VIRTUAL RESIDUE AND GENERALIZED CAYLEY-BACHARACH THEOREM

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ABSTRACT. Using virtual residue, which is a generalization of Grothendieck residue, we generalized Cayley-Bacharach Theorem to the cases with positive dimensions.

1. INTRODUCTION

Let \( C_1, C_2 \subset \mathbb{P}^2 \) be plane curves of degrees \( d \) and \( e \) respectively, meeting in a collection of \( de \) distinct points \( \Gamma := \{p_1, \cdots, p_{de}\} \). Cayley-Bacharach Theorem said that if \( C \subset \mathbb{P}^2 \) is any plane curve of degree \( d + e - 3 \) containing all but one point of \( \Gamma \), then \( C \) contains all of \( \Gamma \), see [3, Theorem CB4]. The extension of Cayley-Bacharach property on projective manifolds had been proved to be related to Fujita conjecture, and the construction of special bundles, see [6], [7], [8] and [9].

By using Grothendieck residue, Griffiths and Harris [5, Chapter5] proved the following generalized Cayley-Bacharach theorem. Let \( \mathcal{M} \) be a compact complex manifold and let \( E \) be a holomorphic bundle over \( \mathcal{M} \) with rank \( E = \dim \mathcal{M} = n \). Let \( \tilde{s} \) be a holomorphic section of \( E \) who zero loci \( Z \) are isolated points. If \( Z \) consists of distinct simple points, then each \( D \in |K \otimes \det E| \) that passes through all but one point of \( Z \) necessarily contains that remaining points, where \( K_{\mathcal{M}} \) is the canonical line bundle of \( \mathcal{M} \).

In this paper we will deal with the cases where some connected components of \( Z \) may have positive dimensions.

Let \( M \) be a complex manifold and let \( V \) be a holomorphic bundle over \( M \) with rank \( V = \dim M = n \). Let \( s \) be a holomorphic section of \( V \) with compact zero loci \( Z \). Given any holomorphic section

\[ \psi \in \Gamma(M, K_M \otimes \det V), \]

using the Koszul complex of \((V,s)\), the authors [2] constructed a closed form \( \eta_\psi \in \Omega^{n,n-1}(M \setminus Z) \) via Griffiths-Harris’s construction [5, Chapter 5]. Then they define the virtual residue as

\[ \text{Res}_Z \frac{\psi}{s} := \left( \frac{1}{2\pi i} \right)^n \int_N \eta_\psi \in \mathbb{C} \]

where \( N \) is a real \( 2n - 1 \) dimensional piecewise smooth compact subset of \( M \) that “surrounds \( Z \)”, in the sense that \( N = \partial T \) for some compact domain \( T \subset M \), which contains \( Z \) and is homotopically equivalent to \( Z \). When \( \dim Z > 0 \), it is a generalization of Grothendieck residue. It vanishes whenever \( M \) is compact by Stokes theorem.

Denote \( \mathcal{A}^{ij} (\wedge^k V \otimes \wedge^l V^*) \) to be the sheaf of smooth \((i,j)\) forms on \( M \) valued in \( \wedge^k V \otimes \wedge^l V^* \). The Hermitian metrics of \( M \) and \( V \) induce a metric on the bundle
which corresponds to the sheaf $\oplus_{i,j,k,l} A^{i,j} (\wedge^k V \otimes \wedge^l V^*)$. Denote this metric by $\langle \cdot, \cdot \rangle (z)$ for $z \in M$ and set $|\alpha| (z) = \sqrt{\langle \alpha, \alpha \rangle (z)}$.

Denote $\Omega^{(i,j)} (\wedge^k V \otimes \wedge^l V^*) := \Gamma (M, A^{i,j} (\wedge^k V \otimes \wedge^l V^*))$ and assign its element $\alpha$ to have degree $|\alpha| = i + j + k - \ell$.

Given $u \in \Omega^{(i,j)} (\wedge^k V)$ and $k \geq \ell$, we define

$$u_{\ell,j} : \Omega^{(p,q)} (\wedge^l V^*) \longrightarrow \Omega^{(p+i,q+j)} (\wedge^{k-l} V) \tag{1.2}$$

where for $\theta \in \Omega^{(p,q)} (\wedge^l V^*)$, the $u_{\ell,j}$ is determined by

$$(u_{\ell,j}, \nu^*) = (-1)^{(i+j)+(p+q)+\ell}(u, \theta \wedge \nu^*), \quad \forall \nu^* \in \Lambda^0 (\wedge^{k-l} V^*).$$

where $(\cdot, \cdot)$ is the dual pairing between $\wedge^k V, \wedge^l V^*$.

Applying the integral representation for the virtual residue $\Res_{\alpha} (\psi) \tag{[2, Theorem 1.1]}$ to the case where $M$ is compact, we have

**Theorem 1.1.** Let $M$ be a compact complex manifold. Pick a Hermitian metric $h$ on $V$ and let $\nabla$ be its associated Hermitian connection with $\nabla^{0,1} = 0$. Let $\xi = -\langle *, s \rangle$ be a smooth section of $V^*$ and

$$S = -|s|^2 + \overline{\partial} \xi \in \oplus_{p=0,1} \Omega^{(p,0)} (\Lambda^p V^*).$$

One has

$$\Res \frac{\psi}{s} = \frac{(-1)^n}{(2\pi i)^n} \int_M (\psi, e^{-S}) = 0. \tag{1.3}$$

Here $\langle \cdot, \cdot \rangle$ is the operation contracting $det V$ with $det V^*$ so that $\psi, e^{-S} \in \Lambda^{*,*}$.

Assuming that all the connected components $Z_i \subset Z = s^{-1}(0)$ are smooth, and $V$ is splitting over $Z_i$, $V|_{Z_i} = V_i \oplus N_i$, where $N_i = N_{Z_i/M}$. Let $j : Z_i \rightarrow M$ be the embedding. Then we evaluate the integral in (1.3) as following

**Theorem 1.2.**

$$\frac{(-1)^n}{(2\pi i)^n} \int_M (\psi, e^{-S}) = \sum_{Z_i} \frac{(-1)^n}{(2\pi i)^n} \int_{Z_i} \frac{\psi}{det N_i ((1 + R_i^V)/ - 2\pi \sqrt{-1})} = 0, \tag{1.4}$$

where $R_i^V := -(ds)^{-1} P^{l_{im}} ds R(., j_* P^V_i) \in T^* Z_i \otimes V_i^* \otimes \End N_i$.

This can be considered as a mathematical interpretation of the residue formula used in [3, (2.11)].

Let $Z_1$ be one of the zero dimension components, then we have the following generalized Cayley- Bacharach Theorem.

**Corollary 1.3.** Under the assumptions of Theorem 1.2. If $\psi$ is vanishing on all components of $Z$ except $Z_1$, then it is vanishing on $Z$.

**Question 1.4.** Can we extend the Cayley- Bacharach Theorem to the cases where all the connect components of $Z$ are positive dimensions?

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2. Localization by section

Let \( Z = \bigcup Z_i \), and \( Z_i \) be smooth connected component. Denote by \( N_i = N_{Z_i} / M \) the normal bundle of \( Z_i \subset M \). Because \( Z_i \) is smooth, the Kuranishi sequence gives us a exact sequence

\[
0 \to T_{Z_i} \to T_{M}|_{Z_i} \xrightarrow{ds} V|_{Z_i} \to V_i \to 0.
\]

Since \( N_i \cong T_{M}|_{Z_i}/T_{Z_i} \), the above sequence gives us the following short exact sequence

\[
0 \to N_i \xrightarrow{ds} V|_{Z_i} \to V_i \to 0.
\]

Assuming that the above exact sequence is splitting, therefore \( V|_{Z_i} = V_i \oplus N_i \cong V_i \oplus \text{Im } ds \). Let \( \psi \in \Gamma(M, K_M \otimes \det V) \), thus it can be viewed as morphism

\[
(2.1) \quad \psi : \det V^* \to K_M.
\]

\( ds \) induced the following isomorphism

\[
(2.2) \quad \det ds : \det N_i \to \det \text{Im } ds.
\]

\( (2.1) \) and \( (2.2) \) induced a morphism

\[
\frac{\psi}{\det ds} : \det V_i^* \to K_{Z_i}.
\]

The correspondent element in \( \Gamma(Z, K_{Z_i} \otimes \det V_i) \) is also denoted by \( \frac{\psi}{\det ds} \).

Let \( h \) be a Hermitian metric on \( V \) such that \( V_i \) and \( N_i \) are orthogonal on \( Z_i \). Let \( g_i \) be a Hermitian metric on \( N_i \) such that \( ds : N_i \cong \text{Im } ds \) is an isometry. Let \( R^V \) be the curvature of the holomorphic Hermitian connection \( \nabla \) on \( (V, h) \). Let \( j : Z_i \to M \) be the embedding. Let \( P^{V_i} \) and \( P^{\text{Im } ds} \) be the natural projections from \( V \) onto \( V_i \) and \( \text{Im } ds \). Let

\[
R^V_i := -(ds)^{-1} P^{\text{Im } ds} R^V j_+ P^{V_i} \in T^*Z_i \otimes V_i^* \otimes \text{End } N_i
\]

\( R^V_i \) is well defined since \( P^{\text{Im } ds} R^V j_+ P^{V_i} = 0 \).

**Theorem 2.1.** Under the above assumptions we have the following formula,

\[
(2.3) \quad \int_M (\psi e^{(2t)^{-1} S}) = \sum \int_{Z_i} \left( \frac{\psi}{\det ds} \right) \left( \frac{1}{\det N_i ((1 + R^V_i)/-2\pi \sqrt{-1})} \right) = 0.
\]

**Proof.** By [2] Proposition 4.14, \( \int_M (\psi e^{(2t)^{-1} S}) \) is independent of \( t \). Therefore we can do the calculation by setting \( t \to 0 \). This method is parallel to the one used in [4]. For arbitrary \( y \in Z_i \), since \( Z_i \) is a complex submanifold with dimension \( m_i \), we can find out holomorphic coordinates \( \{z_i\} \) of the neighborhood \( U \) of \( y \) such that \( y \) corresponds to 0, and \( \{\frac{\partial}{\partial z_k}\}_{k=m_i+1}^{n} \) is an orthonormal basis of the normal bundle \( N_y \). Moreover \( U \cap Z_i = \{p \in U, z_{m_i+1}(p) = \cdots = z_n(p) = 0\} \). Denote \( z' = (z_1, \cdots, z_{m_i}), z'' = (z_{m_i+1}, \cdots, z_n), z = (z', z'') \).

Let \( \{\mu_k(z', 0)\}_{k=1}^{n} \) and \( \{\mu_k(z', 0)\}_{k=m_i+1}^{n} \) be the holomorphic frame for \( V \) and \( \text{Im } ds \) on \( U \cap Z_i \) with

\[
\nabla \frac{\partial}{\partial z_k} s|_y = \frac{\partial s}{\partial z_k}|_y = \mu_k(0) \quad m_i + 1 \leq k \leq n.
\]

Let \( \{\mu_k(z', 0)\}_{k=1}^{n} \) be the correspondent basis of \( V^* \). Define \( \mu_k(z) \) by parallel transport of \( \mu_k(z', 0) \) with respect to \( \nabla \) along the curve \( u \to (z', uz'') \). Identify \( V_z \) with
by identify \( \mu_k(z) \) with \( \mu_k(z', 0) \). Denote by \( W_y(\epsilon) \) the neighborhood of \( y \) in the normal space \( N_i \). Then

\[
\int_{Z_i \cap U} \int_{W_y(\epsilon)/\sqrt{t}} \left( \psi \sqrt{-1} \langle \nabla(-\xi)(\sqrt{t}z) + |s(\sqrt{t}z)|^2 \rangle \right)\]

From now on we set \( z = (0, z'') \), \( v_z = \sum_{j=m_i}^n z_j (\frac{\partial}{\partial z_j}) \) and \( Y = v_z + \bar{v}_z \). The tautological vector field is \( Y = v_z + \bar{v}_z \). Then

\[
\frac{1}{2t} |s(\sqrt{t}z)|^2 = \frac{1}{2} |\nabla_Y s|^2 + O(t^{\frac{1}{2}}) = \frac{1}{2} |z|^2 + O(\sqrt{t}),
\]
and

\[
\bar{\partial}(\ast, s) = \sum_{k=1}^n \langle \mu_k, \nabla s \rangle \mu^k.
\]

Since \( \nabla_Y \mu_k(0) = 0 \), we have

\[
\frac{1}{2t} \bar{\partial}(\ast, s)(\sqrt{t}z) = \frac{1}{2t} \sum_{k=1}^n \langle \mu_k, \nabla s \rangle(\sqrt{t}z) \mu^k(0)
\]

\[
= \frac{1}{2t} \sum_{k=1}^n \left( \langle \mu_k, \nabla s \rangle(0) + \sqrt{t} \langle \mu_k, \nabla Y \nabla s \rangle(0) \right)
\]

\[
+ \frac{t}{2} \left( \langle \nabla_Y \nabla_Y \mu_k, \nabla s \rangle + \langle \mu_k, \nabla_Y \nabla_Y \nabla s \rangle(0) + o(t^{\frac{3}{2}}) \right) \mu^k(0).
\]

Because there is a factor \( t^{n-m_i} \) in (2.4), it should be clear that in the limit, only those monomials in the vertical form

\[
dz_{m_i+1} \wedge \cdots \wedge dz_n \otimes \mu^m(0) \wedge \cdots \wedge \mu^n(0)
\]
whose weight is exactly \( t^{n-m} \) should be kept.

So the second term contribute zero to the integral, and the terms contributes nonzero in the third term are

\[
\frac{1}{4} \sum_{k=1}^n \sum_{j=1}^{m_i} \left( \langle \nabla_Y \nabla_Y \mu_k, \nabla s \rangle(0) + \langle \mu_k, \nabla_Y \nabla_Y \nabla s \rangle(0) \right) d\bar{z}_j \otimes \mu^k(0).
\]

But for \( 1 \leq j \leq m_i \), both \( \nabla \frac{\partial}{\partial z_j} s(0) = 0, \nabla \frac{\partial}{\partial z_j} \nabla s(0) = \nabla \frac{\partial}{\partial z_j} (R^Y(v_z, v_z))s(0) = 0 \). Thus

\[
\nabla_Y \nabla_Y \nabla \frac{\partial}{\partial z_j} s(0) = 2R^Y(\bar{v}_z, \frac{\partial}{\partial z_j}) \nabla s(0) + \nabla \frac{\partial}{\partial z_j} \nabla v_z \nabla s(0).
\]
By previous discussion, as $s$ satisfies all the conditions of the above theorem.

Let $M$ be a compact complex manifold with $\dim M = n$, $L$ be a holomorphic bundle with rank $r < n$. Let $s \in \Gamma(M, L)$ be transversal. Then the zero locus $Z$ of $s$ is smooth with $\dim Z = n - r$. Let $V = L \oplus V_1$, where $V_1$ is a holomorphic bundle with rank $n - r$. $s$ can be considered as a section of $V$. It satisfies all the conditions of the above theorem.

Let $V$ be a holomorphic bundle over a compact complex manifold $M$, with rank $V = \dim M = n$. Let $\psi \in \Gamma(M, K_M \otimes \det V)$, and $s \in \Gamma(M, V)$ be a transversal section with smooth zero loci. Then the zero locus of $s$ is finite points $\{p_i\}$. Assuming that around a neighborhood of $p_i$, $s = \sum s_k e_k$, and $\psi = h(z)dz_1 \wedge \cdots \wedge dz_n \otimes e_1 \wedge \cdots \wedge e_n$. Then we have the following equalities, which recovered the residue theorem in [4 Page 731].

**Corollary 2.3.** $\text{Res}_{\bar{s}} \psi = \sum_{p_i} \frac{h(p_i)}{\det(\Delta^{|p_i|})} = 0$.

Let $M$ be a compact manifold, $V$ be a holomorphic bundle over $M$ that $\text{rk} V = \dim M$, with a section $s \in \Gamma(M, V)$ and the zero loci $Z = \bigcup_{j=1}^{m} Z_i$, where all $Z_i$ are smooth and at least one $Z_i$ is zero dimension. $V$ is splitting on $Z_i$ as in Theorem 2.1. Let $\psi$ be a section of $K_M \otimes \det V$. Then we have the following general Cayley-Bacharach theorem.

**Corollary 2.4.** With the assumptions as above and $\psi$ is vanishing on all components of $Z$ except one of the zero dimension component $Z_i$, then it is vanishing on whole $Z$. 

Note that $\nabla = \nabla^{N_i} \oplus \nabla^{V_i}$ on $Z_i$, where $\nabla^{N_i}$ and $\nabla^{V_i}$ are induced connections.
Proof. Assuming that one of $Z_i$ is point $p$, then by the corollary \[2.3\] \(
abla \frac{\psi}{s} = \frac{h(p)}{\det(\frac{\partial}{\partial z}(p))}\). For $M$ is compact, we have

\[
\text{Res}_{Z_i} \frac{\psi}{s} = \frac{(-1)^n}{(2\pi)^n} \sum \int_{Z_i} \left( \frac{\psi}{\det ds} \frac{1}{\det N_i((1 + R^1)/ - 2\pi \sqrt{-1})} \right) = \frac{h(p)}{\det(\frac{\partial}{\partial z}(p))} = 0.
\]

So $h(p) = 0$. \qed

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