On the topology of desingularizations of Calabi-Yau orbifolds

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1 Introduction

A Calabi-Yau 3-fold is a compact complex 3-manifold \((X,J)\) equipped with a Kähler metric \(g\) and a holomorphic volume form \(\Omega\), which is a nonzero \((3,0)\)-form on \(X\) with \(\nabla\Omega = 0\), where \(\nabla\) is the Levi-Civita connection of \(g\). Calabi-Yau 3-folds are Ricci-flat and have \(\text{Hol}(g) \subseteq SU(3)\), and one can construct many examples of them using Yau’s solution of the Calabi conjecture.

Suppose \(X\) is a Calabi-Yau 3-fold and \(G\) a finite group that acts on \(X\) preserving \(J,g\) and \(\Omega\). Then \(X/G\) is a Calabi-Yau orbifold, where an orbifold is a singular manifold with at most quotient singularities. Often it is possible to find a compact manifold \(Y\) that desingularizes \(X/G\), and carries a family of Calabi-Yau structures that converge to the singular Calabi-Yau structure on \(X/G\) in a well-defined sense, so that the orbifold metric on \(X/G\) may be regarded as the degenerate case in a smooth family of Calabi-Yau metrics on \(Y\).

In this paper we will study Calabi-Yau 3-folds \(Y\) that arise by deforming the complex structure of orbifolds \(X/G\) which have singularities in complex codimension two. We shall focus on the topology of \(Y\), and will show that a single orbifold \(X/G\) can admit several different Calabi-Yau desingularizations \(Y_1, Y_2, \ldots, Y_k\), with different Hodge numbers and Euler characteristics. We give an explanation of this phenomenon in terms of the ‘Weyl group’ of the codimension two singularities, and make some tentative suggestions on how this idea may help to explain our examples in the physical language of String Theory. We also briefly discuss resolutions of 7- and 8-orbifolds with the exceptional holonomy groups \(G_2\) and \(\text{Spin}(7)\).

The two main strategies for desingularizing \(X/G\) to get \(Y\) are called resolution and deformation. A resolution \((Y,\pi)\) of \(X/G\) is a nonsingular complex 3-fold \(Y\) with a proper holomorphic map \(\pi: Y \to X/G\), that induces a biholomorphism between dense open subsets of \(Y\) and \(X/G\). We call \(Y\) a crepant resolution if \(K_Y \cong \pi^*(K_{X/G})\). Since Calabi-Yau manifolds have trivial canonical bundles, to get a Calabi-Yau structure on \(Y\) we must choose a crepant resolution.
A family of deformations of $X/G$ consists of a (singular) complex 4-manifold $Y$ with a proper holomorphic map $f : Y \to \Delta$, where $\Delta$ is the unit disc in $\mathbb{C}$, such that $Y_0 = f^{-1}(0)$ is isomorphic to $X/G$. The other fibres $Y_t = f^{-1}(t)$ for $t \neq 0$ are called deformations of $X/G$. If the $Y_t$ are smooth manifolds for $t \neq 0$, they are called smoothings of $X/G$.

Thus, resolutions and smoothings are two different ways to desingularize a complex orbifold. We can also combine the two by smoothing a partial resolution of $Z$, or by resolving a deformation of $Z$. We shall use the word desingularization to mean any of these processes.

A number of important ideas about the topology of desingularizations $Y$ of a Calabi-Yau orbifold $X/G$ were first proposed by physicists working in String Theory, motivated by physical considerations. In 1985, Dixon et al. [5, p. 684] conjectured that the Euler characteristic of $Y$ should be given by

$$\chi(Y) = \chi(X, G) = \frac{1}{|G|} \sum_{g,h \in G, rh=hg} \chi(X^g \cap X^h),$$

where $X^g$ is the submanifold of $X$ fixed by $g \in G$, and $\chi$ is the Euler characteristic. Vafa [21] and Zaslow [24, p. 312] (see also Batyrev and Dais [2]) found a related formula for the Hodge numbers $h^{p,q}(Y)$ of $Y$, which we will not give.

As far as the author understands, the physicists who made these conjectures believed that their formulae should apply to at least one desingularization $Y$ of $X/G$, but not necessarily to every desingularization; and an example of a desingularization $Y$ for which (1) does not hold appears in a paper by some of the same authors [23, §2].

Much more is known about the case of crepant resolutions $Y$ of $X/G$, particularly in complex dimension three. Roan [16] proves that every 3-dimensional Calabi-Yau orbifold admits a crepant resolution, for which (1) holds. An orbifold $X/G$ can admit several topologically distinct crepant resolutions $Y_1, \ldots, Y_k$. But these resolutions are all related by ‘flops’, and have the same Euler characteristic and Hodge numbers.

Reid [15] and Ito and Reid [7] develop a theory of crepant resolutions of orbifolds which they call the ‘McKay correspondence’, and in dimension 3 they show that every crepant resolution $Y$ of $X/G$ must satisfy (1), and also the Hodge number formulae of Vafa and Zaslow. In dimensions four and above, it is known that some orbifolds admit no crepant resolution (since they have ‘terminal singularities’). However, there is strong evidence that crepant resolutions in all dimensions must satisfy (1) and the Hodge number formulae.

These results still leave open the question of the topology of Calabi-Yau desingularizations $Y$ of $X/G$ that are not crepant resolutions, but involve some deformation of the complex structure. What can we say about this situation? By Schlessinger’s Rigidity Theorem [17], quotient singularities of codimension 3 or more have no nontrivial deformations. But Calabi-Yau orbifolds cannot have singularities in codimension one. So to resolve $X/G$ by deformation, the singularities must be of codimension two.
The simplest sort of codimension two singularities in Calabi-Yau 3-folds are modelled on \( \mathbb{C} \times (\mathbb{C}^2/H) \), where \( H \) is a finite subgroup of \( SU(2) \). The natural way to desingularize this is to use \( \mathbb{C} \times X \), where \( X \) is a Calabi-Yau desingularization of \( \mathbb{C}^2/H \). In fact, Kronheimer \([12, 13]\) shows that we can give \( X \) a metric with holonomy \( SU(2) \) that is asymptotic to the flat metric on \( \mathbb{C}^2/H \) at infinity, making \( X \) into an ALE space. However, ALE spaces are all diffeomorphic to the (unique) crepant resolution of \( \mathbb{C}^2/H \), so in this case too, any desingularization has the topology of a crepant resolution.

Therefore, to find desingularizations \( Y \) which do not have the topology of crepant resolutions, we must consider singularities modelled on \( \mathbb{C}^3/G \), where \( G \) is a finite subgroup of \( SU(3) \) with a nontrivial subgroup \( H \) contained in some \( SU(2) \subset SU(3) \), to produce the codimension two singularities, but where \( G \) is not itself contained in \( SU(2) \). For instance, consider a finite subgroup \( G \) of elements of \( SU(3) \) of the form

\[
\begin{pmatrix}
e^{i\theta} & 0 & 0 \\0 & a & b \\0 & c & d\end{pmatrix}, \quad \text{where} \quad \begin{pmatrix}a & b \\c & d\end{pmatrix} \in U(2) \quad \text{and} \quad ad - bc = e^{-i\theta}. \tag{2}
\]

Let \( H \) be the subgroup of elements \( h \in G \) for which \( e^{i\theta} = 1 \). Then \( H \) is a finite subgroup of \( SU(2) \). It is also easy to show that \( H \) is a normal subgroup of \( G \) and that \( G/H \cong \mathbb{Z}_k \) for some positive integer \( k \), where \( e^{i\theta} \) is a \( k \)-th root of unity for each \( g \in G \).

To desingularize \( \mathbb{C}^3/G \), we may proceed in two stages. The first stage is to choose a desingularization \( X \) of \( \mathbb{C}^2/H \), so that \( \mathbb{C} \times X \) is a resolution of \( \mathbb{C}^3/H \). Then we hope to find an action of \( \mathbb{Z}_k = G/H \) upon \( \mathbb{C} \times X \), which is asymptotic to the prescribed action of \( \mathbb{Z}_k \) on \( \mathbb{C}^3/H \). If we can find such an action, then the second stage is to desingularize \( (\mathbb{C} \times X)/\mathbb{Z}_k \), either by a crepant resolution or a smoothing, to get a desingularization \( Y \) of \( \mathbb{C}^3/G \).

Our key observation is the following: although the diffeomorphism type of \( X \) is uniquely determined by \( H \), the action of \( \mathbb{Z}_k \) on \( X \) and on its cohomology is not always uniquely determined by \( G \). Instead, there can be a finite number of topologically distinct ways for \( \mathbb{Z}_k \) to act on \( X \), depending on the choice of complex structure of \( X \), and on the level of cohomology these actions differ by an element of the Weyl group of the singularity \( \mathbb{C}^2/H \). For one of these \( \mathbb{Z}_k \)-actions the desingularization \( Y \) of \( (\mathbb{C} \times X)/\mathbb{Z}_k \) has the topology of a crepant resolution, but for other choices of the \( \mathbb{Z}_k \)-action \( Y \) does not have this topology, and its Euler characteristic is not given by \( (1) \).

The rest of the paper is set out as follows. Section 2 summarizes the theory of singularities \( \mathbb{C}^2/H \) for \( H \) a finite subgroup of \( SU(2) \), their desingularizations, and the idea of the Weyl group. Section 3 explains some theory about the orbifolds we are interested in, and the topology of their desingularizations, and §4 gives an example of this. Section 5 discusses a second way in which an orbifold can have several Calabi-Yau desingularizations with different topology, and §6 considers another example, the orbifold \( T^6/\mathbb{Z}_2^2 \), which combines both phenomena, and has other interesting features.
In §7 we extend the discussion to the exceptional holonomy groups $G_2$ and $Spin(7)$, and give an example of an isolated singularity $\mathbb{R}^8/G$ which fits into our framework. Finally, section 8 speculates about the interpretation of these examples in String Theory. The author would like to thank Cumrun Vafa and David Morrison for helpful suggestions, and for correcting some of his misunderstandings about String Theory.

2 Kleinian singularities and ALE spaces

The quotient singularities $\mathbb{C}^2/H$, for $H$ a finite subgroup of $SU(2)$, were first classified by Klein in 1884 and are called Kleinian singularities; they are also called Du Val surface singularities, or rational double points. The theory of these singularities and their resolutions is very rich, and has many connections to other areas of mathematics. Most of the following facts are taken from McKay [14], Slodowy [19], and Kronheimer [12, 13]. A good reference on Lie groups, Dynkin diagrams and Weyl groups is Bourbaki [3], in particular the tables on pages 250-270.

There is a 1-1 correspondence between finite subgroups $H \subset SU(2)$ and the Dynkin diagrams of type $A_r$ ($r \geq 0$), $D_r$ ($r \geq 4$), $E_6$, $E_7$ and $E_8$. Let $\Gamma$ be the Dynkin diagram associated to $H$. These Dynkin diagrams appear in the classification of Lie groups, and each one corresponds to a unique compact, simple Lie group; they are the set of such diagrams containing no double or triple edges.

Each singularity $\mathbb{C}^2/H$ admits a unique crepant resolution $(X, \pi)$. The preimage $\pi^{-1}(0)$ of the singular point is a union of a finite number of rational curves in $X$. These curves correspond naturally to the vertices of $\Gamma$. They all have self-intersection $-2$, and two curves intersect transversely at one point if and only if the corresponding vertices are joined by an edge in the diagram; otherwise the curves do not intersect.

These curves give a basis for the homology group $H_2(X, \mathbb{Z})$, which may be identified with the root lattice of the diagram, and the intersection form with respect to this basis is the negative of the Cartan matrix of $\Gamma$. Define $\Delta$ to be $\{ \delta \in H_2(X, \mathbb{Z}) : \delta \cdot \delta = -2 \}$. Then $\Delta$ is identified with the set of roots of the diagram. There are also 1-1 correspondences between the curves and the nonidentity conjugacy classes in $H$, and also the nontrivial representations of $H$; it makes sense to regard the nonidentity conjugacy classes as a basis for $H_2(X, \mathbb{Z})$, and the nontrivial representations as a basis for $H^2(X, \mathbb{Z})$.

By the theory of Lie groups, the Dynkin diagram $\Gamma$ of $\mathbb{C}^2/H$ has a Weyl group $W$, and a representation of $W$ on the root lattice $H_2(X, \mathbb{Z})$ of $\Gamma$. This action of $W$ preserves the subset $\Delta$ and the intersection form on $H_2(X, \mathbb{Z})$, and by duality $W$ also acts on $H^2(X, \mathbb{Z})$. Let $\text{Aut}(\Gamma)$ be the group of automorphisms
the graph $\Gamma$, which is given by
\[
\text{Aut}(\Gamma) = \begin{cases} 
\{1\} & \text{if } \Gamma = A_1, E_7 \text{ or } E_8, \\
\mathbb{Z}_2 & \text{if } \Gamma = A_k \ (k \geq 2), D_k \ (k \geq 5) \text{ or } E_6, \\
S_3 & \text{if } \Gamma = D_4.
\end{cases}
\]

Now the vertices of $\Gamma$ correspond to the basis elements of $H_2(X, \mathbb{Z})$, so that $\text{Aut}(\Gamma)$ acts naturally on $H_2(X, \mathbb{Z})$, preserving the intersection form. But the Weyl group $W$ also acts on $H_2(X, \mathbb{Z})$. It turns out that there is a natural semidirect product $\text{Aut}(\Gamma) \ltimes W$, and the actions of $\text{Aut}(\Gamma)$ and $W$ on $H_2(X, \mathbb{Z})$ combine to give a representation of $\text{Aut}(\Gamma) \ltimes W$ on $H_2(X, \mathbb{Z})$. Define $\rho$ to be the dual representation of $\text{Aut}(\Gamma) \ltimes W$ on both $H^2(X, \mathbb{R})$ and $H^2(X, \mathbb{C})$. The action of each element of $\text{Aut}(\Gamma) \ltimes W$ is induced by a diffeomorphism of $X$, so we can interpret $\text{Aut}(\Gamma) \ltimes W$ as a group of isotopy classes of diffeomorphisms of $X$. However, in general $\text{Aut}(\Gamma) \ltimes W$ is not a group of diffeomorphisms of $X$, nor an isometry group of any of the metrics or complex structures on $X$.

The singularities $\mathbb{C}^2/H$ can be desingularized by deformation as well as by crepant resolution. Klein found that each singularity $\mathbb{C}^2/H$ is isomorphic as an affine complex variety to the zeros of a polynomial on $\mathbb{C}^4$. For example, $\mathbb{C}^2/\mathbb{Z}_k$ may be identified with the set of points $(x, y, z) \in \mathbb{C}^3$ for which $xy - z^k = 0$. The deformations of $\mathbb{C}^2/H$ are constructed by adding terms of lower order in $x, y$ and $z$ to this polynomial. All of the smooth deformations of $\mathbb{C}^2/H$ are diffeomorphic to the unique crepant resolution $X$ of $\mathbb{C}^2/H$.

Each of these desingularizations of $\mathbb{C}^2/H$ carries a special family of metrics with holonomy $SU(2)$. The metrics are asymptotic up to $O(r^{-4})$ to the Euclidean metric on $\mathbb{C}^2/H$, and so are called Asymptotically Locally Euclidean; the complex manifold $X$ with its Kähler metric is called an ALE space. A complete construction and classification of ALE spaces was carried out by Kronheimer [12, 13], and we describe it next.

Let $Y$ be an ALE space asymptotic to $\mathbb{C}^2/H$. Then $Y$ is diffeomorphic to $X$, and carries a geometric structure which is encoded in the Kähler form $\omega$ and the holomorphic volume form $\Omega$ of $X$. Both $\omega$ and $\Omega$ are closed forms, so they define de Rham cohomology classes $\alpha = [\omega] \in H^2(X, \mathbb{R})$ and $\beta = [\Omega] \in H^2(X, \mathbb{C})$. Thus, to each ALE space $Y$ we may associate the pair $(\alpha, \beta) \in H^2(X, \mathbb{R}) \times H^2(X, \mathbb{C})$.

For each pair $(\alpha, \beta) \in H^2(X, \mathbb{R}) \times H^2(X, \mathbb{C})$, Kronheimer [12] defined an explicit, possibly singular ALE space $X_{\alpha, \beta}$ asymptotic to $\mathbb{C}^2/H$, using the hyperkähler quotient construction. Let $U$ be the subset
\[
U = \{(\alpha, \beta) \in H^2(X, \mathbb{R}) \times H^2(X, \mathbb{C}) : \alpha(\delta) \neq 0 \text{ or } \beta(\delta) \neq 0 \text{ for all } \delta \in \Delta\}.
\]

Then $U$ is a dense open subset of $H^2(X, \mathbb{R}) \times H^2(X, \mathbb{C})$. Kronheimer showed that if $(\alpha, \beta) \notin U$ then $X_{\alpha, \beta}$ is an orbifold, and if $(\alpha, \beta) \in U$ then $X_{\alpha, \beta}$ is nonsingular and diffeomorphic to $X$, and the Kähler form $\omega$ and holomorphic volume form $\Omega$ of $X_{\alpha, \beta}$ have cohomology classes $[\omega] = \alpha$ and $[\Omega] = \beta$. The manifolds $X_{\alpha, \beta}$ for $(\alpha, \beta) \in U$ form a family diffeomorphic to $X \times U$.  

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Next, Kronheimer [13] showed that if $Y$ is any ALE space asymptotic to $\mathbb{C}^2/H$ then the associated pair $(\alpha, \beta)$ must lie in $U$, and that if $Y_1$ and $Y_2$ are two ALE spaces asymptotic to $\mathbb{C}^2/H$ that both yield the same pair $(\alpha, \beta)$, then $Y_1, Y_2$ are isomorphic as ALE spaces. Combining these results we see that every ALE space $Y$ asymptotic to $\mathbb{C}^2/H$ is isomorphic to $X_{\alpha, \beta}$ for some pair $(\alpha, \beta) \in U$, so we have a complete description of all ALE spaces.

The group $\text{Aut}(\Gamma) \ltimes W$ associated to $\mathbb{C}^2/H$ acts on Kronheimer’s construction, in the following way. The obvious action of $\text{Aut}(\Gamma) \ltimes W$ on $H^2(X, \mathbb{R}) \times H^2(X, \mathbb{C})$ preserves the subset $U$, so that $\text{Aut}(\Gamma) \ltimes W$ also acts on $U$. The action extends naturally to the hyperkahler quotient construction that Kronheimer uses to construct $X_{\alpha, \beta}$, and this shows that if $w \in \text{Aut}(\Gamma) \ltimes W$ and $(\alpha, \beta) \in U$, then $X_{w \cdot \alpha, w \cdot \beta}$ is isomorphic to $X_{\alpha, \beta}$ as an ALE space.

Moreover, if $w \in W$ rather than $\text{Aut}(\Gamma) \ltimes W$, then there is a unique ALE space isomorphism between $X_{w \cdot \alpha, w \cdot \beta}$ and $X_{\alpha, \beta}$ that is asymptotic to the identity at infinity. One can also show that if $X_{\alpha', \beta'}$ is isomorphic to $X_{\alpha, \beta}$ as an ALE space, then $(\alpha', \beta') = (w \cdot \alpha, w \cdot \beta)$ for some $w \in \text{Aut}(\Gamma) \ltimes W$. If in addition the isomorphism between $X_{\alpha', \beta'}$ and $X_{\alpha, \beta}$ is asymptotic to the identity at infinity, then $w \in W$.

Let $w \in \text{Aut}(\Gamma) \ltimes W$. Now $X_{\alpha, \beta}$ and $X_{w \cdot \alpha, w \cdot \beta}$ are both diffeomorphic to $X$, under diffeomorphisms that are natural up to isotopy. The identification between $X_{\alpha, \beta}$ and $X_{w \cdot \alpha, w \cdot \beta}$ that comes from their isomorphism as ALE spaces can thus be thought of as a diffeomorphism of $X$, up to isotopy. The corresponding isotopy class of diffeomorphisms of $X$ is identified with $w \in \text{Aut}(\Gamma) \ltimes W$, regarding $\text{Aut}(\Gamma) \ltimes W$ as a group of isotopy classes of diffeomorphisms of $X$, as above. In particular, the identification between $X_{\alpha, \beta}$ and $X_{w \cdot \alpha, w \cdot \beta}$ induces the action of $w$ on $H^2(X, \mathbb{R})$ and $H^2(X, \mathbb{C})$, and this is why it is possible for two isomorphic Kähler forms apparently to have two different cohomology classes $\alpha$ and $w \cdot \alpha$.

Here is a heuristic description of what is going on. When we desingularize $\mathbb{C}^2/H$ we replace the singular point by a bunch of 2-spheres, and this introduces nontrivial homology classes in $H_2(X, \mathbb{Z})$. The Weyl group $W$ then acts as a kind of ‘internal symmetry group’ on the new homology classes; we can visualize elements of $W$ as diffeomorphisms of $X$ that are the identity outside a small neighbourhood of the 2-spheres. Elements of $\text{Aut}(\Gamma)$ also act as diffeomorphisms of $X$, but they act nontrivially near infinity.

The ALE spaces $X_{\alpha, \beta}$ for $(\alpha, \beta) \in U$ can be thought of as a family of Kähler structures upon the fixed real 4-manifold $X$. Then $X_{\alpha, \beta}$ and $X_{w \cdot \alpha, w \cdot \beta}$ represent Kähler structures on $X$ that are equivalent under a diffeomorphism $\phi$ of $X$ corresponding to $w$. If $w \neq 1$ then $\phi$ is not isotopic to the identity, and acts nontrivially on $H^2(X, \mathbb{R})$ and $H^2(X, \mathbb{C})$.

### 3 Desingularizing Calabi-Yau orbifolds

We are interested in desingularizing Calabi-Yau orbifolds of dimension 3 whose singularities are modelled upon $\mathbb{C}^3/G$, where $G$ is a finite subgroup of $SU(3)$,
and $\mathbb{C}^3/G$ has singularities in codimension two. If we first understand the different ways of desingularizing such $\mathbb{C}^3/G$, this will give us a local model for how to desingularize more general Calabi-Yau orbifolds $X/G$.

Suppose $G \subset SU(3)$ is finite and $\mathbb{C}^3/G$ has codimension two singularities. Pick $x \in \mathbb{C}^3$ such that $xG$ is a generic point in the codimension two singular set, and let $H$ be $\{h \in G : h(x) = x\}$, the stabilizer subgroup of $x$ in $G$. Then there is a natural orthogonal splitting $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C}^2$, such that $x \neq 0$ lies in $\mathbb{C}$, and $H$ fixes $\mathbb{C}$ and acts on $\mathbb{C}^2$ as a finite, nontrivial subgroup of $SU(2)$. We may write $\mathbb{C}^3/H = \mathbb{C} \times (\mathbb{C}^2/H)$, where $\mathbb{C}^2/H$ is one of the Kleinian singularities of $\S^2$.

We shall restrict our attention to the case that $H$ is a normal subgroup of $G$. If $H$ is not normal then things are more difficult. So suppose that $H$ is normal in $G$, and let $K$ be the quotient group $G/H$. Then $K$ acts naturally on $\mathbb{C} \times \mathbb{C}^2/H$, and $(\mathbb{C} \times \mathbb{C}^2/H)/K = \mathbb{C}^3/G$. Let the notation $X, \Gamma, \Delta, W, \text{Aut}(\Gamma), \rho, U$ and $X_{\alpha,\beta}$ all be as defined in the previous section.

We begin by constructing two natural group homomorphisms $\phi : K \to U(1)$ and $\psi : K \to \text{Aut}(\Gamma)$. Since $H$ is the subgroup of $G$ fixing $\mathbb{C}$ and is normal in $G$, it follows that $G$ preserves the splitting $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C}^2$. Therefore $G$ is a subgroup of $SU(\mathbb{C})$ preserving this splitting, and we may write each element $g \in G$ as a pair $(\sigma, \tau)$, where $\sigma \in U(1), \tau \in U(2)$, and $\sigma \cdot \det \tau = 1$. Then $g \in H$ if and only if $\sigma = 1 \in U(1)$. Define a map $\phi : K \to U(1)$ by $\phi(gH) = \sigma$ for each $g \in G$, where $g = (\sigma, \tau)$. It is easy to see that $\phi$ is well-defined, and a group homomorphism.

Write $C_h$ for the conjugacy class of $h$ in $H$, and let $S_H$ be the set of nonidentity conjugacy classes in $H$. As $H$ is a normal subgroup, $gC_hg^{-1}$ is also a conjugacy class in $H$ for each $g \in G$, which is the identity if and only if $C_h$ is. Thus $C_h \to gC_hg^{-1}$ defines a map from $S_H$ to itself. Define a map from $K \times S_H$ to $S_H$ by $(gH,C_h) \mapsto gC_hg^{-1}$. Then this map is well-defined and is an action of $K$ on $S_H$. But there is a natural correspondence between $S_H$, the nonidentity conjugacy classes in $H$, and the vertices of the Dynkin diagram $\Gamma$. Thus $K$ acts on the vertices of $\Gamma$. In fact $K$ acts by automorphisms of the whole graph, and this defines the group homomorphism $\psi : K \to \text{Aut}(\Gamma)$ that we want.

Now let $(\alpha, \beta) \in U$, so that $X_{\alpha,\beta}$ is a nonsingular ALE space, diffeomorphic to $X$, and asymptotic to $\mathbb{C}^2/H$ as in $\S^2$. Then $\mathbb{C} \times X_{\alpha,\beta}$ desingularizes $\mathbb{C} \times \mathbb{C}^2/H$, and has a natural Calabi-Yau structure. Our goal is to choose $(\alpha, \beta)$ such that $\mathbb{C} \times X_{\alpha,\beta}$ admits a $K$-action preserving this Calabi-Yau structure, which is asymptotic to the natural action of $K$ on $\mathbb{C} \times \mathbb{C}^2/H$. To achieve this, we must work out what conditions $\alpha$ and $\beta$ must satisfy for such a $K$-action to exist.

First consider how $K$ can act on $X$, as a group of diffeomorphisms. The action of $K$ on $\mathbb{C}^2/H$ only determines how $K$ should act ‘near infinity’ in $X$. But elements of $W$ may be visualized as diffeomorphisms of $X$ that are the identity near infinity. Thus the action of $K$ on $X$ may not be uniquely determined by the asymptotic conditions on it, but instead there may be several such actions, differing only by elements of $W$. The data we need to determine how $K$ acts on $X$ is a group homomorphism $\chi : K \to \text{Aut}(\Gamma) \rtimes W$, such that $\pi \circ \chi = \psi$, where $\pi : \text{Aut}(\Gamma) \rtimes W \to \text{Aut}(\Gamma)$ is the natural projection, so that $\chi$ lifts $\psi$ from $\text{Aut}(\Gamma)$ to $\text{Aut}(\Gamma) \rtimes W$. Choose such a homomorphism $\chi$. 

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There always exists at least one such homomorphism, because there is a canonical choice for \( \chi \), which will yield desingularizations of \( \mathbb{C}^3/G \) with the topology of crepant resolutions in the construction below. As \( \text{Aut}(\Gamma) \) is naturally isomorphic to a subgroup of \( \text{Aut}(\Gamma) \times W \), we can regard \( \psi : K \to \text{Aut}(\Gamma) \) as a homomorphism \( K \to \text{Aut}(\Gamma) \times W \), and this is the canonical choice for \( \chi \). However, for many groups \( G, H \) there are other, different choices for \( \chi \), and we may be able to use these to construct resolutions of \( \mathbb{C}^3/G \) which do not have the topology of a crepant resolution.

Since \( \chi : K \to \text{Aut}(\Gamma) \times W \) is a group homomorphism and \( \rho \) is a representation of \( \text{Aut}(\Gamma) \times W \) on \( H^2(X, \mathbb{R}) \) and \( H^2(X, \mathbb{C}) \), we see that \( \rho \circ \chi \) is a representation of \( K \) on \( H^2(X, \mathbb{R}) \) and \( H^2(X, \mathbb{C}) \). So, suppose for the moment that \( K \) acts on \( X \) as a group of diffeomorphisms, with action asymptotic to the prescribed action of \( K \) on \( \mathbb{C}^2/H \), such that the induced action of \( K \) on \( H^2(X, \mathbb{R}) \) and \( H^2(X, \mathbb{C}) \) is \( \rho \circ \chi \).

Next, we choose \( (\alpha, \beta) \in U \) and identify \( X_{\alpha,\beta} \) with \( X \) as a real 4-manifold, so that \( K \) acts on \( X_{\alpha,\beta} \), and so on \( C \times X_{\alpha,\beta} \). What is the condition on the pair \( (\alpha, \beta) \) for this \( K \)-action to preserve the natural Calabi-Yau structure on \( C \times X_{\alpha,\beta} \)? Let the Kähler form and holomorphic volume form of \( C \) be \( \omega \) and \( \Omega \), and let the Kähler form and holomorphic volume form of \( X_{\alpha,\beta} \) be \( \omega' \) and \( \Omega' \), respectively. Then the Kähler form of \( C \times X_{\alpha,\beta} \) is \( \omega + \omega' \), and the holomorphic volume form of \( C \times X_{\alpha,\beta} \) is \( \Omega \wedge \Omega' \). Thus the \( K \)-action on \( C \times X_{\alpha,\beta} \) must preserve both \( \omega + \omega' \) and \( \Omega \wedge \Omega' \).

Write \( g \in G \) as a pair \((\sigma, \tau)\) as above, where \( \sigma \in U(1) \) and \( \tau \in U(2) \). Then \( gH \in K \) acts on \( \omega \) and \( \Omega \) by \( gH \cdot \omega = \omega \) and \( gH \cdot \Omega = \tau \cdot \Omega \). Therefore, the Calabi-Yau structure of \( C \times X_{\alpha,\beta} \) is \( K \)-invariant if \( gH \cdot \omega' = \omega' \) and \( \tau \cdot (gH \cdot \Omega') = \Omega' \). Now the cohomology classes of \( \omega' \) and \( \Omega' \) are \( \alpha \) and \( \beta \) respectively, and \( \sigma = \phi(gH) \) from above. Therefore a necessary condition on the pair \( (\alpha, \beta) \) for the Calabi-Yau structure on \( C \times X_{\alpha,\beta} \) to be \( K \)-invariant is

\[
\rho \circ \chi(gH) \alpha = \alpha \quad \text{and} \quad \phi(gH) \cdot \rho \circ \chi(gH) \beta = \beta \quad \text{for all} \; gH \in K. \tag{5}
\]

It turns out that equation (5) is also a sufficient condition for there to exist a \( K \)-action on \( X_{\alpha,\beta} \) with all the properties we require. In particular, we do not need the assumption we made above about the existence of a suitable action of \( K \) on \( X \) by diffeomorphisms, because equation (5) guarantees this. We now explain why (5) is a sufficient condition. Recall from §2 that if \( w \in \text{Aut}(\Gamma) \times W \) and \( (\alpha, \beta) \in U \) then \( X_{\alpha,\beta} \) and \( X_{w \cdot \alpha, w \cdot \beta} \) are isomorphic as ALE spaces, and that if \( w \in W \) then there is a unique isomorphism which is asymptotic to the identity at infinity.

Extending the arguments used to show this, one can prove the following result. Let \( gH \in K \) and \( (\alpha, \beta) \) and \( (\alpha', \beta') \) lie in \( U \), and consider a map \( \Theta_{gH} : X_{\alpha,\beta} \to X_{\alpha',\beta'} \) that is an isomorphism of Kähler manifolds, is asymptotic to the action of \( gH \) on \( \mathbb{C}^2/H \), multiplies holomorphic volume forms by \( \phi(gH)^{-1} \), and acts on cohomology by \( \rho \circ \chi(gH) \). Then the necessary and sufficient condition for there to exist such a map \( \Theta_{gH} \) is that \( \rho \circ \chi(gH) \alpha = \alpha' \) and \( \phi(gH) \cdot \rho \circ \chi(gH) \beta = \beta' \), and if it exists then \( \Theta_{gH} \) is unique.
But any solution $(\alpha, \beta)$ to (3) satisfies these conditions with \((\alpha', \beta') = (\alpha, \beta)\). Therefore, this result guarantees the existence and uniqueness of a map \(\Theta_{gH}\) from \(X_{\alpha,\beta}\) to itself with the properties above, for each \(gH \in K\). It is then easy to show that the maps \(\Theta_{gH}\) yield an action of \(K\) on \(X_{\alpha,\beta}\) with all the properties we need. We summarize our progress so far in the following Theorem; the final part is left as an exercise for the reader.

**Theorem 3.1** Using the above notation, suppose that \(\chi : K \to \text{Aut}(\Gamma) \ltimes W\) is a group homomorphism such that \(\pi \circ \chi = \psi\), and suppose that \((\alpha, \beta) \in U\) satisfies the condition that \(\rho \circ \chi(gH)\alpha = \alpha\) and \(\phi(gH) \cdot \rho \circ \chi(gH)\beta = \beta\) for all \(gH \in K\). Then there exists a unique action of \(K\) on \(C \times X_{\alpha,\beta}\) that preserves the Calabi-Yau structure of \(C \times X_{\alpha,\beta}\) and is asymptotic to the natural action of \(K\) on \(C \times C^2/H\). The representation of \(K\) on \(H^2(X_{\alpha,\beta}, \mathbb{R})\) induced by this action is \(\rho \circ \chi\), identifying \(X_{\alpha,\beta}\) and \(X\) as real 4-manifolds. Moreover, if \(Y\) is any ALE space asymptotic to \(C^2/H\) such that \(C \times Y\) admits a \(K\)-action preserving the Calabi-Yau structure and asymptotic to the given action on \(C \times C^2/H\), then \(Y\) arises from this construction.

Let us now consider the condition (3) more closely. What it really means is that \(\alpha\) has to be invariant under the action \(\rho \circ \chi\) of \(K\) on \(H^2(X, \mathbb{R})\), but \(\beta\) has to be invariant under the action \(\phi \cdot \rho \circ \chi\) of \(K\) on \(H^2(X, \mathbb{C})\). When \(\phi\) is nontrivial, these two \(K\)-actions are different, and will have different invariant subspaces.

Our construction only works if we are able to choose \((\alpha, \beta) \in U\) satisfying (3). Thus by definition of \(U\), we must find \(K\)-invariant elements \(\alpha\) and \(\beta\) such that either \(\alpha(\delta) \neq 0\) or \(\beta(\delta) \neq 0\) for each \(\delta \in \Delta\). To satisfy this condition in examples, the fact that the two \(K\)-actions are different is important. For instance, it can happen that for some \(\delta \in \Delta\), every element \(\alpha \in H^2(X, \mathbb{R})\) invariant under the \(K\)-action \(\rho \circ \chi\) satisfies \(\alpha(\delta) = 0\). But because \(\beta\) must be invariant under a different \(K\)-action \(\phi \cdot \rho \circ \chi\), there may still exist a suitable element \(\beta\) with \(\beta(\delta) \neq 0\).

The goal of this section is to find Calabi-Yau desingularizations of quotient singularities \(\mathbb{C}^3/G\). We divide the problem into two stages, firstly to desingularize \(\mathbb{C}^3/H\) to get \(\mathbb{C} \times X_{\alpha,\beta}\) with a \(K\)-action, and then secondly to divide by \(K\) and desingularize the result \((\mathbb{C} \times X_{\alpha,\beta})/K\). So far we have discussed only the first stage in this process. But what about the second stage?

In fact, at the second stage three things can happen. Firstly, the \(K\)-action on \(X_{\alpha,\beta}\) may have no fixed points. In this case \((\mathbb{C} \times X_{\alpha,\beta})/K\) has no singularities, and is itself a desingularization of \(\mathbb{C}^3/G\). Secondly, the singularities of \((\mathbb{C} \times X_{\alpha,\beta})/K\) may be isolated points. In this case we must desingularize using a crepant resolution. And thirdly, \((\mathbb{C} \times X_{\alpha,\beta})/K\) may have singularities in codimension 2. In this case we can of course use a crepant resolution, but we are also free to apply the method above again.

That is, the singularities of \((\mathbb{C} \times X_{\alpha,\beta})/K\) are locally modelled on \(\mathbb{C}^3/G'\), where \(G'\) is a finite subgroup of \(SU(3)\) that is isomorphic to a subgroup of \(K\). As the singularities have codimension 2, there is a subgroup \(H'\) in \(G'\) contained in some \(SU(2) \subset SU(3)\). We may use the method above to find...
ways to desingularize $\mathbb{C}^3/G'$, and use these as a local model to desingularize $(\mathbb{C} \times X_{\alpha,\beta})/K$. Note that $|G'| \leq |K|$ as $G'$ is a subgroup of $K$, and $|K| = |G|/|H| < |G|$, so that $|G'| < |G|$. Thus, if we use the method iteratively the size of the quotient groups decreases at each stage, and the process must terminate.

4  An example

We now apply the theory of the previous section to the example of $\mathbb{C}^3/\mathbb{Z}_4$. First in §4.1 we explain how to desingularize $\mathbb{C}^3/\mathbb{Z}_4$ in two topologically distinct ways, and then in §4.2 we use this to desingularize a compact Calabi-Yau orbifold $T^6/\mathbb{Z}_4$.

4.1 Two ways to desingularize $\mathbb{C}^3/\mathbb{Z}_4$

Let $\mathbb{C}^3$ have complex coordinates $(z_1, z_2, z_3)$, and define $\kappa : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$\kappa : (z_1, z_2, z_3) \mapsto (-z_1, iz_2, iz_3).$$

(6)

Define $G = \{1, \kappa, \kappa^2, \kappa^3\}$, so that $G$ is a finite subgroup of $SU(3)$ isomorphic to $\mathbb{Z}_4$. The only fixed point of $\kappa$ and $\kappa^3$ is $(0,0,0)$, but the fixed points of $\kappa^2$ are $(z_1,0,0)$ for all $z_1 \in \mathbb{C}$. Therefore $\mathbb{C}^3/G$ has singularities of codimension two. The subgroup of $G$ fixing the points $(z_1,0,0)$ is $H = \{1, \kappa^2\}$, which is a normal subgroup of $G$, and preserves the obvious splitting $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C}^2$. Thus the theory of §3 applies to $G$ and $H$.

The Kleinian singularity $\mathbb{C}^2/H$ is $\mathbb{C}^2/\{\pm 1\}$. The crepant resolution $X$ of $\mathbb{C}^2/\{\pm 1\}$ has $H_2(X, \mathbb{Z}) = \mathbb{Z}$. The Dynkin diagram $\Gamma$ is $A_1$, with Aut($\Gamma$) = $\{1\}$ and Weyl group $W = \mathbb{Z}_2$. Let $W = \{1, \lambda\}$. Then the generator $\lambda$ of $W$ acts on $H_2(X, \mathbb{Z})$ by multiplication by $-1$. The quotient group $K = G/H$ is $\mathbb{Z}_2$ with generator $\kappa H$. Thus the homomorphism $\chi : K \to \text{Aut}(\Gamma) \times W$ of §3 maps $\mathbb{Z}_2$ to $\mathbb{Z}_2$, and the condition $\pi \circ \chi = \psi$ on $\chi$ is trivial since Aut($\Gamma$) = $\{1\}$. Therefore there are two possibilities for $\chi$, given by

(a) $\chi(H) = 1$, $\chi(\kappa H) = 1$, and (b) $\chi(H) = 1$, $\chi(\kappa H) = \lambda$.

(7)

Since $H^2(X, \mathbb{R}) \cong \mathbb{R}$ and $H^2(X, \mathbb{C}) \cong \mathbb{C}$, the ALE spaces asymptotic to $\mathbb{C}^2/H$ are parametrized by pairs $(\alpha, \beta)$ with $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$. The condition for $(\alpha, \beta) \in U$ is that either $\alpha \neq 0$ or $\beta \neq 0$. Let us calculate the conditions on $(\alpha, \beta)$ for the ALE space $X_{\alpha,\beta}$ to admit a suitable $K$-action, for each possibility (a) and (b) in (7). These conditions involve $\phi$, which is given by $\phi(H) = 1$, $\phi(\kappa H) = -1$. In case (a), with $gH = \kappa H$, equation (8) gives $\alpha = \alpha$ and $\beta = -\beta$, which holds if $\beta = 0$. In case (b) with $gH = \kappa H$, equation (8) gives $\alpha = -\alpha$ and $\beta = \beta$, which holds if $\alpha = 0$.

Thus, section 3 gives two different ways (a) and (b) to choose an ALE space $X_{\alpha,\beta}$ asymptotic to $\mathbb{C}^2/\{\pm 1\}$ together with a $K$-action on $\mathbb{C} \times X_{\alpha,\beta}$ asymptotic to the given action of $K$ on $\mathbb{C}^3/H$. The next step is to desingularize $(\mathbb{C} \times
$X_{\alpha,\beta})/K$ to get a desingularization $Y$ of $\mathbb{C}^3/\mathbb{Z}_4$. Here is what happens in each case.

(a) Let $\alpha \in \mathbb{R}$ be nonzero. Then $K$ acts on $\mathbb{C} \times X_{\alpha,0}$. The fixed points of $\kappa H$ in $\mathbb{C} \times X_{\alpha,0}$ are a copy of $\mathbb{C}\mathbb{P}^1$. Thus $(\mathbb{C} \times X_{\alpha,0})/K$ has singularities in codimension two. These singularities admit no deformations, but they do have a unique crepant resolution $Y_1$, which can be described explicitly using toric geometry. The Betti numbers $b^j = b^j(Y_1)$ of $Y_1$ are

$$b^0 = b^2 = b^4 = 1, \quad b^1 = b^3 = b^5 = b^6 = 0. \quad (8)$$

(b) Let $\beta \in \mathbb{C}$ be nonzero. Then $X_{0,\beta}$ is isomorphic as a complex surface to the hypersurface $x_1x_3 - x_2^2 = \beta$ in $\mathbb{C}^3$. Using these coordinates on $X_{0,\beta}$, the $K$-action on $\mathbb{C} \times X_{0,\beta}$ is given by $\kappa H \cdot (z_1, x_1, x_2, x_3) = (z_1, -z_1, -x_2, -x_3)$. Now this action has no fixed points in $\mathbb{C} \times X_{0,\beta}$, since $(0, 0, 0, 0)$ does not satisfy $x_1x_3 - x_2^2 = \beta$. Thus $Y_2 = (\mathbb{C} \times X_{0,\beta})/K$ is already nonsingular, and is a desingularization of $\mathbb{C}^3/\mathbb{Z}_4$. A careful analysis shows that $Y_2$ retracts onto the subset

$$\{ \pm (0, x_1, x_2, x_3) \in Y_2 : |x_1|^2 + 2|x_2|^2 + |x_3|^2 = 2|\beta| \}, \quad (9)$$

which is a copy of $\mathbb{R}^2$. Thus the fundamental group and cohomology of $Y_2$ and $\mathbb{R}^2$ are isomorphic. So $\pi_1(Y_2) \cong \mathbb{Z}_2$, and the Betti numbers $b^j = b^j(Y_2)$ are

$$b^0 = 1, \quad b^1 = \cdots = b^6 = 0. \quad (10)$$

From (8) and (10) we see that methods (a) and (b) yield desingularizations $Y_1$ and $Y_2$ of $\mathbb{C}^3/\mathbb{Z}_4$ with rather different topology. The reason for this difference is that in case (a), $\kappa H$ acts trivially on $H_2(X_{\alpha,0}, \mathbb{Z})$, but in case (b), $\kappa H$ acts on $H_2(X_{0,\beta}, \mathbb{Z})$ by multiplication by $-1$, so the two $K$-actions are topologically distinct.

### 4.2 An orbifold $T^6/\mathbb{Z}_4$ and how to desingularize it

Let $\mathbb{C}^3$ have complex coordinates $(z_1, z_2, z_3)$, and define a lattice $\Lambda$ in $\mathbb{C}^3$ by

$$\Lambda = \{(a_1 + ib_1, a_2 + ib_2, a_3 + ib_3) : a_j, b_j \in \mathbb{Z}\}. \quad (11)$$

Then $\mathbb{C}^3/\Lambda$ is a 6-torus $T^6$, equipped with a flat Calabi-Yau structure. Let $\kappa$ act on $T^6$ by

$$\kappa : (z_1, z_2, z_3) + \Lambda \mapsto (-z_1, iz_2, iz_3) + \Lambda, \quad (12)$$

as in (8). Then $\kappa$ is well-defined and preserves the Calabi-Yau structure on $T^6$. Let $G = \{1, \kappa, \kappa^2, \kappa^3\}$ be the group generated by $\kappa$, so that $G \cong \mathbb{Z}_4$. Then $T^6/G$
is a compact Calabi-Yau orbifold. To understand the singular set of $T^6/G$, we shall first find the fixed points of $\kappa, \kappa^2$ and $\kappa^3$.

The subset of $T^6$ fixed by $\kappa$ and $\kappa^3$ turns out to be the 16 points

$$\{(z_1, z_2, z_3) + \Lambda : z_1 \in \{0, \frac{1}{2}, \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\}, z_2, z_3 \in \{0, \frac{1}{2}, \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\}\}. \tag{13}$$

And the subset of $T^6$ fixed by $\kappa^2$ is 16 copies of $T^2$, given by

$$\{(z_1, z_2, z_3) + \Lambda : z_1 \in \mathbb{C}, z_2, z_3 \in \{0, \frac{1}{2}, \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\}\}. \tag{14}$$

Twelve of the 16 copies of $T^2$ fixed by $\kappa^2$ are identified in pairs by the action of $\kappa$, and these contribute 6 copies of $T^2$ to the singular set of $T^6/G$. On the remaining 4 copies $\kappa$ acts as $-1$, so these contribute 4 copies of $T^2/\{\pm 1\}$ to the singular set. Each $T^2/\{\pm 1\}$ contains 4 of the 16 points fixed by $\kappa$.

Therefore the singular set of $T^6/G$ consists of 6 copies of $T^2$, with singularities modelled on $T^2 \times \mathbb{C}^2/\{\pm 1\}$, and 4 copies of $T^2/\{\pm 1\}$ each copy of $T^2$ in the singular set, may be desingularized using either method (a) or method (b) above. For each $k = 0, \ldots, 4$, let $Z_k$ be one of the manifolds obtained by desingularizing $T^6/G$ using a crepant resolution for each $T^2$ in the singular set, using method (a) for $k$ of the $T^2/\{\pm 1\}$'s, and using method (b) for the remaining $4-k$ copies of $T^2/\{\pm 1\}$. Then each $Z_k$ is a compact, nonsingular manifold carrying a family of Calabi-Yau structures.

We shall find the Betti numbers of $Z_k$. The Betti numbers of $T^6/G$ are

$$b^0(T^6/\mathbb{Z}_4) = 1, \quad b^1(T^6/\mathbb{Z}_4) = 0, \quad b^2(T^6/\mathbb{Z}_4) = 5, \quad b^3(T^6/\mathbb{Z}_4) = 4. \tag{15}$$

To find the Betti numbers of $Z_k$ we must add on contributions from each component of the singular set. The resolution of each copy of $T^2$ in the singular set adds 1 to $b^2$ and 2 to $b^3$. Desingularizing a $T^2/\{\pm 1\}$ using method (a) adds 5 to $b^2$ and fixes $b^3$, but desingularizing using method (b) fixes $b^2$ and adds 2 to $b^3$. All three processes fix $b^0$ and $b^1$.

Thus we calculate that the Calabi-Yau manifolds $Z_k$ have Betti numbers

$$b^0(Z_k) = 1, \quad b^1(Z_k) = 0, \quad b^2(Z_k) = 11 + 5k, \quad b^3(Z_k) = 24 - 2k, \tag{16}$$

giving Euler characteristic $\chi(Z_k) = 12k$. For $k = 1, 2, 3, 4$ one can show that $Z_k$ is simply-connected, and carries metrics with holonomy $SU(3)$. But $Z_0$ is the quotient of $T^2 \times K3$ by a free $\mathbb{Z}_2$-action, and has fundamental group $\mathbb{Z}_2 \times \mathbb{Z}^2$ and holonomy $\mathbb{Z}_2 \times SU(2)$.

We have found five compact Calabi-Yau manifolds $Z_0, \ldots, Z_4$ that desingularize the same orbifold $T^6/\mathbb{Z}_4$. Now the physicists’ formula \[\square\] for the Euler characteristic of desingularizations of $T^6/\mathbb{Z}_4$ predicts the value 48, which is true when $k = 4$ but false when $k = 0, \ldots, 3$. This is consistent, since $Z_k$ is a crepant resolution of $T^6/\mathbb{Z}_4$ and it is known that \[\square\] holds for crepant resolutions, but
Z_0, \ldots, Z_3 are only smoothings of T^6/Z_4. Similarly, the Hodge number formulae of Vafa and Zaslow hold for Z_4, but not for Z_0, \ldots, Z_3.

Our examples show that the physicists’ Euler characteristic and other similar formulae, do not always hold when a Calabi-Yau orbifold is desingularized by deformation. These are not the first such examples known, as an example was given by Vafa and Witten [23, §2], but so far as the author knows this phenomenon has not been studied before in a systematic way. We shall suggest in §8 how these examples might be explained using string theory.

So far we have not justified our claim that each manifold Z_k carries a family of Calabi-Yau structures, which converge to the singular, flat Calabi-Yau structure on T^6/G in an appropriate sense. One way to prove this quite explicitly is to use the analytic methods in the author’s papers [9], [10] on exceptional holonomy; see in particular [10, Ex. 2, p. 350]. Minor modifications to the technique are needed to resolve singularities of the kind above, and they will be described in the author’s book [11]. The problem can also be approached using algebraic geometry.

5 Simultaneous resolution of nodes

There is another, already well-understood way in which a singular Calabi-Yau 3-fold can have several desingularizations with different topology, which involves nodes in 3-folds. A node, or ordinary double point, is a simple kind of singularity of Calabi-Yau 3-folds, modelled on the origin in

\[ \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \}. \]

There are two ways to desingularize this:

(i) A small resolution replaces the singular point with a rational curve \( \mathbb{CP}^1 \). This can be done in two ways, related by a flop.

(ii) One can deform away the singularity by changing to \( z_1^2 + z_2^2 + z_3^2 + z_4^2 = \epsilon \), for nonzero \( \epsilon \in \mathbb{C} \). Topologically, this replaces the singular point by \( S^3 \).

So in case (i) the node is replaced by \( S^2 \), and in case (ii) by \( S^3 \). Let \( X \) be a singular Calabi-Yau manifold with one node. Let \( Y_1 \) be the real manifold got by replacing the node in \( M \) by \( S^2 \) as in (i), and let \( \Sigma_1 \subset Y_1 \) be the new \( S^2 \). Let \( Y_2 \) be the real manifold got by replacing the node by \( S^3 \) as in (ii), and let \( \Sigma_2 \subset Y_2 \) be the new \( S^3 \). Then \( Y_1 \) is Calabi-Yau if and only if \( [\Sigma_1] \neq 0 \) in \( H_2(Y_1, \mathbb{R}) \), and \( Y_2 \) is Calabi-Yau if and only if \( [\Sigma_2] \neq 0 \) in \( H_3(Y_2, \mathbb{R}) \). If \( X \) is compact and the node the only singular point, then exactly one of \([\Sigma_1]\) and \([\Sigma_2]\) is nonzero. Thus exactly one of \( Y_1 \) and \( Y_2 \) is Calabi-Yau.

Now suppose that \( X \) is a singular Calabi-Yau manifold with \( k \) nodes, for \( k > 1 \). In this case it may be possible to desingularize \( X \) as a Calabi-Yau manifold in two ways: firstly, by performing a small resolution of all the nodes together, and secondly, by deforming \( X \) so that all the nodes disappear. (This
idea was originally due to Clemens). For this to happen, the global topology of $X$ must satisfy certain conditions.

Let $Y$ be a small resolution of $X$, let $\Sigma_1, \ldots, \Sigma_k \subset Y$ be the copies of $S^2$ introduced at each node, and let $[\Sigma_j]$ be the homology classes of the $\Sigma_j$ in $H_2(Y, \mathbb{R})$. Then $Y$ is Kähler, and hence a Calabi-Yau 3-fold, if and only if there is a class in $H_2(Y, \mathbb{R})$ that is positive on each $[\Sigma_j]$. Also, Friedman [6, §8] and Tian [20] prove that $X$ admits smooth deformations $X_t$ if and only if there exist nonzero constants $\lambda_1, \ldots, \lambda_k$ such that $\lambda_1[\Sigma_1] + \cdots + \lambda_k[\Sigma_k] = 0$ in $H_2(Y, \mathbb{R})$. This condition is independent of the choice of small resolution $Y$, and the deformations $X_t$ are Calabi-Yau 3-folds.

Let us now apply these ideas to the case of orbifolds. If $X/G$ is a 3-dimensional Calabi-Yau orbifold, there may exist a partial desingularization $Y$ of $X/G$, which is nonsingular except for a finite number of nodes. One can then try to desingularize $Y$ as above by small resolutions, deformations, or a combination of both, to get a number of topologically distinct Calabi-Yau desingularizations $Z_1, \ldots, Z_k$ of $X/G$, which can have a range of different Betti numbers. Even in simple examples, the topological calculations involved in understanding the possibilities for $Z_1, \ldots, Z_k$ can be long and difficult.

Here too, the singularities of $X/G$ must be of codimension two for this trick to produce desingularizations $Z_j$ by deformation. Let $Y$ be a partial desingularization of $X/G$ with $k$ nodes, let $\tilde{Y}$ be a small resolution of $Y$, and let $\Sigma_1, \ldots, \Sigma_k$ be the copies of $S^2$ in $\tilde{Y}$ introduced by the resolution. To desingularize $Y$ by deformation, we need linear relations on the classes $[\Sigma_j]$ in $H_2(\tilde{Y}, \mathbb{R})$. But these relations only exist if the singularities of $X/G$ are of codimension two, because then the different curves $\Sigma_j$ can be joined together by the part of $\tilde{Y}$ that resolves the codimension two singularities. If the singularities of $X/G$ are not of codimension two, then the nodes in $Y$ are isolated from one another, and there are no suitable linear relations on the $[\Sigma_j]$.

We have now proposed two ways in which an orbifold $X/G$ can admit several topologically distinct Calabi-Yau desingularizations $Z_1, \ldots, Z_k$, firstly the method of §3 using the Weyl group, and secondly the method above involving simultaneous resolution of nodes. What is the relationship between the two?

In fact the two phenomena are genuinely different, and one cannot be explained in terms of the other. The method of §3 is a two stage process, where in the first stage one desingularizes $\mathbb{C}^3/H$ in a $K$-invariant way, and we are free to choose some topological data $\chi$. The method of this section comes in at the second stage, since it may give us several ways to desingularize $(\mathbb{C} \times X_{\alpha, \beta})/K$. In particular, the data $\chi$ still makes sense on a manifold with nodes, and is not changed by the choice of how to resolve the nodes.

### 6 Another example

In this section we consider an example that combines the ideas of §3 and §5, and shows how complicated the business of desingularizing orbifolds can be. First in §6.1 we describe the different ways to desingularize $\mathbb{C}^3/\mathbb{Z}^2$ as a Calabi-Yau
manifold. Then in §6.2 we apply this to study the possible desingularizations of the orbifold $T^6/\mathbb{Z}_2^3$. This turns out to be a complex problem, which we do not solve completely.

### 6.1 Desingularizations of $\mathbb{C}^3/\mathbb{Z}_2^2$

Let $\mathbb{C}^3$ have complex coordinates $(z_1, z_2, z_3)$, and define $\kappa_j : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$
\kappa_1 : (z_1, z_2, z_3) \mapsto (z_1, -z_2, -z_3), \quad \kappa_2 : (z_1, z_2, z_3) \mapsto (-z_1, z_2, -z_3)
$$

and $\kappa_3 : (z_1, z_2, z_3) \mapsto (-z_1, -z_2, z_3).$

(17)

Then $G = \{1, \kappa_1, \kappa_2, \kappa_3\}$ is a subgroup of $SU(3)$ isomorphic to $\mathbb{Z}_2^3$. The singular set of $\mathbb{C}^3/G$ splits into 3 pieces: the points $\pm(z_1, 0, 0)$ coming from the fixed points of $\kappa_1$, the points $\pm(0, z_2, 0)$ from the fixed points of $\kappa_2$, and the points $\pm(0, 0, z_3)$ from the fixed points of $\kappa_3$. Each piece is a copy of $\mathbb{C}/\{\pm1\}$, and they meet at $(0, 0, 0)$.

Define $H_j = \{1, \kappa_j\}$ for $j = 1, 2, 3$. Then $H_1, H_2$ and $H_3$ are normal subgroups of $G$ with $\mathbb{C}^3/H_j \cong \mathbb{C} \times \mathbb{C}^2/\{\pm1\}$. The quotient groups $K_j = G/H_j$ are isomorphic to $\mathbb{Z}_2$, and act upon $\mathbb{C}^3/H_j$. Thus we can apply the method of §3 to $\mathbb{C}^3/G$ in 3 different ways, by starting with $H_1, H_2$ or $H_3$, and these 3 ways correspond to the 3 pieces of the singular set. Now $\mathbb{C}^2/\{\pm1\}$ has Dynkin diagram $\Gamma = A_1$, with $\text{Aut}(\Gamma) = \{1\}$ and Weyl group $W = \{1, \lambda\}$ isomorphic to $\mathbb{Z}_2$.

Thus, by §3, every Calabi-Yau desingularization $Y$ of $\mathbb{C}^3/\mathbb{Z}_2^2$ has three pieces of topological data, the group homomorphisms $\chi_j : K_j \to \text{Aut}(\Gamma) \ltimes W$ for $j = 1, 2, 3$. Here $K_1 = \{H_1, \kappa_2 H_1\}$ and $\text{Aut}(\Gamma) \ltimes W = \{1, \lambda\}$ are both isomorphic to $\mathbb{Z}_2$, so there are two possibilities for $\chi_1$,

(a) $\chi_1(H_1) = 1, \chi_1(\kappa_2 H_1) = 1$, and (b) $\chi_1(H_1) = 1, \chi_1(\kappa_2 H_1) = \lambda$.

(18)

As a shorthand, we shall write $\chi_1 = 1$ to denote case (a) and $\chi_1 = -1$ to denote case (b). Similarly, there are two possibilities for each of $\chi_2$ and $\chi_3$, which we will also write $\chi_2 = \pm 1, \chi_3 = \pm 1$. We can think of $\chi_1, \chi_2$ and $\chi_3$ as describing the topology of $Y$ near infinity.

In this section we will study all the different ways to desingularize $\mathbb{C}^3/\mathbb{Z}_2^2$ as a Calabi-Yau manifold. First we describe the deformations of $\mathbb{C}^3/\mathbb{Z}_2^2$. Let $\gamma : \mathbb{C}^3/\mathbb{Z}_2^3 \to \mathbb{C}^4$ be given by $\gamma((z_1, z_2, z_3)G) = (z_1^2, z_2^2, z_3^2, z_1 z_2 z_3)$. Then $\gamma$ is well-defined, and induces an isomorphism between $\mathbb{C}^3/G$ and the hypersurface

$$
W_{0,0,0,0} = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : x_1 x_2 x_3 - x_4^2 = 0\}
$$

(19)

in $\mathbb{C}^4$. Let $\alpha, \beta_1, \beta_2$ and $\beta_3$ be complex numbers, and define

$$
W_{\alpha, \beta_1, \beta_2, \beta_3} = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : x_1 x_2 x_3 - x_4^2 = \alpha + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3\}.
$$

(20)
Then $W_{\alpha,\beta_1,\beta_2,\beta_3}$ is a deformation of $\mathbb{C}^3/\mathbb{Z}_2^2$. For generic $\alpha, \ldots, \beta_3$ the hypersurface $W_{\alpha,\beta_1,\beta_2,\beta_3}$ is nonsingular, but for some special values of $\alpha, \ldots, \beta_3$ it has singularities. If $W_{\alpha,\beta_1,\beta_2,\beta_3}$ is singular, then it can be resolved with a crepant resolution to make it nonsingular.

Thus, each set of values of $\alpha, \ldots, \beta_3$ may give one or more ways to desingularize $\mathbb{C}^3/\mathbb{Z}_2^2$ as a Calabi-Yau manifold. We will now list the different cases that arise in this way. In each case we will give the values of $\chi_1, \chi_2$ and $\chi_3$, and the Betti numbers $b^2$ and $b^4$ of the desingularization.

(i) $W_{0,0,0,0}$ is isomorphic to $\mathbb{C}^3/\mathbb{Z}_2^2$. It has 4 possible crepant resolutions, which are easily described using toric geometry. Each has $\chi_1 = \chi_2 = \chi_3 = 1$ and Betti numbers $b^2 = 3$ and $b^4 = 0$.

(ii) $W_{0,0,0,0}$ for $\alpha \neq 0$. This is nonsingular, has $\chi_1 = \chi_2 = \chi_3 = 1$, and Betti numbers $b^2 = 2$ and $b^4 = 1$.

(iii) $W_{0,\beta_1,0,0}$ for $\beta_1 \neq 0$. This is isomorphic to $(\mathbb{C} \times X_{0,\beta_1})/K_1$, and has singularities at the points $(0, x_2, x_3, 0)$ for $x_2 x_3 = \beta_1$. It has a unique crepant resolution, by blowing up the singular set, which has $\chi_1 = -1$ and $\chi_2 = \chi_3 = 1$, and Betti numbers $b^2 = 1$ and $b^4 = 1$.

(iv) $W_{\alpha,\beta_1,0,0}$ for $\alpha, \beta_1 \neq 0$. This is nonsingular, and is a smooth deformation of the resolution in (iii), with the same topology and values of $\chi_j$ and $b^k$.

(v), (vi) As (iii) and (iv) but with $\beta_2$ nonzero instead of $\beta_1$, and $\chi_2 = -1$ instead of $\chi_1$.

(vii), (viii) As (iii) and (iv) but with $\beta_3$ nonzero instead of $\beta_1$, and $\chi_3 = -1$ instead of $\chi_1$.

(ix) $W_{\alpha,\beta_1,\beta_2,\beta_3}$ with $\beta_1, \beta_2, \beta_3 \neq 0$ and $\alpha^2 \neq 4\beta_1 \beta_2 \beta_3$. This is nonsingular and has $\chi_1 = \chi_2 = \chi_3 = -1$ and Betti numbers $b^2 = 0$ and $b^4 = 1$.

For a few special values of $\alpha, \ldots, \beta_3$, we cannot resolve $W_{\alpha,\beta_1,\beta_2,\beta_3}$ as a Calabi-Yau manifold with the appropriate asymptotic behaviour, so it does not appear on the above list. Here are the missing cases, with the reason why.

- If exactly one of $\beta_1, \beta_2, \beta_3$ is zero, say $\beta_1$, then $W_{\alpha,0,\beta_2,\beta_3}$ is nonsingular and is topologically equivalent to case (ix). However, we should regard it as being 'singular at infinity'.
- Also, $W_{\alpha,\beta_1,\beta_2,\beta_3}$ with $\beta_1, \beta_2, \beta_3 \neq 0$ and $\alpha^2 = 4\beta_1 \beta_2 \beta_3$ has a single node at $x_j = -\alpha/2\beta_j$ for $j = 1, 2, 3$. However, neither of the small resolutions of it are Kähler manifolds.

Here is what we mean by ‘singular at infinity’. Our goal is to construct Calabi-Yau manifolds that desingularize $\mathbb{C}^3/\mathbb{Z}_2^2$. As with the ALE spaces of §2, we expect these manifolds to be asymptotic to $\mathbb{C}^3/\mathbb{Z}_2^2$ at infinity, and the metrics on them to be asymptotic at infinity to the Euclidean metric on $\mathbb{C}^3/\mathbb{Z}_2^2$, in some suitable sense. However, because the singularities of $\mathbb{C}^3/\mathbb{Z}_2^2$ extend to infinity, things are more complicated than they seem at first.
I have studied this problem, and I have found a good definition for the idea of ALE space in the case of non-isolated singularities, and have also proved an existence result for Calabi-Yau metrics satisfying this definition, by adapting Yau’s proof of the Calabi conjecture. I hope to publish these results in my forthcoming book [11]. In cases (i)-(ix) above, my results do guarantee the existence of Calabi-Yau metrics on the given desingularizations, for suitable choices of the Kähler class.

Roughly speaking, in this case the asymptotic conditions on the metrics are as follows. If at least two of $z_1, z_2, z_3$ are very large, then the metric on the desingularization near the point $(z_1, z_2, z_3)G$ in $\mathbb{C}^3/\mathbb{Z}_2^2$ must be close to the flat metric on $\mathbb{C}^3/\mathbb{Z}_2^2$. But if only one of $z_1, z_2, z_3$ is large, say $z_1$, then the metric in the desingularization near the point $(z_1, z_2, z_3)G$ in $\mathbb{C}^3/\mathbb{Z}_2^2$ must be close to the product Calabi-Yau metric on $\mathbb{C} \times X_{\delta, \epsilon}$, where $z_1$ is the coordinate in $\mathbb{C}$, and $X_{\delta, \epsilon}$ is an ALE space asymptotic to $\mathbb{C}^2/\{\pm 1\}$, which has coordinates $\pm (z_2, z_3)$.

We say the Calabi-Yau metric is singular at infinity if the ALE space $X_{\delta, \epsilon}$ appearing in this asymptotic condition is singular – in this case, if $X_{\delta, \epsilon} = \mathbb{C}^2/\{\pm 1\}$. We have excluded cases like $W_{a,0,\beta_2,\beta_3}$ for $\beta_2, \beta_3 \neq 0$ from our list because they are singular at infinity, so that the singularities $\pm (z_1, 0, 0)$ for $z_1$ very large, effectively remain unresolved. It can be shown, although we will not prove this, that every desingularization of $\mathbb{C}^3/\mathbb{Z}_2^2$ as a Calabi-Yau manifold, that is not ‘singular at infinity’, is modelled on one of cases (i)-(ix) above.

In cases (ii), (iv), (vi) and (viii) above, there are nontrivial conditions upon the Kähler class for the metrics to be nonsingular at infinity. The allowed values for the Kähler class split into several connected components – six components in case (ii) and two components in cases (iv), (vi) and (viii). The connected component of the Kähler class can be regarded as an extra topological choice in the desingularization; but we will not discuss this issue here.

The relationship between cases (i) and (ii) may be understood in terms of the ideas of §5. There exists a partial resolution (in fact, three different partial resolutions) of $\mathbb{C}^3/\mathbb{Z}_2^2$ with a single node. This partial resolution can be resolved by a small resolution, giving one of the four manifolds in case (i). Alternatively, the node can be deformed away, giving case (ii).

Observe that in the possible desingularizations of $\mathbb{C}^3/\mathbb{Z}_2^2$, we can have 0, 1 or 3 of $\chi_1, \chi_2$ and $\chi_3$ equal to $-1$, but we cannot have exactly 2 of $\chi_1, \chi_2$ and $\chi_3$ equal to $-1$. Thus we cannot choose $\chi_1, \chi_2$ and $\chi_3$ independently. The moral is that when we desingularize $\mathbb{C}^3/\mathbb{Z}_2^2$ or other orbifolds in which the codimension 2 singularities split into several pieces, the topological choices for different pieces of the singular set are not in general independent, but are subject to constraints involving all the pieces.
6.2 Classifying the desingularizations of $T^6/\mathbb{Z}_2^2$

Let $\Lambda$ be as in §4.2, so that $\mathbb{C}^3/\Lambda$ is a 6-torus $T^6$ with a flat Calabi-Yau structure. Let $\kappa_1, \kappa_2$ and $\kappa_3$ act on $T^6$ by

\begin{align*}
\kappa_1 : (z_1, z_2, z_3) &\mapsto (z_1, -z_2, -z_3) + \Lambda, \\
\kappa_2 : (z_1, z_2, z_3) &\mapsto (-z_1, z_2, -z_3) + \Lambda,
\end{align*}

and $\kappa_3 = \kappa_1 \kappa_2$, as in (17). Then $G = \{1, \kappa_1, \kappa_2, \kappa_3\}$ acts on $T^6$ preserving the Calabi-Yau structure, and is a group isomorphic to $\mathbb{Z}_2^3$. The quotient $T^6/G$ is a Calabi-Yau orbifold, with singularities modelled on $\mathbb{C}^3/\mathbb{Z}_2^3$. This orbifold was studied by Vafa and Witten (see also §4.2), who showed that $T^6/\mathbb{Z}_2^3$ can be resolved by crepant resolution, in many different ways, to get a Calabi-Yau manifold with $h^{1,1} = 51$ and $h^{2,1} = 3$. But they also found one way to desingularize $T^6/\mathbb{Z}_2^2$ by deformation, to get another Calabi-Yau manifold with $h^{1,1} = 3$ and $h^{2,1} = 115$.

Let us now consider how to describe all the different possible ways to desingularize $T^6/\mathbb{Z}_2^2$ as a Calabi-Yau manifold. We will not be able to offer a complete classification, because the calculations involved are extremely complex. However, we can explain the first steps in this classification, and we will see that there are in fact a large number of different ways to desingularize $T^6/\mathbb{Z}_2^2$, of which the possibilities found by Vafa and Witten represent two extremes.

We shall regard $T^6$ as a product $T^2 \times T^2 \times T^2$. Then $\mathbb{Z}_2$ acts on each copy of $T^2$, so that $\mathbb{Z}_2^3$ acts on $T^6$, and $G$ is a subgroup of this $\mathbb{Z}_2^3$. This $\mathbb{Z}_2$-action on $T^2$ has 4 fixed points $p_1, \ldots, p_4$. The fixed points of $\kappa_1$ on $T^6$ are $T^2 \times p_j \times p_k$ for $j, k = 1, \ldots, 4$, which is 16 copies of $T^2$. For $j, k = 1, \ldots, 4$, define $A_{jk} = T^2/\mathbb{Z}_2 \times p_j \times p_k \subset T^6/\mathbb{Z}_2^2$. Similarly, for $i, k = 1, \ldots, 4$, define $B_{ik} = p_i \times T^2/\mathbb{Z}_2 \times p_k \subset T^6/\mathbb{Z}_2^2$, and for $i, j = 1, \ldots, 4$ define $C_{ij} = p_i \times p_j \times T^2/\mathbb{Z}_2 \subset T^6/\mathbb{Z}_2^2$.

The singular set of $T^6/\mathbb{Z}_2^2$ is the union of these sets $A_{jk}, B_{ik}$ and $C_{ij}$. The $A_{jk}$ come from the fixed points of $\kappa_1$, the $B_{ik}$ from $\kappa_2$, and the $C_{ij}$ from $\kappa_3$. Each of the $A_{jk}, B_{ik}$ and $C_{ij}$ is a copy of $T^2/\mathbb{Z}_2$. They are not disjoint, but for each $i, j, k = 1, \ldots, 4$ the three sets $A_{jk}, B_{ik}$ and $C_{ij}$ intersect in the point $p_{ijk} = p_i \times p_j \times p_k$ in $T^6/\mathbb{Z}_2^2$. The $p_{ijk}$ are the 64 singular points in $T^6/\mathbb{Z}_2^2$ which have a singularity modelled on 0 in $\mathbb{C}^3/\mathbb{Z}_2^3$.

Now, following the method of §3, to desingularize $T^6/\mathbb{Z}_2^2$ we must first choose some topological data about the desingularization, the group homomorphism $\chi$. As we saw above, for the $\mathbb{C}^3/\mathbb{Z}_2^3$ singularity there are 3 pieces of data $\chi_1, \chi_2, \chi_3$ corresponding to the $\kappa_1, \kappa_2$ and $\kappa_3$ singularities, and each $\chi_j$ can take the values $\pm 1$. In our case, a little thought shows that $\chi_1$ gives topological information about the way the singularities $A_{jk}$ are resolved. One can show that $\chi_1$ must be constant on each $A_{jk}$, since $A_{jk}$ is connected, but different $A_{jk}$ can have different values of $\chi_1$. Write $\chi_{1,jk}$ for the value of $\chi_1$ on $A_{jk}$. Then for $j, k = 1, \ldots, 4$, we have $\chi_{1,jk} = \pm 1$.

Similarly, $\chi_2$ gives information on how the $B_{ik}$ are resolved, and we write $\chi_{2,ik}$ for the value of $\chi_2$ on $B_{ik}$, and $\chi_3$ gives information on how the $C_{ij}$ are resolved, and we write $\chi_{3,ij}$ for the value of $\chi_3$ on $C_{ij}$. Thus, to desingularize
$T^6/\mathbb{Z}_2^2$ we must first choose the values of $\chi_{1,ik}, \chi_{2,ik}$ and $\chi_{3,ij}$. These are 48 variables taking the values ±1, so there are $2^{48}$, or about $2.8 \times 10^{14}$ possible choices.

Now, we saw above that the possible desingularizations (i)-(ix) of $\mathbb{C}^3/\mathbb{Z}_2^3$ as a Calabi-Yau manifold allow 0, 1 or 3 of $\chi_1, \chi_2, \chi_3$ to be −1, but not two to be −1. This condition applies at each of the 64 points $p_{ijk}$. Therefore, a necessary condition for the data $\chi_{i,jk}$ to represent a possible Calabi-Yau desingularization is that for each set of values $i, j, k = 1, \ldots, 4$, exactly 0, 1 or 3 of $\chi_{1,ik}, \chi_{2,ik}$ and $\chi_{3,ij}$ are −1, but not two of them.

This condition excludes nearly all of the $2^{48}$ choices for the $\chi_{i,jk}$, but there are still many choices for which this condition is satisfied, although we have not been able to count them. However, we can give four explicit families of solutions to the conditions, and so find a lower limit for their number. For the first family, let $\delta_i, \epsilon_j$ and $\zeta_k$ take the values ±1 for $i, j, k = 1, \ldots, 4$, and define

$$\chi_{1,ik} = \epsilon_j \zeta_k, \quad \chi_{2,ik} = \delta_i \zeta_k \quad \text{and} \quad \chi_{3,ij} = -\delta_i \epsilon_j. \quad (22)$$

Then $\chi_{1,ik} \chi_{2,ik} \chi_{3,ij} = -1$ for all $i, j, k$, and this means that either 1 or 3 of $\chi_{1,ik}, \chi_{2,ik}$ and $\chi_{3,ij}$ are equal to −1, but not 0 or 2. Conversely, any set of values of $\chi_{1,ik}, \chi_{2,ik}$ and $\chi_{3,ij}$ for which this holds may be written in the form (22). There are $2^{12}$ possible values for $\delta_i, \epsilon_j$ and $\zeta_k$, but reversing the sign of all the $\delta_i, \epsilon_j$ and $\zeta_k$ does not change the $\chi_{i,jk}$, so this gives $2^{11} = 2048$ different solutions for the $\chi_{i,jk}$.

For the second family, let $\chi_{2,ik} = \chi_{3,ij} = 1$ for all $i, j, k$, and let $\chi_{1,ik}$ be ±1. Clearly, for all $i, j, k$, either 0 or 1 of $\chi_{1,ik}, \chi_{2,ik}$ and $\chi_{3,ij}$ are equal to −1, so the condition is satisfied. There are $2^{16} = 65536$ possible choices for the $\chi_{1,ik}$. Similarly, by putting $\chi_{1,ik} = \chi_{3,ij} = 1$, and by putting $\chi_{1,ik} = \chi_{2,ik} = 1$, we get two other families of $2^{16}$ choices. In total, allowing for repeated choices, we have found 198651 different sets of values for $\chi_{1,ik}, \chi_{2,ik}$ and $\chi_{3,ij}$ in which the conditions are satisfied. This is a lower limit on the number of solutions, which is probably rather larger than this.

Next, having chosen a set of suitable values of the $\chi_{i,jk}$, we must look for Calabi-Yau desingularizations of $T^6/\mathbb{Z}_2^2$ with this data. There are still further topological choices to make. At each of the 64 points $p_{ijk}$, we must choose one of the desingularizations (i)-(ix) above that is consistent with the values of $\chi_{1,ik}, \chi_{2,ik}$ and $\chi_{3,ij}$ already chosen. For instance, if $\chi_{1,ik} = \chi_{2,ik} = \chi_{3,ij} = 1$ then either case (i) or case (ii) will do. There are also more subtle topological choices to do with Weyl groups and the connected component of the Kähler class, which we will not go into.

Having made all these topological choices, we can finally construct a unique real 6-manifold $Y$ that desingularizes $T^6/\mathbb{Z}_2^2$, which locally has the topology of a Calabi-Yau desingularization. However, many of these 6-manifolds do not admit Calabi-Yau structures desingularizing $T^6/\mathbb{Z}_2^2$. Recall that in §5 we discussed the global topological issues involved in desingularizing a Calabi-Yau manifold with finitely many nodes. In this situation there are some rather similar conditions that must be satisfied for a Calabi-Yau structure to exist on $Y$. 19
But in some special cases we can see quite easily that the Calabi-Yau structures exist. For instance, in the second family above with $\chi_{i,k} = \chi_{3,j} = 1$, if we choose desingularization $(i)$ for $p_{ijk}$ when $\chi_{1,k} = 1$ and desingularization $(iii)$ for $p_{ijk}$ when $\chi_{1,k} = -1$, then one can prove that the resulting manifold has a Calabi-Yau structure, which is a crepant resolution of $(T^2 \times K3)/\mathbb{Z}_2$. Using the same trick with the third and fourth families gives a total of 196606 different sets of values of the $\chi_{i,j,k}$ which do correspond to Calabi-Yau desingularizations; and as there are four topological choices for resolution $(i)$, these will lead to many more manifolds.

Our discussion has shown that the problem of classifying all the possible Calabi-Yau desingularizations of $T^6/\mathbb{Z}_2^2$ is of great complexity. There are a large number of choices to be made, but these choices are subject to many complicated conditions. The author’s feeling is that these conditions are not too restrictive, and the number of different ways of desingularizing $T^6/\mathbb{Z}_2^2$ is probably very large. But $T^6/\mathbb{Z}_2^2$ is in fact one of the simplest and most obvious Calabi-Yau orbifolds that one can think of, and the problems involved in analyzing more complex examples must be even worse!

Finally, we explain how the desingularizations of Vafa and Witten [23, §2] fit into this framework. Crepant resolutions of $T^6/\mathbb{Z}_2^2$ have all $\chi_{i,j,k} = 1$, and each point $p_{ijk}$ is resolved using case $(i)$ above. The nonsingular deformation of $T^6/\mathbb{Z}_2^2$ given by Vafa and Witten has all $\chi_{i,j,k} = -1$, and each point $p_{ijk}$ is resolved using case $(ix)$ above; this leads to a unique manifold. These are two extremes in the possible choices for the $\chi_{i,j,k}$, and there are many possibilities in between.

We can also relate another part of Vafa and Witten’s ideas to our analysis above. Vafa and Witten make their desingularization in two stages, by first deforming to a singular Calabi-Yau manifold with 64 nodes which they say is a ‘mirror partner’ to the crepant resolutions of $T^6/\mathbb{Z}_2^2$, and then by deforming away the nodes to get a nonsingular manifold. Above we explained that if $\beta_1, \beta_2, \beta_3 \neq 0$ and $\alpha^2 = 4\beta_1\beta_2\beta_3$, then $W_{\alpha,\beta_1,\beta_2,\beta_3}$ has a single node, which vanishes under deformation. It seems clear that Vafa and Witten’s singular Calabi-Yau manifold is modelled on this.

7 Exceptional holonomy

Calabi-Yau manifolds can be described as Riemannian manifolds with holonomy group $SU(3)$. Now the exceptional holonomy groups are two special cases in the classification of Riemannian holonomy groups, the holonomy groups $G_2$ in 7 dimensions, and $Spin(7)$ in 8 dimensions. They share many properties with the holonomy groups $SU(3)$. In particular, compact Riemannian 7- and 8-manifolds with holonomy $G_2$ or $Spin(7)$ can be made by desingularizing orbifolds $T^7/G$ or $T^8/G$ in a special way, and this method was used by the author [8, 9, 10, 11] to construct the first known examples.

The exceptional holonomy groups are also important in String Theory – see for instance Shatashvili and Vafa [18] – in the same way that Calabi-Yau
manifolds are. Suppose that $X$ is a nonsingular 7- or 8-manifold with holonomy $G_2$ or $\text{Spin}(7)$, constructed by desingularizing an orbifold $Y/G$. Then, just as with the Euler characteristic formula (1) and the Hodge number formula of Vafa and Zaslow for the case of Calabi-Yau manifolds, String Theory can be used to predict topological information about $X$ from the orbifold $Y/G$. This appears implicitly in Shatashvili and Vafa [18].

For a 7-manifold with holonomy $G_2$, the important Betti numbers are $b_2$ and $b_3$, and String Theory can be used to predict their sum $b_2 + b_3$. For an 8-manifold with holonomy $\text{Spin}(7)$, the important Betti numbers are $b_2$, $b_3$, and $b_4$, and String Theory predicts the linear combinations $2b_2 + b_4$, $b_3$, and $b_4 - b_4$. In particular, the Euler characteristic formula (1) should apply without change to 8-manifolds with holonomy $\text{Spin}(7)$. (For 7-manifolds the Euler characteristic is of course zero.)

Now, using the techniques of [8]-[11] and the ideas of §3-§6, one can construct examples of orbifolds which have a number of topologically distinct resolutions with holonomy $G_2$ or $\text{Spin}(7)$, for some of which the String Theory formulae do not hold. To actually prove this, one needs to use more sophisticated techniques than those of [8]-[10], which will be explained in [11].

In the case of Calabi-Yau manifolds, we have a clear distinction between crepant resolutions, for which the String Theory formulae are always true (at least in dimensions 2 and 3), and deformations, for which (in dimensions 3 and above) the formulae are often false. However, in the geometry of $G_2$ and $\text{Spin}(7)$ the distinction between crepant resolutions and deformations no longer makes sense. So it seems that we cannot separate out a special class of resolutions for which the String Theory formulae can be conjectured, or proved, to hold. Perhaps further developments in String Theory will make the matter clearer.

We shall now present an example of an isolated quotient singularity $\mathbb{R}^8/G$, which has several resolutions within holonomy $\text{Spin}(7)$ for which (1) does not hold. Identify $\mathbb{R}^8$ with $\mathbb{C}^4$, which has complex coordinates $(z_1, z_2, z_3, z_4)$, and the standard Euclidean metric and holomorphic volume form. This induces a flat $\text{SU}(4)$-structure on $\mathbb{R}^8$. But the holonomy groups $\text{SU}(4)$ and $\text{Spin}(7)$ are subgroups of $O(8)$, such that $\text{SU}(4) \subset \text{Spin}(7) \subset O(8)$. Because of this, an $\text{SU}(4)$-structure on an 8-manifold induces a unique $\text{Spin}(7)$-structure on the same manifold.

Define maps $\kappa, \lambda : \mathbb{R}^8 \to \mathbb{R}^8$ by

$$\kappa : (z_1, \ldots, z_4) \mapsto (iz_1, iz_2, iz_3, iz_4), \quad \lambda : (z_1, \ldots, z_4) \mapsto (\overline{z}_2, -\overline{z}_1, \overline{z}_4, -\overline{z}_3).$$

(23)

Then $\kappa$ and $\lambda$ satisfy the relations $\kappa^4 = \lambda^4 = 1$, $\kappa^2 = \lambda^2$ and $\kappa \lambda = \lambda \kappa^3$, and they generate a group $G$ of automorphisms of $\mathbb{R}^8$, which is nonabelian and of order 8. The quotient $\mathbb{R}^8/G$ has one singular point, the origin. It can be shown that $\kappa$ lies in both $\text{SU}(4)$ and $\text{Spin}(7)$, and $\lambda$ lies in $\text{Spin}(7)$ but not in $\text{SU}(4)$. Thus $G$ is a subgroup of $\text{Spin}(7)$, and the subgroup $H_1 = \{1, \kappa, \kappa^2, \kappa^3\}$ of $G$ is also a subgroup of $\text{SU}(4)$. Note that $H_1$ is a normal subgroup of $G$, and $K_1 = G/H_1$ is isomorphic to $\mathbb{Z}_2$. 21
Following the ideas of §3, we will first desingularize $\mathbb{R}^8/H_1$ in a $K_1$-invariant way. As $H_1$ is a subgroup of $SU(4)$, we shall desingularize $\mathbb{C}^4/H_1$ with holonomy $SU(4)$, that is, as a Calabi-Yau manifold. Now $\mathbb{C}^4/H_1$ admits a unique crepant resolution $X$, by blowing up the singular point, in which the singular point is replaced by a copy of $\mathbb{CP}^3$. But the singularities of $\mathbb{C}^4/H_1$ are rigid under deformation, so this crepant resolution is the only way to desingularize $\mathbb{C}^4/H_1$ as a Calabi-Yau manifold.

There exists an explicit 1-parameter family of ALE Calabi-Yau metrics on $X$, which was written down by Calabi [2, p. 285]. Examining these metrics, one can easily verify that the map $\lambda : X \to X$ that is asymptotic to $\lambda$ at infinity. This involution has no fixed points, so $Y_1 = X/\{1, \lambda\}$ is nonsingular. The Calabi-Yau metric on $X$ gives an $SU(4)$-structure, which induces a $Spin(7)$-structure on $X$, and $\lambda'$ preserves this $Spin(7)$-structure (but not the $SU(4)$-structure).

Thus, $Y_1$ is a nonsingular 8-manifold asymptotic to $\mathbb{R}^8/G$, and it carries a 1-parameter family of ALE metrics and $Spin(7)$-structures that converge to the orbifold metric on $\mathbb{R}^8/G$. The holonomy group of these metrics is $\mathbb{Z}_2 \times SU(4)$, which is a subgroup of $Spin(7)$. The fundamental group $\pi_1(Y_1)$ is $\mathbb{Z}_2$, and the Betti numbers of $Y_1$ are $b^1 = b^2 = b^3 = b^4 = 0$ and $b^5 = b^6 = 1$. In particular, this means that the resolution of the singularity adds 1 to the Euler characteristic. But a naive application of String Theory ideas suggests that the resolution should add 5 to the Euler characteristic, which is the number of nonidentity conjugacy classes in $G$.

In fact, we can resolve $\mathbb{R}^8/G$ in three slightly different ways. Define $H_2 = \{1, \kappa\lambda, \kappa^2, \kappa^3\lambda\}$ and $H_3 = \{1, \lambda, \lambda^2, \lambda^3\}$. Then $H_2$ and $H_3$ are also normal subgroups of $G$, and $K_2 = G/H_2$ and $K_3 = G/H_3$ are isomorphic to $\mathbb{Z}_2$. There are many different embeddings of $SU(4)$ as a subgroup of $Spin(7)$, and there exist subgroups $SU(4) \subset Spin(7)$ which contain $H_2$ or $H_3$. Using these $SU(4)$ embeddings, we can construct 8-manifolds $Y_2$ and $Y_3$ desingularizing $\mathbb{R}^8/G$ in the same way as we made $Y_1$. At present, $Y_1, Y_2$ and $Y_3$ are the only ways the author knows to desingularize the singularity $\mathbb{R}^8/G$ within holonomy $Spin(7)$, and in particular it is unknown whether there exists any desingularization for which the naive String Theory predictions hold.

The author believes that there is a generalization of the idea of Weyl group of a quotient singularity, using which one can explain desingularizations $Y_1, Y_2$ and $Y_3$ in the following way, using the method of §3. The Weyl group $W$ of the crepant resolution $X$ of $\mathbb{C}^4/H_1$ should be $\{1, \gamma\} \cong \mathbb{Z}_2$, where $\gamma$ multiplies by $-1$ in $H_2(X, \mathbb{Z})$ and $H_3(X, \mathbb{Z})$, and acts trivially on $H_4(X, \mathbb{Z})$. The group homomorphism $\chi : K_1 \to W$ must be $\chi(H_1) = 1$, $\chi(\lambda H_1) = \gamma$, because this yields the correct action of $\lambda'$ on $H_4(X, \mathbb{Z})$. We then see that the naive String Theory predictions do not hold for $Y_1$ because they implicitly assume that $\chi \equiv 1$, but in fact $\chi$ is nontrivial.
8 Orbifolds and string theory

String Theory is a branch of high-energy theoretical physics in which particles are modelled not as points but as 1-dimensional objects – ‘strings’ – propagating in some background space-time, which is usually curved and may have dimension 10, 11 or 26, depending on the theory. String theorists aim to construct a quantum theory of the string’s motion. The process of quantization is extremely complicated, and fraught with mathematical difficulties that are as yet still poorly understood.

String theorists believe that to each compact Calabi-Yau 3-fold $X$ one can associate a conformal field theory (CFT), which is a Hilbert space with a collection of operators satisfying some relations, to be regarded as the quantum theory of strings moving in $X$. They can then use their understanding of conformal field theories to make conjectures about Calabi-Yau 3-folds, which have often turned out to be true, and mathematically very interesting.

String Theory can be used to study Calabi-Yau orbifolds, and their resolutions. The idea is this. Let $X$ be a compact Calabi-Yau 3-fold and $\mathcal{H}_X$ the associated CFT, and suppose that $G$ is a finite group acting on $X$ preserving the Calabi-Yau structure. Then $G$ also acts on $\mathcal{H}_X$, and Dixon et al. [5] showed that by a complicated process one can construct a nonsingular quotient CFT $\mathcal{H}_{X/G}$ that corresponds to the singular Calabi-Yau orbifold $X/G$.

Then small, smooth deformations of $\mathcal{H}_{X/G}$ as a CFT correspond to desingularizations of $X/G$ as a Calabi-Yau orbifold. Because of this, topological data such as the Hodge numbers of these Calabi-Yau desingularizations of $X/G$ can be extracted from $\mathcal{H}_{X/G}$, which in turn depends only on $X$ and $G$. Therefore Dixon et al. and others were able to make predictions such as [4] about the topology of Calabi-Yau desingularizations of $X/G$.

Now it turns out that the quotient CFT $\mathcal{H}_{X/G}$ is not always uniquely determined. Instead, Vafa and Witten [22, 23] show that there can be several different ways to reassemble the pieces of $\mathcal{H}_X$ to make $\mathcal{H}_{X/G}$, a phenomenon which they call discrete torsion. One of these ways is preferred, corresponding to crepant resolutions of $X/G$, but the other possibilities correspond to certain partial desingularizations of $X/G$ in which a finite number of nodes remain unresolved, as in §5.

It would be interesting to be able to properly explain all the different ways to desingularize a Calabi-Yau orbifold in terms of String Theory, and hence perhaps to get a better understanding of these desingularizations. The outline of this explanation already seems clear: to an orbifold $X/G$ we should be able to associate a finite number of different quotient conformal field theories $\mathcal{H}_{X/G}$, and every Calabi-Yau desingularization of $X/G$ should be associated to a small deformation of one of these theories.

However, there is a problem. The idea of discrete torsion gives a way to construct a small number of possibilities for $\mathcal{H}_{X/G}$, but not enough different theories to explain the wide range of desingularizations possible in examples. For instance, Vafa and Witten [23, §2] find only two possibilities for $\mathcal{H}_{X/G}$ in the case of $T^6/Z_2^2$, but the analysis of §6 suggests that there should be hundreds of
thousands of orbifold theories. One explanation of this is given by Aspinwall and Morrison [1] p. 128. They observe that the CFT corresponding to the Calabi-Yau deformation of $T^6/\mathbb{Z}_2^2$ that they consider, is singular for the orbifold itself. Thus, one cannot construct a nonsingular orbifold CFT $\mathcal{H}_X/G$ corresponding to this family. Presumably this means that one must also consider singular CFT’s $\mathcal{H}_X/G$.

I would like to finish by making two tentative suggestions on how it may be possible to construct a larger set of quotient conformal field theories $\mathcal{H}_X/G$ than is given by the framework of [22, 23]. I am not competent to develop these ideas myself, but I hope that some physicist will do so and will tell me the answer. The first suggestion is very obvious: to regard discrete torsion as living locally on $X/G$ rather than globally in $H^2(G,U(1))$, so that one can make different choices of discrete torsion on different pieces of the singular set of $X/G$.

The second suggestion is more serious: I think the ideas on Weyl groups in §3 should have an interpretation in String Theory, and that this should lead directly to a way to construct more quotient CFT’s $\mathcal{H}_X/G$. In particular, the Weyl group $W$ of a singularity should appear in String Theory as an extra group of symmetries acting on the quotient CFT.

Suppose for instance that $G$ has a normal subgroup $H$ with $K = G/H$, and that the singularities of $X/H$ have Weyl group $W$. Then $\mathcal{H}_{X/H}$ should admit an action of $K \rtimes W$ preserving the CFT structure. Here $W$ acts nontrivially only on the twisted sectors in $\mathcal{H}_{X/H}$. The data $\chi$ of §3 defines a subgroup $\tilde{K}$ of $K \rtimes W$ isomorphic to $K$, and quotienting $\mathcal{H}_{X/H}$ by $\tilde{K}$ should give an orbifold CFT corresponding to $X/G$. The difficult parts of this programme appear to be in understanding the action of $W$ on $\mathcal{H}_{X/H}$, which is not obvious, and in making sense of CFT sectors twisted by elements of $K \rtimes W$ rather than $K$.

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