The averaged null energy condition on holographic evaporating black holes

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Abstract: We examine the averaged null energy condition (ANEC) for strongly coupled fields, along the event horizon of an evaporating black hole by using the AdS/CFT duality. First, we consider a holographic model of a 3-dimensional evaporating black hole with a perturbed 4-dimensional black droplet geometry as the bulk dual, and investigate how negative energy flux going into the boundary black hole horizon appears. We show that the ingoing negative energy flux always appears at the boundary black hole horizon when the horizon area decreases. Second, we test the ANEC in a holographic model whose boundary geometry is a 4-dimensional asymptotically flat spacetime, describing the formation and subsequent evaporation of a spherically symmetric black hole. By applying the “bulk-no-shortcut principle”, we show that the ANEC is always satisfied when the local null energy is averaged with a weight function along the incomplete null geodesic on the event horizon from beginning of the formation to the final instant of the black hole evaporation. Our results indicate that the total ingoing negative energy flux is compensated by a large amount of positive energy flux in the early stage of the black hole formation.

Keywords: AdS-CFT Correspondence, Black Holes

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1 Introduction

Black hole evaporation is one of the most mysterious phenomena in quantum gravity. Hawking’s discovery [1] that a black hole evaporates by emitting thermal radiation is based on quantum field theory in a fixed curved background. There have been a number of calculations of the vacuum expectation value of the stress-energy tensor for quantum fields on various black hole backgrounds. In particular, negative energy flux going into the black hole horizon was found in a two-dimensional model of gravitational collapse for the Unruh vacuum state [2, 3]. This effect is consistent with the thermal radiation from the black hole observed at spatial infinity, according to the energy conservation. Because of the technical difficulties, however, most of the calculations are restricted to free field theories with no interaction.

For interacting fields such as a strongly coupled field, the AdS/CFT correspondence [4] provides a useful tool to investigate the Hawking radiation on a fixed black hole background. So far, several models of AdS bulk solutions associated with boundary black holes have been numerically constructed [5–9]. The black droplet is such an AdS bulk black hole solution in which the boundary black hole horizon is disconnected from the bulk horizons [10]. When there are no non-extremal bulk horizons isolated from the AdS boundary, the black droplet solution is regarded as the gravity dual to the Unruh vacuum state in the boundary field.
theory, as the negative energy induced by a vacuum polarisation effect decays to zero at the spatial infinity [5].

One peculiar behavior in strongly coupled fields in Unruh vacuum state is that there is no negative energy flux through the horizon, except two dimensional conformal field theory [11]. This would be explained by the formation of quantum “hair” around the black hole [5]. Due to its strong attractive self-interaction, the quantum field tends to collapse into the black hole, and it is balanced to the radiation pressure from the black hole. Although this picture is quite interesting, we need further investigations on a more general non-stationary black droplet solution containing a time-dependent boundary black hole, since most of the calculations so far made are based on the assumption that the AdS bulk spacetime is time-independent.

In this paper, we consider evaporating black holes constructed on time-dependent boundaries of some asymptotically AdS bulk spacetime, for which, as shown in section 2, the null-null component of the boundary Ricci tensor \( R_{\mu\nu}l^\mu l^\nu \) is found to be negative, where \( l^\mu \) is the tangent vector to a null geodesic generator of the boundary black hole horizon. This does not immediately imply that the null-null component of the vacuum expectation value of the boundary stress-energy tensor \( \langle T_{\mu\nu} \rangle l^\mu l^\nu \) is necessarily negative because the AdS/CFT correspondence itself does not provide any dynamical equations of motion such as \( R_{\mu\nu}l^\mu l^\nu = 8\pi G \langle T_{\mu\nu} \rangle l^\mu l^\nu \) on the boundary. However, we can still expect that \( \langle T_{\mu\nu} \rangle l^\mu l^\nu \) is negative, or negative energy flux across the horizon appears when the horizon area decreases from the perspective of the conjectured generalized second law (GSL) as follows. The GSL states that the sum of the gravitational and the matter entropy outside the black hole cannot decrease even during the evaporation process [1, 13]. The GSL is widely believed to be true and it was established for super-renormalizable quantum field theories [14] (for a holographic setting, see also [15]). If the GSL is always satisfied for any quantum field theories, ingoing negative energy flux should appear on the horizon when the area decreases, in accord with the energy conservation law. In this paper, we consider two such time-dependent holographic models; one is a 3-dimensional circular symmetric boundary black hole and the other a 4-dimensional spherically symmetric boundary black hole. In the 3-dimensional case, we will explicitly show that ingoing negative energy flux always appears on the horizon when the area decreases, in accord with the GSL picture.

For the 3-dimensional time-dependent model, we start from the 4-dimensional static black droplet solution [16] as our background bulk geometry, whose boundary metric is conformal to the 3-dimensional Banados-Teitelboim-Zanelli (BTZ) black hole [17]. Then, we consider metric perturbations on this background so that the perturbed geometry describes a regular bubble in the bulk and an evaporating black hole on the boundary. We are concerned with what happens when we dynamically perturb the static BTZ black hole with Unruh vacuum. As expected from the above observation, we show that a negative energy flux going into the boundary black hole horizon always appears when the horizon area decreases under the adiabatic approximation. From the perspective of the AdS/CFT

\footnote{On the non-equilibrium thermal state, heat flow between two boundary black holes occurs for a non-stationary four-dimensional bulk solution [12].}
correspondence, the stress-energy tensor on the boundary field theory depends on both of the boundary geometry and the boundary condition deep inside the bulk. Since a regularity condition is imposed on the bubble radius in our model as a natural boundary condition, the stress-energy tensor is only determined by the boundary geometry. So, our results indicate that the negative energy flux reflects the negative value of $R_{\mu\nu}l^\mu l^\nu$ on the boundary black hole horizon.

Another key question regarding the negative energy flux is whether there is a lower bound for the flux. Presumably, the absolute value of the total negative energy flux would be bounded from above by the initial black hole mass by energy conservation law. To evaluate the total amount of the flux, we consider a holographic model in which the AdS boundary metric is conformal to a 4-dimensional asymptotically flat spherically symmetric metric which describes an evaporating black hole formed by a gravitational collapse. The null geodesic generator of the event horizon begins at a regular spacetime point and ends at a zero mass naked singularity, where the evaporation is completed. The behavior of the null geodesic congruence is very similar to the one on the spatially compact $S^3$ universe in which the null geodesic congruence expands from a point on the south pole and shrinks again to a point on the north pole. In such spatially compact universes, it was shown that the averaged null energy condition (ANEC) with a weight function is satisfied along the null geodesic under the “no-bulk-shortcut condition”, stating that no bulk causal curve can travel faster than the boundary achronal null geodesics [18, 19]. When the null energy condition is satisfied in the bulk spacetime, the no-bulk-shortcut condition is satisfied under the assumptions that there are no pathological behavior such as a naked singularity formation in the bulk and the boundary [20]. By applying the no-bulk-shortcut condition to the holographic model of the evaporating black hole, we show that the ANEC is satisfied.

The paper is organized as follows. In section 2, we consider the 3-dimensional evaporating black hole on the boundary of the perturbed 4-dimensional static black droplet solution and derive the negative energy flux going into the boundary black hole horizon. In section 3, we derive the ANEC with a weight function in a background of a 4-dimensional evaporating black hole in the context of the AdS/CFT correspondence. Section 4 is devoted to summary and discussions.

2 3-dimensional evaporating black hole and negative energy flux

In this section, we construct a 4-dimensional asymptotically locally AdS spacetime with a 3-dimensional time-dependent black hole on the AdS boundary, describing the evaporation of a black hole. We assume that the evaporating process of our boundary black hole proceeds very slowly so that the boundary black hole can be viewed as almost static. Then, the corresponding 4-dimensional bulk dual should also be well described by an almost static or stationary black droplet geometry which has no non-extremal bulk horizon isolated from the boundary black hole horizon. We construct such a bulk geometry by perturbing the exact background solution of a static or stationary black droplet.
2.1 A holographic model

We consider an analytic four-dimensional black droplet solution of the four-dimensional vacuum Einstein equations with a negative cosmological constant \[16\], in which there is a bubble of nothing deep inside the bulk, and the boundary metric is conformal to the BTZ black hole. The metric of the 4-dimensional black droplet solution is written by

\[
\begin{align*}
\text{ds}^2 &= \frac{d\rho^2}{f(\rho)} + \frac{\rho^2 r_0^2}{r^2 l^2} \left[\frac{r^2 - r_0^2}{l^2} dt^2 + \frac{l^2 dr^2}{r^2 - r_0^2} + L^2 r^2 \frac{f(\rho)}{\rho^2} d\varphi^2\right], \\
f(\rho) &= \frac{\rho^2}{L^2} + 1 - \frac{\mu_0}{\rho},
\end{align*}
\]

where \( L \) and \( l \) denote the AdS curvature length in the bulk and that in the boundary, respectively, and where \( \mu_0 \) describes the mass parameter given by

\[
\mu_0 := \rho_0 \left(\frac{\rho_0^2}{L^2} + 1\right).
\]

(2.1)

Here, \( \varphi \) is the periodic coordinate with period \( 2\pi \), and the circle along \( \varphi \) smoothly caps off at the bulk radius, \( \rho = \rho_0 \) when the horizon radius \( r_0 \) of the boundary BTZ black hole is given by

\[
r_0 = \frac{2\rho_0 l L}{3\rho_0^2 + L^2}.
\]

(2.2)

In this case, a bubble appears at the radius, \( \rho = \rho_0 \) instead of the non-extremal bulk horizon.

Since the boundary metric of eq. (2.1) at infinity \( \rho \to \infty \) is conformal to the static BTZ black hole, the solution describes a confining vacuum in the dual field theory in the BTZ black hole background. Following the procedure \[21\], the stress-energy tensor is calculated as

\[
T_{tt} = -\frac{\mu_0 r_0^3 L (r^2 - r_0^2)}{16\pi G_4 l^4 r^3}, \quad T_{rr} = \frac{L\mu_0 r_0^3}{16\pi G_4 l^3 (r^2 - r_0^2)}, \quad T_{\varphi\varphi} = -\frac{\mu_0 L r_0^3}{8\pi G_4 l^2 r},
\]

(2.4)

where \( G_4 \) is the 4-dimensional gravitational constant. It is clear that there is no energy flux on the boundary black hole horizon, \( r = r_0 \), even though the energy density outside the BTZ black hole is negative. In the following subsections, we construct a time-dependent boundary evaporating black hole geometry by considering perturbations of the static droplet solution (2.1).

2.2 Metric perturbations in the 4-dimensional bulk

Under the coordinate transformation \((r, t, \varphi) \to (\tilde{r}, \tilde{t}, \tilde{\varphi})\),

\[
r = \frac{r_0}{\tilde{r}}, \quad t = \frac{l^2 \tilde{t}}{r_0}, \quad \varphi = \frac{l}{Lr_0} \tilde{\varphi},
\]

(2.5)

the metric (2.1) becomes the following warped product type metric that includes, as a part, the 2-dimensional de Sitter metric,

\[
\text{ds}^2 = f(\rho) d\varphi^2 + \frac{d\rho^2}{f(\rho)} + \rho^2 \left[-(1 - \tilde{r}^2) d\tilde{t}^2 + \frac{d\tilde{r}^2}{1 - \tilde{r}^2}\right].
\]

(2.6)
Note that the de Sitter horizon at $\bar{r} = 1$ corresponds to the BTZ black hole horizon at $r = r_0$ on the boundary, as can be seen in (2.1).

In order to consider perturbations on this geometry, it is more convenient to take the double wick rotation:

$$\bar{\varphi} = i\tau, \quad \bar{t} = i\phi, \quad \bar{r} = \cos \theta.$$  \hspace{1cm} (2.7)

Then the black droplet metric takes the standard form of the 4-dimensional Schwarzschild AdS metric:

$$ds^2 = g_{ab}(y)dy^a dy^b + \rho^2 \gamma_{ij}(z)dz^i dz^j = -f(\rho)d\tau^2 + \frac{d\rho^2}{f(\rho)} + \rho^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right),$$  \hspace{1cm} (2.8)

where here and hereafter we use the coordinate notation $y^a = (\tau, \rho)$ and $z^i = (\theta, \phi)$. On this background we perform metric perturbations by applying the gauge-invariant formalism of [22]. For our purpose, it is sufficient to consider the scalar (polar)-type metric perturbations, which can be expanded in terms of the scalar harmonic functions $S_k$ on the unit 2-sphere,

$$\hat{D}_i \hat{D}_j S_k + k^2 S_k = 0,$$  \hspace{1cm} (2.9)

where $\hat{D}_i$ denotes the derivative operator associated with the unit 2-sphere metric $\gamma_{ij}dz^i dz^j = d\theta^2 + \sin^2 \theta d\phi^2$. We can express the components $(\delta g_{ab}, \delta g_{ai}, \delta g_{ij})$ of the scalar-type metric perturbations as

$$\delta g_{ab} = \epsilon f_{abk} S_k, \quad \delta g_{ai} = \epsilon \rho f_{aik} \hat{D}_j S_k, \quad \delta g_{ij} = 2\epsilon \rho^2 (H_L k \gamma_{ij} S_k + H_T k S_k),$$  \hspace{1cm} (2.10)

where $\epsilon$ is an infinitesimally small parameter and where $S_{kij}$ is defined by

$$S_{(k)ij} = \frac{1}{k^2} \hat{D}_i \hat{D}_j S_k + \frac{1}{2} \gamma_{ij} S_k.$$  \hspace{1cm} (2.11)

Hereafter we omit the mode index $k$, for the notational simplicity. We introduce the vector field $X_a$ on the 2-dimensional spacetime spanned by $y^a = (\tau, \rho)$ by

$$X_a = \frac{\rho}{k} \left(f_a + \frac{\rho}{k} D_a H_T\right),$$  \hspace{1cm} (2.12)

where $D_a$ is the derivative operator associated with the 2-dimensional metric $g_{ab}$. We find the following combinations are gauge invariant;

$$F = H_L + \frac{1}{2} H_T + \frac{1}{\rho} D^a \rho X_a, \quad F_{ab} = f_{ab} + D_a X_b + D_b X_a.$$  \hspace{1cm} (2.13)

For convenience, we introduce

$$X := F_\tau^\tau - 2F, \quad Y = F_\rho^\rho - 2F.$$  \hspace{1cm} (2.14)

Furthermore, we assume that $\partial_\tau$ is a Killing vector on the perturbed spacetime. Thus, we consider the static perturbations in the background (2.8). This, in turn, implies that we are concerned with dynamical but circular perturbations along $\varphi$ in the original black
droplet background via the double Wick rotation (2.7) and the rescaling (2.5). In this case, \(X, F, \text{ and } F_a^b\) are determined by \(Y\) as

\[
X = Y + 2 \frac{f}{f'} Y, \quad F_\tau = -F_\rho = \frac{f}{f'} Y', \quad F = -\frac{1}{2} \left( Y + \frac{f}{f'} Y' \right),
\]

where the prime denotes the derivative with respect to \(\rho\).

For later convenience, we choose the gauge

\[
f_\rho = f_{\rho\rho} = f_\tau = 0.
\]

(2.16)

Then, \(X_\tau = 0\) for the static perturbation. By eqs. (2.13), (2.15), and (2.16), \(X_\rho, H_T, \text{ and } H_L\) are determined by \(Y\) as

\[
X_\rho = -\frac{1}{2\sqrt{f}} \int \sqrt{f} \frac{Y'}{f'} d\rho, \quad H_T = k^2 \int \frac{X_\rho}{\rho^2} d\rho, \quad H_L = -\frac{1}{2} \left( Y + \frac{fY'}{f'} \right) - \frac{H_T}{2} - \frac{f}{\rho} X_\rho.
\]

(2.17)

The master variable \(Y\) satisfies

\[
Y'' + \alpha Y' + \beta Y = 0,
\]

(2.18)

where

\[
\rho^2 f f' \alpha = 4x - \frac{4\rho^2}{L^2} - 2x^2 + \frac{16}{L^2} x \rho^2 + \frac{4\rho^4}{L^4}, \quad \rho^2 f \beta = -k^2 + 2,
\]

(2.19)

with \(x := \mu_0/\rho\).

### 2.3 The boundary conditions in the bulk

Let us consider appropriate boundary conditions for our perturbations. We first consider boundary conditions from the bulk viewpoint, which we need to impose at the bubble radius, \(\rho = \rho_0\), and at the AdS infinity, \(\rho = \infty\). The former condition comes from the regularity condition [23],

\[
\lim_{\rho \to \rho_0} \frac{\nabla_M(\xi^2) \nabla^M(\xi^2)}{4\xi^2} = \left( \frac{l}{L \rho_0} \right)^2,
\]

(2.20)

where \(\xi^2\) denotes the norm squared of the circular Killing vector \(\partial/\partial \bar{\phi}\). Note that \(\bar{\phi} = i\tau\), and therefore that the r.h.s. is equivalent to square of the surface gravity \(\kappa = l/L \rho_0\) of the Killing horizon with respect to \(\partial/\partial \tau\) in the Schwarzschild-AdS black hole (2.8). Up to \(O(\epsilon)\), this regularity condition reduces to

\[
f_\tau = \alpha_\tau (\rho - \rho_0)^2 + \cdots.
\]

(2.21)

This determines the boundary condition for \(f_\tau\) and it is derived from eqs. (2.13) and (2.17) as

\[
f_\tau = F_\tau + f f' X_\rho = -\frac{f^2}{f'} Y' - \frac{\sqrt{f}}{2} f' \int_{\rho_0}^\rho \frac{Y'}{f'} d\rho,
\]

(2.22)

---

2The right-hand side of (2.20) comes from the fact that \(\bar{\phi}\) has the period \(2\pi L \rho_0/l\) via the rescaling (2.5).

If \(\xi^2\) is chosen to be the norm squared of \(\partial/\partial \varphi\), the r.h.s. should be unit since the \(\varphi\) circle has period \(2\pi\).
which is equivalent to the condition
\[ X_\rho(\rho_0) = 0. \]  
(2.23)

Note that there is another integration constant which remains undetermined for \( H_T \) in eq. (2.17), while the integration constant of \( X_\rho \) is determined by the regularity condition, \( X_\rho(\rho_0) = 0 \). To determine the integration constant for \( H_T \), we impose the following condition at the AdS boundary:
\[ \lim_{\rho \to \infty} \frac{H_T}{\rho^2} = 0. \]  
(2.24)

This condition for \( H_T \), together with our gauge condition (2.16) and the coordinate transformation (2.5), implies that our perturbations deform the boundary BTZ metric to the following form;
\[ ds^2 = e^{A(t,r)} \left( -\frac{r^2 - r_0^2}{l^2} dt^2 + \frac{l^2 dr^2}{r^2} \right) + e^{B(t,r)} r^2 d\phi^2. \]  
(2.25)

Namely, the perturbed boundary black hole is time-dependent and circularly symmetric as intended, but the coordinate location of the event horizon is fixed at \( r = r_0 \) under our gauge.

### 2.4 Boundary conditions on the boundary

Let us turn to boundary conditions from the boundary view point, which are to be imposed at the horizon of the 3-dimensional boundary black hole. For this purpose, we take the double wick rotation (2.7) again and go back to the background geometry (2.6). By doing so, we can now view that our metric perturbations considered in the previous subsection 2.2 are expanded by — instead of the harmonic functions on the 2-sphere defined in (2.9) — the mode functions \( S_k \) on the 2-dimensional de Sitter spacetime spanned by \((\bar{t}, \bar{r})\), which now satisfy the wave equation below,
\[ \left[ -\frac{\partial^2}{\partial \bar{t}^2} + \left( 1 - \bar{r}^2 \right) \left\{ \frac{\partial}{\partial \bar{r}} \left( 1 - \bar{r}^2 \right) \frac{\partial}{\partial \bar{r}} \right\} + \left( 1 - \bar{r}^2 \right) k^2 \right] S_k = 0. \]  
(2.26)

We set \( S_k = e^{-i\omega \bar{t}} \tilde{S}_k(\bar{r}) \) so that \( \tilde{S}_k(\bar{r}) \) satisfies
\[ \left[ \left( 1 - \bar{r}^2 \right)^2 \frac{d^2}{d\bar{r}^2} - 2\bar{r}(1-\bar{r}^2) \frac{d}{d\bar{r}} + k^2(1-\bar{r}^2) + \omega^2 \right] \tilde{S}_k = 0, \]  
(2.27)

and find that the general solution of \( \tilde{S}_k(\bar{r}) \) is represented by the associated Legendre functions, \( P_n^m \) and \( Q_n^m \) as
\[ S_k = a P_n^m(\bar{r}) + b Q_n^m(\bar{r}), \quad m := i\omega, \quad n := \frac{1}{2} \left( -1 + \sqrt{1 + 4k^2} \right), \]  
(2.28)

where
\[ P_n^m(x) = \frac{e^{m\pi i}}{\Gamma(1 - m)} \left( \frac{1 + x}{1 - x} \right)^{m/2} F \left( -n, n + 1, 1 - m; \frac{1 - x}{2} \right), \]
\[ Q_n^m(x) = \frac{\pi}{2 \sin m\pi} \left[ \cos m\pi P_n^m(x) - \frac{\Gamma(n + m + 1)}{\Gamma(n - m + 1)} P_n^{-m}(x) \right]. \]  
(2.29)
Here, note that we assume that $|m|$ is very small, since we consider the case that the time evolution of the bulk geometry is very slow.

Since we are interested in the future horizon, $\bar{r} = 1$ at the right-hand side (r.h.s.) in figure 1, the ingoing boundary condition should be imposed as a regularity condition. In terms of a retarded null coordinate $u$, $S_k$ behaves near the horizon as

$$S_k \sim e^{-i\omega u}, \quad u := \bar{t} - \int \frac{d\bar{r}}{1 - \bar{r}^2} \simeq \bar{t} + \frac{1}{2} \ln (1 - \bar{r}). \quad (2.30)$$

The ingoing mode comes from the left past horizon, $\bar{r} = -1$ in figure 1, as an outgoing wave. So, we shall impose that $S_k$ behaves near the past horizon, $\bar{r} = -1$ at the left-hand side (l.h.s.) of figure 1 as

$$S_k \sim e^{-i\omega u}, \quad u := \bar{t} - \int \frac{d\bar{r}}{1 - \bar{r}^2} \simeq \bar{t} - \frac{1}{2} \ln (1 + \bar{r}). \quad (2.31)$$

From the boundary condition (2.30) and the form of the associated Legendre function (2.28), $S_k \sim P_n^m(x)$ near the future horizon, $\bar{r} = 1$, and then, we obtain

$$b = 0. \quad (2.32)$$

Using the linear transformation formulae for hypergeometric functions, $S_k$ behaves near the past horizon, $\bar{r} = -1$ as

$$S_k \sim P_n^m(\bar{r}) \simeq e^{m\pi i} \left( \frac{1 + \bar{r}}{2} \right)^{\frac{m}{2}} \left[ \frac{\Gamma(m)}{\Gamma(-n+1)} \left( \frac{1 + \bar{r}}{2} \right)^{-m} F \left( 1 + n - m, -n - m, 1 - m; \frac{1 + \bar{r}}{2} \right) \right. \left. + \frac{\Gamma(-m)}{\Gamma(1-m+n)\Gamma(-n-m)} F \left( -n, n + 1, m + 1; \frac{1 + \bar{r}}{2} \right) \right]. \quad (2.33)$$

By imposing the boundary condition (2.31), we find that the first term should be zero, and thus,

$$n = N, \quad k^2 = N(N + 1) \quad (2.34)$$
for an arbitrary positive integer $N$. Note that here, we do not consider a particular class of the mode function, $k = 0$ in which $H_T = 0$ in eq. (2.10). In section 3 we will consider a 4-dimensional time-dependent boundary geometry that describes a black hole formation and evaporation via Hawking radiation, in which the energy flux comes in from past null infinity. Our boundary conditions (2.32) and (2.34) may be viewed as boundary conditions that correspond to such a black hole formation and evaporation induced via some incoming flux from past infinity.

2.5 Negative energy flux on the evaporating black hole

In this section we derive the stress-energy tensor of the boundary field theory by taking the Fefferman-Graham coordinate system and using the AdS/CFT dictionary [21], and then we evaluate it on the horizon. Thanks to the gauge choice (2.16), the Fefferman-Graham coordinates system

$$g_{MN}dx^Mdx^N = \frac{L^2}{z^2} \left( dz^2 + g_{mn}(x,z)dx^m dx^n \right),$$

$$g_{mn}(z,x) = g_{(0)mn}(x) + z^2 g_{(2)mn}(x) + \cdots$$

(2.35)

is derived from

$$\rho(z) = \frac{L}{z} \left( 1 - \frac{1}{4} z^2 + \frac{\mu}{6L} z^3 + \cdots \right).$$

(2.36)

The stress-energy tensor $T_{mn}$ can be read off from the coefficient $g_{(3)mn}(x)$ as

$$T_{mn} = \frac{3L^2}{16\pi G_4} g_{(3)mn},$$

(2.37)

where $G_4$ is the bulk gravitational constant. To derive the coefficient $g_{(3)mn}(x)$, let us expand the master variable $Y$ near the infinity as

$$Y = a_0 + \frac{a_1}{\rho} + \frac{a_2}{\rho^2} + \frac{a_3}{\rho^3} + \cdots.$$  

(2.38)

From eq. (2.18), each coefficient $a_n$ ($n \geq 2$) is determined by $a_0$ and $a_1$ as

$$a_2 = \frac{1}{2} a_0(k^2 - 2)L^2, \quad a_3 = \frac{1}{6} a_1(k^2 - 6)L^2, \quad a_4 = \frac{1}{24} \left\{ a_0(24 - 14k^2 + k^4)L^4 + 18a_1\rho_0(L^2 + \rho_0^2) \right\}, \cdots.$$  

(2.39)

Under the boundary conditions (2.23) and (2.24), $H_T$, $H_L$, and $f_{\tilde{\phi}\tilde{\phi}}$ are expanded as a series of $1/\rho$ from eqs. (2.17) and (2.22) as

$$H_T = \frac{c_1 k^2 L}{4 \rho^2} + \frac{a_1 k^2 L^2}{12 \rho^3} + \cdots,$$

$$H_L = \frac{c_1 - a_0 L}{2L} + \frac{L(k^2 - 2)(a_0 L - c_1)}{8 \rho^2} + \frac{a_1 (k^2 - 2)L^3 - 6c_1 \rho_0 (L^2 + \rho_0^2)}{24L \rho^3} + \cdots,$$

$$f_{\tilde{\phi}\tilde{\phi}} = \frac{c_1 \rho^2}{L^3} + \frac{1}{4} \left\{ a_0 (k^2 - 2) - \frac{2c_1}{L} \right\} - \frac{a_1 (k^2 - 2)}{6 \rho} + \cdots.$$  

(2.40)

(2.41)
where the coefficient $c_1$ is defined as

$$c_1 := \int_{\rho_0}^{\infty} \sqrt{Y'/Y} \; d\rho.$$  \hfill (2.42)

Substituting (2.41) into (2.10) and expanding the metric as a series in $z$ with the help of eq. (2.36), we obtain the boundary metric,

$$g(0)_{mn} dx^m dx^n = r_0^2 \left[ \frac{r^2}{l^2} \left( 1 + \epsilon c_1 S_k \right) - \frac{1}{l^2} \right] + \frac{r_0^2}{l^2} \left( 2dvdr - \frac{r^2}{l^2} dv^2 \right) \right] =: \left( \frac{r_0}{r} \right)^2 ds^2_{\text{dBTZ}},$$  \hfill (2.43)

where

$$v := t + \int \frac{l^2}{r^2 - r_0^2} \; dr.$$  \hfill (2.44)

Note that the stress-energy tensor for the boundary metric $ds^2_{\text{dBTZ}}$ — which stands for the deformed BTZ metric — cannot be proportional to the coefficient $g_{(3)mn}$, as $g_{(0)mn}$ is different from the metric $ds^2_{\text{dBTZ}}$ by a conformal factor, $(r_0/r)^2$. In general, for the conformal transformation

$$\hat{g}_{mn} dx^m dx^n = \Omega^2 g_{mn} dx^m dx^n,$$  \hfill (2.45)

the stress-energy tensor transforms as

$$\hat{T}^m_{\; mn} = \frac{1}{\Omega} T^m_{\; mn},$$  \hfill (2.46)

for the 3-dimensional boundary spacetime. Since we are interested in the stress-energy tensor on the horizon, $r = r_0$, the conformal factor is equal to 1, and then, the null-null component of the stress energy tensor for the metric $ds^2_{\text{dBTZ}}$ can be read off from the formula (2.37) as

$$T_{vv}(v, r_0) = \frac{\epsilon a_1 L}{32\pi G_4} \left( \partial_v^2 S_k - \frac{r_0}{l^2} \partial_v S_k \right).$$  \hfill (2.47)

The Ricci tensor for the deformed BTZ metric $ds^2_{\text{dBTZ}}$ on the horizon is also calculated as

$$R_{vv}(v, r_0) = \epsilon \frac{c_1}{2L} \left( \frac{r_0}{l^2} \partial_v S_k - \partial_v^2 S_k \right).$$  \hfill (2.48)

So, the ratio between $T_{vv}(v, r_0)$ and $R_{vv}(v, r_0)$ is determined by the dimensionless coefficient $\zeta$,

$$\zeta := -\frac{a_1}{c_1}.$$  \hfill (2.49)

As shown in the proposition below, this coefficient $\zeta$ is always positive, except the trivial case, $a_1 = c_1 = 0$.

**Proposition 1.** For the mode functions $S_k$ satisfying eq. (2.26) with (2.34), $\zeta > 0$ for all integers $N > 1$, and $a_1 = c_1 = 0$ for $N = 1$. 

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Proof. In terms of the new variable, \( u = \rho_0/\rho \), the master eq. (2.18) can be rewritten as

\[
(1 - u)\{\hat{\rho}_0^2(1 + u) + (1 + \hat{\rho}_0^2)u^2\}\{2\hat{\rho}_0^2 + (1 + \hat{\rho}_0^2)u^3\}\dot{Y} - 2u\{-4\hat{\rho}_0^2 + 9\hat{\rho}_0^2(1 + \hat{\rho}_0^2)u + (1 + \hat{\rho}_0^2)u^3\}\dot{Y} + (2 - k^2)\{2\hat{\rho}_0^2 + (1 + \hat{\rho}_0^2)u^3\}Y = 0,
\]

\[
\dot{Y}(1) = \frac{1 - \frac{k^2}{1 + 3\hat{\rho}_0^2}}{1 + 3\hat{\rho}_0^2}Y(1),
\]

(2.50)

where \( \hat{\rho}_0 = \rho_0/L \), and the dot denotes the derivative with respect to \( u \). The second equation comes from the regularity on the bubble radius, \( \rho = \rho_0 \). Since the differential equation is linear, we can set \( Y(1) = 1 \) without loss of generality. First, consider the case \( N > 1 \). Then, \( k^2 > 2 \) and hence \( \dot{Y}(1) < 0 \). Now suppose \( Y \) would have a maximum at \( u = u_m \) (0 \( \leq u_m \leq 1 \)). In this case, \( Y(u_m) > 1, \dot{Y}(u_m) = 0, \text{and} \dot{Y}(u_m) \leq 0 \). This is impossible from eq. (2.50). So, \( \dot{Y} < 0 \) for \( 0 \leq u \leq 1 \). In terms of \( u \), the coefficient \( c_1 \) in eq. (2.42) is rewritten by

\[
\frac{c_1}{L} = \int_0^1 \frac{\sqrt{g(u)}}{2 + \left(1 + \frac{1}{\rho_0^2}\right)u^3}du, \quad g(u) := 1 + \frac{u^2}{\rho_0^2} - \left(1 + \frac{1}{\rho_0^2}\right)u^3.
\]

(2.51)

From this, \( g(u) \geq 0 \) and \( c_1 > 0 \), while \( a_1 \geq 0 \) from the expansion (2.38). Thus, we have shown \( \zeta > 0 \). As for the \( N = 1 \) case, the \( Y \) term in eq. (2.50) vanishes, as \( k^2 = 2 \). This is analytically solved and the only regular solution is \( Y = 1 \). In this case, \( a_1 = c_1 = 0 \). \( \square \)

Now let us consider the time dependence of our boundary black hole. By eq. (2.43), the area on the horizon is given by

\[
\mathcal{A} = 2\pi r_0 \sqrt{1 + \epsilon \frac{c_1 S_k}{L}} \sim r_0 \left(1 + \frac{\epsilon c_1 S_k}{2L}\right),
\]

(2.52)

where \( S_k \) is evaluated at \( u = 1 \). By setting \( 0 < m = \bar{\omega} \ll 1 \) and choosing the sign of \( \epsilon \) so that \( \epsilon c_1 S_k > 0 \), the area decreases very slowly, thus describing the evaporating process. In this case, \( \partial_v S_k \) is negligible compared with \( \partial_v S_k \), and hence, \( R_{uv}(v, r_0) \) in eq. (2.48) is negative on the horizon. The above proposition means that the null-null component of the stress-energy tensor is always negative during the evaporating phase. By the energy conservation law, this implies that the energy outside the evaporating black hole increases. This picture agrees with the Hawking radiation, and thus, the entropy outside the horizon would increase, supporting the GSL conjecture.

At first glance, the existence of the negative energy flux would violate the averaged null energy condition (ANEC) on the horizon. In the next section, we examine the ANEC in a background of 4-dimensional evaporating black hole in the context of the AdS/CFT correspondence.

### 3 4-dimensional evaporating black hole and ANEC

In this section, to examine the ANEC, we construct the 5-dimensional bulk geometry dual to a 4-dimensional asymptotically flat, spherically symmetric time-dependent boundary.
black hole, describing the formation by gravitational collapse and subsequent evaporation. Such a bulk spacetime can be constructed, at least locally near the boundary, by taking the Vaidya metric as our boundary geometry and performing the Fefferman-Graham expansion into 5-dimensional bulk. As such, although our construction of the holographic time-dependent black hole is limited, it is sufficient for the purpose of examining the ANEC on our time-dependent boundary black hole by applying the method of refs. [18, 19], based on the no-bulk shortcut principle.

3.1 An asymptotically flat evaporating black hole

We consider an asymptotically flat spherically symmetric evaporating black hole formed by injecting a null dust shell. Such a model of evaporating black hole is well described by the Vaidya model [24] as,

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 2dvdr - \left(1 - \frac{2m(v)}{r}\right)dv^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \(v\) denotes the advanced null coordinate and \(m(v)\) the mass of the black hole. Note that here and hereafter, we use the coordinate index \(x^\mu\) to denote tensor components on the 4-dimensional boundary spacetime. As shown in figure 2, the null dust shell is injected at \(v = v_0\) to form the black hole, of which the event horizon (EH) begins at the center of spherical symmetry [point \(p\) at \((r = 0, v = v_i)\)] and terminates at a zero mass naked singularity at point \(q\) \((r = 0, v = v_f)\), where the evaporation is completed. The apparent horizon (AH) is located outside the EH during the evaporating process, as the area decreases. We assume that the spacetime is flat, i.e., \(m(v) = 0, v < v_0\) before the collapse of the null shell.

Figure 2. The Penrose diagram of a four-dimensional spherically symmetric evaporating black hole which is formed by a collapsing null dust shell. The (red) dotted line and the (green) dashed-dotted line represent, respectively, the event horizon and the apparent horizon. The (blue) solid line represents the collapsing null dust shell, which is characterized in terms of the advanced null coordinate \(v\) as \(v = v_0\).
As is often discussed in the consideration of backreaction effects of the Hawking radiation, for a free massless field, the negative energy flux $F$ flowing into the black hole is determined by the dimensional analysis as

$$F \simeq \frac{A}{m(v)^2},$$

(3.2)

where $A$ is of order unity in Planck units [25]. Assuming that the mass of the black hole is lost by the negative energy flux, $dm/dv \simeq -A/m^2$, one obtains

$$m(v) \sim (v - v_f)^{1/3}.$$  

(3.3)

Generalizing eq. (3.3), we shall consider a holographic model in which the AdS boundary is conformal to the Vaidya metric (3.1) with $m(v)$ satisfying

$$m(v) = m_0 \left(1 - \frac{v_0}{v_f}\right)^{-\alpha} \left(1 - \frac{v}{v_f}\right)^\alpha, \quad m_0 > 0, \quad 0 < \alpha < 1,$$

(3.4)

where $m_0$ is the mass of the black hole just after the collapse of the null dust shell. The radial null geodesic on the EH obeys the equation

$$\frac{dr}{dv} = \frac{1}{2} \left(1 - \frac{2m(v)}{r}\right).$$

(3.5)

Note that the AH is located at the radius $r_{AH}$ where the r.h.s. of the above equation vanishes, i.e., $r_{AH} = 2m(v) = 2m_0(1 - v/v_f)^\alpha$. In the evaporation process $v_0 < v < v_f$, the EH is located inside the AH as depicted in figure 2, which implies that the EH radius $r_{EH}$ is bounded from above by $r_{AH} = 2m(v)$. This can be the case since the quantum field stress-energy will not satisfy the weak energy condition under the evaporation process. In particular, at the last stage of the evaporation, the geometry rapidly changes and it is plausible that the EH radius could be much smaller than that of the AH, close to the final point of the evaporation $v \sim v_f$. In such a case, by assuming $r_{EH} \ll 2m(v)$, we can find the approximated solution of (3.5) near $v = v_f$ as,

$$r_{EH}(v) \simeq \sqrt{\frac{2m_0v_f}{\alpha + 1} \left(1 - \frac{v_0}{v_f}\right)^{-\frac{\alpha}{2}} \left(1 - \frac{v}{v_f}\right)^{(\alpha + 1)/2}}.$$  

(3.6)

Along the radial outgoing null vector

$$k^\mu = (\partial_v)^\mu + \frac{1}{2} \left(1 - \frac{2m(v)}{r}\right) (\partial_r)^\mu,$$

(3.7)

on the EH, the affine parametrized null geodesic generator $\gamma$ is given by

$$K^\mu := h(v)k^\mu,$$

(3.8)

where the factor $h$ obeys the following equation,

$$\frac{dh}{dv} = -h \frac{m}{r^2}.$$  

(3.9)
Near the naked singularity, \( v = v_f, \ r = 0 \), the solution is approximately given by the functions (3.4) and (3.6) as
\[
\frac{dv}{dV} = h \propto \left( 1 - \frac{v}{v_f} \right)^{(\alpha+1)/2},
\]
where \( V \) is the affine parameter of the null geodesic \( \gamma \) with the tangent \( K^\mu \).

### 3.2 The no bulk-short cut principle in the dual bulk spacetime

We briefly recall the statement of the no-bulk-short-cut principle proposed in [18, 19, 26]. Suppose that a pair of boundary points \( p \) and \( q \) are connected by an achronal null geodesic \( \gamma \) on the boundary spacetime. Then, the no-bulk-short-cut principle states that there is no bulk timelike curve \( \lambda \) which connects \( p \) to \( q \). In other words, \( \gamma \) is the fastest causal curve among all causal curves from \( p \) to \( q \), including the bulk causal curves. In a class of spacetime, this principle was shown in the Gao-Wald theorem [20]. By applying this principle to boundary Minkowski spacetime in a holographic setting, the ANEC was proved in [26]. The principle is also applied to the spatially compact universes and the CANEC or the ANEC with a weight function were derived [18, 19]. In this regard, see also [27] for the ANEC in de Sitter and AdS spacetime. Here, we shall apply this principle to the holographic model with the boundary geometry describing the evaporating black hole (3.1) discussed above, whose event horizon is generated by achronal null geodesics \( \gamma \) from \( v = v_i \) to \( v = v_f \). We wish to derive the lower bound of the null-null component of the boundary stress-energy tensor along boundary achronal null geodesics \( \gamma \). Our strategy is very similar to the one performed in refs. [18, 19], which is as follows. Given such a boundary achronal null geodesic \( \gamma \), we consider all the bulk causal curves connected with \( \gamma \) via Jacobi fields, thus considered to be in the neighborhood of \( \gamma \) covered by Fefferman-Graham coordinates. Then, by imposing the no-bulk-short-cut principle that \( \gamma \) be the fastest causal curve among all the causal curves in the neighborhood, we will obtain the lower bound.

The 5-dimensional bulk spacetime in the neighbourhood of the boundary metric (3.1) including the boundary null geodesics is represented by the following Fefferman-Graham coordinates
\[
g_{ab}dx^a dx^b = \frac{L^2}{z^2} \left( dz^2 + g_{\mu\nu}(x,z)dx^\mu dx^\nu \right) := \frac{L^2}{z^2} \hat{g}_{ab} dx^a dx^b, \]
\[
g_{\mu\nu}(z,x) = g_{(0)\mu\nu}(x) + z^2 g_{(2)\mu\nu}(x) + \cdots + z^4 g_{(4)\mu\nu}(x) + h_{(4)\mu\nu} z^4 \ln z^2 + \cdots, \quad (3.11)
\]
where \( g_{(0)\mu\nu}(x) \) corresponds to the boundary metric (3.1), and \( \hat{g}_{ab} \) is the unphysical metric conformal to \( g_{ab} \). The subleading coefficients \( g_{(2)\mu\nu}, g_{(4)\mu\nu}, \) and \( h_{(4)\mu\nu} \) are given by [21],
\[
g_{(2)\mu\nu} = -\frac{1}{2} \left( R_{\mu\nu} - \frac{R}{6} g_{(0)\mu\nu} \right),
\]
\[
g_{(4)\mu\nu} = t_{\mu\nu} + \frac{1}{8} g_{(0)\mu\nu} \left[ (\text{Tr} g_{(2)})^2 - \text{Tr}(g_{(2)}^2) \right] + \frac{1}{2} g_{(2)\mu\alpha} g_{(0) \alpha \beta} g_{(2)\beta\nu} - \frac{1}{4} g_{(2)\mu\nu} \text{Tr}(g_{(2)}),
\]
\[
h_{(4)\mu\nu} = \frac{1}{2} g_{(2)\mu\alpha} g_{(0) \alpha \beta} g_{(2)\beta\nu} - \frac{1}{8} g_{(0)\mu\nu} \text{Tr}(g_{(2)}^2), \quad (3.12)
\]
where $R_{\mu\nu}$, $R$ are the Ricci tensor, Ricci scalar of the metric $g_{(0)\mu\nu}$, and the indices are raised and lowered by the metric $g_{(0)\mu\nu}$. According to the AdS/CFT dictionary [21], the expectation value of the stress-energy tensor $\langle T_{\mu\nu} \rangle$ is determined by $t_{\mu\nu}$ as

$$\langle T_{\mu\nu} \rangle = \frac{1}{4\pi G_5} t_{\mu\nu}.$$  \hspace{1cm} (3.13)

It is convenient to use the following double null coordinates

$$g_{(0)\mu\nu}dx^\mu dx^\nu = -e^{f(U,V)} dU dV + r^2(U,V)(d\theta^2 + \sin^2 \theta d\phi^2),$$  \hspace{1cm} (3.14)

instead of the metric (3.1). On the EH, we can set

$$U = 0, \quad f(0,V) = 0,$$  \hspace{1cm} (3.15)

without loss of generality. In this case, $V$ is the affine parameter of the boundary null geodesic $\gamma$ with the tangent vector $K^\mu$ defined by eq. (3.8). We denote $V_i$ and $V_f$ as the values of the point $p$ ($v = v_i$) and the point $q$ ($v = v_f$), respectively.

Now consider a bulk causal curve $\lambda$ with the tangent vector $K^a$ near the boundary null geodesic $\gamma$, which connects the boundary point $p$ to the boundary point $q$ on $\gamma$. From spherical symmetry, it is sufficient to consider the case $K^\theta = K^\phi = 0$, and $K^a$ can be expanded as a series of small parameter $\epsilon$ as

$$K^a = (K^z, K^U, K^V, K^\theta, K^\phi) = \left( \frac{dz}{dV}, \frac{dU}{dV}, 1, 0, 0 \right),$$

$$z = \epsilon z_1 + \epsilon^2 z_2 + \cdots,$$

$$\frac{dU}{dV} = \epsilon^2 \frac{dU_2}{dV} + \epsilon^3 \frac{dU_3}{dV} + \epsilon^4 \frac{dU_4}{dV} + \cdots.$$  \hspace{1cm} (3.16)

Substituting this into eq. (3.11), the constraint of $\hat{g}_{ab}K^aK^b \leq 0$ is expressed as

$$\hat{g}_{ab}K^aK^b = \epsilon^2 \left( -\frac{dU_2}{dV} + z_1^2 + z_2^2 g_{(2)VV}(0,V) \right)$$

$$+ \epsilon^3 \left( 2z_1z_2 + 2z_1z_2 g_{(2)VV}(0,V) - \frac{dU_3}{dV} \right)$$

$$+ \epsilon^4 \left( z_2^2 + z_2^2 g_{(2)VV}(0,V) + 2z_1z_3 + 2z_1z_3 g_{(2)VV}(0,V) + 2z_1 \ln z_1 h_{VV}(0,V) \right.$$  
$$\left. + 2z_1z_3 g_{(2)VV}(0,V) + 2z_1 \ln z_1 h_{VV}(0,V) \right)$$

$$+ 2z_1^2 g_{(2)UV}(0,V) \frac{dU_2}{dV} - \frac{dU_4}{dV} + z_1^4 g_{(4)VV}(0,V)$$

$$+ z_1^2 \partial_{U} (g_{(2)VV})(0,V) U_2 - (\partial_{U} f)(0,V) U_2 \frac{dU_2}{dV}$$

$$+ 2\epsilon^4 \ln(\epsilon) z_1^4 h_{VV}(0,V) + \cdots$$

$$\leq 0,$$  \hspace{1cm} (3.17)
where the dot denotes the derivative with respect to $V$. At $O(c^2)$, we obtain the inequality of the variation of $U$ from $V_i$ to $V_f$ as
\[
\Delta U_2 = \int_{V_i}^{V_f} \frac{dU_2}{dV} dV \geq \int_{V_i}^{V_f} \left( \dot{z}_1^2 + \dot{z}_1^2 g_{(2)VV}(0,V) \right) dV, \tag{3.18}
\]
where the equality is satisfied for the null curve. The r.h.s. of eq. (3.18) is minimized for $z_1$ obeying
\[
\ddot{z}_1 = g_{(2)VV}(0,V)z_1 = \frac{\dot{r}(0,V)}{r(0,V)} \dot{z}_1, \tag{3.19}
\]
where we have used that on the EH, $R_{VV}(0,V) = -2\dot{r}/r$, under our gauge condition $f(0,V) = 0$ and the holographic formula (3.12) for $g_{(2)VV}$. Therefore the solution satisfying $z_1(p) = z_1(q) = 0$ is simply given by
\[
z_1 = cr(0,V), \tag{3.20}
\]
for some positive constant $c$. Then, the integrand of the r.h.s. of (3.18) is expressed simply as $(z_1 \dot{z}_1)$ and thus, integrating the r.h.s. of eq. (3.18) results in the boundary values of $z_1 \dot{z}_1$. As shown in ref. [19], $z_1$ is identified as the magnitude of a Jacobi field for a null geodesic $\gamma$ in figure 2], the Jacobi field $\dot{z}_1$ from the point $\gamma$ to the point $p$ in figure 2], where the EH is formed. Also, since the EH completely evaporates at $(U = 0, V = V_f)$ [the point $q$ in figure 2], the Jacobi field $\dot{z}_1$ is expected to vanish there too;
\[
z_1(p) = z_1(q) = 0. \tag{3.21}
\]
This guarantees that the bulk null curve $\lambda$ remains in the neighbourhood of the boundary null geodesic $\gamma$ from the point $p$ to the point $q$ when $\epsilon$ is small enough. In fact, one can directly check that the boundary value $\dot{z}_1 z_1(V)$ goes to zero in the limit $V \to V_f$ by utilizing eqs. (3.6) and (3.10), as
\[
\lim_{V \to V_f} \dot{z}_1(V) z_1(V) = \lim_{V \to V_f} h(v) \frac{dz_1}{dv} z_1 \sim \lim_{V \to V_f} \left( 1 - \frac{v}{v_f} \right)^{(3\alpha+1)/2} \to 0. \tag{3.22}
\]
Thus, the r.h.s. of eq. (3.18) must vanish, implying that there is no time delay between $\gamma$ and $\lambda$ at $O(c^2)$ when $\lambda$ is the null curve, as the variation of $U$ between $\gamma$ and $\lambda$ is zero. Actually, $U_2$ is analytically written by $z_1$ as
\[
U_2 = z_1 \dot{z}_1. \tag{3.23}
\]
Similarly, the variation of $U$ from $V_i$ to $V_f$ is minimized for $z_2$ and $z_3$ obeying the same Jacobi equation (3.19), and $\Delta U_3$ is also zero at $O(c^2)$.

As for $O(c^4)$, by noting the fact that for the Vaidya metric (3.1), only non-vanishing component among $g_{(2)\mu\nu}$ in eqs. (3.12) is $g_{(2)VV} = -R_{VV}/2$. Thus, $h_{VV} = 0$, and we obtain
\[
\Delta U_3 \geq \int_{V_i}^{V_f} z_1^4 t_{VV}(0,V) dV + \mathcal{I},
\]
where
\[
\mathcal{I} := \int_{V_i}^{V_f} \left( \frac{q^2}{2} \partial U(g_{(2)VV})(0,V)U_2 - (\partial_U f)(0,V)U_2 \frac{dU_2}{dV} \right) dV
\]
\[
= \frac{c^4}{16} \int_{V_i}^{V_f} r^2 \left[ r^2 \partial_\mu^2 - 2R_{VV} \right] dV, \tag{3.24}
\]
\[
\Delta U_4 \geq \int_{V_i}^{V_f} z_1^4 t_{VV}(0,V) dV + \mathcal{I},
\]
where
\[
\mathcal{I} := \int_{V_i}^{V_f} \left( \frac{q^2}{2} \partial U(g_{(2)VV})(0,V)U_2 - (\partial_U f)(0,V)U_2 \frac{dU_2}{dV} \right) dV
\]
\[
= \frac{c^4}{16} \int_{V_i}^{V_f} r^2 \left[ r^2 \partial_\mu^2 - 2R_{VV} \right] dV, \tag{3.24}
\]
\[
\Delta U_4 \geq \int_{V_i}^{V_f} z_1^4 t_{VV}(0,V) dV + \mathcal{I},
\]
where
\[
\mathcal{I} := \int_{V_i}^{V_f} \left( \frac{q^2}{2} \partial U(g_{(2)VV})(0,V)U_2 - (\partial_U f)(0,V)U_2 \frac{dU_2}{dV} \right) dV
\]
\[
= \frac{c^4}{16} \int_{V_i}^{V_f} r^2 \left[ r^2 \partial_\mu^2 - 2R_{VV} \right] dV, \tag{3.24}
\]
where $\theta_\pm$ and $\mu$ are outgoing and ingoing expansions, and the mass density defined by

$$
\dot{\theta}_+ = \frac{2\dot{r}}{r}, \quad \dot{\theta}_- = \frac{2r'}{r}, \quad \mu := \frac{1}{r^2} + \dot{\theta}_+ \dot{\theta}_- ,
$$

(3.25)

where the dash expresses the derivative with respect to $U$. The second equality in eq. (3.24) is shown in the appendix. Applying the no-bulk-short-cut principle to our case, $\Delta U_4$ should be non-negative, and we finally obtain the following inequality with the weight function $r^4$,

$$
\int_{V_i}^{V_f} r^4 t_{VV} dV \geq \frac{1}{16} \int_{v_i}^{v_f} h \left[ 2R_{VV} - r^2 \mu \theta_+^2 \right] dV .
$$

(3.26)

### 3.3 The ANEC in the evaporating BH

Now, we can derive the ANEC from eq. (3.26). Using the fact that $f(0, V) = 0$ and $\partial_U = -1/(2h)\partial_v$ along the EH, the expansions $\theta_\pm$ and the mass density $\mu$ in eqs. (3.25) are rewritten in terms of the advanced null coordinate, $v$ and $r$ as

$$
\dot{\theta}_+ = \left( 1 - \frac{2m(v)}{r} \right) \frac{h}{r}, \quad \dot{\theta}_- = -\frac{1}{hr}, \quad \mu = \frac{2m}{r^3} .
$$

(3.27)

Substituting eq. (3.27) into the inequality (3.26), we find

$$
\int_{V_i}^{V_f} r^4 t_{VV} dV \geq \frac{1}{4} \int_{v_i}^{v_f} h \left[ \frac{dm}{dv} - \left( 1 - \frac{2m}{r(v)} \right)^2 \frac{m}{2r(v)} \right] dv .
$$

(3.28)

In order to inspect the above inequality closely, let us divide the whole region $v_i \leq v \leq v_f$ into four phases:

- **Phase (I)** $v_i \leq v \leq v_0 - \delta_1$: the geometry is the Minkowski spacetime,
- **Phase (II)** $v_0 - \delta_1 \leq v \leq v_0 + \delta_1$: the null shell is collapsing,
- **Phase (III)** $v_0 + \delta_1 \leq v \leq v_f - \delta_2$: the BH is evaporating,
- **Phase (IV)** $v_f - \delta_2 \leq v \leq v_f$: the final explosion.

Here, we assume that the time evolution of the evaporation is sufficiently slow ($v_f \gg m_0$), and the time durations $\delta_1, \delta_2$ are very small, i.e.,

$$
\delta_1, \delta_2 \ll |v_f - v_0| .
$$

(3.29)

In Phase (I), the r.h.s. of eq. (3.28) vanishes, as $m = 0$. In the collapsing phase (II), the mass rapidly grows during the short time period $\delta_1$ by the condition (3.29), and the second term is negligible compared with the first term. In Phase (III), the time evolution of the mass is very slow, and $r \approx 2m$ in this phase. Let us set the location of the EH as $r = 2m(1 - \eta)$ with $\eta$ assumed to be small so that $|\eta| \ll |dm/dv| \ll 1$ in Phase (III). Then, linearizing eq. (3.5) with respect to $\eta$, one obtains the approximate solution,

$$
\eta \simeq -2 \frac{d}{dv} [2m(1 - \eta)] \simeq -4 \frac{dm}{dv} .
$$

(3.30)
This implies that the second term in the r.h.s. of eq. (3.28) is small compared with the first term, as

$$\left(1 - \frac{2m}{r}\right)^2 \frac{m}{2r} \simeq \frac{\eta^2}{4} \ll \left|\frac{dm}{dv}\right| \simeq \left|\frac{\eta}{4}\right|,$$

(3.31)

under the approximation (3.30).

In the final phase (IV), one might expect that the second term in the r.h.s. of eq. (3.28) could give a large contribution. However, in the case that the time evolution of the mass is very slow, i.e., \(v_f \gg m_0\), it turns out that the contribution from the second term is still small compared with that from the first term in Phase (IV). This can be seen from eq. (3.6) that the ratio of the second term to the first term is at least of \(O(\sqrt{m_0}/v_f)\). Thus, it is sufficient for our purpose to evaluate the first term of the r.h.s. of eq. (3.28) in each phase.

In Phase (I), the spacetime is flat, and then \(h = 1\). In Phase (II), by integrating eq. (3.9), we obtain

$$\int_{v_0+\delta_1}^{v_f} \frac{dh}{h} = \ln h(v_0 + \delta_1) = - \int_{v_0-\delta_1}^{v_0+\delta_1} \frac{m}{r^2} dv = O(\delta_1),$$

(3.32)

which implies \(h \sim 1\) during Phase (II). Thus, we obtain

$$\int_{v_0}^{v_f} h \frac{dm}{dv} dv = \int_{v_0-\delta_1}^{v_0+\delta_1} h \frac{dm}{dv} dv + \int_{v_0+\delta_1}^{v_f} h \frac{dm}{dv} dv$$

$$\simeq \int_{v_0-\delta_1}^{v_0+\delta_1} \frac{dm}{dv} dv + \int_{v_0+\delta_1}^{v_f} \frac{h}{v} \frac{dm}{dv} dv$$

$$= m_0 + \int_{v_0+\delta_1}^{v_f} \frac{dm}{dv} dv.$$

(3.33)

By eq. (3.9), \(dh/dv < 0\) during the evaporating phases (III) and (IV),

$$0 \leq h \leq 1 \quad \text{for} \quad v \geq v_0 + \delta_1.$$

(3.34)

Since \(dm/dv < 0\) during the evaporating phases (III) and (IV), we thus obtain

$$\int_{V_c}^{V_f} r^4 t_{VV} dV \geq \frac{1}{4} \left[ m_0 + \int_{v_0+\delta_1}^{v_f} \frac{h}{v} \frac{dm}{dv} dv \right]$$

$$\geq \frac{1}{4} \left[ m_0 + \int_{v_0+\delta_1}^{v_f} \frac{dm}{dv} dv \right]$$

$$\geq \frac{1}{4} m_0 + \frac{1}{4} [m(v)]_{v_0+\delta_1}^{v_f}$$

$$\geq 0.$$

(3.35)

This shows that the ANEC with the weight function \(r^4\) is satisfied for the evaporating black hole.
4 Summary and discussions

We have investigated two holographic models of evaporating black holes. First, by perturbing the 4-dimensional black droplet solution associated with BTZ black holes on the AdS boundary [16], we have constructed the bulk geometry dual to the boundary field theory in a time-dependent BTZ type black hole with the horizon area decreasing. We found that negative energy flux going into the time-dependent horizon always appears by calculating the boundary stress-energy tensor. This calculation agrees with Hawking’s picture of evaporating black holes [1], where the horizon area decreases by absorbing negative quantum energy flux. According to the energy conservation, the total energy and the matter entropy outside the evaporating black hole should increase by the Hawking radiation. Our result supports the generalized second law (GSL) [1, 13], which states that the total of the gravitational entropy and the matter entropy outside the horizon should not decrease.

The existence of the negative energy flux shown in this paper implies a local violation of the null energy condition (NEC). As first shown in a class of holographic theories [26], ANEC is satisfied along a complete achronal null geodesic in flat spacetimes. In curved spacetime, however, we cannot expect that the ANEC is generically satisfied, except some particular class of curved spacetimes [18, 19]. It was shown for some semi-analytic black droplet solutions [28, 29] that along the incomplete null geodesics, which terminate at a singularity inside the boundary black hole, the ANEC is violated due to the negative divergence of the null energy toward the singularity. In our present holographic model, the null geodesic generators along the event horizon begin from the formation of the boundary black hole, and terminate at the final instant of the black hole evaporation, where a naked singularity appears, as shown in figure 2. Although this situation is very similar to the incomplete null geodesics observed in refs. [28, 29], we can expect the ANEC to hold in our case, as the singularity at the very final moment of the complete black hole evaporation is a weak singularity in the sense that the mass \( m(v) \) becomes zero at the final singularity. In addition, the null geodesic congruence along the horizon expands from the point at the formation of the black hole, and shrinks to zero again at the final zero mass singularity. This behavior of the null geodesic congruence is very similar to that in a spatially compact \( S^3 \) universe, in which a congruence expands from the south pole and then shrinks to zero at the north pole. This peculiar feature enables us to prove the ANEC along the null geodesic generator of the event horizon, just like the case considered in spatially compact universes [18, 19]. This indicates that the total negative energy flux is bounded from below, and is compensated by a positive energy flux in the early stage of the black hole formation.

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A The derivation of eq. (3.26)

In the Vaidya metric (3.1), $R_{\mu\nu} = 0$ except $R_{vv}$. In the double null coordinates (3.14), $R_{UV} = R_{\theta\theta} = 0$ yields

$$
\dot{f} = -\frac{2\dot{r}}{r}, \quad \dot{r} = -\frac{1}{r} \left( \dot{r}r' + \frac{1}{4}e^f \right). \tag{A.1}
$$

By the fact that $f = 0$ and $\partial_U \sim h^{-1} \partial_r$ along the EH where $U = 0$, the asymptotic behavior of $\dot{r}$ and $f'$ in eq. (A.1) becomes

$$
\dot{r} \sim \left(1 - \frac{v f}{v}\right)^{-1}, \quad f' \sim \left(1 - \frac{v f}{v}\right)^{-a-1} \tag{A.2}
$$

from eqs. (3.4), (3.6), and (3.10).

By eq. (3.12), $g_{(2)VV}$ is given by

$$
g_{(2)VV} = -\frac{R_{VV}}{2} = \frac{\dot{r} - \frac{\dot{r}}{r}}{r}. \tag{A.3}
$$

Substituting this into the first term of $I$ in eq. (3.24), we obtain

$$
\begin{align*}
\int z_1^2 \partial_U (g_{(2)VV}) U_2 \, dV &= \int z_1^2 \dot{z}_1 \left[ \frac{\dot{r}}{r} - \frac{\dot{r}'}{r^2} r - \frac{2\dot{r}}{r^3} \left( \dot{r}r' + \frac{1}{4} \right) \right] \, dV \\
&= c^4 \int \dot{r} \left[ r^2 \dot{r}'' - r \dot{r}r'' - \frac{2\dot{r}}{r} \left( \dot{r}r' + \frac{1}{4} \right) \right] \, dV \\
&= c^4 \int r \dot{r} dV. \tag{A.4}
\end{align*}
$$

Here, in the second line, we used the condition $f(0,V) = 0$ along the EH and also eqs. (3.23), (A.1), and (3.20). In the third line, we performed integration by parts for the first term in the second line, and used the fact that the boundary term $r^2 \dot{r} \dot{r}$ vanishes in the limit $v \to v_f$ by eqs. (3.6), (3.10), and (A.2).

Similarly, we evaluate the second term of $I$ in eq. (3.24) as

$$
- \int f' U_2 U_2 \, dV = - \int \frac{\dot{r}'}{r} U_2^2 \, dV = c^4 \int \left( \dot{r}r' + \frac{1}{4} \right) \dot{r}^2 dV, \tag{A.5}
$$

where we performed integration by parts in the first equality and used eq. (A.1). Note that the boundary term $f' U_2^2$ vanishes in the limit $v \to v_f$ by eqs. (3.6), (3.10), and (A.2). To derive the second equality, we used eqs. (A.2) and (3.20). Substituting eqs. (A.4) and (A.5) into $I$ in eq. (3.24) and using eqs. (3.27) and (A.3), we finally obtain the inequality (3.26).

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