DISKS IN EXPANDING FRW UNIVERSES

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ABSTRACT

We construct exact solutions to Einstein equations representing relativistic disks embedded in an expanding FRW universe. We show that the expansion influences kinematical characteristics of the disks, such as rotational curves, surface mass density, etc. The effects of the expansion are exemplified with nonstatic generalizations of Kuzmin-Curzon and Schwarzschild disks.

Subject headings: accretion, accretion disks — cosmology: theory — galaxies: kinematics and dynamics

1. INTRODUCTION

Recently, there has been a renewed interest in the study of relativistic disks (Bičák, Lynden-Bell, & Katz 1993; Bičák & Ledvinka 1993; Bičák, Lynden-Bell, & Pichon 1993), stemming from a desire to address the possible observational evidence of giant black holes sustained by surrounding disks. The gravitational fields of galactic disks can be accurately modeled by the Newtonian potential theory provided that the thickness of the disks are negligible compared to the typical size of their halos. In more sound situations, however, when gravity is strong enough, one must turn to general relativity. A typical situation in which it is believed that Einstein's theory may contribute is in considering the accretion disks around the central black holes in quasars. Disks may also be used to model sheetlike structures, so their study may shed some light on the large-scale inhomogeneities present in the universe (Lemos & Ventura 1994).

In general relativity, solutions representing thin disks can be constructed starting with static, axially symmetric vacuum Weyl metrics. A discontinuity is then introduced in the first derivative of the metric across the azimuthal plane \( z = 0 \), which in turn induces distribution-like terms in the Ricci tensor. Everywhere outside the \( z = 0 \) plane the solution is vacuum, with a nonvanishing surface mass density concentrated on this plane. Since Weyl solutions are static, they should represent static disks. To allow for rotation in general relativity, one must drop the assumption of the orthogonality of the two Killing vector fields associated with Weyl geometry, thus complicating the problem by introducing drag in the inertial frames. Fortunately, Morgan & Morgan (1969a, 1969b, 1970) had the very smart idea of interpreting the "static disks" as being formed out of two equal streams of collisionless particles circulating in opposite directions. Consequently, the total angular momentum vanishes, and the system may be described by a Weyl line element. Interestingly enough, there exists observational evidence for some galaxies with two countercircling stellar components around their center (Rix et al. 1992; Merrifield & Kuijken 1994; Bertola et al. 1996; Kuijken, Fisher, & Merrifield 1996). Another problem in interpreting the solutions described by the Weyl metric arises from the fact that the disks are infinite in their extent. One may argue, however, that infinite disks model the inner portions of galaxies or accretion disks. Once the problems of interpretation are dealt with, it is not difficult to model relativistic disks and study their physical properties, such as velocity profiles, surface mass density, redshifts, and so on.

Evans & de Zeeuw (1992) showed that it is possible to analyze any classical axially symmetric disk as a linear superposition of so-called Kuzmin disks (Kuzmin 1956). From this, Bičák, Lynden-Bell, & Katz (1993, hereafter BLK) constructed families of counterrotating disk spacetimes. Their work was further extended by Bičák, Lynden-Bell, & Pichon (1993, hereafter BLP) and more recently by Lynden-Bell & Pichon (1996). On the other hand, Lemos & Letelier (1993, 1994, 1996), using the well-known technique of superposing different Weyl solutions, have constructed space-times describing disks surrounding Schwarzschild black holes.

The main purpose of this paper is to consider, in the framework of exact solutions in general relativity, the effect of the cosmological expansion on the disk dynamics and the influence of disklike large-scale structures on the cosmological model's kinematics. It is known (Noerdlinger & Petrosian 1971) that the effects of the expansion on bounded gravitational systems is proportional to the ratio \( \rho/\rho_s \), where \( \rho \) is the background energy density and \( \rho_s \) is the mean rest density of the system. Therefore, for galactic and accretion disks fueling quasars, these effects are not expected to be too significant, although the overall expansion may change the disk dynamics. For disklike large-scale structures, however, the expansion may turn out to be of particular importance.

In §2, we propose an algorithm for generating solutions to Einstein equations that may be interpreted as inhomogeneities embedded in a spatially flat isotropic universe. This technique uses a scalar field generalization in a cosmological setting of static Weyl solutions. The scalar field can be split into two parts: a homogenous part, which may be hydrodynamically interpreted as the velocity potential for an adiabatic perfect fluid acting as the source of a FRW expansion, and a highly irregular part, which may be thought of as a local disk inhomogeneity. In §3 we apply the generating procedure of the previous section to the relativistic Kuzmin disks. We then analyze the energy-momentum tensor for the solutions obtained and interpret them as disks embedded in a FRW background. Section 4 studies the dynamics of the disks, with an emphasis on the
Throughout this paper, we use mathematical quantities. We conclude the paper with a discussion and future prospects.

2. EXACT NONSTATIC SOLUTIONS WITH AXIAL SYMMETRY

2.1. The Generating Algorithm

In this section, we obtain nonstatic solutions to Einstein field equations representing compact objects in a cosmological setting. We start with Weyl's line element, which can be written as:

\[
\begin{align*}
\text{ds}^2 &= -e^{2\phi_1} dt^2 + e^{-2\phi_2} \rho^2 d\phi^2 \\
&\quad + e^{-2\phi_3 + 2\phi_2} (d\rho^2 + dz^2).
\end{align*}
\]

(1)

Throughout this paper, we use \( G = c = 1 \). It is well known that vacuum static axially symmetric fields in general relativity can be generated starting with a Newtonian potential (Weyl 1917; Levi-Civita 1919a, 1919b). For space-times obtained in this way, the metric function \( \nu \) in equation (1) may be taken to be any classical solution of the Laplace equation in cylindrical coordinates; the other metric function, \( \zeta \), is obtained by a quadrature.

New, nonstatic metrics can be generated in two stages. First, starting from a vacuum solution of the Einstein equations for a metric given by equation (1), we construct a new static solution with a minimally coupled, massless scalar field \( \phi \) as a source. For these space-times, the energy-momentum tensor takes the form

\[
T_{ab} = \psi_{,a} \psi_{,b} - \frac{1}{3} g_{ab} \psi \psi_{,c} \psi_{,c},
\]

(2)

and the set of Einstein equations is

\[
\begin{align*}
\nu_{zz} + \nu_{\rho\rho} + \rho^{-1} \nu &= 0 \\
\zeta_{zz} + \zeta_{\rho\rho} - \rho^{-1} \zeta + 2 \nu &= - \psi_{,\rho}^2 \\
\zeta_{zz} + \zeta_{\rho\rho} + \rho^{-1} \zeta + 2 \nu &= - \psi_{,z}^2 \\
\rho^{-1} \zeta - 2 \nu \zeta &= - \psi_{,z} \psi_{,\rho}.
\end{align*}
\]

(3) - (6)

along with the Klein-Gordon equation

\[
\psi_{zz} + \psi_{\rho\rho} + \rho^{-1} \psi_{,\rho} = 0.
\]

(7)

It is easy to see that if \( \nu_0 \) and \( \zeta_0 \) solve the vacuum Einstein equations, a solution to equations (3)-(7) is then given by

\[
\begin{align*}
\nu &= \nu_0 + C \log \rho \\
\zeta &= B \nu_0 + E \rho_0 + F \log \rho \\
\psi &= A \nu_0 + D \log \rho,
\end{align*}
\]

(8) - (10)

where the constants are subject to the constraints

\[
\begin{align*}
2C + AD &= E \\
A^2 + 2B &= 2F \\
D^2 + 2C^2 &= 2F.
\end{align*}
\]

(11) - (13)

For the purposes of this work, we will discard the logarithmic terms by setting the constants \( C, F, \) and \( D \) to zero, thus ensuring asymptotic flatness at spatial infinity in a static case. Once the solution with a massless scalar field metric has been constructed, we transform it into a nonstatic solution with a self-interacting scalar field with an exponential potential. This is accomplished by using an algorithm originally developed by Fonarev (1995); see also the generalization by Feinstein, Ibáñez, & Lazkoz (1995).

The new solution, in a synchronous system of coordinates, reads

\[
\begin{align*}
\text{ds}^2 &= -e^{2\psi(\rho, z)} dt^2 + R^2(t)e^{-2\psi(\rho, z)} \\
&\quad \times [\rho^2 d\phi^2 + e^{2\psi(\rho, z)}(d\rho^2 + dz^2)],
\end{align*}
\]

(14)

where the scale factor is \( R(t) = t^2 k^2 \), and the new metric functions are

\[
\begin{align*}
\nu &= \nu_0, \\
\zeta &= \left(1 + \frac{2}{k^2}\right)\zeta_0,
\end{align*}
\]

(15)

the constant \( k \) being the slope of the potential \( V = \Lambda e^{-k^2}, \) and \( \Lambda = (12 - 2k^2)/k^4. \)

The line element in equation (14) is a solution of the Einstein field equations with the energy momentum tensor given by

\[
T_{ab} = \psi_{,a} \psi_{,b} - g_{ab}(\frac{1}{2} \psi_{,\rho}^2 + \Lambda e^{-k^2})
\]

(16)

Furthermore, the new scalar field \( \psi(t, \rho, z) \) splits into a homogeneous and an inhomogeneous part, \( \psi(t, \rho, z) = \psi_{\text{hom}} + \psi_{\text{inh}} \) with \( \psi_{\text{hom}} \) and \( \psi_{\text{inh}} \) given by

\[
\begin{align*}
\psi_{\text{hom}} &= \frac{2}{k} \log t, \\
\psi_{\text{inh}} &= \frac{2}{k} \nu_0.
\end{align*}
\]

(17)

Note that in the particular case where \( k^2 = 6 \), the potential term vanishes, and one is left with a massless scalar field.

2.2. Interpretation of the New Solutions

As long as the seed static metric is asymptotically flat, the newly generated nonstatic solution is asymptotically homogeneous and isotropic, and at spatial infinity the geometry is represented by the FRW metric

\[
\text{ds}^2 = -dt^2 + t^{4/3}(\rho^2 d\phi^2 + d\rho^2 + dz^2).
\]

(18)

We now identify the homogeneous part of the scalar field with the velocity potential of an irrotational perfect fluid, which is the source of the metric in equation (18). The energy-momentum tensor is then

\[
T_{ab}^{\text{FRW}} = (p + \mu) u_a u_b + pg_{ab},
\]

(19)

where the four-velocity of the fluid is given by

\[
u_a = \frac{\psi_{\text{hom},\rho} \delta_a^t}{\sqrt{-(\psi_{\text{hom},t}^2 \psi_{\text{hom},\rho}^2)}},
\]

(20)

so that the pressure \( p \) and the energy density \( \mu \) become

\[
\begin{align*}
\mu &= -\frac{1}{2} \psi_{\text{hom},t} \psi_{\text{hom},\rho} + \Lambda t^{-2}, \\
p &= -\frac{1}{2} \psi_{\text{hom},t} \psi_{\text{hom},\rho} - \Lambda t^{-2}.
\end{align*}
\]

(21)

Substituting the expression for the scalar field, we readily get

\[
p = \frac{4k^2 - 12}{t^2 k^4}, \quad \mu = \frac{12}{t^2 k^4}.
\]

(22)

It is straightforward to see that equation (22) defines the barotropic equation of state,

\[
p = \gamma \mu, \quad \gamma = \frac{k^2 - 3}{3}.
\]

(23)
Note, however, that the scalar field also has an inhomogeneous component. Although the inhomogeneities steadily dilute as the spatial distance increases, they cannot be regarded as negligible in the intermediate regions. It is well known that as long as the gradient of the scalar field is timelike, the energy momentum tensor can be interpreted in terms of a perfect fluid. The asymptotic homogeneity of the solution ensures that at any time, the gradient of the scalar field is timelike outside a certain closed spatial region, which will depend on the mass and compactness of the disk and on the strength of the scalar field. Outside this region, the perfect fluid interpretation holds, and at spatial infinity the fluid becomes homogeneous and isotropic, as mentioned above. On the other hand, the evolution of the local inhomogeneities with time may also be deduced by studying the gradient of the scalar field

$$\psi_{\text{inh},c} = \psi_{\text{hom},c} \psi_{\text{hom},c} + \psi_{\text{inh},c} \psi_{\text{inh},c},$$

(24)

where

$$\psi_{\text{hom},c} \psi_{\text{hom},c} = -\frac{4}{3\gamma + 3} t^{-2} e^{-2\nu}$$

(25)

and

$$\psi_{\text{inh},c} \psi_{\text{inh},c} = \frac{4}{3\gamma + 3} t^{-4(3\gamma + 3)} e^{2\nu - 2\nu} (v^2 + v_{\rho}^2).$$

(26)

We have checked numerically that for models with accelerated expansion ($\gamma < -\frac{1}{3}$), the outer region, where the homogeneous term of the scalar field dominates the term representing the spatial gradients, grows with time. On the other hand, the highly irregular inner region shrinks to zero. This behavior is similar to that found in the inflationary scenario, in that the small-scale inhomogeneities are washed away because of the accelerated expansion.

The highly inhomogeneous region surrounding the disk plane where the perfect fluid interpretation is not valid may be thought of as playing the role of a cushion between the disk and the cosmological fluid, representing a transition region between the two regimes. We therefore suggest that solutions of the type given by equation (14) should be interpreted as describing an inhomogeneity embedded in an expanding universe, on the grounds that on large scales one recovers the Friedmannian behavior, while the local structure for small distances is governed by the seed metric. This interpretation will be reinforced by the results of the following sections.

3. RELATIVISTIC EXPANDING THIN DISKS

We now apply the described algorithm to the well-known family of axially symmetric solutions representing infinitesimally thin disks in a relativistic context, and construct their dynamical counterparts. We are most interested in the influence of the expansion on the kinematical quantities that characterize the disks. We expect that for an expanding universe, the mass-energy density on the disk plane will steadily decrease as time goes by. However, the effect on other quantities, such as velocity or angular momentum, is not intuitively foreseeable. In addition, we believe that the conclusions reached in this work can bring a new perspective to the problem of embedding irregular sources into standard cosmologies.

3.1. Generation of Thin Disk Families

The disk configurations we are dealing with were first studied by Morgan & Morgan (1969a, 1969b, 1970), who describe counterrotating disks with the same number of leftwise and rightwise rotating particles. As was pointed out in the previous section, one can start with the Newtonian potential produced by a thin disk to obtain its relativistic version. For the sake of simplicity, unless otherwise stated, we will focus our analysis on two families of disks, Kuzmin-Curzon disks and generalized Schwarzschild disks (BLK).

Kuzmin-Curzon disks can be used as building blocks for infinite families of more complicated disklike solutions; one needs only to superpose elementary Kuzmin-Curzon disks with different compactness, weighted by a function $W(b)$, as shown by Evans & de Zeeuw (1992). Their work on Newtonian axisymmetric potential-density pairs was extended to relativistic cases by BLK, who gave the general form of the metric for a superposition of Kuzmin-Curzon disks. BLP went a step further in the study of relativistic disks by constructing more involved families of relativistic versions of the Kuzmin-Toomre (Toomre 1963; Nagai & Miyamoto 1976; Evans & de Zeeuw 1992) and Kalnajs-Mestel (Mestel 1963; Kalnajs 1976) disks. The latter family of solutions is particularly interesting because they have long and flat rotation curves for certain values of the parameters, and include the so-called generalized Schwarzschild disks as their lowest order representatives. The potential corresponding to a Kuzmin-Curzon disk is obtained by introducing a discontinuity in the original Curzon metric (Curzon 1924; Chazy 1924) using the transformation $z \rightarrow |z| + b$. This method of introducing the discontinuity is equivalent to placing two mirror particles of mass $M$ at a distance $b$ below and above the $z = 0$ plane on the $z$-axis. The Newtonian potential for the Kuzmin-Curzon disk is

$$v_0^{K-C} = -\frac{M}{\sqrt{\rho^2 + (|z| + b)^2}};$$

(27)

following Weyl (1917), it can be identified with the metric function $v_0^\nu$ in equation (10). This metric function remains a solution of the Laplace equation outside the $z = 0$ plane, but now, due to the discontinuity, there appears to be a nonvanishing surface mass density. Integration of the remaining metric function yields

$$\zeta_0^{K-C} = -\frac{M^2 \rho^2}{\left(\rho^2 + (|z| + b)^2\right)^{3/2}}.$$

(28)

Correspondingly, as shown by BLK, the classical surface mass density $\Sigma(\rho) = 2v_{\rho,|z|=0}^{K-C}$ is

$$\Sigma^{K-C} = \frac{4Mb}{(\rho^2 + b^2)^{3/2}}.$$

(29)

Here, $M$ is the total mass of the disk as measured from infinity and $b$ is a parameter measuring the compactness of the disk. The analogue of a generalized Schwarzschild disk is constructed in the classical Kuzmin picture by substituting the two mirror particles by two rods with constant line density. Then the general relativity solution is given by

$$v_0^{G-S} = \frac{M}{b_{\text{max}} - b_{\text{min}}} \log \frac{\rho_{\text{min}} + |z| + b_{\text{min}}}{\rho_{\text{max}} + |z| + b_{\text{max}}},$$

(30)
and
\[
\tau_0^{\rho-s} = \frac{2M^2}{(b_{\text{max}} - b_{\text{min}})^2} \log \left( \frac{\rho_{\text{min}} + \rho_{\text{max}}}{4\rho_{\text{min}} \rho_{\text{max}}} \right).
\]

In this case, the classical surface mass density \( \Sigma(\rho) \) is given by
\[
\Sigma_0^{\rho-s} = \frac{4M}{(b_{\text{max}} - b_{\text{min}})(\rho^2 + b^2)^{1/2}} \rho_{\text{max}},
\]
where \( \rho_{\text{min}}^2 = \rho^2 + (|z| + b_{\text{min}})^2 \) and \( \rho_{\text{max}}^2 = \rho^2 + (|z| + b_{\text{max}})^2 \), \( b_{\text{max}} - b_{\text{min}} \) being the parameter that measures the compactness of the disk.

3.2. Disks in an Expanding FRW Universe

We start by looking somewhat more carefully at the stress-energy tensor for the solutions given by equation (14) with the metric functions corresponding to a generic thin counterrotating disk.

In order to deal with the discontinuities in the metric across the \( z = 0 \) plane, we consider the metric functions as distributions. The energy-momentum tensor reads (cf. Gregory, & Chamorro, 1990 et al.), the metric coefficients will have square-integrable weak derivatives, and so the usual formula for the Ricci tensor can be interpreted in terms of distributions. The stress-energy tensor reads (cf. Chamorro, Gregory, & Stewart 1987)
\[
T_{\rho\rho} - T_{\xi\xi} = 2R(t)^{-2}(\rho_{\text{eff}} + v^2 - v_{\rho}^2)e^{-2\xi_0 + 2\psi}(33)
\]
\[
T_{\rho\rho} + T_{\xi\xi} = -2R(t)^{-2}(R^2 + 2R\dot{R})e^{-2\psi}(34)
\]
\[
T_{\xi\xi} = -2R(t)^{-2}[(v_{\rho} + v_{\xi_0}^2) + 2\delta(z)v_{\xi} + \rho^{-1}v_{\rho} - \xi'_{\rho\rho} - \xi'_{\xi\xi} - 2\delta(z)\xi'_{\rho\xi} - v^2 - v_{z}^2)e^{-2\psi_0 + 2\psi} - 2R(t)^{-2}(R^2 - 2R\dot{R})e^{-2\psi}(35)
\]
\[
T_{\xi\xi} = -2R(t)^{-2}[v_{\rho\rho} + v_{\xi_0} + 2\delta(z)v_{\xi} + \rho^{-1}v_{\rho} - \xi'_{\rho\rho} - \xi'_{\xi\xi} - 2\delta(z)\xi'_{\rho\xi} - v^2 - v_{z}^2)e^{-2\psi_0 + 2\psi} - 2R(t)^{-2}(R^2 - 2R\dot{R})e^{-2\psi}(36)
\]
\[
T_{\zeta\zeta} = 2R(t)^{-2}[(\theta(z) - \theta(-z)](2v_{\rho}v_{\xi} - \rho^{-1}\xi_{\rho\rho} - v^2)e^{-2\psi_0 + 2\psi}(37)
\]
\[
T_{\rho\rho} = 2R(t)^{-3}\dot{R}v_{\rho} e^{-2\psi}(38)
\]
\[
T_{\xi\xi} = 2R(t)^{-3}[\theta(z) - \theta(-z)]\dot{R}v_{\xi} e^{-2\psi}(39)
\]

The energy-momentum tensor given by equations (33)-(39) splits into a regular and a singular part proportional to the functional \( \delta(z) \), which is interpreted as a thin counterrotating disk. We still separate the regular part into two: a highly inhomogeneous part, vanishing at infinity, that we interpret as an interaction term between the matter composing the disk and the cosmic fluid, and another part representing at infinity an isotropic perfect fluid. Phenomenologically, we write
\[
T^b_a = T^b_{\text{disk}} + T^b_{\text{int}} + e^{-2\psi}T^b_{\text{FRW}},
\]
where
\[
T^b_{\text{FRW}} = 3R(t)^{-2}\dot{R}^2
\]
and
\[
T^b_{\xi\xi} = T^b_{\rho\rho} = -2R(t)^{-2}(R^2 + 2R\dot{R}).
\]

The identification of the other terms is straightforward. The term that we refer to as \( T^b_{\text{disk}} \) is discontinuous across the \( z = 0 \) plane and therefore gives rise to the nonvanishing mass density. Care must be taken in interpreting equation (40), since it has a purely phenomenological character. Obviously, none of the three terms in the right-hand side of equation (40) can be regarded as a true energy-momentum tensor, since they do not satisfy the energy conservation equation \( T^b_{\xi\xi} = 0 \) separately. Note that this decomposition into three terms is similar to that proposed in § 2.2, based on the character of the gradient of the scalar field. Both approaches qualitatively lead to the same physical description.

4. Dynamics of Expanding Disks

In this section, we study the kinematical quantities of relativistic disks and discuss the influence of the expansion. We explicitly prove that the mass-energy density decreases with time, and we also address the question of the effect of the expansion on the flattening of the rotation curves. Special attention will be paid to the comparison between a static disk and its counterpart in a pressure-free universe. Our motivation is the fact that the universe, as observed at present, has negligible pressure. We now look at the surface mass density and streaming velocities of the particles on the disk. The \( \tau^a_\xi \) surface components are obtained by integration across the disk of the \( T^a_\xi \) components of the energy-momentum tensor,
\[
\tau^a_\xi = \int T^b_a R(t)e^{(\xi_0 - \psi)}dz.
\]

As shown by BLK, the surface rest-mass density in a fixed reference system reads \( \sigma_0 = 2\sigma_0(1 - v^2)^{-1/2} \), while the surface mass density in the same reference system is \( \sigma = 2\sigma_0(1 - v^2)^{-1} - \sigma_0(1 - v^2)^{-1/2} \), where \( v \) is the rotational velocity and \( \sigma_0 \) is the proper rest mass density of one stream. We first calculate the surface mass density \( \sigma(\rho, t) \) on the disk plane by integrating the \( T^a_\xi \) term across the disk. Using equations (35) and (36), we obtain
\[
\sigma(\rho, t) = -\tau^a_\xi = -\int_{0-}^{0+} T^a_\xi R(t)e^{(\xi_0 - \psi)}dz
\]
\[
= R(t)^{-1}e^{(\gamma - \xi_0)(4v_{\xi} - 2\xi_{\rho\rho})}|_{\xi=0}.
\]

Substituting \( \xi_{\rho\rho} \) as calculated from equation (6), we obtain after some algebra
\[
\sigma(\rho, t) = 4R(t)^{-1}v_{\rho}\xi e^{(\gamma - \xi_0)[1 - \rho f(\gamma)v_{\rho}]|_{\xi=0},
\]
where \( f(\gamma) = (3\gamma + 3)/(3\gamma + 5) \), and \( \gamma \) is the adiabatic index of the perfect fluid. Note that the static case will be recovered in the limit \( \gamma \to \infty, f(\gamma) \to 1 \). The particular case of a dust-filled universe corresponds to \( \gamma = 0, f(\gamma) = \frac{1}{2} \).

By integrating the \( T^0_\xi \) term obtained from equations (35) and (36), we obtain
\[
v^2\sigma = \tau^0_\xi = \int_{0-}^{0+} T^0_\xi R(t)e^{(\xi_0 - \psi)}dz = 2R(t)^{-1}e^{(\gamma - \xi_0)f(\gamma)}v_{\rho}\xi_{\rho\rho}|_{\xi=0}.
\]

or, alternatively,
\[
v^2\sigma = 4R(t)^{-1}e^{(\gamma - \xi_0)f(\gamma)}v_{\rho}\xi_{\rho\rho}|_{\xi=0}.
\]
Then, from equations (45) and (47) we readily obtain the expression for the square of the streaming velocities,

$$v^2 = \frac{\rho f(\gamma)v_{,\rho}}{1 - \rho f(\gamma)v_{,\rho} |\zeta = 0}.$$  

(48)

In order to emphasize the difference between the flattening of the rotation curves in the static and expanding cases, we also evaluate the derivative of the velocity in the radial direction, obtaining

$$v_{,\rho} = -\frac{f(\gamma)^{1/2}v_{,\zeta}}{2[v_{,\rho}(1 - \rho f(\gamma)v_{,\rho})]^{1/2}} |\zeta = 0.$$  

(49)

Since the classical mass density of the disks, $\Sigma(r) = 4\nu f(\gamma)|\zeta = 0$, is positive everywhere, in order to ensure that the surface energy density $\sigma$ is positive, we must impose the condition $\rho f(\gamma)v_{,\rho} |\zeta = 0 < 1$, as deduced from equation (45). This is equivalent to fulfilling the weak energy condition (cf. BLK). For the particular case of an expanding Kuzmin-Curzon disk, this restriction reduces to

$$\frac{Mf(\gamma)}{b} < \frac{\sqrt{27}}{2},$$

(50)

while for a generalized Schwarzschild disk it becomes

$$\frac{Mf(\gamma)}{(b_{\text{max}} - b_{\text{min}})} < \frac{[1 - (b_{\text{min}}/b_{\text{max}})^2]^{1/2}}{[1 - (b_{\text{min}}/b_{\text{max}})^2]^{3/2}};$$

(51)

therefore, the expanding disks may be denser than their static counterparts. Moreover, if $\rho f(\gamma)v_{,\rho} |\zeta = 0 < \frac{1}{2}$, the dominant energy condition holds and the rotation velocities are subluminal. It is clear that as long as the dominant energy condition holds, a positive value of the mass density is also ensured. To compare the streaming velocities of a static disk to any of its time-evolving counterparts, one may look at the expression of their quotient, which reads

$$\frac{v^2(\gamma)}{v^2(\gamma = \infty)} = f(\gamma)\frac{1 - \rho v_{,\rho}}{1 - \rho f(\gamma)v_{,\rho} |\zeta = 0}.$$  

(52)

Since $f(\gamma) \leq 1$, we conclude that for expanding disks, the velocities are typically everywhere lower than those for static disks (see Fig. 1). To compare our results to those obtained by BLK, we represent the kinematical quantities as functions of the circumferential radius $\rho_c = \rho e^{-\gamma}$, where $2\pi\rho_c$ is the physical circumference of a circle with radius $\rho = \text{const}$. Note that for the solutions studied in this paper, the comoving radius does not depend on the adiabatic index $\gamma$. Yet, more interesting conclusions are obtained from the quotient between the velocity derivatives in the radial direction,

$$\frac{v_{,\rho}(\gamma)}{v_{,\rho}(\gamma = \infty)} = f^{1/2}(\gamma)\frac{1 - \rho v_{,\rho}}{1 - \rho f(\gamma)v_{,\rho} |\zeta = 0}.$$  

(53)

From equation (53), it is apparent that the slope of the rotation curves is less pronounced for expanding disks than for static disks. It can be concluded that the expansion accentuates the flattening of the rotation curves (see Fig. 2). Another question to consider is the effect of the expansion on the mass-energy density on the disk plane. Similarly, as above for velocities and their radial derivatives, we look at the quotient of the mass energy density of an evolving and a static disk:

$$\frac{\sigma(\gamma)}{\sigma(\gamma = \infty)} = \frac{1 - \rho f(\gamma)v_{,\rho} (te^{\gamma})^{-2/(3\gamma + 3)}}{1 - \rho v_{,\rho} |\zeta = 0}.$$  

(54)

Since $\zeta_0$ is negative everywhere on the disk, we conclude that expanding disks are denser than static ones, with additional mass concentrated outside the center, while the central density is not affected by the expansion. Time evolution of this quantity crucially depends on the adiabatic index $\gamma$. To be more specific, the mass-density dilution rate increases, keeps constant, or decreases with time depending on whether the expansion parameter $\gamma$ is smaller than, equal to, or larger than $-\frac{1}{2}$, respectively. It is also interesting to examine the surface rest mass density of the rotating particles as measured in a frame attached to fixed axes. This quantity is given by

$$\sigma_0 = \sigma(1 - v^2)^{1/2}.$$  

(55)

Substituting this expression for $\sigma$ and $v$ in equations (45) and (48), we get

$$\sigma_0(\rho, t) = 4\nu \sigma_0 e^{\gamma - \gamma_0}\frac{[1 - \rho f(\gamma)v_{,\rho}]^{1/2}[1 - 2\rho f(\gamma)v_{,\rho}]^{1/2}}{|\zeta = 0|}. $$

(56)
The latter quantity is everywhere positive, provided that the dominant energy condition holds. Taking the previous results into account, it is straightforward to see that the influence of the expansion on the surface rest mass density $\sigma_0$ is the same as for the mass density $\sigma$. We have furthermore defined a new function, $\Delta \sigma/\sigma \equiv [\sigma(\gamma) - \sigma(\gamma = \infty)]/\sigma(\gamma)$, that gives the contrast in surface mass density between an expanding and a static disk. Figures 3a and 3b represent...
the contrast function $\Delta \sigma / \sigma$, and Figures 4a and 4b give their rest counterparts $\Delta \sigma_0 / \sigma_0$ for both a generalized Schwarzschild disk and a Kuzmin-Curzon disk. It can be seen from these figures that expanding disks with the same mass parameter $M$ are much denser than their static counterparts. Finally, it is worth looking at the specific angular momentum of the counterrotating components of the disk. Bearing in mind that the specific angular momentum of a particle with rest mass $m$ rotating at radius $r$ is defined as

$$j = (p_{\phi}/m) = g_{\phi\phi} d\phi/d\lambda,$$

where $\lambda$ is proper time, then

$$j(\rho, t) = R(t) \rho e^{-\psi} \left( 1 - \rho^2 \right)^{1/2} \big|_{\zeta = 0} ,$$

or, alternatively,

$$j(\rho, t) = R(t) \left[ \rho^2 f(\gamma) v_{\rho,\rho} \right]^{1/2} e^{-\psi} \frac{1}{\left[ 1 - 2pf(\gamma) v_{\rho,\rho} \right]^{1/2}} \big|_{\zeta = 0} .$$

As with the rotation velocities, a smaller $\gamma$ gives a smaller angular momentum, which increases with time. This behavior is depicted in Figures 5a and 5b, where we compare the specific angular momentum in an expanding and a static case for a generalized Schwarzschild disk and a Kuzmin-Curzon disk.

5. CONCLUSIONS AND OUTLOOK

We have constructed exact solutions to Einstein equations that we interpret as representing relativistic disks in a cosmological setting. A self-interacting scalar field serves as a source of the global expansion, yet locally it defines a disklike structure across the $z = 0$ plane. Far away from this plane, the gradient of the scalar field can be identified with the velocity potential of an irrotational perfect fluid with an adiabatic equation of state $p = \gamma \rho$. Near the azimuthal plane, the spatial gradients of the scalar field dominate over the kinetic, and the geometry is highly inhomogeneous. No sharp-cut transition region exists between these two different regimes; however, this probably corresponds to a physically more realistic situation than surface matching between different solutions of gravitational field equations. Moreover, to the best of our knowledge, there exist no such solutions in the case of axial symmetry and an external FRW geometry. Some particular cases of the solutions we have obtained ($k^2 = 6$) may be reinterpreted as disks in the Brans-Dicke theory. Indeed, in this case, the scalar field may be considered as massless (the potential term vanishes), and a simple conformal transformation $ds^2 \rightarrow e^{\phi/2(1+3k/2)} ds^2$ transforms the solution to a Brans-Dicke frame (Tabensky & Taub 1973). The physical interpretation of the solutions is then quite different, presenting relativistic disks in a theory where the gravitational constant varies in space and time. We will not dwell on this point here, but only mention that the analysis of our paper applies equally well to Brans-Dicke disks. Once the solutions are interpreted as local inhomogeneities in a model universe, the way is cleared to look at the effects produced by expansion. We have found that the inhomogeneities occupy larger or smaller regions of the universe depending on the rate of expansion. More specifically, for the accelerated expansion, we find that inhomogeneities disappear with time, while in a decelerated model their growth is unbounded. The effect of the expansion on the kinematical characteristics of the disks is studied as well. We have shown that expansion changes in principle the falloff of the rotational curves, the angular momentum, and the surface mass density of the disks. We have also compared the characteristics of static disks with those in a dust-filled universe. Although we have concentrated our study on disklike objects, it is remarkable that the generating technique used to obtain solutions here is also appropriate for studying other type of sources of astrophysical interest, such as cosmic strings, walls, spherical shells, and so forth.

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REFERENCES

Bertola, F., Cinzano, P., Corsini, E. M., Pizzella, A., Persic, M., & Salucci, P. 1996, ApJ, 458, L67
Bčák, J., & Ledvinka, T. 1993, Phys. Rev. Lett., 71, 1669
Bčák, J., Lynden-Bell, D., & Katz, J. 1993, Phys. Rev. D, 47, 4334 (BLK)
Bčák, J., Lynden-Bell, D., & Pichon, C. 1993, MNRAS, 265, 126 (BLP)
Chamorro, A., Gregory, R., & Stewart, J. M. 1987, Proc. R. Soc. London A, 413, 251
Chazy, J. 1924, Bull. Soc. Math. Paris, 52, 17
Curzon, H. E. J. 1924, Proc. London Math. Soc., 23, 477
Evans, N. W., & de Zeeuw, P. T. 1992, MNRAS, 257, 152
Feinstein, A., Ibáñez, J., & Lazkoz, R. 1995, Classical Quantum Gravity, 12, L57
Fonarev, O. A. 1995, Classical Quantum Gravity, 12, 1739
Kalnajs, A. 1976, ApJ, 205, 751
Kuijken, K., Fisher, D., & Merrifield, M. R. 1996, MNRAS, 283, 543
Kuzmin, G. G. 1956, Astron. Zh., 33, 27
Lemos, J. P. S., & Letelier, P. S. 1993, Classical Quantum Gravity, 10, L75
Lemos, J. P. S., & Letelier, P. S. 1994, Phys. Rev. D, 49, 5135
———. 1996, Int. J. Mod. Phys. D, 5, 53
Lemos, J. P. S., & Ventura, O. S. 1994, J. Math. Phys., 35, 3604
Levi-Civita, T. 1919a, Rend. Acad. Lincei, 28, 101
———. 1919b, Rend. Acad. Lincei, 28, 101
Lynden-Bell, D., & Pichon, C. 1996, MNRAS, 280, 1007
Merrifield, M. R., & Kuijken, K. 1994, ApJ, 432, 575
Mestel, L. 1963, MNRAS, 126, 553
Morgan, T., & Morgan, L. 1969a, Phys. Rev., 183, 1097
———. 1969b, Phys. Rev. E, 188, 2544
———. 1970, Phys. Rev. D, 2, 2576
Nagai, R., & Miyamoto, M. 1976, PASJ, 28, 1
Noerslinger, P. D., & Petrosian, V. 1971, ApJ, 168, 1
Rix, H., Fisher, D., & Illingworth, G. 1992, ApJ, 400, L5
Tabensky, R., & Taub, A. H. 1973, Commun. Math. Phys., 29, 61
Toomre, A. 1963, ApJ, 138, 385
Weyl, H. 1917, Ann. Phys., 54, 307