Equilibrium and dynamics of a trapped superfluid Fermi gas with unequal masses

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Interacting Fermi gases with equal populations but unequal masses are investigated at zero temperature using local density approximation and the hydrodynamic theory of superfluids in the presence of harmonic trapping. We derive the conditions of energetic stability of the superfluid configuration with respect to phase separation and the frequencies of the collective oscillations in terms of the mass ratio and the trapping frequencies of the two components. We discuss the behavior of the gas after the trapping potential of a single component is switched off and show that, near a Feshbach resonance, the released component can still remain trapped due to many-body interaction effects. Explicit predictions are presented for a mixture of 6Li and 40K with resonant interaction.

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The superfluid behavior of dilute interacting Fermi gases at very low temperature is now rather well understood both from the experimental and theoretical point of view [1]. In particular the attractive nature of the interaction between the two different spin components of the gas is known to play a crucial role along the whole BCS-BEC crossover. This includes the case of small and negative values of the scattering length, where the ordinary BCS regime of superfluidity holds, the BEC regime characterized by the formation of molecules in the Bose-Einstein condensed state and the unitary regime where the scattering length takes a divergent value and the attraction results in peculiar many-body effects.

A more recent and intriguing direction is the search for superfluidity in mixtures of Fermi gases belonging to different species, and hence having different masses. Experimentally, the most promising candidates are ultracold mixtures of 40K and 6Li, where the mass ratio is 6.7, near a heteronuclear s-wave Feshbach resonances [2]. Equilibrium configurations of the uniform superfluid phase where the atom densities of the two species are equal but the masses are different have been theoretically investigated in Ref.[3] within BCS mean field theory and, more recently, by quantum Monte Carlo methods [4]. Other recent works have explored the interplay between different masses and different populations of the two components [5,6].

In this Letter we investigate the equilibrium and the dynamic properties of a trapped Fermi gas with unequal masses using local density approximation and developing the hydrodynamic theory of superfluids at zero temperature. We assume that the external potentials for the two components, hereafter called ↑ and ↓, are harmonic and given by $V_{ho}^\sigma(r) = m_\sigma(\omega_{\sigma x}^2 x^2 + \omega_{\sigma y}^2 y^2 + \omega_{\sigma z}^2 z^2)/2$ where $\sigma = \uparrow, \downarrow$ and $m_\uparrow$ and $m_\downarrow$ are the atomic masses. Since the two species have different magnetic and optical properties, the trapping frequencies $\omega_{\sigma \sigma'}$ can be tuned separately.

We consider a mixture of two different fermionic species with equal populations $N_\uparrow = N_\downarrow = N/2$, corresponding to the most favorable condition for Cooper pairing. We assume that the gas is superfluid at zero temperature and we explore configurations where the densities of the two components are equal and move in phase: $n_\uparrow = n_\downarrow = n/2$, $\mathbf{v}_\uparrow = \mathbf{v}_\downarrow = \mathbf{v}$. We will not consider here exotic polarized phases, like the FFLO phase, whose relevance for trapped Fermi gases is still unclear at present.

At equilibrium, where $\nu = 0$, the atomic density profile $n_0(r)$ of the gas is given by the local density (also called Thomas-Fermi) approximation for the chemical potential

$$\mu_0 = \mu(n_0(r)) + \tilde{V}_{ho}(r),$$

where $\mu_0$ is fixed by the normalization condition $\int n_0(r) dr = N$ and $\mu(n) = \partial \epsilon/\partial n$ is the chemical potential of uniform matter, $\epsilon(n)$ being the energy per unit volume of the homogeneous phase. In Eq. (1) we have introduced the effective trapping potential

$$\tilde{V}_{ho}(r) = \frac{1}{2}(V_{ho}^\uparrow + V_{ho}^\downarrow)\frac{m}{2} (\omega_{x x}^2 x^2 + \omega_{y y}^2 y^2 + \omega_{z z}^2 z^2),$$

given by the average of the two potentials and

$$\omega_{x x}^2 = \frac{m_\uparrow \omega_{x x}^2 + m_\downarrow \omega_{x x}^2}{m_\uparrow + m_\downarrow}$$

with $m = (m_\uparrow + m_\downarrow)/2$. In the superfluid phase the densities of the two components are equal, even if the trapping potentials, in the absence of interactions, would give rise to different equilibrium profiles at zero temperature, i.e. even if the oscillator lengths $\hbar/(m_\sigma \omega_{\sigma})^{1/2}$ of the two components do not coincide. If the oscillator lengths are equal, the effective frequencies [9] simply reduce to the geometrical averages $\tilde{\omega}_i = \sqrt{\omega_{\uparrow i}\omega_{\downarrow i}}$.

Let us discuss the behavior of the equation of state along the BCS-BCS crossover and the corresponding...
shape of the density profiles. At unitarity, where the scattering length $a$ diverges, the equation of state takes the universal form \[ \mu = (3\pi^2/2\hbar^2) (1 + \beta)n^{3/2}/4m, \]
where $m_r = m_{1}m_{1}/(m_{1} + m_{1})$ is the reduced mass and $\beta$ is a dimensionless parameter depending on the mass ratio $m_{1}/m_{1}$ and accounting for the interacting effects in this strongly interacting regime. By inserting $\mu = \alpha n^{3/2}$ in Eq. (1), with $\alpha = (3\pi^2/2\hbar^2) (1 + \beta)/4m$, we find that the density distribution of the resonant gas takes the usual form $n(\mathbf{r}) = (\mu_0 - \tilde{V}_{bo}(\mathbf{r}))^{3/2}/\alpha^{3/2}$, the Thomas-Fermi radii $R_i$, where the density vanishes, being given by
\[ R_i = \tilde{\alpha}_{bo}(24N)^{1/6}(1 + \beta)^{1/4} \frac{\omega_{bo}}{\omega_i}, \]
where $\omega_{bo} = (\tilde{\omega}_x\tilde{\omega}_y\tilde{\omega}_z)^{1/3}$ is the geometrical average of the three effective oscillator frequencies and $\tilde{\alpha}_{bo}^2 = \hbar/\omega_{bo}(m_{1}m_{1})^{1/2}$. The value of $\beta$ is known for the special case of equal masses $m_{1} = m_{1}$, where $\beta = -0.58$ [3,8]. Preliminary Monte Carlo calculations suggest that $\beta$ depends very weakly on the mass ratio [9]. In the deep BCS limit, corresponding to a weakly attractive interaction ($n(\mathbf{a})^{1/4} \ll 1$), the equation of state and the Thomas-Fermi radii are given by the same expressions holding at unitarity by simply setting $\beta = 0$.

In the limit of small and positive scattering length, corresponding to $na^{3} \ll 1$, the gas instead corresponds to a BEC diatomic heteronuclear molecules of mass $m_{1} + m_{1} = 2m$ and molecular density $n/2$. In this regime the equation of state is fixed by the repulsive interaction between molecules and takes the usual bosonic form $\mu_{m} = g_{m}n/2$, where the coupling constant $g_{m}$ is related to the molecule-molecule scattering length $a_{m}$ by $g_{m} = 2\pi\hbar^{2}a_{m}/m$. The exact value of the molecular scattering length has been calculated by Petrov et al. [10] as a function of the atom scattering length $a$ and the mass ratio $m_{1}/m_{1}$. The bosonic chemical potential is related to the fermionic one by $\mu_{m} = -\tilde{E}_{b} + 2m$, where $\tilde{E}_{b} = -\hbar^2/2m_{1}a^2$ is the two-body binding energy. From Eq. (1), the density profile is then given by $n(\mathbf{r}) = (\mu_0 - \tilde{V}_{bo}(\mathbf{r}))2m/\pi\hbar^{2}a_{m}$ corresponding to the Thomas Fermi radii
\[ R_{i}^{BCS} = \tilde{\alpha}_{bo}^{BCS} \left( \frac{15N a_{m}}{2\tilde{\alpha}_{bo}^{BCS}} \right)^{1/5} \frac{\omega_i}{\omega_{bo}}, \]
where $\omega_{bo} = (\tilde{\omega}_x\tilde{\omega}_y\tilde{\omega}_z)^{1/3}$ and $\tilde{\alpha}_{bo}^{BCS} = \hbar/\sqrt{2m\omega_{bo}}$. Notice that $\tilde{\alpha}_{bo}^{BCS}$ differs from the oscillator length $\tilde{\alpha}_{bo}$ defined above for the resonant case.

We now take advantage of the fact that the trapping potentials of the two different species can be tuned separately to suggest an experiment pointing out in a direct way the attractive role of the interactions in the presence of a Feshbach resonance. After generating the equilibrium configuration discussed above we switch off the confining potential of a single species, say $\tilde{V}_{1} \rightarrow 0$, corresponding to a change of the trapping frequencies [3] into the new values
\[ \tilde{\omega}_{i,new} = \sqrt{\frac{m_{1}}{m_{1} + m_{1}}} \omega_{1}. \]

The potential can be switched off either adiabatically, bringing the system into a new equilibrium configuration, or suddenly, giving rise to the excitation of collective oscillations (see discussion in the second part of this work).

In the absence of interactions, the $\downarrow$-atoms would fly away leaving an ideal gas of $\uparrow$-fermions trapped in the harmonic potential $\tilde{V}_{1} \rightarrow 0$. This will be also the case in the deep BCS superfluid regime where interactions are too weak to keep the $\uparrow$-fermions confined. In the other (more robust) superfluid regimes, however, the released atoms do not necessarily escape to infinity but can remain trapped due to the attractive interaction with the other species. This statement is obvious in the BEC regime, where each $\downarrow$-fermion forms a bound molecule with a corresponding $\uparrow$-particle. At unitarity, however, the two-body binding energy $E_{b} = -\tilde{E}_{b}/2m_{1}a^{2}$ vanishes meaning that no molecules can exist in vacuum. In this case the trapping of the $\downarrow$ component is a pure many-body effect reflecting the attractive nature of the interatomic force.

It is not difficult to derive explicit conditions for the energetic stability of the new superfluid configuration. The stability is ensured if the energy $E_{S}$ of the configuration where the two components remain trapped and fully overlapped is smaller than the energy $E_{N}$ of the normal state where the gas is phase separated and only the $\downarrow$-atoms are trapped. At unitarity the energy of the trapped superfluid state is given by
\[ E_{S} = \hbar \omega_{bo} (3N)^{4/3} \left( \frac{m_{1} + m_{1}}{8} \right)^{2/3} m_{1} \left( \frac{1 + \beta}{m_{1} m_{1}} \right) \]
and, in the absence of trapping for the $\downarrow$-atoms, one has $\omega_{bo} = \sqrt{m_{1}/(m_{1} + m_{1})}(\tilde{\omega}_{1}(\omega_{1}^{\downarrow}\tilde{\omega}_{1}^{\downarrow}))^{1/3}$. Since the energy of a trapped gas of non-interacting $\uparrow$ fermions is given by $E_{N} = \hbar(\omega_{1}^{\uparrow}\omega_{1}^{\downarrow}\omega_{1}^{\downarrow})^{1/3}(3N)^{4/3}/8$, we find that the superfluid gas, where both components remain trapped, is energetically stable if the condition
\[ (1 + \beta) \left( \frac{m_{1} + m_{1}}{m_{1}} \right) < 1 \]
is satisfied. Taking into account that $\beta$ barely depends on the mass ratio [3] and hence remains close to the equal mass value $\beta = -0.58$, the above condition is always satisfied if $m_{1} \leq m_{1}$, showing that the superfluid remains energetically stable if we release the potential of the heavy species. Conversely, if the condition [3] is violated, the superfluid configuration corresponds to a metastable state which is energetically unstable toward phase separation. It is also interesting to compare the Thomas Fermi radii $R_{i,S}$ and $R_{i,N}$ of the trapped cloud in the (new) superfluid and in the separated normal phase, respectively. A simple calculation
yields $R_{s}/R_{IN} = (1 + \beta)^{1/4}/(m_{1} + m_{1})^{1/4}$, showing that the superfluid phase corresponds to the configuration with smaller radii if and only if Eq. [3] is satisfied.

Let us now discuss the macroscopic dynamic behavior of the superfluid. This is obtained by deriving the hydrodynamic equations of motion in terms of the atom density $n(r, t)$ and the velocity field $v(r, t)$. The Lagrangian $L$ of the system, in the local density approximation (LDA), is given by

$$L = \int d\mathbf{r} \left[ e(n) + (V_{h0} + V_{h1}) \frac{n}{2} \frac{n}{2} \frac{\partial \phi}{\partial t} + \frac{1}{2} (m_{1} + m_{1}) v^{2} n \right]$$

where $\phi$ is the phase of the order parameter $\langle \Psi_1(r, t) \Psi_1^\dagger(r, t) \rangle \equiv |\langle \Psi_1(r) \Psi_1(r) \rangle| e^{i \phi(r, t)}$, $\Psi_1(r, t)$ being the fermionic field operators. The superfluid velocity $v$ and the phase $\phi$ entering the Lagrangian (9) are related by the most important condition

$$v = \frac{\hbar}{m_{1} + m_{1}} \nabla \phi,$$

where $m_{1} + m_{1}$ is the mass of the pair, which can be derived microscopically by noticing the state moving with velocity $v$ is obtained from the steady state by applying the gauge transformation $\Psi_1(r) \rightarrow \Psi_2(r) e^{i m_{1} n \cdot v r / \hbar}$ to the two Fermi field operators.

Taking Eq. (10) into account, the equations of motion of Lagrangian (9) yield the hydrodynamic equations

$$\frac{\partial n}{\partial t} + \nabla (n v) = 0,$$

$$m \frac{\partial v}{\partial t} + \nabla \left( \mu(n) + \tilde{V}_{h0}(r) + \frac{1}{2} m v^{2} \right) = 0. \tag{12}$$

Equations (11) and (12) remain valid even if the effective trapping frequencies depend explicitly on time, i.e. if $\tilde{\omega}_{i} = \tilde{\omega}_{i}(t)$. Remarkably, if the chemical potential has the power law dependence $\mu \propto n^{\gamma}$ on the density and the confining potential is harmonic, the hydrodynamic equations admit a class of exact solutions given by $v(t, r) = \alpha \cdot r$ and $n(t, r) = n_0(x,b_x,y,b_y,z,b_z)/(b_x b_y b_z)$, where $\alpha_i(t), \beta_i(t)$ are time-dependent parameters. From the scaling form of the density, we see that the parameters $\beta_i(t)$ are related to the Thomas Fermi radii $R_i(t)$ of the evolving cloud according to $R_i(t) = \tilde{R}_i(0) b_i(t)$. Inserting the scaling ansatz in Eq. (11) yields the relationships $\alpha_i = \tilde{b}_i/b_i$ and then one obtains a set of coupled ordinary differential equations

$$\tilde{b}_i + \tilde{\omega}_i^{2}(t) b_i = \frac{\tilde{\omega}_i^{2}(0)}{\tilde{b}_i} \frac{1}{(b_x b_y b_z) \gamma}. \tag{13}$$

Equations (13) apply both to resonance, where $\gamma = 2/3$, and to the BEC regime where $\gamma = 1$.

If one suddenly switches off the trapping potential for the $\perp$-component, corresponding to suddenly setting the effective frequencies to the new values $\tilde{\omega}_{i,\perp}$, the superfluid gas will start oscillating around the new equilibrium configuration. The dynamics of the Thomas Fermi radii of the cloud can be calculated from Eq. (13) by substituting $\tilde{\omega}_i(t > 0) = \tilde{\omega}_{i,\text{new}}$ and employing the initial conditions $\tilde{b}_i(0) = 1$ and $b_i(0) = 0$.

For simplicity, let us assume that the two trapping potentials are axis-symmetric ($\omega_{i,\sigma} = \omega_{\mu,\sigma} \equiv \omega_{\perp,\sigma}$). We further assume that the two components have initially equal oscillator lengths (and hence, in the degenerate limit, equal density profiles also in the absence of interaction).
corresponding to $\omega_\perp^2(0) = \omega_\perp \omega_\perp$ and $\omega_\parallel^2(0) = \omega_\parallel \omega_\perp$.

By introducing the dimensionless time $\tau = \dot{\omega}_\perp(0)t$ and setting $b_x = b_y = b_1$, Eqs. (13) take the form

$$\frac{\partial^2 b_1}{\partial \tau^2} + \eta \omega_\perp = \frac{1}{b_1 (b_1^4 b_2^2)^{2/3}},$$

$$\frac{\partial^2 b_2}{\partial \tau^2} + \eta \lambda^2 \omega_\perp = \frac{1}{b_2 (b_1^4 b_2^2)^{2/3}},$$

where $\eta = m_1/(m_1 + m_1)$ and $\lambda = \omega_\perp(0)/\omega_\perp(0)$ is the aspect ratio of the trapping potential [2].

We have solved Equations (14) and (15) for a mixture of $^{40}$K and $^6$Li fermions in an elongated trap with $\lambda = 0.2$. In Fig. 4 we plot the calculated time evolution of the radius of the cloud in the radial direction, after we suddenly switch off the trapping potential of $^{40}$K fermions, corresponding to $\eta = 0.869$. The static solution $b_1 = b_2 = 1/\eta^{1/4}$ of these equations and the corresponding Thomas-Fermi radii $R_{\text{new}} = R_0(0)/\eta^{1/4} = 1.036 R_0(0)$ characterize the new equilibrium conditions after releasing the $\parallel$ trapping potential. Since $\eta$ is close to 1, the initial configuration is close to equilibrium and the resulting oscillation is a linear superposition of the two breathing modes, in the radial and longitudinal directions. For elongated cloud, corresponding to $\lambda < 1$, these are given by $\omega_{\text{rad}} = \sqrt{10/3}\eta^{1/2}\lambda\omega_\perp(0)$ [3].

In Fig. 2 we instead plot the time dependence of the transverse radii of the cloud, after we suddenly switch off the trapping potential of the $^6$Li fermions, corresponding to $\eta = 0.131$ and $R_{\text{new}} = 1.66 R_0(0)$. In this case, the initial configuration is far from equilibrium and non linear effects play an important role. In particular we see that the breathing of the cloud around the equilibrium configuration is no longer symmetric. We emphasize that in this second case the superfluid configuration is energetically unstable. A major question in this case is to understand the decay mechanisms and the role played by the sudden excitation of the collective modes.

In conclusions, we have derived the equilibrium conditions and hydrodynamic equations of a superfluid Fermi gas with unequal masses and investigated the behavior of a gas at unitarity after the release of the trapping potential of a single component. We have shown that, under appropriate conditions, the superfluid phase is energetically stable against phase separation of the two components. As a result, the released fermions remain confined in the trap due to the pairing with the other component, pointing out in a direct and remarkable way the attractive nature of the interatomic forces near a Feshbach resonance.

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