Dynamics of Scalar field in a Brane World

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We study the dynamics of a scalar field in the brane cosmology. We assume that a scalar field is confined in our 4-dimensional world. As for the potential of the scalar field, we discuss three typical models: (1) a power-law potential, (2) an inverse-power-law potential, and (3) an exponential potential. We show that the behavior of the scalar field is very different from a conventional cosmology when the energy density square term is dominated.

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I. INTRODUCTION

A scalar field has played a very important role in cosmology. The problems of the standard Big-Bang theory, such as the horizon, and flatness problems could be resolved by the inflationary scenario [1]. In this scenario, the universe expands quasi-exponentially because of the vacuum energy, whose origin in many models is given by a scalar field (inflaton). Among inflationary scenarios, the chaotic inflation, which has a power-law type potential \( V(\phi) \sim \phi^n : n \) is an integer) is the simplest model. Another example in which, a scalar field plays an important role is a cosmological phase transition and formation of topological defects [2]. The observed dark energy could be also explained by a scalar field whose energy density decreases in time (quintessence) [3,4]. In a typical quintessence model, the potential of the scalar field is assumed to be an inverse-power-law \( V(\phi) \sim \phi^{-\alpha} : \alpha > 0 \). An exponential-type \( V(\phi) \sim \exp[-\lambda \phi] \) is naturally expected in many supergravity theories [5]. Hence those scalar fields are extensively studied by many authors in the conventional Einstein gravity theory. However, recent higher dimensional unification scenario based on a brane world suggests that the gravitational law could be different from the Einstein’s one at the early stage of the universe or at high-energy scale.

In the brane-world scenario, which is based on a superstring or M-theory, our universe is embedded in higher dimensions. Standard-model particles are confined in this four-dimensional hypersurfaces (3-branes), while gravity propagates in a higher-dimensional bulk space [6,7]. Among them, the Randall-Sundrum’s second model is very interesting because it provides a new type of compactification of gravity. Assuming a 3-brane with a positive tension is embedded in 5-dimensional anti de-Sitter bulk spacetime, the conventional four-dimensional gravity theory is recovered in low energy limit, even though the extra dimension is not compact [8]. While, at high energy scale, gravity differs from the conventional Einstein theory [9]. Many authors discussed the geometrical aspects of the brane and its dynamics (For a review, see, [10]), as well as its cosmological implications [11–14]. Based on the Randall-Sundrum’s second model, if we assume that our brane is homogeneous and isotropic (Friedmann-Robertson-Walker universe), the difference from conventional cosmology appears as two new terms in the Friedmann equation, i.e. the quadratic term of energy-momentum and dark radiation [12–14]. As for the dark radiation, its effect is constrained by the nucleosynthesis. It is also diluted away in the inflationary era. Hence, the important change in scalar field dynamics, which we are going to discuss, may be due to the appearance of the term of quadratic energy density.

The purpose of the present paper is to study dynamics of a scalar field confined on the brane, taking into account the effect of the quadratic-energy-density term. We discuss three typical potentials appeared in many cosmological implications; a power-law potential, an inverse-power-law potential, and an exponential potential. The analysis of the quintessential potential will complete our previous discussion [15].

In Sec. II, we present the basic equations assuming the Randall-Sundrum’s second model. Next, in Sec. III, we show cosmological solutions in both scalar-field dominant and radiation dominant eras. These solutions are found analytically by using some approximations.

In Sec. IV, we first study linear perturbations of the solutions, and then in Sec. V, we analyze those global stability. Sec. VI is devoted to summary and discussion.

II. BASIC EQUATIONS

We analyze dynamics of a scalar field in the Randall-Sundrum’s second brane scenario [8], because the model is simple and concrete. It is, however, worthwhile not-
ing that the present result may be also valid in other types of brane world models, in which a quadratic term of energy-momentum tensor generically appears. In the brane world, all matter fields and forces except gravity are confined on the 3-brane in a 5-dimensional spacetime. The extra-dimension is not compactified, but gravity is confined in the brane, resulting in the Newtonian gravity in our world. Since the gravity is confined in the brane, it is described by the intrinsic metric of the 4-dimensional brane spacetime. By use of Israel’s junction condition and assuming $Z_2$ symmetry, the gravitational equations on the 3-brane is given by

\[ (4)G_{\mu\nu} = -(4)\Lambda g_{\mu\nu} + \kappa_4^2 T_{\mu\nu} + \kappa_5^4 \Pi_{\mu\nu} - E_{\mu\nu}, \] (2.1)

where $^{(4)}G_{\mu\nu}$ is the Einstein tensor with respect to the intrinsic metric $g_{\mu\nu}$, $^{(4)}\Lambda$ is the F 4-dimensional cosmological constant, $T_{\mu\nu}$ represents the energy-momentum tensor of matter fields confined on the brane and $\Pi_{\mu\nu}$ is defined by its quadratic form. $E_{\mu\nu}$ is a part of the 5-dimensional Weyl tensor and carries some information about a bulk geometry. $\kappa_4^2 = 8\pi G_4$ and $\kappa_5^4 = 8\pi G_5$ are 4-dimensional and 5-dimensional gravitational constants, respectively. In what follows, we use the 4-dimensional Planck mass $m_4 \equiv \kappa_4^{-1} = (2.4 \times 10^{18}\text{GeV})$ and the 5-dimensional Planck mass $m_5 \equiv \kappa_5^{-2/3}$, which could be much smaller than $m_4$.

Assuming the Friedmann-Robertson-Walker spacetime in our brane world, we find the effective Friedmann equations from Eq.(2.1) as

\[ H^2 + \frac{k}{a^2} = \frac{1}{3}\frac{^{(4)}\Lambda}{m_4^4} + \frac{1}{3m_4^4}\rho + \frac{1}{36m_5^6}\rho^2 + \frac{C}{a^4} \] (2.2)

\[ \dot{H} - \frac{k}{a^2} = -\frac{1}{2m_4^4}(P + \rho) - \frac{1}{12m_5^6}\rho(P + \rho) - \frac{2C}{a^4} \] (2.3)

where $a$ is a scale factor of the Universe, $H = \dot{a}/a$ is its Hubble parameter, $k$ is a curvature constant, $P$ and $\rho$ are the total pressure and total energy density of matter fields, respectively. $C$ is a constant, which term describes "dark" radiation coming from $E_{\mu\nu}$. In what follows, we consider only the flat Friedmann model ($k = 0$) and assume that $^{(4)}\Lambda$ vanishes for simplicity.

As for matter fields on the brane, we consider a scalar field $\phi$ as well as the conventional radiation fluid, i.e. $\rho = \rho_\phi + \rho_r$, where $\rho_\phi$ and $\rho_r$ are the energy densities of scalar field $\phi$, and of radiation fluid. We consider only the early stage of the universe, at which the quadratic term is dominant and then matter fluid can be ignored.

Although a 5-dimensional scalar field living in the bulk may also appear in a brane world scenario, in this paper we only consider a 4-dimensional scalar field confined on the brane. The origin of such a scalar field might be found as a result of condensation of matter fields confined on the brane such as fermions.

Since the energy and momentum of each field on the brane are conserved in the present model, we find the dynamical equation for a scalar field as a conventional one, i.e.

\[ \ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \] (2.4)

where $V$ is a potential of the scalar field. The energy density of the scalar field is

\[ \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi). \] (2.5)

For the energy density of radiation fluid, we have

\[ \dot{\rho}_r + 4H\rho_r = 0, \] (2.6)

which is integrated as $\rho_r \propto a^{-4}$.

As the Universe expands, the energy density decreases. This means that the quadratic term was very important in the early stage of the Universe. Comparing two terms (the conventional energy density term and the quadratic one), we find that the quadratic term dominates when

\[ \rho > \rho_c \equiv 12m_5^6/m_4^4. \] (2.7)

When the quadratic term is dominant, the expansion law of the Universe is modified. For example, the expansion law in radiation dominant era is $a \propto t^{1/4}$ instead of the conventional one $t^{1/2}$. Since we know the behavior of the scalar field in the linear term dominant stage, which is the conventional cosmological model, in this paper we study the behavior of the scalar field only in the quadratic term dominant stage.

The nucleosynthesis gives a constraint on fundamental constants, if the quadratic-energy-density dominant stage exists. The nucleosynthesis must take place in the conventional radiation dominant era to explain the amount of light elements. Hence, assuming that the energy density at $a = a_0$ is dominated by radiation as $\rho_c \sim (\pi^2/30)gT_c^4$, where $g$ is the degree of freedom of particles, the temperature of the universe $T_c$ must be higher than that of nucleosynthesis, i.e. $T_c > T_{NS} \sim 1\text{MeV}$. This constraints implies $m_5 > 1.6 \times 10^4(g/100)^{1/6}(T_{NS}/1\text{MeV})^{2/3} \text{GeV}$.

### III. COSMOLOGICAL SOLUTIONS

There are two interesting limiting cases: one is the scalar-field dominant era and the other is the radiation dominant era. We discuss those solutions separately.

#### A. Solutions in scalar-field dominant era

Assuming the scalar field dominance, we find the Friedmann equation (2.2) as

\[ H = \frac{1}{6m_5^6}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]. \] (3.1)
We consider the following three types of the potential: 
\[ V(\phi) = \mu^{\alpha+4} \phi^{-\alpha} + \mu^4 \exp[-\lambda \frac{\phi}{m_5}] \] and \[ \frac{1}{2} m^2 \dot{\phi}^2, \ \left(\frac{1}{4} \lambda \dot{\phi}^4 \right) \] in order.

**A.1 V(\phi) = \mu^{\alpha+4} \phi^{-\alpha}**

The equation for the scalar field (2.4) is now
\[ \ddot{\phi} + 3H \dot{\phi} - \alpha \mu^{\alpha+4} \phi^{-\alpha-1} = 0. \] (3.2)
Inserting Eq. (3.1) into Eq. (2.2), we find a second order differential equation for \( \phi \). It is easily shown that the asymptotic behaviors of the solutions are classified into three cases: (a) slow rolling (a potential term dominant) solution \( (\alpha < 2) \), (b) a solution in which the potential term balances with the kinetic term \( (\alpha = 2) \), and (c) a kinetic-term dominant solution \( (\alpha > 2) \).

Assuming a slow rolling condition, we find that
\[ H \approx \frac{1}{6m_5^3} \mu^{\alpha+4} \phi^{-\alpha} \text{ and } \dot{\phi} \approx \frac{\alpha}{3H} \mu^{\alpha+4} \phi^{-\alpha-1}, \] (3.3)
which lead to \( \phi \dot{\phi} = 2\alpha m_5^3 \). Then we find the solution
\[ \frac{\dot{\phi}}{m_5} = 2 [\alpha m_5 (t - t_0)]^{1/2}, \] (3.4)
where \( t_0 \) is an integration constant. In order to keep the slow-rolling condition, we obtain \( \alpha < 2 \) because \( \phi \sim t^{-3/2} \ll V' \sim t^{-1} \) and \( \dot{\phi}^2 \sim t^{-1} \ll V \sim t^{-1/2} \).

The universe expands as
\[ a = a_0 \exp \left\{ \left[ H_0 (t - t_0) \right]^{(2-\alpha)/2} \right\}, \] (3.5)
where a constant \( H_0 \) is given by
\[ \frac{H_0}{m_5} = \left[ 3 (2 - \alpha) \left( 2 \sqrt{\alpha} \right) \right]^{-1/2} \left( \frac{\mu}{m_5} \right)^{2(\alpha+4)/12 - \alpha}. \] (3.6)
This solution describes an inflationary evolution, whose expansion rate is slower than that of the conventional exponential inflation, but faster than the power-law type. It may be interesting to discuss a spectrum of density perturbations for such an inflationary scenario.

For \( \alpha = 2 \), we find a power-law solution [13] as
\[ a = a_0 (t/t_0)^p, \] (3.7)
with
\[ p = \frac{1}{6} \left[ 1 + \frac{1}{8} \left( \frac{\mu}{m_5} \right)^6 \right]. \] (3.8)
and
\[ \frac{\dot{\phi}}{m_5} = 2[2m_5l]^{1/2}. \] (3.9)
The scalar field energy density evolves as
\[ \rho \propto t^{-1} \propto \alpha^{-1/p}. \] (3.10)

If \( p > 1 \), i.e. \( \mu > 40^{1/6} m_5 \approx 1.85 m_5 \), we have a power-law inflationary solution [23], which is an attractor of the present system as we will see later. While, if \( p < 1/4 \), i.e. \( \mu < 41/6 m_5 \approx 1.26 m_5 \), this solution is no longer an attractor, leading to the radiation dominant era (see Sec. IV). If \( \alpha > 2 \), a kinetic-term dominant solution gives the asymptotic behavior of the present system (see Sec. IV). If a kinetic term is dominant, we find
\[ H \approx \frac{1}{12 m_5^3} \phi^2 \text{ and } \dot{\phi} + 3H \dot{\phi} \approx 0, \] (3.11)
leading to \( \dot{\phi} \approx -(1/4m_5^3)\phi^3 \), which is integrated as
\[ \frac{\phi}{m_5} = \pm \sqrt{2} m_5^{1/2} (t - t_0)^{1/2} + \frac{\phi_0}{m_5}, \] (3.12)
where \( t_0 \) and \( \phi_0 \) are integration constants. For the solution with + sign, the potential term decreases as \( V \propto t^{-1/2} \), while the kinetic term drops as \( \dot{\phi}^2 \propto t^{-1} \), then the kinetic term is always dominant if \( \alpha > 2 \). In this case, we find that the Universe expands as
\[ a = a_0 (t/t_0)^{1/6}, \] (3.13)
which is exactly the expansion law of the quadratic-term dominant universe with stiff matter.

On the other hand, for the solution with - sign, \( \phi \to 0 \) as \( t \to t_0 \), and then \( V \to \infty \). Then before reaching \( t_0 \), the potential term becomes dominant. However, since the potential-dominant solution is not any attractor, it turns out that the former kinetic-dominant solution is eventually reached (see Sec. IV). Consequently, the kinetic-term dominance is the asymptotic behavior for \( \alpha > 2 \). This guarantees the radiation, if it exists, eventually dominates the scalar-field, which makes initial conditions for a successful quintessence wider [13][14].

**A.2 V(\phi) = \mu^4 \exp[-\lambda \frac{\phi}{m_5}]**

The exponential potential is steeper than the inverse-power-law potential. Then the asymptotic solution is the same as Eqs. (3.12) and (3.13) in the case (1) with \( \alpha > 2 \). Then the radiation dominant era is eventually realized.

**A.3 Chaotic inflationary potential**

The inflationary solution of this model was already obtained [23]. For completeness, we list here the analytic solution including its oscillating phase.

For power-law potential such as \( V = \frac{1}{2} m^2 \phi^2 \) or \( \frac{1}{4} \lambda \dot{\phi}^4 \), we find a stronger inflation than the conventional one [23].

(i) For the model \( V(\phi) = \frac{1}{2} m^2 \phi^2 \), assuming a slow rolling condition, we find that
\[ H \approx \frac{m^2}{12m_5^3} \phi^2 \text{ and } \dot{\phi} \approx -\frac{m^2}{3H} \phi, \] (3.14)
which lead to $\phi' = -4m^3_5$. We then obtain the solution
\begin{equation}
\phi = \pm 2\sqrt{2m^3_5/3} (t_0 - t)^{1/2}, \tag{3.15}
\end{equation}
where $t_0$ and $a_0$ are integration constants.

The slow-rolling condition is broken when $|\ddot{\phi}| \sim |V'|$ or $(1/2)\dot{\phi}^2 \sim V$. Both conditions give the time when inflation ends, i.e. $t_f \sim t_0 - (2m)^{-1}$. In order for inflation to take place in the quadratic energy density dominant stage, $\rho_\phi(\phi_f) > \rho_c$, which implies $m > 3m^3_5/m^2_5$.

After the end of inflation, there appears an oscillation period, which solution is given approximately as
\begin{equation}
\phi = \frac{2m^{3/2}}{ml^{1/2}} \sin mt, \tag{3.17}
\end{equation}
\begin{equation}
a = a_0(t/t_0)^{1/3}. \tag{3.18}
\end{equation}
The scale factor evolves just as the case with a pressureless perfect fluid (dust fluid) at the quadratic term dominant stage. Since the amplitude of $\phi$ decreases more slowly than the conventional case because of the small expansion rate, we expect a sufficient particle production in the preheating phase [18].

(ii) For the model with $V(\phi) = \frac{1}{2} \lambda \phi^4$, the behavior of the scalar field in the inflationary phase is not so different from the case of the previous massive inflaton model. Assuming a slow rolling condition, we find that
\begin{equation}
H \approx \frac{\lambda}{24m^5_5} \dot{\phi}^4 \quad \text{and} \quad \dot{\phi} \approx -\frac{\lambda}{3H} \phi^3, \tag{3.19}
\end{equation}
which lead to $\phi' = -8m^3_5$. We obtain the solution
\begin{equation}
\phi = \pm 4m^{3/2}_5 (t_0 - t)^{1/2}, \tag{3.20}
\end{equation}
\begin{equation}
a = a_0 \exp \left[-\frac{32\lambda m^3_5}{9} (t_0 - t)^3\right]. \tag{3.21}
\end{equation}
where $t_0$ and $a_0$ are integration constants.

The slow-rolling condition is broken at $t = t_f \sim t_0 - 2^{1/3}/(\lambda^{1/3}4m_5)$ when $(1/2)\dot{\phi}^2 \sim V$. Another condition ($|\ddot{\phi}| \sim |V'|$) corresponds to the time $t'_f \sim t_0 - 1/(\lambda^{1/3}4m_5)$, which appears after $t_f$. The scale factor at the end of inflation $a_f \equiv a(t_f) = a_0 \exp[-1/9] \sim a_0$. In order inflation to take place in the quadratic energy density dominant stage, $\rho_\phi(\phi_f) > \rho_c$, which implies $\lambda > (27/32)(m_5/m_4)^6$.

For the oscillating period, it is shown that there is a time-averaged relation between the potential term and the kinetic term of the scalar field, i.e., $\langle \dot{\phi}^2 \rangle = \lambda \langle \phi^4 \rangle$ and then the scalar field behaves approximately as the radiation fluid [17]. Then we find that $a \propto t^{1/4}$ and $\langle \phi \rangle \propto t^{-1/4}$.

Introducing a conformal time $\eta \equiv \int dt a(t)^{-1}$ and the conformal field $\varphi \equiv a\phi$, the equation for $\varphi$ (Eq. (2.4)) is
\begin{equation}
\varphi'' + \lambda \varphi^3 - \frac{a''}{a} \varphi = 0, \tag{3.23}
\end{equation}
where $'$ stands for the derivative with respect to the conformal time. Since the time averaged value of $\varphi$ is almost constant while $(a''/a) \sim \eta^{-2}$, the third term can be ignored. In the conventional cosmology, since $a \sim t^{1/2} \sim \eta$, the third term exactly vanishes. Then we can integrate Eq. (3.23), finding the elliptic function as
\begin{equation}
\varphi = \varphi_0 \cn \left( x - x_0, \frac{1}{\sqrt{2}} \right), \tag{3.24}
\end{equation}
where a constant $\varphi_0$ is an oscillation amplitude of $\varphi$, $x \equiv \sqrt{\Lambda} \eta$ is a dimensionless conformal time variable, and $x_0$ is the value of $x$ when the inflaton field starts to oscillate coherently. This solution is the same as that obtained in the conventional cosmology. However the corresponding time evolution of the time-averaged scalar field $\langle \phi \rangle$ is given as $\langle \phi \rangle \propto \alpha^{-1} \propto t^{-1/4}$, which decreases more slowly than the conventional one. We study this effect at the preheating stage in [18].

### B. Solutions in radiation dominant era

In the radiation dominant era, the scale factor expands as $a \propto t^{1/4}$. We have to solve the equation for a scalar field.

The basic equation is now
\begin{equation}
\ddot{\phi} + \frac{3}{4t} \dot{\phi} + V' = 0. \tag{3.25}
\end{equation}
We shall consider again the same potentials discussed in the previous section in order.

#### B.1 $V(\phi) = \mu^{\alpha+4} \phi^{-\alpha}$

The equation of motion for the scalar field (2.4) is now
\begin{equation}
\ddot{\phi} + \frac{3}{4t} \dot{\phi} - \alpha \mu^{\alpha+4} \phi^{-\alpha-1} = 0. \tag{3.26}
\end{equation}
We find an analytic solution for $\alpha < 6$ [15], that is
\begin{equation}
\phi = \phi_0 \left( \frac{t}{t_0} \right)^{\frac{-2\alpha}{\alpha + 2}}, \tag{3.27}
\end{equation}
with
\begin{equation}
\phi_0^{\alpha + 2} = \frac{2\alpha + 2}{6 - \alpha} \mu^{\alpha + 4} t_0^6, \tag{3.28}
\end{equation}
where $t_0$ and $\phi_0(>0)$ are integration constants. Eq. (3.28) requires $\alpha < 6$.

The energy density of the scalar field evolves as
\[
\rho_\phi = \frac{3\alpha(\alpha + 2)}{6 - \alpha} V_0 \left( \frac{t}{t_0} \right)^{-\frac{3\alpha}{\alpha + 2}},
\]
\[
= \frac{3\alpha(\alpha + 2)}{6 - \alpha} V_0 \left( \frac{a}{a_0} \right)^{-\frac{3\alpha}{\alpha + 2}},
\]
where \( V_0 = V(\phi_0) \) and \( a_0 = a(t_0) \).

The density parameter of the scalar field, which we denote as \( \Omega_\phi \), is
\[
\Omega_\phi = \frac{\rho_\phi}{\rho_r + \rho_\phi} \approx \frac{\rho_\phi}{\rho_r} = \Omega_\phi^{(0)} \left( \frac{a}{a_0} \right)^{\frac{2(2-\alpha)}{\alpha + 2}},
\]
(3.30)
where \( \Omega_\phi^{(0)} = \frac{3\alpha(\alpha + 2)}{8 - \alpha} V_0/\rho_r(t_0) \) is an initial value of the density parameter.

If \( \alpha > 2 \), \( \Omega_\phi \) decreases with time, just contrary to the tracking solution in the conventional quintessence model. The scalar field energy decreases faster than that of the radiation. This is a new interesting feature because the "initial" smallness of a quintessence-field energy could be dynamically explained \([13,16]\). If \( \alpha = 2 \), \( \Omega_\phi \) is constant until the linear term becomes dominant. This is the so-called "scaling" solution. The scalar field energy drops at the same rate as that of the radiation. On the other hand, if \( \alpha < 2 \), \( \Omega_\phi \) increases, and then the universe get into a scalar field energy dominant era discussed above, which is an inflationary solution.

For \( \alpha > 6 \), we find a kinetic-term dominant solution
\[
\phi = \phi_0 \left( \frac{t}{t_0} \right)^{1/4},
\]
(3.31)
where \( \phi_0(>0) \), and \( t_0 \) are integration constants, decided by the initial value only. In this case, they do not include the characteristic scale, because there is no contribution of the potential term for the energy density of the scalar field.

B.2 \( V(\phi) = \mu^4 \exp[-\lambda \frac{\phi}{m_0}] \)

This case is not so interesting, because the kinetic energy dominant solution (3.31) gives an asymptotic behavior just as the same as the case of scalar field dominant era.

B.3 Chaotic inflationary potential

(i) For the model \( V(\phi) = \frac{1}{2} m^2 \phi^2 \), the solution is given approximately as
\[
\phi = \phi_0 \left( \frac{t}{t_0} \right)^{-3/8} \sin mt.
\]
(3.32)
The energy density of the scalar field \( \rho_\phi \), which evolves approximately as \( \rho_\phi \propto t^{-3/4} \propto a^{-3} \), is the same as that of dust fluid, then the scalar field energy will eventually dominate unless its energy is transferred into radiation via a reheating process.

(ii) For the model \( V(\phi) = \frac{1}{4} \lambda \phi^4 \), the behavior is the same as Eq.(3.24), because the time evolution of the scale factor is the same.

We summarize the above solutions in the next table.

| potential | S | R \( (a \propto t^{1/4}) \) |
|-----------|---|---------------------|
| \( \phi^{-\alpha} \) | \( \alpha < 2 \) | inflation: Eqs.(3.3),(3.4) | \( \rightarrow \) power-law: Eqs.(3.7),(3.5) |
|          | \( \alpha = 2 \) | power-law: Eqs.(3.7),(3.5) | \( \times \) | kinetic dominance: Eqs.(3.13),(3.12) |
|          | \( 2 < \alpha < 6 \) | kinetic dominance: Eqs.(3.13),(3.12) | \( \rightarrow \) | kinetic dominance: Eqs.(3.31) |
|          | \( \alpha \geq 6 \) | | |
| \( e^{-\phi^2} \) | | | |
| \( \frac{1}{2} m^2 \phi^2 \) | inflation:Eqs.(3.16),(3.15) | \( \rightarrow \) oscillation:Eqs.(3.18),(3.17) | |
| \( \frac{1}{4} \lambda \phi^4 \) | inflation:Eqs.(3.21),(3.20) | \( \rightarrow \) oscillation:Eqs.(3.22),(3.24) |

TABLE I. The asymptotic behavior in quadratic-energy-density dominated stage. S and R denote scalar-field dominant era \( (\rho_\phi \gg \rho_r) \) and radiation dominant era \( (\rho_r \gg \rho_\phi) \), respectively. The arrows describe the final fate.
IV. STABILITY ANALYSIS

In the previous section, we have listed up several analytic solutions in the quadratic energy density dominated era, which are summarized in Table 1. If those solutions are one of the attractors in the dynamical system, such spacetimes may be naturally realized in the history of the universe. Therefore, it is very important to study their stability.

In this section, we study the stability of the solutions against linear perturbations. Although we may need the detailed analysis for the solutions for the chaotic inflation and the following oscillation, we expect that such solutions are found for a wide range of initial conditions just as the same as conventional chaotic inflationary scenario. Hence we focus only on the stability of solutions of inverse-power-law and exponential potentials.

Since the cosmological solutions are time dependent, we first have to find new variables by which the solutions are described as fixed points in the dynamical system.

A. scalar field dominated stage

1. the power-law solution

First, we investigate the power-law solution (3.7), (3.9) with an inverse-power-law potential with \( \alpha = 2 \), i.e. \( V = \mu^\alpha \phi^{-2} \). The asymptotic behavior of the solution is

\[
\frac{H}{m_5} \sim \frac{p}{(m_5 t)^2}, \quad \frac{V}{m_5^2} \sim \frac{\mathcal{M}_0^2}{m_5 t}, \quad \frac{\dot{\phi}}{m_5^2} \sim \frac{1}{2} \left( \frac{m_5}{t} \right)^{1/2}
\]

where \( \mathcal{M}_0 \equiv (\mu/m_5)^3/2\sqrt{2} \) and \( p = (1 + \mathcal{M}_0^2)/6 \).

For these time dependence, we introduce new variables as

\[
\mathcal{H} \equiv t H, \quad \mathcal{V} \equiv m_5^3 \mathcal{V}, \quad \mathcal{P} \equiv m_5^{-3/2} \dot{\phi}.
\]

In terms of these variables, the basic equations are expressed as

\[
\dot{\mathcal{H}} = \mathcal{H}(1 - 6\mathcal{H} + \mathcal{V}), \quad \dot{\mathcal{V}} = \mathcal{V} - (\sqrt{2}\mathcal{M}_0)^{-1}\mathcal{P}\mathcal{V}^{3/2}, \quad \dot{\mathcal{P}} = \frac{\mathcal{P}}{2}(1 - 6\mathcal{H}) + (\sqrt{2}\mathcal{M}_0)^{-1}\mathcal{V}^{3/2},
\]

with the following constraint,

\[
\mathcal{H} = \frac{1}{12} \mathcal{P}^2 + \frac{1}{6} \mathcal{V},
\]

where a prime denotes a derivative with respect to new time coordinate \( \tau = \ln(m_5 t) \).

These suggest that there are only two independent variables.

For the basic equations, the power-law solution is described by a fixed point \( (\mathcal{H}_0, \mathcal{V}_0, \mathcal{P}_0) = (p, \mathcal{M}_0^2, \sqrt{2}) \).

After eliminating \( \mathcal{H} \) with Eq. (4.10), the perturbation equations around this solution are

\[
\delta\mathcal{V}' = -\frac{1}{2} \delta\mathcal{V} - \frac{\mathcal{M}_0^2}{\sqrt{2}} \delta\mathcal{P}, \quad \delta\mathcal{P}' = \frac{1}{2\sqrt{2}} \delta\mathcal{V} - \left( 1 + \frac{1}{2}\mathcal{M}_0^2 \right) \delta\mathcal{P}.
\]

Because these equations do not include the explicit time-dependence, we can discuss the stability of the solution, by the eigenvalues of these coupled equations.

They are

\[
\left( -1, -\frac{p}{2} \right),
\]

which are all negative, and then this solution is stable against linear perturbations.

2. the kinetic dominant solution

Next, we analyze the stability of the kinetic dominant solution (3.12) with an inverse-power-law potential with \( \alpha > 2 \) and an exponential potential. First we discuss the case with the potential \( V(\phi) = \mu^{\alpha+4}\phi^{-\alpha} \). The asymptotic behavior of the solution is

\[
\frac{H}{m_5} \sim \frac{1}{6(m_5 t)^2}, \quad \frac{V}{m_5^2} \sim \mathcal{V}_0(m_5 t)^{-\alpha/2}, \quad \frac{\dot{\phi}}{m_5^2} \sim \frac{\sqrt{2}}{(m_5 t)^{1/2}},
\]

where \( \mathcal{V}_0 \equiv 8^{-\alpha/2}(\mu/m_5)^{\alpha+4} \).

For such time-dependence, the variables \( \mathcal{H} \) and \( \mathcal{P} \) are given by the same relations (4.4) and (4.6), but \( \mathcal{V} \) is newly introduced as

\[
\mathcal{V} \equiv m_5^{-4} \mathcal{V} e^{\frac{\phi}{m_5}},
\]

using the time coordinate \( \tau = \ln(m_5 t) \). Then we find the following basic equations as,

\[
\mathcal{H}' = \mathcal{H}(1 - 6\mathcal{H} + \mathcal{V} e^{-\frac{\alpha-2}{2}}, \quad \mathcal{V}' = \frac{\alpha}{2} \mathcal{V} \left[ 1 - \frac{\mathcal{P}}{\sqrt{2}} \left( \frac{\mathcal{V}}{\mathcal{V}_0} \right)^{\frac{\alpha}{2}} \right], \quad \mathcal{P}' = \frac{\mathcal{P}}{2}(1 - 6\mathcal{H}) + \frac{\alpha}{2\sqrt{2}} \mathcal{V}_0^{\frac{\alpha+1}{2}} \mathcal{V}^{\frac{\alpha-2}{2}} e^{-\frac{\alpha-2}{2}}.
\]
with the following constraint equation,
\[ H = \frac{1}{6} \left[ \frac{1}{2} \dot{P}^2 + \nu e^{-\frac{\alpha}{2}} \right]. \] (4.21)

From Eqs. (4.18)-(4.20) with (4.21), in the limit of \( \tau \to \infty \), we find the fixed point \((H, \nu, P) = (1/6, \nu_0, \sqrt{2})\), which corresponds to the kinetic-term dominant solution. Eliminating \( H \) with Eq. (4.21), and expanding the above basic equations around this fixed point as \((H, \nu, P) = (1/6 + \delta H, \nu_0 + \delta \nu, \sqrt{2} + \delta P)\), we find that the perturbation equations are
\[ \delta \nu' = -\frac{1}{2} \delta \nu - \frac{\alpha \nu_0}{2 \sqrt{2}} \delta P, \] (4.22)
\[ \delta P' = (\alpha - 2) \frac{\nu_0}{2 \sqrt{2}} e^{-\frac{\alpha}{2} \tau} - \delta P. \] (4.23)

One may think that this system is easily analyzed because the time-dependent part in these equations decays with time. If we estimate the eigenvalues for the asymptotic equations, we find that those are -1/2 and -1, which suggests the system is stable. However, since the equations have explicit time-dependence, we have to be careful to conclude it. Fortunately, in this case, we can easily integrate the above perturbed equations directly as
\[ \delta \nu = \delta \nu_0 e^{-\frac{\tau}{2}} + \frac{\alpha \nu_0}{\sqrt{2}} \delta P_0 e^{-\tau} \]
\[ - \frac{\alpha(\alpha - 2)}{2(\alpha - 3)(\alpha - 4)} \nu_0^2 e^{-\frac{\alpha}{2} \tau}, \] (4.24)
\[ \delta P = \delta P_0 e^{-\tau} - \frac{\alpha(\alpha - 2)}{2(\alpha - 4)} \nu_0 e^{-\frac{\alpha}{2} \tau}, \] (4.25)
if \( \alpha \neq 3, 4 \),
\[ \delta \nu = \frac{3 \nu_0}{\sqrt{2}} \delta P_0 e^{-\tau} - \frac{3 \nu_0}{4} \nu_0^2 e^{-\frac{\tau}{2}} + \delta \nu_0 e^{-\frac{\tau}{2}}, \] (4.26)
\[ \delta P = \delta P_0 e^{-\tau} + \frac{1}{\sqrt{2}} \nu_0 e^{-\frac{\tau}{2}}, \] (4.27)
for \( \alpha = 3 \), and
\[ \delta \nu = \delta \nu_0 e^{-\frac{\tau}{2}} + 2 \sqrt{2} \nu_0 \delta P_0 e^{-\tau} + 2 \nu_0^2 (\tau + 2) e^{-\tau}, \] (4.28)
\[ \delta P = \delta P_0 e^{-\tau} + \frac{1}{\sqrt{2}} \nu_0 \tau e^{-\tau}, \] (4.29)
for \( \alpha = 4 \), where \( \delta \nu_0 \) and \( \delta P_0 \) are the integration constants. These show that \( \delta \nu, \delta P \to 0 \) in the limit of \( \tau \to \infty \), which means that the kinetic dominant solution is stable against linear perturbations for \( V(\phi) = \mu^{4+\alpha} \phi^{-\alpha} \), with \( \alpha > 2 \). Note that for the case of \( 2 < \alpha < 3 \), the perturbations drop more slowly than those expected just from the above eigenvalues.

Next, we consider the case with an exponential potential \( V = \mu^4 e^{-\lambda_0 \phi/m_0} \). The asymptotic behavior of the solution is
\[ \frac{H}{m_5} \sim \frac{1}{6 (m_5 \tau)}, \] (4.30)
\[ \frac{\phi}{m_5} \sim 2 \sqrt{2} (m_5 \tau)^{1/2}, \] (4.31)
\[ \frac{\phi}{m_5^2} \sim \frac{\sqrt{2}}{(m_5 \tau)^{1/2}}. \] (4.32)

In this case, because the potential decreases quite fast as \( V \propto \exp(-2 \sqrt{2} \lambda (m_5 \tau)^{1/2}) \), we adopt new variables
\[ F \equiv m_5^{-3/2} \phi \; t^{-1/2} \] (4.33)
instead of \( V \). The basic equations are given as
\[ H' = H - \frac{1}{24} P'^4 - \frac{1}{12} \left( \frac{\mu}{m_5} \right)^4 P^2 \exp \left( \tau - \lambda e^{\tau/2} F \right), \] (4.34)
\[ F' = \frac{1}{2} F + P, \] (4.35)
\[ P' = \frac{P}{2} (1 - 6 H) + \frac{\mu^4}{m_5^3} \exp \left( \frac{3}{2} \tau - \lambda e^{\tau/2} F \right), \] (4.36)
with the following constraint equation,
\[ H = \frac{1}{12} P^2 + \frac{\mu^4}{6 m_5^6} \exp \left( \tau - \lambda e^{\tau/2} F \right). \] (4.37)

If \( F > 0 \), in the limit of \( \tau \to \infty \), we find a fixed point \( (H_0, F_0, P_0) = (1/6, 2 \sqrt{2}, \sqrt{2}) \), which corresponds to the kinetic-term dominant solution (3.12), (3.13).

Expanding the basic equations around this fixed point as \( (H, F, P) = (1/6 + \delta H, 2 \sqrt{2} + \delta F, \sqrt{2} + \delta P) \), and eliminating \( H \) with Eq. (4.37), we find that the perturbation equations are
\[ \delta F' = -\frac{1}{2} \delta F + \delta P, \] (4.38)
\[ \delta P' = -\delta P + \lambda \frac{\mu^4}{m_5^3} \exp \left( \frac{3}{2} \tau - 2 \sqrt{2} \lambda e^{\tau/2} \right) \]
\[ - \frac{\sqrt{2}}{2} \frac{\mu^4}{m_5^3} \exp \left( \tau - 2 \sqrt{2} \lambda e^{\tau/2} \right). \] (4.39)

In this case the time-dependent term in the equations drops very fast as \( \exp(-2 \sqrt{2} \lambda \exp(\tau/2)) \), we just analyze the asymptotic equations. In the limit of \( \tau \to \infty \), which is
\[ \delta F' = -\frac{1}{2} \delta F + \delta P, \] (4.40)
\[ \delta P' = -\delta P. \] (4.41)

The eigenvalues of the system (4.40), (4.41) are -1/2 and -1, then the kinetic dominant solution is stable against linear perturbations for \( V(\phi) = \mu^4 e^{-\lambda_0 \phi/m_0} \). We can easily check that it is correct even including the time-dependent terms.
3. The inflationary solution

Finally, we study a stability about the inflationary solution (3.4) with the inverse-power-law potential \( V = \mu^{\alpha + 4} \phi^{-\alpha} \) \((\alpha < 2)\). We use \( H, V(\phi) \) and \( \phi \) as our dynamical variables. The asymptotic behavior of these variables for the inflationary solution is

\[
\frac{H}{m_5} \sim \frac{V}{6m_5^2} \sim \frac{1}{6} \nu_0 \left( m_5 t \right)^{-\frac{2}{3}},
\]

\[
\frac{\dot\phi}{m_5^5} \sim \mathcal{P}_0 \left( m_5 t \right)^{-\frac{2}{3}},
\]

where \( \nu_0 \equiv (2\sqrt{\alpha})^{-\alpha} (\mu/m_5)^{\alpha + 4} \) and \( \mathcal{P}_0 \equiv \alpha^2 \). For such time-dependence of the solution, we introduce new variables as

\[
\mathcal{H} \equiv m_5^{-1} H T^{-2/\alpha},
\]

\[
\mathcal{V} \equiv m_5^{-4} V T^{-2/\alpha},
\]

\[
\mathcal{P} \equiv m_5^2 \dot\phi T^{-2/\alpha},
\]

where the new time coordinate \( T \) is defined as

\[
T \equiv (m_5 t)^{1-\frac{2}{3\alpha}}.
\]

Using these variables, the basic equations are now,

\[
\left( 1 - \frac{\alpha}{2} \right) \frac{d\mathcal{H}}{dT} = -\mathcal{H}(6\mathcal{H} - \mathcal{V}) + \frac{\alpha}{2} \mathcal{H} T^{-1},
\]

\[
\left( 1 - \frac{\alpha}{2} \right) \frac{d\mathcal{V}}{dT} = \frac{\alpha}{2} \mathcal{V} \left[ 1 - \frac{\mathcal{P}}{\mathcal{P}_0} \left( \frac{\mathcal{V}}{\mathcal{V}_0} \right)^{1/\alpha} \right] T^{-1},
\]

\[
\left( 1 - \frac{\alpha}{2} \right) \frac{d\mathcal{P}}{dT} = \frac{\mathcal{P}}{2} \left[ \frac{6\mathcal{H}}{\mathcal{V}} - \left( \frac{\mathcal{P}}{\mathcal{P}_0} \right)^{-1} \left( \frac{\mathcal{V}}{\mathcal{V}_0} \right)^{1/\alpha} \right] \mathcal{V}
\]

\[
+ \frac{\mathcal{P}}{2} T^{-1},
\]

with the following constraint equation,

\[
\mathcal{H} = \frac{1}{12} \mathcal{P}^2 T^{-1} + \frac{1}{6} \mathcal{V}.
\]

In the limit of \( T \to \infty \), we find the fixed point \((\mathcal{H}_0, \mathcal{V}_0, \mathcal{P}_0)\), which corresponds to the inflationary solution.

Expanding the basic equations around this fixed point as \((\mathcal{H}, \mathcal{V}, \mathcal{P}) = (\mathcal{H}_0 + \delta \mathcal{H}, \mathcal{V}_0 + \delta \mathcal{V}, \mathcal{P}_0 + \delta \mathcal{P})\), and eliminating \( \mathcal{H} \) with Eq. (1.54), we find that the perturbation equations are

\[
\left( 1 - \frac{\alpha}{2} \right) \frac{d\delta \mathcal{V}}{dT} = -\frac{3}{2} \delta \mathcal{V} T^{-1} - \frac{\alpha \mathcal{V}_0}{2 \mathcal{P}_0} \delta \mathcal{P} T^{-1},
\]

\[
\left( 1 - \frac{\alpha}{2} \right) \frac{d\delta \mathcal{P}}{dT} = \frac{\mathcal{P}_0}{2} \delta \mathcal{V} - \frac{\mathcal{V}_0}{2} \delta \mathcal{P}
\]

\[
+ \frac{1}{2} \mathcal{P}_0 (1 - \frac{1}{2} \mathcal{P}_0) T^{-1}.
\]

If we adopt a naive analysis for the asymptotic equations ignoring decaying time-dependent terms, we find that the eigen values are \(-\nu_0/(2 - \alpha)\) and 0. The zero eigenvalue corresponds to a marginally unstable mode, which appears in a slow-rolling phase of an inflationary solution.

In order to confirm the asymptotic behaviors of \( \delta \mathcal{V} \) and \( \delta \mathcal{P} \), we will integrate the above equations directly. To solve (4.53) and (4.54), we first need to find the following first-order differential equation

\[
\frac{d}{dT} [A(T) \delta \mathcal{P} + B(T) \delta \mathcal{V}] =
\]

\[
\sigma [A(T) \delta \mathcal{P} + B(T) \delta \mathcal{V}] + F(T).
\]

where \( A(T), B(T), \) and \( F(T) \) are chosen appropriately as follows.

We can easily show that if \( A(T), B(T), F(T) \) satisfy

\[
\frac{dA}{dT} - \frac{1}{(2 - \alpha)T A} \left[ \left( \sigma + \frac{\nu_0}{2 - \alpha} \right) A - \frac{\alpha \nu_0}{(2 - \alpha)T} B \right] = 0,
\]

\[
\frac{dB}{dT} + \frac{\mathcal{P}_0}{(2 - \alpha)T} B - \left( \frac{\sigma}{(2 - \alpha)T} + \frac{1}{(2 - \alpha)T^2} \right) B = 0,
\]

and

\[
F = \frac{\mathcal{P}_0}{2 - \alpha} \left( 1 - \frac{1}{2} \mathcal{P}_0 \right) A, T,
\]

is found.

From (4.53) and (4.54), by eliminating \( A, \) we find the second order differential equation for \( B, \) as

\[
\frac{dB}{dT^2} = \left[ \left( 2\sigma + \frac{\nu_0}{2 - \alpha} \right) + \frac{1}{2 - \alpha} \left( 1 - \frac{1}{2} \mathcal{P}_0 \right) \frac{dB}{dT} + \frac{\sigma}{2 - \alpha} \right] \left( \frac{\sigma}{2 - \alpha} + \frac{2\nu_0}{(2 - \alpha)^2} \right) \frac{1}{T}
\]

\[
+ \frac{1}{(2 - \alpha)T^2} \right] \times B = 0.
\]

The solution of this equation is given by two independent hypergeometric functions, which asymptotic behaviors are given by

\[
B_1(T) = e^{\sigma T} \mathcal{T}^{-\frac{\nu_0}{2 - \alpha}},
\]

\[
B_2(T) = e^{[\sigma + \frac{\nu_0}{2 - \alpha}] T} T^{-\frac{\nu_0}{2 - \alpha}}.
\]

From (1.55) and (4.57), we find \( A(T) \) and \( F(T) \) for each solution as

\[
A_1(T) = -\frac{\alpha}{\mathcal{P}_0} e^{\sigma T} \mathcal{T}^{-\frac{\nu_0}{2 - \alpha}},
\]

\[
F_1(T) = -\frac{\alpha}{2 - \alpha} \left( 1 - \frac{1}{2} \mathcal{P}_0 \right) e^{\sigma T} \mathcal{T}^{-\frac{\nu_0}{2 - \alpha}},
\]

\[
A_2(T) = -\frac{\alpha \nu_0}{\mathcal{P}_0} e^{[\sigma + \frac{\nu_0}{2 - \alpha}] T} T^{-\frac{\nu_0}{2 - \alpha}},
\]

\[
F_2(T) = -\frac{\alpha \nu_0}{2 - \alpha} \left( 1 - \frac{1}{2} \mathcal{P}_0 \right) e^{[\sigma + \frac{\nu_0}{2 - \alpha}] T} T^{-\frac{\nu_0}{2 - \alpha}}.
\]
We then obtain two sets of (4.54), which general solution is given by
\begin{align*}
A_1(T) \delta P + B_1(T) \delta V &= e^{\sigma T} \int^T dT' e^{-\sigma T'} F_1(T'), \\
A_2(T) \delta P + B_2(T) \delta V &= e^{\sigma T} \int^T dT' e^{-\sigma T'} F_2(T').
\end{align*}
(4.65)
(4.66)

The asymptotic behaviors of these equations are
\begin{align*}
- \frac{\alpha}{P_0} \delta P + T \delta V &= c_1 T^{-\frac{\alpha}{2}} - \left(1 - \frac{1}{2} P_0^2 \right), \\
- \frac{\alpha \nu_0}{P_0} \delta P + \delta V &= c_2 T^{1+\frac{\alpha}{2}} e^{-\frac{\nu_0}{2} T} - \alpha \left(1 - \frac{1}{2} P_0^2 \right) T^{-1},
\end{align*}
where \(c_1\) and \(c_2\) are integration constants.

Then we obtain the asymptotic solution of \(\delta P\) and \(\delta V\) as
\begin{align*}
\delta P &= \Delta^{-1} \left[ c_1 T^{-\frac{\alpha}{2}} - c_2 T^{\frac{\alpha}{2}} e^{-\frac{\nu_0}{2} T} \right. \\
&\quad \left. + \left( \alpha - 1 \right) \left(1 - \frac{1}{2} P_0^2 \right) \right], \\
\delta V &= \Delta^{-1} \left[ c_2 \frac{\alpha \nu_0}{P_0} T^{1+\frac{\alpha}{2}} e^{-\frac{\nu_0}{2} T} \right. \\
&\quad \left. - \alpha \left( \frac{\alpha - \nu_0}{P_0} T \right) \left(1 - \frac{1}{2} P_0^2 \right) \right],
\end{align*}
where
\[\Delta = \frac{\alpha}{P_0} \nu_0 T - 1.\]
(4.69), (4.70) and (4.71) give the asymptotic behavior of \(\delta P\) and \(\delta V\) as
\begin{align*}
\delta P &\to \frac{(\alpha - 1) P_0}{\alpha \nu_0} \left(1 - \frac{1}{2} P_0^2 \right) T^{-1}, \\
\delta V &\to - \left(1 - \frac{1}{2} P_0^2 \right) T^{-1}
\end{align*}
(4.72)
(4.73)

This power-law decay corresponds to zero eigenvalue. Then the above naive analysis by eigen values is confirmed. We conclude that the inflationary solution is stable against linear perturbations.

**B. radiation dominated stage**

1. **the power-law solution**

We show here that for the scalar field with an inverse-power-law potential model \(V(\phi) = \mu^{4+\alpha} \phi^{-\alpha-4}\), with \(\alpha < 6\), the power-law solution in the radiation dominant stage given by (3.27), (3.28) is stable against the linear perturbations.

Following (3), we introduce the new variables as
\begin{align*}
\tau &\equiv \ln m_5 t, \\
F &\equiv \frac{\phi}{\phi_c},
\end{align*}
where \(\phi_c\) is the exact solution given by (3.27), (3.28),
\[\frac{\phi_c}{m_5} = A(\alpha)(m_5 t)^{\frac{\alpha}{6+\alpha}}.\]
(4.77)

\(A(\alpha)\) is a dimensionless constant, which depends on \(\alpha\) and \(\mu\),
\[A(\alpha) = \left( \frac{2(\alpha+2)^2}{6-\alpha} \right)^{\frac{\alpha}{6+\alpha}} \left( \frac{\mu}{m_5} \right)^{\frac{6+\alpha}{6+\alpha}}.\]
(4.78)

With these changes, Eq.(3.21) becomes,
\[F' = P, \]
\[P' = \frac{\alpha - 14}{4(\alpha+2)} P + \frac{\alpha - 6}{2(\alpha+2)^2} F - \frac{\alpha - 6}{2(\alpha+2)^2} F^{\alpha - 1},\]
(4.79)
(4.80)

where prime denotes the derivative with respect to \(\tau\) and the power-law solution given by (3.27) and (3.28) corresponds to a fixed point \((F, P) = (1, 0)\).

In order to study the stability near the critical point, we here use the linear analysis (3). Expanding the above equations around the fixed point as \((F, P) = (1 + \delta F, \delta P)\), we find that
\begin{align*}
\delta F' &= \delta P, \\
\delta P' &= \frac{\alpha - 6}{2(\alpha+2)} \delta F + \frac{\alpha - 14}{4(\alpha+2)} \delta P.
\end{align*}
(4.81)
(4.82)

Because these equations do not include the explicit time-dependence, we can discuss the stability of the solution by the eigenvalues of these coupled equations.

The eigenvalues are given as
\[-\frac{1}{2} \text{ and } -\frac{6-\alpha}{\alpha + 2},\]
which are all negative for \(\alpha < 6\), and then this solution is stable against linear perturbations.

2. **the kinetic dominant solution**

We show here that for the scalar field with an inverse-power-law potential model \(V(\phi) = \mu^{4+\alpha} \phi^{-\alpha-4}\), with \(\alpha > 6\), and an exponential potential model \(V(\phi) = \mu^4 e^{-\lambda \phi/m_5}\) the kinetic dominant solution in the radiation dominant stage given by (3.31), is stable against the linear perturbations.
At first, we show the case with an inverse-power-law potential model $V(\phi) = \mu^4 e^{-\phi/m_5}$, with $\alpha > 6$. As in the power-law solution in the radiation dominant stage, we introduce the new variables as

$$\tau \equiv m_5 t,$$

$$\mathcal{F} \equiv \frac{\dot{\phi}}{\phi_k},$$

where $\phi_k$ is the kinetic dominant solution given by (3.31),

$$\frac{\dot{\phi}_k}{m_5} = B(m_5 t)^{1/4}.$$

In this case, $B$ is the dimensionless constant, whose value depends on the initial condition. In terms of these variables, the basic equation (3.26) is expressed as,

$$\mathcal{F}' = \mathcal{P},$$

$$\mathcal{P}' = -\frac{1}{4} v + \alpha \left( \frac{\mu}{m_5} \right)^{\alpha+4} B^{-\alpha-2} \mathcal{F}^{-\alpha-1} e^{\frac{\phi}{m_5}} \tau.$$

In the limit of $\mathcal{F} \rightarrow \infty$, we find the critical point $(\mathcal{F}_0, \mathcal{P}) = (\mathcal{F}_0, 0)$, where $\mathcal{F}_0$ is an arbitrary constant. This point corresponds to the kinetic dominant solution. Expanding the basic equations around this critical point as $(\mathcal{F}, \mathcal{P}) = (\mathcal{F}_0 + \delta \mathcal{F}, \delta \mathcal{P})$, we find that the perturbation equations are,

$$\delta \mathcal{F}' = \delta \mathcal{P},$$

$$\delta \mathcal{P}' = -\frac{1}{4} \delta \mathcal{P} + \alpha \left( \frac{\mu}{m_5} \right)^{\alpha+4} B^{-\alpha-2} \mathcal{F}_0^{-\alpha-1} e^{\frac{\phi}{m_5}} \tau.$$

Because these equations have explicit time-dependence, we cannot evaluate the stability with the eigenvalues. However, for $\alpha > 6$, we can easily integrate the above perturbed equations as

$$\delta \mathcal{F} = \frac{16\alpha}{(7 - \alpha)(6 - \alpha)} \left( \frac{\mu}{m_5} \right)^{\alpha+4} B^{-\alpha-2} \mathcal{F}_0^{-\alpha-1} e^{\frac{\phi}{m_5}} \tau - 4\delta \mathcal{P}_0 e^{-\frac{\phi}{m_5}} + \delta \mathcal{F}_0,$$

$$\delta \mathcal{P} = \frac{4\alpha}{7 - \alpha} \left( \frac{\mu}{m_5} \right)^{\alpha+4} B^{-\alpha-2} \mathcal{F}_0^{-\alpha-1} e^{\frac{\phi}{m_5}} \tau + \delta \mathcal{P}_0 e^{-\frac{\phi}{m_5}},$$

for $\alpha \neq 7$, and

$$\delta \mathcal{F} = -28 \left( \frac{\mu}{m_5} \right)^{11} B^{-9} \mathcal{F}_0^{-8} e^{-\frac{\phi}{m_5}} (\tau - 4) - 4\delta \mathcal{P}_0 e^{-\frac{\phi}{m_5}} + \delta \mathcal{F}_0,$$

$$\delta \mathcal{P} = 7 \left( \frac{\mu}{m_5} \right)^{11} B^{-9} \mathcal{F}_0^{-8} \tau e^{-\frac{\phi}{m_5}} + \delta \mathcal{P}_0 e^{-\frac{\phi}{m_5}},$$

for $\alpha = 7$, where $\delta \mathcal{F}_0$ and $\delta \mathcal{P}_0$ are integration constants.

These show that in the limit of $\tau \rightarrow \infty$, even though this is not strictly a fixed point, the kinetic-term dominant solutions are attractors in a sense that these satisfy (3.31).

Next, we investigate the case with an exponential potential model $V(\phi) = \mu^4 e^{-\phi/m_5}$.

In this case, the equation of motion for the scalar field (2.4) is now,

$$\ddot{\phi} + \frac{3}{4t} \dot{\phi} - \lambda \mu \frac{\mu}{m_5} \exp[-\lambda \phi/m_5] = 0.$$

In terms of the variables, (4.84)-(4.86), the basic equation (4.95) is expressed as,

$$\mathcal{F}' = \mathcal{P},$$

$$\mathcal{P}' = -\frac{1}{4} \mathcal{P} + \lambda \left( \frac{\mu}{m_5} \right)^4 \exp \left[ \frac{7}{4} \tau - \lambda \mu \frac{\mu}{m_5} \right].$$

In the limit of $\tau \rightarrow \infty$, we find the critical point $(\mathcal{F}, \mathcal{P}) = (\mathcal{F}_0, 0)$, where $\mathcal{F}_0$ is an arbitrary constant. This point corresponds to the kinetic dominant solution. Expanding the basic equations around this critical point as $(\mathcal{F}, \mathcal{P}) = (\mathcal{F}_0 + \delta \mathcal{F}, \delta \mathcal{P})$, we find that the perturbation equations are,

$$\delta \mathcal{F}' = \delta \mathcal{P},$$

$$\delta \mathcal{P}' \approx -\frac{1}{4} \delta \mathcal{P} + \lambda \left( \frac{\mu}{m_5} \right)^4 \exp \left[ \frac{7}{4} \tau - \lambda \mu \frac{\mu}{m_5} \right].$$

In the limit of $\tau \rightarrow \infty$, the asymptotic behavior of these equations are,

$$\delta \mathcal{F}' = \delta \mathcal{P},$$

$$\delta \mathcal{P}' \approx -\frac{1}{4} \delta \mathcal{P},$$

which can be integrated as

$$\delta \mathcal{F} = \delta \mathcal{F}_0 - 4\delta \mathcal{P}_0 e^{-\tau/4},$$

$$\delta \mathcal{P} \approx \delta \mathcal{P}_0 e^{-\tau/4},$$

where $\delta \mathcal{F}_0$ and $\delta \mathcal{P}_0$ are integration constants.

These show that in the limit of $\tau \rightarrow \infty$, even though this is not strictly a fixed point, the kinetic term dominant solutions are attractors in a sense that these satisfy (3.31).

V. GLOBAL STABILITY OF THE SOLUTIONS

In the previous section, we show that the analytic solutions listed up in Table 1, are stable against the linear perturbations. However, in order to show that these solutions are realized from wide initial conditions, we have to study stability against non-linear perturbations. The we analyze a global stability of the solutions in the phase-space.

For the model with the potential $V(\phi) = \mu^4 e^{-\phi/m_5}$, we study it both numerically and analytically including both the scalar field and the radiation (5V.A). We also show the numerical results for the kinetic dominant solutions (5.12) and the inflationary solutions (3.4) by assuming scalar field dominance (5V.B) and for the power-law solutions ($\alpha \neq 2$) (5.27) and the kinetic dominant solutions (3.31) by assuming radiation dominance (5V.C).
A. the power-law solution ($\alpha = 2$)

Here, we study a global stability of the solutions for $V(\phi) = \mu^6 \phi^{-2}$ in detail, following [2]. We introduce new variables

\[
X = \frac{m_5^{-3/2} \phi}{2\sqrt{3} \sqrt{H}}, \\
Y = \frac{2m_5^{3/2} M_0 \phi^{-1}}{\sqrt{3} \sqrt{H}},
\]

(5.1)

where $M_0 = \mu^3/(2\sqrt{2}m_5^3)$, and new derivative with respect to $\ln a$, which is described by $\dot{\prime}$. The dynamical equations are rewritten as a plane-autonomous system:

\[
X' = F(X,Y) \equiv X(X^2 - 2Y^2 - 1) + 3M_0^{-1}Y^3, \\
Y' = G(X,Y) \equiv -Y(2Y^2 - X^2 + 3M_0^{-1}XY - 2),
\]

(5.2)

and radiation energy is given by the constraint equation

\[
\frac{\rho_r}{6m_5^3 H} + X^2 + Y^2 = 1.
\]

(5.3)

From the constraint equation, the density parameter of a scalar field is given by

\[
\Omega_\phi \equiv \frac{\rho_\phi}{6m_5^3 H} = X^2 + Y^2.
\]

(5.4)

Since radiation energy is non-negative ($\rho_r \geq 0$), we find $0 \leq X^2 + Y^2 \leq 1$, so the evolution of this system is completely described by trajectories within the unit disc. As the system is symmetric under the reflection $(X,Y) \to (X,-Y)$ and the evolution will not go beyond the $Y = 0$-line (which corresponds to $\phi = \infty$ or $H = \infty$), it is enough to discuss the upper half-disc ($Y \geq 0$).

Depending on the value of $M_0$ (or $\mu/m_5$), we have four or five fixed points (critical points) where $X'$ and $Y'$ vanish. The four critical points $(0,0), (1,0), (-1,0),$ and $A(1/\sqrt{1 + M_0^2}, M_0/\sqrt{1 + M_0^2})$ exist on the boundary of the phase space, while the fifth critical point B $(\sqrt{8/9}M_0^{1/2}, \sqrt{2/9}M_0^{1/2})$ appears in the half unit disc only if $M_0 < \sqrt{2}/2$.

The simple analysis by linear perturbations shows that the critical point A is stable if $M_0 > \sqrt{2}/2$, while it becomes a saddle point when $M_0 < \sqrt{2}/2$, and newly appeared critical point B becomes stable. Since the point A locates on the boundary, $\Omega_\phi$ is always unity (scalar-field dominated stage), while the point B appears on the line $Y = X/\sqrt{2}$. The density parameter of this solution is $\Omega_\phi = \sqrt{2}M_0$, which means that the radiation density becomes larger that that of a scalar field if $M_0 < \sqrt{2}/4$.

In order to analyze the global behavior in the phase space of the present dynamical system, we first draw critical curves corresponding to $F(X,Y) = 0$, which corresponds to the curve $C_1$ or $G(X,Y) = 0$, which gives two curves: the straight line $Y = 0$ and hyperbolic curve $C_2$ (see Fig. 1 and Fig. 2).

\[\text{FIG. 1. Schematic phase diagram for the model with } V(\phi) = \mu^6 \phi^{-2} \text{ for the case of } M_0 > \sqrt{2}/2. \text{ The arrows show the directions in which the spacetime evolves. The critical point A is an attractor.}\]

\[\text{FIG. 2. Schematic phase diagram for the model with } V(\phi) = \mu^6 \phi^{-2} \text{ for the case of } M_0 < \sqrt{2}/2. \text{ The arrows show the directions in which the spacetime evolves. The critical point A is a saddle point, while the point B is an attractor.}\]

From the signs of $F(X,Y)$ and $G(X,Y)$, we find the direction of the evolutionary track of the universe in the phase space, which is shown by arrows in Fig. 1 and Fig. 2. Following those arrows, we find whether the critical points are globally stable or unstable. It turns out that the critical points $(1,0)$ and $(-1,0)$ are unstable,
and \((0,0)\) is a saddle point. For \(M_0 > \sqrt{2}/2\), the critical point A is stable, but if \(M_0 < \sqrt{2}/2\), it becomes a saddle point, as we expect from our perturbation analysis, and new critical point B becomes a stable point.

We also show the result of numerical analysis for both cases. We set \(\mu = 2m_5(M_0 = 2\sqrt{2})\) for the case of \(M_0 > \sqrt{2}/2\) and \(\mu = m_5(M_0 = \sqrt{2}/4)\) for the case of \(M_0 < \sqrt{2}/2\).

First, we show that the kinetic dominant solution \((3.12)\) is given by \(V(\phi) = \mu^2 \phi^{-3}\), the similar results are obtained for the model with \(V(\phi) = \mu^{4+\alpha} \phi^{-\alpha}, \alpha > 2\), and \(V(\phi) = \mu^4 \exp[-\lambda \phi/m_5]\). Fig. 3 shows that the kinetic dominance realizes at first, then the solution eventually approaches the critical point \((H = 1/6, P = \sqrt{2})\).

In what follows, we only show numerical analysis for global stability for other cases.

1. the kinetic dominant solution

First, we show that the kinetic dominant solution \((3.12)\) is obtained as an attractor from the various initial conditions (see Fig 4). The kinetic dominant solution is described by \(H = (1/12)P^2\), and the solution of \((3.12)\) is given by \(H = 1/6, P = \sqrt{2}\). \((H\text{ and } P\text{ are defined by Eq. (4.4) and Eq. (4.6), respectively.})\). Even though we show only the case for the model \(V(\phi) = \mu^2 \phi^{-3}\), the similar results are obtained for the model with \(V(\phi) = \mu^{4+\alpha} \phi^{-\alpha}, \alpha > 2\), and \(V(\phi) = \mu^4 \exp[-\lambda \phi/m_5]\). Fig. 4 shows that the kinetic dominance realizes at first, then the solution eventually approaches the critical point \((H = 1/6, P = \sqrt{2})\).

2. the inflationary solution

Next, we show the inflationary solution \((3.12)\) realizes from the various initial conditions numerically (see Fig 4). The slow-rolling condition gives \(H = (1/6)V\), and the solution \((3.12)\) is described by \(H = 1/12, V = 1/2\).
(\(\mathcal{H}\) and \(\mathcal{V}\) are defined by Eq.(1.44) and Eq.(1.45), respectively.) From Fig. 6, we easily find the inflationary solution for a wide range of initial data as expected. Even though we show only the case for the model with \(V(\phi) = \mu^5 \phi^{-1}\), the similar results are obtained for the model with \(V(\phi) = \mu^{4+\alpha} \phi^{-\alpha}, \alpha < 2\).

\[
\mathcal{H} = \frac{1}{(1/6)\mathcal{V}}
\]

![Global stability of the solutions for the model with \(V(\phi) = \mu^5 \phi^{-1}\). This shows that the slow-rolling (\(\mathcal{H} = (1/6)\mathcal{V}\)) realizes first, then the solution eventually approaches the critical point (\(\mathcal{V} = 1/2, \mathcal{H} = 1/12\), which corresponds to the solution (3.4). We set \(\mu = m_5\). As for initial conditions \((\phi/m_5, (\phi/m_5)' \equiv t \times d(\phi/m_5)/dt)\) at \(m_5t = 1.0\), we choose the following seven sets of initial data; (1) (1.0, 0.0), (2) (1.0, 2.0), (3) (1.0, 3.0), (4) (1.5, 3.0), (5) (3.0, 3.0), (6) (1.0 \times 10^5, 3.0), (7) (1.0 \times 10^7, 0.0).

**C. radiation dominated stage**

1. **the power-law solution \((\alpha \neq 2)\)**

Here, we show that the power-law solution (3.27) and (3.28) \((\alpha \neq 2)\) realizes from the distant point in the parameter space numerically (see Fig. 7). The power-law solution is given by \((\phi/m_5, (\phi/m_5)' \equiv t \times d(\phi/m_5)/dt)\) are \((1.0, 0.0)\). Furthermore, \(\rho_\phi/m_5^4 = 1.0\) and \(\rho_\phi/m_5^4 = 1.0\).

![The attractor behavior of the power-law solution. We set \(\mu = m_5\). As for initial condition, we start from \(m_5t = 1.0\). The initial values of \((\mathcal{F}, \mathcal{P})\) are \((4.6 \times 10^{-1}, -1.2 \times 10^{-1})\), which corresponds to the initial values of \((\phi/m_5, (\phi/m_5)' \equiv t \times d(\phi/m_5)/dt)\) are \((1.0, 0.0)\). Furthermore, \(\rho_\phi/m_5^4 = 1.0\) and \(\rho_\phi/m_5^4 = 1.0\).]

2. **the kinetic dominant solution**

Finally, we show the result for the models with \(V(\phi) = \mu^{4+\alpha} \phi^{-\alpha}, \alpha > 6\) and \(V(\phi) = \mu^4 e^{-\lambda_\phi/m_5}\), in the radiation dominant stage (Fig. 8). Because this kinetic dominant solution does not include the characteristic mass scale, we cannot introduce the fixed critical point like the previous solutions. Therefore, we show the ratio of the potential energy of the scalar field to the total energy of the scalar field. From the potential dominant initial condition \((\rho^{(p)}_\phi/\rho^{(tot)}_\phi = 1\) initially), the kinetic term eventually dominates the potential term. \((\rho^{(p)}_\phi/\rho^{(tot)}_\phi \rightarrow 0)\). Even though we show only the case for the model \(V(\phi) = \mu^{11} \phi^{-7}\), the similar results are obtained for the other models \(V(\phi) = \mu^{4+\alpha} \phi^{-\alpha}, \alpha > 6\) and \(V(\phi) = \mu^4 e^{-\lambda_\phi/m_5}\).
FIG. 8. The ratio of the potential energy of the scalar field to the total energy of the scalar field. This shows that even though the potential energy dominates the kinetic energy initially, the kinetic term dominance eventually realizes. We set \( \mu = m_5 \). As for initial condition, we start from \( m_5 t = 1.0 \). The initial values of \( (\phi/m_5, (\phi/m_5)' = t \times (d(\phi/m_5)/dt) \) are \((1.2, 0.0)\). Furthermore, \( \rho_{\phi} / m_5^4 = 2.8 \times 10^{-1} \) and \( \rho_p / m_5^4 = 1.0 \).

VI. SUMMARY AND DISCUSSION

In this paper we have studied the dynamics of a scalar field with the typical potential in a brane world scenario. We have adopted the second Randall-Sundrum brane scenario and assumed that the scalar field is confined on our 4-dimensional spacetime. In this model, because of the quadratic term of the energy density, which is from the consequence of our brane embedded in the extra dimension, the dynamics of the scalar field is modified from a conventional cosmology in the early stage of the universe.

For the power-law potential model, which was discussed in [22, there exist an inflationary solution in the quadratic term dominant stage, which satisfies slow-rolling condition. Similar to the conventional cosmology, the e-folding number more than 60 is obtained for sufficiently large initial values of \( \phi \). However, because the cosmic expansion is faster than that in the conventional cosmology, a shorter time is required for the enough e-folding number.

After the slow-rolling condition is broken, and the oscillating phase starts, the dynamics of the scalar field depends on the power of the scalar field potential. For the model, \( V(\phi) = \frac{1}{2} m^2 \phi^2 \), the energy density of the scalar field decreases as the pressureless perfect fluid and more slowly than that in the conventional case. In the preheating stage, this changes the time dependence of the variable appeared in the Mathieu equation and leads to an efficient particle production via non-perturbative decay of inflaton even for the smaller coupling constant. For the model, \( V(\phi) = \frac{1}{2} \lambda \phi^4 \), the energy density of the scalar field decreases as the radiation fluid. In this case, the solution of the scalar field evolution is represented in terms of the elliptic function and the form of the Lamé equation in the preheating stage is not changed. Although this is the same as the conventional case, because of the slower growth of the scale factor, the particle production becomes a little efficient [18].

For the models with an inverse-power-law potential model \( V = \mu^{4+\alpha} \phi^{-\alpha} \) or an exponential potential model \( V = \mu^4 e^{-\lambda \phi / m_5} \), we find much difference from conventional cosmology. We find analytic solutions in several circumstances, and analyze those stabilities. According to the solutions and those analysis, the asymptotic behavior of the universe is summarized as follows:

- For \( V = \mu^{4+\alpha} \phi^{-\alpha} \), with \( \alpha < 2 \) and with \( \alpha = 2 \) \((\mu > \sqrt[3]{10} m_5)\), an inflationary solution is obtained in the quadratic term dominant stage. Since this is an attractor, such a solution is eventually obtained even though from the radiation-dominant initial condition. The inflationary solutions in these models are milder than that in the conventional cosmology. Because this potential has no minima, in order to recover the Big-Bang cosmology, some unknown reheating mechanisms is required. The gravitational particle production could provide its resolution [23].

For \( V = \mu^{4+\alpha} \phi^{-\alpha} \), with \( 2 < \alpha < 6 \), which we discussed in the previous paper for the quintessence scenario [13][14], the attractor universe is radiation dominant and the power-law solution is the attractor for the scalar field. Our work concerned to quintessence for this model is completed in this paper to show the stability of the solutions. (For the naturalness of the quintessence scenario and the constraints to this model, see [13][14].)

For \( V = \mu^{4+\alpha} \phi^{-\alpha} \), with \( 6 < \alpha < 10 \) and \( V = e^{-\lambda \phi / m_5} \), because the shape of the potential is too steep, the kinetic term of the scalar field dominates the potential term both in the scalar field dominant universe and the radiation dominant universe asymptotically.

In this paper, we have assumed that a scalar field is confined in the brane. However, most models based on a string theory require a dilaton field in the bulk, which may gives very important effects on the behavior of gravity and a scalar field on the brane [19]. The analysis in such models is left to the future work.

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