ORBITS OF FINITE SOLVABLE GROUPS ON CHARACTERS

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Abstract. We prove that if a solvable group $A$ acts coprimely on a solvable group $G$, then $A$ has a “large” orbit in its corresponding action on the set of ordinary complex irreducible characters of $G$. This extends (at the cost of a weaker bound) a 2005 result of A. Moretò who obtained such a bound in case that $A$ is a $p$-group.

1. Introduction

The main purpose of this paper is to generalize a 2005 result of A. Moretò on orbits in a certain group action. While there are many results on the orbits in the action of a group acting via automorphisms on some other group (of which the action of linear groups on their natural modules is a particularly prominent example), Moretò’s result is noteworthy in that it is one of the very few dealing with the action of a group on the set of irreducible characters of another group.

More precisely, let the finite group $A$ act (via automorphisms) on the finite group $G$. Such an action induces an action of $A$ on the set $\text{Irr}(G)$ in an obvious way (where $\text{Irr}(G)$ denotes the set of complex irreducible characters of $G$). When $G$ is elementary abelian, we are back to studying linear group actions and all the known results apply. But for nonabelian $G$ not much is known about this interesting action. Note that when $(|A|, |G|) = 1$, then it is well-known that the orbit sizes of $A$ on $\text{Irr}(G)$ are the same as the orbit sizes in the natural action of $A$ on the conjugacy classes of $G$, and this latter action was of some importance in \[\text{[5]},\] where some specialized results on this action were obtained. But apart from these we are aware only of two major results on the action of $A$ on $\text{Irr}(G)$.

The first such result is due to D. Gluck \[\text{[2]},\] He proved that when $A$ is abelian and $G$ is solvable, then there always exists an “arithmetically large” orbit on $\text{Irr}(G)$ (i.e., an orbit whose size is divisible by “many” different primes).

The second result is the 2005 result \[\text{[10]}\] by Moretò mentioned above. It proves the existence of a “large” orbit on $\text{Irr}(G)$ in case that $A$ is a $p$-group for some prime $p$ and $G$ is solvable such that $(|A|, |G|) = 1$.

In this paper we take the second result to the next level and establish the existence of a large orbit on $\text{Irr}(G)$ in case that $A$ is solvable and $G$ is solvable such that $(|A|, |G|) = 1$. More precisely, our main result is the following.

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Theorem A. Let $A$ and $G$ be finite solvable groups such that $A$ that acts faithfully and coprimely on $G$. Let $b$ be an integer such that $|A : C_A(\chi)| \leq b$ for all $\chi \in \text{Irr}(G)$. Then $|A| \leq b^{49}$.

We make a few observations. First, it has already been observed in [10] that the coprimeness assumption cannot be omitted.

Second, not surprisingly, our bound is weaker than the bound obtained in [10]. While in [10], when $A$ is a $p$-group, the existence of an orbit of size roughly $|A|^{1/19}$ on $\text{Irr}(G)$ is proved, in the more general situation of Theorem A we only get an orbit of size about $|A|^{1/49}$. Both bounds, however, are far from best possible anyway and thus the results are more of a qualitative nature. The true bound is probably close to $|A|^{1/2}$ in both cases.

Third, if we assume $|AG|$ to contain no small primes, the bounds one can get tend to be much better. In [10] it is observed in passing that if odd order is assumed, then the bound will get much better. And here we will explicitly establish a better bound in case that $|G|$ is not divisible by 6 (see Theorem 3.3 below).

Our proof of Theorem A is largely based on the ideas introduced in [10] and extends them to the more general hypothesis of Theorem A. In particular, our proof does not use Moretó’s result, but rather reproves it (with a weaker bound) as a special case. To keep the bound from getting too large, we also make use of a recent strengthening in the solvable case of a result by M. Aschbacher and R. Guralnick [1] on the size of $|G/G'|$ of a linear solvable group; see [22].

2. The abelian quotient of linear groups

We first recall a result due to Aschbacher and Guralnick.

Proposition 2.1. Let $G$ be a finite solvable group that acts faithfully and completely reducibly on a finite vector space $V$. Then $|G : G'| \leq |V|$.

Proof. This is a special case of the much more general [1, Theorem 3].

Next we provide a recent strengthening of Proposition 2.1 by the authors.

Theorem 2.2. Let $G$ be a finite solvable group that acts faithfully and completely reducibly on a finite vector space $V$, and let $B$ be the size of the largest orbit of $G$ on $V$. Then $|G : G'| \leq B$.

Proof. This will appear in [6].

3. Main Theorem

The following result is an extension of [10, Lemma 2.1].

Theorem 3.1. Assume that a solvable $\pi$-group $A$ acts faithfully on a solvable $\pi'$-group $G$. Let $b$ be an integer such that $|A : C_A(\chi)| \leq b$ for all $\chi \in \text{Irr}(G)$. Let $\Gamma = AG$ be the semidirect product. Let $K_{i+1} = F_{i+1}(\Gamma)/F_i(\Gamma)$ and let $K_{i+1,\pi}$ be the Hall $\pi$-subgroup of $K_{i+1}$ for all $i \geq 1$. Let $K_i/\Phi(\Gamma/F_i(\Gamma)) = V_{i1} + V_{i2}$ where $V_{i1}$ is the $\pi$ part of $K_i/\Phi(\Gamma/F_i(\Gamma))$
and $V_2$ is the $\pi'$ part of $K_i/\Phi(\Gamma/F_{i-1}(\Gamma))$ for all $i \geq 1$. Let $K \triangleleft \Gamma$ such that $\Phi(\Gamma/F_{i-1}(\Gamma)) = K$. Let $L_{i+1,\pi} = K_{i+1,\pi} \cap K$. We have that $|C_{L_{i+1,\pi}}(V_{i1})| \leq b^2$, and $|C_{L_{i+1,\pi}}(V_{i1})| \leq b$ if $L_{i+1,\pi}$ is abelian. The order of the maximum abelian quotient of $C_{L_{i+1,\pi}}(V_{i1})$ is less than or equal to $b$ for all $i \geq 1$.

**Proof.** We know that $L_{i+1,\pi}$ acts faithfully and completely reducibly on $V_{i1} + V_{i2}$. Clearly $C_{L_{i+1,\pi}}(V_{i1})$ acts faithfully and completely reducibly on $V_{i2}$ and $L_{i+1,\pi}/C_{L_{i+1,\pi}}(V_{i1})$ acts faithfully and completely reducibly on $V_{i1}$.

Let $L$ be the pre-image of $C_{L_{i+1,\pi}}(V_{i1})$ in $(\Gamma/F_{i-1}(\Gamma))/\Phi(\Gamma/F_{i-1}(\Gamma))/V_{i1}$. Write $L = QV_{i2}$, where $Q \in \text{Hall}_L(L)$. We have to prove that $|Q| \leq b^2$. Clearly $F(L) = V_{i2}$ and $\Phi(L) = 1$. We know by a theorem of Brauer that $Q$ acts faithfully on $V_{i2}$. Replacing $A$ by a conjugate, if necessary, we may assume that $Q \leq A$. It follows from our hypothesis that $|Q : C_Q(\chi)| \leq b$ for all $\chi \in \text{Irr}(G)$.

Now, let $\lambda \in \text{Irr}(V_{i2})$. By [3, Theorem 13.28], there exists $\chi \in \text{Irr}(G)$ lying over $\lambda$ that is $C_Q(\lambda)$-invariant. We claim that $C_Q(\chi) = C_Q(\lambda)$. It is clear that $|Q : C_Q(\lambda)|$ divides the degree of any character of $L$ lying over $\lambda$. Therefore, $|Q : C_Q(\lambda)|$ divides the degree of any character of $QG$ lying over $\lambda$ since the pre-image of $L$ in $\Gamma$ is normal in $\Gamma$. Now, [3, Corollary 8.16] and Clifford’s correspondence [3, Theorem 6.11] yield that there exist $\psi \in \text{Irr}(QG)$ lying over $\chi$, whence over $\lambda$, such that $\psi(1) = |Q : C_Q(\chi)|$. It follows that $C_Q(\chi) = C_Q(\lambda)$, as desired. In particular, $|Q : C_Q(\lambda)| \leq b$. We deduce that for all $\lambda \in \text{Irr}(V_{i2})$, $|Q : C_Q(\lambda)| \leq b$. Now, [4], for instance, implies that $|Q| \leq b^2$. Also, if $Q$ is abelian, then $|Q| \leq b$. The order of the maximum abelian quotient of $Q$ is less than or equal to $b$ by Theorem 2.2. This completes the proof of the theorem.

Now we are ready to prove Theorem A, which we restate.

**Theorem 3.2.** Let $A$ be a solvable $\pi'$-group that acts faithfully on a solvable $\pi'$-group $G$. Let $b$ be an integer such that $|A : C_A(\chi)| \leq b$ for all $\chi \in \text{Irr}(G)$. Then $|A| \leq b^{\#}$.

**Proof.** Let $\Gamma = AG$ be the semidirect product of $A$ and $G$. By Gaschütz’s theorem, $\Gamma/F(\Gamma)$ acts faithfully and completely reducibly on $\text{Irr}(\Gamma/F(\Gamma))$. It follows from [3, Theorem 3.3] that there exists $\lambda \in \text{Irr}(\Gamma/F(\Gamma))$ such that $T = C_T(\lambda) \leq F(\Gamma)$.

Let $K_{i+1} = F_{i+1}(\Gamma)/F_i(\Gamma)$ and let $K_{i+1,\pi}$ be the Hall $\pi'$-subgroup of $K_{i+1}$ for all $i \geq 1$. We know that $K_{i+1,\pi}$ acts faithfully and completely reducibly on $K_{i}/\Phi(F_{i-1}(\Gamma))$. It is clear that we may write $K_{i}/\Phi(F_{i-1}(\Gamma)) = V_{i1} + V_{i2}$ where $V_{i1}$ is the $\pi$ part of $K_{i}/\Phi(F_{i-1}(\Gamma))$ and $V_{i2}$ is the $\pi'$ part of $K_{i}/\Phi(F_{i-1}(\Gamma))$ for all $i \geq 1$. Clearly $C_{K_{i+1,\pi}}(V_{i1})$ acts faithfully and completely reducibly on $V_{i2}$. Thus $|C_{K_{i+1,\pi}}(V_{i1})| \leq b^2$ and the order of the maximum abelian quotient of $C_{K_{i+1,\pi}}(V_{i1})$ is less than or equal to $b$ by Theorem 3.1. Also $K_{i+1,\pi}/C_{K_{i+1,\pi}}(V_{i1})$ acts faithfully and completely reducibly on $V_{i1}$. Since $K_{i+1,\pi}/C_{K_{i+1,\pi}}(V_{i1})$ is nilpotent, $|K_{i+1,\pi}/C_{K_{i+1,\pi}}(V_{i1})| \leq |V_{i1}|^{\beta}/2$ where $\beta = \log(32)/\log(9)$ by [7, Theorem 3.3]. Also the order of the maximum abelian quotient of $K_{i+1,\pi}/C_{K_{i+1,\pi}}(V_{i1})$ is bounded above by $|V_{i1}|$ by Proposition 2.1.

Thus we have the following, $|K_{2,\pi}| \leq b^2$ and the order of the maximum abelian quotient of $K_{2,\pi}$ is bounded above by $b$.

$|K_{3,\pi}| \leq |C_{K_3,\pi}(V_{21})| \cdot |K_{3,\pi}/C_{K_3,\pi}(V_{21})| \leq b^2 \cdot b^3$ and the order of the maximum abelian quotient of $K_{3,\pi}$ is bounded above by $b \cdot b = b^2$. 


$$|K_{4,π}| \leq |C_{K_{4,π}}(V_{31})| \cdot |K_{4,π}/C_{K_{4,π}}(V_{31})| \leq b^2 \cdot b^{2β}$$ and the order of the maximum abelian quotient of $K_{4,π}$ is bounded above by $b \cdot b^2 = b^3$.

$$|K_{5,π}| \leq |C_{K_{5,π}}(V_{41})| \cdot |K_{5,π}/C_{K_{5,π}}(V_{41})| \leq b^2 \cdot b^{3β}$$ and the order of the maximum abelian quotient of $K_{5,π}$ is bounded above by $b \cdot b^3 = b^4$.

$$|K_{6,π}| \leq |C_{K_{6,π}}(V_{51})| \cdot |K_{6,π}/C_{K_{6,π}}(V_{51})| \leq b^2 \cdot b^{4β}$$ and the order of the maximum abelian quotient of $K_{6,π}$ is bounded above by $b \cdot b^4 = b^5$.

$$|K_{7,π}| \leq |C_{K_{7,π}}(V_{61})| \cdot |K_{7,π}/C_{K_{7,π}}(V_{61})| \leq b^2 \cdot b^{5β}$$ and the order of the maximum abelian quotient of $K_{7,π}$ is bounded above by $b \cdot b^5 = b^6$.

$$|K_{8,π}| \leq |C_{K_{8,π}}(V_{71})| \cdot |K_{8,π}/C_{K_{8,π}}(V_{71})| \leq b^2 \cdot b^{6β}.$$  

Next, we show that $|Γ : T|_π \leq b$.

Let $χ$ be any irreducible character of $G$ lying over $λ$. Then every irreducible character of $Γ$ that lies over $χ$ also lies over $λ$ and hence has degree divisible by $|Γ : T|$. But $χ$ extends to its stabilizer in $Γ$ and thus some irreducible character of $Γ$ lying over $λ$ has degree $χ(1)|A : C_{A}(χ)|$. The $π$-part of $|Γ : T|$, therefore, divides $|A : C_{A}(χ)|$, which is at most $b$.

This gives that $|A| \leq b^2 \cdot b^2 \cdot b^3 \cdot b^2 \cdot b^{2β} \cdot b^2 \cdot b^{3β} \cdot b^2 \cdot b^{4β} \cdot b^2 \cdot b^{5β} \cdot b^2 \cdot b^{6β} \cdot b = b^{15+21β} \leq b^{48.124}$.  

**Theorem 3.3.** Let $A$ be a solvable $π$-group that acts faithfully on a solvable $π'$-group $G$. Assume that $2, 3 ∉ π$. Let $b$ be an integer such that $|A : C_{A}(χ)| \leq b$ for all $χ ∈ \text{Irr}(G)$. Then $|A| \leq b^4$.

**Proof.** Let $Γ = AG$ be the semidirect product of $A$ and $G$. By Gaschütz’s theorem, $Γ/Φ(Γ)$ acts faithfully and completely reducibly on $\text{Irr}(Γ)$. It follows from [6] Theorem 3.2 that there exists $λ ∈ \text{Irr}(Γ)$ such that $T = C_Γ(λ) ⊆ K$, $Φ(Γ) ⊆ K ⊆ F_3(Γ)$. The $π$-subgroup of $K$ acts faithfully and completely reducibly on $V_{i+1}/V_i$ and $K_i/Φ(Γ)$ acts faithfully and completely reducibly on $V_{i+1}/V_i$.

Since the image of $K ∩ F_2(Γ)$ in $K_{2,π}$ is abelian. $|(K ∩ F_2(Γ))/F_2(Γ)| \leq b$ and the order of the maximum abelian quotient of $K_{2,π}$ is bounded above by $b$ by Theorem 3.1.

Since $L_{3,π} = K/F_2(Γ)$ is abelian. $|L_{3,π}| \leq |C_{L_{3,π}}(V_{21})| \cdot |L_{3,π}/C_{L_{3,π}}(V_{21})| \leq b \cdot b$ by Theorem 3.1 and Proposition 2.1.

Next, we show that $|Γ : T|_π \leq b$.

Let $χ$ be any irreducible character of $G$ lying over $λ$. Then every irreducible character of $Γ$ that lies over $χ$ also lies over $λ$ and hence has degree divisible by $|Γ : T|$. But $χ$ extends to its stabilizer in $Γ$ and thus some irreducible character of $Γ$ lying over $χ$ has degree $χ(1)|A : C_{A}(χ)|$. The $π$-part of $|Γ : T|$, therefore, divides $|A : C_{A}(χ)|$, which is at most $b$.

This gives that $|A| \leq b \cdot b \cdot b \cdot b \leq b^4$.  

Since when $|A|, |G| = 1$, the orbit sizes of $A$ on $\text{Irr}(G)$ are the same as the orbit sizes in the natural action of $A$ on the conjugacy classes of $G$, the following results immediately follow from the previous ones.

**Theorem 3.4.** Let $A$ be a solvable $π$-group that acts faithfully on a solvable $π'$-group $G$. Let $b$ be an integer such that $|A : C_{A}(C)| \leq b$ for all $C ∈ cl(G)$. Then $|A| \leq b^{49}$.  

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Theorem 3.5. Let $A$ be a solvable $\pi$-group that acts faithfully on a solvable $\pi'$-group $G$. Assume that $2, 3 \notin \pi$. Let $b$ be an integer such that $|A : C_A(C)| \leq b$ for all $C \in \text{cl}(G)$. Then $|A| \leq b^4$.

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