Wilsonian Renormalization Group and the Lippmann-Schwinger Equation with a Multitude of Cutoff Parameters* 

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Abstract The Wilsonian renormalization group approach to the Lippmann-Schwinger equation with a multitude of cutoff parameters is introduced. A system of integro-differential equations for the cutoff-dependent potential is obtained. As an illustration, a perturbative solution of these equations with two cutoff parameters for a simple case of an S-wave low-energy potential in the form of a Taylor series in momenta is obtained. The relevance of the obtained results for the effective field theory approach to nucleon-nucleon scattering is discussed.

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Key words: nucleon-nucleon scattering, effective field theory, renormalization group

1 Introduction

The chiral effective field theory (EFT) approach to few-nucleon systems\cite{10–2} has attracted much attention during the past two and a half decades. The problem of renormalization and power counting in this framework turned out to be highly nontrivial and caused controversial debates in the community. A number of formulations alternative to Weinberg’s original proposal have been suggested to resolve the issue of renormalization, see Refs.\cite{3–8} for review articles. In our recent paper,\cite{9} we have compared a subtractive renormalization approach with the Wilsonian renormalization group (RG) approach\cite{10–11} in the context of the EFT for the two-nucleon system close to the unitary limit. In particular, within the subtractive scheme, we have identified the choices of renormalization conditions corresponding to the Kaplan-Savage-Wise (KSW),\cite{12} see also Refs.\cite{13–14}, and Weinberg\cite{1} power counting schemes. The standard Wilsonian RG method with a single cutoff scale is, on the other hand, only compatible with the KSW counting scheme. We argued that this mismatch is caused by the too restrictive formulation of the Wilsonian RG approach in its conventional form, which does not take into account the full freedom in the choice of renormalization conditions in EFT. This is the origin of the often made (incorrect, see Ref.\cite{9}) statement that the Weinberg power counting scheme for two-nucleon scattering corresponds to the expansion around a trivial fixed point.

In the Wilsonian RG approach one integrates out degrees of freedom with energies higher than some cutoff scale and systematically exploits the cutoff-parameter dependence of coupling constants to ensure that physics at energies below the cutoff scale remains unchanged.\cite{15} In contrast, the Gell-Mann-Low RG equations determine the dependence of various quantities on the scale(s) of renormalization.\cite{16} In renormalizable (in the traditional sense) theories only logarithmic divergences contribute to the renormalization of the coupling constants and, therefore, there is a direct correspondence between the two approaches. On the other hand, in EFTs with non-renormalizable interactions, power-law divergences have to be taken care of and the direct link between the two RG equations is lost. Notice further that in theories with more than one coupling constant, as it is the case in EFTs, each coupling is attributed its own renormalization scale. In the Wilsonian approach one usually introduces a single cutoff scale and studies how various parameters of a theory depend on it. However, in certain cases such as e.g. the few-nucleon problem in chiral EFT, it is advantageous to exploit the freedom of choosing several renormalization scales independently.\cite{9, 17}

In this paper we fill this gap and generalize the Wilsonian RG analysis of low-energy two-particle scattering in

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the framework of the Lippmann-Schwinger (LS) equation, pioneered in Ref. [10], by introducing a multitude of cutoff parameters. We obtain a system of integro-differential RG equations describing the dependence of the potential on several cutoff scales. As an application, we study a perturbative solution of the obtained system of equations for the case of two cutoff parameters by making an ansatz for the potential in the form of a Taylor series expansion in powers of momenta. We demonstrate that the resulting potential indeed obeys the Weinberg power counting for the choice of renormalization conditions suggested in Ref. [9].

Our paper is organized as follows. In the next section we derive the system of RG equations for the case of several cutoff parameters. In Sec. 3, we present the perturbative solution of this system of equations and discuss the obtained results in the context of EFT for two-nucleon scattering. Finally, our main findings are briefly summarized in Sec. 4.

2 The Lippmann-Schwinger Equation with a Multitude of Cutoff Parameters

To introduce a multitude of cutoff parameters and derive the corresponding system of RG equations we start with the fully off-shell LS equation

\[ T(p, q, k) = V(p, q, k) + \int \frac{d^3l}{(2\pi)^3} V(p, l, k) G(k, l) T(l, q, k), \]  

(1)

where \( G(k, l) = \frac{2m}{(k^2 - l^2 + i\epsilon)} \) is the nonrelativistic two-particle Green’s function, and \( k^2/m \) is the kinetic energy in the centre-of-mass frame. We assume that the low-energy dynamics of the system at hand is describable in the framework of the non-relativistic Schrödinger theory, i.e. that the underlying potential \( V(p, q, k) \) is non-singular and well-behaved in the quantum mechanical sense. We regard Eq. (1) as an “underlying” model and follow the philosophy of Wilson’s renormalization group approach. Specifically, we aim at integrating out the high-momentum modes by introducing 2N cutoffs \( \Lambda_1, \Lambda_1, \Lambda_2, \Lambda_2, \ldots, \Lambda_N, \Lambda_N \) such that the off-shell amplitude remains unchanged at low-energies. While in all practical applications one considers Hermitean cutoff potentials, corresponding to \( \Lambda_1 = \Lambda_2 \), to keep our resulting equations in the most general form we do not impose this condition in our derivation. We start by writing the potential \( V(p, q, k) \) as a sum of various contributions (the choice of which depends on the particular problem one is dealing with)

\[ V(p, q, k) \equiv V_{11}(p, q, k) + V_{12}(p, q, k) + \cdots + V_{1N}(p, q, k) + V_{21}(p, q, k) + V_{22}(p, q, k) + \cdots \]

\[ + V_{2N}(p, q, k) + \cdots + V_{N1}(p, q, k) + V_{N2}(p, q, k) + \cdots + V_{NN}(p, q, k) \]

\[ \equiv (1, 1, \ldots, 1) \begin{pmatrix} V_{11}(p, q, k) & V_{12}(p, q, k) & \cdots & V_{1N}(p, q, k) \\ V_{21}(p, q, k) & V_{22}(p, q, k) & \cdots & V_{2N}(p, q, k) \\ \vdots & \vdots & \ddots & \vdots \\ V_{N1}(p, q, k) & V_{N2}(p, q, k) & \cdots & V_{NN}(p, q, k) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \equiv \bar{U} V(p, q, k) \bar{U}. \]  

(2)

Similarly to the potential, we represent the scattering amplitude as

\[ T(p, q, k) = T_{11}(p, q, k) + T_{12}(p, q, k) + \cdots + T_{1N}(p, q, k) + T_{21}(p, q, k) + T_{22}(p, q, k) + \cdots \]

\[ + T_{2N}(p, q, k) + \cdots + T_{N1}(p, q, k) + T_{N2}(p, q, k) + \cdots + T_{NN}(p, q, k) \]

\[ \equiv (1, 1, \ldots, 1) \begin{pmatrix} T_{11}(p, q, k) & T_{12}(p, q, k) & \cdots & T_{1N}(p, q, k) \\ T_{21}(p, q, k) & T_{22}(p, q, k) & \cdots & T_{2N}(p, q, k) \\ \vdots & \vdots & \ddots & \vdots \\ T_{N1}(p, q, k) & T_{N2}(p, q, k) & \cdots & T_{NN}(p, q, k) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \equiv \bar{U} T(p, q, k) \bar{U}. \]  

(3)

We substitute Eqs. (2) and (3) into Eq. (1) and, removing \( \bar{U} \) and \( U \) corresponding to the initial and final states, demand that the following matrix equation is satisfied

\[ T(p, q, k) = V(p, q, k) + \int \frac{d^3l}{(2\pi)^3} V(p, l, k) U G(k, l) \bar{U} T(l, q, k). \]  

(4)

Next, we introduce the cutoff-dependent potential via

\[ V(p, q, k, \Lambda, \Lambda) = V_{11}(p, q, k, \Lambda, \Lambda) \theta(\Lambda_1 - p) \theta(\Lambda_1 - q) + V_{12}(p, q, k, \Lambda, \Lambda) \theta(\Lambda_1 - p) \theta(\Lambda_2 - q) \]

\[ + \cdots + V_{1N}(p, q, k, \Lambda, \Lambda) \theta(\Lambda_1 - p) \theta(\Lambda_N - q) + V_{21}(p, q, k, \Lambda, \Lambda) \theta(\Lambda_2 - p) \theta(\Lambda_1 - q) + V_{22}(p, q, k, \Lambda, \Lambda) \theta(\Lambda_2 - p) \theta(\Lambda_2 - q) \]

\[ + \cdots + V_{2N}(p, q, k, \Lambda, \Lambda) \theta(\Lambda_2 - p) \theta(\Lambda_N - q) + \cdots + V_{N1}(p, q, k, \Lambda, \Lambda) \theta(\Lambda_N - p) \theta(\Lambda_1 - q) + V_{N2}(p, q, k, \Lambda, \Lambda) \theta(\Lambda_N - p) \theta(\Lambda_2 - q) \]

\[ + \cdots + V_{NN}(p, q, k, \Lambda, \Lambda) \theta(\Lambda_N - p) \theta(\Lambda_N - q) \]
In general, Eq. (8) is a system of integro-differential equations. Eq. (8) reduces to the differential RG equation of Ref. [10].

\[
\begin{pmatrix}
V_{11}(p, q, k, \Lambda) & V_{12}(p, q, k, \Lambda) & \ldots & V_{1N}(p, q, k, \Lambda) \\
V_{21}(p, q, k, \Lambda) & V_{22}(p, q, k, \Lambda) & \ldots & V_{2N}(p, q, k, \Lambda) \\
\vdots & \vdots & \ddots & \vdots \\
V_{N1}(p, q, k, \Lambda) & V_{N2}(p, q, k, \Lambda) & \ldots & V_{NN}(p, q, k, \Lambda)
\end{pmatrix}
\begin{pmatrix}
\theta(\Lambda_1 - p) \\
\theta(\Lambda_2 - p) \\
\vdots \\
\theta(\Lambda_N - p)
\end{pmatrix}
= \left(\frac{\partial}{\partial \Lambda_i}\right)_i \Theta(p) V(p, q, k, \Lambda) \Theta(q),
\]

where \( \Lambda \equiv \{\Lambda_i\} \), \( \bar{\Lambda} \equiv \{\bar{\Lambda}_i\} \), \( p \equiv |p| \), \( q \equiv |q| \), and \( \Theta(x) \) is the Heaviside theta function, by requiring that it satisfies the matrix equation

\[
V(p, q, k, \bar{\Lambda}, \Lambda) = V(p, q, k) + \int \frac{d^3l}{(2\pi)^3} V(p, l, k) \left[ iG(k, l) \bar{\Lambda} - \Theta(l) G(k, l) \Theta(l) \right] V(l, q, k, \bar{\Lambda}),
\]

It then follows from Eqs. (4) and (6) that the off-shell low-energy T-matrix \( T(p, q, k) \) can be obtained by solving the following equation

\[
T(p, q, k) = V(p, q, k, \bar{\Lambda}, \Lambda) + \int \frac{d^3l}{(2\pi)^3} V(p, l, k, \bar{\Lambda}, \Lambda) \theta(l) G(k, l) \bar{\Lambda} T(l, q, k).\]

Any solution of Eq. (6) also satisfies the following system of \( 2N \) RG equations

\[
\frac{\partial V(p, q, k, \bar{\Lambda}, \Lambda)}{\partial \Lambda_i} = - \int \frac{d^3l}{(2\pi)^3} V(p, l, k, \bar{\Lambda}, \Lambda) \frac{\partial \Theta(l) G(k, l) \bar{\Lambda}}{\partial \Lambda_i} V(l, q, k, \bar{\Lambda}),
\]

with \( i = 1, \ldots, N \). While the above derivation of Eq. (8) served mainly for the purpose of demonstrating its physical content, it can be directly obtained from Eq. (7) by demanding cutoff independence of \( T(p, q, k) \). Therefore, for \( V(p, q, k, \bar{\Lambda}, \Lambda) \), satisfying Eqs. (8), the off-shell amplitude \( T(p, q, k) \) obtained from the solution of Eq. (7) is cutoff independent and coincides with the solution of Eq. (1) at low energies, i.e. below all cutoffs \( \Lambda_i \) and \( \bar{\Lambda}_i \).

The case of Hermitian cutoff-dependent potentials corresponds to choosing \( \Lambda_i = \bar{\Lambda}_i \) for all \( i \), so that \( \Theta(x) = (\Theta(x))^T \). Furthermore, for a single cutoff parameter, Eq. (8) reduces to the differential RG equation of Ref. [10]. In general, Eq. (8) is a system of integro-differential equations, however in some cases such as e.g. for separable potentials, it can be reduced to a system of differential equations.

### 3 RG Equation with Two Cutoffs

In exact analogy to the previous section, one can obtain a system of RG equations for the LS equation in partial wave basis

\[
T(p, q, k) = V(p, q, k) + \int dV(p, l, k) G(k, l) T(l, q, k),
\]

where \( G(k, l) = m l^2/(2\pi^2 k^2 - l^2 + i\epsilon) \). The corresponding cutoff regularized potential, defined analogously to Eq. (5), satisfies the following system of RG equations

\[
\frac{\partial V(p, q, k, \Lambda)}{\partial \Lambda_i} = - \int dV(p, l, k) \times \frac{\partial \Theta(l) G(k, l) \bar{\Lambda}}{\partial \Lambda_i} V(l, q, k, \Lambda).
\]

Here and in what follows, we restrict ourselves to the case of Hermitian potentials.

As a simple application, we solve the RG equations with two cutoff parameters, \( \Lambda_1 = \bar{\Lambda}_1 \) and \( \Lambda_2 = \bar{\Lambda}_2 < \Lambda_1 \), as a perturbative power series expansion in the small parameters, \( p, q, k, \) and \( \Lambda_2 \). Specifically, we consider the cutoff regularized potential of the form

\[
V(p, q, k, \Lambda) = V_{11}(k, \Lambda) \theta(\Lambda_1 - p) \theta(\Lambda_1 - q) + V_{12}(p, q, k, \Lambda) \theta(\Lambda_1 - p) \theta(\Lambda_2 - q) + V_{21}(p, q, k, \Lambda) \theta(\Lambda_2 - p) \theta(\Lambda_1 - q) + V_{22}(p, q, k, \Lambda) \theta(\Lambda_2 - p) \theta(\Lambda_2 - q)
\]

\[
\equiv (\Theta(p))^T V_2(p, q, k, \Lambda) \Theta(q),
\]

\[\uparrow\]

While we use the sharp cutoff, our results are equally applicable for the theta functions replaced by smooth regulator functions.
where $V_{21}(p, q, k, \Lambda) = V_{12}(q, p, k, \Lambda)$. We look for $V_2$ as a solution to Eq. (10) in the form of a perturbative expansion in small parameters

$$V_2(p, q, k, \Lambda) = V_{10}(p, q, k, \Lambda) + V_{NLO}(p, q, k, \Lambda) + \cdots$$

where $V_{10}(p, q, k, \Lambda)$ is given by Eq. (12) and $V_{NLO}(p, q, k, \Lambda)$ is obtained by matching Eq. (12) into Eq. (10) and solving order-by-order. We write Eq. (15) as a perturbative expansion valid both for the case of a natural and unnaturally large scattering length, $T(k) = T_{LO} + T_{NLO} + \cdots$, where

$$T_{LO}(k) = -\frac{4\pi}{m} \frac{1}{-1/a + (r/2)k^2 + \cdots - ik},$$

$$T_{NLO}(k) = \frac{2\pi}{m} \frac{r k^2}{(-1/a - ik)^2},$$

By demanding that $V_2$ reproduces $T_{LO}$ and the perturbative inclusion of $V_{NLO}$ generates $T_{NLO}$, we obtain

$$c_{10} = \frac{\pi}{2a}, \quad c_{31} = -\frac{\pi}{a} + \frac{\pi r}{c_{20} m}.$$
energy by introducing a multitude of cutoff parameters. We derive a system of integro-differential equations for the cutoff regularized potential, which reduces to the RG equation of Ref. [10] for the case of a single cutoff. As a simple application, we considered a perturbative solution of the system of RG equations in the form of a power series expansion in momenta and energy. We have demonstrated that by introducing two cutoff parameters, one obtains a perturbative expansion of the potential which follows the Weinberg power counting rules,[2] while as shown in Ref. [10], the usage of a single cutoff parameter leads to the power counting of Refs. [12–14]. This simple example demonstrates that the enlargement of the space of the renormalization group parameters by exploiting the full freedom in the choice of renormalization conditions can be advantageously used in the context of the low-energy EFT for nucleon-nucleon scattering. It will be interesting to apply the presented formalism with the multitude of cutoff parameters to the case of the potentials with a long-range interaction. This work is in progress.

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