NOTES ON LATTICE BUMP FOURIER MULTIPLIER OPERATORS ON $L^2 \times L^2$

TOMOYA KATO, AKIHIKO MIYACHI, AND NAOHITO TOMITA

Abstract. Given a smooth bump function, we consider the multiplier formed by taking the linear combination of the translations of the bump function and the corresponding bilinear Fourier multiplier operator. Under certain condition on the bump function, we give a complete characterization of the coefficients of the linear combination for which the corresponding bilinear operator defines a bounded operator from $L^2 \times L^2$ to $L^2$-based amalgam spaces.

1. Introduction

1.1. Background. For $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, the bilinear Fourier multiplier operator $T_\sigma$ is defined by

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{2\pi i x \cdot (\xi + \eta)} \sigma(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) \, d\xi \, d\eta,$$

for $f, g \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$.

We also write $T[\sigma]$ or $T[\sigma(\xi, \eta)]$ to denote $T_\sigma$. If $X, Y, Z$ are function spaces on $\mathbb{R}^n$ equipped with norms $\| \cdot \|_X$, $\| \cdot \|_Y$, $\| \cdot \|_Z$ respectively, then we define

$$\|\sigma\|_{M(X \times Y \to Z)} = \|T_\sigma\|_{X \times Y \to Z} = \sup\{ \|T_\sigma(f, g)\|_Z \mid f \in \mathcal{S} \cap X, g \in \mathcal{S} \cap Y, \|f\|_X = \|g\|_Y = 1 \}.$$

If $\|T_\sigma\|_{X \times Y \to Z} < \infty$, then, with a slight abuse of terminology, we shall say that $T_\sigma$ is bounded from $X \times Y$ to $Z$ and write $T_\sigma : X \times Y \to Z$.

One of the fundamental theorems for the boundedness of bilinear operators $T_\sigma$ in Lebesgue spaces is that if the multiplier $\sigma$ satisfies the condition

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leq C_{\alpha, \beta}(|\xi| + |\eta|)^{-|\alpha| - |\beta|},$$

then $T_\sigma : L^p \times L^q \to L^r$ for all $1 < p, q, r < \infty$ with $1/p + 1/q = 1/r > 0$. This theorem was proved by Coifman-Meyer [4], Kenig-Stein [13], and Grafakos-Torres [9]. This theorem may be considered as a natural extension of the well-known theorem about the linear Fourier multiplier operators satisfying Hörmander-Mihlin type condition. However, if we consider other type of conditions, bilinear Fourier multiplier operators take properties that are different from the linear operators. One of such properties concerns with multipliers satisfying the condition

$$(1.1) \quad |\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leq C_{\alpha, \beta}.$$
$1/p + 1/q = 1/r$ then there exists a multiplier $\sigma$ satisfying the condition (1.1) for which the corresponding operator $T_\sigma$ is not bounded from $L^p \times L^q$ to $L^r$.

The subject of the present paper concerns with bilinear multipliers satisfying the condition (1.1). We shall consider the boundedness of $T_\sigma$ only on $L^2 \times L^2$. The above result of Bényi-Torres [1] implies that we can expect the boundedness of $T_\sigma$ only if we strengthen the condition (1.1). So far several such results are known. As far as the boundedness of operators $T_\sigma$ on $L^2 \times L^2$ is concerned, it seems that the most general result known so far is the theorem given in Kato-Miyachi-Tomita [11], which reads as follows. For nonnegative function $W$ on $\mathbb{Z}^n \times \mathbb{Z}^n$, the class $BS_{0,0}^W(\mathbb{R}^n)$ is defined to be all $C^\infty$ functions $\sigma$ on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the estimate

$$|\partial_\xi^a \partial_\eta^b \sigma(\xi, \eta)| \leq C_{\alpha, \beta} \sum_{\mu, \nu \in \mathbb{Z}^n} W(\mu, \nu) |1_R(\xi - \mu)1_R(\eta - \nu)|,$$

where $R = [-1/2, 1/2]^n$ (this notation is slightly different from the one given in [11]). The theorem of [11, Theorem 1.3] asserts that $T_\sigma : L^2 \times L^2 \to L^r$, $1 \leq r \leq 2$, for all $\sigma \in BS_{0,0}^W(\mathbb{R}^n)$ if and only if

$$\|W\|_B = \sup_{\mu, \nu \in \mathbb{Z}^n} \sum_{\mu, \nu \in \mathbb{Z}^n} W(\mu, \nu) F(\mu) G(\nu) H(\mu + \nu) < \infty,$$

where the sup is taken over all nonnegative sequences $F, G, H \in \ell^2(\mathbb{Z}^n)$ with $\|F\|_2 = \|G\|_2 = \|H\|_2 = 1$. It is known that all nonnegative $W$ in the Lorentz class $\ell^{4,\infty}(\mathbb{Z}^{2n})$ satisfies the condition (1.2); see [11, Proposition 3.4]. In particular $W(\mu, \nu) = (1 + |\mu| + |\nu|)^{-n/2}$ is a typical example of $W$ satisfying (1.2). The $L^2 \times L^2 \to L^1$ estimates for operators in the class $BS_{0,0}^W(\mathbb{R}^n)$ with $W(\mu, \nu) = (1 + |\mu| + |\nu|)^{-n/2}$ is also given in [13], where estimates in other function spaces are given as well. Slavíková [15] also gives the $L^2 \times L^2 \to L^1$ estimates of $T_\sigma$ for $\sigma \in BS_{0,0}^W(\mathbb{R}^n)$ with $W \in \ell^{4,\infty}(\mathbb{Z}^{2n})$ even under restricted smoothness assumptions. Related results are also given in Grafakos-He-Slavíková [7].

Now the theorem [11, Theorem 1.3] is sharp in itself but it is a theorem that is concerned with a class of multipliers not with an individual multiplier. In the present paper, we shall give a theorem that treats individual multiplier, although we strongly restrict the form of multipliers. Here we recall the work of Grafakos and Kalton [8] that considers a problem in the same spirit.

In [8], the authors consider the multiplier $\sigma$ of the form

$$\sigma_A(\xi, \eta) = \sum_{j,k \in \mathbb{Z}} a_{j,k} \phi(2^{-j} \xi) \phi(2^{-k} \eta), \quad \xi, \eta \in \mathbb{R}^n,$$

where $A = (a_{j,k})_{j,k \in \mathbb{Z}}$ is an infinite matrix of complex numbers and $\phi$ is a function in $C_0^\infty(\mathbb{R}^n)$ such that $\sup \phi \subset \{2^{-1} \leq |\xi| \leq 2\}$ and $\sum_{j \in \mathbb{Z}} \phi(2^{-j} \xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. The authors of [8] introduce a norm $H(A)$ for infinite matrices $A$ and prove the inequality

$$c^{-1} H(A) \leq \|T_{\sigma_A}\|_{H^p \times H^q \to L^r} \leq c H(A),$$

where $H^p$, $0 < p < \infty$, denotes Hardy spaces on $\mathbb{R}^n$ and $p, q, r$ are positive real numbers satisfying $1/p + 1/q = 1/r$ (see [8, Theorem 6.5]). Here we do not give the definition of $H(A)$ but we mention that it depends only on $A$ but not on $p, q, r$. As an application of this theorem, the authors give an example of a multiplier $\sigma(\xi, \eta)$ that satisfies the Marcinkiewicz type condition

$$|\partial_\xi^a \partial_\eta^b \sigma(\xi, \eta)| \leq C_{\alpha} |\xi|^{-|\alpha|} |\eta|^{-|\beta|}$$
but the corresponding operator $T_a$ is not bounded in $H^p \times H^q \to L^r$ for any $0 < p, q, r < \infty$ with $1/p + 1/q = 1/r$.

In the present paper, we shall give a theorem that is similar to (1.3) in relation to multipliers satisfying the condition (1.1). Although our result is not directly related to the result of [3], we shall use some ideas given in this paper.

A study of similar nature can be found in Buriánková-Grafakos-He-Honzík [3].

1.2. Main results. Now we shall give the statement of the main result of this paper.

If $a_{\mu,\nu} \in \mathbb{C}$ is given for each $(\mu, \nu) \in \mathbb{Z}^n \times \mathbb{Z}^n$, then we call $A = (a_{\mu,\nu})_{\mu,\nu \in \mathbb{Z}^n}$ a matrix on $\mathbb{Z}^n \times \mathbb{Z}^n$. If $\sup_{\mu,\nu} |a_{\mu,\nu}| < \infty$, then we say $A$ is an $L^\infty$ matrix.

If $A = (a_{\mu,\nu})_{\mu,\nu \in \mathbb{Z}^n}$ is an $L^\infty$ matrix on $\mathbb{Z}^n \times \mathbb{Z}^n$, and if $\Phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, we define the function $\sigma_{A,\Phi}$ by

$$\sigma_{A,\Phi}(\xi, \eta) = \sum_{\mu, \nu \in \mathbb{Z}^n} a_{\mu,\nu} \Phi(\xi - \mu, \eta - \nu), \quad \xi, \eta \in \mathbb{R}^n.$$  

Notice that we always have $\sigma_{A,\Phi} \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. The purpose of the present paper is to show that under certain condition on $\Phi$ we can completely characterize all $L^\infty$ matrices $A$ that satisfy

$$T_{\sigma_{A,\Phi}} : L^2 \times L^2 \to (L^2, \ell^3), \quad 1 \leq q \leq \infty,$$

where $(L^2, \ell^3)$ is the $L^2$-based amalgam space. The precise definition of the amalgam space will be given in the next section. Our characterization of $A$ for (1.4) does not depend on $q \in [1, \infty]$, which would be of independent interest.

We shall introduce a norm for $L^\infty$ matrices. For this, we write $\ell^2 = \ell^2(\mathbb{Z}^n)$ to denote the class of all $F : \mathbb{Z}^n \to \mathbb{C}$ such that

$$\|F\|_{\ell^2} = \left( \sum_{\mu \in \mathbb{Z}^n} |F(\mu)|^2 \right)^{1/2} < \infty.$$  

We also write $\ell^2_0 = \ell^2_0(\mathbb{Z}^n)$ to denote the class of all those $F \in \ell^2$ such that $F(\mu) = 0$ except for a finite number of $\mu \in \mathbb{Z}^n$. The norm for $L^\infty$ matrices is defined as follows.

**Definition 1.1.** If $A = (a_{\mu,\nu})_{\mu,\nu \in \mathbb{Z}^n}$ is an $L^\infty$ matrix, then $\|A\|_B$ denotes the norm of the trilinear functional

$$(\ell^2_0(\mathbb{Z}^n))^3 \ni (F, G, H) \mapsto \sum_{\mu, \nu \in \mathbb{Z}^n} a_{\mu,\nu} F(\mu) G(\nu) H(\mu + \nu) \in \mathbb{C},$$

i.e., $\|A\|_B$ is the sup of

$$\left| \sum_{\mu, \nu \in \mathbb{Z}^n} a_{\mu,\nu} F(\mu) G(\nu) H(\mu + \nu) \right|$$

over all $F, G, H \in \ell^2_0(\mathbb{Z}^n)$ satisfying $\|F\|_{\ell^2} = \|G\|_{\ell^2} = \|H\|_{\ell^2} = 1$.

We also use the following.

**Definition 1.2.** We say that a function $\Phi \in C_0^\infty(\mathbb{R}^d)$ satisfies the condition (A) if there exists a function $\Theta \in C_0^\infty(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \Theta(x) \Phi(x - \alpha) \, dx = \begin{cases} 1 & \text{if } \alpha = 0 \in \mathbb{Z}^d, \\ 0 & \text{if } \alpha \in \mathbb{Z}^d \setminus \{0\}. \end{cases}$$
Obviously any nonzero function $\Phi \in C^\infty_0(\mathbb{R}^d)$ with support included in the cube $[-1/2, 1/2]^d$ satisfies the condition (A). It may not be so obvious that there exists a nonzero $\Phi \in C^\infty_0(\mathbb{R}^d)$ that does not satisfy the condition (A). In Section 5, we shall give some examples of $\Phi$ that satisfy or do not satisfy the condition (A).

The following is the main result of this paper.

**Theorem 1.3.** (1) For every $\Phi \in C^\infty_0(\mathbb{R}^n \times \mathbb{R}^n)$, there exists a constant $c \in (0, \infty)$ depending only on $n$ and $\Phi$ such that

$$\label{1.5} \|\sigma_{A,\Phi}\|_{\mathcal{M}(L^2 \times L^2 \to (L^2, \ell^1))} \leq c\|A\|_B$$

for all $L^\infty$ matrices $A$.  

(2) If $\Phi \in C^\infty_0(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies the condition (A), then exists a constant $c \in (0, \infty)$ depending only on $n$ and $\Phi$ such that

$$\label{1.6} \|\sigma_{A,\Phi}\|_{\mathcal{M}(L^2 \times L^2 \to (L^2, \ell^\infty))} \geq c^{-1}\|A\|_B$$

for all $L^\infty$ matrices $A$.

Since the norms of the amalgam spaces satisfy

$$\|f\|_{(L^2, \ell^\infty)} \leq \|f\|_{(L^2, \ell^q)} \leq \|f\|_{(L^2, \ell^1)}, \quad 1 \leq q \leq \infty,$$

the following is an immediate corollary to the above theorem.

**Corollary 1.4.** If $\Phi \in C^\infty_0(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies the condition (A), then there exists a constant $c \in (0, \infty)$ depending only on $n$ and $\Phi$ such that

$$c^{-1}\|A\|_B \leq \|\sigma_{A,\Phi}\|_{\mathcal{M}(L^2 \times L^2 \to (L^2, \ell^q))} \leq c\|A\|_B$$

for all $L^\infty$ matrices $A$ and all $1 \leq q \leq \infty$.

We recall that the $L^2$-based amalgam spaces satisfy the following inclusion relations:

$$(L^2, \ell^2) = L^2,$$

$$(L^2, \ell^q) \hookrightarrow L^q \text{ if } 1 < q < 2,$$

$$(L^2, \ell^1) \hookrightarrow h^1 \hookrightarrow L^1,$$

$L^q \hookrightarrow (L^2, \ell^q) \text{ if } 2 < q < \infty,$

$L^\infty \hookrightarrow bmo \hookrightarrow (L^2, \ell^\infty),$

where $h^1$ is the local Hardy space and $bmo$ is the local BMO space given by Goldberg [6] (a proof of the embedding $(L^2, \ell^1) \hookrightarrow h^1$ can be found in [12, §2.3]). Thus Corollary 1.4 implies nontrivial results for mapping properties of $T_{\sigma_{A,\Phi}}$ in Lebesgue spaces, $h^1$, or $bmo$. For example, it implies that if $T_{\sigma_{A,\Phi}} : L^2 \times L^2 \to L^2$ then $T_{\sigma_{A,\Phi}} : L^2 \times L^2 \to h^1 \hookrightarrow L^1$.  

2. **Amalgam spaces**

Here we recall the definition of the amalgam spaces. We use the following notation to denote the cubes in $\mathbb{R}^n$:

$$Q = [-1/2, 1/2]^n,$$

$$aQ = [-a/2, a/2]^n, \quad a \in (0, \infty).$$

If $X, Y, Z$ are function spaces equipped with norms, then we write the mixed norm as

$$\|f(x, y, z)\|_{X, Y, Z} = \|\|f(x, y, z)\|_{X, y}\|_{Y, z}.$$
We write $L$ an arbitrary in Kato-Miyachi-Tomita [11]. Some ideas go back to Boulkhemair [2]. For details of amalgam spaces, see Fournier–Stewart [5] or Holland [10].

The integral in the left hand side of (3.1) can be written as

$$
\int_{\mathbb{R}^n} T_{\sigma,A,\Phi}(f,g)(x)h(x)\,dx
$$

with the usual modification when $p$ or $q$ is equal to $\infty$. The amalgam space $(L^p, \ell^q)$ is defined to be the set of all those $f$ satisfying $\|f\|_{(L^p, \ell^q)} < \infty$.

It is obvious that $(L^p, \ell^p) = L^p$. We have

$$(L^{p_1}, \ell^{q_1}) \hookrightarrow (L^{p_2}, \ell^{q_2})$$

if $p_1 \geq p_2$ and $q_1 \leq q_2$.

For $1 \leq p, q < \infty$, the duality

$$(L^p, \ell^q)^* = (L^{p'}, \ell^{q'})$$

holds.

In the present paper, we use only the spaces $(L^p, \ell^q)$ with $p = 2$ and $1 \leq q \leq \infty$. For details of amalgam spaces, see Fournier–Stewart [5] or Holland [10].

### 3. Proof of Theorem 1.2 (1)

In this section, we prove the inequality (1.5). We shall follow the argument given in Kato-Miyachi-Tomita [11]. Some ideas go back to Boukhemair [2].

The proof will be divided into two steps.

In the first step, we assume that $\Phi$ is written as $\Phi(\xi, \eta) = u(\xi)v(\eta)$ with $u, v \in C_0^\infty(\mathbb{R}^n)$. For this $\Phi$, we shall prove that if $T$ is a positive number satisfying $\text{supp } u \subset TQ$ and $\text{supp } v \subset TQ$ then (1.5) holds with $c = c_{n,T}\|u\|_{L^\infty}\|v\|_{L^\infty}$. We assume $A$ is an arbitrary $L^\infty$ matrix on $\mathbb{Z}^n \times \mathbb{Z}^n$. By duality, it is sufficient to show the inequality

$$
(3.1) \quad \left| \int_{\mathbb{R}^n} T_{\sigma,A,\Phi}(f,g)(x)h(x)\,dx \right| \leq c_{n,T}\|u\|_{L^\infty}\|v\|_{L^\infty}\|A\|\|f\|_{L^2}\|g\|_{L^2}\|h\|_{(L^2, \ell^\infty)}
$$

for all $f, g \in \mathcal{S}$ and all $h \in (L^2, \ell^\infty)$ with compact support.

The integral in the left hand side of (3.1) can be written as

$$
\int_{\mathbb{R}^n} T_{\sigma,A,\Phi}(f,g)(x)h(x)\,dx
= \sum_{\mu,\nu \in \mathbb{Z}^n} \int_{\xi,\eta,\xi' \in \mathbb{R}^n} e^{2\pi i \cdot (\xi' + \eta)} a_{\mu,\nu} u(\xi - \mu)v(\eta - \nu) \hat{f}(\xi)\hat{g}(\eta)h(x)\,d\xi d\eta dx
= \sum_{\mu,\nu,\rho \in \mathbb{Z}^n} \int_{\xi,\eta,\xi' \in TQ} e^{2\pi i (x + \rho) \cdot (\xi + \mu + \eta)} a_{\mu,\nu} u(\xi)v(\eta) \hat{f}(\xi + \mu)\hat{g}(\eta + \nu)h(x + \rho)\,d\xi d\eta dx
= (\ast)
$$

We write

$$
\begin{aligned}
e^{2\pi i (x + \rho) \cdot (\xi + \mu + \eta)}
&= e^{2\pi i \cdot (\xi + \eta)} e^{2\pi i \cdot (\mu + \nu)} e^{2\pi i \rho \cdot \xi} e^{2\pi i \rho \cdot \eta}
\end{aligned}
$$

$$
= e^{2\pi i \cdot (\mu + \nu)} e^{2\pi i \rho \cdot \xi} e^{2\pi i \rho \cdot \eta} \sum_{\alpha \in \mathbb{N}[0]} \frac{1}{\alpha!} (2\pi i)^{|\alpha|} x^\alpha \xi^\alpha \sum_{\beta \in \mathbb{N}[0]} \frac{1}{\beta!} (2\pi i)^{|\beta|} x^\beta \eta^\beta
$$
(notice that \( e^{2\pi i p(\mu + \nu)} = 1 \) since \( \rho \cdot (\mu + \nu) \) is an integer). Then we have

\[
(*) = \sum_{\mu, \nu, \rho \in \mathbb{Z}^n} \sum_{\alpha, \beta \in (\mathbb{N} \cup \{0\})^n} \frac{(2\pi i)^{|\alpha|}(2\pi i)^{|\beta|}}{\alpha! \beta!} \\
\times a_{\mu, \nu} \left( \int_{TQ} e^{2\pi i \rho \cdot \xi} u(\xi) \xi^\alpha \widehat{f}(\xi + \mu) \, d\xi \right) \left( \int_{TQ} e^{2\pi i \eta \cdot \eta} \eta^\beta g(\eta + \nu) \, d\eta \right) \\
\times \left( \int_{Q} e^{2\pi i x \cdot (\mu + \nu)} x^{\alpha + \beta} h(x + \rho) \, dx \right).
\]

In order to estimate this, we take the sums \( \sum_{\mu, \nu}, \sum_{\rho}, \) and \( \sum_{\alpha, \beta} \) in this order. First, we estimate the sum \( \sum_{\mu, \nu} \) by using \( \|A\|_B \) and Parseval’s identity to obtain

\[
\left| \sum_{\mu, \nu \in \mathbb{Z}^n} a_{\mu, \nu} \left( \int_{TQ} e^{2\pi i \rho \cdot \xi} u(\xi) \xi^\alpha \widehat{f}(\xi + \mu) \, d\xi \right) \left( \int_{TQ} e^{2\pi i \eta \cdot \eta} \eta^\beta g(\eta + \nu) \, d\eta \right) \right| \\
\times \left( \int_{Q} e^{2\pi i x \cdot (\mu + \nu)} x^{\alpha + \beta} h(x + \rho) \, dx \right) \\
\leq \|A\|_B \left\| \int_{TQ} e^{2\pi i \rho \cdot \xi} u(\xi) \xi^\alpha \widehat{f}(\xi + \mu) \, d\xi \right\|_{\ell_2^\alpha} \left\| \int_{TQ} e^{2\pi i \eta \cdot \eta} \eta^\beta g(\eta + \nu) \, d\eta \right\|_{\ell_2^\beta} \\
\times \left\| \int_{Q} e^{2\pi i x \cdot (\mu + \nu)} x^{\alpha + \beta} h(x + \rho) \, dx \right\|_{\ell_2^{\alpha + \beta}} \\
= (**) \|h(x + \rho)\|_{L_2(Q)}
\]

Next, we estimate the sum \( \sum_{\rho} \) by using the Cauchy-Schwarz inequality, Parseval’s identity, and Plancherel’s theorem to obtain

\[
\sum_{\rho \in \mathbb{Z}^n} (**) \\
\leq \|A\|_B \left\| \int_{TQ} e^{2\pi i \rho \cdot \xi} u(\xi) \xi^\alpha \widehat{f}(\xi + \mu) \, d\xi \right\|_{\ell_2^\alpha} \left\| \int_{TQ} e^{2\pi i \eta \cdot \eta} \eta^\beta g(\eta + \nu) \, d\eta \right\|_{\ell_2^\beta} \\
\times \sup_{\rho} \|x^{\alpha + \beta} h(x + \rho)\|_{L_2^2(Q)} \\
\leq c_n T^{\alpha + |\beta|} \|A\|_B \|u(\xi)\|_{L_2^2(T^*)} \|v(\eta)\|_{L_2^2(T^*)} \|g(\eta + \nu)\|_{L_2^2(T^*)} \\
\times \sup_{\rho} \|x^{\alpha + \beta} h(x + \rho)\|_{L_2^2(Q)} \\
\leq c_n T^{\alpha + |\beta|} \|A\|_B \|u\|_{L_\infty} \|v\|_{L_\infty} \|\widehat{f}(\xi + \mu)\|_{L_2^2(T^*)} \|\widehat{g}(\eta + \nu)\|_{L_2^2(T^*)} \\
\times \sup_{\rho} \|h(x + \rho)\|_{L_2^2(Q)} \\
= c_n T^{\alpha + |\beta|} \|A\|_B \|u\|_{L_\infty} \|v\|_{L_\infty} \|f\|_{L_2^2(\mathbb{R}^n)} \|g\|_{L_2^2(\mathbb{R}^n)} \|h\|_{L_2^2(\mathbb{R}^n)}
\]
(the constant $c_{n,T}$ in different places are not the same). Finally, we estimate the sum $\sum_{\alpha,\beta}$ by using
\[
\sum_{\alpha,\beta \in \mathbb{N}^n} \frac{(2\pi T)^{\alpha}}{\alpha!} \frac{(2\pi T)^{\beta}}{\beta!} = c_{n,T} < \infty
\]
and obtain (3.1).

In the second step, we prove that the inequality (1.5) holds for every $\Phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. We shall use the idea of using Fourier expansion (this idea may trace back to [4]). Take a positive number $T$ satisfying $\text{supp} \Phi \subset 2^{-1}TQ \times 2^{-1}TQ$ and take a function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp} \phi \subset TQ$ and $\phi(\xi) = 1$ on $2^{-1}TQ$. Since $\Phi$ is a smooth function supported in the interior of $TQ \times TQ$, the Fourier series expansion gives
\[
\Phi(\xi, \eta) = \sum_{k,\ell \in \mathbb{Z}^n} b_{k,\ell} e^{2\pi ik \cdot \xi / T} e^{2\pi i \ell \cdot \eta / T}, \quad (\xi, \eta) \in TQ \times TQ,
\]
where $b_{k,\ell}$ is a rapidly decreasing sequence. From the choice of the function $\phi$, we have
\[
\Phi(\xi, \eta) = \sum_{k,\ell \in \mathbb{Z}^n} b_{k,\ell} e^{2\pi ik \cdot \xi / T} e^{2\pi i \ell \cdot \eta / T} \phi(\xi) \phi(\eta), \quad (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n.
\]
Now the estimate proved in the first step yields
\[
\left\| \sum_{\mu,\nu \in \mathbb{Z}^n} a_{\mu,\nu} e^{2\pi i k \cdot (\xi - \mu) / T} e^{2\pi i \ell \cdot (\eta - \nu) / T} \phi(\xi - \mu) \phi(\eta - \nu) \right\|_{M(L^2 \times L^2)} \leq c_{n,T} \|A\|_B
\]
for each $(k, \ell)$ with the constant $c_{n,T}$ independent of $k, \ell$. Thus since $b_{k,\ell}$ is rapidly decreasing we obtain (1.5). This proves the part (1) of Theorem 1.3.

4. Proof of Theorem 1.3 (2)

In this section, we prove the inequality (1.6). We shall first consider $\Phi$ of a special form and then consider general $\Phi$.

4.1. The case of a special $\Phi(\xi, \eta)$. In this subsection we shall prove the inequality (1.6) for the function $\Phi$ defined by $\Phi(\xi, \eta) = \phi(\xi)\phi(\eta)$ with $\phi$ satisfying
\[
\phi \in C_0^\infty(\mathbb{R}^n), \quad \text{supp} \phi \subset Q, \quad \phi(\xi) = 1 \text{ for } \xi \in 2^{-1}Q.
\]
We assume $A$ is an arbitrary $L^\infty$ matrix, and assume $F, G, H \in \ell_2(\mathbb{Z}^n)\) satisfy $\|F\|_2 = \|G\|_2 = \|H\|_2 = 1$. We shall prove that there exist $f, g \in \mathcal{S}$ such that
\[
\|f\|_{L^2} \approx \|g\|_{L^2} \approx 1
\]
and
\[
\|T_{\sigma,\psi}(f, g)\|_{L^2(\mathbb{R}^n)} \gtrsim \sum_{\mu,\nu \in \mathbb{Z}^n} a_{\mu,\nu} F(\mu)G(\nu)H(\mu + \nu),
\]
where the constant in the above $\approx$ and $\gtrsim$ depend only on $n$ and $\phi$. This certainly implies the desired estimate (1.6).

We take a function $\theta \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp} \theta \subset 2^{-1}Q$ and $|F^{-1}\theta(x)| \gtrsim 1$ for $x \in Q$, and define $f, g$ by
\[
\widehat{f}(\xi) = \sum_{\mu \in \mathbb{Z}^n} F(\mu)\theta(\xi - \mu), \quad \widehat{g}(\eta) = \sum_{\nu \in \mathbb{Z}^n} G(\nu)\theta(\eta - \nu).
\]
Obviously $f, g \in \mathcal{S}(\mathbb{R}^n)$. We shall prove that these $f$ and $g$ satisfy (4.1) and (4.2).

The estimate (4.1) is obvious since

$$
\|f\|_{L^2} = \|\hat{f}\|_{L^2} \approx \|F\|_{\ell^2} = 1, \quad \|g\|_{L^2} = \|\hat{g}\|_{L^2} \approx \|G\|_{\ell^2} = 1.
$$

To prove (4.2), notice that our choice of $\phi$ and $\theta$ implies

$$
\phi(\xi - \mu)\hat{f}(\xi) = F(\mu)\theta(\xi - \mu), \quad \phi(\eta - \nu)\hat{g}(\xi) = G(\nu)\theta(\eta - \nu)
$$

for all $\mu, \nu \in \mathbb{Z}^n$. Hence

$$
T_{\sigma, A}(f, g)(x) = \sum_{\mu, \nu \in \mathbb{Z}^n} a_{\mu, \nu} \int e^{2\pi i x \cdot (\xi + \eta)} \phi(\xi - \mu)\phi(\eta - \nu)\hat{f}(\xi)\hat{g}(\eta) d\xi d\eta
$$

$$
= \sum_{\mu, \nu \in \mathbb{Z}^n} a_{\mu, \nu} F(\mu)G(\nu) \int e^{2\pi i x \cdot (\xi + \eta)} \theta(\xi - \mu)\theta(\eta - \nu) d\xi d\eta
$$

$$
= \sum_{\mu, \nu \in \mathbb{Z}^n} a_{\mu, \nu} F(\mu)G(\nu) \int e^{2\pi i x \cdot (\xi + \mu + \eta + \nu)} \phi(\xi)\theta(\eta) d\xi d\eta
$$

$$
= \sum_{\mu, \nu \in \mathbb{Z}^n} a_{\mu, \nu} F(\mu)G(\nu) e^{2\pi i x \cdot (\mu + \nu)} (\mathcal{F}^{-1}\theta(x))^2.
$$

Let $h \in L^2(Q)$ be the function defined by

$$(\mathcal{F}^{-1}\theta(x))^2 h(x) = \sum_{\rho \in \mathbb{Z}^n} H(\rho) e^{-2\pi i \rho x}, \quad x \in Q.$$

Then, since $|\mathcal{F}^{-1}\theta(x)| \gtrsim 1$ on $Q$, Parseval’s identity implies $\|h\|_{L^2(Q)} \approx \|H\|_{\ell^2} = 1$.

We have

$$
\int_Q T_{\sigma, \phi}(f, g)(x) h(x) \, dx
$$

$$
= \sum_{\mu, \nu \in \mathbb{Z}^n} a_{\mu, \nu} F(\mu)G(\nu) \int_Q e^{2\pi i x \cdot (\mu + \nu)} (\mathcal{F}^{-1}\theta(x))^2 h(x) \, dx
$$

$$
= \sum_{\mu, \nu \in \mathbb{Z}^n} a_{\mu, \nu} F(\mu)G(\nu) H(\mu + \nu).
$$

Thus

$$
\|T_{\sigma, \phi}(f, g)\|_{L^2, \ell^\infty} \geq \|T_{\sigma, \phi}(f, g)\|_{L^2(Q)} \gtrsim \int_Q T_{\sigma, \phi}(f, g)(x) h(x) \, dx
$$

$$
= \left| \sum_{\mu, \nu \in \mathbb{Z}^n} a_{\mu, \nu} F(\mu)G(\nu) H(\mu + \nu) \right|,
$$

as desired. Thus the estimate (4.6) is proved for the special $\Phi$.

4.2. The case of general $\Phi(\xi, \eta)$. The argument in this subsection follows the ideas given by Grafakos and Kalton [8, Proposition 6.2 and Lemma 6.3].

In order to simplify notation, we write $X = (L^2, \ell^\infty)$. We also use the following notation: for $m \in L^\infty(\mathbb{R}^n)$, we write $\|m\|_{M(L^2 \to L^2)}$ to denote the $L^2 \to L^2$ operator norm of the linear Fourier multiplier operator $f \mapsto \mathcal{F}^{-1}(m\hat{f})$. By Plancherel’s theorem, we have in fact $\|m\|_{M(L^2 \to L^2)} = \|m\|_{L^\infty}$.

We shall give the argument in a sequence of lemmas.
**Lemma 4.1.** The equality

\[ \|\sigma(-\xi_0, \eta + \eta_0)\|_{\mathcal{M}(L^2 \times L^2 \to X)} = \|\sigma(\xi, \eta)\|_{\mathcal{M}(L^2 \times L^2 \to X)} \]

holds for all \( \sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) and all \( \xi_0, \eta_0 \in \mathbb{R}^n \).

**Proof.** If we define \( E_a(x) = e^{2\pi ia \cdot x} \) for \( a \in \mathbb{R}^n \) and \( x \in \mathbb{R}^n \), then we have the formula

\[ T[\sigma(-\xi_0, \eta + \eta_0)](f, g)(x) = E_{-\xi_0-\eta_0}(x)T[\sigma](E_{\xi_0}f, E_{\eta_0}g)(x). \]

This implies the equality of the lemma since multiplication by the unimodular function \( E_a \) does not change the norms of \( L^2 \) and \( X = (L^2, \ell^\infty) \).

\[ \square \]

**Lemma 4.2.** Let \( \sigma_{\mu, \nu} \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) be given for each \( (\mu, \nu) \in \mathbb{Z}^n \times \mathbb{Z}^n \) and assume the following: (i) \( \sup_{\mu, \nu}\|\sigma_{\mu, \nu}\|_{L^\infty} < \infty \); (ii) there exists a \( K \in (0, \infty) \) such that \( \sup \sigma_{\mu, \nu} \subset (\mu, \nu) + KQ \times KQ \) for all \( \mu, \nu \); (iii) there exists an \( M \in [0, \infty) \) such that

\[ \|\sum_{\mu, \nu \in \mathbb{Z}^n} \alpha_{\mu, \nu} \sigma_{\mu, \nu}\|_{\mathcal{M}(L^2 \times L^2 \to X)} \leq M \]

for all \( \alpha_{\mu} \in \{0, 1\} \) and \( \alpha'_{\nu} \in \{0, 1\} \). Then for each \( \Phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n) \), there exists a constant \( c \) depending only on \( n, K, \Phi \) such that

\[ \left\| \sum_{\mu, \nu \in \mathbb{Z}^n} \Phi(\xi - \mu, \eta - \nu)\sigma_{\mu, \nu} \right\|_{\mathcal{M}(L^2 \times L^2 \to X)} \leq cM. \]

**Proof.** We first consider the case where \( \Phi \) is the product form \( \Phi(\xi, \eta) = u(\xi)v(\eta) \) with \( u, v \in C_0^\infty(\mathbb{R}^n) \). In this case, we shall prove that if \( T \) is a positive number satisfying \( \text{supp } u, \text{supp } v \subset TQ \) then there exists a constant \( c_{n, K, T} \) depending only on \( n, K, T \) such that

\[ \left\| \sum_{\mu, \nu \in \mathbb{Z}^n} u(\xi - \mu)v(\eta - \nu)\sigma_{\mu, \nu} \right\|_{\mathcal{M}(L^2 \times L^2 \to X)} \leq c_{n, K, T}\|u\|_{L^\infty}\|v\|_{L^\infty}M. \]

We take a positive integer \( N \) satisfying \( N > (T + K)/2 \) and we define the equivalence relation for \( \mu, \nu \in \mathbb{Z}^n \) by

\[ \mu \equiv \nu \iff N^{-1}(\mu - \nu) \in \mathbb{Z}^n. \]

Then from the assumptions on the supports of \( \sigma_{\mu, \nu}, u, \) and \( v, \) and from our choice of \( N, \) we see that

\[ \mu \equiv \mu', \nu \equiv \nu', \ u(\xi - \mu')v(\eta - \nu')\sigma_{\mu, \nu}(\xi, \eta) \neq 0, \iff \mu = \mu', \nu = \nu'. \]

Hence, for each fixed \( j, k \in \{0, \ldots, N - 1\}^n \), we can write

\[ \sum_{\mu \equiv j} \sum_{\nu \equiv k} u(\xi - \mu)v(\eta - \nu)\sigma_{\mu, \nu}(\xi, \eta) \]

and hence

\[ \left\| \sum_{\mu \equiv j} \sum_{\nu \equiv k} u(\xi - \mu')v(\eta - \nu')\sigma_{\mu, \nu}(\xi, \eta) \right\|_{\mathcal{M}(L^2 \times L^2 \to X)} \leq \left\| \sum_{\mu' \equiv j} \sum_{\nu' \equiv k} u(\xi - \mu') \right\|_{\mathcal{M}(L^2 \times L^2 \to X)} \left\| \sum_{\nu' \equiv k} v(\eta - \nu') \right\|_{\mathcal{M}(L^2 \times L^2 \to X)}. \]
The $\mathcal{M}(L^2 \times L^2 \to X)$ norm on the right hand side does not exceed $M$ by the assumption (1.3). For the two $\mathcal{M}(L^2 \to L^2)$ norms, Plancherel’s theorem and the assumptions on the supports of $u$ and $v$ yield

$$
\left\| \sum_{\mu \equiv j} u(\xi - \mu') \right\|_{\mathcal{M}(L^2 \to L^2)} = \left\| \sum_{\mu' \equiv j} u(\xi - \mu') \right\|_{L^\infty} \leq c_{n,T} \|u\|_{L^\infty},
$$

$$
\left\| \sum_{\nu' \equiv k} v(\eta - \nu') \right\|_{\mathcal{M}(L^2 \to L^2)} = \left\| \sum_{\nu' \equiv k} v(\eta - \nu') \right\|_{L^\infty} \leq c_{n,T} \|v\|_{L^\infty}.
$$

Thus we obtain

$$
\left\| \sum_{\mu \equiv j} \sum_{\nu \equiv k} u(\xi - \mu')v(\eta - \nu')\sigma_{\mu,\nu}(\xi, \eta) \right\|_{\mathcal{M}(L^2 \times L^2 \to X)} \leq c_{n,T} \|u\|_{L^\infty} \|v\|_{L^\infty} M.
$$

Summing over $(j, k)$, we obtain (4.4).

The case of general $\Phi \in C^\infty_0(\mathbb{R}^n \times \mathbb{R}^n)$ can be deduced from the case of product $\Phi$ by the use of the expansion (3.2) (see the last part of Section 3). Lemma 4.2 is proved.

**Lemma 4.3.** Suppose $\sigma_{\mu,\nu}$, $K$, and $M$ satisfy the assumptions (i), (ii), (iii) of Lemma 4.2. Set

$$
\langle \sigma_{\mu,\nu} \rangle = \int_{\mathbb{R}^n \times \mathbb{R}^n} \sigma_{\mu,\nu}(\xi, \eta) \, d\xi d\eta
$$

and define the $L^\infty$ matrix $B$ by $B = (\langle \sigma_{\mu,\nu} \rangle)_{\mu,\nu \in \mathbb{Z}^n}$. Then for every $\Phi \in C^\infty_0(\mathbb{R}^n \times \mathbb{R}^n)$ there exists a constant $c$ depending only on $n, K, \Phi$ such that

$$
\|\sigma_{B,\Phi}\|_{\mathcal{M}(L^2 \times L^2 \to X)} \leq c M
$$

**Proof.** By Lemma 4.1 we have

$$
(4.5) \quad \left\| \sum_{\mu,\nu} \alpha_\mu \alpha'_\nu \sigma_{\mu,\nu}(\xi + \xi_0, \eta + \eta_0) \right\|_{\mathcal{M}(L^2 \times L^2 \to X)} \leq M
$$

for all sequences $(\alpha_\mu)$ and $(\alpha'_\nu)$ consisting of 0 or 1 and for all $\xi_0, \eta_0 \in \mathbb{R}^n$.

Take an arbitrary function $\Phi \in C^\infty_0(\mathbb{R}^n \times \mathbb{R}^n)$ and take a $T \in (0, \infty)$ such that $\text{supp } \Phi \subset TQ \times TQ$. Then the estimate

$$
(4.6) \quad \left\| \sum_{\mu,\nu} \Phi(\xi - \mu, \eta - \nu)\sigma_{\mu,\nu}(\xi + \xi_0, \eta + \eta_0) \right\|_{\mathcal{M}(L^2 \times L^2 \to X)} \leq c_{n,K,\Phi} M 1_{(T+K)Q}(\xi_0) 1_{(T+K)Q}(\eta_0)
$$

holds for all $\xi_0, \eta_0 \in \mathbb{R}^n$. In fact, if $\xi_0 \notin (T + K)Q$ or $\eta_0 \notin (T + K)Q$, then the function on the left hand side of (4.6) is identically equal to 0 and the estimate is obvious. If $\xi_0 \in (T + K)Q$ and $\eta_0 \in (T + K)Q$, then the support of the function $\sigma_{\mu,\nu}(\xi + \xi_0, \eta + \eta_0)$ is included in $(\mu, \nu) + (T + 2K)Q \times (T + 2K)Q$ and we obtain (4.6) from (4.5) and Lemma 4.2.

Now since the norm in $X = (L^2, c^\infty)$ is a Banach-space norm (satisfying the triangular inequality), we take the integral of (4.6) over $\xi_0, \eta_0 \in \mathbb{R}^n$ to obtain the desired inequality. □
Lemma 4.4. For each \( \Phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n) \), there exists a constant \( c \) depending only on \( n \) and \( \Phi \) such that the inequality
\[
\left\| \sum_{\mu, \nu \in \mathbb{Z}^n} \alpha_\mu \alpha'_\nu \Phi(\xi - \mu, \eta - \nu)\sigma(\xi, \eta) \right\|_{\mathcal{M}(L^2 \times L^2 \to X)} \leq c \| \sigma \|_{\mathcal{M}(L^2 \times L^2 \to X)}
\]
holds for all \( \sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) and all sequences \( (\alpha_\mu) \) and \( (\alpha'_\nu) \) consisting of 0 and 1.

Proof. Let \( (\alpha_\mu) \) and \( (\alpha'_\nu) \) be arbitrary sequences consisting of 0 and 1. If \( \Phi \) is of the form \( \Phi(\xi, \eta) = u(\xi)v(\eta) \), then we can write
\[
\sum_{\mu, \nu} \alpha_\mu \alpha'_\nu u(\xi - \mu)v(\eta - \nu)\sigma(\xi, \eta) = \left( \sum_{\mu} \alpha_\mu u(\xi - \mu) \right) \left( \sum_{\nu} \alpha'_\nu v(\xi - \mu) \right) \sigma(\xi, \eta).
\]
Hence, if \( T \) is a positive number satisfying \( \text{supp} \, u, \text{supp} \, v \subset TQ \), then
\[
\left\| \sum_{\mu, \nu} \alpha_\mu \alpha'_\nu u(\xi - \mu)v(\eta - \nu)\sigma(\xi, \eta) \right\|_{\mathcal{M}(L^2 \times L^2 \to X)} \leq \left( \sum_{\mu} \alpha_\mu \right) \left( \sum_{\nu} \alpha'_\nu \right) \left\| \sigma(\xi, \eta) \right\|_{\mathcal{M}(L^2 \times L^2 \to X)} \leq c_{n,T} \left\| u \right\|_{L^\infty} \left\| v \right\|_{L^\infty} \left\| \sigma(\xi, \eta) \right\|_{\mathcal{M}(L^2 \times L^2 \to X)}.
\]
The case of general \( \Phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) can be reduced to the case of the above \( \Phi \) by the use of Fourier expansion as in the last part of Section 3. \( \square \)

Lemma 4.5. Let \( \tilde{\Phi}, \check{\Phi} \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) and suppose \( \Phi \) satisfies the condition \( (A) \). Then there exists a constant \( c \) depending only on \( n, \Phi \), and \( \check{\Phi} \) such that
\[
\| \sigma \check{\Phi} \|_{\mathcal{M}(L^2 \times L^2 \to X)} \leq c \| \sigma \Phi \|_{\mathcal{M}(L^2 \times L^2 \to X)}
\]
for all \( L^\infty \) matrices \( A = (a_{\mu, \nu}) \).

Proof. Let \( A = (a_{\mu, \nu})_{\mu, \nu \in \mathbb{Z}^n} \) be an arbitrary \( L^\infty \) matrix. Take a function \( \Theta \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n) \). By Lemma 4.4 we have
\[
\left\| \sum_{\mu, \nu} \alpha_\mu \alpha'_\nu \Theta(\xi - \mu, \eta - \nu)\sigma_{A, \Phi}(\xi, \eta) \right\|_{\mathcal{M}(L^2 \times L^2 \to X)} \leq c_{n, \Theta} \| \sigma_{A, \Phi} \|_{\mathcal{M}(L^2 \times L^2 \to X)}
\]
for all sequences \( (\alpha_\mu) \) and \( (\alpha'_\nu) \) consisting of 0 and 1. Hence Lemma 4.3 implies
\[
(4.7) \quad \| \sigma \check{\Phi} \|_{\mathcal{M}(L^2 \times L^2 \to X)} \leq c_{n, \Theta} \| \sigma_{A, \Phi} \|_{\mathcal{M}(L^2 \times L^2 \to X)},
\]
where \( \tilde{A} = (\bar{a}_{\mu, \nu}) \) with
\[
\bar{a}_{\mu, \nu} = \int_{\mathbb{R}^n \times \mathbb{R}^n} \Theta(\xi - \mu, \eta - \nu)\sigma_{A, \Phi}(\xi, \eta) \, d\xi d\eta
\]
and
\[
\check{A} = \sum_{\mu, \nu} a_{\mu, \nu} \int_{\mathbb{R}^n \times \mathbb{R}^n} \Theta(\xi - \mu, \eta - \nu) \check{\Phi}(\xi - \mu, \eta - \nu) \, d\xi d\eta.
\]
If we define
\[
R(\alpha, \beta) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \Theta(\xi, \eta)\Phi(\xi - \alpha, \eta - \beta) \, d\xi d\eta, \quad \alpha, \beta \in \mathbb{Z}^n,
\]
then we have
\[ \bar{a}_{\mu, \nu} = \sum_{\mu', \nu'} a_{\mu', \nu'} R(\mu' - \mu, \nu' - \nu). \]

Since \( \Phi \) satisfies the condition (A), we can choose the function \( \Theta \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) so that we have
\[ R(\alpha, \beta) = \begin{cases} 1 & \text{if } (\alpha, \beta) = (0, 0) \\ 0 & \text{if } (\alpha, \beta) \neq (0, 0). \end{cases} \]

With this choice of \( \Theta \), we have \( \bar{a}_{\mu, \nu} = a_{\mu, \nu} \) and (4.7) is the desired inequality. \( \square \)

Using Lemma 4.5, we can complete the proof of Theorem 1.3 (2). If \( \Phi \) satisfies the condition (A), then by Lemma 4.5 the inequality
\[ \| \sigma A, \tilde{\Phi} \|_{M(L^2 \times L^2 \to X)} \leq c_n \| \Phi, \tilde{\Phi} \|_{M(L^2 \times L^2 \to X)} \]
holds for any \( \tilde{\Phi} \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n) \). In particular the above inequality holds for the special function \( \tilde{\Phi}(\xi, \eta) = \varphi(\xi) \varphi(\eta) \) that was treated in Subsection 4.1. For this \( \tilde{\Phi} \), we have proved
\[ \| \sigma A, \tilde{\Phi} \|_{M(L^2 \times L^2 \to X)} \geq c^{-1} \| A \|_B, \]
which combined with the above inequality implies the same lower bound for \( \| \sigma A, \Phi \|_{M(L^2 \times L^2 \to X)} \). Thus Theorem 1.3 (2) is proved and the proof of Theorem 1.3 is complete.

5. Remarks on the condition (A)

In this section, we give examples of \( \Phi \in C_0^\infty(\mathbb{R}^d) \) that satisfy or do not satisfy the condition (A).

First, we give an example of \( \Phi \in C_0^\infty(\mathbb{R}^d) \) that satisfies the condition (A). This example is essentially the same as the function considered in [8, Lemma 6.3] in a slightly different situation.

First consider the case \( d = 1 \). Let \( \phi \) be a function on \( \mathbb{R} \) such that
\[ \phi \in C_0^\infty(\mathbb{R}), \quad \text{supp } \phi \subset [-1, 1], \quad \phi(0) \neq 0. \]

This \( \phi \) satisfies the condition (A). In fact, it is easy to see that the three functions \( \phi(x - j), j = -1, 0, 1, \) are linearly independent on the open interval \((-1, 1)\). This means that the linear functionals
\[ C_0^\infty((-1, 1)) \ni \theta \mapsto \int_{-1}^{1} \theta(x) \phi(x - j) \, dx \in \mathbb{C}, \quad j = -1, 0, 1, \]
are linearly independent and hence, by an elementary fact of linear algebra, there exists a function \( \theta \in C_0^\infty((-1, 1)) \) such that
\[ \int_{-1}^{1} \theta(x) \phi(x - j) \, dx = \begin{cases} 1 & \text{for } j = 0 \\ 0 & \text{for } j = -1, 1. \end{cases} \]

But the integral on the left hand side is also equal to 0 for all \( j \in \mathbb{Z} \setminus \{-1, 0, 1\} \) since \( \phi(-x - j) = 0 \) on \((-1, 1)\) for those \( j \). Thus \( \theta \) has the property required for the condition (A).

For \( d \geq 2 \), it is easy to see that the function \( \Phi \) defined by
\[ \Phi(x) = \phi(x_1) \cdots \phi(x_d), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \]
with the function \( \phi \) of (5.1) satisfies the condition (A).

Next we give an example of nonzero function that does not satisfy the condition (A).
We first consider the case $d = 1$. Let $\phi$ be the function of (5.1) and set

$$\widetilde{\phi}(x) = \phi(x) + \phi(x - 1).$$

Then

$$\sum_{k \in \mathbb{Z}, k \text{ even}} \widetilde{\phi}(x - k) = \sum_{k \in \mathbb{Z}, k \text{ odd}} \widetilde{\phi}(x - k)$$

and hence

$$\sum_{k \in \mathbb{Z}} (-1)^k \int_{\mathbb{R}} \theta(x) \widetilde{\phi}(x - k) \, dx = 0$$

for all $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$. Hence $\widetilde{\phi}$ does not satisfy the condition (A).

For $d \geq 2$, the function $\widetilde{\Phi}$ defined by

$$\widetilde{\Phi}(x) = \widetilde{\phi}(x_1) \cdots \widetilde{\phi}(x_d), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

satisfies

$$\sum_{\alpha \in \mathbb{Z}^d} (-1)^{\alpha_1 + \cdots + \alpha_d} \widetilde{\Phi}(x - \alpha) = 0$$

and hence $\widetilde{\Phi}$ does not satisfy the condition (A).

**References**

[1] Á. Bényi and R. Torres, Almost orthogonality and a class of bounded bilinear pseudodifferential operators, Math. Res. Lett., 11, (2004), 1–11.

[2] A. Boulkhemair, $L^2$ estimates for pseudodifferential operators, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 22, (1995), 155–183.

[3] E. Buriáková, L. Grafakos, D. He, and P. Honzík, The lattice bump multiplier problem, preprint.

[4] R.R. Coifman and Y. Meyer, Au delà des opérateurs pseudo-différentiels, Astérisque, 57, (1978), 1–185.

[5] J.J.F. Fournier and J. Stewart, Amalgams of $L^p$ and $\ell^q$, Bull. Amer. Math. Soc. (N.S.), 13, (1985), 1–21.

[6] D. Goldberg, A local version of real Hardy spaces, Duke Math. J., 46, (1979), 27–42.

[7] L. Grafakos, D. He, and L. Slavíková, $L^2 \times L^2 \to L^1$ boundedness criteria, Math. Ann., 376 (2020), 431–455.

[8] L. Grafakos and N. J. Kalton, The Marcinkiewicz multiplier condition for bilinear operators, Studia. Math., 146, (2001), 115–156.

[9] L. Grafakos and R. Torres, Multilinear Calderón-Zygmund theory, Adv. Math., 165, (2002), 124–164.

[10] F. Holland, Harmonic analysis on amalgams of $L^p$ and $\ell^q$, J. London Math. Soc. (2), 10, (1975), 295–305.

[11] T. Kato, A. Miyachi, and N. Tomita, Boundedness of bilinear pseudo-differential operators of $S_{0,0}$-type on $L^2 \times L^2$, to appear in J. Pseudo-Differential Operators and Appl., available at [arXiv:1901.07237](https://arxiv.org/abs/1901.07237).

[12] T. Kato, A. Miyachi, and N. Tomita, Boundedness of multilinear pseudo-differential operators of $S_{0,0}$-type on $L^2$-based amalgam spaces, to appear in J. Math. Soc. Japan, available at [arXiv:1908.11641](https://arxiv.org/abs/1908.11641).

[13] C. Kenig and E. M. Stein, Multilinear estimates and fractional integration, Math. Res. Lett. 6 (1999), 1–15.

[14] A. Miyachi and N. Tomita, Calderón-Vaillancourt-type theorem for bilinear operators, Indiana Univ. Math. J., 62, (2013), 1165–1201.

[15] L. Slavíková, Bilinear Fourier multipliers and the rate of decay of their derivatives, J. Approx. Theory, 261 (2021), 105485.
