Binary Decision Diagrams for Bin Packing with Minimum Color Fragmentation

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Abstract

Bin Packing with Minimum Color Fragmentation (BPMCF) is an extension of the Bin Packing Problem in which each item has a size and a color and the goal is to minimize the sum of the number of bins containing items of each color. In this work, we introduce the BPMCF and present a decomposition strategy to solve the problem, where the assignment of items to bins is formulated as a binary decision diagram and an optimal integrated solutions is identified through a mixed-integer linear programming model. Our computational experiments show that the proposed approach greatly outperforms a direct formulation of BPMCF and that its performance is suitable for large instances of the problem.

Keywords. Bin Packing; Binary Decision Diagrams; Integer Programming.

1 Introduction

In this work, we investigate Bin Packing with Minimum Color Fragmentation (BPMCF), an extension of the Bin Packing Problem in which each item is associated with a color and one wishes to identify assignments of items to bins placing items of a common color in the fewest number of bins possible. BPMCF finds application in scenarios where having elements of a certain category (color) in each bin results in costs that should be minimized. In production planning, for example, a company requires expensive modules in order to process orders each of which is of a certain type and has a processing time. Each plant has limited total processing time. In this situation, items correspond to orders and order types correspond to categories. To minimize costs for the modules, the company strives to pack orders of each type in as few bins as possible. Similarly, in logistics, there are scenarios where heterogeneous cargo needs to be transported and differences in cargo necessitate vehicles equipped with specific capabilities, for example temperature or pressure controlling devices. Additionally, in large-group seating problems that arise in corporate events or weddings, attendees form subgroups and event organizers strive to pack individuals of the same type in the fewest number of tables.
Many colored extensions of the bin packing problem have been studied in the literature, but with a different (typically opposite) objective. Examples abound. In the colored bin packing problem (CBPP), commonly colored items are not allowed to be packed next to each other in the same bin (Balogh et al. (2015, 2012), Böhm et al. (2014)). CBPP has also been investigated under the name of class constrained bin packing problem, mostly from a theoretical perspective (Shachnai and Tamir (2001, 2004), Xavier and Miyazawa (2008)). Approximation results have been obtained for the variable class constrained bin packing problem, where bins may have different sizes and the goal is to minimize the sum of the sizes of the bins that have been used (Dawande et al. (2001), Xavier and Miyazawa (2008)). Jansen introduced the bin packing problem with conflicts (BPMCF), a generalization of CBPP where we are given a graph on the items, with edges indicating pairs of elements that cannot be placed in the same bin (Jansen (1999), Jansen and Ohring (1997)). Several algorithms have been introduced in the literature to address the BPMCF (Elhedhli et al. (2011), Gendreau et al. (2004), Muritiba et al. (2010), Sadykov and Vanderbeck (2013)), such as a branch-and-price algorithm for general conflict graphs (Sadykov and Vanderbeck (2013)). Another variant of CBPP is the co-printing problem, in which bins are bounded both in terms of weight and number of colors they may contain (Peeters and Degraeve (2004)); both heuristic and exact algorithms have been proposed to solve this problem (Kochetov and Kondakov (2017), Kondakov and Kochetov (2018)). To the best of our knowledge, BPMCF is yet to be investigated in the literature yet.

In this article, we introduce BPMCF, show how to cast the assignment of items to bins as a binary decision diagram (BDD) (Bryant (1986, 1992)), and present a mixed-integer linear programming (MIP) formulation to solve the problem. BDDs and their multivalued extension have been successfully applied in different applications for optimization (Andersen et al. (2007), Miller and Drechsler (1998), Bergman et al. (2011), Matsumoto et al. (2018)). Decomposition strategies relying on the combination of decision diagrams and integer programming, such as the one employed in this work, have been applied to other optimization problems (Bergman and Cire (2016b), Bergman and Lozano (2018), Bergman and Cire (2018), Lozano et al. (2018)). Our experiments suggest the efficiency of the proposed algorithm, with a clear superiority over a direct MIP formulation of BPMCF.

## 2 Problem overview

A formal definition of BPMCF is presented below:

**Definition 1** (Bin Packing with Minimum Color Fragmentation). Let \( k \in \mathbb{N} \) denote the number of bins, \( B \in \mathbb{N} \) be the capacity of each bin, \( G \subseteq \mathbb{N} \) be a set of colors, and \( \mathcal{O} = \{o_1, o_2, \ldots, o_n\} \) be a set of indivisible items such that, for each \( o \in \mathcal{O} \), \( s(o) \in \mathbb{N} \) denotes its size and \( g(o) \in G \) its color. A feasible solution for the problem consists of a partition of \( \mathcal{O} \) into disjoint sets \( \mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k \) such that \( \sum_{o \in \mathcal{O}_g} s(o) \leq B, \forall i \in [k] \). Let \( n_g \) denote the number of bins containing items of color \( g \). A feasible solution is said to be optimal if it minimizes \( \sum_{g \in G} n_g \).

Set \( \mathcal{O}_g = \{o \in \mathcal{O} : g(o) = g\} \) contains the items in \( \mathcal{O} \) whose color is \( g \), \( \mathcal{O}_s = \{o \in \mathcal{O} : s(o) = s\} \) is the set of items of size \( s \), and set \( \mathcal{O}_{g,s} = \{o \in \mathcal{O}_g : s(o) = s\} = \mathcal{O}_g \cap \mathcal{O}_s \) contains all items of color \( g \) and size \( s \). We use \( B \) to denote the set of bins available for packing. \( \mathcal{K}_g = \{s \in \mathbb{N} : \exists o \in \mathcal{O}_g \text{ with } s(o) = s\} \) is the set of all sizes of items belonging to \( \mathcal{O}_g \). In scenarios where bins may have different sizes, we use \( B_b \) to represent the capacity of bin \( b \).

BPP is NP-hard (Garey and Johnson (1978, 2002)), and the following results shows that BPMCF is also challenging in the general but easy in a special case.
Proposition 1. Deciding whether an arbitrary instance of \textit{BPMCF} with at least 2 bins is feasible is NP-complete.

Proof. The result follows from a reduction from the Partition Problem (\textit{PP}), which is NP-complete (Karp (1972)). Given an instance \( I \) of \textit{PP} with a set \( A \) of elements, with each \( a \in A \) of size \( s'(a) \), we create an instance \( I' \) of \textit{BPMCF} such that each item \( i \) in \( I' \) is associated with an element \( i(a) \) of \( A \), has size \( s(i(a))) = s'(a) \), and group \( g(i(a)) = 1 \), i.e., all items have the same group. Moreover, \( I' \) consists of two bins, each of size \( \frac{1}{2} \sum_{a \in A} s'(a) \). A feasible solution for \( I \) can be directly converted into a solution for \( I' \) and vice-versa, so the reduction follows. \( \square \)

Proposition 2. If each bin can contain at most two items, an optimal solution of \textit{BPMCF} can be computed in polynomial time.

Proof. We can assume without loss of generality that each item \( o \) in \( \mathcal{O} \) is such that there is at least one other item \( o' \) for which \( s(o) + s(o') \leq B \). A feasible solution should contain at least \( q = |\mathcal{O}| - |\mathcal{B}| \) pairs of items being placed in the same bin. For each possible value of \( q \), we create the following instance of the maximum weighted matching problem. Let \( G = (V, E, w) \) be a graph where \( V = V(\mathcal{O}) \cup V' \), each vertex in \( V(\mathcal{O}) \) is associated with an item in \( \mathcal{O} \) and vertices in \( V' \) contains \( |\mathcal{O}| - 2q \) artificial elements. Set \( E \) contains an edge for each pair \( \{u, v\} \subseteq V(\mathcal{O}) \); if \( g(u) = g(v) \), \( w(\{u, v\}) = 2 + |\mathcal{O}| \), and otherwise we have \( w(\{u, v\}) = 1 + |\mathcal{O}| \). \( E \) also contains an edge for each pair \( \{u, v\} \in V(\mathcal{O}) \times V' \), each with weight \( |\mathcal{O}|^2 \).

By construction, any optimal solution contains a set of edges covering all the artificial vertices, and solutions with \( q \) edges covering all the remaining elements have higher value than others with \( q - 1 \) or less. The maximum weighted matching problem can be computed in polynomial time (Edmonds (1965)), and the number of values of \( q \) that need to be inspected is bounded by \( |\mathcal{O}| \), so the result follows. \( \square \)

3 Direct Formulation

The following binary program is the direct formulation of \textit{BPMCF} used for a baseline algorithm in our computational experiments.

\[
\begin{align*}
\text{(IP)} \quad \min \quad & \sum_{(b,g) \in \mathcal{B} \times \mathcal{G}} y_{b,g} \\
\sum_{b \in \mathcal{B}} x_{b,o} &= 1 \quad \forall o \in \mathcal{O} \\
\sum_{o \in \mathcal{O}} s(o)x_{b,o} &\leq B \quad \forall b \in \mathcal{B} \\
x_{b,o} &\leq y_{b,g} \quad \forall (b,g,o) \in \mathcal{B} \times \mathcal{G} \times \mathcal{O}_g \\
x_{b,o} &\in \{0,1\} \quad \forall (b,o) \in \mathcal{B} \times \mathcal{O} \\
y_{b,g} &\in \{0,1\} \quad \forall (b,g) \in \mathcal{B} \times \mathcal{G}
\end{align*}
\]

In (IP), the assignment of each item \( o \) to each bin \( b \) is defined by binary decision variable \( x_{b,o} \). Additionally, we use \( y_{b,g} \) to indicate whether bin \( b \) contains at least one item of color \( g \). The first family of constraints of (IP) asserts that each item is assigned to exactly one bin. The second family of constraints avoids assignments where the sum of the sizes of the selected items exceeds the capacity of the bin. The last set of constraints is used to set \( y_{b,g} \); if \( x_{b,o} = 1 \) for some \( o \in \mathcal{O}_g \), \( y_{b,g} = 1 \), whereas the objective function drives \( y_{b,g} \) to zero otherwise.
4 Binary decision diagram-based algorithm

Our algorithm relies on a decomposition strategy in which the assignment of items to a bin is represented as a BDD and feasible solutions are given by the paths connecting the root node to the terminal node. The structure of a BDD depends solely on the capacity $B$ of the associated bin; we use $D^B$ to refer to the BDD for bins of size of $B$ (dropping $B$ if it is clear from context).

Notation: A BDD $D^B = (N^B, A^B, v^B, d^B)$ is a layered-acyclic graph composed of a set of nodes $N^B$, a set of arcs $A^B$, together with a cost function $v^B : A^B \to \mathbb{R}$ and arc-domain function $d^B : A^B \to \{0, 1\}$ defined on the arcs. Nodes in $N^B$ are partitioned into a set $L^B = \{0, 1, \ldots, |O|, |O| + 1\}$ of layers. For every node $u \in N^B$, $t^B(u)$ denotes the layer where $u$ belongs. Layer 0 contains only the root node $r^B$ of $N^B$, and layer $= |O| + 1$ contains its terminal node $t^B$. Each layer $l \in \{1, \ldots, |O|\}$ is associated with an item $o(l)$; analogously, we define $o(u) = o(l^B(u))$ for each node $u \in N$. We assume the layers are ordered by groups first (any arbitrary ordering of $G$ may be employed) and then arbitrarily in each group. Each arc $a \in A^B$ is directed from a start-node $u^s(a)$ located in layer $l^B(h(a)) \in L^B\setminus t$ to an end-node $u^e(a)$ located in layer $l^B(u^e(a)) = l^B(u^s(a)) + 1$. That is, each arc connects two nodes in consecutive layers. Every node $u \in N^B\setminus \{0^B\}$ has at most two arcs directed out of it with unique arc domains, referred to as the nodes one-arc if $d(a) = 1$ and its zero-arc if $d(a) = 0$.

In our BDD formulation, each arc $a$ represents the decision about the inclusion of the item associated with its start-node $u^s(a)$ determined by the arc-domain of $a$. Every root-to-terminal arc-specified path therefore encodes a collection of items defined by the one-arcs on the path. Formally, let $p = (a^1, \ldots , a^{|O|})$ be a $r^B$ to $t^B$ path. This path corresponds to the collection of items $O(p) := \{o(l) : d(a^l) = 1\}$ and has cost $C(p) := \sum_{i=1}^{|O|} v^B(a^i)$. Each BDD $D^B$ will satisfy the following two properties. (1) There is a one-to-one mapping between collection of items $\tilde{O} \subseteq O$ with $\sum_{o \in \tilde{O}} s(o) \leq B$ and root-to-terminal paths $p$ with $O(p) = \tilde{O}$. (2) For every root-to-terminal path $p$, $C(p)$ equals the number of groups present in $O(p)$ (i.e., $C(p) = |\{g : g(o) = g \text{ for } o \in O(p)\}|$).

Any such BDD is called an exact BDD for a bin of size $B$.

Given an instance of BPMCF, suppose we build such a BDD for each bin; i.e., for $b \in B$, we build $D^B_b$, which is an exact BDD for a bin of size $B_b$. Any collection of paths $p^i$, one from each BDD, for which the family of sets $O(p^i), b \in B$ are mutually exclusive and collectively exhaustive of $O$ will provide a feasible solution where items in $O(p^i)$ are placed in bin $b$. Additionally, $\sum_{b \in B} C(p^i)$ will correspond to the objective function value of the solution. The BPMCF can therefore be solved by finding such a collection of BDDs, and identifying a mutually exclusive and collectively exhaustive collection of paths that minimizes $\sum_{b \in B} C(p^i)$, a problem recently introduced into the literature as the consistent path problem (Bergman and Cire (2018), Lozano et al. (2018), Bergman and Lozano (2018), Raghunathan et al. (2018)).

Construction: Building BDDs for discrete optimization problems has been a major research focus in the last decade (Hadzic et al. (2008), Bergman et al. (2011, 2016a,b, 2012)). Given a bin of size $B$, the family of sets of items that fit in the bin corresponds exactly to the set of feasible solutions to a knapsack problem, where each item $o$ has size $s(o)$ and the knapsack has capacity $B$. The dynamic programming-based construction algorithm of a BDD representing such a set of solutions is well known (Trick (2003), Behle (2008), Bergman and Cire (2016a), Bergman et al. (2018)). We adopt this algorithm, but with additional care required because of the objective function.

We build a BDD iteratively, layer-by-layer, by associating with each node $u$ a state $z(u) \in \mathbb{Z} \cup \mathbb{B}$
that will represent the remaining capacity in the bin for any partial solution defined by paths starting from the root and ending at \( u \) and whether or not any item with the color of the object in layer \( l(u) \) is selected. The root node \( r \) is assigned state \((B, 0)\). The first index represents the remaining capacity of the bin, and the second represents whether or not an item of the same color as the item \( l(o) \) has been selected or not. Recall that the items corresponding to the layers of the BDD are ordered by color.

Having constructed layer \( l \), we build layer \( l+1 \) by iteratively processing the nodes in \( l \). When processing node \( u \) with state \( z(u) = (A, d) \), we calculate the states \( z_0/z_1 \) that will arise from a zero-arc/one-arc directed out of \( u \). \( z_0 = (A, d) \) and \( z_1 = (A - s(l(o)), 1) \). Since a zero-arc dictates the omission of item \( o \), neither \( A \) of \( d \) changes. A one-arc dictates the inclusion of item \( o \) and so the remaining capacity is \( A - s(l(o)) \) and an object with this color is selected.

For any state \( z_0 \) we create a zero-arc \( a_0 \) directed to a node \( u_0 \), and assign that node state \( s(u_0) = z_0 \). If \( A - s(l(o)) < 0 \), we do not create a one-arc, because this would exceed the capacity of the bin. If \( A - s(l(o)) \geq 0 \), we create a one-arc \( a_1 \) directed to a node \( u_1 \), and assign that node state \( s(u_1) = z_1 \). The cost of the arc is 0 for \( a_0 \) and the cost of arc \( a_1 \) is 1 if the second component of \( s(u) \) is 0, and is 0 otherwise. This particular assignment of costs is because only when a one-arc results in the first item of a color being selected should a cost be assigned, as this will indicate that there are items from this color category that are selected for this bin should this arc be selected.

If the item corresponding to layer \( l \) is the last item with color \( g \), we now change the second component of both \( s(u_0) \) and \( s(u_1) \) to 0, because in the subsequent layer \( l + 1 \) no objects of color \( g(o(l + 1)) \) have previously been considered.

Finally, for \( d = 0, 1 \), if any node \( \tilde{u} \) previously added to layer \( l + 1 \) has state \( z_d \), the \( d \)-arc directed out of \( u \) is directed to \( \tilde{u} \). Otherwise, a new node \( \hat{u} \) is created, added to layer \( l + 1 \), with the \( d \)-arc directed from \( u \) to \( \hat{u} \). After constructing all layers, each node in layer \( l = n + 1 \) is merged into a single terminal node.

**Example:** Consider an instance with 5 items of sizes 2,3,2,3,2 and colors 1,1,1,2,2, respectively, with bin capacity 4. A BDD for this bin is presented in Figure 1. Each layer corresponds to an item. The solid / dashed arcs correspond to one-arcs / zero-arcs. The arc costs are specified next to each arc. Note that there can be one arcs with zero cost (e.g., the arc from \( u_5 \) to \( u_9 \)). Also, any solution corresponds to a path. For example, selecting items \( o_1 \) and \( o_3 \) is a feasible solution that corresponds to the arc-directed path \( r - u_2 - u_5 - u_9 - u_{14} - t \).

![Figure 1: Example BDD](image)
Network flow model: The BDDs allow us to formulate the consistent path problem through (ANF), an Arc-based Network Flow MIP to solve the problem.

\[
(\text{ANF}) \quad \min \sum_{b \in B} \sum_{a \in A^b} v(a) y_{b,a} \\
\sum_{a \in A^b; u'(a) = u} y_{b,a} - \sum_{a \in A^b; u'(a) = u} y_{b,a} = 0 \quad \forall b \in B, u \in N \setminus \{0^B, t^B(b)\} \\
\sum_{a \in A^B(b); u'(a) = 0^B b} y_{b,a} = 1 \quad \forall b \in B \\
\sum_{a \in A^B(b); u'(a) = t^B b} y_{b,a} = 1 \quad \forall b \in B \\
\sum_{b \in B} \sum_{a \in A^B; g_{u'(a)} = g, k_{t^B(b)} = k} w(a) y_{b,a} = |O_{g,k}| \quad \forall (g, k) \in G \times K_g \\
y_{b,a} \in \{0, 1\} \quad \forall b \in B, a \in A^B
\]

(ANF) employs binary variables \(y_{b,a}\), which indicate whether arc \(a\) composes the path selected for \(D^b\). The first three families of equalities model the network flow constraints for each bin \(b\) in \(B\). The last family of constraints asserts that each item is picked exactly once, so they are the joint constraints of (ANF).

5 Computational Experiments

All code used to evaluate the model and algorithms presented were implemented in C++ and utilize commercial software Gurobi 8.0.0 to implement IP and ANF (Gurobi Optimization (2018)); we used all default settings of the solver. All experiments were executed on an Intel CPU Core i7-4770 with 3.4 GHz, 32 GB of RAM. Each execution was restricted to a single thread and to a time limit of 30 minutes.

Instance generation: We generated synthetic instances. For each selected combination of \(k\) and \(B\), 10 instances were generated; in each individual instance, all bins have the same capacity. Item sizes are randomly generated according to the following distribution: size 2 with probability 0.4; size 3 with probability 0.3; size 4 with probability 0.2; and size 5 with probability 0.1. This distribution was selected because of the authors’ experience with group seating optimization applications.

Items are generated uniformly and independently at random from the above distribution until 85% of the overall capacity is occupied. We then sequentially assign colors to the items by selecting \(p\) items to form each color class. With probability 0.6 we selected \(p \in \{2, 3, 4\}\) items, and with probability 0.4 we select between \(p \in \{5, 6, 7, 8\}\), in both cases sampled uniformly at random. If only one item remains, we assign it to the last color. We restrict our experiments to scenarios where \(B \geq 8\), as instances with smaller bins can be efficiently solved (see Proposition 2). We generate instances with \(k \in \{10, 20, 30\}\) and \(B \in \{8, 10, 12\}\), and additionally instances with \(k = 50\) and \(B = 12\) to evaluate how well ANF scales.

Results: The results of our experiments are shown in Table 1 in aggregation. Each row corresponds to a configuration of instances with \(k, B\), as indicated by the first and second columns. The next eight columns report solution statistics, first for IP and then for ANF. In sequence, we
| Instances | IP | ANP |
|-----------|----|-----|
| $k$       | $B$ | Time | LB | UB | Gap | Time | LB | UB | Gap |
| 10        | 8   | 59.55° | 10.1 | 10.1 | 0.0 | 0.10° | 10.1 | 10.1 | 0.0 |
| 10        | 10  | 52.55° | 11.2 | 11.4 | 1.6 | 0.50° | 11.4 | 11.4 | 0.0 |
| 10        | 12  | 428.96° | 11.2 | 11.7 | 4.0 | 66.15° | 11.7 | 11.7 | 0.0 |
| 20        | 8   | -    | 16.4 | 21.2 | 22.3 | 0.98° | 21.2 | 21.2 | 0.0 |
| 20        | 10  | -    | 18.9 | 22.6 | 16.2 | 86.05° | 22.6 | 22.6 | 0.0 |
| 20        | 12  | -    | 19.4 | 23.7 | 17.9 | 454.6° | 23.7 | 23.7 | 0.9 |
| 30        | 8   | -    | 23.1 | 31.9 | 27.6 | 37.52° | 31.9 | 31.9 | 0.0 |
| 30        | 10  | -    | 27.7 | 34.1 | 18.5 | 802.51° | 34.0 | 34.1 | 0.2 |
| 30        | 12  | -    | 28.0 | 34.6 | 18.9 | - | 30.4 | 34.8 | 12.4 |
| 50        | 10  | -    | 40.9 | 56.2 | 27.3 | - | 50.0 | 56.2 | 10.9 |

Table 1: Aggregate summary of results.

Figure 2: Cumulative distribution plot comparing BDD with IP.

report the average solution times for those instances that were solved within 1800 seconds, with
the number of instances solved within 1800 seconds in superscript, the average ending lower bound,
the average ending upper bound, and the average gap.

We see a considerable superiority of ANF over IP, both in terms of gap and running time. IP
solves only those instances with $k = 10$ (and only solves 24 of the 30 instances with this $k$) while
ANF solves all instances with $k = 10$ and $k = 20$, and even 10 with $k = 30$. Additionally, the
ending gap and quality of solutions are significantly better, even for those instances unsolved by
both.

A depiction of the solution time and ending gaps is provided in the Figure 2 through a cumulative
distribution plot of performance. For both algorithms, the left half provides a plot with height equal
to the cumulative number of instances solved at the time given on the horizontal axis. In the right
half, the height of the plot corresponds to the number of instances with at most the optimality
gap given on the horizontal axis by the time limit of 1800 seconds. Figure 2 more readily depicts
the overall performance of ANF. After any amount of time, ANF solves more instances than IP,
with smaller gaps at time limit.
6 Conclusion and future work

In this work, we have introduced the *bin packing with minimum color fragmentation* and presented an algorithm consisting of the integration of decision diagrams and mixed-integer linear programming. Namely, we showed how to represent the assignment of items to individual bins as binary decision diagrams and formulated the integration of the sub-problems using a network flow model. Our computational experiments have shown that the proposed algorithm scales well and is clearly superior to a direct formulation of **BPMCF**.

In future work, we intend to investigate the performance of the proposed algorithm in real-world scenarios. Additionally, we also would like to investigate alternative advanced solution approaches that have been successfully applied to other variants of the colored bin packing problem, such as branch and price.

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