BCS model with asymmetric pair scattering: a non-Hermitian, exactly solvable Hamiltonian exhibiting generalized exclusion statistics

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Abstract
We demonstrate the occurrence of free quasi-particle excitations obeying generalized exclusion statistics in a BCS model with asymmetric pair scattering. The results are derived from an exact solution of the Hamiltonian, which was obtained via the algebraic Bethe ansatz utilizing the representation theory of an underlying Yangian algebra. The free quasi-particle excitations are associated with highest weight states of the Yangian algebra, corresponding to a class of analytic solutions of the Bethe ansatz equations.

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1. Introduction
In a recent fast track communication [15] we discussed the phenomenon of generalized exclusion statistics [11, 12, 16] in a non-Hermitian BCS model. The non-Hermitian Hamiltonian results from an asymmetry in the Cooper pair scattering amplitudes, depending on whether the scattering is from higher energy to lower energy states or vice versa. As a result of being non-Hermitian, the Hamiltonian admits complex eigenvalues for particular choices of the coupling parameters. However as the Hamiltonian is a real-valued operator non-real eigenvalues, when they occur, arise as complex conjugate pairs. In this sense the spectrum of the Hamiltonian mirrors the property of admitting an unbroken $\mathcal{PT}$-symmetric phase and a broken $\mathcal{PT}$-symmetric phase which has been discussed in multiple other models, e.g. see papers contained within [4]. It was found from numerical diagonalization of the asymmetric BCS Hamiltonian that the transition between broken and unbroken $\mathcal{PT}$-symmetry occurs on two boundary lines in the phase diagram. Furthermore, it was determined from the exact Bethe ansatz solution of the Hamiltonian that there exist free quasi-particle excitations on the boundary lines which can be characterized as satisfying generalized exclusion statistics. This finding relies on a deep understanding of the representation theory of a Yangian algebra,
denoted \( Y[gl(2)] \), from which the model may be obtained via the quantum inverse scattering method [7]. The objective of this paper is to provide a detailed exposition of these generalized exclusion statistics.

In section 2 we begin by discussing the reduced BCS pairing Hamiltonian in general terms. We specify the asymmetric choice of Cooper pair scattering amplitudes which leads to our model of investigation. In section 3 we review some relevant aspects of the quantum inverse scattering method [13, 20], and the associated techniques of the algebraic Bethe ansatz for the derivation of exact solutions. This algebraic approach is essentially an application of the representation theory for the Yangian algebra \( Y[gl(2)] \), which was developed by Chari and Pressley [6]. In section 4 we illustrate how excitations obeying generalized exclusion statistics are identified in terms of dividing the solutions of the Bethe ansatz equations into two classes. This situation is reminiscent of generalized exclusion statistics in an exactly solved model studied by Fendley and Schoutens [9], and provides an example of deconfined quantum criticality [18, 19]. In part our analysis amounts to addressing an old question regarding the completeness of the Bethe ansatz solution discussed in Bethe’s original work for the XXX chain [5], and more recently debated in [2, 8]. We conjecture a formula for counting the number of deconfined states, and prove its validity in some limiting cases. Concluding remarks are given in section 5. Two appendices are included. The first tables some complete, analytic solutions of the Bethe ansatz equations associated with low-dimensional Hilbert spaces of states. The second describes a mean-field analysis of the Hamiltonian which predicts that the Hamiltonian always has a real eigenspectrum. These latter calculations highlight the need for exact results to accurately characterize the system.

2. The Hamiltonian

The general form for a reduced BCS Hamiltonian as originally discussed in [3] is given by

\[
    H_{\text{BCS}} = \sum_{j=1}^{L} \epsilon_j n_j - \sum_{j,k=1}^{L} G_{jk} c_{kj}^+ c_{kj},
\]

Here, \( j = 1, \ldots, L \) labels a shell of doubly degenerate single particle energy levels with energies \( \epsilon_j \), and \( n_j = c_{kj}^+ c_{kj} + c_{kj}^+ c_{kj} \) is the fermion number operator for level \( j \). The operators \( c_{kj}^\pm, c_{kj}^\dagger \) are the annihilation and creation operators for fermions at level \( j \). The labels \( \pm \) refer to pairs of time-reversed states.

An important feature of the Hamiltonian (1) is the blocking effect. For any unpaired fermion at level \( j \) the action of the pairing interaction is zero since only paired fermions are scattered. This means that basis states for the Hilbert space can be decoupled into products of paired and unpaired fermion states in which the action of the Hamiltonian on the space for the unpaired fermions is automatically diagonal in the basis of number operator eigenstates. In view of this property the pair number operator

\[
    N = \sum_{j=1}^{L} c_{kj}^+ c_{kj},
\]

commutes with (1) and thus provides a good quantum number. Below, \( M \) will be used to denote the eigenvalues of the pair number operator, while \( m \) will denote the eigenvalues of the total fermion number operator \( n = \sum_{j=1}^{L} n_j \).

It is convenient to express the Hamiltonian in terms of realizations of \( L \) copies of the \( su(2) \) algebra in the pseudo-spin representation, through the identification

\[
    S_j^- = c_{j+} c_{j+}, \quad S_j^+ = c_{j+} c_{j+}, \quad S_j^z = \frac{1}{2} (n_j - 1).
\]
The pseudo-spin operators satisfy the following $su(2)$ commutation relations:

$$[S^i_j, S^k_l] = \pm \delta_{jk} S^i_l, \quad [S^i_j, S^k_l] = 2\delta_{jk} S^i_l,$$

Through this correspondence it is possible to identify a particular form of the Hamiltonian (1) as an exactly solvable model associated with the $su(2)$-invariant six-vertex solution of the Yang–Baxter equation [7]. This occurs for the choice

$$G_{jk} = \begin{cases} 
G_+ & j < k \\
G_+ + G_- & j = k \\
\frac{G_-}{2} & j > k 
\end{cases}$$

for arbitrary $G_+$ and $G_-$. For real-valued $G_+ = G_-$ this model corresponds to the case examined long ago by Richardson [21]. In [7] the specific case where $G_+$ and $G_-$ are a complex conjugate pair, leading to a self-adjoint Hamiltonian, was recognized as the Russian doll BCS model of [14] (see also [1]).

Below we will instead consider the case for which $G_+$ and $G_-$ are both real-valued. Although this does not lead to a Hermitian Hamiltonian, nonetheless the spectrum is real-valued in some region of the coupling parameter space. Throughout we will work with a picket fence model whereby the $\epsilon_j$ are uniformly and symmetrically distributed around zero. In particular we write

$$\epsilon_j = \left( j - \frac{L+1}{2} \right) \delta,$$

where the level spacing $\delta$ provides an energy scale for the system. Through numerical diagonalization of the Hamiltonian it is found that there are clearly identifiable regions where the spectrum is real. Illustrative cases are depicted in figure 1. It is evident that the boundary lines separating real-valued and complex-valued spectra are approximately

$$G_+ - G_- = \pm 2\delta$$

for sufficiently large $G_+$ and $G_-$. On the lines of the coupling parameter space given by (6) we will describe a class of free quasi-particle excitations exhibiting generalized exclusion statistics. Our analysis hereafter will be conducted using exact results. In order to present the exact Bethe ansatz solution for the Hamiltonian it is useful to make a change of variable. We parameterize the coupling constants $G_{\pm}$ through variables $\alpha, \eta$ such that

$$\alpha = \frac{1}{2} \ln \left( \frac{G_+}{G_-} \right), \quad \eta = \frac{1}{2} (G_+ - G_-).$$

Inverting these relations gives

$$G_+ = \frac{2\eta e^{\alpha}}{e^\alpha - e^{-\alpha}}, \quad G_- = \frac{2\eta e^{-\alpha}}{e^\alpha - e^{-\alpha}}.$$

To simplify the discussion we will restrict attention to the subspace of unblocked states. The Hamiltonian is block-diagonal on this subspace with sectors determined by the eigenvalues $M$ of the pair number operator $N$. For each $M$ the dimension of the subspace within the space of unblocked states is $L! / (M! (L - M)!)$. The total space of unblocked states has dimension $2^L$, with the exact solution for the energy spectrum given by

$$E = 2 \sum_{j=1}^{M} v_j$$

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Figure 1. The shaded region depicts the values of the coupling parameters $0 \leq G_+, G_- \leq 10$ for which (1), (4), (5) has real spectrum for the following choices: (a) $\delta = 2, L = 8, m = 6, and M = 3$; (b) $\delta = 2, L = 8, m = 4, and M = 2$; (c) $\delta = 2, L = 8, m = 6, and M = 2$, with blocked levels at $\epsilon_2 = -5$ and $\epsilon_6 = 3$; (d) $\delta = 2, L = 8, m = 6, and M = 2$, with blocked levels at $\epsilon_4 = -1$ and $\epsilon_7 = 5$.

where the $v_j$ are solutions of the Bethe ansatz equations \[7\]

\[
e^{\alpha} \prod_{l=1}^{L} (v_k - \epsilon_l + \eta/2) \prod_{j \neq k}^{M} (v_k - v_j - \eta) = \prod_{l=1}^{L} (v_k - \epsilon_l - \eta/2) \prod_{j \neq k}^{M} (v_k - v_j + \eta). \quad (9)
\]

Before proceeding to an analysis of excitations characterized by the exact solution, we first need to recall and develop some results concerning the algebraic structure underlying the exact solution of the model.

3. The $gl(2)$-Yangian, quantum inverse scattering method, and the algebraic Bethe ansatz

The $gl(2)$-Yangian, which is denoted $Y[gl(2)]$, is an infinite-dimensional algebra with generators $\{T_k^l[l] : j, k = 1, 2; l = 1, \ldots, \infty\}$. In order to specify the
algebraic relations, it is customary to use generating functions dependent on a complex variable [6]

\[ T^i_k (u) = \sum_{l=0}^{\infty} T^i_k [l] u^{-l}, \quad u \in \mathbb{C} \]  

(10)

and then impose the commutation relations

\[ (u - v) [T^i_k (u), T^m_l (v)] = \eta (T^i_k (v) T^m_l (u) - T^m_l (u) T^i_k (v)). \]  

(11)

The parameter \( \eta \) in equation (11) is to be identified as the same parameter appearing in equation (7). For later use we note that the algebra admits a homomorphism \( \phi : Y[gl(2)] \to Y[gl(2)] \)

\[ \phi (T^i_k (u)) = \rho (u) T^i_k (u) \]  

(12)

where \( \rho (u) \) is any function of \( u \).

Each irreducible, finite-dimensional, highest weight \( Y[gl(2)] \)-module is characterized by a highest weight vector \( |\Psi \rangle \) such that [6]

\[ T^1_1 (a) |\Psi \rangle = a (a) |\Psi \rangle, \]  

\[ T^2_2 (a) |\Psi \rangle = d (a) |\Psi \rangle, \]  

\[ T^1_1 (a) |\Psi \rangle = 0 \]  

where \( a (u) \) and \( d (u) \) are monic polynomials. Given such a module, we may obtain an equivalent module-action through (12) with \( \hat{a} (u) = \rho (u) a (u) \), \( \hat{d} (u) = \rho (u) d (u) \). The condition on equivalences of module-actions is expressed through the Drinfeld polynomial \( P (u) \) [6] for which

\[ \frac{P (u - \eta)}{P (u)} = \frac{a (u)}{d (u)}. \]  

(13)

It is easily verified that

\[ \frac{a (u)}{d (u)} = \frac{\hat{a} (u)}{\hat{d} (u)}, \]

signifying the equivalence of the actions.

Central elements are generated by the quantum determinant

\[ D (u) = T^1_1 (u) T^2_2 (u + \eta) - T^2_2 (u) T^1_1 (u + \eta) \]

\[ = T^2_2 (u + \eta) T^1_1 (u) - T^1_1 (u) T^2_2 (u + \eta) \]

which commutes with the generating functions (10). The quantum determinant \( D (u) \) takes the eigenvalue \( \mu (u) = a (u) d (u + \eta) \) on each irreducible, finite-dimensional, highest weight module. In contrast to the Drinfeld polynomial the quantum determinant can distinguish equivalent module-actions, but might not distinguish inequivalent ones. For example consider a one-dimensional module with \( a (u) = d (u) = 1 \). Using the homomorphism (12) we then have for \( \hat{a} (u) = \hat{d} (u) = \rho (u) \)

\[ P (u) = 1, \]

\[ \mu (u) = \rho (u) \rho (u + \eta). \]

Alternatively consider a highest weight vector \( |\Psi \rangle \) with the property that there exists \( v \in \mathbb{C} \) such that

\[ a (v) = d (v) = 0. \]

It follows from the algebraic relations (11) that

\[ |\Psi \rangle = T^2_2 (v) |\Psi \rangle \]  

(14)
satisfies
\[ T_1^1(u)|\tilde{\Psi}\rangle = \tilde{a}(u)|\tilde{\Psi}\rangle, \]
\[ T_2^1(u)|\tilde{\Psi}\rangle = \tilde{d}(u)|\tilde{\Psi}\rangle, \]
\[ T_1^1(u)|\tilde{\Psi}\rangle = 0 \]
with
\[ \tilde{a}(u) = \frac{u - v + \eta}{u - v} a(u), \quad \tilde{d}(u) = \frac{u - v - \eta}{u - v} d(u), \]
i.e. $|\tilde{\Psi}\rangle$ satisfies the conditions of a highest weight vector. Here, either $|\tilde{\Psi}\rangle$ vanishes which is necessarily the case if the corresponding module is irreducible, or the module with highest weight vector $|\Psi\rangle$ is reducible. It is straightforward to check that $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$ admit the same eigenvalue for the quantum determinant $D(u)$,
\[ \mu(u) = a(u)d(u + \eta) = \tilde{a}(u)\tilde{d}(u + \eta). \]
The fundamental two-dimensional $Y[gl(2)]$-module $V(u)$ with basis $[|1\rangle, |2\rangle]$ admits the action
\[ T_1^1(u)|1\rangle = (u - \eta/2)|1\rangle, \quad T_1^1(u)|2\rangle = (u + \eta/2)|2\rangle, \]
\[ T_2^1(u)|1\rangle = 0, \quad T_1^1(u)|2\rangle = \eta|1\rangle, \]
\[ T_2^2(u)|1\rangle = \eta|2\rangle, \quad T_2^2(u)|2\rangle = 0, \]
\[ T_3^1(u)|1\rangle = (u + \eta/2)|1\rangle, \quad T_3^2(u)|2\rangle = (u - \eta/2)|2\rangle. \]
Since $Y[gl(2)]$ is a bialgebra there is a co-product $\Delta : Y[gl(2)] \to Y[gl(2)] \otimes Y[gl(2)]$ given by
\[ \Delta(T_\epsilon^i(u)) = \sum_{\epsilon = i}^2 T_\epsilon^i(u) \otimes T_\epsilon^j(u - \epsilon) \]
for arbitrary $\epsilon \in \mathbb{C}$. Iterating the co-product action permits the construction of tensor product modules
\[ V(u; \epsilon_1, \ldots, \epsilon_L) = V(u - \epsilon_1) \otimes \cdots \otimes V(u - \epsilon_L) \]
with highest weight vector
\[ |\Psi\rangle = |1\rangle^\otimes L. \]
For generic choices of the $\epsilon_j$ this module is irreducible with Drinfeld polynomial [6]
\[ P(u) = \prod_{j=1}^L (u - \epsilon_j + \eta/2). \]

By using $Y[gl(2)]$-modules, abstract integrable models can be formulated as follows in the framework of the quantum inverse scattering method [13, 20]. For $a \in \mathbb{C}$ define a transfer matrix as
\[ t(u) = e^{-a} T_1^1(u) + e^a T_2^2(u). \]
From the commutation relations (11) it follows that
\[ [t(u), t(v)] = 0 \quad \forall u, v \in \mathbb{C}. \]
Through the above relation the transfer matrix is a generator of conserved (i.e. mutually commuting) operators for an abstract quantum system. A Hamiltonian can be defined as a...
polynomial function of the conserved operators. For the specific case of the Hamiltonian (1) subject to (4) we refer to [7].

The algebraic Bethe ansatz provides a means to diagonalize the transfer matrix on a highest weight module with highest weight vector \(|\Psi\rangle\). The eigenvectors of the transfer matrix are taken to be of the form

\[ |v_1, v_2, \ldots, v_M\rangle = \prod_{j=1}^{M} T_j^2(v_j)|\Psi\rangle \]

where the ordering in the product is inconsequential, since the \(T_j^2(v_j)\) commute. From the commutation relations (11) it is found that

\[ t(u)|v_1, v_2, \ldots, v_M\rangle = \lambda(u)|v_1, v_2, \ldots, v_M\rangle \]

\[ \sum_{k=1}^{M} \frac{e^{-\alpha} \eta a(v_k)}{u - v_k} \left( \prod_{j \neq k}^{M} \frac{v_k - v_j + \eta}{v_k - v_j} \right) T_j^2(u)|v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_M\rangle \]

\[ + \sum_{k=1}^{M} \frac{e^\alpha \eta d(v_k)}{u - v_k} \left( \prod_{j \neq k}^{M} \frac{v_k - v_j - \eta}{v_k - v_j} \right) T_j^2(u)|v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_M\rangle \]  \hspace{1cm} (20)

Thus, if for all \(k = 1, \ldots, M\),

\[ a(v_k) \prod_{j \neq k}^{M} \left( v_k - v_j + \eta \right) = e^{\alpha} a(v_k) \prod_{j \neq k}^{M} \left( v_k - v_j - \eta \right), \]  \hspace{1cm} (21)

the unwanted terms of the form \(|v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_M\rangle\) cancel. When this occurs either \(|v_1, v_2, \ldots, v_M\rangle\) is an eigenstate of the transfer matrix, or \(|v_1, v_2, \ldots, v_M\rangle\) vanishes. In the case that the state vanishes we refer to the corresponding solution set \(|v_1, v_2, \ldots, v_M\rangle\) as spurious.

In view of the homomorphism (12), we can minimize the occurrence of spurious solutions sets in some instances by an appropriate choice of \(\rho(u)\) to express (21) in the form

\[ P(v_k - \eta) \prod_{j \neq k}^{M} \left( v_k - v_j + \eta \right) = e^{\alpha} P(v_k) \prod_{j \neq k}^{M} \left( v_k - v_j - \eta \right). \]  \hspace{1cm} (22)

For the choice of highest weight state given by (18) we obtain

\[ a(u) = \prod_{l=1}^{L} (u - \epsilon_l - \eta/2), \]

\[ d(u) = \prod_{l=1}^{L} (u - \epsilon_l + \eta/2) \]

and find that (13) holds for (19). Substituting the above expressions into (21) leads to (9).

Our main objective in the remainder of this work is to determine, for the specific case of the Hamiltonian (1) subject to (4), (5) and restricted to the lines (6), those instances where \(|v_1, v_2, \ldots, v_M\rangle\) is an eigenstate and those instances where \(|v_1, v_2, \ldots, v_M\rangle\) vanishes. Through this investigation it will be seen how a class of free quasi-particle excitations exhibiting generalized exclusion statistics is uncovered.
4. Generalized exclusion statistics

Restricting to the lines (6), which is equivalent to setting \( \delta = \eta \), the Bethe ansatz equations (9) are expressible as

\[
\prod_{l=1}^{L-1} \left( v_k - \eta \left( l - \frac{L}{2} \right) \right) \times \left( e^{2\alpha} \left( v_k + \frac{\eta L}{2} \right) \prod_{j \neq k}^M (v_k - v_j) - (v_k - \frac{\eta L}{2}) \prod_{j \neq k}^M (v_k - v_j + \eta) \right) = 0
\]

for each \( k = 1, \ldots, M \). It is instructive to first consider the cases \( M = 1 \) and \( M = 2 \) before addressing the general case.

4.1. The one Cooper pair sector \( M = 1 \)

In this sector the dimension of the Hilbert space is \( L \). Setting \( M = 1 \) and \( v_1 \equiv v \) in (23) leads to

\[
\left( e^{2\alpha} \left( v + \frac{\eta L}{2} \right) - (v - \frac{\eta L}{2}) \right) \prod_{l=1}^{L-1} \left( v - \eta \left( l - \frac{L}{2} \right) \right) = 0.
\]

The above is a polynomial equation of order \( L \), the roots of which can be stated explicitly. There is one root which is \( \alpha \)-dependent, viz.

\[
v = -\frac{\eta L}{2} \left( e^{\alpha} + e^{-\alpha} \right),
\]

while the remaining \( \alpha \)-independent roots are elements from the set

\[
S = \{ \delta \left( k - L/2 \right) : k = 1, \ldots, L - 1 \}.
\]

To check whether any of these roots is spurious we consider the following action which is obtained through iterated use of the co-product (17):

\[
T_1^2(u) |\Psi\rangle = \eta \sum_{k=1}^L \prod_{l=1}^{L-1} \left( u - \epsilon_l - \eta/2 \right) \prod_{l=j+1}^L \left( u - \epsilon_l + \eta/2 \right) S_j^+ |\Psi\rangle
\]

For the \( \alpha \)-dependent root, substitution into the above expression yields an eigenstate of the Hamiltonian. However, at first sight it appears that the roots from \( S \) are spurious. The state \( T_1^2(v) |\Psi\rangle \) vanishes for \( v \in S \) due to the coefficient polynomial \( \prod_{k=1}^{L-1} (v - \delta(k - L/2)) \). Rescaling the state by this polynomial would then lead to the same eigenstate \( \sum_{j=1}^L S_j^+ |\Psi\rangle \) for all \( v \in S \). However, directly applying the Hamiltonian to this state confirms that it is not an eigenstate. The problem with this approach is that rescaling by \( \prod_{k=1}^{L-1} (v - \delta(k - L/2)) \) means the unwanted terms in (20) no longer cancel.

Nonetheless we can confirm that the roots from \( S \) are not spurious. Starting with a generic value for \( \eta \) we have

\[
T_1^2(u) |\Psi\rangle = \eta \sum_{j=1}^L \prod_{k=1}^{j-1} \left( u - \epsilon_k - \eta/2 \right) \prod_{l=j+1}^L \left( u - \epsilon_l + \eta/2 \right) S_j^+ |\Psi\rangle
\]
\[ \mu \Phi_1(\delta(\mu u) \phi_1(\eta L) = \sum_{j=1}^L \mu_{j-1} (u - \epsilon_k - \eta/2) \prod_{\gamma=\mu+1}^L (u - \epsilon_\gamma) \prod_{\beta=1}^\mu (u - \epsilon_\beta - \eta/2) S_j^+ | \Psi \rangle. \]

For \( \mu = 1, \ldots, L - 1 \) let

\[ \chi_\mu(u) = \sum_{j=1}^L \mu_{j-1} (u - \epsilon_k - \eta/2) \prod_{\gamma=\mu+1}^L (u - \epsilon_\gamma + \eta/2) \prod_{\beta=1}^\mu (u - \epsilon_\beta - \eta/2) S_j^+ | \Psi \rangle. \]

Then we define rescaled states

\[ | \Phi_\mu(u) \rangle = \frac{1}{\chi_\mu(u)} T_\mu^\mu(u) | \Psi \rangle \]

\[ = \sum_{j=1}^{\mu-1} \frac{\prod_{k=1}^{j-1} (u - \epsilon_k - \eta/2)}{\prod_{j=\mu+1}^L (u - \epsilon_\gamma + \eta/2)} \prod_{j=\mu+1}^L (u - \epsilon_\gamma + \eta/2) \prod_{\beta=1}^\mu (u - \epsilon_\beta - \eta/2) S_j^+ | \Psi \rangle + S_j^+ | \Psi \rangle \]

\[ + \sum_{j=\mu+1}^L \frac{\prod_{k=1}^{j-1} (u - \epsilon_k - \eta/2)}{\prod_{j=\mu+1}^L (u - \epsilon_\gamma + \eta/2)} \prod_{j=\mu+1}^L (u - \epsilon_\gamma + \eta/2) \prod_{\beta=1}^\mu (u - \epsilon_\beta - \eta/2) S_j^+ | \Psi \rangle. \]

The Bethe ansatz equations (9) for \( M = 1 \) may be written as

\[ \frac{\prod_{k=1}^{j-1} (u - \epsilon_k - \eta/2)}{\prod_{j=\mu+1}^L (u - \epsilon_\gamma + \eta/2)} = e^{2\alpha} \frac{\prod_{k=1}^{j-1} (u - \epsilon_k - \eta/2)}{\prod_{k=1}^L (u - \epsilon_k - \eta/2)}. \]

This then leads to

\[ | \Phi_\mu(u) \rangle = \sum_{j=1}^{\mu-1} \frac{\prod_{k=1}^{j-1} (u - \epsilon_k - \eta/2)}{\prod_{j=\mu+1}^L (u - \epsilon_\gamma + \eta/2)} S_j^+ | \Psi \rangle + S_j^+ | \Psi \rangle \]

\[ + e^{2\alpha} \sum_{j=\mu+1}^L \frac{u - \epsilon_k + \eta/2}{u - \epsilon_k - \eta/2} \prod_{\beta=1}^{j-1} \frac{u - \epsilon_\gamma + \eta/2}{u - \epsilon_\gamma - \eta/2} \prod_{l=\mu+1}^L \frac{u - \epsilon_k + \eta/2}{u - \epsilon_k - \eta/2} S_j^+ | \Psi \rangle. \]

Next let \( \eta = \delta \) giving

\[ | \Phi_\mu(u) \rangle = \sum_{j=1}^\mu S_j^+ | \Psi \rangle + e^{2\alpha} \sum_{j=\mu+1}^L \frac{u - \epsilon_k + \eta/2}{u - \epsilon_k - \eta/2} S_j^+ | \Psi \rangle. \]

\[ | \Phi_\mu(\delta(u - L/2)) \rangle = \sum_{j=1}^\mu S_j^+ | \Psi \rangle + e^{2\alpha} \sum_{j=\mu+1}^L \frac{u - \epsilon_k + \eta/2}{u - \epsilon_k - \eta/2} S_j^+ | \Psi \rangle \]

\[ = \sum_{j=1}^\mu S_j^+ | \Psi \rangle + e^{2\alpha} \frac{\mu - L}{\mu - L} \sum_{j=\mu+1}^L S_j^+ | \Psi \rangle. \]

(25)
It can be verified by direct calculation that the expressions for $|\Phi_\mu(\delta(\mu - L/2))\rangle$, $\mu = 1, \ldots, L - 1$ as given by (25) are eigenstates of the Hamiltonian (1) subject to (4)–(6). Along with the eigenstate associated with the $\alpha$-dependent root, this provides a complete set of $L$ eigenstates for the sector $M = 1$.

4.2. The two Cooper pair sector $M = 2$

In this sector the dimension of the Hilbert space is $L(L - 1)/2$. The first important observation to make from the previous subsection is that when $\eta = \delta$ the roots of the Bethe equations can be clearly demarcated into two groups, those which are $\alpha$-independent and those which are not. This provides three subcases to consider.

4.2.1. Two $\alpha$-independent roots. From the Bethe ansatz equations (23) it is seen that a solution set is formally obtained by choosing any two elements from the set $S$ given by equation (24), including the case when the roots are equal. In this respect the excitations have the character of two free quasi-particles. However some solution sets are spurious. To identify those instances, we first note that the one Cooper pair sector eigenstates (25) are $Y_1(\mathfrak{gl}(2))$ highest weight states of the form (14) with suitable rescaling. Specifically, choosing $v_1 = \delta(\mu - L/2)$ leads us to consider

$$T_1^1(u)|\Phi_\mu(\delta(\mu - L/2))\rangle = \tilde{a}(u)|\Phi_\mu(\delta(\mu - L/2))\rangle,$$

$$T_2^2(u)|\Phi_\mu(\delta(\mu - L/2))\rangle = \tilde{d}(u)|\Phi_\mu(\delta(\mu - L/2))\rangle$$

with

$$\tilde{a}(u) = (u - \delta(\mu - 1 - L/2)) \prod_{l \neq \mu}(u - \delta(l - L/2)),$$

$$\tilde{d}(u) = (u - \delta(\mu + 1 - L/2)) \prod_{l \neq \mu + 1}(u - \delta(l - 1 - L/2))$$

and the associated Drinfeld polynomial

$$P(u) = \prod_{l \neq \mu, \mu + 1}(u - \delta(l - 1 - L/2))$$

such that (13) holds. For this Drinfeld polynomial we find that the set of $\alpha$-independent roots for the Bethe ansatz equations (22), from which we can choose $v_2$, is the restricted set

$$S' = \{\delta(k - L/2) : k = 1, \ldots, L - 1; \ k \neq \mu - 1, \mu, \mu + 1\}.$$

Here we observe the manifestation of generalized exclusion statistics. Having first chosen the Bethe root $v_1 = \delta(\mu - L/2)$ we find that not only can we not choose it again (as is the case for the familiar fermionic exclusion principle), we also cannot choose the ‘neighbouring’ roots $\delta(\mu \pm 1 - L/2)$. Bearing this in mind, a simple counting argument shows that the number of eigenstates in the $M = 2$ sector where both roots are $\alpha$-independent is given by $(L - 2)(L - 3)/2$.

4.2.2. One $\alpha$-independent root. Choosing $v_1 \in S$ the Bethe ansatz equation for $v_2 \notin S$ is the quadratic equation

$$e^{2\alpha}(v_2 + \delta L/2)(v_2 - v_1 - \delta) = (v_2 - \delta L/2)(v_2 - v_1 + \delta).$$
The solution reads
\[ v_2 = \frac{2(1 - e^{2\alpha})v_1 + \delta(1 + e^{2\alpha})(L - 2) \pm \sqrt{D}}{4(1 - e^{2\alpha})} \]
where
\[ D = \delta^2(e^{2\alpha} + 1)^2(L - 2)^2 + 4(e^{2\alpha} - 1)^2(v_1^2 + 2\delta^2L) + 4\delta(e^{2\alpha} - 1)(L + 2)v_1. \]

While there are generally two solutions for \( v_2 \), two special cases need to be re-examined in closer detail. Choosing \( v_1 = \delta(1 - L/2) \) the equation for \( v_2 \) is
\[ e^{2\alpha}(v_2 + \delta L/2)(v_2 - v_1 - \delta) = (v_2 - \delta L/2)(v_2 - v_1 + \delta) \]
\[ e^{2\alpha}(v_2 + \delta L/2)(v_2 - 2\delta + \delta L/2) = (v_2 - \delta L/2)(v_2 + \delta L/2). \]

Formally there are two solutions for \( v_2 \), but the case \( v_2 = -\delta L/2 \) is \( \alpha \)-independent and spurious by the reasoning presented in the preceding subsubsection. The \( \alpha \)-dependent root is
\[ v_2 = \frac{(L/2 + (L/2 - 2)e^{2\alpha})\delta}{1 - e^{2\alpha}}. \]

Similarly, choosing \( v_1 = \delta(L/2 - 1) \) the equation for \( v_2 \) is
\[ e^{2\alpha}(v_2 + \delta L/2)(v_2 - v_1 - \delta) = (v_2 - \delta L/2)(v_2 - v_1 + \delta) \]
\[ e^{2\alpha}(v_2 + \delta L/2)(v_2 - \delta L/2) = (v_2 - \delta L/2)(v_2 - \delta L/2 + 2\delta). \]

In this instance the \( \alpha \)-independent solution \( v_2 = \delta L/2 \) is spurious. The \( \alpha \)-dependent root is
\[ v_2 = \frac{(L/2 - 2) + e^{2\alpha}L/2\delta}{1 - e^{2\alpha}}. \]

These results indicate that the number of eigenstates in the \( M = 2 \) sector, where only one root is \( \alpha \)-independent, is given by \( 2(L - 2) \).

4.2.3. No \( \alpha \)-independent roots. Since the dimension of the Hilbert space for \( M = 2 \) is \( L(L - 1)/2 \), and we have accounted for \( (L - 2)(L - 3)/2 \) solutions for two \( \alpha \)-independent roots and \( 2(L - 2) \) solutions for one \( \alpha \)-independent root, there can only be one non-spurious solution with both roots being \( \alpha \)-dependent. Although we have not been able to derive this result in the sense of producing an explicit general formula for the solution, we have checked the low-dimensional cases \( L = 4, 5, 6 \) with the results tabulated in appendix A. Specifically, we have verified that for these cases our analyses above are in complete agreement with results obtained by direct diagonalization of the Hamiltonian to obtain the eigenspectrum. In particular, direct diagonalization confirms the picture that the \( \alpha \)-independent roots are associated with free quasi-particle excitations with generalized exclusion statistics.

4.3. The case of general \( M \)

For a given \( L \) and \( M \) let \( P \), with \( P \leq M \), denote the number of \( \alpha \)-independent roots within a set of \( M \) roots. Further let \( n(L, M, P) \) denote the number of non-spurious solutions for each set of these quantities. From our previous discussions we have \( n(L, 0, 0) = 1 \), \( n(L, 1, 1) = L - 1 \), \( n(L, 1, 0) = 1 \), \( n(L, 2, 2) = (L - 2)(L - 3)/2 \), \( n(L, 2, 1) = 2(L - 2) \) and \( n(L, 2, 0) = 1 \).

Next we will prove a formula for \( n(L, M, M) \), i.e., the case when all roots are \( \alpha \)-independent. We saw in the previous subsection for \( M = 2 \) that if we choose a particular root
$v_1 = \delta(\mu - L/2)$, the generalized exclusion principle prohibits the choice $v_2 = \delta(\mu \pm 1 - L/2)$ to obtain an eigenstate. This result generalizes for arbitrary $M$ in a straightforward manner, viz. given a solution set for the $M - 1$ Cooper pair sector

$$\{v_j = \delta(\mu_j - L/2) : j = 1, \ldots, M - 1\}$$

we are prohibited from choosing any neighbouring roots $\delta(\mu_j \pm 1 - L/2)$ for $v_M$ to obtain an eigenstate in the $M$ Cooper pair sector. The proof is a matter of iterating the procedure described for $M = 2$ in subsubsection 4.2.1. To determine $n(L, M, M)$ we need to count the number of $\alpha$-independent solution sets which respect this exclusion rule.

Let

$$\begin{bmatrix} p \\ q \end{bmatrix}$$

denote the number of ways that $q$ identical objects can be placed in $p$ boxes such that the objects cannot be placed into adjacent boxes. This quantity satisfies the recursion relation

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p - 1 \\ q \end{bmatrix} + \begin{bmatrix} p - 2 \\ q - 1 \end{bmatrix}$$

with the initial conditions

$$\begin{bmatrix} p \\ 0 \end{bmatrix} = 1 \quad p \geq 0,$n$\begin{bmatrix} 0 \\ q \end{bmatrix} = 0 \quad q > 0,$n$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1.$

The recursion relation can be solved to obtain

$$\begin{bmatrix} p \\ q \end{bmatrix} = \frac{(p + 1 - q)!}{(p + 1 - 2q)!q!}.$$

The solution set $S$ as given by (24) contains $L - 1$ elements, yielding

$$n(L, M, M) = \begin{bmatrix} L - 1 \\ M \end{bmatrix} = \frac{(L - M)!}{(L - 2M)!M!}.$$

Although we have not been able to prove a general result for $n(L, M, P)$, we conjecture that it is given by

$$n(L, M, P) = \frac{(L - M)!M!}{(L - M - P)!(M - P)!(P)2}.$$ (26)

This formula agrees with the subcases $P = M$, and $M = 0, 1, 2$ with $P \leq M$, discussed above. It also satisfies the required condition

$$\sum_{P=0}^{M} n(L, M, P) = \frac{L!}{(L - M)!M!}$$

to account for all states in each sector of $M$ Cooper pairs. The above identity may be proved by considering binomial expansions of the terms in

$$(1 + x)^L = (1 + x)^{L-M} (1 + x)^M.$$
5. Conclusion

The Hamiltonian (1), subject to (4), (5), admits an exact solution. On the subspace of unblocked states the energy eigenvalues are given by (8) where the set \( \{v_1, \ldots, v_M\} \) is a solution of the Bethe ansatz equations (9). The energy expression is a simple sum of the Bethe roots, which leads to a natural interpretation of each Bethe root being associated with a quasi-particle. In this quasi-particle picture the states are generally bound rather than free, since a solution set for \( M \) quasi-particles is not simply the union of one-body solutions due to the coupled nature of (9). In general the roots \( v_j \) will depend on the two coupling parameters \( G_+ \), \( G_- \), or equivalently, the variables \( \alpha \) and \( \eta \) as given by (7).

Setting \( \eta = \delta \) corresponding to (6), where \( \delta \) is the level spacing in (5), we find that the roots of the Bethe ansatz equations divide into two classes, viz, those which are \( \alpha \)-dependent and those which are \( \alpha \)-independent. From this perspective we may identify the lines (6) with deconfined excitations, those being associated with the \( \alpha \)-independent roots which do not occur for general \( \eta \). From the form of the Bethe ansatz equations given by (23), the \( \alpha \)-independent roots were determined explicitly. They are elements of the set \( S \) defined in (24). The corresponding states were found to be highest weight states of the underlying \( Y[gl(2)] \) algebraic structure. These roots are associated with free quasi-particles excitations, since a solution set for \( M \) quasi-particles is a union of one-body solutions. However, a close examination of the states associated with these solutions led to the conclusion that there are spurious solution sets, which give rise to an interpretation of generalized exclusion statistics.

We have conjectured the formula (26) for the number of non-spurious solution sets with \( M \) roots, where \( P \) of the roots are \( \alpha \)-independent. A proof of the result for \( P = M \) was provided, and it was also found that (26) is valid in certain limiting cases. In some respects these properties are similar to ones found in the Haldane–Shastry model [10], as was previously discussed in [15].

It was proposed in [18, 19] that quantum criticality may be identified by deconfined excitations at the critical point, which are not found in phases adjacent to the critical point. It is interesting to note for the present study that the lines (6) on which the deconfined \( \alpha \)-independent excitations occur are in very close agreement with the boundary lines between the unbroken \( \mathcal{PT} \)-symmetric phase and a broken \( \mathcal{PT} \)-symmetric phase. The \( \mathcal{PT} \)-symmetry breaking phase boundary lines are clearly identified in figure 1, which was obtained by direct numerical diagonalization of the Hamiltonian.

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Appendix A. Tables of roots of the Bethe ansatz equations

In this appendix we list the roots of the Bethe ansatz equations for some small sized systems \( L = 4, 5, 6, M = 2 \) and no blocked states, for when the coupling parameters correspond to the lines (6). The first column lists the energy eigenvalues obtained by direct diagonalization, while the second column lists the roots of the Bethe ansatz equations (23) associated with each energy eigenvalue through (8). These results are consistent with conjecture (26) concerning the distribution of roots, and establish that the energy spectrum of the Hamiltonian on the lines (6) is real-valued for these cases.
Table A1. The energy spectrum and associated solutions of the Bethe ansatz equations (23) on the lines (6) for $L = 4$, $M = 2$. While the pairs $(\delta, 0)$, $(0, -\delta)$ are solutions of (23) they do not correspond to an eigenvalue of the Hamiltonian, giving rise to a generalized exclusion principle.

| $E$ | $(v_1, v_2)$ |
|-----|---------------|
| 0   | $(\delta, -\delta)$ |
| $\frac{2(1 + e^{2\theta}) \delta}{1 - e^{2\theta}}$ | $\left(\delta, \frac{2 e^{2\theta} \delta}{1 - e^{2\theta}}\right)$ |
| $\frac{(1 + e^{2\theta} - \sqrt{9 - 14 e^{2\theta} + 9 e^{4\theta}}) \delta}{1 - e^{2\theta}}$ | $\left(0, \frac{1 + e^{2\theta} - \sqrt{9 - 14 e^{2\theta} + 9 e^{4\theta}} \delta}{2(1 - e^{2\theta})}\right)$ |
| $\frac{(1 + e^{2\theta} + \sqrt{9 - 14 e^{2\theta} + 9 e^{4\theta}}) \delta}{1 - e^{2\theta}}$ | $\left(0, \frac{1 + e^{2\theta} + \sqrt{9 - 14 e^{2\theta} + 9 e^{4\theta}} \delta}{2(1 - e^{2\theta})}\right)$ |

Table A2. The energy spectrum and associated solutions of the Bethe ansatz equations (23) on the lines (6) for $L = 5$, $M = 2$. While the pairs $(3\delta/2, \delta/2)$, $(\delta/2, -\delta/2)$, $(-\delta/2, -3\delta/2)$ are solutions of (23) they do not correspond to an eigenvalue of the Hamiltonian, giving rise to a generalized exclusion principle.

| $E$ | $(v_1, v_2)$ |
|-----|---------------|
| $2\delta$ | $\left(\frac{3\delta}{2}, \frac{\delta}{2}\right)$ |
| 0 | $\left(-3\delta, \frac{3\delta}{2}\right)$ |
| $-2\delta$ | $\left(-\frac{3\delta}{2}, -\frac{\delta}{2}\right)$ |
| $\frac{2(2 + e^{2\theta}) \delta}{1 - e^{2\theta}}$ | $\left(\frac{3\delta}{2}, \frac{1 + 5 e^{2\theta} \delta}{2(1 - e^{2\theta})}\right)$ |
| $\frac{2(1 + 2 e^{2\theta}) \delta}{1 - e^{2\theta}}$ | $\left(-\frac{3\delta}{2}, \frac{5 + 3 e^{2\theta} \delta}{2(1 - e^{2\theta})}\right)$ |
| $\frac{(3 - \sqrt{9 - 16 e^{2\theta} + 16 e^{4\theta}}) \delta}{1 - e^{2\theta}}$ | $\left(\frac{3}{2}, \frac{2 + e^{2\theta} - \sqrt{9 - 16 e^{2\theta} + 16 e^{4\theta}} \delta}{2(1 - e^{2\theta})}\right)$ |
| $\frac{(3 + \sqrt{9 - 16 e^{2\theta} + 16 e^{4\theta}}) \delta}{1 - e^{2\theta}}$ | $\left(\frac{3}{2}, \frac{2 + e^{2\theta} + \sqrt{9 - 16 e^{2\theta} + 16 e^{4\theta}} \delta}{2(1 - e^{2\theta})}\right)$ |
| $\frac{(3 e^{2\theta} - \sqrt{16 - 16 e^{2\theta} + 9 e^{4\theta}}) \delta}{1 - e^{2\theta}}$ | $\left(-\frac{3}{2}, \frac{1 + 2 e^{2\theta} - \sqrt{16 - 16 e^{2\theta} + 9 e^{4\theta}} \delta}{2(1 - e^{2\theta})}\right)$ |
| $\frac{(3 e^{2\theta} + \sqrt{16 - 16 e^{2\theta} + 9 e^{4\theta}}) \delta}{1 - e^{2\theta}}$ | $\left(-\frac{3}{2}, \frac{1 + 2 e^{2\theta} + \sqrt{16 - 16 e^{2\theta} + 9 e^{4\theta}} \delta}{2(1 - e^{2\theta})}\right)$ |
| $\frac{8(1 + e^{2\theta}) \delta}{1 - e^{2\theta}}$ | $\left(4 \frac{4 + 4 e^{2\theta} + \sqrt{1 - 18 e^{2\theta} + e^{4\theta}} \delta}{2(1 - e^{2\theta})}\right)$ |
The energy spectrum and associated solutions of the Bethe ansatz equations (23) on the lines (6) for \( L = 6, M = 2 \). While the pairs \((2\delta, \delta)\), \((\delta, 0)\), \((0, -2\delta)\), \((-2\delta, -2\delta)\) are solutions of (23) they do not correspond to an eigenvalue of the Hamiltonian, giving rise to a generalized exclusion principle.

### Appendix B. Mean-field analysis

In the appendix we show that a mean-field analysis in the continuum limit gives the result that the spectrum of the Hamiltonian (1) subject to (4), (5) is real for all values of the coupling parameters. For this reason it is deemed necessary to analyse the model through the exact Bethe ansatz solution.

For a general non-Hermitian Hamiltonian with a real spectrum, where \(|\Psi^r\rangle\) denotes the right ground-state eigenvector and \(|\Psi^l\rangle\) denotes the left ground-state eigenvector, the ground-state energy can be expressed as \( E = \langle \Psi^r | H | \Psi^l \rangle / \langle \Psi^r | \Psi^l \rangle \) provided \( \langle \Psi^r | \Psi^l \rangle \neq 0 \). When this is the case, for any operator \( A \) we define the ground-state expectation value as \( \langle A \rangle = \langle \Psi^l | A | \Psi^r \rangle / \langle \Psi^l | \Psi^r \rangle \). This definition preserves the Hellmann–Feynman theorem.
Given a BCS Hamiltonian of the form (1), we introduce a (real-valued) chemical potential $\mu$ and set
\[
\Delta_j^1 = \sum_{k=1}^{L} G_{jk} (c_k - c_{k+}), \quad \Delta_j^r = \sum_{j=1}^{L} G_{jk} (c^\dagger_j c^r_{j+})
\]
to obtain the mean-field approximation
\[
H_{\text{MF}} = \sum_{j=1}^{L} \epsilon_j n_j - \sum_{j=1}^{L} \Delta_j^1 c_{j+} c_{j+}^\dagger - \sum_{k=1}^{L} \Delta_k^r c_k c_{k+} + \sum_{j,k=1}^{L} G_{jk} (b_j^\dagger b_k) - \mu (n - n).
\]
Setting $\xi_k = \epsilon_k - \mu$, $\mathcal{E}_k = \sqrt{\xi_k^2 + \Delta_k^1 \Delta_k^r}$, by diagonalizing $H_{\text{MF}}$ it is seen that the elementary excitation energies are simply the $\mathcal{E}_k$. The entire energy spectrum is real-valued if the ground-state energy is real-valued and the products $\Delta_k^1 \Delta_k^r$ are non-negative for all $k$.

The right mean-field ground state is given by
\[
|\Psi^1\rangle = \prod_{k=1}^{L} \left( v_k^l I + v_k^r c_k c_{k+} \right) |0\rangle
\]
where $v_k^l / v_k^r = (\mathcal{E}_k - \xi_k) / \Delta_k^r = \Delta_k^1 / (\mathcal{E}_k + \xi_k)$. Analogously for the left ground state,
\[
|\Psi^r\rangle = \prod_{k=1}^{L} \left( u_k^l I + u_k^r c_k c_{k+} \right),
\]
we have $v_k^l / v_k^r = (\mathcal{E}_k - \xi_k) / \Delta_k^l = \Delta_k^r / (\mathcal{E}_k + \xi_k)$. Then $\langle \Psi_1 | \Psi_2 \rangle$ is found to be non-zero provided $\Delta_k^1 \Delta_k^r \neq 0$ for all $k$.

Self-consistency requirements impose that
\[
\Delta_j^1 = \frac{1}{2} \sum_{k=1}^{L} G_{jk} \frac{\Delta_j^1}{\mathcal{E}_k}, \quad \Delta_j^r = \frac{1}{2} \sum_{j=1}^{L} G_{jk} \frac{\Delta_j^r}{\mathcal{E}_j}, \quad (\text{B.1})
\]
\[
|\langle \Psi \rangle = L + \sum_{j=1}^{L} \frac{\mu - \epsilon_j}{\mathcal{E}_j} \quad \text{B.2}.
\]
We refer to (B.1), (B.2) as the gap equations, and to (B.3) as the chemical potential equation. Using these equations allows for the mean-field ground-state energy to be expressed as
\[
E_{\text{MF}} = \sum_{k=1}^{L} \epsilon_k \left( 1 - \frac{\xi_k}{\mathcal{E}_k} \right) - \frac{1}{2} \sum_{k=1}^{L} \frac{\Delta_k^1 \Delta_k^r}{\mathcal{E}_k}.
\]
Note that for a Hermitian Hamiltonian of the form (1), $\Delta_k^1$, $\Delta_k^r$ are a complex conjugate pair, in which case (B.1), (B.2) are equivalent. The question remains whether for non-Hermitian Hamiltonians there exist solutions of (B.1)–(B.3) such that $\Delta_k^1 \Delta_k^r$ is real-valued for all $k$. For the choice (4) we next show that this is the case in the continuum limit. We introduce a cut-off energy $\omega$ by setting $\delta = 2 \omega / (L - 1) = \left( \epsilon_1 - \omega \right)$ and $\epsilon_\delta = \omega$. Letting $G_\pm = 4 \omega g_\pm / L$, $x = \langle n \rangle / (2L)$, in the continuum limit $L \to \infty$, $G_\pm \to 0$, $\delta \to d \epsilon$ we have (B.1)–(B.3) assuming the integral equation forms
\[
\Delta_j^1 (\epsilon) = g - \int_{-\omega}^{\omega} d\epsilon' \frac{\Delta_j^1 (\epsilon')}{\mathcal{E}(\epsilon')} + g + \int_{\epsilon}^{\omega} d\epsilon' \frac{\Delta_j^1 (\epsilon')}{\mathcal{E}(\epsilon')}
\]
\text{(B.4)}
\[ \Delta^2(\epsilon) = g_+ \int_{-\omega}^{\epsilon} \frac{\Delta^l(\epsilon')}{E(\epsilon')} \, d\epsilon' + g_- \int_{\epsilon}^{\omega} \frac{\Delta^l(\epsilon')}{E(\epsilon')} \, d\epsilon' \]  

(B.5)

\[ x = \frac{1}{2} + \frac{1}{4\omega} \int_{-\omega}^{\omega} d\epsilon' \frac{\mu - \epsilon'}{E(\epsilon')} \]  

(B.6)

where \( E(\epsilon) = \sqrt{(\epsilon - \mu)^2 + \Delta^l(\epsilon)\Delta^l(\epsilon')} \). Differentiating (B.4) and (B.5) leads to the conclusion that \( \Delta = \sqrt{\Delta^l(\epsilon)\Delta^l(\epsilon')} \) is constant. With this observation the integrals (B.4)–(B.6) can be evaluated to obtain

\[ \mu = \frac{\omega(g_+^2 + g_-^2)(2x - 1)}{g_+^2 - g_-^2}, \quad \Delta^2 = \frac{16\omega^2 g_+^4 g_-^4 x(1 - x)}{(g_+^2 - g_-^2)^2}, \]

where \( x = 1/(g_+ + g_-) \). Note that \( \Delta^2 > 0 \) for all \( x, g_{\pm} \). Now the elementary excitation spectrum is explicit, viz. \( E(\epsilon) = \sqrt{(\epsilon - \mu)^2 + \Delta^2}, \quad \omega \leq \epsilon \leq \omega \), such that \( \Delta \) is the gap. We obtain the ground-state energy per fermion as

\[ e_{\text{MF}} = \lim_{L \to \infty} \frac{E_{\text{MF}}}{2\pi L} = -\frac{1}{8\pi\omega} \int_{-\omega}^{\omega} d\epsilon \frac{2\epsilon(\epsilon - \mu) + \Delta^2}{\sqrt{(\epsilon - \mu)^2 + \Delta^2}} = -\frac{1}{8\pi\omega} (\omega + \mu)\sqrt{(\omega - \mu)^2 + \Delta^2} + (\omega - \mu)\sqrt{(\omega + \mu)^2 + \Delta^2}. \]

Thus within this mean-field analysis the non-Hermitian Hamiltonian has a real spectrum for all couplings \( g_{\pm} \) and filling fractions \( x \). In the Hermitian limit \( g_{\pm} \to g \) for which \( \chi \to \infty \), both \( \mu \) and \( \Delta \) can be evaluated through use of \( \exp(x) = \lim_{n \to \infty} (1 + x/n)^n \). In particular for half-filling \( x = 1/2 \) we obtain \( \mu = 0 \) and \( \Delta = \omega/\sinh(1/2g) \) which is in agreement with the classic s-wave result obtained in [3, equation (2.40)].

It might be expected that mean-field results are exact in the thermodynamic limit (e.g. see [17] for when \( g_+ = g_- \)). It would be very useful to determine whether or not the energy spectrum is real to leading order in \( L \), with complex terms only appearing in lower order corrections.

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