Existence of a complete holomorphic vector field via the Kähler–Einstein metric

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Abstract
In this paper, we study the existence of a complete holomorphic vector field on a strongly pseudoconvex complex manifold admitting a negatively curved complete Kähler–Einstein metric and a discrete sequence of automorphisms. Using the method of potential scaling, we will show that there is a potential function of the Kähler–Einstein metric whose differential has a constant length. Then, we will construct a complete holomorphic vector field from the gradient vector field of the potential function.

Keywords The Kähler–Einstein metric · Complete holomorphic vector fields

Mathematics Subject Classification 32Q20 · 32M05 · 53C55

1 Introduction
A fundamental problem in several complex variables is to classify bounded pseudoconvex domains in the complex Euclidean space with a noncompact automorphism group, especially with a compact quotient. A typical result is due to B. Wong’s theorem in [13]: a bounded strongly pseudoconvex domain in \( \mathbb{C}^n \) with a noncompact automorphism group is biholomorphic to the unit ball \( \mathbb{B}^n = \{ z \in \mathbb{C}^n : \|z\| < 1 \} \). J.P. Rosay [11] generalized Wong’s theorem: a bounded domain with an automorphism orbit accumulating at a strongly pseudoconvex boundary point is biholomorphically equivalent to the unit ball. This implies that the unit ball is the biholomorphically unique, smoothly bounded domain with a compact

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quotient. There have been several generalizations of the Wong–Rosay theorem. For instance, in Gaussier–Kim–Krantz [5], the unit ball is also characterized among complex manifolds with strongly pseudoconvex boundary. We will consider in this paper the case of complex manifolds without boundary.

Another important work in this study is due to S. Frankel [4]: a bounded convex domain with a compact quotient is symmetric. A key point in Frankel’s work is to show the existence of an 1-parameter family of automorphisms using the scaling method (for the scaling method, see [8]). In his paper [9], the second named author of this paper introduced a method of potential scaling for bounded pseudoconvex domains in the complex Euclidean spaces. This method is to rescale a canonical potential function of the Kähler–Einstein metric by holomorphic automorphisms and then to construct a certain class of potential functions as a rescaling limit. If a rescaling limit satisfies a specified condition, there is an 1-parameter family of automorphisms. In this paper, we will generalize this method to a complex manifold with a negatively curved complete Kähler–Einstein metric.

Let $X^n$ be a complex manifold of dimension $n$. The automorphism group of $X$, denoted by $\text{Aut}(X)$, is the set of self-biholomorphisms of $X$ under the law of the mapping composition. Throughout this paper, the negatively curved complete Kähler–Einstein metric (simply Kähler–Einstein metric) of $X$ means a complete Kähler–Einstein metric of $X$ with Ricci curvature $-(n+1)$, equivalently, a complete Kähler metric $\omega$ of $X$ with the normalized Einstein condition

$$\text{Ric}_\omega = -(n+1)\omega.$$ 

In a remarkable work by Yau [14], every compact complex manifold with a negative anticanonical class admits a negatively curved complete Kähler–Einstein metric. By Cheng-Yau [1] and Mok-Yau [10], a bounded domain in $\mathbb{C}^n$ admits a Kähler–Einstein metric if and only if the domain is pseudoconvex.

In case of bounded pseudoconvex domains in the complex Euclidean space, the Kähler–Einstein metric has a global potential function. Let $X = \Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with the Euclidean coordinates $z = (z^1, \ldots, z^n)$. The negatively curved complete Kähler–Einstein metric $\omega = i \sum g_{\bar{a}b} dz^a \wedge d\bar{z}^b$ of $\Omega$ has the canonical potential function $\log \det(g_{a\bar{b}})$ in the sense that

$$i\partial \bar{\partial} \log \det(g_{a\bar{b}}) = (n+1)\omega.$$ 

since $\text{Ric}_\omega = -i\partial \bar{\partial} \log \det(g_{a\bar{b}})$. If $\Omega$ is strongly pseudoconvex, as shown in Proposition 4.3 in this paper, the length of $\partial \log \det(g_{a\bar{b}})$ with respect to $\omega$ is continuous up to the boundary:

$$\lim_{p \to \partial \Omega} \left\| \partial \log \det(g_{a\bar{b}}) \right\|_{L^2_\omega}(p) = (n+1)^2.$$ 

If a complex manifold $X^n$ without boundary can be biholomorphically imbedded in $\mathbb{C}^n$ as a strongly pseudoconvex domain, then this noncompact $X$ also admits a negatively curved complete Kähler–Einstein metric $\omega$ and a global potential function $\varphi : X \to \mathbb{R}$ with

$$\lim_{p \to \infty} \left\| \partial \varphi \right\|_{L^2_\omega}(p) = (n+1)^2.$$ 

Here $p \to \infty$ means that $p$ tends to a point of infinity in the one-point compactification of $X$. More precisely, a sequence $\{p_j\}$ in a manifold $X$ converges to the point at infinity, denoted by $p_n \to \infty$ if for any compact subset $K$ in $X$, $p_j \in X \setminus K$ for sufficiently large $j$. 
The main result of this paper is to show the existence of an 1-parameter family of automorphisms in this setting.

**Theorem 1.1** Let $X^n$ be a noncompact complex manifold with the complete Kähler $\omega$ with Ricci curvature $-(n + 1)$. Suppose that there exists a global potential function $\varphi : X \to \mathbb{R}$ in the sense of

$$i\partial \bar{\partial} \varphi = (n + 1)\omega$$

If

1. for any sequence $\{p_j\}$ in $X$ converging to the point at infinity,
   $$\lim_{j \to \infty} \|\partial \varphi\|_{\omega}(p_j) = n + 1;$$
2. there are a sequence of automorphisms $\{f_j\}$ and a point $p_0 \in X$ such that $f_j(p_0) \to \infty$,

then $X$ admits a nowhere vanishing complete holomorphic vector field.

A holomorphic vector field $V$ is a holomorphic section to the $(1, 0)$-tangent bundle of $X$. If a holomorphic vector field $V$ is complete, equivalently its real part $\text{Re} \, V = V + \bar{V}$ is complete, then the flow of $\text{Re} \, V$ is an 1-parameter family of holomorphic transformations of $X$.

In Sect. 2, we introduce the method of potential scaling and prove that there is a global potential function $\bar{\varphi} : X \to \mathbb{R}$ satisfying $\|\partial \bar{\varphi}\|_{\omega}^2 \equiv (n + 1)^2$ (Theorem 2.3). Then, we will prove that there is complete holomorphic vector field $V$ tangent to $\bar{\varphi}$ in Sect. 3 (Theorem 3.2). In the last section, we will discuss a boundary behavior of canonical potential functions in strongly pseudoconvex domains.

## 2 Convergence of Kähler potentials

In this section, we will introduce the method of potential scaling as in [9] and will prove that the manifold $X$ in Theorem 1.1 admits a global potential function $\bar{\varphi}$ such that the length of $\partial \bar{\varphi}$ is constant.

Let $X^n$ be a $n$-dimensional complex manifold with the complete Kähler $\omega$ with Ricci curvature $-(n + 1)$, and let $\varphi : X \to \mathbb{R}$ be a global Kähler potential of $\omega$ in the sense of

$$i\partial \bar{\partial} \varphi = (n + 1)\omega.$$ 

Since every holomorphic automorphism $f \in \text{Aut}(X)$ is an isometry of $\omega$,

$$i\partial \bar{\partial} f^* \varphi = f^* i\partial \bar{\partial} \varphi = (n + 1) f^* \omega = (n + 1)\omega,$$ 

so each pulling-back $f^* \varphi = \varphi \circ f$ is also a potential function. The method of potential scaling is to construct a certain potential function as a limit of sequence of potential functions

$$\varphi \circ f_j - (\varphi \circ f_j)(p_0)$$

for some $f_j \in \text{Aut}(X)$ and $p_0 \in X$. We will mainly consider the convergence of the sequence.

When we define
for convenience, we can write (2.2) by
\[
\varphi \circ f_j - (\varphi \circ f_j)(p_0) = \log \frac{\psi \circ f_j}{(\psi \circ f_j)(p_0)}
\]
so it is sufficient to consider the convergence of \(\psi \circ f_j/(\psi \circ f_j)(p_0)\). For the convergence of the sequence, we need the following estimates.

**Lemma 2.1** Suppose that there is a constant \(C > 0\) with
\[
\| \partial \varphi \|_\omega < C \quad \text{on } X. \tag{2.3}
\]
For any compact subset \(K \subset X\) and a point \(p_0 \in X\), there exists a constant \(A = A(K, p_0) > 0\) such that
\[
\frac{1}{A} < \frac{\psi \circ f}{(\psi \circ f)(p_0)} < A \quad \text{on } K
\]
for any \(f \in \text{Aut}(X)\).

**Proof** The automorphism \(f\) is isometric with respect to \(\omega\) so that \(\| \partial (\psi \circ f) \|_\omega^2 = \| \partial \psi \|_\omega^2 \). Since \(\psi\) is nowhere vanishing on \(X\), we have \(\partial \psi = \psi (\partial \log \psi) = \psi \partial \varphi\); hence,
\[
\| \partial (\psi \circ f) \|_\omega^2 = \| \partial \psi \|_\omega^2 \circ f = (\psi \circ f)^2 \left( \| \partial \varphi \|_\omega^2 \circ f \right) < C^2 (\psi \circ f)^2 \tag{2.4}
\]
by (2.3). When we let
\[
\sigma_f = \frac{\psi \circ f}{(\psi \circ f)(p_0)}
\]
for the convenience, the inequality (2.4) implies that
\[
\left\| \partial \sigma_f \right\|_\omega^2 = \frac{1}{(\psi \circ f)^2(p_0)} \| \partial (\psi \circ f) \|_\omega^2 < C^2 (\sigma_f)^2 \text{.}
\]
For a unit speed curve \(\gamma : (-R, R) \to \Omega\) with respect to \(\omega\) with \(\gamma(0) = p_0\), this inequality can be written by \(|(\sigma_f \circ \gamma)'(t)| \leq C |(\sigma_f \circ \gamma)(t)|\). Since \(\sigma_f(p_0) = 1\), Gronwall’s inequality gives
\[
e^{-Cr} \leq |(\sigma_f \circ \gamma)(t)| \leq e^{Cr} \text{.}
\]
As a conclusion, we have that for a point \(p \in X\) with \(d_\omega(p_0, p) < R\) where \(d_\omega\) is the distance associated with \(\omega\), we get
\[
e^{-CR} \leq \sigma_f(p) \leq e^{CR} \text{.}
\]
This is independent of the choice of \(f \in \text{Aut}(X)\). This completes the proof. \(\square\)

Then, we have the convergence of the potential scaling.
Lemma 2.2 Assume (2.3). Then for any sequence \( \{f_j\} \) of automorphisms of \( X \) and a point \( p_0 \in X \), the sequence of potentials

\[
\{ \log \frac{\psi \circ f_j}{(\psi \circ f_j)(p_0)} \}
\]

has a convergent subsequence in the local \( C^\infty \) topology, so a limit is also a potential function.

Proof Let \( Q_j : X \to \mathbb{R} \) be a positively valued function defined by

\[
Q_j = \frac{\psi \circ f_j}{(\psi \circ f_j)(p_0)} \psi(p_0).
\]

For each compact subset \( K \) of \( X \), Lemma 2.1 implies that there is a constant \( A_K > 0 \) such that

\[
\frac{1}{A_K} \frac{\psi(p_0)}{\psi} < Q_j < A_K \frac{\psi(p_0)}{\psi}
\]

for any \( j \). Moreover for the positive constants \( B_K = \sup_K \psi \) and \( C_K = \inf_K \psi \), we have the uniform estimate

\[
0 < \frac{1}{A_K} \frac{\psi(p_0)}{B_K} < Q_j < A_K \frac{\psi(p_0)}{C_K}
\]

on \( K \). (2.5)

Now we consider the convergence of \( \{Q_j\} \). Equation (2.1) can be written by \( i\partial \bar{\partial} \log(\psi \circ f_j) = (n+1)\omega \) so that

\[
i\partial \bar{\partial} \log Q_j = i\partial \bar{\partial} \log(\psi \circ f_j) - i\partial \bar{\partial} \log \psi = 0.
\]

This means \( \log Q_j \) is pluriharmonic. By the \( \partial \bar{\partial} \)-Poincaré lemma, each \( Q_j \) is locally an absolute square of a holomorphic function. For a small coordinate neighborhood \( \mathcal{U} \), we can take a holomorphic function \( \eta_j : \mathcal{U} \to \mathbb{C} \) with \( Q_j = |\eta_j|^2 \) on \( \mathcal{U} \). From Inequality (2.5), we may assume that \( |\eta_j| \) is pinched by two positive constants on \( \mathcal{U} \) so \( \{\eta_j\} \) has a convergent subsequence in the uniform convergence. Hurwitz’s theorem also says that the limit is nowhere vanishing on \( \mathcal{U} \). Since \( \{Q_j\} \) converges subsequentially to a nowhere vanishing function \( Q \) : \( X \to \mathbb{R} \) in the local \( C^\infty \) topology.

As a conclusion, we have a subsequential limit

\[
\frac{\psi \circ f_j}{(\psi \circ f_j)(p_0)} = \frac{\psi}{\psi(p_0)} Q_j \to \frac{\psi}{\psi(p_0)} \tilde{Q}
\]

in the local \( C^\infty \) topology. Since \( i\partial \bar{\partial} \log \tilde{Q} = 0 \), we have

\[
i\partial \bar{\partial} \tilde{\varphi} = i\partial \bar{\partial} \varphi = (n+1)\omega
\]

where

\[
\tilde{\varphi} = \log \frac{\psi}{\psi(p_0)} \tilde{Q}.
\]
This completes the proof.

Now we suppose that there is a constant $C_0 > 0$ such that the value of $\|\partial \varphi\|_\omega$ at the point at infinity is always $C_0$:

$$\lim_{p \to \infty} \|\partial \varphi\|_\omega^2(p) = C_0$$

Then, the length of $\partial \varphi$ is globally bounded: $\|\partial \varphi\|_\omega < C$ on $X$ for some $C > 0$. If there is a sequence of automorphisms $\{f_j\}$ and a point $p_0 \in X$ such that $f_j(p_0) \to \infty$, Lemma 2.2 gives us a potential rescaling limit

$$\varphi \circ f_j - (\varphi \circ f_j)(p_0) \to \tilde{\varphi}$$

where $\tilde{\varphi}$ satisfies $i\partial \bar{\partial} \tilde{\varphi} = (n + 1)\omega$. For any $p \in X$, we have $f_j(p) \to \infty$ by the completeness of $X$ and

$$\|\partial \tilde{\varphi}\|_\omega(p) = \lim_{j \to \infty} \|\partial (\varphi \circ f_j - (\varphi \circ f_j)(p))\|_\omega(p)$$

$$= \lim_{j \to \infty} \|\partial (\varphi \circ f_j)\|_\omega(p) = \lim_{j \to \infty} \|\partial \varphi\|_\omega(f_j(p))$$

by the convergence of the potential scaling. This means that $\|\partial \tilde{\varphi}\|_\omega \equiv C_0$.

As a conclusion, we have

**Theorem 2.3** Let $X^n$ be a noncompact complex manifold with the complete Kähler $\omega$ with Ricci curvature $-(n + 1)$. Suppose that there exists $\varphi : X \to \mathbb{R}$ such that $i\partial \bar{\partial} \varphi = (n + 1)\omega$ and

$$\lim_{p \to \infty} \|\partial \varphi\|_\omega(p) = C_0.$$

for some constant $C_0$. If $f_j(p_0) \to \infty$ for some $f_j \in \text{Aut}(X)$ and $p_0 \in X$, then there is $\tilde{\varphi} : X \to \mathbb{R}$ with

$$i\partial \bar{\partial} \tilde{\varphi} = (n + 1)\omega \quad \text{and} \quad \|\partial \tilde{\varphi}\|_\omega \equiv C_0.$$

This theorem implies that the space $X$ in Theorem 1.1 admits a global potential function $\tilde{\varphi}$ such that $\|\partial \tilde{\varphi}\|_\omega \equiv n + 1$.

### 3 Existence of complete holomorphic vector fields

In this section, we will study the existence of complete holomorphic vector fields on a negatively curved complete Kähler–Einstein manifold. Let $X^n$ be a $n$-dimensional complex manifold with the complete Kähler $\omega$ with Ricci curvature $-(n + 1)$ and suppose that $i\partial \bar{\partial} \varphi = (n + 1)\omega$ on $X$. In a local coordinate function $z = (z^1, \ldots, z^n)$, we can write

$$\omega = ig_{\alpha \bar{\beta}} dz^\alpha \wedge d\bar{z}^\bar{\beta}.$$

We denote the complex conjugate of a tensor by taking the bar on the indices, that is, $\bar{z}^a = z^\alpha \cdot \bar{g}_{\alpha \bar{\beta}} = g_{a \bar{\beta}}$ and so on. We will also use the matrix of the Kähler–Einstein metric $(g_{\alpha \bar{\beta}})$ and its inverse matrix $(g^{\beta \bar{\alpha}})$ to raise and lower indices: $\theta^a = g_{a \bar{\beta}} \theta^{\bar{\beta}}$, $R^{\beta \bar{a} \mu \nu} = g^{\beta \bar{\alpha}} R_{\beta \mu \nu \bar{\alpha}}$.
The Greek index $\alpha, \beta, \ldots$ runs from 1 to $n$, and the summation convention for duplicated indices is always assumed.

Then, $\|\partial \varphi\|^2_\omega$ can be written by

$$\|\partial \varphi\|^2_\omega = \|\partial \varphi dz^\alpha\|^2_\omega = \varphi_a \varphi_\beta g^{a\beta} = \varphi_a \varphi^a$$

where $\varphi_a = \partial \varphi / \partial z^a$ and $\varphi_\beta = \partial \varphi / \partial z^\beta$.

We denote the Kähler connection with respect to $\omega$ by $\nabla$ and denote the covariant derivative with respect to $\partial / \partial z^\alpha$ and $\partial / \partial z^\beta$ by $\nabla_a, \nabla_\beta$, or we will use the semicolon notation.

In order to show the existence of complete holomorphic vector fields, we need the following PDE equation.

**Proposition 3.1** The norm $u = \|\partial \varphi\|^2_\omega$ satisfies the following PDE:

$$\Delta_\omega u = \|\nabla^2 \varphi\|^2_\omega + n(n+1)^2 - (n+1)u$$

where $\nabla^1$ is the $(1, 0)$-part of $\nabla$ and $\Delta_\omega$ is the Laplace–Beltrami operator with non-positive eigenvalues.

**Proof** In the local coordinates $z = (z^1, \ldots, z^n)$, the identity $i\partial \varphi = (n+1)\omega$ implies that

$$\varphi_{a;\bar{\lambda}\bar{\mu}} = \varphi_\bar{\mu} R^\alpha_{a\bar{\alpha}\lambda\bar{\mu}} \quad \varphi_{\bar{\beta};\lambda\bar{\mu}} = 0 \quad (3.1)$$

where $R^\alpha_{a\bar{\alpha}\lambda\bar{\mu}}$ stands for the curvature tensor: $(\nabla_a \nabla_\beta - \nabla_\beta \nabla_a) \partial / \partial z^a = R^\alpha_{a\bar{\alpha}\lambda\bar{\mu}} \partial / \partial z^\beta$. In fact, since $\nabla$ is a Kähler connection of $\omega$, we have

$$\nabla_{\bar{\beta}} \nabla_\lambda \varphi_\alpha = \nabla_\lambda \nabla_{\bar{\beta}} \varphi_\alpha + \varphi_\bar{\mu} R^\alpha_{a\bar{\alpha}\lambda\bar{\mu}} = \nabla_\lambda \varphi_{a;\bar{\lambda}\bar{\mu}} + \varphi_\bar{\mu} R^\alpha_{a\bar{\alpha}\lambda\bar{\mu}} = (n+1) \nabla_\lambda g_{a\bar{\mu}} + \varphi_\bar{\mu} R^\alpha_{a\bar{\alpha}\lambda\bar{\mu}} = \varphi_\bar{\mu} R^\alpha_{a\bar{\alpha}\lambda\bar{\mu}} ,$$

$$\nabla_{\bar{\mu}} \nabla_\beta \varphi_\alpha = \nabla_\beta \nabla_{\bar{\mu}} \varphi_\alpha + \varphi_\lambda R^\alpha_{\bar{\mu}a\bar{\lambda}} = \nabla_\beta \varphi_{\bar{\beta};\lambda\bar{\mu}} + \varphi_\lambda R^\alpha_{\bar{\mu}a\bar{\lambda}} = (n+1) \nabla_\beta g_{\bar{\mu}\lambda} = 0 .$$

Since $u = \varphi_\gamma \varphi^\gamma$,

$$\Delta_\omega u = g^{a\bar{\beta}} \nabla_{\bar{\beta}} \nabla_a (\varphi_\gamma \varphi^\gamma)$$

$$= g^{a\bar{\beta}} \nabla_{\bar{\beta}} \left( \varphi_{\gamma;\alpha} \varphi^\gamma + \varphi_\gamma \varphi^\gamma_{\gamma;\alpha} \right)$$

$$= g^{a\bar{\beta}} \left( \varphi_{\gamma;\alpha} \varphi^\gamma + \varphi_\gamma \varphi^\gamma_{\gamma;\alpha} + \varphi_{\gamma;\beta} \varphi^\gamma + \varphi_\gamma \varphi^\gamma_{\gamma;\beta} \right).$$

Using $(3.1)$, we have the identity, $\varphi_{\gamma;\alpha} \varphi^\gamma = \varphi_\gamma R^\gamma_{\gamma;\alpha} \varphi^\gamma = \varphi_\gamma R^\gamma_{\gamma;\alpha} \varphi^\gamma$. Since $R^\gamma_{\gamma;\alpha} g^{a\beta} = R_{\gamma\delta} = -(n+1)g_{\gamma\delta}$ from the Einstein condition, we have

$$g^{a\bar{\beta}} \varphi_{\gamma;\alpha} \varphi^\gamma = -(n+1) \varphi_\gamma \varphi^\gamma g_{\gamma\delta} = -(n+1)u$$

The second and third terms above can be written by
The last term is vanishing: 
\[ \varphi_{\gamma}^{\alpha} \varphi_{\gamma}^{\beta} = \varphi_{\gamma}^{\beta} \varphi_{\gamma}^{\alpha} = (n + 1)^2 g_{\gamma}^{\beta}g_{\gamma}^{\alpha} = n(n + 1)^2. \]

Theorem 3.2 Let \( \omega \) be the complete Kähler–Einstein metric on \( X \) with Ricci curvature \(-(n + 1)\). Suppose that there exists a global potential \( \varphi \) of \( \omega \) in the sense of \( i\partial\bar{\partial}\varphi = (n + 1)\omega \) such that

\[ \|\partial\varphi\|_{\omega} \equiv n + 1. \]

Then, the vector field

\[ V = i e^{\frac{\varphi}{n+1}} \text{grad}(\varphi) \]  

(3.2)
is a complete holomorphic vector field.

Here \( \text{grad}(\varphi) \) is the \((1, 0)\)-part of the gradient vector field of \( \varphi \) with respect to \( \omega \):

\[ \text{grad}(\varphi) = \varphi^{\alpha} \frac{\partial}{\partial z^{\alpha}} \]

where \( \varphi^{\alpha} = \varphi_{\beta} g^{\beta\alpha} \) (7). When we denote by \( W = i\text{grad}(\varphi) \), then \( V = e^{\frac{\varphi}{n+1}} W \). Since the vector field \( W \) is the turned gradient of \( \varphi \) by the complex structure, \( W \) is tangent to \( \varphi \):

\[ \text{(Re } W\text{)} \varphi = (i\varphi^{\alpha} \frac{\partial}{\partial z^{\alpha}} - i\varphi^{\bar{\alpha}} \frac{\partial}{\partial z^{\bar{\alpha}}}) \varphi = (i\varphi^{\alpha} \varphi_{\alpha} - i\varphi^{\bar{\alpha}} \varphi_{\bar{\alpha}}) = 0. \]

So we have

\[ \text{(Re } W\text{)} e^{\frac{\varphi}{n+1}} = 0. \]

Moreover, \( W \) has constant length \( \|W\|_{\omega} = \|\partial\varphi\|_{\omega} \), so the real tangent vector field \( \text{Re } W \) is complete. Therefore, the following lemma implies that \( V = e^{\frac{\varphi}{n+1}} W \) is also complete.

Lemma 3.3 Let \( Z \) is a complete \((1, 0)\) tangent vector field on \( X \). If there is a nowhere vanishing smooth function \( \rho : X \to \mathbb{R} \) with \( \text{(Re } Z\text{)} \rho \equiv 0 \), then \( \rho Z \) is also complete.

Proof Take an integral curve \( \gamma : \mathbb{R} \to X \) of \( \text{Re } Z \). It satisfies

\[ \text{(Re } Z\text{)} \circ \gamma = \dot{\gamma} \]

Since \( \text{(Re } Z\text{)} \rho \equiv 0 \), the curve \( \gamma \) lies on a level set of \( \rho \) so \( \rho \circ \gamma \equiv c \) for some constant \( c \). For the curve \( \sigma : \mathbb{R} \to X \) defined by \( \sigma(t) = \gamma(ct) \), we have
This means that \( \sigma : \mathbb{R} \to \Omega \) is the integral curve of \( \rho \text{Re} Z \); therefore, \( \rho \text{Re} Z \) is complete.

Now we will prove that \( V \) in (3.2) is holomorphic. The hypothesis and Proposition 3.1 imply that \( \| \nabla'' \varphi \|_\omega^2 = \varphi_{a,\bar{b}} \varphi_{a,\bar{b}} = (n + 1)^2 \). On the other hand, we have

\[
0 = \partial \left( \| \partial \varphi \|_\omega^2 \right) = (\varphi_a \varphi^a)_{\bar{a}} d\bar{z}^a = \left( \varphi_a \varphi^a + \varphi_a \varphi^a_{\bar{a}} \right) d\bar{z}^a = \left( \varphi_a \varphi^a + (n + 1)\varphi_a \varphi^a_{\bar{a}} \right) d\bar{z}^a
\]

It follows that

\[
\varphi_{a,\bar{a}} \varphi^a = -(n + 1)\varphi_a.
\]  

Denote by \( \psi = \exp \varphi \). Recall that \( V \) is defined as follows:

\[
V = i V^a \frac{\partial}{\partial z^a} \quad \text{where} \quad V^a = e^{\frac{1}{n+1} \varphi^a} = \psi^{\frac{1}{n+1}} \varphi^a.
\]

It follows that

\[
\nabla'' V = i V^a \frac{\partial}{\partial z^a} \otimes d\bar{z}^\beta
\]

where \( \nabla'' \) is the \((0, 1)\) part of the Kähler connection \( V \) and

\[
V^a_{a,\bar{b}} = \frac{1}{n + 1} \psi^{\frac{1}{n+1}} (\log \psi)_{\bar{b}} \varphi^a + \psi^{\frac{1}{n+1}} \varphi^a_{a,\bar{b}}
\]

A straightforward computation gives that

\[
\| \nabla'' V \|_\omega^2 = \frac{1}{(n + 1)^2} (\psi^{\frac{1}{n+1}})^2 (\varphi_a \varphi^a)^2 + \frac{1}{n + 1} (\psi^{\frac{1}{n+1}})^2 \varphi_{\bar{a}} \varphi_{\bar{a}} \varphi^a_{a,\bar{b}} + \frac{1}{n + 1} (\psi^{\frac{1}{n+1}})^2 \varphi^a_{a,\bar{b}} \varphi^a_{\bar{b},\bar{a}} + (\psi^{\frac{1}{n+1}})^2 \varphi_{\bar{a}} \varphi_{\bar{a}} \varphi_{\bar{a}} \varphi_{\bar{a}} + (n + 1)^2 \| \partial \varphi \|_\omega^2 + (n + 1)^2.
\]

It follows from Proposition 3.1 and (3.3) that

\[
\| \nabla'' V \|_\omega^2 = (\psi^{\frac{1}{n+1}})^2 \left( \frac{1}{(n + 1)^2} \| \partial \varphi \|_\omega^4 + 2 \frac{(n + 1)}{n + 1} \| \partial \varphi \|_\omega^2 + (n + 1)^2 \right) = 0.
\]

This implies that \( V \) is holomorphic.

Combining Theorem 2.3 and Theorem 3.2, we obtain Theorem 1.1.

**Remark 3.4** Suppose that \( \varphi : X \to \mathbb{R} \) satisfies \( i \partial \bar{\partial} \varphi = (n + 1)\omega \) and

\[
\| \partial \varphi \|_\omega \equiv C
\]
for some constant $C > 0$. The bundle morphism $S : T^{1,0}X \to T^{1,0}X$ defined locally by

$$S = \phi^{\alpha}_{a,\beta} \phi^\beta_{\beta;\alpha} \frac{\partial}{\partial z^\alpha} \otimes dz^\gamma$$

is a positive semidefinite symmetric operator. Equation (3.3) implies that $\text{grad}(\varphi)$ is a field of eigenvectors of $S$ with constant eigenvalue $(n + 1)^2$:

$$S(\text{grad}(\varphi)) = S \left( \phi^\gamma \frac{\partial}{\partial z^\gamma} \right) = \phi^{\alpha}_{a,\beta} \phi^\beta_{\beta;\alpha} \phi^\gamma \frac{\partial}{\partial z^\alpha} = -(n + 1)\phi^{\alpha}_{a,\beta} \phi^\beta_{\beta;\alpha} \frac{\partial}{\partial z^\alpha}$$

$$= (n + 1)^2 \phi^\alpha \frac{\partial}{\partial z^\alpha} = (n + 1)^2 \text{grad}(\varphi).$$

Since the value $\left\| \nabla^2 \varphi \right\|_o^2 = \varphi^{\alpha;\beta} \varphi_{\beta;\alpha} = \varphi^{\alpha}_{a,\beta} \varphi^\beta_{\beta;\alpha}$ coincides with the trace of $S$, we have $\left\| \nabla^2 \varphi \right\|_o^2 \geq (n + 1)^2$. Then, the PDE equation in Proposition 3.1 implies

$$(n + 1)C^2 = \left\| \nabla^2 \varphi \right\|_o^2 + n(n + 1)^2 \geq (n + 1)^3$$

so

$$C \geq n + 1.$$ 

For the case of the unit ball $\mathbb{B}^n = \{ z \in \mathbb{C}^n : ||z|| < 1 \}$ where $z = (z^1, \ldots, z^n)$ is the standard coordinates and $\| \cdot \|$ is the Euclidean norm, the function

$$\varphi = (n + 1) \log \frac{|1 + z^1|^2}{(1 - ||z||^2)}$$

satisfies $i \partial \overline{\partial} \varphi = (n + 1)\omega$ and $\| \partial \varphi \|_o \equiv n + 1$. This is an optimal case of the inequality $C \geq n + 1$.

4 Boundary behavior of the Kähler–Einstein metric on a strongly pseudoconvex domain

In this section, we shall compute the boundary behavior of $\| \partial \varphi \|_o^2$ on a bounded strongly pseudoconvex domain. First, we briefly recall the boundary behavior of the solution of the complex Monge–Ampere equation due to Cheng and Yau [1].

Let $\Omega$ be a smooth bounded strongly pseudoconvex domain in $\mathbb{C}^n$. Then, there exists a defining function $r$ of $\Omega$ satisfying the following conditions:

(I) $r \in C^\infty(\overline{\Omega})$,

(II) $\Omega = \{ z \in \mathbb{C}^n : r(z) < 0 \}$,

(III) $\partial r \neq 0$ on $\partial \Omega$, and

(IV) $(r_{a\beta}) > 0$ in $\Omega$.

Denote by $w = - \log(-r)$. Then, $w$ is a strictly plurisubharmonic function defined in $\Omega$. Easy calculations show that

$$W_{a\beta} = \frac{r_{a\beta}}{-r} + \frac{r_a r_{\beta}}{r^2}, \quad (4.1)$$

and the inverse is

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where

\[(r^a)_\beta = \left(r_{\alpha \beta}\right)^{-1}, \quad r^\alpha = r^{a\beta} r_\beta, \quad \text{and} \quad |\partial r|^2 = r^{a\beta} r_\alpha r_\beta.\]

It is also easy to see that

\[w^{\beta a} w_\alpha w_\beta = \frac{|\partial r|^2}{|\partial r|^2 - r} \leq 1.\]

Thus, the metric \(w_{\alpha \beta}\) is a complete Kähler metric on \(\Omega\). Moreover, we have the following.

\[\det(w_{\alpha \beta}) = \left(\frac{1}{-r}\right)^{n+1} \det(\varphi_{\alpha \beta})(-\varphi + |d\varphi|^2).\]  

(4.3)

Hence, the Ricci tensor of \(w_{\alpha \beta}\) is given by

\[R_{\alpha \beta} = -\frac{\partial^2 F}{\partial z^\alpha \partial \bar{z}^\beta} \log \det(w_{\gamma \delta})\]

\[= -(n + 1)w_{\alpha \beta} - \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} \log \det(r_{\gamma \delta})(-r + |\partial r|^2).\]  

(4.4)

If we denote by \(F = \log \det(r_{\alpha \beta})(-r + |\partial r|^2)\), then \(F\) is a positive smooth function in \(\overline{\Omega}\) satisfying

\[R_{\alpha \beta} + (n + 1)w_{\alpha \beta} = \frac{\partial^2 F}{\partial z^\alpha \partial \bar{z}^\beta}.\]

It is remarkable to note that the function \(F\), which measures how far the metric \(w_{\alpha \beta}\) is from the Kähler–Einstein metric, depends on the defining function \(r\). The Cheng-Yau theorem implies that there exists a solution to the following complex Monge–Ampere equation:

\[\det(w_{\alpha \beta} + u_{\alpha \beta}) = e^{(n+1)u} e^F \det(w_{\alpha \beta})\]

\[\frac{1}{c}(w_{\alpha \beta}) \leq (w_{\alpha \beta} + u_{\alpha \beta}) \leq c(w_{\alpha \beta}).\]  

(4.5)

It is easy to see that \(w_{\alpha \beta} + u_{\alpha \beta}\) is the unique complete Kähler–Einstein metric on \(\Omega\). In [3], Fefferman developed a way to find a good defining function \(r\), which is called an approximate solution of the Monge–Ampere equation. This defining function \(r\) satisfies that

\[F = \log \det(r_{\alpha \beta})(-r + |\partial r|^2) = O(|r|^{n+1})\]

Using this approximate solution, Cheng and Yau computed the boundary behavior of \(u\).

**Theorem 4.1** (Simple version [1]) Let \(\Omega\) be a smooth strongly pseudoconvex domain in \(\mathbb{C}^n\) and let \(r\) be a smooth defining function of \(\Omega\). Suppose that \(F = O(|r|^{n+1})\) and \(u\) is a solution of (4.5). Then

\[|D^\beta u|(x) = O(|r|^{n+1/2 - p - t}) \quad \text{for} \quad \epsilon > 0\]
where $|D^pu|(x)$ is the Euclidean length of the $p$-th derivative of $u$.

In particular, Theorem 4.1 says that

$$|u_\alpha| = O(|r|^{n-1/2-\varepsilon}) \quad \text{and} \quad |u_\beta| = O(|r|^{n-1/2-\varepsilon})$$

for $\varepsilon > 0$ and $1 \leq \alpha, \beta \leq n$. Before computing the boundary behavior of $\|\partial \varphi\|_\omega$, we introduce the following lemma.

**Lemma 4.2** ([2]) There exists a Hermitian $n \times n$ matrix

$$N = (N_{\alpha \beta}) \in \Mat_{\omega \omega} \left( C^\infty (\Omega) \cap C^{n-3/2-\varepsilon} (\overline{\Omega}) \right)$$

with $\|N\| = O(|r|^{n-3/2-\varepsilon})$ for $\varepsilon > 0$, which satisfies that

$$g^{\bar{\beta} a} - w^{\bar{\beta} a} = w^{\bar{\gamma} r} N_{\gamma \delta} w^{\delta a}.$$  

In particular, $g^{\bar{\beta} a} \in C^\infty (\Omega) \cap C^{n-3/2-\varepsilon} (\overline{\Omega})$ and $g^{\bar{\beta} a} = O(|r|)$ for $\varepsilon > 0$.

Now we consider the boundary behavior of $\|\partial \varphi\|_\omega^2$ near the boundary. Note that $\varphi = (n+1)g = (n+1)(w+u)$. It follows from Lemma 4.2 that

$$\|\partial \varphi\|_\omega^2 = \varphi_\alpha \varphi_\beta g^{\alpha \beta} = (n+1)^2 (w+u)_\alpha (w+u)_\beta g^{\alpha \beta}$$

$$= (n+1)^2 (w+u)_\alpha (w+u)_\beta \left( w^{\bar{\beta} a} + w^{\bar{\gamma} r} N_{\gamma \delta} w^{\delta a} \right)$$

$$= (n+1)^2 \left( w_\alpha w_\beta w^{\alpha \beta} + w_\alpha u_\beta w^{\alpha \beta} + u_\alpha w_\beta w^{\alpha \beta} + (w+u)_\alpha (w+u)_\beta w^{\bar{\gamma} r} N_{\gamma \delta} w^{\delta a} \right) .$$

It follows from (4.1), (4.2) and (4.6) that

$$w_\alpha = O(|r|^{-1}) , \quad w^{\bar{\beta} a} = O(|r|) , \quad u_\alpha = O(|r|^{n-1/2-\varepsilon}) , \quad N^{\gamma \delta} = O(|r|^{n-3/2-\varepsilon}) ,$$

thus we have

$$\|\partial \varphi\|_\omega^2 = (n+1)^2 \frac{|\partial r|^2}{|\partial r|^2 - r} + O(|r|).$$

On the other hand, plugging the choice of $F$ and (4.3) into (4.5), it follows that

$$\log \det (g_{\alpha \beta}) = (n+1)g .$$

Therefore, we have

**Proposition 4.3** Let $\Omega$ be a bounded strongly pseudoconvex domain with smooth boundary. Let $\omega = i \sum g_{\alpha \beta} dz^\alpha \wedge d\bar{z}^\beta$ be the unique complete Kähler–Einstein metric on $\Omega$. Then

$$\|\partial \varphi\|_\omega^2 \to (n+1)^2 \quad \text{as} \quad p \to \partial \Omega ,$$
where \( \varphi = \log \det(g_{\alpha\beta}) \).

**Remark 4.4** The boundary behavior of the solution \( u \) of (4.5) is easily generalized to a relatively compact strongly pseudoconvex domain with smooth boundary in a Kähler manifold by [12]. Moreover, it is also localized. More precisely, near a strongly pseudoconvex boundary point of a bounded pseudoconvex domain, one can obtain the same boundary behavior of the solution ( [6]).

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