A Tauberian Theorem for $\ell$-adic Sheaves on $\mathbb{A}^1$

To Wang Yuan on his 80th birthday *

Lei Fu
Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, P. R. China
leifu@nankai.edu.cn

Abstract

Let $K \in L^1(\mathbb{R})$ and let $f \in L^\infty(\mathbb{R})$ be two functions on $\mathbb{R}$. The convolution

$$(K \ast f)(x) = \int_\mathbb{R} K(x - y)f(y)dy$$

can be considered as an average of $f$ with weight defined by $K$. Wiener’s Tauberian theorem says that under suitable conditions, if

$$\lim_{x \to \infty} (K \ast f)(x) = \lim_{x \to \infty} (K \ast A)(x)$$

for some constant $A$, then

$$\lim_{x \to \infty} f(x) = A.$$  

We prove the following $\ell$-adic analogue of this theorem: Suppose $K, F, G$ are perverse $\ell$-adic sheaves on the affine line $\mathbb{A}$ over an algebraically closed field of characteristic $p$ ($p \neq \ell$). Under suitable conditions, if

$$(K \ast F)|_{\eta_\infty} \cong (K \ast G)|_{\eta_\infty},$$

then

$$F|_{\eta_\infty} \cong G|_{\eta_\infty},$$

where $\eta_\infty$ is the spectrum of the local field of $\mathbb{A}$ at $\infty$.

Key words: Tauberian theorem, $\ell$-adic Fourier transformation.

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Introduction

A Tauberian theorem is one in which the asymptotic behavior of a sequence or a function is deduced from the behavior of some of its average. The $\ell$-adic Fourier transform was first introduced by Deligne in the study of exponential sums using $\ell$-adic cohomology theory. It was further developed by Laumon [5]. In this paper, using the $\ell$-adic Fourier transform, we prove an $\ell$-adic analogue of Wiener’s Tauberian theorem in the classical harmonic analysis. Our study shows that many

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results in the classical harmonic analysis have \( \ell \)-adic analogues and this area has not been fully explored. The result in this paper is absolutely not in its final form.

For any \( f_1, f_2 \in L^1(\mathbb{R}) \), their convolution \( f_1 * f_2 \in L^1(\mathbb{R}) \) is defined to be

\[
(f_1 * f_2)(x) = \int_{\mathbb{R}} f_1(x-y)f_2(y)dy.
\]

If we define the product of two functions to be their convolution, then \( L^1(\mathbb{R}) \) becomes a Banach algebra. A function \( f \in L^\infty(\mathbb{R}) \) is called weakly oscillating at \( \infty \) if for any \( \epsilon > 0 \), there exist \( N > 0 \) and \( \delta > 0 \) such that for any \( x_1, x_2 \in \mathbb{R} \) with the properties that \( |x_1|, |x_2| > N \) and \( |x_1 - x_2| < \delta \), we have

\[
|f(x_1) - f(x_2)| \leq \epsilon.
\]

Recall the following theorem (3 VIII 6.5).

**Theorem 0.1** (Wiener’s Tauberian theorem). Let \( K_1 \in L^1(\mathbb{R}) \) and \( f \in L^\infty(\mathbb{R}) \).

(i) If \( \lim_{x \to \infty} f(x) = A \), then

\[
\lim_{x \to \infty} \int_{\mathbb{R}} K_1(x-y)f(y)dy = A \int_{\mathbb{R}} K_1(x)dx.
\]

(ii) Suppose the Fourier transform

\[
\hat{K}_1(\xi) = \int_{\mathbb{R}} K_1(x)e^{i\xi x}dx
\]

of \( K_1 \) has the property \( \hat{K}_1(\xi) \neq 0 \) for all \( \xi \in \mathbb{R} \) and suppose

\[
\lim_{x \to \infty} \int_{\mathbb{R}} K_1(x-y)f(y)dy = A \int_{\mathbb{R}} K_1(x)dx.
\]

Then

\[
\lim_{x \to \infty} \int_{\mathbb{R}} K_2(x-y)f(y)dy = A \int_{\mathbb{R}} K_2(x)dx
\]

for all \( K_2 \in L^1(\mathbb{R}) \). Suppose furthermore that \( f \) is weakly oscillating at \( \infty \). Then we have \( \lim_{x \to \infty} f(x) = A \).

We quickly recall a proof of (ii). Let

\[
I = \{ K \in L^1(\mathbb{R}) \mid \lim_{x \to \infty} \int_{\mathbb{R}} K(x-y)f(y)dy = A \int_{\mathbb{R}} K(x)dx \}.
\]

Then \( I \) is a closed linear subspace of \( L^1(\mathbb{R}) \). If \( K \in I \), then for any \( y \in \mathbb{R} \), the translation \( K_y \) of \( K \) defined by \( K_y(x) = K(x-y) \) lies in \( I \). This implies that \( I \) is a closed ideal of the Banach
algebra $L^1(\mathbb{R})$. Since $\hat{K}_1(\xi) \neq 0$ for all $\xi$, by a theorem of Wiener [3 VIII 6.3], for any $g \in L^1(\mathbb{R})$ such that $\hat{g}$ has compact support, their exists $g_1 \in L^1(\mathbb{R})$ such that $\hat{g} = \hat{g}_1 \hat{K}_1$, which implies that $g = g_1 \ast K_1$. So the closure of the ideal generated by $K_1$ is $L^1(\mathbb{R})$. We have $K_1 \in I$, so we have $I = L^1(\mathbb{R})$. Hence for any $g \in L^1(\mathbb{R})$, such that $\hat{g}$ has compact support, their exists $g_1 \in L^1(\mathbb{R})$ such that $\hat{g} = \hat{g}_1 \hat{K}_1$, which implies that $g = g_1 \ast K_1$. So the closure of the ideal generated by $K_1$ is $L^1(\mathbb{R})$. We have $K_1 \in I$, so we have $I = L^1(\mathbb{R})$. Hence for any $g \in L^1(\mathbb{R})$, we have

$$\lim_{x \to \infty} \int_{\mathbb{R}} K_2(x - y) f(y) dy = A \int_{\mathbb{R}} K_2(x) dx.$$ For any $h > 0$, taking

$$K_2(x) = \begin{cases} \frac{1}{h} & \text{if } x \in [0, h], \\ 0 & \text{if } x \not\in [0, h], \end{cases}$$

we get

$$\lim_{x \to \infty} \frac{1}{h} \int_{x-h}^{x} f(y) dy = A.$$ If $f$ is weakly oscillating at $\infty$, this implies that $\lim_{x \to \infty} f(x) = A$.

In this paper, we study an analogue of the above result for $\ell$-adic sheaves on the affine line. Throughout this paper, $p$ is a prime number, $k$ is an algebraically closed field of characteristic $p$, $\mathbb{F}_p$ is the finite field with $p$ elements contained in $k$, $\ell$ is a prime number distinct from $p$, and $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_{\ell}$ is a fixed nontrivial additive character. Let $\mathbb{A} = \text{Spec } k[x]$ be the affine line. The Artin-Schreier morphism

$$\psi : \mathbb{A} \to \mathbb{A}$$

corresponding to the $k$-algebra homomorphism

$$k[t] \to k[t], \ t \mapsto t^p - t$$

is a finite Galois étale covering space, and it defines an $\mathbb{F}_p$-torsor

$$0 \to \mathbb{F}_p \to \mathbb{A} \xrightarrow{\psi} \mathbb{A} \to 0.$$ Pushing-forward this torsor by $\psi^{-1}$, we get a lisse $\overline{\mathbb{Q}}_{\ell}$-sheaf $\mathcal{L}_\psi$ of rank 1 on $\mathbb{A}$. Let $\mathbb{A}' = \text{Spec } k[x']$ be another copy of the affine line, let

$$\pi : \mathbb{A} \times_k \mathbb{A}' \to \mathbb{A}, \ \pi' : \mathbb{A} \times_k \mathbb{A}' \to \mathbb{A}'$$

be the projections, and let $\mathcal{L}_\psi(xx')$ be the inverse image of $\mathcal{L}_\psi$ under the $k$-morphism

$$\mathbb{A} \times_k \mathbb{A}' \to \mathbb{A}, \ (x, x') \mapsto xx'.$$
corresponding to the $k$-algebra homomorphism

\[ k[t] \to k[x, x'], \ t \mapsto xx'. \]

For any object $K$ in the triangulated category $D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ defined in [2] 1.1, the Fourier transform $\mathcal{F}(K) \in \text{ob} D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ of $K$ is defined to be

\[ \mathcal{F}(K) = R\pi'_!(\pi^* K \otimes L_{\psi}(xx'))[1]. \]

Let

\[ s: \mathbb{A} \times_k \mathbb{A} \to \mathbb{A}, \ (x, y) \mapsto x + y \]

be the $k$-morphism corresponding to the $k$-algebra homomorphism

\[ k[t] \to k[x, y], \ t \mapsto x + y, \]

and let

\[ p_1, p_2: \mathbb{A} \times_k \mathbb{A} \to \mathbb{A} \]

be the projections. For any $K_1, K_2 \in \text{ob} D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$, define their convolution $K_1 * K_2 \in \text{ob} D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ to be

\[ K_1 * K_2 = Rsl(p_1^* K_1 \otimes p_2^* K_2). \]

Let $F \in D_c^b(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$. We say $F$ is a perverse sheaf (confer [1]) if $\mathcal{H}^0(F)$ has finite support, $\mathcal{H}^{-1}(K)$ has no sections with finite support, and $\mathcal{H}^i(K) = 0$ for $i \neq 0, 1$. The Fourier transform of a perverse sheaf on $\mathbb{A}$ is a perverse sheaf on $\mathbb{A}'$.

Let $\mathbb{P} = \mathbb{A} \cup \{\infty\}$ and $\mathbb{P}' = \mathbb{A}' \cup \{\infty'\}$ be the smooth compactifications of $\mathbb{A}$ and $\mathbb{A}'$, respectively. They are projective lines. For any Zariski closed point $x$ (resp. $x'$) in $\mathbb{P}$ (resp. $\mathbb{P}'$), let $\eta_x$ (resp. $\eta_{x'}$) be the generic point of the henselization of $\mathbb{P}$ (resp. $\mathbb{P}'$) at $x$ (resp. $x'$), and let $\bar{\eta}_x$ (resp. $\bar{\eta}_{x'}$) be a geometric point above $\eta_x$ (resp. $\eta_{x'}$). On $\text{Gal}([\bar{\eta}_x/\eta_x]$ (resp. $\text{Gal}([\bar{\eta}_{x'}/\eta_{x'}])$, we have a filtration by ramification subgroups in upper numbering. We can use this filtration to define the breaks of $\overline{\mathbb{Q}}_\ell$-representations of $\text{Gal}([\bar{\eta}_x/\eta_x]$ (resp. $\text{Gal}([\bar{\eta}_{x'}/\eta_{x'}])$. For any perverse sheaf $F$ on $\mathbb{A}$, $\mathcal{H}^{-1}(F)\bar{\eta}_x$ is a $\overline{\mathbb{Q}}_\ell$-representation of $\text{Gal}([\bar{\eta}_x/\eta_x]$. Confer [3] for the definition of the local Fourier transform $\mathcal{F}(x,x')$. 

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**Theorem 0.2** (Tauberian theorem). Let $K \in \text{ob} \, D^b_c(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ be a perverse sheaf on $\mathbb{A}$. Suppose the Fourier transform $\mathcal{F}(K)$ is of the form $L[1]$ for some lisse $\overline{\mathbb{Q}}_\ell$-sheaf $L$ on $\mathbb{A}'$. Let $M, N$ be lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{A}$. Then $K \ast (M[1])$ and $K \ast (N[1])$ are perverse.

(i) If $M_{\eta_{\infty}} \cong N_{\eta_{\infty}}$, then $\mathcal{H}^{-1}(K \ast (M[1]))_{\eta_{\infty}} \cong \mathcal{H}^{-1}(K \ast (N[1]))_{\eta_{\infty}}$.

(ii) Suppose $L$ has rank 1, and all the breaks of $L_{\eta_{\infty}} \otimes \mathcal{F}^{(\infty, \infty)}(M_{\eta_{\infty}})$ and $L_{\eta_{\infty}} \otimes \mathcal{F}^{(\infty, \infty)}(N_{\eta_{\infty}})$ lie in $(1, \infty)$. If $\mathcal{H}^{-1}(K \ast (M[1]))_{\eta_{\infty}} \cong \mathcal{H}^{-1}(K \ast (N[1]))_{\eta_{\infty}}$, then $M_{\eta_{\infty}} \cong N_{\eta_{\infty}}$.

**Remark 0.3.** In Wiener’s Tauberian Theorem 0.1, we have $\hat{K}_1 \in L^1(\mathbb{R})$. This implies that $\hat{K}_1$ is a uniformly continuous function on $\mathbb{R}$. This corresponds to the condition in Theorem 0.2 that $\mathcal{F}(K)$ is of the form $L[1]$ for a lisse sheaf $L$ on $\mathbb{A}'$. There are many perverse sheaves $K$ on $\mathbb{A}$ satisfying this condition. For example, we can start with a lisse sheaf $L$ on $\mathbb{A}'$, and then take $K = a_* \mathcal{F}'(L[1])(1)$, where $\mathcal{F}'$ is the Fourier transform operator defined as above but interchanging the roles of $\mathbb{A}$ and $\mathbb{A}'$, $a : \mathbb{A} \to \mathbb{A}$ is the $k$-morphism corresponding to the $k$-algebra homomorphism

$$k[x] \to k[x], \ x \mapsto -x,$$

and (1) denotes the Tate twist.

**Remark 0.4.** As one can see from the proof of Wiener’s Tauberian Theorem 0.1, the condition $\hat{K}_1(\xi) \neq 0$ for all $\xi$ ensures that for any $g \in L^1(\mathbb{R})$ such that $\hat{g}$ has compact support, there exists $g_1 \in L^1(\mathbb{R})$ such that $\hat{g} = \hat{g}_1 \hat{K}_1$ and $g = g_1 \ast K_1$. So the closure of the ideal generated by $K_1$ in $L^1(\mathbb{R})$. This corresponds to the condition in Theorem 0.2 that $\mathcal{F}(K) = L[1]$ for a lisse sheaf $L$ of rank 1 on $\mathbb{A}'$. Indeed, for any $G \in \text{ob} \, D^b_c(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$, we have

$$\mathcal{F}(G) \cong (\mathcal{F}(G) \otimes L^{-1}) \otimes L \cong (\mathcal{F}(G) \otimes L^{-1}[-1]) \otimes \mathcal{F}(K).$$

It follows that

$$G \cong G_1 \ast K,$$

where $G_1 = a_* \mathcal{F}'(\mathcal{F}(G) \otimes L^{-1})(1)$.

**Remark 0.5.** It is interesting to find a Tauberian theorem in the case where $k$ is of characteristic 0. In this case, the Fourier transform is not available. We need to find a convenient condition on $K$ which ensures that for any $G \in \text{ob} \, D^b_c(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$, their exists $G_1 \in \text{ob} \, D^b_c(\mathbb{A}, \overline{\mathbb{Q}}_\ell)$ such that $G \cong G_1 \ast K$. 
By [3] Theorem II 8.1, the condition $\hat{K}(\xi) \neq 0$ for all $\xi$ in Wiener’s Theorem 0.1 is equivalent to the condition that if $K \ast f = 0$ for some $f \in L^\infty(\mathbb{R})$, then we have $f = 0$. So to obtain a Tauberian theorem for $\ell$-adic sheaves, we may try to find a condition on a perverse sheaf $K$ on $\mathbb{A}$ which ensures that for any $G \in \text{ob} \ D^b_c(\mathbb{A}, \mathbb{Q}_\ell)$ such that $K \ast G = 0$, we have $G = 0$.

1 Proof of the Theorem

Keep the notations in the introduction. Denote by

$$\bar{\pi} : P \times_k P' \rightarrow \mathbb{P}, \ \bar{\pi}' : P \times_k P' \rightarrow \mathbb{P}'$$

the projections, by $\alpha : \mathbb{A} \hookrightarrow \mathbb{P}$ and $\alpha' : \mathbb{A}' \hookrightarrow \mathbb{P}'$ the immersions, and by $\mathcal{L}_\psi(xx')$ the sheaf $(\alpha \times \alpha')_!(\mathcal{L}_\psi(xx'))$ on $P \times_k P'$. For any $\mathbb{Q}_\ell$-representation $V$ of $\text{Gal}(\bar{\eta}_x/\eta_x)$ or $\text{Gal}(\bar{\eta}_x'/\eta_x')$ and any interval $(a, b)$ in $\mathbb{R}$, denote by $V^{(a, b)}$ the largest subspace of $V$ with breaks lying in $(a, b)$.

Lemma 1.1. Let $L$, $U$ and $V$ be $\mathbb{Q}_\ell$-representations of $\text{Gal}(\bar{\eta}_x/\eta_x)$. Suppose either $L^{(1, \infty)} = 0$ or $U^{[0, 1]} = V^{[0, 1]} = 0$.

(i) If $U^{(1, \infty)} \cong V^{(1, \infty)}$, then $(L \otimes U)^{(1, \infty)} \cong (L \otimes V)^{(1, \infty)}$.

(ii) Suppose furthermore that $L$ has rank 1, and all the breaks of $L \otimes U^{(1, \infty)}$ and $L \otimes V^{(1, \infty)}$ lie in $(1, \infty)$. If $(L \otimes U)^{(1, \infty)} \cong (L \otimes V)^{(1, \infty)}$, then $U^{(1, \infty)} \cong V^{(1, \infty)}$.

Proof. We have decompositions

$$L \cong L^{[0, 1]} \bigoplus L^{(1, \infty)}, \quad U \cong U^{[0, 1]} \bigoplus U^{(1, \infty)}.$$

It follows that

$$L \otimes U \cong (L^{[0, 1]} \otimes U^{[0, 1]}) \bigoplus (L^{(1, \infty)} \otimes U^{[0, 1]}) \bigoplus (L \otimes U^{(1, \infty)}).$$

Note that the breaks of $L^{[0, 1]} \otimes U^{[0, 1]}$ lie in $[0, 1]$, and the breaks of $L^{(1, \infty)} \otimes U^{[0, 1]}$ lie in $(1, \infty)$. It follows that

$$(L \otimes U)^{(1, \infty)} \cong (L^{(1, \infty)} \otimes U^{[0, 1]}) \bigoplus (L \otimes U^{(1, \infty)})^{(1, \infty)}.$$

Since either $L^{(1, \infty)} = 0$ or $U^{[0, 1]} = 0$, we have

$$(L \otimes U)^{(1, \infty)} \cong (L \otimes U^{(1, \infty)})^{(1, \infty)}.$$

We have a similar equation for $V$. Our assertion follows immediately. \hfill \square
Lemma 1.2. Let $H$ be a perverse sheaf on $\mathbb{A}$ and let $S \subset \mathbb{A}$ be the set of those closed points $s$ in $\mathbb{A}$ such that either $\mathbb{H}^0(H)_s \neq 0$ or $\mathbb{H}^{-1}(H)$ is not a lisse sheaf near $s$. Then we have

$$
\left(\mathbb{H}^{-1}(\mathcal{F}(H))_{\tilde{\eta}_{\infty'}}\right)^{(1,\infty)} \cong \mathcal{F}(\infty,\infty')\left(\mathbb{H}^{-1}(H)_{\tilde{\eta}_{\infty}}\right),
$$

$$
\left(\mathbb{H}^{-1}(\mathcal{F}(H))_{\tilde{\eta}_{\infty'}}\right)^{(0,1]} \cong \bigoplus_{s \in S} R^0\Phi_{\tilde{\eta}_{s}}\left(\wedge^\ast \alpha^H \otimes \mathcal{L}_{\psi}(xx')\right)_{(s,\infty')}.
$$

Proof. Let $j : \mathbb{A} - S \rightarrow \mathbb{A}$ be the open immersion, and let $\Delta$ be the mapping cone of the canonical morphism $jj^*H \rightarrow H$. Then $\Delta$ has finite support. Hence $\mathbb{H}^i(\mathcal{F}(\Delta))_{\tilde{\eta}_{\infty'}}$, are extensions of $\mathcal{L}_{\psi}(ax')|_{\tilde{\eta}_{\infty'}}$, for some $a \in k$. In particular, they have no subspace with breaks lying in $(1,\infty)$. We have a distinguished triangle

$$
\mathcal{F}(jj^*H) \rightarrow \mathcal{F}(H) \rightarrow \mathcal{F}(\Delta) \rightarrow .
$$

It follows that

$$
\left(\mathbb{H}^{-1}(\mathcal{F}(H))_{\tilde{\eta}_{\infty'}}\right)^{(1,\infty)} \cong \left(\mathbb{H}^{-1}(\mathcal{F}(jj^*H))_{\tilde{\eta}_{\infty'}}\right)^{(1,\infty)}.
$$

By [5] 2.3.3.1, we have

$$
\mathbb{H}^{-1}(\mathcal{F}(jj^*H))_{\tilde{\eta}_{\infty'}} \cong \bigoplus_{s \in S} \mathcal{F}(s,\infty')(\mathbb{H}^{-1}(H)_{\tilde{\eta}_{s}}) \bigoplus \mathcal{F}(\infty,\infty')(\mathbb{H}^{-1}(H)_{\tilde{\eta}_{\infty}}).
$$

(1)

We have

$$
\mathcal{F}(s,\infty')(\mathbb{H}^{-1}(H)_{\tilde{\eta}_{s}}) \cong \mathcal{F}(0,\infty')(\mathbb{H}^{-1}(H)_{\tilde{\eta}_{s}}) \otimes \mathcal{L}_{\psi}(sx')|_{\tilde{\eta}_{\infty'}}.
$$

So by [5] 2.4.3 (i) (b), $\mathcal{F}(s,\infty')(\mathbb{H}^{-1}(H)_{\tilde{\eta}_{s}})$ has breaks lying in $[0,1]$. By [5] 2.4.3 (iii) (b), $\mathcal{F}(\infty,\infty')(\mathbb{H}^{-1}(H)_{\tilde{\eta}_{\infty}})$ has breaks lying in $(1,\infty)$. Taking the part with breaks lying in $(1,\infty)$ on both sides of the equation (1), we get the first equation in the lemma. By [5] 2.3.3.1, we have

$$
\mathbb{H}^{-1}(\mathcal{F}(H))_{\tilde{\eta}_{\infty'}} \cong \bigoplus_{s \in S} R^0\Phi_{\tilde{\eta}_{s}}\left(\wedge^\ast \alpha^H \otimes \mathcal{L}_{\psi}(xx')\right)_{(s,\infty')} \bigoplus \mathcal{F}(\infty,\infty')(\mathbb{H}^{-1}(H)_{\tilde{\eta}_{\infty}}).
$$

(2)

Taking the part with breaks lying in $[0,1]$ on both sides of the equation (2), we get the second equation in the lemma.

The following proposition apparently looks more general than Theorem 0.2.

Proposition 1.3. Let $K \in \text{ob} D^b_c(\mathbb{A}, \mathcal{O})$ be a perverse sheaf on $\mathbb{A}$. Suppose the Fourier transform $\mathcal{F}(K)$ is of the form $L[1]$ for some lisse $\mathcal{O}$-sheaf $L$ on $\mathbb{A}$. Let $F, G \in \text{ob} D^b_c(\mathbb{A}, \mathcal{O})$ be perverse sheaves on $\mathbb{A}$. Then $K \ast F$ and $K \ast G$ are perverse. Suppose furthermore either

$$
\mathbb{H}^{-1}(\mathcal{F}(F))_{\tilde{\eta}_{\infty'}}^{[0,1]} = \mathbb{H}^{-1}(\mathcal{F}(G))_{\tilde{\eta}_{\infty'}}^{[0,1]} = 0,
$$

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or
\[ L^{(1, \infty)}_{\eta_{\infty'}} = 0. \]

(i) If \( \mathcal{H}^{-1}(F)_{\overline{\eta}_{\infty}} \cong \mathcal{H}^{-1}(G)_{\overline{\eta}_{\infty}} \), then \( \mathcal{H}^{-1}(K \ast F)_{\overline{\eta}_{\infty}} \cong \mathcal{H}^{-1}(K \ast G)_{\overline{\eta}_{\infty}} \).

(ii) Suppose \( L \) has rank 1, and all the breaks of
\[ L_{\eta_{\infty'}} \otimes \mathcal{H}^{-1}(\mathcal{F}(F))_{\overline{\eta}_{\infty'}} \] and \( L_{\eta_{\infty'}} \otimes \mathcal{H}^{-1}(\mathcal{F}(G))_{\overline{\eta}_{\infty'}} \)
lie in \((1, \infty)\). If \( \mathcal{H}^{-1}(K \ast F)_{\overline{\eta}_{\infty}} \cong \mathcal{H}^{-1}(K \ast G)_{\overline{\eta}_{\infty}} \), then \( \mathcal{H}^{-1}(F)_{\overline{\eta}_{\infty}} \cong \mathcal{H}^{-1}(G)_{\overline{\eta}_{\infty}} \).

**Proof.** Denote the Fourier transforms of \( K \) and \( F \) by \( \hat{K} \) and \( \hat{F} \), respectively. Let \( a : \mathbb{A} \to \mathbb{A} \) be the \( k \)-morphism corresponding to the \( k \)-algebra homomorphism
\[ k[x] \to k[x], \ x \mapsto -x. \]

By [5] 1.2.2.1 and 1.2.2.7, we have
\[ K \ast F \cong a_*(\mathcal{F}(K \ast F))(1) \]
\[ \cong a_*(\mathcal{F}(K) \otimes \mathcal{F}(F))[1](1) \]
\[ \cong a_*(\mathcal{F}(L \otimes \mathcal{F}(F))(1). \]

So by [5] 1.3.2.3, \( K \ast F \) is perverse. Let \( S' \subset \mathbb{A}' \) be the set of those closed points \( s' \) in \( \mathbb{A}' \) such that either \( \mathcal{H}^0(\mathcal{F}(F)) \not\cong 0 \) or \( \mathcal{H}^{-1}(\mathcal{F}(F)) \) is not a lisse sheaf near \( s' \). By [5] 2.3.3.1, we have
\[ \mathcal{H}^{-1}\left(\mathcal{F}'(L \otimes \mathcal{F}(F))_{\overline{\eta}_{\infty'}} \right)_{\overline{\eta}_{\infty}} \cong \bigoplus_{s' \in S'} R^0\Phi_{\overline{\eta}_{\infty}}(\pi'^*\alpha'_1(L \otimes \mathcal{F}(F)) \otimes \mathcal{O}_s(x'))_{(1, s')} \]
\[ \bigoplus \mathcal{F}'(\mathcal{F}(F))_{\overline{\eta}_{\infty'}} \otimes \mathcal{H}^{-1}(\mathcal{F}(F))_{\overline{\eta}_{\infty'}}. \]

Since \( L \) is lisse on \( \mathbb{A}' \), we have
\[ R^0\Phi_{\overline{\eta}_{\infty}}(\pi'^*\alpha'_1(L \otimes \mathcal{F}(F)) \otimes \mathcal{O}_s(x'))_{(1, s')} \cong L_{\overline{s'}} \otimes R^0\Phi_{\overline{\eta}_{\infty}}(\pi'^*\alpha'_1\mathcal{F}(F) \otimes \mathcal{O}_s(x'))_{(1, s')} \]

Denote also by \( a \) the morphism \( \eta_{\infty} \to \eta_{\infty} \) induced by \( a \). We have
\[ \mathcal{H}^{-1}(K \ast F)_{\overline{\eta}_{\infty}} \cong a_*\left( \bigoplus_{s' \in S'} \mathcal{F}'(L_{\overline{s'}} \otimes R^0\Phi_{\overline{\eta}_{\infty}}(\pi'^*\alpha'_1\mathcal{F}(F) \otimes \mathcal{O}_s(x'))_{(1, s')} \bigoplus \mathcal{F}'(\mathcal{F}(F))_{\overline{\eta}_{\infty'}} \otimes \mathcal{H}^{-1}(\mathcal{F}(F))_{\overline{\eta}_{\infty'}} \right)_{(1, s')}. \]

By Lemma 1.2, we have
\[ \mathcal{H}^{-1}(K \ast F)_{(1, \infty)}^{(1, \infty)} \cong a_*\left( \mathcal{F}'(\mathcal{F}(F))_{\overline{\eta}_{\infty'}} \otimes \mathcal{H}^{-1}(\mathcal{F}(F))_{\overline{\eta}_{\infty'}} \right)_{(1, s')} \]
\[ \mathcal{H}^{-1}(K \ast F)_{(0, 1)}^{(0, 1)} \cong a_*\left( \bigoplus_{s' \in S'} L_{\overline{s'}} \otimes R^0\Phi_{\overline{\eta}_{\infty}}(\pi'^*\alpha'_1\mathcal{F}(F) \otimes \mathcal{O}_s(x'))_{(1, s')} \right)_{(1, s')} \].
Similarly, we have

\[
\mathcal{H}^{-1}(F)_{\tilde{\eta}_{\infty}}^{(1,\infty)} \cong a_*(\mathcal{F}^{(\infty', \infty)}(\mathcal{H}^{-1}(\mathcal{F}(F))_{\tilde{\eta}_{\infty}}))(1),
\]

\[
\mathcal{H}^{-1}(F)_{\tilde{\eta}_{\infty}}^{[0,1]} \cong a_*\left(\bigoplus_{s' \in S'} R^0\Phi_{\tilde{\eta}_{\infty}}(\tilde{\pi}^*\alpha'_1\mathcal{F}(F) \otimes \mathcal{L}(xx'))_{\mathcal{H}(s', s')}\right)(1).
\]

Let \( T' \subset \mathcal{A}' \) be the set of those closed points \( s' \) in \( \mathcal{A}' \) such that either \( \mathcal{H}^0(\mathcal{F}(G))_{s'} \neq 0 \) or \( \mathcal{H}^{-1}(\mathcal{F}(G)) \) is not a lisse sheaf near \( s' \). We have similar equations if we replace \( F \) by \( G \) and \( S' \) by \( T' \).

Suppose \( \mathcal{H}^{-1}(F)_{\tilde{\eta}_{\infty}} \cong \mathcal{H}^{-1}(G)_{\tilde{\eta}_{\infty}} \). From

\[
\mathcal{H}^{-1}(F)_{\tilde{\eta}_{\infty}}^{[0,1]} \cong \mathcal{H}^{-1}(G)_{\tilde{\eta}_{\infty}}^{[0,1]},
\]

we get

\[
a_*\left(\bigoplus_{s' \in T'} R^0\Phi_{\tilde{\eta}_{\infty}}(\tilde{\pi}^*\alpha'_1\mathcal{F}(G) \otimes \mathcal{L}(xx'))_{\mathcal{H}(s', s')}\right)(1) \cong a_*\left(\bigoplus_{s' \in T'} R^0\Phi_{\tilde{\eta}_{\infty}}(\tilde{\pi}^*\alpha'_1\mathcal{F}(F) \otimes \mathcal{L}(xx'))_{\mathcal{H}(s', s')}\right)(1).
\]

Since \( L \) is lisse on \( \mathcal{A}' \), it follows that

\[
a_*\left(\bigoplus_{s' \in T'} L_{s'} \otimes R^0\Phi_{\tilde{\eta}_{\infty}}(\tilde{\pi}^*\alpha'_1\mathcal{F}(F) \otimes \mathcal{L}(xx'))_{\mathcal{H}(s', s')}\right)(1) \cong a_*\left(\bigoplus_{s' \in T'} L_{s'} \otimes R^0\Phi_{\tilde{\eta}_{\infty}}(\tilde{\pi}^*\alpha'_1\mathcal{F}(G) \otimes \mathcal{L}(xx'))_{\mathcal{H}(s', s')}\right)(1),
\]

that is,

\[
\mathcal{H}^{-1}(K \ast F)_{\tilde{\eta}_{\infty}}^{[0,1]} \cong \mathcal{H}^{-1}(K \ast G)_{\tilde{\eta}_{\infty}}^{[0,1]}.
\]

From

\[
\mathcal{H}^{-1}(F)_{\tilde{\eta}_{\infty}}^{(1,\infty)} \cong \mathcal{H}^{-1}(G)_{\tilde{\eta}_{\infty}}^{(1,\infty)},
\]

we get

\[
a_*\left(\mathcal{F}^{(\infty', \infty)}(\mathcal{H}^{-1}(\mathcal{F}(F))_{\tilde{\eta}_{\infty}})(1) \cong a_*\left(\mathcal{F}^{(\infty', \infty)}(\mathcal{H}^{-1}(\mathcal{F}(G))_{\tilde{\eta}_{\infty}})(1).
\]

So we have

\[
\mathcal{F}^{(\infty', \infty)}(\mathcal{H}^{-1}(\mathcal{F}(F))_{\tilde{\eta}_{\infty}},) \cong \mathcal{F}^{(\infty', \infty)}(\mathcal{H}^{-1}(\mathcal{F}(G))_{\tilde{\eta}_{\infty}}).
\]

This is equivalent to

\[
\mathcal{H}^{-1}(\mathcal{F}(F))_{\tilde{\eta}_{\infty}}^{(1,\infty)} \cong \mathcal{H}^{-1}(\mathcal{F}(G))_{\tilde{\eta}_{\infty}}^{(1,\infty)}.
\]
by \[5\] 2.4.3 (iii) (b) and (c). By Lemma 1.1, we have
\[
(L_{\bar{\eta}_{\infty}}, \otimes \mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty}},)^{(1,\infty)} \cong (L_{\bar{\eta}_{\infty}}, \otimes \mathcal{H}^{-1}(\mathcal{F}(G))_{\bar{\eta}_{\infty}},)^{(1,\infty)}.
\] (11)
Hence
\[
\mathcal{F}(\infty,\infty)(L_{\bar{\eta}_{\infty}}, \otimes \mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty}},) \cong \mathcal{F}(\infty,\infty)(L_{\bar{\eta}_{\infty}}, \otimes \mathcal{H}^{-1}(\mathcal{F}(G))_{\bar{\eta}_{\infty}},).
\] (12)
So we have
\[
\mathcal{H}^{-1}(K * F)_{\bar{\eta}_{\infty}} \cong \mathcal{H}^{-1}(K * G)_{\bar{\eta}_{\infty}}.
\] (13)
By equations (6) and (13), we have
\[
\mathcal{H}^{-1}(K * F)_{\bar{\eta}_{\infty}} \cong \mathcal{H}^{-1}(K * G)_{\bar{\eta}_{\infty}}.
\]
The above argument can be reversed. We have the following implications for the above equations:
\[
(3) \iff (4) \implies (5) \iff (6),
\]
\[
(7) \iff (8) \iff (9) \implies (10) \implies (11) \iff (12) \iff (13).
\]
Suppose \(L\) has rank 1, then we have (5) \(\implies\) (4). Suppose furthermore that all the breaks of
\[
L_{\bar{\eta}_{\infty}}, \otimes \mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty}}, \quad \text{and} \quad L_{\bar{\eta}_{\infty}}, \otimes \mathcal{H}^{-1}(\mathcal{F}(G))_{\bar{\eta}_{\infty}},
\]
lie in \((1,\infty)\). Then by Lemma 1.1, we have (11) \(\implies\) (10). If we have \(\mathcal{H}^{-1}(K * F)_{\bar{\eta}_{\infty}} \cong \mathcal{H}^{-1}(K * G)_{\bar{\eta}_{\infty}}\), then (6) and (13) holds. It follows that (3) and (7) holds. We thus have \(\mathcal{H}^{-1}(F)_{\bar{\eta}_{\infty}} \cong \mathcal{H}^{-1}(G)_{\bar{\eta}_{\infty}}\).

\[\square\]

Proof of Theorem 0.2. Theorem 0.2 follows directly from Proposition 1.3 by taking \(F = M[1]\) and \(G = N[1]\). Since \(M\) and \(N\) are lisse, by \[5\] 2.3.3.1 (iii), we have
\[
\mathcal{H}^{-1}(\mathcal{F}(F))_{\bar{\eta}_{\infty}}, \cong \mathcal{F}(\infty,\infty')(M_{\bar{\eta}_{\infty}}).
\]
By \[5\] 2.4.3 (iii) (b), the breaks of \(\mathcal{F}(\infty,\infty')(M_{\bar{\eta}_{\infty}})\) lie in \((1,\infty)\). Using this fact, one checks that the conditions of Proposition 1.3 hold.

\[\square\]
Remark 1.4. Proposition 1.3 is actually not more general than Theorem 0.2. Indeed, if

$$\mathcal{H}^{-1}(\mathcal{F}(F))^{[0,1]}_{\eta_{\infty'}} = 0,$$

then $\mathcal{F}'\mathcal{F}(F)$ is lisse on $\mathbb{A}$ by [5] 2.3.1.3 (ii), and hence $F = M[1]$ for some lisse sheaf $M$ on $\mathbb{A}$. So if we assume the condition

$$\mathcal{H}^{-1}(\mathcal{F}(F))^{[0,1]}_{\eta_{\infty'}} = \mathcal{H}^{-1}(\mathcal{F}(G))^{[0,1]}_{\eta_{\infty'}} = 0,$$

then Proposition 1.3 is exactly Theorem 0.2. If $L^{(1,\infty)}_{\eta_{\infty'}} = 0$, then by the formula [5] 2.3.1.1 (i)$'$, $K$ is a perverse sheaf with finite support. In this case, Proposition 1.3 can be proved directly.

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