A multilevel based reweighting algorithm with joint regularizers for sparse recovery*

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Abstract

Sparsity is one of the key concepts that allows the recovery of signals that are subsampled at a rate significantly lower than required by the Nyquist-Shannon sampling theorem. Our proposed framework uses arbitrary multiscale transforms, such as those build upon wavelets or shearlets, as a sparsity promoting prior which allow to decompose the image into different scales such that image features can be optimally extracted. In order to further exploit the sparsity of the recovered signal we combine the method of reweighted $\ell^1$, introduced by Candès et al., with iteratively updated weights accounting for the multilevel structure of the signal. This is done by directly incorporating this approach into a split Bregman based algorithmic framework. Furthermore, we add total generalized variation (TGV) as a second regularizer into the split Bregman algorithm. The resulting algorithm is then applied to a classical and widely considered task in signal- and image processing which is the reconstruction of images from their Fourier measurements. Our numerical experiments show a highly improved performance at relatively low computational costs compared to many other well established methods and strongly suggest that sparsity is better exploited by our method.

1 Introduction

The field of sparse recovery has a wide range of applications in many different areas such as medical imaging [35], astronomy [5, 33], electron microscopy [38, 37] etc. One of the great successes in this area are the new developments of multiscale sparsifying dictionaries since the invention of wavelets. Indeed wavelets are known to compress natural images very effectively however, they lack in directional sensitivity and are therefore not optimal for certain types of images. Directional systems such as curvelets [13] and shearlets [27, 23] have been able to overcome this deficit and are nowadays widely used. We will briefly recall the concept of wavelets and shearlets in Section 2 as both systems will play an important role in the upcoming content of this paper.

Another milestone in the area of sparse recovery is the development of compressed sensing [14, 19], a novel theory that guarantees the recovery of sparse signals from incomplete measurements under certain assumptions. These signals are typically obtained by solving a convex optimization problem of the form

$$\min_u \|\Psi u\|_1 \quad \text{s.t.} \quad Au = y,$$

where $u$ is the object of interest, $\Psi$ is a sparsifying transform, $A$ is a matrix representing the measurement process, and $y$ are the measured data. If $u$ is already sparse itself it suffices to let

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Ψ to be the identity. Otherwise \(1\) is called Basis Pursuit in the analysis formulation. Very recently, Ahmad and Schniter considered a generalized variant of \(1\) in [3], where they have considered not only one sparsifying transform, but a composition of several sparsifying transforms. Moreover, in order to arrive at a more efficient algorithm, the authors have combined the ideas of reweighted \(\ell^1\) into their algorithmic scheme.

Indeed, the method of reweighted \(\ell^1\) introduced by Candès et al. in [15] further promotes the sparsity of the recovered signal by iteratively updating a weighting matrix in the minimization problem. More precisely, one solves

\[
u_{k+1} = \arg \min_u \|W^k \Psi u\|_1 \quad \text{s.t.} \quad Au = y,
\]

for \(k = 1, 2, \ldots\) and after each iteration \(k\) the diagonal weighting matrix \(W^k\) is updated according to the sparsity structure of the current iterate \(u_k\). The role of \(W^k\) is to mimic the actual sparsity structure of the true signal that one wishes to recover, but obviously is not available. We will recap the ideas of reweighted \(\ell^1\) later in Section 2 in more detail as this will also be one of the main ingredients of the algorithm that we propose in this work.

For the implementation of solvers for minimization problems such as \(1\) there are many different possibilities available in the literature. In this work we focus on the split Bregman algorithm [21, 12] which is closely related to the Alternating Direction Method of Multipliers (ADMM) [6]. Both methods have in common that they transform the constrained problem \(1\) into an unconstrained formulation and by introducing splitting variables they break the original problem down into easier ones. A key ingredient in the so-called soft-thresholding, shrinking or proximal mapping which gives a closed-form solution to some of the subproblems. One of the great advantages of ADMM and split Bregman are that they are easily derived and very flexible in terms of multiple regularizers. Furthermore, as we shall explain in this paper, it can be easily combined with the idea of reweighted \(\ell^1\). Indeed, it turned out that in order to reduce visual artifacts in the solutions of sparse imaging problems that are obtained via \(1\), it can be beneficial to add a second sparsity promoting regularizer, for instance, total variation (TV) [21, 50]. In contrast to multiscale transforms such as those coming from wavelets and shearlets, TV is based on a variational approach. It was initially proposed for denoising problems [45] and is very well established in image processing by now. However, it has been noticed that severe staircasing artifacts may appear in images recovered by TV regularized reconstructions for large noise levels. In order to overcome this issue the authors of [11] have introduced total generalized variation (TGV) which is a generalization of TV to higher order derivatives. Since then TGV has been used in many applications [4, 28] as a regularizer for inverse problems. Moreover, it has been combined with other regularizers, for instance, in [24] the authors have combined it with shearlets.

Our work can be seen as a generalization of the work by Guo et al. in [24] by combining multilevel based regularizers, such as shearlets and wavelets, with TGV in order to solve problems of the form \(1\). Our main contribution is to further exploit the structure of the (multilevel) sparsity by combining iterative reweighting and adaptive multilevel weights. Even though this is somewhat related to the approach in [3], the algorithm and conclusions derived in this work are different. Indeed, in order to solve the resulting multilevel reweighted \(\ell^1\) problem we make use of the flexibility of split Bregman by directly incorporating adaptive multilevel reweighted thresholds into the soft-thresholding step. This results in an algorithm that comes with very little additional computational cost and almost automatically chosen regularization parameters. In particular, as we will show in this paper, the results are greatly improved compared to the non-weighted analogue.

In Figure 1 we give a motivation and a first glance for the possible benefit of combining reweighting methods with well established sparsifying transforms. We show a reconstruction from partial Fourier measurements of an synthetic test image which has parts that are certainly
Figure 1: Reconstructions from 25 radial lines (10.28%) through the k-space origin. **First column**: Original image with zoom. **Second column**: Reconstruction with inverse Fourier transform. Relative error: 0.146. Structured similarity index: 0.735. **Third column**: Reconstruction with redundant Daubechies 4 wavelets and total generalized variation regularizer without reweighting. Relative error: 0.060. Structured similarity index: 0.896. **Fourth column**: Reconstruction with Proposed method. Relative error: 0.031. Structured similarity index: 0.951.

sparse in a wavelet dictionary and parts that should be able to be nicely recovered using TGV. We refer to Section 4 for more details about the numerical implementation. Note that in Figure 1 the proposed method reduces the artifacts while still being able to reconstruct fine details using only 10.28% of the available measurements.

1.1 Outline

In Section 2 we will give a compact overview of all methods that are needed in order to derive our proposed algorithm. In Section 3 we then present our ideas and the method. Finally, in Section 4 we test our method extensively and compare it to other classical and novel algorithms for the recovery from incomplete Fourier measurements.

2 Sparse recovery, convex optimization, and sparsifying transforms

In this section we present a short overview of current concepts and methods that are standard in the area of sparse recovery and are necessary to follow the rest of this work. For more details we refer the interested reader to the mentioned literature.

2.1 Compressed sensing

The problem considered in compressed sensing can be explained by solving a system of linear equations using explicit prior information. Indeed, we are interested in solutions \( u \) of the equation

\[ Au = y, \]  

(3)
where \( y \) is a vector representing the acquired data, \( A \) is a sensing matrix, and \( u \) is the object of interest. The assumption that makes this problem worth to study is that \( y \) should be of very small dimension whereas \( u \) lives in a much higher dimensional space, i.e. \( A \in \mathbb{C}^{m \times n} \) with \( m < n \). Furthermore, the aforementioned prior information that enables us to solve such an underdetermined system is sparsity, i.e. although \( u \) might be drawn from a much higher dimensional space only very few of its entries are nonzero. More precisely, we say a signal \( u \in \mathbb{C}^n \) is \( s \)-sparse if

\[
\| u \|_0 := \# \{ i \in \mathbb{N} : u_i \neq 0, 1 \leq i \leq n \} \leq s.
\]

A common approach to obtain sparse solutions of (3) is to solve the following constrained convex optimization problem

\[
\min_u \| u \|_1 \quad \text{s.t.} \quad A u = y,
\]

see for example \([14, 20]\).

### 2.2 Reweighted \( \ell^1 \)

One of the possibilities to improve on the recovery model is to strengthen the effect of sparsity in the minimization problem. This can be done by the idea of reweighted \( \ell^1 \) introduced by Candès et al. in \([15]\). As this is also one of the key tools for our algorithm we will now briefly state the idea behind this method.

Suppose we are given measurements \( y = A u \in \mathbb{C}^m \) of an \( s \)-sparse signal \( u = (u_1, \ldots, u_n)^T \in \mathbb{C}^n \) with a measurement matrix \( A \in \mathbb{C}^{m \times n} \) for \( m \ll n \). When solving the minimization problem (4) iteratively, one would ask for the following effect: large coefficients should be quickly identified and hence become "cheaper" in the minimization of the objective function in (3), whereas very small coefficients should be neglected since they are most likely to be zero in the true signal. More precisely, let \( u_0 \) be the true signal and define a diagonal weighting matrix \( W \) by

\[
W_{i,i} = \begin{cases} 
\frac{1}{|u_{0,i}|}, & u_{0,i} \neq 0 \\
\infty, & u_{0,i} = 0.
\end{cases}
\]

Now, if the signal \( u_0 \) is \( s \)-sparse, then under some assumptions \([15]\) the weighted \( \ell^1 \)-minimization problem

\[
\min_u \| W u \|_1 \quad \text{s.t.} \quad A u = y,
\]

will find the exact solution. However, since \( u_0 \) is usually unknown such weights are practically infeasible. Therefore in \([15]\) the authors proposed adaptive weights that change at each iteration depending on the previously computed solution \( u^k \) which is an approximation to \( u_0 \). More precisely, the following sequence of minimization problems have been considered

\[
u_{k+1} = \arg \min_u \| W^k u \|_1 \quad \text{s.t.} \quad A u = y,
\]

with weighting matrix

\[
W_{i,i}^k = \frac{1}{|u_{k,i}| + \varepsilon},
\]

where \( \varepsilon > 0 \) is a stability parameter and the initial weighting matrix \( W^0 \) is set to be the identity. In a series of numerical experiments it was shown in \([15]\) that such reweighting methods find sparse solution much faster with significantly reduced errors.
2.3 Sparsifying multilevel transforms

In the previous section we discussed how the concept of sparsity is used to recover signals from possibly highly undersampled data. However, in many applications the signals are not sparse themselves, but only after the application of certain transforms. Such sparsifying transforms are often build upon systems that are equipped with a multiscale structure. Indeed, very recently a new theory on compressed sensing has been developed that is very much motivated by the sparsity structure of multiscale systems [2]. Typical examples of such multiscale transforms are the wavelet transform [17], the shearlet transform [23, 27, 34], and the curvelet transform [13]. For the results of this paper we will only consider the first two transforms. Indeed, both, wavelet reconstructions as well as shearlet reconstructions from an incomplete amount of Fourier measurements are analyzed, for instance, in [1] and [36, 32], respectively. Even though multiscale transform can be generalized to higher dimensions, see [29] for shearlets, we will focus on 2D signals only for the upcoming presentation as we are mainly concerned with the reconstruction of images in this work. However, our algorithm does work for the 3D case as well and some preliminary results with the application to dynamic MRI will be published in an upcoming work.

The wavelet transform is a common tool in the compression of natural images, for instance, the compression algorithm behind JPEG2000 is build on wavelets [17]. It computes the analysis coefficients of the signal of interest, the so-called wavelet coefficients. In Figure 2 we have depicted the sparsity of an MRI test image [22] in a wavelet basis.

![Wavelet Coefficients](image)

Figure 2: Left: Original brain phantom from [22]. Right: Wavelet coefficients with four scales

The multiscale structure of a wavelet basis comes from the use of dyadic scaling matrices of the form

\[
D_j = \begin{pmatrix} 2^j & 0 \\ 0 & 2^j \end{pmatrix}, \quad j = 0, 1, \ldots
\]

Using these matrices together with simple translations one can eventually obtain orthonormal bases for \(L^2(\mathbb{R}^2)\), the space of square integrable functions, of the form

\[
\{ \phi(\cdot - m) : m \in \mathbb{Z}^2 \} \cup \{ \psi^1(D_j \cdot - m) : j \geq 0, m \in \mathbb{Z}^2 \} \\
\cup \{ \psi^2(D_j \cdot - m) : j \geq 0, m \in \mathbb{Z}^2 \} \\
\cup \{ \psi^3(D_j \cdot - m) : j \geq 0, m \in \mathbb{Z}^2 \},
\]

where \(\phi, \psi^1, \psi^2, \psi^3 \in L^2(\mathbb{R}^2)\) with certain regularity properties, see [17] for more details.

The multiscale structure of shearlets are obtained by the use of parabolic scaling matrices of the form

\[
A_{2j} = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}, \quad \tilde{A}_{2j} = \begin{pmatrix} 2^{j/2} & 0 \\ 0 & 2^j \end{pmatrix}, \quad j \in \mathbb{N}.
\]
In addition to the parabolic scaling matrix, a shearlet system is equipped with a directional component that can be obtained by using shear matrices

\[ S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \quad k \in \mathbb{Z}. \]

The so-called cone-adapted shearlet system is then defined as follows.

**Definition 2.1.** Let \( \varphi, \psi, \tilde{\psi} \in L^2(\mathbb{R}^2) \). Then we call \( \Phi(\varphi, c) \cup \Psi(\psi, c) \cup \tilde{\Psi}(\tilde{\psi}, c) \) a cone-adapted shearlet system, where

\[
\Phi(\varphi, c) = \{ \varphi_m : m \in \mathbb{Z}^2 \}, \\
\Psi(\psi, c) = \{ \psi_{j,k,m} : j \geq 0, |k| \leq 2^{j/2}, m \in \mathbb{Z}^2 \}, \\
\tilde{\Psi}(\tilde{\psi}, c) = \{ \tilde{\psi}_{j,k,m} : j \geq 0, |k| \leq 2^{j/2}, m \in \mathbb{Z}^2 \},
\]

and

\[
\varphi_m = \varphi(\cdot - c_1 m), \quad \psi_{j,k,m} = 2^{3/4j} \psi(S_k A_{2^j} \cdot - cm), \quad \tilde{\psi}_{j,k,m} = 2^{3/4j} \tilde{\psi}(S_k^T \bar{A}_{2^j} \cdot - \bar{c} m),
\]

with \( c = (c_1, c_2)^T \in \mathbb{R}_+^2, \bar{c} = (c_2, c_1) \) and the multiplication of \( c \) and \( \bar{c} \) with \( m \) to be understood componentwise.

In contrast to wavelets, there are no known constructions of shearlet orthonormal bases for \( L^2(\mathbb{R}^2) \), however, shearlet systems can be constructed that form a frame \([27, 16]\). Furthermore, similar to wavelets, shearlet systems also sparsify natural images, see Figure 3. In particular, within the model of so-called cartoon-like images \([18]\) shearlets are optimal by fulfilling the optimal sparse approximation rate \([30]\) while wavelets do not.

![Shearlet coefficients at different scales of the brain image used in Figure 2. For better visual difference the contrast at scale 1,2, and 3 has been changed. First: Shearlet coefficients at scale \( j = 0 \). Second: Shearlet coefficients at scale \( j = 1 \). Third: Shearlet coefficients at scale \( j = 2 \). Fourth: Shearlet coefficients at scale \( j = 3 \).](image)

### 2.4 Split Bregman

Split Bregman is a fast algorithm to solve constrained optimization problems by introducing a split variable and solving the resulting decoupled problem with Bregman Iterations. It was proposed in 2009 by Goldstein and Osher and became a popular method since then \([21, 12, 51, 42, 47]\). Even though split Bregman can handle general convex regularizers it is sufficient for our purposes to focus on \( \ell^1 \)-regularized problems in the analysis formulation. We briefly follow the steps in \([41, 21, 51]\) to derive the basic form of the algorithm.

Consider the basis pursuit problem in analysis formulation

\[
\min_u \| W \Psi u \|_1 \quad \text{s.t.} \quad \| y - Au \|_2 \leq \sigma, \tag{6}
\]
for a possibly redundant dictionary $\Psi \in \mathbb{R}^{N_\Psi \times n}$, a measurement matrix $A \in \mathbb{C}^{m \times n}$, a fidelity parameter $\sigma > 0$, and a diagonal weighting matrix with entries $W_{l,l}$ for $l = 1, \ldots, N_\Psi$. Then instead of using a continuation method for enforcing the constraint, i.e. taking $\beta \to \infty$ in

$$ u = \arg \min_u \|W\Psi u\|_1 + \frac{\beta}{2} \|y - Au\|_2^2, $$

problem (6) is transformed into a sequence of unconstrained problems using Bregman iterations

$$\begin{aligned}
    u_{k+1} &= \arg \min_u \|W\Psi u\|_1 + \frac{\beta}{2} \|y - Au + y_k\|_2^2, \\
    y_{k+1} &= y_k + y - Au_{k+1}.
\end{aligned}$$

(7)

We continue by introducing a split variable $d = \Psi u$ for the $\ell^1$-part of the minimization problem in (7) and executing an additional Bregman iteration step to obtain

$$\begin{aligned}
    (u_{k+1}, d_{k+1}) &= \arg \min_{u,d} \|Wd\|_1 + \frac{\beta}{2} \|y - Au + y_k\|_2^2 + \frac{\mu}{2} \|d - \Psi u - b_k\|_2^2, \\
    b_{k+1} &= b_k + \Psi u_{k+1} - d_{k+1}, \\
    y_{k+1} &= y_k + y - Au_{k+1}.
\end{aligned}$$

To solve the $(u, d)$-minimization problem one or multiple nonlinear block Gauss-Seidel iterations are used, which alternate between minimizing with respect to $u$ and $d$. This yields the Split Bregman Algorithm where $x^{\text{cur}}$ denotes the latest available stage of the variable $x$. Note that

```
for $i = 1 : N$ do
    $u_{k+1} = \arg \min_u \frac{\beta}{2} \|y - Au + y_k\|_2^2 + \frac{\mu}{2} \|d^{\text{cur}} - \Psi u - b_k\|_2^2$
    $d_{k+1} = \arg \min_d \|Wd\|_1 + \frac{\mu}{2} \|d - \Psi u^{\text{cur}} - b_k\|_2^2$
end for
```

$b_{k+1} = b_k + \Psi u_{k+1} - d_{k+1},$

$y_{k+1} = y_k + y - Au_{k+1},$

**Algorithm 1**: Split Bregman algorithm

the solution of the $d$-subproblem is explicitly given by soft-thresholding

$$ d_{k+1}(l) = \text{shrink} \left( (\Psi u^{\text{cur}})(l) + b_k(l), \frac{W_{l,l}}{\mu} \right), $$

for $l = 1, \ldots, N_\Psi$ where

$$ \text{shrink} \left( z, \lambda \right) = \begin{cases} 
    \max \left( \|z\| - \lambda, 0 \right) \frac{z}{\|z\|}, & z \neq 0, \\
    0, & z = 0.
\end{cases} $$

In [21] it was furthermore observed that the minimization with respect to $u$ in (1) does not need to be solved to full precision and in many applications only few steps of an iterative method are sufficient.

### 2.5 Total generalized variation

Total variation based methods were initially proposed by Rudin, Osher, and Fatemi in 1992 for image denoising [45] and are now widely used for image reconstruction and compressed sensing, see for example [50, 39]. TV is based on the assumption that the reconstructed image is piecewise constant and therefore gradient sparse. This results in preserving sharp edges.
But for realistic images, which are usually not piecewise constant, this can lead to severe oil painting artifacts or staircasing effects leading to unnatural looking reconstructed images. Among others, Total Generalized Variation (TGV), a generalization of TV, has been proposed to improve on these issues by involving higher order derivatives \[11\]. We will now briefly give a definition of the second order TGV regularizer in \(\mathbb{R}^2\) together with some basic facts. Its general derivation and more details on this subject can be found in \([7, 8, 11, 9, 10]\).

The so-called pre-dual formulation of second order TGV is given by

\[
TGV^2_\alpha(u) = \sup \left\{ \int_\Omega u \, \text{div}^2 v \, dx : v \in C^2_c(\Omega, S^{2 \times 2}), \|v\|_\infty \leq \alpha_0, \|\text{div} v\|_\infty \leq \alpha_1 \right\}, \tag{8}
\]

for \(\alpha = (\alpha_0, \alpha_1) \in \mathbb{R}^2_+, \Omega \subseteq \mathbb{R}^2\) a bounded Lipschitz domain, \(S^{2 \times 2}\) the space of symmetric \(2 \times 2\) matrices, and \(u \in L^1(\Omega, \mathbb{C})\). Thereby the divergences are defined as

\[
(\text{div} v)_i = \sum_{j=1}^2 \frac{\partial v_{ij}}{\partial x_j}, \quad i = 1, 2,
\]

and

\[
\text{div}^2 v = \sum_{i,j=1}^2 \frac{\partial^2 w_{ij}}{\partial x_i \partial x_j},
\]

together with the norms

\[
\|v\|_\infty = \sup_{l \in \Omega} \left( \sum_{i,j=1}^2 |v_{ij}(l)|^2 \right)^{1/2},
\]

and

\[
\|\text{div} v\|_\infty = \sup_{l \in \Omega} \left( \sum_{i=1}^2 (|\text{div} v)_i(l)|^2 \right)^{1/2}.
\]

Under certain conditions, an equivalent and more convenient form of \(TGV^2_\alpha\) is given by the minimum representation as

\[
TGV^2_\alpha(u) = \inf_{v \in \text{BD}(\Omega, C^2)} \alpha_1 \|\nabla u - v\|_1 + \alpha_0 \|E(v)\|_1, \tag{9}
\]

where \(\text{BD}(\Omega, C^2)\) is the space of symmetric tensor fields of bounded deformation and \(E\) the symmetrized derivative defined as

\[
E(v) = \left( \frac{1}{2} \left( \partial_x v_1 + \partial_x v_2 \right), \frac{1}{2} \left( \partial_y v_1 + \partial_y v_2 \right) \right).
\]

In this form \(TGV^2_\alpha\) can be interpreted as balancing the first and second derivatives of \(u\) controlled by the ratio of \(\alpha_0\) and \(\alpha_1\). In \([11]\) and \([28]\) it was observed that the use of TGV as a regularizer indeed leads to reconstructed images with sharp edges but without the staircaising effects of TV. Being furthermore convex and lower semi-continuous with respect to \(L^1\)-convergence, this makes TGV a numerical feasible and therefore a suitable alternative for TV \([7]\).

3 Proposed multilevel based reweighting algorithm with TGV

3.1 Model and discretization

In this section we are aiming to develop an algorithm for solving the multilevel reweighted \(\ell^1\)-problem. In order to do so, the split Bregman approach introduced in Section 2.4 shall be
equipped with an appropriate iteratively reweighted soft-thresholding procedure. For a further reduction of artifacts and improving the reconstruction of piecewise constant as well as smooth regions we not only use the regularizer belonging to the reweighted multilevel decomposition but also TGV as a second regularization term. The idea of combining a multiscale transform such as shearlets with TGV has already been studied in great detail by Guo et al. in [24]. The algorithm therein is build on an ADMM approach to solve the optimization problem and is therefore due to the natural connection to the split Bregman framework also related to our algorithm.

Let $A$ be a measurement operator, let $y$ the measurements of our signal of interest $u$, and let $\sigma > 0$ be a fidelity parameter. The recovery problem can then be stated as

$$\min_u \sum_{j=1}^{\infty} \lambda_j \| W_j \Psi_j u \|_1 + TGV_{\alpha}(u) \quad \text{s.t.} \quad \| y - Au \|_2 \leq \sigma,$$

where $\Psi_j$ corresponds to the $j$-th subband of the multilevel transform $\Psi$, $\lambda_j$ are regularisation parameters accounting for the multilevel structure of $\Psi$ and $W_j$ are diagonal weights. For the sake of clearness we have assumed that there is only one subband per level otherwise an additional index has to be attached to $\Psi_j$ to specify the current subband. Note that after we have established a basic split Bregman framework for solving the minimization problem we will aim to update $\lambda_j$ and $W_j$ iteratively. Using the characterization of $TGV_{\alpha}$ presented in Section 2.3 the objective can be rewritten as

$$\min_{u,v} \sum_{j=1}^{\infty} \lambda_j \| W_j \Psi_j u \|_1 + \alpha_1 \| \nabla u - v \|_1 + \alpha_0 \| E(v) \|_1.$$

(10)

For the discretization, let $u \in \mathbb{C}^{n^2}$ be the vectorized finite-dimensional image of interest which is for simplicity assumed to be of square size. Let $A \in \mathbb{C}^{m \times n^2}$ be the finite dimensional measurement matrix and $y \in \mathbb{C}^m$ the observed data. Furthermore, let $\nabla^f$ and $\nabla^b$ denote a discrete gradient operator with periodic boundary conditions using forward and respectively backward differences. Following [11, 10] we approximate the derivatives in (10) by

$$\nabla u \approx \nabla^f u = \left( \nabla^f_x u \right)$$

and

$$\nabla^b v = \left( \frac{1}{2}(\nabla^b_x v_x + \nabla^b_y v_y) \right).$$

A finite dimensional approximation of (10) is then given by

$$\min_{u,v} \sum_{j=1}^{J} \lambda_j \| W_j \Psi_j u \|_1 + \alpha_1 \| \nabla^f u - v \|_1 + \alpha_0 \| E^b v \|_1,$$

(11)

where $J$ is some fixed a priori chosen maximum scale and $\Psi$ is the discrete transform acting on the vectors. For wavelets and shearlets the discrete transforms are greatly documented in the literature, see Chapter 8 in [48] for wavelets and [31] for shearlets. Let us furthermore introduce the notation

$$\Psi u = (\Psi_j u)_{j=0,\ldots,J} = (\langle \psi_{j,l}, u \rangle)_{j=0,\ldots,J,l=1,\ldots,N_j}$$

(12)

for dividing the analysis coefficients into $J$ subbands, each consisting of $N_j \in \mathbb{N}$ elements. Note that the $\ell^1$-norm in the second summand is thereby defined as

$$\| v \|_1 = \sum_{l=1}^{n^2} (|v_x(l)|^2 + |v_y(l)|^2)^{1/2},$$

9
and for the third summand as
\[ \|e\|_1 = \sum_{l=1}^{n^2} \|e(l)\|_F = \sum_{l=1}^{n^2} \left\| \begin{pmatrix} e(l)_1 & e(l)_2 \\ e(l)_3 & e(l)_4 \end{pmatrix} \right\|_F, \]
where \( \|\cdot\|_F \) is the Frobenius norm of a \( 2 \times 2 \) matrix.

### 3.2 Split Bregman framework

The proposed constrained optimization problem can be casted into the form given in (6) by introducing the variable \( u = (u, v)^T \) together with the matrix
\[ \Psi = \begin{pmatrix} \Psi & 0 \\ \nabla f & -I \\ 0 & \mathcal{E}^b \end{pmatrix}. \]

In order to come up with the explicit form of the resulting split Bregman algorithm as given in Section 2.4, let us split as follows:
\[ \begin{pmatrix} w \\ d \\ t \end{pmatrix} = \begin{pmatrix} \Psi u \\ \nabla f u - v \\ \mathcal{E}^b v \end{pmatrix}. \]

The \((u, v)\)-subproblem of Algorithm (11) is then given by
\[
(u_{k+1}, v_{k+1}) = \arg \min_{u, v} \frac{\beta}{2} \| y - Au + y_k \|_2^2 + \frac{\mu_1}{2} \| u_{\text{cur}} - \Psi u - b^w_k \|_2^2 \\
+ \frac{\mu_2}{2} \| d_{\text{cur}} - (\nabla f u - v) - b^d_k \|_2^2 + \frac{\mu_3}{2} \| t_{\text{cur}} - \mathcal{E}^b v - b^t_k \|_2^2.
\]

We furthermore obtain the subproblems
\[ w^j_{k+1} = \arg \min_{w^j} \lambda_j \| W_j w^j \|_1 + \frac{\mu_1}{2} \| w^j - \Psi_j u_{\text{cur}} - u^j_k \|_2^2, \]
for each subband \( j = 1, \ldots, J \), as well as
\[ d_{k+1} = \arg \min_{d} \alpha_1 \| d \|_1 + \frac{\mu_2}{2} \| d - (\nabla f u_{\text{cur}} - v_{\text{cur}}) - b^d_k \|_2^2, \]
and
\[ t_{k+1} = \arg \min_{t} \alpha_0 \| t \|_1 + \frac{\mu_3}{2} \| t - \mathcal{E}^b v_{\text{cur}} - b^t_k \|_2^2. \]

Note that the regularization parameters \( \lambda_j, \alpha_0, \) and \( \alpha_1 \) have thereby been subsumed into a weighting matrix \( W \). Also we are allowing some more flexibility by incorporating different values for \( \mu_i \) for \( i = 1, 2, 3 \). Furthermore we obtain the following Bregman updates:
\[
\begin{aligned}
\{ b^w_{k+1} &= b^w_k + \Psi u_{k+1} - w_{k+1}, \\
b^d_{k+1} &= b^d_k + (\nabla f u_{k+1} - v_{k+1}) - d_{k+1}, \\
b^t_{k+1} &= b^t_k + \mathcal{E}^b v_{k+1} - t_{k+1},
\end{aligned}
\]
as well as
\[ y_{k+1} = y_k + y - Au_{k+1}. \]
3.3 Solutions of the subproblems

The solution of the subproblem (13) can be obtained by setting the first derivatives with respect to $u$, $v_x$, and $v_y$ to zero. We then obtain the linear system

$$
\begin{pmatrix}
 b_1 & b_4^* & b_5^* \\
 b_3 & b_2 & b_6 \\
 b_5 & b_6 & b_3
\end{pmatrix}
\begin{pmatrix}
 u \\
 v_x \\
 v_y
\end{pmatrix} =
\begin{pmatrix}
 R_1 \\
 R_2 \\
 R_3
\end{pmatrix},
$$

(17)

where $b_i$ are $n^2 \times n^2$ block matrices defined as

$$
b_1 = \beta A^* A + \mu_1 \Psi^* \Psi + \mu_2 (\nabla f)^* \nabla f,
$$

$$
b_2 = \mu_3 (\nabla b_x)^* \nabla b_x + \mu_2 I,
$$

$$
b_3 = \mu_3 (\nabla b_y)^* \nabla b_y + \mu_2 I,
$$

$$
b_4 = -\mu_2 \nabla f_x,
$$

$$
b_5 = -\mu_2 \nabla f_y,
$$

$$
b_6 = \frac{\mu_3}{2} (\nabla b_x)^* \nabla b_y,
$$

and the components of the right hand side are given by

$$
R_1 = \beta A^* (y + y_k) + \mu_1 \Psi^* (w^\text{cur} - b_k^w) + \mu_2 (\nabla f)^* (d^\text{cur} - b_k^d),
$$

$$
R_2 = \mu_2 (b_{k,x}^d - d^\text{cur}_x) + \mu_3 \left( (\nabla b_x)^* (t_1^\text{cur} - b_{k,1}) + (\nabla b_y)^* (t_2^\text{cur} - b_{k,2}) \right),
$$

$$
R_3 = \mu_2 (b_{k,y}^d - d^\text{cur}_y) + \mu_3 \left( (\nabla b_x)^* (t_2^\text{cur} - b_{k,2}) + (\nabla b_y)^* (t_3^\text{cur} - b_{k,3}) \right).
$$

Similar to [21], it was observed in [24], that in many cases the linear system in (17) can be efficiently solved by using the 2D-Fourier transform $F \in \mathbb{C}^{n^2 \times n^2}$. Note that $\nabla f$ and $\nabla b$ are circulant, since they correspond to periodic boundary conditions. Therefore

$$
F \nabla^* \nabla F^*
$$

is a diagonal matrix. For a tight frame $\Psi$ we have

$$
\Psi^* \Psi = a I,
$$

(18)

where $a \in \mathbb{R}$ is the frame bound of $\Psi$. This is for example the case in [24], where the Fast Finite Shearlet Transform ([25, 26]) was used, which forms a Parseval frame for $\mathbb{R}^{n^2}$, i.e. Equation (18) with $a = 1$. Note that also for the non-tight shearlet system of ShearLab [31] the matrix $F \Psi^* \Psi F^*$ is diagonal and can be explicitly computed as described in [24].

In the numerical experiments of this paper we focus on subsampled Fourier measurements. The measurement matrix can then be written as

$$
A = P F,
$$

where $P \in \{0, 1\}^{m \times n^2}$ is selecting or discarding the measurements. In this case

$$
A^* A = F^* P F
$$

is naturally diagonalized by the 2D Fourier transform.
If all blocks \( d_i \) for \( i = 1, \ldots, 6 \) can be diagonalized in this way, the authors of [24] proposed to multiply with a preconditioner matrix from the left to obtain the system

\[
\begin{pmatrix}
\hat{b}_1 & \hat{b}_4^* & \hat{b}_5^* \\
\hat{b}_1 & \hat{b}_2 & \hat{b}_6^* \\
\hat{b}_5 & \hat{b}_6 & \hat{b}_3
\end{pmatrix}
\begin{pmatrix}
F u \\
F v_x \\
F v_y
\end{pmatrix}
= \begin{pmatrix}
FR_1 \\
FR_2 \\
FR_3
\end{pmatrix},
\]

(19)

where each \( \hat{d}_j = F d_j F^* \) is a \( n^2 \times n^2 \) diagonal matrix. A closed form solution can then be obtained by applying Cramer’s rule.

In other applications, such as inpainting or reconstruction from the subsampled cosine transform, the measurement matrix can be written as

\[
A = PT,
\]

(20)

for a unitary matrix \( T \in \mathbb{C}^{n^2 \times n^2} \). In the ADMM model of [24] it is proposed to include an additional split to deal with the fact that \( A^* A \) cannot be diagonalized under \( F \).

However, in many applications a representation of the form (20) is not always possible. For example in Partial Parallel Imaging (PPI) [49, 43] in MRI where the image is to be reconstructed by using subsampled Fourier data from parallel scans of multiple coils. We therefore propose to solve the linear system of equation (17) only approximately. As it was observed in [21], usually only a few steps of an iterative solver are sufficient for the convergence of the resulting algorithm. A proof of this fact can be found in [40]. In order to save memory and computation time, we still want to use that all \( d_j \) are diagonal under \( F \) except for \( \hat{d}_1 \).

Therefore we multiply with the same preconditioner matrix as above and obtain a similar system as in equation (19), with the difference that \( \hat{d}_1 \) is not diagonal anymore. With a few explicitly given steps the block matrix can be brought to lower triangular form. In this way we only need to apply an iterative solver to one \( n^2 \times n^2 \) system involving \( A^* A \) instead of solving the entire block system. In our numerical experiments for PPI, which will be presented in an upcoming work we have observed that using a conjugate gradient method together with a warm start (obtained through the previous iterations of the split Bregman algorithm) only very few steps are necessary for sufficient precision.

Let us now briefly discuss the solution of the other subproblems: As described in Section 2.4 a closed-form solution of (14) is given by

\[
w_{k+1}^j(l) = \text{shrink} \left( \Psi_j u_{\text{cur}}^j(l) + t_{k+1}^j(l), \frac{\lambda_j W_j(l)}{\mu_1} \right),
\]

(21)

for \( l = 1, \ldots, N_j - N_{j-1} + 1 \). For equation (15) we obtain

\[
d_{k+1}(l) = \text{shrink}_2 \left( \nabla f u_{\text{cur}}(l) - v_{\text{cur}}(l) + b^j_k(l), \frac{\alpha_1}{\mu_2} \right),
\]

for \( l = 1, \ldots, n^2 \) and the shrinkage rule

\[
\text{shrink}_2(x, \lambda) = \begin{cases} 
\max(||x||_2 - \lambda, 0) \frac{x}{||x||_2}, & x \neq 0, \\
0, & x = 0.
\end{cases}
\]

Similarly, the solution of (16) is given by

\[
t_{k+1}(l) = \text{shrink}_F \left( \mathcal{E} b^j_{\text{cur}}(l) + b^j_k(l), \frac{\alpha_0}{\mu_3} \right),
\]

for \( l = 1, \ldots, n^2 \) and

\[
\text{shrink}_F(x, \lambda) = \begin{cases} 
\max(||x||_F - \lambda, 0) \frac{x}{||x||_F}, & x \neq 0, \\
0, & x = 0.
\end{cases}
\]
3.4 Combining reweighted $\ell^1$ with multiscale transforms

As we already explained in Section 2, the idea of reweighted $\ell^1$ as proposed by Candès et al. in [15] is to improve the $\ell^1$-norm as a sparsity regularizer by an iterative weighting of the nonzero coefficients. Minimization based on the $\ell^1$-norm penalizes larger coefficients more than smaller ones, in contrast to the $\ell^0$-norm which only counts the number of nonzero coefficients. Hence the idea of reweighted $\ell^1$ is that since larger coefficients of an iterative solution are more likely to be nonzero in the true signal, they should be multiplied with a smaller weight during the optimization process. Put differently, the guiding principle of reweighted $\ell^1$ is that small coefficients of an iterative solution are likely going to be zero in the true signal. However, this principle is not necessarily valid for multiscale sparse signals, i.e. signals that can be sparsely represented under a multiscale transform. The magnitudes of multiscale coefficients naturally decrease with increasing scales, but the high scale nonzero coefficients of an iterative solution are not necessarily less important or more likely zero in the actual signal, if compared to low scale coefficients which are intrinsically larger. In the following section we are aiming to compensate for this misfit by including additional weighting parameters for each level in the transformation.
Suppose \( u \in \mathbb{R}^{n^2} \) is the true signal. Let us recall the notation

\[
\Psi u = (\Psi_j u)_{j=0,\ldots,J} = (\langle \psi_{j,l}, u \rangle)_{j=0,\ldots,J,l=1,\ldots,N_j},
\]

for dividing the analysis coefficients into \( J \) subbands, each consisting of \( N_j \) elements. In [3] a multi dictionary reweighting algorithm was proposed which iteratively updates \( \lambda_j^k \) in

\[
u^{k+1} = \arg\min_u \sum_{j=0}^J \lambda_j^k \| W_j \Psi_j u \|_1 \quad \text{s.t.} \quad \| y - Au \|_2 \leq \sigma, \tag{22}\]

by setting

\[
\lambda_j^k = \frac{N_j}{\varepsilon + \| \Psi_j u^k \|_1}, \tag{23}
\]

and \( W_j = I \) for all iterations of solving (22). It was shown therein that the resulting algorithm can be interpreted as applying a Majorization-Minimization algorithm to the unconstrained formulation of (22) with regularizer

\[
\sum_{j=0}^J N_j \log (\varepsilon + \| \Psi_j u \|_1).
\]

This update rule was proposed in [3] for a composition of multiple different dictionaries instead of just one multiscale dictionary divided into its subbands. In the latter case it seems to be less likely to expect that \( \log (\varepsilon + \| \Psi_j u \|_1) \) promotes the sparsity structure of \( u \) within each of the subbands \( \Psi_j \) sufficiently. Indeed, as it was argued in [15], the log-sum penalty is more sparsity enforcing than the \( \ell_1 \)-norm by putting a larger penalty on small nonzero coefficients. In the case of the \( \ell_1 \)-norm of an entire subband this approach seems to be less effective in promoting the sparsity within each level. It was furthermore proposed in [3] to combine the update rule (23) with the classical elementwise reweighting update

\[
W_j = \text{diag} \left( \frac{1}{\varepsilon + |\langle \psi_{j,l}, u^k \rangle|} \right), \tag{24}
\]

for \( j = 1, \ldots, J \). However, note that this combination is very different to what we are aiming for, since there is even more emphasize put on penalizing the smaller coefficients in higher levels which can happen to delete too many highscale coefficients.

This fact is visualized from a different point of view in Figure 4 where we have depicted the shearlet coefficients of a MRI phantom introduced in [22] together with the reweighting rule we have just discussed in the top-left of the figure. The shearlet coefficients are depicted in blue and the values of \( \lambda_j^k \) for a realistic value \( \mu_1 \) are shown in orange. Note that according to the update rule (21) of the split Bregman algorithm everything below the orange curve would be thresholded.

### 3.5 Multilevel adapted reweighting

Considering the previous discussion one of the disadvantages is that the weights corresponding to higher levels might become too large. This can be prevented, for instance, by choosing the regularization parameters as

\[
\lambda_j = \max \{ |\langle \psi_{j,l}, u \rangle| : l = 1, \ldots, N_j \}, \tag{25}
\]

for \( j = 1, \ldots, J \) and zero otherwise, i.e. if \( j = 0 \). Note that we set \( \lambda_0 = 0 \), since for real life signals the low frequency part is usually not sparse, see also Figure 3. This was also proposed in
Figure 5: Convergence plot: Behavior of the error and structured similarity index for reweighting with true shearlet coefficients with respect to increasing number of iterations. Used signal: phantom from [22]. Reconstructed from 6% of Fourier data with proposed algorithm without TGV (see section 4 for details). In IRL1 we are choosing $W_j$ as in (24) and $\lambda_j$ constant and for the proposed method we additionally define $\lambda_j$ as in (25). **Left figure:** Relative error of in each iteration. **Right figure:** SSIM of each iteration.

[10], where it was shown that this idea can be accomplished more effectively using an analysis prior. For some exotic signals it might happen that the magnitude of the analysis coefficients are very irregular per level, in particular, one could have strong outliers. For such cases it might be better to take a quantile instead of the maximum. However, for our test images this will not happen and thus we have used the maximum to obtain a faster algorithm. Also note that this is a heuristic rule accounting for the unknown constant in the theoretical decay of the multilevel coefficients. A schematic representation from the thresholding perspective of split Bregman can be found in the second image of Figure 4.

Our proposed method combines the classical reweighting of (24) with the above choice for $\lambda_j$. The idea behind this is that we are still using the power of pointwise iterative reweighting, but since our multiscale coefficients naturally come in levels of different orders of magnitude, we apply it to each level separately weighted with $\lambda_j$. That means that within each level we follow the democratic philosophy of reweighting which is that small analysis coefficients of the current iterate $\Psi_j u_{\text{cur}}$ are likely to be zero in $\Psi_j u$. By multiplying with the latter choice of $\lambda_j$ we also gain more control and account for the multilevel structure of $\Psi u$. An artificial experiment using the (in reality unknown) true analysis coefficients demonstrates that this update rule does seem to perform better than standard reweighting without such a compensation of multilevel weights, see Figure [3]. For the signal $u$ we are choosing the phantom of [22]. Using the usually unknown shearlet coefficients for the construction of $\lambda_j$ and $W_j$ for $j = 1, \ldots, J$ as explained above we are reconstructing $u$ from only 6% of its Fourier measurements obtained by radial lines through the k-space origin by using the proposed algorithm of the last section, but without the additional TGV regularizer. Note that both reconstructions approximately start with the same error, which indicates that the set of regularization parameters is chosen equally good. However, it can be said that tuning the iterative reweighted shrinking method within the split Bregman framework without the automatic choice of the subband-weights $\lambda_j$ is rather difficult and highly signal dependent. This example further shows the potential of reweighting as the data is significantly undersampled.
### 3.6 Proposed algorithm

Having explained the idea of our method, we now state the final resulting algorithm that is a composition of the split Bregman framework for solving the constrained optimization problem (11) and the previously explained idea of multilevel weighting and iteratively reweighting, respectively. In contrast to the traditional reweighted \( \ell_1 \) approaches we do not iterate between solving the \( \ell_1 \)-problem up to convergence and updating the weights. We propose to incorporate the multilevel adapted reweighting rule directly into the split Bregman algorithm. This is done in such a way that only the shrinking of the \( w \)-subproblem is changed to a multilevel adapted, iteratively reweighted shrinking rule. Note that by choosing the level-weights \( \lambda_j \) depending on the magnitude of the signal coefficients the resulting method appears to stable towards the alternation of signals.

Another advantage is that we observe in our numerical experiments that the computational complexity stays almost the same. In comparison to the traditional split Bregman approach we are adding only the updates of \( W_j \) and \( \lambda_j \) within each iteration. However, due to the iteratively selected weights this could potentially lead to a much faster algorithm.

**Algorithm 2: Proposed algorithm**

Input:
- Measurement operator \( A \), multilevel transform \( \Psi \),
- regularization parameters: \( \alpha_0, \alpha_1, \mu_1, \mu_2, \mu_3, \beta \),
- iteration numbers \( N \) and \( \text{maxIter} \).

Data:
- Measured data \( y \).

Initialization:
- \( k \leftarrow 0 \); 
- \( u_0 \leftarrow A^* y; \)
- \( y_0, v_0, d_0, w_0, t_0, b_{w,0}, b_{d,0}, b_{t,0} \leftarrow 0 \);

while \( k \leq \text{maxIter} \) do
  for \( i = 1, \ldots, N \) do
    \((u_{k+1}, v_{k+1}) \leftarrow \text{solve linear system (19)}; \)
   for \( j = 1, \ldots, J \) do
    \( \lambda_j = \max \{ |\langle \psi_j,l, u \rangle| : l = 1, \ldots, N_j \}; \)
    \( W_j = \text{diag} \left( \frac{1}{\varepsilon + |\langle \psi_j,l, u \rangle|} \right); \)
    \( w_{k+1}^j(l) \leftarrow \text{shrink} \left( (\langle \psi_j,l, u \rangle cur(l) + b^w_j(l), \frac{\lambda_j W_j(l)}{\mu_1}) \right), \quad l = 1, \ldots, N_j; \)
  end for
  \( d_{k+1}(l) \leftarrow \text{shrink}_2 \left( \nabla f u \text{cur}(l) - v \text{cur}(l) + b^d_k(l), \frac{\alpha_1}{\mu_2} \right), \quad l = 1, \ldots, N_j; \)
  \( t_{k+1}(l) \leftarrow \text{shrink}_F \left( \langle \mathcal{E} b, v \text{cur} \rangle (l) + b^t_k(l), \frac{\alpha_0}{\mu_3} \right), \quad l = 1, \ldots, N_j; \)
  end for
  \( b^w_{k+1} \leftarrow b^w_k + \Psi u_{k+1} - w_{k+1}; \)
  \( b^d_{k+1} \leftarrow b^d_k + (\nabla f u_{k+1} - v_{k+1}) - d_{k+1}; \)
  \( b^t_{k+1} \leftarrow b^t_k + \mathcal{E} b v_{k+1} - t_{k+1}; \)
  \( y_{k+1} \leftarrow y_k + y - A u_{k+1}; \)
  \( k \leftarrow k + 1; \)
end while

return Reconstruction \( u_{\text{maxIter}} \).

We like to comment on two things regarding the algorithm above. First, the weighting matrix \( W_j \) depends on the initialization of an \( \varepsilon > 0 \). The choice of \( \varepsilon \) is rather empirical and in
many cases does not effect the solution. This was already noticed in the beginning of reweighted $\ell^1$ in [15]. This is also the case for our algorithm. As we explain in Section 2 the role $\varepsilon$ is essentially just the maximum threshold that our algorithm will do. It provides stability by preventing a division by zero, but the magnitude of the coefficient is mostly determined by the respective analysis coefficient. It is also common to decrease $\varepsilon$ iteratively as it is assumed that the more iterations one runs, the closer one gets to actual coefficients of the signal. However, this appears unnecessary for us and is not further considered.

Furthermore, we have not incorporated an additional stopping criterion besides a maximum number of iterations. For our algorithm it does not seem to be necessary as the convergence plots suggest, see Figure 5, 7, and 8.

4 Numerics

In this section we will recover numerous signals from their Fourier measurements. This is a typical problem in applied mathematics, which has a wide range of applications. One of the most known applications is magnetic resonance imaging (MRI) where data is collected in the so-called $k$-space which is the Fourier domain, i.e. every point in the k-space can be interpreted as a Fourier coefficient of the object of interest. This is also one of the very first areas where compressed sensing has had a great impact, see, for instance [35].

We will now present some extensive numerical testings that verify the performance of the proposed algorithm for the particular setup where Fourier measurements are taken. However, we like to mention that our algorithm is implemented for very general sampling operators. These can, for example, be binary masks as in inpainting or non-uniform Fourier operator for more sophisticated sampling patterns in MRI.

We have chosen the following three criteria that we wish to analyse our algorithm on:

(N1) Quality,
(N2) Convergence,
(N3) Stability.

For (N1) we compare our algorithm with different existing and established methods that are known to perform well in the recovery problem from Fourier measurements these methods are shown in Table 1. The results are then shown in Section 4.2.

For (N2) we use two quality measurements. First, the relative error which is computed by the formula

$$RE = \frac{\|u^{\text{ref}} - u^{\text{rec}}\|_2}{\|u^{\text{ref}}\|_2},$$

where $u^{\text{ref}}$ is the reference image and $u^{\text{rec}}$ the reconstructed image. Second, we use the structured similarity index that as introduced in [53].

The stability (N3) is verified by the fact that we have chosen the same parameters for each multiscale transform across all experiments. Although an extensive tuning of all parameters for different images might yield superior results we have chosen not to do so. The reason behind is two-fold: First, our algorithm already performance very well with a fixed choice of parameters for all different images used in this section. Second, iterative reweighting combined with the proposed multilevel weighting strategy already suggests the level of thresholds for all coefficients and should therefore be less sensitive to the choice of additional parameters.
4.1 Numerical setup

In this section we give a description of which multiscale transforms we used precisely and the parameters.

For wavelets we have used the undecimated 2D wavelet transform of the Spot package available at

http://www.cs.ubc.ca/labs/scl/spot/

Otherwise differently stated we have used a 4 scale Daubechies-2 wavelet system. The chosen parameters for the proposed algorithm (WIRL1 + TGV) are

- $\mu$: [6e2 1e1 2e1],
- $\alpha$: [1 2],
- $\beta$: 1e4,
- $\varepsilon$: 1e-4.

For shearlets we have used the shearlet transform available at

http://www.shearlab.org/

The discrete shearlet system is generated by using 4 scales and [1 1 2 2] for the directional parameters. The chosen parameters for the proposed algorithm (SIRL1 + TGV) are

- $\mu$: [5e3 1e1 2e1],
- $\alpha$: [1 1],
- $\beta$: 1e5,
- $\varepsilon$: 1e-5.

For all experiments we chose 2 number of block Gauss-Seidel iterations and performed 4 inner iterations before updating $y_k$. All experiments were conducted in MATLAB R2015b with an Intel i3 CPU with 8GB memory.

4.2 Comparison with other methods

Our first experiment shows the recovery from a 256 $\times$ 256 rose image available from the open source framework

http://aforgenet.com/framework/

We took 30 radial lines through the k-space origin ($\approx$ 12, 2% of Fourier data) representing the measurement data. We then compared our results obtained by our algorithm using wavelets as well as shearlets. For both results we have used the proposed iterative multi-level reweighting and a generalized total variation regularizer. The results are compared to RecPF by Yang et al. [50], Co-IRL1 by Ahmad and Schniter [3], PANO by Qu et al. [44], and FFST+TGV by Guo et al. [24]. In order to make the experiments comparable we have used the same scales and number of directions in [24]. Furthermore, for Co-IRL1 we have used two redundant Daubechies wavelet dictionaries with the same number of scales. Moreover, one dictionary consists of Haar wavelets and the second one of Daubechies 2 wavelets.

The final result comparing all methods can be found in Figure 6. It can be observed that the recovery obtained by the proposed method shows the least amount of artifacts.

We will next extensively study our algorithm in terms of the effectiveness and stability.
Figure 6: Different reconstructions from 30 radial lines (≈ 12, 2%) through the k-space origin with relative error and structured similarity index. 50 iterations are used for the reconstruction. See Table 1 for used abbreviations.
### Table 1: Table for abbreviations

| Abbreviation       | Description                                                                 |
|--------------------|-----------------------------------------------------------------------------|
| Fourier inverse    | Fourier inversion of data                                                   |
| RecPF              | Total variation and wavelet regularization from [50]                       |
| FFST+TGV           | Shearlets with TGV from [24]                                                |
| Co-IRL1            | Composite iterative reweighting from [3]                                   |
| PANO               | Patch-based nonlocal operator, [44]                                         |
| TV                 | Total variation with our algorithm                                          |
| TGV                | Total generalized variation with our algorithm                              |
| WL1                | Wavelets without reweighting                                                |
| WIRL1              | Wavelets with proposed reweighting                                         |
| WIRL1+TGV          | Wavelets with proposed reweighting and TGV                                  |
| SL1                | Shearlets without reweighting                                               |
| SR + TGV           | Shearlets with standard reweighting and TGV                                 |
| SIRL1              | Shearlets with proposed reweighting                                         |
| SIRL1+TGV          | Shearlets with proposed reweighting and TGV                                 |

#### 4.3 Convergence, signal independence, and the effect of reweighting

In this section we analyze (N2) for our algorithm. We do this by considering two images, one that is well suited for wavelets and the other one where shearlets perform better. We start with a 256 × 256 phantom that was designed by Guerquin-Kern et al. in [22] for MRI studies. As this image is piecewise constant we have chosen a 4 scale wavelet transform generated by Haar wavelets. We reconstructed the image using our algorithm for TV only, WL1, WIRL1 and WIRL1+TGV. Moreover, as this image is very compressible in a Haar wavelet basis, the recovery allows a much lower sampling rate. In fact, we have only used 21 radial lines which corresponds to 8.73%. It is interesting to mention that the exact solution is returned after almost 80 iterations when WIRL1+TGV. Note that, for 24 lines (≈ 9.83%) wavelets with the proposed iterative reweighting step (WIRL1) will eventually also return the exact solution, see Figure 7. Also note that the TV reconstruction performs worst, although this image should be well suited for TV. Our explanation is that at these extremely low sampling rates a highly redundant transformation is needed to still guarantee recovery.

Our third numerical example concerns the 256 × 256 pepper image, see Figure 8. It has many more structures than the previously considered GLPU phantom. More importantly, it does not consist of piecewise constant areas. This image is particularly well suited for shearlets and thus we have chosen the shearlet transform with four scales. The result for a fixed threshold (SL1), i.e. without the proposed reweighting is significantly worse than the one obtained by the multilevel iterative reweighting method (SIRL1). In this case adding total generalized variation as an additional regularizer does not improve the image quality much. Indeed, the improvement of adding TGV as a regularizer depends strongly on the image. It improves the quality if more piecewise constant areas are present in the image (especially if the background is constant as this is typically the case for MRI images), see for instance Figure 7.

As a final experiment we have taken an MRI image available www.mr-tip.com that has a higher resolution than the previously considered images. In fact the resolution is now 512 × 512 opposed to 256 × 256.

We test our algorithm using wavelets with and without the proposed reweighting method,
Figure 7: Different reconstructions from 24 radial lines through the k-space origin with relative error and structured similarity index. The lower left graphics corresponds to 21 radial lines (≈ 8.73%) and the lower right to 24 radial lines (≈ 9.83%). 100 iterations are used for the reconstruction. See Table 1 for used abbreviations.
Figure 8: Different reconstruction from 30 radial lines ($\approx 12.2\%$) through the k-space origin with relative error and structured similarity index. See Table 1 for used abbreviations. 50 iterations are used for the reconstruction.
as well as TGV. Similarly, we use shearlets. The results are shown in Figure 9. For this picture the visual assessment is most important, as in the original image some noise is present. Both transforms show little, but different artifacts when iterative reweighting is used. However, especially in the case of shearlets it helps to recover certain smaller details that are lost otherwise.

### 4.4 Timings

Although our code is not optimized at all, reconstructions are obtained in reasonable time. Below we display the timings that our algorithm needs in order to compute a reconstruction. The timings are recorded from the experiment shown in Figure 6. As our algorithm is also capable of computing reconstructions using only a TV- or TGV-regularizer we additionally recorded these timings.

Note that TGV is more time consuming than TV, mostly due to the larger system. Moreover, as we already mentioned, the wavelet transform as well as the shearlet transform used in this paper are redundant transforms, i.e. the number of coefficients is in this case significantly larger than the number of pixels available. This explains the much slower performance in comparison to the variational methods.

We also like to mention that the additional cost of reweighting (mostly due to a rather unsophisticated implementation) seems manageable, especially given the benefit of quality that one gets in the recovered images.

### 5 Conclusion

In this paper we presented a novel split Bregman based algorithm that incorporates an iteratively reweighted shrinkage step in order to enhance the quality of image reconstruction where the objective is assumed to be sparse in a multiscale dictionary. We presented an extensive study of the algorithm thereby focusing on the quality of the image that is to be reconstructed as well as the stability of the parameters towards the change of different signals. Although our focus in the numerical experiments was on the reconstruction problem from Fourier measurements, our algorithm can also be applied to other common problems such as denoising and inpainting.

Furthermore, in an upcoming work we will apply this algorithm to real MR data that show that this algorithm also works in practice.

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Figure 9: Different reconstruction from 33 radial lines ($\approx 6.99\%$) through the k-space origin. 50 iterations are used for the reconstruction. See Table [1] for used abbreviations.
code for the Co-IRL1 algorithm.

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