{cases}
\end{align*}
\tag{4.1}
$$

where $u(\theta, r) = u(\theta + 2\pi, r)$, $v(\theta, r) = v(\theta + 2\pi, r)$, $u(\theta, 0) = 0$, $v(\theta, 0) = 0$.

Lemma 4.1. Let $(u, v)$ be a smooth solution of \((4.1)\), then we have

$$
\int_{r_0}^{1} \int_{0}^{2\pi} [(u_{\theta \theta} + v_{\theta})^2 + (v_{\theta \theta} - u_{\theta})^2] d\theta dr + \int_{r_0}^{1} \int_{0}^{2\pi} (u_{r \theta}^2 + v_{r \theta}^2) d\theta dr
\leq \frac{C}{\varepsilon^2} \int_{r_0}^{1} \int_{0}^{2\pi} [f_u u_{\theta \theta} + g_v v_{\theta \theta}] dr d\theta.
\tag{4.2}
$$

Proof. Multiplying $u_{\theta \theta}$ for the first equation and $v_{\theta \theta}$ for the second equation in \((4.1)\), integrating in $\Omega$ and summing two terms together, we arrive at

$$
\begin{align*}
\int_{r_0}^{1} \int_{0}^{2\pi} \left[ -\varepsilon^2 \left( r u_{rr} + \frac{u_{\theta \theta}}{r} + 2 \frac{v_{\theta}}{r} + u_r - \frac{u}{r} \right) + p_\theta \right] u_{\theta \theta} dr d\theta \\
+ \int_{r_0}^{1} \int_{0}^{2\pi} \left[ -\varepsilon^2 \left( r v_{rr} + \frac{v_{\theta \theta}}{r} - 2 \frac{u_{\theta}}{r} + v_r - \frac{v}{r} \right) + r p_r \right] v_{\theta \theta} dr d\theta \\
= \int_{r_0}^{1} \int_{0}^{2\pi} [f_u u_{\theta \theta} + g_v v_{\theta \theta}] dr d\theta.
\end{align*}
$$

Integrating by parts and using the divergence-free condition, we deduce

$$
\int_{r_0}^{1} \int_{0}^{2\pi} [p_\theta u_{\theta \theta} + p_r v_{\theta \theta}] dr d\theta = 0.
$$

Then, we compute the diffusive term. Integrating by parts, we obtain

$$
\begin{align*}
\int_{r_0}^{1} \int_{0}^{2\pi} -\varepsilon^2 u_{rr} u_{\theta \theta} dr d\theta &= -\varepsilon^2 \int_{r_0}^{1} \int_{0}^{2\pi} u_{r \theta} (ru_{\theta})_r dr d\theta = -\varepsilon^2 \int_{r_0}^{1} \int_{0}^{2\pi} r u_{r \theta}^2 dr d\theta, \\
\int_{r_0}^{1} \int_{0}^{2\pi} -\varepsilon^2 v_{rr} v_{\theta \theta} dr d\theta &= -\varepsilon^2 \int_{r_0}^{1} \int_{0}^{2\pi} v_{r \theta} (rv_{\theta})_r dr d\theta = -\varepsilon^2 \int_{r_0}^{1} \int_{0}^{2\pi} r v_{r \theta}^2 dr d\theta,
\end{align*}
$$

and

$$
\begin{align*}
\int_{r_0}^{1} \int_{0}^{2\pi} \varepsilon^2 u_r u_{\theta \theta} dr d\theta &= \int_{r_0}^{1} \int_{0}^{2\pi} \varepsilon^2 v_r v_{\theta \theta} dr d\theta = 0, \\
\int_{r_0}^{1} \int_{0}^{2\pi} \varepsilon^2 u_{r \theta} u_{\theta \theta} dr d\theta + \int_{r_0}^{1} \int_{0}^{2\pi} \varepsilon^2 v_{r \theta} v_{\theta \theta} dr d\theta = -\varepsilon^2 \int_{r_0}^{1} \int_{0}^{2\pi} \frac{u_{\theta}^2 + v_{\theta}^2}{r} dr d\theta.
\end{align*}
$$

Summing all terms, we obtain

$$
\begin{align*}
\int_{r_0}^{1} \int_{0}^{2\pi} \left[ -\varepsilon^2 \left( r u_{rr} + \frac{u_{\theta \theta}}{r} + 2 \frac{v_{\theta}}{r} + u_r - \frac{u}{r} \right) + p_\theta \right] u_{\theta \theta} dr d\theta \\
+ \int_{r_0}^{1} \int_{0}^{2\pi} \left[ -\varepsilon^2 \left( r v_{rr} + \frac{v_{\theta \theta}}{r} - 2 \frac{u_{\theta}}{r} + v_r - \frac{v}{r} \right) + r p_r \right] v_{\theta \theta} dr d\theta
\leq \frac{C}{\varepsilon^2} \int_{r_0}^{1} \int_{0}^{2\pi} [f_u u_{\theta \theta} + g_v v_{\theta \theta}] dr d\theta.
\end{align*}
$$
Hence, we obtain (4.2) and then complete the proof of this lemma. □

4.2. Anisotropic Sobolev embedding.

To obtain the $L^\infty$ estimate, we need the following anisotropic Sobolev embedding.

**Lemma 4.2.** Let $(u, v)$ be smooth function and satisfy $(u, v)|_{r=1} = (u, v)|_{r=r_0} = (0, 0)$, then
\[
\| (u, v) \|_\infty \leq C(\| u_\theta, v_\theta \|_2 + \| u_r, v_r \|_2 + \| u_{r\theta}, v_{r\theta} \|_2).
\] (4.3)

**Proof.** In fact, let
\[
 u(\theta, r) = \sum_{k \in Z} u_k(r)e^{ik\theta}, \quad \forall r \in [r_0, 1],
\]
thus
\[
 |u(\theta, r)| \leq \sum_{k \in Z} |u_k(r)|, \quad \forall (\theta, r) \in [0, 2\pi] \times [r_0, 1].
\]
Notice that $u(\theta, r_0) = 0$, $\forall \theta \in [0, 2\pi]$ gives $u_k(\theta, r_0) = 0$, $\forall \theta \in [0, 2\pi], k \in Z$, hence
\[
 \| u_k \|_\infty \leq \sqrt{2} \| u_k \|_2 \| u_k' \|_2^\frac{1}{2}.
\]
Thus,
\[
 \sum_{k \neq 0} \| u_k \|_\infty \leq \sqrt{2} \sum_{k \neq 0} \| u_k \|_2 \| u_k' \|_2 \leq C\left( \sum_{k \neq 0} |k|^2 \| u_k \|_2^2 \right)^\frac{1}{2} \left( \sum_{k \neq 0} |k|^2 \| u_k' \|_2^2 \right)^\frac{1}{2}.
\]
Moreover, it’s easy to get
\[
 u_\theta(\theta, r) = \sum_{k \in Z} iku_k(r)e^{ik\theta}, \quad u_{r\theta}(\theta, r) = \sum_{k \in Z} ik u_k'(r)e^{ik\theta}, \quad \forall r \in [r_0, 1].
\]
Thus,
\[
 \| u_\theta \|_2^2 = \sum_{k \in Z} |k|^2 \| u_k \|_2^2, \quad \| u_{r\theta} \|_2^2 = \sum_{k \in Z} |k|^2 \| u_k' \|_2^2.
\]
Hence, we obtain
\[
 \sum_{k \neq 0} \| u_k \|_\infty \leq C \| u_\theta \|_2^\frac{1}{2} \| u_{r\theta} \|_2^\frac{1}{2}.
\]
Moreover, due to $u_0(r) = \frac{1}{2\pi} \int_{0}^{2\pi} u(\theta, r)d\theta$ and $u(\theta, r_0) = 0$, $\forall \theta \in [0, 2\pi]$, it’s easy to get
\[
 |u_0(r)| \leq C \| u_r \|_2, \quad \forall r \in [r_0, 1].
\]
Finally, we obtain
\[
 \| u \|_\infty \leq C(\| u_\theta \|_2 + \| u_r \|_2 + \| u_{r\theta} \|_2).
\]
Similarly we can obtain the same result for $v$ and hence we complete the proof of this lemma. □
4.3. Existence for the error equations.

We apply the contraction mapping theorem to prove the existence for the error equations (3.2).

**Proposition 4.3.** There exist \( \varepsilon_0 > 0 \), \( \eta_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0) \), the error equations (3.2) have a unique solution \((u, v)\) which satisfies

\[
\|(u, v)\|_\infty \leq C \varepsilon.
\]

**Proof.** For each smooth function \((u, v)\) which satisfies

\[
\begin{cases}
  u_\theta + (rv)_r = 0, \\
  u(\theta, r) = u(\theta + 2\pi, r), \ v(\theta, r) = v(\theta + 2\pi, r), \\
  u(\theta, 1) = 0, \ u(\theta, r_0) = 0, \\
  v(\theta, 1) = 0, \ v(\theta, r_0) = 0,
\end{cases}
\]

we consider the following linear problem:

\[
\begin{cases}
  -\varepsilon^2 (ru_{rr} + \frac{a_{u\theta}}{2} + 2\frac{u_{\theta\theta}}{2} + u_r - \frac{u}{r}) + \bar{P}_\theta + S_\theta = R_u, \\
  -\varepsilon^2 (rv_{rr} + \frac{a_{v\theta}}{2} + 2\frac{v_{\theta\theta}}{2} + v_r - \frac{v}{r}) + \bar{P}_r + S_\theta = R_v, \\
  \partial_\theta u + \partial_r (rv) = 0, \\
  \bar{u}(\theta, r) = \bar{u}(\theta + 2\pi, r), \ \bar{v}(\theta, r) = \bar{v}(\theta + 2\pi, r), \\
  \bar{u}(\theta, 1) = 0, \ \bar{v}(\theta, 1) = 0, \\
  \bar{u}(\theta, r_0) = 0, \ \bar{v}(\theta, r_0) = 0,
\end{cases}
\]

where

\[
S_\theta := u^a \bar{u}_\theta + v^a \bar{r}_r + \bar{u} u^a_{\theta\theta} + v^a \bar{r} + \bar{v} u^a,
\]

\[
S_\theta := u^a \bar{v}_\theta + v^a \bar{r}_r + \bar{u} v^a_{\theta\theta} + v^a \bar{r} - 2 \bar{u} u^a,
\]

\[
R_u := R^a_u - uu_{\theta} - vv_{rr} - vv, \quad R_v := R^a_v - uu_{\theta} - vv_{rr} - uv^2.
\]

By linear stability estimate (3.2), we deduce that there exist \( \varepsilon_0 > 0 \), \( \eta_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0) \), there holds

\[
\|(u_\theta, v_\theta)\|^2 + \varepsilon^2 \|u_r\|^2 \leq C\|(R_u, R_v)\|^2 + \int_{r_0}^{1} \int_{0}^{2\pi} (R_u \bar{u} + R_v \bar{v}) d\theta dr.
\]

Direct computation gives that

\[
\|(R_u, R_v)\|^2 \leq C\|(R^a_u, R^a_v)\|^2 + C\|(u, v)\|^2 \|(u_r, u_\theta, v_\theta)\|^2.
\]

Moreover, there hold

\[
\int_{r_0}^{1} \int_{0}^{2\pi} R_u \bar{u} d\theta dr = \int_{r_0}^{1} \int_{0}^{2\pi} [R^a_u \bar{u} - uu_\theta \bar{u} - rvu_r \bar{u} - vu \bar{u}] d\theta dr \leq C\|u_r\|_2 \|R^a_u\|_2 + C\|u_\theta\|_2 \|u\|_\infty + C\|u_r\|_2 \|u_\theta\|_2 \|u\|_\infty,
\]

\[
\int_{r_0}^{1} \int_{0}^{2\pi} R_v \bar{v} d\theta dr = \int_{r_0}^{1} \int_{0}^{2\pi} [R^a_v \bar{v} - uu_\theta \bar{v} - rvv_r \bar{v} - u^2 \bar{v}] d\theta dr \leq C\|u_\theta\|_2 \|R^a_u\|_2 + C\|u_\theta\|_2 \|v_\theta\|_2 \|u\|_\infty + C\|u_\theta\|_2 \|u_\theta\|_2 \|v\|_\infty
\]

\[
+ C\|u_\theta\|_2 \|u_r\|_2 \|u\|_\infty.
\]
By the Stoke estimate (4.2), we deduce
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} [\bar{u}R_u + \bar{v}R_v] d\theta dr \right| \leq \frac{1}{2} \|(\bar{u}_\theta, \bar{v}_\theta, \varepsilon \bar{u}_r)\|_2^2 + C\varepsilon^{-2} \|(R_u^a, R_v^a)\|_2^2 + C\varepsilon^{-2} \|(u_\theta, v_\theta, \varepsilon u_r)\|_2^2 \|(u, v)\|_\infty^2.
\]
Thus, there holds
\[
\|(\bar{u}_\theta, \bar{v}_\theta, \varepsilon \bar{u}_r)\|_2^2 \leq C\varepsilon^{-2} \|(R_u^a, R_v^a)\|_2^2 + C\varepsilon^{-2} \|(u_\theta, v_\theta, \varepsilon u_r)\|_2^2 \|(u, v)\|_\infty^2. \tag{4.5}
\]
Then, we consider the following Stokes problem:
\[
\begin{cases}
-\varepsilon^2 (r \ddot{u}_r + \ddot{u}_\theta + \frac{2}{r} \ddot{u} - \frac{\ddot{u}}{r}) + \ddot{p}_\theta = R_u - S\bar{u}, \\
-\varepsilon^2 (r \ddot{v}_r + \ddot{v}_\theta + \frac{2}{r} \ddot{v} - \frac{\ddot{v}}{r}) + r \ddot{p}_r = R_v - S\bar{v}, \\
\partial_\theta \bar{u} + \partial_r (r \bar{v}) = 0, \\
\bar{u}(\theta, r) = \bar{u}(\theta + 2\pi, r), \quad \bar{v}(\theta, r) = \bar{v}(\theta + 2\pi, r), \\
\bar{u}(\theta, 1) = 0, \quad \bar{v}(\theta, 1) = 0, \\
\bar{u}(\theta, r_0) = 0, \quad \bar{v}(\theta, r_0) = 0.
\end{cases}
\]
By the Stoke estimate (4.2), we deduce
\[
\|(\bar{u}_\theta, \bar{v}_\theta, \varepsilon \bar{u}_r)\|_2^2 + \|(\bar{u}_r, \bar{v}_r)\|_2^2 \leq \frac{C}{\varepsilon^2} \int_{r_0}^{1} \int_{0}^{2\pi} [(R_u - S\bar{u})\bar{u}_\theta + (R_v - S\bar{v})\bar{v}_\theta] d\theta dr.
\]
We compute the right hand term by terms.

1) The term \(\int_{r_0}^{1} \int_{0}^{2\pi} R_u \bar{u}_\theta d\theta dr\) and \(\int_{r_0}^{1} \int_{0}^{2\pi} R_v \bar{v}_\theta d\theta dr\): First, there holds
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} R_u \bar{u}_\theta d\theta dr \right| \leq \int_{r_0}^{1} \int_{0}^{2\pi} \left| (R_u^a \bar{u}_\theta - uu_\theta \bar{u}_\theta - rvu_r \bar{u}_\theta - vvu_\theta) \right| d\theta dr,
\]
It is easy to obtain
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} (uu_\theta \bar{u}_\theta + vvu_\theta) d\theta dr \right| \leq \frac{\varepsilon^2}{10C} \|(\bar{u}_\theta, \bar{v}_\theta)\|_2^2 + \|\bar{u}_r\|_2^2 + \|\bar{v}_r\|_2^2 + \frac{C}{\varepsilon^2} \|(u, \theta)\|_\infty^2 \|(u, \theta)\|_2^2.
\]
Integrating by part, we deduce that
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} rvvu_r \bar{u}_\theta d\theta dr \right| \leq \frac{\varepsilon^2}{10C} \|\bar{u}_r\|_2^2 + \frac{C}{\varepsilon^2} \|(u, v)\|_{\infty}^2 \|(u_\theta, v_\theta)\|_2^2 + \int_{r_0}^{1} \int_{0}^{2\pi} \left| (uu_\theta \bar{u}_\theta + vvu_\theta) \right| d\theta dr
\]
\[
\leq \frac{\varepsilon^2}{10C} \|\bar{u}_\theta\|_2^2 + \frac{C}{\varepsilon^2} \|(u, v)\|_{\infty}^2 \|(u_\theta, v_\theta)\|_2^2 + \int_{r_0}^{1} \int_{0}^{2\pi} uu_\theta \bar{u}_\theta d\theta dr
\]
\[
\leq \frac{\varepsilon^2}{10C} \|\bar{u}_\theta\|_2^2 + \frac{C}{\varepsilon^2} \|(u, v)\|_{\infty}^2 \|(u_\theta, v_\theta)\|_2^2 + \int_{r_0}^{1} \int_{0}^{2\pi} uu_\theta \bar{u}_\theta d\theta dr.
\]
Thus, we obtain
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} R_u \bar{u}_\theta \bar{d} \theta d\theta dr \right| \leq C \left\| \bar{u}_\theta \right\|_2 \left\| \partial_\theta R_u^a \right\|_2 \\
+ \frac{\varepsilon^2}{10C} \left\| \bar{v}_\theta + \bar{v}_\theta \right\|_2^2 + \frac{\varepsilon^2}{10C} \left\| \left( \bar{u}_r, \bar{v}_\theta \right) \right\|_2^2 + \frac{C}{\varepsilon^2} \left\| (u, v) \right\|_\infty \left\| (u_\theta, v_\theta) \right\|_2^2.
\]
Similarly,
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} R_v \bar{v}_\theta \bar{d} \theta d\theta dr \right| \leq C \left\| \bar{v}_\theta \right\|_2 \left\| \partial_\theta R_u^a \right\|_2 \\
+ \frac{\varepsilon^2}{10C} \left\| \bar{v}_\theta - \bar{u}_\theta \right\|_2^2 + \frac{\varepsilon^2}{10C} \left\| \left( \bar{u}_r, \bar{v}_\theta \right) \right\|_2^2 + \frac{C}{\varepsilon^2} \left\| (u, v) \right\|_\infty \left\| (u_\theta, v_\theta) \right\|_2^2.
\]
2) The term \( \int_{r_0}^{1} \int_{0}^{2\pi} S_u \bar{u}_\theta \bar{d} \theta d\theta dr \) and \( \int_{r_0}^{1} \int_{0}^{2\pi} S_v \bar{v}_\theta \bar{d} \theta d\theta dr \): 

First, there holds
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} S_u \bar{u}_\theta \bar{d} \theta d\theta dr \right| = \left| \int_{r_0}^{1} \int_{0}^{2\pi} \left[ u^a \bar{u}_\theta + v^a r \bar{u}_r + u^a_\theta + ru^a_\theta \bar{v} + v^a \bar{u} + u^a \bar{v} \right] \bar{u}_\theta \bar{d} \theta d\theta dr \right|.
\]
 Integrating by parts, we deduce that
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} u^a \bar{u}_\theta \bar{d} \theta d\theta dr \right| = \frac{1}{2} \left| \int_{r_0}^{1} \int_{0}^{2\pi} u^a_\theta \bar{u}_\theta \bar{d} \theta d\theta dr \right| \leq C(\eta + \varepsilon) \left\| \bar{u}_\theta \right\|_2^2,
\]
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} r v^a \bar{u}_\theta \bar{d} \theta d\theta dr \right| = \left| \int_{r_0}^{1} \int_{0}^{2\pi} \left( (r v^a)_\theta \bar{u}_r + r v^a \bar{u}_r \right) \bar{u}_\theta \bar{d} \theta d\theta dr \right| \\
\leq C(\eta + \varepsilon) \left\| \bar{u}_\theta \right\|_2 \left\| (\bar{u}_r, \bar{u}_r) \right\|_2,
\]
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} u^a \bar{u}_\theta \bar{d} \theta d\theta dr \right| = \left| \int_{r_0}^{1} \int_{0}^{2\pi} \left[ u^a_\theta \bar{u}_\theta + u^a_\theta \bar{u}_\theta \bar{u} \right] \bar{d} \theta d\theta dr \right| \\
\leq C(\eta + \varepsilon) \left\| \bar{u}_\theta \right\|_2^2 + C \varepsilon \left( \eta + \varepsilon \right) \left\| \bar{u}_\theta \right\|_2 \left\| \bar{u}_r \right\|_2,
\]
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} r u^a_\theta \bar{v}_\theta \bar{d} \theta d\theta dr \right| = \left| \int_{r_0}^{1} \int_{0}^{2\pi} \left( (r u^a_\theta)_\theta \bar{v}_\theta + ru^a_\theta \bar{v}_\theta \bar{u}_\theta \right) \bar{d} \theta d\theta dr \right| \\
\leq C(\eta + \varepsilon) \left\| \bar{v}_\theta \right\|_2 \left( C \left\| \bar{v}_\theta \right\|_2 \left\| \bar{v}_\theta \right\|_2 + C \varepsilon^{-1} \eta \left\| \bar{u}_\theta \right\|_2 \left\| \bar{v}_\theta \right\|_2,
\]
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} v^a \bar{u}_\theta \bar{d} \theta d\theta dr \right| = \left| \int_{r_0}^{1} \int_{0}^{2\pi} \left[ v^a_\theta \bar{u}_\theta + v^a_\theta \bar{u}_\theta \bar{u} \right] \bar{d} \theta d\theta dr \right| \\
\leq C \varepsilon \left( \eta + \varepsilon \right) \left\| \bar{u}_\theta \right\|_2 \left\| \bar{u}_r \right\|_2 + C \varepsilon \left( \eta + \varepsilon \right) \left\| \bar{u}_\theta \right\|_2^2,
\]
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} u^a \bar{v}_\theta \bar{d} \theta d\theta dr \right| = \left| \int_{r_0}^{1} \int_{0}^{2\pi} \left[ u^a_\theta \bar{v}_\theta + u^a_\theta \bar{v}_\theta \bar{u} \right] \bar{d} \theta d\theta dr \right| \leq C \left( \left\| \bar{u}_\theta \right\|_2 \left\| \bar{v}_\theta \right\|_2 + \left\| \bar{u}_\theta \right\|_2^2 \right).
\]
Thus, there exist \( \varepsilon_0 > 0, \eta_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0) \), there holds
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} S_u \bar{u}_\theta \bar{d} \theta d\theta dr \right| \leq \frac{\varepsilon^2}{10C} \left( \left\| \bar{u}_r \right\|_2^2 + C \varepsilon^{-1} \left\| (\bar{u}_\theta, \bar{v}_\theta, \bar{v}_\theta) \right\|_2^2 \right).
\]
Similarly, we can obtain
\[
\left| \int_{r_0}^{1} \int_{0}^{2\pi} S_v \bar{v}_\theta \bar{d} \theta d\theta dr \right| \leq \frac{\varepsilon^2}{10C} \left( \left\| \bar{v}_r \right\|_2^2 + C \varepsilon^{-1} \left\| (\bar{u}_\theta, \bar{v}_\theta, \bar{v}_\theta) \right\|_2^2 \right).
\]
Thus, we deduce that
\[
\|(\tilde{u}, \tilde{v})\|_2^2 \leq C \varepsilon^{-2}\|((\partial_y R_u^a, \partial_\theta R_u^a)\|_2^2 + C \varepsilon^{-3}\|((\tilde{u}_\theta, \tilde{v}_\theta, \varepsilon \tilde{u}_r))\|_2^2 + C \varepsilon^{-4}\|(u, v)\|_\infty^2 \|(u_\theta, v_\theta)\|_2^2. \tag{4.6}
\]

Finally, combining (4.3), (4.5) and (4.6), we obtain the following $L^\infty$ estimate: there exist $\varepsilon_0 > 0$ and $\eta_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\eta \in (0, \eta_0)$, there holds
\[
\|(\tilde{u}, \tilde{v})\|_\infty^2 \leq C \varepsilon^{-4}\|((R_u^a, R_v^a, R_u^a, R_v^a)\|_2^2 + C \varepsilon^{-3}\|((\tilde{u}_\theta, \tilde{v}_\theta, \varepsilon \tilde{u}_r))\|_2^2
\]
\[
+C \varepsilon^{-4}\|(u, v)\|_\infty^2\|(u_\theta, v_\theta)\|_2^2. \tag{4.7}
\]

Set
\[
\|(u, v)\|_Y^2 := \|(u_\theta, v_\theta, \varepsilon u_r)\|_2^2 + \kappa \varepsilon^3\|(u, v)\|_\infty^2,
\]
where $\kappa \ll 1$ is a fixed positive number, and combining the estimates (4.5) and (4.7), we arrive at
\[
\|(\tilde{u}, \tilde{v})\|_Y^2 \leq C \varepsilon^{-2}\|((R_u^a, R_v^a, R_u^a, R_v^a)\|_2^2 + C \varepsilon^{-5}\|(u, v)\|_Y^4. \tag{4.8}
\]

Let $Y = \{(u, v) \in C^\infty: (u, v) \text{ satisfies (4.3)} \text{ and } \|(u, v)\|_Y < +\infty\}$. Thus, due to
\[
\|(R_u^a, R_v^a, \partial_\theta R_u^a, \partial_\theta R_v^a)\|_2 \leq C \varepsilon^4,
\]
there exist $\varepsilon_0 > 0$, $\eta_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\eta \in (0, \eta_0)$, the operator
\[
(u, v) \mapsto (\tilde{u}, \tilde{v})
\]
maps the ball $\{(u, v): \|(u, v)\|_Y^2 \leq 2C^3\varepsilon^6\}$ in $Y$ into itself.

Moreover, for every two pairs $(u_1, v_1)$ and $(u_2, v_2)$ in the ball, we have
\[
\|(\tilde{u}_1 - \tilde{u}_2, \tilde{v}_1 - \tilde{v}_2)\|_Y^2 \leq C \varepsilon^{-5}(\|(u_1, v_1)\|_Y^2 + \|(u_2, v_2)\|_Y^2)\|(u_1 - u_2, v_1 - v_2)\|_Y^2. \tag{4.9}
\]

In fact, set
\[
\tilde{U} := \tilde{u}_1 - \tilde{u}_2, \quad \tilde{V} := \tilde{v}_1 - \tilde{v}_2, \quad \tilde{P} := \tilde{p}_1 - \tilde{p}_2,
\]
then we have
\[
\begin{cases}
-\varepsilon^2(\tilde{r}\tilde{u}_{rr} + \frac{\tilde{V}_{\tilde{u}}}{\tilde{r}} + 2 \frac{\tilde{V}_{\tilde{r}}}{\tilde{r}} + \tilde{U}_{\tilde{r}} - \frac{\tilde{V}}{\tilde{r}}) + \tilde{P}_\theta + S_{\tilde{U}} = R_{\tilde{U}}, \\
-\varepsilon^2(\tilde{r}\tilde{v}_{rr} + \frac{\tilde{V}_{\tilde{v}}}{\tilde{r}} - 2 \frac{\tilde{V}_{\tilde{r}}}{\tilde{r}} - \tilde{V}_{\tilde{r}} - \frac{\tilde{V}}{\tilde{r}}) + r \tilde{P}_r + S_{\tilde{V}} = R_{\tilde{V}}, \\
\partial_\theta \tilde{U} + \partial_r (r \tilde{V}) = 0, \\
\tilde{U}(\theta, r) = \tilde{U}(\theta + 2\pi, r), \quad \tilde{V}(\theta, r) = \tilde{V}(\theta + 2\pi, r), \\
\tilde{U}(\theta, 1) = 0, \quad \tilde{V}(\theta, 1) = 0, \\
\tilde{U}(\theta, r_0) = 0, \quad \tilde{V}(\theta, r_0) = 0,
\end{cases}
\]
where
\[
R_{\tilde{U}} \triangleq R_{u_1} - R_{u_2} = u_2 u_{2\theta} + v_2 r v_{2r} + v_2 u_2 - u_1 u_{1\theta} - v_1 r u_{1r} - v_1 u_1
\]
\[
= (u_2 - u_1) \partial_\theta u_2 + u_1 \partial_\theta (u_2 - u_1)
\]
\[
+ (v_2 - v_1) r \partial_r u_2 + v_1 r \partial_r (u_2 - u_1) + (v_2 - v_1) u_2 + v_1 (u_2 - u_1),
\]
\[
R_{\tilde{V}} \triangleq R_{v_1} - R_{v_2} = u_2 v_{2\theta} + v_2 r v_{2r} - u_2^2 - u_1 v_{1\theta} - v_1 r v_{1r} + u_1^2
\]
\[
= (u_2 - u_1) \partial_\theta v_2 + u_1 \partial_\theta (v_2 - v_1)
\]
\[
+ (v_2 - v_1) r \partial_r v_2 + v_1 r \partial_r (v_2 - v_1) + (u_1 + u_2)(u_1 - u_2).
Thus, following the estimate (4.8) line by line, we obtain (4.9). Hence, there exist $\varepsilon_0 > 0, \eta_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0), \eta \in (0, \eta_0)$, the operator

$$(u, v) \mapsto (\bar{u}, \bar{v})$$

maps the ball $\{(u, v) : \| (u, v) \|^2_Y \leq 2C^3\varepsilon^6\}$ in $Y$ into itself and is a contraction mapping. This complete the proof of this proposition. \hfill \Box

Now we can give the proof of Theorem 1.2.

Proof. Combining Proposition 4.3 and the approximate solution (2.87)-(2.89) of the Navier-Stokes equations (1.3), we easily obtain Theorem 1.2. \hfill \Box

5. Appendix

Appendix A: Constant coefficient periodic PDE

In this appendix we give a brief argument to solve the following problem which also appeared in Appendix A in [6],

\begin{align}
(\theta, \psi) & = u_\varepsilon(1)(\theta, \psi), \\
Q_0(\theta, \psi) & = Q_0(\theta + 2\pi, \psi), \\
Q_0|_{\psi=0} & = f_\eta(\theta), \quad Q_0|_{\psi=-\infty} = 0,
\end{align}

where

$$f_\eta(\theta) = \alpha^2 + 2\alpha \eta f(\theta) + \eta^2 f^2(\theta) - u_\varepsilon^2(1) = 2\alpha \eta f(\theta) + \eta^2 f^2(\theta) - \frac{\eta^2}{2\pi} \int_0^{2\pi} f^2(\theta) d\theta.$$ 

Let $Q_0(\theta, \psi) = \sum_{k \in \mathbb{Z}} e^{ik\theta} Q_{0k}(\psi)$ and substitute it into (5.1), we obtain

\begin{align}

ik Q_{0k} & = u_\varepsilon(1) Q_{0k}', \\
Q_{0k}|_{\psi=0} & = \hat{f}_\eta(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} f_\eta(\theta) d\theta, \\
Q_{0k}|_{\psi=-\infty} & = 0.
\end{align}

It is easy to get

$$Q_{0k}(\psi) = \hat{f}_\eta(k) e^{\alpha_k \psi}$$

with $\alpha_k = \sqrt{\frac{|k|}{2u_\varepsilon(1)}} (1 + \text{sgn} \cdot i)$. Then

$$Q_0(\theta, \psi) = \sum_{k \in \mathbb{Z}} e^{ik\theta} \hat{f}_\eta(k) e^{\alpha_k \psi} \in X$$

and

$$\|Q_0\|_X \leq C\eta.$$ 

(5.2)

Appendix B: Construction of corrector $h(\theta, r)$

In this section, we give a construction of corrector $h(\theta, r)$ defined in subsection 2.5. Firstly, we give a simple lemma which is similar to Lemma 6.1 in Appendix B in [6].

Lemma 5.1. Assume that $K(\theta, r)$ is a $2\pi$-periodic smooth function which satisfies

$$\int_0^{2\pi} K(\theta, r) d\theta = 0, \quad \forall r \in [r_0, 1]; \quad K(\theta, r_0) = K(\theta, 1) = 0,$$

then there exists a $2\pi$-periodic function $h(\theta, r)$ such that

$$\partial_\theta h(\theta, r) = K(\theta, r); \quad h(\theta, r_0) = h(\theta, 1) = 0;$$

$$h(\theta, r) = \sum_{k \in \mathbb{Z}} e^{ik\theta} \hat{f}_\eta(k) e^{\alpha_k \psi}.$$
\[
\int_0^{2\pi} h(\theta, r) d\theta = 0, \quad \|\partial_{\theta} \partial_r^k h\|_2 \leq C\|\partial_{\theta} \partial_r^k K\|_2. \quad (5.3)
\]

Proof. Let
\[
K(\theta, r) = \sum_{n \neq 0} K_n(r) e^{in\theta}, \quad K_n(r_0) = K_n(1) = 0.
\]

Set
\[
h(\theta, r) = \sum_{n \neq 0} \frac{K_n(r)}{in} e^{in\theta}.
\]

It’s easy to justify that \(h(\theta, r)\) satisfies (5.3) which complete the proof. \(\square\)

Next, we construct the corrector \(h(\theta, r)\) by the above lemma. Direct computation gives
\[
u_0^a + (rv^a)_r = \varepsilon^4 \partial_{\theta} h(\theta, r) + K(\theta, r),
\]
where
\[
K(\theta, r) = \varepsilon^4 \chi(r) [Y \partial_Y v_p^{(4)}(\theta, Y) + v_p^{(4)}(\theta, Y)] \\
+ \varepsilon^4 (1 - \chi(r)) [Z \partial_Z \tilde{v}_p^{(4)}(\theta, Z) + \tilde{v}_p^{(4)}(\theta, Z)] \\
+ r\chi'(r) \left( \sum_{i=1}^4 \varepsilon^i v_p^{(i)}(\theta, Y) - \sum_{i=1}^4 \varepsilon^i \tilde{v}_p^{(i)}(\theta, Z) \right).
\]

Notice that \(\chi'(r) = 0, \ r \in [r_0, r_1] \cup [r_2, 1]\) and the property of \((v_p^{(i)}, \tilde{v}_p^{(i)})\), we know that \(K(\theta, r) = O(\varepsilon^4)\) and
\[
K(\theta, 1) = 0, \ K(\theta, r_0) = 0.
\]

Moreover, notice that
\[
\int_0^{2\pi} v_p^{(i)}(\theta, Y)d\theta = 0, \ \forall \ Y \leq 0; \ \int_0^{2\pi} \tilde{v}_p^{(i)}(\theta, Z)d\theta = 0, \ \forall \ Z \geq 0, \ \ i = 1, \cdots, 4,
\]
we deduce that
\[
\int_0^{2\pi} K(\theta, r)d\theta = 0, \ \forall r \in [r_0, 1].
\]

Thus, we can choose \(h(\theta, r)\) by Lemma 5.1 such that
\[
\varepsilon^4 \partial_{\theta} h(\theta, r) + K(\theta, r) = 0, \ h(\theta, 1) = 0, \ h(\theta, r_0) = 0, \ \|\partial_{\theta} \partial_r^k h\|_2 \leq C\varepsilon^{-k}.
\]

Appendix C: Generalized Prandtl-Batchelor theory in an annulus.

In this appendix, we derive a generalized Prandtl-Batchelor theory for the forced steady Navier-Stokes equations on a two-dimensional annulus. For Prandtl-Batchelor theory on a simply connected domain, we refer to [1, 22, 23] or Appendix C in [6].

Consider the two-dimensional forced Navier-Stokes equations (1.1). Let \(w^\varepsilon = u_y^\varepsilon - v_x^\varepsilon\) be the vorticity, then it’s easy to obtain that
\[
(\nu^\varepsilon, -u_x^\varepsilon)^T w^\varepsilon + \nabla \left( p^\varepsilon + \frac{1}{2} |u^\varepsilon|^2 \right) = \varepsilon^2 (\triangle u^\varepsilon + F).
\]

Let \(\Phi^\varepsilon(x, y)\) be the stream function which satisfies \(\partial_y \Phi^\varepsilon = u^\varepsilon, -\partial_x \Phi^\varepsilon = v^\varepsilon\). Assuming that
\[
\Gamma^\varepsilon := \{(x, y) | \Phi^\varepsilon(x, y) = c\}
\]
is a closed curve for any $c \in \Phi^\varepsilon(\Omega)$. By integrating the Navier-Stokes equations over the closed curve $\Gamma^\varepsilon_c$ we deduce that
\[
\int_{\Gamma^\varepsilon_c} \left( (v^\varepsilon, -u^\varepsilon)^T w^\varepsilon + \nabla \left( p^\varepsilon + \frac{1}{2} |u^\varepsilon|^2 \right) \right) d\vec{\tau} = 0,
\]
thus we obtain that
\[
\int_{\Gamma^\varepsilon_c} (\Delta u^\varepsilon + F) d\vec{\tau} = 0.
\]
Assume that $u^\varepsilon \to u_e$ in some strong sense and $u_e$ is the solution of forced steady Euler equations, then there holds
\[
\int_{\Gamma^\varepsilon_c} (\Delta u_e + F) d\vec{\tau} = 0,
\]
where $\Gamma^\varepsilon_c$ is the stream line of the Euler flow.

If the Euler flow $u_e$ has no stagnation point in the annulus, then it’s a rotating shear flow $u_e = u_e(r)\vec{e}_\theta$ and the stream line is circle, see [15]. From this and (5.4), we deduce that
\[
u''_e(r) + \frac{1}{r} u'_e(r) - \frac{1}{r^2} u_e(r) = -\bar{F}_u(r),
\]
where $F_u(\theta, r) = F \cdot \vec{e}_\theta$ and $\bar{F}_u(r) := \frac{1}{2\pi} \int_0^{2\pi} F_u(\theta, r) d\theta$.

Similarly, for a finite channel $\{(x, y) : x \in [0, 2\pi], y_0 \leq y \leq y_1\}$, if the inviscid limit of Navier-Stokes flow is a shear flow $(u_e(y), 0)^T$, then there also holds
\[
u''_e(y) + \frac{1}{2\pi} \int_0^{2\pi} F_1(x, y) dx = 0,
\]
where $F_1$ is the first component of $F$.

Acknowledgments

M.Fei is partially supported by NSF of China under Grant No.11871075, 11971357 and 12271004. Z.Lin is partially supported by the NSF grants DMS-1715201 and DMS-2007457. T.Tao is partially supported by the NSF of China under Grant 11901349. C.Gao and T.Tao are deeply grateful to professor L.Zhang for very valuable suggestion.

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