Non-classical behaviour of coherent states for systems constructed using exceptional orthogonal polynomials

Scott E. Hoffmann\textsuperscript{1}, Véronique Hussin\textsuperscript{2}, Ian Marquette\textsuperscript{1} and Yao-Zhong Zhang\textsuperscript{3}

\textsuperscript{1}School of Mathematics and Physics, 
The University of Queensland, 
Brisbane, QLD, 4072, Australia

\textsuperscript{2}Département de Mathématiques et de Statistique, 
Université de Montréal, Montréal, Québec, H3C 3J7, Canada

We construct the coherent states and Schrödinger cat states associated with new types of ladder operators for a particular case of a rationally extended harmonic oscillator involving type III Hermite exceptional orthogonal polynomials. In addition to the coherent states of the annihilation operator, \( c \), we form the linearised version, \( \tilde{c} \), and obtain its coherent states. We find that while the coherent states defined as eigenvectors of the annihilation operator \( c \) display only quantum behaviour, those of the linearised version, \( \tilde{c} \), have position probability densities displaying distinct wavepackets oscillating and colliding in the potential. The collisions are certainly quantum, as interference fringes are produced, but the remaining evolution indicates a classical analogue.

*scott.hoffmann@uqconnect.edu.au
Coherent states are of widespread interest because they can, in some cases, be the quantum-mechanical states with the most classical behaviour. For the harmonic oscillator, the energy eigenvectors have stationary position probability densities. In contrast, the coherent state vectors, superpositions over all energy eigenvectors, are represented by wavepackets that oscillate back and forth in the $x^2$ potential, keeping the same Gaussian profile at all times. For the free electromagnetic field, the expectations of the field strength operators vanish in state vectors with a definite number of photons. In the coherent state superpositions over all photon number states, the field strengths have nonzero expectations that behave classically and obey the free Maxwell equations.

The coherent states of the isospectral (or almost isospectral) superpartners of the harmonic oscillator have been studied by several authors, with two classes of ladder operators having been used in the construction. The first class is the natural, of the form
\[ b = A a A^\dagger, \quad b^\dagger = A a^\dagger A^\dagger, \] (I.1)
where $A, A^\dagger$ are the supercharges and $a, a^\dagger$ are the ladder operators of the harmonic oscillator. The second class is the intrinsic class, that is a linearized version of the natural, with ladder operators of the form
\[ \tilde{b} = f(H^{-})b, \quad \tilde{b}^\dagger = b^\dagger f(H^{-}), \] (I.2)
where $f$ is chosen so that these ladder operators then satisfy the Heisenberg algebra, like the operators $a$ and $a^\dagger$. Here $H^{-}$ is the supersymmetric partner Hamiltonian of the harmonic oscillator Hamiltonian $H^{(+)}$. Note that a relation of the form of Eq. (I.2), with general function $f$, was recently used to define nonlinear supercoherent states.

Other types of ladder operators have been discussed in connection with Painlevé IV transcendent systems. Authors in [10] constructed a deformation of the harmonic oscillator that obeys a second order polynomial Heisenberg algebra and the Painlevé IV equation. They constructed the Barut-Girardello coherent states and obtained results for the energy expectations. A class of deformations of the harmonic oscillator called generalized isotonic oscillators have also been studied by many authors. In particular coherent states were constructed for natural and intrinsic ladder operators, using the Barut-Girardello and the displacement operator definitions, and non-classical behaviour was observed.

All of these systems are in fact part of more general families of deformations of the harmonic oscillator connected with the recently discovered Hermite exceptional orthogonal polynomials. In this work we will focus on the existence of new ladder operators, outside the natural and intrinsic classes, that connect the basis vectors and provide the infinite-dimensional representations of a polynomial Heisenberg algebra (no singlet ground state). These can be constructed via combinations of supercharges of Darboux-Crum and Krein-Adler type. These are not unique ladder operators connecting the basis states, in that they exist alongside the natural and intrinsic operators, since there is more than one path connecting the $(+)$ state vectors (the eigenvectors of $H^{(+)}$) and the $(-)$ state vectors (the eigenvectors of $H^{(-)}$).

The system considered in this paper is a deformation of the harmonic oscillator obtained with supersymmetric quantum mechanics (SUSY QM) using Hermite exceptional orthogonal polynomials (Hermite EOPs). The ground state wavefunction of the partner Hamiltonian, $H^{(-)}$, is
\[ \phi(x) = \left( \frac{8}{\sqrt{\pi}} \right)^{1/2} e^{-x^2/2} \mathcal{H}_2(x). \] (I.3)
Here the modified Hermite polynomials, real for all $m$ and positive definite for even $m$, are defined by
\[ \mathcal{H}_m(x) = (-i)^m H_m(ix). \] (I.4)
The first three are
\[ \mathcal{H}_0(x) = 1, \quad \mathcal{H}_1(x) = 2x, \quad \mathcal{H}_2(x) = 4x^2 + 2. \] (I.5)
This choice (Eq. (I.3)) then determines the partner potential and all other energy eigenvectors of the $(-)$ system. Note that this choice is the $m = 2$ case of a class of Hamiltonians using Hermite polynomials of order $m$ (even). Note also that $\mathcal{H}_2(x)$ is positive definite, so no singularity is introduced, and $\phi(x)$ is square-integrable. We will see that the partner potential for the $(-)$ system is
\[ V^{(-)}(x) = x^2 - 2 \left( \frac{\mathcal{H}_2''}{\mathcal{H}_2} - \left( \frac{\mathcal{H}_2'}{\mathcal{H}_2} \right)^2 + 1 \right) = x^2 + \frac{16(4x^2 - 2)}{(4x^2 + 2)^2} - 2, \] (I.6)
which displays (in Figure 1) a deep, narrow well around the origin. This feature will dominate the physics of the system.

The focus of this paper will be the new ladder operators constructed in [18], which we will denote by \( c \) and \( c^\dagger \). The action of the raising operator, \( c^\dagger \), is to increase the index of the state vector by 3. So this can be used to form three distinct, orthogonal ladders with the operators acting on three orthogonal, disconnected, subspaces. One of these ladders includes the ground state, given in Eq. (I.3). Note that for the ladder operators \( b, b^\dagger \) and \( \tilde{b}, \tilde{b}^\dagger \) mentioned above, the ground state is a singlet excluded from the ladders.

For comparison, we will also construct the operators \( \tilde{c} \) and \( \tilde{c}^\dagger \), which are linearized versions of \( c \) and \( c^\dagger \), respectively.

In this paper we will construct three distinct, orthogonal coherent states of the \( c \) operator, defined, for each complex value of \( z \), by

\[
| z, c(\mu) \rangle = z | z, c(\mu) \rangle.
\]

Similarly, we will construct the three sets of coherent states that are eigenvectors, with complex eigenvalue \( z \), of the \( \tilde{c} \) operator, defined by

\[
| z, \tilde{c}(\mu) \rangle = z | z, \tilde{c}(\mu) \rangle,
\]

for \( \mu = -3, 1, 2 \).

We will construct the coherent states associated with the new ladder operator, \( c \), and with the linearized version, \( \tilde{c} \), to investigate their physical properties, to compare between the two cases and to compare with the properties of the coherent states of the \( a \) operator.

This paper is organized as follow. In Section II we display the eigenfunctions and spectra of the partner Hamiltonian for our system. In Section III we construct two distinct types of ladder operator, the pair \( c, c^\dagger \) and the linearized pair \( \tilde{c}, \tilde{c}^\dagger \). In Section IV we construct the coherent states corresponding to two different annihilation operators for three cases each. In Section V we investigate the properties of these coherent states. We plot the expectations of total energy as functions of \( |z| \). Then we show the time-dependent position probability densities for particular choices of the parameter \( z \). Lastly, we show the time-dependent probability densities for even and odd Schrödinger cat states formed from our coherent states. In all of these cases, we compare with the familiar harmonic oscillator coherent states. Conclusions follow in Section VI.

## II. CONSTRUCTION AND SOLUTION OF THE PARTNER HAMILTONIANS

We use the scaling scheme \( \hbar \to 1, m_0 \to 1/2, \omega \to 2 \), where \( m_0 \) is the mass of the particle and \( \omega \) is the angular frequency of the oscillator. Then the harmonic oscillator Hamiltonian becomes

\[
H^{(0)} = -\frac{d^2}{dx^2} + x^2,
\]

with energy spectrum

\[
E^{(0)}_\nu = 2\nu + 1, \quad \nu = 0, 1, 2, \ldots
\]
and normalized position eigenfunctions
\[ \psi^{(0)}_{\nu}(x) = \frac{1}{\sqrt{\sqrt{\pi} \nu!}} e^{-x^2/2} H_{\nu}(x) \quad \text{for } \nu = 0, 1, 2, \ldots \] (II.3)

satisfying
\[ H^{(0)} \psi^{(0)}_{\nu}(x) = E^{(0)}_{\nu} \psi^{(0)}_{\nu}(x) \quad \text{for } \nu = 0, 1, 2, \ldots . \] (II.4)

The creation and annihilation operators are, respectively,
\[ a^\dagger = -\frac{d}{dx} + x, \quad a = +\frac{d}{dx} + x, \] (II.5)
satisfying the commutation relation
\[ [a, a^\dagger] = 2. \] (II.6)

So the Hamiltonian can be written
\[ H^{(0)} = a^\dagger a + 1. \] (II.7)

We summarize the SUSY QM construction of the partner Hamiltonians \( H^{(+)} \) and \( H^{(-)} \) and their solutions [21].

The choice (I.3) of the ground state leads to the supercharges
\[ A = +\frac{d}{dx} \frac{\phi'}{\phi} = +\frac{d}{dx} + x + \frac{\mathcal{H}_2}{\mathcal{H}_2}, \]
\[ A^\dagger = -\frac{d}{dx} \frac{\phi'}{\phi} = -\frac{d}{dx} + x + \frac{\mathcal{H}'}{\mathcal{H}_2}. \] (II.8)

Then the partner Hamiltonians are
\[ H^{(+)} = A^\dagger A = -\frac{d^2}{dx^2} + x^2 + 5, \]
\[ H^{(-)} = AA^\dagger = -\frac{d^2}{dx^2} + x^2 + \frac{16(4x^2 - 2)}{(4x^2 + 2)^2} + 3. \] (II.9)

The Hamiltonian \( H^{(+)} \) is just the harmonic oscillator Hamiltonian with a constant energy shift. So its spectrum is
\[ E^{(+)}_{\nu} = 2(\nu + 3) \quad \text{for } \nu = 0, 1, 2, \ldots \] (II.10)
and its energy eigenfunctions are given by (II.3), \( \psi^{(+)}_{\nu}(x) = \psi^{(0)}_{\nu}(x) \).

By supersymmetry the Hamiltonian \( H^{(-)} \) has a set of energy eigenvectors with the same energies as those of \( H^{(+)} \) and, in addition, a ground state with zero energy. So we can write the spectrum as
\[ E^{(-)}_{\nu} = 2(\nu + 3) \quad \text{for } \nu = -3, 0, 1, 2, \ldots \] (II.11)

The energy eigenfunctions of \( H^{(-)} \) are
\[ \psi^{(-)}_{\nu}(x) = \begin{cases} \sqrt{\frac{8}{\sqrt{\pi} \mathcal{H}_2(x)}} e^{-x^2/2} y_0(x) & \text{for } \nu = -3, \\ \frac{1}{\sqrt{\sqrt{\pi} 2^{\nu+1} (\nu+3)!}} e^{-x^2/2} y_{\nu+3}(x) & \text{for } \nu = 0, 1, 2, \ldots \end{cases} \] (II.12)

where
\[ y_0(x) = 1, \]
\[ y_{\nu+3}(x) = -\mathcal{H}_2(x) H_{\nu+1}(x) - 4\mathcal{H}_4(x) H_\nu(x) \quad \text{for } \nu = 0, 1, 2, \ldots \] (II.13)

These functions, \( y_k(x) \), are the Hermite exceptional orthogonal polynomials (Hermite EOPs).

Many of the models for which coherent states have been constructed and studied, such as the harmonic oscillator, the Morse potential [22], the Scarf potential [23] and the infinite well [24], admit the classical analogues of ladder operators and are exactly solvable in classical mechanics [25].
III. LADDER OPERATORS

For the three categories of annihilation operators in the set \( \{a, c, \tilde{c}\} \) and their Hermitian conjugates, we show the definitions and matrix elements of the ladder operators and their (polynomial) Heisenberg algebras. In the \( c \) and \( \tilde{c} \) cases we will find patterns of zero modes that will break the spaces into distinct, orthogonal, subspaces.

A. The ladder operators \( c \) and \( c^\dagger \)

We briefly review the properties of the \( a \) and \( a^\dagger \) operators, for comparison. The annihilation operator, \( a \), and the creation operator, \( a^\dagger \), defined by Eq. (II.5), are ladder operators connecting all of the eigenvectors \( |\nu(+)\rangle \) of \( H^{(+)} \) according to

\[
a|\nu(+)\rangle = \sqrt{2\nu}|\nu-1(+)\rangle, \quad a^\dagger|\nu(+)\rangle = \sqrt{2(\nu+1)}|\nu+1(+)\rangle, \quad \text{for} \ \nu = 0, 1, 2, \ldots
\]

(III.1)

With (from Equations (II.9), (II.11) and (II.7))

\[
H^{(+)} = a^\dagger a + 6,
\]

(III.2)

the ladder operators obey the Heisenberg algebra

\[
[H^{(+)}, a] = -2a, \quad [H^{(+)}, a^\dagger] = +2a^\dagger, \quad a^\dagger a = H^{(+)} - 6, \quad a a^\dagger = H^{(+)} - 4.
\]

(III.3)

For these eigenvectors, the position wavefunctions are \( \langle x | \nu(+)\rangle = \psi_\nu^{(+)}(x) = \psi_\nu^{(0)}(x) \), given by Eq. (II.3). For all the following cases they are \( \langle x | \nu(-)\rangle = \psi_\nu^{(-)}(x) \), given by Eq. (II.12).

In [18] new ladder operators are obtained, not of the natural or intrinsic classes, for the \( H^{(-)} \) eigenvectors. These include nonzero matrix elements between the ground state and the first excited state. This is possible because more than one path (the other being the path for \( b \) and \( b^\dagger \)) can be constructed connecting the \( H^{(+)} \) and \( H^{(-)} \) systems. This, in turn, is possible because of properties of the wavefunctions for the \( H^{(+)} \) system, which are type III Hermite exceptional orthogonal polynomials.

They define first-order supercharges using Darboux-Crum and Krein-Adler terms [18],

\[
A_1 = + \frac{d}{dx} x + \frac{\mathcal{H}_0}{\mathcal{H}_1} - \frac{\mathcal{H}_1'}{\mathcal{H}_1}, \quad A_1^\dagger = - \frac{d}{dx} x + \frac{\mathcal{H}_0'}{\mathcal{H}_0} - \frac{\mathcal{H}_1'}{\mathcal{H}_1},
\]

\[
A_2 = + \frac{d}{dx} x + \frac{\mathcal{H}_1}{\mathcal{H}_1} - \frac{\mathcal{H}_2'}{\mathcal{H}_2}, \quad A_2^\dagger = - \frac{d}{dx} x + \frac{\mathcal{H}_1'}{\mathcal{H}_1} - \frac{\mathcal{H}_2'}{\mathcal{H}_2}.
\]

(III.4)

Then the product \( A_2 A_1 \) maps from the \( H^{(+)} \) eigenvectors to the \( H^{(-)} \) eigenvectors in 2 steps and the operator

\[
c = A_2 A_1 A_1^\dagger
\]

(III.5)

acts as a lowering operator (in steps of 3) in the \( H^{(-)} \) system. Likewise

\[
c^\dagger = A A_1^\dagger A_2^\dagger
\]

(III.6)

acts as a raising operator in the \( H^{(-)} \) system. The operators \( c, c^\dagger \) obey the cubic polynomial Heisenberg algebra

\[
[H^{(-)}, c] = -6c, \quad [H^{(-)}, c^\dagger] = +6c^\dagger, \quad [c, c^\dagger] = Q(H^{(-)} + 6) - Q(H^{(-)}),
\]

(III.7)

where

\[
Q(x) = x(x-8)(x-10).
\]

(III.8)

Their actions on the eigenvectors \( |\nu(-)\rangle \) of \( H^{(-)} \) are given by

\[
c |\nu(-)\rangle = -[2(\nu-1)2(\nu-2)2(\nu+3)]^\dagger |\nu-3(-)\rangle,
\]

\[
c^\dagger |\nu(-)\rangle = -[2(\nu+2)2(\nu+1)2(\nu+6)]^\dagger |\nu+3(-)\rangle.
\]

(III.9)
As noted above, there are nonzero matrix elements of the form
\[
\langle 0 (-) | c^\dagger | -3 (-) \rangle \neq 0. \tag{III.10}
\]

Note that
\[
c | \mu (-) \rangle = 0 \quad \text{for} \quad \mu = -3, 1, 2. \tag{III.11}
\]
So there are three independent, mutually orthogonal, ladder subspaces in this system, with lowest weights \( \mu = -3, 1, 2 \).
In the next section we will thus construct three independent and mutually orthogonal coherent states (for each complex value of \( z \)).

**B. The ladder operators \( \tilde{c} \) and \( \tilde{c}^\dagger \)**

As discussed earlier, we linearize the operators \( c \) and \( c^\dagger \) to produce \( \tilde{c} \) and \( \tilde{c}^\dagger \). First we relabel the basis vectors:

\[
| \kappa, \mu \rangle \equiv | \mu + 3 \kappa (-) \rangle \quad \text{for} \quad \kappa = 0, 1, 2, \ldots \tag{III.12}
\]

for each of the three ladders specified by \( \mu = -3, 1, 2 \). (In what follows, we deal only with (-) state vectors. The label (-) is to be understood for these state vectors.) We want to treat each ladder like a harmonic oscillator ladder, so we define

\[
\tilde{c} | \kappa, \mu \rangle = -\frac{\sqrt{2\kappa}}{\sqrt{8(\mu + 3\kappa - 1)(\mu + 3\kappa - 2)(\mu + 3\kappa + 3)}} c | \kappa, \mu \rangle
\]
\[
= \sqrt{2\kappa} | \kappa - 1, \mu \rangle \quad \text{for} \quad \mu \neq -3, 1, 2 \tag{III.13}
\]

and

\[
\tilde{c} | \kappa, \mu \rangle = 0 \quad \text{for} \quad \mu = -3, 1, 2. \tag{III.14}
\]

As an operator expression, this is

\[
\tilde{c} = -\frac{1}{[(H(-) - 2)(H(-) - 4)(H(-) + 6)]^{3/2}} \sqrt{\frac{H(-) - 2\mu}{3}} c. \tag{III.15}
\]

The algebra is just the modified Heisenberg algebra

\[
[H(-), \tilde{c}] = -6\tilde{c}, \quad [H(-), \tilde{c}^\dagger] = +6\tilde{c}^\dagger, \quad \tilde{c}\tilde{c}^\dagger = \frac{1}{3}(H(-) - 6 - 2\mu) + 2, \quad \tilde{c}^\dagger\tilde{c} = \frac{1}{3}(H(-) - 6 - 2\mu). \tag{III.16}
\]

**IV. CONSTRUCTION OF THE COHERENT STATES**

This section is devoted to constructing the appropriate coherent states for the various ladder operators of Section III. Once these are constructed, we will examine their physical properties, looking for classical or non-classical behaviour, in Section V.

Let us point out that ladder operators of the form \( a^3, (a^\dagger)^3 \) have been investigated \[26\] for a particular case of a model related to the fourth Painlevé transcendent \[20\]. It is believed that our work is the first to treat the coherent states of the \( c, c^\dagger \) operators.

There are more general ladder operators of the form \( c_m, c_m^\dagger \) for \( m \) even and greater than or equal to 2 (as well as the trivial case \( m = 0 \)) \[18, 19\]. These lead to models involving EOPs \[20\] and rational solutions of the fourth Painlevé transcendent related to the generalized Okamoto and Hermite polynomials \[19\]. Our \( c, c^\dagger \) operators correspond to the case \( m = 2 \).

If \( \alpha \) is a lowering operator, one of \( \{a, c, \tilde{c}\} \), the coherent state \( |z, \alpha\rangle \), for \( z \) any complex number, is defined as the normalized solution of

\[
\alpha | z, \alpha \rangle = z | z, \alpha \rangle. \tag{IV.1}
\]
This is called the Barut-Girardello definition of coherent states \([5]\). There are other definitions involving, for example, the displacement operator \([3]\).

The general method of solving such an equation is to first write the coherent state as a superposition over the appropriate part of the spectrum of energy (number) eigenvectors. Then Eq. \(\text{[IV.4]}\) gives a recursion relation which can be solved to give all the coefficients proportional to the lowest weight coefficient. Finally the state vector is normalized to unity.

For the \(a\) operator acting on the eigenvectors of \(H^{(+)}\), the solution is well known to be

\[
|z, a\rangle = e^{-|z|^2/4} \sum_{\nu=0}^{\infty} |\nu (+)\rangle \frac{(z/\sqrt{2})^\nu}{\sqrt{\nu!}}, \quad \text{(IV.2)}
\]

equation a superposition over the entire spectrum.

For the \(c\) operator acting on the \(H^{(-)}\) system, there are three orthogonal coherent states (three subspaces), for each complex value of \(z\), with lowest weights \(\mu = -3, 1, 2\), given by

\[
|z, c(\mu)\rangle = \frac{1}{\sqrt{F^{(\mu)}(z)}} \sum_{k=0}^{\infty} |\mu + 3k\rangle \frac{z^k}{D_k^{(\mu)}}, \quad \text{[IV.3]}
\]

with

\[
F^{(\mu)}(z) = \frac{1}{1} F_3(1; \frac{\mu + 2}{3}, \frac{\mu + 1}{3}, \frac{\mu + 6}{3}; |z|^2/216) \quad \text{[IV.4]}
\]

and

\[
D_k^{(\mu)} = (-)^k 6^{3k/2} |(\frac{\mu + 2}{3})_k (\frac{\mu + 1}{3})_k (\frac{\mu + 6}{3})_k| \Gamma(\frac{3}{2}) \]  
\[
\text{[IV.5]}
\]

(In Eq. \(\text{[IV.3]}\) and in what follows, the \((-)\) label is to be understood on the state vectors.) So

\[
|z, c(\mu)\rangle = \frac{1}{\sqrt{1} F_3(1; \frac{\mu + 2}{3}, \frac{\mu + 1}{3}, \frac{\mu + 6}{3}; |z|^2/216)} \sum_{k=0}^{\infty} |\mu + 3k\rangle (-1)^k \frac{(z/\sqrt{216})^k}{((\frac{\mu + 2}{3})_k (\frac{\mu + 1}{3})_k (\frac{\mu + 6}{3})_k)!} \quad \text{[IV.6]}
\]

In this expression (see \([27]\))

\[
(a)_0 = 1, \quad (a)_k = a(a + 1) \ldots (a + k - 1) = \frac{\Gamma(a + k)}{\Gamma(a)} \quad \text{[IV.7]}
\]

and the generalized hypergeometric functions are defined by

\[
_p F_q(\alpha_1, \ldots, \alpha_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{z^n (\alpha_1)_n \ldots (\alpha_p)_n}{n! (b_1)_n \ldots (b_q)_n} \quad \text{[IV.8]}
\]

For the \(\tilde{c}\) operator acting on the \(H^{(-)}\) system, the three coherent states for each complex value of \(z\) are

\[
|z, \tilde{c}(\mu)\rangle = e^{-|z|^2/4} \sum_{k=0}^{\infty} |\mu + 3k\rangle \frac{(z/\sqrt{2})^k}{\sqrt{k!}} \quad \text{for } \mu = -3, 1, 2. \quad \text{[IV.9]}
\]

V. PROPERTIES OF THE COHERENT STATES

In what follows we consider the properties of the coherent states, \(|z, \alpha\rangle\), denoted by \(\alpha = a, c(\mu), \tilde{c}(\mu)\), for \(\mu = -3, 1, 2\). First we find the expectation value of the total energy in each coherent state as a function of the magnitude, \(|z|\), of the coherent state parameter. Then we calculate and graph the time-dependent position probability densities for each coherent state over one period. Finally we construct even and odd Schrödinger cat states for each of the annihilation operators and plot their position probability densities over one cycle.
Figure 2. Energy expectations as functions of $|z|$ for (a) $a$, (b) $c(\mu)$, (c) $\tilde{c}(\mu)$.

A. Energy expectations

The expectation of energy $H^{(+)}$ in a coherent state of the operator $a$, as a function of the coherent state parameter $z$, is given by

$$\langle z, a | H^{(+)} | z, a \rangle = \sum_{\nu=0}^{\infty} \langle z, a | \nu \rangle (2 \nu + 6) \langle \nu | z, a \rangle = 6 + |z|^2,$$

(V.1)

using the completeness of the energy eigenvectors, the fact that $a^\dagger a$ is diagonal in that basis with eigenvalue $2\nu$ and the defining relation (IV.1) for the coherent states. This is a familiar result.

We anticipate that the corresponding expressions for the energy expectations for our other coherent states will differ from this expression. So we use the energy expectation as one way to characterize the differences between our ladder operators.

Since the energy eigenvalues in all (-) cases are given by Eq. (II.11), we find the expectation of energy as a function of $z$ for $\alpha = c(\mu), \tilde{c}(\mu)$ to be

$$\langle z, \alpha | H^{(-)} | z, \alpha \rangle = 6 + 2 \sum_{\nu} |\langle \nu | z, \alpha \rangle|^2 \nu.$$

(V.2)

The sum over $\nu$ takes different forms in the different cases. Note that the results do not depend on the position wavefunctions of the (-) system.

For $c(\mu)$, $\mu = -3, 1, 2$, this is

$$\langle z, c(\mu) | H^{(-)} | z, c(\mu) \rangle = 6 + 2\mu + \frac{1}{(\mu + 2)(\mu + 1)(\mu + 6)} \frac{3|z|^2}{4} \frac{1}{3} \frac{3}{1} F_3 (2, \frac{\mu+2}{3}, \frac{\mu+1}{3}, \frac{\mu+6}{3}; \frac{|z|^2}{216}),$$

(V.3)

using Equations (IV.9, IV.7, IV.8).

Finally, for $\tilde{c}(\mu)$, $\mu = -3, 1, 2$, this is

$$\langle z, \tilde{c}(\mu) | H^{(-)} | z, \tilde{c}(\mu) \rangle = 6 + 2 \sum_{k=0}^{\infty} e^{-|z|^2/2} \frac{(|z|^2/2)^k}{k!} (\mu + 3k)$$

(V.4)

$$= 6 + 2\mu + 3|z|^2,$$

(V.5)

using Equations (IV.9, IV.7, IV.8). The factor of 3 multiplying $|z|^2$, compared to Eq. (V.1), is understandable as the energy levels involved increase in threes.

The results are plotted in Figure 2. We note that the energy expectation rises far more slowly for the $c(\mu)$ cases than for the cases that have been constructed to be like the harmonic oscillator.
B. Time-dependent position probability densities for the coherent states

The time-dependent position probability density for the \((+)\) system is

\[
\rho(x, t; z, a) = \left| \langle x | e^{-iHt} | z, a \rangle \right|^2 = \sum_{\nu=0}^{\infty} \psi_{\nu}(|+\rangle(x) e^{-i(6+2\nu)t} (\nu \langle + | z, a \rangle \left| \nu \right|)^2 = \sum_{\nu=0}^{\infty} \psi_{\nu}(x) e^{-|x|^2/4} \left( z e^{-i2t/\sqrt{2}} \right)^\nu \left( \frac{\mu^{+2}}{3} \right) \left( \frac{\mu^{+1}}{3} \right) \left( \frac{\mu^{+6}}{3} \right) \frac{1}{\sqrt{\nu!}} \left[ \frac{\nu!}{\mu^{+3k}} \right] \left( \frac{\nu!}{\mu^{+3k}} \right) \frac{1}{\sqrt{k!}} \bigg|^{2}, \tag{V.6}
\]

It can be shown \cite{13} that this expression reduces to

\[
\rho(x, t; z, a) = \frac{1}{\sqrt{\pi}} e^{-(x-z \cos 2t)^2} \tag{V.7}
\]

for \(z\) real.

For the various cases in the \((-)\) system, we have

\[
\rho(x, t; z, c(\mu)) = \sum_{k=0}^{\infty} \psi_{\mu+3k}(x) e^{-|x|^2/4} \left( z e^{-i2t/\sqrt{2}} \right)^k \left( \frac{\mu^{+2}}{3} \right) \left( \frac{\mu^{+1}}{3} \right) \left( \frac{\mu^{+6}}{3} \right) \frac{1}{\sqrt{k!}} \left[ \frac{\nu!}{\mu^{+3k}} \right] \left( \frac{\nu!}{\mu^{+3k}} \right) \frac{1}{\sqrt{k!}} \bigg|^{2}, \tag{V.8}
\]

\[
\rho(x, t; z, \tilde{c}(\mu)) = \sum_{k=0}^{\infty} \psi_{\mu+3k}(x) e^{-|x|^2/4} \left( z e^{-i2t/\sqrt{2}} \right)^k \left( \frac{\mu^{+2}}{3} \right) \left( \frac{\mu^{+1}}{3} \right) \left( \frac{\mu^{+6}}{3} \right) \frac{1}{\sqrt{k!}} \left[ \frac{\nu!}{\mu^{+3k}} \right] \left( \frac{\nu!}{\mu^{+3k}} \right) \frac{1}{\sqrt{k!}} \bigg|^{2}. \tag{V.9}
\]

All of these densities are periodic in time, with period \(\pi\) for \(\alpha = a\) and \(\pi/3\) for \(\alpha = c(\mu), \tilde{c}(\mu)\).

We choose to compare all three of the coherent state categories at the value \(z = 15\). We will see in the next subsection that this value gives interesting behaviour of the Schrödinger cat states.

The results are shown in Figures 3, 4 and 5. In Figure 3 (a) and Figure 5 (a), the \(a\) density takes the basic form that is identified as analogous to classical, with a single wavepacket, like a classical particle, undergoing oscillatory motion in the potential while retaining the same width.

The case \(c(-3)\) (Figure 3 (b)) was the least classical of the three. We see several, comparatively indistinct, wavepackets colliding and oscillating in the potential in our time-dependent simulations.

For the linearised case, \(\tilde{c}(-3)\), in Figure 4, we see three wavepackets, distinct until they collide, when interference fringes are formed (visible as the blackness in Figure 4 (a).) This is the closest to classical behaviour, other than the \(a\) case, that we observe. The cases \(\mu = 1, 2\) gave very similar results to Figure 4 so are not shown.

Note that the special case \(z = 0\), in all cases, gives the classical but trivial result of the time-independent probability density of the lowest weight state.

In Figure 5 we show the time dependence of the position probability densities for the three cases, all for \(z = 15\), confirming that the \(\tilde{c}(-3)\) case displays more classical behaviour than the \(c(-3)\) case. Note in Figure 5 (c) the effect of the narrow, deep part of the potential on the wavepacket.
Figure 4. Position probability densities for $\tilde{c}(-3)$ for (a) $t = 0$, (b) $t = \pi/24$ and (c) $t = \pi/12$ for $|z| = 15$ in all cases.

Figure 5. Time-dependent position probability densities for (a) $a$ (the harmonic oscillator), (b) $c(-3)$ and (c) $\tilde{c}(-3)$, all for $z = 15$.

In Figure 6 we show the probability density at $t = 0$ for $c(-3)$ as a function of $z$. As just noted, the case $z = 0$ is trivially classical, while for increasing $z$ we do not see the well-separated wavepackets of the case $\tilde{c}(-3)$.

C. Time-dependent position probability densities for the even and odd cat states

A Schrödinger cat state is a superposition of two macroscopically distinct states, and is studied in connection with the measurement problem of quantum mechanics [28]. In our case we take the two states as $|+z, \alpha\rangle$ and $|-z, \alpha\rangle$, for each case of $\alpha$. We form two normalised superpositions, one symmetric, the other antisymmetric under the exchange of the two state vectors:

$$|z, \alpha, \pm\rangle = \frac{1}{\sqrt{2}} \{ |+z, \alpha\rangle \pm |-z, \alpha\rangle \} / \sqrt{1 + D(|z|, \alpha)}.$$  \hspace{1cm} (V.10)

We calculated the real quantity

$$D(|z|, \alpha) = \langle +z, \alpha | -z, \alpha \rangle$$  \hspace{1cm} (V.11)

Figure 6. Position probability density at $t = 0$ for $c(-3)$ as a function of $z$. 
Figure 7. The real overlap $D(|z|, \alpha) = \langle +z, \alpha | -z, \alpha \rangle$ for (a) $\alpha = a, \tilde{c}(\mu)$ and (b) for $\alpha = c(\mu)$.

for all of our cases and found

$$D(|z|, \alpha) = e^{-|z|^2} \quad \text{for } \alpha = a, \tilde{c}(\mu),$$

$$D(|z|, c(\mu)) = \frac{\text{$_1F_3$(1; $\mu + 2$, $\mu + 1$, $\mu + 6$; $-|z|^2/216$)}}{\text{$_1F_3$(1; $\mu + 2$, $\mu + 1$, $\mu + 6$; $|z|^2/216$)}}. \quad (V.13)$$

These are plotted in Figure 7. The condition for the two states to be macroscopically distinguishable is

$$D(|z|, \alpha) \ll 1.$$

However, we would like to compare all of the cat states at the same value of $z$, so we choose $z = 15$, which certainly gives distinguishability for the $a$ and $\tilde{c}(\mu)$ cases. (We also considered $z = 100$ for $c(\mu)$, with that result to follow.)

The cat state results are shown in Figures 8-10.

For the $a$ harmonic oscillator case, the characteristic pattern shows two wavepackets oscillating $\pi$ out of phase and interfering at the origin. Note that the period of a cat state will be half the period of one of its coherent states. The even cat state is a superposition only over even values of the index $k$, while the odd state contains only odd $k$. The position wavefunctions are proportional to the Hermite polynomials $H_k(x)$, which are even for even $k$, odd for odd $k$. Thus we see a nodal line at $x = 0$ for the odd cat state.

The $\tilde{c}(-3)$ pattern is, like the densities of Figure 4, the nearest we see to classical behaviour (other than, of course, the harmonic oscillator case). We see 6 wavepackets at any time, except when they collide.

For the $c(-3)$ case, we see patterns distinctly different from the earlier cases. We see a nodal line at $x = 0$ for the odd case, Figure 10 (b). In all cases, a nodal line at $x = 0$ occurs when the appropriate position wavefunction is an odd function of $x$. The $\psi^{(-)}_{-3+31}(x)$ are even for even $k$.

Neither case here can be said to have a classical appearance.

For $c(\mu)$, the cases $\mu = 1, 2$, even and odd, gave similar patterns to those seen for $\mu = -3$, with central nodal lines in the appropriate places, so are not shown.

As mentioned earlier, we graphed the case $c(-3)$, $z = 100$. We saw patterns of many peaks and valleys, distinctly non-classical.
VI. CONCLUSIONS

The goal of this paper was to investigate certain properties of the recently constructed ladder operators, $c$ and $c^{\dagger}$, for a rationally deformed harmonic oscillator system. This investigation was to be done by comparing results for coherent states constructed as eigenstates of the annihilation operator, $c$, and the more well-known operator $a$ (for the harmonic oscillator). We also constructed linearized versions of these operators, $\tilde{c}$ and $\tilde{c}^{\dagger}$, and investigated the same properties for them. We constructed coherent states $|z, c(\mu)\rangle$ with lowest weights $\mu = -3, 1, 2$ and corresponding coherent states for the linearized operator, $\tilde{c}$, $|z, \tilde{c}(\mu)\rangle$. These were compared with the coherent states $|z, a\rangle$.

First we plotted the position probability densities at time $t = 0$ for the same value of the coherent state parameter, $z$, for the various coherent states. The coherent states of the harmonic oscillator (of the annihilation operator $a$), as is well known, behave in a very classical way, with a single wavepacket keeping its shape as it oscillates in the potential. We found that the coherent states labelled $c(\mu)$, of the linearised operator, display three well-defined wavepackets oscillating and colliding in the potential. Certainly at the collision points, when interference fringes are produced, the behaviour is quantum. But for large sections of time, the wavepackets are separate, mimicking a classical system. The coherent states labelled $c(\mu)$, however, show no similar classical regime up to $z = 100$, other than the trivial case $z = 0$ of a stationary position probability distribution.

The even and odd Schrödinger cat states were constructed for each of the cases and the position probability densities plotted. This confirmed our conclusion that the operator $c$ leads only to purely quantum behaviour, with no classical analogue, while the linearised operator, $\tilde{c}$, has coherent states that can be said to have a classical analogue.

We noted that our deformed system is the case $m = 2$ of an infinite class characterized by even $m \geq 0$ [18]. In future work we will investigate coherent states for higher values of $m$.

We also noted that the definition we used for a coherent state as the complex eigenstate of an annihilation operator...
is just one of several different ways to define coherent states in such systems. These different ways all reduce to the same result for the harmonic oscillator, but give different results in more general cases. In a future work we will investigate the Perelomov definition of coherent states [6].

ACKNOWLEDGMENTS

IM and YZZ were supported by Australian Research Council Discovery Projects DP160101376 and DP140101492, respectively. VH acknowledges the support of research grants from NSERC of Canada. SH receives financial support from a UQ Research Scholarship.

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