Multispecies TASEP and combinatorial $R^*$

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Abstract
We identify the algorithm for constructing steady states of the $n$-species totally asymmetric simple exclusion process (TASEP) on an $L$ site periodic chain by Ferrari and Martin with a composition of combinatorial $R$ for the quantum affine algebra $U_q(\hat{sl}_L)$ in crystal base theory. Based on this connection and the factorized form of the $R$ matrix derived recently from the tetrahedron equation, we establish a new matrix product formula for the steady state of the TASEP, which is expressed in terms of corner transfer matrices of the $q$-oscillator valued five-vertex model at $q = 0$.

Keywords: multispecies TASEP, combinatorial $R$, tetrahedron equation, crystal base

1. Introduction

In this paper and the next [19] we launch a new approach and integrable structure on the $(n+1)$-state totally asymmetric simple exclusion process (TASEP) on a one-dimensional (1D) periodic chain $\mathbb{Z}_L$ of $L$ sites. It is called $n$-species TASEP or $n$-TASEP for short. A local spin at each site takes values in $\{0, 1, \ldots, n\}$ and the neighboring pairs $(\alpha, \beta)$ with $\alpha > \beta$ are exchanged to $(\beta, \alpha)$ with a uniform probability. It is a model of non-equilibrium stochastic dynamics in physical, biological and many other systems including traffic flow, etc. See for example [5, 7] and reference therein.

The first basic problem in the $n$-TASEP is to determine its steady state, which exhibits an intriguing non-uniform measure for $n \geq 2$ as in example 3.1. It has been solved in [11] by introducing a companion system on an $n$-tuple of $\{0, 1\}$-sequences called multilne process. Steady states of the multiline process possess a uniform measure and those for the $n$-TASEP

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* Dedicated to the memory of Professor Ryogo Hirota.

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are obtained as the image by a certain projection \( \pi \) from the former to the latter. The map \( \pi \) playing the key role is constructed via a combinatorial procedure on the \( n \)-tuple of \{0, 1\}-sequences which we call the Ferrari–Martin algorithm.

Our first main observation is that the Ferrari–Martin algorithm is most naturally formulated in terms of combinatorial \( R \) in crystal base theory, a theory of quantum group at \( q = 0 \) [16]. More specifically, each \{0, 1\}-sequence in the multiline process is regarded as a crystal \( Bl \) of a \( n \)-antisymmetric tensor representation of the Drinfeld–Jimbo quantum affine algebra \( U_q(\widehat{sl}_L) \). The ‘2-body operation’ of the adjacent \{0, 1\}-sequences described as arrival/service/departure [11] is nothing but the combinatorial \( R \) acting on the tensor product \( B^i \otimes B^m \) of crystals [21]. It is the bijection that arises from the quantum \( R \) matrix at \( q = 0 \) (crystallization) which still satisfies the Yang–Baxter equation. The ‘\( n \)-body operation’ in the multiline process is identified, by using the Yang–Baxter equation, with the corner transfer matrix [3] of the size-\( n \) vertex model whose Boltzmann weights are crystallized to the combinatorial \( R \). See (4.2). We remark that such a size-\( n \) system equipped with the internal symmetry \( U_q(\widehat{sl}_L) \) corresponds to the cross-channel of the original \( n \)-TASEP in the sense that the role of the physical space \( \mathbb{Z}_L \) and the internal space \{0, …, \( n \)\} is interchanged. In particular the cyclic symmetry of the physical space \( \mathbb{Z}_L \) in the \( n \)-TASEP has been incorporated into the Dynkin diagram of \( U_q(\widehat{sl}_L) \). The integrability in the direct-channel, i.e. a relation to a \( U_q(\widehat{sl}_{n+1}) \) spin chain on \( \mathbb{Z}_L \), has been demonstrated for a more general ASEP in [2 section 5].

Our second result is the matrix product representation (4.4) of the steady state probability. It is expressed in terms of the operator \( X_i \) (4.6) which can be regarded as a corner transfer matrix of the five-vertex model (2.20) whose Boltzmann weights take values in a \( q \)-deformed oscillator algebra at \( q = 0 \) (2.16). It is obtained by combining the crystal formulation of the Ferrari–Martin algorithm with the factorized form of the combinatorial \( R \) (2.22). The latter is the \( q = 0 \) corollary of the factorized form of the quantum \( R \) matrix established recently [20]. The origin of the factorization itself goes further back to a 3D generalization of the Yang–Baxter equation known as the tetrahedron equation [24]. In fact our operator \( X_i \), which is different from the earlier one [10], possesses a far-reaching generalization in the framework of a 3D lattice model associated with the tetrahedron equation [19].

The layout of the paper is as follows. In section 2 we recall the background from \( U_q(\widehat{sl}_L) \) and the combinatorial \( R \) necessary in this paper. In section 3 we treat the \( n \)-TASEP and reformulate the Ferrari–Martin algorithm in terms of crystals and combinatorial \( R \). In section 4 a new matrix product formula for the steady state probability is derived by synthesizing the contents in sections 2 and 3. Section 5 contains a remark on a generalization of stochastic dynamics from the viewpoint of crystal base theory and a brief announcement on the so called hat relation. There are notable 3D structures connected to the tetrahedron equation behind several scenes in this paper. They will be fully demonstrated in our forthcoming paper [19].

2. Background facts from \( U_q(\widehat{sl}_L) \)

Let us recapitulate relevant results from the representation theory of quantum affine algebras, matrix product forms of the quantum \( R \) matrix and combinatorial \( R \).

2.1. \( U_q(\widehat{sl}_L) \) and antisymmetric representation

The Drinfel’d–Jimbo quantum affine algebra \( U_q = U_q(\widehat{sl}_L) \) is a Hopf algebra with generators \( e_i, f_i, k_i \) \((i \in \mathbb{Z}_L)\) satisfying the Serre relation and the following [8, 15]
\[ k_i k_j^{-1} = k_j^{-1} k_i = 1, \quad [k_i, k_j] = 0, \quad k_i e_j = q^{a_{ij}} e_j k_i, \quad k_i f_j = q^{-a_{ij}} f_j k_i, \]

\[ [e_i, f_j] = \delta_{ij} k_i - k_i^{-1} q - q^{-1}, \]

\[ e_j^2 e_i - (q + q^{-1}) e_i e_j e_i + e_i e_i^2 = 0, \quad f_j^2 f_i - (q + q^{-1}) f_i f_j f_i + f_i f_i^2 = 0 \]

\[(i - j \equiv \pm 1 \mod L).\]

Here \( a_{ij} = 2\delta_{ij} - \delta_{i,j-1} - \delta_{i,j+1} \) with \( \delta_{ij} = 1 \) if \( i \equiv j \) mod \( L \) and \( \delta_{ij} = 0 \) otherwise. All the indices are to be understood in \( \mathbb{Z}_L \) likewise. We adopt the coproduct \( \Delta : U_q \to U_q \otimes U_q \) of the form

\[ \Delta e_i = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta f_i = 1 \otimes f_i + f_i \otimes k_i^{-1}, \quad \Delta k_i^{\pm 1} = k_i^{\pm 1} \otimes k_i^{\pm 1}. \]

Our \( \Delta \) here is opposite of \([20]\). We assume that \( q \) is a generic complex parameter in sections 2.1–2.3.

We are concerned with the degree \( l \) antisymmetric tensor representation \( \phi_l : U_q(SL_L) \to \text{End} V^l \) with \( 1 \leq l < L \). The representation space \( V^l \) is given by

\[ V^l = \bigoplus_{b \in B}/ C[b], \quad B^l = \{ b = (b_1, \ldots, b_L) \in \{0, 1\}^L | |b| = l \} \tag{2.1} \]

where \( |b| = b_1 + \cdots + b_L \). Thus \( \dim V^l = \binom{L}{l} \). The labeling set \( B^l \) of the basis of \( V^l \) is called the crystal of \( V^l \). Note that the dependence on \( L \) is not indicated in the notation \( B^l \). To save the space, elements of \( B^l \) will often be written as words on \( \{0, 1\} \), e.g. \( (1, 0, 1, 0, 0, 0) \in B^3 \) is just denoted by \( 10110 \), etc.

Generators act on \( V^l \) as \( (g \phi_l) \) for a generator \( g \) is denoted by \( g \) for simplicity

\[ e_i |b\rangle = x^{\delta_{i+1}}|b + e_i|\rangle, \quad f_i |b\rangle = x^{-\delta_{i+1}}|b - e_i|\rangle, \quad k_i^{\pm 1}|b\rangle = q^{\pm (b_{i+1} - b_i)}|b\rangle, \]

where \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^L \) and the right hand sides are to be understood as 0 unless the arrays in the ket stay within \( \{0, 1\}^L \). The representation \( \phi_l \) is irreducible and \( x \) is called a spectral parameter.

### 2.2. Quantum R matrix

Let \( \Delta_{x,y} = (\phi_l \otimes \phi_l) \Delta \) be the tensor product representations on \( V^l \otimes V^m \), where \( 1 \leq l, m < L \) are arbitrary. It is known that they are irreducible if \( z = x/y \) is generic. Moreover there is a linear map \( R(z) = R_{l,m}^{x,y}(z) : V^l \otimes V^m \to V^m \otimes V^l \) called a quantum \( R \) matrix which is uniquely characterized by the intertwining relation with the quantum group \( \Delta_{x,y}(g) R(z) = R(z) \Delta_{x,y}(g) \) for \( g \in U_q(SL_L) \) up to an overall scalar. Let us write its matrix elements on the basis \( |i\rangle \otimes |j\rangle \in V^l \otimes V^m \) as

\[ R(z)(|i\rangle \otimes |j\rangle) = \sum_{a,b} R(z)_{i,j}^{a,b} |b\rangle \otimes |a\rangle, \tag{2.2} \]

where \( (|i|, |j|) = (l, m) \) and the sum extends over \( a, b \) such that \( (|a|, |b|) = (l, m) \). The matrix elements have the support property reflecting the weight conservation:

\[ R(z)_{i,j}^{a,b} = 0 \text{ unless } a + b = i + j \quad (\in \{0, 1, 2\}^L). \tag{2.3} \]

\(^4\) A more standard notation is a one-column semi-standard tableau, the one filled with 1, 3, 4 for this example.
The $R$ matrix satisfies the Yang–Baxter equation [3]

\[
\left( R^{l,m}(z) \otimes 1 \right) \left( 1 \otimes R^{k,m}(zz') \right) \left( R^{k,l}(z') \otimes 1 \right) = \left( 1 \otimes R^{l,m}(zz') \right) \left( R^{k,l}(z') \otimes 1 \right) \left( 1 \otimes R^{k,m}(z) \right)
\]

(2.4)

for any $1 \leq k, l, m < L$. We normalize the $R^{l,m}(z)$ so that

\[
R(z) \left( |e_{l,m} \rangle \otimes |e_{l,m} \rangle \right) = \tilde{q}(z) |e_{l,m} \rangle \otimes |e_{l,m} \rangle, \quad \tilde{q}(z) = \prod_{i=(l+m-L)_+}^{\min(l,m)-1} \left( 1 - (-1)^{i+m} q^{i+m-2} \right),
\]

(2.5)

where $(m)_+ = \max(m, 0)$ and $|e_{l,m} \rangle = \{ e_1 + \cdots + e_l \} \in V^l$. Then all the elements $R(z)_{a_b}^c$ are polynomials in $q$ and $z$.

**Example 2.1.** The $R^{1,1}(z)$ is the well known $R$ matrix for the vector representation of $U_q(\mathfrak{sl}_2)$. The $L = 2$ case corresponds to the six-vertex model [3]. The nonzero elements read

\[
R(z)_{e_i e_j} = 1 - q^2 z, \quad R(z)_{e_i e_j} = q (1 - z), \quad R(z)_{e_i e_j} = \left( 1 - q^2 \right) z^{\theta(i<j)},
\]

where $1 \leq i \neq j \leq L$ and $\theta(\text{true}) = 1$, $\theta(\text{false}) = 0$.

**Example 2.2.** Nonzero elements of $R^{2,1}(z)$ for $U_q(\mathfrak{sl}_3)$ are as follows.

\[
R^{101,100}_{100,101} = R^{101,100}_{101,100} = R^{101,100}_{100,101} = R^{100,101}_{100,101} = R^{100,101}_{100,101} = R^{100,101}_{100,101} = 1 + q^2 z,
\]

\[
R^{011,010}_{010,101} = R^{011,010}_{010,101} = R^{011,010}_{010,101} = q (1 + q z), \quad R^{001,101}_{001,101} = R^{001,101}_{001,101} = q R^{001,101}_{001,101} = -q (1 - q^2) z,
\]

**Example 2.3.** Nonzero elements of $R^{2,1}(z)$ for $U_q(\mathfrak{sl}_3)$ are as follows.

\[
R^{101,100}_{110,100} = R^{101,100}_{110,100} = R^{101,100}_{110,100} = R^{100,101}_{100,101} = R^{100,101}_{100,101} = R^{100,101}_{100,101} = 1 + q^2 z,
\]

\[
R^{011,010}_{010,101} = R^{011,010}_{010,101} = R^{011,010}_{010,101} = q (1 + q z), \quad R^{011,010}_{010,101} = R^{011,010}_{010,101} = q R^{011,010}_{010,101} = -q (1 - q^2) z,
\]

See for example [6] for more information on the quantum $R$ matrix.

### 2.3. Matrix product representation of $R(z)$

In a recent work [20] based on the tetrahedron equation, a matrix product representation of the quantum $R$ matrix $R^{l,m}(z)$ was constructed. Let us quote the result in a form adapted to the present convention.

Consider the $q$-oscillator algebra $A_q$ generated by $a^+, a^-, k$ satisfying the relations

\[
k a^2 = -q z^2 a^2 k, \quad a^+ a^- = 1 - k^2, \quad a^- a^+ = 1 - q^2 k^2.
\]

(2.6)

They may be identified with the operators acting on the Fock space $F = \oplus_{m \geq 0} C |m \rangle$ as

\[
a^+ |m \rangle = |m + 1 \rangle, \quad a^- |m \rangle = \left( 1 - q^2 m \right) |m - 1 \rangle, \quad k |m \rangle = (-q)^m |m \rangle.
\]

(2.7)

We arrange them in the 3D $L$ operator $\mathcal{L} = (\mathcal{L}_{ij}^{\alpha \beta})$ [4] where all the indices belong to $\{0, 1\}$. They are zero except the following:

\[5\]

This ket $|m \rangle \in F$ with $m \in \mathbb{Z}^+_0$ should not be confused with $|b \rangle \in V^l$ with $b \in B^l$. 

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The $\mathcal{L}$ may be viewed as defining a six-vertex model whose Boltzmann weights belong to $A_q$, i.e., $q$-oscillator valued six-vertex model:

$$
\mathcal{L}^{a,b}_{i,j} = \begin{bmatrix}
0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & a^+ & a^- & k & qk
\end{bmatrix}
$$

The relations in (2.6) involving $aa^\pm$ are quantization of the so called free fermion condition [3 eq.(10.16.4)]. One may introduce the third arrow perpendicular to the two arrows for each vertex encoding the action of the corresponding $q$-oscillator generators. It amounts to regarding $a$ as a vertex of the 3D square lattice, hence the name 3D $L$ operator. The resulting 3D classical spin model satisfies the tetrahedron equation [20 eq.(2.13)] involving the 3D $R$ operator obtained by replacing $q$ by $-q$ in [20 eq.(2.2)]. See also [4]. Such a 3D structure is a source of the factorization of the $R$ matrix in theorem 2.4 and ultimately the matrix product form of the steady state probability that we will derive in section 4.

Let $h$ be the operator on $F$ defined by $h| m \rangle = m| m \rangle$. Denote by $F^* = \bigoplus_{m \geq 0} C(m)$ the dual of $F$ with the pairing specified by $\langle m | m' \rangle = (q^2)^m \delta_{m,m'}$ with $(q^2)^m = \prod_{j=1}^m (1 - q^{2j})$. Accordingly we set $\text{Tr} (z^b X) = \sum_{m \geq 0} \frac{z^b (m | X | m)}{q^m}$ for $X \in A_q$. Set $\tilde{q}(z) = q^{-[l-m]} \times (1 - (-1)^{l+m} q^{l-m} z)\tilde{q}(z)$ where $\tilde{q}(z)$ is defined in (2.5). The following result is a special case $1 \leq l \leq 7$ of [20 Th.4.1].

Theorem 2.4. Elements of the $R$ matrix (2.2) with the normalization (2.5) are expressed in the matrix product form

$$
\mathcal{R} (z^{a,b}_{i,j}) = \tilde{q}(z) \text{Tr} \left( z^b h \mathcal{L}^{a,b}_{i,j} \cdots \mathcal{L}^{a,z,b}_g \right),
$$

where $a = (a_1, \ldots, a_L)$ and similarly for $b$, $i$, $j$.

For instance we have

$$
\mathcal{R}^{0,1,100}_{0,1,100} = \tilde{q}(z) \text{Tr} \left( z^b h \mathcal{L}^{0,1,0}_{0,1,0} \mathcal{L}^{1,0}_{1,0} \right) = q^2 \tilde{q}(z) \text{Tr} (z^b k^z) = \frac{q^2 \tilde{q}(z)}{1 + q^2 z}
$$
in agreement with example 2.3 due to $(l, m) = (2, 1)$ and $q(z) = q^{-1} (1 + qz) (1 + q^3 z)$.

We remark that the operators $\epsilon, \delta, A$ used for the $n$-ASEP [22] can be realized in $A_q$ via (2.6) via $\epsilon \mapsto a^+, \delta \mapsto a^-, A \mapsto k^2$.

2.4. Combinatorial $R$

The product set of crystals $B^l$ and $B^m$ is again called a crystal and denoted by $B^l \otimes B^m$. As mentioned before, all the elements of the quantum $R$ matrix $\mathcal{R} = R^{l,m}$ are polynomials in $q$. Therefore it makes sense to set $q = 0$. We thus define $R = R^{l,m} = R^{l,m}(z = 1)|_{q=0}$. It turns out that $R$ becomes a beautiful bijection among the crystals $R^{l,m} : B^l \otimes B^m \rightarrow B^m \otimes B^l$, which is called a combinatorial $R$.

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6 Although $z$ will not be concerned in this paper it is an important ingredient of the combinatorial $R$ in the literature.
Example 2.5. Example 2.2 with (2.2) yields the combinatorial $R^{1,2}: B^1 \otimes B^2 \to B^2 \otimes B^1$ as

\[
\begin{align*}
100 \otimes 110 \to & \ 110 \otimes 100, \\
100 \otimes 101 \to & \ 101 \otimes 100, \\
100 \otimes 011 \to & \ 011 \otimes 100, \\
010 \otimes 110 \to & \ 110 \otimes 010, \\
010 \otimes 011 \to & \ 011 \otimes 010, \\
001 \otimes 101 \to & \ 101 \otimes 001,
\end{align*}
\]

Example 2.6. Example 2.3 with (2.2) yields the combinatorial $R^{2,1}: B^2 \otimes B^1 \to B^1 \otimes B^2$ as

\[
\begin{align*}
100 \otimes 100 \to & \ 100 \otimes 110, \\
101 \otimes 100 \to & \ 100 \otimes 101, \\
011 \otimes 100 \to & \ 100 \otimes 011, \\
110 \otimes 100 \to & \ 100 \otimes 110, \\
101 \otimes 010 \to & \ 010 \otimes 110, \\
011 \otimes 010 \to & \ 010 \otimes 011, \\
011 \otimes 001 \to & \ 001 \otimes 011.
\end{align*}
\]

The maps $R^{2,1}$ and $R^{1,2}$ are both bijections inverse to each other. They are nontrivial in the sense that $R(i \otimes j) \neq j \otimes i$ in general.

Back to the general case, let $R = R^{l,m}: B^l \otimes B^m \to B^m \otimes B^l$ be the combinatorial $R$. Write its matrix element like (2.2):

\[
R(i \otimes j) = \sum_{a,b} R_{ij}^{ab} b \otimes a. \tag{2.11}
\]

Then for any $i \otimes j$, the matrix elements $R_{ij}^{ab}$ are all 0 except exactly one pair $b \otimes a = R^{l,m}(i \otimes j)$ determined from $i$ and $j$. An algorithm for finding it, namely the image $b \otimes a = R(i \otimes j)$, was obtained in [21 Rule 3.10]. We call it Nakayashiki–Yamada (NY)-rule.

**NY-rule for $R^{l,m}$:**

(i) Given $i \otimes j \in B^l \otimes B^m$, draw a tableau for $i$ below and the one for $j$ above. For example, the tableaux for $i \otimes j = 1100100100 \otimes 0010111110 \in B^4 \otimes B^6$ look as (i) below (1 and 0 are denoted by dots and empty box):

(i) \[\begin{array}{cccccccc}
\text{j} &=& 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
\text{i} &=& 1 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}\]

(ii) Pick any dot (call it $d$) in $i$ and connect it to a dot (call it $d'$) in $j$. The partner $d'$ must be the rightmost one among the dots located weakly left of $d$. If there is no such dot, return to the right (periodic boundary condition) and $d'$ is the rightmost one.

(ii-1) Repeat (ii-1) for all the remaining unconnected dots in $i$.

(iii) The image $b \otimes a$ is obtained by shifting the $(m-1)$ unconnected dots in $j$ to the bottom.

In the above example, $b \otimes a = 1110110100 \otimes 0000101110 \in B^6 \otimes B^4$.

Remark 2.7. In the step (ii), the order of picking unconnected dots in $i$ is not unique, but the final result is independent of it due to [21 Prop. 3.17].

In the above diagram (ii), the dots in the bottom tableau are picked for example in the order 2, 1, 3, 4 from the left. The connecting lines are called $H$-line. We remark that NY-rule for $R: B^l \otimes B^m \to B^m \otimes B^l$ for $l \leq m$ implies

\footnote{Actually we explain the algorithm for its inverse since the assumptions $l \leq m$ here and $k \geq l$ in [21] corresponds to the opposite situation.}

\footnote{‘Weakly left’ means exactly above or strictly left. ‘Weakly right’ means similar.
\[ R(i \otimes j) = b \otimes a \Rightarrow i \leq b \text{ and } a \leq j, \]  
\[ (2.12) \]

where \( x \leq y \) is defined by \( y - x \in (\mathbb{Z}_{\geq 0})^l \) for \( x, y \in \mathbb{Z}^l \).

The NY-rule for \( R^{lm} \) with \( l \geq m \) is a dual of the above. One starts from a dot \( d \) in \( j \in B^m \) (upper tableau) and seeks a partner \( d' \) in \( i \in B^l \) (lower tableau). The \( d' \) should be the leftmost dot among those located weakly right of \( d \) in a periodic sense. By construction \( R^{lm} R^{lm} = \text{id}_{B^l \otimes B^m} \) holds for any \( l \) and \( m \). Note that \( R^{lm} \) is trivial at \( l = m \) in the sense that \( R^{jj} = 0 \).

It is customary to depict the relation \( R(i \otimes j) = b \otimes a \) as

This formally looks same as (2.9). Note however the arrows there carry \( 0, 1 \) while those here do the elements from crystals \( B^l, B^m \) which are \( L \)-tuples of \( 0, 1 \).

As the \( q = 0, z = z_\neq 1 \) corollary of (2.4) the combinatorial \( R \) also satisfies the Yang–Baxter equation:

\[ (R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R). \]  
\[ (2.14) \]

For example the action of the both sides on \( 0100 \otimes 0011 \otimes 1101 \in B^4 \otimes B^2 \otimes B^3 \) leads to

\[ \begin{array}{ccc}
0111 & 1100 & 0001 \\
0110 & 1101 & 0001 \\
0110 & 0011 & 1101 \\
0100 & 0011 & 1101 \\
\end{array} \]

Here \( = \) means that starting from the same bottom line one ends up with the same top line despite the different order of applications of the combinatorial \( R \)'s. Combinatorial \( R \)'s constitute the most decent and systematic examples of set theoretical solutions of the Yang–Baxter equation [9] connected to the crystal base of quantum groups, which have numerous applications [12–14, 17, 18, 21].

2.5. Matrix product representation of combinatorial \( R \)

Setting \( q = 0 \) in (2.10) leads to a matrix product representation of the combinatorial \( R \). Let us write it out in terms of the 3D \( L \) operator and \( q \)-oscillator at \( q = 0 \). We will be exclusively concerned with \( R^{lm} \) with \( l < m \).

First we define the \( q = 0 \)-oscillator \( A_q \) to be the algebra generated by \( a^+, a^-, k \) satisfying the relations

\[ k^2 = k, \quad k a^+ = 0, \quad a^- k = 0, \quad a^- a^+ = 1, \quad a^+ a^- = 1 - k. \]  
\[ (2.16) \]

\[ 9 \text{ Although the notations } a^+, k, F, \text{ etc are retained, they are to be distinguished from those for } A_q. \]
They may be identified with the operators acting on the Fock space \( F = \oplus_{m \geq 0} \mathbb{C}|m\rangle \) as

\[
a^+|m\rangle = |m + 1\rangle, \quad a^-|m\rangle = (1 - \delta_{m,0})|m - 1\rangle, \quad k|m\rangle = \delta_{m,0}|m\rangle.
\]

(2.17)

As a \( \mathbb{C} \)-vector space, the \( A_0 \) has a Poincaré–Birkhoff–Witt type basis:

\[1, \quad (a^+_i)^r, \quad (a^-_i)^r, \quad (a^+_i)^{r}\bigtriangleup (a^-_i)^{r}\]

(2.18)

where \( r \geq 1 \) and \( s, t \geq 0 \). We introduce the 3D \( L \) operator at \( q = 0 \) denoted by \( L = (L_{ij}^{ab}) \) with indices from \( \{0, 1\} \). Explicitly they are zero except the following:

\[
L_{0,0}^{0,0} = L_{1,1}^{1,1} = 1, \quad L_{0,1}^{0,1} = k, \quad L_{0,1}^{1,0} = a^+, \quad L_{1,0}^{1,0} = a^-.
\]

(2.19)

\[
L_{i,j}^{a,b} = \begin{array}{cccc}
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{array}
\]

(2.20)

Note that compared with \( L \) (2.8), the ‘vertex’ \( L_{1,0}^{1,0} \) (the rightmost one in (2.9)) is missing due to \( q = 0 \). As the result \( L \) is regarded as a \( q = 0 \)-oscillator valued five-vertex model whose Boltzmann weights belong to \( A_0 \).

Let \( F^* = \oplus_{m \geq 0} \mathbb{C}|m\rangle \) be the dual of \( F \) defined by \( \langle m|\rangle = \delta_{m,m} \). Let \( A_{0}^{\text{fin}} \subset A_0 \) be the vector subspace spanned by (2.18) except 1. Then \( \text{Tr}(X) := \sum_{m \geq 0} \langle m|X|m\rangle \) is finite for any \( X \in A_{0}^{\text{fin}} \). From theorem 2.10 and these definitions we have

**Corollary 2.8.** Elements of the combinatorial \( R = R^{lm}; B^j \otimes B^m \to B^m \otimes B^l \) (2.11) with \( l < m \) are expressed in the matrix product form

\[
R_{ij}^{ab} = \text{Tr} \left( L_{i_1,i_2}^{a_1,b_1} \cdots L_{i_{d-c},i_{d}}^{a_{d-b},b_{d}} \right).
\]

(2.21)

For example \( R_{0,10,10,11}^{10,0,0,11} = \text{Tr} \left( L_{0,1}^{1,0} L_{1,0}^{0,1} \right) = \text{Tr}(a^+a^\dagger k) = 1 \) in agreement with example 2.5. From (2.19) the number of \( k \)'s contained in the above product is given by \( |b| - |i| = |j| - |a| = m - l > 0 \). Therefore \( L_{i_1,i_2}^{a_1,b_1} \cdots L_{i_{d-c},i_{d}}^{a_{d-b},b_{d}} \in A_{0}^{\text{fin}} \) is guaranteed and the trace is convergent. The absence of the vertex \( L_{1,0}^{1,0} \) in (2.19) reflects the fact that there is no dot going up in the two row diagram in the NY-rule of \( R^{lm} \) with \( l < m \). In the product (2.21), one can interpret that \( a^+ \) creates (emits) an \( H \)-line from a dot in the lower tableau and \( a^- \) annihilates (absorbs) an \( H \)-line into a dot in the upper tableau. The state \(|m\rangle \in F \) corresponds to the ‘segment’ where there are \( m \) \( H \)-lines.

The \( L \) operator \( L_{ij}^{a,b} \) (2.19) and the matrix product form of the combinatorial \( R \) (2.21) may be depicted in the 3D picture as follows:
Here the black arrows assigned with the indices carry 0 or 1. The blue ones carry the Fock space \( F \) over which the trace is taken.

3. \( n \)-species TASEP and its steady state

3.1. \( n \)-TASEP

Consider the periodic 1D lattice with \( L \) sites which will be denoted by \( \mathbb{Z}_L \). Each site \( i \in \mathbb{Z}_L \) is assigned with a physical variable (local state) \( \sigma_i \in \{0, 1, \ldots, n\} \). It is interpreted as the species of the particle occupying it or 0 indicating the absence of particles. We assume \( 1 \leq n < L \) throughout. Consider a stochastic model on \( \mathbb{Z}_L \) such that neighboring pairs of local states \((\sigma, \sigma') = (\sigma_i, \sigma_{i+1})\) are interchanged as \( \sigma \sigma' \rightarrow \sigma' \sigma \) if \( \sigma > \sigma' \) with the uniform transition rate. The whole space of states is given by

\[
\left( C^{n+1} \right)^{\otimes L} \cong \bigoplus_{(\sigma_1, \ldots, \sigma_L) \in \{0, \ldots, n\}^L} \mathbb{C}[\sigma_1, \ldots, \sigma_L],
\]

where we suppose that the ket \(|\cdots\rangle \) here can safely be distinguished from \(|b\rangle \in V^l \) for \( b \in B^l \) in section 2.1 and also from \(|m\rangle \in F \) for \( m \in \mathbb{Z}_{\geq 0} \) in (2.7) by the context. Let \( P(\sigma_1, \ldots, \sigma_L; t) \) be the probability of finding the configuration \((\sigma_1, \ldots, \sigma_L)\) at time \( t \), and set

\[
|P(t)\rangle = \sum_{(\sigma_1, \ldots, \sigma_L) \in \{0, \ldots, n\}^L} P(\sigma_1, \ldots, \sigma_L; t)|\sigma_1, \ldots, \sigma_L\rangle.
\]

By \( n \)-species TASEP, or \( n \)-TASEP for short, we mean the stochastic system governed by the continuous-time master equation

\[
\frac{d}{dt}|P(t)\rangle = H|P(t)\rangle,
\]

where the Markov matrix (also called ‘Hamiltonian’ by abuse of terminology despite it is not Hermitian in general) has the form

\[
H = \sum_{i \in \mathbb{Z}_L} h_{i,i+1}, \quad h_{\sigma, \sigma'}[\sigma, \sigma'] = \begin{cases} |\sigma', \sigma\rangle - |\sigma, \sigma'\rangle & (\sigma > \sigma'), \\ 0 & (\sigma \leq \sigma'). \end{cases}
\]

Here \( h_{i,i+1} \) is the local Hamiltonian that acts as \( h \) on the \( i \)th and the \((i + 1)\)th components and as the identity elsewhere. As \( H \) preserves the particle content, it acts on each sector labeled with the multiplicity \( m = (m_0, \ldots, m_n) \in (\mathbb{Z}_{\geq 0})^{n+1} \) of the particles\(^{10} \).

\(^{10} \) \( V(m) \) here should not be confused with the antisymmetric tensor representation \( V^m \) from (2.1).
\[ V(m) = \sum_{\sigma \in S(m)} C|\sigma|, \quad S(m) = \{ \sigma = (\sigma_1, \ldots, \sigma_L) \in [0, \ldots, n]^L | \sum_{j=1}^L \delta_{k,\sigma_j} = m_k, \forall k \}. \] (3.5)

It has the dimension \( \dim V(m) = \frac{L!}{m_0! \cdots m_n!} \). The space of states is decomposed as \((\mathbb{C}^{n+1})^L = \bigoplus_{m_0, \ldots, m_n \in \mathbb{Z}_{\geq 0}} V(m)\), where the sum ranges over \( m_i \in \mathbb{Z}_{\geq 0} \) such that \( m_0 + \cdots + m_n = L \). A sector \( V(m_0, \ldots, m_n) \) such that \( m_j \geq 1 \) for all \( 0 \leq i \leq n \) is called basic. Non-basic sectors are equivalent to a basic sector for \( n' \)-TASEP with some \( n' < n \) by a suitable relabeling of species. Thus we shall exclusively deal with basic sectors in this paper (hence \( n < L \) as mentioned before). For various symmetry and spectral property of \( H \), see [2].

In each sector \( V(m) \) there is a unique state \( \bar{P}(m) \) up to a normalization, called the steady state, satisfying \( H \bar{P}(m) = 0 \).

**Example 3.1.** Up to an overall normalization, the steady states in \( V(1,1,1), V(2,1,1) \) and \( V(1,2,1) \) for \( 2 \)-TASEP are given by

\[
\bar{P}(1,1,1) = 2|012| + |021| + |102| + |210| + |201|,
\bar{P}(2,1,1) = 3|0012| + |0021| + |20102| + |30120| + |20201| + |0210|
+ |1002| + |20102| + |31200| + |32001| + |20210| + |2100|,
\bar{P}(1,2,1) = 2|0112| + |0121| + |0211| + |1012| + |012| + |1102|
+ |2112| + |21201| + |1210| + |2011| + |2101| + |2110|.
\]

Similarly the steady state in \( V(1,1,1,1) \) for 3-TASEP is given by

\[
\bar{P}(1,1,1,1) = 9|0123| + 3|0132| + 3|0213| + 3|0231|
+ 5|0312| + |0321| + 3|1023| + |1032|
+ 5|1203| + 9|1230| + 3|1302| + 3|1320|
+ 3|2013| + 5|2031| + |2103| + 3|2130|
+ 9|2301| + 3|2310| + 9|3012| + 3|3021|
+ 3|3102| + 5|3120| + 3|3201| + |3210|.
\]

One can observe the symmetry under the \( \mathbb{Z}_L \) cyclic shifts which holds in general.

A combinatorial algorithm to construct the steady state \( \bar{P}(m) \in V(m) \) for general basic sectors \( V(m) \) of the \( n \)-TASEP on \( \mathbb{Z}_n \) was obtained by Ferrari–Martin [11]. In the rest of this section we show that it is most naturally formulated in terms of the combinatorial \( R \) in the previous section.

### 3.2. Multiline process

To a basic sector \( V(m) \) of the \( n \)-TASEP with the multiplicity \( m = (m_0, \ldots, m_n) \), we associate another system called multiline process. Define the integers \( l_0, \ldots, l_{n+1} \) by

\[
l_i = m_{n-i+1} + \cdots + m_{n-1} + m_n \quad (0 \leq i \leq n + 1). \quad (3.6)
\]

They satisfy \( 0 = l_0 < l_1 < \cdots < l_n < l_{n+1} = L \) due to \( m_i \geq 1 \) for all \( i \). Given \( L \), the data \( (m_i) \) and \( (l_i) \) can be transformed to each other uniquely. Introduce the sets

\[
B(m) = B^{l_1} \otimes \cdots \otimes B^{l_n} = \{ b_1 \otimes \cdots \otimes b_n | b_j \in B^{l_j} \}, \quad (3.7)
\]
\[ B_s(m) = \{ b_1 \otimes \cdots \otimes b_n \in B(m) \mid b_1 \leq \cdots \leq b_n \} \subset B(m), \quad (3.8) \]

where \( \leq \) is defined after (2.12). The following bijection is elementary and will be utilized later:

\[ \varphi : S(m) \rightarrow B_s(m); \quad \sigma = (\sigma_1, \ldots, \sigma_L) \mapsto \varphi_1(\sigma) \otimes \cdots \otimes \varphi_n(\sigma), \quad (3.9) \]

\[ \varphi_j(\sigma) = \left( \theta(\sigma_i \geq n + 1 - j), \ldots, \theta(\sigma_L \geq n + 1 - j) \right) \in B^{l_j}, \quad (3.10) \]

\[ \varphi^{-1}(b_1 \otimes \cdots \otimes b_n) = b_1 + \cdots + b_n, \quad (3.11) \]

where \( \theta \) is defined in example 2.1. Applying \( \varphi^{-1} \) to (3.9) and using (3.11) we see that

\[ \sigma = \varphi_1(\sigma) + \cdots + \varphi_h(\sigma) \quad (3.12) \]

holds identically. If (3.10) is given as \( \varphi_j(\sigma) = (\gamma_{j,1}, \ldots, \gamma_{j,L}) \), it is also easy to see that

\[ \sigma_k = i \quad \Rightarrow \quad (\gamma_{1,k}, \ldots, \gamma_{n,k}) = (0, \ldots, 0, 1, \ldots, 1) \quad (3.13) \]

for each \( k \in \mathbb{Z}_L \). The sum in (3.11) as elements in \( \mathbb{Z}^L \) is not an operation usually performed in the crystal base theory.

**Example 3.2.** Let \( n = 3 \) and take \( \sigma = (3, 0, 1, 2, 3, 0, 1) \in S(m) \) with \( m = (2, 2, 1, 2) \). The image \( \varphi(\sigma) = \varphi_1(\sigma) \otimes \varphi_2(\sigma) \otimes \varphi_3(\sigma) \in B_s(m) \subset B^2 \otimes B^3 \otimes B^2 \) is given by

\[
\begin{array}{c|cccc}
\sigma & 3 & 0 & 1 & 2 \\
\varphi_1(\sigma) & 1 & 0 & 1 & 1 \\
\varphi_2(\sigma) & 1 & 0 & 0 & 1 \\
\varphi_3(\sigma) & 1 & 0 & 0 & 0 \\
\end{array}
\]

One can check \( \varphi_1(\sigma) \leq \varphi_2(\sigma) \leq \varphi_3(\sigma) \) (3.8), \( \sigma = \varphi_1(\sigma) + \varphi_2(\sigma) + \varphi_3(\sigma) \) (3.12) and (3.13).

The multiline process [11] is a stochastic process on the set \( B(m) \). To describe its dynamics we first introduce a deterministic map

\[ T : B(m) \otimes \mathbb{Z}_L \rightarrow \mathbb{Z}_L \otimes B(m); \quad b_1 \otimes \cdots \otimes b_n \otimes k \mapsto k' \otimes b'_1 \otimes \cdots \otimes b'_n. \quad (3.14) \]

It is defined by sending \( \mathbb{Z}_L \) from the right to the left through \( B^i \otimes \cdots \otimes B^i \) by successively applying the following pairwise exchange rule:

\[ B^i \otimes \mathbb{Z}_L \rightarrow \mathbb{Z}_L \otimes B^i; \quad (x_1, \ldots, x_L) \otimes k \mapsto k' \otimes (x'_1, \ldots, x'_L), \]

\[ k' = k + x_k - 1, \quad x'_i = \begin{cases} \min(x_k, x_{k+1}) & i = k, \\ \max(x_k, x_{k+1}) & i = k + 1, \\ x_i & \text{otherwise}. \end{cases} \quad (3.15) \]

The map \( T \) in (3.14) induces the ‘time evolutions’ on \( B(m) \) by setting

\[ T_k : B(m) \rightarrow B(m); \quad T_k(b_1 \otimes \cdots \otimes b_n) = b'_1 \otimes \cdots \otimes b'_n \quad (k \in \mathbb{Z}_L), \quad (3.16) \]

where the right hand side is the one appearing in (3.14).
Example 3.3. Take \( s = b_1 \otimes b_2 \otimes b_3 = 000010 \otimes 001010 \otimes 0010111 \in B^1 \otimes B^2 \otimes B^3 \). Then \( T_2(s) = s \) and the other cases are given by

\[
\begin{array}{cccccc}
0 & 0 & 2 & 0 & 3 & 1 \\
0 & 0 & 2 & 0 & 3 & 1 \\
0 & 0 & 2 & 0 & 3 & 1 \\
0 & 0 & 2 & 0 & 1 & 3 \\
1 & 0 & 2 & 0 & 3 & 0 \\
\end{array}
\]

Here the same tableaux as for the NY-rule are used. As an example, \( T_3(s) \) has been obtained by applying (3.15) successively as

\[
s \otimes 3 = 000010 \otimes 001010 \otimes 001011 \otimes 3 \\
\Rightarrow 000010 \otimes 001010 \otimes 3 \otimes 000111 \\
\Rightarrow 000010 \otimes 3 \otimes 000110 \otimes 000111 \\
\Rightarrow 2 \otimes 000010 \otimes 000110 \otimes 000111 = 2 \otimes T_3(s).
\]

The sequences above the tableaux will be explained after (3.19).

Now we can define a stochastic dynamics on \( B(m) \) by declaring that each element of it undergoes the \( L \) kinds of time evolutions \( T_1, \ldots, T_L \) with an equal probability. The resulting system is called multiline process [11]. We have not found a crystal theoretic interpretation of \( T_k \). See section 5.2 for a further remark.

3.3. Projection from multiline process to n-TASEP

Fix \( L, n \) and \( m = (m_i) \) and \( (l_i) \) as explained around (3.6). Recall that the sector \( V(m) \) of the \( n \)-TASEP is defined in (3.5) and the set of states of multiline process is \( B(m) = B^{l_1} \otimes \cdots \otimes B^{l_n} \) in (3.7). We introduce a map

\[
\pi_j : B(m) \rightarrow B^{l_j}; \quad s = b_1 \otimes \cdots \otimes b_n \mapsto \pi_j(s) \quad (1 \leq j \leq n) \quad (3.17)
\]

by a composition of the combinatorial \( R \). It is most easily grasped by the diagram

\[
\begin{array}{c}
\downarrow \quad \downarrow \\
b_j & \cdots & b_{j+1} & b_{j+2} & \cdots & b_n \\
\uparrow \quad \uparrow \\
\pi_j(s) \quad \pi_j(s) \\
\end{array}
\]

which is a composition of (2.13). Namely one lets the component \( B^{l_j} \) penetrate through its right neighbor \( B^{l_{j+1}} \otimes \cdots \otimes B^{l_k} \) by successively applying the combinatorial \( R = R^{l_j} \) for \( r = j + 1, \ldots, n \). Note that \( \pi_j(b_1 \otimes \cdots \otimes b_n) \) actually depends only on \( b_j \otimes \cdots \otimes b_n \) and in particular \( \pi_j(b_1 \otimes \cdots \otimes b_n) = b_n \).

From \( l_1 < \cdots < l_n \) (see around (3.6)) and (2.12), we have an increasing sequence \( \pi_1(s) \leq \pi_2(s) \leq \cdots \leq \pi_n(s) \) of elements of the crystals \( B^{l_1}, B^{l_2}, \ldots, B^{l_n} \). In fact one can check

\[
\pi_1(s) \otimes \cdots \otimes \pi_n(s) \in B_\pi(m)
\]

(3.19)

for any \( s \in B(m) \). Sending this by \( q^{-1} \) and applying (3.11), we are able to define a surjective map \( \pi \) from the states \( B(m) \) (3.7) of the multiline process to the states \( S(m) \) (3.5) in the \( n \)-TASEP by

\[
\pi : B(m) \rightarrow S(m); \quad \pi(s) = \pi_1(s) + \cdots + \pi_n(s).
\]

(3.20)

The sequences above the tableaux in example 3.3 show the images in \( S(3, 1, 1, 1) \) under \( \pi \). By comparing the NY-rule and the diagram (3.18) with Fig.2 and Fig 3 in [11] and noting remark 2.7, it can be seen that the map \( \pi \) coincides with the Ferrari–Martin algorithm.
$V^{(L)} : X \to Y$ in [11] upon conventional adjustment. This is our first main observation in this paper.

As a consequence of this identification, theorem 4.1 and theorem 2.2 in [11] lead to propositions 3.4 and 3.5 given below.

**Proposition 3.4.** Let $\tau_i$ be the map on $S(m)$ that changes the $i$th and $(i+1)$th components $(\sigma_i, \sigma_{i+1})$ of $\sigma \in S(m)$ into $(\min(\sigma_i, \sigma_{i+1}), \max(\sigma_i, \sigma_{i+1}))$. Then the following diagram is commutative:

$$
\begin{array}{c}
B(m) \xrightarrow{\tau_i} B(m) \\
\pi \\
\downarrow \\
S(m) \xrightarrow{\tau_i} S(m).
\end{array}
$$

One can easily check the commutativity in example 3.3. See section 5.2 for a crystal theoretical generalization of $T_i$ and $\tau_i$.

**Proposition 3.5.** Up to a normalization, the steady state in the sector $V(m)$ is given by

$$
\hat{P}(m) = \sum_{s \in B(m)} |\pi(s)\rangle = \sum_{\sigma \in S(m)} P(\sigma) |\sigma\rangle,
$$

$$
P(\sigma) = \# \{ s \in B(m) \mid \pi(s) = \sigma \}.
$$

The last condition is equivalent to $q^{-1}(\pi_1(s) \otimes \cdots \otimes \pi_n(s)) = \sigma$ due to (3.11) and (3.20). By taking $\varphi$ further and using (3.9) we find

$$
P(\sigma) = \# \{ s \in B(m) \mid \pi_k(s) = \varphi_k(\sigma) \quad (1 \leq k \leq n) \}.
$$

In the next section we invoke the results in section 2.5 to derive a new matrix product formula for the (unnormalized) steady state probability $P(\sigma)$ of the configuration $\sigma$.

4. Matrix product formula

4.1. Diagram for $\pi_1$: Combinatorial corner transfer matrix

There is a single diagram that represents all of $\pi_1(s), \ldots, \pi_n(s)$ in (3.18) simultaneously. We illustrate it along $n = 3$ case. Given $s = b_1 \otimes b_2 \otimes b_3 \in B(m) = B^1 \otimes B^2 \otimes B^3$, the diagrams (3.18) for $\pi_1$ and $\pi_2$ are

Here each vertex stands for the combinatorial $R$. See (2.13). This defining diagram of $\pi_1(s)$ can be deformed by means of the Yang–Baxter equation (2.14):

$$
\begin{array}{c}
b_1 \\
b_2 \\
b_3
\end{array} \quad \pi_1(s) = \begin{array}{c}
b_3 \\
b_2 \\
b_1
\end{array} \quad \pi_1(s)
$$

(4.1)
Inclining the vertices appropriately in the last diagram we find that \( \pi_1(s), \pi_2(s), \pi_3(s) = b_3 \) can all be included in the left diagram below:

\[
\begin{align*}
\text{For } n = 3 & \\
& b_1 \quad \uparrow \quad \uparrow \pi_1(s) \\
b_2 \quad & \uparrow \pi_2(s) \\
b_3 \quad & \uparrow \pi_3(s)
\end{align*}
\]

\[
\begin{align*}
\text{For } n = 4 & \\
& b_1 \quad \uparrow \quad \uparrow \pi_1(s) \\
b_2 \quad & \uparrow \pi_2(s) \\
b_3 \quad & \uparrow \pi_3(s) \\
b_4 \quad & \uparrow \pi_4(s)
\end{align*}
\]

The result for \( n = 4 \) case is also given together for which \( s \) should be understood as \( s = b_1 \otimes \cdots \otimes b_4 \in B(m) = B^{i_1} \otimes \cdots \otimes B^{i_4} \). General case is clear and similarly obtained by transforming the defining diagrams (3.18) of \( \pi_j(s) \) for \( j = 1, \ldots, n \) into composable forms with the aid of the Yang–Baxter equation as in (4.1).

In (4.2) the arrow starting from \( b_1 \) carries the crystal \( B^L \) and each segment corresponds to an element of it. The only nontrivial events are the combinatorial \( R \)'s indicated by the vertices. The 90° left turns along the arrows do not change the elements. It is consistent with the bottom line since \( \pi_i(s) = b_i \) as noted after (3.18). The inclusion of such turns makes the diagram symmetric in that there are \( n \) incoming arrows horizontally and \( n \) outgoing ones vertically. A further significance will become manifest in [19]. Since the combinatorial \( R \) is a deterministic map, every part of the diagram is fixed uniquely upon choosing the input \( s = b_1 \otimes \cdots \otimes b_n \in B(m) \) from the left.

In this way we are led to the corner transfer matrix [3 ch13] of the vertex model associated with the antisymmetric tensor representations of \( U_q(\mathfrak{sl}_L) \). It is at the combinatorial point \( q = 0 \) where every vertex is crystallized to the combinatorial \( R \). Note that the physical space \( \mathcal{Z}_L \) and the internal degrees of freedom \( \{0, 1, \ldots, n\} \) in the original \( n \)-TASEP have been interchanged; we have a corner transfer matrix on the system of linear size \( n \) whose local interaction is encoded in the quantum group \( U_q(\mathfrak{sl}_L) \) at \( q = 0 \). We stress that such a cross-channel of the problem has been captured precisely by reformulating the Ferrari–Martin algorithm in terms of combinatorial \( R \) and thereby enabling a systematic use of the Yang–Baxter equation.

### 4.2. Factorization into five-vertex model

By the graphical representation (4.2) of \( \pi_j(s) \), the stationary probability (3.22) for the configuration \( \sigma = (\sigma_1, \ldots, \sigma_L) \) is expressed, for instance for \( n = 3 \), as

\[
P(\sigma) = \sum_{b_1 \otimes b_2 \otimes b_3 \in B(m)} b_1 \quad \uparrow \quad \phi_1(\sigma) \\
b_2 \quad \uparrow \phi_2(\sigma) \\
b_3 \quad \uparrow \phi_3(\sigma)
\]

Here each \( \phi_j(\sigma) \) signifies the boundary condition \( \pi_j(b_1 \otimes b_2 \otimes b_3) = \phi_j(\sigma) \) that the element \( \pi_j(b_1 \otimes b_2 \otimes b_3) \in B^L \) making the left turn there should coincide with \( \phi_j(\sigma) \) prescribed from \( \sigma \) by the simple rule (3.10). Each summand of (4.3) means 1 or 0 depending on whether such a boundary condition is satisfied or not for the chosen \( b_1 \otimes b_2 \otimes b_3 \).
Now we are in the position to convert this formula into a matrix product form by invoking corollary 2.8. To each vertex of (4.3), substitute (2.21) or equivalently the graphical form (2.22) of the combinatorial $R$. It amounts to putting the three ‘skewers’ standing upright at the vertices. The result takes the form

$$P(\sigma) = \text{Tr} \left( X_{\sigma_1} \cdots X_{\sigma_3} \right),$$

(4.4)

where the trace extends over $F^\otimes 3$ and the factor $X_{\sigma_i}$ corresponds to the $k$th layer of the skewers. It has the same structure as (4.3) but now each vertex represents an $A_0$-valued Boltzmann weight of the five-vertex model (2.20). Accordingly each arrow carries 0 or 1, and the sum over $b_1 \otimes b_2 \otimes b_3$ should be replaced by the configuration sum of the five-vertex model. What about the boundary condition on them? It should be imposed so as to reflect the table of example 3.2 in the diagram (4.3). In this way we find

$$X_0 = \sum_{[2]} \hspace{1cm} X_1 = \sum_{[1]} \hspace{1cm} X_2 = \sum_{[1]} \hspace{1cm} X_3 = \sum_{[1]}$$

(4.5)

The sums here extend over all the configurations of the five-vertex model under the specified boundary conditions. The resulting objects take values in $A_0^\otimes 3$. Explicitly they read

$$X_0 = 1 \otimes 1 \otimes 1 + a^+ \otimes 1 \otimes 1 + k \otimes a^+ \otimes 1 + a^- \otimes a^+ \otimes a^+ + 1 \otimes a^+ \otimes a^+$$

$$X_1 = k \otimes k \otimes 1 + a^- \otimes k \otimes a^+ + 1 \otimes k \otimes a^+$$

$$X_2 = 1 \otimes a^- \otimes k + a^+ \otimes a^- \otimes k + k \otimes 1 \otimes k$$

$$X_3 = 1 \otimes a^- \otimes a^- + a^+ \otimes a^- \otimes a^- + k \otimes 1 \otimes a^- + a^- \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1$$

Here the edges are colored black or red depending on whether they assume 0 or 1, and the three components in the tensor product are arranged so as to correspond to the vertices at (top left) $\otimes$ (bottom left) $\otimes$ (top right). See (2.20).

Again general $n$ case is similar and clear. By imposing the boundary condition according to (3.13), the operator $X_i (0 \leq i \leq n)$ is given graphically as follows:
There are $n$ incoming arrows from the left and $n$-outgoing ones from the top. It represents an element of $A_0^{\otimes(n-1)/2}$. The formula (4.4) remains the same if the trace is understood as the one over $F^{\otimes n(n-1)/2}$. In general it is normalized so that $\Phi(\sigma_1, \ldots, \sigma_L) = 1$ if $\sigma_1 \geq \cdots \geq \sigma_L$ (which means $\sigma_1 = n, \sigma_L = 0$ as we are in a basic sector). As another example, the configurations contributing to $X_1$ with $n = 4$ are as follows:

Accordingly we have $X_1 = k \cdot l \cdot k \cdot l \cdot k + a^- \cdot a^+ \cdot l \cdot k + 1 \cdot k \cdot l \cdot k + 1 \cdot k \cdot l \cdot a^- \cdot a^+ \cdot k + k \cdot l \cdot 1 \cdot a^- \cdot a^+ \cdot a^- \cdot a^+ \cdot k + a^- \cdot a^+ \cdot a^- \cdot a^+ \cdot k + k \cdot a^- \cdot a^+ \cdot 1 \cdot a^- \cdot k + a^- \cdot l \cdot a^- \cdot k + 1 \cdot l \cdot a^- \cdot 1 \cdot a^- \cdot k$, where $\cdot$ denotes $\otimes$ and $A_0^{\otimes(n-1)/2}$ is ordered as (top line) $\otimes$ (middle line) $\otimes$ (bottom line) where within each line they are ordered from the left to the right.

To summarize, our $X_i$ is a corner transfer matrix of the $q=0$-oscillator valued five-vertex model. The steady state probability $\Phi(\sigma)$ (4.4) is a partition function of the 3D lattice model associated with the 3D $L$ operator at $q = 0$ (2.20). It is a 3D system having the prism shape region (4.6) $\times Z_L$ with the boundary condition on one of the surfaces specified by $\sigma$. The $X_i$ plays the role of a layer-to-layer transfer matrix. In general it does not obey the recursion relation with respect to $n$ of the form $X_{i+1} = \sum_{0 \leq j < n-1} a_{ij}^{(n)} \otimes X_{j+1}^{(n-1)}$ for some $a_{ij}^{(n)} \in A_0^{\otimes n-1}$, as a result it is different from those in [1, 10] except one particular case of $n = 3$ in the latter and the simplest nontrivial case $n = 2$:

$$X_0 = 1 + a^+,$$  $X_1 = k,$  $X_2 = 1 + a^-.$

### 5. Discussion

#### 5.1. Summary

In this paper we have revealed that the Ferrari–Martin algorithm for constructing the steady state of the $n$-TASEP [11] is most naturally formulated in terms of a combinatorial $R$ in crystal base theory. Combined with the factorized form of the combinatorial $R$ originating in the tetrahedron equation [20], it has lead to a new matrix product formula (4.4) for the steady state probability. Our operator $X_i$ (4.6) is a corner transfer matrix of the $q$-oscillator valued five-vertex model at $q = 0$, which is a configuration sum in a cross-channel of the original problem. Whether such a result admits generalizations to similar systems like ASEP [22], open boundary conditions and the large list of factorized $R$ matrices for other quantum groups [20] is a natural question to be investigated.

#### 5.2. Generalized dynamics

Let us comment on a possible generalization of the time evolutions $T_i$ and $\tau_i$ in proposition 3.4 from the viewpoint of crystal base theory. Let $B^{l,s}(1 \leq l < L, s \geq 1)$ be the crystal of the
irreducible $U_q(SL_L)$ module (so-called Kirillov–Reshetikhin module) corresponding to the $l \times s$ rectangular shape Young diagram [18, 23]. Elements of $B^{l,s}$ are labeled with the semi-standard tableaux on the Young diagram with entries $\{1, \ldots, L\}$. The family $B^{l,s}$ includes $B^l$ (2.1) as $B^{l,1} = B^l$, and again there is a bijection $B^{l,s} \otimes B^{l,s'} \rightarrow B^{l,s'} \otimes B^{l,s}$ called the combinatorial $R$. For an arbitrary pair of $(l, s)$ and $(l', s')$, an algorithm for finding the image of the $R$ is known [23] which reduces to the NY-rule for $s = s' = 1$.

Associated with any crystal $B^{r,s}$ we introduce the maps $T = T^{r,s} : B(m) \otimes B^{r,s} \rightarrow B^{r,s} \otimes B(m)$ and $T_a : B(m) \rightarrow B(m)$ for each $u \in B^{r,s}$ by a scheme similar to (3.14) and (3.16). Namely we set $T(b_1 \otimes \cdots \otimes b_n \otimes u) = u' \otimes b'_1 \otimes \cdots \otimes b'_n$ and $T_a(b_1 \otimes \cdots \otimes b_n) = b'_1 \otimes \cdots \otimes b'_n$, where $u' \in B^{r,s}$ and $b'_j \in B^l$ are obtained by sending $u \in B^{r,s}$ to the left through $b_1 \otimes \cdots \otimes b_n$ by a successive application of the combinatorial $R$: $B^l \otimes B^{r,s} \rightarrow B^{r,s} \otimes B^l$. Maps like $T_a$ have been studied extensively as a time evolution with "carrier $u$" in the context of integrable cellular automata known as box-ball systems and their generalizations [14]. Based on computer experiments we propose

**Conjecture 5.1.** For $\sigma \in S(m)$, let $\pi^{-1}(\sigma) \subset B(m)$ denote the preimage of the map $\pi$ (3.20). Then the set $\pi \circ T_a(\pi^{-1}(\sigma))$ consists of a single element for any $u \in B^{r,s}$.

Admitting the conjecture we are able to introduce a new time evolution $\tau_a$ on $S(m)$ by $\tau_a := \pi \circ T_a \circ \pi^{-1}$. The definition is equivalent to replacing $T_a$ and $\tau_a$ in the commutative diagram of proposition 3.4 by $T_a$ and $\tau_a$, respectively. The detail of the dynamics will be illustrated elsewhere. The map $T = T^{r,s}$ is invertible as it is a composition of combinatorial $R$’s.

Now we introduce new stochastic processes on $B(m)$ and $S(m)$ analogous to the multiline process and the $n$-TASEP, respectively. They are defined by letting each state undergo the respective time evolutions $T_a$ and $\tau_a$ with an equal probability with respect to $u \in B^{r,s}$. Then from the invertibility of $T$, exactly the same argument as [11 p 821] proves that the steady states are given by the uniform probability distribution for the former and by (3.21) for the latter. In this way we find a curious feature of the probability distribution (3.21) that it yields a simultaneous steady state for the family of stochastic processes on $S(m)$ associated with the crystals $B^{r,s}$ as well as for the $n$-TASEP.

### 5.3. Hat relations

Our derivation of (4.4) and (4.6) in this paper is based on proposition 3.5 which is attributed to [11]. However it is also possible to establish them directly following the idea in [7]. To explain it let us write the action of the local Hamiltonian (3.4) as $h(\alpha, \beta) = \sum_{0 \leq \gamma, \delta \leq n} h_{\alpha,\beta}^{\gamma,\delta} \chi_{\gamma,\delta}$. Then for the quantity $\text{Tr} \left( \hat{X}_n \cdots \hat{X}_0 \right)$ in (4.4) to give the steady state probability, it is sufficient to construct the operators $\hat{X}_0, \ldots, \hat{X}_n \in \text{End} \left( F^{\otimes (n-1)/2} \right)$ satisfying the so called hat relation [7]:

$$X_\alpha \hat{X}_\beta - \hat{X}_\alpha X_\beta = (hXX)_{\alpha,\beta} := \sum_{0 \leq \gamma, \delta \leq n} h_{\gamma,\delta}^{\alpha,\beta} X_\gamma X_\delta \quad (0 \leq \alpha, \beta \leq n). \quad (5.1)$$

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Theorem 5.2. Up to a freedom $\hat{X}_i \to \hat{X}_i + \text{const}$, such an $\hat{X}_i$ is realized as

$$\hat{X}_i = \sum (\alpha_1 + \cdots + \alpha_n)$$

where the sum ranges over the same set of configurations of the five-vertex model as in $X_i$ (4.6). The only difference is the extra coefficient $\alpha_1 + \cdots + \alpha_n$ depending on the outgoing arrows $\alpha_i = 0, 1$.

For instance one has $\hat{X}_0 = a^+ \otimes 1 \otimes 1 + k \otimes a^+ \otimes 1 + a^- \otimes a^+ \otimes a^+ + 2(1 \otimes a^+ \otimes a^+)$ for $n = 3$ from the graphical representation of $X_0$ in section 4.2. The hat relation (5.1) turns out to be a consequence of its far-reaching generalization into a 3D lattice model obeying the tetrahedron equation. An exposition of the detail is beyond the scope of this paper and will be presented in [19] together with a proof of theorem 5.2.

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References

[1] Arita C, Ayyer A, Mallick K and Prolhac S 2011 Recursive structures in the multispecies J. Phys. A: Math. Theor. 44 335004
[2] Arita C, Kuniba A, Sakai K and Sawabe T 2009 Spectrum in multi-species asymmetric simple exclusion process on a ring J. Phys. A: Math. Theor. 42 345002
[3] Baxter R J 2007 Exactly Solved Models in Statistical Mechanics (Dover: New York)
[4] Bazhanov V V and Sergeev S M 2006 Zamolodchikov’s tetrahedron equation and hidden structure of quantum groups J. Phys. A: Math. Theor. 39 3295–310
[5] Blythe R A and Evans M R 2007 Nonequilibrium steady states of matrix product form: a solver’s guide J. Phys. A: Math. Theor. 40 R333
[6] Date E and Okado M 1994 Calculation of excitation spectra of the spin model related with the vector representation of the quantized affine algebra of type $A_n^{(1)}$ Int. J. Mod. Phys. A 09 399–417
[7] Derrida B, Evans M R, Hakim V and Pasquier V 1993 Exact solution of a 1D asymmetric exclusion model using a matrix formulation J. Phys. A: Math. Gen. 26 1493–517
[8] Drinfeld V G 1987 Quantum groups Proc. Int. Congress of Mathematicians (Berkeley, CA, 1986) vol 1, 2 (Providence, RI: American Mathematical Society) pp 798–820
[9] Drinfeld V G 1992 On some unsolved problems in quantum group theory Quantum Groups (Lecture Notes in Mathematics vol 1510) p 1–8
[10] Evans M R, Ferrari P A and Mallick K 2009 Matrix representation of the stationary measure for the multispecies TASEP J. Stat. Phys. 135 217–39
[11] Ferrari P A and Martin J B 2007 Stationary distributions of multi-type totally asymmetric exclusion processes Ann. Probab. 35 807–32
[12] Hatayama G, Kuniba A, Okado M, Takagi T and Tsuboi Z 2002 Paths, crystals and Fermionic formulae Prog. Math. Phys. 23 205–72
[13] Hong J and Kang S-J 2014 Introduction to quantum groups and crystal bases Graduate Studies in Mathematics vol 42 (Providence, RI: American Mathematical Society)
[14] Inoue R, Kuniba A and Takagi T 2012 Integrable structure of box-ball systems: crystal, Bethe ansatz, ultradiscretization and tropical geometry J. Phys. A. Math. Theor. 45 073001
[15] Jimbo M 1985 A q-difference analogue of $U(\hat{g})$ and the Yang–Baxter equation Lett. Math. Phys. 10 63–9
[16] Kashiwara M 1991 On crystal bases of q-analogue of universal enveloping algebras Duke Math. J. 63 465–516
[17] Kang S-J, Kashiwara M, Misra K C, Miwa T, Nakashima T and Nakayashiki A 1992 Affine crystals and vertex models Int. J. Mod. Phys. A 7 449–84
[18] Kang S-J, Kashiwara M, Misra K C, Miwa T, Nakashima T and Nakayashiki A 1992 Perfect crystals of quantum affine Lie algebras Duke Math. J. 68 499–607
[19] Kuniba A, Maruyama S and Okado M 2015 Multispecies TASEP and the tetrahedron equation in preparation
[20] Kuniba A, Okado M and Sergeev S 2015 Tetrahedron equation and generalized quantum groups J. Phys. A: Math. Theor. 48 304001
[21] Nakayashiki A and Yamada Y 1997 Kostka polynomials and energy functions in solvable lattice models Sel. Math. New Ser. 3 547–99
[22] Prolhac S, Evans M R and Mallick K 2009 The matrix product solution of the multispecies partially asymmetric exclusion process J. Phys. A: Math. Theor. 42 165004
[23] Shimozono M 2002 Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties J. Algebr. Comb. 15 151–87
[24] Zamolodchikov A B 1980 Tetrahedra equations and integrable systems in three-dimensional space Sov. Phys. JETP 79 641–64