NEURAL IDEALS IN SAGEMATH

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ABSTRACT. A major area in neuroscience research is the study of how the brain processes spatial information. Neurons in the brain represent external stimuli via neural codes. These codes often arise from stereotyped stimulus-response maps, associating to each neuron a convex receptive field. An important problem consists in determining what stimulus space features can be extracted directly from a neural code. The neural ideal is an algebraic object that encodes the full combinatorial data of a neural code. This ideal can be expressed in a canonical form that directly translates to a minimal description of the receptive field structure intrinsic to the code. In here, we describe a SageMath package that contains several algorithms related to the canonical form of a neural ideal.

1. Introduction

Due to many recent technological advances in neuroscience, researchers’ ability to collect neural data has increased dramatically. With this comes a need for new methods to process and understand this data. One major question faced by researchers is to determine how the brain encodes spatial features of its environment through patterns of neural activity, as with place cell codes [4]. In the recent paper [2], Curto et al. phrase this question as, “What can be inferred about the underlying stimulus space from neural activity alone?” To answer this question, Curto et al. introduced the neural ring and a related neural ideal, algebraic objects that encode the full combinatorial data of a neural code. They further show that the neural ideal can be expressed in a canonical form that directly translates to a minimal description of the receptive field structure intrinsic to the code.

In this article we describe a SageMath [5] package that implements several algorithms to compute the neural ideal and its canonical form, featuring a new iterative algorithm which proves to be much more efficient than the original canonical form algorithm outlined by Curto et al. in [2]. This package also provides an algorithm to compute the primary decomposition of a pseudo-monomial ideal. Accompanying these functions are others that calculate related objects, such as the Gröbner basis, Gröbner fan, universal Gröbner basis, and the neural ideal itself. In Section 2 we give a short introduction to the algebraic geometry of neural codes. Section 3 describes a new and improved algorithm to compute the canonical form of a neural ideal from a neural code. Finally, Section 4 provides a tour of the functionality within our code, centered on the canonical form algorithm.

2. Background

In this section we give a brief introduction to the neural ring and the neural ideal of a neural code. A more thorough background, including necessary theorems and proofs, can be found in [2].

2.1. Neural Codes and the Neural Ideal. A neural code $C \subseteq \{0, 1\}^n$ is a set of binary strings that represent neural activity. A ‘1’ represents a firing neuron, while a ‘0’ represents an idle neuron. For example, the presence of the word 1011 in a 4-neuron code would indicate an instance when neurons 1, 3, and 4 were firing but neuron 2 was not. Given a neural code $C \subseteq \{0, 1\}^n$, the corresponding neural ideal $J_C$ in the ring $\mathbb{F}_2[x_1, \ldots, x_n]$ is defined by the following set of generators. For any \( v \in \{0, 1\}^n \), consider the polynomial

\[ \text{Date: October 3, 2016.} \]
\[
\rho_v = \prod_{i=1}^{n}(1 - v_i - x_i) = \prod_{\{i : v_i = 1\}} x_i \prod_{\{j : v_j = 0\}} (1 - x_j).
\]

Notice that \(\rho_v(x)\) acts as a characteristic function for \(v\), since it satisfies \(\rho_v(v) = 1\) and \(\rho_v(x) = 0\) for any \(x \neq v \in \{0,1\}^n\). The neural ideal \(J_C \subseteq \mathbb{F}_2[x_1, \ldots, x_n]\) associated to the neural code \(C\) is the ideal generated by the polynomials \(\rho_v\) with \(v \notin C\), that is,

\[
J_C = \langle \rho_v \mid v \notin C \rangle.
\]

2.2. Realizations and the Canonical Form. Many systems of neurons react to stimuli which have a natural geographic association. One example is head direction cells in rats, where each neuron responds to a preferred range of angles; these stimuli come from the 1-dimensional set of possible angles for the head. Another example is place cells (in rats), where each neuron is associated to a place field or region of the rat’s 2-dimensional environment. In such a geographic setup, we would assume that if two neurons are observed to fire together, then the sets of stimuli for these neurons must overlap. The idea of a realization for a code formalizes these notions.

Suppose \(U = \{U_1, \ldots, U_n\}\) is a collection of open sets with each \(U_i \subset X\). Here, \(X \subset \mathbb{R}^n\) represents the space of possible stimuli, and \(U_i\) is the receptive field of the \(i\)th neuron, the set of stimuli which will cause that neuron to fire. We say that \(U\) is a realization for a code \(C\), or that \(C = C(U)\), if

\[
C = \{v \in \{0,1\}^n \mid (\bigcap_{v_i = 1} U_i) \setminus \bigcup_{v_j = 0} U_j\};
\]

that is, \(C\) represents the set of regions defined by \(U\). It turns out that by considering the ideal \(J_C\) for the code \(C\), we can determine complete information about the interaction of the \(U_i\) in any realization \(U\) of \(C\). This is facilitated by the canonical form of the neural ideal \(J_C\).

A polynomial \(f \in \mathbb{F}_2[x_1, \ldots, x_n]\) is a pseudo-monomial if \(f\) has the form

\[
f = \prod_{i \in \sigma} x_i \prod_{j \in \tau} (1 - x_j),
\]

where \(\sigma \cap \tau = \emptyset\). An ideal \(J \subset \mathbb{F}_2[x_1, \ldots, x_n]\) is a pseudo-monomial ideal if \(J\) can be generated by a set of finitely many pseudo-monomials. Let \(J \subset \mathbb{F}_2[x_1, \ldots, x_n]\) be an ideal, and \(f \in J\) a pseudo-monomial. Then \(f\) is a minimal pseudo-monomial of \(J\) if there is no other pseudo-monomial \(g \in J\) with \(\deg(g) < \deg(f)\) such that \(f = gh\) for some \(h \in \mathbb{F}_2[x_1, \ldots, x_n]\). The canonical form of a pseudo-monomial ideal \(J\), denoted \(\text{CF}(J)\), is the set of all minimal pseudo-monomials of \(J\).

The set \(\text{CF}(J)\) is unique for any given pseudo-monomial ideal \(J\) and \(J = \langle \text{CF}(J) \rangle\). It is important to note that even though \(\text{CF}(J)\) is made up of minimal pseudo-monomials, it does not necessarily imply that \(\text{CF}(J)\) is a minimal set of generators for \(J\). The neural ideal \(J_C\) for a neural code \(C\) is a pseudo-monomial ideal since \(J_C = \langle \rho_v \mid v \notin C \rangle\) and each of the \(\rho_v\)’s is a pseudo-monomial. The following theorem describes the set of relations on any realization \(U\) of \(C\) which \(\text{CF}(J)\) provides.

**Theorem 2.1.** Let \(C \subset \{0,1\}^n\) be a neural code, and let \(U = \{U_1, \ldots, U_n\}\) be any collection of open sets in a stimulus space \(X\) such that \(C = C(U)\). Given \(\sigma \subseteq \{1, \ldots, n\}\), let \(U_\sigma = \bigcap_{i \in \sigma} U_i\). Then the canonical form of \(J_C\) is:

\[
J_C = \left\{ x_\sigma \mid \sigma \text{ is minimal w.r.t. } U_\sigma = \emptyset \right\},
\]

\[
\left\{ x_\sigma \prod_{i \in \tau} (1 - x_i) \mid \sigma, \tau \neq \emptyset, \sigma \cap \tau = \emptyset, U_\sigma \neq \emptyset, \bigcup_{i \in \tau} U_i \neq X, \text{ and } \sigma, \tau \text{ are minimal w.r.t. } U_\sigma \subseteq \bigcup_{i \in \tau} U_i \right\},
\]

\[
\left\{ \prod_{i \in \tau} (1 - x_i) \mid \tau \text{ is minimal w.r.t. } X \subseteq \bigcup_{i \in \tau} U_i \right\}.
\]
We call the above three (disjoint) sets of relations comprising \( \text{CF}(J_C) \) the minimal Type 1, Type 2 and Type 3 relations, respectively. Since the canonical form is unique, by Theorem 2.1 any receptive field representation of the code \( C = C(U) \) satisfies the following relationships:

Type 1: \( x_\sigma \in \text{CF}(J_C) \) implies \( U_\sigma = \emptyset \), but all intersections \( U_\gamma \) where \( \gamma \subseteq \sigma \) are non-empty.

Type 2: \( x_\sigma \prod_{i \in \tau} (1-x_i) \in \text{CF}(J_C) \) implies \( U_\sigma \subseteq \bigcup_{i \in \tau} U_i \), but no lower-order intersection is contained in \( \bigcup_{i \in \tau} U_i \) and all the \( U_i \)s are needed for \( U_\sigma \subseteq \bigcup_{i \in \tau} U_i \).

Type 3: \( \prod_{i \in \tau} (1-x_i) \in \text{CF}(J_C) \) implies \( X \subseteq \bigcup_{i \in \tau} U_i \), but \( X \not\subseteq \bigcup_{i \in \gamma} U_i \) for any \( \gamma \subseteq \tau \).

3. The Iterative Algorithm

In [2], the authors provided a first algorithm to obtain the canonical form via the primary decomposition of the neural ideal. Here we present an alternative algorithm. Rather than using the primary decomposition, this algorithm begins with the canonical form for a code consisting of a single codeword, and iterates by adding the remaining code words one by one and adjusting the canonical form accordingly. We describe the process for adding in a new codeword in Algorithm 1.

**Algorithm 1:** Iterative step to update a given canonical form after adding a code word \( c \)

**Input:** \( \text{CF}(J_C) = \{f_1, \ldots, f_k\} \), where \( C \subseteq \{0,1\}^n \) is a code on \( n \) neurons, and a codeword \( c \in \{0,1\}^n \)

**Output:** \( \text{CF}(J_{C \cup \{c\}}) \)

begin

\[ L \leftarrow \{\}, M \leftarrow \{\}, N \leftarrow \{\} \]

for \( x \leftarrow 1 \) to \( k \) do

\[ \text{if } f_i(c) = 0 \text{ then} \]
\[ L \leftarrow L \cup \{f_i\} \]
\[ \text{else} \]
\[ M \leftarrow M \cup \{f_i\} \]

end

for \( f \in M \) do

for \( j \leftarrow 1 \) to \( n \) do

\[ \text{if } f(x_j - c_j) \text{ is not a multiple of an element of } L \text{ and } (x_j - c_j - 1) \nmid f \text{ then} \]
\[ N \leftarrow N \cup \{f(x_j - c_j)\} \]

end

end

return \( L \cup N = \text{CF}(J_{C \cup \{c\}}) \)

end

In summary, each pseudo-monomial \( f \) from \( \text{CF}(J_C) \) for which \( f(c) = 0 \) is automatically in the new canonical form \( \text{CF}(J_{C \cup \{c\}}) \). For those pseudo-monomials \( f \in \text{CF}(J_C) \) with \( f(c) = 1 \), we consider the product of \( f \) with all possible linear terms \( (x_j - c_j) \) (so this product will be 0 when evaluated at \( c \)) as a possible candidate for \( \text{CF}(J_C) \); but we remove any such products which are redundant. Here, a redundant pseudo-monomial is one which is either a multiple of another already known to be in the canonical form, or which is a multiple of a Boolean polynomial.

Certainly, any polynomial \( f \) output by this algorithm will have the property that \( f(v) = 0 \) for all \( v \in C \cup \{c\} \). A proof that this algorithm outputs exactly \( \text{CF}(J_{C \cup \{c\}}) \) is found in the Appendix.
We have developed SageMath code that computes the canonical form using this iterative algorithm; see Section 4 for an in-depth tutorial of our package. We find that this iterative algorithm performs substantially better than the original algorithm from [2]. We have also implemented our algorithm in Matlab [6].

Table 1 displays some runtime statistics regarding our iterative canonical form algorithm. These runtime statistics were obtained by running our SageMath implementation on 100 randomly generated sets of codewords for each dimension \( n = 4, \ldots, 10 \). These computations were performed on SageMath 7.2 running on a Macbook Pro with a 2.8 GHz Intel Core i7 processor and 16 GB of memory. We performed a similar test for our implementation of the original canonical form algorithm in [2] and on the Matlab implementation of our iterative method. However, even in dimension 5 the original algorithm performs poorly. In our tests, we found several codes for which the original algorithm took hundreds or even thousands of seconds to compute the canonical form. For example, the iterative algorithm takes 0.01 seconds to compute the canonical form of the code below, but the original method takes 1 hour and 8 minutes to perform the same computation.

\[
\begin{align*}
&10000, 10001, 01011, 01010, 10010, 01110, 01101, 01100, 11111, \\
&11010, 11011, 01000, 01001, 00111, 00110, 00001, 00010, 00011, 00101.
\end{align*}
\]

We also found several codes on dimension 5 for which the original algorithm halted due to lack of memory. One such code is

\[
\begin{align*}
&01110, 01111, 11010, 11100, 11101, 01011, 01000, 00110, 00001, 00011, 10011, 00100, 00101, 10111.
\end{align*}
\]

The iterative algorithm computes the canonical form of the previous code in 0.0089 seconds. In our Matlab implementation we also found several examples in dimension 6 for which the canonical form took thousands of seconds to be computed.

| Dimension | min       | max       | mean     | median    | std       |
|-----------|-----------|-----------|----------|-----------|-----------|
| 4         | 0.000077  | 0.0034    | 0.0016   | 0.0018    | 0.00076   |
| 5         | 0.000087  | 0.014     | 0.0076   | 0.0082    | 0.0034    |
| 6         | 0.00012   | 0.108     | 0.049    | 0.051     | 0.024     |
| 7         | 0.00012   | 0.621     | 0.298    | 0.323     | 0.135     |
| 8         | 0.000097  | 4.011     | 1.964    | 2.276     | 1.036     |
| 9         | 0.698     | 39.28     | 24.86    | 27.38     | 9.976     |
| 10        | 0.229     | 350.5     | 237.45   | 271.3     | 87.1      |

Table 1. Runtime statistics (in seconds) for the iterative CF algorithm in SageMath.

Our code has many features beyond the computation of the canonical form via the new iterative algorithm. In the following tutorial, we show how to use the code to compute any of the following:

1. The neural ideal.
2. The canonical form for a neural ideal via the new iterative algorithm.
3. The canonical form for a neural ideal via the primary decomposition algorithm.
4. A tailored method to compute the primary decomposition of pseudo-monomial ideals.
5. Gröbner bases and the Gröbner fan of a neural ideal.
6. A method to test whether a code is a simplicial complex.

4. SageMath Tutorial

We decided to develop our code on SageMath due to its open-source nature, extensive functionality, and ease of use [5]. In this tutorial, we’ll begin with installing the NeuralCodes package. Then, we’ll walk through the most important functions of the package.
4.1. **Installation.** We will assume that SageMath is properly installed on the system. In our tutorial, the files, `iterative_canonical.spyx`, `neuralcode.py` and `examples.py`, are downloaded in the folder `NeuralIdeals`. Now, we can load the package by:

```python
sage: load("NeuralIdeals/iterative_canonical.spyx")
```

```python
sage: load("NeuralIdeals/neuralcode.py")
```

```python
sage: load("NeuralIdeals/examples.py")
```

The first file contains the iterative algorithm in Cython, so loading will compile the code. Note that they must be loaded in this order, as the Cython file must be compiled for the tests in `neuralcode.py` to run. The second file, `neuralcode.py`, holds all of the code that we will be demonstrating. The third, `examples.py` has some additional examples that can be loaded with `sage: neuralcodes()`. We expect to include our code in SageMath, but currently it can be easily obtained at [https://github.com/e6-1/NeuralIdeals](https://github.com/e6-1/NeuralIdeals). We also want to note that running the package in the free SageMathCloud ([https://cloud.sagemath.com](https://cloud.sagemath.com)) is even easier. One has to create a new project, upload all three files above and then simply run `load('neuralcode.py')`.

4.2. **Examples.** First, we define a neural code:

```python
sage: neuralCode = NeuralCode(['001', '010', '110'])
```

Now, we can perform a variety of useful operations. We can compute the neural ideal:

```python
sage: neuralIdeal = neuralCode.neural_ideal()
```

```python
sage: neuralIdeal
```

Ideal (x0*x1*x2 + x0*x1 + x0*x2 + x0 + x1*x2 + x1 + x2 + 1, x0*x1*x2 + x1*x2, x0*x1*x2 + x0*x1 + x0*x2 + x0, x0*x1*x2 + x0*x2, x0*x1*x2) of Multivariate Polynomial Ring in x0, x1, x2 over Finite Field of size 2

We can compute the primary decomposition using a custom algorithm:

```python
sage: pm_primary_decomposition(neuralIdeal)
```

```
[Ideal (x2 + 1, x1, x0) of Multivariate Polynomial Ring in x0, x1, x2 over Finite Field of size 2, Ideal (x2, x1 + 1) of Multivariate Polynomial Ring in x0, x1, x2 over Finite Field of size 2]
```

We can compute the canonical form of the neural ideal.

```python
sage: canonicalForm = neuralCode.canonical()
```

```python
sage: canonicalForm
```

Ideal (x1*x2, x1*x2 + x1 + x2 + 1, x0*x1 + x0, x0*x2) of Multivariate Polynomial Ring in x0, x1, x2 over Finite Field of size 2

The method `canonical()` will use the iterative algorithm by default. If we want to use the procedure described in [2], one needs to specify it with `canonical(algorithm="original")`. This procedure uses by default the tailored pseudo-monomial primary decomposition, but one can make this explicit with `canonical(algorithm="original", decomposition_algorithm="pm")`.

```python
sage: neuralCode.canonical(algorithm="original", decomposition_algorithm="pm")
```

Ideal (x1*x2, x0*x1 + x0, x1*x2 + x1 + x2 + 1, x0*x2) of Multivariate Polynomial Ring in x0, x1, x2 over Finite Field of size 2

Besides the tailored pseudo-monomial primary decomposition method, we can also use the standard primary decomposition methods implemented in SageMath such as Shimoyama-Yokoyama and Gianni-Trager-Zacharias with the flags "sy" and "gtz", respectively. Table 2 compares the runtimes (in seconds) for the primary decomposition of neural ideals using these three methods. Each entry in this table is the mean value of the runtimes for 50 randomly generated codes.
Table 2. Comparisons of runtimes for primary decompositions of neural ideals.

| Dimension | Pseudo-Monomial PD | Shimoyama-Yokoyama | Gianni-Trager-Zacharias |
|-----------|--------------------|---------------------|-------------------------|
| 4         | 0.16               | 0.028               | 0.027                   |
| 5         | 0.85               | 0.14                | 0.07                    |
| 6         | 6.43               | 2.98                | 0.27                    |
| 7         | 63.1               | 74.9                | 1.9                     |
| 8         | 578                | 3040                | 45                      |

We found that, in general, the custom pseudo-monomial primary decomposition algorithm does not outperform the Gianni-Trager-Zacharias algorithm. Nevertheless, this procedure is implemented to be used in characteristic 0. It also works in fields of large positive characteristic. But in small characteristic this procedure may not terminate.

We also want to observe that the `canonical()` method returns an Ideal object whose generators are not factored and hence not easy to interpret in our context. In order to obtain the generators in the canonical form in factored form, we use `factored_canonical()`:

```
sage: neuralCode.factored_canonical()
[x2 * x1, (x1 + 1) * x0, (x2 + 1) * (x1 + 1), x2 * x0]
```

From this output we can easily read off the RF structure of the neural code. However, we also provide a different command that parses the output to explicitly describe the RF structure.

```
sage: neuralCode.canonical_RF_structure()
Intersection of U_['2', '1'] is empty
X = Union of U_['2', '1']
Intersection of U_['0'] is a subset of Union of U_['1']
Intersection of U_['2', '0'] is empty
```

We can also compute the Gröbner basis and the Gröbner fan of the neural ideal. Note that we could compute the neural ideal and use the built-in `groebner_basis()` method, but that approach will not impose the conditions of the Boolean ring (i.e. $x^2 + x = 0$). Our method uses the built-in `groebner_basis()` command but also reduces modulo the Boolean equations.

```
sage: neuralCode.groebner_basis()
Ideal (x0*x2, x1 + x2 + 1) of Multivariate Polynomial Ring in x0, x1, x2
over Finite Field of size 2
```

```
sage: neuralIdeal.groebner_basis()
[x0*x2, x1 + x2 + 1, x2^2 + x2]
```

```
sage: neuralCode.groebner_fan()
[Ideal (x1 + x2 + 1, x0*x2) of Multivariate Polynomial Ring in x0, x1, x2
over Finite Field of size 2]
```

A neural code is called convex if its codewords correspond to regions defined by an arrangement of convex open sets in Euclidean space. Convex codes have been observed experimentally in many brain areas. Hence there has been increased interest in understanding what makes a neural code convex \cite{1}. It has also been observed that if a code is a simplicial complex then it is convex. One can check if a code is a simplicial complex with the command `is_simplicial(Codes):`

```
sage: is_simplicial(['001','010','110'])
False
```

```
sage: is_simplicial(['000','001','010','100','110','011','101','111'])
True
```
APPENDIX: PROOF OF ITERATIVE ALGORITHM

Here, we show that the process described in Algorithm 1 gives $\text{CF}(J_{C\cup D(c)})$ from $\text{CF}(J_C)$ and $c$. Throughout, we use the following conventions and terminology: $C$ and $D$ are neural codes on the same number of neurons; so, $C, D \subseteq \{0, 1\}^n$. A monomial $x^\alpha$ is square-free if $\alpha_i \in \{1, 0\}$ for all $i = 1, \ldots, n$. A polynomial is square-free if it can be written as the sum of square-free monomials. For example: $x_1x_2 + x_4 + x_1x_3x_2$ is square-free. There is a unique square-free representative of every equivalence class of $\mathbb{F}_2[x_1, \ldots, x_n]/(x_i(1 - x_i))$. For $h \in \mathbb{F}_2[x_1, \ldots, x_n]$, let $h_R$ denote this unique square-free representative of the equivalence class of $h$ in $\mathbb{F}_2[x_1, \ldots, x_n]/(x_i(1 - x_i))$.

Then, for $\text{CF}(J_C) = \{f_1, \ldots, f_r\}$ and $\text{CF}(J_D) = \{g_1, \ldots, g_s\}$, we define the set of reduced products

$$P(C, D) \overset{\text{def}}{=} \{(f_ig_j)_R \mid i \in [r], j \in [s]\}.$$

Note that as pseudo-monomials are square-free, for each pair $i, j$ we have either $(f_ig_j)_R = 0$ or $(f_ig_j)_R$ is a multiple of both $f_i$ and $g_j$. We define the minimal reduced products as

$$\text{MP}(C, D) \overset{\text{def}}{=} \{h \in P(C, D) \mid h \neq 0 \text{ and } h \neq fg \text{ for any } f \in P(C, D), \deg g \geq 1\}.$$

Lemma 1. If $C, D \subseteq \{0, 1\}^n$, then the canonical form of their union is given by the set of minimal reduced products from their canonical forms: $\text{CF}(J_{C\cup D}) = \text{MP}(C, D)$.

Proof. First, we show $\text{MP}(C, D) \subseteq J_{C\cup D}$. For any $h \in \text{MP}(C, D)$, there is some $f_i \in \text{CF}(J_C)$ and $g_j \in \text{CF}(J_D)$ so $h = (f_ig_j)_R$. In particular, $h \in J_C$ as $h$ is a multiple of $f_i$, and $h \in J_D$ as it is a multiple of $g_j$. Thus $h(c) = 0$ for all $c \in C \cup D$, so $h \in J_{C\cup D}$.

Suppose $h \in \text{CF}(J_{C\cup D})$. Then as $J_{C\cup D} \subset J_C$, there is some $f_i \in \text{CF}(J_C)$ so that $h = h_1f_i$, and likewise there is some $g_j \in \text{CF}(J_D)$ so $h = h_2g_j$ where $h_1, h_2$ are pseudo-monomials. Thus $h$ is a multiple of $(f_ig_j)_R$ and hence is a multiple of some element of $\text{MP}(C, D)$. But as every element of $\text{MP}(C, D)$ is an element of $J_{C\cup D}$, and $h \in \text{CF}(J_{C\cup D})$, this means $h$ itself must actually be in $\text{MP}(C, D)$. Thus, $\text{CF}(J_{C\cup D}) \subseteq \text{MP}(C, D)$. For the reverse containment, suppose $h \in \text{MP}(C, D)$; by the above, $h \in J_{C\cup D}$. It is thus the multiple of some $f \in \text{CF}(J_{C\cup D})$. But we have shown that $f \in \text{MP}(C, D)$, which contains no multiples. So $h = f$ is in $\text{CF}(J_{C\cup D})$. □

Proof of Algorithm. Note that if $c \in C$, then $L = \text{CF}(J_C)$, so the algorithm ends immediately and outputs $\text{CF}(J_C)$; we will generally assume $c \notin C$.

To show that the algorithm produces the correct canonical form, we apply Lemma 1 it suffices to show that the set $L \cup N$ is exactly $\text{MP}(C, \{c\})$. This requires that all products are considered, and that we remove exactly those which are multiples or other elements, or zeros. Note that

$$\text{CF}(J_{\{c\}}) = \{x_i - c_i \mid i \in [n]\}.$$

To see that all products are considered we will look at $L$ and $M$ separately. Let $g \in L$. Since $g(c) = 0$, we know $(g \cdot (x_i - c_i))_R = g$ for at least one $i$. So $g \in \text{MP}(C, \{c\})$. Any other product $(g \cdot (x_j - c_j))_R$ will either be 0, $g$, or a multiple of $g$, and hence will not appear in $\text{MP}(C, \{c\})$. Thus, all products of linear terms with elements of $L$ are considered, and all multiples or zeros are removed. It is impossible for elements of $L$ to be multiples of one another, as $L \subset \text{CF}(C)$.

We also consider all products of elements of $M$ with the linear elements of $\text{CF}(J_{\{c\}})$. We discard them if their reduction would be 0, or if they are a multiple of anything in $L$. If neither holds, we add them to $N$. So it only remains to show that no element of $N$ can be a multiple of any other element in $N$, and no element of $N$ can be a multiple of anything in $L$, and thus that we have removed all possible multiples. First, no element of $N$ may be a multiple of an element of $L$, since if $g \in L$, $f \cdot (x_i - c_i) \in N$, and $f \cdot (x_i - c_i) \cdot p = g$ for some pseudo-monomial $p$, then $f_\parallel g$. But this is impossible as $f, g$ are both in $\text{CF}(J_C)$. Now, suppose $f \cdot (x_i - c_i) = h \cdot g \cdot (x_j - c_j)$ for $f, g \in \text{CF}(J_C)$ and $f \cdot (x_i - c_i), g \cdot (x_j - c_j) \in N$, and $h$ a pseudo-monomial. Then as $f \not\parallel g$ and $g \not\parallel f$, we have $i \neq j$, and so $(x_j - c_j)_R$. But this means $f \cdot (x_j - c_j) = f$ and therefore $f \in L$, which is a contradiction.

So no elements of $N$ may be multiples of one another. □
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