Splitting subspaces and a finite field interpretation of the Touchard-Riordan Formula

AMRITANSHU PRASAD

The Institute of Mathematical Sciences, Chennai, India.
Homi Bhabha National Institute, Mumbai, India.

SAMRITH RAM

Indraprastha Institute of Information Technology Delhi, New Delhi, India.

Abstract. We enumerate the number of $T$-splitting subspaces of dimension $m$ for an arbitrary operator $T$ on a $2m$-dimensional vector space over a finite field. When $T$ is regular split semisimple, comparison with an alternate method of enumeration leads to a new proof of the Touchard-Riordan formula for enumerating chord diagrams by their number of crossings.

1. Introduction

Let $\mathbb{F}_q$ denote a finite field of order $q$, and $m$ be a non-negative integer. Given a positive integer $d$ and a linear operator $T : \mathbb{F}_q^{dm} \to \mathbb{F}_q^{dm}$, an $m$-dimensional subspace $W \subset \mathbb{F}_q^{dm}$ is said to be $T$-splitting if

$$W + TW + \cdots + T^{d-1}W = \mathbb{F}_q^{dm}.$$ 

This definition was proposed by Ghorpade and Ram [8], motivated by the work of Niederreiter [15].

The number of $T$-splitting subspaces is known when $T$ has an irreducible characteristic polynomial [3, 5, 8], is regular nilpotent [2], ...
is regular split semisimple \([18, 19]\), or when the invariant factors satisfy certain degree constraints \([1]\). In this article, we consider the case where \(d = 2\). Our main theorem gives a formula for the number of \(T\)-splitting subspaces of dimension \(m\) for any \(T \in M_{2m}(\mathbb{F}_q)\).

**Main Theorem.** For any linear operator \(T : \mathbb{F}_q^{2m} \rightarrow \mathbb{F}_q^{2m}\), the number of \(m\)-dimensional \(T\)-splitting subspaces of \(\mathbb{F}_q^{2m}\) is given by

\[
\sigma^T = q^{\binom{m}{2}} \sum_{j=0}^{2m} (-1)^j X_j^T q^{\binom{m-j+1}{2}},
\]

where \(X_j^T\) is the number of \(j\)-dimensional \(T\)-invariant subspaces of \(\mathbb{F}_q^{2m}\).

The quantities \(X_j^T\) are easy to compute from the Jordan canonical form of \(T\) with the help of a recursive formula of Ramaré \([20]\). For a detailed discussion see Section 2. When \(T\) is regular split semisimple (i.e., it is similar to a diagonal matrix with distinct diagonal entries), \(X_j^T = \binom{2m}{j}\), so the number of \(T\)-splitting subspaces is

\[
\sigma^T = q^{\binom{m}{2}} \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} q^{\binom{m-j+1}{2}}.
\]

The sum above is the right hand side of the Touchard-Riordan formula

\[(q - 1)^m T_m(q) = \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} q^{\binom{m-j+1}{2}}\]  

for the polynomial \(T_m(q)\) that enumerates chord diagrams on \(2m\) nodes according to their number of crossings (see Section 5). This identity is attributed to Touchard \([24]\) and Riordan \([22]\). A proof using the theory of continued fractions was given by Read \([21]\), and a bijective proof was given by Penaud \([16]\). The polynomials \(T_m(q)\) are moments of the \(q\)-Hermite orthogonal polynomial sequence \([10\text{, Prop. 4.1}]\). Several generalizations and variations of the Touchard-Riordan formula can be found in \([6, 7, 12, 13, 14]\).

When \(T \in M_{2m}(\mathbb{F}_q)\) is regular split semisimple, splitting subspaces can also be enumerated (see Theorem 6.1) using the technique of \([19\text{, Section 4.6}]\) as

\[
\sigma^T = q^{\binom{m}{2}} (q - 1)^m T_m(q).
\]

This gives a completely new self-contained proof of the Touchard-Riordan formula. Thus our main theorem could be viewed as a generalisation of the Touchard-Riordan formula in the setting of finite fields.

A software demonstration of our results using SageMath \([23]\) can be found at \(https://www.imsc.res.in/~amri/splitting_subspaces/\).
2. Enumeration of Invariant Subspaces

Each \( T \in M_n(\mathbb{F}_q) \) gives rise to an \( \mathbb{F}_q[t] \)-module \( M_T \) with underlying vector space \( \mathbb{F}_q^n \) on which \( t \) acts by \( T \). A subspace of \( \mathbb{F}_q^n \) is \( T \)-invariant if and only if it is a submodule of \( M_T \). Let \( \text{Par} \) denote the set of all integer partitions and \( \text{Irr} \mathbb{F}_q[t] \) denote the set of all irreducible monic polynomials in \( \mathbb{F}_q[t] \). By the theory of elementary divisors (see [11, Section 3.9] and [9, Section 1]) there exists a unique function \( c_T : \text{Irr} \mathbb{F}_q[t] \rightarrow \text{Par} \) such that

\[
M_T = \bigoplus_{p \in \text{Irr} \mathbb{F}_q[t]} M_{T_p},
\]

with the \( p \)-primary component \( M_{T_p} \) having structure

\[
M_{T_p} = \bigoplus_i \mathbb{F}_q[t]/(p(t)^{c_T(p)})
\]

where \( c_T(p)_1, c_T(p)_2, \ldots \) are the parts of the partition \( c_T(p) \).

Define the invariant subspace generating function \( f_T \) of \( T \in M_n(\mathbb{F}_q) \) as

\[
f_T(t) = \sum_{j=0}^{n} X_j^T t^j,
\]

where \( X_j^T \) is the number of \( j \)-dimensional \( T \)-invariant subspaces of \( \mathbb{F}_q^n \). Each \( \mathbb{F}_q[t] \)-submodule of \( M_T \) is uniquely expressible as a direct sum of submodules of the primary submodules \( M_{T_p} \). Therefore,

\[
f_T(t) = \prod_{p \in \text{Irr} \mathbb{F}_q[t]} f_{T_p}(t).
\]

For each \( \lambda \in \text{Par} \), let \( f_\lambda(q; t) \) denote the invariant subspace generating function of the nilpotent matrix over \( \mathbb{F}_q \) whose Jordan block sizes are the parts of \( \lambda \). A surprisingly simple recurrence of Ramaré [20, Theorem 3.1] allows for easy computation of \( f_\lambda(q; t) \):

\[
(t - 1) f_\lambda(q; t) = t^{\lambda_1+1} q^{\sum_{j=2}^{\lambda_1} \lambda_j} f_{(\lambda_2, \lambda_3, \ldots)}(q; t/q) - f_{(\lambda_2, \lambda_3, \ldots)}(q; tq),
\]

where \( \lambda_1, \lambda_2, \ldots \) are the parts of \( \lambda \) in weakly decreasing order. The empty partition \( \emptyset \) of 0 can be used as the base case with \( f_\emptyset(q; t) = 1 \). The recurrence (2.3) implies that \( f_\lambda(q; t) \) is a polynomial in \( q \) and \( t \) with integer coefficients.

Since the rings \( \mathbb{F}_q[t]/p(t)^d \) and \( \mathbb{F}_q[u]/u^d \) are isomorphic, the invariant subspace generating function of \( T \in M_n(\mathbb{F}_q) \) is given by

\[
f_T(t) = \prod_{p \in \text{Irr} \mathbb{F}_q[t]} f_{c_T(p)}(q^{\deg p}, t^{\deg p}).
\]
It follows that the polynomial \( f_T(t) \) depends on the polynomials \( p \in \text{Irr}\mathbb{F}_q[t] \) only through their degrees.

**Definition 2.1.** A similarity class type of size \( n \) is a multiset \( \tau \) of pairs of the form \((d, \lambda)\) where \( d \) is a positive integer and \( \lambda \) is a non-empty integer partition such that \( \sum_{(d,\lambda) \in \tau} d|\lambda| = n \) (the sum is taken with multiplicity). The similarity class type of \( T \in M_n(\mathbb{F}_q) \) is the similarity class of size \( n \) given by
\[
\{(\deg(p), c_T(p)) \mid p \in \text{Irr}\mathbb{F}_q[t], c_T(p) \neq \emptyset\}.
\]

**Remark 2.2.** The set of similarity class types of size \( n \) is independent of \( q \). Green [9] introduced similarity class types to organise conjugacy classes of \( GL_n(\mathbb{F}_q) \) in a manner independent of \( q \). This enabled him give a combinatorial description of the character table of \( GL_n(\mathbb{F}_q) \) across all \( q \). For a detailed discussion and a software implementation see [17].

**Example 2.3.**
1. An \( n \times n \) scalar matrix has similarity class type \( \{(1, (1^n))\} \).
2. A regular split semisimple \( n \times n \) matrix has similarity class type \( \{(1, (1))\}, \ldots, (1, (1))\) (with \( n \) repetitions).
3. A regular nilpotent \( n \times n \) matrix has type \( \{(1, (n))\} \).
4. An \( n \times n \) matrix with irreducible characteristic polynomial has type \( \{(n, (1))\} \).

**Theorem 2.4.** Given a similarity class type \( \tau \) of size \( n \) and \( 0 \leq j \leq n \) let
\[
f_\tau(u; t) = \prod_{(d,\lambda) \in \tau} f_\lambda(u^d; t^d).
\]
Then for any prime power \( q \) and any matrix \( T \in M_n(\mathbb{F}_q) \) with similarity class type \( \tau \), \( f_T(t) = f_\tau(q;t) \). In particular, for every \( 0 \leq j \leq n \), there exists a polynomial \( X_j^T(u) \in \mathbb{Z}[u] \) such that \( X_j^T = X_j^T(q) \).

**Proof.** The theorem follows from Eqns. (2.3) and (2.4). \( \square \)

The polynomial \( f_\lambda(q;t) \) is known to have non-negative coefficients [4], hence \( X_j^T(q) \) also has non-negative coefficients.

**Example 2.5.** Let \( \tau_i = \{(1, (1^{m+i})), (m-i, (1))\} \) for \( i = 1, \ldots, m \). Then
\[
f_{\tau_i}(t) = \left( \sum_{k=0}^{m+i} \left[ \begin{array}{c} m+i \\ k \end{array} \right]_q t^k \right) (1 + t^{m-i}).
\]
Consequently,
\begin{equation}
X_j^\tau(q) = \left[ \begin{array}{c} m + i \\ j \end{array} \right]_q + \left[ \begin{array}{c} m + i \\ j - m + i \end{array} \right]_q.
\end{equation}

3. The Existence of a Formula

In this section we establish the existence of a formula for the number \( \sigma^T \) of \( m \)-dimensional \( T \)-splitting subspaces of \( \mathbf{F}_q^{2m} \) in terms of \( X_j^T \), \( j = 0, \ldots, m \) (Corollary 3.5). The main step is Proposition 3.3, which is a special case of a more general recurrence of Chen and Tseng [5, Lemma 2.7].

Given a positive integer \( n \), and \( 0 \leq a \leq n \), let \( \mathbf{a} \) denote the set of \( a \)-dimensional subspaces of \( \mathbf{F}_q^n \). Given a linear operator \( T : \mathbf{F}_q^n \to \mathbf{F}_q^n \) and sets \( X \) and \( Y \) of subspaces of \( \mathbf{F}_q^n \), define
\[
(X,Y)_T := \{ W \in X | W \cap T^{-1}W \in Y \}
\]
\[
[X,Y]_T := \{ (W_1,W_2) | W_1 \in X, W_2 \in Y, \text{ and } W_1 \cap T^{-1}W_1 \supset W_2 \}.
\]
Thus \( (\mathbf{a},\mathbf{b})_T \) denotes the set of \( a \)-dimensional subspaces \( W \) such that \( W \cap T^{-1}W \) has dimension \( b \). We drop the subscript \( T \) from the notation when the operator is clear from the context.

**Example 3.1.** For each \( 0 \leq a \leq n \), \( (\mathbf{a},\mathbf{a}) \) denotes the set of \( a \)-dimensional \( T \)-invariant subspaces of \( \mathbf{F}_q^n \). Hence \( |(\mathbf{a},\mathbf{a})_T| = X^T_a \).

**Example 3.2.** If \( n = 2m \), then \( (\mathbf{m},\mathbf{0})_T \) is the set of \( m \)-dimensional \( T \)-splitting subspaces of \( \mathbf{F}_q^{2m} \).

**Proposition 3.3.** Let \( T : \mathbf{F}_q^n \to \mathbf{F}_q^n \) be a linear map. For all integers \( n \geq a > b \geq 0 \), we have
\[
|(\mathbf{a},\mathbf{b})| = X^T_b \left[ \begin{array}{c} n - b \\ a - b \end{array} \right]_q - X^T_a \left[ \begin{array}{c} a \\ b \end{array} \right]_q
\]
\[
+ \sum_{j=0}^{b-1} |(\mathbf{b},\mathbf{j})| \left[ \begin{array}{c} n - 2b + j \\ a - 2b + j \end{array} \right]_q - \sum_{k=b+1}^{a-1} |(\mathbf{a},\mathbf{k})| \left[ \begin{array}{c} k \\ b \end{array} \right]_q.
\]

**Proof.** Since \( \mathbf{a} = \coprod_{0 \leq a \leq k} (\mathbf{a},\mathbf{k}) \), we have
\[
[a,b] = \coprod_{b \leq k \leq a} [(\mathbf{a},\mathbf{k}),\mathbf{b}].
\]
It follows that
\[
|\lfloor a, b \rfloor| = \sum_{k=b}^{a} |\lfloor (a, k), b \rfloor|
\]
\[
= \sum_{k=b}^{a} |(a, k)| \binom{k}{b}_q
\]
(3.1)
\[
= |(a, b)| + \sum_{k=b+1}^{a} |(a, k)| \binom{k}{b}_q.
\]

Similarly,
\[
\lfloor a, b \rfloor = \prod_{0 \leq j \leq b} \lfloor a, (b, j) \rfloor,
\]
so that
\[
|\lfloor a, b \rfloor| = \sum_{j=0}^{b} |\lfloor a, (b, j) \rfloor|
\]
\[
= \sum_{j=0}^{b} |(b, j)| \binom{n - (2b - j)}{a - (2b - j)}_q.
\]
(3.2)

The proposition follows from Eqs. (3.1) and (3.2), and \(|(a, a)| = X_a^T\).

\[\square\]

**Proposition 3.4.** For all integers integer \(n \geq a \geq b \geq 0\), there exist polynomials \(p_0(t), \ldots, p_a(t) \in \mathbb{Z}[t]\) such that, for every prime power \(q\), and every linear map \(T : \mathbb{F}_q^n \to \mathbb{F}_q^n\),
\[
|\lfloor (a, b)T \rfloor| = \sum_{j=0}^{a} p_j(q)X_j^T.
\]

**Proof.** Proposition 3.3 expands \(|\lfloor (a, b)T \rfloor|\) in terms of \(X_a^T, X_b^T\), and \(|\lfloor (a', b')T \rfloor|\) where either \(a' < a\), or \(a' = a\) and \(a' - b' < a - b\). The coefficients are polynomials in \(q\) that are independent of \(T\). Thus repeated application of Proposition 3.3 will result in an expression of the stated form in finitely many steps.

**Corollary 3.5.** For each non-negative integer \(m\), there exist polynomials \(p_0(t), \ldots, p_m(t) \in \mathbb{Z}[t]\) such that, for every linear map \(T : \mathbb{F}_q^{2m} \to \mathbb{F}_q^{2m}\), the number of \(m\)-dimensional \(T\)-splitting subspaces is given by
\[
\sigma^T = \sum_{j=0}^{m} p_j(q)X_j^T.
\]
(3.3)
4. Proof of the Main Theorem

By Theorem 2.4 and Corollary 3.5, for every similarity class type $\tau$ of size $2m$, there exists $\sigma^\tau(u) \in \mathbb{Z}[u]$ such that, for every prime power $q$ and every $T \in M_{2m}(F_q)$ of type $\tau$, $\sigma^T = \sigma^\tau(q)$. Thus the main theorem can be rephrased as follows.

**Theorem 4.1.** For each similarity class type $\tau$ of size $2m$,

\[
\sigma^\tau(q) = q^{\binom{m}{2}} \sum_{j=0}^{2m} (-1)^j X_j^\tau(q) q^{(m-j+1)}.
\]

**The proof strategy.** Since the lattice of submodules of $M^T$ is self-dual, $X_j^\tau(q) = X_{2m-j}^\tau(q)$. Therefore the right hand side of (4.1) can be rewritten in terms of $X_{0}^\tau(q), \ldots, X_{m}^\tau(q)$, bringing it to the form (3.3).

Suppose $\tau_0, \ldots, \tau_m$ are similarity class types of size $2m$ such that the determinant $(X_j^\tau(q))_{0 \leq i,j \leq m}$ is non-zero. Then the system of equations

\[
\sigma^{\tau_i}(q) = \sum_{j=0}^{m} p_j(q) X_j^{\tau_i}(q), \quad i = 0, \ldots, m
\]

has a unique solution for the $p_j(q)$. Thus, if we prove (4.1) for $\tau = \tau_0, \ldots, \tau_m$, we will have shown that Theorem 4.1 holds in general.

Take $\tau_0 = \{(2m, (1))\}$, the type of a simple matrix (a matrix with irreducible characteristic polynomial), and for $i = 1, \ldots, m$, take $\tau_i = \{(1, (1^{m-i})), (m-i, (1))\}$. The proof of Theorem 4.1 is reduced to the following steps:

**Claim 1.** The formula (4.1) holds for $\tau = \tau_0, \ldots, \tau_m$.

**Claim 2.** The determinant of $X = (X_j^{\tau_i}(q))_{0 \leq i,j \leq m}$ is non-zero.

**Proof of Claim 1.** Consider first $\tau = \tau_0$. It is shown in [3, Theorem 1.4] that

\[
\sigma_{\tau_0}(q) = q^{\binom{m}{2}} (q^{\binom{m+1}{2}} + q^{\binom{m}{2}}).
\]

On the other hand, $\tau_0$ is the type of a simple matrix, so $X_j^{\tau}(q) = 1$ if $j = 0$ or $2m$, and $X_j^{\tau}(q) = 0$ for $0 < j < 2m$. Therefore

\[
\sum_{j=0}^{2m} (-1)^j X_j^{\tau}(q) q^{(m-j+1)} = q^{\binom{m}{2}} (q^{\binom{m+1}{2}} + q^{\binom{m}{2}}) = q^{\binom{m}{2}} (q^{\binom{m+1}{2}} + q^{\binom{m}{2}}),
\]

establishing (4.1) for $\tau_0$.

For $i = 1, \ldots, m$, $\sigma^{\tau_i}(q) = 0$ since any $T \in M_n(F_q)$ of type $\tau_i$ satisfies the hypothesis of the following general lemma.
Lemma 4.2. Let \( l(\lambda) \) denote the number of parts of an integer partition \( \lambda \). If \( W \subset \mathbb{F}_q^n \) is such that \( \sum_{j \geq 0} T^j W = \mathbb{F}_q^n \), then \( \dim W \geq l(c_T(p)) \) for all \( p \in \text{Irr} \mathbb{F}_q[t] \). In particular, if \( T \in M_{2m}(\mathbb{F}_q) \) is such that \( l(c_T(p)) > m \) for some \( p \in \text{Irr} \mathbb{F}_q[t] \), then \( T \) does not admit an \( m \)-dimensional splitting subspace.

Proof. Let \( \Pi_p : M_T \to M_{T_p} \) denote the projection map with respect to the primary decomposition (2.1). Since \( \Pi_p \) commutes with \( T \), \( \sum_{j \geq 0} T^j \Pi_p(W) = \Pi_p(\mathbb{F}_q^n) = M_{T_p} \). In other words, \( \Pi_p(W) \) generates \( M_{T_p} \). The \( \mathbb{F}_q[t] \)-module \( M_{T_p} \) has rank \( l(c_T(p)) \), so any generating set must have at least \( l(c_T(p)) \) elements. Therefore, \( \dim W \geq \dim \Pi_p(W) \geq l(c_T(p)) \).

Now \( X^\tau_i(q) \) is given by (2.5). Therefore, in order to establish (4.1) for \( \tau = \tau_i, i = 1, \ldots, m \), it suffices to prove the following result.

Lemma 4.3. For each positive integer \( m \), \( 1 \leq i \leq m \), and \( 0 \leq k \leq m - i \),

\[
\sum_{j=0}^{2m} (-1)^j \binom{m+i}{j-k} q^{(m-j+1)/2} = 0.
\]

Proof. In the \( q \)-binomial theorem

\[
\sum_{j=0}^{n} \binom{n}{j} q^{j} x^j = \prod_{j=0}^{n-1} (1 + q^j x),
\]

set \( n = m + i \), \( x = -q^{k-m} \), and change the index of summation from \( j \) to \( j + k \) to get

\[
(4.2) \quad (-1)^k \sum_{j=k}^{m+i+k} (-1)^j \binom{m+i}{j-k} q^{(j-k) + (k-m)(j-k)} = 0.
\]

Observe that

\[
\binom{m-j+1}{2} = \frac{m(m+1) + j(j-1) - 2mj}{2}, \quad \text{whereas} \quad \binom{j-k}{2} + (k-m)(j-k) = \frac{k(k+1) - 2(k-m) + j(j-1) - 2mj}{2}.
\]

These two expressions differ by a quantity independent of \( j \). Therefore replacing \( q^{(j-k) + (k-m)(j-k)} \) by \( q^{(m-j+1)/2} \) in (4.2) amounts to multiplication by a non-zero factor that is independent of \( j \). Thus we have

\[
\sum_{j=k}^{m+i+k} (-1)^j \binom{m+i}{j-k} q^{(m-j+1)/2} = 0.
\]
The sum remains unchanged when its range is extended to $0 \leq j \leq 2m$, proving the identity in the lemma.

**Proof of Claim 2.** The non-singularity of $X = (X_j^\tau(q))_{0 \leq i,j \leq m}$ is proved using inequalities satisfied by the degrees of its entries.

**Lemma 4.4.** Let $(a_{ij})_{n \times n}$ be a real matrix such that whenever $i < k$ and $j < k$,

$$a_{ik} - a_{ij} < a_{kk} - a_{kj},$$

Then the sum $S(\sigma) = \sum_{1 \leq i \leq n} a_{i\sigma(i)}$ attains its maximum value precisely when $\sigma$ is the identity permutation.

**Proof.** Let $\sigma$ be a permutation for which the sum $S(\sigma)$ is maximised. We claim that $\sigma(n) = n$. Suppose, to the contrary, that $\sigma(n) = s \neq n$. Let $r = \sigma^{-1}(n)$. Now

$$\sum_{1 \leq i \leq n} a_{i\sigma(i)} = \sum_{i \notin \{r,n\}} a_{i\sigma(i)} + a_{rn} + a_{ns}$$

$$< \sum_{i \notin \{r,n\}} a_{i\sigma(i)} + a_{rs} + a_{nn}$$

by the hypothesis since $r < n$ and $s < n$. If $\pi$ denotes the permutation which agrees with $\sigma$ whenever $i \notin \{r,n\}$ with $\pi(r) = s$ and $\pi(n) = n$, then it is clear that $S(\sigma) < S(\pi)$, contradicting the maximality of $S(\sigma)$. This proves the claim that $\sigma(n) = n$. Therefore

$$S(\sigma) = a_{nn} + \max_{\pi \in S_{n-1}} S(\pi).$$

Similar reasoning applied to the leading principal $(n - 1) \times (n - 1)$ submatrix of $A$ shows that $\sigma(n - 1) = n - 1$. Continuing this line of reasoning it can be seen that $\sigma(i) = i$ for each $i \leq n$, completing the proof.

**Proposition 4.5.** The matrix $X = (X_j^\tau)_{0 \leq i,j \leq m}$ is non-singular.

**Proof.** Since $\tau_0$ is the type of a simple matrix, the first row of $X$ is the unit vector $(1,0,\ldots,0)$. Therefore it suffices to show that the minor $X' = (X_j^\tau)_{1 \leq i,j \leq m}$ is non-singular. Let $a_{ij} = \deg X_j^\tau(q)$. Since $\deg [x]_k = (n - k)k$, by (2.5) we have, for $1 \leq i,j \leq m$,

$$a_{ij} = \max\{j(m + i - j), (j - m + i)(2m - j)\} = j(m + i - j),$$
since \( j(m + i - j) - (j - m + i)(2m - j) = 2(m - i)(m - j) \geq 0 \). If \( i < k \) and \( j < k \),

\[
\begin{align*}
a_{ik} - a_{ij} &= k(m + i - k) - j(m + i - j) \\
&= (k - j)(m + i - k - j) \\
&< (k - j)(m - j) \\
&= a_{kk} - a_{kj}.
\end{align*}
\]

Lemma 4.4 implies that \( \det X' \) has degree \( \sum_{i=1}^{m} a_{ii} > 0 \) and is thus non-singular. \( \square \)

5. Chord Diagrams

A chord diagram on \( n = 2m \) nodes refers to one of many visual representations of a fixed-point-free involution on \([2m]\) (see, e.g., \([16, \text{Fig. 2}]\)). We arrange \( 2m \) nodes along the \( X \)-axis. A circular arc lying above the \( X \)-axis is used to connect each node to its image under the involution. For example, the involution \((1, 4)(2, 6)(3, 5)(7, 8)\) is represented by the diagram

\[1 2 3 4 5 6 7 8\]

The left end of each arc will be called an opening node, and the right end a closing node. In the running example, the opening nodes are \(1, 2, 3, 7\) and the closing nodes are \(4, 5, 6, 8\). A crossing of the chord diagram is a pair of arcs \((i, j), (k, l)\) such that \(i < k < j < l\). The chord diagram above has two crossings, namely \((1, 4), (2, 6)\) and \((1, 4), (3, 5)\). Given a fixed-point-free involution \(\sigma\), let \(v(\sigma)\) denote the number of crossings of its chord diagram. Touchard \([24]\) studied the polynomials

\[
T_m(q) = \sum_{\sigma} q^{v(\sigma)},
\]

where the sum runs over all fixed-point-free involutions of \([2m]\).

We now describe the contribution to \(T_m(q)\) of chord diagrams with a specified set of opening nodes.

**Lemma 5.1.** Given \(1 \leq c_1 < \cdots < c_m \leq 2m\) designated as opening nodes for a chord diagram, and the remaining elements of \([2m]\) designated as closing nodes of a chord diagram, \(c_i\) lies to the left of the \(j\)th closing node if and only if

\[
c_i \leq i + j - 1.
\]
Consequently, the number of opening nodes that lie to the left of the \( j \)th closing node is given by

\[
(5.1) \quad r_j := \# \{i \in [m] | c_i \leq j + i - 1\}.
\]

**Proof.** The node \( c_i \) lies to the left of the \( j \)th closing node of \( \sigma \) if and only if there are at most \( j - 1 \) closing nodes to the left of \( c_i \). In other words, the total number of nodes (opening or closing) up to and including \( c_i \) is at most \( i + j - 1 \), meaning that \( c_i \leq i + j - 1 \). \( \square \)

For every non-negative integer \( n \), let \([n]_q\) denote the \( q \)-integer \( 1 + q + \cdots + q^{n-1} \).

**Lemma 5.2.** For every non-negative integer \( m \), and \( 1 \leq c_1 < \cdots < c_m \leq 2m \),

\[
\sum_{\sigma \text{ has opening nodes } c_1, \ldots, c_m} q^{v(\sigma)} = \prod_{j=1}^{m} [r_j - (j - 1)]_q,
\]

where \( r_j \) is given by \((5.1)\).

**Proof.** Suppose we wish to construct a chord diagram on \( 2m \) nodes with opening nodes \( c_1 < \cdots < c_m \). The remaining nodes \( d_1 < \cdots < d_m \) are closing nodes. By Lemma 5.1, for each \( j \in [m] \), the number of opening nodes to the left of \( d_j \) is \( r_j \). Thus there are \( r_1 \) choices of opening node for the arc ending at \( d_1 \). These choices, taken from right to left, will result in \( 0, 1, \ldots, r_1 - 1 \) crossings with arcs with closing nodes to the right of \( d_1 \). Having chosen the node that is joined to \( d_1 \), the number of opening nodes that are available to \( d_2 \) is \( r_2 - 1 \). Once again, these choices, taken from right to left, will result in \( 0, 1, \ldots, r_2 - 2 \) crossings with arcs with closing nodes to the right of \( d_2 \). Continuing in this manner, we see that the contribution of arcs with opening nodes \( c_1, \ldots, c_m \) to \( T_m(q) \) is \( \prod_{j=1}^{m} [r_j - (j - 1)]_q \). \( \square \)

Summing over all possible sets of opening nodes gives the following result.

**Theorem 5.3.** For every non-negative integer \( m \),

\[
T_m(q) = \sum_{1 \leq c_1 < \cdots < c_m \leq 2m} \prod_{j=1}^{m} [r_j - (j - 1)]_q,
\]

where \( r_j \) is given by \((5.1)\).
6. The Enumeration of Splitting Subspaces

The relationship between the enumeration of splitting subspaces and the polynomials \( T_m(q) \) was discovered in [19, Section 4.6]. It is a special case of one of the main results [19, Theorem 4.8] of that paper. The proof in this special case, being relatively simple, is provided here.

**Theorem 6.1.** Let \( T \in M_{2m}(F_q) \) be a diagonal matrix with distinct diagonal entries. The number of \( T \)-splitting subspaces in \( F_{2^m} \) is

\[
\sigma^T = (q - 1)^m q^{2^{m-1}} T_m(q).
\]

Comparison of Theorem 6.1 with the main theorem gives a new proof of the Touchard-Riordan formula (1.1).

**Proof.** Let \( W \subset F_{2^m} \) be a \( T \)-splitting subspace of \( F_{2^m} \). \( W \) has a unique ordered basis in reduced row echelon form. This is a basis whose elements form the rows of an \( m \times 2m \) matrix such that

1. There exist \( 1 \leq c_1 < \cdots < c_m \leq 2m \) (called the pivots of \( W \)) such that the first non-zero entry of the \( i \)th row lies in the \( c_i \)th column, and equals 1.
2. The only non-zero entry in the \( c_i \)th column lies in the \( i \)th row for \( 1 \leq i \leq m \).

For example, when \( m = 3 \), a subspace with pivots 1, 2, 4 is spanned by a matrix of the form

\[
\begin{pmatrix}
1 & 0 & * & 0 & * & * \\
0 & 1 & * & 0 & * & * \\
0 & 0 & 0 & 1 & * & *
\end{pmatrix},
\]

where each \( * \) represents an arbitrary element of \( F_q \).

Suppose that \( W \) has reduced row echelon form with pivots \( c_1, \ldots, c_m \). By a permutation of coordinates, the pivot columns can be moved to the left to rewrite \( A \) in the block form \((I \mid X)\), where \( I \) denotes the \( m \times m \) identity matrix, and \( X \in M_m(F_q) \). The condition (2) in the definition of row echelon from imposes the vanishing of certain entries of \( X \):

\[ X_{ij} = 0 \text{ if } j < c_i - (i - 1). \]

For the matrix in (6.1), moving the pivot columns to the left results in the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & * & * & * \\
0 & 1 & 0 & * & * & * \\
0 & 0 & 1 & 0 & * & *
\end{pmatrix}.
\]

The above permutation of coordinates also permutes the diagonal entries of \( T \), but it remains a diagonal matrix with distinct diagonal.
entries. Write this matrix in block diagonal form as $\begin{pmatrix} T' & 0 \\ 0 & T'' \end{pmatrix}$, where $T'$ and $T''$ are $m \times m$ diagonal matrices.

Now $W$ is a $T$-splitting subspace if and only if the matrix $\begin{pmatrix} I & X \\ T' & XT'' \end{pmatrix}$ is non-singular. Applying the block row operation $R_2 \rightarrow R_2 - T'R_1$ gives $\begin{pmatrix} I & X \\ 0 & XT'' - T'X \end{pmatrix}$

Thus $W$ is a splitting subspace if and only if $Y = XT'' - T'X$ is non-singular. The entries of $Y$ in terms of the entries of $X$ are given by $y_{ij} = (t''_j - t'_i)x_{ij}$, where $t'_i$ (resp. $t''_i$) is the $i$th diagonal entry of $T'$ (resp. $T''$). Since $T'$ and $T''$ have no diagonal entries in common the map $X \mapsto Y$ is a bijection, and an entry of $X$ is non-zero if and only if the corresponding entry of $Y$ is non-zero. Thus we have the following result.

**Lemma 6.2.** The number of $T$-splitting subspaces with pivots $c_1, \ldots, c_m$ is the number of non-singular matrices $Y \in M_m(F_q)$ such that $Y_{ij} = 0$ if $j < c_i - (i - 1)$.

It remains to enumerate such matrices. The number of potentially non-zero entries in the $j$th column of $Y$ is the number of $i$ such that $c_i \leq i + j - 1$, which is precisely the number $r_j$ from Lemma 5.1. Since $Y$ is non-singular, its first column is non-zero. Thus there are $q^{r_1} - 1 = (q - 1)[r_1]_q$ possibilities for the first column of $Y$. The second column is independent of the first, giving $q^{r_2} - q = (q - 1)q[r_2 - 1]_q$ possibilities. Similarly, given the first $j - 1$ columns of $Y$, the number of possibilities for the $j$th column is $q^{r_j} - q^{j-1} = (q - 1)q^{j-1}[r_j - (j - 1)]_q$. Thus the number of matrices $Y$ satisfying the conditions of Lemma 6.2 is

$$(q - 1)^m q^{\binom{m}{2}} \prod_{j=1}^{m}[r_j - (j - 1)]_q.$$ 

Adding up the contribution of all possible sets of pivots and using Lemma 5.2 gives Theorem 6.1. □
References

[1] D. Aggarwal and S. Ram. *Polynomial matrices, splitting subspaces and Krylov subspaces over finite fields*. 2021. arXiv: 2105.15155 [math.CO].

[2] D. Aggarwal and S. Ram. “Splitting subspaces of linear operators over finite fields”. *Finite Fields and Their Applications* 78 (2022), p. 101982. DOI.

[3] A. Arora, S. Ram, and A. Venkateswarlu. “Unimodular polynomial matrices over finite fields”. *J. Algebraic Combin.* 53.4 (2021), pp. 1299–1312. DOI.

[4] L. M. Butler. *Subgroup lattices and symmetric functions*. Vol. 112. Mem. Amer. Math. Soc. 539. 1994. DOI.

[5] E. Chen and D. Tseng. “The splitting subspace conjecture”. *Finite Fields Appl.* 24 (2013), pp. 15–28. DOI.

[6] J. Cigler and J. Zeng. “A curious *q*-analogue of Hermite polynomials”. *J. Combin. Theory Ser. A* 118.1 (2011), pp. 9–26. DOI.

[7] S. Cortes et al. “Matrix ansatz, lattice paths and rook placements”. 21st *International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009)*. Discrete Math. Theor. Comput. Sci. Proc., AK. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009, pp. 313–324. DOI.

[8] S. R. Ghorpade and S. Ram. “Block companion Singer cycles, primitive recursive vector sequences, and coprime polynomial pairs over finite fields”. *Finite Fields Appl.* 17.5 (2011), pp. 461–472. DOI.

[9] J. A. Green. “The characters of the finite general linear groups”. *Trans. Amer. Math. Soc.* 80 (1955), pp. 402–447. DOI.

[10] M. E. H. Ismail, D. Stanton, and G. Viennot. “The combinatorics of *q*-Hermite polynomials and the Askey-Wilson integral”. *European J. Combin.* 8.4 (1987), pp. 379–392. DOI.

[11] N. Jacobson. *Basic algebra. I*. Second. W. H. Freeman and Company, New York, 1985.

[12] M. Josuat-Vergès. “Rook placements in Young diagrams and permutation enumeration”. *Adv. in Appl. Math.* 47.1 (2011), pp. 1–22. DOI.

[13] M. Josuat-Vergès and J. S. Kim. “Touchard-Riordan formulas, *T*-fractions, and Jacobi’s triple product identity”. *Ramanujan J.* 30.3 (2013), pp. 341–378. DOI.

[14] M. Josuat-Vergès and M. Rubey. “Crossings, Motzkin paths and moments”. *Discrete Math.* 311.18-19 (2011), pp. 2064–2078. DOI.
REFERENCES

[15] H. Niederreiter. “The multiple-recursive matrix method for pseudorandom number generation”. Finite Fields Appl. 1.1 (1995), pp. 3–30. DOI.

[16] J.-G. Penaud. “Une preuve bijective d’une formule de Touchard-Riordan”. Discrete Math. 139.1-3 (1995). Formal power series and algebraic combinatorics (Montreal, PQ, 1992), pp. 347–360. DOI.

[17] A. Prasad. Sage Reference Manual: Similarity class types of matrices with entries in a finite field. Accessed on 27th April 2022. URL.

[18] A. Prasad and S. Ram. Set Partitions, Tableaux, and Subspace Profiles of Regular Diagonal Operators. To appear in the Proceedings of the 34th Conference on Formal Power Series and Algebraic Combinatorics (Bangalore). 2022.

[19] A. Prasad and S. Ram. Set partitions, tableaux, and subspace profiles under regular split semisimple matrices. 2021. arXiv: 2112.00479 [math.CO].

[20] O. Ramaré. “Rationality of the zeta function of the subgroups of abelian p-groups”. Publ. Math. Debrecen 90.1-2 (2017), pp. 91–105. DOI.

[21] R. C. Read. “The chord intersection problem”. Second International Conference on Combinatorial Mathematics (New York, 1978). Vol. 319. Ann. New York Acad. Sci. New York Acad. Sci., New York, 1979, pp. 444–454. DOI.

[22] J. Riordan. “The distribution of crossings of chords joining pairs of $2n$ points on a circle”. Math. Comp. 29 (1975), pp. 215–222. DOI.

[23] The Sage Developers. SageMath, the Sage Mathematics Software System (Version 9.5). 2022. URL.

[24] J. Touchard. “Sur un problème de configurations et sur les fractions continues”. Canad. J. Math. 4 (1952), pp. 2–25. DOI.