1. Introduction

One of the most startling discoveries in low-temperature physics in the last few decades was the identification of a new collective state of interacting 2-dimensional electrons in semiconducting films or heterostructures, for temperatures $T < 1K$. We now know that a whole hierarchy of “fractional quantum Hall” states exists, at special fields such that the “filling fraction” $\nu$ of electrons in the lowest Landau level is equal to a set of simple fractions (the “fractional hierarchy” of states); usually these occur for $\nu < 1$, but have also been observed for $\nu > 1$.

Theoretical approaches to this new state of matter started with Laughlin’s “wave-functions” for the ground state and single quasi-particles; more recent years have seen several attempts at field theoretical descriptions of the low-energy physics. The Laughlin theory describes the ground state and low-energy quasi-particles near the filling fractions $\nu_k = 1/(2k+1)$, with $k = 1, 2, 3\ldots$ Nevertheless for at least a decade theorists have been searching for an effective field theory for the FQHE, for various reasons, amongst which are

(i) The need for a theory which deals not only with filling fractions $\nu_k$, but for the full experimentally observed hierarchy of states.

(ii) The need to calculate correlation functions for the low-energy physics near all the stable filling fractions.

(iii) The need to deal with situations where the quasiparticle density is not particularly low, especially in regions where potential fluctuations make the Laughlin wave-function difficult to apply.

The main difficulty is of course the combination of strong correlations and Landau level degeneracy - switching on the interactions must lead to a non-analytic (in the coupling strength) change in the spectrum.

A bosonic field theory of the FQHE introduced a Chern-Simons field to convert the electrons to bosons, and treated the ensuing field theory in a mean-field approximation; the FQHE ground states at $\nu = \nu_k$ then map onto superfluids, while the quasiparticles map onto vortices. Unfortunately, despite some success, this theory cannot deal with the other stable filling fractions ($\nu \neq \nu_k$).

A more recent fermionic theory, devised by Jain, attaches an even number of artificial “statistical” flux quanta to each electron. This theory describes FQHE states with filling fraction $\nu = \nu_p^m = p/(2mp + 1)$, where $p = \pm 1, \pm 2, \pm 3, \ldots$, and $m = 1, 2, 3, \ldots$: the number of statistical flux quanta attached to each electron is then $2m$. Interestingly, the states generated appear to correspond better to the experimental hierarchy; moreover, if we let $|p| \to \infty$, so that $\nu \to 1/2$, a number of interesting predictions arise. At $\nu = 1/2$ the external magnetic field, $B_{1/2}$, is such that there are two magnetic flux quanta for each electron. Electrons are transformed into composite fermions by attaching flux quanta to each in such a way as to cancel on average the field $B_{1/2}$. At $\nu = p/(2p+1)$ (away from $\nu = 1/2$) the CF’s find themselves in a field $\Delta B = B_{1/2} - B_{1/2} = n_e((2p+1)/p - 2) = n_e/p$. Thus we are led to the observation by Jain that the FQHE of electrons is the integer QHE of composite fermions.

Various methods have been used to determine the cyclotron radius of the charged quasiparticles away from $\nu = 1/2$. These experiments have confirmed that these quasiparticles do in fact behave as if they were in a magnetic field $\Delta B$ (so that the cyclotron radius diverges as $B \to B_{1/2}$), which corresponds precisely to the prediction of CF theory. In addition to this, the theory of CF’s also predicts a gap of size $e\Delta B/m^*$, where $m^*$ is the effective mass of the CF’s. These have been numerous direct measurements of the FQHE gap for very small values of $p$ ($p = 1, 2$) but for slightly larger values of $p$ an indirect approach has been used. The effective mass is found by fitting the Shubnikov-de Haas (SdH) oscillations to the Dingle formula for non-interacting quasi-
particles, parametrized by a variable effective mass and by an energy-independent scattering rate. Du et al. have reported that the effective mass appears to diverge for large values of $p$. However it should be noted that another group has reported different results up to a smaller value of $p$. Nevertheless, as $\Delta B \to 0$, when $\nu \to 1/2$, the gap is observed experimentally to go to zero, i.e., $m^*(\Delta B)$ must diverge more slowly than $1/\Delta B$, if at all.

The apparent success of the Jain theory led to its elaboration into a “CF gauge theory”, where the statistical gauge flux is again introduced via a Chern-Simons field. The consequences of this theory have been examined both at $\nu = 1/2$, in the gapless state, and also away from $\nu = 1/2$, near the filling fractions $\nu = \nu_p = p/(2p+1)$. In both cases one must deal with the long-range dynamic gauge interaction between the CF’s. The exact form of this gauge propagator depends on the form of the original electron-electron interactions. When the electron-electron interactions are unscreened Coulombic the theory of CF’s describes a marginal Fermi liquid at $\nu = 1/2$. There is a well defined Fermi surface but the quasiparticles of the theory cease to be well defined in the vicinity of the Fermi energy, because of a large imaginary self-energy. This coincides with a logarithmic divergence in the effective mass near the Fermi surface.

When the interaction between the electrons is screened, the gauge interaction between the CF’s becomes very severely infra-red divergent. Despite a large theoretical effort to understand this problem at $\nu = 1/2$, using various non-perturbative methods, there is no agreement on the final form of the theory for this case.

Away from $\nu = 1/2$ the gauge theory is easier to control, since the CF gap provides an IR cut-off. Quite surprisingly, in view of the large effort devoted to the $\nu = 1/2$ system, few papers have attempted to apply the CF gauge theory to the FQHE states. Kim et al. calculated both the CF gap and the finite-T compressibility at low T, and Stern and Halperin examined the CF gap and the chemical potential. Simon and Halperin also calculated, using the RPA, an expression for the current-current correlation function. Curnoe and Stannard gave a preliminary examination of the CF spectral weight, and compared this with what one expects for the simpler problem of 2-d electrons, moving in Landau levels, interacting with ordinary phonons. These investigations were all perturbative; and moreover used the approximation $|p| \gg 1$, allowing a semi-classical treatment of the sums over Landau levels. All of these papers studied the CF mass in the FQHE states.

The present paper has two purposes. First, we give a more detailed discussion of the brief results presented in [4]. Second, and more important, we wish to give a more careful treatment of the divergences, present in $\Sigma(\epsilon)$, which come from the Landau level quantization. As noted in [4], these divergences have a rather peculiar structure in the interacting gauge system, which led us to look at the corresponding structure in the simpler electron-phonon problem as well. Here we try and give a self-consistent treatment of this structure, going beyond perturbation theory. As we shall see, it is important to do this, since a proper account of it leads to important changes in the dependence of the FQHE gap in the filling fraction. We give both approximate and analytic results; our method is essentially a sum of rainbow graphs, for CF’s moving in Landau levels, and interacting via a singular gauge field.

Our main results are as follows. On the purely theoretical side, we show how one may calculate the forms of both the self-energy and the quasi-particle spectral function, for the case of both electrons interacting with phonons, and composite fermions interacting with gauge fluctuations; in both cases these calculations are done in finite field, incorporating the singular Landau level effects. It turns out that a self-consistent treatment of these effects is not entirely obvious, and so we show how this may be done.

We then apply our technique to the calculation of the experimentally measurable gap, again for the electron-phonon problem and the composite fermion problem. Our results show that if one does not handle the singular Landau level structure self-consistently, the gap in the experimentally relevant regime is badly overestimated in the case of the FQHE; we give results both for CF’s and the electron-phonon problem.

The paper is organised as follows. In section 2 we discuss the self-energy and spectral function of Landau level electrons coupled to phonons. This problem is studied to throw light on the more complex CF problem. It is also of purely academic interest - as far as we are aware, despite the enormous literature on the electron-phonon system, no previous study has been made of the 1-particle propagator of electrons in Landau levels, coupled to phonons which deals with the singular structure introduced by Landau level quantization. The calculations are carried out for both Debye and Einstein phonons.

In section 3 we describe in detail our lowest-order perturbation calculation of $\Sigma(\epsilon)$ and $A(\epsilon)$ for CF’s in Landau levels, interacting via a gauge field. In both the electron-phonon and CF gauge cases, one may distinguish a strong- and weak-coupling regime. In the weak-coupling regime the Landau level excitations gap is hardly affected, whereas it is strongly renormalised in the strong-coupling regime.

The non-trivial structure of $A(\epsilon)$, the quasiparticle spectral function, leads us in section 4 to introduce an iterative self-consistent calculation of $\Sigma(\epsilon)$, which takes systematic account of the coherent parts of $\Sigma(\epsilon)$. This then allows us to give an exact rainbow graph summation of these coherent parts, and we then use this compute the CF gap over a wide range of coupling constants and filling fractions.

### 2. Phonon Interactions

We begin by examining the self-energy $\Sigma_\nu(\epsilon)$ and the spectral function $A_\nu(\epsilon)$ for electrons coupled to phonons, whilst moving on Landau level $n$ (see Fig. [4]). The structure introduced by the combination of interactions and
Landau level degeneracy turns out to be rather interesting.

In zero field, the spectral functions of the coupled electron-phonon system were first studied by Engelsberg and Schrieffer. Using optical and Debye phonon spectra they calculated the real and imaginary parts of the self-energy using conserving approximations (obeying the Ward identities). They found that the spectral function,

\[ A(\epsilon, p) = \frac{1}{\pi} \text{Im} \{ G(\epsilon, p) \} \]

\[ = \frac{1}{\pi} \frac{\Sigma''(\epsilon, p)}{\left( \Sigma' (\epsilon, p) + \epsilon - \xi_p \right)^2 + \Sigma'' (\epsilon, p)^2}, \]  

(1)
yielded structures that they could identify as well defined q\(p\)'s for two cases: when \(p\) is close to the Fermi surface, and when it is very large. In the intermediate regime there exists only an incoherent “smeared” structure.

In our analysis we first consider a two dimensional system of electrons in a small magnetic field \(B\) which interacts with phonons having a Debye spectrum. In this case we will use a the phonon propagator calculated at \(B = 0\),

\[ D(q, \omega) = \frac{q c_s}{\omega^2 - c_s^2 q^2}. \]  

(2)

which is assumed to have a cutoff at the Debye frequency \(\omega_D\) (\(c_s\) is the speed of sound). Within this same semiclassical limit we adopt a well known approximation for the overlap matrix element, between plane waves and Landau level states (see the appendix for more details):

\[ |\Lambda(n, n', q)|^2 = \frac{1}{q} \left( \frac{eB}{2n} \right)^{1/2}, \]  

(3)

where \(n\) and \(n'\) are the Landau level indices as shown in Fig. 1 and \(n \approx p\) is the number of filled Landau levels.

![Fig. 1. Feynman diagram representing the self-energy.](image)

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The self-energy expression is given in perturbation theory by

\[ \Sigma_n(\epsilon) = \int \frac{d^2 q}{(2\pi)^2} \int_0^\infty \frac{d\omega}{\pi} \text{Im} U(q, \omega) \sum_{n' = 0}^\infty |\Lambda(n, n', q)|^2 \int_0^\infty \frac{d\epsilon'}{\pi} \text{Im} G_{n'}^{0}(\epsilon') \times \left( \frac{1 + n_B(\omega) - n_f(\epsilon')}{\epsilon - \epsilon' - \omega + i\delta} + \frac{n_B(\omega) + n_f(\epsilon')}{\epsilon - \epsilon' + \omega + i\delta} \right), \]  

(4)

where \(G^{0}\) is the bare electron Green function, i.e.,

\[ \text{Im} G^{0}_n(\epsilon) = \pi \delta(\epsilon + (p - 1/2 - n) \omega_c). \]  

(5)

and \(U(q, \omega)\) depends on the electron-phonon coupling (see below). The electronic frequency \(\epsilon\) is measured from the Fermi energy, \(\mu = p_a \omega_c\), and the energy of the highest filled Landau level is \((p - 1/2) \omega_c\); since we are doing a quasi-classical calculation, we assume that the Fermi energy is halfway between Landau levels.

In the rest of this section we first calculate the self-energy and the renormalised cyclotron gap energy for the case of a deformation coupling to Debye phonons; we then do the same for the coupling to optical Einstein phonons.

(a) Debye Phonons: We start with Debye phonons having velocity \(c_s\). The function \(U(q, \omega)\) in (4) is just the effective interaction between electrons; in the Debye model it is given by

\[ U(q, \omega) = \frac{q \Xi_D^2 \hbar}{2 e c_s p a} D(q, \omega). \]  

(6)

where \(D(q, \omega)\) is the phonon Green function (2). The electron-phonon interaction is parametrised by the deformation potential, \(\Xi_D\), and \(p\) is the ion mass density of the material (see the appendix for more details).

Evaluating the integrals in eq. (6), yields, at temperature \(T = 0\):

\[ \Sigma'_n(\epsilon) = \frac{K_D}{\pi} \left( \sum_{n' = p}^\infty \left( -\omega_D + ((n' - p + 1/2) \omega_c - \epsilon) \right) \log \frac{\omega_D + ((n' - p + 1/2) \omega_c - \epsilon)}{(n' - p + 1/2) \omega_c - \epsilon} \right) \\
+ \sum_{n' = 0}^{p - 1} \left( \omega_D + ((n' - p + 1/2) \omega_c - \epsilon) \right) \log \frac{(n' - p + 1/2) \omega_c - \epsilon - \omega_D}{(n' - p + 1/2) \omega_c - \epsilon} \left( n' - p + 1/2 \right) \omega_c - \epsilon \right) \]  

(7)

\[ \Sigma''_n(\epsilon) = K_D \left( \sum_{n' = p}^\infty (\omega_D - \epsilon) \\
+ ((n' - p + 1/2) \omega_c - \epsilon) \theta((n' - p + 1/2) \omega_c - \epsilon) \right) \\
+ \sum_{n' = 0}^{p - 1} (\omega_D - \epsilon) + ((n' - p + 1/2) \omega_c - \epsilon) \theta((-n' + p - 1/2) \omega_c), \]  

(8)

The right hand side is independent of \(n\), and so we will omit the subscript on \(\Sigma(\epsilon)\) henceforth. \(\Sigma(\epsilon)\) is in units of \(\omega_c\) and \(K_D\) is a dimensionless constant,

\[ K_D = \frac{\Xi_D^2 \hbar m \omega_c}{4 \pi c_s^2 p a \hbar} \left( \frac{1}{4 \pi n_c} \right)^{1/2}. \]  

(9)

It is useful to rewrite these equations shifting the sum to start at \(-p\) and then enforcing particle-hole symmetry by truncating the sum at \(p - 1\), the number of filled
Landau levels. (In reality the problem is not particle-hole symmetric, but we are only interested in energies within $\omega_c$ of the Fermi surface, where particle-hole symmetry is almost exact. In the case of composite fermions, this will correspond to having an upper cutoff at the true cyclotron frequency). This gives:

$$\Sigma'(\epsilon) = \frac{K_D}{\pi} \sum_{m=0}^{p-1} \left( (m+1/2)\omega_c - \epsilon \right) \log \left| \frac{(m+1/2)\omega_c - \epsilon + \omega_D}{(m+1/2)\omega_c - \epsilon} \right| - ((m+1/2)\omega_c + \epsilon) \log \left| \frac{(m+1/2)\omega_c + \epsilon + \omega_D}{(m+1/2)\omega_c + \epsilon} \right|$$

(10)

$$\Sigma''(\epsilon) = K_D \sum_{m=0}^{p-1} ((m+1/2)\omega_c - \epsilon) \times \theta((m+1/2)\omega_c - \epsilon + \omega_D) \theta((-m-1/2)\omega_c + \epsilon) + ((m+1/2)\omega_c + \epsilon) \times \theta((m+1/2)\omega_c + \epsilon + \omega_D) \theta((-m-1/2)\omega_c - \epsilon))$$

(11)

These functions are shown together in Fig. 2. The structure caused by the Landau levels is very weak; it can just be seen in $\Sigma''(\epsilon)$ (if $\omega_c/\omega_D$ is larger, it is much more obvious). $\Sigma''(\epsilon)$ is simply a sum of ramp functions, coming from each Landau level; as $\omega_c \to 0$, $\Sigma''(\epsilon)$ becomes parabolic for $\epsilon \ll \omega_D$. The logarithmic singularities in $\Sigma'(\epsilon)$ are quite invisible in Fig. 2.

![Fig. 2](image2)

FIG. 2. The real and imaginary parts of the self-energy for electrons interacting with phonons with a Debye spectrum ($\omega_D = 2k\omega_c$) in a small magnetic field. All energies are in units of $\omega_c$; the coupling is $K_D = 1$.

We wish to find the poles of $G_n(\epsilon)$. Generally there are 0, 1, 2 or 3 roots to the equation

$$\Sigma'(\epsilon) + \epsilon = (n+1/2)\omega_c,$$

(12)

which correspond to peaks in the spectral function, $\text{Im} G_n(\epsilon)$. In this expression both $n$ and $\epsilon$ are measured with respect to the Fermi surface. We are only concerned with the low energy behaviour which corresponds to small $n$ and $\epsilon$. When $n$ is small and $K_D$ is small there is one solution occurring near $\epsilon = (n+1/2)\omega_c$, which appears as a well defined quasiparticle peak (a $\delta$-function for $n = 0$). When $K_D$ is large there are three solutions. Two occur well beyond the Debye frequency (outside of the low energy regime) and the third occurs close to $\epsilon = 0$ yielding a $\delta$-function. Intermediate values of $K_D$ do not in general yield the very narrow peaks characteristic of a dressed particle, since if solutions to (12) do occur they occur at finite $\epsilon$ where the imaginary part of $\Sigma$ is also finite. There may also be features arising from incoherent contributions. Some of these are shown in Fig. 3. We remark that the magnetic field causes slight discontinuities in the slope of the spectral functions at the Landau level energies.

![Fig. 3](image3)

FIG. 3. The imaginary part of the Green function of electrons interacting with Debye phonons for $K_D = .001, .1$ and .25 with $n = 0$ (see the text for definition of the coupling $K_D$).

(b) Einstein Phonons: We now consider the interaction with the more singular Einstein phonon spectrum, with phonon propagator $D(q, \omega) = \omega_E/(\omega^2 - \omega_E^2)$; $\omega_E$ is the “optical” phonon frequency. In zero field, this was first considered by Engelsberg and Schrieffer who found the self-energy to be

$$\Sigma'(\epsilon) = -\frac{K_E\omega_c}{\pi} \log \left| \frac{\epsilon + E_F + \omega_E}{\epsilon + E_F - \omega_E} \right|$$

(13)

$$\Sigma''(\epsilon) = K_E\omega_c$$

for $|\epsilon| > \omega_E$

$$= 0$$

for $|\epsilon| < \omega_E$

(14)

With $\Sigma$ expressed in units of $\omega_c$, $K_E$ is the dimensionless coupling.
\[
K_E \approx \frac{g^2 \alpha (4\pi n_e)^{1/2}}{4E_F\omega_c}. \tag{15}
\]

The effect of the magnetic field is to modify these equations to
\[
\Sigma'(\epsilon) = -\frac{K_E\omega_c^2}{\pi} \sum_{m=0}^{p-1} \left( \frac{1}{(m+1/2)\omega_c + \omega_E - \epsilon} - \frac{1}{(m+1/2)\omega_c + \omega_E + \epsilon} \right) \tag{16}
\]
\[
\Sigma''(\epsilon) = K_E\omega_c^2 \sum_{m=0}^{p-1} (\delta ((m+1/2)\omega_c + \omega_E - \epsilon) + \delta ((m+1/2)\omega_c + \omega_E + \epsilon)) \tag{17}
\]

\[
\Sigma'(\epsilon) \text{ is shown in Fig. 1. There is no virtually no effect of a small magnetic field on the self energies for } |\epsilon| < \omega_E; \text{ beyond this region the smooth functions are replaced by functions with divergences at each Landau level energy. All poles in the spectral function appear as } \delta \text{-functions since the imaginary part of the self-energy vanishes everywhere except at discrete points, but the weights of the poles vary. The new electron spectrum is determined by summing over all } n. \text{ Since the } n \text{ are discrete a renormalised gap can be defined as the energy difference between the highest occupied and lowest unoccupied states, i.e., by the difference between the positions of the } \delta \text{-function peaks of } G_0(\epsilon) \text{ and } G_{-1}(\epsilon).
\]

![Graph showing self-energy](image)

**FIG. 4.** Real part of the self-energy of electrons interacting with phonons with an Einstein spectrum \((\omega_E = 20\omega_c)\). Both axes are in units of \(\omega_c\); the coupling is \(K_E = 1\).

The amount of renormalization of the gap depends on the size of the coefficient of the self energy, \(K_E\). When \(K_E\) is small the poles lie near \((n+1/2)\omega_c\) for \(n\omega_c < \omega_E\), as in the Debye case. The renormalized gap is approximately \(\omega_c\), only slightly increased due to the interactions. As \(K_E\) is increased the gap increases further. The gap may be determined by approximating
\[
\Sigma'(\epsilon) = \frac{\partial \Sigma'(0)}{\partial \epsilon} \epsilon \tag{18}
\]
and then solving \([12]\) for \(n = -1\) and \(n = 0:\)

\[
\frac{\partial \Sigma'(0)}{\partial \epsilon} = -\frac{K_E}{\pi} \sum_{m=0}^{p-1} \frac{2}{(m+1/2)\omega_c + \omega_E} \tag{19}
\]
\[
\approx -\frac{2K_E}{\pi} \log \left( \frac{(p+1/2)\omega_c + \omega_E}{\omega_c/2 + \omega_E} \right) \tag{20}
\]

where the sum over \(m\) has been replaced by an integral. Then the solution to \([12]\) is
\[
\epsilon_\pm \left( 1 - \frac{2K_E}{\pi} \log \left( \frac{(p+1/2)\omega_c + \omega_E}{\omega_c/2 + \omega_E} \right) \right) = \pm \frac{\omega_c}{2} \tag{21}
\]

and the gap is
\[
\omega_c^* = |\epsilon_+ - \epsilon_-| = \omega_c \left| 1 - \frac{2K_D}{\pi} \log \left( \frac{(p+1/2)\omega_c + \omega_E}{\omega_c/2 + \omega_E} \right) \right|^{-1} \tag{22}
\]

This procedure is also applicable to the Debye case when there are well defined quasiparticles for \(n = -1, 0\). In this case the renormalized gap is
\[
\omega_c^* = \omega_c \left| 1 - \frac{K_D}{\pi} \left( \log(2\omega_D + \omega_c) \right) \right. \left. - 2(p+1) \log \left( \frac{(p+1/2)\omega_c + \omega_E}{(p+1/2)\omega_c} \right) \right|^{-1} \tag{23}
\]

We may draw some general conclusions about the size of the gap for different coupling strengths using the forms above. Considering first the Einstein case, for small values of the couplings the gap is renormalized to a larger value. This continues to be the case as the coupling is increased. The divergence in the gap seen in equation \([22]\) is avoided because the gap is always bounded by \(2\omega_E\). This is due to the presence of the logarithmic peaks in the real part of \(\Sigma(\epsilon)\) which ensures that there will always be a solution to \([12]\) for \(|\epsilon| < \omega_E\). For large couplings the gap is reduced. In the Debye case, for small \(K_D\) there are well defined quasi-particle peaks outside the gap but they are not strictly \(\delta\)-functions thanks to small but non-zero \(\text{Im}\Sigma(\epsilon)\). The finite width maintains the size of the gap to be \(\omega_c\), as shown in Fig. 1. For larger values of \(K_D\) the divergence of the gap in \([22]\) is avoided for the same reason, as shown in Fig. 1. For large couplings the gap is reduced because of q.p. poles that lie within \(\omega_c\), as shown in Fig. 1.

As can be seen from Figs. 1 and 2, the effect of Landau quantization on the quasi-particle properties, for realistic values of \(\omega_D, \omega_E\) and \(\omega_c\), is very small, at least at low energies (it is however worth noting that once \(\epsilon > \omega_E\) in the case of Einstein phonons, a rather obvious singular structure appears in the self-energy, see Fig. 1). Nevertheless the calculation of these effects may be useful in systems for which \(\omega_c\) can be made as large as \(\omega_D\) or \(\omega_E\).

Our main reason for looking at the electron-phonon problem is that it is a well-understood example of a Fermi liquid, which nevertheless acquires a non-trivial structure when Landau quantization is introduced. We now turn to composite fermions, which are not so simple - the results just derived will help us to understand the CF results.
3. Composite Fermions and Gauge Fluctuations

We now examine the spectral functions of composite fermions near \( \nu = 1/2 \) in the same manner as for electrons. We assume that the Lagrangian has the usual form for composite fermions interacting with a gauge field:

\[
\mathcal{L} = \int d^2x \psi \overline{\psi} ((-i \partial_0 + a_0) \psi(x) + \frac{1}{2m} \psi^3(x)(-i \partial_i + a_i)^2 \psi(x) + a_\mu D_{\mu\nu}^{-1} a_\nu)
\]

where in this case \( \nabla \times a = \Delta B \). The most singular component of the interaction with the gauge propagator, calculated at \( \Delta B = 0 \), is

\[
U(q, \omega) = -\frac{\tilde{\kappa} \times \tilde{q}}{m} \cdot D_{11}(q, \omega)
\]

\[
D_{11}(q, \omega) = \frac{q}{\chi q^s - i \gamma \omega}
\]

This is the transverse component, calculated within the Coulomb gauge. The exponent \( s \) can have values between 2 and 3. \( s = 2 \) when there are unscreened Coulombic electron-electron interactions and \( s > 2 \) corresponds to screening those interactions (typically by having the 2-d doped semi-conductor near to a conducting plate); the case \( s = 3 \) arises when the interaction is effectively zero-range (of strength \( v \), and range less than the magnetic length \( l \)).

The constants are \( \gamma = \frac{2a}{k_f} \) and \( \tilde{\chi} = \frac{s^2 \tilde{\epsilon}}{2 \pi} \) for

\( s = 2 \) and \( \tilde{\chi} = \frac{1}{2} \chi m + \frac{n}{4 \pi} \) for \( s = 3 \) in this random phase approximation (we denote the dielectric constant by \( \tilde{\epsilon} \)).

In what follows we first discuss the self-energy, and then the energy gap for CF’s, after it has been renormalised by gauge fluctuations (in the absence of gauge fluctuations, the gap is just \( \Delta \omega_c = \Delta B/m \), as discussed in the introduction).

(a) Composite Fermion Self-Energy: The results for the CF self-energy were given, without derivation, in a previous note by us [4]; here we briefly recall these (some details of the derivation appear in the Appendix).

We begin by writing down the real and imaginary parts of \( \Sigma_{CF}(\epsilon) \), at \( T = 0 \) (where they can be written in analytic form). For \( s = 2 \) one has:

\[
\Sigma'(\epsilon) = \frac{K_2 \Delta \omega_c}{\pi} \sum_{m=0}^{p-1} \log \left| \frac{(m + 1/2) \Delta \omega_c - \epsilon}{(m + 1/2) \Delta \omega_c + \epsilon} \right|
\]

\[
\Sigma''(\epsilon) = K_2 \Delta \omega_c \sum_{m=0}^{p-1} \left| \theta((-m - 1/2) \Delta \omega_c - \epsilon) \right|
\]

\[
+ \theta((m - 1/2) \Delta \omega_c) + \theta((m + 1/2) \Delta \omega_c) + (m + 1/2 + \epsilon)^{-\alpha} \theta(m + 1/2 + \epsilon) + (m + 1/2 + \epsilon)^{-\alpha} \theta(m + 1/2 + \epsilon)
\]

\[
K_2 = \frac{E_F \tilde{\epsilon}}{e^2 \sqrt{4\pi n_c}} = \frac{\tilde{\epsilon} \sqrt{4\pi n_c}}{2me^2}
\]

whilst for \( 3 \geq s > 2 \):

\[
\Sigma'(\epsilon) = \text{sgn}(\epsilon) \cot \left( \frac{2\pi}{s} \right) \Sigma''(\epsilon)
\]

\[
\Sigma''(\epsilon) = K_s \sum_{m=0}^{p-1} \left| \theta((-m - 1/2) \Delta \omega_c + \epsilon) \right|
\]

\[
+ \csc \left( \frac{2\pi}{s} \right) K_s \sum_{m=0}^{p-1} \left| \theta((-m - 1/2) \Delta \omega_c + \epsilon) \right|
\]

\[
(\epsilon - E_r + i\delta)
\]

\[
\Sigma(\epsilon) \sim \sum_r \frac{|V_r|^2}{\epsilon - E_r + i\delta}
\]

A rather extraordinary feature of the CF propagator appears once we closely examine \( \Sigma'(\epsilon) \) and \( \Sigma''(\epsilon) \) around the divergences - we notice that \( \Sigma'(\epsilon) \) shows only positive divergence on both sides of each Landau level. The paradox is that typically one would expect \( \Sigma(\epsilon) \) to have the form

\[
\Sigma(\epsilon) \sim \sum_r \frac{|V_r|^2}{\epsilon - E_r + i\delta}
\]

in simple perturbation theory if our starting fermion spectrum is composed of discrete (albeit degenerate) levels at energies \( E_r \), and \( V_r \) is some perturbation. This leads to divergences in \( |\Sigma'(\epsilon)| \), but we note that these divergences change sign each each time one crosses the energies \( E_r \).
This sign change occurs for $\Sigma'(\epsilon)$ in the electron-phonon problem; this is most clearly seen by plotting $\partial\Sigma'(\epsilon)/\partial\epsilon$ for this problem, where one sees a series of positive divergences around each Landau level, with no sign change, as shown in Fig. 7. This is exactly what we would expect from (33), since it yields

$$\frac{\partial\Sigma'(\epsilon)}{\partial\epsilon} \sim - \sum_r \frac{|V_r|^2}{(\epsilon - E_r + i\delta)^2} \tag{34}$$

The behaviour of $\Sigma_{CF}(\epsilon)$ is thus exactly the opposite of what one expects. The explanation of this paradox is that although $\Sigma_{CF}'(\epsilon)$ is strongly peaked (indeed divergent) around each Landau level, and does not change sign as one crosses a Landau level, Nevertheless the peak has such a peculiar shape that its Hilbert transform $\Sigma_{CF}'(\epsilon)$ also does not change sign as one crosses a Landau level. Each Landau level $r$ contributes a term $\sim (\epsilon - r)^{-\alpha} \theta(\epsilon - r)$ to $\Sigma_{CF}'(\epsilon)$, with long tails for $(\epsilon - r) \gg 1$. The Hilbert transform of such a function does not change sign as $\epsilon$ crosses $r$, unless $\alpha > 1/2$; however in the CF gauge theory, $0 \leq \alpha \leq 1/3$.

This explains the paradoxical form of $\partial\Sigma/\partial\epsilon$ for composite fermions - it comes from the very long "tails" which extend out from each Landau level. These have no counterpart in the self-energy of a Fermi liquid, such as the electron-phonon problem discussed above.

(b) Composite Fermion Energy Gap: We now evaluate the renormalised energy gap $\Delta\omega^*_c$, using our perturbative result for $\Sigma'(\epsilon)$ above. In this problem the bare CF’s have a gap $\Delta\omega_c = DB/m$ separating the occupied and unoccupied states. We look again to equation (12) to find the location of the poles after the self energy has been included in the CF propagator. For each value of $n$ there can be many solutions to (12), in particular we note that for each value of $n$ there is exactly one pole within the gap $|\epsilon| < \frac{\Delta\omega_c}{2}$. Since the imaginary part is zero in this region these poles appear as $\delta$-functions in the spectral function. Using exactly the same approximations as in the electron-phonon calculation, (18) to (22), we find that for $s = 2$ the renormalised gap is

$$\Delta\omega^*_c \approx \Delta\omega_c \left| 1 - \frac{2K_2}{\pi} \log(2p + 1) \right|^{-1} \tag{35}$$

and for $s > 2$

$$\Delta\omega^*_c \approx \Delta\omega_c \left| 1 - \frac{4K_2}{\sqrt{3}} p^{-\alpha} \right|^{-1} \tag{36}$$

Similar results have been obtained by Stern and Halperin[2] and by Kim et. al.[1].

However the use of equation (18) to get a solution for (12) is only valid if $\Sigma'(\epsilon) + \epsilon$ varies slowly over $\epsilon$ such that $|\Sigma'(\epsilon) + \epsilon| < \frac{\Delta\omega_c}{2}$, and thus (19) is a poor approximation when $K_2$ is in an intermediate range (≈ 0.5). Moreover, the divergence in (13) at $K_2 = \pi/(2\log(2p + 1))$ is unphysical; instead we expect the actual gap $\Delta\omega^*_c$ to decrease monotonically with $K_2$. In fact, we estimate that in the actual experiments so far done, $K_2 \approx 0.8$, which places it right in the intermediate range, so it is necessary to go beyond the estimates in (35) and (36).

We do this first by solving (12) numerically; as an example we have done this for $p = 50$, as shown in Fig. 8. The numerical results show that for small values of the coefficient the gap is reduced by a very small amount.
For larger values of $K_2$ the gap decreases rapidly. Turning our attention to the spectral function shown in Fig. 2 we see that there is a simple physical interpretation for this. Small values of the coefficient give rise to very narrow double peaks at each cyclotron energy in the spectral function. This is the result of multiple solutions to eq. (12), in fact there are two for each logarithmic peak in $\Sigma(p\epsilon)$. The extreme narrowness of these peaks suggests that these excitations are merely Landau level mixing. This conclusion is supported by the fact that for small values of $K_2$ the gap is only very slightly reduced. As $K_2$ is increased the double peaks are reduced in size and eventually give rise to incoherent structures. In the example we have been considering, $p = 50$, this crossover occurs when $K_2 \approx 0.5$ which is also where the gap starts to decrease. When the coefficient becomes large there is no longer any evidence of simple Landau level mixing.

![Figure 8](image)

**FIG. 8.** First order perturbation results for the effective gap, $\Delta \omega^*_c$, as a function of coupling, $K_2$, for $p = 50$. The solid curve is $|1 - \frac{2\pi}{\ln(2p+1)}|^{-1}$ and points are numerical solutions to eq. (12). The inset shows the same data plotted as a function of $n_e$, which is related to $K_2$ via eq. (29).

From the analysis it is clear that experimentalists should be rather cautious in fitting results for $\Delta \omega^*_c$ or $m^*/m$ to SdH or longitudinal resistivity data - our numerical results differ strongly from the analytic approximations (23) and (28) which have previously appeared in the literature.

However, the numerical results in Figs. 8 and 9 are still based on the perturbative results in (27) - (32) for $\Sigma(p\epsilon)$. We now go a little beyond this lowest-order perturbation theory framework.

**4. Iterative Self-Consistent Results for $\Sigma(p\epsilon)$ and $\Delta \omega^*_c$**

As already explained in refs. 22, the CF problem is essentially a non-perturbative one. The results in refs. 22 address the $\nu = 1/2$ gapless system; an analysis of the $s = 2$ case for $\nu \neq 1/2$ was also given by Stern and Halperin 23 but taking no account of the structure described above.

Here we attempt to improve the results just given, by solving iteratively for the self-energy. We have found a way to do this which handles the coherent pole contributions to $\Sigma(p\epsilon)$ in an exact rainbow summation; the incoherent parts must still be treated approximately.

We start from the iterative equation

$$\Sigma^{(i+1)}(p\epsilon) = \sum_m \int d\epsilon' D(p\epsilon - \epsilon') G_m^{(i)}(\epsilon')$$  \hspace{1cm} (37)\]  

with

$$G_m^{(i)}(\epsilon) = \frac{1}{\epsilon - (m + 1/2)\Delta \omega_c + \Sigma^{(i)}(p\epsilon) - i\delta}.$$  \hspace{1cm} (38)\]

When the iterations converge we will have succeeded in summing over all rainbow graphs. This kind of calculation is usually difficult but in this case the analysis is simplified by the following. First, because the real part of $\Sigma^{(1)}(p\epsilon)$ is bounded on either side of the gap $|\epsilon| < \Delta \omega_c/2$ by logarithmic peaks, we are guaranteed that there are solutions to (38) for all $n$ within the gap. That the imaginary part of $\Sigma^{(1)}(p\epsilon)$ is zero in the gap implies that there are actually $2p$ $\delta$-function peaks in $\sum_{m=-p}^{p-1} G_m^{(1)}(p\epsilon)$ within the gap (exactly one for each $m$). The distance between the peaks resulting from $m = 0$ and $m = -1$ gives a renormalized gap, $\Delta \omega_c^{(1)}$.  

![Figure 9](image)

**FIG. 9.** Imaginary part of the Green function of composite fermions in a finite magnetic field with $n = 0$ and $p = 50$. The upper figure has a coefficient $K_2 = 0.31$ and the lower one has $K_2 = 6.3$. The energy is in units of $\Delta \omega_c$.  

![Figure 8](image)

**FIG. 8.** First order perturbation results for the effective gap, $\Delta \omega^*_c$, as a function of coupling, $K_2$, for $p = 50$. The solid curve is $|1 - \frac{2\pi}{\ln(2p+1)}|^{-1}$ and points are numerical solutions to eq. (12). The inset shows the same data plotted as a function of $n_e$, which is related to $K_2$ via eq. (29).
Next we use $G^{(1)}$ in (37) to generate $\Sigma^{(2)}$. We are particularly interested in the form of $\Sigma^{(2)}$ within the gap because this is where further corrections to the renormalized gap originate. The contributions to $\text{Im}G^{(1)}$ may be split into two parts: a coherent piece, coming from the $\delta$-function peaks within the gap and an incoherent piece in the region outside the gap (we use this terminology for convenience—there may in fact be solutions to (12) in the latter region which correspond to poles in $G^{(0)}$). Upon evaluating $\text{Re}\Sigma^{(2)}$ we notice that the coherent parts of $\text{Im}G^{(1)}$ give rise to exactly the same logarithmic divergences as seen in $\Sigma^{(3)}$, except that the peaks are located at the poles of $\text{Im}G^{(1)}$ and each peak is weighted by a factor

$$w(\epsilon_m) = \left[ 1 + \frac{\partial \Sigma'(\epsilon_m)}{\partial \epsilon} \right]^{-1}$$

(39)

where $\epsilon_m$ are the positions of the poles. However, in most cases the weight coming from the poles within the gap is small compared to the weight of the incoherent part of $\text{Im}G^{(1)}$, which means that there may be significant contributions to $\text{Re}\Sigma^{(2)}$ that do not come from the poles of $\text{Im}G^{(1)}$. Only the coherent parts of $\text{Im}G^{(1)}$ contribute to $\text{Im}\Sigma^{(2)}(\epsilon)$ for $|\epsilon| < \Delta\omega_c/2$, which vanishes within the renormalized gap, $|\epsilon| < \Delta\omega_c^{(1)}/2$.

The corrections arising from coherent parts may be calculated exactly. However, we are forced to examine the incoherent corrections using approximations, beginning with the use of (18). This approximation has the effect of smearing out the logarithmic peaks at the Landau levels. As discussed above, these features are a crucial element within the gap; however outside the gap their role is not as important. They may generate quasiparticles peaks as solutions to (12) but the weight of these peaks is very small. The artificial smoothening of the Landau level structures would greatly simplify a numerical integration of (19). We have used this procedure to calculate the gap as a function of $p$, keeping only results which appear to have converged after the second step. We find corrections result that are as large as 30% of the first order result. These calculations are shown in Fig. 10 as $K_2$ plotted as $\frac{w_m}{m} = \frac{w_m}{m_{\text{ref}}}$ for various couplings. We estimate the actual coupling, in the experiment of Du et. al., to be $K_2 \approx 0.8$ (see the Appendix). We emphasize that the results for small values of $p$ are not meaningful because of the approximation (3), which assumes that $p$ is large, thus there is no overlap with experimental results. Therefore, we draw no conclusions about the exact relation between $m^*/m$ and $p$. However, there is a range of $p$ where $p$ may be considered to be large and $\log(2p+1)$ is not. In this range we expect our results to be valid but to have not yet reached the asymptotic limit of $m^*/m = K_2 \log(2p+1)$. This calculation shows the effective mass to be far more sensitive to the coupling than to $p$, thus it is difficult determine what relation these results have to experiments without knowing the coupling exactly.

FIG. 10. The effective mass, $m^*$ as a function of $B$ and $\nu$ for composite fermions. The lower $x-$axis has been obtained by assuming an electron density $n_e = 2.23 \times 10^{11} \text{cm}^{-2}$. The crosses are numerical calculations which include corrections coming from $\Sigma^{(2)}$ (the self-consistent self-energy). The curves are a guide for the eye.

The information shown in Fig. 10 has been calculated using the fillings shown on the upper axis for the different couplings shown on the plots. This information may be interpreted in two ways. First, we may assume that the density, $n_e$, is fixed and that the different couplings arise by variations of the other parameters in (29). The values of $B$ shown in the lower $x$-axis have been determined using $n_e = 2.2 \times 10^{11} \text{cm}^{-2}$. This allows us to compare to the experimental data of Du et. al. but their data does not go beyond $\nu = 3/7$, which is not within the range of our approximations. (Obviously, we cannot compare the results of Du et. al. at these values either.) Alternatively, we may assume that each curve in Fig. 10 is associated with a different density, $n_e$. Using $\nu = 13$ and $m = .07m_e$ in (28) we replot the results as shown in
Fig. 11. The curves show that the effective mass increases with electron density, which agrees with the observations by Leadley et al., but there is no quantitative agreement for the range of data we have calculated, due to the fact our approximations are not valid for the range of \( \nu \) studied in the experiment.

![Effective mass vs electron density](image)

**FIG. 11.** The effective mass \( m^* \) as a function of electron density. These are the same results as in Fig. 10 except that the x-axis has been obtained using \( n_e = \frac{1}{4 \pi} \left( \frac{2 K_e m_e^2}{e} \right)^2 \) with \( \tilde{e} = 13 \) and \( m = 0.07 m_e \). The crosses are numerical calculations which include corrections coming from \( \Sigma^{(2)} \) (the self-consistent self-energy).

5. Summary

The main point of this paper has been to deal directly with the singular structure induced in \( \Sigma(\epsilon) \) and \( G(\epsilon) \) by Landau level quantization, rather than smoothing over it, as has been done in previous work.

As a result of this we find that the size of the CF gap in the gauge theory of the FQHE is considerably changed, particularly in the intermediate coupling regime which is experimentally relevant. The CF gauge theory is so far the only theory in which it seems feasible to calculate this gap (apart from finite-size numerical calculations), for arbitrary stable filling fractions. To test this theory it is necessary to take account of the singular structure we have discussed. At the present time gap measurements are subject to some uncertainty, but our results indicate a significant discrepancy between claimed experimental results for the gap (or what is the same, \( m^*/m \)) and a self-consistent theory.

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Appendix

In this appendix we give the details of the calculations of the three self-energies considered in the main text.

(a) Debye Phonons

In the self-energy calculations presented in the text we have assumed a model similar to the one described by Ezawa et al.\(^4\). In this model it is assumed that phonons propagate throughout the semiconductor layers without reflection at the boundaries. The electrons are confined to a very thin region which we approximate to be two-dimensional. The usual forms of the electron-phonon interactions for both Debye and Einstein models may be used. For the Debye model, the electron-phonon interaction has the form

\[
H_{\text{int}} = \Xi_D \int d^3 r \psi^*_c(\vec{r}) \psi_c(\vec{r}) \nabla \cdot \vec{u}(r)
\]

where \( \Xi_D \) is the deformation potential, and

\[
\vec{u}(r) = \sum_{\vec{q}} \tilde{c} \sqrt{\frac{\hbar}{2 \omega_{\vec{q}}^d V}} (a_{\vec{q}} e^{i \vec{q} \cdot \vec{r}} + a_{\vec{q}}^\dagger e^{-i \vec{q} \cdot \vec{r}})
\]

is the displacement operator, with ion mass density \( \rho \), cell volume \( V = \alpha V \) and dispersion \( \omega_{\vec{q}} = \omega_{\vec{q}}^d \); and \( \psi_c \) is the electron wavefunction expressed in a Landau level basis,

\[
\psi_c(\vec{r}) = \sum_{n,k} c_{n,k} \left( \frac{1}{2\pi n_l^2} \right)^{1/2} \left( \frac{m \omega_c}{\pi \hbar} \right)^{1/4} H_n(y/l - kl) \exp \left( -\frac{(y/l - kl)^2}{2} + i kx \right) (\delta(z))^{1/2}.
\]

In these expressions \( \omega_c = eB/m \) is the cyclotron frequency, \( m \) is the electron mass and \( l = (\hbar/eB)^{1/2} \) is the magnetic length. This leads to

\[
H_{\text{int}} = \Xi_D \sum_{n,n',k,k',\vec{q}} \frac{\hbar}{2 \omega_{\vec{q}}^d V} \tilde{e}(a_{\vec{q}} + a_{-\vec{q}}^\dagger) c_{n,k}^\dagger c_{n',k'}
\]

where

\[
\Lambda(n,n',k,k',\vec{q}) = \left( \frac{1}{2 n_n' \Gamma} \right)^{1/2} \left( \frac{m \omega_c}{\pi \hbar} \right)^{1/2} \frac{1}{l} \times \int d^3 r H_n(y/l - kl) H_{n'}(y/l - k'l) \delta(z)
\]
\[ \times \exp \left( -\frac{(y/l - kl)^2}{2} - \frac{(y/l - k l')^2}{2} + i q_y y \right) \\
+ i(-k + k' + q_x) x + i q_z z \right) \quad (45) \]

\[ \frac{\delta(-k+k'-q_x)}{l} \exp \left( -\frac{q'^2 l^2}{4} + i q_y (k + k') l^2 \right) \]

\[ \left( \frac{2n'}{2 n l} \right)^{1/2} \left( \frac{-q_x l - i q_y l}{2} \right)^{n-n} L_{n'}^0 \left( -\frac{q'^2 l^2}{2} \right) \quad (46) \]

for \( n \leq n' \) and \( q^2 = q_x^2 + q_y^2 \). The variables \( k, k' \) are of no concern as far as the properties of the electron Green’s functions are concerned because the energy of the non-interacting electrons does not depend on \( k \) and the final form of \( H_{\text{int}} \) does not depend on \( k, k' \) or \( q_x \). Furthermore, in what follows we assume that the phonon operator is isotropic with respect to the \( x-y \) plane. In the limit of large \( n \approx n \) and \( q l' \gg (n - n)/n \) the matrix element \( |\Lambda|^2 \) reduces to

\[ |\Lambda(n, q)|^2 = \frac{1}{\pi q} \left( \frac{e B}{2n} \right)^{1/2} \quad (47) \]

The final form of \( H_{\text{int}} \) is

\[ H_{\text{int}} = \Xi_D \sum_{n,n',q} \sqrt{\frac{\hbar}{2c_\rho V}} (a_q + a^\dagger_{-q}) c_n^\dagger c_{n'} \Lambda(n, n', q) \]

(48)

The self-energy is derived from the second order contribution of \( H_{\text{int}} \)

\[ \Sigma_n(\epsilon) = \Xi_D^2 \left( \frac{\hbar}{2c_\rho p a} \right) \sum_{n'=0} \int \frac{q d^2 q}{(2\pi)^2} \int \frac{\omega d \omega}{\pi} |\Lambda(n', q)|^2 \]

\[ \times \left( \frac{1}{\epsilon - \omega - (n + 1/2)\omega_c - i\delta} \right) \quad (49) \]

where the factor \( 1/a \) comes from the \( q_x \)-integral. We now shift the energies by an amount \( p \omega_c \), which is the chemical potential when it lies halfway between Landau levels. The index is shifted by \( p \). Performing the integrals yields the zero temperature results \( \Xi_1 \) and \( \Xi_2 \). When the self-energy and the energy \( \epsilon \) are expressed in units of the Fermi energy, \( E_F = p \omega_c \), the dimensionless coefficient is

\[ \overline{K}_D = \frac{\Xi_2^2 m E_F}{4 \sqrt{2 \pi c_\rho a}} = \frac{3 \pi \Xi_2^2 E_F m}{2 c_\rho \Delta \pi^3 n_e e A_{\text{m}}} \quad (50) \]

Using typical values for GaAs, i.e., \( \omega_D = (6\pi)^{1/3} c_s / a = 345 \text{K} \), \( \Xi_D = -4.4 \text{eV} \), \( m = 0.07 m_e \), \( M \approx 10^3 m_e \), \( a = 3.6 \text{Å} \), \( n_e = 2.25 \times 10^{11} \text{cm}^{-2} \) and \( E_F \approx 1.5 \times 10^{-3} \text{eV} \), one has \( \overline{K}_D \approx 0.23 \). Equation \( (41) \) shows that there is no magnetic field dependence of the coefficient. The relation between \( K_D \), which is given by \( (41) \), and \( \overline{K}_D \) is \( K_D = \overline{K}_D / p \). The difference arises because \( K_D \) is used when the self-energy and the energy \( \epsilon \) are explicitly given in units of \( \omega_c \), which is appropriate for solving \( \Xi_1 \).

(b) Einstein phonons

Following the approach of Engelsberg and Schreiffer we choose a coupling that is independent of \( q \)

\[ H_{\text{int}} = g \sum_{n,n',q} (a_q + a^\dagger_{-q}) c_n^\dagger c_{n'} \Lambda(n, n', q) \quad (51) \]

This leads to a self-energy

\[ \Sigma_n(\epsilon) = g^2 \sum_{n'} |\Lambda(n, n', q)|^2 \int \frac{d^2 q}{(2\pi)^2} \int_0^\infty \frac{d\omega}{\pi} \]

\[ \times \frac{1}{\epsilon - \omega - (n + 1/2)\omega_c - i\delta} \quad (52) \]

where we have assumed that the normalization of the \( q \)-integral is \( (2\pi)^2/2a \). Equations \( (44) \) and \( (47) \) follow directly; with \( \Sigma \) and the energy \( \epsilon \) expressed in units of \( E_F \), the dimensionless coefficient is

\[ \overline{K}_E = \frac{g^2 (2p)^{1/2} a}{4E_F^2} = g^2 a (4\pi n_e)^{1/2} \quad (53) \]

In this form there is no dependence of the coupling on the magnetic field.

(c) Gauge Fluctuations

The original theory of strongly interacting electrons in a magnetic field takes the form

\[ \mathcal{L} = \psi_\uparrow(x)(-i\partial_t - eA_t)^2 \psi_\uparrow(x) + \psi_\downarrow(x)(i\partial_t - eA_t)^2 \psi_\downarrow(x) \]

\[ + \int d^2 y \psi_\uparrow(x)(x)(x-y) \psi_\downarrow(y) \psi_\uparrow(y) \psi_\downarrow(y) \quad (54) \]

where \( B \equiv \nabla \times A \) is the external magnetic field.

The electrons are transformed into composite fermions by attaching two magnetic flux quanta to each. The constraint that this additional magnetic field, \( b \), is proportional to the electron density is

\[ b \equiv \nabla \times a = 4\pi \psi_\uparrow(x) \psi(x) \quad (55) \]

and is implemented with the use of a Chern-Simons term in \( \mathcal{L} \). In addition, the CF’s experience a gauge field that is the difference between the external field \( A \) and the “statistical” field \( a \), \( A - a = \Delta A \):

\[ \mathcal{L} = \psi_\uparrow(x)(-i\partial_t + a_0)^2 \psi_\uparrow(x) + \psi_\downarrow(x)(i\partial_t - e\Delta A_t)^2 \psi_\downarrow(x) \]

\[ + \frac{a_0 a}{4\pi} \nabla \times a \int d^2 y \nabla \times a(v(x-y) \nabla \times a(y)) \quad (56) \]

where the third term is the Chern-Simons term and we have used the constraint \( (55) \) to rewrite the last term. In this expression \( \psi \) is a fermionic operator representing composite fermions. The theory is completed by allowing fluctuations of the gauge field, \( \delta a \),
\[ \mathcal{L} = \psi^\dagger(x)(-i\partial_t + a_0)\psi(x) \\
+ \psi^\dagger(x)\left(\frac{i\partial_x - e\Delta A_1 - \delta a}{2m}\right)^2\psi(x) + \delta a_i D^{-1}_ij \delta a_j \]  

(57)

The last term is the effective action of the gauge fluctuations, which yields \([2][3]\). The second term yields the free CF Hamiltonian,

\[ H_{CF} = \frac{\psi^\dagger(x)(i\partial_t - e\Delta A_1)^2\psi(x)}{2m} \]  

(58)

and the CF-gauge fluctuation interaction,

\[ H_{int} = \frac{\delta a_i(x)\psi^\dagger(x)(i\partial_t - e\Delta A_1)\psi(x)}{m} \]  

(59)

in the Coulomb gauge, \( q \cdot \delta a = 0 \). The eigenfunctions of \( H_{CF} \) are the same as \((13)\) except that \( \omega_\epsilon \) is replaced by \( \Delta \omega_\epsilon \). The operator \( V_1 \equiv (i\partial_t - e\Delta A_1)/m \) introduces additional complications in the determination of the final form of \( H_{int} \). We are only considering interactions with the transverse component of \( \delta a = \delta a_1 \), corresponding to the transverse component of \( V_1 = \cos \theta V_x + \sin \theta V_y \).

\( V_x \) and \( V_y \) can both be expressed in terms of creation and annihilation operators which act on the harmonic oscillator part of \( \psi \):

\[ V_x = \sqrt{\frac{\Delta \omega_\epsilon}{2m}}(a + a^\dagger) \]  

(60)

\[ V_y = i\sqrt{\frac{\Delta \omega_\epsilon}{2m}}(a - a^\dagger) \]  

(61)

\[ V_1 = \sqrt{\frac{\Delta \omega_\epsilon}{2m}}(ae^{i\theta} + a^\dagger e^{-i\theta}) \]  

(62)

In this case the matrix element will be

\[ \Lambda_{CF}(n, n', q) = \sqrt{\frac{\Delta \omega_\epsilon}{2m}}(\sqrt{n}\Lambda(n, n' - 1, q)e^{i\theta} + \sqrt{n' + 1}\Lambda(n, n' + 1, q)e^{-i\theta}) \]  

(63)

which, in the limit of large \( n \approx n' = p \), yields

\[ |\Lambda_{CF}(n, n', q)|^2 \approx \frac{4p\Delta \omega_\epsilon}{2m}|\Lambda(n, q)|^2. \]  

(64)

The coefficient is

\[ \frac{4p\Delta \omega_\epsilon}{2m} \approx \frac{2E_F}{m} \]  

(65)

Thus we are left with

\[ \Sigma_n(\epsilon) = \frac{2E_F}{m} \int \frac{d^2q}{(2\pi)^2} \int_0^\infty \frac{d\omega}{\pi} \int_0^\infty \frac{d\epsilon'}{\pi} \sum_{n'=0}^\infty |\Lambda(n, n', q)|^2 \]

\[ \text{Im}D(q, \omega)\text{Im}G^\text{GH}_{n'}(\epsilon') \left( \frac{\theta(\epsilon')}{\epsilon - \epsilon' - \omega + i\delta} + \frac{\theta(-\epsilon')}{\epsilon' - \epsilon + \omega + i\delta} \right) \]  

(66)

Just as in the phonon case, a cut-off in the sum at \( n' = 2p \) is needed. Equations \((27)-(29)\) assume that the self-energy has units of \( \Delta \omega_\epsilon \), which is more appropriate for solving the self-consistent equation, \((12)\).

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