Mixed problems for degenerate abstract parabolic equations and applications

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ABSTRACT

Degenerate abstract parabolic equations with variable coefficients are studied. Here the boundary conditions are nonlocal. The maximal regularity properties of solutions for elliptic and parabolic problems and Strichartz type estimates in mixed $L_p$ spaces are obtained. Moreover, the existence and uniqueness of optimal regular solution of mixed problem for nonlinear parabolic equation is established. Note that, these problems arise in fluid mechanics and environmental engineering.

1. Introduction and notations

In this work, the boundary value problems (BVPs) for parameter dependent degenerate differential-operator equations (DOEs) are considered. Namely, linear equations and boundary conditions contain small parameters and are degenerated in some part of boundary. These problems have numerous applications in PDE, pseudo DE, mechanics and environmental engineering. The BVP for DOEs have been studied extensively by many researchers (see e.g. [1-11] and the references therein). The maximal regularity properties for DOEs in Banach space valued function class are investigated e.g. in [4-11]. Nonlinear DOEs studied e.g. in [7,10].

The main objective of the present paper is to discusses the initial and nonlocal BVP for the following nonlinear degenerate parabolic equation

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(x) \frac{\partial^{[2]}u}{\partial x_k^2} + B\left(t, x, u, D^{[1]}u\right) u = F\left(t, x, u, D^{[1]}u\right), \quad (1.1)$$
where $a_k$ are complex valued functions, $B$ and $F$ are nonlinear operators in a Banach space $E$ and

$$D^{[1]} u = \left( \frac{\partial^{[1]} u}{\partial x_1}, \frac{\partial^{[1]} u}{\partial x_2}, ..., \frac{\partial^{[1]} u}{\partial x_n} \right), \quad x = (x_1, x_2, ..., x_n) \in G = \prod_{k=1}^{n} (0, b_k),$$

$$D^{[i]} u = u^{(i)}_k = \left( x^{\alpha_k}_k \frac{\partial}{\partial x_k} \right)^i u(x), \quad 0 \leq \alpha_k < 1.$$

First all of, we consider the nonlocal BVP for the degenerate elliptic DOE

$$\sum_{k=1}^{n} a_k(x) \frac{\partial^{[2]} u}{\partial x_k^2} + A(x) u + \lambda u + \sum_{k=1}^{n} A_k(x) \frac{\partial^{[1]} u}{\partial x_k} = f(x), \quad (1.2)$$

where $a_k$ are complex-valued functions, $\lambda$ is a complex parameters, $A(x)$ and $A_k(x)$ are linear operators.

We prove that for $f \in L_p(G; E)$, $|\arg \lambda| \leq \varphi$, $0 < \varphi \leq \pi$ and sufficiently large $|\lambda|$, problem (1.2) has a unique solution $u \in W^{[2]}_{p,\alpha}(G; E(A), E)$ and the following coercive uniform estimate holds

$$\sum_{k=1}^{n} \left| \lambda \right|^{1-\frac{i}{2}} \left\| \frac{\partial^{[i]} u}{\partial x_k^i} \right\|_{L_p(G; E)} + \left\| Au \right\|_{L_p(G; E)} \leq C \left\| f \right\|_{L_p(G; E)}.$$

Then the above result is used to prove the well-posedness of initial BVP (IBVP) and the uniform Strichartz type estimate for the solution the degenerate abstract parabolic equation with parameters

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(x) \frac{\partial^{[2]} u}{\partial x_k^2} + A(x) u = f(x, t), \quad t \in (0, T), \quad x \in G. \quad (1.3)$$

Finally, via maximal regularity properties of (1.3) and contraction mapping argument, the existence and uniqueness of solution of the problem (1.1) is derived.

Note that, the equation and boundary conditions are degenerate d with the different rate at different boundary edges, in general.

In application, the system of degenerate nonlinear parabolic equations is presented. Particularly, we consider the system that serves as a model of systems used to describe photochemical generation and atmospheric dispersion of ozone and other pollutants. The model of the process is given by initial and BVP for the atmospheric reaction-advection-diffusion system having the form

$$\frac{\partial u_i}{\partial t} = \sum_{k=1}^{3} \left[ a_{ki}(x) \frac{\partial^{[2]} u_i}{\partial x_k^2} + b_{ki}(x) \frac{\partial^{[1]} u_i}{\partial x_k} (u_i \omega_k) \right] + \sum_{k=1}^{3} d_k u_k + f_i(u) + g_i, \quad (0.4)$$

where

$$x \in G_3 = \{ x = (x_1, x_2, x_3), \quad 0 < x_k < b_k, \} ,$$
\[ u_i = u_i(x,t), \ i, \ k = 1, 2, 3, \ u = u(x,t) = (u_1, u_2, u_3), \ t \in (0, T) \]

and the state variables \( u_i \) represent concentration densities of the chemical species involved in the photochemical reaction. The relevant chemistry of the chemical species involved in the photochemical reaction and appears in the nonlinear functions \( f_i(u) \), with the terms \( g_i \), representing elevated point sources, \( a_{ki}(x), b_{ki}(x) \) are real-valued functions. The advection terms \( \omega = \omega(x) = (\omega_1(x), \omega_2(x), \omega_3(x)) \), describe transport from the velocity vector field of atmospheric currents or wind. In this direction the work [12] and references there can be mentioned.

Let \( \gamma = \gamma(x) \) be a positive measurable function on \( \Omega \subset \mathbb{R}^n \) and \( E \) be a Banach space. Let \( L_{p,\gamma}(\Omega; E) \) denote the space of strongly measurable \( E \)-valued functions defined on \( \Omega \) with the norm

\[
\|f\|_{L_{p,\gamma}(\Omega; E)} = \left( \int\int \|f(x)\|_E^p \gamma(x) \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.
\]

For \( \gamma(x) \equiv 1 \) we will denote these spaces by \( L_p(\Omega; E) \).

The Banach space \( E \) is called a \( UMD \)-space if the Hilbert operator

\[
(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} \, dy
\]

is bounded in \( L_p(R,E) \), \( p \in (1, \infty) \) (see. e.g. [13]). \( UMD \) spaces include e.g. \( L_p, l_p \) spaces and Lorentz spaces \( L_{pq}, p, q \in (1, \infty) \).

Let \( \mathbb{C} \) be the set of the complex numbers and

\[
S_\varphi = \{ \lambda : \lambda \in \mathbb{C}, \ |\arg \lambda| \leq \varphi \} \cup \{0\}, \ 0 \leq \varphi < \pi.
\]

Let \( E_1 \) and \( E_2 \) be two Banach spaces. \( L(E_1, E_2) \) denotes the space of bounded linear operators from \( E_1 \) into \( E_2 \). For \( E_1 = E_2 = E \) the space \( L(E_1, E_2) \) will be denoted by \( L(E) \). A linear operator \( A \) is said to be \( \varphi \)-positive in a Banach space \( E \) with bound \( M > 0 \) if \( D(A) \) is dense on \( E \) and

\[
\left\| (A + \lambda I)^{-1} \right\|_{L(E)} \leq M (1 + |\lambda|)^{-1} \quad \text{for any} \ \lambda \in S_\varphi, \ 0 \leq \varphi < \pi, \ \text{where} \ I \ \text{is the identity operator in} \ E.
\]

Let \( E_0 \) and \( E \) be two Banach spaces and \( E_0 \) is continuously and densely embeds into \( E \). Let us consider the Sobolev-Lions type space \( W_{p,\gamma}^{m_i}(a, b; E_0, E) \), consisting of all functions \( u \in L_{p,\gamma}(a, b; E_0) \) that have generalized derivatives \( u^{(m)} \in L_{p,\gamma}(a, b; E) \) with the norm

\[
\|u\|_{W_{p,\gamma}^{m_i}(a, b; E_0, E)} = \|u\|_{L_{p,\gamma}(a, b; E_0)} + \|u^{(m)}\|_{L_{p,\gamma}(a, b; E) < \infty}.
\]

Let \( \gamma = \gamma(x) \) be a positive measurable function on \( (0, 1) \) and

\[
W_{p,\gamma}^{[m]} = W_{p,\gamma}^{[m]}(0, 1; E_0, E) = \{ u : u \in L_p(0, 1; E_0) \}.
\]
\[ u^{[m]} \in L^p(0, 1; E), \|u\|_{W^{[m]}_{p, \gamma}} = \|u\|_{L^p(0, 1; E_0)} + \left\| u^{[m]} \right\|_{L^p(0, 1; E)} < \infty \].

Let
\[ \alpha_k(x) = x^{\alpha_k}, \alpha = (\alpha_1, \alpha_2, ..., \alpha_n). \]

Consider the BVP for the following degenerate partial DOE with parameters

\[ \sum_{k=1}^{n} a_k(x_k) \frac{\partial^{[2]} u}{\partial x_k^2} + A(x) u + \lambda u + \sum_{k=1}^{n} A_k(x) \frac{\partial^{[1]} u}{\partial x_k} = f(x), \quad (2.1) \]

\[ L_{kj} u = \sum_{i=0}^{m_{kj}} \alpha_{kji} u^{[i]}_{x_k} (G_{k0}) + \beta_{kji} u^{[i]}_{k} (G_{kb}) = 0, \ j = 1, 2. \]

where \( a_k \) are complex-valued functions, \( A(x) \) and \( A_k(x) \) are linear operators, \( u = u(x), \alpha_{kji}, \beta_{kji} \) are complex numbers, \( \lambda \) is a complex parameter, \( m_{kj} \in \{0, 1\} \).

Consider the principal part of (2.1), i.e., consider the problem

\[ \sum_{k=1}^{n} a_k(x_k) \frac{\partial^{[2]} u}{\partial x_k^2} + A(x) u + \lambda u = f(x), \quad (2.2) \]

\[ \sum_{i=0}^{m_{kj}} \alpha_{kji} u^{[i]}_{x_k} (G_{k0}) + \beta_{kji} u^{[i]}_{k} (G_{kb}) = 0, \ j = 1, 2. \]

**Condition 2.1 Assume;**
(1) $E$ is an UMD space the Banach space, $0 \leq \alpha_k < 1 - \frac{1}{p_k}$, $p_k \in (1, \infty)$, 
$\alpha_{km_{k1}} \neq 0$, $\beta_{km_{k2}} \neq 0$;

(2) $A(x)$ is a uniformly $R$-positive operator in $E$, $A(x) A^{-1}(\bar{x}) \in C(G; L(E))$, $x \in G$;

(3) $a_k \in C^{(m)}([0, b_k])$ and $a_k (x_k) < 0$ for $x_k \in [0, b_k]$;

(4) $a_k (G_{j0}) = a_k (G_{j0}), A(G_{j0}) A^{-1} (x_0) = A(G_{j0}) A^{-1} (x_0)$, $k, j = 1, 2, \ldots, n$;

(5) $G_{km} = (-1)^{m_1} \alpha_{k1} \beta_{k2} - (-1)^{m_2} \alpha_{k2} \beta_{k1} \neq 0$.

First, we prove the separability properties of the problem (2.2):

\textbf{Theorem 2.1.} Let the Conditions 2.1 hold. Then, problem (2.2) has a unique solution $u \in W_{p, \alpha}^{[2]} (G; E) \times E$ for $f \in L_p (G; E)$, $|\arg \lambda| \leq \varphi$ with sufficiently large $|\lambda|$ and the following coercive uniform estimate holds

$$
\sum_{k=1}^{n} \sum_{l=0}^{2} |\lambda|^{1-\frac{2}{p}} \left\| \frac{\partial^{[l]} u}{\partial x_k^l} \right\|_{L_p (G; E)} + \|Au\|_{L_p (G; E)} \leq C \|f\|_{L_p (G; E)}.
$$

\textbf{Proof.} Consider the BVP

$$
(L + \lambda) u = a_1 (x_1) D^{[2]}_{x_1} u (x_1) + (A (x_1) + \lambda) u (x_1) = f (x_1),
$$

where $L_{1j} u = 0$, $j = 1, 2$, $x_1 \in (0, b_1)$,

where $L_{1j}$ are boundary conditions of type (2.2) considered on $(0, b_1)$. By virtue of Theorem 1 in [8], problem (2.4) has a unique solution $u \in W_{p, \alpha}^{[2]} (0, b_1; E) \times E$ for $f \in L_p (0, b_1; E)$, $|\arg \lambda| \leq \varphi$ with sufficiently large $|\lambda|$ and the coercive uniform estimate holds

$$
\sum_{j=0}^{2} |\lambda|^{1-\frac{2}{p}} \left\| u^{[j]} \right\|_{L_p (0, b_1; E)} + \|Au\|_{L_p (0, b_1; E)} \leq C \|f\|_{L_p (0, b_1; E)}.
$$

Now, let us consider the following BVP

$$
\sum_{k=1}^{2} a_k (x_k) D^{[2]}_{x_1} u (x_1, x_2) + A (x_1, x_2) u (x_1, x_2) + \lambda u (x_1, x_2) = f (x_1, x_2),
$$

where $L_{k1} u = 0$, $L_{k2} u = 0$, $k = 1, 2$, $x_1, x_2 \in G_2 = (0, b_1) \times (0, b_2)$.

Let $\alpha (2) = (\alpha_1, \alpha_2)$. Since $L_p (0, b_2; L_p (0, b_1); E) = L_p (G_2; E)$, the BVP (2.5) can be expressed as

$$
a_2 D^{[2]}_{x_2} u (x_2) + B (x_2) u (x_2) = f (x_2),
$$

for $x_1 \in (0, b_1)$, where $B$ is a differential operator in $L_{p_1} (0, b_1; E)$ for $x_2 \in (0, b_2)$, generated by problem (2.4). By virtue of [1, Theorem 4.5.2], $L_{p_1} (0, b_1; E) \in UMD$ for $p_1 \in (1, \infty)$. Moreover, in view of [10] the operator $B$ is $R$-positive in
Let \( f \in L^p(G^2; E) \), \( |\arg \lambda| \leq \varphi \) with sufficiently large \(|\lambda|\) and (2.3) holds for \( n = 2 \). By continuing this process we obtain the assertion.

**Theorem 2.2.** Let the Conditions 2.1 hold and \( A_k(x) A^{-\left(\frac{1}{2} - \nu\right)}(x) \in C(G; L(E)) \) for \( 0 < \nu < \frac{1}{2} \). Then, problem (2.1) has a unique solution \( u \in W^{[2]}_{p,\alpha}(G; E) \) for \( f \in L^p(G; E) \), \( |\arg \lambda| \leq \varphi \) with sufficiently large \(|\lambda|\) and the coercive uniform estimate holds

\[
\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{-\frac{i}{2}} \left\| \frac{\partial^{[i]} u}{\partial x_k^i} \right\|_{L^p(G; E)} + \|Au\|_{L^p(G; E)} \leq C \|f\|_{L^p(G; E)}. \tag{2.6}
\]

**Proof.** By second assumption and embedding theorem [6] for all \( h > 0 \) we have the following Ehrlich-Nirenberg-Gagliardo type estimate

\[
\|L_1 u\|_{L^p(G; E)} \leq h^\mu \|u\|_{W^{[2]}_{p,\alpha}(G; E)} + h^{-1-\mu} \|u\|_{L^p(G; E)}. \tag{2.7}
\]

Let \( O \) denote the operator generated by problem (2.2) and

\[
L_1 u = \sum_{k=1}^{n} A_k(x) \frac{\partial^{[1]} u}{\partial x_k}.
\]

By using the estimate (2.7) we obtain that there is a \( \delta \in (0, 1) \) such that

\[
\left\| L_1 (O + \lambda)^{-1} \right\|_{B(X)} < \delta.
\]

Hence, from perturbation theory of linear operators we obtain the assertion.

### 3. Abstract Cauchy problem for degenerate parabolic equation

Consider the IBVP for degenerate parabolic equation with parameter:

\[
\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(x_k) \frac{\partial^{[2]} u}{\partial x_k^2} + A(x) u + du = f(x, t), \quad t \in (0, T), \quad x \in G, \tag{3.1}
\]

\[
\sum_{i=0}^{m_{k_j}} \alpha_{k_j} u^{[i]}_{x_k^i} (G_{k_0}, t) + \beta_{k_j} u^{[i]}_{x_k^i} (G_{k_0}, t) = 0, \quad j = 1, 2,
\]

\[
u(x, 0) = 0, \quad t \in (0, T), \quad x^{(k)} \in G_k, \tag{3.2}
\]

where \( u = u(x, t) \) is a solution, \( \delta_k \), \( \beta_k \) are complex numbers, \( a_k \) are complex-valued functions on \( G \), \( A(x) \) is a linear operator in a Banach space \( E \), domains \( G, G_k, G_{k_0}, G_{k_0}, \sigma_{ik} \) and \( x^{(k)} \) are defined in the section 2.
For $p = (p_0, p)$, $G_T = (0, T) \times G$, $L_{p,T}$ $(G_T; E)$ will denote the space of all $E$-valued weighted $p$-summable functions with mixed norm.

**Theorem 3.1.** Suppose the Condition 2.1 hold for $\varphi > \frac{p}{2}$. Then, for $f \in L_{p,T}$ $(G_T; E)$ and sufficiently large $d > 0$ problem (3.1) – (3.2) has a unique solution belonging to $W_{p,\alpha}^1$ $(G_T; E(A), E)$ and the following coercive uniform estimate holds

$$
\left\| \frac{\partial u}{\partial t} \right\|_{L_p^p(G_T; E)} + \sum_{k=1}^2 \left\| \frac{\partial^{[2]} u}{\partial x_k^2} \right\|_{L_p^p(G_T; E)} + \left\| Au \right\|_{L^p_p(G_T; E)} \leq C \| f \|_{L^p_p(G_T; E)}.
$$

**Proof.** The problem (3.1) can be expressed as the following abstract Cauchy problem

$$
\frac{du}{dt} + (O + d) u(t) = f(t), \quad u(0) = 0. \quad (3.3)
$$

By virtue of [10], $O$ is $R$-positive in $X = L_p^p(G; E)$, i.e $O$ is a generator of an analytic semigroup in $X$. Then by virtue of [11, Theorem 4.2], problem (3.3) has a unique solution $u \in W_{p,\alpha}^1 (0, T; D(O), X)$ for $f \in L_{p_0}^p (0, T; X)$ and sufficiently large $d > 0$. Moreover, the following uniform estimate holds

$$
\left\| \frac{du}{dt} \right\|_{L_{p_0}^{p_0}(0, T; X)} + \| Ou \|_{L_{p_0}^{p_0}(0, T; X)} \leq C \| f \|_{L_{p_0}^{p_0}(0, T; X)}.
$$

Since $L_{p_0}^{p_0}(G_T; X) = L_{p}^{p}(G_T; E)$, by Theorem 2.1 we have

$$
\| (O + d) u \|_{L_{p_0}^{p_0}(0, T; X)} = D(O).
$$

Hence, the assertion follows from the above estimate.

**5. Nonlinear degenerate abstract parabolic problem**

In this section, we consider IBVP for the following nonlinear degenerate parabolic equation

$$
\frac{\partial u}{\partial t} + \sum_{k=1}^n a_k (x_k) \frac{\partial^{[2]} u}{\partial x_k^2} + B \left( \left( t, x, u, D^{[1]} u \right) \right) u = F \left( t, x, u, D^{[1]} u \right), \quad (5.1)
$$

$$
\sum_{i=0}^{m_k j} \alpha_{kji} u^{[i]}_k (G_{k0}, t) + \beta_{kji} u^{[i]}_k (G_{kb}, t) = 0, j = 1, 2,
$$

$$
u(x, 0) = 0, \quad t \in (0, T), \quad x \in G, \quad x^{(k)} \in G_k, \quad (5.2)
$$

where $u = u(x, t)$ is a solution, $\alpha_{kji}$, $\beta_{kji}$ are complex numbers, $a_k$ are complex-valued functions on $[0, b_k]$; domains $G_g, G_{k0}, G_{kb}$ and $\sigma_{jk}, x^{(k)}$ are defined in the section 2 and

$$
D^{[i]}_k u = \frac{\partial^{[i]} u}{\partial x_k^i} = \left( x_{ak} \frac{\partial}{\partial x_k} \right)^i u(x, t), \quad 0 \leq a_k < 1.
$$
Let $G_T = (0, T) \times G$. Moreover, we let

$$G_0 = \prod_{k=1}^{n} (0, b_{0k}), \quad G = \prod_{k=1}^{n} (0, b_k), \quad b_k \in (0, b_{0k}),$$

$T \in (0, T_0)$, $B_{ki} = (W^{2,p} (G_k, E (A), E), L^p (G_k; E))_{\eta_{ki} - p}$,

$$\eta_{ki} = \frac{m_{ki} + \frac{1}{p(1 - \alpha_k)}}{2}, \quad |\alpha_{kj} m_{kj}| + |\beta_{kj} m_{kj}| > 0, \quad B_0 = \prod_{k=1}^{n} \prod_{i=0}^{1} B_{ki}.$$

Let

$$\alpha = \alpha (x) = \prod_{k=1}^{n} x_{\alpha_k}.$$

**Remark 5.0.** Under the substitutions

$$\tau_k = \frac{x^1_{\alpha_k}}{1 - \alpha_k}, \quad 0 < \alpha_k < 1, \quad k = 1, 2, \ldots, n$$

the spaces $L_p (G; E)$ and $W^{2,p}_{p,\alpha} (G; E (A), E)$ are mapped isomorphically onto the weighted spaces $L_{p,\tilde{\alpha}} (\tilde{G}; E)$ and $W^{2,p}_{p,\tilde{\alpha}} (\tilde{G}; E (A), E)$, respectively, where

$$\tilde{G} = \prod_{k=1}^{n} (0, b_k), \quad \tilde{b}_k = b_k^{1 - \alpha_k}, \quad \tilde{\alpha} (\tau) = \alpha (x_1 (\tau_1), x_2 (\tau_2), \ldots, x_n (\tau_n)).$$

**Remark 5.1.** By virtue of [28, § 1.8.] and Remark 5.0, operators $u \to \frac{\partial^{[i]} u}{\partial x_k^{[i]}} |_{x_k=0}$ are continuous from $W^{2,p}_{p,\alpha} (G; E (A), E)$ onto $B_{ki}$ and there are the constants $C_1$ and $C_0$ such that for $w \in W^{2,p}_{p,\alpha} (G; E (A), E)$, $W = \{w_{ki}\}$, $w_{ki} = \frac{\partial^{[i]} w}{\partial x_k^{[i]}}$, $i = 1, 2, \ldots, n$

$$\left\| \left. \frac{\partial^{[i]} w}{\partial x_k^{[i]}} \right| \right\|_{B_{ki,\infty}} \leq C_1 \|w\|_{W^{2,p}_{p,\alpha} (G; E (A), E)},$$

$$\left\| W \right\|_{0,\infty} \leq \sup_{x \in G} \sum_{i} |w_k| \leq C_0 \|w\|_{W^{2,p}_{p,\alpha} (G; E (A), E)}.$$

**Condition 5.1.** Suppose the following hold:

(1) $E$ is an UMD space and $0 \leq \alpha_1, \alpha_2 < 1 - \frac{1}{p}, p \in (1, \infty)$;

(2) $a_k$ are continuous functions on $[0, b_k]$, $a_k (x_k) < 0$, for all $x \in [0, b_k]$, $\alpha_{k1}, \alpha_{k2} \not= 0, \beta_{k1}, \beta_{k2} \not= 0, \quad k = 1, 2, \ldots, n$;

(3) there exist $\Phi_{ki} \in B_{ki}$ such that the operator $B (t, x, \Phi)$ for $\Phi = \{\Phi_{ki}\} \in B_0$ is $R$-positive in $E$ uniformly with respect to $x \in G_0$ and $t \in [0, T_0]$; moreover,

$$B (t, x, \Phi) B^{-1} (t^0, x^0, \Phi) \in C (\tilde{G}; L (E)), \quad t^0 \in (0, T), \quad x^0 \in G;$$
(4) \( A = B \{ p^0, x^0, \Phi \} : T_t \times B_0 \to L (E (A), E) \) is continuous. Moreover, for each positive \( r \) there is a positive constant \( L (r) \) such that
\[
\| B (t, x, U) - B (t, x, U) \|_E \leq L (r) \| U - U \|_{B_0} \| A \|_E
\]
for \( t \in (0, T) \), \( x \in G, U, \bar{U} \in B_0, \bar{U} = \{ \bar{u}_{ki} \}, \bar{u}_{ki} \in B_{ki}, \| U \|_{B_0}, \| \bar{U} \|_{B_0} \leq r, v \in D (A); \)

(5) the function \( F : T_t \times B_0 \to E \) such that \( F (\cdot, U) \) is measurable for each \( U \in B_0 \) and \( F (t, x, \cdot) \) is continuous for a.a. \( t \in (0, T) \), \( x \in G \). Moreover, \( \| F (t, x, U) - F (t, x, \bar{U}) \|_E \leq \Psi_r (x) \| U - U \|_{B_0} \) for a.a. \( t \in (0, T) \), \( x \in G \), \( U, \bar{U} \in B_0 \) and \( \| U \|_{B_0}, \| \bar{U} \|_{B_0} \leq r \); \( f (\cdot) = F (\cdot, 0) \in L_p (G_T ; E) \).

The main result of this section is the following:

Theorem 5.1. Let Condition 5.1 hold. Then there is a \( T \in (0, T_0) \) and a \( b_k \in (0, b_{0k}) \) such that problem (5.1) – (5.2) has a unique solution belonging to \( W_{p, 0}^{1,2} (G_T ; E (A), E) \).

Proof. Consider the following linear problem
\[
\frac{\partial w}{\partial t} + \sum_{k=1}^n a_k (x_k) \frac{\partial^2 [w]}{\partial x_k^2} + du = f (x, t), x \in G, t \in (0, T),
\]

\[
\sum_{i=0}^{m_{kj}} \alpha_{kji} w_{x_k}^{[i]} (G_{k0}, t) + \sum_{i=0}^{m_{kj}} \beta_{kji} w_{x_k}^{[i]} (G_{kb}, t) = 0, j = 1, 2,
\]

\( w (x, 0) = 0, t \in (0, T), x \in G, x^{(k)} \in G_k, d > 0. \)

By Theorem 3.1 and in view of Proposition 4.1 there is a unique solution \( w \in W_{p, 0}^{1,2} (G_T ; E (A), E) \) of the problem (5.3) for \( f \in L_p (G_T ; E) \) and sufficiently large \( d > 0 \) and it satisfies the following coercive estimate
\[
\| w \|_{W_{p, 0}^{1,2} (G_T ; E (A), E)} \leq C_0 \| f \|_{L_p (G_T ; E)},
\]

uniformly with respect to \( b \in (0, b_0) \), i.e., the constant \( C_0 \) does not depends on \( f \in L_p (G_T ; E) \) and \( b \in (0, b_0) \) where
\[
A (x) = B (x, 0), f (x) = F (x, 0), x \in (0, b).
\]

We want to solve the problem (5.1) – (5.2) locally by means of maximal regularity of the linear problem (5.3) via the contraction mapping theorem. For this purpose, let \( w \) be a solution of the linear BVP (5.3). Consider a ball
\[
B_r = \{ v \in Y, v - w \in Y_1, \| v - w \|_Y \leq r \}.
\]

For given \( v \in B_r \), consider the following linearized problem
\[
\frac{\partial u}{\partial t} + \sum_{k=1}^n a_k (x_k) \frac{\partial^2 [u]}{\partial x_k^2} + A (x) = F (x, V) + [B (x, 0) - B (x, V)] v,
\]

\[
\sum_{i=0}^{m_{kj}} \alpha_{kji} w_{x_k}^{[i]} (G_{k0}, t) + \sum_{i=0}^{m_{kj}} \beta_{kji} w_{x_k}^{[i]} (G_{kb}, t) = 0,
\]

(5.4)
where \( V = \{ v_{ki} \} \), \( v_{ki} \in B_{ki} \). Define a map \( Q \) on \( B_r \) by \( Qv = u \), where \( u \) is solution of (5.4). We want to show that \( Q(B_r) \subset B_r \) and that \( Q \) is a contraction operator provided \( T \) and \( b_1 \) are sufficiently small, and \( r \) is chosen properly. In view of separability properties of the problem (5.3) we have

\[
\|Qv - w\|_Y = \|u - w\|_Y \leq C_0 \{ \|F(x, V) - F(x, 0)\|_X + \]

\[
\| [B(0, W) - B(x, V)] v \|_X \}.
\]

By assumption (4) we have

\[
\| [B(0, W) v - B(x, V)] v \|_X \leq \sup_{x \in [0, b]} \left\{ \| [B(0, W) - B(x, W)] v \|_{L(E_0, E)} \right\} \]

\[
+ \| B(x, W) - B(x, V)\|_{L(E_0, E)} \| v \|_Y \right\} \leq \]

\[
[\delta(b) + L(R) \| W - V \|_{\infty, E_0}] \left\{ \| v - w \|_Y + \| w \|_Y \right\} \leq \]

\[
\{ \delta(b) + L(R) [C_1 \| v - w \|_Y + \| v - w \|_Y] - [\| v - w \|_Y + \| w \|_Y] \} \leq \delta(b) + L(R) [C_1 r + r] [r + \| w \|_Y] ,
\]

where \( \delta(b) = \sup_{x \in [0, b]} \{ \| B(0, W) - B(x, W)\|_{B(E_0, E)} \} \).

By assumption (5) we get

\[
\|F(x, V) - F(x, 0)\|_E \leq \delta(b) + \]

\[
\|F(x, V) - F(x, W)\|_E + \|F(x, W) - F(x, 0)\|_E \leq \]

\[
[\delta(b) + \mu_R [\| v - w \|_Y + \| w \|_Y] - C_1 \] [r + \| w \|_Y] \leq \mu_R [C_1 r + \| w \|_Y] ,
\]

where \( R = C_1 r + \| w \|_Y \) is a fixed number. In view of above estimates, by suitable choice of \( \mu_R, L_R \) and for sufficiently small \( T \in (0, T_0) \) and \( b_k \in (0, b_{0k}) \) we have

\[
\|Qv - w\|_Y \leq r,
\]

i.e.

\[
Q(B_r) \subset B_r.
\]

Moreover, in a similar way we obtain

\[
\|Qv - Q\tilde{v}\|_Y \leq C_0 \{ \mu_R C_1 + M_a + L(R) [\| v - w \|_Y + C_1 r] + \]

\[
L(R) C_1 [r + \| w \|_Y] [\| v - \tilde{v} \|_Y] + \delta(b) .
\]
By suitable choice of $\mu_R$, $L_R$ and for sufficiently small $T \in (0, T_0)$ and $b_k \in (0, b_0k)$ we obtain $\|Q\psi - Q\bar{\psi}\|_Y < \eta \|\psi - \bar{\psi}\|_Y$, $\eta < 1$, i.e. $Q$ is a contraction operator. Eventually, the contraction mapping principle implies a unique fixed point of $Q$ in $B_r$ which is the unique strong solution $u \in W_{p,\alpha}^{1,2}(G_T; E(A), E)$.

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