Lévy Laplacians, Holonomy Group and Instantons on 4-Manifolds

Boris O. Volkov

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Abstract
A connection between the Yang–Mills gauge fields on 4-dimensional orientable compact Riemannian manifolds and modified Lévy Laplacians is studied. A modified Lévy Laplacian is obtained from the Lévy Laplacian by the action of an infinite dimensional rotation. Under the assumption that the 4-manifold has a nontrivial restricted holonomy group of the bundle of self-dual 2-forms, the following is proved. There is a modified Lévy Laplacian such that a parallel transport in some vector bundle over the 4-manifold is a solution of the Laplace equation for this modified Lévy Laplacian if and only if the connection corresponding to the parallel transport satisfies the Yang–Mills anti-self-duality equations. An analogous connection between the Laplace equation for the Lévy Laplacian and the Yang–Mills equations was previously known.

Keywords Lévy Laplacian · Infinite-dimensional Laplacians · Yang–Mills equations · Instantons · Holonomy group

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1 Introduction

The Lévy Laplacian is an infinite-dimensional differential operator which has not a finite dimensional analog. The equivalence of the Yang–Mills equations and the Laplace equation for the Lévy Laplacian is known (see [1–3]). In this paper, we study a similar connection of a modification of this operator with instantons. Instantons play an important role in the clas-
sical and quantum theory of gauge fields (see [4] and the survey paper [5]). An exposition of the mathematical theory of instantons can be found in [6–9].

Let $A$ be a connection in a vector bundle over a Riemannian manifold $M$ and $F = dA + A \wedge A$ be the associated curvature. The Yang–Mills action functional has the form

$$S_{YM}(A) = \frac{1}{2} \int_M \|F(x)\|^2 Vol(dx).$$

The Euler–Lagrange equations for this action functional are the Yang–Mills equations:

$$D^*_A F = 0,$$

where $D^*_A$ is the adjoint operator to the exterior covariant derivative generated by $A$. Any connection $A$ generates the parallel transport $U^A$ which can be considered as a section in a vector bundle over the Hilbert manifold of $H^1$-curves with the fixed origin in $M$. If $f$ is a smooth section in this infinite-dimensional vector bundle then the value of the Lévy Laplacian $\Delta^{AGV}_L$ can be defined as

$$\Delta^{AGV}_L f = \text{tr}_{AGV} f''$$

where the Lévy trace $\text{tr}_{AGV}$ is a special linear integral functional on some space of infinite-dimensional bilinear forms. Also the Lévy Laplacian $\Delta^{AGV}_L$ can be defined as a Cesàro mean of the second-order directional derivatives (see [10–12]). In this paper, this approach is not used. The Lévy Laplacian $\Delta^{AGV}_L$ is a more complex analog of the classical Lévy Laplacian, which was defined by P. Lévy for the functions on $L_2([0, 1], \mathbb{R})$ (see [13]).

The Lévy Laplacian $\Delta^{AGV}_L$ is connected with the Yang–Mills gauge fields in the following way.

**Theorem A.** The following two assertions are equivalent:

1. the connection $A$ is a solution of the Yang–Mills equations;
2. the parallel transport $U^A$ is a solution of the Laplace equation $\Delta^{AGV}_L U^A = 0$.

Theorem A was proved for $M = \mathbb{R}^d$ by Accardi, Gibilisco and Volkov in [2] and generalized for the base Riemannian manifold by Leandre and Volkov in [3].

If $M$ is a oriented 4-manifold, it is possible to consider the Yang–Mills self-duality equations

$$F = *F$$

or the Yang–Mills anti-self-duality equations

$$F = -*F$$

on a connection $A$, where $*$ is the Hodge star on the manifold $M$. A connection is called an instanton or an anti-instanton if it is a solution of equations (3) or (2) respectively. The Bianchi identities $D_AF = 0$ imply that instantons and anti-instantons are solutions of the Yang–Mills equations (1).

It is a natural question whether the Lévy Laplacian and instantons are related (see [14]). In [15] by author, the family of the modified Lévy Laplacians was introduced which is connected with instantons and anti-instantons. Unlike the usual Laplacian, the Lévy Laplacian $\Delta^{AGV}_L$ is not rotation invariant and any smooth curve $W \in C^1([0, 1], SO(4))$ defines a modified Lévy Laplacian $\Delta^W_L$ which acts on a smooth section $f$ in the infinite dimensional bundle by the formula:

$$\Delta^W_L f = \text{tr}_{AGV} (W^* f'' W).$$

The Lie group $SO(p)$ is not simple only if $p = 4$. In this case, where two normal subgroups $S^3_L$ and $S^3_R$ of $SO(4)$, the Lie algebra $so(4)$ can be decomposed as a direct sum.
of Lie algebras

\[ so(4) = \text{Lie}(S^3_L) \oplus \text{Lie}(S^3_R), \]

where

\[ \text{Lie}(S^3_L) \cong \text{Lie}(S^3_R) \cong so(3). \]

This decomposition corresponds to the decomposition of the bundle of 2-forms into the direct sum of the sub-bundles of self-dual and anti-self-dual 2-forms.

The following theorem was proved for instantons over \( \mathbb{R}^4 \) in [15].

**Theorem B.** Let \( W \in C^1([0, 1], S^3_L) \) (\( W \in C^1([0, 1], S^3_R) \)) and

\[ \dim \text{span}\{ W^{-1}(t)\dot{W}(t) \}_{t \in [0,1]} \geq 2. \]

Let the value of the Yang–Mills action functional \( S_{YM} \) be finite on a connection \( A \). The following two assertions are equivalent:

1. the connection \( A \) on \( \mathbb{R}^4 \) is an instanton (anti-instanton);
2. the parallel transport \( U^A \) is a solution of the Laplace equation for the modified Lévy Laplacian \( \Delta^W_L \):

\[ \Delta^W_L U^A = 0. \]

Theorem B means that under some technical assumptions the anti-self-duality equations on \( \mathbb{R}^4 \) are equal to the Laplace equation for the modified Lévy Laplacian. In the proof of Theorem B, the fact that \( \mathbb{R}^4 \) is not compact was essentially used. In [16] by author, the following theorem was proved for instantons on an orientable Riemannian 4-manifold \( M \).

**Theorem C.** Let \( \{ e_1, e_2, e_3 \} \) be a basis of the Lie algebra \( \text{Lie}(S^3_L) \) (Lie algebra \( \text{Lie}(S^3_R) \)). Let \( W_i(t) = e^{it} e_i \) for \( i \in \{ 1, 2, 3 \} \). The following two assertions are equivalent:

1. the connection \( A \) on \( M \) is an instanton (anti-instanton);
2. the parallel transport \( U^A \) is a solution of three Laplace equations for the modified Lévy Laplacians \( \Delta^W_L U^A = 0 \) for \( i \in \{ 1, 2, 3 \} \).

Theorem C means that under some technical assumptions the anti-self-duality Yang–Mills equations (which are the system of the three nonlinear differential equations of the first order) are equal to the system of the three Laplace equations for the different modified Lévy Laplacians.

It turns out that, in the case of a Riemannian manifold, the situation is influenced by the holonomy group. In this paper, we strengthen the results of [16]. We prove that an analog of Theorem B holds for orientable compact Riemannian manifolds with nontrivial restricted holonomy group \( \text{Hol}_m^0(\Lambda^2_+(T^*M)) \) of the bundle of self-dual 2-forms. (Note that anti-self-dual Yang–Mills equations are conformally invariant and, in the case of the trivial restricted holonomy group \( \text{Hol}_m^0(\Lambda^2_+(T^*M)) \), it is possible to consider a conformally equivalent metric with a nontrivial holonomy group.) We find the sufficient conditions on the curve \( W \in C^1([0, 1], S^3_L) \) such that the following two assertions are equivalent:

1. the connection \( A \) is an instanton;
2. the parallel transport \( U^A \) is a solution of the Laplace equation for the Lévy Laplacian \( \Delta^W_L U^A = 0 \).

These conditions are different for the cases when the holonomy group \( \text{Hol}_m^0(\Lambda^2_+(T^*M)) \) coincides with \( SO(2) \) or \( SO(3) \).

It is known from the works of Atiyah and Donaldson (see [17, 18]) about a one-to-one correspondence between the moduli space of instantons on \( \mathbb{R}^4 \) and the space of based holomorphic maps from \( \mathbb{C}P^1 \) to the loop space. (In [19], a similar correspondence between the
moduli space of Yang–Mills fields and the space of based harmonic maps from \( CP^1 \) to the loop space was conjectured.) It means that there is a connection between the Yang–Mills fields and chiral fields with finite dimensional base manifolds and infinite dimensional target spaces. On the contrary, a parallel transport generated by the Yang–Mills field can be considered as a chiral field with an infinite-dimensional base manifold and a finite-dimensional target space (see [20–22]). In [23], the equations of the motion of the chiral fields on the parallel transport with the divergence associated with the Lévy Laplacian were considered (see also [11, 12]). The approach to the Yang–Mills fields based on the Lévy Laplacian goes back to the paper [23]. Different approaches to the Yang–Mills fields based on the parallel transport but not based on the Lévy Laplacian were used in [24–29]. For a recent development in the study of the Lévy Laplacian in the white noise theory, see [30, 31].

The paper is organized as follows. In Section 2, we give preliminary information about the Yang–Mills equations and instantons on 4-dimensional orientable Riemannian manifolds. In Section 3, we give preliminary information about the parallel transport and the holonomy group of the bundle of self-dual 2-forms. In Section 4, we give the scheme of the definition of the second order differential operators on the space of sections in an infinite dimensional vector bundle. Using this scheme, we define the family of the modified Lévy Laplacians parameterized by the choice of a curve in the group \( SO(4) \). We give the value of the modified Lévy Laplacian on the parallel transport. In Section 5, we formulate and prove the main theorem on the equivalence of the anti-self-duality Yang–Mills equations on a manifold with a nontrivial holonomy group of the bundle of self-dual 2-forms and the Laplace equation for some modified Lévy Laplacian.

## 2 Instantons on 4-Manifold

In this section, we give geometric preliminaries about Yang–Mills connections and instantons. For more information see, for example, [9, 32].

Let \( M \) be a smooth orientable compact 4-dimensional Riemannian manifold with a metric \( g \). We will use the Einstein’s notation for summation over the repeated indices and will raise and lower indices using this metric \( g \). Let \( G \) be a closed Lie group realized as a subgroup of \( SO(N) \). The symbol \( g \) denotes the Lie algebra of \( G \) endowed by the trace product:

\[
(B_1, B_2)_g = -\text{tr}(B_1 B_2), \quad B_1, B_2 \in g.
\]

Let \( E = E(\mathbb{R}^N, \pi, M, G) \) be a vector bundle over \( M \) with the projection \( \pi : E \to M \) and the structure group \( G \). The fiber over \( x \in M \) is \( E_x = \pi^{-1}(x) \cong \mathbb{R}^N \). Let \( P \) be the principle bundle over \( M \) associated with \( E \) and \( \text{ad}(P) = g \times_G M \) be the adjoint bundle of \( P \). A connection \( A(x) = A_\mu(x)dx^\mu \) in the vector bundle \( E \) is a smooth section in \( \Lambda^1(T^*M) \otimes \text{ad}P \). Let \( \nabla \) denotes the covariant derivative generalized by this connection. If \( \phi \) is a smooth section in \( \text{ad}P \) then in local coordinates we have

\[
\nabla_\mu \phi = \partial_\mu \phi + [A_\mu, \phi].
\]

The curvature \( F(x) = \sum_{\mu \leq \nu} F_{\mu \nu}(x)dx^\mu \wedge dx^\nu \) of the connection \( A \) is a smooth section in \( \Lambda^2(T^*M) \otimes \text{ad}P \) defined by the formula \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \).

Let

\[
D_A : C^\infty(M, \Lambda^P(T^*M) \otimes \text{ad}P) \to C^\infty(M, \Lambda^{P+1}(T^*M) \otimes \text{ad}P)
\]

\( \square \) Springer
be the operator of the exterior covariant derivative. It is defined by its action on forms $\alpha \otimes \phi$, where $\alpha$ is a real $p$-form and $\phi$ is a section in $\text{ad}P$, by

$$D_A(\alpha \otimes \phi) = d\alpha \otimes \phi + (-1)^p \alpha \otimes \nabla \phi,$$

where $d$ denotes the operator of the usual exterior derivative. Let

$$D_A^* : C^\infty(M, \Lambda^{p+1}(T^*M) \otimes \text{ad}P) \to C^\infty(M, \Lambda^p(T^*M) \otimes \text{ad}P)$$

be a formally adjoint to the operator $D_A$. We have $D_A^* = -* D_A^*$, where $*$ is the Hodge star on the manifold $M$.

The Yang–Mills action functional has the form

$$S_{YM}(A) = \frac{1}{2} \int_M \|F(x)\|^2 \text{Vol}(dx), \quad (4)$$

where $\text{Vol}$ is the Riemannian volume measure on the manifold $M$ and $\| \cdot \|$ is a norm generated by the trace product on the Lie algebra $\mathfrak{g}$. The Yang–Mills equations on a connection $A$ have the form

$$(D_A^* F) = 0. \quad (5)$$

In local coordinates, we have

$$(D_A^* F)_\nu = -\nabla^\mu F_{\mu\nu}$$

and

$$\nabla\lambda F_{\mu\nu} = \partial\lambda F_{\mu\nu} + [A_{\lambda}, F_{\mu\nu}] - F_{\mu\kappa} \Gamma^\kappa_{\lambda\nu} - F_{\nu\kappa} \Gamma^\kappa_{\lambda\mu},$$

where $\Gamma^\kappa_{\lambda\nu}$ are the Christoffel symbols of the Lévy-Civita connection on $M$. Solutions of the Yang–Mills equations are critical points of the Yang–Mills action functional (4). These critical points are called Yang–Mills connections.

Let $F_- = \frac{1}{2}(F - *)F$ and $F_+ = \frac{1}{2}(F + *)F$ be the anti-self-dual and self-dual parts of the curvature $F$ respectively. (We have $*F_- = -F_-$ and $*F_+ = F_+$. A connection $A$ is called anti-instanton (instanton) if it is a solution of the self-duality (anti-self-duality) Yang–Mills equations:

$$F_- = 0 \ (F_+ = 0). \quad (6)$$

The Yang–Mills action functional can be rewritten in the form

$$S_{YM}(A) = \frac{1}{2} \int_M (\|F_-(x)\|^2 + \|F_+(x)\|^2) \text{Vol}(dx).$$

The Pontryagin number or topological charge

$$p_1(E) = \frac{1}{8\pi^2} \int_M (\|F_+(x)\|^2 - \|F_-(x)\|^2) \text{Vol}(dx)$$

is an integer topological invariant of the vector bundle. The inequality

$$S_{YM}(A) \geq 4\pi^2 |p_1(E)|$$

imply that the instantons and the anti-instantons, if they are exist, are local extrema of this functional.
The gauge transform is a smooth section in $\text{Aut}P$. Such a section $\psi$ acts on the connection by the formula

$$A \rightarrow A' = \psi^{-1}A\psi + \psi^{-1}d\psi$$

(7)

and on the curvature by the formula

$$F \rightarrow F' = \psi^{-1}F\psi.$$  

(8)

The Lagrange function of (4), the Yang–Mills equations (5), the self-duality equations and the anti-self-duality equations are invariant under the action of gauge transform.

The Weyl transformation (conformal transformation) of the metric $g \mapsto g'$ is defined by the formula $g'_{\mu\nu} = e^{2\varphi}g_{\mu\nu}$, where $\varphi$ is smooth function on $M$. In the dimension four, the Yang–Mills functional (4), instantons and anti-instantons are invariant under the Weyl transformation.

Let $W_+$ and $W_-$ be the self-dual and anti-self-dual parts of the Weyl tensor (the conformally invariant part of the Riemann curvature tensor) respectively. An oriented Riemannian 4-manifold is called self-dual or anti-self-dual if $W_- = 0$ or $W_+ = 0$ respectively. It was proved in [6] by Atiyah, Hitchin and Singer, that in the case of the self-dual base manifold, the moduli space of instantons (the factor space of all instantons with the respect to the gauge equivalence) is a non-empty finite dimensional manifold.

Examples of self-dual manifolds (see [6]):

- Conformally flat manifolds. For example, 4-sphere $S^4$ or $S^1 \times S^3$.
- The complex projective plane $\mathbb{C}P^2$ is self-dual.
- The 4-dimensional Calabi–Yau manifold is anti-self-dual with respect to it canonical orientation.

If the intersection form on the manifold is indefinite then there are not instantons and anti-instantons on this manifold. The result by Atiyah, Hitchin and Singer was generalized for manifolds with the positive intersection form by Taubes in [7] (see also [32]). The moduli space of instantons on $\mathbb{R}^4$ was described by Atiyah, Drinfeld, Hitchin and Manin in [8]. In this paper, we show that theorem on equivalence of the anti-self-duality equations and the Laplace equation for the modified Lévy Laplacian is true for anti-self-dual compact manifolds which are not Calabi–Yau.

## 3 Parallel Transport and Holonomy Group

In this section, we give geometric preliminaries about the parallel transport and the holonomy group.

### 3.1 Parallel Transport

For any interval $I$ let the symbol $H^1(I, \mathbb{R}^4)$ denote the Sobolev space of $\mathbb{R}^4$-valued functions on $I$. It is the Hilbert space with scalar product

$$(h_1, h_2)_1 = \int_I (h_1(t), h_2(t))_{\mathbb{R}^4} dt + \int_I (\dot{h}_1(t), \dot{h}_2(t))_{\mathbb{R}^4} dt.$$

Let $\gamma$ be the mapping from $[0, 1]$ to $M$. It is called $H^1$-curve if $\phi_{\alpha} \circ \gamma |_I \in H^1(I, \mathbb{R}^4)$ for any sub-interval $I$, such that $\gamma(I) \subset W_\alpha$. Here $(\phi_{\alpha}, W_\alpha)$ is some coordinate chart of the manifold $M$. Let $\Omega_m$ denote the set of the $H^1$-curves with the origin at $m \in M$. This set can
be endowed with the natural structure of a Hilbert manifold modeled on the Hilbert space $H^1_0([0, 1], \mathbb{R}^d) = \{ \gamma \in H^1([0, 1], \mathbb{R}^d) : \gamma(0) = 0 \}$ (see [25, 33, 34]).

For any $H^1$-curve $\gamma$ an operator $U_{t,s}^A(\gamma) \in \text{Hom}(E_{\gamma(s)}, E_{\gamma(t)})$, where $0 \leq s \leq t \leq 1$, is a solution of the system

$$
\begin{align*}
\frac{d}{dt} U_{t,s}^A(\gamma) &= -A_\mu(\gamma(t)) \gamma^\mu(t) U_{t,s}^A(\gamma), \\
\frac{d}{ds} U_{t,s}^A(\gamma) &= U_{t,s}^A(\gamma) A_\mu(\gamma(s)) \gamma^\mu(s), \\
U_{t,s}^A(\gamma) \big|_{s=t} &= I_N.
\end{align*}
$$

(9)

The operator $U_{1,0}^A \in \text{Hom}(E_{\gamma(0)}, E_{\gamma(1)})$ is a parallel transport along the curve $\gamma$ generated by the connection $A$. Let $\mathcal{E}_m$ be a Hilbert vector bundle over $\Omega_m$ such that its fiber over $\gamma \in \Omega_m$ is the space $\text{Hom}(E_m, E_{\gamma(t)})$. The mapping $\Omega_m : \gamma \mapsto U_{1,0}^A(\gamma)$ is a smooth section in this vector bundle (see [24, 25]). Let $Q_{t,s}(\gamma)$ be a parallel transport with respect to the Levi-Civita connection in the tangent bundle $TM$ along the restriction of the $H^1$-curve $\gamma$ on $[s, t]$.

### 3.2 Holonomy Group

For any $x \in M$ let $\Omega_{m,x} = \{ \gamma \in \Omega_m : \gamma(1) = x \}$. Then $\Omega_{m,m}$ is the space of $H^1$-loops based at $m$. Parallel transports with respect to the Levi-Civita connection along loops based at $m$ generate the holonomy group:

$$
\text{Hol}_m(M) = \{ Q_{1,0}(\gamma) : \gamma \in \Omega_{m,m} \}.
$$

The restricted holonomy group $\text{Hol}_m^0(M)$ based at $m$ is the subgroup of $\text{Hol}_m(M)$ generated by the parallel transports along contractible loops $\gamma \in \Omega_{m,m}$. Let $\text{Hol}_m^0(M)$ denote the restricted holonomy group at $x \in M$ ($\text{Hol}_m^0(M) \cong \text{Hol}_m^0(M)$). Due to $M$ is orientable and compact, $\text{Hol}_m^0(M)$ is a connected closed Lie subgroup of $SO(4)$.

The Levi-Civita connection induces the Riemannian connection on the vector bundle $\Lambda^2(TM) = \Lambda^2_+(TM) \oplus \Lambda^2_-(TM)$. The group $\text{Hol}_m^0(\Lambda^2(TM)) = \{ K \in \text{Aut}(\Lambda^2(T_xM)) : K = H \wedge H, H \in \text{Hol}_m^0(M) \}$ is the restricted holonomy group in $\Lambda^2(TM)$ at $x$. The spaces $\Lambda^2_+(T_xM)$ and $\Lambda^2_-(T_xM)$ are invariant under the action of $K \in \text{Hol}_m^0(\Lambda^2(TM))$.

So

$$
\text{Hol}_m^0(\Lambda^2(TM)) = \text{Hol}_m^0(\Lambda^2_+(TM)) \times \text{Hol}_m^0(\Lambda^2_-(TM)).
$$

The group $\text{Hol}_m^0(\Lambda^2_+(TM))$ is a closed connected Lie subgroup of $SO(3)$. There are three possible cases:

- $\text{Hol}_m^0(\Lambda^2_+(TM))$ is trivial. We can consider the following example. If $M$ is a simply connected manifold, then holonomy groups and restricted holonomy groups coincide. If a simply connected base manifold $M$ is irreducible and $\text{Hol}_m(M) \cong SU(2)$, then due to Berger’s classification theorem $M$ (see [35]) is a Calabi–Yau manifold. In this case, $\text{Hol}_m(\Lambda^2_+(TM))$ is trivial.

- $\text{Hol}_m^0(\Lambda^2_+(TM)) \cong SO(2)$. We can consider the following example. If a simply connected base manifold $M$ is irreducible and $\text{Hol}_m(M) \cong U(2)$, then due to Berger’s and Cartan’s classification theorems $M$ is Kahler. (If $M$ is symmetric, then $M = \mathbb{C}P^2$). In this case, then $\text{Hol}_m(\Lambda^2_+(TM)) \cong SO(2)$. 

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• \( \text{Hol}^0_m(\Lambda^2_+ (TM)) \cong SO(3) \). One can show that if \( \text{Hol}^0_m(M) \cong SO(4) \) or \( \text{Hol}^0_m(M) \cong SO(3) \), then \( \text{Hol}^0_m(\Lambda^2_+ (TM)) \cong SO(3) \). For example, due to Berger’s classification theorem in a general case of the simply connected orientable irreducible nonsymmetric manifold \( M \) we have \( \text{Hol}_m(M) \cong SO(4) \). If \( M = S^4 \) then also \( \text{Hol}_m(M) \cong SO(4) \). The case \( \text{Hol}_m(M) \cong SO(3) \) can be also realized if \( M \) is reducible (for example, if \( M = S^1 \times S^3 \)).

4 Lévy Laplacian

In this section, we provide the general scheme of the definition of the second order differential operator. We provide the definitions of the Lévy Laplacian and the modified Lévy Laplacians. We give the value of the modified Lévy Laplacian on the parallel transport.

4.1 Second Order Differential Operators

Fix an orthonormal basis \( \{e_1, e_2, e_3, e_4\} \) in the tangent space \( T_m M \). We identify

\[
H^1_0([0, 1], \mathbb{R}^4) \ni h(\cdot) = (h^\mu(\cdot)) \leftrightarrow e_\mu h^\mu(\cdot) \in H^1_0([0, 1], T_m M).
\]

Let \( \mathcal{H}^0_1 \) be the tangent bundle over \( \Omega_m \). Its fiber over \( \gamma \in \Omega_m \) is a Hilbert space \( H^1_\gamma(TM) \) of \( H^1 \)-fields \( (H^1 \)-sections in the pullback bundle \( \gamma^* TM \) \) \( X \) along \( \gamma \) such that \( X(0) = 0 \) (see [25, 33, 34]). The scalar product on this space is defined by the formula

\[
G_1(X, Y) = \int_0^1 g(X(t), Y(t))dt + \int_0^1 g(\nabla X(t), \nabla Y(t))dt,
\]

where

\[
\nabla X^\mu(t) = \hat{X}^\mu(t) + \Gamma^\mu_{\lambda\nu}(\gamma(t))X^\lambda(t)\gamma^\nu(t).
\]

For any \( \gamma \in \Omega_m \) the Lévy-Civita connection generates the canonical isometric isomorphism between \( H^1_0([0, 1], \mathbb{R}^4) \) and \( H^1_\gamma(TM) \), which action on \( h \in H^1_0([0, 1], \mathbb{R}^4) \) we will denote by \( \tilde{h} \). This isomorphism acts by the formula

\[
\tilde{h}(\gamma; t) = Q_{t, 0}(\gamma)h(t) = e_\mu(\gamma, t)h^\mu(t),
\]

where \( e_\mu(\gamma, \cdot) = Q_{0, 0}(\gamma)e_\mu \) is the parallel transport of \( e_\mu \) \( (\mu \in \{1, 2, 3, 4\}) \) along \( \gamma \in \Omega_m \) by the Levi-Civita connection. The space \( H^1_\gamma(TM) \) is the tangent space to the Hilbert manifold \( \Omega_m \) at \( \gamma \). Due to isomorphism (12), the tangent bundle \( \mathcal{H}^1_0 \) is trivial.

Let \( \mathcal{H}^1_{0, 0} \) denote the sub-bundle of \( \mathcal{H}^1_0 \) such that the fiber of \( \mathcal{H}^1_{0, 0} \) over \( \gamma \in \Omega_m \) is the space \( \{X \in H^1_\gamma(TM) : X(1) = 0\} \). By canonical isomorphism (12) its fiber is isomorphic to the Hilbert space \( H^1_{0, 0} := H^1_{0, 0}([0, 1], \mathbb{R}^4) = \{\gamma \in H^1_0([0, 1], \mathbb{R}^4) : \gamma(1) = 0\} \).

Let \( X \) be a smooth section in \( \mathcal{H}^1_{0, 0} \). If \( f \) is a smooth section in \( \mathcal{E}_m \), then the derivative \( d_X f \) of \( f \) along the field \( X \) is correctly defined. Due to triviality of the vector bundle \( \mathcal{H}^1_{0, 0} \), for any smooth section \( f \) in \( \mathcal{E}_m \) there exists the section \( \tilde{D} f \) in \( \mathcal{E}_m \otimes H^1_{0, 0} \cong \mathcal{H}^1_{0, 0} \) such that

\[
d_{\tilde{h}}\tilde{D} f(\gamma, h) = < \tilde{D} f(\gamma), h >
\]

for any \( h \in H^1_{0, 0} \). Let \( \mathcal{L}(H^1_{0, 0}, H^1_{0, 0}) \) denote the space of all continuous linear operators in \( H^1_{0, 0} \). There exists the section \( \tilde{D}^2 f \) in \( \mathcal{E}_m \otimes \mathcal{L}(H^1_{0, 0}, H^1_{0, 0}) \) such that

\[
< d_{\tilde{u}}\tilde{D}^2 f(\gamma), v > = < \tilde{D}^2 f(\gamma) u, v >
\]
for any \( u, v \in H^1_{0,0} \).

**Definition 1** Let \( S \) be a linear functional on \( \text{dom} \, S \subset \mathcal{L}(H^1_{0,0}, H^1_{0,0}) \). The domain of the second order differential operator \( D^{2,S} \) associated with \( S \) is the space of all smooth sections \( f \) in \( \mathcal{E}_m \) such that \( \tilde{D}^2 f(\gamma) \in \text{dom} \, S \) for all \( \gamma \in \Omega_m \). The second order differential operator \( D^{2,S} \) acts on \( f \) by the formula

\[
D^{2,S} f(\gamma) = S(\tilde{D}^2 f(\gamma)).
\]

**4.2 Modified Lévy Laplacians**

Let \( T^2_{AGV} \) be the space of all continuous bilinear real-valued functionals on \( H^1_{0,0} \times H^1_{0,0} \) that have the form

\[
Q(u, v) = \int_0^1 \int_0^1 Q^V(t, s) < u(t), v(s) > dt 
+ \int_0^1 Q^L(t) < u(t), v(t) > dt 
+ \frac{1}{2} \int_0^1 Q^S(t) < \dot{u}(t), u(t) > dt + \frac{1}{2} \int_0^1 Q^S(t) < \dot{v}(t), v(t) > dt, \quad u, v \in H^1_{0,0},
\]

where \( Q^V \in L^2([0, 1] \times [0, 1], T^2(\mathbb{R}^4)), \; Q^L \in L^1([0, 1], \text{Sym}^2(\mathbb{R}^4)), \; Q^S \in L^\infty([0, 1], \Lambda^2(\mathbb{R}^4)) \). Here \( T^2(\mathbb{R}^4), \text{Sym}^2(\mathbb{R}^4) \) and \( \Lambda^2(\mathbb{R}^4) \) denote the spaces of all tensors, all symmetrical tensors and all antisymmetrical tensors of type \((0, 2) \) on \( \mathbb{R}^4 \) respectively. The functions \( Q^V(\cdot, \cdot), Q^L(\cdot) \) and \( Q^S(\cdot) \) are called the Volterra integral kernel, the Lévy integral kernel and the singular integral kernel respectively. It was proved in [2] that these kernels are defined in a unique way (see also [12]).

**Definition 2** The Lévy trace is a linear functional \( \text{tr}^{AGV}_L \) acting on \( Q \in T^2_{AGV} \) by

\[
\text{tr}^{AGV}_L Q = \int_0^1 \text{tr} \, Q^L(t) dt.
\]

The Lévy Laplacian \( \Delta^{AGV}_L \) is the second order differential operator \( D^{2,\text{tr}^{AGV}_L} \).

This operator was introduced by Accardi, Gibilisco and Volovich in [2] for the flat case and by Leandre and Volovich in [3] for the case of Riemannian manifold.

Let us introduce the modification of the Lévy trace that is connected with instantons. Let \( W \in C^1([0, 1], SO(4)) \). We can consider \( W \) as an orthogonal operator on \( L^2([0, 1], \mathbb{R}^4) \) acting on \( u \in L^2([0, 1], \mathbb{R}^4) \) by pointwise multiplication:

\[
(Wu)(t) = W(t)u(t).
\]

The space \( H^1_{0,0} \) is invariant under the action of \( W \). Then for any \( Q \in T^2_{AGV} \) a tensor \( W^*QW \in T^2_{AGV} \) is defined by

\[
W^*QW(u, v) = Q(Wu, Wv), \quad u, v \in H^1_{0,0}.
\]

**Definition 3** The modified Lévy trace associated with the curve \( W \in C^1([0, 1], SO(4)) \) is a linear functional \( \text{tr}^W_L \) acting on \( Q \in T^2_{AGV} \) by

\[
\text{tr}^W_L Q = \text{tr}^{AGV}_L (W^*QW).
\]
The modified Lévy Laplacian $\Delta^W_L$ associated with the curve $W \in C^1([0, 1], SO(4))$ is the second order differential operator $D^{2, \Omega^W_L}$.

Where two normal subgroups $S^3_L$ and $S^3_R$ of $SO(4)$, which consist of real matrices

\[
\begin{pmatrix}
a & -b & -c & -d \\
b & a & -d & c \\
c & d & a & -b \\
d & -c & b & a 
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
a & -b & -c & -d \\
b & a & d & -c \\
c & -d & a & b \\
d & c & -b & a 
\end{pmatrix},
\]

where $a^2 + b^2 + c^2 + d^2 = 1$, respectively. The Lie algebras $\text{Lie}(S^3_L)$ and $\text{Lie}(S^3_R)$ of the Lie groups $S^3_L$ and $S^3_R$ consist of matrices of the form

\[
\begin{pmatrix}
0 & -b & -c & -d \\
b & 0 & -d & c \\
c & d & 0 & -b \\
d & -c & b & 0 
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & -b & -c & -d \\
b & 0 & d & -c \\
c & -d & 0 & b \\
d & c & -b & 0 
\end{pmatrix},
\]

where $b, c, d \in \mathbb{R}$, respectively, So it holds that

$so(4) = \text{Lie}(S^3_L) \oplus \text{Lie}(S^3_R)$.

Let the symbols $P_L$ and $P_R$ denote the orthogonal projections in the Lie algebra $so(4)$ on its subalgebras $\text{Lie}(S^3_L)$ and $\text{Lie}(S^3_R)$ respectively. If $W \in C^1([0, 1], SO(4))$ let $L_W(t) = W^{-1}(t)\tilde{W}(t)$. Then $L_W \in C([0, 1], so(4))$. Let $L_W^+(t) = P_L(L_W(t))$ and $L_W^-(t) = P_R(L_W(t))$. Due to the fact that $Q_S(t)$ is anti symmetric, it can be also considered as an element from the algebra $so(4)$. Let $Q^+_S(t) = P_L(Q^S(t))$ and $Q^-_S(t) = P_R(Q^S(t))$.

It can be checked by direct computations that the following proposition holds (see [16]).

**Proposition 1** Let $W \in C^1([0, 1], SO(4))$. It holds

\[
\text{tr}^W_L Q = \int_0^1 \text{tr} Q^L(t) dt - \int_0^1 \text{tr}(L_W(t) Q^S(t)) dt = \\
= \int_0^1 \text{tr} Q^L(t) dt - \int_0^1 \text{tr}(L_W^+(t) Q^S_+(t)) dt - \int_0^1 \text{tr}(L_W^-(t) Q^S_-(t)) dt. \quad (14)
\]

**Example 1** Let $f \in C^\infty(M, \mathbb{R})$. Let $\mathcal{L}_f : \Omega_m \to \mathbb{R}$ be defined by:

\[
\mathcal{L}_f(\gamma) = \int_0^1 f(\gamma(t)) dt.
\]

The functional $\mathcal{L}_f$ belongs to the domain of the Lévy Laplacian $\Delta^AGV_L$ and the singular part of the second derivative of $\mathcal{L}_f$ vanishes. One can show that

\[
\Delta^W_L \mathcal{L}_f(\gamma) = \int_0^1 \Delta_{(M,g)} f(\gamma(t)) dt,
\]

for any $W \in C^1([0, 1], SO(4))$. Here $\Delta_{(M,g)}$ is the Laplace–Beltrami operator on the manifold $M$. 

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4.3 Parallel Transport and Modified Lévy Laplacian

The parallel transport $U_{1,0}^A$ belongs to the domain of the Lévy Laplacian (see [3, 16]). We have

$$<\tilde{D}^2 U_{1,0}^A(\gamma) u, v> = \int_0^1 \int_0^1 K^V (\gamma'; t, s) u(t), v(s) > dt ds + \int_0^1 K^L (\gamma; t) < u(t), v(t) > dt + \frac{1}{2} \int_0^1 K^S (\gamma; t) < \dot{u}(t), \dot{v}(t) > dt + \frac{1}{2} \int_0^1 K^S (\gamma; t) < \dot{v}(t), \dot{u}(t) > dt,$$

where the Lévy kernel $K^L$ and the singular kernel $K^S$ have the form

$$K_{\mu\nu}^L (\gamma; t) = \frac{1}{2} U_{1,1}^A(\gamma)(-\nabla_{e_{\mu}(\gamma, t)} F(\gamma(t)) < e_{\nu}(\gamma, t), \dot{\gamma}(t) > - \nabla_{e_{\nu}(\gamma, t)} F(\gamma(t)) < e_{\mu}(\gamma, t), \dot{\gamma}(t) > ) U_{1,1}^A(t, 0)(\gamma),$$

and

$$K_{\mu\nu}^S (\gamma; t) = U_{1,1}^A(\gamma) F(\gamma(t)) < e_{\mu}(\gamma, t), e_{\nu}(\gamma, t) > U_{1,1}^A(t, 0)(\gamma).$$

**Remark 1** The form of the Volterra kernel in (15) does not play a role in the present paper. The exact expression of this kernel can be found in [16].

The following theorem directly follows from Proposition 1.

**Theorem 1** The value of the modified Lévy Laplacian $\Delta_W^L$ on the parallel transport is

$$\Delta_W^L U_{1,0}^A(\gamma) = -\int_0^1 U_{1,1}^A(\gamma) D^*_A F(\gamma(t)) \dot{\gamma}(t) U_{1,0}^A(\gamma) dt + \int_0^1 U_{1,1}^A(\gamma) \text{tr}(L_W(t) F(\gamma(t)) U_{1,0}^A(\gamma) dt. \quad (16)$$

**Remark 2** Equality (16) can be rewritten in the form

$$\Delta_W^L U_{1,0}^A(\gamma) = -\int_0^1 U_{1,1}^A(\gamma) D^*_A F(\gamma(t)) \dot{\gamma}(t) U_{1,0}^A(\gamma) dt + \int_0^1 U_{1,1}^A(\gamma) \text{tr}(L_W^+(t) F_+(\gamma(t)) U_{1,0}^A(\gamma) dt + \int_0^1 U_{1,1}^A(\gamma) \text{tr}(L_W^-(t) F_-(\gamma(t)) U_{1,0}^A(\gamma) dt. \quad (17)$$

Now we can illustrate the relationship between the modified Lévy Laplacian and instantons. Let $F$ be an instanton. Then $F_+ = 0$ and $D^*_A F = 0$. Hence, the first and the second terms in the right side of (17) vanish. If additionally $W \in C^1([0, 1], S^3_L)$, then $L^+_W = 0$ for all $t \in [0, 1]$ and the right side of (17) is zero.
The first term in the right side of (16) is invariant under reparameterization of the curve \( \gamma \) but the second term is not. Therefore, the following lemma is true.

**Lemma 1** Let \( W \in C^1([0, 1], SO(4)) \). If the parallel transport \( U_{1,0}^A \) is a solution of the Laplace equation for the modified Lévy Laplacian \( \Delta_L^W \):
\[
\Delta_L^W U_{1,0}^A = 0,
\]
then the connection \( A \) satisfies the Yang–Mills equations and for any \( \gamma \in \Omega_m \) the following holds
\[
\int_0^1 U_{1,t}^A(\gamma) \text{tr}(L_w(t)F(\gamma(t))U_{1,0}^A(t))dt = 0.
\]

The complete proof of the lemma can be found in [16].

**Remark 3** In paper [36] by the author, the covariant definition of the Lévy Laplacian on manifolds was introduced. A natural question arises whether it is possible to introduce the covariant analog of the modified Lévy Laplacian that is invariant under Weyl transformation and is invariant under reparametrization of the curves.

## 5 Main Theorem

In this section we give the formulation and the proof of the main theorem.

We can assume that \( \{e_1, e_2, e_3, e_4\} \) is a right-handed basis in \( T_M \) without loss of generality. Let us introduce bivectors in \( \Lambda^2(T_mM) \):

\[
v_1^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4),
\]
\[
v_2^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \mp e_2 \wedge e_4),
\]
\[
v_3^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3).
\]

Let \( v_i^\pm(\gamma, \cdot) \) be the parallel transport of \( v_i^\pm (i \in \{1, 2, 3\}) \) along \( \gamma \in \Omega_m \) by the Riemannian connection. Then \( \{v_1^+(\gamma, t), v_2^+(\gamma, t), v_3^+(\gamma, t)\} \) and \( \{v_1^-(\gamma, t), v_2^-(\gamma, t), v_3^-(\gamma, t)\} \) are orthonormal bases in \( \Lambda^2_+(T_{\gamma(t)}M) \) and \( \Lambda^2_-(T_{\gamma(t)}M) \) respectively for any \( t \in [0, 1] \).

We identify an element from \( so(4) \) with the action on \( T_mM \). Let in the basis \( \{e_1, e_2, e_3, e_4\} \) the operators \( L_w^+(t) \) and \( L_w^-(t) \) have matrices

\[
L_w^+(t) = \begin{pmatrix}
0 & -\omega_{W_1}^+(t) & -\omega_{W_2}^+(t) & -\omega_{W_3}^+(t) \\
\omega_{W_1}^+(t) & 0 & -\omega_{W_3}^+(t) & -\omega_{W_2}^+(t) \\
\omega_{W_2}^+(t) & \omega_{W_3}^+(t) & 0 & -\omega_{W_1}^+(t) \\
\omega_{W_3}^+(t) & -\omega_{W_2}^+(t) & \omega_{W_1}^+(t) & 0
\end{pmatrix},
\]

and

\[
L_w^-(t) = \begin{pmatrix}
0 & -\omega_{W_1}^-(t) & -\omega_{W_2}^-(t) & -\omega_{W_3}^-(t) \\
\omega_{W_1}^-(t) & 0 & -\omega_{W_3}^-(t) & -\omega_{W_2}^-(t) \\
\omega_{W_2}^-(t) & \omega_{W_3}^-(t) & 0 & -\omega_{W_1}^-(t) \\
\omega_{W_3}^-(t) & -\omega_{W_2}^-(t) & \omega_{W_1}^-(t) & 0
\end{pmatrix},
\]

respectively.
Let for any $\gamma \in \Omega_m$ and $t, r \in [0, 1]$ bivectors $v^-_W(\gamma, t, r) \in \Lambda^2_-(T_{\gamma(t)}M)$ and $v^+_W(\gamma, t, r) \in \Lambda^2_+(T_{\gamma(t)}M)$ be defined by

$$v^\pm_W(\gamma, t, r) = \omega^\pm_W(r)v^\pm_1(\gamma, t) + \omega^\pm_W(r)v^\pm_2(r)(\gamma, t).$$

Let $\alpha^\pm_W = \int_0^1 \omega^\pm_W(r)dr$ ($i \in \{1, 2, 3\}$) and $w^\pm_W(\gamma, t) = \alpha^\pm_Wv^\pm_1(\gamma, t) + \alpha^\pm_Wv^\pm_2(r)(\gamma, t) + \alpha^\pm_Wv^\pm_3(r)(\gamma, t)$. It is easy to see that

$$w^\pm_W(\gamma, t) = \int_0^1 v^\pm_W(\gamma, t, r)dr.$$

We will use the following notations for all $\gamma \in \Omega_m$ and $t, r \in [0, 1]$:

$$L(\gamma, t) = U_{1,0}^A(\gamma) - F(\gamma(t))U_{1,0}^A(\gamma),$$
$$L^\pm(\gamma, t) = U_{1,0}^A(\gamma) - F^\pm(\gamma(t))U_{1,0}^A(\gamma),$$
$$L_W(\gamma, t, r) = \text{tr}(\hat{W}(r)W^{-1}(r)L(\gamma, t)).$$

It can be obtained by direct computations that

$$L_W(\gamma, t, r) = \text{tr}(\hat{W}(r)W^{-1}(r)L(\gamma, t)) = \text{tr}(L^+_W(r)L^-(\gamma, t)) + \text{tr}(L^-_W(r)L^+_W(\gamma, t)) = L^-(\gamma, t) < v^-_W(\gamma, t, r) > + L^+(\gamma, t) < v^+_W(\gamma, t, r) >.$$ 

Due to Theorem 1 we have

$$\Delta_L^W U_{1,0}^A(\gamma) = - \int_0^1 U_{1,0}^A(\gamma)D^*_A F(\gamma(t))\gamma(t)U_{1,0}^A(\gamma)dt + U_{1,0}^A(\gamma)\int_0^1 L_W(\gamma, t, t)dt.$$

Lemma 1 implies that if $\Delta_L^W U_{1,0}^A = 0$ then $\int_0^1 L_W(\gamma, t, t)dt$ for any $\gamma \in \Omega_m$.

**Lemma 2** Let $W \in C^1([0, 1], S^3_L)$. If the parallel transport $U_{1,0}^A$ is a solution of the Laplace equation for the Lévy Laplacian $\Delta_L^W \colon \Delta_L^W U_{1,0}^A = 0$, then for any $\gamma \in \Omega_m$ the following holds

$$F^+(\gamma(1)) < w^+_W(\gamma, 1) >= 0.$$ 

**Proof** If $W \in C^1([0, 1], S^3_L)$, then $L_W = 0$ and

$$L_W(\gamma, t, r) = L^+_W(\gamma, t) < v^+_W(\gamma, t, r) >.$$ 

For any $r \in (0, 1]$ let introduce $\gamma_r \in \Omega_m$ by

$$\gamma_r(t) = \begin{cases} \gamma(t/r), & \text{if } 0 \leq t \leq r, \\ \gamma(1), & \text{if } r < t \leq 1. \end{cases}$$

The parallel transport is invariant under reparameterization of the curve $\gamma$ (see [16]). Hence,

$$L_W(\gamma_r, t, t) = \begin{cases} L_W(\gamma, t/r, t), & \text{if } 0 \leq t \leq r, \\ L_W(\gamma, 1, t), & \text{if } r < t \leq 1. \end{cases}$$

Then

$$\int_0^1 L_W(\gamma_r, t, t)dt = \int_0^r L_W(\gamma, t/r, r)dt + \int_r^1 L^+(\gamma(1)) < v^+_W(\gamma, 1, t) > dt.$$ 

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There exists $C > 0$ such that

$$\sup_{r \in [0,1], t \in [0,1]} \|L^W(\gamma, t, r)\| < C.$$ 

Hence

$$\|\int_0^r L^W(\gamma, t/r, t)dt\| \leq Cr$$

and

$$\lim_{r \to 0^+} \int_0^r L^W(\gamma, t/r, t)dt = 0.$$ 

The last equality and (21) together imply

$$\lim_{r \to 0^+} \int_0^1 L^W(\gamma r, t, t) dt = \int_0^1 L^W(\gamma, t, t) dt < w^+_w(\gamma, 1) = 0.$$ 

If $\lambda^+_W U^A_{1,0} = 0$ then $\int_0^1 L^W(\gamma r, t, t) dt = 0$ for any $\gamma \in \Omega_m$ and $r \in (0, 1]$. Hence, it follows from (22) that

$$L_+(\gamma(1)) < w^+_W(\gamma, 1) = 0.$$ 

So we have $F_+(\gamma(1)) < w^+_W(\gamma, 1) = 0$ for any $\gamma \in \Omega_m$. 

In the case $\text{Hol}_m^0(\Lambda^2_+(T M)) \cong U(1)$, we can assume without loss of generality that $v_1^+$ is invariant under any action from this group. Let $V_1 = \text{span}\{v_1^+\}$ and $V_2 = \text{span}\{v_2^+, v_3^+\}$.

Then the spaces $V_1$ and $V_2$ are invariant under any action from $\text{Hol}_m^0(\Lambda^2_+(T M))$. For any $\gamma \in \Omega_m$ bivector $v_1^+(\gamma, 1)$ is invariant under any action from $\text{Hol}_m^0(\gamma(1), \Lambda^2_+(T M))$ and the orbit of $v_2^+(\gamma, 1)$ and $v_3^+(\gamma, 1)$ coincides with

$$\text{Orb}(v_2^+(\gamma, 1)) = \text{Orb}(v_3^+(\gamma, 1)) = \{v_2^+(\gamma, 1) \cos \theta + v_3^+(\gamma, 1) \sin \theta : \theta \in [0, 2\pi)\}.$$ 

Also we can canonically identify $\text{Lie}(S^3_L)$ and $\Lambda^2_+(T_m M)$ by

$$\begin{pmatrix} 0 & -b & -c & -d \\ b & 0 & -d & c \\ c & d & 0 & -b \\ d & -c & b & 0 \end{pmatrix} \leftrightarrow bv_1^+ + cv_2^+ + dv_3^+.$$ 

So we can define orthogonal projections $\text{pr}_{V_1}$ and $\text{pr}_{V_2}$ in $\text{Lie}(S^3_L)$ on the spaces $V_1$ and $V_2$ respectively.

Now we can formulate and prove the main theorem.

**Theorem 2** Let $M$ be an orientable compact Riemannian 4-manifold.

1. If $\text{Hol}_m^0(\Lambda^2_+(T M)) \cong SO(3)$ and $W \in C^1([0, 1], S^3_L)$ such that $\int_0^1 L_W(t) dt \neq 0$;
2. If $\text{Hol}_m^0(\Lambda^2_+(T M)) \cong U(1)$ and $W \in C^1([0, 1], S^3_L)$ such that $\text{pr}_{V_1}(\int_0^1 L_W(t) dt) \neq 0$ and $\text{pr}_{V_2}(\int_0^1 L_W(t) dt) \neq 0$

the following two assertions are equivalent:

1. a connection $A$ is a solution of anti-self-duality equations : $F = - \ast F$,
2. the parallel transport $U^A_{1,0}$ is a solution of the equation: $\Delta^W_L U^A_{1,0} = 0$. 

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Proof Let $\Delta^W_L U^A_{1,0} = 0$. Fix any $x \in M$. Fix an arbitrary $\sigma \in \Omega_{m,x}$. Then the set \{ $w^+_W(\gamma, 1) : \gamma \in \Omega_{m,x}$ \} coincides with the orbit $\operatorname{Orb}(w^+_W(\sigma, 1))$ of $w^+_W(\sigma, 1)$ under the action of the group $\operatorname{Hol}^0_0(\Lambda^2_+(TM))$. Then Lemma 2 implies that $F_+(x) < w > = 0$ for any $w \in \operatorname{Orb}(w^+_W(\sigma, 1))$. Now we can consider two cases.

1. Let $\operatorname{Hol}^0_0(\Lambda^2_+(TM)) \cong SO(3)$ and $\int_0^1 L_W(t)dt \neq 0$. The last condition is equal to $\sum_{i=1}^3 (\alpha^+_{W_i})^2 \neq 0$. In this case, $w^+_W(\sigma, 1) \neq 0$. Due to $\operatorname{Hol}^0_0(\Lambda^2_+(TM)) = SO(3)$, the orbit $\operatorname{Orb}(w^+_W(\sigma, 1))$ is a 2-sphere. There are three linearly independent vectors in $\operatorname{Orb}(w^+_W(\sigma, 1))$.

2. Let $\operatorname{Hol}^0_0(\Lambda^2_+(TM)) \cong U(1)$ and $\mathrm{pr}_V_1(\int_0^1 L_W(t)dt) \neq 0$ and $\mathrm{pr}_V_2(\int_0^1 L_W(t)dt) \neq 0$. In this case, the orbit of $w^+_W(\sigma, 1)$ has a form

$$\operatorname{Orb}(w^+_W(\sigma, 1)) = \{ \alpha^+_{W_1} v^+_{1}(\sigma, 1) + \alpha^+_{W_2} v^+_{2}(\sigma, 1) \cos \theta + \alpha^+_{W_3} v^+_{3}(\sigma, 1) \sin \theta : \theta \in [0, 2\pi) \}.$$ 

The conditions $\mathrm{pr}_V_1(\int_0^1 L_W(t)dt) \neq 0$ and $\mathrm{pr}_V_2(\int_0^1 L_W(t)dt) \neq 0$ mean that $\alpha^+_{W_1} \neq 0$ and $(\alpha^+_{W_2})^2 + (\alpha^+_{W_3})^2 \neq 0$. In this case, there are three linearly independent vectors in $\operatorname{Orb}(w^+_W(\sigma, 1))$.

In the both cases, the equality $F_+(x) < w > = 0$ for any $w \in \operatorname{Orb}(w^+_W(\sigma, 1))$ imply $F_+(x) = 0$. Hence, $A$ is an anti-self-dual connection. The other side of the theorem is trivial.

Remark 4 Let $W \in C^1([0, 1], S^3_R)$. If we change the orientation of the manifold $M$ in Theorem 2 we obtain the similar connection between the Yang–Mills self-duality equations and the Laplace equation for the modified Lévy Laplacian $\Delta^W_L$.

Remark 5 If the simply connected base manifold $(M, g)$ is a Calabi–Yau manifold then Theorem 2 does not hold. As it was mentioned in the introduction there exists the Weyl transformation $g \mapsto \overline{g}$ such that for $(M, \overline{g})$ Theorem 2 holds.

6 Conclusion

In this paper, we have shown that the Yang–Mills self-duality and anti-self-duality equations on a connection in the vector bundle over the Riemannian 4-manifold can be reformulated as a linear differential equation on the parallel transport generated by the connection. The Lévy Laplacian is an infinite dimensional differential operator on the space of sections in the vector bundle over the space of $H^1$-curves in the Riemannian manifold. It can be defined as an integral functional generated by the special form of the second order derivative. The equivalence of the Laplace equation for the Lévy Laplacian and the Yang–Mills equations was previously known (see [2, 3]). The Lévy Laplacian is not rotation invariant and the infinite dimensional rotation $W \in C^1([0, 1], SO(4))$ generates the modified Lévy Laplacian $\Delta^W_L$. The group of 4-dimensional rotations $SO(4)$ has two normal subgroups $S^3_L$ and $S^3_R$. In the previous works [15, 16], a connection between Laplace equations for the modified Lévy Laplacians $\Delta^W_L$ for $W \in C^1([0, 1], S^3_L)$ (for $W \in C^1([0, 1], S^3_R)$) and the
Yang–Mills anti-self-duality (self-duality) equations was discovered. In this paper, we have strengthened the results of these works. Under the assumption that the restricted holonomy group of the bundle of self-dual 2-forms of the 4-manifold is nontrivial, we have found the sufficient conditions on the rotation \( W \in C^1([0, 1], S^3) \) such that the following holds. A connection is an instanton (antiinstanton) if and only if the parallel transport is a solution to the Laplace equation for the modified Lévy Laplacian \( \Delta^W_L \).

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