Stability of periodic orbits controlled by time-delay feedback

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Extended time-delay auto-synchronization (ETDAS) is a promising technique for stabilizing unstable periodic orbits in low-dimensional dynamical systems. The technique involves continuous feedback of signals delayed by multiples of the orbit’s period in a manner that is especially well-suited for fast systems and optical implementation. We show how to analyze the stability of a given implementation of ETDAS without explicit integration of time-delay equations. To illustrate the method and point out some nontrivial features of ETDAS, we obtain the domain of control for a period-one orbit of the driven, damped pendulum.

I. INTRODUCTION

The prospect of “controlling chaos” has generated great interest among physicists over the past several years. As first pointed out by Ott, Grebogi, and Yorke, the existence of many periodic orbits embedded in strange attractors raises the possibility of using very small control signals to obtain various types of regular behavior from intrinsically chaotic systems. This fact has implications both for engineering nonlinear systems in which chaotic fluctuations occur but are undesirable and for the understanding of biological systems in which underlying nonlinear dynamical systems are regulated in ways that are at present poorly understood.

The initial problem that must be faced in developing a control mechanism requiring only small externally imposed perturbations to the system is to design a feedback scheme that allows an unstable periodic orbit (UPO) to be stabilized. Recently several techniques have been introduced for accomplishing this using feedback signals that actually vanish (in the absence of noise) when the system is on the desired orbit. One such technique, sometimes called “time-delayed autosynchronization” (TDAS), involves a control signal formed from the difference between the current state of the system and the state of the system delayed by one period of the UPO. One of us (JESS) and coworkers have showed how to efficiently reuse information generated further in the past, using a technique called “extended time-delayed autosynchronization” (ETDAS). In the extended scheme, discussed in detail below, the control signal consists of a particular linear combination of signals from the system delayed by integer multiples of the UPO’s period. TDAS is a special, limiting case of ETDAS.

ETDAS has several features of practical interest. First, the use of a time-delay in the feedback loop eliminates the need for explicitly determining any information about the underlying dynamics other than the period of the desired orbit. Second, it can be implemented using a continuous feedback loop and hence can be applied to stabilize oscillations that are too fast to be handled by standard techniques based on measurements of the system on a surface of section. Finally, ETDAS provides a natural choice for controlling chaos using all-optical methods, as the feedback signal corresponds precisely to the reflected signal from a Fabry-Perot interferometer with properly adjusted cavity length.

This paper concerns the stability analysis of ETDAS in continuous systems for which the dynamical equations are known. We note that versions of ETDAS that apply to discrete maps can be treated analytically and it is known that ETDAS can stabilize orbits that are uncontrollable using TDAS. For continuous systems, both numerical results and experiments have shown that in order for ETDAS to be successful the feedback gain must lie within a finite, and often narrow, range. As the UPO is modified by changes in a bifurcation parameter this range of successful feedback gain will in general shift. In the space of the feedback gain and a bifurcation parameter, the area for which ETDAS can be successfully applied is known as the domain of control.

The results from the pendulum demonstrate the effectiveness of ETDAS in controlling highly unstable orbits that are impossible to stabilize with TDAS. The analysis also shows that the domain of control is surprisingly complex and depends strongly on the particular choices of the accessible control parameter and the measured signal used to generate the feedback. The domains of control have several features that clearly distinguish the application of continuous control of a period-one orbit from the control of a fixed point of a discrete map.

Two items that may be important in experimental implementations of ETDAS but are not treated here are the effects of noise and the unavoidable small time lag between the measurement of the system and the application of feedback. Preliminary numerical investigations and experiments on fast diode resonators indicate that low noise levels and small time lags shift the bound-
aries of the domain of control only by small amounts. The results we obtain for the idealized situation do indeed give a useful guide to the phenomenology of real systems.

II. STABILITY ANALYSIS

A general form for an $N$-dimensional system governed by ordinary differential equations and controlled via variations in an accessible parameter $\kappa$ is

$$\dot{x}(t) = f(x(t), t; \kappa), \quad \kappa = \kappa_0 + \epsilon(t), \quad (1)$$

where $x(t)$ and $f$ are $N$-dimensional vectors and $\epsilon(t)$ is a small control signal. In ETDAS, the control signal is given by

$$\epsilon(t) = \gamma[\xi(t) - (1 - R) \sum_{k=1}^{\infty} R^{k-1} \xi(t - k\tau)], \quad (2)$$

where $\xi$ is some measured component of $x$, $\tau$ is the period of the desired UPO, and $R \in [0, 1)$ is a real parameter. The case $R = 0$ corresponds to TDAS. Note that when control is successful, the system is synchronized with its own past behavior, so $\xi(t - k\tau) = \xi(t)$ for all $k$ and $\epsilon(t)$ vanishes. Thus to analyze the stability of the controlled system with respect to small perturbations, it is sufficient to consider a linearized form of Eqn. (1) in which both $\epsilon(t)$ and the deviations from the UPO are considered small. Letting $x_0(t)$ be the UPO and $y(t)$ be the deviation from this orbit, $y(t) \equiv x(t) - x_0(t)$, we have

$$\dot{y}(t) = J(t) \cdot y(t) + \epsilon(t) \frac{\partial f}{\partial \kappa}. \quad (3)$$

where $J(x_0(t)) \equiv \frac{\partial f}{\partial x}|_{x_0(t), \kappa_0}$ is the Jacobian of the uncontrolled system. Eqn. (3), with $\epsilon$ given by Eqn. (2), can also be written as

$$\dot{y}(t) = J(t) \cdot y(t) + \gamma M(t) \left[ y(t) - (1 - R) \sum_{k=1}^{\infty} R^{k-1} y(t - k\tau) \right], \quad (4)$$

where $M(x_0(t)) \equiv \left( \frac{\partial f}{\partial \kappa} |_{x_0(t), \kappa_0} \right) \otimes \hat{n}$, is an $N \times N$ dyadic which contains all information about how the control is applied to the system; $\hat{n}$ is a constant unit vector that determines the component of $x$ that enters the control signal via $\xi = \hat{n} \cdot x$, and $\frac{\partial f}{\partial \kappa}$ describes the effect on the dynamics of small changes of the control parameter $\kappa$. Eqn. (4) applies to both periodically driven and autonomous systems. For periodically driven systems, the period $\tau$ is equal to the period of the drive (or an integer multiple of it); for autonomous systems $\tau$ is not known a priori, but it can be repeatedly adjusted until control is achieved. (One scheme for making the adjustment is discussed by Kittel et al. [8]). Our goal is to investigate the stability of the trivial $y = 0$ solution of Eqn. (4), which corresponds to the system remaining on the UPO.

Eqn. (4) is a special case of a form that has been treated in the mathematics literature. [2] It can be written as

$$\dot{y}(t) = \sum_{n=0}^{\infty} A_n(t) \cdot y(t - n\tau), \quad (5)$$

where $y(t)$ is an $N$-dimensional vector and each $A_n(t)$ is an $N \times N$ matrix with elements that are periodic in time with period $\tau$. Specifically, we have

$$A_n(t) = \left\{ \begin{array}{ll} J(t) + \gamma M(t) & \text{for } n = 0 \\
-\gamma(1 - R) R^{n-1} M(t) & \text{for } n = 1, 2, 3, \ldots \end{array} \right. \quad (6)$$

Notice that $J$ and $M$ (and consequently $A_n$) are periodic with period $\tau$ by virtue of the fact that they are evaluated along the UPO.

The general approach to the linear stability of integer time-delay differential equations with periodically varying coefficients has been discussed by Hale and Verduyn Lunel [2]. (The one-dimensional case has also been addressed by Ortega [3]). We briefly outline here a derivation of their central result, which will then be used as the basis for an efficient numerical technique for mapping the domain of stability in the appropriate parameter space.

By virtue of the linearity of Eqn. (5), a general solution can be composed from a sum of periodic modes with exponential envelopes:

$$y_k(t) = p_k(t) \exp(\lambda_k t/\tau), \quad (7)$$

where $p_k(t + \tau) = p_k(t)$ is an $N$-dimensional vector and $\lambda$ is a complex number. For one such mode, one obtains from Eqn. (5) (dropping the subscript $k$)

$$\dot{p}(t) = \left( \sum_{n=0}^{\infty} e^{-n\lambda} A_n(t) - \frac{\lambda}{\tau} \right) \cdot p(t). \quad (8)$$

Equivalently we can write

$$p(t) = e^{-\lambda t/\tau} U(t) \cdot p(0), \quad (9)$$

where the matrix $U(t)$ is the solution of the equation

$$\dot{U}(t) = \sum_{n=0}^{\infty} e^{-n\lambda} A_n(t) \cdot U(t), \quad (10)$$

with $U(0) = I$. Defining the Floquet multiplier $\mu \equiv e^\lambda$ a formal solution for $U(\tau)$ can be written as

$$U(\tau) = T \left[ e^{\int_{0}^{\tau} dt \sum_{n=0}^{\infty} \mu^{-n} A_n(t)} \right], \quad (11)$$

where $[\cdot \cdot \cdot]$ indicates the time-ordered product. The time-ordered exponential is simply a compact notation [14] used to emphasize the way in which $U(\tau)$ depends on
\( \mu \). In general, \( \mathbf{U}(\tau) \) must be obtained by direct numerical integration of Eqn. (10).

In order for \( p(t) \) to be periodic, Eqn. (9) implies that \( p(0) \) must satisfy the equation

\[
(\mu^{-1}\mathbf{U}(\tau) - \mathbf{U}(0)) \cdot p(0) = 0, \tag{12}
\]

which in turn requires the vanishing of the determinant of \( (\mu^{-1}\mathbf{U}(\tau) - \mathbf{U}(0)) \). Substituting for \( \mathbf{U}'s \) one obtains a modified eigenvalue equation for \( \mu \):

\[
\left| \mu^{-1}\mathbf{T} \left[ e^{\int_0^\tau dt \sum_{n=0}^{\infty} \mu^{-n}\mathbf{A}_n(t)} \right] - \mathbf{I} \right| = 0. \tag{13}
\]

Eqn. (7) shows that the trivial solution of Eqn. (5) is asymptotically stable if and only if all \( \mu \) which satisfy Eqn. (13) also satisfy \( |\mu| < 1 \). This is the central result advertised above.

Inserting Eqn. (6) into Eqn. (13) and performing the geometric sum over coefficients of \( \mathbf{M} \), we have

\[
\left| \mu^{-1}\mathbf{T} \left[ e^{\int_0^\tau dt \left( \mathbf{J}(t) + \gamma \frac{\mu^{-1}}{1-\mu^{-1}} \mathbf{M}(t) \right)} \right] - \mathbf{I} \right| = 0, \tag{14}
\]

the defining relation for Floquet multipliers of systems under ETADAS control. The system is linearly (or locally asymptotically) stable if and only if \( |\mu^{-1}| > 1 \) for all \( \mu \) satisfying Eqn. (11), (13). For \( R < 1 \), the determinant on the left hand side of Eqn. (13), which will be denoted \( g(\mu^{-1}) \), has no poles inside the unit circle. Hence, by a well-known theorem in complex analysis, the number of roots of \( g(\mu^{-1}) \) with \( |\mu^{-1}| < 1 \) is equal to the number of times the path traced by \( g(\mu^{-1}) \) winds around the origin as \( \mu^{-1} \) is varied one full time around the unit circle. (14)

The condition for linear stability of the controlled system is that this winding number, which will be denoted \( N \), vanishes. (14)

For generic \( \mathbf{J}(t) \) and \( \mathbf{M}(t) \), the time-ordered integral discussed above cannot be obtained in closed form, and so must be computed numerically by explicitly integrating Eqn. (10). (4) Note, however, that this is an ordinary integration which does not involve any time-delayed quantities. Thus we have avoided the integration of a time-delay differential equation, for which the issue of how to choose initial conditions can be rather delicate.

### III. NUMERICAL PROCEDURES

In a typical situation, the system parameters that can be externally adjusted are largely dictated by physical principles and practical considerations. The problem is therefore to determine whether control can be achieved for a given designation of the control parameter \( \kappa \). In general, we may expect the success of ETADAS in controlling highly unstable orbits to depend upon the choice of which system variable is used to construct the feedback signal; i.e., the choice of \( \mathbf{n} \). We wish to determine the domain of values of \( \mathbf{n}, \gamma, \) and \( R \) for which ETADAS is successful for a given \( \kappa \). If the dynamical equations governing the system are known, the results can provide direct guidance in selecting appropriate parameters for operating the controlled system. The exercise of computing the domain of control for a simple model is also useful in that it reveals qualitative features that should be kept in mind when trying to find a stable regime in a system for which the equations are not known.

In general, the function \( \mathbf{f} \) in Eqn. (10) depends on a “bifurcation parameter”, which we denote \( r \). As \( r \) varies, the properties of a given UPO change, so the stability must be considered separately for different \( r \). We choose to map the domains of control in the plane of the bifurcation parameter, \( r \), and the feedback gain, \( \gamma \), for several discrete choices of \( \mathbf{n} \) and \( R \).

Calculation of the domain of control for a system given by Eqn. (10) and particular choices of \( \kappa, \mathbf{n}, \) and \( R \) involves three distinct numerical tasks:

1. the determination of the desired periodic orbits of the uncontrolled system;
2. the calculation of \( g(\mu^{-1}) \) for a given \( \mu \) on the unit circle and given values of \( r \) and \( \gamma \); and
3. the evaluation of the winding number, \( N \), which determines the number of unstable modes.

The first is easily accomplished using a standard Newton’s method. Using the period of the resulting solution and a point on the orbit, the second task requires straightforward simultaneous integration of the uncoupled dynamical equations and the set of first-order ordinary differential equations (Eqn. (10)) that determine \( \mathbf{U}(\tau) \). In the cases we have studied, a fifth order adaptive step size Runge-Kutta method has proven satisfactory. Finally, \( N \) can be determined by evaluating \( g \) for a sequence of sufficiently closely spaced \( \mu \’s \) around the unit circle and considering the sequence of values of \( \text{arg}(g(\mu^{-1})) \). The necessary number of points in the sequence depends upon the proximity of roots of \( g \) to the unit circle.

With these tools in hand, the boundary of the domain of control may be located by the following method. First, a single point on the boundary must be determined. If the orbit of the uncontrolled system becomes unstable when \( r = r_c \), a point on the boundary of the domain of control can always be found at \((r, \gamma) = (r_c, 0)\) since for \( \gamma = 0 \) the controlled system is identical to the system without feedback control. An entire boundary may then be traced by changing \( r \) in small increments, each time searching over a small interval in \( \gamma \) to locate the boundary. The boundary of the domain of control is identified as the point where the winding number jumps from zero to a positive integer, signalling the entry of a root into the unit circle. (12)

Additional work may be necessary in order to find all the islands of stability, since the domain of control may
not be simply connected. We have encountered this situation only in the case where a given periodic orbit is unstable only over a finite interval in the parameter $r$ with endpoints $r_1$ and $r_2$. In this case, there are two separate regions of stability, each of which can be found starting from $r = r_{1,2}$, $\gamma = 0$. (See the example discussed below.) We have not been able to rigorously rule out the occurrence of more complicated situations in which a domain has a hole in it or an island of stability exists that does not merge into an intrinsically stable regime. We have tested for this by scanning through large intervals of $\gamma$ for selected values of $r$ in two different systems, the pendulum discussed below and a diode-resonator circuit that will be the subject of a future publication \[\text{[11]},\] and have yet to observe such behavior.

As a check of the above analysis and numerical procedure, selected sections of the stability domains presented below were checked with explicit integration of the time-delayed differential equations given by Eqn. \[\text{(1)}\]. Using a fourth order Adams-Bashforth-Moulton algorithm modified for time-delayed equations, we found complete agreement between the two methods to the expected degree of accuracy. We emphasize, however, that our procedure is more reliable and significantly easier to automate.

IV. AN EXAMPLE: THE NONLINEAR PENDULUM

To demonstrate the utility of the stability analysis described above and begin to investigate the structure of domains of control, we calculate stability domains for the damped driven nonlinear pendulum:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\nu x_2 - \sin x_1 + F \cos(\omega t).
\end{align*}
\]

Here $x_1$ is the angle of the pendulum and $x_2$ is its angular velocity; $\nu$, $F$, and $\omega$ describe the damping, drive amplitude and drive frequency, respectively. We fix $\nu = 1/2$ and $\omega = 2\pi/10$, and vary $F$ as the bifurcation parameter.

We choose to study a family of period-one orbits which are unstable between $F \approx 0.987$, and $F \approx 2.046$. These orbits and their largest Floquet multipliers are shown in Fig. 1. Note that some of these orbits are highly unstable, having Floquet multipliers as high as 30. The explicit demonstration (below) that periodic orbits with multipliers this large can be stabilized using continuous time-delay techniques is a new result, though a plausibility argument has been given based on the behavior of discrete maps. \[\text{[6]}\]

ETDAS requires that we measure some accessible quantities in the system and feed back the control signal through small modifications of an accessible parameter. We assume that both the position and velocity of the pendulum are measurable, and that small changes can be made to the amplitude of the drive, $F \rightarrow F + \epsilon(t)$. (Note that $F$ is being used here in a dual role, both as the bifurcation parameter, $r$, and the accessible control parameter, $\kappa$.) The control signal is generated from a linear combination of deviations in position and velocity. In the notation of Eqn. \[\text{(4)}\], we have $\hat{u} = (\sin \phi, \cos \phi)$, where $-\pi/2 \leq \phi \leq \pi/2$ is a parameter that can be chosen to optimize the domain of control. The range $-\pi/2 \leq \phi \leq \pi/2$ is sufficient to describe the entire space of possibilities since the case $\phi + \pi$ is equivalent to $\phi$ with the sign of the feedback gain reversed. Note that $\phi = 0$ corresponds to measuring velocity only, and $\phi = \pi/2$ corresponds to measuring position only. The control matrix $M$ is given by

\[
M(t) = F \cos(\omega t) \begin{pmatrix} 0 & 0 \\ \sin \phi & \cos \phi \end{pmatrix}.
\] (17)

Our numerical implementation of the program outlined above was straightforward. The most significant source of potential errors is the possibility that a root of $g(\mu^{-1})$ crosses into the unit circle but remains extremely close to the boundary and is not picked up in the winding number calculation. This problem can be solved to any desired accuracy by choosing sufficiently many points around the unit circle in evaluating $N$ (or by using adaptive step-size methods). For the present case, 500 equally spaced points were used. This large number was necessary, however, only to obtain accurate results in regimes with very narrow features. In general, the necessary spacing between points is determined by the rate at which the first root enters the unit circle as $\gamma$ is varied across the stability boundary.

The dependence on $R$ of the domain of control is shown in Fig. 2 for $\phi = -\pi/8$ which is representative of all values of $\phi$ that we investigated. A key point that is evident here is that large values of $R$ are necessary in order to control the highly unstable periodic orbits. In addition, the domain of control strongly depends on the choice of $\phi$, as might be expected by analogy with the situation for proportional feedback control of a stationary fixed point. Fig. 3 shows the domain of control for $R = 0.95$ and several choices of $\phi$. The domain extends across the entire unstable region only for $\phi$ within a relatively narrow range near $\phi = 0$, indicating the relative merit of measuring the velocity of the pendulum for our example in which the control signal is applied to the amplitude of the drive. Even for $R$ close to 1, it is not always possible to control highly unstable orbits for arbitrary $\phi$.

The results depicted in Fig. 3 illustrate the complexity of the domain of control. The sharp features and the existence of reentrant behavior for varying $\gamma$ indicate that intuition about the qualitative shapes of these domains may be highly misleading. An interesting example of the counterintuitive phenomena that can occur can be seen in the case $\phi = \pi/8$. As shown in Fig. 4, there is a region for which both $\gamma$ and $-\gamma$ successfully stabilize an UPO. In this region, one has an orbit that is unstable in the absence feedback. Naively, one would expect that if a given form of linear feedback resulted in stabilization of
the orbit, then reversing the sign of the feedback gain would make the orbit even more highly unstable. In the present case, however, the variation of $J$ around the orbit allows the inverted feedback gain to be equally effective in stabilizing the orbit. This can never occur for the case of linear control of a stationary fixed point in a continuous system or a discrete map.

The most important point here is that the experimental determination of the ETDAS domain of control for a given system should be guided as much as possible by calculations on model equations. A coarse scan of parameter space guided by “intuitively reasonable” ideas concerning the possible structure of these domains may well miss regimes of practical interest.

V. CONCLUSIONS

The primary purpose of this paper is to demonstrate the proper technique for analyzing the stability of systems controlled using ETDAS. Generic systems do not permit a complete analytical solution, even for the linear stability problem. The logic of our approach does, however, allow the development of a clean numerical technique. In particular, it avoids the need for integrating delay-differential equations directly and thereby avoids the difficulties associated with guaranteeing that a chosen initial condition lies in the appropriate basin of attraction.

The analysis of an orbit of the nonlinear pendulum confirms the fact that ETDAS can work well in continuous systems and also clearly illustrates the complexity of this linear stability problem. We have also studied a set of equations describing a fast diode resonator and compared our results with experiments. Details will be published together with the experimental results. Here we note only that the effects of noise and a time lag in the feedback loop outside the recursively used delay line do alter the domains of control slightly. Inclusion of these effects and extensions of the technique to spatially extended systems will be addressed in future studies.

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Figure 1. (a) A family of period one orbits of the nonlinear pendulum for drive amplitude $F = 0.96$ to $F = 2.056$, and (b) their largest Floquet multipliers, $\mu_l$. 
Figure 2. The domain of control for $\phi = -\pi/8$ for $R = 0, 0.5, 0.95$. Labels A, B, and C indicate the regions where the controlled system is stable for $R = 0.95$, 0.5, and 0, respectively. The inset is an enlargement of the lower right hand side. The arrows at the top (and the vertical line in the inset) indicate the value of the drive amplitude for which the uncontrolled orbit loses stability.
Figure 3. The domains of control for several values of the measured control signal and $R = 0.95$. (See text for definition of $\phi$.) The arrows at the top indicate the value of the drive amplitude for which the uncontrolled orbit loses stability. The horizontal scale is the same for each plot.
Figure 4. Domain of control for $\phi = \pi/8$ and $R = 0.95$. The two regions within the thick lines contain points for which the sign of the feedback gain can be inverted and the orbit will still be controlled. The arrow indicates the value of the drive amplitude for which the uncontrolled orbit loses stability.