Eulerian Derivation of the Coriolis Force

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In textbooks of geophysical fluid dynamics, the Coriolis force and the centrifugal force in a rotating fluid system are derived by making use of the fluid parcel concept. In contrast to this intuitive derivation to the apparent forces, more rigorous derivation would be useful not only for the pedagogical purpose, but also for the applications to other kinds of rotating geophysical systems rather than the fluid. The purpose of this paper is to show a general procedure to derive the transformed equations in the rotating frame of reference based on the local Galilean transformation and rotational coordinate transformation of field quantities. The generality and usefulness of this Eulerian approach is demonstrated in the derivation of apparent forces in rotating fluids as well as the transformed electromagnetic field equation in the rotating system.

I. INTRODUCTION

In textbooks of geophysical fluid dynamics [e.g., 2, 5, 7, 8] and educational web sites [e.g., 6], the apparent forces—the Coriolis force and the centrifugal force—are derived with the help of the framework of classical mechanics of a point particle; it is first shown that the time derivative of a vector $\mathbf{A}$ is written as $(d\mathbf{A}/dt)_I = (d\mathbf{A}/dt)_R + \Omega \times \mathbf{A}$, where $\Omega$ is a constant angular velocity of the rotating frame of reference; $I$ and $R$ stand for the inertial and rotating frames, respectively. The above relation for vector $\mathbf{A}$ is applied to a fluid parcel’s position $\mathbf{r}$ and then to its velocity $\mathbf{u}$, leading to the relation

$$(d\mathbf{u}_I/dt)_I = (d\mathbf{u}_R/dt)_R + 2\Omega \times \mathbf{u}_R + \Omega \times (\Omega \times \mathbf{r}).$$

Assuming that $(d\mathbf{u}_I/dt)_I$ equals to the local force acting per unit mass on a fluid parcel, the apparent forces in the rotating frame are derived.

The above derivation can be called as a Lagrangian approach since it exploits the concept of the fluid parcel. This Lagrangian derivation seems to be a standard style not only in the field of geophysical fluid dynamics, but also in more general fluid dynamics [e.g., 1, p.140].

The conventional Lagrangian derivation is ingenious and simple enough for introductory courses. But, why do we have to use the (Lagrangian) fluid parcel concept when we just want to derive the (Eulerian) field equation in the rotating frame? It should be possible to derive the apparent forces by a straightforward, rotating coordinate transformation of field quantities and the equation. The purpose of this paper is to introduce such an Eulerian derivation of the apparent forces that can be contrasted with the conventional Lagrangian derivation.

There are three advantages of the Eulerian approach shown in this paper compared to the conventional Lagrangian approach. Firstly, it is general. The Eulerian transformation is derived for any vector field [eqs. (16) and (29)]. Therefore, in addition to the fluid system, it can be used to derive, for example, the Maxwell’s equations in a rotating frame of reference in which the fluid parcel concept is invalid. (The Lagrangian approach does not work unless one could define proper working Lagrangian vector like $\mathbf{A}$ for the electromagnetic field.)

Secondly, physical meaning of the Eulerian derivation is clear. The Eulerian transformation of a vector field is composed of the local Galilean transformation and rotational transformation, as we will see in eq. (17). The transformation of the time derivative of a vector field [shown in eq. (29)] is also described by the local Galilean transformation and rotational transformation.

Thirdly, it is mathematically rigorous. The key of the simplicity of the conventional Lagrangian derivation is eq. (1). But note that the expression of $d\mathbf{u}/dt$ is actually an abbreviated form, when it is finally applied to the fluid equation, of rather complicated terms; $\partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla) \mathbf{u}$. Note that the second term is nonlinear of $\mathbf{u}$. If one substitutes $(d\mathbf{u}_I/dt)_I = \partial \mathbf{u}_I/\partial t + (\mathbf{u}_I \cdot \nabla) \mathbf{u}_I$, and $(d\mathbf{u}_R/dt)_R = \partial \mathbf{u}_R/\partial t + (\mathbf{u}_R \cdot \nabla) \mathbf{u}_R$, into the left-hand and right-hand sides of eq. (1), respectively, the covered complexity of the equation becomes apparent that requires a mathematical proof.

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In our opinion, the conventional Lagrangian derivation of the apparent forces is intuitive and simple, while the Eulerian derivation shown here is rigorous and straightforward. These two derivations would be regarded as a complementary approaches to the understanding of the apparent forces including the Coriolis force.

The authors could not find the Eulerian derivation of this kind in textbooks on fluid and geophysical fluid dynamics.

II. VECTOR FIELDS IN A ROTATING FRAME OF REFERENCE

In order to derive the general expression of a vector field in a rotating frame of reference, we start from the Galilean transformation of a vector field between two inertial frames. Let \( L_I \) and \( L'_I \) be inertial frames with relative velocity \( \mathbf{V} \):

\[
x' = x - \mathbf{V}t, \tag{2}
\]

where \( x \) and \( x' \) are coordinates in \( L_I \) and \( L'_I \), respectively. When a vector field \( \mathbf{a}(x,t) \) is defined in \( L_I \), it is observed in \( L'_I \) as

\[
\mathbf{a}'(x',t) = G^\mathbf{V} \mathbf{a}(x,t), \tag{3}
\]

where \( G^\mathbf{V} \) is a Galilean transformation operator. For example, the transformation of a fluid flow \( \mathbf{u}(x,t) \) is given by

\[
\mathbf{u}'(x',t) = G^\mathbf{V} \mathbf{u}(x,t) \equiv \mathbf{u}(x,t) - \mathbf{V}. \tag{4}
\]

Other examples of the Galilean transformation operator \( G^\mathbf{V} \) are for the magnetic field \( \mathbf{B} \) and the electric field \( \mathbf{E} \):

\[
\mathbf{B}'(x',t) = G^\mathbf{V} \mathbf{B}(x,t) \equiv \mathbf{B}(x,t), \tag{5}
\]

\[
\mathbf{E}'(x',t) = G^\mathbf{V} \mathbf{E}(x,t) \equiv \mathbf{E}(x,t) + \mathbf{V} \times \mathbf{B}(x,t). \tag{6}
\]

These transformations are derived from the Lorentz transformation in the limit of \( V \ll c \), where \( c \) is the speed of light. When a vector field \( \mathbf{F} \) is a function of a vector field \( \mathbf{a} \), \( \mathbf{F} = \mathbf{F}(\mathbf{a}) \), its transformation is given by

\[
\mathbf{F}' = G^\mathbf{V} \mathbf{F}(\mathbf{a},t) \equiv \mathbf{F}(G^\mathbf{V} \mathbf{a},t). \tag{7}
\]

For example, when \( \mathbf{F} = (\mathbf{u} \cdot \nabla)\mathbf{u} \),

\[
(\mathbf{u}' \cdot \nabla')\mathbf{u}' = G^\mathbf{V} (\mathbf{u} \cdot \nabla)\mathbf{u} \equiv [(\mathbf{u}(x,t) - \mathbf{V}) \cdot \nabla](\mathbf{u}(x,t) - \mathbf{V}), \tag{8}
\]

where we have used the equivalence of the operators \( \nabla \) and \( \nabla' \) defined for two inertial frames \( L \) and \( L' \).

Let \( \hat{L}_R \) be a rotating frame of reference with constant angular velocity \( \Omega \) with respect to \( L_I \). For simplicity, we suppose that \( \hat{L}_R \) and \( L_I \) share the same origin and \( z \)-axis, and \( \hat{L}_R \) is rotating around the \( z \)-axis; \( \Omega = (0, 0, \Omega) \). The coordinates \( \mathbf{x} \) and \( \hat{\mathbf{x}} \) of a point observed in \( L_I \) and \( \hat{L}_R \) are related as

\[
\hat{\mathbf{x}} = R^\Omega \mathbf{x}, \tag{9}
\]

where \( R^\Omega \) denotes the rotational transformation with matrix expression

\[
R^\Omega = \begin{pmatrix}
\cos \Omega t & \sin \Omega t & 0 \\
-\sin \Omega t & \cos \Omega t & 0 \\
0 & 0 & 1
\end{pmatrix}. \tag{10}
\]

Suppose a point \( P \) at coordinates \( \mathbf{x} \) that is fixed in the rotating frame \( \hat{L}_R \). The point \( P \) is observed as a circular trajectory in \( L_I \). Let \( \mathbf{x}_t \) and \( \mathbf{x}_{t+\Delta t} \) be two positions of \( P \) in the inertial frame \( L_I \) at two successive time steps \( t \) and \( t + \Delta t \). Equation (9) reads

\[
\hat{\mathbf{x}} = R^\Omega(t+\Delta t) \mathbf{x}_{t+\Delta t} = R^\Omega \mathbf{x}_t. \tag{11}
\]

Since the inverse transformation of the rotation \( R^\Omega \Delta t \) is \( R^{-\Omega \Delta t} \), we obtain

\[
\mathbf{x}_{t+\Delta t} = R^{-\Omega \Delta t} \mathbf{x}_t. \tag{12}
\]
Similarly, from eqs. (5) and (6), we obtain the transformation formulae for the magnetic field and the electric field:

\[ \mathbf{a}(x, t) = \Omega \times \mathbf{x}, \]  

and

\[ \mathbf{a}(x, t) = \Omega \times \mathbf{x}, \]

Applying the above local Galilean transformation for the specific point \( P \) in \( L' \), we obtain

\[ \mathbf{a}(x, t) = \Omega \times \mathbf{x}, \]

where, from eq. (16),

\[ \mathbf{a}(x, t) = \Omega \times \mathbf{x}, \]

Here, we have used the following equation for any vector \( \mathbf{a} \)

\[ \mathbf{a} = \mathbf{u} + \mathbf{x} \times \Omega, \]

In general, the transformation of a vector field \( \mathbf{F} \) that is a function of a vector field \( \mathbf{a} \), \( \mathbf{F} = \mathbf{F}(\mathbf{a}, t) \), is given from eq. (11) as

\[ \mathbf{F}(\mathbf{a}, t) = \Omega \times \mathbf{x} \mathbf{F}(\mathbf{a}, t) = \Omega \times \mathbf{x} \mathbf{F}(\mathbf{a}, t). \]

The next step is to derive the transformation of the time derivative of a vector field, \( \partial \mathbf{a}/\partial t \), where the partial derivative should be taken with fixed coordinates \( \mathbf{x} \) in the rotating frame of reference \( \hat{L}_R \):

\[ \frac{\partial \mathbf{a}}{\partial t}(\mathbf{x}, t) = \lim_{\Delta t \to 0} \frac{\mathbf{a}(\mathbf{x}, t + \Delta t) - \mathbf{a}(\mathbf{x}, t)}{\Delta t}, \]

where, from eq. (10),

\[ \mathbf{a}(\mathbf{x}, t + \Delta t) = \Omega \times \mathbf{x} \mathbf{a}(\mathbf{x}, t + \Delta t, t + \Delta t), \]
and
\[ \mathbf{a}(\mathbf{x}, t) = R_{\Omega}^{\mathbf{a}} G^{\Omega \times \mathbf{x}} \mathbf{a}(\mathbf{x}, t). \] (25)

Substituting eq. (13) into \( \mathbf{a}(\mathbf{x}_{t+\Delta t}, t + \Delta t) \) and expanding it with respect to \( \Delta t \) to the first order,
\[ \mathbf{a}(\mathbf{x}_{t+\Delta t}, t + \Delta t) = \mathbf{a}(\mathbf{x}, t) + \Delta t \left[ (\Omega \times \mathbf{x}) \cdot \nabla \right] \mathbf{a}(\mathbf{x}, t) + \Delta t \frac{\partial \mathbf{a}}{\partial t}(\mathbf{x}, t). \] (26)

From eqs. (23)–(26) with the aid of the following relations
\[ \lim_{\Delta t \to 0} \frac{R_{\Omega}^{\mathbf{a}} \Delta t \mathbf{a} - \Delta t \mathbf{a}}{\Delta t} = \mathbf{a} \times \Omega, \] (27)

and
\[ G^{\Omega \times \mathbf{x} + \Delta t} = G^{\Omega \times \mathbf{x} + \Delta t \Omega \times (\Omega \times \mathbf{x})} = G^{\Omega \times \mathbf{x}} (1 + O[\Delta t]), \] (28)

we get
\[ \frac{\partial \mathbf{a}}{\partial t}(\mathbf{x}, t) = R_{\Omega}^{\mathbf{a}} \left[ \frac{\partial}{\partial t} + (\Omega \times \mathbf{x}) \cdot \nabla \right] G^{\Omega \times \mathbf{x}} \mathbf{a}(\mathbf{x}, t). \] (29)

Here we have used \( \mathbf{x} \) instead of \( \mathbf{x}_{t} \), for brevity. This is the general transformation formula for time derivative of vector field \( \mathbf{a} \) between \( L_{I} \) and \( \hat{L}_{R} \).

A special case of eq. (29) is given when the vector field \( \mathbf{a} \) is Galilean invariant, i.e., \( G = 1 \), such as the magnetic field \( \mathbf{B} \) [see eq. (5)];
\[ \frac{\partial \mathbf{B}}{\partial t}(\mathbf{x}, t) = R_{\Omega}^{\mathbf{B}} \left[ \frac{\partial}{\partial t} + (\Omega \times \mathbf{x}) \cdot \nabla \right] \mathbf{B}(\mathbf{x}, t). \] (30)

The transformation of the fluid flow \( \mathbf{u}(\mathbf{x}, t) \) between \( L_{I} \) and \( \hat{L}_{R} \) is obtained by substituting the Galilean transformation operator \( G^V \) for \( \mathbf{u} \) defined in eq. (11) into eq. (29);
\[ \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t) = R_{\Omega}^{\mathbf{u}} \left[ \frac{\partial}{\partial t} + (\Omega \times \mathbf{x}) \cdot \nabla \right] \mathbf{u}(\mathbf{x}, t) + R_{\Omega}^{\mathbf{u}} (\Omega \times \mathbf{x}) \cdot \nabla \mathbf{u}(\mathbf{x}, t). \] (31)

where we have used eq. (21) with \( \mathbf{a} = \Omega \times \mathbf{x} \). It is interesting that the transformation rule of the fluid flow \( \mathbf{u} \) is exactly the same as that of the magnetic field \( \mathbf{B} \) although \( \mathbf{u} \) is not a Galilean invariant vector.

### III. TRANSFORMATIONS OF THE NAVIER-STOKES EQUATION

The Navier-Stokes equation for an incompressible fluid in the inertial frame \( L_{I} \) is written as
\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \mathbf{f}, \] (32)

where \( \mathbf{f} = -\nabla p + \nu \nabla^2 \mathbf{u} \) with viscosity \( \nu \). The pressure gradient term \( \nabla p \) is Galilean invariant vector, and another term in the total force \( \mathbf{f} \) is transformed as
\[ \nabla^2 \mathbf{u} = R_{\Omega}^{\mathbf{u}} \nabla^2 G^{\Omega \times \mathbf{x}} \mathbf{u} \]
\[ = R_{\Omega}^{\mathbf{u}} \nabla^2 G^{\Omega \times \mathbf{x}} \mathbf{u} \quad \text{[cf. eq. (22)]} \]
\[ = R_{\Omega}^{\mathbf{u}} \nabla^2 (\mathbf{u} + \mathbf{x} \times \Omega) \quad \text{[cf. eq. (17)]} \]
\[ = R_{\Omega}^{\mathbf{u}} \nabla^2 \mathbf{u}. \] (33)
Therefore, the force term $f$ is transformed as a Galilean invariant field:
\[
\hat{f}(\hat{x}, t) = R^\Omega_t f(x, t).
\] (34)

Now, let us derive the transformed form of the Navier-Stokes equation \[32\] in the rotating frame $\hat{L}_R$. Combining eqs. \[20\] and \[31\], we get
\[
\frac{\partial \hat{u}}{\partial t} + (\hat{u} \cdot \nabla)\hat{u} = R^\Omega_t \frac{\partial u}{\partial t} + (u \cdot \nabla)u + 2u \times \Omega + (x \times \Omega) \times \Omega].
\] (35)

The last two terms in the right-hand side are rewritten as follows
\[
R^\Omega_t \{2u \times \Omega + (x \times \Omega) \times \Omega} = 2R^\Omega_t (u \times \Omega) + R^\Omega_t \{ (x \times \Omega) \times \Omega} = 2(R^\Omega_t u) \times \Omega + \{(R^\Omega_t x) \times \Omega} \times \Omega
\]
\[\]
\[= 2\{\hat{u} - R^\Omega_t (x \times \Omega)\} \times \Omega [\text{cf. eq. (17)}] \]
\[+ \{(R^\Omega_t x) \times \Omega} \times \Omega
\]
\[= 2\hat{u} \times \Omega + (\Omega \times \hat{x}) \times \Omega [\text{cf. eq. (3)}],
\] (36)

which leads to
\[
\frac{\partial \hat{u}}{\partial t} + (\hat{u} \cdot \nabla)\hat{u} - 2\hat{u} \times \Omega - (\Omega \times \hat{x}) \times \Omega = R^\Omega_t \frac{\partial u}{\partial t} + (u \cdot \nabla)u.
\] (37)

From eqs. \[32\], \[34\], and \[37\], we finally get
\[
\frac{\partial \hat{u}}{\partial t} + (\hat{u} \cdot \nabla)\hat{u} = \hat{f} + 2\hat{u} \times \Omega + (\Omega \times \hat{x}) \times \Omega.
\] (38)

The second term in the right hand side is the Coriolis force, and the third term is the centrifugal force. Note that even if we have another Galilean invariant force in $f$ in the right hand side of eq. \[32\], e.g., the buoyancy force, the equation form of \[35\] does not change. Note also that the Lorentz force in the magnetohydrodynamics [e.g., \[3\]] is Galilean invariant.

**IV. SUMMARY AND DISCUSSION**

In this paper, we have shown a general algorithm to derive the evolving equation in a constantly rotating frame of reference based on the local Galilean transformation and the rotating coordinate transformation of field quantities. This derivation—Eulerian derivation—is applied in a straightforward way to a rotating fluid system to derive the Coriolis force and the centrifugal force. When the angular velocity of the rotating frame is time dependent, i.e., $\Omega(t) \equiv d\Omega(t)/dt \neq 0$, the instantaneous local reference frame fixed at a position $x$ in the rotating system is not an inertial frame, but rather an accelerating frame with the acceleration rate $A(x, t) = \Omega(t) \times x$. Therefore, another pseudo force, or the inertial force, $-A(x, t)$, per unit mass appears in the equation of motion. This term plays important roles in some rotating fluid dynamics such as precession and mutations \[4\].

The usefulness of the Eulerian derivation becomes evident when we apply it to the derivation of the basic equation in the rotating system for other physical systems rather than the fluid. Take the magnetic field $B$ for the example. From the Maxwell’s equations, the induction equation of $B$ is written as
\[
\frac{\partial B}{\partial t} (x, t) = -\nabla \times E(x, t),
\] (39)
in the inertial frame $L_I$. The $\nabla \times E$ term in the right hand side of this equation is transformed into the following form in the rotating frame $L_R$:
\[
\nabla \times \hat{E}(\hat{x}, t)
\]
\[= R^\Omega_t G^{\Omega \times x} \nabla \times E(x, t) [\text{cf. eq. (10)}]
\]
\[= R^\Omega_t \nabla \times \{ G^{\Omega \times x} E(x, t) \} [\text{cf. eq. (22)}]
\]
\[= R^\Omega_t \nabla \times E(x, t) + (\Omega \times x) \times B [\text{cf. eq. (14)}]
\]
\[= R^\Omega_t \{ \nabla \times E(x, t) - \{(\Omega \times x) \cdot \nabla\} B + \Omega \times B \}.
\] (40)
Here we have used $\nabla \cdot \mathbf{B} = 0$, $\nabla \cdot (\Omega \times \mathbf{x}) = 0$, and eq. (21) in the last step. Comparing eq. (30) with eq. (40), we get

$$\frac{\partial \mathbf{B}}{\partial t}(\hat{\mathbf{x}}, t) = -\nabla \times \hat{\mathbf{E}}(\hat{\mathbf{x}}, t).$$  \hspace{1cm} (41)

Therefore, the induction equation does not change its form in the rotating frame of reference. (There is no “apparent induction” term.) This example clearly illustrates the advantage of Eulerian approach of the transformation of the basic equation in the rotating frame.

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[1] Bachelor, G. K. (1970), An Introduction to Fluid Dynamics, Cambridge Univ. Press, Cambridge.
[2] Cushman-Roisin, B. (1994), Introduction to Geophysical Fluid Dynamics, Prentice-hall, Inc., London.
[3] Davidson, P. A. (2001), An Introduction to Magnetohydrodynamics, Cambridge University Press, Cambridge, UK.
[4] Greenspan, H. P. (1990), The Theory of Rotating Fluids, Breukelen Press, Brookline, MA.
[5] Pedlosky, J. (1979), Geophysical Fluid Dynamics, Springer-Verlag New York Inc., New York.
[6] Price, J. F. (2004), A Coriolis tutorial, URL [http://www.whoi.edu/science/PO/people/jprice/class/ac4.pdf], Version 3.1.4, 1–50.
[7] Salmon, R. (1998), Lectures on Geophysical Fluid Dynamics, Oxford University Press, New York.
[8] Stommel, H. M., and D. W. Moore (1989), An Introduction to the Coriolis Force, Columbia University Press.