Relative equilibria in the unrestricted problem of a sphere and symmetric rigid body

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ABSTRACT
We consider the unrestricted problem of two mutually attracting rigid bodies, an uniform sphere (or a point mass) and an axially symmetric body. We present a global, geometric approach for finding all relative equilibria (stationary solutions) in this model, which was already studied by Kinoshita (1970). We extend and generalize his results, showing that the equilibria solutions may be found by solving at most two non-linear, algebraic equations, assuming that the potential function of the symmetric rigid body is known explicitly. We demonstrate that there are three classes of the relative equilibria, which we call cylindrical, inclined co-planar, and conic precessions, respectively. Moreover, we also show that in the case of conic precession, although the relative orbit is circular, the point-mass and the mass center of the body move in different parallel planes. This solution has been yet not known in the literature.

1 INTRODUCTION
In this paper, we consider the dynamical problem of two rigid bodies interacting mutually according to the Newtonian laws of gravitation. It is well known that if both these bodies are approximated by point-masses, their dynamics are analytically solvable in terms of the integrable Kepler problem. However, even if only one of the bodies is an arbitrary rigid body then the complexity of the model changes dramatically. First of all, we have three more degrees of freedom required to describe the rotational motion of the rigid body. Moreover, because the gravitational potential of a rigid body, in general case, depends on an infinite number of parameters, the two rigid-body dynamics depend also on this infinite set of parameters. Finally, orbital and rotational degrees of freedom are coupled non-linearly, and the resulting system is not analytically solvable.

Because of the mathematical complexity involved, many approximate models were studied in the past. For example, under certain assumptions, one can consider the co-planar two rigid-body problem, see e.g. Beletskij (1975), Barkin (1985), Beletskij & Ponomareva (1990); Goździewski (2002). Another way to simplify the problem relies on truncating series expansions of the gravitational potential of the rigid body (the potential of mutual interactions), see e.g. Markov (1967a,b, 1985, 1988); Shcherbina (1989). In the so-called satellite (or restricted) models, it is assumed that the rotational motion does not influence the relative, Keplerian orbit of the mass centres of the bodies. In such models, the relative orbit is given parametrically, and only the rotational dynamics need to be investigated; for details, see e.g. Markov (1967a,b, 1985, 1988); Maciejewski (1994); Maciejewski & Goździewski (1995).

In this work, we focus on particular version of the model involving a sphere (a point-mass), by considering that the rigid body is axially symmetric. Obviously, this leads to much simplified form of the gravitational potential. However, in general, it still depends on an infinite number of parameters. For the first time this problem was investigated by Kinoshita (1970), and hence it will be called the Kinoshita problem from hereafter. Our aim is to determine all possible classes of the relative equilibria (stationary solutions) in this problem, using a general formalism in Wang et al. (1991) and Maciejewski & Goździewski (1995). Keeping in mind a motivation of the work by Kinoshita, we try to generalize his analysis and to simplify conditions for the existence of these equilibria, avoiding repetition of “of” assuming only the necessary general form of the gravitational potential of the rigid body. In this sense, our paper is also related to the work by Scheeres (2006), who searched for the relative equilibria in the unrestricted two rigid-body problem, and formulated conditions of the equilibria in terms of solutions to two non-linear equations, parameterized by integrals of the total energy and angular momentum. Moreover, we stress that our analysis rely on the very basic vectorial form of the equations of motion, and we consider the full, non-restricted model with symmetric rigid body. The assumption of symmetry implies that the results derived for the general, full two-body model cannot be simply “translated” as a particular case, and, in fact, a special reduction of the system which takes into account the symmetry explicitly is necessary. We note that the relative equilibria of the model with axially symmetric body may be understood as particular periodic orbits in the full two rigid body model considered by Scheeres (2006). This reduction is in fact crucial for a generalization of the results of Kinoshita and to discover a class of equilibria that have been missed in his analysis.

Investigations of the unrestricted two rigid body problem, in a version considered in this work, and its particular stationary solutions, are interesting because they concern a special case of generally unsolved, classic problem of the dynamics. Moreover, the qualitative analysis of this model may help us to answer for important, even “practical” questions on the dynamics and coupling of the rotational and orbital motions in many astronomical systems. These are, for instance, binary asteroids, see Scheeres (2006) and references therein. A deep understanding of the qualitative dynam-
ics are important for the attitude determination and control of large artificial satellites orbiting planets and/or irregular natural moons. Recently, the diversity of extrasolar planets discovered in wide dynamical environments also raises questions on their rotational motion and related long term effects (e.g., Correia et al. 2008). In the context of mathematical complexity of the problem, the relative equilibria are the simplest class of solutions that may be found and investigated analytically, and are helpful to construct local, precise analytic theories of motion in their vicinity in the phase space (e.g., Goździewski & Maciejewski 1999).

A plan of this paper is the following. In Sect. 2 we formulate the mathematical model and the equations of motion in the most general, vectorial form are derived. Section 3 is devoted to define the Kinoshita problem. Next, we introduce the relative equilibria, and we perform a global analysis of their existence and bifurcations (Sect. 4). This part relies on particular, geometric reduction of the equations of motion, and is the primary key for our analysis. Finally, we compare the classic results by Kinoshita with the results obtained through the approach introduced in this work.

2 THE EQUATIONS OF MOTION

Let us consider the gravitational two rigid body problem. We assume that one of the bodies, \( \mathcal{B} \), is a point mass, or an uniform sphere, with mass \( m_1 \). The second one, \( \mathcal{B}_2 \), is a rigid body with mass \( m_2 \) (Fig. 1). The mechanical problem in the most general settings has nine degrees of freedom, so the dynamics of the bodies are described by 18 first-order differential equations. Moreover, the dynamical system possesses symmetries, i.e., it may be shown that the equations of motion are invariant with respect to a six-dimensional group. This fact is a direct consequence of the very basic laws of Newtonian mechanics. Namely, the equations of motion do not depend on particular choice of the inertial reference frame. Thus, we can choose the origin of the inertial frame in an arbitrary point, and this implies that the equations of motion are invariant with respect to the natural action of three dimensional group of translations. Moreover, also the orientation of the inertial frame can be chosen arbitrarily. Hence, the equations of motion are invariant with respect to the natural action of the three-dimensional group of rotations.

The existence of these symmetries makes it possible to reduce the dimension of the phase space of this system and to simplify its analysis. However, there is no unique procedure for such a reduction. Obviously, it should rely on such a transformation of the initial system of the equations of motion (phase-space variables) that the resulting equations form a lower-dimensional sub-system. In fact, the reduced equations of motion of the two rigid body problem were obtained by Wang et al. (1997) (actually, they considered the motion of a rigid body in the central gravitational field), and, with a simpler and more direct method and under general settings by Goździewski & Maciejewski (1999). These papers are good references to step-by-step development of the reduced equations of motion, which is skipped here to save space.

Before we write down these equations of motion, we need to fix the notation. Components of a vector \( \mathbf{x} \) (as a geometric object) in an inertial reference frame will be denoted by \( \mathbf{x} = [x_1, x_2, x_3]^T \). Components of the same vector in the rigid body fixed frame we will denote by the corresponding capital letter, i.e., \( \mathbf{X} = [X_1, X_2, X_3]^T \). Thus, if \( \mathbf{A} \) is the orientation (attitude) matrix of the body with respect to the inertial frame, then we have \( \mathbf{x} = \mathbf{A} \mathbf{X} \).

The scalar product of two vectors \( \mathbf{x} \) and \( \mathbf{y} \) is denoted by \( \mathbf{x} \cdot \mathbf{y} \). It can be calculated in an arbitrary orthonormal frame as follows:

\[
\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^{3} x_i y_i = \langle \mathbf{A} \mathbf{X}, \mathbf{AY} \rangle = \langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{i=1}^{3} X_i Y_i.
\]

We shall also write:

\[
x^2 := \mathbf{x} \cdot \mathbf{x} = (\mathbf{x}, \mathbf{x}) = X^2 := (\mathbf{X}, \mathbf{X}).
\]

In our exposition, we follow Maciejewski (1995), Goździewski & Maciejewski (1999). The reduced equations of motion describe the relative motion of the bodies with respect to a frame fixed in rigid body \( \mathcal{B} \). They have the following form of the so-called Newton–Euler equations:

\[
\begin{align*}
\frac{d}{dt} \mathbf{R} &= \mathbf{R} \times \Omega + \mathbf{P}, \\
\frac{d}{dt} \mathbf{P} &= \mathbf{P} \times \Omega - \frac{\partial U}{\partial \mathbf{R}}, \\
\frac{d}{dt} \mathbf{G} &= \mathbf{G} \times \Omega + \mathbf{R} \times \frac{\partial U}{\partial \mathbf{R}}
\end{align*}
\]

(1)

where \( \mathbf{R} \) is the radius vector directed from the mass centre of \( \mathcal{B} \) to \( \mathcal{B}_2 \), \( \mathbf{P} \) is the relative linear momentum, \( \mathbf{G} \) is the angular momentum of \( \mathcal{B}_2 \), and \( \Omega = \mathbf{I}^{-1} \mathbf{G} \) is its angular velocity; \( U \) is the gravitational potential of the body \( \mathcal{B}_2 \), and \( \mathbf{I} = \text{diag}(A, B, C) \) stands for its tensor of inertia. We assume that the rigid body fixed frame coincides with the principal axes frame. The geometry of the model is illustrated in Fig. 1.

As it was shown in Goździewski & Maciejewski (1999), equations (1) are Hamiltonian with respect to a non-canonical Poisson bracket. They admit two first integrals: the energy integral,

\[
H = \frac{1}{2} \langle \mathbf{P}, \mathbf{P} \rangle + \frac{1}{2} \langle \mathbf{G}, \Omega \rangle + U(\mathbf{R}),
\]

(2)

and the modulus of the total angular momentum \( \mathbf{L} \), which, with respect to the body frame, is given by:

\[
L^2 = \langle \mathbf{L}, \mathbf{L} \rangle, \quad \text{where} \quad \mathbf{L} := \mathbf{R} \times \mathbf{P} + \mathbf{G}.
\]

(3)
The total angular momentum with respect to the inertial frame, i.e.,
\[ I = AL, \]
is a constant vector. The time evolution of the attitude matrix \( A \) is
given by the following kinematic equations:
\[ \frac{d}{dt} A = A \Omega, \]
where, as in Go´zdziewski & Maciejewski (1999), for a vector \( X = [X_1, X_2, X_3]^T \), we denote:
\[ \hat{X} := \begin{bmatrix} 0 & -X_3 & X_2 \\ X_3 & 0 & -X_1 \\ -X_2 & X_1 & 0 \end{bmatrix}. \]

If the rigid body \( \mathcal{B} \) does not possess any additional symmetry,
then the relative equilibria of the considered two rigid body problem
are just the equilibria solutions (critical points) of the reduced
equations (i). They were investigated by many authors, see Wang
et al. (1991); Scheeres (2006) and references therein. The most in-
teresting are just the equilibria solutions (critical points) of the reduced

\[ \Omega = [\Omega_1, \Omega_2, \Omega_3]^T. \]

Let us recall a formal, geometric definition of a relative equilib-
rium, which has the following general form:
\[ [R, P, G] = [A_3 R^*, A_3 P^*, A_3 G^*], \]
where \( A_3 \) is an arbitrary matrix of rotation about the third axis of
the body, i.e., a matrix of the following form:
\[ A_3 = \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]
where \( a^2 + b^2 = 1 \).

Let us notice that \( A_3 I = IA_3 \) (i.e., \( I \) commutes with \( A_3 \)), hence we have,
\[ \frac{d}{dt} R = A_3 \frac{d}{dt} R^* = (A_3 R^*) \times (I^{-1} A_3 G^*) + A_3 P^*. \]

This equation may be rewritten as
\[ (A_3 R^*) \times (I^{-1} A_3 G^*) = (A_3 R^*) \times (A_3 I^{-1} G^*) = A_3 \left[R^* \times I^{-1} G^*\right], \]
where the last equality follows from the sixth identity given in
Proposition 3.1. Hence,
\[ A_3 \frac{d}{dt} R^* = \left[A_3 \left[R^* \times \Omega^* + P^*\right]\right], \]
where \( \Omega^* = I^{-1} G^* \), and, finally,
\[ \frac{d}{dt} R^* = R^* \times \Omega^* + P^*, \]
and this shows that the first equation in (1) is invariant with respect
to variables change (3). In a similar manner, we show that the re-
mainding two equations have the same property.

### 4 RELATIVE EQUILIBRIA

#### 4.1 General theory of the relative equilibria

Let us recall a formal, geometric definition of a relative equilib-
rium of a system with symmetry (see, for instance Marsden &
Ratiu 1994), which is crucial to perform our analysis: a relative
equilibrium is a solution of the system represented by the phase-
space curve which is an orbit of a point under the action of a one-
dimensional subgroup of the symmetry group of the system.

In our case, the symmetry group is SO(2, \( \mathbb{R} \)) which is identi-
cified with matrices of the form (\( \mathbb{F} \)). It is a one-dimensional group.

Let us put \( a = \cos \varphi \) and \( b = \sin \varphi \) in equation (9). With this parameter-
ization, we denote elements of SO(2, \( \mathbb{R} \)) by \( A_3 (\varphi) \). Hence, an orbit of a point \([R_0, P_0, G_0]^T\) under the action of SO(2, \( \mathbb{R} \)) is the following set:
\[ \left\{A_3 (\varphi) R_0, A_3 (\varphi) P_0, A_3 (\varphi) G_0 \right\} \in \mathbb{R}^3 \mid \varphi \in [0, 2\pi) \}, \]
where \( R_0, P_0, G_0 \) are constant vectors. Thus, the relative equilib-
rium of the symmetric Kinoshita problem is a solution to equations (1), which has the following general form:
\[ R(t) = A_3 (N_t) R_0, \]
\[ P(t) = A_3 (N_t) P_0, \]
\[ G(t) = A_3 (N_t) G_0, \]
\[ N(t) = \left[\begin{array}{ccc} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{array}\right], \]
\[ \varphi \in [0, 2\pi). \]
where $N$ is the real number. Notice, that $\Omega(t) = \mathbf{I}^{-1} \mathbf{G}(t) = \mathbf{A}_3(N(t)) \mathbf{Q}_0$, where $\mathbf{Q}_0 = \mathbf{I}^{-1} \mathbf{G}_0$. In other words, the relative equilibrium is a special periodic solution to the equations of motion [1]. From the above formulae we can deduce more useful conclusions on the relative equilibrium:

(i) vectors $\mathbf{R}(t)$, $\mathbf{P}(t)$, $\mathbf{G}(t)$ and $\Omega(t)$ have constant lengths,
(ii) their “third” components $R_3(t)$, $P_3(t)$, $G_3(t)$ and $\Omega_3(t)$ are constant,
(iii) all angles between vectors $\mathbf{R}(t)$, $\mathbf{P}(t)$, $\mathbf{G}(t)$ and $\Omega(t)$ are constant.

Let us notice that the relative equilibrium [10] does not determine an unique set of $[\mathbf{R}_0, \mathbf{P}_0, \mathbf{G}_0]$. In fact, instead of $[\mathbf{R}_0, \mathbf{P}_0, \mathbf{G}_0]$, we can take $[\mathbf{A}(\varphi) \mathbf{R}_0, \mathbf{A}(\varphi) \mathbf{P}_0, \mathbf{A}(\varphi) \mathbf{G}_0]$, with arbitrary constant $\varphi$. One would like to perform a reduction of [1] in such a way that in the reduced system, the relative equilibrium corresponds to an “usual” equilibrium point (i.e., which is understood as the critical point of the equations of motion). It is possible, however, it may lead to singularities. For example, let us introduce three sets of cylindrical coordinates:

$$
\mathbf{R} = (p \cos \varphi, p \sin \varphi, R_3)^T,
\mathbf{P} = (p \cos \psi, p \sin \psi, P_3)^T,
\mathbf{G} = (\rho \cos \psi, \rho \sin \psi, \Omega_3)^T.
$$

Using these coordinates, we may write:

$$
\frac{\partial \mathbf{U}}{\partial \mathbf{R}} = (U_\rho \cos \varphi, U_\rho \cos \varphi, U_3)^T,
$$

where,

$$
U_\rho = \frac{\partial \mathbf{U}}{\partial \rho}, \quad \text{and} \quad U_3 = \frac{\partial \mathbf{U}}{\partial R_3}.
$$

With respect to these new variables, system [1] reads as follows:

$$
\begin{align*}
\frac{d}{dt} \rho &= p \cos (\psi - \varphi) - \rho \omega \sin (\psi - \varphi), \\
\frac{d}{dt} \varphi &= P_3 + p \rho \omega \sin (\psi - \varphi), \\
\frac{d}{dt} \psi &= \rho \omega \sin (\psi - \varphi) - U_\rho \cos (\psi - \varphi), \\
\frac{d}{dt} \psi &= -P_3 \omega \sin (\psi - \varphi) - U_\rho \cos (\psi - \varphi), \\
\frac{d}{dt} \omega &= \frac{\partial}{\partial \varphi} \left[ \frac{1}{(C \Omega_3 - A \Omega) \rho} \right], \\
\frac{d}{dt} \varphi &= \rho \omega \sin (\psi - \varphi) + \rho \omega \cos (\psi - \varphi) - \rho \Omega_3, \\
\frac{d}{dt} \psi &= \left( C \rho \Omega_3 - A \Omega \right)^{-1} \left[ \rho \omega \cos (\psi - \varphi) - (A - C \Omega) \right].
\end{align*}
$$

Hence, finally we have one equation less than in [1] because $\Omega_3$ is constant. Note that the right hand sides of the equations depend on the difference between the angles but not on the angles themselves. Thus we can introduce the following new set of variables:

$$
\alpha = \psi - \varphi, \quad \beta = \psi - \psi.
$$

and then we obtain:

$$
\begin{align*}
\frac{d}{dt} \rho &= p \cos (\beta) - \rho \omega \sin (\alpha), \\
\frac{d}{dt} \varphi &= P_3 + p \rho \omega \sin (\alpha), \\
\frac{d}{dt} \psi &= -P_3 \omega \sin (\alpha + \beta) - U_\rho \cos (\beta), \\
\frac{d}{dt} \omega &= \frac{\partial}{\partial \varphi} \left[ \frac{1}{(C \Omega_3 - A \Omega) \rho} \right], \\
\frac{d}{dt} \beta &= \rho \omega \sin (\alpha + \beta) + U_\rho \cos (\beta) - \rho \Omega_3, \\
\frac{d}{dt} \psi &= \left( C \rho \Omega_3 - A \Omega \right)^{-1} \left[ \rho \omega \cos (\alpha + \beta) - (A - C \Omega) \right].
\end{align*}
$$

The first seven equations in the above set represent a system of equations which flows from the reduction [1] by the symmetry. Now, a relative equilibrium of [1] corresponds to an equilibrium point of the above system. However, this system has singularities when one of variables $\rho$, $\varphi$ or $\psi$ vanishes. Because of this fact it would be very difficult to perform global, qualitative analysis which is our primary goal.

### 4.2 The geometric reduction of the system

A possibility that the reduction may introduce singularities to the equations of motion inspired us and, in fact, forced us to choose a particular approach which we describe below in detail. Clearly, instead of using any local variables, it is much more convenient to derive the necessary and sufficient conditions for the existence of relative equilibria in terms of the original global and non-singular variables $(\mathbf{R}, \mathbf{P}, \mathbf{G})$.

Let us assume that a relative equilibrium is given by [10]. Our aim is to find equations determining $(\mathbf{R}_0, \mathbf{P}_0, \mathbf{G}_0)$, and $N$. At first, we may notice that:

$$
\frac{d}{dt} \mathbf{A}_3(N(t)) = \mathbf{A}_3(N(t)) \mathbf{N},
$$

where $\mathbf{N} = [0, 0, N]^T$. Thus we may easily derive that:

$$
\frac{d}{dt} \mathbf{R}(t) = \frac{d}{dt} \mathbf{A}_3(N(t)) \mathbf{R}_0 = \mathbf{A}_3(N(t)) \mathbf{N} \mathbf{R}_0 = \mathbf{A}_3(N(t)) \mathbf{N} \times \mathbf{R}_0.
$$

On the other hand, we have also:

$$
\frac{d}{dt} \mathbf{R}(t) = \mathbf{R}(t) \times \Omega(t) + \mathbf{P}(t) = \mathbf{A}_3(N(t)) \mathbf{R}_0 \times \Omega(t) + \mathbf{P}_0,
$$

as the system has $\text{SO}(2, \mathbb{R})$ symmetry. So, we have:

$$
\mathbf{N} \times \mathbf{R}_0 = \mathbf{R}_0 \times \Omega(t) + \mathbf{P}_0.
$$

We proceed in a similar way with other components of the phase variables defining the relative equilibrium, i.e., with $\mathbf{P}(t)$ and $\mathbf{G}(t)$. Finally, we obtain the following equations:

$$
\begin{align*}
\mathbf{N} \times \mathbf{R}_0 &= \mathbf{R}_0 \times \Omega(t) + \mathbf{P}_0, \\
\mathbf{N} \times \mathbf{P}_0 &= \mathbf{P}_0 \times \Omega(t) - \frac{\partial \mathbf{U}}{\partial \mathbf{R}_0}, \\
\mathbf{N} \times \mathbf{G}_0 &= \mathbf{G}_0 \times \Omega(t) + \mathbf{R}_0 \times \frac{\partial \mathbf{U}}{\partial \mathbf{R}_0}.
\end{align*}
$$

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This system can be rewritten in a more compact form as follows:
\[
\begin{align*}
\vec{\Omega} \times \mathbf{R}_0 &= \mathbf{P}_0, \\
\vec{\Omega} \times \mathbf{P}_0 &= -\frac{\partial U}{\partial \mathbf{R}_0}, \\
\vec{\Omega} \times \mathbf{G}_0 &= \mathbf{R}_0 \times \frac{\partial U}{\partial \mathbf{R}_0},
\end{align*}
\] (11)

where, \(\vec{\Omega} = \Omega_0 + \mathbf{N}\), is the instantaneous angular velocity of the rigid body. System (11) defines nine equations which must be satisfied by ten unknowns. We found that there are several ways to overcome this inconvenience. In our further procedure, we choose the following approach. We look for the relative equilibria which exist for a given value of the angular momentum of the system, chosen as the model parameter. Because
\[
L^2 = (\mathbf{R} \times \mathbf{P} + \mathbf{G})^2 = (\mathbf{R}_0 \times \mathbf{P}_0 + \mathbf{G}_0)^2,
\]
after the elimination of \(\mathbf{P}_0\), we obtain:
\[
L^2 = \left(\vec{\Omega} \mathbf{R}_0^2 - \mathbf{R}_0 \vec{\Omega} \mathbf{R}_0 + \mathbf{G}_0\right)^2.
\]

Hence, our goal is to find all solutions to the following nonlinear equations:
\[
\begin{align*}
\vec{\Omega} \times \mathbf{R}_0 &= \mathbf{P}_0, \\
\vec{\Omega} \mathbf{R}_0 - \mathbf{R}_0 \vec{\Omega} &= -\frac{\partial U}{\partial \mathbf{R}_0}, \\
\vec{\Omega} \times \mathbf{G}_0 &= \mathbf{R}_0 \times \frac{\partial U}{\partial \mathbf{R}_0}, \\
\left(\vec{\Omega} \mathbf{R}_0^2 - \mathbf{R}_0 \vec{\Omega} \mathbf{R}_0 + \mathbf{G}_0\right)^2 &= L^2.
\end{align*}
\] (12)

In the above equations, \(L\), and the third component of \(\mathbf{G}_0\) are now fixed parameters of the reduced model.

### 4.3 Geometric description of the equilibrium solutions

The equations of motion [1] describe the relative motion of the bodies in the rigid body fixed frame. Apparently, this makes it difficult to obtain the geometrical and physical interpretation. In fact, we can deduce all necessary information quite easily.

First of all, let us recall that the total angular momentum \(I\) is a constant vector. So, we can choose an inertial frame in such a way that its third axis is directed along that vector. If \(\mathbf{R}(t) = A_3(Nt)\mathbf{R}_0\) describes the relative orbit in a relative equilibrium, and \(\mathbf{r}(t)\) is the corresponding equilibrium vector, then we have:
\[
I \cdot \mathbf{r}(t) = \langle \mathbf{r}(t) \rangle = \langle \mathbf{AL}, A_3(Nt)\mathbf{R}_0 \rangle = \langle \mathbf{L}, A_3(Nt)\mathbf{R}_0 \rangle.
\]

Moreover, the following general relation holds true:
\[
\mathbf{L} = A_3(Nt)\left(\mathbf{R}_0 \times \mathbf{P}_0 + \mathbf{G}_0\right),
\]
so, finally, we have:
\[
I \cdot \mathbf{r}(t) = \langle \mathbf{R}_0 \times \mathbf{P}_0 + \mathbf{G}_0, \mathbf{R}_0 \rangle = \langle \mathbf{G}_0, \mathbf{R}_0 \rangle.
\] (13)

We can conclude that in a relative equilibrium, the projection of the relative radius vector onto the total angular momentum vector is constant. An important conclusion follows immediately; if this projection is non-zero, then orbits of the point and the rigid body lie in different planes.

Let \(\mathbf{z}(t)\) denotes the unit vector along the symmetry axis of the body. Then
\[
I \cdot \mathbf{z}(t) = \langle \mathbf{L}, \mathbf{Z} \rangle = \langle A_3(Nt)\mathbf{R}_0 \times \mathbf{P}_0 + \mathbf{G}_0, \mathbf{Z} \rangle = \langle \mathbf{R}_0 \times \mathbf{P}_0 + \mathbf{G}_0, \mathbf{Z} \rangle,
\]
because \(\mathbf{Z} = [0, 0, 1]^T\), and so, \(A_3(Nt)\mathbf{Z} = \mathbf{Z}\). Thus, in a relative equilibrium, the projection of the symmetry axis onto the total angular momentum is also constant.

Let us introduce the orbital reference frame with axes parallel to \(\mathbf{r}(t), \dot{\mathbf{r}}(t)\) and \(\mathbf{c}(t) := \mathbf{r}(t) \times \dot{\mathbf{r}}(t)\). Because the relative orbit in a relative equilibrium is circular, this frame is orthogonal. We show that the projection of the symmetry axis \(\mathbf{z}(t)\) onto the axis of this frame is also constant. In other words, we are going to prove that in the relative equilibrium the symmetry axis of the rigid body has a fixed orientation with respect to the orbital reference frame.

To shorten notation, we shall write \(A_3\) instead of \(A_3(Nt)\). Let \(\mathbf{B} = AA_3\). Then we have:
\[
\frac{d}{dt} \mathbf{B} = \dot{A}A_3 + AA_3\dot{\mathbf{A}} = \dot{\mathbf{A}}\mathbf{A}_3 + AA_3\dot{\mathbf{N}},
\]

But in a relative equilibrium we have \(\mathbf{A} = \mathbf{A}_3\Omega_0\). By the fifth property given in Proposition [5],
\[
\dot{\mathbf{A}} = \mathbf{A}_3\dot{\Omega}_0\mathbf{A}_3^T,
\]

hence we obtain that:
\[
\frac{d}{dt} \mathbf{B} = \dot{\mathbf{B}},
\]

where \(\mathbf{B} = \Omega_0 + \mathbf{N}\) is a constant vector. Now, it is easy to show that:
\[
\begin{align*}
\mathbf{r}(t) &= \mathbf{A}\mathbf{r}(t) = \mathbf{BR}_0, \\
\dot{\mathbf{r}}(t) &= \mathbf{B}(\dot{\mathbf{A}}\times \mathbf{R}_0), \\
(\mathbf{r}(t) \times \dot{\mathbf{r}}(t)) &= \mathbf{B}(\dot{\mathbf{A}}\times \mathbf{R}_0). \\
\mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) &= (\mathbf{B}\mathbf{r}_0) \cdot (\mathbf{B}\dot{\mathbf{r}}_0) = (\mathbf{R}_0, \mathbf{Z}), \\
\mathbf{c}(t) &= (\mathbf{B}(\mathbf{r}(t) \times \dot{\mathbf{r}}(t)), \\
\mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) &= (\mathbf{B}\mathbf{r}_0 \times (\mathbf{B}\dot{\mathbf{r}}_0, \mathbf{Z})).
\end{align*}
\]

The above formulae prove our initial claim.

## 5 CONDITIONS FOR THE RELATIVE EQUILIBRIA

In this section, we show that all relative equilibria can be determined by solutions of certain non-linear system comprising of only two non-linear scalar equations. We shall also prove that there exist three classes of these relative equilibria in the dynamical model which we consider.

In order to simplify notation while working with coordinates, we shall skip index “0” when denoting coordinates of vectors \(\mathbf{R}_0\), \(\mathbf{P}_0\), \(\mathbf{G}_0\) from hereafter. Thus, we set
\[
\begin{align*}
\mathbf{R} &= [R_1, R_2, R_3]^T, \\
\mathbf{P} &= [P_1, P_2, P_3]^T, \\
\Omega &= [\Omega_1, \Omega_2, \Omega_3]^T, \\
\vec{\Omega} &= [\Omega_1, \Omega_2, \Omega_3, \mathbf{N}]^T, \\
\mathbf{G} &= [A\Omega_1, A\Omega_2, C\Omega_3]^T, \\
\frac{\partial U}{\partial \mathbf{R}_0} &= [U_1, U_2, U_3]^T.
\end{align*}
\]

The fact that the body is symmetric implies that:
\[
R_1U_2 - R_2U_1 = 0,
\] (16)
with accord to (17). Taking the vector product of both sides of equation (12) with \( \mathbf{R}_0 \), we obtain:

\[
\langle \tilde{\Omega}, \mathbf{R}_0 \rangle \tilde{\Omega} \times \mathbf{R}_0 = \mathbf{R}_0 \times \frac{\partial U}{\partial \mathbf{R}_0}. \tag{17}
\]

Hence, taking into account (16), we have:

\[
\langle \tilde{\Omega}, \mathbf{R}_0 \rangle / (Z, \tilde{\Omega} \times \mathbf{R}_0) = 0. \tag{18a}
\]

Thus, we have to consider two different cases. If \( \langle \tilde{\Omega}, \mathbf{R}_0 \rangle = 0 \) then from (17) it follows immediately that:

\[
\mathbf{R}_0 \times \frac{\partial U}{\partial \mathbf{R}_0} = 0,
\]

or, \( (Z, \tilde{\Omega} \times \mathbf{R}_0) = 0, \) and this gives

\[
\Omega_1 R_2 - \Omega_2 R_1 = 0.
\]

In the first case, the gradient of \( U \) at point \( \mathbf{R}_0 \) is parallel to the radius vector. According to Scheeres (2006), such a point is called \textit{locally central}. We shall say that a relative equilibrium is locally central if point \( \mathbf{R}_0 \) is locally central.

In our analysis, we assume that the gradient of the potential does not vanish at a considered point \( \mathbf{R}_0 \). Thus, a relative equilibrium is \textit{locally central} if and only if \( \langle \tilde{\Omega}, \mathbf{R}_0 \rangle = 0 \). In fact, if \( \mathbf{R}_0 \) is locally central, then from (17) it follows that either \( \langle \tilde{\Omega}, \mathbf{R}_0 \rangle = 0 \) or \( \tilde{\Omega} \times \mathbf{R}_0 = 0. \) If the second possibility occurs, then, by (12a), \( P_0 = 0, \) and, as equations (11) imply, the gradient of \( U \) at \( \mathbf{R}_0 \) vanishes. But, according to our assumptions, it is impossible.

At the final stage of our analysis, it is convenient to use cylindrical coordinates \( (\rho, \varphi, R_3) \) instead of \( (R_1, R_2, R_3) \). The potential \( U \) expressed with respect to the cylindrical coordinates depends only on \( \rho \) and \( R_3 \); still, as a function of these two arguments it will be denoted by the same symbol \( U \). It will be clear from the context which coordinates are in use.

### 5.1 Locally central relative equilibria

If \( \mathbf{R}_0 \) is locally central, then, as we have shown, \( \langle \tilde{\Omega}, \mathbf{R}_0 \rangle = 0, \) and of course:

\[
\mathbf{R}_0 \times \frac{\partial U}{\partial \mathbf{R}_0} = 0.
\]

Thanks to this fact, the basic system of the equations of motion (12) may be simplified considerably, and it reads as follows:

\[
P_0 = \tilde{\Omega} \times \mathbf{R}_0, \tag{18a}
\]

\[
\mathbf{R}_0 \tilde{\Omega}^2 = \frac{\partial U}{\partial \mathbf{R}_0}, \tag{18b}
\]

\[
\tilde{\Omega} \times G_0 = 0, \tag{18c}
\]

\[
\left( \tilde{\Omega} R_0^2 + G_0 \right)^2 = L^2. \tag{18d}
\]

Equation (18c) is equivalent to the following two scalar equations:

\[
\begin{cases}
A \tilde{\Omega}_3 - C \Omega_3 = 0, \\
A \tilde{\Omega}_2 - C \Omega_2 = 0.
\end{cases}
\]

Therefore, either \( \Omega_1 = \Omega_2 = 0, \) or \( A \tilde{\Omega}_3 - C \Omega_3 = 0. \) We investigate both cases separately.

#### 5.1.1 Case \( \Omega_1 = \Omega_2 = 0 \) (cylindrical precession)

If \( \Omega_1 = \Omega_2 = 0 \) then condition \( \langle \tilde{\Omega}, \mathbf{R}_0 \rangle = 0 \) reduces to \( \tilde{\Omega}_3 R_3 = 0. \) But \( \tilde{\Omega}_3 \neq 0, \) otherwise \( \tilde{\Omega}_3 = 0, \) and, as (18b) shows, the gradient of \( U \) vanishes and then \( R_1 = 0. \) Moreover, \( R_1 \) and \( R_2 \) do not vanish simultaneously, otherwise \( \mathbf{R}_0 = 0. \) Hence, we can safely use cylindrical coordinates \( (\rho, z) \equiv \varphi \) because \( \rho > 0. \) We introduce two functions of \( \rho \) given by:

\[
f(\rho) := U_3(\rho, 0), \quad \text{and} \quad g(\rho) := U_\rho(\rho, 0).
\]

Now, equations (18b) and (18d) lead to the following system:

\[
f(\rho) = 0, \quad \rho \tilde{\Omega}_3^2 = g(\rho), \quad \left[ \tilde{\Omega}^2 + C \tilde{\Omega}_3 \right]^2 = L^2. \tag{19}
\]

Through appropriate simplification, we find that the above system is equivalent to the following ones:

\[
f(\rho) = 0, \quad (L + \rho \tilde{\Omega}_3^2)^2 = \rho^3 g(\rho), \quad \tilde{\Omega}_3 = -\frac{L + \rho \tilde{\Omega}_3^2}{\rho^2}. \tag{20}
\]

If such a solution exists then we define:

\[
R_1 = \rho \cos \varphi, \quad R_2 = \rho \sin \varphi, \quad \quad \alpha^2 = \rho^3 U_\rho, \quad \beta = \frac{\alpha}{\rho^2},
\]

where \( \varphi \in [0, 2\pi] \) can be chosen arbitrarily. Then the relative equilibrium is determined through:

\[
R_0 = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \quad P_0 = \begin{bmatrix} -R_3 \\ 0 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Omega_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad N = \beta - \Omega_3.
\]

From the above formulae it immediately follows that point \( P \) and the mass centre of the rigid body \( \mathcal{B} \) move in one plane. In fact, by (13), we have:

\[
I \cdot \mathbf{r}(t) = (G_0, \mathbf{R}_0) = 0.
\]

Moreover, by (15), we have also:

\[
\mathbf{r}(t) \cdot \mathbf{z}(t) = (\mathbf{R}_0, \mathbf{Z}) = 0, \quad \mathbf{r}(t) \cdot \mathbf{z}(t) = (\mathbf{P}_0, \mathbf{Z}) = 0.
\]

Hence, the axis of symmetry of the rigid body is perpendicular to the plane of the relative orbit, see Fig. 2. We called this kind of the relative equilibrium as the \textit{cylindrical precession} from hereafter. In the paper of Kinoshita, it is named, after Duboshin (1959), as the \textit{float solution}. Regarding existence conditions of this solution, let us notice that system (21) is overdetermined, we have two equations for one variable \( \rho \). However, if the body is symmetric with respect to the equatorial plane, then \( U \) is an even function of \( R_3 \). In this instance, we have:

\[
f(\rho) = \frac{\partial U}{\partial \rho}(\rho, 0) \equiv 0,
\]

identically. Thus, in this case, the cylindrical precession exists if and only if the second equation of system (21) has a solution. Moreover, if the potential is such that there exists \( \rho > 0 \), for which

\footnote{Note that due to the symmetry, the third angular coordinate \( \phi \) is irrelevant here.}
g(ρ) > 0, then we always may find parameters of the problem such that equations (21) are fulfilled.

If the body is not symmetric with respect to the equatorial plane, the first equation of system (21) is not satisfied identically. Thus, a question emerges: does there exist an axially symmetric body which is not symmetric with respect to the equatorial plane, and for which equation:

\[ U_3(ρ, 0) = 0, \]

has a solution ρ > 0? To answer this question, let us consider an axisymmetric rigid body composed of three uniform spheres whose centers of masses are placed at the same line. The masses of these spheres are chosen in a way to guarantee that the potential is not symmetric with respect to the equatorial plane. The resulting function \( f(ρ) \) is depicted in Fig. 2. Since its graph crosses the ρ-axis for ρ > 0, there exist such points which satisfy the first condition of system (21), indeed. Then the parameters of the system can always be chosen to fulfill the last condition of (21). Thus, the cylindrical precessions may exist in the system with the rigid body, which is not symmetric with respect to its equatorial plane.

### 5.1.2 Case AΩ3 = CΩ3 (inclined co-planar precession)

Condition \( AΩ_3 = CΩ_3 \) implies immediately that:

\[ N = \frac{C - A}{A}Ω_3. \]

Moreover, now \( (Ω, R_0) = 0 \), is equivalent to \( (G_0, R_0) = 0 \), because \( AΩ = G_0 \). We have even more, because now equation (18b) can be rewritten in the following form:

\[ P_0 = \frac{1}{A}G_0 \times R_0. \]

and equation (18b) reads as:

\[ L^2 = (Ω R_0^2 + G_0)^2 = \frac{G^2_0}{A^2}(R_0^2 + A)^2. \]

Therefore, equation (18b) can be rewritten as follows:

\[ \frac{L^2}{(R_0^2 + A)^2} R_0 = \frac{\partial U}{\partial R_0}. \]

A solution of this equation gives \( R_0 \). However, this form is not convenient for further analysis because solutions to the above equation, whenever they exist, are not isolated. In terms of the cylindrical coordinates, we may rewrite (24) as follows:

\[
\begin{align*}
\frac{L^2}{(R_0^2 + A)^2} & = U_ρ, \\
R_3 \frac{L^2}{(R_3^2 + A)^2} & = U_3,
\end{align*}
\]

A solution to the above system gives us \( R_0 \). Because, due to the symmetry, we may choose freely the polar angle, then we can also assume, without any loss of generality that \( R_2 = 0 \). Vector \( G_0 \) is perpendicular to \( R_0 \), and its modulus is fixed by (23). In the generic case, \( R_1R_3 \neq 0 \), and we can put:

\[
G_0 = a_1 A[0, 1, 0]^T + a_2 \frac{A}{R_0}[R_3, 0, -R_1]^T, \]

where \( a_1 \) and \( a_2 \) are arbitrary real numbers satisfying the following relation:

\[
a_1^2 + a_2^2 = \frac{L^2}{(R_0^2 + A)^2}. \]

For each choice of \( a_1 \) and \( a_2 \), we have \( G_0 \), and then the corresponding value of \( P_0 \) is given by (22). Thus, a solution to equations (25) gives us one-parameter family of the relative equilibria.

From the above formulae it immediately follows that point \( P \), and the mass centre of the body \( B \) move in one plane. In fact, we have:

\[ I \cdot r(t) = (G_0, R_0) = 0. \]

Moreover, as in the first case of the cylindrical precession, we can determine the orientation of the symmetry axis using (15). We have:

\[
\begin{align*}
r(t) \cdot z(t) & = (R_0, Z) = R_3, \\
r(t) \cdot z(t) & = (P_0, Z) = P_3 = -a_1 R_1, \\
c(t) \cdot z(t) & = (R_0 \times (Ω \times R_0), Z) = -a_2 R_0 R_1.
\end{align*}
\]
Thus, in general, the symmetry axis of the rigid body is inclined with respect to any axis of the orbital reference frame. We shall call this class of relative equilibria as the inclined, co-planar regular precession, and its geometry is illustrated in Fig. 4.

5.2 Non-locally central case (conic precession)

As it has been already established, the non-locally central equilibria are characterized by \( P_3 = R_2 \Omega_1 - R_1 \Omega_2 = 0 \). It appears that in such an instance, instead of using system (12), it is better to analyse equations in which vector \( P \) is not eliminated. Hence, we consider the following system of equations:

\[
\begin{align*}
\tilde{\Omega} \times R_0 &= P_0, \quad (26a) \\
\tilde{\Phi} \times P_0 &= -\frac{\partial U}{\partial R_0}, \quad (26b) \\
\tilde{\Omega} \times G_0 &= R_0 \times \frac{\partial U}{\partial R_0}, \quad (26c) \\
(R_0 \times P_0 + G_0) &= L^2. \quad (26d)
\end{align*}
\]

Our aim is to find all solutions of this system which are not locally central. Thus, we look for solutions for which none of variables \( \rho \), \( R_3 \), \( \Omega_3 \) vanishes. Hence, we can express \( U_1 \) and \( U_2 \) in terms of \( U_\rho \), i.e.,

\[
U_i = \frac{R_i}{\rho}U_\rho, \quad \text{for} \quad i = 1, 2.
\]

Since \( P_3 \) vanishes, from the first two components of equations (26b) it follows that

\[
P_1 = -R_1 \frac{U_\rho}{\Omega_3 \rho}, \quad \text{and} \quad P_2 = R_1 \frac{U_\rho}{\Omega_3 \rho}. \quad (27)
\]

Knowing \( P_1 \) and \( P_2 \), from the first two components of equation (26a), we find:

\[
\Omega_i = R_i \frac{\rho \tilde{\Omega}^2 - U_\rho}{\rho R_3 \Omega_3}, \quad \text{for} \quad i = 1, 2. \quad (28)
\]

Substituting (27), (28) into the third component of equation (26b), we obtain:

\[
(pU_\rho + R_3 U_3) \tilde{\Omega}^2 = U_\rho^2, \quad (29)
\]

Now, let us consider equation (26c). It is easy to see that the third component of this equation is fulfilled identically. Using (28), the first two components can be rewritten in the following form:

\[
R_i \left( A \tilde{\Omega}_3 - C \Omega_3 \right) U_0 U_3 = R_i \left( pU_\rho + R_3 U_3 \right) \left( pU_\rho - R_3 U_3 \right) \tilde{\Omega}_3, \quad \text{for} \quad i = 1, 2.
\]

As \( R_1 \) and \( R_2 \) do not vanish simultaneously, the above system is equivalent to just one equation:

\[
\left( A \tilde{\Omega}_3 - C \Omega_3 \right) U_0 U_3 = \left( pU_\rho + R_3 U_3 \right) \left( pU_\rho - R_3 U_3 \right) \tilde{\Omega}_3. \quad (30)
\]

Next, substituting expressions (27) and (28), equation (26d) is rewritten as:

\[
L^2 = \frac{\left( A + R_3 \right) U_3 + pR_3 U_\rho}{\rho pU_\rho + R_3 U_3} \left( pU_\rho + R_3 U_3 \right)^2 + \left( \frac{pU_\rho}{\Omega_3} + G_3 \right)^2, \quad (31)
\]

Finally, we use (30) to eliminate \( \tilde{\Omega}_3 \) from equations (29) and (31). Thus, eventually, we obtain the final subsystem that makes it possible to determine the phase-space variables of the relative equilibrium, in quite a compact form:

\[
L^2 U_3^2 K = \frac{(U_3^2 + U_\rho^2) \left[ A U_3 + R_3 K \right]^2}{\rho pU_\rho + R_3 U_3}, \quad (32)
\]

where we introduced a new variable:

\[
K \equiv pU_\rho + R_3 U_3.
\]

A solution to the above conditions gives us \( R_0 \). The remaining equilibrium variables can be determined by equations (27), (28), and (29), respectively.

For this solution we can find, after tedious simplifications, that

\[
\mathbf{I} \cdot \mathbf{r}(t) = (G_0, R_0) = \frac{U_\rho \left( A U_3 + R_3 K \right)}{U_3 K} \left( R_3 U_0 - pU_3 \right).
\]

According with our assumptions, \( \rho \neq 0 \), and hence \( U_\rho \neq 0 \). Thus, the right hand side of the above equation vanishes if either \( R_3 U_0 - pU_3 = 0 \), or \(AU_3 + R_3 K = 0 \). In the first case, \( R_0 \) is locally central point which we excluded from our considerations. In the second instance, from equation (32), we obtain that \( L = 0 \), so it is also excluded from our considerations. We conclude that non-locally central relative equilibrium is non-Lagrangian ones.

In order to determine the orientation of the symmetry axis in the orbital frame which is specific for this type of relative equilibria, we can easily find that:

\[
\mathbf{r}(t) \cdot \mathbf{z}(t) = (R_0, Z) = R_3,
\]

\[
\mathbf{r}(t) \cdot \mathbf{z}(t) = 0,
\]

\[
\mathbf{c}(t) \cdot \mathbf{z}(t) = (R_0 \times (\tilde{\Omega} \times R_0), Z) = \frac{pU_\rho}{\Omega_3}.
\]

Thus, in the orbital reference frame, the axis of symmetry always lies in the plane formed by the relative position vector and the normal to the orbital plane. We call this kind of the regular precession as the conic precession from hereafter. Notice, that in the considered case, vector \( c(t) \) is not perpendicular to the orbital planes. Moreover, we have:

\[
\mathbf{I} \cdot \mathbf{z}(t) = (R_0 \times (\tilde{\Omega} \times R_0) + G_0, Z) = \frac{pU_\rho}{\Omega_3} + C \Omega_3.
\]

Thus, the axis of symmetry is inclined to the orbital plane. The geometry of this solution is presented in Fig. 5. Unlike the locally central cases, even for bodies with simple potentials, it is difficult to decide whether equations (32) have a solution. In order to justify that the relative equilibria belonging to the conic precession
class may really exist, we apply the following reasoning. If \( \rho \), as well as \( \Omega_3 \), tends to zero, then the conic precession tends to a particular case of the inclined planar regular precession described in Section 5.1.2. In fact, if \( \rho = \Omega_3 = 0 \), then \( U_\rho = 0 \), so the second equation in (32) is satisfied identically, and the first one coincides exactly with the equation describing corresponding case of the inclined co-planar solution. Thus, we can take this particular solution as the zero-th order approximation (with respect to \( \Omega_3 \)) of the conic precession solution, and we apply the perturbative approach to solve the system (32).

Thus, the zero-th order solution is given through:

\[
\rho = 0, \quad R_3 = R_3^0,
\]

where \( R_3^0 \) is a solution of the following equation

\[
U_3|_{\rho = 0} = \frac{L^2}{(A + R_3^0)} R_3.
\]

To shorten the notation, we define:

\[
U_3|_{\rho = 0, R_3 = R_3^0} = F, \quad U_{pp}|_{\rho = 0, R_3 = R_3^0} = V, \quad U_{33}|_{\rho = 0, R_3 = R_3^0} = W.
\]

Taking into account that

\[
U_\rho|_{\rho = 0} = U_{\rho 3}|_{\rho = 0} = 0,
\]

we can find that the Jacobian at the zeroth order solution has the form of:

\[
J = \begin{bmatrix} 0 & J_{12} \\ J_{21} & 0 \end{bmatrix},
\]

where

\[
J_{12} = -\frac{L^2 R_3^0}{(A + R_3^0)^2} \left[ (3 R_3^0 - A) L^2 + W (A + R_3^0)^2 \right], \quad (33a)
\]

\[
J_{21} = -\frac{L^2 R_3^0}{(A + R_3^0)^2} \left[ V (A + R_3^0)^2 - L^2 A \right], \quad (33b)
\]

The determinant of this Jacobian, given through:

\[
\det J = -J_{12} J_{21},
\]

does not vanish identically. Hence, by the implicit function theorem, system (32) has a solution \( \rho = \rho(\Omega_3), R_3 = R_3(\Omega_3) \), such that \( \rho(0) = 0 \), and \( R_3(0) = R_3^0 \). Moreover, such solution is unique.

We underline an important fact that, even if the body is symmetric with respect to the equatorial plane, the described conic precession does exist. In other words, splitting of the orbital planes can be induced not only by an asymmetric mass distribution, it can be also induced by a proper, particular rotation of the rigid body. Remarkably, the first type of solutions has been known since Abe & Barkin (1979), who constructed them for a class of non-symmetric potentials and further analysed their consequences in the motion model of the Earth-Moon system (Barkin 1980).

## 6 A COMPARISON WITH THE WORK OF KINOSHITA

Our analysis presented in the previous sections lead to somewhat different conclusions than those ones in Kinoshita (1970), and in this section we give an explanation and overview of these discrepancies. Kinoshita found three classes of stationary solutions:

(i) the "float" case, when the axis of symmetry of the rigid body is always perpendicular to the orbital plane,

(ii) the "spoke" case, when the axis of symmetry lies along the relative position vector,

(iii) the "arrow" case, when the axis of symmetry lies in the plane formed by tangential and normal to the orbital plane vectors.

Now we may show that all these solutions are, in our terminology, locally central and Lagrangian, i.e., the point mass \( \mathcal{P} \), and the mass centre of the rigid body \( \mathcal{B} \) move in one plane.

Since for the description of the system there has been chosen the reference frame fixed at mass centre of the point body \( \mathcal{P} \), the potential of the body, besides the distance from the mass centre, depends also on the orientation of the rigid body. Following Kinoshita, we denote a parameter, which describes the orientation, as \( v \). In this designation, \( v \) is an angle under which the point body \( \mathcal{P} \) is "seen" from the mass centre of the rigid one \( \mathcal{B} \). Note an important fact that the "float" and "arrow" types of motion correspond to a case when \( U_v \) vanishes. Let us determine the physical meaning of that condition in terms of our parameterization. For that reason, we will use an expansion of the rigid body potential through series in terms of the Legendre polynomials, the same as in Kinoshita (1970). This representation of the potential of \( \mathcal{B} \) is the following:

\[
U = -\frac{1}{r} \sum_{k=0}^{\infty} J_k \left( \frac{a}{r} \right)^k P_k(v), \quad (34)
\]

where \( v \) is the cosine of the angle between the symmetry axis of the body and the relative radius vector, and \( P_k \) is the Legendre polynomial. For the "float" and the "arrow" types of motion, \( U_v \) vanishes. Hence,

\[
U_v = \frac{1}{r} \sum_{k=0}^{\infty} J_k \left( \frac{a}{r} \right)^k P_k(v) = 0,
\]

for a certain value of \( v \).

In terms of variables used in our paper, the expansion of (34) reads as follows:

\[
U = -\frac{1}{R_0} \sum_{k=0}^{\infty} J_k \left( \frac{a}{R_0} \right)^k P_k \left( \frac{R_3}{R_0} \right).
\]

Let us recall that a point is locally central if and only if

\[
\rho U_3 - R_3 U_\rho = 0.
\]
Thus, we have:

$$pU_3 - R_3U_0 = \frac{\rho}{R_0} \sum_{k=0}^{\infty} P_k \left( \frac{a}{R_0} \right)^k = \frac{\rho}{R_0} U_0 = 0. \quad (35)$$

If $U_0 = 0$ then it means that the considered solution is locally central. Condition $U_0 = 0$ includes all possible locally central cases except of that ones when $p$ vanishes. That is why the “arrow” and the “float” type relative equilibria found by Kinoshita (1970), under assumption that $U_0 = 0$, are locally central. Moreover, it is easy to see, the “spoke” solution is also locally central. In fact, for this solution we have $p = U_0 = 0$. Thus, all solutions found by Kinoshita are locally central.

The “float” equilibrium coincides exactly with our first solution as in both of these cases the axis of symmetry is always perpendicular to the orbital plane. The “arrow” and the “spoke” motions are just particular cases of our second solution. Indeed, the second solution, when $\Omega_0$ vanishes, becomes the “spoke” motion, i.e., the axis of symmetry lies along the relative position vector; if we have $R_3 = 0$, then it coincides with the “arrow” motion. Quite surprisingly, in the paper of Kinoshita, we did not find any other cases corresponding to the inclined planar solution. It seems that investigating the “arrow” case, Kinoshita assumed for simplicity, that the body is symmetric with respect to the equatorial plane. Moreover, then he used generally wrong implication saying that $U_0 = 0$ means $\nu = 0$. Let us notice that in our representation $\nu = 0$ is equivalent to $R_3 = 0$.

To summarize this Section, Kinoshita (1970) did not find the non-Lagrangian solutions because he implicitly assumed that one can always choose an inertial reference frame in such a way that the relative orbit lies in its $(x, y)$-plane. In fact, this is equivalent to a priori assumption that all stationary solutions must be Lagrangian ones.

7 CONCLUSIONS

In this paper, we performed global, geometric analysis of the stationary motions in the unrestricted problem of a point mass and an axially symmetric rigid body (the Kinoshita problem). Our aim was to determine all possible relative equilibria in this problem. We have shown that three types of stationary motions can be distinguished. Two of them, the cylindrical precession and the inclined planar motion, are Lagrangian and are characterized by co-planar orbits of the mass centers of the bodies, whereas the third type of solutions, the conic precession, is non-Lagrangian.

The cylindrical precession the axis of symmetry of the rigid body is always perpendicular to the orbital plane. This type of stationary motion was also found by Kinoshita (1970), where it is called the “float” motion.

In the inclined planar precession, the axis of symmetry of the rigid body is inclined to the orbit. Two special cases of such kind of motion are found by Kinoshita (1970). In the first case the axis of symmetry lies along the relative position vector, and in the second one lies in the plane formed by the tangent and the normal to the orbit. In Kinoshita (1970) these solutions are named as the “spoke” and the “arrow” motions, respectively.

In the conic precession, the axis of symmetry lies in the plane formed by the relative position and the normal to the orbital plane vectors. Moreover, the point and the mass centre of the rigid body move in different parallel planes. This type of stationary motion is completely new in the problem analyzed in this work, although, as we noted above, such solutions have been constructed for specific non-symmetric potentials by Abulnaga & Barkin (1979). We found them thanks to a formulation of the problem through the minimal set of assumptions, basically implying only the form of the gravitational potential of the rigid body, besides the analysis of the problem in the very basic settings of the Newtonian dynamics.

We show that the determination of the stationary motions in the problem of Kinoshita may be reduced to solving at most two non-linear, algebraic equations. Our results can be applied to an arbitrary axially symmetric body, providing that an explicit form of its potential function is given.

In our forthcoming paper (Vereshchagin et al. 2009), which is a direct continuation of this work, we perform the stability analysis of the relative equilibria found here, and we apply our approach to study the dynamics of a few specific models (i.e., choosing explicit form of the gravitational potential of the rigid body).

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