We present results on Bose-Einstein condensation (BEC) on general compact quantum graphs, i.e., one-dimensional systems with a (potentially) complex topology. We first investigate non-interacting many-particle systems and provide a complete classification of systems that exhibit condensation. We then consider models with interactions that consist of a singular part as well as a hardcore part. In this way we obtain generalisations of the Tonks-Girardeau gas to graphs. For this we find an absence of phase transitions which then indicates an absence of BEC.

Keywords: Quantum graphs; Many-particle quantum graphs; Bose-Einstein condensation; Tonks-Girardeau gas

1. Introduction

We present results on Bose-Einstein condensation (BEC) in many-particle systems on compact quantum graphs. Quantum graphs are models of particles moving along the edges of a metric graph. They hence combine the simplicity of a one-dimensional model with the complexity of a graph. One of the major findings about quantum graphs was that the one-particle spectra display correlations that are well described by random-matrix theory. For that reason quantum graphs have become popular models in quantum chaos.

Bose-Einstein condensation, on the other hand, is a quantum mechanical phenomenon of many-particle systems that can be well described within the framework of quantum statistical mechanics. Originally, this condensation was predicted by Einstein for a gas of non-interacting bosons in three dimensions. The conclusion was that below some critical temperature $T_{\text{crit}}$, or above some critical particle density $\rho_{\text{crit}}$, all additional particles brought into the system must condense into the one-particle ground state. In other words, the one-particle ground state becomes macroscopically occupied. As recognised later, the macroscopic occupation of a single-particle state is indeed the underlying mechanism that gives rise to the con-
densation, also in interacting many-particle systems

We here introduce many-particle systems on graphs and study BEC in non-interacting systems as well as in systems with hardcore interactions, thus generalising the Tonks-Girardeau gas\(^8\) to graphs.

2. Preliminaries

In this section we briefly summarise relevant concepts of BEC as well as of many-particle quantum graphs. For more details on BEC see Refs. 7,9 on quantum graphs see Refs. 5,10–12, and on many-particle quantum graphs see Refs. 1,2.

A definition of BEC that also applies to a system of interacting bosons was given in Ref. 7 and employs the reduced one-particle density matrix obtained from the canonical density matrix (at inverse temperature \(\beta = \frac{1}{T}\)),

\[
\rho_N(x, y) = \frac{1}{Z_N(\beta)} \sum_n e^{-\beta E^N_n} \bar{\Psi}_n(x) \Psi_n(y),
\]

by tracing out all degrees of freedom except one. Here \(\Psi_n(x)\) is the \(n\)-th eigenfunction of the \(N\)-particle system with eigenvalues \(E^N_n\), and \(Z_N(\beta) = \sum_n e^{-\beta E^N_n}\) is the canonical partition function. Condensation is defined to occur when the largest eigenvalue of the reduced density matrix is asymptotically of order one as \(N \to \infty\).

Unfortunately, it is in general very difficult to prove (or disprove) BEC in an interacting system in the sense of Penrose-Onsager. Instead we shall employ the connection between BEC and phase transitions in the case of interacting bosons on a graph.

The classical configuration space for a particle on a graph is a compact metric graph, i.e., a finite, connected graph \(\Gamma = (V, E)\) with vertex set \(V = \{v_1, \ldots, v_V\}\) and edge set \(E = \{e_1, \ldots, e_E\}\). The edges are identified with intervals \([0, l_e], e = 1, \ldots, E\), thus assigning lengths to intervals. This then introduces a metric on the graph. Note that a graph is called \textit{compact} when all lengths are finite.

For the one-particle quantum system the Hilbert space is

\[
\mathcal{H}_1 = L^2(\Gamma) = \bigoplus_{e=1}^E L^2(0, l_e).
\]

The quantum dynamics shall be generated by a self-adjoint realisation of the Laplacian \(-\Delta_1\). As a differential expression this acts on each edge-component \(f_e\) of a function \(F = (f_1, \ldots, f_E) \in C_0^\infty(\Gamma)\) as the negative second derivative,

\[-\Delta_1 F = (-f''_1, \ldots, -f''_E).\]

As shown in Ref. 11 one way to arrive at all possible self-adjoint realisations of the Laplacian uses quadratic forms,

\[
Q_1[F] = \sum_{e=1}^E \int_0^{l_e} |f'(x)|^2 \, dx - (F_{be}, L_1 F_{be})_{C^2E},
\]
with domain
\[ D_{Q_1} = \{ F \in H^1(\Gamma); P_1 F_{bv} = 0 \} \]
Here \( F_{bv} = (f_1(0), \ldots, f_E(0), f_1(l_1), \ldots, f_E(l_E))^T \in \mathbb{C}^{2E} \) is the vector of boundary values, \( P_1 \) is an orthogonal projection on \( \mathbb{C}^{2E} \) and \( L_1 \) a self-adjoint endomorphism of \( \ker P_1 \).

The \( N \)-particle Hilbert space is the \( N \)-fold tensor product \( \mathcal{H}_N = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1 \),
\[ \mathcal{H}_N = L^2(\Gamma_N) = \bigoplus_{e_1 e_2 \cdots e_N} L^2(D_{e_1 e_2 \cdots e_N}) , \]
where \( D_{e_1 e_2 \cdots e_N} = (0, l_1) \times \cdots \times (0, l_N) \) are \( N \)-dimensional hypercubes. In order to introduce contact interactions that are formally given by a Hamiltonian
\[ H_N = -\Delta_N + \alpha \sum_{i \neq j} \delta(x_{e_i} - x_{e_j}) \]
we also need to dissect the \( N \)-particle configuration space further by cutting the hypercubes along all hypersurfaces defined by \( x_{e_i} = x_{e_j} \) when \( e_i = e_j \). We denote \( \mathcal{L}^N(\Gamma_N^+) \) when taking this further dissection into account. The bosonic subspace is denoted as \( L^2_B(\Gamma_N^+) \).

As in the one-particle case, we require the \( N \)-particle Hamiltonian to be a self-adjoint realisation of the Laplacian \( -\Delta_N \). As a differential expression it acts as
\[ (-\Delta_N \Psi)_{e_1 \cdots e_N} = - \left( \frac{\partial^2}{\partial x^2_{e_1}} + \cdots + \frac{\partial^2}{\partial x^2_{e_N}} \right) \psi_{e_1 \cdots e_N} , \]
on functions \( \Psi \in C_0^\infty(\Gamma_N) \). Self-adjoint (bosonic) realisations of \( -\Delta_N \) are obtained through suitable quadratic forms. For this, we first define vectors of boundary values
\[ \Psi_{bv}(y) = \left( \sqrt{l_{e_2} \cdots l_{e_N}} \psi_{e_1 \cdots e_N}(0, l_{e_2} y_1, \ldots, l_{e_N} y_{N-1}) \right) , \]
with \( y \in [0, 1]^{N-1} \). The desired quadratic form now reads
\[ Q^{(N)}_B[\Psi] = N \sum_{e_1 \cdots e_N} \int_0^{l_{e_1}} \cdots \int_0^{l_{e_N}} |\psi_{e_1 \cdots e_N, x_{e_1}, \ldots, x_{e_N}}|^2 \, dx_{e_N} \cdots dx_{e_1} - N \int_{[0,1]^{N-1}} \langle \Psi_{bv}, L_N(y) \Psi_{bv} \rangle_{L^2(\mathbb{C}^{2N})} \, dy , \]
and is defined on
\[ D_{Q_B^{(N)}} = \{ \Psi \in H^1_0(\Gamma_N^+); P_N(y) \Psi_{bv}(y) = 0 \text{ for a.e. } y \in [0,1]^{N-1} \} . \]
Here \( \Psi \in H^1_0(\Gamma_N^+) \subset H^1_B(\Gamma_N^+) \) if each component \( \psi_{e_1 e_2 \cdots e_N} \) is in \( H^1 \) and vanishes on the hyperplanes \( x_{e_i} = x_{e_j} \). Moreover, \( P_N, L_N : [0,1]^{N-1} \to M(2E^N, \mathbb{C}) \) are bounded and measurable maps such that \( P_N(y) \) is an orthogonal projection and \( L_N(y) \) is a self-adjoint endomorphism of \( \ker P_N(y) \).

The self-adjoint realisation of \( -\Delta_N \) associated with \( Q^{(N)}_B[\cdot] \) is denoted as \( (-\Delta_N, D_N^{(N)}(P_N, L_N)) \), see Ref. 2. Due to the Dirichlet conditions along the hyperplanes \( x_{e_i} = x_{e_j} \) this operator is a rigorous version of the Hamiltonian \( \mathcal{H}_N \).
the limit $\alpha \to \infty$. Therefore it represents \textit{hardcore interactions}. It is important to note that the coordinate dependence of the maps $P_N$ and $L_N$ leads to (additional) singular many-particle interactions that are localised in the vertices of the graph.

3. BEC in non-interacting systems many-particle

In this section we provide a complete classification of non-interacting many-particle systems on general compact quantum graph in terms of the presence or absence of BEC. Since non-interacting Hamiltonians are entirely determined by corresponding one-particle Hamiltonians ($-\Delta_1, D_1(P_1, L_1)$), it is sufficient to refer to the latter only.

\textbf{Definition 3.1.} Let $\Gamma$ be a compact, metric graph with edge lengths $l_1, ..., l_E$. The thermodynamic limit (TL) consists of the scaling $l_e \mapsto \eta l_e$ and taking the limit $\eta \to \infty$.

As a first result we obtain the following.

\textbf{Lemma 3.1.} If the one-particle Laplacian $(-\Delta_1, D_1(P_1, L_1))$ is such that $L_1$ is negative semi-definite, no BEC occurs in the corresponding free Bose gas at finite temperature ($T > 0$) in the thermodynamic limit.

The proof uses the formalism of the grand-canonical ensemble and, via a bracketing construction, compares the given systems with free Bose gases with Dirichlet- and Neumann boundary conditions in the vertices. The number of particles is not fixed and the particle density is adjusted via the chemical potential $\mu$.

Our main result in this section is now the following.

\textbf{Theorem 3.1.} Let a free Bose gas be given on a quantum graph with a one-particle Laplacian $(-\Delta_1, D_1(P_1, L_1))$ such that $L_1$ has at least one positive eigenvalue. Then, in the thermodynamical limit, there is a critical temperature $T_c > 0$ such that BEC occurs below $T_c$.

In order to prove this statement one shows that the one-particle ground state energy (which is negative since $L_1$ has a positive eigenvalue, compare) remains negative in the thermodynamic limit. Indeed, one uses the lower bound for the one-particle ground state energy of Ref. \cite{Ref10} as well as a suitable Rayleigh quotient to prove that the ground state energy converges to $-L_{\text{max}}^2$, where $L_{\text{max}} > 0$ is the largest positive eigenvalue of $L_1$. One also uses that the number of negative eigenstates is bounded from above \cite{Ref10} (even in the thermodynamic limit) and applies the trace formula for quantum graphs. With standard arguments the Theorem then follows.

4. Interacting many-particle systems

In this final section we consider systems of bosons interacting via singular interactions localised at the vertices of the graph as well as hardcore contact interactions,
i.e., we consider (bosonic) self-adjoint realisations \((-\Delta_N, D_{\infty}^N(P_N, L_N))\) of the \(N\)-particle Laplacian. Our goal is to prove that no phase transitions (in the free energy density) are present, indicating an absence of condensation. For this recall that the free energy density of a sequence of \(N\)-particle Hamiltonians \(H_N\) with discrete spectra \(\{E_{Nn}\}\) is defined as

\[
f(\beta, \mu) = -\lim_{V \to \infty} \frac{1}{\beta V} \log \sum_{N=1}^{\infty} e^{N\beta \mu} \text{Tr}_{H_N} e^{-\beta H_N}.
\]  

(11)

The vanishing of functions in the domain \(D_{\infty}^N(P_N, L_N)\) along the hyperplanes \(x_{e_i} = x_{e_j}\) allows to define a Fermi-Bose mapping, relating the bosonic free-energy density to a fermionic one (which is known explicitly). This leads to the following result.\[1\]

**Theorem 4.1.** Let \((-\Delta_N, D_{\infty}^N(P_N, L_N))_{N \in \mathbb{N}}\) be a family of bosonic Laplacians with repulsive hardcore interactions, indexed by the particle number \(N\). Assume that for this family the operator of multiplication with \(L_N(\cdot)\) on \(L^2([0,1]^{N-1})\) is uniformly bounded with respect to \(N\). Then the bosonic, grand-canonical, free-energy density \(f(\beta, \mu)\) is given by

\[
f(\beta, \mu) = -\frac{1}{\pi \beta} \int_0^{\infty} \log \left(1 + e^{-\beta(k^2 - \mu)}\right) \, dk.
\]  

(12)

This function is smooth and, hence, there exists no phase transition.

It is important to note that Theorem 4.1 holds independently of the singular interactions in the vertices, i.e., independently of the maps \(P_N\) and \(L_N\). Hence, even when BEC occurs without the hardcore interactions, switching on the latter destroys the condensation.

**References**

1. J. Bolte and J. Kerner, *J. Phys. A: Math. Theor.* 46, p. 045206 (2013).
2. J. Bolte and J. Kerner, *J. Phys. A: Math. Theor.* 46, p. 045207 (2013).
3. J. Bolte and J. Kerner, preprint arXiv:1309.6091 (2013).
4. T. Kottos and U. Smilansky, *Physical Review Letters* 79, 4794 (1997).
5. S. Gnutzmann and U. Smilansky, *Taylor and Francis. Advances in Physics* 55, 527 (2006).
6. A. Einstein, *Sitzber. Kgl. Preuss. Akadm. Wiss.* , p. 3 (1925).
7. O. Penrose and L. Onsager, *Physical Review* 104, 576 (1956).
8. M. D. Girardeau, *Journal of Mathematical Physics* 1, 516 (1960).
9. M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac and M. Rigol, *Rev. Mod. Phys.* 83, 1405 (2011).
10. V. Kostrykin and R. Schrader, *Contemporary Mathematics* 415, 201 (2006).
11. P. Kuchment, *Waves Random Media* 14, S107 (2004).
12. J. Bolte and S. Endres, *Ann. H. Poincaré* 10, 189 (2009).