A BEURLING THEOREM FOR ALMOST-INVARIANT
SUBSPACES OF THE SHIFT OPERATOR

ISABELLE CHALENDAR, EVA A. GALLARDO-GUTIÉRREZ,
AND JONATHAN R. PARTINGTON

Abstract. A complete characterization of nearly-invariant subspaces of finite de-
cfect for the backward shift operator acting on the Hardy space is provided in the
spirit of Hitt and Sarason’s theorem. As a corollary we describe the almost-invariant
subspaces for the shift and its adjoint.

1. Introduction

Let $B$ be an infinite-dimensional separable complex Banach space, and $T \in \mathcal{L}(B)$ a
linear bounded operator on $B$. A subspace, that is, a closed linear manifold $M$ is called
invariant if $T(M) \subset M$. Further, $M$ is said to be almost-invariant if there exists a
finite-dimensional subspace $F$ of $B$ such that

$$TM \subset M + F.$$ 

In such a case, the smallest possible dimension of such $F$ is called the defect of the space $M$.

A well-known feature is that the structure of the invariant subspaces of an operator $T$
plays an important role in giving a better understanding of its action on the whole space.
To that aim, Androulakis, Popov, Tcaciuc and Troitsky \[1\] initiated in 2009 the study
of almost-invariant half-spaces of operators $T$ acting on complex Banach spaces. Recall
that a half-space is a space of infinite dimension and infinite codimension. Observe also
that every subspace $M$ of $B$ that is not a half-space is clearly almost-invariant under any
operator.

In 2013, Popov and Tcaciuc \[11\] proved that adjoint operators on dual spaces have
almost-invariant half-spaces; and in particular every operator on a complex infinite-di-
dimensional reflexive Banach space has an almost-invariant half-space. Recently, Sirotkin
and Wallis \[13\] have studied the structure of almost-invariant half-spaces of some opera-
tors, proving, in particular, that every quasinilpotent operator on any infinite dimensional
separable complex Banach space $B$ (not necessarily reflexive) admits an almost-invariant
half-space. A recent preprint of Tcaciuc [14] shows that the same holds for any linear bounded operator acting on $\mathcal{B}$ (not necessarily reflexive).

As Androulakis et al. [1] pointed out, the natural question whether the usual unilateral right shift operator $S$ acting on the Hilbert space $H^2$ has almost-invariant half-spaces has an affirmative answer. It is well known that this operator has even invariant half-spaces. Indeed, by Beurling’s Theorem [3], any shift invariant subspace has the form $\theta H^2$, where $\theta$ is an inner function, that is, an analytic function in the unit disc $D$ with contractive values ($|\theta(z)| \leq 1$ for $z \in D$) such that its boundary values

$$\theta(e^{it}) := \lim_{r \to 1^{-}} \theta(re^{it})$$

(which exists for almost every $e^{it}$ with respect to Lebesgue measure on the unit circle) have modulus one for almost all $e^{it}$. Moreover, every inner function $\theta$ can be factorized, in principle, as a product of two inner functions: one collecting all the zeroes of $\theta$ in $D$ (a Blaschke product), and the other, lacking zeroes in $D$, a singular inner function (i.e., it can be expressed by means of an integral formula involving a singular measure on the unit circle) (see [8], for instance). From here, it is not difficult to see that $M$ is an invariant half-space for $S$ if and only if $M = \theta H^2$ with $\theta$ not a finite Blaschke product.

The aim of this work is studying almost-invariant spaces for the unilateral shift operator in the Hardy space. We will provide a complete characterization in terms of nearly invariant subspaces for the adjoint $S^*$. Recall that a subspace $M$ is nearly invariant for $S^*$ if $S^* f \in M$ whenever $f \in M$ and $f(0) = 0$. This concept can be traced back to Sarason’s work [12] (see also [7], where they were called weakly invariant).

The rest of the paper is organized as follow. In Section 2 we recall some preliminaries and observe that every nearly invariant subspace for $S^*$ is indeed an almost-invariant subspace for $S$. In Section 3 we will prove our main theorem. To that end, we introduce the definition of nearly invariant subspaces with defect $m$ for $S^*$, as a generalization of nearly invariant subspaces, and classify them together with the almost-invariant subspaces. As a consequence we can describe the almost-invariant subspaces for $S$. In Section 4 we discuss the same issues for the bilateral shift on $L^2(\mathbb{T})$. We also provide examples of almost-invariant subspaces for the unilateral and bilateral shifts that do not contain any nontrivial invariant subspaces.

2. A FIRST APPROACH: NEARLY ININVARIANT SUBSPACES FOR $S^*$

Let $D$ denote the open unit disc of the complex plane and $H^2$ the classical Hardy space, that is, the space consisting of analytic functions $f$ on $D$ such that the norm

$$\|f\| = \left( \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi} \right)^{1/2}$$

is finite. A classical result due to Fatou (see [5], for instance) states that the radial limit $f(e^{it}) := \lim_{r \to 1^{-}} f(re^{it})$ exists a.e. on the boundary $T$. In this regard, it is well known
that $H^2$ can be regarded as a closed subspace of $L^2(\mathbb{T})$, and moreover, $L^2(\mathbb{T})$ may be decomposed in the following way

$$L^2(\mathbb{T}) = H^2 \oplus \overline{H_0^2},$$

where $\overline{H_0^2} = \{ f \in L^2(\mathbb{T}) : \overline{f} \in H^2 \text{ and } f(0) = 0 \}$. Note that in the above identity we are identifying $H^2$ through the non-tangential boundary values of the $H^2$ functions. Throughout this paper, $\langle \cdot, \cdot \rangle$ will denote the inner product in $L^2(\mathbb{T})$.

Let $S$ denote the unilateral shift acting on $H^2$, that is, $Sf(z) = zf(z)$, for $z \in \mathbb{D}$. The adjoint $S^*$ is defined in $H^2$ as the operator

$$S^*f(z) = \frac{f(z) - f(0)}{z}, \quad (z \in \mathbb{D}),$$

for $f \in H^2$. As was pointed out in the introduction, Beurling’s Theorem \cite{3} provides a complete characterization of the lattice of the invariant subspaces of $S$; and therefore of the lattice of the invariant subspaces for $S^*$; that is, $K_\theta := (\theta H^2)^\perp$, with $\theta$ an inner function. These spaces are usually referred to as model spaces (we refer to Nikolskii’s monograph \cite{9} for more on the subject).

The concept of nearly invariant subspace for $S^*$, already mentioned and defined in the introduction, was introduced by Sarason in \cite{12}.

**Definition 2.1.** A closed subspace $M \subset H^2$ is said to be nearly invariant for $S^*$ if whenever $f \in M$ and $f(0) = 0$, then $S^*f \in M$.

Nearly invariant subspaces for $S^*$ were characterized by Hitt \cite{7} and Sarason \cite{12}. More precisely, any nontrivial nearly invariant subspace has the form $M = gK$ where $g$ is the element of $M$ of unit norm which has positive value at the origin and is orthogonal to all elements of $M$ vanishing at the origin (the reproducing kernel in $M$ at 0), $K$ is an $S^*$-invariant subspace (so, if nontrivial, $K_\theta$ for some inner function $\theta$), and the operator $M_g$ of multiplication by $g$ is everywhere defined and isometric from $K$ into $H^2$.

Our first observation provides a link between nearly invariant subspaces for $S^*$ and almost-invariant spaces for $S$.

**Proposition 2.2.** Every nearly invariant subspace $M = gK_\theta$ for $S^*$ is almost-invariant for $S$ with defect 1. Moreover, if $\theta$ is not rational, it is an almost-invariant half-space with defect 1.

**Proof.** First, we claim that $SK_\theta \subset K_\theta + \mathbb{C}\theta$. Indeed, the orthocomplement is given by

$$(K_\theta + \mathbb{C}\theta)^\perp = \theta H^2 \cap (\mathbb{C}\theta)^\perp = z\theta H^2;$$

and $\langle z\theta h, zf \rangle = 0$ for any $h \in H^2$ and $f \in K_\theta$. Hence $z\theta H^2 \subset (zK_\theta)^\perp$, as claimed.
On the other hand, since the multiplication operator $M_g$ is everywhere defined and isometric from $K_\theta$ into $H^2$, one has $SM \subset M + Cg\theta$. This shows that $M$ is almost-invariant with defect 1. For the last statement, note that the fact that $M$ is a half-space follows straightforwardly since $\theta$ is not rational. This concludes the proof. □

Our next result will state that the orthocomplement of certain nearly invariant subspaces for $S^*$ are also almost-invariant for $S$ of defect 1. Before stating it, we need the following easy lemma.

**Lemma 2.3.** Let $\psi$ and $\theta$ be non-constant inner functions. Then $(\psi K_\theta)^\perp = \theta \psi H^2 \oplus K_\psi$.

**Proof.** Let $f \in H^2$. Then

$$\langle f, \psi k \rangle = 0 \text{ for all } k \in K_\theta \iff \overline{f\psi} \in \theta H^2 \oplus \overline{K_\psi} \iff f \in \theta \psi H^2 \oplus K_\psi,$$

where the last statement follows since $f \in H^2 \cap \psi \overline{H_0}$ if and only if $f \in K_\psi$. This concludes the proof. □

With Lemma 2.3 in hand, we deduce the following result.

**Proposition 2.4.** Let $\psi$ and $\theta$ be non-constant inner functions. Then $(\psi K_\theta)^\perp$ is an almost-invariant space of defect 1. Moreover, if $\psi$ is not rational (finite Blaschke product); or if $\psi$ is rational but $\theta$ is not a rational inner function, then $(\psi K_\theta)^\perp$ is an almost-invariant half-space of defect 1.

**Proof.** The statement just follows bearing in mind that $SK_\psi \subset K_\psi + C_\psi$ for any inner $\psi$ and the identity $(\psi K_\theta)^\perp = \theta \psi H^2 \oplus K_\psi$ proved in Lemma 2.3. Note that the hypotheses of the last statement ensure that the space has infinite dimension and infinite codimension (so it is a half-space). □

In this regard, we shall show that not every almost-invariant half-space $M$ for $S$ is, indeed, a nearly invariant subspace for $S^*$. In other words, the converse of Proposition 2.2 does not hold.

**Proposition 2.5.** There exist almost-invariant half-spaces for $S$ which are not nearly invariant for $S^*$. More precisely, if $\theta$ is not a rational inner function and $\theta(0) = 0$, then $(\theta K_\theta)^\perp$ is an almost-invariant half-space of defect 1, but not nearly invariant for $S^*$.

**Proof.** Let $\theta$ be an inner function, not rational, and satisfying $\theta(0) = 0$. Let $f = \theta^2$. It follows that $f \in (\theta K_\theta)^\perp$, and $f(0) = 0$. Assume on the contrary that $(\theta K_\theta)^\perp$ is nearly invariant for $S^*$. Then $z \mapsto \frac{f(z)}{z}$ belongs to $(\theta K_\theta)^\perp = \theta^2 H^2 \oplus K_\theta$, by Lemma 2.3. Since $\theta(0) = 0$, there exists an inner function $\theta_1$ such that $\theta(z) = z \theta_1(z)$, and then

$$\frac{\theta^2(z)}{z} = \theta(z) \theta_1(z) = \theta^2(z) h(z) + k(z),$$
for some \( k \in K_\theta \) and \( h \in H^2 \). Since \( k \in K_\theta \cap \theta H^2 \), \( k(z) = 0 \) and then \( h(z) = \frac{1}{z} \), a contradiction. \[ \square \]

3. Classification of nearly invariant subspaces

In order to describe the almost-invariant subspaces for \( S \), let us introduce the definition of nearly invariant subspaces with defect \( m \) for \( S^* \) as a generalization of nearly invariant subspaces.

**Definition 3.1.** A closed subspace \( M \subset H^2 \) is said to be nearly \( S^* \)-invariant with defect \( m \) if and only if there is an \( m \)-dimensional subspace \( F \) (which may be taken to be orthogonal to \( M \)) such that if \( f \in M \), \( f(0) = 0 \) then \( S^* f \in M \oplus F \). We say that \( M \) is \( S^* \) almost-invariant with defect \( m \) if and only if \( S^* M \subset M \oplus F \), where \( \dim F = m \).

Clearly \( S^* \) almost-invariance implies near \( S^* \)-invariance (with the same defect). The work of Hitt [7] shows a connection between the two concepts in the case of \( m = 0 \), as a nearly \( S^* \) invariant subspace has the form \( M = fK \), where \( K \) is an \( S^* \)-invariant subspace and \( f \in H^2 \) satisfies \( \|fk\| = \|k\| \) for all \( k \in K \). See also [4] for a vectorial version.

We shall generalize Hitt’s algorithm to obtain a representation of nearly \( S^* \)-invariant subspaces with defect \( m \) (finite), as follows.

Consider a subspace \( M \) that is nearly \( S^* \)-invariant with defect 1, so that \( F = \langle e_1 \rangle \), say, where \( \|e_1\| = 1 \).

Suppose first that not all functions in \( M \) vanish at 0, and let \( f_0 \in M \) denote the normalized reproducing kernel at 0, so that \( f_0 = k_0/\|k_0\| \), where \( \langle f, k_0 \rangle = f(0) \) for all \( f \in M \). Clearly \( k_0(0) \neq 0 \), so \( f_0(0) \neq 0 \).

For each \( f \in M \) we may write \( f = \alpha_0 f_0 + f_1 \), where \( \alpha_0 \in \mathbb{C} \) and \( f_1(0) = 0 \). So \( S^* f_1 = g_1 + \beta_1 e_1 \) where \( g_1 \in M \) and \( \beta_1 \in \mathbb{C} \).

Thus

\[
(1) \quad f(z) = \alpha_0 f_0(z) + z g_1(z) + \beta_1 z e_1(z), \quad (z \in \mathbb{D}),
\]

and

\[
\|f\|^2 = |\alpha_0|^2 + \|f_1\|^2 = |\alpha_0|^2 + \|g_1\|^2 + |\beta_1|^2.
\]

We may now iterate this, starting with \( g_1 \), to obtain

\[
f(z) = (\alpha_0 + \alpha_1 z + \ldots + \alpha_{n-1} z^{n-1}) f_0(z) + z^n g_n(z) + (\beta_1 z + \ldots + \beta_n z^n) e_1(z),
\]

where

\[
\|f\|^2 = \sum_{k=0}^{n-1} |\alpha_k|^2 + \|g_n\|^2 + \sum_{k=1}^{n} |\beta_k|^2.
\]
Now in fact $\|g_n\| \to 0$ as $n \to \infty$. This can be seen on writing $g_n = P_1S^*P_2g_{n-1}$, where $P_1$ is the orthogonal projection with kernel $\langle e_1 \rangle$ and $P_2$ the orthogonal projection with kernel $\langle f_0 \rangle$. Now the backward shift is a $C_0$ operator, so that $\|S^nh\| \to 0$ for all $h \in H^2$. It follows by applying [2, Lemma 3.3] to the adjoint operators that first $P_1S^*$ is $C_0$ (with finite defect), and then, on applying the same lemma again, that $P_2P_1S^*$ is also $C_0$, and hence $\|g_n\| \to 0$.

Consequently, we may write

$$f(z) = \left( \sum_{k=0}^{\infty} \alpha_k z^k \right) f_0 + \left( \sum_{k=1}^{\infty} \beta_k z^k \right) e_1, \quad (z \in \mathbb{D}),$$

where the sums converge in $H^2$ norm and indeed

$$\|f\|^2 = \sum_{k=0}^{\infty} |\alpha_k|^2 + \sum_{k=1}^{\infty} |\beta_k|^2. \quad (2)$$

We may alternatively express this as saying that $f \in M$ if and only if

$$f(z) = k_0(z)f_0(z) + zk_1(z)e_1(z),$$

where $(k_0, k_1)$ lies in a subspace $K \subset H^2 \times H^2$. Now, recall that $H^2 \times H^2$ can be identified with $H^2(\mathbb{D}; \mathbb{C}^2)$, that is, the space consisting of all analytic functions $F : \mathbb{D} \to \mathbb{C}^2$ such that

$$\|F\| = \left( \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} \|F(re^{i\theta})\|_{\mathbb{C}^2}^2 \, d\theta \right)^{1/2} < \infty.$$

By virtue of (2) we see that $K \subset H^2(\mathbb{D}; \mathbb{C}^2)$ is indeed closed. Moreover, $K$ is invariant under the backward shift $S^* \oplus S^*$, since in the algorithm above,

$$g_1 = S^*k_0f_0 + zk_1(z)e_1 \in M.$$

Conversely, if

$$M = \{k_0f_0 + zk_1e_1 : (k_0, k_1) \in K\},$$

is a closed subspace of $H^2$, where $K$ is invariant under the backward shift, then $M$ is nearly $S^*$-invariant with defect 1.

If all the functions in $M$ vanish at 0, then there is no nontrivial reproducing kernel at 0, but the calculations are simpler, as we may replace (1) with

$$f(z) = z(g_1(z) + \beta_1 e_1(z)), \quad (z \in \mathbb{D}),$$

with $g_1 \in M$ and $\beta_1 \in \mathbb{C}$, where $\|g_1\|^2 + |\beta_1|^2 = \|f\|^2$. The algorithm is then iterated to yield

$$f(z) = \beta_1 ze_1(z) + \beta_2 z^2 e_1(z) + \ldots.$$

For general finite defect $m$ the analogous calculations produce the following result.
Theorem 3.2. Let $M$ be a closed subspace that is nearly $S^*$-invariant with defect $m$. Then:

(i) in the case where there are functions in $M$ that do not vanish at 0,

$$M = \{f : f(z) = k_0(z)f_0(z) + \sum_{j=1}^{m} k_j(z)e_j(z) : (k_0, \ldots, k_m) \in K\},$$

where $f_0$ is the normalized reproducing kernel for $M$ at 0, $\{e_1, \ldots, e_m\}$ is any orthonormal basis for $F$, and $K$ is a closed $S^* \oplus \cdots \oplus S^*$ invariant subspace of the vector-valued Hardy space $H^2(\mathbb{D}; \mathbb{C}^m)$, and $\|f\|^2 = \sum_{j=0}^{m} \|k_j\|^2$.

(ii) In the case where all functions in $M$ vanish at 0,

$$M = \{f : f(z) = z \sum_{j=1}^{m} k_j(z)e_j(z) : (k_1, \ldots, k_m) \in K\},$$

with the same notation as in (i), except that $K$ is now a closed $S^* \oplus \cdots \oplus S^*$ invariant subspace of the vector-valued Hardy space $H^2(\mathbb{D}; \mathbb{C}^m)$, and $\|f\|^2 = \sum_{j=1}^{m} \|k_j\|^2$.

Conversely, if a closed subspace $M \subset H^2$ has a representation as in (i) or (ii), then it is a nearly $S^*$-invariant subspace of defect $m$.

Remark 3.3. If $L$ is a non-trivial invariant subspace for $S^*$ and $x_0 \in H^2 \setminus L$, then it is clear that the subspace $L \oplus \mathbb{C}x_0$ is nearly invariant with defect 1. However, not all such subspaces occur in this way, since the example $M = \theta H^2$, where $\theta(0) = 0$, discussed above, occurs as case (ii) with $m = 1$, $K = H^2$, and $e_1 = S^*\theta$. However $M$ contains no nontrivial invariant subspace for $S^*$.

Note that $K^{\perp}$ can be described using the Lax–Beurling theorem (e.g. [10, Thm 3.1.7]), since it is invariant under $S \oplus \cdots \oplus S$. Indeed $K^{\perp} = \Theta H^2(\mathbb{D}; \mathbb{C}^r)$, where $0 \leq r \leq m + 1$ and $\Theta$ is inner in the matrix-valued version of $H^\infty$, that is $\Theta \in H^\infty(\mathbb{D}; \mathcal{L}(\mathbb{C}^r, \mathbb{C}^{m+1}))$ is an isometry almost everywhere on the unit circle.

Corollary 3.4. A closed subspace $M$ is an almost-invariant subspace for $S^*$ with defect $m$ if and only if it satisfies the conditions of Theorem 3.2 together with the extra condition that $S^*f_0 \in M \oplus F$ in case (i), while case (ii) is unchanged.

Remark 3.5. Note also that $S^*M \subset M \oplus F$ is equivalent to the condition that

$$(S(M \oplus F))^\perp \subset M^\perp = (M \oplus F)^\perp \oplus G,$$

where $G = F \ominus M^\perp$ and $\dim G = \dim F$; this gives an expression for $S$ almost-invariant subspaces too (see also [1]).

Note that it is impossible for a nontrivial subspace $M$ to satisfy $SM = M \oplus F$ with $F$ finite-dimensional, since this would imply that $M \subset SM$, and so $M \subset S^nM$ for all $n \geq 1$, and hence $M = \{0\}$. 


Remark 3.6. We expect a version of Theorem 3.2 to hold in the case of the shift on the vector-valued Hardy space \( H^2(\mathbb{T}; \mathbb{C}^m) \), derived by methods similar to those of [4, Thm. 4.4]. We leave this as a subject for further investigation.

4. Almost invariant subspaces for the bilateral shift

Denote by \( U \) the multiplication by \( t \mapsto e^{it} \) on \( L^2(\mathbb{T}) \). Such operator is called the bilateral shift, it is unitary and \( U^*f(\xi) = \overline{\xi}f(\xi) \) for all \( f \in L^2(\mathbb{T}) \). The famous Lax–Beurling theorem provides a complete description of the closed invariant subspace \( M \) by \( U \), namely:

- if \( UM = M \), then there exists a Borel set \( \Omega \subset \mathbb{T} \) such that \( M = \{ f \in L^2(\mathbb{T}) : f(\xi) = 0 \text{ a.e. on } \Omega \} \);
- if \( UM \subset M \), then there exists \( \theta \in L^\infty(\mathbb{T}) \) such that \( |\theta(\xi)| = 1 \text{ a.e. on } \mathbb{T} \) and \( M = \theta H^2 \).

It follows that one can easily describe the lattice of invariant subspaces of \( U^{-1} = U^* \).

Indeed, since \( UM = M \) is equivalent to \( U^{-1}M = M \) and since \( UM \subset M \) is equivalent to \( U^*M^\perp \subset M^\perp \), the invariant subspaces \( N \) of \( U^* = U^{-1} \) can be described as follows:

- if \( U^*N = N \), then there exists a Borel set \( \Omega \subset \mathbb{T} \) such that \( M = \{ f \in L^2(\mathbb{T}) : f(\xi) = 0 \text{ a.e. on } \Omega \} \);
- if \( U^*N \subset N \), then there exists \( \theta \in L^\infty(\mathbb{T}) \) such that \( |\theta(\xi)| = 1 \text{ a.e. on } \mathbb{T} \) and \( N = \theta H^2 \).

We first investigate almost-invariant subspaces for \( U \) of defect 1. Our first observation shows that the case of the bilateral shift is drastically different from the case of the unilateral shift.

Proposition 4.1. Let \( M \) be a closed subspace of \( L^2(\mathbb{T}) \) such that

\[
(3) \quad U(M) = M \oplus^\perp \mathbb{C}x_0.
\]

Then \( M = \theta H^2_0 \) for some \( \theta \in L^\infty(\mathbb{T}) \) taking unimodular values on the unit circle a.e. Conversely, if \( M = \theta H^2_0 \) as above, then \( U(M) = M \oplus^\perp \mathbb{C}x_0 \) where \( x_0 = \theta \).

Proof. Since \( U^{-1} \) is isometric, it follows that \( M = U^{-1}M \oplus^\perp \mathbb{C}U^{-1}(x_0) \), which implies in particular that \( U^{-1}(M) \subset M \). Our hypothesis implies that \( U^{-1}M \neq M \), and the Lax–Beurling theorem says that there exists a unimodular function \( \theta \) such that \( M = \theta H^2_0 \).

The converse is clear. \( \square \)

We also observe by the same argument that we cannot have \( U(M) = M \oplus F \), with \( \dim F > 1 \), as in the case of \( S \).
The second case is not that easy to deal with. As in Proposition 1 (where this condition is automatically satisfied) we shall add the supplementary condition that $x_0 \in L^\infty(\mathbb{T})$.

Proposition 4.2. Let $M$ be a closed subspace of $L^2(\mathbb{T})$ such that

$$U(M) \subset M \oplus \mathbb{C} \langle x_0 \rangle,$$

where $x_0 \in L^\infty(\mathbb{T})$ with $\|x_0\|_2 = 1$. Then

$$M = \{g + hx_0 : (g, h) \in K\},$$

where $K \subset L^2(\mathbb{T}) \times \overline{H_0^2}$ is a closed subspace invariant under $U \oplus P_\cdot U$, where $P_\cdot : L^2(\mathbb{T}) \to \overline{H_0^2}$ is the orthogonal projection.

Proof. Take $m_0 \in M$; then we can write $U m_0 = m_1 + \lambda_0 x_0$, where $m_1 \in M$ and $\lambda_0 \in \mathbb{C}$. Hence

$$m_0(z) = m_1(z)/z + \lambda_0 x_0/z, \quad (z \in \mathbb{D}),$$

and by orthogonality $\|m_0\|^2 = \|m_1\|^2 + |\lambda_0|^2$.

Repeating this decomposition for $U m_1$, and continuing, we arrive at

$$m_0(z) = \frac{m_n(z)}{z^n} + \left(\frac{\lambda_0}{z} + \ldots + \frac{\lambda_{n-1}}{z^n}\right) x_0(z),$$

with

$$\|m_0\|^2 = \|m_n\|^2 + \sum_{j=0}^{n-1} |\lambda_j|^2.$$

Clearly, letting $n \to \infty$, we see that $\lambda_0/z + \ldots + \lambda_{n-1}/z^n$ converges in $L^2$ norm to some $h \in \overline{H_0^2}$. Hence $m_n(z)/z^n$ also converges in $L^2$, with limit, $g$, say, and we have $\|g\|^2 + \|h\|^2 = \|m_0\|^2$.

The set of pairs $(g, h)$ that can occur is clearly a linear subspace, and the fact that it is closed follows because it is the image of $M$ under an isometric mapping. Moreover, if $m_0$ corresponds to $(g, h)$, then $m_1$ corresponds to $(Ug, P_\cdot Uh)$. □

Note that the adjoint of $U \oplus P_\cdot U$ is $U^* \oplus U^*_{|H_0^2}$, and its invariant subspaces are known thanks to the classical results of Lax–Beurling and Wiener. Thus we have a complete description in this case. Moreover, if $M$ has the representation (5), it is clearly an almost-invariant subspace for $U$ with defect 1.

In general, denote by $\langle x_0 \rangle$ the smallest invariant subspace for $U$ generated by $x_0$. Then the closure of $M \oplus \langle x_0 \rangle$ is invariant by $U$, and therefore $M^\perp \cap \langle x_0 \rangle^\perp$ is a closed invariant subspace for $U^{-1}$.

Obviously this information is useful only in the case where $\langle x_0 \rangle$ is not the whole space, which is a condition that we can reformulate thanks to Helson’s theorem.
Theorem 4.3. Let $x_0 \in L^2(\mathbb{T})$. The following assertions are equivalent:

1. $\langle x_0 \rangle = L^2(\mathbb{T})$;
2. $|x_0(\xi)| > 0$ a.e. on $\mathbb{T}$ and $\int_{\mathbb{T}} \log |x_0(\xi)| d\xi = -\infty$.

Assume that $x_0$ vanishes on a Borel subset $\Omega$ of $\mathbb{T}$ of positive measure, and denote by $\Omega^c$ its complement set in $\mathbb{T}$. Using the Lax–Beurling theorem, it follows that $\langle x_0 \rangle^\perp = \{ f \in L^2(\mathbb{T}) : f(\xi) = 0 \text{ a.e. on } \Omega^c \}$ and then there exists a Borel subset $\Omega_1 \supset \Omega$ of $\mathbb{T}$ such that $M^\perp \cap \langle x_0 \rangle^\perp = \{ f \in L^2(\mathbb{T}) : f(\xi) = 0 \text{ a.e. on } \Omega_1 \}$.

Assume now that $|x_0(\xi)| > 0$ a.e. on $\mathbb{T}$ and that $\log |x_0| \in L^1(\mathbb{T})$. Using the Lax–Beurling theorem, there exists $\theta \in L^\infty(\mathbb{T})$ taking values on the unit circle a.e. such that $\langle x_0 \rangle^\perp = \theta H_0^2$. It follows that $M^\perp \cap \overline{\theta H_0^2} = \theta_1 H_0^2$, with $\theta_1$ of modulus 1 a.e on the unit circle, and such that $\theta_1 H_0^2 \subset \overline{\theta H_0^2}$. This last inclusion is equivalent to $\theta H^2 \subset \theta_1 H^2$, which means that there exists an inner function, say $I$, such that $\theta = I\theta_1$.

Remark 4.4. As in Remark 3.3, we see that we can have $U(M) \subset M \oplus \mathbb{C} x_0$, with $M$ not containing any nontrivial invariant subspace for $U$. For if $M = \overline{\theta H_0^2}$, with $\theta$ unimodular, then we cannot have $M \supset \chi_E L^2(\mathbb{T})$ for any nontrivial subset $E \subset \mathbb{T}$ (clearly), nor $M \supset \phi H^2$ for $\phi$ unimodular, since in the second case we could write $\phi = \overline{\theta g}$, where $g \in H^2$ and is necessarily inner; then $\overline{\theta H_0^2} \supset \overline{\theta \phi g H^2}$, which is a contradiction since the right-hand side contains the function $\theta$.

Finally, we would like to pose the following question:

Characterize the bilateral shift half-invariant subspaces with finite defect in $L^2(\mathbb{T})$.

Acknowledgements

This project was initiated in September 2015, during the second and third authors’ research visit to Institut Camille Jordan, at Université Lyon I. They are grateful for the hospitality and the inspiring environment during their stay.

References

[1] G. Androulakis, A. I. Popov, A. Tcaciuc and V. G. Troitsky, Almost invariant half-spaces of operators on Banach spaces, Integral Equations Operator Theory 65 (2009), no. 4, 473–484.
[2] C. Benhida and D. Timotin, Finite rank perturbations of contractions, Integral Equations Operator Theory, 36 (2000), no. 3, 253–268.
[3] A. Beurling, On two problems concerning linear transformations in Hilbert space, Acta Math. 81 (1949), 239–255.
[4] I. Chalendar, N. Chevrot and J.R. Partington, *Nearly invariant subspaces for backwards shifts on vector-valued Hardy spaces*, J. Operator Theory 63 (2010), no. 2, 403–415.

[5] P. L. Duren, *Theory of $H^p$ spaces* Academic Press, New York, 1970.

[6] H. Helson, *Lectures on Invariant Subspaces* Academic Press, 1964.

[7] D. Hitt, *Invariant subspaces of $H^2$ of an annulus*, Pacific J. Math. 134 (1988), no. 1, 101–120.

[8] K. Hoffman, *Banach spaces of analytic functions*, Dover Publication, Inc., 1988.

[9] N. Nikolski, *Operators, functions, and systems: an easy reading Vol. 1 Hardy, Hankel, and Toeplitz* Mathematical Surveys and Monographs, 92. American Mathematical Society, Providence, RI, 2002.

[10] J.R. Partington, *Linear operators and linear systems*, London Mathematical Society Student Texts, 60. Cambridge University Press, Cambridge, 2004.

[11] A. Popov and A. Tcaciuc, *Every operator has almost-invariant subspaces*, J. Funct. Anal. 265 (2013), no. 2, 257–265.

[12] D. Sarason *Nearly invariant subspaces of the backward shift*, Contributions to operator theory and its applications (Mesa, AZ, 1987), 481–493, Oper. Theory Adv. Appl., 35, Birkhäuser, Basel, 1988.

[13] G. Sirotkin and B. Wallis, *The structure of almost-invariant half-spaces for some operators*, J. Funct. Anal. 267 (2014), no. 7, 2298–2312.

[14] A. Tcaciuc, *The almost-invariant subspace problem for Banach spaces*, preprint, 2017. https://arxiv.org/abs/1707.07836.

Université Paris Est Marne-la-Vallée,
5 bd Descartes, Champs-sur-Marne
77454 Marne-la-Vallée, ceDEX 2, France.
E-mail address: isabelle.chalendar@u-pem.fr

Universidad Complutense de Madrid e ICMAT
Departamento de Análisis Matemático,
Facultad de Ciencias Matemáticas,
Plaza de Ciencias 3
28040, Madrid (SPAIN)
E-mail address: eva.gallardo@mat.ucm.es

School of Mathematics,
University of Leeds,
Leeds LS2 9JT, U.K.
E-mail address: J.R.Partington@leeds.ac.uk