On minimal residual entropy in 0 + 1d non-Fermi liquids

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In the large $N$ limit a physical system might acquire a residual entropy at zero temperature even without ground state degeneracy. At the same time poles in the 2-point function might coalesce and form a branch cut. Both phenomena are related to a high density of states in the large $N$ limit. In this short note we address the question: does a branch cut in the 2-point function always lead to non-zero residual entropy? We argue that for generic fermionic systems in 0 + 1 dimensions in the mean-field approximation the answer is positive: branch cut $1/\tau^{2\Delta}$ in the 2-point function does lead to a lower bound $N \log 2(1/2 - \Delta)$ for the entropy. We also comment on higher-dimensional generalizations and relations to the holographic correspondence.

I. MOTIVATION

The large $N$ limit, when the number of interacting fields is taken to infinity, often leads to nice analytical results. However, the limit itself should be taken with great care. In this paper we concentrate on two particular features of this limit. In $0 + 1$ dimensions (quantum mechanics) and at finite $N$, there is always a finite number of energy levels in the system. Hence any two-point function at zero temperature has a spectral decomposition as a sum over poles:

$$\langle O(\omega)O(0) \rangle \sim \sum_n \frac{M_n}{\omega - \omega_n}. \quad (I.1)$$

However, in the large $N$ limit the poles can coalesce and form a branch cut $\omega^{2\Delta - 1}$ (with some fixed $\Delta$) instead. Examples of such behavior include Sachdev–Ye–Kitaev (SYK) model [1–4], its tensor counterparts [5–9], similar Condo models [10] and Banks–Fischler–Shenker–Susskind (BFSS) matrix model [11, 12]. Such branch cut resembles a non-Fermi liquid behavior of higher-dimensional theories. Another feature of the large $N$ limit is the possibility of non-zero residual entropy at zero temperature without ground state degeneracy. This can happen because the density of states is exponential in $N$. Examples again include SYK model and its tensor cousins. Specifically, in

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SYK/tensor models:

\[ G_{\text{SYK}}(\tau) = \langle \psi_i(\tau)\psi_i(0) \rangle \propto \frac{1}{\tau^{2\Delta}}, \quad 1/J \ll \tau, \quad (I.2) \]

and

\[ S_{0,\text{SYK}}/N = \int_0^{1/2 - \Delta} dx \frac{\pi x}{\tan \pi x}. \quad (I.3) \]

Both of these phenomena happen because the density of states is very high. Of course, we should talk about two different densities: for two-point function we are interested in the density of "single-particles" excitations, whereas for the entropy we should talk about the whole spectrum. One needs extra physical input to relate the two. For example, in SYK it is mean-field approximation. For other theories it might be the requirement of having a holographic gravity dual.

The purpose of this short note is to use a mean-field type approximation to argue that a branch cut in fermionic two-point function does imply a lower bound on residual entropy. Specifically, we consider particle-hole symmetric case and the following branch-cut behavior at zero temperature

\[ G_{\beta=\infty}(\tau) = \langle \psi_i(\tau)\psi_i(0) \rangle \propto \frac{1}{\tau^{2\Delta}}, \quad 1/J \ll \tau, \quad (I.4) \]

where \( J \) is some temperature-independent energy scale. Note that the branch cut extends all the way to \( \tau = \infty \) in Euclidean time. This will be important for our analysis and we discuss this momentarily.

We show that such behavior necessarily implies a lower bound on zero-temperature (generalized) entropy \( \tilde{S}_0 \):

\[ \tilde{S}_0/N \geq \log(2) \left( \frac{1}{2} - \Delta \right). \quad (I.5) \]

and \( \tilde{S}_0 \) is defined as:

\[ \tilde{S}_0/N = \lim_{\beta \to \infty} \sum_n \frac{1}{2} \log G_{\beta}(i\omega_n), \quad (I.6) \]

where the sum is taken over Matsubara frequencies \( \omega_n \). The bound is saturated for generalized free fields (defined by eq. \( \text{(II.20)} \)).

Term "generalized entropy" is discussed in detail in Section \( \text{III} \). We will show that for large \( N \) (i.e. mean-field approximation) this quantity is simply the residual entropy plus certain thermal susceptibilities, eq. \( \text{(III.9)} \). Also in Section \( \text{V} \) we discuss why this definition is natural for holographic theories.

\(^1\)This generalized entropy has nothing to do with black hole generalized entropy.
Notice that the branch cut in the Green function was assumed to be at zero temperature, whereas in the equation above we have to deal with a finite temperature Green function. So we will need a few extra assumptions about $T \to 0$ behavior of the Green function. These assumptions are of dynamical nature and are not related to extra symmetries. Assuming extra symmetries actually make this problem "trivial". Let us briefly explain what we mean. In $1 + 1$D CFT the thermal two-point functions in the infinite volume are actually fixed by a conformal symmetry. The form (I.4) suggest some type of scale symmetry. However, in $0 + 1$D the situation is more subtle as all mappings are conformal. In the SYK/tensor model case the actual symmetry group is $SL(2, R)$ which does determine finite-temperature Green function uniquely from zero temperature answer:

$$G \sim (\sin(\pi \tau / \beta))^{-2\Delta}.$$ 

In turn, it completely fixes $\tilde{S}_0$ and the answer is given by eq. (I.3), the corresponding computation is identical to the SYK one. The comparison between $SL(2, R)$ answer (I.3) and our bound (I.5) is presented on Figure 1.

![Graph](graph.png)

Figure 1. Comparison between the conformal $SL(2, R)$-invariant answer (I.3), (blue) and the bound (I.5), (orange).

Presence of $SL(2, R)$ is something special. For example, in BFSS model one expects to find power-law correlators (I.4), but there is no $SL(2, R)$ symmetry. Also it was recently discussed in \cite{13–15} that the boost symmetry(which is a part of $SL(2, R)$) enforces the theory to be maximally chaotic. Here we do not want to assume the presence of $SL(2, R)$ symmetry and instead derive a lower bound on $\tilde{S}_0$.

It is important to emphasize the position of the branch cut in the two-point function. In eq. (I.4) the range of (Euclidean) time $|\tau|$ is $[1/J, +\infty)$. We assume that it is valid for any time separation bigger than the scale $1/J$. In the frequency domain it results in $\omega^{2\Delta-1}$ behavior for any $\omega$ less than $J$. Famously, matrix model resolvents and Green functions do have a branch cut in the
frequency domain, but they do not have residual entropy. The reason for this, is that the branch cut does not extend all the way to $\omega = 0$. For example in the random-hopping model, also known as $q = 2$ SYK, exact answer for the retarded Green function at finite temperature is

$$G_{R,RH}(\omega) \propto \frac{1}{\omega + \sqrt{\omega^2 - 4J^2}}. \tag{I.7}$$

For $|\omega| \ll J$, $G_{R,RH}(\omega)$ is simply $\propto 1/J$ and does not have a branch cut. This model does not have exponentially big in $N$ density of states.

The logic of this paper is following:

- In Section II we carefully examine the zero temperature limit of $\text{Tr} \log G$ and derive the bound on $\tilde{S}_0$.

- Then in Section III we give a general large $N$ argument of why $\text{Tr} \log G$ captures the residual entropy. Also we give a precise relation between the generalized entropy $\tilde{S}_0$ and standard residual entropy $S_0$.

What are the possible caveats in the proposed bound? Up until now, we implicitly assumed that we are computing the fermionic contribution to the residual entropy. Surprisingly, in Section IV we find that a similar bosonic $\text{Tr} \log G$ contributions can give a negative contribution to the residual entropy. We will give an explicit example with supersymmetric SYK model [16].

A related issue concerns the type of large $N$ limit we study. As a rule, certain couplings have to be rescaled in the large $N$ limit. This will be important for us in Section III when we discuss the origin of generalized entropy. However, it is also possible to have a $N$-dependent number of couplings. In this case the analysis of Section III has to be slightly modified and we might need to include extra bosonic contributions which might lower the entropy, even if the original theory is purely fermionic. This is illustrated in Section IV by supersymmetric SYK model. Section VI is dedicated to a brief discussion of the results and open question. In Appendix A we compute the residual entropy in the complex SYK model [17]. We do this to illustrate that one of our technical assumptions (Assumption 2, how $G_R(\omega)$ approaches the zero-temperature limit) is important.

II. THE BOUND FROM BRANCH-CUT

This Section is dedicated to studying $\text{Tr} \log G$. We assume that it has a power-law decay at certain frequencies and then derive a useful bound for its low-temperature behavior. In the next
Section we explain why it is actually related to the entropy.

Following [10], we can rewrite eq. (I.6) in the real frequency domain. We introduce the Fermi function \( n_F(\omega) = \frac{1}{e^{\omega \beta} + 1} \) and convert the residues into two integrals, right above and right below the real axis:

\[
\frac{1}{2} \sum_n \log G(i\omega_n) = \frac{\beta}{4\pi i} \int_{-\infty}^{+\infty} d\omega \ n_F(\omega) \left( \log \frac{G_R(\omega)}{G_A(\omega)} \right). \tag{II.1}
\]

Now the phase of the Green function is important. Let us define it as

\[
G^> = i\langle \psi_i(t)\psi_i(0) \rangle. \tag{II.2}
\]

For Majorana fermions or in the presence of a particle-hole symmetry:

\[
G^<(t) = -G^>(-t) \quad \text{and} \quad [G^>(t)]^* = -G^>(-t). \tag{II.3}
\]

Hence the retarded and advanced Green functions

\[
G_R(t) = \theta(t) \left( G^> (t) + G^> (-t) \right), \tag{II.4}
\]

\[
G_A(t) = -\theta(-t) \left( G^> (t) + G^> (-t) \right), \tag{II.5}
\]

are purely imaginary. From that we infer that

\[
\text{Re} \ G_R(\omega) - \text{odd function},
\]

\[
\text{Im} \ G_R(\omega) - \text{even function}. \tag{II.6}
\]

Note that it does not imply that \( \text{arg} \ G_R \) is an odd function. Rather, we have a relation:

\[
\text{arg} \ G_R(-\omega) = \pi - \text{arg} \ G_R(\omega). \tag{II.7}
\]

A qualitative sketch of \( \text{arg} \ G_R \) is presented in Figure 2. Also there is a general relation

\[
[G_R(\omega)]^* = G_A(\omega). \tag{II.8}
\]

Therefore we can rewrite eq. (II.1) as

\[
\frac{\beta}{2\pi} \int_{-\infty}^{+\infty} d\omega \ n_F(\omega) \left( \text{arg} \ G_R(\omega) \right). \tag{II.9}
\]

Our main assumption about the Green’s function behavior is the following:
Assumption 1. There is temperature-independent scale $\Lambda_{uv}$ below which $G_R$ has a branch-cut singularity:

$$G_R(\omega) = \text{const} \cdot i e^{i \pi (\frac{1}{2} - \Delta)} \cdot \frac{1}{\omega^{1-2\Delta}}, \quad 1/\beta \ll |\omega| \ll \Lambda_{uv}. \quad (\text{II.10})$$

This is essentially saying that there is a branch-cut at $T = 0$ below energy $3\Lambda_{uv}$. The choice of the phase is consistent with eq. (II.6).

Let us return to the computation of the integral (II.9). At very large frequencies $\omega \gg \Lambda_{uv}$ we are expecting free fermion behavior $G_R \sim -1/\omega$. So the integral is convergent. Contribution from $[\Lambda_{uv}, +\infty]$ vanishes exponentially for $\beta \to +\infty$, since

$$\beta \int_{\Lambda_{uv}}^{+\infty} d\omega \; n_F(\omega) = \log \left(1 + e^{-\beta \Lambda_{uv}}\right). \quad (\text{II.11})$$

We need to understand if the interval $[-\infty, -\Lambda_{uv}]$ gives any $\beta^0$ contribution. This question is related to how fast does $G_R(\omega)$ approach its form at zero temperature. If the approach is faster than $1/\beta$ when obviously there are no $\beta^0$ contributions. Hence our next assumption is

Assumption 2. The retarded Green function $G_R(\omega, T)$ at finite temperature approaches its form at zero temperature $G_R(\omega, T=0)$ faster than $T$. In other words,

$$\lim_{T \to 0} \frac{|G_R(\omega, T) - G_R(\omega, 0)|}{T} = 0. \quad (\text{II.12})$$

Roughly this Assumption can be restated as follows: at low energies we have a conformal Green function (II.10). Behavior at higher energies is captured by some conformal perturbation theory with marginal or irrelevant operators. Then we need to require that $\Delta = 1$ operator is not present in this perturbation theory. To demonstrate that this assumption is important, in Appendix A we will compute the residual entropy in complex SYK model where this statement does not hold. We will explicitly see that the interval $[-\infty, -\Lambda_{uv}]$ does give an extra contribution.\(^4\)

So we are left with

$$\frac{\beta}{2\pi} \int_{-\Lambda_{uv}}^{\Lambda_{uv}} d\omega \; n_F(\omega) \left(\arg G_R(\omega)\right). \quad (\text{II.13})$$

The argument of $G_R(\omega)$ goes from $\pi \Delta$ to $\pi - \pi \Delta$ - Figure 2. Equation (II.13) contains a constant shift to the ground state energy. We can see this by noticing that changing the phase of $G_R$ by $\phi_0$ \(^3\)

\(^3\) $\Lambda_{uv}$ does not have to be a coupling constant, it just sets the UV scale. For example in SYK, we can have several terms in the Hamiltonian: $H_{\text{SYK}} \propto J_4 \psi^4 + J_6 \psi^6$. In this case conformal solution $1/\sqrt{\tau}$ is valid up to energies $\Lambda_{uv} = \min(J_4, J_6^3/J_4^2)$.

\(^4\) One underappreciated fact about Majorana SYK is that $|G_R(\omega, T) - G_R(\omega, 0)|$ does go to zero as $T^2$. This can be seen explicitly at large $q$. At finite $q$ this statement can be guessed for $|\omega| \ll J$ from the expansion $\sin^{-2\Delta}(\pi t) = t^{-2\Delta} + \frac{1}{2\pi} t^2 2^{-2\Delta} \Delta T^2 + O(T^4)$ and then checked with numerics for any $\omega$.\(^4\)
Assumption about the power-law form (II.10) implies that \( \arg G \) develops two long plateaus. In the main text we argue that the region \( |\omega| \gtrsim \Lambda_{uv} \) contributes to the ground state energy only.

will change the integral by

\[
\phi_0 \beta \Lambda_{uv} + \mathcal{O}(e^{-\beta \Lambda_{uv}}). \tag{II.14}
\]

In order to eliminate this shift, we need to make sure \( \arg G(\omega) \to 0 \) for \( \omega \to -\Lambda_{uv} \), since \( n_F(\omega) \) does not decrease in this direction. Thanks to the eq. (II.10) we know that this shift is \( \pi \Delta \). So we simply need to take the limit \( \beta \to +\infty \) in

\[
\tilde{S}_0 = \lim_{\beta \to +\infty} \frac{\beta}{2\pi} \int_{-\Lambda_{uv}}^{\Lambda_{uv}} d\omega \ n_F(\omega) \left( \arg G(\omega) - \pi \Delta \right). \tag{II.15}
\]

The crucial observation for obtaining the bound is the following. We can define an odd function of the frequency \( \tilde{\arg G} \):

\[
\tilde{\arg G} = \arg G - \frac{\pi}{2}. \tag{II.16}
\]

Now we need to study

\[
\tilde{S}_0 = \lim_{\beta \to +\infty} \frac{\beta}{2\pi} \int_{-\Lambda_{uv}}^{\Lambda_{uv}} d\omega \ n_F(\omega) \left( \tilde{\arg G}(\omega) + (1 - 2\Delta) \frac{\pi}{2} \right). \tag{II.17}
\]

Notice if we substitute \( n_F(\omega) \) by \( n_F(|\omega|) \), the part with \( \tilde{\arg G} \) will cancel out (since \( \tilde{\arg G} \) is odd). However, this will decrease the value of the integral only if \( \tilde{\arg G} + \pi(1 - 2\Delta)/2 \) is positive everywhere. This is true if the following assumption holds:

**Assumption 3.** \( \arg G(\omega, T = 0) \) is a monotonic function of \( \omega \in [-\Lambda_{uv}, \Lambda_{uv}] \). Or at least \( \arg G - \pi \Delta \) is positive in the interval \( [-\Lambda_{uv}, \Lambda_{uv}] \).

This assumption can be justified by considering \( G_R(\omega) \) at zero temperature and finite \( N \). Its spectral decomposition is given by

\[
G_R(\omega) = \sum_n \left| \langle 0 | \psi_i | n \rangle \right|^2 \frac{1}{\omega - E_n + i0}. \tag{II.18}
\]
In this form $\text{arg} \, G_R$ is just a bunch of spikes at $\omega = E_n$ with their heights determined by $|\langle 0 | \psi_i | n \rangle|^2$.

In reasonable physical systems we expect that this matrix element decays with energy. Therefore, as $N$ grows, the spikes with form a continuous and monotonic curve\(^5\). More generally, it is possible to derive a universal bound on the derivative of $\text{arg} \, G_R$ with respect to the frequency [18]. It would be interesting to use this bound to drop the assumption above.

Finally, we obtain the bound:

$$\widetilde{S}_0 \geq \lim_{\beta \to +\infty} \frac{\beta}{2\pi} \int_{-\Lambda_{uv}}^{\Lambda_{uv}} d\omega \, n_F(|\omega|) (1 - 2\Delta) \frac{\pi}{2} = \log(2) \left( \frac{1}{2} - \Delta \right).$$  \hfill (II.19)

This is the bound for one fermion.

Let us comment that the bound is saturated if $G_R(\omega)$ is proportional to $\omega^{2\Delta - 1}$ for all $\omega$. This is the case of the so-called generalized free fields (GFF). They are characterized by the feature that thermal 2-point function can be obtained via the method of images:

$$G_{\beta = \infty}^{GFF}(\tau) = \sum_n (-1)^n G_{\beta = \infty}^{GFF}(\tau + n\beta).$$  \hfill (II.20)

Holographic theories typically have this property.

### III. $\text{Tr} \log G$ AND (GENERALIZED) ENTROPY

In this Section we relate $\text{Tr} \log G$ to the actual entropy. The argument below is based on [17] but it appeals to large $N$ only, without referring to holography.

Suppose we have a system of $N$ Majorana fermions without spacial structure: every fermion interacts with every other one. The partition function can be written as a path integral in the imaginary time:

$$Z_N = \int D\psi_i \exp \left( - \int d\tau \, \psi_i \partial_\tau \psi_i - H(\psi_1, \psi_2, \ldots) \right).$$  \hfill (III.1)

Let us add another fermion $\psi_0$ to the system. Since the system does not have a spacial structure it is not ambiguous to do that. Now we have

$$Z_{N+1} = \int D\psi_i D\psi_0 \exp \left( - \int d\tau \, \psi_i \partial_\tau \psi_i + \psi_0 \partial_\tau \psi_0 - H(\psi_1, \psi_2, \ldots) - \psi_0 \xi(\psi_1, \ldots) \right).$$  \hfill (III.2)

\(^5\)For example, in Majorana SYK $\text{arg} \, G_R \propto \arctan \left( \frac{\cot(\pi \Delta) \tanh(\omega/2)}{\tanh(\omega/2)} \right)$.
Upon integrating out the original $N$ fermions we get a non-local action for $\psi_0$

$$Z_{N+1} = Z_N \int D\psi_0 \exp \left( \int d\tau_1 d\tau_2 \psi_0(\tau_1)G^{-1}(\tau_1 - \tau_2)\psi_0(\tau_2) + \int d\tau_1 \ldots d\tau_4 K_4(\tau_1, \ldots, \tau_4)\psi_0(\tau_1)\psi_0(\tau_2)\psi_0(\tau_3)\psi_0(\tau_4) + \ldots \right) \quad (\text{III.3})$$

So far we have not used large $N$ anywhere. Now is the time to use it. In the large $N$ limit, interaction terms like $K_4$ will give a connected contributions to $n$-point correlation functions.

We are expecting that they are suppressed in the large $N$ limit compared to the disconnected contribution.

The addition of a single fermion produces

$$\frac{\partial \log Z}{\partial N} = \frac{1}{2} \text{Tr} \log G. \quad (\text{III.4})$$

However, this is not the end of the story, as taking the large $N$ limit almost always involves tuning some of the couplings. We assume that the couplings we tune are $J_1, J_2, \ldots$. Hence the full answer for the derivative with respect to $N$ is

$$\frac{d \log Z}{dN} = \frac{1}{2} \text{Tr} \log G + \sum_{\alpha} \frac{\partial \log Z}{\partial J_{\alpha}} \frac{\partial J_{\alpha}}{\partial N}. \quad (\text{III.5})$$

In the large $N$ limit we expect that $\log Z = -\beta NF$, hence the left hand side is exactly the free energy per fermion. At low temperatures

$$F = E_0 - TS_0 + O(T^2). \quad (\text{III.6})$$

In the right hand side we see the desired $\text{Tr} \log G$ term, but we also have extra terms. Assuming that the Hamiltonian has the form

$$H = H_0 + \sum_{\alpha} J_{\alpha} O_{\alpha}, \quad (\text{III.7})$$

we can rewrite eq. (III.5) as

$$\frac{d \log Z}{dN} = \frac{1}{2} \text{Tr} \log G - \beta \sum_{\alpha} \langle O_{\alpha} \rangle_{\beta} \frac{\partial J_{\alpha}}{\partial N}. \quad (\text{III.8})$$

Then $T^0$ term in the $\text{Tr} \log G$ will compute the following expression

$$\widetilde{S}_0 = S_0 + \sum_{\alpha} \left. \frac{\partial J_{\alpha}}{\partial N} \frac{\partial \langle O_{\alpha} \rangle}{\partial T} \right|_{T=0} \quad (\text{III.9})$$

We will call this quantity generalized entropy.
There is one interesting special case: suppose that the only coupling which gets rescaled with $N$ is $J$ which is also the only dimensionful coupling in the theory. Then it is obvious that $S_0$ does not depend on $J$ and $S_0$ can be extracted from the $\text{Tr} \log G$ term. This is the case of the original SYK model. Interestingly, it is also true for the BFSS matrix model. However, BFSS model does not have zero temperature entropy.

IV. POSSIBLE CAVEATS: $N$-DEPENDENT NUMBER OF COUPLINGS AND BOSONS

In this Section we describe the situations when the proposed bound (I.5) might be violated. So far we have considered only the contribution from fermions. Interestingly, bosons with a branch cut in their 2-point function can contribute negatively to entropy. The physical reason for this is that if the fermions can form a boson, it can condensate and lower the entropy \(^6\).

Also, notice that in Section III we have taken into account the possibility of $N$-dependent coupling strength, but not $N$-dependent number of couplings. As we will see in a moment, such situation can lead to extra bosons, thus lowering the entropy. Let us start from such example.

A. $N$-dependent number of couplings

Let us say more precisely what we mean by the $N$-dependent number of couplings. This correspond to the situation, when increasing $N$ by one (adding one extra fermion) leads to an extra term involving $N$ ”old” fermions only:

$$Z_{N+1} = \int D\psi_i D\psi_0 \exp \left( -\int d\tau \psi_i \partial_\tau \psi_i + \psi_0 \partial_\tau \psi_0 - H(\psi_1, \psi_2, \ldots) - \psi_0 \xi(\psi_1, \ldots) - \eta(\psi_1, \ldots, \psi_N) \right).$$

One such example is $\mathcal{N} = 1$ supersymmetric SYK model [16]. Its Hamiltonian reads as

$$\mathcal{L}_{SUSY SYK} = \sum_{i=1}^{N} \psi_i \partial_\tau \psi_i + \sum_{i,j,m,n=1}^{N} C^l_{ij} C^l_{mn} \psi^i \psi^j \psi^m \psi^n,$$

where $C^l_{ij}$ has three indices and each of them goes from 1 to $N$. Moreover its components are independent identically distributed Gaussian variables. Increasing $N$ to $N + 1$ leads to an extra term

$$\eta(\psi_1, \ldots, \psi_N) = \sum_{1 \leq i,j,m,n \leq N} C^0_{ij} C^0_{mn} \psi^i \psi^j \psi^m \psi^n.$$

\(^6\)The author is grateful to Leon Balents for suggesting this picture. Similarly, it was observed in [19, 20] that dressing photon degrees of freedom can lower the entropy.
This type of interaction can be split into halves by introducing an extra bosonic field $b^l$:

$$\mathcal{L}_{SUSY\ \text{SYK}} = \sum_i \psi_i \partial_\tau \psi_i - \sum_l \frac{1}{2} b^l + \sum_{ijkl} C_{ijkl}^l b^l \psi_i \psi_j. \quad (IV.4)$$

Now the number of couplings is not $N$-dependent in the above sense. As we will show now, such emergent bosons are capable of decreasing the entropy.

**B. Bosons**

Following the logic of Section III, the contribution to the generalized entropy from a boson can be captured by the determinant:

$$\Delta \tilde{S}_0 = -\frac{1}{2} \text{Tr} \log G_b = -\frac{\beta}{2\pi} \mathcal{P} \int_{-\infty}^{+\infty} d\omega \ n_B(\omega) \ \text{arg} \ G_b(\omega). \quad (IV.5)$$

Interestingly, this can contribute *negatively* to $\tilde{S}_0$. Specifically, we assume a similar non-Fermi liquid like behavior plus $SL(2,R)$ symmetry:

$$G_{b,SL(2,R)}(\tau) \sim \left(\frac{1}{\sin \left(\pi \tau / \beta \right)}\right)^{2\Delta_b}, \quad (IV.6)$$

then

$$\text{arg} \ G_{b,SL(2,R)}(\tau) = \arctan(\cot(\Delta_b \pi) \ \tanh(\beta \omega / 2)) + \text{const} . \quad (IV.7)$$

For SUSY SYK $\Delta_b = 2/3$. In order to remove the contribution to the ground state energy one needs to shift the phase such that the integral is convergent. In order to compute $\mathcal{P}$ integral one can symmetrize with respect to $\omega$. This way one obtains

$$\Delta \tilde{S}_{0,SL(2,R)} = \frac{1}{4\pi} \int_{-\infty}^{+\infty} d\omega \left[ \arctan(\cot(\Delta_b \pi) \ \tanh(\omega / 2)) \ \coth(\omega / 2) - \arctan(\cot(\pi \Delta_b)) \right]. \quad (IV.8)$$

It is easy to compute this integral numerically can see that in the range $1/2 < \Delta_b < 1$ it is always negative. For example, for SUSY SYK it is $-0.0161376$.

In the fermionic case it was possible to obtain the bound simply from asymptotic of $\text{arg} \ G$. Here it is not the case. For example, consider the following model behavior:

$$\text{arg} \ G_b = \arctan(\cot(\Delta_b \pi) \ \tanh(a \beta \omega / 2)) + \text{const} . \quad (IV.9)$$

By varying parameter $a$ it is possible to change the entropy by arbitrary amount. However, the above example corresponds to unphysical Green’s function $\left(1/\sin \left(\alpha^{-1} \pi \tau / \beta \right)\right)^{2\Delta_b}$ which does not satisfy KMS condition. So maybe it is possible to obtain the bound in the bosonic case, but it will require more sophisticated analysis which includes KMS condition.
V. A COMMENT ON HOLOGRAPHIC THEORIES

In holographic correspondence \[21\] we expect that at large \(N\) some (boundary) theories are dual to weakly coupled gravity in higher dimension with Anti-de Sitter (AdS) boundary conditions. From this standpoint the behavior \(G_{\text{boundary}} \sim 1/\tau^{2\Delta}\) at short times is typical for holographic states: small time separations probe the geometry in the near-AdS region where one expects the conformal answer. Large time separations probe the geometry further where the answer might be complicated. This is illustrated by Figure 3. The bound we derived is not sensitive to the geometry deep inside.

The interaction strength in the bulk in governed by \(1/N\). Therefore from holographic point of view, the thermodynamics at large \(N\) should be determined by bulk geometry(like area of the horizons) plus 1-loop determinants of matter:

\[-\beta F = S_{\text{cl gravity}} + \text{Tr log } G_{\text{bulk}} + O(1/N).\]  

Therefore, in holography we directly arrive at the statement that the free energy contains \(\text{Tr log } G_{\text{bulk}}\) term.

However, this is not a full story in the holographic case. Typically in holographic theories the number of bulk fields does not scale with \(N\), so the determinant part is subleading compared to the large Bekenstein–Hawking entropy of black hole horizon(which does scale with \(N\)).

Moreover, eq. (V.1) instructs us to consider the Green’s function in the dual bulk geometry, instead of the boundary ones. For free fields in AdS these \(\text{Tr log}\) are not equal. In general, bulk matter can be quantized with two boundary conditions (Dirichlet and Neumann) and the following relation holds \[22\]:

\[\text{Tr log } G^\text{Dirichlet}_{\text{bulk}} - \text{Tr log } G^\text{Neumann}_{\text{bulk}} = \text{Tr log } G_{\text{boundary}}.\]  

(V.2)
So the result we obtained in this paper is not about a single bulk theory, but rather the difference in entropies between the Dirichlet and Neumann quantizations. It would be interesting to extend the results of this paper to bulk Green functions.

It is worth discussing $N$ dependence. In BFSS\textsuperscript{7} there are actually a lot of operators having a power-law decay

$$
\langle O(t)O(t') \rangle \propto \frac{1}{|t - t'|^{2\nu + 1}}, \quad \lambda^{-1/3} \ll |t - t'| \ll \lambda^{-1/3} N^{10/21},
$$

(V.3)

and so having the branch-cuts in 2-point functions\cite{23–26}. However, they are $SU(N)$ singlets and their number does not scale with $N$. Correspondingly, BFSS model is not expected to have a residual entropy proportional to $N$. There is a version of BFSS model where $SU(N)$ symmetry is ungauged\cite{27} so one can ask questions about the correlation function of non-singlet operators. However, it has been argued\cite{27, 28} that such operators do not have a power-law decay (non-singlets are gapped). Correspondingly is there is no residual entropy even in the ungauged theory\cite{28}.

\section{VI. DISCUSSION}

In this paper we argued that a branch-cut $\omega^{2\Delta - 1}$ extending all the way to $\omega = 0$ in a fermionic two-point function at zero temperature puts a lower bound on residual entropy (or generalized entropy more generally). We used a few assumptions. The main one being of a mean-field type to justify the extraction of the entropy from $\text{Tr} \log G$. Also we needed a technical Assumption 2 to control how fast the thermal Green function approaches zero temperature limit. This assumption is violated in the complex SYK which results in a different residual entropy, which we explain in Appendix A. In conventional quantum field theories it might be challenging to estimate how $G$ approaches $T = 0$ limit, but in holographic theories this is simply controlled by how fast the spacetime metric approaches $T = 0$ limit. In this case the bound is still useful, as computing the whole matter determinant in a curved background is generally very difficult. Another input we needed is Assumption 3 about the behavior of $\arg G_R$. In the main text we briefly commented that this property is related to how fast certain matrix elements decay with energy. It would be very interesting to make these arguments more sharp.

Finally, let us comment on possible higher-dimensional generalizations. We again can try to

\textsuperscript{7}Strictly speaking, our results are not applicable for BFSS, as there one should compute $\text{Tr} \log G_{\text{bulk}}^{\text{Dirichlet}}$ rather than $\text{Tr} \log G_{\text{boundary}}$. 

look at the $\text{Tr} \log G$ term. In this case $\text{Tr} \log G$ obviously contains a sum over spacial momenta:

$$\text{Tr} \log G \propto \sum_{n,p} \log G(i\omega_n, p). \quad (VI.1)$$

If there is a branch-cut $\omega^2 \Delta^{-1}$ in the frequency domain for all spacial momenta $p$, then the results of this paper naturally lead a lower bound on residual entropy density. In other words, to extensive residual entropy. Unfortunately, the only such theory known to the author is a family of SYK chains. Chiral SYK [29] does not have this property. Another interesting 2d analogue of SYK [30] has non-standard fermion kinetic terms and so the results of this paper are not applicable. Reference [31] discusses the possibility of sub-extensive residual entropy. This is realizable in our framework: if for some fraction of momenta $p$ in the sum (VI.1) there is a branch cut, then we can easily obtain a lower bound. The resulting lower bound will be sub-extensive in volume with the exact volume-dependence determined by the corresponding fraction of momenta with branch cuts. Presumably, even for very small momenta the branch cut will terminate at $\omega \sim \rho^\alpha$, $\alpha = \text{const}$. However, if $\rho$ itself is some inverse power of the volume then the arguments in this paper might be applicable. We leave the detailed study of this possibility for future work.

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Appendix A: UV piece in complex SYK

In complex SYK model in the presence of the chemical potential $\mu$ it is more convenient to fix the spectral asymmetry parameter $\mathcal{E}$ which is related to the chemical potential as

$$\mu = \mu_0 + 2\pi T \mathcal{E}. \quad (A.1)$$

---

8We thank S. Sachdev for pointing this out.
Hence, the grand canonical thermodynamic potential $\Omega$ is defined by

$$d\Omega = (S - 2\pi \mathcal{E}Q)dT - 2\pi TQd\mathcal{E}. \quad (A.2)$$

Therefore we will compute $\mathcal{G} = S - 2\pi \mathcal{E}Q$ instead of $S$.

Usually $\mathcal{G}$ at zero temperature, $\mathcal{G}_0 = \mathcal{G}(T = 0)$, is obtained by first obtaining the following answer for the derivative [10]:

$$\frac{d\mathcal{G}_0}{d\mathcal{E}} = -2\pi Q. \quad (A.3)$$

In [17] it was argued that the answer for the entropy can be obtained by directly computing the Tr log. However, the actual calculation in [17] lifted this Tr log to $AdS_2$ space with an electric field which made the computation quite involved. In this Section we perform a simple direct (boundary) computation of the entropy.

Let us start from specifying the details of the model. We will list only the properties we need for the entropy calculation. We refer to [17] for a detailed discussion.

The Hamiltonian of the simplest complex SYK model is given by

$$H_{\text{complex SYK}} = \sum_{j_1 < \cdots < j_{q/2}; k_1 < \cdots < k_{q/2}} J_{j_1 \ldots j_{q/2}; k_1 \ldots k_{q/2}} \bar{\psi}_{j_1} \ldots \bar{\psi}_{j_{q/2}} \psi_{k_1} \ldots \psi_{k_{q/2}}. \quad (A.4)$$

At low energies the model has the following Matsubara two-point function in imaginary time

$$G(\tau) = -\langle T\psi_j(\tau)\bar{\psi}_j(0) \rangle = c_1 \exp \left(2\pi \mathcal{E} \left(\frac{1}{2} - \frac{\tau}{\beta}\right) \right) \left(\sin \left(\frac{\pi \tau}{\beta}\right) \right)^{-2\Delta}, \quad (A.5)$$

and in Euclidean frequencies $\omega_n = \frac{2\pi}{\beta} \left(n + \frac{1}{2}\right)$:

$$G(\pm i\omega_n) = \mp i c_2 e^{\mp i\theta} \Gamma \left(\frac{\beta \omega_n}{2\pi} + \Delta \pm i\mathcal{E} \right) \Gamma \left(-\frac{\beta \omega_n}{2\pi} + \Delta \mp i\mathcal{E} \right) \sin \left(\frac{\beta \omega_n}{2\pi} + \pi \Delta \pm i\mathcal{E} \right), \quad (A.6)$$

where

$$\Delta = \frac{1}{q}, \quad (A.7)$$

and $c_{1,2}$ are positive real constants.

Notice that for non-zero $\mathcal{E}$, the Green function does not approach its zero-temperature limit faster than $T$:

$$\exp \left(2\pi \mathcal{E} \left(\frac{1}{2} - \frac{\tau}{\beta}\right) \right) \sin^{-2\Delta}(\pi t) =$$

$$= e^{\mathcal{E}\pi \tau^{-2\Delta} - 2\pi \mathcal{E} e^{\mathcal{E}\pi \tau^{1-2\Delta} T} + \frac{1}{3} e^{\mathcal{E}\pi \tau^{2-2\Delta} (6\mathcal{E}^2 + \Delta) T^2 + O(T^3)}. \quad (A.8)$$

---

9We are omitting bilinear terms which make the Hamiltonian particle-hole symmetric. These terms do not matter in the large $N$ limit.
Looking at eq. (A.6) we see that $E$ simply shifts the frequency in the real domain, such that the retarded Green’s function is given by
\[
G_R(\omega) = -ie^{i\theta} \Gamma \left( \frac{i\beta \omega}{2\pi} + \Delta + iE \right) \Gamma \left( -\frac{i\beta \omega}{2\pi} + \Delta - iE \right) \sin \left( \frac{i\beta \omega}{2} + \pi \Delta + i\pi E \right),
\]
(A.9)
and its phase is
\[
\arg G_R = -\theta + \arctan \left( \cot(\pi \Delta) \tanh \left( \frac{\beta \omega}{2} + \pi E \right) \right).
\]
(A.10)
The last piece of information we will need is the relation between $\theta$ and $E$:
\[
e^{-2i\theta} = \frac{\cos (\pi \Delta + i\pi E)}{\cos (\pi \Delta - i\pi E)}.
\]
(A.11)
Now we have enough information to compute the entropy via $\text{Tr} \log G$. First of all, now we have $G = \langle \psi \bar{\psi} \rangle$ and $\bar{G} = \langle \bar{\psi} \psi \rangle$ which differ by $E \rightarrow -E$. But we still have the relation $G_R^* = G_A$, hence in the real frequency domain we have
\[
\frac{\beta}{2\pi} \int_{-\infty}^{\infty} d\omega \ n_F(\omega) \left( \arg G_R(\omega, E) + \phi_0 \right) + [E \rightarrow -E],
\]
(A.12)
where the constant shift $\phi_0$ is needed to make the integral convergent at $\omega \rightarrow -\infty$.

Again, the integral can be separated into two parts: UV part $|\omega| \gtrsim \Lambda_{uv}$ where Green’s functions can be approximated by their non-interacting form and IR part $|\omega| \lesssim \Lambda_{uv}$ where we can use the conformal answer. UV part does not contribute. However, what is $\Lambda_{uv}$ in our case?

In the particle-hole symmetric case (for example, in the original SYK model with quartic interaction only), $\Lambda_{uv} \propto J_4$. However, as can be directly seen from eq. (A.10), the spectral asymmetry parameter $E$ acts by shifting the frequency. Therefore physically the UV cut-off depends on $E$. We can trust the conformal expression (A.10) in the frequency range $[-\Lambda_{uv} + 2\pi E/\beta, \Lambda_{uv} - 2\pi E/\beta]$, where $\Lambda_{uv}$ is some fixed scale which is $\beta$ and $E$ independent. We see that the actual UV cut-off becomes $E$ and $\beta$ dependent.

The above discussion is important only for large negative $\omega$ where the integral in not suppressed by $n_F(\omega)$. As follows from the above discussion, $G_R(\omega, E)$ and $G_R(\omega, -E)$ have different cut-offs and, moreover, the cut-off has $1/\beta$ piece. Previously we argued that the shift $\phi_0$ is responsible for the ground state energy. Here it does contribute to the entropy. Specifically, notice that the phase shift $\theta$ is $E$-odd according to eq. (A.11). So it will not cancel out after adding $-E$ contribution. We conclude that the entropy does receive the contribution
\[
G_{UV} = +2E\theta,
\]
(A.13)
from the UV region because the UV cut-off is IR sensitive.

The rest of the entropy comes from IR region and can be computed as

\[
G_{\text{IR}} = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \arctan \left( \frac{\cot(\pi \Delta) \tan \left( \frac{\omega}{2} + \pi \mathcal{E} \right)}{\omega + 1} \right) + \frac{(1 - 2\Delta)\pi/2}{\omega + 1} + [\mathcal{E} \to -\mathcal{E}].
\]  
(A.14)

To check that we have obtained the correct answer we can compute the derivative:

\[
\frac{\partial G_{\text{UV}}}{\partial \mathcal{E}} = 2\theta + \mathcal{E} \pi (\tan(\pi \Delta - i\pi \mathcal{E}) + \tan(\pi \Delta + i\pi \mathcal{E})),
\]  
(A.15)

\[
\frac{\partial G_{\text{IR}}}{\partial \mathcal{E}} = -\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{\pi \sin(2\pi \Delta)}{\cos(2\pi \Delta) - \cosh(2\pi \mathcal{E} + \omega)} \frac{1}{\omega + 1} + [\mathcal{E} \to -\mathcal{E}].
\]  
(A.16)

The last integral can be computed explicitly. Combining everything together, we get

\[
\frac{\partial G_0}{\partial \mathcal{E}} = \frac{\partial}{\partial \mathcal{E}} (G_{\text{UV}} + G_{\text{IR}}) = 2\theta - i\pi \left( \frac{1}{2} - \Delta \right) (\tan(\pi \Delta + i\pi \mathcal{E}) - \tan(\pi \Delta - i\pi \mathcal{E})).
\]  
(A.17)

This is the correct expression for the derivative [17].

[1] Subir Sachdev and Jinwu Ye. Gapless spin fluid ground state in a random, quantum Heisenberg magnet. *Phys. Rev. Lett.*, 70:3339, 1993. doi:10.1103/PhysRevLett.70.3339

[2] Subir Sachdev. Bekenstein-Hawking Entropy and Strange Metals. *Phys. Rev.*, X5(4):041025, 2015. doi:10.1103/PhysRevX.5.041025

[3] Alexei Kitaev. A simple model of quantum holography. http://online.kitp.ucsb.edu/online/entangled15/kitaev/, http://online.kitp.ucsb.edu/online/entangled15/kitaev2/. Talks at KITP, April 7, 2015 and May 27, 2015.

[4] A. Georges, O. Parcollet, and S. Sachdev. Quantum fluctuations of a nearly critical heisenberg spin glass. *Phys. Rev. B*, 63:134406, March 2001. doi:10.1103/PhysRevB.63.134406 URL http://link.aps.org/doi/10.1103/PhysRevB.63.134406

[5] Razvan Gurau. Colored Group Field Theory. *Commun. Math. Phys.*, 304:69–93, 2011. doi:10.1007/s00220-011-1226-9

[6] Razvan Gurau and Vincent Rivasseau. The 1/N expansion of colored tensor models in arbitrary dimension. *Europhys. Lett.*, 95:50004, 2011. doi:10.1209/0295-5075/95/50004.

[7] Edward Witten. An SYK-Like Model Without Disorder. 2016.

[8] Igor R. Klebanov and Grigory Tarnopolsky. Uncolored random tensors, melon diagrams, and the Sachdev-Ye-Kitaev models. *Phys. Rev.*, D95(4):046004, 2017. doi:10.1103/PhysRevD.95.046004

[9] Igor R. Klebanov and Grigory Tarnopolsky. On Large N Limit of Symmetric Traceless Tensor Models. *JHEP*, 10:037, 2017. doi:10.1007/JHEP10(2017)037

[10] Olivier Parcollet, Antoine Georges, Gabriel Kotliar, and Anirvan Sengupta. Overscreened multichannel SU(N) Kondo model: Large-N solution and conformal field theory. *Phys. Rev. B*, 58:3794, 1998.
[11] Tom Banks, W. Fischler, S. H. Shenker, and Leonard Susskind. M theory as a matrix model: A Conjecture. *Phys. Rev.*, D55:5112–5128, 1997. [doi:10.1103/PhysRevD.55.5112](https://doi.org/10.1103/PhysRevD.55.5112).

[12] Nissan Itzhaki, Juan Martin Maldacena, Jacob Sonnenschein, and Shimon Yankielowicz. Supergravity and the large N limit of theories with sixteen supercharges. *Phys. Rev.*, D58:046004, 1998. [doi:10.1103/PhysRevD.58.046004](https://doi.org/10.1103/PhysRevD.58.046004).

[13] Mike Blake, Hyunseok Lee, and Hong Liu. A quantum hydrodynamical description for scrambling and many-body chaos. *JHEP*, 10:127, 2018. [doi:10.1007/JHEP10(2018)127](https://doi.org/10.1007/JHEP10(2018)127).

[14] Mike Blake, Richard A. Davison, Sašo Grozdanov, and Hong Liu. Many-body chaos and energy dynamics in holography. *JHEP*, 10:035, 2018. [doi:10.1007/JHEP10(2018)035](https://doi.org/10.1007/JHEP10(2018)035).

[15] Mike Blake and Hong Liu. On systems of maximal quantum chaos. *JHEP*, 05:229, 2021. [doi:10.1007/JHEP05(2021)229](https://doi.org/10.1007/JHEP05(2021)229).

[16] Wenbo Fu, Davide Gaiotto, Juan Maldacena, and Subir Sachdev. Supersymmetric SYK models. 2016.

[17] Yingfei Gu, Alexei Kitaev, Subir Sachdev, and Grigory Tarnopolsky. Notes on the complex Sachdev-Ye-Kitaev model. *JHEP*, 02:157, 2020. [doi:10.1007/JHEP02(2020)157](https://doi.org/10.1007/JHEP02(2020)157).

[18] Pengfei Zhang, Yingfei Gu, and Alexei Kitaev. An obstacle to sub-AdS holography for SYK-like models. *JHEP*, 21:094, 2020. [doi:10.1007/JHEP03(2021)094](https://doi.org/10.1007/JHEP03(2021)094).

[19] P. Romatschke. Fractional Degrees of Freedom at Infinite Coupling in Large Nf QED in 2+1 Dimensions. *Phys. Rev. Lett.*, 123(24):241602, 2019. [doi:10.1103/PhysRevLett.123.241602](https://doi.org/10.1103/PhysRevLett.123.241602).

[20] Chunxiao Liu, Gábor B. Halász, and Leon Balents. Symmetric u(1) and z2 spin liquids on the pyrochlore lattice. *Physical Review B*, 104(5), Aug 2021. ISSN 2469-9969. [doi:10.1103/physrevb.104.054401](https://doi.org/10.1103/physrevb.104.054401) URL http://dx.doi.org/10.1103/PhysRevB.104.054401.

[21] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Int. J. Theor. Phys.*, 38:1113–1133, 1999. [doi:10.1023/A:1026654312961](https://doi.org/10.1023/A:1026654312961) [Adv. Theor. Math. Phys.2,231(1998)].

[22] Thomas Hartman and Leonardo Rastelli. Double-trace deformations, mixed boundary conditions and functional determinants in AdS/CFT. *JHEP*, 01:019, 2008. [doi:10.1088/1126-6708/2008/01/019](https://doi.org/10.1088/1126-6708/2008/01/019).

[23] Yasuhiro Sekino and Tamiaki Yoneya. Generalized AdS / CFT correspondence for matrix theory in the large N limit. *Nucl. Phys. B*, 570:174–206, 2000. [doi:10.1016/S0550-3213(99)00793-2](https://doi.org/10.1016/S0550-3213(99)00793-2).

[24] Yasuhiro Sekino. Supercurrents in matrix theory and the generalized AdS / CFT correspondence. *Nucl. Phys. B*, 602:147–171, 2001. [doi:10.1016/S0550-3213(01)00126-2](https://doi.org/10.1016/S0550-3213(01)00126-2).

[25] Ingmar Kanitscheider, Kostas Skenderis, and Marika Taylor. Precision holography for non-conformal branes. *JHEP*, 09:094, 2008. [doi:10.1088/1126-6708/2008/09/094](https://doi.org/10.1088/1126-6708/2008/09/094).

[26] Masanori Hanada, Jun Nishimura, Yasuhiro Sekino, and Tamiaki Yoneya. Monte Carlo studies of Matrix theory correlation functions. *Phys. Rev. Lett.*, 104:151601, 2010. [doi:10.1103/PhysRevLett.104.151601](https://doi.org/10.1103/PhysRevLett.104.151601).

[27] Juan Maldacena and Alexey Milekhin. To gauge or not to gauge? *JHEP*, 04:084, 2018. [doi:10.1007/JHEP04(2018)084](https://doi.org/10.1007/JHEP04(2018)084).
[28] Evan Berkowitz, Masanori Hanada, Enrico Rinaldi, and Pavlos Vranas. Gauged And Ungauged: A Nonperturbative Test. *JHEP*, 06:124, 2018. doi:10.1007/JHEP06(2018)124.

[29] Biao Lian, S. L. Sondhi, and Zhenbin Yang. The chiral SYK model. *JHEP*, 09:067, 2019. doi:10.1007/JHEP09(2019)067.

[30] Gustavo Turiaci and Herman Verlinde. Towards a 2d QFT Analog of the SYK Model. *JHEP*, 10:167, 2017. doi:10.1007/JHEP10(2017)167.

[31] Debanjan Chowdhury, Antoine Georges, Olivier Parcollet, and Subir Sachdev. Sachdev-Ye-Kitaev models and beyond: Window into non-Fermi liquids. *Rev. Mod. Phys.*, 94(3):035004, 2022. doi:10.1103/RevModPhys.94.035004.