$U_q(\widehat{sl}_n)$-analog of the XXZ chain with a boundary.

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Abstract

We study $U_q(\widehat{sl}_n)$ analog of the XXZ spin chain with a boundary magnetic field $h$. We construct explicit bosonic formulas of the vacuum vector and the dual vacuum vector with a boundary magnetic field. We derive integral formulas of the correlation functions.
1 Introduction

In the standard treatment of quantum integrable systems, one starts with a finite box and impose periodic boundary conditions, in order to ensure integrability. Recently, there has been increasing interest in exploring other possible boundary conditions compatible with integrability.

For the free fermionic models, there have been obtained many explicit formulas of the correlation functions with non-periodic boundary conditions. In this category, the work on the two dimensional Ising model by B.M. McCoy and T.T. Wu [1] are among the earliest. They derived the spin-spin correlation functions with a boundary field, by combinatorial arguments. For an impenetrable Bose gas model, T. Kojima derived the ground state correlation functions [2] and the time dependent correlation functions [3] with Dirichlet or Neumann conditions.

In this paper we are interested in the non free fermion model. For the non free fermion model, E.K. Sklyanin [4] began a systematic approach to open boundary problem, so-called open boundary Bethe Ansatz. He formulated the transfer matrix to open boundary problem, and derived the Bethe Ansatz equations. Jimbo et al. [5] united Sklyanin’s open boundary Bethe Ansatz and Kyoto school’s method [6] - so called representation theory approach to solvable models. They studied XXZ model with a boundary, which are governed by the quantum affine symmetry $U_q(\hat{sl}_2)$. They constructed explicit bosonic formulas of the vacuum vector with an arbitrary boundary magnetic field. They derived integral formulas of the boundary magnetizations. H. Ozaki [7] studied $U_q(\hat{sl}_3)$ analog of the XXZ chains with a boundary. He constructed explicit bosonic formulas of the vacuum vector for the special boundary conditions. In this paper we studied $U_q(\hat{sl}_n)$ analog of the XXZ chains with an arbitrary boundary magnetic field $h$.

Our results are new even for $U_q(\hat{sl}_3)$ case.

The Hamiltonian of our model is given by

$$H_B = \sum_{k=1}^{\infty} \left\{ \cosh(\gamma) \sum_{a,b=0}^{n-1} e^{(k+1)}_{aa} e^{(k)}_{bb} + \sinh(\gamma) \sum_{a,b=0}^{n-1} \text{sgn}(b-a) e^{(k+1)}_{aa} e^{(k)}_{bb} \right. $$

$$- \sum_{a,b=0 \atop a \neq b}^{n-1} e^{(k+1)}_{ab} e^{(k)}_{ba} \bigg\} + h \sum_{a=L}^{M-1} e^{(1)}_{aa} - 2 \sinh(\gamma) \left\{ \sum_{a=0}^{L-1} e^{(1)}_{aa} - \sum_{a=M}^{n-1} e^{(1)}_{aa} \right\}, \quad (1)$$

where $0 \leq \gamma < +\infty$ and $0 \leq L \leq M \leq n - 1$. The Hamiltonian acts on the semi infinite tensor products of $\mathbb{C}^n$. We construct explicit bosonic formulas of the vacuum vector and the dual vacuum vector with a boundary magnetic field. Using bosonization of the vacuum, the dual vacuum and the vertex operators [9], we derive integral formulas of the correlation functions.

Now a few words about the organization of this paper. In Section 2, we formulate our problem. In Section 3, we construct explicit bosonic formulas of the vacuum vector and the dual vacuum vector. In Section 4, we derive
integral formulas of the correlation functions. In Appendix we summarize the
bosonizations of the Vertex operators, for reader's convenience.

2 Formulation

The purpose of this section is to formulate our problem.

2.1 Notation

We fix a real number $-1 < q < 0$ and an integer $n \in \mathbb{N} \setminus \{0, 1\}$. In the sequel, we denote $(q^k - q^{-k})/(q - q^{-1})$ by $[k]$. Let $P$ be a free Abelian group on letters $\Lambda_1, \cdots, \Lambda_{n-1}, \delta$.

$$P = \oplus_{i=0}^{n-1} \mathbb{Z} \Lambda_i \oplus \mathbb{Z} \delta.$$ 

We call $P$ the weight lattice. Let $h_1, \cdots, h_{n-1}, d$ be an ordered basis of $P^* = \text{Hom}(P, \mathbb{Z})$ dual to $\Lambda_1, \cdots, \Lambda_{n-1}, \delta$.

$$\langle \Lambda_i, h_j \rangle = \delta_{ij}, \langle \Lambda_i, d \rangle = 0, \langle \delta, h_j \rangle = 0, \langle \delta, d \rangle = 1.$$ 

Let us set the simple roots as

$$\alpha_0 = -\Lambda_{n-1} + 2\Lambda_0 - \Lambda_1 + \delta, \alpha_j = -\Lambda_{j-1} + 2\Lambda_j - \Lambda_{j+1}, \ (j = 1, \cdots, n-1).$$

The projection to classical lattice is given by

$$\bar{\Lambda}_i = \Lambda_i - \Lambda_0, \ \delta = 0.$$ 

The invariant bilinear form on $(\cdot|\cdot) : P \times P \to \mathbb{Z}$ by

$$(\alpha_i|\alpha_j) = -\delta_{i,j-1} + 2\delta_{i,j} - \delta_{i,j+1}, \ (\delta|\delta) = 0.$$ 

The quantum affine algebras $U_q(\hat{sl}_n)$ are algebras with 1 over $\mathbb{C}$, defined by the generators $e_i, f_i, t_i^{\pm 1} = q^{\pm h_i}, q^d, (i = 0, \cdots, n-1)$ through the following defining relations:

$$t_i t_j = t_j t_i, \quad t_i e_j t_i^{-1} = q^{\langle \alpha_j, h_i \rangle} e_j, \quad t_i f_j t_i^{-1} = q^{-\langle \alpha_j, h_i \rangle} f_j,$$

$$[e_i, f_j] = \delta_{i,j} t_i - t_i^{-1}.$$ 

$$\sum_{k=0}^{b} (-1)^k \begin{bmatrix} b \\ k \end{bmatrix} e_i^k e_j e_i^{b-k} = 0, \quad \sum_{k=0}^{b} (-1)^k \begin{bmatrix} b \\ k \end{bmatrix} f_i^k f_j f_i^{b-k} = 0,$$

where we have set

$$b = 1 - \langle \alpha_i, h_j \rangle, \quad B = \begin{bmatrix} b \\ k \end{bmatrix} = \frac{[b]!}{[k]![b-k]!}, \quad [k]! = [k][k-1] \cdots [1].$$

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Let us set $U_q'(sl_n)$ be the subalgebra of $U_q(sl_n)$ generated by $t_i, e_i, f_i, (i = 0, \cdots , n - 1)$. Let us set $U_q(sl_n)$ be the subalgebra of $U_q(sl_n)$ generated by $t_i, e_i, f_i, (i = 1, \cdots, n - 1)$. We denote the irreducible highest weight $U_q$-module with highest weight $\lambda$ by $V(\lambda)$. Let $V$ be a finite dimensional representation of $U_q(sl_n)$. The evaluation module $V_2 = V \otimes \mathbb{C}[z, z^{-1}]$ in the homogeneous picture is the following $U_q(sl_n)$-module defined by
\[
e_0(v \otimes z^m) = (f_1 v) \otimes z^{m+1}, e_j(v \otimes z^m) = (e_j v) \otimes z^m, \quad (j = 1, \cdots, n),
\]
\[
f_0(v \otimes z^m) = (e_1 v) \otimes z^{-m-1}, f_j(v \otimes z^m) = (f_j v) \otimes z^m, \quad (j = 1, \cdots, n),
\]
\[
t_0 = t_1^{-1}, t_1(v \otimes z^m) = (t_1 v) \otimes z^m, d = \frac{dz}{z}.
\]

### 2.2 Solvable Model

Fix the number $i \in \{0, 1, \cdots, n - 1\}$. Let $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_{n-1}$ be a basic representation of $U_q(sl_n)$. Let the R-matrix $R^{(i)}(z_1/z_2) \in \text{End}(V_{z_1} \otimes V_{z_2})$ be an intertwiner of $U_q(sl_n)$ in the homogeneous picture,

\[
PR^{(i)}(z_1/z_2) : V_{z_1} \otimes V_{z_2} \rightarrow V_{z_2} \otimes V_{z_1}.
\]

Fixing the normalization constant, the R-matrix $R(z)$ is given by

\[
R^{(i)}(z)_{m,j}^{k,l} = \frac{1}{\kappa^{(i)}(z)} \begin{cases}
\frac{(1 - q^2 z)\sqrt{z}}{1 - q^2 z} \delta_{m,k} \delta_{j,l} \sqrt{z} \, \text{sgn}(k-l) \\ \delta_{m,k},
\end{cases}
\]

$$
+ \frac{(1 - z)q}{1 - q^2 z} \delta_{m,k} \delta_{j,k}, \quad 0 \leq k \neq l \leq n - 1,
$$

$$
0 \leq k = l \leq n - 1.
$$

Here we have set

\[
\kappa^{(i)}(z) = z^{\delta_{i,0}} \left( \frac{q^2 z^{-1}; q^{2n}}{q^{2n} z^{-1}; q^{2n}} \right) \left( \frac{q^2 z; q^{2n}}{q^{2n} z; q^{2n}} \right) \left( \frac{q^2 z^{-1}; q^{2n}}{q^{2n} z^{-1}; q^{2n}} \right),
\]

where $(z;p)_\infty = \prod_{n=1}^{\infty} (1 - z p^n)$.

The R-matrix $R^{(i)}(z)$ satisfies the Yang-Baxter equation. Let us fix the integer number $0 \leq L \leq M \leq n - 1$ and $r \in \mathbb{R}$. Let us set the reflection K-matrix $K^{(i)}(z) \in \text{End}(V_z)$ [4], [3], by

\[
K^{(i)}(z) = \begin{pmatrix}
\frac{k_0(z)}{k_0(1/z)} & \frac{k_1(z)}{k_1(1/z)} & \cdots & \frac{k_{n-1}(z)}{k_{n-1}(1/z)} \\
\frac{k_1(z)}{k_1(1/z)} & \cdots & \frac{k_{n-1}(z)}{k_{n-1}(1/z)} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{k_{n-1}(z)}{k_{n-1}(1/z)} & \cdots & \frac{k_1(z)}{k_1(1/z)} & \frac{k_0(z)}{k_0(1/z)}
\end{pmatrix},
\]

where we have set

\[
k_0(z) = \cdots = k_{L-1}(z) = z,
\]

\[
k_L(z) = \cdots = k_{M-1}(z) = 1 - rz,
\]

4
The scalar functions $\varphi^{(i)}(z)$ are given by (15), (18), (19), (20), (28), (29), and (30). The reflection matrix $K^{(i)}(z)$ satisfies the Boundary Yang-Baxter equation.

$$k_M(z) = \cdots = k_{n-2}(z) = k_{n-1}(z) = 1.$$

Note. de Vega and Ruiz [8] found the special diagonal solutions for the case $L = 0, 1 \leq M \leq n - 1$ and $0 \leq L = M \leq n - 1$. Ozaki [7] found the general diagonal solutions as the same arguments as in [8].

Graphically, an elements of the R-matrix $R^{(i)}(z)_{kl}^{mj}$ is the picture described in Figure 1. An element of the reflection K-matrix $K^{(i)}(z)_k^j$ is the picture described in Figure 2.

**Figure 1**

The Type-I Vertex operator $\Phi^{\mu,V}_{\lambda}(z)$ is an intertwining operator of $U_q(\hat{sl}_n)$ defined by

$$\Phi^{\mu,V}_{\lambda}(z) : V(\lambda) \to V(\mu) \otimes V_z.$$

The Type-I dual Vertex operator is an intertwing operator of $U_q(\hat{sl}_n)$ defined by

$$\Phi^{\lambda}_{\mu,V}(z) : V(\mu) \otimes V_z \to \hat{V}(\lambda).$$

Let us set the components of the vertex operators $\Phi^{\mu,V}_{\lambda,j}(z)$ as follows.

$$\Phi^{\mu,V}_{\lambda,j}(z)|u\rangle = \sum_{j=1}^{n-1} \Phi^{\mu,V}_{\lambda,j}(z)|u\rangle \otimes v_j, \quad \text{for} \quad |u\rangle \in V(\lambda).$$

Let us set the components of the vertex operators $\Phi^{\lambda}_\mu,V,j(z)$ as follows.

$$\Phi^{\lambda}_\mu,V,z(|u\rangle \otimes v_j) = \Phi^{\lambda}_\mu,V,j(z)|u\rangle, \quad \text{for} \quad |u\rangle \in V(\mu).$$
Only $\Phi^{(\Lambda_j),\Lambda_j}(z)$, $\Phi^{(\Lambda_{j+1}),\Lambda_j}(z)$ are nontrivial. We take the following normalizations.

\[ \Phi^{(\Lambda_j),\Lambda_j}(z)|\Lambda_j\rangle \otimes v_i = |\Lambda_j\rangle \otimes v_i + \cdots, \]

\[ \Phi^{(\Lambda_{j+1}),\Lambda_j}(z)|\Lambda_j\rangle \otimes v_i = |\Lambda_{j+1}\rangle + \cdots, \]

where $|\Lambda_j\rangle$ is the highest vector of $V(\Lambda_j)$. Let us consider the product of the vertex operators. Let us fix the following notation.

\[ \Phi^{(\Lambda_i-2),\Lambda_i}(z_1) \Phi^{(\Lambda_i-1),\Lambda_i}(z_2) = \sum_{j_1,j_2=0}^{n-1} \Phi^{(\Lambda_i-2),\Lambda_i}(z_1) \Phi^{(\Lambda_i-1),\Lambda_i}(z_2) \otimes v_{j_1} \otimes v_{j_2}. \]

The vertex operators satisfy

\[ R^{(i)}(z_1/z_2) \Phi^{(\Lambda_{i-2}),\Lambda_{i-1}}(z_1) \Phi^{(\Lambda_{i-1}),\Lambda_i}(z_2) = \Phi^{(\Lambda_{i-1}),\Lambda_i}(z_1) \]

In the sequel, we use the abbreviations

\[ \Phi^{(i,i+1)}(z) = \Phi^{(\Lambda_i),\Lambda_i}(z), \quad \Phi^{*(i,i+1)}(z) = \Phi^{(\Lambda_{i+1}),\Lambda_i}(z) \]

Graphically, the vertex operator is the picture described in Figure 3. The dual vertex operator is the picture described in Figure 4.

\[ \Phi_j(z) = \]

\[ \Phi_j^*(z) = \]

We define the normalized transfer matrix by

\[ T_B^{(i)}(z) = g_n \sum_{j=0}^{n-1} \Phi_j^{*(i,i-1)}(z^{-1}) K^{(i)}(z) \Phi_j^{(i-1,i)}(z), \]

where we have used

\[ g_n = \frac{(q^2;q^2^n)_\infty}{(q^{2n};q^{2n})_\infty}. \]
Graphically, the transfer matrix $T^{(i)}_B(z)$ in the semi-infinite chain, is the picture in Figure 5. It describes a semi-infinite two-dimensional lattice, with alternating spectral parameter.

![Figure 5](image)

The renormalized Hamiltonian $H^{(i)}_B$ in (1) is then defined by

$$
\frac{d}{dz} T^{(i)}_B(z) \bigg|_{z=1} = \frac{q}{1-q^2} H^{(i)}_B + \text{const},
$$

where we set

$$
h = \frac{1 - q^2}{q} \times \frac{r + 1}{r - 1}, \quad q = e^{-\gamma}.
$$

Here the right hand side $H^{(i)}_B$ acts on the space $H^{(i)}$, where $H^{(i)}$ is the span of vectors $|p\rangle = \otimes_{k=1}^{\infty} v_{p(k)}$, called paths, labelled by maps $p : \mathbb{Z} \geq 1 \rightarrow \mathbb{Z}/n\mathbb{Z}$ satisfying the asymptotic boundary condition

$$
p(k) = k + i \in \{0, 1, 2, \cdots, n-1\} = \mathbb{Z}/n\mathbb{Z}, \quad \text{for } k \gg 1.
$$

We have identified the highest weight module $V(\Lambda_i)$ and the path space $H^{(i)}$, following the strategy proposed in Ref. [6]. In order to diagonalize the Hamiltonian $H^{(i)}_B$ in (1), we diagonalize the transfer matrix $T^{(i)}_B(z)$. Using the Boundary Yang-Baxter equations, we have

$$
[T^{(i)}_B(z), T^{(i)}_B(z')] = 0, \quad T^{(i)}_B(1) = \text{id}, \quad T^{(i)}_B(z)T^{(i)}_B(z^{-1}) = \text{id}.
$$

This commuting relation of the transfer matrix asserts the integrability of this problem.

3 Vacuum Vectors

The purpose of this section is to construct the explicit bosonic formulas of the vacuum vector $|i\rangle_B$ such that

$$
T^{(i)}_B(z)|i\rangle_B = |i\rangle_B, \quad (i = 0, \cdots, n - 1),
$$
which is realized as

$$|i\rangle_B = e^{F_i}|i\rangle,$$

where $|i\rangle$ is the highest weight vector of $V(\Lambda_i)$, and $F_i$ is a quadratic in the boson operators. Multiplying the vertex operator $\Phi^{(i-1,i)}(z^{-1})$ from the left, and using the inversion relation,

$$g_n \Phi^{(i-1,i)}_j(z) \Phi^*_{j+1}^{(i-1,i)}(z) = id, \quad g_n = \frac{(q^2;q^2)_{\infty}}{(q^2n;q^2)_{\infty}},$$

we know the eigenvalue problem (4) is equivalent to

$$K^{(i)}(z) \Phi^{(i-1,i)}_j(z) |i\rangle_B = \Phi^{(i-1,i)}_j(z^{-1}) |i\rangle_B. \quad (5)$$

We construct the dual vacuum vectors $B\langle i|$ such that

$$B\langle i| T^{(i)}_B(z) = B\langle i|, \quad (i = 0, \cdots, n-1), \quad (6)$$

which is realized as

$$B\langle i| = \langle i| e^{G_i},$$

where $\langle i|$ is the lowest weight vector of the restricted dual module $V^*(\Lambda_i)$, and $G_i$ is a quadratic in the boson operators. As the same argument as the vacuum vectors, we know the eigenvalue problem (4) is equivalent to

$$K^{(i)}(z) B\langle i| \Phi^*_{j+1}^{(i,i-1)}(z^{-1}) = B\langle i| \Phi^*_{j}^{(i,i-1)}(z). \quad (7)$$

The scalar factor of the refrection matrix $\varphi^{(i)}(z)$ are given by (15), (18), (19), (20), (22), (23), and (30).

Note. H. Ozaki [7] constructed the vacuum vector $|0\rangle_B$ for $U_q(\widehat{sl_3})$, $V(\Lambda_0)$, (a) $L = M = 2$ or (b) $L = 0, M = 2$ cases. Our results are new for $U_q(\widehat{sl_3})$-case.

### 3.1 Vacuum

Let us consider the vacuum vector $|i\rangle_B$. Since the total spin is conserved, it should be a linear combination of the states created by the oscillators $a_s(-k)$ over the highest weight vector $|i\rangle$. We make the ansatz that it has the following form.

$$|i\rangle_B = e^{F_i}|i\rangle,$$

where

$$F_i = \sum_{s,t=1}^{n-1} \sum_{k=1}^{\infty} \alpha_{s,t}(k) a_s(-k) a_t(-k) + \sum_{s=1}^{n-1} \sum_{k=1}^{\infty} \beta_s^{(i)}(k) a_s(-k).$$
The operator $e^{F_i}$ has the effect of a Bogoliubov transformation,

$$e^{-F_i}a_j(k)e^{F_i} = a_j(k) + \sum_{s,t=1}^{n-1} \alpha_{s,t}(k) \left( \frac{[a_j|a_s]k}{k} - a_t(-k) + (s \leftrightarrow t) \right) + \sum_{s=1}^{n-1} \beta_s^{(i)}(k) \frac{[a_j|a_s]k}{k}.$$ 

Using the bosonic formulas of the vertex operators, we have the $(n-1)$-th component of the equation (3) as follows,

$$\varphi^{(i+1)}(z) = z \frac{e^{-\frac{i\pi}{2}i(\tilde{A}_{n-1}\tilde{A}_{i+1})}}{e^{P(z)}q^{Q(z)}e^{F_{i+1}}|i\rangle} = (z \leftrightarrow z^{-1}).$$

Comparing the bosonic parts of the both sides, we have

$$\alpha_{s,n-1}(k) = - \frac{1}{2} \frac{sk}{|k|^2[nk]} q^{2(n+2)k} \langle 1 \rangle \quad (1 \leq s \leq n - 1).$$

Comparing the bosonic part of the $j$-th component of the equation (3), we have

$$\alpha_{s,t}(k) = - \frac{kq^{2(n+1)k}}{2[k]} \times I_{s,t}(k). \quad (8)$$

Here the matrix $(I_{s,t}(k))_{1 \leq s,t \leq n-1}$ is the inverse matrix of the A-type Cartan matrix $([a_s|a_t]k)_{1 \leq s,t \leq n-1}$. More explicitly

$$I_{s,t}(k) = \frac{sk|(n-t)k}{|k|^2[nk]} = I_{t,s}(k), \quad (1 \leq s \leq t \leq n - 1). \quad (9)$$

Using the explicit formulas of $\alpha_{s,t}(k)$, we have the simple formulas of the action of the basic operators to the vacuum vectors.

$$e^{Q(z)|i\rangle}_B = h^{(i)}(z)e^{P(1/z)|i\rangle}_B,$$

$$e^{S_j(w)|i\rangle}_B = g_j^{(i)}(w)e^{R_j(q^{2n+1}/w)|i\rangle}_B, \quad (1 \leq j \leq n - 1),$$

where

$$h^{(i)}(z) = \exp \left( -\frac{1}{2} \sum_{k=1}^{\infty} \frac{(n-1)k}{|nk|^2} q^k z^{-2k} - \sum_{k=1}^{\infty} \frac{k}{k} \beta^{(i)}_{n-1}(k) q^{-(2n+1)/2} z^{-k} \right),$$

and

$$g_j^{(i)}(w) = \exp \left( -\frac{1}{2} \sum_{k=1}^{\infty} \frac{2k}{k} q^{2(n+3)k} - \sum_{k=1}^{\infty} \sum_{s=1}^{n-1} \beta_s^{(i)}(k) q^{k/2} w^{-k} \right).$$

The $(n-1)$-th component of the equation (3) reduces to

$$\varphi^{(i)}(z) = z^{b_{i-1}} h^{(i)}(z^{-1}), \quad (0 \leq i \leq n - 1). \quad (10)$$
When we find the functions $g_j^{(i)}(w)$, $(1 \leq j \leq n-1)$, we can determine both $\beta_j^{(i)}(k)$, $(1 \leq j \leq n-1)$, and $h^{(i)}(z)$.

First we consider the case $|0\rangle_B$, and $0 \leq L < M \leq n - 1$. We show the following pair of $g_j^{(0)}(q^{n+1}w)$ give the vacuum vector.

\[
g_j^{(0)}(q^{n+1}w) = \begin{cases} 
(1 - 1/w^2)(1 - q^{-n+2M-L}/rw), & j = L \\
(1 - 1/w^2)(1 - q^{-M}r/w), & j = M \\
(1 - 1/w^2), & j \neq L, M.
\end{cases}
\]  

(11)

The $(n - 2)$-th component of the equation (8) reduces to

\[
\oint \frac{dz}{2\pi i w_{n-1}} \frac{z^{-1}k_{n-2}(z)w_{n-1}g_j^{(0)}(q^{n+1}w_{n-1})}{(1 - qw_{n-1}/z)(1 - qz/w_{n-1})(1 - q/(zw_{n-1}))} 
\times e^{P(z) + P(1/z) + R_{n-1}^{-1}(q^{n+1}w_{n-1}) + R_{n-1}^{-1}(q^{n+1}/w_{n-1})} |0\rangle_B = (z \leftrightarrow z^{-1}),
\]

where the contour encircles $w = 0, qz^\pm 1$ but not $q^{-1}z^\pm 1$. Because the bosonic part of this equation is invariant under the change of variable $w_{n-1} \rightarrow w_{n-1}'$ and $z \rightarrow z^{-1}$, this equation reduces to the following integrand relation.

\[
g_j^{(0)}(q^{n+1}w) g_{n-2}^{(0)}(q^{n+1}/w) = -w^{-2}k_{n-2}(z)/z(1 - qz/w) - k_{n-2}(1/z)z(1 - q/(zw)) - w^{-2}k_{n-2}(z)/z(1 - qzw) - k_{n-2}(1/z)z(1 - qw/z).
\]

Therefore we have

\[
g_j^{(0)}(q^{n+1}w) = \begin{cases} 
1 - 1/w^2, & \text{for } k_{n-2}(z) = 1 \\
(1 - 1/w^2)(1 - rz/w), & \text{for } k_{n-2}(z) = 1 - rz.
\end{cases}
\]

The $(n - k)$-th component of the equation (8) reduces to

\[
\oint \frac{dw_{n-1}}{2\pi i w_{n-1}} \cdots \oint \frac{dw_{n-k+1}}{2\pi i w_{n-k+1}} \frac{w_{n-k+1}g_j^{(0)}(q^{n+1}w_{n-1}) \cdots g_j^{(0)}(q^{n+1}w_{n-k+1})}{D(z, w_{n-1})D(w_{n-1}, w_{n-2}) \cdots D(w_{n-k+1}, w_{n-k+1})} 
\times e^{P(z) + P(1/z) + R_{n-1}^{-1}(q^{n+1}w_{n-1}) + \cdots + R_{n-k}^{-1}(q^{n+1}w_{n-k+1}) + R_{n-k}^{-1}(q^{n+1}/w_{n-k+1})} |0\rangle_B = (z \leftrightarrow z^{-1}),
\]

where we have set

\[
D(w_1, w_2) = (1 - qw_1/w_2)(1 - qw_2/w_1)(1 - qw_1w_2)(1 - q/(w_1w_2)).
\]

Here the contour of the integral $\oint \frac{dw_j}{2\pi i w_j}$ encircles $0$ and $q^{\pm 1}w_{j+1}$ but not $q^{-1}w_{j+1}$.

$(w_n = z)$ Because the bosonic part of this equation and the function $D(w_j, w_{j+1})$ are invariant under the change of variables $w_j \rightarrow w_j^{-1}$ and $z \rightarrow z^{-1}$, this equation reduces to the following integrand relations.

\[
g_j^{(0)}(q^{n+1}w) g_j^{(0)}(q^{n+1}/w) = \begin{cases} 
-w^{-2} (1 - q^{-n+2M-L}/rw), & j \neq L, M \\
-w^{-2} (1 - q^{-n+2M-L}w/r), & j = L \\
w^{-2} (1 - q^{-n-M}rw), & j = M.
\end{cases}
\]

(10)
Therefore we have the relation (11).

As the same arguments as the above, we can construct \( g_j^{(0)}(q^{n+1}w) \) for the case \( 0 \leq L = M \leq n - 1 \). We have

\[
g_j^{(0)}(q^{n+1}w) = \begin{cases} 
1 - 1/w^2, & j \neq L, \\
1 - 1/w^4, & j = L.
\end{cases}
\]  

(12)

Now we have solved the problem for \( |0\rangle_B \) case. The coefficients of the bosonic operators are given by (8) and

\[
\beta_j^{(0)}(k) = (q^{n+3/2}k - q^{(n+1/2)k})\theta_k \sum_{s=1}^{n-1} \hat{I}_{j,s}(k)
\]  

(13)

\[
+ \begin{cases} 
-\hat{I}_{j,L}(k)q^{2M-L+1/2}k - \hat{I}_{j,M}(k)q^{(2n-M+1/2)k}z^k, & (0 \leq L < M \leq n - 1) \\
-2(-1)^{k/2}\theta_k \hat{I}_{j,L}(k)q^{(n+1/2)k}, & (0 \leq L = M \leq n - 1).
\end{cases}
\]

Here we have used the symmetric matrix \( \hat{I}_{s,t}(k) \) defined by

\[
\hat{I}_{s,t}(k) = \begin{cases} 
0, & (st = 0), \\
I_{s,t}(k), & (1 \leq s, t \leq n - 1),
\end{cases}
\]  

(14)

where we have used the inverse matrix of the A-type Cartan matrix \( I_{s,t}(k) \) (3).

We have used

\[
\theta_k = \begin{cases} 
1, & \text{for } k = \text{even}, \\
0, & \text{for } k = \text{odd}.
\end{cases}
\]

Using the explicit formulas of \( \beta_n^{(0)}(k) \), we have the scalar factor of the refrection matrix.

\[
\varphi^{(0)}(z) = \frac{(q^{2n+2}z^2;q^{4n})_\infty}{(q^{4n}z^2;q^{4n})_\infty} \prod_{k=1}^{n-1} \frac{-q^{2(n+1)/2}z^{2k} \theta_k \hat{I}_{j,L}(k)q^{(n+1/2)k}z^k}{(rq^{2n+2}z^2;q^{4n})_\infty (r^{-1}q^{2M-L+1/2}z^2;q^{4n})_\infty},
\]  

(15)

\[
\times \begin{cases} 
0 \leq L < M \leq n - 1, \\
0 \leq L = M \leq n - 1.
\end{cases}
\]

As the same arguments as the above, we can construct \( |i\rangle_B \), \( 1 \leq i \leq n - 1 \).

For \( |i\rangle_B \) case, the integrand functions satisfy the following relations.

\[
g_j^{(1)}(q^{n+1}w) = w^{2q_i-j} g_j^{(0)}(q^{n+1}w).
\]

We have

\[
g_j^{(L)}(q^{n+1}w) = \begin{cases} 
(1 - 1/w^2), & j = L, \\
(1 - q^{n-M+1/2}L/w^2)(1 - q^{-n-M+1/2}L/w^2), & j = M, \\
(1 - 1/w^2), & j \neq L, M,
\end{cases}
\]  

\[
\begin{pmatrix} \begin{array}{ccc} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{array} \end{pmatrix}
\]  

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The refraction matrix is given by
\[
g_j^{(M)}(q^{n+1}w) = \begin{cases} 
(1 - 1/w^2)(1 - q^{-n+2M-L}/(rw)), & j = L, \\
(1 - 1/w^2)/(1 - q^{M-M-n}/(rw)), & j = M, \\
(1 - 1/w^2)/(1 - q^{M-M-n}/(rw)), & j \neq L, M,
\end{cases} \quad \left\{ \begin{array}{l}
1 \leq i \leq n-1 \\
i = M \\
0 \leq L < M \leq n-1
\end{array} \right.,
\]
and
\[
g_j^{(i)}(q^{n+1}w) = \begin{cases} 
(1 - 1/w^2), & j = L, \\
(1 - 1/w^2)/(1 - q^{n-M}/w), & j = M, \\
(1 - 1/w^2)/(1 + 1/w^2), & j = i,
\end{cases} \quad \left\{ \begin{array}{l}
1 \leq i \leq n-1 \\
i \neq L, M, M \leq n-1
\end{array} \right.,
\]
\[
g_j^{(i)}(q^{n+1}w) = (1 - 1/w^2), \quad \left\{ \begin{array}{l}
1 \leq i \leq n-1 \\
0 \leq L = M = i \leq n-1
\end{array} \right.
\]
The coefficients of the bosonic operators are given by
\[
\beta_j^{(i)}(k) = (q^{(n+3/2)k} - q^{(n+1/2)k}) \theta_k \sum_{s=1}^{n-1} \hat{I}_{j,s}(k)
\]
\[
+ \begin{cases} 
\hat{I}_{j,L}(k)q^{(2n-M+L+1/2)k} - \hat{I}_{j,M}(k)q^{(2n-M+1/2)k}, & (i = L) \\
-\hat{I}_{j,L}(k)q^{(2M-L+1/2)k} - \hat{I}_{j,M}(k)q^{(M+1/2)k}, & (i = M)
\end{cases} \quad (i \neq L, M),
\]
and
\[
\beta_j^{(i)}(k) = (q^{(n+3/2)k} - q^{(n+1/2)k}) \theta_k \sum_{s=1}^{n-1} \hat{I}_{j,s}(k)
\]
\[
+ 2(-1)^{k/2} \theta_k q^{(n+1/2)k}(-\hat{I}_{j,L}(k) + \hat{I}_{j,M}(k)), \quad \left\{ \begin{array}{l}
1 \leq i \leq n-1 \\
0 \leq L < M \leq n-1
\end{array} \right.
\]
where we have used the matrix \(\hat{I}_{s,t}(k)\) defined in (14). The scalar factor of the refraction matrix is given by
\[
\varphi^{(i)}(z) = z^{-1}(q^{2n+2z}; q^{4n})_\infty \times \begin{cases} 
(q^{2n+2z}; q^{2n})_\infty, & (i = L) \\
(q^{2n+2z}; q^{2n})_\infty, & (i = M)
\end{cases} \quad \left\{ \begin{array}{l}
1 \leq i \leq n-1 \\
0 \leq L < M \leq n-1
\end{array} \right.
\]
Using the bosonic formulas of the vertex operators, and comparing the dual vacuum vectors, we have used the element of the matrix (9). Using the explicit formulas from the previous subsection, we construct the bosonic formulas for the dual vacuum vectors. We make the ansatz that the dual vacuum has the following form.

\[ \varphi^{(i)}(z) = z^{-(q^{2n+2}z^2, q^{4n})_\infty} \left( \frac{1}{(q^{2n}z^2, q^{4n})_\infty} \right)^{n-1} \left( \frac{1}{(q^{2n-2M}z^2, q^{4n})_\infty} \right)^{n-1} \]

\[ \times \left( 1 \leq i \leq n-1, i \neq L, M \right) \]

\[ 0 \leq L < M \leq n-1 \] (19)

and

\[ \varphi^{(i)}(z) = z^{-(q^{2n+2}z^2, q^{4n})_\infty} \left( \frac{1}{(q^{2n}z^2, q^{4n})_\infty} \right)^{n-1} \left( \frac{1}{(q^{2n-2L}z^2, q^{4n})_\infty} \right)^{n-1} \]

\[ \times \left( 1 \leq i \leq n-1 \right) \]

\[ 0 \leq L < M \leq n-1 \] (20)

Let us consider the action of Type-II vertex operators. Using the bosonic expression of the vacuum \( |i\rangle_B \), we have

\[ \Psi_n^{-1}(z)|i\rangle_B = z^{2k} |(q^{2n}z^2, q^{4n})_\infty |i\rangle_B \]

For \( L = 0 \leq M \leq n-1 \), A. Doikou and R.I. Nepomechie derived the boundary S-matrix by the Bethe Ansatz method. Their result is expressed by the \( q \)-gamma function. By changing variables

\[ z^\pm = (-1)^\pm (q^2)^{-1/2} \lambda/n \]

their results coincide to ours.

### 3.2 Dual Vacuum

Let us consider the dual vacuum vector \( B|\langle i| \). As the same arguments as the previous subsection, we construct the bosonic formulas for the dual vacuum vectors. We make the ansatz that the dual vacuum has the following form.

\[ B|\langle i| = |i| e^{G_i} \]

where

\[ G_i = \sum_{s,t=1}^{n-1} \sum_{k=1}^{\infty} \gamma_{s,t}(k) a_s(k) a_t(k) + \sum_{s=1}^{n-1} \sum_{k=1}^{\infty} \delta_s^{(i)}(k) a_s(k). \] (21)

Using the bosonic formulas of the vertex operators, and comparing the both sides of the equation (9), we have

\[ \gamma_{s,t}(k) = \frac{-kq^{-2k}}{2[k]} \times I_{s,t}(k), \] (22)

where we have used the element of the matrix (3). Using the explicit formulas of \( \gamma_{s,t}(k) \), we have the simple formulas of the action of the basic operators to the dual vacuum vectors.

\[ B|\langle i| e^{P^s(z)} = h^{s(i)}(z) B|\langle i| e^{Q^s(1/z)}, \]

\[ B|\langle i| e^{S^s_j(w)} = g^{s(i)}(w) B|\langle i| e^{R^s_j(q^2/w)}, \quad (1 \leq j \leq n-1), \]

\[ B|\langle i| e^{S^s_j(w)} = g^{s(i)}(w) B|\langle i| e^{R^s_j(q^2/w)}, \quad (1 \leq j \leq n-1), \]

\[ B|\langle i| e^{S^s_j(w)} = g^{s(i)}(w) B|\langle i| e^{R^s_j(q^2/w)}, \quad (1 \leq j \leq n-1), \]
Therefore the coefficients of the bosonic operators are given by (22) and
\[ h^{*(i)}(z) = \exp\left( -\frac{1}{2} \sum_{k=1}^{\infty} \frac{(n-1)k}{nk} q^k z^{2k} + \sum_{k=1}^{\infty} \frac{k}{k} \delta^{(i)}_1(k) q^{3k/2} z^k \right), \]
and
\[ g_j^{*(i)}(w) = \exp\left( -\frac{1}{2} \sum_{k=1}^{\infty} \frac{[2k]}{[k]k} q^{-k} w^{2k} - \sum_{k=1}^{\infty} \sum_{s=1}^{n-1} \delta^{(i)}_s(k) \frac{[(a_j,a_s)k]}{k} q^{k/2} w^k \right). \]

The 0-th component of the equation (3) reduces to
\[ \phi^{(i)}(z) = k_0(z)^{-1} h^{*(i)}(z), \quad (0 \leq i \leq n - 1). \]
(23)

Let us consider the case \( B(0) \). As the same arguments as the previous subsection, the integrand relations reduce to the following relations.
\[
\begin{cases}
\frac{g_j^{(0)}(qw)}{g_j^{(0)}(q/w)} = \begin{cases}
-\frac{w^2}{(1-q^L rw)}, & j \neq L, M, \\
-\frac{(1-q^L rw)}{(1-q^L M w/r)}, & j = L, \quad (0 \leq L < M \leq n - 1), \\
-\frac{(1-q^{M-2L}/rw)}{(1-q^{M-2L}/rw)}, & j = M,
\end{cases}
\end{cases}
\]

Therefore we have
\[
\begin{cases}
g_j^{*(0)}(qw) = \begin{cases}
\frac{(1-w^2)}{(1-w^2)}, & j \neq L, M, \\
\frac{(1-w^2)}{(1-w^2)}, & j = L, \quad (0 \leq L < M \leq n - 1), \\
\frac{(1-q^{2L-M}rw)}{(1-q^{2L-M}rw)}, & j = M,
\end{cases}
\end{cases}
\]

As the same arguments we have
\[
\begin{cases}
g_j^{*(0)}(qw) = \begin{cases}
\frac{(1-w^2)}{(1-w^2)}, & j \neq L, \\
\frac{(1-w^2)}{(1-w^2)}, & j = L, \quad (0 \leq L = M \leq n - 1).
\end{cases}
\end{cases}
\]

Therefore the coefficients of the bosonic operators are given by (22) and
\[ \delta_j^{(i)}(k) = -(q^{-k/2} - q^{-3k/2}) \sum_{s=1}^{n-1} \hat{I}_{j,s}(k) \]
\[ + \begin{cases}
-q^{(-L-3/2)k} \hat{I}_{j,L}(k) - q^{(2L-M-3/2)k} \hat{I}_{j,M}(k), & (0 \leq L < M \leq n - 1), \\
-2(-1)^{k/2} q^{-3k/2} \theta_k \hat{I}_{j,L}(k), & (0 \leq L = M \leq n - 1).
\end{cases} \]

(24)

For the other boundary conditions \( B(i), \quad (1 \leq i \leq n - 1) \), the integrand relations reduce to the following relation.
\[ \frac{g_j^{*(i)}(qw)}{g_j^{*(i)}(q/w)} = w^{2\delta_j^{(i)}} \frac{g_j^{*(0)}(qw)}{g_j^{*(0)}(q/w)}. \]
(25)

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Therefore we have

\[
g_j^{(i)}(qw) = \begin{cases} 
(1 - w^2), & j = L, M, \\
(1 - w^2)(1 - q^L w), & j = L, \\
\frac{(1 - w^2)(1 - q^M w)}{1 - q^{2L-M} w}, & j = M, \\
\end{cases} \quad \left( 1 \leq i \leq n - 1, \ i = L \right),
\]

\[
g_j^{(i)}(qw) = \begin{cases} 
(1 - w^2), & j = L, M, \\
\frac{(1 - w^2)}{1 - q^{-L} w/r}, & j = L, \\
\frac{(1 - w^2)(1 - q^{-M} w/r)}{1 - q^{2L-M} w}, & j = M, \\
\end{cases} \quad \left( 1 \leq i \leq n - 1, \ i = L \right),
\]

and

\[
g_j^{(i)}(qw) = \begin{cases} 
(1 - w^2), & j = L, i, \\
(1 - w^2)(1 + w^2), & j = i, \\
\frac{(1 - w^2)}{1 - q^{-L} w/r}, & j = L, \\
\end{cases} \quad \left( 1 \leq i \leq n - 1, \ i \neq L \right),
\]

\[
g_j^{(i)}(qw) = (1 - w^2), \quad \left( 0 \leq i \leq n - 1, \ 0 \leq L = M = i \leq n - 1 \right)
\]

Therefore the coefficients of the bosonic operators are given by (22) and

\[
d_{j}^{(i)}(k) = -(q^{-k/2} - q^{-3k/2}) \theta_k \sum_{s=1}^{n-1} \hat{I}_{j,s}(k) \quad (26)
\]

\[
+ \begin{cases} 
q^{(L-3/2)k} \theta_k \hat{I}_{j,L}(k) - q^{(2L-M-3/2)k} \theta_k \hat{I}_{j,M}(k), & i = L \\
q^{(-L-3/2)k} \theta_k \hat{I}_{j,L}(k) + q^{((M-2L-3/2)k} \theta_k \hat{I}_{j,M}(k), & i = M \\
-2(1)^{k/2} q^{-3k/2} \theta_k \hat{I}_{j,i}(k), & i \neq L, M \\
\end{cases} \quad \left( 1 \leq i \leq n - 1, \ 0 \leq L < M \leq n - 1 \right)
\]

and

\[
d_{j}^{(i)}(k) = -(q^{-k/2} - q^{-3k/2}) \theta_k \sum_{s=1}^{n-1} \hat{I}_{j,s}(k) \quad (27)
\]

\[
+ 2(1)^{k/2} q^{-3k/2} \theta_k \hat{I}_{j,i}(k) (\hat{I}_{j,i}(k) - \hat{I}_{j,L}(k)), \quad \left( 1 \leq i \leq n - 1, \ 0 \leq L = M \leq n - 1 \right). \]

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The scalar factors of the reflection matrix \( \varphi^{(i)}(z) \) are given by

\[
\varphi^{(i)}(z) = \frac{1}{k_0(z)} \frac{(q^{2n+2}z^2;q_{4n})_\infty}{(q^{4n}z^2;q_{4n})_\infty}, \quad (0 \leq L = M = i \leq n-1),
\]

\[
\varphi^{(0)}(z) = \frac{1}{k_0(z)} \frac{(q^{2n+2}z^2;q_{4n})_\infty}{(q^{4n}z^2;q_{4n})_\infty} \times \left\{ \begin{array}{ll}
(rq^{2L}z^2;q_{4n})_\infty ((r^{-1}z;q_{2n})_\infty)^{1-\delta_{L,0}} & (0 \leq L < M \leq n-1), \\
(rq^{2n-2M+2L}z^2;q_{4n})_\infty ((r^{-1}q^{2n-2L}z^2;q_{2n})_\infty)^{1-\delta_{L,0}} & (0 \leq L = M \leq n-1), \\
((-q^{2L}z^2;q_{4n})_\infty)^{1-\delta_{L,0}} & (0 \leq L = M \leq n-1).
\end{array} \right.
\]

\[
\varphi^{(i)}(z) = \frac{1}{k_0(z)} \frac{(q^{2n+2}z^2;q_{4n})_\infty}{(q^{4n}z^2;q_{4n})_\infty} \times \left\{ \begin{array}{ll}
(rq^{2n-2M+2L}z^2;q_{4n})_\infty ((r^{-1}q^{2n-2L}z^2;q_{2n})_\infty)^{1-\delta_{L,0}} & (0 \leq L = M = i \leq n-1), \\
(r^{-1}q^{2n-2M+2L}z^2;q_{4n})_\infty ((r^{-1}q^{2n-2L}z^2;q_{2n})_\infty)^{1-\delta_{L,0}} & (0 \leq L = M = i \leq n-1), \\
((-q^{2n-2L}z^2;q_{4n})_\infty)^{1-\delta_{L,0}} & (0 \leq L = M = n-1, i \neq L).
\end{array} \right.
\]

4 Correlation functions

In the previous section we have constructed the bosonic formulas of the vacuum and the dual vacuum. In this section we consider an application of these bosonic formulas. We have constructed both vacuum and dual vacuum for the same transfer matrix, in the following cases.

1. \( \Lambda_i \), \( (0 \leq i \leq n-1) \), \( 0 \leq L = M = i \leq n-1 \),
2. \( \Lambda_i \), \( (0 \leq i \leq n-2) \), \( 0 \leq L = i \leq M \leq n-1 \),
3. \( \Lambda_i \), \( (1 \leq i \leq n-1) \), \( 0 \leq L < M = i \leq n-1 \).

Therefore we can derive the vacuum expectation value for the above cases.

Let \( L \) be a linear operator on the \( m \)-fold tensor product of the \( n \)-dimensional vector space \( V \otimes \cdots \otimes V \). The corresponding local operator \( L \) acting on our space of states \( V(\Lambda_i) \) can be defined in terms of the Type-I vertex operators, in exactly the same way as in the bulk theory. Explicitly, if \( E_{j,k}(m) \) is the spin operator at the \( m \)-site

\[
E_{j,k}(m) = E_{j,k} \otimes id \otimes \cdots \otimes id,
\]
the corresponding local operator $E^{(i)}_{j,k}(m)$ is given by

$$E^{(i)}_{j,k}(m) = g_n^m \sum_{j_1 \cdots j_{m-1}=0}^{n-1} \Phi^*_{j_1} (1) \cdots \Phi^*_{j_{m-1}} (1) \Phi^*_{j} (i-m+1, i-m+2) \cdots \Phi^*_{j_{1}} (1),$$

where we have used

$$g_n = \frac{(q^2; q^{2n})_\infty}{(q^{2n}, q^{2n})_\infty}.$$

Therefore the boundary magnetization is given by

$$\sum_{j=0}^{n-1} \omega^j P^{(i)}_j (1), \quad (31)$$

where $\omega$ is an $n$-th primitive root of 1, and we have used the one-point function $P^{(i)}_j (z)$ with a spectral parameter $z$, defined by

$$P^{(i)}_j (z) = g_n B(i) \Phi^*_{j} (i-1) (z) \Phi_{j} (i-1, i)(z) |i\rangle B.$$ \quad \quad (32)

In order to evaluate the expectation value $\langle \Phi_{j} (32) \rangle$, we invoke the bosonization formulas of various quantities. By normal-ordering the product of vertex operators, for $j = 0, n - 1$, we have

$$P^{(i)}_j (z) = q^i (1 - q^2)^n \oint \frac{dw_1}{2\pi i w_1} \cdots \oint \frac{dw_{n-1}}{2\pi i w_{n-1}} \frac{1}{w_i}$$

$$\times \prod_{i=1}^{n-2} \frac{1}{(1 - qw_i/w_{i+1})(1 - qw_{i+1}/w_i)} I(z, w_1/q, \cdots, w_j/q, q^{n+1}w_{j+1}, \cdots, q^{n+1}w_{n-1})$$

$$\times \begin{cases} 
\frac{w_1}{z} (1 - q^{n+1}w_1/z)(1 - qz/w_{n-1})(1 - q^{n}/z), & (j = 0), \\
\frac{1}{w_{n-1}} (1 - qz/w_1)(1 - qw_1/z)(1 - q^{n+3}/w_{n-1}), & (j = n - 1).
\end{cases} \quad (33)$$

where the contours of integrals are taken as $|qw_i/w_{i+1}|, |qw_{i+1}/w_i| < 1, (1 \leq l \leq n-2)$ and, for $j = 0$ case we add the conditions $|q^{n+1}w_1/z|, |qz/w_{n-1}|, |qw_{n-1}/z| < 1$, for $j = n - 1$ case, we add the conditions $|qz/w_1|, |qw_1/z|, |q^{n+3}/w_{n-1}| < 1$. For $1 \leq j \leq n - 2$, we have

$$P^{(i)}_j (z) = q^i (1 - q^2)^n \oint \frac{dw_1}{2\pi i w_1} \cdots \oint \frac{dw_{n-1}}{2\pi i w_{n-1}} \frac{1}{w_i}$$

$$\times \prod_{i=1}^{n-2} \frac{1}{(1 - qw_i/w_{i+1})(1 - qw_{i+1}/w_i)} I(z, w_1/q, \cdots, w_j/q, q^{n+1}w_{j+1}, \cdots, q^{n+1}w_{n-1})$$

$$\times \frac{w_jw_{j+1}}{z} (1 - qz/w_1)(1 - qw_1/z)(1 - q^{n+3}w_{j+1}/w_j)(1 - qz/w_{n-1})(1 - qw_{n-1}/z), \quad (34)$$
where the contour of integrals is taken as $|qw_l/w_{l+1}|, |qw_{l+1}/w_l| < 1$, $(1 \leq l \neq j \leq n - 2)$ and $|qz/w_1|, |qw_1/z|, |q^{n+3}w_{j+1}/w_j|, |qz/w_{n-1}|, |qw_{n-1}/z| < 1$.

Here we have set

$$I(z, w_1, \cdots, w_{n-1}) \times B\langle i|v \rangle_B$$

$$= B\langle i| \exp \left( \sum_{k=1}^{n-1} \sum_{p=1}^{n-1} \sum_{l=1}^{n-1} I_{p,l}(k) x_{l}(k) a_p(-k) \right)$$

$$\times \exp \left( \sum_{k=1}^{n-1} \sum_{p=1}^{n-1} \sum_{l=1}^{n-1} I_{p,l}(k) y_{l}(k) a_p(k) \right) |i\rangle_B, \quad (35)$$

where

$$x_{j}(k) = \begin{cases}  
\frac{q^{k/2}}{[2k]} z^{-k} + \frac{q^{k/2}}{[k]} w_{1}^{k} - \frac{q^{k/2}}{[2k]} w_{2}^{k}, & (j = 1), \\
\frac{q^{k/2}}{[2k]} z^{-1} w_{j-1}^{k} + \frac{q^{k/2}}{[k]} w_{1}^{k} - \frac{q^{k/2}}{[2k]} w_{j+1}^{k}, & (2 \leq j \leq n - 2), \\
\frac{q^{k/2}}{[2k]} z w_{n-2}^{k} - \frac{q^{k/2}}{[2k]} w_{n-1}^{k}, & (j = n - 1),
\end{cases}$$

and

$$y_{j}(k) = \begin{cases}  
\frac{q^{-k/2}}{[2k]} z^{k} + \frac{q^{-k/2}}{[k]} w_{1}^{k} - \frac{q^{-k/2}}{[2k]} w_{2}^{k}, & (j = 1), \\
\frac{q^{-k/2}}{[2k]} z w_{j-1}^{k} + \frac{q^{-k/2}}{[k]} w_{1}^{k} - \frac{q^{-k/2}}{[2k]} w_{j+1}^{k}, & (2 \leq j \leq n - 2), \\
\frac{q^{-(2n+1)k/2}}{[2k]} z w_{n-2}^{k} - \frac{q^{-(2n+1)k/2}}{[2k]} w_{n-1}^{k}, & (j = n - 1),
\end{cases}$$

Here $I_{p,l}(k)$ is defined in (34). To calculate the vacuum expectation values (46), we use the coherent states. Let us define the coherent state by

$$|\xi_1 \cdots \xi_{n-1}\rangle_i = \exp \left( \sum_{p=1}^{n-1} \sum_{k=1}^{\infty} \frac{k}{[k][2k]} \xi_{p}(k) a_p(-k) \right) |i\rangle,$$

and

$$i\langle \xi_1 \cdots \xi_{n-1}| = \langle i| \exp \left( \sum_{p=1}^{n-1} \sum_{k=1}^{\infty} \frac{k}{[k][2k]} \bar{\xi}_{p}(k) a_p(k) \right).$$

The coherent states enjoy

$$a_p(k)|\xi_1 \cdots \xi_{n-1}\rangle_i = \sum_{j=1}^{n-1} \frac{[a_p|a_j]\bar{k}}{[2k]} \xi_{j}(k)|\xi_1 \cdots \xi_{n-1}\rangle_i,$$

$$i\langle \xi_1 \cdots \xi_{n-1}|a_p(-k) = i\langle \xi_1 \cdots \xi_{n-1}| \sum_{j=1}^{n-1} \frac{[a_p|a_j]\bar{k}}{[2k]} \bar{\xi}_{j}(k).$$

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Here we have used
\[ id = \int_{-\infty}^{\infty} \left( \frac{-1}{2\pi i} \right)^{n-1} \prod_{i=1}^{n-1} \prod_{k>0} \frac{k[(i+1)k] d\xi_i(k) d\bar{\xi}_i(k)}{[2k]^2} \]
\[ \times \exp \left( -\sum_{i,j=1}^{n-1} \sum_{1 \leq k \leq \infty} \frac{[a_i][a_j]k}{[k][2k]} \xi_i(k) \bar{\xi}_j(k) \right) |\xi_1 \cdots \xi_{n-1}| i \bar{\xi}_1 \cdots \bar{\xi}_{n-1}|, \]

where the integration is taken over the entire complex plane with the measure
\[ d\xi d\bar{\xi} = -2idxdy \text{ for } \xi = x + iy. \]
Using this completeness relation, we have the following.

\[ I(z, w_1, \cdots, w_{n-1}) = \prod_{k=1}^{\infty} \exp \left( \frac{1}{q^{2nk} - 1} \frac{[2k]^2}{[k][nk]} \right) \]
\[ \times \left\{ \sum_{l=1}^{n-1} [l][n-l]k \left( -q^{2nk} x_1(k)y_l(k) + \frac{q^{-2k}}{2} x_1(k)^2 + \frac{q^{2(n+1)k}}{2} y_l(k)^2 \right) \right. \]
\[ + \sum_{1 \leq l_1 < l_2 \leq n-1} [l_1][n-l_2]k \left( -q^{2nk} x_{l_1}y_{l_2} - q^{2nk} x_{l_2}y_{l_1} + q^{-2k} x_{l_1} x_{l_2} + q^{2(n+1)k} y_{l_1} y_{l_2} \right) \]
\[ + \sum_{l=1}^{n-1} [l][n-l]k \left( \bar{\beta}^{(i)}_{l_1}(k)(q^{-2k} x_{l_1}(k) - y_{l_1}(k)) + \bar{\delta}^{(i)}_{l_1}(k)(q^{2(n+1)k} y_{l_1}(k) - x_{l_1}(k)) \right) \]
\[ + \sum_{1 \leq l_1 < l_2 \leq n-1} [l_1][n-l_2]k \left( \bar{\delta}^{(i)}_{l_1}(k)(q^{2(n+1)k} y_{l_1}(k) - x_{l_1}(k)) + \bar{\ barred}^{(i)}_{l_2}(k)(q^{2(n+1)k} y_{l_2}(k) - x_{l_2}(k)) \right) \} \]

and

\[ B^{(i)} B \]
\[ = \prod_{k=1}^{n-1} \left( \frac{1}{\sqrt{1 - q^{2nk}}} \right)^{n-1} \prod_{k=1}^{\infty} \exp \left( \frac{1}{q^{2nk} - 1} \frac{[2k]^2}{[k][nk]} \right) \]
\[ \left\{ \sum_{l=1}^{n-1} [l][n-l]k \left( -\tilde{\beta}^{(i)}_{l_1}(k)\tilde{\delta}^{(i)}_{l_1}(k) + \frac{q^{-2k}}{2} \tilde{\beta}^{(i)}_{l_1}(k)^2 + \frac{q^{2(n+1)k}}{2} \tilde{\delta}^{(i)}_{l_1}(k)^2 \right) \right. \]
\[ + \sum_{1 \leq l_1 < l_2 \leq n-1} [l_1][n-l_2]k \left( -\tilde{\beta}^{(i)}_{l_1}\tilde{\delta}^{(i)}_{l_1} - \tilde{\delta}^{(i)}_{l_1}\tilde{\beta}^{(i)}_{l_1} + q^{-2k} \tilde{\beta}^{(i)}_{l_1}\tilde{\delta}^{(i)}_{l_1} + q^{2(n+1)k} \tilde{\delta}^{(i)}_{l_1}\tilde{\delta}^{(i)}_{l_1} \right) \} \}

Here we have used
\[ \tilde{\beta}^{(i)}_{l_1}(k) = \sum_{s=1}^{n-1} \frac{[a_s][a_s]k}{[2k]} \beta^{(i)}_{s}(k), \quad \tilde{\delta}^{(i)}_{l_1}(k) = \sum_{s=1}^{n-1} \frac{[a_s][a_s]k}{[2k]} \delta^{(i)}_{s}(k). \]
The sum in the right-hand sides are evaluated as follows.

The Norm of the vacuum vectors:

1. \( \Lambda_i \), \( 0 \leq i \leq n - 1 \), \( 0 \leq L = M = i \leq n - 1 \) case.
   \[
   B (i| i) B = \frac{1}{\sqrt{(q^4 n; q^4 n)^{\infty}}} \prod_{j=1}^{n-1} \left\{ \frac{\sqrt{(q^{4n+2-2j}; q^{4n})_{\infty}(q^{4n-2-2j}; q^{4n})_{\infty}}}{(q^{4n-2j}; q^{4n})_{\infty}} \right\}^{j(n-j)}. \tag{38}
   \]

2. \( \Lambda_i \), \( 0 \leq i \leq n - 2 \), \( 0 \leq L = i < M \leq n - 1 \) case.
   \[
   B (i| i) B = \frac{1}{\sqrt{(q^4 n; q^4 n)^{\infty}}} \prod_{j=1}^{n-1} \left\{ \frac{\sqrt{(q^{4n+2-2j}; q^{4n})_{\infty}(q^{4n-2-2j}; q^{4n})_{\infty}}}{(q^{4n-2j}; q^{4n})_{\infty}} \right\}^{j(n-j)} \\
   \times \prod_{s=1}^{M-L} \frac{(q^{4n-2L+2s-2}; q^{4n})_{\infty}}{(q^{4n-2s+2}; q^{4n})_{\infty}}, \quad (0 \leq L = i < M \leq n - 1), \tag{39}
   \]

3. \( \Lambda_i \), \( 1 \leq i \leq n - 1 \), \( 0 \leq L < M = i \leq n - 1 \) case.
   \[
   B (i| i) B = \frac{1}{\sqrt{(q^4 n; q^4 n)^{\infty}}} \prod_{j=1}^{n-1} \left\{ \frac{\sqrt{(q^{4n+2-2j}; q^{4n})_{\infty}(q^{4n-2-2j}; q^{4n})_{\infty}}}{(q^{4n-2j}; q^{4n})_{\infty}} \right\}^{j(n-j)} \\
   \times \prod_{s=1}^{M-L} \frac{(q^{4n+2M-2L-2s+2}; q^{4n})_{\infty}}{(q^{4M-4L-2s+2}; q^{4n})_{\infty}}, \quad (0 \leq L < M = i \leq n - 1). \tag{40}
   \]

Integrand of Correlation Functions:

1. \( \Lambda_i \), \( 0 \leq i \leq n - 1 \), \( 0 \leq L = M = i \leq n - 1 \) case.
   \[
   I(z, w_1, \cdot \cdot \cdot, w_{n-1}) = J(z, w_1, \cdot \cdot \cdot, w_{n-1}), \quad (0 \leq L = M = i \leq n - 1). \tag{41}
   \]

2. \( \Lambda_i \), \( 0 \leq i \leq n - 2 \), \( 0 \leq L = i < M \leq n - 1 \) case.
   \[
   I(z, w_1, \cdot \cdot \cdot, w_{n-1}) = J(z, w_1, \cdot \cdot \cdot, w_{n-1}) \times \frac{(1 - rz)}{(1 - rq^{-M+1} w_M)}, \quad (i = L = 0 < M \leq n - 1), \tag{42}
   \]
   and

   \[
   I(z, w_1, \cdot \cdot \cdot, w_{n-1}) = J(z, w_1, \cdot \cdot \cdot, w_{n-1}) \tag{43}
   \]
   \[
   \times \frac{q^{L-1} w_L}{(q^{2n+L-1} w_L)_{\infty} (q^{2n+L-1} w_L)_{\infty}} \frac{(q^{2n+L-1} w_M)_{\infty}}{(q^{2n-M+1} w_M)_{\infty}} \frac{(q^{2n+L-1} w_M)_{\infty}}{(q^{2n-M+1} w_M)_{\infty}} \times \frac{q^{2n+2L-2} w_{L^2}}{(q^{2n+2L-2} w_{L^2})_{\infty}} \times \frac{q^{2n+2L-2} w_{L^2}}{(q^{2n+2L-2} w_{L^2})_{\infty}}, \quad (1 \leq L = i < M \leq n - 1). \]
(3) $A_i \ (1 \leq i \leq n - 1), \ 0 \leq L < M = i \leq n - 1.$

$$I(z, w_1, \cdots, w_{n-1}) = J(z, w_1, \cdots, w_{n-1}) \times \frac{(1 - 1/(rz))}{(1 - q^{M+1}/(rz))}, \ (0 = L < M = i \leq n - 1),$$

and

$$J(z, w_1, \cdots, w_{n-1}) = J(z, w_1, \cdots, w_{n-1})$$

$$\times \frac{(q^{2M-L-1} - w_1)_{\infty}(q^{2M-L+1} - w_1)_{\infty}(q^{M-2L-1} - w_1)_{\infty}(q^{2n-2L+M+1} - w_1)_{\infty}}{(q^{L-1} - w_1)_{\infty}(q^{2n-1} - w_1)_{\infty}(q^{M-1} - w_1)_{\infty}(q^{M+1} - w_1)_{\infty}}$$

$$\times \frac{\prod_{j=1}^{n} (w_j; q^n)_{\infty}(q^{n+1} - w_j)_{\infty}}{(z; q^n)_{\infty}(q^n; q^n)_{\infty}}$$

(33) and (34), (1), (41), (42), (43), (44), (45), and (46).

Here we have set the function $J(z, w_1, \cdots, w_{n-1})$ by

$$J(z, w_1, \cdots, w_{n-1}) = (q^{2n})_{\infty}(q^{2n+2})_{\infty}(q^{n+1}; q^n)_{\infty}(q^n z; q^n)_{\infty}(q^{n+1}; q^n)_{\infty}$$

$$\times \sqrt{(q^{2n} z^2)_{\infty}(q^{2n} z^2)_{\infty}(q^{2n} z^2)_{\infty}}$$

$$\times \prod_{j=1}^{n-1} \frac{(q^{n+1} - w_j; q^n)_{\infty}(q^{n+1} - w_j; q^n)_{\infty}}{(w_j; q^n)_{\infty}(q^{n+2} w_j^{-1}; q^n)_{\infty}}$$

$$\times \prod_{j=1}^{n-2} \frac{q^n z w_{j+1} w_{j+2}^{-1}}{(q^n z w_{j+1}^{-1})_{\infty}(q^{n+1} w_{j+1}^{-1})_{\infty}(q^{n+1} w_{j+1}^{-1})_{\infty}}$$

Here we have used the abbreviation

$$z_{\infty} = (z; q^{2n})_{\infty}.$$

We summarize the main result of this section. For the asymptotic boundary conditions $V(A_i), \ (i = 0, \cdots, n - 1),$ let us consider the following boundary conditions,

(1) $0 \leq L = M = i \leq n - 1, \ (2) 0 \leq L = i < M \leq n - 1, \ (3) 0 \leq L < M = i \leq n - 1.$

Then the boundary magnetization is given by

$$\sum_{j=0}^{n-1} \omega^j P_j^{(i)}(1),$$

where the correlation function $P_j^{(i)}(z)$ is given in (14) and (14), the integrand function of the correlation function $P_j^{(i)}(z)$ is given in (11), (12), (13), (14), (15), and (16), and $\omega$ is an $n$-th primitive root of 1.
Acknowledgements  We want to thank to Professor M. Jimbo, Professor T. Miwa and Professor R. Nepomechie for their interests to this work. We want to thank to S. Yamasita for useful discussion, to improve our manuscript. This work is partly supported by the Grant from Research Institute of Science and Technology, Nihon University, and the Grant from the Ministry of Education, Science, Sports, and Culture, Japan (11740099).

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5 Appendix

For reader’s convenience, we summarize the bosonizations of the vertex operators [6]. Let \(C[\hat{P}]\) be the \(C\)-algebra generated by symbols \(\{e^{\alpha_2}, \cdots, e^{\alpha_{n-1}}, e^{\bar{\alpha}_{n-1}}\}\)
which satisfy the following defining relations:

\[ e^{\alpha_i} e^{\alpha_j} = (-1)^{[\alpha_i,\alpha_j]} e^{\alpha_j} e^{\alpha_i}, \quad (2 \leq i, j \leq n-1), \]

\[ e^{\alpha_i} e^{\bar{n}_{i-1}} = (-1)\delta_{i,n-1} e^{\bar{n}_{i-1}} e^{\alpha_i}, \quad (2 \leq i \leq n-1). \]

For \( \alpha = m_2\alpha_2 + \cdots + m_{n-1}\alpha_{n-1} + m_n\bar{n}_{n-1} \), we denote \( e^{m_2\alpha_2} \cdots e^{m_{n-1}\alpha_{n-1}} e^{m_n\bar{n}_{n-1}} \) by \( e^\alpha \). Let \( \mathbb{C}[Q] \) be the \( \mathbb{C} \)-algebra generated by the symbols \( \{e^{\alpha_1}, \ldots, e^{\alpha_{n-1}}\} \) which satisfy the following defining relations:

\[ e^{\alpha_i} e^{\alpha_j} = (-1)^{[\alpha_i,\alpha_j]} e^{\alpha_j} e^{\alpha_i}, \quad (1 \leq i, j \leq n-1). \]

Let the boson be the \( \mathbb{C} \)-algebra generated by the symbols \( a_s(k), \ (s \in \{0, 1, \ldots, n-1\}, k \in \mathbb{Z}) \) which satisfy the following defining relations:

\[ [a_s(k), a_t(l)] = \delta_{s,t} \frac{[(a_s [a_t] k)] [k]}{k}. \]

The highest weight module \( V(\Lambda_i) \) is realized as

\[ V(\Lambda_i) = \mathbb{C}[a_s(-k), \ (s \in \{0, 1, \ldots, n-1\}, k \in \mathbb{Z} \geq 0)] \otimes \mathbb{C}[Q] e^{\bar{n}_{i-1}}. \]

Here the actions of the operators \( a_s(k), \partial_s, e^\alpha \) on \( V(\Lambda_i) \) are defined as follows:

\[ a_s(k) f \otimes e^\beta = \begin{cases} a_s(k) f \otimes e^\beta, \quad (k < 0), \\ [a_s(k), f] \otimes e^\beta, \quad (k > 0). \end{cases} \]

\[ \partial_s f \otimes e^\beta = (\alpha | \beta) f \otimes e^\beta. \]

\[ e^\alpha f \otimes e^\beta = f \otimes e^\alpha e^\beta. \]

The bosonizations of the vertex operators are given by

\[ \Phi_{n-1}^{(i+1,1)}(z) = e^{P(z)} e^Q(z) e^{\bar{n}_{n-1}} (e^{n+1}z) \delta_{n,1} + \frac{n-i-1}{n} (-1)(\delta_{n,1} - \frac{n-i-1}{(n-1)+\frac{1}{2}((n-i)(n-i-1)}, \]

\[ \Phi_{j-1}^{(i+1,1)}(z) = [\phi_{j-2}^{(i+1,1)}(z), f_j] = \oint \frac{dw_j}{2\pi i} [\phi_{j-2}^{(i+1,1)}(z), e^{R_j(w_j)} e^{S_j(w_j)} e^{-\alpha_j} w_j^{-\partial_{\alpha_j}}] q, \]

\[ \Phi_0^{(i+1,1)}(z) = e^{P^*(z)Q^*(z)} e^{\bar{n}_1} ((-1)^{n-1}q) \delta_{n,1} + \frac{1}{2} q^{i} (-1)^{n+\frac{1}{2}i+1}, \]

\[ \Phi_j^{(i+1,1)}(z) = [f_j, \phi_{j-1}^{(i+1,1)}(z)]_{q=1} = \oint \frac{dw_j}{2\pi i} [\phi_{j-1}^{(i+1,1)}(z), e^{-R_j(q) \alpha_j} e^{-S_j(q) w_j} w_j^{-\partial_{\alpha_j}}] q, \]

\[ \Psi_{n-1}^{(i+1,1)}(z) = e^{-P^*(z)Q(z)} e^{-\bar{n}_{n-1}} e^{(q+1)z} - \delta_{n,1} + \frac{1}{2} (-1)^{n+\frac{1}{2}i+1}, \]

\[ \Psi_j^{(i+1,1)}(z) = [e_j, \phi_{j-1}^{(i+1,1)}(z)]_{q=1} = \oint \frac{dw_j}{2\pi i} [\phi_{j-1}^{(i+1,1)}(z), e^{-R_j(q) \alpha_j} e^{-S_j(q) w_j} w_j^{-\partial_{\alpha_j}}] q, \]

\( i = 0, \ldots, n-1, \quad j = 1, \ldots, n-1, \)
where we have used

\[
\begin{align*}
P(z) &= \sum_{k=1}^{\infty} a_{n-1}^*(-k)q^{\frac{2n+3}{2}k}z^k, & Q(z) &= \sum_{k=1}^{\infty} a_{n-1}^*(k)q^{-\frac{2n+1}{2}k}z^{-k}, \\
P^*(z) &= \sum_{k=1}^{\infty} a_1^*(-k)q^{\frac{1}{2}k}z^k, & Q^*(z) &= \sum_{k=1}^{\infty} a_1^*(k)q^{-\frac{1}{2}k}z^{-k}, \\
R_j^-(w) &= -\sum_{k=1}^{\infty} \frac{a_j(-k)}{[k]}q^{\frac{1}{2}w}k^w, & S_j^-(w) &= \sum_{k=1}^{\infty} \frac{a_j(k)}{[k]}q^{\frac{1}{2}w}k^{-w}, \\
a_{n-1}^*(k) &= \sum_{l=1}^{n-1} \frac{-[lk]}{[k][nk]}a_l(k), & a_1^*(k) &= \sum_{l=1}^{n-1} \frac{-(a_l-1)k}{[k][nk]}a_l(k).
\end{align*}
\]