The Quantum Aspects of Relativistic Fermion Systems with Particle Condensation

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A consistent local approach to the study of interacting relativistic fermion systems with a condensation of bare particles in its ground or vacuum state, which may have a finite matter density, is developed. The attention is paid to some of the not so well explored quantum aspects that survive the thermodynamic limit. A 4-vector local field, called the primary statistical gauge field, and a statistical blocking parameter are introduced for a consistent treatment of the problem. The effects of random fluctuations of the fields on local observables are discussed. It is found that quasiparticle contributions are not sufficient to saturate local observables. The property of the primary statistical gauge field are discussed in some detail. Two models for the strong interaction are then introduced and studied using the general framework developed. Four possible phases for these models are found. The possibility of spontaneous CP violation and local fermion creation in two of the four phases is revealed. The implications of the finding on our understanding of some of the strong interaction processes are discussed.

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I. INTRODUCTION

Fermions are the fundamental building blocks of the observable universe, which, at certain level, is supposed to be understandable in terms of quantum field theories (QFTs) like QCD, standard model of electroweak interacting, etc.. They are, however, less familiar to us theoretically compared to bosons partly due to their lack of classical correspondences. An understanding of Nature therefore requires a more direct understanding of the behavior of fermions.

Represented by anticommuting Grassmann numbers in the path integration formulation of a QFT, fermions are harder to handle in numerical simulations (like the lattice ones) than bosons. It is therefore desirable to integrate the fermionic degrees of freedom out (in the path integration sense) analytically. This turns out to be an easy task, at least formally, in most of the cases which deals with a Lagrangian density that is only quadratic in the fermion fields. The result is usually a much more complicated effective action for the bosonic fields of the system to be functionally integrated by various means like a numerical calculation or simulation, an approximated analytic computation, modeling and possibly the mixture of all of them. The fermion loop effects in numerical simulations are in some sense less well understood than their bosonic counterparts in a theory at present. The problems become more sever in the presence of a finite chemical potential in lattice simulations. This situation calls for more analytic efforts to understand the effects of fermionic quantum fluctuations in an interacting system since it indicates that our understanding of the problem is still insufficient.

The traditional treatment of the finite density problems (at finite temperature) in statistical mechanics is based upon the grand canonical ensemble in which the partition function is \( Z = \text{Tr} e^{-\beta (\hat{H} - \mu c \hat{N})} \) with \( \beta \) the inverse temperature, \( \hat{H} \) the Hamiltonian and \( \hat{N} \) the particle number of the system. A global chemical potential \( \mu c \) is introduced to select, among all possible particle numbers, the corresponding set of particle numbers that are different from each other by a finite quantity in the thermodynamic limit. Since only those extensive quantities that are proportional to the (infinite) volume of the system are relevant, the above-mentioned differences are irrelevant to bulk thermodynamic quantities. This makes the grand canonical ensemble equivalent to the canonical ensemble \( Z / Z \) in which the number of particles are kept fixed. The usually calculated quantities, called the apparent particle number here, are expressed as \( N_{\text{app}} = \beta^{-1} \partial \ln Z / \partial \mu c \) is formally identical to what is called the absolute particle number \( N_{\text{abs}} = \int d^3x \text{Tr} \hat{\rho}(x, t = 0) e^{-\beta (\hat{H} - \mu c \hat{N})} / Z \). It can be realized that the identification between \( N_{\text{app}} \) and \( N_{\text{abs}} \) is not mathematically warranted in the thermodynamic limit since the particle number \( \hat{N} \) is a macroscopic operator with eigenvalues proportional to the volume of the system. we thus expect that the formal equivalence between \( N_{\text{app}} \) and \( N_{\text{abs}} \) and many other physical observables so computed to be broken down under certain conditions especially when the quantum fluctuation effects are taken into account. In order to characterize the deviation of the quantity \( N_{\text{app}} \) from \( N_{\text{abs}} \), a dark component for each physical local observables in a relativistic QFT is introduced. For example, the dark component of the fermion number density operator is defined as \( \Delta \rho = (N_{\text{abs}} - N_{\text{app}}) / \Omega \) with \( \Omega \) the volume of the system. The questions to be assessed are whether or not \( \Delta \rho = 0 \) and what is the origin of the dark component when \( \Delta \rho \neq 0 \).

Such a possibility is important to study because some of the questions posted for the vacuum state of a relativistic system governed by certain QFT are different from the ones asked for the condensed matter system in which the
quantities under study like the average particle number density is finite with its absolute values playing no direct physical role in physical processes and in which the spacetime resolution (energy) of the observation is usually low. The vacuum state of a relativistic system, especially the trivial one, is characterized by its nothingness nature, namely, all physical observables in the trivial vacuum state are by definition zero. In the non-trivial vacua of the system, certain physical quantities develop finite values which need to be evaluated correctly. Those quantities like the fermion (baryon) number density and associated energy density should in principle manifest themselves in gravitational process at the macroscopic level. In addition, global quantities have no direct physical meaning in a classical relativistic system according to principle of relativity. It is expected that the apparent quantities like \( \hat{N}_{\text{app}} \) are not sufficient ones in the study of the vacuum state of a relativistic system. Rather, one should go back to the absolute quantities like \( N_{\text{abs}} \) defined above. Therefore, for a better marriage between relativity and quantum mechanics, instead of a global chemical potential \( \mu_{\text{ch}} \), a localized quantity called the primary statistical gauge field \( \mu^\alpha(x) \) seems to be necessary, which leads to a local theory for the finite density problem. The fundamental derivative of the logarithm of the new partition functional with respect to \( \mu^\alpha(x) \) at certain spacetime location does give the absolute particle number density since it is a finite number.

The principle of locality has far reaching consequences in the development of modern physics. Implied in Maxwell’s equations for electrodynamics, it motivated the birth of relativity in which it is raised to a principle that governs all physical laws in classical physics. Localities in quantum field theories are implemented in most of the theories regarded as fundamental like the quantum electrodynamics and quantized non-abelian Yang-Mills gauge theories of the standard model. Locality in the quantum field theories, including the fundamental ones, for non-vanishing matter and energy density is not, as a matter of fact, fully implemented. Such theories contain inconsistencies, at least at the conceptual level, that have to be removed.

Locality of a symmetry generates a gauge one. For the \( U(1) \) invariance corresponding to the conservations of fermion number, a new local symmetry called the statistical gauge invariance with the gauge field \( \mu^\alpha(x) \) is induced after its localization.

The introduction of a primary statistical gauge field is expected to produce a series of new problems and opportunities. One of the goals of this paper is to solve these problems and to explore such opportunities so that to develop a consistent framework beyond the quasiparticle picture using which the problems related to the vacuum state of a relativistic fermion system can be systematically tackled.

One of the most important problems in understanding a system governed by a QFT that contains interaction is to determine the phase structure of its vacuum. The vacuum is a state of the system that has the lowest energy that can be different from the trivial one for interacting systems. The non-trivial vacuum of an interacting system covered by this study are the ones that contain macroscopic condensation of particles in various form. It has zero overlap with the corresponding trivial one when the thermodynamic limit is taken. Such a state can not be reached by a perturbative computation, which is local in nature and can only change finite number of particles. A large set of non-trivial vacua of interest are expected to be describable in terms of a set of parameters that characterize the macroscopic condensation of bare particles. Some of such parameters are called the order parameters since they are indicators of a spontaneous breaking down of certain global symmetries of the system. They are accompanied by massless Goldstone bosons that generate long range orders which stabilizes the symmetry breaking states. Others, together with the order parameters, specify the macroscopic condensation of bare particles in the non-trivial vacuum in a more detailed way. For example, the vacuum of the light quark system (up and down quarks) is known to be condensed with quark-antiquark pairs. This phenomenon, which induces the spontaneous breaking down of a chiral symmetry, is shown to happen in the lattice QCD simulation\(^3\) and is supported by experimental facts due to the success of the partial conservation of axial vector current (PCAC) relationship and resulting current algebra\(^3\).

For a system with a large mass gap (compared to the typical excitation mass scale of the system), the condensation of bare particles is unlikely to occur in its vacuum state. So, for the purpose of this paper, I consider light (compared to the typical excitation mass scale of the system) fermion systems. They can be approximately represented by a massless fermion system.

An interaction amongst massless fermions and antifermions of the right sign and magnitude is expected to generate a non-vanishing number of fermion-antifermion pairs from the bare vacuum. Under certain conditions, the number of such pairs can become macroscopic (or proportional to the volume of the system) in the thermodynamic limit. In such a case, it is expected to has a phase transition. Such a phase transition was show to happen to the Nambu Jona-Lasinio (NJL) model in the quasiparticle approximation (the meaning of which is going to be specified in the following sections). It is also shown in Ref.\(^3\) that fermion pairs (and antifermion pairs) can condense to lead to a different phase (\(\beta\) phase) that belongs to the same chiral symmetry breaking chain, namely \( SU(2)_L \times SU(2)_R \rightarrow SU(2)_V \).

In general, the condensation of fermion pairs in the vacuum of an interacting fermionic system belongs to one of three categories: 1) condensation of correlated fermion and antifermion pairs 2) condensation of correlated fermion pairs (and possibly some correlated antifermion pairs 3) condensation of correlated antifermion pairs (and possibly
some correlated fermion pairs). The spin, flavor and color combination of the condensed pairs determines the nature of the symmetry breaking channel of the non-trivial vacuum on a finer basis. Since a macroscopic number of fermions and antifermions are pumped out of the bare vacuum in the non-trivial vacuum, it is natural to ask what are the effects on its energy density by considering not only the contributions from the long distance or and time interval correlated quasiparticle excitations, but also, at least partially, the contributions from some of the transient and short distance quantum fluctuations within the system. To investigate the later effects within a consistent QFT, certain new theoretical concepts and tools in describing and interpreting the related physics, to which this work puts its emphasis on, is proven necessary.

The paper is divided into three major parts. The first part consists of sections I and VII, which gives an introduction and a summary. In the second part, which consists of sections II, III and IV, a consistent general approach to the relativistic fermionic system at both zero and finite density is developed. The third part includes sections V and VI, in which two 4–fermion interaction models for the strong interaction are introduced and studied using the method developed in part two; some novel properties of these models are revealed using the new tools.

The more detailed arrangement of the paper is given in the following. In section II, the general framework used to handle the fermionic system is discussed. An 8–component “real” representation for the fermion fields is adopted. The distinction between the Minkowski and Euclidean spacetime formulation of the problem is emphasized. It is pointed out that in the Euclidean spacetime formulation of the problem, additional contributions due to certain quantum penetration of field configurations to classically forbidden region, to which quasiparticle approximation in Minkowski spacetime can not access, can be included in the effective action. I also motivate the need for a distinction between local and global observables in a relativistic QFT. The existence of the dark component is demonstrated. Section III is devoted a tentative local approach to the finite density problem for a Lagrangian density that conserves the fermion number. Such a formalism is used to study the question of whether or not a state with non-vanishing fermion number density can has a lower energy (density) when the vacuum of the system is in a phase different from the trivial one. Spontaneous CP violating stationary points in the Euclidean spacetime is shown to exist in a phase, called the α phase of a massless fermion system, with non-vanishing ψ ψ vacuum expectation value. It is argued that such a result is at least physically not acceptable. Based on these findings, a consistent theory is developed in section IV for a relativistic fermion system in which a phase transition with particle condensation has occurred. A statistical blocking parameter ε is introduced. The question of the statistical gauge invariance of the system due to the original global U(1) symmetry related to fermion number conservation is addressed. To demonstrate the relevance of the theory, two models for strong interaction are introduced in section V. Their phase structures are then studied. The vacuum of both of them has three different phases. One is the trivial (bare) vacuum, which is called the O phase; the second is the above mentioned α phase with fermion and anti-fermion pairs condensation; the third kind of phases, called the β phase and ω phase respectively, are phases that spontaneously break the global U(1) symmetry related to the fermion number because of a condensation of fermion pairs and antifermion pairs. These phases are further analyzed in section VI by using the formalism developed in section IV. A spontaneous CP violation is found to be present in the β and ω phases. Also, the spontaneous creation of matter, which is relevant to Cosmology, is shown to appear naturally in the β and ω phases. Finally, the main results are summarized and discussed in section VII which also contains an outlook.

II. GENERAL DISCUSSIONS

A. Fermion representation

The fermion field is represented by an 8–component “real” spinor Ψ. It is essential for a consistent formulation of a relativistic finite density field theory in a functional (or path integration) approach developed in this work (see also Appendix A) and for other aspects of the QFTs. In this representation, Ψ satisfies the reality condition

$$\Psi(x) = \Psi^T(-x)\Omega_0,$$  \hspace{1cm} (2.1)

where the Ω0 matrix is

$$\Omega_0 = O_1 C = \left( \begin{array}{cc} 0 & -C^{-1} \\ C & 0 \end{array} \right)$$  \hspace{1cm} (2.2)

with C the charge conjugation operator. The matrices O1 is one of the three Pauli matrices O1,2,3 acting on the upper and lower 4 components of Ψ.
For the case of the number of flavors $n_f$ for the fermions is less than three, the symmetry transformation of the 8 component $Ψ$ can be made to be the same as the 4 component representation $ψ$ by imposing a slightly different “reality” condition on $Ψ$ given by Eq. (2.1) with $Ω_o$ replaced by

$$Ω = \begin{pmatrix} 0 & -C^{-1} \rho^{-1} \\ C \rho & 0 \end{pmatrix}.$$  

(2.3)

Here $ρ = 1$ if $n_f = 1$ and $ρ = iτ_2$ if $n_f = 2$. More detailed properties of this representation for the case of $n_f = 2$ are discussed in Refs. [10,11]. They will not be repeated here.

If $n_f \geq 3$, the reality condition Eq. (2.3) is no longer valid. In such a case, the general “reality” condition is given by the original Eq. (2.1) and the representation of $Ψ$ under flavor $SU(n_f)$ transformation generated by

$$T_a = \begin{cases} t_a O_3 & \text{If } t_a \text{ is symmetric} \\ t_a & \text{If } t_a \text{ is antisymmetric} \end{cases}$$  

(2.4)

have to be adopted. Here $a = 1, 2, \ldots, n_f^2 - 1$ and $t_a$ is the generator for the symmetry transformation in the 4 component representation of $ψ$. The set of matrices $T_a$ and $t_a$ ($a = 1, 2, \ldots, n_f$) belong to equivalent adjoint representation of flavor $SU(n_f)$ transformation.

It should be emphasized that the intention of introducing an 8–component representation for the fermions is different from the one related to the doubling of degrees of freedom in the closed time path approach [10,11] to non-equilibrium (also equilibrium) problems or thermal field dynamical [12] approach to equilibrium problems in two aspects: 1) this work is devoted to the study of zero temperature physics where, in the language of the closed time path approach, all (also equilibrium) problems or thermal field dynamical [12] approach to equilibrium problems in two aspects: 1) this work is devoted to the study of zero temperature physics where, in the language of the closed time path approach, all dynamical fields considered in this work lies on a single time axis running from negative infinity to positive infinity (as opposed to two time axis that form a closed loop) 2) there is no doubling of degrees of freedom for the fermions here since the constraint Eq. (2.1) is systematically implemented in the formalism. In case the present formalism is to be extended to finite temperature [3] or to the non-equilibrium situations, a doubling of the components of the 8–component “real” representation has to be made in addition.

**B. Minkowski spacetime formulation**

The generating functional can be written as

$$e^{W[J,\bar{\eta},\eta]} = \int \prod_i D[f_i] D[\Psi] e^{i \int d^4x (\bar{\Psi} i S^{-1}_\mu \Psi + \mathcal{L}_B[f] + \bar{\eta} \Psi + \eta \bar{\Psi} + \sum_k J_k f_k)},$$  

(2.5)

where $J = \{J_1, J_2, \ldots, J_n\}$ are a collection of external fields coupled to the corresponding boson fields $f = \{f_1, f_2, \ldots, f_n\}$, $\mathcal{L}_B[f]$ is the Lagrangian density for the boson fields $f$, and $\eta$, $\bar{\eta}$ are external Grassmann fields coupled to the fermion fields $\Psi$, $\bar{\Psi}$. Here $f_i$ can be either real or complex. $W[J, \bar{\eta}, \eta]$ generates the Green functions of the fermion or the boson fields. The possible gauge fixing conditions, which can be implemented by multiplying $\prod_i D[f_i]$ a set of $δ$ functions or by introducing ghost fields [13], are unimportant to this work and are suppressed in the sequel.

The bosonic part of the Lagrangian density will not be specified in this investigation. This allows the results of this work to be useful in a wide class of problems. For example, in case of QCD, the boson fields are 8 gluon fields $B_a^\mu(x)$ ($a = 1, 2, \ldots, 8$) with the full Lagrangian density in the 8 component representation for quarks provided in Appendix [3].

In the study of the vacuum properties, fermion degrees of freedom can be eliminated first

$$e^{W[J,\bar{\eta},\eta]} = \int \prod_i D[f_i] e^{\frac{i}{2} kn Det_\tau i S^{-1}_\mu[f] + \frac{i}{2} \bar{\Psi} S_{\mu [f]} [\eta] + \frac{i}{2} \int d^4x (\mathcal{L}_B[f] + \sum_k J_k f_k)},$$  

(2.6)

Here “Det” denotes functional determinant. The proper vertex generating functional for the boson fields is then

$$\Gamma[f] = W[J,0,0] - i \sum_k J_k f_k.$$  

(2.7)

The stationary configuration $f$ determined by the equation

$$\frac{d \Gamma[f]}{df_i} = 0 \quad (i = 1, 2, \ldots, n)$$  

(2.8)
with \( f_i \) \((i = 1, 2, \ldots, n)\) spacetime independent determines the phase structure of the vacuum. In general, \( \Gamma[f] \) is difficult to compute directly. It is useful to define an effective action \( S_{\text{eff}} \) for the boson fields as

\[
S_{\text{eff}}[f] = -i \frac{1}{2} \ln \det \gamma_0 i S_F^{-1}[f] + \frac{1}{2} S \ln S_F^{-1}[0] + \int d^4 x L_B[f]
\]

\[
= -i \frac{1}{2} S p \ln S_F^{-1}[f] + \frac{1}{2} S p \ln S_F^{-1}[0] + \int d^4 x L_B[f],
\]

where \( S p \) denotes the functional trace and a constant (infinite) term independent of the boson fields are subtracted. Eq. (2.9) becomes

\[
e^{W[f, \pi, \eta]} = \int \prod \mathcal{D}[f_i] e^{i S_{\text{eff}}[f] + \frac{1}{2} \gamma_0 S F[f] \eta + i \int d^4 x \sum k j_k f_k}.
\]

Starting from \( S_{\text{eff}}[f] \), either systematic improvement beyond the Gaussian approximation or numerical simulations like lattice computation can be made. A more detailed discussion of the local quantum fluctuations around the mean field is given in Appendix B. A formal relation between \( \Gamma[f] \) and \( S_{\text{eff}}[f] \) is developed in Ref. [14] and discussed in Appendix D. When the fluctuation in \( f \) is only treated at an one loop level, the vertex functional \( \Gamma[f] \) is (see Appendix D)

\[
\Gamma[f] = i S_{\text{eff}}[f] - \frac{1}{2} S p \ln D G^{-1}[f],
\]

where \( D \) is the bare propagator for \( f \) and \( \delta^2 S_{\text{eff}}/\delta f \delta f \) is symbolically denoted as \( G^{-1}[f] \). Under such an approximation, the solution to the equation

\[
\frac{\delta S_{\text{eff}}[f]}{\delta f} + \frac{i}{2} S p G[f] \frac{\delta G^{-1}[f]}{\delta f} = 0 \quad (j = 1, 2, \ldots, n)
\]

determines the vacuum phase structure of the system. It will not be further discussed in this paper since the results of this paper depend only on some of the bosonic fields \( \{f_i\} \) as a solution to Eq. (2.8) being different from zero.

The effective action \( S_{\text{eff}}[f] \) can be expressed in terms of the spectra of the operator \( \gamma_0 i S_F^{-1}[f] \), which is Hermitian. The eigenequation of interest is

\[
\gamma^{\alpha} i S_{F}^{-1}[f] \Psi_{\lambda} = \lambda[f] \Psi_{\lambda}
\]

with \( \Psi_{\lambda} \) the eigenvector. In the time independent case, \( S_{\text{eff}}[f] \) in terms of \( \lambda \) is

\[
S_{\text{eff}}[f] = -i T \sum_{j=0}^{\infty} \int_{C} \frac{dp^0}{2 \pi} \ln \frac{\lambda_{p^0, \xi}[f]}{\lambda_{p^0, \xi}[0]} + \int d^4 x L_B[f],
\]

where \( T \) is the temporal dimension of the system \((T \to \infty)\), \( p^0 \) represents the energy of the eigenvector \( \Psi_{\lambda} \) and \( \xi \) is a collection of other quantum numbers that completely determines a single eigenvector \( \Psi_{\lambda} \). The order in which \( p^0 \) integration and \( \xi \) summation is carried out is important in general since they both are divergent before the subtraction. The symbol

\[
\sum_{\xi} \frac{dp^0}{2 \pi} (\ldots)
\]

is understood as that neither the sum over \( \xi \) nor the \( p^0 \) integration is not done first but they are done in a covariant way. The order between them depends on situations which are discussed in the following and in Appendix A.

Due to the logarithmic function, the integrand in the complex \( p^0 \) plane is multivalued, the physical contour \( C \) with Feynman–Mathews–Salam causal structure [10] for the \( p^0 \) integration is shown in Fig. 3. The conventional computation of the energy density of the vacuum with a non-covariant cutoff, which is called the quasiparticle approximation, can be represented by a distortion of contour to curve I of Fig. 3. It corresponds to a summation of the energies of individual stationary quasiparticle orbits in the negative energy Dirac sea with a fixed (time-independent) background configuration for \( f \).
C. Euclidean spacetime formulation

The path integration representation of the generating functional $W[J, \bar{\pi}, \eta]$ on the right hand side of Eq. 2.3 contains ambiguities associated with the non-specification of the initial and final field configurations. For the transition amplitude between a given pair of initial and final field configurations, the contributing intermediate states can in principle be different from the vacuum state interested here. The lowest energy configuration corresponding to the vacuum is automatically projected out in an Euclidean spacetime computation with a sufficiently large Euclidean time for a large set of proper initial and final field configurations.

In addition, the Minkowski effective action Eq. 2.3 has extrema (or is stable) only at configurations of the boson fields that are of classical nature. An important class of configurations corresponding to pure quantum mechanical effects, namely the tunneling effects through potential barrier or penetration into the classically forbidden field configurations are expected to be missing in a steepest-descent or Gaussian approximation. The Euclidean action obtained by replacing time variable $t$ by $-it$ (with $i^2 = -1$) can be stable at either the time independent configurations, which are the same ones as in the Minkowski approach, or the configurations that correspond to the tunneling or penetration that are quantum mechanical in origin. It is important to realize that the later stable configurations are always absent in the set of stable configurations of the Minkowski effective action despite they constitute an important contribution to the energy density of the system. The Euclidean spacetime formulation has been used to find important stable field configurations called instantons in non-Abelian gauge theories. They correspond to stable finite action field configurations due to the tunneling between gauge field configurations of different winding numbers. Other applications of finding the tunneling effects by using the Euclidean spacetime formulation and their connection to the WKB method in quantum mechanics can be found in e.g. Ref. \[13\]. It will not be elaborated here. The effects that are interested in this study are the ones that survive the thermodynamic limit and are thus non-finite action effects despite they can be decomposed into a collection of random finite action ones (see Appendix \[3\]).

The generating functional for the Green functions in the Euclidean spacetime formulation can be expressed as

$$e^{W[J, \bar{\pi}, \eta]} = \prod \int D[f] e^{S_{eff}^E[f] + \frac{i}{2} \bar{\pi} \mathcal{E}^E[f] \eta + \int d^4x \sum_k J_k f_k}$$

which also serves as a definition of the Euclidean effective action $S_{eff}^E[f]$. The Euclidean propagator $S_F^E[f]$ is found by using the rules discussed in the following.

For a fermion system, the simple replacement $t \rightarrow -it$ is not sufficient to obtain a consistent Euclidean spacetime formalism from the corresponding Minkowski one due to the fact that a fermion is not a scalar particle. More sophisticated set of transformations are needed. Formally, an Euclidean effective action for fermions can be obtained by making a continuous change of the metric \[14\]. The result of change for a Dirac particle can be summarized by

$$g_{\mu\nu}^E = \{-, -, -, -\}, \quad \gamma_5^E = i\gamma_5, \quad \gamma_0^E = -i\gamma_0, \quad \gamma_i^E = \gamma_i$$

and the effective action Eq. 2.3 in the Euclidean spacetime formulation becomes

$$S_{eff}^E[f] = \frac{1}{2} \left\{ \text{SpLn} \left[ i\gamma_5 (S_F^E[f])^{-1} \right] - \text{SpLn} \left[ i\gamma_5 (S_F^E[0])^{-1} \right] \right\} + \int d^4x E \mathcal{L}_B[f^E],$$

where the Euclidean propagator $S_F^E[f]$ is also obtained by using substitution rules listed in Eqs. 2.16. In terms of the eigenvalue of the hermitian operator $i\gamma_5 (S_F^E[f])^{-1}$ that satisfies

$$i\gamma_5 (S_F^E[f])^{-1} \Psi_\lambda = \lambda f \Psi_\lambda.$$  

The Euclidean correspondence of Eq. 2.14 is then

$$S_{eff}[f] = -\frac{i}{2} T \int_{\mathcal{C}} \frac{dp^0}{2\pi} \ln \frac{\lambda_{p^0, \xi}^E[f]}{\lambda_{p^0, \xi}^E[0]} + \int d^4x \mathcal{L}_B[f],$$

where the superscript "$E$" on top of a quantity in Euclidean spacetime is suppressed in there and in the following whenever no confusion is thought to occur. It can be demonstrated that Eq. 2.19 can be obtained from Eq. 2.14 by a distortion of the $p^0$ integration contour to curve II shown in Fig. 2.
D. Local and global observables

The physical observables in a classical theory consistent with relativity are local ones like the charge density, energy density etc.. Unlike in the non-relativistic world, the global observables like the total charge and the energy of the system are not directly measurable. In a given frame, however, the global observables can be defined as a spatial integration of the local observables on an equal time hypersurface in the Minkowski spacetime. The meaning of the global observables can only be defined operationally. The particular value of a global observable has to be determined by the following procedure. First a synchronization of the clocks of a group of observers on each spacetime point should be carried out, then let each observer measure the corresponding density at his/her location in spacetime at the same time and finally sum (integrate) each observers finding to obtain the value of the global observable.

The global observables in quantum theories, including the relativistic QFTs, are regarded as physical observables. Let us consider certain density observable \( \hat{\rho}(x) \) in a QFT. The corresponding global observable \( \hat{Q} \) is defined simply as

\[
\hat{Q} = \int_{\Sigma} d^3 x \hat{\rho}(x)
\]

with \( \Sigma \) the equal-time hypersurface in certain reference frame. Then the matrix elements of \( \hat{\rho}(x) \) and \( \hat{Q} \), namely, \( \langle f | \hat{\rho}(x) | i \rangle \) and \( \langle f | \hat{Q} | i \rangle \) define the corresponding observables.

Since both the local observable \( \hat{\rho}(x) \) and the classically not defined global one \( \hat{Q} \) are defined in a relativistic QFT, one can naturally ask the following question, namely, is there any difference between \( O = \langle \Omega | \hat{Q} | \Omega \rangle \) and \( O' = \int d^3 x \langle \Omega | \hat{\rho}(x) | \Omega \rangle \) measured in a state \( \Omega \)? It can be shown that this is a relevant question for a relativistic QFT.

A measurement of the total charge or charge density of a system normally consists of supplying an external global field or local field coupled to the corresponding observables in a known way and then measure the response of the system like the force that the external field exerts on the system or the amount of increase of some proper defined "potential" of the system, which allows the observer to deduce the total charge or charge density measured. Let us consider the measurements, in the above sense, of the global and the local observables in the vacuum state of the system expressed symbolically as

\[
O_0 = \lim_{j \to 0} \langle 0_j | \hat{Q} | 0_j \rangle = \lim_{j \to 0} \langle 0_j | \int_{\Sigma} d^3 x \hat{\rho}(x) | 0_j \rangle,
\]

\[
O'_0 = \int_{\Sigma} d^3 x \lim_{\delta j(x) \to 0} \langle 0_{\delta j(x)} | \hat{\rho}(x) | 0_{\delta j(x)} \rangle,
\]

where \( J \) is a global external field coupled to \( \hat{Q} \) and \( \delta j(x) \) is a local external field taking non-vanishing value only at \( x \) that couples to \( \hat{\rho}(x) \) and \( | 0_j \rangle \) and \( | 0_{\delta j(x)} \rangle \) are the corresponding vacuum states. The first one \( O_0 \) corresponds to a measurement of \( \hat{Q} \) directly; the second one corresponds to the integration of a (infinite) set of measurements of \( \hat{\rho}(x) \) on a space-like hypersurface. These two measurements can in principle be different in a system governed by a QFT because of the random local quantum fluctuations of the fields that are not suppressed in the thermodynamic limit.

Such a potential difference guarantees the possible existence of the dark component in a QFT. The fact that localized random quantum fluctuations are not suppressed in an interacting QFT, especially in some of the non-trivial phases of the system, are discussed in more details in Appendix where it is also shown that the conventionally used quasiparticle picture in many-body theory and QFT is not sufficient to saturate the local observables due to the existence of the dark component. Such a deviation from the conventional physical picture based upon quasiparticles gets less and less significant as the resolution of our observation respect to spacetime gets lower and lower compared to the typical size of the localized random fluctuations \( f_a \) of the system. In such a case these fluctuations are more and more suppressed and the contribution of the dark component becomes smaller and smaller resulting in 1) an emergency of a quasiparticle dominated picture for the system and 2) the validity of the results obtained based upon a global chemical potential \( \mu_{\text{ch}} \) in the grand canonic ensemble in low resolution (energy) observations.

III. A NAIVE LOCAL QUANTUM FINITE DENSITY FIELD THEORY AND ITS PROBLEMS

A. A tentative formalism for the local finite density fermion field theory

In order to develop a theoretical framework consistent with the requirement of locality, a new quantity called primary statistical gauge field \( \mu^a(x) \) is introduced in the following. The motivation for its introduction is discussed
in the introduction and in the above sections. The other reason that it is treated as a local variable is due to the fact that total fermion number has limited meaning in a relativistic system that are generated in the past not infinite long ago. The total number of fermions in such a system is not an observable since there are regions outside the horizon that are classically non-detectable even in principle due to the constraint of causality.

1. The asymptotic grand canonic ensemble

The generating functional corresponding to Eq. 2.5 for a fermionic system with finite density can be formally written as

\[ e^{W[J,\eta,\eta,\mu]} = \int \prod_i D[f_i] D[\Psi] e^{i \int d^4x (\frac{1}{2} \Psi S^\dagger f \Psi + \mu_\alpha j^\alpha + \mathcal{L}_B[f] + ...)}, \]  

(3.1)

where "..." denotes the source terms and the fermion number current \( j^\mu(x) \) is

\[ j^\mu(x) = \frac{1}{2} \Psi(x) \gamma^\mu O_3 \Psi(x). \]  

(3.2)

In the Minkowski spacetime, quantity on the left hand side of the above equation is the transition amplitude of the system from properly weighted initial field configurations \( \phi_i \) at \( t = -\infty \) to final field configurations \( \phi_f \) at \( t = +\infty \), namely,

\[ e^{W[J,\eta,\eta,\mu]} = \sum_{\{\phi_f,\phi_i\}} W[\phi_f,\phi_i] \langle \phi_f, t = +\infty | \phi_i, t = -\infty \rangle, \]  

(3.3)

where \( W[\phi_f,\phi_i] \) is the weight functional discussed in the following.

These initial and final fields are considered free fields. The interaction terms are adiabatically switched on at certain large negative time \(-T\) and switched off at certain large positive time \( T\). Such a technical manipulation does not affect the actual local physical observables like the energy density, fermion number density, etc. at time \( t \) that is far from both \(-T\) and \( T\) due to locality.

Since the particle content for free fields at \( t = \pm\infty \) is clear, which allows a straightforward statistical interpretation in terms of number of particles in each single particle state of the system. It is natural to assume that the initial (final) state are in the grand canonic ensemble with the weight \( W[\phi_f,\phi_i] \) for the summation determined by the factor \( \lim_{\beta \to \infty} \exp[-\beta(E_0 - \mu N)] \) where \( E_0 \) is the total energy and \( N \) is the total fermion number of the (free) system. \( \mu \) agrees with the time component of the spacetime independent part of the primary statistical gauge field \( \mu^\alpha(x) \). Such an ensemble is called the asymptotic grand canonic ensemble here.

The asymptotic grand canonic ensemble differs from the grand canonic ensemble for it allows for a local approach to the relativistic finite density problem and the existence of the dark component discussed in the above sections and in Appendix B. Its predictions, however, approaches that of the grand canonic ensemble in the non-relativistic situations in which the spacetime resolution or energy in a measurement is low. This point is demonstrated in Appendix B at the mean field level.

Two points must be pointed out. The first one is that although the time interval \( 2T \) between the (adiabatic) switching on and off of the interaction terms are let to go to infinity in the final result, the thermodynamic limit, in which the spacetime box that contains the system approaches infinity, is taken first. Therefore \( T \) is still infinitesimal compared to the temporal size of the system. The second one is that, as a result, for the conserved quantities, like the total fermion number, the extreme value of it picked out \footnote{We are interested in the zero temperature case here. The form of the weight functional for free fields at finite temperature can be found using the method given in, e.g., Refs. \cite{10,11}, which involves two time axis: one runs from \(-\infty\) to \(+\infty\) on real \( t \) axis; the other lies below it on the complex \( t \) plan. In the zero temperature limit, only the contributions from the real \( t \) axis is nonzero, which leads to Eq. 3.1.} in the asymptotic grand canonic ensemble with a fixed value of \( \mu \) is unmodified. The later point is elaborated in the following.
Before continuing the development of the formalism, it is important to reveal a property of the asymptotic grand canonic ensemble related to the $U(1)$ symmetry corresponding to the fermion number conservation. The conservation of $\rho(x)$ due to the $U(1)$ symmetry can be derived from the Noether’s theorem at the classical level, namely,

$$\partial_{\mu}\rho^{\mu}(x) = 0.$$  \hspace{1cm} (3.4)

Those interaction Lagrangian densities for which Eq. (3.4) remains true at the quantum level are considered despite the fact that this symmetry is not explicitly related to a gauge symmetry for which a superselection sector in the Hilbert space of the system exists. The reason to consider such a class of models is because for a quark system, the fermion number is identical to the baryon number, which is conserved to a high precision due to the lack of any convincing evidence of the proton decay in observation at the present.

For an uniform system, Eq. (3.4) implies that

$$\frac{\partial \bar{p}}{\partial g_i} = 0 \quad (i = 1, 2, \ldots)$$ \hspace{1cm} (3.5)

in the asymptotic grand canonic ensemble with $\{g_i\}$ representing a set of interaction coupling constants and the derivative taken by keeping $\mu^\alpha$ unchanged. For an uniform system, the mean local fermion density is a function both of the coupling constants $\{g_i\}$ and $\mu^\alpha$, which is now spacetime independent. Eq. (3.4) implies that the mean local fermion density of the system $\bar{p}$ is only a function of $\mu^\alpha$, it is independent of the interaction coupling constants $\{g_i\}$. This property allows us to find out the relationship between $\bar{p}$ and $\mu^\alpha$ easily by considering a non-interacting system. It is discussed in Appendix A. The result for a massless fermion with $n_f$ flavors and $n_c$ colors is

$$\bar{p} = \frac{n_f n_c}{3\pi^2} \mu^3,$$ \hspace{1cm} (3.6)

where $\mu \equiv \sqrt{\mu^2}$. The simplicity of Eq. (3.6) in the asymptotic grand canonic ensemble is due to the fact that the interaction Lagrangian density conserves fermion number. It implies that for an interacting massless fermion system, local quantity $\bar{p}$ is non-zero as long as $\mu$ is non-zero even when a phase transition in its vacuum state that generates a finite gap for the lowest excitation of the system has happened. Such a qualitative behavior is required following the discussion given in Appendix B. Eq. (3.6) is however an exact relation in the asymptotic grand canonic ensemble.

After an exploration of the implications of the $U(1)$ symmetry of the class of models under consideration in the asymptotic grand canonic ensemble, we are in a position to further develop the formalism for the investigation of the density fluctuations of the vacuum after a phase transition. Since Eq. (3.1) can be obtained from Eq. (2.5) by the replacement

$$i\bar{\theta} \to i\bar{\theta} + \dot{\mu} O_3$$ \hspace{1cm} (3.7)

the effective action $S_{eff}$ of the system is changed to

$$S_{eff}[f, \mu] = -\frac{T}{2} \left( \int C \frac{dp^0}{2\pi} \ln \frac{\lambda_{\rho,\xi}[f; \mu]}{\lambda_{\rho,\xi}[0; \mu]} + \sum_{\xi} \int C \frac{dp^0}{2\pi} \ln \frac{\lambda_{\rho,\xi}[0; \mu]}{\lambda_{\rho,\xi}[0, 0]} \right) + \int d^4x L_D[f].$$ \hspace{1cm} (3.8)

with the eigenvalues $\lambda$ obtained from the following equation

$$\gamma^0 \bar{S}^{-1}_F[f; \mu] \Psi = \lambda[f; \mu] \Psi,$$ \hspace{1cm} (3.9)

where $S_F^{-1}[f; \mu]$ is derived from the corresponding one in Eq. (2.13) by making the substitution Eq. (3.5). Since the question interested in this study is related to a comparison of the energies of states with different fermion densities when the interaction is present, I consider the following effective action obtained from $S_{eff}$ by a Legendre transformation

$$\bar{S}_{eff} = S_{eff} - \int d^4x \mu_{\alpha} F^\alpha.$$ \hspace{1cm} (3.10)

It is a canonic functional of $F^\alpha$ with $\mu^\alpha$ implicitly depending on it. For an uniform system like the vacuum, the second logarithmic term in Eq. (3.8) is calculated in such a way as not to over-count the already known (and included) quantum fluctuations of the free fields. The result is given in Appendix A (Eq. A7), it is
\[-i \frac{T}{2} \sum_{\xi} \int_{\mathcal{C}} \frac{dp^0}{2\pi} \ln \frac{\lambda_{\rho^0,\xi}[0;\mu]}{\lambda_{\rho^0,\xi}[0,0]} = \int d^4x (\mu \overline{\pi} - \pi) \] (3.11)

in the rest frame of the density. Together with Eqs. (3.6) and (3.10) for an uniform system takes the form
\[
\tilde{S}_{\text{eff}} = -i \frac{T}{2} \sum_{\xi} \int_{\mathcal{C}} \frac{dp^0}{2\pi} \ln \frac{\lambda_{\rho^0,\xi}[f;\mu]}{\lambda_{\rho^0,\xi}[0,0]} - \frac{n_f n_c}{4\pi^2} \int d^4x \mu^4 + \int d^4x \mathcal{L}_B[f].
\] (3.12)

The effective potential to be minimized for an uniform system is then defined by
\[
V_{\text{eff}} = -\lim_{V_3T \to \infty} \frac{\tilde{S}_{\text{eff}}}{V_3T} = -\lim_{V_3T \to \infty} \frac{1}{V_3|G_0|^2} \frac{\lambda^2}{\pi^2} \ln \frac{\lambda_{\rho^0,\xi}[f;\mu]}{\lambda_{\rho^0,\xi}[0,0]} + \frac{n_f n_c}{4\pi^2} \mu^4 - \mathcal{L}_B[f],
\] (3.13)

where volume $V_3 = L^3$ with $L$ the spatial dimension of the system.

**B. Euclidean Instability of the $\alpha$ Phase and Its CP Problem**

The $\alpha$ phase for a massless fermion system can be shown to be realized by using the Nambu Jona–Lasinio (NJL) model [17]. In the 8–dimensional “real” representation for the fermion spinor, the NJL model with 2 flavors ($n_f = 2$) and 3 colors ($n_c = 3$) are given in appendix B. The NJL model is studied extensively in the literature by assuming $\mu^\alpha = 0$ in the vacuum so far. It’s possible that such a plausible assumption is in fact incorrect.

With Eq. (3.13), the question of whether or not the $\alpha$ phase is stable against fluctuations in $\mu^\alpha$ can be studied. In the $\alpha$ phase, Eq. (3.13) takes the following form
\[
V_{\text{eff}} = 6i \frac{d^4p}{(2\pi)^4} \ln \left(1 - \frac{\sigma^2}{p_+^2}\right) \left(1 - \frac{\sigma^2}{p_-^2}\right) + \frac{1}{4G_0} \sigma^4 + \frac{3}{2\pi^2} \mu^4,
\] (3.14)

where $p_+^\mu = (p^0 + \mu^0, \mathbf{p} + \mu)$ and $p_-^\mu = (p^0 - \mu^0, \mathbf{p} - \mu)$.

1. Quasiparticle approximation and its problem

In the quasiparticle approximation, the $p^0$ integration contour is the one shown in Fig. 3. The localized quantum fluctuations of the order parameter $\sigma$ are not included following such a contour. With the help of Appendix A, Eq. (3.14) can be shown to have the following form
\[
V_{\text{eff}} = \frac{3}{2\pi^2} \mu^4 + \frac{1}{4G_0} \sigma^4 - \frac{6}{\pi^2} \int_{k_F}^{\Lambda_3} dp \lambda(\mathbf{p})^2 \left(\sqrt{\mathbf{p}^2 + \sigma^2} - |\mathbf{p}|\right),
\] (3.15)

where $k_F = \theta(\mu - \sigma)\sqrt{\mu^2 - \sigma^2}$ with $\theta(x)$ the step function and $\Lambda_3$ is the cutoff in the 3–momentum that defines the model. It is a monotonic increasing function of $\mu$. So the stationary point for $V_{\text{eff}}$ is at the position $\mu = 0$, as expected.

There is however an inconsistency related to the fermion number in the quasiparticle approximation to the $\alpha$ phase. Eq. (3.15) holds for any phase of a model Lagrangian density for which the fermion number is conserved. On the other hand, the average fermion number for an uniform system in the quasiparticle approximation is
\[
\overline{\pi} = \frac{2}{\pi^2} \theta(\mu - \sigma)(\mu^2 - \sigma^2)^{3/2}
\] (3.16)

which differs from Eq. (3.15) and the results of Appendix A even qualitatively. The reason, as is mentioned generally in section I, is because the quasiparticle approximation is related to a Gaussian approximation to the generating functional of the model in the Minkowski spacetime formulation. Such an approximation can only take into account of the contributions from propagating modes or quasiparticles. The quasiparticles are not fundamental building blocks, namely, the bare particles, of the system; they are composite objects representing coherent excitations of infinite pairs of bare particles and anti-particles. In the phase that quasiparticles propagate, the excitations correspond to the bare particles of the system damp in spacetime when produced. They are not, however, absent in a time interval that is sufficiently short. They form an important component, which is called the dark component (see Appendix A), that contributes to the fermion number density and other relevant local physical quantities. The contribution of such a dark component is expected to be taken into account, at least partially, if we formulate our semi-classical approximation in the Euclidean spacetime to sample the important contributions of pure quantum mechanical configurations.
2. Euclidean stationary points and the CP problem

In the Euclidean spacetime, Eq. 3.14 can be evaluated by using the \( p^0 \) integration contour shown in Fig. 2 together with a covariant cutoff. This operation corresponds to the replacement rule given by Eq. 2.10 plus \( \mu^0 \to -i\mu^0 \), which is equivalent to the change \( p^0 \to ip^0 \) and \( \mu^0 \to \mu^0 \) in the expression for the Minkowski effective action after the tracing over spin, isospin and color degrees of freedom is taken. The time component of \( b \) is regarded as an external field and is related to the fermion density given by Eq. 3.6. The result is

\[
V_{\text{eff}}/\Lambda^4 = -\frac{3}{\pi^3} \int_0^1 dy y^3 \int_0^1 dx \sqrt{1 - x^2} \ln \left( \frac{y^2 - \tilde{\mu}^2 + \tilde{\sigma}^2}{(y^2 - \mu^2)^2 + 4\mu^2 y^2 x^2} \right) + \frac{1}{16\pi\alpha_0} \tilde{\sigma}^2 + \frac{3}{2\pi^2} \tilde{\mu}^4
\]

(3.17)

\( \tilde{\sigma} \equiv \sigma/\Lambda, \tilde{\mu} \equiv \mu/\Lambda, \alpha_0 = G_0\Lambda^2/4\pi \) and with \( \Lambda \) the covariant cutoff in the Euclidean momentum space.

In the \( \alpha \) phase with non-zero \( \sigma \), the dependence of \( V_{\text{eff}}/\Lambda^4 \) on \( \mu \) is plotted in Fig. 3. The minima of \( V_{\text{eff}} \) is not located at \( \mu = 0 \) for any finite \( \sigma \) but some finite \( 0 < |\mu| < \sigma \). This is physically not acceptable.

There are two problems related to such a vacuum with non-vanishing fermion density if the dominate phase of the universe is in the \( \alpha \) phase.

The first one is related to the strong CP problem. Since in such a background field as \( \mu^0 \), the two CP conjugate eigenstates of a neutral particle has different energies; it implies the occupation number for one eigenstate can be larger than the other in physical processes at sufficiently low energy. This in turn will cause much too large CP violation phenomena not observed in nature. Let us consider the only system, namely the physical strong interaction vacuum is considered to be in the \( \alpha \) phase.

Facing these two serious problems, a solution is needed to be looked for. There are at least two alternatives. The first one, which is likely to be correct, is that there are something missing in the computation procedure used so far. The second one is that our notion that the dominate phase of the universe is in the \( \alpha \) phase is wrong. The later alternative is unlikely to be correct since there is a large body of empirical facts that support such a notion.

With this consideration in mind, I turn next to an investigation of the first alternative.

C. A Fock space inspection of the vacuum structure of the \( \alpha \) phase and the blocking effects

It is known that in the \( \alpha \) phase of the NJL model treated in the mean field approximation, the vacuum can be related to the bare one by an unitary transformation before the thermodynamic limit \( L^3 \to \infty \) is taken. It can be explicitly written as

\[
|\text{vac}\rangle = \prod_{p,h} e^{\tilde{O}(p,h)}|0\rangle
\]

(3.18)

with \( p \) the three momentum, \( h \) the helicity label and operator \( \tilde{O} \) expressed in term of the creation operators \( (a_{ph}^\dagger, b_{ph}^\dagger) \) and annihilation operators \( (a_{ph}, b_{-ph}) \) of the bare fermions as

\[
\tilde{O}(p,h) = \frac{i}{2} \theta_p \left[ a_{ph}^\dagger b_{ph}^\dagger - b_{-ph} a_{ph} \right],
\]

(3.19)

where \( \cos \theta_p = |p|/E_p \) and \( E_p = \sqrt{|p|^2 + \sigma^2} \). The annihilation operators for the quasiparticles \( \alpha_{ph} \) and the anti-quasiparticle \( \beta_{ph} \) of the system are related to the bare ones through a Bogoliubov transformation

\[
\alpha_{ph} = \cos \left( \frac{1}{2} \theta_p \right) a_{ph} - \sin \left( \frac{1}{2} \theta_p \right) b_{ph}^\dagger,
\]

\[
\beta_{ph} = \sin \left( \frac{1}{2} \theta_p \right) a_{-ph}^\dagger + \cos \left( \frac{1}{2} \theta_p \right) b_{ph}.
\]

(3.20)

(3.21)
From Eqs. 3.18, 3.19 two properties can be seen: 1) the \( \alpha \) phase vacuum is a superposition of states in the Fock space of bare fermions with increasing number of fermion and anti-fermion pairs and 2) after the thermodynamic limit \( L^3 \to \infty \) is taken the overlap between the bare vacuum \(|0\rangle \) and the \( \alpha \) phase vacuum \(|\text{vac}\rangle \) tends to zero. This implies 1) the occupation of fermions and anti-fermions in the \( \alpha \) phase vacuum can have blocking effects on the creation operation of bare particle states and 2) the true \( \alpha \) phase vacuum can not be reached by perturbative iterations starting from some preassumed state of the system without introducing some kind of macroscopic variables into the path integration formalism.

IV. A CONSISTENT LOCAL QUANTUM FINITE DENSITY FIELD THEORY

A. The statistical blocking parameter

1. The motivating Fock space study

I develop a theory that takes into account both the fermion–antifermion pair condensation and the resulting blocking effects in this section. As mentioned in the previous section, the bare vacuum with zero pairs of fermion and antifermions is not a suitable initial state that starts the path integration computation since it has zero overlap with the true vacuum state of the system after the phase transition. Instead, the starting state is of the following kind

\[
|\phi_0\rangle = \sum_{n,\xi} C^n_{\xi} |(f\bar{f})_{\xi}^n\rangle = \sum_{n,\xi} C^n_{\xi} |n\xi\rangle
\]

(4.1)

with \( n \) the number of the fermion–antifermion pairs and \( \xi \) other quantum numbers that completely specify the state. The diagonal matrix element of the evolution operator of the system can be expressed in terms of path integration over the dynamical fields of the system

\[
\langle \phi_0, t = +\infty | \phi_0, t = -\infty \rangle = \lim_{t_f \to \infty} \lim_{t_i \to -\infty} \langle \phi_0 | e^{-i\hat{H}(t_f - t_i)} | \phi_0 \rangle = \sum_n \sum_{\xi,\xi'} C^n_{\xi} C^{'n}_{\xi'} \langle n\xi', t = +\infty | n\xi, t = -\infty \rangle + \ldots ,
\]

(4.2)

where \( \hat{H} \) is the total Hamiltonian of the system and \( \ldots \) represents the off diagonal contributions to the transition amplitude between states with different pairs of fermion and antifermion. For a discussion of the eigenstate of the total Hamiltonian like the vacuum state, the off diagonal contributions with macroscopically different initial and final fermion–antifermion pairs are expected to be effectively absent in the final result after the thermodynamic limit.

The transition amplitude \( \langle n\xi', t_f | n\xi, t_i \rangle \) with the external fields present is then written in terms of path integration, namely,

\[
\langle n\xi', t_f = +\infty | n\xi, t_i = -\infty \rangle = N \int D[\psi] \prod_i D[f_i] \epsilon^i \int d^4x (\frac{\epsilon}{2} \bar{\psi}if\psi + \epsilon \bar{\psi}f\psi + \bar{\psi} \Psi + \int \sum_k J_k f_k),
\]

(4.3)

where \( N \) is a constant. The formal manipulations, which express the above functional integration over fermion degrees of freedom by an effective action \( S_{eff} \), remain mostly unchanged, except \( S_{eff} \) depends now on a new statistical parameter \( \epsilon \)

\[
\langle n\xi', t_f = +\infty | n\xi, t_i = -\infty \rangle = N' \int \prod_i D[f_i] \epsilon^i S_{eff}(f, \mu, \epsilon) + \frac{\epsilon}{2} \bar{\psi}if\psi \epsilon + \int \sum_k J_k f_k.
\]

(4.4)

For a stationary situation, the effective action \( S_{eff}[f, \mu, \epsilon] \) is given by Eq. 2.14. The constraint that both of the initial and the final states considered are configurations with both of the bare fermion and antifermion states below energy \( \epsilon \) filled is implemented by a distortion of the contour for \( p^0 \) integration from the one in Fig. 3 to that of Fig. 4.

In the thermodynamic limit of \( L^3 \to \infty \), the sum over \( n \) in Eq. 4.3 can be replaced by an integration over \( \epsilon \), the statistical blocking parameter, namely

\[
\sum_n \to \int d\epsilon M(\epsilon)
\]

(4.5)
with \( M(\epsilon) \) the integration measure of the transformation. Eq. 4.2 becomes

\[
\langle \phi_0, t = +\infty | \phi_0, t = -\infty \rangle = \int d\epsilon M(\epsilon) \sum_{\mathcal{C}, \mathcal{C}'} \tilde{C}_\mathcal{C}^* \tilde{C}_\mathcal{C}' \langle n\xi', t = +\infty | n\xi, t = -\infty \rangle \\
= \int d\epsilon \prod_i D[f_i] e^{iS_{\text{eff}}[f, \mu, \epsilon]} + \frac{i}{\hbar} \mathcal{S}_F[f] \eta + i \int d^4x (\sum_k J_k f_k - V_0(\mu, \epsilon))
\]

(4.6)

with \( \epsilon x \int d^4x V_0(\mu, \epsilon) \) the leading piece of the measure for the \( \epsilon \) integration in the thermodynamic limit. When the volume \( L^4 \) of the system becomes increasingly large, the integrand of the \( \epsilon \) integration becomes increasingly sharp at the extrema positions of the argument of the exponential in the above equation. This is due to the fact that \( \epsilon \) couples to macroscopic variables that are proportional to the volume; it has no quantum fluctuation in the thermodynamic limit. Due to this reason, the detailed form of the measure \( M(\epsilon) \) but its leading piece in Eq. 4.3 is irrelevant in the thermodynamic limit. So the weighted sum

\[
\sum_{\{\phi_0\}} \mathcal{W}[\phi_0, \phi_0] \langle \phi_0, t = +\infty | \phi_0, t = -\infty \rangle = \int \prod_i D[f_i] e^{iS_{\text{eff}}[f, \mu, \epsilon]} + \frac{i}{\hbar} \mathcal{S}_F[f] \eta + i \int d^4x (\sum_k J_k f_k - V_0(\mu, \epsilon))
\]

(4.7)

with \( \epsilon \) taking the value of one of the extrema of the argument of the exponential. The conventional method, which may turn out to be not sufficient in symmetry breaking phases, corresponds to a special case, namely, the \( \epsilon = 0 \) one. The relevance of introducing \( \epsilon \) will be discussed in the following sections. It can be seen that a possible finite \( \epsilon \) in the final result is perturbatively non-reachable.

2. The determination of \( V_0(\mu, \epsilon) \)

Eq. 4.7 tells us that the generating functional of an interacting fermionic system should be written as

\[
e^{\mathcal{W}[\mathcal{J}, \mathcal{S}_F, \mu, \epsilon]} = \int \prod_i D[f_i] e^{iS_{\text{eff}}[f, \mu, \epsilon]} + \frac{i}{\hbar} \mathcal{S}_F[f] \eta + i \int d^4x (\sum_k J_k f_k - V_0(\mu, \epsilon))
\]

(4.8)

It is normalized by the condition

\[
\mathcal{W}[0, 0, 0, 0] = 0.
\]

(4.9)

For a stationary situation, \( \mathcal{S}_{\text{eff}} \) corresponding to Eq. 3.10 can be expressed (see also Eq. 2.14) as

\[
\mathcal{S}_{\text{eff}}[f, \mu, \epsilon] = -i \frac{T}{2} \left( \sum_{\mathcal{C}} \int \frac{dp^0}{2\pi} \ln \lambda_{\rho^0, \xi}[f, \mu] + \sum_{\xi} \int C_\xi \frac{dp^0}{2\pi} \ln \lambda_{\rho^0, \xi}[0, \mu] - \sum_{\xi} \int C_\xi \frac{dp^0}{2\pi} \ln \lambda_{\rho^0, \xi}[0, 0] \right)
\]

\[
+ \int d^4x [\mathcal{L}_B[f] - V_0(\mu, \epsilon)],
\]

(4.10)

where \( C \) represents the \( p^0 \) integration contour shown in Fig. 3 and \( C_0 \) represents the \( p^0 \) integration contour given by Fig. 1. \( V_0(\mu, \epsilon) \) satisfies \( V_0(\mu, 0) = \int d^4x \mu \mathcal{J}^0 \), so that it agrees with Eq. 3.10 in this special case. From Appendix A, it is shown that the left hand side of the following equation is finite for an uniform system, namely,

\[
- \frac{T}{2} \left( \sum_{\xi} \int C_\xi \frac{dp^0}{2\pi} \ln \lambda_{\rho^0, \xi}[0, \mu] - \sum_{\xi} \int C_\xi \frac{dp^0}{2\pi} \ln \lambda_{\rho^0, \xi}[0, 0] \right)
\]

\[
= \int d^4x \left[ (\mu + \mathcal{P}^0_+) + \mu^- \mathcal{P}^0^- \right] - (\mathcal{C}^0_+ + \mathcal{C}^0_- - \mathcal{E})
\]

(4.11)

The basic assumption of the theory is then the following choice for \( V_0(\mu, \epsilon) \), namely,

\[
V_0(\mu, \epsilon) = \int d^4x \left( \mu + \mathcal{P}^0_+ + \mu^- \mathcal{P}^0^- \right)
\]

(4.12)
which completely specifies Eq. 4.7. For an uniform system, the corresponding effective potential in a Hartree–Fock approximation to be minimized is

\[
V_{\text{eff}} = \lim_{V_3 \to \infty} \frac{1}{V_3} \int \frac{d\rho^0}{2\pi} \ln \frac{\lambda_{\rho^0, \xi}[f; \mu]}{\lambda_{\rho^0, \xi}[0; \mu]} + \frac{n_f n_c}{4\pi^2} \left( \mu^4 + 2\epsilon^4 + 12\mu^2 \epsilon^2 \right) - \mathcal{L}_B[f]
\]

(4.13)

with \( C \) denoting the \( p^0 \) integration contour chosen.

In order to preserve the causal structure of the original Minkowski \( p^0 \) integration contour, the Euclidean effective action is obtained by distorting the Minkowski contour given in Fig. 6 to the one labeled “II” in Fig. 7.

**B. More on the primary statistical gauge field**

1. **Statistical gauge invariance, physical states and conservation of fermion number**

In the process of introducing the primary statistical gauge field \( \mu^\alpha \), the original global \( U(1) \) symmetry corresponding to the fermion number conservation is replaced, in a certain sense, by a local symmetry. This local symmetry originates from the fact that the eigenvalues \( \lambda_{\rho^0, \xi}[f; \mu] \), which satisfies the eigenequation

\[
\gamma^0 iS^{-1}_F[f; \mu] \Psi_\lambda = \lambda[f; \mu] \Psi_\lambda,
\]

(4.14)

is invariant under the following gauge transformation

\[
\Psi_\lambda(x) \to e^{i\phi(x)O_3} \Psi_\lambda(x) \quad (4.15)
\]

\[
\mu^\alpha(x) \to \mu^\alpha(x) + \partial^\alpha \phi(x) \quad (4.16)
\]

with \( \phi(x) \) an arbitrary function of the spacetime that decreases to zero sufficiently fast at the spacetime infinity.

The introduction of a local field \( \mu^\alpha(x) \) is expected to introduce infinite extra degrees of freedom, which should be eliminated in certain way. Albeit the full effective action given by Eq. 4.10, which depends on \( \mu^\alpha \mu_\alpha \geq 0 \), is not invariant under the gauge transformations given by Eqs. 4.15 and 4.16, the primary statistical gauge invariance of the quantum fluctuation or the connected part of \( \tilde{S}_{\text{eff}} \) requires further investigation.

Let us find the connection of the primary statistical gauge invariance to the conservation of fermion number by quantize the system governed by the Lagrangian density

\[
L' = L + \mu^\alpha j_\alpha
\]

(4.17)

in Eq. 4.8. Here \( L \) is the original Lagrangian density before introducing \( \mu^\alpha \) and \( j_\alpha \) is the fermion number current density given by Eq. 3.2. The fermion field \( \Psi \) and the boson fields \( \{f_i\} \) are quantized as usual [4]; they shall not be repeated here.

What is needed to be found here is the conjugate variable \( \pi_\alpha^u \) of \( \mu^\alpha \). For that purpose, Eq. 4.17 can be treated as the Hamiltonian density, namely

\[
\mathcal{H}(\mu) = L'.
\]

(4.18)

Before the quantization, it follows from the Hamiltonian dynamics that

\[
\partial_0 \pi_{ui} = -\frac{\partial \mathcal{H}(\mu)}{\partial \mu_i} = j_i,
\]

(4.19)

\[
\partial_0 \mu_i = -\frac{\partial \mathcal{H}(\mu)}{\partial \pi_{ui}} = 0,
\]

(4.20)

(4.21)

where \( i, j = 1, 2, 3 \) labels the spatial components of 4-vectors.

The quantization of \( \mu_i \) is then implemented by the Dirac quantization condition

\[
[\pi_{ui}(\mathbf{x}, t), \bar{\mu}_j(\mathbf{x}', t)] = -i\delta^{(3)}(\mathbf{x} - \mathbf{x}') \delta_{ij}.
\]

(4.22)

The statistical “electric field” \( \pi_{ui} \) commutes with all elementary fields in the Lagrangian density but the one listed above at equal-time. Note that all quantities with a hat “\(^\wedge\)" on top denote operators in the following.
After the quantization, the set of gauge transformation, in which $\phi(x)$ is independent of time, is represented by
\begin{equation}
\hat{\Psi} \rightarrow U[\phi]\hat{\Psi}U^\dagger[\phi] = e^{i\phi_0}\hat{\Psi}
\end{equation}
\begin{equation}
\hat{\mu}_\alpha \rightarrow U[\phi]\hat{\mu}_\alpha U^\dagger[\phi] = \hat{\mu}_\alpha - \partial_\alpha \phi
\end{equation}
with
\begin{equation}
U[\phi] = e^{-i\int d^3x (\hat{\rho} + \nabla \cdot \hat{\pi}_u) \phi},
\end{equation}
where the space integration is at any specific time at which a transformation of an operator is considered. The operator algebra between dynamical fields is then represented in a Hilbert space. Since the quantity $\hat{\rho}$ is introduced into the original theory, it is expected that this Hilbert space contains states that are not physical or are redundant. The physical states are selected within the full Hilbert space by requiring that they satisfy
\begin{equation}
\langle \text{Phys}' | U[\phi] \rangle \text{ Phys} = Z e^{-i\phi}
\end{equation}
under the time independent gauge transformation discussed above, with the common phase factor $\Phi$ restricted to those functions that are independent of time (it is explained in the following). Taking the time derivative of Eq. 4.26, and using dynamical equation Eq. 4.13, one obtains
\begin{equation}
\langle \text{Phys}' | (\partial_0 \hat{\rho} + \nabla \cdot \hat{j}) \rangle \text{ Phys} = \langle \text{Phys}' | \partial_0 \hat{\rho}_\mu \rangle \text{ Phys} = 0
\end{equation}
which is the conservation of fermion number current in physical processes.

The time independent and spatially localized gauge transformation considered is non-trivial one. It selects amongst those states in the extended Hilbert space the physical ones. This can be understood if one consider the commutation relation between $\hat{\rho} + \nabla \cdot \hat{\pi}_u$ and the Hamiltonian of the system.
\begin{equation}
[\hat{\rho} + \nabla \cdot \hat{\pi}_u, \hat{H}] = i \frac{d}{dt} (\hat{\rho} + \nabla \cdot \hat{\pi}_u) = \partial_\mu \hat{j}_\mu
\end{equation}
with $\hat{H}$ the total Hamiltonian of the system. It is zero due to the conservation of fermion number current. It means that $\hat{\rho} + \nabla \cdot \hat{\pi}_u$ is independent of time when the matrix elements between physical states are taken. The states in the extended Hilbert space can be divided into subspaces labeled by a complex (time independent) function of the spatial coordinates according to the matrix elements of $\hat{\rho} + \nabla \cdot \hat{\pi}_u$ between themselves. For those eigenstates of the Hamiltonian of the system, it can be written as
\begin{equation}
\langle \phi^k_i | (\hat{\rho} + \nabla \cdot \hat{\pi}_u) \rangle \phi^l_j = \delta_{E_iE_j}N_{ij}\varsigma
\end{equation}
with $\varsigma$ the space dependent complex function, $| \phi^k_i \rangle$ a state in the physical space that has energy $E_k$, $\delta_{E_iE_j}$ taking zero or unity value if $E \neq E'$ or $E' = E$ (assuming that $E$ is discrete before thermodynamic limit is taken) and $N_{ij}$ independent of spacetime coordinates. Eqs. 4.28 and 4.29 mean that the physical states are the ones that have vanishing matrix elements on the commutator of the Hamiltonian and $\hat{\rho} + \nabla \cdot \hat{\pi}_u$. Therefore they are expressible by a superselection sector in the extended Hilbert space defined and labeled by a complex function $\varsigma$ of spatial coordinates. Such a definition of the physical states for the primary statistical gauge theory is less restrictive than the one used in dynamical gauge theory like QED [2] where due to the existence of the dynamical part for the gauge fields at the tree level, the physical states are restricted to the subspace with $\varsigma \equiv 0$ only. In fact, for the $\beta$ and $\omega$ phases discussed in the following, in which the $U[\phi]$ is a time independent gauge transformation, the fermion number conservation is spontaneously broken down, $\varsigma$ can not be zero due to the fact that before taking into account of the dynamical gauge fields (that of the photon), the massless Goldstone boson corresponding to the spontaneous symmetry breaking has to be considered as a physical excitation. But if $\varsigma$ is chosen to be zero, the massless Goldstone boson belongs to unphysical states [2]. Such a situation actually opens up the possibility for the spontaneous CP violation to be discussed in the following. The choices made for the physical states is a constraint invariant under time evolution due to Eq. 4.28. It shows that definition for physical states remains true at all times and no transition to unphysical states and between the superselection sectors is possible in physical processes.

---

2It is the quantum version of the “classical” constraint equation $\rho + \nabla \cdot \pi_u = \varsigma$.
3Eq. 4.28 may not be restrictive enough. Somewhat more restrictive constraint can be suggested. It consists of decomposing $\hat{\rho} + \nabla \cdot \hat{\pi}_u$ into superposition of holomorphic $\phi^k_i(\hat{\rho} + \nabla \cdot \hat{\pi}_u)_{(-)}$ and antiholomorphic $\phi^k_i(\hat{\rho} + \nabla \cdot \hat{\pi}_u)_{(+)},$components [2]. The physical states are those ones that are eigenstates of $\phi^k_i(\hat{\rho} + \nabla \cdot \hat{\pi}_u)_{(-)}$ with a common eigenvalue $\varsigma$. 

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The primary statistical gauge field $\mu^\alpha$ is non-dynamical at the tree level. At the quantum level a dynamics for $\mu^\alpha$ is generated due to the fermion quantum fluctuations. The relevant effective action for the primary statistical gauge field can be obtained from Eq. 4.10 by a “Legendre transformation” to a form without the contribution of $V_0$, namely

$$S_{\text{eff}}[f, \mu, \epsilon] = -i \frac{T}{2} \left( \sum_\xi \int \frac{dp_0}{2\pi} \ln \lambda_{\rho^\alpha, \xi}[f, \mu] + \sum_\xi \int \frac{dp_0}{2\pi} \ln \lambda_{\rho^\alpha, \xi}[0, \mu] - \int \frac{dp_0}{2\pi} \ln \lambda_{\rho^\alpha, \xi}[0, 0] \right) + \int d^4x \mathcal{L}_B[f],$$

(4.30)

which is a canonic functional of $\mu^\alpha$. The quadratic term for slow varying $\mu^\alpha$ (in spacetime or at long distances) generated from the fermion determinant is of the form

$$S_{\text{eff}}^{(\mu)} = \frac{1}{2} \int d^4x d^4x' \mu_\alpha(x) \pi^{\alpha\beta}(x, x') \mu_\beta(x') + \frac{n_f\epsilon_c}{\pi^2} \int d^4x x^2 \mu^2,$$

(4.31)

where $\mu'_\alpha = \mu_\alpha - \overline{\mu}_\alpha$ with $\overline{\mu}_\alpha$ shifted $\mu_\alpha$ and the last term is from the corresponding one in Eq. 4.30. The first term is generated from the fermion determinant. If the electromagnetic interaction between the fermions are not considered, $\pi^{\alpha\beta}(x, x')$ is given by

$$\pi^{\alpha\beta}_0(x, x') = i\langle 0 | \{ T j^\alpha(x) j^\beta(x') \} | 0 \rangle$$

$$= Z^{(\mu)}(\partial^2_\xi g^{\alpha\beta} - \partial^\alpha \partial^\beta \delta(x - x') + i\langle 0 | j^\alpha(x) | 0 \rangle \langle 0 | j^\beta(x') | 0 \rangle$$

(4.32)

in the normal phase; and,

$$\pi^{\alpha\beta}_0(x, x') = i\langle 0 | \{ T j^\alpha(x) j^\beta(x') \} | 0 \rangle$$

$$= \Pi^{(\mu)}(g^{\alpha\beta} - \partial^\alpha \partial^\beta \Pi^{(\mu)}) \delta(x - x') + i\langle 0 | j^\alpha(x) | 0 \rangle \langle 0 | j^\beta(x') | 0 \rangle,$$

(4.33)

in the phase where the $U(1)$ symmetry corresponding to the fermion number conservation is spontaneously broken down since there exists a massless pole in the matrix element of $j^\alpha$ (see Ref. 8 for a more detailed discussion). Here $Z^{(\mu)}$ and $\Pi^{(\mu)}$ are functions of $x$ and $x'$.

In the normal phase, the effective action for slow varying $\mu'_\alpha$ is

$$S_{\text{eff}}^{(\mu)} = \frac{1}{2} \int d^4x \left[ -\frac{Z^{(\mu)}}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2} \left( i\langle 0 | j^\alpha | 0 \rangle \langle 0 | j^\beta | 0 \rangle + 2g^{\alpha\beta} \frac{n_f\epsilon_c}{\pi^2} \epsilon^2 \right) \mu'_\alpha \mu'_\beta \right]$$

(4.34)

with $f^{\alpha\beta} = \partial^\alpha \mu^\beta - \partial^\beta \mu^\alpha$, since the eigenvalues $\lambda_{\rho^\alpha, \xi}$ are invariant under the gauge transformation given by Eqs. 4.12 and 4.16. In the phase where $U(1)$ is spontaneously broken down and before considering electromagnetic interaction,

$$S_{\text{eff}}^{(\mu)} = \frac{1}{2} \int d^4x \left\{ i\langle 0 | j^\alpha | 0 \rangle \langle 0 | j^\beta | 0 \rangle + g^{\alpha\beta} \left( \Pi^{(\mu)} + 2 \frac{n_f\epsilon_c}{\pi^2} \epsilon^2 \right) \mu'_\alpha \mu'_\beta + \ldots \right\}.$$

(4.35)

In both of the situations with slow varying $\mu'_\alpha$, $Z^{(\mu)}$ and $\Pi^{(\mu)}$ are approximately constants.

If the electromagnetic interaction is considered, then there is a $f_{\alpha\beta} f^{\alpha\beta}$ term in Eq. 4.33 and the $\Pi^{(\mu)}$ term is absent, even in a spontaneous $U(1)$ symmetry breaking phase. This is because when the electromagnetic interaction between the fermions are considered, $\pi^{\alpha\beta}$ can be decomposed into a connected and disconnected part or $\pi^{(c)}_{\mu\nu} + i\langle 0 | j_\mu | 0 \rangle \langle 0 | j_\nu | 0 \rangle$ with the connected part given by

$$\pi^{(c)}_{\mu\nu} = \pi^{(c)}_{0\mu\nu} + \left( \pi^{(c)}_{0} G_0 \pi^{(c)}_{0} \right)_{\mu\nu} + \left( \pi^{(c)}_{0} G_0 \pi^{(c)}_{0} G_0 \pi^{(c)}_{0} \right)_{\mu\nu} + \ldots$$

(4.36)

and

$$\pi^{(c)}_{0\mu\nu} = (q^2 g_{\mu\nu} - q_\mu q_\nu) \pi^{(c)}_{0},$$

$$G_0^{\mu\nu} = \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) \frac{i}{q^2},$$

(4.37)
where $G_{0}^{\mu\nu}$ is the propagator of the bare photon. $\pi^{(c)}_{0\mu\nu} \to 0$ in the $q^{\mu} \to 0$ limit even when $\pi^{(c)}_{0\mu\nu}$ contains a massless pole (in $q^{2}$). Whatever the case, it can be easily seen that the full two point proper vertex for $\mu^{\prime}_{\alpha}$ is non-vanishing in the $q^{\mu} \to 0$ limit if either $\epsilon \neq 0$ or $\langle 0|j_{\mu}|0 \rangle \neq 0$ or both in the vacuum state of the system. Therefore a long range order for $\mu^{\prime}_{0}$ is possible only if both $\epsilon = 0$ and $\mu = 0$ in the vacuum. The spatial component $\mu^{\prime}_{\alpha}$ of $\mu^{\prime}_{\alpha}$ is short ranged in the phase where $\epsilon \neq 0$ and $\mu = 0$. It is however long ranged in the phase where $\epsilon = 0$.

3. Topological configurations

The discussion given above shows that after the introduction of a primary statistical gauge field $\mu_{\alpha}$, the representation Hilbert space of the operator algebra is extended. Such an extended Hilbert space includes not only the physical states, but also non-physical ones. The physical states are the ones that satisfy Eq. 4.29, which is expected to be satisfied in all consistent computations of physical observable where fermion number current conservation is preserved.

The extension of the representing Hilbert space gives us additional leverage to project out the collective excitations and configurations not easily discernible in the conventional approach.

Due to the statistical gauge invariance under the local transformation given by Eqs. 4.15 and 4.16, the physically non-trivial configurations of the system depend, in the path integration sense, only on the flux density or statistical “magnetic field” $b$ defined as

$$b = \nabla \times \mu.$$

Therefore the general form of the generating functional Eq. 4.3 can be written more precisely by imposing certain gauge fixing condition or as

$$e^{W[f,J,\eta,\eta,\mu,\epsilon]} = \int D[b] J(b) \prod_{i} D[f] e^{\frac{i}{\hbar} S_{eff}[f,\mu,\epsilon]} + \frac{1}{2} \eta S_{F}[f,\eta] + \frac{i}{\hbar} \int d^{4}x \left( \sum_{k} J_{k} f_{k} - V_{0}(\mu,\epsilon) \right),$$

where $J(b)$ is the integration measure. Eq. 4.39 is equivalent to Eq. 4.8. It is however useful for us to find out collective stationary configurations that are not easily found in a common approach to the effective action.

For a configuration in which $b$ is non-vanishing only in a localized region, quantization of flux appear, namely

$$\int_{\Sigma} dS \cdot b = \oint_{\partial\Sigma} \cdot d\mu = 2n\pi, \quad (n = 0, \pm 1, \pm 2, \ldots),$$

where $\Sigma$ is the surface that contains the $b$ and line integration is around the edge $\partial\Sigma$ of $\Sigma$. The quantization results, as it is well known, by imposing the uniqueness condition on the eigenfunctions $\Psi_{\lambda}[f,\mu]$.

4. A new macroscopic parameter and long range order

Using the primary statistical gauge field $\mu_{\alpha}$, a new macroscopic parameter that characterizes the vacuum state of the system can be introduced. It is defined as

$$\hat{O}_{\Sigma} = e^{i \oint_{\Sigma} \cdot d\mu},$$

where $\Sigma$ is a large 2-dimensional surface area in space at certain time and the line integration $\oint_{\Sigma}$ is along the edge of the area $\Sigma$.

It is known [23] that the vacuum expectation value of $\hat{O}_{\Sigma}$ provides another one of the macroscopic parameters for a more detailed characterization of the phase structure of the system. For example, in a disordered system in which the correlation between the complex phase of the fermions at different space points becomes short ranged, condensation of vortices or monopoles of the type of configurations with non-vanishing $n$ in Eq. 4.40 can derive a Kosterlitz–Thouless [24] type of phase transition.
V. TWO MODELS FOR STRONG INTERACTION AND THEIR VACUUM PHASE STRUCTURE

The fundamental theory for the strong interaction is considered to be QCD with its Lagrangian density given in Appendix III. The current available method of studying QCD from first principle is the lattice QCD simulation. Albeit great progresses are made, the lattice computation are limited by the small lattice sizes and by the limitation in the computer power. Model approaches, which are simpler than the full QCD calculation and has large overlap with it in the low and intermediate energy regions can be and have been used to obtain much of the physical pictures that are supposed to happen in the system governed by QCD Lagrangian density. Since the mass of the light quarks have values much smaller than the typical scale of the hadronic spectrum of order 1 GeV, certain subset of the behaviors of the light quark system can be simulated by an interacting massless fermion systems that possesses the basic symmetries of the QCD Lagrangian density. It is expected that we can learn some of the important possible behaviors of the light quark system by using models due to our (relatively) increased theoretical analytic power. The spontaneous chiral symmetry breaking down in hadronic systems was historically discussed before the birth of the conception of quark and QCD Lagrangian density. This phenomenon is only latter justified by the QCD (lattice) calculation.

I discuss in this paper the physical properties of strong interaction vacuum related to the fluctuation of baryon number using two model Lagrangian densities that possess the chiral SU(2)$_L \times$ SU(2)$_R$ symmetry of massless QCD and has 3 colors (n$_c = 3$). For a full investigation of the possible phases of the vacuum of a relativistic massless fermion system, these models are so chosen that they allow not only the quark–antiquark condensation that is widely discussed in the literature but also the rarely studied phases that are induced by a condensation of quark–quark (or antiquark–antiquark) pairs. One of these possibilities is studied in detail in Ref. [7,8].

In order to simplify our discussion, I consider two model Lagrangian densities that are half bosonized. Both of them can be viewed as been the descendents of two four quark interaction models after introducing the auxiliary fields in a Fierz invariant way [8].

This section, which mainly serves the purpose of introducing the models, contains the determination of their vacuum phase structure using the conventional approaches. A more detailed study using the refined method developed in sections III and IV is given in the next section.

A. Model I and the $\omega$ phase

The first model is defined by the following Lagrangian density

$$L_1 = \frac{1}{2} \bar{\Psi} \left[i\partial - \sigma - i\vec{\pi} \cdot \gamma \bar{\lambda} O_3 \gamma \chi^c \gamma_0 O(+(-)) - \gamma \chi \gamma_0 \chi \right] \Psi - \frac{1}{4G_0} (\sigma^2 + \vec{\pi}^2) + \frac{1}{2G_3'} \chi^c \chi^c,$$

(5.1)

where $\sigma$, $\vec{\pi}$, and $\chi^c$ are auxiliary fields with $(\chi^c)\dagger = -\chi^c$ and $G_0$, $G_3'$ are coupling constants of the model. $\mathcal{A}_c$ and $\mathcal{A}_c^\prime$ ($c = 1,2,3$) act on the color space of the quark; they are

$$\mathcal{A}_{c_1c_2} = -\epsilon^{c_1c_2}, \quad \mathcal{A}_{c_1c_2} = \epsilon^{c_1c_2},$$

(5.2)

with $\epsilon^{abc}$ ($a,b,c = 1,2,3$) the total antisymmetric Levi–Civitá tensor. Here the quark spinor is represented by the 8-dimensional Dirac spinor and $O_{(\pm)}$ are raising and lowering operators respectively in the upper and lowering 4 components of $\Psi$.

The effective action is given by Eq. 2.14 with the auxiliary fields independent of spacetime, the effective potential has the following form

$$V_{eff} = -\lim_{L,T \to \infty} \frac{1}{L^3T} S_{eff}$$

$$= \frac{1}{2} \lim_{L,T \to \infty} \frac{1}{L^3T} \sum_{\lambda_n} \ln \lambda_n + \frac{1}{4G_0} \sigma^2 - \frac{1}{2G_3'} \chi^c \chi^c,$$

(5.3)

where $\lambda_n$ and $\lambda^{(0)}_n$ correspond to the eigenvalues of the two Hermitian operators defined in Eq. 2.13 with and without the auxiliary fields shifted respectively. Since the auxiliary fields do not depend on spacetime, the eigenvalues $\lambda_n$ and $\lambda^{(0)}_n$ can be labeled by the 4-momentum of the corresponding eigenstates $\Psi_{\lambda_n}$. The result is

$$V_{eff} = \frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \left( 8 \sum_{i=1}^{4} \ln \frac{\lambda_i(p)}{\lambda_i^{(0)}(p)} + 4 \sum_{i=1}^{4} \ln \frac{\lambda_i'(p)}{\lambda_i^{(0)}(p)} + \frac{1}{4G_0} \sigma^2 - \frac{1}{2G_3'} \chi^c \chi^c, \right.

(5.4)$$
where $\lambda_i(p)$, $\lambda'_i(p)$ are eigenvalues of states with color different, the same as $\chi^c$ respectively and factors 8, 4 correspond to their degeneracy. It has the following explicit form

$$V_{\text{eff}} = 4i \int \frac{d^4p}{(2\pi)^4} \ln \left[ \left( 1 - \frac{\sigma^2 + \chi^2}{p^2} \right)^2 - \frac{\sigma^2}{p^2} \left( 1 - \frac{\sigma^2 - \chi^2}{p^2} \right)^2 \right] + \frac{1}{4G_0} \sigma^2 + \frac{1}{2G_3} \chi^2, \quad (5.5)$$

where $\chi^2 \equiv -\nabla \chi^c$.

In the Minkowski spacetime formulation, the contour for the $p^0$ integration is shown in Fig. 3. One of the most commonly used ones, which is called quasiparticle path, is shown in Fig. 2. It can be shown that the resulting effective potential is the change of the energy density of the quasiparticles, which is obtained from summing over the quasiparticle energies determined by the poles of $S_F[f]$, relative to that of the bare particle in the truncated Dirac sea. It does not have a covariant form due to the fact that a non-covariant cutoff in the 3-momentum space has to be introduced. The Euclidean path shown in Fig. 2 can be used to obtain a covariant expression for the effective action. The Euclidean covariant cutoff scheme does not result in deriving physically undesirable results in other covariant approaches [2]. As discussed in section 2.2, it can also be used to include quantum effects (see Appendix B) that are beyond the quasiparticle approximation. The differences between these two paths are however not very important for the purpose of this subsection. Since different kind of cutoffs are used in these two approaches, a comparison between them is difficult. Nevertheless, I shall adopt the Euclidean path shown in Fig. 2. An Euclidean path is however important for discussions to be followed. The resulting expression is

$$V_{\text{eff}}(\sigma, \chi) = -4 \int \frac{d^4p}{(2\pi)^4} \ln \left[ \left( 1 + \frac{\sigma^2 + \chi^2}{p^2} \right)^2 + \frac{\sigma^2}{p^2} \left( 1 + \frac{\sigma^2 - \chi^2}{p^2} \right)^2 \right] + \frac{1}{4G_0} \sigma^2 + \frac{1}{2G_3} \chi^2, \quad (5.6)$$

where $\Lambda$ is the Euclidean cutoff introduced to define the model. A numerical evaluation shows that the minima of $V_{\text{eff}}(\sigma, \chi)$ is located on either the $\sigma$ axis ($\chi = 0$) or the $\chi$ axis ($\sigma = 0$). Explicit expression for $V_{\text{eff}}(\sigma, 0)$ and $V_{\text{eff}}(0, \chi)$ are found to be

$$v_{\text{eff}}(\sigma, 0) = 3f(\frac{\sigma^2}{\Lambda^2}) + \frac{1}{16\pi\alpha_0} \frac{\sigma^2}{\Lambda^2}, \quad (5.7)$$

$$v_{\text{eff}}(0, \chi) = 2f(\frac{\chi^2}{\Lambda^2}) + \frac{1}{16\pi\alpha_3} \frac{\chi^2}{\Lambda^2}, \quad (5.8)$$

where the dimensionless effective potential $v_{\text{eff}}$ is defined by $V_{\text{eff}} \equiv \Lambda^4 v_{\text{eff}}$, $\alpha_0 = G_0\Lambda^2/4\pi$ and $\alpha_3 = G_3\Lambda^2/8\pi$ with

$$f(x) = \frac{1}{8\pi^2} \left[ -x + \ln \left( 1 + \frac{1}{x} \right) x^2 - \ln(1 + x) \right]. \quad (5.9)$$

The values of $\sigma^2$ and $\chi^2$ at the minima of Eqs. (5.7) and (5.8) determine the vacuum of the system in the one loop Hartree–Fock approximation for the fermions (i.e. Eq. (2.1) without the second bosonic one loop term). The phase structure of the model is shown in the $\alpha_0$-$\alpha_3^\gamma$ plane in Fig. 1. Three kinds of phases for the vacuum are possible. The first phase, which is called the $O$ phase, is the bare vacuum. The second phase, or the $\alpha$ phase, has non-vanishing vacuum expectation value of $\psi \bar{\psi}$; the chiral $SU(2)_L \times SU(2)_R$ symmetry is spontaneously broken down to a $SU(2)_V$ flavor symmetry. The third phase, called the $\omega$ phase of the vacuum, has non-vanishing diquark and antidiquark condensation characterized by a non-vanishing $\chi^2$; chiral symmetry is unbroken in this phase.

The phase transitions across the boundary between the $O$ and the $\alpha$ phases ($\alpha_0 = \pi/12$ and $\alpha_3^\gamma < \pi/8$) and the one between the $O$ and the $\omega$ phases ($\alpha_0 < \pi/12$ and $\alpha_3^\gamma = \pi/8$) are of second order. The phase transition between the $\alpha$ and the $\omega$ phases ($\alpha_0 > \pi/12$ and $\alpha_3^\gamma > \pi/8$) is of first order. The Meissner effects for the electromagnetic field are expected in the $\omega$ phase. The basic physics of it is discussed in more detail in Refs. [7,8,13] for model II of the following. I shall relegate such a discussion for this model to other work.

The Minkowski propagator for the quarks in the $\omega$ phase can be found by an inversion of the operator $S_F^{-1}[f]$ in the action, namely, the $\chi^c$ component

$$S_F = i [i\partial + \gamma^5 A \nabla] \chi^c O_{(+)} - \gamma^5 A^\mu \nabla_{(\mu} O_{(-)} ]^{-1}. \quad (5.10)$$

In the momentum space, it can be expressed explicitly as
where a quark of “type I” is the one that has color different from $\chi^c$ (or $\overline{\chi_c}$) and a quark of “type II” is the one that has the same color as $\chi^c$ (or conjugate to that of $\chi^c$). It can be seen that a quark of type II has no gap for its excitation and an excitation of a quark of type I has a finite gap $\sqrt{\chi^2}$.

B. Model II and the $\beta$ phase

The second model Lagrangian density is

$$L_2 = \frac{1}{2\beta} \left[ -i\theta - \sigma - i\tilde{\sigma} \gamma^5 O_3 + \left( \phi_{\mu \nu}^c \gamma^\mu \gamma^5 A_c + \tilde{\phi}_{\mu \nu}^c \gamma^\mu \tilde{A}_c \right) O_{\alpha} - \left( \overline{\phi}_{\mu \nu}^c \gamma^\mu \gamma^5 A^c + \overline{\tilde{\phi}}_{\mu \nu}^c \gamma^\mu \tilde{A}^c \right) O_{\alpha} \right] \Psi,$$

$$- \frac{1}{4G_0} \left( \frac{1}{2G_3} \overline{\phi}_{\mu \nu}^c \phi_{\mu \nu}^c + \frac{1}{2G_3} \overline{\tilde{\phi}}_{\mu \nu}^c \tilde{\phi}_{\mu \nu}^c \right),$$

(5.12)

where $\sigma$, $\tilde{\sigma}$, $\phi_{\mu \nu}^c$, $\overline{\phi}_{\mu \nu}^c$, $\tilde{\phi}_{\mu \nu}^c$, and $\overline{\tilde{\phi}}_{\mu \nu}^c$ are auxiliary fields. It is symmetric under the chiral $SU(2)_L \times SU(2)_R$ group transformation. This model is discussed in detail in Refs. [7,8]. It is written down here for references in the later sections. The Euclidean effective potential for this model used for a determination of the vacuum phase structure in a Hartree–Fock approximation is found to be

$$V_{\text{eff}} = -\frac{1}{2} \int \frac{d^4p}{(2\pi)^{4}} \left[ \sum_{i=1}^{4} p_i \left( \lambda_i(p) \right) \lambda_i^{(0)}(p) + \sum_{i=1}^{4} \lambda_i^{(0)}(p) \lambda_i(p) \right] + \frac{1}{4G_0} \sigma^2 + \frac{1}{2G_3} \phi_{\mu \nu}^c \phi_{\mu \nu}^c \right]$$

$$+ \frac{1}{4G_0} \sigma^2 + \frac{1}{2G_3} \phi_{\mu \nu}^c \phi_{\mu \nu}^c,$$

(5.13)

A numerical evaluation of Eq. (5.13) shows that the absolute minimum of $V_{\text{eff}}(\sigma^2, \phi^2)$ is located at either $\sigma^2 \neq 0$ and $\phi^2 = 0$ or $\sigma^2 = 0$ and $\phi^2 \neq 0$ in the spontaneous symmetry breaking phases. The phase diagram for this model is presented in Fig. 5. Three kinds of phases for the vacuum are possible. The first phase, which is identical to the $O$ phase discussed above, is the bare vacuum. The second phase, which is the same as the $\alpha$ phase introduced above, has non-vanishing vacuum expectation value of $\Psi\overline{\Psi}$; the chiral $SU(2)_L \times SU(2)_R$ symmetry is spontaneously broken down to a $SU(2)_V$ flavor symmetry in this phase. The third phase is labeled as the $\beta$ phase. It has non-vanishing diquark and antidiquark condensation characterized by a non-vanishing $\phi^2$; the chiral symmetry is spontaneously broken down to a flavor symmetry, in the same way as in the $\alpha$ phase.

The phase transition is of second order across $O$ phase and the $\alpha$ phase boundary ($\alpha_0 = \pi/12$ and $\alpha_3 < \pi/4$). It is first order phase transition across the $\alpha$ and $\beta$ phase boundary ($\alpha_0 > \pi/12$ and $\alpha_3 > \pi/4$). There is a second order phase transition across the $O$ and $\beta$ phase boundary ($\alpha_0 < \pi/12$ and $\alpha_3 = \pi/4$).

In the $\beta$ phase, the propagators for the quarks are found [9] to be

$$S_F(p) = \begin{cases} \frac{1}{(1 - iO_2 \frac{\hat{\phi} \cdot \gamma^5 A_c}{p^2 - \phi^2}) \left( \frac{p^2 - \phi^2}{p^2} - 2(p \cdot \phi)^2 \right)} & \text{for quark of type I} \\ \frac{i \hat{\phi}}{p^2} & \text{for quark of type II} \end{cases},$$

(5.14)

where $\phi^2 = \overline{\phi}_{\mu \nu} \phi_{\mu \nu} = -\phi_{\mu \nu}^c \phi_{\mu \nu}^c$. In order to simplify the computation, a special choice for the complex phase of the non-vanishing auxiliary fields $\phi_{\mu}$ and $\overline{\phi}_{\mu}$ is made, namely, $\overline{\phi}_{\mu} = -\phi_{\mu}^c$. 

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VI. THE VACUUM STRUCTURE OF THE $\alpha$, $\beta$ AND $\omega$ PHASES

The possible phases of two model Lagrangian densities are studied in the previous section using the conventional approach to effective potential in the Hartree–Fock approximation. A more detailed characterization of these vacua and their properties is given in the following using the general framework developed in section [V].

A. The $\alpha$ phase

The effective potential for the $\alpha$ phase in the full theory given in section [V] can be obtained by computing the right hand side (r.h.s.) of Eq. 3.14 using the $p^0$ integration contour $C$ in Fig. 8, namely

$$V_{\text{eff}}(\mu, \epsilon) = 6i \int_C \frac{d^4p}{(2\pi)^4} \ln \left( 1 - \frac{\sigma^2}{p^2} - 1 - \frac{\sigma^2}{p^2} \right) + \frac{1}{4G_0} \sigma^2 + \frac{3}{2\pi^2} \mu^4. \quad (6.1)$$

The effective potential has only a trivial minimum located at $\mu = \epsilon = 0$ when the integration contour is chosen to be the quasiparticle one shown in Fig. 8. This contour, which selects only the quasiparticle contributions, violate the conservation of baryon number explicitly; it is demonstrated in section [II]. It is necessary to turn the contour $C$ into the complex plane to find the Euclidean stationary points of the field configurations in a way that preserves the causal structure of Fig. 8. Such a contour is also displayed in Fig. 8.

A numerical study shows that the minima of $V_{\text{eff}}$ for a given value of non-zero $\sigma$ is located either at $\mu \neq 0$ and $\epsilon = 0$ or at $\mu = 0$ and $\epsilon \neq 0$. Fig. 8 shows $V_{\text{eff}}$ along three different directions in the $\mu - \epsilon$ plane. The absolute minimum of $V_{\text{eff}}$ is located at $\epsilon = \epsilon_0 \neq 0$ and $\mu = 0$. This is the result I regard as natural since it avoids the not observed CP violation associated with the $\alpha$ phase found in section [V]. It also agrees with the physical picture for the $\alpha$ phase, which is condensed with correlating quark– antiquark pairs.

With a non-vanishing $\epsilon$, let us first assess the nature of the primary statistical gauge field excitation. The effective action for the primary statistical gauge field $\mu'^\alpha$ at long distances is given by (see Eq. 4.34)

$$S_{\text{eff}}^{(\mu)} = \int d^4x \left[ -\frac{Z(\mu)}{4} f_{\mu\nu} f^{\mu\nu} + \left( \frac{6}{\pi^2} \epsilon^2 \right) [\mu'^\alpha]^2 \right]. \quad (6.2)$$

Since $\epsilon^2 \neq 0$ in the $\alpha$ phase, the $\mu'_\alpha$ excitation is massive (or short ranged) in the static limit and is now stable against quantum fluctuations. This agrees also with our observations since no corresponding long range force and large CP violation are observed at the present-time condition.

Due to the presence of a finite $\epsilon$, the response of the system to external (or internal) excitations is different from the familiar one we have learned from the conventional approaches. When Dirac’s view for the bare vacuum of fermions is taken, namely, the bare vacuum corresponds to a state in which the negative energy states are filled and the positive states empty, the vacuum where $\epsilon \neq 0$ is the state in which all single particle states with energy $E \leq -\epsilon$ and $0 \leq E < \epsilon$ are filled whereas other single particle states empty. Since for an uniform system, the value of the mass $\sigma$ for the quasiparticle is always larger than $\epsilon$ in the models considered, the presence of a finite $\epsilon$ in the vacuum of the $\alpha$ phase appears to have no effects if the quasiparticle can propagate long enough without suffering from further scattering. The presence of $\epsilon$ only provides a virtual possibility for an uniform system that the local fluctuations of the fields can feel. So, it has genuine physical effects on local observables even in uniform systems according to the discussion in Appendix [V].

For a non-uniform system in which the energy of the operator in Eq. 3.9 could be smaller than $\epsilon$, the presence of $\epsilon$ may have a real effect on the dynamical processes. For example, in a chiral soliton in which the energy of the lowest orbits for the valence fermions moves with the size of the soliton, the presence of $\epsilon$ will limit the range of change of the soliton’s possible size which gives an extra stability of such solitons.

To put the above argumentation in a more concrete context, let us consider a situation in which the lowest energy valence fermion state lies within the region $-\epsilon \leq E < 0$ for one size and shape, then a change in its size or shape that moves $E$ upward can be continued freely only until $E = 0$ since the $0 \leq E < \epsilon$ states are filled and the next available state is the one with energy $E = \epsilon$, which can only be reached by a discontinuous change in the size and shape of the soliton. If the nucleon can be regarded as a chiral soliton, this mechanism can prevent it from dissolving inside a nucleus if the lowest energy valence (constituent) quark states inside the nucleon lies between $(-\epsilon, 0)$. Other implications of a non-vanishing $\epsilon$ are worth to be studied in future works.

Perhaps other interesting implications of a non-vanishing $\epsilon$ are on the particle production and dissipation processes in non-equilibrium situations like the heavy ion collision.
B. The $\beta$ and $\omega$ phases

According to the physical picture discussed so far, the baryon number content of the $\beta$ and $\omega$ phases ought to be different from the $\alpha$ phase due to the fact that, instead of quark and antiquark pairs, quark pairs and antiquark pairs (or diquark and antidiquark) are condensed in the vacuum. This is reflected in the fact that in these two vacua, the expectation value of $\chi^{c}$ (together with its conjugate field $\chi^{c}$) and $\phi_{c}^{a}$ (together with its conjugate field $\overline{\phi}_{c}^{a}$) are non-vanishing.

In the $\beta$ and $\omega$ phases, where $\sigma = 0$, the effective potentials for model I and II are found, using Eq. (4.13) to be of the following forms

$$V_{\text{eff}} = -\frac{4}{(2\pi)^{4}} \int d^{4}p \left( 1 + \frac{\chi^{4} - 2\chi^{2} p_{+} \cdot p_{-}}{p_{+}^{2} p_{-}^{2}} \right) + \frac{1}{2G} \chi^{2} + \frac{3}{2\pi^{2}} (\mu^{4} + 2\epsilon^{4} + 12\mu^{2}\epsilon^{2})$$

For Model I (6.3)

$$V_{\text{eff}} = -\frac{4}{(2\pi)^{4}} \int d^{4}p \left( 1 + \phi^{4} - 2\phi^{2} p_{+} \cdot p_{-} \right) - \frac{4}{p_{+}^{2} p_{-}^{2}} - \frac{1}{2G} \phi^{2} + \frac{3}{2\pi^{2}} (\mu^{4} + 2\epsilon^{4} + 12\mu^{2}\epsilon^{2})$$

For Model II (6.4)

Numerical evaluations show that the local minima of the above effective potentials are located either at $\mu \neq 0$ and $\epsilon = 0$ or $\mu = 0$ and $\epsilon \neq 0$. The corresponding $V_{\text{eff}}$ in the $\mu\epsilon$ plane along three different directions are plotted in Figs. 6 and 10 respectively. The absolute minima of both effective potential are located at $\mu \neq 0$ and $\epsilon = 0$. That is, in the $\beta$ and $\omega$ phases where quark pair or diquark condense, the vacua of the systems are the ones with finite density of baryons with the baryon density given by Eq. (5.6).

Such a phase also spontaneously violates the CP invariance of the system’s original Lagrangian density due to the existence of a non-vanishing CP odd order parameter $\mu^{a}$ in these phases. In addition, a pattern in which baryonic matter and antimatter are separated in space for the $\beta$ and $\omega$ phases of the vacuum are energetically favored. The superselection sector in the Hilbert space in which the physical states stays in for such a CP violating phase can then be determined. Assuming that the statistical “electric” field $\pi_{a}$ is finite, then Eq. (4.23) tells us that $\zeta = \pi$ when the spacetime approaches infinity. Due to the translational invariance of the vacuum state, it is natural to require that $\zeta = \pi$ at all spacetime position. In this way the physical states labeled by $\zeta$ in the CP violating phase of the system are determined.

One of the interesting properties of the $\beta$ and $\omega$ phases is that these phases are expected to have off diagonal long range order ODLRO since the spontaneous breaking down of the $U(1)$ symmetry corresponding to baryon number conservation. Macroscopic quantum phenomena manifest in those phases possessing ODLRO. The quantum nature of these phases leads to behaviors of the system not expected from our daily experiences. Some of these behaviors are observed in the superfluid state of $^4$He. Could it allows a quantum mechanical jump (transition) from an initially zero baryon density $\beta$ or $\omega$ phases of the vacuum state to the lowest energy state of the system inside relatively large regions: some of them contain baryonic matter and others contain antibaryonic matter? Such a kind of transition is forbidden in classical picture since it violates the the relativistic causality. It is allowed in the quantum measurement processes according to the standard interpretation of quantum mechanics, in which a collapse of the wave function of the system occurs after a measurement. It remains to be understood in the future more detailed researches. If it is permitted, then each of these regions can has a finite size at the moment of the transition or, in another word, each of them can has a size large than its event horizon.

This property of the $\beta$ and $\omega$ phases could provide a mechanism for the baryogenesis in the early universe in a matter–antimatter symmetric universe. It was shown to be impossible for such a baryogenesis mechanism to be compatible with observations if the process of baryon–antibaryon separation is classical. The discovery of the possible $\beta$ and $\omega$ phases for the strong interaction vacuum in this study might provide a theoretical basis for a reconsideration of the idea of matter antimatter symmetric universe. Some of the other more detailed consequences of this picture, which is beyond scope of this paper, is worthy of exploring.

$^4$ODLRO is absent in the normal phases of matter where classical picture is supposed to emerge for macroscopic systems due to decoherence characterized by a diagonalization of the effective density matrix of the system interested during the time evolution (see, for example, Refs. [27]).
The $\mu'_0$ degrees of freedom is non-propagating in the $\beta$ and $\omega$ phases due to a non-vanishing (uniform) baryon number density is present in these two phases. According to Eq. 4.35,

$$S_{\text{eff}}^{(\mu)} = \frac{i}{2} \int d^4 x \left[ (\overline{\sigma}^2) \mu'_0^2 + \ldots \right].$$

(6.5)

It shows that the $\mu'_0$ fluctuation is damped by a non-oscillating Gaussian factor with the width proportional to the inverse square of the baryon number density in these phases (i.e., $\overline{\sigma}^2$). The spatial component of $\mu^\alpha$ in the $\beta$ and $\omega$ phases is however long ranged. This can also be realized through an inspection of Eq. 4.35.

The question of whether or not there is a condensation of statistical mono-poles in these two phases, which is discussed in a general term in subsection IV B should be studied in future more detailed works.

C. The excitations of the primary statistical gauge field

In the above discussion it can be seen that in the $\alpha$, $\beta$ and $\omega$ phases, either $\epsilon$ or $\mu$ (it is equivalent to $\langle 0 | j | 0 \rangle$) is nonvanishing. From the discussion presented in the subsection IV B, it can be concluded that there is no long range order for the time component of the primary statistical gauge field $\mu^\alpha$ at long distances or $\rho^0$ corresponds to at most a massive excitation in the non-trivial vacuum discussed in this paper.

The situation for the spatial component of $\mu^\alpha$ is different between the $\alpha$ and $\beta$ or $\omega$ phases. In the $\alpha$ phase, the excitation related to $\mu^\alpha$ is short ranged. In the $\beta$ and $\omega$ phases, the excitations related to $\mu$ are long ranged. These excitations can therefore generate a statistical “magnetic force” between different particles within the $\beta$ or $\omega$ phases of the vacuum. The consequences of such a statistical “magnetic force” on the evolution of the system is worthy of studying.

VII. DISCUSSION AND OUTLOOK

It is found that at least three interwinding new theoretical elements are necessary to be brought into a consistent treatment of the problem. The first one is the general existence of the so called dark component in an interacting system originated from the transient and short distance quantum fluctuations of the system, which is measured by the difference between the absolute value and the apparent value of some conserved quantities like the baryon number density, energy density, etc. of the system. The second one is related to the recognition of the existence of the so called blocking effects in the non-trivial phases of a system. The third one is related to the necessity of introducing a primary statistical gauge field coupled to the fermion (baryon) number current density of the system. By introducing these three elements into the formulation of the problem, the door to go beyond the physical pictures limited by the approximated concept of quasiparticles is open, which allows us to explore new physical possibilities.

A systematic path integration formalism for the investigation of the quantum aspects of an interacting fermion system (or sector) sampled by Euclidean spacetime stationary configurations in which a condensation of fermion pairs (fermion pairs, antifermion pairs, and fermion–antifermion pairs) is present is developed based upon the asymptotic grand canonical ensemble. Two statistical parameters, namely the primary statistical gauge field $\mu^\alpha$ and the statistical blocking parameter $\epsilon$ are introduced to allow a finer characterization of the vacuum structure of the system. In addition, it is shown that the asymptotic grand canonical ensemble reduces to the grand canonical ensemble as the spacetime resolution of observation is sufficiently lowered. Such a behavior is a necessary condition for the usefulness of describing the macroscopic properties of an interacting system in terms of particles in certain domain of energy and for the smooth approach to the well established results in non-relativistic condensed matter systems at low energies. Combined with the Euclidean approach to the effective action, some of the quantum effects that survive the thermodynamic limit can be included. The present approach, which uses quantum field theoretical language, is consistent with thermodynamics and can be extended to finite temperature case $\beta$.

Firstly, the dark component for local observables like the fermion number density does exist in interacting theories in which the direct association of the field theoretic definition of fermion number density with the number of “free particles” per unit volume becomes obscure especially when a phase transition inside the system has occurred. This conclusion is also applicable to other local observables like the energy density, which may have implications on the dark matter problem in Cosmology since it implies that the apparent matterless space at the macroscopic level is capable of revealing itself of matter effects in low energy gravitational processes when $\mu$ is below the baryonic particle production threshold even after the energy density for the $\mu = 0$ state is subtracted. This is because gravitational fields couple locally to the source matter fields which contain the random quantum fluctuation generated dark component.
Further researches in that direction in the context of understanding the cosmological baryogenesis and dark matter problem is an interesting direction to be explored.

In addition, the picture that a nucleon is made of three valence quarks (quasiparticles) need to be modified when there is additional close-by virtual phases for the hadronic vacuum state that has slightly higher energy density than the actual one. The implications of such a finding can be explored in observables related to a nucleon. Some of them are studied in Refs. [3,18], other related problems concerning a nucleon, like the understanding of the origin of the Gottfried sum rule violation in deep inelastic scattering confirmed in a recent measurement [31], the small-x behavior of nucleon structure functions in deep inelastic scattering [22,23], etc.

Secondly, it is important to take into account the fermionic blocking effects due to the presence of a macroscopic population of bare particles in the nontrivial vacuum phases of a system. The blocking effects are generally included in the theory by introducing the statistical blocking parameter $\epsilon$, which is non-zero for a system’s certain vacuum phases.

The effects of the blocking can not be progressively generated using perturbative expansion starting from a field configuration with $\epsilon = 0$. In the $\alpha$ phase of the strong interaction vacuum discussed here, $\epsilon = 0$ configurations are inconsistent ones since it is known that the $\alpha$ phase of the strong interaction vacuum is macroscopically populated with the current quarks and antiquarks, which changes the available states for an current quark due to Pauli principle. The discovery of the blocking effects has hitherto unnoticed implications related to, e.g., the stability of a nucleon in a nucleus and nuclear matter, the mechanism for particle production in a heavy ion collision, new ways of (quasi)particle dissipation in a strong interacting system, etc.

Thirdly, the statistical gauge degrees of freedom of the system represents certain collective mode of the system that has a dynamics of its own. There are two components for the primary statistical gauge field: the first one is the classical configurations which serves as a background field; the second one is the local fluctuations of it around the classical configurations.

The classical configurations, which is a spacetime independent background $\mu^\alpha$ in the case studied here, play the role of the chemical potential in the the conventional non-relativistic approach. It determines the asymptotic grand canonic ensemble of the system. It can also has non-trivial topological configurations corresponding to different quantized statistical “magnetic flux” which can determine the phase structure of the system on a finer basis. Once present, the statistical “magnetic field” affects the dynamical evolution of the system that can result in 1) material pattern formation and 2) providing the seed for the galactic magnetic field in the early universe. Whether or not such an idea is actually relevant to comprehend what happened in the the early universe can be studied in further works.

The local fluctuations of it around the background extended configuration represent the corresponding dynamical excitations of the system. The necessary condition for the existence of long range statistical gauge correlation in various possible phases of the system is discussed.

The statistical gauge degrees of freedom are also introduced in condensed matter physics in the context of the half filled Hubbard model, which serves as one of the preferred models that is expected to describe the phenomena of “high temperature superconductivity” in certain materials [35,36]. The motivation for introducing the statistical gauge degrees of freedom there is quite different from the ones in this work. Here, the statistical gauge degrees of freedom are introduced a priori based upon locality and Lorentz invariance with the intention of describing the relativistic fermionic systems. Since the approach in this paper is applicable to all fermionic system, it is quite interesting to see whether the non-relativistic reduction of the problem can lead to some form of statistical gauge degrees of freedom for condensed matter systems at low energies, including the ones that Hubbard model describes. Nevertheless, many techniques in treating the statistical gauge degrees of freedom in condensed matter physics are expected to be applicable or a least adaptable here.

One of the differences at formal level between the approach here and the ones used in condensed matter physics manifests in the different criterion for the selection of physical states within the full representing Hilbert space of the problem. The physical states in the statistical gauge theories developed so far in condensed matter physics is invariant under infinitesimal local gauge transformations, which is realized by the requirement that the operator form of the “Gauss law” annihilates all physical states in the superselection sector of the Hilbert space [37]. Such a strict enforcement of the statistical gauge invariance on the physical state vector of the system is neither necessary nor desirable for the statistical gauge invariant systems since it would exclude all finite density state from the physical sector of the system if one requires that the statistical “electric field” (denoted by $\pi_a$ here) is finite. Such a situations is certainly unacceptable. Albeit this embarrassment can be circumvented in the 2+1 dimension [37], it is not expected to be easily implemented in higher dimensions. For the statistical gauge transformations, the statistical gauge invariance can be implemented by a imposing a less restrictive conditions on the physical superselection sector of the Hilbert space. Instead of requiring that the physical states are invariant under the gauge transformation, the gauge invariance on observables can be implemented by requiring that all states in a physical superselection sector of the Hilbert space change a (coordinate dependent) common phase. The mathematical form for such a requirement is represented by Eq. [29]. The physical superselection sectors are then a functional determined by the common function $\varsigma$. With such
a generalization of the “Gauss law” for the theory, both the requirements of the finiteness of the statistical “electric field” $\pi_a$ and of the fact that finite density states are actually physical states can be met consistently.

The formalism is then applied to two half bosonized model Lagrangian densities. Four possible phases for the vacuum state of the interacting relativistic chiral symmetric systems are found. The first phase, called the $O$ phase, correspond to the bare vacuum state of the system. Fermion–antifermion pairs condense in the second phase, named the $\alpha$ phase, of its vacuum. The $\alpha$ phase has the following properties: 1) the chiral $SU(2)_L \times SU(2)_R$ symmetry of the Lagrangian density of the system is spontaneously broken down to a $SU(2)_V$ symmetry 2) the baryon number density is zero 3) statistical blocking effects exists 4) the statistical gauge correlation is short ranged due to the presence of the statistical blocking effects. The third and fourth possible phases of the vacuum are called the $\omega$ phase and $\beta$ phase respectively. Fermion pairs and antifermion pairs condense in the $\omega$ and $\beta$ phases. It is found that in these two phases of the vacuum: 1) the original chiral $SU(2)_L \times SU(2)_R$ symmetry of the Lagrangian density of the system remains unbroken in the $\omega$ phase and is spontaneously broken down to a $SU(2)_V$ symmetry in the $\beta$ phase 2) the baryon number density is different from zero or can be locally generated by separating fermion and antifermion rich region spontaneously 3) the $U(1)$ symmetry corresponding to electromagnetism is spontaneously broken down to generate “massive photon” excitations (see [8]) 4) no statistical blocking effects in these two phases 5) the spatial components of the statistical gauge excitation is long ranged; the quantum fluctuation in the time component of the primary statistical gauge field is Gaussian damped 6) off diagonal long range order exists in these two phases to give rise to macroscopic quantum behavior for the system, which is suppressed in the normal phase of the system.

The implication of the finding presented in this paper on physical processes of strong interaction phenomena that are currently being or going to be observed or are in need to be explained theoretically remains to be investigated in the future.

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FIG. 1. The $p^0$ integration contour for the effective action. Here the filled circle represents the possible discrete spectra of $S_{P^{-1}}[f]$ and the thick lines extending to positive and negative infinity represent the branch cuts of the logarithmic function. The $\pm i\pi$ above or under the thick lines denote the imaginary parts of the logarithmic function on the physical sheet of the $p^0$ plane.

FIG. 2. Contour I is commonly used in literature where a non-covariant cutoff in 3-momentum space is applied. It is called the quasiparticle contour in this paper. Contour II is the one for the Euclidean effective action in the conventional approach.

FIG. 3. The $\mu$ dependence of $V_{\text{eff}}$ of the conventional approach in the $\alpha$ phase. It has two minima at non-zero $\mu$. Here $\Lambda$ is the Euclidean momentum cutoff that defines the model.

FIG. 4. The phase boundaries between the $\alpha$, $\omega$ and the $O$ phases. The chiral symmetry is unbroken in both the $\omega$ phase and the $O$ phase. The $\alpha$ phase breaks the chiral symmetry spontaneously down to a flavor symmetry.

FIG. 5. The phase boundaries between the $\alpha$, $\beta$ and $O$ phases. The chiral symmetry is unbroken in the $O$ phase. The $\alpha$ phase and $\beta$ phase break the chiral symmetry spontaneously down to a flavor symmetry.

FIG. 6. The $p^0$ integration contour for the effective action for transition amplitudes between states in which both the fermion and antifermion states with absolute value of their energy below $\epsilon$ filled. Here the filled circle represents the possible discrete spectra of $S_{P^{-1}}[f]$ and the thick lines extending to positive and negative infinity represent the branch cuts of the logarithmic function. It also represents the $p^0$ integration contour of the full theory in its original Minkowski spacetime form.

FIG. 7. The quasiparticle $p^0$ integration contour for the full theory is labeled by “I”. The Euclidean $p^0$ integration contour for the full theory, which preserves the causal structure of the original one, is labeled by “II”.

FIG. 8. The dependences of $V_{\text{eff}}(\mu, \epsilon)$ in the $\alpha$ phase on $\mu$ and $\epsilon$ along different directions in the $\mu$-$\epsilon$ plane. The direction in which $V_{\text{eff}}$ has the smallest value is in the $\mu = 0$ direction. Here $v_{\text{eff}} = V_{\text{eff}}/\Lambda^4$ and $x$ variable is either $\mu/\Lambda$ or $\epsilon/\Lambda$.

FIG. 9. The dependences of $V_{\text{eff}}(\mu, \epsilon)$ in the $\omega$ phase on $\mu$ and $\epsilon$ along different directions in the $\mu$-$\epsilon$ plane. The direction in which $V_{\text{eff}}$ has the smallest value is in the $\epsilon = 0$ direction. Here $v_{\text{eff}} = V_{\text{eff}}/\Lambda^4$ and $x$ variable is either $\mu/\Lambda$ or $\epsilon/\Lambda$.

FIG. 10. The dependences of $V_{\text{eff}}(\mu, \epsilon)$ in the $\beta$ phase on $\mu$ and $\epsilon$ along different directions in the $\mu$-$\epsilon$ plane. The direction in which $V_{\text{eff}}$ has the smallest value is in the $\epsilon = 0$ direction. Here $v_{\text{eff}} = V_{\text{eff}}/\Lambda^4$ and $x$ variable is either $\mu/\Lambda$ or $\epsilon/\Lambda$.

FIG. 11. The dependence of $p_{\omega}^{1/3}$ on the spacetime independent part of the primary statistical gauge field $\mu$. Here $\alpha = \pi A$. The unit for the dimensional quantities are $\text{GeV}$. Solid lines represent the case of free theory with mass 0 and 0.5 respectively. Other lines represent the results for the massless NJL model with different strength of local fluctuations characterized by the $\alpha$ of the order parameter.
APPENDIX A: FREE FERMION SYSTEMS WITH $n_f$ FLAVORS AND $n_c$ COLORS

1. With primary statistical gauge field only

The Lagrangian density for a massive fermionic system with the primary statistical gauge field $\mu^\alpha$ (see Eq. 3.1) included has a form

$$\mathcal{L} = \frac{1}{2} \bar{\Psi} (i\partial + \mu \gamma_5 - m) \Psi,$$

(A1)

where $m$ is the mass of the fermion.

The generating functional $W[\bar{\eta}, \eta, \mu]$ for such a system can be written as

$$e^{W[\bar{\eta}, \eta, \mu]} = \int D[\Psi] e^{\frac{i}{2} \bar{\Psi} C \gamma_0 \gamma_5 \gamma^\alpha \gamma^\beta \gamma_5 \gamma_0 \gamma^\beta \bar{S}^0 \eta + \frac{i}{2} \bar{\eta} \gamma_0 \gamma_5 \gamma^\alpha \gamma^\beta \gamma_5 \gamma_0 \gamma^\beta \eta} \times e^{\frac{i}{2} \bar{\Psi} C \gamma_0 \gamma_5 \gamma^\alpha \gamma^\beta \gamma_5 \gamma_0 \gamma^\beta \eta} (\bar{\Psi} \gamma_5 \gamma^\alpha \gamma^\beta \gamma_5 \gamma_0 \gamma^\beta \eta)$$

(A2)

with $\bar{\eta}$ and $\eta$ Grassmann external fields and

$$iS^{(0)}_F = i\bar{\Psi} + \mu \gamma_5 \eta - m.$$  

(A3)

This equation allows us to write

$$iW[0,0,\mu] = \frac{i}{2} \bar{S}^0 \ln \frac{iS^{(0)}_F}{\eta} + \text{const},$$

(A4)

which for an uniform $\mu^\alpha = (\mu,0)$, takes the following form

$$iW[0,0,\mu]/V_4 = -\imath n_f n_c \int_C \frac{d^4 p}{(2\pi)^4} \ln (p_+^2 - m^2)(p_-^2 - m^2) + \text{const},$$

(A5)

where $V_4 = L^3 T$ with $L^3 \to \infty$ the spatial volume and $T \to \infty$ the temporal extension of the system, $p_+^\mu = (p^0 + \mu, \mathbf{p})$, $p_-^\mu = (p^0 - \mu, \mathbf{p})$ and $C$ denotes the $p^0$ integration contour in the complex $p^0$ plane shown in Fig. 2. $W[0,0,\mu]$ can be further specified by requiring $W[0,0,0] = 0$. Such a $W[0,0,\mu]$ is

$$iW[0,0,\mu]/V_4 = -\imath n_f n_c \int_C \frac{d^4 p}{(2\pi)^4} \ln \frac{(p_+^2 - m^2)(p_-^2 - m^2)}{(p^2 - m^2)^2}. $$

(A6)

Since for free fields, the asymptotic grand canonical ensemble is the grand canonical ensemble and the quasiparticle approximation is actually an exact one (Appendix B), the integration contour for $p^0$ integration above can be chosen as the one shown in Fig. 2. Since the quantum fluctuations of the free fields are also exactly known and included already, they should not be sampled using the Euclidean approach. To avoid over counting of the quantum fluctuations of the free field, the $p^0$ integration should be done first and then the spatial component of $p^\mu$. The adoption of the 8-component spinor for the fermion field makes it equivalent whether the $p^0$ integration is carried out on the real axis or is on the imaginary axis 4 provided that it is done first. The result is

$$iW[0,0,\mu]/V_4 = \frac{\imath n_f n_c}{\pi^2} \int_0^\mu d\mathbf{p} (\mu - E_\mathbf{p}) = \mu \bar{\eta} - \tau = -\Omega/V_4,$$

(A7)

where $\tau$ is the internal energy density and $\bar{\eta}$ is the fermion number density of the system and $E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}$. It is evident that $iW[0,0,\mu]$ correspond to the negative of the grand-potential $\Omega(\mu)$ at zero temperature in a many body system.

Before ending of this subsection, it is worth mentioning that had we adopted the usual 4-component representation for a Dirac spinor, we would not have obtained Eq. 5 by using the path integration method with the primary statistical gauge field developed here. Part of the reasons is due to the non-symmetric way of introducing the primary statistical gauge field in the 4-component representation. This issue is discussed in 4 in more details.
2. The full theory

According to section \[\text{II}\] the generating functional \(W[\eta, \eta, \mu, \epsilon]\) for an uniform free fermion system in the full theory can be written as

\[
iW[\eta, \eta, \mu, \epsilon]/V_4 = -in_f n_c \int_{f.q.p.} \frac{d^4 p}{(2\pi)^4} \ln(p_+^2 - m^2)(p_-^2 - m^2) + \text{const}
\]  \(\text{(8)}\)

with “f.q.p.” denoting the full quasiparticle \(p^0\) integration contour in the complex \(p^0\) plane shown in Fig. \[\text{III}\]. The constant in the above equation is so chosen that \(W[0,0,0,0] = 0\), which means that

\[
iW[0,0,\mu,\epsilon]/V_4 = -in_f n_c \int_{f.q.p.} \frac{d^4 p}{(2\pi)^4} \ln(p_+^2 - m^2)(p_-^2 - m^2) - 2 \int_{q.p.} \frac{d^4 p}{(2\pi)^4} \ln(p^2 - m^2) \]

where “q.p.” denoting the quasiparticle \(p^0\) contour shown in Fig. \[\text{IV}\]. It is found, after some algebra, that

\[
iW[0,0,\mu,\epsilon]/V_4 = (\mu + \overline{\mu}) - (\overline{\pi}_+ + \pi_- - \pi),
\]  \(\text{(9)}\)

where \(\mu_\pm = \mu \pm \epsilon\) and

\[
\overline{\pi}_+ = \frac{n_f n_c}{\pi^2} \int_0^{\mu_+} \frac{d|p||p|^2},
\]  \(\text{(10)}\)

\[
\pi_- = \frac{n_f n_c}{\pi^2} \int_0^{\mu_-} d|p||p|^2 E_p.
\]  \(\text{(11)}\)

APPENDIX B: THE EXISTENCE OF THE DARK COMPONENTS FOR LOCAL OBSERVABLES

1. Cluster decomposition and the origin of the dark component

For simplicity, the two flavor half bosonized NJL model (see Appendix \[\text{E}\]) is used for our discussion. The model Lagrangian density with a primary statistical gauge field \(\mu^\alpha(x)\) is

\[
\mathcal{L} = \frac{1}{2} \overline{\Psi} (i\partial_\mu + \mu O_3 - \sigma - i\gamma_5 \vec{\pi} \cdot \vec{\tau}) \Psi - \frac{1}{2G_0} (\sigma^2 + \pi^2).
\]  \(\text{(B1)}\)

After performing the path integration over the fermion fields \(\Psi\) and \(\overline{\Psi}\), the generating functional \(W[J]\) can be written as

\[
e^{iW[J,\mu]} = \int D[\sigma, \overline{\sigma}, \mu] e^{iS_{eff}[\sigma, \overline{\sigma}, \mu] + i \int d^4xf \cdot J},
\]  \(\text{(B2)}\)

where “\(J\)” and “\(f\)” represent, collectively, the external fields and auxiliary fields respectively. The effective action \(S_{eff}\) is given by

\[
S_{eff}[\sigma, \overline{\sigma}, \mu] = -i \frac{1}{2} \text{SpLn} S^{-1}_F[\sigma, \overline{\sigma}, \mu] S_F[0,0,0] + \frac{1}{2G_0} \int d^4x (\sigma^2 + \pi^2),
\]  \(\text{(B3)}\)

where \(S_F\) denotes the functional trace. The operator \(iS_F^{-1}[\sigma, \overline{\sigma}, \mu] = i\partial_\mu + \mu O_3 - \sigma - i\gamma_5 \vec{\pi} \cdot \vec{\tau}\) is the inversed propagator of the fermions in the background auxiliary fields \(\sigma(x)\) and \(\overline{\sigma}(x)\).

Let us define \(R[\sigma, \overline{\sigma}, \mu, x] \equiv \text{Tr}\gamma^0(x) [S_F[\sigma, \overline{\sigma}, \mu] x]\), where the trace “\(\text{Tr}\)” is over the internal degrees of freedom of the fermions. The fermion number density of the model in vacuum state is

\[
\sum_{\{f\}} \mathcal{W}[f,\mu] |f, t = +\infty \rangle \langle f, t = -\infty | = \frac{1}{\delta \mu^{\rho}(x)} = \frac{1}{Z} \int D[\sigma, \overline{\sigma}] e^{iS_{eff}[\sigma, \overline{\sigma}, \mu]},
\]  \(\text{(B4)}\)

where \(Z = \int D[\sigma, \overline{\sigma}] e^{iS_{eff}[\sigma, \overline{\sigma}, \mu]}\) and \(f\) denotes the collection of \(\{\sigma, \overline{\sigma}\}\) and \(\mathcal{W}[f,\mu]\) is the weight functional of the asymptotic grand canonic ensemble discussed in the main text. The Minkowski spacetime is not suitable to study the
properties of the vacuum state using Eq. [B4] since the initial and final auxiliary field configurations are not specified. The usual procedure to project out the contributions of the vacuum state is to go to the Euclidean spacetime in which the vacuum state has lowest energy.

In the mean field approximation, the vacuum phase is determined by minimizing the effective potential $V_{eff}(\sigma, \mu) = -S_{eff}[\sigma, 0, \mu]/\Omega$ with $\Omega \to \infty$ the spacetime volume of the system, $\sigma$, $\mu^\alpha$ spacetime independent and $\bar{\pi}$ assumed zero. Due to the chiral symmetry, the assumption $\bar{\pi} = 0$ does not result in a loss generality. The phase of the system is determined by the condition $\delta V_{eff}(\sigma, \mu)/\delta \sigma = 0$. The solution for $\bar{\sigma}$ is non-zero after the coupling constant $G_0$ is greater than a critical value $G_{0c}$. A non-vanishing $\bar{\sigma}$ generates an effective mass for the fermions, which act as quasiparticles.

The mean field fermion number density for the vacuum is obtained from Eq. [B4] by ignoring the functional integration over $\sigma$ and $\bar{\pi}$ and let $\sigma = \bar{\sigma}$. The result is (see Appendix [A])

$$\bar{\rho}_{MF} = \rho(\bar{\sigma}, 0, \mu; x) = \frac{n_1 n_f}{3\pi^2} \theta(\mu - \bar{\sigma}) \left[\mu^2 - \bar{\sigma}^2\right]^{3/2},$$  \hspace{1cm} \text{(B5)}$$

which is non-zero only when $\mu > \bar{\sigma}$ just like the fermion number density of free massive particles with mass $m = \bar{\sigma}$. Such a behavior of $\bar{\rho}$ is also predicted in the finite density field theory based on a global chemical potential $\mu_{ch}$ with $N_{app}$ discussed in the introduction behave in the same way as $\Omega \bar{\rho}_{MF}$. This will be discussed in the following. Therefore, the quasiparticle contributions saturate the fermion number density in the field theoretical approach to finite density problems based on a global chemical potential.

The contributions of quantum fluctuations around the mean field $\bar{\sigma}$ are formally included in Eq. [B4]. The results are commonly expressed as loop corrections to the fermion number density vertex, which is not attempted here.

Instead of performing a loop expansion computation of the fermion number density, the effects of the quantum fluctuations can be evaluated non-perturbatively by “doing” the path integration.

To proceed, the system under consideration is first putted in a Euclidean spacetime box of length $L$ in each direction and with periodic boundary conditions at its boundary surfaces. The thermodynamic limit is defined as the limit of $L \to \infty$. In the thermodynamic limit, the extremal configuration dominates the path integral among those configurations of $\sigma$ and $\bar{\pi}$ that give divergent action in the thermodynamic limit. The contributing finite action quantum fluctuation configurations, the number of which is proportional to the spacetime volume $\Omega = L^4$, are further classified into two categories: 1) correlated localized configurations, which are defined as the ones that approaches to the mean field configurations in the spacetime infinity and 2) correlated extended configurations, which are the ones that remain different from the mean field configurations at the spacetime infinity.

For a given system, whether or not a configuration is a correlated extended configurations or is of localized ones is determined by dynamics. The on-shell amplitudes, are solutions of the “classical equation of motion” in the Euclidean spacetime

$$\frac{\delta S_{eff}^E[f]}{\delta f(x)} = 0,$$  \hspace{1cm} \text{(B6)}$$

with $S_{eff}^E$, the Euclidean effective action, $f$ representing $\sigma$ or $\bar{\pi}$ fields. The superscript $E$ shall be suppressed in the following. The set of the extended solutions to Eq. [B6] are correlated ones. Albeit there are plenty of extended on-shell amplitudes in the Minkowski spacetime, there is no known one in the Euclidean one. I shall assume the absence of them. The degree of correlation of an arbitrary extended configuration at different spacetime points is determined by the degree of their deviation from the extended on-shell amplitudes for propagating excitations of the system.

The correlation between the off-shell configurations at two different space time points decreases exponentially. For example, consider the field–field correlations or propagators

$$\langle 0 | T \hat{f}(x_1) \hat{f}(x_2) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x_1-x_2)} \frac{1}{p^2 + m^2}$$  \hspace{1cm} \text{(B7)}$$

with $T$ denoting time ordering. The off-shellness of these configurations is measured by their mass $m$. The correlation of these configurations at two different locations $x_1$ and $x_2$ (in the Euclidean spacetime) decreases as fast as $\exp(-rm)/r$ with $r = \|x_1 - x_2\|$. So they can be decomposed into a superposition of localized ones with sizes of order $1/m$. For these set of configurations, one can divide the spacetime into cells with a dimension sufficiently larger than

$^5$In the sense that it can be decomposed into localized ones.
their correlation length. Then the contribution of this set of field configurations to the partition functional Eq. B\(B_{10}\) can be cluster decomposed to

\[ Z[J] \approx \prod_k z_k[J], \quad W[J] \approx \sum_k w_k[J] \]  

(B8)

with \(w_k[J] \equiv \ln z_k[J], z_k[J]\) and \(w_k[J]\) the corresponding partition functional of the \(k\)th cell. The full partition of the \(k\)th cell is

\[ z_k[0] \sim \int D_k[f] e^{-S_{\text{eff}}^{(k)}[\bar{\sigma} + f]}, \]  

(B9)

where \(S_{\text{eff}}^{(k)}[f]\) is the effective action of the \(k\)th cell and the path integration of \(f\) is over those ones that equal to the mean field value \(\bar{f}\) outside of the \(k\)th cell but with arbitrary amplitudes inside the finite volume. For a given theory, instead of arbitrary division of the spacetime, it is expected that there is an optimal one with minimum volume \(\bar{\omega}\) for each cell and yet has an error below a predetermined one. We shall assume that such an optimal division of the spacetime into cells has already been found in the following discussion.

Let us evaluate the contributions of the uncorrelated localized quantum fluctuations of the \(\sigma\) and \(\bar{\sigma}\) fields to the vacuum fermion number density using Eqs. B\(B_{4}\) and B\(B_{8}\) in the Euclidean spacetime.

In the phase where the chiral SU(2)\(_L\) \(\times\) SU(2)\(_R\) symmetry is spontaneously broken down, the “chiral angle” variable represented by \(\bar{\sigma}\) in the phase where \(\langle 0 | \bar{\sigma} | 0 \rangle = 0\) becomes massless following the Goldstone theorem. The correlation length of the \(\bar{\sigma}\) field becomes divergent in the chiral symmetric limit. Therefore the field configurations of the Goldstone boson degrees of freedom contain the dominating on-shell components that can be included by doing a loop expansion as usual. Such a loop expansion contains no infrared divergences. Because the fermion number density \(\rho(\sigma, \bar{\sigma}, \mu; x)\) under study is chiral symmetric, which means that it does not depend on a spacetime independent global “chiral angle”; it depends only on the derivatives of the “chiral angle” variables. The absence of a dependence of the fermion number density on a global “chiral angle” guarantees the absence of the infrared divergences in the quantum corrections from the Goldstone bosons. The configurations of the “chiral angle”, being on the edge of their shell and extended configurations in nature, are uniform and infinitesimal in amplitudes since it contains no infrared divergences and has a number of distinct modes proportional to the volume \(\Omega\) of the system. They can not modify the qualitative features of the quasiparticles. So, the \(\bar{\sigma}\) variable in the vacuum fermion number density can be eliminated. It is treated as zero in the following discussions.

The quantum fluctuation in the “mass” term, namely the chiral radius or order parameter represented by \(\sigma\) (when \(\langle 0 | \bar{\sigma} | 0 \rangle = 0\)) has different characteristics due to the fact that it contains no on-shell Euclidean configurations. These off-shell configurations have only short range correlations in spacetime.

If the spacetime is divided into cells with their dimension optimally determined, then the path integration within each cell can be done independently. This gives us a cluster decomposed partition functional of the form given by Eq. B\(B_{8}\) with the partition functional for each cell computed independently of each other.

The cluster decomposition property of the partition functional of the system reduces the full fermion number density of the vacuum given by Euclidean form of Eq. B\(B_{4}\) to

\[ \rho_{\text{vac}} = \frac{1}{z_k[0]} \int D_k[\sigma'] \rho[\bar{\sigma} + \sigma', 0, \mu; x] e^{-S_{\text{eff}}^{(k)}[\bar{\sigma} + \sigma', \mu]}, \]  

(B10)

where the cell labeled by \(k\) is the one that contains the spacetime point \(x\), \(\int D_k[\sigma']\) denotes integration over field configurations that approach the mean field value outside the cell and \(S_{\text{eff}}^{(k)}\) is the Euclidean effective action of the cell.

Eq. B\(B_{10}\) is still too complicated to evaluate analytically. We make a further simplification by assuming that the functional integration of \(\sigma'\) within a spacetime cell can be replaced by an ordinary integration over the spacetime averaged value of \(\sigma'\) within that cell. It can be achieved by keeping the average of \(\sigma'\) fixed while integrate over the rest degrees of freedom. The non-trivial part of it is in the assumption that after eliminating the rest of the degrees of freedom, the resulting effective potential \(V_{\text{eff}}\) remains, at least in form, the same as the original one. This procedure is in the same spirit as the renormalization group analysis. In this way, Eq. B\(B_{10}\) reduces to

\[ \rho_{\text{vac}} = \frac{1}{z[0]} \int_{-\infty}^{\infty} d\delta \rho[\bar{\sigma} + \delta \sigma, 0, \mu; x] e^{-\bar{\omega} V_{\text{eff}}[\bar{\sigma} + \delta \sigma, \mu]}, \]  

(B11)

where \(\delta \sigma = < \sigma' >\) is the spacetime average of \(\sigma'\) within the cell and the same reduction is also made to \(z[0]\). The finiteness of \(\bar{\omega}\) result in different qualitative behavior for \(\rho_{\text{vac}}\) as a function of \(\mu\). To explicitly see the difference, let us expand \(V_{\text{eff}}\) around \(\bar{\sigma}\), keeping only the leading quadratic term
\[ V_{\text{eff}}(\sigma + \delta \sigma) = V_{\text{eff}}(\sigma) + A(\sigma)\delta \sigma^2 + \ldots \] (B12)

and using Eq. 33 for the trace term. If \( \pi A \) is sufficiently large, the result can be written as

\[ \rho_{\text{vac}} = \frac{2}{\pi} \sqrt{\frac{\omega A}{\pi}} \int_{-\mu - \sigma}^{-\mu + \sigma} d\delta \sigma e^{-\omega A \delta \sigma^2} \left[ \mu^2 - (\sigma + \delta \sigma)^2 \right]^{3/2}. \] (B13)

It is clearly non-zero for any finite \( \mu \) below \( \sigma \) since \( A \) is finite. The \( \mu \) dependence of \( \rho_{\text{vac}} \) differs from the form given by Eq. 33 as long as \( G_0 \) is finite. Eq. 33 is recovered only in the limit of \( G_0 \to 0 \), which represents the free field case. The dependence of \( \rho_{\text{vac}} \) on \( \mu \) for a set of different values of \( \alpha = \pi A \) is plotted in Fig. 11. Instead of a sharp rise in \( \rho_{\text{vac}} \) at \( \mu = \sigma, \rho \) is non-vanishing all the way to \( \mu = 0 \). This component of the fermion number density can certainly not be attributed to the contributions of the quasiparticles. It is found therefore that there is a dark component for the fermion number density that cannot be accounted for by the quasiparticle contributions.

The NJL model has only one non-trivial vacuum phase since there is one independent absolute minimum in its effective potential. If the effective potential of the system contains a second local minimum with a higher energy density than the absolute minimum, then there are contributions from the local minimum to the fermion number density. The existence of such a contribution is another direct consequence of the cluster decomposability of the partition functional of the system. Models with a second minimum are studied in the literature. Those that have only one order parameter are represented by the Friedberg–Lee model \([30]\) in which the two minima of the effective potential correspond to confinement and deconfinement phase of the model. Those that have two order parameters are introduced in section V.

In the presence of a higher virtual phase that is separated from the actual phase of the system by a potential barrier, the fermion number density of the system is saturated by both the quasiparticles of the first phase and those ones of the second virtual phase together with their corresponding dark component discussed above. It has a form

\[ \rho_{\text{vac}} = \rho_{\text{vac}1} + e^{-\pi A} \rho_{\text{vac}2}, \] (B14)

where \( \Delta = V_{\text{eff}}^{(2)} - V_{\text{eff}}^{(1)} > 0 \) is the difference in energy density between the virtual phase and the actual phase, \( \rho_{\text{vac}1} \) and \( \rho_{\text{vac}2} \) are the vacuum fermion number density of the actual vacuum phase and that of the virtual vacuum phase respectively and \( \pi A \) is the optimal volume of the spacetime cell between which the order parameters of the system are uncorrelated. The contributions of the virtual phase to the fermion number density or to any local physical observables are non-perturbative effects. Attempts had been made in Refs. 33, 34 to search for possible other virtual phases like the \( \beta \) or \( \omega \) phase of the strong interaction vacuum.

2. The reemergence of the quasiparticle picture

The fermion number density discussed above are defined on a spacetime point. Such a precision is non-achievable in realistic observations. The physical observables can be represented by a “coarse-grained averaging” of the form \( \overline{\rho}_{\text{vac}} = \Delta N_{\text{vac}}/\Delta \Omega \) with \( \Delta \Omega \) the smallest volume in spacetime that the observation apparatus can resolve and \( \Delta N_{\text{vac}} \) the average number of fermions due to the coherent response of the system to an external classical field \( K \) within that volume. \( \Delta N_{\text{vac}} \) is not in general identical to \( \int_{\Delta \Omega} d^4x \rho_{\text{vac}} \) when \( \Delta \Omega >> \pi \). It is obtained from the partition functional \( W[J, \mu] \) by adding the external field \( K \) to \( \mu \). \( K \) has a constant non-zero value only within spacetime volume \( \Delta \Omega \), then \( \Delta N_{\text{vac}} = \partial W[0, \mu + K]/\partial K|_{K=0} \). In case when \( \Delta \Omega << \pi \), namely the precision of the observation is much higher than the correlation length of the order parameter, the observed fermion number density \( \overline{\rho}_{\text{vac}} \) behaves in the same way as \( \rho_{\text{vac}} \). On the other hand, if \( \Delta \Omega >> \pi \), then the smallest spacetime cell that contributes to \( \Delta N_{\text{vac}} \) is a region of volume \( \Delta \Omega \) rather than \( \pi \). In this case, \( \overline{\rho}_{\text{vac}} \) is obtained from \( \rho_{\text{vac}} \) by substituting \( \Delta \Omega \) for \( \pi \). The effects of the dark component in the observed fermion number density are reduced as a result. When the resolution is sufficiently low, which means \( \Delta \Omega A >> 1 \), the quasiparticle picture with only one vacuum, namely the actual one, reemerges in the response of the system to \( K \). Therefore the dark component of the fermion number density is of transient nature. In most of the situations encountered in non-relativistic condensed matter system, the condition \( \Omega A >> 1 \) is satisfied so that a global chemical potential approach is sufficient.

3. The observation of the dark component at low energies

On the other hand, the \( K \) field radiated by the vacuum fermion number density is of the form \( K(x) = \int d^3x' G(x, x') \rho_{\text{vac}}(x') \) with \( G(x, x') \) the Green function for \( K(x) \). In a local QFT, it is \( \rho_{\text{vac}} \) that is the source.
for $K(x)$ rather than $\mathcal{F}_{vac}$, no matter how slow the resulting $K(x)$ varies in spacetime. Thus the effects of the dark component are indirectly observable even in low energy processes.

APPENDIX C: QCD LAGRANGIAN DENSITY WITH QUARK FIELD IN THE REAL 8 COMPONENT REPRESENTATION

The QCD Lagrangian density for strong interaction can be written as

$$\mathcal{L}_{QCD} = -\frac{1}{2} \text{Tr} G^{\mu
u} G_{\mu\nu} + \frac{1}{2} \bar{\Psi} (i\partial - igA - m_0) \Psi,$$

where

$$G^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu], \quad A^\mu = \sum_{a=1}^{8} B_a^\mu \Lambda^a$$

and

$$\Lambda^a = \frac{1}{2} [(1 + O_3)\lambda^a - (1 - O_3)\lambda^{aT}]$$

with $m_0$ current quark mass, $g$ the QCD coupling constant, $B^a$ the bosonic gauge fields, and $\lambda^a$ the Gell-Mann matrices.

It is easy to verify that $\Lambda^a$ ($a = 1, 2, \ldots, 8$) satisfy the same commutation relation with each other as the Gell–Mann matrices $\lambda^a$ do, namely,

$$[\Lambda^a, \Lambda^b] = i f^{abc} \Lambda^c,$$

where $f^{abc}$ (with $a, b, c = 1, 2, \ldots, 8$) are the set of group structure constants of $SU(3)$. Therefore $\Lambda^a$ belong to the same adjoint representation of $SU(3)$ as the one that $\lambda^a$ belong.

APPENDIX D: A FORMAL CONNECTION BETWEEN $S_{eff}[f]$ AND $\Gamma[f]$

A formal approach to relate $S_{eff}[f]$ and $\Gamma[f]$ can be found using the method developed in Ref. [14] by identifying $S_{eff}[f]$ as $I[\phi]$. Assuming that $f$ are all real, which does not loss any generality since a complex field can be regarded as two real fields, the relation between $\Gamma[f]$ and $S_{eff}[f]$ can be established through a generalized vertex functional $\tilde{\Gamma}[f, G]$, which is defined by

$$\tilde{\Gamma}[f, G] = iS_{eff}[f] - \frac{1}{2} Sp \ln D G^{-1} - \frac{1}{2} Sp (D^{-1} G - 1) + \Gamma_2[f, G],$$

where $\Gamma_2[f, G]$ is the contributions of all two particle irreducible graphs with lines representing $G$ in the shifted background fields $f$ and vertices obtainable from $S_{eff}[f]$ by expanding it around a set of shifted $\{f_i\}$ and keeping terms cubic in $f$ or higher. Here $D$ is the bare propagator for the boson fields and $D^{-1}$ is symbolically given by

$$D^{-1} = \frac{\delta^2 S_{eff}[f]}{\delta f \delta f}.$$

The proper vertex function $\Gamma[f]$ equals to $\tilde{\Gamma}[f, G]$ with $G$ satisfying the equation

$$\frac{\delta \tilde{\Gamma}[f, G]}{\delta G} \bigg|_{G=G_G} = 0.$$

In the Euclidean spacetime formulation, Eq. (D1) becomes

$$\Gamma_G[f, G] = S_{eff}[f] - \frac{1}{2} Sp \ln D G^{-1} - \frac{1}{2} Sp (D^{-1} G - 1) + \Gamma_2[f, G],$$

where all the quantities above are evaluated in the Euclidean spacetime.
APPENDIX E: THE NAMBU JONA–LASINIO MODEL

For the simplicity of the discussion, I consider a two flavor NJL model \[17\]. In the conventional 4-dimensional representation for the fermion fields, it takes the following form

\[ \mathcal{L} = \bar{\psi}i\slashed{\partial}\psi + G_0 \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}\gamma^5\vec{\tau}\psi)^2 \right] \]  \hspace{1cm} (E1)

with $G_0$ the coupling constant and $\vec{\tau}$ the three Pauli matrices in the isospin space. It has a $SU(2)_L \times SU(2)_R$ chiral symmetry. Due to the non-linear 4-fermion interaction term, it is not directly solvable. One of the best way to tackle the non-linear 4-fermion interaction model is to introduce auxiliary fields \[8\]. For this model two sets of auxiliary fields, namely $\sigma$ and $\vec{\pi}$ are necessary. The model Lagrangian density after the introduction of the auxiliary fields $\sigma$ and $\vec{\pi}$ is of the following form

\[ \mathcal{L} = \frac{1}{2} \bar{\Psi} \left( i\slashed{\partial} - \sigma - i\vec{\pi} \cdot \vec{\tau}\gamma^5 O_3 \right) \Psi - \frac{1}{4G_0} (\sigma^2 + \pi^2) \]  \hspace{1cm} (E2)

which is written in terms of the 8-dimensional representation $\Psi$ for the Dirac spinors for fermions.

The effective potential for this model can be computed using the Euclidean contour shown in Fig. 2. It has the following form

\[ V^{\text{eff}}(\sigma^2) = \frac{\Lambda^4}{4\pi} \left[ \frac{1}{4} \left( \frac{1}{\alpha_0} - \frac{6}{\pi} \right) \frac{\sigma^2}{\Lambda^2} + 3 \ln \left( 1 + \frac{\Lambda^4}{\sigma^4} \right) \frac{\sigma^4}{\Lambda^4} - \frac{3}{2\pi} \ln \left( 1 + \frac{\sigma^2}{\Lambda^2} \right) \right] \]  \hspace{1cm} (E3)

with $\alpha_0 = G_0\Lambda^2/4\pi$.

It has two phases. The system is in the first one, namely the $O$ phase, when $\alpha_0 \leq \pi/12$. It is in the second one called the $\alpha$ phase when the coupling constant $\alpha_0 > \pi/12$, where the original chiral $SU(2)_L \times SU(2)_R$ symmetry is spontaneously broken down to a $SU(2)_V$ flavor (isospin) symmetry and quark–antiquark pair condenses. Within the approximation adopted, the phase transition across the $\alpha_0 = \pi/12$ point is second order.
FIG. 1.
FIG. 2.
\[ V_{eff}/\Lambda^4 = a_0 = 0.8 \]

FIG. 3.
FIG. 4.
\[ \beta \text{ phase} \]
\[ SU(2)_L \times SU(2)_R \to SU(2)_V \]

\[ \alpha \text{ phase} \]
\[ SU(2)_L \times SU(2)_R \to SU(2)_V \]
FIG. 6.
FIG. 7.
FIG. 8.
FIG. 9.
FIG. 10.
FIG. 11.