Generalized Tsallis Thermostatistics of Magnetic Systems

Fevzi BÜYÜKKILIÇ, Uğur TIRNAKLI* and Doğan DEMİRHAN

Department of Physics, Faculty of Science, Ege University
35100 Bornova, İzmir-TURKEY

Boltzmann-Gibbs statistics fails to study the systems having the conditions (i) the spatial range of the microscopic interactions are long-ranged, (ii) the time range of the microscopic memory is long-ranged and (iii) the system evolves in a (multi)fractal space-time. These kind of systems are said to be nonextensive and a nonextensive formalism of statistics must be needed for them. Recently a generalized thermostatistics is proposed by C. Tsallis to handle the nonextensive systems and up to now, not only the generalization of some of the conventional concepts have been investigated but the formalism has also been properous in some of the physical applications. In this study, our effort is to introduce Tsallis thermostatistics in some details and to give a brief review of the magnetic systems which have been studied in the frame of this formalism.

I. INTRODUCTION

The aim of Statistical Thermodynamics is to establish a bridge between microscopic and macroscopic behaviours of all types of systems in Nature. Standard Boltzmann-Gibbs (BG) statistics (or in other words extensive statistical thermodynamics) is essentially derived from the standard Shannon entropy

\[ S_1 = -k_B \sum_{i=1}^{W} p_i \log p_i \]  

*e-mail : tirnakli@fenfak.ege.edu.tr
which is an extensive and concave quantity (the sub index 1 of $S$ will be defined afterwards). Although BG statistics provides a suitable tool which enables us to handle a large number of physical systems satisfactorily, it has two basic restrictions:

- the range of the microscopic interactions must be small compared to the linear size of the macroscopic systems (short-range interactions)
- the time range of the microscopic memory must be small compared to the observation time (Markovian processes).

In the case of a breakdown in one and/or the other of these restrictions, BG statistics fails. Here, "to fail" is used to imply the divergences of the sums or integrals in the expressions of the relevant thermodynamical quantities.

These kind of violations are met for a long time in gravitation [1], magnetic systems [2], anomalous diffusion [3] and surface tension problems [4]. On the other hand, recently C. Tsallis have pointed out that [5], although not yet clearly identified, the same or analogous type of problems might be present in long-range Casimir-like systems [6], in granular matter [7] and two-dimensional turbulence [8,9]. More precisely, the situation could be classified in a general manner as follows [5]:

- For an Euclidean-like space-time, if the forces and/or the memory are long-ranged, as far as we are interested in an equilibrium state, the BG statistics is weakly violated, therefore BG formalism can be used. On the other hand, whenever a meta-equilibrium state is considered, the BG statistics is strongly violated, hence another formalism must be needed.

- For a (multi)fractal space, BG formalism is strongly violated again and a new formalism is needed.

The way out from these problems seems to be Nonextensive Statistical Thermodynamics which must be a generalization of the BG statistics in a manner that allows a correct description of the nonextensive physical systems as well.
Nonextensive formalisms are much in vogue nowadays in Physics and keep growing along two apparently different lines, namely Generalized Tsallis Thermostatistics (GTT) and Quantum Group-like Approaches (QGA). Although they seem to be very different from each other, in a recently published paper [10], it is noticed that there exist a connection between them. Since this is not the scope of the present paper, we’re contented to imply that there have been attempts of finding some verifications of this connection or establishing new relations by investigating the physical systems within both GTT and QGA [11,12].

From now on, we focus our attention to GTT and the following Section is devoted to a review of the formalism.

II. GENERALIZED TSALLIS THERMOSTATISTICS

GTT, which has been proposed in 1988 [13] as a possible theoretical tool for discussing nonextensive physical systems, basically relies upon two postulates:

1) The generalized entropy is assumed to be defined as

\[ S_q = k \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} \]  

where \( k \) is a positive constant, \( q \in \mathbb{R} \), \( W \) is the total number of microscopic configurations and \( \{p_i\} \) are the probabilities of the microstates of the systems. This expression is a nonextensive quantity and transforms to the standard extensive Shannon entropy given by Eq.(1) if and only if \( q = 1 \).

2) The \( q \)-expection value of an observable \( O \), whose value in state \( i \) is \( O_i \), is given by

\[ < O >_q = \sum_{i=1}^{W} p_i^q O_i \]  

It is clear that the validity of these postulates is confirmed (or rejected) by comparisons to the experiments and by the conclusions which they yield. It is easy to verify that GTT has the following properties:

- \( S \geq 0 \) for any arbitrary set \( \{p_i\} \) and for any value of \( q \) (positivity).
• For microcanonical ensemble (i.e. \( p_i = 1/W \), \( \forall i \)), \( S_q \) attains its extremal value: 
\[ S_q = (W^{1-q} - 1)/(1 - q) \] (equiprobability).

• \( S_q \) is concave for \( q > 0 \) and convex for \( q < 0 \) (concavity).

• The optimization of \( S_q \) under the constraints 
\[ \sum_i p_i = 1 \quad \text{and} \quad U_q = \sum_i p_i^q \varepsilon_i, \] (4)
leads to the canonical probability distribution 
\[ p_i = \frac{1}{Z_q} [1 - (1 - q)\beta \varepsilon_i]^{1/(1-q)} \] (5)
where the generalized partition function is defined to be 
\[ Z_q \equiv \sum_i [1 - (1 - q)\beta \varepsilon_i]^{1/(1-q)} \] (6)
and \( \beta = 1/kT \).

• If \( A \) and \( B \) are two independent systems (i.e. \( p_{ij}^{AB} = p_i^A p_j^B \)), \( S_q \) obeys the following additivity rule: (pseudo-additivity) 
\[ S_q(A \cup B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B) \] (7)
which implies \( S_q \) is superadditive (entropy of the whole system is greater than the sum of its parts) for \( q < 1 \) and subadditive (entropy of the whole system is smaller than the sum of its parts) for \( q > 1 \).

• Under general conditions [14], \( dS_q/dt \) is non-negative for \( q > 0 \), vanishes for \( q = 1 \) and non-positive for \( q < 1 \). (\( H \)-theorem).

• If we partition \( W \) microstates into two subsets \( x \) and \( y \) having respectively \( W_x \) and \( W_y \) microstates (\( W_x + W_y = W \)), corresponding probabilities can be written as
Thus, it is straightforward to verify the following additivity rule: (Shannon additivity)

$$S_q(p_1, \ldots, p_W) = S_q(p_x, p_y) + p_x^q S_q\left(p_1, \ldots, p_{W-1}, \frac{p_w}{p_x}\right) + p_y^q S_q\left(\frac{p_{W+1}}{p_y}, \ldots, p_W\right).$$  \hspace{1cm} (8)

- In this formalism, the Legendre-transform structure of Thermodynamics is invariant for all values of $q$ [15].

The last property is very important and deserves a further comment. As it has been pointed out in [16], this property indicates that the entire formalism of Thermodynamics can be extended to be nonextensive without losing its Legendre-transform structure. In addition to this, amongst a large number of entropic forms, Tsallis formalism seems to be the unique which preserves the standard Legendre-transform structure.

From the year 1988 up to present days, numerous concepts of statistical thermodynamics have been accomplished to generalize in the frame of Tsallis formalism. Amongst them, the specific heat of the harmonic oscillator [17], one-dimensional Ising model [18,19], the Ehrenfest theorem [20], the von Neumann equation [21], quantum statistics [22], the fluctuation-dissipation theorem [23], Langevin and Focker-Planck equations [24], the Bogolyubov inequality [25], classical equipartition theorem [26], paramagnetic systems [27], infinite-range spin-1/2 Ising ferromagnet [28], Callen identity [29], mean-field Ising model [30,31], Planck radiation law [32,33], quantum uncertainty [34], anisotropic rigid rotator [35], Haldane exclusion statistics [36], Bose-Einstein condensation [37], the generalized transmissivity for spin-1/2 Ising ferromagnet [38], self-dual planar Ising ferromagnet [39] and localized-spins ideal paramagnet [40] could be enumerated.

GTT has also been successfully used to overcome the failure of BG statistics in some of the physical applications. The establishment of finite mass for the astrophysical systems in the frame of polytropic structures [41], the calculation of the specific heat of the unionized hydrogen atom [42], the derivation of Levy-like anomalous diffusion [43-45], the construction of a comprehensive thermodynamic description of $d = 2$ Euler turbulence [9] and solar
neutrino problem [46] are the examples that could be mentioned herewith. A detailed review of the subject can be found in [47], and the investigation of the formalism from the mathematical point of view is now available in [48].

After introducing GTT in some details, the remaining part of this paper includes a brief review of the magnetic systems which have been studied within this formalism.

III. REVIEW OF MAGNETIC SYSTEMS IN THE FRAME OF GTT

Up to now, some of the magnetic systems have been investigated within GTT and in this Section our goal is to review these systems briefly rather than to discuss them in details.

A. Ising Chain

The investigation of the Ising chain has been the first example of magnetic systems discussed within GTT. In his first paper [18] related to the subject, Andrade has evaluated the partition function, the internal energy and the specific heat of the Ising chain by making use of the probability distribution given in [13]. He has also discussed the extensivity of the thermodynamical quantities and the influence of the ground state energy. In the calculations a parameter $r$, defined as $r = 1/q - 1$ with $r = 1, 2, \ldots$, has been introduced. Since $q \to 1$ if $r \to \infty$, the increasing values of $r$ provide a comparison to the standard values of BG statistics. Such kind of comparison has been attempted recently [49]. Although the internal energy and the specific heat could be obtained only for the values of $r$ up to 3 because of the cumbersome nature of the expressions, it has been sufficient to make a comparison to the conventional expressions of the Ising chain. Other two interesting results of Andrade's work are those (i) for the negative energy levels, the expressions of the specific heat deviate from the expected behaviour for intensive quantities in BG statistics, namely they are proportional to $1/N^2$ in the limit of large number of spins and (ii) for the positive energy levels, the specific heat presents oscillations. In his second paper on the same subject [19], Andrade has reanalyzed the behaviour of the Ising chain within GTT by making use of the probability
B. Infinite-range Ising Ferromagnet

In order to define infinite-range Ising ferromagnet, let us write down the Hamiltonian

\[ H = -2 \sum_{(i,j)} J_{ij} s_i s_j \quad (s_i = \mp 1) \quad (9) \]

where

\[ J_{ij} = \frac{J}{r_{ij}^{d+\delta}} \quad (J > 0 ; \ d + \delta \geq 0) \quad (10) \]

\( r_{ij} \) is the distance between sites \( i \) and \( j \) and the sum runs over all distinct pairs of sites on a \( d \)-dimensional simple hypercubic lattice. Infinite-range Ising ferromagnet can be obtained in the case of \( d + \delta = 0 \). For the internal energy of this system in the frame of BG statistics, it can be written at \( T = 0 \) (\( T \) being the temperature)

\[ -\frac{U_1}{J} = N \sum_j \frac{1}{r_{ij}^{d+\delta}} \quad (11) \]

At long distances this sum can be replaced by an integral which diverges for \( \delta \leq 0 \). This fact has first pointed out by Hiley and Joyce [2], and it becomes obvious that BG statistics fails for long-range Ising ferromagnet when \( \delta \leq 0 \), hence a nonextensive formalism, such as GTT, must be needed.

Nobre and Tsallis have numerically investigated the infinite-range Ising ferromagnet within GTT [28]. They have evaluated the specific heat of the system numerically for \( q = 1, q < 1 \) and \( q > 1 \). Furthermore, they have exhibited that the thermodynamical limit is well defined. The most remarkable two conclusions of this paper can be quoted by the words of Nobre and Tsallis:

- In complete analogy with the well known \( C_1/Nk_B \) vs. \( k_B T/N \) curves within BG statistics \((q = 1)\), the curves \( N^2 C_q/2^{(1-q)N} k \) vs. \( kT/N^2 J \) within \( q \neq 1 \) statistics tend
to a numerically well defined thermodynamical limit; this is the first time that the existence of such a limit is exhibited for an interacting model

- Analogously with the Landau-like phase transition which is known to exist for \( q = 1 \), a nontrivial divergence, at a finite rescaled temperature, in the rescaled specific heat is observed for \( 0 < q < 1 \) (not for \( q > 1 \)) whenever the energy spectrum includes a positive portion. Although no evidence for phase transitions has been obtained for \( q > 1 \), these should not be excluded without further studies. Indeed, the specific heat critical exponent \( \alpha \) is herein shown to be positive for \( q < 1 \), and it is known to be zero for \( q = 1 \) (Mean-field Approach). Consequently, it could well be that it is negative for \( q > 1 \), thus producing a soft thermal dependence of the specific heat (as herein observed!).

C. Distribution Function of a Paramagnetic System

In [27], Büyükkılıç and Demirhan have attempted to establish a similarity between the random walk problem and a paramagnetic system such a way as to the distance covered in the random walk here corresponds to the total magnetic moment of a paramagnetic system. Similar to the set of the points visited in the random walk, the statistical ensemble of the orientation of the magnetic moments in a paramagnetic system could be considered as a set of self similar elements, i.e. exhibiting a multifractal structure. When the Shannon entropy is maximized by taking the ordinary constraints, namely (i) normalization of the probability and (ii) the finiteness of the mean square total magnetic moment \( < M^2 > \), the Levy distributions of the multifractal structures are not obtained. In this case, the maximum entropy formalism should be modified by taking another entropy definition instead of Shannon one and new constraints appropriate with the new entropy. Following the way used in [43], Büyükkılıç and Demirhan have used the integral form of the Tsallis entropy (in \( k \) units)
\[ S_q = -\frac{1 - \int \rho^q(M) dM}{1 - q} \]  

where \( \rho^q(M) dM \) is the probability of the total magnetic moment of the ensemble to be in the interval \( M \) and \( M + dM \); with the appropriate constraints (i) normalization of the probability and (ii) the finiteness of \( \langle M^2 \rangle_q \) which is the \( q \)-expectation value of \( M^2 \) in the sense of Eq.(3). By using the undetermined Lagrange multipliers method, it is straightforward to obtain the distribution function which requires to have the same asymptotic functional form with the Levy distribution, which behaves for large \( M \) as \( \rho(M) \sim M^{-1-\gamma} \) (0 < \( \gamma \) < 2). Hence, by equating the asymptotic solutions one ends up with

\[ q = \frac{3 + \gamma}{2 + \gamma} \]  

In particular, the limiting cases provide an interval 1 < \( q \) < 3 as it has been given in [43].

**D. Single-site Callen Identity**

In a recent work [29], Sarmento has taken into account of single-site Callen identity from the GTT point of view, and accomplished to calculate the critical temperature of the Ising ferromagnet within mean field approximation. When the Hamiltonian in Eq.(9) is used, it is easy to obtain the standard single-site Callen identity (i.e. the standard expectation value of the spin variable at the lattice site) such that :

\[ \langle s_i \rangle_1 = \langle \tanh(\beta E_i) \rangle \]  

where \( E_i \approx \sum_j J_{ij} s_j \). By approximating the thermal average of the hyperbolic tangent by the hyperbolic tangent of the thermal average, standard mean field approximation can be obtained :

\[ \langle s_i \rangle_1 \approx \tanh \left[ \beta \sum_j J_{ij} \langle s_i \rangle_1 \right] . \]  

The derivation of the generalized single-site Callen identity is started by seperating the Ising Hamiltonian into two parts
\[ H = H_i + H' \]  

(16)

where \( H_i \) includes all contributions associated with the site \( i \) and \( H' \) is the other part which does not depend on site \( i \). After some algebra, within GTT, one can find

\[ \langle s_i \rangle_q = \left( \frac{1 - f_q}{1 + f_q} \right)_q \]  

(17)

where

\[ f_q = \left[ \frac{1 - \beta(1 - q)(E_i + H')}{1 - \beta(1 - q)(-E_i + H')} \right]^{q/(1-q)} \]  

(18)

which is the generalized single-site Callen identity. After establishing this expression, he focuses his attention to the calculation of the critical temperature \( T_c \) for a \( z \)-coordinated spin-1/2 Ising ferromagnet with coupling constant \( J \). In the \( T \to T_c \) limit, the expression for the critical temperature is found to be

\[ \frac{kT_c}{J} = qz \]  

(19)

which transforms to the well known standard result \( k_B T_c/J = z \) in the \( q \to 1 \) limit.

E. Mean-field Ising Model

The authors of the present paper have recently developed a mean-field approximation to the Ising model within GTT [30,31]. In their study, they have used the generalized Bogolyubov inequality obtained in [25]. In the calculations, a trial Hamiltonian

\[ H_0 = -\lambda \sum_i s_i \]  

(20)

which corresponds to a system composed of \( N \) non-interacting single spins, is considered. From the partition function corresponding to a system with just one spin, the free energy of the \( N \) non-interacting spins system has been obtained by making use of approximating the partition function of the system with \( N \) independent spins as the product of \( N \) partition functions for systems with single spin. This "factorization scheme" is the same as the one
used in [22] to obtain the quantal distribution functions. By the help of this approximation, after some algebra, generalized mean-field magnetization and generalized mean-field free energy have been established respectively:

\[
\langle s \rangle_0^q = \frac{1 + \beta Jz (1 - q) \langle s \rangle_0^q}{1 + \beta Jz (1 - q) \langle s \rangle_0^q} - \frac{1 - \beta Jz (1 - q) \langle s \rangle_0^q}{1 + \beta Jz (1 - q) \langle s \rangle_0^q} \]

(21)

\[
(F_{mf})_q = \frac{(F_0)_q [1 + \beta (1 - q)\frac{1}{2} Jz N(\langle s \rangle_0^2) + \frac{1}{2} Jz N(\langle s \rangle_0^2)]}{1 + \beta (1 - q) Jz N(\langle s \rangle_0^2)}
\]

(22)

where

\[
(F_0)_q = \frac{\left\{1 + \beta Jz (1 - q) \langle s \rangle_0^q\right\}^{1/(1-q)} + \left[1 - \beta Jz (1 - q) \langle s \rangle_0^q\right]^{1/(1-q)} N^{(1-q)}}{\beta (1 - q)} - 1.
\]

(23)

By investigating the graphical solutions of the magnetization and determining the minima of the free energy for various values of \( q \), an interval has been established for the Tsallis \( q \) index. This result is found to be the same as the interval obtained for the paramagnetic free spin systems in [27], namely \( 1 < q < 3 \). The other remarkable derivation of this work is that the critical temperature of the generalized mean-field Ising model has been evaluated such that:

\[
\frac{kT_c}{J} = qz
\]

(24)

which coincides with the results calculated in [29,39].

A few magnetic systems, which have been investigated within GTT, are also present, however we are contented to indicate what the subjects of these papers are about:

1) For the spin-1/2 Ising ferromagnet, a transmissivity variable which extends that defined for thermal magnetic systems has been proposed. By using this generalized transmissivity as well as duality arguments, the \( q \)-dependence of the critical temperature corresponding to the square lattice has been calculated [38].
2) In [39], the authors have calculated the phase diagram and the correlation length critical exponent \( \nu \) for the Ising ferromagnet in a self-dual hierarchical lattice which mimics the square lattice.

3) In order to illustrate the existence of a well behaved thermodynamical limit for the canonical ensemble, the ideal paramagnet has been discussed numerically. In addition to this, generalized Schottky anomaly and generalized Curie law are calculated and data collapse is exhibited for \( N \gg 1 \) [40].

**IV. ACKNOWLEDGMENTS**

The authors would like to thank Ayşe Erzan for encouraging them to prepare this manuscript. The authors are also deeply indebted to Constantino Tsallis for providing some of the references herein as well as the bibliography which contains a complete list of the works on this subject.

[1] P. T. Landsberg, *J. Stat. Phys.*, 35 (1984) 159 ; A. M. Salzberg, *J. Math. Phys.*, 6 (1965) 158 ; L. G. Taff, *Celestial Mechanics*, (John Wiley, New York, 1985) p.437.

[2] B. J. Hiley and G. S. Joyce, *Proc. Phys. Soc.*, 85 (1965) 494 ; S. A. Cannas, *Phys. Rev.*, B (1996), in press.

[3] M. F. Shlesinger and B. D. Hughes, *Physica A109* (1981) 597 ; E. W. Montroll and M. F. Shlesinger, *J. Stat. Phys.*, 32 (1993) 209.

[4] J. O. Indekeu, *Physica A183* (1992) 439 ; J. O. Indekeu and A. Robledo, *Phys. Rev.*, E47 (1993) 4607.

[5] C. Tsallis, *Physica A221* (1995) 277.
[6] P. G. de Gennes, *C. R. Acad. Sci. Paris II* 292 (1981) 701; T. W. Burkhardt and E. Eisenriegler, *Phys. Rev. Lett.* 74 (1995) 3189.

[7] M. Ammi, T. Travers and J. P. Troadec, *J. Phys.*, D20 (1987) 424; C. H. Liu and S. R. Nagel, *Phys. Rev. Lett.*, 68 (1992) 2301.

[8] R. H. Kraichman and D. Montgomery, *Rep. Prog. Phys.*, 43 (1980) 547.

[9] B. M. Boghosian, *Phys. Rev.*, E53 (1996), in press.

[10] C. Tsallis, *Phys. Lett.*, A195 (1994) 329.

[11] S. F. Özeren, F. Büyükkılıç, U. Tunakli and D. Demirhan, "Statistical Mechanical Properties of q-deformed Landau Diamagnetism", preprint (1996).

[12] S. F. Özeren, U. Tunakli, F. Büyükkılıç and D. Demirhan, "Generalization of the Landau Diamagnetism within Tsallis Thermostatistics", preprint (1996).

[13] C. Tsallis, *J. Stat. Phys.*, 52 (1988) 479.

[14] A. M. Mariz, *Phys. Lett.*, A165 (1992) 409; J. D. Ramshaw, *Phys. Lett.*, A175 (1993) 169; 171.

[15] E. M. F. Curado and C. Tsallis, *J. Phys.*, A24 (1991) L69; Corrigenda: A24 (1991) 3187; A25 (1992) 1019.

[16] C. Tsallis, *Quimica Nova* 17 (6) (1994) 468.

[17] N. Ito and C. Tsallis, *Nuovo Cimento* 11 (1989) 907.

[18] R. F. S. Andrade, *Physica* A175 (1991) 285.

[19] R. F. S. Andrade, *Physica* A203 (1994) 486.

[20] A. Plastino and A. R. Plastino, *Phys. Lett.*, A177 (1993) 177.

[21] A. R. Plastino and A. Plastino, *Physica* A202 (1994) 438.
[22] F. Büyükkılıç and D. Demirhan, Phys. Lett., A181 (1993) 24; F. Büyükkılıç, D. Demirhan and A. Güleç, Phys. Lett., A197 (1995) 209.

[23] E. P. da Silva, C. Tsallis and E. M. F. Curado, Physica A199 (1993) 137; 203 (1994) E160; A. Chame and E. V. L. de Mello, J. Phys., A27 (1994) 3663.

[24] D. A. Stariolo, Phys. Lett., A185 (1994) 262.

[25] A. Plastino, C. Tsallis, J. Phys., A26 (1993) L893.

[26] A. R. Plastino, A. Plastino and C. Tsallis, J. Phys., A27 (1994) 5707.

[27] F. Büyükkılıç and D. Demirhan, Z. Phys., B99 (1995) 137.

[28] F. D. Nobre and C. Tsallis, Physica A213 (1995) 337.

[29] E. F. Sarmento, Physica A218 (1995) 482.

[30] F. Büyükkılıç, D. Demirhan and U. Tırnaklı, Tr. J. Phys., 20 (1996) 75.

[31] F. Büyükkılıç, D. Demirhan and U. Tırnaklı, "Generalization of the Mean-field Ising Model within Tsallis Thermostatistics", Physica A (1996), in press.

[32] C. Tsallis, F. C. Sa Barreto and E. D. Loh, Phys. Rev., E52 (1995) 1447.

[33] U. Tırnaklı, F. Büyükkılıç and D. Demirhan, "Generalized Distribution Functions and an Alternative Generalization of the Planck Radiation Law", submitted to Physica A (1996).

[34] A. Plastino and M. Portesi, Physica A (1996), in press.

[35] S. Curilef and C. Tsallis, Physica A215 (1995) 542.

[36] A. K. Rajagopal, Phys. Rev. Lett., 74 (1995) 1048; Physica B212 (1995) 309; G. Kaniadakis, A. Lavagno and P. Quarati, "Generalized Fractional Statistics", preprint (1995).

[37] S. Curilef, Phys. Lett., A (1996), in press.

[38] L. R. da Silva and H. E. Stanley, "Duality-based Approximation for the Critical Point of the
Square Lattice Ising Ferromagnet within Tsallis Statistics”, preprint (1996).

[39] S. A. Cannas and C. Tsallis, Z. Phys., B100 (1996) 623.

[40] F. D. Nobre and C. Tsallis, "Localized-spins Ideal Paramagnet within Nonextensive Thermodynamics”, preprint (1996).

[41] A. R. Plastino and A. Plastino, Phys. Lett., A174 (1993) 384.

[42] L. S. Lucena, L. R. da Silva and C. Tsallis, Phys. Rev., E51 (1995) 6247.

[43] P. A. Alemany and D. H. Zanette, Phys. Rev. E49 (1994) R956.

[44] C. Tsallis, S. V. F. Levy, A. M. C. Souza and R. Maynard, Phys. Rev. Lett. 75 (1995) 3589 ; C. Tsallis, A. M. C. Souza and R. Maynard, in "Levy Flights and Related Phenomena in Physics”, Eds. M. F. Shlesinger, U. Frisch and G. M. Zaslavsky (Springer, Berlin, 1995), page 269.

[45] G. Kaniadakis, A. Lavagno and P. Quarati, Phys. Lett. B369 (1996) 308.

[46] C. Tsallis, in "New Trends in Magnetism, Magnetic Materials and Their Applications”, Eds. J. L. Moran-Lopez and J. M. Sanchez (Plenum Press, New York, 1994), page 451 ; in "Chaos, Solitons and Fractals”, Eds. G. Marshall (Pergamon Press, 1994), page 539.

[47] G. A. Raggio, J. Math. Phys., 36 (1995) 4785 ; G. R. Guerberoff, P. A. Purgy and G. A. Raggio, J. Math. Phys., (1996), in press ; G. R. Guerberoff and G. A. Purgy, J. Math. Phys., (1996), in press.

[48] U. Tornahl, D. Demirhan and F. B"uy"ukkil"uc, "Comparison of the Standard Statistical Thermodynamics (SST) with the Generalized Statistical Thermodynamics (GST) Results for the Ising Chain”, preprint (1995).