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A De Bruijn–Erdős theorem for chordal graphs

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Abstract

A special case of a combinatorial theorem of De Bruijn and Erdős asserts that every noncollinear set of $n$ points in the plane determines at least $n$ distinct lines. Chen and Chvátal suggested a possible generalization of this assertion in metric spaces with appropriately defined lines. We prove this generalization in all metric spaces induced by connected chordal graphs.

1 Introduction

It is well known that

(i) every noncollinear set of $n$ points in the plane determines at least $n$ distinct lines.

As noted by Erdős [11], theorem (i) is a corollary of the Sylvester–Gallai theorem (asserting that, for every noncollinear set $S$ of finitely many points in
the plane, some line goes through precisely two points of \( S \); it is also a special case of a combinatorial theorem proved later by De Bruijn and Erdős [10].

Theorem (i) involves neither measurement of distances nor measurement of angles: the only notion employed here is incidence of points and lines. Such theorems are a part of ordered geometry [7], which is built around the ternary relation of betweenness: point \( b \) is said to lie between points \( a \) and \( c \) if \( b \) is an interior point of the line segment with endpoints \( a \) and \( c \). It is customary to write \([abc]\) for the statement that \( b \) lies between \( a \) and \( c \). In this notation, a line \( \overline{uv} \) is defined — for any two distinct points \( u \) and \( v \) — as

\[
\{u, v\} \cup \{p : [puv] \vee [upv] \vee [uvp]\}.
\] (1)

In terms of the Euclidean metric \( \text{dist} \), we have

\[
[abc] \iff \\
\text{a, b, c are three distinct points and } \text{dist}(a, b) + \text{dist}(b, c) = \text{dist}(a, c). \quad (2)
\]

In an arbitrary metric space, equivalence (2) defines the ternary relation of metric betweenness introduced in [12] and further studied in [1,3,8]; in turn, (1) defines the line \( \overline{uv} \) for any two distinct points \( u \) and \( v \) in the metric space. The resulting family of lines may have strange properties. For instance, a line can be a proper subset of another: in the metric space with points \( u, v, x, y, z \) and

\[
\text{dist}(u, v) = \text{dist}(v, x) = \text{dist}(x, y) = \text{dist}(y, z) = \text{dist}(z, u) = 1, \\
\text{dist}(u, x) = \text{dist}(v, y) = \text{dist}(x, z) = \text{dist}(y, u) = \text{dist}(z, v) = 2,
\]

we have

\[
\overline{vy} = \{v, x, y\} \quad \text{and} \quad \overline{xy} = \{v, x, y, z\}.
\]

Chen [4] proved, using a definition of \( \overline{uv} \) different from (1), that the Sylvester–Gallai theorem generalizes in the framework of metric spaces. Chen and Chvátal [5] suggested that theorem (i) too, might generalize in this framework:

(ii) True or false? Every metric space on \( n \) points, where \( n \geq 2 \), either has at least \( n \) distinct lines or else has a line that consists of all \( n \) points.
They proved that

- every metric space on \( n \) points either has at least \( \lg n \) distinct lines or else has a line that consists of all \( n \) points

and noted that the lower bound \( \lg n \) can be improved to \( \lg n + \frac{1}{2} \lg \lg n + \frac{1}{2} \lg \frac{\pi}{2} - o(1) \).

Every connected undirected graph induces a metric space on its vertex set, where \( \text{dist}(u, v) \) is defined as the smallest number of edges in a path from vertex \( u \) to vertex \( v \). Chiniforooshan and Chvátal [6] proved that

- every metric space induced by a connected graph on \( n \) vertices either has \( \Omega(\frac{n^2}{7}) \) distinct lines or else has a line that consists of all \( n \) vertices;

we will prove that the answer to (ii) is ‘true’ for all metric spaces induced by connected chordal graphs.

**Theorem 1.** Every metric space induced by a connected chordal graph on \( n \) vertices, where \( n \geq 2 \), either has at least \( n \) distinct lines or else has a line that consists of all \( n \) vertices.

For graph-theoretic terminology, we refer the reader to Bondy and Murty[2].

## 2 The proof

Given an undirected graph, let us write \([abc]\) to mean that \( a, b, c \) are three distinct vertices such that \( \text{dist}(a, b) + \text{dist}(b, c) = \text{dist}(a, c) \); this is equivalent to saying that \( b \) is an interior vertex of a shortest path from \( a \) to \( c \).

**Lemma 1.** Let \( s, x, y \) be vertices in a finite chordal graph such that \([sxy]\). If \( \overline{sx} = \overline{sy} \), then \( x \) is a cut vertex separating \( s \) and \( y \).

**Proof.** The set of all vertices \( u \) such that \( \text{dist}(s, u) = \text{dist}(s, x) \) separates \( s \) and \( y \). Among all its subsets that separate \( s \) and \( y \), choose a minimal one and call it \( C \). Since \( x \) is an interior vertex of a shortest path from \( s \) to \( y \), it belongs to \( C \). To prove that \( C \) includes no other vertex, assume, to the contrary, that \( C \) includes a vertex \( u \) other than \( x \).

Our graph with \( C \) removed has distinct connected components \( S \) and \( Y \) such that \( s \in S \) and \( y \in Y \); the minimality of \( C \) guarantees that each of its vertices
has at least one neighbour in $S$ and at least one neighbour in $Y$. Since each of $u$ and $x$ has at least one neighbour in $S$, there is a path from $u$ to $x$ with at least one interior vertex and with all interior vertices in $S$. Let $P$ be a shortest such path; note that $P$ has no chords except possibly the chord $ux$. Similarly, there is a path $Q$ from $u$ to $x$ with at least one interior vertex, and with all interior vertices in $Y$, that has no chords except possibly the chord $ux$. The union of $P$ and $Q$ is a cycle of length at least four; since this cycle must have a chord, vertices $u$ and $x$ must be adjacent. In turn, the union of $Q$ and $ux$ is a chordless cycle, and so $Q$ has precisely two edges. This means that some vertex $v$ in $Y$ is adjacent to both $u$ and $x$.

Write $i = \text{dist}(s, x)$ and $j = \text{dist}(x, y)$. Since all vertices $t$ with $\text{dist}(s, t) < i$ belong to $S$ and since $v$ has no neighbours in $S$, we must have $\text{dist}(s, v) > i$; since $\text{dist}(x, v) = 1$, we conclude that $\text{dist}(s, v) = i + 1$ and that $v \in \overline{sx}$. Since $\overline{sx} = \overline{sy}$, it follows that $v \in \overline{sy}$. Since $\text{dist}(v, x) = 1$ and $\text{dist}(x, y) = j$, we have $\text{dist}(v, y) \leq j + 1$. From $\text{dist}(s, v) = i + 1$, $\text{dist}(s, y) = i + j$, $\text{dist}(v, y) \leq j + 1$, $i \geq 1$, $j \geq 1$, and $v \in \overline{xy}$, we deduce that $\text{dist}(v, y) = j - 1$.

Since $\text{dist}(u, v) = 1$, it follows that $\text{dist}(u, y) \leq j$; since $\text{dist}(s, u) = i$ and $\text{dist}(s, y) = i + j$, we conclude that $\text{dist}(u, y) = j$ and $u \in \overline{sy}$. Since $\text{dist}(s, u) = i$, $\text{dist}(s, x) = i$, and $\text{dist}(u, x) = 1$, we have $u \notin \overline{sx}$. But then $\overline{sx} \neq \overline{sy}$, a contradiction.

A vertex of a graph is called simplicial if its neighbours are pairwise adjacent.

**Lemma 2.** Let $s, x, y$ be three distinct vertices in a finite connected chordal graph. If $s$ is simplicial and $\overline{sx} = \overline{sy}$, then $\overline{xy}$ consists of all the vertices of the graph.

**Proof.** Since $\overline{sx} = \overline{sy}$, we have $y \in \overline{sx}$, and so $[ysx]$ or $[syx]$ or $[sxy]$; since $s$ is simplicial, $[ysx]$ is excluded; switching $x$ and $y$ if necessary, we may assume that $[sxy]$. Given an arbitrary vertex $u$, we have to prove that $u \in \overline{xy}$. Let $P$ be a shortest path from $s$ to $u$ and let $Q$ be a shortest path from $u$ to $y$. Lemma 1 guarantees that $x$ is a cut vertex separating $s$ and $y$, and so the concatenation of $P$ and $Q$ must pass through $x$. This means that $[sxu]$ or $[uxy]$ (or both). If $[uxy]$, then $u \in \overline{xy}$; to complete the proof, we may assume that $[sxu]$, and so $u \in \overline{sx}$.

Since $\overline{sx} = \overline{sy}$, we have $[usy]$ or $[suy]$ or $[suy]$; since $s$ is simplicial, $[usy]$ is excluded. If $[suy]$, then $[sxu]$ implies $[xyu]$; if $[suy]$, then $[sxy]$ implies $[xyu]$; in either case, $u \in \overline{xy}$.

4
Proof of Theorem 1 Consider a connected chordal graph on \( n \) vertices where \( n \geq 2 \). By a theorem of Dirac \([9]\), this graph has at least two simplicial vertices; choose one of them and call it \( s \). We may assume that the lines \( sz \) with \( z \neq s \) are pairwise distinct (else some line consists of all \( n \) vertices by Lemma 2). Since the graph is connected and has at least two vertices, \( s \) has at least one neighbour; choose one and call it \( u \). If \( u \) is the only neighbour of \( s \), then every path from \( s \) to another vertex must pass through \( u \), and so \( su \) consists of all \( n \) vertices. If \( s \) has a neighbour \( v \) other than \( u \), then line \( uv \) is distinct from all of the \( n - 1 \) lines \( sz \) with \( z \neq s \): since \( s, u, v \) are pairwise adjacent, we have \( s \not\in uv \). □

3 Related theorems

In Theorem 1, ‘connected chordal graph’ can be replaced by ‘connected bipartite graph’:

- every metric space induced by a connected bipartite graph on \( n \) vertices, where \( n \geq 2 \), has a line that consists of all \( n \) vertices.

In fact, \( xy \) consists of all \( n \) vertices whenever \( x \) and \( y \) are adjacent. To prove this, consider an arbitrary vertex \( u \). Since the graph is bipartite, \( \text{dist}(u, x) \) and \( \text{dist}(u, y) \) have distinct parities; since \( \text{dist}(x, y) = 1 \), they differ by at most one. We conclude that \( \text{dist}(u, x) \) and \( \text{dist}(u, y) \) differ by precisely one, and so \( u \in xy \).

In Theorem 1, ‘connected chordal graph’ can be also replaced by ‘sufficiently large graph of diameter two’: Chiniforooshan and Chvátal \([6]\) proved that

- every metric space on \( n \) points where each nonzero distance equals 1 or 2 has \( \Omega(n^{4/3}) \) distinct lines and this bound is tight.

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