Investigation rectifier circuits based on the mathematical models in the two-dimensional space

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Abstract. The mathematical model for the rectifier circuit using semiconductor diodes is setup in this paper. The properties of the rectifier circuit presented by the ordinary differential equation containing a control parameter K. When K is large enough, the studied equation gives a trajectory approximating to a trajectory of the rectifier circuit above. The theorem about the approximation of these solutions with arbitrary small error (this error can be controlled by increasing K). The usefulness of this model is illustrated via concrete example. This study can to get more profound results in further and investigate an optimal process for an assembly line of rectifiers in electrical engineering.

1. Introduction

As we know, mathematics is the tool for describing changes in each domain as dynamic systems, through which one can indicate their characteristics. One of the problems that attracts attention is to study by mathematical modeling an operation of rectifier circuits [1-3].

In [3], the characteristics of the rectifier circuit using diodes is modelled by differential inclusions which has the following form

\[ \dot{x} \in f(t, x) - N_Q(x), x \in Q \]

where a set Q is a cone in the space \( \mathbb{R}^n \); \( x(t) \) is an unknown function whose values belong to \( \mathbb{R}^n \) at moment \( t \) and the set \( N_Qx \) is called the normal cone which is defined by

\[ N_Qx = \{ z \in \mathbb{R}^n : (z, y - x) \leq 0, \forall y \in Q \}. \]

Obviously, we see that differential inclusions are usually more difficult and complicated than differential equations. Therefore, we propose a form of an ordinary differential equation whose right hand side is a continuous function in order to find a solution approximated to a solution of (1).

Now we consider a special case of the set \( Q \). Let \( n_1, n_2 \in \mathbb{R}^2 \) be two fixed normalized vectors in the plane. As convex closed subset \( Q \subset \mathbb{R}^2 \) we take the intersection of the two half-planes\( Q_1, Q_2 \):

\[ Q_i = \{ z \in \mathbb{R}^2 : (n_i, z) \leq 0 \} (i = 1, 2). \]

To avoid trivial results, we suppose that the vectors \( n_1 \) and \( n_2 \) are not collinear, and denote by \( \omega \in (0, \pi) \) the smaller angle between them. In this paper the mathematical model is defined by the
following ordinary equation

\[ \dot{z} = f(t, P_\omega z) - K \max \{(n_1, z)_+,(n_2, z)_+\} \sum_{k \in M(z)} n_k. \]  

(3)

Where, an element \( P_\omega z \) is called projection which is a point of best approximation to \( z \) in \( Q \):

\[ \| z - P_\omega z \| = \text{dist}(z, Q) = \inf \{ \| z - x \| : x \in Q \}; \]  

(4)
as usual \( u_+ := \max \{u, 0\} \) and \( M(z) \) contains the index \( k \) for which \( \max\{(n_1, z)_+, (n_2, z)_+\} = (n_k, z) \).

The contribution of this paper is to evaluate the error between the solution obtained by solving (3) and the initial condition \( x(t_0) = z(t_0) = x_0 \in Q \). Then (3) takes the simpler form

\[ (x(t) - z(t)) \leq \frac{C e^{L(t-t_0)}}{\sqrt{L \min \{2 \sin \omega_0, 1\} \sin \omega_0 \sqrt{R}}}. \]  

(7)

holds for \( t \in [t_0, T] \) and \( \omega_0 = \min \{\omega, \pi - \omega\} / 2 \).

**Proof.** Let us denote the cone \( K = N_Q(x) \) and the adjoint cone to \( K \) is denoted by \( K^* \). As in [1] there exists \( u = P_{K^*} f(t, x) \) such that \( \hat{x} = f(t, x) - u \). Now, we estimate the term \( p(x, z) \) in two cases:

\[ p(x, z) = \frac{1}{2} \frac{d}{dt} \| x - z \|^2 = (\dot{x} - \dot{z}, x - z). \]

1st case: The maximum \( \max\{(n_1, z)_+, (n_2, z)_+\} \) is realized by only one of the scalar products, say \( (n_1, z)_+ \). Then (3) takes the simpler form \( \dot{z} = f(t, P_\omega z) - K(z - P_\omega z) \). If \( P_\omega z \in Q \), then

\[ p(x, z) = (f(t, x) - f(t, P_\omega z), x - z). \]  

(8)

From the Lipschitz condition (5) and since the operator of projection is nonexpansive (see [4], p.109), we also obtain \( f(t, x) - f(t, P_\omega z), x - z \leq L \| x - z \|^2 \). This together with (8) implies

\[ p(x, z) \leq L \| x - z \|^2 + (u, z - x) + K(z - P_\omega z, x - P_\omega z) + K(z - P_\omega z, P_\omega z - z). \]

The projection \( P_\omega z \) (see (4)) is equivalent to the inequality \( \langle z - P_\omega z, x - P_\omega z \rangle \leq 0 \). Therefore, \( p(x, z) \leq L \| x - z \|^2 + (u, z - x) \). Next, \( p(x, z) \leq L \| x - z \|^2 + (u, z - P_\omega z) + (u, P_\omega z - x) \).

But the definition of the normal cone (see (2)), shows that \( (u, P_\omega z - x) \leq 0 \). Consequently, we get \( p(x, z) \leq L \| x - z \|^2 + (u, z - P_\omega z) \). As was shown in [5], we have \( \| z - P_\omega z \| \leq C/K \). On the other hand, since \( u = P_{K^*} f(t, x) \), we obtain

\[ p(x, z) \leq L \| x - z \|^2 + \frac{c^2}{K}. \]  

(9)

If \( P_\omega z \notin Q \), then we can check \( \| z - P_\omega z \| = \frac{\| z - P_\omega z \|}{\sin \alpha} \leq \frac{c}{K \sin \omega_0} \), where \( \alpha \) is the angle between the vectors \( \overline{P_\omega z} \) and \( \overline{P_\omega z P_\omega z} \). It is easy to see that \( \omega_0 \leq \alpha \leq \pi / 2 \). Combining this with (9), we have
\[ p(x, z) \leq L\|x - z\|^2 + \frac{c^2}{K\sin \omega_0}. \] (10)

2\text{nd} case: The maximum \( \max\{(n_1, z)_+, (n_2, z)_+\} \) is realized by both scalar products, i.e. \( \max\{(n_1, z)_+, (n_2, z)_+\} = (n_1, z)_+ = (n_2, z)_+ \). Then (3) rewrite in the form
\[ \dot{z} = f(t, P_0 z) - K(z - P_0 z + z - P_0 z). \]

In case \( z \in Q \) we have \( z = P_0 z = P_0 z \), then we get
\[ p(x, z) \leq L\|x - z\|^2. \] (11)

On the other hand, in case \( z \not\in Q \) the point \( z \) lies on the bisectrix of the angle between the vectors \( P_0 z P_0 z \) and \( P_0 z P_0 z \). But then we get
\[ z - P_0 z + z - P_0 z = 2 \frac{z - P_0 z}{\|z - P_0 z\|} \|z - P_0 z\| \cos \frac{\omega}{2} = 2(z - P_0 z) \sin^2 \left( \frac{\pi - \omega}{2} \right). \]

So the equation (3) is represented as follows \( \dot{z} = f(t, P_0 z) - 2K(z - P_0 z) \sin^2((\pi - \omega)/2) \). Similarly as the 1\text{st} case, we can obtain \( p(x, z) \leq L\|x - z\|^2 + \frac{c^2}{2K\sin^2((\pi - \omega)/2)}. \) Consequently,
\[ p(x, z) \leq L\|x - z\|^2 + \frac{c^2}{2K\sin \omega_0}. \] (12)

So in this case we have obtained (12) from (11). Combining (12) with (10) we see that the absolutely continuous function \( \phi(t) = \|x(t) - z(t)\|^2 = 2 \int_0^t p(x(s), z(s))ds \) satisfies the differential inequality (see [6], p. 5-6) \( \phi(t) \leq 2L\phi(t) + \frac{c^2}{K\min(2\sin \omega_0, 1)\sin \omega_0} \), and solving this as in the proof of Theorem 1 we arrive at (7).

3. An example and numerical analysis
Consider the full–wave rectifier with circuit feed and load sketched in figure 1.

By \( I_1 \) we denote the current of the feed chain \( eL_1 R_1 \), by \( I_2 \) we denote that of the load chain \( L_2 R_2 \).

The choice of positive directions is indicated by arrows. The input voltage of the feed chain is denoted by \( U_1 \), that of the load chain by \( U_2 \). Then we arrive at the system of equations
\[ \begin{cases} U_{L_1} + R_1 I_1 + U_1 = e(t) \\ U_{L_2} + R_2 I_2 + U_2 = 0. \end{cases} \] (13)

On the other side, \( U_{L_1} = L_1 I_1 \) and \( U_{L_2} = L_2 I_2 \). Therefore, the equation will be rewritten as following
\[ \begin{cases} L_1 I_1 + R_1 I_1 + U_1 = e(t) \\ L_2 I_2 + R_2 I_2 + U_2 = 0. \end{cases} \] (14)

We suppose that we are dealing with ideal diodes which means that the currents \( i_k \) and the voltages \( u_k \) \( (k = 1, 4) \) between the anodes and cathodes satisfy as follows
\[ i_k \geq 0, u_k \leq 0, i_k u_k = 0, k = 1, 4 \] (15)

According to the Kirchhoff formula, we also have:
\[ \begin{cases} I_1 = i_1 - i_2 = i_3 - i_4; I_2 = i_1 + i_4 = i_2 + i_3 \geq 0 \\ U_1 = u_4 - u_4 = u_3 - u_2; U_2 = u_1 + u_2 = u_3 + u_4 \leq 0 \end{cases} \] (16)

Therefore, \( I_2 - I_1 = i_4 + i_2 \geq 0; I_2 + I_1 = i_1 + i_3 \geq 0 \)
\[ \begin{cases} U_2 - U_1 = u_2 + u_4 \leq 0; U_2 + U_1 = u_4 + u_3 \leq 0 \end{cases} \] Then we can verify easily
\[
\begin{align*}
I_2 &\geq |I_1| > 0, \\
U_2 &\leq -|U_1| < 0. 
\end{align*}
\]  

(17)

Following this, if we use following symbols \( I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \) and \( U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \), then:

\[
(I, U) = I_1 U_1 + I_2 U_2 \\
= \frac{1}{4} (i_1 - i_2 + i_3 - i_4)(u_1 - u_2 + u_3 - u_4) \\
+ \frac{1}{4} (i_1 + i_2 + i_3 + i_4)(u_1 + u_2 + u_3 + u_4).
\]

From here and from the system (15), we can say:

\[
(I, U) = \frac{1}{2} (i_1 u_3 + i_3 u_4 + i_2 u_4 + i_4 u_2) \leq 0.
\]

(18)

Now, we will prove that:

\[
(I, U) = 0.
\]

(19)

In order to prove that the equation (19) is correct, we will prove that all figures in the right part of the equation (18) are equal to zero. It’s real, because if one of those figures is smaller than 0, assume \( i_1 u_3 < 0 \), then \( i_1 u_3 < 0 \Leftrightarrow \begin{cases} i_1 > 0 \Rightarrow u_1 = 0 \\ u_3 < 0 \Rightarrow i_3 = 0 \Rightarrow i_2 = 0 \Rightarrow u_2 = 0. \end{cases} \)

From this, we have, \( U_2 = u_1 + u_2 = 0 \). But this contradict with (17); means, \( i_1 u_3 = 0 \). Having similar demonstration, we also have the remaining figures in the right part of the equation (19) are equal to 0.

After that, we will use the following transformation:

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left( \frac{I_1}{I_2} \right) \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \quad \text{and} \quad \mu = \left( \frac{\mu_1}{\mu_2} \right) = \left( \frac{U_1}{U_2} \right) \begin{pmatrix} L_1^{-1} \\ L_2^{-1} \end{pmatrix}
\]

(20)

From the transformation (20) and from (19) we have:

\[
(x, \mu) = 0.
\]

(21)

Then the system (14) will be following:

\[
\dot{x} + Ax + \mu = E(t),
\]

(22)

with \( A = \begin{pmatrix} R_1/L_1 & 0 \\ 0 & R_2/L_2 \end{pmatrix} \) and \( E(t) = \begin{pmatrix} e(t)/\sqrt{L_1} \\ 0 \end{pmatrix} \).

Let the closed convex set \( Q \subset \mathbb{R}^2 \) is the form \( Q = Q_1 \cap Q_2 \), where \( Q_1, Q_2 \) is the half-plane with outer normal, respectively, \( n_1 = \frac{L_2}{\sqrt{L_1 + L_2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and \( n_2 = \frac{L_1}{\sqrt{L_1 + L_2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \). Now, from (17), (20) and (21) it follows that \( x \) belongs to cone \( Q \subset \mathbb{R}^2 \) and \( \mu \in N_Q(x) \). Next, we denote \( f(t, x) = E(t) - Ax \), then the equation (14) can be represented by the differential inclusion in the form of (1). On the other hand, we consider the mathematical model writing by (3). Here, the angle \( \omega \) between vectors \( n_1 \) and \( n_2 \) may be calculated as follows \( \cos \omega = \frac{n_1 \cdot n_2}{||n_1||||n_2||} = \frac{L_1 - L_2}{L_1 + L_2} \). Consequently, \( \sin \frac{\omega}{2} = \frac{L_2}{\sqrt{L_1 + L_2}} \) and \( \sin \left( \frac{\pi}{2} - \frac{\omega}{2} \right) = \frac{L_1}{\sqrt{L_1 + L_2}} \).

So we obtain:

\[
\omega_0 = \min\{\omega, \pi - \omega\}/2 = \arcsin \left( \frac{\min\{L_1, L_2\}}{(L_1 + L_2)} \right).
\]

In order to apply Theorem 1, suppose that the function \( f(t, x) = E(t) - Ax \) satisfies conditions (4) and (5). Then we get the following estimate
\[ \|x(t) - z(t)\| \leq \frac{2Ce^{L(t-t_0)}}{\sqrt{\min\{L_1L_2/L_1+L_2,1\}}\sqrt{\min\{L_1L_2/(L_1+L_2)\}}^{1/2}}. \] (23)

The analysis is carried out with the help of the program Mathematica. Let, we consider the problem (13), where: \( L_1 = L_2 = 1, R_1 = 0.1, R_2 = 0.2, e(t) = 3\cos 2t. \)

If we take \( K = K_1 = 10^{10}, \) the solution of (3) approximates the solution of (1), and the trajectory looks like sketched in figure 2.

To reproduce the estimate (23) for this choice, we have \( L = 1/5. \) From figure 2 it implies \( ||x|| \leq \sqrt{\frac{2}{5}}, \) consequently \( ||f(t,x)|| \leq 3 + \sqrt{\frac{2}{5}} =: C. \) Putting into (23), we obtain \( \|x(t) - z(t)\| \leq \frac{15 + \sqrt{2}}{\sqrt{5}} \cdot \frac{1}{10^5}. \) On the other hand, if we choose another large value for \( K, \) the trajectories of (3) for \( K_1 = K \) and \( K_2 = 2K \) essentially coincide. This means that the smooth approximation gives a stable result which is basically independent of \( K, \) if \( K \) is sufficiently large.

Figure 1. The full–wave rectifier.

Figure 2. The trajectory of the full–wave rectifier.

4. Conclusion

This paper proposed a mathematical model (3) containing a control parameter \( K \) to approximate the characteristics of the rectifier using semiconductor diodes modelling by an exact model (1) in some special case. This means that for sufficiently large \( K, \) there is no difference between solutions of (1) and (3). We can therefore adjust \( K \) to ensure any desired precision for which, the model (3) can be considered as equivalent to the exact model (1). We have also considered a concrete case to illustrate the result of the study.

The results can be used, for example, when designing software for the following tasks: studying and characterization of the data flows in an IP-based network [7], information security risk estimation for cloud infrastructure [8].

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