Logarithmic Regret for Reinforcement Learning with Linear Function Approximation

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Abstract

Reinforcement learning (RL) with linear function approximation has received increasing attention recently. However, existing work has focused on obtaining $\sqrt{T}$-type regret bound, where $T$ is the number of steps. In this paper, we show that logarithmic regret is attainable under two recently proposed linear MDP assumptions provided that there exists a positive sub-optimality gap for the optimal action-value function. In specific, under the linear MDP assumption (Jin et al. 2019), the LSVI-UCB algorithm can achieve $\hat{O}(d^3H^5/\text{gap}_{\min} \cdot \log(T))$ regret; and under the linear mixture model assumption (Ayoub et al. 2020), the UCRL-VTR algorithm can achieve $\hat{O}(d^2H^5/\text{gap}_{\min} \cdot \log^3(T))$ regret, where $d$ is the dimension of feature mapping, $H$ is the length of episode, and $\text{gap}_{\min}$ is the minimum of sub-optimality gap. To the best of our knowledge, these are the first logarithmic regret bounds for RL with linear function approximation.

1 Introduction

Designing efficient algorithms that learn and plan in sequential decision-making tasks with large state and action spaces has become a central task of modern reinforcement learning (RL) in recent years. RL often assumes the environment as a Markov Decision Process (MDP), described by a tuple of state space, action space, reward function, and transition probability function. Due to a large number of possible states and actions, traditional tabular reinforcement learning methods such as Q-learning (Watkins, 1989), which directly access each state-action pair, are computationally intractable. A common approach to cope with high-dimensional state and action spaces is to utilize feature mappings such as linear functions or neural networks to map states and actions to a low-dimensional space.

Recently, a large body of literature has been devoted to provide regret bounds for online RL with linear function approximation. These works can be divided into two main categories. The first category of works is of model-free style, which directly parameterizes the action-value function as a linear function of some given feature mapping. For instance, Jin et al. (2020) studied the episodic MDPs with linear MDP assumption, which assumes that both transition probability function and reward function can be represented as a linear function of a given feature mapping. Under this assumption, the LSVI-UCB algorithm can achieve $\hat{O}(d^3H^5/\text{gap}_{\min} \cdot \log(T))$ regret. However, obtaining logarithmic regret bounds has been an open problem in this setting.

In this paper, we show that logarithmic regret is attainable under two recently proposed linear MDP assumptions. The first assumption is the linear MDP assumption (Jin et al. 2019), which assumes that both transition probability function and reward function can be represented as a linear function of a given feature mapping. Under this assumption, the LSVI-UCB algorithm can achieve $\hat{O}(d^3H^5/\text{gap}_{\min} \cdot \log(T))$ regret. The second assumption is the linear mixture model assumption (Ayoub et al. 2020), which assumes that both transition probability function and reward function can be represented as a linear function of a given feature mapping. Under this assumption, the UCRL-VTR algorithm can achieve $\hat{O}(d^2H^5/\text{gap}_{\min} \cdot \log^3(T))$ regret.
assumption, Jin et al. (2020) showed that the action-value function is a linear function of the feature mapping and proposed a model-free LSVI-UCB algorithm to obtain a $\tilde{O}(d^{3}H^{5}T)$ regret, where $d$ is the dimension of the feature mapping, $H$ is the length of the episode, and $T$ is the number of steps. The second category of works is in model-based style, which parameterizes the underlying transition probability function as a linear function of a given feature mapping. Ayoub et al. (2020) studied RL algorithms with the linear mixture model assumption, which assumes that the transition probability function can be represented as a linear function. Ayoub et al. (2020) proposed a model-based UCRL-VTR algorithm with a regret $\tilde{O}(d\sqrt{H^{5}T})$. Zhou et al. (2020) studied the linear kernel MDP for discounted MDPs and proposed a similar $\sqrt{T}$-type regret. Although these $\sqrt{T}$-type regrets are easy to interpret and understand, they do not consider any additional problem-dependent structure of the underlying MDP. This limitation prevents them from providing a tighter and preciser regret analysis.

A large number of studies on problem-dependent regret bounds have considered a quantity called sub-optimality gap (or gap), which is defined as the strictly positive value gap of the optimal action-value function between the optimal action and the rest ones. Sub-optimality gap has been well-studied in the bandit literature (Bubeck and Cesa-Bianchi, 2012; Slivkins et al., 2019; Lattimore and Szepesvári, 2020), which can be regarded as a special instance of RL problems. For general RL, previous works have considered the tabular MDP with gap condition and provided corresponding gap-dependent regret bounds (Simchowitz and Jamieson, 2019; Yang et al., 2020). However, as far as we know, there does not exist such gap-dependent regret results for RL with linear function approximation. Therefore, a natural question arises:

*Can we derive problem-dependent regret bounds for RL with linear function approximation?*

We answer the above question affirmatively in this paper. In detail, we consider a problem-dependent but algorithm-independent structure quantity called $\text{gap}_{\text{min}}$, which is the minimum action sub-optimality gap of the optimal action-value function. We show that LSVI-UCB proposed in Jin et al. (2020) achieves a $\tilde{O}(d^{3}H^{5}/\text{gap}_{\text{min}} \cdot \log(T))$ regret, and UCRL-VTR proposed by Ayoub et al. (2020) achieves a regret of order $\tilde{O}(d^{2}H^{5}/\text{gap}_{\text{min}} \cdot \log^{3}(T))$. To the best of our knowledge, this is the first problem-dependent $\log T$-type regret achieved by RL with linear function approximation. At the core of our proof is to represent the total regret as a summation of sub-optimality between the estimated value functions and the optimal value functions, divide them into different groups, and analyze them correspondingly. Our results suggest that the dependence on $T$ in regrets can be drastically decreased from $\sqrt{T}$ to $\log T$ when considering the problem structure for both model-free and model-based RL algorithms with linear function approximation.

**Notation** We use lower case letters to denote scalars, and use lower and upper case bold face letters to denote vectors and matrices respectively. For a vector $x \in \mathbb{R}^{d}$, we denote by $\|x\|_{1}$ the Manhattan norm and denote by $\|x\|_{2}$ the Euclidean norm. For two sequences $\{a_{n}\}$ and $\{b_{n}\}$, we write $a_{n} = O(b_{n})$ if there exists an absolute constant $C$ such that $a_{n} \leq C b_{n}$. We use $\tilde{O}(\cdot)$ to further hide the logarithmic factors. For logarithmic regret, we use $\tilde{O}(\cdot)$ to hide all logarithmic terms except $\log T$.

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$^{2}$Linear kernel MDP is the same as linear mixture model.
2 Related Work

Tabular episodic MDP There are a series of work focusing on regret or sample complexity of online RL on tabular episodic MDPs. They can be recognized as model-free methods or model-based methods, which depend on whether they explicitly estimate the model (transition probability function) or not. Dann and Brunskill (2015) proposed a UCFH algorithm that adopts a variant of extended value iteration and obtains a polynomial sample complexity. Azar et al. (2017) proposed a UCB-VI algorithm which adapts the Bernstein-style exploration bonus with a \( \tilde{O}(\sqrt{H SAT}) \) regret which matches the lower bound proposed in Jaksch et al. (2010); Osband and Van Roy (2016) up to logarithmic factors. Zanette and Brunskill (2019) proposed an EULER algorithm which utilizes the problem-dependent bound and achieves a \( \tilde{O}(\sqrt{H SAT}) \) regret. For model-free algorithms, Strehl et al. (2006) proposed a delayed Q-learning algorithm with a sublinear regret. Later Jin et al. (2018) proposed a Q-learning with UCB algorithm which achieves \( \tilde{O}(\sqrt{H^3 SAT}) \) regret. Recently, Zhang et al. (2020) proposed a UCB-advantage algorithm with an improved regret \( \tilde{O}(\sqrt{H^2 SAT}) \), which matches the information-theoretic lower bound (Jaksch et al., 2010; Osband and Van Roy, 2016) up to logarithmic factors.

Logarithmic regret bound for RL A line of works focus on providing log\( T \)-style regret bound for RL algorithms based on problem-dependent quantities. It has been shown that such a log\( T \) dependence is unavoidable according to the lower bound results shown in Ok et al. (2018). For the upper bounds, Auer and Ortner (2007) showed that the UCRL algorithm achieves logarithmic regret in the average reward setting, while the regret bound depends on both the hitting time and the policy sub-optimal gap. Tewari and Bartlett (2008) proposed OLP algorithm for average-reward MDP and showed that OLP achieves logarithmic regret \( O(C(P) \log(T)) \) where \( C(P) \) is an explicit MDP-dependent constant. Both results in Auer and Ortner (2007) and Tewari and Bartlett (2008) are asymptotic, which required the number of steps \( T \) is large enough. For non-asymptotic bounds, Jaksch et al. (2010) proposed UCRL2 algorithm for average-reward MDP with regret \( O(D^2 S^2 A \log(T)/\Delta) \), where \( D \) is the diameter and \( \Delta \) is the policy sub-optimal gap. In episodic MDPs, Simchowitz and Jamieson (2019) proposed a model-based StrongEuler algorithm with a logarithmic regret, and showed a lower bound of regret for MDP dependent on the minimal action sub-optimality gap. Recently, (Yang et al., 2020) showed that the model-free algorithm optimistic Q-learning achieves \( O(SA^H \log(SAT)/\text{gap}_{\text{min}}) \) regret.

Linear function approximation Recently, there emerges a large body of literature on solving MDP using RL with linear function approximation (Jin et al., 2020; Yang and Wang, 2019b; Wang et al., 2019b; Modi et al., 2019; Jiang et al., 2017; Zanette et al., 2020b; Du et al., 2019a; Yang and Wang, 2019a; Du et al., 2019b; Zanette et al., 2020a; Cai et al., 2019; Wang et al., 2020; Jia et al., 2020; Ayoub et al., 2020; Weisz et al., 2020; Zanette et al., 2020c; Zhou et al., 2020). In the setting of linear function approximation, these results can be categorized based on their assumptions on MDP. The first line of works consider to parameterize the action-value function with feature mapping with the linear MDP assumption. Jin et al. (2020) proposed LSVI-UCB algorithm with \( \tilde{O}(\sqrt{d^3 H^3 T}) \) regret. Wang et al. (2019b) proposed USVI-UCB algorithm in a weaker assumption called “optimistic closure” and achieved \( \tilde{O}(H \sqrt{d^3 T}) \) regret. Zanette et al. (2020b) improved the regret to \( \tilde{O}(d H \sqrt{T}) \) by considering a global planning oracle. Besides these works, Du et al. (2019b) proposed DMQ algorithm which obtains \( \epsilon \)-suboptimal policy using polynomial number of trajectories with additional low-variance assumption. Jiang et al. (2017) and Zanette et al. (2020c) focused on weaker assumption which called low inherent Bellman error and proposed algorithms with polynomial sample complexity. The second line of works consider to parameterize
the transition probability function with feature mapping with the linear mixture model assumption. Jia et al. (2020) and Ayoub et al. (2020) proposed UCLR-VTR algorithm for episodic MDPs which achieves $\hat{O}(d\sqrt{HT}T)$ regret. Cai et al. (2019) proposed policy optimization algorithm OPPO which achieves $\hat{O}(d^3HT^2\gamma T)$ regret. Furthermore, Zhou et al. (2020) focused on discounted MDP setting and proposed a UCLK algorithm, which achieves $\hat{O}(d\sqrt{T}/(1-\gamma)^2)$ regret.

3 Preliminaries

We consider episodic Markov Decision Processes (MDP) which can be denoted by a tuple $\mathcal{M}(S, A, H, r, P)$. Here, $S$ is the state space, $A$ is the finite action space, $H$ is the length of each episode, $P = \{P_h\}_{h=1}^H$ is the transition probability function and $r = \{r_h\}_{h=1}^H$ is the reward function. Furthermore, $r_h : S \times A \rightarrow [0, 1]$ is the reward function at step $h$ and $P_h(s'|s, a)$ is the transition probability function at step $h$ which denotes the probability for state $s$ to transfer to state $s'$ with action $a$ at step $h$.

A policy $\pi : S \times [H] \rightarrow A$ is a function which maps a state $s$ and the step number $h$ to an action $a$. For any policy $\pi$ and step $h \in [H]$, we denote the action-value function $Q_h^\pi(s, a)$ and value function $V_h^\pi(s)$ as follows

$$Q_h^\pi(s, a) = r_h(s, a) + \mathbb{E}\left[\sum_{h'=h+1}^{\infty} r_{h'}(s_{h'}, \pi(s_{h'}, h'))|s, \pi, h\right], V_h^\pi(s) = Q_h^\pi(s, \pi(s, h)),$$

where $s_h = s, a_h = a$ and $s_{h'+1} \sim P_h(\cdot|s', a_h')$. We define the optimal value function $V_h^\star$ and the optimal action-value function $Q_h^\star$ as $V_h^\star(s) = \sup_\pi V_h^\pi(s)$ and $Q_h^\star(s, a) = \sup_\pi Q_h^\pi(s, a)$. By definition, the value function $V_h^\pi(s)$ and action-value function $Q_h^\pi(s, a)$ are bounded in $[0, H]$. For simplicity, for any function $V : S \rightarrow \mathbb{R}$, we denote $[\mathbb{E}V](s, a) = \mathbb{E}_{s' \sim P(\cdot|s, a)}V(s')$. Therefore, for each $h \in [H]$ and policy $\pi$, we have the following Bellman equation, as well as the Bellman optimality equation:

$$Q_h^\pi(s, a) = r_h(s, a) + [\mathbb{E}[Q_{h+1}^\pi](s, a), Q_h^\star(s, a) = r_h(s, a) + [\mathbb{E}[V_{h+1}^\pi](s, a)$$

where $V_{H+1}^\pi = V_{H+1}^\pi = 0$. Furthermore, we define the total regret in the first $K$ episodes as follows.

**Definition 3.1.** For any policy $\pi$, we define its regret on MDP $M(S, A, H, r, P)$ in the first $K$ episodes as the sum of the suboptimality for $k = 1, \ldots, K$, i.e.,

$$\text{Regret}(K) = \sum_{k=1}^{K} V_1^\pi(s_1^k) - V_1^\pi(s_1^k),$$

where $\pi_k$ is the policy in episodes $k$.

In this paper, we focus on the action sub-optimality gap condition (Simchowitz and Jamieson, 2019; Du et al., 2019b, 2020; Yang et al., 2020; Mou et al., 2020) and linear function approximation (Jin et al., 2020; Ayoub et al., 2020; Jia et al., 2020; Zhou et al., 2020).

**Definition 3.2** (Positive action sub-optimality gap). For each $s \in S, a \in A$ and step $h \in [H]$, the action sub-optimality gap $\text{gap}_h(s, a)$ is defined as

$$\text{gap}_h(s, a) = V_h^\star(s) - Q_h^\pi(s, a),$$

and the minimal action sub-optimality gap is defined as

$$\text{gap}_{\min} = \min_{s, a} \text{gap}_h(s, a) : \text{gap}_h(s, a) \neq 0\right\}.$$

(3.2)
4 Model-free RL with Gap Condition

In this section we focus on model-free RL algorithms with linear function approximation. We make the following linear MDP assumption (Jin et al., 2020; Yang and Wang, 2019a) where the probability transition kernels and the reward functions are assumed to be linear with respect to a given feature mapping \( \phi : S \times A \to \mathbb{R}^d \).

**Assumption 4.1.** MDP \( \mathcal{M}(S, A, H, r, \mathbb{P}) \) is a linear MDP such that for any step \( h \in [H] \), there exists an unknown vector \( \mu_h \), unknown measures \( \theta_h = (\theta_h^{(1)}, \ldots, \theta_h^{(d)}) \) and a known feature mapping \( \phi : S \times A \to \mathbb{R}^d \), where for each \((s, a) \in S \times A \) and \( s' \in S \),

\[
\mathbb{P}_h(s'|s, a) = \langle \phi(s, a), \theta_h(s') \rangle, r(s, a) = \langle \phi(s, a), \mu_h \rangle.
\]

For simplicity, we assume that the unknown vector \( \mu_h \) and feature \( \phi(s, a) \) satisfy \( \| \phi(s, a) \|_2 \leq 1 \), \( \| \mu_h \|_2 \leq \sqrt{d} \) and \( \| \theta_h(S) \| \leq \sqrt{d} \).

**Remark 4.2.** Under Assumption 4.1, it can be shown that for any policy \( \pi \), the action-value function \( Q^*_h(s, a) \) is a linear function \( \langle \phi(s, a), \theta^*_h \rangle \) with respect to the feature mapping \( \phi \) through the Bellman equation (3.1). This suggests that to estimate the unknown optimal action-value function \( Q^*_h \), we only need to estimate its corresponding parameter \( \theta^*_h \).

**Remark 4.3.** Though the probability transition kernel and the reward function are linear with respect to \( \phi(s, a) \), the degree of freedom of measure \( \theta_h \) is \( |S| \times d \). This suggests that when state space \( S \) is large, it is computationally intractable to estimate the probability transition kernel \( \mathbb{P} \).

4.1 Algorithm

We analyze the LSVI-UCB algorithm proposed in Jin et al. (2020), which is showed in Algorithm 1. At a high level, Algorithm 1 treats the optimal action-value function \( Q^*_h \) as a linear function of the feature \( \phi \) and an unknown parameter \( \theta^*_h \). The goal of Algorithm 1 is to solve for \( \theta^*_h \). Since Algorithm 1 directly estimates the action-value function, it is a model-free algorithm. Algorithm 1 uses the least-square value iteration to estimate the \( \theta^*_h \) for each \( h \) with additional exploration bonuses. In Line 5, Algorithm 1 computes \( w^k_h \), the estimate of \( \theta^*_h \), by solving a regularized least-square problem:

\[
w^k_h \leftarrow \arg\min_{w^k_h \in \mathbb{R}^d} \lambda \|w^k_h\|^2 + \sum_{i=1}^{k-1} (\phi(s^i_h, a^i_h)^T w^k_h - r_h(s^i_h, a^i_h) - \max_a Q^k_{h+1}(s^i_{h+1}, a))^2.
\]

In line 6, Algorithm 1 computes the action-value function \( Q^k_h(s, a) \) by \( w^k_h \) and adds an UCB bonus to make sure the estimate of action-value function \( Q^k_h(s, a) \) is an upper bound of the optimal action-value function \( Q^*_h(s, a) \). In line 9, a greedy policy with respect to estimated action-value function \( Q^k_h(s, a) \) is used to choose action and transit to the next state.

4.2 Main Theorem

In this subsection, we present our regret analysis for LSVI-UCB. For simplicity, we denote \( T = KH \), which is the total number of steps.
Algorithm 1 Least Square Value-iteration with UCB (LSVI-UCB) (Jin et al., 2020)

1: for episodes $k = 1, \ldots, K$ do
2:  Received the initial state $s^i_1$.
3:  for step $h = H, \ldots, 1$ do
4:     $\Lambda^k_h = \sum_{i=1}^{k-1} \phi(s^i_h, a^i_h)\phi(s^i_h, a^i_h)^\top + \lambda \cdot I$
5:     $w^k_h = (\Lambda^k_h)^{-1}\sum_{i=1}^{k-1} \phi(s^i_h, a^i_h)[r_h(s^i_h, a^i_h) + \max_a Q^k_{h+1}(s^i_{h+1}, a)]$
6:     $Q^k_h(s, a) = \min \left\{ \beta \sqrt{\phi(s, a)^\top (\Lambda^k_h)^{-1}\phi(s, a)} + w^k_h \phi(s, a), H \right\}$
7:  end for
8:  for step $h = 1, \ldots, H$ do
9:     Take action $a^k_h \leftarrow \arg\max_a Q^k_h(s^k_h, a)$ and receive next state $s^k_{h+1}$
10: end for
11: end for

Theorem 4.4. Under Assumption 4.1, there exists a constant $C$ such that, if $\lambda = 1$, $\beta = 78dH \sqrt{\log(2dT/\delta)}$ in Algorithm 1, then the expected regret for Algorithm 1 in first $T$ steps is upper bounded by

$$\mathbb{E}[\text{Regret}(K)] \leq \frac{4Cd^3H^5 \log(2dT/\delta)}{\text{gap}_\text{min}}, t + 1,$$

where $\delta$, $t$ are defined as follows:

$$\delta = \frac{1}{2K(K + 1)H^3 \log(H/\text{gap}_\text{min})}, t = \log \left( \frac{Cd^3H^4 \log(2dT/\delta)}{\text{gap}_\text{min}^2} \right).$$

The regret bound in Theorem 4.4 is independent with the size of the state space $S$, action space $A$, and is only logarithmic with the number of steps $T$, which suggests that Algorithm 1 is sample efficient for MDPs with the large state and action spaces. To our knowledge, this is the first theoretical result that achieves logarithmic regret for model-free RL with linear function approximation. Besides, the UCB bonus parameter $\beta$ depends on $T$ logarithmically. When the number of steps $T$ is unknown at the beginning, we can use the “doubling trick” (Besson and Kaufmann, 2018) to learn $T$ adaptively, and the regret will only be increased by a constant factor.

5 Model-based RL with Gap Condition

In this section we focus on model-based RL with linear function approximation. We make the following the linear mixture MDP assumption (Jia et al., 2020; Ayoub et al., 2020; Zhou et al., 2020), which assumes that the unknown transition probability function is an aggregation of several known basis models.

Assumption 5.1. MDP $\mathcal{M}(S, A, H, r, P)$ is a linear mixture MDP and there exists an unknown vector $\theta^* \in \mathbb{R}^d$ with $\|\theta\|_2 \leq C_{\theta}$, such that

$$P(s'|s, a) = \sum_{i=1}^{d} \theta^*_i P_i(s'|s, a), \quad (5.1)$$
where \( P_1, \ldots, P_d \) are known basis models such that \( \sup_{i \in [d], (s, a) \in S \times A} \| P_i (\cdot | s, a) \|_1 \leq 1 \). Moreover, the reward function \( r \) is deterministic and known.

Assumption 5.1 does not require each basis model \( P_i \) to be a probability transition model, and the reward function does not need to be a linear function. For simplicity, we denote feature \( \phi(s'|s, a) = [P_1(s'|s, a), \ldots, P_d(s'|s, a)] \) and we have \( P(s'|s, a) = \phi(s'|s, a)^\top \theta^* \).

### 5.1 Algorithm

In this subsection, we analyze the model-based UCRL with the Value-Targeted Model Estimation (UCRL-VTR) algorithm proposed in Jia et al. (2020); Ayoub et al. (2020), which is shown in Algorithm 2. At a high level, unlike Algorithm 1 which treats the action-value function as a linear function, Algorithm 2 treats the transition probability function as a linear function of the feature mapping \( \phi(\cdot | \cdot, \cdot) \) and an unknown parameter \( \theta^* \) and aims to solve for \( \theta^* \). This makes Algorithm 2 a model-based algorithm since it estimates the underlying transition model rather than directly estimating the optimal action-value function. To estimate \( \theta^* \), Algorithm 2 computes the estimate \( \theta_{k+1} \) by solving the following regularized least-square problem in Line 12:

\[
\theta_{k+1} \leftarrow \arg\min_{\theta \in \mathbb{R}^d} H^2 d \| \theta \|_2^2 + \sum_{i=1}^k \left( \phi_{V_{i, h+1}}^i(s_{h+1}^i, a_{h+1}^i)^\top \theta - V_{i, h+1}^i(s_{h+1}^i) \right)^2,
\]

where for any value function \( V : S \to \mathbb{R} \), we denote \( \phi_V(s, a) = \int_{s'} \phi(s'|s, a) V(s') ds' \in \mathbb{R}^d \). The close-form solution to \( \theta_{k+1} \) can be computed by considering the accumulated covariance matrix \( \Sigma_{k+1} \) in line 8 and 9. To guarantee exploration, in line 14, Algorithm 2 computes the action-value function \( Q_{h+1}^k(s, a) \) by \( \theta_{k+1} \) and adds a UCB bonus to make sure the estimate of action-value function \( Q_{h+1}^k(s, a) \) is an upper bound of the optimal action-value function \( Q^*_h(s, a) \). Algorithm 2 follows the greedy policy induced by the estimated action-value function \( Q_{h+1}^k(s, a) \) in line 14.

### 5.2 Regret Analysis

In this subsection, we propose our regret analysis for UCRL-VTR. For simplicity, we denote \( T = KH \), which is the total number of steps.

**Theorem 5.2.** Under Assumption 5.1, the expected regret for Algorithm 2 in first \( T \) steps is upper bounded by

\[
\mathbb{E} \left[ \text{Regret}(K) \right] \leq \frac{2048 C_\theta^2 d^2 H^5 \log^3(2dT/\delta)}{\text{gap}_{\min}} + 1,
\]

where \( \delta, \ell \) are defined as follows:

\[
\delta = \frac{1}{2K(K+1)H^3 \log(H/\text{gap}_{\min})}, \quad \ell = \log \left( \frac{512 C_\theta^2 d^2 H^4 \log^3(2dT/\delta)}{\text{gap}_{\min}^2} \right).
\]

The regret bound in Theorem 5.2 depends on the gap inversely. It is independent of the size of the state, action space \( S, A \), and is logarithmic in the number of steps \( T \), similar to that of Theorem 4.4. This suggests that model-based RL with linear function approximation also enjoys a \( \log T \)-type regret considering the problem structure.
Algorithm 2 UCRL with Value-Targeted Model Estimation (UCRL-VTR) (Jia et al., 2020; Ayoub et al., 2020)

1: Set $\Sigma_1^k = H^2dI$, $b_1^k = 0$
2: Set $\beta_k = 16d^2H^2\log(1 + Hk)\log^2((k + 1)^2/H/\delta)$.
3: for episodes $k = 1, \ldots, K$ do
4:   Received the initial state $s_1^k$
5:   for step $h = 1, \ldots, H$ do
6:      Take action $a_h^k \leftarrow \arg\max_a Q_h^k(s_h^k, a)$ and receive next state $s_{h+1}^k$
7:      Update value matrix $\Sigma$ and vector $b$:
8:      $\Sigma_{h+1}^k \leftarrow \Sigma_h^k + (\phi_{V_{h+1}^k}(s_h^k, a_h^k))^\top \phi_{V_{h+1}^k}(s_h^k, a_h^k)$
9:      $b_{h+1}^k = b_h^k + V_{h+1}^k(s_{h+1}^k) \cdot \phi_{V_{h+1}^k}(s_h^k, a_h^k)$
10: end for
11: Set $\Sigma_{k+1}^k \leftarrow \Sigma_{H+1}^k$, $b_{k+1}^k \leftarrow b_{H+1}^k$
12: Compute $\theta_{k+1} \leftarrow (\Sigma_{k+1}^k)^{-1}b_{k+1}^k$
13: for step $h = H, \ldots, 1$ do
14:      $Q_{h+1}^k(s, a) = r(s, a) + \phi_{V_{h+1}^k}(s, a)^\top \theta_{k+1} + \sqrt{\beta_{k+1} \phi_{V_{h+1}^k}(s, a)^\top (\Sigma_{k+1}^k)^{-1} \phi_{V_{h+1}^k}(s, a)}$
15: end for
16: end for

6 Proof of the Main Results

6.1 Proof of the Main Theorem

In this subsection, we give a brief overview of our proof. The proof can be divided into three main steps.

Step 1: Regret decomposition

Our goal is to upper bound the total regret $\text{Regret}(K)$. Following the regret decomposition procedure proposed in Simchowitz and Jamieson (2019); Yang et al. (2020), we rewrite the sub-optimality $V_h^*(s_h^k) - V_{\pi_h}^*(s_h^k)$ as follows:

$$V_h^*(s_h^k) - V_{\pi_h}^*(s_h^k) = (V_h^*(s_h^k) - Q_h^*(s_h^k, a_h^k)) + (Q_h^*(s_h^k, a_h^k) - V_h^*(s_h^k))$$

$$= \text{gap}_h(s_h^k, a_h^k) + \mathbb{E}_{s' \sim P(|s_h^k, a_h^k})[V_h^*(s') - V_{\pi_h}^*(s')]$$

(6.1)

where $\text{gap}_h(s, a) = V_h^*(s) - Q_h^*(s, a)$. Taking expectation on both sides of (6.1) gives

$$\mathbb{E}[V_h^*(s_h^k) - V_{\pi_h}^*(s_h^k)] = \mathbb{E}[\text{gap}_h(s_h^k, a_h^k) + V_h^*(s_{h+1}) - V_{\pi_h}^*(s_{h+1})]$$

(6.2)

Taking summation of (6.2) over all $k \in [K], h \in [H]$, we have

$$\mathbb{E}[\text{Regret}(K)] = \mathbb{E}\left[\sum_{k=1}^K \sum_{h=1}^H \text{gap}_h(s_h^k, a_h^k)\right].$$

(6.3)

(6.3) suggests that the total expected regret can be represented as a summation of gaps through time steps. Therefore, to bound the total regret, it suffices to bound each gap separately, which leads to our next proof step.
Step 2: Bound the number of sub-optimalties

Recall the range of action sub-optimality gap $\text{gap}_h(s^k_h, a^k_h)$ is $[\text{gap}_{\min}, H]$. Therefore, to bound the summation of gaps, it suffices to divide the range $[\text{gap}_{\min}, H]$ into several intervals and count the number of gaps falling into each interval. Such a division is also used in Yang et al. (2020) which is similar to the “peeling technique” widely used in local Rademacher complexity analysis (Bartlett et al., 2005). Formally speaking, we divide the interval $[\text{gap}_{\min}, H]$ to $N = \lceil \log(H/\text{gap}_{\min}) \rceil$ intervals $[2^{i-1}\text{gap}_{\min}, 2^i\text{gap}_{\min}](i \in [N])$. Therefore, for each gap falls into $[2^{i-1}\text{gap}_{\min}, 2^i\text{gap}_{\min}](i \in [N])$, it can be upper bounded by $2^i\text{gap}_{\min}$. Meanwhile, we have the following inequality by considering $V_h^*(s^k_h) - Q_h^*(s^k_h, a^k_h)$, which is the upper bound of gap $\text{gap}_h(s^k_h, a^k_h)$:

$$V_h^*(s^k_h) - Q_h^*(s^k_h, a^k_h) \geq \text{gap}_h(s^k_h, a^k_h) \geq 2^{i-1}\text{gap}_{\min},$$

which suggests that to count how many gaps belong to the interval $[2^{i-1}\text{gap}_{\min}, 2^i\text{gap}_{\min}](i \in [N])$, we only need to count the number of sub-optimalties $V_h^*(s^k_h) - Q_h^*(s^k_h, a^k_h)$ belonging to the interval. The following lemma is our main technical lemma. It is inspired by Jin et al. (2020), and it shows that the number of sub-optimalties can indeed be upper bounded.

**Lemma 6.1.** There exist a constant $C$ such that, for $h \in [H]$, $n \in N$, with probability at least $1 - (K + 1)\delta$, we have

$$\sum_{k=1}^{K} \mathbb{1} \left[ V_h^*(s^k_h) - Q_h^*(s^k_h, a^k_h) \geq 2^n\text{gap}_{\min} \right] \leq \frac{Cd^3H^4\log(2dT/\delta)}{4^n\text{gap}_{\min}^2} \log \left( \frac{Cd^3H^4\log(2dT/\delta)}{4^n\text{gap}_{\min}^2} \right).$$

Step 3: Summation of total error

Lemma 6.1 gives an upper bound of the number of action sub-optimality gaps in each interval $[2^{i-1}\text{gap}_{\min}, 2^i\text{gap}_{\min})$. We further give the following upper bound for the gaps in each intervals:

$$\sum_{\text{gap}_h(s^k_h, a^k_h) \in [2^{i-1}\text{gap}_{\min}, 2^i\text{gap}_{\min})} \text{gap}_h(s^k_h, a^k_h) \leq \sum_{k=1}^{K} 2^n\text{gap}_{\min} \mathbb{1} \left[ \text{gap}_h(s^k_h, a^k_h) \in [2^{i-1}\text{gap}_{\min}, 2^i\text{gap}_{\min}) \right]$$

$$\leq \sum_{k=1}^{K} 2^n\text{gap}_{\min} \mathbb{1} \left[ V_h^*(s^k_h) - Q_h^*(s^k_h, a^k_h) \geq 2^{i-1}\text{gap}_{\min} \right].$$

Thus, by adapting the upper bound of the number of sub-optimalties proposed in Lemma 6.1, we have the following lemma:

**Lemma 6.2.** There exist a constant $C$ such that, for $h \in [H]$, with probability at least $1 - 2(K + 1)\log(H/\text{gap}_{\min})\delta$, we have

$$\sum_{k=1}^{K} \left( V_h^*(s^k_h) - Q_h^*(s^k_h, a^k_h) \right) \leq \frac{4Cd^3H^4\log(2dT/\delta)}{\text{gap}_{\min}} \log \left( \frac{Cd^3H^4\log(2dT/\delta)}{4^n\text{gap}_{\min}^2} \right).$$

Lemma 6.2 suggests that with high probability, the summation of gap in step $h$ is logarithmic in the number of steps $T$ and the dependency in gap is $1/\text{gap}_{\min}$. This leads to our final proof of our main theorem.
Proof of Theorem 4.4. We define the high probability event $\Omega$ as follows.

$$\Omega = \{\text{Lemma 6.2 holds for all } h \in [H]\}. \tag{6.4}$$

Given the event $\Omega$, Lemma 6.2 holds for all $h \in [H]$. Therefore we have

$$\mathbb{E}[\text{Regret}(K)] = \mathbb{E}\left[\sum_{k=1}^{K} \sum_{h=1}^{H} \text{gap}_h(s^k_h, a^k_h)\right] \leq \mathbb{E}\left[\sum_{k=1}^{K} \sum_{h=1}^{H} V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h)\right] \Pr[\text{traj} \in \Omega] + KH^2 \cdot \Pr[\text{traj} \notin \Omega]$$

$$\leq \frac{4Cd^3H^5 \log(2dHK/\delta)}{\text{gap}_{\min}} \times \log \left(\frac{Cd^3H^4 \log(2dHK/\delta)}{\text{gap}_{\min}^2}\right) + 2K(K+1)H^3 \log(H/\text{gap}_{\min}) \delta,$$

where the first inequality holds due to $V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h) \leq H$ and the second holds due to Lemma 6.2. Finally, choosing $\delta = 1/(2K(K+1)H^3 \log(H/\text{gap}_{\min}))$ completes the proof. \qed

6.2 Proof of the Key Technical Lemma

In this subsection, we propose the proof to the main technical lemma, Lemma 6.1. Our proof follows the idea of error decomposition proposed in Wang et al. (2019a); Yang et al. (2020), that is, to upper bound the summation of sub-optimalities by considering their summation of the exploration bonuses. The key difference between our proof and that of Wang et al. (2019a); Yang et al. (2020) is the choice of exploration bonus. Wang et al. (2019a); Yang et al. (2020) considered the tabular MDP setting and adapted a $1/\sqrt{n}$-type bonus term, while we consider the linear function approximation setting and adapt a linear bandit-style exploration bonus (Dani et al., 2008; Abbasi-Yadkori et al., 2011; Li et al., 2010) as suggested in line 6. The following lemmas guarantee that our constructed $Q^k_h$ is indeed the UCB of the optimal action-value function:

Lemma 6.3 (Lemma B.4 in Jin et al. (2020)). With probability at least $1 - \delta$, for any policy $\pi$ and all $s \in \mathcal{S}, a \in \mathcal{A}, h \in [H], k \in [K]$, we have

$$\langle \phi(s,a), \mathbf{w}^k_h \rangle - Q^*_h(s,a) = [\mathbb{P}_h(V^k_h - V^\pi_{h+1})](s,a) + \Delta,$$

where $|\Delta| \leq \beta \sqrt{\phi(s,a)^T \Lambda^k_h}^{-1} \phi(s,a)$

Lemma 6.4 (Lemma B.5 in Jin et al. (2020)). With probability at least $1 - \delta$, for all $s \in \mathcal{S}, a \in \mathcal{A}, h \in [H], k \in [K]$, we have

$$Q^k_h(s,a) \geq Q^*_h(s,a).$$

We also need the following technical lemma, which gives us a slightly stronger upper bound for the summation of exploration bonuses:
Lemma 6.5. For any subset \( C = \{c_1, ..., c_k\} \subseteq [K] \) and any \( h \in [H] \), we have
\[
\sum_{i=1}^{k} (\phi_h^{ci})^\top (\Lambda_h^{ci})^{-1} \phi_h^{ci} \leq 2d \log \left( \frac{\lambda + k}{\lambda} \right),
\]
where \( \phi_h^{ci} \) is abbreviation of \( \phi_h^{ci}(s_h, a_h) \).

With the lemmas above, we begin to prove Lemma 6.1.

Proof of Lemma 6.1. We fix a \( h \) in this proof. Let \( k_0 = 0 \), and for \( i \in [N] \), we denote \( k_i \) as the minimum index of the episode where the sub-optimality at step \( h \) is no less than \( 2^i \text{gap}_{\min} \):
\[
k_i = \min \{ k : k > k_{i-1}, V_h^*(s_h^k) - Q_h^{\pi_h^k}(s_h^k, a_h^k) \geq 2^i \text{gap}_{\min} \}.
\]
(6.5)

For simplicity, we denote \( K' \) as the number of episodes such that the sub-optimality of this episode at step \( h \) is no less than \( 2^i \text{gap}_{\min} \). Formally speaking, we have
\[
K' = \sum_{k=1}^{K} \mathbb{1} \left[ V_h^*(s_h^k) - Q_h^{\pi_h^k}(s_h^k, a_h^k) \geq 2^i \text{gap}_{\min} \right].
\]

From now we only consider the episodes whose sub-optimality is no less than \( 2^i \text{gap}_{\min} \). We first lower bound the summation of difference between the estimated action-value function \( Q_h^{k_i} \) and the action-value function induced by the policy \( \pi_{k_i} \), which can be represented as follows:
\[
\sum_{i=1}^{K'} (Q_h^{k_i}(s_h^k, a_h^k) - Q_h^{\pi_h^{k_i}}(s_h^k, a_h^k)) \\
\geq \sum_{i=1}^{K'} \left( Q_h^{k_i}(s_h^k, \pi_h^*(s_h^k, h)) - Q_h^{\pi_h^{k_i}}(s_h^k, a_h^k) \right) \\
\geq \sum_{i=1}^{K'} \left( Q_h^*(s_h^k, \pi_h^*(s_h^k, h)) - Q_h^{\pi_h^{k_i}}(s_h^k, a_h^k) \right) \\
= \sum_{i=1}^{K'} (V_h^*(s_h^k) - Q_h^{\pi_h^{k_i}}(s_h^k, a_h^k)) \\
\geq 2^i \text{gap}_{\min} K',
\]
(6.6)

where the first inequality holds due to policy \( \pi_{k_i} \), the second inequality holds due to Lemma 6.4 and the last inequality holds due to the definition of \( k_i \) in (6.5). On the other hand, we upper bound
\[
\sum_{i=1}^{K'} (Q_h^{k_i}(s_h^k, a_h^k) - Q_h^{\pi_h^{k_i}}(s_h^k, a_h^k))
\]
as follows. For any \( h' \in [H], k \in [K] \), we have
\[
Q_h^k(s_h^k, a_h^k) - Q_{h'}^{\pi_h^{k_i}}(s_h^k, a_h^k)
\]
\[
= \langle \phi(s_h^k, a_h^k), w_h^k \rangle - Q_h^{\pi_h^{k_i}}(s_h^k, a_h^k) + \beta \sqrt{\phi(s_h^k, a_h^k)^\top (\Lambda_h^k)^{-1} \phi(s_h^k, a_h^k)}
\]
\[
\leq [p_h(V_h^{k+1} - V_h^{\pi_h^{k_i}})] (s_h^k, a_h^k) + 2 \beta \sqrt{\phi(s_h^k, a_h^k)^\top (\Lambda_h^k)^{-1} \phi(s_h^k, a_h^k)}
\]
\[
= V_h^{k+1}(s_h^{k+1}) - V_{h'}^{\pi_h^{k_i}}(s_h^{k+1}) + \epsilon_h^k + 2 \beta \sqrt{\phi(s_h^k, a_h^k)^\top (\Lambda_h^k)^{-1} \phi(s_h^k, a_h^k)}
\]
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where we use the fact that 
where the first inequality holds due to Cauchy-Schwarz inequality and the second inequality holds 
from (6.6) and (6.11). Finally, combining (6.6) and (6.11), we can derive the following constraint 

\[
\sum_{i=1}^{K'} \sum_{h'=h}^{H} 2 \beta \sqrt{\phi(s_{k_i}^{k_i}, a_{k_i})^\top \Lambda_h \phi(s_{k_i}^{k_i}, a_{k_i})} + \sum_{i=1}^{K'} \sum_{h'=h}^{H} \epsilon_{k_i}^{k_i}
\]

It therefore suffices to bound \( I_1 \) and \( I_2 \) separately. For \( I_1 \), we have

\[
I_1 = \sum_{i=1}^{K'} \sum_{h'=h}^{H} 2 \beta \sqrt{\phi(s_{k_i}^{k_i}, a_{k_i})^\top \Lambda_h \phi(s_{k_i}^{k_i}, a_{k_i})} \leq 2 \beta \sqrt{K'} \sum_{h'=h}^{H} \sqrt{\sum_{i=1}^{K'} \phi(s_{k_i}^{k_i}, a_{k_i})^\top \Lambda_h \phi(s_{k_i}^{k_i}, a_{k_i})} \leq 2H \beta \sqrt{K'} \sqrt{2d \log(K' + 1)},
\]

where the first inequality holds due to Cauchy-Schwarz inequality and the second inequality holds due to Lemma 6.5. For \( I_2 \), by Lemma C.1, for each \( k \in [K] \), with probability at least \( 1 - \delta \) we have

\[
\sum_{i=1}^{K'} \sum_{j=h}^{H} \left( \mathbb{P}(V_{j+1}^{k_i}(s_j^k, a_j^k)) - (V_{j+1}^{k_i}(s_j^k) - V_{j+1}^{k_i}(s_j^k)) \right) \leq \sqrt{2KH^2 \log(2/\delta)},
\]

where we use the fact that \( \mathbb{P}(V_{j+1}^{k_i}(s_j^k, a_j^k)) - (V_{j+1}^{k_i}(s_j^k) - V_{j+1}^{k_i}(s_j^k)) \) forms a martingale difference sequence. Taking a union bound for all \( k \in [K] \) gives that, with probability at least \( 1 - K' \delta \),

\[
\sum_{i=1}^{K'} \sum_{j=h}^{H} \left( \mathbb{P}(V_{j+1}^{k_i}(s_j^k, a_j^k)) - (V_{j+1}^{k_i}(s_j^k) - V_{j+1}^{k_i}(s_j^k)) \right) \leq \sqrt{2K'H^2 \log(2/\delta)},
\]

Substituting (6.9) and (6.10) into (6.8), we obtain that with probability at least \( 1 - (K + 1) \delta \),

\[
\sum_{i=1}^{K'} \left( Q_h^{k_i}(s_h^k, a_h^k) - Q_h^{k_i}(s_h^k, a_h^k) \right) \leq \sqrt{2K'H^2 \log(2/\delta)} + 2H \beta \sqrt{K'} \sqrt{2d \log(K' + 1)}.
\]

By now, we have obtained both the lower and upper bounds for \( \sum_{i=1}^{K'} (Q_h^{k_i}(s_h^k, a_h^k) - Q_h^{k_i}(s_h^k, a_h^k)) \) from (6.6) and (6.11). Finally, combining (6.6) and (6.11), we can derive the following constraint on \( K' \):

\[
2^\alpha \text{gap}_{\min} K' \leq \sqrt{2K'H^2 \log(2/\delta)} + 2H \beta \sqrt{2K'd \log(K' + 1)}.
\]
Solving out $K'$ from (6.12), we conclude that there exists a constant $C$ such that

$$K' \leq \frac{Cd^3H^4\log(2dHK/\delta)}{4^n\text{gap}_{\min}^2} \times \log \left( \frac{Cd^3H^4\log(2dHK/\delta)}{4^n\text{gap}_{\min}^2} \right),$$

which ends our proof.

\section{Conclusion}

In this paper, we analyze the RL algorithms with function approximation by considering a specific problem-dependent quantity $\text{gap}_{\min}$. We show that two existing algorithms LSVI-UCB and UCRL-VTR attain $\log T$-type regret instead of $\sqrt{T}$-type regret under their corresponding linear function approximation assumptions. It remains unknown whether the dependence of the length of the episode $H$ and dimension $d$ is optimal or not, and we leave it as future work.

\appendix

\section{Proof of Theorem 5.2}

\begin{lemma} \text{ (Section D.1.1 in Jia et al. (2020))} \end{lemma}

With probability at least $1 - \delta$, for all $s \in S, a \in A, h \in [H], k \in [K]$, we have

$$Q_h^k(s, a) \geq Q_h^*(s, a).$$

\begin{lemma} \text{ (Section D.1.2 in Jia et al. (2020))} \end{lemma}

With probability at least $1 - \delta$, for all $s \in S, a \in A, h \in [H], k \in [K]$, we have

$$V_h^k(s_h^k) - V_h^{\pi_k}(s_h^k) \leq [\mathbb{P}(V_{h+1}^k - V_{h+1}^{\pi_k})(s, a) + \Delta,$$

where $|\Delta| \leq 2\sqrt{3k} \sqrt{\phi(s, a)^\top (\Lambda_h^k)^{-1} \phi(s, a)}$

\begin{lemma} \end{lemma}

For any subset $C = \{c_1, \ldots, c_k\} \subseteq [K]$ and any $h \in [H]$, we have

$$\sum_{i=1}^k (\phi_{h,c_i}^\top (\Lambda_{h,c_i}^1)^{-1} \phi_{h,c_i}^{ci} \leq 2d \log \left( \frac{\lambda + k}{\lambda} \right),$$

where $\phi_{h,c_i}^{ci}$ is abbreviation of $\phi_{h,c_i}(s_{h,c_i}^c, a_{h,c_i}^c)$.

\begin{lemma} \end{lemma}

For $h \in [H], n \in N$, with probability at least $1 - (K+1)\delta$, we have

$$\sum_{k=1}^K [V_h^*(s_h^k) - Q_h^*(s_h^k, a_h^k)] \geq 2^n\text{gap}_{\min}] \leq \frac{512C_0d^2H^4\log^3(2dT/\delta)}{4^n\text{gap}_{\min}^2} \log \left( \frac{512C_0d^2H^4\log^3(2dT/\delta)}{4^n\text{gap}_{\min}^2} \right).$$

\begin{lemma} \end{lemma}

For $h \in [H]$, with probability at least $1 - 2(K+1)\log(H/\text{gap}_{\min})\delta$, we have

$$\sum_{k=1}^K (V_h^*(s_h^k) - Q_h^*(s_h^k, a_h^k)) \leq \frac{2048C_0^2d^2H^4\log^3(2dT/\delta)}{\text{gap}_{\min}} \log \left( \frac{512C_0^2d^2H^4\log^3(2dT/\delta)}{\text{gap}_{\min}} \right).$$
Proof of Theorem 5.2. For each \( h \in [H], k \in [K] \), we have
\[
V^*_h(s^k_h) - V^\pi_h(s^k_h) = (V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h)) + (Q^*_h(s^k_h, a^k_h) - V^\pi_h(s^k_h))
\]
where \( \text{gap}_h(s, a) = V^*_h(s) - Q^*_h(s, a) \). Taking expectation of (A.1), we have
\[
\mathbb{E}[V^*_h(s^k_h) - V^\pi_h(s^k_h)] = \mathbb{E}[\text{gap}_h(s^k_h, a^k_h) + V^*_h(s^k_{h+1}) - V^\pi_h(s^k_{h+1})].
\] (A.2)
Taking a summation of (A.2) over all \( k \in [K], h \in [H] \) and we define high probability event \( \Omega \)
\[
\Omega = \{\text{Lemma A.5 holds for all } h \in [H]\}
\]
as Lemma A.5 holds for all \( h \in [H] \), then we have
\[
\mathbb{E}[\text{Regret}(K)]
\leq \mathbb{E} \left[ \sum_{k=1}^{H} \sum_{h=1}^{K} \text{gap}_h(s^k_h, a^k_h) \right]
\leq \mathbb{E} \left[ \sum_{k=1}^{H} \sum_{h=1}^{K} V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h) \right] \text{Pr[traj } \in \Omega \} + \frac{KH^2 \cdot \text{Pr[traj } \notin \Omega \}}{1 - \epsilon}
\leq \frac{2048C^2d^2H^4 \log(2dHK/\delta)}{\text{gap}_{\min}} \text{log} \left( \frac{512C^2d^2H^4 \log(2dHK/\delta)}{\text{gap}_{\min}^2} \right) + 2K(K + 1)H^3 \text{log}(H/\text{gap}_{\min}),
\]
where the first inequality holds due to \( V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h) \leq H \) and the second inequality holds due to Lemma A.5. Finally, choosing \( \delta = 1/(2K(K + 1)H^3 \text{log}(H/\text{gap}_{\min})) \) and we complete the proof of Theorem 5.2. \( \square \)

B Proof of Lemma in Previous Section

Proof of Lemma 6.2. By the definition of \( \text{gap}_{\min} \) in (3.2), for each \( h \in [H], k \in [K] \), we have
\[
V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h) = 0 \text{ or } V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h) \geq \text{gap}_{\min}.
\]
Thus, we divide the interval \([\text{gap}_{\min}, H]\) to \( N = \lceil \text{log}(H/\text{gap}_{\min}) \rceil \) intervals: \([2^i \text{gap}_{\min}, 2^{i+1} \text{gap}_{\min})\) \( (i \in [N]) \) and with probability at least 1 - 2\((K + 1)\text{log}(H/\text{gap}_{\min})\)\(, \) we have
\[
\sum_{k=1}^{K} (V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h)) \leq \sum_{i=1}^{N} \sum_{k=1}^{K} \mathbb{I} \left[ \text{gap}_{\min} \geq V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h) \geq 2^{i-1} \text{gap}_{\min} \right] \times 2^i \text{gap}_{\min}
\leq \sum_{i=1}^{N} \sum_{k=1}^{K} \mathbb{I} \left[ V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h) \geq 2^{i-1} \text{gap}_{\min} \right] \times 2^i \text{gap}_{\min}
\leq \sum_{i=1}^{N} \sum_{k=1}^{K} \mathbb{I} \left[ V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h) \geq 2^{i-1} \text{gap}_{\min} \right] \times 2^i \text{gap}_{\min}
\leq \sum_{i=1}^{N} \frac{4Cd^3H^4 \log(2dHK/\delta)}{2^i \text{gap}_{\min}} \times \log \left( \frac{Cd^3H^4 \log(2dHK/\delta)}{4^{i-1} \text{gap}_{\min}} \right)
\]
where the first inequality holds due to $\Lambda'_k \preceq \Lambda_k$.

and we have

\[ \sum_{i=1}^{k} (\phi_i(s^c_i, a^c_i)) \odot (\Lambda'_i)^{-1}(\phi_i(s^c_i, a^c_i)) \leq \sum_{i=1}^{k} (\phi_i)^\top (\Lambda'_{i-1})^{-1} \phi_i \leq 2 \log \left( \frac{\det(\Lambda'_k)}{\det(\phi_0)} \right) \leq 2d \log \left( \frac{\lambda + k}{\lambda} \right), \]

where the first inequality holds due to $\Lambda'_i \preceq \Lambda_k$, the second inequality holds due to Lemma C.2 and the last inequality holds due to $\|\Lambda'_k\| = \|\lambda I + \sum_{i=1}^{k} (\phi_i)^\top \phi_k\| \leq \lambda + k$. \hfill \(\blacksquare\)

**Proof of Lemma A.3.** For simplicity, we denote

\[ \phi'_i = \phi_i(s^c_i, a^c_i), \Lambda'_i = \lambda I + \sum_{j=1}^{i} (\phi'_i)^\top \phi'_i, \]

and we have

\[ \sum_{i=1}^{k} (\phi'_i)^\top (\Lambda'_i)^{-1} \phi'_i \leq \sum_{i=1}^{k} (\phi_i)^\top (\Lambda'_{i-1})^{-1} \phi_i \leq 2 \log \left( \frac{\det(\Lambda'_k)}{\det(\phi_0)} \right) \leq 2d \log (1 + k), \]

where the first inequality holds due to $\Lambda'_i \preceq \Lambda_k$, the second inequality holds due to Lemma C.2 and the last inequality holds due to $\|\Lambda'_k\| = \|\lambda I + \sum_{i=1}^{k} (\phi_k)^\top \phi_k\| \leq H^2d + k$. \hfill \(\blacksquare\)

**Proof of Lemma A.4.** We denote $k_0 = 0$ and for $i \in [N]$, we denote

\[ k_i = \min \{ k | k > k_{i-1}, V_h(s^k_i) - Q_h^* (s^k_i, a^k_i) \geq 2^i \text{gap}_{\min} \}. \] \hfill (B.1)

For simplicity, we denote

\[ K' = \sum_{k=0}^{K} 1 \left[ V_h(s^k) - Q_h^* (s^k, a^k) \geq 2^k \text{gap}_{\min} \right], \]

and we have

\[ \sum_{i=1}^{K'} (Q_{k_i}(s^k_i, a^k_i) - Q_{k_i}^* (s^k_i, a^k_i)) \geq \sum_{i=1}^{K'} (Q_{k_i}(s^k_i, \pi^*_h(s^k_i, h)) - Q_{k_i}^* (s^k_i, a^k_i)) \]

\[ \leq \sum_{i=1}^{K'} (Q_{k_i}(s^k_i, \pi^*_h(s^k_i, h)) - Q_{k_i}^* (s^k_i, a^k_i)) \]
where the first inequality holds due to policy $\pi_k$, the second inequality holds due to Lemma A.1 and the last inequality hold due to definition of $k_i$ in (B.1).

In other hand, for $h' \in [H], k \in [K]$, we have

$$Q_k^h(s_h', a_h') - Q_{\pi_k}^h(s_h', a_h') = V_k^h(s_h') \leq \mathbb{P}(V_{k+1} - V_{\pi_k}^h)(s_h', a_h') + 2\sqrt{\beta_k} \sqrt{\phi(s_h', a_h')^\top (\Lambda_k)^{-1} \phi(s_h', a_h')}
$$

$$= V_{k+1}^h(s_h', a_h') - V_{\pi_k}^h(s_h', a_h') + 2\sqrt{\beta_k} \sqrt{\phi(s_h', a_h')^\top (\Lambda_k)^{-1} \phi(s_h', a_h')}
$$

$$= Q_{k+1}^h(s_h', a_h') - Q_{\pi_k}^h(s_h', a_h') + \epsilon_k^h + 2\sqrt{\beta_k} \sqrt{\phi(s_h', a_h')^\top (\Lambda_k)^{-1} \phi(s_h', a_h')},
$$

(B.3)

where

$$\epsilon_k^h = \mathbb{P}(V_{k+1} - V_{\pi_k}^h)(s_h', a_h') - (V_{k+1}^h(s_h', a_h') - V_{\pi_k}^h(s_h', a_h'))
$$

and the inequality holds due to Lemma A.2. Take summation for (B.3) over all $k_i$ and $h \leq h' \leq H$, we have

$$\sum_{i=1}^{K'} (Q_h^k(s_h^k, a_h^k) - Q_{\pi_k}^h(s_h^k, a_h^k)) \leq \sum_{i=1}^{K'} \sum_{h'=h}^H \sum_{i=1}^{K'} \sum_{h'=h}^H \epsilon_k^h .
$$

(B.4)

For term $I_1$, we have

$$I_1 = \sum_{i=1}^{K'} \sum_{h'=h}^H \sum_{i=1}^{K'} \sum_{h'=h}^H 2\sqrt{\beta_k} \sqrt{\phi(s_h', a_h')^\top (\Lambda_k)^{-1} \phi(s_h', a_h')}
$$

$$\leq 2\sqrt{\beta_K} \sqrt{K'} \sum_{h'=h}^H \sum_{i=1}^{K'} \phi(s_h', a_h')^\top (\Lambda_k)^{-1} \phi(s_h', a_h')
$$

$$\leq 2H \sqrt{\beta_K} \sqrt{K'} \sqrt{2d \log(K' + 1)},
$$

(B.5)

where the first inequality holds due to Cauchy-Schwarz inequality and the second inequality holds due to Lemma A.3.

For each $k \in [K]$, by Lemma C.1, with probability at least $1 - \delta$, we have

$$\sum_{i=1}^{K} \sum_{h=H}^H \left( \mathbb{P}(V_{j+1}^k - V_{\pi_k}^h)(s_j^k, a_j^k) - (V_{j+1}^k(s_j^k) - V_{\pi_k}^h(s_j^k)) \right) \leq 2kH^2 \log(2/\delta).
$$
Thus, taking a union bound for all $k \in [K]$, with probability at least $1 - K\delta$, we can bound term $I_2$ as follows:

$$
\sum_{i=1}^{K'} \sum_{j=h}^{H} \left( \mathbb{P}(V^{k_i}_{j+1} - V^{\pi_k}_{j+1}) (s^{k_i}_j, a^{k_i}_j) - (V^{k_j}_{j+1}(s^{k_j}_{j+1}) - V^{\pi_k}_{j+1}(s^{k_j}_{j+1})) \right) \leq \sqrt{2K'H^2 \log(2/\delta)}.
$$

(B.6)

Substituting (B.5) and (B.6) into (B.4), with probability at least $1 - (K + 1)\delta$, we have

$$
\sum_{i=1}^{K'} (Q_h^k(s^{k_i}_h, a^{k_i}_h) - Q_h^\pi_k(s^{k_i}_h, a^{k_i}_h)) \leq \sqrt{2KH^2 \log(2/\delta)} + 2H\sqrt{\beta_K} \sqrt{K'} \sqrt{2\log(K' + 1)}.
$$

(B.7)

Combining (B.7) and (B.2), we have

$$
2^n\text{gap}_\text{min} K' \leq \sqrt{2KH^2 \log(2/\delta)} + 2H\sqrt{\beta_K} \sqrt{K'} \sqrt{2\log(K' + 1)},
$$

which implies

$$
K' \leq \frac{512C^2_d d^2 H^4 \log^3(2dHK/\delta)}{4^n \text{gap}_\text{min}^2} \log \left( \frac{512C^2_d d^2 H^4 \log^3(2dHK/\delta)}{4^n \text{gap}_\text{min}^2} \right).
$$

Proof of Lemma A.5. By the definition of $\text{gap}_\text{min}$ in (3.2), for each $h \in [H], k \in [K]$, we have $V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h) = 0$ or $V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h) \geq \text{gap}_\text{min}$. Thus, we divide the interval $[\text{gap}_\text{min}, H]$ to $N = \lceil \log(H/\text{gap}_\text{min}) \rceil$ intervals: $[2^{-i-1}\text{gap}_\text{min}, 2^i\text{gap}_\text{min}) (i \in [N])$ and with probability at least $1 - 2(K + 1)\log(H/\text{gap}_\text{min})\delta$, we have

$$
\sum_{k=1}^{K} (V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h)) \leq \sum_{i=1}^{N} \sum_{k=1}^{K} \mathbb{I} \left( 2^i \text{gap}_\text{min} \geq V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h) \geq 2^{i-1} \text{gap}_\text{min} \right) \times 2^i \text{gap}_\text{min}
$$

\leq \sum_{i=1}^{N} \sum_{k=1}^{K} \mathbb{I} \left( V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h) \geq 2^{i-1} \text{gap}_\text{min} \right) \times 2^i \text{gap}_\text{min}

\leq \sum_{i=1}^{N} 2048C^2_d d^2 H^4 \log^3(2dHK/\delta) \log \left( \frac{512C^2_d d^2 H^4 \log^3(2dHK/\delta)}{4^{i-1} \text{gap}_\text{min}^2} \right)
$$

\leq \frac{2048C^2_d d^2 H^4 \log^3(2dHK/\delta)}{\text{gap}_\text{min}^2} \log \left( \frac{512C^2_d d^2 H^4 \log^3(2dHK/\delta)}{\text{gap}_\text{min}^2} \right),
$$

where the first inequality holds due to $\text{gap}_\text{min} \leq V^*_h(s^k_h) - Q^*_h(s^k_h, a^k_h) \leq V^*_h(s^k_h) - Q^*_{\pi_k}(s^k_h, a^k_h)$ or $V^*_h(s^k_h) - Q^*_{\pi_k}(s^k_h, a^k_h) = 0$, the third inequality holds due to Lemma A.1, the fourth inequality holds due to Lemma A.4. \qed
C Auxiliary Lemmas

Lemma C.1 (Azuma–Hoeffding inequality (Cesa-Bianchi and Lugosi, 2006)). Let \{x_i\}_{i=1}^n be a martingale difference sequence with respect to a filtration \{\mathcal{G}_i\} satisfying |x_i| \leq M for some constant M, x_i is \mathcal{G}_{i+1}-measurable, \mathbb{E}[x_i|\mathcal{G}_i] = 0. Then for any 0 < \delta < 1, with probability at least 1 – \delta, we have

\[ \left| \sum_{i=1}^n x_i \right| \leq M \sqrt{2n \log(2/\delta)}. \]

Lemma C.2 (Lemma 11 in Abbasi-Yadkori et al. (2011)). Let \{X_t\}_{t=1}^{+\infty} be a sequence in \mathbb{R}^d, \mathbb{V} a d \times d positive definite matrix and define \mathbb{V}_t = \mathbb{V} + \sum_{i=1}^t X_t^\top X_t. If \|X_t\|_2 \leq L and \lambda_{\text{min}}(\mathbb{V}) \geq \max(1, L^2), then we have

\[ \sum_{i=1}^t X_i^\top (\mathbb{V}_{i-1})^{-1} X_i \leq 2 \log \left( \frac{\det \mathbb{V}_t}{\det \mathbb{V}} \right). \]

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