EMERGENT PLANCK MASS AND DARK ENERGY FROM AFFINE GRAVITY

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We introduce a novel model of affine gravity, which implements the no-scale scenario. Namely, the Planck mass and Hubble constant emerge dynamically in our model, through the mechanism of spontaneous breaking of scale invariance. This naturally gives rise to inflation, thus introducing a new inflationary mechanism. Moreover, the time direction and nondegenerate metric emerge dynamically as well, which allows considering the usual General Relativity as an effective theory. We show that our model is phenomenologically viable, both from the perspective of the direct tests of gravity and from the standpoint of cosmological evolution.

Keywords: general relativity, affine gravity, dark energy, spontaneous symmetry breaking, scale invariance

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1. Introduction

The power of General Relativity (GR) comes from the fact that it combines profound physical ideas with an elegant mathematical apparatus. By generalizing the latter, one may hope to gain a deeper insight into gravity. One of the possible routes to that goal is known as affine gravity [1]–[9]. The idea of this approach is based on the fact that any gravitational theory must include the connection field, but not (necessarily) the metric. The only independent field in affine gravity is the connection, while the metric is introduced as the momentum canonically conjugate to the connection. This suggests a new look on the independent variables in gravity, which, for example, might be important for its quantization. However, it is known that affine gravity is phenomenologically equivalent to GR with a nonzero cosmological constant term [10]–[12]. In this context, the standard affine gravity does not provide new insights into the physics of gravity.

The aim of this paper is to introduce a novel model of affine gravity, which we call scalar–affine gravity. In comparison with the standard affine gravity and GR, this model has the following features.

First, scalar–affine gravity is scale invariant and realizes the no-scale scenario of [13]–[18]. Namely, we start with a Lagrangian including only dimensionless constants. Then all physically important quantities, including the Planck mass and Hubble parameter, emerge dynamically through the mechanism of spontaneous symmetry breaking (SSB). In particular, this naturally gives rise to inflation. None of the previously considered models, including Born–Infeld-type gravities, have these features.
Second, the vacuum solution in scalar–affine gravity is a de Sitter-like space. Specifically, the metric is the same as in the de Sitter space, but the background connection is modified due to nonmetricity. Such a configuration of the vacuum is achieved by introducing a scalar field nonminimally coupled to gravity. Remarkably, the scalar field effectively acts as dark energy, thus providing an alternative to the cosmological constant term.

Finally, the time direction and the metric also emerge dynamically in our model. Namely, we start with the general notion of a 4-dimensional manifold with an affine connection. Then, due to SSB, a non-degenerate metric appears and one of the directions becomes distinguishable, playing the role of the time direction. In this context, our model realizes the idea in [19], [20] that the metric can appear as a field breaking the general covariance to the Poincaré subgroup. We strengthen this approach by showing that the corresponding SSB also induces spontaneous origination of the time direction.

The model that we introduce has several drawbacks, which we discuss in the main part of the paper. However, we believe that the mentioned features of scalar–affine gravity can rekindle the interest in the study of such models.

This paper is organized as follows. In Sec. 2, we give an overview of affine gravity and introduce the notation. In Sec. 3, we present our model and discuss its features. In Sec. 4, we show that our model is phenomenologically viable, both from the perspective of the direct tests of gravity and from the standpoint of cosmological evolution. Finally, in Sec. 5, we discuss the results and possible ways of generalizing the model.

2. Affine gravity

2.1. Overview of affine gravity. Depending on the independent variables, all gravitational theories can be classified into three categories [12].

The most often encountered are metric theories of gravity. In such theories, the metric is the only source of gravity and the connection is fixed to be given by the Christoffel symbol of the metric. For example, GR belongs to this type of theories.

Metric–affine theories, also known as the first-order (Palatini) formulations of gravity, form the second category. In this approach, the metric and the connection are regarded as independent variables. It is known that for a given \( f(R) \) theory, the metric and the metric–affine approaches yield different physical observables [21].

The third class of gravitational theories comprises the affine theories. In such models, the only source of gravity is the connection, while the metric is introduced as the momentum canonically conjugate to the connection [3], [7]. Affine theories of gravity were first formulated by Einstein and Eddington [1], [2], further developed by Schrödinger [3]–[6], and later by Ferraris and Kijowski [7], [8].

In affine gravity, the only available tensor is the Riemann curvature tensor

\[
R^\sigma_{\mu \nu \rho} = 2 \partial_{[\mu} \Gamma^\sigma_{\nu \rho]} + 2 \Gamma^\sigma_{[\mu} \Gamma^\lambda_{\nu \rho]}, \quad (2.1)
\]

Here and hereafter, we use parentheses and square brackets to respectively denote symmetrization and antisymmetrization in the corresponding indices with proper weights. We also fix the dimension of the manifold to 4 and prescribe Greek letters to take values from 0 to 3. The contractions of the Riemann tensor give rise to the Ricci tensor

\[
R_{\mu \nu} \equiv R^\sigma_{\mu \nu \sigma} = 2 \partial_{[\mu} \Gamma^\sigma_{\nu \lambda]} + 2 \Gamma^\rho_{[\mu} \Gamma^\lambda_{\nu \rho]}, \quad (2.2)
\]

and to the homothetic curvature tensor

\[
S_{\mu \nu} \equiv R^\sigma_{\mu \nu \sigma} = 2 \partial_{[\mu} \Gamma^\sigma_{\nu \rho]}, \quad (2.3)
\]
The homothetic curvature coincides with the antisymmetric part of the Ricci tensor in the case of a symmetric connection, and is an independent field otherwise. These are the building blocks of affine theories of gravity.

Throughout the paper, we assume that the connection is torsion-free and, for the reasons to become clear shortly, use the symmetric part of the Ricci tensor. Then, in the absence of additional fields, the only diffeomorphism-invariant Lagrangian, up to multiplication by an arbitrary constant, is given by

$$
\mathcal{L} = \sqrt{-\text{det}(R_{(\mu\nu)}(\Gamma))}.
$$

This is the Lagrangian of the simplest affine gravity model.

Because the Lagrangian above has a square root form, affine gravity can be considered a subclass of Born–Infeld gravity [22]. However, in most theories of the Born–Infeld type, the metric is considered an independent primary field. Because the question of identifying proper independent degrees of freedom might be of crucial importance (for example, for constructing quantum gravity), we prefer to distinguish affine gravity as a separate class.

There are two main ways to show that affine gravity is equivalent to GR in the first-order formalism. The first approach employs the fact that in the first-order formalism, the GR Lagrangian does not include derivatives of the metric. This allows regarding the metric as a Lagrange multiplier and integrating it out. In the case of a nonzero cosmological constant, the resulting Lagrangian is that of affine gravity. Indeed, we consider GR with a nonzero cosmological constant term,

$$
\mathcal{L}_\Lambda = \sqrt{-g(M_{Pl}^2 R + 2\Lambda)},
$$

where $M_{Pl}$ is the Planck mass. Then by taking the determinant of the Einstein equations, we obtain

$$
\sqrt{-g(M_{Pl}^2 R + 2\Lambda)} = 2\frac{M_{Pl}^2}{\Lambda} \sqrt{-\text{det} \, R_{(\mu\nu)}},
$$

which proves the equivalence.

The second way exploits the definition of the metric as the momentum canonically conjugate to the connection. We demonstrate this approach with a concrete example, closely following [7]. We consider the Lagrangian

$$
\mathcal{L}_{\text{ex}} = \frac{2}{m^2 \phi^2} \sqrt{-\text{det}(\partial_\mu \phi \partial_\nu \phi - M_{Pl}^2 R_{(\mu\nu)})},
$$

where $m^2$ is some constant and $\phi$ is a scalar field. We introduce $g^{\mu\nu}_\lambda$ as the momentum canonically conjugate to the connection,

$$
g^{\mu\nu}_\lambda = \frac{\delta \mathcal{L}}{\delta \partial_\rho \Gamma^{\lambda}_{\mu\nu}}.
$$

Further, by introducing $g^{\mu\nu}$ as

$$
g^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta R_{(\mu\nu)}},
$$

we can express $g^{\mu\nu}_\lambda$ via $g^{\mu\nu}$ only,

$$
g^{\mu\nu}_\lambda = \delta^{(\mu}_{\lambda} g^{\nu)} - \delta^{\mu}_{\lambda} \delta^{\nu}_{\rho} g^{\rho\nu}.
$$

We note that $g^{\mu\nu}$ is automatically symmetric in its indices due to the symmetrization of $R_{\mu\nu}$ in the Lagrangian, and that its mass dimension equals two.
Lagrangian (2.7) is a function of the connection, the field \(\psi\), and their derivatives. Then, because the derivatives of the connection enter Lagrangian (2.4) only via the Ricci tensor, its full differential can be written as
\[
dL = J^\mu_{\nu\lambda} d\Gamma^\lambda_{\mu\nu} + g^\mu\nu dR_{(\mu\nu)} + J d\psi + P^\mu d\partial_\mu \psi
\]
for some \(J^\mu_{\nu\lambda}\), \(J\) and \(P^\mu\). To demonstrate the equivalence to GR, we perform the Legendre transformation by contact-deforming Lagrangian (2.7) in the gravitational sector,
\[
L = \mathcal{L} - g^{\mu\nu} R_{(\mu\nu)}.
\]
In this “Hamiltonian” formulation of gravity, \(g^{\mu\nu}\) should be regarded as an independent field, while \(R_{(\mu\nu)}\) as a function of \(g^{\mu\nu}\) and \(\Gamma^\lambda_{\mu\nu}\). The full differential of (2.12) is thus
\[
dL = J^\mu_{\nu\lambda} d\Gamma^\lambda_{\mu\nu} - R_{(\mu\nu)} dg^{\mu\nu} + J d\psi + P^\mu d\partial_\mu \psi.
\]
By definition, this implies
\[
R_{(\mu\nu)} = -\frac{\delta L}{\delta g^{\mu\nu}}.
\]
Then, after introducing the usual metric \(g_{\mu\nu}\) as
\[
\bar{g}^{\mu\nu} = -M^2_{\text{Pl}} \sqrt{-g} g^{\mu\nu}
\]
and using the formula for the compound derivative, Eq. (2.14) can be rewritten in the form of the Einstein equations [7].

It remains to note that the variation of the action with respect to \(\Gamma^\lambda_{\mu\nu}\) yields the metric-compatibility condition,
\[
\nabla_\lambda (\sqrt{-g} \bar{g}^{\mu\nu}) = 0,
\]
and that \(\psi\) obeys the Klein–Gordon equation for a scalar field with mass \(m^2\). Thus, Lagrangian (2.7) can be replaced by the equivalent Lagrangian of a usual massive field coupled to the standard GR.

### 2.2. Prospects and problems of affine gravity.

If affine gravity is equivalent to GR, one may wonder what are the benefits of using the affine formalism. We believe that there are two major benefits.

First, affine gravity illustrates that the metric might not be a fundamental field [7]. Instead, in this approach it appears as the momentum conjugate to the connection, whose equations of motion are precisely the Einstein equations. As we discuss in Sec. 5.2, this change of perspective on the fundamental variables allows approaching gravity as a gauge theory of the connection. The point above also makes affine gravity more natural than GR from the geometrical perspective. Indeed, the notion of the connection is of primary importance for comparing vectors at different points. On the other hand, the metric, although it has a clear physical meaning, is always an additional structure. In contrast to GR, affine gravity exploits only the primary geometrical object, the connection, and by Eq. (2.9) unambiguously defines the physical metric.

The second reason is that affine gravity allows a wide range of modifications. For example, one can choose to not symmetrize the \(R_{\mu\nu}\) indices in Lagrangian (2.4), thus obtaining a nonsymmetric metric. Previously, this was regarded as a possibility to incorporate electromagnetism and strong interactions [1], [4] into a unified geometrical model. However, the studies showed that this and similar ideas cannot be successfully implemented. Nonetheless, possible extensions of affine gravity are the subject of ongoing research [23]–[27], including newly introduced polynomial affine gravity [28], [29] and affine-based models of inflation [30], [31].
At this point, we also make contact with the idea that gravity should be described by a spin-2 field with spin-2 gauge invariance or general covariance [32]–[35], [20]. An important remark is that in this approach, the graviton does not have to be a fundamental particle. Instead, it might be some composite or even auxiliary field. Hence, gravity theories based on the ideas in [32]–[35], [20] should be regarded as effective theories of gravity, and affine gravity is consistent with these ideas.

A major problem in affine gravity is that it does not allow a unified description of the Standard Model and gravity. Namely, we can add the Lagrangian of the Standard Model to the Hamiltonian of gravity, Eq. (2.12), but it is unknown how (and if) such a full theory can be translated back to the Lagrangian formulation of gravity, i.e., in the form similar to Eq. (2.4). In particular, it is unknown how fermions should be introduced in the Lagrangian formulation.

3. Scalar–affine gravity

We introduce the scalar–affine gravity. As the name of the model suggests, we introduce a scalar field $\varphi$, which is a scalar tensor density of weight $w$. Accordingly, its covariant derivative is given by

$$\nabla_\mu \varphi = \partial_\mu \varphi - w \Gamma^a_{\mu a} \varphi.$$  \hspace{1cm} (3.1)

We note that this introduces a nonminimal coupling between $\varphi$ and gravity. The Lagrangian of our model is

$$\mathcal{L} = \sqrt{-\det L_{\mu\nu}} + \alpha \varphi^{-1} \left( L_{\mu\nu} = R_{\mu\nu} + \frac{c}{2} \frac{\nabla_\mu \varphi \nabla_\nu \varphi}{\varphi^2} \right).$$ \hspace{1cm} (3.2)

where $\alpha$ and $c$ are some constants. By redefining $\varphi$, we can set $w = 1$ and $\alpha = 1$, which is used in what follows.\footnote{For a general $w$, this procedure may require redefining $\varphi$ to be a purely imaginary field. In that case, by $\varphi$'s value one should understand its absolute value.}

We discuss this Lagrangian. First, we note that in order to compensate the transformation of the volume element under an arbitrary change of coordinates, the Lagrangian must be a scalar density of weight +1. This implies that $L_{\mu\nu}$ must be an absolute tensor, i.e., a density of weight zero. Then the only allowed first-order kinetic term for $\varphi$ is the one introduced in Lagrangian (3.2).

Further, we note that our model is scale-invariant: it does not include dimensional constants. In terms of symmetry transformations, it is invariant under the transformations

$$\Gamma^p_{\mu\nu}(x) \rightarrow \lambda \Gamma^p_{\mu\nu}(\lambda x), \quad \varphi(x) \rightarrow \lambda^4 \varphi(\lambda x),$$ \hspace{1cm} (3.3)

where $\lambda$ is some constant. These transformation rules are direct analogues of the standard, metric-based formulation of scale invariance. Indeed, keeping the definition of the metric as the momentum canonically conjugate to the connection, Eq. (2.9), the metric $g^{\mu\nu}$ transforms under (3.3) as

$$g^{\mu\nu}(x) \rightarrow \lambda g^{\mu\nu}(\lambda x).$$ \hspace{1cm} (3.4)

Taken together with the transformation of $\varphi$, this gives the standard definition of scale invariance.

Lagrangian (3.2) is not the most general one that can be written using the available tensors. Possible additional terms fall into two categories.

First, one can add a second-order derivative of $\varphi$ to $L_{\mu\nu}$,

$$L_{\mu\nu} \rightarrow L_{\mu\nu} + c_1 \nabla_\mu \frac{\nabla_\nu \varphi}{\varphi}.$$ \hspace{1cm} (3.5)
with some constant $c_1$. This term, however, violates the equivalence to GR. Indeed, because it contains derivatives of the connection, the contact-deformed Lagrangian, Eq. (2.12), cannot be written in the form similar to (2.13). This breaks down identities (2.14), which are, in fact, the Einstein equations. Because there is no phenomenological evidence for considering such modified theories of gravity, we forbid terms like (3.5) in the Lagrangian.

Second, one can introduce more root structures into the Lagrangian, such as

$$
L_{\text{add}} = \sqrt{-\det c_2 \nabla_\mu \varphi \nabla_\nu \varphi} \varphi^2, \quad (3.6)
$$

where $c_2$ is some constant. We expect that in a complete theory of affine gravity, some symmetry would restrict the Lagrangian to be a certain function of $R_{\mu\nu}$ and other fields. In particular, it would forbid the additional square-root terms. At present, we do not know the corresponding symmetry, but assume that the terms like the one above are forbidden. Finally, one can also consider the presented model as an illustrative one, showing the possibilities of the approach.

Theories most similar to the introduced one were considered in [22], [36]. However, none of them is scale invariant or includes coupling between the matter fields and the affine connection. In this perspective, scalar-affine gravity represents a new model, not studied anywhere previously.

We now qualitatively discuss the dynamics of our model. The $\varphi^2$ term appearing in the denominator of the kinetic term for $\varphi$ initiates spontaneous breaking of scale invariance. Indeed, the $\varphi$ equations of motion are given by

$$
- \nabla_\lambda \left( g^{\mu\nu} \frac{\delta L_{\mu\nu}}{\delta \nabla_\lambda \varphi} \right) + g^{\mu\nu} \frac{\delta L_{\mu\nu}}{\delta \varphi} + 1 = 0. \quad (3.7)
$$

Because this equation contains a term with a negative power of $\varphi$, as well as a constant term, we have $\varphi \neq 0$ and $\varphi \neq \infty$ on the solution. In turn, this forces $g^{\mu\nu} \neq 0$. This is the mechanism for the emergence of a nondegenerate metric and dimensional parameters in our model.

To study the dynamics of the model, we replace Lagrangian (3.2) with the equivalent one

$$
L_{\text{equiv}} = -4 \sqrt{-\det \tilde{g}^{\mu\nu}} + \tilde{g}^{\mu\nu} L_{\mu\nu} + \varphi. \quad (3.8)
$$

Here, it is understood that $\tilde{g}^{\mu\nu}$ and the connection are independent variables. Two Lagrangians (3.2) and (3.8) are equivalent because by integrating $\tilde{g}^{\mu\nu}$ out from (3.8) we arrive at the initial Lagrangian. In particular, because the $\tilde{g}^{\mu\nu}$ equations of motion coincide with those for $g^{\mu\nu}$, we omit tildes over $\tilde{g}^{\mu\nu}$ in what follows. Thus, by introducing $g^{\mu\nu}$ as an independent field, we eliminate the square root structure and arrive at a commonly known first-order formulation of gravity.

As we have noted, $\varphi \neq 0$ on the solution. This allows us to redefine the fields as

$$
\varphi = \varphi_0 e^\pi, \quad g^{\mu\nu} = -\frac{\varphi_0^2}{\varphi} \sqrt{-g} g^{\mu\nu} \quad (3.9)
$$

where $\varphi_0$ is some constant of unit mass dimension and $g^{\mu\nu}$ is a nondegenerate symmetric absolute rank-2 tensor. Now $\pi$ can be thought of as the dilaton and $g_{\mu\nu}$ as the metric. We use this parameterization of the fields in what follows.

It is convenient to start the analysis of the model by considering the equations of motion in the gravitational sector. The variation of the action with respect to $\Gamma_\mu^{\lambda\nu}$ yields

$$
\nabla_\lambda g^{\mu\nu} - \delta^{[\mu}_\lambda \nabla_\sigma g^{\nu]}_{\sigma} = cg^{\sigma(\mu} \delta^{\nu)}_\lambda \nabla_\sigma \varphi. \quad (3.10)
$$
Importantly, this equation implies
\[
\left(1 + \frac{c}{3}\right)\Gamma^\sigma_{\mu\nu} = \partial_{\mu} \ln \sqrt{-g} + \frac{c}{3} \partial_{\mu} \ln \varphi.
\] (3.11)

For the special value \(c = -3\), this reduces to
\[
\varphi = \kappa \varphi_0 \sqrt{-g},
\] (3.12)

where \(\kappa\) is some constant. Thus, in this special case, \(\varphi\) is not an independent degree of freedom, and the total number of the degrees of freedom coincides with that in GR. This is our motivation for fixing \(c = -3\).

As we demonstrate below, this is consistent with the \(\varphi\) equation of motion and ensures that the Planck mass is a constant (as a function of time).

The solution of Eq. (3.10) is given by
\[
\Gamma^\lambda_{\mu\nu} = \left\{\begin{array}{l}
\lambda \\
\mu \\
\nu
\end{array}\right\} - \frac{1}{2}(v_{\nu} \delta^\lambda_{\mu} + v_{\mu} \delta^\lambda_{\nu} - 3g_{\mu\nu}g^{\lambda\rho}v_{\rho}), \quad v_{\mu} = \frac{\nabla_{\mu} \varphi}{\varphi},
\] (3.13)

where \(\{\lambda_{\mu\nu}\}\) is the Christoffel symbol of the metric \(g_{\mu\nu}\). As we see, there is some nonzero nonmetricity due to the coupling of \(\varphi\) to the connection.

Having established these facts, we proceed to solving the Einstein equations. For this, we first obtain the energy–momentum tensor for \(\varphi\). From the \(\varphi\) equation of motion, we know that \(\nabla_{\mu} \varphi \neq 0\). Therefore, we can choose coordinates in which the \(\nabla_{\nu} \varphi\) direction coincides with the 0 axis,
\[
\frac{\nabla_{\mu} \varphi}{\varphi} = (-2v, 0, 0, 0),
\] (3.14)

where \(v\) is an arbitrary function of the coordinates. We let \(T^\varphi_{\mu\nu}\) denote the energy–momentum tensor plus terms proportional to \(v^2\) coming from the Einstein tensor due to nonmetricity. Then
\[
T^\varphi_{\mu\nu} = -3v^2g_{\mu\nu}.
\] (3.15)

We see that \(\varphi\) effectively acts as dark energy. Further, by using the redefinition of fields in Eq. (3.9), we see that \(\det g^{\mu\nu}\) in Lagrangian (3.8) also acts as a cosmological constant term. Therefore, the solution is an Einstein manifold, with the metric
\[
g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t)).
\] (3.16)

We note that the \(-1\) element of the metric correlates with the \(\nabla_{\mu} \varphi\) direction. This is must be the case because each of them defines a distinguished direction, and these must coincide. Thus, the time direction emerges dynamically in our model, with \(\nabla_{\mu} \varphi\) defining the proper time axis.

For metric (3.16), the nonzero components of the connection are
\[
\Gamma^0_{ij} = a^2(H + 3v)\gamma_{ij}, \quad \Gamma^i_{0j} = (H + v)\delta^i_j, \quad \Gamma^0_{00} = -v,
\] (3.17)

where \(H \equiv a'/a\), Latin indices stay for spatial components, \(i = 1, 2, 3\), the prime denotes differentiation with respect to time, and \(\gamma_{ij} = a^{-2}g_{ij}\) is the flat metric. From the Einstein equations (or, equivalently, by varying the action with respect to \(g^{\mu\nu}\)), we then deduce
\[
H' = 0,
\] (3.18a)
\[
3(H^2 + 3Hv + v') = 2\varphi_0^2.
\] (3.18b)

Hence, \(H\) is a constant while \(v\) might be time-dependent.
We now consider the $\varphi$ equation of motion. After taking into account that $\varphi$ follows the dynamics of the determinant of the metric, Eq. (3.12), this equation of motion takes the form

$$6(3Hv + v') = -\kappa \varphi_0^2. \quad (3.19)$$

This and Eq. (3.18b) form a full set of equations of scalar–affine gravity. Any of the dimensional parameters (e.g., $\varphi_0$) can be chosen as a reference unit. Then these equations define the ratios of the other dimensional parameters to the reference one. Thus, scalar–affine gravity realizes the no-scale scenario.

By comparing Eqs. (3.9) and (2.15) and applying the arguments from Sec. 2, we conclude that $\varphi_0$ is the Planck mass. Then we have\(^2\)

$$\left( \frac{H}{M_{Pl}} \right)^2 = \frac{\kappa + 4}{6}. \quad (3.20)$$

As we demonstrate below, for the mass of elementary particles to be much smaller than the Planck mass, we must have $\kappa \ll 1$. In this case, the Hubble constant is of the order of the Planck mass. This naturally gives rise to inflation as a consequence of spontaneous breaking of scale invariance, which is accompanied by the emergence of the metric. This seems to be an interesting starting point for the “birth” of the Universe. However, at present, we do not know how to end the inflation, and leave this question for future study.

We now discuss how fields can be added into the model with the example of a scalar field $\chi$. Because we already have a scalar density field $\varphi$ in the theory, $\chi$ can be introduced as an absolute tensor. Then requiring the Lagrangian to be a sum of the square root term and a potential, the most general Lagrangian is given by

$$\mathcal{L} = \sqrt{-\det(L_{\mu\nu} + \frac{q}{2}\partial_\mu \chi \partial_\nu \chi)} + \varphi(1 + \beta \chi^2), \quad (3.21)$$

where $\beta$ is some constant and $q = \pm 1$. Assuming that $\beta > 0$ (thus $\chi$ does not participate in SSB), the equivalent first-order Lagrangian, with the metric as an auxiliary field, is

$$\mathcal{L}_{\text{equiv}} = -4\sqrt{-\det g_{\mu\nu}} g^{\mu\nu}L_{\mu\nu} + \varphi + g^{\mu\nu} \frac{q}{2}\partial_\mu \chi \partial_\nu \chi + \beta \varphi \chi^2. \quad (3.22)$$

We also note that, unlike the usual fields in common field theories, the $\chi$ mass dimension is zero. To normalize it conventionally, we redefine it as

$$\chi \rightarrow \varphi_0^{-1} \chi. \quad (3.23)$$

Now the $\chi$ mass dimension is unity and its kinetic term is properly normalized. From the Lagrangian, we then read the $\chi$ mass

$$m_\chi^2 = \beta \kappa M_{Pl}^2. \quad (3.24)$$

Assuming $\beta$ to be of the order of unity, elementary particles are much lighter than the Planck mass only in the regime $\kappa \ll 1$.

4. Phenomenological validity

4.1. Linearized limit. Because of the nonzero nonmetricity, scalar–affine gravity might not be phenomenologically viable even in the linearized limit. In the Appendix, we show that this is not the case; the equations of motion in all helicity sectors coincide with that of GR. Hence, our model describes phenomenologically viable gravitational waves and reproduces Newton’s law.

\(^2\kappa = -4\) corresponds to a special solution for which $H = 0$ and $v$ is time-dependent. We do not consider this case here.
4.2. Cosmology. As we have mentioned previously, our model does not provide means for ending inflation. However, if we assume that inflation has somehow finished, scalar-affine gravity gives rise to phenomenologically viable cosmology. Indeed, after SSB took place, we can introduce all fields of the Standard Model into the theory as the corresponding representations of the stability group of the metric. It then remains to show that the standard conservation laws hold in our model. We prove this statement below.

We first discuss the $\Gamma_{\mu \nu}^\lambda$ and $\varphi$ equations of motion in the presence of matter. We assume that matter fields do not couple to the covariant derivative of $\varphi$ or to the connection. This guarantees that the mechanism of spontaneous breaking of scale invariance introduced earlier is unaffected by the usual matter. Then the $\Gamma_{\mu \nu}^\lambda$ equations of motion remain unchanged, with the solution given by Eq. (3.13), and $\varphi$ is given by Eq. (3.12) at any time. In particular, in the leading order in the magnitude of fields, the $\varphi$ equation of motion is given by Eq. (3.19).

Further, as usual, we assume that the Universe can be described in the hydrodynamic limit and seek de Sitter-like solutions of the Friedmann equations. These are

\begin{align*}
(00) : & \quad 3\varphi' + 9\varphi H - 2\varphi_0^{c/2} + 3H^2 = 8\pi M_{\text{Pl}}^{-2} \rho, \\
(ii) : & \quad 3\varphi' + 9\varphi H - 2\varphi_0^{c/2} + 3H^2 + 2H' = -8\pi M_{\text{Pl}}^{-2} p,
\end{align*}

(4.1a, b)

where $\rho$ and $p$ are respectively the matter energy density and pressure. As follows from Eq. (3.19), the first three terms in the left-hand side of both equations are constant. Then by introducing the “dark energy” density and pressure,

\begin{equation}
-\rho_{\text{de}} = p_{\text{de}} = 3\varphi' + 9\varphi H - 2\varphi_0^{c/2},
\end{equation}

(4.2)

we can reduce the Friedmann equations to the usual ones in $\Lambda$CDM cosmology. An immediate consequence of these equations is the usual conservation law in the matter sector,

\begin{equation}
\rho' + 3H(p + \rho) = 0.
\end{equation}

(4.3)

This implies that the cosmological evolution of matter densities as functions of $a^2$ is the same as in $\Lambda$CDM cosmology.

Interestingly, in our model, the connection is time-dependent during the dust- and radiation-dominating epochs. For example, for the dust-dominating epoch, we obtain

\begin{equation}
v = \frac{\kappa}{18} M_{\text{Pl}}^2 t.
\end{equation}

(4.4)

Because the connection depends on $v$, the latter also becomes time-dependent. If $\kappa \ll 1$, this has little effect for cosmological evolution. Nonetheless, because fermions do couple to the connection, this might have observable phenomenological effects. We comment on this question in the next section.

We end this section by noting that scalar-affine gravity admits a Schwarzschild-like solution. Namely, the metric is the same as in the Schwarzschild solution, but, due to the nonmetricity, the background connection is modified. This fact might have important phenomenological consequences. However, before studying them, we should first provide more solid ground for the foundation of our model. Correspondingly, we leave a detailed study of this question for the future.

5. Discussion

5.1. Possible physical implications. Although scalar-affine gravity cannot account for all phenomenological data, it has some remarkable and promising features. First, as we have already mentioned, it realizes the no-scale scenario: all physical quantities are generated dynamically through the SSB mechanism. Combined with the fact that scalar-affine gravity is phenomenologically viable, this might provide a foundation for a scale-invariant theory of gravity. In particular, the presence of the additional field $\varphi$ might improve the UV behavior of gravity.
Second, as we have shown, scalar–affine gravity naturally gives rise to inflation. Namely, we do not need to postulate any specific form of the potential: the scalar-density field $\varphi$ effectively acts as dark energy, thus giving rise to the expanding Universe. Although our model does not provide means for ending the inflation, this problem, presumably, can be solved by extending the field content of the model.

Third, the time direction and Lorentz invariance (in the flat limit) emerge dynamically in our model. In this regard, scalar–affine gravity is an alternative to the standard idea of the emergent Lorentz invariance [37]–[39], as well as to the Lorentz-violating gravity [40]–[43], including, in particular, the Hořava gravity [44], [45].

Finally, scalar–affine gravity features a nonzero nonmetricity: although the metric is of the de Sitter form, the connection is not canonical. In particular, because $\Gamma^0_{00} \neq 0$ and fermions couple to the connection, this implies that all fermions are massive, with the mass of the order of $v$. This provides us with an alternative to the $\nu$MSM mechanism for the generation of neutrino masses [46]. However, to discuss this topic, we first need to fully embed the Standard Model into our theory, including, in particular, the Higgs field.

In this context, we also note that scalar–affine gravity does not provide means for explaining the signature of the metric. Namely, we could change the minus sign before the determinant to the plus sign in Lagrangian (3.2). In this case, the metric would have a Euclidean signature. It would be interesting to study whether there exists a mechanism for fixing the minus sign in Lagrangian (3.2). This might be closely related to the idea of dynamical generation of the Lorentzian signature of the metric by introducing an additional scalar field [47]–[49].

5.2. Gravity as a gauge theory of connection. One of the approaches to constructing a theory of gravity lies in formulating it as a usual gauge theory [50]–[54]. Affine theories of gravity allow approaching this problem from a new standpoint. Namely, we can try to construct the theory of gravity as a gauge theory of the connection. Below, we consider this idea and discuss how it might be used as a foundation of scalar–affine gravity.

Because the basic field of affine gravity is the connection, it is natural to recall Einstein’s $\lambda$-transformations [55]

$$\Gamma(x)^{\lambda}_{\mu\nu} \rightarrow \Gamma(x)^{\lambda}_{\mu\nu} + \partial_{\mu}\lambda(x)\delta^{\lambda}_{\nu},$$

(5.1)

where $\lambda(x)$ is an arbitrary function. The Riemann curvature tensor is known to be invariant under such transformations. In fact, GR has an even larger symmetry, namely,

$$\Gamma^{\lambda}_{\mu\nu} \rightarrow \Gamma^{\lambda}_{\mu\nu} + \xi_{\mu}\delta^{\lambda}_{\nu},$$

(5.2)

where $\xi_{\mu}$ is an arbitrary function [56], [57]. This symmetry is known as (a subgroup of) projective invariance [58]. To reveal its geometrical meaning, we consider some geodesic [4]. In affine geometry, there is no notion of a predefined metric. Accordingly, the geodesic equation can be defined only as a parallel transport of a vector, remaining parallel to itself,

$$\frac{d^2x^\mu}{dp^2} + \Gamma^{\mu}_{\sigma\rho} \frac{dx^\sigma}{dp} \frac{dx^\rho}{dp} = \chi(p)\frac{dx^\mu}{dp},$$

(5.3)

where $p$ is the parameter along the geodesic and $\chi(p)$ is some function. Because of the form of the second term in the left-hand side of (5.3), only the symmetric part of the connection, denoted as $\Upsilon^{\lambda}_{\mu\nu}$, is relevant. Then the general transformation of the connection leaving a geodesic invariant (up to a reparameterization) is given by [4], [5], [58]

$$\Upsilon^{\lambda}_{\mu\nu} \rightarrow \Upsilon^{\lambda}_{\mu\nu} + \delta^{\lambda}_{\nu}V_{\mu} + \delta^{\lambda}_{\mu}V_{\nu},$$

(5.4)

where $V_{\nu}$ is an arbitrary function. These transformations are known as projective transformations and are wider than transformations (5.2).
Projective symmetry might form the basis for formulating the scalar–affine gravity as a gauge-type theory. Indeed, in the absence of a predefined notion of the metric, Eq. (5.3) is the only possible definition of a geodesic. If we consider geodesics as primary objects, all of their symmetries must be symmetries of the full theory as well. Accordingly, we expect the general theory of scalar–affine gravity to incorporate the projective symmetry. To demonstrate why this idea is promising, we consider Einstein’s $\lambda$-transformations for the trace of the connection,

$$\Gamma^\sigma_{\mu\nu} \rightarrow \Gamma^\sigma_{\mu\nu} + 4\partial_\mu \lambda.$$  (5.5)

They mimic the transformation law of the 4-potential $A_\mu$ under the U(1) gauge symmetry. The analogy goes further: the homothetic curvature tensor $S_{\mu\nu}$, Eq. (2.3), has the same structure as the electromagnetic tensor $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. This suggests that the trace of the connection may be somehow identified with the 4-potential, and $\lambda$-transformations with the corresponding gauge invariance [59]. Such an extension might be of interest for extending the Standard Model, with U(1) as a symmetry of some hidden or yet unobserved sector [8], [23], [26], [29], [59]–[62].

To conclude, we would like to say that there are a number of open questions in scalar–affine gravity. Nonetheless, it provides new ways for approaching long-standing problems in gravity. We believe that this makes it worth considering and studying further.

Appendix: Linearized limit

To verify the phenomenological validity of scalar–affine gravity, we show that the linearized limit of our model coincides with that of GR. We parameterize the fluctuations of the metric as

$$g_{00} = \eta_{00} + h_{00}, \quad g_{ij} = a^2(\eta_{ij} + h_{ij}), \quad g_{0i} = h_{0i},$$  (A.1)

where $h_{\mu\nu}$ are fluctuations of the metric. In the remainder of this appendix, the sum is taken with respect to the flat metric $\delta_{ij}$, and we do not distinguish between upper and lower indices. We use the 3 + 1 decomposition

$$h_{00} = 2\Phi,$$  \hspace{1cm} (A.2a)

$$h_{0i} = \partial_i Z + Z_i^T,$$  \hspace{1cm} (A.2b)

$$h_{ij} = -2\Psi \delta_{ij} + 2\partial_i \partial_j E + \partial_i W_j^T + h_i^T h_j^T,$$  \hspace{1cm} (A.2c)

where, as usual,

$$\partial_i Z_i^T = 0, \quad \partial_i W_i^T = 0, \quad \partial_i h_{ij}^T = 0, \quad h_i^T T_i = 0,$$  \hspace{1cm} (A.3)

and impose the gauge $h_{0i} = 0$.

For our choice of the constant $c = -3$, $\phi$ follows the dynamics of the determinant of the metric. Hence, their fluctuations are the same. However, we also need to obtain a formula governing the fluctuations of the covariant derivatives of $\phi$. For the $v_\mu$ fluctuations, $v_\mu = v^\text{vac}_\mu + u_\mu$, we then obtain the equation

$$- \left( a^3 \partial_0 + 3a^2 \partial_0 a \right) \left( u_0 + \frac{\nu}{2} (h + h_{00}) \right) + a \partial_i u_i = 0,$$  \hspace{1cm} (A.4)

where $h = h_i^i$. The solution of this equation is

$$u_0 = -\frac{\nu}{2} (h - h_{00}) - u, \quad \dot{u} + 3Hu + a^{-2} \partial_i u_i = 0.$$  \hspace{1cm} (A.5)
We now consider the equations of motion for the perturbations of the metric. As follows from Eq. (3.13), the terms quadratic in $h_{\mu\nu}$ are suppressed if
\[ h_{..} \ll 1, \quad vh_{..} \ll \partial h_{..}, \quad (A.6) \]
where dots stay for some indices. The first condition is standard. The second constraint appears due to the nonzero nonmetricity and its validity should be verified after obtaining the solution of the equations of motion.

We use Cadabra software [63], [64] for obtaining the equations of motion for the perturbed metric. The result is that the equations of motion in the spin-2 and spin-1 sectors are the same as those in GR on the de Sitter background. Hence, gravitational waves are the same in our model as in GR, and vector perturbations are stable.

In the spin-0 sector in the gauge $E = 0$, the equations of motion are
\[
\begin{align*}
(00) & : 3H(2H + 3v)\Phi + 27Hv\Psi + 3(u' + 3Hu + a^{-2}\partial_i u_i) - 6H\Psi' + 2a^{-2}\Delta \Psi = 0, \\
(0i) & : \partial_i (H\Phi - \Psi') = 0, \\
(ij) & : \partial_i \partial_j (\Phi + \Psi) - \delta_{ij} \left( \Delta (\Phi + \Psi) + a^2 (3H(2H + 3v)\Phi + 27Hv\Psi + 2H\Phi' - 6H\Psi' - 2\Psi'') \right) = 0, \\
\end{align*}
\]
where $\Delta = \partial_i \partial_i$. After taking Eq. (A.5) into account, these equations become the same as in GR. Thus, scalar–affine gravity is fully equivalent to GR in the linearized limit. In particular, it follows from Eqs. (A.7c) that $\Psi + \Phi = 0$. Hence, our model passes experimental constraints on the difference of these potentials [65]. In particular, by introducing a point-like particle of mass $M$ at the origin of the coordinates, we recover Newton’s law. Requirement (A.6) is then fulfilled if
\[ H \ll \frac{M_{Pl}^2}{M}, \quad (A.8) \]
which holds in all cases of physical interest.

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