No-scalar-hair theorem for spherically symmetric reflecting stars

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Abstract

It is proved that spherically symmetric compact reflecting objects cannot support static bound-state configurations made of scalar fields whose self-interaction potential $V(\psi^2)$ is a monotonically increasing function of its argument. Our theorem rules out, in particular, the existence of massive scalar hair outside the surface of a spherically symmetric compact reflecting star.
I. INTRODUCTION

The no-hair theorem of Bekenstein [1] (see also [2]) has revealed the intriguing fact that, in accord with Wheeler’s celebrated conjecture [3, 4], asymptotically flat black holes cannot support regular self-gravitating static matter configurations made of massive scalar fields in their external spacetime regions [5–9]. This important physical characteristic of black holes is often attributed in the physics literature to the fact that the boundary of a classical black hole (its horizon) acts as a one-way membrane that irreversibly absorbs matter and radiation fields.

One naturally wonders whether this no-scalar-hair behavior is a unique property of black holes? In the present paper we shall explore the possibility of extending the no-scalar-hair theorem to the regime of regular (that is, horizonless) curved spacetimes. In particular, we here raise the following physically intriguing question: Can regular compact reflecting objects (that is, reflecting stars [10] which possess no event horizons) support self-gravitating massive scalar field configurations in their exterior spacetime regions?

In order to address this interesting question, in the present study we shall replace the standard ingoing (absorbing) boundary condition which characterizes the behavior of classical fields at the horizon of a black hole [1, 6] by a reflecting (repulsive) boundary condition at the surface of the horizonless compact star. Our theorem, to be proved below, reveals the intriguing fact that horizonless compact reflecting stars share the characteristic no-scalar-hair property with asymptotically flat black holes.

II. THE NO-SCALAR-HAIR THEOREM FOR SPHERICALLY SYMMETRIC COMPACT REFLECTING STARS

We consider a static spherically symmetric compact reflecting object (a reflecting star [10]) of radius $R$. Using the Schwarzschild coordinates $(t, r, \theta, \phi)$, the line element of the corresponding curved spacetime can be expressed in the form [11, 12]

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

(1)

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$. An asymptotically flat spacetime is characterized by the asymptotic behaviors $\nu \sim O(r^{-1})$ and $\lambda \sim (r^{-1})$ for $r \to \infty$. 
The compact star is non-linearly coupled to a real scalar field $\psi$ with a general self-interaction potential $V = V(\psi^2)$ whose action is given by \cite{11,13}

$$S = S_{EH} - \frac{1}{2} \int \left[ \partial_\alpha \psi \partial^\alpha \psi + V(\psi^2) \right] \sqrt{-g} d^4x .$$  

(2)

We shall assume a positive semidefinite self-interaction potential for the scalar field which is a monotonically increasing function of its argument. That is,

$$V(0) = 0 \quad \text{with} \quad \dot{V} \equiv \frac{d[V(\psi^2)]}{d(\psi^2)} \geq 0 .$$  

(3)

Note that the physically interesting case of a free massive scalar field with $\dot{V} = \mu^2 \geq 0$ is covered by this form of the self-interaction scalar potential.

From the action (2) one finds the characteristic equation \cite{11}

$$\partial_\alpha \partial^\alpha \psi - \dot{V} \psi = 0$$  

(4)

for the self-interacting scalar field. Substituting the line-element (1) of the spherically symmetric curved spacetime into (1), one obtains the characteristic radial differential equation \cite{14}

$$\psi'' + \frac{1}{2} \left( \frac{4}{r} + \nu' - \lambda' \right) \psi' - e^\lambda \dot{V} \psi = 0$$  

(5)

for the self-interacting static scalar field.

The energy density of the self-interacting scalar field (2) is given by \cite{11}

$$\rho = -T_t^t = \frac{1}{2} \left[ e^{-\lambda}(\psi')^2 + V(\psi^2) \right] .$$  

(6)

An asymptotically flat (finite mass) spacetime is characterized by an energy density $\rho$ which approaches zero asymptotically faster than $1/r^3$ \cite{15}:

$$r^3 \rho(r) \to 0 \quad \text{for} \quad r \to \infty .$$  

(7)

Taking cognizance of Eqs. (3), (6), and (7), one deduces that the scalar field eigenfunction is characterized by the asymptotic boundary condition

$$\psi(r \to \infty) \to 0$$  

(8)

at spatial infinity. In addition, we shall assume that the scalar field vanishes on the surface $r = R$ of the compact reflecting star \cite{16,17}:

$$\psi(r = R) = 0 .$$  

(9)
Taking cognizance of the boundary conditions (8) and (9), one concludes that the characteristic eigenfunction $\psi$ of the scalar field must have (at least) one extremum point, $r = r_{\text{peak}}$, between the surface $r = R$ of the reflecting star and spatial infinity [that is, in the interval $r_{\text{peak}} \in (R, \infty)$]. At this extremum point the eigenfunction $\psi$ of the external scalar field is characterized by the relations

$$\{\psi' = 0 \text{ and } \psi \cdot \psi'' < 0\} \text{ for } r = r_{\text{peak}}. \quad (10)$$

Substituting (3) and (10) into the l.h.s of (5), one finds the inequality

$$\psi'' + \frac{1}{2} \left( \frac{4}{r} + \nu' - \lambda' \right) \psi' - e^{\lambda} \dot{V} \psi < 0 \text{ for } r = r_{\text{peak}}, \quad (11)$$

in contradiction with the characteristic relation (5) of the self-interacting scalar field.

**III. SUMMARY**

In this compact analysis, we have proved that if a spherically symmetric compact reflecting star [10] can support self-gravitating massive scalar field configurations, then the corresponding scalar field eigenfunction $\psi$ must have an extremum point outside the reflecting surface of the star. At this extremum point, the scalar field eigenfunction is characterized by the inequality (11). However, one realizes that this inequality is in contradiction with the characteristic identity (5) for the self-interacting scalar field. Thus, there is no solution for the external scalar eigenfunction except the trivial one, $\psi \equiv 0$ [18, 19].

We thus conclude that spherically symmetric compact reflecting objects cannot support static bound-state configurations made of scalar fields whose self-interaction potential $V(\psi^2)$ is a monotonically increasing function of its argument. In particular, our theorem rules out the existence of asymptotically flat massive scalar hair (regular self-gravitating massive scalar field configurations) outside the surface of a spherically symmetric (horizonless) compact reflecting star.

Our compact theorem therefore reveals the interesting fact that horizonless compact reflecting stars share the no-scalar-hair property with the more familiar asymptotically flat absorbing [20] black holes.
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[5] As noted in [6], this interesting no-hair property of black holes can be extended to the regime of self-gravitating scalar fields whose self-interaction potential $V(\psi^2)$ is a monotonically increasing function of its argument [see Eqs. (2) and (3) below].
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[10] We use here the term ‘reflecting star’ to describe a physical compact object for which the external scalar field vanishes on its surface.

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[12] We shall use natural units in which $G = c = 1$.

[13] Here $S_{EH}$ is the Einstein-Hilbert action.

[14] Here a prime $'$ denotes a derivative with respect to the radial coordinate $r$.

[15] S. Hod, Phys. Lett. B 739, 383 (2014) [arXiv:1412.3808].

[16] It is worth noting that in the vast physics literature that deals with the famous ‘black-hole bomb’ mechanism of Press and Teukolsky [17], one usually places a reflecting surface around a black hole in order to prevent the scalar field from escaping to infinity. On the other hand, in the present study the role of the reflecting surface is to prevent the scalar field from entering the central horizonless compact star.

[17] W. H. Press and S. A. Teukolsky, Nature 238, 211 (1972); W. H. Press and S. A. Teukolsky, Astrophys. J. 185, 649 (1973).

[18] As nicely emphasized by the anonymous referee, the result of the present paper can be framed in the familiar context of standard quantum mechanics. In particular, a stationary state of a standard one dimensional quantum mechanical problem is also characterized by the relation (10), which implies that the energy of the stationary quantum state is bounded from below by the potential energy at the corresponding extremum point. Note that in the terminology of quantum mechanics, our static scalar field corresponds to a zero-energy state, whereas the effective radial potential is positive [see (19)].

[19] As pointed out by the anonymous referee, it would be interesting to check whether a reflecting star can support a stationary complex scalar field configuration around it.

[20] It is worth emphasizing again that black holes, as opposed to the compact reflecting stars discussed in the present analysis, are characterized by the presence of absorbing one-way membranes (event horizons).