Optimal nonequilibrium entanglement of nanomechanical oscillators

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We investigate nonequilibrium entanglement generation in a chain of harmonic oscillators with time–dependent linear coupling. We use optimal control theory to determine the coupling modulation that leads to maximum logarithmic negativity for a pair of opposite oscillators and show that it corresponds to a synchronization of the eigenmodes of the chain. We further analytically relate the maximum attainable entanglement to the irreversible work done to produce it, thus bridging nonequilibrium entanglement production and nonequilibrium thermodynamics.

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Entanglement is a puzzling feature of quantum theory. Entangled states are routinely created and manipulated in dozens of laboratories around the world: Controlled multiparticle entanglement has for example been achieved for up to eight qubits in ion traps \[1,2\]. In addition, entangled pairs of photons have been prepared over large distances of several kilometers in optical fibers \[3,4\] as well as in free space \[5,6\]. Due to their unique properties, entangled states have lead to many applications in quantum communication and quantum information theory that are not possible at the classical level, such as quantum cryptography \[7\] and teleportation \[8\]. And yet, the true nature of entanglement remains elusive, especially in many–body systems \[8\].

For quantum many–body systems at thermal equilibrium, macroscopic entanglement has recently been related to thermodynamic quantities like the magnetic susceptibility \[10\], the internal energy \[11,12\], and the heat capacity \[13\]. Some of these theoretical considerations have been successfully confirmed in experiments with quantum spin systems \[13,14,15,16\]. In the above examples, equilibrium thermodynamic functions can thus be regarded as entanglement witnesses that can be used to detect and characterize entangled states. The situation, however, is much different away from equilibrium and the question of how to relate entanglement production to irreversible thermodynamics is largely open.

Our purpose in this paper is to determine the maximum attainable entanglement generated in a nonequilibrium transformation and express the optimal value in terms of the irreversible work done during the process. The concept of irreversible work plays a fundamental role in the study of nonequilibrium thermodynamics of small systems \[17\]. To be specific, we shall consider a chain of quantum harmonic oscillators with time–dependent linear coupling. This system is of direct relevance to the study of nonequilibrium thermodynamics of small systems \[17\].

In the following, we investigate nonequilibrium entanglement generation in the harmonic chain from the point of view of irreversible thermodynamics. We use optimal control theory \[22\] to determine the modulation of the coupling coefficient that leads to maximum entanglement between distant pairs of oscillators. We find that optimal entanglement corresponds to a synchronization of the eigenfrequencies of the linear chain. Moreover, we show that the optimal value can greatly surpass that obtained from a sudden switch of the coupling strength. By further using recent findings on the nonequilibrium work distribution of a single oscillator \[24\], we are able to relate the optimal entanglement to the irreversible work performed during the nonadiabatic transformation.

Harmonic chain. We consider a closed chain of \(N\) harmonic oscillators with identical frequency \(\omega_0\) and linear nearest–neighbour coupling with variable intensity \(c(t)\):

\[
H = \frac{1}{2} \sum_{n=1}^{N} \left( p_n^2 + \omega_0^2 q_n^2 + c(t)(q_{n+1} - q_{n})^2 \right). \tag{1}
\]

To analyze the Hamiltonian \(\text{(1)}\), it is convenient to introduce the normal mode coordinates \(Q_s\) and \(P_s\) via,

\[
q_n = \sum_{s=1}^{N} e^{2\pi is/N} \frac{Q_s}{\sqrt{N}}, \quad p_n = \sum_{s=1}^{N} e^{2\pi is/N} \frac{P_s}{\sqrt{N}}. \tag{2}
\]

The Hamiltonian of the chain can then be rewritten as a sum of independent oscillators with time–modulated
frequencies, \( \omega^2_s(t) = \omega_0^2 + 4c(t) \sin^2(\pi s/N), \)
\[
H = \frac{1}{2} \sum_{s=1}^{N} \left( P_s P_s^\dagger + \omega^2_s(t) Q_s Q_s^\dagger \right),
\]
where we have used the property, \( Q_{-s} = Q_s^\dagger, P_{-s} = P_s^\dagger \).

The linear Heisenberg equations of motion for \( P_s(t) \) and \( Q_s(t) \) can readily be written down and solved \[22]\.

The Gaussian states of the linear chain are fully characterized by the symmetric \( 2N \times 2N \) covariance matrix \( \Gamma \), whose elements are defined by \( \Gamma_{q_n q_m} = 2 \text{Re}(q_n q_m^*), \Gamma_{p_n p_m} = 2 \text{Re}(p_n p_m^*) \).

We have omitted the first moments, since they do not influence the entanglement properties. Assuming for the time being that the chain is initially in the ground state, and that therefore \( \Gamma_{q_n q_m}(0) = \Gamma_{p_n p_m}(0) = \delta_{n,m}, \Gamma_{q_n p_m}(0) = 0 \), the matrix elements can be written in terms of the second moments of the normal modes as,
\[
\langle q_n q_m \rangle = \frac{1}{N} \sum_{s=1}^{N} e^{2\pi i s(n-m)/N} \langle Q_s^\dagger Q_s \rangle,
\]
using \( \langle Q_s^\dagger Q_s \rangle = \langle q_s q_s^\dagger \rangle \delta_{r,s} \). Similar expressions are obtained for \( \langle p_n p_m \rangle \) and \( \langle p_n q_m \rangle \). The covariance matrix is hence entirely determined by the second moments of the normal coordinates. It is known that a time-dependent oscillator is squeezed when its frequency is changed nonadiabatically, while an adiabatic transformation leads to no squeezing \[23]\.

We can therefore express the quadratures of the normal oscillators in terms of the squeezing parameter \( r_s \) and the rotation angle \( \theta_s \) in the form,
\[
\langle Q_s^\dagger Q_s \rangle = \frac{1}{2\omega_s} (e^{-2r_s \cos^2 \theta_s + e^{2r_s \sin^2 \theta_s}}),
\]
\[
\langle P_s P_s^\dagger \rangle = \frac{\omega_s}{2} (e^{-2r_s \cos^2 \theta_s + e^{-2r_s \sin^2 \theta_s}}),
\]
\[
\langle Q_s P_s^\dagger \rangle = \sinh(2r_s) \sin \theta_s \cos \theta_s.
\]

The time dependence of \( \omega_s, r_s \) and \( \theta_s \) is controlled by the linear coupling coefficient \( c(t) \). Standard thermodynamics considers infinitesimal changes that take place infinitely slowly. In the spirit of finite–time thermodynamics \[20]\, we will in the sequel consider infinitesimal changes that can occur at arbitrary rate. We will thus assume that the coupling intensity is small, \( c(t) \ll \omega^2_0 \), but otherwise arbitrary. Moreover, without loss of generality, we will set the frequency \( \omega_0 = 1 \).

Maximum entanglement. The entanglement between any two oscillators \( n \) and \( m \) in the chain can be characterized with the help of the logarithmic negativity \[20]\,
\[
E_N = -\frac{1}{2} \sum_{i=1}^{4} \log_2 [\min(1, |l_i|)].
\]
Here, \( l_i \) are the symplectic eigenvalues of the covariance matrix of the partially transposed reduced density operator of the two oscillators. In the present investigation, we choose two opposite oscillators, \( m = N/2 + n \). The latter configuration corresponds to the largest possible distance in the chain. It can also be shown to correspond to the largest value of the logarithmic negativity due to symmetry reasons. It is worth noting that in this case the exponential factor in Eq. (1) simplifies to \( e^{-2\pi r s(n-m)/N} = (-1)^n \), thus distinguishing even and odd normal modes. Furthermore, in view of the translational invariance along the chain, the logarithmic negativity does not depend on the particular value of \( n \).

The symplectic eigenvalues of the \( 4 \times 4 \) reduced covariance matrix \( \Gamma_{m,n} \) can be computed explicitly using the parametrization \[16\]. A careful examination of their structure reveals that for moderate squeezing the logarithmic negativity \( \Theta \) is maximum when the angles \( \theta_s \) take special values. These optimal angles are given by \( \theta_s = \pi/4 (\text{mod} \pi) \) for \( s \) even and \( \theta_s = 3\pi/4 (\text{mod} \pi) \) for \( s \) odd. Hence, for fixed squeezing, maximum entanglement is achieved when both even and odd oscillators are synchronized. From Eqs. (15), it follows that, at the optimal angles, \( \langle Q_s Q_s^\dagger \rangle = \cosh(2r_s)/2\omega_s, \langle P_s P_s^\dagger \rangle = \cosh(2r_s)\omega_s/2 \) and \( \langle Q_s P_s^\dagger \rangle = (-1)^s \sinh(2r_s)/2 \). Combining Eqs. (14) and (15), with \( \omega_s \approx \omega_0 \), we then find that \( \langle q_n^2 \rangle = \langle p_n^2 \rangle = a \) and \( \langle q_n p_m \rangle = \langle p_n q_m \rangle = b \). As a result, the symplectic eigenvalues in Eq. (16) are simply given by,
\[
|l_1| = \sqrt{|a+b-(c+d)| |a\pm b+c\pm d|},
\]
with \( c = \langle q_n p_m \rangle \) and \( d = \langle p_n q_m \rangle \). We therefore obtain that the maximum attainable logarithmic negativity is:
\[
E_{N,\text{max}} = -\frac{1}{2} \log_2 \left[ \frac{2}{N} \sum_{s, \text{odd}} e^{-2r_s} \left( \frac{2}{N} \sum_{s, \text{even}} e^{-2r_s} \right) \right].
\]
Equation (18) shows that entanglement is zero for vanishing squeezing and increases with increasing squeezing. By comparing the simplified symplectic eigenvalues (17) with their general expression, we further find that the optimal angles are valid provided the condition \( c \sum_s \exp(2r_s)/4 \ll N \) is satisfied. As we shall discuss below; this condition is not very restrictive.

Optimal control theory. Optimal control theory (OCT) is a powerful method that allows to determine the function that optimizes a given cost functional \[22\]. It can be regarded as a generalization of the calculus of variations and has a long tradition in finite–time thermodynamics \[20\]. We apply Pontryagin’s principle to find the function \( c(t) \) that optimizes the entanglement between opposite oscillators. To this end, we take as the cost functional to minimize, the argument of the maximum attainable logarithmic negativity (8), \( \sum_{s, \text{odd}} \exp(-2r_s) \sum_{s, \text{even}} \exp(-2r_s) \). It is clear that the minimum of this quantity will lead to maximum entanglement. The constraints given by the equations of motion of the normal modes (2) enter the optimization problem in the form of Lagrange multipliers. The optimal function \( c(t) \) is eventually obtained by steepest descent.
The results of the numerical implementation of optimal control theory for a chain of $N = 8$ oscillators are summarized in Figs. 1 and 2.

Figure 1 shows the logarithmic negativity $E_N$ of oscillators 1 and 5 as a function of time for various coupling intensities $c(t)$. The shaded curve at the bottom corresponds to a sudden switch from $c = 0$ to $c = 0.05$. The oscillatory curve in the center depicts the entanglement generated using OCT for the same parameters. The upper curve, on the other hand, is the maximum attainable entanglement at zero temperature, Eq. (8), when the angles of the normal oscillators are perfectly synchronized at all times. We observe that the entanglement produced by the sudden switch is one order of magnitude smaller than the maximum attainable value in the present situation. Moreover, we note that the optimal logarithmic negativity (almost) reaches the maximum attainable value $E_N^{\text{max}}$. Figure 2 shows the angles $\theta_s$ of the normal oscillators, corresponding to the optimal modulation, in two short time intervals: the first around the peak of highest entanglement (denoted by B in Fig. 1) and the second around a minimum of the logarithmic negativity (interval A). The (nearly perfect) synchronization of the even and odd modes at the center of the interval is clearly visible in case B, while it is absent in case A. The optimal modulation $c(t)$ is plotted in Fig. 3 and consists of three different parts. The first part is a sequence of square pulses between $c = 0$ and $c = 0.05$. The latter is the result of the optimization procedure that tries to maximally squeeze each normal oscillator in order to minimize the argument of Eq. (8); the maximal entanglement $E_N^{\text{max}}$ is determined by the duration of this sequence. In the second part, the coupling $c(t)$ is kept constant allowing the angles to synchronize; the synchronization time is directly controlled by the value of $c$: a larger coupling will lead to a faster synchronization. Since each normal mode evolves with a different frequency, the synchronization of the angles cannot persist in time. However, by stopping the modulation of $c(t)$ (dashed line in Fig. 3), once the biggest entanglement has been attained, the peak value can be maintained (see dashed line in Fig. 1).

**Dissipative work.** We next relate the nonequilibrium entanglement produced by a nonadiabatic switching of the coupling to the average dissipated work. The dissipated work is a central concept in thermodynamics and is intimately connected to the irreversible entropy production $\dot{F}$, it is defined as the difference between the total work and the reversible work (or free energy difference), $W_{\text{dis}} = W - \Delta F$. For an adiabatic transformation, $W_{\text{dis}} = 0$ and the total work is given by $\Delta F$. The dissipated work for a frequency-modulated normal oscillator of the chain can be simply expressed in terms of the frequency and the squeezing parameter at time $t$:

$$W_{\text{dis},s} = \omega_s \sinh^2 r_s .$$

The above equation indicates that there is a one-to-one correspondence between the dissipated work $W_{\text{dis},s}$ and the squeezing parameter $r_s$: small squeezing yields small dissipated work, $W_{\text{dis},s} \approx \omega_s r_s^2$, whereas large squeezing leads to an exponential increase of the dissipated work, $W_{\text{dis},s} \approx \omega_s \exp(2 r_s)/4$. The total work dissipated in the chain is given by the sum over all eigenmodes, $W_{\text{dis}} = \sum_s W_{\text{dis},s}$. By combining Eqs. (8) and (9), we can now directly relate the maximum attainable logarithmic negativity to the irreversible work required to generate it. This relationship is of importance for sev-
FIG. 3: Optimal coupling modulation $c(t)$ that generates the logarithmic negativity shown in the center of Fig. [4] eral reasons: First, from a fundamental point of view, it bridges nonequilibrium entanglement production with nonequilibrium thermodynamics; second, it gives us the maximum amount of entanglement that can be produced by dissipating a given amount of work, in the ideal situation where all the phases are synchronized. One could for instance dissipate considerably more work by highly squeezing few oscillators in the chain, irrespective of their angles, and produce entanglement far below the maximum possible value. In that respect, this relationship is similar in spirit to the Carnot efficiency of a heat engine that gives the maximum efficiency that can be achieved for given temperatures, in an ideal cycle.

Effect of temperature. Our discussion so far was limited to a chain at zero temperature. Finite temperatures can easily be taken into account by multiplying the initial covariance matrix $\Gamma(0)$ by the factor $f(T) = \coth(\omega_0/2T)$. The new symmetric eigenvalues are then merely multiplied by the factor $f(T)$. As a consequence, the maximum attainable logarithmic negativity is still given by its expression [8], with the argument now multiplied by $f(T)$. Since $f(T) \geq 1$, temperature reduces the maximal value of entanglement, as expected. Figure [5] shows the maximal logarithmic negativity for $T = 0.5$. A good estimate of the highest temperature $T_m$ at which entanglement vanishes can be obtained by replacing the squeezing parameters of the different eigenmodes by the average squeezing, $R = \sum_s r_s/N$. We then find that entanglement survives up to $T_m = \omega_0/\ln(\coth(2R))$.

Discussion. One of the main limitations that have to be considered in real systems is the validity of the harmonic approximation. As the oscillators are squeezed, their energy increases and anharmonicities may appear. For the optimal modulation of Fig. [3] with a squeezing sequence up to $t = 20$, $R = 0.4$ and $W_{\text{dis}} = 1.2$ (in units of $\hbar$). This leads to an increase of 30% of the total energy of the chain from $E_0 = 4$ to $E_0 + W_{\text{dis}} = 5.2$. A tenfold energy increase is achieved with a squeezing sequence up to $t = 85$, yielding $R = 1$ and $E_{N_{\text{max}}} = 1.6$. These parameters are well within the bounds of the criterion given below Eq. [8]. A second limitation is environment induced decoherence. Fortunately, as shown in Ref. [22], entanglement in the chain is particularly robust against external noise sources at low temperatures.

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