ON THE CROSSING NUMBER OF SOME COMPLETE MULTIPARTITE GRAPHS

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ABSTRACT. In this paper, we find the crossing number of the complete multipartite graphs $K_{1,1,1,n}$, $K_{1,2,2,n}$, $K_{1,1,2,n}$ and $K_{1,4,n}$. Our proof depends on Kleitman’s results for the complete bipartite graphs [D. J. Kleitman, The crossing number of $K_{5,n}$, J. Combinatorial Theory 9 (1970) 315-323].

1. INTRODUCTION

In this paper, we consider $G$ as a 1-complex, that is the union of vertices and edges and, if two edges $e$ and $e'$ intersect, $e$ intersects with $e'$ in one of the endpoints. An immersion $\phi$ of $G$ into the 2-dimensional Euclidean space $\mathbb{R}^2$ is said to be good, if the following conditions are satisfied:

(i) $\phi(\phi^{-1}(\phi(V)))$ and $\phi(e)$ are one to one, where $e$ is an edge.

(ii) For any point $p$ in $\mathbb{R}^2$, $\phi^{-1}(p)$ consists of at most two points.

(iii) $\phi(e_i) \cap \phi(e_j)$ consists of at most one point for distinct edges $e_i$ and $e_j$.

Note that $\phi(\hat{e}_i) \cap \phi(\hat{e}_j) = \emptyset$ for adjacent edges $e_i$ and $e_j$, where $\hat{e}$ is the interior of $e \in E$. We will call the image $\phi(G)$ of a good immersion $\phi$ a drawing of $G$.

Let $A$ and $B$ be subsets of $E$. Then the cardinality of $\{\phi(a) \cap \phi(b) \mid a \in A, b \in B\}$ is denoted by $cr_\phi(A, B)$. Especially, $cr_\phi(A, A)$ will be denoted by $cr_\phi(A)$. We call $cr_\phi(E)$ the crossing of $\phi$. The crossing number $cr(G)$ of a graph $G$ is the minimum crossing number among all good immersions.

Let $A$ be a nonempty subset of $V$ or of $E$, for a graph $G$. Then $\langle A \rangle$ denotes the subgraph of $G$ induced by $A$. The set of edges which are incident with a vertex $v$ is denoted by $E(v)$. For a complete $k$-partite graph $K_{a_1,a_2,...,a_k}$ with the partition $(A_1,A_2,...,A_k)$ and the edge set $E$, where $|A_i| = a_i$, we will write $E_{A_i,A_j}$ for the edge sets of $\langle A_i \cup A_j \rangle$.

We note the following formulas, which can be shown easily.

\begin{align*}
(1) \quad cr_\phi(A \cup B) &= cr_\phi(A) + cr_\phi(B) + cr_\phi(A, B) \\
(2) \quad cr_\phi(A, B \cup C) &= cr_\phi(A, B) + cr_\phi(A, C)
\end{align*}

where $A$, $B$ and $C$ are mutually disjoint subsets of $E$.

A good, updated survey on crossing numbers is [8]. A longstanding problem in the theory of crossing number is Zarankiewicz’s conjecture, which asserts that the crossing number of the complete graphs $K_{m,n}$ is given by

$$cr(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$ 

It is only known to be true for $m \leq 6$ [7]; and for $m = 7$ and $n \leq 10$ [9]. Recently, in [2], E. deKlerk et al. give a new lower bound for the crossing number of $K_{m,n}$ and $K_n$. In the following, $Z(m,n)$ will denote the right member of (3).
It is natural to ask generalize the Zarankiewicz’s conjecture and ask: What is the crossing number for the complete multipartite graph? In this paper, using Kleitman’s result, we will determine the crossing number of the multipartite graphs $K_{1,1,1,1,n}$, $K_{1,2,2,n}$, $K_{1,1,1,2,n}$ and $K_{1,4,n}$ as follows: $cr(K_{1,1,1,1,n}) = Z(4,n) + n$; $cr(K_{1,2,2,n}) = Z(5,n) + \lfloor \frac{3n}{2} \rfloor$; $cr(K_{1,1,1,2,n}) = Z(5,n) + 2n$; $cr(K_{1,4,n}) = Z(5,n) + 2\lfloor \frac{n}{2} \rfloor$.

The arguments used to obtain the crossing numbers of the graphs except $K_{1,4,n}$ are borrowed heavily from Asano [1], who determines the crossing number of $K_{1,3,n}$ and $K_{2,3,n}$ as follows: $cr(K_{1,3,n}) = Z(4,n) + \lfloor \frac{n}{2} \rfloor$; $cr(K_{2,3,n}) = Z(5,n) + n$. In addition to Asano’s paper, our proof is based on the results of Harborth in [6], who have presented all the non-isomorphic drawings of $K_{2,2}$, $K_{2,3}$ and $K_{3,3}$. To obtain the crossing numbers of $K_{1,4,n}$, we use the basic counting argument. On the other hand, the crossing number of $K_{1,3,n}$ can be determined by this counting argument.

2. Preliminary Results

In this section, we provide some results which can be used to obtain the required crossing numbers. Lemma 2.1 and 2.2 will be used to obtain the crossing number of $K_{1,2,2,n}$.

Lemma 2.1. For any good immersion $\phi$ of $K_{1,2,2}$, there is at most 1 region whose boundary contains at least 4 vertices in $V$.

Proof. From [6], we know that there are 6 non-isomorphic drawings of $K_{2,3}$, namely, the drawings in Figure 1.

![Diagram of K2,3 drawings](image)

Figure 1.

To obtain a drawing of $K_{1,2,2}$ from these drawings of $K_{2,3}$, we have to choose a vertex from the partition of $K_{2,3}$ containing 3 vertices and draw edges connecting this vertex and the other two vertices in the partition. For $2 \leq i \leq 6$, $D_i$ has no more than 1 region whose boundary contains at least 4 vertices. Therefore, $D_1$ is the only drawing of $K_{2,3}$ which can be used to obtain a possible counterexample of Lemma 2.1.

![Counterexample](image)

Figure 2.
In $D_1$, we have to choose a vertex from the partition of $K_{2,3}$ with 3 vertices (that is, the white vertices in $D_1$) and draw edges connecting this vertex and the other two vertices in the partition. Up to isomorphism, the only possible drawing of $K_{1,2,2}$ is as in Figure 2, which proves the lemma. □

**Lemma 2.2.** There are 3 non-isomorphic drawings of $K_{1,2,2}$ such that it contains a region whose boundary contains more than 4 vertices in $V$.

**Proof.** From [6] again, we know that there are 6 non-isomorphic drawings of $K_{2,3}$, namely, the drawings in Figure 1. To obtain a drawing of $K_{1,2,2}$ from these drawings of $K_{2,3}$ such that there is a region whose boundary contains more than 4 vertices, the only candidates are $D_2$ and $D_5$. To preserve the region whose boundary containing more than 4 vertices, the only possible drawings of $K_{1,2,2}$, up to isomorphism, are shown in Figure 3.

![Figure 3](image)

3. **Crossing Number of $K_{1,1,1,n}$**

In this section, we will prove

**Theorem 3.1.** The crossing number of the complete 5-partite graph $K_{1,1,1,n}$ is given by

$$\text{cr}(K_{1,1,1,n}) = Z(4, n) + n.$$ 

**Proof.** Let $(X,Y,S,T,Z)$ be partition of $K_{1,1,1,n}$ such that $X = \{x_1\}$, $Y = \{y_1\}$, $S = \{s_1\}$, $T = \{t_1\}$ and $Z = \bigcup_{i=1}^{n}\{z_i\}$. To show that $\text{cr}(K_{1,1,1,n}) \leq Z(4, n) + n$, see Figure 4 for $n = 4$ and it can be generalized to $n$.

![Figure 4](image)
Therefore it suffices to show that

\[ cr(K_{1,1,1,1,n}) \geq Z(4,n) + n, \tag{4} \]

Note that \( K_{1,1,1,1,1} \) is isomorphic to the complete graph of order 5, \( K_5 \). Therefore, \( cr(K_{1,1,1,1,1}) = cr(K_5) = 1 \), which shows that (4) is true for \( n = 1 \).

Now suppose \( n \geq 1 \). If \( cr(K_{1,1,1,1,1,n}) < Z(4,n) + n \), then there exists a good immersion \( \phi \) of \( K_{1,1,1,1,1,n} \) such that

\[ cr_\phi(E) < Z(4,n) + n. \tag{5} \]

Let \( W = E_{XY} \cup E_{XS} \cup E_{XT} \cup E_{YS} \cup E_{YT} \cup E_{ST} \). Then, by (1) and (2), we have

\[ cr_\phi(E) = cr_\phi(W) + cr_\phi(\bigcup_{i=1}^{n} E(z_i)) + cr_\phi(W, \bigcup_{i=1}^{n} E(z_i)) \tag{6} \]

This gives \( cr_\phi(W, E(z_i)) = 0 \) for some \( i \). By reordering of \( i \), we may suppose

\[ cr_\phi(W, E(z_1)) = 0. \tag{8} \]

Then \( \phi(W) \) divides \( \mathbb{R}^2 \) into regions and the condition (8) implies that \( \phi(X \cup Y \cup S \cup T) \) is contained in the boundary of one of the regions. Denote \( F = W \cup E(z_1) \). Then, without loss of generality, we may assume that \( \phi(F) \) has the drawing as in Figure 5.

![Figure 5](image_url)

For \( i \geq 2 \), no matter which regions \( \phi(z_i) \) is drawn,

\[ cr_\phi(F, E(z_i)) \geq 2. \tag{9} \]

Also, by (11) and (2), we have

\[ cr_\phi(E) = cr_\phi(F) + cr_\phi(\bigcup_{i=2}^{n} E(z_i)) + \sum_{i=2}^{n} cr_\phi(F, E(z_i)). \tag{10} \]
We will prove (12) by induction on $n$. For $n = 1$, $K_{1,2,2,1}$ contains $K_{3,3}$ and it is clear that $K_{3,3}$ is nonplanar, therefore $cr(K_{1,2,2,1}) \geq 1$. Therefore (12) is true for $n = 1$. For $n = 2$, consider a good immersion $\phi$ of $K_{1,2,2,2}$. Note that $\langle E_{XY} \cup E_{XU} \cup E_{YU} \rangle \cong K_{1,2,2}$, by Lemma 2.1 any drawings of $K_{1,2,2}$ has at most one region which contains at least 4 vertices. We will consider three cases:

**Case A.** If all the regions of the drawing $\phi(\langle E_{XY} \cup E_{XU} \cup E_{YU} \rangle)$ whose boundary
contains less than 4 vertices in \(\phi(X \cup Y \cup U)\), then \(cr_\phi(E(z_i)), E_{XY} \cup E_{XU} \cup E_{YU}) \geq 2\) for \(i = 1, 2\) which implies that \(cr_\phi(E) \geq 4\). Therefore \((12)\) is true for this case.

**Case B.** Suppose that the drawing \(\phi((E_{XY} \cup E_{XU} \cup E_{YU}))\) contains a unique region, \(f\), whose boundary contains exactly 4 vertices in \(\phi(X \cup Y \cup U)\). If there exists \(z_i\), say \(z_1\), such that \(\phi(z_1)\) is not contained in \(f\), then \(cr_\phi(E(z_1), E_{XY} \cup E_{XU} \cup E_{YU}) \geq 2\). Note also that \(cr_\phi(E(z_2), E_{XY} \cup E_{XU} \cup E_{YU}) \geq 1\) since all drawings of \(K_{1,2,2}\) have only one region which contains at least 4 vertices in \(\phi(V)\). This gives \(cr_\phi(E) \geq 3\). Now suppose that all \(z_i\) whose image under \(\phi\) are contained in \(f\).

If \(cr_\phi(E(z_i), E_{XY} \cup E_{XU} \cup E_{YU}) > 1\) for some \(i\), then the proof is the same as before. If \(cr_\phi(E(z_i), E_{XY} \cup E_{XU} \cup E_{YU}) = 1\) for \(i = 1, 2\), then both \(\phi(z_1)\) and \(\phi(z_2)\) must lies in \(f\). Thus \(cr_\phi(E(z_1), E(z_2)) \geq 1\) (see Figure 7). This gives \(cr_\phi(E) \geq 3\). Therefore \((12)\) is true for this case.

![Figure 7](image1)

![Figure 8](image2)

**Case C.** If the drawing \(\phi((E_{XY} \cup E_{XU} \cup E_{YU}))\) contains a unique region, \(f\), which contains more than 4 vertices, then, by Lemma 2.2 the only possible drawings of \(\phi((E_{XY} \cup E_{XU} \cup E_{YU}))\) are shown in Figure 3. If \(\phi((E_{XY} \cup E_{XU} \cup E_{YU})) = D_1\) or \(D_3\), then \(cr_\phi(E) \geq 3\). If \(\phi((E_{XY} \cup E_{XU} \cup E_{YU})) = D_2\), we may assume that there exists \(z_i\), say \(z_1\), such that \(\phi(z_1)\) lies in \(f\), that is the unbounded region of \(D_2\) and \(cr_\phi(E(z_1), E_{XY} \cup E_{XU} \cup E_{YU}) = 0\) (otherwise, \(cr_\phi(E) \geq 3\)). Then \(\phi((E(z_1) \cup E_{XY} \cup E_{XU} \cup E_{YU}))\) must be drawn as in Figure 8.

However, no matter which region \(z_2\) is placed, we have \(cr_\phi(E(z_2)), E(z_1) \cup E_{XY} \cup E_{XU} \cup E_{YU}) \geq 1\) which implies that \(cr_\phi(E) \geq 3\). Therefore \((12)\) is true for this case.

This shows that \((12)\) is true for \(n = 2\). Now consider \(n \geq 3\). Suppose \((12)\) is true for all value less than \(n\) and is not true for \(n\). Then there exists a good immersion \(\phi\) of \(K_{1,2,2,n}\) such that

\[
(13)\quad cr_\phi(E) < Z(5, n) + \left[ \frac{3n}{2} \right].
\]

Let \(W = E_{XY} \cup E_{XU} \cup E_{YU}\).

Note that \((6)\) is also true for \(\phi\). Since \(\bigcup_{i=1}^{n} E(z_i) \cong K_{5,n}\), by \((3)\), we have

\[
(14)\quad cr_\phi\left(\bigcup_{i=1}^{n} E(z_i)\right) \geq Z(5, n).
\]

If \(cr_\phi(W, E(z_i)) \geq 2\) for all \(i\), by \((6)\) and \((14)\), we have \(cr_\phi(E) \geq Z(5, n) + 2n\) which contradict to \((13)\). Therefore \(cr_\phi(W, E(z_i)) \leq 1\) for some \(i\). We will consider two cases:
Case 1. If \( cr_\phi(W, E(z_i)) = 0 \) for some \( i \).

By reordering, we may assume that \( cr_\phi(W, E(z_1)) = 0 \). \( \phi((W)) \) divides \( \mathbb{R}^2 \) into regions and the condition \( cr_\phi(W, E(z_1)) = 0 \) implies that \( \phi(X \cup Y \cup U) \) is contained in the boundary of one of the regions. Denote \( F = W \cup E(z_1) \). Then Figure 9 has shown all the possible drawings of \( \phi|_F \).

![Figure 9](image)

For each drawing \( D_i \) (1 \( \leq \) i \( \leq \) 3), if \( \phi(z_j) \) (2 \( \leq \) j \( \leq \) n) is located in the region I to V, we have

\[
\text{(15)} \quad cr_\phi(F, E(z_j)) \geq 4.
\]

If \( \phi(z_j) \) (2 \( \leq \) j \( \leq \) n) is located in the region other than I to V, we have

\[
\text{(16)} \quad cr_\phi(W, E(z_j)) \geq 3.
\]

Let \( l \) be the number of \( \phi(z_i) \) (2 \( \leq \) i \( \leq \) n) such that it is not located in the region I to V. Note that \( cr_\phi(W) \geq 2 \) and combining (9), (13), (14) and (16), we have

\[
\text{(17)} \quad l \leq \lfloor \frac{n-2}{2} \rfloor.
\]

Note that the number of \( \phi(z_i) \) (2 \( \leq \) i \( \leq \) n) such that it is located in the region I to V is \( n - l - 1 \). Note that (11) is also true for \( \phi \). Note also that \( \bigcup_{i=2}^{n} E(z_i) \cong K_{5,n-1} \), by (3), we have

\[
\text{(18)} \quad cr_\phi\left(\bigcup_{i=2}^{n} E(z_i)\right) \geq Z(5, n-1).
\]

Since \( cr_\phi(F) \geq 2 \) in \( D_i \) (1 \( \leq \) i \( \leq \) 3), by (10), (15), (16), (17) and (18), we have

\[
\text{cr}_\phi(E) \geq 2 + Z(5, n-1) + 4(n-1-1)+3l \geq Z(5, n)+\frac{3n}{2} \] which contradicts to (13).

Case 2. If \( cr_\phi(W, E(z_i)) \geq 1 \) for all \( i \).

Since \( cr_\phi(W, E(z_i)) \leq 1 \) for some \( i \), \( cr_\phi(W, E(z_i)) = 1 \) for some \( i \). Now by reordering, we may assume that \( cr_\phi(W, E(z_1)) = 1 \). Then there is at least one region in \( \phi((W)) \) containing at least 4 vertices in \( \phi(X \cup Y \cup U) \). Then, by Lemma 2.1, \( \phi((W)) \) has a unique region whose boundary contains at least 4 vertices. We denote the unique region by \( f \) and its boundary by \( \partial f \).
Suppose $\partial f$ contains more than 4 vertices in $\phi(X \cup Y \cup U)$. Then by Lemma 2.2 there are 3 possible drawings of $\phi(W)$, which are shown in Figure 3 and $f$ is the unbounded region for each $D_i$. Let $F = W \cup E(z_1)$. Under the condition $cr_\phi(W, E(z_1)) = 1$, $\phi(z_1)$ must be drawn in $f$. However, if $\phi(z_1)$ is drawn in $f$, $cr_\phi(W, E(z_1)) \neq 1$.

Therefore, $\partial f$ must contain exactly 4 vertices in $\phi(X \cup Y \cup U)$. For the $z_i$ such that $cr_\phi(W, E(z_i)) = 1$, $\phi(z_i)$ must lie in $f$. Suppose the vertices in $\phi(X \cup Y \cup U)$ contained in $\partial f_i$, in the clockwise manner, are $a_1, a_2, a_3, a_4$ and $a_5$ is the vertex of $\phi(X \cup Y \cup U)$ which is not contained in $f$. For $cr_\phi(W, E(z_i)) = 1$, $\phi(z_i)$ is drawn in $f$. Since $\phi(z_ia_5)$ cross $\partial f$, $\phi(z_ia_5)$, $1 \leq j \leq 4$, does not cross. See Figure 10 for the case $\phi(z_ia_5)$ cross the boundary of $f$ between $a_1$ and $a_2$. Note that the boundary of $f$ between $a_j$ and $a_{j+1}$ (1 ≤ $j$ ≤ 4 and mod 4 for $j+1$) may not be the image of a single edge of $W$, but it must be composed of some portions of the image of the edges of $W$ under $\phi$.

![Figure 10](image1.png)  
![Figure 11](image2.png)

Let $l$ be the number of $z_i$ such that $cr_\phi(W, E(z_i)) = 1$. Also, let $l_j$ (1 ≤ $j$ ≤ 4) be the number of $z_i$ such that $cr_\phi(W, E(z_i)) = 1$ and $\phi(z_ia_5)$ crosses the boundary of $f$ between $a_j$ and $a_{j+1}$ (mod 4 for $j+1$) where 1 ≤ $j$ ≤ 4. Then we have $\sum_{j=1}^{4} l_j = l$.

We may assume $l_1 + l_3 \geq l_2 + l_4$. Also we may assume $l_1 \geq l_3$.

Now by reordering, we may assume that $cr_\phi(W, E(z_1)) = 1$ such that $\phi(z_1a_5)$ crosses the boundary of $f$ between $a_1$ and $a_2$. Let $F = W \cup E(z_1)$, then $\phi(F)$ must be drawn as in Figure 10. For $z_i \neq z_1$ such that $cr_\phi(W, E(z_i)) = 1$, if $\phi(z_ia_5)$ crosses the boundary between $a_1$ and $a_2$, then $cr_\phi(F, E(z_i)) \geq 5$; if $\phi(z_ia_5)$ crosses the boundary between $a_2$ and $a_3$ or the boundary between $a_4$ and $a_1$, we have $cr_\phi(F, E(z_i)) \geq 4$; if $\phi(z_ia_5)$ crosses the boundary between $a_3$ and $a_4$ (see Figure 11), we have $cr_\phi(F, E(z_i)) \geq 3$.

Since $l$ is the number of $z_i$ where 1 ≤ $i$ ≤ $n$ such that $cr_\phi(W, E(z_i)) = 1$, the number of $z_i$ where 1 ≤ $i$ ≤ $n$ such that $cr_\phi(W, E(z_i)) \geq 2$ is $n - l$. From 6, 13 and 14, we have

\begin{equation}
(19) \quad l \geq \left\lceil \frac{n}{2} \right\rceil + 1.
\end{equation}

We will consider two subcases:

**Case A.** Suppose for all $z_i$ (2 ≤ $i$ ≤ $n$) such that $cr_\phi(W, E(z_i)) = 2$ we have

\begin{equation}
(20) \quad cr_\phi(F, E(z_i)) \geq 3.
\end{equation}
Note that (10) is also true for $\phi$. Then by (10), (18), (20), the fact that $cr_{\phi}(F) \geq 1$ and our previous discussion, we have

$$cr_{\phi}(E) \geq 1 + Z(5, n - 1) + \sum_{cr_{\phi}(W, E(z_i)) = 1}^{1} cr_{\phi}(F, E(z_i)) + \sum_{cr_{\phi}(W, E(z_i)) \geq 2}^{4} cr_{\phi}(F, E(z_i)) \geq 1 + Z(5, n - 1) + 3(n - l) + 5(l_1 - 1) + 4(l_2 + l_4) + 3l_3 \geq Z(5, n - 1) + 3n + l - 4 \geq Z(5, n) + \left\lfloor \frac{3n}{2} \right\rfloor$$

where the third inequality follows from the fact that $l_1 \geq l_3$ and $\sum_{j=1}^{4} l_j = l$ and the last inequality follows from (19). However, $cr_{\phi}(E) \geq Z(5, n) + \left\lfloor \frac{3n}{2} \right\rfloor$ contradicts to (18).

**Case B.** Suppose there exists $z_i (2 \leq i \leq n)$ such that $cr_{\phi}(F, E(z_i)) = cr_{\phi}(W, E(z_i)) = 2$. We may assume $cr_{\phi}(F, E(z_2)) = cr_{\phi}(W, E(z_2)) = 2$ which implies that $$cr_{\phi}(E(z_1), E(z_2)) = 0.$$ For the vertices $z_1, z_2, z_k \in Z$, $\langle E(z_i) \cup E(z_j) \cup E(z_k) \rangle$ is homeomorphic to $K_{5,3}$. Hence by (11) and (21), we have

$$cr_{\phi}(E(z_i) \cup E(z_j), E(z_k)) + cr_{\phi}(E(z_i), E(z_j)) \geq 4.$$ Therefore, by using (21) and (22), we have

$$cr_{\phi}(E(z_1) \cup E(z_2), E(z_k)) \geq 4.$$ Let $E' = E - (E(z_1) \cup E(z_2))$. Then $\langle E' \rangle = K_{1,2,2,n-2}$ and

$$cr_{\phi}(E) = cr_{\phi}(E') + cr_{\phi}(E(z_1) \cup E(z_2)) + \sum_{i=3}^{n} cr_{\phi}(E(z_1) \cup E(z_2), E(z_i)).$$

On the other hand, if $cr_{\phi}(E(z_1), E(z_2)) = 0$, we have

$$cr_{\phi}(W, E(z_1) \cup E(z_2)) \geq 3.$$ To see this, note that under the condition $cr_{\phi}(E(z_1), E(z_2)) = 0$, $\phi(\langle E(z_1) \cup E(z_2) \rangle)$ must be drawn as in Figure 12 where ▲ is the vertex in $X$, ● are the vertices in $Y$ and ○ are the vertices in $U$.

![Figure 12](image)

Naming the vertices of $\phi(\langle E(z_1) \cup E(z_2) \rangle)$ in Figure 12 from left to right $a_1, a_2, \ldots, a_5$. In $F_1, a_1a_3, a_1a_4, a_2a_4, a_3a_5$ have to cross with the edges in $W$ at least once.
Therefore, \( cr(\phi(W, E(z_1) \cup E(z_2))) \geq 4 \). In \( F_2 \), \( a_1a_3, a_1a_4, a_2a_5 \) have to cross with the edges in \( W \) at least one. Therefore, \( cr(\phi(W, E(z_1) \cup E(z_2))) \geq 3 \). In \( F_3 \), \( a_1a_3, a_1a_4, a_2a_4, a_2a_5, a_3a_5 \) have to cross with the edges in \( W \) at least once. Therefore, \( cr(\phi(W, E(z_1) \cup E(z_2))) \geq 3 \). This proves (25).

Then, by the induction assumption and (23), (24), (25), \( cr(\phi(E)) \geq Z(5, n - 2) + \left\lceil \frac{3(n - 2)}{2} \right\rceil + 4(n - 2) \geq Z(5, n) + \left\lceil \frac{3n}{2} \right\rceil \), which contradicts to (13).

\[ \square \]

5. Crossing number of \( K_{1,1,1,2,n} \)

In this section, the crossing number of \( K_{1,1,1,2,n} \) is found. More precisely, we have

**Theorem 5.1.** The crossing number of the complete 5-partite graph \( K_{1,1,1,2,n} \) is given by

\[ cr(K_{1,1,1,2,n}) = Z(5, n) + 2n. \]

**Proof.** Let \( (X, Y, S, T, Z) \) be partition of \( K_{1,1,1,2,n} \) such that \( X = \{x_1\}, Y = \{y_1\}, S = \{s_1\}, T = \{t_1, t_2\} \) and \( Z = \bigcup_{i=1}^{n}\{z_i\} \). To show that \( cr(K_{1,1,1,2,n}) \leq Z(5, n) + 2n \), see Figure 13 for \( n = 4 \), and it can be generalized to all \( n \).

![Figure 13](image-url)
containing at least 4 vertices on its boundary, which is impossible. Therefore, \( \text{[20]} \) is true for \( n = 1 \).

![Figure 14.](image)
![Figure 15.](image)
![Figure 16.](image)
![Figure 17.](image)

For \( n = 2 \), suppose that \( \text{[20]} \) is not true, that is, \( cr(K_{1,1,1,2,2}) < 4 \). Then there exists a good immersion \( \phi \) of \( K_{1,1,1,2,2} \) such that \( cr_{\phi}(E) < 4 \). Let \( Q = E_{ST} \cup E_{SZ} \cup E_{TZ} \). Since \( (Q) \cong K_{1,2,2} \), by Lemma \( \text{[2.1]} \) \( \phi((Q)) \) has at most one region whose boundary contains at least 4 vertices in \( \phi(S \cup T \cup Z) \). In \( \phi((Q)) \), if all the regions whose boundary contains less than 4 vertices in \( \phi(S \cup T \cup Z) \), then we have \( cr_{\phi}(E(x_1), Q) \geq 2 \) and \( cr_{\phi}(E(y_1), Q) \geq 2 \) which implies that \( cr_{\phi}(E) \geq 4 \).

Therefore \( \phi((Q)) \) has a unique region whose boundary contains at least 4 vertices in \( \phi(S \cup T \cup Z) \). Denote the unique face by \( f \) and it boundary by \( \partial f \). We will consider three cases:

**Case A.** If \( cr_{\phi}(Q) = 0 \), since \( K_{1,2,2} \) is 3-connected, it has an unique embedding in the plane, namely, the drawing in Figure 15.

If \( \phi(x_1) \) does not lie in \( f \), then \( cr_{\phi}(E(x_1), Q) \geq 2 \). If \( \phi(x_1) \) lie in \( f \), then \( cr_{\phi}(E(x_1), Q) \geq 1 \). Since \( cr_{\phi}(E) < 4 \), \( cr_{\phi}(E(x_1), Q) = 1 \) or \( cr_{\phi}(E(y_1), Q) = 1 \). Without loss of generality, we may assume that \( cr_{\phi}(E(x_1), Q) = 1 \) which implies \( \phi(x_1) \) lies in \( f \). Denote \( K = E(x_1) \cup Q \), then the only possible drawing of \( \phi((K)) \) is shown in Figure 16. However, no matter which region \( \phi(y_1) \) is contained in will result in \( cr_{\phi}(E) \geq 4 \).

**Case B.** Suppose that \( cr_{\phi}(Q) \geq 1 \) and \( \partial f \) contains more than 4 vertices in \( \phi(S \cup T \cup Z) \). By Lemma \( \text{[2.2]} \) the possible drawings of \( \phi((Q)) \) are shown in Figure 3. Note that \( f \) is just the unbounded region.

If \( \phi(x_1) \) does not lie in \( f \), then \( cr_{\phi}(E(x_1), Q) \geq 3 \). Since \( cr_{\phi}(E) < 4 \), \( cr_{\phi}(E(x_1), Q) \leq 1 \) or \( cr_{\phi}(E(y_1), Q) \leq 1 \). Without loss of generality, we may assume that \( cr_{\phi}(E(x_1), Q) \leq 1 \) which implies \( \phi(x_1) \) lies in \( f \). If \( \phi(x_1) \) lies in \( f \), under the condition \( cr_{\phi}(E(x_1), Q) \leq 1 \), we have \( cr_{\phi}(E(x_1), Q) = 0 \). Denote \( K = E(x_1) \cup Q \), then the possible drawings of \( \phi((K)) \) are shown in Figure 9. However, no matter which region \( \phi(y_1) \) lies will result in \( cr_{\phi}(E) \geq 4 \), contradicted to \( cr_{\phi}(E) < 4 \).

**Case C.** Suppose that \( cr_{\phi}(Q) \geq 1 \) and \( \partial f \) contains exactly 4 vertices in \( \phi(S \cup T \cup Z) \). Since \( cr_{\phi}(E) \leq 3 \), then \( cr_{\phi}(E(x_1), Q) \leq 1 \) or \( cr_{\phi}(E(y_1), Q) \leq 1 \) which implies that \( \phi(x_1) \) or \( \phi(y_1) \) lies in \( f \). We may assume that \( \phi(x_1) \) lies in \( f \). Note that if \( \phi(x_1) \) lies in \( f \), then \( cr_{\phi}(E(x_1), Q) \geq 1 \). Therefore \( cr_{\phi}(E(x_1), Q) = 1 \). On the other hand, if \( \phi(y_1) \) does not lie in \( f \), then \( cr_{\phi}(E(y_1), Q) \geq 2 \) which implies \( cr_{\phi}(E) \geq 4 \). Thus, both \( \phi(x_1) \) and \( \phi(y_1) \) lie in \( f \).

Suppose the vertices in \( \phi(S \cup T \cup Z) \) contained in \( \partial f \), in the clockwise manner, are \( a_1, a_2, a_3, a_4 \) and \( a_5 \) is the vertex of \( \phi(S \cup T \cup Z) \) which is not contained in \( f \). Since \( \phi(x_1 a_5) \) must cross \( \partial f \), under the condition \( cr_{\phi}(E(x_1), Q) = 1 \), \( \phi(x_1 a_j) \) where \( 1 \leq j \leq 4 \), does not cross the edges in \( Q \). See Figure 17 for the case \( \phi(x_1 a_5) \) cross the boundary of \( f \) between \( a_1 \) and \( a_2 \). Note that the boundary of \( f \) between \( a_j \) and \( a_{j+1} \) (\( 1 \leq j \leq 4 \) and mod 4 for \( j + 1 \)) may not be the image of a single
edge of $Q$, but it must be composed of some portions of the image of the edges of $Q$ under $\phi$.

Now by reordering, we may suppose $\phi(x_1a_5)$ cross the boundary of $f$ between $a_1$ and $a_2$. However, if $\phi(y_1)$ lies in $f$, $cr_\phi(E(x_1) \cup Q, E(y_1)) \geq 2$. This gives $cr_\phi(E) \geq 4$, contradicted to $cr_\phi(E) < 4$.

This shows that (26) is true for $n = 2$. Now we can assume that $n \geq 3$. Suppose (26) is true for all value less than $n$ and is not true for $n$. Then there exists a good immersion $\phi$ of $K_{1,1,1,2,n}$ such that

$$cr_\phi(E) < Z(5, n) + 2n.$$  \hspace{1cm} (27)

Let $W = E_{XY} \cup E_{XS} \cup E_{XT} \cup E_{YS} \cup E_{YT} \cup E_{ST}$. Note that (6) and (14) is also true for $\phi$.

If $cr_\phi(W, E(z_i)) \geq 2$ for all $i$, by (6) and (14), we have $cr_\phi(E) \geq Z(5, n) + 2n$ which contradict to (27). Therefore $cr_\phi(W, E(z_i)) \leq 1$ for some $i$. We will consider the following two cases:

Case 1.: $cr_\phi(W, E(z_i)) = 0$ for some $i$;
Case 2.: $cr_\phi(W, E(z_i)) \geq 1$ for all $i$.

Case 1. By reordering, we may assume that $cr_\phi(W, E(z_1)) = 0$. $\phi(W)$ divides $\mathbb{R}^2$ into regions and the condition $cr_\phi(W, E(z_1)) = 0$ implies that $\phi(X \cup Y \cup S \cup T)$ is contained in the boundary of one of the regions. Figure 18 shows all the possible drawings of $\phi(W)$.

![Figure 18](image)

Denote $F = W \cup E(z_1)$. Then Figure 19 has shown all the possible drawings of $F$. Therefore, for each drawing $D_i$ ($1 \leq i \leq 2$) of $F$, no matter which regions of $z_j$ ($2 \leq j \leq n$) is located, we have

$$cr_\phi(F, E(z_j)) \geq 4.$$  \hspace{1cm} (28)

![Figure 19](image)

Note that (10) and (18) are true for $\phi$. Since $cr_\phi(F) \geq 3$ in $D_i$ ($1 \leq i \leq 3$), by (10), (18), (28), we have $cr_\phi(E) \geq 3 + Z(5, n - 1) + 4(n - 1) \geq Z(5, n) + 2n$ which
contradicts to (27).

**Case 2.** Since $cr_\phi(W, E(z_i)) \leq 1$ for some $i$, we have $cr_\phi(W, E(z_i)) = 1$ for some $i$. We may assume that

\begin{equation}
(29) \quad cr_\phi(W, E(z_1)) = 1.
\end{equation}

Therefore there must be at least 4 vertices of $\phi(X \cup Y \cup S \cup T)$ is contained in the boundary of one of the regions in $\phi(W)$. Note that $K_{1,1,1,2}$ can be obtained by deleting an edge in $K_{1,1,1,1} \cong K_5$. Therefore, the drawings of $\phi(W)$ can be obtained by deleting an edge in a drawing of $K_5$ such that one of the regions contains at least 3 vertices. One can check that the only possible drawings of $K_5$ such that one of the regions contains at least 3 vertices are shown in Figure 20.

![Figure 20](image)

To obtain the possible drawing of $\phi(W)$, we have to choose an edge and delete it such that the resulting the drawing has a region containing at least 4 vertices. One can check that the only possible drawings of $\phi(W)$ are shown in Figure 18 and 21.

![Figure 21](image)

Note that it is impossible for $\phi(W)$ drawn as in Figure 18, that is, it is impossible for $\phi(W) = D_1, D_2$. If $\phi(W) = D_1$ or $D_2$, it is impossible for $cr_\phi(W, E(z_1)) = 1$ for any good drawing. Therefore $\phi(W)$ must be drawn as in Figure 21. Denote $F = W \cup E(z_1)$. Then by (29), $\phi(F)$ must be drawn as in Figure 22.

If $\phi(F) = F_1, F_5, F_6$ or $F_7$, then we must have $cr_\phi(F, E(z_j)) \geq 4$. Then following the same proof in Case 1, we can obtain a contradiction.

Therefore we can assume that $\phi(F) = F_2, F_3$ or $F_4$. Then if $z_j$ ($2 \leq j \leq n$) is located in the region which does not mark with *, we have

\begin{equation}
(30) \quad cr_\phi(F, E(z_j)) \geq 4.
\end{equation}
If $z_j$ ($2 \leq j \leq n$) is located in the region marked with $\ast$, we have

$$cr_\varphi(F, E(z_j)) \geq 3.$$  \hspace{1cm} (31)

We claim that it is impossible for $cr_\varphi(F, E(z_j)) = 3$. Suppose not, we may assume that $z_2$ is located in the region marked with $\ast$ such that $cr_\varphi(F, E(z_2)) = 3$. Then we must have

$$cr_\varphi(W, E(z_2)) = 3$$

and $cr_\varphi(E(z_1), E(z_2)) = 0$. Hence, from (3), (1), (2), we have

$$cr_\varphi(E(z_1) \cup E(z_2), E(z_k)) + cr_\varphi(E(z_1), E(z_2)) \geq 4.$$  \hspace{1cm} (33)

Then by (32) and (33), we have

$$cr_\varphi(E(z_1) \cup E(z_2), E(z_k)) \geq 4 \quad \text{for} \quad 3 \leq k \leq n.$$  \hspace{1cm} (34)

Let $E' = E - (E(z_1) \cup E(z_2))$. Then by (11), (22),

$$cr_\varphi(E) = cr_\varphi(E') + cr_\varphi(E(z_1) \cup E(z_2)) + cr_\varphi(W, E(z_1))$$

$$+ cr_\varphi(W, E(z_2)) + \sum_{k=3}^{n} cr_\varphi(E(z_1) \cup E(z_2), E(z_k)).$$  \hspace{1cm} (35)
Then by the fact that \( \langle E' \rangle \cong K_{1,1,1,2,n-2} \) and induction assumption, (29), (32), (34), (35), we have \( cr_\phi(E) \geq Z(5,n-2) + 2(n-2) + 1 + 3 + 4(n-2) \geq Z(5,n) + 2n \) which contradicts (27). This proves our claim.

By the claim and (31), we know that if \( z_j \ (2 \leq j \leq n) \) is located in the region marked with *, we have
\[
(36) \quad cr_\phi(F,E(z_j)) \geq 4.
\]
Then by (30) and (36), we know that no matter which region \( z_j \) is located, we have \( cr_\phi(F,E(z_j)) \geq 4 \). Then following the same proof in Case 1, we can obtain a contradiction. \( \square \)

6. Crossing number of \( K_{1,4,n} \)

In this section, we will prove

**Theorem 6.1.** The crossing number of the complete 3-partite graph \( K_{1,4,n} \) is given by
\[
cr(K_{1,4,n}) = Z(5,n) + 2\lceil \frac{n}{2} \rceil.
\]

**Proof.** Let \( (X,Y,Z) \) be partition of \( K_{1,4,n} \) such that \( X = \{x_1\}, Y = \bigcup_{i=1}^{4} \{y_i\} \) and \( Z = \bigcup_{i=1}^{n} \{z_i\} \). To show that \( cr(K_{1,4,n}) \leq Z(5,n) + 2\lceil \frac{n}{2} \rceil \), see Figure 23 for \( n = 4 \), and it can be easily generalized to \( n \).

Therefore it suffices to show that
\[
(37) \quad cr(K_{1,4,n}) \geq Z(5,n) + 2\lceil \frac{n}{2} \rceil.
\]

We will prove (37) by induction on \( n \). It is clear that (37) is true for \( n = 1 \). For \( n = 2 \), since \( K_{1,4,2} \) contains \( K_{3,4} \), \( cr(K_{1,4,2}) \geq cr(K_{3,4}) = 2 \) by (3). Therefore, (37) is true for \( n = 2 \). Now we can assume that \( n \geq 3 \). Suppose (37) is true for all value
less than \( n \) and is not true for \( n \). Thus there exists a good immersion \( \phi \) of \( K_{1,4,n} \) such that
\[
(38) \quad cr_{\phi}(E) < Z(5, n) + 2\left\lfloor \frac{n}{2} \right\rfloor.
\]

Note that the drawing of \( K_{1,4,n}, E \), includes \( n \) drawings of \( K_{1,4,n-1} \), each obtained by suppressing one vertex \( z_i \) in \( Z \). Each crossing between \( u_i, z_j \) and \( u_k z_l \) where \( u_i, u_k \in X \cup Y \) and \( z_j, z_l \in Z \) will occur in \( n-2 \) of these drawings, namely, those in which neither \( z_i \) nor \( z_l \) is suppressed. Also, each crossing between \( u_i, u_j, u_k \in X \cup Y \) and \( z_l \in Z \) will occur in \( n-1 \) of these drawings, namely, those in which \( z_l \) is not suppressed. Therefore, by the fact that \( cr_{\phi}(E_XY) = 0 \), it follows that
\[
(39) \quad (n-1)cr_{\phi}(E_XY, \bigcup_{i=1}^{n} E(z_i)) + (n-2)cr_{\phi}(\bigcup_{i=1}^{n} E(z_i)) \geq n \ cr_{\phi}(K_{1,4,n-1})
\]

Then by induction assumption and (39), we have
\[
(40) \quad cr_{\phi}(E) \geq Z(5, n-1) + 2\left\lfloor \frac{n-1}{2} \right\rfloor \ - \ \sum_{i=1}^{n} \frac{cr_{\phi}(E_XY, E(z_i))}{n-2}.
\]

By (1) and (2), we have
\[
(41) \quad cr_{\phi}(E) = cr_{\phi}(E_XY) + cr_{\phi}(\bigcup_{i=1}^{n} E(z_i)) + \sum_{i=1}^{n} cr_{\phi}(E_XY, E(z_i)).
\]

Since \( \bigcup_{i=1}^{n} E(z_i) \cong K_{5,n} \), by (3), we have
\[
(42) \quad cr_{\phi}(\bigcup_{i=1}^{n} E(z_i)) \geq Z(5, n).
\]

Therefore, by (38), (41) and (42), we have
\[
(43) \quad \sum_{i=1}^{n} cr_{\phi}(E_XY, E(z_i)) \leq 2\left\lfloor \frac{n}{2} \right\rfloor - 1.
\]

Combining (40), (43) and the fact that \( cr_{\phi}(E) \) is an integer, we have
\[
(44) \quad cr_{\phi}(E) \geq Z(5, n) + 2\left\lfloor \frac{n}{2} \right\rfloor - 1.
\]

By (38), (44) and the fact that \( cr_{\phi}(E) \) is an integer, we have
\[
(45) \quad cr_{\phi}(E) = cr_{\phi}(\bigcup_{i=1}^{n} E(z_i)) + \sum_{i=1}^{n} cr_{\phi}(E_XY, E(z_i)) = Z(5, n) + 2\left\lfloor \frac{n}{2} \right\rfloor - 1.
\]

We may assume that \( \phi((E_XY)) \) is drawn as in Figure 24.
For 1 \leq j \leq 4, let A_j be the set of \( z_i, 1 \leq i \leq n \), such that \( x_1z_i \) lies between the edges \( y_j \) and \( y_{j+1} \) (mod 4 for \( j + 1 \)). See Figure 25 for \( z_i \in A_4 \).

For \( n \) is even, we may assume that \( |A_1| \leq |A_3| \). For \( n \) is odd, we have \( |A_1| \neq |A_3| \) or \( |A_2| \neq |A_4| \). (Otherwise, \( n \) is even). Then we may assume \( |A_1| < |A_3| \). Therefore, we have

\[
|A_1| \leq |A_3| - \left[ \frac{n}{2} \right] + \left[ \frac{n}{2} \right]
\]

We are going to obtain a drawing of \( K_{5,n+1} \) from \( \phi(E) \) and then obtain a contradiction. To obtain a drawing of \( K_{5,n+1} \) from \( \phi(E) \), we draw a new vertex, denoted it by \( z_{n+1} \), near the vertex \( x_1 \) and lying in the region between the edges \( x_1y_1 \) and \( x_1y_2 \), as shown in Figure 26.

For 1 \leq j \leq 4, draw the edge \( z_{n+1}y_j \) next to the edge \( x_1y_j \) and draw the edge \( z_{n+1}x_1 \) without crossing any edges in \( E \). Then remove the edges \( x_1y_j \) from \( E \) where 1 \leq j \leq 4. See Figure 27.

Now we have a drawing of \( K_{5,n+1} \) with \( \{x_1, y_1, y_2, y_3, y_4\} \) as the partition with 5 vertices and \( \bigcup_{i=1}^{n+1}\{z_i\} \) as the partition with \( n + 1 \) vertices, denote the immersion of \( K_{5,n+1} \) by \( \phi' \).

Note that if \( z_i \in A_1 \) (see Figure 28), then

\[
\text{cr}_{\phi'}(E(z_i), E(z_{n+1})) = \text{cr}_{\phi}(E(z_i), E_{XY}) + 2.
\]

If \( z_i \in A_2 \cup A_4 \) (see Figure 29), then

\[
\text{cr}_{\phi'}(E(z_i), E(z_{n+1})) = \text{cr}_{\phi}(E(z_i), E_{XY}) + 1.
\]

If \( z_i \in A_3 \) (see Figure 30), then

\[
\text{cr}_{\phi'}(E(z_i), E(z_{n+1})) = \text{cr}_{\phi}(E(z_i), E_{XY}).
\]

Note also that

\[
\text{cr}_{\phi'}(\bigcup_{i=1}^{n}E(z_i)) = \text{cr}_{\phi}(\bigcup_{i=1}^{n}E(z_i))
\]

Then by (11), (2), (47), (48), (19) and (50), we have the crossing number of \( \phi' \) is

\[
\text{cr}_{\phi}(\bigcup_{i=1}^{n}E(z_i)) + \sum_{i=1}^{n}\text{cr}_{\phi}(E_{XY}, E(z_i)) + 2|A_1| + |A_2| + |A_4|
\]

\[
\leq Z(5,n) + 2\left[ \frac{n}{2} \right] - 1 + |A_1| + |A_2| + |A_3| + |A_4| - \left[ \frac{n}{2} \right] + \left[ \frac{n}{2} \right]
\]

\[
= Z(5,n) + 2\left[ \frac{n}{2} \right] - 1 + n - \left[ \frac{n}{2} \right] + \left[ \frac{n}{2} \right],
\]
where the second inequality follows from (15), (49), and the last equality follows from the fact that \(|A_1| + |A_2| + |A_3| + |A_4| = n\). However, since \(Z(5, n) + 2 \lfloor \frac{n}{2} \rfloor - 1 + n - \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor < Z(5, n + 1)\), we obtained a drawing of \(K_{5,n+1}\) with crossing number less than \(Z(5, n + 1)\), which is contradicted to (7).

\[\square\]

7. Crossing number of \(K_{1,3,n}\)

In this section, we use the basic counting argument to determine the crossing number of \(K_{1,3,n}\) which is obtained by Asano in citeAsano.

**Theorem 7.1.** The crossing number of the complete 3-partite graph \(K_{1,3,n}\) is given by

\[cr(K_{1,3,n}) = Z(4, n) + \lfloor \frac{n}{2} \rfloor.\]

**Proof.** As proved in [1], one has \(cr(K_{1,3,n}) \leq Z(4, n) + \lfloor \frac{n}{2} \rfloor\). (Actually, to show this, one can try to draw \(K_{1,3,n}\) with such crossing numbers in a similar way in the previous sections.) Therefore it suffices to show that

\[cr(K_{1,3,n}) \geq Z(4, n) + \lfloor \frac{n}{2} \rfloor.\]

We will prove (51) by induction on \(n\). It is clear that (51) is true for \(n = 1\). For \(n = 2\), note that \(K_{1,3,2}\) contains \(K_{3,3}\), which gives \(cr(K_{1,3,2}) \geq cr(K_{3,3}) = 1\). Therefore (51) is true for \(n = 2\). Now we can assume that \(n \geq 3\). Suppose (51) is true for all value less than \(n\) and is not true for \(n\). Then there exists a good immersion \(\phi\) of \(K_{1,3,n}\) such that

\[cr(\phi(E)) < Z(4, n) + \lfloor \frac{n}{2} \rfloor.\]

Note that (5) and (7) are also true for \(\phi\). Therefore, (4), (7), (52) and the fact that \(cr(\phi(E_{XY})) = 0\) gives

\[\sum_{i=1}^{n} cr(\phi(E_{XY}, E(z_i))) \leq \lfloor \frac{n}{2} \rfloor - 1.\]

Now, we will apply the counting argument. Note that the drawing of \(K_{1,3,n}, E\), includes \(n\) drawings of \(K_{1,3,n-1}\), each obtained by suppressing one vertex \(z_i\) in \(Z\). Each crossing between \(u_i z_j\) and \(u_k z_l\) where \(u_i, u_k \in X \cup Y\) and \(z_j, z_l \in Z\) will occur in \(n - 2\) of these drawings, namely, those in which neither \(z_j\) nor \(z_l\) is suppressed. Also, each crossing between \(u_i u_j\) and \(u_k z_l\) where \(u_i, u_j, u_k \in X \cup Y\) and \(z_l \in Z\) will occur in \(n - 1\) of these drawings, namely, those in which \(z_l\) is not suppressed. Therefore, by the fact that \(cr(\phi(E_{XY})) = 0\), it follows that

\[(n - 1)cr(\phi(E_{XY}) \cup E(z_i)) + (n - 2)cr(\phi(\bigcup_{i=1}^{n} E(z_i))) \geq n \ cr(K_{1,3,n-1}).\]

Therefore, by induction assumption, (4), (54) and the face that \(cr(\phi(E_{XY})) = 0\), we have

\[cr(\phi(E)) \geq Z(4, n) + \lfloor \frac{n}{2} \rfloor - \sum_{i=1}^{n} \frac{cr(\phi(E_{XY}, E(z_i)))}{n - 2}.\]
Since \( n \geq 3 \) and \( cr_\phi(E) \) is an integer, combining (53) and (55), we have \( cr_\phi(E) \geq Z(4, n) + \lfloor \frac{n}{2} \rfloor \), which contradicts (52). \( \square \)

8. Conclusion

We conclude this paper by stating the following conjectures:

Conjecture 8.1.

\[
\begin{align*}
    cr(K_{1,1,3,n}) &= Z(5, n) + \lfloor \frac{3n}{2} \rfloor; \\
    cr(K_{2,4,n}) &= Z(6, n) + 2n.
\end{align*}
\]

One can easily check that the conjectural values are the upper bounds. By using similar methods in this paper, we can proved that the conjecture is true for many drawings of them.

Note added in the proof: Conjecture 8.1 was solved in [4] and [5] respectively.

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