Uhrig dynamical control of a three-level system via non-Markovian quantum state diffusion

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Abstract

In this paper, we use the quantum state diffusion (QSD) equation to implement the Uhrig dynamical decoupling to a three-level quantum system coupled to a non-Markovian reservoir comprising of infinite numbers of degrees of freedom. For this purpose, we first reformulate the non-Markovian QSD to incorporate the effect of the external control fields. With this stochastic QSD approach, we demonstrate that an unknown state of the three-level quantum system can be universally protected against both coloured phase and amplitude noises when the control-pulse sequences and control operators are properly designed. The advantage of using non-Markovian QSD equations is that the control dynamics of open quantum systems can be treated exactly without using Trotter product formula and be efficiently simulated even when the environment is comprised of infinite numbers of degrees of freedom. We also show how the control efficacy depends on the environment memory time and the designed time points of applied control pulses.

(Some figures may appear in colour only in the online journal)

1. Introduction

Few-level atomic and molecular systems play crucial roles in quantum control and quantum information processing. For example, qubits have great advantages in certain computational tasks compared to classical bits because of quantum coherence and quantum entanglement. However, for an open quantum system, mutual effects due to the coupling between the system and environment are inevitable and result in very complex reduced dynamics including dissipation, fluctuation, decoherence and disentanglement [1–6].

Inspired from the Hahn spin echo in nuclear magnetic resonance [7], dynamical decoupling (DD) was shaped into a useful tool to mitigate the decoherence of a quantum system coupled to an environment [8–14]. Recently, different aspects of DD have been studied with significant progress; it has been shown that by using aperiodic control pulses, the so-called Uhrig dynamical decoupling (UDD) scheme, a single qubit in the pure dephasing spin-boson model can maintain its coherence to the $N$th order by using only $N$ or $N + 1$ pulses [15–17]. Furthermore, [18] constructed two layers of nesting UDD that can protect a single qubit against both dephasing and relaxation. In addition, the multi-layer nesting UDD and continuous DD schemes are designed to not only preserve quantum coherence and the entanglement of two-qubit systems [19–23], but also to protect multi-qubit systems [24] and quantum gates [25]. Studies on higher-order effects have also been performed for non-ideal pulses [26, 27], as well as optimized pulses under different environment noises [28–30]. In experiments, the remarkable performance of UDD in prolonging the life time of a single-qubit state in various models has been studied [31–34].
The purpose of this paper is to develop a non-Markovian quantum state diffusion (QSD) approach that can naturally incorporate the external control fields in the framework of Uhrig’s dynamical coupling approach. We present a UDD scheme with new control operators that may be used to control an unknown state of the three-level system (spin-1 system) against dissipation and pure phase decoherence. Our strategy is first to derive a time-local non-Markovian QSD equation for a single three-level dephasing model and dissipative model under the external DD control fields, then we show that the non-Markovian QSD can be systematically solved to simulate Uhrig control dynamics of a three-level system under the influence of dephasing and dissipative noises. One of the important features of our stochastic approach is that we do not make any assumptions on the system–environment coupling and the size of the environment. In this way, we can consistently solve the control dynamics without invoking the Trotter product formalism.

If quantum systems are coupled to small environments, it is straightforward to use the direct numerical simulations to solve the combined system and environment such that the UDD scheme can be efficiently implemented [20]. When the open system is coupled to a large environment consisting of finite or infinite numbers of degrees of freedom, one has to invoke an efficient quantum approach to deal with the open system dynamics allowing both non-Markovian and Markov environments to be dealt with. Since the control pulses applied to the open quantum system are typically implemented in a very short time scale and high intensity that is much stronger than the system–environment coupling, it is usually assumed there is no coupling between the system and the environment when the control pulses are applied. Therefore, in the previous work, the dynamics of the controlled open system behaves like the quantum jump process [15, 17, 35]. In this paper, without the above assumption, we treat the total system plus environment with control as an integrated and consistent entirety to solve its exact dynamics by using the non-Markovian QSD equation initially proposed in [36]. Derived directly from an underlying microscopic model irrespective of environment memory time and coupling strength, the stochastic QSD equation is a useful approach for solving several models exactly [37–41]. As shown below, the QSD equation can provide a systematic tool to deal with the non-Markovian quantum open system under the UDD control fields. The advantage of using non-Markovian QSD is its numerical power and its versatility in dealing with varied environmental sizes ranging from a few degrees of freedom to infinite numbers of degrees of freedom in arbitrary non-Markovian regimes.

The organization of this paper is as follows. In section 2, we present the physical models and our control strategy for a three-level atomic system interacting with a phase noise environment. We introduce a modified non-Markovian QSD equation with a time-dependent coupling operator and apply it to the dynamical control of the three-level system. In section 3, we present the nesting sequences to control the three-level system coupled to the dissipative environment. We derive a set of dynamic equations for the coefficients of the QSD equation. Lastly, section 4 includes discussions and concludes the paper.

2. Modified QSD equation for dephasing noise

In this section, we study the DD scheme for an open quantum system involving a three-level atom coupled linearly to a general bosonic environment consisting of a set of bosonic operators $b_n, b_n^\dagger$ satisfying $[b_n, b_m^\dagger] = \delta_{mn}$. The total Hamiltonian may be written as (setting $\hbar \equiv 1$):

$$H_{\text{tot}} = H_{\text{sys}} + H_{\text{env}} + H_{\text{int}},$$

with the three terms:

$$H_{\text{sys}} = \omega J_z,$$
$$H_{\text{env}} = \sum_n \alpha_n b_n^\dagger b_n,$$
$$H_{\text{int}} = \sum_n J_n (g_n b_n^\dagger + g_n^* b_n),$$

where the system has three energy levels $E_0 = -\omega, E_1 = 0$ and $E_2 = \omega$, $J_z = [2|2] - [0|0]$ is the coupling operator and $g_n$ are the coupling constants between the three-level atom and the environmental modes. This is a pure dephasing model where there are no population transitions between the energy levels as the interaction Hamiltonian commutes with the system Hamiltonian.

For this pure dephasing noise, it is not difficult to show that the iterative pulses of Uhrig’s type can preserve an arbitrary initial state. More precisely, for the DD scheme, the control sequence consisting of $N$ instantaneous pulses over a duration of time $T$ can be described by the following control Hamiltonian:

$$H_{\text{ctr}}(t) = \sum_{j=1}^{N} \frac{\pi}{2} \delta(t - T_j) P,$$

where we choose the UDD time intervals [15, 16]:

$$T_j = T \sin^2 \left( \frac{\pi}{2N+2} \right), \quad j = 1, 2, \ldots, N.$$  

(4)

The control operator $P$ for a single three-level pure dephasing model may be determined by using these two criteria [20]: $P^2 = I$ and $[J_z, P] = 0$. Therefore, it is easy to check that the following control operator satisfies the required conditions:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$  

(5)

In the rotating reference frame with respect to $H_{\text{ctr}}$, we can obtain an effective Hamiltonian in the interaction picture. Using the commutation relation $[P, [P, J_z]] = 4J_z$, we have

$$\exp \left[ -i \int_0^t ds H_{\text{ctr}}(s) \right] (J_z) \exp \left[ i \int_0^t ds H_{\text{ctr}}(s) \right]$$

$$= \exp \left[ -i \frac{\pi}{2} \text{Step}(t) P \right] (J_z) \exp \left[ i \frac{\pi}{2} \text{Step}(t) P \right]$$

$$= p(t) J_z,$$

(6)

where $\text{Step}(t) = 1$ when $t \in [T_j, T_{j+1})$ and $p(t) = \pm 1$, which changes sign at the time points $T_j$. Now, the total Hamiltonian describing the control-pulse sequence plus the system and
environment in such a “toggling frame” [42–44] can be written as
\[
\tilde{H} = \exp \left[ -i \int_0^t H_{in}(s) \, ds \right] H_{in} \exp \left[ i \int_0^t H_{in}(s) \, ds \right]
\]
where the effective system Hamiltonian and system–environment interaction Hamiltonian are given by
\[
\tilde{H}_{\text{sys}} = \omega p(t) J_z,
\]
\[
H_{\text{int}} = \sum_a p(t) J_z (g_a b^a_n + g^*_a b_n^a).
\]
(7)
The exact dynamics for the three-level atomic system under both the environmental noise and control pulses can be compactly described by the non-Markovian QSD equation derived from the above total Hamiltonian in the toggling frame:
\[
\frac{\partial}{\partial t} \psi_t = -i \tilde{H}_{\text{sys}} \psi_t + L^s \psi_t - L^{t'} \int_0^t ds \alpha(t-s) \delta \frac{\partial \psi_t}{\partial z^s},
\]
(9)
where \(L = p(t) J_z \) is the modified system Lindblad operator incorporating both the effects of environment and the external control pulses. Note that the correlation function \(\alpha(t-s) = \sum_a |g_a|^2 e^{-i \omega_a (t-s)}\) is arbitrary and \(z^s = -i \sum_a g_a z^a e^{i \omega_a t}\) is a complex Gaussian process satisfying \(M[z^s] = M[z^s] = 0\) and \(M[z^s z^a] = \alpha(t,s)\). Here \(M[.]\) denotes the statistical average over the classical Gaussian noise \(z\). When \(\alpha(t,s) = \Gamma_0 \delta(t-s)\), the noise \(z^s\) reduces to the memoryless white noise. It should be noted that the above non-Markovian QSD equation represents a new type of QSD with a time-dependent Lindblad operator.

In order to solve the non-Markovian QSD equation, we may rewrite the functional derivative term as \(\frac{1}{\sqrt{2}} \psi_t(z^s) = O(t,s,z^s) \psi_t(z^s)\), where \(O(t,s,z^s)\) is a time-dependent operator acting on the system Hilbert space. The equation of motion for \(O(t,s,z^s)\) can be obtained by using the consistency condition [45]:
\[
\frac{\partial}{\partial t} O = \left[ -i \tilde{H}_{\text{sys}} + L^s - L^{t'} \tilde{O}, O \right] - L^{t'} \delta \frac{\partial \tilde{O}}{\partial z^s},
\]
(10)
where \(\tilde{O}(t,z^s) = \int_0^t ds \alpha(t,s) O(t,s,z^s)\).

For the three-level dephasing model under DD control described by equation (7), it can be shown that the exact \(O\) operator is simply given by
\[
O(t,s,z^s) = f(t,s) J_z,
\]
(11)
with \(f(t,s) = p(s)\). Therefore, the explicit non-Markovian QSD equation can be compactly written into
\[
\frac{\partial}{\partial t} \psi_t(z) = (-i \omega + z^s + F(t) J_z) p(t) J_z \psi_t(z),
\]
(12)
where \(F(t) = \int_0^t ds \alpha(t,s) p(s)\). By calculating the statistical average over many realizations of the trajectory generated by the stochastic process \(z^s\), one can recover the density operator of the three-level system:
\[
\rho_t = M[|\psi_t(\tau)\rangle \langle \psi_t(\tau)|] = \int \frac{dz^s}{\pi} e^{-|z^s|^2} |\psi_t(z^s)\rangle \langle \psi_t(z^s)|.
\]
(13)
It is known that the exact QSD equation can be applicable to an open system model with an arbitrary correlation function \(\alpha(t,s)\). In order to investigate how the environment memory time affects the effectiveness of the DD control, in the following numerical simulations, we model the environmental noise as the Ornstein–Uhlenbeck process with the correlation function although our approach is valid for an arbitrary correlation function:
\[
\alpha(t,s) = \frac{\gamma}{2} e^{-|t-s|/\tau}
\]
(14)
where \(\gamma\) represents essentially the environmental bandwidth, hence the environment memory time scale can be represented by the parameter \(1/\gamma\). The advantage of choosing the Ornstein–Uhlenbeck process is that the Markov limit is simply dictated by the single parameter \(\gamma\). When \(\gamma \to \infty\), \(\alpha(t,s) \to \delta(t-s)\) recovering the Markov limit. Typically, a finite (small) \(\gamma\) represents a non-Markovian regime. Note that, for the Ornstein–Uhlenbeck noise, the equation of motion for \(F(t)\) is simply given by
\[
\frac{d}{dt} F(t) = \frac{\gamma}{2} p(t) - \gamma F(t).
\]
(15)
With (15), the QSD equation (12) is fully determined. We first compute the fidelity and the angular momentum time evolution with different numbers of control pulses. The plots are shown in figure 1.

Without the UDD control, as shown in figure 1(a), the three-level system under the pure dephasing relaxation will evolve into a complete mixed final state. Here we choose the environment memory parameter \(\gamma = 1\) which stands for a moderate non-Markovian regime [41]. In figure 1(a), the dynamics of \(J_z\) and \(J_\phi\) is shown to exhibit a few oscillations before reaching their final values.

However, with the UDD control sequence applied to this three-level system, it is shown that the system’s fidelity for the given initial state is well protected as illustrated in figures 1(b) and (c). Clearly, a better control result can be achieved if more controlled pulses are used in the control scheme.

The control processes may be better understood by inspecting the shapes of \(J_z\) and \(J_\phi\) curves. At each UDD time point, the effect of the control pulse is simply to change the sign of operator acting on the system given by equation (8). As a result, after each single control pulse, the mean angular momentum is modified towards its opposite direction. On average, both \(J_z\) and \(J_\phi\) are effectively preserved in the presence of the noise as illustrated by the temporal evolution of the three-level quantum system.

It is interesting to know how the effectiveness of the UDD control is affected by the environment memory times [49]. The results are shown in figure 2. Clearly, the transition of dynamics from non-Markovian to Markov regimes is dictated by environment memory time \(\tau = 1/\gamma\). In the case of small \(\gamma\), which stands for a long environment memory time, the more pronounced control-pulse effects on the three-level system are expected. In fact, as seen from the example, a longer coherence time may be preserved when the environment has a long memory time. Consequently, the fidelity will be efficiently protected. In contrast, when the system approaches the Markov limit as the memory time \(\tau\) becomes shorter and shorter (i.e., \(\gamma \gg 1\)), the ineffectiveness of the control pulses can be easily observed. This is easy to understand as the rapid coherence
denotes the fidelity, the red dotted line denotes the green dashed line denotes the sequences. We choose the initial state of the single-qutrit dephasing model under different UDD control (solid line and the green dash–dotted line in figure 2). This has before the external control pulses take effect (see the black decay has rendered the system effectively a classical ensemble before the external control pulses take effect (see the black solid line and the green dash–dotted line in figure 2). This has clearly demonstrated that an engineered environment with a longer memory time typically conduces to a better control of the system when the UDD pulses are applied.

Another interesting situation is that the initial state of the three-level system is a mixed state. Since any initial mixed state can be represented as $\rho_0 = \sum_{k=0}^N c_k \psi_0 \langle \psi_0 |$, then the density matrix at time $t$ is simply expressed in terms of QSD solutions:

\[
\rho_t = \sum_{k=0}^N c_k \mathcal{M} \left[ | \psi_{0,k} \rangle \langle \psi_{0,k} | \right], \quad \text{where} \quad | \psi_{0,k} \rangle \quad \text{is governed by the QSD equation (12)}.
\]

As an illustration, we consider the initial state as a Werner-like state [46, 47]:

\[
\rho_0 = \frac{1 - M}{2} I + \frac{3M - 1}{2} | \psi_0 \rangle \langle \psi_0 |.
\]

where $| \psi_0 \rangle = \frac{1}{\sqrt{3}} (| 0 \rangle + | 1 \rangle + | 2 \rangle)$. The parameter $M \in [1/3, 1]$ describes the ‘degree of mixture’ of the initial qutrit state $\rho_0$, when $M = 1/3$ it is a maximally mixed state, while when $M = 1$ it reduces to the pure state $| \psi_0 \rangle$.

The time evolution of fidelity is illustrated in figure 3. The left 3D picture (a) plots fidelity dynamics without UDD control. It is seen that the fidelity decays faster for an initial qutrit state with a higher degree of purity. However, when applying only ten UDD pulses, the fidelity over the time scale considered here is well protected as shown in figure 3(b).

3. Modified QSD equation for dissipative noise

In this section we will consider a three-level system coupled to a dissipative noise [41], the total Hamiltonian of the system plus environment can be described as

\[
H_{\text{tot}} = \omega J_z + \sum_n \omega_n b_n^\dagger b_n + \sum_{n} (g_{n,J} b_n^\dagger + g_{n,J}^* J + b_n),
\]

where $J_+ = \sqrt{2} (| 0 \rangle \langle 1 | + | 1 \rangle \langle 2 | )$ and $J_- = \sqrt{2} (| 1 \rangle \langle 2 | - | 2 \rangle \langle 0 | )$. Clearly, without control, the initial qutrit state will lose its coherence quickly by dissipation. In order to suppress the decoherence and protect the fidelity of the initial state of the three-level system, it is shown that two layers of DD control sequences are necessary. The control Hamiltonian can be written as

\[
H_{\text{ct}} = H_{c1} + H_{c2}
\]

\[
= \sum_{j=1}^{N_1} \pi \delta (t - T_j) P + \sum_{j=1}^{N_2} \sum_{k=1}^{N_1} \pi \delta (t - T_{jk}) Q,
\]

where $H_{c1}$ is the outer layer UDD sequence consisting of number $N_1$ of $P$ pulses as in equation (5) of section 2. And $H_{c2}$ is the inner layer UDD sequence of $Q$ pulses which would be applied at the time points [18]:

\[
T_{jk} = T_j + (T_{j+1} - T_j) \sin^2 \left( \frac{k \pi}{2 N_2 + 2} \right).
\]
In the case of dissipative noise, it can be shown that the two control operators $P$ and $Q$ are given by

$$
P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

(20)

where the new control operator $Q$ can be derived from the following criteria [20]: $Q^2 = 1, \{P, Q\} = [J_-, Q] = 0$ and $[J_+, Q] = [J_z, Q] = 0$.

In order to solve the system dynamics of the three-level dissipative model under the UDD control scheme, we need to use the rotating frame transformations twice, one for the control Hamiltonian $H_{c1}$ and the other for the control Hamiltonian $H_{c2}$, respectively (see appendix A for details). The final effective total Hamiltonian in the new bogging frame are given by

$$\check{H} = \tilde{H}_{\text{sys}} + \sum_n \omega_n b_n^\dagger b_n + \sum_n (g_n b_n^\dagger L + g_n^* b_n L^\dagger),$$

(21)

where $\tilde{H}_{\text{sys}} = p(t)\omega J_z$ and the effective Lindblad operator is

$$L = l_1(t)J_z + l_2(t)J_+,$$

(22)

where the coefficients are

$$l_1(t) = q(t) \frac{1 + p(t)}{2},$$

$$l_2(t) = q(t) \frac{1 - p(t)}{2}.$$  

(23)

Here the piecewise functions $p(t) = \pm 1$ (the values change at time points $T_j$) and $q(t) = \pm 1$ (the values change at time points $T_k$). Again, we see that the effect of the external control field is represented by the time-dependent Lindblad operator.

For the effective Hamiltonian given in equation (21), we can establish a modified QSD equation (9) with time-dependent Lindblad operators. Once the correlation function $\alpha(t, s)$ is given, the modified QSD equation for the three-level system under control can be solved numerically. For simplicity, we use the perturbation $O$ operator [48] corresponding to the weak noise approximation (for more details, see appendix B),

$$O(t, s) = f_1(t, s)J_+ + f_2(t, s)J_- + f_3(t, s)J_z,$$

(24)

$$\tilde{O}(t, s) = \int_0^t ds \alpha(t, s)O(t, s)$$

$$= F_1(t, s)J_+ + F_2(t, s)J_-$$

$$+ F_3(t, s)J_z,$$

(25)

where $F_i(i = 1, 2, 3, 4) = \int_0^\infty ds \alpha(t, s) f_i$. It is noted that the QSD equation derived here is valid for an arbitrary correlation function. In our numerical simulations, we always choose the Ornstein–Uhlenbeck type of correlation function $\alpha(t, s) = \frac{\gamma}{2} e^{-\gamma s}$. By substituting equations (24) and (25) into equation (10), we can derive a set of differential equations for the coefficients as

$$\frac{d}{dt} F_1 = \frac{\gamma}{2} l_1 + ((ip\omega - l_1 F_3 + l_2 F_4 - \gamma) F_1,$$

$$\frac{d}{dt} F_2 = \frac{\gamma}{2} l_2 + ((ip\omega - l_1 F_3 + l_2 F_4 - \gamma) F_2,$$

$$\frac{d}{dt} F_3 = ((ip\omega - 3l_1 F_1 + l_1 F_3 - l_2 F_4 - \gamma) F_3 + (l_1 F_1 - l_2 F_2 + l_3 F_4),$$

$$\frac{d}{dt} F_4 = ((ip\omega - 3l_1 F_2 + l_1 F_3 - l_2 F_4 - \gamma) F_4 + (l_1 F_1 - l_2 F_2 + l_3 F_4) F_2, $$

(26)

with $F_i(0) = 0$. Finally, the QSD equation is given by

$$\frac{d}{dt} \psi_t = [-i p \omega J_z + (l_1 J_+ + l_2 J_+) \zeta_t$$

$$- l_1 F_1 J_+ J_+ - l_2 F_2 J_- J_-$$

$$- l_1 F_1 J_+ J_+ - l_2 F_2 J_- J_- - l_3 F_3 J_+ J_+ - l_4 F_4 J_+ J_+, \psi_t.$$  

(27)

The numerical results with 2000 realizations are plotted in figure 4. For the zero-temperature environment, without any control pulses, the spontaneous emission always causes the three-level system to decay into the ground state. So in figure 4(a), when $t \rightarrow \infty$, we get $\langle J_z \rangle \rightarrow 0$ and $\langle J_+ \rangle \rightarrow 1$. Beyond the single UDD control sequence for the dephasing model, by applying two layers of nesting UDD control sequences, we can also successfully resist the dissipation (figures 4(a)–(c)). For the two-layer nesting UDD sequences, the total number of control pulses is $N_{\text{tot}} = N_1 + (N_1 - 1)N_2$. So in the case illustrated by figure 4(b), $N_1 = N_2 = 10$, there are in total 100 pulses applied to the dissipative model. In comparison to the dephasing model with 40 control pulses (figure 1(c)), we find that the dissipative model needs more control pulses to achieve the same degree of protection. Also let us compare figures 4(c) and (d). There are 37 pulses in the former case and 45 pulses in the latter case. However, the fidelity evolution and the angular momentum evolution for case (c) are much better.
Figure 4. Ensemble average of fidelity and $\langle J \rangle$ over 2000 trajectories of the single-qutrit dissipative model under different UDD control sequences. We choose the initial state $|\psi_0\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$ and the environment memory parameter $\gamma$ represents the fidelity, the red dotted line denotes $|\psi_0\rangle$, the green dashed line denotes $\langle J_x \rangle$, and the blue dot–dashed line denotes $\langle J_z \rangle$.

Figure 5. Fidelity evolution of the single-qutrit dissipative model under control with a different environment memory parameter $\gamma$.

We apply two-layer nesting UDD control sequences with the outer layer $N_1 = 20$ and the inner layer $N_2 = 10$. The initial state $|\psi(0)\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$.

Figure 6. Fidelity time evolution of a single-qutrit dissipative model with the initial state $\rho_0 = \frac{i}{\sqrt{3}}J_3 + \frac{1}{2\sqrt{3}}|\psi_0\rangle\langle\psi_0|$, where $M$ is the degree of mixing and $|\psi(0)\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$. The environment memory index $\gamma = 1$.

4. Concluding remarks

With both theoretical analysis and numerical simulation, we have shown that the modified quantum state diffusion (QSD) approach with time-dependent Lindblad operators can be a useful tool for implementing the UDD scheme in situations where the open system is coupled to a large non-Markovian environment. For a non-Markovian three-level open system, we show that a new control strategy in the DD scheme can efficiently protect an unknown three-level quantum state against both dephasing and dissipation noises. This new set of control operators only works for ladder-type three-level quantum systems with equal distance eigenvalues. Our three-level system represents a spin-1 or angular momentum system which is of interest in many physically interesting cases such as quantum cryptography and quantum entanglement [53]. It is interesting to note that in a more general three-level system with different energy spacings such as V-type or $\lambda$-type atoms, a universal effective control via UDD combined with non-Markovian QSD is still possible, but it becomes much more complicated technically and multi-nesting sequences with different control operators have to be employed. Moreover, the exact QSD will be difficult to find, but still we use approximate non-Markovian QSD to simulate the Uhrig control dynamics [21, 37, 41].

One of the advantages of using non-Markovian QSD is its versatility in solving large environments with arbitrary finite memory times. As an illustrative example, the explicit control dynamics measured by time-dependent angular momentum and fidelity in the non-Markovian regime is solved by using the modified non-Markovian QSD equations. Several scenarios for reinforcing the effectiveness of the regulation and control of three-level systems are considered such as increasing the number of control pulses or engineering the environment to modify the environment memory time scale. Our method also allows an interesting extension to non-perturbative dynamical decoupled in a non-Markovian regime.
In summary, with the versatility and the capability of the non-Markovian QSD equation, the control processes can be substituted into the system dynamics consistently and be studied in detail. More general extensions to multi-state atomic systems are of importance where the non-Markovian QSD is known to be more powerful numerically. It is also feasible to investigate the non-ideal control-pulse cases by using modified non-Markovian QSD equations [51, 52].

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Appendix A. Toggling frame of the dissipative three-level model under UDD control

In this appendix, we show the derivation of the interaction Hamiltonian in the toggling frame. To start with, we list the total Hamiltonian of the single three-level model with two layers of UDD control-pulse sequences according to equations (17) and (18) in section 3:

\[
H_{\text{tot}} = H_s + H_{\text{c}1} + H_{\text{c}2} + H_e + H_i,
\]

where

\[
H_s = \alpha J_z,
\]

\[
H_{\text{c}1} = \sum_{j=1}^{N} \frac{\pi}{2} \delta(t - T_j) P,
\]

\[
H_{\text{c}2} = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\pi}{2} \delta(t - T_{jk}) Q,
\]

\[
H_e = \sum_{n} \omega_n b_n^\dagger b_n,
\]

\[
H_i = \sum_{n} \left[ (g_n b_n + g_n^* b_n^\dagger) J_z + i (g_n b_n - g_n^* b_n^\dagger) J_y \right] + \mathcal{B}_f J_x + \mathcal{B}_s J_y.
\]

Rotating the reference frame with respect to \(H_{\text{c}1}\), the total Hamiltonian in the interaction picture is given by

\[
H_{\text{tot}}^{(1)} = e^{-i J_z^{\dagger} \text{d} t H_{\text{tot}}} (H_s + H_{\text{c}2} + H_e + H_i) e^{i J_z^{\dagger} \text{d} t H_{\text{tot}}}.
\]

By using the commutation relations,

\[
[P, Q] = [P, J_z] = 0,
\]

\[
[P, P_{J_z}] = 4 J_z,
\]

we have

\[
e^{-i J_z^{\dagger} \text{d} t H_{\text{tot}}} P e^{i J_z^{\dagger} \text{d} t H_{\text{tot}}} = Q,
\]

\[
e^{-i J_z^{\dagger} \text{d} t H_{\text{tot}}} J_s e^{i J_z^{\dagger} \text{d} t H_{\text{tot}}} = J_s,
\]

\[
e^{-i J_z^{\dagger} \text{d} t H_{\text{tot}}} J_y e^{i J_z^{\dagger} \text{d} t H_{\text{tot}}} = p(t) J_y
\]

\[
e^{-i J_z^{\dagger} \text{d} t H_{\text{tot}}} J_x e^{i J_z^{\dagger} \text{d} t H_{\text{tot}}} = p(t) J_x,
\]

where the piecewise function \(p(t) = \pm 1\) with the value changes at time points \(T_j\). Therefore, the total Hamiltonian in the new frame becomes

\[
H_{\text{tot}}^{(1)} = H_s^{(1)} + H_e + H_{\text{c}2} + H_i^{(1)},
\]

\[
H_{i}^{(1)} = f(t) \alpha J_z,
\]

\[
H_{c}^{(1)} = B_f J_x + f(t) B_s J_y.
\]

Next we rotate the reference frame again with respect to \(H_{\text{c}2}\), the total Hamiltonian in this new interaction picture is

\[
H_{\text{tot}}^{(2)} = e^{-i J_z^{\dagger} \text{d} t H_{\text{tot}}^{(1)}} (H_s^{(1)} + H_e + H_{i}^{(1)}) e^{i J_z^{\dagger} \text{d} t H_{\text{tot}}^{(1)}}.
\]

Now the commutation relations are

\[
[Q, J_z] = 0,
\]

\[
[Q, [Q, J_z]] = 4 J_z,
\]

\[
[Q, [Q, J_z]] = 4 J_z,
\]

so we can get

\[
e^{-i J_z^{\dagger} \text{d} t H_{\text{tot}}^{(1)}} J_x e^{i J_z^{\dagger} \text{d} t H_{\text{tot}}^{(1)}} = q(t) J_x,
\]

\[
e^{-i J_z^{\dagger} \text{d} t H_{\text{tot}}^{(1)}} J_y e^{i J_z^{\dagger} \text{d} t H_{\text{tot}}^{(1)}} = q(t) J_y,
\]

\[
e^{-i J_z^{\dagger} \text{d} t H_{\text{tot}}^{(1)}} J_x e^{i J_z^{\dagger} \text{d} t H_{\text{tot}}^{(1)}} = J_z,
\]

and the \(q(t) = \pm 1\) with the value changes at time points \(T_{jk}\). So using the rotating reference frame twice, the total Hamiltonian becomes

\[
H_{\text{tot}}^{(2)} = H_s^{(1)} + H_e + H_{i}^{(2)}
\]

\[
= q(t) \left[ \sum_n (g_n b_n^\dagger + g_n^* b_n) J_z + i p(t) (g_n b_n - g_n^* b_n^\dagger) J_y \right]
\]

\[
= \sum_n (g_n b_n^\dagger L + g_n^* b_n L^\dagger)
\]

where the time-dependent Lindblad operator is

\[
L = q(t) \left[ 1 + p(t) \right] J_z + q(t) \left[ 1 - p(t) \right] J_y.
\]

Finally, the effective Hamiltonian of the dissipative three-level model under two-layer UDD control sequences in the toggling frame is

\[
H_{\text{tot}}^{(2)} = p(t) \alpha J_z + \sum_n \omega_n b_n^\dagger b_n + \sum_n (g_n b_n^\dagger L + g_n^* b_n L^\dagger).
\]

This is equation (21) used in section 3.

Appendix B. Dynamical equation for the O operator

In the non-Markovian case, the linear stochastic Schrödinger equation was derived in [36], it reads

\[
\frac{\partial}{\partial t} \psi_s = -i H_{\text{sys}} \psi_t + L^\dagger \psi_t - L T_0^t \text{d}s \alpha(t - s) \frac{\delta \psi_s}{\delta z^*}.
\]

It is noted that the above exact equation contains a time-local term. In order to find a time-local QSD equation, one can introduce a time-dependent also noise-dependent operator \(O(t, s, z^*)\), defined as

\[
\frac{\delta}{\delta z^*} \psi_s(z^*) = O(t, s, z^*) \psi_t(z^*),
\]
which can be determined from the consistency condition,
\[ \frac{\delta}{\delta z^*} \frac{\partial \psi_j}{\partial t} = \frac{\delta}{\delta z} \frac{\partial \psi_j}{\partial t}. \]  \hspace{1cm} (B.3)

So we can derive the formal evolution equation for the operator
\[ O(t, s, z^*), \]
\[ \frac{\partial}{\partial t} O = \left[ -i H_{\text{sys}} + L z^* - L^* \overline{O}, O \right] - L^* \frac{\delta}{\delta z^*} \frac{\partial}{\partial t} \overline{O}, \]  \hspace{1cm} (B.4)

where \( \overline{O}(t, z^*) \equiv \int_0^t ds \alpha(t, s) O(t, s, z^*) \). This equation of motion for the operator has to be solved with the initial condition,
\[ O(t, s = t, z^*) = L. \]
\hspace{1cm} (B.5)

For the three-level dephasing model under DD control described by equation (7) in section 2, one can easily derive the exact \( O \) operator
\[ O(t, s, z^*) = f(t, s) J_z \]
\hspace{1cm} (B.6)

with \( f(t, s) = \rho(s) \). However, in section 3, for the second example with the three-level dissipative system and the control field, the explicit \( O \) operator cannot be determined. In this case, we have to use a perturbative expansion in terms of noise \( z^* \). This is called a weak noise perturbation, which means we choose the \( O \) operator containing noise-free terms,
\[ O(t, s) = f_1(t, s) J_z + f_2(t, s) J_+ + f_3(t, s) J_- + f_4(t, s) J_{J_+}, \]
\[ \overline{O}(t, s) = \int_0^t ds \alpha(t, s) O(t, s) \]
\[ = F_1(t, s) J_z + F_2(t, s) J_+ + F_3(t, s) J_- + F_4(t, s) J_{J_+}, \]
\hspace{1cm} (B.7)

where \( F_i(1 = 1, 2, 3, 4) = \int_0^t ds \alpha(t, s) f_i \). Of course, the QSD equation still contains noise explicitly (more details are given in [48]). By substituting the above two equations into equation (B.4), we can derive a set of differential equations for the coefficients as
\[ \frac{\partial}{\partial t} f_1 = (\gamma \rho + l_2 F_2 - l_1 F_3 + l_2 F_4) f_1 - l_2 F_1 f_2, \]
\[ \frac{\partial}{\partial t} f_2 = (-\gamma \rho + l_1 F_1 - l_1 F_3 + l_2 F_4) f_2 - l_2 F_1 f_1, \]
\[ \frac{\partial}{\partial t} f_3 = (\gamma \rho - l_1 F_1 - l_1 F_3 - l_2 F_4) f_3 + (l_1 F_1 - 2 l_2 F_2 + 2 l_2 F_3) f_1 - 2 l_2 F_1 f_2 - l_2 F_1 f_1, \]
\[ \frac{\partial}{\partial t} f_4 = (-\gamma \rho - l_2 F_2 + l_3 F_3 - l_2 F_4) f_4 - (l_1 F_1 - 2 l_2 F_2 + 2 l_2 F_3) f_1 + 2 l_2 F_1 f_2 - l_2 F_1 f_1, \]
\hspace{1cm} (B.8)

with the initial conditions:
\[ f_1(t, s = t) = l_1(t), \]
\[ f_2(t, s = t) = l_2(t), \]
\[ f_3(t, s = t) = 0, \]
\[ f_4(t, s = t) = 0. \]
\hspace{1cm} (B.9)

In this paper, we choose the Ornstein–Uhlenbeck type of correlation function \( \alpha(t, s) = \frac{1}{2} e^{-|t-s|}, \) so the equations of motion for \( F_i \) are
\[ \frac{\partial}{\partial t} F_1 = \frac{\gamma}{2} (l_1 + (i \rho \omega - l_1 F_3 + l_2 F_4 - \gamma) F_1), \]
\[ \frac{\partial}{\partial t} F_2 = \frac{\gamma}{2} (l_2 + (-i \rho \omega - l_1 F_3 + l_2 F_4 - \gamma) F_2), \]
\[ \frac{\partial}{\partial t} F_3 = (i \rho \omega - 3 l_1 F_1 + l_1 F_3 - l_2 F_4 - \gamma) F_3 + (l_1 F_1 - l_2 F_2 + l_2 F_4) F_1, \]
\[ \frac{\partial}{\partial t} F_4 = (-i \rho \omega - 3 l_2 F_2 + l_1 F_3 - l_2 F_4 - \gamma) F_4 + (l_1 F_1 - l_2 F_2 + l_2 F_4) F_2, \]
\hspace{1cm} (B.10)

with \( F_i(0) = 0 \). Consequently, the \( O \) operator and the QSD equation are fully determined.

References

[1] Paz J P and Zurek W H 1999 Environment-induced decoherence and the transition from quantum to classical 72nd Les Houches Summer School on 'Coherent Matter' (July–August) pp 3–64 (arXiv:quant-ph/0001001)
[2] Anastopoulos C and Hu B L 2000 Phys. Rev. A 62 033821
[3] Zyczkowski K, Horodecki P, Horodecki M and Horodecki R 2001 Phys. Rev. A 65 012101
[4] Yu T and Eberly J H 2002 Phys. Rev. B 66 193306
[5] Anastopoulos C, Shresta S and Hu B L 2009 Quantum Inform. Process. 8 549
[6] Helm J and Strunz W T 2009 Phys. Rev. A 80 042108
[7] Hahn E L 1950 Phys. Rev. 80 580
[8] Viola L and Lloyd S 1998 Phys. Rev. A 58 2733
[9] Viola L, Knill E and Lloyd S 1999 Phys. Rev. Lett. 82 2417
[10] Wu L A, Byrd M S and Lidar D A 2002 Phys. Rev. Lett. 89 127901
[11] Khodjasteh K and Lidar D A 2005 Phys. Rev. Lett. 95 180501
[12] Khodjasteh K and Lidar D A 2007 Phys. Rev. A 75 062310
[13] Carr H Y and Purcell E M 1954 Phys. Rev. 94 630
[14] Meiboom S and Gill D 1958 Rev. Sci. lnstrum. 29 688
[15] Uhrig G S 2007 Phys. Rev. Lett. 98 100504
[16] Uhrig G S 2008 New J. Phys. 10 083024
[17] Yang W and Liu R B 2008 Phys. Rev. Lett. 101 180403
[18] West J R, Fong B H and Lidar D A 2010 Phys. Rev. Lett. 104 130501
[19] Gordon G, Erez N and Kurizki G 2007 J. Phys. B: At. Mol. Opt. Phys. 40 575–95
[20] Mukhtar M, Soh W T, Saw T B and Gong J 2010 Phys. Rev. A 82 052338
[21] For an interesting application of an approximate non-Markovian QSD to a qubit system see Kao J T, Hung J T, Chen P and Mou C Y 2010 Phys. Rev. A 82 060210
[22] Pan Y, Xi Z R and Gong J 2011 J. Phys. B: At. Mol. Opt. Phys. 44 175501
[23] Chaudhry A Z and Gong J 2012 Phys. Rev. A 85 012315
[24] Wang Z Y and Liu R B 2011 Phys. Rev. A 83 022306
[25] West J R, Lidar D A, Fong B H and Gyure M F 2010 Phys. Rev. Lett. 105 230503
[26] Uhrig G S and Pasini S 2010 New J. Phys. 12 045001
[27] Pasini S, Karbach P and Uhrig G S 2011 Europhys. Lett. 96 10003
[28] Gordon G, Kurizki G and Lidar D A 2008 Phys. Rev. Lett. 101 010403
[29] Chen K and Liu R B 2010 Phys. Rev. A 82 052324
[30] Pasini S and Uhrig G S 2010 Phys. Rev. A 81 012309
[31] Biercuk M J, Uys H, VanDevender A P, Shiga N, Itano W M and Bollinger J J 2009 Nature 458 996
[32] Du J, Rong X, Zhao N, Wang Y, Yang J and Liu R B 2009 Nature 461 1265
[33] de Lange G, Wang Z H, Ristè D, Dobrovitski V V and Hanson R 2010 Science 330 60
[34] Ajoy A, Alvarez G A and Suter D 2011 Phys. Rev. A 83 032303
[35] Dalibard J, Castin Y and Mølmer K 1992 Phys. Rev. Lett. 68 580
[36] Strunz W T, Diósi L and Gisin N 1999 Phys. Rev. Lett. 82 1801
[37] Strunz W T, Diósi L, Gisin N and Yu T 1999 Phys. Rev. Lett. 83 4909
[38] Broadbent C J, Jing J, Yu T and Eberly J H 2012 Ann. Phys. 327 1962
[39] Strunz W T and Yu T 2004 Phys. Rev. A 69 052115
[40] Yu T 2004 Phys. Rev. A 69 062107
[41] Jing J and Yu T 2010 Phys. Rev. Lett. 105 240403
[42] Viola L, Lloyd S and Knill E 1999 Phys. Rev. Lett. 83 4888
[43] Uhrig G S and Lidar D A 2010 Phys. Rev. A 82 012301
[44] Ng H K, Lidar D A and Preskill J 2011 Phys. Rev. A 84 012305
[45] Diósi L, Gisin N and Strunz W T 1998 Phys. Rev. A 58 1699
[46] Werner R F 1989 Phys. Rev. A 40 4277
[47] Shu W and Yu T 2011 J. Phys. B: At. Mol. Opt. Phys. 44 225501
[48] Yu T, Diósi L, Gisin N and Strunz W T 1999 Phys. Rev. A 60 91
[49] Escher B M, Bensky G, Clausen J, Kurizki G and Davidovich L 2011 J. Phys. B: At. Mol. Opt. Phys. 44 154015
[50] Liu B H et al 2011 Nature Phys. 7 931
[51] Jing J, Wu L A, You J Q and Yu T 2012 arXiv:1202.5056 [quant-ph]
[52] Wang Z M, Wu L A, Jing J, Shao B and Yu T 2012 Phys. Rev. A 86 032303
[53] Mair A, Vaziri A, Weih G and Zeilinger A 2001 Nature 412 313