Idempotent functors that preserve cofiber sequences and split suspensions

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We show that an $f$–localization functor $L_f$ commutes with cofiber sequences of $(N-1)$–connected finite complexes if and only if its restriction to the collection of $(N-1)$–connected finite complexes is $R$–localization for some unital subring $R \subseteq \mathbb{Q}$. This leads to a homotopy theoretical characterization of the rationalization functor: the restriction of $L_f$ to simply connected spaces (not just the finite complexes) is rationalization if and only if $L_f(S^2)$ is nontrivial and simply connected, $L_f$ preserves cofiber sequences of simply connected finite complexes and for each simply connected finite complex $K$, there is a $k$ such that $\Sigma^k L_f(K)$ splits as a wedge of copies of $L_f(S^n)$ for various values of $n$.

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Introduction

Let $f : P \to Q$ be a continuous function from one CW complex to another. The $f$–localization functor $L_f$ is the universal example of a homotopically idempotent functor from spaces to spaces which carries $f$ to a weak equivalence. These functors are the primary—conceivably the only$^1$—examples of homotopically idempotent functors.

The $R$–localization functors are among the earliest defined and best behaved examples of localization functors, defined for (unital) subrings $R \subseteq \mathbb{Q}$; we denote them by $L_R$. There are various constructions of these, but all of them agree on simply connected spaces. They were constructed with the intention of lifting to spaces the algebraic operation of $R$–localization applied to homotopy and homology groups. This prescribed algebraic effect guarantees two nice homotopy theoretical properties of $R$–localization: $L_R$ carries cofiber sequences of simply connected spaces to cofiber sequences and similarly for fiber sequences.

$^1$Casacuberta, Scevenels and Smith have shown [2] that it is impossible to prove in ZFC that all homotopically idempotent functors take the form $L_f$ for some $f$; but in the presence of Vopěnka’s Principle (which is thought, but not known, to be consistent with ZFC), every homotopically idempotent functor is of the form $L_f$ for some well-chosen map $f$.
Rationalization is the special case $R = \mathbb{Q}$, and here the theory is most powerful. The central theorems of Quillen [6] and Sullivan [8] show that the homotopy theory of simply connected rational spaces (i.e., simply connected spaces for which $X \to L_\mathbb{Q}(X)$ is a weak equivalence) is perfectly modeled by the algebraic homotopy theories of DGLAs and of CDGAs, respectively. Rational spaces have two additional homotopy theoretic properties beyond those enjoyed by all simply connected $R$–local spaces: their suspensions split as wedges of copies of $L_\mathbb{Q}(S^n)$ for various values of $n$ and, dually, their loop spaces split as products of copies of $K(\mathbb{Q}, n)$.

Many noncomputational theorems about rationalizations which were first proved via the algebraic machinery provided by Quillen and Sullivan can also be proved by appealing to the basic homotopy theoretical properties noted above. For example, the celebrated Mapping Theorem for Lusternik–Schnirelmann category was originally proved using Sullivan models by Félix and Halperin [4], but later a much simpler proof was found that was based on the splitting of loop spaces; see Félix and Lemaire [5]. Examples like this led the author to wonder if every localization functor $L_f$ that satisfies the four homotopy theoretical properties listed above must be related to rationalization in some way.

In fact, rationalization is determined—overdetermined!—by these properties: it is the only $f$–localization functor of simply connected spaces that preserves cofiber sequences and splits suspensions. Furthermore, the restriction of $R$–localization to simply connected finite complexes is characterized simply by the fact that it commutes with cofiber sequences. These are the main results of this paper.

We write $\mathcal{K}(N)$ for the collection of all $(N - 1)$–connected finite complexes.

**Theorem 1** Let $f: P \to Q$ be a map of CW complexes. Let $N \geq 2$ and suppose $L_f(S^N)$ is simply connected and not contractible. Then the following are equivalent:

1. there is a subring $R \subseteq \mathbb{Q}$ and a natural transformation $\xi: L_R \to L_f$ such that $\xi_K$ is a weak equivalence for every $K \in \mathcal{K}(N)$;

2. $L_f$ commutes with cofiber sequences\(^2\) of spaces in $\mathcal{K}(N)$.

Taking $N = 2$ in Theorem 1, we obtain a characterization of the rationalization of simply connected spaces in terms of the elementary notions of homotopy theory.

\(^2\)There are natural comparison maps $\xi_\alpha: C_{L_f(\alpha)} \to L_f(C_{\alpha})$; $L_f$ commutes with the cofiber sequence $X \to Y \to C_{\alpha}$ if $\xi_\alpha$ is a weak equivalence.
Theorem 2  The restriction of a localization functor $L_f$ to simply connected spaces is rationalization if and only if the following three conditions hold:

- $L_f(S^2)$ is simply connected and not weakly contractible;
- $L_f$ commutes with cofiber sequences of simply connected finite complexes;
- if $K$ is a simply connected finite complex, then for some $k \in \mathbb{N}$, the suspension $\Sigma^k L_f(K)$ splits as a wedge of copies of $L_f(S^n)$ for various values of $n$.

Interestingly, the hypotheses of Theorem 2 only concern finite complexes, but the conclusion applies to all simply connected spaces. One of the referees has pointed out that we do not need the full power of the third property: all that is required is that the $f$–localization of certain Moore spaces $M(\mathbb{Z}/p, n)$ should split as wedges of localized spheres after repeated suspension.

We conclude this introduction with a bit of speculation. Our proof of Theorem 1 comes very close to showing that $L_f$ commutes with cofiber sequences (and hence restricts to $R$–localization on $K(N)$) if and only if $L_f$ ‘respects the smash and suspension structure of spheres’—that is, if and only if $\Sigma L_f(S^n)$ and $L_f(S^n) \wedge L_f(S^m)$ are $f$–local for all $m, n \geq N$. Using a relative version of the theory of resolving classes, this can be proved under the assumption that $f$ factors up to homotopy through a finite dimensional complex (we have not included that proof here). Is it true for all maps $f$?

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1 Preliminaries

We will work in a fixed convenient category $\mathcal{T}_*$ of pointed topological spaces; for example $\mathcal{T}_*$ could be the category of compactly generated weak Hausdorff spaces. If so inclined, the reader may pretend that this paper was written simplicially.

We write $\text{conn}(X) = n$ if $X$ is $n$–connected but not $(n + 1)$–connected.

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3See the author [7] for an account of the absolute theory.
1.1 Localization functors

Let \( f: P \to Q \) be a map between pointed CW complexes. A pointed space \( X \) is said to be \( f \)-local if the induced map

\[
f^*: \text{map}_*(Q, X) \to \text{map}_*(P, X)
\]

of pointed mapping spaces is a weak homotopy equivalence; a map \( q: X \to Y \) is called an \( f \)-equivalence if for every \( f \)-local space \( Z \) the induced map

\[
q^*: \text{map}_*(Y, Z) \to \text{map}_*(X, Z)
\]

is a weak homotopy equivalence. A map \( i: X \to L \) is said to be an \( f \)-localization of \( X \) if \( L \) is \( f \)-local and \( i \) is an \( f \)-equivalence.

The following important properties follow easily from the definitions.

**Lemma 3**

(a) An \( f \)-equivalence between \( f \)-local spaces is a weak equivalence.

(b) Let \( i: X \to L \) be \( f \)-localization and let \( g: X \to Z \) with \( Z \) \( f \)-local. Then in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Z \\
\downarrow{i} & & \downarrow{\gamma} \\
L & \xleftarrow{\delta} & \text{ } \text{ }
\end{array}
\]

there is a map \( \gamma \), unique up to homotopy, making the triangle commute up to homotopy.

A **coaugmented functor** is a functor \( F: \mathcal{T}_* \to \mathcal{T}_* \) equipped with a natural transformation (its coaugmentation) \( \iota: \text{id} \to F \). A coaugmented functor \( F \) is **homotopically idempotent** if for every \( X \) the maps

\[
\iota_{F(X)}, F(\iota_X): F(X) \to F(F(X))
\]

are homotopic to one another and both are weak equivalences. An \( f \)-localization functor is a homotopically idempotent functor \( L_f \) such that, for every space \( X \), the coaugmentation \( \iota_X: X \to L_f(X) \) is an \( f \)-localization of \( X \).

The main existence theorem is as follows [3, Theorem 1.A.3].

**Theorem 4** (Bousfield, Farjoun) For any map \( f: P \to Q \) between CW complexes, there exists an \( f \)-localization functor \( L_f \).

Here are some basic properties of \( f \)-localization.
Proposition 5  Let $f : P \to Q$ be a map of CW complexes.

(a) If $q : X \to Y$ is an $f$–equivalence, then so is $q \wedge \text{id}_Z$ for any space $Z$.

(b) $L_f(q)$ is a weak equivalence if and only if $q$ is an $f$–equivalence.

(c) For any $X$ and $Y$, the natural map $L_f(X \times Y) \to L_f(X) \times L_f(Y)$ is a weak equivalence.

Proof  Parts (a) and (b) are [3, Example 1.D.5 and Proposition 1.C.5 ], respectively. Part (c) is [3, 1.A.8(e.4)].

If $q : X \to Y$ is an $f$–equivalence, then every $f$–local space is also $q$–local, and so the transformation $j_{L_f}$ in the square

$\begin{array}{ccc}
\text{id} & \to & L_q \\
\downarrow & & \downarrow \text{L}_q(\text{id}) \\
L_f & \to & \text{L}_q \circ L_f
\end{array}$

evaluates to a weak equivalence $j_{L_f(X)} : L_f(X) \to L_q(L_f(X))$ for every space $X$. Thus we say that $\iota : \text{id} \to L_f$ factors through $j : \text{id} \to L_q$ ‘up to weak homotopy equivalence.’ Since $L_q \circ L_f(X) \sim L_f(X)$ for all $X$, the functor $L_q \circ L_f$ is a perfectly good choice of $f$–localization functor. In this situation we will abuse notation by silently redefining $L_f$ and referring to $L_q(\iota)$ as a comparison transformation $\xi : L_q \to L_f$.

Proposition 6  Suppose $q$ is an $f$–equivalence. If $L_q(X)$ is $f$–local, then the comparison $\xi_X : L_q(X) \to L_f(X)$ is a weak equivalence.

Proof  Any $q$–equivalence is a fortiori an $f$–equivalence. Since $X \to L_q(X)$ is a $q$–equivalence by definition and $L_q(X)$ is $f$–local by hypothesis, we have weak equivalences $L_f(X) \to L_f(L_q(X)) \leftarrow L_q(X)$.

1.2 $R$–localization

We record here the definition and basic properties of localization with respect to a subring $R \subseteq \mathbb{Q}$. Write $p_n : S^n \to S^n$ for the degree $p$ map from the $n$–sphere to itself ($p$ may be any integer, but we usually take $p$ to be a prime). Now for a subring $R \subseteq \mathbb{Q}$, let $\mathcal{P}(R)$ denote the set of all primes that are invertible in $R$, and define

$q_R = \bigvee_{p \in \mathcal{P}(R)} p_1 : \bigvee_{p \in \mathcal{P}(R)} S^1 \to \bigvee_{p \in \mathcal{P}(R)} S^1.$
The localization $L_{qR}$ is called $R$–localization; we abbreviate it $L_R$ and refer to a $q_R$–equivalence as an $R$–equivalence. There are other (different) definitions for $R$–localization, but they all agree up to weak homotopy equivalence on simply connected spaces [3, 6.1.E.2–3].

We will need the following simple result.

**Lemma 7** If $X$ is $(N - 1)$–connected, then $X \rightarrow L_{\Sigma N - 1 q_R}(X)$ is $R$–localization.

It is easy to detect simply connected $R$–local spaces using their standard algebraic invariants. An abelian group $G$ is said to be $R$–local if the map $G \otimes \mathbb{Z} \mathbb{Z} \rightarrow G \otimes \mathbb{Z} R$ induced by the inclusion $\mathbb{Z} \hookrightarrow R$ is an isomorphism.

**Proposition 8** (Sullivan [8]) Let $R \subseteq \mathbb{Q}$ be a unital subring. If $X$ is simply connected, then the following are equivalent:

1. $X$ is $R$–local;
2. $\pi_n(X)$ is $R$–local for all $n \geq 2$;
3. $\widetilde{H}_n(X; \mathbb{Z})$ is $R$–local for all $n \geq 2$.

Together with the exactness of the $R$–localization of abelian groups, Proposition 8 implies the two basic homotopy theoretical properties of $R$–localization of spaces.

**Theorem 9** (Sullivan [8]) If $R \subseteq \mathbb{Q}$, then $L_R$ commutes with both fiber sequences and cofiber sequences of simply connected spaces.

## 2 Proof of Theorem 1

Since it is well known that (1) implies (2), we suppose (2) and prove (1). The plan is to prove that $S^n \rightarrow L_f(S^n)$ is $R$–localization for $n \geq N$ and then work by induction on the cells of a finite CW complex.

### 2.1 The connectivity of $L_f(S^n)$

We begin by showing that $L_f(S^n)$ is simply connected for $n \geq N$. Applying (2) to the cofiber sequences

$$
S^n \rightarrow * \rightarrow S^{n+1}, \quad S^n \rightarrow S^m \rightarrow S^m \vee S^{n+1},
$$

we have...
reveals that $L_f(S^{n+1}) \sim \Sigma L_f(S^n)$ and $L_f(S^m \vee S^{n+1}) \sim L_f(S^m) \vee L_f(S^{n+1})$, as long as $m, n \geq N$. The first of these provides, by induction, weak equivalences $S^m \wedge L_f(S^n) \sim L_f(S^m \wedge S^n)$ for all $m \geq 1$ and $n \geq N$. Next (with both $m, n \geq N$) apply $L_f$ to the cofiber sequence

$$S^m \vee S^{n+1} \to S^m \times S^{n+1} \to S^m \wedge S^{n+1}$$

to obtain

$$L_f(S^m \wedge S^{n+1}) \sim L_f(S^m \times S^{n+1})/L_f(S^m \vee S^{n+1})$$
$$\sim (L_f(S^m) \times L_f(S^{n+1}))/L_f(S^m) \vee L_f(S^{n+1}))$$
$$\sim L_f(S^m) \wedge L_f(S^{n+1}),$$

using Proposition 5(c). This shows, in particular, that $L_f(S^n)$ is simply connected for $n \geq N$ and that $L_f(S^m) \wedge L_f(S^{n+1}) \sim L_f(S^{n+m+1})$ is $f$–local for $n, m \geq N$.

We use these preliminary results to construct the dotted arrows in the diagram

$$\begin{array}{ccc}
S^m \wedge S^{n+1} & \xrightarrow{id \wedge \iota_{S^n+1}} & S^m \wedge L_f(S^{n+1}) \\
\iota_{S^m \wedge S^{n+1}} & & \iota \\
L_f(S^m \wedge S^{n+1}) & \sim & L_f(S^m) \wedge L_f(S^{n+1})
\end{array}$$

for $m, n \geq N$. Proposition 5(a) implies that the solid arrows in the diagram are $f$–equivalences. Thus the composite $S^m \wedge S^{n+1} \to L_f(S^m) \wedge L_f(S^{n+1})$ is an $f$–equivalence to an $f$–local space, and therefore is $f$–localization. It follows from Lemma 3(b) that (up to homotopy) there is a unique map $\epsilon_{m,n+1}$ making the diagram commute up to homotopy. We define $\iota$ to be the composite in the diagram; but since $id \wedge \iota_{S^{n+1}}$ is $f$–localization, $\iota$ is also uniquely determined up to homotopy by the commutativity of the upper left triangle. The (homotopy) commutativity of the diagram implies that $\epsilon_{m,n+1}$ and $\iota$ are $f$–equivalences. Since the three spaces in the lower triangle are $f$–local, the maps in that triangle are weak equivalences by Lemma 3(a).

Now we determine the connectivity of $L_f(S^n)$. Write $\text{conn}(L_f(S^N)) = c$; since $L_f(S^N)$ is simply connected by hypothesis, we have

$$\text{conn}(L_f(S^n)) = \text{conn}(\Sigma^{n-N} L_f(S^N)) = c + (n-N)$$

for all $n \geq N$, so we may work with $n > N$, and we have the equation

$$\text{conn}(L_f(S^n) \wedge L_f(S^n)) = 2(c + (n-N)) + 1.$$
Since $L_f(S^n)$ is simply connected, the homotopy commutative diagram of weak equivalences

$$
\begin{array}{cccc}
S^n \land L_f(S^n) & \sim & L_f(S^n) \land L_f(S^n) & \sim \\
\sim & \sim & \sim & \sim \\
& & & \\
L_f(S^{2n}) & \sim & L_f(S^n) \land S^n & \\
\end{array}
$$

shows that

$$n + (c + (n - N)) = \text{conn}(\Sigma^n L_f(S^n)) = \text{conn}(L_f(S^n) \land L_f(S^n)) = 2(c + (n - N)) + 1,$$

so $c = N - 1$ and consequently

$$\text{conn}(L_f(S^n)) = n - 1, \quad \text{for all } n \geq N.$$

### 2.2 $L_f(S^n)$ Is a Moore space $M(R, n)$

Next we show that there is an abelian group $R$ such that $L_f(S^n)$ is the Moore space $M(R, n)$ for every $n$. This is done by determining $A_* = \Sigma^{-n} \tilde{H}_*(L_f(S^n); \mathbb{Z})$, which is canonically identified with $\Sigma^{-m} \tilde{H}_*(L_f(S^m); \mathbb{Z})$ for $m \geq N$ using the weak equivalences established in Section 2.1.

We expand the previous diagram to a three-dimensional diagram, writing $L = L_f(S^n)$, with $n > N$:

$$
\begin{array}{cccc}
S^n \land S^n & \sim & S^n \land S^n & \sim & S^n \land S^n \\
S^n \land L & \sim & L \land L & \sim & L \land S^n \\
S^n \land S^n & \sim & S^n \land S^n & \sim & S^n \land S^n \\
L_f(S^n \land L) & \sim & L_f(S^{2n}) & \sim & L_f(L \land S^n), \\
\end{array}
$$

in which the unmarked arrows are $f$–localizations, and the vertical arrows are weak equivalences. Applying integral homology to the front face of this diagram results in
the diagram

\[
\begin{array}{ccc}
\mathbb{Z} \otimes \mathbb{Z} A_* & \rightarrow & A_* \otimes \mathbb{Z} A_* \\
\cong & \kappa & \cong \\
A_* & \cong & \Sigma^{-2n} \tilde{H}_* (L \land L; \mathbb{Z}) \\
\cong & \kappa & \cong \\
\rightarrow & \rightarrow & \rightarrow \\
A_* & \rightarrow & A_* \\
\end{array}
\]

in which we use \( \kappa \) to denote the Künneth exterior product map. The commutativity of the diagram implies that \( \kappa: A_* \otimes A_* \rightarrow \Sigma^{-2n} \tilde{H}_* (L \land L; \mathbb{Z}) \) is an isomorphism (\( \kappa \) is injective by the Künneth theorem). Write \( A_+ \) for the positive degree part of \( A_* \). Since \( (\mathbb{Z} \otimes \mathbb{Z} A_+) \cap (A_+ \otimes \mathbb{Z} A_*) = 0 \) in \( A_* \otimes \mathbb{Z} A_* \), it must be that \( A_+ \cong \mathbb{Z} \otimes \mathbb{Z} A_+ = 0 \) or else the vertical composite could not be injective in positive degrees. Thus \( A_* \) is some abelian group \( R \) concentrated in degree zero.

### 2.3 The structure of \( R \)

We complete the determination of \( S^n \rightarrow L_f (S^n) \) by showing that \( R \) is isomorphic to a unital subring of \( \mathbb{Q} \) and the induced map on homology is simply the inclusion \( \mathbb{Z} \hookrightarrow R \).

The vertical map \( \mu: A_* \otimes A_* \rightarrow A_* \) in the diagram of Section 2.2 endows the graded abelian group \( A_* \) with a multiplication having a two-sided unit \( \mathbb{Z} \rightarrow A_* \) induced by \( S^n \rightarrow L_f (S^n) \). If \( n > N \) is even, this multiplication is commutative and associative. To see this, write \( T: S^n \land S^n \rightarrow S^n \land S^n \) for the twist map that switches smash factors.

Working through the proof in Section 2.1 that \( L_f (S^n \land S^n) \sim L_f (S^n) \land L_f (S^n) \) reveals that \( L_f (T) \) may be identified with the twist map \( T: L_f (S^n) \land L_f (S^n) \rightarrow L_f (S^n) \land L_f (S^n) \). Since we take \( n \) to be even, \( T \simeq \text{id} \), and we have the homotopy commutative diagram

\[
\begin{array}{ccc}
S^n \land S^n & \xrightarrow{T} & S^n \land S^n \\
\downarrow \iota \land \iota & & \downarrow \iota \land \iota \\
L_f (S^n) \land L_f (S^n) & \xrightarrow{T} & L_f (S^n) \land L_f (S^n) \\
\downarrow \epsilon_{n,n} & & \downarrow \epsilon_{n,n} \\
L_f (S^{2n}) & \xrightarrow{\text{id}} & L_f (S^{2n}).
\end{array}
\]
which implies that the multiplication $\mu: A_\ast \otimes_\mathbb{Z} A_\ast \to A_\ast$ is commutative. The proof of associativity is similar, where the map $T$ is replaced with the associator map $A: S^n \wedge (S^n \wedge S^n) \cong (S^n \wedge S^n) \wedge S^n$; we omit the details.

Since the multiplication $\mu: R \otimes_\mathbb{Z} R \to R$ is an isomorphism, $R$ is a solid ring (defined and studied by Bousfield and Kan in [1]). Applying the Künneth theorem to compute $A_1$ (which we know to be zero), we find that $\text{Tor}_0(R, R) = 0$.

**Lemma 10** If $R$ is a solid ring with $\text{Tor}_0(R, R) = 0$, then $R$ is isomorphic to a subring of $\mathbb{Q}$.

**Proof** Let $T \subseteq R$ be the torsion subgroup. We cannot have $T = R$, for then $R \cong \mathbb{Z}/d$ by [1, Lemma 3.6], and this forces $\text{Tor}_0(R, R) \neq 0$. Thus we apply [1, Lemma 3.10] to the short exact sequence

$$0 \to T \to R \to R/T \to 0$$

to discover that $R/T$ is isomorphic to a subring $S \subseteq \mathbb{Q}$ (and is therefore a flat $\mathbb{Z}$–module). But also $T$ must be a sum of cyclic groups, which implies that if $T \neq 0$, then $\text{Tor}_0(T, T) \neq 0$. Since $R \cong S \oplus T$ as abelian groups, we may compute

$$\text{Tor}_0(R, R) \cong \text{Tor}_0(S, S) \oplus \text{Tor}_0(S, T) \oplus \text{Tor}_0(T, S) \oplus \text{Tor}_0(T, T)$$

$$\cong \text{Tor}_0(T, T),$$

showing that if $\text{Tor}_0(R, R) = 0$, then $T = 0$ and $R \cong S \subseteq \mathbb{Q}$. \hfill \square

Lemma 10 implies that $L_f(S^n) \sim M(R, n)$ where $R$ is a unital subring of $\mathbb{Q}$. As we observed in the beginning of this section, the unit $\mathbb{Z} \to R$ of this subring is induced by $t_{S^n}: S^n \to L_f(S^n)$. It follows that $t_{S^n}$ is $R$–localization for $n \geq N$.

### 2.4 Finishing the proof of Theorem 1

Write $q = \Sigma^{N-1} q_R$. To finish the proof of Theorem 1, we show that $L_f(q)$ is a weak equivalence, so that there is a comparison transformation $\xi: L_q \to L_f$. Our work so far implies $\xi_{S^n}$ is a weak equivalence for $n \geq N$ and since both $L_q$ and $L_f$ commute with cofiber sequences of finite $(N-1)$–connected complexes, an easy induction shows that $\xi_K$ is a weak equivalence for every finite $(N-1)$–connected complex $K$.

Let $p$ be a prime that is invertible in $R$, and consider $L_f(p_n)$ with $n \geq N$. This map is understood using the diagram

$$\begin{array}{cccccc}
S^n & \to & S^n \\
\downarrow t & & \downarrow t \\
M(R, n) & \to & M(R, n)
\end{array}$$

$$\begin{array}{cccccc}
& & \mathcal{L}_f(p_n) & \to & \\
\downarrow t & & \downarrow t & & \downarrow \text{id}/p \\
M(R, n) & \to & M(R, n) & \to & M(R, n).
\end{array}$$
Since everything in this diagram is in the stable range, we have

\[(\text{id}/p) \circ i \circ p_n \simeq (p \cdot \iota)/p \simeq \iota.\]

Now Lemma 3(b) implies that \((\text{id}/p) \circ L_f(p_n) \simeq \text{id}_{M(R,n)}\), so that \(L_f(p_n)\) is a weak equivalence provided \(p\) is invertible in \(R\).

Write \(q = \Sigma^{N-1} q_R = \sqrt[p \in \mathcal{P}(R)]{p_N}\). Since \(L_f(p_N)\) is a weak equivalence for every \(p \in \mathcal{P}(R)\), we see that every \(f\)–local space is also \(q\)–local, and so there is a comparison transformation \(\xi: L_q \rightarrow L_f\). Applied to \(S^n\) with \(n \geq N\) we have

\[
\begin{array}{ccc}
S^n & \xrightarrow{\xi_S} & L_f(S^n) \\
L_q(S^n) & \xrightarrow{\xi} & L_f(S^n)
\end{array}
\]

in which both diagonal maps are \(R\)–localizations, forcing \(\xi_{S^n}\) to be a weak equivalence. Finally, suppose \(\xi_K\) is a weak equivalence for all CW complexes with at most \(k\) cells. If \(L\) has at most \(k + 1\) cells, then there is a cofiber sequence \(S^n \rightarrow K \rightarrow L \rightarrow S^{n+1} \rightarrow \Sigma K\) in which \(K\) and \(\Sigma K\) have at most \(k\) cells. Applying \(\xi\) we obtain the diagram

\[
\begin{array}{ccccccc}
L_q(S^n) & \xrightarrow{\xi_S} & L_q(K) & \xrightarrow{\xi_K} & L_q(L) & \xrightarrow{\xi_L} & L_q(S^{n+1}) & \xrightarrow{\xi_{S^{n+1}}} & L_q(\Sigma K) \\
\downarrow{\xi_S} & & \downarrow{\xi_K} & & \downarrow{\xi_L} & & \downarrow{\xi_{S^{n+1}}} & & \downarrow{\xi_{\Sigma K}} \\
L_f(S^n) & \xrightarrow{\xi_S} & L_f(K) & \xrightarrow{\xi_K} & L_f(L) & \xrightarrow{\xi_L} & L_f(S^{n+1}) & \xrightarrow{\xi_{S^{n+1}}} & L_f(\Sigma L).
\end{array}
\]

The inductive hypothesis shows that the maps \(\xi_{S^n}\), \(\xi_{K}\), \(\xi_{S^{n+1}}\) and \(\xi_{\Sigma K}\) are weak equivalences. Since all the spaces are simply connected and the rows are cofiber sequences, \(\xi_L\) is also a weak equivalence, completing the proof of Theorem 1. \(\square\)

## 3 Proof of Theorem 2

First of all, it is well known that the three conditions of Theorem 2 hold for the rationalization of simply connected spaces.

To prove the converse, suppose \(L_f\) satisfies those conditions. Then Theorem 1 gives, for some \(R \subseteq \mathbb{Q}\), a comparison transformation \(\xi: L_{\Sigma q_R} \rightarrow L_f\) which evaluates to weak equivalences on simply connected finite complexes. But \(R\) must be \(\mathbb{Q}\), for if the prime \(p\) is not invertible in \(R\), then the Moore space \(M(\mathbb{Z}/p, n)\) is \(R\)–local and cannot split as copies of \(L_f(S^{n\alpha}) = M(R, n\alpha)\), even after repeated suspension.
Let $\mathcal{L}$ denote the collection of $f$–local spaces. We know that $S^n_{\mathbb{Q}} \in \mathcal{L}$ for $n \geq 2$, and in particular $K(\mathbb{Q}, 2n + 1) \sim S^{2n+1}_{\mathbb{Q}} \in \mathcal{L}$ for all $n \geq 1$; then we obtain that $K(\mathbb{Q}, 2n) \sim \Omega K(\mathbb{Q}, 2n + 1) \in \mathcal{L}$ by [3, 1.A.8(e.1)]. Therefore

$$\prod_{I} K(\mathbb{Q}, n) \sim K \left( \prod_{I} \mathbb{Q}, n \right) \in \mathcal{L}$$

for all index sets $I$. Since every rational vector space $V$ is a retract of a vector space of this form, $K(V, n) \in \mathcal{L}$ for any rational vector space $V$. We deduce that every simply connected rational Postnikov piece is in $\mathcal{L}$, and since $\mathcal{L}$ is closed under homotopy limits [3, 1.A.8(e.3)], every simply connected rational space is $f$–local.

Lemma 7 implies that if $X$ is simply connected, then $X \to L_{\Sigma \mathbb{Q}}(X)$ is rationalization, and in particular that $L_{\Sigma \mathbb{Q}}(X)$ is a simply connected rational space, and hence is $f$–local. Therefore Proposition 6 implies that $\xi_X : L_{\Sigma \mathbb{Q}}(X) \to L_f(X)$ is a weak equivalence for simply connected $X$, which completes the proof of Theorem 2. \[\square\]

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