Exotic Smoothness and Physics

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March 29, 1994

Abstract

The essential role played by differentiable structures in physics is reviewed in light of recent mathematical discoveries that topologically trivial space-time models, especially the simplest one, $\mathbb{R}^4$, possess a rich multiplicity of such structures, no two of which are diffeomorphic to each other and thus to the standard one. This means that physics has available to it a new panoply of structures available for space-time models. These can be thought of as source of new global, but not properly topological, features. This paper reviews some background differential topology together with a discussion of the role which a differentiable structure necessarily plays in the statement of any physical theory, recalling that diffeomorphisms are at the heart of the principle of general relativity. Some of the history of the discovery of exotic, i.e., non-standard, differentiable structures is reviewed. Some new results suggesting the spatial localization of such exotic structures are described and speculations are made on the possible opportunities that such structures present for the further development of physical theories.

To be published in Journal of Mathematical Physics.
1 Introduction

Almost all modern physical theories are stated in terms of fields over a smooth manifold, typically involving differential equations for these fields. Obviously the mechanism underlying the notion of differentiation in a space-time model is indispensable for physics. However, the essentially local nature of differentiation has led us to assume that the mathematics underlying calculus as applied to physics is trivial and unique. In a certain sense this is true: all smooth manifolds of the same dimension are locally differentiably equivalent, and thus physically indistinguishable in the local sense. However, only fairly recently have we learned that this need not be true in the global sense. Mathematicians have discovered that certain common, even topologically trivial, manifolds may not be globally smoothly equivalent (diffeomorphic) even though they are globally topologically equivalent (homeomorphic). This is a strikingly counter-intuitive result which presents new, non-topological, global possibilities for space-time models. The fact that the dimension four plays a central role in this story only enhances the speculation that these results may be of some physical significance.

Briefly, until some twelve years ago it was known that any smooth manifold that was topologically equivalent to $\mathbb{R}^n$ for $n \neq 4$ must necessarily be differentiably equivalent to it. It is probably safe to say most workers expected that the outstanding case of $n = 4$ would be resolved the same way when sufficient technology was developed for the proof. Thus, it was quite a shock when the work of Freedman and Donaldson showed that there are an infinity of differentiable structures on topological $\mathbb{R}^4$, no two of which are equivalent, i.e., diffeomorphic, to each other. The surprising feature of this result is the fact that while each of these manifolds is locally smoothly equivalent to any other, this local equivalence cannot be continued to a global one, even though the manifold model, $\mathbb{R}^4$, is clearly topologically trivial. Among this infinity of smooth manifolds homeomorphic to $\mathbb{R}^4$ is the standard one, with smoothness inherited simply from the smooth product structure $\mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1$. This standard one we denote by the same symbol as we have used for the topological model, $\mathbb{R}^4$. The other smooth manifolds are called variously “fake,” “non-standard,” or “exotic.” The last is the term used in this paper and such a manifold is denoted by $\mathbb{R}^4_{\Theta}$.

The fact that $\mathbb{R}^4_{\Theta}$’s arise only in the physically significant case of dimension four makes the result even more intriguing to physicists. Unfortunately, however, these proofs have been more existential than constructive in the sense of a practical coordinate patch presentation. This lack makes progress in the field difficult. For example, no global expression can be obtained for any non-scalar field, such as the metric. However, certain existence, and non-existence, results can be obtained and some of these will be reviewed in this paper.

The intent of this paper is to provide a review of the mathematical technology necessary to grasp the significance of some of these results and to point out their possible implications for physics. In doing so, we draw extensively on two earlier papers, [3],[4]. We begin with a review of some basic differential topology and discuss its relevance to the expression of physical theories. In particular, we recall the notions of topological and smooth manifolds and the important difference between them. This difference is often overlooked and the smoothness structure is simply assumed to be some “standard” one naturally associated with the topological structure. Along the way, the concept of a differentiable structure will be defined, involving first the notion of an atlas and then the equivalence class under diffeomorphisms. The distinction between “different” and “non-diffeomorphic” differentiable structures is so important that it will be exemplified in terms of simple real models and then a toy model in which an explicit “exotic” complex structure is explored as an alternative to the standard one for two-dimensional vacuum Maxwell’s equations.

Next, a brief review of the discovery of exotic smoothness structures will be given, starting with the explicit case of Milnor seven-spheres. After this, the somewhat tortuous path leading to the discovery of $\mathbb{R}^4_{\Theta}$’s is reviewed. This story is much more involved and calls upon a much wider variety of mathematical fields than that of the Milnor spheres. Finally, the development of tools enabling the definition of a candidate for “spatially localized” exotic structures will be described and discussed. We close with some conjectures and speculations.
2 Manifolds: Topological and Smooth

Our primitive construct for modelling space-time was for a long time the standard Euclidean manifold, \( \mathbb{R}^4 \), a special case of the standard manifold, \( \mathbb{R}^n \). This latter is defined as the set of points, \( p \), each of which can be identified with an \( n \)-tuple of real numbers, \( \{ p^\alpha \} \). The topology is induced by the usual product topology of the real line. What is important for our discussion is that this standard model implicitly imparts to each space-time point an identifiable, objective, existence, independent of any choice of coordinatization. In fact, this topological notion is prior to the existence of any coordinate.

A thorough understanding of this fact is absolutely necessary to follow the intricacies involved with the definition of differentiable structures. Historically, of course, Einstein and others simply identified the list \( \{ p^\alpha \} \) as a list of some physical coordinates thereby choosing what we will call the standard differentiable structure. By choosing this standard \( \mathbb{R}^4 \), we limit physics to the trivial Euclidean topology, eliminating the rich and potentially very useful set of physical possibilities inherent in non-trivial topologies. Also, and more importantly for the purposes of this paper, this choice also precludes investigation of the myriad possibilities opened up by the recent discoveries in differential topology. Consequently, we now proceed to review some of the basic tools of this field.

First, recall the notion of a topological manifold. A topological space, \( M \), is a topological manifold if it is locally homeomorphic to \( \mathbb{R}^n \). Thus, \( M \) must be covered by a family of open sets, \( \{ U_a \} \), each of which is homeomorphic to \( \mathbb{R}^n \). Such a family is called a (topological) atlas, \( \mathcal{A} = \{ U_a, x^a_i \} \), where the \( x^a_i \) are the local \( \mathbb{R}^n \) homeomorphisms. Each \( (U_a, x^a_i) \) in the atlas thus serves as topological coordinate patch, or chart, locally identifying points with ordered sets of \( n \) real numbers in a topological manner. Unlike the case of smooth manifolds defined below, note that no further assumptions need be made concerning the transition functions between two overlapping patches. They will necessarily be local homeomorphisms in standard \( \mathbb{R}^n \). Also note that all manifolds used in this paper are assumed to be Hausdorff.

To this point, no notion of differentiation has been introduced. Thus, if \( M \) is a topological manifold with elements (points) \( p \), and \( f \) is a real-valued map, \( f : M \to \mathbb{R} \), there is not yet any structure available to do calculus with \( f \). Of course, the existence of the local charts identifying points, \( p \), with \( n \)-tuples, \( x^a_i \), of real numbers would seem to provide a clue. For example, we can re-express \( f \) locally as \( f_a : x_a(U_a) \to \mathbb{R} \) given by

\[
    f_a(x^i) = f(p), \quad \text{where} \quad x^i = x^a_i(p). \tag{1}
\]

We can then proceed to do calculus on the invariantly defined \( f \) in terms of the well-known, standard, real variable calculus, applied to its coordinate patch form, \( f_a \). Using this representation, \( f \) will be smooth, or differentiable in the \( C^\infty \) sense if and only if \( f_a \) is \( C^\infty \) in the standard real variable sense. However, to be useful, this definition should be independent of the local chart, \( \{ U_a, x^a_i \} \) used in its definition. The condition for this consistency is contained in the definition of a smooth atlas of charts. Given a topological manifold, \( M \), a topological atlas \( \mathcal{A} = \{ U_a, x^a_i \} \) covering \( M \) is said to be a smooth atlas if for every pair for which \( W = U_a \cap U_b \) is not empty, the local homeomorphism of \( \mathbb{R}^n \) defined by \( x_a x_b^{-1} \) is smooth in the standard sense in \( \mathbb{R}^n \). This forms the basis for differential topology and seems to contain the minimal structure needed to do physics expressed in differential equations.

Clearly any given \( M \) could conceivably support many different smooth atlases. Two such, say \( \mathcal{A} \) and \( \mathcal{A}' \), are said to be compatible if their union is also a smooth atlas. This leads naturally to a set-theoretic type of ordering and the notion of a maximal atlas. Using this we now define a central notion in differential topology. A differentiable structure, \( \mathcal{D}(M) \), is a maximal smooth atlas on \( M \), and defines \( M \) as a smooth, or differentiable, manifold. Clearly any atlas in a maximal one defines the differentiable structure, so the maximalization procedure is often not explicitly carried out. It turns out that most topological manifolds can support more than one \( \mathcal{D} \), so there are many smooth manifolds over any given topological one. A natural question to ask, and certainly an important one for physics, is whether

\footnote{This “objective reality” assigned to points in space-time models is, of course, strongly reminiscent of the old pre-relativity ether models. However, we are not concerned with such questions here. Many people have raised this issue. Some thoughts on it are contained in my article, “Roles of Space-Time Models,” p27, in Quantum Theory and Gravitation, ed. A.R. Marlow, Academic Press, 1980.}
or not two different smooth manifolds with the same topology are really equivalent in an appropriate differentiable sense.

This question is answered in terms of the notion of a diffeomorphism. First, a map, \( f : M \rightarrow M' \) of one smooth manifold into another is said to be differentiable if it is smooth in the standard \( \mathbb{R}^n \) sense when expressed in the local smooth charts. \( f \) is a diffeomorphism if it is a differentiable homeomorphism. From the viewpoint of differential topology two smooth manifolds are equivalent if they are diffeomorphic. The same is true for physics, since a diffeomorphism is the mathematical embodiment of the notion of a global re-coordinatization of the manifold. Thus, the principle of general relativity demands that the physics of two different, but diffeomorphic, manifolds be regarded as identical. However, is the differentiable structure determined by the topology? That is,

**Fundamental Question:** Can two homeomorphic manifolds support truly different, i.e., non-diffeomorphic, differentiable structures?

If so, these manifolds describe space-time models which while identical both in the global topological and local smooth senses are not physically equivalent.

### 3 Simple Examples and a Metaphor

This question opens the door to new possibilities, other than topological, for global structures and their effects to occur in physical space-time models. However, the difference between different differentiable structures, and non-diffeomorphic ones, is a somewhat elusive concept, so in this section some simple examples will be considered. Also, a toy model involving the more easily managed complex structures on the plane will be explored. This allows us to look explicitly at the distinction between different complex structures and those which produce manifolds which are not biholomorphic to each other.

First, consider the problem of establishing a differentiable structure on the simple topological manifold, \( \mathbb{R}^1 \). Recall the distinctive notation to identify the “objectively existing” points of this space, \( \mathbb{R}^1 = \{p\} \) where, in this case each point, \( p \), is simply one real number. A natural maximal atlas and thus a differentiable structure, \( D \), is generated simply by the one global chart with coordinate, \( x(p) = p \). This is the standard differentiable structure for \( \mathbb{R}^1 \). In this differentiable structure, a function of \( p \) is smooth if and only if it is a smooth function of \( p \) in the usual sense. However, there are many other possible differentiable structures. For example, one such is generated by the same global chart, but with coordinate \( y(p) = p^3 \). Clearly this leads to a different differentiable structure, say \( \mathcal{D} \), since the combination \( y \cdot x^{-1} : p \rightarrow p^3 \), which is not smooth. So, the topological manifold, \( \mathbb{R}^1 \) has at least two different differentiable structures on it, \( D \) and \( \mathcal{D} \). However, it is easy to show that these different structures would not lead to different physics (or smooth mathematics for that matter) since they are actually diffeomorphic. This result can be explicitly demonstrated by defining the homeomorphism, \( f \), of \( \mathbb{R}^1 \) onto itself by \( f(p) = p^3 \). This is a diffeomorphism since its expression in the two charts is \( y \cdot f \cdot x^{-1} : p \rightarrow (p^3)^\frac{1}{3} = p \) is clearly smooth. Thus, \( D \) and \( \mathcal{D} \), while being different differentiable structures, are in fact diffeomorphic. Actually, it is well known and fairly easy to prove that \( \mathbb{R}^1 \) can support only one differentiable structure up to diffeomorphisms. Thus, no new global physics on one-dimensional models can be encountered without changing the topology itself. At first glance, the idea that \( f(p) = p^3 \) should be a diffeomorphism may be somewhat troubling. However, the issue is that the original numbers \( p \) do not of themselves define a basis for differentiation, unless an explicit assumption is made. This assumption is that the \( D \) is the differentiable structure to be used. However, there is nothing any more natural to this assumption than there is to the assumption that our space-time geometry should be flat.

From this simple example it is quite tempting to conjecture that while topologically trivial spaces such as \( \mathbb{R}^n \) can support an infinity of different differentiable structures, none of these will lead to any new physics since each will be diffeomorphic to the standard one. If this were true, then there would likely be no new physics from differential topology since all global structures would have to be relegated to the topology. As discussed in the introduction this is in fact the case for all \( n \neq 4 \). However, the surprising discovery that this conjecture is not true precisely for the physically significant dimension \( n = 4 \), means that potential new global physical tools are available, apart from topological ones. Unfortunately, the
absence of any effective coordinate patch presentation of any exotic smoothness makes the physical exploration difficult. However, we do have access to a manageable model which can serve illustrate the relationships that are important, but difficult to make explicit, in the smooth case. This is provided by the notion of complex structures and related holomorphisms. The problem of complex structures on \( \mathbb{R}^2 \) is relatively easy, compared to that of the differentiable structures on \( \mathbb{R}^4 \), so we can explicitly explore the relationship between the mathematical structure and its physical implications.

Consider then the two-dimensional physics of vacuum electrostatic vector fields \( \mathbf{E}(x, y) \) described by component functions \( E_x(x, y) \) and \( E_y(x, y) \). The Maxwell vacuum electrostatic equations are just the Cauchy conditions for the real and (negative) imaginary parts of an analytic function of the complex variable \( z \equiv x + i y \). The most general solution to the Maxwell equations in this formalism can be obtained simply from the complex equation,

\[
\text{Vacuum 2-D Maxwell } \Leftrightarrow E_x - i E_y = F(z),
\]

where \( F(z) \) is an arbitrary analytic function.

These facts are well-known and apart from a few illustrative boundary value problems, do not seem to lead to any significant physical consequences or further insights, probably because the introduction of a complex structure on the space model is possible only for two-dimensional problems. However, the fact that all structures can be explicitly described make this an excellent example to illustrate the concepts in the real, smooth case.

Complex structures are defined in a manner analogous to that of smooth structures, with complex analyticity replacing differentiability, and biholomorphisms replacing diffeomorphisms. Thus a complex structure, \( \text{CS} \), on a two-dimensional manifold, \( M \), is defined by covering \( M \) with an atlas of charts, \( U_a \), together with maps, \( z_a \) taking \( U_a \) (smoothly and invertibly) onto open balls in \( \mathbb{R}^2 \) identified with the complex plane \( \mathbb{C} \) in the “standard” way, so that the local coordinates can be expressed \( z_a : U_a \rightarrow x_a(p) + iy_a(p) \in \mathbb{C}, \) for \( pxU_a \in M \). Furthermore, where defined, \( z_a \circ z_a^{-1} \) must be analytic in \( \mathbb{C} \) in the usual complex sense. The charts, \( U_a \), are sometimes called “coordinate patches” and, for \( pxU_a \subset M \), the value \( z_a(p) \in \mathbb{C} \) is the “coordinate of \( p \) relative to the patch \( U_a \).” A complex valued function, \( F : V \rightarrow \mathbb{C} \), for some neighborhood \( V \subset M \), is “analytic”, or “holomorphic”, if it is complex analytic when expressed in the local coordinates, \( z_a \), over the \( U_a \) covering \( V \), that is, \( F \circ z_a^{-1} \) is analytic (where defined) in the usual sense on \( \mathbb{C} \).

Two such structures on a given \( M \), say \( \text{CS}' \) given by \( \{U_a', z'_a\} \) and \( \text{CS} \) given by \( \{U_a, z_a\} \) are analytically equivalent (biholomorphic) if and only if there exists a homeomorphism, \( F, \) of \( M \) onto itself such that when \( F \) is expressed in terms of the local charts it is a local biholomorphism in the standard sense. Thus, \( z_a \circ F \circ z_a^{-1} \) and \( z'_a \circ F^{-1} \circ z_a^{-1} \) must both be holomorphic where defined. Note that it is not necessary that the \( \text{CS} \) coordinates themselves be analytic in terms of \( \text{CS}' \), but only when combined with a homeomorphism. This is an important distinction. To clarify this point, consider \( U_1 = U_1' = \mathbb{R}^2 \), with \( z_1(x, y) = x + iy \) and \( z'_1(x, y) = x - iy \). Then clearly the primed coordinate is not analytic in terms of the unprimed one. However, these are equivalent complex structures since the homeomorphism, \( F(x, y) = (x, -y) \) satisfies the above condition for equivalence. That is,

\[
z_1 \circ F \circ z_1^{-1} : z = x + iy \rightarrow (x, -y) \rightarrow (x, y) \rightarrow x + iy = z.
\]

Thus, to the extent that the complex structure determines the physics, the physical content of \( \text{CS} \) is identical to that of \( \text{CS}' \). For example, (2) could be expressed in either \( \text{CS} \) or \( \text{CS}' \) with the same physical content since the homeomorphism \( F \) takes us from one expression of a physical electrostatic field to another. In this complex model, the invariance of the physical theory under the transition from \( \text{CS} \) to \( \text{CS}' \) is precisely analogous to the diffeomorphism invariance required in the real differentiable structure case as formulated in the principle of general relativity. It is easy to think that there is something preferred about the standard complex structure so that notions of analyticity must be settled in terms of \( z = x + iy \) only. However, this is clearly not so, but the prejudice that there should be is reminiscent of that in the real differentiable structure case.

Of course, this model would be trivial for our purposes if the standard \( \text{CS} \) were unique up to biholomorphisms. However, the standard complex structure is not unique! There is precisely one other
inequivalent one, that is, one not related by a biholomorphism. One presentation of this second structure, $CS_1$, can be defined by

$$(x, y) \rightarrow z_1 = e^{-1/r} r(x + i y) \in \mathbb{C}.$$  

(4)

The key point here is that $|z_1|$ is bounded. It is now easy to show that $CS_0$ is not equivalent to $CS_1$. In fact, if it were then there would exist a function, $F(x, y) = (F_x(x, y), F_y(x, y))$, of the plane onto itself such that $e^{-i/r} (F_x(x, y) + i F_y(x, y))$ would be a global analytic function of $x + i y$ in the usual sense. Clearly however this cannot be since this function is non-constant, but bounded on the entire plane, violating a well known property of global analytic functions for the standard $CS_0$.

Now we can state the physical implications of the choice of structure, complex in this case: If the physical theory is expressed by the statement that the $x$ and $y$ components of the electrostatic vacuum two-dimensional field are real and (negative) imaginary parts of a function analytic relative to the chosen complex structure, then $CS_0$ and $CS_1$ lead to different fields, with physically measurable differences since non-constant electrostatic fields in the $CS_0$ case cannot be bounded, whereas they are in the $CS_1$ case. In other words, to the extent that (2) is the statement of a physical theory, then in principle, experiment could distinguish $CS_0$ from $CS_1$. However, experiment cannot distinguish $CS_0$ from $CS'$ described earlier, since these are biholomorphic.

Of course, this discussion is intended to be illustrative rather than of any likely physical significance itself. The description of electrostatic field theory in terms of analyticity requirements is certainly not the basis of a general physical theory. In fact, it could be argued that changing the complex structure results in a changed metric and that the correct theory should include this metric. However, this model does illustrate explicitly the difference between merely different and inequivalent structures, non-biholomorphic in our case, non-diffeomorphic in our smooth case. For this complex model, $CS_1$ plays the role of an exotic differentiable structure in the real case. While $CS_1$ was easily displayed we do not have yet have this luxury for a $R^4_o$. However, we now proceed to review the path to these strange structures.

4 Some History of Exotic Differentiable Structures

An early and fairly easily accessible example of an exotic differentiable structure on a simple topological space was provided in 1956 by Milnor. He was able to use an extension of the Hopf fibering of spheres to construct an exotic seven-sphere, $\Sigma_7$. Consider the $S^3$ bundles over $S^4$,

$$S^3 \rightarrow M^7 \xrightarrow{p} S^4$$  

(5)

with $Spin(4)$ (the covering group of the rotation group, $SO(4)$) acting on $S^3$, as bundle group. In its original form, the Hopf fibering makes use of the fact that the topological seven sphere can be expressed in terms of a pair of quaternions, $S^7 = \{(q_1, q_2) : |q_1|^2 + |q_2|^2 = 1\}$. The projection to $S^4$ is through the quaternion projective map: $\{q_1 : q_2\} \rightarrow S^4$. The natural kernel of this map is of course the set of unit quaternions, $S^3 = SU(2) \subset Spin(4)$. For a wider class of bundles a classification is provided by $\pi_3(Spin(4)) \approx Z + Z$, as discussed in §18 of [3]. This construction can be described in terms of the normal form for $M^7$ in which the base $S^4$ is covered by two coordinate patches, say upper and lower hemispheres. The overlap is then $R^4 \times S^3$ which has $S^3$ as a retract. Thus, the bundle transition functions are defined by their value on this subset, defining a map from $S^3$ into $Spin(4)$ and thus generating an element of $\pi_3(Spin(4))$. The group action of $Spin(4)$ on the fiber $S^3$ can be conveniently described in the well-known quaternion form, $u \rightarrow u' = vuv^*$, where $u, v, w$ are all unit quaternions and $v^*$ is quaternion conjugate of $w$. Thus $u \in S^3$ and $(v, w) \in SU(2) \times SU(2) \approx Spin(4)$. Standard $S^7$ is obtained from the element of $\pi_3(Spin(4))$ generated by $(v, 1)$, so that the group action reduces to one $SU(2) \approx S^3$ and the bundle is in fact an $SU(2)$ principal one. In fact, this is precisely the
principle $SU(2)$ Yang-Mills bundle over compactified space-time, $S^4$. For more details on classifying sphere bundles, see §20 and, from the physics viewpoint, §20.

Milnor’s breakthrough in 1956 involved his proving that $M^7$ for the transition function map, an element of $\pi_3(Spin(4))$, given by

$$u \to (u^h, \pi^j) \in Spin(4),$$

with $h + j = 1$ and $h - j = k$, and $k^2 \not\equiv 1 \mod 7$, is in fact exotic, i.e., homeomorphic to $S^7$, but not diffeomorphic to it. Clearly, the constructive part is fairly easy, but the proof of the exotic nature of the resulting sphere is more involved, drawing from several important results in differential topology including the Thom bordism result, cohomology theory, Pontrjagin classes, etc. Later, Kervaire and Milnor expanded on these results, leading to a good understanding of the class of exotic spheres in dimensions seven and greater.

For the purposes of this paper, the importance of Milnor’s discovery is that it provides an explicit example of a topologically simple space, $S^7$, for which the topology does not determine the smoothness. Furthermore, its construction is explicit and easily understood. From the physics viewpoint, its most important application may be in terms of some “exotic Yang-Mills” models as discussed below.

Unfortunately, the path to $\mathbb{R}^4_S$ is much less easy to describe than are the Milnor spheres. The techniques required span a variety of mathematical disciplines and the following survey is necessarily brief. The reader is referred to mathematical reviews for more details.

The first step involves a topological tool important in classifications of four manifolds. The intersection form of a compact oriented manifold without boundary, is obtained by the Poincare duality pairing of homology classes in $H_{n-k}$ and $H_k$ can be simply represented in dimension $n = 4 = 2 + 2$ by a symmetric square matrix of determinant $\pm 1$. This form basically reflects the way in which pairs of oriented two-dimensional closed surfaces fill out the full (oriented) four-space at their intersection points. Physicists are perhaps more familiar with deRham cohomology involving exterior forms for which this intersection pairing is the volume integral of the exterior product of a pair of closed two-forms representing the individual cohomology classes, which again makes sense only in dimension four. Unfortunately, deRham cohomology necessarily involves real coefficients and is thus too coarse for our applications, which need integral homology theory. At any rate, this integral intersection form, $\omega$, plays a central role in classifying compact four manifolds. Whitehead used it to prove that one-connected closed 4-manifolds are determined up to homotopy type by the isomorphism class of $\omega$. Later, Freedman proved that $\omega$ together with the Kirby-Siebenmann invariant classifies simply-connected closed 4-manifolds up to homeomorphism. For our purposes, the important result was that there exists a topological four manifold, $|E_8|$, having intersection form $\omega = E_8$, the Cartan matrix for the exceptional lie algebra of the same name.

As it stands, Freedman’s work is in the topological category, and does not address smoothness questions. The theorem of Rohlin states that the signature of a closed connected oriented smooth 4-manifold must be divisible by 16, so that $|E_8|$ cannot exist as a smooth manifold since its signature is 8. Next, Donaldson’s theorem provides the crucial (for our purposes) generalization of this result to establish that $|E_8 \oplus E_8|$ cannot be endowed with any smooth structure, even though its signature is 16. The work of Donaldson is based on the moduli space of solutions to the $SU(2)$ Yang-Mills equations on a four-manifold, which first occur in physics literature.

Having established some algebraic machinery, the next step involves an algebraic variety, the Kummer surface, $K$, a real four-dimensional smooth manifold in $CP^3$. It is known that

$$K = | - E_8 \oplus - E_8 \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |.$$  

The last part of this intersection form is easily seen to be realizable by $3(S^2 \times S^2)$, which is smooth. Thus, Donaldson’s theorem implies that it is impossible to do smooth surgery on $K$ in just such a way as to excise the smooth $3(S^2 \times S^2)$, leaving a smooth (reversing orientation) $|E_8 \oplus E_8|$. In the following, we refer to these two parts as $V_1$ (smoothable) and $V_2$ (not smoothable) respectively, so smooth $K = V_1 \cup V_2$. In investigating the failure of this smooth surgery Freedman found the first fake $R^4_S$. Using a topological
S^3 to separate V_1 from V_2. Donaldson’s result showed that this S^3 cannot be smoothly embedded, since otherwise V_2 would have a smooth structure. However, by further surgery, it is found that this dividing S^3 is also topologically embedded in a topological R^4 and actually includes a compact set in its interior. Thus we have

**Existence of exotic R^4**: This topological R^4 contains a compact set which cannot be contained in any smoothly embedded S^3. This surprising result then implies that this manifold is indeed an R^4 since in any diffeomorphic image of R^4 every compact set is included in the interior of a smooth sphere.

5 Some Properties of R^4’s

After this basic existence theorem, many developments have occurred, some of which are summarized in the book by Kirby[12]. Unfortunately, none of the uncountable infinity of R^4’s has been presented in explicit atlas of charts form, so most of the properties can only be described indirectly, through existence or non-existence type of theorems. In this paper we restrict the discussion to those topics of possible physical applications.

The defining feature of the original R^4, as discussed above can be summarized in coordinate form as follows: R^4 = the *topological* product R^1 × R^1 × R^1 × R^1. Thus, it can be described *topologically* as the set R^4 = {p^α}, α = 1,...,4. However, these coordinates cannot be globally smooth, since otherwise the product would be smooth and the manifold diffeomorphic to standard R^4. Or, in terms of the property used in the discovery, if {p^α} were globally smooth, since every compact set is contained in some |p|^2 = ∑(p^α)^2 = R^2 for sufficiently large R, then every compact set would be contained in some smoothly embedded S^3, contrary to the existence statement above. In fact, this absence of “sufficiently large” smooth three-spheres is a strange and defining characteristic of R^4’s. It shows, among other things that the exoticness necessarily extends to infinity. Nevertheless, the possibility of confining the exoticness to a time-like tube is shown in Theorem 2 below.

Even though the global topological coordinates cannot be globally smooth, it can be shown that in some diffeomorphic copy they can be made smooth locally. Thus

**Theorem 1** There exists a smooth copy of each R^4, for which the global C^0 coordinates are smooth in some neighborhood. That is, there exists a smooth copy, R^4 = \{p^α\}, for which p^α ∈ C^∞ for |p| < ℵ.

The proof of this involves the Annulus Theorem[13] to C^0 tie some local smooth coordinates to the global C^0 \{p^α\} across some annulus.

What this gives is a local smooth coordinate patch, on which standard differential geometry can be done, but which cannot be extended indefinitely. The obstruction should be physically interesting. Why cannot certain local fields be continued globally in the absence of any topological obstructions?

Theorem 1 leads naturally to the following construction. By puncturing R^4 in the neighborhood where the topological coordinates are smooth, we get a “semi-exotic” cylinder, i.e. R^4−0 ≃ R^1 × S^3, where × denotes topological but not smooth product. By “semi-exotic” we mean that the product is actually smooth for a semi-infinite extent of the first coordinate. This might be a very interesting cosmological model for physics, which after the big bang is R^1 × S^3. Here we would run into an obstruction to continuing the smooth product structure at some finite time (first coordinate) for some unknown, but potentially very interesting, reason.

An even more interesting possibility to consider would involve localizing the “fakeness” in some sense. One version that comes to mind would happen if we could smoothly glue two such semi-exotic cylinders at their exotic ends. Of course a second gluing at their smooth ends would then give an exotic smoothness on the topological product, S^1 × S^3. The existence of such an S^1 × S^3 is apparently not known.

However, some results on “localization of exoticness” can be obtained. In fact, the main result of another paper[4] can be summarized informally:
Result 1  There exists exotic smooth manifolds with $\mathbb{R}^4$ topology which are standard at spatial infinity, so that the exoticness can be regarded as spatially confined.

A more precise statement of this result is provided in Theorem 2 below. The resulting manifold structures have the property that everything looks normal at space-like infinity but the standard structure cannot be continued all the way in to spatial origin.

This result could have great significance in all fields of physics, not just relativity. Some model of space-time underlies every field of physics. It has now been proven that we cannot infer that space is necessarily smoothly standard from investigating what happens at space-like infinity, even for topologically trivial $\mathbb{R}^4$. It seems very clear that this is potentially very important to all of physics since it implies that there is another possible obstruction, in addition to material sources and topological ones, to continuing external vacuum solutions for any field equations from infinity to the origin. Of course, in the absence of any explicit coordinate patch presentation, no example can be displayed. However, this leads naturally to a conjecture, informally stated:

Conjecture 1  This localized exoticness can act as a source for some externally regular field, just as matter or a wormhole can.

The full exploration of this conjecture will require more detailed knowledge of the global metric structure than is available at present. The notions of domains of dependence, Cauchy surfaces, etc., necessary for such studies cannot be fully explored with present differential geometric information on exotic manifolds. However, a beginning can be made with certain general existence results as established and discussed below.

We now state and sketch the proof of

Theorem 2  There exists smooth manifolds which are homeomorphic but not diffeomorphic to $\mathbb{R}^4$ and for which the global topological coordinates $(t, x, y, z)$ are smooth for $x^2 + y^2 + z^2 \geq \epsilon^2 > 0$, but not globally. Smooth metrics exists for which the boundary of this region is timelike, so that the exoticness is spatially confined.

To arrive at this result we make use of techniques developed by Gompf[14] which lead to the construction of a large topological variety of exotic four-manifolds, some of which would appear to have considerable potential for physics. Gompf’s “end-sum” process provides a straightforward technique for constructing an exotic version, $M$, of any non-compact four-manifold whose standard version, $M_0$, can be smoothly embedded in standard $\mathbb{R}^4$. Recall that we want to construct $M$ which is homeomorphic to $M_0$, but not diffeomorphic to it. First construct a tubular neighborhood, $T_0$, of a half ray in $M_0$. $T_0$ is thus standard $\mathbb{R}^4 = [0, \infty) \times \mathbb{R}^3$. Now consider a diffeomorphism, $\phi_0$ of $T_0$ onto $N_0 = [0, 1/2) \times \mathbb{R}^3$ which is the identity on the $\mathbb{R}^3$ fibers. Do the same thing for some exotic $\mathbb{R}^4_\Theta$ with the important proviso that it cannot be smoothly embedded in standard $\mathbb{R}^4$. Such manifolds are known in infinite abundance [14]. Then construct a similar tubular neighborhood for this $\mathbb{R}^4_\Theta$, $T_1$, with diffeomorphism, $\phi_1$, taking it onto $N_1 = [1, 1/2) \times \mathbb{R}^3$. The desired exotic $M$ is then obtained by forming the identification manifold structure

$$M = M_0 \cup_{\phi_0} ([0, 1] \times \mathbb{R}^3) \cup_{\phi_1} \mathbb{R}^4_\Theta. \tag{8}$$

The techniques of forming tubular manifolds and defining identification manifolds can be found in standard differential topology texts, such as [15] or [16].

Informally, what is being done is that the tubular neighborhoods are being smoothly glued across their “ends”, each $\mathbb{R}^3$. The proof that the resulting $M$ is indeed exotic is then easy: $M$ contains $\mathbb{R}^4_\Theta$ as a smooth sub-manifold. If $M$ were diffeomorphic to $M_0$ then $M$, and thus $\mathbb{R}^4_\Theta$, could be smoothly embedded in standard $\mathbb{R}^4$, contradicting the assumption on $\mathbb{R}^4_\Theta$. Finally, it is clear that the constructed $M$ is indeed homeomorphic to the original $M_0$ since all that has been done topologically is the extension of $T_0$.

In order to relate these constructions to possible physical applications, and to complete the proof, let us now introduce $(t, x, y, z)$ as the global topological coordinates. Let the tubular neighborhoods used
in the end sum techniques be generated by the continuous (but not globally smooth) $t$-curves, and that the standard $\mathbb{R}^4$ corresponds to $t < 0$. Then “stuffing” the upper $\mathbb{R}^4$ into the tube results in a manifold which we label $M_3$, having the property that $(t, x, y, z)$ are smooth for $t < 0$ and for $x^2 + y^2 + z^2 > \epsilon^2$, all $t$ for some positive $\epsilon$. An obvious doubling of this property leads to $M_4$ for which $(t, x, y, z)$ are smooth for all $x^2 + y^2 + z^2 > \epsilon^2$, for all $t$. The smoothness properties of the $M_4$ can also be stated in terms of products. Global $C^0$ coordinates, $(t, x, y, z)$, are smooth in the exterior region $[\epsilon, \infty) \times S^2 \times \mathbb{R}^1$, while the closure of the complement of this is clearly an exotic $B^3 \times \mathbb{R}^1$. Since the exterior component is standard, a wide variety, including flat, of Lorentz metrics can be imposed. Picking only those for which $\partial/\partial t$ is timelike in this region provides a natural sense in which the world-tube confining the exotic part is “spatially localized.” The smooth continuation of such a metric to the full metric is then guaranteed by Lemma 1 below and the discussion following it below. This completes the proof.

6 Some Geometry on $R^4_{\Theta}$'s

Some very basic, if sketchy, information about differential geometry on $R^4_{\Theta}$ can be obtained. For example,

**Theorem 3** There can be no geodesically complete metric (of any signature) with non-positive sectional curvature on $R^4_{\Theta}$.

Proof: If there were such a metric, the Hadamard-Cartan theorem could be used to show that the exponential map would provide a diffeomorphism of the tangent space at a point onto $R^4_{\Theta}$. In particular, there can be no flat geodesically complete metric. For more discussion on exotic geometry, see [17]. Natural questions then arise concerning the nature of the obstructions to continuing the solutions to the differential equations expressing flatness in the natural exponential coordinates. In physics, obstructions to continuation of solutions are often of considerable significance, e.g., wormhole sources. However, up to now, such obstructions generally have been a result of either topology, (incompleteness caused by excision), or some sort of curvature singularity. Neither of these is present here. This problem is particularly interesting for those $R^4_{\Theta}$'s which cannot be smoothly embedded in standard $\mathbb{R}^4$, which thus cannot be geodesically completed with a flat metric.

Consider now what can be said about the continuation of a Lorentz signature metric from some local chart. It turns out that any Lorentzian metric on a closed submanifold, $A$, some smooth continuation to all of $M$ under certain conditions. For example, we have

**Lemma 1** If $M$ is any smooth connected 4-manifold and $A$ is a closed submanifold for which $H^4(M, A; \mathbb{Z}) = 0$, then any smooth time-orientable Lorentz signature metric defined over $A$ can be smoothly continued to all of $M$.

Proof: This is basically a question of the continuation of cross sections on fiber bundles. Standard obstruction theory is usually done in the continuous category, but it has a natural extension to the smooth class, [6]. First, we note that any time-orientable Lorentz metric is decomposable into a Riemannian one, $g$, plus a non-zero time-like vector field, $v$. The continuation of $g$ follows from the fact that the fiber, $Y_S$, of non-degenerate symmetric four by four matrices is $q$-connected for all $q$. From standard obstruction theory, this implies that $g$ can be continued from $A$ to all of $M$ without any topological restrictions. On the other hand, the fiber of non-zero vector fields is the three-sphere which is $q$-connected for all $q < 3$, but certainly not 3-connected ($\pi_3(S^3) = \mathbb{Z}$). Again from standard results, [6], any obstruction to a continuation of $v$ from $A$ to all of $M$ is an element of $H^4(M, A; \mathbb{Z})$. Thus, the vanishing of this group is a sufficient condition for the continuation of $v$, establishing the Lemma.

What is missing from this result, of course, is that the continued metric satisfy the vacuum Einstein equations and that it be complete in an appropriate Lorentzian sense. Of course, any smooth Lorentzian metric satisfies the Einstein equation for some stress-energy tensor, but this tensor must be shown to be

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2Here the “exotic” can be understood as referring to the product which is continuous but cannot be smooth. See the discussion around Lemma 2 below.
physically acceptable. Unfortunately, these issues cannot be resolved without more explicit information on the global exotic structure than is presently available. However, we can be more specific about the conditions under which some smooth Lorentzian metric can be globally continued on an exotic manifold from some local coordinate presentation.

In the applications in this paper, \( M \) is non-compact, so \( H^4(M; \mathbb{Z}) = 0 \). Using the exact cohomology sequence generated by the inclusion \( A \to M \), we can develop several easily satisfied sufficient conditions on \( A \) to meet the conditions of Lemma 1. One easy one is to require \( H^3(A; \mathbb{Z}) = 0 \). Another would be to establish that the map, \( H^3(M; \mathbb{Z}) \to H^3(A; \mathbb{Z}) \) is an epimorphism. For example, if \( A \) is simply a closed miniature version of \( \mathbb{R}^2 \times S^2 \) itself, i.e., \( A = D^2 \times S^2 \), then \( H^3(A; \mathbb{Z}) = 0 \) so the continuation of a smooth Lorentzian metric is ensured. The spaces \( \mathbb{R}^2 \times S^2 \), each have the topology of the Kruskal presentation of the Schwarzschild metric. Using the standard Kruskal notation \( \{(u, v, \omega); u^2 - v^2 < 1, \omega \in S^2\} \) constitute global topological coordinates. From Theorem 1, these can be smooth over the closure of some open set, say \( A \), homeomorphic \( D^2 \times S^2 \), but \( (u, v) \) cannot be continued as smooth functions over the entire range: \( u^2 - v^2 < 1 \). Over \( A \) then we can solve the vacuum Einstein equations as usual to get the Kruskal form. From Lemma 1, some smooth metric can be continued from this over the entire manifold. However, whatever it is, it cannot be the standard Kruskal one. The obstruction to continuation of the metric occurs not for any reasons associated with the development of singularities in the coordinate expression of the metric, or for any topological reasons, but simply because the coordinates, \( (u, v, \omega) \), cannot be continued smoothly beyond some proper subset, \( A \), of the full manifold. This establishes

**Theorem 4** On some smooth manifolds which are topologically \( \mathbb{R}^2 \times S^2 \), the standard Kruskal metric cannot be smoothly continued over the full range, \( u^2 - v^2 < 1 \).

An interesting variation occurs when \( A \) contains a trapped surface, so a singularity will inevitably develop from well-known theorems. However, if \( A \) does not contain a trapped surface what will happen is not known.

Another way to study this metric is in terms of the original Schwarzschild \((r, t)\) coordinates for \( M_4 \). For this model the coordinates \((t, r, \omega)\) are smooth for all of the closed sub-manifold \( A \) defined by \( r \geq \epsilon > 2M \) but cannot be continued as smooth over the entire \( M \) or over any diffeomorphic (physically equivalent) copy. In this case \( A \) is topologically \([\epsilon, \infty) \times S^2 \times \mathbb{R}^1\), so again \( H^3(A; \mathbb{Z}) = 0 \) and the conditions of lemma 1 are met. Hence there is some smooth continuation of any exterior Lorentzian metric in \( A \), in particular, the Schwarzschild metric, over the full \( \mathbb{R}_+^4 \). Whatever this metric is, it cannot be Schwarzschild since the manifolds are not diffeomorphic. An interesting feature of this model is that the manifold is “asymptotically” standard in spite of the well known fact that exotic manifolds are badly behaved “at infinity”. However, we note that this model is asymptotically standard only as \( r \to \infty \), but certainly not as \( t \to \infty \).

These models are clearly highly suggestive for investigation of alternative continuation of exterior solutions into the tube near \( r = 0 \). We often first discover an exterior, vacuum solution, and look to continue it back to some source. This is a standard problem. In the stationary case, we typically have a local, exterior solution to an elliptic problem, and try to continue it into the origin but find we can’t as a vacuum solution unless we have a topology change (e.g., a wormhole), or unless we add a matter source, changing the equation. Now, we are led to consider a third alternative, can exotic smoothness serve as a source for some exterior metric?

Of course, the discussion of stationary solutions involves the idea of time foliations, which cannot exist globally for these exotic manifolds, at least not into standard factors. In fact,

**Lemma 2** \( \mathbb{R}^4_\Theta \) cannot be written as a smooth product, \( \mathbb{R}^1 \times_{\text{smooth}} \mathbb{R}^3 \). Similarly \( \mathbb{R}^2 \times_\Theta S^2 \) cannot be written as \( \mathbb{R}^1 \times_{\text{smooth}} (\mathbb{R}^1 \times S^2) \).

Clearly, if either factor decomposition were smooth, the original manifold would be standard, since the factors are necessarily standard from known lower dimensional results, establishing the lemma. It is not now possible to establish the more general result for which the second factor is simply some smooth three manifold without restriction.
Of course, the lack of a global time foliation of these manifolds means that such models are inconsistent with canonical approach to gravity, quantum theory, etc. However, it is worth noting that all experiments yield only local data, so we have no a priori basis for excluding such manifolds.

These discussions lead naturally to a consideration of what can be said about Cauchy problems. Consider then the manifold, $M_3$, for which the global $(t, x, y, z)$ coordinates are smooth for all $t < 0$ but not globally. Now consider, the Cauchy problem $R_{\alpha\beta} = 0$, with flat initial data on $t = -1$. This is guaranteed to have the complete flat metric as solution in the standard, $\mathbb{R}^4$ case. However, because of Theorem 3, this cannot be true for $M_3$. What must go wrong in the exotic case, of course, is that $t = -1$ is no longer a Cauchy surface. However, Lemma 1 can again be applied here to guarantee the continuation of some Lorentzian metric over the full manifold since here $A = (-\infty, -1] \times \mathbb{R}^3$ so clearly $H^3(A; \mathbb{Z}) = 0$.

Finally, consider a cosmological model based on $\mathbb{R}^1 \times \Theta S^3$. In this case, we can start with a standard cosmological metric for some time, so here $A = (-\infty, 1] \times S^3$. Clearly, $H^3(A; \mathbb{Z})$ does not vanish in this case, but it can be shown that the inclusion induced map $H^3(M; \mathbb{Z}) \rightarrow H^3(A; \mathbb{Z})$ is onto, so the conditions of Lemma 1 are met. Thus some smooth Lorentzian continuation will indeed exist, leading to some exotic cosmology on $\mathbb{R}^1 \times S^3$.

7 Conclusion and Conjectures

What are the possible physical implications of the existence of the exotic spaces? First, consider the $\Sigma_7^\phi$, which can be explicitly constructed. Perhaps they could be considered as possible models for exotic Yang-Mills theory. Some $\Sigma_7^\phi$ are $SU(2)$ bundles, but not principle ones, since their groups must be $Spin(4)$. This would contrast with standard Yang-Mills structure[18] in which the total space is $S^7$ regarded as a principle $SU(2)$ bundle. Next, $\Sigma_7^\phi$ can be used as toy space-time models, serving as the base manifolds for various geometric and other field theories. Various questions of physical interest can then be asked on these models and the answers compared to those obtained from standard $S^7$.

For example, the non-existence of a constant curvature metric on $\Sigma_7^\phi$ has already been thoroughly explored[3]. The analysis of such differential geometric problems on $\Sigma_7^\phi$ as compared to $S^7$ should give some indication of the type of results that could come from physics on $\mathbb{R}^4_\phi$ as compared to that on standard $\mathbb{R}^4$.

There are also many obvious questions concerning the physical implications of doing general relativity on $\mathbb{R}^4_\phi$. The entire problem of developing a manifold from a coordinate patch piece on which an Einstein metric is known, still has many unanswered aspects even in standard smoothness. Recall for example the evolution of our understanding of the appropriate manifold to support the (vacuum) Schwarzschild metric. Originally, the solution was expressed using $(t, r, \theta, \phi)$ coordinates as differentiable outside of the usual “coordinate singularities” well known for spherical coordinates. However, the Schwarzschild metric form itself in these coordinates exhibits another singularity on $r = 2m$, sometimes referred to as the “Schwarzschild singularity.” Later work, culminating in the Kruskal representation, showed that the Schwarzschild singularity could be regarded as merely another coordinate one in the same sense as is the $z$-axis for $(r, \theta, \phi)$. This example helps to illustrate that in general relativity our understanding of the physical significance of a particular metric often undergoes an evolution as various coordinate representations are chosen. In this process, the topology and differentiable structure of the underlying manifold may well change. In other words, as a practical matter, the study of the completion of a locally given metric often involves the construction of the global manifold structure in the process. Could any conceivable local Einstein metric lead to an $\mathbb{R}^4_\phi$ by such a process?

Of course, local coordinate patch behavior is of great importance to physics, so another set of physically interesting questions would relate to the coordinate patch study of $\mathbb{R}^4_\phi$. This may be too difficult of a task for present mathematical technology, but some questions may be reasonable. For example, can some $\mathbb{R}^4_\phi$’s be covered by only a finite number of coordinate patches? If so, what is the minimum number? What are the intersection properties of the coordinate patch set which makes it non-standard?
A directly physical set of questions to be considered would stem from attempts to embed known solutions to the Einstein equations in $\mathbb{R}^4_\Theta$, then asking what sort of obstruction intervenes to prevent their indefinite, complete, continuation, in this space. Examples of non-topological obstructions to the continuation of a wide class of Einstein metrics was discussed in Section 6. What is the physical significance of these obstructions?

Finally, it is indeed true that the existence of $\mathbb{R}^4_\Theta$'s does not in any way change the local physics of general relativity or any other field theory. However, it has long been known that global questions can have profound effects on a physical theory. Until recently, physicists have thought of global matters almost exclusively as being of purely topological significance, whereas we now know that at least in the physically important case of $\mathbb{R}^4$, there are very exciting global questions related to differentiability structures, the way in which local physics is patched together smoothly to make it global. Certainly, the $\mathbb{R}^4_\Theta$'s are essentially just “other” manifolds. However, there are an infinity of them which have never been remotely considered in the physical context of classical space-time physics on Einstein’s original model, $\mathbb{R}^4$. It would be surprising indeed if none of these had any conceivable physical significance.

I am very grateful to Duane Randall and Robert Gompf for their invaluable assistance in this work.
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