Two-parametric $\delta'$-interactions: approximation by Schrödinger operators with localized rank-two perturbations

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Abstract
We construct a norm resolvent approximation to the family of point interactions $f(+0) = \alpha f(-0) + \beta f'(0)$, $f'(0) = \alpha^{-1} f'(-0)$, $f'(-0) = \alpha^{-1} f'(-0)$ by Schrödinger operators with localized rank-two perturbations coupled with short range potentials. In particular, a new approximation to the $\delta'$-interactions is obtained.

Keywords: 1D Schrödinger operator, point interaction, $\delta$'-interaction, $\delta$'-potential, solvable model, finite rank perturbation

(Some figures may appear in colour only in the online journal)

1. Introduction

Schrödinger operators with pseudo-potentials that are distributions supported on discrete sets (such potentials are usually termed point interactions) have received considerable attention from many researchers over recent decades. The point interactions have been widely and extensively investigated from various points of view and the study of solvable models based on the concept of zero range quantum interactions has a long and interesting history. General references for this fascinating area are [1, 2]. Historically the point interactions were introduced in quantum mechanics as limits of families of squeezed potentials. The main purpose was to find solvable models describing with admissible fidelity the real quantum processes governed by Hamiltonians with localized potentials. However the connection between real short-range interactions and point interactions is very complex and ambiguously determined. This is certainly the reason why the ‘inverse’ problem—how to approximate a given point interaction by regular Hamiltonians with localized perturbations—is also important.

In the one-dimensional case, among all zero range interactions, the $\delta'$-interactions, along with $\delta$ potentials, are most studied in this kind of research. The $\delta'$-interaction at the origin, of
strength $\beta$, is described by the self-adjoint operator $f \mapsto -f''$ in $L^2_2(\mathbb{R})$ restricted to functions in $W^2_2(\mathbb{R} \setminus \{0\})$ obeying the interface conditions

$$\begin{pmatrix} f(+0) \\ f'(+0) \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f(-0) \\ f'(-0) \end{pmatrix}. \quad (1)$$

This operator is widely accepted as a model for the pseudo-Hamiltonian

$$H = -\frac{d^2}{dx^2} + \beta \langle \delta'(x), \cdot \rangle \delta'(x).$$

However, as shown in [3], no self-adjoint regularization $-\frac{d^2}{dx^2} + \beta \langle \varphi_\epsilon, \cdot \rangle \varphi_\epsilon$ of $H$ provides an approximation to point interactions (1). Here the sequence of smooth functions $\varphi_\epsilon$ converges in the sense of distributions to the first derivative of the Dirac delta function. Šeba [29] was the first to approximate the $\delta'$-interactions in strong resolvent sense by the operators $-\frac{d^2}{dx^2} + \lambda_\epsilon \langle \varphi_\epsilon, \cdot \rangle \varphi_\epsilon$ with an infinitely small $\epsilon \to 0$ coupling constant $\lambda_\epsilon$. This result can be improved to convergence in the norm resolvent sense; see for instance [15], where the problem on metric graphs was studied. Families of non-self-adjoint Schrödinger operators with nonlocal perturbations which converge to the $\delta'$-interactions were constructed by Albeverio and Nizhnik [5].

Exner, Neidhardt and Zagrebnov [16] obtained very subtle potential approximations to the $\delta'$-interactions in the norm resolvent topology: the family of potentials was built as a triple of $\delta$-like potentials, shrinking to the origin, with non-trivial dependence between coupling constants and separation distances. The paper became a mathematical justification of the result previously obtained by Cheon and Shigehara [11], who built the approximation in terms of three $\delta$ functions with the renormalized strengths and disappearing distances. In the context of the ‘three delta approximation’, it is worth mentioning the works of Albeverio, Fassari and Rinaldi [10, 17] and the recent publication of Zolotaryuk [33]; see also [12] for the case of quantum graphs. The reader also interested in the literature on other aspects of $\delta'$-interactions and approximations of point interactions by local and non-local perturbations is referred to [6–9, 14, 25–28].

In this paper, we study Schrödinger operators with localized rank-two perturbations coupled with short range $\delta$-like potentials. A careful asymptotic analysis of these operators shows that some part of the set of limit operators which can be obtained in the norm resolvent topology, as the support of perturbation shrinks to the origin, deals with the 2-parametric family of point interactions

$$\begin{pmatrix} f(+0) \\ f'(+0) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} f(-0) \\ f'(-0) \end{pmatrix}. \quad (2)$$

In fact, we built a norm resolvent approximation to these point interactions. In particular, we obtained a new approximation to the classic $\delta'$-interaction that corresponds to the case $\alpha = 1$. In [4], the point interactions (2) appeared as exactly solvable $\mathcal{P}$-Hermitian Hamiltonians obtained as operator realizations of $\mathcal{P}$-symmetric zero-range singular perturbations

$$-\frac{d^2}{dx^2} + a \langle \delta, \cdot \rangle \delta + b \langle \delta', \cdot \rangle \delta + c \langle \delta, \cdot \rangle \delta' + d \langle \delta', \cdot \rangle \delta'$$

of the free Schrödinger operator; see also the recent publication [14].

It is worth to note that the $\delta'$-interactions should not be confused with the $\delta$ potentials. The Schrödinger operators with $(a\delta' + b\delta)$-like potentials
have been recently investigated in [13, 20–24, 30–32]. For instance, it has been proved in [23] that $H_\varepsilon$ converge, as $\varepsilon \to 0$, in the norm resolvent sense to the operator $f \mapsto -f''$ in $L^2(\mathbb{R})$ restricted to functions in $W^2_2(\mathbb{R} \setminus \{0\})$ such that

$$
\left( f(0^+) \begin{pmatrix} \mu & 0 \\ \nu & \mu^{-1} \end{pmatrix} f'(0^-) \right),
$$

if potential $V_2$ possesses a zero energy resonance, and to the direct sum $S_- \oplus S_+$ of the half-line Schrödinger operators $S_\pm = -d^2/dx^2$ on $\mathbb{R}_\pm$ subject to the Dirichlet boundary condition at the origin, otherwise. The spectral properties of models with point interactions (4) as well as the scattering coefficients were studied in [18, 19].

### 2. Statement of problem and main result

Let us consider the Schrödinger operator

$$S_0 = -\frac{d^2}{dx^2} + V(x)$$

in $L^2(\mathbb{R})$, where potential $V$ is a real-valued, measurable and locally bounded. We also assume that $V$ is bounded from below in $\mathbb{R}$. Let $\varphi_1$ and $\varphi_2$ be real functions of compact support in $L^2(\mathbb{R})$. We introduce the rank-two operators

$$B_\varepsilon \psi(x) = \varphi_1 \left( \frac{x}{\varepsilon} \right) \int_{\mathbb{R}} \varphi_2 \left( \frac{s}{\varepsilon} \right) \psi(s) \, ds + \varphi_2 \left( \frac{x}{\varepsilon} \right) \int_{\mathbb{R}} \varphi_1 \left( \frac{s}{\varepsilon} \right) \psi(s) \, ds$$

acting in $L^2(\mathbb{R})$, and the family of self-adjoint operators

$$S_\varepsilon = S_0 + \varepsilon^{-3} B_\varepsilon + \varepsilon^{-1} q \left( \frac{x}{\varepsilon} \right).$$

Here $q$ is also a real-valued, measurable and bounded function of compact support. The perturbation of operator $S_0$ has a small support shrinking to the origin as the small positive parameter $\varepsilon$ goes to zero. For this reason, $\text{dom} S_\varepsilon = \text{dom} S_0$.

From now on, the inner scalar product and norm in $L^2(\mathbb{R})$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. We denote by

$$f^{(-1)}(x) = \int_{-\infty}^{x} f(s) \, ds, \quad f^{(-2)}(x) = \int_{-\infty}^{x} (x-s) f(s) \, ds$$

the first and second antiderivatives of a function $f$. The antiderivatives are well-defined for measurable functions of compact support, for instance. In addition, if $f$ has zero mean, then $f^{(-1)}$ is also a function of compact support. In this paper, we will consider only the case when $\varphi_1$ and $\varphi_2$ are functions of zero means, i.e.

$$\int_{\mathbb{R}} \varphi_j \, dx = 0, \quad j = 1, 2.$$  \hfill (5)

Therefore $\varphi_1^{(-1)}$ and $\varphi_2^{(-1)}$ have compact supports and the function

$$\omega = \| \varphi_2^{(-1)} \| \cdot \varphi_1^{(-2)} - \| \varphi_1^{(-1)} \| \cdot \varphi_2^{(-2)}$$



\hfill (6)
is constant in some neighbourhoods of negative and positive infinities (see figure 1).

We introduce notation

\[ \kappa = \lim_{x \to +\infty} \omega(x), \]

\[ a_0 = \int_R q \, dx, \quad a_1 = \int_R q \omega \, dx, \quad a_2 = \int_R q \omega^2 \, dx. \]

We will denote by \( V \) the space of \( L_2(\mathbb{R}) \)-functions \( f \) such that \( f(x) = f_-(x) \) if \( x < 0 \) and \( f(x) = f_+(x) \) if \( x > 0 \) for some \( f_- \) and \( f_+ \) belonging to the domain of \( S_0 \). Let us consider the operator \( S_{\alpha\beta}f = -f'' + Vf \),

\[ \text{dom} S_{\alpha\beta} = \{ f \in V: f(+0) = \alpha f(-0) + \beta f'(-0), \quad f'(+0) = \alpha^{-1} f'(-0) \}. \]

Our main result is the following theorem.

**Theorem 1.** Let \( \varphi_1, \varphi_2 \) and \( q \) be integrable, real-valued functions with compact supports. Suppose that

(i) \( \varphi_1, \varphi_2 \) have zero means, antiderivatives \( \varphi_1^{(-1)}, \varphi_2^{(-1)} \) are orthogonal in \( L_2(\mathbb{R}) \), and

\[ \| \varphi_1^{(-1)} \| \cdot \| \varphi_2^{(-1)} \| = 1; \]

(ii) potential \( q \) satisfies conditions

\[ a_0a_2 = a_1^2, \quad a_2 \neq \kappa a_1. \]

Then the operator family \( S_\varepsilon \) converges as \( \varepsilon \to 0 \) in the norm resolvent sense to operator \( S_{\alpha\beta} \), where

\[ \alpha = \frac{a_2 - \kappa a_1}{a_2}, \quad \beta = \frac{\kappa^2}{a_2 - \kappa a_1}. \]

There is a wide class of functions \( \varphi_1, \varphi_2 \) and \( q \) satisfying the assumptions in theorem 1. Moreover, for any pair \( (\alpha, \beta) \) of real numbers with \( \alpha \neq 0 \), there exists a family of operators \( S_\varepsilon \) such that \( S_\varepsilon \to S_{\alpha\beta} \) as \( \varepsilon \to 0 \) in the norm resolvent sense. The sole exception is the point interactions with matrix

\[ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad \alpha \neq 1, \]

because \( \beta \) vanishes together with \( \kappa \) and therefore \( \alpha = 1 \) by (10). Note that the last point interactions correspond to the case of \( \delta' \)-potentials and can be approximated by operators (3) with \( V_1 = 0 \) provided \( V_2 \) possesses a zero energy resonance [21, 24].
It is a simple matter to choose \( \varphi_1 \) and \( \varphi_2 \). For instance, it is enough to take two orthonormal in \( L_2(\mathbb{R}) \) functions \( \eta_1 \) and \( \eta_2 \) of compact support belonging to \( W_2^1(\mathbb{R}) \), and then set \( \varphi_1 = \eta_1^\prime \) and \( \varphi_2 = \eta_2^\prime \). With these functions in hands, we can construct \( \omega \) and calculate \( \kappa = \omega(+\infty) \). If \( \kappa = 0 \), then \( \alpha = 1 \) and \( \beta = 0 \), and the limiting operator is the free Schrödinger operator on the line. Suppose now that \( \kappa \) is different from zero. Note that \( \omega = \eta_1^{(-1)} - \eta_2^{(-1)} \) is a continuous and non-constant function, by the orthogonality of \( \eta_1 \) and \( \eta_2 \). Hence \( 1, \omega, \omega^2 \) are linearly independent functions on each interval \([-r, r]\). Then for any \( (a_0, a_1, a_2) \in \mathbb{R}^3 \) there exists a potential \( q \) of compact support for which equalities (7) hold. Given \( \alpha \neq 1 \) and \( \beta \neq 0 \), we choose \( q \) such that

\[
 a_0 = \frac{(1 - \alpha)^2}{\alpha \beta}, \quad a_1 = \frac{\kappa(1 - \alpha)}{\alpha \beta}, \quad a_2 = \frac{\kappa^2}{\alpha \beta}.
\]

This triple of numbers satisfies (9) and (10). The same is true for the case of the classic \( \delta' \)-interactions when \( \alpha = 1 \) and \( \beta \neq 0 \), if we set \( a_0 = 0, a_1 = 0 \) and \( a_2 = \beta^{-1}\kappa^2 \).

In this paper we do not consider the case \( \kappa = 0 \). We also note that the orthogonality of \( \varphi_1^{(-1)} \) and \( \varphi_2^{(-1)} \) of course implies the linear independence of \( \varphi_1 \) and \( \varphi_2 \). Hence operator \( B_\varepsilon \) has actually rank two.

3. Half-bound states

Let us consider the operator

\[
 B = -\frac{d^2}{dx^2} + \langle \varphi_2, \cdot \rangle \varphi_1 + \langle \varphi_1, \cdot \rangle \varphi_2, \quad \text{dom} \, B = W_2^2(\mathbb{R})
\]

in space \( L_2(\mathbb{R}) \).

**Definition 1.** We say that the operator \( B \) possesses a half-bound state provided there exists a nontrivial solution \( \psi \) of the equation

\[
 -\psi'' + \langle \varphi_2, \psi \rangle \varphi_1 + \langle \varphi_1, \psi \rangle \varphi_2 = 0
\]

that is bounded on the whole line.

Let us introduce notation

\[
 n_j = \| \varphi_j^{(-1)} \|, \quad m_j = \int_\mathbb{R} x \varphi_j \, dx, \quad j = 1, 2.
\]

Now we prove the first of two key lemmas for the proof of main theorem.

**Lemma 1.** Under assumption (i) of theorem 1, the operator \( B \) possesses the 2-dimensional space of half-bound states generated by the constant function and function \( \omega \), given by (6).

**Proof.** Any solution of (11) can be written as

\[
 \psi(x) = c_1 \varphi_1^{(-2)}(x) + c_2 \varphi_2^{(-2)}(x) + c_3 x + c_4,
\]

where the constants \( c_k \) are connected via two conditions

\[
 \begin{align*}
 \langle \varphi_2^{(-2)}, \varphi_1 \rangle c_1 + \langle \varphi_1^{(-2)}, \varphi_1 \rangle - 1) c_2 + m_1 c_3 &= 0, \\
 \langle \varphi_1^{(-2)}, \varphi_2 \rangle - 1) c_1 + \langle \varphi_2^{(-2)}, \varphi_2 \rangle c_2 + m_2 c_3 &= 0.
\end{align*}
\]

\[
(12)
\]
These conditions can be easy derived from (11) in view of the linear independence of \( \varphi_1 \) and \( \varphi_2 \). In general, \( \varphi_1^{(-2)} \) and \( \varphi_2^{(-2)} \) do not belong to \( L_2(\mathbb{R}) \). But it will cause no confusion if we use the scalar products \( \langle \varphi_1^{(-2)}, \varphi_j \rangle \) as notation for the integrals \( \int_{\mathbb{R}} \varphi_1^{(-2)} \varphi_j \, dx \), which are finite, because of compact supports of \( \varphi_j \).

Since \( \psi = c_3 x + c_4 \) in some neighbourhood of negative infinity, the constant \( c_3 \) must be zero, because we are looking for bounded solutions. Also, the constant function is a half-bound state, since \( \varphi_1 \) and \( \varphi_2 \) have zero means. Therefore if any other (linearly independent) half-bound state exists, then it has the form

\[
\psi(x) = c_1 \varphi_1^{(-2)}(x) + c_2 \varphi_2^{(-2)}(x),
\]

where vector \( \vec{c} = (c_1, c_2) \) must be a nontrivial solution of the linear system \( A \vec{c} = 0 \) with matrix

\[
A = \begin{pmatrix}
\langle \varphi_1^{(-2)}, \varphi_1 \rangle & \langle \varphi_2^{(-2)}, \varphi_1 \rangle - 1 \\
\langle \varphi_1^{(-2)}, \varphi_2 \rangle - 1 & \langle \varphi_2^{(-2)}, \varphi_2 \rangle
\end{pmatrix}.
\]

The system is obtained from (12) by putting \( c_3 = 0 \).

Since \( \varphi_1^{(-1)} \) and \( \varphi_2^{(-1)} \) are compactly supported, we obtain

\[
\int_{\mathbb{R}} \varphi_j^{(-2)} \varphi_j \, dx = \varphi_j^{(-2)} \varphi_j \bigg|_{-\infty}^{+\infty} - \int_{\mathbb{R}} \varphi_j^{(-1)} \varphi_j^{(-1)} \, dx = -\langle \varphi_1^{(-1)}, \varphi_j^{(-1)} \rangle.
\]

According to the assumptions, antiderivative \( \varphi_1^{(-1)} \) and \( \varphi_2^{(-1)} \) are orthogonal in \( L_2(\mathbb{R}) \). From this we have \( \langle \varphi_1^{(-2)}, \varphi_2 \rangle = \langle \varphi_2^{(-2)}, \varphi_1 \rangle = 0 \), \( \langle \varphi_1^{(-2)}, \varphi_1 \rangle = -n_1^2 \) and \( \langle \varphi_2^{(-2)}, \varphi_2 \rangle = -n_2^2 \). Hence

\[
A = \begin{pmatrix}
n_1^2 & 1 \\
1 & n_2^2
\end{pmatrix}.
\]

Matrix \( A \) is degenerate by (8) and thereby system \( A \vec{c} = 0 \) admit a non trivial solution \( \vec{c} = (n_2, -n_1) \). Hence, the function \( \omega = n_2 \varphi_1^{(-2)} - n_1 \varphi_2^{(-2)} \) is a half-bound state of \( B \).

**4. Auxiliary statements**

Without loss of generality we can assume that the supports of \( \varphi_1 \), \( \varphi_2 \) and \( q \) lie in interval \( I = [-1, 1] \). Then

\[
\varphi_j^{(-k)}(-1) = 0
\]

for \( k = 0, 1, 2 \) and \( j = 1, 2 \). Also,

\[
\varphi_1^{(-1)}(1) = 0, \quad \varphi_2^{(-1)}(1) = 0, \quad \varphi_1^{(-2)}(1) = -m_1, \quad \varphi_2^{(-2)}(1) = -m_2,
\]

because from (5) we have

\[
\varphi_j^{(-2)}(1) = \int_{-\infty}^{1} (1 - x) \varphi_j(x) \, dx = \int_{\mathbb{R}} \varphi_j(x) \, dx - \int_{\mathbb{R}} x \varphi_j(x) \, dx = -m_j.
\]
Then we also deduce that $\omega(-1) = \omega'(1) = 0$ and $\omega(1) = \varpi$, where

$$\varpi = n_1m_2 - n_2m_1. \quad (15)$$

Next, a half-bound state $\psi$ of $B$ is now constant outside $I$ as a bounded solution of equation $\psi'' = 0$, and therefore the restriction of $\psi$ to $I$ is a non-trivial solution of the boundary value problem

$$-\psi'' + (\varphi_2, \psi) \varphi_1 + (\varphi_1, \psi) \varphi_2 = 0, \quad t \in I, \quad \psi'(-1) = 0, \quad \psi'(1) = 0, \quad (16)$$

where $(\cdot, \cdot)$ is the scalar product in $L_2(I)$. Given $h \in L_2(I)$ and $a, b \in C$, we consider the nonhomogeneous problem

$$-v'' + (\varphi_2, v) \varphi_1 + (\varphi_1, v) \varphi_2 = h, \quad t \in I, \quad v'(-1) = a, \quad v'(1) = b. \quad (17)$$

Owing to lemma 1, homogeneous problem (16) has a 2-dimensional space of solutions. Therefore problem (17) is in general unsolvable.

**Proposition 1.** Under assumption (i) of theorem 1, the nonhomogeneous boundary value problem (17) admits a solution if and only if

$$a - b = (1, h), \quad a = (1 - \varpi^{-1} \omega, h). \quad (18)$$

Then among all solutions of (17) there exists a unique one such that

$$v(-1) = 0, \quad v(1) = 0. \quad (19)$$

In addition, this solution satisfies the estimate

$$\|v\|_{W^2(I)} \leq C\|h\|_{L_2(I)}, \quad (20)$$

where the constant $C$ does not depend on $h$.

**Proof.** Conditions (18) can be easily obtained by multiplying equation (17) by $1$ and $\varpi$ in turn and then integrating by parts twice in view of the boundary conditions. Though the sufficiency of (18) follows from the Fredholm alternative, we will prove it directly by explicit construction of the desired solution.

We look for a partial solution of (17) in the form

$$v_0 = k_1\varphi_1(-1) + k_2\varphi_2(-1) - h^{(-2)} + at,$$

where $k_1, k_2$ are arbitrary constants and $h^{(-2)}(t) = \int_{-1}^t (t - s)h(s)\,ds$. Function $v_0$ satisfies boundary conditions (17) for all $k_1$ and $k_2$. In fact,

$$v_0'(-1) = k_1\varphi_1(-1)(-1) + k_2\varphi_2(-1)(-1) - h^{(-1)}(-1) + a = a,$$

by (13). From (14) and the first solvability condition in (18) we see

$$v_0'(1) = k_1\varphi_1(-1)(1) + k_2\varphi_2(-1)(1) - h^{(-1)}(1) + a = a - (1, h) = b,$$

since $h^{(-1)}(1) = (1, h)$. Direct substitution $v_0$ into equation (17) yields
\[
\begin{pmatrix}
    n_1^2 & 1 \\
    1 & n_2^2
\end{pmatrix}
\begin{pmatrix}
    k_1 \\
    k_2
\end{pmatrix}
= \begin{pmatrix}
    am_1 - (\varphi_1, h^{(-2)}) \\
    am_2 - (\varphi_2, h^{(-2)})
\end{pmatrix},
\]

(see the proof of lemma 1). According to (8) we have \(n_1 n_2 = 1\), and then the system can be written as

\[
\begin{pmatrix}
    n_1 & n_2 \\
    n_1 & n_2
\end{pmatrix}
\begin{pmatrix}
    k_1 \\
    k_2
\end{pmatrix}
= \begin{pmatrix}
    g_1 \\
    g_2
\end{pmatrix},
\]

(21)

where \(g_1 = n_2 (am_1 - (\varphi_1, h^{(-2)}))\) and \(g_2 = n_1 (am_2 - (\varphi_2, h^{(-2)}))\). Therefore the system is consistent if and only if \(g_1 = g_2\). But this equality is equivalent to the second solvability condition in (18). Indeed, recalling now (15), we have

\[
g_2 - g_1 = a(n_1 m_2 - n_2 m_1) + (n_2 \varphi_1 - n_1 \varphi_2, h^{(-2)})
= a \cdot + (\omega'', h^{(-2)}) = \varphi (a - (1 - \varphi^{-1}(\omega, h)) = 0,
\]

because integrating by parts twice gives

\[
(\omega'', h^{(-2)}) = - \varphi h^{(-1)}(1) + (\omega, h) = - \varphi (1, h) + (\omega, h) = - \varphi (1 - \varphi^{-1}(\omega, h)).
\]

The vector \(\vec{k} = (n_2 g_1, 0)\) solves (21) and then \(v_0 = n_2 g_1 \varphi_1^{(-2)} - h^{(-2)} + a t\) is a solution of (17). With the aid of \(v_0\) we can construct a solution \(v\) satisfying conditions (19). We set

\[
v = v_0 + a - \varphi^{-1}(v_0(1) + a) \omega, \text{ i.e.}
\]

\[
v = n_2 g_1 \varphi_1^{(-2)} - h^{(-2)} + a(t + 1) - \frac{1}{\varphi} (n_2 m_1 g_1 - h^{(-2)}(1 + 2a) \omega.
\]

Estimate (20) looks strange at first sight, because a solution of (17) is bounded by the right-hand side \(h\) of the equation only without regard for right-hand sides \(a\) and \(b\) in the boundary conditions. But by virtue of solvability conditions (18), numbers \(a\) and \(b\) can be expressed via function \(h\):

\[
a(h) = (1 - \varphi^{-1}(\omega, h)), \quad b(h) = - \varphi^{-1}(\omega, h).
\]

Therefore for each \(h \in L_2(\mathcal{I})\) there exists a unique boundary data \((a(h), b(h))\) such that problem (17) is solvable. So regarding \(a\) and \(g_1\) as linear functionals in \(L_2(\mathcal{I})\), we have the bounds

\[
|a(h)| = |(1 - \varphi^{-1}(\omega, h))| \leq C_1 \Vert h \Vert_{L_2(\mathcal{I})},
\]

\[
|g_1(h)| = |n_2| \cdot m_2(a(h) - (\varphi_1, h^{(-2)})) \leq C_2 (|a(h)| + \Vert h^{(-2)} \Vert_{L_2(\mathcal{I})}) \leq C_3 \Vert h \Vert_{L_2(\mathcal{I})}.
\]

From this and explicit formula for \(v\) we immediately deduce

\[
\Vert v \Vert_{W_2^2(\mathcal{I})} \leq C_4 (|a(h)| + |g_1(h)| + \Vert h^{(-2)} \Vert_{W_2^2(\mathcal{I})}) \leq C_5 \Vert h \Vert_{L_2(\mathcal{I})},
\]

since the operator \(L_2(\mathcal{I}) \ni h \mapsto h^{(-2)} \in W_2^2(\mathcal{I})\) is bounded.

In the end of the section, we record some technical assertion. Let \([g]_a\) denote the jump \(g(a + 0) - g(a - 0)\) of function \(g\) at a point \(a\).
Proposition 2. Let $U$ be the real line with two removed points $x = -\varepsilon$ and $x = \varepsilon$, i.e. $U = \mathbb{R} \setminus \{-\varepsilon, \varepsilon\}$. Assume that function $g \in W^2_{2,\text{loc}}(U)$ along with its first derivative has jump discontinuities at points $x = -\varepsilon$ and $x = \varepsilon$. There exists a function $\rho \in C^\infty(U)$ such that $g + \rho$ belongs to $W^2_{2,\text{loc}}(\mathbb{R})$ and
\[
|\rho^{(k)}(x)| \leq C \left( |[g]_{-\varepsilon}| + |[g']_{-\varepsilon}| + |[g']_{-\varepsilon}| + |[g']_{\varepsilon}| \right)
\] (23)
for $|x| \geq \varepsilon$, $k = 0, 1, 2$, where the constant $C$ does not depend on $g$ and $\varepsilon$. Moreover, $\rho$ is a function of compact support and $\rho$ vanishes in $(-\varepsilon, \varepsilon)$.

Proof. Let us introduce functions $w_0$ and $w_1$ that are smooth outside the origin, have compact supports contained in $(0, \infty)$, and such that $w_0(+0) = 1$, $w_0'(+0) = 0$, $w_1(+0) = 0$ and $w_1'(+0) = 1$. We set
\[
\rho(x) = [g]_{-\varepsilon} w_0(-x - \varepsilon) - [g']_{-\varepsilon} w_1(-x - \varepsilon)
\]
\[
- [g]_{\varepsilon} w_0(x - \varepsilon) - [g']_{\varepsilon} w_1(x - \varepsilon).
\]
By construction, $\rho$ has a compact support and vanishes in $(-\varepsilon, \varepsilon)$. An easy computation also shows that
\[
[\rho]_{-\varepsilon} = -[g]_{-\varepsilon}, \quad [\rho]_{\varepsilon} = -[g]_{\varepsilon}, \quad [\rho']_{-\varepsilon} = -[g']_{-\varepsilon}, \quad [\rho']_{\varepsilon} = -[g']_{\varepsilon}.
\]
Therefore $g + \rho$ is continuous on $\mathbb{R}$ along with the first derivative and consequently belongs to $W^2_{2,\text{loc}}(\mathbb{R})$. Finally, the explicit formula for $\rho$ makes it obvious that inequality (23) holds. $\square$

5. Proof of theorem 1

Given $f \in L_2(\mathbb{R})$ and $\zeta \in C \setminus \mathbb{R}$, we must compare two elements $u_\varepsilon = (S_\varepsilon - \zeta)^{-1} f$ and $u = (S_{\alpha\beta} - \zeta)^{-1} f$, and show that the difference $u_\varepsilon - u$ is infinitely small in $L_2(\mathbb{R})$, as $\varepsilon \to 0$, uniformly on $f$. The basic idea of the proof is to construct a suitable approximation to $u_\varepsilon$. For $\varepsilon > 0$, we introduce the sequence of functions
\[
y_\varepsilon(x) = \begin{cases} u(x) & \text{if } |x| > \varepsilon, \\ \psi \left( \frac{x}{\varepsilon} \right) + \varepsilon v_\varepsilon \left( \frac{x}{\varepsilon} \right) & \text{if } |x| < \varepsilon, \end{cases}
\] (24)
where $\psi(t) = u(-0) + \varepsilon t^{-1}(u(+0) - u(-0)) \omega(t)$ is a restriction of a half-bound state of operator $B$ such that
\[
\psi(-1) = u(-0), \quad \psi(1) = u(+0);
\] (25)
function $v_\varepsilon$ solves the problem
\[
-v''_\varepsilon + (\varphi_2, v_\varepsilon) \varphi_1 + (\varphi_1, v_\varepsilon) \varphi_2 = \varepsilon f(xt) - q \psi(t), \quad t \in \mathbb{Z},
\] (26)
\[
-v''_\varepsilon(-1) = u'(-0) + \xi_\varepsilon(f), \quad v''_\varepsilon(1) = u'(0) + \eta_\varepsilon(f)
\] (27)
with some numbers $\xi_\varepsilon$ and $\eta_\varepsilon$ depending on $f$.

First we record some estimates on $u$ and $\psi$. We observe that $(S_{\alpha\beta} - \zeta)^{-1}$ is a bounded operator from $L_2(\mathbb{R})$ to $\text{dom} S_{\alpha\beta}$ equipped with the graph norm. Since potential $V$ is locally
bounded, the latter space is a subspace of $W^2_{2,\text{loc}}(\mathbb{R} \setminus \{0\}) \cap \mathcal{V}$. Hence there exists a constant independent of $f$ such that
\[ \|u\|_{W^2_{2,\text{loc}}((-r), (t) \setminus \{0\})} \leq c\|f\|, \]
for any $r > 0$, and thus $\|u\|_{C^1(\mathbb{R})} + \|u\|_{C^1([0, t])} \leq c\|f\|$, by the Sobolev embedding theorem. In particular, we have
\[ |u(-0)| + |u(+0)| + \|u'(-0)| + \|u'(0)| \leq c\|f\|. \]
It follows from the last bound that
\[ \|\psi\|_{L^2(\mathcal{Z})} \leq c_1 (|u(0)| + |u(0)|) \leq c_2\|f\|. \] (28)

Next, there exists a constant being independent of $\varepsilon$ and $u$ such that
\[ |u^{(k)}(-\varepsilon) - u^{(k)}(0)| + \|\psi^{(k)}(\varepsilon) - u^{(k)}(0)^{2} \| \leq C\varepsilon^{1/2}\|f\| \] (29)
for $k = 0, 1$, since
\[ |u^{(k)}(\pm \varepsilon) - u^{(k)}(\pm 0)| \leq \int_{0}^{\pm \varepsilon} |u^{(k+1)}(\pm x)| \, dx \leq c\varepsilon^{1/2}\|u\|_{W^2_{2}((-1, 1) \setminus \{0\})}. \]

In view of proposition 1, for each $f \in L^2(\mathbb{R})$ there exists a unique pair $(\xi_\varepsilon, \eta_\varepsilon)$ such that problem (26), (27) admits a solution. We conclude from (22) that
\[ \xi_\varepsilon(f) = \left(\frac{\varepsilon^{-1}}{\omega} - 1, q\psi\right) - u'(-0) + \varepsilon\left(1 - \frac{\varepsilon^{-1}}{\omega}, f'(\varepsilon \cdot)\right), \]
\[ \eta_\varepsilon(f) = \frac{\varepsilon^{-1}}{\omega}(q\omega, \psi) - u'(0) - \varepsilon\frac{\varepsilon^{-1}}{\omega}(\omega, f(\varepsilon \cdot)). \] (30)

Then (26), (27) has a solutions $v_\varepsilon$ such that $v_\varepsilon(-1) = 0, v_\varepsilon(1) = 0$ and
\[ \|v_\varepsilon\|_{W^2_{2}(\mathcal{Z})} \leq c_1 (\|\psi\|_{L^2(\mathcal{Z})} + \varepsilon\|f(\varepsilon \cdot)\|_{L^2(\mathcal{Z})}) \leq c_2\|f\|. \] (31)

by (20). Here we employed (28) and the obvious inequality
\[ \|f(\varepsilon \cdot)\|_{L^2(\mathcal{Z})} \leq c\varepsilon^{-1/2}\|f\|. \] (32)

Function $v_\varepsilon$ given by (24) does not belong to the domain of $S_\varepsilon$, because it is in general discontinuous at points $x = -\varepsilon$ and $x = \varepsilon$. Although $v_\varepsilon$ has points of discontinuity, we will show that its jumps and jumps of its first derivative at these points are small as $\varepsilon \to 0$. Recalling (25), boundary conditions (16) and (27) we see at once that
\[ [y_\varepsilon][-\varepsilon] = u(-0) - u(-\varepsilon), \quad [y_\varepsilon'_][-\varepsilon] = u'(-0) - u'(-\varepsilon) + \xi_\varepsilon(f), \]
\[ [y_\varepsilon][\varepsilon] = u(0) - u(+0), \quad [y_\varepsilon'][\varepsilon] = u'(0) - u'(0) + \eta_\varepsilon(f). \] (33)

The following lemma is the second key point of the proof. From the technical point of view, it states that the jumps of $y_\varepsilon$ and $y'_\varepsilon$ are small only for $q$ satisfying condition (9) and $\alpha, \beta$ given by (10). But in essence, the lemma demonstrates a subtle connection between the half-bound states of $B$, potential $q$ and point interactions (2).

**Lemma 2.** Suppose that $u = (S_{\alpha, \beta} - \zeta)^{-1}f$ with $\alpha$ and $\beta$ given by (10). Under the assumptions of theorem 1, sequences $\xi_\varepsilon(f)$ and $\eta_\varepsilon(f)$ are infinitesimal as $\varepsilon \to 0$ and the estimate
\[ |\xi_\varepsilon(f)| + |\eta_\varepsilon(f)| \leq C\varepsilon^{1/2}\|f\| \] (34)
holds for all $f \in L^2(\mathbb{R})$ and a constant $C$ which does not depend on $f$. 

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Proof. We will show that the terms in (30), which do not depend on $\varepsilon$, are equal to zero, i.e.

$$\left(\varepsilon^{-1} \omega - 1, q \psi \right) - u'(-0) = 0, \quad \varepsilon^{-1} (q \omega, \psi) - u'(0) = 0. \quad (35)$$

Recall that $u$ is a unique solution of the problem

$$-u'' + (V - \zeta)u = f \quad \text{in} \ \mathbb{R} \setminus \{0\},$$

$$u(+0) = \alpha u(-0) + \beta u'(-0), \quad u'(+0) = \alpha^{-1} u'(-0). \quad (37)$$

Then half-bound state $\psi$ in (24) can be written as

$$\psi(t) = u(-0) + \varepsilon^{-1} \left( \left( \alpha - 1 \right) u(-0) + \beta u'(-0) \right) \omega(t). \quad (38)$$

First, we consider the case when constant $a_1$ in (9) is different from zero. Then we also have $a_0 \neq 0$ and $a_2 \neq 0$. With this, we obtain

$$(q, \psi) = \left( q, u(-0) + \varepsilon^{-1} \left( (\alpha - 1) u(-0) + \beta u'(-0) \right) \omega \right)$$

$$= a_0 u(-0) + \varepsilon^{-1} a_1 \left( (\alpha - 1) u(-0) + \beta u'(-0) \right)$$

$$= \left( \varepsilon^{-1} a_1 (\alpha - 1) + a_0 \right) u(-0) + \varepsilon^{-1} a_1 \beta u'(-0)$$

$$= a_1 \varepsilon \left( \alpha - \frac{a_1 - \varepsilon a_0}{a_1} \right) u(-0) + \frac{a_1 \beta}{\varepsilon} u'(-0) = \frac{a_1 \beta}{\varepsilon} u'(-0),$$

because

$$\alpha = \frac{a_2 - \varepsilon a_1}{a_2} = \frac{a_0 a_2 - \varepsilon a_0 a_1}{a_0 a_2} = \frac{a_1 - \varepsilon a_0}{a_1}$$

in view of identity $a_0 a_2 = a_1^2$. In the same manner, we deduce

$$(q \omega, \psi) = a_1 u(-0) + \varepsilon^{-1} a_2 \left( (\alpha - 1) u(-0) + \beta u'(-0) \right)$$

$$= a_2 \varepsilon \left( \alpha - \frac{a_2 - \varepsilon a_1}{a_2} \right) u(-0) + \frac{a_2 \beta}{\varepsilon} u'(-0) = \frac{a_2 \beta}{\varepsilon} u'(-0),$$

by the choice of $\alpha$ in (10). By the above, we have

$$\left( \varepsilon^{-1} \omega - 1, q \psi \right) - u'(-0) = \varepsilon^{-1} (q \omega, \psi) - (q, \psi) - u'(-0)$$

$$= \left( \frac{a_2 \beta}{\varepsilon^2} - \frac{a_1 \beta}{\varepsilon} - 1 \right) u'(-0) = \left( \beta \cdot \frac{a_2 - \varepsilon a_1}{\varepsilon^2} - 1 \right) u'(-0) = 0,$$

by the choice of $\beta$. Since $\alpha^{-1} = \varepsilon^{-2} a_2 \beta$, we find

$$\varepsilon^{-1} (q \omega, \psi) - u'(0) = \varepsilon^{-2} a_2 \beta u'(-0) - u'(0) = \alpha^{-1} u'(-0) - u'(0) = 0,$$

by the second boundary condition in (37). Note that the first condition (37) is already used in (38). Therefore identities (35) hold and

$$\xi_{\varepsilon}(f) = \varepsilon \left( 1 - \varepsilon^{-1} \omega, f(\varepsilon \cdot) \right), \quad \eta_{\varepsilon}(f) = -\varepsilon \varepsilon^{-1} (\omega, f(\varepsilon \cdot)). \quad (39)$$
Finally, then, we consider the last formulae and inequality (32) we immediately deduce (34).

Now, we consider the case $a_1 = 0$ which corresponds to the classic $\delta^2$-interaction with $\alpha = 1$ and $\beta = x^2 a_2^{-1}$. Consequently (9) implies $a_0 = 0$. Also, (38) reduces to $\psi(t) = u(0) + x^{-1} \beta u(0) \omega(t)$, since $u^\prime(0) = u^\prime(+0) = u^\prime(0)$. A direct calculation shows that $(g, \psi) = 0$ and $(q \omega, \psi) = x^{-1} a_2 \beta u^\prime(0)$. Therefore

$$
\xi_e(f) = \varepsilon (1 - x^{-1} \omega, f(\varepsilon \cdot)) + x^{-1} (q \omega, \psi) - u^\prime(0)
$$

$$
= \varepsilon (1 - x^{-1} \omega, f(\varepsilon \cdot)) + x^{-2} a_2 \beta u^\prime(0) - u^\prime(0) = \varepsilon (1 - x^{-1} \omega, f(\varepsilon \cdot)),
$$

$$
\eta_e(f) = -\varepsilon x^{-1} (\omega, f(\varepsilon \cdot)) + x^{-1} (q \omega, \psi) - u^\prime(0)
$$

$$
= -\varepsilon x^{-1} (\omega, f(\varepsilon \cdot)) + x^{-2} a_2 \beta u^\prime(0) - u^\prime(0) = -\varepsilon x^{-1} (\omega, f(\varepsilon \cdot)),
$$

and $\xi_e(f)$ and $\eta_e(f)$ also have the form (39), which completes the proof.

Returning now to the jumps (33), we see at once that

$$
|\{y_2\} - \varepsilon| + |\{y_2\} - \varepsilon| + |\{y_2\} - \varepsilon| \leq C \varepsilon^{1/2}||f||,
$$

by (29) and (34). Owing to proposition 2 there exists a corrector $\rho_e$ such that $Y_e = y_e + \rho_e$ belongs to $W^2_{\text{loc}}(\mathbb{R})$. Moreover, $\rho_e$ has a compact support, $\rho_e(x) = 0$ for $x \in (-\varepsilon, \varepsilon)$, and

$$
|\rho_e(x)| + |\rho_e''(x)| \leq C \varepsilon^{1/2}||f|| \quad \text{for } |x| \geq \varepsilon.
$$

(40)

Since $Y_e$ belongs to the domain of $S_e$, we now can compute $F_e = (S_e - \zeta) Y_e$ in order to estimate the accuracy of approximation. From (36) it follows that

$$
F_e(x) = -\frac{d^2}{dx^2} + (V(x) - \zeta)(u(x) + \rho_e(x)) = f(x) - \rho_e''(x) + (V(x) - \zeta)\rho_e(x)
$$

for all $x$ such that $|x| > \varepsilon$. In the case $|x| < \varepsilon$, we have

$$
F_e(x) = -\frac{d^2}{dx^2}(Y_e \left(\frac{x}{\varepsilon}\right)) + (V(x) - \zeta)Y_e \left(\frac{x}{\varepsilon}\right)
$$

$$
= \varepsilon^{-3} \int_{-\varepsilon}^{\varepsilon} (\varphi_1 \left(\frac{x}{\varepsilon}\right) \varphi_2 \left(\frac{x}{\varepsilon}\right) + \varphi_2 \left(\frac{x}{\varepsilon}\right) \varphi_1 \left(\frac{x}{\varepsilon}\right)) Y_e \left(\frac{x}{\varepsilon}\right) \, dx
$$

$$
= \varepsilon^{-2} \left( -\varphi'' \left(\frac{x}{\varepsilon}\right) + (\varphi_2, \psi) \varphi_1 \left(\frac{x}{\varepsilon}\right) + (\varphi_1, \psi) \varphi_2 \left(\frac{x}{\varepsilon}\right) \right)
$$

$$
+ \varepsilon^{-1} \left( -\rho_e'' \left(\frac{x}{\varepsilon}\right) + (\varphi_2, v_e) \varphi_1 \left(\frac{x}{\varepsilon}\right) + (\varphi_1, v_e) \varphi_2 \left(\frac{x}{\varepsilon}\right) + q \left(\frac{x}{\varepsilon}\right) \psi \left(\frac{x}{\varepsilon}\right) \right)
$$

$$
+ q \left(\frac{x}{\varepsilon}\right) \varphi_2 \left(\frac{x}{\varepsilon}\right) + (V(x) - \zeta)Y_e \left(\frac{x}{\varepsilon}\right) = f(x) + q \left(\frac{x}{\varepsilon}\right) v_e \left(\frac{x}{\varepsilon}\right) + (V(x) - \zeta)Y_e \left(\frac{x}{\varepsilon}\right).
$$

by (16) and (26). Therefore $(S_e - \zeta)Y_e = f + r_e$, where

$$
r_e(x) = \begin{cases} -\rho_e''(x) + (V(x) - \zeta)\rho_e(x) & \text{if } |x| > \varepsilon, \\ q \left(\frac{x}{\varepsilon}\right) \varphi_e \left(\frac{x}{\varepsilon}\right) + (V(x) - \zeta)Y_e \left(\frac{x}{\varepsilon}\right) & \text{if } |x| < \varepsilon. \\
\end{cases}
$$

For any $g \in L^2(\mathcal{I})$ we have $||g(\varepsilon^{-1})||_{L^2(-\varepsilon, \varepsilon)} = \varepsilon^{1/2}||g||_{L^2(\mathcal{I})}$. Using this equality together with the facts that $V$ is local bounded, $\rho_e$ has a compact support, and $Y_e = y_e$ on $(-\varepsilon, \varepsilon)$, we obtain

$$
||r_e|| \leq c_1 \max_{|x| > \varepsilon} ||\rho_e''|| + (\zeta - V)\rho_e + ||q(\varepsilon^{-1})v_e(\varepsilon^{-1})\cdot + (V - \zeta)Y_e(\varepsilon^{-1})||_{L^2(-\varepsilon, \varepsilon)}
$$

$$
\leq c_2 \max_{|x| > \varepsilon} ||\rho_e|| + ||\rho_e''|| + c_3 \varepsilon^{1/2} \left(||q(\varepsilon^{-1})v_e||_{L^2(\mathcal{I})} + ||y||_{L^2(\mathcal{I})} + ||v_e||_{L^2(\mathcal{I})}\right)
$$

$$
\leq c_2 \max_{|x| > \varepsilon} ||\rho_e|| + ||\rho_e''|| + c_4 \varepsilon^{1/2} \left(||q(\varepsilon^{-1})v_e||_{L^2(\mathcal{I})} + ||v_e||_{L^2(\mathcal{I})}\right).
$$
Combining estimates (28), (31) and (40) yields the bound
\[ \|r_\varepsilon\| \leq c\varepsilon^{1/2}\|f\|. \]
We can also apply the similar considerations to the difference
\[ Y_\varepsilon(x) - u(x) = \begin{cases} \rho_\varepsilon(x) & \text{if } |x| > \varepsilon, \\ \psi(\frac{x}{\varepsilon}) + \varepsilon v_\varepsilon(\frac{x}{\varepsilon}) - u(x) & \text{if } |x| < \varepsilon \end{cases} \]
and obtain the estimate
\[ \|Y_\varepsilon - u\| \leq c\varepsilon^{1/2}\|f\|. \]
The last bound means that a non-zero contribution in the $L_2$-norm of $Y_\varepsilon$, as $\varepsilon \to 0$, is produced by $u = (S_{\alpha\beta} - \zeta)^{-1}f$ only. Next, from $(S_\varepsilon - \zeta)Y_\varepsilon = f + r_\varepsilon$ we have
\[ (S_\varepsilon - \zeta)^{-1}f = Y_\varepsilon - (S_\varepsilon - \zeta)^{-1}r_\varepsilon. \]
Finally we conclude that
\[ \|(S_\varepsilon - \zeta)^{-1}f - (S_{\alpha\beta} - \zeta)^{-1}f\| = \|Y_\varepsilon - u - (S_\varepsilon - \zeta)^{-1}r_\varepsilon\| \leq \|Y_\varepsilon - u\| + \|(S_\varepsilon - \zeta)^{-1}r_\varepsilon\| \leq \|Y_\varepsilon - u\| + |\text{Im } \zeta|^{-1}\|r_\varepsilon\| \leq C\varepsilon^{1/2}\|f\|, \]
by (41) and (42). The last bound establishes the norm resolvent convergence of $S_\varepsilon$ to operator $S_{\alpha\beta}$ with $\alpha$ and $\beta$ given by (10), which is the desired conclusion.

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