On the $i\phi^3$ $\mathcal{P}\mathcal{T}$-symmetric Scalar Field Theory

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Abstract

In this work, we show that, for the $i\phi^3$ scalar field theory, there exists a contradiction between the assumption that the field is real and the fact that the quantized as well as the classical fields have to satisfy the Klein-Gordon equation. In solving the Klein-Gordon equation for the theory under investigation, we realized that the field is a pure imaginary solitary wave which spoils out the non-Hermiticity of the theory. Thus, instead of being non-Hermitian, the $i\phi^3$ scalar field theory is a kind of a Hermitian-Lee-Wick theory which suffers from the existence of the famous ghost states and instability problems. We applied a Canonical transformation to obtain a Non-Hermitian and non-$\mathcal{PT}$-symmetric representation which leads to the invalidity of the previous trials in the literature to cure the ghost states problem. Moreover, the solitonic solution is a non-topological one which is a very strange result to appear for a one component field theory. To account for this strange result, we conjecture that the $i\phi^3$ scalar field theory has an equivalent Hermitian and non-Lee wick theory that have a conserved Noether current.

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The existence of challenging problems in physics makes the investigation of new trends in physics more than important. Among the new ideas that deserve the draw of our attention is the emergence of possible physical applications of some non-Hermitian models \[1–14\]. Some of these models show up interesting properties like asymptotic freedom in the $\mathcal{PT}$-symmetric $-\phi^4$ scalar field theory \[13–17\]. This property by itself strongly recommends the employment of the $\mathcal{PT}$-symmetric $-\phi^4$ theory to play the role of a strongly interacting Higgs mechanism in the standard model of particle interactions \[18\]. However, unlike the quantum mechanical non-Hermitian models, the progress in the study of non-Hermitian field theoretic models is slow due to the existence of two main technical problems. The first problem is that the complex contour method applied successfully in quantum mechanical problems \[19–24\] is hard (if not impossible) to be applied for the quantum field models. The second problem is the lack of existence of a simple algorithm for the calculation of the metric operator for field theories \[25\]. In fact, for the calculation of physical observables in a non-Hermitian theory (with real spectrum), the metric operator calculation is indispensable. Concerning the first problem, we have shown that the famous effective field approach can be applied successfully for such theories \[13\]. For the second problem, we introduced a new ansatz for the metric operator which is local in the fields and can be extended to higher orders easily \[14, 26\].

A prototype example of a $\mathcal{PT}$-symmetric \[42\] field theory is the $i\phi^3$ scalar field theory. Against all the previous treatments carried out to study the model \[14, 22, 25\], in this work, we show that the model is in fact a Lee-Wick \[27, 28\] Hermitian theory. The problem is manifested by the priori assumption of the reality of the field without any reference to the solution of the corresponding Klein-Gordon equation. In fact, the quantized as well as the classical fields have to satisfy the Klein-Gordon equation for the theory under investigation. For the $i\phi^3$ scalar field theory, it is well known that \[29\],

$$\dot{\pi} = -i [\pi, H] = (\nabla^2 - m^2 - 3ig\phi) \phi,$$

(1)

where $\pi$ is the canonical momentum field conjugate to $\phi$. In other words, the quantized field $\phi$ has to satisfy the klein-Gordon equation. This important realization, will lead to a new vision which is introduced in this work for the first time about the treatment of the $\mathcal{PT}$-symmetric $i\phi^3$ scalar field theory. The new vision will prove the validity of our ansatz in Ref. \[14\] as well as will invalidate the other trials carried out for the $C$ operator calculations \[22, 25\] of the $\mathcal{PT}$-symmetric $i\phi^3$ scalar field theory.
The backbone for the calculation of observables in non-Hermitian theories with real spectra is the metric operator $\eta$. This operator has the property $H^\dagger = \eta H \eta^{-1}$, where $H$ is a Hamiltonian operator and $H^\dagger$ is its adjoint [11, 12]. Here $\eta$ is Hermitian, invertible and positive definite metric operator. In fact, $\eta$ is not unique [30–35] and thus one can find different forms for $\eta$ for the same Hamiltonian operator $H$. While the metric operator calculations go easily in the quantum mechanical case, the calculations are very complicated for the quantum field versions. Moreover, the accomplished calculations which so far are believed to be valid are non-local and can be done in a closed form but for first order approximation and for 1 + 1 space-time dimensions only [22, 25].

The lack of existence of a simple form for metric operator in field theory may lead to the following question; is the extension from the quantum mechanical version to the quantum field version can go straight as done in the literature [22, 25]? To answer this question, let us consider the quantum mechanical theory with Hamiltonian $H$ such that:

$$H = \frac{p^2}{2} + \frac{1}{2}m^2x^2 + igx^3,$$

where $m$ is the mass parameter and $g$ is the coupling constant. In the literature, it is assumed that $x$ is real and thus the Hamiltonian is non-Hermitian but $\mathcal{PT}$-symmetric which means that the spectrum is real. For the quantum field version of this theory, we have the Hamiltonian form:

$$H = H_0 + gH_I$$

$$H_0 = \frac{1}{2} \int d^3x \left( \pi^2 + (\nabla \phi)^2 + m^2\phi^2 \right),$$

$$H_I = i \int d^3x \phi^3,$$ (2)

where $\phi$ is the field operator and $\pi$ is its canonical conjugate. In fact, $\phi$ does not represent an observable and one cannot have a priori assumption that it is real as in the case of the quantum mechanical version where $x$ can be taken real as it represents an observable. In fact, a priori assumption of the reality of the field $\phi$ in the $\mathcal{PT}$-symmetric field theory with cubic interaction will lead to a contradiction. To explain this, we consider the equation

$$\dot{\pi} = -i [\pi, H],$$ (3)

which is nothing but the klein-Gordon equation for the field $\phi$. In other words, the quantized
field for the $i\phi^3$ scalar field theory has to satisfy the Klein-Gordon equation of the form:

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + m^2 \phi + 3ig\phi^2 = 0. \quad (4)$$

In the rest frame and in $1 + 1$ space-time dimensions, this equation can take the form:

$$- \frac{d^2 \phi}{dx^2} + m^2 \phi + 3ig\phi^2 = 0. \quad (5)$$

in multiplying by $\frac{d\phi}{dx}$, one can have the relation between $\phi$ and $x$ as

$$x = \int \frac{1}{\sqrt{m^2\phi^2 + 2ig\phi^3}} d\phi, \quad (6)$$

or

$$\phi(x) = \frac{2im^2}{g} \frac{e^{mx}}{(e^{mx} + 1)^2},$$

$$= \frac{im^2}{2g} \text{sech}^2 \left( \frac{mx}{2} \right). \quad (7)$$

For the time dependent solution, one can boost the static solution above to get the result:

$$\phi(x, t) = \frac{2im^2}{g} \frac{e^{\gamma m(x-vt)}}{(e^{\gamma mx} + 1)^2},$$

$$= \frac{im^2}{2g} \text{sech}^2 \left( \frac{\gamma m(x-vt)}{2} \right). \quad (8)$$

where $\gamma$ is the Lorentz factor and $v$ is the velocity. One can easily check that the above solution satisfies the Klein-Gordon equation given by:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + m^2 \phi + 3ig\phi^2 = 0.$$

Apart from the operator characteristic of the field which is employed as Fourier coefficients
in the Fourier transform of the field, the important information in this solution is that the
quantized field is a pure imaginary solitary wave as shown in Fig.1. This realization is
important as solitary waves are candidates for the description of Hadrons as they retain
their shapes after collision [36].

The anti-soliton solution, $\phi^\dagger = \frac{-im^2}{2g} \text{sech}^2 \left( \frac{mx}{2} \right)$, belongs to the theory with opposite sign of the coupling ($-ig\phi^3$). With the soliton for $ig\phi^3$ theory and the antisoliton solution for $-ig\phi^3$ theory, both theories merge into an equivalent one Lee-Wick theory. Since the soliton
solutions $\phi$ and $\phi^\dagger$ have the boundary conditions;

$$\phi_{\pm \infty} = 0, \quad \phi^\dagger_{\pm \infty} = 0,$$
they are non-topological ones. This is a very strange result because non-topological solitons are characterizing theories with more than one component [36]. Accordingly, one may conjecture that the theory under investigation has an equivalent Hermitian theory which includes both $\phi$ and $\phi^+$ with a conserved Noether current.

In view of the features of the obtained solution in Eq.(8), the Hamiltonian in Eq.(2) is $\mathcal{PT}$-symmetric as well as Hermitian and thus the metric operator is the Identity operator which means that all the previous trials to calculate the $C$ operator for the theory at hand were invalid [22, 25]. However, this Hamiltonian suffers from the existence of ghost states (as a Lee-Wick theory) and stability problems. In fact, the presence of such problems does not mean that the theory is physically unacceptable. For instance, Bender et al. have treated the swanson model and the Lee model for which such kind of problems exist [37, 38].

To start the algorithm of curing the ghost states and instability problems in the $i\phi^3$ model, we use the map

$$\zeta(x) = \int dx \exp \left(\frac{i\ln b}{2} (\pi(x) \phi(x) + \phi(x) \pi(x))\right),$$

where $b$ is a constant $C-number$. In using Baker–Campbell–Hausdorff formula, one can obtain the following relations;

$$\zeta(x) \phi(y) \zeta^{-1}(x) = b\phi(x),$$
$$\zeta(x) \pi(y) \zeta^{-1}(x) = \frac{\pi(x)}{b},$$

Accordingly,

$$\zeta(x) H(y) \zeta^{-1}(x) = -\frac{1}{2} \int d^3x \left( (\nabla\phi)^2 + \pi^2 + m^2\phi \right) + \int d^3x \phi^3,$$

where $b = -i$. One may concludes that this form is equivalent to the form in the literature with the assumption that the field is real from the very beginning. In fact, this conclusion is not correct as the Hamiltonian in Eq.(9) is no longer $\mathcal{PT}$-symmetric and thus the $C$-operator regime followed in the literature collapses [22, 25]. As the theory has a real spectrum, one can find an equivalent Hermitian as well as positive definite theory via the calculation of the metric operator.

In Refs. [14, 26], we introduced a new ansatz for the metric operator for $i\phi^3$ scalar field theory and showed that the $Q_1$ operator has the form;
\[ Q_1 = C_1 \int d^dz \pi^3(z) + C_2 \int d^dz \phi(z) \pi(z) \phi(z) + C_3 \int d^dz \nabla \phi(z) \pi(z) \nabla \phi(z), \]  
\tag{10} \]

where \( C_i \) are real parameters to be adapted for \( Q_1 \) to satisfy Eq. (11) below. Note that, the relation \( H^\dagger = \eta H \eta^{-1} \) has to exist and thus the first order correction for the metric operator can be obtained from the relation;

\[-2H_I = [-Q_1, H_0], \]  
\tag{11} \]

where

\[ \eta = \exp(-Q), \]
\[ Q = Q_0 + gQ_1 + g^2Q_2 + + g^3Q_3 + \ldots \]

Now, back to the soliton solution, in 1+1 space-time dimensions, one can obtain its mass as;

\[ M = -\frac{1}{4} \frac{m^6}{m} \int_{-\infty}^{\infty} \frac{\cosh^2 \frac{1}{2} x - 1}{g^2 \cosh^6 \frac{1}{2} x} dx \]
\[ = -\frac{2}{15} \frac{m^5}{g^2}, \]

with the classical energy given by \( E = \gamma M \). This means that the soliton bears a particle characteristics. In fact, this is not the first time to have negative masses for a \( P\mathcal{T} \)-symmetric theory \[39\] but its appearance here in the classical theory is a reminiscent of the ghost-states but certainly it would be cured in the quantized one via the use of the metric operator. Also, this results agrees with the negative central charge of a closely related model studied in Ref. [40].

For Higher space-time dimensions one gets the the solution;

\[ \phi(x, y, z) = \frac{im^2}{2g} \text{sech}^2 \left( \frac{m(x + y + z)}{2\sqrt{3}} \right). \]

Although this is finite everywhere in the space, the classical energy is infinite for the higher dimensions which agrees well with Derrick’s theorem \[41\].

For the quantization of the classical sliton, one can expand the quantized field around the static solution and use the traditional quantum field methods to calculate the amplitudes \[36\], which will be a research topic for us in the near future.
To conclude, we discovered that the now known as $\mathcal{PT}$-symmetric and non-Hermitian scalar field theories are in fact a kind of Hermitian Lee-Wick theories. For that, we have solved the Klein-Gordon equation for the $i\phi^3$ scalar field theory. For this theory, both the classical and the quantized fields have to verify the Klein-Gordon equation. We found that the field is pure imaginary and thus spoils the non-Hermiticity of the theory. Accordingly, the theory is Hermitian but suffers from existence of instability and ghost states problems. To treat these problems, we applied a Canonical transformation to transform the now Hermitian $i\phi^3$ theory into a non-Hermitian theory with positive kinetic term. Though the form obtained looks like the traditional one in the literature, a crucial difference exists as our form is not $\mathcal{PT}$-symmetric and thus the the $C$-operator regime followed in the literature to cure the ghost states problem is no longer working. Another reason that this regime is not working is that we have shown that the integration $\int d^dx \nabla^2 \phi(x) \pi^2$ vanishes while it has been set to non-zero value in the literature.

The soliton solutions obtained in this work are non-perturbative since they are singular at the limit $g \to 0$. Accordingly, in quantizing the theory around the classical solutions, the physical quantities receives non-perturbative corrections and thus turn them more accurate than the quantization around the trivial vacuum. Moreover, the solitons are non-topological and in order to account for the stability of the solitons we conjecture that the mother (equivalent) Hermitian theory has a conserved Noether current. In fact, the current algorithms in the literature can not lead to such kind of expectations and the theory deserves more careful analysis.

We think that the novel trend followed in this work will lead to a progress in building up a concrete formulation of a strongly interacting scalar Higgs mechanism.

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FIG. 1: The field $-i\psi$ (solitary wave) as a function of $x$ at $t = 0$, $t = 0.5$ and $t = 1$ for $v = 0.7$ from left to right respectively.