A LICHEROWICZ-HITCHIN VANISHING THEOREM FOR FOLIATIONS

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Abstract. We establish a generalization of the Lichnerowicz-Hitchin vanishing theorem to the case of foliations. As a consequence, we show that there is no foliation of positive leafwise scalar curvature on any torus. Our proof, which is inspired by the analytic localization techniques developed by Bismut and Lebeau, applies to give a purely geometric proof of the Connes vanishing theorem which also extends the Lichnerowicz vanishing theorem to the case of foliations.

0. Introduction

A classical theorem of Lichnerowicz [15] states that if a closed spin manifold of dimension $4k$ admits a Riemannian metric of positive scalar curvature, then the Hirzebruch
A-genus\(^1\) of this manifold vanishes. Hitchin [13] extended it to the case of all dimensions.

In this paper, we generalize this Lichnerowicz-Hitchin result to the case of foliations.

To be more precise, let \(M\) be a smooth manifold, let \(F\) be an integrable subbundle of the tangent vector bundle \(TM\) of \(M\). Let \(g^F\) be a Euclidean metric on \(F\). Then \(g^F\) determines a leafwise scalar curvature \(k^F \in C^\infty(M)\) as follows: for any \(x \in M\), the integrable subbundle \(F\) determines a leaf \(\mathcal{F}_x\) passing through \(x\) such that \(F|_{\mathcal{F}_x} = T\mathcal{F}_x\). Thus, \(g^F\) determines a Riemannian metric on \(\mathcal{F}_x\). Let \(k^{F_x}\) denote the scalar curvature of this Riemannian metric. We define

\[
(0.1) \quad k^F(x) = k^{F_x}(x).
\]

On the other hand, for a closed spin manifold \(M\), let \(\widehat{A}(M)\) be defined by that if \(\dim M = 8k + 4i\) with \(i = 0\) or \(1\), then \(\widehat{A}(M) = 3 + (-1)^{i+1}\widehat{A}(M)\); if \(\dim M = 8k + i\) with \(i = 1\) or \(2\), then \(\widehat{A}(M) \in \mathbb{Z}_2\) is the Atiyah-Milnor-Singer \(\alpha\) invariant\(^2\), while in other dimensions one takes \(\widehat{A}(M) = 0\).

The main result of this paper can be stated as follows.

**Theorem 0.1.** Let \(F\) be an integrable subbundle of the tangent bundle of a closed spin manifold \(M\). If there exists a metric \(g^F\) on \(F\) such that \(k^F > 0\) over \(M\), then \(\widehat{A}(M) = 0\).

When taking \(F = TM\), one recovers the Lichnerowicz-Hitchin theorem.

**Remark 0.2.** Theorem 0.1 maybe viewed as a non-existence result. For example, take any \(8k + 1\) dimensional closed spin manifold \(M\) such that \(\widehat{A}(M) \neq 0\). Then by a result of Thurston [22], there always exists a codimensional one foliation on \(M\). However, by our result, there is no metric on the associated integrable subbundle of \(TM\) with positive leafwise scalar curvature.

Combining Theorem 0.1 with the well-known results of Gromov-Lawson [10] and Stolz [21], one gets the following purely geometric consequence.

**Corollary 0.3.** Let \(F\) be an integrable subbundle of the tangent bundle of a closed simply connected manifold \(M\) with \(\dim M \geq 5\). If there exists a metric \(g^F\) on \(F\) such that \(k^F > 0\) over \(M\), then \(M\) admits a Riemannian metric of positive scalar curvature.

**Remark 0.4.** That whether the existence of \(g^F\) with \(k^F > 0\) implies the existence of \(g^{TM}\) with \(k^{TM} > 0\) is a longstanding open question in foliation theory (cf. [25, Remark C14]), which admits an easy positive answer in the very special case where \((M, F)\) carries a transverse Riemannian structure (cf. [8, page 8]). An approach to this question for codimension one foliations is outlined in [8, page 193].

Clearly, if the question in Remark 0.4 had a positive answer, then Theorem 0.1 would be a direct consequence of the original Lichnerowicz-Hitchin theorem. The point here is that, conversely, while a direct geometric solution to this question is not available yet, the index theoretic results such as Theorem 0.1 can be used to study this purely geometric question.

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\(^1\)Cf. [21, pp. 13] for a definition.

\(^2\)Cf. [14, Section 2.7] for a definition.
On the other hand, our proof of Theorem 0.1 applies to give a direct geometric proof of the following celebrated vanishing theorem of Connes, where instead of assuming $TM$ being spin, one assumes that $F$ is spin. This new proof provides a positive answer to a longstanding question in index theory (cf. [12, Page 5 of Lecture 9]).

**Theorem 0.5. (Connes [6])** Let $F$ be a spin integrable subbundle of the tangent bundle of a compact oriented manifold $M$. If there is a metric $g^F$ on $F$ such that $k^F > 0$ over $M$, then $\hat{A}(M) = 0$.

Recall that the proof given in [6] for Theorem 0.5 uses in an essential way the noncommutative geometry. It is based on the Connes-Skandalis longitudinal index theorem [7] as well as the techniques of cyclic cohomology. Thus it relies on the spin structure on $F$, and we do not see how to adapt it to prove Theorem 0.1.

However, we will make use of a geometric trick in [6], which is the construction of a fibration\footnote{Which will be called a Connes fibration in what follows.} over an arbitrary foliation, in our proof of Theorems 0.1 and 0.5. The key advantage of this fibration is that the lifted (from the original) foliation is almost isometric, i.e., very close to the Riemannian foliation on which we have seen that the question in Remark 0.4 admits an easy positive answer. On the other hand, this fibration is noncompact, which makes the proofs of both Theorems 0.1 and 0.5 highly nontrivial.

Roughly speaking, the Connes fibration over a foliation $(M,F)$ is a fibration $\pi : M \to M$ where for any $x \in M$, the fiber $\pi^{-1}(x)$ is the space of Euclidean metrics on the quotient space $T_x M/F_x$. The integrable subbundle $F$ of $TM$ lifts to an integrable subbundle $\mathcal{F}$ of $TM$, and $(M,\mathcal{F})$ carries an almost isometric structure in the sense of [6, Section 4]. Take any metric on the transverse bundle $TM/F$, which by definition determines an embedded section $s : M \hookrightarrow M$. The induced fibration $s \circ \pi : M \to s(M)$ looks like a vector bundle, and Connes obtained his theorem by examining the corresponding Riemann-Roch property in noncommutative frameworks.

Our proof of Theorem 0.1 is different. It is inspired by the index theoretic analytic localization techniques developed by Bismut-Lebeau [4, Sections 8 and 9], and can be thought of as a kind of transgression.

To be more precise, let $T^V \mathcal{M}$ be the vertical tangent bundle of the Connes fibration $\pi : \mathcal{M} \to M$. Taking a splitting $TM = \mathcal{F} \oplus T^V \mathcal{M} \oplus F^\perp$, then $F^\perp \simeq \pi^*(TM/F)$ carries a natural metric $g^{F^\perp}$. If one lifts $g^F$ to a metric $g^\mathcal{F}$ on $\mathcal{F}$, then for any $\beta > 0$, $\varepsilon > 0$, one can consider the rescaled metric $g^\mathcal{F}_{\beta,\varepsilon} = \beta^2 g^\mathcal{F} \oplus g^{T^V \mathcal{M}} \oplus g^{F^\perp}_{2\varepsilon}$.

Since $TM$ is assumed to be spin, $\mathcal{F} \oplus F^\perp \simeq \pi^*(TM)$ is also spin. Thus one can construct a Dirac type operator $\mathcal{D}^{\mathcal{F}}_{\beta,\varepsilon}$ acting on $\Gamma(S(\mathcal{F} \oplus F^\perp) \otimes \Lambda^*(T^V \mathcal{M}))$, where $S(\cdot)$ (resp. $\Lambda^*(\cdot)$) is the notation for spinor bundle (resp. exterior algebra bundle).

Now take a sufficiently small open neighborhood $U$ of $s(M)$ in $\mathcal{M}$. Inspired by [4], for any $\beta$, $\varepsilon$, $T > 0$, we construct an isometric embedding (see Section 2 for more details)

$$J_{T,\beta,\varepsilon} : \Gamma \left( S \left( \mathcal{F} \oplus F^\perp \right) \right) \to \Gamma \left( S \left( \mathcal{F} \oplus F^\perp \right) \otimes \Lambda^* \left( T^V \mathcal{M} \right) \right)$$

\footnote{Called a sub-Dirac operator in [18].}
such that for any $\sigma \in \Gamma(S(F \oplus F^\perp)|_{s(M)})$, $J_{T,\beta,\varepsilon}\sigma$ has compact support in $U$. Let $E_{T,\beta,\varepsilon}$ be the $L^2$-completion of the image space of $J_{T,\beta,\varepsilon}$. Let $p_{T,\beta,\varepsilon}: L^2(S(F \oplus F^\perp) \otimes \Lambda^*(T^V M)) \to E_{T,\beta,\varepsilon}$ be the orthogonal projection. Then one finds that the operator

$$J_{T,\beta,\varepsilon}^{-1}p_{T,\beta,\varepsilon}D_{\beta,\varepsilon}^M J_{T,\beta,\varepsilon}: \Gamma\left( S\left( F \oplus F^\perp \right) \big|_{s(M)} \right) \to L^2\left( S\left( F \oplus F^\perp \right) \big|_{s(M)} \right)$$

is elliptic, formally self-adjoint and homotopic to the Dirac operator on $s(M) \simeq M$. Thus Theorem 0.1 will follow if one can show that for certain values of $\beta, \varepsilon$ and $T$, this operator is invertible. Indeed, this is exactly what we will establish in this paper.

Moreover, by combining the above invertibility with the techniques of Lusztig [19] and Gromov-Lawson [9], one obtains the following result which generalizes the corresponding result of Schoen-Yau [20] and Gromov-Lawson [9] for the case of $F = T(T^n)$.

**Theorem 0.6.** There exists no foliation $(T^n, F)$ on any torus $T^n$ such that the integrable subbundle $F$ of $T(T^n)$ carries a metric of positive scalar curvature over $T^n$.

Now if we assume that $F$ is spin instead of that $TM$ being spin, we can replace $S(F \oplus F^\perp)$ in (0.2) by $S(F) \otimes \Lambda^*(F^\perp)$ and consider the corresponding sub-Dirac operators in the sense of [18]. In this way, we get a purely geometric proof of Theorem 0.5.

We would like to mention that the idea of constructing sub-Dirac operators has also been used in [16] to prove a generalization of the Atiyah-Hirzebruch vanishing theorem for circle actions [1] to the case of foliations.

This paper is organized as follows. In Section 1, we discuss the case of almost isometric foliations and carry out the local computation. We also introduce the sub-Dirac operator in this case and prove Theorem 0.5 in the case where the underlying foliation is compact. In Section 2, we work on noncompact Connes fibrations and carry out the proofs of Theorems 0.1, 0.5 and 0.6.

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1. **Adiabatic limit and almost isometric foliations**

In this section, we discuss the geometry of almost isometric foliations in the sense of Connes [6]. We introduce for this kind of foliations a rescaled metric and show that the leafwise scalar curvature shows up from the limit behavior of the rescaled scalar curvature. We also introduce in this setting the sub-Dirac operator inspired by the original construction given in [18]. Finally, by combining the above two procedures, we prove a vanishing result when the almost isometric foliation under discussion is compact.

This section is organized as follows. In Section 1.1 we recall the definition of the almost isometric foliation in the sense of Connes. In Section 1.2 we introduce a rescaling of the given metric on the almost isometric foliation and study the corresponding limit behavior of the scalar curvature. In Section 1.3 we study Bott type connections on
certain bundles transverse to the integrable subbundle. In Section 1.3 we introduce the so-called sub-Dirac operator and compute the corresponding Lichnerowicz type formula. In Section 1.5 we prove a vanishing result when the almost isometric foliation is compact and verifies the conditions in Theorem 0.5.

1.1. Almost isometric foliations. Let \((M,F)\) be a foliated manifold, where \(F\) is an integrable subbundle of \(TM\), i.e., for any smooth sections \(X, Y \in \Gamma(F)\), one has

\[
[X, Y] \in \Gamma(F).
\]

Let \(G\) be the holonomy groupoid of \((M,F)\) (cf. [23]).

Let \(TM/F\) be the transverse bundle. We make the assumption that there is a proper subbundle \(E\) of \(TM/F\) and choose a splitting

\[
TM/F = E \oplus (TM/F)/E.
\]

Let \(q_1, q_2\) denote the dimensions of \(E\) and \((TM/F)/E\) respectively.

**Definition 1.1.** (Connes [6, Section 4]) If there exists a metric \(g_{TM/F}\) on \(TM/F\) with its restrictions to \(E\) and \((TM/F)/E\) such that the action of \(G\) on \(TM/F\) takes the form

\[
\begin{pmatrix}
O(q_1) & 0 \\
A & O(q_2)
\end{pmatrix},
\]

where \(O(q_1), O(q_2)\) are orthogonal matrices of ranks \(q_1, q_2\) respectively, and \(A\) is a \(q_2 \times q_1\) matrix, then we say that \((M,F)\) carries an almost isometric structure.

Clearly, the existence of the almost isometric structure does not depend on the splitting (1.2). We assume from now on that \((M,F)\) carries an almost isometric structure as above.

Now choose a splitting \(TM = F \oplus F^\perp\). We can and we will identify \(TM/F\) with \(F^\perp\).

Let \(g_{TM}\) be a metric on \(TM\) so that we have the orthogonal splitting

\[
TM = F \oplus F_1^\perp \oplus F_2^\perp,
\quad g_{TM} = g_F \oplus g_{F_1^\perp} \oplus g_{F_2^\perp}.
\]

Let \(\nabla_{TM}\) be the Levi-Civita connection associated to \(g_{TM}\).

From the almost isometric condition (1.3), one deduces that for any \(X \in \Gamma(F), U_i, V_i \in \Gamma(F_i^\perp), i = 1, 2\), the following identities, which may be thought of as infinitesimal versions of (1.3), hold (cf. [18, (A.5)]):

\[
\langle [X, U_i], V_i \rangle + \langle U_i, [X, V_i] \rangle = X \langle U_i, V_i \rangle,
\]

\[
\langle [X, U_2], U_1 \rangle = 0.
\]

Equivalently,

\[
\langle X, \nabla_{U_i}^{TM} V_i + \nabla_{V_i}^{TM} U_i \rangle = 0,
\]

\[
\langle \nabla_{X}^{TM} U_2, U_1 \rangle + \langle X, \nabla_{U_2}^{TM} U_1 \rangle = 0.
\]

In this paper, for simplicity, we also make the following assumption. This assumption holds by the Connes fibration to be dealt with in the next section.
In particular, one has
\text{(1.10)}
\[ [U_2, V_2] \in \Gamma \left( F_2^\perp \right). \]

1.2. Adiabatic limit and the scalar curvature. It has been shown in [18, Proposition A.2] that an almost isometric foliation in the sense of Definition [13, 2.1] is an almost Riemannian foliation in the sense of [13, Definition 2.1]. Thus many computations in what follows are contained implicitly in [18] (see also [17]).

For convenience, we recall the standard formula for the Levi-Civita connection that for any \( X, Y, Z \in \Gamma(TM) \),
\text{(1.8)}
\[ 2 \langle \nabla^TM_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle + \langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle. \]

For any \( \beta, \varepsilon > 0 \), let \( g_{\beta, \varepsilon}^{TM} \) be the rescaled Riemannian metric on \( TM \) defined by
\text{(1.9)}
\[ g_{\beta, \varepsilon}^{TM} = \beta^2 g_F^{TM} \oplus \frac{1}{\varepsilon^2} g_{F_1}^{TM} \oplus g_{F_2}^{TM}. \]
We will always assume that \( 0 < \beta, \varepsilon \leq 1 \). We will use the subscripts and/or superscripts \( \beta, \varepsilon \) to decorate the geometric data associated to \( g_{\beta, \varepsilon}^{TM} \). For example, \( \nabla^{TM, \beta, \varepsilon} \) will denote the Levi-Civita connection associated to \( g_{\beta, \varepsilon}^{TM} \). When the corresponding notation does not involve \( \beta, \varepsilon \), we will mean that it corresponds to the case of \( \beta = \varepsilon = 1 \).

Let \( p, p_1^\perp, p_2^\perp \) be the orthogonal projections from \( TM \) to \( F, F_1^\perp, F_2^\perp \) with respect to the orthogonal splitting \text{(1.4)}. Let \( \nabla^{F, \beta, \varepsilon}, \nabla^{F_1^\perp, \beta, \varepsilon}, \nabla^{F_2^\perp, \beta, \varepsilon} \) be the Euclidean connections on \( F, F_1^\perp, F_2^\perp \) defined by
\text{(1.10)}
\[ \nabla^{F, \beta, \varepsilon} = p \nabla^{TM, \beta, \varepsilon} p, \quad \nabla^{F_1^\perp, \beta, \varepsilon} = p_1^\perp \nabla^{TM, \beta, \varepsilon} p_1^\perp, \quad \nabla^{F_2^\perp, \beta, \varepsilon} = p_2^\perp \nabla^{TM, \beta, \varepsilon} p_2^\perp. \]
In particular, one has
\text{(1.11)}
\[ \nabla^F = p \nabla^{TM} p, \quad \nabla^{F_1^\perp} = p_1^\perp \nabla^{TM} p_1^\perp, \quad \nabla^{F_2^\perp} = p_2^\perp \nabla^{TM} p_2^\perp. \]

By \text{(1.8)}-\text{(1.11)} and the integrability of \( F \), the following identities hold for \( X \in \Gamma(F) \):
\text{(1.12)}
\[ \nabla^{F, \beta, \varepsilon} = \nabla^F, \quad p \nabla^{TM, \beta, \varepsilon} p_1^\perp = p \nabla^{TM} p_1^\perp, \quad i = 1, 2, \]
\[ p_1^\perp \nabla^{TM, \beta, \varepsilon} p = \beta^2 \varepsilon^2 p_1^\perp \nabla^T p, \quad p_2^\perp \nabla^{TM, \beta, \varepsilon} p = \beta^2 p_2^\perp \nabla^{TM} p. \]

From \text{(1.5)}-\text{(1.9)}, we deduce that for \( X \in \Gamma(F), U_i, V_i \in \Gamma(F_i^\perp), i = 1, 2, \)
\text{(1.13)}
\[ \left\langle \nabla_{U_i}^{TM, \beta, \varepsilon} V_1, X \right\rangle = \left\langle \nabla_{U_i}^{TM} V_1, X \right\rangle = \frac{1}{2} \left\langle [U_1, V_1], X \right\rangle, \]
while
\text{(1.14)}
\[ \left\langle \nabla_{U_2}^{TM, \beta, \varepsilon} V_2, X \right\rangle = \left\langle \nabla_{U_2}^{TM} V_2, X \right\rangle = \frac{1}{2} \left\langle [U_2, V_2], X \right\rangle = 0. \]
Equivalently, for any \( U_i \in \Gamma(F_i^\perp), i = 1, 2, \)
\text{(1.15)}
\[ p_1^\perp \nabla_{U_1}^{TM, \beta, \varepsilon} p = \beta^2 \varepsilon^2 p_1^\perp \nabla_{U_1}^{TM} p, \quad p_2^\perp \nabla_{U_2}^{TM, \beta, \varepsilon} p = 0. \]
Similarly, one verifies that

\begin{align}
\left\langle \nabla^{TM,\beta,\varepsilon}_{U_1} X, U_2 \right\rangle &= \frac{1}{2} \left\langle [U_1, X], U_2 \right\rangle - \frac{\beta^2}{2} \left\langle [U_1, U_2], X \right\rangle , \\
\left\langle \nabla^{TM,\beta,\varepsilon}_{U_2} X, U_1 \right\rangle &= \varepsilon^2 \frac{2}{2} \left\langle [U_1, X], U_2 \right\rangle + \frac{\beta^2 \varepsilon^2}{2} \left\langle [U_1, U_2], X \right\rangle .
\end{align}

For convenience of the later computations, we collect the asymptotic behavior of various covariant derivatives in the following lemma. These formulas can be derived by applying (1.5)-(1.9). The inner products appear in the lemma correspond to \( \beta = \varepsilon = 1 \).

**Lemma 1.3.** The following formulas hold for \( X, Y, Z \in \Gamma(F), U_i, V_i, W_i \in \Gamma(F^+_i) \) with \( i = 1, 2 \), when \( \beta > 0, \varepsilon > 0 \) are small,

\begin{align}
\left\langle \nabla^{TM,\beta,\varepsilon}_X Y, Z \right\rangle &= O(1), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_X Y, U_1 \right\rangle = O(\beta^2 \varepsilon^2), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_X Y, U_2 \right\rangle = O(\beta^2), \\
\left\langle \nabla^{TM,\beta,\varepsilon}_X U_1, Y \right\rangle &= O(1), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_X U_1, V_1 \right\rangle = O(1), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_X U_1, U_2 \right\rangle = O(1), \\
\left\langle \nabla^{TM,\beta,\varepsilon}_X U_2, Y \right\rangle &= O(1), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_X U_2, U_1 \right\rangle = O(1), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_X U_2, V_2 \right\rangle = O(1), \\
\left\langle \nabla^{TM,\beta,\varepsilon}_{U_1} X, Y \right\rangle &= O(1), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_{U_1} X, V_1 \right\rangle = O(\beta^2 \varepsilon^2), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_{U_1} X, U_2 \right\rangle = O(1), \\
\left\langle \nabla^{TM,\beta,\varepsilon}_{U_1} V_1, X \right\rangle &= O(1), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_{U_1} V_1, W_1 \right\rangle = O(1), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_{U_1} V_1, U_2 \right\rangle = O\left( \frac{1}{\varepsilon^2} \right), \\
\left\langle \nabla^{TM,\beta,\varepsilon}_{U_1} U_2, X \right\rangle &= O\left( \frac{1}{\beta^2} \right), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_{U_1} U_2, V_1 \right\rangle = O(1), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_{U_1} U_2, V_2 \right\rangle = O(1), \\
\left\langle \nabla^{TM,\beta,\varepsilon}_{U_2} X, Y \right\rangle &= O(1), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_{U_2} X, U_1 \right\rangle = O(\varepsilon^2), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_{U_2} X, U_2 \right\rangle = 0, \\
\left\langle \nabla^{TM,\beta,\varepsilon}_{U_2} U_1, X \right\rangle &= O\left( \frac{1}{\beta^2} \right), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_{U_2} U_1, V_1 \right\rangle = O(1), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_{U_2} U_1, U_2 \right\rangle = O(1), \\
\left\langle \nabla^{TM,\beta,\varepsilon}_{U_2} V_2, X \right\rangle &= 0, \quad \left\langle \nabla^{TM,\beta,\varepsilon}_{U_2} V_2, U_1 \right\rangle = O(\varepsilon^2), \quad \left\langle \nabla^{TM,\beta,\varepsilon}_{U_2} V_2, U_2 \right\rangle = O(1).
\end{align}
In what follows, when we compute the asymptotics of various covariant derivatives, we will simply use the above asymptotic formulas freely without further notice.

Let $R^{TM, \beta, \varepsilon} = (\nabla^{TM, \beta, \varepsilon})^2$ be the curvature of $\nabla^{TM, \beta, \varepsilon}$. Then for any $X, Y \in \Gamma(TM)$, one has the following standard formula,

\begin{equation}
R^{TM, \beta, \varepsilon}(X, Y) = \nabla_X^{TM, \beta, \varepsilon} \nabla_Y^{TM, \beta, \varepsilon} - \nabla_Y^{TM, \beta, \varepsilon} \nabla_X^{TM, \beta, \varepsilon} - \nabla^{TM, \beta, \varepsilon}_{[X,Y]}.
\end{equation}

Let $R^F = (\nabla^F)^2$ be the curvature of $\nabla^F$. Let $k^{TM, \beta, \varepsilon}$, $k^F$ denote the scalar curvature of $g^{TM, \beta, \varepsilon}$, $g^F$ respectively. Recall that $k^F$ is defined in (1.1). The following formula for $k^F$ is obvious,

\begin{equation}
k^F = - \sum_{i,j=1}^{\text{rk}(F)} \langle R^F(f_i, f_j) f_i, f_j \rangle,
\end{equation}

where $f_i$, $i = 1, \cdots, \text{rk}(F)$, is an orthonormal basis of $F$. Clearly, when $F = TM$, it reduces to the usual definition of the scalar curvature $k^{TM}$ of $g^{TM}$.

**Proposition 1.4.** If Condition (C) holds, then when $\beta > 0$, $\varepsilon > 0$ are small, the following formula holds uniformly on any compact subset of $M$,

\begin{equation}
k^{TM, \beta, \varepsilon} = \frac{k^F}{\beta^2} + O\left(1 + \frac{\varepsilon^2}{\beta^2}\right).
\end{equation}

**Proof.** By (1.1), (1.12), (1.26) and Lemma 1.3, one deduces that when $\beta > 0$, $\varepsilon > 0$ are very small, for any $X, Y \in \Gamma(F)$, one has

\begin{equation}
\langle R^{TM, \beta, \varepsilon}(X, Y)X, Y \rangle = \langle \nabla^X_{TM, \beta, \varepsilon} (p + p_1^1 + p_2^1) \nabla^X_{TM, \beta, \varepsilon} X, Y \rangle
- \langle \nabla^X_{TM, \beta, \varepsilon} (p + p_1^1 + p_2^1) \nabla^X_{TM, \beta, \varepsilon} X, Y \rangle
- \langle \nabla^{TM, \beta, \varepsilon}_{[X,Y]} X, Y \rangle
= \langle R^F(X, Y)X, Y \rangle - \beta^2 \varepsilon^2 \langle p_1^1 \nabla^X_{TM, \beta, \varepsilon} X, \nabla^X_{TM, \beta, \varepsilon} Y \rangle
- \beta^2 \varepsilon^2 \langle p_2^1 \nabla^X_{TM, \beta, \varepsilon} X, \nabla^X_{TM, \beta, \varepsilon} Y \rangle
+ \beta^2 \varepsilon^2 \langle p_1^1 \nabla^X_{TM, \beta, \varepsilon} X, \nabla^X_{TM, \beta, \varepsilon} Y \rangle
+ \beta^2 \varepsilon^2 \langle p_2^1 \nabla^X_{TM, \beta, \varepsilon} X, \nabla^X_{TM, \beta, \varepsilon} Y \rangle
= \langle R^F(X, Y)X, Y \rangle + O(\beta^2).
\end{equation}

For $X \in \Gamma(F)$, $U \in \Gamma(F_1)$, by (1.5)-(1.26), one finds that when $\beta, \varepsilon > 0$ are small,

\begin{equation}
\langle R^{TM, \beta, \varepsilon}(X, U)X, U \rangle = \langle \nabla^X_{TM, \beta, \varepsilon} (p + p_1^1 + p_2^1) \nabla^X_{TM, \beta, \varepsilon} X, U \rangle
- \langle \nabla^X_{TM, \beta, \varepsilon} (p + p_1^1 + p_2^1) \nabla^X_{TM, \beta, \varepsilon} X, U \rangle
- \langle \nabla^{TM, \beta, \varepsilon}_{(p + p_1^1 + p_2^1)[X,U]} X, U \rangle
= \beta^2 \varepsilon^2 \langle \nabla^X_{TM, \beta, \varepsilon} p \nabla^X_{TM, \beta, \varepsilon} X, U \rangle
+ \beta^2 \varepsilon^2 \langle \nabla^X_{TM, \beta, \varepsilon} p_1^1 \nabla^X_{TM, \beta, \varepsilon} X, U \rangle
- \varepsilon^2 \langle p_1^1 \nabla^X_{TM, \beta, \varepsilon} X, \nabla^{TM, \beta, \varepsilon} U \rangle
- \beta^2 \varepsilon^2 \langle \nabla^X_{TM, \beta, \varepsilon} p_1^1 \nabla^X_{TM, \beta, \varepsilon} X, U \rangle
+ \varepsilon^2 \langle p_2^1 \nabla^X_{TM, \beta, \varepsilon} X, \nabla^{TM, \beta, \varepsilon} U \rangle
- \beta^2 \varepsilon^2 \langle \nabla^X_{TM, \beta, \varepsilon} p_2^1 \nabla^X_{TM, \beta, \varepsilon} X, U \rangle
- \langle \nabla^{TM, \beta, \varepsilon}_{(p + p_1^1 + p_2^1)[X,U]} X, U \rangle
= O(\beta^2 + \varepsilon^2).
\end{equation}
Similarly, for $X \in \Gamma(F)$, $U \in \Gamma(F^+_2)$, one has that when $\beta > 0$, $\varepsilon > 0$ are small,

\[
(1.31) \quad \langle R^{TM,\beta,\varepsilon}(X,U),X,U \rangle = \left< \nabla^{TM,\beta,\varepsilon}_X (p + p_1^+ + p_2^+) \nabla^{TM,\beta,\varepsilon}_U X,U \right> \\
- \left< \nabla^{TM,\beta,\varepsilon}_U (p + p_1^+ + p_2^+) \nabla^{TM,\beta,\varepsilon}_X U, U \right> - \left< \nabla^{TM,\beta,\varepsilon}_{(p+p_1^++p_2^+)[X,U]} X,U \right> \\
= \beta^2 \left< \nabla^{TM}_X p \nabla^{TM}_X X,U \right> - \frac{1}{\varepsilon^2} \left< p_1^+ \nabla^{TM,\beta,\varepsilon}_X X, \nabla^{TM,\beta,\varepsilon}_X U \right> + \beta^2 \left< \nabla^{TM,\beta,\varepsilon}_X p_2^+ \nabla^{TM}_X X,U \right> \\
- \beta^2 \left< \nabla^{TM}_U p \nabla^{TM}_X X,U \right> - \beta^2 \left< \nabla^{TM,\beta,\varepsilon}_U p_{1}^+ \nabla^{TM}_X X,U \right> - \beta^2 \left< \nabla^{TM,\beta,\varepsilon}_U p_2^+ \nabla^{TM}_X X,U \right> \\
- \beta^2 \left< \nabla^{TM}_{p[X,U]} X,U \right> - \beta^2 \left< \nabla^{TM}_{p_2^+[X,U]} X,U \right> = O \left( \beta^2 + \varepsilon^2 \right).
\]

For $U$, $V \in \Gamma(F^+_2)$, one verifies that

\[
(1.32) \quad \langle R^{TM,\beta,\varepsilon}(U,V),U,V \rangle = \left< \nabla^{TM,\beta,\varepsilon}_U (p + p_1^+ + p_2^+) \nabla^{TM,\beta,\varepsilon}_V U,V \right> \\
- \left< \nabla^{TM,\beta,\varepsilon}_V (p + p_1^+ + p_2^+) \nabla^{TM,\beta,\varepsilon}_U U,V \right> - \left< \nabla^{TM,\beta,\varepsilon}_{(p+p_1^++p_2^+)[U,V]} U,V \right> \\
= \beta^2 \varepsilon^2 \left< \nabla^{TM}_U p \nabla^{TM,\beta,\varepsilon}_V U,V \right> + \left< \nabla^{TM}_U p_{1}^+ \nabla^{TM}_V U,V \right> - \varepsilon^2 \left< p_1^+ \nabla^{TM,\beta,\varepsilon}_V U, \nabla^{TM,\beta,\varepsilon}_U V \right> \\
- \beta^2 \varepsilon^2 \left< \nabla^{TM}_V p \nabla^{TM,\beta,\varepsilon}_U U,V \right> - \left< \nabla^{TM}_V p_{1}^+ \nabla^{TM}_U U,V \right> + \varepsilon^2 \left< p_1^+ \nabla^{TM,\beta,\varepsilon}_U U, \nabla^{TM,\beta,\varepsilon}_V V \right> \\
- \left< \nabla^{TM,\beta,\varepsilon}_{p[U,V]} U,V \right> - \left< \nabla^{TM}_{p_2^+[U,V]} U,V \right> - \left< \nabla^{TM,\beta,\varepsilon}_{p_2^+[U,V]} U,V \right> \\
= -\varepsilon^2 \left< p_2^+ \nabla^{TM,\beta,\varepsilon}_V U, \nabla^{TM,\beta,\varepsilon}_U V \right> + \varepsilon^2 \left< p_2^+ \nabla^{TM,\beta,\varepsilon}_U U, \nabla^{TM,\beta,\varepsilon}_V V \right> + O(1) = O \left( \frac{1}{\varepsilon^2} \right),
\]

from which one gets that when $\beta > 0$, $\varepsilon > 0$ are small,

\[
(1.33) \quad \varepsilon^2 \langle R^{TM,\beta,\varepsilon}(U,V),U,V \rangle = O(1).
\]

For $U$, $V \in \Gamma(F^+_2)$, one verifies directly that

\[
(1.34) \quad \langle R^{TM,\beta,\varepsilon}(U,V),U,V \rangle = \left< \nabla^{TM,\beta,\varepsilon}_U (p + p_1^+ + p_2^+) \nabla^{TM,\beta,\varepsilon}_V U,V \right> \\
- \left< \nabla^{TM,\beta,\varepsilon}_V (p + p_1^+ + p_2^+) \nabla^{TM,\beta,\varepsilon}_U U,V \right> - \left< \nabla^{TM,\beta,\varepsilon}_{(p+p_1^++p_2^+)[U,V]} U,V \right> \\
= \beta^2 \left< \nabla^{TM}_U p \nabla^{TM,\beta,\varepsilon}_V U,V \right> - \frac{1}{\varepsilon^2} \left< p_1^+ \nabla^{TM,\beta,\varepsilon}_V U, \nabla^{TM,\beta,\varepsilon}_U V \right> + \left< \nabla^{TM}_U p_{2}^+ \nabla^{TM}_V U,V \right> \\
- \beta^2 \left< \nabla^{TM}_V p \nabla^{TM,\beta,\varepsilon}_U U,V \right> + \frac{1}{\varepsilon^2} \left< p_1^+ \nabla^{TM,\beta,\varepsilon}_U U, \nabla^{TM,\beta,\varepsilon}_V V \right> - \left< \nabla^{TM}_V p_{2}^+ \nabla^{TM}_U U,V \right> \\
- \left< \nabla^{TM}_{p_2^+[U,V]} U,V \right> = O(1).
\]
For \( U \in \Gamma(F_1^+) \), \( V \in \Gamma(F_2^+) \), one verifies directly that,

\[
R^{TM, \beta, \varepsilon}(U, V)U, V) = \left\langle \nabla_U^{TM, \beta, \varepsilon} \left( p + p_1^++ p_2^+ \right) \nabla_V^{TM, \beta, \varepsilon}U, V \right\rangle
- \left\langle \nabla_V^{TM, \beta, \varepsilon} \left( p + p_1^++ p_2^+ \right) \nabla_U^{TM, \beta, \varepsilon}U, V \right\rangle
- \left\langle \nabla^{TM, \beta, \varepsilon}U, V \right\rangle
\]

\[
= -\beta^2 \left\langle p \nabla_v^{TM, \beta, \varepsilon}U, \nabla_u^{TM, \beta, \varepsilon}V \right\rangle - \frac{1}{\varepsilon^2} \left\langle p_1^+ \nabla_v^{TM, \beta, \varepsilon}U, \nabla_u^{TM, \beta, \varepsilon}V \right\rangle
+ \left\langle \nabla_u^{TM, \beta, \varepsilon}U, \nabla_v^{TM, \beta, \varepsilon}V \right\rangle
- \left\langle \nabla^{TM, \beta, \varepsilon}U, V \right\rangle
\]

\[
+ \frac{1}{\varepsilon^2} \left\langle U, \nabla^{TM, \beta, \varepsilon}V \right\rangle = O \left( \frac{1}{\varepsilon^2} + \frac{1}{\beta^2} \right),
\]

from which one gets that when \( \beta > 0, \varepsilon > 0 \) are small,

\[
\varepsilon^2 \left\langle R^{TM, \beta, \varepsilon}(U, V)U, V \right\rangle = \left\langle R^{TM, \beta, \varepsilon}(V, U)V, U \right\rangle = O \left( 1 + \frac{\varepsilon^2}{\beta^2} \right).
\]

From (1.27), (1.29)-(1.31), (1.33), (1.34) and (1.36), one gets (1.28). \( \square \)

1.3. **Bott connections on \( F_1^+ \) and \( F_2^+ \).** From (1.5) and (1.7)-(1.10), one verifies directly that for \( X \in \Gamma(F) \), \( U, V \in \Gamma(F_i^+) \), \( i = 1, 2 \), one has

\[
\left\langle \nabla_X^{F_i^+, \beta, \varepsilon}U, V \right\rangle = \langle [X, U], V \rangle \frac{\beta _2 \varepsilon^2}{2} \langle [U, V], X \rangle,
\]

\[
\left\langle \nabla_X^{F_i^+, \beta, \varepsilon}U, V \right\rangle = \langle [X, U], V \rangle \frac{\beta _2 \varepsilon^2}{2} \langle [U, V], X \rangle.
\]

By (1.37), one has that for \( X \in \Gamma(F) \), \( U, V \in \Gamma(F_i^+) \), \( i = 1, 2 \),

\[
\lim_{\varepsilon \to 0^+} \nabla_X^{F_i^+, \beta, \varepsilon}U = \nabla_X^{F_i^+}U := p_i^+ [X, U].
\]

Let \( \nabla_X^{F_i^+} \) be the connection on \( F_i^+ \) defined by the second equality in (1.38) and \( \nabla_X^{F_i^+}U = \nabla_X^{F_i^+}U \) for \( U \in \Gamma(F_i^+) = \Gamma(F_1^+ \oplus F_2^+) \). In view of (1.38) and [5], we call \( \nabla_X^{F_i^+} \) a Bott connection on \( F_i^+ \) for \( i = 1 \) or 2. Let \( \tilde{R}_i^+ \) denote the curvature of \( \nabla_X^{F_i^+} \) for \( i = 1, 2 \).

The following result holds without Condition (C).

**Lemma 15.** For \( X, Y \in \Gamma(F) \) and \( i = 1, 2 \), the following identity holds,

\[
\tilde{R}_i^+(X, Y) = 0.
\]

**Proof.** We proceed as in [24 Proof of Lemma 1.14]. By (1.38) and the standard formula for the curvature (cf. [24 (1.3)]), for any \( U \in \Gamma(F_i^+) \), \( i = 1, 2 \), one has,

\[
\tilde{R}_i^+(X, Y)U = \nabla_X^{F_i^+} \nabla_Y^{F_i^+}U - \nabla_Y^{F_i^+} \nabla_X^{F_i^+}U - \nabla_{[X,Y]}^{F_i^+}U
\]

\[
= p_i^+ \left( [X, [Y, U]] + [Y, [U, X]] + [U, [X, Y]] \right) - p_i^+ \left( [X, (Id - p_i^+) [Y, U]] \right)
\]

\[
- p_i^+ \left( [Y, (Id - p_i^+) [U, X]] \right)
\]

\[
= -p_i^+ \left( [X, (p_i^+ + p_2^+ - p_1^+) [Y, U]] - p_i^+ \left( [Y, (p_i^+ + p_2^+ - p_1^+) [U, X]] \right),
\]

where the last equality follows from the Jacobi identity and the integrability of \( F \).
Now if $i = 1$, then by (1.5), one has $U \in \Gamma(F^\perp_1)$ and
\begin{equation}
(1.41) \quad p^\perp_1 [X, p^\perp_2 [Y, U]] = p^\perp_1 [Y, p^\perp_2 [U, X]] = 0.
\end{equation}
While if $i = 2$, still by (1.5), one has $U \in \Gamma(F^\perp_2)$ and
\begin{equation}
(1.42) \quad p^\perp_1 [Y, U] = p^\perp_1 [U, X] = 0.
\end{equation}
From (1.40)-(1.42), one gets (1.39). The proof of Lemma 1.5 is completed. \hfill \Box

**Remark 1.6.** For $i = 1, 2$, let $R^{F^\perp_i, \beta, \varepsilon}$ denote the curvature of $\nabla^{F^\perp_i, \beta, \varepsilon}$. From (1.37)-(1.39), one finds that for any $X, Y \in \Gamma(F)$, when $\beta > 0, \varepsilon > 0$ are small, the following identity holds:
\begin{equation}
(1.43) \quad R^{F^\perp_i, \beta, \varepsilon}(X, Y) = O \left( \beta^2 \varepsilon^2 \right).
\end{equation}
On the other hand, for $i = 1, 2$, and $U_i, V_i, W_i, Z_i \in \Gamma(F^\perp_i)$, by using (1.5), (1.7), (1.8), (1.10) and (1.20), one verifies directly that when $\beta > 0, \varepsilon > 0$ are small, the following identities, which will be used later, hold,
\begin{align}
\beta^{-1} \varepsilon \left( R^{F^\perp_i, \beta, \varepsilon}(X, U_1) V_1, W_1 \right) &= O \left( \beta^{-1} \varepsilon \right), \quad (1.44) \\
\beta^{-1} \left( R^{F^\perp_2, \beta, \varepsilon}(X, U_2) V_2, W_2 \right) &= O \left( \beta^{-1} \varepsilon \right), \quad (1.45) \\
\beta^{-1} \left( R^{F^\perp_1, \beta, \varepsilon}(X, U_2) V_1, W_1 \right) &= O \left( \beta^{-1} \varepsilon \right), \quad (1.46) \\
\varepsilon^2 \left( R^{F^\perp_1, \beta, \varepsilon}(U_1, V_1) W_1, Z_1 \right) &= O \left( \varepsilon^2 \right), \quad (1.47) \\
\left( R^{F^\perp_2, \beta, \varepsilon}(U_2, V_2) W_2, Z_2 \right) &= O \left( 1 \right), \quad (1.48) \\
\varepsilon \left( R^{F^\perp_1, \beta, \varepsilon}(U_1, U_2) V_1, W_1 \right) &= O \left( \varepsilon \right), \quad (1.49) \\
\left( R^{F^\perp_1, \beta, \varepsilon}(U_2, V_2) V_1, W_1 \right) &= O \left( 1 \right), \quad (1.50) \\
\beta^{-1} \varepsilon \left( R^{F^\perp_2, \beta, \varepsilon}(X, U_1) V_2, W_2 \right) &= O \left( \beta^{-1} \varepsilon \right), \quad (1.51) \\
\varepsilon \left( R^{F^\perp_2, \beta, \varepsilon}(U_1, U_2) V_2, W_2 \right) &= O \left( \varepsilon \right), \quad (1.52) \\
\varepsilon^2 \left( R^{F^\perp_2, \beta, \varepsilon}(U_1, V_1) V_2, W_2 \right) &= O \left( \varepsilon^2 \right). \quad (1.53)
\end{align}
1.4. Sub-Dirac operators associated to spin integrable subbundles. Following [15 \S 2b], we assume now that \( TM, F, F_i^\perp, i = 1, 2 \), are all oriented and of even rank, with the orientation of \( TM \) being compatible with the orientations on \( F, F_i^\perp \) and \( F_i^\perp \) through \([14]\). We further assume that \( F \) is spin and carries a fixed spin structure.

Let \( S(F) = S_+(F) \oplus S_-(F) \) be the Hermitian bundle of spinors associated to \((F, g_F)\). For any \( X \in \Gamma(F)\), the Clifford action \( c(X) \) exchanges \( S_\pm(F) \).

Let \( i = 1 \) or \( 2 \). Let \( \Lambda^*(F_i^\perp) \) denote the exterior algebra bundle of \( F_i^{\perp*} \). Then \( \Lambda^*(F_i^\perp) \) carries a canonically induced metric \( g^{\Lambda^*(F_i^\perp)} \) from \( g_{F_i^\perp} \). For any \( U \in F_i^\perp \), let \( U^* \in F_i^{\perp*} \) correspond to \( U \) via \( g_{F_i^\perp} \). For any \( U \in \Gamma(F_i^\perp) \), set
\[
(1.54) \quad c(U) = U^* \wedge -i_U, \quad \bar{c}(U) = U^* \wedge +i_U,
\]
where \( U^* \wedge \) and \( i_U \) are the exterior and interior multiplications by \( U^* \) and \( U \) on \( \Lambda^*(F_i^\perp) \).

Denote \( q_i = \text{rk}(F_i^\perp) \), \( q_i = \text{rk}(F_i^{\perp*}) \).

Let \( h_1, \ldots, h_{q_i} \) be an oriented orthonormal basis of \( F_i^\perp \). Set
\[
(1.55) \quad \tau \left( F_i^\perp, g_{F_i^\perp} \right) = \left( \frac{1}{\sqrt{-1}} \right)^{\frac{q_i(q_i + 1)}{2}} c(h_1) \cdots c(h_{q_i}).
\]
Then
\[
(1.56) \quad \tau \left( F_i^{\perp*}, g_{F_i^{\perp*}} \right)^2 = \text{Id}_{\Lambda^*(F_i^\perp)}.
\]
Set
\[
(1.57) \quad \Lambda^*_\pm(F_i^\perp) = \left\{ h \in \Lambda^*(F_i^\perp) : \tau \left( F_i^{\perp*}, g_{F_i^{\perp*}} \right) h = \pm h \right\}.
\]
Since \( q_i \) is even, for any \( h \in F_i^\perp \), \( c(h) \) anti-commutes with \( \tau(F_i^\perp, g_{F_i^\perp}) \), while \( \bar{c}(h) \) commutes with \( \tau(F_i^{\perp*}, g_{F_i^{\perp*}}) \). In particular, \( c(h) \) exchanges \( \Lambda^*_\pm(F_i^\perp) \).

Let \( \tilde{\tau}(F_i^\perp) \) denote the \( \mathbb{Z}_2 \)-grading of \( \Lambda^*(F_i^\perp) \) defined by
\[
(1.58) \quad \tilde{\tau}(F_i^\perp) \big|_{\Lambda^\text{even}(F_i^\perp)} = \pm \text{Id} \big|_{\Lambda^\text{even}(F_i^\perp)}.
\]

Now we have the following \( \mathbb{Z}_2 \)-graded vector bundles over \( M \):
\[
(1.59) \quad S(F) = S_+(F) \oplus S_-(F),
\]
\[
(1.60) \quad \Lambda^*(F_i^\perp) = \Lambda^*_+(F_i^\perp) \oplus \Lambda^*_-(F_i^\perp), \quad i = 1, 2,
\]
and
\[
(1.61) \quad \Lambda^*(F_i^\perp) = \Lambda^\text{even}(F_i^\perp) \oplus \Lambda^\text{odd}(F_i^\perp), \quad i = 1, 2.
\]

We form the following \( \mathbb{Z}_2 \)-graded tensor product, which will play a role in Section 2:
\[
(1.62) \quad W \left( F, F_1^\perp, F_2^\perp \right) = S(F) \hat{\otimes} \Lambda^*(F_1^\perp) \hat{\otimes} \Lambda^*(F_2^\perp),
\]
with the \( \mathbb{Z}_2 \)-grading operator given by
\[
(1.63) \quad \tau_W = \tau_{S(F)} \cdot \tau \left( F_1^\perp, g_{F_1^\perp} \right) \cdot \tilde{\tau} \left( F_2^\perp \right),
\]
where \( \tau_{S(F)} \) is the \( \mathbb{Z}_2 \)-grading operator defining the splitting in \([13, 14]\). We denote by
\[
(1.64) \quad W \left( F, F_1^\perp, F_2^\perp \right) = W_+ \left( F, F_1^\perp, F_2^\perp \right) \oplus W_- \left( F, F_1^\perp, F_2^\perp \right)
\]
the $\mathbb{Z}_2$-graded decomposition with respect to $\tau_W$.

Recall that the connections $\nabla^F$, $\nabla^{F_1^+}$ and $\nabla^{F_2^+}$ have been defined in (1.11). They lift canonically to Hermitian connections $\nabla^{S(F)}$, $\nabla^{\Lambda^+(F_1^+)}$, $\nabla^{\Lambda^+(F_2^+)}$ on $S(F)$, $\Lambda^+(F_1^+)$, $\Lambda^+(F_2^+)$ respectively, preserving the corresponding $\mathbb{Z}_2$-gradings. Let $\nabla^{W(F;F_1^+,F_2^+)}$ be the canonically induced connection on $W(F,F_1^+,F_2^+)$ which preserves the canonically induced Hermitian metric on $W(F,F_1^+,F_2^+)$, and also the $\mathbb{Z}_2$-grading of $W(F,F_1^+,F_2^+)$.

For any vector bundle $E$ over $M$, by an integral polynomial of $E$ we will mean a bundle $\phi(E)$ which is a polynomial in the exterior and symmetric powers of $E$ with integral coefficients.

For $i = 1, 2$, let $\phi_i(F_i^\perp)$ be an integral polynomial of $F_i^\perp$. We denote the complexification of $\phi_i(F_i^\perp)$ by the same notation. Then $\phi_i(F_i^\perp)$ carries a naturally induced Hermitian metric from $g^{F_i^\perp}$ and also a naturally induced Hermitian connection $\nabla^{\phi_i(F_i^\perp)}$ from $\nabla^{F_i^\perp}$.

Let $W(F,F_1^+,F_2^+) \otimes \phi_1(F_1^+) \otimes \phi_2(F_2^+)$ be the $\mathbb{Z}_2$-graded vector bundle over $M$,

$$(1.65) \quad W(F,F_1^+,F_2^+) \otimes \phi_1(F_1^+) \otimes \phi_2(F_2^+) = W_+ (F,F_1^+,F_2^+) \otimes \phi_1(F_1^+) \otimes \phi_2(F_2^+) + W_- (F,F_1^+,F_2^+) \otimes \phi_1(F_1^+) \otimes \phi_2(F_2^+).$$

Let $\nabla^{W \otimes \phi_1 \otimes \phi_2}$ denote the naturally induced Hermitian connection on $W(F,F_1^+,F_2^+) \otimes \phi_1(F_1^+) \otimes \phi_2(F_2^+)$ with respect to the naturally induced Hermitian metric on it. Clearly, $\nabla^{W \otimes \phi_1 \otimes \phi_2}$ preserves the $\mathbb{Z}_2$-graded decomposition in (1.65).

Let $S$ be the End($TM$)-valued one form on $M$ defined by

$$(1.66) \quad \nabla^{TM} = \nabla^F + \nabla^{F_1^+} + \nabla^{F_2^+} + S.$$

Let $e_1, \ldots, e_{\dim M}$ be an orthonormal basis of $TM$. Let $\nabla^{F,\phi_1(F_1^+)\otimes\phi_2(F_2^+)}$ be the Hermitian connection on $W(F,F_1^+,F_2^+) \otimes \phi_1(F_1^+) \otimes \phi_2(F_2^+)$ defined by that for any $X \in \Gamma(TM)$,

$$(1.67) \quad \nabla^X_{\phi_1(F_1^+)\otimes\phi_2(F_2^+)} = \nabla^X_{\phi_1 \otimes \phi_2} + \frac{1}{2} \sum_{i,j=1}^{\dim M} \langle S(X)e_i, e_j \rangle c(e_i) c(e_j).$$

Let the linear operator $D^{F,\phi_1(F_1^+)\otimes\phi_2(F_2^+)} : \Gamma(W(F,F_1^+,F_2^+) \otimes \phi_1(F_1^+) \otimes \phi_2(F_2^+)) \to \Gamma(W(F,F_1^+,F_2^+) \otimes \phi_1(F_1^+) \otimes \phi_2(F_2^+))$ be defined by (compare with [18, Definition 2.2])

$$(1.68) \quad D^{F,\phi_1(F_1^+)\otimes\phi_2(F_2^+)} = \sum_{i=1}^{\dim M} c(e_i) \nabla_{e_i}^{F,\phi_1(F_1^+)\otimes\phi_2(F_2^+)}.$$

We call $D^{F,\phi_1(F_1^+)\otimes\phi_2(F_2^+)}$ a sub-Dirac operator with respect to the spin vector bundle $F$.

One verifies that $D^{F,\phi_1(F_1^+)\otimes\phi_2(F_2^+)}$ is a first order formally self-adjoint elliptic differential operator. Moreover, it exchanges $\Gamma(W_+(F,F_1^+,F_2^+) \otimes \phi_1(F_1^+) \otimes \phi_2(F_2^+))$. We denote by $D^+_+(F_1^+;F_2^+)$ the restrictions of $D^{F,\phi_1(F_1^+)\otimes\phi_2(F_2^+)}$ to $\Gamma(W_+(F,F_1^+,F_2^+) \otimes \phi_1(F_1^+) \otimes \phi_2(F_2^+))$. Then one has

$$(1.69) \quad \left(D^+_+(F_1^+;F_2^+)\right)^* = D^+_-(F_1^+;F_2^+).$$

**Remark 1.7.** As in [18, (2.21)], when $F_1^+$, $F_2^+$ are also spin and carry fixed spin structures, then $TM = F \oplus F_1^+ \oplus F_2^+$ is spin and carries an induced spin structure from the
spin structures on $F$, $F^\perp_1$ and $F^\perp_2$. Moreover, one has the following identifications of \( Z_2 \)-graded vector bundles (cf. \[14\]) for \( i = 1, 2 \),

\[
(1.70) \quad \Lambda^* (F^\perp_i) \oplus \Lambda^* (F^\perp_i) = S_+ (F^\perp_i) \otimes S (F^\perp_i)^* \oplus S_- (F^\perp_i) \otimes S (F^\perp_i)^*,
\]

\[
(1.71) \quad \Lambda^\text{even} (F^\perp_i) \oplus \Lambda^\text{odd} (F^\perp_i) = \left( S_+ (F^\perp_i) \otimes S_+ (F^\perp_i)^* \oplus S_- (F^\perp_i) \otimes S_-(F^\perp_i)^* \right) \oplus \left( S_+ (F^\perp_i) \otimes S_- (F^\perp_i)^* \oplus S_- (F^\perp_i) \otimes S_+ (F^\perp_i)^* \right).
\]

By (1.55)-(1.68), (1.70) and (1.71), \( D^{F, \phi_1(F^\perp_i) \otimes \phi_2(F^\perp_i)} \) is simply the twisted Dirac operator

\[
(1.72) \quad D^{F, \phi_1(F^\perp_i) \otimes \phi_2(F^\perp_i)} : \Gamma \left( (S(TM) \otimes S (F^\perp_i)^*) \otimes \phi_1 (F^\perp_i) \otimes \phi_2 (F^\perp_i) \right) \rightarrow \Gamma \left( (S(TM) \otimes S (F^\perp_i)^*) \otimes \phi_1 (F^\perp_i) \otimes \phi_2 (F^\perp_i) \right),
\]

where for \( i = 1, 2 \), the Hermitian (dual) bundle of spinors \( S(F^\perp_i)^* \) associated to \( (F^\perp_i, g^{F^\perp_i}) \) carries the Hermitian connection induced from \( \nabla^{F^\perp_i} \).

The point of (1.68) is that it only requires \( F \) being spin. While on the other hand, (1.72) allows us to take the advantage of applying the calculations already done for usual (twisted) Dirac operators when doing local computations.

**Remark 1.8.** It is clear that the definition in (1.68) does not require that \( F \) being an integrable subbundle of \( TM \).

Let \( \Delta^{F, \phi_1(F^\perp_i) \otimes \phi_2(F^\perp_i)} \) denote the Bochner Laplacian defined by

\[
(1.73) \quad \Delta^{F, \phi_1(F^\perp_i) \otimes \phi_2(F^\perp_i)} = \sum_{i=1}^{\dim M} \left( \nabla_{e_i}^{F, \phi_1(F^\perp_i) \otimes \phi_2(F^\perp_i)} \right)^2 - \frac{1}{2} \sum_{i=1}^{\dim M} \nabla^{TM}_{e_i} \nabla^{TM}_{e_i}.
\]

Let \( k^{TM} \) be the scalar curvature of \( g^{TM} \), \( R^{F^\perp_i} \) \((i = 1, 2)\) be the curvature of \( \nabla^{F^\perp_i} \). Let \( R^{\phi_1(F^\perp_i) \otimes \phi_2(F^\perp_i)} \) be the curvature of the tensor product connection on \( \phi_1(F^\perp_i) \otimes \phi_2(F^\perp_i) \) induced from \( \nabla^{\phi_1(F^\perp_i)} \) and \( \nabla^{\phi_2(F^\perp_i)} \).

In view of Remark 1.7 the following Lichnerowicz type formula holds:

\[
(1.74) \quad \left( D^{F, \phi_1(F^\perp_i) \otimes \phi_2(F^\perp_i)} \right)^2 = -\Delta^{F, \phi_1(F^\perp_i) \otimes \phi_2(F^\perp_i)} + \frac{k^{TM}}{4} + \frac{1}{2} \sum_{i,j=1}^{\dim M} c(e_i) c(e_j) R^{\phi_1(F^\perp_i) \otimes \phi_2(F^\perp_i)} (e_i, e_j) + \frac{1}{8} \sum_{i,j,s,t=1}^{\dim M} \left( R^{F^\perp_i} (e_i, e_j) e_t, e_s \right) c(e_i) c(e_j) \hat{c}(e_s) \hat{c}(e_t).
\]

When \( M \) is compact, by the Atiyah-Singer index theorem \[2\] (cf. \[14\]), one has

\[
(1.75) \quad \text{ind} \left( D^{F, \phi_1(F^\perp_i) \otimes \phi_2(F^\perp_i)} \right) = 2^{\frac{\dim M}{2}} \left\langle \hat{A}(F) \hat{L} (F^\perp_i) e (F^\perp_i) \text{ch} \left( \phi_1 (F^\perp_i) \right) \text{ch} \left( \phi_2 (F^\perp_i) \right), [M] \right\rangle,
\]
where \( \widehat{\mathcal{L}}_k \) is the Hirzebruch \( \widehat{\mathcal{L}} \)-class (cf. [14] (11.18') of Chap. III]) of \( F_1^+ \), and \( e(F_2^+) \) is the Euler class (cf. [24, §3.4]) of \( F_2^+ \), and “ch” is the notation for the Chern character (cf. [24, §1.6.4]).

### 1.5. A vanishing theorem for almost isometric foliations

In this subsection, we assume \( M \) is compact and prove a vanishing theorem. Some of the computations in this subsection will be used in the next section where we will deal with the case where \( M \) is non-compact.

Let \( f_1, \ldots, f_q \) be an oriented orthonormal basis of \( F \). Let \( h_1, \ldots, h_{q_1} \) (resp. \( e_1, \ldots, e_{q_2} \)) be an oriented orthonormal basis of \( F_1^+ \) (resp. \( F_2^+ \)).

Let \( \beta > 0 \), \( \varepsilon > 0 \) and consider the construction in Section 1.4 with respect to the metric \( g_{\beta, \varepsilon}^{TM} \) defined in (1.9). We still use the superscripts “\( \beta, \varepsilon \)” to decorate the geometric data associated to \( g_{\beta, \varepsilon}^{TM} \). For example, \( D_{F_1}^{\phi_1(F_1^+) \otimes \phi_2(F_2^+), \beta, \varepsilon} \) now denotes the sub-Dirac operator constructed in (1.68) associated to \( g_{\beta, \varepsilon}^{TM} \). Moreover, it can be written as

\[
D_{F_1}^{\phi_1(F_1^+) \otimes \phi_2(F_2^+), \beta, \varepsilon} = \beta^{-1} \sum_{i=1}^{q} c(f_i) \nabla_{f_i} F_{\phi_1(F_1^+) \otimes \phi_2(F_2^+), \beta, \varepsilon} + \varepsilon \sum_{j=1}^{q_1} c(h_j) \nabla_{h_j} F_{\phi_1(F_1^+) \otimes \phi_2(F_2^+), \beta, \varepsilon} + \sum_{s=1}^{q_2} c(e_s) \nabla_{e_s} F_{\phi_1(F_1^+) \otimes \phi_2(F_2^+), \beta, \varepsilon}.
\]

By (1.76), the Lichnerowicz type formula (1.74) for \( (D_{F_1}^{\phi_1(F_1^+) \otimes \phi_2(F_2^+), \beta, \varepsilon})^2 \) takes the following form (compare with [18, Theorem 2.3]),

\[
(D_{F_1}^{\phi_1(F_1^+) \otimes \phi_2(F_2^+), \beta, \varepsilon})^2 = -\Delta_{F_1}^{\phi_1(F_1^+) \otimes \phi_2(F_2^+), \beta, \varepsilon} + \frac{k^{TM, \beta, \varepsilon}}{4} + \frac{1}{2\beta^2} \sum_{i,j=1}^{q} c(f_i) c(f_j) R_{\phi_1(F_1^+) \otimes \phi_2(F_2^+), \beta, \varepsilon} (f_i, f_j) + \varepsilon \sum_{i,j=1}^{q_1} c(h_i) c(h_j) R_{\phi_1(F_1^+) \otimes \phi_2(F_2^+), \beta, \varepsilon} (h_i, h_j) + \frac{1}{\beta} \sum_{i,j=1}^{q_2} c(e_i) c(e_j) R_{\phi_1(F_1^+) \otimes \phi_2(F_2^+), \beta, \varepsilon} (e_i, e_j) + \frac{1}{8\beta^2} \sum_{i,j=1}^{q} \sum_{s,t=1}^{q_1} \langle R_{F_1^+}^{\beta, \varepsilon} (f_i, f_j) h_t, h_s \rangle c(f_i) c(f_j) \widehat{\mathcal{L}} (h_s) \widehat{\mathcal{L}} (h_t) + \frac{\varepsilon}{8} \sum_{i,j=1}^{q_1} \sum_{s,t=1}^{q_1} \langle R_{F_1^+}^{\beta, \varepsilon} (h_i, h_j) h_t, h_s \rangle c(h_i) c(h_j) \widehat{\mathcal{L}} (h_s) \widehat{\mathcal{L}} (h_t) + \frac{1}{8} \sum_{i,j=1}^{q_2} \sum_{s,t=1}^{q_1} \langle R_{F_1^+}^{\beta, \varepsilon} (e_i, e_j) h_t, h_s \rangle c(e_i) c(e_j) \widehat{\mathcal{L}} (h_s) \widehat{\mathcal{L}} (h_t) + \frac{\varepsilon}{4\beta} \sum_{i,j=1}^{q_1} \sum_{s,t=1}^{q_1} \langle R_{F_1^+}^{\beta, \varepsilon} (f_i, h_j) h_t, h_s \rangle c(f_i) c(h_j) \widehat{\mathcal{L}} (h_s) \widehat{\mathcal{L}} (h_t).
\]
\[ + \frac{1}{4\beta} \sum_{i=1}^{q} \sum_{j=1}^{q_1} \sum_{s=1}^{q_1} \left\langle R^{F_1^+, \beta, \varepsilon} (f_i, e_j) h_t, h_s \right\rangle c (f_i) c (e_j) \widehat{c} (h_s) \widehat{c} (h_t) \]
\[ + \frac{\varepsilon}{4} \sum_{i=1}^{q_1} \sum_{j=1}^{q_1} \sum_{s=1}^{q_1} \left\langle R^{F_1^+, \beta, \varepsilon} (h_i, e_j) h_t, h_s \right\rangle c (h_i) c (e_j) \widehat{c} (h_s) \widehat{c} (h_t) \]
\[ + \frac{1}{8\beta^2} \sum_{i,j=1}^{q} \sum_{s=1}^{q_1} \sum_{t=1}^{q_1} \left\langle R^{F_2^+, \beta, \varepsilon} (f_i, f_j) e_t, e_s \right\rangle c (f_i) c (f_j) \widehat{c} (e_s) \widehat{c} (e_t) \]
\[ + \frac{\varepsilon^2}{8} \sum_{i,j=1}^{q} \sum_{s=1}^{q_1} \sum_{t=1}^{q_1} \left\langle R^{F_2^+, \beta, \varepsilon} (h_i, h_j) e_t, e_s \right\rangle c (h_i) c (h_j) \widehat{c} (e_s) \widehat{c} (e_t) \]
\[ + \frac{1}{8} \sum_{i,j=1}^{q} \sum_{s=1}^{q_1} \sum_{t=1}^{q_1} \left\langle R^{F_2^+, \beta, \varepsilon} (e_i, e_j) e_t, e_s \right\rangle c (e_i) c (e_j) \widehat{c} (e_s) \widehat{c} (e_t) \]
\[ + \frac{\varepsilon}{4} \sum_{i=1}^{q} \sum_{j=1}^{q_1} \sum_{s=1}^{q_1} \sum_{t=1}^{q_1} \left\langle R^{F_2^+, \beta, \varepsilon} (f_i, h_j) e_t, e_s \right\rangle c (f_i) c (h_j) \widehat{c} (e_s) \widehat{c} (e_t) \]
\[ + \frac{1}{4\beta} \sum_{i=1}^{q} \sum_{j=1}^{q_1} \sum_{s=1}^{q_1} \sum_{t=1}^{q_1} \left\langle R^{F_2^+, \beta, \varepsilon} (f_i, e_j) e_t, e_s \right\rangle c (f_i) c (e_j) \widehat{c} (e_s) \widehat{c} (e_t) \]
\[ + \frac{\varepsilon}{4} \sum_{i=1}^{q_1} \sum_{j=1}^{q_1} \sum_{s=1}^{q_1} \sum_{t=1}^{q_1} \left\langle R^{F_2^+, \beta, \varepsilon} (h_i, e_j) e_t, e_s \right\rangle c (h_i) c (e_j) \widehat{c} (e_s) \widehat{c} (e_t) . \]

By (1.28), (1.43)–(1.53) and (1.77), we get that when \( \beta > 0, \varepsilon > 0 \) are small,
\[ (1.78) \quad \left( D^{F, \phi_1 (F_1^+) \otimes \phi_2 (F_2^+) \beta, \varepsilon} - \Delta^{F, \phi_1 (F_1^+) \otimes \phi_2 (F_2^+) \beta, \varepsilon} + \frac{k^F}{4\beta^2} + O \left( \frac{1}{\beta} + \frac{\varepsilon^2}{\beta^2} \right) . \]

**Proposition 1.9.** If \( k^F > 0 \) over \( M \), then for any Pontrjagin classes \( p(F_1^+) \), \( p' (F_2^+) \) of \( F_1^+, F_2^+ \) respectively, the following identity holds,
\[ (1.79) \quad \left\langle \hat{A} (F) p (F_1^+) e (F_2^+) p' (F_2^+) , [M] \right\rangle = 0 . \]

**Proof.** Since \( k^F > 0 \) over \( M \), one can take \( \beta > 0, \varepsilon > 0 \) small enough so that the corresponding terms in the right hand side of (1.78) verifies that
\[ (1.80) \quad \frac{k^F}{4\beta^2} + O \left( \frac{1}{\beta} + \frac{\varepsilon^2}{\beta^2} \right) > 0 \]
over \( M \). Since \( -\Delta^{F, \phi_1 (F_1^+) \otimes \phi_2 (F_2^+) \beta, \varepsilon} \) is nonnegative, by (1.69), (1.78) and (1.80), one gets
\[ (1.81) \quad \text{ind} \left( D^{F, \phi_1 (F_1^+) \otimes \phi_2 (F_2^+) \beta, \varepsilon} \right) = 0 . \]

From (1.75) and (1.81), we get
\[ (1.82) \quad \left\langle \hat{A} (F) \hat{L} (F_1^+) c (\phi_1 (F_1^+)) e (F_2^+) c (\phi_2 (F_2^+)) , [M] \right\rangle = 0 . \]

Now as it is standard that any Pontrjagin class of \( F_1^+ \) (resp. \( F_2^+ \)) can be expressed as a rational linear combination of classes of the form \( \hat{L} (F_1^+) c (\phi_1 (F_1^+)) \) (resp. \( c (\phi_2 (F_2^+)) \)), one gets (1.79) from (1.82). \( \square \)
Remark 1.10. Recall that $F^\perp = F_1^\perp \oplus F_2^\perp$. It is proved in [18, Theorem 2.6] that if the conditions in Proposition 1.9 hold, then $\langle \tilde{\mathcal{A}}(F)p(F^\perp), [M] \rangle = 0$. Here if one changes the $\mathbb{Z}_2$-grading in the definition of the sub-Dirac operator by replacing $\tilde{\tau}(F_2^\perp)$ in (1.63) by $\tau(F_2^\perp, g F_2^\perp)$, then one can prove that under the same condition as in Proposition 1.9,

\begin{equation}
\langle \tilde{\mathcal{A}}(F)p(F_1^\perp) p'(F_2^\perp), [M] \rangle = 0
\end{equation}

for any Pontrjagin classes $p(F_1^\perp)$, $p'(F_2^\perp)$ of $F_1^\perp$, $F_2^\perp$.

Remark 1.11. Formulas (1.79) and (1.83) hold indeed without Condition (C) in Definition 1.2. This can be checked if we set $\varepsilon = \sqrt{\beta}$.

2. Connes fibration and vanishing theorems

In this Section we prove Theorems 0.1, 0.5 and 0.6. We will make use of the Connes fibration which has indeed played an essential role in Connes’ original proof of Theorem 0.5 given in [6].

This Section is organized as follows. In Section 2.1, we recall the construction of the Connes fibration over a foliation. In Section 2.2, we introduce a coordinate system near the embedded submanifold from the original foliation into the Connes foliation. In Section 2.3, we give an adiabatic limit estimate of the sub-Dirac operator on the Connes fibration. In Section 2.4, we embed the smooth sections over the embedded submanifold to the space of smooth sections, having compact support near the embedded submanifold, on the Connes fibration. In Section 2.5, we state an important estimate result which will be proved in Sections 2.6, 2.8. In Sections 2.9, 2.12, we complete the proofs of Theorems 0.1, 0.5 and 0.6 respectively.

2.1. The Connes fibration. We start by recalling the original construction in [6].

Let $(M, F)$ be a compact foliation, where $F$ is an integrable subbundle of the tangent vector bundle $TM$ of a closed manifold $M$. We make the assumption that $TM, F$ are oriented, then $TM/F$ is also oriented. We further assume that $F$ is spin and carries a fixed spin structure.

For any oriented vector space $E$ of rank $n$, let $\mathcal{E}$ be the set of all Euclidean metrics on $E$. It is well known that $\mathcal{E}$ is the homogeneous space $GL(n, \mathbb{R})^+/SO(n)$ (with $\dim \mathcal{E} = \frac{n(n+1)}{2}$), which carries a natural Riemannian metric of nonpositive sectional curvature (cf. [11]). In particular, any two points of $\mathcal{E}$ can be joined by a unique geodesic.

Following [6, Section 5], let $\pi : \mathcal{M} \to M$ be the fibration over $M$ such that for any $x \in M$, $\mathcal{M}_x = \pi^{-1}(x)$ is the space of Euclidean metrics on the linear space $T_xM/F_x$. Clearly, $\mathcal{M}$ is noncompact.

Let $T^V \mathcal{M}$ denote the vertical tangent bundle of the fibration $\pi : \mathcal{M} \to M$. Then it carries a natural metric $g^{T^V \mathcal{M}}$ such that any two points $p, q \in \mathcal{M}_x$, with $x \in M$, can be joined by a unique geodesic in $\mathcal{M}_x$. 
By using the Bott connection \(^5\) on \(TM/F\), one can lift \(F\) to an integrable subbundle \(\mathcal{F}\) of \(TM\). Moreover, \(\mathcal{F}\) is spin and carries a spin structure induced from that of \(F\).

For any \(v \in \mathcal{M}\), \(T_v\mathcal{M}/(\mathcal{F}_v \oplus T^v\mathcal{M})\) identifies with \(T_{\pi(v)}\mathcal{M}/F_{\pi(v)}\) under the projection \(\pi : \mathcal{M} \to M\). By definition, \(v\) determines a metric on \(T_{\pi(v)}\mathcal{M}/F_{\pi(v)}\); thus it also determines a metric on \(T_v\mathcal{M}/(\mathcal{F}_v \oplus T^v\mathcal{M})\). In this way, \(TM/(\mathcal{F} \oplus T^V\mathcal{M})\) carries a canonically induced metric.

Let \(\mathcal{F}_1^\perp\) be a subbundle of \(TM\), which is transversal to \(\mathcal{F} \oplus T^V\mathcal{M}\), such that we have a splitting \(TM = (\mathcal{F} \oplus T^V\mathcal{M}) \oplus \mathcal{F}_1^\perp\). Then \(\mathcal{F}_1^\perp\) can be identified with \(TM/(\mathcal{F} \oplus T^V\mathcal{M})\) and carries a canonically induced metric \(g_{\mathcal{F}_1^\perp}\). We also denote \(T^V\mathcal{M}\) by \(\mathcal{F}_2^\perp\).

Let \(g^\mathcal{F}\) be a Euclidean metric on \(F\), then it lifts to a Euclidean metric \(g^\mathcal{F}\) on \(\mathcal{F}\). Let \(g^{TM}\) be the Riemannian metric on \(TM\) defined by the following orthogonal splitting,

\[
(2.1) \quad TM = \mathcal{F} \oplus \mathcal{F}_1^\perp \oplus \mathcal{F}_2^\perp, \quad g^{TM} = g^\mathcal{F} \oplus g_{\mathcal{F}_1^\perp} \oplus g_{\mathcal{F}_2^\perp}.
\]

By \(^6\) Lemma 5.2, \((\mathcal{M}, \mathcal{F})\) admits an almost isometric structure in the sense of Definition \(^{1,3}\) with the metrics given in \(^{1,4}\) and/or \(^{2,1}\). In particular, \(^{1,5}\) holds\(^7\).

Take a metric on \(TM/F\). This is equivalent to taking an embedded section \(s : M \hookrightarrow \mathcal{M}\) of the Connes fibration \(\pi : \mathcal{M} \to M\). Then we have a canonical inclusion \(s(M) \subset \mathcal{M}\), as well as an induced fibration \(s \circ \pi : \mathcal{M} \to s(M)\).

**Definition 2.1.** By a Connes fibration over \((M, F)\) we mean a fibration \(\pi : \mathcal{M} \to M\) such that (i) there exists a splitting \(TM = T^V\mathcal{M} \oplus T^H\mathcal{M}\), where \(T^V\mathcal{M}\) is the vertical tangent bundle of the fibration, such that \(F\) lifts to an integrable subbundle \(\mathcal{F} \subset T^H\mathcal{M}\); (ii) if we denote \(T^V\mathcal{M} = \mathcal{F}_2^\perp\), then there exists a splitting \(T^H\mathcal{M} = \mathcal{F} \oplus \mathcal{F}_1^\perp\) as well as Euclidean metrics \(g_{\mathcal{F}_1^\perp}, g_{\mathcal{F}_2^\perp}\) on \(\mathcal{F}_1^\perp, \mathcal{F}_2^\perp\) such that the foliation \((\mathcal{M}, \mathcal{F})\) carries an associated almost isometric foliation structure in the sense of Section \(^1,3\) (iii) there exists a smooth (embedded) section \(s : M \hookrightarrow \mathcal{M}\).

One of the specific features of a Connes fibration is that since \(\mathcal{F}_2^\perp = T^V\mathcal{M}\) is the vertical tangent bundle of a fibration, the following identity holds:

\[
(2.2) \quad [U, V] \in \Gamma(\mathcal{F}_2^\perp) \quad \text{for} \quad U, V \in \Gamma(\mathcal{F}_2^\perp).
\]

That is, Condition (C) in Definition \(^{1,2}\) holds for \((\mathcal{M}, \mathcal{F})\). Combining with \(^{1,1}\) and the second identity in \(^{1,3}\), one sees that \(\mathcal{F} \oplus \mathcal{F}_2^\perp\) is also an integrable subbundle of \(TM\).

---

\(^5\)Indeed, the Bott connection on \(TM/F\) determines an integrable lift \(\widetilde{F}\) of \(F\) in \(T\mathcal{M}\), where \(\mathcal{M} = \text{GL}(TM/F)^+\) is the \(\text{GL}(q_1, \mathbb{R})^+\) (with \(q_1 = \text{rk}(TM/F)\)) principal bundle of oriented frames over \(M\).

Now as \(\mathcal{M}\) is a principal \(\text{SO}(q_1)\) bundle over \(\mathcal{M}\), \(\widetilde{F}\) determines an integrable subbundle \(\mathcal{F}\) of \(TM\).

\(^6\)We will use notations similar to those in Section \(^1\) with the only difference that when dealing with the Connes fibration, we use calligraphic letters.

\(^7\)In fact, for any \(X \in \Gamma(F)\), let \(X' \in \Gamma(\mathcal{F})\) denote the lift of \(X\). Let \(\varphi_t\) (with \(t\) close to zero) be the one parameter family of diffeomorphisms on \(\mathcal{M}\) generated by \(X'\). Then each \(\varphi_t\) acts on the complete transversal to \(F\) in \(\mathcal{M}\). The differential of \(\varphi_t\), when acting on the complete transversal, maps each \((\mathcal{F}_1^\perp + \mathcal{F}_2^\perp)_x (x \in \mathcal{M})\) to \((\mathcal{F}_1^\perp + \mathcal{F}_2^\perp)_{\varphi_t(x)}\) and verifies \(^6\) Lemma 5.2. By taking derivative at \(t = 0\), one gets \(^{1,5}\).
For any \( \beta > 0, \varepsilon > 0 \), let \( g_{\beta, \varepsilon}^{TM} \) be the Riemannian metric on \( TM \) defined as in (1.9). By (1.8), (1.9) and (2.2), the following identity holds for the Connes fibration,

\[
\nabla_{\mathcal{F}^+_{2, \beta, \varepsilon}} = \nabla_{\mathcal{F}^+_{2}}.
\]

Equivalently, for any \( X \in TM \) and \( U, V \in \Gamma(\mathcal{F}^+_{2}) \), one has \( \langle \nabla_{X}^{\mathcal{F}^+_{2, \beta, \varepsilon}} U, V \rangle = \langle \nabla_{X}^{\mathcal{F}^+_{2}} U, V \rangle \).

### 2.2. A coordinate system near \( s(M) \)

Let \( s(M) \subset \mathcal{M} \) be the image of the embedded section \( s : M \hookrightarrow \mathcal{M} \). Consider the induced fibration \( s \circ \pi : \mathcal{M} \to s(M) \). In what follows, for any \( x \in s(M) \), we will denote the fiber \( \mathcal{M}_{\pi(x)} \) simply by \( \mathcal{M}_{x} \).

For any \( x \in s(M) \), \( Z \in T_{x}\mathcal{M}_{x} = \mathcal{F}^+_{2}|_{x} \) with \( |Z| \) sufficiently small, let \( \exp^{\mathcal{M}_{x}}(tZ) \) be the geodesic in \( \mathcal{M}_{x} \) such that \( \exp^{\mathcal{M}_{x}}(0) = x, \frac{d\exp^{\mathcal{M}_{x}}(tZ)}{dt}|_{t=0} = Z \).

Let \( 0 < c \leq 1 \) which will be fixed later in (2.167).

For any \( \alpha > 0 \), let \( \psi_{c} : U_{c}(\mathcal{F}^+_{2}) = \{(x, Z) : x \in s(M), Z \in \mathcal{F}^+_{2}|_{x}, |Z| < \alpha \} \to \mathcal{M} \) be defined such that for any \( x \in s(M), Z \in T_{x}\mathcal{M}_{x} \) with \( |Z| < \alpha \),

\[
\psi_{c} : (x, Z) \mapsto \exp^{\mathcal{M}_{x}}(cZ).
\]

Clearly, \( \psi_{c} \) is a diffeomorphism from \( U_{c}(\mathcal{F}^+_{2}) \) to its image, when \( \alpha \) is sufficiently small, which we fix it now. In case of no confusion, we will also use the notation \( (x, Z) \) to denote its image \( \psi_{c}(x, Z) \). In particular, \( (x, 0) = x \). We also denote the geodesic \( \exp^{\mathcal{M}_{x}}(tZ) \) by \( tZ \).

On \( \psi_{c}(U_{c}(\mathcal{F}^+_{2})) \approx U_{c}(\mathcal{F}^+_{2}) \), the volume form \( dv_{\mathcal{M}} \) can be written as

\[
dv_{\mathcal{M}}(x, Z) = k_{c}(x, Z)dv_{\mathcal{F}^+_{2, x}}(Z)dv_{s(M)}(x),
\]

where \( dv_{\mathcal{F}^+_{2, x}} \) is the volume form on \( \mathcal{F}^+_{2, x} = \mathcal{F}^+_{2}|_{x} \) which in turn determines the corresponding volume form on \( \mathcal{M}_{x} \cap \psi_{c}(U_{c}(\mathcal{F}^+_{2})) \), \( dv_{s(M)} \) is the volume form on \( s(M) \) with respect to the restricted metric, and \( k_{c}(x, Z) > 0 \) is the function determined by (2.4) and (2.5).

In what follows, we will also denote \( dv_{\mathcal{F}^+_{2, x}} \) by \( dv_{\mathcal{M}_{x}} \).

### 2.3. Adiabatic limit near \( s(M) \)

For simplicity, we assume that \( q = \text{rk}(\mathcal{F}) \) and \( q_{1} = \text{rk}(\mathcal{F}^+_{1}) \) are divisible by 8. Then all the spinor bundles and exterior algebras have real structures. So we can work on the category of real spaces.

Recall that for \( \beta > 0 \) and \( \varepsilon > 0 \), \( g_{\beta, \varepsilon}^{TM} \) is the Riemannian metric on \( TM \) defined by

\[
g_{\beta, \varepsilon}^{TM} = \beta^{2}g_{F}^{\mathcal{F}} \oplus \frac{1}{\varepsilon^{2}}g_{\mathcal{F}^+_{1}}^{\mathcal{F}^+_{1}} \oplus g_{\mathcal{F}^+_{2}}^{\mathcal{F}^+_{2}},
\]

and that \( D_{\mathcal{F}^+_{2}, \phi_{1}(\mathcal{F}^+_{1}), \beta, \varepsilon} \) is the sub-Dirac operator constructed in (1.68) with respect to \( g_{\beta, \varepsilon}^{TM} \)\(^8\). By (2.6) one has

\[
dv_{(TM, g_{\beta, \varepsilon}^{TM})} = \frac{\beta^{q}}{\varepsilon^{q}}dv_{(TM, g^{TM})}.
\]

\(^{8}\)As \( \mathcal{F}^+_{2}|_{s(M)} \) need not be orthogonal to \( Ts(M) \), \( k_{c}(x, 0) \) need not be constant on \( s(M) \) (compare with (8.22)). The subscript “c” here simply indicates that this function depends on \( c \).

\(^{9}\)In this section, we will not consider the twisted bundle \( \phi_{2}(\mathcal{F}^+_{2}) \), as it does not affect the final result.
For simplicity, from now on, by $L^2$-norms we will mean the $L^2$-norms with respect to the volume form $du(T,M,g_T,M)$, i.e., for any $s \in \Gamma(W(F,F_1^+,F_2^+) \otimes \phi_1(F_1^+))$ with compact support, one has

$$(2.8) \quad \|s\|_0^2 := \int_M \langle s,s \rangle_{\beta,\varepsilon} du(T,M,g_T,M),$$

where the subscripts "$\beta, \varepsilon$" indicate that the pointwise inner product is induced from $g_T^{TM}$.

From (2.7) and (2.8), one sees that the operators which are formally self-adjoint with respect to the usual $L^2$-norm, which is associated with the volume form $du(T,M,g_T^{TM})$, is still formally self-adjoint with respect to the $L^2$-norm defined in (2.8).

By (1.78), one knows that when $\beta, \varepsilon > 0$ are sufficiently small, the following identity holds on $U_\alpha(F_2^+)$:

$$(2.9) \quad \left( D^{F,\phi_1(F_1^+)_{\beta,\varepsilon}} \right)^2 = -\Delta^{F,\phi_1(F_1^+)_{\beta,\varepsilon}} + \frac{k^F}{4\beta^2} + O \left( \frac{1}{\beta} + \varepsilon^2 \right).$$

Let $h_1, \cdots, h_{\dim M}$ be an oriented orthonormal basis of $(T,M,g_T^{TM})$. Then for any $s \in \Gamma(W(F,F_1^+,F_2^+) \otimes \phi_1(F_1^+))$ having compact support, the following identity holds:

$$(2.10) \quad \left\langle -\Delta^{F,\phi_1(F_1^+)_{\beta,\varepsilon}} s, s \right\rangle = \sum_{i=1}^{\dim M} \left\| \nabla_{h_i}^{F,\phi_1(F_1^+)_{\beta,\varepsilon}} s \right\|_0^2.$$

On the other hand, for any $\sigma \in \Gamma((S(F)\hat{\otimes}\Lambda^*(F_1^+) \otimes \phi_1(F_1^+))|_{s(M)})$, similarly as in (2.8), we define its $L^2$-norm by

$$(2.11) \quad \|\sigma\|_0^2 := \int_{s(M)} \langle \sigma, \sigma \rangle_{\beta,\varepsilon} du(s(M)), $$

where, as in (2.5), $du(s(M))$ is the volume form on $s(M)$ associated to the restricted metric from $g_T^{TM}|_{s(M)}$.

In what follows, we will also denote $du(T,M,g_T^{TM})$ by $du_M$ as in (2.5).

### 2.4. An embedding from sections on $s(M)$ to sections on $M$.

Recall that $\Lambda^*(F_2^+) = \bigoplus_{i=0}^{r_k(F_2^+)} \Lambda^i(F_2^+)$, with $\Lambda^0(F_2^+) = \mathbb{R}$. Let

$$(2.12) \quad Q : \Lambda^*(F_2^+) \to \Lambda^0(F_2^+) = \mathbb{R}$$

denote the corresponding orthogonal projection. Let

$$(2.13) \quad i_Q : \Lambda^0(F_2^+) \hookrightarrow \Lambda^*(F_2^+)$$

denote the canonical inclusion. In view of (1.62) and (1.65), the projection $Q$ and the embedding $i_Q$ induce the following canonical orthogonal projection and embedding, which we will denote by the same notations,

$$(2.14) \quad Q : W(F,F_1^+,F_2^+) \otimes \phi_1(F_1^+) \to S(F)\hat{\otimes}\Lambda^*(F_1^+) \otimes \phi_1(F_1^+),$$

$$(2.15) \quad i_Q : S(F)\hat{\otimes}\Lambda^*(F_1^+) \otimes \phi_1(F_1^+) \hookrightarrow W(F,F_1^+,F_2^+) \otimes \phi_1(F_1^+).$$
Let $Q\nabla F,\phi_1(F^1_+)_{\beta,\varepsilon}$ be the induced connection on $S(F)\otimes\Lambda^*(F^1_+)\otimes\phi_1(F^1_+)$ defined by
\begin{equation}
(2.16)\quad Q\nabla F,\phi_1(F^1_+)_{\beta,\varepsilon} = Q\nabla F,\phi_1(F^1_+)_{\beta,\varepsilon}i_Q.
\end{equation}
Clearly, $Q\nabla F,\phi_1(F^1_+)_{\beta,\varepsilon}$ is a Euclidean connection.

Let $\sigma \in \Gamma((S(F)\otimes\Lambda^*(F^1_+)\otimes\phi_1(F^1_+))|_{s(M)})$. For any $(x, Z) \in U_{\alpha}(F^1_2)$, let $\tau\sigma(x, Z) \in (S(F)\otimes\Lambda^*(F^1_+)\otimes\phi_1(F^1_+))|_{\psi(x, Z)}$ be the parallel transport of $\sigma(x)$ along the geodesic $(x, tZ)$, $0 \leq t \leq c$, with respect to the connection $Q\nabla F,\phi_1(F^1_+)_{\beta,\varepsilon}$.

Let $\gamma$ be a smooth function on $\mathbb{R}$ such that $\gamma(1) = 1$ if $b \leq \frac{\alpha}{3}$, while $\gamma(0) = 0$ if $b \geq \frac{2\alpha}{3}$.

For $T > 0$, $x \in s(M)$, set
\begin{equation}
(2.17)\quad \alpha_T(x) = \int_{M_x} \exp \left( -T|Z|^2 \right) \gamma^2(|Z|) dv_{M_x}(Z).
\end{equation}
Clearly, $\alpha_T(x)$ is constant on $s(M)$, which we will denote by $\alpha_T$.

Inspired by [4, Definition 9.4], for $T > 0$, let
\begin{equation}
J_{T,\beta,\varepsilon} : \Gamma ((S(F)\otimes\Lambda^*(F^1_+)\otimes\phi_1(F^1_+))|_{s(M)}) \longrightarrow \Gamma (W(F, F^1_1, F^1_2) \otimes\phi_1(F^1_+))
\end{equation}
be the embedding defined by
\begin{equation}
(2.18)\quad J_{T,\beta,\varepsilon} : \sigma \mapsto (J_{T,\beta,\varepsilon}\sigma)|_{\psi_{(x, Z)}} = (k_{c}(x, Z)\alpha_T)^{-\frac{\delta}{2}} \gamma(|Z|) \exp \left( -\frac{T|Z|^2}{2} \right) i_Q(\tau\sigma(x, Z)).
\end{equation}

By the definition of $\gamma$, one sees that $J_{T,\beta,\varepsilon}$ is well-defined. Moreover, in view of (2.3), (2.6), (2.11), (2.17) and (2.18), one sees that $J_{T,\beta,\varepsilon}$ is an isometric embedding.

Clearly, any $J_{T,\beta,\varepsilon}\sigma$ has compact support in $M_{2\beta/3}$. Let $E'_{T,\beta,\varepsilon}$ denote the image of $\Gamma((S(F)\otimes\Lambda^*(F^1_+)\otimes\phi_1(F^1_+))|_{s(M)})$ under $J_{T,\beta,\varepsilon}$. Let $p_{T,\beta,\varepsilon}$ denote the orthogonal projection from the $L^2$-completion of $\Gamma(W(F, F^1_1, F^1_2) \otimes\phi_1(F^1_+))$ to the $L^2$-completion of $E'_{T,\beta,\varepsilon}$, which we denote by $E_{T,\beta,\varepsilon}$.

2.5. An estimate for $\|p_{T,\beta,\varepsilon}D^F,\phi_1(F^1_+)_{\beta,\varepsilon}p_{T,\beta,\varepsilon}\|^2_0$. Let $f_1, \ldots, f_{q+1}$ be an orthonormal basis of $(F \oplus F^1_1)|_{s(M)}$ with respect to $g^F_0 \oplus g^{F^1_1}_0$, where $f_1, \ldots, f_q$ is an orthonormal basis of $F|_{s(M)}$ and thus $f_{q+1}, \ldots, f_{q+1}$ is an orthonormal basis of $F^1_1|_{s(M)}$. Let $e_1, \ldots, e_{q_2}$ be an orthonormal basis of $F^1_2|_{s(M)}$ with respect to $g^{F^1_2}_0$.

For any $f \in (F \oplus F^1_1)|_{s(M)}$ (resp. $e \in F^1_2|_{s(M)}$), let $\tau f \in \Gamma(F \oplus F^1_1)$ (resp. $\tau e \in \Gamma(F^1_2)$) be such that for any $(x, Z) \in U_{\alpha}(F^1_2)$, $\tau f|_{\psi_{(x, Z)}}$ (resp. $\tau e|_{\psi_{(x, Z)}}$) is the parallel transport of $f_x$ (resp. $e_x$) along the geodesic $(x, tZ)$, $0 \leq t \leq c$, with respect to the Euclidean connection $(p + p^1_1)\nabla_{T,M,\beta,\varepsilon}(p + p^1_1)$ (resp. $\nabla_{F^1_2,\beta,\varepsilon}$).

Clearly, $\beta^{-1}\tau f_i$, $1 \leq i \leq q$, $\varepsilon \tau f_j$, $q + 1 \leq j \leq q + q_1$ and $\varepsilon \tau e_k$, $1 \leq k \leq q_2$ form an orthonormal basis of $(TM, g^{TM}_{\beta,\varepsilon})$.

Let $c_{\beta,\varepsilon}()$ be the Clifford action associated to $g^{TM}_{\beta,\varepsilon}$. For any $X, Y \in TM$, one has
\begin{equation}
(2.19)\quad c_{\beta,\varepsilon}(X)c_{\beta,\varepsilon}(Y) + c_{\beta,\varepsilon}(Y)c_{\beta,\varepsilon}(X) = -2\langle X, Y \rangle g^{TM}_{\beta,\varepsilon}.
\end{equation}
By (1.68), one has

\begin{equation}
D^F,\phi_1(F^+_{1}),\beta,\varepsilon = \beta^{-1} \sum_{i=1}^{q} c_{\beta,\varepsilon} \left( \beta^{-1} \tau f_i \right) \nabla^F_{\tau f_i} \phi_1(F^+_{1}),\beta,\varepsilon + \varepsilon \sum_{i=q+1}^{q+q_1} c_{\beta,\varepsilon} \left( \varepsilon \tau f_i \right) \nabla^F_{\tau f_i} \phi_1(F^+_{1}),\beta,\varepsilon + \sum_{s=1}^{q_2} c_{\beta,\varepsilon} \left( \tau e_s \right) \nabla^F_{\tau e_s} \phi_1(F^+_{1}),\beta,\varepsilon.
\end{equation}

From now on, we make the assumption that the leafwise scalar curvature $k^F$ of $g^F$ verifies that $k^F \geq \eta > 0$ over $M$.

We state a key asymptotic estimate result for $\|p_{T,\beta,\varepsilon} D^F,\phi_1(F^+_{1}),\beta,\varepsilon p_{T,\beta,\varepsilon}\|_0^2$, when $T \to +\infty$ and $\beta, \varepsilon > 0$ being small, as follows.

**Proposition 2.2.** There exist $C' > 0$ not depending on $c > 0$, $C'' > 0$, $0 < \delta$, $\beta_0$, $\varepsilon_0 < 1$ and $T_0 > 0$ such that for any $0 < \beta \leq \beta_0$, $0 < \varepsilon \leq \varepsilon_0$ and $T \geq T_0$, there exists $C_{\beta,\varepsilon} > 0$ for which the following inequality holds for any $\sigma \in \Gamma((S(F) \otimes \Lambda^*(F^+_{1}) \otimes \phi_1(F^+_{1}))|_{s(M)})$:

\begin{equation}
\left\|p_{T,\beta,\varepsilon} D^F,\phi_1(F^+_{1}),\beta,\varepsilon J_{T,\beta,\varepsilon}\sigma\right\|_0^2 \geq \left( \frac{\eta - c C'}{4 \beta^2} - C'' \left( \frac{1}{\beta} + \frac{\varepsilon \delta}{\beta^2} + \frac{\varepsilon}{\beta^4} \right) \right) \int_{s(M)} |\sigma|^2 dv_{s(M)} + \frac{\varepsilon \delta}{8 \beta^2} \sum_{k=1}^{q} \int_{s(M)} |Q_{\nabla^F_{f_k}} \phi_1(F^+_{1}),\beta,\varepsilon (\tau \sigma)|^2 dv_{s(M)} + \frac{\varepsilon}{8} \sum_{k=q+1}^{q+q_1} \int_{s(M)} |Q_{\nabla^F_{f_k}} \phi_1(F^+_{1}),\beta,\varepsilon (\tau \sigma)|^2 dv_{s(M)} - \frac{C_{\beta,\varepsilon}}{\sqrt{T}} \int_{s(M)} |\sigma|^2 + \sum_{k=1}^{q+q_1} |Q_{\nabla^F_{f_k}} \phi_1(F^+_{1}),\beta,\varepsilon (\tau \sigma)|^2 dv_{s(M)}.
\end{equation}

The basic idea of the proof of Proposition 2.2 is very natural. Indeed, since $p_{T,\beta,\varepsilon} : L^2(W(F, F^+_{1}, F^+_{0}) \otimes \phi_1(F^+_{1})) \to E_{T,\beta,\varepsilon}$ is an orthogonal projection, for any $\sigma \in \Gamma((S(F) \otimes \Lambda^*(F^+_{1}) \otimes \phi_1(F^+_{1}))|_{s(M)})$, one has

\begin{equation}
\left\|p_{T,\beta,\varepsilon} D^F,\phi_1(F^+_{1}),\beta,\varepsilon J_{T,\beta,\varepsilon}\sigma\right\|_0^2 = \left\|D^F,\phi_1(F^+_{1}),\beta,\varepsilon J_{T,\beta,\varepsilon}\sigma\right\|_0^2 - \left\|(1 - p_{T,\beta,\varepsilon}) D^F,\phi_1(F^+_{1}),\beta,\varepsilon J_{T,\beta,\varepsilon}\sigma\right\|_0^2.
\end{equation}

In view of (2.7) and (2.8), the operator $D^F,\phi_1(F^+_{1}),\beta,\varepsilon$ is formally self-adjoint with respect to the $L^2$-norm in (2.22). Thus, the first term in the right hand side of (2.22) can be estimated by using (2.9) and (2.10). So we need to estimate the second term in the right hand side of (2.22), to make it as small as possible.

In what follows, for brevity, we also write $\nabla^F,\beta,\varepsilon$ for $\nabla^F,\phi_1(F^+_{1}),\beta,\varepsilon$.

Set

\begin{equation}
I_1 = \sum_{i \neq j, \ 1 \leq i, j \leq q} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} \left( \beta^{-1} \tau f_i \right) \nabla^F_{\beta^{-1} \tau f_i} J_{T,\beta,\varepsilon}\sigma, c_{\beta,\varepsilon} \left( \beta^{-1} \tau f_j \right) \nabla^F_{\beta^{-1} \tau f_j} J_{T,\beta,\varepsilon}\sigma \right\rangle,
\end{equation}
(2.24) \[ I_2 = \sum_{i \neq j, \ p+1 \leq i, j \leq q+q_1} \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\varepsilon \tau f_i) \tilde{\nabla}_{\varepsilon \tau f_i}^F \beta, \varepsilon J_{T, \beta, \varepsilon} \sigma, c_{\beta, \varepsilon}(\varepsilon \tau f_j) \tilde{\nabla}_{\varepsilon \tau f_j}^F \beta, \varepsilon J_{T, \beta, \varepsilon} \sigma \right\rangle, \]

(2.25) \[ I_3 = \sum_{i \neq j, \ 1 \leq i, j \leq q_2} \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tau e_i) \tilde{\nabla}_{\tau e_i}^F \beta, \varepsilon J_{T, \beta, \varepsilon} \sigma, c_{\beta, \varepsilon}(\tau e_j) \tilde{\nabla}_{\tau e_j}^F \beta, \varepsilon J_{T, \beta, \varepsilon} \sigma \right\rangle, \]

(2.26) \[ I_4 = 2 \sum_{i=1}^{q} \sum_{j=q+1}^{q+q_1} \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} \left( \beta^{-1} \tau f_i \right) \tilde{\nabla}_{\beta^{-1} \tau f_i}^F \beta, \varepsilon J_{T, \beta, \varepsilon} \sigma, c_{\beta, \varepsilon}(\varepsilon \tau f_j) \tilde{\nabla}_{\varepsilon \tau f_j}^F \beta, \varepsilon J_{T, \beta, \varepsilon} \sigma \right\rangle, \]

(2.27) \[ I_5 = 2 \sum_{i=1}^{q} \sum_{j=1}^{q_2} \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\varepsilon \tau f_i) \tilde{\nabla}_{\varepsilon \tau f_i}^F \beta, \varepsilon J_{T, \beta, \varepsilon} \sigma, c_{\beta, \varepsilon}(\tau e_j) \tilde{\nabla}_{\tau e_j}^F \beta, \varepsilon J_{T, \beta, \varepsilon} \sigma \right\rangle, \]

(2.28) \[ I_6 = 2 \sum_{i=q+1}^{q+q_1} \sum_{j=1}^{q_2} \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tau e_i) \tilde{\nabla}_{\tau e_i}^F \beta, \varepsilon J_{T, \beta, \varepsilon} \sigma, c_{\beta, \varepsilon}(\tau e_j) \tilde{\nabla}_{\tau e_j}^F \beta, \varepsilon J_{T, \beta, \varepsilon} \sigma \right\rangle. \]

By (2.20) and (2.23)-(2.28), one has

(2.29) \[ \left\| (1 - p_{T, \beta, \varepsilon}) D_{\beta, \varepsilon}^{F, \phi_1(F \mathbf{1}), \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma \right\|_0^2 = \sum_{k=1}^{6} I_k + \sum_{i=1}^{q} \left\| (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} \left( \beta^{-1} \tau f_i \right) \tilde{\nabla}_{\beta^{-1} \tau f_i}^F \beta, \varepsilon J_{T, \beta, \varepsilon} \sigma \right\|_0^2 \]

\[ + \sum_{i=q+1}^{q+q_1} \left\| (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\varepsilon \tau f_i) \tilde{\nabla}_{\varepsilon \tau f_i}^F \beta, \varepsilon J_{T, \beta, \varepsilon} \sigma \right\|_0^2 \]

\[ + \sum_{i=1}^{q_2} \left\| (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tau e_i) \tilde{\nabla}_{\tau e_i}^F \beta, \varepsilon J_{T, \beta, \varepsilon} \sigma \right\|_0^2. \]

Naturally, we need to study the behaviour when \( T \to +\infty \) of each term in the right hand side of (2.29). Due to the Gaussian factor \( \exp(-T|Z|^2/2) \) in (2.18), one sees as in [1, Chapters 8 and 9] that when \( T \to +\infty \), all terms in (2.29) localize onto \( s(M) \). All one need is to choose the rescaling factors \( \beta, \varepsilon \) conveniently such that the estimate goes as desired. For this the geometric nature of the Connes fibration plays an essential role.

The fact that the right hand side of (2.29) has nine terms, with each term further splits into four or even more terms in the process of estimation, partly explains the length of the computations, which are purely routine and elementary.

2.6. Estimates of the terms \( I_k, 1 \leq k \leq 6, \text{ Part I} \). Before going on, we set a notational convention: in what follows, by \( O(|Z|^2) \) and \( O\left(\frac{1}{\sqrt{T}}\right) \), we will mean \( O_{\beta, \varepsilon}(|Z|^2) \) and \( O_{\beta, \varepsilon}\left(\frac{1}{\sqrt{T}}\right) \), i.e., the associated estimating constants may depend on \( \beta > 0 \) and \( \varepsilon > 0 \). While for other \( O(\cdots) \) terms, the corresponding estimating constants will not depend on \( \beta > 0 \) and \( \varepsilon > 0 \), unless there appear the subscripts “\( \beta \)” and/or “\( \varepsilon \)” which will indicate that the corresponding estimating coefficient will depend on \( \beta \) and/or \( \varepsilon \).
For brevity, let $f_T$ be the smooth function on $\mathcal{M}$ defined by that on any $(x, Z) \simeq \psi_c(x, Z)$, one has,

$$f_T(x, Z) = (k_c(x, Z)\alpha_T)^{-\frac{1}{2}} \gamma(|Z|) \exp \left( -\frac{T|Z|^2}{2} \right).$$

Then one can rewrite $J_{T,\beta,\varepsilon}\sigma$ in (2.18) as

$$J_{T,\beta,\varepsilon}\sigma(x, Z) = f_T(x, Z)i_Q(\tau\sigma(x, Z)).$$

From now on, in case of no confusion, we will omit $i_Q$.

**Lemma 2.3.** (i) For any $\sigma \in \Gamma((S(\mathcal{F}) \hat{\otimes} \Lambda^1(\mathcal{F}_1^+)) \otimes \phi_1(\mathcal{F}_1^+))|_{s(M)}$ and any $f \in C^\infty(\mathcal{M})$ with $\text{Supp}(f) \subset \psi_c(U_a(\mathcal{F}_2^+))$, one has

$$p_{T,\beta,\varepsilon}(f\tau\sigma)(x, Z) = \left( \int_{\mathcal{M}_x} f_T(x, Z')f(x, Z')k_c(x, Z')dv_{\mathcal{M}_x}(Z') \right) (J_{T,\beta,\varepsilon}\sigma)(x, Z);$$

(ii) For any $u \in \Gamma(W(\mathcal{F}, \mathcal{F}_1^+, \mathcal{F}_2^+) \otimes \phi_1(\mathcal{F}_1^+))$ with $\text{Supp}(u) \subset \psi_c(U_a(\mathcal{F}_2^+))$, one has

$$p_{T,\beta,\varepsilon}(f_Tu)(x, Z) = J_{T,\beta,\varepsilon}\left( (Qu)|_{s(M)} \right) + p_{T,\beta,\varepsilon}(O_{\beta,\varepsilon}(|Z|)).$$

**Proof.** Take any $u \in \Gamma((W(\mathcal{F}, \mathcal{F}_1^+, \mathcal{F}_2^+) \otimes \phi_1(\mathcal{F}_1^+))|_{\psi_c(U_a(\mathcal{F}_2^+)))}$. Then for any $(x, Z) \in U_a(\mathcal{F}_2^+)$, $(Qu)|_{\psi_c(x,z)}$ determines a unique element $u' \in (S(\mathcal{F}) \hat{\otimes} \Lambda^1(\mathcal{F}_1^+) \otimes \phi_1(\mathcal{F}_1^+))|_x$ such that $(\tau u')|_{\psi_c(x,z)} = (Qu)|_{\psi_c(x,z)}$. We denote this element by $\tau^{-1}((Qu)|_{x,z})$.

Then one verifies easily that (compare with [4] (9.6) and (9.13))

$$p_{T,\beta,\varepsilon}u(x, Z) = f_T(x, Z) \left( \tau \int_{\mathcal{M}_x} f_T(x, Z')k_c(x, Z') \tau^{-1}((Qu)|_{x,z})dv_{\mathcal{M}_x}(Z') \right)(x, Z).$$

Formulas (2.32) and (2.33) follow from (2.34) easily. \hfill \Box

**Lemma 2.4.** For any $X \in \Gamma((\mathcal{F} \oplus \mathcal{F}_1^+)|_{s(M)})$, one has

$$p_{T,\beta,\varepsilon}c_{\beta,\varepsilon}(\tau X) = c_{\beta,\varepsilon}(\tau X)p_{T,\beta,\varepsilon}.$$

**Proof.** For any $\sigma \in \Gamma((S(\mathcal{F}) \hat{\otimes} \Lambda^1(\mathcal{F}_1^+) \otimes \phi_1(\mathcal{F}_1^+))|_{s(M)})$ and $X \in \Gamma((\mathcal{F} \oplus \mathcal{F}_1^+)|_{s(M)})$, we claim that

$$c_{\beta,\varepsilon}(\tau X)\tau\sigma = \tau(c_{\beta,\varepsilon}(X)|_{s(M)}).$$

Indeed, it is easy to verify that

$$Q \nabla^{\mathcal{F}_{\beta,\varepsilon}}_Z (c_{\beta,\varepsilon}(\tau X)|_{s(M)})\tau\sigma = Q \left( c_{\beta,\varepsilon} \left( \nabla^{T\mathcal{M},\beta,\varepsilon}_Z(\tau X) \right) \tau\sigma \right) + c_{\beta,\varepsilon}(\tau X)Q \nabla^{\mathcal{F}_{\beta,\varepsilon}}_Z(\tau\sigma) + c_{\beta,\varepsilon}(\tau X)(p + p_1) \nabla^{T\mathcal{M},\beta,\varepsilon}_Z(\tau X)\tau\sigma = 0.$$

From (2.37), one sees that $c_{\beta,\varepsilon}(\tau X)|_{s(M)}$ is the parallel transport of $(c_{\beta,\varepsilon}(\tau X)|_{s(M)}) = c_{\beta,\varepsilon}(X)|_{s(M)}$, from which (2.36) follows.
Now for any \( \sigma \in \Gamma(\{(S) \wedge \Lambda^\ast(F^\perp) \otimes \phi_1(F^\perp)\}|_{s(M)}) \) and \( u \in \Gamma(W(F, F^\perp_1, F^\perp_2) \otimes \phi_1(F^\perp)) \) with \( \text{Supp}(u) \subset \psi_c(U_{\alpha}(F^\perp_2)) \), one verifies via (2.36) that

\[
\langle pt_{\beta,\varepsilon}c_{\beta,\varepsilon}(\tau X), J_{T,\beta,\varepsilon}\sigma \rangle = \langle u, c_{\beta,\varepsilon}(\tau X)J_{T,\beta,\varepsilon}\sigma \rangle
\]

\[
= -\langle u, J_{T,\beta,\varepsilon}(c_{\beta,\varepsilon}(X)\sigma) \rangle = -\langle pt_{\beta,\varepsilon}u, c_{\beta,\varepsilon}(\tau X)J_{T,\beta,\varepsilon}\sigma \rangle
\]

\[
= \langle c_{\beta,\varepsilon}(\tau X)pt_{\beta,\varepsilon}u, J_{T,\beta,\varepsilon}\sigma \rangle,
\]

from which (2.35) follows. \( \square \)

For any \( X \in \Gamma((F \oplus F^\perp_1)|_{s(M)}) \), by (2.35), one finds

\[
(1 - pt_{\beta,\varepsilon})c_{\beta,\varepsilon}(\tau X) = c_{\beta,\varepsilon}(\tau X) (1 - pt_{\beta,\varepsilon}).
\]

Let \( f_1', \ldots, f_q' \) be lifted from corresponding elements on \( M \). That is, there is an orthonormal basis \( \tilde{f}_1, \ldots, \tilde{f}_q \) of \( (F, g^F) \) such that

\[
f_i' = \pi^*\tilde{f}_i, \quad 1 \leq i \leq q.
\]

**Lemma 2.5.** The following asymptotic formulas at \((x, Z)\) (i.e., \( \psi_c(x, Z) \)) with \( x \in s(M) \), \( Z \in F^\perp_2|_x \) hold near \( s(M) \): (i) if \( 1 \leq i \leq q \), then

\[
\tau f_i = f_i' + \sum_{m=1}^{q+q_1} O(\varepsilon^2|Z|) f_m' + O(|Z|^2);
\]

(ii) if \( q + 1 \leq i \leq q + q_1 \), then

\[
\tau f_i = f_i' + \sum_{j=1}^{q} O\left(\frac{|Z|}{\beta^2}\right) f_j' + \sum_{m=q+1}^{q+q_1} O(|Z|) f_m' + O(|Z|^2).
\]

**Proof.** We write

\[
\tau f_i = f_i' + \sum_{k=1}^{q+q_1} \langle \tau f_i - f_i', f_k' \rangle f_k'.
\]

Since

\[
(p + p^\perp) \nabla_T M, \beta, \varepsilon(\tau f_i) = 0,
\]

by (2.4) one has for \( 1 \leq i, k \leq q \) that

\[
\langle \tau f_i - f_i', f_k' \rangle_{(x, Z)} = Z \left( \langle \tau f_i, f_k' \rangle_{(x, Z)} \right) + O(|Z|^2)
\]

\[
= c \left( \langle \tau f_i, \nabla_T M, \beta, \varepsilon f_k' \rangle_{(x, Z)} \right) + O(|Z|^2) = c \left( \langle f_i, \nabla_T M, \beta, \varepsilon f_k' \rangle \right)_x + O(|Z|^2),
\]

while for \( 1 \leq i \leq q, q + 1 \leq k \leq q + q_1 \), one has, by (1.6), (1.8),

\[
\langle \tau f_i - f_i', f_k' \rangle_{(x, Z)} = Z \left( \langle \tau f_i, f_k' \rangle_{(x, Z)} \right) + O(|Z|^2)
\]

\[
= \beta^2 \varepsilon^2 \left( \langle f_i, \nabla_T M, \beta, \varepsilon f_k' \rangle \right)_x + O(|Z|^2) = O(\varepsilon^2|Z|) + O(|Z|^2).
\]
Now by (2.40), one has that for any $e \in \Gamma(F_{p}^{1})$ and $1 \leq i \leq q$,
\begin{equation}
(2.47) \quad [e, f'_{i}] \in \Gamma(F_{p}^{1}),
\end{equation}
from which one verifies that for any $e \in \Gamma(F_{p}^{1})$ and $1 \leq i, k \leq q$,
\begin{equation}
(2.48) \quad \langle f'_{i}, \nabla^{TM, \beta, \epsilon} f'_{k} \rangle = \langle e, \nabla^{TM, \beta, \epsilon} f'_{k} \rangle = 0.
\end{equation}

From (2.43), (2.45), (2.46) and (2.48), one gets (2.41).

Then, one has
\begin{equation}
(2.49) \quad \langle \tau_{m} - f'_{m}, f'_{k} \rangle_{(x, Z)} = Z \left( \langle \tau_{m}, f'_{k} \rangle_{(x, Z)} \right) + O \left( |Z|^{2} \right)
\end{equation}
\begin{equation*}
= \frac{c}{\beta^{2} \epsilon^{2}} \left( \langle f_{m}, \nabla^{TM, \beta, \epsilon} f'_{k} \rangle_{x} \right) + O \left( |Z|^{2} \right) = O \left( \frac{|Z|}{\beta^{2}} \right) + O \left( |Z|^{2} \right),
\end{equation*}
while for $q + 1 \leq m, k \leq q + q_{1}$, one has
\begin{equation}
(2.50) \quad \langle \tau_{m} - f'_{m}, f'_{k} \rangle_{(x, Z)} = Z \left( \langle \tau_{m}, f'_{k} \rangle_{(x, Z)} \right) + O \left( |Z|^{2} \right)
\end{equation}
\begin{equation*}
= c \left( \langle f_{m}, \nabla^{TM, \beta, \epsilon} f'_{k} \rangle_{x} \right) + O \left( |Z|^{2} \right) = O \left( |Z| \right) + O \left( |Z|^{2} \right).
\end{equation*}

By proceeding as in (2.45), one sees that for $q + 1 \leq m \leq q + q_{1}, 1 \leq k \leq q$,
\begin{equation}
(2.51) \quad \sum_{i=1}^{q+q_{1}} \left| Q \nabla_{\tau X}^{F, \phi_{1}(F_{1}^{+}), \beta, \epsilon}(\tau \sigma) \right|_{\psi_{c}(x, Z)}^{2} + \sum_{j=1}^{q_{2}} \left| Q \nabla_{\tau X}^{F, \phi_{1}(F_{1}^{+}), \beta, \epsilon}(\tau \sigma) \right|_{\psi_{c}(x, Z)}^{2}
\leq C_{\beta, \epsilon} \left( \sum_{i=1}^{q+q_{1}} \left| Q \nabla_{\tau X}^{F, \phi_{1}(F_{1}^{+}), \beta, \epsilon}(\tau \sigma) \right|^{2} + |\sigma|^{2} \right).
\end{equation}

Lemma 2.6. There exists $C_{\beta, \epsilon} > 0$ such that the following estimate holds near $s(M)$ for $|Z| \leq 2\alpha/3$: for any $\sigma \in \Gamma((S(F) \hat{\otimes} \Lambda^{*}(F_{1}^{+}) \otimes \phi_{1}(F_{1}^{+}))|_{s(M)})$, one has
\begin{equation}
(2.52) \quad \left( \left. Q \nabla_{\tau X}^{F, \phi_{1}(F_{1}^{+}), \beta, \epsilon}(\tau \sigma), \tau \sigma' \right|_{\beta, \epsilon} \right) = \tau X \left( \langle \tau \sigma, \tau \sigma' \rangle_{\beta, \epsilon} - \langle \tau \sigma, Q \nabla_{\tau X}^{F, \phi_{1}(F_{1}^{+}), \beta, \epsilon}(\tau \sigma') \rangle_{\beta, \epsilon} \right) \nabla_{\tau X}^{F, \phi_{1}(F_{1}^{+}), \beta, \epsilon}(\tau \sigma) \nabla_{\tau X}^{F, \phi_{1}(F_{1}^{+}), \beta, \epsilon}(\tau \sigma')_{\beta, \epsilon}.
\end{equation}

Proof. For any $X \in (TM)|_{s(M)}$ and $\sigma, \sigma' \in \Gamma((S(F) \hat{\otimes} \Lambda^{*}(F_{1}^{+}) \otimes \phi_{1}(F_{1}^{+}))|_{s(M)})$, one verifies that,
\begin{equation}
(2.53) \quad \left( \frac{\tau f_{i}}{|\tau f_{i}|_{\beta, \epsilon}} \right) = \frac{\tau f_{i}}{|\tau f_{i}|_{\beta, \epsilon}}.
\end{equation}

Then, one has $\tau f_{i} = \beta^{-1} f_{i}$ if $1 \leq i \leq q$, while $\tau f_{i} = \epsilon f_{i}$ if $q + 1 \leq i \leq q + q_{1}$.
Let $1 \leq i, j \leq q + q_1$ be such that $i \neq j$. By (2.39) one deduces that

\begin{equation}
\langle (1 - pt, \beta, \varepsilon) c_{\beta, \varepsilon} (\tilde{\tau} f_i) \tilde{\nabla}_{\tilde{f}_j}^{F, \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma, (1 - pt, \beta, \varepsilon) c_{\beta, \varepsilon} (\tilde{\tau} f_j) \tilde{\nabla}_{\tilde{f}_j}^{F, \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma \rangle
\end{equation}

\begin{equation}
= \langle c_{\beta, \varepsilon} (\tilde{\tau} f_i) (1 - pt, \beta, \varepsilon) \tilde{\tau} f_i (ft) \tau \sigma, c_{\beta, \varepsilon} (\tilde{\tau} f_j) (1 - pt, \beta, \varepsilon) \tilde{\tau} f_j (ft) \tau \sigma \rangle
\end{equation}

\begin{equation}
+ \langle c_{\beta, \varepsilon} (\tilde{\tau} f_i) (1 - pt, \beta, \varepsilon) \tilde{\tau} f_i (ft) \tau \sigma, c_{\beta, \varepsilon} (\tilde{\tau} f_j) (1 - pt, \beta, \varepsilon) \tilde{\tau} f_j (ft) \tilde{\nabla}_{\tilde{f}_j}^{F, \beta, \varepsilon} (\tau \sigma) \rangle
\end{equation}

\begin{equation}
+ \left\langle 1 - pt, \beta, \varepsilon \right\rangle c_{\beta, \varepsilon} (\tilde{\tau} f_i) f_t \tilde{\nabla}_{\tilde{f}_j}^{F, \beta, \varepsilon} (\tau \sigma), c_{\beta, \varepsilon} (\tilde{\tau} f_j) (1 - pt, \beta, \varepsilon) \tilde{\tau} f_j (ft) \tau \sigma \right\rangle
\end{equation}

\begin{equation}
+ \left\langle 1 - pt, \beta, \varepsilon \right\rangle c_{\beta, \varepsilon} (\tilde{\tau} f_i) f_t \tilde{\nabla}_{\tilde{f}_j}^{F, \beta, \varepsilon} (\tau \sigma), (1 - pt, \beta, \varepsilon) c_{\beta, \varepsilon} (\tilde{\tau} f_j) f_t \tilde{\nabla}_{\tilde{f}_j}^{F, \beta, \varepsilon} (\tau \sigma) \right\rangle.
\end{equation}

By (2.31) and (2.32), one has for any $1 \leq i \leq q + q_1$,

\begin{equation}
(1 - pt, \beta, \varepsilon) \tau f_i (ft) \tau \sigma = \left( \tau f_i (ft) - f_t \int_{M_x} f_t \tau f_i (ft) k_c \mathrm{dv}_{M_x} \right) \tau \sigma.
\end{equation}

For any $1 \leq i \leq q + q_1$, set

\begin{equation}
\rho_{T, \beta, \varepsilon, i} = \tau f_i (ft) - f_t \int_{M_x} f_t \tau f_i (ft) k_c \mathrm{dv}_{M_x}.
\end{equation}

By (2.30), one has

\begin{equation}
\tau f_i (ft) (x, Z) = \left( -\frac{\tau f_i (k_c)}{2k_c^{3/2}} + \frac{\tau f_i (\gamma)}{k_c^{1/2}} \alpha_t - \frac{T \tau f_i (Z^2)}{2k_c^{1/2}} \alpha_t \right) \exp \left( -\frac{T|Z|^2}{2} \right).
\end{equation}

Let $Z = \sum_{i=1}^{q_2} z_i e_i \in F_2^+ | s(M)$. Let $a_{ik}^j \in C^\infty (s(M))$ be defined by

\begin{equation}
\tau f_i (z_j) = \tau f_i (z_j) | s(M) + \sum_{k=1}^{q_2} a_{ik}^j z_k + O (|Z|^2).
\end{equation}

By (2.30), (2.56) - (2.58) and Lemma 2.3 when $T > 0$ is large enough, if $1 \leq i \leq q$,

\begin{equation}
\rho_{T, \beta, \varepsilon, i} (x, Z) = -\frac{T \tau f_i (|Z|^2)}{2} f_t (x, Z) + \frac{\tau f_i (\gamma)}{k_c^{1/2}} \left( 1 - \gamma \right) \exp \left( -\frac{T|Z|^2}{2} \right)
\end{equation}

\begin{equation}
+ \frac{1}{2} \left( \sum_{j=1}^{q_2} a_{ij}^j + O (|Z|) + O (|Z|^2) + O \left( \frac{1}{\sqrt{T}} \right) \right) f_t (x, Z),
\end{equation}

while for $q + 1 \leq i \leq q + q_1$, one has

\begin{equation}
\rho_{T, \beta, \varepsilon, i} (x, Z) = -\frac{T \tau f_i (|Z|^2)}{2} f_t (x, Z) + \frac{\tau f_i (\gamma)}{k_c^{1/2}} \left( 1 - \gamma \right) \exp \left( -\frac{T|Z|^2}{2} \right)
\end{equation}

\begin{equation}
+ \frac{1}{2} \left( \sum_{j=1}^{q_2} a_{ij}^j + O \left( \frac{|Z|}{\beta^2} \right) + O (|Z|^2) + O \left( \frac{1}{\sqrt{T}} \right) \right) f_t (x, Z).
\end{equation}

We now start to estimate (2.54).

For the first term in the right hand side of (2.54), by (2.55), and (2.56), for $i \neq j$,

\begin{equation}
\langle c_{\beta, \varepsilon} (\tilde{\tau} f_i) (1 - pt, \beta, \varepsilon) \tau f_i (ft) \tau \sigma, c_{\beta, \varepsilon} (\tilde{\tau} f_j) (1 - pt, \beta, \varepsilon) \tau f_j (ft) \tau \sigma \rangle
\end{equation}

\begin{equation}
= \langle c_{\beta, \varepsilon} (\tilde{\tau} f_i) c_{\beta, \varepsilon} (\tilde{\tau} f_j) \rho_{T, \beta, \varepsilon, i} \rho_{T, \beta, \varepsilon, j} \tau \sigma, \tau \sigma \rangle = 0,
\end{equation}

\end{document}
as $c_{\beta, \epsilon}(\tilde{f}_i) c_{\beta, \epsilon}(\tilde{f}_j)$ is skew-adjoint.

For the second and the third terms in the right hand side of (2.54), by (2.39), one finds that for $i \neq j$,

\begin{equation}
\int_{\mathcal{M}_x} \left( (1 - p_{T, \beta, \epsilon}) c_{\beta, \epsilon}(\tilde{f}_i) f_T \nabla_{\tilde{f}_i}^{F, \beta, \epsilon}(\tau \sigma), (1 - p_{T, \beta, \epsilon}) c_{\beta, \epsilon}(\tilde{f}_j) f_T \nabla_{\tilde{f}_j}^{F, \beta, \epsilon}(\tau \sigma) \right) k_c \, dv_{\mathcal{M}_x}
\end{equation}

\begin{equation}
\int_{\mathcal{M}_x} \left( (1 - p_{T, \beta, \epsilon}) c_{\beta, \epsilon}(\tilde{f}_i) f_T \nabla_{\tilde{f}_i}^{F, \beta, \epsilon}(\tau \sigma), (1 - p_{T, \beta, \epsilon}) c_{\beta, \epsilon}(\tilde{f}_j) f_T \nabla_{\tilde{f}_j}^{F, \beta, \epsilon}(\tau \sigma) \right) k_c \, dv_{\mathcal{M}_x}
\end{equation}

\begin{equation}
= \left( c_{\beta, \epsilon}(\tilde{f}_i) c_{\beta, \epsilon}(\tilde{f}_j) f_T \nabla_{\tilde{f}_i}^{F, \beta, \epsilon}(\tau \sigma), (1 - Q) \nabla_{\tilde{f}_j}^{F, \beta, \epsilon}(\tau \sigma), c_{\beta, \epsilon}(\tilde{f}_j) (1 - Q) \nabla_{\tilde{f}_j}^{F, \beta, \epsilon}(\tau \sigma) \right)_x
\end{equation}

\begin{equation}
+ \frac{1}{\sqrt{T}} \left| \sigma \right|^2_x + O \left( \frac{1}{\sqrt{T}} \right) \sum_{j=1}^{q+q_1} |Q \nabla_{\tilde{f}_j}^{F, \beta, \epsilon}(\tau \sigma)|^2_x.
\end{equation}

By definition (cf. (1.66) and (1.67)), one has on $s(M)$ that

\begin{equation}
(1 - Q) \left( \nabla_{\tilde{f}_i}^{F, \beta, \epsilon} \right) Q = \frac{\beta}{2} \sum_{k=1}^{q} \sum_{j=1}^{q_2} \left( \nabla_{\tilde{f}_i}^{T, M, \beta, \epsilon} e_j, f_k \right) c_{\beta, \epsilon}(e_j) c_{\beta, \epsilon}(\beta^{-1} f_k)
\end{equation}

\begin{equation}
+ \frac{\epsilon^{-1}}{2} \sum_{k=q+1}^{q+q_1} \sum_{j=1}^{q_2} \left( \nabla_{\tilde{f}_i}^{T, M, \beta, \epsilon} e_j, f_k \right) c_{\beta, \epsilon}(e_j) c_{\beta, \epsilon}(\epsilon f_k).
\end{equation}

By (2.48), one has for $1 \leq i, k \leq q$ that

\begin{equation}
\left( \nabla_{\tilde{f}_i}^{T, M, \beta, \epsilon} e_j, f_k \right) = 0.
\end{equation}

Also, by (1.5) and (1.8), one finds that when $1 \leq i \leq q, q + 1 \leq k \leq q + q_1$,

\begin{equation}
\epsilon^{-1} \left( \nabla_{\tilde{f}_i}^{T, M, \beta, \epsilon} e_j, f_k \right) = O(\epsilon).
\end{equation}
From (2.53) and (2.63)-(2.66), one gets that if \( 1 \leq i, j \leq q \) with \( i \neq j \), then

\[
\int_{M_x} \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{F, \beta, \varepsilon} (\tau \sigma), (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_j) f_T \tilde{\nabla}_{\tilde{\tau} f_j}^{F, \beta, \varepsilon} (\tau \sigma) \right\rangle k_c \, dv_{M_x}
= \left( O \left( \frac{\varepsilon^2}{\beta^2} \right) + O \left( \frac{1}{\sqrt{T}} \right) \right) |\sigma|^2_x + O \left( \frac{1}{\sqrt{T}} \right) \sum_{j=1}^{q+q_1} \left| Q \tilde{\nabla}_{\tilde{\tau} f_j}^{F, \beta, \varepsilon} (\tau \sigma) \right|^2_x.
\]

If \( q + 1 \leq i \leq q + q_1 \), \( q + 1 \leq k \leq q \), then one has by (1.21) that

\[
\beta \left\langle \nabla_{\tilde{f}_i}^{T, M, \beta, \varepsilon} e_j, f_k \right\rangle = O \left( \frac{1}{\beta} \right),
\]
while if \( q + 1 \leq i, k \leq q + q_1 \), one has

\[
e^{-1} \left\langle \nabla_{\tilde{f}_i}^{T, M, \beta, \varepsilon} e_j, f_k \right\rangle = O \left( e^{-1} \right).
\]

Combining with (2.63)-(2.66), one gets that if \( q + 1 \leq i \leq q + q_1 \), \( 1 \leq j \leq q \), then

\[
\int_{M_x} \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{F, \beta, \varepsilon} (\tau \sigma), (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_j) f_T \tilde{\nabla}_{\tilde{\tau} f_j}^{F, \beta, \varepsilon} (\tau \sigma) \right\rangle k_c \, dv_{M_x}
= \left( O \left( \frac{\varepsilon (\beta + \varepsilon)}{\beta^2} \right) + O \left( \frac{1}{\sqrt{T}} \right) \right) |\sigma|^2_x + O \left( \frac{1}{\sqrt{T}} \right) \sum_{j=1}^{q+q_1} \left| Q \tilde{\nabla}_{\tilde{\tau} f_j}^{F, \beta, \varepsilon} (\tau \sigma) \right|^2_x.
\]

Also, when \( q + 1 \leq i, j \leq q + q_1 \) with \( i \neq j \), one gets

\[
\int_{M_x} \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{F, \beta, \varepsilon} (\tau \sigma), (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_j) f_T \tilde{\nabla}_{\tilde{\tau} f_j}^{F, \beta, \varepsilon} (\tau \sigma) \right\rangle k_c \, dv_{M_x}
= \left( O \left( \frac{(\beta + \varepsilon)^2}{\beta^2} \right) + O \left( \frac{1}{\sqrt{T}} \right) \right) |\sigma|^2_x + O \left( \frac{1}{\sqrt{T}} \right) \sum_{j=1}^{q+q_1} \left| Q \tilde{\nabla}_{\tilde{\tau} f_j}^{F, \beta, \varepsilon} (\tau \sigma) \right|^2_x.
\]

Now we consider the terms \( I_5 \) and \( I_6 \). By (2.27) and (2.28), we need to consider the following term for \( 1 \leq j \leq q + q_1 \) and \( 1 \leq k \leq q_2 \):

\[
\left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_i) \tilde{\nabla}_{\tilde{\tau} f_i}^{F, \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma, (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tau e_k) \tilde{\nabla}_{\tau e_k}^{F, \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma \right\rangle
= \left\langle c_{\beta, \varepsilon} (\tilde{\tau} f_i) (1 - p_{T, \beta, \varepsilon}) \tilde{\tau} f_i (f_T) \tau \sigma, c_{\beta, \varepsilon} (\tau e_k) \tau e_k (f_T) \tau \sigma \right\rangle
+ \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_i) f_T \nabla_{\tilde{\tau} f_i}^{F, \beta, \varepsilon} \tau \sigma, c_{\beta, \varepsilon} (\tau e_k) f_T \tilde{\nabla}_{\tau e_k}^{F, \beta, \varepsilon} (\tau \sigma) \right\rangle
+ \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{F, \beta, \varepsilon} (\tau \sigma), c_{\beta, \varepsilon} (\tau e_k) \tau e_k (f_T) \tau \sigma \right\rangle
+ \left\langle c_{\beta, \varepsilon} (\tilde{\tau} f_i) (1 - p_{T, \beta, \varepsilon}) \tilde{\tau} f_i (f_T) \tau \sigma, c_{\beta, \varepsilon} (\tau e_k) f_T \tilde{\nabla}_{\tau e_k}^{F, \beta, \varepsilon} (\tau \sigma) \right\rangle.
\]
Lemma 2.7. For any $U \in \Gamma(\mathcal{F}_2^\perp|s(M))$, the following identity holds on $s(M)$,

\[
(2.74) \quad \left( Q \tilde{\nabla}_{U}^{F,\beta,\varepsilon}(\tau\sigma) \right)_{|s(M)} = 0.
\]

Proof. By construction, one has

\[
(2.75) \quad Q \tilde{\nabla}_{Z}^{F,\phi_1(F^1_{\pm}),\beta,\varepsilon}(\tau\sigma) = 0.
\]

Taking the derivative with respect to $z_i$, one gets

\[
(2.76) \quad \left( Q \tilde{\nabla}_{e_i}^{F,\phi_1(F^1_{\pm}),\beta,\varepsilon}(\tau\sigma) \right)_{|s(M)} = 0.
\]

Formula (2.74) follows from (2.76). \qed

For the second term in the right hand side of (2.72), one obtains by (2.33), (2.51) and Lemma 2.7 that for any $x \in s(M)$, one has

\[
(2.77) \quad \int_{\mathcal{M}_x} \left( 1 - p_{T,\beta,\varepsilon} \right) c_{\beta,\varepsilon} \left( \tilde{\tau} f_i \right) f_T \tilde{\nabla}_{e_i}^{F,\beta,\varepsilon}(\tau\sigma), c_{\beta,\varepsilon} \left( \tau e_k \right) f_T \tilde{\nabla}_{e_k}^{F,\beta,\varepsilon}(\tau\sigma) \right)_{(x,z)} k_c \, dv_{\mathcal{M}_x}
\]

\[
= \left. \left( c_{\beta,\varepsilon} \left( \tilde{\tau} f_i \right) (1 - Q) \tilde{\nabla}_{e_i}^{F,\beta,\varepsilon}(\tau\sigma), c_{\beta,\varepsilon} \left( e_k \right) (1 - Q) \tilde{\nabla}_{e_k}^{F,\beta,\varepsilon}(\tau\sigma) \right) \right|_{x} + O \left( \frac{1}{\sqrt{T}} \right) |\sigma|^2_x + O \left( \frac{1}{\sqrt{T}} \right) \sum_{j=1}^{q+q_1} Q \tilde{\nabla}_{e_j}^{F,\beta,\varepsilon}(\tau\sigma) | \binom{Q}{x} |^2.
\]

By (1.6) and (2.2), one knows that for any $U, V \in \Gamma(\mathcal{F}_2^\perp)$ and $X \in \Gamma(\mathcal{F})$, one has

\[
(2.78) \quad \left. \langle \nabla_{U}^{T,M,\beta,\varepsilon} V, X \right\rangle = 0.
\]

Similar to (2.64), one has by (2.78) that, on $s(M)$,

\[
(2.79) \quad (1 - Q) \left( \tilde{\nabla}_{e_k}^{F,\beta,\varepsilon} \right) Q = \frac{\beta}{2} \sum_{s=1}^{q} \sum_{j=1}^{q_2} \left( \nabla_{e_k}^{T,M,\beta,\varepsilon} e_j, f_s \right) c_{\beta,\varepsilon} \left( e_j \right) c_{\beta,\varepsilon} \left( \beta^{-1} f_s \right)
\]

\[
+ \frac{\varepsilon^{-1}}{2} \sum_{s=q+1}^{q+q_1} \sum_{j=1}^{q_2} \left( \nabla_{e_k}^{T,M,\beta,\varepsilon} e_j, f_s \right) c_{\beta,\varepsilon} \left( e_j \right) c_{\beta,\varepsilon} \left( \varepsilon f_s \right)
\]

\[
\quad = \frac{\varepsilon^{-1}}{2} \sum_{s=q+1}^{q+q_1} \sum_{j=1}^{q_2} \left( \nabla_{e_k}^{T,M,\beta,\varepsilon} e_j, f_s \right) c_{\beta,\varepsilon} \left( e_j \right) c_{\beta,\varepsilon} \left( \varepsilon f_s \right).
\]

\footnote{By the “parity consideration” here we mean that if a term $A$ involves an odd number of Clifford actions $c(U)$ with $U \in \mathcal{F}_2^\perp$, then one has the obvious fact that $Q A Q = 0$, etc. The “degree consideration” appears in the later text is based on the same reasoning.}
From (2.64), (2.77), (2.79) and the easy parity consideration, one gets that for $1 \leq i \leq q + q_1$, $1 \leq k \leq q_2$,

$$
(2.80) \quad \int_{\mathcal{M}_z} \left\langle (1 - p_{\tau, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_{i}) f_{T} \tilde{\nabla}_{\tilde{\tau} f_{i}}^{F, \beta, \varepsilon} (\tau \sigma), c_{\beta, \varepsilon} (\tau e_{k}) f_{T} \tilde{\nabla}_{\tau e_{k}}^{F, \beta, \varepsilon} (\tau \sigma) \right\rangle_{x, Z} k_{c} d\nu_{\mathcal{M}_z} = O \left( \frac{1}{\sqrt{T}} \right) \left| \sigma \right|_{x}^{2} + O \left( \frac{1}{\sqrt{T}} \right) \sum_{j=1}^{q+q_1} \left| Q_{\tilde{\tau} f_{j}}^{F, \phi_{1}(\tau e_{k}), \beta, \varepsilon} (\tau \sigma) \right|_{x}^{2}.
$$

For the third term in the right hand side of (2.72), if $1 \leq i \leq q + q_1$, one has by an easy degree consideration,

$$
(2.81) \quad \left\langle (1 - p_{\tau, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_{i}) f_{T} \tilde{\nabla}_{\tilde{\tau} f_{i}}^{F, \beta, \varepsilon} (\tau \sigma), c_{\beta, \varepsilon} (\tau e_{k}) f_{T} (f_{T}) \tau \sigma \right\rangle = \left\langle c_{\beta, \varepsilon} (\tilde{\tau} f_{i}) f_{T} (1 - Q) \tilde{\nabla}_{\tilde{\tau} f_{i}}^{F, \beta, \varepsilon} (\tau \sigma), c_{\beta, \varepsilon} (\tau e_{k}) f_{T} (f_{T}) \tau \sigma \right\rangle.
$$

As in (2.64), one has

$$
(2.82) \quad (1 - Q) \left( \tilde{\nabla}_{\tilde{\tau} f_{i}}^{F, \beta, \varepsilon} \right) Q = \frac{1}{2\beta} \sum_{k=1}^{q} \sum_{j=1}^{q_2} \left\langle \nabla_{\tilde{\tau} f_{i}}^{T, \beta, \varepsilon} (\tau e_{j}), \tau f_{k} \right\rangle_{\beta, \varepsilon} c_{\beta, \varepsilon} (\tau e_{j}) c_{\beta, \varepsilon} (\beta^{-1} \tau f_{k})
$$

$$
+ \frac{\varepsilon}{2} \sum_{k=q+1}^{q+q_1} \sum_{j=1}^{q_2} \left\langle \nabla_{\tilde{\tau} f_{i}}^{T, \beta, \varepsilon} (\tau e_{j}), \tau f_{k} \right\rangle_{\beta, \varepsilon} c_{\beta, \varepsilon} (\tau e_{j}) c_{\beta, \varepsilon} (\varepsilon \tau f_{k}),
$$

where the subscripts “$\beta$”, “$\varepsilon$” are to emphasize that the pointwise inner product is the one with respect to $g_{\beta, \varepsilon}^{T \mathcal{M}}$.

From (2.82), one finds

$$
(2.83) \quad \left\langle c_{\beta, \varepsilon} (\tilde{\tau} f_{i}) f_{T} (1 - Q) \tilde{\nabla}_{\tilde{\tau} f_{i}}^{F, \beta, \varepsilon} (\tau \sigma), c_{\beta, \varepsilon} (\tau e_{k}) f_{T} (f_{T}) \tau \sigma \right\rangle = \frac{1}{2\beta} \sum_{m=1}^{q} \sum_{j=1}^{q_2} \left( \int_{s(\mathcal{M})} \left\langle c_{\beta, \varepsilon} (f_{i}) c_{\beta, \varepsilon} (e_{j}) c_{\beta, \varepsilon} (\beta^{-1} f_{m}) \sigma, c_{\beta, \varepsilon} (e_{k}) \sigma \right\rangle d\nu_{s(\mathcal{M})} \right.
$$

$$
\cdot \int_{\mathcal{M}_z} \left\langle \nabla_{\tilde{\tau} f_{i}}^{T, \beta, \varepsilon} (\tau e_{j}), \tau f_{m} \right\rangle_{\beta, \varepsilon} f_{T} \tau e_{k} (f_{T}) k_{c} d\nu_{\mathcal{M}_z} (Z) \right)
$$

$$
+ \frac{\varepsilon}{2} \sum_{m=q+1}^{q+q_1} \sum_{j=1}^{q_2} \left( \int_{s(\mathcal{M})} \left\langle c_{\beta, \varepsilon} (f_{i}) c_{\beta, \varepsilon} (e_{j}) c_{\beta, \varepsilon} (\varepsilon f_{m}) \sigma, c_{\beta, \varepsilon} (e_{k}) \sigma \right\rangle d\nu_{s(\mathcal{M})} \right.
$$

$$
\cdot \int_{\mathcal{M}_z} \left\langle \nabla_{\tilde{\tau} f_{i}}^{T, \beta, \varepsilon} (\tau e_{j}), \tau f_{m} \right\rangle_{\beta, \varepsilon} f_{T} \tau e_{k} (f_{T}) k_{c} d\nu_{\mathcal{M}_z} (Z) \right).
$$
Clearly, when \( i \neq m \), \( c(f_i)c(f_m) \) is skew-adjoint, thus

\[
(2.84) \quad \langle c_{\beta,\varepsilon}(f_i) c_{\beta,\varepsilon}(f_m) \sigma, \sigma \rangle = 0.
\]

By (2.30), one has

\[
(2.85) \quad \tau e_k(f_T)(x,Z) = \left( -\frac{\tau e_k(k_c)}{2k_c^{3/2}\sqrt{\alpha_T}} + \frac{\tau e_k(\gamma)}{k_c^{1/2}\sqrt{\alpha_T}} - \frac{T\tau e_k(|Z|^2)\gamma}{2k_c^{1/2}\sqrt{\alpha_T}} \right) \exp \left( -\frac{T|Z|^2}{2} \right).
\]

By (2.3), one knows that \( \tau e_k \) does not depend on \( \beta \) and \( \varepsilon \).

From Lemma 2.5 and (2.65), one gets that for \( 1 \leq i, m \leq q, 1 \leq j \leq q_2 \),

\[
(2.86) \quad \left\langle \nabla_{\tau f_i} \tau e_k(f_T), \tau f_m \right\rangle_{\beta,\varepsilon,(x,Z)} = \left\langle \nabla_{\tau f_i} \tau e_k(f_T), \tau f_m \right\rangle_{\beta,\varepsilon} + \sum_{k=q+1}^{q+q_1} O(\varepsilon^2|Z|) f_k f_m + O(|Z|^2) = O(\varepsilon^2|Z|) + O(|Z|^2).
\]

From (2.83) and (2.86), one gets

\[
(2.87) \quad \frac{1}{\beta} \int_{M_z} \left\langle \nabla_{\tau f_i} \tau e_k(f_T), \tau f_m \right\rangle_{\beta,\varepsilon} f_T \tau e_k(f_T) k_c dv_{M_z}(Z) = O\left( \frac{\varepsilon^2}{\beta} \right) + O\left( \frac{1}{\sqrt{T}} \right).
\]

From (2.81), (2.83), (2.84) and (2.87), one finds that when \( 1 \leq i \leq q, 1 \leq k \leq q_2 \),

\[
(2.88) \quad \int_{M_z} \left( 1 - p_{T,\beta,\varepsilon} \right) c_{\beta,\varepsilon}(\tilde{\tau} f_i) f_T \nabla_{\tau f_i} \tau e_k(f_T) \tau e_k(f_T) f_m k_c dv_{M_z}(Z)
\]

\[= O\left( \frac{\varepsilon^2}{\beta^2} \right) + O\left( \frac{1}{\sqrt{T}} \right) |\sigma|^2_x.
\]

Now for \( q + 1 \leq i, m \leq q + q_1 \) and \( 1 \leq j \leq q_2 \), one has

\[
(2.89) \quad \left\langle \nabla_{\tau f_i} \tau e_k(f_T), \tau f_m \right\rangle_{\beta,\varepsilon,(x,Z)} = \left\langle \nabla_{\tau f_i} \tau e_k(f_T), \tau f_m \right\rangle_{\beta,\varepsilon} + \sum_{j=1}^{q} O\left( \frac{|Z|}{\beta^2} \right) f_j + \sum_{k=q+1}^{q+q_1} O(|Z|) f_k + O(|Z|^2)
\]

\[= O\left( \frac{1}{\varepsilon^2} \right) + O\left( \frac{1}{\beta^2 + \frac{1}{\varepsilon^2}} |Z| \right) + O(|Z|^2).
\]

By using (2.81), (2.83), (2.85) and (2.89), one finds that when \( q + 1 \leq i \leq q + q_1, 1 \leq k \leq q_2 \),

\[
(2.90) \quad \int_{M_z} \left( 1 - p_{T,\beta,\varepsilon} \right) c_{\beta,\varepsilon}(\tilde{\tau} f_i) f_T \nabla_{\tau f_i} \tau e_k(f_T) \tau e_k(f_T) f_m k_c dv_{M_z}(Z)
\]

\[= O\left( 1 + \frac{\varepsilon^2}{\beta^2} \right) + O\left( \frac{1}{\sqrt{T}} \right) |\sigma|^2_x.
\]
For the fourth term in the right hand side of (2.72), one verifies easily that

\[
\left\langle c_{\beta,\varepsilon} (\tau f_i) (1 - p_{T,\beta,\varepsilon}) \tau f_i (f_T) \tau \sigma, c_{\beta,\varepsilon} (\tau e_k) f_T \tilde{\nabla}^{F,\beta,\varepsilon}_{\tau e_k} (\tau \sigma) \right\rangle
\]

\[
= \left\langle c_{\beta,\varepsilon} (\tau f_i) (1 - p_{T,\beta,\varepsilon}) \tau f_i (f_T) \tau \sigma, c_{\beta,\varepsilon} (\tau e_k) f_T (1 - Q) \tilde{\nabla}^{F,\beta,\varepsilon}_{\tau e_k} (\tau \sigma) \right\rangle
\]

\[
= \left\langle c_{\beta,\varepsilon} (\tau f_i) \rho_{T,\beta,\varepsilon,\tau,\sigma}, c_{\beta,\varepsilon} (\tau e_k) f_T (1 - Q) \tilde{\nabla}^{F,\beta,\varepsilon}_{\tau e_k} (\tau \sigma) \right\rangle.
\]

As in (2.82), one has

\[
(1 - Q) \tilde{\nabla}^{F,\beta,\varepsilon}_{\tau e_k} (\tau \sigma) = \frac{1}{2\beta} \sum_{j=1}^{q_2} \sum_{m=1}^{q} \left\langle \nabla^{T,M,\beta,\varepsilon}_{\tau e_k} (\tau e_j), \tau f_m \right\rangle_{\beta,\varepsilon} c_{\beta,\varepsilon} (\tau e_j) c_{\beta,\varepsilon} (\beta^{-1} \tau f_m) \tau \sigma
\]

\[
+ \frac{\varepsilon}{2} \sum_{j=1}^{q_2} \sum_{m=q+1}^{q+q_1} \left\langle \nabla^{T,M,\beta,\varepsilon}_{\tau e_k} (\tau e_j), \tau f_m \right\rangle_{\beta,\varepsilon} c_{\beta,\varepsilon} (\tau e_j) c_{\beta,\varepsilon} (\varepsilon \tau f_m) \tau \sigma.
\]

By Lemma 2.5, (2.2) and (2.78), one verifies that for \(1 \leq m \leq q\), one has

\[
\left\langle \nabla^{T,M,\beta,\varepsilon}_{\tau e_i} (\tau e_j), \tau f_m \right\rangle_{\beta,\varepsilon}_{|x,Z|} = \left\langle \nabla^{T,M,\beta,\varepsilon}_{\tau e_i} (\tau e_j), f'_m + \sum_{k=q+1}^{q+q_1} O \left( \frac{e^2 |Z|}{m} \right) f'_k \right\rangle_{\beta,\varepsilon} + O \left( |Z|^2 \right)
\]

\[
= O \left( \frac{e^2 |Z|}{m} \right) + O \left( |Z|^2 \right),
\]

while for \(q + 1 \leq m \leq q + q_1\), one has,

\[
\left\langle \nabla^{T,M,\beta,\varepsilon}_{\tau e_i} (\tau e_j), \tau f_m \right\rangle_{\beta,\varepsilon}_{|x,Z|} = \left\langle \nabla^{T,M,\beta,\varepsilon}_{\tau e_i} (\tau e_j), f'_m + \sum_{j=1}^{q} O \left( \frac{|Z|}{\beta^2} \right) f'_j + \sum_{k=q+1}^{q+q_1} O \left( |Z| |f'_k \right) \beta,\varepsilon\right\rangle_{\beta,\varepsilon}
\]

\[
+ O \left( |Z|^2 \right) = O \left( 1 \right) + O \left( |Z| \right) + O \left( |Z|^2 \right).
\]

From Lemma 2.5, (2.59), (2.60) and (2.91)-(2.94), one gets that for \(1 \leq i \leq q\) and \(1 \leq k \leq q_2\), and also using the parity consideration,

\[
\frac{1}{\beta} \left\langle c_{\beta,\varepsilon} (\beta^{-1} \tau f_i) (1 - p_{T,\beta,\varepsilon}) \tau f_i (f_T) \tau \sigma, c_{\beta,\varepsilon} (\tau e_k) f_T \tilde{\nabla}^{F,\beta,\varepsilon}_{\tau e_k} (\tau \sigma) \right\rangle
\]

\[
= \left( O \left( \frac{e^2}{\beta^2} \right) + O \left( \frac{1}{\sqrt{T}} \right) \right) \int_{s(M)} |\sigma|^2 u_{s(M)},
\]

while for \(q + 1 \leq i \leq q + q_1\) and \(1 \leq k \leq q_2\), one has

\[
\varepsilon \left\langle c_{\beta,\varepsilon} (\varepsilon \tau f_i) (1 - p_{T,\beta,\varepsilon}) \tau f_i (f_T) \tau \sigma, c_{\beta,\varepsilon} (\tau e_k) f_T \tilde{\nabla}^{F,\beta,\varepsilon}_{\tau e_k} (\tau \sigma) \right\rangle
\]

\[
= \left( O \left( \frac{e^2}{\beta^2} \right) + O \left( \frac{1}{\sqrt{T}} \right) \right) \int_{s(M)} |\sigma|^2 u_{s(M)}.
\]
Now we consider the term for $1 \leq i, k \leq q_2$ with $i \neq k$,

\begin{equation}
(2.97) \quad \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tau e_i) \nabla_{\tau e_i} F_{T,\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma, c_{\beta,\varepsilon} (\tau e_k) \nabla_{\tau e_k} F_{T,\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\rangle \\
= \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tau e_i) \tau c_{\beta,\varepsilon} (\tau e_k) (f_T) \tau \sigma \right\rangle \\
+ \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tau e_i) f_T \nabla_{\tau e_i} F_{T,\beta,\varepsilon} (\tau \sigma), c_{\beta,\varepsilon} (\tau e_k) f_T \nabla_{\tau e_k} F_{T,\beta,\varepsilon} (\tau \sigma) \right\rangle \\
+ \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tau e_i) f_T \nabla_{\tau e_i} F_{T,\beta,\varepsilon} (\tau \sigma), c_{\beta,\varepsilon} (\tau e_k) f_T \nabla_{\tau e_k} F_{T,\beta,\varepsilon} (\tau \sigma) \right\rangle.
\end{equation}

For the first term in the right hand side of (2.97), one has, as $i \neq k$,

\begin{equation}
(2.98) \quad \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tau e_i) \tau c_{\beta,\varepsilon} (\tau e_k) (f_T) \tau \sigma \right\rangle \\
= - \left\langle (\tau e_k) (f_T) \tau c_{\beta,\varepsilon} (\tau e_i) \tau c_{\beta,\varepsilon} (\tau e_k) \tau \sigma \right\rangle = 0.
\end{equation}

For the second term in the right hand side of (2.97), one has by (2.33) and Lemma 2.7 that for any $x \in s(M)$,

\begin{equation}
(2.99) \quad \int_{\mathcal{M}_x} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tau e_i) f_T \nabla_{\tau e_i} F_{T,\beta,\varepsilon} (\tau \sigma), c_{\beta,\varepsilon} (\tau e_k) f_T \nabla_{\tau e_k} F_{T,\beta,\varepsilon} (\tau \sigma) \right\rangle_{(x, z)} k_c d\mathcal{M}_x \\
= \int_{\mathcal{M}_x} f_T^2 \left\langle (1 - Q) c_{\beta,\varepsilon} (\tau e_i) (1 - Q) \nabla_{\tau e_i} F_{T,\beta,\varepsilon} (\tau \sigma), c_{\beta,\varepsilon} (\tau e_k) (1 - Q) \nabla_{\tau e_k} F_{T,\beta,\varepsilon} (\tau \sigma) \right\rangle_{(x, z)} k_c d\mathcal{M}_x \\
+ O \left( \frac{1}{\sqrt{T}} \right) |\sigma|^2_x + O \left( \frac{1}{\sqrt{T}} \right) \sum_{i=1}^{q+q_1} \left| Q \nabla_{f_i}^2 \phi,(f_{1+})_{\beta,\varepsilon} (\tau \sigma) \right|^2_x \\
= \left\langle (1 - Q) c_{\beta,\varepsilon} (e_i) (1 - Q) \nabla_{\tau e_i} F_{T,\beta,\varepsilon} (\tau \sigma), c_{\beta,\varepsilon} (e_k) (1 - Q) \nabla_{\tau e_k} F_{T,\beta,\varepsilon} (\tau \sigma) \right\rangle_x \\
+ O \left( \frac{1}{\sqrt{T}} \right) |\sigma|^2_x + O \left( \frac{1}{\sqrt{T}} \right) \sum_{i=1}^{q+q_1} \left| Q \nabla_{f_i}^2 \phi,(f_{1+})_{\beta,\varepsilon} (\tau \sigma) \right|^2_x.
\end{equation}

Now, one has by (2.79) that for any $1 \leq i \leq q_2$, at $x \in s(M)$,

\begin{equation}
(2.100) \quad (1 - Q) c_{\beta,\varepsilon} (e_i) (1 - Q) \nabla_{\tau e_i} F_{T,\beta,\varepsilon} Q \\
= \varepsilon^{-1} \sum_{j=1, j \neq i}^{q_2} \sum_{m=q+1}^{q+q_1} \left\langle \nabla_{\tau e_i}^{T,\beta,\varepsilon} e_j, f_m \right\rangle c_{\beta,\varepsilon} (e_i) c_{\beta,\varepsilon} (e_j) c_{\beta,\varepsilon} (\varepsilon f_m).
\end{equation}

For $q + 1 \leq m \leq q + q_1$, one has, by (2.2),

\begin{equation}
(2.101) \quad \left\langle \nabla_{\tau e_i}^{T,\beta,\varepsilon} e_j, f_m \right\rangle = O \left( \varepsilon^2 \right).
\end{equation}

From (2.99)–(2.101), one gets that for $x \in s(M)$,

\begin{equation}
(2.102) \quad \int_{\mathcal{M}_x} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tau e_i) f_T \nabla_{\tau e_i} F_{T,\beta,\varepsilon} (\tau \sigma), c_{\beta,\varepsilon} (\tau e_k) f_T \nabla_{\tau e_k} F_{T,\beta,\varepsilon} (\tau \sigma) \right\rangle_{(x, z)} k_c d\mathcal{M}_x \\
= \left( O \left( \varepsilon^2 \right) + O \left( \frac{1}{\sqrt{T}} \right) \right) |\sigma|^2_x + O \left( \frac{1}{\sqrt{T}} \right) \sum_{i=1}^{q+q_1} \left| Q \nabla_{f_i}^2 \phi,(f_{1+})_{\beta,\varepsilon} (\tau \sigma) \right|^2_x.
\end{equation}
For the third term in the right hand side of (2.97), since \( i \neq k \), by (2.92) and a simple parity consideration, one has that

\[
\left\langle (1 - p_{T, \beta, \epsilon}) c_{\beta, \epsilon} (\tau e_i) f_T \bar{\nabla}^F_{\tau e_i} (\tau \sigma), c_{\beta, \epsilon} (\tau e_k) f_T (\tau \sigma) \rightangle = 0.
\]

Similarly, for the fourth term in the right hand side of (2.97), one has

\[
\left\langle (1 - p_{T, \beta, \epsilon}) c_{\beta, \epsilon} (\tau e_i) f_T (\tau \sigma), c_{\beta, \epsilon} (\tau e_k) f_T \bar{\nabla}^F_{\tau e_k} (\tau \sigma) \rightangle = 0.
\]

By (2.25), (2.72), (2.73), (2.80), (2.88) and (2.95), one gets

\[
\int_{s(M)} |\sigma|^2 dv_{s(M)} + O \left( \frac{1}{\sqrt{T}} \right) \int_{s(M)} \sum_{i=1}^{q+q_1} \left| Q_{\nabla^F_{\tau f_i}} \right|^2 dv_{s(M)}.
\]

Similarly, by (2.27), (2.72), (2.73), (2.80), (2.88) and (2.95), one gets

\[
\int_{s(M)} |\sigma|^2 dv_{s(M)} + O \left( \frac{1}{\sqrt{T}} \right) \int_{s(M)} \sum_{i=1}^{q+q_1} \left| Q_{\nabla^F_{\tau f_i}} \right|^2 dv_{s(M)},
\]

while by (2.28), (2.72), (2.73), (2.80), (2.90) and (2.96), one gets

\[
\int_{s(M)} |\sigma|^2 dv_{s(M)} + O \left( \frac{1}{\sqrt{T}} \right) \int_{s(M)} \sum_{i=1}^{q+q_1} \left| Q_{\nabla^F_{\tau f_i}} \right|^2 dv_{s(M)}.
\]

2.7. Estimates of the terms \( I_k \), \( 1 \leq k \leq 6 \), Part II. In this subsection, we deal with the term left in (2.62). First of all, by Lemma 2.6 it is easy to see that the last term in (2.62) verifies the following estimate,

\[
\left\langle c_{\beta, \epsilon} (\tilde{\tau} f_i) c_{\beta, \epsilon} (\tilde{\tau} f_j) f_T p_{T, \beta, \epsilon} (\tilde{\tau} f_i (f_T) \tau \sigma), Q_{\nabla^F_{\tilde{\tau} f_j}} \bar{\nabla}^F_{\tilde{\tau} f_j} (\tau \sigma) \rightangle = O \left( \frac{1}{\sqrt{T}} \right) \int_{s(M)} |\sigma|^2 dv_{s(M)} + O \left( \frac{1}{\sqrt{T}} \right) \int_{s(M)} \sum_{i=1}^{q+q_1} \left| Q_{\nabla^F_{\tilde{\tau} f_i}} \right|^2 dv_{s(M)}.
\]

Thus we need to deal with the following term:

\[
\left\langle c_{\beta, \epsilon} (\tilde{\tau} f_i) c_{\beta, \epsilon} (\tilde{\tau} f_j) f_T f_T \tau \sigma, Q_{\nabla^F_{\tilde{\tau} f_j}} \bar{\nabla}^F_{\tilde{\tau} f_j} (\tau \sigma) \rightangle = \int_{M} \tilde{\tau} f_i (f_T) f_T \left( c_{\beta, \epsilon} (\tilde{\tau} f_i) c_{\beta, \epsilon} (\tilde{\tau} f_j) \tau \sigma, Q_{\nabla^F_{\tilde{\tau} f_j}} \bar{\nabla}^F_{\tilde{\tau} f_j} (\tau \sigma) \right) dv_M.
\]

In view of (2.57), we need to examine the first order terms (in \( Z \)) of the inner product term in the right hand side of (2.109).
By (2.36), one has, for $Z = \sum_{i=1}^{q_2} z_k \tau e_k$, the following pointwise formula on $\mathcal{M}$,

\[(2.110)\quad Z \left\langle c_{\beta,\varepsilon} (\bar{\tau} f_i) c_{\beta,\varepsilon} (\bar{\tau} f_j) \tau \sigma, Q \hat{\nabla}_{\bar{\tau} f_j} F, \beta, \varepsilon (\tau \sigma) \right\rangle = \left\langle c_{\beta,\varepsilon} (\bar{\tau} f_i) c_{\beta,\varepsilon} (\bar{\tau} f_j) \tau \sigma, Q \nabla_Z F, \beta, \varepsilon (\tau \sigma) \right\rangle \]

\[= \left\langle c_{\beta,\varepsilon} (\bar{\tau} f_i) c_{\beta,\varepsilon} (\bar{\tau} f_j) \tau \sigma, \left( Q R^F, \beta, \varepsilon (Z, \bar{\tau} f_j) + Q \nabla_{[Z, \bar{\tau} f_j]} F, \beta, \varepsilon \right) \tau \sigma \right\rangle ,\]

where $Q R^F, \beta, \varepsilon$ is the curvature of $Q \hat{\nabla}_{\bar{\tau} f_j} F, \beta, \varepsilon$.

From Lemma (2.6) (2.4) and (2.110), one has, at $(x, Z) \simeq \psi_c (x, Z) \in \mathcal{M}$,

\[(2.111)\quad \left\langle c_{\beta,\varepsilon} (\bar{\tau} f_i) c_{\beta,\varepsilon} (\bar{\tau} f_j) \tau \sigma, Q \hat{\nabla}_{\bar{\tau} f_j} F, \beta, \varepsilon (\tau \sigma) \right\rangle = \tau \left( \frac{Q \hat{\nabla}_{\bar{\tau} f_j} F, \beta, \varepsilon (\tau \sigma)}{|s(M)|} \right) + O \left( |Z|^2 \right) \left( |\sigma|^2_2 + \sum_{i=1}^{q_1+q_1} |Q \nabla^F c_{\beta,\varepsilon} (\tau \sigma)|^2_2 \right).

Clearly,

\[(2.112)\quad Q R^F, \beta, \varepsilon = Q R^F, \phi_1 (F_1^\perp), \beta, \varepsilon + Q \nabla^F c_{\beta,\varepsilon} (1 - Q) \nabla^F, \phi_1 (F_1^\perp), \beta, \varepsilon Q.

Recall that $f_1', \ldots, f_{q_1+q_1}'$ is an orthonormal basis of $\mathcal{F} \oplus F_1^\perp$ with respect to $g^F \oplus g^{F_1^\perp}$ not depending on $\beta$ and $\varepsilon$, such that $f_1', \ldots, f_q'$ is an orthonormal basis of $\mathcal{F}$ verifying (2.40).

By definition (cf. (1.67)), one has

\[(2.113)\quad \left( Q R^F, \phi_1 (F_1^\perp), \beta, \varepsilon \right) (Z, \tau f_j) = \frac{1}{4 \beta^2} \sum_{s,t=1}^{q} \left\langle R^T M, \beta, \varepsilon (Z, \tau f_j) \tau f_s, \tau f_t \right\rangle \beta, \varepsilon c_{\beta,\varepsilon} (\beta^{-1} \tau f_s) c_{\beta,\varepsilon} (\beta^{-1} \tau f_t)

\[+ \frac{\varepsilon^2}{4q} \sum_{s,t=q+1}^{q+q_1} \left\langle R^T M, \beta, \varepsilon (Z, \tau f_j) \tau f_s, \tau f_t \right\rangle \beta, \varepsilon c_{\beta,\varepsilon} (\varepsilon \tau f_s) c_{\beta,\varepsilon} (\varepsilon \tau f_t)

\[+ \frac{\varepsilon}{2 \beta} \sum_{s=1}^{q} \sum_{t=q+1}^{q+q_1} \left\langle R^T M, \beta, \varepsilon (Z, \tau f_j) \tau f_s, \tau f_t \right\rangle \beta, \varepsilon c_{\beta,\varepsilon} (\beta^{-1} \tau f_s) c_{\beta,\varepsilon} (\varepsilon \tau f_t)

\[+ \frac{\varepsilon^2}{4} \sum_{s,t=q+1}^{q+q_1} \left\langle Q R^F, \beta, \varepsilon (Z, \tau f_j) f_s', f_t' \right\rangle \beta, \varepsilon c_{\beta,\varepsilon} (\varepsilon f_s') c_{\beta,\varepsilon} (\varepsilon f_t') + R^F, \phi_1 (F_1^\perp), \beta, \varepsilon (Z, \tau f_j).\]
If $1 \leq j, s, t \leq q$, one verifies, by (2.48) that $1$

\[(2.114) \quad \frac{1}{\beta^2} \langle R^{T,M,\beta,\varepsilon}(Z, \tau f_j) \tau f_s, \tau f_t \rangle_{\beta, \varepsilon} = \langle R^{T,M,\beta,\varepsilon}(f'_s, f'_t) Z, f'_j \rangle + O \left( |Z|^2 \right)
\]

\[\quad = \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle - \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle - \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle + O \left( |Z|^2 \right)
\]

\[\quad = - \left\langle p \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle - \frac{1}{\beta^2 \varepsilon^2} \left\langle p_{1} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle \frac{1}{\beta^2} \left\langle p_{2} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle
\]

\[\quad + \left\langle p \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle + \frac{1}{\beta^2 \varepsilon^2} \left\langle p_{1} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle \frac{1}{\beta^2} \left\langle p_{2} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle
\]

\[\quad + f'_s \left( \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle \right) - f'_t \left( \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle \right) - \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle + O \left( |Z|^2 \right)
\]

\[\quad = O \left( \varepsilon^2 |Z| \right) + O \left( |Z|^2 \right).
\]

If $1 \leq j \leq q$ and $q + 1 \leq s, t \leq q + q_1$, one has, in view of (1.20),

\[(2.115) \quad \varepsilon^2 \langle R^{T,M,\beta,\varepsilon}(Z, \tau f_j) \tau f_s, \tau f_t \rangle_{\beta, \varepsilon} = \beta^2 \varepsilon^2 \langle R^{T,M,\beta,\varepsilon}(f'_s, f'_t) Z, f'_j \rangle + O \left( |Z|^2 \right)
\]

\[\quad = \beta^2 \varepsilon^2 \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle - \beta^2 \varepsilon^2 \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle - \beta^2 \varepsilon^2 \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle + O \left( |Z|^2 \right)
\]

\[\quad = - \beta^2 \varepsilon^2 \left\langle p \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle - \left\langle p_{1} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle - \varepsilon \left\langle p_{2} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle
\]

\[\quad + \beta^2 \varepsilon^2 \left\langle p \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle + \left\langle p_{1} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle + \varepsilon \left\langle p_{2} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle
\]

\[\quad + \beta^2 \varepsilon^2 f'_s \left( \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle \right) - \beta^2 \varepsilon^2 f'_t \left( \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle \right) - \beta^2 \varepsilon^2 \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle + O \left( |Z|^2 \right)
\]

\[\quad = O \left( \varepsilon^2 |Z| \right) + O \left( |Z|^2 \right).
\]

If $1 \leq j, t \leq q$ and $q + 1 \leq s \leq q + q_1$, one has

\[(2.116) \quad \frac{\varepsilon}{\beta} \langle R^{T,M,\beta,\varepsilon}(Z, \tau f_j) \tau f_s, \tau f_t \rangle_{\beta, \varepsilon} = \beta \varepsilon \langle R^{T,M,\beta,\varepsilon}(f'_s, f'_t) Z, f'_j \rangle + O \left( |Z|^2 \right)
\]

\[\quad = \beta \varepsilon \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle - \beta \varepsilon \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle - \beta \varepsilon \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle + O \left( |Z|^2 \right)
\]

\[\quad = - \beta \varepsilon \left\langle p \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle - \frac{1}{\beta} \varepsilon \left\langle p_{1} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle + \frac{\varepsilon}{\beta} \left\langle p_{2} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle
\]

\[\quad + \beta \varepsilon \left\langle p \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle + \frac{1}{\beta} \varepsilon \left\langle p_{1} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle + \frac{\varepsilon}{\beta} \left\langle p_{2} \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, \nabla_{f'_j}^{T,M,\beta,\varepsilon} f'_j \right\rangle
\]

\[\quad + \beta \varepsilon f'_s \left( \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle \right) - \beta \varepsilon f'_t \left( \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle \right) - \beta \varepsilon \left\langle \nabla_{f'_j}^{T,M,\beta,\varepsilon} Z, f'_j \right\rangle + O \left( |Z|^2 \right)
\]

\[\quad = O \left( \frac{\varepsilon |Z|}{\beta} \right) + O \left( |Z|^2 \right).
\]
If \( q + 1 \leq j \leq q + q_1 \) and \( 1 \leq s, t \leq q \), one has

\[
(2.117) \quad \frac{1}{\beta^2} \langle R_{\tau f_s, \tau f_t}^M, \beta, \varepsilon (Z, \tau f_j) \rangle + O (|Z|^2) = \frac{1}{\beta^2} \langle R_{\tau f_s, \tau f_t}^M, \beta, \varepsilon (f'_s, f'_t) \rangle Z, f'_j \rangle + O (|Z|^2)
\]

\[
= \frac{1}{\beta^2} \langle \nabla_{f'_s}^M, \beta, \varepsilon Z, f'_j \rangle - \frac{1}{\beta^2} \langle \nabla_{f'_t}^M, \beta, \varepsilon Z, f'_j \rangle - \frac{1}{\beta^2} \langle \nabla_{[f'_s, f'_t]}^M, \beta, \varepsilon Z, f'_j \rangle + O (|Z|^2)
\]

\[
= - \frac{1}{\beta^2} \langle p \nabla_{f'_s}^M, \beta, \varepsilon Z, \nabla_{f'_t}^M, \beta, \varepsilon f'_j \rangle - \frac{1}{\beta^2} \langle p \nabla_{f'_t}^M, \beta, \varepsilon Z, \nabla_{f'_s}^M, \beta, \varepsilon f'_j \rangle - \frac{1}{\beta^2} \langle p \nabla_{[f'_s, f'_t]}^M, \beta, \varepsilon Z, \nabla_{f'_t}^M, \beta, \varepsilon f'_j \rangle + \frac{1}{\beta^2} \langle p \nabla_{[f'_s, f'_t]}^M, \beta, \varepsilon Z, \nabla_{f'_s}^M, \beta, \varepsilon f'_j \rangle
\]

\[
= O \left( \frac{|Z|}{\beta^2} \right) + O (|Z|^2).
\]

If \( q + 1 \leq j, s, t \leq q + q_1 \), one has

\[
(2.118) \quad \varepsilon^2 \langle R_{\tau f_s, \tau f_t}^M, \beta, \varepsilon (Z, \tau f_j) \rangle + O (|Z|^2)
\]

\[
= \langle \nabla_{f'_s}^M, \beta, \varepsilon Z, f'_j \rangle - \langle \nabla_{f'_t}^M, \beta, \varepsilon Z, f'_j \rangle - \langle \nabla_{[f'_s, f'_t]}^M, \beta, \varepsilon Z, f'_j \rangle + O (|Z|^2)
\]

\[
= - \beta^2 \langle p \nabla_{f'_s}^M, \beta, \varepsilon Z, \nabla_{f'_t}^M, \beta, \varepsilon f'_j \rangle - \langle p \nabla_{f'_t}^M, \beta, \varepsilon Z, \nabla_{f'_s}^M, \beta, \varepsilon f'_j \rangle - \varepsilon^2 \langle p \nabla_{[f'_s, f'_t]}^M, \beta, \varepsilon Z, \nabla_{f'_t}^M, \beta, \varepsilon f'_j \rangle + \varepsilon^2 \langle p \nabla_{[f'_s, f'_t]}^M, \beta, \varepsilon Z, \nabla_{f'_s}^M, \beta, \varepsilon f'_j \rangle + f'_s \langle \nabla_{f'_s}^M, \beta, \varepsilon Z, f'_j \rangle - f'_t \langle \nabla_{f'_t}^M, \beta, \varepsilon Z, f'_j \rangle - \langle \nabla_{[f'_s, f'_t]}^M, \beta, \varepsilon Z, f'_j \rangle + O (|Z|^2)
\]

\[
= O (|Z|) + O (|Z|^2).
\]

If \( q + 1 \leq j, t \leq q + q_1 \) and \( 1 \leq s \leq q \), one has

\[
(2.119) \quad - \frac{\varepsilon}{\beta} \langle R_{\tau f_s, \tau f_t}^M, \beta, \varepsilon (Z, \tau f_j) \rangle + O (|Z|^2)
\]

\[
= \beta \varepsilon \langle \nabla_{Z}^M, \beta, \varepsilon f'_t, f'_s \rangle - \beta \varepsilon \langle \nabla_{Z}^M, \beta, \varepsilon f'_t, f'_s \rangle - \beta \varepsilon \langle \nabla_{[Z, f'_t]}^M, \beta, \varepsilon f'_t, f'_s \rangle + O (|Z|^2)
\]

\[
= - \beta \varepsilon \langle p \nabla_{f'_t}^M, \beta, \varepsilon f'_t, \nabla_{Z}^M, \beta, \varepsilon f'_s \rangle - \frac{\varepsilon}{\beta} \langle p \nabla_{f'_t}^M, \beta, \varepsilon f'_t, \nabla_{Z}^M, \beta, \varepsilon f'_s \rangle - \frac{\varepsilon}{\beta} \langle p \nabla_{f'_t}^M, \beta, \varepsilon f'_t, \nabla_{Z}^M, \beta, \varepsilon f'_s \rangle + \frac{\varepsilon}{\beta} \langle p \nabla_{f'_t}^M, \beta, \varepsilon f'_t, \nabla_{Z}^M, \beta, \varepsilon f'_s \rangle + \beta \varepsilon \langle \nabla_{Z}^M, \beta, \varepsilon f'_t, f'_s \rangle - \beta \varepsilon f'_t \langle \nabla_{Z}^M, \beta, \varepsilon f'_t, f'_s \rangle - \beta \varepsilon \langle \nabla_{[Z, f'_t]}^M, \beta, \varepsilon f'_t, f'_s \rangle + O (|Z|^2)
\]

\[
= O \left( \frac{\varepsilon |Z|}{\beta} \right) + O (|Z|^2).
\]
If $1 \leq j \leq q + q_1$ and $q + 1 \leq s$, $t \leq q + q_1$, one has

$$\left(2.120\right) \quad \left\langle R^{F_1, \alpha \varepsilon}_{f_1} (Z, \tau f_j) f_t', f_s' \right\rangle$$

$$\left(2.121\right) \quad \left\langle R^{F_1, \alpha \varepsilon}_{f_1} (Z, \tau f_j) f_t', f_s' \right\rangle = \left\langle \nabla^{F_1, \alpha \varepsilon}_{f_1, f_t'} \nabla^{F_1, \alpha \varepsilon}_{f_1, f_s'} - \left\langle \nabla^{F_1, \alpha \varepsilon}_{f_1, f_t'} \nabla^{F_1, \alpha \varepsilon}_{f_1, f_s'} - \left\langle \nabla^{F_1, \alpha \varepsilon}_{f_1, f_t'} \nabla^{F_1, \alpha \varepsilon}_{f_1, f_s'} + O \left(|Z|^2\right) \right\rangle \right\rangle$$

$$\left(2.122\right) \quad \left\langle R^{F_1, \alpha \varepsilon}_{f_1} (Z, \tau f_j) f_t', f_s' \right\rangle = O \left(\varepsilon^2 |Z| \right) + O \left(|Z|^2\right),$$

By $(2.115)$ and $(2.120)$, one sees that when $1 \leq j \leq q$, $q + 1 \leq s$, $t \leq q + q_1$, one has

$$\left(2.123\right) \quad (1 - Q) \nabla^{F, \phi_1}_{Z, \beta \varepsilon} = O \left(\varepsilon |Z| \right) + O \left(|Z|^2\right).$$

Similarly, one has

$$\left(2.124\right) \quad Q \nabla^{F, \phi_1}_{Z, \beta \varepsilon} (1 - Q) = O \left(\varepsilon |Z| \right) + O \left(|Z|^2\right).$$

On the other hand, by $(2.64)$-$(2.66)$, one finds that for $1 \leq j \leq q$,

$$\left(2.125\right) \quad (1 - Q) \nabla^{F, \phi_1}_{f_1, \beta \varepsilon} = O \left(\varepsilon \right) + O_{\beta, \varepsilon} \left(|Z| \right).$$

Similarly,

$$\left(2.126\right) \quad Q \nabla^{F, \phi_1}_{f_1, \beta \varepsilon} (1 - Q) = O \left(\varepsilon \right) + O_{\beta, \varepsilon} \left(|Z| \right).$$

While for $q + 1 \leq j \leq q + q_1$, by $(2.64)$, $(2.68)$ and $(2.69)$, one has

$$\left(2.127\right) \quad (1 - Q) \nabla^{F, \phi_1}_{f_1, \beta \varepsilon} = O \left(\beta^{-1} + \varepsilon^{-1} \right) + O_{\beta, \varepsilon} \left(|Z| \right).$$

Similarly,

$$\left(2.128\right) \quad Q \nabla^{F, \phi_1}_{f_1, \beta \varepsilon} (1 - Q) = O \left(\beta^{-1} + \varepsilon^{-1} \right) + O_{\beta, \varepsilon} \left(|Z| \right).$$

From $(2.112)$-$(2.128)$, one gets that if $1 \leq i$, $j \leq q + q_1$, then the following identity holds at $(x, Z)$ near $s(M)$,

$$\left(2.129\right) \quad \left\langle c_{\beta \varepsilon} (\tau f_i) c_{\beta \varepsilon} (\tau f_j) Q \nabla^{F, \phi_1}_{f_1, \beta \varepsilon} (Z, \tau f_j) \tau \sigma \right\rangle = O \left(\frac{\varepsilon}{\beta^2 |Z|} \right) + O \left(|Z|^2\right) |\sigma|^2.$$

Now we examine the term

$$\left(2.130\right) \quad \left\langle c_{\beta \varepsilon} (\tau f_i) c_{\beta \varepsilon} (\tau f_j) \tau \sigma, Q \nabla^{F, \phi_1}_{f_1, \beta \varepsilon} (\tau \sigma) \right\rangle$$

in $(2.111)$. 

Write $Z = \sum_{k=1}^{q_2} z_k \tau e_k$. Then one has, by (2.44),

\[(p + p_1^\perp) [Z, \tau f_j] = -(p + p_1^\perp) \nabla_{\tau f_j}^{T_M, \beta, \varepsilon} Z = -\sum_{k=1}^{q_2} z_k \left( p + p_1^\perp \right) \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k).
\]

(2.130)  \[ (p + p_1^\perp) \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k) = \sum_{s=1}^{q} \left\langle \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k), f'_s \right\rangle f'_s + \sum_{s=q+1}^{q+q_1} \left\langle \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k), f'_s \right\rangle f'_s
\]

\[= \sum_{s=1}^{q} O_{\beta, \varepsilon}(\|Z\|) f'_s + \sum_{s=q+1}^{q+q_1} \left( O(\varepsilon^2) + O_{\beta, \varepsilon}(\|Z\|) \right) f'_s.
\]

By (2.130) and (2.131), for $1 \leq j \leq q$, one has,

\[(p + p_1^\perp) \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k) = \sum_{s=1}^{q} \left\langle \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k), f'_s \right\rangle f'_s + \sum_{s=q+1}^{q+q_1} \left\langle \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k), f'_s \right\rangle f'_s
\]

\[= \sum_{s=1}^{q} O_{\beta, \varepsilon}(\|Z\|) f'_s + \sum_{s=q+1}^{q+q_1} \left( O(\varepsilon^2) + O_{\beta, \varepsilon}(\|Z\|) \right) f'_s.
\]

Similarly, for $1 \leq k \leq q_2, q+1 \leq j \leq q + q_1$, one has

\[(p + p_1^\perp) \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k) = \sum_{s=1}^{q} \left\langle \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k), f'_s \right\rangle f'_s + \sum_{s=q+1}^{q+q_1} O_{\beta, \varepsilon}(\|Z\|) f'_s.
\]

Thus, for $q + 1 \leq j \leq q + q_1$, one has

\[\varepsilon Q_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k) = \sum_{s=1}^{q} \left\langle \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k), f'_s \right\rangle f'_s + \sum_{s=q+1}^{q+q_1} O_{\beta, \varepsilon}(\|Z\|) f'_s.
\]

Thus for $1 \leq k \leq q_2, q+1 \leq j \leq q + q_1$, one has by (2.130) and (2.135),

\[\varepsilon Q_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k) = -\varepsilon \sum_{k=1}^{q_2} \sum_{s=q+1}^{q+q_1} \left( z_k \left\langle \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k), f'_s \right\rangle = \sum_{s=q+1}^{q+q_1} \left\langle \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k), f'_s \right\rangle f'_s = p_1^\perp \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k).
\]

The following formula, which holds on $s(M)$ for any $1 \leq i \leq q$, is useful in relation with (2.136),

\[\sum_{k=1}^{q_2} \sum_{s=q+1}^{q+q_1} f_i(z_k) \left\langle \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k), f'_s \right\rangle f'_s = \sum_{s=q+1}^{q+q_1} \left\langle \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k), f'_s \right\rangle f'_s = p_1^\perp \nabla_{\tau f_j}^{T_M, \beta, \varepsilon}(\tau e_k).
\]
Now for any $1 \leq j \leq q + q_1$, one has

\begin{equation}
\tag{2.138}
\left. p_2^+ [Z, \tau f_j] = p_2^+ \nabla^T Z (\tau f_j) - \nabla^Z \tau f_j Z \right.
= \sum_{k=1}^{q_2} \langle \nabla^T Z (\tau f_j), \tau e_k \rangle \tau e_k - \sum_{k=1}^{q_2} \tau f_j (z_k) \tau e_k - \sum_{k=1}^{q_2} z_k \nabla^Z \tau f_j (\tau e_k).
\end{equation}

From (2.138) and Lemmas 2.6, 2.7, one finds

\begin{equation}
\tag{2.139}
Q \nabla^X,\phi_1(F^+),\beta,\varepsilon(Z, \tau f_j) \langle \tau \sigma \rangle = - \sum_{k=1}^{q_2} \tau f_j (z_k) Q \nabla^Z \phi_1(F^+),\beta,\varepsilon(Z, \tau f_j) \langle \tau \sigma \rangle + O \left( \langle Z \rangle^2 \right) \left( \langle \sigma \rangle + \sum_{k=1}^{q_2} \left| Q \nabla^Z \phi_1(F^+),\beta,\varepsilon(Z, \tau f_j) \langle \tau \sigma \rangle \right| \right).
\end{equation}

For another section $\sigma'$ on $s(M)$, one has

\begin{equation}
\tag{2.140}
Q \nabla^X,\phi_1(F^+),\beta,\varepsilon(Z, \tau e_k) \langle \tau \sigma \rangle = \left\langle Q \nabla^Z \phi_1(F^+),\beta,\varepsilon(Z, \tau e_k) \langle \tau \sigma \rangle, \tau \sigma' \right\rangle
= \left\langle Q R^X,\phi_1(F^+),\beta,\varepsilon(Z, \tau e_k) \langle \tau \sigma \rangle, \tau \sigma' \right\rangle + \left\langle Q \nabla^Z \phi_1(F^+),\beta,\varepsilon(Z, \tau e_k) \langle \tau \sigma \rangle, \tau \sigma' \right\rangle.
\end{equation}

As in (2.138), one verifies

\begin{equation}
\tag{2.141}
[Z, \tau e_k] = - \nabla^Z \tau e_k Z = - \sum_{j=1}^{q_2} \tau e_k (z_j) \tau e_j - \sum_{j=1}^{q_2} z_j \nabla^Z \tau e_k (\tau e_j).
\end{equation}

Clearly,

\begin{equation}
\tag{2.142}
\tau e_k (z_j) = \delta_{kj} + O \langle |Z| \rangle.
\end{equation}

By Lemma 2.7 and (2.140)-(2.142), one deduces that

\begin{equation}
\tag{2.143}
\left\langle Q \nabla^X,\phi_1(F^+),\beta,\varepsilon(Z, \tau \sigma \rangle, \tau \sigma' \right\rangle = \frac{1}{2} \left\langle Q R^X,\phi_1(F^+),\beta,\varepsilon(Z, \tau e_k) \langle \tau \sigma \rangle, \tau \sigma' \right\rangle + O \langle |Z| \rangle^2
= \frac{1}{2} \sum_{m=1}^{q_2} z_m \left\langle Q R^X,\phi_1(F^+),\beta,\varepsilon(Z, \tau e_m) \langle \tau \sigma \rangle, \tau \sigma' \right\rangle + O \langle |Z| \rangle^2.
\end{equation}

From (2.139) and (2.143), one gets

\begin{equation}
\tag{2.144}
\left. c_{\beta,\varepsilon} (\bar{\tau} f_j) c_{\beta,\varepsilon} (\bar{\tau} f_j) \tau \sigma, Q \nabla^Z \phi_1(F^+),\beta,\varepsilon(Z, \tau \sigma \rangle \right)_{(x, Z)}
= - \frac{1}{2} \left\langle c_{\beta,\varepsilon} (\bar{\tau} f_j) c_{\beta,\varepsilon} (\bar{\tau} f_j) \tau \sigma, Q R^X,\phi_1(F^+),\beta,\varepsilon(Z, \nabla^Z \tau f_j Z) \tau \sigma \right\rangle_{(x, Z)} + O \langle |Z| \rangle^2.
\end{equation}

From (2.112), (2.123), (2.124) and (2.144), one gets

\begin{equation}
\tag{2.145}
\left. c_{\beta,\varepsilon} (\bar{\tau} f_j) c_{\beta,\varepsilon} (\bar{\tau} f_j) \tau \sigma, Q \nabla^Z \phi_1(F^+),\beta,\varepsilon(Z, \tau \sigma \rangle \right)_{(x, Z)}
= - \frac{1}{2} \left\langle c_{\beta,\varepsilon} (\bar{\tau} f_j) c_{\beta,\varepsilon} (\bar{\tau} f_j) \tau \sigma, R^X,\phi_1(F^+),\beta,\varepsilon(Z, \nabla^Z \tau f_j Z) \tau \sigma \right\rangle_{(x, Z)} + O \langle \varepsilon^2 |Z| \rangle + O \langle |Z| \rangle^2.
\end{equation}
As in (2.113), we have

\begin{equation}
(2.146) \quad \left( Q R^{F, \phi (F^+_1), \beta, \varepsilon} Q \right) \left( \tau_{e_m}, \tau_{e_k} \right) \\
= \frac{1}{4\beta^2} \sum_{s, t=1}^{q} \langle R^{T, M, \beta, \varepsilon}(\tau_{e_m}, \tau_{e_k}) f_s, f_t \rangle_{\beta, \varepsilon} c_{\beta, \varepsilon} (\varepsilon - 1) \tau_{f_s} \left( \varepsilon - 1 \beta_{f_s} \right) \\
+ \frac{\varepsilon}{4} \sum_{s, t=q+1}^{q+q_1} \langle R^{T, M, \beta, \varepsilon}(\tau_{e_m}, \tau_{e_k}) f_s, f_t \rangle_{\beta, \varepsilon} c_{\beta, \varepsilon} (\varepsilon - 1) \tau_{f_s} \left( \varepsilon - 1 \beta_{f_s} \right) \\
- \frac{2\varepsilon^2}{4} \sum_{s, t=q+1}^{q+q_1} \langle R^{F^+_1, \beta, \varepsilon}(\tau_{e_m}, \tau_{e_k}) f'_s, f'_t \rangle_{\beta, \varepsilon} c_{\beta, \varepsilon} (\varepsilon - 1) \tau_{f'_s} \left( \varepsilon - 1 \beta_{f'_s} \right) + \langle R^{\phi (F^+_1), \beta, \varepsilon}(\tau_{e_m}, \tau_{e_k}) \rangle.
\end{equation}

If \( 1 \leq s, t \leq q \), one has, in view of (2.48) and (2.78), that

\begin{equation}
(2.147) \quad \frac{1}{\beta^2} \langle R^{T, M, \beta, \varepsilon}(\tau_{e_m}, \tau_{e_k}) f'_s, f'_t \rangle_{\beta, \varepsilon} = \langle R^{T, M, \beta, \varepsilon}(\tau_{e_m}, \tau_{e_k}) f'_s, f'_t \rangle + O_{\beta, \varepsilon}(|Z|) \\
= \langle \nabla_{\tau_{e_m}}^{T, M, \beta, \varepsilon} f'_s, f'_t \rangle - \langle \nabla_{\tau_{e_k}}^{T, M, \beta, \varepsilon} f'_s, f'_t \rangle - \langle \nabla_{\tau_{e_m}}^{T, M, \beta, \varepsilon} f'_s, f'_t \rangle + O_{\beta, \varepsilon}(|Z|) \\
= -\frac{1}{\beta^2} \langle p \nabla_{\tau_{e_k}}^{T, M, \beta, \varepsilon} f'_s, \nabla_{\tau_{e_m}}^{T, M, \beta, \varepsilon} f'_t \rangle - \frac{1}{\beta^2} \langle p \nabla_{\tau_{e_k}}^{T, M, \beta, \varepsilon} f'_s, \nabla_{\tau_{e_m}}^{T, M, \beta, \varepsilon} f'_t \rangle + O_{\beta, \varepsilon}(|Z|) \\
+ \tau_{e_m} \langle \nabla_{\tau_{e_k}}^{T, M, \beta, \varepsilon} f'_s, f'_t \rangle - \tau_{e_k} \langle \nabla_{\tau_{e_m}}^{T, M, \beta, \varepsilon} f'_s, f'_t \rangle + O_{\beta, \varepsilon}(|Z|) \\
= O \left( \frac{\varepsilon^2}{\beta^2} \right) + O_{\beta, \varepsilon}(|Z|).
\end{equation}

If \( 1 \leq s \leq q, q + 1 \leq t \leq q + q_1 \), one has

\begin{equation}
(2.148) \quad \varepsilon \langle R^{T, M, \beta, \varepsilon}(\tau_{e_m}, \tau_{e_k}) f'_s, f'_t \rangle_{\beta, \varepsilon} = \frac{1}{\beta^2} \langle R^{T, M, \beta, \varepsilon}(\tau_{e_m}, \tau_{e_k}) f'_s, f'_t \rangle + O_{\beta, \varepsilon}(|Z|) \\
= \frac{1}{\beta^2} \langle \nabla_{\tau_{e_m}}^{T, M, \beta, \varepsilon} f'_s, f'_t \rangle - \frac{1}{\beta^2} \langle \nabla_{\tau_{e_k}}^{T, M, \beta, \varepsilon} f'_s, f'_t \rangle - \frac{1}{\beta^2} \langle \nabla_{\tau_{e_m}}^{T, M, \beta, \varepsilon} f'_s, f'_t \rangle + O_{\beta, \varepsilon}(|Z|) \\
= -\beta \langle p \nabla_{\tau_{e_k}}^{T, M, \beta, \varepsilon} f'_s, \nabla_{\tau_{e_m}}^{T, M, \beta, \varepsilon} f'_t \rangle - \beta \langle p \nabla_{\tau_{e_k}}^{T, M, \beta, \varepsilon} f'_s, \nabla_{\tau_{e_m}}^{T, M, \beta, \varepsilon} f'_t \rangle + \beta \langle p \nabla_{\tau_{e_m}}^{T, M, \beta, \varepsilon} f'_s, \nabla_{\tau_{e_k}}^{T, M, \beta, \varepsilon} f'_t \rangle + \beta \langle p \nabla_{\tau_{e_m}}^{T, M, \beta, \varepsilon} f'_s, \nabla_{\tau_{e_k}}^{T, M, \beta, \varepsilon} f'_t \rangle + O_{\beta, \varepsilon}(|Z|) \\
+ \frac{1}{\beta^2} \tau_{e_m} \langle \nabla_{\tau_{e_k}}^{T, M, \beta, \varepsilon} f'_s, f'_t \rangle - \frac{1}{\beta^2} \tau_{e_k} \langle \nabla_{\tau_{e_m}}^{T, M, \beta, \varepsilon} f'_s, f'_t \rangle + O_{\beta, \varepsilon}(|Z|) \\
= O \left( \frac{\varepsilon}{\beta} \right) + O_{\beta, \varepsilon}(|Z|).
If \( q + 1 \leq s, t \leq q + q_1 \), one has, in view of (2.23),

\[
(2.149) \quad \varepsilon^2 \left< R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) f_s, f_t \right>_{\beta, \varepsilon} = \varepsilon^2 \left< R^{TM, \beta, \varepsilon}(f'_s, f'_t) \tau e_m, \tau \epsilon_k \right> + O_{\beta, \varepsilon} (|Z|)
\]

\[
= \varepsilon^2 \left< \nabla_{f'_s} R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) \nabla_{f'_t} \right> - \varepsilon^2 \left< \nabla_{f'_t} R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) \nabla_{f'_s} \right> - \varepsilon^2 \left< \nabla_{[f'_s, f'_t]} R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) \nabla_{f'_s} \right> - O_{\beta, \varepsilon} (|Z|)
\]

\[
= -\varepsilon^2 \varepsilon^2 \left< p \nabla_{f'_s} R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) \nabla_{f'_t} \right> - \varepsilon^2 \left< p_1 \nabla_{f'_t} R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) \nabla_{f'_s} \right> - \varepsilon^2 \left< p_2 \nabla_{f'_s} R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) \nabla_{f'_t} \right> + O_{\beta, \varepsilon} (|Z|)
\]

\[
= \left< p_1 \nabla_{f'_t} R^{TM, \beta, \varepsilon}(\tau e_k, \tau \epsilon_m) \nabla_{f'_s} + \varepsilon^2 \left( -\varepsilon^2 \varepsilon^2 \left< p \nabla_{f'_s} R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) \nabla_{f'_t} \right> - \varepsilon^2 \left< p_1 \nabla_{f'_t} R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) \nabla_{f'_s} \right> - \varepsilon^2 \left< p_2 \nabla_{f'_s} R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) \nabla_{f'_t} \right> + O_{\beta, \varepsilon} (|Z|) \right>.
\]

From (2.149), one gets that for \( q + 1 \leq s, t \leq q + q_1 \), one has

\[
(2.150) \quad \left< R^{T, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) f_s, f_t \right> = \left< R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) f'_s, f'_t \right> + \beta \varepsilon^2 \left< p \nabla_{\tau e_m} \nabla_{\tau e_k} f'_s, f'_t \right> + \varepsilon^2 \left< p_1 \nabla_{\tau e_k} \nabla_{\tau e_m} f'_s, f'_t \right> + \varepsilon^2 \left< p_2 \nabla_{\tau e_m} \nabla_{\tau e_k} f'_s, f'_t \right> + O_{\beta, \varepsilon} (|Z|)
\]

\[
= \left< p_1 \nabla_{f'_t} R^{TM, \beta, \varepsilon}(\tau e_k, \tau \epsilon_m) \nabla_{f'_s} + \varepsilon^2 \left( -\varepsilon^2 \varepsilon^2 \left< p \nabla_{f'_s} R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) \nabla_{f'_t} \right> - \varepsilon^2 \left< p_1 \nabla_{f'_t} R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) \nabla_{f'_s} \right> - \varepsilon^2 \left< p_2 \nabla_{f'_s} R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) \nabla_{f'_t} \right> + O_{\beta, \varepsilon} (|Z|) \right>.
\]

From (2.23), (2.54), (2.61), (2.62), (2.67), (2.108), (2.109), (2.111), (2.129), (2.132), (2.137) and (2.149) - (2.150), one finds that there exists \( C_1 > 0 \) not depending on \( c > 0 \) such that

\[
(2.151) \quad |I_1| \leq \left( \frac{c C_1}{\beta^2} \sum_{i=1}^{q} \sum_{t=q+1}^{q+q_1} \left| p_1 \nabla_{f'_t} R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) \nabla_{f'_s} \right|^2 + O \left( \frac{\varepsilon}{\beta^2} \right) \left| \tau e_m, \tau \epsilon_k \right| \right) \int_{s(M)} \alpha^2 d\sigma \left| \tau e_m, \tau \epsilon_k \right|
\]

\[
+ \sum_{k=q+1}^{q+q_1} O \left( \frac{\varepsilon}{\beta^2} \right) \int_{s(M)} \alpha \left| Q_{\tau e_m, \tau e_k} \phi_1(F^+_t), \beta, \varepsilon \right| \alpha \left| \tau e_m, \tau \epsilon_k \right| d\sigma \left| \tau e_m, \tau \epsilon_k \right|
\]

\[
+ O \left( \frac{1}{\sqrt{T}} \right) \int_{s(M)} \sum_{i=1}^{q+q_1} \alpha \left| Q_{\tau e_m, \tau e_k} \phi_1(F^+_t), \beta, \varepsilon \right| \left| \tau e_m, \tau \epsilon_k \right|^2 d\sigma \left| \tau e_m, \tau \epsilon_k \right|
\]

\[
= \left( \frac{c C_1}{\beta^2} \sum_{i=1}^{q} \sum_{t=q+1}^{q+q_1} \left| p_1 \nabla_{f'_t} R^{TM, \beta, \varepsilon}(\tau e_m, \tau \epsilon_k) \nabla_{f'_s} \right|^2 + O \left( \frac{\varepsilon}{\beta^2} \right) \left| \tau e_m, \tau \epsilon_k \right| \right) \int_{s(M)} \alpha^2 d\sigma \left| \tau e_m, \tau \epsilon_k \right|
\]

\[
+ O \left( \frac{\varepsilon}{\beta^2} \right) \int_{s(M)} \sum_{k=q+1}^{q+q_1} \left| Q_{\tau e_m, \tau e_k} \phi_1(F^+_t), \beta, \varepsilon \right| \left| \tau e_m, \tau \epsilon_k \right| d\sigma \left| \tau e_m, \tau \epsilon_k \right|
\]

\[
+ O \left( \frac{1}{\sqrt{T}} \right) \int_{s(M)} \sum_{i=1}^{q+q_1} \left| Q_{\tau e_m, \tau e_k} \phi_1(F^+_t), \beta, \varepsilon \right| \left| \tau e_m, \tau \epsilon_k \right|^2 d\sigma \left| \tau e_m, \tau \epsilon_k \right|
\]
(2.152) \[ I_2 = \left( O \left( \frac{\beta + \varepsilon}{\beta^2} \right) + \left( \frac{1}{\sqrt{T}} \right) \right) \int_{s(M)} |\sigma|^2 dv_{s(M)} \\
+ \sum_{i=1}^{q} O \left( \frac{\varepsilon^2}{\beta^2} \right) \int_{s(M)} |\sigma| \cdot \left| Q_{\nabla^F_{f_i}}^{\phi_1(F^+_i),\beta,\varepsilon}(\tau \sigma) \right| dv_{s(M)} \\
+ \sum_{k=q+1}^{q+q_1} O \left( \frac{\varepsilon}{\beta^2} \right) \int_{s(M)} |\sigma| \cdot \left| Q_{\nabla^F_{f_k}}^{\phi_1(F^+_k),\beta,\varepsilon}(\tau \sigma) \right| dv_{s(M)} \\
+ O \left( \frac{1}{\sqrt{T}} \right) \int_{s(M)} \sum_{i=1}^{q+q_1} \left| Q_{\nabla^F_{f_i}}^{\phi_1(F^+_i),\beta,\varepsilon}(\tau \sigma) \right|^2 dv_{s(M)} \\
= \left( O \left( \frac{\beta + \varepsilon}{\beta^2} \right) + \left( \frac{1}{\sqrt{T}} \right) \right) \int_{s(M)} |\sigma|^2 dv_{s(M)} \\
+ \left( O \left( \frac{\varepsilon^2}{\beta^2} \right) + O \left( \frac{1}{\sqrt{T}} \right) \right) \int_{s(M)} \sum_{i=1}^{q} \left| Q_{\nabla^F_{f_i}}^{\phi_1(F^+_i),\beta,\varepsilon}(\tau \sigma) \right|^2 dv_{s(M)} \\
+ \left( O \left( \frac{\varepsilon}{\beta^2} \right) + O \left( \frac{1}{\sqrt{T}} \right) \right) \int_{s(M)} \sum_{k=q+1}^{q+q_1} \left| Q_{\nabla^F_{f_k}}^{\phi_1(F^+_k),\beta,\varepsilon}(\tau \sigma) \right|^2 dv_{s(M)}.

\]

From (2.24), (2.54), (2.61), (2.62), (2.71), (2.108), (2.109), (2.111), (2.129), (2.136), (2.137) and (2.145)- (2.150), one gets

(2.153) \[ I_4 = \left( O \left( \frac{\varepsilon}{\beta^3} + \frac{\varepsilon}{\beta} \sum_{i=1}^{q} \sum_{t=q+1}^{q+q_1} \left| p_1^T M_{\beta,\varepsilon} \left( \nabla^F_{f_i} Z \right) \right|^2 \right) + \left( \frac{1}{\sqrt{T}} \right) \right) \int_{s(M)} |\sigma|^2 dv_{s(M)} \\
+ \sum_{i=1}^{q} O \left( \frac{\varepsilon}{\beta^3} \right) \int_{s(M)} |\sigma| \cdot \left| Q_{\nabla^F_{f_i}}^{\phi_1(F^+_i),\beta,\varepsilon}(\tau \sigma) \right| dv_{s(M)} \\
+ \sum_{i=1}^{q+q_1} O \left( \frac{\varepsilon}{\beta} \right) \int_{s(M)} \left| c_{\beta,\varepsilon} (\beta^{-1} \varepsilon f_i) c_{\beta,\varepsilon} (\varepsilon f_i) \sigma, Q_{\nabla^F_{f_i}}^{\phi_1(F^+_i),\beta,\varepsilon}(\nabla^F_{f_i} Z) (\tau \sigma) \right| dv_{s(M)} \\
+ \sum_{k=q+1}^{q+q_1} O \left( \frac{\varepsilon^3}{\beta} \right) \int_{s(M)} |\sigma| \cdot \left| Q_{\nabla^F_{f_k}}^{\phi_1(F^+_k),\beta,\varepsilon}(\tau \sigma) \right| dv_{s(M)} \\
+ O \left( \frac{1}{\sqrt{T}} \right) \int_{s(M)} \sum_{i=1}^{q+q_1} \left| Q_{\nabla^F_{f_i}}^{\phi_1(F^+_i),\beta,\varepsilon}(\tau \sigma) \right|^2 dv_{s(M)},
\]
from which one gets

\[(2.154)\]

\[
|I_4| \leq \left( \frac{4c}{\beta^2} + O \left( \frac{\varepsilon}{\beta} \right) \right) \sum_{i=1}^{q} \sum_{t=q+1}^{q+q_1} |p_1^T_{f_i} \nabla^{T,M,\beta,\varepsilon}_{f_i} \left( \nabla^{F^*_i}_{f_i} Z \right)|^2 + O \left( \frac{\varepsilon}{\beta^4} \right) \int_{s(M)} |\sigma|^2 dv_{s(M)}
\]

\[+ O \left( \frac{\varepsilon}{\beta^2} \right) \int_{s(M)} \sum_{i=1}^{q} \left| Q \nabla_{f_i}^{F,\phi_1(F^*_i),\beta,\varepsilon} (\tau \sigma) \right|^2 dv_{s(M)}
\]

\[+ \left( \frac{c \varepsilon^2}{16} + O (\varepsilon^3) \right) \int_{s(M)} \sum_{k=q+1}^{q+q_1} \left| Q \nabla_{f_k}^{F,\phi_1(F^*_i),\beta,\varepsilon} (\tau \sigma) \right|^2 dv_{s(M)}
\]

\[+ O \left( \frac{1}{\sqrt{T}} \right) \int_{s(M)} \left( |\sigma|^2 + \sum_{k=1}^{q+q_1} \left| Q \nabla_{f_k}^{F,\phi_1(F^*_i),\beta,\varepsilon} (\tau \sigma) \right|^2 \right) dv_{s(M)}.
\]

We conclude this subsection by the following simple and useful result.

**Lemma 2.8.** There exists $C_2 > 0$ not depending on $c > 0$ such that the following formula holds on $s(M)$:

\[(2.155)\]

\[
\sum_{i=1}^{q} \sum_{t=q+1}^{q+q_1} \left| p_1^T_{f_i} \nabla^{T,M,\beta,\varepsilon}_{f_i} \left( \nabla^{F^*_i}_{f_i} Z \right) \right|^2 \leq C_2.
\]

*Proof.* From (1.22) and (2.137), one gets (2.155). \(\square\)

### 2.8. Proof of Proposition 2.2

From (2.105), (2.106), (2.107), (2.151), (2.152), (2.154) and (2.155), one has

\[(2.156)\]

\[
\sum_{i=1}^{6} |I_i| \leq \left( \frac{c C_1 C_2 + 4c C_2}{\beta^2} + O \left( \frac{\varepsilon}{\beta^4} \right) \right) \int_{s(M)} |\sigma|^2 dv_{s(M)}
\]

\[+ O \left( \frac{\varepsilon}{\beta^2} \right) \int_{s(M)} \sum_{i=1}^{q} \left| Q \nabla_{f_i}^{F,\phi_1(F^*_i),\beta,\varepsilon} (\tau \sigma) \right|^2 dv_{s(M)}
\]

\[+ \left( \frac{c \varepsilon^2}{16} + O (\varepsilon^3) \right) \int_{s(M)} \sum_{k=q+1}^{q+q_1} \left| Q \nabla_{f_k}^{F,\phi_1(F^*_i),\beta,\varepsilon} (\tau \sigma) \right|^2 dv_{s(M)}
\]

\[+ O \left( \frac{1}{\sqrt{T}} \right) \int_{s(M)} \left( |\sigma|^2 + \sum_{k=1}^{q+q_1} \left| Q \nabla_{f_k}^{F,\phi_1(F^*_i),\beta,\varepsilon} (\tau \sigma) \right|^2 \right) dv_{s(M)},
\]

where $C_1 > 0$, $C_2 > 0$ do not depend on $c > 0$. 
From (2.39), (2.10), (2.22), (2.29), (2.39) and (2.156), one deduces that

\[
(2.157) \quad \left\| pt_{,\beta,\varepsilon} D^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon} J_{T,\beta,\varepsilon}\sigma \right\|_{0}^{2} \geq \left\langle \left( \frac{k_{F}}{4\beta^{2}} + O \left( \frac{1}{\beta} + \frac{\varepsilon^{2}}{\beta^{2}} \right) \right) \right\rangle \left( J_{T,\beta,\varepsilon}\sigma, J_{T,\beta,\varepsilon}\sigma \right) \\
+ \frac{1}{\beta^{2}} \sum_{i=1}^{q} \left\| pt_{,\beta,\varepsilon} \nabla_{\tau f_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon} J_{T,\beta,\varepsilon}\sigma \right\|_{0}^{2} + \varepsilon \sum_{i=1}^{q+1} \left\| pt_{,\beta,\varepsilon} \nabla_{\tau f_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon} J_{T,\beta,\varepsilon}\sigma \right\|_{0}^{2} \\
+ \sum_{i=1}^{q} \nabla_{\tau e_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon} J_{T,\beta,\varepsilon}\sigma \right\|_{0}^{2} - \sum_{i=1}^{q} \left( 1 - pt_{,\beta,\varepsilon} \right) c_{\beta,\varepsilon}(\tau e_{i}) \nabla_{\tau e_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon} J_{T,\beta,\varepsilon}\sigma \right\|_{0}^{2} \\
- \left( \frac{c C_{1} C_{2} + 4c C_{2}}{\beta^{2}} + O \left( \frac{\varepsilon}{\beta^{4}} \right) \right) \int_{s(M)} \left\| q_{\nabla_{f_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon}(\tau \sigma) \right\|_{0}^{2} \\
+ O \left( \frac{1}{\sqrt{T}} \right) \int_{s(M)} \left[ \left\| \nabla_{\tau e_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon} J_{T,\beta,\varepsilon}\sigma \right\|_{0}^{2} - \left\| c_{\beta,\varepsilon}(\tau e_{i}) \nabla_{\tau e_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon} J_{T,\beta,\varepsilon}\sigma \right\|_{0}^{2} = 0. \\
\right. \\
\]

Clearly, for any \( 1 \leq i \leq q_{2} \), one has

\[
(2.158) \quad \left\| \nabla_{\tau e_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon} J_{T,\beta,\varepsilon}\sigma \right\|_{0}^{2} - \left\| (1 - pt_{,\beta,\varepsilon}) c_{\beta,\varepsilon}(\tau e_{i}) \nabla_{\tau e_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon} J_{T,\beta,\varepsilon}\sigma \right\|_{0}^{2} \\
\geq \left\| \nabla_{\tau e_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon} J_{T,\beta,\varepsilon}\sigma \right\|_{0}^{2} - \left\| c_{\beta,\varepsilon}(\tau e_{i}) \nabla_{\tau e_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon} J_{T,\beta,\varepsilon}\sigma \right\|_{0}^{2} = 0. \\
\]

Also recall that we have assumed that \( k_{F} \geq \eta \) over \( M \), which implies that \( k_{F} \geq \eta > 0 \) over \( U_{s}(F_{r}^{2}) \). Then one has

\[
(2.159) \quad \left\langle \left( \frac{k_{F}}{4\beta^{2}} + O \left( \frac{1}{\beta} + \frac{\varepsilon^{2}}{\beta^{2}} \right) \right) \right\rangle \left( J_{T,\beta,\varepsilon}\sigma, J_{T,\beta,\varepsilon}\sigma \right) \geq \int_{s(M)} \left( \frac{\eta}{4\beta^{2}} + O \left( \frac{1}{\beta} + \frac{\varepsilon^{2}}{\beta^{2}} \right) \right) \left\| q_{\nabla_{f_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon}(\tau \sigma) \right\|_{0}^{2} \\
\right. \\
\]

For \( 1 \leq i \leq q + q_{1} \), by (2.18) and (2.30)-(2.32), one has,

\[
(2.160) \quad pt_{,\beta,\varepsilon} \nabla_{\tau f_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon} J_{T,\beta,\varepsilon}\sigma = pt_{,\beta,\varepsilon} \left( \tau f_{i} (f_{T}) \tau \sigma + f_{T} \nabla_{\tau f_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon}(\tau \sigma) \right) \\
= \left( \int_{M_{x}} f_{T} \tau f_{i} (f_{T}) k_{e} \, dv_{M_{x}} \right) J_{T,\beta,\varepsilon}\sigma + pt_{,\beta,\varepsilon} \left( f_{T} Q \nabla_{\tau f_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon}(\tau \sigma) \right). \\
\right. \\
\]

From (2.33) and Lemma 2.6 one deduces that the following formula holds for any \( 1 \leq i \leq q + q_{1} \),

\[
(2.161) \quad \left\| pt_{,\beta,\varepsilon} \left( f_{T} Q \nabla_{\tau f_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon}(\tau \sigma) \right) \right\|_{0}^{2} = \int_{s(M)} \left\| Q \nabla_{\tau f_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon}(\tau \sigma) \right\|_{0}^{2} \\
+ O \left( \frac{1}{\sqrt{T}} \right) \int_{s(M)} \left\| q_{\nabla_{f_{i}}^{F,\phi_{1}(F_{r}^{1}),\beta,\varepsilon}(\tau \sigma) \right\|_{0}^{2} \\
\right. \\
\]
If $1 \leq i \leq q$, by (2.11) and (2.57), one gets

$$
\int_{\mathcal{M}_x} f_T \tau f_i (f_T) k_c \, dv_{\mathcal{M}_x} = O\left(1\right) + O\left(\frac{1}{\sqrt{T}}\right).
$$

If $q + 1 \leq i \leq q_1$, by (2.42) and (2.57), one gets

$$
\int_{\mathcal{M}_x} f_T \tau f_i (f_T) k_c \, dv_{\mathcal{M}_x} = O\left(\frac{1}{\beta^2}\right) + O\left(\frac{1}{\sqrt{T}}\right).
$$

Recall the following obvious inequality,

$$
|a + b|^2 \geq \frac{|a|^2}{2} - |b|^2.
$$

By (2.160)-(2.164), one gets that for $0 < \delta \leq 1$ sufficiently small,

$$
\int_{s(M)} \left( O\left(\frac{\varepsilon^2}{\beta^2} + \frac{\varepsilon^2}{\beta^4}\right) + O\left(\frac{1}{\sqrt{T}}\right) \right) \, dv_{s(M)} = \frac{1}{4} \int_{s(M)} \sum_{q=1}^{q+q_1} \left| Q_{\nabla^F,\phi_1(\mathcal{F}_1^+),\beta,\varepsilon}(\sigma) \right|^2 \, dv_{s(M)} + O\left(\frac{1}{\sqrt{T}}\right) \sum_{i=1}^{q+q_1} \int_{s(M)} \left| Q_{\nabla^F,\phi_1(\mathcal{F}_1^+),\beta,\varepsilon}(\tau \sigma) \right|^2 \, dv_{s(M)}.
$$

From (2.157)-(2.159) and (2.165), one deduces that

$$
\left| p_{T,\beta,\varepsilon} D_{F,\phi_1(\mathcal{F}_1^+),\beta,\varepsilon} J_{T,\beta,\varepsilon} \right|_0^2 \geq \left( \eta \frac{c}{4\beta^2} - \frac{c C_1 + 4c C_2}{\beta^2} + O\left(\frac{1}{\beta} + \frac{\varepsilon^2}{\beta^2} + \frac{\varepsilon^2}{\beta^4}\right) \right) \int_{s(M)} \left| \sigma \right|^2 \, dv_{s(M)}
$$

where $C_1 > 0$, $C_2 > 0$ do not depend on $c > 0$.

From (2.166), one gets (2.21).

The proof of Proposition 2.2 is completed.
2.9. **Proof of Theorem 0.5** We assume first that \( \dim M \) and \( \text{rk}(F) \) are divisible by 8.

Recall that \( C' > 0 \) is the constant, not depending on \( c > 0 \), determined in Proposition 2.2. Set

\[
(2.167) \quad c = \min \left\{ \frac{\eta}{2C'}, 1 \right\}.
\]

Then (2.21) becomes

\[
(2.168) \quad \left\| p_{T, \beta, \varepsilon}^* D^{f \phi_1(F^\perp_1), \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma \right\|_0^2 \geq \left( \frac{\eta}{8\beta^2} - \frac{c^\delta}{\beta^2} \right) \int_{s(M)} |\sigma|^2 dv_{s(M)}
\]

\[
+ \frac{\varepsilon^2}{8\beta^2} \sum_{k=1}^{q} \int_{s(M)} |Q_{\nabla_{f_k}^*}^{f \phi_1(F^\perp_1), \beta, \varepsilon} (\tau \sigma)|^2 dv_{s(M)} + \frac{\varepsilon^2}{8} \sum_{k=q+1}^{q+q_1} \int_{s(M)} |Q_{\nabla_{f_k}^*}^{f \phi_1(F^\perp_1), \beta, \varepsilon} (\tau \sigma)|^2 dv_{s(M)}
\]

\[
- \frac{(\varepsilon^2 + \sum_{k=1}^{q+q_1} \int_{s(M)} (Q_{\nabla_{f_k}^*}^{f \phi_1(F^\perp_1), \beta, \varepsilon} (\tau \sigma))^2 dv_{s(M)})}{\sqrt{T}}.
\]

Take \( \varepsilon = \beta^3 \) in (2.168). One sees that when \( \beta > 0 \) is sufficiently small, one has

\[
(2.169) \quad \left\| p_{T, \beta, \varepsilon}^* D^{f \phi_1(F^\perp_1), \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma \right\|_0^2 \geq \left( \frac{\eta}{10\beta^2} - \frac{(\varepsilon^2 + \sum_{k=1}^{q+q_1} \int_{s(M)} (Q_{\nabla_{f_k}^*}^{f \phi_1(F^\perp_1), \beta, \varepsilon} (\tau \sigma))^2 dv_{s(M)})}{\sqrt{T}} \right) \int_{s(M)} |\sigma|^2 dv_{s(M)}
\]

By fixing \( \beta > 0 \) and taking \( T > 0 \) large enough, one sees that there exist \( \beta_0 > 0 \), \( T_0 > 0 \) such that the following inequality holds

\[
(2.170) \quad \left\| p_{T_0, \beta_0}^* D^{f \phi_1(F^\perp_1), \beta_0} J_{T_0, \beta_0} \sigma \right\|_0^2 \geq \left( \frac{\eta}{16\beta_0^2} - \frac{(\varepsilon^2 + \sum_{k=1}^{q+q_1} \int_{s(M)} (Q_{\nabla_{f_k}^*}^{f \phi_1(F^\perp_1), \beta_0} (\tau \sigma))^2 dv_{s(M)})}{\sqrt{T}} \right) \int_{s(M)} |\sigma|^2 dv_{s(M)}
\]

where as we take now \( \varepsilon = \beta^3 \), we omit the subscript \( \varepsilon \).

Let \( \beta_0 > 0, T_0 > 0 \) be fixed as in (2.170). Let

\[
(2.171) \quad D_{s(M), \beta_0, T_0} : \Gamma \left( [S(F) \otimes \Lambda^* (F^\perp_1) \otimes \phi_1 (F^\perp_1)]_{s(M)} \right) \to \Gamma \left( [S(F) \otimes \Lambda^* (F^\perp_1) \otimes \phi_1 (F^\perp_1)]_{s(M)} \right)
\]

be the operator defined by

\[
(2.172) \quad D_{s(M), \beta_0, T_0} = J_{T_0, \beta_0}^{-1} p_{T_0, \beta_0}^* D^{f \phi_1(F^\perp_1), \beta_0} p_{T_0, \beta_0} J_{T_0, \beta_0}.
\]
Since $J_{T_0, \beta_0}$ is an isometry, by (2.170) and (2.172), $D_{s(M), \beta_0, T_0}$ is a formally self-adjoint elliptic operator. Moreover,

(2.173) \[ \ker (D_{s(M), \beta_0, T_0}) = 0. \]

Let

(2.174) \[ D_{s(M), \beta_0, T_0, +} : \Gamma \left( \left( (S(F) \otimes \Lambda^* (F^1))_+ \otimes \phi_1 (F^1) \right)|_{s(M)} \right) \]

\[ \rightarrow \Gamma \left( \left( (S(F) \otimes \Lambda^* (F^1))_- \otimes \phi_1 (F^1) \right)|_{s(M)} \right) \]

be the obvious restriction, then by (2.173), one has

(2.175) \[ \text{ind} (D_{s(M), \beta_0, T_0, +}) = 0. \]

Since $F|_{s(M)} \simeq s_* F \subset Ts(M)$ and $F^1|_{s(M)} \simeq s_* (TM/F) \simeq Ts(M)/s_* F$, by (2.172) and (2.174), one sees that $D_{s(M), \beta_0, T_0, +}$ is homotopic to the corresponding sub-Dirac operator on $s(M)$ (and thus on $(M, F)$) constructed in [18, Definition 2.2]. Thus, they have the same index. In particular, by the Atiyah-Singer index theorem [2], one gets (compare with (1.75) and (18.244))

(2.176) \[ \text{ind} (D_{s(M), \beta_0, T_0, +}) = 2^{\frac{p}{2}} \left< \hat{A}(F) \hat{L}(TM/F) \text{ch} (\phi_1 (TM/F)), [M] \right>. \]

By (2.175) and (2.176), one gets

(2.177) \[ \left< \hat{A}(F) \hat{L}(TM/F) \text{ch} (\phi_1 (TM/F)), [M] \right> = 0, \]

for any $\phi_1 (TM/F)$. This implies that for any Pontrjagin class $p(TM/F)$ of $TM/F$, one has

(2.178) \[ \left< \hat{A}(F) p(TM/F), [M] \right> = 0. \]

By taking $p(TM/F) = \hat{A}(TM/F)$, one gets the vanishing of $\hat{A}(M)$.

Now if one of dim $M$ and rk$(F)$ is not divisible by 8, then we simply work on $M \times \cdots \times M$ (8 times) to get the result.

The proof of Theorem 0.5 is completed.

2.10. **Proof of Theorem 0.1 the case of dimension $4k$.** Without loss of generality we assume that dim $M$, rk$(F)$ are divisible by 8. We assume first that $F$ and thus $F^\perp \simeq TM/F$ are also oriented.

Since $M$ is spin, $F \oplus F^1 = \pi^* (TM)$ is spin over $\mathcal{M}$. Thus, we consider directly the sub-Dirac operator (and its deformations) acting on smooth sections of $S(F \oplus F^1) \otimes \Lambda^* (F^2) \otimes \phi_1 (F^1)$, where $S(F \oplus F^1)$ is the corresponding bundle of spinors. Then everything in the previous subsections still works, and we get the vanishing result

(2.179) \[ \left< \hat{A}(TM) \text{ch}(\phi_1(TM/F)), [M] \right> = 0, \]

from which one deduces that

(2.180) \[ \left< \hat{A}(F) p(TM/F), [M] \right> = 0 \]

for any Pontrjagin class $p(TM/F)$ of $TM/F$. In particular, Theorem 0.1 holds.
Now if $F$ is not orientable, then one can consider the double covering of $M$ with respect to $w_1(F)$, the first Stiefel-Whitney class of $F$. Then one applies Theorem 0.1 to this double covering and get the same result on $M$ by the multiplicativity of the $\hat{A}$-genus.

2.11. Vanishing of the mod 2 index. In this subsection, we will prove Theorem 0.1 for the case of $\dim M = 8k + 1$. The proof for the case of $\dim M = 8k + 2$ is similar.

So from now on we assume that $\dim M = 8k + 1$ ($k \geq 1$). Then $\dim M = \dim M + \frac{q_1(q_1+1)}{2}$, where $q_1 = \dim M - \text{rk}(F)$ is the codimension of $F$. We assume first that $F$ and thus $F^\perp$ are oriented. Set $\tilde{\mathcal{M}} = \mathcal{M} \times \mathbb{R}\frac{7q_1(q_1+1)}{2}$, then one has

$$\dim \tilde{\mathcal{M}} \equiv 1 \mod 8.$$  

(2.181)

Recall that in this dimension one can construct real spinor representations (cf. [14]). We lift everything to $\tilde{\mathcal{M}}$ and use "\tilde{~}" to decorate the obvious modifications. We assume that $TM$, $F$ and $F^\perp \cong TM/F$ are oriented and that $M$ is spin and carries a fixed spin structure. Then $\tilde{F} \oplus \tilde{F}_1^\perp$ carries a canonical spin structure.

As in Section 2.10 we consider the sub-Dirac operator

$$D^{\tilde{\mathcal{M}}} : \Gamma \left( S \left( \tilde{F} \oplus \tilde{F}_1^\perp \right) \otimes \Lambda^* \left( \tilde{F}_2^\perp \right) \right) \to \Gamma \left( S \left( \tilde{F} \oplus \tilde{F}_1^\perp \right) \otimes \Lambda^* \left( \tilde{F}_2^\perp \right) \right).$$  

(2.182)

For any $\beta > 0$, let $D^{\tilde{\mathcal{M}},\beta}$ denote the sub-Dirac operator in (2.182) with respect to the deformed metric $g^{(\tilde{\mathcal{M}},\beta)}$.

Let $e_1, \ldots, e_{\tilde{q}_2}$ be an oriented orthonormal basis of $\tilde{F}_2^\perp$. Let $e_1, \ldots, e_{\tilde{q}_2}$ be the dual basis. Recall that here $\tilde{q}_2 = 4q_1(q_1 + 1)$.

Let $L$ be the trivial real line bundle generated by the element $1 + e_1 \wedge \cdots \wedge e_{\tilde{q}_2} \in \Lambda^*(\tilde{F}_2^\perp)$. We may also view $L$ as a sub-line bundle of $\Lambda^*(\tilde{F}_2^\perp)$.

Let $Q_L : \Lambda^*(\tilde{F}_2^\perp) \to L$ denote the orthogonal projection from $\Lambda^*(\tilde{F}_2^\perp)$ to $L$.

Let $s' : M \to \mathcal{M} = \mathcal{M} \times \mathbb{R}\frac{7q_1(q_1+1)}{2}$ be the embedding defined by $s'(x) = s(x) \times \{0\}$.

For any $T > 0$, $0 < \beta \leq 1$, let $J_{T,\beta}$ be defined as in (2.183), with respect to the embedding $s'$.

Let

$$J_{T,\beta}^L : \Gamma \left( S \left( \tilde{F} \oplus \tilde{F}_1^\perp \right) \right) \bigg| _{s'(M)} \to \Gamma \left( S \left( \tilde{F} \oplus \tilde{F}_1^\perp \right) \otimes \Lambda^* \left( \tilde{F}_2^\perp \right) \right)$$  

(2.183)

be defined by

$$J_{T,\beta}^L : \sigma \mapsto (J_{T,\beta}\sigma) \frac{1 + e_1 \wedge \cdots \wedge e_{\tilde{q}_2}}{\sqrt{2}}.$$  

(2.184)

Then $J_{T,\beta}^L$ is still an isometric embedding. Let $p_{T,\beta}^L$ be the orthogonal projection from the $L^2$-completion of $\Gamma(S(\tilde{F} \oplus \tilde{F}_1^\perp) \otimes \Lambda^*(\tilde{F}_2^\perp))$ to the $L^2$-completion of $\text{Im}(J_{T,\beta}^L)$.

One verifies directly that

$$Q_L \left( c_\beta(e_1) \cdots c_\beta(e_{\tilde{q}_2}) \left( 1 + e_1 \wedge \cdots \wedge e_{\tilde{q}_2} \right) \right) = 1 + e_1 \wedge \cdots \wedge e_{\tilde{q}_2}.$$  

(2.185)

Let $h_1, \ldots, h_{\dim \tilde{\mathcal{M}}}$ be an oriented orthonormal basis of $T\tilde{\mathcal{M}}$. Set

$$\hat{\tau} = c(h_1) \cdots c(h_{\dim \tilde{\mathcal{M}}}).$$  

(2.186)

12Recall that we now take $\varepsilon = \beta^3$. 


Let \( \widetilde{D}^{\widetilde{M},\beta}_T : \Gamma((S(\widetilde{F} \oplus \widetilde{F}_1^\perp))|_{s'(M)}) \to \Gamma((S(\widetilde{F} \oplus \widetilde{F}_1^\perp))|_{s'(M)}) \) be defined by

\[
(2.187) \quad \widetilde{D}^{\widetilde{M},\beta}_T = (J^L_{T,\beta})^{-1} p^L_{T,\beta} \tilde{D}^{\tilde{M},\beta} p^L_{T,\beta} J^L_{T,\beta}.
\]

Since \( \dim M = \text{rk}(\widetilde{F} \oplus \widetilde{F}_1^\perp) \equiv 1 \mod 8 \), one can consider real spinor bundle \( S(\widetilde{F} \oplus \widetilde{F}_1^\perp) \) as well as the real exterior algebra \( \Lambda^*(\widetilde{F}_2^\perp) \). Thus one can view \( \widetilde{D}^{\tilde{M},\beta}_T \) as a real operator acting on \( \Gamma((S(\widetilde{F} \oplus \widetilde{F}_1^\perp))|_{s'(M)}) \). Moreover, by (2.7), (2.181), (2.186) and (2.187), one verifies that \( \widetilde{D}^{\tilde{M},\beta}_T \) is a real formally skew-adjoint elliptic operator which is homotopic to the corresponding real skew-adjoint Dirac operator on \( s'(M) \) defined in [3]. Thus \( \dim(\ker \widetilde{D}^{\tilde{M},\beta}_T) \mod 2 \) is a smooth invariant which, by the homotopy invariance of the mod 2 index in the sense of Atiyah-Singer, can be identified with the Atiyah-Milnor-Singer \( \alpha \)-invariant (cf. [3, Section 3]).

In summary, we have

\[
(2.188) \quad \alpha(M) = \text{ind}_2 \left( \widetilde{D}^{\tilde{M},\beta}_T \right) \equiv \dim \left( \ker \widetilde{D}^{\tilde{M},\beta}_T \right) \mod 2.
\]

Now by using (2.184), (2.185) and by proceeding as in the previous subsections, one can show that there exist \( 0 < \beta_1 \leq 1 \) and \( T_1 \geq 1 \) such that for any \( T \geq T_1 \), one has

\[
(2.189) \quad \dim \left( \ker \widetilde{D}^{\tilde{M},\beta_1}_T \right) = 0.
\]

Combining with (2.188), we get

\[
(2.190) \quad \alpha(M) = 0.
\]

Thus Theorem 0.1 holds.

If \( F \) and thus \( F_1^\perp \) are not orientable, then we take \( \tilde{M} \) to be the direct sum of the orientation line bundle \( o(\hat{F}_2^\perp) \) and the trivial vector bundle \( \mathbb{R}^{7s(q+1)}-1 \) over \( M \). Then one sees that \( \hat{F}_2^\perp \) is still orientable, and one can then proceed as above to complete the proof of Theorem 0.1.

2.12. Proof of Theorem 0.6. From our positivity result (2.170), which can well be used to replace the Lichnerowicz positivity used in [9, Proof of Theorem 2.1], one can proceed as in [9] to get the same conclusion of [9, Theorem 2.1] and [9, Corollary 2.2] under the condition of Theorem 0.6. In particular, Theorem 0.6 holds.

We leave the details and other immediate generalizations to interested readers.

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