Next to leading-order gravitational wave emission and the dynamical evolution of close binary systems with spin

Dörte Hansen
Institute of Theoretical Physics, Friedrich-Schiller-University Jena, Max-Wien-Platz 1, D-07743 Jena, Germany
E-mail: D.Hansen@uni-jena.de

Received 28 November 2007, in final form 28 April 2008
Published 5 August 2008
Online at stacks.iop.org/CQG/25/165011

Abstract
Close binary systems with spinning components are considered. Finite size effects due to rotational deformation are taken into account. The dynamical evolution and next to leading-order gravitational wave forms are calculated, taking into account the orbital motion up to the first post-Newtonian approximation. The analysis presented here will be relevant mainly for white dwarf binaries and, to some extent, also to compact binaries with a fast spinning neutron star component.

PACS numbers: 04.25.Nx, 04.30.–w, 04.30.Tv, 95.85.Sz

1. Introduction
Inspiring compact binary systems are among the most promising sources for the emission of gravitational waves detectable with the present day’s gravitational wave interferometers. Earth-bound gravitational wave detectors such as LIGO, VIRGO, GEO600 and TAMA are most sensitive at frequencies of about 10–1000 Hz. This corresponds roughly to the last 10 min of the inspiral before the final plunge. In that regime Newtonian mechanics is not valid and the post-Newtonian approximation must be applied. However, the analysis presented applies mainly to close white dwarf binaries. These systems are not observable by the present day’s Earth-bound detectors, but galactic white dwarf binaries are expected to be promising sources for the LISA space-observatory. LISA is designed to cover the frequency band from about $10^{-1}$ to $10^{-4}$ Hz. However, in order to actually detect gravitational waves using matched-filtering techniques, highly accurate templates are required. To that end it is essential to take into account post-Newtonian corrections to the equations of motion (EoM), but it is also necessary to consider higher-order corrections to the famous quadrupole formula of gravitational waves.
In the recent past the study of close compact binaries was often based on the assumption that the stars can be treated as pointlike, non-spinning objects up to the last stable orbit. This is certainly true for black hole binaries. Using this assumption it is possible to derive an analytic, so-called quasi-Keplerian solution for the conservative part of the EoM of non-spinning black hole binaries up to the third post-Newtonian approximation [1]. For a spinning black hole binary system things are much more complicated. In general, due to post-Newtonian spin–orbit coupling neither the orbital angular momentum nor the spin is conserved. An analytic solution for spinning black hole binaries has been derived for special cases only even at leading order (see e.g. [2]).

Dissipative effects due to radiation back reaction first appear at the order \((v/c)^5\), which is of order 2.5 pN beyond the Newtonian dynamics. Gravitational back reaction is responsible for the emission of gravitational waves and thus for the energy loss of a binary system.

While much progress has been made in the investigation of black hole binaries, little has been done to understand the influence of Newtonian perturbation to the dynamics of close binary systems of neutron stars or white dwarfs. Newtonian perturbations arise due to the finite size of the binary components. Within the framework of post-Newtonian analysis it is often argued that finite size effects are negligible during most of the inspiral process and will not become important until the last few orbits before the final plunge [3]. However, recently there is a growing interest in the role of finite size effects, mainly from numerical relativists. In fact, though finite size effects due to stellar rotation and oscillation in compact binary systems are very small they can well be of the same order of magnitude as the first post-Newtonian corrections to the orbital dynamics. Moreover, these secular effects accumulate over a large number of orbits and thus, seen at longer terms, lead to significant phase shifts compared with the gravitational waves emitted by a point-mass binary.

The influence of stellar oscillations on the dynamical evolution and leading-order gravitational wave emission has been investigated by Kokkotas and Schäfer [4] and Lai and Ho [5] for nonrotating, polytropic neutron stars and by Lai et al [6] and Hansen [7] for Riemann-S binaries. In these approaches the analysis was based on Newtonian theory, the 2.5 pN radiation reaction terms being the only post-Newtonian terms included. Recently Flanagan and Hinderer [8] demonstrated that the low-frequency part of the gravitational wave signal could be used to obtain some information on the equation of state (EoS) of the components of neutron star binaries. At distances well above the last stable orbit the influence of tidal corrections to the phase of the gravitational waves is small, but very clean and depends only on the Love number of the neutron star. Flanagan and Hinderer showed how to extract the Love number out of the low-frequency signal, thus gaining information on the neutron star’s EoS.

In this paper, we shall study the influence of finite size effects on the dynamics and gravitational wave emission beyond the leading-order approximation. Basically, perturbations of the point-particle dynamics arise due to stellar oscillations (mainly tidally driven) and rotational deformation. In this work, we shall restrict ourselves to systems in which the apsidal motion due to rotational deformation is much larger than that caused by stellar oscillations.

To begin with, let us note that there is, strictly speaking, no spinning point particle. A spinning object automatically gains a finite size and thus a non-vanishing quadrupole moment (see e.g. [9]). It has been long known that the Newtonian coupling of the stellar quadrupole moment to the orbital motion leads to an apsidal motion. This apsidal motion has been observed for a couple of close main sequence star binaries to great accuracy (see e.g. Claret and Willems [10]). In all cases the apsidal motion due to finite size effects is considerably larger than the relativistic periastron advance. Finite size effects may also play an important
role in binary pulsars such as PSR B 1259-63 [11]. This binary consists of a 47 ms pulsar and 
a Be star, whose spin-induced quadrupole deformation leads to an apsidal motion.

For neutron star binaries the contribution of the Newtonian quadrupole coupling to 
the orbital dynamics has been derived by Poisson [12]. He found that, at least at 2 pN 
approximation, Newtonian perturbations due to the coupling of the stellar quadrupole to the 
orbital motion cannot be neglected. As we shall argue here, it might well be that this effect 
is even larger. However, our investigations are probably most relevant for close white dwarf 
binary systems. As we shall demonstrate it is well possible that the perturbations introduced 
by the coupling of the stellar quadrupole moment to the orbital motion is of the same order of 
magnitude as the first post-Newtonian corrections. This assumption is made throughout the 
paper. The analysis applies to close binary systems, whose spinning component is a white 
dwarf (WD) or a fast rotating neutron star (NS). In order to simplify calculations it is assumed 
that the spin is perpendicular to the orbital plane. In section 2, the orbital evolution of a spinning 
compact binary is studied up to first post-Newtonian order in the point-particle dynamics. The 
EoM as well as a parametric, quasi-Keplerian solution are derived. In section 3 the next to 
leading-order gravitational waveforms are calculated explicitly. The long-time evolution and 
the influence of the quadrupole coupling to the inspiral process is discussed in section 4.

2. The 1 pN orbital motion including spin effects due to rotational deformation

In 1985, Damour and Deruelle [13] succeeded in deriving an analytic solution to the 1 pN 
EoM of a point-particle binary. This so-called quasi-Keplerian solution exhibits a remarkable 
similarity to the well-known Kepler parametrization in Newtonian theory. Adopting the 
strategy outlined by Damour and Deruelle Wex considered, at Newtonian order, a binary 
consisting of a pulsar and a spinning main sequence star [14]. Treating the Newtonian coupling 
between the rotationally deformed star and the orbital dynamics as a small perturbation he 
derived a quasi-Keplerian solution up to first order in the deformation parameter \( q \), which will 
be introduced in the following. In this paper, we shall extend his investigations, taking into 
account the orbital dynamics up to first post-Newtonian approximation. In order to derive an 
analytic solution we shall further assume that the modifications induced by finite size effects 
are of the same order of magnitude as the 1 pN corrections to the orbital dynamics. That is, 
we restrict ourselves to close binary systems consisting of a fast spinning neutron star or white 
dwarf and a non-spinning compact object. Of course, all results can be applied to a mean 
sequence star-compact star binary in the Newtonian limit.

The rotational deformation of a spinning star of mass \( m \) can be described by some 
parameter \( q \), which is defined as [15]

\[
mq := \frac{1}{2} \int dV' \rho(r') [r'^2 - 3(\hat{s} \cdot r')^2] = \Delta I, \tag{1}
\]

where \( m = \int \rho(r') dV' \) is the stellar mass and \( \Delta I \) is the difference of the moments of inertia 
parallel and perpendicular to the spin axis \( \hat{s} \), respectively. In particular, for rotating fluids, \( q \) 
is given by (see e.g. [16])

\[
q = \frac{2}{3} k R^2 \hat{\Omega}^2, \quad \hat{\Omega} = \frac{\Omega}{\sqrt{Gm/R^3}}. \tag{2}
\]

Here \( R \) denotes the polar radius and \( \Omega \) the angular velocity of the rotating star. The constant 
of apsidal motion \( k \) strongly depends on the density distribution. It vanishes if all mass 
is concentrated in the center, while for a homogeneous sphere it takes its maximal value 
\( k_{\text{max}} = 0.75 \). As for neutron stars (NS) and white dwarfs (WD), it is well known that both can
be described by a polytropic EoS with polytropic index $n = 0.5, \ldots, 1$ for white dwarfs and $n \approx 1$ for neutron stars, respectively. It has been shown that, for polytropes, the constant of apsidal motion is given by (see e.g. [17])

$$k = \frac{1}{2} (\Delta_2(n) - 1),$$

where the function $\Delta_2(n)$ has been introduced by Chandrasekhar [18].

The quadrupole deformation gives rise to a quadrupole coupling, which modifies the orbital dynamics as well as the gravitational wave emission of the binary. Neglecting contributions arising from tidally induced stellar oscillations the quadrupole coupling can be described by a Hamiltonian

$$H_q = \frac{GM\mu q}{2r^3} \left[ \frac{3(\hat{s} \cdot r)^2}{r^2} - 1 \right], \quad (3)$$

where $r$ is the orbital separation, while $\mathcal{M}$ and $\mu$ denote total and reduced mass, respectively. As we shall see in the following this contribution will lead to a periastron advance already at Newtonian order.

In general, the direction of the spin axis is not conserved, which complicates the analysis enormously. If, however, the spin is parallel to the total angular momentum and thus perpendicular to the orbital plane, equation (3) takes a rather simple form, namely

$$H_q = -\frac{GM\mu q}{2r^3}. \quad (4)$$

We shall restrict to this assumption throughout the remainder of this paper.

Before we proceed it is important to compare the perturbation introduced by quadrupole coupling with the first post-Newtonian correction terms to the EoM. To that end, let us consider a spinning binary system in a circular orbit. Not taking into account the $q$-coupling the orbital energy reads as

$$E_{\text{orb}} = -\frac{GM\mu}{2r_0} + \frac{7}{8} \frac{G^2M^2\mu}{r_0^3c^2} \equiv E_N + E_{pN}. \quad (5)$$

Comparing $E_{pN}$ with the coupling energy we find that the $q$-term offers a contribution comparable to the 1 pN orbital perturbation if

$$q \approx \frac{GM}{c^2} \equiv r_S.$$ 

If $q/r_0$ is much larger than the Schwarzschild radius $r_S$ the Newtonian quadrupole contribution will clearly dominate, while for $q/r_0 \ll GM/c^2$ the leading perturbation to the Keplerian orbit comes from the post-Newtonian correction terms. The value of $q$ crucially depends on the density distribution (via $k$) and on the angular velocity of the rotating star. The later one is bounded by the mass-shedding limit. For a Newtonian star with a polytropic EoS one can show that the mass-shedding limit is given by [19]

$$\Omega_{\text{max}} = \left( \frac{2}{3} \right)^{3/2} \sqrt{\frac{Gm}{R^3}}, \quad (6)$$

where $R$ is the polar radius of the star. Thus, for polytropic stars $q$ is bounded by

$$q_{\text{max}} = \frac{2}{3} k R^2 \Omega_{\text{max}}^2 = \left( \frac{2}{3} \right)^4 k R^2.$$ 

For a typical neutron star with polytropic index $n = 1$ and polar radius $R = 10$ km the maximal value of $q$ is $q_{\text{max}} = 5.1$ km$^2$, while for a white dwarf with $n = 1.5$ and a radius

1 Note, that $\dot{E} = H$. 

4
of 1000 km this value is given by \( q_{\text{max}} = 28563 \text{ km}^2 \). On the other hand, it is well known that post-Newtonian analysis is valid outside the innermost stable circular orbit only. For finite size binaries the orbital separation should be considerably larger than this value. To be more precise, since we do not consider mass overflow the spinning star should remain well inside its Roche volume throughout this calculation. Taking this into account it becomes clear that for close NS–BH and NS–NS binaries the contribution of the quadrupole coupling is by a factor 100 or more smaller than the first post-Newtonian correction. However, at least for sufficiently fast rotation the \( q \)-term gives rise to perturbations which could be considerably larger than the 1.5 pN order. For spinning white dwarfs things can be different. In that case the finite size contribution due to rotational deformation can be equal or even larger than the pN contribution.

These preliminary considerations showed that there might exist close binary systems where finite-size effects introduced by rotational deformation are of the same order of magnitude as the first post-Newtonian corrections. Employing this assumption we shall now derive the quasi-Keplerian parametrization at the first post-Newtonian order including leading-order quadrupole coupling (further on denoted as \( q \)-coupling). Introducing the reduced energy and angular momentum,

\[
E = \frac{v^2}{2} - \frac{GM}{r} \left[ 1 + \frac{q}{2r^2} \right] + \frac{1}{c^2} \left[ \frac{3}{8} (1 - 3\nu) v^4 + \frac{GM}{2c} \left( (3 + \nu) v^2 + \nu \dot{r}^2 + \frac{GM}{r} \right) \right],
\]

(7)

Since the spin is parallel to the orbital angular momentum \( J \) the orbital plane is invariant in space. We are thus encouraged to introduce polar coordinates \( r \) and \( \phi \) in the usual way. Inserting \( v^2 = \dot{r}^2 + r^2 \dot{\phi}^2 \) into equation (7) one finds, after a little algebra, the 1 pN exact EoM to be

\[
\dot{\phi} = \frac{J}{r^2} \left[ 1 - \frac{1 - 3\nu}{c^2} E - \frac{GM}{c^2 r} (4 - 2\nu) \right],
\]

(8)

\[
\dot{r}^2 = A + \frac{2B}{r} + \frac{C}{r^2} + \frac{D}{r^3},
\]

(9)

where

\[
A = 2E \left[ 1 + \frac{3}{2} (3\nu - 1) \frac{E}{c^2} \right],
\]

\[
B = GM \left[ 1 + (7\nu - 6) \frac{E}{c^2} \right],
\]

\[
C = -J^2 + \frac{1}{c^2} [2 (1 - 3\nu) E J^2 + (5\nu - 10) G^2 M^3],
\]

\[
D = GMq + (8 - 3\nu) \frac{GMJ^2}{c^2},
\]

(10)

are constants. In the standard approach of Damour and Deruelle it is crucial that \( D \) is of order \( O(c^{-2}) \) and thus a small quantity. However, in our case \( D \) depends not only on \( c \) but it is also linear in the deformation parameter \( q \). If the spinning component is governed by a soft EoS the correction to the orbital motion induced by the \( q \)-coupling is much larger than the
1 pN corrections. This is usually the case for main-sequence star binaries (see e.g. Claret and Willems [10]). On the other hand, for more compact stars in close binary systems the contribution of the $q$-coupling can be of the same order as the 1 pN orbital correction. Upon this assumption we can apply Damour and Deruelle’s strategy straightforwardly, deriving a quasi-Keplerian solution up to linear order of $q$. This yields

$$r = a_r (1 - e_r \cos u), \quad u - e_r \sin u = n(t - t_0),$$  \hspace{1cm} (11)

$$\varphi = 2(\kappa + 1) \arctan \left[ \frac{1 + e_\varphi}{1 - e_\varphi} \tan \frac{u}{2} \right].$$  \hspace{1cm} (12)

where $n, a_r, e_r, e_t$ and $e_\varphi$ depend on the coefficients defined above as

$$a_r = - \frac{G M}{E} - \frac{C}{2 J^2}, \quad e_\varphi = e_i \left[ 1 - \frac{A}{B} \left( 2(v - 2) \frac{G M}{c^2} + \frac{D}{J^2} \right) \right],$$  \hspace{1cm} (13)

$$e_r = e_i \left[ 1 - \frac{D A}{2 B J^2} \right], \quad e_t = \sqrt{1 - \frac{A}{B} \left( C + B D \right) \frac{J^2}{c^2} \left( 1 - \frac{e_\varphi}{e_r} \right)},$$  \hspace{1cm} (14)

and $n = \sqrt{-A^3/B^2}$, and $\kappa$ is given by

$$\kappa = \frac{3 G^2 M^2}{J^2} \left[ \frac{1}{c^2} + \frac{q}{2 J^2} \right].$$  \hspace{1cm} (15)

As it turns out, the parameter which has to be small for this solution to hold is $\delta \equiv q/J^2 \propto 1/c^2$. Expressing the parameters of the quasi-Keplerian solution in terms of the 1 pN conserved energy and $\delta$, we find

$$a_r = - \frac{G M}{2 E} - \frac{1}{2} \frac{G M \delta}{4 c^2} (v - 7),$$

$$n^2 = - \frac{8 E^3}{G^2 M^2} \left[ 1 - \frac{v - 15}{2} \frac{E}{c^2} \right],$$

$$e_r = e_i \left[ 1 + E \left\{ \frac{\delta}{c^2} + \frac{8 - 3 v}{c^2} \right\} \right],$$

$$e_\varphi = e_i \left[ 1 - E \left\{ \frac{\delta}{c^2} + \frac{v}{c^2} \right\} \right].$$

2.1. Hamiltonian formulation

The quasi-Keplerian solution given above describes the dynamics of a spinning compact binary with Newtonian quadrupole coupling at the first post-Newtonian order. However, according to GR the system loses energy due to the emission of gravitational waves, beginning at the order $(v/c)^5$ in post-Newtonian approximation schemes. There are basically two ways to study the dynamical evolution of the binary system. In one approach the gravitational wave emission is considered as a secular effect and the dissipative terms do not enter the EoM. This allows the derivation of the quasi-Keplerian parametrization given above. Here we shall follow the other approach, where the radiation reaction terms are included into the EoM. In a first step we derive the Hamiltonian formulation to the conservative system. Energy loss due to emission of gravitational waves is incorporated by a time-dependent radiation reaction Hamiltonian in a second step.
The total Hamiltonian of a spinning compact binary system up to first post-Newtonian approximation reads as

\[ H_{1pN} = H_N + H_q + H_{qN}, \]  

where \( H_N \) and \( H_{qN} \) denote the point-particle Hamiltonian at Newtonian and first post-Newtonian order, respectively. If we would extend our analysis up to 2 pN, other spin-dependent terms would be present: at 1.5 pN order the relativistic spin–orbit coupling enters into the Hamiltonian. It is, among others, responsible for the Lense–Thirring effect. At the level of the second post-Newtonian approximation the relativistic spin–spin coupling leads to a precession of the orbital plane. Both, incorporating spin–orbit as well as investigating the spin–spin coupling, is beyond this paper, which is devoted to the study of the influence of certain finite size effects on the dynamical evolution and gravitational wave emission of the binary system.

In the center-of-mass system the 1 pN Hamiltonian, including \( q \)-coupling, reads as

\[ H_{1pN} = \frac{1}{2\mu} \left[ p_r^2 + \frac{p_\phi^2}{r^2} \right] - \frac{GM\mu}{r} - \frac{GM\mu}{2r^3} q + \frac{1}{c^2} \left[ \frac{3\nu - 1}{8\mu^3} \left( p_r^4 + 2p_r^2p_\phi^2 + \frac{p_\phi^4}{r^4} \right) \right] \]

\[ - \frac{GM}{2r} \left( (3 + 2\nu) \frac{p_r^2}{\mu} + (3 + \nu) \frac{p_\phi^2}{\mu r^2} \right) + \frac{GM^2\mu}{2r^2} \].

(17)

Note that \( H_{1pN} = \mathcal{E} = \mu E \), since \( H_{1pN} \) is conserved at the first post-Newtonian order. As has been already mentioned before, the quadrupole interaction term is de facto a Newtonian correction to the point-particle Hamiltonian. This is important to keep in mind when, for instance, calculating the orbital evolution and gravitational wave emission of binary pulsars with a main sequence star companion, such as PSR B1259-63 [14]. For the Hamiltonian equations that govern the time evolution of the binary system one finds

\[ \dot{r} = \frac{p_r}{\mu} \left[ 1 + \frac{1}{\mu^2c^2} \left( \frac{3\nu - 1}{2} \left( p_r^2 + \frac{p_\phi^2}{r^2} \right) - \frac{GM\mu^2}{r} (3 + 2\nu) \right) \right], \]

\[ \dot{\phi} = \frac{p_\phi}{\mu r^2} \left[ 1 + \frac{1}{\mu^2c^2} \left( \frac{3\nu - 1}{2} \left( p_r^2 + \frac{p_\phi^2}{r^2} \right) - \frac{GM\mu^2}{r} (3 + \nu) \right) \right], \]

\[ \dot{p}_r = \frac{p_r^2}{\mu r^3} - \frac{GM\mu}{r^2} \left( 1 + \frac{3q}{2r^2} \right) + \frac{1}{c^2} \left( \frac{3\nu - 1}{2\mu} \left( \frac{p_r^2}{r^2} + \frac{p_\phi^2}{r^2} \right) \right) \]

\[ + \frac{GM^2\mu}{2r^2} \left( (3 + 2\nu) p_r^2 + (3 + \nu) \frac{p_\phi^2}{r^2} \right), \]

\[ \dot{p}_\phi = 0. \]

At this point, let us include leading-order dissipative effects into that scheme. In general, the leading-order energy dissipation of a matter distribution is governed by the time-dependent radiation reaction Hamiltonian [20]

\[ H_{\text{rad}}(t) = \frac{2G}{5c^3} I_{ij}^{(3)}(t) \int dV \left[ \frac{\pi_i \pi_j}{\rho} + \frac{1}{4\pi G} \partial_t U \partial_j U \right], \]

(19)

where \( I_{ij} \) is the symmetric tracefree mass quadrupole tensor of the matter distribution, \( \rho \) is the coordinate rest-mass density, \( \pi_i \) the momentum density and \( U \) is the gravitational potential that satisfies the Poisson equation with source term \( \rho \). Since we are treating the \( q \)-dependent terms as being formally of first post-Newtonian order, the components of the binary can be
considered as pointlike objects throughout the calculation of $H_{\text{reac}}(t)$. Thus in the center-of-mass frame the radiation reaction Hamiltonian describing the leading-order energy dissipation due to gravitational wave emission can be described by

$$H_{\text{reac}}(t) = \frac{2G}{5c^5} I^{(3)}_{ij}(t) \left[ \frac{p_i p_j}{\mu} - G M \mu \frac{x^i x^j}{r^3} \right].$$

(20)

It is crucial to consider $I^{(3)}_{ij}(t)$ as a function of time, and not as a function of generalized coordinates and momenta, when calculating the radiation reaction part of the EoM according to

$$(p_i)_{\text{reac}} = -\frac{\partial H_{\text{reac}}}{\partial q^i}, \quad (q^i)_{\text{reac}} = \frac{\partial H_{\text{reac}}}{\partial p_i}.$$  

Only afterward $I^{(3)}_{ij}$ can be expressed as a function of $p_r, p_\phi, r$ and $\phi$. Explicitly, the calculation yields (see e.g. [4])

$$(p_r)_{\text{rad}} = \frac{8G^2 p_r}{3r^4c^5} \left( \frac{G M^2 v^2}{5} - \frac{p_\phi^2}{v r} \right),$$

$$(p_\phi)_{\text{rad}} = -\frac{8G^2 p_\phi}{5vr^3c^5} \left( \frac{2G M^2 v^3}{r} + 2 \frac{p_\phi^2}{r^2} - p_r^2 \right),$$

$$(r)_{\text{rad}} = \frac{8G^2 r^2}{15vr^2c^3} \left( 2p_r^2 + 6 \frac{p_\phi^2}{r^2} \right),$$

$$(\phi)_{\text{rad}} = -\frac{8G^2 p_r p_\phi}{3vr^4c^5}.$$  

(21)

For numerical calculations it is useful to introduce scaled variables such that $G = c = 1$. The corresponding scaling is given by

$$p_r = \mu c \tilde{p}_r, \quad p_\phi = \frac{G M \mu}{c} \tilde{p}_\phi, \quad r = \frac{G M}{c^2} \tilde{r}, \quad H = \mu c^2 \tilde{H}, \quad q = \frac{G^2 M^2}{c^4} \tilde{q}.$$  

Applying this, the Hamiltonian equations governing the evolution of the binary system including leading-order radiation back reaction read as

$$\dot{\tilde{r}} = \tilde{p}_r \left[ 1 + \frac{3v - 1}{2} \left( \frac{\tilde{p}_r^2 + \tilde{p}_\phi^2}{\tilde{r}^2} \right) - \frac{3 + 2v}{\tilde{r}} \right] - \frac{8v}{15} \tilde{r} \left[ 2\tilde{p}_r^2 + 6 \frac{\tilde{p}_\phi^2}{\tilde{r}^2} \right],$$

(22)

$$\dot{\tilde{\phi}} = \frac{\tilde{p}_\phi}{\tilde{r}^2} \left[ 1 + \frac{3v - 1}{2} \left( \frac{\tilde{p}_r^2 + \tilde{p}_\phi^2}{\tilde{r}^2} \right) - \frac{3 + v}{\tilde{r}} \right] - \frac{8v}{3} \tilde{p}_r \tilde{p}_\phi,$$

(23)

$$\dot{\tilde{p}}_r = \frac{\tilde{p}_r^2}{\tilde{r}^3} - \frac{1}{\tilde{r}^2} \left[ 1 + \frac{3v}{2\tilde{r}^2} \right] \left( \frac{\tilde{p}_r^2 + \tilde{p}_\phi^2}{\tilde{r}^2} \right) - \frac{3 + 2v}{2\tilde{r}^2} \tilde{p}_r^2 - \frac{3}{2} (3 + v) \frac{\tilde{p}_\phi^2}{\tilde{r}^4} + \frac{1}{\tilde{r}^3} + \frac{8v}{5} \tilde{p}_r \left[ \frac{1}{\tilde{r}^2} - \frac{\tilde{p}_\phi^2}{\tilde{r}^2} \right],$$

(24)

$$\dot{\tilde{p}}_\phi = -\frac{8v}{5} \tilde{p}_\phi \left[ \frac{2}{\tilde{r}} + 2 \frac{\tilde{p}_r^2}{\tilde{r}^2} - \tilde{p}_\phi^2 \right].$$

(25)

Neither the total energy nor the orbital angular momentum is conserved, as indicated by equation (25).
The time evolution of binary systems described by equations (22)–(25) is fully determined by the initial values of three parameters: the semi-major axis $a_r$, the orbital eccentricity $e_r$, and the deformation parameter $q$. Starting the numerical integration in the periastron, i.e. at $\varphi(0) = 0$, the initial values for $r$ and $p_r$ follow immediately as

$$r(0) = r_0 = a_r(0) (1 - e_r(0)), \quad p_r(0) = 0.$$ 

To determine the initial value for $p_{\varphi}$ we use that, at the beginning of the integration, the total energy of the system is given by the conservative part of the Hamiltonian, or, using the reduced energy, $E(0) = E = H_{1PN}/\mu$. It then follows from equation (17) that

$$p_{\varphi}(0)^2 = 2r_0^2 E_{1PN} + 2G\mathcal{M}r_0 \left[ 1 + \frac{q}{2r_0^2} \right] + \frac{1}{c^2} \left[ (1 - 3\nu)r_0^2 (E_{orb}^N)^2 + 4(1 - \nu)G\mathcal{M}r_0 E_{orb}^N + (6 - \nu)G^2\mathcal{M}^2 \right],$$

where $E_{orb}^N$ is the Newtonian energy of the orbit.

3. Higher-order gravitational wave emission

In the previous sections we have investigated the dynamical evolution of a spinning close binary system at the first post-Newtonian approximation. Now we shall turn our attention to the gravitational waves emitted by the system. Far away from the source spacetime can be assumed to be asymptotically flat, that is, the metric is locally Minkowskian. In fact, in asymptotically flat spacetimes the gravitational waves emitted by an isolated binary system are expected to obey a multipole expansion of the form (see e.g. [21])

$$h_{ij}^{TT} = \frac{G}{Dc^4} P_{ij km}(N) \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left[ \left( \frac{1}{c} \right)^{l-2} \frac{4}{l!} T_{ij km,\mathcal{A}_l}(t - D/c) N_{\mathcal{A}_l} \right. \left. + \left( \frac{1}{c} \right)^{l-1} \frac{8l}{(l + 1)!} \epsilon_{pq(k} j_{l m,p \mathcal{A}_l}(t - D/c) N_{q} N_{\mathcal{A}_l} \right],$$

where $h_{ij}^{TT}$ is the symmetric-tracefree (STF) part of the metric perturbation $h_{ij}$, the brackets denote symmetrization and $D$ is the source–observer distance. The unit vector $N$ points from the binary to the observer and $A_l = a_1 a_2 \cdots a_l$ ($a_i = 1, 2, 3$) is a multi-index. $T_{ij}$ and $T_{ij,\mathcal{A}_l}$ are the STF mass and current multipole moments that parametrize the radiation field in a Cartesian coordinate frame. However, if the direction of the angular momentum is conserved, it is more suitable to use STF-multipole moments $I_{lm}$ and $S_{lm}$, $m = -l, \ldots, l$, that are irreducibly defined with respect to the axis of angular momentum. Their relations to the Cartesian multipole components are given in equations (A.1) and (A.2) in the appendix. From equation (26) it is then derived that the radiation field $h_{ij}^{TT}$, expressed in terms of time derivatives of $I_{lm}$ and $S_{lm}$, is given by

$$h_{ij}^{TT} = \frac{G}{Dc^4} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left[ \left( \frac{1}{c} \right)^{l-2} I_{ij lm}^{\mathcal{C}} (t - D/c) T_{ij}^{E2,lm}(\Theta, \Phi) \right. \left. + \left( \frac{1}{c} \right)^{l-1} S_{ij lm}^{\mathcal{C}} (t - D/c) T_{ij}^{B2,lm}(\Theta, \Phi) \right],$$

where $T_{ij}^{E2,lm}$ and $T_{ij}^{B2,lm}$ are the so-called pure-spin tensor–spherical harmonics of electric and magnetic types. These harmonics are orthonormal on the unit sphere. In fact, introducing

$\text{Note that } p_{\varphi} \rightarrow \mu p_{\varphi} \text{ and } p_r \rightarrow \mu p_r.$
unit vectors $\hat{\Phi}$ and $\hat{\Theta}$, they can be decomposed into a term proportional to $(\hat{\Theta} \otimes \hat{\Theta} - \hat{\Phi} \otimes \hat{\Phi})$ and $(\hat{\Phi} \otimes \hat{\Phi} + \hat{\Theta} \otimes \hat{\Theta})$, respectively. That way, if the $T_{E,2m}^{E,2m}$ are known, one obtains the polarization states $h_+$ and $h_\times$ of the radiation field from equation (27) without any further calculations. The pure-spin tensor–spherical harmonics needed here are given by equations (A.7)–(A.16) in the appendix.

In section 2 the point-particle contribution was taken into account up to first post-Newtonian approximation, and the quadrupole coupling term, though present already at Newtonian order, was assumed to be of the same order as the 1 pN corrections to the point-particle dynamics. That means, we have to extend our analysis beyond the leading-order gravitational wave formula. Considering the dynamics up to 1 pN requires the application of the multipole expansion (27) up to $l = 4$ for the mass multipole moments and up to $l = 3$ for the current multipole moments. Explicitly, neglecting all higher-order terms, equation (27) is reduced to

$$h_{+ \times} = \frac{G}{D c^4} \left\{ \sum_{m=-2}^{2} T_{E,2m}^{E,2m} + \sum_{m=-3}^{2} T_{E,3m}^{E,3m} + \sum_{m=-2}^{2} S_{E,2m}^{E,2m} \right\}$$

or, in a more convenient form,

$$h_{+ \times} = \frac{1}{c^2} h_{+\times}^{(0)} + \frac{1}{c} h_{+\times}^{(1)} + \frac{1}{c^2} h_{+\times}^{(2)}.$$  

Note that according to equation (28) except for $T_{(2)2m}$ all other time derivatives and multipole moments are required only at leading order.

Let us start by noting that, in the center-of-mass system, the mass and current multipole moments of a two-body system are given by

$$I_{ij} = \mu x_i x_j \left[ 1 + \frac{29}{42} (1 - 3v^2) \frac{v^2}{c^2} - \frac{5 - 8v}{7c^2} \frac{GM}{r} \right] + \frac{\mu (1 - 3v)}{21c^2} \left\{ -12 (r \cdot v) x_i x_j + r^2 (v_i v_j) \right\} + I_{ij}^s,$$

$$I_{ijk} = -\mu \sqrt{1 - 4v^2} x_i x_j x_k,$$

$$I_{ijkl} = \mu (1 - 3v) x_i x_j x_k x_l,$$

$$J_{ij} = -\mu \sqrt{1 - 4v^2} \delta_{ab} x_i x_j x_a x_b,$$

$$J_{ijk} = \mu (1 - 3v) \delta_{ab} x_i x_j x_a x_b,$$

where brackets denote the STF-part of the correspondent tensor (see appendix A). At this point it is necessary to consider, for a moment, the contribution of the stellar mass-quadrupole moment. Since one component of the binary is spinning and thus automatically gains a finite size there could be, in principle, a contribution of $I_{ij}$ to the gravitational wave emission of the system. This contribution is, however, very small, unless the energy stored in the internal stellar degrees of freedom, e.g. in the oscillations of the star, is comparable to the orbital energy. From now on we shall assume that $I_{ij}$ is either trivial or can be neglected compared to all other terms present in the calculation.

5 In this representation, $h_+$ is the $(\hat{\Theta} \otimes \hat{\Theta} - \hat{\Phi} \otimes \hat{\Phi})$ part of equation (27), while $h_\times$ is the $(\hat{\Phi} \otimes \hat{\Phi} + \hat{\Theta} \otimes \hat{\Theta})$ part.

6 From now on we omit the superscript $TT$ for notational convenience.
Using the Newtonian equation of motion for a point-particle binary\(^7\)
\[
\dot{\mathbf{v}} = -\frac{G M}{r^3} \mathbf{r}
\]
to calculate \(\mathcal{T}^{(3)}\), \(\mathcal{F}^{(4)}\), \(\mathcal{F}^{(2)}\), and \(\mathcal{F}^{(3)}\), and the 1 pN equation of motion in the form
\[
\dot{v} = -\frac{G M}{r^3} r \left( 1 + \frac{3v}{2r^2} \right) + \frac{G M}{r^3 c^2} \left[ r \left\{ \frac{G M}{r} (4 + 2v) - v^2 (1 + 3v) + \frac{3v(r \cdot v)^2}{r^2} \right\} 
\right.
\]
\[
+ \left(4 - 2v)(r \cdot v) v \right]
\]
to calculate \(\mathcal{T}^{(2)}_{ij}\), one finds, after some lengthy calculations (see also [22])
\[
\mathcal{T}^{(2)}_{ij} = 2\mu v_i v_j \left[ 1 + \frac{9}{14} (1 - 3v) \frac{v^2}{c^2} + \frac{54v - 25GM}{21 r^2 c^2} \right] 
\]
\[
+ 2\mu x_i x_j \left[ \frac{25 + 9v}{7} \frac{GM}{r^2 c^2} (r \cdot v) - \mu x_i x_j \frac{GM}{r^3} \left[ 2 \left( 1 + \frac{3v}{2r^2} \right) \right] \right.
\]
\[
+ \frac{61 + 48v}{21} \frac{v^2}{c^2} - \frac{2}{7c^2} (1 - 3v) \frac{(r \cdot v)^2}{r^2} - (10 - 9\mu) \frac{GM}{r^2 c^2} \right].
\]
\[
\mathcal{T}^{(3)}_{ijk} = -\mu \sqrt{1 - 4v^2} \left[ \frac{9}{5} \frac{GM}{r^5} (r \cdot v)x_i x_j x_k - \frac{21}{r^3} v_i x_j x_k + 6v_i v_j v_k \right].
\]
\[
\mathcal{T}^{(4)}_{ijkl} = 4\mu (1 - 3v) \left[ 6v_i v_j v_k v_l - 48 \frac{GM}{r^3} v_i v_j x_k x_l + 42 \frac{GM}{r^5} (r \cdot v) v_i x_j x_k x_l \right]
\]
\[
+ \frac{GM}{r^3} x_i x_j x_k x_l \left[ 7 \frac{GM}{r^5} + 3 \frac{v^2}{r^2} - 15 \frac{(r \cdot v)^2}{r^4} \right].
\]
while the time derivatives of the current multipole moments read as
\[
\mathcal{J}^{(2)}_{ij} = \mu \sqrt{1 - 4v^2} \frac{GM}{r^3} \epsilon_{abj} x_i x_a v_b,
\]
\[
\mathcal{J}^{(3)}_{ijk} = 2\mu \frac{GM}{r^3} (1 - 3v) \left[ -4\epsilon_{abj} (k x_i x_j x_a x_b) + \frac{3}{r^2} \frac{GM}{r^3} \epsilon_{abj} (k x_i x_j x_a v_b) \right].
\]
In particular, within our approximation the \(q\)-coupling is relevant for the second time derivative of the mass quadrupole tensor only. Using the relation between the two classes of multipole moments given by equations (A.1) and (A.2) we get the following expressions for the time derivatives of \(I^{lm}\) and \(S^{lm}\), now in polar coordinates:
\[
I^{(2)20} = 4\mu \left[ \frac{3\pi}{5} \left[ -2 v^2 - \frac{GM}{r} \left( 1 + \frac{3v}{2r^2} \right) \right] + \frac{1}{c^2} \left\{ \frac{G^2 M^2}{r^2} (v - 10) \right\} \right.
\]
\[
+ \frac{9}{7} (3v - 1) v^4 + \frac{GM}{7r} ((37 - 20v) r^2 \phi^2 - (15 + 32v) r^2 \phi^2) \right].
\]
\[
I^{(2)21} = 0.
\]
\(^7\) Remember that \(q\) is treated as a 1 pN quantity formally.
\[ I^{(2)22} = \sqrt{\frac{8\pi}{5}} \mu e^{-2\nu} \left[ 2 \left( \frac{r^2 - r^2 \dot{\psi}^2}{r} - \frac{GM}{r} \left( 1 + \frac{3\dot{\psi}}{2r^2} \right) - 2i\dot{\psi} \right) \right. \\
+ \frac{1}{c^2} \left( 10 - v \right) \frac{G^2M^2}{r^2} + \frac{GM}{21r} \left( 3(15 + 32v)r^2 - (11 + 156v)r^2 \dot{\psi}^2 \right) \\
+ \frac{9}{7} (1 - 3v)(r^4 - r^4 \dot{\psi}^4) - i\dot{\psi}^2 \left( \frac{10}{21} (5 + 27v) \frac{GM}{r} + \frac{18}{7} (1 - 3v)v^2 \right) \right], \]
\[ I^{(3)30} = 0, \]
\[ I^{(3)31} = 4\nu(m_1 - m_2) \sqrt{\frac{\pi}{35}} e^{-\nu} \left[ \dot{r} \left( 2 \frac{GM}{r} - \nu^2 \right) + i\dot{\psi} \left( \frac{7GM}{6r} - 6\dot{\psi}^2 + 2r^2 \dot{\psi}^2 \right) \right], \]
\[ I^{(3)33} = 2\nu(m_1 - m_2) \sqrt{\frac{\pi}{21}} e^{-\nu} \left[ 2 \left( \frac{r^2 - 2GM}{r} - 3r^2 \dot{\psi}^2 \right) \dot{r} \\
+ i\dot{\psi} \left[ \frac{GM}{r} - 6\dot{\psi}^2 + 2r^2 \dot{\psi}^2 \right] \right], \]
\[ I^{(4)40} = \frac{2}{21} (1 - 3v)\mu \sqrt{\frac{\pi}{5}} \left[ \frac{7G^2M^2}{r^2} - \frac{GM}{r} (18r^2 + 13r^2 \dot{\psi}^2) + 6\dot{\psi}^4 \right]. \]
\[ I^{(4)41} = I^{(4)43} = 0, \]
\[ I^{(4)42} = \frac{2}{63} \sqrt{2\pi} (1 - 3v)\mu e^{-2\nu} \left[ -7 \frac{G^2M^2}{r^2} + \frac{GM}{r} (18r^2 - 3r^2 \dot{\psi}^2) \\
+ 6r^4 \dot{\psi}^4 - 6\dot{\psi}^4 + 3i\dot{\psi} \left( 4\dot{\psi}^2 - 9 \frac{GM}{r} \right) \right], \]
\[ I^{(4)44} = \frac{2}{9} \sqrt{\frac{\pi}{14}} (1 - 3v)\mu e^{-4\nu} \left[ \frac{7G^2M^2}{r^2} + \frac{GM}{r} (-18r^2 + 51r^2 \dot{\psi}^2) + 6\dot{\psi}^4 \\
- 36r^2 \dot{\psi}^2 + 6r^4 \dot{\psi}^4 + i\dot{\psi} \left( 54 \frac{GM}{r} - 24r^2 + 24r^2 \dot{\psi}^2 \right) \right], \]
\[ S^{(2)20} = S^{(2)22} = 0, \]
\[ S^{(2)21} = \frac{8}{3} \sqrt{\frac{2\pi}{5}} (m_1 - m_2)GM\nu \psi e^{-\nu}, \]
\[ S^{(3)30} = -4 \sqrt{\frac{\pi}{105}} (1 - 3v)GM\mu \dot{\psi}, \]
\[ S^{(3)31} = S^{(3)33} = 0, \]
\[ S^{(3)32} = \frac{2}{3} \sqrt{\frac{2\pi}{7}} (1 - 3v)GM\mu e^{-2\nu}(\dot{r} - 4i\dot{\psi}). \]

Here we used that $\sqrt{1 - 4\nu} = (m_1 - m_2)/M$. Since only $S^{(2)21}$, $I^{(3)31}$ and $I^{(3)33}$ contribute to the first correction to the leading-order quadrupole formula $h^{(1)}_{+,x}$, it becomes clear that $h^{(1)}_{+,x}$...
vanishes for equal-mass binary systems. In that case the first nontrivial correction to $h_{1,\times}^{(0)}$ is of order $1/c^2$.

It is possible to derive, after some straightforward but rather lengthy calculations, analytic expressions for the time derivatives of the multipole moments in terms of the leading order, more useful to express the polarization states of the gravitational radiation field in terms of generalized coordinates and velocities or—in Hamiltonian formulation—in terms of generalized coordinates and momenta. Inserting above relations in equation (28) one finds, at leading order,

$$\frac{Dc^4}{G} h_{\times}^{(0)} = (1 + \cos^2 \Theta) \mu \left[ \cos 2(\Phi - \varphi) \left( r^2 - r^2 \varphi^2 - \frac{G \mathcal{M}}{r} \left( 1 + \frac{3q}{2r^2} \right) \right) + 2r \dot{r} \varphi \sin 2(\Phi - \varphi) \right] - \mu \sin^2 \Theta \left[ r^2 + r^2 \varphi^2 - \frac{G \mathcal{M}}{r} \left( 1 + \frac{3q}{2r^2} \right) \right], \quad (55)$$

$$\frac{Dc^4}{G} h_{\times}^{(0)} = \mu \cos \Theta \left[ 2r \dot{r} \varphi \cos 2(\Phi - \varphi) - \sin 2(\Phi - \varphi) \left( r^2 - r^2 \varphi^2 - \frac{G \mathcal{M}}{r} \left( 1 + \frac{3q}{2r^2} \right) \right) \right].$$

Note that, in order to emphasize the character of the $q$-coupling, the $q$-dependent terms have been included into the leading-order component of the radiation field. Defining $\Delta m \equiv m_1 - m_2$ the first correction terms read as

$$\frac{Dc^5}{G} h_{\times}^{(1)} = \frac{\Delta m}{\mathcal{M}} \mu \sin \Theta \left[ \frac{4}{3} G \mathcal{M} \varphi \sin(\Phi - \varphi) + \frac{3 \cos^2 \Theta - 1}{2} \left\{ \left( \frac{2G \mathcal{M}}{r} - v^2 \right) r \cos(\Phi - \varphi) - \left( v^2 - \frac{7G \mathcal{M}}{6r} \right) r \varphi \sin(\Phi - \varphi) \right\} - \frac{1 + \cos^2 \Theta}{4} \left\{ 2r \cos 3(\Phi - \varphi) \left( r^2 - 3r^2 \varphi^2 - 2 \frac{G \mathcal{M}}{r} \right) - r \varphi \sin 3(\Phi - \varphi) \left( \frac{7G \mathcal{M}}{r} - 6r^2 + 2r^2 \varphi^2 \right) \right\} \right], \quad (56)$$

$$\frac{Dc^5}{G} h_{\times}^{(1)} = \frac{\sin 2\Theta \Delta m}{2 \mathcal{M}} \mu \left[ \cos \varphi \cos(\Phi - \varphi) \left\{ \frac{5G \mathcal{M}}{2r} - v^2 \right\} + \dot{r} \sin(\Phi - \varphi) \left\{ v^2 - 2 \frac{G \mathcal{M}}{r} \right\} + \frac{r \varphi}{2} \cos 3(\Phi - \varphi) \left\{ \frac{7G \mathcal{M}}{r} - 6r^2 + 2r^2 \varphi^2 \right\} + \dot{r} \sin 3(\Phi - \varphi) \left\{ r^2 - 3r^2 \varphi^2 - 2 \frac{G \mathcal{M}}{r} \right\} \right], \quad (57)$$

Note that these expressions depend on the mass difference and vanish for equal-mass binaries. The next corrections to the gravitational waveforms read as

$$\frac{Dc^6}{G} h_{\times}^{(2)} = \frac{1 + \cos^2 \Theta}{2} \mu \left[ \cos 2(\Phi - \varphi) \left\{ (10 - v) \frac{G^2 \mathcal{M}^2}{r^2} + \frac{9}{7}(1 - 3v)(r^2 - r^4 \varphi^4) + \frac{G \mathcal{M}}{r} \left\{ \frac{15 + 32v}{7} r^2 - \frac{11 + 156v}{21} r^2 \varphi^2 \right\} \right\} + \frac{r \varphi}{7} \sin 2(\Phi - \varphi) \left\{ \frac{10}{3}(5 + 27v) \frac{G \mathcal{M}}{r} + 18(1 - 3v) v^2 \right\} \right]$$

13
\[
\begin{align*}
&+ \frac{\sin^2 \Theta}{2} \mu \left[ (v - 10) \frac{G^2 M^2}{r^2} - \frac{9}{7} (1 - 3v)v^4 + \frac{G M}{7r} ((37 - 20v)r^2 \psi^2 \\
&- (15 + 32v)r^2) \right] \\
&- \frac{1 - 3v}{56} \mu (7 \cos^4 \Theta - 8 \cos^2 \Theta + 1) \left[ \frac{7G^2 M^2}{r^2} - \frac{GM}{r} (18r^2 + 13r^2 \psi^2) + 6v^2 \right] \\
&+ \frac{1 - 3v}{42} \mu (7 \cos^4 \Theta - 6 \cos^2 \Theta + 1) \left[ \cos 2(\Phi - \varphi) \left\{ 6(r^2 \psi^4 - \dot{r}^4) \right\} \\
&+ \frac{GM}{r} (18r^2 - 3r^2 \psi^2) - 7 \frac{G^2 M^2}{r^2} \right] - \frac{2 \dot{r} \psi \sin 2(\Phi - \varphi)}{27 \frac{G M}{r}} \\
&+ \frac{1 - 3v}{24} \mu \sin^2 \Theta (1 + \cos^2 \Theta) \left[ \cos 4(\Phi - \varphi) \left\{ \frac{G^2 M^2}{r^2} + 6\dot{r}^4 - 36r^2 \dot{r}^2 \psi^2 \\
&+ 6r^4 \psi^4 + \frac{GM}{r} (51r^2 \psi^2 - 18r^2) \right\} - \frac{2 \dot{r} \psi \sin 4(\Phi - \varphi)}{24 \frac{G M}{r}} \right] \\
&+ 24 \dot{r}^2 \psi^2 \right] - \frac{1 - 3v}{3} GM \mu \psi (2 \cos^2 \Theta - 1) \left\{ 4r \psi \cos 2(\Phi - \varphi) \\
&- \dot{r} \sin 2(\Phi - \varphi) \right\} \right]. \tag{58}
\end{align*}
\]

\[
\frac{De^6}{G} h^{(2)}_x = \mu \cos \Theta \left[ \frac{\dot{r} \psi}{7} \cos 2(\Phi - \varphi) \left\{ \frac{10}{3} (5 + 27v) \frac{G M}{r} + 18v^2 (1 - 3v) \right\} \\
- \sin 2(\Phi - \varphi) \left\{ (10 - v) \frac{G^2 M^2}{r^2} + \frac{G M}{r} \left( \frac{15 + 32v}{7}r^2 - \frac{11 + 156v}{21}r^2 \psi^2 \right) \right\} \\
+ \frac{9}{7} (1 - 3v)(\dot{r}^2 - r^4 \dot{\psi}^2) \right\} + (1 - 3v) \mu \cos \Theta \left\{ -\frac{1}{2} GM \dot{r} \psi \sin^2 \Theta \\
+ \frac{3 \cos^2 \Theta - 1}{6} GM \psi \left\{ \dot{r} \cos 2(\Phi - \varphi) + 4r \psi \sin 2(\Phi - \varphi) \right\} \\
- \frac{\sin^2 \Theta}{12} \left\{ \dot{r} \psi \cos 4(\Phi - \varphi) \left( \frac{54}{GM} - 24\dot{r}^2 + 24r^2 \psi^2 \right) \\
+ \sin 4(\Phi - \varphi) \left( 7 \frac{G^2 M^2}{r^2} + \frac{GM}{r} (51r^2 \psi^2 - 18r^2) + 6\dot{r}^4 + 6r^4 \dot{\psi}^4 \\
- 36r^2 \dot{r}^2 \psi^2 \right) \right\} - \frac{7 \sin^2 \Theta - 5}{42} \left\{ 3 \dot{r} \psi \cos 2(\Phi - \varphi) \left( 4\dot{r}^2 - 9 \frac{GM}{r} \right) \\
+ \sin 2(\Phi - \varphi) \left( -7 \frac{G^2 M^2}{r^2} + \frac{GM}{r} (18r^2 - 3r^2 \psi^2) + 6r^4 \dot{r}^4 - 6\dot{r}^4 \right) \right\} \right]. \tag{59}
\]

The polarization states of the gravitational radiation field, expressed in terms of generalized coordinates and momenta, can be found in appendix B.

4. Discussion

It has been long known that finite size effects introduce a periastron shift already at the level of Newtonian theory. For a couple of main sequence star binaries the total apsidal motion \( \dot{\psi}_\text{tot} \) has been determined from observational evidence. Compared with the contribution \( \dot{\psi}_\text{rel} \) predicted
by GR it became obvious that in all systems the dominant contribution to $\dot{\phi}_{\text{tot}}$ comes from Newtonian perturbations (for an overview see e.g. [10]). This is due to the ‘soft’ equations of state governing the stellar matter of main sequence stars. For compact star binaries Newtonian perturbations are often neglected. In particular, it is often argued that the effect of the spin-induced quadrupole is too small unless the compact star (e.g. a neutron star) is rotating near the mass-shedding limit [3]. However, even for NS–NS or NS–BH binaries this argument does not hold completely. It has been shown in previous sections that for close NS–NS binaries the potential energy introduced by the $q$-coupling could be considerably larger than the corresponding 1.5 pN orbital correction terms, though it is smaller by a factor 100 or more than the 1 pN contribution. The non-relativistic periastron shift induced by the quadrupole coupling accumulates over a large number of periods (figures 3 and 4). In order to obtain highly accurate templates it is thus desirable to take into account these corrections properly, at least for fast spinning neutron stars and white dwarfs in close binary systems.

With the space-bound laser interferometric detector LISA at hand the frequency band accessible to observations will be extended to much lower frequencies ($10^{-1}$ to $10^{-4}$ Hz). That means, a much larger number of possible sources will become accessible for observation. Among these sources will be close white dwarf binary systems, and it is this class of binaries to which the analysis shown in this paper is most relevant. It has been argued by Willems et al in a recent paper [23] that—contrary to prevailing opinions—there might exist a class of eccentric galactic double white dwarfs, which are formed by interactions in tidal clusters. Willems et al showed that tides and stellar rotation strongly dominate the periastron shift at orbital frequencies $\gtrsim 1$ mHz. The phase shifts induced by these Newtonian perturbations are much larger than the general relativistic corrections then. They conclude that it is essential to include phase shifts generated by Newtonian perturbations into the signal templates in order to not bias LISA surveys against eccentric double white dwarfs. Generally, neglecting the contribution to $\dot{\phi}_{\text{tot}}$ induced by rotational deformation will lead to an overestimation of the total mass derived from $\dot{\phi}_{\text{tot}}$.

In this paper, the competing influences of the rotational deformation and the 1 pN correction terms were studied. In particular, we succeeded in calculating a 1 pN quasi-

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{h_+-component of a non-spinning, equal-mass binary system with $\tilde{\alpha}_r = 40$, $\epsilon_r = 0.3$. Plotted are the waveforms according to the leading-order quadrupole formula and the 1 pN-corrected waveform. Observer-dependent parameters: $\Phi = \pi/2$, $\Theta = \pi/4$.}
\end{figure}
Figure 2. $h_x$-component of a non-spinning, equal-mass binary system. All parameters are the same as in figure 1.

Figure 3. Influence of the quadrupole coupling on the gravitational wave emission. The 1 pN-correct $h_\times$-component emitted by a non-spinning binary with semi-major axis and eccentricity $a_0 = 40$ and $e_0 = 0.3$, respectively, is compared with the corresponding waveform emitted by a close binary with $\tilde q = 4$. The masses are $m_1 = 3m_2$. Observer-dependent angles are $\Phi_1 = \pi/2$, $\Theta_1 = \pi/4$.

Keplerian solution, which takes into account finite size effects up to linear order in the quadrupole deformation parameter $q$. The results given in section 2 are valid as long as $q/J^2$ is of the order $O(c^{-2})$. For white dwarf binaries or binary pulsars such as PSR 1259-63 the periastron shift induced by rotational deformation is possibly much larger than the general relativistic contribution. In that case, equations (11) and (12) still apply in the limit $v/c \to 0$. In section 3 the polarization states of the gravitational radiation field are calculated beyond the leading-order approximation. For non-spinning compact binaries the corresponding waveforms are shown in figures 1 and 2. In these figures waveforms calculated
using the leading-order expressions $h_{+,x}^{(0)}$ are compared to the 1 pN correct waveforms with the next to leading-order corrections $h_{+,x}^{(1)}$ and $h_{+}^{(2)}$ taken into account. The first correction, $h_{+}^{(1)}$, is nontrivial only for different mass binaries, i.e. for equal-mass binaries the first non-vanishing correction to the leading-order formula appears at the order $O(c^{-2})$.

The influence of the $q$-coupling on the gravitational waveforms is shown in figures 3 and 4. As expected, the spin-induced quadrupole moment leads to a phase shift compared to the pure point-particle GW emission. Moreover, the quadrupole deformation of the spinning compact objects speeds up the inspiral process, as has been shown in figures 5 and 6 for an equal-mass binary in a slightly elliptic orbit.
Figure 6. $h_+$-component of the gravitational wave field emitted by an equal-mass binary with $\tilde{q} = 4$ during the inspiral process (initial values $\tilde{a}_r(0) = 50$, $\epsilon_r(0) = 0.3$). Observer-dependent parameters $\Phi = \pi/2$, $\Theta = \pi/4$.

More analysis is needed in order to fully understand the imprint of finite size effects onto the gravitational wave pattern of close compact binary systems beyond the leading order. In particular it would be highly desirable to include the stellar oscillation modes into the calculations. From previous works it is expected that in these cases so-called tidal resonances will have an important impact on the inspiral process and the gravitational wave emission of the binary [4, 5, 7].

Acknowledgments

I am grateful to Gerhard Schäfer for helpful discussions and careful reading of the manuscript. This work is supported by the Deutsche Forschungsgemeinschaft (DFG) through SFB/TR7 ‘Gravitationswellenastrophysik’.

Appendix A. Useful relations

The mass and current multipole moments $I^m_l$ and $S^m_l(m = -l, \ldots, l)$ that are irreducibly defined with respect to the orbital angular momentum axis are related to $I_{A_l}$ and $J_{A_l}$ according to

\[ I^m_l(t) = \frac{16\pi}{(2l+1)!!} \sqrt{\frac{(l+1)(l+2)}{2(l-1)!}} I_{A_l}(t) Y^m_l^{*, A_l}, \quad (A.1) \]

\[ S^m_l(t) = -\frac{32\pi l}{(l+1)(2l+1)!!} \sqrt{\frac{(l+1)(l+2)}{2(l-1)!}} J_{A_l} Y^m_l^{*, A_l}, \quad (A.2) \]

where, for $m \geq 0$,

\[ Y^m_l = (-1)^m (2l - 1)!! \sqrt{\frac{2l + 1}{4\pi(l - m)!(l + m)!}} (\delta_{i_1}^l + i\delta_{j_1}^l) \cdots (\delta_{i_m}^l + i\delta_{j_m}^l) \delta_{i_{m+1}}^l \cdots \delta_{i_l}^l, \quad (A.3) \]
and
\[ Y_{lm}^{in} = (-1)^m Y_{lm}^{jm*} \quad \text{for} \ m < 0. \quad (A.4) \]

The complex conjugates are given by
\[ I_{lm}^{im*} = (-1)^m I_{l-m}^m, \quad S_{lm}^{im*} = (-1)^m S_{l-m}^m. \quad (A.5) \]

The pure-spin tensor–spherical harmonics are orthonormal on the unit sphere. For the complex conjugate the following relation holds:
\[ T^{E/B,lm*} = (-1)^m T^{E/B,l-m}. \quad (A.6) \]

Defining
\[ \Upsilon_+ \equiv \hat{\Theta} \otimes \hat{\Phi} - \hat{\Phi} \otimes \hat{\Theta}, \quad \Upsilon_- \equiv \hat{\Theta} \otimes \hat{\Phi} + \hat{\Phi} \otimes \hat{\Theta}, \]

the expressions needed in the paper read as
\[ T_{E,22}^{22} = \sqrt{\frac{5}{128\pi}} e^{2\Phi} [(1 + \cos^2 \Theta) \Upsilon_+ + 2i \cos \Theta \Upsilon_-]. \quad (A.7) \]
\[ T_{E,20}^{20} = \sqrt{\frac{15}{64\pi}} \sin^2 \Theta \Upsilon_+. \quad (A.8) \]
\[ T_{B,21}^{21} = \sqrt{\frac{5}{32\pi}} \sin \Theta e^{i\Phi} [i \Upsilon_+ - \cos \Theta \Upsilon_-]. \quad (A.9) \]
\[ T_{E,33}^{22} = -\sqrt{\frac{21}{256\pi}} \sin \Theta e^{i\Phi} [(1 + \cos^2 \Theta) \Upsilon_+ + 2i \cos \Theta \Upsilon_-]. \quad (A.10) \]
\[ T_{E,31}^{21} = \sqrt{\frac{35}{256\pi}} \sin \Theta e^{i\Phi} [(3 \cos^2 \Theta - 1) \Upsilon_+ + 2i \cos \Theta \Upsilon_-]. \quad (A.11) \]
\[ T_{B,32}^{22} = -\sqrt{\frac{7}{128\pi}} e^{2\Phi} [2i(2 \cos^2 \Theta - 1) \Upsilon_+ - \cos \Theta (3 \cos^2 \Theta - 1) \Upsilon_-]. \quad (A.12) \]
\[ T_{B,30}^{20} = \sqrt{\frac{105}{64\pi}} \cos \Theta \sin^2 \Theta \Upsilon_-. \quad (A.13) \]
\[ T_{E,44}^{24} = \sqrt{\frac{63}{512\pi}} \sin^2 \Theta e^{4i\Phi} [(1 + \cos^2 \Theta) \Upsilon_+ + 2i \cos \Theta \Upsilon_-]. \quad (A.14) \]
\[ T_{E,42}^{22} = \sqrt{\frac{9}{128\pi}} e^{2\Phi} [(7 \cos^4 \Theta - 6 \cos^2 \Theta + 1) \Upsilon_+ + i \cos \Theta (7 \cos^2 \Theta - 5) \Upsilon_-]. \quad (A.15) \]
\[ T_{E,40}^{20} = -\sqrt{\frac{45}{256\pi}} (7 \cos^4 \Theta - 8 \cos^2 \Theta + 1) \Upsilon_+. \quad (A.16) \]

A.1. Symmetric-tracefree tensors

Throughout this paper symmetric-tracefree third and fourth rank tensors are used. Symmetrizing a tensor of rank \( p \) requires to take the properly weighted sum over all index permutations,
\[ T_{(i_1 \ldots i_p)} = T_{\text{symm}}^{i_{1 \ldots i_p}} \equiv \frac{1}{p!} \sum_{\text{permut}} T_{i_1 \ldots i_p}. \quad (A.17) \]
The tracefree part of the tensor $T_{i_1 \cdots i_p}$ is calculated according to [24]

$$T_{\langle i_1 \cdots i_p \rangle} = \frac{p!}{2} \sum_{k=0}^{[p/2]} \alpha_k \delta(i_1 i_2) \cdots \delta_{i_{2k-1} i_{2k}} T_{\text{symm}}^{i_{2k+1} \cdots i_p a_1 \cdots a_{2k+1}} \alpha_1 \cdots \alpha_k,$$  \(A.18\)

with

$$\alpha_k = \frac{p!}{(2p-1)!!} (-1)^k (2p-2k-1)!! (p-2k)!! (2k)!!.$$  \(A.19\)

In particular, third and fourth rank STF tensors are given by

$$T_{\langle abc \rangle} = T(abc) - \frac{1}{5} \left[ \delta_{ab} T(cii) + \delta_{bc} T(aii) + \delta_{ac} T(bii) \right] T_{\langle iij \rangle},$$  \(A.20\)

$$T_{\langle abcd \rangle} = T(abcd) - \frac{1}{7} \left[ \delta_{ab} T(cdii) + \delta_{ac} T(bdii) + \delta_{ad} T(bcii) + \delta_{bc} T(adii) + \delta_{bd} T(acii) + \delta_{cd} T(abii) \right] + \frac{1}{35} \left[ \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc} + \delta_{ab} \delta_{cd} \right] T_{\langle iij \rangle}.$$  \(A.21\)

Appendix B. Expressions for $h_+$ and $h_\times$ in terms of generalized coordinates and momenta

The expressions for the leading and next to leading-order contribution to the polarization states of the gravitational wave field, $h_+$ and $h_\times$, read as

$$h^{(0)}_+ = \frac{G \mu}{D c^3} \left\{ \cos 2(\Phi - \varphi) \left[ \frac{p_r^2 + p_\varphi^2}{r^2} - \frac{GM \mu^2}{r} \left( 1 + \frac{3q}{2r^2} \right) \right] + \frac{2p_r p_\varphi}{r} \sin 2(\Phi - \varphi) \right\},$$  \(B.1\)

$$h^{(1)}_+ = \frac{G A m}{D c^5} \frac{\Delta}{M \mu^2} \sin \Theta \left[ \frac{p_\varphi}{r} \sin(\Phi - \varphi) \left\{ \frac{4 GM \mu^2}{3 r} + \frac{3 \cos^2 \Theta - 1}{2} \left( \frac{7 GM \mu^2}{6 r} - p_r^2 - \frac{p_\varphi^2}{r^2} \right) \right\} + \frac{3 \cos^2 \Theta - 1}{2} \left. p_r \cos(\Phi - \varphi) \left\{ \frac{2 GM \mu^2}{3 r} - p_r^2 - \frac{p_\varphi^2}{r^2} \right\} \right. + \frac{1 + \cos^2 \Theta}{2} \left. \left( \frac{p_r}{r} \cos 3(\Phi - \varphi) \left( p_r^2 - 3 \frac{p_\varphi^2}{r^2} - 2 \frac{GM \mu^2}{r} \right) - \frac{p_\varphi}{r} \sin 3(\Phi - \varphi) \left( \frac{7 GM \mu^2}{6 r} - 3 p_r^2 + \frac{p_\varphi^2}{r^2} \right) \right\} \right\}.$$  \(B.2\)

$$h^{(2)}_+ = \frac{G \mu}{D c^6} \left\{ \frac{\sin^2 \Theta}{14} \left[ \frac{G^2 M^2}{r^2} (v - 10) - \frac{5(3v - 1)}{\mu^4} \left( \frac{p_r^2 + \frac{p_\varphi^2}{r^2}}{r^2} \right)^2 \right] + \frac{G M}{\mu^2 r} \left( 3(23 + 8v)p_r^2 + (121 + 8v)\frac{p_\varphi^2}{r^2} \right) \right\}$$
\[+ \frac{1 + \cos^2 \Theta}{14} \cos 2(\Phi - \varphi) \left[ 7 \frac{G^2 M^2}{r^2} (10 - \nu) + 5(3\nu - 1) \left( \frac{p^2}{\mu^2} - \frac{p^2}{\mu^2 r^2} \right) \right] + \frac{G M}{\mu^2 r} \left\{ \frac{241 - 72 v p^2}{5} p^2 - 3(23 + 8\nu) p^2 \right\} \]

\[+ \frac{1 + \cos^2 \Theta}{7} \sin 2(\Phi - \varphi) p_r p_\varphi \left[ \frac{5(3\nu - 1)}{\mu^2} \left( \frac{p^2_r + p^2_\varphi}{r^2} \right) - \frac{227 - 9v G M}{3} \right] \]

\[+ \frac{1 - 3\nu}{24} \sin^2 \Theta (1 + \cos^2 \Theta) \left[ \cos 4(\Phi - \varphi) \left\{ \frac{7 G^2 M^2}{r^2} - \frac{G M}{\mu^2 r} \left( 18 p^2_r - 51 \frac{p^2_\varphi}{r^2} \right) \right\} \right] \]

\[+ 6 \left( \frac{p^4_r}{\mu^4 r^4} - \frac{25 p^2_r p^2_\varphi}{\mu^4 r^4} + \frac{p^4_\varphi}{\mu^4 r^4} \right) - \frac{p_r p_\varphi}{\mu^2 r} \sin 2(\Phi - \varphi) \left\{ 12 \left( \frac{p^2_r}{\mu^2} + \frac{p^2_\varphi}{\mu^2 r^2} \right) - 27 \frac{G M}{r^2} \right\} \]

\[- \frac{1 - 3\nu}{56} (7 \cos^4 \Theta - 6 \cos^2 \Theta + 1) \left[ \cos 2(\Phi - \varphi) \left\{ -\frac{7 G^2 M^2}{r^2} + \frac{G M}{\mu^2 r} \left( 18 p^2_r - 3 \frac{p^2_\varphi}{r^2} \right) \right\} \right] \]

\[- \frac{G M}{\mu^2 r} \left( 18 p^2_r + 13 \frac{p^2_\varphi}{r^2} \right) \]

\[- \frac{1 - 3\nu}{3} \frac{G M p_r}{\mu^2 r^2} \left\{ \frac{4 p^2}{r} \cos 2(\Phi - \varphi) - p_r \sin 2(\Phi - \varphi) \right\} \right] \]. (B.3)

For \( h_\times \) one finds

\[ h^{(0)}_\times = \frac{2 G^2}{D c^4} \cos \Theta \left\{ \sin 2(\Phi - \varphi) \right\} - \frac{p^2_r}{\mu^2} + \frac{p^2_\varphi}{\mu^2 r^2} + \frac{G M}{r} \left( 1 + \frac{3q}{2r^2} \right) \}

\[+ 2 \frac{p_r p_\varphi}{\mu^2 r} \cos 2(\Phi - \varphi) \right\}, \quad (B.4)\]

\[ h^{(1)}_\times = \frac{G}{D c^5 M \mu^2} \frac{\Delta m}{2} \cos 2(\Phi - \varphi) \left\{ \frac{p^2}{r} \cos \Phi - \varphi \right\} \left\{ \frac{G M \mu^2}{2} \cos \frac{\varphi}{r^2} - \frac{p^2}{r^2} \right\} \]

\[+ \frac{p_r p_\varphi}{r^2} \cos 3(\Phi - \varphi) \left\{ \frac{G M \mu^2}{r} - 6 p^2_r + 2 \frac{p^2_\varphi}{r^2} \right\} \]

\[+ \frac{p_r p_\varphi}{r^2} \cos 3(\Phi - \varphi) \left\{ p^2_r - 3 \frac{p^2_\varphi}{r^2} - 2 \frac{G M \mu^2}{r} \right\} \}. \quad (B.5)\]
\[
\begin{align*}
\frac{h_{xx}^{(2)}}{Dc^5} & = G\mu \cos \Theta \left[ \sin 2(\Phi - \varphi) \left\{ \frac{5}{7} \left( 1 - 3\nu \right) \left( p_r^4 - p_q^4 \right) \right\} \\
& + \frac{GM}{\mu^2 r} \left( \frac{3}{7} (23 + 8\nu) p_r^2 - \frac{241 - 72\nu}{21} p_q^2 \right) - \frac{G^2 M^2}{r^2} (10 - \nu) \right] \\
& + \frac{p_r p_q}{\mu^2 r} \cos 2(\Phi - \varphi) \left\{ \frac{10}{7} (3\nu - 1) \left( p_r^2 + p_q^2 \right) \frac{2}{21} (227 - 9\nu) \frac{GM}{r} \right\} \\
& - \frac{1 - 3\nu}{12} \left[ \frac{p_r^4}{\mu^4} - 6 \frac{p_r^2 p_q^2}{\mu^4 r^2} + \frac{p_q^4}{\mu^4 r^2} \right] - \frac{1 - 3\nu}{2 r^2} \sin^2 \Theta \cos 4(\Phi - \varphi) \left\{ \frac{54}{r} \frac{GM}{\mu^2} + 24 \left( \frac{p_q^2}{\mu^2 r^2} - \frac{p_r^2}{\mu^2} \right) \right\} \\
& - \frac{1 - 3\nu}{12} \sin^2 \Theta \sin 4(\Phi - \varphi) \left\{ \frac{7 G^2 M^2}{r^2} + \frac{GM}{r^2} \left( \frac{18 p_r^2}{r^2} - 27 \frac{p_q^2}{r^2} \right) \right\} \\
& + \frac{6}{2} \left\{ \frac{p^4}{\mu^4} - 6 \frac{p^2 p_q^2}{\mu^4 r^2} + \frac{p_q^4}{\mu^4 r^2} \right\} - \frac{1 - 3\nu}{2} \sin^2 \Theta \frac{GM}{r} p_r p_q \cos 2(\Phi - \varphi) \left\{ \frac{12}{r} \frac{p_r^2}{\mu^2} + \frac{p_q^2}{\mu^2 r^2} \frac{27}{r} \frac{GM}{r} \right\} \\
& + \sin 2(\Phi - \varphi) \left\{ - \frac{7 G^2 M^2}{r^2} + \frac{GM}{r^2} \frac{18 p_r^2}{r^2} - \frac{6}{r^2} \frac{p_q^2}{\mu^4 r^2} \right\} + \frac{1 - 3\nu}{6} \frac{GM}{r} \cos 2(\Phi - \varphi) + \frac{4 p_q^2}{\mu^2 r^2} \sin 2(\Phi - \varphi) \right]\right].
\end{align*}
\]

Appendix C. Higher-order gravitational wave forms: analytic expressions

Using the quasi-Keplerian parametrization derived in section 2 it is possible to calculate analytic expressions for the time derivatives of the STF multipole moments entering in the multipole expansion of \(h^{TT}_{ij}\) in equation (27). Defining \(F(u) \equiv 1 - e_r \cos u\) one obtains

\[
S^{(2)21} = \frac{3}{5} \left( m_1 - m_2 \right) \nu (-E)^{3/2} \frac{\sqrt{1 - e^2_r}}{F(u)^2},
\]

\[
S^{(3)30} = -16\mu E^2 \sqrt{\frac{\pi}{105}} (1 - 3\nu) \frac{e_r \sin u \sqrt{1 - e^2_r}}{F(u)^3},
\]

\[
S^{(3)32} = \frac{8}{5} \left( m_1 - m_2 \right) \mu E^2 e^{-2\nu} \sqrt{1 - e^2_r} \frac{e_r \sin u - 4i \sqrt{1 - e^2_r}}{F(u)^3},
\]

while the time derivatives of the mass multipole moments read

\[
I^{(2)20} = -16\mu E \sqrt{\frac{\pi}{15}} \left[ 1 - \frac{1}{F(u)} \left\{ 1 - \frac{q}{2a^2 F(u)^2} \right\} + \frac{E}{F(u)} \delta \\
+ \frac{E}{14c^2} \left\{ 3(3\nu - 1) - \frac{51\nu - 115}{F(u)^2} + \frac{2(19\nu - 4)}{F(u)^2} + 4(\nu - 26) \frac{1 - e^2_r}{F(u)^2} \right\} \right],
\]

\( (C.4) \)
\[ I^{(2)22} = 4 \sqrt{\frac{8\pi}{5}} \mu E e^{-2i\varphi} \left[ -1 + \frac{3}{F(u)} - 2e_r^2 \sin^2 u \frac{F(u)}{(1 - e_r^2 \sin^2 u)^2} + 2i e_r \frac{\sqrt{1 - e_r^2 \sin^2 u}}{F(u)^2} \left( 1 + e_r^2 \frac{\cos u}{1 - e_r^2} \right) \right] + \frac{5q}{2a_r^2 F(u)^3} \]

\[ - E \delta \left\{ \frac{3}{F(u)} + 4e_r^2 \sin^2 u \frac{F(u)}{F(u)^3} + 2i e_r \frac{\sqrt{1 - e_r^2 \sin^2 u}}{F(u)^2} \left( 1 + e_r^2 \frac{\cos u}{1 - e_r^2} - e_r \cos u \right) \right\} \]

\[ + \frac{E}{42c^2} \left\{ 9(3v - 1) - \frac{3(51v - 115)}{F(u)} + \frac{42(8v - 25) - 18e_r^2(3v - 1)}{F(u)^2} - 4(111v - 254) - \frac{1 - e_r^2}{F(u)^3} \right\} \]

\[ + 2i e_r \sin u \frac{(253 - 171v - 3(23v - 87)e_r \cos u)}{\sqrt{1 - e_r^2 F(u)^3}} + (213v - 505)e_r^2 + 9(3v - 1)e_r^3 \cos u) \right] \].

\[ I^{(3)31} = 8 \sqrt{\frac{2\pi}{35}} (m_1 - m_2) v(-E) \frac{1}{3} e^{-i\varphi} \left[ e_r \sin u \frac{F(u)}{F(u)} - \frac{\sqrt{1 - e_r^2}}{F(u)} \left( 1 - \frac{5}{6} \right) \right]. \]

\[ I^{(3)33} = 8 \sqrt{\frac{2\pi}{21}} (m_1 - m_2) v(-E) \frac{1}{3} e^{-3i\varphi} \left[ - e_r \sin u \left( 1 + 4 \left( 1 - e_r^2 \right) \right) \right] \]

\[ + \frac{i}{\sqrt{1 - e_r^2}} \left\{ 3 - \frac{5}{2} \frac{F(u)}{F(u)^3} + \frac{4(1 - e_r^2)}{F(u)^2} \right\} \].

\[ I^{(4)40} = \frac{8}{21} \sqrt{\frac{\pi}{5}} (1 - 3v) \mu E^2 \left[ 6 - \frac{6}{F(u)} - \frac{5}{F(u)^2} + \frac{5(1 - e_r^2)}{F(u)^3} \right]. \]

\[ I^{(4)42} = \frac{8}{65} \sqrt{2\pi} (1 - 3v)^2 \mu E^2 e^{-2i\varphi} \left[ - \frac{6}{F(u)} + \frac{7 - 12e_r^2}{F(u)^2} + \frac{3(1 - e_r^2)}{F(u)^3} \right] \]

\[ - 3i e_r \frac{\sqrt{1 - e_r^2 \sin^2 u}}{F(u)^2} \left\{ 4 + \frac{1}{F(u)} \right\}. \]

\[ I^{(4)44} = \frac{4}{9} \sqrt{\frac{2\pi}{7}} (1 - 3v)^2 \mu E^2 e^{-4i\varphi} \left[ 6 - \frac{6}{F(u)} + \frac{43 - 48e_r^2}{F(u)^2} - \frac{27(1 - e_r^2)}{F(u)^3} + \frac{48(1 - e_r^2)^2}{F(u)^4} \right] \]

\[ + 6i e_r \frac{\sqrt{1 - e_r^2 \sin^2 u}}{F(u)^2} \left\{ 4 + \frac{1}{F(u)} + \frac{8(1 - e_r^2)}{F(u)^2} \right\}. \]

References

[1] Gopakumar A, Memmesheimer R-M and Schäfer G 2004 Third post-Newtonian accurate generalized quasi-Keplerian parametrization for compact binaries in eccentric orbits Phys. Rev. D 70 104011

[2] Königsdörffer C and Gopakumar A 2005 Post-Newtonian accurate parametric solution to the dynamics of spinning compact binaries in eccentric orbits: the leading order spin–orbit interaction Phys. Rev. D 71 024059

[3] Lai D and Wiseman A G 1996 Innermost stable circular orbit of inspiraling neutron-star binaries: tidal effects, post-Newtonian effects, and the neutron-star equation of state Phys. Rev. D 54 3958

[4] Kokkotas K D and Schäfer G 1995 Tidal and tidal–resonant effects in coalescing binaries Mon. Not. R. Astron. Soc. 275 301
[5] Ho W C G and Lai D 1999 Resonant tidal excitations of rotating neutron stars in coalescing binaries *Mon. Not. R. Astron. Soc.* **308** 153

[6] Rasio F A, Lai D and Shapiro S L 1994 Hydrodynamic instability and coalescence of binary neutron stars *Astrophys. J.* **420** 811

[7] Hansen D 2006 Dynamical evolution and leading order gravitational wave emission of Riemann-S binaries *Gen. Rel. Grav.* **38** 1173

[8] Flanagan É E and Hinderer T 2008 Constraining neutron-star tidal Love numbers with gravitational-wave detectors *Phys. Rev. D* **77** 021502(R)

[9] Schäfer G 2004 Gravitomagnetic effects *Gen. Rel. Grav.* **36** 2223

[10] Claret A and Willems B 2002 New results on the apsidal-motion test to stellar structure and evolution including the effects of dynamic tides *Astron. Astrophys.* **388** 518

[11] Johnston S et al 1992 PSR 1259-63—a binary radio pulsar with a Be star companion *Astrophys. J.* **387** L37

[12] Poisson E 1998 Gravitational waves from inspiraling compact binaries: the quadrupole-moment term *Phys. Rev. D* **57** 5287

[13] Damour T and Deruelle N 1985 General relativistic celestial mechanics of binary systems: I. The post-Newtonian motion *Ann. Inst. Henri Poincaré* **43** 107

[14] Wex N 1998 A timing formula for main-sequence star binary pulsars *Mon. Not. R. Astron. Soc.* **298** 67

[15] Barker B M and O’Connell R F 1970 Derivation of the equations of motion of a gyroscope from the quantum theory of gravitation *Phys. Rev. D* **2** 1428

[16] Bildsten L, Lai D and Kaspi V A 1995 Spin–orbit interaction in neutron star/main-sequence binaries and implications for pulsar timing *Astrophys. J.* **452** 819

[17] Cowling T G 1938 On the motion of the apsidal line in close binary systems *Mon. Not. R. Astron. Soc.* **98** 734

[18] Chandrasekhar S 1933 The equilibrium of distorted bodies: II. The tidal problem *Mon. Not. R. Astron. Soc.* **93** 449

[19] Shapiro S L and Teukolsky S A 1983 *Black Holes, White Dwarfs and Neutron Stars* (New York: Wiley)

[20] Schäfer G 1990 Reduced Hamiltonian formalism for general-relativistic adiabatic fluids and applications *Astron. Nachr.* **311** 215

[21] Thorne K S 1980 Multipole expansion of gravitational radiation *Rev. Mod. Phys.* **52** 299

[22] Junker W and Schäfer G 1992 Binary systems: higher order gravitational radiation damping and wave emission *Mon. Not. R. Astron. Soc.* **254** 146

[23] Vecchio A, Willems B and Kalogera V 2007 Probing white dwarf interiors with LISA: periapsron precession in eccentric double white dwarfs Preprint arXiv:0706.3700

[24] Pirani F A E 1965 *Introduction to Gravitational Radiation Theory, Lectures on General Relativity* (Brandeis Summer Institute in Theoretical Physics vol 1) ed F A E Pirani, A Trautmann and H Bondi (Englewood Cliffs, NJ: Prentice-Hall) chapter 2, p 287