C₀-semigroups and Local Spectral Theory

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ABSTRACT: Let \((T(t))_{t\geq 0}\) be a C₀-semigroup of operators on a Banach space \(X\). In this paper, we show that if there exists \(t_0 > 0\) such that \(T(t_0)\) has the SVEP then \(A\) has the SVEP and if \(\sigma_p(A)\) has empty interior, then \(T(t)\) has the SVEP for all \(t \geq 0\). Also, some local spectral properties for C₀ semigroups and theirs generators and some stabilities results are also established.

Key Words: C₀-semigroup, Local spectrum, SVEP, stability.

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1. Introduction

The semigroups can be used to solve a large class of problems commonly known as the Cauchy problem,

\[
\left\{ \begin{array}{l}
u'(t) = Au(t) \quad \text{for all } t \geq 0, \\
u(0) = u_0.
\end{array} \right.
\]

on a Banach space \(X\). Here \(A\) is a given linear operator with domain \(D(A)\) and the initial value \(u_0\). The solution of the previous Cauchy problem will be given by \(u(t) = T(t)u_0\) for an operator semigroup \((T(t))_{t\geq 0}\) on \(X\). In this paper, we will focus on a special class of linear semigroups called C₀ semigroups which are semigroups of strongly continuous bounded operators. Precisely, a one-parameter family \((T(t))_{t\geq 0}\) of operators on a Banach space \(X\) is called a C₀-semigroup of operators if

1. \(T(0) = I\),
2. \(T(t+s) = T(t)T(s), \forall t, s \geq 0\),
3. \(\lim_{t \to 0} T(t)x = x, \forall x \in X\).

\((T(t))_{t\geq 0}\) has a unique infinitesimal generator \(A\) defined in domain \(D(A)\) by,

\[
Ax = \lim_{t \to 0} \frac{T(t)x - x}{t}, \forall x \in D(A),
\]

\[D(A) = \{ x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists} \}\].

Also, \(T(t)\) are linear and continuous operators on \(X\) for all \(t \geq 0\), and \(A\) is a closed operator, see [4,8]. In order to understand the behavior of the solutions in terms of the data concerning \(A\), one seeks information about the spectrum of \(T(t)\) in terms of the spectrum of \(A\). Unfortunately the spectral mapping theorem \(e^{t\sigma(A)} = \sigma(T(t)) \setminus \{0\}\) often fails, sometimes in dramatic ways. However, the inclusion

\[
e^{t\sigma(A)} \subseteq \sigma(T(t)) \setminus \{0\}\]

is always true. The aim of this paper is to develop a local spectral theory for C₀ semigroups.

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2. Preliminaries

Throughout, $X$ denotes a complex Banach space, let $A$ be a closed operator on $X$ with domain $D(A)$. We denote by $A^*$, $R(A)$, $N(A)$, $R^\infty(A) = \bigcap_{n \geq 0} R(A^n)$, $\sigma_K(A)$, $\sigma_{su}(A)$, $\sigma(A)$, respectively the adjoint, the range, the null space, the hyper-range, the semi-regular spectrum, the surjectivity spectrum and the spectrum of $A$. Recall that for a closed operator $A$ and $x \in X$, the local resolvent of $A$ at $x$, $\rho_A(x)$ defined as the union of all open subset $U$ of $\mathbb{C}$ for which there is an analytic function $f : U \rightarrow D(A)$ such that the equation $(A - \mu I)f(\mu) = x$ holds for all $\mu \in U$. The local spectrum $\sigma_A(x)$ of $A$ at $x$ is defined as $\sigma_A(x) = \mathbb{C} \setminus \rho_A(x)$. Evidently $\sigma_A(x) \subseteq \sigma_{su}(A) \subseteq \sigma(A)$, $\rho_A(x)$ is open and $\sigma_A(x)$ is closed.

Let $f(z) = \sum_{i=0}^{\infty} x_i (z - \mu)^i$ ( in a neighborhood of $\mu$) be the Taylor expansion of $f$. It is easy to see that $\mu \in \rho_A(x)$ if and only if there exists a sequence such that $(x_i)_{i \geq 0} \subseteq D(A)$, $(A - \mu)x_0 = x$, $(A - \mu)x_{i+1} = x_i$, and $\sup_i ||x_i||^{\frac{1}{i+1}} < \infty$, see [5,7].

For any arbitrary closed set $\Omega$ in the complex field, the spectral subspace associated to $\Omega$ is:

$$X_A(\Omega) = \{x \in X : \sigma_A(x) \subseteq \Omega\}$$

$X_A(\Omega)$ is a hyperinvariant subspace of $A$ not always closed, see [6].

Next, let $A$ be a closed operator, $A$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP) if for every open disc $D_{\lambda_0} \subseteq \mathbb{C}$ centered at $\lambda_0$, the only analytic function $f : D_{\lambda_0} \rightarrow D(A)$ which satisfies the equation $(A - zI)f(z) = 0$ for all $z \in D_{\lambda_0}$ is the function $f \equiv 0$. $A$ is said to have the SVEP if $A$ has the SVEP for every $\lambda \in \mathbb{C}$. Denote by $S(A) = \{\lambda \in \mathbb{C} : A$ has not the SVEP at $\lambda\}$.

$X_A(\emptyset) = \{0\}$ implies $S(A) = \emptyset$ [1]. If $A$ is bounded, then $X_A(\emptyset)$ is closed if and only if $X_A(\emptyset) = \{0\}$ if and only if $S(A) = \emptyset$ [6].

Note that $\mu \in S(A)$ if and only if there exists a sequence $(x_i)_{i \geq 0} \subseteq D(A)$ not all of them equal to zero such that $(A - \mu)x_{i+1} = x_i$, with $x_0 = 0$ and $\sup_i ||x_i||^{\frac{1}{i+1}} < \infty$, see [5].

Let $(T(t))_{t \geq 0}$ be a $C_0$ semigroup with generator $A$, we introduce the following operator acting on $X$ and depending on the parameters $\lambda \in \mathbb{C}$ and $t \geq 0$.

$$B_\lambda(t)x = \int_0^t e^{(\lambda - \mu)T(s)}xds, \text{ for all } x \in X.$$ 

It is well known that $B_\lambda(t)$ is a bounded operator on $X$ and we have ([4,8]):

$$(e^{\lambda t} - T(t))^n x = (\lambda - A)^n B^n_\lambda(t)x, \text{ for all } x \in X \text{ and all } n \in \mathbb{N}$$

$$(e^{\lambda t} - T(t))^n x = B^n_\lambda(t)(\lambda - A)^n x, \text{ for all } x \in D(A^n) \text{ and all } n \in \mathbb{N};$$

$$R^\infty(e^{\lambda t} - T(t)) \subseteq R^\infty(\lambda - A);$$

$$N(\lambda - A)^n \subseteq N(e^{\lambda t} - T(t))^n.$$ 

Recall that some spectral inclusions for various reduced spectra are studied in [3], [4] and [8]. The authors proved that

$$e^{\nu(T)} \subseteq \nu(T(t))$$

where $\nu \in \{\sigma_{ap}, \sigma_K\}$, approximate point spectrum and semi-regular spectrum, also we have equality where $\nu \in \{\sigma_p, \sigma_r\}$ point spectrum and residual spectrum. In the next two sections, we will prove a spectral inclusion for local spectrum and a framing of $S(\cdot)$ which characterizes it. Some related stability results are also presented.

3. Local Spectral Theory

**Theorem 3.1.** For the generator $A$ of a strongly continuous semigroup $(T(t))_{t \geq 0}$, then for all $t \geq 0$, we have

$$e^{tS(A)} \subseteq S(T(t)) \setminus \{0\} \subseteq e^{t \text{ int}(\sigma_p(A))}$$
Let \( t > 0 \). Then \( T(t) \) has SVEP at \( e^{\lambda_0 t} \). Let us show that \( \lambda_0 \notin S(A) \). Let \( D_{\lambda_0} \) the open disc centered at \( \lambda_0 \), \( f : D_{\lambda_0} \to D(A) \) an analytic function such that for all \( \mu \in D_{\lambda_0} \), \((\mu - A)f(\mu) = 0\). Show that \( f \equiv 0 \).

Consider the analytic function \( \varphi_t : \mu \in D_{\lambda_0} \to e^{t\mu} \). For all \( \mu \in D_{\lambda_0} \), \( \varphi_t(\mu) = te^{t\mu} \neq 0 \). By the inverse function theorem, there exists a neighborhood \( V \subseteq D_{\lambda_0} \), \( \varphi_t(V) \) is open and the function \( \varphi_t : V \to \varphi_t(V) \) is bijective. The function \( \varphi_t^{-1} : \varphi_t(V) \to V \) is analytic and therefore the function \( g : z \in \varphi_t(V) \to f(\varphi_t^{-1}(z)) \) is analytic. Moreover, for all \( z \in \varphi_t(V) \), there exists a \( \mu \in V \) such that \( z = e^{t\mu} \). Furthermore,

\[
(z - T(t))g(z) = (\mu - A)B_{\mu}(t)f(\varphi_t^{-1}(z)) = (\mu - A)B_{\mu}(t)f(\mu) = B_{\mu}(t)(\mu - A)f(\mu) = 0.
\]

Thus \( g \equiv 0 \), then \( f \equiv 0 \) on \( V \), hence \( f \equiv 0 \) on \( D_{\lambda_0} \). Hence \( \lambda_0 \notin S(A) \).

On the other hand \( S(T(t)) \setminus \{0\} \subseteq \text{int}(\sigma_p(T(t)) \setminus \{0\}) = \text{int}(e^{t\sigma_p(A)}) \subseteq e^{t\text{int}(\sigma_p(A))} \). So the proof is complete.

In the following, we give a sufficient condition to show that the spectral subspace \( X_A(\emptyset) \) is closed for all \( t > 0 \).

**Corollary 3.2.** Let \( (T(t))_{t \geq 0} \) be a \( C_0 \)-semigroup, with generator \( A \), then:
\( X_{T(t)}(\emptyset) = \{0\} \) for some \( t \geq 0 \) implies that \( A \) has the SVEP.

**Proof.** Let \( t \geq 0 \) such that \( X_{T(t)}(\emptyset) = \{0\} \), that implies that \( S(T(t)) = \emptyset \), by theorem 3.1 we have \( S(A) = \emptyset \).

**Corollary 3.3.** Let \( (T(t))_{t \geq 0} \) be a \( C_0 \)-semigroup, with infinitesimal generator \( A \).

1. If \( T(t_0) \) has the SVEP for some \( t_0 \geq 0 \), then \( A \) has the SVEP.
2. If \( \sigma_p(A) \) has empty interior, then \( T(t) \) has the SVEP for all \( t \geq 0 \).

**Example 3.4.** We consider the left translation group \( T(t)_{t \in \mathbb{R}} \) on \( X = C_0(\mathbb{R}) \). Then \( \sigma(A) = i\mathbb{R} \) and \( \sigma(T(t)) = \{z \in \mathbb{C} : |z| = 1\} \), so \( A \) has the SVEP. According to corollary 3.3, \( T(t) \) has the SVEP for all \( t \geq 0 \). Then \( \sigma_{su}(T(t)) = \{z \in \mathbb{C} : |z| = 1\} \).

**Example 3.5.** A \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) is called periodic if there exists \( t_0 > 0 \) such that \( T(t_0) = I \), so \( T(t_0) \) has the SVEP. From corollary 3.3, the infinitesimal generator \( A \) of \( (T(t))_{t \geq 0} \) has the SVEP.

To continue the development of a spectral theory for semigroups and their generators, we prove that the formula \((1.1)\) holds for local spectrum.

**Theorem 3.6.** Let \( (T(t))_{t \geq 0} \) be a \( C_0 \)-semigroup on \( X \) with infinitesimal generator \( A \). The following spectral inclusion holds:

\[
e^{t\sigma_A(x)} \subseteq \sigma_{T(t)}(x) \setminus \{0\}, \text{ for all } t \geq 0 \text{ and } x \in X.
\]

**Proof.** Let \( e^{\lambda t} \notin \sigma_{T(t)}(x) \), then there exists \( (x_i)_{i \geq 0} \subseteq X \), such that

\[
(e^{\lambda t} - T(t))x_0 = x, \quad (e^{\lambda t} - T(t))x_i = x_{i-1} \text{ and sup} \|x_i\|^\dagger < \infty.
\]

Let \( y_i = B_{\lambda}^{i+1}(t)x_i \), then \( (y_i)_{i \geq 0} \subseteq D(A) \) and \( y_0 = B_{\lambda}(t)x_0 \). We have:

\[
(\lambda - A)y_i = (\lambda - A)B_{\lambda}(t)B_{\lambda}^{i}(t)x_i = (e^{\lambda t} - T(t))B_{\lambda}^{i}(t)x_i = B_{\lambda}^{i}(t)(e^{\lambda t} - T(t))x_i = B_{\lambda}^{i}(t)x_{i-1} = y_{i-1}.
\]
and 

\[ \sup \|y(t)\|_1^+ < \infty \]

So that \( \lambda \notin \sigma_A(x) \)

\[ \square \]

**Remark 3.1.** The spectral inclusion for local spectrum is strict. Indeed, let \( (T(t))_{t \geq 0} \) be a quasi-nilpotent \( C_0 \) semigroup with infinitesimal generator \( A \), and \( 0 \neq x \in X \). We have \( \sigma_{T(t)}(x) = \{0\} \), but \( e^{t\sigma_A(x)} = \emptyset \).

4. Stability Results.

Let \( (T(t))_{t \geq 0} \) ba a \( C_0 \)-semigroup on \( X \) with infinitesimal generator \( A \). \( (T(t))_{t \geq 0} \) is said to be strongly stable if \( \lim_{t \to \infty} \|T(t)x\| = 0 \) for all \( x \in X \). We say that \( (T(t))_{t \geq 0} \) is uniformly stable if \( \lim_{t \to \infty} \|T(t)\| = 0 \).

In [2], A. Elkoutri and M. A. Taoudi showed that \( (T(t))_{t \geq 0} \) is strongly stable if \( \sigma_K(A) \cap i\mathbb{R} = \emptyset \). In the following, we give a stability result for strongly continuous semigroups using the local spectrum:

**Proposition 4.1.** Let \( A \) be the generator of a bounded strongly continuous semigroup \( (T(t))_{t \geq 0} \). If \( \sigma_A(x) \cap i\mathbb{R} = \emptyset \) for all \( x \in X \), then \( (T(t))_{t \geq 0} \) is strongly stable.

**Proof.** If \( \sigma_A(x) \cap i\mathbb{R} = \emptyset \), for all \( x \in X \). Then,

\[ \sigma_{su}(A) \cap i\mathbb{R} = \bigcup_{x \in X} \sigma_A(x) \cap i\mathbb{R} = \bigcup_{x \in X} (\sigma_A(x) \cap i\mathbb{R}) = \emptyset. \]

As \( \sigma_K(A) \cap i\mathbb{R} \subseteq \sigma_{su}(A) \cap i\mathbb{R} = \emptyset \), then \( \sigma_K(A) \cap i\mathbb{R} = \emptyset \). According to [2, corollary 2.1], \( (T(t))_{t \geq 0} \) is strongly stable.

\[ \square \]

**Proposition 4.2.** Let \( A \) be the generator of a bounded strongly continuous semigroup \( (T(t))_{t \geq 0} \). Then, the following assertions are equivalent:

1. \( (T(t))_{t \geq 0} \) is uniformly stable;
2. for all \( x \in X \), there exists \( t_0 > 0 \) such that \( \sigma_{T(t_0)}(x) \cap \Gamma = \emptyset \)

where \( \Gamma \) stands for the unit circle of \( \mathbb{C} \).

**Proof.** According to [2, corollary 2.2] and [3, Theorem 3.2], it suffices to show that \( \sigma_{T(t_0)}(x) \cap \Gamma = \emptyset \) implies that \( \sigma_K(T(t_0)) \cap \Gamma = \emptyset \). Indeed: If \( \sigma_{T(t_0)}(x) \cap \Gamma = \emptyset \) for all \( x \in X \), then

\[ \sigma_{su}(T(t_0)) \cap \Gamma = \bigcup_{x \in X} \sigma_{T(t_0)}(x) \cap \Gamma = \bigcup_{x \in X} (\sigma_{T(t_0)}(x) \cap \Gamma) = \emptyset. \]

As \( \sigma_K(T(t_0)) \cap \Gamma \subseteq \sigma_{su}(T(t_0)) \cap \Gamma = \emptyset \), then \( \sigma_K(T(t_0)) \cap \Gamma = \emptyset \).

\[ \square \]

**Example 4.3.** Consider the Heat equation in \( L^p(0, \pi) \).

\[
\begin{cases}
\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x), (t,x) \in \mathbb{R}^+ \times (0, \pi) \\
u(t,0) = 0 = u(t,\pi), t \geq 0 \\
u(0,x) = f(x) \quad x \in (0, \pi).
\end{cases}
\]

Let \( p > 2 \). On \( X = L^p(0, \pi) \) consider the operator defined by

\[ Af(x) = f''(x) \]

with domain \( D(A) = W^{2,p}(0, \pi) \cap W_0^{1,p}(0, \pi) \), \( x \in (0, \pi) \) where

\[ W_0^{1,p} = \{ f \in W^{1,p}(0, \pi) : f(0) = 0 = f(\pi) \} \].
The operator $A$ is self-adjoint. For each $f \in W^{2,p}(0, \pi) \cap W^{1,p}_0(0, \pi)$ the unique solution of the equation is given by

$$u(t, x) = (T(t)f)(x).$$

The spectrum of $A$ is $\sigma(A) = \{-n^2; n \geq 1\}$. Since $\text{int}(\sigma(A)) = \emptyset$, then $A$ has the SVEP. So, $T(t)$ has the SVEP for all $t > 0$. Since $\sigma(x) \cap \mathbb{R} = \emptyset$ for all $x \in X$, then $(T(t))_{t \geq 0}$ is strongly stable.

**Example 4.4.** On the Banach space $X := L^3(\frac{a}{2}, 1]$ define the operator:

$$Af := -f' - (\mu + b)f \quad \text{with} \quad D(A) := \{ f \in W^{1,1}(\frac{a}{2}, 1] : f(\frac{a}{2}) = 0 \},$$

where $\mu$ is a positive continuous function on $[\frac{a}{2}, 1]$ and $b$ a continuous function with $b(s) > 0$ for $s \in (\alpha, 1)$, $b(s) = 0$ otherwise. The operator $A$ generates a $C_0$ semigroup $(T(t))_{t \geq 0}$ on $X$ given by:

$$T(t)f(s) = \begin{cases} e^{-\int_{s-t}^{s-\alpha}(\mu(\tau)+b(\tau))d\tau} \cdot f(s-t) & \text{for } s-t > \frac{a}{2}, \\ 0 & \text{elsewhere.} \end{cases}$$

The spectrum $A$ is empty. Hence $A$ has the SVEP, so $T(t)$ has the SVEP for all $t > 0$. Furthermore, $(T(t))_{t \geq 0}$ is a nilpotent semigroup, so $\sigma(T(t)) = \{0\}$. Hence $\sigma_s(T(t)) = \{0\}$, where $\sigma_s = \sigma_s, \sigma_{\alpha p}, \sigma_k, \sigma_c$. Since, $T(t)$ has the SVEP, then $\sigma_{T(t)}(x) = \{0\}$ for all $x \in X \setminus \{0\}$, so $\sigma_{T(t)}(x) \cap \Gamma = \emptyset$, thus $(T(t))_{t \geq 0}$ is uniformly stable.

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