LOCAL GROUPS IN DELONE SETS

NIKOLAY DOLBILIN

Abstract. In the paper, we prove that in an arbitrary Delone set $X$ in 3D space, the subset $X_6$ of all points from $X$ at which local groups has axes of the order not greater than 6 is also a Delone set. Here, under the local group at point $x \in X$ is meant the symmetry group $S_x(2R)$ of the cluster $C_x(2R)$ of $x$ with radius $2R$, where $R$ (according to Delone’s theory of the ‘empty sphere’) is the radius of the largest ‘empty’ ball, that is, the largest ball free of points of $X$.

The main result (Theorem 2.1) seems to be the first rigorously proved statement on absolutely generic Delone sets which implies substantial statements for Delone sets with strong crystallographic restrictions. For instance, an important observation of Shtogrin on the boundedness of local groups in Delone sets with equivalent $2R$-clusters (Theorem 1.7) immediately follows from Theorem 2.1.

In the paper, the ‘crystalline kernel conjecture’ (Conjecture 1) and its two weaker versions (Conjectures 2 and 3) are suggested. According to Conjecture 1, in a quite arbitrary Delone set, points with locally crystallographic axes (of order 2, 3, 4, or 6) only inevitably constitute essential part of the set. These conjectures significantly generalize the famous statement of Crystallography on the impossibility of (global) 5-fold symmetry in a 3D lattice.

1. Introduction and basic definitions

This paper grew out of the local theory for regular systems, i.e for Delone sets with very strong requirements. On the other hand, here we consider an arbitrary Delone sets in $\mathbb{R}^3$ without any additional assumptions. For example, for a Delone set, we do not suppose a typical condition of the local theory such as the sameness of clusters of certain radius as we did it in numerous papers (see e.g., [2], [6], [8]). Another feature of the paper is as follows, for a Delone set, we consider local groups operated over clusters of radius, namely $2R$ (for definitions see below).

In [9], A.L. Mackay says: "In an infinite crystal there may be extra elements of symmetry which operate over a limited range. These may be seen by non-space-group extinctions in diffraction pattern. . . . The local operations need not to be 'crystallographic'.” Nevertheless, in this paper, we show that the last statement for regular systems (i.e. for sets with transitive groups) should be significantly revised. If we fix for a Delone sets in 3D space, of type $(r, R)$ the range of action of local groups as $2R$ (for details see below), then, it turns out, one can obtain interesting results on properties of such groups. To accurately formulate the results and open hypotheses we will need several definitions and notations.

Euclidean distance between points $x$ and $x'$ in euclidean space $\mathbb{R}^d$ is denoted by $|x, x'|$. Let $d(z, X)$ denote distance from $z \in \mathbb{R}^d$ to set $X \subset \mathbb{R}^d$, i.e. $d(z, X) := \inf_{x \in X} |z, x|$.

Definition 1.1. [Delone set] Given positive real numbers $r$ and $R$, a point subset $X$ of $\mathbb{R}^d$ is called a Delone set of type $(r, R)$ if the following two conditions hold:
(1) an open $d$-ball $B_x^o(r)$ of radius $r$ centered at any point $z$ of space contains at most one point of $X$;
(2) a closed d-ball $B_z(R)$ of radius $R$ centered at an arbitrary point $z$ of space contains at least one point of $X$.

It is clear that a Delone set $X$ of type $(r, R)$ is obviously a Delone set of type $(r', R')$ if $r' \leq r$ and $R' \geq R$. Therefore, we can adopt the convention in designating the parameters $(r, R)$ to a Delone set $X$ as follows. For a given Delone set $X$, we choose as $r$ the largest possible value satisfying Condition (1) of Definition 1.1 and choose as $R$ the smallest value satisfying Condition (2).

We will need also the following interpretations of the parameters $r$ and $R$:

$$\inf_{x, x' \in X} |x x'| = 2r, \quad \sup_{z \in \mathbb{R}^d} d(z, X) = R. \tag{1}$$

Thus, the value of $r$ equals the half of the smallest (infimum) inter-point distance in $X$. The value of $R$ is a distance from the most remote from $X$ point of space $\mathbb{R}^d$ to the set $X$.

In the local theory of Delone sets, the key concepts are that of a cluster.

**Definition 1.2** ($\rho$-cluster). Let $x$ be a point of a Delone set $X$ of type $(r, R)$, $\rho \geq 0$, and let $B_z(\rho)$ be a ball with radius $\rho$ centered at point $z \in \mathbb{R}^d$. We call a point set

$$C_X(\rho) := X \cap B_z(\rho)$$

the cluster of radius $\rho$ at point $x$ or simply the $\rho$-cluster at $x$.

**Definition 1.3** (equivalent clusters). Two clusters $C_x(\rho)$ and $C_{x'}(\rho)$ of the same radius $\rho$ at points $x$ and $x'$ are said to be equivalent if there is an isometry $g \in \text{Iso}(\mathbb{R}^3)$ such that

$$g(x) = x' \quad \text{and} \quad g(C_x(\rho)) = C_{x'}(\rho). \tag{2}$$

**Definition 1.4** (cluster group). Given a point $x \in X$ and its $\rho$-cluster $C_x(\rho)$, a group $S_x(\rho)$ of all isometries $s \in \text{Iso}(\mathbb{R}^d)$ which leave the $x$ fixed and the cluster $C_x(\rho)$ invariant is called the cluster group:

$$S_x(\rho) := \{ s \in \text{Iso}(\mathbb{R}^d) \mid s(x) = x, s(C_x(\rho)) = C_x(\rho) \}.$$  

Groups of equivalent clusters $C_x(\rho)$ and $C_{x'}(\rho)$ are conjugate in the full group $\text{Iso}(d)$ of isometries: $S_x(\rho) = g^{-1} S_{x'}(\rho) g$, where $g$ is determined by conditions of Definition 1.3. It is clear that as the radius $\rho$ increases, the cluster $C_x(\rho)$ expands but the cluster group $S_x(\rho)$ never increases and sometimes can contract only. It is clear that if $0 \leq \rho < 2r$ group $C_x(\rho) = O_x(3)$? i.e. the full point group of all isometries that leave point $x$ fixed. On the other and, it is well-know that the $S_x(2R)$

Since the main result grew up ideologically from the local theory of regular systems, here, we briefly recall basic concepts of this theory. Modern in form, the following definitions of regular system and crystal are equivalent to those that go back to E.S. Fedorov.

**Definition 1.5** (regular system, crystal). A Delone set $X$ is called a regular system if it is an orbit of some point $x$ with respect to a certain space group $G \subset \text{Iso}(d)$, i.e.

$$X = G \cdot x = \{ g(x) \mid g \in G \};$$

a Delone set $X$ is a crystal if $X$ is a union of several orbits: $X = \bigcup_{i=1}^{m} G \cdot x_i$.

We emphasize that the notion of a regular system is an essential case of the crystal, i.e. a multi-regular system, and generalizes the lattice concept. In fact, a lattice is a a particular case of a regular system when $G$ is a group of translations generated by $d$ linearly independent
translations. Moreover, due to a celebrated theorem by Schoenflies and Bieberbach, any regular system is the union of congruent and mutually parallel lattices.

The local theory of regular systems began with the Local criterion in [2].

**Theorem 1.6** (Local Criterion, [2]). A Delone set \(X\) is a regular system if and only if there is some \(\rho_0 > 0\) such that the following two conditions hold:

1) all \(\rho_0 + 2R\)-clusters are mutually equivalent;
2) \(S_x(\rho_0) = S_x(\rho_0 + 2R)\) for \(x \in X\).

In [3], [4], this criterion has been generalized for crystals, i.e., multi-regular systems.

From now on, we restrict ourselves only to the 3D case. One of central problems of the local theory of regular systems is to search for an upper (and lower) bound for the regularity radius, i.e., a minimum value \(\hat{\rho}_3 > 0\) such that equivalence of \(\hat{\rho}_3\)-clusters in a Delone set \(X\) implies the regularity of the set \(X \subset \mathbb{R}^3\). In [6], [8] is given a proof of the upper bound \(\hat{\rho}_3 \leq 10R\). The long proof starts with selection of a special finite list of finite subgroups of \(O(3)\). Groups of this list have a chance to occur in Delone sets with equivalent \(2R\)-clusters as local groups \(S_x(2R)\). The list of selected groups is provided by Theorem 1.7 found by Shtogrin in the late 1970’s but published only in 2010 ([10]).

**Theorem 1.7** ([10]). If in a Delone set \(X \subset \mathbb{R}^3\) all \(2R\)-clusters are mutually equivalent, then the order of any rotational axis of \(S_x(2R)\) does not exceed 6.

Quite recently, [7], it was realized for the first time that an important statement about groups in Delone sets with significant requirements on equivalent clusters may follow from a certain statement true for pretty general Delone sets.

Namely, in [7], Theorem 1.8 is proved. Given an arbitrary Delone set \(X \subset \mathbb{R}^3\) and \(x \in X\), let the maximal order of rotational axis in group \(S_x(2R)\) be denoted by \(n_x\).

**Theorem 1.8.** In a Delone set \(X \subset \mathbb{R}^3\) there is at least one point \(x\) with \(n_x \leq 6\).

It is obvious that Theorem 1.8 immediately implies Theorem 1.7. In fact, the subset \(X_6\) of all points in a Delone set \(X\) with \(n_x \leq 6\) is always very rich. Due to Theorem 2.1 the subset \(X_5\) is a Delone set itself.

2. The main result and conjectures

**Theorem 2.1** (Main result). Given a Delone set \(X \subset \mathbb{R}^3\) of type \((r, R)\), let \(X_6 \subseteq X\) be the subset of all points \(x \in X\) such that the maximal order \(n_x\) of a rotation axis in \(S_x(2R)\) does not exceed 6. Then \(X_6\) is a Delone set of a certain type \((r', R')\), where \(r \leq r' \leq R' = kR\) for some \(k\) independent on \(X\).

At the moment, we do not care on the value \(kR\) of the upper bound for the parameter \(R'\) of the \(X_6\). For us, so far it is more important to establish that the subset \(X_6\) is always a Delone subset.

From now on, we will focus on clusters \(C_x(2R)\) of radius \(2R\) and their groups \(S_x(2R)\). As well-known, for a Delone set \(X\), the \(2R\)-clusters all are full-dimensional (i.e., the dimension of their convex hulls is \(d\)). Hence, the cluster groups \(S_x(2R)\) are necessarily finite. At the same time, we emphasize that the value of \(2R\) is the smallest value of radius that guarantees the finiteness of the cluster group of radius \(2R\) for any Delone set with parameter \(R\). In other words, for an arbitrary \(\varepsilon > 0\) one can present a Delone set \(X\) with parameter \(R\) such that for some \(x \in X\) in \(S_x(2R - \varepsilon)\) is infinite.
By virtue of the above, we will single out group $S_x(2R)$ and call it a local group at $x$.

Theorem 2.1 immediately implies Theorem 1.7 which concerns a Delone set $X$ with mutually equivalent $2R$-clusters. Really, for such a Delone set $X$, the local groups at all points are pairwise conjugate and existence of points $x$ with $n_x \leq 6$ implies the same inequality for all others.

Now, among points of $X_6 \subseteq X$ we select points $x$ with the condition $n_x \neq 5$, i.e., all points of $X$ whose local groups contain axes of only ‘crystallographic’ orders 2, 3, 4, or 6. We call the subset of all such points in $X$ a crystalline kernel of $X$ and denote by $K$.

**Conjecture 1** (Crystalline kernel conjecture). The crystalline kernel $K$ of a Delone set $X$ is a Delone subset with some parameter $R' \leq kR$, where $k$ is some constant which does not depend on $X$.

Let $Y$ denote the subset of all points $x \in X$ at which local groups $S_x(2R)$ do not contain the pentagonal axis. It is clear that $K \subseteq Y$ and if $K$ is a Delone set then $Y$ is a Delone set too. Therefore Conjecture 1 if proven, immediately implies the following two Conjectures 2 and 3.

**Conjecture 2** (5-gonal symmetry conjecture). Given a Delone set $X \subset \mathbb{R}^3$, the subset $Y$ of points $x$, whose groups $S_x(2R)$ are free of 5-fold axes, is also a Delone set.

In its turn, Conjecture 2 enforces the following statement that seems to be much easier proved.

**Conjecture 3** (Weak 5-gonal symmetry conjecture). Given a Delone set $X \subset \mathbb{R}^3$ with mutually equivalent $2R$-clusters, the local group $S_x(2R)$ contains no 5-fold axis.

It is obvious that these hypotheses relate to a celebrated crystallographic theorem on the impossibility of the global 5-fold symmetry in a three-dimensional lattice. Conjectures 1–3 significantly reinforce a classical statement on famous crystallographic restrictions.

It is well-known that for a 3-dimensional lattice even in the group $S_x(r_1)$, there are no the 5-fold symmetry, where $r_1 = 2r \leq 2R$ is the minimum inter-point distance in the lattice. In contrast to lattices, in regular systems in 3D space, the pentagonal symmetry can locally manifest itself on clusters of a certain radius less than $2R$. So, for instance, there are regular systems such that even group $S_x(r_3)$ contains the 5-fold axis, but the local group $S_x(2R)$ does not. Here, in the systems, $r_3$ ($r_1 < r_2 < r_3 < 2R$) is the 3rd inter-point distance in $X$. But, it is still unknown whether there are regular systems with 5-fold symmetrical $2R$-clusters.

Since the regularity radius for dimension 3 is not less than $6R$ (see [11]), among Delone sets $X$ with mutually equivalent $2R$-clusters, there are non-regular and even non-crystallographic sets. Thus, even the weakest Conjecture 3 concerns also a wide class of those non-regular sets.

### 3. Proof of Theorem 2.1

*Proof.* Generally speaking, in the local group $S_x(2R) \subset O(3)$, $x \in X$, there are several axes of maximal order $n_x$. Bearing in mind the well-known list of all finite subgroups of $O(3)$, we see that more than one axes of the maximal order $n_x$ in $S_x(2R)$ cannot happen provided $n_x > 5$. Let $\ell_x$ be one of those axes. Since $\text{rk}(C_x(2R)) = 3$, i.e. the convex hull of the $2R$-cluster is 3-dimensional, in $C_x(2R)$, there are necessarily points off the $\ell_x$. 

Since \( X_6 \) is a subset of \( X \), the minimal inter-point distance \( 2\tilde{r} \) in \( X_6 \) (in fact, the infimum of such distances) is not less than \( 2r = \inf_{x,x' \in X} |x, x'| \). In order to prove that \( X_6 \) is a Delone set with a certain parameter \( \tilde{R} \), we will prove that the distance from a given point \( z \in \mathbb{R}^3 \) to the nearest point of \( X_6 \) does not exceed \( \tilde{R} : \min_{x \in X} |z, x| \leq \tilde{R} \) (due to interpretations (1) for the parameters \( r \) and \( R \) in Section 1. We will be looking for the point \( x \in X_6 \) nearest to \( z \in \mathbb{R}^3 \) by walking along a special finite point sequence in \( X_6 \).

**Definition 3.1.** A sequence of points \([x_1, x_2, \ldots, x_m, \ldots] \subset X \) (finite or infinite, no matter') is termed an off-axial chain if the following condition holds for any \( i = 1, 2, \ldots \):

the point \( x_{i+1} \) is the nearest point to \( x_i \) among all points of \( X \) which are off the axis \( \ell_{x_i} \), where the axis \( \ell_{x_i} \) means an axis of the local group \( S_{x_i}(2R) \) of the maximal order \( n_{x_i} \).

In case the subgroup of all (orientation-preserving) rotations of \( S_{x_i}(2R) \) is trivial (that is, in the local group at \( x_i \) no axes through \( x_1 \), any nearest to \( x_i \) point of \( X \) can be chosen as \( x_{i+1} \).

Note that for any point \( x \in X \), there are off-axial sequences \([x_1(= x), x_2, x_3, \ldots] \).

**Lemma 3.2.** Given a Delone set \( X \) and an off-axial sequence \([x_1, x_2, \ldots, x_m, \ldots] \subset X \), assume that \( x_i \notin X_6 \), \( \forall i \in \overline{1, m} \). Then for \( i \in \overline{1, m} \) the following holds:

\[
|x_i, x_{i+1}| < 0.87^{i-1} \cdot 2R \quad \text{and} \quad |x_1, x_m| < 7.7 \cdot 2R = 15.4R, \text{ for all } m.
\] (3)

**Proof.** (of Lemma 3.2). Let \([x_1, x_2, \ldots, x_m, \ldots] \) be an off-axial chain and assume that it belongs \( X \setminus X_6 \). Recall that, by construction, in this chain, the length \( r_i^* \) of each link \( x_i, x_{i+1} \) is less than \( 2R \). Hence \( x_{i+1} \in C_{x_i}(2R) \). Therefore, the rotation \( g_{x_i} \in S_{x_i}(2R) \) can be applied to point \( x_{i+1} \) too.

By assumption, \( x_i \notin X_6 \), that is, \( n_{x_i} \geq 7 \). Let \( x_2 \) be the nearest to \( x_1 \) point \( x_2 \) which is off the axis \( \ell_{x_1} \), also let \( g_{x_1} \) be a rotation around axis \( \ell_{x_1} \) by angle \( 2\pi/n_{x_1} \). Since \( r_i^* \leq 2R \) the cluster \( C_{x_1}(2R) \) necessarily contains vertices of a regular \( n_{x_1} \)-gon \( P_1 \) which is generated by the rotation \( g_{x_1} \in S_{x_1}(2R) \) applied to the point \( x_2 \). The polygon \( P_1 \) is located in a plane orthogonal to the \( \ell_{x_1} \). The center of \( P_1 \) is on \( \ell_{x_1} \) (Figure 1).

Denote the side-length of \( P_1 \) by \( a_1 \). Since the circumradius of \( P_1 \) does not exceed \( r_{x_1}^* \) and \( n_{x_1} \geq 7 \) we have for \( a_1 \) and \( r_{x_2}^* \) the following estimate

\[
r_{x_2}^* \leq a_1 \leq 2r_{x_1}^* \sin \frac{\pi}{n_{x_1}} \leq 2r_{x_1}^* \sin \frac{\pi}{7} < 0.87 \cdot 2R.
\] (4)

\[\text{Figure 1. Polygon } P_1 \text{ and the beginning of an off-axial chain } [x_1, x_2, \ldots]\]
Assuming now that \( x_2 \notin X_6 \), i.e. \( n_{x_2} \geq 7 \), we will construct the next point \( x_3 \) in the off-axial chain \( [x_1, x_2, \ldots] \) and obtain upper estimates for \( r_{x_3} \) and \( a_2 \) (see inequalities (5) below). The rotation \( g_{x_2} \) about the axis \( \ell_{x_2} \) is assumed to belong to the local group \( S_{x_2}(2R) \). Since \( x_3 \in C_{x_2}(2R) \), the \( g_{x_2} \) can be applied to the point \( x_3 \). Hence the cluster \( C_{x_2}(2R) \) necessarily contains vertices of a regular \( n_{x_2} \)-gon \( P_2 \) generated by rotation \( g_{x_2} \) applied to the point \( x_3 \). Denote the side-length of \( P_2 \) by \( a_2 \) and note that \( a_2 \leq r_{x_2}^* \leq 2R \).

Point \( x_3 \), like a vertex of the regular \( n_{x_2} \)-gon \( P_2 \), has two adjacent vertices in \( P_2 \) at distance \( a_2 \) from vertex \( x_3 \). Vertex \( x_3 \) and two adjacent vertices of \( P_2 \) form a non-collinear triple. Therefore, no matter how the axis \( \ell_{x_2} \) passes through the point \( x_3 \), anyway, at least one of two neighboring points is off the axis \( \ell_{x_2} \). It follows that the distance \( r_{x_3}^* \) from \( x_3 \) to the nearest point \( x_4 \in X \setminus \ell_{x_2} \) does not exceed \( a_2 \). Since we bear in mind that in the set \( X \setminus \ell_{x_2} \), there may be points nearer to the \( x_3 \) than distance \( a_1 \) we have \( r_{x_3}^* \leq a_2 \).

Since \( n_{x_2} \geq 7 \), by the same argument as above, we have

\[
r_{x_3}^* < a_2 \leq 2r_{x_2}^* \sin \frac{\pi}{n_{x_2}} \leq 2r_{x_2}^* \sin \frac{\pi}{7} < 0.87 \, r_{x_1}^* < 0.87^2 \cdot 2R. \tag{5}
\]

This reasoning can be repeated over and over again. Under condition \( n_{x_i} \geq 7 \), \( \forall i \in \overline{1, m} \), we get the off-axial chain \( [x_1, x_2, \ldots] \) for which the sequence of inter-point distances \( r_{x_i}^* = |x_i x_{i+1}| \) is dominated by a geometric progression

\[
r_{x_{i+1}}^* < 0.87 \, r_{x_i}^* < (0.87)^i \, r_{x_1}^* < (0.87)^i \, 2R. \tag{6}
\]

Thus, we obtain the required inequalities (3). Lemma 3.2 is proved. \( \square \)

Now we are going to complete the proof of Theorem 2.1. Given a Delone set \( X \), from Lemma 3.2 it follows that any off-axial chain provided \( n_{x_i} \geq 7 \) is finite. Moreover its length \( m \) can be bounded from above: \( m < M := \log(\frac{R}{r})/\log \frac{1}{0.87} \). This implies that in the chain \( [x_1, x_2, \ldots] \) there are points \( x_m \) with \( n_{x_m} \leq 6 \), i.e. \( x_m \in X_6 \). By Lemma 3.2 the segment-length \( |x_1, x, x_m| < 15.4 \).

Now we set up upper boundedness of the distance from an arbitrary point \( z \) of space to the nearest point of the subset \( X_6 \). Let the nearest to \( z \) point of \( X \) be \( x_1 \) (see Figure 1) and \( x_m \in X_6 \) (by Lemma 3.2). Then \( |z, x_1| \leq R \) and we get

\[
\min_{x \in X_6} |z, x| \leq |z, x_m| \leq |z, x_1| + |x_1, x_m| = 16.4 \, R. \tag{7}
\]

In other words, we proved that the subset \( X_6 \) is a Delone set with some parameter \( \tilde{R} \leq 15.4 \, R \). Theorem 2.1 is proved. \( \square \)

4. Concluding remarks

We emphasize that the upper bound for the parameter \( \tilde{R} \), established here, is far from optimal one. We believe that we will soon be able to present a sharper bound \([11]\). The purpose of this paper was to present the result, in our opinion, of a new type. The result suggests a few conjectures which should be interesting both in itself and in context of the theory of quasicrystals. So, for instance, in Penrose patterns, in structures of real Shechtman quasicrystals, the centers of \( 2\tilde{R} \)-clusters with local 5-fold symmetry constitute a rich Delone subset. At the same time, in these known quasicrystalline structures, there are also Delone subsets of points with local crystallographic axes (including identical) \( ^\dagger \) However, according
to Conjecture 1, not only in these structures but in any other possible Delone sets, points with local crystallographic axes inevitably constitute an essential part of the structure.

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REFERENCES

[1] I. Baburin, M. Bouniaev, N. Dolbilin, N. Yu. Erokhovets, A. Garber, S. V. Krivovichev, and E. Schulte, On the Origin of Crystallinity: a Lower Bound for the Regularity Radius of Delone Sets, Acta Crystallographica, A74:6 (2018), 616–629.

[2] B. N. Delone, N. P. Dolbilin, M. I. Shtogrin, R. V. Galiulin, A local criterion for regularity of a system of points, Dokl. Akad. Nauk SSSR, 227:1 (1976), 19–21.

[3] N. P. Dolbilin, M. I. Shtogrin, A local criterion for a crystal structure, Abstracts of the IXth All-Union Geometrical Conference, Kishinev, 1988, p. 99 (in Russian).

[4] N. P. Dolbilin, J. C. Lagarias, M. Senechal, Multiregular point systems, it Discrete Comput. Geom., 20:4 (1998), 477 – 498.

[5] N. Dolbilin, Delone Sets: Local Identity and Global Symmetry, Discrete Geometry and Symmetry, Springer Volume dedicated to the 60th anniversary of Professors Karoly Bezdek and Egon Schulte, Springer Proc. Math. Statist., 234, Springer, Cham, 2018, 109–125.

[6] N. P. Dolbilin, Delone sets in \( \mathbb{R}^3 \) with 2R-regularity conditions, Topology and Physics, Volume dedicated to the 80th anniversary of of academician. Sergey Petrovich Novikov, Proc. Steklov Inst. Math., 302 (2018), 161–185.

[7] N. P. Dolbilin, From Local Identity to Global Order, Materials of the XIII Lupanov International Seminar, MSU, June 17-22, 2019, p.13-22.

[8] N. Dolbilin, A. Garber, U. Leopold, E. Schulte, On the regularity radius of Delone sets in \( \mathbb{R}^3 \), 2019, arXiv:1909.05805.

[9] A. L. Mackay, Generalized Crystallography, Comp & Maths and Appls., Vol. 128, 1/2 (1986), 21–37.

[10] M. I. Shtogrin, On a bound of the order of a spider’s axis in a locally regular Delone system (in Russian), Abstracts of the International Conference “Geometry, Topology, Algebra and Number Theory, Applications” dedicated to the 120-th anniversary of Boris Nikolaevich Delone (1890-1980), Moscow, August 16–20, 2010, 168–169, http://delone120.mi.ras.ru/delone120abstracts.pdf.

[11] N.P. Dolbilin, M.I. Shtogrin (in preparation).

Nikolay Dolbilin, 
Steklov Mathematical Institute, 
Gubkina str.8, Moscow, Russia 119991
DOLBILIN@MI-RAS.RU