ON THE HEISENBERG ALGEBRA ASSOCIATED WITH THE RATIONAL $R$-MATRIX

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Abstract. We associate a deformation of Heisenberg algebra to the suitably normalized Yang $R$-matrix and we investigate its properties. Moreover, we construct new examples of quantum vertex algebras which possess the same representation theory as the aforementioned deformed Heisenberg algebra.

1. Introduction

The notion of vertex algebra, which was introduced by Borcherds [2], presents a remarkable connection between mathematics and theoretical physics. Starting with Belavin, Polyakov and Zamolodchikov [1], such objects were extensively studied by physicists in connection with conformal symmetries of two-dimensional quantum field theory. On the other hand, the theory of vertex algebras led to important new methods, techniques and results in multiple areas of mathematics such as affine Kac–Moody Lie algebras, automorphic forms, finite simple groups and $W$-algebras; see, e.g., the books by E. Frenkel and Ben-Zvi [5], I. Frenkel, Lepowsky and Meurman [7] and Kac [9].

Let $\mathfrak{h}$ be an abelian Lie algebra over $\mathbb{C}$ equipped with the nondegenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$. The affine Lie algebra $\hat{\mathfrak{h}}$ is defined on the complex space

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} C$$

with the Lie brackets given by

$$[a(r), b(s)] = \langle a, b \rangle r \delta_{r+s,0} C \quad \text{and} \quad [C, x] = 0$$

for all $r, s \in \mathbb{Z}$, $a, b \in \mathfrak{h}$ and $x \in \hat{\mathfrak{h}}$, where $a(r)$ denotes the element $a \otimes t^r$. The corresponding Heisenberg Lie algebra $\hat{\mathfrak{h}}_*$ is defined as a subalgebra

$$\hat{\mathfrak{h}}_* = \prod_{n \in \mathbb{Z} \setminus \{0\}} (\mathfrak{h} \otimes t^n) \oplus \mathbb{C} C \subset \hat{\mathfrak{h}}.$$

As with many other infinite-dimensional Lie algebras, Heisenberg Lie algebras play an important role in the theory of vertex algebras and their representations; see, e.g., [6, 8, 13, 18]. In this paper, we study certain deformation of the universal enveloping algebra of the Heisenberg Lie algebra, along with the underlying quantum vertex algebra theory.

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The notion of quantum vertex algebra was introduced by Etingof and Kazhdan in [4], where they also associated the quantum affine vertex algebras to the rational, trigonometric and elliptic $R$-matrix of type $A$. The $S$-locality of these quantum vertex algebras, i.e. the quantum version of the locality property for vertex algebras, possesses the form of the so-called quantum current commutation relation, which goes back to Reshetikhin and Semenov-Tian-Shansky [17]. In this paper, we continue the study [11] of the interplay between the quantum current commutation relation associated with the rational $R$-matrix of type $A$ and quantum vertex algebra theory. Motivated by the form of the Heisenberg Lie algebra defining relation (1.2), we investigate quantum (vertex) algebras defined by the relations which come from the aforementioned quantum current commutation relation by extracting the quadratic terms and terms containing only the central element $C$ or the unit 1. Such relations can be expressed in the form similar to (1.2) as

$$y_1(u)y_2(v) + S_{12}(u - v, C) = y_2(v)y_1(u) + S_{21}(v - u, C),$$

where $y(z)$ denotes the $N \times N$ matrix of formal power series of the algebra generators and $S(z, C)$ is a certain product of two copies of the suitably normalized Yang $R$-matrices. The precise meaning of (1.3) is explained in Subsection 2.1.

In Section 2, we construct quantum vertex algebras $H_N^+$ whose braiding map $S = S(z)$ is governed by $S(z, c)$, where $c \in \mathbb{C}$, so that the form of their $S$-locality property resembles (1.3). Moreover, we show that $H_N^+$ contains a quantum vertex subalgebra $V_{H}(c)$ whose classical limit $\hbar \to 0$ coincides with the level $c$ Heisenberg vertex algebra associated to $\hat{\mathfrak{h}}_c$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}_N$.

In Section 3, we study certain associative algebras $H(C)$ and $H(C)_*$ defined over $\mathbb{C}[[\hbar]]$, whose defining relations are found by taking the diagonal entries of (1.3). In particular, we establish the Poincaré–Birkhoff–Witt theorem for these algebras. Roughly speaking, $H(C)$ and $H(C)_*$ can be regarded as deformations of universal enveloping algebras $U(\mathfrak{h})$ and $U(\mathfrak{h}_c)$, respectively. Next, we turn to their representation theory and, following the classical theory, we introduce the notion of restricted module. Furthermore, we construct examples of such modules by generalizing the well-known canonical realization of the affine Lie algebra $\hat{\mathfrak{h}}$. Finally, we show that the (irreducible) modules for the quantum vertex algebra $V_{H}(c)$ coincide with the level $c$ (irreducible) restricted $H(C)$-modules.

In the end, we should mention that the problem of associating quantum vertex algebras to certain deformed Heisenberg Lie algebras, which differ from those considered in this paper, was studied by Li [15].

2. QUANTUM VERTEX ALGEBRAS

In Subsections 2.1 and 2.2, we introduce the data which is required to define a structure of quantum vertex algebra over a certain quotient $H_N^+$ of the $\hbar$-adically completed algebra of polynomials in infinitely many variables. In Subsection 2.3, we present the main result of this section, i.e. the aforementioned construction of quantum vertex algebra $H_N^+$, and in Subsection 2.4 we give its proof. Finally, in Subsection 2.5, we discuss certain quantum vertex subalgebra $V_{H}(c) \subset H_N^+$ which is a deformation of the Heisenberg vertex algebra.

2.1. Creation and annihilation operators. Let $N \geq 2$ be an integer and $\hbar$ a formal parameter. The Yang $R$-matrix $R(u) = R_{12}(u) \in \text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N[\hbar/\hbar]$ is defined by

$$R(u) = I - \frac{\hbar}{u} P,$$

where $P$ is a projector on an irreducible representation of $U(\mathfrak{sl}_N)$.
where $I$ is the identity and $P$ the permutation operator,

\[ I = \sum_{i,j=1}^{N} e_{ii} \otimes e_{jj} \quad \text{and} \quad P = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji}, \quad (2.1) \]

and $e_{ij}$ are the matrix units. Let $C$ be another formal parameter. There exists a unique formal power series

\[ G(u, C) = 1 + \frac{C + N \frac{h^2}{u^2}}{N} - \frac{C(C + N) \frac{h^3}{u^3}}{N} + \ldots \in \mathbb{C}[C][[h/u]] \]

such that

\[ \text{tr}_i (G(u, C)R(u)R(-u - hC) - I) = \text{tr}_2 (G(u, C)R(u)R(-u - hC) - I) = 0, \quad (2.2) \]

where $\text{tr}_i$ denotes the trace taken over the $i$-th tensor factor. Fix $c \in \mathbb{C}$ and define a power series $S(u, c) \in u^{-2} \text{End} \mathbb{C}^{N} \otimes \text{End} \mathbb{C}^{N}[[h/u]]$ by

\[ S(u, c) = h^{-2} (G(u, c)R(u)R(-u - hc) - I). \quad (2.3) \]

We shall often omit the second argument $c$ and write $S(u)$ instead of $S(u, c)$. One easily checks that $S(u)$ is well-defined, i.e. that the expression $G(u, c)R(u)R(-u - hc) - I$ possesses a zero of order two at $h = 0$, so that (2.3) does not contain any negative powers of the parameter $h$. Moreover, we have

\[ S(u) \in \frac{c}{N u^2} (I - NP) + \frac{h}{u^3} \text{End} \mathbb{C}^{N} \otimes \text{End} \mathbb{C}^{N}[[h/u]]. \quad (2.4) \]

Consider the $h$-adically completed algebra of polynomials in variables $x_{ij}^{(r)}$,

\[ \mathcal{P} = \mathbb{C}[x_{ij}^{(r)} : i, j = 1, \ldots, N, r \geq 1][[h]]. \]

Let $\mathcal{I}$ be the $h$-adically complete ideal in $\mathcal{P}$ generated by the elements

\[ x_{11}^{(r)} + x_{22}^{(r)} + \ldots + x_{NN}^{(r)} \quad \text{with} \quad r = 1, 2, \ldots. \quad (2.5) \]

We define the associative algebra $H_N^+$ over the ring $\mathbb{C}[[h]]$ as the quotient

\[ H_N^+ = \mathcal{P}/\mathcal{I}. \]

It is clear that $H_N^+$ is topologically free, i.e. a torsion-free, separated and $h$-adically complete $\mathbb{C}[[h]]$-module; see, e.g., [10, Ch. XVI] for more information on $\mathbb{C}[[h]]$-modules.

It will be convenient to arrange the elements $x_{ij}^{(r)}$ into matrices of formal power series,

\[ x^+(u) = \sum_{i,j=1}^{N} e_{ij} \otimes x_{ij}^+(u) \in \text{End} \mathbb{C}^{N} \otimes H_N^+[[u]], \quad \text{where} \quad x_{ij}^+(u) = \sum_{r \geq 1} x_{ij}^{(r)} u^{r-1}. \]

Also, generalizing the above formula, for any integer $n \geq 1$ we write

\[ x_{[n]}^+(u) = x_1^+(u_1) \ldots x_n^+(u_n) \quad \text{and} \quad x_{[n]}^+(z + u) = x_1^+(z + u_1) \ldots x_n^+(z + u_n), \quad (2.6) \]

where $u = (u_1, \ldots, u_n)$ is a family of variables, $z$ a single variable and

\[ x_k^+(u) = \sum_{i,j=1}^{N} 1^{\otimes (k-1)} \otimes e_{ij} \otimes 1^{\otimes (n-k)} \otimes x_{ij}^+(u) \quad \text{with} \quad k = 1, \ldots, n. \quad (2.7) \]

Hence the coefficients of the expressions in (2.6) belong to $(\text{End} \mathbb{C}^{N})^{\otimes n} \otimes H_N^+$. Throughout the paper we shall often use the notation as in (2.7), where the subscripts indicate factors in the tensor product algebra.
Let $V$ be a $\mathbb{C}[[h]]$-module. We denote by $V((u))_h$ the $\mathbb{C}[[h]]$-module of all series
\[ a(u) = \sum_{r \in \mathbb{Z}} a_r u^{-r-1} \in V[[u^{\pm 1}]] \quad \text{such that} \quad a_r \to 0 \text{ when } r \to \infty \quad (2.8) \]
with respect to the $h$-adic topology. Furthermore, we denote by $V[u^{-1}]_h$ the $\mathbb{C}[[h]]$-module of all series as in $(2.8)$ such that, in addition, $a(u)$ belongs to $V[[u^{-1}]]$. Such notation naturally extends to the multiple variable case, so that we write, e.g., $V((u_1, \ldots, u_n))_h$.
Observe that if $V$ is a topologically free $\mathbb{C}[[h]]$-module, hence isomorphic to $V_0[[h]]$ for some complex space $V_0$, then $V((u))_h$ is topologically free as well. Moreover, $V((u))_h$ can be then identified with $V_0((u))[[h]]$, which is the $h$-adic completion of $V((u))$.

The next proposition can be easily proved by using $(2.2)$ and the defining relations for the algebra $H_N^+$,
\[ x_{11}^{(-r)} + x_{22}^{(-r)} + \ldots + x_{NN}^{(-r)} = 0 \quad \text{for all} \quad r = 1, 2, \ldots . \quad (2.9) \]

**Lemma 2.1.** For any $c \in \mathbb{C}$ there exists a unique operator
\[ x^-(u) = \sum_{i,j=1}^N e_{ij} \otimes x^-_{ij}(u), \quad \text{where} \quad x^-_{ij}(u) = \sum_{r \geq 1} x_{ij}^{(r-1)} u^{-r}, \]
which belongs to $\text{End} \mathbb{C}^N \otimes \text{Hom}(H_N^+, H_N^+[u^{-1}]_h)$, such that $x^-(u)1 = 0$ and for any integer $n \geq 1$ and the variables $u = (u_1, \ldots, u_n)$ we have
\[ x_0^-(u_0) x_0^+(u)^n(u) = - \sum_{j=1}^n S_{0j}(u_0 - u_j) x_1^+(u_1) \ldots x_{j-1}^+(u_{j-1}) x_{j+1}^+(u_{j+1}) \ldots x_n^+(u_n). \quad (2.10) \]

Regarding the identity $(2.10)$, note that, in accordance with $(2.7)$, $x_0^-(u_0)$ is applied to the first and $x_0^+(u)^n(u)$ on the next $n$ tensor factors of $\text{End} \mathbb{C}^N \otimes (\text{End} \mathbb{C}^N)^{\otimes n} \otimes H_N^+$. Furthermore, in $(2.10)$, as well as in the rest of the paper, we use the expansion convention where the expressions of the form $(x_1 + \ldots + x_n)^r$ with $r < 0$ are expanded in the nonnegative powers of the variables $x_2, \ldots, x_n$. Hence, for example, we have
\[(u_0 - u_j)^r = \sum_{k=0}^r \binom{r}{k} u_0^{-k} (-u_j)^k \in \mathbb{C}[u_0^{-1}][[u_j]] \quad \text{for} \quad r < 0. \]

By employing $(2.10)$ one can prove
\[ x_1^-(u_1) x_2^+(u_2) = x_2^+(u_2) x_1^-(u_1). \quad (2.11) \]
Moreover, the equalities in $(2.2)$ imply
\[ x_{11}^{(r-1)} + x_{22}^{(r-1)} + \ldots + x_{NN}^{(r-1)} = 0 \quad \text{for all} \quad r = 1, 2, \ldots . \quad (2.12) \]

We now regard $x^+(u)$ as an operator on $H_N^+$, where its action is given by the multiplication.

By $(2.10)$ for $n = 1$ we have
\[ x_1^-(u_1) x_2^+(u_2) - x_2^+(u_2) x_1^-(u_1) = -S(u_1 - u_2). \quad (2.13) \]

Let us combine $x^+(u)$ and $x^-(u)$ into a single operator series
\[ x(u) = x^+(u) + x^-(u) \in \text{End} \mathbb{C}^N \otimes \text{Hom}(H_N^+, H_N^+[u^{-1}]_h). \quad (2.14) \]

Since $PS(u) = S(u)P$, the identities $(2.11)$ and $(2.13)$ imply
\[ x_1(u_1) x_2(u_2) - x_2(u_2) x_1(u_1) = -S(u_1 - u_2) + S(u_2 - u_1), \quad (2.15) \]
while (2.9) and (2.12) imply
\[ x_{11}^{(r)} + x_{22}^{(r)} + \ldots + x_{NN}^{(r)} = 0 \quad \text{for all} \quad r \in \mathbb{Z}. \] (2.16)

Note that equality (2.15) can be also written as
\[ x_1(u_1)x_2(u_2) + S(u_1 - u_2) = x_2(u_2)x_1(u_1) + S(u_2 - u_1). \] (2.17)

In accordance with our expansion convention, the left hand side of (2.17) belongs to
\[ (\text{End } \mathbb{C}^N)^{\otimes 2} \otimes \text{Hom}(H_N^+, H_N^+((u_1))((u_2))_h) \]
and the right hand side to
\[ (\text{End } \mathbb{C}^N)^{\otimes 2} \otimes \text{Hom}(H_N^+, H_N^+((u_2))((u_1))_h), \]
so that the both sides are elements of \((\text{End } \mathbb{C}^N)^{\otimes 2} \otimes \text{Hom}(H_N^+, H_N^+((u_1, u_2))_h)\). We write
\[ x_{[2]}(u) = x_{[2]}(u_1, u_2) = x_1(u_1)x_2(u_2) + S(u_1 - u_2). \] (2.18)

Our next goal is to generalize (2.18) to an arbitrary number of factors. Let \(n\) be a positive integer. For \(n = 1\) we set \(x_{[1]}(u) = x(u)\). Suppose \(n > 1\). For any \(k = 0, \ldots, \lfloor n/2 \rfloor\) denote by \(I^n_k\) the family of all sets of \(k\) ordered pairs \((p, q) \in \{1, \ldots, n\}^2\) such that \(p < q\) and such that the coordinates of all pairs which belong to the same set are mutually distinct. For any \(i \in I^n_k\) we denote by \(i'\) the set of all integers in \(\{1, \ldots, n\}\) which do not appear in \(i\).

**Example 2.2.** For \(n = 4\) and \(k = 0, 1, 2\) we have \(I^4_0 = \emptyset\), \(I^4_1 = \{\{(1, 2)\}, \{(1, 3)\}, \{(1, 4)\}, \{(2, 3)\}, \{(2, 4)\}, \{(3, 4)\}\}, \)
\(I^4_2 = \{\{(1, 2), (3, 4)\}, \{(1, 3), (2, 4)\}, \{(1, 4), (2, 3)\}\}. \)

Note that for all \(i \in I^n_k\) we have \(i' = \emptyset\). As for the elements of \(I^n_1\) we have, e.g.,
\(\{(1, 2)\}' = \{3, 4\}, \quad \{(1, 4)\}' = \{2, 3\}, \quad \{(2, 3)\}' = \{1, 4\}. \)

For \(u = (u_1, \ldots, u_n)\) write \(S_{i_p,j_p} = S_{i_p,j_p}(u_{i_p} - u_{j_p})\) for all \(p = 1, \ldots, k\). Define
\[ x_{[n]}(u) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{i=(i_1,j_1),\ldots,(i_k,j_k)\in I^n_k \atop l_1<\ldots<l_{n-2k} \leq i'} S_{i_1,j_1} \ldots S_{i_k,j_k} x_{l_1}(u_{l_1}) \ldots x_{l_{n-2k}}(u_{l_{n-2k}}), \] (2.19)
where the second sum is \(x_1(u_1) \ldots x_n(u_n)\) for \(k = 0\) and the indices denote the tensor factors of \((\text{End } \mathbb{C}^N)^{\otimes n} \otimes H_N^+\) on which the corresponding elements are applied.

**Example 2.3.** Clearly, (2.19) coincides with (2.18) for \(n = 2\), while for \(n = 3, 4\) we get
\[
\begin{align*}
  x_{[3]}(u) &= x_1(u_1)x_2(u_2)x_3(u_3) + S_{12}x_3(u_3) + S_{13}x_2(u_2) + S_{23}x_1(u_1), \\
  x_{[4]}(u) &= x_1(u_1)x_2(u_2)x_3(u_3)x_4(u_4) + S_{12}x_3(u_3)x_4(u_4) + S_{13}x_2(u_2)x_4(u_4) \\
  &\quad + S_{14}x_2(u_2)x_3(u_3) + S_{23}x_1(u_1)x_4(u_4) + S_{24}x_1(u_1)x_3(u_3) \\
  &\quad + S_{34}x_1(u_1)x_2(u_2) + S_{12}S_{14} + S_{13}S_{24} + S_{14}S_{23}. 
\end{align*}
\]

Obviously, (2.19) belongs to \((\text{End } \mathbb{C}^N)^{\otimes n} \otimes \text{Hom}(H_N^+, H_N^+((u_1) \ldots (u_n))_h)\). However, by using commutation relation (2.17) one can verify the following stronger statement:

**Proposition 2.4.** For any integer \(n \geq 1\) we have
\[ x_{[n]}(u_1, \ldots, u_n) \in (\text{End } \mathbb{C}^N)^{\otimes n} \otimes \text{Hom}(H_N^+, H_N^+((u_1, \ldots, u_n))_h). \]
Let $z$ be a single variable. Due to Proposition 2.4 we can define

$$x_{[n]}(z + u) = x_{[n]}(z + u_1, \ldots, z + u_n) := x_{[n]}(z_1, \ldots, z_n)\big|_{z_1 = z + u_1, \ldots, z_n = z + u_n},$$  \quad (2.20)

In addition, the given element satisfies

$$x_{[n]}(z + u) \in (\text{End } \mathbb{C}^N)^{\otimes n} \otimes \text{Hom}(H_N^+, H_N^+((z))_h[[u_1, \ldots, u_n]]).$$  \quad (2.21)

Note that $x_{[n]}(z + u)$ differs from

$$x_{[n]}(u + z) = x_{[n]}(u_1 + z, \ldots, u_n + z) := x_{[n]}(z_1, \ldots, z_n)\big|_{z_1 = u_1 + z, \ldots, z_n = u_n + z}$$

(which is also well-defined by Proposition 2.4) as $x_{[n]}(u + z)$ should be expanded in the nonnegative powers of $z$.

**Remark 2.5.** As we demonstrate later on, the coefficients of the matrix entries of $x^+(u)$ and $x^-(u)$ can be regarded as deformations of certain creation and annihilation operators, respectively, while the operators $x_{[n]}(u)$ take place of the normal-ordered products.

2.2. **Braiding map.** From now on, the tensor products of $\mathbb{C}[[h]]$-modules are understood as $h$-adically completed. Define

$$T(z) = S(z) - S(-z) \in \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}[z^{-1}]_h.\quad (2.22)$$

Note that $T(z_1 - z_2)$ is not equal to $S(z_1 - z_2) - S(z_2 - z_1)$, since $S(z_2 - z_1)$ is to be expanded in nonnegative powers of $z_1$. However, for any integer $n \geq 0$ there exists an integer $r \geq 0$ such that

$$(z_1 - z_2)^r T(z_1 - z_2) = (z_1 - z_2)^r (S(z_1 - z_2) - S(z_2 - z_1)) \quad \text{mod } h^n.\quad (2.23)$$

Let $m, n \geq 1$ be integers. For any $k = 0, \ldots, \min\{m, n\}$ let $I^{n,m}_{k} \subseteq I^{n+m}_k$ be the family of all sets of $k$ ordered pairs $(p, q) \in \{1, \ldots, n\} \times \{n + 1, \ldots, n + m\}$ such that the coordinates of all pairs which belong to the same set are mutually distinct. For $i \in I^{n,m}_k$ we denote by $i'$ the set of all integers in $\{1, \ldots, n + m\}$ which do not appear in $i$.

**Example 2.6.** For $n = 3$, $m = 2$ and $k = 0, 1, 2$ we have $I^{3,2}_0 = \emptyset$,

$$I^{3,2}_1 = \{ \{(1, 4)\}, \{(1, 5)\}, \{(2, 4)\}, \{(2, 5)\}, \{(3, 4)\}, \{(3, 5)\} \},$$

$$I^{3,2}_2 = \{\{ (1, 4), (2, 5) \}, \{ (1, 4), (3, 5) \}, \{ (1, 5), (2, 4) \}, \{ (1, 5), (3, 4) \}, \{ (2, 4), (3, 5) \}, \{ (2, 5), (3, 4) \} \}.$$

Also, for example, we have

$$\{(1, 4)\}' = \{2, 3, 5\}, \quad \{(3, 5)\}' = \{1, 2, 4\}, \quad \{(2, 4), (3, 5)\}' = \{1\}.$$  

Let $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_m)$ be families of variables. Consider the expression

$$x_{[n]}^{+13}(u)x_{[m]}^{+24}(v) = x_{1+n+m}^+(u_1, \ldots, x_{n+n+m+1}^+(u_n, x_{n+1+n+m+2}^+(v_1, \ldots, x_{n+m+n+m+2}^+(v_m))(2.24)$$

with coefficients in

$$\frac{1}{\text{(End } \mathbb{C}^N)^{\otimes n}} \otimes \frac{2}{(\text{End } \mathbb{C}^N)^{\otimes m}} \otimes \frac{3}{H_N^+} \otimes \frac{4}{H_N^+}$$

and superscripts $1, 2, 3, 4$ indicating the tensor factors as in (2.25). If $n = 0$ or $m = 0$ we define the corresponding empty product in (2.24) to be the unit $1 \in H_N^+$.  


Lemma 2.7. There exists a unique $\mathbb{C}[[h]]$-module map

$$S(z) : H_N^+ \otimes H_N^+ \to H_N^+ \otimes H_N^+[z^{-1}]_h$$

such that for any integers $m, n \geq 0$ and the variables $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$

$$S(z)x_{[n]}^{13}(u)x_{[m]}^{24}(v) = \sum_{k=0}^{\min\{m,n\}} \sum_{i=(i_1,j_1), \ldots, (i_k,j_k) \in \mathcal{I}^{m,n}_{k}} T_{i_1,j_1} \cdots T_{i_k,j_k} x_{l_1}^+ \cdots x_{l_{n+m-2k}}^+ \cdots x_{l_{n+m-2k}}^+,$$  \hspace{1cm} (2.26)

where $T_{i_p,j_p}$ denotes $T_{i_p,j_p}(z + u_{i_p} - v_{j_p-n})$ and

$$x_{l_p}^+ = \begin{cases} x_{l_p+n+m+1}(u_{i_p}) & \text{for } l_p = 1, \ldots, n, \\ x_{l_p+n+m+2}(v_{j_p-n}) & \text{for } l_p = n+1, \ldots, n+m. \end{cases}$$

The given map is of the form $S = 1 + O(h)$ and satisfies the Yang-Baxter equation,

$$S_{13}(z_1)S_{13}(z_1 + z_2)S_{23}(z_2) = S_{23}(z_2)S_{13}(z_1 + z_2)S_{13}(z_1) \hspace{1cm} (2.27)$$

and the unitarity condition,

$$S_{21}(z) = S^{-1}(-z). \hspace{1cm} (2.28)$$

Proof. It is clear that (2.26) uniquely determines a well-defined $\mathbb{C}[[h]]$-module map on $H_N^+ \otimes H_N^+$. Furthermore, its image belongs to $H_N^+ \otimes H_N^+[z^{-1}]_h$ as $T(z)$ is in $\text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N[z^{-1}]_h$. In fact, (2.4) and (2.22) imply that $T(z)$ belongs to $h \text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N[z^{-1}]_h$. Hence, as the only summand on the right-hand side of (2.26) which does not contain a copy of $T$ is $x_{[n]}^{13}(u)x_{[m]}^{24}(v)$, the given map is of the form $S = 1 + O(h)$.

Regarding the Yang-Baxter equation (2.27), it is sufficient to show that the operators $S_{12}(z_1)$, $S_{13}(z_1 + z_2)$ and $S_{23}(z_2)$ are mutually commutative. However, this follows directly from the definition (2.26). Indeed, choose any $A, B \in \{S_{12}(z_1), S_{13}(z_1 + z_2), S_{23}(z_2)\}$ such that $A \neq B$ and then consider the actions of $AB$ and $BA$ on the expression

$$x := x_{[n]}^{14}(u)x_{[m]}^{25}(v)x_{[k]}^{36}(w) = x_{[n]}^{14}(u_1, \ldots, u_n)x_{[m]}^{25}(v_1, \ldots, v_m)x_{[k]}^{36}(w_1, \ldots, w_k)$$

with coefficients in

$$(\text{End} \mathbb{C}^N)^{\otimes n} \otimes (\text{End} \mathbb{C}^N)^{\otimes m} \otimes (\text{End} \mathbb{C}^N)^{\otimes k} \otimes H_N^+ \otimes H_N^+ \otimes H_N^+. \hspace{1cm} (2.29)$$

By (2.26), both of the actions produce a sum of certain elements of the form

$$T_{i_1,j_1} \cdots T_{i_r,j_r} x_{a_1}^+b_1 \cdots x_{a_{n+m+k-2r}}^+b_{n+m+k-2r} \hspace{1cm} (2.30)$$

with coefficients in (2.29) such that all factors in (2.30) mutually commute. Finally, note that every such term of the form (2.30) appears as a summand in $ABx$ if and only if it appears as a summand in $BAx$, which implies $AB = BA$.

Let us prove that $S(z)$ possesses the unitarity property (2.28). Note that $T(z)$ is an odd function, i.e. we have $T(z) = -T(-z)$; recall (2.22). Moreover, as $R_{12}(z) = R_{21}(z)$ we have $T_{12}(z) = T_{21}(z)$ and, consequently, $S_{12}(z) = S_{21}(z)$. Therefore, by (2.26) we have

$$S_{21}(-z)x_{[n]}^{13}(u)x_{[m]}^{24}(v) = \sum_{k=0}^{\min\{m,n\}} \sum_{i=(i_1,j_1), \ldots, (i_k,j_k) \in \mathcal{I}^{m,n}_{k}} (-1)^k T_{i_1,j_1} \cdots T_{i_k,j_k} x_{l_1}^+ \cdots x_{l_{n+m-2k}}^+ \cdots x_{l_{n+m-2k}}^+. \hspace{1cm} (2.31)$$
By combining the identities \((2.26)\) and \((2.31)\) one easily checks that all terms of the form 
\[
T_{i_1,j_1} \cdots T_{i_k,j_k} x_{i_1}^+ \cdots x_{i_{k+m-2k}}^+
\]
with \(k > 0\)
\[
S_{21}(-z)S(z)x_{[n]}^{+13}(u)x_{[m]}^{+24}(v) \quad \text{and} \quad S(z)S_{21}(-z)x_{[n]}^{+13}(u)x_{[m]}^{+24}(v)
\]
cancel, so that both expressions in \((2.32)\) are equal to \(x_{[n]}^{+13}(u)x_{[m]}^{+24}(v)\), as required. \(\Box\)

2.3. Quantum vertex algebra \(H_N^+\). Let us recall the Etingof–Kazhdan definition of quantum vertex algebra; see [4, Subsect. 1.4].

**Definition 2.8.** A quantum vertex algebra is a quadruple \((V, Y, 1, S)\) such that

1. \(V\) is a topologically free \(\mathbb{C}[[h]]\)-module.
2. \(Y\) is a \(\mathbb{C}[[h]]\)-module map (the vertex operator map)
\[
Y: V \otimes V \to V((z))_h
\]
\[
u \otimes v \mapsto Y(z)(u \otimes v) = Y(u, z)v = \sum_{r \in \mathbb{Z}} u_r vz^{-r-1}
\]
which satisfies the weak associativity: for any \(u, v, w \in V\) and \(n \in \mathbb{Z}_{\geq 0}\) there exists \(s \in \mathbb{Z}_{\geq 0}\) such that
\[
(z_0 + z_2)\ast Y(u, z_0 + z_2)Y(v, z_2)w - (z_0 + z_2)\ast Y(Y(u, z_0)v, z_2)w \in h^nV[[z_0^\pm 1, z_2^\pm 1]].
\]
3. \(1\) is a distinct element of \(V\) (the vacuum vector) such that
\[
Y(1, z)v = v, \quad Y(v, z)1 \in V[[z]] \quad \text{and} \quad \lim_{z \to 0} Y(v, z)1 = v \quad \text{for all} \quad v \in V.
\]
4. \(S = S(z)\) is a \(\mathbb{C}[[h]]\)-module map \(V \otimes V \to V \otimes V \otimes \mathbb{C}((z))[[h]]\) (the braiding) of the form \(S = 1 + O(h)\) which satisfies the Yang–Baxter equation \((2.27)\), the unitarity condition \((2.28)\), the shift condition \((2.29)\), the \(S\)-locality: for any \(u, v \in V\) and \(n \in \mathbb{Z}_{\geq 0}\) there exists \(r \in \mathbb{Z}_{\geq 0}\) such that
\[
(z_1 - z_2)^r Y(z_1)(1 \otimes Y(z_2))(S(z_1 - z_2)(u \otimes v) \otimes w)
\]
\[
- (z_1 - z_2)^r Y(z_2)(1 \otimes Y(z_1))(v \otimes u \otimes w) \in h^nV[[z_1^{\pm 1}, z_2^{\pm 1}]] \quad \text{for all} \quad w \in V
\]
and the hexagon identity:
\[
S(z_1)(Y(z_2) \otimes 1) = (Y(z_2) \otimes 1)S_{23}(z_1)S_{13}(z_1 + z_2).
\]

**Remark 2.9.** We should mention that Definition 2.8 slightly differs from the original in [4]. We included both weak associativity \((2.33)\) and hexagon identity \((2.37)\) in our definition in order to emphasize the importance of both of these properties for quantum vertex algebra theory. However, any one of them can be omitted from the definition and then proved using the remaining axioms; see [4, Prop. 1.4] and [3, Prop. 3.14].

We now use the data from the previous subsections, in particular, Lemmas 2.1 and 2.7, to define a quantum vertex algebra structure over \(H_N^+\).

**Theorem 2.10.** For any \(c \in \mathbb{C}\) there exists a unique structure of quantum vertex algebra on \(H_N^+\) such that the vacuum vector is the unit \(1 \in H_N^+\), the braiding \(S(z)\) is given by \((2.26)\) and the vertex operator map is given by
\[
Y(x_{[n]}^+(u_1, \ldots, u_n), z) = x_{[n]}(z + u_1, \ldots, z + u_n).
\]
The theorem is proved by directly verifying the constraints imposed by Definition 2.8. We dedicate the next subsection to its proof.

2.4. A proof of Theorem 2.10. It is clear that the $h$-adically completed polynomial algebra $H_N^+$ is topologically free. Moreover, the matrix entries of the coefficients of all $x_{[n]}^+(u_1, \ldots, u_n)$ along with $1$ span an $h$-adically dense $\mathbb{C}[\![h]\!]$-submodule of $H_N^+$ so that (2.38) uniquely determines the vertex operator map. However, we have to prove that the vertex operator map is well-defined by (2.38). It is sufficient to check that $Y(z)$ maps the ideal of defining relations $[x_{[n]}^+(u), x_{[n]}^+(v)] = 0$ and (2.9) for $H_N^+$ to itself. Regarding the first family of relations, for any integers $n > j > 0$ consider the expression

$$x_{[n]}^{+(j)}(u) := x_{[n]}^+(u_1) \ldots x_{[n]}^+(u_{j-1}) x_{[n]}^+(u_{j+1}) x_{[n]}^+(u_j) x_{[n]}^+(u_{j+2}) \ldots x_{[n]}^+(u_n). \quad (2.39)$$

We will show that its image under $Y(z)$ coincides with the right-hand side of (2.38), which implies the desired conclusion. First, we write (2.19) as

$$x_{[n]}(u) = x_{[n]}(u)_{j,j+1} + x_{[n]}(u)^{j,j+1} + x_{[n]}(u)_0,$$  \hspace{1cm} \text{where} \hspace{1cm} \text{(2.40)}

- $x_{[n]}(u)_{j,j+1}$ denotes the sum of all $S_{11,j} \ldots S_{jk} x_{[1]}(u_1) \ldots x_{[n-2k]}(u_{n-2k})$ in (2.19) which contain $x_j(u_j)x_{j+1}(u_{j+1})$;
- $x_{[n]}(u)^{j,j+1}$ denotes the sum of all $S_{11,j} \ldots S_{jk} x_{[1]}(u_1) \ldots x_{[n-2k]}(u_{n-2k})$ in (2.19) which contain $S_{j+1}(u_j - u_{j+1})$;
- $x_{[n]}(u)_0 = x_{[n]}(u) - x_{[n]}(u)_{j,j+1} - x_{[n]}(u)^{j,j+1}$ are the remaining terms.

Next, we compare $x_{[n]}(u)$ with

$$x_{[n]}^{(j)}(u) := P_{j+1} x_{[n]}(u_1, \ldots, u_{j-1}, u_{j+1}, u_j, u_{j+2}, \ldots, u_n) P_{j+1},$$

where $P_{j+1}$ denotes the permutation operator from (2.1) applied on the tensor factors $j$ and $j+1$. As with (2.40), we write $x_{[n]}^{(j)}(u)$ as

$$x_{[n]}^{(j)}(u) = x_{[n]}^{(j)}(u)_{j,j+1,j} + x_{[n]}^{(j)}(u)^{j,j+1,j} + x_{[n]}^{(j)}(u)_0,$$  \hspace{1cm} \text{where} \hspace{1cm} \text{(2.41)}

- $x_{[n]}^{(j)}(u)_{j,j+1,j}$ denotes the sum of all $S_{11,j} \ldots S_{jk} x_{[1]}(u_1) \ldots x_{[n-2k]}(u_{n-2k})$ in $x_{[n]}^{(j)}(u)$ which contain $x_{j+1}(u_{j+1})x_j(u_j)$;
- $x_{[n]}^{(j)}(u)^{j,j+1,j}$ denotes the sum of all $S_{11,j} \ldots S_{jk} x_{[1]}(u_1) \ldots x_{[n-2k]}(u_{n-2k})$ in $x_{[n]}^{(j)}(u)$ which contain $S_{j+1,j}(u_j - u_{j+1})$;
- $x_{[n]}^{(j)}(u)_0 = x_{[n]}^{(j)}(u) - x_{[n]}^{(j)}(u)_{j,j+1,j} - x_{[n]}^{(j)}(u)^{j,j+1,j}$ are the remaining terms.

Clearly, $x_{[n]}(u)_0 = x_{[n]}^{(j)}(u)_0$. Furthermore, as $S_{12}(z) = S_{21}(z)$, the identity (2.17) implies

$$x_{[n]}(u)_{j,j+1} + x_{[n]}(u)^{j,j+1} = x_{[n]}^{(j)}(u)_{j,j+1,j} + x_{[n]}^{(j)}(u)^{j,j+1,j}.$$  \hspace{1cm} \text{By combining these two observations with (2.40) and (2.41) we find that}  \hspace{1cm} \text{(2.42)}

Finally, by replacing the variables $(u_1, \ldots, u_n)$ with $(z + u_1, \ldots, z + u_n)$ in (2.42) we get

$$Y(x_{[n]}^{+(j)}(u), z) = Y(x_{[n]}^{+(j)}(u), z),$$

as required. As for the remaining family of relations (2.9), this is a direct consequence of (2.2) and (2.16). Indeed, due to the aforementioned equalities, for any $i = 1, \ldots, n$ by applying the partial trace $\text{tr}_i$ to the $i$-th tensor factor of any summand in (2.19) produces zero. Thus, we conclude that the vertex operator map is well-defined by (2.38).
Regarding the vacuum vector axioms (2.34), $Y(1,z)v = v$ clearly holds for all $v \in H_N^+$ while the remaining properties follow from the identity
\[
x_{[n]}(u_1, \ldots, u_n)1 = x_{[n]}^+(u_1, \ldots, u_n),
\]
which can be directly verified using Lemma 2.1 and commutation relations (2.13). Finally, the image of the vertex operator map belongs to $H_N^+ \otimes H_N^+(z)$ by (2.21).

As for the braiding, let us verify the shift condition (2.35). The remaining requirements on $S(z)$, which are imposed by Definition 2.8, hold by Lemma 2.7. First, note that by applying (2.38) on the vacuum vector $1$ and then taking the coefficient with respect to $z$ we get, due to (2.43),
\[
Dx_{[n]}^+(u_1, \ldots, u_n) = \left(\sum_{r=1}^n \frac{\partial}{\partial u_r}\right) x_{[n]}^+(u_1, \ldots, u_n).
\]
Furthermore, by (2.34) we have $D1 = 0$. Using these observations along with (2.26) we compute the action of $(D \otimes 1)S(z)$ and $S(z)(D \otimes 1)$ on
\[
x := x_{[n]}^{+13}(u)x_{[m]}^{+24}(v) = x_{[n]}^{+13}(u_1, \ldots, u_n)x_{[m]}^{+24}(v_1, \ldots, v_m)
\]
as follows. We get
\[
(D \otimes 1)S(z)x = (D \otimes 1) \sum_{k=0}^{\min\{m,n\}} \sum_{i=(i_1,j_1), \ldots \ldots (i_k,j_k) \in I_{l_1}^{n,m}} \sum_{l_1 < \cdots < l_{n+m-2k}} T_{i_1,j_1} \cdots T_{i_k,j_k} x_{i_1}^+ \cdots x_{i_{n+m-2k}}^+,
\]
where all $T_{i_p,j_p}$ and $x_{i_p}^+$ are defined as in the statement of Lemma 2.7. Finally, by comparing the expressions (2.44) and (2.45) and using the identity
\[
\left(\sum_{r=1}^n \frac{\partial}{\partial u_r}\right) T_{i_1,j_1} \cdots T_{i_k,j_k} = \frac{\partial}{\partial z} T_{i_1,j_1} \cdots T_{i_k,j_k},
\]
which follows from
\[
\frac{\partial}{\partial u_{i_p}} T_{i_p,j_p}(z + u_{i_p} - v_{j_p-n}) = \frac{\partial}{\partial z} T_{i_p,j_p}(z + u_{i_p} - v_{j_p-n}),
\]
we conclude that the difference of (2.44) and (2.45) is equal to $-\frac{\partial}{\partial z} S(z)x$, as required. Hence the shift condition holds.

To finish the proof it remains to check that the vertex operator map satisfies the weak associativity (2.33) and $S$-locality (2.36); recall Remark 2.9. These properties are verified in Lemmas 2.11 and 2.12 below. Alternatively, one could prove the hexagon identity (2.37) instead of the weak associativity. However, we find the direct proof of the hexagon identity to be more technical, so we prove the latter property instead.
Let \( u = (u_1, \ldots, u_n) \) be a family of variables and \( z \) a single variable. In order to simplify the notation, we denote the families \((z + u_1, \ldots, z + u_n)\) and \((u_1 + z, \ldots, u_n + z)\) by \( z + u \) and \( u + z \) respectively. For example, (2.38) can be written briefly as
\[
Y(x^+_{[n]}(u), z) = x^+_{[n]}(z + u).
\]

**Lemma 2.11.** **Vertex operator map** (2.38) **satisfies the weak associativity** (2.33).

**Proof.** Choose any element \( y \in H^+_N \) and integers \( t, r_1, \ldots, r_n, s_1, \ldots, s_m \in \mathbb{Z}_{\geq 0} \). We start by considering the image of
\[
x^+_{[n]}(u)x^+_{[m]}(v) = x^+_{n+m+1}(u_1) \ldots x^+_{n+m+1}(u_n)x^+_{n+m+2}(v_1) \ldots x^+_{n+m+2}(v_m)
\]
(2.46) under the second summand in (2.33). By applying \( Y(z_2)(Y(z_0) \otimes 1) \) to (2.46) we get
\[
Y(x^+_{[n]}(z_0 + u)x^+_{[m]}(v), z_2).
\]
(2.47)
By (2.13) the argument of the vertex operator map can be expressed as
\[
x^+_{[n]}(z_0 + u)x^+_{[m]}(v) = \sum_{k=0}^{\min\{m,n\}} \sum_{i=(i_1,j_1), \ldots, (i_k,j_k) \in I^m_k} (-1)^k S_{i_1,j_1} \ldots S_{i_k,j_k} x^+_{l_1} \ldots x^+_{l_n+m-2k}
\]
(2.48) for \( S_{i_p,j_p} = S_{i_p,j_p}(z_0 + u_{i_p} - v_{j_p} - n) = S_{i_p,j_p}(w_{i_p} - w_{j_p}) \) and \( x^+_l = x^+_l(n+m+1)(w_p) \), where
\[
w_{i_p} = \begin{cases} 
  z_0 + u_{i_p} & \text{for } l_p = 1, \ldots, n, \\
  v_{i_p} - n & \text{for } l_p = n + 1, \ldots, n + m.
\end{cases}
\]
Therefore, by applying \( Y(z_2) \) to (2.48) and using (2.38) we conclude that (2.47) equals
\[
\sum_{k=0}^{\min\{m,n\}} \sum_{i=(i_1,j_1), \ldots, (i_k,j_k) \in I^m_k} (-1)^k S_{i_1,j_1} \ldots S_{i_k,j_k} x^+_{[n+m-2k]}(z_2 + w_1, \ldots, z_2 + w_{n+m-2k}),
\]
(2.49) where the expression \( x^+_{[n+m-2k]}(z_2 + w_1, \ldots, z_2 + w_{n+m-2k}) \) is applied on the tensor factors \( l_1, \ldots, l_{n+m-2k} \) and \( n + m + 1 \) of
\[
(\text{End} \mathbb{C}^N)^{\otimes n} \otimes (\text{End} \mathbb{C}^N)^{\otimes m} \otimes H^+_N.
\]
(2.50)
Define
\[
x^+_{x_p} = \begin{cases} 
  z_0 + z_2 + u_{i_p} & \text{for } l_p = 1, \ldots, n, \\
  z_2 + v_{i_p} - n & \text{for } l_p = n + 1, \ldots, n + m.
\end{cases}
\]
Clearly, there are only finitely many summands in (2.49). Hence by Proposition 2.4 there exists a nonnegative integer \( s \) such that the coefficients of
\[
u_{i_1}' \ldots u_{i_{n}}' e_1^{s_1'} \ldots e_{m}^{s_{m}'} \quad \text{with} \quad r_i' = 0, \ldots, r_i, \quad \text{and} \quad s_j' = 0, \ldots, s_j,
\]
(2.51)
where \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \), in
\[
(z_0 + z_2)^s S_{i_1,j_1} \ldots S_{i_k,j_k} x^+_{[n+m-2k]}(z_2 + w_1, \ldots, z_2 + w_{n+m-2k}) \quad \text{and} \quad (z_0 + z_2)^s S_{i_1,j_1} \ldots S_{i_k,j_k} x^+_{[n+m-2k]}(x_{l_1}, \ldots, x_{l_{n+m-2k}})
\]
(2.52)
(2.53)
coincide modulo $h^t$. Note that $x_{l^p} = z_2 + w_{l^p}$. However, the variable $z_0$ comes first from the left in $x_{l^p}$ for $l^p = 1, \ldots, n$, thus indicating that different expansions are applied in (2.52) and (2.53). Hence the expressions in (2.52) and (2.53) do not need to be equal.

For any choice of indices $l_1, \ldots, l_{n+m-2k}$ as in (2.49) introduce the function

$$\sigma := \sigma_{l_1, l_2, \ldots, l_{n+m-2k}} : \{1, \ldots, n + m - 2k\} \to \{l_1, \ldots, l_{n+m-2k}\}$$

$p \mapsto l_p$.

Henceforth all the given expressions are regarded modulo $u_{l_1}^{r_1 + 1} \ldots u_{l_m}^{r_m + 1} v_1^{s_1 + 1} \ldots v_m^{s_m + 1} h^t$, i.e. we work with the coefficients of (2.51) modulo $h^t$. Consider the action of the product of $(z_0 + z_2)^s$ and (2.49) on the element $y$. By rewriting all terms $x_{n+m-2k}(z_2 + w_l, \ldots, z_2 + w_{n+m-2k})$ of (2.49) using formula (2.19) we get

$$(z_0 + z_2)^s \sum_{k=0}^{\min(m,n)} \sum_{i=(i_1,j_1), \ldots, (i_k,j_k)} \in l_1^{n+m} \sum_{l_1 \cdots \leq l_{n+m-2k} \in l^t} (-1)^k S_{i_1,j_1} \cdots S_{i_k,j_k} \times \sum_{r=0}^{[(n+m-2k)/2]} \sum_{a=\{(a_1,b_1), \ldots, (a_r,b_r)\} \in \sigma((l_1^{n+m-k})} S_{a_1 b_1} \cdots S_{a_r b_r} x_{c_1} \cdots x_{c_{n+m-2k-2}} y, \quad (2.54)$$

where $\sigma = \sigma_{l_1, l_2, \ldots, l_{n+m-2k}}$, $S_{a\cdot b\cdot} = S_{a\cdot b\cdot}(w_{a\cdot} - w_{b\cdot})$ and $x_{c\cdot} = x_{c\cdot n+m+1}(z_2 + w_{c\cdot})$. All summands in (2.54) are of the form

$$(-1)^k S_{i_1,j_1} \cdots S_{i_k,j_k} S_{a_1 b_1} \cdots S_{a_r b_r} x_{c_1} \cdots x_{c_{n+m-2k-2}} y. \quad (2.55)$$

However, due to the sign $(-1)^k$, all summands (2.55) in (2.54) which contain at least one copy of $S_{p\cdot q\cdot}$ with $p \in \{1, \ldots, n\}$ and $q \in \{n+1, \ldots, n+m\}$ cancel, so that (2.54) equals

$$(z_0 + z_2)^s \sum_{r=0}^{[(n+m)/2]} \sum_{a=\{(a_1,b_1), \ldots, (a_r,b_r)\} \in \sigma((l_1^{n+m-k}))} S_{a_1 b_1} \cdots S_{a_r b_r} x_{c_1} \cdots x_{c_{n+m-2k}} y. \quad (2.56)$$

Finally, we conclude by (2.19) that (2.56) coincides with

$$(z_0 + z_2)^s x_{[n]}^{13}(z_2 + z_0 + u) x_{[m]}^{23}(z_2 + v) y. \quad (2.57)$$

Now consider the image of (2.46) under the first summand in (2.33). By applying $(z_0 + z_2)^s Y(z_0 + z_2)(1 \otimes Y(z_2))$ to (2.46) and using (2.38) we get

$$(z_0 + z_2)^s x_{[n]}^{13}(z_0 + z_2 + u) x_{[m]}^{23}(z_2 + v) y. \quad (2.58)$$

By the choice of the integer $s$, the coefficients of the variables (2.51) in (2.57) and (2.58) coincide modulo $h^t$, so the weak associativity follows. \qed

**Lemma 2.12.** Vertex operator map (2.38) possesses the $S$-locality property (2.36).

**Proof.** As with the proof of Lemma 2.11, we verify the $S$-locality directly, by comparing the images of (2.46) under the first and the second summand in (2.36). Let $t, r_1, \ldots, r_m, s_1, \ldots, s_m \geq 0$ be arbitrary integers. Henceforth all the expressions are regarded modulo

$$u_{l_1}^{r_1 + 1} \ldots u_{l_m}^{r_m + 1} v_1^{s_1 + 1} \ldots v_m^{s_m + 1} h^t, \quad (2.59)$$
i.e. we consider only the coefficients of the monomials (2.51) modulo $h^t$. Applying the first summand in (2.36), $Y(z_1)(1 \otimes Y(z_2))S(z_1 - z_2)$ to (2.46) and using (2.26) we get

\[
Y(z_1)(1 \otimes Y(z_2)) \sum_{k=0}^{\min\{m,n\}} \sum_{i=(i_{1,j_1},\ldots,i_{k,j_k}) \in I_k^{n,m}} T_{i_{1,j_1}} \cdots T_{i_{k,j_k}} x_{i_1}^+ \cdots x_{i_{n+m-2k}}^+ ,
\]

where

\[
T_{i_{p,j_p}} = S_{i_{p,j_p}} (z_1 - z_2 + u_{i_p} - v_{j_p-n}) - S_{i_{p,j_p}} (-z_1 + z_2 - u_{i_p} + v_{j_p-n}) ,
\]

\[
x_{i_p}^+ = \begin{cases} x_{i_p+n+m+1}(u_{i_p}) & \text{for } l_p = 1, \ldots, n, \\ x_{i_p+n+m+2}(v_{i_p-n}) & \text{for } l_p = n + 1, \ldots, n + m . \end{cases}
\]

Next, using (2.38), we rewrite (2.60) as

\[
\sum_{k=0}^{\min\{m,n\}} \sum_{i=(i_{1,j_1},\ldots,i_{k,j_k}) \in I_k^{n,m}} T_{i_{1,j_1}} \cdots T_{i_{k,j_k}} x_{i_{n-k}}^{13} x_{i_{m-k}}^{23} ,
\]

where the terms

\[
x_{i_{n-k}}^{13} = x_{i_{n-k}}(z_1 + u_{l_1}, \ldots, z_1 + u_{n-k})
\]

and

\[
x_{i_{m-k}}^{23} = x_{i_{m-k}}(z_2 + v_{l_{n-k+1}-n}, \ldots, z_2 + v_{n+m-2k-n})
\]

are applied on the tensor factors

\[
l_1, \ldots, l_{n-k}, n + m + 1 \quad \text{and} \quad l_{n-k+1}, \ldots, l_{n+m-2k}, n + m + 1
\]

of (2.50), respectively.

For every $T_{i_{p,j_p}}$, as given by (2.61), introduce the element

\[
U_{i_{p,j_p}} = S_{i_{p,j_p}} (z_1 - z_2 + u_{i_p} - v_{j_p-n}) - S_{i_{p,j_p}} (-z_1 + z_2 - u_{i_p} + v_{j_p-n}) .
\]

It is clear that $T_{i_{p,j_p}}$ and $U_{i_{p,j_p}}$ do not coincide due to different expansions. However, by (2.23) we can choose an integer $s \geq 0$ such that all products $(z_1 - z_2)^s U_{i_{1,j_1}} \cdots U_{i_{k,j_k}}$ modulo (2.59) are well-defined and such that we have

\[
(z_1 - z_2)^s T_{i_{1,j_1}} \cdots T_{i_{k,j_k}} = (z_1 - z_2)^s U_{i_{1,j_1}} \cdots U_{i_{k,j_k}} \mod (2.59).
\]

Let us turn to the second summand, $Y(z_2)(1 \otimes Y(z_1))$ in (2.36). By (2.38), its action on the expression $x_{i_{n-k}}^{13}(v)x_{i_{m-k}}^{23}(u)$ produces

\[
x_{i_{n-k}}^{13}(z_2 + v) x_{i_{m-k}}^{23}(z_1 + u) = x_{i_{n-k}}^{13}(z_2 + v_1, \ldots, z_2 + v_m) x_{i_{m-k}}^{23}(z_1 + u_1, \ldots, z_1 + u_n) .
\]

We now multiply (2.67) by a sufficiently large power $(z_1 - z_2)^r$, where $r \geq s$, so that in the given expression, when regarded modulo (2.59), we can employ commutation relation (2.15) to move each term $x(z_2 + v_q)$ to the right of all $x(z_1 + u_p)$. Thus we get

\[
\sum_{k=0}^{\min\{m,n\}} \sum_{i=(i_{1,j_1},\ldots,i_{k,j_k}) \in I_k^{n,m}} (z_1 - z_2)^r U_{i_{1,j_1}} \cdots U_{i_{k,j_k}} x_{i_{n-k}}^{13} x_{i_{m-k}}^{23} ,
\]

where $x_{i_{n-k}}^{13}$ and $x_{i_{m-k}}^{23}$, as given by (2.63) and (2.64), are applied on the tensor factors (2.65) of (2.50), respectively, and the whole expression is regarded modulo (2.59). Finally, let $y \in \mathbb{H}_N^\lambda$ be arbitrary. Apply (2.62) and (2.68) on $y$ and, furthermore, multiply the
implies that this is over its $\mathcal{S}$-locality follows.

2.5. Quantum vertex algebra $\mathcal{V}_H(c)$. Consider the topologically free subalgebra of $H^+_N$ generated by all $x_i^{(-r)}$ for $i = 1, \ldots, N$ and $r = 1, 2, \ldots$ and 1. By using (2.19) and (2.26) one easily checks that this algebra is closed under the actions of the vertex operator map (2.38) and the braiding map (2.26). Therefore, Theorem 2.10 implies that this is a quantum vertex subalgebra of $H^+_N$. We denote this quantum vertex algebra by $\mathcal{V}_H(c)$. Clearly, as a $\mathbb{C}[[\hbar]]$-module $\mathcal{V}_H(c)$ coincides with the quotient of the $h$-adically complete algebra of polynomials

$$\mathbb{C}[x_i^{(-r)}] : i = 1, \ldots, N, r \geq 1[[\hbar]]$$

over its $h$-adically complete ideal generated by the elements in (2.5).

We now follow the exposition in [12, Chap. 6] to recall the construction of the Heisenberg vertex algebra. Let $\mathfrak{h}$ be an abelian Lie algebra with generators $a_1, a_2, \ldots, a_N$ and the defining relations $a_1 + \ldots + a_N = 0$. It is equipped with the nondegenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ given by

$$\langle a_i, a_j \rangle = \delta_{ij} - \frac{1}{N} \quad \text{for all} \ i, j = 1, \ldots, N. \quad (2.69)$$

The corresponding affine Lie algebra $\hat{\mathfrak{h}}$ is defined on the complex vector space (1.1) via Lie brackets give by (1.2). We organize all $a(r) = a \otimes t^r$ in the power series

$$a(z) = \sum_{r \in \mathbb{Z}} a(r) z^{-r-1} \quad \text{for all} \ a \in \mathfrak{h}.$$

Also, we write

$$a^+(z) = \sum_{r \leq -1} a(r) z^{-r-1}.$$

Consider the subalgebras $\hat{\mathfrak{g}}(\pm) = \mathfrak{h} \otimes t^\pm \mathbb{C}[t^\pm] \subset \hat{\mathfrak{h}}$. For any $c \in \mathbb{C}$ let

$$\mathcal{V}_\mathfrak{h}(c) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t, t^{-1}])} \mathbb{C}_c,$$

be the induced $\hat{\mathfrak{g}}$-module, where $\mathfrak{h}$ and $\mathfrak{g}$ act trivially on $\mathbb{C}_c = \mathbb{C}$ and $\mathfrak{c}$ acts as the scalar $c$. By the Poincaré–Birkhoff–Witt theorem $\mathcal{V}_\mathfrak{h}(c)$ is isomorphic, as a vector space, to the underlying space of the universal enveloping algebra $U(\mathfrak{h}(\pm))$. We can regard $\mathfrak{h}$ as a subspace of $\mathcal{V}_\mathfrak{h}(c)$ via the map

$$\mathfrak{h} \ni a \mapsto a(-1)1 \in \mathcal{V}_\mathfrak{h}(c), \quad (2.70)$$

where $1 = 1 \in \mathbb{C} \subset \mathcal{V}_\mathfrak{h}(c)$. Using defining relations (1.2) one can prove induction the following identity describing the action of $a(z)$ in $\mathcal{V}_\mathfrak{h}(c)[[z, z^{-1}]]$ with $a \in \mathfrak{h}$ on $\mathcal{V}_\mathfrak{h}(c)$:

$$a(z) b^+_1(z) \ldots b^+_n(z) = a^+(z) b^+_1(z) \ldots b^+_n(z) + \sum_{b_1, \ldots, b_n} \frac{c \langle a, b_k \rangle}{(z - z_k)^2} b^+_1(z) \ldots b^+_k(z) b^+_{k+1}(z) b^+_{k+2}(z) \ldots b^+_n(z) \quad (2.71)$$

for any $b_1, \ldots, b_n \in \mathfrak{h}$. The space $\mathcal{V}_\mathfrak{h}(c)$ can be equipped with the vertex algebra structure; cf. \[8, 16]:

**Theorem 2.13.** For any $c \in \mathbb{C}$ there exists a unique vertex algebra structure on $\mathcal{V}_\mathfrak{h}(c)$ such that

$$Y(a, z) = a(z) \in \text{End} \mathcal{V}_\mathfrak{h}(c)[[z, z^{-1}]] \quad \text{for all} \quad a \in \mathfrak{h}.$$
We now discuss the classical limit of the quantum vertex algebra \( V_\mathcal{H}(c) \). First, note that both \( V_\mathcal{H}(c) \) and \( V_\mathfrak{h}(c) \equiv U(\mathfrak{h}^{(1)}) \) can be naturally regarded as commutative associative algebras. Denote by \( \tilde{x}_{ii}^{(r)} \) the image of \( x_{ii}^{(r)} \) in the quotient \( V_\mathcal{H}(c)_0 := V_\mathcal{H}(c)/\hbar V_\mathcal{H}(c) \). The assignments

\[
\tilde{x}_{ii}^{(r)} \mapsto a_i(-r)
\]

with \( r = 1, 2, \ldots \) and \( i = 1, \ldots, N \) define an isomorphism

\[
V_\mathcal{H}(c)_0 \to V_\mathfrak{h}(c)
\]

of complex commutative associative algebras. In particular, for all \( i = 1, \ldots, N \) map (2.72) identifies \( \tilde{x}_{ii}^{(-1)} \) with \( a_i = a_i(-i)1 \in \mathfrak{h} \subset V_\mathfrak{h}(c) \); recall (2.70). As the classical limit of quantum vertex algebra is vertex algebra (see [4]), let \( \hat{Y} = \hat{Y}(z) \) be the classical limit of (2.38), i.e. the vertex operator map of the vertex algebra \( V_\mathcal{H}(c)_0 \). By Theorem 2.13 the vertex operator map of \( V_\mathfrak{h}(c) \) is uniquely determined by its action on the elements of \( \mathfrak{h} \). Therefore, in order to show that (2.72) is the vertex algebra isomorphism, it is sufficient to check that it maps

\[
\hat{Y}(\tilde{x}_{ii}^{(-1)}, z)\tilde{x}_{j_1j_1}^{(r_1)} \ldots \tilde{x}_{j_nj_n}^{(r_n)} \to Y(a_i, z)a_{j_1}(-r_1) \ldots a_{j_n}(-r_n)
\]

for all \( n \geq 0 \), \( i, j_1, \ldots, j_n = 1, \ldots, N \) and \( r_1, \ldots, r_n = 1, 2, \ldots \).

Note that by (2.4) the classical limit of the matrix entry \( e_{ii} \otimes e_{jj} \) of \( S(z) \) equals

\[
\frac{c}{z^2} (\frac{1}{n} - \delta_{ij}) = -\frac{c}{z^2} \langle a_i, a_j \rangle.
\]

Hence, by (2.38), the classical limit of the matrix entries \( e_{ii} \otimes e_{j_1j_1} \otimes \cdots \otimes e_{j_nj_n} \) in (2.10) is equal to

\[
\hat{Y}(\tilde{x}_{ii}^{(-1)}, z)\tilde{x}_{j_1j_1}^{(r_1)} \ldots \tilde{x}_{j_nj_n}^{(r_n)} = \tilde{x}_{ii}^{(z)} \tilde{x}_{j_1j_1}^{(z_1)} \ldots \tilde{x}_{j_nj_n}^{(z_n)}
\]

\[
+ \sum_{k=1}^{n} \frac{c \langle a_i, a_{j_k} \rangle}{(z - z_k)^2} \tilde{x}_{j_1j_1}^{(z_1)} \ldots \tilde{x}_{j_{k-1}j_{k-1}}^{(z_{k-1})} \tilde{x}_{j_{k+1}j_{k+1}}^{(z_{k+1})} \ldots \tilde{x}_{j_nj_n}^{(z_n)}.
\]

Finally, by comparing the coefficients in (2.71) and (2.74) we find that the second term in (2.73) coincides with the image of the first term under the map (2.72).

**Proposition 2.14.** For any \( c \in \mathbb{C} \) the classical limit \( V_\mathcal{H}(c)_0 \) of the quantum vertex algebra \( V_\mathcal{H}(c) \) is the Heisenberg vertex algebra \( V_\mathfrak{h}(c) \).

### 3. Quantum Heisenberg Algebra

In Subsection 3.1, we introduce certain algebras \( H(C) \) and \( H(C)_* \). Moreover, we construct examples of their modules, which we use in Subsection 3.2 to prove the Poincaré–Birkhoff–Witt theorem for these algebras. Finally, in Subsection 3.3, we establish an equivalence between \( V_\mathcal{H}(c) \)-modules and certain class of \( H(C) \)-modules.

#### 3.1. Algebras \( H(C) \) and \( H(C)_* \).

The series \( S(u, C) \), as defined by (2.3), can be written in the form

\[
S(u, C) = \sum_{i,j,k,l=1}^{N} e_{ij} \otimes e_{kl} s_{ijkl}(u, C) \quad \text{for some} \quad s_{ijkl}(u, C) \in \mathbb{C}[C, u^{-1}][[\hbar]].
\]

To simplify the notation we write \( s_{ij}(u, C) := s_{iijj}(u, C) \). Note that \( s_{ij}(u, C) = s_{ji}(u, C) \).

Define the algebra \( H(C) \) as the \( \hbar \)-adically complete associative algebra over the commutative ring \( \mathbb{C}[[\hbar]] \) generated by the central element \( C \) and the elements \( y_i^{(r)} \), where
i = 1, \ldots, N \text{ and } r \in \mathbb{Z}, \text{ subject to defining relations written in terms of the generator series}

\[ y^i(u) = \sum_{r \in \mathbb{Z}} y_i^{(r)} u^{-r-1} \quad \text{for } i = 1, \ldots, N. \]  

(3.1)

The relations are given by \( Cy = yC \) for all \( y \in \mathbb{H}(C) \) along with

\[
y^i(u) y^j(v) + s_{ij}(u - v, C) = y^j(v) y^i(u) + s_{ij}(v - u, C),
\]

(3.2)

\[
y^1(u) + \ldots + y^N(u) = 0,
\]

(3.3)

where \( i, j = 1, \ldots, N \). It shall be useful to arrange the generators into the diagonal matrix

\[
y(u) = \sum_{i=1}^{N} e_{ii} \otimes y^i(u) \in \text{End } \mathcal{C}^N \otimes \mathbb{H}(C)[[u^{\pm 1}]].
\]

(3.4)

Also, we shall use the notation

\[
\text{diag} \left( \sum_{i_1, \ldots, i_n=1}^{N} \sum_{j_1, \ldots, j_n=1}^{N} e_{i_1 j_1} \otimes \ldots \otimes e_{i_n j_n} \right) = \sum_{i_1, \ldots, i_n=1}^{N} e_{i_1 i_1} \otimes \ldots \otimes e_{i_n i_n}
\]

to extract the diagonal matrix entries. Write \( \overline{S}(u, C) = \text{diag } S(u, C) \). The defining relations (3.2) and (3.3) can be equivalently written in the matrix form as

\[
y_1(u) y_2(v) + \overline{S}(u - v, C) = y_2(v) y_1(u) + \overline{S}(v - u, C) \quad \text{and} \quad \text{tr } y(u) = 0.
\]

Let \( W \) be an \( \mathbb{H}(C) \)-module. Then, \( W \) is said to be of level \( c \) if the action of \( C \) on \( W \) is scalar multiplication by \( c \in \mathbb{C} \). Moreover, \( W \) is said to be restricted if it is a topologically free \( \mathbb{C}[[h]] \)-module and the action of (3.1) on \( W \) satisfies

\[
y^i(z) \in \text{Hom}(W, W((z))_h) \quad \text{for all } i = 1, \ldots, N.
\]

If \( W \) is restricted module of level \( c \), the matrix of generators \( y(u) \), as given by (3.4), can be regarded as element of \( \text{End } \mathcal{C}^N \otimes \text{Hom}(W, W((u))_h) \). Now denote by \( u \) the family of variables \((u_1, \ldots, u_n)\). Following (2.19) one can introduce the elements

\[
y_{[n]}(u) = y_{[n]}(u_1, \ldots, u_n) \in \text{End } \mathcal{C}^N \otimes \text{Hom}(W, W((u_1, \ldots, u_n))_h)
\]

(3.5)

for any \( n = 1, 2, \ldots \) by

\[
y_{[n]}(u) = \sum_{k=0}^{[n/2]} \sum_{i=\{(i_1, j_1), \ldots, (k, j_k)\}} \mathcal{S}_{i_1 j_1} \ldots \mathcal{S}_{i_k j_k} y_1(u_{i_1}) \ldots y_{n-2k}(u_{n-2k}),
\]

where \( \mathcal{S}_{ij} = \mathcal{S}_{ji}(u_i - u_j, c) \). As with (2.20), we introduce the elements

\[
y_{[n]}(z + u) \in \text{End } \mathcal{C}_N \otimes \text{Hom}(H^+_N, H^+_N((z))_h[[u_1, \ldots, u_n]])
\]

(3.6)

by

\[
y_{[n]}(z + u) = y_{[n]}(z + u_1, \ldots, z + u_n) := y_{[n]}(z_1, \ldots, z_n) \big|_{z_1 = z + u_1, \ldots, z_n = z + u_n}.
\]

Note that due to (2.15) and (2.16) the assignments

\[
C \mapsto c \quad \text{and} \quad y_i^{(-r)} \mapsto x_i^{(-r)}, \quad \text{where } i = 1, \ldots, N \quad \text{and} \quad r \in \mathbb{Z},
\]

(3.7)

define a structure of \( \mathbb{H}(C) \)-module of level \( c \) on \( \mathcal{V}_H(c) \). By (2.4) the series \( s_{ij}(u, c) \) belong to \( \mathbb{C}[u^{-1}][[h]] \) for all indices \( i \) and \( j \), so (2.10) implies that \( \mathcal{V}_H(c) \) is restricted. This construction can be generalized as follows. First, observe that by (2.4) we have \( \text{Res}_u S(u) = 0 \).
Let Corollary 3.2.

The action of the generators (3.8) on

\[ \mathcal{G}_{\mathbb{C}} \] on

is of the form

\[ \mathcal{G}_{\mathbb{C}} : j = 1, \ldots, N - 1, r = 1, 2, \ldots \]
The algebra $H(C)$ may be regarded as an $h$-adic deformation of the universal enveloping algebra $U(\hat{h})$ of the commutative affine Lie algebra $\hat{h}$; see, in particular, the Poincaré–Birkhoff–Witt theorem for $H(C)$ below. Motivated by the classical theory, one can introduce the corresponding deformed Heisenberg algebra $H(C)_*$ in parallel with the definition of $H(C)$. More specifically, $H(C)_*$ is the $h$-adically complete associative algebra over the commutative ring $\mathbb{C}[[h]]$ generated by the central element $C$ and the elements $y_i^{(r)}$, where $i = 1, \ldots, N$ and $r \in \mathbb{Z} \setminus \{0\}$, subject to defining relations written in terms of the generator series

$$y_i^{(r)}(u) = \sum_{r \in \mathbb{Z} \setminus \{0\}} y_i^{(r)} u^{-r-1} \quad \text{for } i = 1, \ldots, N.$$ 

Relations are given by $C y = y C$ for all $y \in H(C)_*$ along with

$$y_i^{(u)} y_j^{(v)} + s_{ij}(u-v, C) = y_j^{(v)} y_i^{(u)} + s_{ij}(v-u, C),$$

$$y_i^{(u)} + \ldots + y^N(u) = 0,$$

where $i, j = 1, \ldots, N$. The notion of restricted module of level $c$ for $H(C)_*$ can be introduced in parallel with the corresponding definition for $H(C)$. Furthermore, the suitable modifications of Proposition 3.1 and Corollary 3.2 hold for the Heisenberg algebra as well.

### 3.2. Poincaré–Birkhoff–Witt theorem

For $c \in \mathbb{C}$ denote by $H(c)$ the quotient of the algebra $H(C)$ over its $h$-adically closed ideal generated by $C - c$. It is clear from the defining relations (3.2) and (3.3) that the algebra $H(0)$ is isomorphic to the $h$-adically completed polynomial algebra in variables $\hat{y}_j^{(r)}$, where $j = 1, \ldots, N - 1$ and $r \in \mathbb{Z}$. Therefore, with $H(0)$ being a quotient of $H(C)$, the ordered monomials in elements of $\mathcal{G}_{H(C)}$, with respect to any linear ordering, form a linearly independent subset in $H(C)$.

Let us introduce a linear ordering on the set of generators $\mathcal{G}_{H(C)}$. Set

$$C \prec \hat{y}_j^{(r)} \quad \text{for all } j = 1, \ldots, N - 1 \text{ and } r \in \mathbb{Z}. \quad (3.9)$$

Next, define

$$\hat{y}_s^{(s)} \prec \hat{y}_j^{(r)} \quad \text{if } s < r \quad \text{or} \quad s = r \text{ and } i < j. \quad (3.10)$$

Let $\mathcal{M}_{H(C)}$ be the family of all increasing monomials in generators, i.e. the monomials of the form $m_n \cdots m_1$, where $n \geq 0$ and $m_1, \ldots, m_n \in \mathcal{G}_{H(C)}$, such that $m_n \prec \ldots \prec m_1$. We extend the ordering from $\mathcal{G}_{H(C)}$ to $\mathcal{M}_{H(C)}$ as follows. For any two distinct monomials $m = m_n \cdots m_1$ and $m' = m_n' \cdots m_1'$ in $\mathcal{M}_{H(C)}$ we write $m \prec m'$ if there exists an index $0 \leq u \leq n + n' + 1$ such that $m_1 = m_1', \ldots, m_{u-1} = m_{u-1}'$ and such that one of the following conditions holds:

$$n < u \leq n' \quad \text{or} \quad u \leq n, n' \quad \text{and} \quad m_u \prec m_u'.$$

One easily checks that this defines a linear ordering on $\mathcal{M}_{H(C)}$.

The following theorem provides a topological basis of $H(C)$ in the $h$-adic sense.

**Theorem 3.3.** The set $\mathcal{M}_{H(C)}$ forms a topological basis of $H(C)$.

**Proof.** Note that defining relations (3.2) for the algebra $H(C)$, along with (3.8), imply

$$\hat{y}_i^{(r)} \hat{y}_j^{(r)} = \hat{y}_j^{(r)} \hat{y}_i^{(r)} \quad \text{for all } i, j = 1, \ldots, N - 1, r \in \mathbb{Z}. \quad (3.11)$$
Consider the family $F$ of all increasing monomials in $C$ and $y_j^{(r)}$, $j = 1, \ldots, N - 1$, $r \in \mathbb{Z}$, where the ordering is defined analogously to (3.9) and (3.10), i.e. we have

$$C \prec y_j^{(r)} \quad \text{and} \quad y_i^{(s)} \prec y_j^{(r)} \quad \text{if} \quad s < r \quad \text{or} \quad s = r \quad \text{and} \quad i < j.$$

Here we assume that $F$ also contains the empty monomial, that is the unit 1. By using defining relations (3.2) and (3.3) one easily checks that the $\mathbb{C}[[h]]$-span of $F$ forms an $h$-adically dense $\mathbb{C}[[h]]$-submodule of $H(C)$. However, by employing (3.8) and (3.11), we can express every monomial in $F$ as a $\mathbb{C}$-linear combination of some elements of $\mathcal{M}_{H(C)}$. Thus, we conclude that the $\mathbb{C}[[h]]$-span of $\mathcal{M}_{H(C)}$ is $h$-adically dense $\mathbb{C}[[h]]$-submodule of $H(C)$ as well. Hence it remains to prove that $\mathcal{M}_{H(C)}$ is linearly independent over $\mathbb{C}[[h]]$.

Assume that in $H(C)$ we have a relation of the form

$$\sum_{i \in I} p_i(C)m^i = 0, \quad (3.12)$$

where $I$ is a finite nonempty set, $p_i(x) \in \mathbb{C}[[h]][x]$ are nonzero polynomials and $m^i \in \mathcal{M}_{H(C)}$ are distinct monomials of the form $m^i = m_i^+ m_i^-$, where $m_i^-$ (respectively $m_i^+$) are monomials in $y_j^{(r)}$ for $j = 1, \ldots, N - 1$ and $r > 0$ (respectively $r < 0$).

1. Suppose that all $p_i(x)$ belong to $\mathbb{C}[x]$. Let $i_0 \in I$ be such that for all $i \in I$ we have $m_{i_0}^- \prec m_i^-$ or $m_{i_0}^- = m_i^-$. Denote by $J \subset I$ the set of all indices $j$ such that $m_j^- = m_{i_0}^-$. Let $c \in \mathbb{C}\setminus\{0\}$ be such that $p_j(c) \neq 0$ for all $j \in J$. Due to Corollary 3.2, we can choose an element $v \in \mathcal{V}_H(c, 0)$ such that

$$m_j^- v \in \mathbb{C}\setminus\{0\} + h\mathcal{V}_H(c, 0) \quad \text{for all} \quad j \in J \quad \text{and} \quad m_i^- v \in h\mathcal{V}_H(c, 0) \quad \text{for all} \quad i \in I \setminus J. \quad (3.13)$$

If we act with

$$\sum_{i \in I} p_i(C)m^i = \sum_{j \in J} p_j(C)m_j^+ m_j^- + \sum_{i \in I \setminus J} p_i(C)m_i^+ m_i^-$$

on $v$, from (3.13) follows the relation

$$\sum_{j \in J} p_j(c)(m_j^- v)m_j^+ = 0 \mod h. \quad (3.14)$$

Note that all $p_j(c)(m_j^- v)$ belong to $\mathbb{C}\setminus\{0\}$ modulo $h$. Since all $m_j^+$ for $j \in J$ are mutually distinct ordered monomials in generators of $\mathcal{V}_H(c, 0)$, they are linearly independent, so that (3.14) produces a contradiction.

2. In general case, i.e. for $p_i(x) \in \mathbb{C}[[h]][x]$, choose a minimal integer $m \geq 0$ so that there exists a nonempty subset of indices $K \subset I$ such that

$$p_k(x) \in h^m\mathbb{C}[[h]][x]\setminus\{0\} + h^{m+1}\mathbb{C}[[h]][x] \quad \text{for all} \quad k \in K,$$

$$p_i(x) \in h^{m+1}\mathbb{C}[[h]][x] \quad \text{for all} \quad i \in I \setminus K.$$

We now suitably adapt the arguments from (1). Let $j_0 \in K$ be such that for all $k \in K$ we have $m_j^- \prec m_k^-$ or $m_j^- = m_k^-$. Denote by $J \subset K$ the set of all indices $j$ such that $m_j^- = m_{j_0}^-$. Choose $c \in \mathbb{C}\setminus\{0\}$ such that $h^{-m}p_j(c) \in \mathbb{C}\setminus\{0\} + h\mathbb{C}[[h]]$ for all $j \in J$. As before, due to Corollary 3.2, we can choose $v \in \mathcal{V}_H(c, 0)$ such that (3.13) holds. By applying (3.12) on $v$ and then multiplying the equality by $h^{-m}$ we get

$$\sum_{j \in J} (h^{-m}p_j(c))(m_j^- v)m_j^+ = 0 \mod h. \quad (3.15)$$
Since \((h^{-m}p_j(c))(m_j^-v)\) belongs to \(\mathbb{C}\{0\}\) modulo \(h\) for all \(j \in J\), (3.15) leads to contradiction as in (1).

So far we have proved that all monomials in \(\mathcal{M}_{\text{H}(C)}\) which do not contain elements \(\hat{y}_j(0)\) are linearly independent over \(\mathbb{C}[[h]]\). Assume that in (3.12) all \(m^t\) are mutually distinct elements of \(\mathcal{M}_{\text{H}(C)}\) of the form \(m^t = m_j^+m_i^0m_j^-\), where \(m_j^\pm\) are as before and \(m_i^0\) are ordered monomials in \(\hat{y}_j(0)\) for \(j = 1, \ldots, N - 1\). Such equality can be then written as

\[
\sum_{j \in J} q_j(C, \hat{y}_1(0), \ldots, \hat{y}_{N-1}(0))m_j^+m_j^- = 0
\]

for some \(J \subseteq I\) and nonzero polynomials \(q_j(x_1, \ldots, x_N) \in \mathbb{C}[[h]][x_1, \ldots, x_N]\), so that all monomials \(m_j^+m_j^-\) with \(j \in J\) are mutually distinct. Indeed, polynomials \(q_j\) are found by

\[
q_j(C, \hat{y}_1(0), \ldots, \hat{y}_{N-1}(0)) = \sum_{i \in I, m_i^+ = m_j^-} p_i(C)m_i^0.
\]

However, we can choose \(c \in \mathbb{C} \setminus \{0\}\) and \(\alpha \in \mathfrak{h}\) so that the action of \(q_j(C, \hat{y}_1(0), \ldots, \hat{y}_{N-1}(0))\) on \(\mathcal{Y}_{\text{H}}(c, \alpha)\) is nonzero for all \(j \in J\); recall Corollary 3.2. Thus, the linear independence can be again established by arguing as in (1) and (2).

Denote by \(\mathcal{M}_{\text{H}(C)_*}\), the family of all increasing monomials, with respect to the linear order \(\prec\) defined by (3.9) and (3.10), in generators \(C\) and \(\hat{y}_j^{(r)}\), where \(j = 1, \ldots, N - 1\) and \(r \in \mathbb{Z} \setminus \{0\}\), of \(\text{H}(C)_*\). The following corollary is clear.

**Corollary 3.4.** The set \(\mathcal{M}_{\text{H}(C)_*}\) forms a topological basis of \(\text{H}(C)_*\).

### 3.3. Equivalence of \(\text{H}(C)\)-modules and \(\mathcal{Y}_{\text{H}(c)}\)-modules.

The notion of module for quantum vertex algebra was introduced by Li in parallel with the notion of vertex algebra module; see [14, Def. 2.23]. By Lemma 3.6 below, it coincides with the notion of module given by the following definition.

**Definition 3.5.** Let \((V, Y, 1, S)\) be a quantum vertex algebra. A \(\text{V-module}\) is a pair \((W, Y_W)\), where \(W\) is a topologically free \(\mathbb{C}[[h]]\)-module and \(Y_W(z)\) a \(\mathbb{C}[[h]]\)-module map

\[
Y_W(z): V \otimes W \rightarrow W((z))_h
\]

\[
v \otimes w \mapsto Y_W(z)(v \otimes w) = Y_W(v, z)w = \sum_{r \in \mathbb{Z}} v_r w z^{-r-1}
\]

which satisfies \(Y_W(1, z)w = w\) for all \(w \in W\) and the \(S\)-Jacobi identity

\[
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_W(z_1)(1 \otimes Y_W(z_2))(u \otimes v \otimes w) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y_W(z_2)(1 \otimes Y_W(z_1)) \left( S(-z_0)(v \otimes u) \otimes w \right) = z_2^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_W(Y(u, z_0)v, z_2)w \quad \text{for all} \quad u, v \in V \quad \text{and} \quad w \in W.
\]

Let \(W_1\) be a topologically free \(\mathbb{C}[[h]]\)-submodule of \(W\). A pair \((W_1, Y_{W_1})\) is said to be a \(\text{V-submodule}\) of \(W\) if \(v, w \in W_1\) for all \(v \in V\), \(w \in W_1\) and \(r \in \mathbb{Z}\), where by \(Y_{W_1}\) we denote the restriction and corestriction of \(Y_W\),

\[
Y_{W_1}(z) = Y_W(z)|_{V \otimes W_1}: V \otimes W_1 \rightarrow W_1((z))[[h]].
\]
The next lemma is well-known and follows by an argument similar to [14, Rem. 2.16]; cf. also [11, Lemma 1.3]. It is a quantum vertex algebra analogue of [12, Thm. 4.4.5].

**Lemma 3.6.** Let \((V, Y, 1, S)\) be a quantum vertex algebra. Suppose \(W\) is a topologically free \(\mathbb{C}[[h]]\)-module such that there exists a \(\mathbb{C}[[h]]\)-module map

\[
Y_W(z) : V \otimes W \rightarrow W((z))_h
\]

\[
v \otimes w \mapsto Y_W(z)(v \otimes w) = Y_W(v, z)w = \sum_{r \in \mathbb{Z}} v_r w z^{-r-1}
\]

which satisfies \(Y_W(1, z)w = w\) for all \(w \in W\) and the weak associativity: for any \(u, v \in V, w \in W\) and \(n \in \mathbb{Z}_{>0}\) there exists \(s \in \mathbb{Z}_{>0}\) such that

\[
(z_0 + z_2)^s Y_W(u, z_0) Y_W(v, z_2) w - (z_0 + z_2)^s Y_W(Y(u, z_0)v, z_2) w \in h^n W[[z_0^{\pm 1}, z_2^{\pm 1}]].
\]

Then \((W, Y_W)\) is a \(V\)-module. In particular, it possesses the \(S\)-locality property: for any \(u, v \in V\) and \(n \in \mathbb{Z}_{>0}\) there exists \(s \in \mathbb{Z}_{>0}\) such that

\[
(z_1 - z_2)^s Y_W(z_1)(1 \otimes Y_W(z_2))(S(z_1 - z_2)(u \otimes v) \otimes w)
- (z_1 - z_2)^s Y_W(z_2)(1 \otimes Y_W(z_1))(v \otimes u \otimes w) \in h^n W[[z_1^{\pm 1}, z_2^{\pm 1}]] \quad \text{for all}\ w \in W.
\]

The goal of this section is to establish an equivalence between \(H(c)\)-modules and \(\mathcal{V}_H(c)\)-modules. The next theorem is our first result in this direction.

**Theorem 3.7.** Let \(W\) be an (irreducible) restricted \(H(C)\)-module of level \(c \in \mathbb{C}\). There exists a unique structure of (irreducible) \(\mathcal{V}_H(c)\)-module on \(W\) given by

\[
Y_W(\text{diag } x_{i_1}^+(u_1, \ldots, u_n), z) = y_{i_1}(z + u_1, \ldots, z + u_n) \quad \text{with} \quad n \geq 0.
\]

**Proof.** Let \(W\) be a restricted \(H(C)\)-module of level \(c\). First of all, we have to check that (3.19), along with \(Y_W(1, z) = 1_W\), defines a \(\mathbb{C}[[h]]\)-module map on \(\mathcal{V}_H(c)\). It is sufficient to show that the ideals of relations

\[
\left[ x_{ij}^{(-r)}, x_{ij}^{(-s)} \right] = 0 \quad \text{and} \quad x_{i_1}^{(-r)} + \cdots + x_{N N}^{(-r)} = 0, \quad \text{where} \quad i, j = 1, \ldots, N, r, s \geq 1
\]

are mapped to itself; recall Subsection 2.5. This can verified by using the defining relations (3.2) and (3.3) for \(H(C)\) and arguing as in the corresponding part of the proof of Theorem 2.10; see Subsection 2.4. Furthermore, the map \(Y_W(\cdot, z)\) is uniquely determined by (3.19) as all monomials \(x_{i_1}^{(-r_1)} \cdots x_{i_m}^{(-r_m)}\) along with 1 span an \(h\)-adically dense \(\mathbb{C}[[h]]\)-submodule of \(\mathcal{V}_H(c)\). Since \(W\) is restricted, the image of \(\mathcal{V}_H(c)\) under the map \(v \mapsto Y_W(v, z)\) belongs to \(\text{Hom}(W, W((z))_h)\) due to (3.6). Hence, in order to establish a structure of \(\mathcal{V}_H(c)\)-module on \(W\) via (3.19), it is sufficient to verify the weak associativity (3.17); recall Lemma 3.6. However, given the fact that the expressions \(x_{\{i\}}(u)\) and \(y_{\{i\}}(u)\) are defined by analogous formulae, (2.19) and (3.5) respectively, the weak associativity can be proved by suitably adapting the arguments from the proof of Lemma 2.11. Thus we conclude that (3.19) defines a structure of \(\mathcal{V}_H(c)\)-module on \(W\).

Finally, suppose that \(W\) is an irreducible \(H(C)\)-module of level \(c\). Let \(W_1 \subseteq W\) be a \(\mathcal{V}_H(c)\)-submodule of \(W\). By employing (3.19) we find

\[
y_i(z)w = Y_W(x_i^{(-1)}, z)w \in W_1[[z^{\pm 1}]] \quad \text{for all} \ w \in W_1 \text{ and } i = 1, \ldots, N,
\]

which implies \(H(C)W_1 \subseteq W_1\). Thus, \(W_1\) is equal to 0 or \(W\), so that \(W\) is irreducible as a \(\mathcal{V}_H(c)\)-module as well. \(\square\)
Theorem 3.8. Let $W$ be an (irreducible) $\mathcal{V}_H(c)$-module for some $c \in \mathbb{C}$. There exists a unique structure of (irreducible) restricted $H(C)$-module of level $c$ on $W$ such that

$$y_i'(z) = Y_W(x_i(1), z) \quad \text{for all} \quad i = 1, \ldots, N.$$  \hfill (3.20)

Proof. Let $W$ be a $\mathcal{V}_H(c)$-module. The map $Y_W(\cdot, z)$ satisfies the $\mathcal{S}$-Jacobi identity (3.16),

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_W(z_1)(1 \otimes Y_W(z_2))$$

$$- z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_W(z_2)(1 \otimes Y_W(z_1)) (\mathcal{S}(-z_0)P \otimes 1)$$

$$= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_W(z_2)(Y(z_0) \otimes 1)$$  \hfill (3.21)

on $\mathcal{V}_H(c) \otimes \mathcal{V}_H(c) \otimes W$. First, by applying the identity on

$$\text{diag} \left( x_{i3}^+(0) \otimes x_{24}^+(0) \otimes w \right) = \sum_{i,j=1}^N e_{ii} \otimes e_{jj} \otimes x_{ii}^{-1} \otimes x_{jj}^{-1} \otimes w,$$

where $w \in W$ is arbitrary, and using the definitions of the corresponding maps, recall (2.26), (2.38) and (3.20), we get

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_W(\text{diag} x_{i3}^+(0), z_1) Y_W(\text{diag} x_{23}^+(0), z_2) w$$

$$- z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) (Y_W(\text{diag} x_{23}^+(0), z_2) Y_W(\text{diag} x_{i3}^+(0), z_1) w + \tilde{T}_{12}(-z_0, c) \otimes w)$$

$$= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) (Y_W(\text{diag}(x_{i3}(z_0)x_{23}^+(0)), z_2) w - \tilde{S}_{12}(z_0, c) \otimes w),$$  \hfill (3.22)

where $\tilde{T}(z, c) = \tilde{S}(z, c) - \tilde{S}(-z, c)$; recall (2.22). Next, by the property of the delta function we have

$$\left( z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) \right) \tilde{S}_{12}(z_0, c) \otimes w = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) \tilde{S}_{12}(z_0, c) \otimes w.$$

Finally, by adding equalities (3.22) and (3.23) and employing (2.22) we obtain

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) (Y_W(\text{diag} x_{i3}^+(0), z_1) Y_W(\text{diag} x_{23}^+(0), z_2) + \tilde{S}_{12}(z_0, c) ) w$$

$$- z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) (Y_W(\text{diag} x_{23}^+(0), z_2) Y_W(\text{diag} x_{i3}^+(0), z_1) + \tilde{S}_{12}(-z_0, c) ) w$$

$$= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_W(\text{diag}(x_{i3}(z_0)x_{23}^+(0)), z_2) w.$$

Taking the residue with respect to the variable $z_0$ produces the identity

$$Y_W(\text{diag} x_{i3}^+(0), z_1) Y_W(\text{diag} x_{23}^+(0), z_2) + \tilde{S}_{12}(z_1 - z_2, c)$$

$$- Y_W(\text{diag} x_{23}^+(0), z_2) Y_W(\text{diag} x_{i3}^+(0), z_1) w + \tilde{S}_{12}(z_2 - z_1, c) = 0 \hfill (3.24)$$
which holds when applied on $w$. With $w$ being an arbitrary element of $W$, we conclude by (3.24) that the map (3.20) satisfies defining relation (3.2). As for the other defining relation (3.3), it is also satisfied by the map (3.20) due to (2.16). In addition, Definition 3.5 implies that the image of $Y_W(\cdot, z)$ belongs to $\text{Hom}(W, W((z))_h)$. Therefore, formula (3.20) defines a structure of restricted $H(C)$-module of level $c$ on $W$, as required.

As for the second assertion of the theorem, suppose that $W$ is an irreducible $V_H(c)$-module. Let $W_1 \subseteq W$ be a $H(C)$-submodule of $W$. By using (3.19) we get

$$Y_W(\text{diag} x^r_i(u), z)w = y_{[n]}(z + u)w \in W_1[[z^{\pm 1}, u_1, \ldots, u_n]] \quad (3.25)$$

for all $n \geq 1$ and $w \in W_1$, where $u = (u_1, \ldots, u_n)$; cf. Remark 3.9 below. This implies that $W_1$ is a $V_H(c)$-submodule, so we conclude that $W_1$ equals 0 or $W_1$. Hence $W$ is an irreducible $H(C)$-module, as required.

Remark 3.9. We should say that in the end of the proof of Theorem 3.8 we use the fact that the $V_H(c)$-module map $Y_W$ coming from (3.20) takes the form (3.25). This is easily verified by using the so-called iterate formula,

$$Y_W(Y(u, z_0)v, z_2)w = \text{Res}_{z_1} \left( z_0^{-1} \delta \left( \frac{z_1 - z_0}{z_0} \right) Y_W(z_1)(1 \otimes Y_W(z_2))(u \otimes v \otimes w) \right. 
- \left. z_0^{-1} \delta \left( \frac{z_0 - z_1}{z_0} \right) Y_W(z_2)(1 \otimes Y_W(z_1))(S(-z_0)(v \otimes u) \otimes w) \right),$$

which is obtained by taking the residue $\text{Res}_{z_1}$ of the $S$-Jacobi identity (3.16). More specifically, due to the iterate formula, it is sufficient to check that the $h$-adic completion of the $\mathbb{C}[[h]]$-module $S \subseteq V_H(c)$, which contains the elements $x_{-1}^{(-1)}, \ldots, x_{-N}^{(-1)}$ and $1$ and satisfies $u, v \in S$ for all $u, v \in S$ and $r \in \mathbb{Z}$, coincides with $V_H(c)$. However, one can prove by induction over $n$ that all monomials $x_{-r_1}^{(-r_1)} \cdots x_{-r_n}^{(-r_n)}$ with $i_1, \ldots, i_n = 1, \ldots, N$ and $r \geq 1$ belong to $S$, which implies such conclusion. Indeed, for $n = 1$ this follows by extracting the coefficients of $Y(x_{i_1}^{(-1)}, z)1$. Suppose that the statement holds for all monomials of length less than or equal to $n$. By (2.19) the coefficients of matrix entries of

$$\text{diag} x^r_{[n+1]}(z, v_1, \ldots, v_n) \quad \text{and}$$
$$\text{diag} x^{13}(z)x^{23}_{[n]}(v_1, \ldots, v_n) = \text{diag} Y(x^{13}(0), z)x^{23}_{[n]}(v_1, \ldots, v_n)$$

coincide modulo monomials of length less than or equal to $n$, so the induction assumption implies that the statement holds for the monomials of length $n + 1$ as well.

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