First Law of Black Rings Thermodynamics in Higher Dimensional Chern-Simons Gravity

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The physical process version and the equilibrium state version of the first law of black ring thermodynamics in n-dimensional Einstein gravity with Chern-Simons term were derived. This theory constitutes the simplest generalization of the five-dimensional one admitting a stationary black ring solutions. The equilibrium state version of the first law of black ring mechanics was achieved by choosing any cross section of the event horizon to the future of the bifurcation surface.

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I. INTRODUCTION

Various attempts to unify the forces of Nature involve a great resurgence of the consideration of spacetimes of dimensionality greater than four. Consequently, there occurs also continuously growing interest in n-dimensional black holes and its mathematical structure. Namely, the uniqueness theorem for static n-dimensional black holes was quite well established [1]. Recently, the proof of the rigidity theorem in higher dimensional gravity for non-extremal black holes was provided [2]. On the other hand, for stationary axisymmetric n-dimensional solutions the situation is far from obvious. It was shown [3] that even in five-dimensional spacetime there is the so-called black ring solution having $S^2 \times S^1$ topology of the event horizon. It has the same mass and angular momentum as a spherical five-dimensional stationary axisymmetric black hole. If one assumes the topology of black hole event horizon as $S^3$ the uniqueness proof can be established (see Ref.[4] for the vacuum case and Ref.[5] for the stationary axisymmetric self-gravitating $\sigma$-model). In literature, there has been black ring solution possessing both electric and magnetic dipole charges [6, 7], also static black ring solution has been found in five-dimensional Einstein-Maxwell-dilaton gravity [8] and systematically derived in [9] both in asymptotically flat and non-asymptotically flat case. There are also a supersymmetric generalizations of these objects [10] (for a review of a black ring story see [11] and references therein).

As the interest of these object systematically grows we shall in our paper study the first law of black ring thermodynamics in the case of n-dimensional gravity coupled to Chern-Simons (CS) term. In five-dimensional case of Einstein gravity with CS term, i.e., a minimal supergravity theory, the black ring solution was given in Ref.[12]. We shall consider the simplest generalization of this theory to the case of n-dimensions. First, we look for the physical process version of the first law of black ring thermodynamics. In black hole physics it was realized changing a stationary black hole by some infinitesimal physical process, e.g., by throwing matter into black hole. When one assumes that the final state of black hole settles down to a stationary one, we can extract the changes of black hole’s parameters and in this way conceive the idea about the first law of black hole mechanics. The physical process version of the first law of black hole thermodynamics was studied in Einstein and Einstein-Maxwell (EM) theory in Refs.[14, 15] and in Einstein-Maxwell axion-dilaton (EMAD) gravity being the low-energy limit of the heterotic string theory in Ref.[16]. Having in mind the assumption about spherical topology of n-dimensional black hole the physical process version of the first law of thermodynamics was treated in Ref.[17] in the case of Einstein gravity coupled to $(n-2)$-gauge form field strength. This kind of derivation of the first law of thermodynamics was also elaborated in the case black rings in higher dimensionality containing $(p+1)$-form field strength and dilaton field [18], being the simplest generalization of five-dimensional one in which stationary black ring solution has been provided [19].

The other attitude to the problem of the first law of black hole thermodynamics is the so-called equilibrium state version. It was studied in the seminal paper of Bardeen, Carter and Hawking [20]. This attitude is based on taking into account the linear perturbations of a stationary electrovac black hole to another one. Arbitrary asymptotically flat perturbations of a stationary black hole were considered in Ref.[21], while the first law of black hole thermodynamics valid for an arbitrary diffeomorphism invariant Lagrangian with metric and matter fields possessing stationary and axisymmetric black hole solutions were given in Refs.[21]-[24]. The cases of higher curvature terms and higher derivative terms in the metric were considered in [25], while the situation when the Lagrangian is an arbitrary function of metric, Ricci tensor and a scalar field was elaborated in Ref.[26]. In Ref.[27] of a charged rotating black hole where fields were not smooth through the event horizon was treated.
The first law of black hole thermodynamics was also provided in the case of \( n \)-dimensional black holes. The equilibrium state version was studied in Ref.\[28\]. Some of the works assume that four-dimensional black hole uniqueness theorem extends to higher dimensional case are devoted to the problem \[29\].

In Ref.\[30\], the authors using the notion of bifurcate Killing horizons and taking into account dipole charges were managed to find the first law of black hole thermodynamics for black ring solutions. In the higher dimensional gravity containing \((p+1)\)-form field strength and dilaton fields the first law of black ring mechanics choosing an arbitrary cross section of the event horizon to the future of the bifurcation surface was derived in Ref.\[31\].

The first law of black hole thermodynamics was also provided in the case of \( n \)-dimensional black holes. In five-dimensional case the solutions were elaborated the physical process version of the first law. We shall derive this law by choosing the arbitrary cross section of the black ring event horizon to the future of the bifurcation surface. It allows one to treat fields which are not necessary smooth through the event horizon. Our requires only that the pull-back of the fields in the future of bifurcation surface be smooth.

II. PHYSICAL PROCESS VERSION OF THE FIRST LAW OF BLACK RING MECHANICS

In this Sec. we shall consider the simplest higher dimensional generalization of minimal five-dimensional supergravity theory where a stationary black ring solutions have been found. In five-dimensional case the solutions were sufficiently complicated that a first law of thermodynamics could not be found by inspection \[13\]. In the case under consideration the action will be subject to the relation

\[
\mathbf{L} = \epsilon \left( (n)R - F_{\mu\nu}F^{\mu\nu} - \gamma \epsilon^{a_1a_2b_1b_2\ldots m_1m_2}A_a F_{a_1a_2} \ldots F_{m_1m_2} \right),
\]

where \( \gamma \) corresponds to CS coupling constant, \( \epsilon \) is the \( n \)-dimensional volume element, \( F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]} \). We remark that above Lagrangian applies to an odd dimensional spacetimes \((n = 2d + 1)\) for which it includes the CS term.

The equations of motion for the \( n \)-dimensional gravity with CS term yield

\[
G_{\mu\nu} - T_{\mu\nu}(F) = 0,
\]

\[
\nabla_\mu F^{\mu\nu} - \frac{\gamma}{4}(n - 1)\epsilon^{a_1a_2b_1b_2\ldots m_1m_2}F_{a_1a_2} \ldots F_{m_1m_2} = 0,
\]

while the energy momentum tensor consist only of the Maxwell field contribution, i.e.,

\[
T_{\mu\nu}(F) = 2F_{\mu\beta}F_{\nu}^\beta - \frac{1}{2}g_{\mu\nu}F^2.
\]

To deal with the problem of the physical version of the first law of black rings thermodynamics we shall first begin with the explicit expressions for the variation of mass and angular momentum and perform variation of the Lagrangian \[11\] evaluating the variations of the adequate fields, which implies

\[
\delta \mathbf{L} = \epsilon \left( G_{\mu\nu} - T_{\mu\nu}(F) \right) \delta g^{\mu\nu} - \left( 4\nabla_\alpha F^{\alpha\beta} - \frac{\gamma}{4}(n - 1)\epsilon^{a_1a_2b_1b_2\ldots m_1m_2}F_{a_1a_2} \ldots F_{m_1m_2} \right) \delta A_\beta + d\mathbf{\Theta}.
\]

In our paper we denote fields in the underlying theory by \( \psi_\alpha \), while their variations by \( \delta \psi_\alpha \). Having in mind relation \[5\] we get the symplectic \((n - 1)\)-form \( \Theta_{j_1\ldots j_{n-1}}[\psi_\alpha, \delta \psi_\alpha] \) of the form as

\[
\Theta_{j_1\ldots j_{n-1}}[\psi_\alpha, \delta \psi_\alpha] = \epsilon_{j_1\ldots j_{n-1}} \left[ \omega^\mu - 4F^{\mu\beta} \delta A_\beta - 2\gamma(n - 2)\epsilon^{a_1a_2\ldots m_1m_2}A_a F_{a_1a_2} \ldots F_{m_1m_2} \delta A_\alpha \right],
\]

where \( \omega^\mu = \nabla^\alpha \delta g_{\alpha\mu} - \nabla_\mu \delta g^{\alpha\beta} \).

In the next step one ought to find the Noether \((n - 1)\)-form with respect to this above mentioned Killing vector. Namely, we look for the form subject to the relation \( J_{j_1\ldots j_{n-1}} = \epsilon_{m_1\ldots j_{n-1}} \mathcal{J}^m[\psi_\alpha, \mathcal{L}_\xi \psi_\alpha] \). Thus we have finally left with

\[
J_{j_1\ldots j_{n-1}} = d \left( Q_{GR} + Q_{FCS} \right)_{j_1\ldots j_{n-1}} + 2\epsilon_{j_1\ldots j_{n-1}} \left( G^\delta - T^\delta(F) \right) \xi^\eta
\]

\[
+ \epsilon_{m_1\ldots j_{n-1}} \xi^d A_d \left[ -4\nabla_\beta F^{\beta\alpha} + \gamma(n - 1)\epsilon^{m_1a_1a_2b_1b_2\ldots m_1m_2}F_{a_1a_2} F_{b_1b_2} \ldots F_{m_1m_2} \right].
\]
where we denoted by $Q_{j_1 \ldots j_{n-2}}^{GR}$ the expression as follows:

$$Q_{j_1 \ldots j_{n-2}}^{GR} = -\epsilon_{j_1 \ldots j_{n-2}ab} \nabla^a \xi^b,$$  

and by $Q_{j_1 \ldots j_{n-2}}^{FCS}$ the relation of the following form:

$$Q_{j_1 \ldots j_{n-2}}^{FCS} = \epsilon_m \delta_{j_1 \ldots j_{n-2}} \left( \frac{2 F^{\delta m}}{(n-2)!} - \frac{\gamma(n-2)}{(n-2)!} \delta_{mab_1 \ldots m_{1,2}} A_a F_{b_1 b_2 \ldots F_{m_{1,2}}} \right) \xi_d A^d.$$  

Having in mind that $\mathcal{J}[\xi] = dQ[\xi] + \xi^a C_a$, where $C_a$ is an $(n-1)$-form constructed from dynamical fields, i.e., from $g_{\mu \nu}$ and $F_{\mu \nu}$ gauge field. One can identify $Q_{j_1 \ldots j_{n-1}} = (Q_{j_1 \ldots j_{n-2}}^{GR} + Q_{j_1 \ldots j_{n-2}}^{FCS})_{j_1 \ldots j_{n-1}}$ with the Noether charge for the considered theory. It reveals then that $C_a$ reduces to the following:

$$C_{a j_1 \ldots j_{n-1}} = 2 \epsilon_{m j_1 \ldots j_{n-1}} \left[ G^m_a - T^m_a(F) \right] + \epsilon_{m j_1 \ldots j_{n-1}} A_a \left( -4 \nabla_{\beta} F^{\beta m} + \gamma(n-1) \epsilon^{m a b_1 b_2 \ldots m_{1,2}} F_{a a_1 a_2} F_{b_1 b_2 \ldots F_{m_{1,2}}} \right).$$

The case when $C_{a} = 0$ is responsible for the source-free Eqs. of motion but on the contrary, when this is not the case, one gets the following:

$$G_{\mu \nu} - T_{\mu \nu}(F) = T_{\mu \nu}(\text{matter}),$$

$$\nabla_{\beta} F^{\beta \mu} = \frac{\gamma(n-1)}{4} \epsilon^{m a b_1 b_2 \ldots m_{1,2}} F_{a a_1 a_2} F_{b_1 b_2 \ldots F_{m_{1,2}}} + j^{\mu}(\text{matter}).$$

Let us assume further that $(g_{\mu \nu}, F_{\alpha \beta})$ are solutions of source-free equations of motion and $(\delta g_{\mu \nu}, \delta F^{\alpha \beta})$ are the linearized perturbations satisfying Eqs. of motion with sources $\delta T_{\mu \nu}(\text{matter})$ and $\delta j^{\mu}(\text{matter})$. It enables us to conclude that

$$\delta C_{a j_1 \ldots j_{n-1}} = \epsilon_{m j_1 \ldots j_{n-1}} \left( 2 \delta T^m_a(\text{matter}) + j^m(\text{matter}) A_a \right).$$

The Killing vector field $\xi_a$ describes a symmetry of the background matter field. By virtue of it the formula for a conserved quantity related with the Killing vector field $\xi_a$ may be expressed as

$$\delta H_\xi = - \int_{\Sigma} \epsilon_{m j_1 \ldots j_{n-1}} \left[ 2 \delta T^m_a(\text{matter}) \xi^a A_a \delta j^m(\text{matter}) \right] + \int_{\partial \Sigma} \left[ \delta Q(\xi) - \xi_\epsilon \Theta \right].$$

As in Ref. [15], if one takes $\xi^a$ to be an asymptotic time translation $t^a$, then it enables us to identify $M = H_t$ and in particular we obtain the variation of the ADM mass. Thus in this picture we get the following

$$\alpha \delta M = - \int_{\Sigma} \epsilon_{m j_1 \ldots j_{n-1}} \left[ 2 \delta T^m_a(\text{matter}) t^a A_a \delta j^m(\text{matter}) \right] + \int_{\partial \Sigma} \left[ \delta Q(t) - t_\epsilon \Theta \right],$$

where we denoted by $\alpha = \frac{\Sigma}{\Sigma}$. On the other hand, if we turn our attention to the Killing vector fields $\phi_{(i)}$ which are responsible for the rotation in the adequate directions, one gets the relations for angular momenta written in the form as

$$\delta J_{(i)} = \int_{\Sigma} \epsilon_{m j_1 \ldots j_{n-1}} \left[ 2 \delta T^m_a(\text{matter}) \phi_{(i)}^a A_a \phi_{(i)}^a \delta j^m(\text{matter}) \right] + \int_{\partial \Sigma} \left[ \delta Q(t) - \phi_{(i)} \Theta \right].$$

Next, we proceed to the physical process version of the first law of black ring thermodynamics. Let us us assume that $(g_{\mu \nu}, F_{\alpha \beta})$ are solutions to the source free Einstein equations. Moreover, let $\eta_a$ denotes the event horizon Killing vector field of the form

$$\eta^\mu = t^\mu + \sum_i \Omega_{(i)} \phi_{(i)}^\mu.$$
Suppose, now that one perturbs the black ring by dropping into it some matter. Furthermore, assume that the black ring will be not destroyed in the process of it and it settles down to a stationary solution [15]. We shall compute changes of a mass and angular momenta using relations [15, 16, 17] and find the change of the horizon’s area using the Raychaudhuri equation. In what follows we shall assume that Σ₀ is an asymptotically flat hypersurface which terminating on the black ring event horizon. Then, one takes into account the initial data on Σ₀ for a linearized perturbations of \((\delta g_{\mu\nu}, \delta F_{\alpha\beta})\) with \(\delta T_{\mu\nu}(\text{matter})\) and \(\delta j^\mu(\text{matter})\). We require that \(\delta T_{\mu\nu}(\text{matter})\) and \(\delta j^\mu(\text{matter})\) vanish at infinity and the initial data for \((\delta g_{\mu\nu}, \delta F_{\alpha\beta})\) disappear in the vicinity of the black ring horizon \(\mathcal{H}\) on the hypersurface Σ₀. These requirements provide that for the initial time Σ₀, the considered black hole is unperturbed. Hence the perturbations vanish near the internal boundary \(\partial \Sigma_0\), one gets from relations [15] and [16] that the following is fulfilled:

\[
\alpha \delta M - \sum_i \Omega_{(i)} \delta J^{(i)} = \int_{\Sigma_0} \epsilon_{\mu j_1...j_{n-1}} \left[ 2\delta T^{\mu}_{\alpha}(\text{matter}) \eta^\alpha + \eta^\alpha A_{\alpha} \delta j^\mu(\text{matter}) \right] - \int_{\mathcal{H}} \gamma^\alpha k_{\alpha} \epsilon_{j_1...j_{n-1}},
\]

where \(\epsilon_{j_1...j_{n-1}} = n^\delta \epsilon_{\delta j_1...j_{n-1}}\) while \(n^\delta\) is a future directed unit normal to the hypersurface Σ₀. On the other hand, \(k_{\alpha}\) is tangent vector to the affinely parametrized null geodesics generators of the black ring event horizon.

In the last term of relation (18) we have replaced \(n^\delta\) for \(k_{\delta}\). It can be done because of the fact that the current \(\gamma^\alpha\) is conserved as well as by the assumption that all the matter falls into black ring.

Before considering the integrals over the black ring event horizon let take into account the following relation:

\[
\mathcal{L}_\eta A_{\alpha} \epsilon^{mcb_{12}...m_{12}} F_{b_1 b_2} \ldots F_{m_1 m_2} = \eta^k F_{kc} \epsilon^{mcb_{12}...m_{12}} F_{b_1 b_2} \ldots F_{m_1 m_2} = \nabla_c \left( A_j \eta^j \epsilon^{mcb_{12}...m_{12}} F_{b_1 b_2} \ldots F_{m_1 m_2} \right).
\]

Because of the fact that \(\eta_{\alpha}\) is symmetry of the background solution the first term of the left-hand side of Eq. (19) is equal to zero. Then, let us turn to \(n\)-dimensional Raychaudhuri equation of the form as follows:

\[
\frac{d\theta}{d\lambda} = \frac{\theta^2}{(n-2)} - \sigma_{ij} \sigma^{ij} - R_{\mu\nu} \xi^\mu \xi^\nu,
\]

where \(\lambda\) is an affine parameter corresponding to vector \(k_{\alpha}\), \(\theta\) is the expansion and \(\sigma_{ij}\) is shear. Shear and expansion vanish in the stationary background. Inspection of Eq. (20) provides that \(R_{\alpha\beta} k^\alpha k^\beta |_{\mathcal{H}} = 0\), which in turn implies that \(F_{\mu\nu} \partial_{\alpha} k^\alpha |_{\mathcal{H}} = 0\). By the antisymmetry of the \(U(1)\)-gauge field tensor one reveals that \(F_{\alpha\beta} k^\alpha \sim k_\beta\). It provides the pull-back of \(F_{\alpha\beta} k^\alpha\) to the black hole ring horizon vanishes. On the other hand this fact leads immediately to the conclusion that \(F_{\alpha\beta} k^\alpha\) is a closed one-form. Using the Hodge decomposition theorem it may be written as a sum of closed and harmonic form. Due to the fact that the field Eqs. are satisfied the exact form is equal to zero and the only contribution stems from the harmonic part of the considered one-form. As in Refs. [15, 17] by means of the duality between homology and cohomology it follows that the surface terms will be of the form of a constant relating to the harmonic part of the one-form and the variation of a local charge. We arrive at he following:

\[
\alpha \delta M - \sum_i \Omega_{(i)} \delta J^{(i)} + \Phi_e \delta Q_e + \Phi_m \delta q_m = 2 \int_{\mathcal{H}} \delta T_{\mu\nu}(\text{matter}) \xi^\mu k_{\nu},
\]

The right-hand side of Eq. (21), can be found by the same procedure as described in Refs. [15, 17], i.e., having in mind \(n\)-dimensional Raychaudhuri Eq. and using the fact that the null generators of the event horizon of the perturbed black ring coincide with the null generators of the unperturbed stationary black ring, one arrives at the expression

\[
\kappa \delta A = \int_{\mathcal{H}} \delta T_{\mu\nu}(\text{matter}) \xi^\nu k_{\mu},
\]

where \(\kappa\) is the surface gravity.

The physical process version of the first law of black rings mechanics may be written as

\[
\alpha \delta M - \sum_i \Omega_{(i)} \delta J^{(i)} + \Phi_e \delta Q_e + \Phi_m \delta q_m = 2 \kappa \delta A.
\]
III. EQUILIBRIUM STATE VERSION OF THE FIRST LAW OF BLACK RING MECHANICS

In this section we shall derive the first law of black rings dynamics in n-dimensional CS gravity by choosing an arbitrary cross section of the event horizon to the future of the bifurcation sphere. As was remarked in [21] this attitude enables one to treat fields which are not necessarily smooth through the horizon. The only requirement is that the pull-back of these fields in the future of the bifurcation surface be smooth. Let us consider the spacetime with asymptotic conditions at infinity and equipped with the Killing vector field $\xi_\mu$, which introduces an asymptotic symmetry. It is turned out that there exists a conserved quantity $H_\xi$ [24], which yields

$$\delta H_\xi = \int_\infty \left( \delta Q(\xi) - \xi \Theta \right).$$

(24)

$\delta$ is the variation which has no effect on $\xi_\alpha$ because of the fact that the Killing vector field is treated as a fixed background and it ought not to be varied in expression [24].

In our considerations we were bound to the case of stationary axisymmetric black ring solution so the Killing vector field will be given by Eq.(17). We shall consider an asymptotically hypersurface $\Sigma$ ending on the part of the event horizon $H$. In our considerations of the first law of black ring dynamics we shall compare variations between two neighbouring states of the considered object. In general, there is a freedom which points can be chosen to correspond when one compares two slightly different solutions. In what follows we choose this freedom [19] to make $S_H$ the same of the two solutions (freedom of the general coordinate transformation) as well as we consider the case when the null vector remains normal to $S_H$. Of course, the stationarity and axisymmetricity of the solution will be preserved which in turn causes that $\delta \eta$ and $\delta g_{\mu\nu}$ will be equal to zero. It yields that the variation of the Killing vector field $\eta_\mu$ is of the form $\delta \eta^\mu = \sum_i \delta \Omega(i) \phi^{\mu(i)}$.

As in the previous section let us assume that $(g_{\mu\nu}, F_{\alpha\beta})$ are solutions of the equations of motion and $(\delta g_{\mu\nu}, \delta F_{\alpha\beta})$ are the linearized perturbations satisfying Eqs. of motion. We shall require that the pull-back of $F_{\alpha\beta}$ to the future of the bifurcation surface be smooth, but not necessary smooth on it. We also assume that the fields and their variations fall off sufficiently rapid at infinity. Having it all in mind we can write the relation

$$\alpha \delta M - \sum_i \Omega(i) \delta J^{(i)} = \int_{S_H} \left( \delta Q(\eta) - \eta \Theta \right).$$

(25)

The same arguments as quoted in the previous section help us to conclude that the following is satisfied:

$$\int_{S_H} Q^{FCS}_{j_1...j_{n-2}}(\eta) = \int_{S_H} Q^F_{j_1...j_{n-2}}(\eta) + \int_{S_H} Q^{CS}_{j_1...j_{n-2}}(\eta)$$

$$= \Phi_c Q_c + \Phi_m q_m.$$

(26)

The variation $\tilde{\delta}$ of $Q^F_{j_1...j_{n-2}}(\eta)$ implies

$$\tilde{\delta} \int_{S_H} Q^F_{j_1...j_{n-2}}(\eta) = \delta \int_{S_H} Q^F_{j_1...j_{n-2}}(\eta) - \int_{S_H} \left( \Phi_c Q_c \right)$$

$$+ \frac{2}{(n-2)!} \int_{S_H} \sum_i \delta \Omega(i) \phi^{\mu(i)} A_\mu \epsilon_{\mu\alpha j_1...j_{n-2}} F^{\alpha\mu}.$$  

(27)

We can express the volume element $\epsilon_{\mu\alpha j_1...j_{n-2}}$ by the volume element on $S_H$ and by the vector $N^\alpha$, which is ingoing future directed null normal to $S_H$, together with the normalization $N^\alpha \eta_\alpha = -1$. It can be easily verify that this relation and Eq.(27) enables us to write

$$\delta \Phi_c Q_c = \frac{4}{(n-2)!} \int_{S_H} \epsilon_{j_1...j_{n-2}} F^{\alpha\mu} N_\mu \eta_d \delta A_d$$

$$- \frac{2}{(n-2)!} \int_{S_H} \epsilon_{\mu\alpha j_1...j_{n-2}} F^{\alpha\mu} \sum_i \delta \Omega(i) \phi^{\mu(i)} A_\mu.$$  

(28)

We take now into account symplectic $(n-1)$-form for the potential $A_\mu$. Due to the fact that on the event horizon of black ring $F_{\mu\alpha} \eta^\mu \sim \eta_\alpha$ and expressing the volume element $\epsilon_{\mu\alpha j_1...j_{n-2}}$ in the same form as in the above case, one gets

$$\int_{S_H} \eta^{j_1} \Theta^F_{j_1...j_{n-2}} = \frac{4}{(n-2)!} \int_{S_H} \epsilon_{j_1...j_{n-2}} F^{\alpha\mu} N_\mu \eta_d \delta A_d.$$  

(29)
Using Eq.\((25)\) together with the expressions \((29)\) we finally conclude
\[
\bar{\delta} \int_{S_\mathcal{H}} Q_{j_1 \ldots j_{n-2}}^F(\eta) - \eta^{j_1} \Theta_{j_1 \ldots j_{n-1}}^F = \Phi_c \delta Q_c. \tag{30}
\]
The same procedure as above applied to Chern-Simons term provides the following:
\[
\bar{\delta} \int_{S_\mathcal{H}} Q_{j_1 \ldots j_{n-2}}^{CS}(\eta) = \delta \left( \Phi_m \eta_m \right) \tag{31}
\]
while variation of the potential multiplied by the local charge gives us
\[
\delta \Phi_m \eta_m = \int_{S_\mathcal{H}} \epsilon_{j_1 \ldots j_{n-2}} \frac{2\gamma(n-2)}{(n-2)!} \epsilon^{maab_2 \ldots m_1 m_2} A_a F_{b_1 b_2} \ldots F_{m_1 m_2} \sum_i \delta \Omega(i) \delta^{d(i)} A_d \tag{32}
\]
Consequently it is easy to see that
\[
\int_{S_\mathcal{H}} \eta^{j_1} \Theta_{j_1 \ldots j_{n-1}}^{CS}(\eta) = \int_{S_\mathcal{H}} \epsilon_{j_1 \ldots j_{n-2}} \frac{2\gamma(n-2)}{(n-2)!} \eta_m \epsilon^{m_1 m_2 b_1 \ldots b_2} A_a F_{b_1 b_2} \ldots F_{m_1 m_2} \eta^d \delta A_d \tag{33}
\]
By virtue of Eq.\((31)\) and \((33)\) we can finally conclude the following:
\[
\bar{\delta} \int_{S_\mathcal{H}} Q_{j_1 \ldots j_{n-2}}^{GR}(\eta) - \xi^{j_1} \Theta_{j_1 \ldots j_{n-1}}^{GR} = \Phi_m \delta q_c. \tag{34}
\]
Consider, next, the contribution connected with gravitational field \((19)\). It implies
\[
\int_{S_\mathcal{H}} Q_{j_1 \ldots j_{n-2}}^{GR}(\eta) = 2\kappa A, \tag{35}
\]
where \(A = \int_{S_\mathcal{H}} \epsilon_{j_1 \ldots j_{n-2}}\) is the area of the black ring horizon. In terms of the above derivations one obtains
\[
\bar{\delta} \int_{S_\mathcal{H}} Q_{j_1 \ldots j_{n-2}}^{GR}(\eta) = 2\delta \left( \kappa A \right) + 2 \sum_i \delta \Omega(i) J^{(i)}, \tag{36}
\]
where \(J^{(i)} = \frac{1}{2} \int_{S_\mathcal{H}} \epsilon_{j_1 \ldots j_{n-2}ab} \nabla^a \phi^{(i)b}\) is the angular momentum connected with the Killing vector field \(\phi^{(i)}\) responsible for the rotation in the adequate directions. Conducting the calculations as in Ref.\((19)\) it could be found that the integral over the black ring horizon from the product \(\xi^{j_1} \Theta_{j_1 \ldots j_{n-1}}^{GR}(\eta)\) can be written as
\[
\int_{S_\mathcal{H}} \eta^{j_1} \Theta_{j_1 \ldots j_{n-1}}^{GR}(\eta) = 2A \delta \kappa + 2 \sum_i \delta \Omega(i) J^{(i)}. \tag{37}
\]
From Eq.\((36)\) and \((37)\) one immediately obtains the expression
\[
\bar{\delta} \int_{S_\mathcal{H}} Q_{j_1 \ldots j_{n-2}}^{GR}(\eta) - \xi^{j_1} \Theta_{j_1 \ldots j_{n-1}}^{GR} = 2\kappa \delta A. \tag{38}
\]
Summing it all up, namely taking into account Eqs.\((30)\) and \((38)\), we find that the equilibrium state version of the first law of black rings mechanics in Einstein \(n\)-dimensional gravity with CS term, can be determined by the formula:
\[
\alpha \delta M - \sum_i \Omega(i) \delta J^{(i)} + \Phi_c \delta Q_c + \Phi_m \delta q_m = 2\kappa \delta A. \tag{39}
\]
IV. CONCLUSIONS

This paper provides the first law of black ring thermodynamics both for the physical process version and equilibrium state one. We considered $n$-dimensional gravity with CS term being the simplest generalization of the minimal five-dimensional supergravity in which a stationary black rings solutions were achieved. In five-dimensional case the solutions were sufficiently complicated that a first law could not be obtained by inspection (in Ref. [30] the first law of black ring thermodynamics was found by means of the ADM formalism method [20]).

Deriving the physical process version we changed infinitesimally a stationary black ring solution by throwing matter into it. Taking into account that the black ring settles down to a stationary state we derive the first law of thermodynamics. The same form of the law we obtained considering equilibrium state version. We choose an arbitrary cross section of the black ring event horizon to the future of bifurcation surface, contrary to the previous derivations based on taking into considerations bifurcation surface as the boundary of the hypersurface extending to spatial infinity. In general, as was remarked in Ref. [27] this attitude enables one to treat fields which are not necessary smooth through the event horizon. The only requirement is that their pull-back in the future of the bifurcation surface be smooth.

[1] G.W. Gibbons, D. Ida, and T. Shiromizu, Prog. Theor. Phys. Suppl. 148, 284 (2003),
G.W. Gibbons, D. Ida, and T. Shiromizu, Phys. Rev. D 66, 044010 (2002),
G.W. Gibbons, D. Ida, and T. Shiromizu, Phys. Rev. Lett. 89, 041101 (2002),
M. Rogatko, Class. Quantum Grav. 19, L151 (2002),
M. Rogatko, Phys. Rev. D 67, 084025 (2003),
M. Rogatko, ibid. 70, 044023 (2004),
M. Rogatko, ibid. 73, 124027 (2006).

[2] S. Hollands, A. Ishibashi and R. M. Wald, A Higher Dimensional Stationary Rotating Black Hole Must be Axisymmetric, gr-qc 0605106 (2006).

[3] R. Emparan and H. S. Reall, Phys. Rev. Lett. 88, 101101 (2002).

[4] Y. Morisawa and D. Ida, Phys. Rev. D 69, 124005 (2004).

[5] M. Rogatko, Phys. Rev. D 70, 084025 (2004).

[6] H. Elvang, Phys. Rev. D 68, 124016 (2003).

[7] H. Elvang and R. Emparan, JHEP 11, 035 (2003).

[8] H. K. Kunduri and J. Lucietti, Phys. Lett. B 609, 143 (2005).

[9] S. Yazadjiev, Class. Quantum Grav. 22, 3875 (2005).

[10] S. Yazadjiev, Phys. Rev. D 72, 104014 (2005).

[11] I. Bena and P. Kraus, Phys. Rev. D 70, 046003 (2004).

[12] R. Emparan and H. S. Reall, Black Rings, hep-th 0608012 (2006).

[13] H. Elvang, R. Emparan, and P. Figueras, JHEP 02, 031 (2005).

[14] R. M. Wald, Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics, University of Chicago Press (Chicago, 1994).

[15] S. Gao and R. M. Wald, Phys. Rev. D 64, 084020 (2001).

[16] M. Rogatko, Class. Quantum Grav. 19, 3821 (2002).

[17] M. Rogatko, Phys. Rev. D 71, 104004 (2005).

[18] M. Rogatko, Phys. Rev. D 72, 074008 (2005), Erratum ibid. 72, 089901 (2005).

[19] J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. 31, 161 (1973).

[20] D. Sudarsky and R. M. Wald, Phys. Rev. D 46, 1453 (1992).

[21] R. M. Wald, Phys. Rev. D 48, R3427 (1993).

[22] V. Iyer and R. M. Wald, Phys. Rev. D 50, 846 (1994).

[23] V. Iyer and R. M. Wald, Phys. Rev. D 52, 4430 (1995).

[24] V. Iyer, Phys. Rev. D 55, 3411 (1997).

[25] T. Jacobson, G. Kang, and R. C. Myers, Phys. Rev. D 49, 6587 (1994),
T. Jacobson, G. Kang, and R. C. Myers, Phys. Rev. D 52, 3518 (1995).

[26] J. Koga and K. Maeda, Phys. Rev. D 58, 064020 (1998).

[27] S. Gao, Phys. Rev. D 68, 044016 (2003).

[28] G. W. Gibbons, M. J. Perry, and C. N. Pope, Class. Quantum Grav. 22, 1503 (2005),
M. Korzynski, J. Lewandowski, and T. Pawlowski, ibid. 22, 2001 (2005),
M. Rogatko, Phys. Rev. D 71, 024031 (2005).

[29] R. C. Myers and M. J. Perry, Ann. Phys. 172, 304 (1986),
J.P. Gauntlett, R.C. Myers, and P.K. Townsend, Class. Quantum Grav. 16, 1 (1999),
P.K. Townsend and M. Zamaklar, Class. Quantum Grav. 18, 5269 (2001).
[30] K. Copsey and G. T. Horowitz, Phys. Rev. D 73, 024015 (2005).
[31] M. Rogatko, Phys. Rev. D 73, 024022 (2005).
[32] R. M. Wald and A. Zoupas, Phys. Rev. D 61, 084027 (2000).