Graded algebraic structure in the canonical formulation of $N = 3$ chiral supergravity

Motomu Tsuda *

Laboratory of Physics, Saitama Institute of Technology
Okabe-machi, Saitama 369-0293, Japan

Abstract

We focus on $N = 3$ chiral supergravity (SUGRA) which is the lowest $N$ theory involving a spin-1/2 field, and derive the Ashtekar’s canonical formulation of $N = 3$ SUGRA starting with the chiral Lagrangian constructed by closely following the standard SUGRA. The polynomiality of constraints in terms of canonical variables and the graded algebraic structure of constraints are discussed in the canonical formulation. In particular, we show the polynomiality of the rescaled right- and left-handed SUSY constraints by a nonpolynomial factor. And also we show the graded algebraic structure of Osp(3/2) in the constraint algebra by calculating the Poisson brackets of Gauss, SU(2) gauge and right-handed SUSY constraints, although the algebra among only those three types of constraints does not closed.

*e-mail: tsuda@sit.ac.jp
1. Introduction

Nonperturbative quantum gravity was extensively developed in the framework of the Ashtekar’s canonical formulation (ACF) of general relativity (GR) [1], which was formulated as an (complexified) SU(2) gauge formulation of GR, and in the loop quantum gravity (LQG) [3, 4]. In those developments, the unification of gravity and gauge fields and the fermionic matter contribution to the nonperturbative quantum gravity have also been discussed; gravity and gauge fields (Maxwell or Yang-Mills fields) were discussed in [5]-[7], while gravity and spin-1/2 fields were studied in [5, 8]. In [9] the Einstein-Maxwell-Dirac theory was also considered.

On the other hand, supersymmetry (SUSY) in both linear realization [10] and nonlinear realization [11] is an important notion in order to construct an unified theory beyond the standard model. From this viewpoint it is useful to investigate the nonperturbative aspects of the supergravity (SUGRA) theory as the supersymmetric extension of the above works [1]-[9]. In fact, the extension of ACF and LQG to SUGRA has been achieved by many authors with the following several points which have to be discussed; namely,

(a) the construction of a chiral \(^2\) Lagrangian in first-order form which leads to the ACF of SUGRA,

(b) the polynomiality of constraints in terms of canonical variables in the ACF of SUGRA,

(c) the graded algebraic structure of constraints (in addition to the closure of the constraint algebra)

and

(d) quantization and exact solutions of quantum constraints (under reality conditions)

were mainly discussed in the extension.

The results for (a)-(d) is well-known, in particular, up to the extended \(N = 2\) chiral SUGRA, in which many aspects of ACF and LQG are maintained as the supersymmetric extension of [1]-[7]. Indeed, as for (a), chiral Lagrangians in first-order form were constructed for both the \(N = 1\) [12, 13] and \(N = 2\) [15, 16]

\(^1\)The canonical formulation of GR based on the real-valued SU(2) connection variable was also formulated in [2].

\(^2\)In this paper, “chiral” means that only right-handed (or left-handed) spinor fields are coupled to the self-dual spin connection in the kinetic terms of spinor fields.
theories, where a consistency problem arising from the use of a complex self-dual spin connection which couples to spinor (spin-3/2) fields was solved for \( N = 1 \) \([12, 17, 18]\) and for \( N = 2 \) \([16]\).

The points of (b)-(d) are the problems in the canonical formulation of the chiral SUGRA (i.e., the ACF of SUGRA), in which two types of SUSY constraints, right- and left-handed SUSY constraints, appear in addition to Gauss-law, \( U(1) \) gauge (for \( N = 2 \)), vector and Hamiltonian constraints as the result of invariances of the chiral Lagrangian. Particularly, (c) and (d) show the nonperturbative structures of the chiral SUGRA. Indeed, as for (b), in the \( N = 1 \) theory \([12]\) all the constraints are written in polynomial form in terms of canonical variables. In the \( N = 2 \) theory \([14, 15, 19]\), although the left-handed SUSY constraint (and also the Hamiltonian constraint) has a nonpolynomial factor as in the case of the ACF of the Einstein-Maxwell theory \([6]\), the rescaled left-handed SUSY constraint by multiplying this factor becomes polynomial.

The simple graded algebraic structure with respect to (c) was first pointed out for the \( N = 1 \) theory through \([20, 21]\); the \( SU(2) \) algebra generated by the Gauss-law constraint is graded by means of the right-handed SUSY constraint \([20, 21]\), and all the constraints were also rewritten in a very simple polynomial form \([21]\) by using graded connection and momentum variables associated with the graded algebra (Lie superalgebra), \( Osp(1/2) \) (or \( GSU(2) \)) \([22]\). The \( Osp(2/2) \) (or \( G^2SU(2) \)) graded algebraic structure \([23]\) for the \( N = 2 \) theory among only the Gauss, \( U(1) \) gauge and right-handed SUSY constraints was pointed out in \([24]\) from the viewpoint of the canonical formulation of the BF theory as a topological field theory \([25]\), and also in \([19]\) from the straightforward derivation of the canonical formulation of \( N = 2 \) chiral SUGRA. As for exact solutions of quantum constraints in (d), two main results of pure gravity, i.e., Wilson loops \([26]\) and the exponential of the Chern-Simons form \([27]\), were discussed both in the \( N = 1 \) \([21]\) and \( N = 2 \) \([14, 19, 24]\) theories. In addition, for the \( N = 1 \) theory, based on the irreducible representation of \( Osp(1/2) \), the spin network state \([28]\) for SUGRA was constructed in \([29]\).

In contrast with the above situation in \( N = 1 \) and \( N = 2 \) chiral SUGRA, many open questions exist for \( N \geq 3 \) chiral SUGRA except for the construction of the chiral Lagrangian; indeed, chiral Lagrangians were constructed for \( N = 3, 4 \) theories based on the two-form SUGRA \([30]\), while for \( N = 3, 4 \) and 8 theories based on the standard SUGRA \([31]\), in which Lagrangians do not contain any auxiliary fields as introduced in the two-form SUGRA and SUSY transformation parameters are not constrained. In this paper we focus on \( N = 3 \) chiral SUGRA as the supersymmetric extension of gravity and spin-1/2 fields \([5, 8, 9]\), since it is the lowest \( N \) theory involving a spin-1/2 field. We derive the ACF of \( N = 3 \) SUGRA by using the chiral Lagrangian constructed in \([31]\), and we explicitly discuss on the problems (b) and (c), i.e., the polynomiality of constraints in terms of canonical variables.
and the graded algebraic structure of constraints for $N = 3$ chiral SUGRA. In particular, we show the polynomiality of the rescaled right- and left-handed SUSY constraints by the nonpolynomial factor which appears in the ACF of $N = 2$ SUGRA [14, 15, 19]. The graded algebraic structure of Osp(3/2) (see, for example, Ref.[32]) in the constraint algebra is also pointed out by calculating the Poisson brackets of the Gauss, SU(2) gauge and right-handed SUSY constraints. However, we show that the algebra among only those three types of constraints does not closed, although the constraint algebra among all the constraints is expected to be closed.

This paper is organized as follows. In Sec.2 we present a globally O(3) invariant Lagrangian in $N = 3$ chiral SUGRA which is slightly modified from the Lagrangian given in [31]. In Sec.3 a chiral Lagrangian for gauged $N = 3$ chiral SUGRA is introduced by extending the internal, global O(3) invariance to local one in order to discuss the graded algebraic structure of the Gauss, SU(2) gauge and right-handed SUSY constraints. The canonical formulation of the gauged $N = 3$ chiral SUGRA is derived in Sec.4, and we discuss on the problems (b) and (c) in the above arguments. We summarize our results in Sec.5.

2. Globally O(3) invariant chiral Lagrangian

In this section we present the Lagrangian of $N = 3$ chiral SUGRA [31]. Corresponding to the spin contents $(2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, 1, 1, \frac{1}{2})$ in the $N = 3$ theory, let us denote the fundamental variables as $e_i^\mu$ for a (real) tetrad, $\psi^{(I)}_\mu$ for three (Majorana) Rarita-Schwinger (spin-3/2) fields, $A^{(I)}_\mu$ for Maxwell fields, $\chi$ for a (Majorana) spin-1/2 field and $A_{ij\mu}^{(+)}$ for a (complex) self-dual spin connection which satisfies $(1/2)\epsilon_{ij}{}^{kl}A_{kl\mu}^{(+)} = iA_{ij\mu}^{(+)}$. In the first-order formalism, the $N = 3$ chiral Lagrangian density in terms of the above fundamental variables, which is globally O(3) invariant, is written as follows; namely, we have

$$
\mathcal{L}_{N=3}^{(+)} = -\frac{i}{2} \epsilon_{\mu \nu \rho \sigma} e_i^\mu e_j^\nu R_{ij\rho\sigma}^{(+)\mu} - \epsilon^{\mu \nu \rho \sigma} \tilde{\psi}_{R\mu}^{(I)} \gamma_\rho D_\sigma^{(+)} \psi^{(I)}_R
$$

$$
- \frac{e}{2} (F^{(I)}_{\mu \nu})^2 - i e \bar{\chi} \gamma^\mu D_\mu^{(+)} \chi_R
$$

$$
+ \frac{1}{4\sqrt{2}} \tilde{\psi}_{\mu}^{(J)} \{ e (F^{(I)}_{\mu \nu} + \tilde{F}^{(I)}_{\mu \nu}) + i \gamma_5 (\tilde{F}^{(I)}_{\mu \nu} + \tilde{\tilde{F}}^{(I)}_{\mu \nu}) \} \psi_{\nu}^{(K)} \epsilon^{(I)(J)(K)}
$$

$$
+ \frac{1}{2} e \left( \tilde{F}^{(I)}_{\mu \nu} - \frac{i}{2} \tilde{\psi}_{\mu}^{(I)} \gamma_{\nu} \chi \right) \tilde{\psi}_{\lambda}^{(I)} S^{\mu \nu} \gamma^\lambda \chi
$$

Greek letters $\mu, \nu, \ldots$, are spacetime indices, Lattin letters $i, j, \ldots$, are local Lorentz indices and $(I), (J), \ldots (= (1), (2), (3))$, denote O(3) internal indices. We take the Minkowski metric $\eta_{ij} = \text{diag}(-1, +1, +1, +1)$ and the totally antisymmetric tensor $\epsilon_{ijkl}$ is normalized as $\epsilon_{0123} = +1$. We define $\epsilon_{\mu \nu \rho \sigma}$ and $\epsilon^{\mu \nu \rho \sigma}$ as tensor densities which take values of $+1$ or $-1$. 

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the Maxwell kinetic term in Eq.(2.1), where \( F^{4 \mu} \) is defined by

\[ F^{4 \mu} = \frac{i}{8} \epsilon^{\mu \nu \rho \sigma} \left( \bar{\psi}^{(J)} L_{\mu} \bar{\psi}^{(K)} R_{\nu} \right) \psi^{(L)} A_{\rho} \epsilon^{(I)(J)(K)} \epsilon^{(I)(L)(M)} \]

and \( e \) denotes \( \det(e^i_\mu) \), the covariant derivative \( D^+(\mu) \) and the curvature \( R^{(ij)}_{\mu \nu} \) with respect to the \( A^{(ij)}_{\mu \nu} \) are defined by

\[ D^+(\mu) = \partial_\mu + \frac{i}{2} A^{(ij)}_{\mu \nu} \sigma^{ij}, \]

\[ R^{(ij)}_{\mu \nu} = 2 \partial_\mu A^{(ij)}_{\nu \nu} + A^{(ij)}_{k \nu} A^{(+)(kj)}_{\nu \nu}. \]

In addition, \( \epsilon^{(I)(J)(K)} \) is a totally antisymmetric tensor and \( F^{(I)}_{\mu \nu} \) means the Abelian field strength, i.e., \( F^{(I)}_{\mu \nu} = 2 \partial_\mu A^{(I)}_{\nu} \).

In Eq.(2.1), we have used \( \tilde{F}^{(I)}_{\mu \nu} = (1/2) \epsilon^{\mu \nu \rho \sigma} F^{(I)}_{\rho \sigma} \) and \( \hat{F}_{\mu \nu} \) is defined as

\[ \hat{F}^{(I)}_{\mu \nu} = F^{(I)}_{\mu \nu} - \frac{1}{\sqrt{2}} \epsilon^{(I)(J)(K)} \bar{\psi}^{(J)}_\mu \psi^{(K)}_\nu. \]

Following the case of \( N = 2 \) chiral SUGRA [19], we have also used \( (F^{-}(I))_{\mu \nu} \) as the Maxwell kinetic term in Eq.(2.1), where \( F^{-}(I)_{\mu \nu} = (1/2)(F^{(I)}_{\mu \nu} + i e F^{(I)}_{\mu \nu}) \).

Here we explain the role of the last two four-fermion contact terms in Eq.(2.1) with respect to the Rarita-Schwinger and spin-1/2 fields [31], which will be denoted by \( \Psi_{4-\text{fermi}} \) of Eq.(2.7) below. Those terms are pure imaginary but are necessary to reproduce the Lagrangian of the standard \( N = 3 \) SUGRA [33, 34] in the second-order formalism as follows. Indeed, in the first-order formalism, the \( N = 3 \) chiral Lagrangian density (2.1) differs from that of the standard \( N = 3 \) SUGRA by the following imaginary terms,

\[ (L^{(+)}_{N=3} - L^{(+)\text{standard SUGRA}}) [\text{first order}] = \Psi_{\text{kin}} + \Psi_{\text{CS-boundary}} + \Psi_{4-\text{fermi}} \]

with \( \Psi_{\text{kin}}, \Psi_{\text{CS-boundary}} \) and \( \Psi_{4-\text{fermi}} \) being defined by

\[ \Psi_{\text{kin}} = -i \left\{ \epsilon^{\mu \nu \rho \sigma} (T^{(I)}_{\mu \nu} + i \bar{\psi}^{(I)}_\mu \gamma_\lambda \psi^{(I)}_\nu) T^{(I)}_{\rho \sigma} + 2 e \bar{\chi} \gamma_5 \gamma^\nu \chi T^{(I)}_{\mu \nu} \right\}, \]

\[ \Psi_{\text{CS-boundary}} = -i \left( \frac{e}{4} \epsilon^{\mu \nu \rho \sigma} (T^{(I)}_{\nu \rho} + i \bar{\psi}^{(I)}_\rho \gamma_\lambda \psi^{(I)}_\sigma) - e \bar{\chi} \gamma_5 \gamma^\mu \chi \right. \]

\[ \left. + 2 \epsilon^{\mu \nu \rho \sigma} A^{(I)}_{\rho} \partial_\mu A^{(I)}_{\sigma} \right\}, \]

\[ \Psi_{4-\text{fermi}} = i \left\{ \epsilon^{\mu \nu \rho \sigma} (\bar{\psi}^{(J)}_{L_{\mu}} \bar{\psi}^{(K)}_{R_{\nu}}) \psi^{(L)}_{R_{\rho}} \psi^{(M)}_{L_{\sigma}} \epsilon^{(I)(J)(K)} \epsilon^{(I)(L)(M)} \right\}

\[ + \frac{1}{8} \epsilon(\bar{\psi}^{(I)}_\mu \gamma^\mu \psi^{(I)}_\nu) \bar{\chi} \gamma_5 \gamma^\nu \chi, \]
where the torsion tensor is $T_{\mu \nu}^i = -2D_{[\mu}e_{\nu]}^i$ with $D_{\mu}e_{\nu}^i = \partial_{\mu}e_{\nu}^i + A_{i \mu \nu}^j e_{\nu}^j$. The terms in $\Psi_{\text{kin}}$ and $\Psi_{\text{CS-boundary}}$ appear from the chiral gravitational Lagrangian density and from the kinetic terms of the Rarita-Shwinger, Maxwell and spin-$1/2$ fields in Eq.(2.1). The imaginary boundary term $\Psi_{\text{CS-boundary}}$ of Eq.(2.6) corresponds to a Chern-Simons boundary term given in [18] as a generating function of the canonical transformation. However, the four-fermion contact terms in $\Psi_{4-\text{fermi}}$ of Eq.(2.7) do not appear in $N = 1$ chiral SUGRA [12, 18], and those are the non-minimal terms required from the invariance under first-order SUSY transformations [16, 31].

In the second-order formalism, i.e., when we solve the equation $\delta \mathcal{L}_{N=3}^{(+)} / \delta A_{ij \mu}^{(+)} = 0$ with respect to $A_{ij \mu}^{(+)}$ and we use the obtained solution in Eq.(2.4), e.g., we substitute the solution for the torsion tensor,

$$T_{\rho \mu \nu} = -\frac{i}{2}\bar{\psi}_{\mu}^{(I)}\gamma_{\rho}\psi_{\nu}^{(I)} + \frac{1}{4}\epsilon_{\mu \nu \rho \sigma}\bar{\chi}\gamma_{\rho}\gamma_{\sigma}\chi,$$

into Eq.(2.4), the $\Psi_{\text{kin}}$ of Eq.(2.5) becomes

$$\Psi_{\text{kin}} = -\frac{i}{32}\epsilon_{\mu \nu \rho \sigma}(\bar{\psi}_{\mu}^{(I)}\gamma_{\lambda}\psi_{\nu}^{(J)})\bar{\psi}_{\rho}^{(J)}\gamma_{\lambda}\psi_{\sigma}^{(J)} \mid_{(I) \neq (J)}$$

$$-\frac{1}{8}\epsilon(\bar{\psi}_{\mu}^{(I)}\gamma_{\mu}\psi_{\nu}^{(I)})\bar{\chi}\gamma_{\nu}\chi,$$

which do not vanish by itself in contrast with the $N = 1$ chiral SUGRA. On the other hand, the first term in $\Psi_{4-\text{fermi}}$ of Eq.(2.7) can be rewritten as

$$\frac{i}{8}\epsilon_{\mu \nu \rho \sigma}(\bar{\psi}_{\mu}^{(I)}\gamma_{\lambda}\psi_{\nu}^{(J)})\bar{\psi}_{\rho}^{(J)}\gamma_{\lambda}\psi_{\sigma}^{(J)} \mid_{(I) \neq (J)}$$

$$= \frac{i}{32}\epsilon_{\mu \nu \rho \sigma}(\bar{\psi}_{\mu}^{(I)}\gamma_{\lambda}\psi_{\nu}^{(J)})\bar{\psi}_{\rho}^{(J)}\gamma_{\lambda}\psi_{\sigma}^{(J)} \mid_{(I) \neq (J)}$$

by using Fierz transformations, and so $\Psi_{4-\text{fermi}}$ (i.e., the last two terms in Eq.(2.1)) exactly cancels out the terms of Eq.(2.9). Therefore, in the second-order formalism, the $N = 3$ chiral Lagrangian density (2.1) is reduced to the Lagrangian density of the $N = 3$ standard SUGRA up to imaginary boundary terms as

$$\mathcal{L}_{N=3}^{(+)}[\text{second order}] = \mathcal{L}_{N=3 \text{ standard SUGRA}[\text{second order}]}$$

$$+ \frac{1}{8}\partial_{\mu}(\epsilon_{\mu \nu \rho \sigma}\bar{\psi}_{\nu}^{(I)}\gamma_{\rho}\psi_{\sigma}^{(I)}) - ie\bar{\chi}\gamma_{\mu}\chi$$

$$-4ie_{\mu \nu \rho \sigma}A_{\nu}^{(I)}\partial_{\rho}A_{\sigma}^{(I)}.$$

\footnote{Note that a boundary term quadratic in the Maxwell field $A_{\mu}^{(I)}$ appears in Eq.(2.6) since we choose $(F_{\mu \nu}^{(I)(-)})^2$ as the Maxwell kinetic term in Eq.(2.1).}
3. Gauging the \( \text{O}(3) \) invariance

In order to discuss the graded algebraic structure in the canonical formulation of \( N = 3 \) chiral SUGRA in the next section, let us extend the internal, global \( \text{O}(3) \) invariance of the chiral Lagrangian density (2.1) to local one [35]. This method, i.e., gauging the \( \text{O}(3) \) invariance, is the same as the case of \( N = 2 \) SUGRA [35] except for the introduction of the non-Abelian field strength. Indeed, after introducing a minimal coupling of \( \psi^{(I)}_{R\mu} \) with \( A^{(I)}_{\mu} \), it requires to replace the Abelian field strength \( F^{(I)}_{\mu\nu} \) with the non-Abelian one,

\[
F^{n(I)}_{\mu\nu} = F^{(I)}_{\mu\nu} + \lambda \epsilon^{(I)(J)(K)} A^{(J)}_{\mu} A^{(K)}_{\nu} \tag{3.1}
\]

with the gauge coupling constant \( \lambda \). Furthermore the minimal coupling automatically requires a spin-3/2 mass-like term and a cosmological term in order to ensure the SUSY invariance of the Lagrangian; these three terms added to Eq.(2.1) are then written as

\[
\mathcal{L}_{\text{cosm}} = \frac{\lambda}{2} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}^{(I)}_{\mu} \gamma_5 \gamma_\rho \psi^{(K)}_{\sigma} A^{(J)}_{\epsilon} \epsilon^{(I)(J)(K)} \\
- \sqrt{2} i \kappa^{-1} \lambda \epsilon \left( \bar{\psi}^{(I)}_\mu \gamma_\mu \psi^{(I)}_\nu \right) \\
- \Lambda \kappa^{-2} e \tag{3.2}
\]

where the cosmological constant \( \Lambda \) is related to \( \lambda \) as \( \Lambda = -6\kappa^{-2} \lambda^2 \). Note that the first term of Eq.(3.2) is comparable with the kinetic term of \( \psi^{(I)}_{R\mu} \) in Eq.(2.1), because

\[
\frac{\lambda}{2} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}^{(I)}_{\mu} \gamma_5 \gamma_\rho \psi^{(K)}_{\sigma} A^{(J)}_{\epsilon} \epsilon^{(I)(J)(K)} = -\lambda \epsilon^{\mu\nu\rho\sigma} \bar{\psi}^{(I)}_{R\mu} \gamma_\rho \psi^{(K)}_{R\nu} A^{(J)}_{\epsilon} \epsilon^{(I)(J)(K)}. \tag{3.3}
\]

We denote the gauged \( N = 3 \) chiral Lagrangian density as the sum of Eqs.(2.1) and (3.2), in which \( F^{(I)}_{\mu\nu} \) is replaced by \( F^{n(I)}_{\mu\nu} \) of Eq.(3.1); namely, we define

\[
\mathcal{L}^{(+)}_{N=3} \text{gauged} = \mathcal{L}^{(+)}_{N=3} [F^{(I)}_{\mu\nu} \rightarrow F^{n(I)}_{\mu\nu}] + \mathcal{L}_{\text{cosm}}. \tag{3.4}
\]

Let us give two comments on the above gauged chiral Lagrangian. First, from the discussion in Sec.2, it is obvious that in the second-order formalism the gauged \( N = 3 \) chiral Lagrangian density (3.4) is reduced to the gauged Lagrangian density of the \( N = 3 \) standard SUGRA [35] up to imaginary boundary terms as

\[
\mathcal{L}^{(+)}_{N=3} \text{gauged}[\text{second order}] = \mathcal{L}^{\text{standard SUGRA}}_{N=3}[\text{second order}] + \frac{1}{8} \partial_\mu \left( \epsilon^{\mu\nu\rho} \bar{\psi}^{(I)}_\rho \gamma_\nu \psi^{(I)}_{\sigma} - ie \bar{\chi} \gamma_5 \gamma_\mu \chi \right) \\
- \frac{i}{2} \partial_\mu \left( \epsilon^{\mu\nu\rho} \left( A^{(I)}_\nu \partial_\rho A^{(I)}_\sigma + \frac{\lambda}{3} \epsilon^{(I)(J)(K)} A^{(I)}_\nu A^{(J)}_\rho A^{(K)}_\sigma \right) \right). \tag{3.5}
\]
Secondly, because of Eq. (3.5), the $\mathcal{L}_{N=3}^{(+)}$ gauged of Eq. (3.4) in the second-order formalism is invariant under the SUSY transformation of the standard gauged $N = 3$ SUGRA given in [35] by

$$\delta e^i_\mu = i \bar{\alpha}^{(I)} \gamma^j \psi^{(I)}_j,$$
$$\delta A^{(I)}_{\mu} = \sqrt{2} \epsilon^{(I)(J)(K)} \bar{\alpha}^{(J)} \psi^{(K)}_{\mu} + i \bar{\alpha}^{(I)} \gamma_\mu \chi,$$
$$\delta \psi^{(I)}_\mu = 2 \{ D_\mu [A(e, \psi^{(I)}_j)] \alpha^{(I)} + \lambda \epsilon^{(I)(J)(K)} A^{(J)}_{\mu} \alpha^{(K)} \} - i \frac{1}{8} (\bar{\chi} \gamma^5 \chi) \gamma^\lambda \gamma_\mu \alpha^{(I)}$$
$$- \frac{1}{\sqrt{2}} \epsilon^{(I)(J)(K)} \bar{F}^{(J)}_{\rho \sigma} \gamma_{\mu} \alpha^{(K)}$$
$$+ \frac{1}{2 \sqrt{2}} \epsilon^{(I)(J)(K)} \{ (\bar{\psi}^{(J)}_\mu \gamma_\nu \chi) \gamma^\nu \alpha^{(K)} - (\bar{\psi}^{(J)}_\mu \gamma_5 \gamma_\nu \chi) \gamma_5 \gamma^\nu \alpha^{(K)} \}$$
$$- \sqrt{2} i \lambda \gamma_\mu \alpha^{(I)}.$$

(3.6)

where $A_{ij\mu}(e, \psi^{(I)}_j)$ in $\delta \psi^{(I)}_\mu$ is defined as the sum of the Ricci rotation coefficients $A_{ij\mu}(e)$ and $K_{ij\mu}$ which is expressed as

$$K_{\mu \rho \sigma} = \frac{i}{4} (\bar{\psi}^{(I)}_\mu \gamma_\rho \psi^{(I)}_\nu + \bar{\psi}^{(I)}_\nu \gamma_\rho \psi^{(I)}_\mu - \bar{\psi}^{(I)}_\mu \gamma_5 \gamma_\rho \psi^{(I)}_\nu) + \frac{1}{8} \epsilon_{\mu \rho \sigma} \bar{\chi} \gamma_5 \gamma^\sigma \chi.$$ (3.7)

On the other hand, in the first-order formalism, the SUSY invariance of $\mathcal{L}_{N=3}^{(+)}$ may be realized by introducing the right- and left-handed SUSY transformations as in the case of $N = 1$ chiral SUGRA [12, 36].

4. Canonical formulation of $N = 3$ chiral SUGRA

In this section, we derive the canonical formulation of $N = 3$ chiral SUGRA (the ACF of $N = 3$ SUGRA) by means of the (3+1) decomposition of spacetime, starting with the gauged $N = 3$ chiral Lagrangian density (3.4). The gauge condition for the tetrad $e^i_\mu$ in the (3+1) decomposition of spacetime which we shall follow is the same as that of [19]. Namely, we assume that the topology of spacetime $M$ is $\Sigma \times R$ for some three-manifold $\Sigma$ so that a time coordinate function $t$ is defined on $M$. Then the time component of the tetrad can be defined as

$$e^t_\mu = N n^i + N^a e^i_a.$$ (4.1)

As for the indices of the canonical formulation, Latin letters $a, b, \ldots$ are used as the spatial part of the spacetime indices $\mu, \nu, \ldots$, and capital letters $I, J, \ldots$ are used as the spatial part of the local Lorentz indices $i, j, \ldots$. Two-component spinor indices $A, B, \ldots$ and $A', B', \ldots$ are also used. As for the conventions of the two-component spinor formulation and the other several conventions in the canonical formulation, we shall follow those of [19].
Here $n^i$ is the timelike unit vector orthogonal to $e_{ia}$, i.e., $n^i e_{ia} = 0$ and $n^i n_i = -1$, while $N$ and $N^a$ denote the lapse function and the shift vector, respectively. Furthermore, we give a restriction on the tetrad with the choice $n_i = (-1, 0, 0, 0)$ in order to simplify the Legendre transform of Eq.(3.4). Once this choice is made, $e_{ia}$ becomes tangent to the constant $t$ surfaces $\Sigma$ and $e_{0a} = 0$. Therefore we change the notation $e_{ia}$ to $E_{ia}$ below. We also take the spatial restriction of the totally antisymmetric tensor density $\epsilon_{\mu\nu\rho\sigma}$ as $\epsilon_{abc} = \epsilon_{tabc}$, while $\epsilon_{IJK} = \epsilon_{0IJK}$.

From the $(3+1)$ decomposition of Eq.(3.4) under the above gauge condition of the tetrad, the kinetic terms in the canonical formulation which define canonical variables are obtained as

$$L_{N=3}^{(+)}[\text{kinetic terms}] = \tilde{E}_I^a \dot{A}_a^I - \tilde{\pi}^{(I)} A^a_0 \dot{\bar{\psi}}^{(I)}_a + \tilde{\pi}^{(I)} A^a_0 \dot{\chi}, \quad (4.2)$$

where $A^{" I}_a := -2A_0^{(+) I}_a$ and the momentum variables $(\tilde{\pi}^{(I)} A^a, \tilde{\pi}^{(I)} a, \tilde{\pi} A)$ are defined by

$$\tilde{\pi}^{(I)} A^a = \frac{\delta L^{(+)}}{\delta \dot{A}^{(I)} A^a} = -\sqrt{2} i \epsilon^{abc} E^T_c \bar{\psi}_b^{(I)} \sigma_{IAA'}, \quad (4.3)$$

$$+\tilde{\pi}^{(I)} a = \frac{\delta L^{(+)}}{\delta \dot{\chi}} = \tilde{\pi}^{(I)} a + i \tilde{B}^{(I)} a \quad (4.4)$$

$$\tilde{\pi} A = \frac{\delta L^{(+)}}{\delta \dot{\chi}} = \sqrt{2} E \bar{\chi}^A n_{AA'} \quad (4.5)$$

with

$$\tilde{\pi}^{(I)} a = \frac{e}{2N^2} \eta^{ab} \left\{ 2 \left( F_{tb} - N^d F_{db} \right) + \sqrt{2} \left( \bar{\psi}_t^{(J)} \psi_b^{(K)} \right) \right. \right.$$ \left. \left. - \frac{i}{2\sqrt{2}} \epsilon^{abc} \bar{\psi}_b^{(J)} \gamma_5 \psi_c^{(K)} \epsilon^{(I)(J)(K)} \right\} \right.

$$- \frac{ie}{2N^2} \eta^{ab} \left( E^T_b \psi_tA^{(I)} \gamma_5 \chi - N^d E^T_d \psi_b^{(I)} \gamma_5 \chi - N^d \psi_b^{(I)} \gamma_0 \chi \right) \right.

$$- \frac{1}{2} \epsilon^{abc} F_{bc}^{(I)} \bar{\psi}_b^{(I)} \gamma_5 \chi. \quad (4.6)$$

$$\tilde{B}^{(I)} a = \frac{1}{2} \epsilon^{abc} F_{bc}^{(I)}. \quad (4.7)$$

We have used the Majorana spinors $\psi_\mu^{(I)}$ in Eq.(4.6) for simplicity.

Furthermore, we obtain constraints which reflect the invariance of the gauged $N = 3$ chiral Lagrangian density (3.4) from the variation of the Lagrangian with
respect to Lagrange multipliers. In this paper, let us explicitly show the Gauss, SU(2) gauge, right-handed SUSY and left-handed SUSY constraints in terms of the canonical variables. Varying Eq.(3.4) by Lagrange multipliers $\Lambda^I_t$, $A^{(I)}_t$, $\psi^{(I)A}_t$ and $\rho^{(I)A}_t$, in which $\Lambda^I_t$ and $\rho^{(I)A}_t$ are defined by

$$
\Lambda^I_t = -2A^{(+)}_0 t, \quad \rho^{(I)A}_t = E^{-1}\bar{\psi}^{(I)}_A n^{AA'},
$$

(4.8)

yields those four types of constraints as

$$
G^{(I)} = \frac{\delta L^{(+)}(I)}{\delta \Lambda^I_t} = D_a \bar{E}^a_I - \frac{i}{\sqrt{2}} \bar{\pi}^{(I)A}_a \sigma^A_B \psi^{(I)B}_a - \frac{i}{\sqrt{2}} \bar{\pi} A \sigma^A_B \chi^B = 0,
$$

(4.9)

$$
g^{(I)} = \frac{\delta L^{(+)}(I)}{\delta A^{(I)}_t} = \partial_a \bar{\pi}^{(I)A}_a + \lambda \epsilon^{(I)(J)(K)} A^{(J)}_a + \bar{\pi}(K)_a + \lambda \psi^{(I)A}_a \bar{\pi}(K)_A \epsilon^{(I)(J)(K)}
$$

(4.10)

$$
R S^{(I)}_A = \frac{\delta L^{(+)}(I)}{\delta \rho^{(I)A}_t} = D_a \bar{\pi}^{(I)A}_a + \frac{1}{\sqrt{2}} + \bar{\rho} \psi^{(I)B}_a \epsilon^{AB} \epsilon^{(I)(J)(K)}
$$

(4.11)

$$
L S^{(I)}_A = \frac{\delta L^{(+)}(I)}{\delta \rho^{(I)A}_t} = -\sqrt{2} \bar{E}^a_I \bar{E}^a_I + \left( 2 (D_a \psi^{(I)(J)} B e^{(I)(J)(K)} A^{(J)}_a \psi^{(K)}_a B e^{(I)(J)(K)} e_{BC} + \frac{i}{\sqrt{2}} \lambda \epsilon^{AB} \bar{\pi}^{(I)C} B e_{BC} \right)
$$

(4.12)

where the $\Phi^a$ in Eqs.(4.11) and (4.12) is a quantity expressed by the canonical variables in polynomial form as

$$
\Phi^a = \epsilon^{abc} \left( \Gamma_{bc}^{(K)} + \frac{1}{\sqrt{2}} \epsilon_{CD} \psi^{(L)C}_b \psi^{(M)D} e^{(K)(L)(M)} \right)
$$

(4.13)

\begin{align*}
+ i^{+} \bar{\pi}^{(K)A}_a + i^{a} \bar{\rho}^{(K)} A \chi^C,
\end{align*}

and the covariant derivatives on $\Sigma$ are defined as

$$
D_a \bar{E}^a_I = \partial_a \bar{E}^a_I + i \epsilon_{IJK} A^{(J)}_a \bar{E}^K_a,
$$

10
\[ D_a \tilde{\pi}^{(I) A} = \partial_a \tilde{\pi}^{(I) A} - \frac{i}{\sqrt{2}} A^B a \tilde{\pi}^{(I) B} a. \] (4.14)

Obviously, the Gauss and SU(2) gauge constraints of Eqs. (4.9) and (4.10) are polynomial with respect to the canonical variables, while both the right- and left-handed SUSY constraints of Eqs. (4.11) and (4.12) are not polynomial because of the factor \( E^{-2} \). But the rescaled right- and left-handed SUSY constraints by the nonpolynomial factor, i.e., \( E^{2 R} S_A^{(I)} \) and \( E^{2 L} S_A^{(I)} \) become polynomial because of the relation, \( E^2 = (1/6) \epsilon_{abc} \epsilon^{IKL} \tilde{E}^a \tilde{E}^b \tilde{E}^c \).

In order to discuss the graded algebraic structure in the canonical formulation of \( N = 3 \) chiral SUGRA, we calculate the Poisson brackets of the Gauss, SU(2) gauge and right-handed SUSY constraints of Eqs. from (4.9) to (4.11) by using the non-vanishing Poisson brackets among the canonical variables,

\[
\begin{align*}
\{ A^I_a(x), \tilde{E}^a_B(y) \} &= \delta^I_I \delta^a_a \delta^B_B (x - y), \\
\{ \psi_d^{(I) a}(x), \tilde{\pi}^{(J) b}(y) \} &= - \delta^{(I) (J)} \delta^a_b \delta^B_B \delta^b^d \delta^{(x - y)}, \\
\{ A^a_I(x), + \tilde{\pi}^{(J) b}(y) \} &= \delta^{(I) (J)} \delta^a_b \delta^{(x - y)}, \\
\{ \chi^A(x), \tilde{\pi}^B(y) \} &= - \delta^A_B \delta^3 (x - y). 
\end{align*}
\] (4.15)

In fact, when we define the smeared functions,

\[
\begin{align*}
G_I[\Lambda^I] &= \int_\Sigma d^3 x \, \Lambda^I G_I, \\
g^{(I)}[a^{(I)}] &= \int_\Sigma d^3 x \, a^{(I)} g^{(I)}, \\
(E^2 R S_A^{(I)})[\xi^{(I) A}] &= \int_\Sigma d^3 x \, \xi^{(I) A} (E^2 R S_A^{(I)})(4.16)
\end{align*}
\]

for convenience of the calculation, the Poisson brackets of \( G_I, g^{(I)} \) and \( R S_A^{(I)} \) are obtained as

\[
\begin{align*}
\{ G_I[\Lambda^I], G_J[\Gamma^J] \} &= G_I[\Lambda^J], \\
\{ G_I[\Lambda^I], g^{(I)}[a^{(I)}] \} &= 0, \\
\{ g^{(I)}[a^{(I)}], g^{(J)}[b^{(J)}] \} &= \lambda g^{(I)}[a^{(I)}], \\
\{ G_I[\Lambda^I], (E^2 R S_A^{(I)})[\xi^{(I) A}] \} &= (E^2 R S_A^{(I)})[\xi^{(I) A}], \\
\{ g^{(I)}[a^{(I)}], (E^2 R S_A^{(I)})[\xi^{(I) A}] \} &= \lambda (E^2 R S_A^{(I)})[\xi^{(I) A}], \\
\{(E^2 R S_A^{(I)})[\xi^{(I) A}], (E^2 R S_B^{(J)})[\eta^{(J) B}] \} &= \lambda E^2 G_I[\Lambda^{n I}] + E^4 g^{(I)}[a^{(n I)}] \\
&\quad + E^2 R S_A^{(I)}[\eta^{(I) A}] + E^2 S_A^{(I)}[\eta^{(n) A}], \\
\end{align*}
\] (4.22)

where the smeared function,

\[
(E^2 L S_A^{(I)})[\xi^{(I) A}] = \int_\Sigma d^3 x \, \xi^{(I) A} (E^2 L S_A^{(I)}),
\] (4.23)
has also been used in Eq.(4.22). In Eqs. from (4.17) to (4.22), $\Lambda^I$, $\Lambda''^I$, $\xi^{(I)A}$, $\xi''^{(I)A}$, $a^{(I)}$ and $a''^{(I)}$, are defined as the field-independent parameters by

$$
\Lambda^I = i \epsilon^{IJK} \Lambda_J \Gamma_K, \\
\Lambda''^I = 2i \xi^{(I)A} \eta^{(J)B} \sigma_{AB} \delta^{(I)(J)}, \\
\xi^{(I)A} = \frac{i}{\sqrt{2}} \Lambda_I \xi^{(I)B} \sigma_{IB} A, \\
\xi''^{(I)A} = \epsilon^{(I)(J)(K)} a^{(J)} \xi^{(K)A}, \\
a^{(I)} = \epsilon^{(I)(J)(K)} a^{(J)} \eta^{(K)} B, \\
a''^{(I)} = \frac{1}{\sqrt{2}} \epsilon^{(I)(J)(K)} \xi^{(J)A} \eta^{(K)} B \epsilon_{AB},
$$

(4.24)

and also $\eta^{(I)A}$ and $\eta''^{(I)A}$ are defined as the field-dependent parameters by

$$
\eta^{(I)A} = -\frac{i}{2} \left( \xi^{(I)A} \eta^{(J)B} + \xi^{(J)B} \eta^{(I)A} \right) \epsilon^{abc} (\sigma^I \sigma^J) B^C \tilde{E}_I \tilde{E}_J \tilde{r}^{(J)} C^c, \\
\eta''^{(I)A} = \frac{1}{2} \epsilon^{(I)(J)(K)} \xi^{(J)B} \eta^{(K)C} \epsilon_{BC} \tilde{r}^{A}.
$$

(4.25)

Except for the last two terms in the r.h.s. of Eq.(4.22), the resultant Poisson bracket (4.17)-(4.22) shows that the SU(2) $\times$ SU(2) algebra of Eqs. from (4.17) to (4.19) is graded by means of $R_S^{(I)}$, i.e., the algebra of the Gauss, SU(2) gauge and right-handed SUSY constraints includes the graded algebra (Lie superalgebra), Osp(3/2) [32]. This is expressed in terms of the generators $(J_I, J^{(I)}; J_{A}^{(I)})$ which correspond to those three types of constraints as

$$
[J_I, J_J] = i \epsilon_{IJ} \Gamma_K J_K, \\
[J^{(I)}, J^{(J)}] = \lambda \epsilon^{(I)(J)(K)} J^{(K)}, \\
[J_I, J^{(I)}] = \frac{i}{\sqrt{2}} \sigma_{IAB} J_B^{(I)}, \\
[J^{(I)}, J_{A}^{(J)}] = \lambda \epsilon^{(I)(J)(K)} J_{A}^{(K)}, \\
[J_{A}^{(I)}, J_{B}^{(J)}] = 2i \lambda \delta^{(I)(J)} \sigma_{AB} J_I + \frac{1}{\sqrt{2}} \epsilon^{(I)(J)(K)} \epsilon_{AB} J^{(K)}, \\
[J_I, J^{(I)}] = 0,
$$

(4.26)

where the structure constants are determined from Eqs.(4.17)-(4.22). However, in contrast with the case of the $N = 1$ and $N = 2$ theories, the algebra among only the Gauss, SU(2) and right-handed SUSY constraints does not closed, in particular, by the last term in Eq.(4.22) which is proportional to the left-handed SUSY constraint $L_S^{(I)}$, although the algebra among all the constraints which appear in the canonical formulation is expected to be closed.
5. Conclusions

In this paper we have derived the canonical formulation of $N = 3$ chiral SUGRA (the ACF of $N = 3$ SUGRA), starting with the gauged $N = 3$ chiral Lagrangian density (3.4), in which the Maxwell kinetic term is modified as $(F^\alpha(-)^{ij}(\tilde{f})_{\mu\nu})^2$. We have shown the explicit form of the Gauss, SU(2) gauge, right-handed SUSY and left-handed SUSY constraints in terms of the canonical variables, and we have also discussed that both the right- and left-handed SUSY constraints are not polynomial because of the nonpolynomial factor $E^{-2}$, but the rescaled constraints by this factor become polynomial. In addition, by calculating of the Poisson brackets among the Gauss, SU(2) gauge and right-handed SUSY constraints following the case of the canonical formulation of $N = 1$ and $N = 2$ chiral SUGRA, we have shown the graded algebraic structure of Osp(3/2) [32] in the constraint algebra. However, in contrast with the case of the $N = 1$ and $N = 2$ theories, the algebra among only those three types of constraints does not closed, in particular, by the term which is proportional to the left-handed SUSY constraint, although the algebra among all the constraints is expected to be closed as a whole.

We are now trying to canonically quantize the theory and to derive exact solutions of quantum constraints, e.g., based on the introduction of graded variables associated with the Osp(3/2) algebra.

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