Study of isospin nonconservation in the framework of spectral distribution theory

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ABSTRACT

The observed isospin-symmetry breaking in light nuclei are caused not only by the Coulomb interaction but by the isovector one and two body plus isotensor two body nuclear interactions as well. Spectral distribution theory which treats nuclear spectroscopy and other structural properties in a statistical framework was earlier applied to isospin conserving Hamiltonians only. In this paper we extend that to include the nuclear interactions non-scalar in isospin and work out examples in sd shell to calculate the linear term in the isobaric mass-multiplet equation originating from these non-scalar parts.

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1 Introduction

In conventional nuclear structure physics, isospin is considered to be a good symmetry with the isospin operator $T$ commuting with the nuclear one and two body Hamiltonian ($H_{\text{nuc}}(1)$ and $H_{\text{nuc}}(2)$) so that the many nucleon energy eigenstates are labeled by specific isospin quantum numbers $T$ and they have a degeneracy of $(2T + 1)$. The states of the multiplet i.e. $T_z = -T, -T + 1, \ldots, T - 1, T$ belong to different nuclei and are often called the Isobaric Analog States (IAS). This follows from the postulate of Charge Independence of the interaction meaning that the proton-proton, proton-neutron and neutron-neutron interactions are identical. However extensive experimental evidence gathered by now confirms that isospin symmetry is broken by not only the electromagnetic Coulomb interaction but also by a small amount of charge-dependent nuclear interaction [1]. The difference of energies of the ground states (or specific low-lying states) of two mirror nuclei i.e. nuclei with neutron number ($N$) and proton number ($Z$) interchanged, give a measure of the isospin nonconservation. A number of examples, particularly of light nuclei are available with reasonably accurate measurements of the Mirror Energy Difference (MED) [2] given by $\Delta E = E(J, T_z = -T) - E(J, T_z = +T)$ with $T=1/2, 1, \ldots$. Actually Wigner [3] had postulated an isobaric mass multiplet equation (IMME) to take care of the splitting, in general, going upto the quadratic term as

$$M(\alpha, T, T_z) = a(\alpha, T) + b(\alpha, T)T_z + c(\alpha, T)T_z^2$$

for multiplets for given $\alpha$ and $T$. The coefficients $a$, $b$ and $c$ can be calculated by theory or extracted from experimental results. On the theoretical side one has constructed one and two body Hamiltonians no longer isoscalar but with isovector and isotensor parts as well and performed shell model [4] and other calculations. The parameters of the one plus two body interaction used in the shell model calculations were then best fitted to reproduce the experimental data [1].

In contrast to microscopic structure models like the nuclear shell model a theory to describe the statistically averaged energy spectra and transition strengths of nuclear excitations and decays have been developed over the years based on the results of random matrix ensembles embedded in many nucleon spaces. This theory [5] [6] [7] avoids the diagonalisation of large Hamiltonian matrices and use the low moments of the Hamiltonian by evaluating the traces of low powers of the Hamiltonian in many nucleon spaces. The basic result of this theory is the fact that the averaged density of energy eigenstates goes fast towards a Gaussian with the increase in the number of valence nucleons in large dimensional shell model spaces. This result is seen to be true even for each configuration when one considers configuration partitioning as long as the configuration dimension is large. Though in principle this theory is not meant for considering individual states, even then one can invert the problem and locate low-lying discrete states in energy like the ground state and compare the values with experimental binding energies. This has yielded reasonable success for light nuclei not only for the stable ones [8] [9] [10] but for the neutron-rich nuclei as well [11]. But to date all applications of spectral distribution theory have been done considering isospin conserving interaction only. In this work we plan to extend this to include isovector and isotensor Hamiltonians and describe how the relevant traces are first evaluated in spaces with many neutrons and protons and then through subtraction of these traces one projects out the ones in states with fixed isospin. This is then used to estimate the coefficient $b$ in IMME. In section 2 some basic features of the spectral distribution theory with respect to the calculation of
low-lying spectra are described. Section 3 involves spectral distribution investigations including the non-scalar Hamiltonians and presents some results for sd shell nuclei. Section 4 gives some concluding remarks.

2 Methods of Spectral Distribution Theory

The Gaussian nature of the density of states can be demonstrated for the non-interacting case, i.e. for a one-body Hamiltonian, by using the Central Limit Theorem (CLT) neglecting the Pauli blocking effect. For the interacting case i.e. with (1+2)-body Hamiltonian, one needs to carry out an averaging over the Embedded Gaussian Orthogonal Ensemble (EGOE) \[12\] \[13\] to get the Gaussian result. With 'm' valence particles distributed over 'N' single particle states giving the dimension of the shell model space dimension \(d(m)\) (equal to \(NC_m\)), the normalised density of states \(\rho_m(E)\) is then a function only of the centroid \(\epsilon(m)\) and width \(\sigma(m)\) of the Gaussian. But for using the result for real nuclei one needs to partition the space into subspaces with fixed isospin and for better accuracy, do a configuration partitioning, by distributing the 'm' particles in 'l' shell model orbits, giving rise to the normalised configuration-isospin densities, \(\rho_{m,T}(E)\) and dimensions \(d(m,T)\). However the Gaussian form of the eigenvalue density is seen to be true in the \((m,T)\) subspaces too as long as dimension of each subspace is large enough. The intensities in the subspaces then just add up to give the total density.

\[
I_{m,T}(E) = \sum_m I_{m,T}(E) = \sum_m d(m,T)\rho_{m,T}(E)
\]

The asymptotic Gaussian result is observed for almost all realistic Hamiltonians and the calculated higher cumulants are found to be small.

The ground state energy \(\bar{E}_g\) of a nucleus with the number of valence nucleons \(m\) and isospin \(T\) is given by the Ratcliff prescription \[14\]

\[
\sum_m \int_{-\infty}^{\bar{E}_g} I_{m,T}(E)dE = d_0/2
\]

where \(d_0\) is the degeneracy of the ground state. Thus the energy where the integrated area below the level density from the low energy side reaches half the ground state degeneracy, is taken to be the energy of the ground state.

The total Hamiltonian under consideration here is given by

\[
H = \sum_\nu H^{(\nu)} = H^{(0)} + H^{(1)} + H^{(2)}
\]

where \(H^{(0)}\), \(H^{(1)}\) and \(H^{(2)}\) are the isoscalar, isovector and isotensor (i.e. tensor of rank two in isospin) parts. For handling this Hamiltonian one first writes all the three parts in proton-neutron(p,n) formalism. The proton and neutron single particle matrix elements are given by \[4\]

\[
\epsilon^p_r = \epsilon^0_r + \epsilon^1_r/2 \quad \text{and} \quad \epsilon^n_r = \epsilon^0_r - \epsilon^1_r/2
\]

where \(\epsilon^0_r\) and \(\epsilon^1_r\) are the scalar and vector single particle matrix elements.
elements for orbit ‘r’. The values of the isovector and isotensor single particle energies and two body matrix elements were obtained by Ormand and Brown [4] for the 0p, pds, 1s – 0d, df and 0f – 1p shells by a least squared fit to the observed ‘b’ and ‘c’ coefficients of the IMME. We use the same values here though we limit ourselves to the study of the coefficient ‘b’ only. The relation between the two body matrix elements in the (p,n) formalism and the fixed isospin formalism is wellknown [15] and so we mention it here briefly. The proton-proton, neutron-neutron along with the neutron-proton or proton-neutron ‘T=1’ part matrix elements are [4]

\[ V_{rstu;J}^{pp} = V_{rstu;J,T=1}^0 + V_{rstu;J,T=1}^1/2 + V_{rstu;J,T=1}^2/6 \]  

(5)

\[ V_{rstu;J}^{nn} = V_{rstu;J,T=1}^0 - V_{rstu;J,T=1}^1/2 + V_{rstu;J,T=1}^2/6 \]  

(6)

\[ V_{rstu;J}^{pm}(T = 1) = V_{rstu;J,T=1}^0 - V_{rstu;J,T=1}^2/3 \]  

(7)

where \( V_{rstu;J,T}^{\nu} \) are the two body matrix elements in the isospin formalism. The \( \nu = 0, 1, 2 \) stand for the scalar, vector and rank 2 tensor matrix elements respectively. Then the total neutron-proton or proton-neutron two body matrix elements become [4]

\[ V_{rstu;J}^{pm} = [(1 + \delta_{rs})(1 + \delta_{tu})/2]^{1/2}(V_{rstu;J(T = 1)}^{pm} + V_{rstu;J,T=0}^0) \]  

(8)

In the spectral distribution method the traces of one operator or a product of operators in second quantized form are obtained by the contraction of the creation and destruction operators and then summing over all states. Alternatively one can write expressions for the trace equivalent operators in a closed form using the scalars of the relevant group involved [15] [17]. For example, the trace equivalent (TE) Hamiltonian that reproduces centroids in the ‘pn’ subspaces is given in terms of the scalars of \( \Sigma_r[U(N_r/2) + U(N_r/2)] \) subgroup by

\[ H^{pn}(TE) = \sum_r \epsilon_p n_r^p + \sum_r \epsilon_n n_r^n + \sum_r W_{rr}^{pp} n_r^p(n_r^p - 1)/2 + \sum_{r \neq s} W_{rs}^{pp} n_r^p n_s^p 
+ \sum_r W_{rr}^{nn} n_r^n(n_r^n - 1)/2 + \sum_{r \neq s} W_{rs}^{nn} n_r^n n_s^n + \sum_{r,s} W_{rs}^{pn} n_r^p n_s^n \]  

(9)

where \( n_r^p \) and \( n_r^n \) are the proton and neutron number operators in orbit ‘r’, \( W_{rr}^{pp} \) and \( W_{rr}^{nn} \) are the averaged two body proton-proton and neutron-neutron matrix elements in orbit ‘r’ and \( W_{rs}^{pp} \), \( W_{rs}^{nn} \) and \( W_{rs}^{pn} \) the averaged two body proton-proton, neutron-neutron and proton-neutron two body matrix elements with one particle in orbit ‘r’ and the other in orbit ‘s’.

For calculating the variances we write the norm of the full Hamiltonian as

\[ < H^2 >= < H^{(0)} H^{(0)} > + 2 < H^{(0)} H^{(1)} > + 2 < H^{(0)} H^{(2)} > \]  

(10)
where \(< ... \>\) denote averages in the relevant spaces and we neglect the terms quadratic involving \(H^{(1)}\) or \(H^{(2)}\) as the terms non-scalar in isospin terms are small. Once the centroids and variances in all the proton-neutron configuration spaces are evaluated for a fixed number of particles, corresponding averages in fixed isospin configuration spaces are then projected out by a procedure involving the \((p,n)\) traces including the relevant Clebch-Gordan coefficients for non-scalar Hamiltonians. Briefly the method is as follows \[18\] \[19\].

If we consider a space with \(m_p\) protons and \(m_n\) neutrons then the eigenvalue of \(T_z\) is \(T_0 = (m_n - m_p)/2\). Then the isospin quantum number has values \(T_0\), \(T_0 + 1\),...,\((m_n + m_p)/2\). The trace of operator \(O\) in the \((p,n)\) spaces is given as a sum over the traces of the reduced matrix elements of \(O^{\lambda_T}\) in the isospin spaces with different \(T\) values

\[
<< O >>^{m_n=m-k,m_p=k} = \sum_{i=0}^{k} (m - 2i + 1)^{-1/2} (C^{m/2-i}_{m/2-k} \lambda_T m/2-i) << O^{\lambda_T} >>^{m,T=m/2-i}
\]

(11)

where \(<< ... >>^{m_n,m_p}\) and \(<< ... >>^{m,T}\) are the traces in the \((p,n)\) spaces and isospin spaces respectively. The above \((k+1)\) equations for \(k=0,1,...,k\) when inverted give one traces of the reduced matrix elements in \((m,T)\) spaces which are required.

3 Applications to some nuclei in the \(1s - 0d\) shell

The spectral distribution results are obtained in configuration spaces with fixed number of valence particles and isospin. For that calculations are first carried out in proton-neutron configuration spaces where instead of the shell model space of \('l'\) orbits one considers \('2l'\) orbits with the first \('l'\) as proton orbits and the next \('l'\) orbits as neutron orbits. The single particle energies(spe) similarly have \('2l'\) values the first \('l'\) being proton single particle energies and the next \('l'\) as neutron single particle energies. When one has only isoscalar interaction the proton and neutron spe-s for a specific orbit are the same but once one includes T-nonconserving interactions they are no longer the same as can be seen from their expressions given in the last section. In the \(1s - 0d\) shell there are 6 orbits in this pn-formalism with 6 distinct spe-s. The two body matrix elements in the pn-formalism are given, instead of the \(V_{rstu}^{JT}\) where \((JT)\) stand for angular momentum and isospin and \('r'\),\('s'\),\('t'\) and \('u'\) for the orbits, in the form of \(V_{rstu;i}^{pp}\), \(V_{rstu;j}^{nn}\) and \(V_{rstu;j}^{pn}\) s given by the expressions in the last section. For the isoscalar part of the interaction in \(1s - 0d\) shell we use the highly successful Wildenthal’s mass dependent interaction. For the nonscalar one and two body parts we use the values of Ormand and Brown \[4\] as already mentioned because the shell model results done with them show good agreement with experiment.

We consider three examples in the \(1s - 0d\) shell nuclei and calculate the ground state energies by spectral distributions with the total Hamiltonian as well as with only the isoscalar part. From their difference one can estimate the parameter \('b'\) of IMME coming from the nuclear interactions neglecting the quadratic terms in the mass multiplet equation. We add to that the Coulomb contribution to \('b'\) taking values from figures 9 and 10 of \[1\] for all the 3 cases.

Table 1 and Table 2 give the values for the centroids and widths the isoscalar as well as the total Hamiltonian for nuclei with 4 and 5 valence particles in \(1s - 0d\) shell with isospin 1 and 3/2 respectively. They show all possible fixed \((m,T)\) configurations. All the results show a very
interesting aspect: once one includes the isospin violating parts in the interaction the centroid of each configuration move down by a few MeV whereas the width remains essentially the same. Thus the whole configuration energy state densities just shift in energy as the neutron number increases. This is mainly because the nonzero isovector single particle energies contribute to the centroids but has very little contribution in spreading the states in the spaces around the centroids. Table 3 shows our results for the ground state energies (GSE) by spectral distributions with and without the isospin violating parts and the value of the parameter ‘b’ calculated for the nuclear interactions. The GSE values are with respect to $^{16}O$ as the closed core. The Coulomb contributions are then added to these values. The results agree reasonably well with the observed values which are also included in Table 3. Keeping in mind that the spectral distribution theory is a statistical framework for the global properties of the nuclei the agreement is satisfactory. We also note that in earlier works to predict the ground state energies more accurately one considered corrections including low-lying excited states in the spectra as well as by including small non-zero skewness and excess for the averaged configuration density distributions [8] [9] [10], but here as the difference in energy of two states of nuclei with the same number of valence particles is involved those corrections are unimportant.

The calculations also point out to a major simplification of the problem. The changes in the values of the centroid almost fully come from the one body isovector part. This we have found to be true for all the cases considered and is due to the smallness of the overall multiplicative constants for the two body matrix elements of both the isovector and isotensor two body terms given by Ormand-Brown [4]. Thus neglecting the nonscalar two body parts will be a good approximation. Then one can easily write trace equivalent Hamiltonians for the total Hamiltonian by adding a one body term proportional to the vector isospin operator $T$ to the trace equivalent isoscalar Hamiltonians [16] to carry out other spectral distribution studies.

4 Conclusion

In this paper we show that spectral distribution theory can be applied to problems with Hamiltonians which have non-isoscalar parts also. We plan to work out in future other examples going beyond the $1s-0d$ shell and compare the spectral distribution results with the experimental ones. Also replacing the isovector and isotensor interactions by just a one body isovector part needs detailed future study.

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| Fixed T configuration | Dimension | Isoscalar H centroid (MeV) | Isoscalar H width (MeV) | Total H centroid (MeV) | Total H width (MeV) |
|-----------------------|-----------|-----------------------------|-------------------------|------------------------|-------------------|
| (400)                 | 105       | -23.91                      | 4.62                    | -27.31                 | 4.63              |
| (310)                 | 360       | -18.07                      | 4.46                    | -21.45                 | 4.47              |
| (301)                 | 180       | -21.53                      | 4.54                    | -24.91                 | 4.55              |
| (220)                 | 366       | -12.19                      | 4.50                    | -15.55                 | 4.51              |
| (211)                 | 408       | -15.97                      | 4.50                    | -19.33                 | 4.51              |
| (202)                 | 81        | -21.07                      | 4.33                    | -24.46                 | 4.34              |
| (130)                 | 144       | -5.87                       | 4.67                    | -9.19                  | 4.68              |
| (121)                 | 264       | -10.07                      | 4.66                    | -13.42                 | 4.67              |
| (112)                 | 120       | -15.41                      | 4.32                    | -18.77                 | 4.33              |
| (103)                 | 12        | -22.40                      | 3.80                    | -25.79                 | 3.81              |
| (040)                 | 15        | 0.23                        | 5.14                    | -3.10                  | 5.16              |
| (031)                 | 48        | -3.95                       | 5.01                    | -7.28                  | 5.02              |
| (022)                 | 34        | -9.86                       | 4.57                    | -13.21                 | 4.59              |
| (013)                 | 8         | -17.07                      | 3.73                    | -20.42                 | 3.74              |

Table 1: Centroids and widths of the isoscalar Hamiltonian compared to the centroids and widths of the total Hamiltonian (including the isovector and istensor parts) in all fixed-T configurations with 4 particles in $1s-0d$ shell with $T=1$.

| Fixed T configuration | Dimension | Isoscalar H centroid (MeV) | Isoscalar H width (MeV) | Total H centroid (MeV) | Total H width (MeV) |
|-----------------------|-----------|-----------------------------|-------------------------|------------------------|-------------------|
| (500)                 | 84        | -31.22                      | 5.30                    | -36.33                 | 5.31              |
| (410)                 | 480       | -25.03                      | 4.91                    | -30.10                 | 4.93              |
| (401)                 | 240       | -28.12                      | 5.33                    | -33.21                 | 5.35              |
| (320)                 | 740       | -19.06                      | 4.87                    | -24.11                 | 4.88              |
| (311)                 | 880       | -22.32                      | 5.11                    | -27.38                 | 5.12              |
| (302)                 | 150       | -27.33                      | 4.90                    | -32.40                 | 4.91              |
| (230)                 | 444       | -12.80                      | 4.99                    | -17.84                 | 5.00              |
| (221)                 | 910       | -16.47                      | 5.18                    | -21.51                 | 5.19              |
| (212)                 | 384       | -21.54                      | 4.78                    | -26.56                 | 4.79              |
| (203)                 | 30        | -27.92                      | 4.12                    | -33.01                 | 4.12              |
| (140)                 | 96        | -6.45                       | 5.29                    | -11.47                 | 5.30              |
| (131)                 | 336       | -10.31                      | 5.45                    | -15.34                 | 5.46              |
| (122)                 | 240       | -15.80                      | 4.94                    | -20.82                 | 4.95              |
| (113)                 | 48        | -22.29                      | 3.92                    | -27.34                 | 3.93              |
| (050)                 | 4         | -0.60                       | 6.23                    | -5.58                  | 6.25              |
| (041)                 | 32        | -4.33                       | 6.00                    | -9.32                  | 6.01              |
| (032)                 | 36        | -10.21                      | 5.29                    | -15.20                 | 5.31              |
| (023)                 | 12        | -17.06                      | 4.21                    | -22.06                 | 4.22              |

Table 2: Centroids and widths of the isoscalar Hamiltonian compared to the centroids and widths of the total Hamiltonian (including the isovector and istensor parts) in all fixed-T configurations with 5 particles in $1s-0d$ shell with $T=3/2$. 
Table 3: The parameter ‘b’ of IMME coming from nuclear interactions from evaluation of the ground state energies (GSE) calculated by spectral distributions. The Coulomb contribution is taken from ref [1].

| (m,T) | GSE for Iso-scalar H (MeV) | GSE for Total H (MeV) | b from Total H (MeV) | b for Coulomb (MeV) | Total | | | |
|-------|---------------------------|----------------------|---------------------|-------------------|-------|-------|-------|-------|-------|
| (4,1) | -33.83                    | -37.26               | -3.43               | -0.7              | 4.13  | 4.21  |       |       |
| (5,3/2)| -42.16                    | -47.30               | -3.42               | -0.9              | 4.32  | 4.44  |       |       |
| (6,1) | -62.88                    | -66.22               | -3.34               | -1.1              | 4.44  | 4.60  |       |       |

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