Diagram expansions in classical stochastic field theory

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A diagram approach to classical nonlinear stochastic field theory is introduced. This approach
is intended to serve as a link between quantum and classical field theories, resulting in an indepen-
dent constructive characterisation of the measure in Feynman path integrals in terms of stochastic
differential equations for the paths.

I. INTRODUCTION

Diagrams are commonly associated with either Feynman’s work on quantum electrodynamics, or with Matsubara
and Keldysh techniques in statistical mechanics, so that diagrams appear to be something to do with subtle problems
in relativistic quantum field theory or the theory of phase transitions. This view is totally misleading. The concept
of diagrams is very general and can be applied to virtually any problem in physics. By encoding details of a physical
problem as expressions for the lines and vertices, a solution to this problem is readily produced following general
diagram rules.

There exists an additional dimension to the diagram approach not yet fully appreciated, namely its ability to
establish links between approaches based on inconsistent techniques, like quantum and classical field theories (QFT
and CFT). The diagram approach to the QFT is common knowledge. In this paper, we develop a diagram approach
to the theory of stochastic differential equations (SDEs), which is the formal ground of the CFT. The central formal
result of the paper is a general relation which we call the causal Wick’s theorem. It plays the same role in deriving
diagram series for SDEs as Wick’s theorem in QFT plays in deriving the Feynman (say) diagram series. We then
find that any SDE is formally solved by diagram series of a certain structure, which we call causal diagram series.
In these series, the propagator is determined by the linear part of the SDE, and vertices by the nonlinearity and noise
sources. This result may be interesting in itself since it allows one to extend methods specific to diagram approaches
(e.g., the Dyson equations) to SDEs.

The formal relation between diagram series in the QFT and in the CFT will be investigated in papers to follow.
Here we note only that the (pseudo-)stochastic measure on the solutions of an SDE is exactly the measure in certain
Feynman path integrals: solutions to the SDE are the Feynman paths. This may result in better understanding of
the path-integral approach and also in the possibility of characterising path integrals constructively, in well-defined
mathematical terms, and, indeed in a way independent of perturbation theory.

From an even more fundamental perspective, we encounter a very important link between regularisations in the
diagrams and stochastic calculus in SDEs. Technically, we find that regularisations in a diagram series which make
diagrams convergent affect the respective SDE in such a way that it becomes mathematically defined within the normal
calculus. We also find some indication that a renormalisation procedure, which formally consists of (i) regularisation
and (ii) a limiting procedure removing this regularisation in a certain way, appears from the point of view of the
respective SDE as a limiting procedure specifying the stochastic calculus. This link between renormalisation and
stochastic calculus suggests more extensive mathematical investigation, which may lead to better understanding of
the formal grounds of quantum field theory.

II. CLASSICAL STOCHASTIC SELF-ACTION PROBLEM

A. Stochastic calculus and regularisations in SDEs

We consider a c-number field \( \psi(r,t) \), which satisfies a generic equation with a source \( s(r,t) \),
\[
\mathcal{L}\psi(r,t) = s(r,t), \tag{1}
\]
where \( \mathcal{L} \) is a differential operator. The source \( s \) is dependent on the field \( \psi \) and this dependence may include
randomness; that is, in general (1) is a stochastic differential equation for the field \( \psi(r,t) \). To be specific, we assumed
that $\psi$ is a classical field in one-dimensional space; to consider other situations, e.g., a multi-mode field, or a field in three-dimensional space, one should simply replace in the relations below $\int dr$ by $\sum_r$ (i.e., $r$ is then a mode index), or $\int d^3r$, respectively. For a single-mode field, the variable $r$ and the summation should simply be dropped. We shall often resort to this last case for simplicity of examples.

Formally solving (1) turns it into an integral equation, (assuming $s(r,t) \to 0$ as $t \to -\infty$)

$$\psi(r,t) = \int_{-\infty}^{\infty} dt' \int dr' G(r,r',t-t')s(r',t') + \psi_0(r,t),$$

where $G(r,r',t-t')$ is the retarded Green’s function of the equation (1),

$$L G(r,r',t-t') = \delta(r-r')\delta(t-t'), \quad G(r,r',t-t') = 0, t < t',$$

so that integration in (2) is in fact from minus infinity to $t$. It obeys the free version of the equation (1), $L \psi_0(r,t) = 0$.

Note that the existence of a retarded Green’s function is not guaranteed for an arbitrary $L$; conditions (3) single out equations that may have physical meaning.

Although these definitions (as well as results below) are quite general, three cases are of major practical importance: the nonrelativistic Schrödinger equation,

$$L = i\partial_t + \frac{1}{2}\partial_r^2,$$

the wave equation,

$$L = \partial_t^2 - \partial_r^2,$$

and the relativistic Klein-Gordon equation,

$$L = \partial_t^2 - \partial_r^2 + 1,$$

where units were chosen so as to remove dimensional constants. Spatial Fourier transformation,

$$\psi(r,t) = \int dk e^{ikr} \psi_k(t),$$

reduces these equations to their single-mode cases:

$$L = i\partial_t - k^2 \frac{2}{2}, \quad G_k(t) = -i\theta(t)e^{-ik^2t/2},$$

$$L = \partial_t^2 + k^2, \quad G_k(t) = \theta(t)\sin kt,$$

$$L = \partial_t^2 + k^2 + 1, \quad G_k(t) = \theta(t)\sin t\sqrt{k^2 + 1},$$

where the corresponding Green’s functions are also given.

These Green’s functions all have a singularity at zero time, so problems arise if the source $s$ is a singular function as well. E.g., if $L = i\partial_t - k^2/2$ and $s$ contains white noise, defining mathematically the inhomogeneous Schrödinger equations (1) and (2) requires specification of a stochastic calculus. Another way around this problem is found by assuming that the Green’s function and/or the source are regularised, e.g., properly smoothed, and a corresponding limiting procedure is employed for restoring their singular values. The choice of stochastic calculus then becomes a result of this regularisation. For example, regularising the source by introducing finite correlation times means that normal calculus holds at all stages of the regularisation procedure, so that in the end the Stratonovich calculus is recovered.

As we shall argue now, regularising $G$ may lead to Ito calculus. Consider, for example, the equation,

$$i\partial_t \psi(t) = \varepsilon \psi(t)W'(t),$$

where $\varepsilon$ is a constant and $W'$ is the derivative of the Wiener process. It can only be defined in the sense of the theory of generalised functions,
\[ \int dtW'(t)\varphi(t) = -\int dtW(t)\varphi'(t), \]

where \( \varphi(t) \) is a “good” function. It would suffice for \( \varphi \) to have finite support and be continuously differentiable, so that \( \varphi' \) is a function of finite variation. It is then clear that for \( \psi W' \) to be defined, it would suffice for \( \psi \) to be continuously differentiable, but this is certainly not consistent with Eq. (12).

The standard way around this problem is to define \( \int \psi W'dt = \int \psi dW \) as a stochastic integral so that the equation \( \psi = \psi(0) - \varepsilon \int_0^t \psi dW \) is defined. Instead, we replace an undefined integral equation, \( \psi = \psi_0 - \varepsilon \int_{-\infty}^t \psi W'dt' \), by a regularised integral equation,

\[ \psi(t) = \psi_0 + \int G_{\text{reg}}(t-t')\varepsilon(t')W'(t')dt'. \]

(14)

\( G_{\text{reg}}(t) \) is a regularised retarded Green’s function, which is (i) causal, \( G_{\text{reg}}(t) = 0, t \leq 0 \), (ii) a given number of times continuously differentiable, and (iii) in a certain sense close to \( G(t) \). For example, \( (k \geq 1) \)

\[ G_{\text{reg}}(t) = -i\theta(t)\left(1 - e^{-\Gamma t}\right)^{k+1} \]

is \( k \) times continuously differentiable. \( \Gamma \) here is an arbitrary positive parameter, and the final limit \( \Gamma \to \infty \) is implied. We have also introduced a truncating factor \( \varepsilon(t) \) into the white noise factor \( W' \): \( \varepsilon(t) \leq \varepsilon \) is infinitely differentiable, and a negative \( T' \) exists such that \( \varepsilon(t) = \varepsilon \) if \( t > T' \) and \( \varepsilon(t) = 0 \) if \( t < \varepsilon T' \); this is necessary to have a consistent \( \text{in-} \) formulation. Eq. (14) is now consistent with the assumption of \( \psi \) being \( k \) times continuously differentiable. If this is indeed the case, the factor \( G_{\text{reg}}(t-t')\varepsilon(t') \) at a given \( t > \varepsilon T' \) is (at least) continuously differentiable and has a finite support \( \{ t' : 2T' < t' < t \} \), so that the integral on the right of Eq. (14) is defined: then, continuity of its \( k \)-th derivative by \( t \) follows from the fact that \( W' \) is, loosely speaking, no more singular than a \( \delta \)-function. (For \( t < \varepsilon T' \), (14) reduces to \( \psi = \psi_0 \).

An indication that in the limit \( \Gamma \to \infty \) a solution to Eq. (14) approaches a solution to the Ito differential equation (12) may be seen from the following considerations. As a generalised function, \( \varepsilon(t')W'(t') \) may be approximated by a discrete sum of \( \delta \)-functions,

\[ \varepsilon(t')W'(t') \approx \sum_{k=1}^{\infty} \varepsilon_k \delta(t-t_k), \]

where \( t_k = 2T' + k\Delta t \), and \( \Delta t \) is a discretisation scale. Eq. (14) then has a unique solution,

\[ \psi(t) = \psi_0 + \sum_{k=1}^{m} \varepsilon_k G_{\text{reg}}(t-t_k)\psi(t_k), \]

(17)

where \( \psi(t_k) \) may be found recurrently,

\[ \psi(t_m) = \psi_0 + \sum_{k=1}^{m-1} \varepsilon_k G_{\text{reg}}(t_m-t_k)\psi(t_k). \]

(18)

The sum in (17) is in fact finite, so that \( \psi(t) \) inherits all the “goodness” of \( G_{\text{reg}}(t) \). It is now easy to see that if \( \Gamma \Delta t \gg 1 \), the integral in (14) coincides with the partial sum \( -i \sum \varepsilon(t_k)\psi(t_k) [W(t_{k+1}) - W(t_k)] \); so that in the limit \( \Delta t \to 0 \) we recover the Ito integral \( -i \int_{2T'} dW(t')\psi(t') \varepsilon(t') \). From the practical point of view, this “proves” the hypothesis that in the limit \( \Gamma \to \infty \) the Ito calculus is recovered, since any real calculation implies time discretisation. In order to prove it mathematically, one should commute the limits: first \( \sum_{k=1}^{\infty} \varepsilon_k \delta(t-t_k) \to \varepsilon(t')W'(t') \) and second \( \Gamma \to \infty \), while the above considerations imply the opposite order of the limits.

Below we shall see that \( G(r, r', t-t') \) becomes the propagator in the diagram series so that regularisation it prevents ultraviolet divergences in the diagrams (this is nothing but the Pauli-Villars regularisation known in QFT [3]). The choice of stochastic calculus is hence connected to regularisations in the diagram series and hence to renormalisations. Investigation of this connection in full is a major undertaking; it should also include the continuous-space limit which we do not even try to tackle. At this stage, we just assume that \( G \) and/or \( s \) are regular (or regularised if necessary), so that all our expressions make sense. Note that this means that the integral equation (3) rather than the differential equation (1) is considered: the latter only emerges when regularisations are removed.
B. Characterisation of the stochastic sources and fields

In order to have a fully defined stochastic differential equation, the functional dependence of the source on the full (local or microscopic) field, \( \psi(r,t) \), is needed. Such a characterisation of the source is usually given by the physical model considered. Then to solve equations [8] or [9] means finding the field \( \psi \) as function of the in-field \( \psi_0(r,t) \).

For example, in a homogenous linear medium, \( s(r,t) = \chi \psi(r,t) \), where \( \chi \) is the linear susceptibility of the medium. In general, when randomness is also included, the source is a stochastic variable whose properties are determined by a probability distribution, \( P(s|\psi) \), of the function \( s \) conditioned on the full field \( \psi(r,t) \). It will be convenient to distinguish between real and complex fields. We will start from the former case and will generalize the results to complex fields at the end. We introduce a characteristic functional \( S(\alpha|\psi) \) corresponding to \( P(s|\psi) \) as,

\[
S(\alpha|\psi) = \mathbb{E}^{\alpha|\psi} = \int Ds \, e^{\alpha s} P(s|\psi),
\]

Here, \( \alpha(r,t) \) is an arbitrary “good” real function and \( \int Ds \) denotes a functional integration over the “trajectories” \( s(r,t) \). We use an abbreviated notation in which \( \alpha s \) denotes \( \int dx \alpha(x) s(x) \), where \( x = \{r,t\} \) and \( \int dx = \int drdt \).

Differentiating \( S \) over \( \alpha(x) \) produces multi-space-time averages of the source, conditioned on the full field,

\[
\frac{s(x_1) \cdots s(x_n)}{x_1} = \frac{\partial^n}{\partial \alpha(x_1) \cdots \partial \alpha(x_n)} S(\alpha|\psi) \big|_{\alpha=0}.
\]

To specify the functional dependence of the source on \( \psi \) it is convenient to introduce generalised susceptibilities (or just susceptibilities) \( \chi^{(m,n)}(x_1, \cdots, x_m; x'_1, \cdots, x'_n) \). They are coefficients in the series expressing cumulants of the local source \( s(x) \) in terms of the powers of the local field \( \psi(x) \).

\[
\frac{s(x)}{x} = \chi^{(1,0)}(x) + \int dx' \chi^{(1,1)}(x;x') \psi(x') \\
+ \frac{1}{2} \int dx' dx'' \chi^{(1,2)}(x;x',x'') \psi(x') \psi(x'') + \cdots,
\]

(21a)

\[
\frac{s(x)s(x')}{x} = \frac{s(x)}{x} \frac{s(x')}{x'} + \chi^{(2,0)}(x,x') + \int dx'' \chi^{(2,1)}(x,x';x'') \psi(x'') + \cdots,
\]

(21b)

With them the generating functional can be given a simple form

\[
S(\alpha|\psi) = \exp \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \alpha^m \chi^{(m,n)} \psi^n,
\]

(22)

where

\[
\alpha^m \chi^{(m,n)} \psi^n = \int dx_1 \cdots dx_m dx'_1 \cdots dx'_n \alpha(x_1) \cdots \alpha(x_m) \chi^{(m,n)}(x_1, \cdots, x_m; x'_1, \cdots, x'_n) \psi(x'_1) \cdots \psi(x'_n).
\]

(23)

The susceptibilities should obey the causality condition,

\[
\chi^{(m,n)}(x_1, \cdots, x_m; x'_1, \cdots, x'_n) = 0, \quad \max(t_1, \cdots, t_m) > \max(t'_1, \cdots, t'_n);
\]

(24)

this condition can also be formulated as the latest argument of a susceptibility is always an output one; hence \( \chi^{(0,n)} = 0 \) for all \( n \).

The quantity \( \chi^{(1,0)}(x) \) is a given source (non-random). For \( m = 1 \) we find susceptibilities proper: the linear one, \( \psi^{(1,1)} \), and the nonlinear ones, \( \psi^{(1,n)} \) for \( n > 1 \). If \( \chi^{(m,n)} = 0 \) for \( m > 1 \), the dependence of the source on the local field is not stochastic, and equation [8] is not stochastic either; it is linear, if \( \chi^{(1,n)} = 0 \) for \( n > 1 \), and otherwise nonlinear. Non-zero \( \chi^{(m,n)} \) for \( m > 1 \) introduce stochasticity: non-zero \( \chi^{(m,0)} \) are cumulants of a given random source, while \( \chi^{(m,n)} \) for both \( m > 1 \) and \( n > 0 \) describe how the statistics of the source depends on the field.

In practice, as a rule, only a finite number of generalised susceptibilities are non-zero. To be specific, we restrict ourselves to the case of only \( \chi^{(1,0)} \), \( \chi^{(1,1)} \), \( \chi^{(1,2)} \), \( \chi^{(2,0)} \) and \( \chi^{(2,1)} \) non-zero; these are exactly the susceptibilities shown...
explicitly in Eqs. (21). This case corresponds to a source which is Gaussian if conditioned on the full field (this reservation is important). Also, the susceptibilities are commonly local: e.g., \( \chi^{(1-1)}(x; x') \sim \delta(x - x') \), so that the source at the point \( x \) depends only on the local field at the same point. The causality condition (24) is then satisfied automatically. Locality gives more physical sense to the equations, but leads to mathematical problems. In this paper, we assume that these problems are overcome by regularisations.

Solving equations (1) and (2) means finding the field dependence on the in-field \( \psi_0 \). It is immediately obvious that this is equivalent to finding the dependence of the source on the in-field, i.e., given \( P(s|\psi) \) finding the probability distribution \( \Pi(s|\psi_0) \). Then, once the “macroscopic” probability distribution \( \Pi(s|\psi_0) \) is known, solving for the field is straightforward. To characterise the field, we introduce the characteristic functional of the multi-space-time field averages,

\[
\Phi(\zeta) = \Phi(\zeta|\psi_0) = \left. e^{\zeta \psi} \right|_{\psi_0},
\]

so that

\[
\psi(x_1) \cdots \psi(x_n) = \left. \psi(x_1) \cdots \psi(x_n) \right|_{\psi_0} = \frac{\partial^n}{\partial \zeta(x_1) \cdots \partial \zeta(x_n)} \Phi(\zeta)|_{\zeta=0}.
\]

Since \( \psi = Gs + \psi_0 \), the field statistics are effectively that of the source, and we have,

\[
\Phi(\zeta) = \int Ds \, e^{\zeta(Gs + \psi_0)} \Pi(s|\psi_0) = e^{\zeta \psi_0} \Sigma(\zeta|\psi_0),
\]

where \( [\zeta G](x) = \int dx' \zeta(x')G(x', x) \). Here we have introduced the characteristic functional \( \Sigma(\alpha|\psi_0) \) corresponding to \( \Pi(s|\psi_0) \):

\[
\Sigma(\alpha|\psi_0) = \left. e^{\alpha \psi} \right|_{\psi_0} = \int Ds \, e^{\alpha s} \Pi(s|\psi_0).
\]

In the complex case, our definitions must be generalised to allow one to consider averages containing the fields and sources as well as their complex conjugates. Thus the notation \( \Pi(s|\psi_0) \) used for the real case, must be replaced for complex fields by \( \Pi(s, s^*|\psi_0, \psi_0^*) \); the functional integration in the complex case will be denoted as \( \int D\psi D\psi^* \) and \( \int DsDs^* \), etc. The characteristic functionals \( S, \Sigma \) and \( \Phi \) are defined as,

\[
S(\alpha, \alpha^\dagger|\psi, \psi^*) = \int DsDs^* e^{\alpha^\dagger s + \alpha s^*} P(s, s^*|\psi, \psi^*),
\]

\[
\Sigma(\alpha, \alpha^\dagger|\psi_0, \psi_0^*) = \int DsDs^* e^{\alpha^\dagger s + \alpha s^*} \Pi(s, s^*|\psi_0, \psi_0^*),
\]

\[
\Phi(\zeta, \zeta^\dagger) = \Phi(\zeta, \zeta^\dagger|\psi_0, \psi_0^*) = \int DsDs^* e^{\zeta^\dagger (Gs + \psi_0) + \zeta (G^* s^* + \psi_0^*)} \Pi(s, s^*|\psi_0, \psi_0^*),
\]

where \( \alpha(x), \alpha^\dagger(x), \zeta(x) \) and \( \zeta^\dagger(x) \) are arbitrary “good” functions (\( ^\dagger \) here is just a notation, not relevant to Hermitian conjugation). One can also consider \( \alpha, \alpha^\dagger \) and \( \zeta, \zeta^\dagger \) as pairs of complex-conjugated functions, \( \alpha^\dagger = \alpha^*, \zeta^\dagger = \zeta^* \). Then, e.g.,

\[
\left. s^*(x_1) \cdots s^*(x_m)s(x_1') \cdots s(x_n') \right|_{\psi, \psi^*} = \frac{\partial^{m+n}}{\partial \alpha(x_1) \cdots \partial \alpha(x_m) \partial \alpha^\dagger(x_1') \cdots \partial \alpha^\dagger(x_n')} \times S(\alpha, \alpha^\dagger|\psi, \psi^*)|_{\alpha=\alpha^\dagger=0}.
\]

The rest of the above relations change accordingly.

C. Relation between \( P(s|\psi) \) and \( \Pi(s|\psi_0) \)

Our goal now is to find a formal solution to Eq. (1) in the form of a relation between the “microscopic” and “macroscopic” probability distributions, \( P(s|\psi) \) and \( \Pi(s|\psi_0) \). Assume discretisation of the time axis, \( s(t), \psi(t) \rightarrow s(t_k), \psi(t_k) \), where \( t_k = k\Delta t, k = -\infty, \cdots, \infty \); a final limit of \( \Delta t \rightarrow 0 \) is implied. We omit the spatial variable as irrelevant. With discretisation, averages of the source at the given field are given by a functional integration \( (t, t', \cdots t'') \) are among \( t_k \),
\[
\overline{s(t)s(t') \cdots s(t'')} |_\psi = \int s(t)s(t') \cdots s(t'') P(s|\psi) \prod_{k=-\infty}^{\infty} ds(t_k).
\]

The distribution \(P(s|\psi)\) is causal, i.e., \(s(t_k)\) depends on \(\psi(t_m)\) only for \(m \leq k\) (which is equivalent to the above causality condition for susceptibilities). This allows one to introduce reduced probability distributions,

\[
P_m(s \leq t_m | \psi \leq t_m) = \int P(s|\psi) \prod_{k=m+1}^{\infty} ds(t_k),
\]

where

\[
s \leq t_m = \{s(t_k) : k \leq m\},
\]

\[
\psi \leq t_m = \{\psi(t_k) : k \leq m\}.
\]

The fact that \(P_m\) depends on \(\psi \leq t_m\) and not on the whole \(\psi\) leads to the following causality condition for the averages,

\[
\frac{\partial}{\partial \psi(t_m)} \overline{s(t)s(t') \cdots s(t'')} |_\psi = 0, \quad t_m > \max(t, t', \cdots t'').
\]

The unravelling of the source statistics in time is described by the conditional probability distribution,

\[
P_m(s(t_m) | s \leq t_{m-1}, \psi \leq t_m) = \frac{P_m(s \leq t_m | \psi \leq t_m)}{P_{m-1}(s \leq t_{m-1} | \psi \leq t_{m-1})}
\]

In turn,

\[
P_m(s \leq t_m | \psi \leq t_m) = \prod_{k=-\infty}^{m} P_k(s(t_k) | s \leq t_{k-1}, \psi \leq t_k)
\]

Since we always observe a system only for finite times, \(P_m\) with \(m\) large enough is actually as good as \(P(s|\psi)\), so that we can write

\[
P(s|\psi) = \prod_{k=-\infty}^{\infty} P_k(s(t_k) | s \leq t_{k-1}, \psi \leq t_k)
\]

(More rigorously, this relation implies \(w\)-limits for both plus and minus infinity.)

The advantage of the “unravelling” representation \((31)\) for \(P(s|\psi)\) is that it is perfectly designed so as to accept the dynamical relation between the source and the field, Eq. \((2)\). With time discretisation, it is understood as,

\[
\psi(t_m) = \psi_0(t_m) + \Delta t \sum_{k=-\infty}^{\infty} G(t_m - t_k) s(t_k).
\]

Note that since \(G(t)\) is both causal and regular (or regularised), \(G(0) = 0\) and hence the latest source value to contribute to \(\psi(t_m)\) is \(s(t_{m-1})\), i.e., \(\psi(t_m)\) depends on \(s \leq t_{m-1}\) (\(\psi_0\) is either non-random or uncorrelated with \(s\)). Then, both \(s \leq t_{m-1}\) and \(\psi \leq t_m\) in the conditional probability \(P_m(s(t_m) | s \leq t_{m-1}, \psi \leq t_m)\) are effectively non-random and we can write

\[
\Pi_m(s(t_m) | s \leq t_{m-1}, (\psi_0) \leq t_m) = P_m(s(t_m) | s \leq t_{m-1}, (Gs + \psi_0) \leq t_m).
\]

Here, \(\Pi_m(s(t_m) | s \leq t_{m-1}, (\psi_0) \leq t_m)\) is the probability distribution for the source at \(t_m\), conditioned on its own prehistory and the \(in\)-field. Relation \((2)\) solves the self-action problem for the source, expressing its actual (macroscopically observable) statistics in terms of the (microscopic) relations characterising the system. For the multi-time probability distribution for the source, conditional on the \(in\)-field, we have,

\[
\Pi(s|\psi_0) = \prod_{m=-\infty}^{\infty} \Pi_m(s(t_m) | s \leq t_{m-1}, (\psi_0) \leq t_m),
\]

\[
6
\]
and hence,
\[ \Pi(s|\psi_0) = P(s|Gs + \psi_0). \quad (44) \]
This relation originates in Eq. (11) (regularised!) and causality.

The above derivation does not depend on whether we deal with a real or complex field. However, following the above convention regarding notation, in the complex case it will be written as
\[ \Pi(s, s^*|\psi_0, \psi_0^*) = P(s, s^*|Gs + \psi_0, G^* s^* + \psi_0^*). \quad (45) \]
Relation (44) applies in the real case.

D. Causal Wick’s theorems

Our aim is now to rewrite relation (44) in terms of the characteristic functionals. To this end, we first rewrite this relation using the shift operator,
\[ \Pi(s|\psi_0) = \exp \left( \frac{\partial}{\partial \psi} G_s \right) P(s|\psi)|_{\psi=\psi_0}. \quad (46) \]
Here, we again use condensed notation, \( \frac{\partial}{\partial \psi} G_s = \int dx dx' \frac{\partial}{\partial \psi(x)} G(x; x') s(x') \); note that the spatial dependence of the fields is restored. Then,
\[ \Sigma(\alpha|\psi_0) = \int Ds \exp \left( \alpha s + \frac{\partial}{\partial \psi} G_s \right) P(s|\psi)|_{\psi=\psi_0} \]
\[ = \exp \left( \frac{\partial}{\partial \psi} G_0 \frac{\partial}{\partial \alpha} \right) \int Ds e^{\alpha s} P(s|\psi)|_{\psi=\psi_0} = \exp \left( \frac{\partial}{\partial \psi} G_0 \frac{\partial}{\partial \alpha} \right) S(\alpha|\psi_0)|_{\psi=\psi_0}. \quad (47) \]
It is worth noting why we have to write, \( \frac{\partial}{\partial \psi} (\cdots)_{\psi=\psi_0} \), rather than just \( \frac{\partial}{\partial \psi} \). The problem is that \( \psi_0 \) is a solution to a free equation, i.e., \( \frac{\partial}{\partial \psi} G_s \) is a derivative with constraints, whereas when applying the shift operator, \( \exp \left( \frac{\partial}{\partial \psi} G_s \right) \), to \( P(s|\psi) \) in order to turn it into \( P(s|Gs + \psi) \), one has to assume that \( \psi \) is arbitrary. Since \( P(s|\psi) \) is indeed defined for an arbitrary \( \psi \), the whole situation is consistent.

Relation (47) is startlingly reminiscent of relations known in QFT that express Wick’s theorem for bosonic operators as a differential operation, and we shall call it the causal Wick’s theorem. Below we shall see that it plays the same role in deriving diagram series for the SDEs as Wick’s theorem proper plays in deriving diagram series for interacting bosonic fields. The assumptions it is based on are: (i) causality, (ii) Eq. (2) and (iii) the condition \( G(r, r', 0) = 0 \). Note that the last condition is meaningful only for regularised SDEs.

For the functional \( \Phi(\zeta) \), relations (47) and (27) result in,
\[ \Phi(\zeta) = \exp \left( \frac{\partial}{\partial \psi} G \frac{\partial}{\partial \alpha} \right) \exp (\zeta \psi) S(\alpha|\psi)|_{\alpha=0, \psi=\psi_0}. \quad (48) \]
This relation will be referred to as a generating formula for causal diagram series.

In the complex case, relations expressing \( \Sigma \) and \( \Phi \) over \( S \) (i.e., the causal Wick’s theorem and the generating formula for causal series) are found to be,
\[ \Sigma(\alpha, \alpha^\dagger|\psi_0, \psi_0^*) = \exp \left( \frac{\partial}{\partial \psi} G_0 \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \psi^\dagger} G^* \frac{\partial}{\partial \alpha^\dagger} \right) S(\alpha, \alpha^\dagger|\psi, \psi^\dagger)|_{\psi=\psi_0, \psi^\dagger=\psi_0^*}. \quad (49) \]
\[ \Phi(\zeta, \zeta^\dagger) = \exp \left( \frac{\partial}{\partial \psi} G_0 \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \psi^\dagger} G^* \frac{\partial}{\partial \alpha^\dagger} \right) \exp (\zeta^\dagger \psi + \zeta \psi^\dagger) \times \]
\[ S(\alpha, \alpha^\dagger|\psi, \psi^\dagger)|_{\alpha=\alpha^\dagger=0, \psi=\psi_0, \psi^\dagger=\psi_0^*}. \quad (50) \]
In these relations, \( \psi, \psi^\dagger \) is a pair of “good” functions which can be either arbitrary or complex conjugated. We have assumed that \( S(\alpha, \alpha^\dagger|\psi, \psi^\dagger) \) may be regarded as an analytic function of separately \( \psi \) and \( \psi^* \); in all practical examples \( S = \exp(\text{Polynomial of } \psi, \psi^*) \) (cf Eq. 21), so that this assumption is valid. Note, without going in detail, that assuming \( \psi, \psi^\dagger \) “good” is consistent only with a regular \( G(x; x') \), so that relations (49) and (50) imply regularisations; this certainly applies to (47) and (48) as well.
III. DIAGRAM EXPANSIONS AND SDES

In QFT, relations similar to the causal Wick’s theorems are used in order to express Wick’s theorem proper in a compact form as a differential operation. Then, if this form of Wick’s theorem is plugged into the standard perturbation approach, relations for Green’s functions of the interacting bosonic fields are found which are structurally identical to relations or for classical averages (this identity is straightforward for non-relativistic quantum fields, but becomes more involved in relativistic QFT). In principle, this identity is all that is necessary in order to establish the link between the quantum and classical field theories. For this purpose, diagram series (which both in QFT and CFT may be found by expanding generating formulae in power series) are redundant.

Moreover, establishing the quantum/classical link in terms of the diagram series, strictly speaking, degrades the approach since it brings over the problem of convergence of the diagram series and confines the link to the perturbation theory (whereas, e.g., for the anharmonic oscillator linking the generating formulae leads to exact results).

Despite all this, we believe introducing diagram series in CFT may be beneficial. First, diagrams are common language in QFT, and introducing them in CFT eliminates the seeming incompatibility between the q- and c-number techniques. Second, diagram expansions in CFT are not introduced as a computational tool. They are regarded only as a leading consideration, visualising certain structural properties of an SDE and/or the respective generating relation. These properties should then be proved directly. (Similarly, in QFT analysing diagram series leads to relations for observable quantities, like Dyson equations. As a rule, these relations may be derived independently.) Hence rigor of the diagram approach is not an issue. Third, in view of the encountered link between regularisations and stochastic calculus, introducing diagrams related to SDEs becomes a must: after all, those are diagrams that diverge!

A. Causal diagram series

For simplicity, we confine ourselves to a real SDE, with the noise source specified by the set of susceptibilities shown explicitly in Eqs. (21). Examples of series related to complex SDEs will be presented elsewhere.

Graphically, the quantities $\psi_0(x)$ and $G(x; x')$ will be denoted as lines,

$$G(x; x') = \left\{ \begin{array}{c} x' \\ \longrightarrow \end{array} \right\}, \quad (51a)$$
$$\psi_0(x) = \left\{ \begin{array}{c} x \\ \longrightarrow \end{array} \right\}, \quad (51b)$$

while $\zeta(x)$ and the susceptibilities $\chi^{(m,n)}(x)$ as vertices,

$$\zeta(x) = \left\{ \begin{array}{c} x \\ - \longrightarrow \end{array} \right\}, \quad (52a)$$
$$\chi^{(1,0)}(x) = \left\{ \begin{array}{c} \bullet \\ \longrightarrow \end{array} \right\}, \quad (52b)$$
$$\chi^{(1,1)}(x; x') = \left\{ \begin{array}{c} x' \bullet \longrightarrow x \\ \longrightarrow \end{array} \right\}, \quad (52c)$$
$$\chi^{(1,2)}(x; x', x'') = \left\{ \begin{array}{c} x'' \bullet \longrightarrow x' \\ \longrightarrow \\ \longrightarrow \end{array} \right\}, \quad (52d)$$
$$\chi^{(2,0)}(x, x') = \left\{ \begin{array}{c} x \\ \longrightarrow \end{array} \right\}, \quad (52e)$$
$$\chi^{(2,1)}(x, x'; x'') = \left\{ \begin{array}{c} x'' \bullet \longrightarrow x' \\ \longrightarrow \end{array} \right\}. \quad (52f)$$

The “time arrow”, which is drawn below each diagram, distinguishes graphically input and output arguments of the lines and vertices: The argument of $\psi_0(x)$ is regarded as a line output, as well as the “future” argument (i.e., $x$) in $G(x; x')$; the “past” argument (i.e., $x'$) in $G(x; x')$ is regarded as a line input. A generalised susceptibility,
\(\chi^{(m,n)}(x_1, \ldots, x_m; x'_1, \ldots, x'_n)\), is a quantity with \(n\) vertex inputs and \(m\) vertex outputs. By definition, the argument of \(\zeta(x)\) is a vertex input.

We now expand all exponents in the generating relation \((48)\) in power series, and consider a certain term in these series. (In detail, this procedure is discussed in the appendix.) We see that \(\alpha(x)\) always occurs convolved with a vertex output, while \(\frac{\partial}{\partial \alpha(x)}\) is always convolved with a line input. Since \(\frac{\partial \alpha(x)}{\partial \alpha(x')} = \delta(x - x')\), differentiating by \(\alpha\)’s leaves all vertex outputs pairwise convolved with line inputs. No free vertex outputs or line inputs may remain; terms with unequal number of these give zero. Similarly, derivatives \(\frac{\partial}{\partial \psi(x)}\) leave vertex inputs convolved with the line outputs of the propagators \(G(x; x')\); “surviving” \(\psi\)’s become \(\psi_0\)’s. As a result, we find all vertex inputs pairwise convolved with line outputs, and no free arguments remain. Graphically, convolved input-output pairs are denoted by connecting respective ends of the lines to the vertices, e.g.,

\[
\begin{align*}
\{ \quad \quad \} &= \int dx \zeta(x) \psi_0(x), \quad (53a) \\
\{ \quad \quad \} &= \int dx dx' \zeta(x) G(x; x') \chi^{(1,0)}(x'), \quad (53b) \\
\{ \quad \quad \} &= \int dx_1 \cdots dx_5 \zeta(x_1) \zeta(x_2) G(x_1; x_3) G(x_2; x_4) \times \\
& \quad \chi^{(2,1)}(x_3, x_4; x_5) \psi_0(x_5), \quad (53c) \\
\{ \quad \quad \} &= \int dx_1 \cdots dx_6 \zeta(x_1) G(x_1; x_2) \chi^{(1,2)}(x_2; x_3, x_4) \times \\
& \quad G(x_3; x_5) G(x_4; x_6) \chi^{(2,0)}(x_5, x_6). \quad (53d)
\end{align*}
\]

Expressions of such structure will be called causal diagrams. Note that the “time arrow” applies to all elements in a diagram.

In general, a causal diagram is a product of the basic elements—lines and vertices, where some line inputs and outputs are pairwise convolved with, respectively, vertex outputs and inputs. A diagram containing free arguments is called incomplete; e.g., \((51)\) and \((52)\) are legitimate incomplete diagrams. A diagram without free arguments (cf \((53)\)) is complete. As is shown in the appendix, the functional \(\Phi(\zeta)\) is expressed as a sum of all complete causal diagrams which may be built of the elements \((51a-52f)\), with certain coefficients; the rules for calculating these coefficients may be found in textbooks. Incomplete diagrams occur, e.g., in diagram expansions for the field averages. Formally, these are found by “stripping” the \(\zeta\) vertices from the complete diagrams. The simplest example of such a diagram is \((51)\): it contributes to \(\psi(r, t)\) and is found by stripping the \(\zeta\) vertex from \((53a)\).

It is also worth noting that whereas the “time flow” in diagrams is from left to right, in analytical expressions, as a rule, time increases from right to left. Eg, in the diagram in \((53b)\) the \(\zeta\) vertex is on the right, whereas in the analytical expression \(\zeta G_{\chi^{(1,0)}} = \int dx dx' \zeta(x) G(x; x') \chi^{(1,0)}(x')\) the natural position of the function \(\zeta(x)\) is on the left. The diagram notation thus commonly has an inverted order of objects compared with the analytical notation.

B. Connected diagrams and field cumulants

A diagram that graphically consists of a number of separate sub-diagrams without common elements is called disconnected; otherwise, a diagram is connected. A disconnected diagram is a product of its connected components. In the above examples, all diagrams are connected; however, full causal diagram series for the functional \(\Phi(\zeta)\) contain all possible complete diagrams, connected as well as disconnected.

To get rid of disconnected diagrams, one should describe the field in terms of its cumulants rather than averages. Formally, the field cumulants \(C^{(n)}\), \(n = 0, 1, \ldots\), are defined as,

\[
\Phi(\zeta) = \exp \sum_{n=1}^{\infty} \frac{1}{n!} \zeta^n C^{(n)}, \quad (54)
\]

where \((n = 1, 2, \cdots)\)
\[
\zeta^n C^{(n)} = \int dx_1 dx_2 \cdots dx_n C^{(n)}(x_1, x_2, \ldots, x_n) \zeta(x_1) \zeta(x_2) \cdots \zeta(x_n).
\] (55)

In particular,
\[
\psi(x) = C^{(1)}(x),
\]
\[
\psi(x) \psi(x') = C^{(2)}(x, x') + C^{(1)}(x) C^{(1)}(x'),
\]
\[
\psi(x) \psi(x') \psi(x'') = C^{(3)}(x, x', x'') + C^{(2)}(x, x') C^{(1)}(x'') + C^{(2)}(x, x'') C^{(1)}(x') + C^{(2)}(x', x'') C^{(1)}(x) + C^{(1)}(x) C^{(1)}(x') C^{(1)}(x''),
\]
\[
\vdots
\]

i.e., expression of a particular field average by the cumulants corresponds to its all possible factorisations. Characterisation of the field statistics by cumulants is more economical than characterisation by field averages. E.g., if \( \psi(x) \) is non-random, \( C^{(1)}(x) = \psi(x) \), and \( C^{(n)} = 0, n > 1 \); if \( \psi(x) \) is random and Gaussian, its cumulants vanish for \( n > 2 \); whereas all field averages are non-zero even for non-random \( \psi \).

There exists a general theorem (Mayer’s first theorem) stating that the diagram expansion of the logarithm \( \Phi(\zeta) \) contains only connected diagrams, so that
\[
\Phi(\zeta) = \exp[\text{conn} \Phi(\zeta)],
\]
where \( \text{conn} \Phi(\zeta) \) is given by diagram series where all disconnected diagrams are dropped while connected ones retain their coefficients. Comparing this with the definition of the field cumulants, we see that
\[
\text{conn} \Phi(\zeta) = \ln \Phi(\zeta) = \sum_{n=1}^{\infty} \frac{1}{n!} \zeta^n C^{(n)}.
\] (60)

Hence (cf the examples below) a diagram expansion for \( \frac{1}{n!} \zeta^n C^{(n)} \) contains all connected complete diagrams with exactly \( n \) \( \zeta \)-vertices, occurring with the coefficients they had in the series for \( \Phi(\zeta) \). Correspondingly, the expansion for \( C^{(n)}(x_1, \ldots, x_n) \) contains all connected incomplete causal diagrams with exactly \( n \) free line outputs.

C. Diagram structures corresponding to certain types of equations

1. Emission of given sources

To start with, we consider the simplest possible stochastic problem of radiation of a given random source, \( s(x) = s_0(x) \). For simplicity (and also to stay within (21)), we assume that it is Gaussian, and described by the cumulants \( s_0(x) = \chi^{(1,0)}(x) \) and \( s_0(x) s_0(x') - s_0(x) s_0(x') = \chi^{(2,0)}(x, x') \) (conditioning on the full field is irrelevant for a given source). This problem is readily solved,
\[
\psi(x) = \psi_0(x) + \psi'(x),
\]
\[
\psi'(x) = \int dx' G(x; x') s_0(x'),
\] (62)

where \( \psi'(x) \) is the emitted field. For the field cumulants we have,
\[
C^{(1)}(x) = \psi_0(x) + \int dx' G(x; x') \chi^{(1,0)}(x') = \left\{ x \right\} + \left\{ \dot{x} \right\},
\]
\[
C^{(2)}(x, x') = \int dx'' dx''' G(x; x'') G(x'; x''') \chi^{(2,0)}(x'', x''') = \left\{ \dot{x'} \right\},
\] (63b)

so that,
\[
\ln \Phi(\zeta) = \zeta C^{(1)} + \frac{1}{2} \zeta^2 C^{(2)} = \left\{ \begin{array}{c}
\bullet
\end{array} \right\} + \left\{ \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \right\} + \frac{1}{2} \left\{ \begin{array}{c}
\bullet

\end{array} \right\} + \left\{ \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\begin{array}{c}
\bullet
\end{array}
\end{array} \right\} + \cdots + \left\{ \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \right\} + \cdots. \tag{64}
\]

It is easy to see that these are indeed the three (and only three) connected complete diagrams that can be built of the available graphical elements \(\psi_0\), \(G\), \(\chi^{(1,0)}\), and \(\chi^{(2,0)}\), with right coefficients (cf the appendix). Hence expressions (63) and (64) are exactly that one would find from the diagrammatic approach. Note, however, that the logic of the diagrammatic solution is reversed: relation (64) is found summing all legitimate complete connected diagrams, while relations (63) follow.

2. Linear susceptibility

Consider now an equation with a non-zero linear susceptibility, \(\chi^{(1,1)}(x; x') \neq 0\). For simplicity, we assume that the only other non-zero susceptibility is \(\chi^{(1,0)}(x)\), i.e., we consider radiation of a given non-random source into a linear medium. The following considerations are nevertheless applicable in a general case of any set of non-zero susceptibilities.

It is easy to see that all connected diagrams containing only the vertices \(\chi^{(1,1)}(x; x')\) and \(\chi^{(1,0)}(x)\) are chains,

\[
\ln \Phi(\zeta) = \left\{ \begin{array}{c}
\bullet
\end{array} \right\} + \left\{ \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \right\} + \left\{ \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \right\} + \cdots + \left\{ \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \right\} + \cdots. \tag{65}
\]

On their “past” end, the chains are terminated either by the line \(\psi_0\),

\[
\left\{ \begin{array}{c}
\bullet
\end{array} \right\} = \zeta \psi_0, \tag{66}
\]

\[
\left\{ \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \right\} = \zeta G\chi^{(1,1)} \psi_0, \tag{67}
\]

\[
\left\{ \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \right\} = \zeta G\chi^{(1,1)} G\chi^{(1,1)} \psi_0, \tag{68}
\]

\[
\vdots \tag{69}
\]

or by the vertex \(s_0\),

\[
\left\{ \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \right\} = \zeta G\chi^{(1,0)}, \tag{70}
\]

\[
\left\{ \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \right\} = \zeta G\chi^{(1,1)} G\chi^{(1,0)}, \tag{71}
\]

\[
\left\{ \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \right\} = \zeta G\chi^{(1,1)} G\chi^{(1,1)} G\chi^{(1,0)}, \tag{72}
\]

\[
\vdots \tag{73}
\]

The coefficients of the chains are all equal to one. (Formally, there is an additional class of connected diagrams, namely, loops,

\[
\text{Tr}G\chi^{(1,1)} + \frac{1}{2} \text{Tr}G\chi^{(1,1)} G\chi^{(1,1)} + \frac{1}{3} \text{Tr}G\chi^{(1,1)} G\chi^{(1,1)} G\chi^{(1,1)} + \cdots. \tag{74}
\]

They are zero due to causality conditions and regularisations.)

There are two possibilities as to how \(\chi^{(1,1)}\) can appear in a diagram: (i) between two \(G\)s, and (ii) between a \(\psi_0\) and \(G\); it is the former that is responsible for infinite number of chains. Then, consider the sum of chains,
\[ G'(x; x') = \left\{ \begin{array}{c} \begin{array}{c} x' \\ \rightarrow \\ x \end{array} \end{array} \right\} + \left\{ \begin{array}{c} \begin{array}{c} x' \\ \rightarrow \\ x \end{array} \end{array} \right\} + \left\{ \begin{array}{c} \begin{array}{c} x' \\ \rightarrow \\ x \end{array} \end{array} \right\} + \cdots \]  
\[ = G(x; x') + \int dx'' dx''' G(x; x'') \chi^{(1,1)}(x''; x''') G(x''''; x') + \cdots. \]  
(75)  
(76)  

It obeys an integral (Dyson) equation,
\[ G' = G + G \chi^{(1,1)} G'. \]  
(77)  

Acting on it by the operator \( \mathcal{L} \), and using that \( \mathcal{L} G = \mathds{1} \), we find,
\[ \mathcal{L}' G' = \mathds{1}, \]  
(78)  

where \( \mathcal{L}' = \mathcal{L} - \chi^{(1,1)} \). Thus partial summation of the chains in diagrams corresponds to shifting the linear susceptibility from the source to the “free” equation.

Replacing \( G \rightarrow G' \) allows one to drop all diagrams containing a \( \chi^{(1,1)} \) vertex between two \( G \) lines. As to the \( \chi^{(1,1)} \) vertices placed between \( \psi_0 \)s and \( G \)s, these are only found in the combination,
\[ \psi'_0(x) = \left\{ \begin{array}{c} \begin{array}{c} x' \\ \rightarrow \\ x \end{array} \end{array} \right\} + \left\{ \begin{array}{c} \begin{array}{c} x' \\ \rightarrow \\ x \end{array} \end{array} \right\}, \]  
(79)  

where the propagator is now \( G' \). It is easy to see that \( \mathcal{L} \psi_0 = 0 \) results in \( \mathcal{L}' \psi'_0 = 0 \). Hence by redefining the graphical notation,
\[ \left\{ \begin{array}{c} \begin{array}{c} x' \\ \rightarrow \\ x \end{array} \end{array} \right\} = \psi'_0(x), \]  
(80)  
\[ \left\{ \begin{array}{c} \begin{array}{c} x' \\ \rightarrow \\ x \end{array} \end{array} \right\} = G'(x; x'), \]  
(81)  
\[ \left\{ \begin{array}{c} \begin{array}{c} x' \\ \rightarrow \\ x \end{array} \end{array} \right\} = 0, \]  
(82)  

(and dropping the primes) we arrive at an equivalent problem where the linear susceptibility is included into the operator \( \mathcal{L} \). Below we always assume this to be the case.

Note that instead of redefining the \textit{in}-field, one could equally redefine the \( \chi^{(1,0)} \) vertex,
\[ \left\{ \begin{array}{c} \begin{array}{c} x' \\ \rightarrow \\ x \end{array} \end{array} \right\} + \left\{ \begin{array}{c} \begin{array}{c} x' \\ \rightarrow \\ x \end{array} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \begin{array}{c} x' \\ \rightarrow \\ x \end{array} \end{array} \right\}, \]  
(83)  

leaving \( \psi_0 \) unchanged. Then \( \psi_0 \) is no longer a solution to the “free” equation, but, firstly, when deriving diagrams using Eq. (48) this fact is irrelevant, secondly, \( \mathcal{L} \psi_0 = 0 \), and hence \( \mathcal{L}' \psi'_0 = 0 \), may anyway be not the case after regularisations.

3. Nonlinear non-stochastic equations

We now consider an equation with quadratic nonlinearity, \( \chi^{(1,2)} \neq 0 \). We assume that this equation is non-stochastic, \( \chi^{(2,0)} = \chi^{(2,1)} = 0 \), and that \( \chi^{(1,1)} = 0 \) (or included in the operator \( \mathcal{L} \)). The available graphical elements hence are \( \psi_0, G, \chi^{(1,0)} \) and \( \chi^{(1,2)} \); it is then easy to see that, with two exceptions, all connected diagrams that one can build using these elements are trees branching into the past:
\[ \ln \Phi(\zeta) = \left\{ \begin{array}{c} \begin{array}{c} \bullet \\ \rightarrow \bullet \\ \rightarrow \bullet \end{array} \end{array} \right\} + \left\{ \begin{array}{c} \begin{array}{c} \bullet \\ \rightarrow \bullet \\ \rightarrow \bullet \end{array} \end{array} \right\} + \frac{1}{2} \left\{ \begin{array}{c} \begin{array}{c} \bullet \\ \rightarrow \bullet \\ \rightarrow \bullet \end{array} \end{array} \right\} + \frac{1}{2} \left\{ \begin{array}{c} \begin{array}{c} \bullet \\ \rightarrow \bullet \\ \rightarrow \bullet \end{array} \end{array} \right\} + \cdots. \]  
(84)
The futuremost vertex in the trees is $\zeta$, branching occurs at $\chi^{(1,2)}$ vertices, and branches are ultimately terminated by $\psi_0$ lines or $\chi^{(1,0)}$ vertices. The series contain all such trees, each with a coefficient $1/g$, where $g$ is the order of the group of symmetry of the tree.

4. Stochastic differential equations

Consider firstly a special case when $\chi^{(m,n)} = 0$ for $n > 1$. Connected diagrams are then trees branching into the future:

$$\ln \Phi(\zeta) = \left\{ \begin{array}{c} \text{Trees} + \frac{1}{2} \end{array} + \frac{1}{2} \right\} + \left\{ \text{Diagrams with loops} \right\} + \left\{ \begin{array}{c} \text{Diagrams with branches} \end{array} \right\} + \cdots. \right.$$

(85)

It is clear that diagrams with arbitrary number of $\zeta$ vertices may be found in this series. Hence, despite the fact that the local source conditioned on the full radiated field is Gaussian, the radiated field itself is non-Gaussian; moreover, it has non-zero cumulants $C^{(n)}$ for all $n$. The number of connected diagrams is infinite, yet the diagram series retain certain “tameness”: e.g., the number of diagrams contributing to each field cumulant, $C^{(n)}$, is still finite.

The final (and crucial) step leading to truly nontrivial series is combining nonlinearity and noise. Among the connected diagrams produced by relation (85), we find all trees, both “nonlinear” (84) and “stochastic” (83); on the top of that, we find a totally new class of netlike diagrams which are often called diagrams with loops:

$$\ln \Phi(\zeta) = \text{Trees} + \frac{1}{2} \left\{ \begin{array}{c} \text{Trees} \end{array} + \frac{1}{2} \right\} + \left\{ \begin{array}{c} \text{Diagrams with loops} \end{array} \right\} + \left\{ \begin{array}{c} \text{Diagrams with branches} \end{array} \right\} + \cdots. \right.$$

(86)

Now, the number of connected diagrams contributing to any field cumulant is infinite.

IV. NONLINEAR STOCHASTIC RESPONSE PROBLEM

The above CFT approach is a radiation problem, when one looks for properties of fields observable under given conditions; the quantum analog of it is, e.g., Glauber’s quantum coherence theory. An alternative formulation is a response problem, when one is interested in the dependence of the field properties on external influences; in quantum mechanics, this is mainly associated with Kubo’s linear response theory. As was shown in, the formal diagram solution to the quantum nonlinear response problem has a structure identical to that of a classical nonlinear stochastic response problem. We therefore briefly outline how the above diagram expansions are adjusted to the “response” viewpoint.

Following the spirit of Kubo’s linear reaction theory, we add a given external source to $s(r, t)$ in the Eqs (4) and (3), $s(r, t) \rightarrow s(r, t) + s_{\text{ext}}(r, t)$, and consider how the properties of the field $\psi$ depend on an infinitesimally small perturbation $s_{\text{ext}}$. We define nonlinear stochastic response functions as,
The response functions obey the causality condition,
\[ R^{(m,n)}(x_1, \ldots, x_m; x'_1, \ldots, x'_n) = 0, \quad \max(t_1, \ldots, t_m) < \max(t'_1, \ldots, t'_n). \] (88)

For sufficiently small \( s_{\text{ext}} \), (and assuming we are not at a phase transition point)
\[ \Phi(\zeta, s_{\text{ext}}) = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \chi^m R^{(m,n)} s_{\text{ext}}^n, \] (89)

where \( \Phi(\zeta, s_{\text{ext}}) \) is the characteristic functional of the field averages in presence of the external source, so that \( \Phi(\zeta) = \Phi(\zeta, 0) \).

In diagram terms, adding the external source does not bring much new. One simply replaces \( \chi^{(1,0)} \rightarrow \chi^{(1,0)} + s_{\text{ext}} \):
\[ \Phi(\zeta, s_{\text{ext}}) = \Phi(\zeta) \chi^{(1,0)} + s_{\text{ext}}, \] (90)

This means considering two types of \( \chi^{(1,0)} \) vertices and summing all diagrams thus emerging. If for simplicity we assume that \( \chi^{(1,0)} = 0 \), the connection between the “radiation” and “response” series becomes trivial: To find the latter, take the former with
\[
\{ \bullet x \} = s_{\text{ext}}(x). \] (91)

Diagram expansions for the response functions are found by stripping the diagrams of the \( \zeta \) and \( \chi^{(1,0)} = s_{\text{ext}} \) vertices,
\[
R^{(1,1)}(x; x') = \begin{cases} \bullet x \rightarrow x' \rightarrow x \end{cases} + \begin{cases} \bullet x' \rightarrow x \rightarrow x \end{cases} + \frac{1}{2} \begin{cases} x' \rightarrow \circ \rightarrow x \rightarrow x \end{cases} + \cdots, \] (92a)
\[
R^{(2,1)}(x, x'; x'') = \begin{cases} x'' \rightarrow x' \rightarrow x \end{cases} + \begin{cases} x'' \rightarrow x'' \rightarrow x \end{cases} + \begin{cases} x'' \rightarrow x \rightarrow x' \rightarrow x \end{cases} + \cdots, \] (92b)
\[
R^{(1,2)}(x; x', x'') = \begin{cases} x' \rightarrow \circ \rightarrow x \rightarrow \circ \rightarrow x' \rightarrow x'' \rightarrow x' \rightarrow x \end{cases} + \cdots, \] (92c)

etc. It is easy to prove that the causality condition \( (88) \) holds not only for an \( R^{(m,n)} \) as a whole, but also for each particular diagram in its expansion.

**V. CONSTRUCTING AN SDE FOR A GIVEN CAUSAL DIAGRAM SERIES**

Assume a diagram series is given (derived in a certain q-number approach, say). This series appears as a causal one, i.e., is generated by an expression like \( [48] \), and the causality conditions hold for all graphical elements. Formally, the generalised susceptibilities \( \chi^{(m,n)} \) then provide one with a complete and unambiguous description of the equivalent classical stochastic process. In practice, however, it would be more convenient to deal with an explicit SDE, written in terms of noise sources which are independent of the field. This leads us into what can be termed the inverse problem of the causal diagram techniques: How to explicitly write an SDE to which a causal diagram series corresponds?

With no stochasticity present, the relation between a causal diagram series and the corresponding DE is straightforward: A diagram series with only single-output vertices, \( \chi^{(1,n)} \), solves the equation,
Basically, any SDE should look identical to this equation, with the only difference that (some of) the susceptibilities are random. It is clear that,

$$
\Phi(\zeta) = \overline{\Phi_0(\zeta)},
$$

(94)

where $\Phi(\zeta)$ is the diagram series solving (93) regarded as an SDE, $\Phi_0(\zeta)$ is the series solving (93) as a non-stochastic equation, and the upper bar here denotes averaging over the random susceptibilities in (93). The key to the inverse problem is in the relation between $\Phi_0(\zeta)$ and $\Phi(\zeta)$.

Consider, to begin with, an example of a given random source. The SDE is then

$$
\mathcal{L}\psi(x) = s_0(x).
$$

(95)

For simplicity, we assume that $s_0$ is Gaussian with zero average, $\overline{s_0(x)} = 0$ and $\overline{s_0(x)s_0(x')} = \chi^{(2,0)}(x, x')$. Then,

$$
\Phi_0(\zeta) = \exp \left\{ \begin{array}{c}
\bullet \longrightarrow \bullet
\end{array} \right\},
$$

(96)

$$
\Phi(\zeta) = \exp \left( \frac{1}{2} \begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array} \right) \cdot \cdot \cdot
$$

(97)

On the other hand, expanding $\Phi_0(\zeta)$ in a diagram series and performing the averaging as required by (94) in each of the diagrams separately, we find:

$$
\Phi(\zeta) = 1 + \begin{array}{c}
\begin{array}{c}
\bullet \longrightarrow \bullet
\end{array}
\end{array} + \frac{1}{2} \begin{array}{c}
\begin{array}{c}
\bullet \longrightarrow \bullet
\end{array}
\end{array} + \frac{1}{6} \begin{array}{c}
\begin{array}{c}
\bullet \longrightarrow \bullet
\end{array}
\end{array} + \frac{1}{24} \begin{array}{c}
\begin{array}{c}
\bullet \longrightarrow \bullet
\end{array}
\end{array} + \cdot \cdot \cdot
$$

$$
= 1 + 0 + \frac{1}{2} \begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array} + 0 + \frac{1}{8} \begin{array}{c}
\begin{array}{c}
\bullet \longrightarrow \bullet
\end{array}
\end{array} + \cdot \cdot \cdot
$$

(98)

We see that the graphical operation reflecting $s_0$ becoming random and Gaussian is a pairwise merging of the $\chi^{(1,0)}$ vertices into $\chi^{(2,0)}$ vertices; diagrams with an odd number of the $\chi^{(1,0)}$ vertices become zero. Note that whereas itself the merging of the vertices reflects stochasticity, the fact that it is pairwise is clearly due to the Gaussian statistics: if $s_0$ were non-Gaussian, then non-zero $\chi^{(m,0)}$ vertices would each result from merging of $m\ \chi^{(1,0)}$ vertices.

Consider now an SDE with a multiplicative noise,

$$
\mathcal{L}\psi(x) = \int dx' \eta(x; x') \psi(x') + s'(x),
$$

(99)

where $\eta(x; x')$ is random and $s'(x)$ contains non-stochastic terms (it may also contain other noise sources provided they are not correlated with $\eta$). For simplicity, we again assume that $\eta(x; x')$ is Gaussian and $\overline{\eta(x; x')} = 0$, so that it is specified by the average $\overline{\eta(x_1; x'_1)\eta(x_2; x'_2)}$. The diagram series for equation (100) can be found by averaging those for an equation with a linear susceptibility $\chi^{(1,1)}(x; x') = \eta(x; x')$. This is most intuitive if done directly in the generating expression (100):

$$
\Phi(\zeta) = \exp \left( \frac{\partial}{\partial \psi} G \frac{\partial}{\partial \alpha} \right) \exp \left\{ \begin{array}{c}
\bullet \longrightarrow \bullet
\end{array} \right\} \exp (\zeta \psi) \left. S'(\alpha | \psi) \right|_{\alpha=0, \psi=\psi_0}
$$

$$
= \exp \left( \frac{\partial}{\partial \psi} G \frac{\partial}{\partial \alpha} \right) \exp \left( \frac{1}{4} \begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array} \right) \exp (\zeta \psi) \left. S'(\alpha | \psi) \right|_{\alpha=0, \psi=\psi_0},
$$

(100)
where $S'(\alpha|\psi)$ contains vertices originating in $s'(x)$, and the following graphical notation is used,

\[
\begin{align*}
\{ \begin{array}{c} \longrightarrow \end{array} \} &= \psi(x), \\
\{ \begin{array}{c} \longrightarrow \end{array} \begin{array}{c} \rightarrow \end{array} \} &= \alpha(x), \\
\{ \begin{array}{c} \longrightarrow \end{array} \begin{array}{c} \bullet \end{array} \} &= \alpha \chi^{(1,1)}\psi, \\
\{ \begin{array}{c} \times \end{array} \} &= \alpha^2 \chi^{(2,2)}\psi^2.
\end{align*}
\]

(The quantities $\alpha^m \chi^{(m,n)}\psi^n$ are defined in Eq. (23). Note that we have departed from the case of only the five susceptibilities shown explicitly in Eqs. (21) being non-zero.) Hence the graphical representation of the linear susceptibility becoming random is pairs of the $\chi^{(1,1)}$ vertices merging into the $\chi^{(2,2)}$ vertices, e.g.,

\[
\frac{1}{2} \left\{ \begin{array}{c} \longrightarrow \rightarrow \end{array} \right\} \rightarrow \frac{1}{4} \left\{ \begin{array}{c} \times \begin{array}{c} \bullet \end{array} \end{array} \right. \right\}.
\]

Similarly, if a certain pair of susceptibilities ($\chi^{(1,0)}$ and $\chi^{(1,2)}$, say) become random and correlated (implying Gaussian), then averaging $\Phi_0$ we get,

\[
\Phi(\zeta) = \exp \left( \frac{\partial}{\partial \psi} G \frac{\partial}{\partial \alpha} \right) \exp \left( \frac{1}{2} \left\{ \begin{array}{c} \bullet \end{array} \right\} + \frac{1}{2} \left\{ \begin{array}{c} \longrightarrow \rightarrow \end{array} \right\} \right) \exp \left( \zeta \psi \right) S'(\alpha|\psi) |_{\alpha=0,\psi=\psi_0}
\]

\[
= \exp \left( \frac{\partial}{\partial \psi} G \frac{\partial}{\partial \alpha} \right) \exp \left( \frac{1}{2} \left\{ \begin{array}{c} \longrightarrow \ \bullet \ \rightarrow \end{array} \right\} + \frac{1}{48} \left\{ \begin{array}{c} \times \ \bullet \ \times \end{array} \right\} + \frac{1}{4} \left\{ \begin{array}{c} \times \ \rightarrow \ \times \end{array} \right\} \right)
\]

\[
\times \exp \left( \zeta \psi \right) S'(\alpha|\psi) |_{\alpha=0,\psi=\psi_0},
\]

where

\[
\begin{align*}
\{ \begin{array}{c} \bullet \end{array} \} &= \alpha \chi^{(1,0)} , \\
\{ \begin{array}{c} \longrightarrow \ \bullet \ \rightarrow \end{array} \} &= \alpha \chi^{(1,2)}\psi^2, \\
\{ \begin{array}{c} \longrightarrow \ \bullet \ \rightarrow \end{array} \} &= \alpha^2 \chi^{(2,0)}, \\
\{ \begin{array}{c} \times \ \bullet \ \times \end{array} \} &= \alpha^2 \chi^{(2,4)}\psi^4, \\
\{ \begin{array}{c} \times \ \rightarrow \ \times \end{array} \} &= \alpha^2 \chi^{(2,2)}\psi^2.
\end{align*}
\]

and
χ(2,0)(x_1, x_2) = χ(1,0)(x_1)χ(1,0)(x_2), \quad (113a)

χ(2,4)(x_1, x_2, x'_1, x'_2, x'_3, x'_4) = \text{Symm} \chi(1,2)(x_1; x'_1, x'_2)\chi(1,2)(x_2; x'_3, x'_4), \quad (113b)

χ(2,2)(x_1, x_2; x'_1, x'_2) = \text{Symm} \chi(1,1)(x_1; x'_1)\chi(1,1)(x_2; x'_2). \quad (113c)

Symm here denotes summation over all different terms obtained by permutations of the input and output arguments, cf. (103). We see that whereas randomness results in self-mergings (χ(1,0) + χ(1,0) → χ(2,0) and χ(1,2) + χ(1,2) → χ(2,4)), correlations manifest themselves as cross-mergings (χ(1,0) + χ(1,2) → χ(2,2)).

This way, if we take equation (113), derive a diagram series for it, and then assume that the susceptibilities χ^{(1,n)} are random, the new series is found from the initial series by merging certain vertices in the diagrams. Hence in order to recover the initial non-stochastic equation, the merged vertices should be factorised into the products of the initial vertices. Formally, this means solving functional equations like (105) or (113), with given stochastic vertices, in order to find statistics of the susceptibilities in (113). This is nothing but the well-known problem of moments in probability theory.

It is obvious that there exist causal diagram series that do not correspond to any SDE in the true meaning of the word. E.g., χ^{(2,0)}(x; x') = −δ(x−x') would require σ_0(x)σ_0(x') = −δ(x−x'); this is certainly impossible in probability theory. We therefore have to adopt the concept of pseudo-probability [12] and consider pseudo-stochastic differential equations (PSDE) as well as stochastic ones. (This is anyway inevitable in quantum stochastics since the measure in Feynman path integrals is as a rule nonpositive.) Even with this generalisation, it is not clear if the inverse problem has a general solution consistent with the causality conditions for the susceptibilities.

However, this clearly is the case for an important class of problems, namely, local Markovian problems. In terms of the susceptibilities, this means that they are non-zero only if all their arguments (both input and output) coincide. E.g., let χ^{(2,2)}(x_1, x_2; x'_1, x'_2) = χδ(x_1 − x'_1)δ(x_2 − x'_2)δ(x_1 − x_2), where χ is a real constant, positive or negative. Then, the inverse problem is solved by χ^{(1,1)}(x; x') = √|χ|η(x)δ(x−x'), where for χ > 0 η(x) is a standardised Gaussian δ-correlated noise source, η(x)η(x') = δ(x−x'); whereas for χ < 0 η(x) is a standardised Gaussian δ-correlated pseudo-stochastic source, η(x)η(x') = −δ(x−x'). Note that one can equally find an alternative solution using factorisation (113). This shows that a solution to the inverse problem is in general non-unique.

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APPENDIX:

We now show that relation (133) can formally be expanded in a series of the causal diagrams. To this end, we re-write it as,

\[ \Phi(\zeta) = \sum_{p,q,r,s,t,u=1}^{\infty} \frac{1}{|\zeta|^p||\eta||^q} \exp\mathcal{D}\{\bullet\bullet\bullet\}^p\{\bullet\bullet\}^q\{\bullet\bullet\\}^r \times \]
we notice that, 

That is, $D$

where

$$D = D_1 + D_2 = \frac{\partial}{\partial \psi} \psi_0 + \frac{\partial}{\partial \psi} G \frac{\partial}{\partial \alpha}.$$  

(A2)

and

$$\{ \bullet \} = \zeta \psi,$$

$$\{ \bullet \} = \alpha^2 \chi^{(2,1)} \psi.$$  

(A3)

(A4)

(A5)

The rest of the graphical notation used in (A1) was introduced in Chapter V. Note that we have returned to the case of only the five susceptibilities shown explicitly in Eqs. (21) being non-zero.

Rewriting the product of connected diagrams in (A1) as a disconnected diagram, the general term in the series (A1) is found to be,

$$\text{const} \times \exp D \{ D_{pqrst} \} |_{\alpha=\psi=0}.$$  

(A6)

Here $\{ D_{pqrst} \}$ is a disconnected diagram with $p+q+r+s+t+u$ connected components, where $\{ \bullet \}$ is repeated $p$ times, $\{ \bullet \}$ is repeated $q$ times, etc. To find a graphical representation of the operation $\exp D \{ D_{pqrst} \} |_{\alpha=\psi=0}$, we notice that,

$$\{ D_{pqrst} \} = \int dx_1 \cdots dx_m dx_1' \cdots dx_n' \times$$

$$\psi(x_1) \cdots \psi(x_m) \alpha(x_1') \cdots \alpha(x_n') \Xi(x_1', \ldots, x_n'; x_1, \ldots, x_m),$$

where $m = p + r + t + 2u$, $n = q + r + 2s + 2t + u$, and $\Xi(x_1', \ldots, x_n'; x_1, \ldots, x_m)$ is a product of the generalised susceptibilities and $\zeta$s; $x_1, \ldots, x_m$ and $x_1', \ldots, x_n'$ are, respectively, input and output arguments, each belonging to a certain susceptibility factor (or to a $\zeta$) in this product. $\Xi$ contains neither os nor $\psi$s, hence we have to understand only how $\exp D$ acts on the product of the os and $\psi$s.

Consider firstly the operator $\exp D_1$, where $D_1 = \frac{\partial}{\partial \psi} \psi_0$. It is readily seen that

$$D_1 \psi(x_1) \cdots \psi(x_m) = \sum_{i=1}^m \psi(x_1) \cdots \psi(x_{i-1}) \psi_0(x_i) \psi(x_{i+1}) \cdots \psi(x_m).$$  

(A8)

That is, $D_1$ turns a product of $m$ $\psi$s into $m$ terms, in each of which one factor $\psi$ is replaced by $\psi_0$. Then, $\frac{1}{2!} D_1^2$ will turn the product of $m$ $\psi$s into $m(m-1)/2$ terms, in each of which two factors $\psi$ are replaced by $\psi_0$s; the factor $\frac{1}{2!}$ exactly compensates for two possible orders of replacement leading to each term. It is now easy to realise (and prove by induction) that $\frac{1}{k!} D_1^k$ will turn the product of $m$ $\psi$s into $m! \binom{m+k-1}{k}$ different terms, in each of which $k$ factors $\psi$ are replaced by $\psi_0$.

At $k = m$, we shall find one term with all $\psi$s replaced by $\psi_0$s, $\frac{1}{m!} D_1^m \psi(x_1) \cdots \psi(x_m) = \psi_0(x_1) \cdots \psi_0(x_m)$, and $D_1^k \psi(x_1) \cdots \psi(x_m) = 0, k > m$. Thus,

$$\exp D_1 \psi(x_1) \cdots \psi(x_m) = \left( 1 + D_1 + \frac{1}{2!} D_1^2 + \cdots + \frac{1}{m!} D_1^m \right) \psi(x_1) \cdots \psi(x_m)$$

$$= \text{Sum of all different products of } k \text{ } \psi_0 \text{s and } m-k \text{ } \psi \text{s, } 0 \leq k \leq m.$$  

(A9)

Note that this result is based primarily on relation (A8); the rest of the derivation is mere combinatorics.

Similarly, how the operator $\exp D_2$, where $D_2 = \frac{\partial}{\partial \psi} G \frac{\partial}{\partial \alpha}$, acts on a product of os and $\psi$s, is based primarily on the relation,
\[
D_2 \psi(x_1) \cdots \psi(x_m) \alpha(x'_1) \cdots \alpha(x'_n) = \sum_{i=1}^{m} \sum_{j=1}^{n} G(x_i; x_j) \psi(x_1) \cdots \psi(x_{i-1}) \psi(x_{i+1}) \cdots \psi(x_m) \times \alpha(x'_i) \cdots \alpha(x'_{j-1}) \alpha(x'_{j+1}) \cdots \alpha(x'_n).
\]

(A10)

This relation reads: in the product \( \psi(x_1) \cdots \psi(x_m) \alpha(x'_1) \cdots \alpha(x'_n) \), select a pair \( \psi(x_i), \alpha(x'_j) \), and replace it by \( G(x_i; x'_j) \); \( D_2 \psi(x_1) \cdots \psi(x_m) \alpha(x'_1) \cdots \alpha(x'_n) \) is then found as a sum of all different terms obtained this way. Then, \( \frac{1}{\hbar} D_2^k \psi(x_1) \cdots \psi(x_m) \alpha(x'_1) \cdots \alpha(x'_n) \) is equal to the sum of all different terms, found by replacing \( k \) such pairs by the \( G \)s in the initial product; if \( k > \min(m,n) \), one finds zero. Finally,

\[
\exp D_2 \psi(x_1) \cdots \psi(x_m) \alpha(x'_1) \cdots \alpha(x'_n)
= \text{Sum of all different products of } k \text{ } G \text{s, } m - k \ psi \text{ and } n - k \ alpha, \ 0 \leq k \leq \min(m,n).
\]

(A11)

(This clearly follows the pattern of Wick’s theorem for bosonic operators, replacing operator pairs by the “junction” \( G \), cf. [3][4].) Now there is no problem in understanding the action of the operator \( \exp D = \exp(D_1 + D_2) \):

\[
\exp D \psi(x_1) \cdots \psi(x_m) \alpha(x'_1) \cdots \alpha(x'_n)
= \text{Sum of all different products of } k \text{ } G \text{s, } l \ psi \text{ and } m - k - l \ alpha, \ 0 \leq k \leq n, 0 \leq l + k \leq m.
\]

(A12)

The graphical recipe corresponding to this relation is quite transparent: (i) represent the product \( \psi(x_1) \cdots \psi(x_m) \alpha(x'_1) \cdots \alpha(x'_n) \) as a disconnected diagram with \( m \) elements \( \{101\} \) and \( n \) elements \( \{102\} \), then (ii) write a sum of all different diagrams obtainable by replacing randomly chosen \( \psi \)s by \( \psi_0 \)s and pairs \( \alpha, \psi \) by \( G \)s,

\[
\begin{align*}
\{ \_ \quad \_ \} & \rightarrow \{ \_ \quad \_ \}, \\
\{ \_ \quad \_ \quad \_ \} & \rightarrow \{ \_ \quad \_ \quad \_ \}.
\end{align*}
\]

(A13)

(A14)

For example,

\[
\exp D \alpha(x) \psi(x') = \exp D \left\{ \begin{array}{c}
\_ \quad \_ \\
\_ \quad \_ \\
\end{array} \right\} = \left\{ \begin{array}{c}
\_ \quad \_ \\
\_ \quad \_ \\
\end{array} \right\} + \left\{ \begin{array}{c}
\_ \quad \_ \\
\_ \quad \_ \\
\end{array} \right\},
\]

(A15)

\[
\exp D \alpha(x) \psi(x') \psi(x'') = \exp D \left\{ \begin{array}{c}
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\end{array} \right\}
= \left\{ \begin{array}{c}
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\end{array} \right\} + \left\{ \begin{array}{c}
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\end{array} \right\} + \left\{ \begin{array}{c}
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\end{array} \right\} + \\
\left\{ \begin{array}{c}
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\end{array} \right\} + \left\{ \begin{array}{c}
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\end{array} \right\},
\]

(A16)

Then, setting \( \alpha = \psi = 0 \) cancels all terms with dashed lines, e.g.,

\[
\exp D \left\{ \begin{array}{c}
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\end{array} \right\} |_{\alpha = \psi = 0} = \left\{ \begin{array}{c}
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\end{array} \right\},
\]

(A17a)

\[
\exp D \left\{ \begin{array}{c}
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\end{array} \right\} |_{\alpha = \psi = 0} = \left\{ \begin{array}{c}
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\end{array} \right\} + \left\{ \begin{array}{c}
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\_ \quad \_ \\
\end{array} \right\},
\]

(A17b)
\[
\exp D \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\} |_{\alpha=\psi=0} = \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\} + \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\}.
\]

Thus, the graphical recipe for calculating \( \exp D \psi(x_1) \cdots \psi(x_m) \alpha(x'_1) \cdots \alpha(x'_n) |_{\alpha=\psi=0} \) is that for calculating \( \exp D \psi(x_1) \cdots \psi(x_m) \alpha(x'_1) \cdots \alpha(x'_n) |_{\alpha=\psi=0} \), plus (iii) retain only terms without dashed lines. Note that the number of resulting \( \psi \)'s and \( G \)'s in all such terms is fixed \((n \text{ and } m - n, \text{ respectively})\), while their arguments vary from term to term.

It is now obvious that to calculate \( \{A\} \) graphically, one should remove pairs of dashed lines \( \alpha, \psi \) in the diagram \( \{D_{pqrsut}\} \), installing instead solid lines connecting inputs of the vertices to outputs, so that no free outputs remain; then, all remaining dashed \( \psi \)'s should be replaced by solid \( \psi \)'s, connected to inputs of the vertices. Terms where the number of outputs in \( \{D_{pqrsut}\} \) exceeds the number of inputs do not contribute. Each term in series \( \{A\} \) is thus equal to either a sum of causal diagrams or zero, e.g.,

\[
\exp D \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\} |_{\alpha=\psi=0} = 0,
\]

\[
\exp D \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\} |_{\alpha=\psi=0} = \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\},
\]

\[
\frac{1}{2} \exp D \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\} |_{\alpha=\psi=0} = \frac{1}{2} \exp D \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\} |_{\alpha=\psi=0} = \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\},
\]

\[
\exp D \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\} |_{\alpha=\psi=0} = \exp D \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\} |_{\alpha=\psi=0} = \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\},
\]

\[
\frac{1}{4} \exp D \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\} |_{\alpha=\psi=0} = \frac{1}{4} \exp D \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\} |_{\alpha=\psi=0} = \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\} + \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\} = \frac{1}{2} \left\{ \begin{array}{c}
\bullet \\
\circ \\
\star \\
\times
\end{array} \right\}.
\]

The last example shows that, in general, not all diagrams obtained from a particular \( \{D_{pqrsut}\} \) are different (unlike in sums \( \{A\} \)): this is due to possible symmetry of the vertex product \( \Xi \). All such equal diagrams count when calculating the coefficient with which a diagram appears in the series. It is obvious that any given causal diagram, connected or disconnected, is produced by a unique term in series \( \{A\} \), namely, by the one containing the same set of vertices as the diagram sought. It is then follows that all complete causal diagrams are found in the resulting diagram series.