Regular black holes with sub-Planckian curvature

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Abstract

We construct a sort of regular black holes with a sub-Planckian Kretschmann scalar curvature. The metric of this sort of regular black holes is characterized by an exponentially suppressing gravity potential as well as an asymptotically Minkowski core. In particular, with different choices of the potential form, they can reproduce the metric of Bardeen/Hayward/Frolov black hole at large scales. The heuristical derivation of this sort of black holes is performed based on the generalized uncertainty principle over curved spacetime which includes the effects of tidal force on any object with finite size which is bounded below by the minimal length.

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I. INTRODUCTION

It is widely believed that quantum gravitational effects would remove the curvature singularity of a black hole. Before a complete theory of quantum gravity could be established, people have extensively investigated various non-singular black holes at the phenomenological level, which usually are not solutions to the vacuum Einstein equations. Instead some exotic matter fields must be introduced which in general violate some energy conditions in general relativity.

Regular black holes were originally proposed to avoid the singularity of ordinary black holes\cite{1–11}, which includes the well known Bardeen black hole, Hayward black hole as well as Frolov black hole\cite{1–4}. The Bardeen black hole is the first regular black hole that obeys the weak energy condition. It was originally proposed as a counterexample to prove the possibility of singularities in black hole space without the need to assume global Cauchy hypersurfaces or strong energy conditions\cite{12}. Later, in order to describe the formation and evaporation of black holes, the Hayward black holes were constructed\cite{2, 13}. Ref.\cite{14, 15} constructed a general class of black holes and included Bardeen and Hayward black hole. The most important property of these regular black holes is that its Kretschmann scalar curvature is finite everywhere. In particular, for Bardeen/Hayward/Frolov black holes, the singularity which appears at $r = 0$ in ordinary black holes now is replaced by the de-Sitter spacetime. That is to say, the asymptotic metric of those regular black holes near the center ($r \to 0$) becomes

$$F(r) = 1 - \frac{r^2}{l^2},$$

which is nothing but the solution to the Einstein equation with the cosmological constant $\Lambda = 3/l^2$, where $l$ is a constant related to the Planck length $l_p$ and $\Lambda$ may be understood as the effective cosmological constant at small distance\cite{2}. Therefore, they are also called as regular black holes with de-Sitter core. In Ref.\cite{16}, the authors analyzed the general conditions that would lead to a finite Kretschmann scalar and originally proposed a black hole solution with an asymptotically Minkowski core. In contrast to all the previous regular black holes as mentioned above, as $r \to 0$, the function $F(r)$ in the metric becomes $F(r) = 1$ such that the singularity of the black hole is replaced by Minkowski spacetime rather than de-Sitter spacetime. Thus we call this solution as the regular black hole with Minkowski core. This kind of regular black hole is featured by an exponentially suppressing potential.
and a vanishing Hawking temperature at the final stage of evaporation, in contrast to the phenomenon that the Hawking temperature becomes divergent for classical black holes, as well to the phenomenon that the Hawking temperature takes a maximal value staying at the Planck energy level for semi-classical black holes where the quantum effects of gravity such as the generalized uncertainty principle (GUP)[17–20] or modified dispersion relations (MDR) are taken into account[21–24]. Thus, this regular black hole solution provides more realistic picture for the final stage of black hole evaporation and more reasonable behavior of the black hole remnant. Subsequently this solution has been generalized into a kind of regular black holes with different forms of the exponential potential in [25–30]. Nevertheless, we notice that for the regular black holes proposed in [16], the maximal value of Kretschmann scalar curvature depends on the mass of black hole, which is in contrast to Bardeen/Hayward/Frolov black holes where the maximal value of Kretschmann scalar curvature is mass independent. Explicitly, once the parameter $\alpha$ is given in [16], which is supposed to be fixed by the quantum effects of gravity, the maximal value is proportional to the square of the mass. It means that Kretschmann scalar curvature is not bounded above, but can exceed the Planck mass density ($M_p^4$) easily by increasing the mass of the black hole, which of course is not reasonable or expectable from quantum gravity point of view. Or in another word, it implies that the metric for previous regular black holes with Minkowski core makes sense only for black holes with small mass at the Planck scale, or for the final stage of black hole evaporation. Additionally, the regular black holes with Minkowski cores also have interesting feature. The stress-energy tensor vanishes at the centre. This implies that the physics of this region is greatly simplified compared to the black holes with de-Sitter core[28, 31].

The purpose of this paper is twofold. Firstly, we intend to construct a new sort of regular black holes with Minkowski core, getting rid of the shortcoming that the Kretschmann scalar curvature is not bounded by the Planck mass density, such that the metric is applicable to the black hole with arbitrarily large mass, and thus applicable to all the stage of black hole evaporation from the beginning to the end. Secondly, we intend to disclose a closer relation between regular black holes with Minkowski core and those with de-Sitter core. Specifically, we will demonstrate that with different choices of the potential form, this sort of black holes can reproduce the metric of Bardeen/Hayward/Frolov black hole at large scales. In this situation, these two kinds of black holes may have distinct behavior only at the late stage
of evaporation.

This paper is organized as follows. In next section we will present the general setup for this sort of regular black holes with spherical symmetry, and then in section three we find the condition that Kretschmann scalar curvature can be bounded above and then show a specific example in comparison with the solution proposed in [16]. In section four and five, we will demonstrate that for specifical choice of the potential from, the regular black holes have the similar behavior as Bardeen/Hayward black holes at large scales, respectively. Finally, we will argue how the consideration of generalized uncertainty principle over curved spacetime could lead to this sort of black holes in a heuristical manner. We point out that the effect of tidal force on any object with finite size bounded by the minimal length plays an essential role in this modification.

II. THE GENERAL SETUP FOR STATIC REGULAR BLACK HOLES WITH SPHERICAL SYMMETRY

Firstly, to construct the regular black hole at the phenomenological level, the key point is to obtain a finite value for Kretschmann scalar curvature such that the singularity is avoided. For this purpose we consider a static spherically symmetric black hole with a general form of the metric

$$ds^2 = -(1 + 2\phi_1) dt^2 + (1 + 2\phi_2)^{-1} dr^2 + r^2 d\Omega^2,$$

(2)

where \(\phi_1 = \phi_1(r), \phi_2 = \phi_2(r)\) are gravitational potentials. As shown in [16], it was found that to get rid of the singularity, one essential condition is that as \(r \to 0\), the asymptotic behaviors of modified gravitational potential must take the form

$$\phi_1 \to r^l, l \geq 2,$$

$$\phi_2 \to r^s, s \geq 2.$$

(3)

For simplicity, in this paper we consider the case \(\phi_1 = \phi_2\) such that the metric takes the following form

$$ds^2 = -F(r) dt^2 + \frac{1}{F(r)} dr^2 + r^2 d\Omega^2,$$

(4)

with

$$F(r) = 1 + 2\psi(r).$$

(5)
Typically, if $\psi(r) = -GM/r$, then it is nothing but the Schwarzschild black hole. Now, in order to include the strong quantum effects of gravity which would modify the singularity behavior at the Planck scale, we assume $F(r)$ would be a general form, with the requirement that it goes back to Schwarzschild black hole at asymptotical infinity. The location of the horizon $r_h$ is determined by $g^{rr} = F(r_h) = 0$.

From Einstein field equations $G^{\mu\nu} = 8\pi T^{\mu\nu}$, one can derive the effective stress-energy tensor which is given by

$$T^{\mu\nu} = \frac{1}{4\pi r} \text{diag}\{\frac{\psi}{r}, \psi, \frac{\psi'}{r}, \frac{\psi'}{r}, \psi, \psi' + \frac{r}{2} \psi'', \psi' + \frac{r}{2} \psi''\}.$$  

Furthermore, it is straightforward to derive Kretschmann scalar curvature which is given by

$$K = R^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda} = \frac{16\psi(r)^2}{r^4} + \frac{16\psi'(r)^2}{r^2} + 4\psi''(r)^2. \tag{6}$$

The Hawking temperature of the black hole and the luminosity are respectively given by

$$T = \frac{F'(r_h)}{4\pi} = \frac{\psi'(r_h)}{2\pi}, \quad L = \frac{\sigma r_h^2 F'(r_h)^4}{64\pi^3} = \frac{\sigma r_h^2 \psi'(r)^4}{4\pi^3}. \tag{7}$$

Now we propose a new sort of regular black holes with asymptotically Minkowski core which is constructed with an exponentially suppressing form of the gravitational potential, where $\psi(r)$ is specified as

$$\psi = -\frac{GM}{r} e^{-\alpha_0 (GM)^x l_p^x/r^n}, \tag{8}$$

with $\alpha_0 > 0$, $n > x \geq 0$ and $n \geq 1$. In addition, all the parameters $\alpha_0$, $n$ and $x$ are understood as dimensionless. Throughout this paper, for simplicity we ignore the factor difference of $G$ and $l_p$ by setting $G = l_p^2 = 1$.

In comparison with previous regular black holes with Minkowski core in [16], the exponential form in our formalism may depend on the mass of black hole, which plays an essential role in suppressing the maximal value of Kretschmann scalar curvature to be sub-Planckian. Moreover, for specific values of $x$ and $n$, one can establish a one-to-one correspondence between the regular black hole with Minkowski core and that with de-Sitter core at large scale. We will explicitly demonstrate this feature in next sections. Obviously, if $x = 0$, then it goes back to the regular black hole proposed by Li et.al. [16].
III. THE REGULAR BLACK HOLE WITH SUB-PLANCKIAN KRETSCHMANN SCALAR CURVATURE

In this section we will construct a regular black hole with sub-Planckian Kretschmann scalar curvature by setting $x = 1$ and $n = 2$.

First of all, it is helpful to understand why all the previous regular black holes with $x = 0$ lead to a huge Kretschmann scalar curvature whenever the mass of black hole becomes large. For instance, for the regular black hole proposed by Li et al. [16], the gravity potential is $\psi = -\frac{M}{r} e^{-\alpha_0/r^2}$, and Kretschmann scalar curvature is given by

$$K = \frac{16M^2 e^{-2\alpha_0/r^2}}{r^6} (3 - \frac{14\alpha_0}{r^2} + \frac{33\alpha_0^2}{r^4} - \frac{20\alpha_0^3}{r^6} + \frac{4\alpha_0^4}{r^8}),$$

which is finite everywhere. In particular, near the center $r \to 0$, $K \to 0$, which is in contrast to the standard Schwarzschild black hole in which $K$ becomes divergent. However, one notices that once $\alpha_0$ is fixed as it should be from the quantum gravity point of view, the maximum value of $K$ is proportional to $M^2$, which means this value may easily exceed the Planck mass density for large black holes, namely $K > 1$ with the unit of $m_p^4$. We show this behavior in the right plot of Fig.1. Of course this feature is not satisfactory if one insists that all the reasonable quantities should be sub-Planckian due to the quantum gravity effects.

On the other hand, if we assume that the exponential factor could be mass dependent, then such situation would change dramatically. As a matter of fact, if we set $n = 2$ and replace $\alpha_0$ by $\alpha_0 M^x$, then one obtains $K$ with the similar expression as Eq.(9), simply replacing $\alpha_0$ by $\alpha_0 M^x$. A simple algebra shows that now the maximal value of $K$ is proportional to $M^{2-3x}/\alpha_0^3$. Therefore, if we demand $x \geq 2/3$, then the maximal value would inversely be proportional to the mass. It is this observation that leads us to propose such generalized model. Without loss of generality, we will set $x = 1$ in this section and compare this regular black holes with the previous one [16], which is obtained by setting $x = 0$.

Firstly, we present the basic properties of the regular black hole when we take $x = 1$. From $F(r_h) = 0$, one obtains the relation between the location of horizon $r_h$ and the mass $M$ as

$$2M = r_h e^{\alpha_0 M/r_h^2}.$$
Then the radius of the horizon $r_h$ can be expressed as a function of mass

$$r_h = 2M \sqrt{\frac{\theta}{W(\theta)}}, \quad \theta = -\frac{\alpha_0}{2M},$$

where $W(\theta)$ is the Lambert-W function. As discussed in [28, 32], the real-valued $W(\theta)$ with negative arguments has two branches, corresponding to the outer and inner horizons of the black hole, respectively. The inner horizon of the black hole is located at $r = 2M \sqrt{\frac{\theta}{W^{-1}(\theta)}}$ while the outer horizon is located at $r = 2M \sqrt{\frac{\theta}{W_0(\theta)}}$. In this paper, we are concerned with the thermodynamical properties of the black hole which are closely related to the outer horizon, thus we will concentrate on the analysis of the outer horizon, where the Lambert-W function is given by:

$$W_0(\theta) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} \theta^n,$$

with $W_0(\theta) \geq -1$ [33].

The above equation is quite similar to the one for $x = 0$, but now the variable $\theta = -\frac{\alpha_0}{2M}$, rather than $\theta = -\frac{\alpha_0}{2M^2}$ in [16]. Moreover, as pointed out in [16], a real $W$ requires $\theta \geq -e^{-1}$ such that we have the lowest bound for the mass of the black hole

$$M \geq \frac{e\alpha_0}{2}.$$ 

When this bound is saturated, the black hole is characterized by the minimal radius of the horizon $r_h = \sqrt{e\alpha_0}$, which may be treated as the remnant of the black hole evaporation, as
we will elaborate it as below. We remark that when $M = \frac{e\alpha_0}{2}$, the inner and outer horizons of the black hole are merged and the black hole becomes extremal since the temperature goes to zero. Such a remnant may be viewed as a candidate for the dark matter[20].

Now we compare the distinct behavior of Kretschmann scalar curvature for these two regular black holes, namely $x = 0$ and $x = 1$. We plot $K(r)$ as the function of the radius for different masses $M$ in Fig[1]. As we expect, the maximal value of Kretschmann scalar curvature $K_{max}$ increases dramatically with the increase of the mass for $x = 0$, while for $x = 1$, it decreases with $M$ indeed. We further find that the relation $K_{max} \propto M^2$ for $x = 0$, and $K_{max} \propto 1/M$ for $x = 1$ can be justified by numerical analysis. Therefore, for $x = 1$ we may fix the parameter $\alpha_0$ to have $K_{max} < 1$ for $M = M_{min}$, then it is guaranteed that Kretschmann scalar curvature is always bounded from above for arbitrary mass $M$, as we illustrate in the right plot of Fig[1].

Next it is also interesting to take a look at the thermodynamical behavior of these two regular black holes. In parallel with the analysis presented in [16], it is straightforward to derive the Hawking temperature $T$, the heat capacity $C \equiv \frac{4M}{dT}$ as well as the luminosity $L$ as the function of the mass $M$

$$T = \frac{W_0 + 1}{8\pi M} \sqrt{\frac{W_0}{\theta}},$$
$$C = -\frac{16\pi M^2(W_0 + 1)}{2 + W_0(5 + W_0)} \sqrt{\frac{\theta}{W_0}},$$
$$L = \frac{\sigma(W_0 + 1)^4 W_0}{256\pi^3 M^2 \theta}. \quad (14)$$

The temperature as well as the luminosity has the same expression for cases $x = 0$ and $x = 1$, but we stress that since the variable $\theta$ has a different relation with the mass, they exhibit distinct behavior with the mass $M$, as illustrated in Fig[2]. For $x = 0$, the temperature goes to zero at the final stage $W_0 \rightarrow -1$ ($M = \sqrt{\frac{e\alpha_0}{2}}$ or $r_h = \sqrt{2\alpha_0}$), and the temperature always reaches its maximal value $T_{max} = 1/(6\pi \sqrt{6\alpha_0})$ at $W_0 = -1/3$ ($M = \sqrt{\frac{4\alpha_0^3}{2}} \alpha_0$ or $r_h = \sqrt{6\alpha_0}$); while for $x = 1$, the temperature goes to zero at the final stage $W_0 \rightarrow -1$ as well ($M = \frac{e\alpha_0}{2}$ or $r_h = \sqrt{e\alpha_0}$), but reaches its maximal value at $W_0 = (-5 + \sqrt{17})/2$. We also find $T_{max} \propto 1/\alpha_0$ and $M|_{T=T_{max}} \propto \alpha_0$, implying that during the evaporation, the temperature may reach its maximal value at larger scale and then go down to zero in a longer time period if $\alpha_0 > 1$. In addition, the remnants at zero temperature correspond to the extremal limit of the black hole.
Finally, we remark that for both black holes, the capacity is vanishing at the final stage of evaporation, which is obvious to see in Eq. (14) as $W \to -1$. In addition, the capacity of both black holes undergoes a transition form $C < 0$ to $C > 0$ at $T = T_{\text{max}}$.

IV. THE REGULAR BLACK HOLE CORRESPONDING TO BARDEEN BLACK HOLE AT LARGE SCALE

In this section we present the regular black hole corresponding to Bardeen black hole at large scales by appropriately choosing the parameter $x$ with $n = 2$. Recall that for Bardeen black hole, the maximum of Kretschmann scalar curvature $K$ is independent of the mass of black hole. Remarkably, as we analyzed in the previous section, if we set $x = 2/3$, then the maximum value of $K$ is independent of mass $M$ as well. By virtue of this observation, we intend to consider the regular black hole with $x = 2/3$ and $n = 2$. In this case, the radius of the horizon $r_h$ is given by

$$r_h = 2M \sqrt{\frac{\theta}{W_0(\theta)}}, \quad \theta = -\frac{\alpha_0}{2M^{4/3}}. \tag{15}$$

The low bound for the mass of the black hole is

$$M \geq \left(\frac{e}{2}\right)^{3/4} \alpha_0^{3/4}. \tag{16}$$

The Hawking temperature $T$ and the luminosity $L$ maintain the same form as shown in Eq. (14), while the heat capacity $C$ becomes

$$C = -\frac{24\pi M^2(W_0 + 1)}{3 + W_0(8 + W_0)} \sqrt{\frac{\theta}{W_0}}. \tag{17}$$
Next we show that this regular black hole reproduces the Bardeen metric at large scales. On one hand, for large \( r \gg \sqrt{\alpha_0 M^{1/3}} \), the function \( F(r) \) in the metric behaves
\[
F(r) = 1 + 2\psi(r) = 1 - \frac{2M}{r} e^{-\alpha_0 M^{2/3} / r^2} \approx 1 - \frac{2M}{r} (1 - \frac{\alpha_0 M^{2/3}}{r^2} + ...). \tag{18}
\]
On the other hand, for the Bardeen regular black hole, the gravitational potential \( \psi(r) \) is specified as
\[
\psi(r) = -\frac{Mr^2}{\left( \frac{2}{3} \alpha_0 M^{2/3} + r^2 \right)^{3/2}}. \tag{19}
\]
Therefore, at large scales the function \( F(r) \) behaves
\[
F(r) = 1 + 2\psi(r) = 1 - \frac{2Mr^2}{\left( \frac{2}{3} \alpha_0 M^{2/3} + r^2 \right)^{3/2}} \approx 1 - \frac{M}{r} (1 - \frac{\alpha_0 M^{2/3}}{r^2} + ...), \tag{20}
\]
which is identical to the regular black hole with \( x = 2/3 \) at large scale, indeed.

Nevertheless, these two black holes have different cores. One has asymptotically Minkowski core, while the other has asymptotically de-Sitter core. Thus it is interesting to compare the behavior of Kretschmann scalar curvature as well as thermodynamical properties for these two regular black holes. For compactness, we intend to summarize their key differences as the following list

- **The features of Kretschmann scalar curvature** \( K \). For Bardeen black hole with the above potential, Kretschmann scalar curvature is given by
\[
K = \frac{1296M^2}{(2\alpha_0 M^{2/3} + 3r^2)^7} \left( 32\alpha_0^4 M^{8/3} - 162\alpha_0 M^{2/3} r^6 + 423\alpha_0^2 M^{4/3} r^4 - 24\alpha_0^3 M^2 r^2 + 81r^8 \right). \tag{21}
\]
We plot Kretschmann scalar curvature \( K \) as the function of the radius for these regular black holes in Fig.3. Obviously, for both black holes the maximum value of \( K \) is independent of mass \( M \). The location of \( K_{max} \) is always fixed at the center for Bardeen black hole, independent of the parameter \( \alpha_0 \), while for regular black hole with \( x = 2/3 \), \( K \) is always zero at the center, and the location of \( K_{max} \) moves to larger radius with the increase of the parameter \( \alpha_0 \).

- **The features of thermodynamics.** The Hawking temperature \( T \) of Bardeen black hole is given by
\[
T = \frac{3\sqrt{3}M \left( 3r_h^3 - 4\alpha_0 M^{2/3} r_h \right)}{2\pi \left( 3r_h^2 + 2\alpha_0 M^{2/3} \right)^{5/2}}. \tag{22}
\]

\(^1\) We remark that when the dimension is restored, the expression of \( \psi(r) \) is \( \psi(r) = -\frac{GMr^2}{\left( \frac{2}{3} \alpha_0 (GM)^{2/3} r^2 + r^2 \right)^{3/2}} \).
Moreover, to form a black hole with horizon, the mass of the black hole is bounded from below

$$M \geq \left( \frac{9\alpha_0}{4} \right)^3.$$  \hspace{1cm} \text{(23)}

When this bound is saturated, the black hole is characterized by the minimal radius

$$r_h = \frac{3}{2} \sqrt{3\alpha_0^3}.$$ 

We plot the Hawking temperature $T$ as the functions of $M$ for two black holes in Fig.4 respectively. Both black holes are characterized by a vanishing temperature at the final stage of evaporation. However, for Bardeen black holes, we remark that there is no maximal value for the Hawking temperature and its capacity exhibits a monotonic behavior with the mass with $C = dM/dT > 0$. 

FIG. 3: Kretschmann scalar curvature $K$ as the functions of the radial coordinate $r$ for Bardeen regular black hole(left) and $x = 2/3$(right) respectively.

FIG. 4: The Hawking temperature $T$ as the functions of $M$ for Bardeen black hole (left) and for the regular black hole with $x = 2/3$(right).
V. THE REGULAR BLACK HOLE CORRESPONDING TO HAYWARD BLACK HOLE AT LARGE SCALE

In a quite parallel way, we construct a regular black hole corresponding to Hayward black hole at large scales. With the general form of gravity potential as Eq.(8), we find the maximal value of Kretschmann scalar curvature $K$ behaves as $K \propto \frac{M^2}{(\alpha_0 M^2)^{n/3}}$. Thus if we set $n = 3$, then $x = 1$ leads to a mass-independent curvature as well. In this section we demonstrate that this regular black hole has the same behavior as Hayward black hole at large scales indeed.

We start with the gravitational potential $\psi(r)$ as

$$\psi = -\frac{M}{r} e^{-\alpha/r^3}. \quad (24)$$

Then Kretschmann scalar curvature is given by

$$K = \frac{12e^{-\frac{2\alpha}{r^3}} M^2}{r^6} (4 - \frac{32\alpha}{r^3} + \frac{132\alpha^2}{r^6} - \frac{108\alpha^3}{r^9} + \frac{27\alpha^4}{r^{12}}). \quad (25)$$

If we set $\alpha = \alpha_0 M$, then it goes to the curvature with $x = 1$. The maximal value of $K$ is proportional to $1/\alpha_0^2$. The radius of the horizon $r_h$ is given by

$$r_h = 2M \left( \frac{\theta}{W_0(\theta)} \right)^{1/3}, \quad \theta = -\frac{3\alpha_0}{8M^2}. \quad (26)$$

The mass of the black hole is bounded as

$$M \geq \frac{1}{2} \sqrt{\frac{3\alpha_0}{2}}, \quad (27)$$

and thus the minimal radius of the horizon is $r_h = \sqrt{\frac{3}{2}\alpha_0 e^{1/3}}$. The Hawking temperature $T$ is given by

$$T = \frac{r_h^3 - 3\alpha_0 M}{4\pi r_h^2} = 1 + \frac{W_0}{8M\pi} \left( \frac{W_0}{\theta} \right)^{1/3}. \quad (28)$$

Similar to the analysis in previous section, we show that this regular black hole reproduces the Hayward metric at large scales. On one hand, for large $r \gg (\alpha_0 M)^{1/3}$, the function $F(r)$ in the metric behaves as

$$F(r) = 1 + 2\psi(r) = 1 - \frac{2M}{r} e^{-\alpha_0 M/r^3} \simeq 1 - \frac{2M}{r} (1 - \frac{\alpha_0 M}{r^3} + \ldots). \quad (29)$$
On the other hand, for Hayward regular black hole, the gravitational potential $\psi(r)$ is specified as

$$
\psi = -\frac{Mr^2}{r^3 + \alpha_0 M}.
$$

(30)

Therefore, at large scales the function $F(r)$ behaves

$$
F(r) = 1 + 2\psi(r) = 1 - \frac{2Mr^2}{r^3 + M\alpha_0} \approx 1 - \frac{2M}{r^3}(1 - \frac{\alpha_0 M}{r^3} + ...),
$$

(31)

which is identical to the regular black hole with $x = 1$ and $n = 3$ at large scale, indeed.

As well, we compare the features of these two black holes as follows.

- The features of Kretschmann scalar curvature $K$. For Hayward black hole, Kretschmann scalar curvature is given by

$$
K = \frac{48M^2 (r^{12} - 4Mr^6\alpha_0 + 18M^2r^6\alpha_0^2 - 2m^3r^3\alpha_0^3 + 2M^4\alpha_0^4)}{(r^3 + M\alpha_0)^6}.
$$

(32)

We plot Kretschmann scalar curvature $K$ as the function of the radius for these regular black holes in Fig.5. Again, for both black holes the maximum value of $K$ is independent of mass $M$. The location of $K_{\text{max}}$ is always fixed at the center for Hayward black hole, independent of the parameter $\alpha_0$, while for regular black hole with $x = 1$, $K$ is always zero at the center, and the location of $K_{\text{max}}$ moves to larger radius with the increase of the parameter $\alpha_0$.

- The features of thermodynamics. The mass of Hayward black hole is bounded by

$$
M \geq \frac{3}{4}\sqrt{\frac{3\alpha_0}{2}}.
$$

(33)

The minimal radius of the horizon is $r_h = \sqrt{\frac{3\alpha_0}{2}}$. The Hawking temperature $T$ is given by

$$
T = \frac{Mr_h (r_h^3 - 2M\alpha_0)}{2\pi (r_h^3 + M\alpha_0)^2}.
$$

(34)

We plot the Hawking temperature $T$ as the functions of $M$ for two black holes in Fig.6 respectively. Both black holes are characterized by a vanishing temperature at the final stage of evaporation. It is also interesting to notice that the mass dependent behavior of the temperature is quite similar for these two black holes, even at the final stage of the evaporation.

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2 We remark that when the dimension is restored, the expression of $\psi$ is $\psi = -\frac{GM_\text{r}^2}{r^3 + \alpha_0 GM_\text{r}^2}$. 

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VI. THE REGULAR BLACK HOLE WITH GENERAL EXPONENTIAL FORM

In this section, we consider the generalized form of the black hole with the gravitational potential as specified in Eq. (8). In Table I, we summarize the dependent behavior of different quantities on $\alpha_0$ with factors $x$ and $n$. We emphasize that $\alpha_0$ is dimensionless and does not depend on $x$ for given $n$.

The special case happens whenever $x = n$. It is obvious to see that the mass of the black hole is not bounded by $\alpha_0$ any more, and the temperature becomes divergent when mass goes to zero, just like the Schwarzschild black hole. Thus, to form a regular black hole with remnant, we require that $x < n$. Moreover, if we expect the maximal value of Kretschmann scalar curvature $K$ is bounded above with the increase of the mass $M$, we find $x \geq n/3$. In particular, when $x = n/3$, $K_{\text{max}}$ is mass independent. Thus, we conclude that
| $K_{Max}$ | $r_{h Min}$ | $M_{Min}$ | $T_{Max}$ | $M_{TMax}$ |
|------------|------------|-----------|-----------|-----------|
| $M^{2-6x/n}n^{-6/n}$ | $\alpha_{0}^{1/(n-z)}$ | $\alpha_{0}^{1/(n-z)}$ | $\alpha_{0}^{1/(x-n)}$ | $\alpha_{0}^{1/(n-z)}$ |

TABLE I: The dependent behavior of different quantities on $\alpha_{0}$, where $r_{h Min}$ and $M_{Min}$ are the minimal values of the horizon and the mass at $T = 0$ respectively, while $M_{TMax}$ is the mass corresponding to $T = T_{Max}$.

to form a regular black hole with sub-Planckian Kretschmann scalar curvature, we demand $n > x \geq n/3$.

On the other hand, given the regular black hole with the gravitational potential as specified in Eq.(8), we may construct the corresponding regular black hole with asymptotically de-Sitter core which has the same behavior at large scales. The function $F(r)$ in the metric reads as

$$F(r) = 1 + 2\psi(r) = 1 - \frac{2Mr^{2z-1}}{(r^{n} + x\alpha_{0}Mx)^{1/x}},$$

which can be viewed as the generalization of Bardeen and Haywald black holes. In addition, we may give a remark about Frolov black hole and its correspondence. For Frolov black hole, the gravitational potential reads as

$$\psi(r) = 1 - \frac{2Mr^{2}}{\alpha_{0}^{3} + \alpha_{0}M + r^{3}}.$$  

At large scale, the black hole is identical to the regular black hole with potential $\psi(r) = -\frac{M}{r} e^{-\frac{\alpha_{0}M + \alpha_{0}x}{r^{3}}}$.

Finally, we briefly discuss the effective stress-energy tensor and the energy condition. The effective stress-energy tensor corresponding to Eq.(8) is given by

$$T_{0}^{0} = T_{1}^{1} = \frac{1}{4\pi r^{2}}\alpha_{0}nMr^{-n}\psi(r),$$
$$T_{2}^{2} = T_{3}^{3} = \frac{\alpha_{0}nM^{2x-2(n+1)}\psi(r)(\alpha_{0}nM^{x} + (-n-1)r^{n})}{8\pi}.$$  

The strong energy condition (SEC) is violated when $-T_{0}^{0} + T_{1}^{1} + T_{2}^{2} + T_{3}^{3} < 0$, namely $r < \left(\frac{\alpha_{0}nM^{x}}{n+1}\right)^{\frac{1}{3}}$. We notice that only for $x = 0$, the region with the violation of SEC is mass independent, while for other allowed values of $x$, the region becomes larger with the increase of the mass.
VII. HEURISTICAL UNDERSTANDING ON MODIFIED GRAVITY POTENTIAL

In this section we will present a heuristical understanding on the origin of such exponentially suppressing potential, closely following the strategy proposed in [16, 34]. As pointed out in [34], when the quantum effects of gravity are taken into account, the same observed results such as COW phase shift in gedanken experiment [35] can be understood by two equivalent pictures. One is that the classical gravitational field strength remains unchanged by quantum object, but the usual Heisenberg’s uncertainty principle obeyed by the object is forced to be a generalized one

\[ [\hat{x}, \hat{p}] = iz(\hat{p}), \] (38)

where \( z(\hat{p}) \) could be a general function of the momentum operator which reflects the effects of gravity on this quantum object. Typically, the widely considered one in literature is \( z(\hat{p}) = 1 + \alpha \hat{p}^2 \), which leads to a quantum theory with the minimal observable length. Alternatively, one can assume that the quantum objects still obey the usual Heisenberg’s uncertainty relation, namely the quantum theory is retained, but introduce an effective gravitational field strength to count in the interacting effects of gravity and the quantum object. It turns out that in latter point of view, the space time with Schwarzschild metric will be modified to be one characterized by an effective Newton constant \( G' = G/z \). Namely, we obtain a modified Schwarzschild metric as [16, 34]

\[ ds^2 = -\left(1 - \frac{2MG}{r^2z}\right)dt^2 + \left(1 - \frac{2MG}{r^2z}\right)^{-1}dr^2 + r^2d\Omega^2, \] (39)

which becomes the starting point that we propose the modified metric for regular black holes in this paper. However, rather than considering the ordinary GUP associated with \( z(\hat{p}) = 1 + \alpha \hat{p}^2 \), we introduce a general exponential function with \( z(\hat{p}) = e^{\alpha \hat{p}^a} \), which has also previously considered in [36]. Obviously, when \( a = 2 \) and \( \alpha \hat{p}^2 \ll 1 \), this goes back to \( z(\hat{p}) = 1 + \alpha \hat{p}^2 + \ldots \). In this sense, such an exponential form includes the non-perturbative effects of quantum gravity because in the expansion of weak momentum it recovers the quadratic form of the momentum.

Next, we need to evaluate the momentum uncertainty of any quantum object with \( m \) in the gravitational field. We know, for an observable quantity, \( \hat{p}^2 \geq \Delta \hat{p}^2 \). The key point is that whenever the quantum effects of gravity are taken into account, the quantum probe must
be characterized by a position uncertainty bounded below by the Planck length, namely $\Delta x \geq 1$. Therefore, it must experience the gravitational tidal force in curved spacetime, which leads to

$$(\Delta p)^2 \geq \frac{\Delta p}{\Delta x} = \frac{F \Delta t}{\Delta x} = \frac{2GMm\Delta t\Delta x}{\Delta x r^3} \geq \frac{2GM\Delta E\Delta t}{r^3} \geq \frac{2GM}{r^3},$$  

(40)

where $\Delta E$ and $\Delta t$ are the characteristic energy of the probe like a photon and the time in the process of detection, which is supposed to have a photon-particle collision. To avoid new particle pair production, $\Delta E \leq m$ is assumed. Thus, we have

$$z = e^{\alpha p^a} \sim e^{\alpha (\frac{2GM}{r^3})^{a/2}}.$$  

(41)

In particular, if we set $\alpha = \alpha_0/2$ and $a = 2$, then by identifying $G' = G/z$ and plugging it into the metric, we find the corresponding metric is nothing but the one of regular black hole with $n = 3$ and $x = 1$. Or, if we set $\alpha = 2^{3/2}\alpha_0$ and $a = 4/3$, then it exactly gives rise to the regular black hole with $n = 2$ and $x = 2/3$. In general if we set $\alpha \propto \alpha_0 M^{x-n/3}$ and $a = 2n/3$, then

$$z = e^{\alpha p^{2n/3}} \sim e^{\frac{\alpha_0 M^x}{r^{3n}}},$$  

(42)

leading to the potential given in Eq.(43). In parallel, if one considers the leading term as the ordinary GUP, then

$$z = 1 + \alpha p^{2n/3} \sim 1 + \frac{\alpha_0 M^x}{r^n}.$$  

(43)

Identifying $G' = G/z$ and plugging it into the metric, one obtains a sort of regular black holes with de-Sitter core including Bardeen black hole as well as Hayward black hole.

**VIII. CONCLUSION AND DISCUSSION**

In this paper we have introduced an exponentially suppressing gravitational potential to construct a sort of regular black holes with asymptotically Minkowski core. In contrast to all previous regular solutions with Minkowski core, the Kretschmann scalar curvature is not only finite everywhere, but also bounded above by the Planck mass density, regardless of the mass of the black hole. Without doubt, from the viewpoint of quantum gravity, the spacetime with sub-Planckian curvature is more realistic and applicable to the whole process of evaporation. We think this is a dramatic improvement in comparison with the previous
regular black holes with Minkowski core. In the thermodynamical aspect, all these regular
black holes are characterized by a vanishing Hawking temperature at the late stage of evapo-
ration and the remnant has a minimal mass at the Planck scale, which may be viewed as the
candidate of dark matter. As shown in [20, 37, 38], the remnant of regular black holes may
be viewed as a candidate for dark matter. The contribution of the remnant characterised by
\( \alpha_0 \) is constrained by the observed density of dark matter. Thus, the more detailed detection
of dark matter would potentially provide constraints on the choices of the parameter value
\( \alpha_0 \) in future. Furthermore, we have demonstrated that for specifical choice of the poten-
tial from, the regular black holes have the similar behavior as Bardeen/Hayward/Frolov
black holes at large scales, respectively. Therefore, we have established a one-to-one corre-
spondence between the regular black holes with Minkowski core and those with de-Sitter
core. This correspondence provides us a scheme to construct new regular black holes with
de-Sitter core as well. Theoretically, these two different sorts of black holes may be as-
cribed to the different forms of the modified gravitational potential, which are supposed
to contain some non-perturbative corrections due to quantum gravity effects such that the
singularity could be avoided. We remark that these effects should be non-perturbative in
the sense that the modified gravitational potential could be expanded as the polynomials of
the distance and the leading term is the standard classical gravitational potential, similar
to 1-loop corrections in perturbative calculations [39, 42]. But 1-loop correction is obtained
with perturbative method and is not strong enough to get rid of the singularity. More-
over, the exponential form for black holes with Minkowski core implies that regular black
holes with Minkowski cores have stronger non-perturbative corrections compared to those
with de-Sitter cores. We comment that in this paper we have mainly considered the pa-
rameters leading to the Bardeen/Hayward/Frolov black hole at large scales such that the
Kretschmann scalar curvature of the regular black hole with a Minkowski core has the same
feature as those with de-Sitter core, namely \( K_{max} \) is mass independent. Definitely, one may
consider the regular black holes with other parameters whose Kretschmann scalar curvature
may be mass dependent but maintain sub-Planckian. We also remark that the choices of
these parameters are not random, but different choices do imply there are different forms of
the modified gravitational potential due to the quantum gravity effects. The precise form of
the correction terms remains to be studied through further research on quantum theory of
gravity and its phenomenology. In addition, the regular black hole with de-Sitter core has
a Kretschmann scalar curvature that reaches its maximum value at the center of the black hole, while the regular black hole with Minkowski core does not. Instead, the Kretschmann scalar curvature is always zero at the center and reaches its maximum value at a point between the center and the horizon. This distinction may result in observable effects in the future and has been theoretically investigated in [32, 43]. It is found that the shadows of black holes with Minkowski cores have larger deformations than those with de-Sitter cores. Moreover, the radius of the photon sphere in regular black hole with de Sitter cores is larger than the black hole with Minkowski cores. It is desirable to distinguish these two types of black holes by observation in future.

Finally, we have provided a theoretical understanding on the origin of such exponentially suppressing potential appeared in the black hole metric in a heuristic manner. Two essential ingredients are taken into account. Firstly, any quantum object has a size larger than the minimal length at Planck scale due to the effects of gravity. Secondly, any object with a finite size must be affected by the gravitational tidal force. Moreover, the exponentially suppressing form of the gravity potential in the metric results from the generalized function $z(p)$ with a novel exponential form of the momentum in GUP, which is in contrast to the ordinary one with the quadratic form of the momentum. This generalization may reflect the non-pertubative effects of quantum gravity. It is also this relation that paves a bridge between the effective metric of regular black holes with asymptotically Minkowski core with that with asymptotically de-Sitter core.

Of course, in this paper the regular black holes that we have constructed are static, we expect the Hawking radiation may be added in such that they can be extended to dynamical ones such as Vaidya-like solutions to investigate the formation and evaporation of regular black holes with Minkowski core, as demonstrated for the black hole with de-Sitter core in [2]. The whole picture about the black hole evaporation will help us to understand the final fate of black hole evolution and disclose the mystery of black hole information loss paradox.

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