ALGEBRAIC ORBIFOLD QUANTUM PRODUCTS

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1. Introduction

The purpose of this note is to give an overview of our work on defining algebraic counterparts for W. Chen and Y. Ruan’s Gromov-Witten Theory of orbifolds. This work will be described in detail in a subsequent paper.

The presentation here is generally based on lectures given by two of us at the Orbifold Workshop in Madison, Wisconsin. Following the spirit of the workshop, our presentation is intended to be understandable not only to Algebraic geometers, but also practitioners of the other disciplines represented there, including differential geometers and mathematical physicists.

We make a special effort to make our constructions as canonical as we can, systematically using the language of algebraic stacks. Our constructions are based on the theory of twisted stable maps developed in [N-V3], but requires making explicit some details which were not studied in the paper [N-V3]. Apart from the pleasure we take in understanding these details, our efforts bear some concrete fruits; in particular, we are able to define the Chen-Ruan stringy product in degree 0 (the so called stringy cohomology) with integer coefficients.

We work over the field \(\mathbb{C}\) of complex numbers (although our discussion works just as well over any field of characteristic 0).

2. Stacks and their moduli spaces

There are two, quite different, ways in which “orbifolds” or “stacks” arise.

2.1. Groupoids in schemes. This is the way most differential geometers, as well as many algebraic geometers, are introduced to the subject, since it is in some sense concrete and geometric: one thinks about an object which is locally modeled on a quotient of a variety by the action of an algebraic group. Then one needs to define a good notion of maps between such objects - this is the difficult part of the picture.

Concretely, one is given a “relation”, namely a morphism \(R \to U \times U\), where each of the projections \(R \to U\) is étale (or, more generally, smooth), with the extra data of an “inverse” \(R \to R\) and “composition” \(R \times_U R \to R\) satisfying natural assumptions which the reader may guess. This is what one called a groupoid.

One then defines morphisms of groupoids, Morita equivalences, and finally one defines the category of orbifolds using groupoids up to Morita equivalence.

In the theory of stacks, there is a natural algebraic stack \([U/R]\) associated to a groupoid. Two groupoids are Morita equivalent if and only if the associated stacks are isomorphic; therefore the groupoid appears as a presentation of the stack. The stack is a more intrinsic object.

Date: October 25, 2018.
Research of D.A. partially supported by NSF grant DMS-0070970.
Research of T.G. partially supported by an NSF post-doctoral research fellowship.
Research of A.V. partially supported by the University of Bologna, funds for selected research topics.
For instance, the quotient stack $[U/G]$ of a scheme $U$ by the action of a finite group $G$ is the stack associated to the groupoid given by $R = G \times U$.

A particularly simple and useful case is the classifying stack $BG = [pt/G]$ of the group $G$ (where $pt$ stands for the point $\text{Spec } \mathbb{C}$).

2.2. Moduli problems. This is the way stacks arise in a more abstract, categorical incarnation, but still extremely natural.

One can encode the data of a “moduli problem” in the category of families of objects which we want to parametrize. For instance, the data of “moduli of curves of genus $g$” is encoded in the category $\mathcal{M}_g$ whose objects are

$$\text{Ob}(\mathcal{M}_g) = \left\{ \begin{array}{c} C \\ S \end{array} \right. | \begin{array}{c} S \text{ is a scheme,} \\ C \to S \text{ is a smooth family of curves of genus } g \\ \text{and} \\ \end{array} \right\},$$

and the arrows from one object $C \to S$ to another $C' \to S'$ are the fiber diagrams

$$\begin{array}{ccc} C & \to & C' \\ \downarrow & & \downarrow \\ S & \to & S' \end{array}.$$

There is also a structural functor $\mathcal{M}_g \to \text{Sch}$ which sends a family of curves $C \to S$ to the underlying scheme $S$.

Formally, an algebraic stack is a category $\mathcal{M}$ with a functor $\mathcal{M} \to \text{Sch}$, satisfying certain assumptions which guarantee that it is “somewhat close to being representable by a scheme”. These assumptions imply, in particular, that an algebraic stack is always the stack associated to a groupoid in schemes $[U/R]$. If the automorphism group of every object in the category is finite, then in fact $\mathcal{M}$ is locally given as the quotient $[U/G]$ of a scheme by a finite group. Such a stack is called a Deligne–Mumford stack.

If $\mathcal{M}$ is a stack of a moduli problem, then exactly as in the example, the objects are families $C \to S$, and the arrows are fiber diagrams

$$\begin{array}{ccc} C & \to & C' \\ \downarrow & & \downarrow \\ B & \to & S' \end{array}.$$

The functor $\mathcal{M} \to \text{Sch}$ simply sends a family $C \to S$ to its base scheme $S$.

Simple examples include: $\mathcal{M}_g$, the moduli stack of smooth curves of genus $g$, and $BG$, the moduli stack of principal homogeneous $G$-spaces, say for a finite group $G$.

In introductory texts about moduli theory, one learns about the moduli functor of a moduli problem - in the example on $\mathcal{M}_g$ above, the functor sends a scheme $S$ to the set of isomorphism classes of all smooth families $C \to S$ of curves of genus $g$. It may be the result of an unfortunate historical tradition, that often the discussion of the category, which faithfully encodes the moduli problem in question, is delayed till after the shortcomings of the moduli functor are discovered.

2.3. Coarse moduli spaces. Every Deligne–Mumford stack $\mathcal{M}$ has an associated algebraic space called the coarse moduli space $\underline{M}$ (for a general proof see [K-M]):

- In case $\mathcal{M}$ is given via a groupoid as $[U/R]$, the coarse moduli space is the orbit space $U/R$.
- In case $\mathcal{M}$ is the moduli stack of a moduli problem, then $\underline{M}$ is the usual coarse moduli space parametrizing objects modulo isomorphisms.
3. Twisted stable maps

3.1. Stable maps into a projective variety. Recall that, when $X$ is a projective variety, one has a projective moduli space $\overline{M}_{g,n}(X, \beta)$ of $n$-pointed stable maps of genus $g$ and image class $\beta$. This is always the coarse moduli space of a Deligne–Mumford stack $\overline{M}_{g,n}(X, \beta)$, whose objects over a scheme $T$ are triples:

$$(C \to T, \{s_1, \ldots, s_n : T \to C\}, f : C \to X).$$

Here

1. $C \to T$ is a family of prestable curves of arithmetic genus $g$,
2. $s_i : T \to C$ are sections of $C \to T$ with disjoint images lying in the smooth locus, and
3. $f : C \to X$ is a morphism, such that the group of automorphism of fibers of $C \to T$ commuting with $f$ and fixing all the sections $s_i$ is finite.

It is no secret in this workshop that the spaces (and stacks) of stable maps have been useful in symplectic geometry, enumerative geometry and mathematical physics through Gromov–Witten invariants. They have also served as a stepping stone for constructing objects of interest in algebraic geometry (such as other moduli spaces). For a recent application in higher dimensional geometry see [G-H-S].

3.2. Stable maps into a stack. Since here we are in the business of enlarging our vocabulary of geometric objects from “varieties” to “orbifolds” (or “stacks”), one may wonder if it is worthwhile to replace the projective variety $X$ in the definition of a stable map by a stack $\mathcal{X}$.

Let us consider two examples of stacks which were mentioned above:

1. Fix an integer $h > 1$, and say $\mathcal{M} = \overline{M}_h$. Then a map of a curve $C$ to $\mathcal{M}$ is the same as an object of $\mathcal{M}$ over $C$, in other words a family $X \to C$ of stable curves of genus $h$ over $C$. Thus the moduli space of all such things is in particular a moduli space of a certain type of surfaces with extra structure (given by a map to a curve). For an algebraic geometer, it is evident that moduli of surfaces are of interest. So having a space of stable maps into $\overline{M}_g$ is useful at least as a construction tool. See [G-H-S].
2. Fix a finite group $G$, and say $\mathcal{M} = B\mathcal{G}$. Then a map of a curve $C$ to $\mathcal{M}$ is a family $P \to C$ of principal homogeneous $G$-spaces, in other words, a principal $G$-bundle. The moduli of these objects are closely related to Hurwitz schemes, level structures and admissible covers, which have been a subject of interest in algebraic geometry for years. See [K-C-V].

It is from this point of view, of using stable maps as a basic construction tool, that two of us approached the problem of constructing a suitable space of stable maps into a stack, see [R-V3]. Admittedly, Gromov–Witten theory for stacks seemed to us a distant possibility until the work of W. Chen and Y. Ruan on the subject was made public.

3.3. The non-compactness problem. One still needs to give a definition of what one means by a “stable map into a stack”. As a first attempt, one can define stable maps into a stack $\mathcal{X}$ just as stable maps into a projective variety were defined. The only problem with this is, that the stack of usual stable maps into $\mathcal{X}$ is not necessarily compact.

This phenomenon is already manifest with the simplest series of examples where $\mathcal{X} = B\mathcal{G}$, so that maps to $\mathcal{X}$ correspond to principal $G$-bundles, see [R-V3]. Say $G = (\mathbb{Z}/m\mathbb{Z})^4$, $n \geq 2$, and $g = 2$. A smooth curve $C$ of genus 2 carries a connected principal $G$-bundle $P \to C$, since its first homology group is $\mathbb{Z}^4$. But if one takes a one-parameter family $P_t : C_t \to C$, $t \neq 0$ of such bundles where the underlying curve $C_t$ degenerates, as $t$ approaches 0, to a nodal curve $C_0$ of geometric genus 1, then there is nothing to which the one-parameter family of bundles can
Degenerate. This is so simply because the first homology of the limiting curve $C_0$ is $\mathbb{Z}^3$, and thus $C_0$ carries no connected principal $G$-bundle.

An example of a similar phenomenon with $X = \overline{M}_{1,1}$ is described in [R-V1].

### 3.4. Adding orbispace structure on curves.

A simple analysis, detailed in [R-V2], [R-V3], of what goes wrong in the example above, reveals that the problem is centered above the node of the degenerate curve $C_0$, and, moreover, at the node there is a natural orbispace structure $C_0$ which does afford a limiting connected principal bundle $P_0 \to C_0$.

One then realizes that, in order to have a compact moduli space (or stack) of maps into a stack $X$, one needs to allow the source curves to acquire an orbispace structure as well. This may seem as a surprise at first, but we have come to believe that this is completely natural and beautiful. In a way, nature imposes this solution upon us, so why fight the elements?

### 3.5. Twisted curves.

Let us describe in some detail the orbispace curves - which we call twisted curves, underlying these new maps. We restrict ourselves here to balanced twisted curves, the only type that is relevant to our discussion in this paper.

#### 3.5.1. Nodes.

First, consider the structure at a node. The orbispace curve here is locally the quotient $[U/\mu_r]$ of a nodal curve $U = \{xy = f(t)\}$ by the following action of the cyclic group $\mu_r$ of $r$-th roots of 1:

$$ (x, y) \mapsto (\zeta x, \zeta^{-1} y). $$

The coarse curve $C$ underlying the orbispace curve $\mathcal{C}$ is, locally, the schematic quotient $U/\mu_r$ defined by $uv = f(t)^r$, where $u = x^r$ and $v = y^r$.

This kind of orbispace structure by itself would suffice for solving the compactification problem we discussed above. But we also know that it is useful to be able to describe the “boundary” of moduli by gluing marked curves into nodal ones. Reversing this line of thought, it is useful to understand what kind of structure one obtains when separating a node into two marked points on a marked curve.

#### 3.5.2. Markings.

We are led to consider an orbispace structure along a marking. Here the orbispace curve $C \to T$ is locally the quotient $[U/\mu_r]$ of a smooth curve $U$, with local coordinate $z$ defining the marking, by the following action of $\mu_r$:

$$ z \mapsto \zeta z. $$

The coarse curve $C$ is locally a smooth curve with coordinate $w = z^r$.

The integer $r$ in these local descriptions is called the index of the orbifold curve at the point in question.

#### 3.5.3. Global structure of a marking.

Before imposing on the reader the formal definition of a twisted curve, we need to put a word of caution. Having given the local description of a family of twisted curves $C \to T$ at a marking, one might be misled to believe that there is automatically globally a section $T \to C$. This is not the case in general. This fact is going to be important later when we consider evaluation maps.

Take the coarse family of curves $C \to T$, with its section $T \to C$, having image $\Sigma^C \subset C$. The reduction $\Sigma^C \subset C$ of the inverse image of $\Sigma^C$ in the twisted curve $\mathcal{C}$ is canonically a gerbe over $T$ banded by the group $\mu_r$. This means, in particular, that locally $\Sigma^C$ is isomorphic to the stack-theoretic quotient of $\Sigma^C$ by the trivial action of $\mu_r$, but globally there is an obstruction for such a quotient description. The obstruction, which naturally lies in $H^2(T, \mu_r)$, comes in our case from $H^1(T, \mathbb{G}_m) = \text{Pic}(T)$ in the following manner:
Let $L = s^* N$ be the pullback to $T$ of the normal bundle of $\Sigma^C \subset C$. Then $\Sigma^C$ is the moduli stack of $r$-th roots of $L$, whose objects over a scheme $S$ are triples
\[(g : S \to T, \quad M, \quad \sigma : M^{\otimes r} \xrightarrow{\sim} g^* L),\]
where
- $g : S \to T$ is a morphism,
- $M$ is a line bundle on $S$, and
- $\sigma$ is an isomorphism of line bundles.

The group $\mu_r$ acts on every object here by fiberwise multiplication on $M$. All automorphisms arise this way.

It is clear from this description that $\Sigma^C \to T$ has a section if and only if the line bundle $L$ has an $r$-th root over $T$.

Incidentally, there is also a simple prescription for obtaining $C$ from the coarse curve $\mathcal{C}$, at least away from the nodes. We will not need that description in the present exposition.

3.5.4. Finally, here is a formal definition of twisted curves, as in [R-V3]:

**Definition 3.5.5.** A twisted nodal $n$-pointed curve over a scheme $T$ is a diagram
\[
\begin{array}{ccc}
\Sigma^C & \subset & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{C} & \to & T
\end{array}
\]
where
1. $\mathcal{C}$ is a tame Deligne-Mumford stack, proper over $T$, which étale locally is a nodal curve over $T$;
2. $\Sigma^C = \cup_{i=1}^n \Sigma^C_i$, where $\Sigma^C_i \subset \mathcal{C}$ are disjoint closed substacks in the smooth locus of $\mathcal{C} \to S$;
3. $\Sigma^C_i \to T$ are étale gerbes;
4. the morphism $\mathcal{C} \to \mathcal{C}$ exhibit $\mathcal{C}$ as the coarse moduli scheme of $\mathcal{C}$; and
5. $\mathcal{C} \to \mathcal{C}$ is an isomorphism over $C_{\text{gen}}$.

The notation $C_{\text{gen}}$ in the definition above stands for the generic locus, namely the complement of the nodes and markings on $C$.

One categorical issue we have to mention is the following: the collection of all stacks naturally forms a 2-category, and therefore families of twisted curves also form a 2-category. However, the fact that each twisted curve has a dense open subset which is a scheme, can be shown to imply that the 2-category of twisted curves is equivalent to a category (so it has a chance of being a stack).

3.6. Twisted stable maps. We can now define a twisted stable map as follows (see [R-V3]):

**Definition 3.6.1.** Let $\mathcal{X}$ be a Deligne–Mumford stack, $\mathcal{X} \to X \subset \mathbb{P}^N$ a projectively embedded coarse moduli scheme.

An $n$-pointed twisted stable map $f : \mathcal{C} \to \mathcal{X}$ to $\mathcal{X}$ of genus $g$, degree $d$, is a diagram
\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{\mathcal{f}} & X
\end{array}
\]
where

1. $\mathcal{C}$ is a twisted marked curve, with coarse moduli space $\mathcal{C} \to C$,
2. $\mathcal{C} \to X$ a stable map of genus $g$ and degree $d$, and
3. $\mathcal{C} \to \mathcal{X}$ representable

With this definition, we have the following theorem:

**Theorem 3.6.2.** The category of $n$-pointed twisted stable maps of genus $g$ and degree $d$ to $\mathcal{X}$ is a proper Deligne–Mumford stack $\mathcal{K}_{g,n}(\mathcal{X}, d)$ admitting a projective coarse moduli space $\mathcal{K}_{g,n}(\mathcal{X}, d)$.

Of course, the stacks $\mathcal{K}_{g,n}(\mathcal{X}, d)$ can be further decomposed using a numerical equivalence class, or a homology class $\beta \in H_1(X, \mathbb{Z})$ instead of the degree.

The notation using the letter $\mathcal{K}$ rather than $\mathcal{M}$ was chosen originally to avoid confusion when inserting something like $\mathcal{M}_\gamma$ as an argument in place of $\mathcal{X}$. In this note we will introduce the notation $\mathcal{M}_{g,n}(\mathcal{X}, d)$ for a certain variant of $\mathcal{K}_{g,n}(\mathcal{X}, d)$ (namely the stack of twisted stable maps with trivialized markings).

The stacks $\mathcal{K}_{g,n}(\mathcal{X}, d)$ can also be decomposed in terms of the type of the markings. To avoid cumbersome notation, we will not do this explicitly. Note, however, that the index $r_i$ of the $i$-th marking is a locally constant function $\mathcal{K}_{g,n}(\mathcal{X}, d) \to \mathbb{Z}$. As usual with locally constant functions, $r_i$ induces a linear operator on homology, cohomology, and Chow groups.

Also, as in the case of usual stable maps, there is a stabilization morphism $\mathcal{K}_{g,n}(\mathcal{X}, *) \to \mathcal{K}_{g,n}(\mathcal{Y}, *)$ associated to a morphism of stacks $\mathcal{X} \to \mathcal{Y}$. This is going to be useful in this paper in the special case where $\mathcal{Y}$ is a point.

Theorem 3.6.2 is closely related to Proposition 2.3.8 in Chen and Ruan’s paper [CR 2]. There are several differences: Chen and Ruan work with pseudo-holomorphic maps into smooth symplectic orbifolds in general. In [RV3] we work only with algebraic Deligne–Mumford stacks. On the other hand we allow the target stack $\mathcal{X}$ to be singular (and allow tame stacks in positive characteristics). Chen and Ruan’s moduli space is a differential orbifold, whereas we work systematically with stacks.

Some words on the proof of the theorem in [RV3]: often, the construction of a moduli stack (or space) is a standard process based on the existence of Hilbert and Quot schemes. Unfortunately, there is no well-developed machinery of universal constructions analogous to Hilbert and Quot schemes for stacks. The proof in [RV3] constructs the stack $\mathcal{K}_{g,n}(\mathcal{X}, d)$ “with bare hands” using deformation theory and algebraization, following the list of axioms in M. Artin’s paper [A]. Some of the methods developed in [RV3] might serve as building block for a future theory of universal constructions for stacks.

4. Gromov-Witten theory of varieties and stacks

4.1. Quantum intersection theory. In standard treatments of Gromov-Witten theory in symplectic geometry, as well as in [CR 2], one defines the quantum product in terms of “$n$-point Gromov-Witten numbers”, using the inverse matrix of the “2-point product” on cohomology, both for defining the product and proving associativity. In the context of algebraic cycles on varieties, let alone stacks, this cannot be done in general, since there is no analogue of Poincaré duality. A formalism which circumvents this issue was described in the paper [GP]:

Let $X$ be a smooth projective variety. Consider the correspondence
\[
\overline{M}_{g,n+1}(X,\beta) \overset{\epsilon_{n+1}}{\to} X
\]
\[
e_{1,\ldots,n} \downarrow \\
X^n
\]
where \(e_{1,\ldots,n} : \overline{M}_{g,n+1}(X,\beta)\) is the evaluation map at the first \(n\) points, and \(e_{n+1}\) is the last evaluation map.

Define a map
\[
A^*(X)^n \longrightarrow A^*(X) \\
\gamma_1 \times \cdots \times \gamma_n \mapsto \langle \gamma_1, \ldots, \gamma_n, * \rangle_{g,\beta},
\]
where
\[
\langle \gamma_1, \ldots, \gamma_n, * \rangle_{g,\beta} = (e_{n+1})_* \left( e_{1,\ldots,n}^*(\gamma_1 \times \cdots \times \gamma_n) \cap [\overline{M}_{g,n+1}(X,\beta)]^v \right)
\]
Here we used the notation \([\overline{M}_{g,n+1}(X,\beta)]^v\) for the virtual fundamental class of \(\overline{M}_{g,n+1}(X,\beta)\), defined using the cotangent complex of deformation theory, see, e.g., [B-F], [L-T].

Gromov–Witten invariants of smooth projective varieties satisfy a number of relations which are coded by a number of well known axioms. In particular, the genus 0 invariants can be combined to give the multiplication law of the associative quantum cohomology ring. For instance, the so-called “small” quantum product is defined by
\[
\gamma_1 \ast \gamma_2 = \sum_{\beta \in A^1(X)} \langle \gamma_1, \gamma_2, * \rangle_{0,\beta} q^\beta.
\]

The definition using pullback and pushforward avoids entirely the use of the inverse matrix of the multiplication table in cohomology. Of course, in this setting we identify “homology” and “cohomology” cycles on the smooth variety \(X\). This means that, for calculations in enumerative geometry the inverse matrix would be used anyway.

To prove the axioms of Gromov–Witten invariants, and, in particular, to prove associativity of the quantum product, one needs to compare the fiber product
\[
\overline{M}_{g_1,n_1+1}(X,\beta_1) \times_X \overline{M}_{g_2,n_2+1}(X,\beta_2),
\]
along with its virtual fundamental class, with a divisor in
\[
\overline{M}_{g_1+g_2,n_1+n_2}(X,\beta_1 + \beta_2),
\]
as we will illustrate later for \(g = 0\).

In order to generalize this picture, we will need
1. an analogue of \(A^*(X)\) for a smooth Deligne–Mumford stack,
2. an analogue of the evaluation maps \(e_{1,\ldots,n}\) and \(e_{n+1}\),
3. an analogue of the virtual fundamental class, and
4. an analogue of the divisor description of the fibered product.

### 4.2. Intersection theory of a stack.

The first ingredient in the “non-orbifold” case which needs a revision in the stack situation is the intersection theory \(A^*(X)\).

Two important ingredients in intersection theory of smooth varieties are
1. a ring \(A^*(X)\), functorial under pull-back along a morphism \(f : X \to Y\), and
2. push-forward maps \(f : X \to Y\) for \(f\) proper, satisfying the projection formula.
It has been observed long ago that such a perfect theory with integer coefficients cannot exist for Deligne–Mumford stacks. The problem lies with the pushing forward. Let \( G \) be a finite group of \( r \) elements, and let \( p = \text{Spec} k \) be a point. Consider the fiber diagram

\[
\begin{array}{ccc}
G & \to & p \\
\phi \downarrow & & \downarrow \psi \\
p & \to & BG.
\end{array}
\]

Since the degree of \( G \to p \) is the order \( r \) of \( G \) (i.e. \( \phi_*[G] = r[p] \)), and since \( p \to BG \) is proper and flat, it follows by the projection formula that the degree of \( \psi : p \to BG \) is \( r \) as well. But we also have the map \( \pi : BG \to P \), since \( p \) is the coarse moduli space of \( BG \), and of course \( \pi \circ \psi = \text{id} : p \to p \). So the degree of \( \pi \) must be \( 1/r \), so \( r\pi_*[BG] = [p] \), which is impossible since the generator \([p]\) is not divisible by \( r \).

In [G] and [V], Gillet and Vistoli defined an analogous intersection theory with rational coefficients for smooth Deligne-Mumford stacks. Basically

1. as a group, \( A^*(X) \) is isomorphic to \( A^*(X)_\mathbb{Q} \), where \( X \) is the coarse moduli space, and it has the structure of a commutative ring, but
2. push forward is a bit more subtle, and takes into account the degrees of stabilizers.

This construction can be generalized in the sense of [F], Chapter 17 (see also [B-F]) to a bivariant intersection theory of Deligne–Mumford stacks - we will use this when proving associativity of the quantum product.

In [K], A. Kresch defined an intersection theory \( A^*(X) \) with integer coefficients but where push-forward maps are defined only for representable morphisms \( f : \mathcal{X} \to \mathcal{Y} \). We recall that a morphism of algebraic stacks is representable if and only if for any geometric object \( \xi \in \mathcal{X}(\text{Spec} \mathbb{C}) \), the map \( f : \text{Aut} \xi \to \text{Aut} f(\xi) \) is a monomorphism. We will use Kresch’s theory when we define an integer-coefficients version of Chen and Ruan’s stringy cohomology ring of \( \mathcal{X} \).

The constructions of Kresch and Vistoli are related by the fact that \( A^*(X) \otimes \mathbb{Q} = A^*(X)_{\mathbb{Q}} \).

4.3. \( A^*(X) \) is too small. Both constructions discussed above are not quite sufficient for what we need. Even for global quotients, it has long been observed that the \( K \)-theory of \( \mathcal{X} = [V/G] \), namely the equivariant \( K \)-theory of \( V \), is bigger than the \( K \)-theory of \( V/G \), even with rational coefficients. Indeed, the decomposition theorem says that the equivariant \( K \)-ring \( K_0(V//G) \) is a product of \( K_0(V^g/C(g)) \), suitably twisted by cyclotomic rings. Thus any reasonable Riemann-Roch formula must take into account contributions from all the fixed-point sets. These contributions are lost in \( A^*(X)_{\mathbb{Q}} \). A detailed theory of Riemann-Roch type theorems for stacks, in which such contributions are introduced, was developed in Toen’s work [T].

In enumerative geometry, T. Graber studied in [G] the number of hyperelliptic curves of genus \( g \) and degree \( d \) in \( \mathbb{P}^2 \) through \( 3d + 1 \) general points. It is explained there that the cohomology of \( \text{Sym}^2 \mathbb{P}^2 \) does not carry enough information to encode all these invariants, and therefore the Hilbert scheme \( \text{Hilb}_2(\mathbb{P}^2) \) was used instead. The extra information in \( \text{Hilb}_2(\mathbb{P}^2) \) is completely accounted for by contributions of fixed-point sets. It is now known (see [B-K-R]) that the derived categories of \( \text{Hilb}_2(\mathbb{P}^2) \) and the stack symmetric square \( ((\mathbb{P}^2)^2)/(\mathbb{Z}/2\mathbb{Z})) \) are equivalent, and therefore their \( K \) groups are isomorphic.

In [CR2], Chen and Ruan were motivated by string theory to introduce similar contributions of fixed-point sets. We follow their approach, with a slight modification which avoids any choice of roots of 1.
4.4. **The inertia stack and variants.** The stringy cohomology of $\mathcal{X}$ according to Chen-Ruan is, as a vector space, the cohomology of another smooth stack, namely the inertia stack

$$\mathcal{X}^{\text{Chen-Ruan}}_1 = I_{\mathcal{X}} = \{(x, g) \mid x \in \text{Ob}(\mathcal{X}), \ g \in \text{Aut} \ x\}$$

$$= \{(x, H, g) \mid x \in \text{Ob}(\mathcal{X}), \ H \subset \text{Aut} \ x, \ g \text{ a generator of } H\}$$

A slightly more canonical Gromov-Witten formalism arises if instead one takes a Galois twist of this:

$$\mathcal{X}_1 = \{\text{HomRep}(\mathcal{B}\mu_r, X)\}$$

i.e., the stack of representable morphism from a constant cyclotomic gerbe to $\mathcal{X}$.

Of course, over $\mathbb{C}$ there is a natural isomorphism of $\mu_r$ with $\mathbb{Z}/r\mathbb{Z}$, which gives an isomorphism $\mathcal{X}^{\text{Chen-Ruan}}_1 \simeq \mathcal{X}_1$. But over other fields the group-schemes are different, in which case one has to use $\mu_r$: with $\mathbb{Z}/r\mathbb{Z}$ one cannot even define the Gromov-Witten invariants of $\mathcal{X}$.

There is an interesting variant of this stack, which we denote by $\overline{\mathcal{X}}_1$. It arises as a rigidification of $\mathcal{X}_1$. The point is that the group $\mu_r$ acts on every object of $\mathcal{X}_1$ in the component corresponding to $\mu_r$, and the rigidification process (see $\text{[R-C-V]}$) allows one to remove this action from the picture (just as the Picard functor is obtained by removing the $\mathbb{C}^*$ action from the stack of line bundles).

$$\overline{\mathcal{X}}_1 = \{(x, H, \chi) \text{ up to } \mu_r\}$$

$$= \text{stack of cyclotomic gerbes in } \mathcal{X}$$

The geometric objects of $\mathcal{X}_1$ and $\overline{\mathcal{X}}_1$ are the same $(x, H, \chi)$, but the automorphisms are different: in the first case an automorphism is an element of the centralizer $C(H)$, in the second it is an element of $C(H)/H$.

We note that

- there are obvious inclusions $\mathcal{X} \subset \mathcal{X}_1$ and $\mathcal{X} \subset \overline{\mathcal{X}}_1$.
- There is also a “forgetful map” $\mathcal{X}_1 \to \mathcal{X}$, but there is in general no map from $\overline{\mathcal{X}}_1$ to $\mathcal{X}$.
- Finally the rigidification map $\pi_1 : \mathcal{X}_1 \to \overline{\mathcal{X}}_1$ exhibits $\mathcal{X}_1$ as the universal gerbe over $\overline{\mathcal{X}}_1$.

This implies that the coarse moduli space $X_1$ of $\mathcal{X}_1$ is also the moduli space of $\mathcal{X}_1$.

The total stringy Chow group of $\mathcal{X}$ is defined as the total Chow group $A^*(X_1)$. We can also consider the integral version $A^*(X_1)$ defined by Kresch.

There is a useful endomorphism on

$$r : A^*(X_1) \to A^*(X_1),$$

induced by the locally constant function

$$r : X_1 \to \mathbb{Z}$$

with value $r$ on the piece corresponding to $\text{HomRep}(\mathcal{B}\mu_r, \mathcal{X})$.

4.5. **Evaluation maps.** Given a family of $n$-pointed twisted stable maps parametrized by a scheme $S$:

$$(C \to S, f : C \to \mathcal{X})$$

we have, for each $i$ in the range $1 \leq i \leq n$, a map of the marking $(\Sigma_i)$ into $\mathcal{X}$. This means that $S$ parametrizes a family of cyclotomic gerbes in $\mathcal{X}$. By definition this gives a morphism
(\bar{e}_i)_S : S \to \mathcal{X}_1$, and since this is true for every twisted stable map, we get a map of moduli stacks

$$\bar{e}_i : \mathcal{K}_{g,n}(\mathcal{X}, \beta) \to \mathcal{X}_1.$$  

Passing to moduli spaces we get

$$e^{\text{coarse}}_i : \mathcal{K}_{g,n}(\mathcal{X}, \beta) \to \mathcal{X}_1.$$  

Chen and Ruan \cite{CR2} use this latter map $e^{\text{coarse}}_i$ of coarse moduli space. It so happens that the formalism of Gromov-Witten theory basically goes through (with an important twist to be introduced soon) if one works with this map as if it came from a map $e^{\text{orb}}_i : \mathcal{K}_{g,n}(\mathcal{X}, \beta) \to \mathcal{X}_1$: it is a fact of life that such a “virtual map” does not exist in general. However, as long as one works with cohomology or intersection theory with rational coefficients, then one can define a “cohomological pull-back”:

$$(e^{\text{orb}}_i)_* = r^{-1} \cdot e^{\text{orb}}_i(\pi_1)_*.$$  

(here again $r$ denotes the operator which multiplies a class on a component of index $r$ by the integer $r$). A similar trick works also for pushforward

$$(e^{\text{orb}}_i)_* = r \cdot (\pi_1)_* (e_i)_*.$$  

One needs to be a little careful dealing with the fibered products that arise in the proof of associativity.

In order to avoid the confusion of working with such “weighted maps” we can simply work with the map $\bar{e}_i : \mathcal{K}_{g,n}(\mathcal{X}, \beta) \to \mathcal{X}_1$, changing the products by a suitable factor. For the sake of exposition, it may be a bit more transparent to use $\mathcal{X}_1$ and work with the following formalism:

Define $\mathcal{M}_{g,n}(\mathcal{X}, \cdot)$ to be the stack of twisted stable maps with sections of all the gerbes. This is simply the fibered product of the $n$ universal gerbes over $\mathcal{K}_{g,n}(\mathcal{X}, \cdot)$.

Now we have evaluation maps

$$\mathcal{M}_{g,n}(\mathcal{X}, \cdot) \xrightarrow{\bar{e}_i} \mathcal{X}_1 \quad \downarrow \quad \mathcal{K}_{g,n}(\mathcal{X}, \cdot) \xrightarrow{e_i} \mathcal{X}_1.$$  

There is an obvious involution $\iota$ on $\mathcal{X}_1$ and $\mathcal{X}_1$ via $\chi \mapsto \chi^{-1}$. We define the twisted evaluation map $\bar{e}_i = \iota \circ e_i$. The twisted evaluation map comes about when one glues two twisted stable maps along two matkings to form a balanced twisted stable map.

We can work with $\mathcal{M}_{g,n}(\mathcal{X}, \cdot)$, as long as we remember to account for the degree $1/(r_1 \cdots r_n)$ of the map $\mathcal{M}_{g,n}(\mathcal{X}, \cdot) \to \mathcal{K}_{g,n}(\mathcal{X}, \cdot)$. Note that this degree is only locally constant, and varies from one connected component to another.

4.6. Deformation and obstruction. We want to define an appropriate virtual fundamental class for $\mathcal{M}_{g,n}(\mathcal{X}, \cdot)$, generalizing the situation of the space of maps into a smooth projective variety.

The first obstacle one needs to overcome is mostly psychological - does deformation theory as we know it work in the situation of a stack?

The answer is yes - and already in the literature. The point is, that Illusie’s work on deformation theory \cite{Ill} works by deforming a ringed topos (essentially the category of sheaves on a site), as soon as it is reasonable enough to allow the definition of a cotangent complex. The fact that a satck has such a topos is discussed in \cite{L-MB}, Chapter 12, and the existence of a cotangent complex is discussed in \cite{L-MB}, Chapter 17 (in fact the case of Deligne–Mumford stacks can be more easily deduced from the case of schemes).
So, as in the case if schemes, the infinitesimal deformations of a twisted curve \( C \) are measured by \( \text{Ext}^1(\Omega^1_{\mathcal{C}}, \mathcal{O}_C) \), and if one wants to deform some additional non-twisted marking, then infinitesimal deformations are in \( \text{Ext}^1(\Omega^1_{\mathcal{C}}(\log D), \mathcal{O}_C) \), where \( D \) is the divisor given by these markings. Obstructions would lie in the corresponding \( \text{Ext}^2 \) group, which turns out to be zero for twisted pointed curves (see [R-C-V]).

Similarly, infinitesimal deformations of a map \( f : \mathcal{C} \to \mathcal{X} \) fixing the structure of \( \mathcal{C} \) (as well as of \( \mathcal{X} \)) are measured by \( \text{Hom}(f^*\Omega^1_{\mathcal{X}}, \mathcal{O}_C) = H^0(\mathcal{C}, f^*T_{\mathcal{X}}) \). Obstructions lie in \( H^1 \).

Finally, these can be put together in the following standard manner:

Consider the complex
\[
\mathbb{L}_f = [f^*\Omega^1_{\mathcal{X}} \to \Omega^1_{\mathcal{C}}(\log D)],
\]
with the term on the right positioned in degree 0. Then infinitesimal deformations of \( f : \mathcal{C} \to \mathcal{X} \) (allowing \( \mathcal{C} \) and the markings to deform) are measured by
\[
\text{Ext}^1(\mathbb{L}_f, \mathcal{O}_C).
\]
Obstructions lie in the corresponding \( \text{Ext}^2 \) group.

Now, the formalism of the virtual fundamental classes (see [L-T], [B-F], [Kr]) automatically allows one to define a virtual fundamental class \([\mathcal{K}_{g,n}(\mathcal{X}, \cdot)]^v \). There is similarly a class \([\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \cdot)]^v \), which is just the pullback of \([\mathcal{K}_{g,n}(\mathcal{X}, \cdot)]^v \). As we indicated before, to get the degrees right we need to replace the latter by a multiple: \([\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \cdot)]^w = r_1 \cdots r_n [\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \cdot)]^v \).

Since deformations of twisted marked curves are unobstructed, it is known (see [B-F], [B-L]) that this virtual fundamental class can also be constructed "relatively to the deformation space of the marked curve"; namely, in terms of \( \text{Ext}^* (f^*\Omega^1_{\mathcal{X}}, \mathcal{O}_C) \) only. This is useful when showing associativity of the quantum product.

4.7. The product. We can now define the Gromov-Witten operation just as it is done for smooth projective varieties. Consider the diagram

\[
\overline{\mathcal{M}}_{g,n+1}(\mathcal{X}, \cdot) \xrightarrow{\tilde{e}_{n+1}} \mathcal{X}_1 \\
\downarrow e_{1,\ldots,n} \\
\mathcal{X}_1^n
\]

Define
\[
A^*(\mathcal{X}_1)^n \longrightarrow A^*(\mathcal{X}_1) \\
\gamma_1 \times \cdots \times \gamma_n \mapsto \langle \gamma_1, \ldots, \gamma_n, * \rangle_{g,\beta} ,
\]
where
\[
\langle \gamma_1, \ldots, \gamma_n, * \rangle_{g,\beta} = \tilde{e}_{n+1} \ast (e_{1,\ldots,n}^*(\gamma_1 \times \cdots \times \gamma_n) \cap [\overline{\mathcal{M}}_{g,n+1}(\mathcal{X}, \beta)]^w) .
\]

Again - the reason we are using the twisted evaluation map \( \tilde{e}_{n+1} \) has to do with the fact that the twisted curves are balanced, and will become more explicit below.

5. Associativity

As in the now "classical" case of smooth varieties, the most subtle property of the quantum product is its associativity. Here we go through the main steps in the proof. We revisit these steps in the next section when discussing the stringy cohomology ring of \( \mathcal{X} \).
5.1. **Associativity: the product diagram.** For simplicity of the discussion, we restrict from now on to the case where \( g = 0 \) and \( n = 3 \).

We want to prove that the “3-point” quantum product is associative. Since it is commutative, it suffices to show

**Theorem 5.1.1.**

\[
\sum_{\beta_1 + \beta_2 = \beta} \langle\langle \gamma_1, \gamma_2, * \rangle_0, \beta_1, \gamma_3, * \rangle_0, \beta_2 = \sum_{\beta_1 + \beta_2 = \beta} \langle\langle \gamma_1, \gamma_3, * \rangle_0, \beta_1, \gamma_2, * \rangle_0, \beta_2. 
\]

Fix two homology classes \( \beta_i \in H_1(X, \mathbb{Z}) \), and for simplicity write \( \mathcal{M}_i = \overline{\mathcal{M}}_{0, 3}(X, \beta_i) \) for the corresponding stacks of twisted stable maps. The corresponding evaluation maps are denoted \( e^1_i : \mathcal{M}_1 \to X_1 \) and \( e^2_i : \mathcal{M}_2 \to X_1 \).

The relevant diagram is the following:

\[
\begin{array}{ccc}
\mathcal{M}_2 & \xrightarrow{e^2_2} & X_1 \\
\downarrow e^2_1 & & \downarrow e^2_1 \\
\mathcal{M}_1 & \xrightarrow{e^1_2} & X_1 \\
\mathcal{M}_1 & \xrightarrow{e^1_1} & \mathcal{M}_1
\end{array}
\]

As a first step we complete the top left corner of the diagram:

\[
\begin{array}{ccc}
\mathcal{M}_1 \times \mathcal{M}_2 & \xrightarrow{p_2} & \mathcal{M}_2 \\
p_1 \downarrow & \downarrow e^2_1 & \downarrow \delta_X \\
\mathcal{M}_1 & \xrightarrow{e^1_2} & \mathcal{M}_1 \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{M}_1 \times \mathcal{M}_2 & \xrightarrow{p_1 \times p_2} & \mathcal{M}_1 \times \mathcal{M}_2 \\
\downarrow e^1_1 \times e^2_1 & \downarrow \delta_X & \downarrow \delta_X \\
X_1 & \xrightarrow{\delta_X} & X_1 \times X_1
\end{array}
\]

Consider the following morphisms from \( \mathcal{M}_1 \times X_1 \times \mathcal{M}_2 \) to \( X_1 \):

\[
\begin{align*}
e^1_1 & = e^1_1 \circ p_1 \\
e^2_1 & = e^2_1 \circ p_1 \\
e^3_1 & = e^2_1 \circ p_2 \\
e^4_1 & = e^2_1 \circ p_2
\end{align*}
\]

A formal push-pull argument detailed in [Gar-P] gives:

\[
\langle\langle \gamma_1, \gamma_2, * \rangle_0, \beta_1, \gamma_3, * \rangle_0, \beta_2 = (e^4_1)^* \left( (e^3_1)^* \gamma_1 \cup (e^2_1)^* \gamma_2 \cup (e^3_1)^* \gamma_3 \right) \cap \delta^!( [\mathcal{M}_1]^w \times [\mathcal{M}_2]^w )
\]

As we will see, the class \( \delta^!( [\mathcal{M}_1]^w \times [\mathcal{M}_2]^w ) \) has an interpretation as a weighted virtual fundamental class of \( \mathcal{M}_1 \times X_1 \times \mathcal{M}_2 \) with respect to a natural obstruction theory. In analysing the weight, the following locally constant function becomes useful:

\[
r_\times = p^1_1 r^1_3.
\]

By definition of the fibered product we also have

\[
r_\times = p^2_2 r^2_1.
\]

Again, this function induces an endomorphism of \( A_*(\mathcal{M}_1 \times X_1 \mathcal{M}_2) \), also denoted \( r_\times \).
5.2. **Associativity: the divisor diagram.** The stack $\mathcal{M}_1 \times \mathcal{M}_2$ can be viewed as the stack of pairs of 3-pointed genus 0 twisted stable maps with *balanced gluing data* along $\hat{e}_1^3$ and $e_1^2$, and trivialization of all the gerbes. Writing $\beta = \beta_1 + \beta_2$, this means that there is a gluing morphism

$$gl : \bigcup_{\beta_1 + \beta_2 = \beta} \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_{0,4}(X, \beta),$$

just as in the case of usual stable maps.

Consider the stabilization morphism $st : \mathcal{M}_{0,4}(X, \beta) \to \overline{\mathcal{M}}_{0,4}$ and the divisor $D := (12|34) \subset \overline{\mathcal{M}}_{0,4}$ (corresponding to the reducible stable 4-pointed curve of genus 0 where the two first points are on the same component). We have the top and bottom of the following fiber diagram:

$$
\begin{array}{cccc}
\bigcup_{\beta_1 + \beta_2 = \beta} \mathcal{M}_1 \times \mathcal{M}_2 & \to & \mathcal{M}_{0,4}(X, \beta) & \to & \overline{\mathcal{M}}_{0,4} \\
\downarrow g & & \downarrow & & \downarrow \\
\mathcal{M}_{0,4}^{orb} & \supset & D \times \mathcal{M}_{0,4}^{orb} & \supset & D^{orb} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{\mathcal{M}}_{0,4} & \supset & D
\end{array}
$$

The notation $D(\mathcal{X})$ stands for the substack of reducible *twisted* stable maps to $\mathcal{X}$ with the first two marking separated from the last two by a node. We also denote by $e_i : \overline{\mathcal{M}}_{0,4}(\mathcal{X}, \beta) \to \mathcal{X}_1$ the evaluation maps.

On the second row we inserted the Artin stack $\overline{\mathcal{M}}_{0,4}^{orb}$ of *quasistable twisted curves*, i.e. connected proper twisted curves without a stability requirement. This is a smooth stack which is neither separated nor of finite type, yet it comes in handy just as the usual stack of quasistable curves was handy in [Be]. We denote by

$$j : D^{orb} \subset \overline{\mathcal{M}}_{0,4}^{orb}$$

the embedding of the substack of reducible quasistable curves with the first two marking separated from the last two by a node.

Some relatively simple facts we have are:

1. the map $g : \bigcup_{\beta_1 + \beta_2 = \beta} \mathcal{M}_1 \times \mathcal{M}_2 \to D(\mathcal{X})$ has locally constant degree $1/r_x$. In fact $\bigcup_{\beta_1 + \beta_2 = \beta} \mathcal{M}_1 \times \mathcal{M}_2$ is the universal gerbe over the node of the universal curve of $D(\mathcal{X})$.

2. We have an equality of divisors on $\overline{\mathcal{M}}_{0,4}^{orb}$:

$$[D \times \overline{\mathcal{M}}_{0,4}^{orb}] = r_x [D^{orb}].$$

3. $e_i^* = e_i \circ gl$.

5.3. **Associativity: compatibility of virtual fundamental classes.** A more subtle issue is the following:

**Proposition 5.3.1.**

\[ \delta'( [\mathcal{M}_1]^w \times [\mathcal{M}_2]^w ) = r_x^2 \cdot g^* (j^! [\overline{\mathcal{M}}_{0,4}(X)]^w) \]
**Proof.** By definition of $[\mathcal{M}_1]^w, [\mathcal{M}_2]^w, \overline{[\mathcal{M}_{0,4}(X)]}^w$, and $r_x$ we need to show $$\delta^i([\mathcal{M}_1]^w \times [\mathcal{M}_2]^w) = g^*(j^![[\mathcal{M}_{0,4}(X)]^w].$$

Denote $$\mathcal{Y} = \mathcal{M}_1 \times_{X_1} \mathcal{M}_2.$$ By the pullback property of virtual fundamental classes ([B-F], Proposition 7.2) the right-hand side coincides with the obstruction class $[\mathcal{Y}, E_f]$, where $(\pi : \mathcal{C} \to \mathcal{Y}, f : \mathcal{C} \to X)$ is the pullback of the universal family on $\overline{[\mathcal{M}_{0,4}(X)]}$.

The left-hand side involves the class $[\mathcal{M}_1]^w \times [\mathcal{M}_2]^w$, which, by the product property of virtual fundamental classes ([B-F], Proposition 7.4), coincides with the obstruction class $[\mathcal{M}_1 \times \mathcal{M}_2, E_{f12}]$ where $f_{12}$ is the disjoint union of the pullback of the universal twisted stable maps $(\pi_1 : \mathcal{C}_1 \to \mathcal{M}_1, f_1 : \mathcal{C}_1 \to X)$ on $\mathcal{M}_1$ and $(\pi_2 : \mathcal{C}_2 \to \mathcal{M}_2, f_2 : \mathcal{C}_2 \to X)$ on $\mathcal{M}_2$.

Denote by $f'(p_1 \times p_2)^*\mathcal{C}_{12} \to X$ the pullback of the disjoint union family.

The equality in the proposition can now be rewritten as $$[\mathcal{Y}, E_f] = (\delta_{x_1})^! [\mathcal{M}_1 \times \mathcal{M}_2, E_{12}].$$

To prove this equality, consider the normalization exact sequence of sheaves on $\mathcal{C}$:

$$0 \to f^*T_X \to \nu_*f'^*T_X \to (f^*T_X)|_{\Sigma} \to 0.$$ 

This induces a distinguished triangle on $\mathcal{Y}$:

$$\mathbb{R}\pi_*f^*T_X \to \mathbb{R}\pi'_*f'^*T_X \to \pi_*((f^*T_X)|_{\Sigma}) \to \mathbb{R}\pi_*f^*T_X[1].$$

We use the terminology of [B-F], Proposition 7.5. The following Lemma says that $E_f$ and $E_{12}$ are compatible over $\delta_{x_1}$, which by [B-F], Proposition 7.5 exactly implies that $[\mathcal{Y}, E_f] = (\delta_{x_1})^! [\mathcal{M}_1 \times \mathcal{M}_2, E_{12}].$ 

**Lemma 5.3.2.** $\pi_*((f^*T_X)|_{\Sigma}) \simeq p_1^*(\tilde{e}_3^*)^*T_{x_1}$

**Proof of the Lemma.** Over a geometric point $y$ of $\mathcal{Y}$, we can identify the fiber of $\Sigma$ with $E\mathbf{B}_{\mu}$. The point $y$ maps to a geometric point $x$ of $X$, with stabilizer $G$, and we can locally describe $X$ around $x$ as $[U/G]$. The pullback $T$ of the tangent space of $X$ to $Y$ has a natural action of $\mu$, and the fiber of $\pi_*((f^*T_X)|_{\Sigma})$ is naturally the space of invariants $T\mu$. This is naturally isomorphic to the tangent space of $X_1$ at the point $\tilde{e}_3(p_1(y))$, which is what we needed.

### 5.4. Associativity: end of proof.

Let us apply the Proposition to prove associativity. Denote

$$c = (e_1^X)^*\gamma_1 \cup (e_2^X)^*\gamma_2 \cup (e_3^X)^*\gamma_3 \in A^*(\overline{[\mathcal{M}_{0,4}(X)]}).$$

First, it is an easy exercise to show that $j^![[\mathcal{M}_{0,4}(X)]^w = (\delta_{\overline{[\mathcal{M}_{0,4}(X)]}^w})^!([\mathcal{M}_{0,4}(X)]^w \times [D^{orb}]).$ A push-pull argument shows that $$l_* (j^![\overline{[\mathcal{M}_{0,4}(X)]}^w \cap l^*c) = (s^*j_*[D^{orb}] \cap ([\overline{[\mathcal{M}_{0,4}(X)]}^w \cap c)).$$
We therefore obtain
\[
\sum \langle \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle \rangle_{0, \beta_2} = (\bar{e}_4) \cdot r_2 \cdot l_g \cdot (\overline{\mathcal{M}_{0,4}(X)}^w \cap l^*c)
\]
\[
= (\bar{e}_4) \cdot r_3 \cdot l_g \cdot (\overline{\mathcal{M}_{0,4}(X)}^w \cap l^*c)
\]
\[
= (\bar{e}_4) \cdot r_3 \cdot l_g \cdot (s^*j_3[D_{orb}] \cap (\overline{\mathcal{M}_{0,4}(X)}^w \cap c))
\]
\[
= (\bar{e}_4) \cdot (s^*(c_1(\mathcal{O}_{\overline{\mathcal{M}_{0,4}}}(D)) \cap (\overline{\mathcal{M}_{0,4}(X)}^w \cap c))
\]

The latter expression is clearly independent of the way the markings are grouped, which is what we need for the associativity theorem:
\[
\sum \langle \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle \rangle_{0, \beta_2} = \sum \langle \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle \rangle_{0, \beta_2}.
\]

\[\clubsuit\]

6. Stringy Chow rings with integral coefficients

We now concentrate on the part of quantum Chow ring of \(X\) involving only \(g = 0, n = 3\) and \(\beta = 0\). Following a suggestion of Y. Ruan during a discussion at the conference, we call it the stringy Chow ring of \(X\).

6.1. The main result. A-priori, one gets a new ring structure on the group \(A^*_st(X)_Q := A^*(X_1)_Q\). It is easy to see that this is well defined whenever \(X\) is smooth and separated, without any properness assumptions. Our main claim is the following:

**Theorem 6.1.1.** Let \(X\) be a smooth and separated Deligne–Mumford stack, not necessarily proper. There is a commutative, associative ring structure on Kresch’s Chow group \(A^*_st(X) := A^*(X_1)\) lifting the stringy chow ring structure (with rational coefficients) on \(A^*(X_1)_Q\).

Moreover, let \(j : X \hookrightarrow X_1\) be the embedding. Then the group homomorphism \(j_* : A^*(X) \to A^*(X_1) = A^*_st(X)\) is a ring homomorphism, in particular \(j_*1\) is the identity in \(A^*_st(X)\).

The proof of this theorem is quite a bit more subtle than the rational case.

6.2. Refined evaluation and the product law. First, we need to define the product. The crucial point is that in the case \(g = 0, n = 3\) and \(\beta = 0\) we have refined evaluation maps:

**Lemma 6.2.1.** Each evaluation maps \(\bar{e}_i : \mathcal{K}_{0,3}(X, 0) \to X_1\) admits a lifting \(e_i : \mathcal{K}_{0,3}(X, 0) \to X_1\).

The morphism \(e_i\) is representable and finite.

**Proof.** The existence of \(e_i\) stems from the following: given a family of twisted stable maps \((\mathcal{C} \to S, \mathcal{C} \to X)\) in \(\mathcal{K}_{0,3}(X, 0)(S)\), the coarse curve is just \(\mathbb{P}^1 \times S\), and the markings can be identified as \(\{0\} \times S\), \(\{1\} \times S\) and \(\{\infty\} \times S\). Constructing a lifting \(e_i\) is tantamount to constructing an \(r_i\)-th root of the normal bundle of the \(i\)-th marking, functorially in \(S\), which is the same as constructing an \(r_i\)-th root of the tangent space of \(\mathbb{P}^1\) restricted to the \(i\)-th marking. But a line bundle over a point obviously has an \(r_i\)-th root.

The morphism \(e_i\) is easily seen to be proper and with finite fibers. Representability is an easy calculation of stabilizers.

\[\clubsuit\]

We note that \(\mathcal{K}_{0,3}(X, 0)\) is always smooth. The obstruction class \(O_{0,3}\) defining \([\mathcal{K}_{0,3}(X, 0)]^v\) is given by the top Chern class of the bundle
\[
E_{0,3} = \mathbb{R}^1(\pi_{0,3})_*f_{0,3}^*T_X.
\]
We can now define the product as follows:

**Definition 6.2.2.** Let \( \gamma_1, \gamma_2 \in A^*(\mathcal{X}_1) \). Define
\[
\gamma_1 \circ \gamma_2 = \hat{e}_{3*} \left( e_1^*(\gamma_1) \cdot e_2^*(\gamma_2) \cdot O_{0,3} \right)
\]

As things stand, this definition depends on the choice of the liftings \( e_i \). However, it is not difficult to show that the group homomorphisms \( e_{i*} \) and \( e_{i}^* \) are independent of the choices, and therefore the product \( \gamma_1 \circ \gamma_2 \) is independent of the choices.

6.3. **Associativity.** Next we need to prove associativity. The discussion in section 5.1 goes through word for word and one obtains (with notation analogous to that in section 5.1)

\[
(\gamma_1 \circ \gamma_2) \circ \gamma_3 = (e_4^*)_* \left( (e_1^*)^* \gamma_1 (e_2^*)^* \gamma_2 (e_3^*)^* \gamma_3 \right) \delta^1 \left( [K_1]^v \times [K_2]^v \right)
\]

We can now follow the arguments of section 5.2. As before, we have a gluing morphism
\[
gl : K_1 \times_{X_1} K_2 \rightarrow D(X) \hookrightarrow K_{0,4}(X, 0).
\]
This time, the morphism \( g \) is a \( \mu_{r_*} \)-bundle.

There is a little problem with item 3 at the end of section 5.2: the morphism \( \bar{\epsilon}_i : K_{0,4}(X, 0) \rightarrow X_1 \) does not lift to \( X_1 \), since the normal bundle to the \( i \)-th section of the universal curve over \( \overline{M}_{0,4} \) has degree 1, and thus has no \( r_i \)-th root unless \( r_i = 1 \). However, we restrict to be the inverse image
\[
\bar{K}_{0,4}(X, 0)
\]
of the open set \( \mathbb{A}^1 = \overline{M}_{0,4} \smallsetminus (14|23) \subset \overline{M}_{0,4} \), then a section does exist. We denote by \( gl : K_1 \times_{X_1} K_2 \rightarrow \bar{K}_{0,4}(X, 0) \) the gluing map. So we can define
\[
e_i : \bar{K}_{0,4}(X, 0) \rightarrow X_1, \quad i = 1, \ldots, 4
\]
and if we do so carefully then
\[
e_i \circ gl = e_i^x.
\]

Continuing with the arguments, the proof of Proposition 5.3.1 goes through and gives
\[
\delta^1 \left( [K_1]^v \times [K_2]^v \right) = g^*(j^*[K_{0,4}(X, 0)]^v).
\]
We also have that \( K_{0,4}(X, 0) \rightarrow K_{0,4}^{orb} \) is flat, therefore
\[
g^*(j^*[K_{0,4}(X, 0)]^v) = gl^*[K_{0,4}(X, 0)]^v.
\]
This implies that
\[
(\gamma_1 \circ \gamma_2) \circ \gamma_3 = e_4^x gl^* \left( e_{1*}^{e_1*} e_{2*}^{e_2*} e_{3*}^{e_3*} \bar{K}_{0,4}(X, 0)]^v \right).
\]

Unfortunately, \( e_4 \) is neither proper nor representable, so we cannot replace \( e_4^x \) by \( e_{4*} gl^* \). We remark that Chen and Ruan utilize a certain identification of the coarse moduli space \( K_{0,4}(X, 0) = \overline{M}_{0,4} \times X_3 \), which does not lift to the level of stacks.

Let \( \tau : \bar{K}_{0,4}(X, 0) \rightarrow \bar{K}_{0,4}(X, 0) \) be the automorphism induced by the involution \( (2, 3) \) on the markings, and denote \( gl_{\tau} = \tau \circ gl \). One immediately sees that associativity follows from the following:

**Claim.**
\[
e_4^x gl^* = e_4^x (\tau \circ gl)^* : A^*(\bar{K}_{0,4}(X, 0)) \rightarrow A^*X_1.
\]
To show this equality it is useful to refine the divisor diagram. We deal with one connected component of $\tilde{K}_{0,4}(\mathcal{X}, 0)$ at a time, so let $\tilde{K}$ be such a component. We denote $\times := gl^{-1}\tilde{K}$ and similarly $\times_{\tau} := gl^{-1}\tau \tilde{K}$. One can show that the locally constant function $r_{\times}$ is constant on $\times$ and $\times_{\tau}$, and we denote these constants by $r_{\times}$ and $r_{\times_{\tau}}$ respectively.

We have a fiber diagram

$$
\begin{array}{ccc}
\tilde{K} & \supset & \tilde{D}(X) \\
\downarrow & & \downarrow \\
\mathcal{M}_{0,4} & \supset & \mathcal{B}_{\mu_{r_{\times}}} \leftarrow \text{Spec } \mathbb{C}
\end{array}
$$

and a similar diagram for the $\tau$ version.

Here $\mathcal{M}_{0,4}$ denotes the open stack of twisted 4-pointed curves over $\mathbb{A}^1$ with stable 4-pointed coarse curve, with markings of indices $r_1, \ldots, r_4$, and nodes of index $r_\times$ over $\{0\} \subset \mathbb{A}^1$ and index $r_{\times_{\tau}}$ over $\{1\} \subset \mathbb{A}^1$. It is simply obtained by endowing $\mathbb{A}^1$ with orbifold structure of index $r_{\times}$ at $\{0\} \subset \mathbb{A}^1$ and index $r_{\times_{\tau}}$ at $\{1\} \subset \mathbb{A}^1$. It thus contains two divisors $\mathcal{G}_0$ and $\mathcal{G}_1$ over $\{0\}$ and $\{1\}$ in $\mathbb{A}^1$, isomorphic to $\mathcal{B}_{\mu_{r_{\times}}}$ and $\mathcal{B}_{\mu_{r_{\times_{\tau}}}}$, respectively.

The square on the right is cartesian since $\text{Spec } \mathbb{C} \to \mathcal{B}_{\mu_{r_{\times}}}$ is the universal principal bundle.

Ignoring the middle column, we can extend the fiber diagram as follows:

$$
\begin{array}{ccc}
\tilde{K} & \xleftarrow{gl} & \times \\
\downarrow & & \downarrow \\
\mathcal{M}_{0,4} \times \mathcal{X}_1 & \leftarrow & \mathcal{X}_1 \leftarrow \text{Spec } \mathbb{C}
\end{array}
$$

(and a similar diagram with $\tau$).

Considering the projection formula for the upper square, associativity follows if we prove that

$$(p \times id)^* = (p_{\tau} \times id)^*: A^*(\mathcal{M}_{0,4} \times \mathcal{X}_1) \to A^*(\mathcal{X}_1)$$

is independent

Given $\xi \in A^*(\mathcal{M}_{0,4} \times \mathcal{X}_1)$, we can restrict it to the open set over $\mathcal{M}_{0,4} \approx (\mathcal{G}_0 \cup \mathcal{G}_1) \approx \mathbb{A}^1 \sim \{0, 1\}$:

$$
\begin{array}{ccc}
\mathcal{M}_{0,4} & \to & \mathcal{G}_0 \cup \mathcal{G}_1 \\
\downarrow & & \downarrow \\
\mathbb{A}^1 \sim \{0, 1\} & \approx & \mathbb{A}^1
\end{array}
$$

The usual exact sequence says that this restriction extends to a class on $\mathbb{A}^1 \times \mathcal{X}_1$, which can be pulled back to a class $\xi'$ on $\mathcal{M}_{0,4} \times \mathcal{X}_1$. On the one hand, $A^*(\mathcal{X}_1 \times \mathbb{A}^1) = A^*(\mathcal{X}_1)$, and therefore $(p \times id)^*\xi'$ is independent of the choice of $p$. On the other hand, $\xi$ and $\xi'$ differ by a class supported on $\mathcal{G}_0 \cup \mathcal{G}_1$. It thus suffices to show that

$$(p \times id)^*\xi = 0$$

whenever $\xi$ is supported on $(\mathcal{G}_0 \cup \mathcal{G}_1) \times \mathcal{X}_1$. It suffices to treat one of the two. Now say $\iota: \mathcal{G}_0 \times \mathcal{X}_1 \hookrightarrow \mathcal{M}_{0,4} \times \mathcal{X}_1$ is the embedding of the gerbe over 0, with normal sheaf $N_0$. Then

$$
iota^*\iota_*\xi = c_1(N_0)\xi.$$

Pulling back to $\mathcal{X}_1 = \text{Spec } \mathbb{C} \times \mathcal{X}_1$, we have

$$(p \times id)^*\iota_*\xi = (p \times id)^*c_1(N_0)\xi.$$
A simple computation shows that
\[(p \times id)^* c_1(N_0) = 0,\]
therefore \[(p \times id)^* \iota_* \xi = 0.\]

7. Remarks on grading, expected dimension, divisors

7.1. Degree shifting. When we read about virtual fundamental classes on moduli spaces of “usual” stable maps, we often find it stressed that the virtual fundamental class has the “expected dimension”. In our discussion here we (somewhat perversely) avoided any discussion of the expected dimension, and evidently it is not at all necessary for proving the associativity of the quantum product. On the other hand, the existence of such dimension and its properties are of interest.

An interesting feature of the expected dimension of twisted stable maps is, that it depends on the component of the moduli space in a subtle way, determined by the type of the evaluation maps, using the so called “degree shifting number” or “age” of the image component in \(X_1\).

The complex \(\mathbb{R} \pi_{g,n,\beta} \star f^* T X\) was used for defining the relative obstruction theory, and thus the class \([K_{g,n}(\mathcal{X}, \beta)]^v\). There is a naïve approximation for the degree of the top Chern class, using the usual Grothendieck–Riemann–Roch formula. However, it is well known that the usual Grothendieck–Riemann–Roch formula does not hold for Deligne–Mumford stacks; it is explained in Toen’s work [To] that one needs to introduce terms corresponding to pullback of the sheaf to the twisted sectors to correct the formula.

The issue can be understood if one considers a smooth marked twisted curve \(C\) with a representable map \(f : C \to X\). Let \(V = f^* T X\). If \(\pi : C \to C\) is the map to the coarse curve, then \(\chi(C,V) = \chi(C, \pi_* V)\), since \(\pi_*\) is an exact functor. Also, \(\deg_C \pi_* V = \deg_C \pi^* \pi_* V\). Thus the failure of Riemann–Roch is exactly the degree of the torsion sheaf \(V/\pi^* \pi_* V\) supported at the twisted markings.

A neighborhood of a twisted marking locally looks like \([\text{Spec } \mathbb{C}[[t]]/\mu_r]\). We describe the action as follows: we let \(\mu_r\) act on the tangent space of \(C\) via the fundamental character. We can choose the parameter \(t\) to be an eigenvector, and since it generates the cotangent space at \(x\), the group \(\mu_r\) acts on \(t\) via the inverse of the fundamental character:

\[t \mapsto \zeta_r^{-1} t.\]

The group \(\mu_r\) is identified via \(f\) with a subgroup \(H\) of the stabilizer \(G_x\) of a point of \(X\). Let the tangent space \(V_x\) of \(X\) at \(x\) have basis \((v_1, \ldots, v_n)\) consisting of \(\mu_r\)-eigenvectors. The degree of \(V/\pi^* \pi_* V\) is then exactly \(\frac{1}{d} \sum_i k_i\) where

\[k_i = \min\{l | v_i \cdot t^l \text{ is invariant}\}.\]

In other words, \(\mu_r\) acts on \(v_i\) via

\[v_i \mapsto \zeta_r^{k_i} v_i,\]

with \(0 \leq k_i < r\). Thus the contribution of a marking to the degree depends only on the component of \(X_1\) where it evaluates. This contribution \(\frac{1}{d} \sum_i k_i\) is known as the degree shifting number ([CR2]) or the age (Miles Reid’s terminology). If the connected components of \(X_1\) are \(X_1^i\), we denote this value by \(a(X_1^i)\).

As in [CR1], one defines a grading on the group \(A_{\text{st}}^*(\mathcal{X})\) as follows:

\[A_{\text{st}}^m(\mathcal{X}) = \bigoplus_i A^{m-a(X_1^i)}(X_1^i)\]
An argument similar to that of Chen and Ruan shows that this makes the stringy Chow ring with integer coefficients $A^m_{st}(\mathcal{X})$ into a $\mathbb{Z}$-graded ring.

7.2. Forgetting a marking. The Gromov–Witten invariants of projective varieties satisfy the so-called “divisor axiom”, which comes from the fact that $\overline{M}_{g,n+1}(X, \beta)$ is the universal curve over $\overline{M}_{g,n}(X, \beta)$. This still works for stacks when we restrict to the case where the $(n + 1)$-st marking is untwisted. In our setup there is no “forgetful map” for a twisted marking. There is, however, a way to change the formalism which allows for such forgetful maps. One way to do this entails

1. systematically breaking up $\overline{M}_{g,n}(X, \beta)$ according to the types of the markings, with components $\overline{M}_{g,n}(X, \beta, \tau)$, where $\tau \in \pi_0(X_1)^n$ is an $n$-tuple of connected components of $X_1$ indicating where each evaluation map $e_i$ lands;
2. for each index set $I \subset \{1, \ldots, n\}$ of twisted markings in $\overline{M}_{g,n}(X, \beta, \tau)$ of size $|I| = l$ introducing a new moduli stack $\overline{M}_{g,n-l}(X, \beta, \tau') = [\overline{M}_{g,n}(X, \beta, \tau)/S_l]$. The right hand side is the quotient of $\overline{M}_{g,n}(X, \beta, \tau)$ by the action of the symmetric group on $l$ letters $S_l$ acting on the indices of $I$. The point is that the only thing we can “forget” about a collection of twisted markings is their order. The notation $\tau'$ on the left denotes the image of $\tau$ in $\pi_0(X_1)^n/S_l$ obtained by forgetting the order of the indices in $I$.

At this point we do not know which formalism carries more computational or theoretical advantages.

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