ON THE CRITICAL $p$-LAPLACE EQUATION

GIOVANNI CATINO, DARIO D. MONTICELLI, ALBERTO RONCORONI

Abstract. In this paper we provide the classification of positive solutions to the critical $p$–Laplace equation on $\mathbb{R}^n$, for $1 < p < n$, possibly having infinite energy. If $n = 2$, or if $n = 3$ and $\frac{3}{2} < p < 2$ we prove rigidity without any further assumptions. In the remaining cases we obtain the classification under energy growth conditions or suitable control of the solutions at infinity. Our assumptions are much weaker than those already appearing in the literature. We also discuss the extension of the results to the Riemannian setting.

Key Words: quasilinear elliptic equations, qualitative properties, manifolds with nonnegative Ricci curvature

AMS subject classification: 35J92, 35B33, 35B06, 58J05, 53C21

1. Introduction

Given $n \geq 2$ and $1 < p < n$ we consider positive solutions of the well-known critical $p$–Laplace equation

$$\Delta_p u + u^{p^*-1} = 0 \quad \text{in } \mathbb{R}^n,$$

where $\Delta_p$ is the usual $p$–Laplace operator and $p^*$ is the Sobolev exponent, explicitly

$$\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u), \quad \text{and} \quad p^* = \frac{np}{n-p}.$$

The critical $p$–Laplace equation has been the object of several studies in the differential geometry and in the PDE’s communities, indeed problem (1.1) is related to the study of the critical points of the Sobolev inequality (see e.g. the survey [24]) and, for $p = 2$, to the Yamabe problem (see e.g. the survey [18]). An interesting and challenging problem is the classifications of solutions to (1.1): one can show that the following functions

$$U_{\lambda,x_0}(x) := \left( \frac{\lambda^{\frac{1}{n-1}} \left( \frac{1}{n} \left( \frac{n-p}{p-1} \right) \frac{n-1}{p} \right)^{\frac{n-p}{p}}}{\lambda^{\frac{n-p}{p-1}} + |x-x_0|^{\frac{n-p}{p-1}}} \right)^{\frac{n-p}{p}}, \quad \lambda > 0, \ x_0 \in \mathbb{R}^n. \quad (1.2)$$

form a 2–parameters family of solutions to (1.1)$^1$. Hence the natural question is the following:

**Given a positive solution to (1.1), is it of the form (1.2)?**

The functions described in (1.2) are usually called Aubin-Talenti bubbles, since in two independent papers Aubin, in [3], and Talenti, in [28], prove that the functions (1.2) realize the equality in the sharp Sobolev inequality in $\mathbb{R}^n$.

---

$^1$If one chooses a different normalization constant in the numerator of the functions $U_{\lambda,x_0}$, then one obtains a constant $k \neq 1$ as a coefficient of $u^{p^*-1}$ in equation (1.1).
It is well known (see e.g. [10] and [8]) that there exist multiple sign-changing, non-radial, finite energy solutions to
\[ \Delta_p u + u|u|^{p^*-2} = 0 \text{ in } \mathbb{R}^n. \]
In this paper we focus on non-negative weak solutions to (1.1) which, by the maximum principle for quasilinear equations (see e.g. [30]), are either zero or positive.

Turning back to the classification results of positive solutions to (1.1) in the seminal paper [4] (see also [22] and [12] for previous important results) the authors consider the semilinear case (i.e. \( p = 2 \)) and they prove that positive smooth solutions to (1.1) with \( p = 2 \) are given by the Aubin-Talenti bubbles (1.2) (see also [6] and [19]). The proof is based on a refinement of the method of moving planes (introduced in [1] in the context of constant mean curvature hypersurfaces and transposed in [26] and in [13] to study of qualitative properties of solutions of the PDE’s) and on the Kelvin transform.

The quasilinear case (i.e. \( 1 < p < n, p \neq 2 \)) is more complicated since, for example, the Kelvin transform is not available. Nevertheless the method of moving planes has been exploited in [9, 25, 31] to prove the following classification result:

**Theorem 1.1.** Let \( u \) be a positive weak solution of equation (1.1) with finite energy, i.e.
\[ u \in D^{1,p}([\mathbb{R}^n]) := \{ u \in L^{p^*}([\mathbb{R}^n]) : \nabla u \in L^p([\mathbb{R}^n]) \}, \]
then \( u(x) = U_{\lambda,x_0}(x) \) for some \( \lambda > 0 \) and \( x_0 \in \mathbb{R}^n \).

We mention that this result has been recently generalized in [7] in the anisotropic setting (see also [11]) and in convex cones of \( \mathbb{R}^n \) (see also [20]) and in [5, 17, 21] in the Riemannian setting (see Appendix A for a more detailed discussion).

As far as we know trying to prove the same result without the assumption that \( u \) has finite energy is an open and challenging problem for \( p \neq 2 \); in this paper we deal with this problem obtaining a classification result of all positive weak solutions of (1.1) in dimensions \( n = 2, 3 \) for \( \frac{n}{2} < p < 2 \), while for different values of \( n \) and \( p \) we require that \( u \) satisfies suitable conditions at infinity (which are weaker than the finite energy assumption).

Before stating our results we recall the variational nature of the critical \( p \)-Laplace equation (1.1): the energy associated to (1.1) is given by
\[ E_{[\mathbb{R}^n]}(u) := \frac{1}{p} \int_{\mathbb{R}^n} |\nabla u|^p + \frac{1}{p^*} \int_{\mathbb{R}^n} u^{p^*}, \]
indeed it is well-known that the Euler-Lagrange equation associated to this energy functional is (1.1). We define the energy on a general open set \( \Omega \subseteq \mathbb{R}^n \) and we split it as follows:
\[ E_{\Omega}(u) = E_{\Omega}^{\text{kin}}(u) + E_{\Omega}^{\text{pot}}(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p + \frac{1}{p^*} \int_{\Omega} u^{p^*}. \]

With these notations our first theorem is a rigidity result under a growth assumption of the energy on annuli, indeed we have the following:

**Theorem 1.1.** Let \( u \) be a positive weak solution of equation (1.1). If one of the following holds
(i) \( 1 < p \leq \frac{2n}{n+1} \) and
\[ E_{A_R}(u) = O \left( R^{\frac{n}{n-1}} \right), \] or
(ii) \( \frac{2n}{n+1} < p < 2 \) and
\[ E_{A_R}(u) = O \left( R^{2-(p)(n-p)} \right), \] or
(iii) $p > 2$ and $u(x) \leq C|x|^\alpha$ as $|x| \to \infty$ for some $\alpha \geq 0$ and
\[
E_{AR}(u) = O\left(R^k\right) \quad \text{for some} \quad k < \frac{2(n-p)}{2 + (n-3)p} - \alpha \frac{p(n(p-2)+p)}{(n-p)[2 + (n-3)p]},
\]
where $A_R := B_{2R} \setminus B_R$, then $u(x) = U_{\lambda,x_0}(x)$ for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$.

In particular this result implies the classification of solutions $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, simply by observing that
\[
u \in \mathcal{D}^{1,p}(\mathbb{R}^n) \iff E_{\mathbb{R}^n}(\nu) < \infty.
\]
Here one also has to recall that positive solutions $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ have the following behavior:
\[
u(x) \leq \frac{C}{1 + |x|^{\frac{n-p}{p-1}}} \quad \text{and} \quad |\nabla \nu(x)| \leq \frac{C}{1 + |x|^{\frac{n-p}{p-1}}},
\]
for all $x \in \mathbb{R}^n$ and some $C > 0$, as it was shown in [31, Theorem 1.1]. We explicitly note that we are not using these estimates in our estimates.

Note also that, if $\nu$ is a positive weak solution of equation (1.1), by Lemmas 2.7 and 2.9 for every $\alpha > 0$ one has
\[
E_{AR}^{\text{pot}}(\nu) = O(R^\alpha) \iff E_{AR}(\nu) = O(R^\alpha) \iff E_{AR}^{\text{kin}}(\nu) = O(R^\alpha).
\]
See Remark 2.10 for the proof.

For suitable choices of $n$ and $p$ we can show rigidity results without any further assumptions on the solution.

**Theorem 1.2.** Let $\nu$ be a positive weak solution of equation (1.1). If one of the following holds
(i) $n = 2$ and $1 < p < 2$, or
(ii) $n = 3$ and $\frac{3}{2} < p < 2$,
then $\nu(x) = U_{\lambda,x_0}(x)$ for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$.

In our last two classification theorems, where we consider general $n$ and $p$, we rely on conditions on the behavior of the solution at infinity, which are much weaker than all the results already available in the literature. For $1 < p < 2$ we have he following:

**Theorem 1.3.** Let $\nu$ be a positive weak solution of equation (1.1) with
\[
u(x) \leq C|x|^\alpha \quad \text{as} \quad |x| \to \infty,
\]
for some
\[
\alpha < \tilde{\alpha} := \frac{(3p-n)(n-p)}{p(n-2p)}.
\]
If one of the following holds
(i) $n = 3$ and $1 < p \leq \frac{3}{2}$, or
(ii) $n \geq 4$ and $1 < p < 2$,
then $\nu(x) = U_{\lambda,x_0}(x)$ for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$.

For $2 < p < n$, on the other hand, we have:

**Theorem 1.4.** Let $\nu$ be a positive weak solution of equation (1.1) with
\[
u(x) \leq C|x|^\alpha \quad \text{as} \quad |x| \to \infty.
\]
Let
\[
\tilde{\alpha} := \frac{2(n-p)}{p(p-2)}, \quad \tilde{\alpha} := \frac{(n-p)^2}{(p-2)(p-1)}, \quad \tilde{\alpha} := \frac{(3p-n)(n-p)}{p(n-2p)}, \quad \tilde{\alpha} := \frac{(3p-n)(n-p)}{p(n-3p+2)}.
\]
Assume that one of the following holds
(i) $n = 3$, and $2 < p < 3$ and $\alpha < \tilde{\alpha}$;
(ii) \( n = 4 \), and
\[
2 < p < \bar{p} \quad \text{and} \quad \alpha < \hat{\alpha},
\]
or
\[
\bar{p} \leq p < 4 \quad \text{and} \quad \alpha < \hat{\alpha};
\]
(iii) \( n = 5 \) or \( n = 6 \), and
\[
2 < p < \frac{n+2}{3} \quad \text{and} \quad \alpha < \bar{\alpha},
\]
or
\[
\frac{n+2}{3} \leq p < \bar{p} \quad \text{and} \quad \alpha < \bar{\alpha},
\]
or
\[
\bar{p} \leq p < n \quad \text{and} \quad \alpha < \hat{\alpha};
\]
(iv) \( n \geq 7 \) and
\[
2 < p \leq \frac{n}{3} \quad \text{and} \quad \alpha < \bar{\alpha},
\]
or
\[
\frac{n}{3} < p < \frac{n+2}{3} \quad \text{and} \quad \alpha < \bar{\alpha},
\]
or
\[
\frac{n+2}{3} \leq p < \bar{p} \quad \text{and} \quad \alpha < \hat{\alpha},
\]
or
\[
\bar{p} \leq p < n \quad \text{and} \quad \alpha < \hat{\alpha},
\]
where \( \bar{p} = \frac{n-2+\sqrt{n^2-4n+12}}{2} \). Then \( u(x) = U_{\lambda,x_0}(x) \) for some \( \lambda > 0 \) and \( x_0 \in \mathbb{R}^n \).

In particular we have the following classifications:

**Corollary 1.5.** Let \( u \) be a positive bounded weak solution of equation (1.1) on \( \mathbb{R}^n \), with \( n \leq 6 \), or \( n \geq 7 \) and \( p > \frac{n}{2} \). Then \( u(x) = U_{\lambda,x_0}(x) \) for some \( \lambda > 0 \) and \( x_0 \in \mathbb{R}^n \).

**Corollary 1.6.** Let \( u \) be a positive weak solution of equation (1.1) on \( \mathbb{R}^n \) with
\[
 u(x) \leq C |x|^{-\frac{n-p}{p}}.
\]
Then \( u(x) = U_{\lambda,x_0}(x) \) for some \( \lambda > 0 \) and \( x_0 \in \mathbb{R}^n \).

We stress the fact that all the limiting exponents in Theorems 1.3 and 1.4 are strictly larger than \( -\frac{n-p}{p} \) in the ranges of \( n \) and \( p \) where they are used. Actually they are strictly positive under the hypothesis of Corollary 1.5. Note that the exponent \( -\frac{n-p}{p} \) is the threshold decay in order for a radial solution to have finite energy.

As an auxiliary result, which may have an independent interest, we prove a gradient estimate for positive solutions to (1.1) which is instrumental in the proofs of Theorems 1.1 and 1.4 in the case \( 2 < p < n \).

**Proposition 1.7.** Let \( u \) be a positive weak solution of equation (1.1) with \( 1 < p < n \). Then, for every \( 0 < \varepsilon < \frac{p-1}{n-p} \) it holds
\[
|\nabla u| \leq C \left( \frac{\|u\|_{C^1(B_{2R}(x_0))}^{\frac{1}{n-p} + \varepsilon} + R^{-\varepsilon\frac{n-p}{n-1}}}{R^{\frac{n-1}{n-p} - \varepsilon}} \right) u^{\frac{n-1}{n-p} - \varepsilon} \quad \text{on} \ B_R(x_0)
\]
for some \( C = C(n,p,\varepsilon) > 0 \), for every \( R > 0 \) and every \( x_0 \in \mathbb{R}^n \).
This estimate is sharp for the positive solutions \( U_{\lambda,x_0}(x) \).

Most of the available classification results for the critical \( p \)-Laplace equation are based on a careful application of the moving plane technique. Interesting exceptions can be found in [5, 7, 17] where the authors, exploiting integral estimates obtained through test functions arguments, prove the classification via the vanishing of a suitable traceless tensor field depending on the solutions and their derivatives. Similar estimates have been used by Gidas and Spruck [14] and Serrin and Zou [27] in the subcritical case. In this paper we adopt a similar approach; the starting point in the proof of our classification results is the key integral estimate in Corollary 2.4 which in turn is obtained adapting arguments in [27] to the critical case.

One of the nice features of our approach is that it can be quite easily extended to the Riemannian setting, as it was shown in the case \( p = 2 \) in [5]. We review all the needed steps in order to adapt our arguments to the case of a Riemannian manifold \((M^n, g)\) in the Appendix A, where we sketch the proof of the following:

**Theorem 1.8.** Let \( u \) be a positive weak solution with regularity (2.1)–(2.4) on a complete non-compact Riemannian manifold \((M^n, g)\) such that

(i) \( \text{Ric} \geq 0 \), if \( 1 < p < 2 \), or

(ii) \( \text{Sec} \geq 0 \), if \( 2 < p < n \).

Then, under the hypotheses of Theorems 1.1-1.2-1.3-1.4, \((M^n, g)\) is isometric to \( \mathbb{R}^n \) with the Euclidean metric and \( u(x) = U_{\lambda,x_0}(x) \) for some \( \lambda > 0 \) and \( x_0 \in \mathbb{R}^n \).

The paper is organized as follows: in Section 2 we collect some useful preliminary results that will be needed in the proof of our main theorems, in particular in Corollary 2.4 we prove a key integral estimate that will be the starting point in the proofs of the main results; in Section 3 we prove the sharp gradient estimate in Proposition 1.7; in Section 4 we give the proof of Theorem 1.1; in Section 5 we show Theorems 1.3 and 1.4. Finally, in Appendix A we discuss the generalizations to the Riemannian setting.

## 2. Preliminaries

### 2.1. The key integral estimate.

In this part we collect some well-known facts about equation (1.1): the definition of weak solutions and of sub/super-solutions to (1.1) and the regularity theory related to the \( p \)-Laplace equation. Moreover we show the main integral estimate that we are going to use to prove our rigidity results.

**Definition 2.1.** A weak solution of (1.1) is a function \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty_{\text{loc}}(\mathbb{R}^n) \) such that

\[
\int_{\mathbb{R}^n} |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle - \int_{\mathbb{R}^n} u^{p-1} \psi = 0 \quad \forall \psi \in W^1_0(\mathbb{R}^n),
\]

where \( W^1_0(\mathbb{R}^n) \) denotes the set of compactly supported functions of \( W^1(\mathbb{R}^n) \).

Moreover, a weak subsolution of (1.1) is a function \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty_{\text{loc}}(\mathbb{R}^n) \) such that

\[
\int_{\mathbb{R}^n} |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle - \int_{\mathbb{R}^n} u^{p-1} \psi \leq 0 \quad \forall \psi \in W^1_0(\mathbb{R}^n),
\]

such that \( \psi \) is non-negative. Finally, \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty_{\text{loc}}(\mathbb{R}^n) \) is a weak supersolution of (1.1) if the opposite inequality holds.
Thanks to the classical regularity theory we have that any weak solution of (1.1), with \(1 < p \leq 2\), satisfies

\[
u \in W^{2,2}_{\text{loc}}(\mathbb{R}^n) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)\quad \text{(2.1)}
\]

for some \(\alpha \in (0, 1)\) and

\[
|\nabla u|^{p-2}\nabla u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n).
\]

(2.2)

On the other hand, for \(p > 2\),

\[
u \in W^{2,2}_{\text{loc}}(\mathbb{R}^n \setminus \Omega_{cr}) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)
\]

for some \(\alpha \in (0, 1)\) and

\[
|\nabla u|^{p-2}\nabla^2 u \in L^2_{\text{loc}}(\mathbb{R}^n \setminus \Omega_{cr}),
\]

(2.3)

where

\[
\Omega_{cr} = \{x \in \mathbb{R}^n | \nabla u(x) = 0\}.
\]

We note that by a bootstrap argument, any weak solution is actually \(C^\infty\) on \(\Omega_{cr}^c\). These results can be found, for example, in [27, Section 8] and also in the recent paper [2] (see also references therein).

Here, also in order to help the reader, we adopt the same notation as in [27, Chapter II]. If \(u\) is a positive solution of (1.1) we define the vector fields

\[
u := |\nabla u|^{p-2}\nabla u, \quad \upsilon := u \frac{n(p-1)}{n-p} |\nabla u|^{p-2}\nabla u.
\]

(2.5)

The tensors \(\nabla \upsilon\) and \(\nabla \upsilon\) are well defined and actually smooth only on \(\Omega_{cr}^c\) when \(1 < p < 2\), while they are well defined and continuous on the whole of \(\mathbb{R}^n\) if \(p \geq 2\). It is then convenient to redefine them to be 0 on \(\Omega_{cr}\), for \(1 < p < 2\). Then we set

\[
U := \begin{cases} \nabla u, & \text{in } \Omega_{cr}^c, \\ 0, & \text{in } \Omega_{cr}\end{cases}, \quad V := \begin{cases} \nabla \upsilon, & \text{in } \Omega_{cr}^c, \\ 0, & \text{in } \Omega_{cr}\end{cases}
\]

(2.6)

Note that of course \(\nabla \upsilon \equiv U\) and \(\nabla \upsilon \equiv V\) on \(\mathbb{R}^n\) if \(p > 2\). Moreover, we recall the definition of the traceless tensor

\[
\hat{V} := V - \frac{\text{tr}V}{n} \text{Id}_n
\]

where \(\text{Id}_n\) is the identity tensor.

Using this notation, we have the following fundamental estimate.

**Proposition 2.2.** Let \(1 < p < n\), \(p \neq 2\), and \(u\) be a positive weak solution of equation (1.1). Then for every \(0 \leq \phi \in C^\infty_0(\mathbb{R}^n)\) we have

\[
\int u \frac{(n-1)p}{n-p} |\nabla \phi|^2 \leq -\int u \frac{(n-1)p}{n-p} \langle \nabla \phi, \nabla \phi \rangle
\]

Remark 2.3. We remark that the result holds with the equality sign when \(p > 2\). It also holds when \(p = 2\) with the equality sign, replacing \(V\) with \(\nabla \upsilon\) (see Proposition 6.2 in [27]).

**Proof.** In case \(p > 2\) the result follows from Proposition 6.2 in [27]. When \(1 < p < 2\), due to problems of regularity, one can first use Proposition 7.1 in [27], where a truncation of \(|\nabla u|\) is introduced in order to deal with the critical set of \(u\), and then pass to the limit as the relevant parameter \(\varepsilon\) tends to 0 to conclude. We provide here some details to help the reader.

We start with the case \(p > 2\). Formula (6.16) in Proposition 6.2 in [27] with

\[
a = -\frac{n(p-1)}{n-p}, \quad b = \frac{p(n-1)}{n-p}, \quad q = \frac{np}{n-p} - 1
\]

(2.7)
reads as
\[
\int (u^b I + \psi) \phi = -\int \langle \omega, \nabla \phi \rangle.
\]
Here
\[
\omega := u^{(\alpha-1)p} \left( \mathbf{v} \cdot \mathbf{V} - \frac{1}{n} \mathbf{v} \mathrm{tr} \mathbf{V} \right) = u^{(\alpha-1)p} \mathbf{v} \cdot \mathbf{\tilde{V}}.
\]
The expression $\mathbf{v} \cdot \mathbf{V}$ is interpreted as the vector with components $(\mathbf{v} \cdot \mathbf{V})_i = v_j V_{ij}$, for $i = 1, \ldots, n$, where we use the Einstein convention of summation over repeated indices. Moreover
\[
I := |\mathbf{V}|^2 - \frac{1}{n} \mathrm{tr}(\mathbf{V})^2 \equiv |\mathbf{\tilde{V}}|^2,
\]
and
\[
\psi := u^{b+2a+q-1} \left( A + q \hat{A} \right) |\nabla u|^p + B u^{b+2a-2} |\nabla u|^p + C \mathrm{div} \left( u^{b+2a-1} |\nabla u|^p \mathbf{u} \right) \equiv 0,
\]
since by our choices of $a, b, q$ one easily computes $\left( A + q \hat{A} \right) = B = C = 0$, using their explicit expressions in [27]. Thus the result immediately follows, with the equality sign.

The case $1 < p < 2$ is more involved. For any fixed $\varepsilon \in (0, 1)$, following [27] we set
\[
|\nabla u|_\varepsilon = \max\{|\nabla u|, \varepsilon\}, \quad \mathbf{u}_\varepsilon = |\nabla u|_\varepsilon^{-2} \nabla u, \quad \mathbf{v}_\varepsilon = u^{\frac{n(p-1)}{n-p}} \mathbf{u}_\varepsilon, \quad \mathbf{V}_\varepsilon = \nabla \mathbf{v}_\varepsilon.
\]
By Proposition 7.1 in [27] we obtain
\[
\int (u^b I_\varepsilon + \psi_\varepsilon) \phi = -\int \langle \omega_\varepsilon, \nabla \phi \rangle + O(\varepsilon^{2(p-1)}), \quad (2.8)
\]
with
\[
I_\varepsilon = \mathrm{tr}(\mathbf{V}_\varepsilon \mathbf{V}) - \frac{1}{n} \mathrm{tr} \mathbf{V}_\varepsilon \mathrm{tr} \mathbf{V}\]
\[
\omega_\varepsilon = \left( \mathbf{v}_\varepsilon \cdot \mathbf{V} - \frac{1}{n} \mathbf{v}_\varepsilon \mathrm{tr} \mathbf{V} \right) u^{\frac{(\alpha-1)p}{n-p}} = \mathbf{v}_\varepsilon \cdot \mathbf{\tilde{V}} u^{\frac{(\alpha-1)p}{n-p}}
\]
\[
\psi_\varepsilon = u^{b+2a+q-1} \left( \hat{A} + q \hat{A} \right) \Gamma_\varepsilon + \hat{A} u^{b+2a-1} |\nabla u|^p \mathrm{div} \mathbf{u}_\varepsilon
\]
\[
+ B u^{b+2a-2} |\nabla u|^p \Gamma_\varepsilon + C \mathrm{div} \left( u^{b+2a-1} |\nabla u|^p \mathbf{u}_\varepsilon \right),
\]
where
\[
\Gamma_\varepsilon = \langle \mathbf{u}_\varepsilon, \nabla u \rangle = |\nabla u|_\varepsilon^{-2} |\nabla u|^2,
\]
see formulas (7.6) and (6.15) in [27]. Using their explicit expressions provided in [27], one can easily see that choosing $a, b, q$ as in (2.7) we get $B = C = \hat{A} + \hat{A} + q \hat{A} = 0$.

Since $u \in C^1$ and positive, $\mathbf{u}_\varepsilon$ and $\mathbf{v}_\varepsilon$ converge to $\mathbf{u}$ and $\mathbf{v}$ in $L^2_{\text{loc}}$ respectively. Since $\mathbf{V}$ and $\mathbf{\tilde{V}}$ are in $L^2_{\text{loc}}$ and $u \in C^1$ and positive, $\omega_\varepsilon$ converges to $\omega$ in $L^1_{\text{loc}}$ as $\varepsilon$ tends to 0.

Moreover, $\Gamma_\varepsilon$ converges to $|\nabla u|^p$ in $L^2_{\text{loc}}$ and $|\nabla u|^p \mathrm{div} \mathbf{u}_\varepsilon$ converges weakly in $L^2_{\text{loc}}$ to $|\nabla u|^p \mathrm{div} \mathbf{u} = -|\nabla u|^p u^q$, see the proof of Proposition 7.2 in [27]. Then we obtain that $\psi_\varepsilon$ converges weakly to 0 in $L^2_{\text{loc}}$ as $\varepsilon$ tends to 0.

Finally we consider the term $I_\varepsilon$. Let
\[
\Omega_\varepsilon = \{ x \in \mathbb{R}^n | 0 < |\nabla u| < \varepsilon \},
\]
then $I_\varepsilon = I \geq 0$ almost everywhere on $\Omega_\varepsilon^c$ and $I_\varepsilon = I = 0$ on $\Omega_{\text{cr}}$. Thus
\[
\int \phi u^b I_\varepsilon = \int_{\Omega_\varepsilon^c} \phi u^b I + \int_{\Omega_\varepsilon} \phi u^b I_\varepsilon.
\]
By formula (7.22) in [27] we have
\[ \liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \phi u^b I_\varepsilon \geq 0, \]
while by the monotone convergence theorem we conclude that
\[ \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \phi u^b I = \int \phi u^b I. \]
Passing to the limit as \( \varepsilon \) tends to 0 in (2.8) we have
\[ \int u^b I \phi \leq - \int \langle \omega, \nabla \phi \rangle, \]
which is the desired inequality. \( \square \)

An easy consequence of the previous proposition is the following key integral estimate, from which we will deduce our main rigidity results.

**Corollary 2.4.** Let \( 1 < p < n, \ p \neq 2, \) and \( u \) be a positive weak solution of equation (1.1). Then for every \( 0 \leq \eta \in C^\infty_0(\mathbb{R}^n) \) and \( l \geq 2 \) we have
\[ \int u^{(n-1)/p} |\nabla|^{2p-1} \eta^l \leq C \int u^{(2-p)/(n-p)} |\nabla u|^{2(p-1)} |\nabla \eta|^2 \eta^l \]
and
\[ \int u^{(n-1)/p} |\nabla|^{2} \eta^l \leq C \left( \int_{\text{supp} |\nabla \eta|} u^{(n-1)/p} |\nabla|^{p} \eta^l \right)^{\frac{1}{2}} \left( \int u^{(2-p)/(n-p)} |\nabla u|^{2(p-1)} |\nabla \eta|^2 \eta^{l-2} \right)^{\frac{1}{2}}. \]

**Proof.** We consider \( \phi = \eta^l \) in Proposition 2.2. Using Cauchy-Schwarz and Young’s inequalities and the definition of \( v \) we get
\[ \int u^{(n-1)/p} |\nabla|^{2} \eta^l \leq C \int u |\nabla|^{2p-1} |\nabla \eta|^{l-1} \]
\[ \leq \frac{1}{2} \int u^{(n-1)/p} |\nabla|^{2} \eta^l + C \int u^{(2-p)/(n-p)} |\nabla u|^{2(p-1)} |\nabla \eta|^2 \eta^{l-2} \]
and the first inequality follows. The second part of the statement can be obtained similarly, using Hölder’s inequality instead. \( \square \)

**Remark 2.5.** A similar estimate appears in [5] in the case \( p = 2. \)

### 2.2. Some a priori estimates.
We collect here some general lemmas concerning the behavior of positive solutions of the equation (1.1), that we will need in the proofs of our main theorems. The first is a lower bound for positive \( p \)-superharmonic functions.

**Lemma 2.6 ([27, Lemma 2.3]).** Let \( u \) be a positive weak solution of
\[ \Delta_p u \leq 0 \]
on \( \mathbb{R}^n \setminus K \) with \( K \) compact and \( 1 < p < n. \) Then there exist positive constants \( \rho, A > 0 \) such that
\[ u(x) \geq \frac{A}{|x|^{\frac{n-p}{p-1}}} \text{ for all } x \in B^c_\rho. \]

The next lemma provides bounds for the kinetic energy in terms of the potential energy for positive weak subsolutions of equation (1.1). In particular, solutions \( u \in L^{p^*}(\mathbb{R}^n) \) automatically have finite energy.
Lemma 2.7. Let \( u \) be a positive weak solution of
\[-\Delta_p u \leq u^{p^*-1} \text{ in } \mathbb{R}^n.\]
Then, for every \( \varepsilon > 0 \) there exists a constant \( C = C(n, p) > 0 \) such that for every \( R > 0 \)
\[\int_{B_{2R} \setminus B_R} |\nabla u|^p \leq C \left( 1 + \varepsilon^{-\frac{n}{n-p}} \right) \int_{B_{5R/2} \setminus B_{R/2}} u^{p^*} + C \varepsilon,\]
Moreover, there exists \( C = C(n, p) > 0 \) such that for every \( R > 0 \)
\[\int_{B_R} |\nabla u|^p \leq C \int_{B_{2R}} u^{p^*} + C \left( \int_{B_{2R}} u^{p^*} \right)^{\frac{n-p}{n}}.\]
In particular, if \( u \in L^p(\mathbb{R}^n) \), then \( |\nabla u| \in L^p(\mathbb{R}^n) \), i.e. \( E_{\mathbb{R}^n}(u) < \infty \).

Remark 2.8. A estimate similar to the second part of the statement appears in the proof of Theorems 1.2 and 1.4 in [5] in the case \( p = 2 \).

Proof. Testing the weak formulation given in Definition 2.1 with \( u^q \eta^q \), with \( q > 1 \) and where \( \eta \in C_0^\infty(\mathbb{R}^n) \), we obtain
\[\int u^\frac{mq}{n-p} \eta^q \geq \int |\nabla u|^p \eta^q + q \int u|\nabla u|^{p-2} (\nabla u, \nabla \eta) \eta^{q-1}.\]
i.e., from Cauchy-Schwarz and Young inequalities
\[\int |\nabla u|^p \eta^q \leq \int u^\frac{mq}{n-p} \eta^q + q \int u|\nabla u|^{p-1} |\nabla \eta| \eta^{q-1} \leq \int u^\frac{mq}{n-p} \eta^q + \frac{1}{2} \int |\nabla u|^p \eta^q + C \int |\nabla \eta|^p u^p \eta^{q-p} \leq C \left( 1 + \varepsilon^{-\frac{n}{n-p}} \right) \int u^\frac{mq}{n-p} \eta^q + \frac{1}{2} \int |\nabla u|^p \eta^q + \varepsilon \int |\nabla \eta|^n \eta^{q-n},\]
for every \( \varepsilon > 0 \). Let \( q > n \), for any \( R > 1 \), we choose \( \eta \in C_0^\infty(\mathbb{R}^n) \) such that \( \eta \equiv 1 \) in \( B_{2R} \setminus B_R \), \( \eta \equiv 0 \) in \( B_{R/2} \cup B_{5R/2}^c \), \( 0 \leq \eta \leq 1 \) on \( \mathbb{R}^n \) and \( \eta \) satisfies
\[|\nabla \eta|^2 \leq CR^{-2} \text{ in } (B_{5R/2} \setminus B_{2R}) \cup (B_R \setminus B_{R/2}).\]
Hence, for every \( \varepsilon > 0 \), we get
\[\int_{B_{2R} \setminus B_R} |\nabla u|^p \leq C \left( 1 + \varepsilon^{-\frac{n}{n-p}} \right) \int_{B_{5R/2} \setminus B_{R/2}} u^\frac{mq}{n-p} + C \varepsilon,\]
which is the first part of the statement. In order to prove the second part, let \( q > p \) and for any \( R > 1 \) choose \( \eta \in C_0^\infty(\mathbb{R}^n) \) such that \( \eta \equiv 1 \) in \( B_R \), \( \eta \equiv 0 \) in \( B_{2R}^c \), \( 0 \leq \eta \leq 1 \) on \( \mathbb{R}^n \) and \( \eta \) satisfies
\[|\nabla \eta|^2 \leq CR^{-2} \text{ in } B_{2R} \setminus B_R.\]
From (2.9) using Hölder inequality we get
\[\int_{B_R} |\nabla u|^p \leq C \int_{B_{2R}} u^\frac{mq}{n-p} + C \left( \int_{B_{2R}} u^\frac{mq}{n-p} \right)^{\frac{n-p}{n}} |B_{2R} \setminus B_R|^2 \leq C \int_{B_{2R}} u^\frac{mq}{n-p} + C \left( \int_{B_{2R}} u^\frac{mq}{n-p} \right)^{\frac{n-p}{n}}.\]
\[\square\]
Similar to the previous lemma, the following provides bounds for the potential energy in terms of the kinetic energy for positive weak supersolutions of the equation (1.1). In particular, solutions with \( \nabla u \in L^p(\mathbb{R}^n) \) automatically have finite energy. Since the proof is similar to the previous one, we will omit it.

**Lemma 2.9.** Let \( u \) be a positive weak solution of
\[
-\Delta_p u \geq u^{p^*-1} \quad \text{in } \mathbb{R}^n.
\]
Then, for every \( \varepsilon > 0 \) there exists a constant \( C = C(n, p) > 0 \) such that for every \( R > 0 \)
\[
\int_{B_{2R} \setminus B_R} u^{p^*} \leq C \left( 1 + \varepsilon^{-\frac{p}{n(p-1)}} \right) \int_{B_{5R/2} \setminus B_{R/2}} |\nabla u|^p + C \varepsilon,
\]
Moreover, there exists \( C = C(n, p) > 0 \) such that for every \( R > 0 \)
\[
\int_{B_R} u^{p^*} \leq C \int_{B_{2R}} |\nabla u|^p + C \left( \int_{B_{2R}} |\nabla u|^p \right)^\frac{n(p-1)}{n(p-1)+p}.
\]
In particular, if \( \nabla u \in L^p(\mathbb{R}^n) \), then \( u \in L^{p^*}(\mathbb{R}^n) \), i.e. \( E_{2\mathbb{R}^n}(u) < \infty \).

**Remark 2.10.** As already observed in the introduction, if \( u \) is a positive weak solution of equation (1.1), by Lemmas 2.7 and 2.9 for every \( \alpha > 0 \) one has
\[
E_{A_R}^{\text{pot}}(u) = O(R^{\alpha}) \quad \iff \quad E_{A_R}(u) = O(R^{\alpha}) \quad \iff \quad E_{A_R}^{\text{kin}}(u) = O(R^{\alpha}).
\]
Indeed by Lemma 2.7 there exists \( C > 0 \) such that, for every \( R > 0 \), we have
\[
E_{A_R}^{\text{pot}}(u) \leq E_{A_R}(u) \leq (C + 1) \int_{B_{5R/2} \setminus B_{R/2}} u^{p^*} + 1 \leq (C + 1) \left( E_{A_{5R/2}}^{\text{pot}}(u) + E_{A_R}^{\text{pot}}(u) + E_{A_{2R}}^{\text{pot}}(u) \right) + 1.
\]
Similarly by Lemma 2.9 for every \( R > 0 \)
\[
E_{A_R}^{\text{kin}}(u) \leq E_{A_R}(u) \leq (C + 1) \int_{B_{5R/2} \setminus B_{R/2}} |\nabla u|^p + 1 \leq (C + 1) \left( E_{A_{5R/2}}^{\text{kin}}(u) + E_{A_R}^{\text{kin}}(u) + E_{A_{2R}}^{\text{kin}}(u) \right) + 1.
\]
Thus we conclude.

### 3. A sharp gradient estimate

In this section we will prove the sharp gradient estimate in Proposition 1.7. To the best of our knowledge this result is new and we believe that it may have independent interest.

We begin by defining the (second order part of the) linearized \( p \)-Laplace operator (see e.g. [16, 29])
\[
P_f(w) := |\nabla f|^{p-2} \Delta w + (p - 2)|\nabla f|^{p-4} \nabla^2 w(\nabla f, \nabla f).
\]
Observe that \( P_f(f) = \Delta_p f \). The following inequality follows from the extension to \( p \)-Laplace operator of the classical Bochner formula (see e.g. [16, 29]).
Lemma 3.1. Given $x \in \mathbb{R}^n$, a domain $U$ containing $x$ and a function $f \in C^3(U)$, if $|\nabla f|(x) \neq 0$, at $x$ it holds

$$\frac{1}{p} P_f(|\nabla f|^p) \geq \frac{1}{n} (\Delta_p f)^2 + \frac{n}{n-1} \left( \frac{1}{n} \Delta_p f - (p - 1)|\nabla f|^{p-4}\nabla^2 f(\nabla f, \nabla f) \right)^2$$

$$+ |\nabla f|^{p-2} \left[ \langle \nabla f, \nabla \Delta_p f \rangle - (p - 2) \frac{\Delta_p f}{|\nabla f|^2} \nabla^2 f(\nabla f, \nabla f) \right]$$

Proof. It follows combining the p-Bochner formula [29, Proposition 3.1] with the sharp estimate [29, Lemma 3.2].

3.1. Proof of Proposition 1.7. Let $u$ be a positive weak solution of equation (1.1) with $1 < p < n$ and define $f := u^a > 0$, $a \in \mathbb{R} \setminus \{0\}$. By regularity we know that $f$ is smooth where $|\nabla f| > 0$. Thus, where $|\nabla f| > 0$, we have

$$\nabla f = au^{-1} \nabla u$$

and,

$$\Delta_p f = \text{div} (|\nabla f|^{p-2} \nabla f)$$

$$= a|a|^{p-2} \text{div} (u^{(a-1)(p-1)} |\nabla u|^{p-2} \nabla u)$$

$$= a|a|^{p-2} u^{(a-1)(p-1)} \Delta_p u + a|a|^{p-2} (a-1)(p-1)u^{(a-1)(p-1)-1} |\nabla u|^p$$

$$= -a|a|^{p-2} u^{(a-1)(p-1)+q} + a|a|^{p-2} (a-1)(p-1)u^{(a-1)(p-1)-1} |\nabla u|^p$$

with $q = p^* - 1$. Using Lemma 3.1, where $|\nabla f| > 0$, we obtain

$$\frac{1}{p} P_f(|\nabla f|^p) \geq \frac{1}{n} (\Delta_p f)^2 + \frac{n}{n-1} \left( \frac{1}{n} \Delta_p f - (p - 1)|\nabla f|^{p-4}\nabla^2 f(\nabla f, \nabla f) \right)^2$$

$$+ |\nabla f|^{p-2} \left[ \langle \nabla f, \nabla \Delta_p f \rangle - (p - 2) \frac{\Delta_p f}{|\nabla f|^2} \nabla^2 f(\nabla f, \nabla f) \right]$$

$$\geq \frac{1}{n-1} (\Delta_p f)^2 + |\nabla f|^{p-2} \langle \nabla f, \nabla \Delta_p f \rangle - \frac{2(p-1)+(n-1)(p-2)}{(n-1)} |\nabla f|^{p-4} \Delta_p f \nabla^2 f(\nabla f, \nabla f)$$

$$= \frac{1}{n-1} (\Delta_p f)^2 + |\nabla f|^{p-2} \langle \nabla f, \nabla \Delta_p f \rangle + c_1 |\nabla f|^{-2} \Delta_p f \langle \nabla |\nabla f|^p, \nabla f \rangle$$

$$\geq \frac{(1-a)(p-1)+a(n-p)}{a(n-1)} |\nabla f|^{2p} - c_2 f^{\frac{(a-1)(p-1)+q}{a}} |\nabla f|^{p-2}$$

$$+ \left( c_1 |\nabla f|^{-2} \Delta_p f + c_3 f^{-1} |\nabla f|^{p-2} \langle \nabla |\nabla f|^p, \nabla f \rangle ight)$$

with

$$c_1 = -\frac{2(p-1)+(n-1)(p-2)}{p(n-1)}, \quad c_2 = |a|^{p-2} \left[ \frac{n+1}{n-1} (a-1)(p-1) + q \right],$$

$$c_3 = \frac{(a-1)(p-1)}{a}, \quad c_4 = -a|a|^{p-2} c_1, \quad c_5 = \frac{(a-1)(p-1)}{a} c_1 + c_3.$$

We choose $a$

$$a := -\frac{p-1}{n-p} + \varepsilon$$

(3.1)
for a given $0 < \varepsilon < \frac{B - 1}{n - p}$. Then there exists $\lambda = \lambda(\varepsilon) > 0$ such that, where $|\nabla f| > 0$, we have
\[
\frac{1}{p} P_f(|\nabla f|^p) \geq \lambda f^{-2} |\nabla f|^{2p} - c_2 f^{(a - 1)(p - 1) + q + 1} - 1 |\nabla f|^p
\]
\[
+ \left( c_4 f^{(a - 1)(p - 1) + q + 3} |\nabla f|^{-2} + c_5 f^{-1} |\nabla f|^{p - 2} \right) \langle \nabla |\nabla f|^p, \nabla f \rangle.
\] (3.2)

If $|\nabla f|$ achieves its maximum in $B_{2R}$ at some point $\bar{x} \in B_{2R}$, we have
\[
\nabla |\nabla f|^p = 0 \quad \text{and} \quad P_f(|\nabla f|^p) \leq 0 \quad \text{at} \ \bar{x}
\]
and (3.2) implies, at $\bar{x}$,
\[
0 \geq f^{-2} |\nabla f|^p \left( \lambda |\nabla f|^p - c_2 f^{(a - 1)(p - 1) + q + 1} \right) = f^{-2} |\nabla f|^p \left( \lambda |\nabla f|^p - c_2 u^{(a - 1)(p - 1) + q + 1} \right).
\]

Moreover, since $q + 1 = \frac{\sup}{n - p}$ we have
\[
\theta := (a - 1)(p - 1) + q + a = \frac{\theta}{n - p} + p\varepsilon > 0.
\] (3.3)

We obtain
\[
\sup_{B_{2R}} |\nabla f| \leq C \sup_{B_{2R}} u^\theta \iff |\nabla u(x)| \leq C \left( \sup_{B_{2R}} u^{\theta + \varepsilon} \right) u(x)^{\frac{n - 1}{n - p} - \varepsilon}
\]
for all $x \in B_{2R}$.

On the other hand, if $|\nabla f|$ does not achieve its maximum at some point $\tilde{x}$, we have to employ a cutoff argument. For a given $0 < \delta < \frac{1}{2}$ there exist nonnegative cutoff functions $\phi = \phi(|x|)$ with $\phi \equiv 1$ on $B_{2R}$, $\phi \equiv 0$ on $B_{2R}^c$, $0 \leq \phi \leq 1$ on $\mathbb{R}^n$ and such that
\[
|\nabla \phi| \leq \frac{C}{R} \phi^{1 - \delta}, \quad |\nabla^2 \phi| \leq \frac{C}{R^2} \phi^{1 - 2\delta},
\] (3.4)
on $B_{2R} \setminus B_R$ for some $C > 0^2$. Let
\[
H := \phi |\nabla f|^p
\]
and $\bar{x}$ be a maximum point of $H$. We can assume that $\phi(\bar{x}) > 0$ and $|\nabla f|(\bar{x}) > 0$. At $\bar{x}$ we have
\[
\nabla H = 0, \quad P_f H \leq 0,
\]
Therefore, at $\bar{x}$, we have
\[
\nabla H = 0 \iff \nabla |\nabla f|^p = -\phi^{-2} H \nabla \phi.
\] (3.5)

Moreover, using (3.5), we have
\[
\nabla_i \nabla_j H = \phi^{-1} H \nabla_i \nabla_j \phi - 2 \phi^{-2} H \nabla_i \phi \nabla_j \phi + \phi \nabla_i \nabla_j |\nabla f|^p
\]
and thus
\[
\Delta H = \phi^{-1} H \Delta \phi - 2 \phi^{-2} H |\nabla \phi|^2 + \phi \Delta |\nabla f|^p.
\]

Using the definition of $P_f$, at $\bar{x}$ we obtain
\[
0 \geq |\nabla f|^{4 - p} P_f H
\]
\[
= |\nabla f|^{2\phi - 2} \left( \phi \Delta \phi - 2 |\nabla \phi|^2 \right) H + \phi |\nabla f|^2 \Delta |\nabla f|^p
\]
\[
\quad + (p - 2) \phi^{-2} \left[ \phi^2 \nabla^2 \phi |\nabla f|^2 H - 2 \langle \nabla \phi, \nabla f \rangle^2 H + \phi^3 \nabla^2 |\nabla f|^p (\nabla f, \nabla f) \right]
\]
\[
\geq \phi |\nabla f|^{4 - p} P_f |\nabla f|^p - \frac{C}{R^2} \phi^{-2\delta} |\nabla f|^2 H,
\]
\[\text{We observe that such cutoff functions can be obtained setting } \phi(x) = \psi \left( \frac{|x|}{R} \right)^{1/\delta}, \text{ where } \psi \in C^2([0, \infty)) \text{ is such that } \psi \equiv 1 \text{ in } [0, 1), \psi \equiv 0 \text{ in } [2, \infty) \text{ and } 0 \leq \psi \leq 1.\]
i.e.

\[ 0 \geq \phi^{1+2\delta} P_f |\nabla f|^p - \frac{C}{R^2} \phi^{-\frac{n-2}{p-1}} H^{\frac{2(p-1)}{p}}. \]  

(3.6)

From (3.2), we have

\[ \frac{1}{p} P_f(|\nabla f|^p) \geq \lambda f^{-2} \phi^{-2} H^2 - c_2 f^{-2} u^\theta \phi^{-1} H - \left(c_4 f^{-1} u^\theta |\nabla f|^2 + c_5 f^{-1} |\nabla f|^{p-2}\right) \phi^{-2} \langle \nabla \phi, \nabla f \rangle H, \]

where \( \theta > 0 \) is defined in (3.3). We get

\[ \frac{1}{p} P_f(|\nabla f|^p) \geq \lambda f^{-2} \phi^{-2} H^2 - C f^{-2} u^\theta \phi^{-1} H - \frac{C}{R} f^{-1} u^\theta \phi^{-1-\delta+\frac{1}{p}} H^{\frac{p}{p-1}} - C \frac{C}{R} \phi^{-1-\delta-\frac{n-2}{p}} H^{\frac{2(p-1)}{p}}. \]

Using Lemma 2.6, on \( B_{2R} \) we have

\[ f = u^\alpha \leq CR^{-n-\frac{n-p}{p-1}} = CR^{1-\varepsilon n-\frac{n-p}{p-1}} \iff -\frac{1}{R} \geq \frac{CR^{-\frac{n-p}{p-1}}}{f}. \]

Therefore

\[ \frac{1}{p} P_f(|\nabla f|^p) \geq f^{-2} \left( \lambda \phi^{-2} H^2 - \phi^{-1} u^\theta H - \phi^{-1-\delta+\frac{1}{p}} u^\theta H^{\frac{n-1}{p}} - \phi^{-1-\delta-\frac{n-2}{p}} H^{\frac{2(p-1)}{p}} \right). \]

Using (3.6) we obtain

\[ 0 \geq \lambda \phi^{-2} H^2 - \phi^{-1} u^\theta H - \phi^{-1-\delta+\frac{1}{p}} u^\theta H^{\frac{n-1}{p}} - \phi^{-1-\delta-\frac{n-2}{p}} H^{\frac{2(p-1)}{p}} \]
\[ - C \phi^{-1-\delta-\frac{n-2}{p}} H^{\frac{2(p-1)}{p}} - C \phi^{-1-2\delta-\frac{n-2}{p}} H^{\frac{2(p-1)}{p}}. \]

Now, choosing \( \delta < \min\{\frac{1}{p}, \frac{1}{2}\} \), since \( 0 \leq \phi \leq 1 \), at \( \bar{x} \) we obtain

\[ 0 \geq \lambda H^2 - C u^\theta H - C u^\theta R^{-\frac{n-2}{p}} H^{\frac{n-1}{p}} - CR^{-\frac{n-2}{p}} H^{\frac{2(p-1)}{p}} \]
\[ \geq \frac{\lambda}{2} H^2 - C \left( u^\theta + R^{-\frac{n-2}{p}} H^{\frac{n-1}{p}} \right), \]

where we used Young’s inequality\(^3\). This clearly implies

\[ H \leq H(\bar{x}) \leq C \left( u^\theta(\bar{x}) + R^{-\frac{n(p-\rho)}{p-1}} \right) \quad \text{on } B_{2R} \]

for every \( R > 0 \), and in particular

\[ |\nabla f|^p \leq C \left( \sup_{B_{2R}} u^\theta + R^{-\frac{n(p-\rho)}{p-1}} \right) \quad \text{on } B_{R}, \]

i.e.

\[ |\nabla u| \leq C \left( \sup_{B_{2R}} u^\frac{1}{n-p} + \varepsilon + R^{-\frac{n-\rho}{p-1}} \right) u^{\frac{n-1}{n-p} \varepsilon} \quad \text{on } B_{R} \]

which is the thesis.

\[ \square \]

In case \( u \) is controlled by a power of the distance at infinity, one can obtain the following point-wise estimate on the gradient of \( u \) in terms of \( u \). In \cite[Corollary 2.2]{17} the authors obtained a logarithmic gradient estimate in case \( p = 2 \).

\[^3\text{We remark that on the third term } u^\theta R^{-\frac{p-\rho}{p-1}} H^{\frac{n-1}{p}} \text{ we used the following generalization of the classical Young’s inequality:} \]
\[ abc \leq \varepsilon a^r + k_1(\varepsilon)b^s + k_2(\varepsilon)c^t, \]

for all \( a, b, c \geq 0, \varepsilon > 0 \) and where \( r, s, t > 1 \) are such that \( \frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1 \).
Corollary 3.2. Let $u$ be a positive weak solution of equation (1.1) with $1 < p < n$. Assume
\[ u(x) \leq C|x|^\alpha \]
for all $x \in B_1$, for some $\alpha \in \mathbb{R}$. Then, for every $0 < \varepsilon < \frac{p-1}{n-p}$ it holds
\[ |\nabla u(x)| \leq C \left( |x|^{\frac{1}{n-p-\varepsilon}} + |x|^{-\varepsilon (\alpha-1)} \right) u(x)^{\frac{n-1}{n-p-\varepsilon}} \]
for some $C = C(n, p, \varepsilon, \alpha) > 0$, for every $x \in B_1$.

Proof. It is sufficient to apply Proposition 1.7 on $B_{|x|/4}(x)$ at the point $x$. \hfill \Box

4. Rigidity with energy control

In this section we prove Theorem 1.1.

4.1. Proof of Theorem 1.1 (i). Let $u$ be a positive weak solution of equation (1.1) with $1 < p \leq \frac{2n}{n+1}$. From Corollary 2.4 with $l = 2$ we have
\[ \int u^{(n-1)p \over n-p} |\nabla|^2 \eta^2 \leq C \int u^{(2-p)n-p \over n-p} |\nabla u|^{2(p-1)} |\nabla \eta|^2. \tag{4.1} \]
for every $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta \geq 0$. For any $R > 1$, we choose $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\eta \equiv 1$ in $B_R$, $\eta \equiv 0$ in $B_{2R}^\epsilon$, $0 \leq \eta \leq 1$ on $\mathbb{R}^n$ and $\eta$ satisfies
\[ |\nabla \eta|^2 \leq CR^{-2} \text{ in } A_R = B_{2R} \setminus B_R. \]
We show that the integral on the righthand side of (4.1) is uniformly bounded in $R$. Since $p \leq \frac{2n}{n+1} < 2$, we have
\[ \frac{(2-p)n-p}{n-p} \geq 0 \quad \text{and} \quad 2(p-1) < p. \]
If $1 < p < \frac{2n}{n+1}$, using Hölder inequality we obtain
\[ \int u^{(2-p)n-p \over n-p} |\nabla u|^{2(p-1)} |\nabla \eta|^2 \leq \frac{C}{R^2} \left( \int_{A_R} u^{n \over np} \right)^{2(p-1) \over n} \left( \int_{A_R} |\nabla u|^p \right)^{2 \over p} |A_R|^{1 \over n} \]
\[ \leq \frac{C}{R} E_{AR}^\text{pot}(u)^{(2-p)n-p \over np} E_{AR}^\text{kin}(u)^{2(p-1) \over p} \]
\[ \leq \frac{C}{R} E_{AR}(u)^{n-1 \over n}. \]
If $p = \frac{2n}{n+1}$, similarly we have
\[ \int u^{(2-p)n-p \over n-p} |\nabla u|^{2(p-1)} |\nabla \eta|^2 = \int |\nabla u|^{2(n+1) \over n+1} |\nabla \eta|^2 \]
\[ \leq \frac{C}{R^2} \left( \int_{A_R} |\nabla u|^{2n \over n+1} \right)^{n-1 \over n} |A_R|^{1 \over n} \]
\[ \leq \frac{C}{R} E_{AR}^\text{kin}(u)^{n-1 \over n} \]
\[ \leq \frac{C}{R} E_{AR}(u)^{n-1 \over n}. \]
Thanks to the energy assumptions, in both cases we have that the righthand side of (4.1) is uniformly bounded in $R$. Hence
\[ \int_{\mathbb{R}^n} u^{(n-1)p \over n-p} |\nabla|^2 < \infty, \]
and by the second inequality in Corollary 2.4, passing to the limit as \( R \) tends to infinity, we obtain
\[
\int_{\mathbb{R}^n} u^{(2-p)n-p} |\nabla|^2 = 0,
\]
i.e.
\[
\nabla = \nabla v - \frac{\nabla v}{n} \text{Id}_n \equiv 0 \quad \text{in } \Omega^c.
\]
Let \( \Omega_0 \subseteq \Omega^c \) be a connected component of \( \Omega^c \). Since \( 0 < u \in C^{1,\alpha}_0(\mathbb{R}^n) \) then
\[
v = u - \frac{n-p}{n} \in C^{1,\alpha}_0(\mathbb{R}^n).
\]
Since
\[
v = -\left(\frac{n-p}{p}\right)^{p-1} |\nabla v|^{p-2} \nabla v
\]
we get
\[
\text{div } v = -\left(\frac{n-p}{p}\right)^{p-1} \Delta_p v
\]
\[
= u \frac{n(p-1)}{n-p} \Delta_p u - \frac{n(p-1)}{n-p} u \frac{p(n-1)}{n-p} |\nabla u|^p
\]
\[
= -u^{\frac{p}{n-p}} - \frac{n(p-1)}{n-p} u \frac{p(n-1)}{n-p} |\nabla u|^p \in C^{0,\alpha}_0(\mathbb{R}^n).
\]
By standard elliptic regularity, we have \( v \in C^{2,\alpha}_0(\Omega_0) \), \( u \in C^{2,\alpha}(\Omega_0) \) and then \( \text{div } v \in C^{1,\alpha}_0(\Omega_0) \). Differentiating (4.2), we get
\[
\partial_i (\text{div } v) = n \partial_i (\text{div } v).
\]
Therefore \( \text{div } v = \text{const on } \Omega_0 \) and thus \( v = C(x - x_0) \), for some \( C \in \mathbb{R} \) and some \( x_0 \in \mathbb{R}^n \). Thus
\[
v = C_1 + C_2 |x - x_0|^{\frac{2}{p-1}}
\]
on \( \Omega_0 \), for some \( C_1, C_2 > 0 \). Then \( u(x) = U_{A, x_0}(x) \) on \( \Omega_0 \) for some \( \lambda > 0 \) and \( x_0 \in \mathbb{R}^n \). Since the argument above holds whenever \( \nabla u \neq 0 \), we must have \( \Omega_0 = \mathbb{R}^n \setminus \{x_0\} \) and the result follows.

4.2. Proof of Theorem 1.1 (ii). Let \( u \) be a positive weak solution of equation (1.1) with \( \frac{2n}{n+1} < p < 2 \). From Corollary 2.4 with \( l = 2 \) we have
\[
\int u^{(n-1)p} |\nabla|^2 \eta^2 \leq C \int u^{\frac{(2-p)n-p}{n-p}} |\nabla u|^{2(p-1)} |\nabla \eta|^2.
\]
We choose the same cutoff functions as in (i). Since \( \frac{2n}{n+1} < p < 2 \), we have
\[
\frac{(2-p)n-p}{n-p} < 0 \quad \text{and} \quad 2(p-1) < p.
\]
Thanks to Lemma 2.6 we have \( u \geq CR^{\frac{n-p}{p-1}} \) on \( A_R \) and using Hölder inequality we get
\[
\int u^{\frac{(2-p)n-p}{n-p}} |\nabla u|^{2(p-1)} |\nabla \eta|^2 \leq CR^{\frac{(n-1)(2-p)}{p-1}} \left( \int_{A_R} |\nabla u|^p \right)^{\frac{2(p-1)}{p}} |A_R|^{\frac{2-p}{p}}
\]
\[
\leq CR^{\frac{(n-p)(2-p)}{p(p-1)}} E_{A_R}(u)^{\frac{2(p-1)}{p}}
\]
\[
\leq CR^{\frac{(n-p)(2-p)}{p(p-1)}} E_{A_R}(u)^{\frac{2(p-1)}{p}}
\]
Thanks to the energy assumption, we have that the righthand side of (4.3) is uniformly bounded in \( R \). Hence

\[
\int_{\mathbb{R}^n} u^{(n-1)p\over n-p} |\nabla|^2 = 0,
\]

and the conclusion follows as in the proof of Theorem 1.1 (i).

4.3. Proof of Theorem 1.1 (iii). Let \( u \) be a positive weak solution of equation (1.1) with \( 2 < p < n \) and assume

\[
|u(x)| \leq C|x|^\alpha,
\]
as \( |x| \to \infty \) for some \( \alpha \geq 0 \). From Corollary 2.4 with \( l = 2 \) we have

\[
\int u^{(n-1)p\over n-p} |\nabla|^2 \eta^2 \leq C \int u^{(2-p)n-p\over n-p} |\nabla|^2 |2(p-1)|\eta|^2. \tag{4.4}
\]

We choose the same cutoff functions as in (i). We have

\[
\int u^{(2-p)n-p\over n-p} |\nabla|^2 |2(p-1)|\eta|^2 = \int u^{(2-p)n-p\over n-p} |\nabla|^\theta |\nabla|^2 |2(p-1)-\theta|\eta|^2
\]

\[
\leq CR\left(\frac{1}{n-p}+\varepsilon\right)\alpha\theta \int u^{(2-p)n-p+\theta(n-1)-(n-p)\theta\over n-p} |\nabla|^2 |2(p-1)-\theta|\eta|^2,
\]

for every \( \theta > 0 \) and every \( \varepsilon > 0 \) small enough, where we used the gradient estimate in Corollary 3.2. We assume

\[
p-2 < \theta < 2(p-1) \quad \text{and} \quad \theta > \frac{(p-2)n+p}{n-1-\varepsilon(n-p)}
\]

and we apply Hölder inequality to obtain

\[
\int u^{(2-p)n-p\over n-p} |\nabla|^2 |2(p-1)|\eta|^2
\]

\[
\leq CR\left(\frac{1}{n-p}+\varepsilon\right)\alpha\theta \int u^{(2-p)n-p+\theta(n-1)-(n-p)\theta\over n-p} |\nabla|^2 |2(p-1)-\theta|\eta|^2
\]

\[
\leq CR\left(\frac{1}{n-p}+\varepsilon\right)\alpha\theta \int u^{(2-p)n-p+\theta(n-1)-(n-p)\theta\over n-p} |\nabla|^2 |2(p-1)-\theta|\eta|^2
\]

\[
\leq CR\left(\frac{1}{n-p}+\varepsilon\right)\alpha\theta \int u^{(2-p)n-p+\theta(n-1)-(n-p)\theta\over n-p} |\nabla|^2 |2(p-1)-\theta|\eta|^2
\]

Under our assumptions, \( E_{AR}(u) \leq CR^k \) for some \( k > 0 \). Then

\[
\int u^{(2-p)n-p\over n-p} |\nabla|^2 |2(p-1)|\eta|^2 \leq CR\left(\frac{1}{n-p}+\varepsilon\right)\alpha\theta \int u^{(2-p)n-p+\theta(n-1)-(n-p)\theta\over n-p} |\nabla|^2 |2(p-1)-\theta|\eta|^2
\]

Since \( p-2 < \frac{(p-2)n+p}{n-1-\varepsilon(n-p)} < 2(p-1) \), by choosing \( \theta \) close to \( \frac{(p-2)n+p}{n-1-\varepsilon(n-p)} \) and \( \varepsilon \) close to 0, thanks to the energy assumption, we have that the righthand side of (4.4) is uniformly bounded in \( R \). Hence

\[
\int_{\mathbb{R}^n} u^{(n-1)p\over n-p} |\nabla|^2 = 0,
\]

and the conclusion follows as in the proof of Theorem 1.1 (i).
5. RIGIDITY WITH CONTROL AT INFINITY

In this section we prove Theorems 1.3 and 1.4. We start by proving the following weak energy estimate which shows that a weighted energy has controlled growth on balls.

**Lemma 5.1.** Let $u$ be a positive weak solution of equation (1.1) for some $1 < p < n$ and let $t < -1$. Then, for every $R > 1$, we have
\[
\int_{B_R} u \frac{np+p(n-p)}{n-p} + \int_{B_R} u^t |\nabla u|^p \leq CR^\beta,
\]
for some $C = C(n, p, t) > 0$ and
\[
\beta := \begin{cases} \frac{t(n-p)}{p} & t + p > 0 \\ \frac{t(n-p)}{p-1} & t + p \leq 0. \end{cases}
\]

**Proof.** We choose $\eta \in C_0^\infty(\mathbb{R}^n)$ be such that $\eta \equiv 1$ in $B_R$, $\eta \equiv 0$ in $B^c_{2R}$, $0 \leq \eta \leq 1$ on $\mathbb{R}^n$ and $\eta$ satisfies
\[
|\nabla \eta|^2 \leq CR^{-2} \quad \text{in} \ A_R = B_{2R} \setminus B_R.
\]
Testing the weak formulation given in 2.1 with $u^{l+1} \eta^l$, for $l$ sufficiently large, we obtain
\[
- \int u \frac{np+p(n-p)}{n-p} \eta^l \geq (t + 1) \int u^t |\nabla u|^p \eta^l - l \int u^{l+1} |\nabla u|^{p-2} (\nabla u, \nabla \eta) \eta^{l-1}
\]
\[
\geq (t + 1) \int u^t |\nabla u|^p \eta^l - l \int u^{l+1} |\nabla u|^{p-1} |\nabla \eta| \eta^{l-1}
\]
If $t + p > 0$, we get
\[
- \int u \frac{np+p(n-p)}{n-p} \eta^l \geq (t + 1) \int u^t |\nabla u|^p \eta^l - \varepsilon \int u \frac{np+p(n-p)}{n-p} \eta^l - \varepsilon \int u^t |\nabla u|^p \eta^l
\]
\[
- C_\varepsilon \int |\nabla \eta| \frac{np+p(n-p)}{p} \eta^l - C_\varepsilon \int u \frac{np+p(n-p)}{n-p} \eta^l - C_\varepsilon R \frac{np+p(n-p)}{p} |B_{2R}|
\]
for some $\varepsilon > 0$, where we used Cauchy-Schwarz and Young’s inequalities. Choosing $\varepsilon$ small enough, we conclude.

On the other hand, if $t + p \leq 0$, we get
\[
- \int u \frac{np+p(n-p)}{n-p} \eta^l \geq (t + 1) \int u^t |\nabla u|^p \eta^l - \varepsilon \int u^t |\nabla u|^p \eta^l - C_\varepsilon \int u^{t+p} |\nabla \eta|^{p} \eta^{l-p}
\]
\[
\geq (t + 1) \int u^t |\nabla u|^p \eta^l - C_\varepsilon R \frac{(t+p)(n-p)}{p-1} \eta^{l-p}
\]
\[
\geq (t + 1) \int u^t |\nabla u|^p \eta^l - C_\varepsilon R \frac{(t+1)(n-p)}{p-1}
\]
for some $\varepsilon > 0$, where we used Lemma 2.6. Choosing $\varepsilon$ small enough, we conclude.

\[\square\]

---

4We remark that on the third term $u^{t+1} |\nabla u|^{p-1} |\nabla \eta| \eta^{l-1}$ we used the following generalization of the classical Young’s inequality:
\[
abc \leq \varepsilon a^r + \varepsilon b^s + k(\varepsilon)c^t,
\]
for all $a, b, c \geq 0$, $\varepsilon > 0$ and where $r, s, t > 1$ are such that $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1$. 
5.1. **Proof of Theorems 1.2 and 1.3.** Let $u$ be a positive weak solution of equation (1.1) with $1 < p < 2$ and let $\gamma > 0$. Let $\eta$ as in the proof of the previous lemma. From Corollary 2.4 with $l = 2$ we have

$$
\int u^{(n-1)p}_{n-p} |\nabla|^{2} \eta^{2} \leq C \int u^{(2-p)n-p}_{n-p} |\nabla u|^{2(p-1)} |\nabla \eta|^{2}.
$$

(5.1)

We choose $\eta \in C_{0}^{\infty}(\mathbb{R}^{n})$ be such that $\eta \equiv 1$ in $B_{R}$, $\eta \equiv 0$ in $B_{2R}^{c}$, $0 \leq \eta \leq 1$ on $\mathbb{R}^{n}$ and $\eta$ satisfies $|\nabla \eta|^{2} \leq CR^{-2}$ in $A_{R} = B_{2R} \setminus B_{R}$.

Since $p < 2$, then $2(p - 1) < p$ and, by Hölder inequality we obtain

$$
\int u^{(2-p)n-p}_{n-p} |\nabla u|^{2(p-1)} |\nabla \eta|^{2}
\leq \sup_{A_{R}} u^{\gamma} \int u^{(2-p)n-p}_{n-p} - \gamma |\nabla u|^{2(p-1)} |\nabla \eta|^{2}
\leq \frac{1}{R^{2}} \sup_{A_{R}} u^{\gamma} \int u^{(2-p)n-p}_{n-p} - \frac{2(p-1)}{p} \gamma \left(\int A_{R} u |\nabla u|^{p} \right)^{\frac{2(p-1)}{p}}
\leq \frac{1}{R^{2}} \sup_{A_{R}} u^{\gamma} \left(\int A_{R} u^{\gamma} |\nabla u| \right)^{\frac{2(p-1)}{p}} \left(\int A_{R} u^{\frac{(2-p)n-p - 2\gamma(n-p) - 2(p-1)(n-p)}{(n-p)(2-p)}} \right)^{\frac{2}{2p}}.
$$

We choose

$$
t = \bar{t} := - \frac{p}{n - p} - \gamma
$$

in order to have

$$
\frac{(2 - p)np - p^{2} - \gamma p(n - p) - 2\bar{t}(p - 1)(n - p)}{(n-p)(2-p)} = \frac{np + \bar{t}(n-p)}{n-p}.
$$

We observe that

$$
\bar{t} < -1 \iff (\gamma - 2)p < (\gamma - 1)n.
$$

(5.2)

Then, from Lemma 5.1, we obtain

$$
\int u^{(2-p)n-p}_{n-p} |\nabla u|^{2(p-1)} |\nabla \eta|^{2} \leq \frac{C}{R^{2}} \sup_{A_{R}} u^{\gamma} \left(\int A_{R} u^{\gamma} |\nabla u| \right)^{\frac{2(p-1)}{p}} \left(\int A_{R} u^{\frac{(2-p)n-p - 2\gamma(n-p) - 2(p-1)(n-p)}{(n-p)(2-p)}} \right)^{\frac{2}{2p}}.
$$

(5.3)

5.2. **Proof of Theorem 1.2 (i).** Let $n = 2$, $1 < p < 2$ and choose $\gamma = 0$. In particular (5.2) is satisfied and $\bar{t} + p = -\frac{p(p-1)}{2-p} < 0$. Then, $\beta = 2$ and from (5.3) we get

$$
\int u^{\frac{4-3p}{2-p}} |\nabla u|^{2(p-1)} |\nabla \eta|^{2} \leq C;
$$

for every $R > 0$. Hence, arguing as in the proof of Theorem 1.1 (i), from Corollary 2.4 the conclusion follows.

5.3. **Proof of Theorem 1.2 (ii).** Let $n = 3$. If $\frac{3}{2} < p < 2$ we again choose $\gamma = 0$. In particular (5.2) is satisfied and $\bar{t} + p = \frac{p(2-p)}{3-p} > 0$. Then, $\beta = 1$ and from (5.3) we get

$$
\int u^{\frac{6-4p}{3-p}} |\nabla u|^{2(p-1)} |\nabla \eta|^{2} \leq CR^{-1} \longrightarrow 0,
$$

as $R$ tends to $\infty$. Hence, arguing as in the proof of Theorem 1.1 (i), from Corollary 2.4 the conclusion follows.
5.4. **Proof of Theorem 1.3 (i).** If $1 < p \leq \frac{3}{2}$ we assume

$$u(x) \leq C|x|^\alpha,$$

as $|x| \to \infty$, for some $\alpha < \bar{\alpha} := \frac{3(p-1)(3-p)}{p(3-2p)}$. If $\gamma$ satisfies (5.2), from (5.3) we get

$$\int u^{\frac{6-4p}{3+p}} |\nabla u|^{(p-1)|\nabla \eta|^2} \leq CR^{3-2+\alpha \gamma}.$$

We choose $\alpha$ and $\gamma$ such that

$$\begin{cases} (\gamma - 2)p < 3(\gamma - 1) \\ \beta - 2 + \alpha \gamma \leq 0 \end{cases} \iff \begin{cases} \gamma > \frac{3-2p}{3-p} \\ \alpha \leq \frac{2-\beta}{\gamma}. \end{cases}$$

For $\gamma$ close to $\frac{3-2p}{3-p}$, we have $\bar{\beta} + p$ close to $p - 1 > 0$. Then

$$\beta = 1 + \frac{\gamma(n - p)}{p} \quad \text{and} \quad \alpha \leq \frac{p(p + 1) - n \gamma}{p \gamma}.$$

Since the right-hand side in the second inequality is decreasing in $\gamma$, it is sufficient to have

$$\alpha < \frac{(3p-1)(3-p)}{p(3-2p)} = \bar{\alpha}.$$

Letting $R \to \infty$, from (5.1) we obtain

$$\int_{\mathbb{R}^n} u^{\frac{2n}{n-2p}} |\tilde{V}|^2 = 0,$$

and the conclusion follows as in the proof of Theorem 1.1 (i).

5.5. **Proof of Theorem 1.3 (ii).** Let $n \geq 4$, $1 < p < 2$ and assume

$$u(x) \leq C|x|^\alpha,$$

as $|x| \to \infty$, for some $\alpha < \bar{\alpha} := \frac{(3p-n)(n-p)}{p(n-2p)}$. If $\gamma$ satisfies (5.2), from (5.3) we get

$$\int u^{\frac{(2-p)n-p}{n-2p}} |\nabla u|^{(p-1)|\nabla \eta|^2} \leq CR^{3-2+\alpha \gamma}.$$

We choose $\alpha$ and $\gamma$ such that

$$\begin{cases} (\gamma - 2)p < n(\gamma - 1) \\ \beta - 2 + \alpha \gamma \leq 0 \end{cases} \iff \begin{cases} \gamma > \frac{n-2p}{n-p} \\ \alpha \leq \frac{2-\beta}{\gamma}. \end{cases}$$

For $\gamma$ close to $\frac{n-2p}{n-p}$, we have $\bar{\beta} + p$ close to $p - 1 > 0$. Then

$$\beta = 1 + \frac{\gamma(n - p)}{p} \quad \text{and} \quad \alpha \leq \frac{p(p + 1) - n \gamma}{p \gamma}.$$

Since the right-hand side in the second inequality is decreasing in $\gamma$, it is sufficient to have

$$\alpha < \frac{(3p-n)(n-p)}{p(n-2p)} = \bar{\alpha}.$$

Letting $R \to \infty$, from (5.1) we obtain

$$\int_{\mathbb{R}^n} u^{\frac{(n-1)n}{n-2p}} |\tilde{V}|^2 = 0,$$

and the conclusion follows as in the proof of Theorem 1.1 (i).
5.6. **Proof of Theorem 1.4.** Let \( p > 2 \) and assume
\[
    u(x) \leq C|x|^\alpha,
\]
as \( |x| \to \infty \). We choose \( \eta \in C_0^\infty(\mathbb{R}^n) \) be such that \( \eta \equiv 1 \) in \( B_R \), \( \eta \equiv 0 \) in \( B_{2R}^c \), \( 0 \leq \eta \leq 1 \) on \( \mathbb{R}^n \) and \( \eta \) satisfies
\[
    |
    \nabla \eta |^2 \leq CR^2 \quad \text{in} \quad A_R = B_{2R} \setminus B_R.
\]

**Case 1:** \( \alpha \geq 0 \). We have
\[
\int u \eta^{(2(p-1)) |\nabla \eta|^2} = \int u \eta^{(2(p-1)-\theta)} |
\nabla u|^2 \leq \frac{C}{R^2} \sup_{A_R} u^{\gamma} \int u \eta^{(2(p-1)+\theta)} |
\nabla u|^2 \leq CR \left( \frac{1}{n-p} + \varepsilon \right)^{\alpha\theta} \sup_{A_R} u^{\gamma} \int u \eta^{(2(p-1)+\theta) \left( u^{\bar{t}2(p-1)-\theta} \mid \nabla u \mid^p \right)^\frac{2(p-1)-\theta}{p}} \]
for every \( t \in \mathbb{R}, \theta > 0, \gamma > 0 \) and every \( \varepsilon > 0 \) small enough, where we used the gradient estimate in Corollary 3.2 with \( \alpha \geq 0 \). We assume
\[
    p - 2 < \theta < 2(p-1)
\]
and we apply Hölder inequality to obtain
\[
\int u \eta^{(2(p-1)) |\nabla \eta|^2} \leq CR \left( \frac{1}{n-p} + \varepsilon \right)^{\alpha\theta} \sup_{A_R} u^{\gamma} \left( \int u \eta^{\bar{t}2(p-1)-\theta} \mid \nabla u \mid^p \right)^\frac{2(p-1)-\theta}{p} \]
We choose
\[
    t = \bar{t} := \frac{p + \theta + \varepsilon(n-p)\theta}{n-p} - \gamma
\]
in order to have
\[
\left\{ \frac{(2-p)n-p-\gamma(n-p)+\theta(n-1)-\varepsilon(n-p)\theta}{n-p} - \bar{t}2(p-1)-\theta \right\} \frac{p}{2-p+\theta} = \frac{np+\bar{t}(n-p)}{n-p}.\]
We observe that
\[
\bar{t} < -1 \iff (\gamma-2)p < (\gamma-1)n + \theta + \varepsilon(n-p)\theta.
\]
Then, from Lemma 5.1, we obtain
\[
\int u \eta^{(2(p-1)) |\nabla \eta|^2} \leq CR \left( \frac{1}{n-p} + \varepsilon \right)^{\alpha\theta} \sup_{A_R} u^{\gamma} \left( \int u \eta^{\bar{t}2(p-1)-\theta} \mid \nabla u \mid^p \right)^\frac{2(p-1)-\theta}{p} \]
\[
\leq C \sup_{A_R} u^{\gamma} R^{\beta+\left( \frac{1}{n-p} + \varepsilon \right)\alpha\theta-2} \leq CR^{\beta+\left( \frac{1}{n-p} + \varepsilon \right)\alpha\theta-2+\alpha\gamma}
\]
We aim at finding $\theta$ and $\gamma$ satisfying (5.4) and (5.5) such that

$$\beta + \left(1 \over n - p + \varepsilon\right) \alpha \theta - 2 + \alpha \gamma \leq 0.$$  \hfill (5.7)

**Case 1.1:** If $p \geq \frac{n+2}{3}$, we choose $\theta$ close to $p - 2$, $\gamma = 0$ and $\varepsilon > 0$ small enough. Then $\bar{t} + p$ is close to

$$-p^2 + (n - 2)p + 2 \over n - p$$

If $\frac{n+2}{3} \leq p < \frac{n-2+\sqrt{n^2-4n+12}}{2} := \hat{p}$, then $\bar{t} + p > 0$ and hence $\beta = -\hat{t}(n-p)$. A simple computation shows that (5.7) is satisfied if

$$\alpha < 2(n-p) \over p(p-2) =: \hat{\alpha}.$$  

On the other hand, if $p \geq \hat{p}$, then $\bar{t} + p < 0$ and hence $\beta = -\hat{t}(n-p) / p-1$, which is close to $3p-n-2 \over p-1$. Thus (5.7) is satisfied if

$$\alpha < (n-p)^2 \over (p-2)(p-1) =: \check{\alpha}.$$  

**Case 1.2:** If $2 < p < \frac{n+2}{3}$, we choose again $\theta$ close to $p - 2$, $\gamma$ close to $\frac{n-3p+2}{n-p}$ and $\varepsilon > 0$ small enough. Then $\bar{t} + p$ is close to $p - 1 > 0$ and hence $\beta = -\hat{t}(n-p)$ and it is close to $\frac{n-p}{p}$. Hence, in order to verify (5.7) it is sufficient to choose

$$\alpha < (3p-n)(n-p) \over p(n-2p) =: \check{\alpha}.$$  

In particular this case occurs only if $\frac{n}{3} < p < \frac{n+2}{3}$.

**Case 2:** $\alpha < 0$. We have

$$\int u^{(2-p)n-p \over n-p} |\nabla u|^{2(p-1)} |\nabla \eta|^2 \leq \int u^{(2-p)n-p \gamma(n-p)} \over n-p} |\nabla u|^\theta |\nabla u|^{2(p-1)-\theta} |\nabla \eta|^2$$

$$\leq CR^{-2} \sup_{A_R} u^\gamma \int u^{(2-p)n-p - \gamma(n-p)} \over n-p} |\nabla u|^\theta |\nabla u|^{2(p-1)-\theta}$$

$$= CR^{-2} \sup_{A_R} u^\gamma \int u^{(2-p)n-p - \gamma(n-p)} \over n-p} |\nabla u|^\theta |\nabla u|^{2(p-1)-\theta}$$

for every $t \in \mathbb{R}$, $\theta > 0$, $\gamma \geq 0$ and every $\varepsilon > 0$ small enough, where we used the gradient estimate in Corollary 3.2 with $\alpha < 0$. We assume

$$p - 2 < \theta < 2(p-1)$$  \hfill (5.8)

and we apply Hölder inequality to obtain

$$\int u^{(2-p)n-p \over n-p} |\nabla u|^{2(p-1)} |\nabla \eta|^2$$

$$\leq CR^{-2} \sup_{A_R} u^\gamma \left( \int_{A_R} u^{(2-p)n-p - \gamma(n-p)} \over n-p} |\nabla u|^\theta |\nabla u|^{2(p-1)-\theta} \right)^{p-2 \over p} \cdot \left( \int_{A_R} u^t |\nabla u|^p \right)^{2(p-1)-\theta \over p}.$$
We choose
\[ t = \tilde{t} := -\frac{p + \theta + \varepsilon(n - p)\theta}{n - p} - \gamma \]
in order to have
\[ \left\{ \frac{(2 - p)n - p - \gamma(n - p) + \theta(n - 1) - \varepsilon(n - p)\theta}{n - p} - \frac{\tilde{t}[2(p - 1) - \theta]}{p} \right\} \frac{p}{2 - p + \theta} = \frac{np + \tilde{t}(n - p)}{n - p}. \]

We observe that
\[ \tilde{t} < -1 \iff (\gamma - 2)p < (\gamma - 1)n + \theta + \varepsilon(n - p)\theta. \]

Then, from Lemma 5.1, we obtain
\[
\int u \frac{(2-p)n-p}{n-p} |\nabla u|^{2(p-1)}|\nabla \eta|^2 \leq CR^{-2} \sup_{A_R} u^\gamma \left( \int_{A_R} u^\frac{2(p-1)-\theta}{p} \right)^{\frac{1}{p}} \left( \int_{A_R} u^{np+\tilde{t}(n-p)} \right)^{\frac{1}{p}} \leq C \sup_{A_R} u^\gamma R^{\beta-2} \leq CR^{\beta-2+\alpha\gamma}
\]

We aim at finding \( \theta \) and \( \gamma \) satisfying (5.8) and (5.9) such that
\[ \beta - 2 + \alpha\gamma < 0. \]

**Case 2.1:** If \( p \geq \frac{n+2}{3} \) the choice \( \theta \) close to \( p - 2 \), \( \gamma = 0 \) and \( \varepsilon > 0 \) small enough gives that \( \beta < 2 \) both if \( \tilde{t} + p > 0 \) and if \( \tilde{t} + p \leq 0 \). Since \( \alpha < 0 \), (5.11) is satisfied.

**Case 2.2:** If \( 2 < p < \frac{n+2}{3} \), we choose again \( \theta \) close to \( p - 2 \), \( \gamma \) close to \( \frac{n-3p+2}{n-p} \) and \( \varepsilon > 0 \) small enough. Then \( \tilde{t} + p \) is close to \( p - 1 > 0 \) and hence \( \beta = -\frac{\tilde{t}(n-p)}{p} \) and it is close to \( \frac{2-p}{p} \). Hence a simple computation shows that, in order to verify (5.11), it is sufficient to choose
\[ \alpha < \frac{(3p-n)(n-p)}{p(n-3p+2)} := \tilde{\alpha}. \]

To conclude we observe that we have
\[
\int u \frac{(2-p)n-p}{n-p} |\nabla u|^{2(p-1)}|\nabla \eta|^2 \leq CR^{-\delta}
\]
for some \( \delta > 0 \) if one of the assumptions
(i) \( n = 3 \), and \( 2 < p < 3 \) and \( \alpha < \tilde{\alpha} \);
(ii) \( n = 4 \), and \( 2 < p < \tilde{p} \) and \( \alpha < \tilde{\alpha} \);

or
\[ \tilde{p} \leq p < 4 \text{ and } \alpha < \tilde{\alpha}; \]

(iii) \( n = 5 \) or \( n = 6 \), and \( 2 < p < \frac{n+2}{3} \) and \( \alpha < \tilde{\alpha} \);

or
\[ \frac{n+2}{3} \leq p < \tilde{p} \text{ and } \alpha < \tilde{\alpha}; \]

or
\[ \tilde{p} \leq p < n \text{ and } \alpha < \tilde{\alpha}; \]
(iv) \( n \geq 7 \) and

\[
2 < p \leq \frac{n}{3} \quad \text{and} \quad \alpha < \tilde{\alpha},
\]

or

\[
\frac{n}{3} < p < \frac{n+2}{3} \quad \text{and} \quad \alpha < \tilde{\alpha},
\]

or

\[
\frac{n+2}{3} \leq p < \hat{p} \quad \text{and} \quad \alpha < \tilde{\alpha},
\]

or

\[
\hat{p} \leq p < n \quad \text{and} \quad \alpha < \tilde{\alpha},
\]

hold. Then letting \( R \to \infty \), from (5.1) we obtain

\[
\int_{\mathbb{R}^n} u^{(n-1)p} |\nabla u|^{2} = 0,
\]

and the conclusion follows as in the proof of Theorem 1.1 (i).

Appendix A. Riemannian setting

In this Appendix we consider the case of a complete, non-compact (without boundary) Riemannian manifold \((M^n, g)\) of dimension \( n \geq 2 \). We will emphasize the main differences with respect to Euclidean case when dealing with the same issues.

We consider positive weak solutions of

\[
\Delta_p u + u^{p-1} = 0 \quad \text{in } M^n
\]

(A.1)

where \( \Delta_p \) is the usual \( p \)-Laplace-Beltrami operator with respect to the metric \( g \). Moreover, we denote with Ric and Sec the Ricci and the sectional curvatures of \((M^n, g)\), respectively.

When \( \text{Ric} \geq 0 \) equation (A.1) has been recently studied, in the semilinear case \( p = 2 \), in [5] and [17]; in particular, in [5] the authors prove that the only positive classical solutions to (A.1) are given by the Aubin-Talenti bubbles (1.2) with \( p = 2 \) and the Riemannian manifold is isometric to the Euclidean space, provided \( n = 3 \) or \( u \) has finite energy or \( u \) satisfies suitable conditions at infinity. Furthermore, when the manifold is a Cartan-Hadamard manifold, i.e. complete and simply connected Riemannian manifold with non positive sectional curvature, in [21] the authors prove that all the positive energy minimizing solutions to (A.1) are given by the Aubin-Talenti bubbles (1.2) and the Riemannian manifold is isometric to the Euclidean space, assuming the validity of an optimal isoperimetric inequality on \( M^n \).

In this appendix we deal with the quasilinear case, i.e. \( 1 < p < n \), and we show that the analogue of Theorems 1.1-1.2-1.3-1.4 hold, provided that the Riemannian manifold \((M^n, g)\) satisfies:

(i) \( \text{Ric} \geq 0 \), if \( 1 < p < 2 \), or

(ii) \( \text{Sec} \geq 0 \), if \( 2 < p < n \).

Of course, if \( n = 2 \), both conditions are replaced by non-negativity of the scalar/Gauss curvature. The main differences with respect to the Euclidean case are the following:

• By assumption, we choose \( u \) to be a positive weak solution having the regularity given in (2.1)–(2.4).

• The estimate in Corollary 2.4 becomes the following

\[
\int u^{(n-1)p} |\nabla u|^{2} \eta^l + \int u^{(n-1)p} \text{Ric}(v, v) \eta^l \leq C \int u^{(2-p)\eta^l} |\nabla u|^{2(p-1)} |\nabla \eta|^{2} \eta^{l-2},
\]

(A.2)

the Ricci tensor appears when computing

\[
\text{div} (v \cdot \nabla v) = \nabla_j (\nabla_i v_j v_i) = \nabla_j \nabla_i v_j v_i + \nabla_i v_j \nabla_j v_i
\]

\[
= \nabla_i \nabla_j v_j v_i - \text{Ric}(v, v) + |\nabla v|^2
\]
since $\nabla_j v_i$ is symmetric. This identity is used in the proof of Propositions 6.2 and 7.1 in [27], which are used in our proof of Corollary 2.4.

- Let $r(x) := \text{dist}(x, p)$ be the geodesic distance from a fixed point $p \in M$. Due to the presence of the curvature, when $\text{Ric} \geq 0$ the Laplacian comparison implies that the function $r^{-\frac{n-2}{n-p}}$ is a weak $p-$subharmonic function, i.e. its $p-$Laplacian is non-negative. Hence, Lemma 2.6 holds true also in this setting.

- Lemma 2.7 and Lemma 2.9 still hold, provided

$$\text{Vol}(B_R) \leq CR^n$$

for every $R > 0$, which is ensured by Bishop-Gromov volume comparison.

- The Bocher formula in Lemma 3.1 becomes the following (see e.g. [29])

$$\frac{1}{p} P_f(\nabla f)^p \geq \frac{1}{n} (\Delta_p f)^2 + \frac{n}{n-1} \left( \frac{1}{n} \Delta_p f - (p-1) |\nabla f|^{p-2} \nabla f(\nabla, \nabla f) \right)^2$$

$$+ |\nabla f|^{p-2} \left[ (\nabla f, \nabla \Delta_p f) - (p-2) \frac{\Delta_p f}{|
abla f|^2} \nabla^2 f(\nabla, \nabla f) \right]$$

$$+ |\nabla f|^{2(p-2)} \text{Ric}(\nabla f, \nabla f) .$$

Thus, Lemma 3.1 still holds thanks to the condition $\text{Ric} \geq 0$.

- Following the proof of the gradient estimate in Proposition 1.7 (in particular, the estimate above (3.6)), one can observe that the same arguments works if we can construct a family of smooth cut-off functions $\phi$ defined on $B_{2R}$ such that

$$|\nabla \phi| \leq \frac{C}{R} \phi^{1-\delta}, \quad |\nabla f|^2 \Delta \phi + (p-2) \nabla^2 \phi(\nabla f, \nabla f) \geq - \frac{C}{R^2} \phi^{1-2\delta} |\nabla f|^2 . \quad (A.3)$$

Assume that $(M^n, g)$ has Sec $\geq 0$. Let $r(x) := \text{dist}(x, p)$ be the geodesic distance from a fixed point $p \in M$ and let $\psi \in C^2([0, \infty))$ be such that $\psi \equiv 1$ in $[0, 1)$, $\psi \equiv 0$ in $[2, \infty)$, $\psi' \leq 0$ and $0 \leq \psi \leq 1$. Since

$$\nabla^2 \psi(r) = \psi' \nabla^2 r + \psi'' dr \otimes dr,$$

by standard Hessian comparison (see e.g. [23]) we known that, outside the cutlocus of $p$, one has

$$\nabla^2 r \leq \frac{n-1}{r} g_{ij},$$

and thus the function $\psi(r)$ satisfies $\psi' \leq 0$ and

$$|\nabla \psi(r)| \leq C, \quad |\nabla f|^2 \Delta \psi + (p-2) \nabla^2 \psi(\nabla f, \nabla f) \geq - \frac{C}{r} |\nabla f|^2 .$$

Therefore, the function $\phi(r) = \psi \left( \frac{r}{R} \right)^{1/\delta}$ satisfies (A.3) (outside the cutlocus of $p$). To overcome the lack of regularity in the cutlocus of $p$ one can use the so called Calabi trick. Therefore all the arguments in the proof Proposition 1.7 go through and we obtain the following extension to the Riemannian setting:

**Proposition A.1.** Let $(M^n, g)$ be a complete Riemannian manifold with nonnegative sectional curvature. Let $u$ be a positive weak solution of equation (1.1) with $1 < p < n$. Then, for every $0 < \varepsilon < \frac{2}{n-p}$ it holds

$$|\nabla u| \leq C \left( \sup_{B_{2R}(x_0)} u^{n-1-p-\varepsilon} + R^{-\varepsilon} u^{n-p-1} \right) u^{\frac{n-1}{n-p}-\varepsilon} \quad \text{on } B_R(x_0)$$

for some $C = C(n, p, \varepsilon) > 0$, for every $R > 0$ and every $x_0 \in M^n$.

We explicitly note that this estimate is used only in the case $2 < p < n$. 
Remark A.2. Gradient estimates for positive $p$–harmonic functions have been obtained in [16] and [32]. In particular in [32] the authors managed to avoid imposing conditions on the sectional curvature, using integral estimates based on a Moser iteration argument, which only uses nonnegative Ricci curvature and first derivatives of the distance function. We expect that the same argument could work in our setting.

- In all the proofs of Theorems 1.1-1.2-1.3-1.4 we used the fact that volume of geodesic balls has at most Euclidean growth, which is guaranteed by our curvature assumptions (i) and (ii), as already observed.
- The final step in the proofs of Theorems 1.1-1.2-1.3-1.4, in the Riemannian setting, goes as follows. From (A.2), we obtain
\[ \int_{M} u^{(n-1)p \over n-p} |\nabla|^{2} + \int_{M} u^{(n-1)p \over n-p} \text{Ric}(v, v) = 0 \]
i.e.
\[ \nabla = \nabla v - \frac{\text{div} v}{n} g \equiv 0 \text{ in } \Omega_{cc}, \quad \text{Ric}(v, v) \equiv 0 \text{ in } M. \] (A.4)
Let $\Omega_0 \subseteq \Omega_{cc}$ be a connected component of $\Omega_{cc}$. Arguing as in the proof of Theorem 1.1, by elliptic regularity, we have $\text{div} v \in C^{1,\alpha}_{\text{loc}}(\Omega_0)$. Differentiating the first identity in (A.4), we get
\[ \nabla_i \text{div} v = n \nabla_j \nabla_i v_j = n \nabla_i \text{div} v - \text{Ric}_{ij} v_j = n \nabla_i \text{div} v. \]
Therefore $\text{div} v = \text{const}$ on $\Omega_0$. Hence, the vector field $v$ is homotetic, i.e. it satisfies
\[ \nabla_i v_j + \nabla_j v_i = \lambda g_{ij}, \quad \lambda \in \mathbb{R}. \]
Therefore, by a classical result of Kobayashi [15], we have that $\Omega_0$ must be locally Euclidean and we conclude as in the proof of Theorem 1.1.

Acknowledgments. The first author is member of the GNSAGA, Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni of INdAM. The second and the third authors are members of GNAMPA, Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni of INdAM.

References
[1] A.D. Alexandrov. Uniqueness theorems for surfaces in the large. I (Russian), Vestnik Leningrad Univ. Math., 11 (1956), 5–17.
[2] C. A. Antonini, G. Ciraolo, A. Farina. Interior regularity results for inhomogeneous anisotropic quasilinear equations. Preprint.
[3] T. Aubin. Problèmes isopérimétriques et espaces de Sobolev. J. Differential Geometry 11 (1976), no. 4, 573–598.
[4] L. Caffarelli, B. Gidas, J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. Comm. Pure Appl. Math. 42 (1989), no. 3, 271–297.
[5] G. Catino, D. D. Monticelli. Semilinear elliptic equations on manifolds with nonnegative Ricci curvature. Preprint.
[6] W. X. Chen, C. Li. Classification of solutions of some nonlinear elliptic equations. Duke Math. J. 63 (1991), no. 3, 615–622.
[7] G. Ciraolo, A. Figalli, A. Roncoroni. Symmetry results for critical anisotropic p-Laplacian equations in convex cones. Geom. Funct. Anal. 30 (2020), 770–803.

[8] M. Clapp, L. R. Rios. Entire nodal solutions to the pure critical exponent problem for the p–Laplacian. J. Differential Equations 265 (2018) 891–905.

[9] L. Damascelli, S. Merchán, L. Montoro, B. Sciuini. Radial symmetry and applications for a problem involving the $-\Delta_p(\cdot)$–operator and critical nonlinearity in $\mathbb{R}^n$. Adv. Math., (10)265 (2014), 313–335.

[10] W. Y. Ding. On a conformally invariant elliptic equation on $\mathbb{R}^n$. Comm. Math. Phys. 107 (2) (1986) 331–335.

[11] F. Esposito, G. Riey, B. Sciuini, D. Vuono. Anisotropic Kelvin transform and the Sobolev anisotropic critical equation. Preprint

[12] B. Gidas, W. M. Ni, L. Nirenberg. Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$. Mathematical analysis and applications, Part A, pp. 369–402, Adv. in Math. Suppl. Stud., 7a, Academic Press, New York-London, 1981.

[13] B. Gidas, W. M. Ni, L. Nirenberg. Symmetry and related properties via the maximum principle. Comm. Math. Phys. 68 (3) 209–243 (1979).

[14] B. Gidas, J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. Comm. Pure Appl. Math. 34 (1981), no. 4, 525–598.

[15] S. Kobayashi. A theorem on the affine transformation group of a Riemannian manifold. Nagoya Math. J. 9 (1955), 39–41.

[16] B. Kotschwar, L. Ni. Local gradient estimates of p-harmonic functions, $1/H$-flow, and an entropy formula. Ann. Scient. Ec. Norm. Sup. 42 (2009), 1–36.

[17] M. Fogagnolo, A. Malchiodi, L. Mazzieri. A note on the critical Laplace Equation and Ricci curvature. Preprint.

[18] J.M. Lee, T.H. Parker. The Yamabe problem. Bull. Amer. Math. Soc., 17 (1987), 37–91.

[19] Y. Li, L. Zhang. Liouville-type theorems and Harnack-type inequalities for semilinear elliptic equations. J. Anal. Math. 90 (2003), 27–87.

[20] P. L. Lions, F. Pacella, M. Tricarico. Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions. Indiana Univ. Math. J. (2)37 (1988), 301–324.

[21] M. Muratori, N. Soave. Some rigidity results for Sobolev inequalities and related PDEs on Cartan–Hadamard manifolds. to appear in Ann. Sc. Norm. Super. Pisa Cl. Sci.

[22] M. Obata. The conjectures on conformal transformations of Riemannian manifolds. J. Differential Geometry, 6 (1971), 247–258.

[23] P. Petersen. Riemannian Geometry. Graduate Texts in Mathematics, vol. 171. Springer–Verlag, New York, 1998.

[24] A. Roncoroni. An overview on extremals and critical points of the Sobolev inequality in convex cones. To appear in Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.

[25] B. Sciuini. Classification of positive $D^{1,p}(\mathbb{R}^n)$-solutions to the critical $p$-Laplace equation in $\mathbb{R}^n$. Advances in Mathematics, 291 (2016), 12–23.

[26] J. Serrin. A symmetry problem in potential theory. Arch. Rat. Mech. Anal., 43 (1971), 304–318.

[27] J. Serrin, H. Zou. Cauchy–Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities. Acta Math., (1) 189 (2002), 79–142

[28] G. Talenti. Best constant in Sobolev inequality. Ann. Mat. Pura Appl. (4) 110 (1976), 353–372.

[29] D. Valtorta. Sharp estimate on the first eigenvalue of the $p$-Laplacian. Nonlinear Anal., Theory Methods Appl. 75 (2012), 4974–4994.

[30] J. L. Vázquez. A strong maximum principle for some quasilinear elliptic equations. Appl. Math. Optim. 12 (3) (1984) 191–202.
[31] J. Vétois. *A priori estimates and application to the symmetry of solutions for critical p-Laplace equations*. cJ. Differential Equations, 260 (2016), 149–161.

[32] X. Wang, L. Zhang. *Local gradient estimate for p–harmonic functions on Riemannian manifolds*. Comm. Anal. Geo. 19 (2011), 759–771.

G. Catino, Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133, Milano, Italy.
Email address: giovanni.catino@polimi.it

D. Monticelli, Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133, Milano, Italy.
Email address: dario.monticelli@polimi.it

A. Roncoroni, Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133, Milano, Italy.
Email address: alberto.roncoroni@polimi.it