On certain global conformal invariants and 3-surface twistors of initial data sets

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The Chern–Simons functionals built from various connections determined by the initial data \( h_{\mu\nu}, \chi_{\mu\nu} \) on a 3-manifold \( \Sigma \) are investigated. First it is shown that for asymptotically flat data sets the logarithmic fall off for \( h_{\mu\nu} \) and \( r\chi_{\mu\nu} \) is the necessary and sufficient condition of the existence of these functionals. The functional \( Y_{(k,l)} \), built in the vector bundle corresponding to the irreducible representation of \( SL(2, \mathbb{C}) \) labelled by \((k,l)\), is shown to be determined by the Ashtekar–Chern–Simons functional and its complex conjugate. \( Y_{(k,l)} \) is conformally invariant precisely in the \( l = k \) (i.e. tensor) representations. An unexpected connection with twistor theory is found: \( Y_{(k,k)} \) can be written as the Chern–Simons functional built from the 3-surface twistor connection, and the not identically vanishing spinor parts of the 3-surface twistor curvature are given by the variational derivatives of \( Y_{(k,k)} \) with respect to \( h_{\mu\nu} \) and \( \chi_{\mu\nu} \). The time derivative \( \dot{Y}_{(k,k)} \) of \( Y_{(k,k)} \) is another global conformal invariant of the initial data set, and for vanishing \( \dot{Y}_{(k,k)} \), in particular for all Petrov III. and N spacetimes, the Chern–Simons functional is a conformal invariant of the whole spacetime.

1. Introduction

In a recent joint paper with Robert Beig the conformal invariant \( Y[h_{\mu\nu}] \) of Chern and Simons [1], defined for closed orientable Riemannian 3-manifolds \((\Sigma, h_{\mu\nu})\), was generalized for triples \((\Sigma, h_{\mu\nu}, \chi_{\mu\nu})\), where \( \chi_{\mu\nu} \) is a symmetric tensor field [2]. (For the sake of simplicity we call such a triple an initial data set and \( \chi_{\mu\nu} \) the extrinsic curvature even if we don’t use any field equation, not even any constraints for \( h_{\mu\nu} \) and \( \chi_{\mu\nu} \); and even if \((\Sigma, h_{\mu\nu}, \chi_{\mu\nu})\) is not assumed to be imbedded in any spacetime.) Similarly to \( Y[h_{\mu\nu}] \), the new \( Y_0[h_{\mu\nu}, \chi_{\mu\nu}] \) was defined as the integral of the Chern–Simons 3-form, built from the connection on an appropriate vector bundle over \( \Sigma \), modulo \( 16\pi^2 \). In the former case the connection was the Levi-Civita connection determined by \( h_{\mu\nu} \) on the tangent bundle of \( \Sigma \), whilst in the latter it was the real Sen connection determined by \( h_{\mu\nu} \) and \( \chi_{\mu\nu} \) on a trivializable Lorentzian vector bundle, i.e. the vector bundle constructed by the vector representation of \( SL(2, \mathbb{C}) \), over \( \Sigma \). If \( \Sigma \) is a spacelike hypersurface in a Lorentzian spacetime, then this vector bundle is just the spacetime tangent bundle pulled back to \( \Sigma \). Thus, roughly speaking, we retain the four dimensional Lorentzian character of the geometry of the initial data set infinitesimally, i.e. at the level of tangent spaces, even in a 3+1 decomposition of spacetime. \( Y[h_{\mu\nu}] \) is known to be invariant with respect to conformal rescalings of the 3-metric \( h_{\mu\nu} \), and \( Y_0[h_{\mu\nu}, \chi_{\mu\nu}] \) turned out to be invariant with respect to changes of \( h_{\mu\nu} \) and \( \chi_{\mu\nu} \) corresponding to spacetime conformal rescalings. The functional derivative of \( Y[h_{\mu\nu}] \) with respect to \( h_{\mu\nu} \) is known to be the Cotton–York tensor, and hence the stationary points of \( Y[h_{\mu\nu}] \) are the locally conformally flat Riemannian 3-manifolds. The variational derivatives of \( Y_0[h_{\mu\nu}, \chi_{\mu\nu}] \) yield two symmetric trace-free tensor fields, \( B_{\mu\nu} \) and \( H_{\mu\nu} \), whose vanishing characterizes the local isometric imbeddability of \((\Sigma, h_{\mu\nu}, \chi_{\mu\nu})\) into some conformal Minkowski spacetime. \( H_{\mu\nu} \) is the conformal magnetic curvature, while \( B_{\mu\nu} \) is the natural generalization of the Cotton–York tensor for non-vanishing \( \chi_{\mu\nu} \). The nontriviality of these invariants is shown by a result of Meyerhoff [3], namely that in the Riemannian case...
for certain hyperbolic manifolds $Y[h_{\mu\nu}]$ takes values which are dense on the circle $S^1 = \mathbb{R}$ modulo $16\pi^2$. In [2] the analogous functional $Y_\pm[h_{\mu\nu}, \chi_{\mu\nu}]$ based on the complex self-dual/anti-self-dual Ashtekar connection, i.e. based on the bundle constructed by the self-dual/anti-self-dual representation of $SL(2, \mathbb{C})$, was also considered and was shown not to be conformally invariant. In fact, the stationary points of $Y_\pm[h_{\mu\nu}, \chi_{\mu\nu}]$ are precisely those data sets that can be locally isometrically embedded into the Minkowski spacetime.

From physical points of view it would be desirable to be able to define the conformal invariant $Y_0 := Y_0[h_{\mu\nu}, \chi_{\mu\nu}]$ for asymptotically flat initial data sets too, because these data sets are thought to represent the gravitational field of localized objects. In particular, this may provide a useful tool in studying the structure of spacelike and null infinity (see e.g. [4,5]), since it serves as a natural foliation of the space of initial data for the conformally equivalent spacetimes. As we mentioned above, the critical points of $Y_0$ are precisely those data sets that can be imbedded into some conformally flat spacetime. But, as Tod proved [6], this imbeddability is equivalent to the complete integrability of the 3-surface twistor equation. Taking into account the conformal invariance of $Y_0$, one might conjecture that there is a hidden connection between our previous construction and various 3-surface twistor concepts. In particular $Y_0$ might be a functional of the 3-surface twistor connection. Furthermore, it might be interesting even from twistor theoretical points of view to clarify the properties of the Chern–Simons functional built from the 3-surface twistor connection. Since the functionals $Y_0, Y_\pm := Y_\pm[h_{\mu\nu}, \chi_{\mu\nu}]$ have different conformal properties, the question arises whether new nontrivial invariants can be obtained by considering other vector bundles, i.e. representations of $SL(2, \mathbb{C})$, or not. (Recently another interesting connection was introduced in the canonical description of general relativity, the so-called Barbero connection [7,8]. Although the Barbero–Chern–Simons functional has several interesting properties, e.g. it is conformally invariant precisely in the tensor representations for any real value of the Barbero–Immirzi parameter, we will not consider that in the present paper.) Further interesting issue is the question of the time evolution of these functionals, i.e. how they change as the function of time if the data is evolved in time (e.g. by Einstein’s field equations).

In the present paper we investigate (further) the properties of $Y_0, Y_\pm$, and the Chern–Simons functional built from the 3-surface twistor connection. In the first two subsections we review the main points of the construction of $Y_0$ and discuss how $h_{\mu\nu}$ and $\chi_{\mu\nu}$ determine it uniquely. (This issue was not exhaustively discussed in [2].) Moreover, we improve several points of the presentation and correct some minor numerical errors.) We define $Y_0$ for asymptotically flat initial data sets by determining the weakest possible fall-off conditions for $h_{\mu\nu}$ and $\chi_{\mu\nu}$. Since however the Chern–Simons functional is defined in the tetrad rather than the metric theory, a new technique was needed to determine the asymptotic behaviour of the initial data. This technique can also be used to derive the fall-off and asymptotic gauge conditions in the canonical analysis of the (tetrad or triad) general relativity. We will see that the weakest possible fall-off conditions are much weaker than those coming from general relativity, namely logarithmic fall-off for $h_{\mu\nu}$ and $r\chi_{\mu\nu}$, and hence the Chern–Simons conformal invariant is well defined for asymptotically flat initial data sets for Einstein’s theory. These fall-off conditions ensure the existence of the Ashtekar–Chern–Simons functional as well. For later use (especially in the twistorial approach) the construction is rewritten in the spinor representation in subsection 2.3. In this representation the real Sen– and the complex Ashtekar–Chern–Simons functional can be treated simultaneously. Finally, we clarify how the Chern–Simons functional depends on the representation of the structure group by showing that the construction, based on a general finite dimensional representation, doesn’t give anything new, that is simply a combination of the Ashtekar–Chern–Simons functional and its complex conjugate, or, equivalently, the Sen–Chern–Simons and the Ashtekar–Chern–Simons functionals. The conformally invariant functionals correspond precisely to the tensor representations.

In section three the potential relation to twistor theory will be clarified. In subsection 3.1 the unitary spinor forms of the Sen operator and the concept of 3-surface twistors will be reviewed. Although most of that subsection is essentially a review (mainly to fix the notations, to present the tools for the next subsection
and to retain the coherence and readability of the paper), it contains several new elements, e.g. the unitary spinor form of the full Ricci and Bianchi identities and the tensors $H_{\mu
u}, B_{\mu\nu}$, too. Then, in subsection 3.2, we calculate the 3-surface twistor connection and curvature explicitly, and show that the tensors $H_{\mu\nu}, B_{\mu\nu}$ above represent the non-vanishing components of the 3-surface twistor curvature. Thus $H_{\mu\nu}$ and $B_{\mu\nu}$ have natural twistorial interpretation. Then the Sen–Chern–Simons functional will be shown to be just twice the Chern–Simons functional built from the 3-surface twistor connection.

Section four is devoted to the problem of time evolution of the Chern–Simons functionals. We derive a formula by means of which we can compare these functionals on arbitrary two spacelike hypersurfaces and determine the conditions of their hypersurface–independence. These conditions are satisfied for a large class of algebraically general and special spacetimes, including all the Petrov III, and N. type metrics, yielding two new global invariants for these spacetimes. One of them is a global \textit{conformal invariant}. Finally, we discuss the properties of the imaginary part of the Ashtekar–Chern–Simons functional, a proposal for the natural time variable in cosmological spacetimes, and we will see that $\text{Im} Y_\pm$ is monotonic only for a very limited class of spacetimes. Finally we calculate the Chern–Simons functional for the general closed homogeneous Bianchi cosmologies with simply-transitive group actions, and, in particular, for the vacuum Kasner, the general Robertson–Walker and the special anisotropic Barrow solutions. This is the only point where Einstein’s equations are used in the present paper. These examples show the usefulness and the nontriviality of the generalizations $Y_0, Y_\pm$ of the (Riemannian) conformal invariant of Chern and Simons for initial data sets.

Our general spinor–twistor reference is [9], and that of differential geometry is [10]. In particular, the wedge product of forms is defined to be the anti-symmetric part of the tensor product, the signature of the metric. We changed our previous notations slightly. We use several types of indices, both abstract and concrete (name) indices, whose range will be explained when they appear first.

2. The Chern–Simons functional of asymptotically flat initial data sets

2.1 The general Chern–Simons functional

Let $\Sigma$ be a connected orientable 3-manifold, which is asymptotically Euclidean in the sense that for some compact set $K \subset \Sigma$ the complement $\Sigma - K$ is diffeomorphic to $\mathbb{R}^3 - B$, where $B$ is a closed ball in $\mathbb{R}^3$. This complement represents the ‘asymptotic end’ of $\Sigma$. Greek indices from the second half of the Greek alphabet, e.g. $\mu, \nu, \ldots$, will be abstract tensor indices referring to $\Sigma$ in general, but in the present subsection they denote concrete coordinate indices, too. Let $G$ be a Lie group, $\mathcal{G}$ its Lie algebra, and $\pi : P \to \Sigma$ a trivializable principal bundle over $\Sigma$ with structure group $G$. Let $E^a$ a $k$ dimensional vector space over $K=\mathbb{R}$ or $\mathbb{C}$, $\rho : G \to GL(E^a)$ a linear representation of $G$ on $E^a$, $\rho_* : \mathcal{G} \to gl(E^a)$ the corresponding representation of the Lie algebra, and $\pi : E^a(\Sigma) \to \Sigma$ the associated (trivializable) vector bundle. Because of the trivializability $E^a(\Sigma)$ admits $k$ global cross sections, $E^a_\underline{a}, \underline{a} = 1, \ldots, k$, such that at each point $p \in \Sigma$ $\{E^a_\underline{a}|_p\}$ spans the fibre $\pi^{-1}(p) \subset E^a(\Sigma)$. Such a collection of cross sections of $E^a(\Sigma)$ will be called a global frame field. Thus small Roman indices are abstract ‘internal bundle’ indices, while the underlined small Roman indices are name indices. The global cross section $\sigma : \Sigma \to P$ of the principal bundle defines a transformation $\rho \circ \sigma$ of the
global frame fields (‘globally defined local gauge transformations’); i.e. it is a $k \times k$ matrix valued function $\Lambda^a_{\mu b}$ on $\Sigma$ acting on a global frame field as $E^a_{\mu} \mapsto E^a_{\mu} \Lambda^a_{\mu b}$.

Any connection on $P$ defines a connection on $E^a(\Sigma)$, whose connection coefficients with respect to a global frame field form a $\rho_*(G) \subset g\ell(k, \mathbb{K})$-valued 1-form $A^a_{\mu b}$ on $\Sigma$, and the curvature of this connection is the $\rho_*(G)$-valued 2-form $-F^a_{\mu \nu \rho} := \partial_{[\mu} A^a_{\nu \rho]} + A^a_{\nu [\mu} A^b_{\rho \nu]} A^b_{\nu \mu} - A^a_{\nu [\mu} A^b_{\rho \nu]} A^b_{\rho \mu]}$. Using the matrix notation for the underlined (‘internal name’) indices, the Chern–Simons function al is well known to be defined by

$$Y[A] := \int_{\Sigma} \text{Tr} \left( F_{[\mu \nu} A_{\rho]} + \frac{2}{3} A_{[\mu A_{\nu} A_{\rho]} \right). \quad (2.1.1)$$

To ensure the existence of this integral we must impose certain fall-off conditions on the connection coefficients.

Let $(r, \theta, \phi)$ be the standard polar coordinates on $\Sigma - K \approx \mathbb{R}^3 - B$, let $\{E^a_{\mu}\}$ be a fixed global frame field and determine the fall-off condition for $A^a_{\mu b}$, with respect to these coordinates and global frame field, implied by the existence of $Y[A]$. Since on the asymptotic end the integrand of (2.1.1) takes the form $-2\text{Tr}(A_{\mu} \partial_{\nu} A_{\rho} + \frac{1}{3} A_{[\mu} A_{\nu} A_{\rho]}),$ where $\mu, \nu, \ldots = r, \theta, \phi$ and $\epsilon_{\mu \nu} = \epsilon_{\nu \rho}$ is the alternating Levi-Civita symbol, it seems natural to impose the following fall-off conditions

$$A^a_{\mu b}(r, \theta, \phi) = A^a_{\mu b}(\theta, \phi) + o \left( \frac{1}{r^a} \right),$$
$$A^a_{\mu b}(r, \theta, \phi) = A^a_{\mu b}(\theta, \phi) + o \left( \frac{1}{r^b} \right), \quad \tau = \theta, \phi, \quad (2.1.2)$$
for some $a + b > 1, \ b > 0$,

where a function $f(r)$ is said to behave at infinity like $o(r^{-a})$ if $\lim_{r \to \infty} (r^a f(r)) = 0$. If the $A^a_{\mu b}$ component has $1/r$ fall-off, then the logarithmic fall-off for the tangential components $A^a_{\rho b}$ is the necessary and sufficient condition of the existence of (2.1.1). If therefore $\mathcal{A}$ denotes the set of all the connection 1-forms on $E^a(\Sigma)$ satisfying the fall-off condition (2.1.2) then $Y : \mathcal{A} \to \mathbb{K}$ becomes well defined. If $A^a_{\mu b}(u)$ is a smooth 1 parameter family of connections in $\mathcal{A}$ then the derivative of the Chern–Simons functional $Y[A(u)]$ with respect to $u$, i.e. the ‘variation’ of $Y[A]$, is $\delta Y[A] := (\frac{d}{du} Y[A(u)])|_{u=0} = 2 \int_{\Sigma} \text{Tr} F_{[\mu \nu} \partial_{\rho] A_{\mu} A_{\nu} A_{\rho]],}$ where $\delta A^a_{\mu b} := (\frac{d}{du} A^a_{\mu b}(u)))|_{u=0}$, the ‘variation’ of the connection 1-form. Thus the fall-off condition (2.1.2) ensure the functional differentiability of $Y[A]$ with respect to the connection 1-form, and the functional derivative is essentially the curvature.

Under a gauge transformation $\Lambda : \Sigma \to \rho(G)$ the connection 1-form transforms as $A^a_{\mu b} \mapsto A^a_{\mu b} := \Lambda^a_{\mu c}(A^c_{\nu b} \Lambda^\nu_{\rho} + \partial_{\nu} \Lambda^\nu_{\rho})$, where $\Lambda^a_{\mu b}$ is defined by $\Lambda^a_{\mu b} \Lambda^b_{\rho} = \delta^a_{\rho}$. Thus the gauge transformations preserve the fall-off properties of the connection 1-forms, i.e. they don’t take a connection 1-form $A^a_{\mu b}$ out of $\mathcal{A}$, if

$$\Lambda^a_{\mu b}(r, \theta, \phi) = o \Lambda^a_{\mu b} + \frac{\Lambda^a_{\mu b}(\theta, \phi)}{r^c} + o \left( \frac{1}{r^c} \right), \quad c \geq \max \{a - 1, b\}, \quad (2.1.3)$$

where $o \Lambda^a_{\mu b}$ is a constant $\rho(G)$-matrix. Under these gauge transformations $Y[A]$ transforms as

$$Y[A] - Y[A'] = \frac{2}{3} \int_{\Sigma} \Lambda^a_{\mu b} (\partial_{\rho} \Lambda^\rho_{\nu}) A^c_{\nu b} (\partial_{\nu} \Lambda^\nu_{\rho}) A^d_{\rho c} (\partial_{\rho} \Lambda^\rho_{\mu}) \frac{1}{3!} \delta^{\mu \rho \nu \sigma} +$$
$$+ 2 \int_{\Sigma} \partial_{\mu} \left( A^a_{\mu b} \Lambda^b_{\rho} (\partial_{\rho} \Lambda^\rho_{\mu}) \right) \frac{1}{3!} \delta^{\mu \rho \nu \sigma}, \quad (2.1.4)$$

where the second term, the integral of an exact 3-form, vanishes as a consequence of the fall-off conditions. For small gauge transformations (i.e. for gauge transformations $\Lambda : \Sigma \to \rho(G)$ homotopic to the identity transformation) the integrand of the first term in (2.1.4) is also exact, and hence the right hand side of (2.1.4) is zero, but for large gauge transformations (i.e. which are not small) the right hand side is $16\pi^2 N$
for some integer $N$ depending on the homotopy class of $\Lambda : \Sigma \to \rho(G)$. This implies that $Y[A]$ modulo $16\pi^2$ is gauge invariant, and if $Y[A]$ is complex valued then its imaginary part $\text{Im} Y[A]$ in itself is gauge invariant. If $\psi : \Sigma \to \Sigma$ is any smooth proper map then $Y[\psi^* A] = \text{deg}(\psi)Y[A]$, where $\text{deg}(\psi)$ is the degree of $\psi$ [11]. Since however $\text{deg}(\psi)$ is one for orientation preserving diffeomorphisms, $Y[A]$ is invariant with respect to them.

### 2.2 The Sen–Chern–Simons functional of initial data sets

Let $\pi : L \to \Sigma$ be a trivializable principal fiber bundle over $\Sigma$ with structure group $SO_0(1,3)$, the connected component of the Lorentz group $O(1,3)$, $\rho_0$ its defining representation on the four dimensional real vector space $V^a$, and $V^a(\Sigma)$ the associated vector bundle and $V_a(\Sigma)$ its dual vector bundle. Let $E^a_\Sigma : \Sigma = 0, \ldots, 3$, be a global frame field in $V^a(\Sigma)$ with given ‘space’ and ‘time’ orientation, and let $\vartheta^a_\Sigma$ be the dual global frame field in the dual bundle. Thus small Roman indices are abstract ‘internal’ Lorentzian indices, i.e. they refer to the Lorentzian vector bundle, while underlined small Roman indices are Lorentzian name indices. If $\eta_{ab} := \text{diag}(1, -1, -1, -1)$, then $g_{ab} := \eta_{ab} \vartheta^a_\Sigma \vartheta^b_\Sigma$ is a Lorentzian fibre metric on $V^a(\Sigma)$, and $E^a_\Sigma$ becomes a $g_{ab}$-orthonormal global frame field. The global cross sections of $L$ define global gauge transformations taking $g_{ab}$-orthonormal global frame fields into $g_{ab}$-orthonormal global frame fields. $g_{ab}$ identifies $V^a(\Sigma)$ with its dual $V_a(\Sigma)$. Let $\Theta : T \Sigma \to V^a(\Sigma) : (p, v^\mu) \mapsto (p, v^\mu \Theta^a_\mu(p))$ be an imbedding of $T \Sigma$ into $V^a(\Sigma)$ such that $h_{\mu \nu} := \Theta^a_\mu \Theta^b_\nu g_{ab}$ is a global frame field into a $\Theta$-compatible frame field $[2]$. One can raise and lower the indices of $\Theta^a_\mu$ by $h^{\mu \nu}$ and $g_{ab}$, respectively, and e.g. $\Theta^a_\mu$ defines an imbedding of $T \Sigma$ into $V_a(\Sigma)$. Let $t^a$ be the section of $V^a(\Sigma)$ which is orthogonal to $\Theta(T \Sigma)$, i.e. $v^\mu \Theta^a_\mu t^a = 0$ for any section $v^\mu$ of $T \Sigma$, and has unit norm with respect to $g_{ab}$. The orientation of $t^a$ is chosen to be compatible with the ‘time’ orientation of the global frame fields, e.g. to be ‘future’ directed. Then $P^a_p := \delta^a_b - t^a t_b$ is the projection of the fibre $V^a_p$ onto $\Theta(T \Sigma)$ for any $p \in \Sigma$, and hence any section $X^a$ of $V^a(\Sigma)$ can be decomposed in a unique way as $X^a = N t^a + N^a$, where $N^a = P^a_b N^b$ is called the shift and $N$ is the lapse part of $X^a$. This decomposition defines a vector bundle isomorphism between the Whitney sum of the trivial line bundle over $\Sigma$ and $T \Sigma$, and the Lorentzian vector bundle: $i : (\Sigma \times \mathbf{R}) \oplus T \Sigma \to V^a(\Sigma) : (p, (N, N^a)) \mapsto (p, N t^a + N^a \Theta^a_\mu)$. Any $h_{\mu \nu}$-orthonormal frame field $e^a_i$, $i = 1, 2, 3$, in $T \Sigma$ defines a $g_{ab}$-orthonormal global frame field $\{t^a, e^a_i\}$ in $V^a(\Sigma)$ by $e^a_i := e^a_i \Theta^a_\mu$. Such a frame field in $V^a(\Sigma)$ will be said to be compatible with the imbedding $\Theta$, and the set of all such $\Theta$-compatible frame fields defines a reduction $SO_0(1,3) \to SO(3)$ of the gauge group (‘time gauge’). Since the quotient $SO_0(1,3)/SO(3)$ is homeomorphic to $\mathbf{R}^3$, there always exist small gauge transformations taking a global frame field into a $\Theta$-compatible frame field $[2]$.

Any connection on $\pi : L \to \Sigma$ defines a $g_{ab}$-compatible covariant derivative $D_\mu$ on $V^a(\Sigma)$, which can be characterized completely by its action on pointwise independent sections of $V^a(\Sigma)$, e.g. by $\chi_{\mu a} := D_\mu t^a$ and $D_\mu (e^a_i \Theta^a_\mu)$. We call $D_\mu$ the real Sen connection on $V^a(\Sigma)$ if the next three conditions are satisfied:

i. $D_\mu g_{ab} = 0$,

ii. $\chi_{\mu a} := (D_\mu t^a) \Theta^a_\mu = \chi_{(\mu a)}$,

iii. $D_\mu (e^a_i \Theta^a_\mu) P^a_b = (D_\mu e^a_i) \Theta^a_\mu$, where $D_\mu$ is the Levi-Civita covariant derivative on $T \Sigma$ determined by the metric $h_{\mu \nu}$.

For fixed bundle isomorphism $i$ and tensor fields $h_{\mu \nu}$ and $\chi_{\mu a}$ these conditions uniquely determine the derivative $D_\mu$. The $D_\mu$-derivative of the section $X^a = N t^a + N^a$ is $D_\mu X^a = (D_\mu N)^a t^a + (D_\mu N^b) P^a_b + (\chi_{\mu a} t^a - t^a \chi_{\mu a}) X^b$. Thus it seems useful to define the action of the Levi-Civita derivative $D_\mu$ on sections $v^a$ of $V^a(\Sigma)$ satisfying $v^a = v^b P^a_b$ by $D_\mu v^a := D_\mu (v^b \Theta^a_\mu) \Theta^a_\mu$ (see requirement iii. above), and then to extend its action to any section of $V^a(\Sigma)$ by demanding $D_\mu t^a = 0$, since then both $D_\mu$ and $D_\mu$ would be defined on the same vector bundle and one could compare them. Their difference is $(D_\mu - D_\mu) X^a = (\chi_{\mu a} t^a - t^a \chi_{\mu a}) X^b$. For the sake of later convenience let us introduce $V_{\mu \nu a} := \chi_{\mu \rho \sigma} \chi_{\nu \omega} - \chi_{\mu \omega} \chi_{\nu \rho}$ and its traces $V_{\mu \nu} := V^{a} a_{\mu \nu}$ and $V = V^{\rho \rho}$. They have all the algebraic symmetries of the Riemann and Ricci tensors in three dimensions. The curvature of $D_\mu$
has the form $F^{ab}_{\mu\nu} = \Theta^a_{\rho b}(R^\rho_{\mu\nu} + V^\rho_{\mu\nu}) + (\tau^a_{\rho b} - \tau^b_{\rho a})(D_\mu \chi_\nu^a - D_\nu \chi_\mu^a)$. Here $R^\rho_{\mu\nu}$ is the curvature tensor of $(\Sigma, h_{\mu\nu})$. The ‘Ricci part’ of the curvature is $F^{ab}_{\mu\nu}\Theta^c_\rho = \Theta^a_\rho(R^\rho_{\mu\nu} + V^\rho_{\mu\nu}) - \tau^a_{\rho b}(D_\mu \chi_\nu^a - D_\nu \chi_\mu^a)$, which, contrast to Riemannian 3-manifolds, doesn’t determine the full curvature of $D_\mu$. What remains undetermined is the term that can be represented by $H_{\mu\nu} := -\kappa_{\mu\nu}D^\rho\chi_\rho^a$. If $(M, g_{ab})$ is a Lorentzian spacetime and $\theta : \Sigma \to M$ is an embedding such that $\theta(\Sigma)$ is spacelike, then $V^a(\Sigma)$ can be identified with the pull back to $\Sigma$ of the spacetime tangent bundle $TM$ along $\theta$ and $\Theta^a_\rho$ is the differential of $\theta$. The Sen connection $D_\mu$ introduced here is $\Theta^a_\rho D_\alpha$, the pull back to $\Sigma$ of the derivative $D_\alpha := P^i_b\nabla_b$ of Sen [12], and its curvature is just the pull back to $\Sigma$ of the spacetime curvature tensor: $F^a_{\mu\nu} = (4)R^b_{\rho\mu\nu}\Theta^a_\rho D_\nu$. Then the tensor $H_{\mu\nu}$ becomes the pull back to $\Sigma$ of the magnetic part $H_{ab} := \frac{1}{2}\varepsilon_{abc}\varepsilon_{df}C_{ef\beta\gamma}\varepsilon^{\alpha\beta\gamma\delta}$ of the spacetime Weyl tensor. (Note that we use the convention in which the relation between the three and four dimensional volume forms is $\varepsilon_{abc} = \varepsilon_{abcd}\varepsilon^{\alpha\beta\gamma\delta}$.) On the other hand, in general the electric part of the spacetime Weyl tensor, $E_{ab} := C_{ef\beta\gamma}\varepsilon^{\alpha\beta\gamma\delta}$, cannot be expressed by the geometric data on $\Sigma$. It contains the spatial-spatial part of the spacetime Einstein tensor and the spacetime curvature scalar too: $E_{ab} = -(R_{ab} + V_{ab} - \frac{1}{2}(4)G_{cd}P_a^cP_b^d + \frac{1}{4}h_{ab}(R + V + \frac{1}{2}(4)R)$. The ‘constraint parts’ of the spacetime Einstein tensor are $(4)G_{ab}\eta^{\alpha\beta} = -\frac{1}{4}(R + V)$ and $(4)G_{ab}\eta^{\alpha\beta} = -D_\alpha(\chi^\alpha - \chi^\delta_\delta)$. The connection coefficients of $D_\mu$ with respect to any pair of dual global frame fields are $\Gamma^a_{\mu\nu} := \varepsilon^a_{\beta\gamma}\rho D_\nu\rho_{\beta\gamma}$, and, following the general prescription of the previous subsection, we can form the Chern–Simons functional built from the real Sen connection. This $Y[\mathcal{W}_{\mu\nu}]$ will be called the Sen–Chern–Simons functional. Since the difference of the Chern–Simons 3-form in one gauge and in another gauge obtained by a small gauge transformation is an exact 3-form, the Sen–Chern–Simons functional can always be calculated in the time gauge. In the frame field compatible with the imbedding $\Theta$ the connection coefficients are $\Gamma^a_{\mu\nu} = -\kappa_{\mu\nu}\varepsilon^a_{\beta\gamma}$ and $\Gamma^a_{\lambda j} = \zeta^a_\lambda D_\mu\varepsilon^\mu_j$, the Ricci rotation coefficients of $D_\mu$. Here $\{\varepsilon^a_{\beta\gamma}\}$ is the global frame field in $T^*\Sigma$ dual to $\{e^\mu_i\}$. The curvature 2-form, also in the time gauge, is given by $F^a_{\jmath k} = \varepsilon^a_{\beta\gamma}\rho D_\jmath\rho_{\beta\gamma}$ and $F^a_{\mu\nu} = e^a_j(D_\mu \chi^\alpha_\nu - D_\nu \chi^\alpha_\mu)$. Since however these expressions depend only on the triad field $\{e^\mu_i\}$ and the tensor field $\chi^\alpha_{\mu\nu}$ and independent of the vector bundle isomorphism $i$ (or even the imbedding $\Theta$), the Sen–Chern–Simons functional will be completely determined by $\{e^\mu_i\}$ and $\chi_{\mu\nu}$. Finally, since $Y[e^\mu_i, \chi_{\mu\nu}]$ modulo $16\pi^2$ is gauge invariant, it is a functional only of $h_{\mu\nu}$ and $\chi_{\mu\nu}$ and will be denoted by $Y_0[h_{\mu\nu}, \chi_{\mu\nu}]$. Next determine the fall-off properties of $h_{\mu\nu}$ and $\chi_{\mu\nu}$ implied by the general fall-off conditions (2.1.2). Let $\rho h_{\mu\nu}$ be a fixed negative definite metric on $\Sigma$ such that the asymptotic end $\Sigma - K$, together with the restriction of $\rho h_{\mu\nu}$ to $\Sigma - K$, is isometric to the standard flat geometry on $\mathbb{R}^3 - B$. Let $\{\varepsilon^a_{\beta\gamma}\}$ be a $\rho h_{\mu\nu}$-orthonormal dual frame field which is constant on $\Sigma - K$, i.e. $\rho D_\mu\varepsilon^a_{\beta\gamma} = 0$, and let the orientation of these frame fields be chosen to be that of the $h_{\mu\nu}$-orthonormal ‘physical’ frames $\{\zeta^a_\lambda\}$. Since both $\{\varepsilon^a_{\beta\gamma}\}$ and $\{\zeta^a_\lambda\}$ are bases with the same orientation, they can be combined from each other, i.e. for some globally defined $GL(3, R)$-valued function $\Phi^a_\lambda$ on $\Sigma$ with positive determinant we have $\zeta^a_\lambda = \rho \varepsilon^a_{\beta\gamma}\Phi^a_\lambda$. If $\Phi^a_\lambda$ is defined by $\Phi^a_\lambda\Phi^a_\mu = \delta^a_\lambda$, then $e^\mu_i = \rho \varepsilon^a_{\beta\gamma}\Phi^a_\lambda$. $\Phi^a_\lambda$ can be decomposed in a unique way as $\Phi^a_\lambda = S^a_k\Lambda^k_\lambda$, where $\Lambda^k_\lambda$ is a rotation matrix, $\Lambda^k_\lambda\Lambda^1_\lambda = \eta^k_\lambda$ and $S_l^k := \eta^{lk}\eta^k_\lambda = S^{l\lambda}$). $\Lambda^k_\lambda$ represents the pure gauge, while $S_l^k := S^{l\lambda} - \delta^k_\lambda$ is the metric ‘deformation’ content of $\Phi^a_\lambda$. In fact, the ‘physical’ metric is a quadratic expression of the symmetric part: $h_{\mu\nu} = \rho \varepsilon^a_{\beta\gamma}\rho \varepsilon^a_{\beta\gamma}\Phi^a_\lambda S^a_k S^b_l \eta^{k\lambda} = \rho h_{\mu\nu} + \rho \varepsilon^a_{\beta\gamma}\rho \varepsilon^a_{\beta\gamma}(2S_{lj} + 2S^k_l S^j_k \eta^{k\lambda})$. The matrices in the decomposition $\Phi^a_\lambda = S^a_k\Lambda^k_\lambda$ are transposed inverses of the corresponding matrices in $\Phi^a_\lambda$: $S^a_k \Lambda^j_\lambda = \delta^a_j$ and $S^a_k S^j_k = \delta^a_j$. Note that although $\Lambda^k_\lambda = \eta^{jk}\Lambda^1_k \eta^j_\lambda$, $S^j_k$ is not $\eta^{jk} \Phi^a_\lambda \eta^j_\lambda$. Then first calculate the connection coefficients $\Gamma^a_{\mu\nu}$. They are

$$
\Gamma^a_{\mu\nu} = -\rho D_\nu\Phi^a_\lambda \Phi^{k,\lambda} + \Phi^{k,\lambda} \Phi^a_\lambda \eta^{m,\lambda} 0 e^a_{\beta\gamma} e^\mu_i e^\nu_j \frac{1}{2} \rho D_\mu h_{\nu\rho} + \rho D_\nu h_{\rho\mu} + \rho D_\rho h_{\nu\mu} \rho D_\nu h_{\mu\rho} .
$$

Since we are interested in the fall-off of $h_{\mu\nu}$ and of $\chi_{\mu\nu}$ implied by (2.1.2), we may write $S^a_j = \delta^a_j - 2j^a_i$ and retain in (2.2.1) only the terms which are zeroth and first order in $s_{ij}$. We get $(\rho \varepsilon^a_{\beta\gamma}\rho \varepsilon^a_{\beta\gamma})(\rho \varepsilon^a_{\beta\gamma}\rho \varepsilon^a_{\beta\gamma})\Gamma^a_{\mu\nu} = -\rho \varepsilon^a_{\beta\gamma}\Phi^a_\lambda 0 \varepsilon^a_{\beta\gamma} (\partial_\nu S^k_k) + \rho \varepsilon^a_{\beta\gamma}(\partial_\nu S^k_k) 0 \varepsilon^a_{\beta\gamma} - (\partial_\nu S^k_k) 0 \varepsilon^a_{\beta\gamma}$. The evaluation of this equation for the various
components yields the following results: First, the fact that the connection coefficients $\Gamma_{ij}^k$ come from a metric links the powers ‘a’ and ‘b’ in (2.1.2): a = b + 1. Second, the fall-off rate of the metric is just that of the tangential components of the connection: $s_{ij}(r, \theta, \phi) = s_{ij}(\theta, \phi)/r^b + o(r^{-b})$. Third, the gauge part must also tend to a constant gauge transformation with the same fall-off: $\Lambda_{ij}(r, \theta, \phi) = 0\Lambda_{ij} + \Lambda_{ij}(\theta, \phi)/r^b + o(r^{-b})$.

Finally, taking into account these results, the equation $\chi_{\mu\nu} = -\Gamma_{ij}^k h^{\nu}_k$ yields the fall-off $e_i^\mu e_j^\nu \chi_{\mu\nu}(r, \theta, \phi) = \chi_{ij}(\theta, \phi)/r^b + o(r^{-b})$. Thus the Sen–Chern–Simons functional is well defined for those asymptotically flat initial data sets $(\Sigma, h_{\mu\nu}, \chi_{\mu\nu})$ for which both $h_{\mu\nu} - 0 h_{\mu\nu}$ and $r \chi_{\mu\nu}$ fall off like $r^{-b}$ for some positive $b$. But since $b$ may be arbitrarily small, the weakest possible fall-off for $h_{\mu\nu} - 0 h_{\mu\nu}$ and $r \chi_{\mu\nu}$ is logarithmic.

The variational derivative of $Y_0$ with respect to $h_{\mu\nu}$ and $\chi_{\mu\nu}$ has been calculated [2]:

$$\frac{\delta Y_0}{\delta \chi_{\mu\nu}} = 8 \sqrt{|h|} H^{\mu\nu}, \quad \frac{\delta Y_0}{\delta h_{\mu\nu}} = -4 \sqrt{|h|} \left( B^{\mu\nu} + \chi^{(\mu} H^{\nu)} \right),$$

(2.2.2)

where $H_{\mu\nu}$ is the conformal magnetic curvature introduced above, and $B_{\mu\nu} := -\varepsilon_{\rho\nu} D^\rho (R^\mu + V^\mu) + \tfrac{1}{2} \chi^\rho (\varepsilon_{\rho\nu}) R^\mu \chi - D^\rho \chi^\nu$. Both $H_{\mu\nu}$ and $B_{\mu\nu}$ are symmetric and trace-free, and $B_{\mu\nu}$ reduces to the Cotton–York tensor of $(\Sigma, h_{\mu\nu})$ if $\chi_{\mu\nu}$ is vanishing. We have shown that the stationary point of $Y_0[h_{\mu\nu}, \chi_{\mu\nu}]$ (i.e. for which $B_{\mu\nu} = 0$ and $H_{\mu\nu} = 0$) are precisely the data sets $(\Sigma, h_{\mu\nu}, \chi_{\mu\nu})$ that can be locally isometrically imbedded into a conformally flat spacetime with first and second fundamental forms $h_{\mu\nu}$ and $\chi_{\mu\nu}$, respectively.

The spacetime conformal rescaling of $(\Sigma, h_{\mu\nu}, \chi_{\mu\nu})$ by the pair of functions $\Omega : \Sigma \to (0, \infty)$, $\hat{\Omega} : \Sigma \to \mathbb{R}$ was defined by the data set $(\Sigma, \hat{h}_{\mu\nu}, \hat{\chi}_{\mu\nu})$, where $\hat{h}_{\mu\nu} := \Omega^2 h_{\mu\nu}$ and $\hat{\chi}_{\mu\nu} := \Omega \chi_{\mu\nu} + \hat{\Omega} h_{\mu\nu}$. To preserve the fall-off properties of $h_{\mu\nu}$ and $\chi_{\mu\nu}$ we should impose $\lim_{r \to \infty} \hat{\Omega} = 0$ and $\lim_{r \to \infty} \Omega = 0$. In [2] we calculated the transformation of the connection coefficients $\Gamma_{\mu\nu}^k$, the curvature $F^a h_{\mu\nu}$ and the Sen–Chern–Simons functional under spacetime conformal rescalings. If we define $\Upsilon_a := \Omega^{-1}(\hat{\Omega} \Upsilon_a + \Theta_{a}^{\nu} D_{\nu} \Omega)$, then in the present notations $Y[\Gamma_{\mu\nu}^k]$ transforms as

$$Y[\hat{\Gamma}_{\mu\nu}^k] - Y[\Gamma_{\mu\nu}^k] = -\int_{\Sigma} D_{\rho} \left( \varepsilon_{\rho\nu} \Upsilon_a E_{a}^{\nu \rho} \partial_{\nu} \phi \right) d\Sigma,$$

(2.2.3)

where $d\Sigma$ is the metric volume element on $\Sigma$. Thus $Y_0[h_{\mu\nu}, \chi_{\mu\nu}]$ is invariant with respect to the allowed spacetime conformal rescalings of the initial data sets. (From the right hand side of the corresponding formula in [2] the minus sign is missing.) Under the spacetime conformal rescaling the tensors $H_{\mu\nu}$ and $B_{\mu\nu}$ transform as $\hat{H}_{\mu\nu} = H_{\mu\nu}$ and $\hat{B}_{\mu\nu} = \Omega^{-1} (B_{\mu\nu} + \Omega^{-1} \hat{\Omega} H_{\mu\nu})$.

Finally, the diffeomorphism invariance of $Y_0$ implies that for any smooth vector field $N^\mu$, generating 1-parameter families of diffeomorphisms which preserve the fall-off properties of $h_{\mu\nu}$ and $\chi_{\mu\nu}$, the integral $\int_{\Sigma} \left( \frac{\delta Y_0}{\delta h_{\mu\nu}} \frac{L}{N} h_{\mu\nu} + \frac{\delta Y_0}{\delta \chi_{\mu\nu}} L \chi_{\mu\nu} \right) d\Sigma$ is vanishing, where $L\chi$ denotes the Lie derivative along $N^\mu$. This yields a divergence identity for $H_{\mu\nu}$ and $B_{\mu\nu}$.

2.3 The spinor representation and the Ashtekar–Chern–Simons functional

Let $\pi : S \to \Sigma$ be a trivializable principal fibre bundle over $\Sigma$ with structure group $SL(2, \mathbb{C})$, $\rho$ its defining representation on the two complex dimensional vector space $S^A$, and $S^A(\Sigma)$ the (trivializable) associated vector bundle. Because of its trivializability there always exist globally defined frame fields $\left\{ \varepsilon_A^A \right\}$, $A = 0, 1$. Thus the capital Roman indices are abstract spinor indices, while the underlined capital Roman indices are name indices referring to a spinor basis. The dual, complex conjugate and the dual-complex conjugate bundles will be denoted by $S_A(\Sigma)$, $S^A(\Sigma)$ and $\bar{S}_A(\Sigma)$, respectively, in which the global frame fields are $\left\{ \varepsilon_A^A \right\}$, $\left\{ \bar{\varepsilon}^A_A \right\}$ and $\left\{ \varepsilon_A^A \right\}$. If $\varepsilon_{AB}$ is the alternating Levi-Civita symbol, then $\varepsilon_{AB} := \varepsilon_A^A \varepsilon_B^B$ defines a symplectic fibre metric on $S^A(\Sigma)$, i.e. with respect to which $\left\{ \varepsilon_A^A \right\}$ is normalized (or spin frame). By means
of \( \varepsilon_{AB} \) and its inverse \( \varepsilon^{AB} \), defined by \( \varepsilon^{AC}\varepsilon_{BC} = \delta^A_B \), \( S^A(\Sigma) \) can be identified with \( S_A(\Sigma) \), and, by the complex conjugate metric \( \varepsilon_{AB}' \) and its inverse \( \varepsilon^{AB}' \), \( \tilde{S}^A(\Sigma) \) can be identified with \( \tilde{S}_A(\Sigma) \).

Because of the trivializability of the bundles \( V^a(\Sigma) \) and \( S^A(\Sigma) \) the well known isomorphism of the Lorentzian vector space and the space of Hermitian spinors, explained e.g. in [9], can be ‘globalized’ on the whole of \( \Sigma \). Namely, there exists a bundle isomorphism between the complexified Lorentzian vector bundle and the tensor product of \( S^A(\Sigma) \) with its complex conjugate bundle, \( \vartheta : V^a(\Sigma) \otimes C \rightarrow S^A(\Sigma) \otimes \tilde{S}^A(\Sigma) : (p, X^a) \mapsto (p, X^a \theta_a^A) \), satisfying \( \theta_a^A \theta_b^B + \theta_b^B \theta_a^A = \varepsilon_{AB} \). Therefore \( \theta_a^A \) links the fibre metrics \( \varepsilon_{AB} \) on \( S^A(\Sigma) \) and \( g_{ab} \) on \( V^a(\Sigma) \) and defines an \( SL(2, C) \)-spinor structure on \( \Sigma \) in the sense that \( \pi : S \rightarrow \Sigma \) is the universal covering bundle of \( \pi : L \rightarrow \Sigma \) and \( g_{ab} \) maps the right action of \( SL(2, C) \) on the former to the right action of \( SO_0(1, 3) \) on the latter. \( S^A(\Sigma) \) is the bundle of \( SL(2, C) \) spinors on \( \Sigma \). The image of \( V^a(\Sigma) \) in \( S^A(\Sigma) \otimes \tilde{S}^A(\Sigma) \) is the bundle of Hermitian spinors. Thus, if \( E_{AA}^a \) is the inverse of \( \theta_a^A \), then \( E_{AA'}^a \lambda^A \lambda'^A \) is real, null (with respect to the Lorentzian metric) and is either future or past directed for any spinor \( \lambda^A \). We choose \( \theta_a^A \) such that \( E_{AA'}^a \lambda^A \lambda'^A \) be future directed. Therefore the normal section \( t_a \) of \( V_a(\Sigma) \) defines a positive definite Hermitian fibre metric on \( S^A(\Sigma) \) by \( G_{AA'} := \sqrt{2} t_a E_{AA'}^a \). As a consequence of the normalization \( G_{AA'} \) is compatible with the symplectic metric: \( \varepsilon^{AB} G_{AA'} G_{BB'} = \varepsilon_{AB} \), and hence \( \varepsilon^{AB} \varepsilon^{A'B'} G_{BB'} \) is just the inverse \( G_{A'A} \) of \( G_{AA'} \). If \( \{ \varepsilon_A \} \) is a spin frame field and \( \{ \varepsilon'_A \} \) its complex conjugate then \( E_{AA'}^a := E_{AA'}^a \varepsilon^A \varepsilon^{A'} \sigma_{AB}^{A'B'} \) is a \( g_{ab} \)-orthonormal global frame field in \( V^a(\Sigma) \), where \( \sigma_{AB}^{A'B'} \) are the \( SL(2, C) \) Pauli matrices.

Since any connection on a principal bundle determines a unique connection on each of its covering bundles (see e.g. Theorem 6.2 of Ch. II. in [10], which can be generalized to cover this case), the connection on \( \pi : L \rightarrow \Sigma \) defines a connection on \( \pi : S \rightarrow \Sigma \). Thus \( D_\mu \) on \( V^a(\Sigma) \) determines a covariant derivative, denoted also by \( D_\mu \), both on \( S^A(\Sigma) \) and \( \tilde{S}^A(\Sigma) \), and \( D_\mu \) annihilates both \( \varepsilon_{AB} \) and \( \varepsilon_{AB}' \). This spinor covariant derivative is fixed completely by the requirement \( (D_\mu \lambda^A) \bar{\mu}^{A'} + \lambda^A (D_\mu \bar{\mu}^{A'}) E_{AA'}^a = D_\mu (\lambda^A \bar{\mu}^{A'} E_{AA'}^a) \). Applying this equation to the spinors of the spin frame field we get the connection between the spinor connection coefficients \( \Gamma^{A}_{B\mu} := \epsilon^{A}_{A'B} D_\mu \epsilon^{A}_{B} \) and the Lorentzian connection coefficients: \( \delta^{A}_{B'} \Gamma^{A}_{B\mu} + \delta^{A'}_{B} \Gamma^{A'}_{B'\mu} = \sigma^{A}_{B'} \Sigma_{B}^{B'} \sigma^{B}_{B'\mu} \). Then for the curvature \( F^{A}_{B\mu} \) of \( D_\mu \) on \( S^A(\Sigma) \), as can be expected, \( \delta^{A}_{B'} F^{A}_{B\mu} + \delta^{A'}_{B} F^{A'}_{B'\mu} = \delta^{A}_{B'} E_{B'B'}^{A} F^{A}_{B'\mu} \) holds. Now we are in the position to be able to form the Chern–Simons 3-form and functional \( Y[\Gamma^A_{B\nu}] \) built from the \( SL(2, C) \)-connection \( \Gamma^{A}_{B\mu} \) on \( S^A(\Sigma) \), for which we get the following simple result:

\[
Y[\Gamma^A_{B\nu}] = 2 \left( Y[\Gamma^{A}_{B\nu}] + Y[\Gamma^{A'}_{B'\nu}] \right) = 2 \left( Y[\Gamma^{A}_{B\nu}] + Y[\Gamma^{A'}_{B'\nu}] \right).
\]

We note that the fall-off conditions for the initial data determined in the previous subsection imply the existence of the functional \( Y[\Gamma^A_{B\nu}] \), too. Thus the Sen–Chern–Simons functional is only the (four times the) real part of the Chern–Simons functional built from the \( SL(2, C) \) spinor connection of the initial data set on \( S^A(\Sigma) \).

In [2] we also considered the self-dual/anti-self-dual representation \( \rho_{\pm} \) of \( SO_0(1, 3) \) and the Chern–Simons functional on the corresponding vector bundle. That vector bundle was \( \pm \Lambda^2(\Sigma) \), the bundle of self-dual/anti-self-dual 2-forms. \( E_{AA'}^a \) defines an isomorphism between \( -\Lambda^2(\Sigma) \) and the bundle \( S(\Lambda^2(\Sigma)) \) of the symmetric second rank unprimed spinors, and between \( +\Lambda^2(\Sigma) \) and \( \tilde{S}(\Lambda^2(\Sigma)) \). If \( \{ \varepsilon_A \} \) is a spin frame field in \( S_A(\Sigma) \) then \( \varepsilon_{AB} := \varepsilon^{A}_{A'B'} \varepsilon^A_{B'} \), \( i = 1, 2, 3 \), form a global frame field in \( S(\Lambda^2(\Sigma)) \), where \( \varepsilon_{AB}^i \) are the \( SU(2) \) Pauli matrices. This basis is orthonormal with respect to the scalar product \( (z, w) = z_{AB} w_{CD} \varepsilon^{AC} \varepsilon^{BD} \) on \( S(\Lambda^2(\Sigma)) \), inherited from the scalar product \( (Z, W) = \frac{1}{2} Z_{ab} W_{cd} \gamma^{ac} \gamma^{bd} \) on \( \Lambda^2(\Sigma) \).

(The frame field and the scalar product on \( -\Lambda^2(\Sigma) \) that we used in [2] was \( \sqrt{2} \) times and four times the ‘natural’ choice \( \varepsilon^{A}_{A'B'} \varepsilon^A_{B'} \theta_a^A \theta_b^B \) and scalar product above, respectively.) The derivative \( D_\mu \) can naturally be extended to these bundles, whose connection coefficients in the basis \( \{ \varepsilon_{AB} \} \) are \( \Lambda^2_{i} := \varepsilon^{A}_{AB} D_\mu \varepsilon^{A}_{B} = -i \sqrt{2} \varepsilon^{A}_{AB} D_\mu \varepsilon^{A}_{B} \), and similarly in the complex conjugate basis \( \{ \varepsilon_{A'B'} \} \) they are \( \Lambda_{i} := \varepsilon^{A}_{A'B'} D_\mu \varepsilon^{A}_{B'} = -i \sqrt{2} \varepsilon^{A}_{A'B'} D_\mu \varepsilon^{A}_{B'} \), respectively.)
\[ \varepsilon_{ij}^{A}B^{k}D_{\mu}^{\nu} = \varepsilon_{ij}^{A}B^{k}D_{\mu}^{\nu} \]  

Then the corresponding curvature 2-forms are 
\[ -F^{ij}_{\mu\nu} = -i\sqrt{2}\varepsilon_{ij}^{A}B^{FAB}_{\mu\nu}, \]  
and \[ +F^{ij}_{\mu\nu} = F^{ij}_{\mu\nu}. \]  
The Chern–Simons functional built from \( \pm A_{ij}^{A} \), which we called in [2] the self-dual/anti-self-dual Ashtekar–Chern–Simons functional, can now be reexpressed by the \( SL(2, \mathbb{C}) \) connection and curvature:

\[
Y[-A_{ij}^{A}] = 4Y[\Phi_{B^{F}_{ji}}], \quad Y[+A_{ij}^{A}] = 4Y[\Phi_{B^{F}_{ji}}] = 4Y[\Phi_{B^{F}_{ji}}].
\]  

(2.3.2)

Thus, as could be expected, the Ashtekar–Chern–Simons functional is essentially the Chern–Simons functional built from the \( SL(2, \mathbb{C}) \)-spinor connection, and the Sen–Chern–Simons functional is just its real part.

The functional derivative of \( Y_{\pm} := Y_{\pm}[h_{\mu
u}, \chi_{\mu
u}] \), defined by \( Y[\pm A_{ij}^{A}] \) modulo 16\( \pi^{2} \), with respect to \( \chi_{\mu
u} \) and \( h_{\mu
u} \) are

\[
\frac{\delta Y_{\pm}}{\delta h_{\mu
u}} = 8\sqrt{|h|}\left(R^{\mu\nu} - \frac{1}{2}R \eta^{\mu\nu} + V^{\mu\nu} - \frac{1}{2}V h^{\mu\nu}\right),
\]

\[
\frac{\delta Y_{\pm}}{\delta \chi_{\mu
u}} = -4\sqrt{|h|}\left(B^{\mu\nu} + \chi^{(\mu} h^{\nu)\rho} \pm 4i\sqrt{|h|}\left(\varepsilon^{\rho\omega\nu}(D_{\rho}H_{\omega}) + \frac{1}{2}D^{(\mu}(D_{\rho}\chi^{\nu)} - D^{\nu)\chi}\right) - \frac{1}{2}h^{\mu\nu}D_{\rho}(D_{\rho}\chi^{\nu)} - D^{\nu)\chi}\right)\delta h_{\mu\nu}. (2.3.3)
\]

The stationary points of these functionals were shown to be those data sets that can be locally isometrically imbedded into the Minkowski spacetime with first and second fundamental forms \( h_{\mu\nu} \) and \( \chi_{\mu\nu} \), respectively. Under spacetime conformal rescalings \( Y_{\pm} \) are not invariant, because e.g. under the infinitesimal conformal rescaling \( \delta h_{\mu
u} = 2h_{\mu\nu}\delta\Omega, \delta\chi_{\mu\nu} = \chi_{\mu\nu}\delta\Omega + h_{\mu\nu}\delta\Omega \), they transform as \( \delta Y_{\pm} = \pm 8i \int_{\Sigma}(R + V)\delta\Omega - D_{\mu}(D_{\nu}\chi^{\mu\nu} - D^{\mu}\chi)\delta \Omega d\Sigma \). (Unfortunately the formulae corresponding to (2.3.3) in [2] contain trivial numerical and sign errors, and consequently the expression for \( \delta Y_{\pm} \) given there is also erroneous.)

Next, let us consider general finite dimensional irreducible representations of \( SL(2, \mathbb{C}) \), the corresponding associated vector bundles, the connection on them, and the Chern-Simons functional. It is well known that any finite dimensional irreducible representation of \( SL(2, \mathbb{C}) \) is characterized by a pair \( (k, l) \) of non-negative integers and the representation space is \( S^{(A_{1}...A_{k})} \otimes \overline{S}^{(B_{1}...B_{l})} \), the tensor product of the space of the totally symmetric spinors of rank \( k \) and of the totally symmetric primed spinors of rank \( l \). Thus a basis in this space has the form \( \varepsilon_{A_{1}...A_{k}}^{B_{1}...B_{l}} = \varepsilon_{A_{i}...A_{k}}^{B_{i}...B_{l}}, i = 1,...,(k + 1)(l + 1) \). The dual basis in the dual space \( S^{(A_{1}...A_{k})} \otimes \overline{S}^{(B_{1}...B_{l})} \) is denoted by \( \varepsilon_{A_{1}...A_{k}}^{B_{1}...B_{l}} \). These bases can also be expressed by the tensor product bases: \( \varepsilon_{A_{1}...A_{k}}^{B_{1}...B_{l}} = \sigma^{A_{1}...A_{k}}_{A_{1}...A_{k}} \sigma^{B_{1}...B_{l}}_{B_{1}...B_{l}} = \sigma^{(A_{1}...A_{k})}_{A_{1}...A_{k}} \sigma^{B_{1}...B_{l}}_{B_{1}...B_{l}} \), where the combination coefficients (the well known Clebsch–Gordan coefficients) are completely symmetric both in their unprimed and primed spinor name indices and satisfy the duality conditions \( \sigma^{A_{1}...A_{k}}_{A_{1}...A_{k}} \sigma^{B_{1}...B_{l}}_{B_{1}...B_{l}} = \delta^{A_{1}...A_{k}}_{A_{1}...A_{k}} \delta^{B_{1}...B_{l}}_{B_{1}...B_{l}} \) and \( \sigma^{A_{1}...A_{k}}_{A_{1}...A_{k}} \sigma^{B_{1}...B_{l}}_{B_{1}...B_{l}} = \sigma^{(A_{1}...A_{k})}_{A_{1}...A_{k}} \sigma^{B_{1}...B_{l}}_{B_{1}...B_{l}} \).

Then the connection coefficients of the connection on the associated vector bundle \( S^{(A_{1}...A_{k})}(B_{1}...B_{l})(\Sigma) \), determined by the connection \( \mathcal{D}_{\mu} \) on \( S^{A}(\Sigma) \), are

\[
A_{ij}^{A} := \varepsilon_{A_{1}...A_{k}}^{B_{1}...B_{l}} \mathcal{D}_{\mu}^{A_{1}...A_{k}} B_{i}^{B_{1}...B_{l}} = \]
\[
k \sigma^{A_{1}...A_{k}}_{A_{1}...A_{k}} \sigma^{B_{1}...B_{l}}_{B_{1}...B_{l}} \sigma^{(A_{1}...A_{k})}_{A_{1}...A_{k}} \sigma^{B_{1}...B_{l}}_{B_{1}...B_{l}} + l \sigma^{A_{1}...A_{k}}_{A_{1}...A_{k}} \sigma^{B_{1}...B_{l}}_{B_{1}...B_{l}} \sigma^{(A_{1}...A_{k})}_{A_{1}...A_{k}} \sigma^{B_{1}...B_{l}}_{B_{1}...B_{l}} + E_{B_{1}...B_{l}}^{A_{1}...A_{k}} B_{i}^{B_{1}...B_{l}} \mathcal{D}_{\mu}^{A_{1}...A_{k}} B_{i}^{B_{1}...B_{l}},
\]  

(2.3.4)

and there is a similar relation between the curvature 2-forms \( F^{ij}_{\mu\nu} \) and \( F^{E}_{\mu\nu} \). Then, using the duality conditions for the Clebsch–Gordan coefficients, the Chern–Simons functional \( Y[A_{ij}^{A}] \) defined on the vector bundle \( S^{(A_{1}...A_{k})}(B_{1}...B_{l})(\Sigma) \) can be computed easily:
Thus this is a combination of $Y[\Gamma^A_{\mu\nu}]$ and its complex conjugate with integers depending on the representation. The representations in which the Chern–Simons functional is conformally invariant are precisely the tensor representations; i.e. for which $l = k$. Therefore the higher order irreducible representations do not give anything new. The Sen–Chern–Simons and the anti-self-dual/self-dual Ashtekar–Chern–Simons functionals correspond to the $(1,1)$, $(2,0)$ and $(0,2)$ cases, and by an appropriate choice for the normalization the Clebsch–Gordan coefficients reduce to the $SL(2,\mathbb{C})$ Pauli matrices, the $SU(2)$ Pauli matrices and their complex conjugate, respectively.

Finally, since $SL(2,\mathbb{C})$ is semisimple, any finite dimensional representation $\rho$ of that is the direct sum of irreducible representations $\rho_1, \ldots, \rho_n$ (i.e. it is completely reducible). But then the associated vector bundle $E(\Sigma)$ defined by $\rho$ is the Whitney sum of the vector bundles corresponding to $\rho_1, \ldots, \rho_n$, and the connection on $E(\Sigma)$ is the sum of the connections of the constituent bundles. Hence the Chern–Simons functional defined on $E(\Sigma)$ is the sum of the Chern–Simons functionals of the constituent bundles.

As is usual in the recent spinor approaches in general relativity in the rest of this paper we will not write out the isomorphisms $E^a_{\ ;A^I}$, $\bar{\vartheta}_{\mu}^A$ and the Pauli matrices explicitly. Any Lorentzian tensor index, e.g. $a$ and $\sigma$, can be freely replaced by the corresponding pair of spinor indices, i.e. by $A\lambda$ and $A\bar{\lambda}$, respectively.

### 3. Relation to 3-surface twistors

#### 3.1 The unitary form of $D_\mu$ and the 3-surface twistors

The positive definite Hermitian metric defines the bundle maps $\bar{S}_A(\Sigma) \rightarrow S_X(\Sigma) : (p, \mu_A) \mapsto (p, G_X^A\mu_A)$ and $S^A(\Sigma) \rightarrow S^X(\Sigma) : (p, \mu^A) \mapsto (p, -G_X^A\mu^A)$. They are $\mathbb{C}$-linear isomorphisms taking the symplectic fibre metrics into the symplectic fibre metrics, and making possible to use only the unprimed spinors. In particular, the complex conjugation is represented by the $\mathbb{C}$-anti-linear operation $\dagger : S_A(\Sigma) \rightarrow S_A(\Sigma) : (p, \lambda_A) \mapsto (p, \lambda_A) := (p, G^A\lambda_A)$, whose action can obviously be extended to arbitrary spinors. If we convert the primed name indices into unprimed ones in an analogous way, then these bundle maps take the dual complex conjugate and complex conjugate frames, $\{\varepsilon^A_A\}$ and $\{\bar{\varepsilon}^A_A\}$, into the frames $\{\varepsilon_X^A\}$ and $\{\bar{\varepsilon}^A_X\}$, respectively. Every Lorentzian index corresponds to a pair of unprimed spinor indices, e.g. $\varepsilon^a$ to $X^A$; and $X^a$ is proportional to the normal section iff $X^aX^b = X^aX^b$ and $X^a = P^a_{\;b}X^b$ iff $X^{AB} = X^{(AX)}$. Therefore the order of the unitary spinor indices, e.g. $AX$ above, is important unless the corresponding Lorentzian index is purely spatial. For example the unitary spinor form of the normal section $t_a$ and of the imbedding $\Theta^a_\mu$ are $t_{AX} := G_X^A t_{\mu A^I} = \frac{1}{\sqrt{2}}X^a$ and $\Theta^AX_\mu := -\Theta^A_{\mu} G^A_{\chi} = \Theta^\chi_{(AX)}$, respectively. The unitary spinor form of other important tensor fields, namely the metric, the corresponding volume form and the extrinsic curvature are $h_{AXBY} = -\varepsilon_{A(B\varepsilon_Y)X}, \varepsilon_{AXBYCZ} = \frac{1}{\sqrt{2}}(\varepsilon_{A(B\varepsilon_Y)(C\varepsilon_Z)X} + \varepsilon_{X(B\varepsilon_Y)(C\varepsilon_Z)A})$ and $\chi_{\mu AX} := G_X^A\chi_{\mu A^I}$, respectively. A $(2r,2s)$ type spinor $T_{B_1X_1,...,B_rX_r}^{A_1X_1,...,A_sX_s}$ is the unitary spinor form of a real Lorentz tensor iff $T_{B_1X_1,...,B_rX_r}^{A_1X_1,...,A_sX_s} = (-)^{s+s}T_{B_1X_1,...,B_rX_r}^{A_1X_1,...,A_sX_s}$. In particular, the reality of $\Theta^a_\mu$ implies $\Theta^A_{\mu} = -\Theta^A_{\mu}$. The square of the adjoint operation is $\xi_{B_1X_1,...,B_rX_r}^{A_1X_1,...,A_sX_s} = (-)^{s+s}\xi_{B_1X_1,...,B_rX_r}^{A_1X_1,...,A_sX_s}$. The action of the Sen derivative on the spinor fields can be written as $D_\mu\lambda^A = D_\mu\lambda^A - \frac{1}{\sqrt{2}}\chi_{\mu AB}\lambda^B$. Note, however, that although both $D_\mu$ and $D_\mu$ annihilate $\varepsilon_{AB}$ and $D_\mu$ annihilates $G_{AA}$, the Sen derivative doesn’t annihilate the Hermitian metric: $D_\mu G_{AA} = \frac{1}{\sqrt{2}}\chi_{\mu AA'} \neq 0$. Thus the operation of taking the unitary form of a spinor and the Sen–derivative are not commuting. Consequently $D_\mu\lambda^A$ differs from $(D_\mu\lambda^A)^1$, and hence it seems useful to introduce two Sen operators on the spinor fields, the self-dual and the anti-self-dual ones: $\pm D_\mu\lambda^A := D_\mu\lambda^A \pm \frac{1}{\sqrt{2}}\chi_{\mu AB}\lambda^B$. Then $(\mp D_\mu\lambda^A)^1 = \mp D_\mu\lambda^A$, i.e. they are adjoint to each other. (For a more detailed discussion of the theory of unitary spinors in relativity see e.g. [12-16].)
One can define the unitary spinor form of the Sen operators too: \( \pm D_{AX} \lambda_B := \Theta_{AX}^\mu \pm \partial_\mu \lambda_B = D_{AX} \lambda_B \mp \frac{1}{\sqrt{2}} \chi_{AXB} \lambda_C \), for which \((\pm D_{AB} \lambda^C)\) = \(\mp D_{AB} \lambda^C\). The commutator of two such operators is

\[
\pm D_{X(A} \pm D_{B)} \chi_C = -\chi^D \pm \Phi_{DCAB} - \pm \Phi_{EF} \pm D_{EF} \chi_C, \tag{3.1.1}
\]

where

\[
\pm \psi^{CD}_{AB} := \pm \frac{1}{\sqrt{2}} \left( \chi^{CD}_{AB} - \chi^D \delta^C_{AB} \right), \tag{3.1.2}
\]

\[
\pm \Phi_{ABCD} := \frac{1}{2} \left( R_{ABCD} + \frac{R}{2} \varepsilon_{(AC} \varepsilon_{DB)} \right) + \frac{1}{2} D_{(C} \lambda^{X}_{D)AB} - \frac{1}{4} \left( \chi^C_{EF} B \chi_{DEFA} + \chi^D_{EF} B \chi_{CEFA} \right). \tag{3.1.3}
\]

and \( R_{ABCD} \) is the unitary spinor form of the Ricci tensor of \((\Sigma, h_{\mu \nu})\). They represent the ‘torsion’ and the curvature of \( \pm D_\mu \), respectively. The latter is related to the curvature 2-form \( \pm F^A_{\mu \nu} \) of \( \pm D_\mu \) by \( \pm \Phi_{ABCD} = \frac{1}{2} \pm F^A_{ABCD} \pm RS \). Their algebraic symmetries are \( \pm \psi_{ABCD} = \pm \psi_{(AB)(CD)} = \pm \psi_{CDAB} \) and \( \pm \Phi_{ABCD} = \pm \Phi_{(AB)(CD)} \), and their contractions are

\[
\pm \psi_{ACB}^C = \pm \frac{1}{\sqrt{2}} \varepsilon_{AB}, \tag{3.1.4}
\]

\[
\pm \Phi_{ACB}^C = -\varepsilon_{AB} \frac{1}{8} \left( R + \chi^2 - \chi_{\mu \nu} \chi^{\mu \nu} \right) \pm \frac{1}{2 \sqrt{2}} \left( D_\mu \chi^\nu_{AB} - D_{AB} \chi \right). \tag{3.1.5}
\]

Thus \( \pm \Phi_{ABCD} \) is not symmetric in the pairs \( AB \) and \( CD \). The unitary spinor form of the tensor fields \( H_{\mu \nu} \) and \( B_{\mu \nu} \) of subsection 2.2 is

\[
H_{AXBY} = \frac{i}{\sqrt{2}} \left( D_{E(A} \chi^E_{X)BY} + D_{E(B} \chi^E_{Y)AX} \right) = \sqrt{2} D_{E(A} \chi^E_{XBY} = \frac{i}{\sqrt{2}} \left( -\Phi_{AXBY} + \Phi_{BYAX} - \Phi_{AXBY} - \Phi_{BYAX} \right), \tag{3.1.6}
\]

\[
B_{AXBY} = \frac{i}{2 \sqrt{2}} \left( D_{EA} \left( \pm \Phi_{EXY} + \pm \Phi_{BYX} \right) + D_{EX} \left( \pm \Phi_{EABY} + \pm \Phi_{BYAE} \right) + D_{EB} \left( \pm \Phi_{EAYX} + \pm \Phi_{AXY} \right) + D_{EY} \left( \pm \Phi_{EBA} + \pm \Phi_{AXB} \right) + \chi_{AX} \left( B^E \left( D_{CD} \chi_{XECD} - D_{Y}E \right) \chi \right) + \chi_{BY} \left( A^E \left( D_{CD} \chi_{XECD} - D_{X}E \right) \chi \right) \right) \pm \frac{i}{\sqrt{2}} \left( D_{E(A} H_{X)EBY} + D_{E(B} H_{Y)EAX} \right). \tag{3.1.7}
\]

The second Bianchi identity for \( \pm F^A_{\mu \nu} \) then takes the form

\[
D_{AB} \pm \Phi^{CDAB} \pm \sqrt{2} \chi_{(C} \Phi^{D)} EAB = 0. \tag{3.1.8}
\]

The first Bianchi identity doesn’t give any further algebraic symmetry for \( \pm \Phi_{ABCD} \).

Let \((M, g_{ab})\) be a Lorentzian spacetime manifold with spinor structure, and recall [9] that a contravariant 1-valence twistor field \( Z^a \) is a pair \((\omega^a, \pi^a)\) of spinor fields such that, under the conformal rescaling \( \varepsilon_{AB} \mapsto \hat{\varepsilon}_{AB} := \Omega \varepsilon_{AB} \), the spinor fields transform as \( \omega^A \mapsto \hat{\omega}^A := \omega^A \) (i.e. \( \omega^A \) has zero conformal weight) and \( \pi^A \mapsto \hat{\pi}^A := \pi^A + i \omega^A \gamma_{AA'} \ln \Omega \). The spinor parts of \( Z^a \) will also be denoted by \( Z^A \) and \( Z_{A'} \). \( Z^a \) may be defined only on a submanifold of \( M \), e.g. on a (say, spacelike) hypersurface \( \theta(\Sigma) \)
or on a spacelike 2-surface \( \theta(\Sigma) \). If \( \omega^A \) is any spinor field on \( M \) (or on \( \theta(\Sigma) \)) with zero conformal weight, and if \( \pi_{A'} \) is defined by \( \frac{1}{\sqrt{2}} \bar{\nabla} \omega^A_{A'} \) (or on \( \theta(\Sigma) \)) by \( \frac{2}{\sqrt{3}} i \bar{\nabla} \omega^A_{A'} \), where \( \bar{\nabla} \omega^a_b = \) the three dimensional Sen connection \([12]\), then \( Z^\alpha := (\omega^A, \pi_{A'}) \) turns out to be a twistor field on \( M \) (or, respectively, on \( \theta(\Sigma) \)) in the sense above. Thus any such spinor field \( \omega^A \) on \( M \) (or on \( \Sigma \)) determines a twistor field (‘geometric twistor fields’). (If \( \omega^A \) is a spinor field on \( \theta(\Sigma) \) with zero conformal weight and \( \pi_{A'} := i \Delta_{A'A'} \omega^A \), where \( \Delta_a := \Pi_b^a \nabla_b \) is the projection of \( \nabla_a \) to \( \theta(\Sigma) \), the two dimensional Sen connection \([17]\), then \( Z^\alpha := (\omega^A, \pi_{A'}) \) is a twistor field on \( \theta(\Sigma) \) in the sense above. This case, however, will not be considered in the present paper.)

A geometric twistor field \( Z^\alpha \) on \( M \) is called a global twistor, or simply twistor, if its primary spinor part \( \omega^A \) is a solution of the 1-valence twistor equation \( \nabla^{A'}(A\omega^B) = 0 \). It is known \([9]\) that if \( \omega^A \) is a nonzero solution of the twistor equation, then it is a 4-fold principal spinor of the Weyl spinor and \( \omega^A \bar{\omega}^{A'} \) is a future pointing conformal Killing vector; and, conversely \([18]\), if \( \omega^A \) is a 4-fold principal spinor of \( \psi_{ABCD} \) and \( \omega^A \bar{\omega}^{A'} \) is a future pointing conformal Killing vector then for some real function \( f \) the spinor \( \exp(\im i f) \omega^A \) is a solution of the twistor equation. Furthermore the twistor equation is completely integrable, i.e. admits the maximal number of solutions, namely four, if and only if \( (M, g_{ab}) \) is conformally flat.

If \( Z^\alpha := (\omega^A, \pi_{A'}) \) is any twistor field on the spacelike hypersurface \( \theta(\Sigma) \), then its secondary part \( \pi_{A'} \) can equivalently be represented by the unprimed spinor \( \pi_{A'} := G^{A'}_{A} \pi_{A} \). Furthermore, by \( G_{A'B}^{A} \nabla_{A'A'} \omega_{B} = \frac{1}{\sqrt{2}} \varepsilon_{A'Z}(t^\prime \nabla_{\omega_{B}}) - \bar{\nabla}_{(Z^A} \omega_{B)} \) the full 3+1 decomposition of the twistor equation is

\[
\bar{\nabla}_{(A'B)} \omega_{C)} = 0, \quad t^\prime \nabla_{\omega_{A}} = \frac{\sqrt{2}}{3} \bar{\nabla}_{A'B} \omega_{B}.
\]

The first of these, i.e. the spatial part of the twistor equation, is called the 3-surface twistor equation \([6,19]\).

A geometric twistor field \( Z^\alpha = (\omega^A, \pi_{A'}) \) on \( \theta(\Sigma) \) is called a (global) 3-surface twistor if its primary spinor part \( \omega^A \) is a solution of \((3.1.9.a)\). As it was proved in \([6]\), it is completely integrable, and hence admits four \( \mathbb{C} \)-linearly independent solutions, if and only if \( \Sigma \) with its first and second fundamental forms can also be imbedded into a conformally flat spacetime. The self-dual Sen operator \( \pm \bar{D}_{AB} \) appears in the 3+1 decomposition of the complex conjugate twistor equation \( \nabla^{A'}(A\omega^{B'}) = 0 \).

### 3.2 The 3-surface twistor connection and Chern–Simons functional

The basic idea \([9]\) of the twistor parallel transport on \( M \) is to consider the global twistors as constant twistor fields with respect to the twistor connection. This idea was used to introduce the notion of 3-surface twistor connection on (spacelike or timelike) hypersurfaces \([6]\), by means of which the complete integrability of the 3-surface twistor equation could be characterized as the vanishing of the 3-surface twistor curvature. Since we need the explicit form both of the twistor connection and curvature we first recall the main points of the construction of the 3-surface twistor connection in the unitary spinor formalism on an arbitrary triple \( (\Sigma, h_{\mu\nu}, \chi_{\mu\nu}) \), then calculate the connection coefficients and the curvature, and finally we calculate the Chern–Simons 3-form built from this connection.

The 3-surface twistor equation and its complex conjugate can be rewritten as

\[
\pm \bar{D}_{AB} \omega^{C} + i \delta^{C}_{(A'B)} = 0. \tag{3.2.1}
\]

Taking the \( \pm \bar{D}_{AB} \)-derivative of its contraction \( \pi_{C} = \frac{2}{\sqrt{2}} i \bar{D}_{CD} \omega^{D} \), using the commutator \((3.1.1)\) and the 3-surface twistor equation \((3.2.1)\), and finally adding the resulting equation to itself after appropriate permutations of its indices we get

\[
\pm \bar{D}_{AB} \pi_{C} \pm \sqrt{2} \pi_{D}X^{D}_{CAB} - 2i \omega^{D} \Phi_{D(AB)C} = 0. \tag{3.2.2}
\]
The twistor field $Z^\alpha = (\omega^A, \pi_X)$ on $\Sigma$ is a global 3-surface twistor (or conjugate twistor according to the sign $\mp$) if and only if the spinor fields $\omega^A$ and $\pi_X$ satisfy the system of equations (3.2.1), (3.2.2). Thus, recalling that $\mp \mathcal{D}_{AB}$ can be expressed by $\pm \mathcal{D}_{AB}$ and the extrinsic curvature, it seems natural to define the covariant derivative of any twistor field $Z^\alpha$, defined on $\Sigma$, by the pair

$$\pm \mathcal{D}_{MN} Z^\alpha := \left( \pm \mathcal{D}_{MN} \omega^A + i \delta^A_{(M} \pi_N), \mp \mathcal{D}_{MN} \pi_X - 2i \omega^B \pm \Phi_{B(MN)X} \right).$$

(3.2.3)

In fact, $Z^\alpha$ is a global 3-surface twistor/conjugate twistor on $\Sigma$ iff $v^\mu \mp \mathcal{D}_\mu Z^\alpha = 0$ for any tangent vector $v^\mu$ of $\Sigma$, under spacetime conformal rescalings $v^\mu \mp \mathcal{D}_\mu Z^\alpha$ transform as twistor fields on $\Sigma$, and $\pm \mathcal{D}_\mu Z^\alpha$ are determined by $(\Sigma, h_{\mu\nu}, \chi_{\mu\nu}).$ (If $\Sigma$ were imbedded in $(M, g_{\mu\nu})$ as a spacelike hypersurface then we could define $\mathcal{D}_\mu Z^\alpha := P^\mu \nabla_\mu Z^\alpha$, the projection to $\Sigma$ of the spacetime twistor covariant derivative as a derivative analogous to the Sen connection. That derivative, however, would depend not only on the data set $(\Sigma, h_{\mu\nu}, \chi_{\mu\nu})$ but on the spatial–spatial part of the spacetime Ricci tensor too.) The 3-surface twistor connection coefficients are defined by $\pm \mathcal{A}_{\mu\nu}^A := E^A_{\mu\nu} \mp \mathcal{D}_\mu E^A_\nu$ in the dual twistor frame fields $\{E^A_\mu\}, \{E^A_\nu\}$, which may be chosen to be determined by the spin frame fields $\{\varepsilon^A_\mu\}, \{\varepsilon^A_\nu\}$ through $E^A_\mu = (\varepsilon^A_\mu, 0)$, $E^A_\nu = (0, \varepsilon^A_\nu)$ and $E^B_\mu = (\varepsilon^B_\mu, 0)$, $E^B_\nu = (0, \varepsilon^B_\nu)$). They are represented by the following spinor parts

$$\pm \mathcal{A}_{\mu\nu}^A = \left( \pm \mathcal{A}_{\mu\nu}^A \mp \mathcal{A}_{\mu\nu}^A \right) = \left( \begin{array}{c} \pm \Gamma^A_{\mu\nu} \\ \mp \Gamma^A_{\mu\nu} \end{array} \right),$$

(3.2.4)

Here $\pm \Gamma^A_{\mu\nu} := \pm \mathcal{A}_{\mu\nu}^A \mp \mathcal{A}_{\mu\nu}^A$ are the connection coefficients of $\pm \mathcal{D}_\mu$ in the dual spin frame fields. Next, using formulae (3.1.2)-(3.1.8) and (3.2.3)-(3.2.4), a rather laborious calculation yields the 3-surface twistor curvature. The only non-vanishing spinor parts of that curvature are

$$\pm R_{\nu} = \pm 2i H^Y \varepsilon_{RS} \Theta^M \Theta^N \varepsilon^S \varepsilon^N,$$

(3.2.5)

$$\pm R_{\nu} = \left( \sqrt{2} B_{\nu} + D_{E} E_{F} + D_{M} H_{N} \right) + \varepsilon_{BN} \left( D_{E} E_{F} + D_{M} H_{N} \right) + \varepsilon_{BX} \left( D_{E} E_{F} + D_{M} H_{N} \right) + \varepsilon_{BM} \left( D_{E} E_{F} + D_{M} H_{N} \right).$$

(3.2.6)

Thus the tensors $H_{\mu\nu}, B_{\mu\nu}$ have natural twistorial interpretation since they represent the nonvanishing components of the 3-surface twistor curvature. The flatness of the 3-surface twistor connection (3.2.3) is therefore equivalent to the vanishing of $H_{\mu\nu}$ and $B_{\mu\nu}$, i.e., as Tod recognized first, to the local isometric imbeddability of $(\Sigma, h_{\mu\nu}, \chi_{\mu\nu})$ into some conformally flat spacetime. It is an easy calculation to show that under spacetime conformal rescalings the spinor expressions above do, in fact, transform as the spinor parts of a (1,1) twistor (valued 2-form).

By (3.1.5-6) and (3.2.4-6) the (dual of the) Chern–Simons 3-form built from the 3-surface twistor connection $\pm \mathcal{A}^A_{\mu\nu}$ is

$$\varepsilon^{\mu\nu\rho} \left( \pm \mathcal{A}^A_{\mu\nu} \pm \mathcal{A}^A_{\nu\rho} \mp \mathcal{A}^A_{\mu\rho} \mp \mathcal{A}^A_{\nu\rho} \right) =$$

$$\varepsilon^{\mu\nu\rho} \left( \left( \mp \Gamma^A_{\mu\nu} - F^E_{\mu\nu} \mp \Gamma^A_{\mu\nu} \mp \Gamma^A_{\nu\rho} + \frac{2}{3} \Gamma^A_{\mu\rho} \mp \Gamma^A_{\nu\rho} \mp \Gamma^A_{\mu\rho} \mp \Gamma^A_{\nu\rho} \right) + \left( \mp \Gamma^A_{\mu\nu} + F^E_{\mu\nu} \mp \Gamma^A_{\mu\nu} \mp \Gamma^A_{\nu\rho} + \frac{2}{3} \Gamma^A_{\mu\rho} \mp \Gamma^A_{\nu\rho} \mp \Gamma^A_{\mu\rho} \mp \Gamma^A_{\nu\rho} \right) \right) \right) =$$

$$\varepsilon^{\mu\nu\rho} \left( \left( \Gamma^A_{\mu\nu} F^E_{\Delta\mu\nu} + \frac{2}{3} \Gamma^A_{\mu\nu} \Gamma^E_{\Delta\mu\nu} \Gamma^C_{\rho\Delta} \right) + \left( \Gamma^A_{\mu\nu} F^E_{\Delta\mu\nu} + \frac{2}{3} \Gamma^A_{\mu\nu} \Gamma^E_{\Delta\mu\nu} \Gamma^C_{\rho\Delta} \right) \right).$$

(3.2.7)
Thus the Chern–Simons functional built from the 3-surface twistor connection is just half of the real Sen–Chern–Simons functional, yielding a pure twistorial interpretation and a manifest conformally invariant form of the latter (and of \( Y_{(k,k)} \) for any \( k \in \mathbb{N} \), too). Thus the stationary points of the 3-surface twistor Chern–Simons functional, with respect to both the 3-surface twistor connection and the fields \( h_{\mu \nu} \) and \( \chi_{\mu \nu} \) of the initial data, are just the initial data for which the 3-surface twistor connection is flat. Similarly to the higher dimensional spinor representations, we expect that the Chern–Simons functional built from the 3-surface twistor connections for higher valence twistors will essentially coincide with (3.2.7).

4. Time evolution

4.1 Comparison theorem for \( Y[\Gamma_{\Delta B}] \)

Next consider a one parameter family of initial data on a fixed 3-manifold, \((\Sigma, h_{\mu \nu}(t), \chi_{\mu \nu}(t))\), and we ask how the Chern–Simons functional, built from the connection \( \mathcal{D}_\rho \) in some representation \( \rho \), varies as a function of \( t \). Obviously, this problem can be reinterpreted as the question of its time evolution if \( \theta_t : \Sigma \to M, t \in \mathbb{R} \), is a foliation of a Lorentzian spacetime \((M, g_{ab})\) with spacelike hypersurfaces. On the typical hypersurface \( \Sigma \) this foliation is represented by a lapse function \( N(t) : \Sigma \to (0, \infty) \). By (2.3.5) it is enough to consider only the time evolution of \( Y[\Gamma_{\Delta B}] \). In the present subsection we derive a formula by means of which we can compare the Chern–Simons functional on two different spacelike hypersurfaces in a given globally hyperbolic spacetime. This result is independent of any field equation. We take into account Einstein’s field equations only in the next subsection.

Since \( M \) is diffeomorphic to \( \Sigma \times \mathbb{R} \), \( M \) admits a spinor structure, and let \( S^A(M) \) be the (trivializable) spinor bundle and \( \{ \varepsilon^A \} \) a (globally defined) normalized spinor dyad. Let \( \Sigma_t := \theta_t(\Sigma), t_a \) its timelike unit normal and let \( \nabla_e \) be the connection on the spacetime spinor bundle \( S^A(M) \). Then the connection 1-form of \( \nabla_a \) in the dyad \( \{ \varepsilon^A \} \) is \( ^{(4)} \Gamma_{aB}^A := \varepsilon^A_B \nabla_a \varepsilon_B^B \), and the curvature 2-form is \( ^{(4)} R_{\underline{E} \underline{F} \underline{G} \underline{H}}^{\underline{A}} := \varepsilon^A_B \varepsilon^B_C \varepsilon^C_D \varepsilon^D_A R_{\underline{E} \underline{F} \underline{G} \underline{H}}^{\underline{A}} \underline{B}_{\underline{C} \underline{D}} \). Their pull back to \( \Sigma \) along the imbedding \( \theta_t \) are just the connection and curvature forms, \( \Gamma_{aB}^A(t) \) and \( F_{\underline{E} \underline{F} \underline{G} \underline{H}}^{\underline{A}}(t) \), of the Sen connection in the spinor representation at ‘time’ \( t \), respectively. Thus the pull back to \( \Sigma \) of the spacetime Chern–Simons 3-form \( ^{(4)} R_{\underline{A} \underline{B} \underline{C} \underline{D}}^{\underline{E}}(\theta_t) \) along \( \theta_t \) is the Chern–Simons 3-form built from \( \Gamma_{aB}^A(t) \) on \( S^A(\Sigma) \). On the other hand, the exterior derivative of the spacetime Chern–Simons 3-form, contracted with the volume 4-form, is

\[
\frac{1}{2} ^{(4)} R_{\underline{A} \underline{B} \underline{E} \underline{F}}^{\underline{C} \underline{D}} + \frac{1}{2} ^{(4)} R_{\underline{A} \underline{B} \underline{G} \underline{H}}^{\underline{C} \underline{D}} + \frac{1}{2} \delta_{\underline{E} \underline{F}}^{\underline{G} \underline{H}} = 4E_{ab} H^{ab} - \frac{1}{2} \delta_{\underline{E} \underline{F}}^{\underline{G} \underline{H}} G_{ab} (4) G^{ab} + \frac{1}{6} (4) R^2 ,
\]

where we used the expressions for \( E_{ab} \) and \( H_{ab} \) given in subsection 2.2. Thus applying the Stokes theorem to the spacetime Chern–Simons 3-form on the spacetime domain bounded by the \( \Sigma_0 \) and \( \Sigma_t \) hypersurfaces and assuming that the integrals exist, we get

\[
Y[\Gamma_{\Delta B}(t)] - Y[\Gamma_{\Delta B}(0)] = \int_0^t \int_{\Sigma_t} \frac{1}{2} ^{(4)} R_{\underline{A} \underline{B} \underline{E} \underline{F}}^{\underline{C} \underline{D}} + \frac{1}{2} ^{(4)} R_{\underline{A} \underline{B} \underline{G} \underline{H}}^{\underline{C} \underline{D}} N \mathsf{d} \Sigma_t \, dt. \tag{4.1.2}
\]

Note that by (4.1.2) we can compare \( Y[\Gamma_{\Delta B}] \) on any two, maybe intersecting, hypersurfaces \( \Sigma' \) and \( \Sigma'' \) if there is a hypersurface \( \Sigma \) and there is a foliation \( \Sigma' \) between \( \Sigma' \) and \( \Sigma \) and a foliation \( \Sigma'' \) between \( \Sigma'' \) and \( \Sigma \). The real part of the right hand side in (4.1.1) transforms as \( 4E_{ab} H^{ab} \mapsto 4\Omega^{-1} E_{ab} H^{ab} \) under spacetime conformal rescalings. Thus, recalling that by its very definition, \( Nt^a \nabla_a t = 1 \), the lapse function \( N \) transforms as \( N \mapsto \Omega N \), the time derivative of the Chern–Simons functionals defined in the tensor representations is also conformally invariant. In general neither the real nor the imaginary part has definite sign. If however the
4.2 On the time evolution via Einstein’s equations, on a ‘natural time variable’ and examples

Instead of the Stokes theorem of the previous subsection we could start with a choice for the lapse and shift and could evolve $Y[\Gamma_{\Delta B}(t)]$ in time by using (2.3.3) and evolving the initial data via the 3+1 form of Einstein’s equations. What we would obtain, however, is just (the time derivative of) (4.1.2) if in which the shift integrate to zero because of the diffeomorphism invariance of $Y_\pm$, furthermore, although in general the conformal electric curvature $E_{\mu\nu}$ cannot be expressed by the intrinsic and extrinsic geometrical data on $\Sigma$, by Einstein’s equations it becomes an expression of the geometric data and the energy-momentum tensor on $\Sigma$. Explicitly $E_{ab} = -(R_{ab} + V_{ab} - \frac{1}{2}h_{ab}(R + V)) - \frac{1}{2}\kappa(\sigma_{ab} - \frac{1}{2}\sigma^e \sigma_{eb})$, where we used the standard 3+1 decomposition, $T_{ab} = \mu t_a t_b + 2J(a_{b}) + \sigma_{ab}$, of the energy-momentum tensor.

Recently the imaginary part of the Ashtekar–Chern–Simons functional as a natural internal time variable was suggested in the configuration space of cosmological models [21]. Then the question arises as whether one can introduce a natural internal time variable in the spacetime, at least with respect to a foliation $\theta_i$, given geometrically in cosmological spacetimes. This would have to be monotonic with respect to the coordinate time $t$ above. (For such earlier suggestions see, for example, [22,23].) In fact, for the $t =$ constant hypersurfaces of the maximal spacelike symmetry of the closed Robertson–Walker metrics (4.1.1) is $-6\alpha(-3\dot{a}(\dot{a}^2 + 1)$, where by Einstein’s equations the scale function $a(t)$ satisfies $3a^{-1}\ddot{a} = \Lambda - \frac{\kappa}{2}(\mu + 3\rho)$ and $3(\dot{a}^2 + 1) = a^2(\kappa\mu + \Lambda)$, and the pressure $p$ is defined by $\sigma_{ab} = -p\sigma_{ab}$. Therefore Im$Y_\pm$ is in fact monotonic if the strong energy condition is satisfied and e.g. $\Lambda \leq 0$. (For a discussion of the role of the cosmological constant in the dynamics of the Robertson–Walker spacetimes in general see e.g. [24].) It would therefore be interesting to see $Y[\Gamma_{\Delta B}]$ itself. We calculate this by calculating the spinor Chern–Simons invariant for the more general homogeneous Bianchi cosmological spacetimes using the technique of [25,26], where the 3-space is still assumed to be a group manifold. To ensure the existence of the integral of the Chern–Simons 3-form we must assume that the typical Cauchy hypersurfaces are compact. The only three compact 3-dimensional manifolds admitting Lie group structure are the torus $S^1 \times S^1 \times S^1$, the 3-sphere $S^3 \approx SU(2)$ and the projective space $S^3/Z_2 \approx SO(3)$, for which the structure constants can be written as $c_{ij}^k = c_{l(1)2}^{k1} \varepsilon_{ijl}$. Here $c = 0$ for the torus and $c = 1$ for $SU(2)$ and $SO(3)$. Since from the point of view of the Chern–Simons functional the only difference between the $SU(2)$ and $SO(3)$ cases is that the integration domain for $SO(3)$ is half of that for $SU(2)$, we calculate $Y[\Gamma_{\Delta B}]$ only for the torus and the sphere. Thus let $\{\sigma_{\mu}^1\}$ be a left invariant 1-form basis in which the structure constants are the ones given above, i.e. $dz^i = \eta^{ij} c_{ij}^k \sigma^k \wedge \sigma^i$. (With the choice $\eta^{k1} \varepsilon_{kij}$ for the structure constant on $SU(2)$ the basis $\{\sigma_{\mu}^1\}$ will be orthonormal with respect to the unit sphere metric inherited from $\mathbb{R}^4$ through the canonical imbedding.) Let $h_{ij}$ and $\chi_{ij}$ be the components of the metric and extrinsic curvature in the basis $\{\sigma_{\mu}^1\}$, respectively, and $h^{ij}$ the inverse and $h$ the determinant of $h_{ij}$. Then a direct calculation yields
\[ Y[\Gamma^A_B] = -\frac{i}{3} \left\{ (\chi_{ij} h^{ij})^3 - 3(\chi_{ij} h^{ij}) (\chi_{kl} h^{lm} \chi_{mn} h^{nk}) + 2(\chi_{ij} h^{jk} - \chi_{kl} h^{lm} \chi_{mn} h^{ni}) \right\} \sqrt{|\lambda| V_{0\lambda}(\Sigma)} - \\
- \frac{ic}{\sqrt{|\lambda|}} \chi_{ij} \left\{ -8\pi h_{kl} \eta^{ij} + 4\eta^{ij} (h_{kl} \eta^{jk} + 2h^{ij} (h_{kl} \eta^{km} \eta^{ln} h_{mn}) - h^{ij} (h_{kl} \eta^{jk})^2) - \\
- 2\pi c \left( 6\chi_{ij} h^{jk} \chi_{kl} \eta^{il} + \eta^{ij} (\chi_{kl} h^{jk} - \chi_{kl} h^{km} h^{ln} \eta^{mn}) - 4(\chi_{ij} h^{jk}) (h_{kl} \eta^{jk}) - \\
- \frac{4}{3|\lambda|} \left[ 10h_{ij} \eta^{jk} h_{kl} \eta^{il} h_{mn} \eta^{ni} - 9(h_{ij} \eta^{jk}) (h_{kl} \eta^{km} h_{mn} \eta^{nk}) + 2(h_{ij} \eta^{jk})^3 \right] \right\}. \] (4.2.2)

Here \( V_{0\lambda}(\Sigma) := \int_0^\infty \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \), the (non-dynamical) left invariant group volume, which is \( 2\pi^2 \) for \( c = 1 \) and it is chosen to be \( 8\pi^3 \) for \( c = 0 \).

The general \( c = 0 \) vacuum solution is \( ds^2 = (dt)^2 - t^2 p_1 (dx^1)^2 - t^2 p_2 (dx^2)^2 - t^2 p_3 (dx^3)^2 \), where \( \sigma^i = D_\mu x^i \) and \( x^i \in [0,2\pi] \), and \( p_1 + p_2 + p_3 = 1, p_1^2 + p_2^2 + p_3^2 = 1 \) (the spatially closed Kasner solution [20]). Then (4.2.2) gives \( Y[\Gamma^A_B] = -\frac{16\pi^3}{3} (p_1^2 + p_2^2 + p_3^2 - 1) \), a purely imaginary constant. Thus this is an algebraically general spacetime for which the spinor Chern–Simons functional is an invariant for the whole spacetime, and, in particular, \( \Im Y_{\pm} \) is certainly not a time function. This result shows that the generalization of the (Riemannian) conformal invariant \( Y[h_{\mu\nu}] \) of Chern and Simons for initial data sets is not trivial: \( Y_{\pm}[h_{\mu\nu}, \chi_{\mu\nu}] \) depends essentially on \( \chi_{\mu\nu} \), and, apart from the permutations of the Kasner exponents \( (p_1, p_2, p_3) \), characterizes the vacuum Bianchi I. cosmological spacetimes completely. (To see this it may help the use of the global explicit parameterization \( 3p_1 = 1 - \cos \alpha + \sqrt{3} \sin \alpha, 3p_2 = 1 - \cos \alpha - \sqrt{3} \sin \alpha, 3p_3 = 1 + 2 \cos \alpha, \) where \( \alpha \in [0,2\pi] \).)

For \( c = 1 \) the left invariant basis may be chosen such that \( 2\sigma^1 = \sin \psi D_\mu \theta - \cos \psi \sin \theta D_\mu \phi, 2\sigma^2 = \cos \psi D_\mu \theta + \sin \psi \sin \theta D_\mu \phi \) and \( 2\sigma^3 = -D_\mu \psi - \cos \theta D_\mu \psi \), where \( \psi \in [0,4\pi], \phi \in [0,2\pi], \theta \in [0,\pi] \) are the standard Euler angle coordinates on the 3-sphere. Then for the Robertson–Walker line element we get \( Y[\Gamma^A_B] = 8\pi^2 - 4\pi^2 i a (a^2 + 3) \). Thus the conformal invariant \( Y_0 \) for any closed Robertson–Walker initial data set is zero, but \( \Im Y_{\pm} \) is monotonic provided, as we saw, the strong energy condition is satisfied. A non-isotropic solution with the stiff equation of state, \( p = \mu \), and vanishing cosmological constant was found by Barrow [27]. It has the form \( ds^2 = (dt)^2 - a^2(t)(t^2 - b^2(t))((\sigma^1)^2 + (\sigma^2)^2) \) with the scale functions given by \( a^2(\tau) = 4A \sech(A\tau) \) and \( b^2(\tau) = A^{-1} B \cosh(A\tau) \sech^2(\frac{1}{2} B(\tau + \tau_0)) \), where \( A, B \) and \( \tau_0 \) are constants and the parameter \( \tau \) is defined implicitly by \( \frac{dt}{d\tau} = \kappa a(\tau)^2 \). Then the energy density is \( \kappa \mu = 16(B^2 - A^2)a^{-2}b^{-4} \) and the metric becomes isotropic if \( a = b \), whenever \( B = 2A \) (and \( \tau_0 = 0 \)). For the \( t = \) const. hypersurface in this spacetime we find

\[ Y[\Gamma^A_B] = 8\pi^2 \left\{ 1 + \frac{(a^2 - b^2)}{b^2} \right\} + \frac{1}{4b^4} \left( \sqrt{16A^2 - a^4} - \sqrt{4B^2 - a^2 b^2} \right)^2 + \\
+ \frac{4\pi^2}{a^2 b^2} \sqrt{16A^2 - a^4} \left( \sqrt{16A^2 - a^4} - 2\sqrt{4B^2 - a^2 b^2} \right)^2 + \\
+ \frac{16\pi^2}{b^2} \left( 1 - \frac{a^2}{4b^2} - \frac{b^2}{a^2} \right) \sqrt{16A^2 - a^4} \left( \sqrt{4B^2 - a^2 b^2} \right). \]

Therefore the anisotropy of the geometry, characterized by the differences \( \sqrt{16A^2 - a^4} - \sqrt{4B^2 - a^2 b^2} \) and \( a^2 - b^2 \) of the extrinsic curvature and the intrinsic metric, respectively, contribute both to the real and imaginary parts of \( Y_{\pm} \), too, i.e. \( Y_0 \) is also a non-trivial generalization of the Riemannian \( Y[h_{\mu\nu}] \). Both the real and imaginary parts of \( Y[\Gamma^A_B] \) are changing in time but, in general, \( \Im Y[\Gamma^A_B] \) is not monotonic even if the energy condition \( A^2 < B^2 \) is assumed to hold.

Returning to the general formula (4.1.1), neither its real nor its imaginary part has definite sign even if Einstein’s equations are taken into account and the dominant energy condition is satisfied. Moreover the indefinite expression \( E_{ab}E^{ab} - H_{ab}H^{ab} \) is still present even in vacuum. Since however for the analogous
expression in Ricci-flat Riemannian 4-geometries we get $E_{ab}E^{ab} + H_{ab}H^{ab}$, which is positive definite (and in a Lorentzian spacetime this would be just the well known Bel–Robinson tensor contracted with the unit normal $t^a$), one could hope to obtain positive definite imaginary term for the Wick rotated Ashtekar, i.e. the Barbero connection [7,8]. In fact, the same analysis can be repeated for the Barbero connection, obtaining a formula analogous to (4.1.1). But that is much more complicated and neither the real nor the imaginary part appears to have a definite sign either. Hence $\text{Im} Y_\pm$ can be interpreted as a natural time variable in the spacetime only for a very limited class of cosmological models.

Finally, to have an asymptotically flat example for the Chern–Simons functional, we can compute $Y[\Gamma^4_{\mu\nu}]$ for the Reissner–Nordström spacetime. For the 3-manifold $\Sigma$ we choose a maximally extended $t = \text{const.}$ spacelike hypersurface, consisting of two asymptotically flat ‘ends’ and joining together at the surface of bifurcation of the event horizon. It is easy to find a globally defined triad $\{e^a_i\}$ on $\Sigma$, using the fact that $\Sigma$ with the induced metric $h_{\mu\nu}$ is globally conformally flat (see e.g. [28]). However, just because of its global conformal flatness and the fact that this hypersurface is time symmetric, the whole $Y[\Gamma^4_{\mu\nu}]$ is zero. The analogous calculation for the Kerr–Newman solution would be much more complicated.

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