Molecular topological invariants of certain chemical networks

Abstract: Topological descriptors are the graph invariants that are used to explore the molecular topology of the molecular/chemical graphs. In QSAR/QSPR research, physico-chemical characteristics and topological invariants including Randić, atom-bond connectivity, and geometric arithmetic invariants are utilized to correlate and estimate the structure relationship and bioactivity of certain chemical compounds. Graph theory and discrete mathematics have discovered an impressive utilization in the area of research. In this article, we investigate the valency-depended invariants for certain chemical networks like generalized Aztec diamonds and tetrahedral diamond lattice. Moreover, the exact values of invariants for these categories of chemical networks are derived.

Keywords: atom-bond connectivit index, geometric-arithmetic index, generalized Aztec diamond, tetrahedral diamond lattice

1 Introduction

There are many diverse applications of mathematics in electronic and electrical engineering. It relies upon what specific region of electronic and electrical designing you are interested, for instance, there is a mass
\( N_g(\alpha) = \{ \beta \in V(B) : \alpha \beta \in E(B) \} \). All the terminologies used in this manuscript are acquired from books (Diudea et al. 2001, Gutman and Polansky, 1986).

Estrada et al. (1998) gave the concept of renowned degree related topological invariant atom-bond connectivit and described as:

\[
ABC(B) = \sum_{\alpha \beta \in E(B)} \sqrt{d(\alpha) + d(\beta) - 2} \frac{d(\alpha) d(\beta)}{d(\alpha) + d(\beta)}
\]  

(1)

The other notable degree dependend invariant is geometric-arithmetic index established by Vukičević et al. (2009) and interpreted as:

\[
GA(B) = \sum_{\alpha \beta \in E(B)} 2 \sqrt{d(\alpha) d(\beta)} \frac{d(\alpha) + d(\beta)}{d(\alpha) + d(\beta)}
\]  

(2)

In both these indices, first we find the possible degree of all the vertices and then partitined the edges of \( B \) depending upon the degree of every end vertex adjacent to edge.

Later, Ghorbani and Hosseinzadeh (2010) presented the 4th kind of \( ABC \) invariant by generalizing the idea of \( ABC \) index. It is interpreted as:

\[
ABC_4(B) = \sum_{\alpha \beta \in E(B)} \sqrt{S_\alpha + S_\beta - 2} \frac{S_\alpha S_\beta}{S_\alpha + S_\beta}
\]

(3)

Graovac et al. (2011) generalized the geometric index by defining the fifth kind of \( GA \) index in a same way as \( ABC_4 \) index. The index is written as:

\[
GA_5(B) = \sum_{\alpha \beta \in E(B)} 2 \sqrt{S_\alpha S_\beta} \frac{S_\alpha + S_\beta}{S_\alpha + S_\beta}
\]

(4)

In both these indices, first we find the possible degree of all vertices and then partitined the edges of \( B \) depending upon the degree of vertices adjacent to every end vertex of each edge.

The first Zagreb index \( M_1 \) and the second Zagreb index \( M_2 \) (Reti et al., 2019) for a graph \( G \) can be defined as:

\[
M_1(G) = \sum_{v \in V(G)} (d(v))^2 = \sum_{uv \in E(G)} (d(u) + d(v)),
\]

\[
M_2(G) = \sum_{uv \in E(G)} d(u)d(v)
\]

The neighborhood first and second Zegreb indices (first defined by Reti et al. (2019)) are as follows:

\[
NM_1(G) = \sum_{v \in V(G)} (S_v)^2,
\]

\[
NM_2(G) = \sum_{uv \in E(G)} S_u S_v
\]

For more discussion about invariants, see: Ahmad et al. (2017), Alikhani et al. (2014), Akhter and Imran (2016a, 2016b), Akhter et al. (2018, 2019), Bača et al. (2015), Baig et al. (2015a, 2015b), Guirao et al. (2020), Hameed et al. (2020), Hayat and Imran (2014, 2015a, 2015b, 2015c), Imran et al. (2020), Iranmanesh and Zeraatkar (2010), Lin et al. (2014), Manzoor (2015), and Yang et al. (2019).

In this article, we derive the certain degree related molecular topological invariants for chemical networks like generalized Aztec diamonds, tetrahedral diamond lattice, and certain infinite classes of nanostar dendrimers. We compute the analytical formulas for above families of chemically applicable networks.

2 ABC, GA, ABC_4, GA_5, NM_1, and NM_2 indices of generalized Aztec diamonds

In this part of the article, we discuss the degree-based topological descriptors for the generalized Aztec diamonds.

For any two graphs \( W \) and \( F \), the tensor product of the graphs \( W \) and \( F \) is interpreted by \( W \otimes F \). The vertex set of \( W \otimes F \) is \( V(W) \times V(F) \) and \( E(W \otimes F) = \{ (w, f)(s, e) : ws \in E(W) \) and \( fe \in E(F) \} \).

The tensor product of two paths \( L_p \) and \( L_q \) is described by \( L_p \otimes L_q \). It is a graph on \( p \times q \) vertices with vertex set is defined as:

\[
\{(x_1, y_1) : 1 \leq x_1 \leq p, 1 \leq y_1 \leq q\}
\]

and an edge between the \( (x_1, y_1) \) and \( (x_2, y_2) \) exists if and only if:

\[
|x_1 - x_2| + |y_1 - y_2| = 1.
\]

The graph \( L_p \otimes L_q \) is known as a generalized Aztec diamond with vertex cardinality \( pq \). It is depicted in Figure 1. In the next theorem, the \( ABC \) index for generalized Aztec diamond graphs has been computed.
Theorem 2.1

The $ABC$ index of generalized Aztec diamond graph $G = L_p \otimes L_q$ for $p, q \geq 2$, is:

$$ABC(L_p \otimes L_q) = \frac{\sqrt{6}}{2}pq + \left(1 - \frac{3\sqrt{3}}{4}\right)2\sqrt{2}p + \left(1 - \frac{3\sqrt{3}}{4}\right)2\sqrt{2}q + 2\sqrt{q} + 2\left(\sqrt{3} + \sqrt{2} - 6\sqrt{2} + \frac{9\sqrt{6}}{4}\right).$$

Proof: Firstly, we identify that the graph $G$ has vertices having degree one, two, and four. Thus, the graph $G$ has only edges with end vertices having degree one, two, or four. So possible edges are of type (1,4), (2,2), (2,4), and (4,4), where by edge of type $(d(m), d(n))$, we mean end vertices of the edge with degrees $m$ and $n$. In Table 1, all the edges of type (1,4), (2,2), (2,4), and (4,4) are counted. Now, by applying the information shown in Table 1, the $ABC$ index of $L_p \otimes L_q$ is derived as:

$$ABC(G) = \sum_{\alpha \beta \in E(G)} \frac{d(\alpha) + d(\beta) - 2}{d(\alpha)d(\beta)}.$$

$$ABC(L_p \otimes L_q) = 4\sqrt{1 + 4 - \frac{2 + 2 - 2}{2 \times 2}} + 4\sqrt{2 + 4 - \frac{2 + 4 - 2}{2 \times 4}} + 4(p + q - 6)\sqrt{4 + 4 - \frac{2 + 4 - 2}{2 \times 4}} + 2(p - 3)(q - 3).$$

Table 1: The cardinality of edges of type $(d(m), d(n))$ of $L_p \otimes L_q$, where $m$ and $n$ are degrees of adjacent vertices of every edge.

| $(d(m), d(n))$ with $mn \in E(G)$ | Quantity of edges |
|-----------------------------------|------------------|
| (1,4)                            | 4                |
| (2,2)                            | 4                |
| (2,4)                            | $4(p + q - 6)$   |
| (4,4)                            | $2(p - 3)(q - 3)$|

After simplification of above calculations, we acquire the following result of $ABC$ index:

$$ABC(L_p \otimes L_q) = \frac{\sqrt{6}}{2}pq + \left(1 - \frac{3\sqrt{3}}{4}\right)2\sqrt{2}p + \left(1 - \frac{3\sqrt{3}}{4}\right)2\sqrt{2}q + 2\sqrt{q} + 2\left(\sqrt{3} + \sqrt{2} - 6\sqrt{2} + \frac{9\sqrt{6}}{4}\right).$$

In Theorem 2.2, we have computed the $GA$ invariant of generalized Aztec diamond.

Theorem 2.2

For

$$p, q \geq 2, GA(L_p \otimes L_q) = 2pq + 2\left(\frac{4\sqrt{2}}{3} - 3\right)p + 2\left(\frac{4\sqrt{2}}{3} - 3\right)q + 2\left(\frac{63}{5} - \frac{6\sqrt{2}}{3}\right).$$

Proof: Again, by using the information given in Table 1, the $GA$ invariant of generalized Aztec diamond is derived follows; since:

$$GA(G) = \sum_{\alpha \beta \in E(G)} 2\frac{d(\alpha)d(\beta)}{d(\alpha) + d(\beta)}.$$

$$GA(L_p \otimes L_q) = 4 \times 2\sqrt{\frac{1 \times 4}{1 + 4}} + 4 \times 2\sqrt{\frac{2 \times 2}{2 + 2}} + 4\left(p + q - 6\right)\sqrt{\frac{2 \times 4}{2 + 4}} + 2\left(p - 3\right)\sqrt{\frac{2 \times 4}{4 + 4}}.$$
From an easy simplification, we acquired:

\[ GA(L_p \otimes L_q) = 2pq + 2 \left( \frac{4\sqrt{2}}{3} - 3 \right) p + 2 \left( \frac{4\sqrt{2}}{3} - 3 \right) q + 2 \left( \frac{63}{5} - \frac{6\sqrt{2}}{3} \right). \]

Similarly, the \( ABC_s \) index of the generalized Aztec diamond can be computed by using Table 1.

**Theorem 2.3**

For \( p, q \geq 2 \), the \( ABC_s \) index of \( L_p \otimes L_q \) is calculated as:

\[ ABC_s(L_p \otimes L_q) = \sqrt{\frac{30}{8}} pq + \left( \sqrt{3} + \sqrt{14} - \frac{5\sqrt{30}}{8} \right) \]

\[ p + \left( \sqrt{3} + \sqrt{14} - \frac{3\sqrt{30}}{4} \right) q + \left( \frac{2\sqrt{11}}{3} + \frac{2\sqrt{10}}{3} + \frac{2\sqrt{30}}{3} + \frac{2\sqrt{22}}{3} + \frac{68\sqrt{30}}{16} + 8\sqrt{2 - 10\sqrt{3 - 10\sqrt{14}}} \right). \]

**Proof:** In the \( ABC_s \) index, we first find possible degrees of all the vertices connected to end vertices of each edge. An edge with sum of the degrees of end vertices is \( m \) and \( n \), and is interpreted as \( (S_m, S_n) \) – type edges. Thus, the possible edges are of type type (4,9), (6,6), (9,8), (8,12), (12,16), (16,16), (6,12), (9,16), and (12,12). In Table 2, all the edges of type (4,9), (6,6), (9,8), (8,12), (12,16), (16,16), (6,12), (9,16), and (12,12) are counted. Now, by applying the information interpreted in Table 2, the \( ABC_s \) invariant of \( L_p \otimes L_q \) is computed as follows; since:

\[ ABC_s(G) = \sum_{(S_m, S_n) \in E(G)} \frac{S_m + S_n - 2}{S_m S_n}. \]

By an easy calculation, the above equation can be written as:

\[ ABC_s(L_p \otimes L_q) = \sqrt{\frac{30}{8}} pq + \left( \sqrt{3} + \sqrt{14} - \frac{5\sqrt{30}}{8} \right) \]

\[ p + \left( \sqrt{3} + \sqrt{14} - \frac{3\sqrt{30}}{4} \right) q + \left( \frac{2\sqrt{11}}{3} + \frac{2\sqrt{10}}{3} + \frac{2\sqrt{30}}{3} + \frac{2\sqrt{22}}{3} + \frac{68\sqrt{30}}{16} + 8\sqrt{2 - 10\sqrt{3 - 10\sqrt{14}}} \right). \]

The following theorem gives the \( GA_5 \) index of generalized Aztec diamond \( L_p \otimes L_q \).

**Theorem 2.4**

For \( p, q \geq 2 \), the \( GA_5 \) index of generalized Aztec diamond \( G = L_p \otimes L_q \) is:

\[ GA_5(G) = \sum_{uv \in E(G)} \frac{S_u S_v}{S_u + S_v}. \]

\[ GA_5(L_p \otimes L_q) = \sqrt{\frac{4 \times 9}{4}} \times 4 + \frac{2 \sqrt{16 \times 6}}{6 + 6} \times 4 + \frac{2 \sqrt{9 \times 8}}{9 + 8} \]

\[ \times 8 + \frac{2 \sqrt{8 \times 12}}{8 + 12} \times 6 \left( p + q - 10 \right) + \frac{2 \sqrt{6 \times 12}}{6 + 12} \times 4 + \frac{2 \sqrt{16 \times 16}}{16 + 16} \times 4 \left( p - 6 \right) \left( q - 5 \right) + 8 \]

\[ + \frac{2 \sqrt{12 \times 12}}{12 + 12} \times 4 \left( p + q - 10 \right) + \frac{2 \sqrt{9 \times 16}}{9 + 16} \times 4 + \frac{2 \sqrt{12 \times 12}}{12 + 12} \times 4 \]

**Proof:** By applying the information given in Table 2 on Eq. 2, we get:

\[ GA_5(G) = \sum_{uv \in E(G)} \frac{S_u S_v}{S_u + S_v}. \]

\[ GA_5(L_p \otimes L_q) = \sqrt{\frac{4 \times 9}{4}} \times 4 + \frac{2 \sqrt{16 \times 6}}{6 + 6} \times 4 + \frac{2 \sqrt{9 \times 8}}{9 + 8} \]

\[ \times 8 + \frac{2 \sqrt{8 \times 12}}{8 + 12} \times 6 \left( p + q - 10 \right) + \frac{2 \sqrt{6 \times 12}}{6 + 12} \times 4 + \frac{2 \sqrt{16 \times 16}}{16 + 16} \times 4 \left( p - 6 \right) \left( q - 5 \right) + 8 \]

\[ + \frac{2 \sqrt{12 \times 12}}{12 + 12} \times 4 \left( p + q - 10 \right) + \frac{2 \sqrt{9 \times 16}}{9 + 16} \times 4 + \frac{2 \sqrt{12 \times 12}}{12 + 12} \times 4 \]

**Table 2:** The cardinality of edges of type \((S_m, S_n)\) of \( L_p \otimes L_q \), where \( m \) and \( n \) are sum of degrees of all neighboring vertices of end vertices of each edge.

| \((S_m, S_n)\) with \( mn \in E(G)\) | Number of edges |
|-----------------|----------------|
| (4, 9)          | 4              |
| (6,6)           | 4              |
| (6,12)          | 8              |
| (8,12)          | 4(p + q - 10)  |
| (9,8)           | 8              |
| (9,16)          | 4              |
| (12,16)         | 4              |
| (12,16)         | 4(p + q - 10)  |
| (16,16)         | 2(p - 6)(q - 5) + 8 |
After simplification, we get:

\[
GA_x(L_p \otimes L_q) = 2pq + 2 \left( \frac{2\sqrt{6}}{5} + \frac{2\sqrt{3}}{7} - 3 \right)
\]

\[
p + 4 \left( \frac{2\sqrt{6}}{5} + \frac{2\sqrt{3}}{7} - 3 \right)
\]

\[
q + 4 \left( \frac{2\sqrt{2}}{17} - 4\sqrt{6} + \frac{2\sqrt{2}}{3} - \frac{20\sqrt{3}}{7} + 6631 \right)
\]

**Theorem 2.5**

For \( p, q \geq 2 \), the \( NM_1 \) index of \( L_p \otimes L_q \) is calculated as:

\[
NM_1(L_p \otimes L_q) = 256pq - 608p - 608q + 6748.
\]

**Proof:** In the \( NM_1 \) index, we first find possible degrees sum of all the vertices. The possible degree sum of a vertex is either, 4 or 6 or 8 or 9 or 12. There are 4 vertices (red color in Figure 1) of degree sum equal to 4. There are 8 vertices (green color in Figure 1) of degree sum equal to 6. There are 4 vertices (blue color in Figure 1) of degree sum equal to 8 and 12. There are 4 vertices (black color in Figure 1) of degree sum equal to 16. Now, by applying this information, the \( NM_1 \) index of \( L_p \otimes L_q \) is computed as follows:

\[
NM_1(L_p \otimes L_q) = 4(4^2) + 8(6^2) + 2(p + q - 8)(8^2) + 4(9^2) + 2(p + q - 8)(12^2) + (m - 4)(n - 4)(16^2)
\]

\[
= 256(p - 4)(q - 4) + 416(p + q - 8) + 676
\]

\[
= 256pq - 608p - 608q + 6748
\]

The following theorem gives the \( NM_1 \) index of generalized Aztec diamond \( L_p \otimes L_q \).

**Theorem 2.6**

For \( p, q \geq 2 \), the \( NM_2 \) index of \( L_p \otimes L_q \) is:

\[
NM_2(L_p \otimes L_q) = 512pp - 1408p - 1408q + 8144.
\]

**Proof:** Since:

\[
NM_2(G) = \sum_{uv \in E(G)} S_u S_v.
\]

Therefore, by applying this information given in Table 2, the \( NM_2 \) index of \( L_p \otimes L_q \) is computed as follows:

\[
NM_2(L_p \otimes L_q) = 4(6^2) \times 4 + (6 \times 6)
\]

\[
= 512pp - 1408p - 1408q + 8144.
\]

### 3 ABC, GA, ABC₄, GA₅, NM₁, and NM₂ indices of tetrahedral diamond lattice

The structure of tetrahedral diamond lattice (Manuel et al., 2015) of \( t \) dimension is constructed by \( t \) layers, every one interpreted as \( l \). The starting layer has just a single vertex and the next one have 4 vertices, which is isomorphic to \( S_4 \). For \( t \geq 3 \), every layer \( l \) has \( \sum_{k=1}^{l-2} k \) hexagons with 3 pendent vertices. According to depth first labeling, we can construct every next layer for vertices. More precisely, we can use that layer \( l \) presented the labels form \( \sum_{k=1}^{l-1} k^2 + 1 \) to \( \sum_{k=1}^{l-1} k^2 + 1 \).

The vertex set of tetrahedral diamond of dimension \( t \) has \( \sum_{k=1}^{l-1} k^2 \) vertices and the edge set has \( \frac{2}{3}(t^2 - t) \) edges. There is no any odd cycle in Tetrahedral diamond, so it is a bipartite graph.

The tetrahedral diamond lattice of dimension \( t \) is constructed with the help of \( l \)-layers in the following ways. The vertex in the first (starting) layer with label 1 is connected with a vertex in layer 2 with label 3. The layers \( l \) and \( l+1, l \geq 2 \), are connected as:
The vertex having label $\sum_{k=1}^{l-1} k^2 + (\sum_{m=1}^{i} 2(l-m)) + (2f-1)+1$ in layer $l$ is connected to the vertex which has label $\sum_{k=1}^{l} k^2 + 2f$ in layer $l+1$ for all $1 \leq f \leq l$.

(ii) The vertex with label $\sum_{k=1}^{l-1} k^2 + (\sum_{m=1}^{i} 2(l-m)) + (2f-1)+1$ in layer $l$ is connected to the vertex which has label $\sum_{k=1}^{l} k^2 + (\sum_{m=1}^{i} 2(l-m)) + 3 + (2f)$ in layer $l+1$ where $1 \leq i \leq l-1$ and $1 \leq f \leq l-i$.

Figure 2 depicts a 5-dimension tetrahedral diamond. In Theorem 3.1, the value of $ABC_t$ invariant of tetrahedral diamond lattice is derived.

Theorem 3.1

For $t \geq 1$, the $ABC$ index of tetrahedral diamond lattice $G$ with dimension $t$ is:

$$ABC(G) = \sqrt{15}t^4 + \left(6\sqrt{2} - 5\sqrt{15}\right)t + 2\left(\sqrt{3} - 6\sqrt{2} + 3\sqrt{15}\right) + \sum_{k=5}^{t} (k-4)(k-3)\sqrt{6}.$$ 

$Proof$: The graph $G$ contains only vertices of degree one, two, three, and four. The edges of the graph $G$ of dimension $t$ are of the form $(1,4)$, $(2,4)$, $(3,4)$, and $(4,4)$. The construction of graph indicates that the number of edges of type $(1,4)$ are four for every layer. There are $6t-12$ vertices of degree two, and two edges are induced by each vertex. So, number of $(2,4)$-type edges is $12(t-2)$. Also by induction argument, the number of $(3,4)$-type edges is:

$$9(t-2)(t-3) + 3t - 2(t-2)(t-3) - 6(t-2)(t-3) = 2(t^3 - 9t^2 + 26t - 24).$$

Note that the quantity of $(4,4)$-type edges can be obtained by removing all edges of type $(1,4)$, $(2,4)$, $(3,4)$ from $G$. Thus, the quantity of $(4,4)$-type edges is:

$$|E(G)| - ((1,4) + (2,4) + (3,4)) = \frac{2}{3}(t^3 - t) - (4 + 12t - 24 + 6t^2 - 30t + 36) = \frac{2}{3}(t^3 - 6t^2 + 26t - 24).$$

From Table 3, the formula for $ABC$ index can be deduced to:

$$ABC(G) = 4\sqrt{1+4-2 \over 1 \times 4} + 12(t-2)\sqrt{2+4-2 \over 2 \times 4} + 6(t-2)(t-3)\sqrt{3+4-2 \over 3 \times 4} + 2(t^3 - 9t^2 + 26t - 24)\sqrt{4+4-2 \over 4 \times 4}.$$ 

Table 3: The quantity of $(m, n)$-type edges of tetrahedral diamond $G$, where $m$ and $n$ are degrees of the adjacent vertices of each edge

| $(d(m), d(n))$ with $mn \in E(G)$ | Number of edges |
|-----------------------------|----------------|
| (1,4)                       | 4              |
| (2,4)                       | $12(t-2)$      |
| (3,4)                       | $6(t-2)(t-3)$  |
| (4,4)                       | $\frac{2}{3}(t^3 - 9t^2 + 26t - 24)$ |
After computation, we get:

\[
ABC(G) = \sqrt{15}t^2 + \left(6\sqrt{2} - 5\sqrt{15}\right)t + 2\left(\sqrt{3} - 6\sqrt{2} + 3\sqrt{15}\right)
\]

\[
+ \sum_{k=5}^{n} (k-4) (k-3) \sqrt{6}.
\]

**Theorem 3.2**

The \(GA\) index of \(G\) (tetrahedral diamond lattice) of dimension \(t\) where \(t \geq 1\) is given by:

\[
GA(G) = 24t^2 + 4\left(2\sqrt{2} - 30 \frac{\sqrt{3}}{7}\right)t + 16\left(\frac{1}{5} + 9 \frac{\sqrt{3}}{7} - \sqrt{2}\right)
\]

\[
+ 2 \sum_{k=5}^{t} (k-4) (k-3).
\]

**Proof:** The \(GA\) index formula of tetrahedral diamond lattice \(G\) is deduced by using the information from Table 3 as:

\[
GA(G) = 24t^2 + 4\left(2\sqrt{2} - 30 \frac{\sqrt{3}}{7}\right)t + 16\left(\frac{1}{5} + 9 \frac{\sqrt{3}}{7} - \sqrt{2}\right)
\]

\[
+ 2 \sum_{k=5}^{t} (k-4) (k-3).
\]

After an easy computation, we acquire:

\[
ABC_4(G) = \left(\frac{24}{\sqrt{10}} - \frac{20}{\sqrt{6}} - 5 \frac{23}{29}\right)t + \left(\frac{4}{\sqrt{6}} + \frac{23}{29}\right)
\]

\[
t^2 + 3\left(\frac{8}{\sqrt{6}} + \frac{1}{\sqrt{7}} + 2 \frac{13}{14} + 2 \frac{23}{29}\right)
\]

\[
+ \sum_{k=5}^{n} (k-4)(k-3) \frac{9}{2\sqrt{39}}.
\]

**Table 4:** The quantity of \((S_m, S_n)\)-type edges, where \(m\) and \(n\) are sum of degrees of all neighboring vertices of the end vertices of each edge

| \((S_m, S_n)\) with \(mn \in E(G)\) | Number of edges |
|----------------------------------------|-----------------|
| (4,7)                                  | 4               |
| (7,8)                                  | 12              |
| (8,10)                                 | 12(t-3)         |
| (12,10)                                | 4(t-2)(t-3)     |
| (12,13)                                | 2(t-2)(t-3)     |
| (16,13)                                | 2 \sum_{k=5}^{n} (k-4)(k-3) |
In upcoming theorem, we compute the $ABC_4$ index for tetrahedral diamond lattice.

**Theorem 3.4**

The $GA_5$ index for the tetrahedral diamond lattice $G$ is given by:

$$GA_5(G) = 4\left(\frac{20}{3} - \frac{10\sqrt{30}}{11}\right) + t^{8}\left(\frac{\sqrt{30}}{11} + \frac{\sqrt{39}}{25}\right)$$

$$+ t^2 + 4\left(\frac{4\sqrt{7}}{11} + \frac{12\sqrt{14}}{15} - \frac{20}{11} + \frac{12\sqrt{30}}{25} + \frac{12\sqrt{39}}{15}\right)$$

$$+ \frac{16\sqrt{13}}{29} \sum_{k=5}^{4} (k-4)(k-3).$$

**Proof:** By applying the information from Table 4 in Eq. 5, we derive the $GA_5$ index as:

$$GA_5(G) \approx 792t^2 - 3000t + 2656 + 416\sum_{k=5}^{4} (k-4)(k-3).$$

The result for $NM_1$ index of tetrahedral diamond lattice $G$ is:

$$NM_1(G) = 792t^2 - 3000t + 2656 + 416\sum_{k=5}^{4} (k-4)(k-3).$$

**Proof:** By applying the information from Table 4 in Eq. 5, we derive the $GA_5$ index as:

$$NM_1(G) = \sum_{v \in V(G)} (S_v)^2$$

$$= 4(4^2) + 4(7^2) + (6t-12)(8^2) + (6t-18)(10^2)$$

$$+ \frac{(t-2)(3t-5)(12^2) + (t-2)(t-3)(t-4)(16^2)}{3}$$

$$= 85t^3 - 552t^2 + 2410t - 2047.$$

In upcoming theorem, we compute the $GA_5$ index for tetrahedral diamond lattice.

**Theorem 3.6**

The $NM_2$ index for the tetrahedral diamond lattice $G$ is given by:

$$NM_2(G) = 792t^2 - 3000t + 2656 + 416\sum_{k=5}^{4} (k-4)(k-3).$$

**Proof:** By applying the information from Table 4 in Eq. 5, we derive the $GA_5$ index as:

$$NM_2(G) = \sum_{uv \in E(G)} S_u S_v$$

$$= (4x7)4 + (8x7)12 + (8x10)12(t-3)$$

$$+ (12x10)4(t-2)(t-3) + (12x13)$$

$$\times 2(t-2)(t-3) + (16x13)2\sum_{k=5}^{4} (k-4)(k-3)$$

$$= 792t^2 - 3000t + 2656 + 416\sum_{k=5}^{4} (k-4)(k-3).$$

**4 Conclusion**

In this article, some degree depended invariants namely $ABC, GA, ABC_4, GA_5, NM_1,$ and $NM_2$ indices are computed for the networks constructed from generalized Aztec diamonds and tetrahedral diamond lattices. We have found the exact values of these parameters for the said vertex in the graph is either 4 or 7 or 8 or 10 or 12 or 16. The computation gives that:

$$S_4 = 4, S_7 = 4, S_8 = 6t - 12, S_{10} = 6t - 18, S_{12}$$

$$= \frac{(t-2)(3t-5)}{2}, S_{16} = \frac{(t-2)(t-3)(t-4)}{3}.$$

Using this information $NM_1$ index of tetrahedral diamond lattice $G$ can be calculated as follows:

$$NM_1(G) = \sum_{v \in V(G)} (S_v)^2$$

$$= 4(4^2) + 4(7^2) + (6t-12)(8^2) + (6t-18)(10^2)$$

$$+ \frac{(t-2)(3t-5)(12^2) + (t-2)(t-3)(t-4)(16^2)}{3}$$

$$= 85t^3 - 552t^2 + 2410t - 2047.$$
classes of graphs. This, we believe will work in the favour of researcher working in network science to investigate the underlying topologies of given networks.

In future, it is interesting to design certain new networks and then investigate their topological properties and compute their exact values which can be fruitful in understanding of the underlying topologies.

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