Connection between Measurement Disturbance Relation and Multipartite Quantum Correlation

Jun-Li Li\textsuperscript{1}, Kun Du\textsuperscript{1} and Cong-Feng Qiao\textsuperscript{1, 2}\textsuperscript{*}

\textsuperscript{1}Department of Physics, University of the Chinese Academy of Sciences,
YuQuan Road 19A, Beijing 100049, China
\textsuperscript{2}CAS Center for Excellence in Particle Physics, Beijing 100049, China

Abstract

It is found that the measurement disturbance relation (MDR) determines the strength of quantum correlation and hence is one of the essential facets of the nature of quantum nonlocality. In reverse, the exact form of MDR may be ascertained through measuring the correlation function. To this aim, an optical experimental scheme is proposed. Moreover, by virtue of the correlation function, we find that the quantum entanglement, the quantum non-locality, and the uncertainty principle can be explicitly correlated.

1 Introduction

Quantum nonlocality and Heisenberg’s uncertainty principle \cite{1} are two essential concepts in quantum mechanics (QM). The nonclassical information shared among different parts forms the basis of quantum information and is responsible for many counterintuitive features of QM, e.g., quantum cryptography \cite{2} and quantum teleportation \cite{3}. From information theory people have put forward certain principles to specify the quantum correlation, including nontrivial communication complexity \cite{4}, information causality \cite{5}, entropic uncertainty relations \cite{6}, local orthogonality \cite{7}, and global exclusivity \cite{8, 9}.

\textsuperscript{*}corresponding author; qiaocf@ucas.ac.cn
Note these principles stem from the notion of information; they mainly concern the bi-
partite correlation. It has been shown that understanding multipartite intrinsic structure
is indispensable to the determination of quantum correlation, and it is rather difficult to
derive the Hilbert space structure from information quantities alone [10].

Heisenberg’s uncertainty principle has a deep impact on quantum measurement, and it
reflects the mutual influence of measurement precision and disturbance on a quantum sys-
tem only for the measurement disturbance relation (MDR). The well-known Heisenberg-
Robertson uncertainty relation reads [11]

\[ \Delta A \Delta B \geq |\langle C \rangle| \] (1)

with \( C = \frac{1}{i} [A, B] \) and the standard deviation \( \Delta X = \sqrt{\langle \psi | X^2 | \psi \rangle - \langle \psi | X | \psi \rangle^2} \). Note that
in (1), only the properties of two observables in the ensemble of a quantum state are
involved, and the relation is independent of any specific measurement. The MDR has
been intensively studied both theoretically [12–16] and experimentally [17–20]. However,
the implication of the MDR for quantum information and quantum measurement is still
unclear [21–23]. In practice, there are different forms of MDR, most of which have under-
gone experimental checks and still survive [19, 20]. Therefore, determining the impacts of
various MDRs on quantum physics, or ascertaining the right form of MDR, is currently
an urgent task.

In this work, we propose a scheme for transforming any MDR to some constraint
inequalities of multipartite correlation functions. In this way, the attainable strength of
correlations in multipartite state may be considered to be the physical consequence of the
restriction on the quantum measurement imposed by the MDR. The structure of the paper
is as follows. In Sec. 2, typical versions of the MDR are presented, and their essential
differences are illustrated by embedding the MDR into a coordinate system. In Sec. 3, we
transform the various MDRs to constraint inequalities on bipartite correlation functions in a tripartite state with the help of the nonfactorable state. Then it is shown that the constraint inequalities must be held for all the tripartite and multipartite entangled states. Detailed examples and an experimental setup for the verification of the various MDRs based on our scheme are given for a three-qubit system. The concluding remarks are given in Sec. 4.

2 The MDR in QM

2.1 The quantum measurement and its disturbance

A quantum measurement process may be generally implemented by coupling a meter system $|\phi\rangle$ with the original system $|\psi\rangle$. The measurement result $M$ is obtained from the readout of the meter system. As the physical observables are represented by Hermitian operators in QM, following the definition in [12], the measurement precision of physical quantity $A$ and the corresponding disturbance of quantity $B$ are defined as expectation values of the mean squares:

$$
\epsilon(A)^2 \equiv \langle \phi | \langle \psi | [A - A_1 \otimes I_2]^2 | \psi \rangle | \phi \rangle , 
$$

$$
\eta(B)^2 \equiv \langle \phi | \langle \psi | [B - B_1 \otimes I_2]^2 | \psi \rangle | \phi \rangle .
$$

Here $A = U^\dagger (I_1 \otimes M_2)U$, $B = U^\dagger (B_1 \otimes I_2)U$, the subscripts 1 and 2 signify that the operators are acting on states $|\psi\rangle$ and $|\phi\rangle$, respectively, $U$ is a unitary interaction between $|\psi\rangle$ and $|\phi\rangle$, and $I$ is the identity operator. The measurement operator $M$ may be set to $A$ if it has the same possible outcomes as operator $A$, i.e., $A = U^\dagger (I_1 \otimes A_2)U$ [15, 16]. Note that in addition to this operator formalism, which we will work with, there are also other types of definitions for the measurement precision and disturbance, e.g., the probability distribution formalism [21].
The MDR indicates that there is a fundamental restriction on the measurement precision $\epsilon(A)$ and the reaction (disturbance) $\eta(B)$ when two incompatible physical observables $A$ and $B$ are about to be measured. By the definitions of (2) and (3), typical MDR representatives are as follows:

\[ \epsilon(A)\eta(B) \geq |\langle C \rangle|, \]
\[ \epsilon(A)\eta(B) + \epsilon(A)\Delta B + \eta(B)\Delta A \geq |\langle C \rangle|, \]
\[ \epsilon(A)\eta(B) + \epsilon(A)\Delta B + \eta(B)\Delta A \geq |\langle C \rangle|, \]
\[ \epsilon(A)(\Delta B + \Delta B) + \eta(B)(\Delta A + \Delta A) \geq 2|\langle C \rangle|, \]
\[ \Delta B^2\epsilon(A)^2 + \Delta A^2\eta(B)^2 + 2\epsilon(A)\eta(B)\sqrt{\Delta A^2\Delta B^2 - \langle C \rangle^2} \geq \langle C \rangle^2. \]

Here $C = [A, B]/2i$, and $\Delta X$ are the standard deviations of operators $X = A, B, A, B$ evaluated in the quantum state $|\psi\rangle$. Equations (4)-(8) correspond to Heisenberg-type (He), Ozawa’s (Oz) [12], Hall’s (Ha) [13], Weston et al.’s (We) [14], and Branciard’s (B1) [15] MDRs, respectively. Equation (8) can be refined, in the specific qubit case, as

\[ \epsilon(A)^2[1 - \epsilon(A)^2/4] + \eta(B)^2[1 - \eta(B)^2/4] \]
\[ + 2\epsilon(A)\eta(B)\sqrt{1 - \langle C \rangle^2}\sqrt{1 - \eta(B)^2/4}\sqrt{1 - \epsilon(A)^2/4} \geq \langle C \rangle^2. \]

[Equation (9) is abbreviated as B2 bellow.] So far, the Heisenberg-type MDR has been found to be violated, while others have undergone various sorts of trials in experiment and still survive [19, 20]. Finding a stricter constraint on $\epsilon(A)$ and $\eta(B)$ is currently a hot topic in physics [24]. Besides focusing on the natures of different MDRs, it is also important to know what different physical consequences they would have on quantum information science.
Fig. 1. Illustration of different MDRs for the same kinds of quantum states with identical ensemble properties of $\Delta A$, $\Delta B$, $\langle C \rangle$. The allowed values of precision $\epsilon(A)$ and disturbance $\eta(B)$ fill the shaded areas, which correspond to (a) Heisenberg-type, (b) Ozawa’s, and (c) Branciard’s MDRs. The essential differences among those MDRs lie in the forbidden areas for $\epsilon(A)$ and $\eta(B)$ which are characterized by the minimal distances to the origin, $r_{\text{He}}$, $r_{\text{Oz}}$, and $r_{\text{B1}}$, with subscripts stand for the corresponding MDRs.

2.2 The essential difference among various MDRs

It is obvious that the above representative MDRs differ in tightness. Here we propose a method for quantitative study of MDRs. We first transform MDRs into coordinate space and then express them as relation functions of $\epsilon(A)$ and $\eta(B)$. For the sake of convenience and without loss of generality, we take three typical MDRs, the Heisenberg type Eq. (4), Ozawa’s Eq. (5), and Branciard’s Eq. (8), as examples.

In the Heisenberg-type MDR (4), the measurement-dependent $[\epsilon(A), \eta(B)]$ and measurement-independent ($\langle C \rangle$) quantities are on different sides of the inequality. The allowed region (AR) for $\epsilon(A)$ and $\eta(B)$ is above the hyperbolic curve $\epsilon(A)\eta(B) = |\langle C \rangle|$ in quadrant I [see Fig. 1(a)]. The forbidden region for the values of $\epsilon(A)$ and $\eta(B)$ is enclosed by the curve and two axes, which may be characterized by the radius of the circle centered at the origin and tangent with the hyperbola. This radius represents the minimal distance from the AR to the origin of the coordinates, $\epsilon(A)^2 + \eta(B)^2 \geq r_{\text{He}}^2$, where, for the Heisenberg-type MDR, $r_{\text{He}}^2 = f_{\text{He}}(|\langle C \rangle|) = 2|\langle C \rangle|$. 

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For inequality (5), substituting \( \epsilon(A) = \epsilon'(A) - \Delta A \) and \( \eta(B) = \eta'(B) - \Delta B \), we have

\[
\epsilon'(A)\eta'(B) \geq \Delta A \Delta B + |\langle C \rangle|.
\] (10)

This is a displaced hyperbola of the Heisenberg type [see Fig. 1(b)]. The AR for \( \epsilon(A) \) and \( \eta(B) \) may be obtained from (10), and its minimal distance to the origin can also be expressed as

\[
\epsilon(A)^2 + \eta(B)^2 \geq r_{Oz}^2 = f_{Oz}(\Delta A, \Delta B, \langle C \rangle).
\] (11)

Inequality (8) can be reformulated as

\[
(\epsilon(A) \quad \eta(B)) \left( \begin{array}{c}
\frac{\Delta B^2}{\sqrt{\Delta A^2 \Delta B^2 - \langle C \rangle^2}} \\
\frac{\sqrt{\Delta A^2 \Delta B^2 - \langle C \rangle^2}}{\Delta A^2}
\end{array} \right) \left( \begin{array}{c}
\epsilon(A) \\
\eta(B)
\end{array} \right) \geq \langle C \rangle^2.
\] (12)

Different from Heisenberg-type and Ozawa’s MDRs, (12) is an ellipse of \( \epsilon(A) \) and \( \eta(B) \) centered at the origin. Similar to (11), in this case the values of \( \epsilon(A) \) and \( \eta(B) \) in the AR satisfy

\[
\epsilon(A)^2 + \eta(B)^2 \geq r_{B1}^2 = f_{B1}(\Delta A, \Delta B, \langle C \rangle),
\] (13)

where \( r_{B1} \) is the minor axis of the ellipse with regard to the parametric condition \( \Delta A \Delta B \geq |\langle C \rangle| \). For the convenience of comparison, the MDRs in Eqs. (4), (5), (8) are shown in Fig. 1 with the same values of \( \Delta A, \Delta B, \) and \( \langle C \rangle \). The values of \( \epsilon(A) \) and \( \eta(B) \) in the AR fill up the shaded areas, and the unshaded parts are then the forbidden regions.

To summarize, all of the MDRs (including those yet to be discovered) have the shortest distance \( r_q \) from their AR to the origin as a function of \( \Delta A, \Delta B, \) and \( \langle C \rangle \):

\[
\epsilon(A)^2 + \eta(B)^2 \geq r_q^2 = f_q(\Delta A, \Delta B, \langle C \rangle).
\] (14)

Here \( f_q \) relies only on the ensemble properties of a quantum state, i.e., \( \Delta A, \Delta B, \) and \( \langle C \rangle \), which are independent of measurement processes (expressions of \( f_q \) for typical MDRs are presented in Appendix A). Thus, the shortest distance \( r_q \) from the AR to the origin is independent of the measurement process and represents the essence of each MDR.
3 The constraint of MDR on quantum correlation

3.1 A nonfactorable bipartite quantum state

Although variant MDRs may be distinguished by $r_q$, the physical consequences of different $r_q$ in quantum information theory are far from obvious. To this end, in this work we present a scheme to examine MDR. For two Hermite operators $A$ and $B$ with $[A, B] = 2iC$, we may construct a nonfactorable bipartite state $|\psi_{12}\rangle$ satisfying

$$A_1 \otimes I_2 |\psi_{12}\rangle = I_1 \otimes A'_2 |\psi_{12}\rangle, \quad B_1 \otimes I_2 |\psi_{12}\rangle = I_1 \otimes B'_2 |\psi_{12}\rangle,$$

(15)

where $A' = UAU^\dagger$ and $B' = VBV^\dagger$ are unitary transformations of $A$ and $B$, respectively, and hence have the same eigenvalues. The subscripts 1 and 2 indicate the corresponding particles being acted on. We shall show that for Hermitian operators $A$ and $B$, there always exists such a nonfactorable state $|\psi_{12}\rangle$.

The Hermitian operators $A$ and $B$ may be expressed in spectrum decomposition as

$$A = \sum_{i=1}^{N} \alpha_i |\alpha_i\rangle\langle \alpha_i|, \quad B = \sum_{i=1}^{N} \beta_i |\beta_i\rangle\langle \beta_i|.$$

(16)

There is a unitary transformation matrix $W$ between the two orthogonal bases, $|\beta_i\rangle = \sum_{\mu=1}^{N} |\alpha_\mu\rangle w_{\mu i}$, where $w_{\mu i}$ are the matrix elements of $W$. The following proposition holds.

**Proposition 1** If the unitary transformation matrices $U$ and $V$ are congruence equivalent, that is $U = WVW^T$, then $|\psi_{12}\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |\alpha_i\rangle|\alpha'_i\rangle$ satisfies

$$A_1 \otimes I_2 |\psi_{12}\rangle = I_1 \otimes A'_2 |\psi_{12}\rangle, \quad B_1 \otimes I_2 |\psi_{12}\rangle = I_1 \otimes B'_2 |\psi_{12}\rangle.$$

(17)

Here $A'|\alpha'_i\rangle = \alpha_i |\alpha'_i\rangle$, and the subscripts of the operators stand for the particles they act on.
Proof: Given that the bipartite state $|\psi_{12}\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |\alpha_i\rangle|\alpha'_i\rangle$ satisfies the first equality of Eq. (17), we need to prove that the second equality of Eq. (17) is also satisfied. The state $|\psi_{12}\rangle$ may be expressed in the basis of $|\beta_i\rangle$ and $|\beta'_i\rangle$ as

$$|\psi_{12}\rangle = \sum_{i,j=1}^{N} \gamma_{ij}^{(b)} |\beta_i\rangle|\beta'_j\rangle,$$

where $|\beta_i\rangle$, $|\beta'_i\rangle$ are the eigenvectors of $B$, $B'$ with the same eigenvalue $\beta_i$, $\gamma_{ij}^{(b)} \in \mathbb{C}$. We have

$$|\alpha_i\rangle = \sum_j |\alpha'_{ij}\rangle u_{ji}^\dagger,$$

$$|\beta_i\rangle = \sum_j |\alpha_{ij}\rangle w_{ji},$$

$$|\beta'_i\rangle = \sum_j |\beta_{ji}\rangle v_{ji},$$

or, more succinctly,

$$|\beta'_i\rangle = \sum_{j,k,\nu} u_{\nu kj}^\dagger w_{ji} v_{\mu j} \gamma_{ij}^{(b)} |\alpha'_{\mu\nu}\rangle$$

with $v_{ji}$, and $u_{\nu kj}^\dagger$ being matrix elements of $V$ and $U^\dagger$. Therefore, $|\psi_{12}\rangle$ may also be expressed as

$$|\psi_{12}\rangle = \sum_{i,l} \gamma_{il}^{(a)} |\beta_i\rangle|\beta'_l\rangle = \sum_{i,j,k,l,\mu,\nu} w_{\mu i} u_{\nu k}^\dagger w_{kj} v_{ji} \gamma_{il}^{(b)} |\alpha_{\mu\nu}\rangle |\alpha'_{\mu\nu}\rangle,$$

where

$$\sum_{i,j,k,l} w_{\mu i} u_{\nu k}^\dagger w_{kj} v_{ji} \gamma_{il}^{(b)} = U^\dagger W V T^{(b)^T} W^T = \Gamma^{(a)^T}.$$

Here $\gamma_{ij}^{(a)}$, $\gamma_{ij}^{(b)}$ are the matrix elements of $\Gamma^{(a)}$, $\Gamma^{(b)}$ and the superscript T is the transpose of a matrix. Because $|\psi_{12}\rangle = \frac{1}{\sqrt{N}} \sum_i |\alpha_i\rangle|\alpha'_i\rangle$, we have $\gamma_{ij}^{(a)} = \delta_{ij} / \sqrt{N}$, and

$$\Gamma^{(b)^T} = V^\dagger W^\dagger U T^{(a)^T} W^* = \frac{1}{\sqrt{N}} V^\dagger W^\dagger U W^*$$

$$= \frac{1}{\sqrt{N}} V^\dagger W^\dagger W V W^T W^* = \frac{1}{\sqrt{N}},$$

where the congruence relation $U = W V W^T$ is employed. That is,

$$|\psi_{12}\rangle = \frac{1}{\sqrt{N}} \sum_i |\alpha_i\rangle|\alpha'_i\rangle = \frac{1}{\sqrt{N}} \sum_i |\beta_i\rangle|\beta'_i\rangle,$$
and therefore,

\[ A_1 \otimes I_2 |\psi_{12}\rangle = I_1 \otimes A'_2 |\psi_{12}\rangle, \quad B_1 \otimes I_2 |\psi_{12}\rangle = I_1 \otimes B'_2 |\psi_{12}\rangle. \tag{24} \]

Q.E.D.

Proposition 1 indicates that the state \( |\psi_{12}\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |\alpha_i\rangle |\alpha'_i\rangle \) satisfies both equalities of Eq. (15) while the transformation matrices \( U \) and \( V \) satisfy \( U = WVW^T \).

### 3.2 The constraint of MDR on quantum correlation

With the nonfactorable bipartite quantum state in Eq. (15), we have the following theorem.

**Theorem 2** A set of tripartite states may be obtained by an interaction \( U_{13} \) of particle 1 in the state \( |\psi_{12}\rangle \) with a third particle (particle 3), \( \Psi = \{|\psi_{123}\rangle |\psi_{123}\rangle = U_{13}|\psi_{12}\rangle |\phi_3\rangle, U_{13}^\dagger U_{13} = I\} \). The various MDRs imply the following relationship for states \( |\psi_{123}\rangle \in \Psi \):

\[ E(A'_2, A_3) + E(B'_2, B_1) \leq \frac{1}{2}(\langle A'_2^2 \rangle + \langle A_3^2 \rangle + \langle B'_2^2 \rangle + \langle B_1^2 \rangle - \gamma_q). \tag{25} \]

Here \( E(X_i, Y_j) = \langle X_i \otimes Y_j \rangle \) is the bipartite correlation function with \( X, Y \in \{A, A', B, B'\} \), and \( \gamma_q = \text{Max}\{\sum_i |2\langle p_i |\psi_{12}\rangle|^2 f_q^{(i)}\} \) is independent of \( U_{13} \) where \( |p_i\rangle \) is an arbitrary set of orthogonal bases; \( f_q^{(i)} \) represents the function \( f_q \) evaluated under quantum states \( |\psi^{(i)}_1\rangle = 2\langle p_i |\psi_{12}\rangle / \sqrt{2\langle p_i |\psi_{12}\rangle} \).

**Proof:** A set of quantum states \( |\psi^{(i)}_1\rangle \) of particle 1 may be prepared by projecting particle 2 in the bipartite entangled state \( |\psi_{12}\rangle \) proposed in Proposition 1 with a set of complete and orthogonal bases \( |p_i\rangle \):

\[ |\psi^{(i)}_1\rangle = \frac{2\langle p_i |\psi_{12}\rangle}{\sqrt{2\langle p_i |\psi_{12}\rangle}}. \tag{26} \]
Substituting \( |\psi_{1}^{(i)}\rangle \) in the definition of measurement precision and disturbance [i.e., Eqs. (2) and (3)], we have

\[
\epsilon^{(i)}(A)^2 = \langle \phi_{3} | [A - A_{1} \otimes I_{3}]^{2} |\psi_{1}^{(i)}\rangle |\phi_{3}\rangle , \tag{27}
\]

\[
\eta^{(i)}(B)^2 = \langle \phi_{3} | [B - B_{1} \otimes I_{3}]^{2} |\psi_{1}^{(i)}\rangle |\phi_{3}\rangle . \tag{28}
\]

Here \( |\phi_{3}\rangle \) describes the meter system. Further, we may write

\[
|2\langle p_{i} |\psi_{12}\rangle|^{2}\epsilon^{(i)}(A)^2 = \langle \phi_{3} | [U_{13}^{\dagger}(I_{1} \otimes I_{2} \otimes A_{3})U_{13} - A_{1} \otimes I_{2} \otimes I_{3}]^{2} |\psi_{12}\rangle |\phi_{3}\rangle , \tag{29}
\]

\[
|2\langle p_{i} |\psi_{12}\rangle|^{2}\eta^{(i)}(B)^2 = \langle \phi_{3} | [U_{13}^{\dagger}(B_{1} \otimes I_{2} \otimes I_{3})U_{13} - B_{1} \otimes I_{2} \otimes I_{3}]^{2} |\psi_{12}\rangle |\phi_{3}\rangle . \tag{30}
\]

where \( P_{2}^{(i)} = |p_{i}\rangle_{2} \langle p_{i}| \) is a projecting operator acting on particle 2. Using the complete relation \( \sum_{i} |p_{i}\rangle \langle p_{i}| = 1 \),

\[
\sum_{i} |2\langle p_{i} |\psi_{12}\rangle|^{2}\epsilon^{(i)}(A)^2 = \langle \phi_{3} | [U_{13}^{\dagger}(I_{1} \otimes I_{2} \otimes A_{3})U_{13} - A_{1} \otimes I_{2} \otimes I_{3}]^{2} |\psi_{12}\rangle |\phi_{3}\rangle , \tag{31}
\]

\[
\sum_{i} |2\langle p_{i} |\psi_{12}\rangle|^{2}\eta^{(i)}(B)^2 = \langle \phi_{3} | [U_{13}^{\dagger}(B_{1} \otimes I_{2} \otimes I_{3})U_{13} - B_{1} \otimes I_{2} \otimes I_{3}]^{2} |\psi_{12}\rangle |\phi_{3}\rangle . \tag{32}
\]

According to Proposition \( \mathbb{I} \)

\[
\sum_{i} |2\langle p_{i} |\psi_{12}\rangle|^{2}\epsilon^{(i)}(A)^2 = \langle \phi_{3} | [U_{13}^{\dagger}(I_{1} \otimes I_{2} \otimes A_{3})U_{13} - I_{1} \otimes A_{2}' \otimes I_{3}]^{2} |\psi_{12}\rangle |\phi_{3}\rangle , \tag{33}
\]

\[
\sum_{i} |2\langle p_{i} |\psi_{12}\rangle|^{2}\eta^{(i)}(B)^2 = \langle \phi_{3} | [U_{13}^{\dagger}(B_{1} \otimes I_{2} \otimes I_{3})U_{13} - I_{1} \otimes B_{2}' \otimes I_{3}]^{2} |\psi_{12}\rangle |\phi_{3}\rangle , \tag{34}
\]

which give (note the interaction \( U_{13} \) commutes with the operators of particle 2)

\[
\sum_{i} |2\langle p_{i} |\psi_{12}\rangle|^{2}\epsilon^{(i)}(A)^2 = \langle \psi_{123} | (A_{3} - A_{2}')^{2} |\psi_{123}\rangle , \tag{33}
\]

\[
\sum_{i} |2\langle p_{i} |\psi_{12}\rangle|^{2}\eta^{(i)}(B)^2 = \langle \psi_{123} | (B_{1} - B_{2}')^{2} |\psi_{123}\rangle . \tag{34}
\]

Here \( |\psi_{123}\rangle = U_{13}|\psi_{12}\rangle |\phi_{3}\rangle . \)

For each quantum state \( |\psi_{1}^{(i)}\rangle \), every MDR has its own AR region, and its minimal distance to the origin is

\[
\epsilon^{(i)}(A)^2 + \eta^{(i)}(B)^2 \geq f_{q}^{(i)}(\Delta A, \Delta B, \langle C \rangle) , \tag{35}
\]

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where the superscript of $f_q^{(i)}(\Delta A, \Delta B, \langle C \rangle)$ specifies that the argument of the function is evaluated under the quantum state $|\psi^{(i)}_1\rangle$ and $q$ stands for He, Oz, B1, etc. The sum of Eq. (33) and Eq. (34) gives

$$\sum_i |2\langle p_i|\psi_{12}\rangle|^2 [\epsilon^{(i)}(A)^2 + \eta^{(i)}(B)^2] = \langle \psi_{123}|(A_3 - A'_2)^2 + (B_1 - B'_2)^2|\psi_{123}\rangle. \quad (36)$$

Applying (35) to (36),

$$\langle \psi_{123}|(A_3 - A'_2)^2 + (B_1 - B'_2)^2|\psi_{123}\rangle = \sum_i |2\langle p_i|\psi_{12}\rangle|^2 [\epsilon^{(i)}(A)^2 + \eta^{(i)}(B)^2] \geq F_q(|\psi^{(i)}_1\rangle, \Delta A, \Delta B, \langle C \rangle). \quad (37)$$

Here $F_q(|\psi^{(i)}_1\rangle, \Delta A, \Delta B, \langle C \rangle) \equiv \sum_i |2\langle p_i|\psi_{12}\rangle|^2 f_q^{(i)}(\Delta A, \Delta B, \langle C \rangle)$. Note that the establishment of inequality (37) depends on Eq. (35) but not on the choice of projection bases $|p_i\rangle$. Therefore the inequality (37) should also hold as we optimize the bases $|p_i\rangle$ to get a maximum value of $F_q$, that is,

$$\langle \psi_{123}|(A_3 - A'_2)^2 + (B_1 - B'_2)^2|\psi_{123}\rangle \geq \gamma_q, \quad (38)$$

with $\gamma_q = \text{Max}\{\sum_i |2\langle p_i|\psi_{12}\rangle|^2 f_q^{(i)}\}$. Expanding the quadratic terms, Eq. (38) turns into the constraint on correlation $E(A_3, A'_2) + E(B_1, B'_2)$,

$$E(A_3, A'_2) + E(B_1, B'_2) \leq \frac{1}{2}(\langle A_3^2 \rangle + \langle A'_2^2 \rangle + \langle B_1^2 \rangle + \langle B'_2^2 \rangle - \gamma_q), \quad (39)$$

with $E(X_i, Y_j) = \langle \psi_{123}|X_i Y_j|\psi_{123}\rangle$ and $X, Y \in \{A, A', B, B'\}$. Q.E.D.

The theorem may be summarized as follows: (1) When a pair of incompatible observables $A, B$ is given, the bipartite entangled state $|\psi_{12}\rangle$ exists, and one particle of this state may interact with a third particle $|\phi_3\rangle$ by $U_{13}$. (2) Different forms of MDR may give different constraints on the bipartite correlations that can be shared with a third particle. (3) Each constraint inequality, characterized by $\gamma_q$, is independent of the interaction.
U_{13}, and is satisfied by all the quantum states in the set \( \Psi \). The constraint in the form of Eq. (25) is satisfied by all tripartite pure systems; for details see discussions on the universality of Theorem 2 presented in Appendix B.

The measurement process of Theorem 2 is illustrated in Fig. 2. According to Theorem 2, when one of the entangled particles interacts with a third particle, the maximal quantum correlation that may be shared with the third party is not determined by the interaction but by the MDR. From Eq. (25) it is clear that the larger the forbidden area of measurement precision and disturbance is, the less correlation the MDR predicts. The generalization of Theorem 2 to incorporate multipartite states may be realized by incorporating successive measurements with more meter systems. Other generalizations are also possible as the measurement processes might be implemented in different scenarios.

### 3.3 Physical consequences of MDR in a three-qubit system

Here we give a detailed example for three-qubit system to show that different MDRs indeed give different constraints on bipartite quantum correlations.

In a qubit system, two incompatible operators may be set to \( A = Z = \sigma_z \) and \( B =...\)
$X = \sigma_x$. The nonfactorable state can be generally constructed as

$$|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle), \quad (40)$$

where $\sigma_z|\pm\rangle = \pm|\pm\rangle$ and we have chosen $A' = A$, $B' = B$. It is easy to verify

$$A_1|\psi_{12}\rangle = A_2|\psi_{12}\rangle, \quad B_1|\psi_{12}\rangle = B_2|\psi_{12}\rangle. \quad (41)$$

Then let particle 1 interact with particle 3, $|\phi_3\rangle = \cos \theta_3 |+\rangle + \sin \theta_3 |-\rangle$, in arbitrary form. Suppose the interaction is a controlled NOT (CNOT) gate between particles 1 and 3,

$$|\psi_{123}\rangle = U_{\text{CNOT}} |\psi_{12}\rangle (\cos \theta_3 |+\rangle + \sin \theta_3 |-\rangle)
= \frac{1}{\sqrt{2}} ([|++\rangle (\cos \theta_3 |+\rangle + \sin \theta_3 |-\rangle) + ||--\rangle (\cos \theta_3 |-\rangle + \sin \theta_3 |+\rangle]) . \quad (42)$$

Here particle 1 is the control qubit, and particle 3 is the target qubit. According to Theorem 2 we have

$$E(A'_2, A_3) + E(B_1, B'_2) = E(Z_2, Z_3) + E(X_1, X_2)
\leq \frac{1}{2} (\langle Z_3^2 \rangle + \langle Z_2^2 \rangle + \langle X_1^2 \rangle + \langle X_2^2 \rangle - \gamma_q) = 2 - \frac{\gamma_q}{2}, \quad (43)$$

under the condition $Z^2 = X^2 = 1$ and with the real parameter

$$\gamma_q = \operatorname{Max} \left[ \sum_i | \langle p_i |\psi_{12}\rangle |^2 f_q^{(i)} \right] . \quad (44)$$

We may choose any set of complete and orthogonal bases $|p_i\rangle$ to test the MDR. Generally, we need to optimize the choice of the bases in order to obtain the maximum value of $\gamma_q$. For a qubit system, by choosing $|p_1\rangle = (|+\rangle + i|--\rangle)/\sqrt{2}$ and $|p_2\rangle = (|+\rangle - i|--\rangle)/\sqrt{2}$, which are the eigenvectors of $\sigma_y$ with eigenvalues of +1 and −1, respectively, we have

$$|\psi_1^{(1)}\rangle = \frac{2 \langle p_1 |\psi_{12}\rangle}{| 2 \langle p_1 |\psi_{12} \rangle |} = \frac{1}{\sqrt{2}} (|+\rangle - i|--\rangle), \quad (44)$$

$$|\psi_1^{(2)}\rangle = \frac{2 \langle p_2 |\psi_{12}\rangle}{| 2 \langle p_2 |\psi_{12} \rangle |} = \frac{1}{\sqrt{2}} (|+\rangle + i|--\rangle). \quad (45)$$
Because for quantum states $|\psi^{(i)}_1\rangle$, [Eqs. (44) and (45)] we have $\langle \sigma_{z,x} \rangle = 0$, $\Delta \sigma_z = \Delta \sigma_x = \sqrt{\langle \sigma_y \rangle}$ (see Appendix A), the functions $f^{(i)}_q$ in Eq. (35) reach the maximum. That is

$$f^{(1)}_q = \kappa_q |\langle \psi^{(1)}_1|C|\psi^{(1)}_1\rangle|, \quad f^{(2)}_q = \kappa_q |\langle \psi^{(2)}_1|C|\psi^{(2)}_1\rangle|,$$

where $\kappa_{He} = 2$, $\kappa_{B2} = (4 - 2\sqrt{2})$, $\kappa_{B1} = 1$, $\kappa_{We} = 0.59$, $\kappa_{Ha} = 2/5$, $\kappa_{Oz} = (2 - \sqrt{2})^2$, and $C = [A, B]/2i = \sigma_y$. Therefore,

$$\gamma_q = \left[ \sum_{i=1}^{2} |2\langle p_i|\psi_{12}\rangle|^2 \kappa_q \left| \langle \psi^{(i)}_1|C|\psi^{(i)}_1\rangle \right| \right] = \kappa_q .$$

Substituting $\gamma_q$ into Eq. (42), we have

- **He**: $E(Z_2, Z_3) + E(X_1, X_2) \leq 1$
- **B2**: $E(Z_2, Z_3) + E(X_1, X_2) \leq \sqrt{2}$
- **B1**: $E(Z_2, Z_3) + E(X_1, X_2) \leq \frac{3}{2}$
- **We**: $E(Z_2, Z_3) + E(X_1, X_2) \leq 1.71$
- **Ha**: $E(Z_2, Z_3) + E(X_1, X_2) \leq \frac{9}{5}$
- **Oz**: $E(Z_2, Z_3) + E(X_1, X_2) \leq 2\sqrt{2} - 1$

while the QM prediction is

$$E(Z_2, Z_3) + E(X_1, X_2) = \langle \psi_{123}|I_1 \otimes Z_2 \otimes Z_3|\psi_{123}\rangle + \langle \psi_{123}|X_1 \otimes X_2 \otimes I_3|\psi_{123}\rangle$$

$$= \cos(2\theta_3) + \sin(2\theta_3).$$

The constraints from MDR and QM on correlation functions are illustrated in Fig. 3(a).

The constraints on correlation functions tend to unveil a more intrinsic nature of the nonlocal system when we transform them into a Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [25]. Equation (42) gives the measurement precision of $Z$ and the disturbance on $X$ for the qubit system

$$E(Z_2, Z_3) + E(X_1, X_2) \leq 2 - \frac{\gamma_q}{2} .$$
Fig. 3. The supremum for correlation functions predicted by different MDRs. (a) The supremum imposed by different MDRs on the sum of bipartite correlation functions $E(X_1, X_2)$ of particles 1 and 2 and $E(Z_2, Z_3)$ of particle 2 and 3. (b) Constraints of the sum of CHSH Bell operators for particles 1 and 2 and 2 and 3 imposed by the MDRs. Here the abbreviations He, Oz, Ha, We, B1, and B2 indicate the MDRs of Eqs. (4)-(9), respectively. The QM prediction (black line) contradicts the constraint of the Heisenberg-type MDR.

Similarly, for the measurement precision of $X$ and the disturbance on $Z$, we have

$$E(X_2, X_3) + E(Z_1, Z_2) \leq 2 - \frac{\gamma_q}{2}. \quad (51)$$

Combining Eq. (50) with Eq. (51), we have

$$E(Z_2, Z_3) + E(X_1, X_2) + E(X_2, X_3) + E(Z_1, Z_2) \leq 4 - \gamma_q. \quad (52)$$

Based on the method introduced in Ref. [16], here we introduce two additional directions, $\vec{c} = \frac{1}{\sqrt{2}}(1, 0, 1), \vec{d} = \frac{1}{\sqrt{2}}(1, 0, -1)$, in the real space of the $z-x$ plane,

$$\vec{z} = \vec{a} = (0, 0, 1) = \frac{1}{\sqrt{2}}(\vec{c} - \vec{d}) , \quad \vec{x} = \vec{b} = (1, 0, 0) = \frac{1}{\sqrt{2}}(\vec{c} + \vec{d}). \quad (53)$$

Equation (52) can then be reexpressed as

$$E(\hat{a}_2, \hat{c}_3) - E(\hat{a}_2, \hat{d}_3) + E(\hat{b}_1, \hat{c}_2) + E(\hat{b}_1, \hat{d}_2) + E(\hat{b}_2, \hat{c}_3) + E(\hat{b}_2, \hat{d}_3) + E(\hat{a}_1, \hat{c}_2) - E(\hat{a}_1, \hat{d}_2) \leq \sqrt{2}(4 - \gamma_q), \quad (54)$$
where \( \hat{a} = \vec{\sigma} \cdot \vec{Z} = Z \), \( \hat{b} = \vec{\sigma} \cdot \vec{X} = X \), \( \hat{c} = \vec{\sigma} \cdot \vec{C} \), \( \hat{d} = \vec{\sigma} \cdot \vec{D} \). Through some rearrangement, Eq. (54) now turns into a more transparent form

\[
E(\hat{a}_2, \hat{c}_3) - E(\hat{a}_2, \hat{d}_3) + E(\hat{b}_2, \hat{c}_3) + E(\hat{b}_2, \hat{d}_3) + E(\hat{b}_1, \hat{c}_2) + E(\hat{b}_1, \hat{d}_2) + E(\hat{a}_1, \hat{c}_2) - E(\hat{a}_1, \hat{d}_2) \leq \sqrt{2}(4 - \gamma_q). \tag{55}
\]

This is just the constraint on two CHSH Bell operators for particles 1 and 2 and particles 2 and 3,

\[
B_{\text{CHSH}}^{(23)} + B_{\text{CHSH}}^{(12)} \leq 2\sqrt{2}(2 - \frac{\gamma_q}{2}). \tag{56}
\]

Therefore, for MDRs in the qubit system we have

- \( \text{He} : B_{\text{CHSH}}^{(23)} + B_{\text{CHSH}}^{(12)} \leq 2\sqrt{2} \),
- \( \text{B2} : B_{\text{CHSH}}^{(23)} + B_{\text{CHSH}}^{(12)} \leq 4 \),
- \( \text{B1} : B_{\text{CHSH}}^{(23)} + B_{\text{CHSH}}^{(12)} \leq 3\sqrt{2} \),
- \( \text{We} : B_{\text{CHSH}}^{(23)} + B_{\text{CHSH}}^{(12)} \leq 3.42\sqrt{2} \),
- \( \text{Ha} : B_{\text{CHSH}}^{(23)} + B_{\text{CHSH}}^{(12)} \leq \frac{18\sqrt{2}}{5} \),
- \( \text{Oz} : B_{\text{CHSH}}^{(23)} + B_{\text{CHSH}}^{(12)} \leq 8 - 2\sqrt{2} \),

while the QM prediction is \( \langle B_{\text{CHSH}}^{(23)} \rangle^2 + \langle B_{\text{CHSH}}^{(12)} \rangle^2 \leq 8 \). \tag{58}

The results from the MDR and QM prediction are shown in Fig. 3(b). Here we see that the second Branciard MDR [Eq. (9)] gives the same supremum as Eq. (58) on the sum of two Bell operators.

From Fig. 3(a) we notice that, while the supremum from the Heisenberg-type MDR is 1 in the given configuration, the QM prediction is \( \sqrt{2} \), the largest value for QM (see
Appendix C). We conclude that, for every MDR which can be expressed in the operator formalism, there will be a concrete constraint on the quantum correlations in the multipartite state. The MDR manifests itself as a principle determining the strength of the correlations, which may be shared with other particles through interaction. In the form of the CHSH inequality in Fig. 3(b), the MDR also provides a physical origin of the monogamy of the entanglement in multipartite entangled states. On the other hand, the exact form of the monogamy relation for entanglement may also be used in the reverse to obtain the exact form of MDR.

3.4 Experimental verification of the MDR

Except for the fundamental physical implications for multipartite correlations, the MDR’s unique constraint on the bipartite correlation function in the tripartite state makes the experimental test on MDR applicable in various physical systems, e.g., atoms, ions, and even higher energy particles, through measurement of correlation functions [16]. One schematic optical experimental setup for a qubit system is shown in Fig. 4. A pair of polarization-entangled photons, $|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)$, is generated.
by spontaneous parametric down conversion (SPDC). The meter system of the photon state may be tuned into the state $|\phi_3\rangle = \cos \theta_3 |H\rangle + \sin \theta_3 |V\rangle$ by wave plates (WP). Then it interacts with photon 1 via a CNOT operation resulting in the tripartite state $|\psi_{123}\rangle = \frac{1}{\sqrt{2}} ([|HH\rangle (\cos \theta_3 |H\rangle + \sin \theta_3 |V\rangle) + |VV\rangle (\cos \theta_3 |V\rangle + \sin \theta_3 |H\rangle)]$. We measure the correlation functions $E(Z_2, Z_3), E(X_1, X_2)$ under $|\psi_{123}\rangle$ where $\{|H\rangle, |V\rangle\}, \{|H\rangle \pm |V\rangle\}$ are eigen bases for $Z, X$. Taking the measured values in Eq. (48), the validity of the MDRs will be verified [Fig. 3(a)].

4 Conclusions

In this work, we have shown that the strength of correlation, which can be shared with a third particle through its interaction with one of the particles in the entangled bipartite system, is not determined by the interaction employed but by the fundamental measurement principle of QM. In this sense, the multipartite nonlocality may be considered to be the physical consequence of the uncertainty principle when quantum measurement is involved, and hence, the essential elements of the quantum theory, i.e., the quantum entanglement, quantum nonlocality, and uncertainty principle, are distinctly correlated in our scheme. This is heuristic and implies that these essential elements should and could be investigated jointly. For instance, the limit on measurement in quantum metrology [23] may be modified by taking into account the multipartite entanglement and MDR; the intricate correlation structures in multipartite entanglement [27], on the other hand, indicate that more investigations on the unexplored features of quantum measurement are necessary. In order to ascertain the exact form of MDR through measuring the correlation function, an optical experimental scheme has been proposed. Finally, it should be mentioned that although the analysis of measurement precision and disturbance in this work is based on the definitions of (2) and (3), it is also applicable to other operator-type definitions.
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Appendix

A  Functions \( f_q(\Delta A, \Delta B, \langle C \rangle) \)

For the Heisenberg-type MDR \( \epsilon(A)\eta(B) \geq |\langle C \rangle| \), we know that the shortest distance from the AR to the origin of the coordinate is

\[
 f_{\text{He}}(\langle C \rangle) = 2|\langle C \rangle| .
\]

Ozawa’s MDR [Eq. (5)] may be rewritten as

\[
 [\epsilon(A) + \Delta A][\eta(B) + \Delta B] \geq \Delta A\Delta B + |\langle C \rangle| .
\]

The shortest distance from the AR to the origin for this MDR, \( f_{\text{Oz}} \), may be solved by minimizing the value \( \sqrt{\epsilon(A)^2 + \eta(B)^2} \) under the constraints of \([\epsilon(A) + \Delta A][\eta(B) + \Delta B] \geq \Delta A\Delta B + |\langle C \rangle| \) and \( \Delta A\Delta B \geq |\langle C \rangle| \). It is found that when \( \Delta A = \Delta B = \sqrt{|\langle C \rangle|} \) (i.e., the ideal minimum uncertainty states for \( A, B \)), \( f_{\text{Oz}} \) gets its maximum value of \( (2 - \sqrt{2})|\langle C \rangle| \).

As expressed in Eq. (12), Branciard’s MDR in Eq. (8) is an ellipse centered at the origin. The minimal distance from the AR to the origin for this MDR is equal to the minor axis of the ellipse, which is

\[
 f_{\text{B1}}(\Delta A, \Delta B, \langle C \rangle) = \frac{1}{2}[\Delta A^2 + \Delta B^2 - \sqrt{(\Delta A + \Delta B)^2 - 4\langle C \rangle^2}] .
\]

When \( \Delta A = \Delta B = \sqrt{|\langle C \rangle|} \), \( f_{\text{B1}} \) gets the maximum value of \( |\langle C \rangle| \).

Along the same line, the corresponding expressions of \( f_q \) can all be solved for other MDRs. We get, after numerical evaluations, that the maximum attainable value from \( r^2_{\text{B2}} = f_{\text{B2}} \) form Eq. (9) is \((4 - 2\sqrt{2})|\langle C \rangle| \) (at \( \Delta A = \Delta B = \sqrt{|\langle C \rangle|} \)). The values of Eqs. (6) and (7) may also be obtained under their optimal measurements, i.e., \( (\Delta A)^2 = (\Delta A)^2 + \epsilon(A)^2 \) \([13]\), where \( r^2_{\text{Ha}} = 2|\langle C \rangle|/5 \) and \( r^2_{\text{We}} \approx 0.591|\langle C \rangle| \) when \( \Delta A = \Delta B = \sqrt{|\langle C \rangle|} \).
B  The generality of Theorem 2

Here we explicitly demonstrate that for all tripartite entangled states, there is a constraint on correlations in the form of Eq. (25) derived from the MDRs.

First, for arbitrary incompatible observables $A$ and $B$, Eq. (25) is satisfied by the set of quantum states $\Psi = \{|\psi_{123}\rangle \rangle | \psi_{123}\rangle = U_{13}|\psi_{12}\rangle|\phi_3\rangle, U_{13}^\dagger U_{13} = I\}$. All their local unitary equivalent states $|\Psi_{123}\rangle = U_1 \otimes U_2 \otimes U_3|\psi_{123}\rangle \in \Psi$ because

$$|\Psi_{123}\rangle = U_1 \otimes U_2 \otimes U_3|\psi_{123}\rangle$$

$$= U_2 \otimes U_{13}'|\psi_{12}\rangle|\phi_3\rangle = U_{13}' V_1|\psi_{12}\rangle|\phi_3\rangle$$

where $U_{13}' = (U_1 \otimes U_3) U_{13}, U_{13}'' = U_{13}' (V_1 \otimes I_3)$, and we have used the fact that the unitaries $U_2|\psi_{12}\rangle = V_1|\psi_{12}\rangle$ always exist for the quantum state $|\psi_{12}\rangle = \frac{1}{\sqrt{N}} \sum_i |\alpha_i\rangle|\alpha'_i\rangle$.

On the other hand, an arbitrary invertible operator $\Lambda$ makes the transformation $|\tilde{\psi}_{12}\rangle = \Lambda_2|\psi_{12}\rangle$ by acting on particle 2. A set of states $|\tilde{\psi}_1^{(i)}\rangle = 2\langle q_i | \tilde{\psi}_{12} \rangle / |2\langle q_i | \tilde{\psi}_{12}\rangle|$ is prepared via a complete projection basis $\{|q_i\rangle\}$. The measurement precision and disturbance for such states are

$$\tilde{\epsilon}^{(i)}(A)^2 = \langle \phi_3 | \langle \tilde{\psi}_1^{(i)} | [A - A_1 \otimes I_3]^2 | \tilde{\psi}_1^{(i)} \rangle | \phi_3 \rangle ,$$

$$\tilde{\eta}^{(i)}(B)^2 = \langle \phi_3 | \langle \tilde{\psi}_1^{(i)} | [B - B_1 \otimes I_3]^2 | \tilde{\psi}_1^{(i)} \rangle | \phi_3 \rangle ,$$

where the tilde indicates the precision and disturbance are evaluated under the states projected from $|\tilde{\psi}_{12}\rangle$ and $A$ and $B$ have been defined in Eqs. (2) and (3). Summing over the complete basis $\{|q_i\rangle\}$,

$$\sum_i |2\langle q_i | \tilde{\psi}_{12}\rangle|^2 \tilde{\epsilon}^{(i)}(A)^2 = \langle \phi_3 | \langle \tilde{\psi}_{12} | (U_{13}' A_3 U_{13} - A_1)^2 | \tilde{\psi}_{12} \rangle | \phi_3 \rangle ,$$

$$\sum_i |2\langle q_i | \tilde{\psi}_{12}\rangle|^2 \tilde{\eta}^{(i)}(B)^2 = \langle \phi_3 | \langle \tilde{\psi}_{12} | (U_{13}' B_1 U_{13} - B_1)^2 | \tilde{\psi}_{12} \rangle | \phi_3 \rangle .$$
Here, taking the measurement precision as an example, we have

\[
\sum_i |2\langle q_i | \tilde{\psi}_{12} \rangle|^2 \tilde{\epsilon}^{(i)}(A)^2 = \langle \phi_3 | (U_{13}^\dagger A_3^2 U_{13} + A_1^2) | \tilde{\psi}_{12} \rangle |\phi_3 \rangle - \\
\langle \phi_3 | (U_{13}^\dagger A_3 U_{13} A_1 + A_1 U_{13}^\dagger A_3 U_{13}) | \tilde{\psi}_{12} \rangle |\phi_3 \rangle \\
= \langle \psi_{123} | A_0^2 \Lambda_2^1 A_2 | \psi_{123} \rangle + \langle \psi_{123} | A_2^0 \Lambda_2^1 A_2 | \psi_{123} \rangle - \\
\langle \psi_{123} | A_3 (\Lambda_2^1 A_2 + A_2' \Lambda_2^1 A_2) | \psi_{123} \rangle,
\]

where \(|\psi_{123}\rangle = U_{13} |\psi_{12}\rangle |\phi_3 \rangle\). Since \(\langle \psi_{123} | A_0^2 \Lambda_2^1 A_2 | \psi_{123} \rangle = \langle \psi_{123} | A_2^0 \Lambda_2^1 A_2 | \psi_{123} \rangle\),

\[
2 \sum_i |2\langle q_i | \tilde{\psi}_{12} \rangle|^2 \tilde{\epsilon}^{(i)}(A)^2 = \langle \psi_{123} | A_3^2 \Lambda_2^1 A_2 + \Lambda_2^1 A_2 A_2^0 + A_2^0 \Lambda_2^1 A_2 + \Lambda_2^1 A_2 A_2^0 \\
- 2A_3 A_2 \Lambda_2^1 A_2 - 2\Lambda_2^1 A_2 A_3 A_2' | \psi_{123} \rangle.
\]

Along the same line,

\[
2 \sum_i |2\langle q_i | \tilde{\psi}_{12} \rangle|^2 \tilde{\eta}^{(i)}(B)^2 = \langle \psi_{123} | B_1^2 \Lambda_2^1 A_2 + \Lambda_2^1 A_2 B_1^2 + B_2^0 \Lambda_2^1 A_2 + \Lambda_2^1 A_2 B_2^0 \\
- 2B_1 B_2' \Lambda_2^1 A_2 - 2\Lambda_2^1 A_2 B_1 B_2' | \psi_{123} \rangle.
\]

Summing over the above two equations, defining \(\tilde{F}_q \equiv \sum_i |2\langle q_i | \tilde{\psi}_{12} \rangle|^2 \tilde{f}^{(i)}_q\), where \(\tilde{f}^{(i)}_q \leq \tilde{\epsilon}^{(i)}(A)^2 + \tilde{\eta}^{(i)}(B)^2\) are the same function as that of Eq. \((35)\) but evaluated under states \(|\tilde{\psi}_{123}^{(i)}\rangle\), we have

\[
\tilde{F}_q^2 \leq |\langle \psi_{123} | [(A_3 - A_2')^2 + (B_1 - B_2')^2] \Lambda_2^1 A_2 | \psi_{123} \rangle|^2 \\
\leq \langle \psi_{123} | \Lambda_2^1 A_2 [(A_3 - A_2')^2 + (B_1 - B_2')^2] \Lambda_2^1 A_2 | \psi_{123} \rangle \\
\times \langle \psi_{123} | [(A_3 - A_2')^2 + (B_1 - B_2')^2] | \psi_{123} \rangle,
\]

(59)

where the equality holds when \(\Lambda^\dagger \Lambda = M\) in the second inequality. The right-hand side of Eq. \((59)\) does not depend on the choice of \(|q_i \rangle\); therefore,

\[
\langle \tilde{\psi}_{123} | (A_3 - A_2')^2 + (B_1 - B_2')^2 | \tilde{\psi}_{123} \rangle \xi \geq \frac{\gamma_q^2}{N},
\]

(60)
where $|\tilde{\psi}_{123}\rangle = \Lambda^\dagger_2 \Lambda_2 |\psi_{123}\rangle / \sqrt{N}$, the normalization factor $N = \langle \psi_{123} | (\Lambda^\dagger_2 \Lambda_2)^2 |\psi_{123}\rangle$, and 
$\tilde{\gamma}_q = \text{Max} \{ \tilde{F}_i \}$ over $\{|q_i\}$, $\xi = \langle \psi_{123} | (A_3 - A'_3)^2 + (B_1 - B'_1)^2 |\psi_{123}\rangle$. However, according to Eq. (36), $\xi = \sum |2 \langle p_i |\psi_{12} \rangle |^2 [\epsilon(i) (A)^2 + \eta(i) (B)^2]$ does not depend on the matrix $\Lambda$. For varying $\Lambda$, $\xi$ may be chosen as a constant whose value is $\gamma_q$ which is determined by the fact that Eq. (60) should reduce to Eq. (38) when $\Lambda^\dagger \Lambda = I$. Therefore, Eq. (60) may be rewritten as

$$\langle A'_2 A_3 + B'_2 B_1 \rangle \leq \frac{1}{2} [\langle A'_2 \rangle^2 + \langle A_3^2 \rangle + \langle B'_2 \rangle^2 + \langle B_1^2 \rangle - (\tilde{\gamma}_q^2) / (\gamma_q N) ] ,$$

which is a fundamental constraint on the bipartite correlations for the quantum state $|\tilde{\psi}_{123}\rangle$ and is the same as that of Eq. (25) for $|\psi_{123}\rangle$ due to the MDR.

The set of states satisfying Eq. (61) may be formulated as (up to a normalization)

$$\tilde{\Psi} = \{ |\tilde{\psi}_{123}\rangle | \tilde{\psi}_{123}\rangle = \Lambda^\dagger_2 \Lambda_2 |\psi_{123}\rangle , \text{Det} [\Lambda] \neq 0 , |\psi_{123}\rangle \in \Psi \} .$$

Defining the Hermitian operator as $\Lambda^\dagger \Lambda = H$, $|\tilde{\psi}_{123}\rangle$ may be generally expressed as

$$|\tilde{\psi}_{123}\rangle = \frac{1}{\sqrt{N}} \sum_i H |\alpha_i^\dagger\rangle_2 U_{13} U^\dagger_3 U_3 \left( |\alpha_i\rangle_1 \sum_{j=1}^{N} \gamma_j |\alpha_j\rangle_3 \right)$$

$$= \frac{1}{\sqrt{N}} \sum_i H |\alpha_i^\dagger\rangle_2 U_{13} U^\dagger_3 (|\alpha_i\rangle_1 |\alpha_1\rangle_3 )$$

$$= \frac{1}{\sqrt{N}} \sum_{l,m,n,i} ( |\alpha_i\rangle_1 \langle \alpha_i | \otimes |\alpha'_m\rangle_2 \langle \alpha'_m | \otimes |\alpha_n\rangle_3 (\alpha_n |) H |\alpha_i^\dagger\rangle_2 U'_{13} (|\alpha_i\rangle_1 |\alpha_1\rangle_3 )$$

$$= \frac{1}{\sqrt{N}} \sum_{l,m,n,i} h_{mi} \psi_{lin} |\alpha_i\rangle_1 |\alpha'_m\rangle_2 |\alpha_n\rangle_3 ,$$

where we have used $|\phi_3\rangle = \sum_j \gamma_j |\alpha_j\rangle , U_3 |\phi_3\rangle = |\alpha_1\rangle ; U'_{13} = U_{13} U^\dagger_3$ is another unitary interaction matrix of $N^2 \times N^2 , h_{mi} = 2 \langle \alpha_m | H |\alpha_i\rangle_2 , \text{and} \psi_{lin} = u'_{lin} = 1 \langle \alpha_i |_3 (\alpha_n |U'_{13} |\alpha_i\rangle_1 |\alpha_1\rangle_3$.

The number of free real parameters in $|\tilde{\psi}_{123}\rangle$ includes $2N^3 - N - N(N - 1)$ real parameters from $u'_{lin}$ and $N^2$ from the Hermitian operator $H$. The total number of $2N^3$ equals the number of real parameters of the quantum state of $(N \times N \times N)$-dimensional tripartite states. Thus, for all the tripartite states, Eq. (25) or Eq. (61) is satisfied.
C Maximal value of \( E(Z_2, Z_3) + E(X_1, X_2) \) for three-qubit states

An arbitrary three-qubit state may be expressed as

\[
|\varphi_{123}\rangle = a_1|++\rangle + a_2|+-\rangle + a_3|+--\rangle + a_4|--\rangle + a_5|--+\rangle + a_6|++-\rangle + a_7|--+\rangle + a_8|--\rangle .
\]

Here \( a_i \in \mathbb{C} \), and the normalization requires \( \sum_{i=1}^{8} |a_i|^2 = 1 \). The QM prediction of \( E(Z_2, Z_3) + E(X_1, X_2) \) for this arbitrary state takes the following form:

\[
E(Z_2, Z_3) + E(X_1, X_2) = \langle \varphi_{123}|I_1 \otimes Z_2 \otimes Z_3|\varphi_{123}\rangle + \langle \varphi_{123}|X_1 \otimes I_2 \otimes I_3|\varphi_{123}\rangle
\]

\[
= |a_1|^2 + |a_4|^2 + |a_5|^2 + |a_8|^2 - |a_2|^2 - |a_3|^2 - |a_6|^2 - |a_7|^2 + a_1^*a_7 + a_2^*a_8 + a_3^*a_5 + a_4^*a_6 + a_1a_7^* + a_2a_8^* + a_3a_5^* + a_4a_6^*
\]

\[
= |\vec{r}_1|^2 - |\vec{r}_2|^2 + |\vec{r}_1|^* \cdot |\vec{r}_2|^* ,
\]

where \( \vec{r}_1 \equiv \{a_1, a_4, a_5, a_8\}^T \), \( \vec{r}_2 \equiv \{a_7, a_6, a_3, a_2\}^T \) and \( |\vec{r}_1|^2 + |\vec{r}_2|^2 = 1 \). We may set \( |\vec{r}_1| = \cos \theta \), \( |\vec{r}_2| = \sin \theta \), and according to the Cauchy-Schwarz inequality \( |\vec{r}_1^* \cdot \vec{r}_2| \leq |\vec{r}_1||\vec{r}_2| \), we have

\[
|\vec{r}_1|^2 - |\vec{r}_2|^2 + |\vec{r}_1^* \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_2^* | \leq |\vec{r}_1|^2 - |\vec{r}_2|^2 + 2|\vec{r}_1^* \cdot \vec{r}_2| \cos \phi \leq \cos(2\theta) + \sin(2\theta) \cos \phi \leq \sqrt{2}. \quad (64)
\]

Here \( |\vec{r}_1^* \cdot \vec{r}_2| \cos \phi \) are the real part of the complex number \( \vec{r}_1^* \cdot \vec{r}_2 \).
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