BEYOND THE FAN–HOFFMAN INEQUALITY*

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Abstract. We establish a singular value inequality inspired by the Fan–Hoffman inequality and resolve the \( j \)-conjecture of Yang and Zhang.

Key words. Singular value, Eigenvalue, \( j \)-conjecture.

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1. Introduction. Majorization inequalities for the eigenvalues and singular values of matrices are commonplace. Inequalities that compare the \( j \)th eigenvalue or singular value of one matrix to that of another are more special. One such inequality is the Fan–Hoffman inequality \[2\] which asserts that 

\[ \lambda_j(\Re(A)) \leq \sigma_j(|A|) \text{ for } j = 1, \ldots, n, \]

for an \( n \times n \) complex matrix \( A \). The notation \( A^* \) denotes the adjoint matrix of \( A \), \( \Re(A) \) denotes the hermitian part \( \frac{1}{2}(A + A^*) \) of \( A \), and \( |A| = (A^*A)^{1/2} \) the absolute value of \( A \). For a hermitian matrix \( H \), \( \lambda_j(H) \) denotes the \( j \)th eigenvalue taken in decreasing order and \( \sigma_j(A) = \lambda_j(|A|) \), the \( j \)th singular value of \( A \). For more details on these concepts, the reader may consult Bhatia \[1\].

In a related paper \[3\], R.C. Thompson establishes the following result.

**Theorem 1.** For any pair \( A, B \) of \( n \times n \) complex matrices, there exist unitary \( n \times n \) matrices \( U \) and \( V \) such that \(|A + B| \leq U^*|A|U + V^*|B|V \) where \( \leq \) is understood in the positive semidefinite sense.

An immediate consequence is the following.

**Corollary 2.** For any \( n \times n \) complex matrix \( A \) and any \( n \times n \) contraction \( C \), we have

\[ \sigma_j(C + A) \leq \lambda_j(I + |A|) \text{ for } j = 1, \ldots, n. \]

Here, we have denoted by \( I \) the \( n \times n \) identity matrix.

By a contraction \( C \), we mean a complex matrix with its spectral norm \( \|C\| \leq 1 \). It is this corollary that we intend to generalize by establishing the following.

**Theorem 3.** For any \( n \times n \) complex matrix \( A \), any \( n \times n \) contraction \( C \) and any positive nondecreasing function \( f : [0, \|A\|] \to (0, \infty) \), the inequality

\[ \sigma_j((C + A)f(|A|)) \leq \lambda_j((I + |A|)f(|A|)) \text{ for } j = 1, \ldots, n, \]

holds.

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For a hermitian matrix $H$ with spectral decomposition

$$H = \sum_{j=1}^{n} \lambda_j(H)e_j^*e_j,$$

with $\{e_j, j = 1, \ldots, n\}$ an orthonormal basis, we have denoted

$$f(H) = \sum_{j=1}^{n} f(\lambda_j(H))e_j^*e_j,$$

the standard symbolic calculus for hermitian matrices. Note that $(I + |A|)f(|A|)$ is hermitian since $I + |A|$ and $f(|A|)$ are commuting hermitian matrices.

As a consequence, we will establish the $j$-conjecture of Yang and Zhang [4].

**Theorem 4.** Let $A$ be a strict contraction. Then

$$\lambda_j\left(\Re((I - A)^{-1})\right) \geq \lambda_j\left((I + |A|)^{-1}\right), \text{ for } j = 1, 2, \ldots, n.$$

### 2. Proofs of the results.

**Proof.** Proof of Theorem 3.

Since $|A|$ has only finitely many eigenvalues, we may always extend $f$ to a continuous positive non-decreasing function defined on $(0, \infty)$ which we do without change of notation. The function $g$ given by $g(t) = (1 + t)f(t)$ is continuous and strictly increasing tending to $\infty$ at $\infty$. The inverse function $h$ of $g$ is also a strictly increasing continuous function.

Equivalent to 1.1, we will show

$$\lambda_j(B(C^*C + C^*A + A^*C + A^*A)B) \leq \lambda_j((I + |A|)B)^2, \quad (2.1)$$

where $B = f(|A|)$. We will assume that (2.1) is false and find a contradiction. Then there exists a real number $\nu$ such that

$$\lambda_j(B(C^*C + C^*A + CA^* + A^*A)B) > \nu^2 \quad \text{and} \quad \nu > \lambda_j((I + |A|)B).$$

Also we have $B = f(|A|) \geq f(0)I$ so that $f(0) \leq \lambda_n(B) \leq \lambda_j(B) \leq \lambda_j((I + |A|)B)$. Hence by the intermediate value theorem, there exists $\mu$ such that $(1 + \mu)f(\mu) = g(\mu) = \nu$. Hence, there is a linear subspace $E$ of dimension $j$ such that

$$\xi^*B(C^*C + C^*A + CA^* + A^*A)B\xi > (1 + \mu)^2f(\mu)^2||\xi||^2,$$

for all nonzero $\xi \in E$.

Let

$$(I + |A|)B = \sum_{k=1}^{n} \lambda_k((I + |A|)B)\eta_k\eta_k^*,$$

be the spectral decomposition of $(I + |A|)B$ where the $\eta_k$ are the mutually orthogonal unit eigenvectors. Let $F$ be the linear subspace of dimension $n + 1 - j$ spanned by $\{\eta_k; j \leq k \leq n\}$. On $F$ we have $(I + |A|)B \leq \nu I$. 

Since \( h \) is increasing and \( f \) is nondecreasing
\[
|A| = h(|I + |A||B|) \leq h(\nu I) = \mu I \quad \text{and} \quad |B| = f(|A|) \leq f(\mu)I,
\]
on \( F \). Note also that \( F \) is invariant under both \( |A| \) and \( B \) since \( |A| = h(|I + |A||B|) \), \( B = f(|A|) \) and using the symbolic calculus of hermitian matrices.

By dimensionality, the intersection \( E \cap F \) is forced to be nonzero. We choose a unit vector \( \xi \in E \cap F \). Then
\[
(1 + \mu)^2 f(\mu)^2 < \xi^* B(C^* C + C^* A + CA^* + A^* A)B \xi
\]
\[
\leq \|CB\xi\|^2 + 2\Re \* BC^* AB \xi + \xi^* B|A|^2 B \xi
\]
\[
\leq \|B\xi\|^2 + 2\|BC\xi\||AB\xi\| + \mu^2 f(\mu)^2
\]
\[
\leq \xi^* B^2 \xi + 2\|CB\xi\||AB\xi\| + \mu^2 f(\mu)^2
\]
\[
\leq f(\mu)^2 + 2\|B\xi\||AB\xi\| + \mu^2 f(\mu)^2
\]
\[
\leq (1 + \mu^2)f(\mu)^2 + 2\sqrt{\xi^* B^2 \xi} \sqrt{\xi^* B^2 \xi} \leq (1 + \mu^2)f(\mu)^2 + 2\sqrt{f(\mu)^2} \sqrt{\xi^* B^2 \xi} \leq (1 + \mu^2)f(\mu)^2 + 2\sqrt{\mu^2} f(\mu)^2 = (1 + \mu^2)f(\mu)^2.
\]

This contradiction establishes 2.1.

\textbf{Proof. Proof of Theorem 4}

We replace \( A \) by \(-A\) which is also a strict contraction, and we will therefore prove that
\[
\lambda_j \left( \Re((I + A)^{-1}) \right) \geq \lambda_j \left( (I + |A|)^{-1} \right), \quad \text{for } j = 1, 2, \ldots, n.
\]
We have
\[
2\Re((I + A)^{-1}) = (I + A^*)^{-1} + (I + A)^{-1}
\]
\[
= (I + A^*)^{-1} (2I + A^* + A)(I + A)^{-1}
\]
\[
= (I + A^*)^{-1} ((I + A^* + A + A^* A) + (I - A^* A))(I + A)^{-1}
\]
\[
= I + (I + A^*)^{-1} (I - A^* A)(I + A)^{-1}.
\]
On the other hand, we have
\[
2(I + |A|)^{-1} = I + (I + |A|)^{-\frac{1}{2}} (I - |A|)(I + |A|)^{-\frac{1}{2}}.
\]
Therefore, it suffices to show that
\[
\lambda_j \left( (I + A^*)^{-1} (I - A^* A)(I + A)^{-1} \right) \geq \lambda_j \left( (I + |A|)^{-\frac{1}{2}} (I - |A|)(I + |A|)^{-\frac{1}{2}} \right),
\]
or equivalently
\[
\sigma_j \left( (I - |A|)^{\frac{1}{2}} (I + A)^{-1} \right) \geq \sigma_j \left( (I - |A|)^{\frac{1}{2}} (I + |A|)^{-\frac{1}{2}} \right),
\]
or since \( \sigma_j(X) = (\sigma_{n+1-j}(X^{-1}))^{-1} \) that
\[
\sigma_{n+1-j} \left( (I + A)(I - |A|^2)^{-\frac{1}{2}} \right) \leq \sigma_{n+1-j} \left( (I + |A|)^{\frac{1}{2}} (I - |A|)^{-\frac{1}{2}} \right),
\]
Beyond the Fan–Hoffman inequality

But

\[(I + |A|)^{\frac{1}{2}}(I - |A|)^{-\frac{1}{2}} = (I + |A|)(I - |A|)^{2^{-\frac{1}{2}}},\]

so finally, it remains to show that

\[\sigma_{n+1-j}\left((I + A)f(|A|)\right) \leq \sigma_{n+1-j}\left((I + |A|)f(|A|)\right),\]

where \(f(t) = (1 - t^2)^{-\frac{1}{2}}\). But \(f\) is a positive increasing function on \([0, \|A\|]\) and the result follows from Theorem 3 taking \(C = I\).

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