How Much Information can be Obtained by a Quantum Measurement?

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How much information about an unknown quantum state can be obtained by a measurement? We propose a model independent answer: the information obtained is equal to the minimum entropy of the outputs of the measurement, where the minimum is taken over all measurements which measure the same “property” of the state. This minimization is necessary because the measurement outcomes can be redundant, and this redundancy must be eliminated. We show that this minimum entropy is less or equal than the von Neumann entropy of the unknown states. That is a measurement can extract at most one meaningful bit from every qubit carried by the unknown states.

I. INTRODUCTION

Quantum mechanics has at its core a fundamental statistical aspect. Suppose you are given a single quantum particle in a state $|\Psi\rangle$ unknown to you. There is no way to find what $|\Psi\rangle$ is - to find it out you need an infinite ensemble of quantum particles, all prepared in the same state. Indeed, the different properties which characterize the state are, in general, complementary to one another; measuring one disturbs the rest. Only if an infinite ensemble is given can one find out the state. But infinite ensembles don’t exist in practice. Given a finite ensemble of identically prepared particles, how well can one estimate the state? The problem is a fundamental one for understanding the very basis of quantum mechanics. It has been investigated by many authors, see for instance \cite{1} \cite{2}, and it constitutes probably the oldest problem in what is at present called “quantum information”. Here we approach this problem from a new point of view which, we think, leads to a deeper understanding.

What is the optimal way to estimate the quantum state given a finite ensemble? As such the question is not well posed. Indeed, since we cannot completely determine the state, i.e. completely determine all its properties, we must decide which particular property we want to determine. For an ensemble of spins, for example, estimating as well as possible the mean value of the $z$ spin component is, obviously, a different question than estimating as well as possible the mean value of the $x$ spin component.

But things are in fact even more complicated. The apparent benign words “as well as possible” in the previous paragraph are not well defined. Indeed, “as well as possible” actually means “as well as possible given a specific measure of what “well” means”. Obviously, one can imagine many different measures. For example, suppose that a source emits states $|\psi_i\rangle$ with probability $p_i$. The problem is to design a measurement at the end of which we must guess which state was emitted. Let the guess be $|\phi_j^{\text{guess}}\rangle$, and let the measure of success (fidelity) be

$$F_{ij} = |\langle \phi_j^{\text{guess}} | \psi_i \rangle|^2,$$

i.e. the absolute value square of the scalar product in between the true state $|\psi_i\rangle$ and the guess $|\phi_j^{\text{guess}}\rangle$. The goal is to optimize the measurement such that it yields the highest average fidelity

$$F = \sum_{i,j} p_i F_{ij} p(j|i).$$

where $p(j|i)$ is the probability to make guess $j$ if the state is $|\psi_i\rangle$. On the other hand, one can imagine another fidelity function, such as

$$F_{ij}' = |\langle \phi_j^{\text{guess}} | \psi_i \rangle|^4.$$

Or one could try to optimize the mutual information

$$I = - \sum_i p_i \ln p_i + \sum_j p_j \sum_i p(i|j) \ln p(i|j).$$

or any other measure.
The important point to notice about the above different problems is that the different fidelities (2-4) not only define different scales according to which we measure the degree of success in estimating the state, but also, implicitly, define which property of the state we are actually estimating. If all the different fidelities where to lead to the same optimal measurements, we could say that we learn the same property about the state but just expressed in a different way. However the different fidelities will in general lead to different optimal measurements which means that in each case we learn a different property about the system.

To summarize, in general each particular estimation problem is completely different from the other, they measure different properties and their degree of success is measured on different scales, with the scales also defining implicitly what exactly is the property we estimate.

That one can learn different properties is a fact of life inherent to quantum mechanics. But there is no reason not to use the same scale to gauge how successful we have been in learning the property we decided to measure. The aim of this paper is to propose such a universal scale, and in the process to introduce a novel approach to quantum state estimation.

**II. MAIN IDEA**

The central point of our approach starts from a simple but fundamental question: what do we actually learn from a measurement on a state? Let us illustrate this question by an example. We shall contrast two situations. Consider a source which emits spin 1/2 particles. In the first case the particles are polarized with equal probability along either the $+z$ ($|\uparrow_z\rangle$) or $-z$ ($|\downarrow_z\rangle$) directions. In the second case the states are polarized along random directions uniformly distributed on the sphere. Suppose we want to identify the states as well as possible according to the fidelity eq. (2).

In the first case it is obvious that a measurement along $\sigma_z$ perfectly identifies the state, hence the fidelity is $F=1$. In the second case, it has been shown [3] that the measurement along $\sigma_z$ is also optimal. But in this case the states cannot be identified perfectly, and the fidelity is only $F=2/3$.

Nevertheless the two situations seem extremely similar. In both cases we perform the same measurement. And in both cases before we perform the measurement we know that the outcomes of the measurement are either +1 or -1, and the a priori probabilities of the two outcomes are equal. When we perform the measurement this uncertainty is resolved. Hence in both cases the measurement yields 1 bit of information. Our main idea is to interpret this quantity as the information we extract from the state. Incidentally we note that in both cases this information (the Shannon information of the outcomes) equals the von Newmann entropy of the unknown states (both are equal to 1).

This idea might seem paradoxical at first sight because in one case we completely recognize the state whereas in the other case we recognize it badly. To understand let us introduce a classical source that decides which quantum state is emitted from the quantum source (see figure 1). In the first case the classical source must only specify one bit (either $+z$ or $-z$) to determine which state is emitted. In the second case it must provide a direction $\mathbf{n}$ in (ie. an infinite number of bits) in order to specify the state $|\uparrow_{\mathbf{n}}\rangle$. In both cases one extracts one bit of information. In the first case this means that the classical information supplied by the source is completely recovered. In the second case information is lost. However it is now clear that the loss does not occur during the measurement, but during the first step, where classical information is converted into quantum.
FIG. 1. Chain of events leading to a quantum state estimation problem. The classical source specifies which state should be sent. The quantum source then emits the corresponding state. Finally the measuring device tries to identify the emitted state.

To summarize, the quantum state estimation problem as presented in figure 1 consists of a chain of events which starts with a classical source that tells the quantum source what state to emit, and ends with the measurement. The fidelity measures the overall performance of the chain since it is proportional to the scalar product $\mathcal{M}_{\text{guess}} \cdot \mathcal{M}_{\text{output}}$. On the other hand the number of bits in the output characterizes how much information is extracted by the measurement. Therefore in this article we shall focus on the latter quantity.

### III. MAIN RESULT

The preceding discussion suggests that the Shannon information of the outcomes

$$I_{\text{output}}^S = - \sum_j p_j \ln p_j,$$

(5)

where $p_j = \sum_i p(j|i)$ is the probability of outcome $j$, measures how much information is extracted from the state. This idea however has to be refined.

The main problem is that there may be redundancies in the outputs of the measurement. As a trivial example, a measurement could be accompanied by the flip of a coin, and the outcomes of the measurement would consist of both the outcomes of the measurement proper and the outcomes of the coin flip. This adds one bit to the entropy of the outputs without telling anything about the system. In less trivial examples involving POVM’s and ancillas, redundancies can arise in a less obvious way, and it is not immediate how they can be identified and eliminated.

Our main result is that no matter what property of the system one wants to measure, when the redundancy is eliminated, the remaining Shannon information of the outputs has a universal upper bound which is the von Neumann entropy of the quantum source:

$$I_{\text{output}}^S(\text{no redundancy}) \leq I_{\text{input}}^V,$$

(6)

where $I_{\text{input}}^V = -Tr \rho \ln \rho$ is the Shannon information of the quantum source and $\rho$ is the density matrix of the quantum source $\rho = \sum_i p_i |\psi_i\rangle\langle \psi_i|.$

One does not always attain equality in eq. (6). Indeed some questions are more informative about the system than others. Less informative questions can be answered by measurements whose output entropy is smaller. More informative questions require measurements with more entropy. But the most detailed questions can always be answered in $I_{\text{input}}^V$ bits.

### IV. STRATEGY

The main problem we face in deriving eq. (6) is to eliminate the redundancy. In order to do this we shall proceed in several steps.
1. The first step is to decide which property we are interested in. We may fix the property directly (for instance decide to measure the average of $\sigma_z$) or implicitly by choosing a fidelity. In the rest of this paper we shall adopt the second approach.

2. We then look at optimal measurements, that is measurements which maximize the fidelity. In general there is an entire class of such measurements.

3. We perform a second optimization. Namely among the optimal measurements we look for the measurements which minimize $I_{\text{output}}^S$.

This double optimization strategy has already been considered for some particular cases in \cite{4,5,6}.

One expects that this strategy yields measurements which have no spurious redundancy. However as we will find out later through some examples, redundancies cannot be completely eliminated by the above procedure and we will have to further modify it.

These further modifications are motivated by the classical and quantum theory of information \cite{7,8} which suggest the idea of performing measurements on blocks of quantum states, rather than on individual particles. Thus we shall allow the measuring device to accumulate a large number $L$ of input states before making a collective measurement on the $L$ states simultaneously. It is in the context of these collective measurements that we make the two optimizations (points 2 and 3 above) and thereby eliminate the spurious redundancies.

We want to emphasize that this procedure cannot increase the fidelity since the subsequent particles are completely uncorrelated. However by considering measurements on large blocks we can hope to reduce the redundancy of the measurement, i.e., the entropy of the outcomes, by making “better use” of each outcome.

Two technicalities have to be taken into account. First of all we must take care not to modify the definition of fidelity as we go from measurements on single particles to block measurements. That is the fidelity must still be the fidelity of each state individually, rather than the fidelity for the whole block. Second we should not require the measurement to absolutely maximize the fidelity, since then using block measurements does not help to reduce the entropy (this follows once more from the fact that the subsequent states are completely uncorrelated). However, following the ideas of information theory, we shall only require that the measurement has a fidelity approaching arbitrarily closely the optimum. In this framework we shall prove eq. \eqref{eq:3}.

To summarize, there is no best way of estimating an unknown quantum state. Different measurements will learn about different properties of the state, and it is up to us to choose which property we want to learn about. However once we fix the property we want to learn about, we show that quantitatively one cannot learn more than $I_{\text{input}}^{VN} = -Tr\rho\ln\rho$ bits about this property. That is a measurement can extract at most one meaningful bit from each qubit coming from the source.

V. EXAMPLES

Before embarking on a proof of our result, we give two examples which illustrate the main points that must be taken into account in the proof.

In the first example there are two possible input states $|\psi_1\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ and $|\psi_2\rangle = \alpha|\uparrow\rangle - \beta|\downarrow\rangle$ which occur with equal probability. The density matrix of the source is $\rho = \alpha^2|\uparrow\rangle\langle\uparrow| + \beta^2|\downarrow\rangle\langle\downarrow|$ which is different from the identity for $\alpha \neq \beta$ Therefore the von Neumann entropy of the input states $I_{\text{input}}^{VN} < 1$ qubit.

In this example we use a fidelity defined as follows: after each measurement one must guess whether the state is $|\psi_1\rangle$ or $|\psi_2\rangle$. In case of a correct guess one receives a score of $+1$, and for an incorrect guess one receives a score of $-1$. The aim is to maximize the average score. The techniques of section \ref{VII} can be used to show that the optimal measurement is a von Neumann measurement of $\sigma_x$, see figure 2. The two outcomes of this measurement occur with equal probability, and hence $I_{\text{output}}^S = 1 > I_{\text{input}}^{VN}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2}
\caption{The two input states $|\psi_1\rangle, |\psi_2\rangle = \alpha|\uparrow\rangle \pm \beta|\downarrow\rangle$. The optimal measurement is a measurement of the spin in the $x$ direction.}
\end{figure}
In this example, a natural first step in eliminating the redundancy is to project blocks of input states onto their probable subspace. This projection succeeds with arbitrarily high probability, and affects the input states arbitrarily little. But it reduces the dimensionality of the Hilbert space of the input states from $2^N$ to $2^{NI_{\text{input}}}$. Hence if we can prove that there is a von-Neumann measurement restricted to the probable subspace that is optimal, we will have proved our claim. However the construction of such a von-Neumann measurement is non-trivial, as is illustrated in the next example.

In our second example there is no “most probable” subspace because the density matrix of the inputs is completely random. In this example there are three input states $|\psi_1\rangle = |\uparrow\rangle$, $|\psi_2\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{\sqrt{3}}{2}|\downarrow\rangle$, $|\psi_3\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{\sqrt{3}}{2}|\downarrow\rangle$, each occurring with equal probability $p_i = 1/3$. The density matrix of these states is $\rho = I/2$ and their entropy is $I_{\text{V N input}} = 1$ qubit. The fidelity is defined as above: after the measurement one must guess which was the input state. If the guess is correct one scores +1 point, if the guess is incorrect, one scores −1 points. The aim is to maximize the average score (fidelity).

Using the techniques of section VII, one can show that the elements of an optimal POVM are necessarily proportional to the three projectors $|\psi_1\rangle\langle\psi_1|$, $|\psi_2\rangle\langle\psi_2|$, $|\psi_3\rangle\langle\psi_3|$, see figure 3. Therefore the optimal POVM whose output entropy is minimum is $\{\frac{2}{3}|\psi_1\rangle\langle\psi_1|, \frac{1}{3}|\psi_2\rangle\langle\psi_2|, \frac{1}{3}|\psi_3\rangle\langle\psi_3|\}$. In this case $I_{\text{output}}^S = \ln 3 > 1$ bits. The other optimal measurements have larger $I_{\text{output}}^S > \ln 3$ bits. One can also show that there is no measurement on blocks of $L$ input states whose fidelity is strictly equal to the optimum and whose output entropy is less then $L \ln 3$ bits. However if one only requires that the fidelity is arbitrarily close to the maximum, then in the asymptotic limit ($L \to \infty$) the output entropy can be made arbitrarily close to $L$ bits, thereby attaining the bound eq. (6). The main difficulty of the proof will be to construct such a measurement on large blocks whose output entropy is equal to $L$ bits and whose fidelity is arbitrarily close to the optimal fidelity.

VI. PLAN OF THE PROOF

The main part of this paper is devoted to proving the bound eq. (6). In section VII we introduce a large class of fidelities, and derive some properties of the optimal measurements. In section VIII we show how to generalize these fidelities to measurements on large blocks of input states. At the end of section VIII we are in a position to state with precision a first version of our main result, eq. (6). In section IX we extend the notion of fidelity and state a slightly more general version of our result. In section XI we show how to construct a measurement on large blocks which has little redundancy. In section XII we derive an intermediate result concerning the fidelity of the measurement constructed in section XI. If the states are uniformly distributed in Hilbert space (ie. the density matrix is proportional to the identity, $\rho = I/d$), then this intermediate result already proves our main claim eq. (6). When the states are not uniformly distributed in Hilbert space, we must first project blocks of states onto the probable subspace before using the intermediate result of section XII. This is done in section XIII and completes the proof of eq. (6).

VII. FIDELITY

Let us consider the general setup described in figure 1. The states emitted by the quantum source $|\psi_i\rangle$ belong to a Hilbert space of dimension $d$. They occur with probability $p_i$. Their density matrix is $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ with $Tr\rho = 1$. 

FIG. 3. The three input states $\psi_1, \psi_2, \psi_3$ in the second example. The optimal measurement is a POVM whose elements are projectors onto the three states $\psi_1, \psi_2, \psi_3$. 

Using the techniques of section VII, one can show that the elements of an optimal POVM are necessarily proportional to the three projectors $|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|, |\psi_3\rangle\langle\psi_3|$, see figure 3. Therefore the optimal POVM whose output entropy is minimum is $\{\frac{2}{3}|\psi_1\rangle\langle\psi_1|, \frac{1}{3}|\psi_2\rangle\langle\psi_2|, \frac{1}{3}|\psi_3\rangle\langle\psi_3|\}$. In this case $I_{\text{output}}^S = \ln 3 > 1$ bits. The other optimal measurements have larger $I_{\text{output}}^S > \ln 3$ bits. One can also show that there is no measurement on blocks of $L$ input states whose fidelity is strictly equal to the optimum and whose output entropy is less then $L \ln 3$ bits. However if one only requires that the fidelity is arbitrarily close to the maximum, then in the asymptotic limit ($L \to \infty$) the output entropy can be made arbitrarily close to $L$ bits, thereby attaining the bound eq. (6). The main difficulty of the proof will be to construct such a measurement on large blocks whose output entropy is equal to $L$ bits and whose fidelity is arbitrarily close to the optimal fidelity.
The most general measurement on the input states is a POVM with $M$ element: $a_j \geq 0$, $\sum_{j=1}^M a_j = I_d$.

We introduce the fidelity in the following way. To each outcome $j$ of the measurement we associate a state $|\phi_j^{\text{guess}}\rangle$ which is our “guess” as to what the input state was. The correctness of this guess is measured by a function, the fidelity, which depends on the input state and the guessed state $f(|\phi_j^{\text{guess}}\rangle,|\psi_i\rangle)$. For instance $f$ could have the form eq. (1) or (3). The mean fidelity is then:

$$F = \sum_i \sum_j p_i \sum_j p(j|i)f(|\phi_j^{\text{guess}}\rangle,|\psi_i\rangle).$$

(7)

where the probability to obtain outcome $j$ if the state is $|\psi_i\rangle$ is

$$p(j|i) = \langle \psi_i | a_j | \psi_i \rangle$$

(8)

An optimal measurement is one which maximizes the mean fidelity $F$.

This is a rather general formulation of the state estimation problem. However the fidelity is not the most general one could consider. To see this let us consider the optimization of $F$. When we make the optimization, we must compare the value of $F$ for different POVM’s, however the guessed states $|\phi_j^{\text{guess}}\rangle$ are kept fixed. That is the guessing strategy is fixed once and for all, and we try to optimize the measurement for fixed guessing strategy. The advantage of formulating the fidelity in this way is technical: it ensures that the fidelity depends linearly on the POVM elements.

We shall show in section [X] how to extend our result to more general fidelities for which the guessed states $|\phi_j^{\text{guess}}\rangle$ are not kept fixed.

We summarize here the main properties of optimal measurements for the fidelity eq. (6), see also [2] [3].

First of all note that we can always take the optimal POVM to consist of one dimensional projectors $b_j = |b_j\rangle\langle b_j|$ (The $b_j$ are not normalized). Indeed refining a POVM can only increase the fidelity. This can be seen formally in the following way: suppose the $a_j$ are an optimal POVM, but not necessarily made out of one dimensional projectors. Then each $a_j$ can always be decomposed as $a_j = \sum_k |b_{jk}\rangle\langle b_{jk}|$ since it is a positive operator. Inserting this into the expression for $F$ one sees that the $b_{jk}$ (to which we associate the guessed state $|\phi_j^{\text{guess}}\rangle$) are also optimal.

Thus we can optimize $F$ in the class of POVM’s whose elements are one dimensional projectors $|b_j\rangle\langle b_j|$. These projectors are subject to the unitarity condition $\sum_j |b_j\rangle\langle b_j| = I_d$. This can be implemented by introducing $d^2$ Lagrange multipliers $\lambda_{\mu\nu}$ which we group into one operator $\hat{\lambda}$:

$$F = \sum_i \sum_j p_i \sum_j \langle \psi_i | b_j | \psi_i \rangle f(\psi_i, \phi_j^{\text{guess}}) - \text{Tr}[\hat{\lambda}(\sum_j |b_j\rangle\langle b_j| - I_d)]$$

(9)

where $\hat{F}_j = \sum_i p_i |\psi_i\rangle\langle \psi_i | f(\psi_i, \phi_j^{\text{guess}})$. If we vary this with respect to $|b_j\rangle$, we obtain the equations

$$\hat{F}_j - \hat{\lambda} = 0.$$  

(10)

Inserting this into eq. (10) shows that $F = \text{Tr} \hat{\lambda}$.

Eq. (10) is the essential equation to find optimal measurements explicitly. For instance consider the first example of section [X]. There are two input states $\psi_1$ and $\psi_2$ and two guessed states $\phi_1^{\text{guess}} = |\psi_1\rangle$ and $\phi_2^{\text{guess}} = |\psi_2\rangle$. If the input state is $|\psi_1\rangle$ and one guesses $\phi_1^{\text{guess}}$, then $f = +1$, whereas if the input state is $|\psi_2\rangle$ and one guesses $\phi_1^{\text{guess}}$, then $f = -1$, hence $\hat{F}_1 = \frac{1}{2}(|\psi_1\rangle\langle \psi_1| - |\psi_2\rangle\langle \psi_2|) = +\alpha \beta \sigma_x$. Similarly $\hat{F}_2 = \frac{1}{2}(|\psi_2\rangle\langle \psi_2| - |\psi_1\rangle\langle \psi_1|) = -\alpha \beta \sigma_x$. The task is then to find an operator $\hat{\lambda}$ such that null eigenvectors of $\hat{F}_{1,2} - \hat{\lambda}$ can satisfy the completeness relation. The only possibility is $\hat{\lambda} = \alpha \beta I$. Therefore the optimal measurement is along the $x$ axis, and $F_{\text{max}} = 2\alpha \beta$. The second example of section [X] can be treated along similar lines.

An important consequence of eq. (10) is an explicit expression for the value of $F$ if the measurement is not optimal. Consider a measurement $a_j'$ which is not optimal, but each positive operator $a_j'$ is “close” to the corresponding operator $b_j$ of the optimal measurement. We then decompose the operator $a_j'$ in terms of its components along $|b_j\rangle$: $a_j' = X_j |b_j\rangle\langle b_j| + Y_j |b_j\rangle\langle b_j| + Y^*_j |b_j^\perp\rangle\langle b_j| + z_j$ where the state $|b_j^\perp\rangle$ is orthogonal to $|b_j\rangle$ and the operator $z_j$ obeys $z_j |b_j\rangle = 0$, $\langle b_j | z_j = 0$. Inserting this decomposition into the expression for $F$, we obtain

$$F(a') = \text{Tr} \hat{\lambda} + \sum_j \text{Tr}[\hat{F}_j - \hat{\lambda}]a_j'$$

6
there exists a measurement on sequences of $L(7)$. Given any $\varepsilon > 0$, where $C$ is some positive constant independent of $j$. This expresses in a simple way how much the fidelity differs from its maximal value in terms of how much the measurement differs from the optimal measurement.

VIII. FIDELITY FOR MEASUREMENTS ON LARGE BLOCKS

As discussed above it is necessary to also consider measurements on large blocks of $L$ input states $|\psi_1 \ldots \psi_L\rangle$. The fidelity for measurements on large blocks is

$$F_L = \sum_{i_1 \ldots i_L} p_{i_1} \ldots p_{i_L} \sum_{j=1}^N \langle \psi_{i_1} \ldots \psi_{i_L} | A_j | \psi_{i_1} \ldots \psi_{i_L} \rangle \frac{1}{L} \sum_{k=1}^L f(\psi_{i_k}, \phi_j^{guess}) ,$$

where $A_j$ is the measurement on the $L$ input states. The guessed state is the product $|\Phi_j^{guess}\rangle = |\phi_{1j}^{guess} \ldots \phi_{Lj}^{guess}\rangle$. The fidelity is taken to be the average of the fidelities for each state $|\psi_{i_1}\rangle, \ldots, |\psi_{i_L}\rangle$. This ensures that eq. (12) is just the average of the fidelities eq. (7), as can be seen by rewriting $F_L$ as

$$F_L = \frac{1}{L} \sum_{k=1}^L \sum_{i_k} p_{i_k} \langle \psi_{i_k} | A_j^{(k)} | \psi_{i_k} \rangle f(\psi_{i_k}, \phi_j^{guess}) ,$$

where the operators $A_j^{(k)}$ are the operators $A_j$ restricted to the space of particle $k$:

$$A_j^{(k)} = Tr_{l \neq k} \left( \prod_{l \neq k} \rho_l \right) A_j .$$

Note that a possible measurement that maximizes $F_L$ is built out of the measurement $\{a_i\}$ which maximize eq. (6):

$$A_j = a_{j_1} \otimes \ldots \otimes a_{j_L} .$$

This measurement has $M^L$ outcomes. And in general $M$ will be larger than $2^{I_{input}^N}$.

Our main result is that one can always construct optimal measurements with $2^{I_{input}^N}$ outcomes per input state which also maximize $F$. Stated with precision we shall prove the following result:

Consider a state estimation problem in which the unknown state $|\psi_i\rangle$ have density matrix $\rho = \sum |\psi_i\rangle \langle \psi_i|$, and von Neumann entropy $I_{input}^N = -Tr \rho \ln \rho$. The quality of the state estimation is measured by a fidelity of the form eq. (3). Given any $\epsilon > 0$ and $\eta > 0$, then there exists $L_0$ such that for any $L \geq L_0$, and any $N$ larger than $2^{L(I_{input}^N + \eta)}$, there exists a measurement on sequences of $L$ input states which has $N$ outcomes and attains a fidelity $F_L \geq F_{max} - \epsilon$.

The Shannon entropy of the outputs per input state, $I_{output}^N$, can therefore be made equal or less then $I_{input}^N + \eta$.

It is this result that will be proven in sections XI to XIII.

IX. OTHER FIDELITIES

Our main result, as stated with precision at the end of the preceding section, applies only to fidelities of the form eq. (6) with fixed guessed states. In this section we enquire whether it can be generalized to other fidelities?

As a first generalization, we consider fidelities of the form eq. (6), but for which both the POVM elements $\{a_j\}$, and the guessed states are undetermined and must be varied to find the optimum estimation strategy. That is whereas in section VII the specification of an estimation strategy consisted only of the POVM elements $\{a_j\}$, it now consists of the set $\{a_j, \phi_j^{guess}\}$ which comprises both the POVM elements and the guessed states. An example of such more general fidelities was considered in [6]. The unknown states $|\psi_i\rangle$ where taken to be $n$ spin 1/2 particles all polarized
along the same direction $\Omega$ and the fidelity was taken to be the scalar product of one spin polarized along $\Omega$ with one spin polarized along the guessed direction $f = |\langle \uparrow_\Omega | \uparrow_\Omega^{\text{guess}} \rangle|^2$.

It is easy to show that our main result eq. (1) also applies to such more general fidelities for which both the POVM elements and the guessed states can be varied. First note that one can always find an optimal estimation strategy with only a finite number $M$ of outcomes \cite{10,11}. Associated to each outcome is a guessed state $\phi_j^{\text{guess(OPT)}}$, $j = 1, \ldots, M$.

Let us now consider the subclass of estimation strategies $\{a_j, \phi_j^{\text{guess(OPT)}}\}$ for which the guessed states are fixed to be an optimal set and only the POVM elements can vary. Note that the optimal fidelity for this subclass is equal to the optimal fidelity for the more general estimation strategy since the guessed states are taken to be optimal. Since for this subclass only the POVM elements can vary, we are in the conditions of section VII and VIII. The result stated at the end of section VIII therefore applies. Hence there exists a measurement on large blocks whose output entropy is less or equal to the von Neumann entropy of the input states and whose fidelity is greater than the optimal fidelity minus $\epsilon$. This shows that our main result also holds for these more general fidelities.

One can however construct even more general fidelities (for instance by taking the fidelity to be non linear in the POVM elements). For such more general fidelities it is an open question whether our claim also applies. One example of such more general fidelities is the mutual information eq. (4). For this particular example our claim also holds. This is discussed in the next section.

X. RELATION TO THE CLASSICAL CAPACITY OF A QUANTUM CHANNEL

In the state estimation problem as presented in figure 1, the classical source specifies in a completely random manner which quantum state is emitted. The task of the measurement is to recognize as well as possible which state was emitted by the quantum source. It is instructive to compare this to the problem of classical communication through a quantum channel \cite{10,11}. In this case the classical source chooses a controlled subset of all possible sequences (called code words) in such a way that they can be recognized (almost) perfectly by the receiver. He can then communicate classical information reliably through the quantum channel. The relation between the two problems is that in the communication problem the receiver must recognize the code words, so he is confronted with a state estimation problem, although it is a particular one.

For this reason the two problems are related both conceptually and formally. On the conceptual side, a corollary of our main result is an alternative proof of Holevo’s upper bound on the classical capacity of a quantum channel \cite{10} in the case where the quantum channel consists of pure states. Indeed if the message is to be transmitted faithfully, Bob must recognize the code words with high fidelity. We can now view the code words as the states $|\psi_i\rangle$ that are emitted by the quantum source in figure 1. The von Neumann entropy of the words is less than $n I_{\text{VN}}(\rho)$ where $n$ is the number of letters in a word and $I_{\text{VN}}(\rho)$ is the von Neumann entropy of the letters. Recall now that the question answered in this paper is to find, among all the measurements which recognize the input words with high fidelity, those whose output has the minimum entropy. Clearly this minimum entropy is an upper bound to the capacity of the channel. We have shown that it is less or equal to the von Neumann entropy of the channel. Thus the quantum channel has a classical capacity less than $I_{\text{VN}}(\rho)$ bits per word, confirming Holevo’s result.

On the formal side, the techniques we have used to construct a measurement which minimizes the entropy of the outputs are closely related and inspired by the techniques used to construct a decoding measurement which maximizes the capacity of the channel \cite{11}. There is however a very important difference with the communication problem. Indeed in that case one can easily build a measurement with a small number of outcomes (corresponding to a few code words, i.e. a small capacity), and the task is to try to maximize the number of outcomes of the measurement while continuing to recognize the code words faithfully. In this paper we can easily build a measurement with a high fidelity (i.e. which is optimal), but with a large redundancy in the output. The difficulty is to minimize the number of outcomes (the redundancy) while keeping the measurement optimal. Nevertheless the mathematical technique that we use in section XI to decrease the number of outcomes without substantially modifying the measurement is related to the techniques used in \cite{11}.

XI. ELIMINATING REDUNDANCY

Our aim in this section is to construct a measurement with less outcomes than the optimal measurement eq. (13). The next two sections will be devoted to prove that this measurement does not diminish the fidelity. This measurement
is very similar to the measurement used in [11] to decode a classical message sent through a quantum communication channel.

We start from the optimal POVM acting on one input state and decomposed into one dimensional projectors \( b_i = |b_i\rangle \langle b_i| \). We express it in terms of the normalized operator \( \tilde{b}_i = |\tilde{b}_i\rangle \langle \tilde{b}_i| = b_i / \text{Tr}(b_i) \) as \( b_i = \beta_i \tilde{b}_i \). (Throughout the text we shall denote normalized operators by \( \tilde{\cdot} \)). The \( \beta_i \) sum to \( \sum_i \beta_i = d \) obtained by taking the trace of the completeness relation.

We now construct \( N \) operators acting on the space of \( L \) input states:

\[
\tilde{B}_j = |\tilde{B}_j\rangle \langle \tilde{B}_j| = \tilde{b}_{j_1} \otimes \cdots \otimes \tilde{b}_{j_L}
\]

where each \( \tilde{b}_{j_k} \) is chosen randomly and independently from the set \( \tilde{b}_1, \ldots, \tilde{b}_M \) with probabilities \( p_1 = \beta_1/d, \ldots, p_M = \beta_M/d \).

The \( |\tilde{B}_j\rangle \) span a subspace \( H_B \) of the Hilbert space of the \( L \) input states. In this subspace the operator \( B = \sum_j \tilde{B}_j \) is strictly positive, hence we can construct the operators

\[
C_j = |C_j\rangle \langle C_j| = B^{-1/2} \tilde{B}_j B^{-1/2}.
\]

The \( C_j \) are positive operators, which sum up to the identity in \( H_B \): \( \sum_{j=1}^N C_j = \Pi_B \) where \( \Pi_B \) is the projector onto \( H_B \). The POVM we shall use consists of the \( C_j \) and the projector onto the complementary subspace \( C_0 = I_{d^L} - \Pi_B \) (\( I_{d^L} \) is the identity on the Hilbert space of the \( L \) input states).

Our strategy in the next sections will be to compute the average fidelity \( F_L \), where the average is taken over possible choices of \( B_j \) in eq. \([13]\). We shall show that the average of \( F_L \) satisfies our main result stated at the end of section \([VIII]\). Therefore there necessarily are some choices of \( B_j \) which also satisfy our main result.

But first we derive some important properties of the \( C_j \). We shall obtain mean properties, where the mean is the average over choices of \( B_j \) in eq. \([13]\).

- The mean of \( \tilde{B}_j \) is \( \langle \tilde{B}_j \rangle = I_{d^L} / d^L \).
- The mean of \( B \) is:

\[
\overline{B} = \sum_{j=1}^N \overline{B_j} = \frac{N}{d^L} I_{d^L}.
\]

This motivates our writing

\[
B = \frac{N}{d^L} (I_{d^L} + \Delta)
\]

and subsequently making expansions in \( \Delta \).

- The dimension of \( H_B \) is

\[
dim_{H_B} = \sum_j \text{Tr} C_j = \sum_j \text{Tr} B^{-1} \tilde{B}_j = \frac{d^L}{N} \sum_j \text{Tr} \left( I_{d^L} + \Delta \right) \tilde{B}_j \\
\geq \frac{d^L}{N} \sum_j \text{Tr} (I_{d^L} - \Delta) \tilde{B}_j.
\]

Furthermore

\[
\text{Tr} \Delta \tilde{B}_j = \text{Tr} \left( \frac{d^L}{N} B - I_{d^L} \right) \tilde{B}_j \\
= \text{Tr} \left[ \frac{d^L}{N} (\tilde{B}_j + \sum_{k \neq j} \tilde{B}_k \tilde{B}_j) - \tilde{B}_j \right]
\]

where we have used the fact that \( \tilde{B}_j^2 = \tilde{B}_j \). We now take the average of this expression. Using the fact that for \( k \neq j \), \( \tilde{B}_k \) and \( \tilde{B}_j \) are independent, the average of \( \tilde{B}_k \tilde{B}_j (k \neq j) \) is the product of the averages \( \overline{B_k \tilde{B}_j} = \overline{B_j} \overline{B_k} = I_{d^L} / d^L \). And hence \( \sum_{k \neq j} \tilde{B}_k \tilde{B}_j = (N - 1) I_{d^L} / d^L \). Putting all together, we find \( \overline{\text{Tr} \Delta \tilde{B}_j} = \frac{d^L-1}{d} \) and
\[ d^L \geq \frac{\dim H_B}{N} \geq d^L \left(1 - \frac{d^L - 1}{N}\right). \]  

(22)

This shows that if \( N \) is slightly larger than the dimension of the Hilbert space \( d^L \), then the \( C_j \) (\( j \neq 0 \)) fill the Hilbert space.

- Finally we need to know how much the \( C_j \) differ from the \( \tilde{B}_j \). We write \( |C_j\rangle = \alpha_j |\tilde{B}_j\rangle + |B^L_j\rangle \) and compute \( \alpha_j^2 \):

\[
\alpha_j^2 = \text{Tr} C_j \tilde{B}_j \\
= \text{Tr} \tilde{B}_j B^{-1/2} \tilde{B}_j B^{-1/2} \\
= \left( \text{Tr} \tilde{B}_j B^{-1/2}\right)^2 \\
\geq \frac{d^L}{N} \left(1 - \frac{1}{2} \text{Tr} \tilde{B}_j \Delta\right)^2. 
\]

(23)

Hence

\[
\alpha_j^2 \geq \frac{d^L}{N} \left(1 - \text{Tr} \tilde{B}_j \Delta\right) \\
= \frac{d^L}{N} \left(1 - \frac{d^L - 1}{N}\right). 
\]

(24)

This is then used to compute the average of \( \langle B^L_j | B^L_j \rangle \):

\[
\langle B^L_j | B^L_j \rangle = \text{Tr} C_j - \text{Tr} C_j B_j \leq \frac{d^L}{N} \frac{d^L - 1}{N}, 
\]

(25)

which shows that the \( C_j \) are arbitrarily close to the \( \tilde{B}_j \) when \( N > d^L \).

**XII. AN INTERMEDIATE RESULT**

In this section we shall prove the following intermediate result:

Suppose that the input states \( |\psi_i\rangle \) belong to a Hilbert space of dimension \( d \) and have a density matrix \( \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \). Denote by \( \rho_{\text{max}} \) the largest eigenvalue of \( \rho \). Consider measurements on blocks of \( L \) input states. Give yourself any positive number \( \eta > 0 \). Let \( N \) be any integer larger than \( 2^{L(2 \ln d + \ln \rho_{\text{max}} + \eta)} \). Then there exist measurements with \( N \) outcomes with a fidelity \( F_L \geq F_{\text{max}} - R2^{-L\eta} \) where \( R \) is a positive constant.

In the next section we shall combine this intermediate result with the concept of probable subspace of a long sequence of states to prove our claim in full generality.

To prove this intermediate result, we proceed as follows:

Let \( \{b_j = |b_j\rangle \langle b_j|\} \) be a POVM that maximizes the fidelity \( F \) eq. (9). Using the algorithm of eq. (10) to (17) we construct a measurement \( C_j \), \( j = 0, \ldots, N \) acting on the space of \( L \) copies of the input states.

Let us consider the fidelity for the measurement \( C_j \):

\[
F_L = \sum_{j=0}^{N-1} \frac{1}{L} \sum_{k=1}^{L} \sum_{i_k} p_{i_k} \langle \psi_{i_k} | C_j^{(k)} | \psi_{i_k}\rangle f(\psi_{i_k}, \phi_{i_k}^{\text{guess}}) 
\]

(26)

where the \( C_j^{(k)} = \text{Tr}_{l \neq k} \left( \prod_{l \neq k} \rho_l \right) C_j \) are defined as in eq. (14).

We can decompose \( C_j^{(k)} \) (for \( j \neq 0 \)) according to its components along \( |\tilde{b}_{jk}\rangle \): \( C_j^{(k)} = X_{jk} |\tilde{b}_{jk}\rangle \langle \tilde{b}_{jk}| + Y_{jk} |\tilde{b}_{jk}\rangle \langle b^L_{jk}| + Y_{jk}^\ast |b^L_{jk}\rangle \langle \tilde{b}_{jk}| + z_{jk} \) where \( z_{jk} |\tilde{b}_{jk}\rangle = 0, |\tilde{b}_{jk}| z_{jk} = 0 \). Inserting this expression in eq. (26), and using eq. (14), yields

\[
F_L \geq \frac{1}{L} \sum_{k=1}^{L} \left( F_{\text{max}} - C \sum_{j=1}^{N} \text{Tr} z_{jk} - C \text{Tr} C_0^{(k)} \right) 
\]

(27)
The coefficients $C_k$ to whether when restricted to the space of the
$k$th particle, it is equal to $|b_j\rangle$ or not: $|B_j^\perp\rangle = |\tilde{b}_j\rangle|\phi\rangle + |\tilde{b}_j^\perp\rangle|\chi\rangle$. Inserting this into the trace which yields $C_j$, we obtain
\[
C_j = Tr_{l\neq k}(\prod_{l'\neq k} \rho_{l'})(I_{d^L} - \Pi_B) \\
\leq (\rho_{\text{max}})^{L-1} Tr(I_{d^L} - \Pi_B) = (\rho_{\text{max}})^{L-1}(d^L - \text{dim } H_B) \tag{28}
\]
where $\rho_{\text{max}}$ is the largest eigenvalue of $\rho$.

To estimate $Trz_{jk}$ we recall the decomposition of $|C_j\rangle = \alpha_j|\tilde{B}_j\rangle + |B_j^\perp\rangle$. We can further decompose $|B_j^\perp\rangle$ according to whether when restricted to the space of the $k$th particle, it is equal to $|b_j\rangle$ or not: $|B_j^\perp\rangle = |\tilde{b}_j\rangle|\phi\rangle + |\tilde{b}_j^\perp\rangle|\chi\rangle$.

We now take the average of this expression over all possible choices of $b_j$.

We first recall the properties of the probable subspace $\text{[8] [9]}$. Consider a long sequence of
input states, then this intermediate result directly implies our main claim. Indeed when $\rho = I/d$, then $Tr_L \geq F_{\text{max}} - R2^{-L\eta}$ if $N \geq 2L(\text{ln}d + \eta)$. When the input states are not uniformly distributed in Hilbert space, we must use the notion of probable Hilbert space of a long sequence to prove our main result. This is done in the next section.

\section{XIII. MEASUREMENTS ON PROBABLE SUBSPACES}

We now combine the result of the previous section with the notion of probable subspace of large blocks of states.

We first recall the properties of the probable subspace $\text{[8] [9]}$. Consider a long sequence of $L'$ input states $|\psi_{l_1}...\psi_{l'}\rangle$.

The density matrix of these states is $\rho = \prod_{k=1}^{L'} \rho_k$. The projector $\Pi$ onto the probable subspace has the properties that given $\epsilon' > 0, \eta' > 0$, and for $L'$ sufficiently large,

1. $Tr\Pi \rho \geq 1 - \epsilon'$, i.e. the probability to be in the probable subspace is arbitrarily close to 1.

2. $\Pi$ and $\rho$ commute, i.e. the eigenvectors of $\rho$ are either eigenvectors of $\Pi$ or of $1 - \Pi$. And furthermore the eigenvectors which are common to $\Pi$ and $\rho$ have eigenvalues comprised between $2^{L'(-H-\eta')} \leq (\rho_{L'}) \leq 2^{L'(-H+\eta')}$

3. From these two properties it follows that the dimension of the probable Hilbert space is bounded by $(1 - \epsilon')2^{L'(-H-\eta')} \leq Tr\Pi \leq 2^{L'(H+\eta')}$

Therefore if $N \geq 2L(\text{ln}d + \eta + \rho_{\text{max}} + \eta)$, then $Tr_L \geq F_{\text{max}} - R2^{-L\eta}$ where $R = 2C/\rho_{\text{max}}$. This proves the intermediate result.

Note that if the input states are uniformly distributed in Hilbert space, i.e. $\rho = I/d$, then this intermediate result directly implies our main claim. Indeed when $\rho = I/d$, $\rho_{\text{max}} = 1/d$, then $Tr_L \geq F_{\text{max}} - R2^{-L\eta}$ if $N \geq 2L(\text{ln}d + \eta) = 2L(\text{ln}N + \eta)$. When the input states are not uniformly distributed in Hilbert space, we must use the notion of probable Hilbert space of a long sequence to prove our main result. This is done in the next section.
Let us now show that measurements restricted to the probable subspace are arbitrarily close to optimal. Suppose that \( A_j \) is a measurement that optimizes the state determination problem eq. (12) for sequences of \( L' \) input states (for instance the measurement eq. (13)). Consider the POVM consisting of the operators \( A'_j = \Pi A_j \Pi \) (to which we associate the unmodified guessed states \( \phi_{j,k}^{guess} \)) and the operator \( I - \Pi \) (to which we associate the minimal value of the fidelity \( f_{\text{min}} \)). The fidelity for this measurement is

\[
F'_L = \sum_{i_1, \ldots, i_{L'}} p_{i_1} \cdots p_{i_{L'}} \sum_{j=1}^N \langle \psi_{i_1} \cdots \psi_{i_{L'}} | \Pi A_j \Pi | \psi_{i_1} \cdots \psi_{i_{L'}} \rangle \frac{1}{L'} \sum_{k=1}^{L'} f(\psi_{i_k}, \phi_{jk})
+ \sum_{i_1, \ldots, i_{L'}} p_{i_1} \cdots p_{i_{L'}} \langle \psi_{i_1} \cdots \psi_{i_{L'}} | 1 - \Pi | \psi_{i_1} \cdots \psi_{i_{L'}} \rangle f_{\text{min}}
\geq F_{\text{max}}
- \sum_{i_1, \ldots, i_{L'}} p_{i_1} \cdots p_{i_{L'}} \sum_{j=1}^N \langle \psi_{i_1} \cdots \psi_{i_{L'}} | A_j - \Pi A_j \Pi | \psi_{i_1} \cdots \psi_{i_{L'}} \rangle \frac{1}{L'} \sum_{k=1}^{L'} f(\psi_{i_k}, \phi_{jk})
+ f_{\text{min}} \text{Tr}(1 - \Pi) .
\]  

(33)

We bound the second term by

\[
| \sum_{i_1, \ldots, i_{L'}} p_{i_1} \cdots p_{i_{L'}} \sum_{j=1}^N \langle \psi_{i_1} \cdots \psi_{i_{L'}} | A_j - \Pi A_j \Pi | \psi_{i_1} \cdots \psi_{i_{L'}} \rangle | \frac{1}{L'} \sum_{k=1}^{L'} f(\psi_{i_k}, \phi_{jk}) |
\leq f_{\text{max}} \sum_{j=1}^N | \sum_{i_1, \ldots, i_{L'}} p_{i_1} \cdots p_{i_{L'}} \langle \psi_{i_1} \cdots \psi_{i_{L'}} | (A_j - \Pi A_j \Pi) | \psi_{i_1} \cdots \psi_{i_{L'}} \rangle |
\]

\[
= f_{\text{max}} \sum_{j=1}^N [\text{Tr}(\rho(A_j - \Pi A_j \Pi))]
\]

\[
= f_{\text{max}} \text{Tr}(\rho(\Pi \rho ) \sum_{j=1}^N A_j) = f_{\text{max}} \text{Tr}(\rho(I - \Pi))
\]

\[
\leq \epsilon' f_{\text{max}}
\]  

(34)

where \( f_{\text{max}} \) is the maximum value of the fidelity and we have used the fact that \( \rho - \Pi \rho \Pi \) is a positive operator, and therefore that \( \text{Tr}(\rho(A_j - \Pi A_j \Pi)) \geq 0 \) which allows us to remove the absolute value sign and put the sum over \( j \) inside the trace.

Putting everything together we have

\[
F'_L \geq F_{\text{max}} - \epsilon'(f_{\text{max}} - f_{\text{min}}).
\]  

(35)

This shows that the restriction of the measurement to the probable Hilbert space diminishes the fidelity by an arbitrarily small amount \( \epsilon'(f_{\text{max}} - f_{\text{min}}) \).

We can now build a measurement which satisfies our main result as stated at the end of section VIII. We decompose the input states into blocks of \( L' \) states. On each of these blocks we first carry out a partial measurement \( \Pi \) and \( I - \Pi \) to know whether it is in the probable subspace or not. If the result is \( I - \Pi \) the sequence is discarded. The sequences which pass the test are kept.

We now take the sequences which have passed the test as the input states in the intermediate result. These sequences belong to a Hilbert space of dimension \( \text{dim } H_{\text{probable}} \leq 2^{L(L' \text{input} + \eta')} \) and the largest eigenvalue of their density matrix is \( \rho_{\text{max}} \leq 2^{L(-L' \text{input} + \eta')} \). To apply the intermediate result, we take an integer \( L \) and an \( \eta > 0 \). Then there exists a measurement on blocks of \( L \) sequences which has a number of possible outcomes equal to any integer \( N \) larger than \( 2^{L(L' \text{input} + \eta') + \eta} \). Let \( R \) be a positive constant.

Let us calculate the entropy \( I_{\text{outputs}}^S \) of the outputs of this measurement. We need less than \( I_{v} = -\epsilon' \ln \epsilon' - (1 - \epsilon') \ln(1 - \epsilon') \) bits to describe whether or not the input state passes the first test of belonging to the probable Hilbert space or not. If it does then we need less than \( \ln N \) bits to encode the output of the measurement on the \( L \) blocks of probable sequences. Therefore the total number of bits we need to describe the output of this measurement on \( LL' \) elementary input states is \( I_{\text{output}}^S \leq \ln N + LL' \). Replacing \( N \) by its bound, we have \( I_{\text{output}}^S \leq LL'(I_{\text{input}}^V + (3\eta' + \eta/L' + I_v/L') \). Since \( \epsilon' \) and \( \eta' \) can be chosen arbitrarily small, and \( L' \) arbitrarily large, our claim is proven.
XIV. CONCLUSION

In this paper we have obtained a quantitative estimate of how much information can be obtained by a quantum measurement. We considered optimal measurements, that is measurements which maximize a fidelity function. We then enlarged the set of optimal measurements in two ways. First we considered optimal measurements that act collectively on large blocks of input states rather than measurements restricted to act on each state separately. Secondly we did not require the fidelity of the measurements to be exactly equal to the optimal fidelity, but only that it be arbitrarily close to the optimal fidelity. In this context we showed that whatever property of a quantum system one wants to learn about, one can learn at most one bit of information about every qubit of quantum information carried by the unknown quantum system. That is, the Shannon entropy of the outcomes of optimal measurements can always be made equal or less than the von Neumann entropy of the unknown quantum states.

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[1] C. W. Helstrom, Quantum Detection and Estimation Theory, New York, Academic Press, 1976
[2] A. S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory, North Holland, Amsterdam, 1982
[3] S. Massar and S. Popescu, Phys. Rev. Lett. 74 (1995) 1259
[4] R. Derka, V. Buzek, A. K. Ekert, Phys. Rev. Lett. 80 (1998) 1571
[5] J. I. Latorre, P. Pascual, R. Tarrach, Phys. Rev. Lett. 81 (1998) 1351
[6] G. Vidal, J. I. Latorre, P. Pascual, R. Tarrach, quant-ph/9812068
[7] C. E. Shannon, Bell. Syst. Tech. J. 27 (1948) 379
[8] B. Schumacher, Phys. Rev. A 51 (1995) 2738
[9] R. Jozsa and B. Schumacher, J. Mod. Opt. 41 (1994) 2343
[10] A. S. Holevo, Probl. Peredachi Inf. 9 (1973) 3 [Probl. Inf. Transm. (USSR) 9 (1973) 177]
[11] P. Hausladen, R. Jozsa, B. Schumacher, M. Westmoreland, W. K. Wooters, Phys. Rev. A 54 (1996) 1869