Global Strong Solutions to Compressible Navier-Stokes System with Degenerate Heat Conductivity and Density-Depending Viscosity

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Abstract

We consider the compressible Navier-Stokes system where the viscosity depends on density and the heat conductivity is proportional to a positive power of the temperature under stress-free and thermally insulated boundary conditions. Under the same conditions on the initial data as those of the constant viscosity and heat conductivity case ([Kazhikhov-Shelukhin. J. Appl. Math. Mech. 41 (1977)], we obtain the existence and uniqueness of global strong solutions. Our result can be regarded as a natural generalization of the Kazhikhov’s theory for the constant heat conductivity case to the degenerate and nonlinear one under stress-free and thermally insulated boundary conditions.

Keywords: Compressible Navier-Stokes system; Density-depending viscosity; Degenerate heat conductivity; Stress-free

1 Introduction

The compressible Navier-Stokes system which describes the one-dimensional motion of a viscous heat-conducting perfect polytropic gas is written in the Lagrange variables in the following form (see [4,20]):

\[ v_t = u_x, \]

\[ u_t + P_x = \left( \mu \frac{u_x}{v} \right)_x, \]

\[ \left( e + \frac{1}{2} u^2 \right)_t + (Pu)_x = \left( \kappa \theta_x + \mu uu_x \right)_x, \]

where \( t > 0 \) is time, \( x \in \Omega = (0,1) \) denotes the Lagrange mass coordinate, and the unknown functions \( v > 0, u \) and \( P \) are, respectively, the specific volume of the gas, fluid velocity, and pressure. In this paper, we concentrate on ideal polytropic gas, that is, \( P \) and \( e \) satisfy

\[ P = R\theta/v, \quad e = c_v\theta + \text{const}, \]

\[ \kappa \theta_x + \mu uu_x \]

where \( \kappa \) is the heat conductivity, \( \mu \) is the viscosity, \( c_v \) is the specific heat at constant volume, and \( R \) is the gas constant. The system is subject to the initial conditions

\[ v|_{t=0} = v_0, \quad u|_{t=0} = u_0, \quad P|_{t=0} = P_0 \]

and the boundary conditions

\[ v|_{x=0} = v_0, \quad P|_{x=1} = P_1, \quad u|_{x=1} = u_1, \]

where \( v_0, u_0, P_0, v_1, u_1, P_1 \) are given functions.

Our result can be regarded as a natural generalization of the Kazhikhov’s theory for the constant heat conductivity case to the degenerate and nonlinear one under stress-free and thermally insulated boundary conditions.
where both specific gas constant $R$ and heat capacity at constant volume $c_v$ are positive constants. We also assume that $\mu$ and $\kappa$ satisfy

$$
\mu = \tilde{\mu} \left( 1 + v^{-\alpha} \right), \quad \kappa = \tilde{\kappa} \theta^\beta,
$$

with constants $\tilde{\mu}, \tilde{\kappa} > 0$ and $\alpha, \beta \geq 0$. The system (1.1)-(1.5) is supplemented with the initial conditions

$$
(v, u, \theta)(x, 0) = (v_0, u_0, \theta_0)(x), \quad x \in (0, 1),
$$

under stress-free and thermally insulated boundary conditions

$$
\left( \frac{\mu}{\nu} u_x - P \right) (0, t) = \left( \frac{\mu}{\nu} u_x - P \right) (1, t) = 0, \quad \theta_x(0, t) = \theta_x(1, t) = 0,
$$

and the initial data (1.6) should be compatible with the boundary conditions (1.7).

The boundary conditions (1.7) describe the expansion of a finite mass of gas into vacuum. One also considers other kind of boundary conditions

$$
u(0, t) = u(1, t) = 0, \quad \theta_x(0, t) = \theta_x(1, t) = 0,
$$

which mean that the gas is confined into a fixed tube with impermeable gas.

For constant coefficients ($\alpha = \beta = 0$) with large initial data, Kazhikhov and Shelukhin [14] first obtained the global existence of solutions under boundary conditions (1.8). From then on, significant progress has been made on the mathematical aspect of the initial boundary value problems, see [1–3, 16–18] and the references therein. Moreover, much effort has been made to generalize this approach to other cases. Motivated by the fact that in the case of isentropic flow a temperature dependence on the viscosity translates into a density dependence, there is a body of literature (see [3, 6–8, 13, 19] and the references therein) studying the case that $\mu$ is independent of $\theta$, and heat conductivity is allowed to depend on temperature in a special way with a positive lower bound and balanced with corresponding constitution relations.

Kawohl [13], Jiang [10,11] and Wang [22] established the global existence of smooth solutions for (1.1)-(1.3), (1.6) with boundary condition of either (1.7) or (1.8) under the assumption $\mu(v) \geq \mu_0 > 0$ for any $v > 0$ and $\kappa$ may depend on both density and temperature. However, it should be mentioned here that the methods used there relies heavily on the non-degeneracy of both the viscosity $\mu$ and the heat conductivity $\kappa$ and cannot be applied directly to the degenerate and nonlinear case ($\alpha \geq 0, \beta > 0$). Under the assumption that $\alpha = 0$ and $\beta \in (0, 3/2)$, Jenssen-Karper [9] proved the global existence of a weak solution to (1.1)-(1.7). Later, for $\alpha = 0$ and $\beta \in (0, \infty)$, Pan-Zhang [19] obtain the global strong solutions. In [9,19], they only consider the case of non-slip and heat insulated boundary conditions. Recently, for the case of stress-free and heat insulated boundary condition, Duan-Guo-Zhu [5] obtain the global strong solutions of (1.1)-(1.7) under the condition that

$$
(v_0, u_0, \theta_0) \in H^1 \times H^2 \times H^2.
$$

In fact, one of the main aims of this paper is to prove the existence and uniqueness of global strong solutions to (1.1)-(1.7) for $\alpha \geq 0$ and $\beta > 0$ with the conditions on the initial data:

$$
(v_0, u_0, \theta_0) \in H^1,
$$

which is similar as those of [14]. Then we state our main result as follows.
Theorem 1.1. Suppose that
\[ \alpha \geq 0, \quad \beta > 0, \]
and that the initial data \((v_0, u_0, \theta_0)\) satisfies
\[ (v_0, u_0, \theta_0) \in H^1(0, 1), \]
and
\[ \inf_{x \in (0, 1)} v_0(x) > 0, \quad \inf_{x \in (0, 1)} \theta_0(x) > 0. \]
Then, the initial-boundary-value problem \((1.1)-(1.7)\) has a unique strong solution \((v, u, \theta)\) such that for each fixed \(T > 0\),
\[
\begin{align*}
\frac{v}{t} &\in L^\infty(0, T; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1)), \\
u_t, \theta_t, u_{xx}, \theta_{xx} &\in L^2((0, 1) \times (0, T)),
\end{align*}
\]
and
\[ \inf_{(x,t) \in (0,1) \times (0,T)} v(x,t) \geq C^{-1}, \quad \inf_{(x,t) \in (0,1) \times (0,T)} \theta(x,t) \geq C^{-1}, \]
where \(C\) is a positive constant depending only on the data and \(T\).

A few remarks are in order.

Remark 1.1. Our Theorem 1.1 can be regarded as a natural generalization of the Kazhikhov-Shelukhin’s result \([14]\) for the constant heat conductivity case to the degenerate and nonlinear one under stress-free and thermally insulated boundary conditions.

Remark 1.2. Our result improves Duan-Guo-Zhu’s result \([5]\) where they need the initial data satisfy \((1.9)\) which are stronger than \((1.11)\).

We now comment on the analysis of this paper. After modifying slightly the method due to Kazhikhov-Shelukhin \([14]\), we obtain a key representation of \(v\) (see \((2.1)) which can be used to obtain directly not only the lower bound of \(v\) (see \((2.12)) but also a pointwise estimate between \(v\) and \(\theta\) (see \((2.16))\). A direct consequence of this pointwise estimate between \(v\) and \(\theta\) (see \((2.16)) is the bound on \(L^\infty(0, T; L^1(0, 1))\)-norm of \(v\) (see \((2.17)) which play an important role in getting the upper bound of \(v\) but cannot be obtained directly from \((1.1)) due to the stress-free boundary condition \((1.7)\). Next, we multiply the momentum equation \((1.2)\) by \(\theta^1 + \beta \theta\) and make full use of the stress-free boundary condition to find that the \(L^2((0, 1) \times (0, T))\)-norm of \(u_{xx}\) can be bounded by the \(L^2((0, 1) \times (0, T)); \theta^1 + \beta \theta\) \(\theta_{xx}\) (see \((2.23)) which indeed can be obtained by multiplying the equation of \(\theta\) (see \((2.13)) by \(\theta^1 + \beta \theta\) and using Gronwall’s inequality (see \((2.32))\). Once we get the bounds on \(L^2((0, 1) \times (0, T))\)-norm of both \(u_{xx}\) and \(u_t\) (see \((2.27))\), the desired estimates on \(\theta_t\) and \(\theta_{xx}\) can be obtained by standard method (see \((2.33))\). The whole procedure will be carried out in the next section.

2 Proof of Theorem 1.1

We first state the following local existence result which can be proved by using the principle of compressed mappings (c.f. \([12, 15, 21]\)).
Lemma 2.1. Let (1.10) - (1.12) hold. Then there exists some $T > 0$ such that the initial-boundary-value problem (1.1) - (1.7) has a unique strong solution $(v, u, \theta)$ satisfying

\[
\begin{cases}
v, u, \theta \in L^\infty(0, T; H^1(0, 1)), \\
v_t \in L^\infty(0, T; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1)), \\
u_t, \theta_t, v_{xt}, u_{xx}, \theta_{xx} \in L^2((0, 1) \times (0, T)).
\end{cases}
\]

Then, the proof of Theorem 1.1 is based on the use of a priori estimates (see (2.15), (2.21), (2.27), and (2.33) below) the constants in which depend only on the data of the problem. The estimates make it possible to continue the local solution to the whole interval $[0, \infty)$. Without loss of generality, we assume that $\bar{\mu} = \bar{\kappa} = R = c_v = 1$.

Next, we derive the following representation of $v$ which is essential in obtaining the time-depending upper and lower bounds of $v$.

Lemma 2.2. We have the following expression of $v$

\[
v(x, t) = B_0(x)D_1(x, t)D_2(x, t) \left\{ 1 + \frac{k(\alpha)}{B_0(x)} \int_0^t \frac{\theta(x, \tau)}{D_1(x, \tau)D_2(x, \tau)} d\tau \right\},
\]

where

\[
B_0(x) = \begin{cases} 
\exp \left( \ln v_0(x) - \frac{1}{\alpha v_0(x)} \right), & \text{if } \alpha > 0, \\
v_0(x), & \text{if } \alpha = 0,
\end{cases} \tag{2.2}
\]

\[
D_1(x, t) = \exp \left\{ k(\alpha) \int_0^x (u(y, t) - u_0(y)) \, dy \right\}, \tag{2.3}
\]

\[
D_2(x, t) = \begin{cases} 
\exp \left\{ \frac{1}{\alpha v(x, t)} \right\}, & \text{if } \alpha > 0, \\
1, & \text{if } \alpha = 0,
\end{cases} \tag{2.4}
\]

and

\[
k(\alpha) = \begin{cases} 
1, & \text{if } \alpha > 0, \\
1/2, & \text{if } \alpha = 0.
\end{cases} \tag{2.5}
\]

Proof. First, it follows from (1.2) that

\[
u_t = \left( \frac{\mu}{v} u_x - P \right)_x, \tag{2.6}
\]

Integrating this over $(0, x)$ and using (1.7) gives

\[
\left( \int_0^x u \, dy \right)_t = \frac{\mu}{v} u_x - P. \tag{2.6}
\]

Then, on the one hand, if $\alpha > 0$, since $u_x = v_t$, we have

\[
\left( \int_0^x u \, dy \right)_t = \left( \ln v - \frac{1}{\alpha v^\alpha} \right)_t - \frac{\theta}{v}. \tag{2.7}
\]

Integrating (2.7) over $(0, t)$ yields

\[
\ln v - \frac{1}{\alpha v^\alpha} - \ln v_0 + \frac{1}{\alpha v_0^\alpha} - \int_0^t \frac{\theta}{v} \, d\tau = \int_0^x (u - u_0) \, dy,
\]
which implies
\[ v(x, t) = B_0(x)D_1(x, t)D_2(x, t) \exp \left\{ k(\alpha) \int_0^t \frac{\theta(x, \tau)}{v(x, \tau)} d\tau \right\}, \quad (2.8) \]

with \( D_1(x, t), D_2(x, t) \) and \( B_0(x) \) as in (2.2)-(2.4) respectively. On the other hand, if \( \alpha = 0 \), it follows from (2.6) that
\[ \int_0^x u dy = 2(\ln v)_t - \frac{\theta}{v}. \]

Integrating this over \((0, t)\) leads to
\[ 2 \ln v - 2 \ln v_0 = \int_0^x (u - u_0) dy + \int_0^t \frac{\theta}{v} d\tau, \]

which shows (2.8) still holds for \( \alpha = 0 \). Finally, denoting
\[ g(x, t) = k(\alpha) \int_0^t \frac{\theta(x, \tau)}{v(x, \tau)} d\tau, \]

we have by (2.8)
\[ g_t = \frac{k(\alpha)\theta(x, t)}{v(x, t)} = \frac{k(\alpha)\theta(x, t)}{B_0(x)D_1(x, t)D_2(x, t) \exp\{g\}}, \]

which gives
\[ \exp\{g\} = 1 + \frac{k(\alpha)}{B_0(x)} \int_0^t \frac{\theta(x, \tau)}{D_1(x, \tau)D_2(x, \tau)} d\tau. \]

Putting this into (2.8) yields (2.1) and finishes the proof of Lemma 2.2.

With Lemma 2.2 at hand, we are in a position to prove the lower bounds of \( v \) and \( \theta \).

**Lemma 2.3.** It holds
\[ \min_{(x,t) \in [0,1] \times [0,T]} v(x, t) \geq C^{-1}, \quad \min_{(x,t) \in [0,1] \times [0,T]} \theta(x, t) \geq C^{-1}, \quad (2.9) \]

where (and in what follows) \( C \) denotes generic positive constant depending only on \( \beta, \alpha, T, \| (v_0, u_0, \theta_0) \|_{H^1(0,1)}, \inf_{x \in (0,1)} v_0(x), \) and \( \inf_{x \in (0,1)} \theta_0(x) \).

**Proof.** First, integrating (1.3) over \((0, 1)\) and using (1.7) immediately leads to
\[ \int_0^1 \left( \theta + \frac{1}{2} u^2 \right) (x, t) dx = \int_0^1 \left( \theta + \frac{1}{2} u^2 \right) (x, 0) dx, \quad (2.10) \]

which in particular gives
\[ \left| \int_0^x u dy \right| \leq \int_0^1 |u| dy \leq \left( \int_0^1 u^2 dy \right)^{\frac{1}{2}} \leq C. \]

Combining this with (2.3) implies
\[ C^{-1} \leq D_1(x, t) \leq C, \quad (2.11) \]
which together with (2.1) yields that for any \((x, t) \in [0, 1] \times [0, T]\),
\[
v(x, t) \geq B_0(x)D_1(x, t)D_2(x, t) \geq C^{-1},
\]
due to
\[
D_2(x, t) \geq 1, \quad C^{-1} \leq B_0(x) \leq C.
\]
Finally, we rewrite (1.3) as
\[
\theta_t + \frac{\theta}{v} u_x = \left( \frac{\theta^\beta \theta_x}{v} \right)_x + \frac{\mu_2^2}{v}.
\]
(2.13)
For \(r > 2\), multiplying the above equality by \(\theta^{-r}\) and integrating the resultant equality over \((0, 1)\) yields that
\[
\frac{1}{r - 1} \frac{d}{dt} \int_0^1 (\theta^{-1})^{r-1} dx + \int_0^1 \frac{\mu_2^2}{v \theta^r} dx + r \int_0^1 \frac{\theta^\beta \theta_x^2}{v \theta^{r+1}} dx
\]
\[
= \int_0^1 \frac{u_x}{v \theta^{r-1}} dx
\]
\[
\leq \frac{1}{2} \int_0^1 \frac{\mu_2^2}{v \theta^r} dx + \frac{1}{2} \int_0^1 \frac{1}{\mu v \theta^{-2}} dx
\]
\[
\leq \frac{1}{2} \int_0^1 \frac{\mu_2^2}{v \theta^r} dx + C \| \theta^{-1} \|^r_{L^{r-1}},
\]
where in the second inequality we have used \(\mu v = v + v^{1-\alpha} > v \geq C^{-1}\). Combining (2.14) with Gronwall’s inequality yields
\[
\sup_{0 \leq t \leq T} \| \theta^{-1}(. , t) \|_{L^{r-1}} \leq C,
\]
with \(C\) independent of \(r\). Letting \(r \to \infty\) proves the second inequality of (2.9). Thus, the proof of Lemma 2.3 is finished. 

Lemma 2.4. There exists a positive constant \(C\) such that for each \((x, t) \in [0, 1] \times [0, T]\),
\[
C^{-1} \leq v(x, t) \leq C.
\]
(2.15)

Proof. First, it follows from (2.9) and (2.14) that for any \((x, t) \in [0, 1] \times [0, T]\),
\[
1 \leq D_2(x, t) \leq C,
\]
which together with (2.1) and (2.11) yields that for any \((x, t) \in [0, 1] \times [0, T]\),
\[
C^{-1} \leq v(x, t) \leq C + C \int_0^t \theta(x, \tau) d\tau.
\]
(2.16)
Integrating this with respect to \(x\) over \((0, 1)\) and using (2.10) leads to
\[
\sup_{0 \leq t \leq T} \int_0^1 v(x, t) dx \leq C.
\]
(2.17)
Next, for $\eta \in (0, 1)$ and $\varepsilon \in (0, 1)$, integrating (2.13) multiplied by $\theta^{-\eta}$ over $(0, 1) \times (0, T)$, we get by (2.10) and (2.9)

\[
\int_0^T \int_0^1 \eta \theta \beta \theta^2 \log^2 \ dx dt + \int_0^T \int_0^1 \mu v \theta \ dx dt = \int_0^T \int_0^1 \frac{\mu v \theta}{\theta^{\eta-1}} \ dx dt
\]

[continued...]

where in the first inequality we have used

\[
\int_0^1 \theta^{-\eta} \ dx \leq C.
\]

Finally, using (2.17), we obtain that for $\eta = \min\{1, \beta\}/2$,

\[
\int_0^T \max_{x \in [0, 1]} \theta \ dx dt \leq C + C \int_0^T \int_0^1 |\theta_x| \ dx dt
\]

[continued...]

which together with (2.18) yields that

\[
\int_0^T \max_{x \in [0, 1]} \theta \ dx dt \leq C,
\]

(2.19)

and that for $\eta \in (0, 1)$

\[
\int_0^T \int_0^1 \theta^\beta - 1 - \eta \theta^2 \ dx dt \leq C(\eta).
\]

(2.20)

Combining (2.16) with (2.19) finishes the proof of Lemma 2.4.

Lemma 2.5. There exists a positive constant $C$ such that

\[
\sup_{0 \leq t \leq T} \int_0^1 v_x^2 \ dx \leq C.
\]

(2.21)

Proof. First, we rewrite (1.2) as

\[
\left( \frac{\mu v_x}{v} - u \right)_t = \left( \frac{\theta}{v} \right)_x.
\]

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The proof of Lemma 2.5 is finished.\
\
Next, it follows from (2.10) and (2.15) that for (2.22), we obtain after using Gronwall’s inequality and (2.25) that where in the last inequality we have used (2.20). Finally, adding (2.23) and (2.24) to (2.21) gives
\[
\frac{1}{2} \int_0^1 \left( \frac{\mu v_x}{v} - u \right)^2 dx - \frac{1}{2} \int_0^1 \left( \frac{\mu v_x}{v} - u \right)^2 (x,0) dx + \int_0^1 \frac{\mu v_x^2}{v^3} dx dt = \int_0^T \int_0^1 \theta v_x u dx dt + \int_0^T \int_0^1 \theta x \left( \frac{\mu v_x}{v} - u \right) dx dt \triangleq I_1 + I_2.\
\]

Then, on the one hand, Cauchy’s inequality, (2.10), (2.15), and (2.19) lead to
\[
|I_1| \leq \frac{1}{2} \int_0^T \int_0^1 \theta v_x^2 dx dt + \frac{1}{2} \int_0^T \int_0^1 \mu \theta v_x^2 dx dt \leq C \int_0^T \max_{x \in [0,1]} \theta dt + \frac{1}{2} \int_0^T \int_0^1 \mu \theta v_x^2 dx dt \leq C + \frac{1}{2} \int_0^T \int_0^1 \mu \theta v_x^2 dx dt.\
\]

On the other hand, we deduce from (2.15), (2.20), and (2.19) that for \( \eta = \min\{1, \beta\}/2, \)
\[
|I_2| \leq C \int_0^T \int_0^1 \theta^{\beta - 1 - \eta} \theta_x^2 dx dt + C \int_0^T \int_0^1 \theta^{1+\eta-\beta} \left( \frac{\mu v_x}{v} - u \right)^2 dx dt \leq C + C \int_0^T \max_{x \in [0,1]} \theta^2 \int_0^1 \left( \frac{\mu v_x}{v} - u \right)^2 dx dt.\
\]

Next, it follows from (2.10) and (2.15) that for \( \eta = \min\{1, \beta\}/2, \)
\[
\int_0^T \max_{x \in [0,1]} (\theta^{1+\beta} + \theta^2) dt \\
\leq C \int_0^T \max_{x \in [0,1]} \theta^{2+\beta-\eta} dt \\
\leq C + C \int_0^T \left( \int_0^1 \theta^{\frac{2+\beta-\eta}{2}} x dx \right)^2 dt \\
\leq C + C \int_0^T \left( \int_0^1 \theta^{\frac{\beta-\eta}{2}} |x| dx \right)^2 dt \\
\leq C + C \int_0^T \left( \int_0^1 \theta^{\beta} \frac{v}{v\theta v^{1+1}} dx \right) \left( \int_0^1 v \theta dx \right) dt \\
\leq C + C \int_0^T \int_0^1 \theta^{\beta} \frac{v}{v\theta v^{1+1}} dx dt \\
\leq C,
\]
where in the last inequality we have used (2.20). Finally, adding (2.23) and (2.24) to (2.22), we obtain after using Gronwall’s inequality and (2.25) that
\[
\sup_{0 \leq t \leq T} \int_0^1 \left( \frac{\mu v_x}{v} - u \right)^2 dx + \int_0^1 \frac{\mu v_x^2}{v^3} dx dt \leq C,
\]
which together with (2.10) and (2.15) gives
\[
\sup_{0 \leq t \leq T} \int_0^1 v_x^2 dx \leq C + C \sup_{0 \leq t \leq T} \int_0^1 \left( \frac{\mu v_x}{v} - u \right)^2 dx \\
\leq C,
\]
The proof of Lemma 2.5 is finished. \( \square \)
Lemma 2.6. There exists a positive constant $C$ such that

$$
\sup_{0 \leq t \leq T} \int_0^1 u_x^2 \, dx + \int_0^T \int_0^1 (u_t^2 + u_{xx}^2) \, dx \, dt \leq C. \tag{2.27}
$$

Proof. First, integrating $(\frac{\mu}{v} u_x - P)_x$ multiplied by $(\frac{\mu}{v} u_x - P)_x$ over $(0, 1)$, we obtain by integration by parts and $(2.13)$ that

$$
\int_0^1 (\frac{\mu}{v} u_x - P)^2 \, dx
\quad = \quad - \int_0^1 (\frac{\mu}{v} u_x - P) u_{tx} \, dx
\quad = \quad - \frac{1}{2} \int_0^1 \frac{\mu}{v} (u_x^2)_t \, dx + \left( \int_0^1 P u_x \, dx \right)_t - \int_0^1 P_t u_x \, dx
\quad = \quad - \frac{1}{2} \left( \int_0^1 \frac{\mu}{v} u_x^2 \, dx \right)_t + \frac{1}{2} \int_0^1 \frac{\mu}{v} u_x^2 \, dx + \left( \int_0^1 P u_x \, dx \right)_t
\quad + \frac{2}{2} \int_0^1 \frac{\theta}{v^2} u_x^2 \, dx + \int_0^1 \frac{\theta^3 \theta_x}{v} (\frac{u_x}{v}) \, dx - \int_0^1 \frac{\mu u_x^3}{v^2} \, dx,
$$

which in particular gives

$$
\left( \int_0^1 \frac{\mu}{v} u_x^2 - P u_x \right)_t \, dx + \int_0^1 \left( \frac{\mu}{v} u_x^2 + \frac{\theta^2}{v^2} \right) \, dx
\leq C \int_0^1 \left| u_{xx} \right| \left| v_x \right| \left| u_x \right| + \left| v_x \right| |\theta| + \theta^3 |\theta_x| \, dx + C \int_0^1 |\theta_x| |\theta| \, dx
\quad + C \int_0^1 \left| \theta_x \right| |\theta| \left| u_x \right| \, dx + C \int_0^1 \left( u_x^2 + \theta^2 \right) \, dx + C \int_0^1 \left( u_x^3 + u_x^2 \theta \right) \, dx
\quad \leq \frac{1}{4} \int_0^1 \frac{\mu}{v} u_x^2 \, dx + C \int_0^1 \theta^2 \left( \int_0^1 u_x^2 \, dx \right)^2 + C \max_{x \in [0,1]} (u_x^2 + \theta^2) \left( 1 + \int_0^1 v_x^2 \, dx + \int_0^1 \theta \, dx \right)
\quad \leq \frac{1}{2} \int_0^1 \frac{\mu}{v} u_x^2 \, dx + C \int_0^1 \theta^2 \left( \int_0^1 u_x^2 \, dx \right)^2 + C \max_{x \in [0,1]} \theta^2 + C \left( \int_0^1 u_x^2 \, dx \right)^2 + C, \tag{2.28}
$$

where in the last inequality we have used $(2.26)$ and

$$
\max_{x \in [0,1]} u_x^2 \leq C(\varepsilon) \int_0^1 u_x^2 \, dx + \varepsilon \int_0^1 u_x^2 \, dx, \tag{2.29}
$$

for $\varepsilon > 0$ small enough. Then, integrating $(2.13)$ over $(0, 1) \times (0, T)$ yields that

$$
\int_0^T \int_0^1 \frac{\mu u_x^2}{v} \, dx \, dt = \int_0^1 \theta \, dx - \int_0^1 \theta_0 \, dx + \int_0^T \int_0^1 \frac{\theta}{v} u_x \, dx \, dt
\quad \leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{\mu u_x^2}{v} \, dx \, dt + C \int_0^T \int_0^1 \frac{\theta^2}{\mu v} \, dx \, dt
\quad \leq C + \frac{1}{2} \int_0^T \int_0^1 \frac{\mu u_x^2}{v} \, dx \, dt + C \int_0^T \max_{x \in [0,1]} \theta(x, t) \, dt
$$
which together with (2.19) and (2.15) gives
\[ \int_0^T \int_0^1 u_x^2 \, dx \, dt \leq C. \] (2.30)

Next, integrating (2.13) multiplied by \( \theta^{1+\beta} \) over \((0,1)\) leads to
\[ \frac{1}{2 + \beta} \left( \int_0^1 \theta^{2+\beta} \, dx \right)_t + (1 + \beta) \int_0^1 \frac{\theta^{2+\beta}}{v} \, dx \]
\[ = - \int_0^1 \frac{\theta^{2+\beta} u_x}{v} \, dx + \int_0^1 \frac{u \theta^{1+\beta} u_x^2}{v} \, dx \]
\[ \leq C \int_0^1 \theta^{3+\beta} \, dx + C \int_0^1 \theta^{1+\beta} u_x^2 \, dx \]
\[ \leq C \max_{x \in [0,1]} \int_0^1 \theta^{2+\beta} \, dx + C \max_{x \in [0,1]} \theta^{1+\beta} \int_0^1 u_x^2 \, dx. \] (2.31)

Choosing \( C_2 \geq C_1 + 1 \) suitably large such that
\[ C_2 \theta^{2+\beta} + \mu v - 1 u_x^2 \geq 4(2 + \beta) \theta v - 1 |u_x|, \]
adding (2.31) multiplied by \( C_2 \) to (2.28), and choosing \( \varepsilon \) sufficiently small, we obtain from Gronwall’s inequality, (2.30), and (2.25) that
\[ \sup_{0 \leq t \leq T} \int_0^1 \left( \theta^{2+\beta} + u_x^2 \right) \, dx + \int_0^T \int_0^1 u_x^2 \, dx \, dt + \int_0^T \int_0^1 \theta^{2+\beta} \theta_x^2 \, dx \, dt \leq C. \] (2.32)

Finally, rewriting (1.2) as
\[ u_t = \frac{\mu u_{xx}}{v} + \left( \frac{\mu}{v} \right)' \frac{u_{xx}}{v} u - \frac{\theta_x}{v} + \frac{\theta v_x}{v^2}, \]
we deduce from (2.15), (2.32), (2.26), (2.30), (2.29), and (2.25) that
\[ \int_0^T \int_0^1 u_x^2 \, dx \, dt \leq C \int_0^T \int_0^1 \left( u_x^2 + u_x^2 v_x^2 + \theta_x^2 + \theta^2 v_x^2 \right) \, dx \, dt \]
\[ \leq C + C \int_0^T \max_{x \in [0,1]} \theta^2 \, dt \]
\[ \leq C, \]
which together with (2.32) finishes the proof of Lemma 2.6. \( \square \)

**Lemma 2.7.** There exists a positive constant \( C \) such that
\[ \sup_{0 \leq t \leq T} \int_0^1 \theta_x^2 \, dx + \int_0^T \int_0^1 \left( \theta_t^2 + \theta_{xx}^2 \right) \, dx \, dt \leq C. \] (2.33)

**Proof.** First, multiplying (2.13) by \( \theta^\beta \theta_t \) and integrating the resultant equality over \((0,1)\) yields
\[ \int_0^1 \theta^\beta \theta_t^2 \, dx + \int_0^1 \frac{\theta^\beta + u_x \theta_t}{v} \, dx \]
\[ = - \int_0^1 \left( \frac{\theta^\beta \theta_t}{v} \right)' \theta^\beta \theta_t \, dx + \int_0^1 \frac{\mu u_x^2 \theta^\beta \theta_t}{v} \, dx \]
\[ = - \frac{1}{2} \frac{d}{dt} \int_0^1 \left( \frac{\theta^\beta \theta_t}{v} \right)^2 \, dx + \frac{1}{2} \int_0^1 \left( \theta^\beta \theta_x \right)^2 \left( \frac{1}{v} \right)' \, dx + \int_0^1 \frac{\mu u_x^2 \theta^\beta \theta_t}{v} \, dx \]
\[ = - \frac{1}{2} \frac{d}{dt} \int_0^1 \left( \frac{\theta^\beta \theta_t}{v} \right)^2 \, dx - \frac{1}{2} \int_0^1 \left( \theta^\beta \theta_x \right)^2 u_x \, dx + \int_0^1 \frac{\mu u_x^2 \theta^\beta \theta_t}{v} \, dx, \]
which combined with the Hölder inequality, (2.21), and (2.32) leads to

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \left( \frac{\theta^2 \theta_x}{v} \right)^2 dx + \int_0^1 \theta^2 \theta_t^2 dx
\]

\[
= -\frac{1}{2} \int_0^1 \frac{\theta^2 \theta_x}{v^2} u_x dx + \int_0^1 \frac{\theta \theta_t}{v} u_x^2 dx - \int_0^1 \frac{\theta \theta_t}{v} u_x dx
\]

\[
\leq C \int_0^1 \theta^2 \theta_x^2 u_x dx + \frac{1}{2} \int_0^1 \theta^2 \theta_t^2 dx + C \int_0^1 u_x^4 \theta^2 dx + C \int_0^1 \theta^2 + u_x^2 dx
\]

\[
\leq C \max_{x \in [0,1]} |u_x| \int_0^1 \theta^2 \theta_x^2 dx + \frac{1}{2} \int_0^1 \theta^2 \theta_t^2 dx + C \max_{x \in [0,1]} \left( u_x^2 \theta^2 + \theta^{2+2} \right)
\]

\[
\leq \frac{1}{2} \int_0^1 \theta^2 \theta_x^2 dx + C \left( \int_0^1 \theta^2 \theta_x^2 dx \right) + C \max_{x \in [0,1]} \left( u_x^2 + \theta^{2+2} \right) + C.
\]

It follows from (2.21), (2.29), (2.27), and the Hölder inequality that

\[
\int_0^T \max_{x \in [0,1]} u_x^4 dt \leq C \int_0^T \int_0^1 u_x^2 dx dt + C \int_0^T \int_0^1 |u_x^3 u_{xx}| dx dt
\]

\[
\leq C \int_0^T \max_{x \in [0,1]} u_x^2 \int_0^1 u_x^2 dx dt
\]

\[
+ C \int_0^T \max_{x \in [0,1]} u_x^2 \left( \int_0^1 u_x^2 dx \right) \frac{1}{2} \left( \int_0^1 u_{xx}^2 dx \right) \frac{1}{2} dt
\]

\[
\leq C \int_0^T \int_0^1 \left( u_x^2 + u_{xx}^2 \right) dx dt + \frac{1}{2} \int_0^T \max_{x \in [0,1]} u_x^4 dt
\]

\[
\leq C + \frac{1}{2} \int_0^T \max_{x \in [0,1]} u_x^4 dt
\]

which combined with (2.34), (2.32), and the Gronwall inequality yields

\[
\sup_{0 \leq t \leq T} \int_0^1 \left( \frac{\theta^2 \theta_x}{v} \right)^2 dx + \int_0^T \int_0^1 \theta^2 \theta_t^2 dx dt \leq C,
\]

where we have used

\[
\max_{x \in [0,1]} \theta^{2+2} \leq C + C \left( \max_{x \in [0,1]} |\theta^{+1} (x, t) - (\int_0^1 \theta dx)^{+1}| \right)^2
\]

\[
\leq C + C \left( \int_0^1 |\theta^2 \theta_x| dx \right)^2
\]

\[
\leq C + C \int_0^1 (\theta^2 \theta_x)^2 dx.
\]

Combining (2.36) with (2.37) implies for all \((x, t) \in (0, 1) \times (0, T)\)

\[
\theta(x, t) \leq C.
\]

Meanwhile, both (2.36) and (2.34) lead to

\[
\sup_{0 \leq t \leq T} \int_0^1 \theta_x^2 dx + \int_0^T \int_0^1 \theta_t^2 dx dt \leq C.
\]
Finally, it follows from (2.13) that
\[
\frac{\theta^\beta}{\theta v} \frac{\theta_{xx}}{v} = -\frac{\beta}{v} \theta^{\beta-1} \frac{\theta_{x}^2}{v} + \frac{\theta^\beta v_x}{v^2} - \frac{\mu u_x^2}{v} + \frac{\theta u_x}{v} + \theta_t,
\]
which together with (2.39), (2.9), (2.38), (2.35), and (2.21) gives
\[
\int_0^T \int_0^1 \theta^2_{xx} dx dt \leq C \int_0^T \int_0^1 (\theta^4_x + v_x^2 \theta^2_x + u_x^4 + \theta^2 u_x^2 + \theta^2_t) dx dt \\
\leq C + C \int_0^T \max_{x \in [0,1]} \theta_x^2 dt \\
\leq C + C \int_0^T \int_0^1 \theta^2_x dx dt + \frac{1}{2} \int_0^T \int_0^1 \theta^2_{xx} dx dt.
\]
Combining this with (2.39) gives (2.33) and finishes the proof of Lemma 2.7.  

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