HOMOLOGICAL REALIZATION OF RESTRICTED KOSTKA POLYNOMIALS

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ABSTRACT. In this paper we give two realizations of restricted Kostka polynomials for $\mathfrak{sl}_2$. Firstly we identify restricted Kostka polynomials with characters of the zero homology of the current algebra with coefficients in the certain modules. As a corollary we reobtain the alternating sum formula. Secondly we show that restricted Kostka polynomials are $q$-multic太平ities of the decomposition of the certain integrable $\hat{\mathfrak{sl}}_2$-modules to the irreducible components. This allows to write a kind of fermionic formula for the Virasoro unitary characters.

INTRODUCTION

For $\mathbf{m} \in \mathbb{Z}_{\geq 0}^k$ let $K_{l, \mathbf{m}}(q)$ be a Kostka polynomial for $\mathfrak{sl}_2$ and $K_{l, \mathbf{m}}^{(k)}(q)$ be a level $k$ restricted Kostka polynomial (we follow the notations in [FJKLM]; see also [SS]). Let $V_{\mathbf{m}}$ be the fusion product,

$$V_{\mathbf{m}} = \pi_1 \otimes \cdots \otimes \pi_l \otimes \cdots \otimes \pi_k \otimes \cdots \otimes \pi_k,$$

where $\pi_i$ is irreducible $(i+1)$-dimensional representation of $\mathfrak{sl}_2$ (see [FL, FF1, CP]). Recall that $K_{l, \mathbf{m}}(q)$ is the $q$-multiplicity of $\pi_l$ in $V_{\mathbf{m}}$ (see [FJKLM]). Therefore, $K_{l, \mathbf{m}}(q)$ is equal to the character of the $\mathfrak{sl}_2$-invariants in $V_{\mathbf{m}} \otimes \pi_l$. We consider the induced module $\text{Ind}_{\mathfrak{sl}_2}^{\mathfrak{sl}_2 \otimes \mathbb{C}[t]} \pi_l$. Note that $V_{\mathbf{m}} \otimes \text{Ind}_{\mathfrak{sl}_2}^{\mathfrak{sl}_2 \otimes \mathbb{C}[t]} \pi_l$ is a free $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-module, and therefore, the relative homology

$$H_0(\mathfrak{sl}_2 \otimes \mathbb{C}[t], \mathfrak{sl}_2; V_{\mathbf{m}} \otimes \text{Ind}_{\mathfrak{sl}_2}^{\mathfrak{sl}_2 \otimes \mathbb{C}[t]} \pi_l)$$

is isomorphic to the $\mathfrak{sl}_2$-invariants in the tensor product $V_{\mathbf{m}} \otimes \pi_l$. We obtain

$$K_{l, \mathbf{m}}(q) = \text{ch}_q H_0(\mathfrak{sl}_2 \otimes \mathbb{C}[t], \mathfrak{sl}_2; V_{\mathbf{m}} \otimes \text{Ind}_{\mathfrak{sl}_2}^{\mathfrak{sl}_2 \otimes \mathbb{C}[t]} \pi_l).$$

We now replace the induced module in (1) by its quotient, which is isomorphic to some irreducible integrable representation of $\hat{\mathfrak{sl}}_2$. Namely, let $L_{l, k}, 0 \leq l \leq k$ be the set of irreducible level $k$ integrable $\hat{\mathfrak{sl}}_2$-modules. We show that

$$K_{l, \mathbf{m}}^{(k)}(q) = \text{ch}_q H_0(\mathfrak{sl}_2 \otimes \mathbb{C}[t], \mathfrak{sl}_2; V_{\mathbf{m}} \otimes L_{l, k}^*),$$

and the higher homology vanish:

$$H_p(\mathfrak{sl}_2 \otimes \mathbb{C}[t], \mathfrak{sl}_2; V_{\mathbf{m}} \otimes L_{l, k}^*) = 0, \quad p > 0.$$
Formulas (2), (3) and the BGG resolution allows us to reobtain the alternating sum formula
\[ K_{l,m}^{(k)}(q) = \sum_{i \geq 0} q^{(k+2)i^2+(l+1)i} K_{2(k+2)i+l,m}(q) - \sum_{i \geq 0} q^{(k+2)i^2-(l+1)i} K_{2(k+2)i-l-2,m}(q). \]
The left-hand side is equal to \( \sum_{p \geq 0} (-1)^p \text{ch}_p H_p(\mathfrak{sl}_2 \otimes \mathbb{C}[t], \mathfrak{sl}_2; V_m \otimes L_{l,k}) \), while the right-hand side coincides with the Euler characteristics of the certain complex, counting \( H_*(\mathfrak{sl}_2 \otimes \mathbb{C}[t], \mathfrak{sl}_2; V_m \otimes L_{l,k}) \).

We now describe the realization of restricted Kostka polynomials as \( q \)-multiplicities. Let \( L_{m,k} \) be an integrable module induced from the fusion product \( V_m \):

\[ L_{m,k} = \left( \text{Ind}_{\mathfrak{sl}_2 \otimes \mathbb{C}[t]}^{\mathfrak{sl}_2} V_m \right) / (K - k, e(z)^{k+1}), \]

where \( K \) is the central element. We show that this module coincides with the inductive limit of a fusion products (see [FP2]). Recall that there exists an embedding \( V_m \rightarrow V_{(m_1, \ldots, m_{l-1}, m_{k+2})} \). We prove that \( L_{m,k} = \lim_{m \rightarrow \infty} V_{(m_1, \ldots, m_{k+2})} \). Consider the decomposition of \( L_{m,k} \) into the direct sum of irreducible modules

\[ L_{m,k} = \bigoplus_{l=0}^{k} N_{l,m} \otimes L_{l,k} \]

\( (N_{l,m} \) is spanned by highest weight vectors of the weight \( l \)). It was proved in [FP2] that the following equality is true in the level \( k \) Verlinde algebra \( V^{(k)}(1) \)

\[ [1]^{m_1}[2]^{m_2} \cdots [k]^{m_k} = [0] \dim N_{0,m} + \cdots + [k] \dim N_{k,m}, \]

where \( [l] \) corresponds to the \((l+1)\)-dimensional representation of \( \mathfrak{sl}_2 \). We show that the character of \( N_{l,m} \) coincides with a reversed Kostka polynomial \( \hat{K}_{l,m}^{(k)}(q) = q^{h(m)} K_{l,m}^{(k)}(q) / q^{-1} \) for the certain function \( h : \mathbb{Z}_k^* \rightarrow \mathbb{Z}_{\geq 0} \). This agrees with the fact that \( K_{l,m}^{(k)}(1) \) are the structure constants of \( V^{(k)} \) (see [HKKOTY, FJKLM]).

We now apply the decomposition of \( L_{m,k} \) to the coset construction (see [GKO]) to obtain a finitization of the characters of the minimal unitary Virasoro models. Our finitization is expressed in terms of Kostka polynomials (see also [ABF, B, S]). We give some details below. (For the connection of the branching functions and Kostka polynomials in a more general settings see [SS]. See also [K] for the \( \mathfrak{sl}_N \) case and \( W_N \) instead of the Virasoro algebra).

The decomposition of \( L_{m,k} \) can be applied to the study of the coset constructions

\[ (\widehat{\mathfrak{sl}_2})_{k_1} \otimes \cdots \otimes (\widehat{\mathfrak{sl}_2})_{k_n} / (\widehat{\mathfrak{sl}_2})_{k_1+\cdots+k_n}. \]

For example, we can get fermionic and bosonic formulas for the corresponding conformal theories. In this paper we are dealing with the simplest case \( n = 2 \) and \( k_1 = 1 \).

Consider the decomposition of the tensor product \( L_{i,1} \otimes L_{j,k} = \bigoplus_{l=0}^{k+1} N_l \otimes L_{l,k+1} \). Each \( N_l \) is a representation of the Virasoro algebra. Namely, each \( N_l \) is isomorphic to the certain minimal model \( M_{r,s} (k+2, k+3) \) (see [KW]). We prove that \( L_{i,1} \otimes L_{j,k} = \lim_{N \rightarrow \infty} L_{(1)^{N(j+1)}, k+1} \), where \( L_{(1)^{m_1\cdots m_k}, k} = L_{m,k} \). This gives a finitization of the characters of \( N_l \) in terms of Kostka polynomials. We also show that the Rocha-Caridi formula for the characters of the minimal models (see [RC]) is a corollary from the alternating sum formula. There also exists the fermionic formula for restricted Kostka polynomials (see [SS, FJKLM]). The limit of this
The fermionic formula for Kostka polynomials naturally appears as $q$-multiplicity in the decomposition of $L_{m,k}$. Namely, in [FF2] the defining relations in principal subspaces were described. This gives the fermionic formula for $L_{m,k}$.

We use a certain space of coinvariants to find the highest weight vectors of the weight $l$ in the decomposition (5). This leads to the fermionic formula for the character of $N_{l,m}$.

We finish the introduction with a discussion of possible generalizations. Let $\hat{g}$ be an affine Kac-Moody algebra. Fix $\lambda = (\lambda_1, \ldots, \lambda_k)$ to be a vector of the highest weights of the irreducible representations $\pi_{\lambda_i}$ of $\hat{g}$. Let $V_{\lambda}$ be the corresponding fusion product $\pi_{\lambda_1} \ast \cdots \ast \pi_{\lambda_k}$ (the adjoint graded space of the tensor product of the evaluation representations, which is conjecturally independent of the evaluation parameters, see [FL, FF1, K, CL]). Let $L_\mu$ be a level $k$ irreducible integrable highest weight representation of $\hat{g}$ with highest weight $\mu$. Define

$$K^{(k)}_{\mu,\lambda}(q) = H_0(\hat{g} \otimes \mathbb{C}[t], \hat{g}; V_{\lambda} \otimes L_\mu).$$

(As above, $\hat{g} \otimes \mathbb{C}[t]$ are the generating operators in $L_\mu$). We conjecture that this definition of the restricted Kostka polynomials coincides with one in [FKLM], as far as in [SW].

Another possibility is to use the induced module as in (4). Namely, restricted Kostka polynomials can be defined as the $q$-multiplicities of the irreducible components in the decomposition of

$$\left( \text{Ind}_{\hat{g}}^g \mathbb{C}[t] \otimes V_{\lambda} \right) / \langle K - k, I \rangle.$$

Here $I \hookrightarrow U(\hat{g})$ is the ideal which vanishes in any highest weight integrable level $k$ $\hat{g}$-module.

Our paper is organized as follows:

In Section 1, we settle our notations and collect the main properties of the fusion products.

In Section 2, we give a homological realization of restricted Kostka polynomials and prove the vanishing theorem for the higher homology.

In Section 3, we derive the alternating sum formula from the BGG resolution.

Section 4 is devoted to the decomposition of $L_{m,k}$ into the direct sum of irreducible modules.

In Section 5, we obtain the finitization of the Virasoro unitary characters in terms of Kostka polynomials and describe the coinvariants approach to the fermionic formula.

In Appendix, the connection between the ABF and Kostka polynomials finitizations of the unitary characters is studied.

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1. Preliminaries

1.1. Fusion products. In this section, we fix our notations and recall the main results in [FL, FF1, FF2].

Let $e, h, f$ be the standard basis of $\mathfrak{sl}_2$, and let $\tilde{\mathfrak{sl}}_2$ be the affine Kac-Moody algebra,

$$\tilde{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where $K$ is a central element, and $[d, x_i] = -ix_i$. We set $x_i = x \otimes t^i$ for $x \in \mathfrak{sl}_2,
\ i \in \mathbb{Z}$.

Let $\pi_i$ be $(l+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$. We fix some $n$-tuple $(z_1, \ldots, z_n)$ of pairwise distinct complex numbers and consider the tensor product

$$\pi_{a_1}(z_1) \otimes \cdots \otimes \pi_{a_n}(z_n)$$

of the evaluation representations of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$. Then the fusion product

$$\pi_{a_1} \ast \cdots \ast \pi_{a_n}$$

is an adjoint graded module with respect to the filtration

$$F_s = \text{span}(x_1^{(1)} \cdots x_i^{(t)}v_A, \ x^{(j)} \in \mathfrak{sl}_2, i_1 + \cdots + i_t \leq s, )$$

where $A = (a_1, \ldots, a_n)$ and $v_A$ is the tensor product of the lowest weight vectors of $\pi_{a_i}(z_i)$.

Suppose now that $a_i \leq k$ for any $i$. Then we also use a following notation. Let $m = (m_1, \ldots, m_k)$ be some $k$-tuple of non-negative integers. Set

$$V_m = \bigotimes_{i_1 \leq m_1, \ldots, i_k \leq m_k} \pi_{i_1} \ast \cdots \ast \pi_{i_k}.$$ 

We recall that $V_m$ is cyclic with respect to the algebra $\mathbb{C}[e_0, e_1, \ldots]$. Denote the corresponding cyclic vector by $v_m$ (note that this vector coincides with $v_A$). We use a notation $u_m$ for a cyclic vector with respect to the algebra $\mathbb{C}[f_0, f_1, \ldots]$. Note that $v_m$ ($u_m$) is the vector of the minimal (maximal) $h_0$-eigenvalue. We recall that $V_m$ is bigraded, namely

$$V_m^\alpha = \{ v \in V_m : h_0v = \alpha v \}, \quad V_m^{\alpha, s} = V_m^\alpha \cap \text{span} \{ e_{i_1} \cdots e_{i_p} v_m : i_1 + \cdots + i_p = s \}. $$

For $v \in V_m^{\alpha, s}$ set $\deg_x v = \alpha$, $\deg_y v = s$, and define the character of a homogeneous subspace $V \hookrightarrow V_m$ by

$$\text{ch} V = \sum_{\alpha, s} \dim(V \cap V_m^{\alpha, s}) z^\alpha q^s, \quad \text{ch}_q V = \sum_{\alpha, s} \dim(V \cap V_m^{\alpha, s}) q^s.$$ 

We now recall some exact sequences of fusion products.

Let $1 \leq a_1 \leq \cdots \leq a_n$. Then there exist an exact sequences of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-modules:

$$0 \to \pi_{a_2-a_1} \ast \cdots \ast \pi_{a_n} \to \pi_{a_1} \ast \pi_{a_2} \ast \cdots \ast \pi_{a_n} \to \pi_{a_1-1} \ast \pi_{a_2+1} \ast \cdots \ast \pi_{a_n} \to 0$$

and

$$0 \to \pi_{a_1} \ast \cdots \ast \pi_{a_{n-2}} \otimes \pi_{a_n-a_{n-1}} \to \pi_{a_1} \ast \cdots \ast \pi_{a_n} \to \pi_{a_1} \ast \cdots \ast \pi_{a_{n-2}} \ast \pi_{a_{n-1}-1} \ast \pi_{a_{n}+1} \to 0.$$
Now suppose that $a_i = a_{i+1}$. Then we also have an exact sequence of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-modules

\begin{equation}
0 \rightarrow \pi_{a_1} \otimes \cdots \otimes \pi_{a_{i-1}} \otimes \pi_{a_{i+2}} \otimes \cdots \otimes \pi_{a_n} \rightarrow \pi_{a_1} \otimes \cdots \otimes \pi_{a_n} \rightarrow \\
\rightarrow \pi_{a_1} \otimes \cdots \otimes \pi_{a_{i-1}} \otimes \pi_{a_{i+1}} \otimes \pi_{a_{i+2}} \otimes \cdots \otimes \pi_{a_n} \rightarrow 0.
\end{equation}

We note that each of (10), (11), and (12) contains the piece

\[ \pi_{a_1} \otimes \cdots \otimes \pi_{a_n} \rightarrow \pi_{a_1} \otimes \cdots \otimes \pi_{a_{i-1}} \otimes \pi_{a_{i+1}} \otimes \pi_{a_{i+2}} \otimes \cdots \otimes \pi_{a_n} \]

in some special case. For the general case see [FP3].

We now recall some facts about a subspace

\[ \mathbb{C}[e_1, e_2, \ldots] \cdot v_A \hookrightarrow \pi_{a_1} \otimes \cdots \otimes \pi_{a_n}. \]

Note that $\mathbb{C}[e_1, e_2, \ldots] \cdot v_A$ is invariant with respect to the subalgebra $\mathfrak{a}_1 \hookrightarrow \mathfrak{sl}_2 \otimes \mathbb{C}[t]$ generated by $e_1$ and $f_0$. Let $\mathfrak{a}_2 \hookrightarrow \mathfrak{sl}_2 \otimes \mathbb{C}[t]$ be the subalgebra generated by $e_0$ and $f_1$. Fix an isomorphism $\iota : \mathfrak{a}_1 \rightarrow \mathfrak{a}_2$ sending $e_1$ to $e_0$ and $f_0$ to $f_1$. Then we have an isomorphism of $\mathfrak{a}_2$-modules

\begin{equation}
\mathbb{C}[e_1, e_2, \ldots] \cdot v_A \cong \pi_{a_1} \otimes \cdots \otimes \pi_{a_{n-1}},
\end{equation}

where the action of $\mathfrak{a}_2$ on the left-hand side is a composition of $\iota^{-1}$ and the natural action of $\mathfrak{a}_1$. In addition we have an exact sequence of $\mathfrak{a}_1$-modules

\begin{equation}
0 \rightarrow \mathbb{C}[e_1, e_2, \ldots] \cdot v_A \rightarrow \pi_{a_1} \otimes \cdots \otimes \pi_{a_n} \rightarrow \pi_{a_1} \otimes \cdots \otimes \pi_{a_n-1} \rightarrow 0.
\end{equation}

We describe an inductive limits of fusion products. Using (12) we obtain a sequence of embeddings

\begin{equation}
V(m_1, \ldots, m_k) \hookrightarrow V(m_1, \ldots, m_k+2) \hookrightarrow V(m_1, \ldots, m_k+4) \hookrightarrow \cdots
\end{equation}

We denote the inductive limit of (15) by $L_{m,k}$. This space can be endowed with the structure of a level $k$ integrable $\mathfrak{sl}_2$-module (the action of the affine algebra is compatible with the natural action of the annihilation operators $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$). We consider the decomposition

\begin{equation}
L_{m,k} = L_{0,k} \otimes N_{0,m} \oplus \cdots \oplus L_{k,k} \otimes N_{k,m},
\end{equation}

where $L_{i,k}$, $0 \leq i \leq k$ are level $k$ irreducible highest weight representations of $\widehat{\mathfrak{sl}}_2$ with highest weight vectors $v_i,k$: $h_0 v_i,k = iv_i,k$, $K v_i,k = kv_i,k$, $d v_i,k = 0$. Then the dimensions of $N_{i,m}$ are given in terms of the level $k$ Verlinde algebra $V^{(k)}$ for $\mathfrak{sl}_2$. Let $[0], [1], \ldots, [k]$ be a basis of $V^{(k)} ([t])$ corresponds to the $(l+1)$-dimensional representation of $\mathfrak{sl}_2$. Introduce the notation

\begin{equation}
[m_1 \ldots [2]^{m_2} \ldots [k]^{m_k} = [0]c_{0,m} + \cdots + [k]c_{k,m}.
\end{equation}

The following theorem is proved in [FP2].

**Theorem 1.1.** We have an equality

\begin{equation}
\dim N_{i,m} = c_{i,m}.
\end{equation}

We finish this subsection with a remark on our characters notations. For any homogeneous $V \hookrightarrow L_{m,k}$ set

\[ \text{ch} V(z, q) = \sum_{\alpha, s} z^\alpha q^s \dim \{ v \in V : h_0 v = \alpha v, dv = sv \}, \quad \text{ch}_q V = \text{ch} V(1, q). \]
We recall that the character \(\tilde{\text{ch}}_q V_m\) is given by (18). We also need the "reversed" character, coming from the embedding \(j: V_m \hookrightarrow L_{m,k}\). Namely, set

\[
\tilde{\text{ch}}_q V_m(q) = \text{ch}_q(j \cdot V_m).
\]

Obviously, \(\tilde{\text{ch}}_q V_m(q) = q^{h(m)} \text{ch}_q V_m(q^{-1})\) for some \(h: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}\) (see Lemma 5.5 for the computation of \(h(m)\)).

### 1.2. The Weyl group and Kostka polynomials

We first settle our notations concerning \(\mathfrak{sl}_2\) (see [Kac]). Let \(h = \text{span}\{h_0, K, d\} \hookrightarrow \mathfrak{sl}_2\) be the Cartan subalgebra, \(\mathfrak{n}\) be the nilpotent subalgebra, \(n = \mathfrak{sl}_2 \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathfrak{f}_0\), and \(u = \mathfrak{sl}_2 \otimes t^{-1}\mathbb{C}[t^{-1}]\).

Let \(s_0, s_1 \in W\) be simple reflections, where \(W\) is the Weyl group of \(\mathfrak{sl}_2\). Let \(\rho \in \mathfrak{h}^*\) be the element, defined by \(\rho(\alpha_i') = 1\), \(i = 0, 1\), where \(\alpha_0'\) and \(\alpha_1'\) are the simple coroots. We set \(w \ast \alpha = w(\alpha + \rho) - \rho\) for the shifted action of the Weyl group on \(\mathfrak{h}^*\).

Define \((i, k, m) \in \mathfrak{h}^*\) by \((i, k, m) h_0 = i\), \((i, k, m) K = k\), and \((i, k, m) d = m\). Then \(\rho = (1, 2, 0)\) and

\[
(s_0 s_1)^n(i, k, m) = (-i - 2k + 2, k, m + k - i + 1)
\]

The following lemma gives the shifted action of an arbitrary element of \(W\) on \(\mathfrak{h}^*\).

**Lemma 1.1.**

a. \((s_0 s_1)^n(i, k, m) = (-i - 2(n + 1)(k + 2), k, m + (n + 1)(k + 2) - (n + 1)(i + 1))\).

b. \((s_0 s_1)^n(i, k, m) = (i + 2n(k + 2), k, m + n^2(k + 2) + n(i + 1))\).

c. \((s_0 s_1)^n(i, k, m) = (-i - 2 - 2n(k + 2), k, m + n^2(k + 2) + n(i + 1))\).

d. \((s_1 s_0)^n(i, k, m) = (i - 2n(k + 2), k, m + n^2(k + 2) - n(i + 1))\).

In our paper we use notations for Kostka polynomials as in [FJKLM]. Let us recall the connection between the notations in [SS] and [FJKLM]. For \(m \in (\mathbb{N} \cup 0)^k\) we set

\[
|m| = \sum_{i=1}^{k} im_i, \quad 2\|m\| = -|m| + \sum_{1 \leq i, j \leq k} \min(i, j)m_i m_j.
\]

Now let \(0 \leq l \leq k\). Denote

\[
\lambda = \left(\frac{|m| + l}{2}, \frac{|m| - l}{2}\right), \quad R(m) = (k_{m_k}, \ldots, 1_{m_1}).
\]

We have

\[
K_{i,m}(q) = q^{\|m\|} K_{R(m)}(q^{-1}), \quad K_{i,m}^{(k)}(q) = q^{\|m\|} K_{R(m)}^{k}(q^{-1}),
\]

where the right-hand side stands for Kostka and level-restricted Kostka polynomials in the notations of [SS].

### 2. Homological realization of restricted Kostka polynomials

In this section, we consider the fusion product \(V_m\) as \(\mathfrak{sl}_2 \otimes \mathbb{C}[t^{-1}]\)-module via the isomorphism

\[
\mathfrak{sl}_2 \otimes \mathbb{C}[t^{-1}] \rightarrow \mathfrak{sl}_2 \otimes \mathbb{C}[t] \quad x_i \mapsto x_{-i}.
\]

Our goal is to show that

\[
\text{ch}_q H_{\rho}(n, V_m \otimes L_{i,k})^\alpha = \delta_{0,p} K_{i,m}^{(k)}(q),
\]

where \(H_{\rho}(n, V_m \otimes L_{i,k})^\alpha\) denotes an eigenspace of the operator \(h_0\) with an eigenvalue \(\alpha\). To prove this statement in the case \(p = 0\) we use the theorem in [FJKLM]:
\textbf{Theorem 2.1.} \( \text{ch}_q V_m / \langle h_0 + l, e_0, e_{-l+1} \rangle = K_{l,m}^{(k)}(q) \).

\textbf{Lemma 2.1.} \( \text{ch}_q H_0(n, V_m \otimes L_{l,k})^0 = K_{l,m}^{(k)}(q) \).

\textbf{Proof.} In view of Theorem 2.1 it is enough to prove that
\[ \text{ch}_q H_0(n, V_m \otimes L_{l,k})^0 = \text{ch}_q V_m / \langle h_0 + l, e_0, e_{-l+1} \rangle. \]

We recall the first terms of the BGG-resolution
\[ (21) \quad 0 \leftarrow L_{l,k} \leftarrow M_{l,(k,0),0} \leftarrow M_{-l+2,0} \oplus M_{l+2+2,0} \leftarrow \cdots, \]
where \( M_\alpha \) is the Verma module with a highest weight \( \alpha \). We set \( v_\alpha \) to be a highest weight vector of \( M_\alpha \). Note that the differential \( \partial \) is given by
\[ (22) \quad \partial v_{l+2,0} = f_{l+2}^1 v_{l,0}, \quad \partial v_{l+2+2,0} = e_{-l+1} v_{l,0}. \]

We tensor (21) by \( V_m \) and obtain the free resolution of the \( n \)-module \( L_{l,k} \otimes V_m \)
\[ 0 \leftarrow L_{l,k} \otimes V_m \leftarrow M_{l,(k,0),0} \otimes V_m \leftarrow (M_{-l+2,0} \oplus M_{l+2+2,0}) \otimes V_m \leftarrow \cdots. \]

Therefore, \( H_0(n, V_m \otimes L_{l,k})^0 \) is isomorphic to the homology of the complex
\[ \left[ \mathbb{C} \otimes U(n) \left( M_{l,(k,0),0} \otimes V_m \right) \right]^0 \leftarrow \left[ \mathbb{C} \otimes U(n) \left( M_{l,(k,0),0} \otimes V_m \oplus M_{l+2+2,0} \otimes V_m \right) \right]^0, \]
where \( U(n) \) is the universal enveloping algebra. We note that \( M_{l,(k,m)} \) is a free \( U(n) \)-module with one generator with \( z \)-degree \( i \). Hence, because of (22)
\[ H_0(n, V_m \otimes L_{l,k})^0 \simeq V_m / \langle f_{l+1}^1, e_{-l+1}, h_0 + l \rangle. \]

But
\[ [V_m / \langle f_{l+1}^j \rangle]^{-l} \simeq [V_m / \langle e_0 \rangle]^{-l}. \]

This finishes the proof of the lemma.

Our next step is the proof of the statement in the case \( m_1 + \cdots + m_k = 1 \), i.e., when \( V_m \) is a single representation \( \pi_n \). For this, we first recall the homology of \( u \).

\textbf{Lemma 2.2.} \( H_p(u, L_{l,k}) \) is isomorphic to \( \pi_{p(k+2)+l} \) for even \( p \) and to \( \pi_{p(k+2)+k-l} \) for odd \( p \) as a representation of \( \mathfrak{sl}_2 \).

\textbf{Proof.} We recall (see [GL, Kumu]) that \( H_p(u, L_{l,k}) \) is irreducible \( \mathfrak{sl}_2 \)-module with the highest weight \( (w_p * (l, k, 0))(h_0) \), where \( w_p = s_0 s_1 s_0 \cdots \). But Lemma [GL] gives that
\[ (w_p * (l, k, 0))(h_0) = p(k+2)+l+1 \text{ for even } p \text{ and } (w_p * (l, k, 0))(h_0) = p(k+2)+k-l+1 \text{ for odd } p. \]

\textbf{Proposition 2.1.} \( H_p(n, \pi_n \otimes L_{l,k})^0 \) is one-dimensional if
\[ p \text{ is even and } n = p(k+2) + l \quad \text{ or } \quad p \text{ is odd and } n = p(k+2) + k-l, \]
and vanishes otherwise.
Proof. We note that \( u \) is an ideal in \( n \). Consider the Hochschild-Serre spectral sequence (see \([CE]\)) with
\[
E^2_{p,q} = H_p(n/u, H_q(u, \pi_n \otimes L_{l,k})).
\]
We first note that \( u \) acts trivially on \( \pi_n \). Therefore,
\[
H_q(u, \pi_n \otimes L_{l,k}) \simeq \pi_n \otimes H_q(u, L_{l,k}).
\]
In addition, \( n/u \) is one-dimensional algebra \( \mathbb{C} f_0 \). We obtain that
\[
\dim H_p(n, \pi_n \otimes L_{l,k}) = \dim \operatorname{coker} \left( [\pi_n \otimes H_p(u, L_{l,k})]^0 \right) + \dim \ker \left( [\pi_n \otimes H_{p-1}(u, L_{l,k})]^0 \right).
\]
This gives that \( \dim H_p(n, \pi_n \otimes L_{l,k}) \) vanishes unless \( n + 1 = \dim H_p(u, L_{l,k}) \), and in this case (because of Lemma 2.2) the dimension is equal to 1 for \( n = p(k + 2) + l \) with even \( p \) or \( n = p(k + 2) + k - l \) with odd \( p \). Proposition is proved.

From the proof of the proposition we obtain a following corollary.

**Corollary 2.1.** \( H_p(n, \pi_n \otimes L_{l,k})^{-1} = 0 \) for any \( p, n \geq 0 \).

**Proof.** We note that for any \( n, n_1 \geq 0 \) the operator
\[
f_0 : [\pi_n \otimes \pi_{n_1}]^1 \to [\pi_n \otimes \pi_{n_1}]^{-1}
\]
is an isomorphism. Now our corollary follows from the formula \([26]\). \( \square \)

**Corollary 2.2.** For any \( m \in (\mathbb{N} \cup 0)^{k'} \) we have \( H_p(n, V_m \otimes L_{l,k})^{-1} = 0, p \geq 0 \).

**Proof.** Our corollary follows from Corollary 2.1 and a fact that \( V_m \) has a filtration such that each quotient is irreducible finite-dimensional \( \mathfrak{sl}_2 \)-module. \( \square \)

**Remark 2.1.** We note that Corollary 2.2 \( p = 0 \) follows from the exact sequence \([21]\). In fact, in the same way as in the proof of Lemma 2.7 we get
\[
H_0(n, V_m \otimes L_{l,k})^{-1} = V_m / (e^{-l+1}, f_0^{l+1}, h_0 + l + 1).
\]
But for any \( v \in V_m \otimes L_{l,k} \) with \( h_0 v = -(l + 1) v \) there exists \( v_1 \in V_m \otimes L_{l,k} \) such that \( v = f_0^{l+1} v_1 \). Therefore, \( H_0(n, V_m \otimes L_{l,k})^{-1} = 0 \).

To prove that \( H_p(n, V_m \otimes L_{l,k}) \) for \( p > 0 \) we need one more technical lemma. Let \( m \) be some \( k' \)-tuple with \( m_s \neq 0 \) and \( m_{s+1} = \cdots = m_k = 0 \). Introduce the notation for \( k' \)-tuple \( m^i = (m_1, \ldots, m_s, 1, 0, \ldots, 0) \). We set
\[
\tilde{V}_{m^i} = U(n) \cdot v_m \Rightarrow V_m
\]
(recall that in this section fusion products are considered as modules over the generating operators \( x_i, i \leq 0 \)).

**Lemma 2.3.** \( H_p(n, \tilde{V}_{m^i} \otimes L_{l,k})^\alpha \simeq H_p(n, V_{m^i} \otimes L_{k-l,k})^{k-s-\alpha} \).

**Proof.** Consider the Lie algebra automorphism \( \phi : n \to n, \phi(e_{-1}) = f_0, \phi(f_0) = e_{-1} \). We note that \( \phi \) induces an automorphism of the universal enveloping algebra \( U(n) \). Denote this automorphism by the same letter. Let \( v_{l,k} \) be a highest weight vector of \( L_{l,k} \) and \( v_m, u_m \in V_m \) be lowest and highest (with respect to the operator
We note that because of Lemma 2.1 we only need to prove that $h_0$ is an isomorphism of $n$-modules, where the action of $n$ on $V_m \otimes L_{k-l,k}$ is a composition of $\phi$ and a standard action.

We need to show that $I$ identifies $(V_m \otimes L_{l,k})^\alpha$ and $(V_m \otimes L_{k-l-1,k})^{k-s-\alpha}$. Note that $[h_0, \phi(f_0)] = 2\phi(f_0)$ and $[h_0, \phi(e_{-1})] = -2\phi(e_{-1})$. Therefore, for $x \in U(n)$ with $[h_0, x] = \beta x$ ($\beta \in \mathbb{C}$) we have $[h_0, \phi(x)] = -\beta \phi(x)$. In addition, in view of $h_0v_m = (-\sum_{i=1}^{k} im_i)v_m$ and $h_0u_m = (\sum_{i=1}^{k} im_i)u_m$, we obtain

\begin{equation}
(24) \ h_0(v_m \otimes v_{l,k}) = (l - \sum_{i=1}^{k'} im_i)v_m \otimes v_{l,k},
\end{equation}

\begin{equation}
\ h_0(I(v_m \otimes v_{l,k})) = h_0(u_m \otimes v_{k-l,k}) = \left(\sum_{i=1}^{k'} im_i + k - l\right)v_m \otimes v_{l,k} = (-l + \sum_{i=1}^{k'} im_i + k - s)v_m \otimes v_{l,k}.
\end{equation}

This finishes the proof of the lemma. \hfill \Box

**Corollary 2.3.** Let $m \in \mathbb{Z}_{\geq 0}^{k}, m_{k+1} \neq 0$. Then $H_p(n, V_m \otimes L_{l,k})^0 = 0$ for any $p \geq 0$.

**Proof.** We set $m^1 = (m_1, \ldots, m_{k+1} - 1)$, $m^2 = (m_1, \ldots, m_k + 1, m_{k+1} - 1)$. Recall the exact sequence (14) of $n$-modules

\[ 0 \to V_{m^1} \to V_m \to V_{m^2} \to 0. \]

We note that the map $V_m \to V_{m^2}$ is defined by $u_m \mapsto u_{m^2}$ (because $V_m$ is cyclic $n$-module with the cyclic vector $u_m$). In addition, $\deg_z u_{m^2} = \deg_z u_m - 1$. Therefore, for any $\alpha$ we obtain an exact sequence

\[ 0 \to \tilde{V}_{m}^\alpha \to V_m^\alpha \to V_{m^2}^\alpha \to 0. \]

This gives an exact sequence of homology

\begin{equation}
(25) \ 0 \leftarrow H_0(n, V_{m^2} \otimes L_{l,k})^{\alpha - 1} \leftarrow H_0(n, V_m \otimes L_{l,k})^\alpha \leftarrow H_0(n, \tilde{V}_m \otimes L_{l,k})^\alpha \leftarrow H_1(n, V_{m^2} \otimes L_{l,k})^{\alpha - 1} \leftarrow \ldots
\end{equation}

Now let $\alpha = 0$. Then because of Lemma 2.2 and Corollary 2.2 we obtain $H_p(n, V_{m^2} \otimes L_{l,k})^{-1} = 0$, $H_p(n, \tilde{V}_m \otimes L_{l,k})^0 \approx H_p(n, V_m \otimes L_{k-l,k})^{k-l-1} = 0$. In view of the exact sequence (25) our corollary is proved. \hfill \Box

We now prove the main theorem of this section.

**Theorem 2.2.** Let $m \in \mathbb{Z}_{\geq 0}^k$. Then

\[ \text{ch}_q H_p(n, V_m \otimes L_{l,k})^0 = \delta_{0,p} K_{l,m}^{(k)}(q). \]

**Proof.** Because of Lemma 2.1 we only need to prove that $H_p(n, V_m \otimes L_{l,k})^0$ vanishes for $p > 0$. We use the induction on $m$. We order a $k$-tuples by the rule

\[ m > n \text{ if } \sum_{i=1}^{k} m_i > \sum_{i=1}^{k} n_i \text{ or } \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} n_i \text{ and } \prod_{i=1}^{k} i^{m_i} > \prod_{i=1}^{k} i^{n_i}. \]
For $\sum_{i=1}^{k} m_i = 1$ our theorem follows from Proposition 2.1. Now let $V_m = \pi_{a_1} \ast \cdots \ast \pi_{a_n}$ and $1 \leq a_1 \leq \cdots \leq a_n$. We recall the exact sequence \[ 0 \rightarrow \pi_{a_2-a_1} \ast \pi_{a_3} \ast \cdots \ast \pi_{a_n} \rightarrow \pi_{a_1} \ast \pi_{a_2} \ast \cdots \ast \pi_{a_n} \rightarrow \pi_{a_1-1} \ast \pi_{a_2+1} \ast \pi_{a_3} \ast \cdots \ast \pi_{a_n} \rightarrow 0. \]

We denote the first fusion product in (26) by $V_m^{(1)}$ and the third one by $V_m^{(2)}$. From (26) we obtain a long exact sequence

\[ \cdots \rightarrow H_0(n, V_m^{(2)} \otimes L_{l,k}) \rightarrow H_0(n, V_m \otimes L_{l,k}) \rightarrow H_0(n, V_m^{(1)} \otimes L_{l,k}) \rightarrow \cdots \]

We note that $m^{(1)} < m$, and therefore, $H_p(n, V_m^{(1)} \otimes L_{l,k})^0$ vanishes for $p > 0$ by induction assumption. In addition, $m^{(2)} < m$, and $V_m^{(2)}$ is either the fusion product of the representations of dimension at most $k + 1$ or one of the fused representations is of the dimension $k + 2$. In the latter case Corollary 2.3 gives the vanishing of the higher homology. Hence, because of the exact sequence (27), the theorem is proved.

\section*{Corollary 2.4.}

We have \[ \text{ch}_q H_p(\mathfrak{sl}_2 \otimes \mathbb{C}[t^{-1}], \mathfrak{sl}_2; V_m \otimes L_{l,k}) = \delta_{0,p} K^{(k)}_{l,m}(q). \]

\textbf{Proof.} We note that as $\mathfrak{sl}_2$-module $V_m \otimes L_{l,k}$ decomposes into the direct sum of finite-dimensional representations. Therefore,

\[ \text{ch}_q H_p(\mathfrak{sl}_2 \otimes \mathbb{C}[t^{-1}], \mathfrak{sl}_2; V_m \otimes L_{l,k}) = \text{ch}_q(H_p(u, V_m \otimes L_{l,k})^{\mathfrak{sl}_2}) \]

(the right-hand side is a subspace of $\mathfrak{sl}_2$-invariants). In view of the Hochschild-Serre spectral sequence we also obtain

\[ H_p(n, V_m \otimes L_{l,k})^0 = H_0(\mathbb{C}f_0, H_p(u, V_m \otimes L_{l,k}))^0 = H_p(u, V_m \otimes L_{l,k})^{\mathfrak{sl}_2}. \]

Corollary is proved.

\textbf{Remark 2.2.} We note that our theorem concerns only the case of a homology with the coefficients in $\pi_{a_1} \ast \cdots \ast \pi_{a_n} \otimes L_{l,k}$ with $a_i \leq k$. For the general $a_i$ the corresponding homology is not concentrated in one dimension.

\section*{3. The BGG resolution and alternating sum formula for Kostka polynomials}

In this section, we give a homological interpretation of the alternating sum formula (see [SS], [EJKLM])

\[ K^{(k)}_{l,m}(q) = \sum_{i \geq 0} q^{(k+2)i^2+(l+1)i} K_{2(k+2)i+l,m}(q) - \sum_{i > 0} q^{(k+2)i^2-(l+1)i} K_{2(k+2)i-l-2,m}(q). \]

We note that in view of Theorem 2.2 the left-hand side coincides with the Euler characteristics $\sum_{p \geq 0} (-1)^p \text{ch}_q H_p(n, V_m \otimes L_{l,k})^0$. The idea is that the right-hand side is also the Euler characteristics of the complex, counting the homology $H_p(n, V_m \otimes L_{l,k})^0$.

Consider the BGG-resolution of $L_{l,k}$ (see [BGG], [Kum])

\[ 0 \leftarrow L_{l,k} \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots, \quad F_p = \bigoplus_{l(w) = p} M(w \ast (l,k,0)), \]

where $M(w \ast (l,k,0))$ are the $l$-th order Kostka polynomials.
where \( l(w) \) is a length of the element of the Weyl group of \( \mathfrak{sl}_2 \). Tensoring (28) with \( V_m \) we obtain the resolution for \( V_m \otimes L_{l,k} \). Therefore, the following complex counts

\[
H_*(n, V_m \otimes L_{l,k})^0
\]

(30) \( 0 \leftarrow (\mathbb{C} \otimes U(n) (F_0 \otimes V_m))^0 \leftarrow (\mathbb{C} \otimes U(n) (F_1 \otimes V_m))^0 \leftarrow \cdots \)

Recall that \( F_p \) is a free \( U(n) \)-module. Therefore, we can rewrite (30) as

\[
0 \leftarrow ((\mathbb{C} \otimes U(n) F_0) \otimes V_m)^0 \leftarrow ((\mathbb{C} \otimes U(n) F_1) \otimes V_m)^0 \leftarrow \cdots
\]

**Lemma 3.1.** \( \text{ch}_q((\mathbb{C} \otimes U(n) F_p) \otimes V_m)^0 = \sum_{l(w)=p} q^{(w \ast (l,k,0))} \text{ch}_q V_m^{-(w \ast (l,k,0))h_0}. \)

**Proof.** \( F_p \) is a free \( U(n) \)-module with a generators labeled by \( w \) such that \( l(w) = p \). In addition, the \( z \)-degree of the generator equals to \( (w \ast (l,k,0))h_0 \). This proves our lemma.

**Lemma 3.2.** \( K_{l,m}(q) = \text{ch}_q V_m/\langle e_0^{l+1}, h_0 + l \rangle \), i.e., \( K_{l,m}(q) \) is a \( q \)-multiplicity of \( \pi_l \) in decomposition of \( V_m \) as \( \mathfrak{sl}_2 \)-module to the irreducible components.

**Proof.** Follows from Theorem 2.1 and a fact \( \lim_{k \to \infty} K_{l,m}^{(k)} = K_{l,m}(q) \).

**Corollary 3.1.** \( K_{l,m}(q) = \text{ch}_q V_m^l - \text{ch}_q V_m^{l+2} \).

**Proposition 3.1.**

(31) \[
\sum_{p \geq 0} (-1)^p \text{ch}_q((\mathbb{C} \otimes U(n) F_p) \otimes V_m)^0 =
\sum_{p \geq 0} q^{(k+2)p^2+(l+1)p} K_{2(k+2)p+l,m} - \sum_{p > 0} q^{(k+2)p^2-(l+1)p} K_{2(k+2)p-l-2,m}.
\]

**Proof.** In view of Lemma 3.1 and Lemma 1.1, we obtain that the left-hand side of (31) is equal to

\[
\sum_{p \geq 0} \left( q^{(p+1)^2(k+2)-(p+1)(l+1)} \text{ch}_q(V_m)^{2p(k+2)-l} + q^{2(k+2)+p(l+1)} \text{ch}_q(V_m)^{-2p(k+2)-l} \right) -
\sum_{p \geq 0} \left( q^{(p+1)^2(k+2)-(p+1)(l+1)} \text{ch}_q(V_m)^{-2(p+1)(k+2)+l-k} + q^{2(k+2)+p(l+1)} \text{ch}_q(V_m)^{(2p+1)(k+2)-k+l} \right) =
\sum_{p \geq 0} q^{2(k+2)+p(l+1)} \left( \text{ch}_q(V_m)^{2p(k+2)+l} - \text{ch}_q(V_m)^{2p(k+2)+l+2} \right) +
\sum_{p \geq 1} q^{2(k+2)-p(l+1)} \left( \text{ch}_q(V_m)^{2p(k+2)-l} - \text{ch}_q(V_m)^{2p(k+2)-l-2} \right) =
\sum_{p \geq 0} q^{2(k+2)+p(l+1)} K_{2p(k+2)+l,m}(q) - \sum_{p \geq 1} q^{2(k+2)-p(l+1)} K_{2p(k+2)-l-2,m}(q),
\]

where Corollary 3.1 is used.

As a corollary we obtain the alternating sum formula (28).
4. The decomposition of \( L_{m,k} \)

We introduce the notations \( \widetilde{K}_{l,m}(q) \) and \( \widetilde{K}_{l,m}^{(k)}(q) \) for the "reversed" Kostka polynomials:

\[
(32) \quad \widetilde{K}_{l,m}^{(k)}(q) = q^{h(m)}K_{l,m}^{(k)}(q^{-1}), \quad \widetilde{K}_{l,m}(q) = q^{h(m)}K_{l,m}(q^{-1}),
\]

where \( h(m) = \max\{\deg v, \ v \in V_m\} \) (see Lemma 5.5 for the computation of \( h(m) \)). Therefore, "reversed" polynomials are Kostka polynomials in notations in \([SS]\) up to a power of \( q \).

We recall the decomposition (16)

\[
L_{m,k} = L_{0,k} \otimes N_{0,m} \oplus \cdots \oplus L_{k,k} \otimes N_{k,m}.
\]

**Lemma 4.1.** \( \text{ch}_q N_{l,m} = \widetilde{K}_{l,m}^{(k)}(q) \).

**Proof.** We first note that \( K_{l,m}^{(1)}(1) = \dim N_{l,m} \), because both sides are structure constants of the Verlinde algebra (see \([15]\)).

We show that \( \text{ch}_q N_{l,m} = \text{ch}_q V_m/(f_0, h_0 - l, f_1^{k-l+1}) \). Recall that \( L_{m,k} = \lim_{s \to \infty} V_{m(s)} \), where \( m(s) = (m_1, \ldots, m_{k-1}, m_k + 2s) \). We denote

\[
N_{l,m}^s = \{ v \in L_{m,k} : e_0 v = f_1 v = 0, h_0 v = lv, v \in V_m \}.
\]

Note that \( N_{l,m}^s \) is a subspace of the space of highest weight vectors of the weight \( l \). Therefore, any \( v \in N_{l,m}^s \) is not the element of \( (f_0, h_0 - l, f_1^{k-l+1})V_m \) (because for a highest weight vector \( v_{l,k} \in L_{l,k} \) we have \( v_{l,k} \notin (f_0, h_0 - l, f_1^{k-l+1})L_{l,k} \)). We thus obtain that

\[
\text{ch}_q N_{l,m}^s \leq \text{ch}_q V_{m(s)}/(f_0, h_0 - l, f_1^{k-l+1})
\]

(the difference of the right-hand side and left-hand side is a polynomial with non-negative coefficients). In addition, there exists \( s_0 \) such that \( N_{l,m}^{s_0} = N_{l,m} \). Hence, \( \text{ch}_q N_{l,m} \leq \widetilde{K}_{l,m}^{(k)}(q) \). Let \( c_{l,m} \) denote the structure constants of the level \( k \) Verlinde algebra:

\[
[1]^{m_1} \cdots [k]^{m_k} = \sum_{l=0}^{k} c_{l,m}[l].
\]

In view of \([k]^2 = 0\) we get \( \text{ch}_q N_{l,m}(1) = c_{l,m} = \widetilde{K}_{l,m}^{(k)}(1) \) (for the second equality see \([15]\)). Therefore, there exists \( s_0 \) such that

\[
\text{ch}_q N_{l,m} = \widetilde{K}_{l,m}^{(k)}(q).
\]

To complete the proof we need to show that

\[
(33) \quad \widetilde{K}_{l,m}^{(k)}(q) = \widetilde{K}_{l,m}^{(k)}(1)(q).
\]

We recall an exact sequence of \( \mathfrak{sl}_2 \otimes \mathbb{C}[t]\)-modules

\[
0 \to V_m \to V_m^{(1)} \to V_{(m_1, \ldots, m_{k-2}, m_{k-1}+1, m_k, 1)} \to 0.
\]

Because of Corollary 2.2 the corresponding long exact sequence of \( n \)-homology is of the form

\[
0 \leftarrow 0 \leftarrow H_0(n, V_m^{(1)} \otimes L_{l,k})^0 \leftarrow H_0(n, V_m \otimes L_{l,k})^0 \leftarrow 0 \leftarrow \cdots .
\]

Because of Lemma 2.4 the equation (33) is shown. \( \square \)
We finish this section with the identification of $L_{m,k}$ with the induced module from the fusion product $V_m$. We first need one lemma.

**Lemma 4.2.** Let $1 \leq a_1 \leq \cdots \leq a_n$ and $a_n \geq k + 1$. Then $\pi_{a_1} \cdots \pi_{a_n}/\langle e_0, h_0 + l, e_1^{k-l+1} \rangle = 0$ for any $0 \leq l \leq k$.

**Proof.** We prove our lemma by induction on a pair $(n, \prod_{i=1}^n a_i)$. We set $(n_1, s_1) \geq (n_2, s_2)$ if $n_1 > n_2$ or $n_1 = n_2$ and $s_1 > s_2$. Let $n = 1$. Then for any $0 \leq l \leq k$ we have $\pi_a/\langle h_0 + l, e_0 \rangle = 0$ if $a \geq k + 1$. We now consider an exact sequence of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-modules

$$0 \rightarrow \pi_{a_2} \pi_{a_2} \cdots \pi_{a_n} \rightarrow \pi_{a_1} \cdots \pi_{a_n} \rightarrow \pi_{a_1} \cdots \pi_{a_n} \rightarrow 0.$$ 

By induction assumption our lemma is true for the submodule and for the quotient module. Therefore, it also holds for $\pi_{a_1} \cdots \pi_{a_n}$. □

**Proposition 4.1.** Let $m \in \mathbb{Z}_{\geq 0}$. Then

$$L_{m,k} = \left(\text{Ind}_{\mathfrak{sl}_2 \otimes \mathbb{C}[t]}^{\hat{\mathfrak{sl}}_2} V_m \right)/\langle (e(z)^{k+1}, K-k) \rangle,$$

where the right-hand side is a quotient of the induced module (with fixed $K = k$) by the action of the coefficients of the series $e(z)^{k+1} = (\sum_i e_i z^i)^{k+1}$. In addition

$$\left(\text{Ind}_{\mathfrak{sl}_2 \otimes \mathbb{C}[t]}^{\hat{\mathfrak{sl}}_2} \pi_{a_1} \cdots \pi_{a_n} \right)/\langle (z)^{k+1}, K-k \rangle = 0,$$

if $a_1 \leq \cdots \leq a_n$ and $a_n \geq k + 1$.

**Proof.** Note that the right-hand side of (34) is a level $k$ integrable $\hat{\mathfrak{sl}}_2$-module (because $(z)^k$ acts by 0). Therefore, it can be decomposed into the direct sum of irreducible modules $L_{l,k}$. We show (34) by checking that the $q$-multiplicity of $L_{l,k}$ is equal to $\text{ch}_q V_m/\langle e_0, h_0 + l, e_1^{k-l+1} \rangle$.

Consider $\mathfrak{sl}_2$ homomorphisms from $L_{l,k}$ to the right-hand side of (34). They coincide with the homomorphisms $L_{l,k} \rightarrow \text{Ind}_{\mathfrak{sl}_2 \otimes \mathbb{C}[t]}^{\hat{\mathfrak{sl}}_2} V_m/\langle K-k \rangle$, which are labeled by the elements of the quotient $V_m/\langle e_0, h_0 + l, e_1^{k-l+1} \rangle$ (because of the highest weight condition for $L_{l,k}$). The first part of our proposition is verified.

We note that (35) can be checked in the same manner, taking into account Lemma 4.2. □

5. Virasoro Unitary Models

5.1. The alternating sum formula. We first recall the coset construction (see [GKO]). Consider the decomposition of the tensor product $L_{i,1} \otimes L_{j,k}$ into the sum of irreducible $\mathfrak{sl}_2$-modules

$$L_{i,1} \otimes L_{j,k} = \bigoplus_{l=0}^{k+1} N_l \otimes L_{l,k+1},$$

where $N_l$ is spanned by a highest weight vectors of the weight $l$. Let $L^{(1)}_i, L^{(2)}_i$ and $L^{\text{diag}}_i$ be the Sugawara operators, acting on $L_{i,1}, L_{j,k}$ and $L_{i,1} \otimes L_{j,k}$. Then operators

$$L_i = L^{(1)}_i \otimes \text{Id} + \text{Id} \otimes L^{(2)}_i - L^{\text{diag}}_i$$

form the Virasoro algebra, which acts on the tensor product $L_{i,1} \otimes L_{j,k}$ with the central charge $\frac{3}{3+2} + \frac{3k}{k+2} - \frac{3(k+1)}{k+3} = \frac{k^2+5k}{(k+2)(k+3)}$. The important property is that $L_i$
commute with the diagonal action of $\mathfrak{sl}_2$. Therefore, each $N_1$ is a representation of the Virasoro algebra. Using the alternating sum formula we derive a formula for the character of $N_1$. This formula coincides with the Rocha-Caridi formula for the character of the minimal model $M_{j_{1+1}, t+1}(k + 2, k + 3)$ (see [KW]).

We first recall the Rocha-Caridi formula for the character of $M_{r,s}(p, p')$ (see [RC]). Here $p, p'$ are relatively prime numbers and $1 \leq r \leq p - 1$, $1 \leq s \leq p' - 1$. Let $t = \frac{q}{p}$. Then the central element $c$ of the Virasoro algebra acts on $M_{r,s}(p, p')$ as a scalar $13 - 6(t + \frac{1}{t})$. Let $\Delta_{r,s} = \frac{(rt-s)^2-(t-1)^2}{4t}$ and

\[
\chi_{r,s} = \text{ch} M_{r,s}(p, p') = \sum_{d \in \mathbb{Z}_{\geq 0}} q^d \dim \{ v : L_0 v = dv \}.
\]

We set

\[
(q)_n = \prod_{\alpha=1}^{n} (1 - q^\alpha), \quad (q)_{\infty} = \prod_{\alpha=1}^{\infty} (1 - q^\alpha), \quad \left[ \frac{m}{n} \right]_q = \frac{(q)_m}{(q)_n(q)_{m-n}}.
\]

Then

\[
\chi_{r,s} = \frac{q^{\Delta_{r,s}}}{(q)_{\infty}} \left( \sum_{n \in \mathbb{Z}} q^{pp'n^2 +(p'r-ps)n} - \sum_{n \in \mathbb{Z}} q^{pp'n^2 +(p'r+ps)n+rs} \right).
\]

We note that in the case $(p, p') = (k + 2, k + 3)$ the central charge is equal to $\frac{k(k+2)(k+3)}{k(k+2)(k+3)}$.

We now recall the embedding of $V_m$ and $L_{m,k}$ into the tensor product of the level one irreducible modules. In what follows we use the notation $L_{(1^{m_1} \ldots k^{m_k}),k}$ for $L_{m,k}$. Let $v(p) \in L_{0,1} \oplus L_{1,1}$ be the set of extremal vectors, $h_0 v(p) = -pv(p)$.

Then we have the isomorphisms

\begin{align*}
(36) & \quad V_m \simeq U(\mathfrak{sl}_2 \otimes \mathbb{C}[t]) \cdot (v(m_1 + \cdots + m_k) \otimes v(m_2 + \cdots + m_k) \otimes \ldots \otimes v(m_k)), \\
(37) & \quad L_{m,k} \simeq U(\mathfrak{sl}_2) \cdot (v(m_1 + \cdots + m_k) \otimes v(m_2 + \cdots + m_k) \otimes \cdots \otimes v(m_k)).
\end{align*}

This gives an embedding of $V_m$ and $L_{m,k}$ into the tensor product $L_{i_1,1} \otimes \cdots \otimes L_{i_k,1}$, where $i_0 = 0, 1$. For example, in view of $L_{j,k} = L_{(j),k}$ we obtain

\[
L_{j,k} \simeq U(\mathfrak{sl}_2) \cdot (v(1)^{\otimes j} \otimes v(0)^{\otimes (k-j)}) \hookrightarrow L_{i_1,1}^{\otimes j} \otimes L_{0,1}^{\otimes (k-j)}.
\]

**Lemma 5.1.** We have an isomorphism of $\mathfrak{sl}_2$-modules

\[
(38) \quad L_{i_1,1} \otimes L_{j,k} \simeq \lim_{n \to \infty} U(\mathfrak{sl}_2) \cdot (v(2n + i) \otimes v(1)^{\otimes j} \otimes v(0)^{\otimes (k-j)}).
\]

**Proof.** We first note that $v(2n + i) \otimes [v(1)^{\otimes j} \otimes v(0)^{\otimes (k-j)}]$ is the tensor product of extremal vectors of $L_{i_1,1}$ and $L_{j,k}$. In addition,

\[
U(\mathfrak{sl}_2) \cdot (v(2n + i) \otimes v(1)^{\otimes j} \otimes v(0)^{\otimes (k-j)}) = U(\mathfrak{sl}_2) \cdot (v(2n + i + 2s) \otimes v(2s + 1)^{\otimes j} \otimes v(2s)^{\otimes (k-j)})
\]

for any integer $s$. Therefore, to prove our lemma it suffices to show that

\[
v(2n + i) \otimes v(1)^{\otimes j} \otimes v(0)^{\otimes (k-j)} \hookrightarrow U(\mathfrak{sl}_2) \cdot (v(2n + 2 + i) \otimes v(1)^{\otimes j} \otimes v(0)^{\otimes (k-j)})
\]

(in this case all the products of the extremal vectors of $L_{i_1,1}$ and $L_{j,k}$ are the elements of $\mathfrak{sl}_2$).

We recall that $e_{N-1} v(N) = v(N - 2)$ and $e_{N} v(N) = 0$. Hence,

\[
e_{i+2n-1}(v(2n + i) \otimes v(1)^{\otimes j} \otimes v(0)^{\otimes (k-j)}) = (v(2n - 2 + i) \otimes v(1)^{\otimes j} \otimes v(0)^{\otimes (k-j)}).
\]
Lemma is proved.

□

Corollary 5.1. \( L_{i,1} \otimes L_{j,k} \simeq \lim_{n \rightarrow \infty} L_{(1^{2n+i-1}(j+1)),k+1}. \)

Proof. Because of the formula (37)
\[
U(sl_2) \cdot (v(2n + i) \otimes v(1) \otimes v(0) \otimes (k-j)) \simeq L_{(1^{2n+i-1}(j+1)),k+1}.
\]

Consider the decomposition
\[
L_{(1^N(j+1)),k+1} = \bigoplus_{l=0}^{k+1} N_{l,(1^N(j+1))} \otimes L_{l,k+1}.
\]

Corollary 5.2. We have

\[
\text{ch}_q N_l = \lim_{n \rightarrow \infty} \text{ch}_q N_{l,(1^{2n+i-1}(j+1))}.
\]

We now compute the limit from the above corollary. Because of Lemma 4.1
\[
\text{ch}_q N_{l,(1^N(j+1))} = \bar{K}_{l,(1^N(j+1))}(q).
\]

Recall that \( L_0 v_{i,k} = \frac{(i+2)}{4(i+2)} v_{i,k} \) for the highest weight vector \( v_{i,k} \in L_{i,k}. \) In view of the alternating sum formula (28) and the formula (32) we obtain

\[
\text{ch}_q N_{l,(1^{2n+i-1}(j+1))} = \frac{(i+2)}{4(i+2)} + \frac{(i+2)}{4(i+2)} - \frac{(i+2)}{4(i+2)} \frac{\bar{K}_{l,(1^{2n+i-1}(j+1))}(q) =}{\sum_{p \geq 0} q^{-(k+3)p^2+(l+1)p}\bar{K}_{2(k+3)p+l,(1^{n+i-1}(j+1))(q)} -}
\]
\[
\sum_{p > 0} q^{-(k+3)p^2+(l+1)p}\bar{K}_{2(k+3)p-l,2,(1^{n+i-1}(j+1))(q)}
\]

We want to compute the limit of the above expression while \( n \rightarrow \infty. \)

Lemma 5.2. Let \( a(q) = \sum_{i \geq 0} a_i q^i. \) We write \( a(q) = O(q^N) \) if \( a_i = 0 \) for \( i < N. \) Then
\[
\bar{K}_{2s+i+j,(1^{2n+i+1}(j+1))}(q) - (\text{ch}_q L_{i,1}^{2s+i} - \text{ch}_q L_{i,1}^{2s+i+2j+2}) = O(q^{n+s(s+i-1)}).
\]

Proof. Consider the embeddings
\[
V_{(1^{2n+i})} \otimes \pi_j \hookrightarrow V_{(1^{2n+i+1}(j+1))} \hookrightarrow V_{(1^{2n+i+2})} \otimes \pi_j,
\]
where the first embedding comes from (11) and the second from (36). We note that (40) means that \( \lim_n \rightarrow \infty V_{(1^{2n+i+1}(j+1))} \simeq L_{i,1} \otimes \pi_j. \) Note that
\[
\text{ch}_q V^{2s+i}_{(1^{2n+i})} = q^{s(s+i)} \left[ \begin{array}{c} 2n+i \\ n-s \\ j \end{array} \right]_q.
\]

Therefore,
\[
\text{ch}_q V^{2s+i}_{(1^{2n+i})} - \text{ch}_q L_{i,1}^{2s+i} = q^{s(s+i)} \left( \left[ \begin{array}{c} 2n+i \\ n-s \\ j \end{array} \right]_q - \frac{1}{(q)_\infty} \right) = O(q^{n+s(s+i-1)}).
\]

We obtain that
\[
\text{ch}_q (V_{(1^{2n+i+1})} \otimes \pi_j)^{2s+i+j} - \text{ch}_q (L_{i,1} \otimes \pi_j)^{2s+i+j} = O(q^{n+s(s+i-1)}).
\]
To finish the proof it suffices to use (40) and the formula
\[
\tilde{K}_{2s+i+j, (12n+i+j+1)(j+1)}(q) = \tilde{\chi}_q V_{(12n+i+j+1)(j+1)}^{2s+i+j} - \tilde{\chi}_q V_{(12n+i+j+1)(j+1)}^{2s+i+j+2},
\]
\[\square\]

We derive from this lemma that in [39] we can replace reversed Kostka polynomials by the difference of the characters of the weight subspaces of \(L_{i,1}\).

Let \(i = 0\). Then for such \(l\) that \(j + l\) is even we obtain
\[
\lim_{n \to \infty} \chi N_{1, (12n+i+j+1)(j+1)} = q^{(j+2)\frac{l(l+1)}{2}} \times \\
\left( \sum_{p \geq 0} q^{-(k+3)p^2-(l+1)p} \frac{1}{(q)_{\infty}} (q^{(k+3)p+(l-j)/2} - q^{(k+3)p+(l+j+2)/2}) - \right.
\]
\[
\sum_{p > 0} q^{-(k+3)p^2+(l+1)p} \frac{1}{(q)_{\infty}} (q^{(k+3)p-(l+j+2)/2} - q^{(k+3)p-(l-j)/2})
\]
\[
= q^{\frac{(j+2)(l+j+1)}{2}} \sum_{p \in \mathbb{Z}} q^{p^2(k+3)p+(k+3)(j+1)+(l+1)(j+1)} \left( \sum_{p \in \mathbb{Z}} q^{p^2(k+3)p+(k+3)(j+1)-(k+2)(l+1)} \right)
\]
\[
= \chi M_{j+1, l+1}(k+2, k+3)
\]

One can repeat the same computation for \(i = 1\). We obtain
\[
L_{i,1} \otimes L_{j,k} = \bigoplus_{l=0}^{k+1} L_{i,k+1} \otimes M_{j+1, l+1}(k+2, k+3),
\]
where the sum is taken over \(l\) such that \(l + i + j\) is even.

5.2. The fermionic formula. Recall (see [SS, FJKLM]) that
\[
K_{i,m}^{(k)}(q) = \sum_{s \geq 0, m \geq 0, 2|n| = |m| - 1} q^{s A_s + v_s} \left[ A(m - 2s) - v + s \right]_q,
\]
where \(A_{\alpha, \beta} = \min(\alpha, \beta)\), \(v_\alpha = \max(0, \alpha - k + l)\), \(|m| = \sum_{\alpha=1}^k m_\alpha\) and for two vectors \(m, n \in \mathbb{Z}_0^k\) we set 
\[
[m]_q = \left[ \prod_{\alpha=1}^k \left[ m_\alpha \over n_\alpha \right]_q \right].
\]
We now explain how this formula naturally appears as a \(q\)-multiplicity.

Lemma 5.3. Denote by \(e(z)^s \) the coefficient in front of the power \(z^s\) in the series \((\sum_{j \in \mathbb{Z}} e_j z^j)^s\). Then
\[
\chi_q L_{m,k}/e_s, s \leq 0; e(z)^{k-l+1}, s \leq k - l + 1; h_0 + l = K_{i,m}^{(k)}(q).
\]
Proof. We recall an embedding $V_m \hookrightarrow L_{m,k}$. Let $v_m$ be a lowest weight vector of $V_m$. We consider the principal subspace

$$W = \mathbb{C}[e_N, e_{N-1}, \ldots] \cdot v_m,$$

where $N$ is fixed by $e_N v_m \neq 0$ and $e_{N+1} v_m = 0$. It is proved in [FF2] that the defining relations in $W$ are $e(z)^{k+1} v_m = 0$ and

$$e(z^i) v_m = z^{Ni-im_1-\cdots-m_i} q(z^{-1}), \quad i = 1, \ldots, k,$$

where $q$ is some series. This means that the dual space of the quotient

$$W/\langle e_s, s \leq 0; e^{(z)^k-l+1}s, s \leq k-l+1; h_0+l \rangle$$

can be identified with a subspace of symmetric polynomials satisfying the conditions

1. The number of variables is $s = \frac{1}{2} \left( \sum_{i=1}^k i m_i - l \right)$,
2. $f(z, \ldots, z, z_{i+1}, \ldots, z_s) = 0$,
3. $\deg_z f(z, \ldots, z, z_{i+1}, \ldots, z_s) \leq \sum_{i=1}^k \min(a_i) m_i - a$,
4. $f(0, z_2, \ldots, z_s) = 0$,
5. $f(z, z_{i+1}, \ldots, z_s) = z^{k-l+2}$.

But this space of symmetric polynomials coincides with the dual space $(V_m/\langle h_0 + l, e_0, e^{k-l+1} \rangle)^*$ from [FJKLM], and the character of the latter coincides with the corresponding restricted Kostka polynomial.

We recall that $L_{m,k} = \lim_{N \to \infty} V_{(m_1, \ldots, m_{k-1}, m_k+2N)}$. Now our lemma follows from the equality (33). □

5.4. Lemma. The $q$-multiplicity of $L_{m,k}$ in the decomposition of $L_{m,k}$ is equal to the character of

$$L_{m,k}/\langle e_s, s \leq 0; e(z)^{k-l+1}s, s \leq k-l+1; h_0+l \rangle.$$

Proof. We first note that the character of the space of the highest weight vectors in $L_{m,k}$ of the weight $l$ is less or equal then the character of the quotient (42). But their dimensions coincide. □

5.3. The limit of the fermionic formula. We want to find the limit

$$\lim_{N \to \infty} \hat{K}_{l,1^{(N+1)}}^{(k+1)}(q), \quad N+1 = l+j \mod 2.$$

Up to a power of $q$ this limit coincides with the unitary Virasoro character.

5.5. Lemma. Let $p(m) = \# \{ \alpha = 1, \ldots, k : m_\alpha + \cdots + m_k \text{ is odd} \}$. Define

$$h(m) = \max \{ \deg_q v : v \in V_m \}.$$

Then $h(m) = \frac{mAm_p(m)}{4}$. 

Proof. We use the embedding (36). Note that $h(1^N) = \frac{N^2}{4} - \frac{p(1^N)}{4}$. Therefore, it follows from (36) that
\begin{equation}
(43) \quad h(m) \leq \frac{(m_1 + \cdots + m_k)^2}{4} + \cdots + \frac{m_k^2}{4} - \frac{p(m)}{4} = \frac{mAm - p(m)}{4}.
\end{equation}

To prove that in (38) we have an equality it suffices to show that $$ v(i_1) \otimes \cdots \otimes v(i_k) \in U(sl_2 \otimes \mathbb{C}[t]) \cdot (v(m_1 + \cdots + m_k) \otimes \cdots \otimes v(m_k)) $$
(i_\alpha = 0 if $m_\alpha + \cdots + m_k$ is even and $i_\alpha = 1$ otherwise). But this follows from the formula
\begin{equation}
\frac{1}{s!} c_{m_1, \ldots, m_k}^s (v(m_1 + \cdots + m_k) \otimes \cdots \otimes v(m_k)) = v(m_1 + \cdots + m_k - 2) \otimes \cdots \otimes v(m_s + \cdots + m_k - 2) \otimes v(m_{s+1} + \cdots + m_k) \otimes \cdots \otimes v(m_k),
\end{equation}
where $s$ is determined by $m_1 = \cdots = m_{s-1} = 0$ and $m_s \neq 0$. □

In view of Lemma 5.5 and a fact $a \begin{bmatrix} a \\ b \end{bmatrix} q^{-1} = q^{-b(a-b)} a \begin{bmatrix} a \\ b \end{bmatrix} q$ we get:
\begin{equation}
q^{\frac{p(m) - 1}{2}} R_{l,m}^{(q)} (q) = q^{\frac{mAm}{2}} \sum_{\alpha, \beta} q^{-sAs-vs} \left[ A(\frac{m-2s}{2}) - v + s \right]_q =
\end{equation}
\begin{equation}
\sum_{\alpha, \beta} q^{\frac{mAm}{2} - sAs-vs - s(A(m-2s) - v+s)} \left[ A(\frac{m-s}{2}) - v + s \right]_q.
\end{equation}

Now let $m = (1^N(j+1)) \in \mathbb{Z}_k^j, N + 1 = j + l \mod 2$. We first rewrite the power of $q$ in the last line of the above formula using the relation $2|s| = N + j + 1 - l$:
\begin{equation}
\frac{(m)}{2} - sA(\frac{m}{2}) - s = \sum_{\alpha, \beta} s_\alpha s_\beta + \frac{1}{4} (N^2 + j + 1 + 2N) - N \sum_{\alpha} s_\alpha - \sum_{\alpha} \min(\alpha, j+1) s_\alpha.
\end{equation}
Replacing $s_1$ by $\frac{N + j + 1 - l}{2} - \alpha s_\alpha$ we get
\begin{equation}
\sum_{\alpha, \beta} s_\alpha ((j + 1 - l)(\alpha - 1) + \max(0, \alpha - j - 1)) + \frac{(l-j)^2 + j}{4}.
\end{equation}

Now we consider the binomial coefficient $\left[ A(\frac{m-2s}{2}) - v + s \right]_q$. This is the product
\begin{equation}
\prod_{\alpha=1}^{k+1} \left[ N + \min(\alpha, j+1) - 2 \sum_{\beta=1}^{k+1} \min(\alpha, \beta) s_\beta - v_\alpha + s_\alpha \right].
\end{equation}
Let $\alpha > 1$. Then

\[
\left[ N + \min(\alpha, j + 1) - 2s_1 - 2\sum_{\beta=2}^{k+1} \min(\alpha, \beta)s_\beta - v_\alpha + s_\alpha \right]_q = \\
\left[ 2\sum_{\beta=2}^{k+1} (\beta - \min(\alpha, \beta))s_\beta + l - j - 1 + \min(\alpha, j + 1) - v_\alpha + s_\alpha \right]_q.
\]

Now let $\alpha = 1$. In this case the binomial coefficient depends on $N$. We want to know the limit of this expression while $N \to \infty$.

\[
\left[ N + 1 - s_1 - 2\sum_{\beta=2}^{k+1} s_\beta - v_1 \right]_q = \\
\left[ \frac{N+l+1}{2} + \sum_{\beta=2}^{k+1} (\beta - 2)s_\beta - v_1 \right]_q \rightarrow \frac{1}{(2\sum_{\beta=2}^{k+2} (\beta - 1)s_\beta + l - j - v_1)_q!}.
\]

We obtain the following proposition.

**Proposition 5.1.** Fix $0 \leq j \leq k$, $0 \leq l \leq k + 1$. Then

\[
\text{ch} M_{j+1,l+1}(k+2, k+3) = q^{\Delta_{j+1,l+1}} \sum_{t=(t_2, \ldots, t_{k+1}) \in \mathbb{Z}_{\geq 0}^k} q^{tB_t + ut} \\
\prod_{\alpha=2}^{k+1} \left[ 2\sum_{\beta=2}^{k+1} (\beta - \min(\alpha, \beta))t_\beta + l - j - 1 + \min(\alpha, j + 1) - v_\alpha + t_\alpha \right]_q \\
(2\sum_{\beta=2}^{k+2} (\beta - 1)t_\beta + l - j - v_1)_q!,
\]

where $B_{\alpha, \beta} = \max(\alpha, \beta)(\min(\alpha, \beta) - 1)$, $u_\alpha = (j + 1 - l)(\alpha - 1) + \max(0, \alpha - j - 1)$.

**Proof.** In view of a formulas (44), (45) and (46) we only need to find the power of $q$ in front of the sum. Note that $p(1^3(j + 1)) = j + i$, where $i = 0, 1$ and $i + N + 1$ is even. Now it is enough to mention that

\[
\frac{i(i + 1)}{12} + \frac{j(j + 2)}{4(k + 2)} - \frac{l(l + 2)}{4(k + 3)} - \frac{i + j}{4} + \frac{(l - j)^2 + j}{4} = \\
\frac{((k + 3)j - (k + 2)(l + 1) - 1}{4(k + 2)(k + 3)} = \Delta_{j+1,l+1}(k+2, k+3).
\]

(The first three terms come from the action of the Sugawara operators, the fourth one from $p(m)$ and the last from the formula (44)).

---

**Appendix A. The ABF finitization and Kostka polynomials**

In this appendix we study the connection between ABF finitization of the minimal Virasoro unitary characters (see [ABF]) and Kostka polynomials $K^{(k)}_{\tau, (1^\infty(j+1))}$, generalizing the case $j = 0$ in [JMT].
We first recall the ABF finitization. Fix some $a, b \in \mathbb{Z}$. For $N \geq 0$ such that $N \equiv b - a \mod 2$, define a polynomial

$$
\hat{\chi}_{b,a}(r,r+1) \bigl( q; N \bigr) = \sum_{n \in \mathbb{Z}} q^{r(r+1)n^2+(r+1)b+ra} \left[ \frac{N}{2} - (r+1)n \right]_q
$$

In view of the Rocha-Caridi formula it is obvious that

$$
\lim_{N \to \infty} q^{\Delta_{b,a}} \hat{\chi}_{b,a}(r,r+1) \bigl( q; N \bigr) = \text{ch}_{M_{b,a}}(r,r+1).
$$

Lemma A.1.

$$
q^{\frac{(a-b)^2}{4}} \hat{\chi}_{b,a}(r,r+1) \bigl( q^{-1}; N \bigr) = \sum_{n \in \mathbb{Z}} q^{r(r+1)n^2-na} \left[ \frac{N}{2} - (r+1)n \right]_q
$$

Proof. We just use the identity

$$
\left[ \frac{N}{k} \right]_{q^{-1}} = q^{-k(N-k)} \left[ \frac{N}{k} \right]_q.
$$

Proposition A.1.

$$
q^{(l-j)^2} K_{l,(1^N(j+1))}^{(k)}(q) = 
\sum_{s=0}^{\frac{(N+1)l}{2}} q^{\frac{(k+1, k+2)}{4}} \chi_{j+1-2s,l+1}(q^{-1}; N+1) - q^{\frac{N^2}{4}} \sum_{s=0}^{j+1} \chi_{j-2s,l+1}(q^{-1}; N).
$$

Proof. In view of Corollary 3.1 and the alternating sum formula we have

$$
K_{l,m}^{(k)}(q) = 
\sum_{p \geq 0} q^{(k+2)p^2+(l+1)p} K_{2(k+2)p+l,m}^{(k+1)}(q) - \sum_{p > 0} q^{(k+2)p^2-(l+1)p} K_{2(k+2)p-l-2,m}^{(k+1)}(q) = 
\sum_{p \in \mathbb{Z}} q^{(k+2)p^2+(l+1)p} (\text{ch}_q V_{m}^{2(k+2)p+l} - \text{ch}_q V_{m}^{2(k+2)p+l+2}).
$$

Now let $m = (1^N(j+1))$. We show by induction on $j$ that

$$
\text{ch}_q V_{(1^N(j+1))}^l = \left[ \frac{N+1}{2} \right]_q + \sum_{s=0}^{j-1} q^{\frac{N+j+1-l-2s}{2}} \left[ \frac{N}{2} \right]_q.
$$
For \( j = 0 \) we have \( \text{ch}_q V_{(1^N+1)}^l = \left[ \frac{N+1}{N+1-l} \right]_q \). For the induction procedure we use \( \text{(14)} \). This gives

\[
\text{ch}_q V_{(1^N(j+1))}^l = \text{ch}_q V_{(1^N j)}^l + q^{N+1-j}\text{ch}_q V_{(1^N)}^{l-j-1} = 
\left[ \frac{N+1}{N-j+1-l} \right]_q + \sum_{s=0}^{j-1} q^{N+1-j-2s} \left[ \frac{N}{N+j+1-l-2s} \right]_q.
\]

Using the identity

\[
q^b \left[ \begin{array}{c} a \\ b \end{array} \right]_q + \left[ \begin{array}{c} a \\ b-1 \end{array} \right]_q = \left[ \begin{array}{c} a+1 \\ b \end{array} \right]_q
\]

we rewrite \( \text{(14)} \) in the following way

\[
\text{ch}_q V_{(1^N(j+1))}^l = \sum_{s=0}^{j} \left[ \frac{N+1}{N+j+1-l-2s} \right]_q - \sum_{s=0}^{j-1} \left[ \frac{N}{N+j-1-l-2s} \right]_q.
\]

Therefore, we obtain

\[
K_{l,(1^N(j+1))}^{(k)}(q) = \sum_{p \in \mathbb{Z}} q^{(k+2)p^2+(l+1)p} \times
\sum_{s=0}^{j} \left( \left[ \frac{N+1}{N+j+1-l-2s} - (k+2)p \right]_q \right) - \sum_{p \in \mathbb{Z}} q^{(k+2)p^2+(l+1)p} \times
\sum_{s=0}^{j-1} \left( \left[ \frac{N}{N+j-1-l-2s} - (k+2)p \right]_q \right).
\]

We show that

\[
\text{(48)} \quad \sum_{p \in \mathbb{Z}} q^{(k+2)p^2+(l+1)p} \left( \left[ \frac{N+1}{N+j+1-l-2s} - (k+2)p \right]_q \right) = q^{(N+1)^2-(l+1)^2} X_{j+1-2s,l+1}^{(k+1,k+2)}(q^{-1}; N+1).
\]

We rewrite the left-hand side of \( \text{(48)} \) as

\[
\sum_{p \in \mathbb{Z}} q^{(k+2)p^2-(l+1)p} \left[ \frac{N+1}{N+j+1-l-2s} + (k+2)p \right]_q - \sum_{p \in \mathbb{Z}} q^{(k+2)p^2+(l+1)p} \left[ \frac{N+1}{N+j+1-l-2s} + (k+2)p \right]_q =
\sum_{p \in \mathbb{Z}} q^{(k+2)p^2-(l+1)p} \left[ \frac{N+1}{N+j+1-l-2s} + (k+2)p \right]_q - \sum_{p \in \mathbb{Z}} q^{(k+2)p^2+(l+1)p} \left[ \frac{N+1}{N+j+1-l-2s} + (k+2)p \right]_q,
\]
which coincides with the right-hand side of (45). In the same way one can prove that

\[
\sum_{p \in \mathbb{Z}} q^{(k+2)p^2 + (l+1)p} \left[ \frac{N}{2} - (k + 2)p \right]_q = q^{N^2 - (l-j)^2} \lambda_{j-2s,l+1}^{(k+1,k+2)} \left( q^{-1}; N \right).
\]

Proposition is proved. \(\square\)

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