An upper bound on the fluctuations of a second class particle

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Abstract

This note proves an upper bound for the fluctuations of a second-class particle in the totally asymmetric simple exclusion process. The proof needs a lower tail estimate for the last-passage growth model associated with the exclusion process. A stronger estimate has been proved for the corresponding discrete time model, but not for the continuous time model we work with. So we take the needed estimate as a hypothesis. The process is assumed to be initially in local equilibrium with a slowly varying macroscopic profile. The macroscopic initial profile is smooth in a neighborhood of the origin where the second-class particle starts off, and the forward characteristic from the origin is not a shock. Given these assumptions, the result is that the typical fluctuation of the second-class particle is not of larger order than \( n^{2/3}(\log n)^{1/3} \), where \( n \) is the ratio of the macroscopic and microscopic space scales. The conjectured correct order should be \( n^{2/3} \). Landim et al. have proved a lower bound of order \( n^{5/8} \) for more general asymmetric exclusion processes in equilibrium. Fluctuations in the case of shocks and rarefaction fans are covered by earlier results of Ferrari–Fontes and Ferrari–Kipnis.

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1 Introduction and result

We study the motion of a second class particle in a totally asymmetric simple exclusion process on the one-dimensional integer lattice \( \mathbb{Z} \). This process describes the evolution of indistinguishable particles that randomly jump to the right on the lattice, one step at a time. Jumps to already occupied sites are prohibited. The state of the process at time \( t \geq 0 \) is the configuration \( \eta(t) = (\eta_i(t) : i \in \mathbb{Z}) \in \{0, 1\}^\mathbb{Z} \) of occupation numbers, where \( \eta_i(t) = 1 \) if site \( i \) is occupied by a particle at time \( t \), and \( \eta_i(t) = 0 \) if site \( i \) is vacant at time \( t \).

The process is constructed on a probability space on which are defined the initial particle configuration \( \eta(0) = (\eta_i(0))_{i \in \mathbb{Z}} \), and independently of \( \eta(0) \), a collection \( \{D_i : i \in \mathbb{Z}\} \) of mutually independent rate 1 Poisson point processes on the time axis \((0, \infty)\). \( D_i \) is the random set of time points (or epochs) when a jump from site \( i \) to site \( i + 1 \) is attempted. Such a jump is executed at an epoch \( t \) of \( D_i \) if immediately prior to time \( t \) site \( i \) is occupied and site \( i + 1 \) is vacant. In other words \( \eta_i(t-) = 1 \) and \( \eta_{i+1}(t-) = 0 \), and then after the jump \( \eta_i(t) = 0 \) and \( \eta_{i+1}(t) = 1 \).

The dynamics can be represented by the generator \( L \) that acts on bounded cylinder functions \( f \) on the state space \( \{0, 1\}^\mathbb{Z} \):

\[
L f(\eta) = \sum_{i \in \mathbb{Z}} \eta_i (1 - \eta_{i+1}) [f(\eta^{i,i+1}) - f(\eta)].
\]

Here \( \eta^{i,i+1} \) is the configuration that results from the jump of a single particle from site \( i \) to site \( i + 1 \).

The position of a second class particle is defined as follows. Let \( X(0) \in \mathbb{Z} \) be a random initial position, and suppose that initially \( \eta_{X(0)}(0) = 0 \). Define a second initial configuration \( (\tilde{\eta}_i(0) : i \in \mathbb{Z}) \) that differs from \( \eta(0) \) only at site \( X(0) \): \( \tilde{\eta}_{X(0)}(0) = 1 \) and \( \tilde{\eta}_i(0) = \eta_i(0) \) for \( i \neq X(0) \). Run the processes \( \eta(t) \) and \( \tilde{\eta}(t) \) so that they read the same Poisson jump time processes \( \{D_i\} \). Then there is always a unique site \( X(t) \) at which the two processes differ: \( \tilde{\eta}_{X(t)}(t) = \eta_{X(t)}(t) + 1 \) and \( \tilde{\eta}_i(t) = \eta_i(t) \) for \( i \neq X(t) \). This defines the position \( X(t) \) of the second class particle.

We refer to the literature for further details of the construction of these processes. See Chapters III.1–2 in [9].

Now we consider the hydrodynamic limit setting. Assume given a measurable function \( 0 \leq \rho_0(x) \leq 1 \) on \( \mathbb{R} \). Suppose we have a sequence \( \eta^n = (\eta^n_i)_{i \in \mathbb{Z}} \) of random initial configurations that satisfy, for all finite \( a < b \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=\lceil na \rceil + 1}^{\lceil nb \rceil} \eta^n_i = \int_a^b \rho_0(x) dx \quad \text{in probability.} \tag{1}
\]

From this assumption follows a hydrodynamic limit. Let \( \eta^n(t) \) denote the process with
The initial configuration $\eta^n$. Then for all finite $a < b$ and $0 \leq t < \infty$,

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{i=\lfloor na \rfloor + 1}^{\lfloor nb \rfloor} \eta^n_i(nt) = \int_a^b \rho(x,t)dx \text{ in probability. (2)}
$$

The macroscopic profile $\rho(x,t)$ is the unique entropy solution of the nonlinear scalar conservation law

$$
\rho_t + f(\rho) \frac{\partial}{\partial x} = 0, \quad \rho|_{t=0} = \rho_0,
$$

with flux function

$$
f(\rho) = \rho(1-\rho). \quad (4)
$$

To describe the behavior of a second class particle in this scaling, we construct the solution $\rho(x,t)$ of (3) via the Lax-Oleinik formula, and then show how the characteristics of (3) are defined in this setting.

Let $g$ be the nonincreasing, nonnegative convex function on $\mathbb{R}$ defined by

$$
g(x) = \sup_{0 \leq \rho \leq 1} \{f(x) - x\rho\} = \begin{cases} -x, & x < -1 \\ (1/4)(1-x)^2, & -1 \leq x \leq 1 \\ 0, & x \geq 1. \end{cases} \quad (5)
$$

Define an antiderivative $u_0$ of $\rho_0$ by

$$
u_0(0) = 0 \quad \text{and} \quad u_0(b) - u_0(a) = \int_a^b \rho_0(x)dx \quad \text{for all } a < b. \quad (6)
$$

For $x \in \mathbb{R}$ set $u(x,0) = u_0(x)$, and for $t > 0$

$$
u(x,t) = \sup_{y \in \mathbb{R}} \left\{ u_0(y) - tg \left( \frac{x-y}{t} \right) \right\} \quad (7)
$$

The supremum is attained at some $y \in [x-t, x+t]$. The function $u$ is uniformly Lipschitz on $\mathbb{R} \times [0, \infty)$, nonincreasing in $t$ and nondecreasing in $x$. (7) is the Hopf-Lax formula. It defines $u(x,t)$ as the unique viscosity solution of the Hamilton-Jacobi equation

$$
u_t + f(u_x) = 0, \quad u(x,0) = u_0(x).
$$

Define the minimal and maximal Hopf-Lax maximizers in (7) by

$$
y^- (x,t) = \inf \left\{ y \geq x - t : u(x,t) = u_0(y) - t g \left( \frac{x-y}{t} \right) \right\}
$$

and

$$
y^+ (x,t) = \sup \left\{ y \leq x + t : u(x,t) = u_0(y) - t g \left( \frac{x-y}{t} \right) \right\}.
$$
The entropy solution of (3) is defined by the Lax-Oleinik formula:

$$\rho(x, t) = -g'(\frac{x - y^\pm(x, t)}{t}).$$

(8)

This definition makes sense a.e. because for a fixed $t$, $y^-(x, t) = y^+(x, t)$ for all but countably many $x$. The derivative $u(x, t)$ exists and equals $\rho(x, t)$ for all $(x, t)$ such that $y^-(x, t) = y^+(x, t)$. A point $(x, t)$ for $t > 0$ is a shock if $y^-(x, t) \neq y^+(x, t)$. Equivalently, $\rho$ is not continuous at $(x, t)$.

The minimal and maximal forward characteristics are defined for $b \in \mathbb{R}$, $t > 0$, as

$$w^-(b, t) = \inf\{x : y^+(x, t) \geq b\} \quad \text{and} \quad w^+(b, t) = \sup\{x : y^-(x, t) \leq b\}.$$

(9)

The forward characteristics $w^\pm(b, t)$ are Filippov solutions of the ordinary differential equation $dx/dt = f'(\rho(x, t))$, $x(0) = b$. If $\rho_0$ is continuous at $b$, the forward characteristic is unique, in other words $w(b, t) = w^\pm(b, t)$.

Now return to the hydrodynamic limit setting where a sequence of processes $\eta^n(t)$ is assumed to satisfy (1). Let $X_n(t)$ be the position of a second class particle in process $\eta^n(t)$. Assume the initial location is always the origin: $X_n(0) = 0$ a.s. Assume also that the forward characteristic from 0 is unique, so $w(0, t) = w^\pm(0, t)$. Then we have the law of large numbers

$$\lim_{n \to \infty} \frac{1}{n} X_n(nt) = w(0, t) \quad \text{in probability.}$$

(10)

Original references for the hydrodynamic limits above are [11, 15], and for the second class particle limit [3, 12, 16]. See [7, 18] for general treatments of the macroscopic behavior of the exclusion process, and [2] for the basic p.d.e. theory used above.

The result of our note is on the fluctuations from the limit (10). First a brief mention of known results. Let $\nu_{\lambda, \rho}$ denote the product measure on the state space $\{0, 1\}^\mathbb{Z}$ under which particles are present with density $\lambda$ ($\rho$) to the left (right) of the origin. Put a second class particle initially at the origin. Now there is only a single process, not a sequence. Consider two cases.

(i) If $\lambda < \rho$ then the characteristic from the origin, $w(0, t) = t(1 - \lambda - \rho)$, is a shock. In this case $X(t)$ has diffusive fluctuations. It satisfies a central limit theorem with explicitly known variance in the scale $\sqrt{t}$. This result is from Ferrari and Fontes [4], and is covered in Liggett’s monograph [9].

(ii) If $\lambda > \rho$ then the characteristic from the origin is not unique, and we have $w^-(0, t) = t(1 - 2\lambda)$, $w^+(0, t) = t(1 - 2\rho)$. $X(t)$ has macroscopic fluctuations: $t^{-1}X(t)$ converges weakly to a uniform distribution on $[w^-(0, t), w^+(0, t)]$. This result is due to Ferrari and Kipnis [3].

In particular, there is currently no definitive result in the equilibrium case $\lambda = \rho$. In this case it is conjectured that the typical fluctuations are of order $t^{2/3}$. Recent results
on the $t^{1/3}$ fluctuations for growth models provide corroboration for this conjecture, see [10]. Landim et al. [8] proved a $t^{5/4}$ lower bound on the variance in the following weak sense: there exists a constant $C > 0$ such that for small enough $\lambda > 0$,

$$
\int_0^{\infty} e^{-\lambda t} \text{Var}(X(t)) dt \geq C \lambda^{-9/4}.
$$

A large deviation bound for the second class particle in the equilibrium situation was proved in [17], and used to prove central limit theorems for additive functionals of the exclusion process.

We prove an upper bound on the size of the typical fluctuation in the limit (10). Make the following assumptions on the macroscopic profile around the origin:

$$
\rho_0 \text{ is continuously differentiable in a neighborhood of the origin, and } 0 < \rho_0(0) < 1. \text{ Suppose } t > 0 \text{ is such that } \rho_0'(0) < 1/(2t). \tag{11}
$$

Smoothness around the origin forces the forward characteristic to be unique, so we can denote this value by $w(0, t) = w^\pm(0, t)$. The assumptions $0 < \rho_0(0) < 1$ and $\rho_0'(0) < 1/(2t)$ are for technical convenience: the first implies that $w(0, t)$ lies in the interior of $(-t, t)$, and the second that the mapping from $x \to y^\pm(x, t)$ is Lipschitz both ways for $x$ close enough to $w(0, t)$.

Secondly, we need to assume that $w(0, t)$ is not a shock:

$$
y^-(w(0, t), t) = y^+(w(0, t), t). \tag{12}
$$

The limit (10) required only assumption (1). We need to strengthen this in order to have sharper control over initial fluctuations. We assume that the initial particle configurations are in local equilibrium with macroscopic profile $\rho_0$:

$$
\text{For each fixed } n, \text{ the random variables } (\eta_i^n)_{i \neq 0} \text{ are mutually independent, and for each } i, E[\eta_i^n] = n \int_{(i-1)/n}^{i/n} \rho_0(x) dx. \tag{13}
$$

The second class particle starts at the origin, so the origin is left empty in the initial exclusion configurations $[\eta_0^n(0) = 0]$.

The proof of our fluctuation bound relies on three ingredients. (i) A bound on the typical lower tail fluctuation of the last-passage growth model associated with the asymmetric exclusion process. (ii) A sharper bound on the upper tail deviations of the growth model. (iii) A variational representation for the location of the second class particle.

We face a dilemma: ingredients (ii) and (iii), the upper tail bound and the variational representation, have been proved for the totally asymmetric exclusion process in continuous time considered here [14, 16]. But point (i), the lower tail bound, has presently been proved only for discrete time exclusion with geometric waiting times.
The estimate we need is just a little more than what follows from Johansson’s distributional limit [6, eqn. (1.22)] for continuous time exclusion. Baik et al. [1] have proved a much stronger bound for discrete time exclusion, and there should be no doubt that an analogous estimate is true for continuous time too. For this reason we feel comfortable in taking the lower tail estimate we need as an extra hypothesis.

Consider the following last-passage growth model. Let \( N = \{1, 2, 3, \ldots\} \) be the set of natural numbers, and let \( \{u_{i,j} : (i, j) \in \mathbb{N}^2\} \) be i.i.d. exponential mean 1 random variables. Set

\[
H(M, N) = \max_\sigma \sum_{(i,j) \in \sigma} u_{i,j},
\]

where the maximum is over lattice paths

\[
\sigma = \{(1,1) = (i_1, j_1), (i_2, j_2), \ldots, (i_{M+N-1}, j_{M+N-1}) = (M, N)\}
\]
in \( \mathbb{N}^2 \) that take only up-right steps:

\[
(i_{m+1}, j_{m+1}) - (i_m, j_m) = (0, 1) \text{ or } (1, 0) \text{ for each } m.
\]

Johansson [6, Theorem 1.6] proved a distributional limit for \( H([n\alpha], [n\beta]) \) as \( n \to \infty \), \( (\alpha, \beta) \in \mathbb{R}_+^2 \). What we need for our proof is the following estimate.

**Hypothesis H.** Suppose \( \alpha_n \to \alpha > 0 \) and \( \beta_n \to \beta > 0 \) are convergent sequences of positive numbers such that, for a constant \( B < \infty \),

\[
|\alpha_n - \alpha| + |\beta_n - \beta| \leq Bn^{-1/3} (\log n)^{1/3} \text{ for all } n.
\]

Let \( \varepsilon > 0 \). Then, if \( C \) is fixed large enough,

\[
P\left\{ H([n\alpha_n], [n\beta_n]) < n(\sqrt{\alpha_n} + \sqrt{\beta_n})^2 - Cn^{1/3}\right\} \leq \varepsilon \text{ for all } n.
\]

The connection of \( H(M, N) \) to exclusion is this: start the exclusion process with all sites to the left of site 1 occupied, and all sites to the right of site 0 empty. Then the first time when there are \( j \) particles to the right of site \( i \) is distributed as \( H(i + j, j) \) for \( j > (-i) \lor 0 \). Rost [13] treated the hydrodynamic limit of this particular exclusion process without the last-passage formulation. There seems to be no first paper to cite as the original source of the last-passage connection. This author began using the last-passage representation in 1995, by which time it was certainly known by a number of people. In one conversation A. Gandolfi was credited with making this observation.

An alternative course for this paper would be to reprove ingredients (ii) and (iii) for discrete time exclusion, following [14, 16]. Then in place of (16) we could use the Baik et al. [1] estimate for the discrete time growth model. Our theorem would then be for discrete time exclusion. We chose the present course since continuous time exclusion is the process that most people prefer to work with. And also to avoid extra work, since in any case our approach does not quite get the optimal order.

We now state the theorem.
Theorem 1.1 Assume (11), (12), and (13), and assume Hypothesis H. Let \( \eta^n(t) \) be the totally asymmetric simple exclusion process with initial configuration \( \eta^n \). Let \( X_n(t) \) be the location of a second class particle in the process \( \eta_n(t) \), started at the origin \( X_n(0) = 0 \). Let \( \varepsilon > 0 \). Then there exists a constant \( b < \infty \) such that, for all \( n \),

\[
P \left\{ \left| X_n(nt) - nw(0,t) \right| \geq bn^{2/3} \left( \log n \right)^{1/3} \right\} \leq \varepsilon.
\]

The rest of the note contains the proof. We start with properties of the Hopf-Lax formula. Then we explain the variational coupling representation of the totally asymmetric exclusion process. This representation is used in conjunction with two probability estimates, one from [14], the other from hypothesis (16).

2 Properties of the macroscopic profile

Recall the connections (6) and (8). Abbreviate \( r = w(0,t) \) throughout the proof. Let

\[
I(x,t) = \{ y \in \mathbb{R} : u(x,t) = u_0(y) - t g((x-y)/t) \}
\]

denote the set of maximizers in the Hopf-Lax formula (7).

Lemma 2.1 Suppose \( u_0 \) is twice continuously differentiable in a neighborhood of 0. Assume that \( y^\pm(r, t) = 0 \) and \( u_0'(0) < 1/(2t) \). Let \( a_0 > 0 \). Then there exist \( \delta_1 > 0 \) and \( 0 < c_0 < \infty \) such that the following holds: if \( x \in [r - \delta_1, r + \delta_1] \), \( y \in I(x,t) \), and \( \eta \in [-a_0, a_0] \), then

\[
u_0(y) - t g \left( \frac{x-y}{t} \right) \geq u_0(\eta) - t g \left( \frac{x-\eta}{t} \right) + c_0(\eta - y)^2. \tag{17}
\]

Furthermore, if \( x_0 \in [r - \delta_1, x) \) is distinct from \( x \in [r - \delta_1, r + \delta_1] \), and \( y_0 \in I(x_0,t) \), then

\[
c_0(x - x_0) \leq y - y_0 \leq c_0^{-1}(x - x_0). \tag{18}
\]

Proof. We first prove (18). By the assumption \( y^\pm(r, t) = 0 \), \( I(x,t) \to \{0\} \) as \( x \to r \). So for \( x \) close enough to \( r \) and \( y \in I(x,t) \),

\[
u_0'(y) = -g'((x-y)/t) = (t-x+y)/(2t), \tag{19}
\]

and similarly for \( x_0 \) and \( y_0 \). This shows \( y \neq y_0 \) if \( x \neq x_0 \). Hence

\[
\frac{x - x_0}{y - y_0} = 1 - 2t \cdot \frac{u_0'(y) - u_0'(y_0)}{y - y_0} = 1 - 2tu_0''(\theta)
\]
by the mean value theorem, where \( \theta \in (y_0, y) \). If \( y, y_0 \) are close enough to 0, \( 1 - 2tu''_0(\theta) \) is bounded and bounded away from 0. This can be achieved by taking \( x, x_0 \in [r - \delta_1, r + \delta_1] \) for small enough \( \delta_1 \). This proves \([18]\).

By the assumptions, we can choose \( \alpha, \delta_2 > 0 \) so that \( u_0 \) is \( C^2 \) and \( u''_0 < 1/(2t) - \alpha \) on \((-3\delta_2, 3\delta_2)\). Shrink the \( \delta_1 > 0 \) chosen in the previous paragraph so that \( I(x, t) \subseteq (-\delta_2, \delta_2) \) for all \( x \in [r - \delta_1, r + \delta_1] \). By continuity and compactness, there exists \( \delta_3 > 0 \) such that

\[
0 \leq u_0(y) - tu \left( \frac{x - y}{t} \right) \leq u_0(\eta) - tu \left( \frac{x - \eta}{t} \right) + \delta_3
\]

for all \( x \in [r - \delta_1, r + \delta_1], y \in I(x, t) \) and \( 2\delta_2 \leq |\eta| \leq a_0 \). Thus for these \( \eta \), \([17]\) holds with \( c_0 = \delta_3/(a_0 + \delta_2)^2 \).

Keeping still \( x \in [r - \delta_1, r + \delta_1] \) and \( y \in I(x, t) \) so that \( y \in (-\delta_2, \delta_2) \), for \( \eta \in (-2\delta_2, 2\delta_2) \) use Taylor’s theorem with the Lagrange form of the remainder term \([13\, \text{p. 195}]\):

\[
u_0(\eta) - tu \left( \frac{x - \eta}{t} \right) = u_0(y) - tu \left( \frac{x - y}{t} \right) + \frac{1}{2}(\eta - y)^2 \left\{ u''_0(y + \theta(\eta - y)) - \frac{1}{2t} \right\} \\
\leq u_0(y) - tu \left( \frac{x - y}{t} \right) - \frac{\alpha}{2}(\eta - y)^2.
\]

Here \( \theta \in (0, 1) \) depends on \( \eta \) and \( y \), but the upper bound on \( u''_0 \) works in all cases because \( y + \theta(\eta - y) \in (-2\delta_2, 2\delta_2) \). 

3 The variational coupling

We summarize briefly the variational coupling representation of the process and the second class particle \([14, 15, 16]\). Now the exclusion process is constructed in terms of a process \( z(t) = (z_i(t))_{i \in \mathbb{Z}} \) of labeled particles that move on \( \mathbb{Z} \) subject to the constraint

\[
0 \leq z_{i+1}(t) - z_i(t) \leq 1 \quad \text{for all } i \in \mathbb{Z} \text{ and } t \geq 0.
\]

In the graphical construction, \( z_i \) attempts to jump one step to the left at epochs of \( D_i \). A jump is suppressed if it leads to a violation of \([20]\). We start the process so that \( z_0(0) = 0 \). The connection between the exclusion \( \eta(t) \) and the process \( z(t) \) is that

\[
\eta_i(t) = z_i(t) - z_{i-1}(t).
\]

This equation is used both ways: given the initial configuration \( \eta(0) \), define initial \( z(0) \) by \( z_0(0) = 0 \) and by \([21]\). Construct the process \( z(t) \) by the graphical representation. And then define \( \eta(t) \) by \([21]\).
Construct a family \( \{ w^k(t) : k \in \mathbb{Z} \} \) of auxiliary processes. The initial configuration \( w^k(0) \) depends on the initial position \( z_k(0) \) through a global shift:

\[
w^k_i(0) = \begin{cases} 
    z_k(0), & i \geq 0 \\
    z_k(0) + i, & i < 0.
\end{cases}
\]  

(22)

The processes \( \{ w^k(t) \} \) are coupled to each other and to \( z(t) \) through the Poisson processes \( \{ D_i \} \), so that particle \( w^k_i \) attempts jumps to the left at epochs of \( D_{i+k} \). The key variational coupling property says that for all \( i \in \mathbb{Z} \) and \( t \geq 0 \),

\[
z_i(t) = \sup_{k \in \mathbb{Z}} w^k_{i-k}(t) \quad \text{a.s.}
\]

(25)

It is convenient to decompose \( w^k(t) \) into a sum of the initial position defined by (22) and the increment determined by the Poisson processes. Define a family of processes \( \{ \xi^k(t) \} \) by

\[
\xi^k_i(t) = z_k(0) - w^k_i(t) \quad \text{for } i \in \mathbb{Z}, \ t \geq 0.
\]

(23)

We can think of \( \xi^k \) as a growth model on the upper half plane, so that \( \xi^k_i \) gives the height of the interface above site \( i \). Its dynamics are specified by saying that \( \xi^k_i \) advances one step up at each epoch of \( D_{k+i} \), provided these inequalities are preserved:

\[
\xi^k_i(t) \leq \xi^k_{i-1}(t) \quad \text{and} \quad \xi^k_i(t) \leq \xi^k_{i+1}(t) + 1.
\]  

(24)

The connection of \( \xi^k \) with the exclusion process is this: start the exclusion process so that all sites from \( k \) to the left are occupied, and all sites from \( k+1 \) to the right are vacant. Then \( \xi^k_i(t) \) is the number of particles to the right of site \( k+i \) at time \( t \).

In terms of \( \xi \), the variational coupling can be expressed as

\[
z_i(t) = \sup_{k \in \mathbb{Z}} \{ z_k(0) - \xi^k_{i-k}(t) \}
\]  

(25)

Now suppose \( \eta_{X(0)}(0) = 0 \) and put a second class particle initially at the (possibly random) location \( X(0) \). Then the later location of the second class particle satisfies

\[
X(t) = \inf \{ i \in \mathbb{Z} : z_i(t) = z_k(0) - \xi^k_{i-k}(t) \text{ for some } k \geq X(0) \}.
\]  

(26)

We conclude this overview with two monotonicity properties. First, the coupling of the \( \xi^k \) processes through common Poisson clocks gives us this inequality:

\[
\xi^k_{i-k}(t) \leq \xi^k_{i-l}(t) \quad \text{for all } i, \text{ if } k \leq l.
\]  

(27)

Second, the variational coupling has this property.
Lemma 3.1 The following statement holds almost surely. Suppose $z_i(t) = w_{i-k}^k(t)$, and $j > i$. Then there exists an index $m$ such that $m \geq k$ and $z_j(t) = w_{j-m}^m(t)$.

Proof. First check that the statement is true at $t = 0$, by the definition of the family $\{w^k\}$ and the restriction $0 \leq z_{i+1} - z_i \leq 1$.

To prove it up to time $t_0$, fix indices $i_0 << 0 << i_1$ so that the Poisson jump time processes $D_l$ are empty up to time $t_0$ for $l \in \{i_0, i_0 + 1, i_1, i_1 + 1\}$. Since the positions $z_l$ and $w_{n-l}^n$ for $l \in \{i_0, i_0 + 1, i_1, i_1 + 1\}$ and $n \in \mathbb{Z}$ do not change up to time $t_0$, the statement of the lemma holds for $(i, j) = (i_0, i_0 + 1)$ and $(i_1, i_1 + 1)$ and $0 \leq t \leq t_0$. Now prove it simultaneously for pairs $(i, j)$, $i_0 < i < j \leq i_1$, by induction on the finitely many jump times in $\bigcup_{i_0 < i < i_1} D_l \cap (0, t_0]$.

Since the indices $i_0 << 0 << i_1$ can be chosen arbitrarily far away from the origin, this way the lemma is proved for all $i < j$ up to time $t_0$. □

4 Fluctuation bounds for $\xi$

Here is the lower tail bound for $\xi$ that we need. Recall the last-passage model $H(M, N)$ discussed in the introduction. Its precise connection with $\xi$ is

$$P\{\xi_i(t) < j\} = P\{H(i+j, j) > t\}.$$  (28)

Proposition 4.1 Let $t > 0$ and $\varepsilon > 0$. Then there is a finite constant $C > 0$ such that, for all $x \in [-t + \varepsilon, t - \varepsilon]$, all small enough $h > 0$ and all $n$,

$$P\left\{\xi_{[nx]}(nt) \leq ntg\left(\frac{x}{t}\right) - 2nh\right\} \leq \exp\left\{-n\left(\frac{4 \sqrt{2}}{3} \cdot \frac{\sqrt{t-x}}{t+x} \cdot h^{3/2} - Ch^{5/2}\right)\right\}.$$  (29)

Proof. According to Theorems 4.2 and 4.4 in [14],

$$\sup_n \frac{1}{n} \log P\{H([nr], [nw]) > nt\} = \lim_{n \to \infty} \frac{1}{n} \log P\{H([nr], [nw]) > nt\} = -\Psi_{w,t}(r)$$

where the rate function is defined by

$$\Psi_{w,t}(r) = \sqrt{(t - r - w)^2 - 4rw} - 2r \cosh^{-1}\left(\frac{t + r - w}{2 \sqrt{tw}}\right) - 2w \cosh^{-1}\left(\frac{t + w - r}{2 \sqrt{tw}}\right)$$  (30)

for values $r, w, t \geq 0$ that satisfy $\sqrt{w} + \sqrt{r} \leq \sqrt{t}$. Note that the limiting value is

$$(\sqrt{w} + \sqrt{r})^2 = \lim_{n \to \infty} \frac{1}{n} H([nw], [nr]).$$  (31)

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In case the reader wishes to compare with the source [14], note that the last-passage model is indexed slightly differently there. Our variable $w$ appears as a negative variable $x$ in [14]. Estimate (29) was in principle covered later by Johansson’s [6] bigger results, but the rate function is not explicitly available in [6].

Expanding $\Psi_{w,t}(\cdot)$ close to the value $u = (\sqrt{t} - \sqrt{w})^2$ gives

$$\Psi_{w,t}(u - h) = \frac{4}{3} \cdot \frac{w^{1/4}}{t^{1/4}u^{1/2}} \cdot h^{3/2} + O(h^{5/2}) \quad (32)$$

where the $O$-term is uniform over $w$ and $u$ bounded away from 0, and $h > 0$ is small enough. Finally for large enough $n$ so that $nh > 1$,

$$P \left\{ \xi_{nx}(nt) \leq ntg(x/t) - 2nh \right\} \leq \epsilon$$

for all $n$. The corresponding bound on the right,

$$P \left\{ \xi_{nx}(nt) \geq ntg(x/t) - nh \right\} \leq \epsilon$$

follows by an argument so similar it is not worth repeating.

Let $\varepsilon_0 = \varepsilon/4$. The assumption $0 < u_n'(0) = \rho_0(0) < 1$ entails that

$$-t + 3\alpha_1 \leq r \leq t - 3\alpha_1 \quad (34)$$

for some $\alpha_1 > 0$ [by (19) applied to $(x, y) = (r, 0)$].

**Lemma 5.1** For sufficiently large $n$,

$$P \left\{ z^n_{[nr - bu_n]}(nt) = z^h_k - \xi^n_{[nr - bu_n] - k}(nt) \right\} \leq \varepsilon_0. \quad (35)$$
Proof. By Lemma \[\text{(3.1)}\] if 
\[z^n_{[nr-bu_n]}(nt) = z^n_k - \xi^n_{[nr-bu_n]-k}(nt)\]
for some \(k \geq \alpha_1 n\), then also 
\[z^n_{[nr]}(nt) = z^n_m - \xi^n_{[nr]-m}(nt)\]
for some \(m \geq \alpha_1 n\). By passing to the hydrodynamic limit and by compactness, this would imply the existence of \(\eta \geq \alpha_1\) such that 
\[u(r, t) = u_0(\eta) - t\log(r - \eta)/t.\]
See [16, Prop. 5.1] for more details on this type of an argument. But now we have a contradiction, because by assumption 0 is the unique maximizer for \(u(r, t)\) in the Hopf-Lax formula \[\text{(3.3)}\]. \(\blacksquare\)

So now, except on an event of probability less than \(\varepsilon_0\), the inequality

\[X_n(nt) \leq nr - bu_n\]  \hspace{1cm} (36)

implies that

\[z^n_{[nr-bu_n]}(nt) = z^n_k - \xi^n_{[nr-bu_n]-k}(nt)\]  \hspace{1cm} (37)

holds for some \(0 \leq k \leq \alpha_1 n\), if \(n\) is large enough. The follows from combining variational formula \[\text{(28)}\] with Lemmas \[\text{(3.1)}\] and \[\text{(3.2)}\].

Let \(y_n \in I(r - bu_n n^{-1}, t)\) be a Hopf-Lax maximizer for the point \((r - bu_n n^{-1}, t)\). Note that \(y_n < 0\), and \(y_n \nearrow 0\) as \(n \nearrow \infty\). Together with \[\text{(17)}\], the variational equation \[\text{(24)}\] implies that

\[z^n_{[ny_n]} - \xi^n_{[nr-bu_n]-[ny_n]}(nt) \leq z^n_k - \xi^n_{[nr-bu_n]-k}(nt)\]  \hspace{1cm} (38)

for some \(0 \leq k \leq \alpha_1 n\). For each \(n\), set \(k_j = \lfloor j n^{1/3} \rfloor\) for \(0 \leq j \leq \alpha_1 n^{2/3}\). For \(k_j \leq k \leq k_{j+1}\), we have \(z^n_k \leq z^n_{k_{j+1}}\), and \[\text{(27)}\] gives

\[\xi^n_{[nr-bu_n]-k_j}(nt) \leq \xi^n_{[nr-bu_n]-k}(nt).\]

From \[\text{(38)}\] we then get an intermediate conclusion that, on the event \[\text{(36)}\],

\[\xi^n_{k_j}_{[nr-bu_n]-k_{j}}(nt) \leq \xi^n_{[ny_n]}(nt) + z^n_{k_{j+1}} - z^n_{[ny_n]}\]

for some \(0 \leq j \leq \alpha_1 n^{2/3}\), except for an event of probability less than \(\varepsilon_0\).

Now we estimate the random variables on the right-hand side of the last inequality. By \[\text{(34)}\] and \[\text{(18)}\], for large \(n\) the point \(r - bu_n n^{-1} - y_n\) lies in \([-t + \alpha_1, t - \alpha_1]\). Having this point in the interior of \((-t, t)\) permits us to apply the hypothesis made in the Introduction. Note that \(bu_n n^{-1}\) is of order \(O(n^{-1/3} \log n)^{1/3}\), and so is \(y_n\) by \[\text{(15)}\]. Thus \[\text{(17)}\] is satisfied. Then by \[\text{(16)}\] and \[\text{(28)}\], we may fix a constant \(a_1 < \infty\) such that, for all \(n\),

\[P \{ \xi^n_{[ny_n]}(nt) \leq nt \log (r - bu_n n^{-1} - y_n) + a_1 n^{1/3}\} \geq 1 - \varepsilon_0.\]

For the initial state, if we increase \(a_1\) sufficiently, then

\[P \{ |z^n_{k_j} - z^n_{[ny_n]} - nu_0(k_j/n) + nu_0(y_n)| \leq a_1 (k_j - ny_n)^{1/2} (\log n)^{1/2}\]

for all \(0 \leq j \leq \alpha_1 n^{2/3}\} \geq 1 - \varepsilon_0\]
for all $n$. This is proved by an exponential Chebychev’s inequality, with separate arguments for the two sides.

Now outside an event of probability less than $3\varepsilon_0$, it follows from (38) that
\[
\xi_{[nt-bu_n]-k_j}(nt) \leq ntg(t^{-1}(r-bu_n n^{-1} - y_n)) + n(u_0(k_{j+1}/n) - u_0(y_n)) + a_1 n^{1/3} + a_1(k_{j+1} - ny_n)^{1/2}(\log n)^{1/2}
\]
(39)
for some $0 \leq j < \alpha_1 n^{2/3}$.

Set $x_j = r - bu_n n^{-1} - k_j n^{-1}$. Use (17) to write
\[
tg\left(\frac{r}{t} - \frac{bu_n}{nt} - \frac{y_n}{t}\right) + u_0(k_{j+1}/n) - u_0(y_n) \leq tg\left(\frac{r}{t} - \frac{bu_n}{nt} - \frac{k_{j+1}}{nt}\right) - a_2\left(\frac{k_{j+1}}{n} - y_n\right)^2
= tg(x_{j+1}/t) - a_2\left(\frac{k_{j+1}}{n} - y_n\right)^2,
\]
for a constant $a_2$. Then (38) gives
\[
\xi_{[nx_{j}]}^{k_j}(nt) \leq ntg(x_j/t) + a_1 n^{1/3} + a_1(k_{j+1} - ny_n)^{1/2}(\log n)^{1/2} - a_1 n(k_{j+1}/n - y_n)^2 + ntg(x_{j+1}/t) - ntg(x_j/t).
\]
(40)
By the Lipschitz property of $g$,
\[
ntg(x_{j+1}/t) - ntg(x_j/t) \leq a_3 n^{1/3},
\]
for a constant $a_3 < \infty$. Use (18) in the form
\[
y_n < -\delta_1bu_n n^{-1} = -\delta_1 b n^{-1/3}(\log n)^{1/3}.
\]
Substitute in $k_j = jn^{1/3}$ and $u_n = n^{2/3}(\log n)^{1/3}$, and use the inequalities $(p+q)^{1/2} \leq p^{1/2} + q^{1/2}$ and $(p+q)^2 \geq p^2 + q^2$ valid for $p, q \geq 0$. After these steps, the right-hand side of (40) is bounded above by
\[
ntg(x_j/t) + (a_1 + a_3)n^{1/3} + a_1(j + 1)^{1/2}n^{1/6}(\log n)^{1/2} + a_1 n^{1/2} |y_n|^{1/2}(\log n)^{1/2} - a_2(j + 1)^2 n^{-1/3} - a_2 ny_n^2
= ntg(x_j/t) + \left\{(a_1 + a_3)n^{1/3} - (a_2 b^2 \delta_1^2/4)n^{1/3}(\log n)^{2/3}\right\}
+ \left\{(a_1(j + 1)^{1/2}n^{1/6}(\log n)^{1/2} - a_2(j + 1)^2 n^{-1/3} - (a_2 b^2 \delta_1^2/4)n^{1/3}(\log n)^{2/3}\right\}
+ \left\{a_1 n^{1/2} |y_n|^{1/2}(\log n)^{1/2} - a_2(4/n)y_n^2\right\}
- (a_2 b^2 \delta_1^2/4)n^{1/3}(\log n)^{2/3}
\]
By fixing $b$ large enough relative to the other constants $\delta_1$ and $a_1-a_3$, the expressions in braces can be made nonpositive for all $n$ and $0 \leq j \leq \alpha_1 n^{2/3}$. Thus (40) implies that for some $0 \leq j \leq \alpha_1 n^{2/3}$,
\[
\xi_{[nx_j]}^{k_j}(nt) \leq ntg(x_j/t) - (a_2 b^2 \delta_1^2/4)n^{1/3}(\log n)^{2/3}.
\]
Recall that this event is a consequence of (36), except on an event of probability less than $3\varepsilon_0$, if $n$ is large enough. Set $c_2 = a_2b^2\delta_1^2/4$. We can summarize the development in the inequality

$$P\{X_n(nt) \leq nr - bu_n\} \leq 3\varepsilon_0$$

$$+ P\{\xi_{[nx_j]}^k(nt) \leq ntg(x_j/t) - c_2n^{1/3}(\log n)^{2/3} \}
\text{for some } 0 \leq j \leq \alpha_1n^{2/3}\}.$$  

Apply Prop. 4.1 to the last probability above with $h = (c_2/2)n^{-2/3}(\log n)^{2/3}$. This probability is less than $\varepsilon_0$, if $b$ (and hence $c_2$) is fixed large enough, and if $n \geq n_0$ for some $n_0$ that depends on $c_2$. The cutoff $n_0$ is required to make $h$ small enough for Prop. 4.1, after $c_2$ has been first fixed large enough. The values $n < n_0$ can then be accounted for by increasing $b$ suitably. This completes the proof of (33).

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