On trigonometric sums with random frequencies

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Abstract

We prove that if \( I_k \) are disjoint blocks of positive integers and \( n_k \) are independent random variables with uniform distribution on \( I_k \), then

\[
N^{-1/2} \sum_{k=1}^{N} (\sin 2\pi n_k x - \mathbb{E}(\sin 2\pi n_k x))
\]

has, with probability 1, a mixed Gaussian limit distribution relative to the interval \((0,1)\) equipped with Lebesgue measure. We also investigate the case when \( n_k \) have continuous uniform distribution on disjoint intervals \( I_k \) on the positive axis.

1 Introduction

Salem and Zygmund \cite{7} proved that if \( (n_k) \) is a sequence of positive integers satisfying the Hadamard gap condition

\[
n_{k+1}/n_k \geq q > 1 \quad (k = 1, 2, \ldots)
\]

then the sequence \( \sin 2\pi n_k x \), \( k \geq 1 \) obeys the central limit theorem, i.e.

\[
N^{-1/2} \sum_{k=1}^{N} \sin 2\pi n_k x \xrightarrow{d} N(0, 1/2)
\]

with respect the probability space \((0,1)\) equipped with Borel sets and Lebesgue measure. Here the exponential growth condition \((1.1)\) can be weakened, but as Erdős

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[3] showed, there exists a sequence \((n_k)\) growing faster than \(e^{\sqrt{k}}\) such that the CLT (1.2) fails. On the other hand, using random constructions one can find slowly growing sequences \((n_k)\) satisfying (1.2). Salem and Zygmund [8] proved that if \(\xi_1, \xi_2, \ldots\) are independent random variables on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) taking the values 0 and 1 with probability \(1/2 - 1/2\) and \((n_k)\) denotes the set of indices \(j\) such that \(\xi_j = 1\), then with \(\mathbb{P}\)-probability 1, the CLT (1.2) holds. For this sequence \((n_k)\) we have \(n_k \sim 2k\) and by the theorem of "pure heads" we have \(n_{k+1} - n_k = O(\log k)\). Berkes [1] showed that if \(N = \bigcup_{k=1}^{\infty} I_k\) where \(I_1, I_2, \ldots\) are disjoint intervals of positive integers such that \(|I_k| \to \infty\), and \(n_1, n_2, \ldots\) are independent random variables on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that \(n_k\) is uniformly distributed on \(I_k\), then with \(\mathbb{P}\)-probability 1, \(\sin 2\pi n_k x\) satisfies the CLT (1.2). Thus, given any positive sequence \(\omega_k \to \infty\), there exists an increasing sequence \((n_k)\) of positive integers such that \(n_{k+1} - n_k = O(\omega_k)\) and \(\sin 2\pi n_k x\) satisfies (1.2). In [1] the question was raised if the CLT (1.2) can hold for any sequence \((n_k)\) with \(n_{k+1} - n_k = O(1)\). Bobkov and Götze [2] showed that the answer to this question is negative, and in particular, if in the construction in [1] we choose \(|I_k| = d\) for \(k = 1, 2, \ldots\), then with probability 1, the limit distribution of \(N^{-1/2} \sum_{k=1}^{N} \sin 2\pi n_k x\) is mixed normal. On the other hand, Fukuyama [4] showed, using another type of random construction, that for any \(0 < \sigma^2 < 1/2\) there exists a sequence \((n_k)\) of integers with bounded gaps \(n_{k+1} - n_k\) such that (1.2) holds with a limiting normal distribution with variance \(\sigma^2\). The purpose of the present paper is to return to the random models in [1], [2] and investigate the case of constant block sizes \(|I_k| = d\), allowing arbitrary gaps between the blocks. We will prove the following result.

**Theorem 1.** Let \(I_1, I_2, \ldots\) be disjoint blocks of consecutive positive integers with size \(d\) and let \(n_1, n_2, \ldots\) be a sequence of independent random variables on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) such that \(n_k\) is uniformly distributed over \(I_k\). Let \(\lambda_k(x) = \mathbb{E}(\sin 2\pi n_k x)\). Then \(\mathbb{P}\)-almost surely

\[
\frac{1}{\sqrt{N}} \sum_{k=1}^{N} (\sin 2\pi n_k x - \lambda_k(x)) \overset{d}{\to} N(0, g) \tag{1.3}
\]

over the probability space \(((0, 1), \mathcal{B}, \lambda)\), where \(\mathcal{B}\) is the Borel \(\sigma\)-algebra in \((0, 1)\), \(\lambda\) is the Lebesgue measure,

\[
g(x) = \frac{1}{2} \left(1 - \frac{\sin^2 d\pi x}{d^2 \sin^2 \pi x}\right) \tag{1.4}
\]

and \(N(0, g)\) denotes the distribution with characteristic function \(\int_0^1 e^{-g(x)t^2} dx\).

Here \(g \geq 0\) and \(N(0, g)\) is the distribution of \(\sqrt{g}\zeta\), where \(\zeta\) is a standard normal random variable on \((0, 1)\), independent of \(g\). Clearly, \(N(0, g)\) is a variance mixture of zero mean Gaussian distributions.

Note that \(\sum_{k=1}^{N} \lambda_k(x) = \mathbb{E}(\sum_{k=1}^{N} \sin 2\pi n_k x)\) is the averaged version of \(\sum_{k=1}^{N} \sin 2\pi n_k x\), a nonrandom trigonometric sum and Theorem 1 states that the fluctuations of the
random trigonometric sum \( \sum_{k=1}^{N} \sin 2\pi n_k x \) around its nonrandom average part always have a mixed normal limit distribution. If \( \bigcup_{k=1}^{m} I_k = \mathbb{N} \), i.e. there are no gaps between the blocks \( I_k \), then \( \sum_{k=1}^{n} \lambda_k(x) = O(1) \) for any fixed \( x \) and thus (1.3) holds without \( \lambda_k(x) \), yielding the result of Bobkov and Götze [2]. Letting \( \Delta_k \) denote the number of integers between \( I_k \) and \( I_{k+1} \) (the "gaps"), we will see that the CLT (1.3) also holds with \( \lambda_k(x) = 0 \) if \( \Delta_k \) is nondecreasing and \( \Delta_k = O(k^\gamma) \) for some \( \gamma < 1/4 \). If \( \Delta_k \) grows at least exponentially, then so does the sequence \( (A_k) \), where \( A_k \) denotes the smallest integer of \( I_k \). Now

\[
\lambda_k(x) = \frac{\sin d\pi x}{d\sin \pi x} \sin 2\pi(A_k + d/2 - 1/2)x
\]

and from the CLT of Salem and Zygmund [7] it follows that the limit distribution of \( N^{-1/2} \sum_{k=1}^{N} \lambda_k(x) \) is \( N(0, g^*) \), where

\[
g^*(x) = \frac{\sin^2 d\pi x}{2d^2 \sin^2 \pi x}.
\]

By Theorem 1, the limit distribution of \( N^{-1/2} \sum_{k=1}^{N} (\sin 2\pi n_k x - \lambda_k(x)) \) is \( N(0, g) \) with \( g \) in (1.4) and the convolution of these two mixed Gaussian laws is \( N(0, 1/2) \), which is exactly the limit distribution of \( N^{-1/2} \sum_{k=1}^{N} \sin 2\pi n_k x \) by the theorem of Salem and Zygmund, since \( (n_k) \) grows exponentially. Thus the pure Gaussian limit distribution of \( N^{-1/2} \sum_{k=1}^{N} \sin 2\pi n_k x \) is obtained as the combination of two mixed Gaussian distributions \( N(0, g) \) with \( g \) in (1.4) and \( N(0, g^*) \) with \( g^* \) in (1.6).

It is worth noting that for any fixed \( x \in (0, 1) \), \( \sin 2\pi n_k x - \lambda_k(x) \) are independent, uniformly bounded mean zero random variables on \( (\Omega, \mathbb{A}, \mathbb{P}) \) and

\[
\mathbb{E}(\sin 2\pi n_k x - \lambda_k(x))^2 = \mathbb{E}(\sin^2 2\pi n_k x) - \lambda^2_k(x)
\]

\[
= \frac{1}{d} \sum_{j \in I_k} \sin^2 2\pi j x - \left( \frac{1}{d} \sum_{j \in I_k} \sin 2\pi j x \right)^2 = g(x)
\]

by elementary calculations. Thus by the law of the iterated logarithm we have for any fixed \( x \in (0, 1) \) with \( \mathbb{P} \)-probability 1

\[
\limsup_{N \to \infty} \frac{1}{\sqrt{2N \log \log N}} \sum_{k=1}^{N} (\sin 2\pi n_k x - \lambda_k(x)) = \sqrt{g(x)}.
\]

By Fubini’s theorem, with \( \mathbb{P} \)-probability 1 relation (1.7) holds for almost every \( x \in (0, 1) \) with respect to Lebesgue measure, yielding the LIL corresponding to (1.3). Actually, the previous argument also shows that for any fixed \( x \in (0, 1) \) we have (1.3) over the probability space \( (\Omega, \mathbb{A}, \mathbb{P}) \), with \( N(0, g) \) replaced by \( N(0, g(x)) \). However, Fubini’s theorem does not work for distributional results and thus we cannot interchange the role of \( x \in (0, 1) \) and \( \omega \in \Omega \) and we will need an elaborate argument in Section 2 to prove Theorem 1.

Formula (1.4) shows that for any \( 0 < x < 1 \) we have \( \lim_{x \to \infty} g(x) = 1/2 \) and thus for large \( d \) the sequence \( \sin 2\pi n_k x - \lambda_k(x) \) nearly satisfies the ordinary CLT and LIL
with limit distribution $N(0, 1/2)$ and $\limsup = 1/2$, just as lacunary trigonometric series with exponential gaps. Formally, this is not surprising since for large $d$ the expected gaps $\mathbb{E}(n_{k+1} - n_k)$ in our sequence are large. As the pictures of $g$ for $d = 3$ and $d = 10$ below show, however, the near CLT and LIL actually hold for relatively small values of $d$ such as $d = 10$. Thus the reason of the near CLT and LIL is not solely large gaps in the sequence $(n_k)$ but the random fluctuations of the sequence $(n_k)$ as well.

The analogue of Theorem 1 is valid also in the case when $n_1, n_2, \ldots$ have continuous uniform distribution over the intervals $I_1, I_2, \ldots$. To formulate the result, define the probability measure $\mu$ on the Borel sets of $\mathbb{R}$ by

$$\mu(A) = \frac{1}{\pi} \int_{A} \left( \frac{\sin x}{x} \right)^2 \, dx, \quad A \subset \mathbb{R}.$$ 

**Theorem 2.** Let $n_1, n_2, \ldots$ be a sequence of independent random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $n_k$ has continuous uniform distribution on the interval $[A_k, A_k + B]$, where $A_{k+1} - A_k \geq B + 2$, $k = 1, 2, \ldots$ Let $\lambda_k(x) = \mathbb{E}(\sin n_k x)$. Then $\mathbb{P}$-almost surely

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N} (\sin n_k x - \lambda_k(x)) \xrightarrow{d} F$$

(1.8)

with respect to the probability space $(\mathbb{R}, \mathcal{B}, \mu)$, where the characteristic function of $F$ is

$$\phi(\lambda) = \int_{-\infty}^{+\infty} \exp \left( -\frac{\lambda^2}{4} \left( 1 - \frac{4 \sin^2(Bx/2)}{B^2 x^2} \right) \right) \, d\mu(x).$$

(1.9)

**2 Proofs**

We will give the proof of Theorem 2, where the calculations are slightly simpler. Let

$$\varphi_k(x) = \sin n_k x - \mathbb{E}(\sin n_k x)$$
and
\[ T_N = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \varphi_k(x). \]

By \( A_{k+1} - A_k \geq B + 2 \) and the fact that
\[ \int_{-\infty}^{+\infty} \cos \alpha x \left( \frac{\sin x}{x} \right)^2 \, dx = 0 \quad \text{for} \ |\alpha| > 2 \]  
(2.10)
(see e.g. Hartman [5]) it follows that for every fixed \( \omega \in \Omega \) the functions \( \varphi_k \) are orthogonal over \( L^2(\mathbb{R}) \) and thus elementary algebra shows that the \( L^2(\mathbb{R}) \) norm of \( |T_M - T_{N^3}| \) is at most \( C/\sqrt{N} \) for \( N^3 \leq M \leq (N+1)^3 \) with an absolute constant \( C \).

Hence to prove (1.8) it suffices to show that \( T_{N^3} \xrightarrow{d} F \) \( \mathbb{P} \)-a.s.

A simple calculation shows that
\[ \lambda_k(x) = \mathbb{E}(\sin n_k x) = \frac{1}{B} \int_{A_k}^{A_k+B} \sin tx \, dt = \frac{1}{B} (\cos A_k x - \cos(A_k + B)x) \]
\[ = \frac{2 \sin(Bx/2)}{Bx} \sin(A_k + B/2)x \]  
(2.11)
and
\[ \mathbb{E}(\cos 2n_k x) = \frac{1}{B} \int_{A_k}^{A_k+B} \cos 2tx \, dt = \frac{\sin Bx}{Bx} \cos(2A_k + B)x. \]

Thus
\[ \mathbb{E}\varphi_k^2(x) = \mathbb{E}(\sin^2 n_k x) - \lambda_k^2(x) = \frac{1}{2} (1 - \mathbb{E}(\cos 2n_k x)) - \lambda_k^2(x) \]
\[ = \frac{1}{2} - \frac{\sin Bx}{2Bx} \cos(2A_k + B)x - \frac{4 \sin^2(Bx/2)}{B^2x^2} \sin^2(A_k + B/2)x \]
\[ = \left( \frac{1}{2} - \frac{2 \sin^2(Bx/2)}{B^2x^2} \right) + \left( \frac{2 \sin^2(Bx/2)}{B^2x^2} - \frac{\sin Bx}{2Bx} \right) \cos(2A_k + B)x. \]

From (2.10), \( A_{k+1} - A_k \geq B + 2 \) and elementary trigonometric identities it follows that the functions \( \cos(2A_k + B)x \) are orthogonal in \( L^2(\mathbb{R}) \) and thus the Rademacher-Menushov convergence theorem implies that \( \sum_{k=1}^{\infty} k^{-1} \cos(2A_k + B)x \) converges \( \mu \)-almost everywhere. Consequently, the Kronecker lemma implies
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \cos(2A_k + B)x = 0 \quad \mu \text{-a.e.} \]

and thus
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \mathbb{E}\varphi_k^2(x) = \frac{1}{2} \left( 1 - \frac{4 \sin^2(Bx/2)}{B^2x^2} \right) \quad \mu \text{-a.e.} \]

Since for fixed \( x \) \( \varphi_k^2(x) - \mathbb{E}\varphi_k^2(x), k = 1, 2, \ldots \) are independent, uniformly bounded, zero mean random variables, the strong law of large numbers yields
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} (\varphi_k^2(x) - \mathbb{E}\varphi_k^2(x)) = 0 \quad \mathbb{P} \text{-a.s.} \]
and thus we conclude that for \( \mu \)-a.e. \( x \) we have \( \mathbb{P} \)-almost surely

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \varphi_k^2(x) = \frac{1}{2} \left( 1 - \frac{4 \sin^2(Bx/2)}{B^2 x^2} \right). \tag{2.12}
\]

By Fubini’s theorem, \( \mathbb{P} \)-almost surely the last relation holds for \( \mu \)-almost all \( x \in \mathbb{R} \). Fix \( \lambda \in \mathbb{R} \). Using \( |\varphi_k(x)| \leq 2 \) and

\[
\exp(z) = (1 + z) \exp\left( \frac{z^2}{2} + o(z^2) \right) \quad z \to 0
\]

we get

\[
\exp\left( \frac{i \lambda}{\sqrt{N}} \varphi_k(x) \right) = \left( 1 + \frac{i \lambda}{\sqrt{N}} \varphi_k(x) \right) \exp\left( -\frac{\lambda^2 \varphi_k^2(x)}{2N} + o\left( \frac{\lambda^2 \varphi_k^2(x)}{N} \right) \right)
\]

as \( N \to \infty \), uniformly in \( x \) and the implicit variable \( \omega \in \Omega \). Thus the characteristic function

\[
\phi_{T_N}(\lambda) = \int_{-\infty}^{\infty} \exp\left( \frac{i \lambda}{\sqrt{N}} \sum_{k=1}^{N} \varphi_k(x) \right) d\mu(x) = \int_{-\infty}^{\infty} \exp\left( \frac{i \lambda}{\sqrt{N}} \sum_{k=1}^{N} \varphi_k(x, \omega) \right) d\mu(x)
\]

of \( T_N \) with respect to the probability space \( (\mathbb{R}, \mathcal{B}, \mu) \) can be written as

\[
\phi_{T_N}(\lambda) = \prod_{k=1}^{N} \left( 1 + \frac{i \lambda}{\sqrt{N}} \varphi_k(x) \right)
\]

\[
\times \exp\left( -(1 + o(1)) \frac{\lambda^2}{2N} \sum_{k=1}^{N} \varphi_k^2(x) \right) \frac{1}{\pi} \left( \sin \frac{x}{x} \right)^2 dx.
\]

For simplicity let

\[
\hat{g}(x) = \frac{1}{2} \left( 1 - \frac{4 \sin^2(Bx/2)}{B^2 x^2} \right).
\]

Using \( 1 + x \leq e^x \) and \( |\varphi_k(x)| \leq 2 \) we get

\[
\left| \prod_{k=1}^{N} \left( 1 + \frac{i \lambda}{\sqrt{N}} \varphi_k(x) \right) \right| = \prod_{k=1}^{N} \left( 1 + \frac{\lambda^2}{N} \varphi_k^2(x) \right)^{1/2}
\]

\[
\leq \exp\left( \frac{\lambda^2}{2N} \sum_{k=1}^{N} \varphi_k^2(x) \right) \leq e^{2\lambda^2}
\]

and thus the dominated convergence theorem and (2.12) imply \( \mathbb{P} \)-almost surely

\[
\phi_{T_N}(\lambda) = \prod_{k=1}^{\infty} \left( 1 + \frac{i \lambda}{\sqrt{N}} \varphi_k(x) \right) \exp\left( -\lambda^2 \hat{g}(x)/2 \right) \frac{1}{\pi} \left( \sin \frac{x}{x} \right)^2 dx + o(1).
\]
Since the characteristic function \( \phi(\lambda) \) of \( F \) in (1.8) is given by (1.9), to prove that \( T_N \overset{d}{\to} F \) \( \mathbb{P} \)-a.s., it remains to show that letting

\[
\Gamma_N = \int_{-\infty}^{+\infty} \left[ \prod_{k=1}^{N} \left( 1 + \frac{i\lambda}{\sqrt{N}} \varphi_k(x) \right) - 1 \right] \exp \left( -\lambda^2 g(x)/2 \right) \frac{1}{\pi} \left( \frac{\sin x}{x} \right)^2 dx,
\]

we have

\[
\Gamma_N \overset{\mathbb{P}}{\to} 0.
\]

Clearly

\[
\mathbb{E} |\Gamma_N|^2 = \mathbb{E} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \prod_{k=1}^{N} \left( 1 + \frac{i\lambda}{\sqrt{N}} \varphi_k(x) \right) - 1 \right] \left[ \prod_{k=1}^{N} \left( 1 - \frac{i\lambda}{\sqrt{N}} \varphi_k(y) \right) - 1 \right]
\times \exp \left( -\lambda^2 g(x)/2 \right) \exp \left( -\lambda^2 g(y)/2 \right) d\mu(x)d\mu(y).
\]

(2.14)

Now using the independence of the \( \varphi_k \) and \( \mathbb{E} \varphi_k(x) = \mathbb{E} \varphi_k(y) = 0 \) we get

\[
\mathbb{E} \left[ \prod_{k=1}^{N} \left( 1 + \frac{i\lambda}{\sqrt{N}} \varphi_k(x) \right) - 1 \right] \left[ \prod_{k=1}^{N} \left( 1 - \frac{i\lambda}{\sqrt{N}} \varphi_k(y) \right) - 1 \right]
= \mathbb{E} \left[ \prod_{k=1}^{N} \left( 1 + \frac{i\lambda}{\sqrt{N}} \varphi_k(x) \right) \left( 1 - \frac{i\lambda}{\sqrt{N}} \varphi_k(y) \right) \right] - 1
= \mathbb{E} \left[ \prod_{k=1}^{N} \left( 1 + \frac{i\lambda}{\sqrt{N}} \varphi_k(x) - \frac{i\lambda}{\sqrt{N}} \varphi_k(y) + \frac{\lambda^2}{N} \varphi_k(x) \varphi_k(y) \right) \right] - 1
= \prod_{k=1}^{N} \left( 1 + \frac{\lambda^2}{N} \Psi_k(x,y) \right) - 1,
\]

where \( \Psi_k(x,y) = \mathbb{E} \varphi_k(x) \varphi_k(y) \). Thus interchanging the expectation with the double integral in (2.14) we get

\[
\mathbb{E} |\Gamma_N|^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \prod_{k=1}^{N} \left( 1 + \frac{\lambda^2}{N} \Psi_k(x,y) \right) - 1 \right] \times
\times \exp \left( -\lambda^2 g(x)/2 - \lambda^2 g(y)/2 \right) d\mu(x)d\mu(y)
\leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \prod_{k=1}^{N} \left( 1 + \frac{\lambda^2}{N} \Psi_k(x,y) \right) - 1 \right] d\mu(x)d\mu(y).
\]

Using \( |\Psi_k(x,y)| \leq 4 \) and \( |\log(1 + x) - x| \leq Cx^2 \) for all \( |x| \leq 1 \) and some constant \( C > 0 \), one deduces for all sufficiently large \( N \),

\[
\left| \log \prod_{k=1}^{N} \left( 1 + \frac{\lambda^2}{N} \Psi_k(x,y) \right) - \sum_{k=1}^{N} \frac{\lambda^2}{N} \Psi_k(x,y) \right| \leq \frac{16C\lambda^4}{N}.
\]

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Thus letting 
\[ G_N(x, y) := \sum_{k=1}^{N} \frac{\lambda_k^2}{N} \Psi_k(x, y) \]
we get, using \( G_N(x, y) \leq 4\lambda^2 \), that 
\[ \prod_{k=1}^{N} \left( 1 + \frac{\lambda_k^2}{N} \Psi_k(x, y) \right) = \exp \left\{ G_N(x, y) + O(\lambda^4/N) \right\} = 1 + O(\lambda^2) + O(1/N). \]
Thus 
\[ \mathbb{E} |\Gamma_N|^2 \leq C_1 \left( \frac{1}{N^2} + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |G_N(x, y)| \, d\mu(x) \, d\mu(y) \right) \]
for some constant \( C_1 \). In view of \( A_{k+1} - A_k \geq B + 2 \) and (2.10), for any \( \lambda_1 \in [A_k, A_k + B] \), \( \lambda_2 \in [A_l, A_l + B] \), \( k \neq l \), \( \sin \lambda_1 x \) and \( \sin \lambda_2 x \) are orthogonal in \( L^2_{\mu}(\mathbb{R}) \), which implies that \( \varphi_k \) and \( \varphi_\ell \) are also orthogonal in \( L^2_{\mu}(\mathbb{R}) \). Since \( \Psi_k(x, y) \Psi_l(x, y) = \mathbb{E} \varphi_k(x) \varphi_l(x) \varphi_k(y) \varphi_\ell(y) \), it follows that 
\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi_k(x, y) \Psi_l(x, y) \, d\mu(x) \, d\mu(y) = 0 \quad \text{for} \quad k \neq l \]
and thus by the Cauchy-Schwarz inequality the last integral in (2.15) is \( O(N^{-1/2}) \). Hence \( \mathbb{E} |\Gamma_N|^2 = O(N^{-1/2}) \) and thus \( \sum_{N \in \mathbb{N}} \mathbb{E} |\Gamma_N|^2 < \infty \), implying \( \sum_{N \in \mathbb{N}} |\Gamma_N|^2 < \infty \) and \( \Gamma_{N^3} \to 0 \, \mathbb{P}\text{-a.s.} \), completing the proof of (1.8).

In conclusion we prove the claim made after Theorem 1, namely that if the size of the gaps \( \Delta_k \) between the blocks \( I_k \) is nondecreasing and satisfies 
\[ \Delta_k = O(k^{\gamma}), \quad \gamma < 1/4 \quad (2.16) \]
then 
\[ N^{-1/2} \sum_{k=1}^{N} \lambda_k(x) \longrightarrow 0 \quad \text{a.s.} \]
and thus (1.3) holds with \( \lambda_k(x) = 0 \). Since we proved our main limit theorem in the continuous case of Theorem 2, we prove our claim also in the context of Theorem 2 in which case we also assume that the intervals \([A_k, A_k + B]\) have integer endpoints. In view of (2.11) it suffices to show that 
\[ N^{-1/2} \sum_{k=1}^{N} e^{iA_k x} \longrightarrow 0 \quad \text{a.s.} \quad (2.17) \]
and here nothing changes if we replace \( x \) by \( 2\pi x \). In the case of constant \( \Delta_k \) we have \( A_k = Dk + D^* \) for some constants \( D > 0 \) and \( D^* \) and (2.17) is obvious by an explicit
computation of the sum. Thus we can assume $\Delta_k \uparrow \infty$, and then also $A_{k+1} - A_k \uparrow \infty$. Recalling that the $A_k$ are integers, let us break the sum $\sum_{k=1}^{N} e^{2\pi i A_k x}$ into subsums

$$Z_{N,r} = \sum_{k \leq N, A_{k+1} - A_k = r} e^{2\pi i A_k x}, \quad r = 1, 2, \ldots .$$  \tag{2.18}$$

Clearly $Z_{N,r}$ consists of $M_r$ consecutive terms of $\sum_{k=1}^{N} e^{2\pi i A_k x}$ for some $M_r \geq 0$ and thus in the case $M_r \geq 1$ we have for some integer $P_r \geq 0$,

$$|Z_{N,r}| = \left| \sum_{j=0}^{M_r-1} e^{2\pi i (P_r + jr)x} \right| = \left| \sum_{j=0}^{M_r-1} e^{2\pi i jrx} \right| \leq \frac{1}{|e^{2\pi i rx} - 1|} \leq \frac{C}{\langle rx \rangle},$$

except when $rx$ is an integer, where $C$ is an absolute constant and $\langle t \rangle$ denotes the distance of $t$ from the nearest integer. From a well known result in Diophantine approximation theory (see e.g. Kuipers and Niederreiter [6], Definition 3.3. on p. 121 and Exercise 3.5 on page 130), for every $\varepsilon > 0$ and almost all $x$ in the sense of Lebesgue measure we have $(nx) \geq cn^{-1(1+\varepsilon)}$ for some constant $c = c(x) > 0$ and all $n \geq 1$. This shows that $Z_{N,r} = O(r^{1+\varepsilon})$ a.e. and since by (2.16) the largest $r$ actually occurring in breaking $\sum_{k=1}^{N} e^{2\pi i A_k x}$ into a sum of $Z_{N,r}$’s is at most $C_1 N^{\gamma}$, we have

$$\left| \sum_{k=1}^{N} e^{2\pi i A_k x} \right| \leq C_2 \sum_{r \leq C_1 N^{\gamma}} r^{1+\varepsilon} = o(\sqrt{N}) \quad \text{a.e.}$$

by $\gamma < 1/4$, upon choosing $\varepsilon$ small enough.

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