Time-delayed Nonlinear Feedback Controllers to Suppress the Principal Parameter Excitation

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ABSTRACT Six different time-delayed controllers are introduced within this article to explore their efficiencies in suppressing the nonlinear oscillations of a parametrically excited system. The applied control techniques are the linear and nonlinear versions of the position, velocity, and acceleration of the considered system. The time-delay of the closed-loop control system is included in the proposed model. As the model under consideration is a nonlinear time-delayed dynamical system, the multiple scales homotopy method is utilized to derive two nonlinear algebraic equations that govern the vibration amplitude and the corresponding phase angle of the controlled system. Based on the obtained algebraic equations, the stability charts of the loop-delays are plotted. The influence of both the control gains and loop-delays on the steady-state vibration amplitude is examined. The obtained results illustrated that the loop-delays can play a dominant role in either improving the control efficiency or destabilizing the controlled system. Accordingly, two simple objective functions are introduced in order to design the optimum values of the control gains and loop-delays in such a way that improves the controllers’ efficiency and increases the system robustness against instability. The efficiency of the proposed six controllers in mitigating the system vibrations is compared. It is found that the cubic-acceleration feedback controller is the most efficient in suppressing the system vibrations, while the cubic-velocity feedback controller is the best in bifurcation control when the loop-delay is neglected. However, the analytical and numerical investigations confirmed that the cubic-acceleration controller is the best either in vibration suppression or bifurcation control when the optimal time-delay is considered. It is worth mentioning that this may be the first article that has been dedicated to introducing an objective function to optimize the control gains and loop-delays of nonlinear time-delayed feedback controllers.

INDEX TERMS Principal parametric resonance, Linear and nonlinear feedback control, Optimum time-delays, Objective function, Vibration control, Bifurcation control, Stability.

I. INTRODUCTION

Vibration control of mechanical structures has been the foremost subject for scientists and engineers for a long time. Different vibration reduction techniques that are either based on passive or active control strategies have been introduced. One of the important engineering structures is the cantilever beam system. Dynamical behaviours of many vital mechanical structures such as the wings of the aircraft, the robot manipulators, and blades in wind-turbine and helicopter rotors can be modeled as a cantilever beam system. Therefore, an enormous number of scientific articles concerning the vibration analysis and/or control for different configurations of the cantilever
beam system is published annually. Yabuno et al. [1] studied the nonlinear vibrations control of a parametrically excited cantilever beam system passively using a pendulum as a vibration absorber. The authors reported that static friction between the pendulum and the beam system at the supporting point can play an important role in suppressing the parametric oscillations. Pai et al. [2] introduced an active control method to suppress the primary response vibrations of the cantilever beam system. They investigated the vibration suppression efficiency of both the saturation and position-feedback controllers. Based on the theoretical and the experimental investigations, the authors concluded that the position-controller has high efficiency in mitigating the transient vibrations; meanwhile, the nonlinear saturation controller is the best in suppressing the beam steady-state vibration. Ashour and Nayfeh [3] designed an adaptive controller based on the well-known saturation phenomenon to suppress the primary resonance vibration of the cantilever beam system. The authors made their controller adaptive via incorporating an efficient frequency measuring circuit that guarantees that the internal resonance between the beam system and controller is in the ratio of 1:2. Rechdaoui and Azrar [4] applied the position-velocity controller to mitigate the subharmonic and superharmonic resonance vibrations of a piezoelectric–elastic–piezoelectric sandwich beam. Warminski et al. [5] introduced four different control techniques (that are linear-position, cubic position, positive position, and saturation controllers) to control the nonlinear vibrations of a cantilever beam system at the primary resonance case. They concluded that the saturation controller is ineffective in the case of parametrically excited systems. El-Ganaini et al. [6] explored the vibration suppression efficiency of the positive position feedback controller in suppressing the primary resonance vibrations for the same system that was studied in Ref. [5]. The authors illustrated that the proposed controllers can eliminate the system vibrations when the controller's natural frequency is tuned to be equal to the excitation frequency.

On the other hand, the efficiency of different time-delayed feedback controllers is investigated in order to mitigate or suppress the nonlinear vibrations of the cantilever beam system. Macarri [7] utilized the linear time-delayed position-velocity controller to mitigate the nonlinear vibrations of a cantilever beam system at the primary resonance case. The author concluded that the efficiency of the proposed controller can be enhanced at specific values of the loop-delay. Alhazza et al. [8-11] discussed the nonlinear vibrations control of the cantilever beam system applying the linear and nonlinear time-delayed feedback controllers when excited either parametrically or externally.

Peng et al. [12] studied the nonlinear vibration control of the cantilever beam system at primary resonance case using the linear-position, velocity, and accelerations time-delayed controller. The authors reported that time-delay can enhance the control performance at specific values of the loop-delay. Saeed et al [13-15] applied the time-delayed position-velocity feedback controller to suppress the primary resonance vibrations of nonlinear dynamical systems. The authors explained a simple technique to design the optimum values of the loop-delays to enhance the controller’s efficiency. Also, they showed the possibility of the position-controller to act as a velocity-controller and vice versa at specific values of the loop-delays. The influences of the time-delay on more advanced control algorithms such as the saturation and the positive position feedback controllers are investigated [16-19]. The main conclusion is that the loop-delay can play a dominant role in either stabilizing or destabilizing the system under control.

The vibration mitigation of the parametrically excited structures cannot be fully controlled by using the linear state feedback controllers such as the position, velocity, and acceleration [20]. Moreover, the advanced control strategies such as the saturation and the positive position controllers cannot work properly in this case [5]. Accordingly, Oueini and Nayfeh [20] explored for the first time the efficiency of the cubic-velocity feedback controller in mitigating the nonlinear vibrations of a cantilever beam system when subjected to a principal parametric excitation. The analytical and experimental findings confirmed the effectiveness of the applied control law in both vibration suppression and bifurcation control. In addition, Chen [21] investigated the cubic-position, linear-velocity, and cubic-velocity feedback controllers to mitigate the nonlinear oscillations of a parametrically excited cantilever beam system. The author reported that the combination of the linear and nonlinear velocity controllers has the best suppression efficiency. Pratiher [22] applied the cubic-velocity feedback controller to suppress the nonlinear vibrations of a cantilever beam system having end mass when subjected to either primary or parametric excitations.

Within this work, six time-delayed controllers are proposed to control the nonlinear oscillations of the cantilever beam system that is subjected to principal parametric excitation. The applied controllers are the linear and nonlinear forms of the beam position, velocity, and
acceleration. The efficiency of the applied controllers in suppressing the system nonlinear oscillations is compared. Moreover, the influence of loop-delays on both the system stability and the controllers’ efficiency is explored. It is found that the cubic-acceleration feedback controller is the most efficient in suppressing the system vibrations, while the cubic-velocity feedback controller is the best in bifurcation control when the loop-delay is neglected. In addition, the performed analytical and numerical investigations confirmed that the cubic-acceleration controller is the best either in vibration suppression or bifurcation control when the optimal time-delay is considered. Additionally, we proposed a simple method to design the optimum values of the loop-delays in such a way that improves the controllers’ efficiency and guarantees the system stability at the same time.

Comparing the current work with the previously published articles, the cubic-velocity feedback controller has been extensively investigated as the best control algorithm in the case of the parametrically excited systems [20-22]. However, the obtained results in the current work have proved that the cubic-acceleration controller is the best either in vibration suppression or bifurcation control when the optimal time-delay is considered. Moreover, there is no specific method that has discussed the optimal design of the loop-delays values in all previously published articles; but nevertheless within this work it is reported that the optimum time-delays values that can improve the system robustness against instability and enhance the linear-controllers efficiency in suppression the system vibrations should be selected in such a way that maximizes the linear damping function 

$$\mu_{Eq} = \mu - \frac{\beta_1}{2\omega^2} \sin(\omega \tau_1) + \frac{\beta_3}{2\omega} \cos(\omega \tau_2) + \frac{\beta_5}{2} \sin(\omega \tau_3)$$

Besides, the optimum control gains and loop-delays that can enhance the nonlinear-controllers efficiency in suppression the system nonlinear oscillations should be selected in such a way that maximizes the nonlinear damping function 

$$\mu_N = -\frac{\beta_2}{\omega} \sin(\omega \tau_1) + \omega^2 \beta_4 \cos(\omega \tau_2) + \omega^5 \beta_6 \sin(\omega \tau_3)$$.

![FIGURE 1. (a) Cantilever beam system under longitudinal excitation, and (b) The controlled system block diagram.](image)

II. MATHEMATICAL FORMULATION AND NONLINEAR ANALYSIS

The nonlinear temporal equation that governs the first mode vibrations of a vertical composite beam system having an end mass $M$ and excited periodically along the $X$ axis as shown in Fig. 1a is given as follows (Ref. [23]):
\[ \ddot{q} + 2\mu \omega \dot{q} + \omega^2 q + \alpha_0 q^3 + \alpha_2 \dot{q}^2 + \alpha_3 \ddot{q}^2 \]

\[ = \eta f \Omega^2 q \cos(\Omega t) - F_v \]  

where \( F_v \) is the control force. Six different control algorithms that consist of the linear and nonlinear versions of the position, velocity, and acceleration controllers are introduced. Accordingly, the control force can be expressed as:

\[ F_v = \beta_7 q(t - \tau_1) + \beta_8 \dot{q}^3(t - \tau_2) + \beta_9 \ddot{q}(t - \tau_3) \]

\[ + \beta_4 \dot{q}^3(t - \tau_2) + \beta_5 \ddot{q}(t - \tau_3) + \beta_6 \dddot{q}(t - \tau_4) \]  

(2)

where \( \beta_1 \) and \( \beta_2 \) are the linear and nonlinear gains of the position feedback controllers, \( \beta_3 \) and \( \beta_4 \) are the linear and nonlinear gains of the velocity feedback controllers, and \( \beta_5 \) and \( \beta_6 \) are the linear and nonlinear gains of the acceleration feedback controllers. Fig. 1b illustrates the engineering implementation of the proposed control method, where the Macro-Fiber Composite (MFC) sensor measures the instantaneous oscillations of the vertical beam system as shown in Fig. 1a. Using an analog-to-digital converter, the measured signal \( q(t) \) is fed into a digital computer that works as a controller. The proposed control algorithms (i.e. position, velocity, and acceleration controllers) that are installed on the digital computer manipulate the received control signal according to the designed control law. The manipulated signals (i.e. \( \beta_1 q(t - \tau_1), \beta_2 \dot{q}^3(t - \tau_2), \beta_3 \ddot{q}(t - \tau_3), \beta_4 \dot{q}^3(t - \tau_2), \beta_5 \dddot{q}(t - \tau_3), \) and \( \beta_6 \dddot{q}(t - \tau_4) \)) are then fed again into the MFC actuator via a power amplifier as in Fig. 1a, which ultimately mitigates the oscillation amplitude of the considered system. Accordingly, to report the efficient control algorithm among the proposed six controllers, and to design the optimum control parameters, a nonlinear mathematical investigation of the whole system model (i.e. Eqs. (1) and (2)) is introduced using the multiple scales homotopy approach as in section II. Moreover, a comprehensive discussion of the obtained results is presented in section III and IV.

**A. PARAMETRIC RESONANCE CASE (\( \Omega = 2\omega + \sigma \))**

As the controlled system dynamics is governed by a nonlinear time-delayed differential equation, the multiple scales homotopy approach is applied. Accordingly, the homotopy equation of the given equation of motion may be written as [24, 25]:

\[ H(q, \rho) = L(q) + \rho N(q), \quad \rho \in [0,1] \]  

(3)

where \( L(q) \) and \( N(q) \) are the linear and nonlinear parts of the given differential equation, respectively. \( \rho \) is defined as an embedded artificial Homotopy parameter. Accordingly, one gets

\[ L(q) = \ddot{q} + \omega^2 q, \]  

(4)

\[ N(q) = 2\mu \omega \dot{q} + \alpha_0 q^3 + \alpha_2 \dot{q}^2 + \alpha_3 \ddot{q}^2 \]

\[ - \eta f \Omega^2 q \cos(\Omega t) + \left( \beta_7 q(t - \tau_1) + \beta_8 \dot{q}^3(t - \tau_2) + \beta_9 \ddot{q}(t - \tau_3) \right. \]

\[ + \beta_4 \dot{q}^3(t - \tau_2) + \beta_5 \ddot{q}(t - \tau_3) + \beta_6 \dddot{q}(t - \tau_4) \]  

(5)

Therefore, the proposed homotopy problem in Eq. (3) can be expressed as follows:

\[ H(q, \rho) = \ddot{q} + \omega^2 q + \rho \left( 2\mu \omega \dot{q} + \alpha_0 q^3 + \alpha_2 \dot{q}^2 + \alpha_3 \ddot{q}^2 \right. \]

\[ + \alpha_2 q^3 + \alpha_3 \dot{q}^2 - \eta f \Omega^2 q \cos(\Omega t) + \left( \beta_7 q(t - \tau_1) + \beta_8 \dot{q}^3(t - \tau_2) + \beta_9 \ddot{q}(t - \tau_3) \right. \]

\[ + \beta_4 \dot{q}^3(t - \tau_2) + \beta_5 \ddot{q}(t - \tau_3) + \beta_6 \dddot{q}(t - \tau_4) \)\]

(6)

Without any loss of generality, two time-scales may be considered. Typically, the displacement may be expanded as follows:

\[ q(t, \rho) = q_0(T_0, T_1) + \rho q_1(T_0, T_1) + O(\rho^3), \]  

(7)

\[ q(t - \tau, \rho) = q_0(T_0 - \tau, T_1 - \rho \tau) + \rho q_1(T_0 - \tau, T_1 - \rho \tau) + O(\rho^3). \]  

(8)

where \( T_0 = t, \) and \( T_1 = \rho t. \) It follows that the derivatives \( \frac{dq}{dt} \) and \( \frac{d^2q}{dt^2} \) can be expressed in terms of the time scales \( T_0 \) and \( T_1 \) as follows:

\[ \frac{dq}{dt} = D_0 + \rho D_1, \quad \frac{d^2q}{dt^2} = D_0^2 + 2 \rho D_0 D_1, \]

(9)

where \( D_j = \frac{\partial}{\partial T_j}, \quad j = 0, 1 \)

Substituting Eqs. (7) to (9) into Eq. (6), then equating coefficients that have the same powers of \( \rho, \) yields

\[ \left( D_0^2 + \omega^2 \right) q_0 = 0 \]  

(10)

\[ \left( D_0^2 + \omega^2 \right) q_1 = -2D_0 D_1 q_0 - 2\mu \omega D_0 q_0 - \alpha_0 q_0^3 \]

\[ - \alpha_2 q_0^3 (D_0 q_0)^2 - \alpha_3 q_0^2 D_0^2 q_0 \]

\[ + \eta f \Omega^2 q \cos(\Omega T_0) \]

\[ - \beta_7 q_0(T_0 - \tau_1, T_1 - \rho \tau_1) \]

\[ - \beta_8 q_0^3(T_0 - \tau_1, T_1 - \rho \tau_1) \]

\[ - \beta_9 \ddot{q}_0(T_0 - \tau_1, T_1 - \rho \tau_1) \]

\[ - \beta_4 q_0^3(T_0 - \tau_2, T_1 - \rho \tau_2) \]

\[ - \beta_5 \ddot{q}_0(T_0 - \tau_2, T_1 - \rho \tau_2) \]

\[ - \beta_6 \dddot{q}(T_0 - \tau_3, T_1 - \rho \tau_3) \]

\[ - \beta_6 \dddot{q}(T_0 - \tau_3, T_1 - \rho \tau_3) \]  

(11)

The solution of Eq. (9) may be written as follows:

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where $cc$ denotes the complex conjugate of the preceding term. Substituting Eq. (12) into Eq. (11), one finds

$$D_i q_i + \omega^2 q_i = \left(-2i\omega D_i A - 2i\mu A^2 A - 3\alpha A^2 \tilde{A} \right)$$

$$+ \frac{\eta f}{2} (2\omega + \sigma)^2 A e^{i\omega T_i} - \beta A e^{-i\omega T_i}$$

$$-3\beta_1^2 A^2 \tilde{A} e^{-i\omega T_i} - i\omega \beta A e^{-i\omega T_i}$$

$$+ 3\omega^6 \rho_1^6 A^2 \tilde{A} e^{-i\omega T_i} + \beta_1 \omega^2 A e^{-i\omega T_i}$$

$$\left(-\alpha_1 + \alpha_2 \omega^2 + \alpha_3 \omega^2\right)$$

$$+ 3\omega^6 \beta_4 \omega^2 A e^{-i\omega T_i} + \beta_1 \omega^2 A e^{-i\omega T_i}$$

$$+ \frac{\eta f}{2} A (2\omega + \sigma)^2 e^{i\omega T_i} + cc$$

(13)

In order to obtain a uniform valid expansion, the secular terms must be removed. The elimination of these terms requires a cancellation of the coefficients of the exponentials $e^{\pm i\omega T_i}$. Accordingly, one can get the following solvability condition:

$$-2i\omega D_i A + A \left(-2i\mu A^2 - \beta A e^{-i\omega T_i} - i\omega \beta A e^{-i\omega T_i}\right)$$

$$+ \omega^2 \beta e^{-i\omega T_i} + A^2 \tilde{A} \left(-3\alpha_1 - \omega^2 \alpha_2 + 3\omega^2 \alpha_3\right)$$

$$-3\beta_1^2 e^{-i\omega T_i} - 3\omega^6 \beta_4 e^{-i\omega T_i} + 3\omega^6 \beta_6 e^{-i\omega T_i}\right)$$

$$+ A \frac{\eta f}{2} (2\omega + \sigma)^2 e^{i\omega T_i} = 0$$

(14)

According to the solvability condition given in Eq. (14), the solution of Eq. (13) can be expressed as follows:

$$q_i(T_0, T_i) = \frac{-1}{8\omega^2} A e^{3i\omega T_i} \left(-\alpha_1 + \alpha_2 \omega^2 + \alpha_3 \omega^2\right)$$

$$- \beta_1 e^{-i\omega T_i} + i\omega \beta_4 e^{-i\omega T_i} + \omega^6 \beta_6 e^{-i\omega T_i}\right)$$

$$- \frac{\eta f}{16\omega^2} A (2\omega + \sigma)^2 e^{i\omega T_i} + cc$$

(15)

Multiplying Eq. (13) by $\rho$ with making $\rho$ tend to unity, we have

$$-2i\omega \frac{\Delta A}{dt} + A \left(-2i\mu A^2 - \beta A e^{-i\omega T_i} - i\omega \beta A e^{-i\omega T_i}\right)$$

$$+ \omega^2 \beta e^{-i\omega T_i} + A^2 \tilde{A} \left(-3\alpha_1 - \omega^2 \alpha_2 + 3\omega^2 \alpha_3\right)$$

$$-3\beta_1^2 e^{-i\omega T_i} - 3\omega^6 \beta_4 e^{-i\omega T_i} + 3\omega^6 \beta_6 e^{-i\omega T_i}\right)$$

$$+ A \frac{\eta f}{2} (2\omega + \sigma)^2 e^{i\omega T_i} = 0$$

(16)

The solutions of the first-order nonlinear differential equation with a complex coefficient given in (16) can be solved as in Ref. [26]. Accordingly, the function $A(t)$ may be expressed in a polar form as follows:

$$A(t) = \frac{1}{2} a(t) e^{i\beta(t)}$$

(17)

where $a(t)$, and $\beta(t)$ are two real functions of the time. They represent vibration amplitude and the modified phase-angle of the cantilever beam system, respectively. Substituting Eq. (17) into Eq. (16), and then separating the real and the imaginary parts, one gets the following amplitude-phase modulation equations:

$$\dot{a} = -a\dot{\omega} \left(\mu - \beta_1 \cos(\omega \tau_1) + \frac{\beta_1}{2\omega} \cos(\omega \tau_2)\right)$$

$$+ \frac{\beta_1}{4\omega} \cos(\omega \tau_3) + \frac{\beta_1}{2\omega} \sin(\omega \tau_1)$$

$$+ \frac{\beta_1}{2\omega} \cos(\omega \tau_3) + \frac{\alpha^2}{4\omega} \left(-3\alpha_1 - \omega^2 \alpha_2\right)$$

$$+ \frac{\beta_1}{8\omega} \cos(\omega \tau_3) - 3\omega^6 \beta_4 \cos(\omega \tau_3) - 3\omega^6 \beta_4 \sin(\omega \tau_3)$$

$$+ \frac{\eta f}{2\omega} (2\omega + \sigma)^2 \cos(\phi).$$

(18)

$$\dot{\phi} = \left(\sigma - \beta_1 \cos(\omega \tau_1) - \beta_1 \sin(\omega \tau_1)\right)$$

$$+ \omega \beta_5 \cos(\omega \tau_2) + \frac{\alpha^2}{4\omega} \left(-3\alpha_1 - \omega^2 \alpha_2\right)$$

$$+ \omega \beta_5 \cos(\omega \tau_3) + \frac{\beta_1}{8\omega} \cos(\omega \tau_2)$$

$$+ \omega \beta_5 \cos(\omega \tau_3) + \frac{\beta_1}{8\omega} \cos(\omega \tau_2)$$

(19)

where $\phi = \sigma t - 2\beta$. At steady-state vibrations, we have $\dot{a} = \dot{\phi} = 0$. Therefore, we can get the following nonlinear algebraic equations that govern the steady-state oscillations amplitude and the corresponding phase angle of the beam system from Eqs. (18) and (19) as :

$$q_i(T_0, T_i) = \frac{-1}{8\omega^2} A e^{3i\omega T_i} \left(-\alpha_1 + \alpha_2 \omega^2 + \alpha_3 \omega^2\right)$$

$$- \beta_1 e^{-i\omega T_i} + i\omega \beta_4 e^{-i\omega T_i} + \omega^6 \beta_6 e^{-i\omega T_i}\right)$$

$$- \frac{\eta f}{16\omega^2} A (2\omega + \sigma)^2 e^{i\omega T_i} + cc$$

(15)
\[-\frac{1}{4\omega} a\eta f(2\omega + \sigma)^2 \sin \phi = a(-\omega \mu + \frac{\beta_1}{2\omega} \sin(\omega \tau_1)) - \frac{\beta_2}{2} \cos(\omega \tau_2) - \frac{\beta_3}{2} \omega \sin(\omega \tau_3)) - \frac{3a^3}{8} \left(\omega^2 \beta_4 \cos(\omega \tau_2)\right) - \frac{\beta_1}{\omega} \sin(\omega \tau_1) + \omega^5 \beta_5 \sin(\omega \tau_3)) \]  

(21)

By combining Eqs. (20) and (21), we can get the following frequency-response equation

\[
\frac{1}{16\omega^2} a^2 \eta^2 f^2 (2\omega + \sigma)^4 \left[ a \left(4 \omega \sigma - 4 \beta_1 \cos(\omega \tau_1) \right) - 4\omega\beta_3 \sin(\omega \tau_2) + 4\omega^2 \beta_3 \cos(\omega \tau_3) + \frac{a^3}{8\omega} (-3\alpha_{11}) \right]
- \omega^2 \alpha_2 + 3\omega^2 \alpha_3 - 3\beta_2 \omega \sin(\omega \tau_3) - 3\omega^3 \beta_4 \sin(\omega \tau_2)
+ 3\omega^6 \beta_5 \cos(\omega \tau_2) \right]^2 + a \left(-\omega \mu + \frac{\beta_1}{2\omega} \sin(\omega \tau_1) \right)
- \frac{\beta_1}{2} \cos(\omega \tau_2) - \frac{\beta_3}{2} \omega \sin(\omega \tau_3) \right)
- \frac{3a^3}{8} \left(\omega^2 \beta_4 \cos(\omega \tau_2)\right)
- \frac{\beta_1}{\omega} \sin(\omega \tau_1) + \omega^5 \beta_5 \sin(\omega \tau_3)) \right]^2 \]  

(22)

Utilizing the frequency-response equation (i.e. Eq. (22)), we can investigate the influence of each control parameter (i.e. \(b_j, \tau_k, j = 1,2, \ldots, 6, k = 1,2,3, \ldots \)) on the system vibration amplitude via plotting the oscillation amplitude \(a\) versus the detuning parameter \(\sigma\) (i.e. plotting the frequency response-curve) as illustrated in section III. Moreover, to investigate the stability of the obtained response-curve, the linearized stability may be examined around the fixed points (equilibrium points). Consequently, one may assume a steady-state solution as follows:

\[
a = a_{i0} + a_{i1}, \quad \phi = \phi_{i0} + \phi_{i1} \Rightarrow \dot{a} = \dot{a}_{i1}, \quad \dot{\phi} = \dot{\phi}_{i1} \]  

(23)

where \(a_{i0}\) and \(\phi_{i0}\) are the equilibrium solution, while \(a_{i1}\) and \(\phi_{i1}\) are small perturbation. Substituting Eq. (23) into Eqs. (18) and (19) with expanding for the small deviation \(a_{i1}\) and \(\phi_{i1}\) keeping the linear terms only. This procedure results in the following linear dynamical system:

\[
\begin{pmatrix}
\dot{a}_{i1} \\
\dot{\phi}_{i1}
\end{pmatrix} = 
\begin{pmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{pmatrix}
\begin{pmatrix}
a_{i1} \\
\phi_{i1}
\end{pmatrix}
\]  

(24)

where \(J_{jk}, j = 1,2, k = 1,2\) are given in the appendix. Based on the above square matrix (Jacobian matrix), one can obtain the following characteristic equation

\[
\lambda^2 - (s_1 + s_2) \lambda + (s_1 s_2 - s_3 s_4) = 0
\]  

(25)

where \(s_1, s_2, s_3, s_4\) are given in the appendix. Consequently, the necessary and sufficient conditions for solution stability may be written as follows:

\[
s_1 + s_2 < 0 \quad \text{and} \quad s_1 s_2 - s_3 s_4 > 0
\]  

(26)

III. BIFURCATION DIAGRAMS AND NUMERICAL SIMULATIONS

The efficiency of the six applied controllers in mitigating the vertical beam vibrations is explored within this section via solving the derived frequency-response equation (i.e. Eq. (22)) in terms of either of the detuning parameter \(\sigma\) or the excitation force \(f\). Moreover, the stability of the obtained solution is checked according to Eq. (26). In all the obtained response curves the solid line denotes the stable solutions, while the dotted line represents the unstable solutions. Besides the response curve obtained by solving Eq. (22), a numerical simulation for the system original equation (Eq. (1)) is performed to confirm the accuracy of the obtained response curves. The obtained analytical and numerical results are given based on the selected system parameters \(\omega = 3.06, \mu = 0.05, \alpha_1 = 14.4, \alpha_2 = \alpha_3 = 3.27, \eta = 1.5, f = 0.1, \Omega = 2\omega + \sigma, \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = 0, \quad \text{and} \quad \tau_1 = \tau_2 = \tau_3 = 0.0\) as given in Ref. [23]. The following subsections are organized in such a way that section III.A is dedicated to investigating the oscillatory behaviours of the uncontrolled vertical beam system, while section III.B is devoted to comparing the vibration reduction efficiency of the proposed six controllers, where the loop-delay is neglected. The rest of the subsections are intended to explore the dynamical behaviours of the six controllers when the loop-delay is considered.

A. UNCONTROLLED SYSTEM

The beam system frequency response-curve and force response-curve are shown in Fig. 2. It is clear from the figure that the oscillation amplitude is a monotonic increasing function of the excitation force \(f\). However, Fig. 2a illustrates that the system can be excited with a considerable oscillation amplitude depending on both the excitation frequency \(\Omega\) and the excitation amplitude \(f\), where the figure shows that the system will respond with
the trivial solution (i.e. no-oscillations) when $\sigma \notin [-2.5, 1.3]$ as long as $f \leq 0.1$. Numerical simulation of Fig. 2a at $\sigma = 0.0$ (i.e. when $\Omega = 2\omega$) at three different levels of the excitation force $f$ is illustrated in Fig. 3. The figure illustrates that the beam vibration amplitude evolving as the excitation force $f$ increases from $f = 0.05$ to 0.1. By comparing Fig. 3 and Fig. 2a, one can deduce the excellent agreement between the numerical and the analytical results.

B. THE CONTROLLED SYSTEM WITHOUT TIME- DELAYS

In this section, the effect of the six control gains ($\beta_j$, $j = 1, 2, ..., 6$) on the oscillatory behaviours of the cantilever beam system is investigated when the loop-delay is zero (i.e. $\tau_j = 0.0, j = 1, 2, 3$). By examining Fig. 4, it is clear from Figs. 4a and 4e that both the linear-position and the linear-acceleration controllers are responsible for shifting the system frequency response-curve either to the left or the right, depending on the feedback gains $\beta_1$ and $\beta_5$. Accordingly, it is possible to avoid the system's high oscillation amplitude via changing its natural frequency away from the excitation frequency and utilizing the linear-position and/or the linear-acceleration feedback controller. In addition, Figs. 4a and 4e confirm that the linear-acceleration feedback controller is more efficient than the linear-position controller in mitigating the system nonlinear vibrations with small control gains. However, the control method that depends on avoiding the resonant frequency via changing the system’s natural frequency is not the optimal technique, where the excitation frequency is unmeasurable most of the time. It is clear from Figs. 4b, 4c, 4d, and 4f that the nonlinear-position, the linear-velocity, the nonlinear-velocity, and the nonlinear-acceleration controllers are responsible for mitigating the beam vibrations via increasing the system viscous damping without affecting its natural frequency. By comparing Figs. 4b, 4c, 4d, and 4f, we can deduce that the least efficient controller is the nonlinear-position controller, while the most efficient one is the nonlinear-acceleration controller. According to Fig. 4, one can confirm that the optimum control technique that can damp the high oscillation amplitude of a parametrically excited system with a very small feedback gain (i.e. $\beta_6 = 0.25$) is the cubic-acceleration feedback controller.

Numerical simulations for temporal vibrations of the controlled system according to Figs. 4b, 4c, 4d, and 4f when $\sigma = 0$ (i.e. when $\Omega = 2\omega$) are illustrated in Fig. 5 via solving the system original equations (i.e. Eq. (1)) numerically. Fig. 5a simulates the system temporal oscillations when the cubic-position feedback gain $\beta_2$ increases from $\beta_2 = -1$ to $\beta_2 = 1$ along the time span $t \in [0, 450]$ with fixing $\beta_6 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = 0.0$. It is clear from the figure that the increase of the cubic-position feedback gain decreases the system temporal vibrations, which is in conformity with Fig. 4b. The nonlinear temporal vibration of the beam system is illustrated in Fig. 5b, when increasing the linear-velocity feedback gain from $\beta_3 = 0$ to $\beta_3 = 0.5$ according to Fig. 4c. The figure illustrates the effect of the online increase in $\beta_3$ on the system vibration amplitude according to Fig. 4c when $\sigma = 0$. Comparing Fig. 5b with Fig. 4c, we can find an excellent agreement between the two figures, where the oscillation amplitude is a monotonic decreasing function of $\beta_3$. According to Fig. 4d when $\sigma = 0$ the nonlinear vibration of the beam system is illustrated in Fig. 5c at three different values of the cubic-velocity feedback control gain (i.e. $\beta_4 = 0.0, 0.25, 0.5$), while Fig. 5d simulates the system temporal vibrations according to Fig. 4f when $\sigma = 0$ at $\beta_6 = 0.0, 0.25$, and 0.5. Generally, it is clear from Fig. 5 that increasing the control gains $\beta_2, \beta_3, \beta_4$, and $\beta_6$ decreases the oscillation amplitudes of the considered system. By comparing Figs. 5b, 5c, and 5d, we can deduce that the cubic-acceleration feedback controller is the most efficient controller where the system oscillation amplitude is close to zero at the feedback gain $\beta_6 = 0.5$. Accordingly, the cubic-acceleration feedback controller is the best control method for suppressing the nonlinear oscillations of a parametrically excited system.

It worthy to mention that this is the first time to investigate efficiency of the cubic-acceleration feedback controller in suppressing the nonlinear vibrations of a parametrically excited system, where the obtained analytical and numerical results (i.e. Fig. 4f & Fig. 5d) confirmed that this type of control is the best one compared to the other five control methods. Accordingly, it is expected for such a control method to draw considerable attention from scientists and engineers that are specialists in the field of vibration control.

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The response curve of the system before control: (a) the system frequency-response curve at three different values of the excitation force $f$, and (b) the system force-response curve at four different values of the detuning parameter $\sigma$.

The temporal oscillation of the cantilever beam system according to Fig. 2 when $\sigma = 0$ (i.e. $\Omega = 2\omega$) at the three different values of the excitation force $f = 0.05, 0.075$, and $0.1$.

The effect of the six controllers on the system frequency-response curve when the loop-delays are zeros (i.e. $\tau_1 = \tau_2 = \tau_3 = 0$): (a) the influence of the linear-position gain $\beta_1$, (b) the influence of the cubic-position gain $\beta_2$, (c) the influence of the linear-velocity gain $\beta_3$, (d) the influence of the cubic-velocity gain $\beta_4$, (e) the influence of the linear-acceleration gain $\beta_5$, and (f) the influence of the cubic-acceleration gain $\beta_6$. 
C. TIME-DELAYED LINEAR CONTROLLERS

The influence of the loop-delays on the vibration suppression efficiency of the proposed linear controllers (i.e. linear position, velocity, and accelerations) is explored within this section with the aid of the derived amplitude-phase modulation equations (i.e. Eqs. (18) and (19)). To investigate the performance of the time-delayed linear-position controller in suppressing the nonlinear vibrations of considered system, the stability chart is obtained via plotting the control gain $\beta_1$ versus the time-delay $\tau_1$ to illustrate the stable and unstable solutions regions in $\beta_1 - \tau_1$ plane as shown in Fig. 6a. It is clear from Fig. 6a that the unstable solutions region is repeated periodically along $\tau_1$ –axis either for the negative or positive values of $\beta_1$. Accordingly, it is possible to select the time-delay $\tau_1$ according to Fig. 6a either for the positive or negative
values of $\beta_1$ in such a way that guarantees the system stability. Fig. 6b shows the system frequency-response curve when $\beta_1 = 1.5$ at three different values of the loop-delay $\tau_1$ that are selected within a stable region according to Fig. 6a (i.e., $\tau_1 = 1, 1.5$, and 2.0). Fig. 6c illustrates the system frequency-response curve at the three different values of the loop-delay $\tau_1 = 2, 2.5$, and 3.0 that are selected within the stable solution region when $\beta_1 = -1.5$. It is clear from Figs. 6b and 6c that the loop-delay can play a dominant role in minimizing the system vibration amplitude as in the case of $\tau_1 = 1.5$ when $\beta_1 = 1.5$ and $\tau_1 = 2.5$ when $\beta_1 = -1.5$. Coming back to Fig. 6a, and examining the position of the optimum vibration control parameters (i.e. $(\beta_1, \tau_1) = (1.5, 1.5), (-1.5, 2.5)$), we find these points in the center of a stable solution region that is located between two consecutive unstable regions.

Fig. 7 shows the system temporal oscillations and the corresponding phase trajectories according to Fig. 6b at $\sigma = 0.0$ when $\tau = 1.5, 2$. By comparing these two figures, one can notice the good agreement between the analytical results in Fig. 6b and the corresponding numerical simulations given in Fig. 7. Moreover, Fig. 7a illustrates the high effect of the loop-delay in suppressing the system nonlinear vibrations, where the system vibration at $\tau = 1.5$ (i.e. at optimal time-delay value) is half of that at $\tau = 2.0$.

Fig. 8 illustrates the stability chart and the corresponding frequency-response curves for the linear velocity feedback controller, while Fig. 9 shows the stability chart and the corresponding frequency-response curves for the linear-acceleration feedback controller. It is clear that Fig. 8a and Fig. 9a are a repetition of Fig. 6a, where the stable solutions region is repeated periodically along the time-delay axes. Moreover, Figs. 8b, 8c, 9b, and 9c confirm that the optimum time-delay values should be selected within the center of the stable solutions region.

Now, it is possible to simply interpret how the time-delayed linear feedback controllers can stabilize or destabilize the system under control according to Figs. 6, 8, 9, with the aid of Eqs. (18) and (19). It is clear from Eqs. (18) and (19) that the connection of the linear time-delayed controllers (i.e. linear-position, linear-velocity, and linear-acceleration) to the beam system has modified the linear damping coefficient $\mu$ to the equivalent damping term $\mu_{Eq}$. Moreover, the detuning parameter $\sigma$ has been modified to the equivalent detuning term $\sigma_{Eq}$, where $\mu_{Eq}$ and $\sigma_{Eq}$ are given as follows (See Eqs. (18) and (19)):

$$\mu_{Eq} = \mu - \frac{\beta_1}{2\omega^2} \sin(\omega \tau_1) + \frac{\beta_3}{2\omega} \cos(\omega \tau_2)$$

$$\sigma_{Eq} = \sigma - \frac{\beta_1}{\omega} \sin(\omega \tau_1) - \beta_3 \sin(\omega \tau_2) + \omega \beta_5 \cos(\omega \tau_3)$$

It is clear from Eqs. (27) and (28) that $\mu_{Eq}$ and $\sigma_{Eq}$ are periodic functions on the time-delays $\tau_1, \tau_2$, and $\tau_3$. Moreover, at zero time-delays (i.e. $\tau_1 = \tau_2 = \tau_3 = 0$) the controlled system has the equivalent linear damping term $\mu_{Eq} = \mu + \frac{\beta_3}{2\omega^2}$ and the equivalent detuning term $\sigma_{Eq} = \sigma - \frac{\beta_3}{\omega} + \omega \beta_5 = \Omega - 2(\omega + \frac{\beta_1}{2\omega} - \frac{\omega}{2} \beta_5)$. This means that the linear-velocity controller is responsible for increasing the system linear damping coefficient to $\mu + \frac{\beta_3}{2\omega^2}$, while the linear-position and linear-acceleration controllers are responsible for modifying the system natural frequency to $\omega + \frac{\beta_1}{2\omega} - \frac{\omega}{2} \beta_5$. However, including the time-delays in the control loop results in different dynamical behaviours for the proposed linear-controllers. One can find from Eqs. (27) and (28) that $\mu_{Eq} = \mu - \frac{\beta_1}{2\omega^2} + \frac{\beta_3}{2}$ and $\sigma_{Eq} = \sigma - \beta_3$ if we let $\tau_1 = \tau_2 = \tau_3 = \frac{\pi}{2\omega}$. This means that the linear-position and linear-acceleration controllers act as a velocity controller, while the linear-velocity controller acts as a position-controller at $\tau_1 = \tau_2 = \tau_3 = \frac{\pi}{2\omega}$. Based on this explanation, to get the highest control efficiency for the proposed linear-controllers, the control gains $(\beta_1, \beta_2, \beta_3)$ and the time-delays $(\tau_1, \tau_2, \tau_3)$ should be selected in such a way that only maximizes the objective function $\mu_{Eq}$. Comparing the obtained results in Figs. 6b, 6c, 8b, 8c, 9b, and 9c with the equivalent damping term that is given by Eq. (27), we can report that the best vibration suppression condition has occurred at the maximum value of $\mu_{Eq}$ as summarized in Table 1.
FIGURE 6. The system frequency-response curve in the case of the time-delayed linear-position controller: (a) stable and unstable solutions region in $\beta_1 - \tau_1$ plane, (b) the system frequency-response curve when $\beta_1 = 1.5$ at $\tau_1 = 1, 1.5, 2$, and (c) the system frequency-response curve when $\beta_1 = -1.5$ at $\tau_1 = 2, 2.5, 3$.

FIGURE 7. The temporal oscillation of the cantilever beam system according to Fig. 6b when $\sigma = 0, \beta_1 = 1.5$, and $\tau_1 = 1, 1.5, 2.0$: (a) the system temporal oscillations, (b) the corresponding phase trajectory when $\tau_1 = 1.5$, and (c) the corresponding phase trajectory when $\tau_1 = 2.0$.

FIGURE 8. The system frequency-response curve in the case of the time-delayed linear-velocity controller: (a) stable and unstable solutions region in $\beta_3 - \tau_2$ plane, (b) the system frequency-response curve when $\beta_3 = 0.5$ at $\tau_2 = 1.5, 2, 2.5$, and (c) the system frequency-response curve when $\beta_3 = -0.5$ at $\tau_2 = 0.5, 1, 1.5$. 

D. TIME-DELAYED NONLINEAR CONTROLLERS

The effect of the time-delays ($\tau_1$, $\tau_2$, and $\tau_3$) on the efficiency of the nonlinear controllers (i.e., cubic-position, cubic-velocity, and cubic-acceleration) is discussed within this section with the aid of Eq. (18) and (19). It is clear from Eq. (18) that the nonlinear controllers (i.e., $\beta_2$, $\beta_4$, and $\beta_6$) do not appear as a linear damping term as in the case of the linear controllers. However, the nonlinear controllers’ gains appear in Eq. (18) as a coefficient for the cubic form of the vibration amplitude $a^3$. Moreover, Refs. [20, 21] approved that the cubic-velocity feedback controller can suppress the nonlinear vibration of the parametrically excited systems via adding nonlinear damping to such systems. Accordingly, the coefficient of $a^3$ in Eq. (18) (i.e., $\mu_N = -\frac{\beta_2}{\omega} \sin(\omega \tau_1) + \omega^2 \beta_4 \cos(\omega \tau_2) + \omega^5 \beta_6 \sin(\omega \tau_3)$) can be considered as the nonlinear damping term that has been added due to the nonlinear controllers, where $\mu_N = \omega^2 \beta_4$ at the zero time-delays. Based on this deduction, each nonlinear controller efficiency is investigated via selecting the control gain ($\beta_2$, $\beta_4$, and $\beta_6$) and the times-delays ($\tau_1$, $\tau_2$, and $\tau_3$) in such a way that maximizes the equivalent nonlinear damping term $\mu_N$. Fig. 10 shows the cantilever beam frequency-response curves at three different values of the loop-delays when $\beta_2 = 1.0$ and $-1.0$. It is clear from Fig. 10a that the optimum vibration suppression efficiency occurs when $(\beta_2, \tau_1) = (1.0, 1.54)$, while Fig. 10b confirms that the optimum vibration control efficiency happens at $(\beta_2, \tau_1) = (-1.0, 0.51)$. It is important to mention that the optimal values $(\beta_2, \tau_1) = (1.0, 1.54)$ and $(-1.0, 0.51)$ are selected in such a way that maximizes $\mu_N = -\frac{\beta_2}{\omega} \sin(\omega \tau_1) + \omega^2 \beta_4 \cos(\omega \tau_2) + \omega^5 \beta_6 \sin(\omega \tau_3)$. The effect of the time-delay $\tau_2$ on the vibration suppression efficiency of the cubic-velocity controller is investigated in Fig. 11, while the influence of $\tau_3$ on the cubic-acceleration
efficiency is explored in Fig. 13. It is clear from Fig. 11 that the optimum vibration suppression efficiency occurs at $(\beta_4, \tau_2) = (0.5, 0.0)$ and $(-0.5, 1.026)$, while Fig. 13 confirms that the optimum vibration control efficiency happens at $(\beta_6, \tau_3) = (0.2, 0.51)$ and $(-0.2, 1.54)$. Numerical simulations for the system original equations according to Fig. 11a when $\sigma = 0.0$ are illustrated in Fig. 12 at $\tau_2 = 0, 0.038$. By comparing Fig. 11a and 12 one can notice the excellent agreement between the analytical and numerical results. Moreover, Fig. 12 illustrates a huge influence of the loop-delay $\tau_2$ in improving the vibration suppression efficiency.

Based on the above deduction, to get the highest vibration suppression efficiency for the proposed nonlinear controllers, the control gains $(\beta_2, \beta_4, \beta_6)$ and the time-delays $(\tau_1, \tau_2, \tau_3)$ should be selected in such a way that maximizes the objective function $\mu_N$. By comparing the obtained results in Figs. 10, 11, and 13 with the nonlinear damping function $\mu_N = -\frac{\beta_2}{\omega} \sin(\omega \tau_1) + \omega^2 \beta_4 \cos(\omega \tau_2) + \omega^5 \beta_6 \sin(\omega \tau_3)$, one can report that the best vibration suppression condition has occurred at the maximum value of $\mu_N$ as summarized in Table 2.

**FIGURE 10.** The system frequency-response curve in the case of the time-delayed cubic-position controller: (a) the system frequency-response curve when $\beta_2 = 1.0$ at $\tau_1 = 1.0, 1.54, 2.0$, and (b) the system frequency-response curve when $\beta_2 = -1.0$ at $\tau_1 = 0, 1.0, 0.51, 0.9$.

**FIGURE 11.** The system frequency-response curve in the case of the time-delayed cubic-velocity controller: (a) the system frequency-response curve when $\beta_4 = 0.5$ at $\tau_2 = 0.0, 0.25, 0.38$, and (b) the system frequency-response curve when $\beta_4 = -0.5$ at $\tau_2 = 0.5, 0.64, 1.26$. 
IV. CUBIC-VELOCITY CONTROLLER VERSUS CUBIC-ACCELERATION CONTROLLER

Based on the above discussions, the efficiency of both the cubic-velocity and cubic-acceleration controllers in suppressing the nonlinear oscillation of a parametrically excited system is compared either when the loop-delays is considered or neglected.

A. WITHOUT TIME-DELAYS

Figure 14 shows the system frequency-response curves in the case of both the cubic-velocity and cubic-acceleration controllers for two values of the feedback gains $\beta_4$ and $\beta_6$, when the time-delays $\tau_2 = \tau_3 = 0.0$. Fig. 14a compares the system frequency-response curve of the two controllers when $\beta_4 = \beta_6 = 0.25$, while Fig. 14b illustrates the same frequency-response curve when $\beta_4 =$
\( \beta_6 = 0.5 \). It is clear from Figs. 14a and 14b that the system maximum oscillation amplitude in the case of the cubic-acceleration controller is less than half of the maximum oscillation amplitude in the case of the cubic-velocity controller, which confirms the preference of the cubic-acceleration controller over the cubic-velocity one in mitigating the nonlinear oscillation of the parametrically excited systems. In addition, Figs. 14a shows that the system may have a multiple-solutions interval and exhibit a jump-phenomenon either in the case of the cubic-velocity or the cubic-acceleration controllers. However, Figs. 14b illustrates that the increase of the control gains from \( \beta_4 = \beta_6 = 0.25 \) (as in Fig. 14a) to \( \beta_4 = \beta_6 = 0.5 \) has eliminated the multiple-solutions interval, and the Jump-phenomenon has disappeared in the case of the cubic-velocity controller, while the system still suffers from the multiple-solutions and the jump-phenomenon in the case of the cubic-acceleration controller. Therefore, the cubic-acceleration feedback controller has higher efficiency than the cubic-velocity one in controlling the motion bifurcation.

**B. WITH THE OPTIMUM TIME-DELAyS**

The efficiency of the cubic-velocity and the cubic-acceleration controllers in both suppressing the system vibrations and eliminating the motion bifurcation is compared within this section when the optimum loop-delays are considered. Fig. 15 illustrates the system frequency-response curves either in the case of the time-delayed cubic-velocity controller or the time-delayed cubic-acceleration controller for the positive (i.e. \( \beta_4 = \beta_6 = 0.25 \)) and negative (i.e. \( \beta_4 = \beta_6 = -0.25 \)) control gains. Figs. 15a compares the system frequency-response curves in the case of the two controllers when \( \beta_4 = \beta_6 = 0.25, \tau_2 = 0.0 \) and \( \tau_3 = 0.51 \), while Figs. 15b illustrates the response curves when \( \beta_4 = \beta_6 = -0.25, \tau_2 = 1.026 \) and \( \tau_3 = 1.54 \). It is clear from Figs. 15a and 15b that the system maximum oscillation amplitude in the case of the time-delayed cubic-velocity controller is about five times the maximum oscillation amplitude in the case of the time-delayed cubic-acceleration controller. Moreover, the system may suffer from motion bifurcation in the case of the time-delayed cubic-velocity controller that has disappeared in the case of the time-delayed cubic-acceleration controller. Accordingly, the time-delayed cubic-acceleration controller is the best either in vibration suppression or bifurcation control even with a small feedback gain. In addition to improving the controller performance using the loop-delay as a new control parameter, it also makes us able to design either negative-feedback or positive-feedback controllers to control the targeted system.

![FIGURE 14](image_url)

**FIGURE 14.** The system frequency-response curve in the case of the cubic-velocity controller and the cubic-acceleration controller when the loop-delays are zeros: (a) the system frequency-response curve when \( \beta_4 = \beta_6 = 0.25 \), and (b) the system frequency-response curve when \( \beta_4 = \beta_6 = 0.5 \).
V. CONCLUSIONS

Within this article, six time-delayed state feedback controllers are introduced to control the nonlinear vibrations of the cantilever beam system that is subjected to principal parametric excitation. The proposed controllers are the linear and nonlinear forms of the system position, velocity, and acceleration. The time-delays in the control loop are included in the mathematical model. Accordingly, the time-delayed nonlinear differential equation that simulates the whole system dynamics is analyzed utilizing the multiple scales homotopy method. The frequency-response equation that governs the controlled system vibration amplitude is derived. Stability charts for the loop-delays are obtained. The different response curves for the system before and after control are plotted using the obtained frequency-response equation. The influences of both the six control gains ($\beta_j, j = 1, 2, ..., 6$) and loop-delays ($\tau_j, j = 1, 2, 3$) on the vibration amplitude of the considered system are explored. According to the above discussion, one can conclude the following important remarks:

1. The cubic-position and cubic-acceleration feedback controllers act as damping controllers that can mitigate the system vibrations without affecting the motion bifurcation behaviours.
2. The linear-velocity and the cubic velocity feedback controllers can control both the vibrations amplitude and the bifurcation behaviours of the considered system.
3. The linear-acceleration feedback controller acts as a linear–position controller that can shift the controlled system frequency-response curve either to the right or to the left via modifying the system natural frequency.
4. The most efficient controller in suppressing the system nonlinear vibrations with a small control gain is the cubic-acceleration feedback controller when loop-delay is neglected.
5. The most effective controller in eliminating the motion bifurcation of the cantilever beam system is the cubic-velocity feedback controller when loop-delay is neglected.
6. The existence of time-delays in the control loop could play a dominant role in either improving or degrading the performance of the proposed controllers.
7. The optimum time-delays values that can improve the system robustness against instability and enhance the linear-controllers efficiency in suppression the system vibrations should be selected in such a way that maximizes the linear damping function $\mu_{Eq} = \mu - \frac{\beta_1}{2\omega^2} \sin(\omega\tau_1) + \frac{\beta_3}{2\omega} \cos(\omega\tau_2) + \frac{\beta_5}{2} \sin(\omega\tau_3)$.
8. The optimum control gains and the loop-delays that can enhance the nonlinear-controllers efficiency in suppressing the system nonlinear oscillations should be selected in such a way that maximizes the nonlinear damping function $\mu_N = -\frac{\beta_2}{\omega} \sin(\omega\tau_1) + \omega^2 \beta_4 \cos(\omega\tau_2) + \omega^3 \beta_6 \sin(\omega\tau_3)$.
9. Based on the concluded points (4) to (8), the optimal control method for the parametrically excited systems is the time-delayed cubic-acceleration feedback controller with the optimum time-delays values.
10. In addition to improving the controller performance using the loop-delay as a new control parameter, it also makes us able to design either negative-feedback or positive-feedback controllers to control the targeted system.

**APPENDIX**

**Coefficients of the Jacobin matrix in Eq. (24):**

\[
J_{11} = -\omega_0 + \frac{\beta_1}{2\omega} \sin(\omega \tau_1) - \frac{\beta_2}{2\omega} \cos(\omega \tau_2) - \frac{\beta_3}{2\omega} \omega \sin(\omega \tau_3) \]

\[
-9 \frac{a_1^2}{8} \left( \omega^2 \beta_4 \cos(\omega \tau_2) - \frac{\beta_2}{\omega} \sin(\omega \tau_1) + \omega^5 \beta_6 \sin(\omega \tau_3) \right)
\]

\[
+ \frac{1}{4\omega} \eta f(2\omega + \sigma)^2 \sin(\phi_0)
\]

\[
J_{12} = \frac{-a_1}{4\omega} \eta f(2\omega + \sigma)^2 \cos(\phi_0)
\]

\[
J_{21} = \frac{a_1}{2\omega} \left(-3\alpha_1 - \omega^2 \alpha_2 + 3\omega^2 \alpha_3 - 3\beta_2 \cos(\omega \tau_1) \right)
\]

\[
-3\omega^2 \beta_4 \sin(\omega \tau_2) + 3\omega^6 \beta_6 \cos(\omega \tau_3)
\]

\[
J_{22} = \frac{-1}{2\omega} \eta f(2\omega + \sigma)^2 \sin(\phi_0)
\]

**Coefficients of Eq. (25):**

\[
s_1 = (-\omega_0 + \frac{\beta_1}{2\omega} \sin(\omega \tau_1) - \frac{\beta_2}{2\omega} \cos(\omega \tau_2) - \frac{\beta_3}{2\omega} \omega \sin(\omega \tau_3) \]

\[
-9 \frac{a_1^2}{8} \left( \omega^2 \beta_4 \cos(\omega \tau_2) - \frac{\beta_2}{\omega} \sin(\omega \tau_1) + \omega^5 \beta_6 \sin(\omega \tau_3) \right)
\]

\[
+ \frac{1}{4\omega} \eta f(2\omega + \sigma)^2 \sin(\phi_0)
\]

\[
s_2 = \frac{-1}{2\omega} \eta f(2\omega + \sigma)^2 \sin(\phi_0)
\]

\[
s_3 = \frac{a_1}{2\omega} \left(-3\alpha_1 - \omega^2 \alpha_2 + 3\omega^2 \alpha_3 - 3\beta_2 \cos(\omega \tau_1) \right)
\]

\[
-3\omega^2 \beta_4 \sin(\omega \tau_2) + 3\omega^6 \beta_6 \cos(\omega \tau_3)
\]

\[
s_4 = \frac{1}{4\omega} \eta f(2\omega + \sigma)^2 \cos(\phi_0)
\]

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