SOLITONS IN A 3d INTEGRABLE MODEL

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Abstract

Equations of motion for a classical 3d discrete model, whose auxiliary system is a linear system, are investigated. The Lagrangian form of the equations of motion is derived. The Lagrangian variables are a triplet of “tau functions”. The equations of motion for the Triplet of Tau functions are Three Trilinear equations. Simple solitons for the trilinear equations are given. Both the dispersion relation and the phase shift reflect the triplet structure of equations.

1 Introduction

Recently there was proved the exact integrability of one discrete 3d model, classical as well as quantum \[1, 2\]. This model was formulated originally in terms of the pure time evolution as the map of the dynamical variables from time \(t\) to time \(t+1\). Namely, for \(t\) fixed the system of the dynamical variables are the system of

\[
\begin{bmatrix}
  u_{\alpha,a,b}, w_{\alpha,a,b}
\end{bmatrix}, \quad \alpha = 1, 2, 3, \quad a, b \in Z_M,
\]

(1)

where \(M\) is the finite spatial size of the system, and the Poisson brackets in the classical case are

\[
\{ u_{\alpha,a,b}, w_{\alpha',a',b'} \} = \delta_{\alpha,\alpha'} \delta_{a,a'} \delta_{b,b'} u_{\alpha,a,b} w_{\alpha,a,b}.
\]

(2)

The evolution was formulated as the explicit form of functions \(f, g,\)

\[
[u, w] = [u(t), w(t)] \mapsto [u(t+1), w(t+1)] = [f(u, w), g(u, w)],
\]

(3)

such that the Poisson brackets are conserved by this map. The existence of the complete set of involutive integrals of motion was proved in \[1, 3\].

The map \(t \mapsto t+1\) may be considered as a sort of the Hamiltonian equations of motion for this classical discrete model. In this paper we will derive their Lagrangian counterpart. Following \[3\], one may guess à priori that the Lagrangian variable is Hirota’s tau function and it might obey Hirota’s discrete bilinear equations \[4\]. But we will show that this is not true in general, although equations we will derive resemble Hirota’s ones.

In this note we recall the reader the Hamiltonian form of the equations of motion first, then introduce the analogous of the tau function and derive the Lagrangian form of the equations of motion, and finally give the multi-soliton solution of them.

1
2 Hamiltonian equations of motion

In this paper we investigate the equations of motion, so we need no the evolution operator and may choose more appropriate coordinate system. Actually, the coordinates we will use here are the light cone frame with respect to previous $t, a, b$. Let $e_1, e_2, e_3$ be three translations making

$$e_1 : (t, a, b) \mapsto (t + 1, a, b),$$
$$e_2 : (t, a, b) \mapsto (t + 1, a + 1, b),$$
$$e_3 : (t, a, b) \mapsto (t + 1, a, b + 1).$$

$e_1, e_2, e_3$ are the elementary orthonormal shifts of the three dimensional cubic lattice. So for any site of the cubic lattice $p$ given, the following eight sites form the elementary cube of the lattice:

$$\begin{pmatrix}
    p, & p + e_2 + e_3, & p + e_1 + e_3, & p + e_1 + e_2, \\
    p + e_1 + e_2 + e_3, & p + e_1, & p + e_2, & p + e_3.
\end{pmatrix}$$

Equations of motion for $u_{\alpha, p}, w_{\alpha, p}$ may be extracted from the form of $f, g$ in the definition of the evolution in \[1, 2\], and in our coordinates look like (i)

$$\begin{align*}
    w_{1, p+e_1} &= \frac{w_{1, p} w_{2, p} + u_{3, p} w_{2, p} + \kappa_3 u_{3, p} w_{3, p}}{w_{3, p}}, \\
    u_{1, p+e_1} &= \frac{\kappa_2 u_{1, p} u_{2, p} w_{2, p}}{\kappa_1 u_{1, p} w_{2, p} + \kappa_3 u_{2, p} w_{3, p} + \kappa_1 \kappa_3 u_{1, p} w_{3, p}},
\end{align*}$$

(ii)

$$\begin{align*}
    w_{2, p+e_2} &= \frac{w_{1, p} w_{2, p} w_{3, p}}{w_{1, p} w_{2, p} + u_{3, p} w_{2, p} + \kappa_3 u_{3, p} w_{3, p}}, \\
    u_{2, p+e_2} &= \frac{u_{1, p} u_{2, p} u_{3, p}}{u_{2, p} u_{3, p} + u_{2, p} w_{1, p} + \kappa_1 u_{1, p} w_{1, p}},
\end{align*}$$

(iii)

$$\begin{align*}
    w_{3, p+e_3} &= \frac{\kappa_2 u_{2, p} w_{2, p} w_{3, p}}{\kappa_1 u_{1, p} w_{2, p} + \kappa_3 u_{2, p} w_{3, p} + \kappa_1 \kappa_3 u_{1, p} w_{3, p}}, \\
    u_{3, p+e_3} &= \frac{u_{2, p} u_{3, p} + u_{2, p} w_{1, p} + \kappa_1 u_{1, p} w_{1, p}}{u_{1, p}}.
\end{align*}$$

Describe in a couple of words the idea of the derivation of equations (6-8). These relations may be obtained as the zero curvature condition of the following system. Let $\varphi_p$ be an auxiliary variable assigned to the sites of the cubic lattice. Consider
three orthogonal plaquettes of the cube endowed by the following relations

\begin{align*}
0 &= f_{1.p} \overset{\text{def}}{=} \varphi_p w_{1.p} - \varphi_{p+e_2+e_3} u_{1.p} + \varphi_{p+e_2} \kappa_1 u_{1.p} w_{1.p} , \\
0 &= f_{2.p} \overset{\text{def}}{=} \varphi_p + \varphi_{p+e_1+e_2} \kappa_2 u_{2.p} w_{2.p} + \varphi_{p+e_1} w_{2.p} - \varphi_{p+e_3} u_{2.p} , \\
0 &= f_{3.p} \overset{\text{def}}{=} - \varphi_p u_{3.p} + \varphi_{p+e_1+e_3} w_{3.p} + \varphi_{p+e_2} \kappa_3 u_{3.p} w_{3.p} .
\end{align*}

Easy to see that the number of linear equations is three times greater then the number of \( \varphi_p \). This means that the coefficients of the system of linear equations must obey extra conditions. As an example consider a cube. Six plaquettes give six linear relations for eight corner \( \varphi_p \):

\begin{equation}
f_{\alpha.p} = f_{\alpha.p+e_\alpha} = 0 , \quad \alpha = 1, 2, 3 .
\end{equation}

Demand that between these six relations there are only four linearly independent, so that the cube relations fix only four of eight corner linear variables \( \varphi_p \) (e.g. we just may express \( \varphi_{p+e_1+e_2} , \varphi_{p+e_2+e_3} , \varphi_{p+e_1+e_3} \) and \( \varphi_{p+e_1+e_2+e_3} \) via independent \( \varphi_p , \varphi_{p+e_1} , \varphi_{p+e_2} \) and \( \varphi_{p+e_3} \) and nothing more). This demand gives equations of motion immediately.

## 3 Lagrangian equations of motion

Three simple relations between the equations of motion are to be mentioned:

\begin{align*}
\begin{cases}
w_{1.p} w_{2.p} &= w_{1.p+e_1} w_{2.p+e_2} , \\
u_{2.p} u_{3.p} &= u_{2.p+e_2} u_{3.p+e_3} , \\
u_{1.p}/w_{3.p} &= u_{1.p+e_1}/w_{3.p+e_3} .
\end{cases}
\end{align*}

Following [3], a parametrization of \( u, w \) in terms of tau functions must turn relations (11) into tautologies. Thus without lost of generality (we deal with the infinite system, avoiding hence all the boundary problems)

\begin{align*}
\begin{cases}
w_{1.p} = \frac{\tau_{3.p+e_2}}{\tau_{3.p}} , & u_{1.p} = \frac{\tau_{2.p}}{\tau_{2.p+e_3}} , \\
w_{2.p} = \frac{\tau_{3.p}}{\tau_{3.p+e_1}} , & u_{2.p} = \frac{\tau_{1.p}}{\tau_{1.p+e_3}} , \\
w_{3.p} = \frac{\tau_{2.p}}{\tau_{2.p+e_1}} , & u_{3.p} = \frac{\tau_{1.p+e_2}}{\tau_{1.p}} .
\end{cases}
\end{align*}

As usual, \( \tau_{\alpha.p} \) may contain a quadratic and linear pre-exponent,

\begin{equation}
\tau_{\alpha.p} = e^{\frac{1}{2} \left( p . q_{\alpha.p} \right)} + \left( q_{\alpha.p} \right) \cdot \tau_{\alpha.p} ,
\end{equation}

\( 3 \)
where \( q_\alpha \) and diagonals of \( Q_\alpha \) are arbitrary, but all non-diagonal elements of \( Q_\alpha \) must coincide (in the natural basis \( e_\alpha \)).

Substituting now the parametrization \((12)\) to \((18)\) and taking \((13)\) into account, we obtain Three Trilinear relations for Three \( \tau \) functions:

\[
\begin{align*}
\tau_1 \tau_2, p + e_2, e_3 \tau_3, p & = \tau_1, p \tau_2, p + e_3 \tau_3, p + e_2 \\
& + s_2 \tau_1, p + e_2 \tau_2, p + e_3 \tau_3, p \\
& + s_3^{-1} \tau_1, p + e_3 \tau_2, p \tau_3, p + e_2 , \\
\tau_2 \tau_1, p \tau_2, p + e_1 + e_3 \tau_3, p & = \tau_1, p + e_3 \tau_2, p \tau_3, p + e_1 \\
& + s_3 \tau_1, p \tau_2, p + e_3 \tau_3, p + e_1 \\
& + s_1^{-1} \tau_1, p + e_3 \tau_2, p + e_1 \tau_3, p , \\
\tau_3 \tau_1, p \tau_2, p \tau_3, p + e_1 + e_2 & = \tau_1, p + e_2 \tau_2, p + e_1 \tau_3, p \\
& + s_1 \tau_1, p + e_2 \tau_2, p \tau_3, p + e_1 \\
& + s_2^{-1} \tau_1, p \tau_2, p + e_1 \tau_3, p + e_2 .
\end{align*}
\]

Here we gather all \( \kappa \)-s and all the pre-exponents into arbitrary \( s_\alpha, r_\alpha, \alpha = 1, 2, 3 \). One may change them in any appropriate way.

Because of the possibility to introduce the Poisson brackets \((2)\) in the system of \( u_\alpha, p, w_\alpha, p \), the original equations of motion \((13)\) may be regarded as a kind of the Hamiltonian equations of motion. Contrary to that, the tautological parametrization of \((11)\) resembles the introduction of a velocity instead of a momentum, so that the number of variables decreases twice, so the trilinear relations \((14-16)\) are nothing but the Lagrangian form of the equations of motion. \((14\text{--}17)\) may be reduced to Hirota’s bilinear discrete equation in the limit

\[
\kappa_1 << \kappa_2 = \kappa_3 << 1 .
\]

### 4 Solitons

Now describe the simple solitons of the trilinear relations \((14\text{--}16)\).

Let \((\alpha, \beta, \gamma)\) be any cyclic permutation of \((1, 2, 3)\). Fix \( r_\alpha \) via

\[
r_\alpha = 1 + s_\beta + \frac{1}{s_\gamma} .
\]

Then let the exponent

\[
W = e^{i (k, p)} = e^{i k_1 p_1 + i k_2 p_2 + i k_3 p_3}
\]

for \( s_\alpha \) given, is defined by

\[
e^{i k_\alpha} = \frac{\lambda_\alpha ((\lambda_\alpha - \lambda_\gamma) + s_\alpha (\lambda_\alpha - \lambda_\beta))}{\lambda_\beta (\lambda_\alpha - \lambda_\gamma) + s_\alpha \lambda_\gamma (\lambda_\alpha - \lambda_\beta)} .
\]
This parametrization of the dispersion relation we will exhibit as \( W = W(\lambda) \). For the dispersion curve \( W = W(\lambda) \) given, let for the shortness
\[
W^{(k)}_\alpha = \lambda^{(k)}_\alpha W(\lambda^{(k)}) \tag{21}
\]
The phase shift \( D \) is defined by
\[
D(\lambda, \lambda') = \frac{d(\lambda, \lambda') \cdot d(\lambda^{-1}, \lambda'^{-1})}{d(\lambda^{-1}, \lambda') \cdot d(\lambda, \lambda'^{-1})} \tag{22}
\]
where
\[
d(\lambda, \lambda') = \det \begin{pmatrix}
1 & 1 & 1 \\
\lambda_1 & \lambda_2 & \lambda_3 \\
\lambda'_1 & \lambda'_2 & \lambda'_3
\end{pmatrix}, \tag{23}
\]
and \( \lambda^{-1} \) stands for \( \{\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}\} \). For the shortness let
\[
D^{(k,l)} = D(\lambda^{(k)}, \lambda^{(l)}) \tag{24}
\]
One soliton solution is \( \tau^{(1)}_{\alpha, p} = 1 - W_\alpha \). Two solitons are
\[
\tau^{(2)}_{\alpha, p} = 1 - W^{(1)}_\alpha - W^{(2)}_\alpha + D^{(1,2)} W^{(1)}_\alpha W^{(2)}_\alpha \tag{25}
\]
Three solitons are
\[
\tau^{(3)}_{\alpha, p} = 1 - W^{(1)}_\alpha - W^{(2)}_\alpha - W^{(3)}_\alpha \\
+ D^{(1,2)} W^{(1)}_\alpha W^{(2)}_\alpha + D^{(1,3)} W^{(1)}_\alpha W^{(3)}_\alpha + D^{(2,3)} W^{(2)}_\alpha W^{(3)}_\alpha \\
- D^{(1,2)} D^{(1,3)} D^{(2,3)} W^{(1)}_\alpha W^{(2)}_\alpha W^{(3)}_\alpha \tag{26}
\]
And so on, common expression is usual. Remarkable is that the dispersion relation for the exponents \( e^{ik\alpha} \) may be parametrized by their origin \( \lambda_\alpha \), \([27]\), and parameters \( s_\alpha \) are defined by an arbitrary linear pre-exponent.

### 5 Discussion

System \([14-16]\) contains a set of bilinear relations as its necessary conditions. Looking for a solution of trilinear relations in a form of holomorphic functions, we get the following set of necessary relations
\[
\Lambda_{\alpha, p+e_\alpha} \tau_{\alpha, p} = \tau_{\beta, p+e_\alpha} \tau_{\gamma, p} + s_\alpha \tau_{\beta, p} \tau_{\gamma, p+e_\alpha} \tag{27}
\]
where as before \( (\alpha, \beta, \gamma) \) are any cyclic permutation of \( (1, 2, 3) \), and auxiliary functions \( \Lambda_{\alpha, p} \) are supposed to be holomorphic. Thus \([14-13]\) have a host of solutions similar to that of Hirota’s equation. Namely, identifying \([27]\) with a set of Fay’s or Plücker’s relations, we may obtain elliptic or determinant solution of \([14-16]\).
Probably, the exception is a Bethe – ansatz – type solution of (27), because in this case (14-16) do not follow from (27) tautologically. Because of all these do not contain anything so surprising as the solitons, we do not give any explicit formula here. Mention only two aspects.

First one is the symmetry group of (14-16). For Hirota’s equation all the symmetries with respect to permutations and reflections of the space directions are obvious. In the case of (14-16) the Cube group is less trivial because of it acts on the parameters sα, rα. As an example give the realisation of the complete reflection P. For a solution τα,p of the trilinear equations with some sα, rα given let

\[
\tau_{α,p} = τ_{α,p} - e_α, \quad \overline{τ}_α = \frac{r_β}{r_γ} \frac{s_α}{s_β s_γ}, \quad \overline{r}_α = \frac{s_β}{s_γ} \frac{r_β r_γ}{r_α}.
\]

(28)

Then the set of “overlined” objects obey the same set of the trilinear relations (14-16) but with reflected e_α → −e_α for all α.

Mention also the possibility to construct the complete solution of original finite evolution model in terms of the elliptic functions on a finite genus curve. In such case one may consider a non-homogeneous and non-equidistant version of the trilinear relations with a boundary conditions taken into account. The number of the parameters (moduli, initial divisor and homogeneousless data) will coincide with the number of variables of the initial state. This will be the subject of a separate paper.

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