1 Introduction

In this paper we discuss and solve a class of 1 + 1 dimensional matrix field theories in the light-cone Hamiltonian approach that are obtained from a dimensional reduction of 3 + 1 dimensional Yang-Mills theory. Recently a similar procedure has been used to formulate a conjectured for M theory [1].

The strategy in formulating the model is to retain all of the essential degrees of freedom of higher dimensional QCD. We start by considering QCD$_{3+1}$ coupled to Dirac adjoint fermions [2]. The virtual creation of fermion-antifermion pairs is not suppressed in the large-$N$ limit – in contrast to the case for fermions in the fundamental representation [3, 4, 5] – and so one may study the structure of bound states beyond the valence quark (or quenched) approximation.

The gauge group of the theory is actually $SU(N)/Z_N$, which has nontrivial topology and vacuum structure. For the particular gauge group $SU(2)$ this has been discussed elsewhere [4]. While this vacuum structure may in
fact be relevant for a discussion of condensates, for the purposes of this calculation it will be ignored.

In the first section we formulate the $3+1$ dimensional $SU(N)$ Yang-Mills theory and then perform dimensional reduction to obtain a $1+1$ dimensional matrix field theory. The light-cone Hamiltonian is then derived for the light-cone gauge $A_- = 0$ following a discussion of the physical degrees of freedom of the theory. We then discuss the exact massless solutions of the boundstate integral equations. These massless states have constant wavefunctions in momentum space and are therefore fundamental excitations of the theory.

2 Definitions

We first consider $3+1$ dimensional $SU(N)$ Yang-Mills coupled to a Dirac spinor field whose components transform in the adjoint representation of $SU(N)$:

$$\mathcal{L} = \text{Tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} (\bar{\Psi} \gamma^\mu \overset{\leftrightarrow}{D}_\mu \Psi) - m \bar{\Psi} \Psi \right], \quad (1)$$

where $D_\mu = \partial_\mu + ig [A_\mu, \cdot]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu]$. We also write $A_\mu = A^a_\mu \tau^a$ where $\tau^a$ is normalized such that $\text{Tr}(\tau^a \tau^b) = \delta_{ab}$. The projection operators $\Lambda_L, \Lambda_R$ permit a decomposition of the spinor field $\Psi = \Psi_L + \Psi_R$,

$$\Lambda_L = \frac{1}{2} \gamma^+ \gamma^-, \quad \Lambda_R = \frac{1}{2} \gamma^- \gamma^+ \quad \text{and} \quad \Psi_L = \Lambda_L \Psi, \quad \Psi_R = \Lambda_R \Psi. \quad (2)$$

Inverting the equation of motion for $\Psi_L$, we find

$$\Psi_L = \frac{1}{2i D_-} \left[ i \gamma^i D_i + m \right] \gamma^+ \Psi_R \quad (3)$$

where $i = 1, 2$ runs over transverse space. Therefore $\Psi_L$ is not an independent degree of freedom.

Dimensional reduction of the $3+1$ dimensional Lagrangian (1) is performed by assuming (at the classical level) that all fields are independent of the transverse coordinates $x^\perp = (x^1, x^2)$: $\partial_{\perp \mu} A_\mu = 0$ and $\partial_{\perp} \Psi = 0$. In the resulting $1+1$ dimensional field theory, the transverse components

\footnote{We use the conventions $\gamma^\pm = (\gamma^0 \pm \gamma^3)/\sqrt{2}$, and $x^\pm = (x^0 \pm x^3)/\sqrt{2}$.}
$A_\perp = (A_1, A_2)$ of the gluon field will be represented by the $N \times N$ complex matrix fields $\phi_\pm$:

$$\phi_\pm = A_1 \mp i A_2 / \sqrt{2}. \quad (4)$$

Here, $\phi_-$ is just the Hermitian conjugate of $\phi_+$. When the theory is quantized, $\phi_\pm$ will correspond to $\pm 1$ helicity bosons (respectively).

The components of the Dirac spinor $\Psi$ are the $N \times N$ complex matrices $u_\pm$ and $v_\pm$, which are related to the left and right-moving spinor fields according to

$$\Psi_R = \frac{1}{2\pi} \begin{pmatrix} u_+ \\ 0 \\ 0 \\ u_- \end{pmatrix}, \quad \Psi_L = \frac{1}{2\pi} \begin{pmatrix} 0 \\ v_+ \\ v_- \\ 0 \end{pmatrix} \quad (5)$$

Adopting the light-cone gauge $A_- = 0$ allows one to explicitly rewrite the left-moving fermion fields $v_\pm$ in terms of the right-moving fields $u_\pm$ and boson fields $\phi_\pm$, by virtue of equation (3). We may therefore eliminate $v_\pm$ dependence from the field theory. Moreover, Gauss' Law

$$\partial^2 A_+ = g \left( i[\phi_+, \partial_- \phi_-] + i[\phi_-, \partial_- \phi_+] + \{u_+, u_\dagger_-\} + \{u_-, u_\dagger_+\} \right) \quad (6)$$

permits one to remove any explicit dependence on $A_+$, and so the remaining physical degrees of freedom of the field theory are represented by the helicity $\pm \frac{1}{2}$ fermions $u_\pm$, and the helicity $\pm 1$ bosons $\phi_\pm$. There are no ghosts in the quantization scheme adopted here. In the light-cone frame the Poincaré generators $P^-$ and $P^+$ for the reduced $1 + 1$ dimensional field theory are given by

$$P^+ = \int_{-\infty}^{\infty} dx^- Tr \left[ 2\partial_- \phi_+ \cdot \partial_- \phi_+ + \frac{i}{2} \sum_h \left( u_\dagger_h \cdot \partial_- u_h - \partial_- u_\dagger_h \cdot u_h \right) \right] \quad (7)$$

$$P^- = \int_{-\infty}^{\infty} dx^- Tr \left[ m_\phi^2 \phi_+ \cdot \phi_- - \frac{g^2}{2} J^+ \frac{1}{\partial^2} J^+ + \frac{ig^2}{2} [\phi_+, \phi_-]^2 + \sum_h F_\dagger_h \frac{1}{i\partial_-} F_h \right] \quad (8)$$

where the sum $\sum_h$ is over $h = \pm$ helicity labels, and

$$J^+ = i[\phi_+, \partial_- \phi_-] + i[\phi_-, \partial_- \phi_+] + \{u_+, u_\dagger_-\} + \{u_-, u_\dagger_+\} \quad (9)$$

$$F_\pm = \mp s g [\phi_\pm, u_\pm] + \frac{m}{\sqrt{2}} u_\pm \quad (10)$$

We have generalized the couplings by introducing the variables $t$ and $s$, which do not spoil the $1 + 1$ dimensional gauge invariance of the reduced theory;
the variable \( t \) will determine the strength of the quartic-like interactions, and the variable \( s \) will determine the strength of the Yukawa interactions between the fermion and boson fields, and appears explicitly in equation (10). The dimensional reduction of the original 3 + 1 dimensional theory yields the canonical values \( s = t = 1 \).

Renormalizability of the reduced theory also requires the addition of a bare coupling \( m_b \), which leaves the 1 + 1 dimensional gauge invariance intact. In all calculations, the renormalized boson mass \( \tilde{m}_b \) will be set to zero.

Canonical quantization of the field theory is performed by decomposing the boson and fermion fields into Fourier expansions at fixed light-cone time \( x^+ = 0 \):

\[
 u_\pm = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, b_\pm(k)e^{-ikx^-} \quad \text{and} \quad \phi_\pm = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2|k|}} \, a_\pm(k)e^{-ikx^-}
\]

where \( b_\pm = b_\pm^a \tau^a \) etc. We also define

\[
b_\pm(-k) = d_\mp^\dagger(k), \quad a_\pm(-k) = a_\mp^\dagger(k), \tag{12}\]

where \( d_\pm \) correspond to antifermions. Note that \((b_\pm^3)^{ij}\) should be distinguished from \(b_\pm^a \tau^a_{ij}\), since in the former the quantum conjugate operator \( \dagger \) acts on (color) indices, while it does not in the latter. The latter formalism is sometimes customary in the study of matrix models. The precise connection between the usual gauge theory and matrix theory formalism may be stated as follows:

\[
b_{\pm ji}^\dagger = b_{\mp ji}^a \tau^{a*}_{ji} = b_{\pm ji}^a \tau^{a}_{ji} = (b_{\pm ji})^\dagger
\]

The commutation and anti-commutation relations (in matrix formalism) for the boson and fermion fields take the following form in the large-\( N \) limit \((k,\tilde{k} > 0; \, h, h' = \pm)\):

\[
 \left[a_{hij}(k), a_{h'kl}^\dagger(\tilde{k}) \right] = \left\{b_{hij}(k), b_{h'kl}^\dagger(\tilde{k}) \right\} = \left\{d_{hij}(k), d_{h'kl}^\dagger(\tilde{k}) \right\} = \delta_{hh'} \delta_{ji} \delta_{ik} \delta(k-\tilde{k}), \tag{13}\]

where have used the relation \( \tau_{ij}^a \tau_{kl}^a = \delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \). All other (anti)commutators vanish.

The Fock space of physical states is generated by the color singlet states, which have a natural ‘closed-string’ interpretation. They are formed by a color trace of the fermion, antifermion and boson operators acting on the vacuum state \(|0\rangle\). Multiple string states couple to the theory with strength \(1/N\), and so may be ignored.
3 The Light-Cone Hamiltonian

The current-current term \( J^+ \frac{1}{p^-} J^+ \) in equation (8) in momentum space, for example, takes the form

\[
P_{J^+,J^+} = \frac{g^2}{2\pi} \int_{-\infty}^{\infty} dk_1 dk_2 dk_3 dk_4 \frac{\delta(k_1 + k_2 - k_3 - k_4)}{(k_3 - k_1)^2} \frac{\text{Tr}}{2} \left[ \sum_{h,h'} : \{ b_h^\dagger(k_1), b_h(k_3) \} : \{ b_{h'}^\dagger(k_2), b_{h'}(k_4) \} : \right.
\]

\[
+ \frac{(k_1 + k_3)(k_2 + k_4)}{4\sqrt{|k_1||k_2||k_3||k_4|}} : [a_+^\dagger(k_1), a_+(k_3)] : [a_+^\dagger(k_2), a_+(k_4)] : 
\]

\[
+ \frac{(k_2 + k_4)}{2\sqrt{|k_2||k_4|}} \sum_h : \{ b_h^\dagger(k_1), b_h(k_3) \} : [a_+^\dagger(k_2), a_+(k_4)] : 
\]

\[
+ \frac{(k_3 + k_1)}{2\sqrt{|k_1||k_3|}} \sum_{h'} : [a_+^\dagger(k_1), a_+(k_3)] : \{ b_{h'}^\dagger(k_2), b_{h'}(k_4) \} : \right] (14)
\]

The explicit form of remaining terms of the Hamiltonian (14) in terms of the operators \( b_\pm, d_\pm \) and \( a_\pm \) is straightforward to calculate, but too long to be written down here. It should be stressed, however, that several \( 2 \rightarrow 2 \) parton processes are suppressed by a factor \( 1/N \), and so are ignored in the large-\( N \) limit. No terms involving \( 1 \leftrightarrow 3 \) parton interactions are suppressed in this limit, however.

One can show that this Hamiltonian conserves total helicity \( h \), which is an additive quantum number. Moreover, the number of fermions minus the number of antifermions is also conserved in each interaction, and so we have an additional quantum number \( N \). States with \( N = \text{even} \) will be referred to as \( \text{boson} \) boundstates, while the quantum number \( N = \text{odd} \) will refer to \( \text{fermion} \) boundstates. The cases \( N = 0 \) and 3, are analogous to conventional mesons and baryons (respectively).

4 Exact Solutions

For the special case \( s = m = \tilde{m}_b = 0 \), the only surviving terms in the Hamiltonian (8) are the current-current interactions \( J^+ \frac{1}{p^-} J^+ \) and the four gluon interaction \( [\phi_+, \phi_-]^2 \). The current-current interaction includes four gluon interaction, four Fermion interaction and two gluon, two Fermion interactions.
This theory has infinitely many massless bound states, and the partons in these states are either fermions or antifermions. States with bosonic \( a_\pm \) quanta are always massive. One also finds that the massless states are pure, in the sense that the number of partons is a fixed integer, and there is no mixing between sectors of different parton number. In particular, for each integer \( n \geq 2 \), one can always find a massless bound state consisting of a superposition of only \( n \)-parton states. A striking feature is that the wavefunctions of these states are \textit{constant}, and so these states are natural generalizations of the constant wavefunction solution appearing in t’Hooft’s model [7].

We present an explicit example below of such a constant wavefunction solution involving a three fermion state with total helicity \(+\frac{3}{2}\) which is perhaps the simplest case to study. We apply \( P^- \) to the state and show that the result vanishes. Massless states with five or more partons appear to have more than one wavefunction which are non-zero and constant, and in general the wavefunctions are unequal. It would be interesting to classify all states systematically, and we leave this to future work. One can, however, easily count the number of massless states. In particular, for \( N = 3 \), \( h = +\frac{3}{2} \) states, there is one three-parton state, 2 five-parton states, 14 seven-parton states and 106 nine-parton states that yield massless solutions.

Let us now consider the action of the light-cone Hamiltonian \( P^- \) on the three-parton state

\[
|b_+ b_+ b_+\rangle = \int_0^\infty dk_1 dk_2 dk_3 \delta(\sum_{i=1}^3 k_i - P^+) \times \\
\begin{align*}
& f_{b_+ b_+ b_+}(k_1, k_2, k_3) \frac{1}{N^{3/2}} \text{Tr}[b_+\dagger(k_1) b_\dagger(k_2) b_\dagger(k_3)] |0\rangle
\end{align*}
\]

The quantum number \( N \) is 3 in this case, and ensures that the state \( P^- |b_+ b_+ b_+\rangle \) must have at least three partons. In fact, one can deduce the following:

\[
P^- |b_+ b_+ b_+\rangle = \int_0^\infty dk_1 dk_2 dk_3 \delta(\sum_{i=1}^3 k_i - P^+) \times \\
\left\{-\frac{g^2 N}{2\pi} \int_0^\infty d\alpha d\beta \frac{\delta(\alpha + \beta - k_1 - k_2)}{\delta(\alpha - k_1)^2} [f_{b_+ b_+ b_+}(\alpha, \beta, k_3) - f_{b_+ b_+ b_+}(k_1, k_2, k_3)]
\right. \\
\left. \frac{1}{N^{3/2}} \text{Tr} \left[ b_+\dagger(\alpha) b_+\dagger(\beta) b_\dagger(k_3) \right] |0\rangle
\right. \\
+ \frac{g^2 N}{2\pi} \int_0^\infty d\alpha d\beta d\gamma \sum_h \frac{\delta(\alpha + \beta + \gamma - k_1)}{\delta(\alpha + \beta)^2} f_{b_+ b_+ b_+}(\alpha + \beta + \gamma, k_2, k_3) \frac{1}{N^{5/2}}
\]
\[
\text{Tr} \left[ \{ b^\dagger_h(\alpha), d^\dagger_{-h}(\beta) \} b^\dagger_+(k_2) b^\dagger_+(k_3) - \{ b^\dagger_h(\alpha), d^\dagger_{-h}(\beta) \} b^\dagger_+(k_2) b^\dagger_+(k_3) b^\dagger_+(\gamma) \right] |0\rangle
\]

\[
+ \frac{g^2 N}{4\pi} \int_0^\infty d\alpha d\beta d\gamma \sum_h \frac{\delta(\alpha + \beta + \gamma - k_1)}{\sqrt{\alpha\beta(\alpha + \beta)^2}} f_{b_+ b_+ b_+} \frac{1}{N^{5/2}}(\alpha + \beta + \gamma, k_2, k_3)
\]

\[
\text{Tr} \left[ [a^\dagger_h(\alpha), a^\dagger_{-h}(\beta)] b^\dagger_+(\gamma) b^\dagger_+(k_2) b^\dagger_+(k_3) - [a^\dagger_h(\alpha), a^\dagger_{-h}(\beta)] b^\dagger_+(k_2) b^\dagger_+(k_3) b^\dagger_+(\gamma) \right] |0\rangle
\]

\[
+ \text{ cyclic permutations } \right \}
\]

(16)

The five-parton states above correspond to virtual fermion-antifermion and boson-boson pair creation. The expression (16) vanishes if the wavefunction \( f_{b_+ b_+ b_+} \) is constant.

One of the exact seven particle wave functions is

\[
\int_0^\infty dk_1 dk_2 dk_3 dk_4 dk_5 dk_6 dk_7 \theta\left(\sum_{i=1}^7 k_i - P^+\right) \frac{1}{N^{7/2}} \frac{1}{\sqrt{2}}
\]

\[
\text{Tr}[b^\dagger_+(k_1) b^\dagger_+(k_2) b^\dagger_+(k_3) b^\dagger_+(k_4) d^\dagger_+(k_5) d^\dagger_+(k_6) d^\dagger_+(k_7)]
\]

(17)

5 Conclusions

We have presented a non-perturbative Hamiltonian formulation of a class of 1 + 1 dimensional matrix field theories, which may be derived from a classical dimensional reduction of QCD_{3+1} coupled to Dirac adjoint fermions. We choose to adopt the light-cone gauge \( A_- = 0 \), and solve the boundstate integral equations in the large-\( N \) limit. Different states may be classified according to total helicity \( h \), and the quantum number \( \mathcal{N} \), which defines the number of fermions minus the number of antifermions in a state.

For a special choice of couplings we find an infinite number of pure massless states of arbitrary length. The wavefunctions of these states are always constant, and are the analogs of the t’Hooft pion. Sometimes the wavefunctions involves several (possibly different) constants. In this model we have both fermion and boson states of this type. The state explicitly shown here are all fermions.

The techniques employed here are not specific to the choice of field theory, and are expected to have a wide range of applicability, particularly in the light-cone Hamiltonian formulation of supersymmetric field theories[8, 9].
6 acknowledgments

The work was supported in part by a grant from the US Department of Energy. Travel support was provided in part by a NATO collaborative grant.

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