Abstract

Semiclassical Hamiltonian field theory is investigated from the axiomatic point of view. A notion of a semiclassical state is introduced. An "elementary" semiclassical state is specified by a set of classical field configuration and quantum state in this external field. "Composed" semiclassical states viewed as formal superpositions of "elementary" states are nontrivial only if the Maslov isotropic condition is satisfied; the inner product of "composed" semiclassical states is degenerate. The mathematical proof of Poincare invariance of semiclassical field theory is obtained for "elementary" and "composed" semiclassical states. The notion of semiclassical field is introduced; its Poincare invariance is also mathematically proved.
1 Introduction

Different approaches to semiclassical field theory have been developed. Most of them were based on the functional integral technique: physical quantities were expressed via functional integrals which were evaluated with the help of saddle-point or stationary-phase technique. Since energy spectrum and S-matrix elements can be found from the functional integral [1, 2], this approach appeared to be useful for the soliton quantization theory [1, 2, 3, 4, 5].

Another important partial case of the semiclassical field theory is the theory of quantization in a strong external background classical field [1] or in curved space-time [1]: one decomposes the field as a sum of a classical c-number component and a quantum component. Then the theory is quantized.

The one-loop approximation [1, 2, 10, 11], the time-dependent Hartree-Fock approximation [1, 2, 3] and the Gaussian approximation developed in [14, 15, 16, 17] may be also viewed as examples of applications of semiclassical conceptions.

On the other hand, the axiomatic field theory [18, 19, 20] tells us that main objects of QFT are states and observables. The Poincare group is represented in the Hilbert state space, so that evolution, boosts and other Poincare transformations are viewed as unitary operators.

The purpose of this paper is to introduce the semiclassical analogs of such QFT notions as states, fields and Poincare transformations. The analogs of Wightman Poincare invariance and field axioms for the semiclassical field theory are to be formulated and checked.

Unfortunately, ”exact” QFT is mathematically constructed for a restricted class of models only (see, for example, [21, 22, 23, 24]). Therefore, formal approximate methods such as perturbation theory seem to be ways to quantize the field theory rather than to construct approximations for the exact solutions of QFT equations. The conception of field quantization within the perturbation framework is popular [25, 26]. One can expect that the semiclassical approximation plays an analogous role.

To construct the semiclassical formalism based on the notion of a state, one should use the equation-of-motion formulation of QFT rather than the usual $S$-matrix formulation. It is well-known that additional difficulties such as Stueckelberg divergences [27] and problems associated with the Haag theorem [28, 29, 30] arise in the equation-of-motion approach. There are some ways to overcome them. The vacuum divergences can be eliminated in the perturbation theory with the help of the Faddeev transformation [29]. Stueckelberg divergences can be treated analogously [30] (exactly solvable models with Stueckelberg divergences have been suggested recently [31, 32]). These investigations are important for the semiclassical Hamiltonian field theory [33].

The semiclassical approaches are formally applicable to the quantum field theory models if the Lagrangian depends on the fields $\varphi$ and the small parameter $\lambda$ as follows (see, for example, [4]):

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 - \frac{1}{\lambda} V(\sqrt{\lambda} \varphi),$$  

where $V$ is an interaction potential. To illustrate the formal semiclassical ansatz for the state vector, use the functional Schrodinger representation (see, for example, [12, 13, 16, 17]). States at fixed moment of time are represented as functionals $\Psi[\varphi(\cdot)]$ depending on fields $\varphi(x), x \in \mathbb{R}^d$, the field operator $\hat{\varphi}(x)$ is the operator of multiplication by $\varphi(x)$, while the canonically conjugated momentum $\hat{\pi}(x)$ is represented as a differentiation operator $-i\delta / \delta \varphi(x)$. The functional Schrodinger equation reads

$$i \frac{d\psi^t}{dt} = \mathcal{H} \psi^t,$$

where

$$\mathcal{H} = \int dx \left[ -\frac{\delta^2}{2} \frac{\delta^2}{\delta \varphi(x) \delta \varphi(x)} + \frac{1}{2} (\nabla \varphi)^2(x) + \frac{m^2}{2} \varphi^2(x) + \frac{1}{\lambda} V(\sqrt{\lambda} \varphi(x)) \right].$$

The simplest semiclassical state corresponds to the Maslov theory of complex germ in a point [34, 35, 36]. It depends on the small parameter $\lambda$ as

$$\psi^t[\varphi(\cdot)] = e^{\frac{1}{\lambda} S^t} e^{\frac{i}{\lambda} \int d\pi \Pi^t(\varphi) \sqrt{\lambda} - \Phi^t(\varphi)} f^t\left(\varphi(\cdot) - \frac{\Phi^t(\cdot)}{\sqrt{\lambda}}\right) \equiv (K_{S^t, \Pi^t, \Phi^t} f^t)[\varphi(\cdot)],$$
where $S^t$, $\Pi^t(x)$, $\Phi^t(x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^d$ are smooth real functions which rapidly damp with all their derivatives as $x \to \infty$, $f^t[\phi(\cdot)]$ is a $t$-dependent functional.

As $\lambda \to 0$, the substitution (1.3) satisfies eq. (1.2) in the leading order in $\lambda$ if the following relations are obeyed. First, for the "action" $S^t$ one finds,

$$\frac{dS^t}{dt} = \int d\mathbf{x}[\Pi^t(x)\dot{\Phi}^t(x) - \frac{1}{2}(\Pi^t(x))^2 - \frac{1}{2}(\nabla\Phi^t(x))^2 - \frac{m^2}{2}(\Phi^t(x))^2 - V(\Phi^t(x))],$$

(1.4)

Second, $\Pi^t$, $\Phi^t$ obeys the classical Hamiltonian system

$$\dot{\Phi}^t = \Pi^t, -\dot{\Pi}^t = (-\Delta + m^2)\Phi^t + V'(\Phi^t),$$

(1.5)

Finally, the functional $f^t$ satisfies the functional Schrödinger equation with the quadratic Hamiltonian

$$if^t[\phi(\cdot)] = \int d\mathbf{x} \left[ -\frac{1}{2}\frac{\delta^2}{\delta\phi(\mathbf{x})}\delta\phi(\mathbf{x}) + \frac{1}{2}(\nabla\phi(\mathbf{x}))^2 + \frac{m^2}{2}\phi^2(\mathbf{x}) + \frac{1}{2}V''(\Phi^t(\mathbf{x}))\phi^2(\mathbf{x}) \right] f^t[\phi(\cdot)].$$

(1.6)

There are more complicated semiclassical states that also approximately satisfy the functional Schrödinger equation (1.2). These ansätze correspond to the Maslov theory of Lagrangian manifolds with complex germs [34, 35, 36]. They are discussed in section 5.

However, the QFT divergences lead to the following difficulties.

It is not evident how one should specify the class of possible functionals $f$ and introduce the inner product on such a space via functional integral. This class was constructed in [33]. In particular, it was found when the Gaussian functional

$$f[\phi(\cdot)] = \text{const} \exp\left(\frac{i}{2} \int d\mathbf{x} d\mathbf{y} \phi(\mathbf{x}) \phi(\mathbf{y}) \mathcal{R}(\mathbf{x}, \mathbf{y})\right)$$

(1.7)

belongs to this class. The condition on the quadratic form $\mathcal{R}$ which was obtained in [33] depends on $\Phi$, $\Pi$ and differs from the analogous condition in the free theory. This is in agreement with the statement of [37, 38] that nonequivalent representations of the canonical commutation relations at different moments of time should be considered if QFT in the strong external field is investigated in the leading order in $\lambda$. However, this does not lead to non-unitarity of the exact theory: the simple example has been presented in [32].

Another problem is to formulate the semiclassical theory in terms of the axiomatic field theory. Section 2 deals with formulation of axioms of relativistic invariance and field for the semiclassical theory. Section 3 is devoted to construction of $\text{Poincaré}$ transformations. In section 4 the notion of semiclassical field is investigated. More complicated semiclassical states are constructed in section 5. Section 6 contains concluding remarks.

## 2 Axioms of semiclassical field theory

In the Wightman axiomatic approach the main object of QFT is a notion of a state space [18, 19, 20]. Formula (1.3) shows us that in the semiclassical field theory a state at fixed moment of time should be viewed as a set $(S, \Pi(\cdot), \Phi(\cdot), f[\phi(\cdot)])$ of a real number $S$, real functions $\Pi(x)$, $\Phi(x)$, $x \in \mathbb{R}^d$ and a functional $f[\phi(\cdot)]$ from some class. This class depends on $\Pi$ and $\Phi$. Superposition of semiclassical states $(S_1, \Pi_1, \Phi_1, f_1)$ and $(S_2, \Pi_2, \Phi_2, f_2)$ is of the semiclassical type (1.3) if and only if $S_1 = S_2$, $\Phi_1 = \Phi_2$, $\Pi_1 = \Pi_2$.

Thus, one introduces [33, 34] the structure of a vector bundle (called as a "semiclassical bundle" in [10]) on the set of semiclassical states of the type (1.3). The base of the bundle being a space of sets $(S, \Pi, \Phi)$ ("extended phase space" [33]) will be denoted as $\mathcal{X}$. The fibers are classes of functionals which depend on $\Phi$ and $\Pi$. Making use of the result concerning the class of functionals [33], one makes
the bundle trivial as follows. Consider the $\Phi$, $\Pi$-dependent mapping $V$ which defines a correspondence between functionals $f$ and elements of the Fock space $\mathcal{F}$:

$$V : \Psi \mapsto f, \quad \Psi \in \mathcal{F}, \quad f = f[\phi(\cdot)].$$

as follows. Let $\hat{\mathcal{R}}(x, y)$ be an $\Phi$, $\Pi$-dependent symmetric function such that its imaginary part is a kernel of a positively definite operator and the condition of ref. [33] (see eq. (3.73) of subsection 3.6) is satisfied. By $\hat{\mathcal{R}}$ we denote the operator with kernel $\hat{\mathcal{R}}$, while $\hat{\Gamma}$ has a kernel $i^{-1}(\hat{\mathcal{R}} - \hat{\mathcal{R}}^*)$. The vacuum vector of the Fock space corresponds to the Gaussian functional $\langle i \rangle$. The operator $V$ is uniquely defined from the relations

$$V^{-1} \phi(x) V = i(\hat{\Gamma}^{-1/2}(A^+ - A^-))(x),$$

$$V^{-1} \frac{i}{\delta \phi(x)} V = i(\hat{\mathcal{R}}\hat{\Gamma}^{-1/2}A^+ - \hat{\mathcal{R}}^*\hat{\Gamma}^{-1/2}A^-)(x).$$

Here $A^\pm(x)$ are creation and annihilation operators in the Fock space.

**Definition 2.1.** A semiclassical state is a point on the trivial bundle $\mathcal{X} \times \mathcal{F} \to \mathcal{X}$.

An important postulate of QFT is Poincaré invariance. This means that a representation of the Poincare group in the state space should be specified. For each Poincare transformation of the form

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu, \quad \mu, \nu = 0, d$$

which is denoted as $(a, \Lambda)$, the unitary operator $U_{a,A}$ should be specified. The group property

$$U_{(a_1, \Lambda_1)}U_{(a_2, \Lambda_2)} = U_{(a_1, \Lambda_1)(a_2, \Lambda_2)}$$

with

$$(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2).$$

should be satisfied.

Formulate an analog of the Poincare invariance axiom for the semiclassical theory. Suppose that the Poincare transformation $U_{a,A}$ takes any semiclassical state $(X, f)$ to a semiclassical state $(\hat{X}, \hat{f})$ in the leading order in $\lambda^{1/2}$. Denote $\hat{X} = u_{a,A}X, \ f = U(u_{a,A}X \leftarrow X)f$.

**Axiom 1 (Poincare invariance)**

1) the mappings $u_{a,A} : \mathcal{X} \to \mathcal{X}$ are specified, the group properties for them $u_{a_1, \Lambda_1}u_{a_2, \Lambda_2} = u_{(a_1, \Lambda_1)(a_2, \Lambda_2)}$ are satisfied;  
2) for all $X \in \mathcal{X}$ the unitary operators $U_{a,A}(u_{a,A}X \leftarrow X) : \mathcal{F} \to \mathcal{F}$, obeying the group property

$$U_{a_1, \Lambda_1}(u_{a(1), \Lambda_1})(a_2, \Lambda_2)X \leftarrow u_{a(2), \Lambda_2}x)U_{a_2, \Lambda_2}(u_{a(2), \Lambda_2}X \leftarrow X) = U_{(a_1, \Lambda_1)(a_2, \Lambda_2)}(u_{a(1), \Lambda_1})(a_2, \Lambda_2)X \leftarrow X)$$

are specified.

An important feature of QFT is the notion of a field: it is assumed that an operator distribution $\hat{\phi}(x, t)$ is specified. Investigate it in the semiclassical theory. Applying the operator $\phi(x)$ to the semiclassical state (1.3), we obtain an analogous state:

$$e^{\frac{i}{\hbar}S_t}e^{\frac{i}{\hbar}\int dx \Pi(x)\sqrt{\Phi(x)}\Phi(x)} \hat{f}(x) - \frac{\Phi(x)}{\sqrt{\lambda}},$$

where

$$\hat{f}(\phi(\cdot)) = (\lambda^{-1/2}\Phi'(x) + \phi(x)) \hat{f}([\phi(\cdot)]$$

As $\lambda \to 0$, one has

$$\hat{\phi}(x, t) = \lambda^{-1/2}\phi'(x) + \hat{\phi}(x, t : X),$$

where $\hat{\phi}(x, t : X)$ is a $\Pi, \Phi$-dependent operator in $\mathcal{F}$, $\Phi'(x) \equiv \Phi(x : X)$ is a solution to the Cauchy problem for eq.(1.5). The field axiom can be reformulated as follows.
Axiom 2. For each $X \in \mathcal{X}$ the operator distribution $\hat{\phi}(x, t; X) : \mathcal{F} \rightarrow \mathcal{F}$ is specified.

An important feature of the relativistic quantum field theory is the property of Poincare invariance of fields. The operator distribution $\hat{\phi}(x, t)$ should obey the following property

$$U_{a, \Lambda} \hat{\phi}(x) = \hat{\phi}(\Lambda x + a)U_{a, \Lambda}.$$ 

Apply this identity to a semiclassical state $(X, f)$. In leading orders in $\lambda^{1/2}$, one obtains:

$$\lambda^{-1/2}\Phi(x : X)(u_{a, \Lambda}X, U_{a, \Lambda}(u_{a, \Lambda}X \triangleleft X)f) + (u_{a, \Lambda}X, U_{a, \Lambda}(u_{a, \Lambda}X \triangleleft X)f) = \lambda^{-1/2}\Phi(\Lambda x + a : u_{a, \Lambda}X)(u_{a, \Lambda}X, U_{a, \Lambda}(u_{a, \Lambda}X \triangleleft X)f)$$

$$+(u_{a, \Lambda}X, \hat{\phi}(\Lambda x + a : u_{a, \Lambda}X)U_{a, \Lambda}(u_{a, \Lambda}X \triangleleft X)f).$$

Therefore, we formulate the following axiom.

Axiom 3. (Poincare invariance of fields). The following properties are satisfied:

$$\Phi(x : X) = \Phi(\Lambda x + a : u_{a, \Lambda}X); \quad (2.3)$$

$$\hat{\phi}(\Lambda x + a : u_{a, \Lambda}X)U_{a, \Lambda}(u_{a, \Lambda}X \triangleleft X) = U_{a, \Lambda}(u_{a, \Lambda}X \triangleleft X)\hat{\phi}(x : X). \quad (2.4)$$

3 Construction of the Poincare transformations

This section is devoted to the problem of relativistic invariance of the semiclassical field theory. The axiom 1 will be checked. The mappings $u_{a, \Lambda}$ and unitary operators $U_{a, \Lambda}$ are to be specified, the group property is to be justified.

3.1 Heuristic definition

Consider some special cases of Poincare transformations $(a, \Lambda)$. The transformation $(a, 1)$ is called translation. If $a^0 = 0$, this is a spatial translation, while the $a = 0$-case corresponds to the time translation or evolution. The transformation $(0, \Lambda) \equiv \Lambda$ is called as a Lorentz transformation. If $\Lambda = \Lambda_{\tau}$,

$$\Lambda_{\tau} = \begin{pmatrix} \cosh \tau & -\sinh \tau & 0 \\ -\sinh \tau & \cosh \tau & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.1)$$

the Lorentz transformation is called as $x^1$-boost. If

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix}$$

the Lorentz transformation is called as a spatial rotation. Let $L_n$ be such a spatial rotation that

$$L_n \mathbf{n} = \mathbf{e}_1, \quad (3.2)$$

where $\mathbf{e}_1$ is a spatial vector of the form $(1, 0, 0, \ldots)$. The transformation

$$\Lambda_{\phi \mathbf{n}} = L_n^{-1} \Lambda_{\phi} L_n \quad (3.3)$$

will be called as $\mathbf{n}$-boost. It does not depend on choice of $L_n$. Namely, let $L_n^{(1)}$ and $L_n^{(2)}$ be spatial rotations obeying eq. (3.2). Then $L_n^{(1)}(L_n^{(2)})^{-1}\mathbf{e}_1 = \mathbf{e}_1$. Therefore, the transformations $\Lambda_{\phi}$ and $L_n^{(1)}(L_n^{(2)})^{-1}$ commute. Thus, $(L_n^{(1)})^{-1}\Lambda_{\phi} L_n^{(1)} = (L_n^{(2)})^{-1}\Lambda_{\phi} L_n^{(2)}$. 

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Lemma 3.1. Let \((a, \Lambda)\) be a Poincare transformation. It is uniquely presented as
\[
(a, \Lambda) = (a, 1) \Lambda_\phi L,
\]
where \(\Lambda_\phi\) is a boost, \(L\) is a spatial rotation.

Proof. It follows from (2.2) that \((a, \Lambda) = (a, 1)\Lambda\). Let us show that \(\Lambda = \Lambda_\phi L\) and this decomposition is unique. Consider the transformation \(x' = \Lambda x\):
\[
x'^0 = ax^0 + \beta_i x^i, \\
x'^i = \gamma^i x^0 + \Lambda^i_k x^k.
\]

Let \(L(1)\) be such a rotation that \(L(1)\mathbf{\tilde{\gamma}} = ||\mathbf{\tilde{\gamma}}||\mathbf{e}_1\). Consider the transformation \(y^i = (L(1))_j^i x'^j, \ y^0 = x^0\). One has
\[
y^0 = ax^0 + \beta_i x^i; \\
y^i = ||\mathbf{\tilde{\gamma}}||x^0 + A^i_k x^k; \\
\tilde{\gamma}^j = A^j_\beta x^\beta.
\]

where \(\beta, \rho = 2, d, i, k = 1, d\). Since \(a^2 - ||\mathbf{\tilde{\gamma}}||^2 = 1\), set \(a = \cosh \phi, ||\mathbf{\tilde{\gamma}}|| = \sinh \phi\). Denote \(z' = \Lambda^{-1} y\). One has
\[
z'^0 = x^0 + \tilde{\beta}_i x^i, \\
z'^i = B^i_j x^j.
\]

Therefore, \(\tilde{\beta}_i = 0\). This is a rotation \(L(2)\) then. Thus, \(\Lambda = (L(1))^{-1} \Lambda_\phi L(1)(L(1))^{-1} L(2) = \Lambda_{\phi N} L\) with \(N = \mathbf{\tilde{\gamma}}/||\mathbf{\tilde{\gamma}}||\). This decomposition is unique. Lemma 3.1 is proved.

To construct mappings \(u_{a,\Lambda}\) and operators \(U_{a,\Lambda}\), one may consider first the partial cases (time evolution, spatial translations, \(x^i\) or \(n\)-boost, spatial rotations) and then use the group property.

First of all, let us consider the representation of the Poincare group \(\tilde{U}_{a,\Lambda}\) in the functional Schrodinger representation. Formally, they are related with \(U_{a,\Lambda}\) by the relation
\[
\tilde{U}_{a,\Lambda}(u_{a,\Lambda} X \leftarrow X) = V_{u,\Lambda} X U_{a,\Lambda}(u_{a,\Lambda} X \leftarrow X) V_X^{-1}.
\]

To construct operators \(\tilde{U}_{a,\Lambda}\) and mappings \(u_{a,\Lambda}\), let us use formal expressions for the Poincare transformations in the ”exact” field theory. Namely, the (formally) unitary operator \(U_{a,\Lambda}\) corresponding to the Poincare transformation
\[
(a, \Lambda) = (a^0, 0)(a, 0) \exp(\alpha^k l^0 k) \exp(\frac{1}{2} \theta_{sm} l^m l^s)
\]

where \(\theta_{sm} = -\text{theta}_{ms}\),
\[
(\lambda^\mu)^\alpha = -g^{\lambda \alpha} \delta^\mu_\beta + g^{\mu \alpha} \delta^\lambda_\beta,
\]

has the form
\[
U_{a,\Lambda} = \exp[i \mathcal{P}^0 a^0] \exp[-i \mathcal{P}^j a^j] \exp[i \alpha^k \mathcal{M}^{0k}] \exp[\frac{i}{2} \mathcal{M}^{lm} \theta_{lm}].
\]

The momentum and angular momentum operators entering to formula (3.3) have the well-known form (see, for example, 23)
\[
\mathcal{P}^\mu = \int d^4x T^{\mu 0}(x), \quad \mathcal{M}^{\mu \lambda} = \int d^4x [x^{\mu} T^{\lambda 0}(x) - x^{\lambda} T^{\mu 0}(x)],
\]

where formally
\[
T^{00} = \frac{1}{2} \pi^2 + \frac{1}{2} \partial_i \hat{\phi} \partial_i \hat{\phi} + \frac{m^2}{2} \hat{\phi}^2 + \frac{1}{\lambda} V(\sqrt{\lambda} \hat{\phi}), \quad T^{k0} = -\partial_k \hat{\phi} \hat{\pi}.
\]

We are going to apply the operator (3.5) to the semiclassical state (1.3). Note that the operators \(\mathcal{P}^\mu\) and \(\mathcal{M}^{\mu \nu}\) (3.6) depend on field \(\hat{\phi}\) and momentum \(\hat{\pi}\) semiclassically,
\[
\mathcal{P}^\mu = \frac{1}{\lambda} \mathcal{P}^\mu(\sqrt{\lambda} \hat{\pi}(\cdot), \sqrt{\lambda} \hat{\phi}(\cdot)), \quad \mathcal{M}^{\mu \nu} = \frac{1}{\lambda} \mathcal{M}^{\mu \nu}(\sqrt{\lambda} \hat{\pi}(\cdot), \sqrt{\lambda} \hat{\phi}(\cdot)),
\]
It is convenient to consider the more general problem (cf. [35]). Let us find as \( \lambda \to 0 \) the state
\[
\exp(-iA)K_{s,\Pi,\Phi}f^0, 
\]
where \( K_{s,\Pi,\Phi} \) has the form (1.3),
\[
\mathcal{A} = \frac{1}{\lambda}A(\sqrt{\xi}(\cdot), \sqrt{\lambda}\hat{\phi}(\cdot)).
\]
Note that the state functional (3.7) may be viewed as a solution to the Cauchy problem of the form
\[
i\frac{\delta\Psi^\tau}{\delta\tau} = \frac{1}{\lambda}A(\sqrt{\xi}(\cdot), \sqrt{\lambda}\hat{\phi}(\cdot))\Psi^\tau, \quad \Psi^0[\phi(\cdot)] = (K_{s,\Pi,\phi}f^0)[\phi(\cdot)]
\]
at \( \tau = 1 \). Let us look for the asymptotic solution to eq.(3.8) in the following form:
\[
\Psi^\tau[\phi(\cdot)] = (K_{s,\Pi,\phi}f^\tau)[\phi(\cdot)].
\]
Substitution of functional (3.8) to eq.(3.8) gives us the following relation:
\[
\left[-\frac{1}{\lambda}\left(\hat{S}^\tau - \int dx\Pi^\tau(x)\hat{\Phi}^\tau(x)\right) - \frac{1}{\sqrt{\lambda}}\int dx(\Pi^\tau(x)\phi(x) + \hat{\Phi}^\tau(x)i\frac{\delta}{\delta\phi(x)}) + i\frac{\partial}{\partial\tau} f^\tau[\phi(\cdot)]\right] = \frac{1}{\lambda}A(\Pi^\tau(\cdot) - i\sqrt{\lambda}\frac{\delta}{\delta\phi(x)}, \Phi^\tau(\cdot) + \sqrt{\lambda}\phi(\cdot))f^\tau[\phi(\cdot)].
\]
Considering the terms of the orders \( O(\lambda^{-1}) \), \( O(\lambda^{-1/2}) \) and \( O(1) \) in eq.(3.10), we obtain
\[
\hat{S}^\tau = \int dx\Pi^\tau(x)\hat{\Phi}^\tau(x) - A(\Pi^\tau(\cdot), \Phi^\tau(\cdot)), \quad \hat{\Phi}^\tau(x) = \frac{\delta A(\Pi^\tau(\cdot), \Phi^\tau(\cdot))}{\delta\Pi(x)}, \quad \Pi^\tau(x) = -\frac{\delta A(\Pi^\tau(\cdot), \Phi^\tau(\cdot))}{\delta\Phi(x)}.
\]
\[
i\frac{\delta f^\tau[\phi(\cdot)]}{\delta\tau} = \left(\int dx dy \left[1 \frac{\delta}{\delta\phi(x)\delta\Pi(y)} \frac{\delta^2 A}{\delta\phi(x)\delta\Pi(y)} \frac{1}{2} \frac{\delta}{\delta\phi(y)} + \frac{\delta^2 A}{\delta\phi(x)\delta\phi(y)}\phi(y)\right] + A_1\right) f^\tau[\phi(\cdot)].
\]
Here \( A_1 \) is a c-number quantity which counts the ordering of the operators \( \hat{\phi} \) and \( \hat{\pi} \) and is relevant to the renormalization problem.

We see that for the cases \( \mathcal{A} = -\mathcal{P}^0 a^0 \), \( \mathcal{A} = \mathcal{P}^i a^i \), \( \mathcal{A} = -\mathcal{S}^k \mathcal{M}^0 \), \( \mathcal{A} = \frac{1}{2}\theta^{sm}\mathcal{M}^{sm} \) the mapping \( u_{a,\Lambda} \) takes the initial condition for the system (3.11), (3.12) to the solution of the Cauchy problem for this system at \( \tau = 1 \). The operators \( \hat{U}_{a,\Lambda} \) transforms the initial condition for eq.(3.13) to the solution at \( \tau = 1 \).

### 3.2 Poincare invariance of the classical theory

The purpose of this subsection is to find explicit forms of mappings \( u_{a,\Lambda} \). Consider some special cases.

#### 3.2.1 Spatial rotations

For this case, \( a = 0 \), \( \Lambda = L = \exp(\frac{1}{2}\theta^{sm}\theta_{sm}) \), so that \( L^k_l = (\exp\theta)^k_l \), where \( \theta \) is an antisymmetric matrix with elements \( \theta_{kl} \). One has \( \mathcal{A} = \frac{1}{2}\theta_{sm}\mathcal{M}^{sm} \), so that
\[
A[\Pi, \Phi] = \frac{1}{2} \int dx\Pi(x)\theta_{sm}(x^s\partial_m - x^m\partial_s)\Phi(x)
\]
with \( \partial_m \equiv \frac{\partial}{\partial x^m} \). System (3.11), (3.12) takes the form
\[
\hat{\Phi}^\tau(x) = \frac{1}{2}\theta_{sm}(x^s\partial_m - x^m\partial_s)\Phi^\tau(x), \quad \Pi^\tau(x) = \frac{1}{2}\theta_{sm}(x^s\partial_m - x^m\partial_s)\Pi^\tau(x),
\]
\[
\hat{S}^\tau = 0.
\]
Eqs. (3.15) can be represented as
\[
\frac{\partial}{\partial \tau} \Phi^\tau (\exp(\frac{\tau}{2} \theta sm^s) x) = 0, \quad \frac{\partial}{\partial \tau} \Pi^\tau (\exp(\frac{\tau}{2} \theta sm^s) x) = 0.
\]
Therefore, \( \Phi^1 (Lx) = \Phi^0 (x) \), \( \Pi^1 (Lx) = \Pi^0 (x) \), so that
\[
\Phi^1 (x) = \Phi^0 (L^{-1} x), \quad \Pi^1 (x) = \Pi^0 (L^{-1} x), \quad S^1 = S^0.
\] (3.16)

### 3.2.2 Spatial translations

For this case, \( a^0 = 0 \), \( \Lambda = 1 \), so that \( \mathcal{A} = P^j a^j \) and
\[
A[\Pi, \Phi] = - \int dxa^k \partial_k \Phi(x) \Pi(x).
\] (3.17)
System (3.11), (3.12) takes the form
\[
\dot{\Phi}^\tau (x) = - a^k \partial_k \Phi^\tau (x), \quad \dot{\Pi}^\tau (x) = - a^k \partial_k \Pi^\tau (x), \quad \dot{S}^\tau = 0.
\] (3.18)
Therefore,
\[
\Phi^\tau (x) = \Phi(x - a \tau), \quad \Pi^\tau (x) = \Pi(x - a \tau), \quad S^\tau = S^0.
\] (3.19)

### 3.2.3 Evolution transformation

Let \( a^0 = -t, \ a = 0, \ \Lambda = 1 \). Then \( A[\Pi, \Phi] \) is a classical Hamiltonian, so that \( u_{t,0,1} \) is a mapping taking the initial condition for system (1.4), (1.5) to the solution of the corresponding Cauchy problem.

### 3.2.4 The n-boost

Let \( \Lambda = \Lambda_n \) have the form (3.3), \( a = 0 \). Then \( \mathcal{A} = n^k \mathcal{M}^k \), so that
\[
A[\Pi, \Phi] = \int d\mathbf{x} n^k x^k \left[ \frac{1}{2} \Pi^2 (x) + \frac{1}{2} (\nabla \Phi)^2 (x) + m^2 \frac{1}{2} \Phi^2 (x) + V(x, \Phi(x)) \right]
\] (3.20)
System (3.11), (3.12) takes the form:
\[
\dot{\Phi}^\tau (x) = n^k x^k \Pi^\tau (x), \\
- \dot{\Pi}^\tau (x) = - \nabla x^k n^k \nabla \Phi^\tau (x) + n^k x^k (m^2 \Phi^\tau (x) + V' (x, \Phi^\tau (x))), \\
\dot{S}^\tau = \int dx [\Pi^\tau (x) \dot{\Phi}^\tau (x) - x^k n^k (\frac{1}{2} (\Pi^\tau (x))^2 + \frac{1}{2} (\nabla \Phi^\tau (x))^2 + m^2 (\Phi^\tau (x))^2 + V(\Phi^\tau (x))].
\] (3.21)

### 3.2.5 General formulas

Let \((a, \Lambda)\) be an arbitrary Poincare transformation. It happens that the mapping \( u_{a, \Lambda} : (S, \Pi, \Phi) \mapsto (\tilde{S}, \tilde{\Pi}, \tilde{\Phi}) \) has the following form. Let \( \Phi(x, t) \equiv \Phi(x) \) be a solution of the Cauchy problem
\[
\partial_\mu \partial^\mu \Phi(x) + m^2 \Phi(x) + V'(\Phi(x)) = 0, \\
\Phi(x, 0) = \Phi(x), \quad \partial_\mu \Phi(x, t)|_{t=0} = \Pi(x).
\] (3.22)
Denote
\[
\tilde{\Phi}(x) = \Phi(\Lambda^{-1} (x - a)).
\]
It appears that
\[
\tilde{\Phi}(x) = \Phi(x, 0), \quad \tilde{\Pi}(x) = \partial_\mu \tilde{\Phi}(x)|_{t=0} = \partial_\mu \Phi(x)|_{t=0}, \\
\tilde{S} = S + \int dx [\theta(a^0) \theta(-(\Lambda x + a)^0) - \theta(-a^0) \theta((\Lambda x + a)^0)] \\
\times [\frac{1}{2} \partial_\mu \Phi(x) \partial^\mu \Phi(x) - \frac{m^2}{2} \Phi^2 (x) - V(\Phi(x))].
\] (3.23)
First of all, show that eqs. (3.23) are correct for the partial cases mentioned above. For spatial translations and rotations formulas (3.23) give \( \tilde{\delta} x \) eqs.(1.4), (1.5). Let us check formulas (3.23) for the partial case, \( x \). This coincides with eqs.(3.16), (3.19). For evolution transformation, eqs.(3.23) are in agreement with formulas (3.23) for the general case, it is sufficient to prove the following lemma.

Lemma 3.2. For transformation (3.23), the group property is satisfied.

Proof. Let \( u_{a_1, \Lambda_1} \), \( u_{a_2, \Lambda_2} \) be Poincare transformations. Let \( u_{a_1, \Lambda_1}u_{a_2, \Lambda_2} \) be a differentiable functional of \( \Phi \). Any Poincare transformation can be obtained as a composition of considered partial cases. To check formulas (3.23) for the general case, it is sufficient to prove the following lemma.

Let us find the coefficients from eqs.(3.15), (3.19), (1.5), (3.21). Let \( \tilde{\delta} x \) be Poincare transformations. Show that \( \Phi(\Lambda x) = \Phi(\Lambda \tilde{x}) \). Elements of Lie algebra of the Poincare group can be identified with sets \( \tau \), \( \Phi \). According to formula (A.2) of Appendix A, one can introduce the operators \( \Phi(\Lambda x) = \Phi(\Lambda \tilde{x}) \).

Definition 3.1. \( \mathcal{X} \) is a space of sets \( (S, \Pi, \Phi) \) of a number \( S \) and functions \( \Pi, \Phi \in \mathcal{S}(\mathbb{R}^d) \) such that there exists a unique solution of the Cauchy problem (3.22) such that the functions \( \Phi(\Lambda x + a) |_{x^0 = 0} \) and \( \partial_{\mu} \Phi(\Lambda x + a) |_{x^0 = 0} \) are of the class \( \mathcal{S}(\mathbb{R}^d) \) for all \( a, \Lambda \).

We see that the transformation \( u_{a, \Lambda} : \mathcal{X} \to \mathcal{X} \) is defined.

3.2.6 Infinitesimal properties

According to formula (A.2) of Appendix A, one can introduce the operators \( \delta[A] \) on the space of differentiable functionals \( F \) of \( S, \Pi, \Phi \) for each element \( A \) of the Poincare algebra. This operator plays an important role in analysis of algebraic properties of the representation \( U_{a, \Lambda} \).

Elements of Lie algebra of the Poincare group can be identified with sets \( (b^\mu, \theta^{\mu
u}) \), \( \theta^{\mu
u} = -\theta^{\nu\mu} \). The curve on the Poincare group with the tangent vector \( (b, \theta) \) can be chosen to be

\[
(a(\tau), \Lambda(\tau)) = (\tau b, \exp(\tau \theta^{\mu k} l_{0k}) \exp(\tau \frac{1}{2} \theta^{km} l_{km})).
\]

The operator \( \delta[(b, \theta)] \) is a linear combination of \( b, \theta \):

\[
\delta[(b, \theta)] = \frac{1}{2} \theta_{km} \delta^{im} - b^k \delta^i_p - \theta_{0m} \delta^m_B + b^0 \delta_H.
\]

Let us find the coefficients from eqs.(3.15), (3.19), (3.21). Let \( F \) be a differentiable functional of \( S, \Pi, \Phi \).

1. Let \( \theta = 0, b^0 = 0, b \neq 0 \). Then

\[
-b^k \delta^i_p F = -b^k \int d\mathbf{x} \left( \frac{\delta F}{\delta \Phi(\mathbf{x})} \partial_k \Phi(x) + \frac{\delta F}{\delta \Pi(x)} \partial_k \Pi(x) \right)
\]
2. For \( \theta^{0k} = 0, \ b = 0 \), one has
\[
\frac{1}{2} \theta_{lm} \delta F = \frac{1}{2} \theta_{lm} \int dx ((x^l \partial_m - x^m \partial_l) \Phi(x) \frac{\delta F}{\delta \Phi(x)}) + (x^l \partial_m - x^m \partial_l) \Pi(x) \frac{\delta F}{\delta \Pi(x)}.
\]

3. For \( b^0 \neq 0, \ b = 0, \ \theta = 0 \)
\[
b^0 \delta H F = -b^0 \int dx \left[ \Pi(x) \frac{\delta F}{\delta \Phi(x)} - (-\Delta \Phi(x) + m^2 \Phi(x) + V'(\Phi(x))) \frac{\delta F}{\delta \Pi(x)} \right] - \frac{b^0}{2} \int dx \left[ \frac{1}{2} \Pi^2(x) - \frac{1}{2} (\nabla \Phi(x))^2 - \frac{m^2}{2} \Phi^2(x) - V(\Phi(x)) \right].
\]

4. For boost transformation \( \theta^{0k} \neq 0, \ \eta_{lm} = 0, \ b = 0 \) and
\[
-\theta_{0m} \delta^m = \theta_{0m} \int dx \left[ x^m \Pi(x) \frac{\delta F}{\delta \Phi(x)} - (-\partial_x x^m \partial_l \Phi(x) + x^m m^2 \Phi(x) + x^m V'(\Phi(x))) \frac{\delta F}{\delta \Pi(x)} \right] + \theta_{0m} \frac{\delta F}{\delta \Pi(x)} \int dx \left[ \frac{1}{2} \Pi^2(x) - \frac{1}{2} (\nabla \Phi(x))^2 - \frac{m^2}{2} \Phi^2(x) - V(\Phi(x)) \right].
\]

The introduced operators obey usual properties of the Poincare algebra \( [A.3] \):
\[
[i \delta^k_p, i \delta^s_H] = 0; \quad [i \delta^l_m, i \delta^s_p] = i (g^{ms} i \delta^l_p - g^{ls} i \delta^m_p); \quad [i \delta^l_p, i \delta^s_p] = 0.
\]

\[
[i \delta^k_m, i \delta^s_H] = 0; \quad [i \delta^l_B, i \delta^s_p] = -i g^{ks} i \delta^H; \quad [i \delta^l_m, i \delta^s_H] = -i (g^{lr} i \delta^m - g^{mr} i \delta^m + g^{ms} i \delta^m - g^{ls} i \delta^m);
\]
\[
[i \delta^l_B, i \delta^s_B] = -i (g^{lk} i \delta^B - g^{mk} i \delta^B);
\]
\[
[i \delta^l_B, i \delta^l_B] = -\delta^k_P, \quad [i \delta^l_B, i \delta^l_B] = \delta^k_M.
\]

### 3.3 Semiclassical Poincare transformations in the functional representation

We have formally found the operators \( \tilde{U}_{a,A} \). However, it is not easy to check the group property. Therefore, construct the representation of the Poincare algebra according to Appendix A. Then we will check the algebraic property. The group property will be a corollary of the results of Appendix A.

Let us construct the operators \( \tilde{H}(\xi, \phi) : S, \Pi, \Phi) \) \( [A.7] \) for elements of the Poincare algebra:
\[
\tilde{H}(\xi, \phi) : S, \Pi, \Phi) = \frac{1}{2} \theta_{lm} \tilde{M}^{lm} + b^k \tilde{P}^k + \theta_{0m} \tilde{B}^m - b^0 H.
\]

Consider some cases.

#### 3.3.1 Spatial rotations

For this case, \( a = 0, \ \Lambda = L = \exp(\frac{1}{2} \theta_{sm} l^{sm} \Phi) \), \( A = -\frac{1}{2} \theta_{sm} M^{sm} \), \( A \) has the form \( \mathcal{A}, \mathcal{A}' \). Therefore, eq. \( (7.13) \) takes the form
\[
i f^r = \frac{1}{2} \theta_{sm} \int dx (x^s \partial_m - x^m \partial_s) \phi(x) \frac{1}{i} \frac{\delta}{\delta \phi(x)} f^r.
\]

It follows from eq. \( (A.7) \) that
\[
\tilde{M}^{sm} = - \int dx [(x^s \partial_m - x^m \partial_s) \phi(x)] \frac{1}{i} \frac{\delta}{\delta \phi(x)}.
\]
3.3.2 Spatial translations

Let \( a^0 = 0, \, a = b \tau, \, \Lambda = 1 \). Then \( \mathcal{A} = P^k a^k \), \( A \) has the form \( (3.17) \). Eq.\( (3.13) \) takes the form:

\[
i \dot{j}^\tau = -b^k \int d\mathbf{x} \partial_k \phi(\mathbf{x}) \frac{1}{i} \frac{\delta}{\delta \phi(\mathbf{x})} f^\tau.
\]

Therefore,

\[
\tilde{P}^k = - \int d\mathbf{x} \partial_k \phi(\mathbf{x}) \frac{1}{i} \frac{\delta}{\delta \phi(\mathbf{x})}.
\] (3.25)

3.3.3 Evolution transformation

Let \( a^0 = -\tau, \, a = 0, \, \Lambda = 1 \). Then \( A \) is a classical Hamiltonian, eq.\( (3.13) \) takes the form \( (1.6) \). \( \tilde{H} \) takes the form

\[
\tilde{H} = \int d\mathbf{x} \left[ -\frac{1}{2} \frac{\delta^2}{\delta \phi(\mathbf{x}) \delta \overline{\phi}(\mathbf{x})} + \frac{1}{2} (\nabla \phi)^2(\mathbf{x}) + \frac{m^2}{2} \phi^2(\mathbf{x}) + \frac{1}{2} V''(\Phi(\mathbf{x})) \phi^2(\mathbf{x}) \right].
\] (3.26)

3.3.4 The n-boost

Let \( a = 0, \, \Lambda = \Lambda_n \) have the form \( (3.3) \), so that \( \theta_{k0} = -n^k = -\theta_{0k} \). Then \( \mathcal{A} = n^k \mathcal{M}^k \), \( A \) has the form \( (3.20) \), so that eq.\( (3.13) \) takes the form:

\[
i \dot{j}^\tau = \int d\mathbf{x} n^k x^k \left[ -\frac{1}{2} \frac{\delta^2}{\delta \phi(\mathbf{x}) \delta \overline{\phi}(\mathbf{x})} + \frac{1}{2} (\nabla \phi)^2(\mathbf{x}) + \frac{m^2}{2} \phi^2(\mathbf{x}) + \frac{1}{2} V''(\Phi(\mathbf{x})) \phi^2(\mathbf{x}) \right] f^\tau.
\]

Therefore,

\[
\tilde{B}^m = \int d\mathbf{x} x^m \left[ -\frac{1}{2} \frac{\delta^2}{\delta \phi(\mathbf{x}) \delta \overline{\phi}(\mathbf{x})} + \frac{1}{2} (\nabla \phi)^2(\mathbf{x}) + \frac{m^2}{2} \phi^2(\mathbf{x}) + \frac{1}{2} V''(\Phi(\mathbf{x})) \phi^2(\mathbf{x}) \right].
\] (3.27)

Note that the divergences in these operators are to be eliminated by adding \( c \)-number quantities to them.

3.3.5 Properties of infinitesimal transformations

For operators

\[
\tilde{\mathcal{M}}^{ms} = \mathcal{M}^{ms} + i \delta_{\mathcal{M}}^{ms}, \quad \tilde{\mathcal{P}}^m = \mathcal{P}^m + i \delta_{\mathcal{P}}^m, \quad \tilde{\mathcal{X}}^0 = \mathcal{H}^{m} + i \delta_{\mathcal{H}}, \quad \tilde{\mathcal{M}}^{k0} = \tilde{\mathcal{B}}^{k} + i \delta_{\mathcal{B}}^{k}
\]

the commutation relations of the Poincare algebra

\[
[\tilde{\mathcal{P}}, \tilde{\mathcal{P}}^\mu] = 0; \quad [\tilde{\mathcal{M}}^{\lambda \mu}, \tilde{\mathcal{M}}^{\sigma \nu}] = i (g^{\mu \sigma} \tilde{\mathcal{P}}^{\lambda} - g^{\lambda \sigma} \tilde{\mathcal{P}}^{\mu})
\]

\[
[\tilde{\mathcal{M}}^{\lambda \mu}, \tilde{\mathcal{M}}^{\rho \sigma}] = -i (g^{\lambda \rho} \tilde{\mathcal{M}}^{\mu \sigma} - g^{\mu \rho} \tilde{\mathcal{M}}^{\lambda \sigma} + g^{\rho \sigma} \tilde{\mathcal{M}}^{\lambda \mu} - g^{\lambda \sigma} \tilde{\mathcal{M}}^{\mu \rho})
\] (3.28)

should be satisfied (eq.\( (A.11) \)). The formal check of these relations is straightforward. However, the functional representation is ill-defined, so that one should use the Fock representations and perform a renormalization.

Find a relationship between operators \( H(A : X) \) and \( \tilde{H}(A : X) \) being generators of representation \( U_{\alpha, \Lambda} \) in Fock and Schrodinger pictures. Let \( g(\tau) \) be a curve on the Poincare group with tangent vector \( A \). Eq.\( (3.4) \) implies

\[
\tilde{U}_{g(\tau)}[X] = V_{u_g(\tau)} X U_{g(\tau)}[X] V_X^{-1}.
\]

Differentiating this relation by \( \tau \) at \( \tau = 0 \), we find

\[
-i \tilde{H}(A : X) = \delta[A] V_X V_X^{-1} - i V_X H(A : X) V_X^{-1}.
\]
Therefore,

\[ H(A : X) - i\delta[A] = V_X^{-1}(\tilde{H}(A : X) - i\delta[A])V_X. \]

We see that operators

\[ \tilde{H}(A : X) \equiv H(A : X) - i\delta[A] = -\frac{1}{2}\theta_{lm}\tilde{M}^{lm} + b^k\tilde{P}^k + \theta_{nm}\tilde{B}^n - b^0\tilde{H} \]

with

\[ \tilde{M}^{ms} = V_X^{-1}(\tilde{M}^{ms} + i\delta_{M}^{ms})V_X, \]
\[ \tilde{P}^m = V_X^{-1}(\tilde{P}^m + i\delta_{H}^m)V_X, \]
\[ \tilde{B}^0 = V_X^{-1}(\tilde{B}^0 + i\delta_{B}^0)V_X \]

formally obey commutation relations (3.28). However, the divergences and renormalization problem should be taken into account.

### 3.4 Poincare transformations in the Fock representation

The purpose of this subsection is to construct Poincare transformations \( U_{a,\Lambda}(u_{a,\Lambda}X \leftarrow X) \) in the Fock space. First of all, we calculate the explicit form of generators. Then we will renormalize the obtained expressions and check the conditions of Appendix A. Then we will construct operators \( U_{a,\Lambda} \) and check the group property.

First of all, investigate some properties of the operator \( V_X \).

#### 3.4.1 Some properties of the operator \( V \)

Remind that the operator \( V \) taking the Fock space vector \( \Psi \in \mathcal{F} \) to the functional \( f[\phi(\cdot)] \) is defined from the relation

\[ V : |0 \rangle \mapsto c \exp\left[\frac{i}{2} \int dx dy \tilde{R}(x, y)\phi(x)\phi(y) \right] \]

and from formulas (2.1) which can be rewritten as

\[ V A^+(x) V^{-1} = A^+(x) \equiv (\tilde{\Gamma}^{-1/2}\tilde{R}^{*}\phi - \tilde{\Gamma}^{-1/2}\frac{i}{\delta\phi}(x), \]
\[ V A^-(x) V^{-1} = A^-(x) \equiv (\tilde{\Gamma}^{-1/2}\tilde{R}\phi - \tilde{\Gamma}^{-1/2}\frac{i}{\delta\phi}(x). \]

|\( c |\) can be formally found from the normalization condition

\[ |c|^2 \int D\phi | \exp\left[\frac{i}{2} \int dx dy \phi(x)\tilde{R}(x, y)\phi(y) \right] |^2 = 1 \]

The argument can be chosen to be arbitrary, for example,

\[ Argc = 0. \]

**Proposition 3.3.** The operator \( V \) is defined from the relations (3.29) - (3.32) uniquely.

Namely, any element of the Fock space can be presented \( [1] \) via its components, vacuum state and creation operators as

\[ \Psi = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int dx_1...dx_n \Psi_n(x_1, ..., x_n) A^+(x_1)...A^+(x_n)|0 > \]

Specify \( [1] \)

\[ V \Psi = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int dx_1...dx_n \Psi_n(x_1, ..., x_n) A^+(x_1)...A^+(x_n)V|0 > . \]

---

1. The problem of divergence of the series is related with the problem of correctness of the functional Schrodinger representation. It is not investigated here.
Since the operators $A^\pm(x)$ satisfy usual canonical commutation relations and $A^-(x)|0> = 0$, we obtain $VA^\pm(x) = A^\pm(x)V$.

The operator $V$ depend on $\mathcal{R}$. It is useful to find an explicit form of the operator $V^{-1}\delta V$.

**Proposition 4.4.** The following property is satisfied:

$$V^{-1}\delta V = -iA^+\hat{\Gamma}^{-1/2}\delta\hat{\mathcal{R}}\hat{\Gamma}^{-1/2}A^+ - i\frac{1}{2}A^+\hat{\Gamma}^{-1/2}\delta\hat{\mathcal{R}}\hat{\Gamma}^{-1/2}A^- + A^+\hat{\Gamma}^{-1/2}\delta\hat{\mathcal{R}}\hat{\Gamma}^{-1/2}A^- + i\frac{1}{4}Tr[\delta(\hat{\mathcal{R}} + \hat{\mathcal{R}}^*)\hat{\Gamma}^{-1}]$$

(3.33)

The notations of the type $A^\pm\hat{\mathcal{B}}A^-$ are used for the operators like $\int dxdy A^+(x)\hat{\mathcal{B}}(x,y)A^-(y)$, where $\hat{\mathcal{B}}(x,y)$ is a kernel of the operator $\hat{\mathcal{B}}$.

To check formula (3.33), consider the variation of the formula (2.1) if $\mathcal{R}$ is varied:

$$[A^\pm(x); V^{-1}\delta V] = (\hat{\Gamma}^{1/2}\delta\hat{\mathcal{R}}\hat{\Gamma}^{-1/2}A^\pm)(x) - i(\hat{\Gamma}^{-1/2}\delta\hat{\mathcal{R}}\hat{\Gamma}^{-1/2}A^\pm)(x) + i(\hat{\Gamma}^{-1/2}\delta\hat{\mathcal{R}}^*\hat{\Gamma}^{-1/2}A^-)(x)$$

Therefore, formula (3.33) is correct up to an additive constant. To find it, note that

$$\delta V|0> = [\frac{i}{2}\int dxdy\phi(x)\delta\hat{\mathcal{R}}(x,y)\phi(y) + \delta lnc]V|0>$$

This relation and formula (2.1) imply

$$<0|V^{-1}\delta V|0> = \frac{i}{2}Tr(\delta\hat{\mathcal{R}}\hat{\Gamma}^{-1}) + \delta lnc.$$  

It follows from the normalization conditions (3.31) and (3.32) that $c = (det\hat{\Gamma})^{1/4}$. Therefore, $\delta lnc = \frac{1}{4}Tr\delta\hat{\mathcal{R}}\hat{\Gamma}^{-1}$. Thus, $<0|V^{-1}\delta V|0> = \frac{i}{4}Tr\delta(\hat{\mathcal{R}} + \hat{\mathcal{R}}^*)\hat{\Gamma}^{-1}$. Formula (3.33) is checked.

### 3.4.2 Explicit forms of Poincare generators

Proposition 3.4 allows us to find an explicit forms of Poincare generators. For the simplicity, we consider the case when the quadratic form $\mathcal{R}$ is invariant under spatial translations and rotations:

$$\hat{\mathcal{R}}(x,y : u(a,L)X) = \hat{\mathcal{R}}(L^{-1}(x - a), L^{-1}(y - a) : X).$$

(3.34)

This property implies that

$$[\partial_k; \hat{\mathcal{R}}] = \delta_k^\mu\hat{\mathcal{R}}; \quad [\partial_k; \hat{\Gamma}^{1/2}] = \delta_k^\mu\hat{\Gamma}^{1/2};$$

$$[(x^k\partial_l - x^l\partial_k); \hat{\mathcal{R}}] = \delta_{kl}^\mu\hat{\mathcal{R}}; \quad [(x^k\partial_l - x^l\partial_k); \hat{\Gamma}^{1/2}] = \delta_{kl}^\mu\hat{\Gamma}^{1/2}. (3.35)$$

1. **Spatial translations and rotations.**

Eqs. (3.35), proposition 3.4 and relation (2.1) imply that formally

$$\hat{P}^k = -iA^+\partial_kA^- + i\delta^k_\mu; \quad M^{kl} = -iA^+(x^k\partial_l - x^l\partial_k)A^- + i\delta_{kl}^\mu.$$  

(3.36)

These operators do not contain any divergences.

2. **Time evolution.**

Proposition 3.4 and eqs. (2.1) imply that the operator $\hat{H}(X)$ is also quadratic with respect to creation and annihilation operators,

$$\hat{H}(X) = i\delta_H + \frac{1}{2}A^-\mathcal{H}^-\Hat(X)A^- + A^+(\hat{\omega} + \mathcal{H}(X))A^- + \frac{1}{2}A^+\mathcal{H}^+(X)A^+ + \hat{\mathcal{H}}.$$  

Here $\mathcal{H}^{\pm\pm}(X)$ and $\mathcal{H}(X)$ are the following operators:

$$\mathcal{H}^{++}(X) = \hat{\Gamma}^{-1/2}[\delta_H\hat{\mathcal{R}} - \hat{\mathcal{R}}\hat{\Gamma} - (-\Delta + m^2 + V''(\Phi(x)))\hat{\Gamma}^{-1/2}; \quad \mathcal{H}^+(X) = (\mathcal{H}^{++})^\dagger;$$

$$\mathcal{H}(X) = \hat{\Gamma}^{-1/2}(\hat{\mathcal{R}}\hat{\Gamma}^* + (-\Delta + m^2 + V''(\Phi(x))) - \frac{1}{2}\delta_H(\hat{\mathcal{R}} + \hat{\mathcal{R}}^*) + i\frac{1}{2}[\delta_H\hat{\Gamma}^{1/2}; \hat{\Gamma}^{1/2}])\hat{\Gamma}^{-1/2} - \hat{\omega}.$$
while
\[ \hat{\omega} = \sqrt{-\Delta + m^2} \]

is a (\( \Pi, \Phi \))-independent self-adjoint operator.

The divergent number \( \overline{H} \) is formally equal to
\[ \overline{H} = \frac{1}{2} Tr[\hat{\Gamma}^{-1/2}(\hat{R}\hat{R}^*) + (-\Delta + m^2 + V''(\Phi(x)) - \frac{1}{2}\delta_H(\hat{R} + \hat{R}^*))\hat{\Gamma}^{-1/2}] \]. (3.37)

3. The n-boost

For boost transformation, we obtain that
\[ \hat{B}^k(X) = i\delta_B^k + \frac{1}{2} B^k(X)A^+ + A^+(L_k + B^k(X))A^+ + \frac{1}{2} A^+ B^{k++}(X)A^+ + \overline{B}^k. \]

Here
\[ B^{k++}(X) = \hat{\Gamma}^{-1/2}[\delta_k^B \hat{R} - \hat{R} x^k \hat{R} - (-\partial_i x^k \partial_i + x^k m^2 + x^k V''(\Phi(x)))\hat{\Gamma}^{-1/2}; \]
\[ B^{k--} = (B^{k++})^+, \]
\[ B = \hat{\Gamma}^{-1/2}[\hat{R} x^k \hat{R}^* + (-\partial_i x^k \partial_i + x^k m^2 + x^k V''(\Phi(x))) - \frac{1}{2}\delta^k_B(\hat{R} + \hat{R}^*) + \]
\[ \frac{1}{2}[\delta^k_B \hat{\Gamma}^{1/2}, \hat{\Gamma}^{1/2}]\hat{\Gamma}^{-1/2} - L_k, \] (3.38)

while
\[ L_k = \frac{1}{2} \hat{\omega}^{-1/2}[\hat{\omega} x^k \hat{\omega} + (-\partial_i x^k \partial_i + x^k m^2)\hat{\omega}^{-1/2} \]

is a boost generator in the free theory which is a self-adjoint operator.

The divergent term is
\[ \overline{B}^k = \frac{1}{2} Tr[\hat{\Gamma}^{-1/2}[\hat{R} x^k \hat{R}^* + (-\partial_i x^k \partial_i + x^k m^2 + x^k V''(\Phi(x))) - \frac{1}{2}\delta^k_B(\hat{R} + \hat{R}^*)]] \] (3.39)

3.4.3 Check of algebraic conditions and renormalization

Let us write down the requirements which are sufficient for satisfying the properties H1-H6 of the Appendix A. Since the Poincare generators are quadratic with respect to creation and annihilation operators, we will use the results of Appendix B.

1. Let
\[ \hat{K} = \hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}. \]

This is a bounded self-adjoint operator without zero eigenvalues. Therefore, \( \hat{K}^{-1} \equiv T^{1/2} \) is a (non-bounded) self-adjoint operator and
\[ T = \hat{\omega}^{1/4}(x^2 + 1)\hat{\omega}^{1/4}(x^2 + 1)\hat{\omega}^{1/4}. \]

By \( \mathcal{D} \subset \mathcal{F} \) we denote the domain \( \{ \psi \in \mathcal{F} \mid ||\psi||_T^2 < \infty \} \).

**Lemma 3.5.** For self-adjoint operators
\[ A_k = L_k, \quad A_{d+k} = -i\partial_k, \quad A_{2d+kd+i} = -i(x^k \partial_i - x^i \partial_k), \quad A_{2d+d^2+1} = \hat{\omega} \]
the following properties are satisfied: 1. \( ||T^{-1/2}A_j T^{-1/2}|| < \infty, ||A_j T^{-1/2}|| < \infty. \)
2. \( ||T^{1/2} e^{iA_j} T^{-1/2}|| \leq C, ||T e^{-iA_j} T^{-1/2}|| \leq C, t \in [0, t_1]. \)

**Proof.** The first part of lemma is justified as follows. One should check that the following norms are finite:
\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]
\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]
\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}(\hat{k} \hat{x}^a - \hat{k}^a \hat{x}^i)\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]
\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]
\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]
\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]
\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]
\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]
\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||; \]
\[ ||\hat{\omega}^{-1/4}(x^2 + 1)^{-1}\hat{\omega}^{-1/4}||. \]
where $\hat{k}^j = -i\partial/\partial x^j$. This statement is a corollary of the following lemma.

**Lemma 3.6.** The operators

\[
[\hat{\omega}^\alpha; (x^2 + 1)^{-1}]; \quad [\hat{\omega}^\alpha; x^s(x^2 + 1)^{-1}]; \quad [\hat{\omega}^\alpha; x^l x^s(x^2 + 1)^{-1}]
\]

are bounded if $\alpha \leq 1$.

This lemma is a corollary of Lemma C.29 of Appendix C.

To prove the second statement of lemma 3.5, represent it in the following form:

\[
||e^{iA_1^j T^{1/2}} e^{-iA_1^j T^{-1/2}}|| \equiv ||T^{1/2}_j(t) T^{-1/2}|| \leq C; \quad ||T_j(t) T^{-1}|| \leq C. \quad (3.40)
\]

It is necessary to investigate the Poincare transformation properties of the operators $\hat{z}^j$ and $\hat{k}^j$.

**Lemma 3.7.** The following relations are satisfied:

\[
e^{i\omega t} \hat{x}^j e^{-i\omega t} = \hat{x}^j + \hat{k}^j \omega t, \quad e^{i\omega t} \hat{k}^j e^{-i\omega t} = \hat{k}^j;
\]

\[
e^{ik^j x} \hat{x}^j e^{-ik^j x} = \hat{x}^j + a^j; \quad e^{ik^j x} \hat{k}^j e^{-ik^j x} = \hat{k}^j;
\]

\[
e^{i\theta} e^{ik^j x} (\hat{x}^j e^{-i\theta} - \hat{k}^j e^{-i\theta}) = (e^{-i\theta} \hat{x})^j;
\]

\[
e^{i\theta} e^{ik^j x} (\hat{x}^j e^{-i\theta} - \hat{k}^j e^{-i\theta}) = (e^{-i\theta} \hat{k})^j;
\]

\[
e^{iL^1^j \tau} \hat{k}^j e^{-iL^1^j \tau} = \hat{k}^j, \quad l \geq 2; \quad e^{iL^1^j \tau} \hat{k}^j e^{-iL^1^j \tau} = \hat{k}^j \cosh \tau - \hat{\omega} \sinh \tau.
\]

The operators $X^1(\tau) = e^{iL^1^j \tau} \hat{x}^j e^{-iL^1^j \tau}$ have the following Weyl symbols:

\[
X^1 = \frac{\omega_k}{\omega_k \cosh \tau - k^1 \sinh \tau} x^1; \quad X^\alpha = x^\alpha + \frac{k^\alpha \sinh \tau x^1}{\omega_k \cosh \tau - k^1 \sinh \tau}
\]

To check the properties, it is sufficient to show that they are satisfied at $\tau = 0$ and show that the derivatives of left-hand and right-hand sides of these relations coincide.

Making use of commutation relations $[x^s, f(\hat{k})] = i\partial f(\hat{k})/\partial k$ and result of lemma 3.6, we find that operators (3.40) are bounded uniformly with respect to $t \in [0, t_1]$. Lemma 3.5 is proved.

Let the following conditions on $R$ be imposed.

Let $h(\alpha)$ be an arbitrary smooth curve on the Poincare group.

P1. The property (3.34) is satisfied.

P2. The $\alpha$-dependent operator functions $TB^{k^+}(\alpha)(X)$ and $TH^{k^+}(\alpha)(X)$ are continuous in the Hilbert-Schmidt topology $\| \cdot \|_2$.

P3. The $\alpha$-dependent operator functions $B^{k^+}(\alpha)(X)$ and $H^{k^+}(\alpha)(X)$ are continuously differentiable with respect to $\alpha$ in the Hilbert-Schmidt topology.

P4. The $\alpha$-dependent operator functions $B^k(\alpha)(X)$, $\mathcal{H}(\alpha)(X)$, $TB^k(\alpha)(X) T^{-1}$, $T^{1/2} B^k(\alpha)(X) T^{-1/2}$, $T \mathcal{H}(\alpha)(X) T^{-1}$, $T^{1/2} \mathcal{H}(\alpha)(X) T^{-1/2}$ are strongly continuous.

P5. The $\alpha$-dependent operator functions $T^{-1/2} \mathcal{H}(\alpha)(X) T^{-1/2}$, $T^{-1/2} B^k(\alpha)(X) T^{-1/2}$, $\mathcal{H}(\alpha)(X) T^{-1}$, $B^k(\alpha)(X) T^{-1}$ are continuously differentiable with respect to $\alpha$ in the operator norm $\| \cdot \|$ topology.

P6. The functions $\mathcal{P}(\alpha)(X)$ and $\overline{B^k}(\alpha)(X)$ are continuous.

**Lemma 3.6.** Let the properties P1-P6 be satisfied. Then properties H1, H2, H4-H6 are also satisfied.

**Proof.** Property H1 is a corollary of estimations performed in lemmas B.1, B.2, B.3 of Appendix B. Property H4 is a corollary of theorem B.15. Properties H2 and H5 are obtained from lemma B.4. Making use of the results of lemmas B.1, B.2, B.3 and property $\|U_B^\tau \Psi - \Psi\|_{1} \to \tau \to 0$ obtained in theorem B.15, we find that property H6 is satisfied. Lemma is proved.

2. Let us check the commutation relations (3.9), i.e. property H3. Note that the divergences arise in terms $\overline{B^k}$ and $\mathcal{P}$ only, so that we suppose them to be arbitrary and then find the conditions that provide Poincare invariance.
Let
\[ \hat{H}_k = \frac{1}{2} A^+ H_k^+ A^+ + A^+ \mathcal{H}^+ A^- + \frac{1}{2} A^- \mathcal{H}^- A^- + \Pi_k + i \delta_k \]
be arbitrary quadratic Hamiltonians. Then the property \([\hat{H}_1, \hat{H}_2] = \hat{H}_3\) under condition \([i \delta_1, i \delta_2] = i \delta_3\) means that
\begin{align*}
\mathcal{H}^+ & = -i[H_1^+ - H_2^+ + H_1^+(H_2^-)^* - H_2^+(H_1^-)^*] + \delta_1 \mathcal{H}^+_2 - \delta_2 \mathcal{H}^+_1. \\
\mathcal{H}^- & = -i\{H_2^+ (H_1^+)^* - H_1^+(H_2^+)^* + [H_1^+; (H_2^-)^*]\} + \delta_1 \mathcal{H}^-_2 - \delta_2 \mathcal{H}^-_1,
\end{align*}
(3.41) (3.42)
\[ \Pi_3 = -i \frac{1}{2} Tr[\mathcal{H}^+_2 (H_1^-)^* - \mathcal{H}^-_1 (H_2^+)^*] + \delta_1 \Pi_2 - \delta_2 \Pi_1. \] (3.43)

Relations (3.41), (3.42), (3.43) are treated in sense of bilinear forms on \(D(T)\).

Consider now the commutation relations.

1. The relations
\[ [\bar{p}^k, \bar{p}^l] = 0, \quad [\hat{M}^{lm}, \bar{p}^s] = i(g^{ms} \bar{p}^l - g^{ls} \bar{p}^m] \]
are satisfied automatically since
\[ [\partial_k, \partial_l] = 0, \quad -[x^l \partial_m - x^m \partial_l; \partial_s] = g^{ms} \partial_l - g^{ls} \partial_m. \]

2. The relation
\[ [\hat{M}^{lm}, \hat{M}^{rs}] = -i(g^{lr} \hat{M}^{ms} - g^{mr} \hat{M}^{ls} + g^{ms} \hat{M}^{lr} - g^{ls} \hat{M}^{mr}) \]
is also satisfied.

3. For the relation
\[ [\bar{p}^k, \bar{p}^0] = 0 \]
eqs (3.41) - (3.43) takes the form
\[ \delta_p^k \mathcal{H}^+ - [\partial_k; \mathcal{H}^+] = 0, \quad \delta_p^k \mathcal{H}^- - [\partial_k; \mathcal{H}^-] = 0, \]
\[ \delta_p^k \Pi = 0. \] (3.45)

4. For the relation
\[ [\hat{M}^{kl}, \bar{p}^0] = 0, \]
eqs (3.41) - (3.43) are written as
\[ \delta_M^{kl} \mathcal{H}^+ - [x^k \partial_l - x^l \partial_k; \mathcal{H}^+] = 0; \quad \delta_M^{kl} \mathcal{H}^- - [x^k \partial_l - x^l \partial_k; \mathcal{H}^-] = 0; \]
\[ \delta_M^{kl} \Pi = 0. \] (3.47)

5. Consider the relation
\[ [\hat{M}^{k0}, \bar{p}^s] = -i g^{ks} \bar{p}^0. \]
We write egs (3.41) - (3.43) as follows:
\[ [\partial_s, B^{k+}] - \delta_p^s B^{k+} = -g^{ks} \mathcal{H}^+, \quad [\partial_s, B^{k-}] - \delta_p^s B^{k-} = -g^{ks} \mathcal{H}^-, \]
\[ \delta_p^s B^k = g^{ks} \Pi. \] (3.49)

6. The commutation relation
\[ [\hat{M}^{lm}, \hat{M}^{k0}] = -i(g^{lk} \hat{M}^{m0} - g^{mk} \hat{M}^{l0}) \]
is equivalent to
\[ [x^l \partial_m - x^m \partial_l; B^{k+}] - \delta_M^{lm} B^{k+} = g^{lk} B^{m+} - g^{mk} B^{l+}, \]
\[ [x^i \partial_m - x^m \partial_i; B^{k+}] - \delta^m_M B^{k+} = g^{ik} B^{m+} - g^{mk} B^{i+}, \quad (3.50) \]
\[ -\delta^m_M B^k = g^{ik} B^m - g^{mk} B^i. \quad (3.51) \]

7. The most nontrivial commutation relations are
\[ [\dot{M}^{k0}; \dot{p}^0] = i\dot{P}^k, \quad [\dot{M}^{k0}; \dot{M}^{l0}] = -i\dot{M}^{kl} \]

They can be rewritten as follows:
\[ 0 = -i\{B^{k+} - H^{++} + H^{++}(B^{k+} - B^{l+}) + B^{l+}(H^{++} - B^{k+})\} + \delta^B_B H^{++} - \delta^B_B B^{k+}; \]
\[ 0 = -i\{B^{l+} - H^{++} + H^{++}(B^{l+} - B^{k+}) + B^{k+}(H^{++} - B^{l+})\} + \delta^B_B H^{++} - \delta^B_B B^{k+}; \]
\[ 0 = -i \frac{1}{2} Tr[H^{++}(B^{k+})^* - B^{k+}(H^{++})^*] + \delta^B_B H^{++} - \delta^B_B B^{k+}; \]
\[ 0 = -i \frac{1}{2} Tr[B^{l+}(B^{k+})^* - B^{k+}(B^{l+})^*] + \delta^B_B H^{++} - \delta^B_B B^{k+}. \quad (3.53) \]

3. Properties (3.44), (3.46), (3.50), (3.51) are obvious corollaries of relations (3.35). Properties (3.52) and (3.54) are checked by nontrivial but also direct computations.

To justify the properties (3.45), (3.47), (3.49), (3.50), (3.51), (3.53), let us extract divergences from \( \bar{H} \) and \( \bar{B}^k \). Notice that formally
\[ \bar{H} = \bar{H}_{\text{reg}} + \frac{1}{4} Tr \hat{\Gamma}, \quad \bar{B}^k = \bar{B}_{\text{reg}}^k + \frac{1}{4} Tr x^k \hat{\Gamma} \]

with
\[ \bar{H}_{\text{reg}} = -\frac{1}{4} Tr[H^{++} + H^{-}]; \quad \bar{B}_{\text{reg}}^k = -\frac{1}{4} Tr[B^{k+} + B^{k-}] \quad (3.56) \]

Expressions (3.56) are well-defined if we impose the following additional condition.

P7. The operators \( B^{k+} \) and \( H^{++} \) are of the trace class and \( \text{Tr} B^{k+}(u_{\alpha}(X)) \) and \( \text{Tr} H^{++}(u_{\alpha}(X)) \) are continuous functions of \( \alpha \).

To perform a renormalization of the semiclassical theory, one should substitute the divergent terms \( \text{Tr} \hat{\Gamma} \) and \( \text{Tr} x^k \hat{\Gamma} \) by finite terms which will be denoted as \( \text{Tr} R \hat{\Gamma} \) and \( \text{Tr} R x^k \hat{\Gamma} \),
\[ \bar{H} = \bar{H}_{\text{reg}} + \frac{1}{4} \text{Tr} R \hat{\Gamma}, \quad \bar{B}^k = \bar{B}_{\text{reg}}^k + \frac{1}{4} \text{Tr} R x^k \hat{\Gamma}. \]

Relations (3.45), (3.47), (3.49), (3.51), (3.53), (3.53) are straightforwardly checked under the following condition.

P8. The quantities \( \text{Tr} R \hat{\Gamma} \) and \( \text{Tr} R x^k \hat{\Gamma} \) obey the following properties:
\[ \delta^B_B \text{Tr} R \hat{\Gamma} = 0; \quad \delta^B_B M \text{Tr} R \hat{\Gamma} = 0; \]
\[ \delta^B_B \text{Tr} R x^k \hat{\Gamma} = -\delta^B_B \text{Tr} R x^k \hat{\Gamma}; \quad \delta^B_B M \text{Tr} R x^k \hat{\Gamma} = \delta^B_B M \text{Tr} R x^k \hat{\Gamma} \]
\[ \text{Tr} [x^i (\delta^B_B \hat{\Gamma} - \hat{\Delta} x^i \hat{\Gamma} - \hat{\Gamma} x^i \hat{\Delta}) - x^i (\delta^B_B \hat{\Gamma} - \hat{\Delta} x^i \hat{\Gamma} - \hat{\Gamma} x^i \hat{\Delta})] = \delta^B_B M \text{Tr} R x^k \hat{\Gamma} - \delta^B_B \text{Tr} R x^k \hat{\Gamma} = 0; \]
\[ \text{Tr} [x^i (\delta^B_B \hat{\Gamma} - \hat{\Delta} x^i \hat{\Gamma} - \hat{\Gamma} x^i \hat{\Delta}) - (\delta^B_B \hat{\Gamma} - \hat{\Delta} x^i \hat{\Gamma} - \hat{\Gamma} x^i \hat{\Delta})] = \delta^B_B M \text{Tr} R \hat{\Gamma} - \delta^B_B \text{Tr} R x^k \hat{\Gamma} = 0, \]
where \( \hat{\Delta} = \frac{1}{2} (\hat{R} + \hat{R}^*) \).

Note also that property P6 can be substituted by the following property.

P9. The functions \( \text{Tr} R \Gamma(u_{\alpha}(X)) \) and \( \text{Tr} R x^k \Gamma(u_{\alpha}(X)) \) are continuous.

Thus, we have formulated the conditions of invariance of the semiclassical field theory under Poincare algebra.
3.5 Construction of Poincare transformations

Let us construct now the operators $U_{a, \Lambda}$.

1. First of all, consider the case of spatial rotations, $a = 0$, $\Lambda = L$. Denote $U_{0,L} \equiv \mathcal{V}_L$ for this case. Let $L(\tau) = \exp(\frac{1}{2} \theta_{sm} \theta_{lm})$. Then the operator $\mathcal{V}_{L(\tau)}$ transforms the initial condition for the equation

$$i \dot{\Psi}^\tau = \frac{i}{2} \theta_{sm} A^\tau(x^s \partial_m - x^m \partial_s)A^{-\tau} \Psi^\tau, \quad \Psi^\tau \in \mathcal{D}$$

(3.57)

to the solution. The operator $\mathcal{V}_{L(\tau)}$ is uniquely defined from the relations

$$\mathcal{V}_L |0> = |0>, \quad \mathcal{V}_L A^{\pm}(x) \mathcal{V}_L^{-1} = A^{\pm}(Lx).$$

(3.58)

The group property for operators $\mathcal{V}_L$ is obviously satisfied.

2. Let $(a, \Lambda)$ is an $x^1$-boost: $a = 0$, $\Lambda = \Lambda_r$ has the form (3.1). Then the operator $U_{0,\Lambda_r} \equiv W_\tau$ takes the initial condition for the Cauchy problem for the equation

$$i \dot{\Psi}^\tau = B^1(S^\tau, \Phi^\tau, \Pi^\tau) \Psi^\tau$$

(3.59)

to the solution of this Cauchy problem. Here $S^\tau, \Phi^\tau, \Pi^\tau$ are obtained from system (3.21).

3. Let $L$ be such a rotation that $Le_1 = e_1$. Then there exists a matrix smooth function $L(t)$ such that $L(0) = 1$, $L(1) = L$, $L(t)e_1 = e_1$. One has $L(t)\Lambda_{\tau_1}L(\tau)^{-1}\Lambda_{\tau_1}^{-1} = 1$ for all $\tau \in \mathbb{R}$, so that condition (A.17) of lemma A.8 is satisfied. Therefore,

$$W_\tau \mathcal{V}_L = \mathcal{V}_L W_\tau,$$

(3.60)

provided that $Le_1 = e_1$.

4. Let $|n| = 1$, $L_n$ be such a spatial rotation that $L_n e_1 = e_1$. Introduce the operator $W_{\phi n}$ as follows,

$$W_{\phi n} = \mathcal{V}_L^{-1} W_{\phi} \mathcal{V}_L.$$

(3.61)

It corresponds to the Lorentz transformation

$$\Lambda_{\phi n} = L_n^{-1} \Lambda_{\phi} L_n.$$

Show that definition (3.61) is correct. Let $L_n^{(1)} e_1 = e_1$, $L_n^{(2)} e_1 = e_1$. One has $L_n^{(1)}(L_n^{(2)})^{-1}e_1 = e_1$, so that formula (3.60) implies that

$$(\mathcal{V}_L^{(1)})^{-1} W_{\phi} \mathcal{V}_L^{(1)} = (\mathcal{V}_L^{(2)})^{-1} W_{\phi} \mathcal{V}_L^{(2)}.$$

Note that the following property is satisfied.

Let $L$ be a spatial rotation. Then

$$L \Lambda_{\phi} L^{-1} = \Lambda_{\phi}, \quad \mathcal{V}_L W_{\phi} \mathcal{V}_L^{-1} = W_{\phi}.$$

(3.62)

Namely, let $L_n$ be such a rotation that $L_n e_1 = e_1$, where $n = \phi / ||\phi||$. Then

$$L \Lambda_{\phi} L^{-1} = L L_n^{-1} \Lambda_{\phi} L_n L^{-1} = \Lambda_{\phi},$$

$$\mathcal{V}_L W_{\phi} \mathcal{V}_L^{-1} = \mathcal{V}_L \mathcal{V}_L^{-1} W_{\phi} \mathcal{V}_L \mathcal{V}_L^{-1} = W_{\phi}.$$

5. Let $\Lambda$ be an arbitrary Lorentz transformation. Lemma 3.1 implies that it can be uniquely decomposed as follows, $\Lambda = \Lambda_{\phi} L$. Set

$$U_\Lambda = W_{\phi} \mathcal{V}_L.$$

(3.63)

Let us check the group property. Let

$$\Lambda_1 = \Lambda_{\phi_1} L_1, \quad \Lambda_2 = \Lambda_{\phi_2} L_2$$
be Lorentz transformations. Then

$$\Lambda_1 \Lambda_2 = \Lambda_{\bar{\phi}_1} \Lambda_{L_1 \bar{\phi}_2} L_2 = \Lambda_{\bar{\phi}_3} L(\bar{\phi}_1, L_1 \bar{\phi}_2) L(\bar{\phi}_1, L_1 \bar{\phi}_2) L_2,$$

where $\bar{\phi}_3$ and $L$ are defined from the construction of proof of lemma 3.1. One should check that

$$U_{\Lambda_1} U_{\Lambda_2} = U_{\Lambda_1 \Lambda_2},$$

i.e.

$$W_{\phi_1} W_{\phi_2} = W_{\phi_3(\phi_1, \phi_2)} L(\phi_1, \phi_2),$$

(3.64)

where $\phi_2 = L_1 \phi_2$,

$$\Lambda_{\phi_1, \phi_2} = \Lambda_{\phi_3(\phi_1, \phi_2)} L(\phi_1, \phi_2)$$

(3.65)

Consider the functions $\alpha \phi_1$ and $\alpha \phi_2$ instead of fixed vectors $\phi_1$ and $\phi_2$. Then

$$\Lambda_{\alpha \phi_1, \alpha \phi_2} \Lambda_{\phi_3(\alpha \phi_1, \alpha \phi_2)} L(\alpha \phi_1, \alpha \phi_2) = 1$$

(3.66)

so that we can apply the result of lemma A8:

$$W_{\alpha \phi_1} W_{\alpha \phi_2} W_{\phi_3(\alpha \phi_1, \alpha \phi_2)} L(\alpha \phi_1, \alpha \phi_2) = 1.$$
since the group properties for translations and Lorentz transformations have been already checked. We see that it is sufficient to justify the property

$$U_{(\Lambda a, 1)} = U_{(0, \Lambda)} U_{(a, 1)} U_{(0, \Lambda)}^{-1}$$

(3.72)

Since the operator $U_{(0, \Lambda)}$ was defined as

$$U_{(0, \Lambda)} = \mathcal{V}_{L_1} W_\phi \mathcal{V}_{L_2}$$

if $\Lambda = L_1 \phi L_2$, where $L_1, L_2$ are rotations, while

$$U_{(a, 1)} = U_{(a, 0, 1)} U_{(0, a, 1)}^{-1},$$

it is sufficient to check eq. (3.72) for the following cases:

1. $a^0 = 0, \Lambda = L$;
2. $a = 0, \Lambda = L$;
3. $a^0 = 0, \Lambda = \Lambda_f$;
4. $a = 0, \Lambda = \Lambda_f$.

Lemma A.8 imply all these properties. Therefore, we have proved the following statement.

**Theorem 3.6.** Let conditions H1-H5, H7-H9 be satisfied. Then the operators $U_{(a, \Lambda)}(u_{a, \Lambda}X \leftrightarrow X)$ defined by eqs. (3.71), (3.70), (3.63) are unitary and satisfy the group property.

### 3.6 Choice of the operator $\mathcal{R}$

Let us choose operator $\mathcal{R}$ in order to satisfy properties P1-P5, P7. We will use the notions of Appendix C (subsection C.5). First, we construct such an asymptotic expansion of a Weyl symbol $\mathcal{R}_N$ that for $\mathcal{R} = \mathcal{R}_N$

$$deg[\delta_B \mathcal{R} - \mathcal{R} \ast x^l \ast \mathcal{R} - x^l(\omega_k^2 + V''(\Phi(x)))] > \max\{d/2, d - 1\};$$

$$deg[\delta_H \mathcal{R} - \mathcal{R} \ast \mathcal{R} - (\omega_k^2 + V''(\Phi(x)))] > \max\{d/2, d - 1\}.$$  

(3.73)

Next, we will construct another asymptotic expansion of a Weyl symbol $\mathcal{R}$ which obeys the condition $\text{Im} \mathcal{R} > 0$ and approximately equals to $\mathcal{R}_N$ at large $|k|$, so that eqs. (3.73) are satisfied. This will imply that properties P1-P5, P7 are satisfied.

Let us define the expansions $\mathcal{R}_N$ with the help of the following recursive relations. Set

$$\mathcal{R}_0 = i\omega_k;$$

$$\mathcal{S}_0 = -\delta_H \mathcal{R}_0 + \mathcal{R}_0 \ast \mathcal{R}_0 + \omega_k^2 + V''(\Phi(x));$$

$$\mathcal{R}_{n+1} = \mathcal{R}_n + \frac{i}{2\omega_k} \mathcal{S}_n.$$  

(3.74)

**Lemma 3.9.** The following relation is satisfied:

$$deg \mathcal{S}_n = n.$$

**Proof.** For $n = 0, \mathcal{S}_0 = V''(\Phi(x))$, so that statement of lemma is satisfied. Suppose that statement of lemma is justified for $n < N$. Check it for $n = N$. One has

$$\mathcal{S}_N = \mathcal{S}_{N-1} + \mathcal{R}_N \ast \left(\frac{i}{2\omega_k} \mathcal{S}_{N-1}\right) + \left(\frac{i}{2\omega_k} \mathcal{S}_{N-1}\right) \ast \mathcal{R}_N + \left(\frac{i}{2\omega_k} \mathcal{S}_{N-1}\right) \ast \left(\frac{i}{2\omega_k} \mathcal{S}_{N-1}\right) - \frac{i}{2\omega_k} \delta_H \mathcal{S}_{N-1}.$$
Since
\[ \text{deg} \left[ \left( \frac{i}{2\omega k} \delta_{S_{N-1}} \right)^* \left( \frac{i}{2\omega k} \delta_{S_{N-1}} \right) - \frac{i}{2\omega k} \delta_{H} \delta_{S_{N-1}} \right] \geq \text{deg} S_{N-1} + 1 = N \]
and
\[
S_{N} = S_{N-1} + R_N \ast \left( \frac{i}{2\omega k} S_{N-1} \right) + \left( \frac{i}{2\omega k} S_{N-1} \right) \ast R_N \cong \\
S_{N-1} + R_N \left( \frac{i}{2\omega k} S_{N-1} \right) + \left( \frac{i}{2\omega k} S_{N-1} \right) R_N = 0
\]
up to terms of the degree \( N \), one finds
\[ \text{deg} S_{N} = N. \]
Lemma 3.9 is proved.

Denote
\[ X_n' = -\delta_B R_n + R_n \ast x' \ast R_n + x' (\omega_k^2 + V^{''}(\Phi(x))). \]

**Lemma 3.10.** The following property is obeyed:
\[ \delta_B S_n - \delta_H X_n' = -X_n' \ast R_n - R_n \ast X_n' + S_n \ast x' \ast R_n + R_n \ast x' \ast S_n. \] (3.75)

**Proof.** Denote
\[ F_n' = \delta_B S_n - \delta_H X_n' + X_n' \ast R_n + R_n \ast X_n' - S_n \ast x' \ast R_n - R_n \ast x' \ast S_n. \]
One has
\[ F_n' = (\delta_B - x' \delta_H) V^{''}(\Phi(x)) + |\delta_B; \delta_H| R_n - [x' (\omega_k^2 + V^{''}(\Phi(x))) - x' (\omega_k^2 + V^{''}(\Phi(x))) - x' \ast R_n] \ast R_n \ast x' \ast R_n. \]
It follows from the definition of the Weyl symbol that
\[ x' \ast f(x, k) = (x' + \frac{i}{2} \partial_{x'} f(x, k) \]
One also has
\[ (\delta_B - x' \delta_H) V^{''}(\Phi(x)) = 0. \]
Thus,
\[ F_n' = [\delta_H; \delta_B] R_n + ik' \ast R_n - R_n \ast ik' = \frac{\partial R_n}{\partial x'} - \delta_B R_n. \]
However, the property
\[ \frac{\partial R_n}{\partial x'} = \delta_B R_n \]
which means that eq. (3.34) is satisfied is checked by induction. Lemma 3.10 is proved.

**Lemma 3.11.** The following properties are satisfied:
1. \( \text{deg} X_n' = n \).
2. \( \text{deg} (X_n' - x' \ast S_n) \geq n + 1 \).

**Proof.** It follows from the results of Appendix C that \( X_n' \) is an asymptotic expansion of a Weyl symbol. Let \( \text{deg} X_n' = \alpha \).

Suppose that \( \alpha < n \). Then the left-hand side of eqs. (3.74) is of the degree \( \alpha \), the degree of the right-hand side of eq. (3.74) is greater than or equal to \( \alpha - 1 \). In the leading order in \( 1/|k| \) the right-hand side has the form one has \( -2i\omega_k x' \ast S_n \) and its degree should be greater than or equal to \( \alpha \). Therefore, \( \text{deg} X_n' \geq \alpha + 1 \). We obtain a contradiction.

Suppose \( \alpha > n \). Then the left-hand side of eq. (3.74) is of the degree \( n \), the right-hand side in the leading order in \( 1/|k| \) has the form \( 2i\omega_k x' \ast S_n \), so that \( \text{deg} S_n \) should obey the inequality \( \text{deg} S_n \geq n + 1 \). We also obtain a contradiction.
Thus, $\alpha = \nu$. In the leading order in $1/|k|$ one has
\[
0 \simeq -2i\omega_k (X'_n - x^l S_n)
\]
up to terms of the degree $n$, so that $\deg(X'_n - x^l S_n) \geq n + 1$. Lemma 3.11 is proved.

We see that for $N \geq \max \{ d/2, d - 1 \}$ the properties (3.73) are satisfied.

**Lemma 3.12.** Let $R^{(1)}$ and $R^{(2)}$ be asymptotic expansions of Weyl symbols, $\deg R^{(1)} = \deg R^{(2)} = -1$ and $\deg(R^{(1)} - R^{(2)}) = N + 1$. Then
\[
\deg(X^{(1)} - X^{(2)}) = N
\]
and
\[
\deg(S^{(1)} - S^{(2)}) = N.
\]

**Proof.** Denote $R^{(1)} - R^{(2)} = D$. Then
\[
X^{(1)} - X^{(2)} = -\delta B + R^{(1)} * x^l * D + D * x^l * R^{(1)} + D * D * x^l * D
\]
We see that $\deg(X^{(1)} - X^{(2)}) = N$. The second statement is checked analogously. Lemma 3.12 is proved.

Let us construct such an asymptotic expansion $R$ that $\deg(R - R_N) = N + 1$ and $\Im R > 0$. We will look for $R$ as follows (cf. [34]),
\[
R = A + i\omega_k^{1/4} * \exp B * \omega_k^{1/4} * \exp B * \omega_k^{1/4},
\]
where $A$ and $B$ are real asymptotic expansions. Then
\[
\Gamma^1/2 = \omega_k^{1/4} * \exp B * \omega_k^{1/4};
\]
\[
\Gamma^{-1/2} = \omega_k^{-1/4} * \exp(-B) * \omega_k^{-1/4}
\]
are also asymptotic expansions of Weyl symbols. Choose $A$ and $B$ to be polynomials,
\[
A = \sum_{s=1}^{S_1} \frac{A_s(x, k/\omega_k)}{\omega_k^{2s}}, \quad B = \sum_{s=1}^{S_2} \frac{B_s(x, k/\omega_k)}{\omega_k^{2s}},
\]
where $S_1 = \lfloor N/2 \rfloor$, $S_2 = \lfloor N+1/2 \rfloor$.

**Lemma 3.13.** There exists unique functions $A_1, \ldots, A_{S_1}, B_1, \ldots, B_{S_2}$ such that $\deg(R - R_N) = N + 1$.

**Proof.** It follows from recursive relations (3.74) that
\[
\Re R_N = \sum_{s=1}^{\infty} A_{N,s}(x, k/\omega_k)
\]
\[
\Im R_N = \omega_k + \sum_{s=1}^{\infty} \frac{C_{N,s}(x, k/\omega_k)}{\omega_k}
\]
Therefore, $A_s = A_{N,s}$, so that $A$ is uniquely defined. Denote
\[
B_s = \frac{B_s(x, k/\omega_k)}{\omega_k^{2s}}.
\]
Show that $B_s$ is uniquely defined. In the leading order in $1/|k|$, one has
\[
\Im R \simeq \omega_k + 2B_1/\omega_k,
\]
so that $B_1 = C_{N,1}/2$. Suppose that one can choose $B_1, \ldots, B_{S_1}$ in such a way that the degree of the asymptotic expansion of a Weyl symbol
\[
E_{N,s} = \Im R_N - \omega_k^{1/4} * \exp(B_1 + \ldots + B_{s-1}) * \omega_k^{1/2} * \exp(B_1 + \ldots + B_{s-1}) * \omega_k^{1/4}
\]

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satisfies the inequality 
\[ \text{deg} F_{N,s} \geq 2s - 1. \]

Choose \( B_s \) in such a way that \( \text{deg} F_{N,s} \geq 2s - 1 \). One has
\[
F_{N,s+1} = \text{Im} R_N - \omega_k^{1/4} \sum_{l_1=0}^{\infty} \frac{(B_1 + \ldots + B_{s+1})l_1!}{l_1!} \ast \omega_k^{1/2} \ast \sum_{l_2=0}^{\infty} \frac{(B_1 + \ldots + B_{s+1} + B_s)l_2!}{l_2!} \ast \omega_k^{1/4}.
\]

Up to terms of the degree \( 2s + 1 \), one has
\[
F_{N,s+1} \simeq \text{Im} R_N - \omega_k^{1/4} \ast (\sum_{l_1=0}^{\infty} \frac{(B_1 + \ldots + B_{s+1})l_1}{l_1!} + B_s) \ast \omega_k^{1/2} \ast (\sum_{l_2=0}^{\infty} \frac{(B_1 + \ldots + B_{s+1})l_2}{l_2!} + B_s) \ast \omega_k^{1/4}.
\]

Since
\[
F_{N,s} = \frac{1}{\omega_k^{2s-1}} \sum_{l=0}^{\infty} \frac{F_{N,s,l}(x, k/\omega_k)}{\omega_k^l},
\]
one finds that
\[
B_s = \frac{1}{2\omega_k^{2s}} F_{N,s,0}(x, k/\omega_k)
\]
is uniquely defined. Lemma 3.13 is proved.

Thus, we have constructed the operator \( R \) such that properties (3.73) are satisfied. We obtain the following theorem.

**Theorem 3.14.** Properties P1-P5, P7 are satisfied.

This theorem is a direct corollary of the results of Appendix C. Property P1 is satisfied because of construction of the operator \( R \). Properties P2-P5, P7 are corollaries of Theorems C.31, C.32, C.33, properties (3.73) and lemmas C.8, C.9, C.19.

### 3.7 Regularization and renormalization of a trace

The purpose of this subsection is to specify functionals \( TR_R \Gamma \) and \( TR_R^{k^4} \Gamma \) of arguments \( \Phi, \Pi \) in order to satisfy properties P8,P9. We want the renormalized trace to satisfy properties like these:

(i) \( TR_R \hat{A} = TR \hat{A} \) if \( \hat{A} \) is of the trace class;
(ii) \( TR_R (A + \lambda B) = TR_R \hat{A} + \lambda TR_R \hat{B} \);
(iii) \( TR_R [\hat{A}; \hat{B}] = 0 \);
(iv) \( TR_R \hat{A}_n \to 0 \) if \( A_n \to 0 \)

for such class of operators that is as wide as possible. Under these conditions, properties P8 and P9 are satisfied. However, one cannot specify such a renormalized trace. Namely, one should have
\[
TR_R [\hat{v}_j; W(\frac{k_j}{\omega_k} f(x))] = 0,
\]
where \( f \in S(\mathbb{R}^d) \). \( W(A) \) is a Weyl quantization of the function \( A \) (see appendix C). Property (3.74) means that
\[
TR_R W(i \frac{\partial}{\partial k_j} \frac{k_j}{\omega_k} f(x)) = 0.
\]

Therefore,
\[
\delta_{ij} TR_R W(\frac{f(x)}{\omega_k}) - l TR_R W(\frac{k_i k_j}{\omega_k^{l+2}} f(x)) = 0.
\]

Choose \( l = d \). Consider \( i = j \) in eq.(3.77) and perform the summation over \( i \). Making use of the relation \( \omega_k^2 - k_i k_i = m^2 \), we find
\[
\delta_{ij} TR_R W(m^2 \omega_k^{d-2} f(x)) = 0.
\]
However, the operator with Weyl symbol \( m^2 f(x) \omega_k^{-d-2} \) is of the trace class. Its trace is nonzero, provided that \( \int dx f(x) \neq 0 \).

However, we can introduce a notion of a trace for asymptotic expansions of Weyl symbols. The trace will be specified not only by operator but also by its asymptotic expansion which is not unique (see remark after definition C.6).

Let \( A = (A, \tilde{A}) \) be asymptotic expansion of a Weyl symbol. Suppose that the coefficients \( A_l \) of the formal asymptotic expansion

\[
\tilde{A} \equiv \sum_{l=0}^{\infty} \omega_k^{-l} A_l(x, \kappa/\omega_k)
\]

are polynomial in \( \kappa/\omega_k \). One formally has

\[
Tr_R A = \sum_{l=0}^{L_0} \int \frac{d\kappa dx}{(2\pi)^d \omega_k^{\alpha+l}} A_l(x, \kappa/\omega_k) + \int \frac{d\kappa dx}{(2\pi)^d} \left( A(x, \kappa) - \sum_{l=0}^{L_0} \frac{1}{\omega_k^{\alpha+l}} A_l(x, \kappa/\omega_k) \right).
\]

(3.78)

For \( \alpha + l_0 + 1 > d \), the last integral in the right-hand side of eq.(3.78) converges. To specify trace, it is sufficient then to specify values of integrals

\[
I_{k_1\ldots k_n}^{s,n} = \int \frac{d\kappa}{\omega_k^{s-1}} \frac{k_{i_1}}{\omega_k} \ldots \frac{k_{i_n}}{\omega_k}
\]

(3.79)

for \( s \leq d \) which are divergent. We will define the quantities (3.79), making use of the following argumentation.

1. We are going to specify to specify trace in such a way that

\[
Tr_R \frac{\partial}{\partial k_i} A = 0.
\]

(3.80)

Let

\[
A = \frac{1}{\omega_k^{s-1}} \frac{k_{i_1}}{\omega_k} \ldots \frac{k_{i_n}}{\omega_k}
\]

property (3.80) implies the following recursive relations

\[
\sum_{s=1}^{n+1} \delta_{ij_i} I_{j_1\ldots j_{s-1}j_{s+1}\ldots j_n}^{s,n} = (s + n) I_{i_1\ldots i_n}^{s,n+2}.
\]

(3.81)

Therefore, \( I^{s,n} = 0 \) for odd \( n \), while for even \( n \) \( I^{s,n} \) is defined from eqs.(3.81), for example, \( I_{ij}^{s,2} = \frac{1}{2} \delta_{ij} I^{s,0} \).

Therefore, it is sufficient to define integrals

\[
I^{s,0} = \int d\kappa \omega_k^{-s}.
\]

(3.82)

Let us use the approach based on the dimensional regularization [43, 44]. It is based on considering integrals (3.82) at arbitrary dimensionality of space-time. Expression (3.82) appears to be a meromorphic function of \( d \). Substracting the poles corresponding to sufficiently small positive integer values of \( d \), we obtain a finite expression.

Formally, one has

\[
I^{s,0} = \frac{1}{\Gamma(s/2)} \int_0^{\infty} d\alpha \alpha^{s/2-1} \int d\kappa \kappa^{\alpha(k^2 + m^2)} = \frac{\pi^{d/2}}{\Gamma(s/2)} \frac{\Gamma\left(\frac{s-d}{2}\right)}{m^{s-d}}.
\]

If \( \frac{s-d}{2} = -N \) is a nonpositive integer number, one should modify the definition of \( I^{s,0} \). Change \( d \rightarrow d - 2\varepsilon \). One finds:

\[
I^{s,0} \sim \frac{\pi^{d/2}(-1)^N}{\Gamma(s/2)m^{s-d}N!}(1 + \varepsilon(-ln(\pi m^2) + \Gamma(1) + 1 + \ldots + N^{-1})) + O(\varepsilon).
\]
In the MS renormalization scheme [14], one should omit the term $O(\varepsilon^{-1})$. There is also an renormalization scheme in which one omits also a fixed term of order $O(1)$. Let us omit the term $-\ln(\pi m^2) + \Gamma'(1)$. We obtain the following renormalized value of the integral:

$$I_{\text{ren}}^s = \frac{\pi^{d/2}}{\Gamma(s/2) m^{s-d}} \frac{(-1)^N}{N!} (1 + \ldots + 1/N),$$

provided that $N = \frac{d-s}{2}$ is a nonnegative integer number.

Therefore, we have defined the renormalized trace of an asymptotic expansion of a Weyl symbol by formula (3.78), provided that the coefficient functions are polynomials in $k/\omega_k$.

Let us investigate properties of the renormalized trace. Some properties are direct corollaries of definition (3.78).

**Lemma 3.15.** The following properties are satisfied:

(i) $Tr_R(A + \lambda B) = 0$;
(ii) $Tr_R \frac{\partial A}{\partial \alpha} = 0$; $Tr_R \frac{\partial A}{\partial \alpha} = 0$;
(iii) Let $E - \lim_{n \to \infty} A_n = A$. Then $lim_{n \to \infty} Tr_R A_n = Tr_R A$.
(iv) Let $deg A > d$. Then $Tr_R A = Tr A$.

**Corollary.** The property AP9 is satisfied.

Let us check that $Tr_R (A * B - B * A) = 0$. First of all, prove the following statement.

**Lemma 3.16.** $Tr_R A * B = Tr_R AB$.

**Proof.** Making use of eq.(C.8), we find

$$\hat{\Gamma}(x,k) = \int \frac{dp_1 dp_2 dy_1 dy_2}{(2\pi)^2} f_0 \left[ a \frac{\partial}{\partial \alpha} A(x+y_1;k + \alpha \frac{p_2}{2}) B(x+y_2,k_2 - \alpha \frac{p_1}{2}) \right] e^{-ip_1 y_1 - ip_2 y_2} = -\frac{1}{2} \int \frac{dp_1 dp_2 dy_1 dy_2}{(2\pi)^2} f_0 \left[ \frac{\partial}{\partial x} A(x+y_1;k + \alpha \frac{p_2}{2}) \frac{\partial}{\partial x} B(x+y_2,k_2 - \alpha \frac{p_1}{2}) \right] e^{-ip_1 y_1 - ip_2 y_2} = -\frac{i}{2} \frac{\partial C^j(x,k)}{\partial k^j}$$

with

$$C^j(x,k) = \int \frac{dp_1 dp_2 dy_1 dy_2}{(2\pi)^2} f_0 \left[ A(x+y_1;k + \alpha \frac{p_2}{2}) \frac{\partial}{\partial x} B(x+y_2,k_2 - \alpha \frac{p_1}{2}) - B(x+y_2,k_2 - \alpha \frac{p_1}{2}) \frac{\partial}{\partial x} A(x+y_1;k + \alpha \frac{p_2}{2}) \right] e^{-ip_1 y_1 - ip_2 y_2}.$$

One also has

$$\hat{A} * \hat{B} - \hat{A} \hat{B} = -\frac{i}{2} \frac{\partial \hat{C}^j}{\partial k^j}$$

with

$$\hat{C}^j(x,k) = \sum_{s=0}^{\infty} \sum_{l_1 l_2 \geq 0} I_{l_1 l_2}^{s=0} \left( \frac{-i^{l_1} l_2}{2! l_1 l_2} \frac{1}{\partial^{l_1 + l_2} A} \left[ \frac{\partial^{l_1 + l_2} A}{\partial k^{l_1} \partial k^{l_2} \partial x^{s_1} \partial x^{s_2} \partial x^{s_1} \partial x^{s_2} \partial x^{s_1} \partial x^{s_2} \partial x^{s_1} \partial x^{s_2}} \right] \frac{\partial^{l_1 + l_2} B}{\partial k^{l_1} \partial k^{l_2} \partial x^{s_1} \partial x^{s_2} \partial x^{s_1} \partial x^{s_2} \partial x^{s_1} \partial x^{s_2} \partial x^{s_1} \partial x^{s_2}} \right].$$

Analogously to Appendix C, one finds that $(C^j, \hat{C}^j) \equiv \hat{C}^j$ is an asymptotic expansion of a Weyl symbol. It follows from lemma 3.15 that $Tr_R \frac{\partial C^j}{\partial \alpha} = 0$. We obtain statement of lemma 3.16.

**Lemma 3.17.** For $deg B \geq 2$, $Tr_R x^k \omega_k \ast \hat{B} = Tr_R x^k \omega_k B$ and $Tr_R \omega_k \ast \hat{B} = Tr_R \omega_k B$.

The proof is analogous.

**Corollary 1.** The following relations are satisfied:

1. $Tr_R (A * B - B * A) = 0$;
2. $Tr_R (x^k \omega_k \ast \hat{B} - \hat{B} \ast x^k \omega_k) = 0$;
3. $Tr_R (\omega_k \ast \hat{B} - \hat{B} \ast \omega_k) = 0$.

**Corollary 2.** Property P8 is satisfied.

Thus, we have constructed functionals $Tr_R x^k \Gamma = Tr_R x^k \hat{\Gamma}$ and $Tr_R \hat{\Gamma} = Tr_R \Gamma$ such that properties P8 and P9 are satisfied.
Note that the "finite renormalization" \cite{25} can be also be made. One can add quantities $\Delta Tr_R x^k \hat{\Gamma}$ and $\Delta Tr_R \hat{\Gamma}$ to renormalized traces in such a way that

$$
\delta_k^P \Delta Tr_R \hat{\Gamma} = 0; \quad \delta_k^M \Delta Tr_R \hat{\Gamma} = 0; \\
\delta_l^P \Delta Tr_R x^k \hat{\Gamma} = -\delta_{kl} \Delta Tr_R \hat{\Gamma}; \quad \delta_l^M \Delta Tr_R x^k \hat{\Gamma} = \delta_{kl} \Delta Tr_R x^m \hat{\Gamma} - \delta_{mk} \Delta Tr_R x^l \hat{\Gamma}; \\
\delta_l^B \Delta Tr_R x^k \hat{\Gamma} - \delta_k^B \Delta Tr_R x^l \hat{\Gamma} = 0; \\
\delta_l^B \Delta Tr_R \hat{\Gamma} - \delta^H \Delta Tr_R x^l \hat{\Gamma} = 0.
$$

This corresponds to the possibility of adding the finite one-loop counterterm to the Lagrangian.

## 4 Semiclassical field

An important feature of QFT is a notion of field. In this section we introduce the notion of a semiclassical field and check its Poincare invariance.

### 4.1 Definition of semiclassical field

First of all, introduce the notion of a semiclassical field $\tilde{\phi}(x, t : X)$ in the functional Schrodinger representation. At $t = 0$, this is the operator of multiplication by $\phi(x)$. For arbitrary $t$, one has

$$
\tilde{\phi}(x, t : X) = U_{-t}(X \leftarrow u_t X)\phi(x) U_t(X \leftarrow X),
$$

where $U_t(x \leftarrow X)$ is the operator transforming the initial condition for the Cauchy problem to the solution to the Cauchy problem.

The field operator in the Fock representation is related with $\tilde{\phi}$ by the transformation (2.1),

$$
\hat{\phi}(x, t : X) = V_X^{-1}\tilde{\phi}(x, t : X) V_X.
$$

Making use of eq.(3.4), one finds

$$
\hat{\phi}(x, t : X) = (U_H^t(X))^{-1} \hat{\phi}(x, u_t X) U_H^t(X)
$$

(4.1)

Here $\hat{\phi}(x : X) = i(\Gamma^{-1/2}(A^+ - A^-))(x)$, while

$$
U_H^t(X) \equiv U_{a, \Lambda}(u_{a, \Lambda} X \leftarrow X), \quad \Lambda = 1, a = 0, a^0 = -t.
$$

Let us define $\hat{\phi}$ mathematically as an operator distribution.

Let $S(R^d)$ be a space of complex smooth functions $u : R^d \rightarrow C$ such that

$$
\|u\|_{l,m} = \max_{\alpha_1 + \ldots + \alpha_d \leq l, x \in R^d} (1 + |x|^m) \sup_{\alpha_1 + \ldots + \alpha_d} \left| \frac{\partial^{\alpha_1 + \ldots + \alpha_d}}{\partial x^{\alpha_1} \ldots \partial x^{\alpha_d}} u(x) \right| \rightarrow_{k \rightarrow \infty} 0.
$$

We say that the sequence $\{u_k\} \in S(R^d)$, $k = 1, \infty$ tends to zero if $\|u_k\|_{l,m} \rightarrow_{k \rightarrow \infty} 0$ for all $l, m$.

**Definition 4.1.** (cf.\cite{20}). 1. An operator distribution $\phi$ defined on $\mathcal{D} \in \mathcal{F}$ is a linear mapping taking functions $f \in S(R^d)$ to the linear operator $\phi[f] : \mathcal{D} \rightarrow \mathcal{F}$,

$$
\phi : f \in S(R^d) \mapsto \phi[f] : \mathcal{D} \rightarrow \mathcal{F},
$$

such that $\|\phi[f_n] \Phi\| \rightarrow_{n \rightarrow \infty} 0$ if $f_n \rightarrow_{n \rightarrow \infty} 0$.

2. A sequence of operator distributions $\phi_n$ is called convergent to the operator distribution $\phi$ if

$$
\|\phi_n[f] \Phi - \phi[f] \Phi\| \rightarrow_{n \rightarrow \infty} 0
$$

for all $\Phi \in \mathcal{D}$, $f \in S(R^d)$. 25
We will write
\[ \phi[f] \equiv \int dx \phi(x)f(x), \quad x \in \mathbb{R}^d. \]

Consider the mapping \( f \mapsto \phi_t\{f\}, \ f \in S(\mathbb{R}^d) \) of the form
\[ \phi_t\{f : X\} = \int dx \phi(x,t : X)f(x). \]

**Lemma 4.1.** \( \phi_t \) is an operator distribution being continous with respect to \( t \).

**Proof.** One has
\[
\phi_t\{f : X\} = (U_H^t(X))^{-1}i \int dx (A^+(x)(\hat{\Gamma}^{-1/2}f)(x) - A^-(x)(\hat{\Gamma}^{-1/2}f)(x))U_H^t(X).
\]

It follows from lemma B.3 and theorem B.15 that this operator distribution is defined on
\[ \mathcal{D} = \{ \Psi \in \mathcal{F} | \ ||\Psi||_H^1 < \infty \} \]
and continous with respect to \( t \). Lemma 4.1 is proved.

Consider the mapping \( f \mapsto \phi[f], \ f \in S(\mathbb{R}^{d+1}) \) of the form
\[
\phi[f : X] = \int dt \phi_t\{f(\cdot,t) : X\}.
\]

**Lemma 4.2.** \( \phi \) is an operator distribution.

The proof is analogous to lemma 4.1.

4.2 Poincare invariance of the semiclassical field

4.2.1 Algebraic properties

To check the property of Poincare invariance, notice that it is sufficient to check it for partial cases: spatial translations, rotations, evolution, boost, since any Poincare transformation can be presented as a composition of these transformations. Let \( g_B(\tau) = (a(\tau), \Lambda(\tau)) \) be a one-parametric subgroup of Poincare group corresponding to the element \( B \) of the Poincare algebra. The Poincare invariance property can be rewritten as
\[
\hat{\phi}[f : X] = (U_H^t(X))^{+}\phi[v_{g_B(\tau)}F : U_{g_B(\tau)}X]U_B^t[X], \quad (4.2)
\]
where
\[
(v_{g_B(\tau)}f)(x) = f(\Lambda^{-1}(\tau)(x - a(\tau))).
\]

Obviously, \( v_{g_1}v_{g_2} = v_{g_1g_2} \).

Let us check relation (4.2). It is convenient to reduce the group property to an algebraic property. The formal derivative with respect to \( \tau \) of the right-hand side of eq.(4.2) is
\[
(U_B^{[\tau u]}(X))^{+}\{i[H(B : u_{g_B(\tau)}X); \phi[v_{g_B(\tau)}f : u_{g_B(\tau)}X]] + \frac{\partial}{\partial \tau}\phi[v_{g_B(\tau)}f : u_{g_B(\tau)}X]U_B^t(X) \} \quad (4.3)
\]
If the quantity (4.3) vanishes, the property (4.2) will be satisfied since it is obeyed at \( \tau = 0 \). Making use of the group property \( g_B(\tau + \delta \tau) = g_B(\delta \tau)g_B(\tau) \), we find that vanishing of expression (4.3) is equivalent to the property:
\[
\frac{\partial}{\partial \tau}|_{\tau=0}\phi[v_{g_B(\tau)}f : u_{g_B(\tau)}X] - i[\phi[f : X] ; H(B : X)] = 0. \quad (4.4)
\]
We obtain the following lemma.

**Lemma 4.3.** Let the bilinear form (4.4) vanish on \( \mathcal{D} \). Then the property (4.2) is satisfied on \( \mathcal{D} \).
Proof. Consider the matrix element

\[ \chi^\tau = (U_B^\tau(X)\Psi_1, \phi[v_{gb}(\tau)f : u_{gb}(\tau)X]U_B^\tau(X)\Psi_2) - (\Psi_1, \hat{\phi}[f : X]\Psi_2), \]

where \( \Psi_1, \Psi_2 \in D \). Show it to be differentiable with respect to \( \tau \). Let us use an auxiliary lemma.

Lemma 4.4. Let \( \Psi \in D \). Then the vector \( \phi[v_{gb}(\tau)f : u_{gb}(\tau)X] \Psi \) is strongly continuously differentiable with respect to \( \tau \).

Proof. One has:

\[ \frac{\phi[v_{gb}(\tau+\delta\tau)f : u_{gb}(\tau+\delta\tau)X] - \phi[v_{gb}(\tau)f : u_{gb}(\tau)X]}{\delta \tau} \Psi = \phi[v_{gb}(\tau)f : u_{gb}(\tau)X] \Psi + \frac{\partial}{\partial \tau} \phi[v_{gb}(\tau)f : u_{gb}(\tau)X] \Psi. \]

The first term tends to \( \phi[v_{gb}(t)u_{gb}(t)] : u_{gb}(\tau)X] \Psi \) because of estimations of lemma B.3. The second term tends to \( \partial [B]\phi[v_{gb}(\tau)f : u_{gb}(\tau)X] \Psi \) because of construction of operator \( \hat{\Gamma}^{-1/2} \) and formula (A.13) of Appendix A. Lemma 4.4 is proved.

To prove lemma 4.3, notice that

\[ \chi^{\tau+\delta\tau} - \chi^\tau = (U_B^{\tau+\delta\tau}(X)\Psi_1, U_B^{\tau+\delta\tau}(X) - U_B^\tau(X) \Psi_2) \]

\[ + ((U_B^{\tau+\delta\tau}(X) - U_B^\tau(X))\Psi_1; \phi[v_{gb}(\tau)f : u_{gb}(\tau)X]U_B^\tau(X) \Psi_2). \]

This quantity tends as \( \delta \tau \to 0 \) to the matrix element of the bilinear form (4.3) and vanishes under condition (4.4). Lemma 4.3 is proved.

4.2.2 Check of invariance

One should check property (4.2) for spatial translations and rotations, evolution and boost transformations.

For spatial translations and rotations, property (4.2) reads:

\[ \hat{\phi}(x, t : X) = U_{01,\alpha, L}^{-1}\phi(Lx + a, t : u_{01,\alpha, L}X)U_{01,\alpha, L} \]

It follows from commutativity of \( U_{01,\alpha, L} \) and \( U_t \) and eqs. (3.68), (3.58), (3.34) that property (4.3) is satisfied.

For evolution operator, property (1.2) is rewritten as:

\[ \hat{\phi}(x, t : X) = (U_H^t(X))^{-1}\phi(x, t - \tau : u_r X)U_H^t(X) \]

Relation (1.6) is a direct corollary of definition (1.1) and group property for evolution operators.

Consider now the n-boost transformation. Check property (1.3). It can be presented as

\[ [\hat{B}^k; \hat{\phi}(x, t; X)] = -i(x^k \frac{\partial}{\partial t} + t \frac{\partial}{\partial x^k})\hat{\phi}(x, t; X) \]

or

\[ [B^k(X) ; (U_H^t(X))^{-1}\phi(x : u_t X)U_H^t(X)] = \frac{i}{\delta \tau} \{ (U_H^t(X))^{-1}\phi(x : u_t X)U_H^t(X) \} \]

\[ -i(x^k \frac{\partial}{\partial t} + t \frac{\partial}{\partial x^k})(U_H^t(X))^{-1}\phi(x : u_t X)U_H^t(X) \]

\[ \frac{i}{\delta \tau} \{ (U_H^t(X))^{-1}\phi(x : u_t X)U_H^t(X) \} \]

\[ \frac{i}{\delta \tau} \{ (U_H^t(X))^{-1}\phi(x : u_t X)U_H^t(X) \} \]

\[ \frac{i}{\delta \tau} \{ (U_H^t(X))^{-1}\phi(x : u_t X)U_H^t(X) \} \]

Let us make use of property (A.13). For partial case it can be presented as

\[ B^k(X)(U_H^t(X))^{-1} = i(U_H^t(X))^{-1}(\delta_B^k U_H^t(X))(U_H^t(X))^{-1} + (U_H^t(X))^{-1}[B^k(u_t X) - tP^k(u_t X)] \]

or

\[ U_H^t(X)B^k(X) = i(\delta_B^k U_H^t(X)) + [B^k(u_t X) - tP^k(u_t X)]U_H^t(X). \]
Making use of relations (4.8), (4.9), we take relation (4.7) to the form

\[ \tilde{B}_k(Y) - \tilde{P}_k(Y) = x^k[H(Y); \phi(x : Y)] - it \frac{\partial \phi(x : Y)}{\partial x^k}, \]

where \( Y = u_tX \). The property

\[ i \frac{\partial \phi(x : Y)}{\partial x^k} = [\tilde{P}_k(Y); \phi(x : Y)] \]

is a corollary of relation (3.35). The relation

\[ [\tilde{B}_k(Y) - x^k \tilde{H}(Y); \phi(x : Y)] = 0 \]

is also checked by direct calculation.

Thus, we have obtained the following result.

**Theorem 4.5.** The invariance property (2.4) is satisfied.

## 5 Composed semiclassical states

### 5.1 Semiclassical states in quantum mechanics

The most famous semiclassical approach to quantum mechanics is the WKB-approach. It is the following. One investigates the behavior of solutions of semiclassical equation of the form

\[ i\varepsilon \frac{\partial \psi_t(x)}{\partial t} = H_t(x, \frac{\partial}{\partial x}) \psi_t(x), \quad t \in \mathbb{R}, x \in \mathbb{R}^d \]

as \( \varepsilon \to 0 \). Here the Weyl quantization is considered. The initial condition is chosen to be

\[ \psi_0(x) = \varphi_0(x)e^{\frac{i}{\varepsilon} S_0(x)} \]

where \( S_0 \) is a real function. The WKB-result \[34\] is that the solution of eq.(5.1) at time moment \( t \) has the same form (5.2) up to \( \mathcal{O}(\varepsilon) \),

\[ ||\psi_t - \varphi_te^{\frac{i}{\varepsilon} S_t}|| = \mathcal{O}(\varepsilon), \]

provided that \( S_t(x) \) is a solution to the Cauchy problem for the Hamilton-Jacobi equation

\[ \frac{\partial S_t}{\partial t} + H_t(x, \frac{\partial S_t}{\partial x}) = 0, \]

while \( \varphi_t(x) \) obeys the transport equation

\[ \frac{\partial \varphi_t(x)}{\partial t} + \frac{\partial H_t(x, \frac{\partial S_t}{\partial x})}{\partial x} \frac{\partial \varphi_t(x)}{\partial x} = 0, \]

and initial condition \( \varphi_0 \).

However, we are not obliged to choose the initial condition for eq.(5.1) in a form (5.2). There are other substitutions to eq.(5.1) that conserve their forms under time evolution as \( \varepsilon \to 0 \). For example, the wave function

\[ \psi_t(x) = e^{\frac{i}{\varepsilon} S_t} e^{\frac{i}{\varepsilon} P_t(x - Q_t)} f_t(\frac{x - Q_t}{\sqrt{\varepsilon}}) \equiv (K_{S_t,P_t,Q_t,f_t})(x), \quad f_t \in S(\mathbb{R}^d) \]

used in the Maslov complex-WKB theory \[34, 35\] also approximately satisfies eq.(5.1) if

\[ i \dot{f}_t(\xi) = \left[ \frac{1}{2} i \frac{\partial}{\partial \xi_i} \frac{\partial H}{\partial P_j} + \frac{1}{2} \frac{\partial^2 H}{\partial Q_i \partial Q_j} \xi_i + \frac{1}{2} \frac{\partial^2 H}{\partial Q_i \partial Q_j} \xi_i \xi_j \right] f_t(\xi). \]
Therefore, for the initial condition $K_{S_0,P_0,Q_0}^\varepsilon f_0$ the solution for the Cauchy problem for eq. (5.1) will be asymptotically equal to $K_{S_0,P_0,Q_1}^\varepsilon f_1$ up to terms of the order $O(\varepsilon)$.

The wave function (5.2) rapidly oscillates with respect to all variables. The wave function (5.3) rapidly damps at $x - Q_t >> O(\sqrt{\varepsilon})$. One should come to the conclusion that there exists a wave function asymptotically satisfying eq. (5.1) which oscillates with respect to one group of variables and damps with respect to other variables. The construction of such states is given in the Maslov theory of Lagrangian manifolds with complexes germ [41, 42]. Let $\alpha \in \mathbb{R}^k$, $(P(\alpha), Q(\alpha)) \in \mathbb{R}^{2d}$ be a $k$-dimensional surface in the 2d-dimensional phase space, $S(\alpha)$ be a real function, $f(\alpha, \xi)$, $\xi \in \mathbb{R}^d$ is a smooth function. Set $\psi(x)$ to be not exponentially small if and only if the distance between point $x$ and surface $Q(\alpha)$ is of the order $\leq O(\sqrt{\varepsilon})$. Otherwise, set $\psi(x) \approx 0$. If $\min_\alpha |x - Q(\alpha)| = |x - Q(\alpha)| = O(\sqrt{\varepsilon})$, set

$$
\psi(x) = c_\varepsilon e^{i\varepsilon S(\alpha)} e^{i\varepsilon P(\alpha)(x - Q(\alpha))} f(\alpha, \frac{x - Q(\alpha)}{\sqrt{\varepsilon}}), \quad (5.5)
$$

One can note that wave functions (5.2) and (5.3) are partial cases of the wave function (5.5). Namely, for $k = 0$ the manifold $(P(\alpha), Q(\alpha))$ is a point, so that the functions (5.5) coincide with (5.3). Let $k = n$. If the surface $(P(\alpha), Q(\alpha))$ is in the general position, for $x$ in some domain one has $x = Q(\alpha)$ for some $\alpha$. Therefore,

$$
\psi(x) = c_\varepsilon e^{i\varepsilon S(\alpha)} f(\alpha, 0).
$$

We obtain the WKB-wave function. Thus, WKB and wave-packet asymptotic formulas (5.2) and (5.3) are partial cases of the wave function (5.5) appeared in the theory of Lagrangian manifolds with complex germ.

The lack of formula (5.5) is that the dependence on $\alpha$ on $x$ is implicit and too complicated. However, under certain conditions formula (5.5) is invariant if $\alpha$ is shifted by a quantity of the order $O(\sqrt{\varepsilon})$. In this case, the point $\alpha$ can be chosen in arbitrary way such that the distance of $x$ and $Q(\alpha)$ is of the order $O(\sqrt{\varepsilon})$.

Namely,

$$
e^{i\varepsilon S(\alpha + \sqrt{\varepsilon} \beta)} e^{i\varepsilon P(\alpha + \sqrt{\varepsilon} \beta)(x - Q(\alpha + \sqrt{\varepsilon} \beta))} f(\alpha + \sqrt{\varepsilon} \beta, \frac{x - Q(\alpha + \sqrt{\varepsilon} \beta)}{\sqrt{\varepsilon}}) \approx e^{i\varepsilon S(\alpha)} e^{i\varepsilon P(\alpha)(x - Q(\alpha))} f(\alpha, \frac{x - Q(\alpha)}{\sqrt{\varepsilon}}) \quad (5.6)
$$

if

$$
\frac{\partial S}{\partial \alpha_i} = P \frac{\partial Q}{\partial \alpha_i} \quad (5.7)
$$

$$
e^{i\beta(\xi \partial \alpha - \frac{1}{2} \xi \partial \alpha \partial \alpha f)} f = f \quad (5.8)
$$

To obtain eqs. (5.4) and (5.8), one should expand left-hand side of eq. (5.4). Considering rapidly oscillating factors, we obtain eq. (5.4). Conditions (5.7), (5.8) simplify the check [31] that the wave function (5.3) approximately satisfies eq. (5.1) if the functions $S, P, Q, f$ are time-dependent. They should satisfy eqs. (5.4). One can show that conditions (5.7), (5.8) are invariant under time evolution.

The form (5.3) of the semiclassical state appeared in the theory of Lagrangian manifolds with complex germ is not convenient for generalization to systems of infinite number of degrees of freedom. It is much more convenient to consider to consider wave function (5.3) as an "elementary" semiclassical state and wave function (5.5) as a "composed" semiclassical state presented as a superposition of elementary semiclassical states:

$$
\psi(x) = C_\varepsilon \int d\alpha e^{i\varepsilon S(\alpha)} e^{i\varepsilon P(\alpha)(x - Q(\alpha))} g(\alpha, \frac{x - Q(\alpha)}{\sqrt{\varepsilon}}), \quad (5.9)
$$

where $g(\alpha, \xi)$ is a rapidly damping function as $\xi \to \infty$. Superpositions of such type were considered in [13, 10, 17]; the general case was investigated in [30, 48]. The composed semiclassical states for the abstract semiclassical theory were studied in [40].
To show that expression (5.9) is in agreement with formula (5.3), notice that the wave function (5.9) is exponentially small if the distance between $x$ and the surface $Q(\alpha)$ is of order $O(\sqrt{\varepsilon})$. Let $\min |x - Q(\alpha)| = O(\sqrt{\varepsilon})$ and $|x - Q(\pi)| = O(\sqrt{\varepsilon})$. Consider the substitution $\alpha = \pi + \sqrt{\varepsilon}$. We find

$$
\psi(x) = C_\varepsilon e^{\varepsilon^{k/2}} \int d\beta e^{-i\varepsilon S(\pi + \beta \sqrt{\varepsilon})} e^{\frac{i}{\varepsilon} P(\pi + \beta \sqrt{\varepsilon})(x - Q(\pi + \beta \sqrt{\varepsilon}))} g(\pi + \beta \sqrt{\varepsilon}, \frac{x - Q(\pi + \beta \sqrt{\varepsilon})}{\sqrt{\varepsilon}})
$$

If the condition (5.7) is not satisfied, this is an integral of a rapidly oscillating function. It is exponentially small. Under condition (5.8) one can consider a limit $\varepsilon \to 0$ and obtain the expression (5.3), provided that

$$
c_\varepsilon = C_\varepsilon e^{\varepsilon^{k/2}}
$$

and

$$
f(\pi, \xi) = \int d\beta e^{-i\varepsilon S(\pi + \beta \sqrt{\varepsilon})} e^{\frac{i}{\varepsilon} P(\pi + \beta \sqrt{\varepsilon})(x - Q(\pi + \beta \sqrt{\varepsilon}))} g(\pi, \xi) = (2\pi)^k \prod_{s=1}^k \delta(\frac{\partial P_m}{\partial \alpha_s} \xi_m - \frac{\partial Q_m}{\partial \alpha_s} 1 \frac{\partial}{\partial \xi_m}) g(\pi, \xi). \quad (5.10)
$$

Integral representation (5.9) simplifies substitution of the wave function to eq.(5.1) and estimation of accuracy. Namely, the integrand entering to eq.(5.9) is an asymptotic solution to eq.(5.1), provided that $\text{eqs.} (5.4)$ are satisfied. Using the linearity property, we obtain that the wave function (5.9) approximately satisfies eq. (5.10) $\Rightarrow$. Properties (5.7), (5.10) are shown to be invariant under time evolution $\Rightarrow$.

It follows from eq.(5.10) that the function $f$ is invariant under the following change of the function $g$ ("gauge transformation"):

$$
g(\alpha, \xi) \to g(\alpha, \xi) + (\frac{\partial P_m}{\partial \alpha_s} \xi_m - \frac{\partial Q_m}{\partial \alpha_s} 1 \frac{\partial}{\partial \xi_m}) \chi_s(\alpha, \xi). \quad (5.11)
$$

Thus, the semiclassical state is specified at fixed $S(\alpha)$, $P(\alpha)$, $Q(\alpha)$ not by the function $g$ but by the class of equivalence of functions $g$: two functions are equivalent if they are related by the transformation (5.11).

This fact can be also illustrated if we evaluate the inner product $||\psi||^2$ as $\varepsilon \to 0$:

$$
||\psi||^2 = C_\varepsilon^2 \int \gamma d\alpha \gamma \int dx e^{-i\varepsilon S(\alpha)} e^{\frac{i}{\varepsilon} P(\alpha)(x - Q(\alpha))} g^*(\alpha, \frac{x - Q(\alpha)}{\sqrt{\varepsilon}}) e^{\frac{i}{\varepsilon} S(\gamma)} e^{\frac{i}{\varepsilon} P(\gamma)(x - Q(\gamma))} g(\gamma, \frac{x - Q(\gamma)}{\sqrt{\varepsilon}}),
$$

The integral over $x$ is not exponentially small if $\alpha - \gamma = O(\sqrt{\varepsilon})$. After substitution $\gamma = \alpha + \beta \sqrt{\varepsilon}$, $x - Q(\alpha) = \xi \sqrt{\varepsilon}$ and considering the limit $\varepsilon \to 0$, we find

$$
||\psi||^2 \approx C_\varepsilon^2 e^{\frac{k \alpha}{2}} \int d\alpha (g(\alpha, \cdot), \prod_{s=1}^k 2\pi \delta(\frac{\partial P_m}{\partial \alpha_s} \xi_m - \frac{\partial Q_m}{\partial \alpha_s} 1 \frac{\partial}{\partial \xi_m}) g(\alpha, \cdot)). \quad (5.12)
$$

The $k$-dimensional surface $\{(S(\alpha), P(\alpha), Q(\alpha))\}$ ("isotropic manifolds") in the extended phase space has the following physical meaning. Consider the average value of a semiclassical observable $A(x, -i\varepsilon \partial / \partial x)$. As $\varepsilon \to 0$, one has

$$
(\psi, A(x, -i\varepsilon \partial / \partial x) \psi) \approx C_\varepsilon^2 e^{\frac{k \alpha}{2}} \int d\alpha A(Q(\alpha), P(\alpha)) (g(\alpha, \cdot), \prod_{s=1}^k 2\pi \delta(\frac{\partial P_m}{\partial \alpha_s} \xi_m - \frac{\partial Q_m}{\partial \alpha_s} 1 \frac{\partial}{\partial \xi_m}) g(\alpha, \cdot)).
$$

We see that only values of the corresponding classical observable on the surface $\{(Q(\alpha), P(\alpha))\}$ are relevant for calculations fo average values as $\varepsilon \to 0$. This means that the Blokhintsev-Wigner density function (Weyl symbol of the density matrix) corresponding to the composed semiclassical state is proportional to the delta function on the manifold $\{(Q(\alpha), P(\alpha))\}$.

Therefore, elementary semiclassical states describe evolution of a point particle, while composed semiclassical states (including WKB-states) describe evolution of the more complicated objects - isotropic manifolds.
5.2 Composed semiclassical states in quantum field theory

5.2.1 Construction of semiclassical states

Analogously to quantum mechanical formula (5.9), consider the superposition of the "elementary" quantum field semiclassical states (1.3) of the form

\[ \psi[\varphi(\cdot)] = \int \frac{d\alpha}{\sqrt{4\pi^k}} e^{\frac{i}{\hbar} S(\alpha)} e^{\frac{i}{\hbar} \Pi(\alpha; x)[\varphi(x)\sqrt{X} - \Phi(\alpha, x)]} g(\alpha, \varphi(\cdot) - \frac{\Phi(\alpha, \cdot)}{\sqrt{X}}) \]  

(5.13)

where \( \alpha \in \mathbb{R}^k, S(\alpha), \Pi(\alpha; x), \Phi(\alpha; x) \) are smooth functions. Calculate (formally) the functional integral for \( (\psi, \psi) \):

\[ (\psi, \psi) = \int \frac{d\alpha d\beta}{\sqrt{4\pi^k}} \int D\varphi e^{-\frac{i}{\hbar} S(\alpha)} e^{-\frac{i}{\hbar} \Pi(\alpha; x)[\varphi(x)\sqrt{X} - \Phi(\alpha, x)]} g(\alpha, \varphi(\cdot) - \frac{\Phi(\alpha, \cdot)}{\sqrt{X}}) g^*(\beta, \varphi(\cdot)) e^{\frac{i}{\hbar} S(\beta)} e^{\frac{i}{\hbar} \Pi(\beta; x)[\varphi(x)\sqrt{X} - \Phi(\beta, x)]} \]

(5.14)

After substitution \( \gamma = \alpha + \sqrt{\lambda}\beta, \varphi(\cdot) = \frac{\Phi(\alpha, \cdot)}{\sqrt{\lambda}} + \phi(\cdot) \) we obtain as \( \lambda \to 0 \):

\[ (\psi, \psi) \simeq \int d\alpha d\beta e^{\frac{i}{\hbar} S(\alpha)} e^{\frac{i}{\hbar} \Pi(\alpha; x)[\Phi(\alpha, x)\sqrt{X} - \Phi(\alpha, x)]} g(\alpha, \varphi(\cdot)) e^{\frac{i}{\hbar} S(\beta)} e^{\frac{i}{\hbar} \Pi(\beta; x)[\Phi(\beta, x)\sqrt{X} - \Phi(\beta, x)]} g^*(\beta, \varphi(\cdot)) \]

(5.15)

The condition

\[ \frac{\partial S}{\partial \alpha_s} = \int d\Pi(\alpha, x) \frac{\partial \Phi(\alpha, x)}{\partial \alpha_s} \]

should be satisfied. Otherwise, the integral (5.14) will be exponentially small as \( \lambda \to 0 \), so that state (5.13) will be trivial. Under condition (5.15), one has

\[ (\psi, \psi) \to_{\lambda \to 0} \int d\alpha d\beta \int D\varphi g^*(\alpha, \varphi(\cdot)) e^{\frac{i}{\hbar} S(\alpha)} e^{\frac{i}{\hbar} \Pi(\alpha; x)[\Phi(\alpha, x)\sqrt{X} - \Phi(\alpha, x)]} g(\alpha, \varphi(\cdot)) \]

(5.16)

To specify the composed semiclassical state in the functional representation, one should:

(i) specify the smooth functions \( S(\alpha), \Pi(\alpha, x), \Phi(\alpha, x) \equiv X(\alpha) \) obeying eq.(5.15) (determine the \( k \)-dimensional isotropic manifold in the extended phase space \( \mathcal{X} \));

(ii) specify the \( \alpha \)-dependent functional \( g(\alpha, \varphi(\cdot)) \).

The inner product of composed semiclassical states is given by expression (5.16).

Since the inner product (5.16) may vanish for nonzero \( g \), one should factorize the space of composed semiclassical states. Such functionals \( g \) that obey the property

\[ \int d\alpha (g^*(\alpha, \cdot) \prod_i \delta(\int d\Pi(\alpha, x) \frac{\partial \Phi(\alpha, x)}{\partial \alpha_l} \phi(x) - \frac{\partial \Phi(\alpha, x)}{\partial \alpha_l} \phi(x)) g(\alpha, \cdot) = 0 \]

(5.17)

should be set to be equal to zero, \( g \sim 0 \).

One can define the Poincare transformation of the composed semiclassical state as follows. The transformation of \( (S(\alpha), \Pi(\alpha, \cdot), \Phi(\alpha, \cdot)) \) is \( u_{a,\Lambda}(S(\alpha), \Pi(\alpha, \cdot), \Phi(\alpha, \cdot)) \). The transformation of \( g(\alpha, \varphi(\cdot)) \) is

\[ \tilde{U}_{a,\Lambda}(u_{a,\Lambda}(S, \Pi, \Phi) \leftrightarrow (S, \Pi, \Phi)) g(\alpha, \varphi(\cdot)) \]

One should check that the inner product entering to eq.(5.17) is invariant under Poincare transformations. This will also imply that equivalent states are taken to equivalent.

Since the functional Schrodinger representation is not well-defined, let us consider the Fock representation. One should then specify the \( \alpha \)-dependent Fock vector \( Y(\alpha) = V^{-1} g(\alpha, \cdot) \) instead of the \( \alpha \)-dependent functional \( g(\alpha, \varphi(\cdot)) \). Making use of formulas (2.1), we find that the inner product (5.16) takes the form

\[ \left( \begin{array}{c} \Lambda^k \\ Y(\cdot) \end{array} \right), \left( \begin{array}{c} \Lambda^k \\ Y(\cdot) \end{array} \right) = \int d\alpha d\beta (Y(\alpha), e^{\frac{i}{\hbar} S(\alpha, x) \Lambda^k} e^{\frac{i}{\hbar} \Pi(\alpha, x) A^+(x) - B^+_s(\alpha, x) A^s_\alpha(x)} Y(\alpha)) \]

(5.18)
where
\[ B_s(\alpha, \cdot) = \hat{\Gamma}^{-1/2}(\hat{R} \frac{\partial \Phi(\alpha, \cdot)}{\partial \alpha_s} - \frac{\partial \Pi(\alpha, \cdot)}{\partial \alpha_s}), \] (5.19)

\( \hat{\Gamma} = \hat{\Gamma}(\Phi(\alpha, \cdot), \Pi(\alpha, \cdot)), \) \( \hat{R} = \hat{R}(\Phi(\alpha, \cdot), \Pi(\alpha, \cdot)). \) If the isotropic manifold \((\Phi(\alpha, \cdot), \Pi(\alpha, \cdot))\) is non-degenerate, the functions \( B_s(\alpha, x) \) are linearly independent.

The transformation of the composed semiclassical state \( \{X(\alpha)\}_{Y(\cdot)} \) is
\[ \{u_{a,\Lambda}(\alpha) \leftarrow X(\alpha)\}_{Y(\cdot)}. \]

Let us investigate some properties of the inner product (5.18) in order to check its invariance under Poincare transformations.

### 5.2.2 Constrained Fock space

The purpose of this subsection is to investigate the properties of the inner product
\[ < Y_1, Y_2 > = \int d\beta(Y_1, \exp) \sum_{s=1}^{k} \beta_s \int d\mathbf{x}(B_s(\mathbf{x})A^+(\mathbf{x}) - B^*_s(\mathbf{x})A^-(\mathbf{x}))|Y_2) \] (5.20)

for the fock vectors \( Y_1, Y_2. \) Suppose the functions \( B_1, ..., B_k \) to be linearly independent. Since the inner product (5.20) resembles the inner products for constrained systems \[ 49], \] we will call the space under construction as a constrained Fock space.

First of all, investigate the problem of convergence of the integral (5.20). Note that the operator
\[ U[B] = \exp[\int d\mathbf{x}(B(\mathbf{x})A^+(\mathbf{x}) - B^*(\mathbf{x})A^-(\mathbf{x}))] \]
is a well-defined unitary operator \[ 41], \] provided that \( B \in L^2(\mathbb{R}^d), \) and obey the relations
\[ A^-(\mathbf{x})U[B] = U[B](A^-(\mathbf{x}) + B(\mathbf{x})); \]
\[ A^+(\mathbf{x})U[B] = U[B](A^+(\mathbf{x}) + B^*(\mathbf{x})). \]

**Lemma 5.1.** (cf. \[ 36]). The following estimation is satisfied:
\[ ||B||^m(\{Y_1, U[B]Y_2\}) \leq \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} ||Y_1||^{k/2}||Y_2||^{(m-k)/2}. \] (5.21)

**Proof.** One has
\[ \int d\mathbf{x}B^*(\mathbf{x})A^-(\mathbf{x}); U[B] = ||B||^2U[B], \]
so that
\[ ||B||^2(\{Y_1, U[B]Y_2\}) = \int d\mathbf{x}B(\mathbf{x})A^+(\mathbf{x})Y_1, U[B]Y_2 - (Y_1, U[B] \int d\mathbf{x}B^*(\mathbf{x})A^-(\mathbf{x})Y_2). \]

Applying this identity \( m \) times, we obtain:
\[ ||B||^{2m}(\{Y_1, U[B]Y_2\}) = \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{k!(m-k)!} (\int d\mathbf{x}B(\mathbf{x})A^+(\mathbf{x}))^kY_1, U[B](\int d\mathbf{x}B^*(\mathbf{x})A^-(\mathbf{x}))^{m-k}Y_2). \]

Making use of the result of lemma B.3,
\[ \| \int d\mathbf{x}B(\mathbf{x})A^\pm(\mathbf{x})Y_{l} \|_l \leq ||B|| ||Y||_l^{l+1/2}, \]
we find:
\[ ||B||^{2m}(\{Y_1, U[B]Y_2\}) \leq \sum_{s=0}^{m} \frac{m!}{s!(m-s)!} ||B||^{m}||Y_1||_{s/2}||Y_2||_{(m-s)/2}. \]
Lemma 5.1 is proved.

Corollary 1. Let \( B_1, ..., B_k \) be linearly independent functions. Then for some constant \( C_1 > 0 \) the following estimation is satisfied:
\[
|\beta|^m |(Y_1, U[\sum_s \beta_s B_s] Y_2)| \leq C_1^m \|Y_1\|_{m/2} \|Y_2\|_{m/2}.
\]

Proof. It is sufficient to notice that for linearly independent \( B_1, ..., B_k \) the matrix \( (B_m, B_s) \) is not degenerate, so that \( \frac{1}{2} \sum_s \beta_m B_m \|B_s\|^2 \geq C_1^{-1} |\beta|^2 \) for some \( C_1 \). Applying the property \( \|Y\|_{s/2} \leq \|Y\|_{m/2} \) for \( s \leq m \), making use of eq. (5.21), we prove corollary 1.

Corollary 2. Let \( \|Y_1\|_{m/2} < \infty, \|Y_2\|_{m/2} < \infty \) for some \( m > k \), Then the integrand entering to eq. (5.20) obeys the relation
\[
|\langle Y_1, U[\sum_s \beta_s B_s] Y_2 \rangle| \leq \frac{\text{const}}{\|\beta\|+1}^m \tag{5.22}
\]
and integral (5.20) converges.

Corollary 3. Let \( Y_1, n, Y_2, n \) be such sequences of Fock vectors that \( \|Y_1, n\|_{m/2} \to n \to \infty 0, \|Y_2, n\|_{m/2} \leq C \) for some \( m > k \). Then \( Y_1, Y_2 \to n \to \infty 0 \).

Lemma 5.2. Let \( \|Y\|_m < \infty \) for some \( m > k \) and \( \text{Im}(B_s, B_t) = 0 \). Then \( \langle Y, Y \rangle \geq 0 \).

Proof. Introduce the following ”regularized” inner product
\[
\langle Y, Y \rangle_{\varepsilon} = \int d\beta e^{-\varepsilon |\beta|^2} \langle Y, U[\sum_s \beta_s B_s] Y \rangle.
\]
It follows from estimation (5.22) and the Lesbegue theorem [52] that
\[
\langle Y, Y \rangle_{\varepsilon} \to 0 \text{ as } \varepsilon \to 0.
\]
It is sufficient then to prove that \( \langle Y, Y \rangle_{\varepsilon} \geq 0 \). One has:
\[
e^{-\varepsilon |\beta|^2} = (4\pi \varepsilon)^{k/2} \int d\beta e^{-2\varepsilon |\beta-\beta'|^2 - 2\varepsilon |\beta''|^2}
\]
Therefore,
\[
\langle Y, Y \rangle_{\varepsilon} = \int d\beta d\beta'' (4\pi \varepsilon)^{k/2} e^{-2\varepsilon ||\beta'|^2 + |\beta''|^2} \langle U[\sum_s \beta_s'' B_s] Y, U[\sum_s \beta_s' B_s] Y \rangle, \tag{5.23}
\]
here the shift of variable \( \beta = \beta' - \beta'' \) is made. We have also taken into account that
\[
U[\sum_s \beta_s' B_s] U[- \sum_s \beta_s'' B_s] = U[\sum_s (\beta_s' - \beta_s'') B_s],
\]
provided that the operators
\[
\int dx (B_s(x) A^+(x) - B^*_s(x) A^-(x))
\]
commute (i.e. \( \text{Im}(B_s, B_t) = 0 \). Formula (5.23) is taken to the form
\[
\langle Y, Y \rangle_{\varepsilon} = \| \int d\beta (4\pi \varepsilon)^{k/4} e^{-2\varepsilon |\beta|^2} U[\sum_s \beta_s B_s] Y \|^2 \geq 0.
\]
Lemma 5.2 is proved.

The expression (5.20) depends on \( k \) functions \( B_1, ..., B_k \). However, one may perform linear substitutions of variables \( \beta \), so that only the subspace \( \text{span}\{B_1, ..., B_k\} \) is essential.

Definition 5.1. A \( k \)-dimensional subspace \( L_k \in L^2(\mathbb{R}^d) \) is called as a \( k \)-dimensional isotropic plane if \( \text{Im}(B', B'') = 0 \) for all \( B', B'' \in L_k \).
Let $L_k$ be a $k$-dimensional isotropic plane with an invariant under shifts measure $d\sigma$. Let $B_1,\ldots,B_k$ be a basis on $L_k$. One can assign then coordinates $\beta_1,\ldots,\beta_n$ to any element $B \in L_k$ according to the formula $B = \sum_s \beta_s B_s$. The measure $d\sigma$ is presented as $d\sigma = ad\beta_1\ldots ad\beta_k$ for some constant $a$. Consider the inner product
\[ <Y_1,Y_2>_L = a \int d\beta(Y_1,U[\sum_s \beta_s B_s]Y_2) = \int d\sigma(Y_1,U[B]Y_2), \]
(5.24)

$||Y_{1,2}||_{[k/2+1]} \leq \infty$. This definition is invariant under change of basis.

By $F_{[k/2+1]}$ we denote space of such Fock vectors $Y$ that $||Y||_{[k/2+1]} < \infty$. We say that $Y \perp_k 0$ if $<Y,Y>_L = 0$. Thus, the space $F_{[k/2+1]}$ is divided into equivalence classes. Introduce the following inner product on the factor-space $F_{[k/2+1]}/\sim$:
\[ <[Y_1],[Y_2]>_L = <Y_1,Y_2>_L \]
(5.25)

for all $Y_1 \in [Y_1], Y_2 \in [Y_2]$. This definition is correct because of the following statement.

**Lemma 5.3.** Let $<Y,Y>_L = 0$. Then $<Y,Y'>_L = 0$ for all $Y'$.

The proof is standard (cf., for example, [52]). One has
\[ 0 \leq <Y' + \sigma Y, Y' + \sigma Y>_L = <Y',Y'>_L + \sigma <Y,Y'>_L + \sigma Y >_L \]
for all $\sigma \in C$, so that $<Y,Y'>_L = 0$.

**Definition 5.2.** A constrained Fock space $F(L_k,d\sigma)$ is the completeness of the factor-space $F_{[k/2+1]}/\sim$ with respect to the inner product (5.23),
\[ F(L_k) = F_{[k/2+1]}/\sim. \]

### 5.2.3 Transformations of constrained Fock vectors

Let us investigate evolution of constrained Fock vectors. Consider the Cauchy problem for eq.(B.3) (Appendix B):
\[ i\hat{\Psi}_t = \left[ \frac{1}{2} A^+ H_t^+ A^+ A^+ H_t^- A^- + \frac{1}{2} A^- H_t^- A^- + H_t \right] \hat{\Psi}_t. \]
(5.26)

**Lemma 5.4.** Let $\Psi_0 \in F_m$. Then $\Psi_t \in F_m$.

**Proof.** Analogously to proof of lemma B.11, one has
\[ ||U_t \Psi_0|| = ||U_t^{-1}(A^+ A^- + 1)^m U_t \Psi_0|| = ||(1 + (A^+ G_t^T + A^- F_t^+))(F_t A^+ + G^*_t A^-))^m \Psi_0|| \]
It follows from Lemmas B.2, B.3 that
\[ ||(1 + (A^+ G_t^T + A^- F_t^+))(F_t A^+ + G^*_t A^-))\Psi|| \leq ||\Psi|| + ||F_t|| \frac{3}{2} ||\Psi|| + \sqrt{2} ||G_t^T F_t|| ||\Psi||_{t+1} + ||F_t F_t^* + G^*_t G_t|| ||\Psi||_{t+1} + ||F_t G_t^*|| ||\Psi||_{t+1} \]
\[ \leq C ||\Psi||_{t+1} \]
with
\[ C = 1 + ||F_t|| \frac{3}{2} + \sqrt{2} ||G_t^T F_t|| ||\Psi||_{t+1} + ||F_t F_t^* + G^*_t G_t|| + ||F_t G_t^*||. \]

Applying this estimation, we obtain by induction:
\[ ||U_t \Psi_0||_m \leq C^m ||\Psi_0||_m. \]

Lemma is proved.
Let $L_k$ be a $k$-dimensional isotropic plane with invariant measure $d\sigma$. Define its evolution transformation $L'_k$ as follows. Let $(B_1, \ldots, B_k)$ be a basis on $L_k$. Let $B'_s$ be solutions to the Cauchy problems

\[i\dot{B}'_s = \mathcal{H}'_{-}B'_s + \mathcal{H}'_{+}B'_s^*; \quad -iB'^*_{s} = \mathcal{H}'_{-}B'^*_{s} + \mathcal{H}'_{+}B'_s;\]
\[B'_0 = B_s; B'^*_{0} = B'^*_{s}.\]

(5.27)

they can be expressed as

\[B'_s = F'_s B^*_{s} + G'_s B_s; \quad B'^*_{s} = F'_s B^*_{s} + G'_s B'_s.\]

(5.28)

**Lemma 5.5.** Let $\text{Im}(B_t, B_t') = 0$. Then $\text{Im}(B'_t, B_t') = 0$.

**Proof.** One has:

\[2i\text{Im}(B'_t, B'_t) = (B'_t, B'_t) - (B'_t, B'_t) = (F'_t B'_t + G'_t B_t, F'_t B^*_{t} + G'_t B^*_{t}) - (F'_t B'_t + G'_t B_t, F'_t B^*_{t} + G'_t B'_t) = (B'_t, B'_t) - (B'_t, B'_t) = 0.

because of relations (B.22) of Appendix B.

Therefore, $L'_k$ is also an isotropic plane. Define the measure $d\sigma'$ on $L'_k$ as follows. For the choice of coordinates $\beta_1, \ldots, \beta_k$ on $L'_k$ according to the formula $B = \sum s \beta_k B'_s$, set $d\sigma = d\beta_1 \ldots d\beta_k$, where $a$ does not depend on $t$.

**Lemma 5.6.** The inner product $<\cdot, \cdot>_{L'_k}$ is invariant under time evolution:

\[<\Psi_t, \Psi_t>_{L'_k} = <\Psi_0, \Psi_0>_{L_k}.

**Proof.** By definition, one has

\[<\Psi_t, \Psi_t>_{L'_k} = a \int d\beta (\Psi_t, U[\sum s \beta_s B'_s] \Psi_t) = a \int d\beta (\Psi_0, U_t^+ U[\sum s \beta_s B'_s] U_t \Psi_0).

Eq. (5.28) implies that

\[U[\sum s \beta_s B'_s] = \exp \sum s \beta_s d\mathbf{x} (A^+_t(\mathbf{x}) B_s(\mathbf{x}) - A^-_t(\mathbf{x}) B^*_s(\mathbf{x})).\n
Making use of the relation

\[U^+ \Lambda_t^- = \Lambda_t^\pm(\mathbf{x}),\]

we obtain statement of lemma 5.6.

It follows from lemma 5.6 that operator $U_t$ takes equivalent states to equivalent. Therefore, it can be reduced to the factorspace $\mathcal{F}_{[k/2+1]}/\sim$. Since it is unitary, it can be extended to $\mathcal{F}(L_k)$.

### 5.2.4 Definition of composed semiclassical state and its Poincare transformation

Let us formulate a definition of a composed semiclassical state.

Let $\Lambda^{k}$ be a smooth $k$-dimensional manifold $(S(\alpha) < \Pi(\alpha, \cdot), \Phi(\alpha, \cdot))$ in the extended phase space with measure $d\Sigma$ such that the property (5.13) is satisfied. Such manifolds are called isotropic.

Specify an isotropic plane $L_k(\alpha) \equiv L_k(\alpha : \Lambda^{k})$ as follows. Let $B_s$ have the form (5.19). The subspace $\text{span}\{B_1, \ldots, B_k\}$ does not depend on particular choice of coordinates $\alpha_1, \ldots, \alpha_k$. Consider the quantity

\[(B_s, B_l) - (B_l, B_s) = \left(\frac{\partial \phi}{\partial \alpha_1}, (\hat{\mathcal{R}}^* - \hat{\mathcal{R}}) \hat{\mathcal{I}}^{-1} \frac{\partial \Pi}{\partial \alpha_1}\right) - \left(\frac{\partial \phi}{\partial \alpha_1}, (\hat{\mathcal{R}}^* - \hat{\mathcal{R}}) \hat{\mathcal{I}}^{-1} \frac{\partial \Pi}{\partial \alpha_1}\right).

(5.29)

Differentiating (5.15) with respect to $\alpha_l$, we obtain that quantity (5.29) vanishes. Thus, plane $L_k(\alpha)$ is isotropic.
Introduce the following measure \( d\sigma(\alpha) \) on \( L_k(\alpha) \):

\[
d\sigma(\alpha) = \frac{D\Sigma(\alpha)}{D\alpha} d\beta_1 ... d\beta_k,
\]

where \( \beta_1, ..., \beta_k \) are coordinates on \( L_k(\alpha) \) which are determined as \( B = \sum_s \beta_s B_s \).

Definition (5.30) is invariant under change of coordinates. Namely, let \( (\alpha_1', ..., \alpha_k') \) be another set of local coordinates chosen instead of \( (\alpha_1, ..., \alpha_k) \). Then

\[
B'_i = \sum_{s=1}^k \frac{\partial \alpha_s}{\partial \alpha'_i} B_s,
\]

so that property \( \sum_i \beta'_i B'_i = \sum_s \beta_s B_s \) implies that coordinate sets \( \beta \) and \( \beta' \) should be related as follows:

\[
\beta_s = \sum_i \frac{\partial \alpha_s}{\partial \alpha'_i} \beta'_i.
\]

Therefore, for the choice of coordinates \( \alpha' \) one has

\[
d\sigma' = \frac{D\Sigma}{D\alpha'} d\beta'_1 ... d\beta'_k = \frac{D\Sigma}{D\alpha'} \frac{D\alpha'}{D\alpha} d\beta_1 ... d\beta_k = d\sigma.
\]

The invariance property is checked.

Introduce the vector (Hilbert) bundle \( \pi_\Lambda^k \) as follows. The base of the bundle is the isotropic manifold \( \Lambda^k \). The fibre that corresponds to the point \( \alpha \in \Lambda^k \) is \( H_\tau = F(L_k(\alpha)) \). Composed semiclassical states are introduced as sections of bundle \( \pi_\Lambda^k \).

**Definition 5.2.** A composed semiclassical state is a set of isotropic manifold \( \Lambda^k \) and section \( Z \) of the bundle \( \pi_\Lambda^k \), such that the inner product

\[
< (\Lambda^k, Z), (\Lambda^k, Z) > = \int_{\Lambda^k} d\Sigma(Z(\alpha), Z(\alpha)) F(L_k(\alpha))
\]

converges.

Poincare transformation of isotropic manifold \( \Lambda^k = \{X(\alpha)\} \) is determined as

\[
u_{a,\Lambda}\{X(\alpha)\} = \{u_{a,\Lambda}X(\alpha)\}.
\]

Section \( \{Z(\alpha)\} \) is transformed as follows. Let \( Z(\alpha) = [Y(\alpha)] \), define \( U_{a,\Lambda}Z(\alpha) = [U_{a,\Lambda}Y(\alpha)] \). This definition is correct because of the resulta of previous subsubsection, provided that

\[
L_k(\alpha : u_{a,\Lambda}\Lambda^k) = U_{a,\Lambda}L_k(\alpha : \Lambda^k).
\]

(5.31)

It is sufficient to prove property (5.31) for partial cases: spatial translations and rotations, time evolution and \( x^1 \)-boost.

1. **Spatial translations.**

The functions \( \Phi, \Pi \) obey eqs. (3.18), while eq. (5.26) has the form (3.67). Eqs. (5.27) for \( B^i_s \) take the form

\[
\dot{B}^i_s = -a^k \partial_k B^i_s.
\]

This equation is satisfied because of eqs. (3.18), (5.19) and property (3.35).

2. **Spatial rotations.**

The functions \( \Phi, \Pi \) obey eqs. (3.13), while eq. (5.26) has the form (5.22). Eqs. (5.27) take the form

\[
\dot{B}^i_s = \frac{1}{2} \theta_{sm}(x^s \partial_m - x^m \partial_s) B^i_s.
\]

This equation is satisfied because of eqs. (3.13), (5.19) and property (3.35).
3. Time evolution.

The functions $\Phi, \Pi$ obey eq. (1.5), while eq. (5.26) has the form (3.69). Eq. (5.27) takes the form:

$$i\dot{B}_j^t = (\mathcal{H} + \hat{\omega})B_j^t + \mathcal{H}^{++}B_j^{t*},$$

where $\mathcal{H}$ and $\mathcal{H}^{++}$ have the form (3.69). It is satisfied because of eqs. (1.5) and (5.19).

4. The $x^1$-boost.

The functions $\Phi, \Pi$ obey eq. (5.13) with $n = (1, 0, ..., 0)$, while eq. (5.26) has the form (3.59). Eq. (5.27) has the form

$$i\dot{B}_j^t = (L_k + B^k_j)B_j^t + B^{k++}B_j^{t*},$$

where $B^k_j$ and $B^{k++}$ have the form (3.38). It is satisfied because of eqs. (3.21) and (5.19).

Thus, the composed semiclassical states and their Poincare transformations are introduced.

6 Conclusions

In this paper a notion of a semiclassical state is introduced. "Elementary" semiclassical state are specified by a set $(X, \Psi)$ of classical field configuration $X$ (point on the infinite-dimensional manifold $\mathcal{X}$, see section 2 and subsection 3.2) and element $\Psi$ of the space $\mathcal{F}$. Set of all "elementary" semiclassical states may be viewed as a semiclassical bundle.

The physical meaning of classical field $X$ is evident. Discuss the role of $\Psi$. In the soliton quantization language [1, 2] $\Psi$ specifies whether the quantum soliton is in the ground or excited state. For the Gaussian approach [14, 15, 16, 17], $\Psi$ specifies the form of the Gaussian functional, while for QFT in the strong external classical field [6, 7] $\Psi$ is a state of a quantum field in the classical background.

The "composed" semiclassical states have been also introduced (section 5). They can be viewed as superpositions of "elementary" semiclassical states and are specified by the functions $(X(\tau), \Psi(\tau))$ defined on some domain of $\mathbb{R}^k$ with values on the semiclassical bundle.

Not arbitrary superposition of elementary semiclassical states is nontrivial. The isotropic condition (5.15) should be satisfied. Moreover, the inner product of the "composed" semiclassical states (eq. (5.18)) is degenerate, so that there is a "gauge freedom" (5.17) in specifying composed semiclassical states.

The composed semiclassical states are used [36] in soliton quantization, since there are translation zero modes and solitons can be shifted. They are useful if there are conserved integrals of motion like charges. The correspondence between composed and elementary semiclassical states in QFT resembles the relationship between WKB and wave packet approximations in quantum mechanics.

An important feature of QFT is the property of Poincare invariance. In this paper an explicit check of this property is presented for semiclassical QFT. The Poincare transformations of elementary and composed semiclassical states have been constructed as follows. First, the simplest Poincare transformations like spatial translations and rotations, evolution and boost are considered. The infinitesimal transformations are investigated, the Lie algebraic commutation relations have been checked and the group properties have been justified.

For the "composed" states, conservation of the degenerate inner product and isotropic condition under Poincare transformation have been checked.

An important feature of QFT is a notion of field. In this paper this notion is introduced for semiclassical QFT. The property of Poincare invariance of semiclassical field is checked.

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A Symmetries of the semiclassical theory under Lie groups

Let the semiclassical field theory be symmetric under Lie group transformations. Let $\mathcal{G}$ be a Lie group. Let $\mathcal{X}$ be a smooth (maybe, infinite-dimensional) manifold, $\mathcal{H}$ be a Hilbert space. Suppose that () a smooth mapping $u : \mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X}$ is specified; the smooth mappings $u_g : \mathcal{X} \rightarrow \mathcal{X}$ associated with
the mapping $u$ of the form $u_g X = u(g, X)$ satisfy the group property $u_{g_1} u_{g_2} = u_{g_1 g_2}$;

\((\alpha)\) for each $X \in \mathcal{X}$ unitary operators $U_g [X] \equiv U_g (u_g X \leftarrow X) : \mathcal{F} \to \mathcal{F}$ are specified; the group property

$$U_{g_1} (u_{g_1 g_2} X \leftarrow u_{g_2} X) U_{g_2} (u_{g_2} X \leftarrow X) = U_{g_1 g_2} (u_{g_1 g_2} X \leftarrow X)$$

(A.1)

are satisfied.

Let us investigate the properties of infinitesimal transformations.

By $T_e \mathcal{G}$ we denote the tangent space to the Lie group $\mathcal{G}$ at $g = e$. Let $A \in T_e \mathcal{G}$, $g(\tau)$ be a smooth curve on the group $\mathcal{G}$ with the tangent vector $A$ at the point $g(0) = e$. Introduce the differential operator $\delta[A]$ on the space of differentiable functionals $\mathcal{F}$ on $\mathcal{X}$:

$$\langle \delta[A] F \rangle(X) = \frac{d}{d\tau} |_{\tau=0} F(u_{g(\tau)} X).$$

(A.2)

**Lemma A.1.** The quantity $\langle A.2 \rangle$ does not depend on the choice of the curve $g(\tau)$ with the tangent vector $A$.

2. The following property

$$\delta[A_1 + \alpha A_2] = \delta[A_1] + \alpha \delta[A_2], \alpha \in \mathbb{R}, A_1, A_2 \in T_e \mathcal{G}$$

is satisfied.

**Proof.** Let $g_1(\tau)$ and $g_2(\tau)$ be smooth curves on the Lie group $\mathcal{G}$ such that $g_1(0) = e$, $g_2(0) = e$. One has

$$\frac{d}{d\tau} |_{\tau=0} F(u_{g_1(\tau)g_2(\tau)} X) = \frac{d}{d\tau} |_{\tau=0} F(u_{g_1(\tau)} X) \frac{d}{d\tau} |_{\tau=0} F(u_{g_2(\tau)} X).$$

(A.3)

since

$$\frac{F(u_{g_1(\tau)g_2(\tau)} X)}{d\tau} |_{\tau=0} = \int d\xi \frac{\partial}{\partial \xi} F(u_{g_1(\hat{\tau})u_{g_2(\tau)} X}) |_{\hat{\tau}=\xi} + \frac{F(u_{g_2(\tau)} X) - F(X)}{\tau}.$$

Let $g(\tau)$ and $\tilde{g}(\tau)$ be curves on $\mathcal{G}$ with the tangent vector $A$. Choose $g_1(\tau) = g(\tau)$, $g_2(\tau) = g^{-1}(\tau) \tilde{g}(\tau)$. Since $g_2(\tau) - e = O(\tau^2), \frac{d}{d\tau} |_{\tau=0} F(u_{g_2(\tau)} X) = 0$. It follows from eq(A.3) that $\frac{d}{d\tau} |_{\tau=0} F(u_{g(\tau)} X) = \frac{d}{d\tau} |_{\tau=0} F(u_{\tilde{g}(\tau)} X)$. Thus, definition $\langle A.2 \rangle$ is correct.

Let $g_1(\tau), g_2(\tau)$ be curves with tangent vectors $A$ and $B$ correspondingly. Then $g_1(\tau) g_2(\tau)$ is a curve with the tangent vector $A + B$. Eq.(A.3) implies that

$$\delta[A + B] = \delta[A] + \delta[B].$$

Finally, consider a curve $g(\tau)$ with the tangent vector $A$ and the curve $\tilde{g}(\tau) = g(\alpha \tau)$ with the tangent vector $\alpha A$. One has

$$\frac{d}{d\tau} |_{\tau=0} F(u_{\tilde{g}(\tau)} X) = \alpha \frac{d}{d\tau} |_{\tau=0} F(u_{g(\tau)} X).$$

Thus, $\delta[\alpha A] = \alpha \delta[A]$. Lemma A.1 is proved.

Let $g \in \mathcal{G}, A \in T_e \mathcal{G}, h(\tau)$ be a curve on $\mathcal{G}$ with tangent vector $B$ at $h(0) = e$. Then the tangent vector for the curve $g(\tau) h(\tau) g^{-1}(\tau)$ at $h(0) = e$ does not depend on the choice of the curve $h(\tau)$. Denote it by $g B g^{-1}$.

Define the operator $W_g$ on the space of functionals $\mathcal{F}$ as $W_g F[X] = F[u_{g^{-1}} X]$.

**Lemma A.2.** The following property is satisfied:

$$W_g \delta[B] W_{g^{-1}} F = \delta[gBg^{-1}] F.$$

(A.4)

The proof is straightforward.

Let $g = g(\tau)$ be a curve with the tangent vector $A$ at $g(0) = e$. Differentiating expression $\langle A.4 \rangle$ by $\tau$ at $\tau = 0$, we obtain:
Lemma A.3. The following relation is satisfied:
\[ ([\delta[A], \delta[B]] + \delta([A, B]))F = 0. \]  \tag{A.5}

Here \([A, B]\) is the Lie-algebra commutator for the group \(G\).

Proof. Let \(g(\tau)\) be a smooth curve on the Lie group \(G\) with tangent vector \(A\) at \(g(0) = e\). Make use of the property (A.4):
\[ W_{g(\tau)} \delta[B] W_{g^{-1}(\tau)} = \delta[g(\tau)B g^{-1}(\tau)]F, \]  \tag{A.6}
rewrite definition (A.2) as
\[ \delta[A] = \frac{d}{d\tau} \big|_{\tau=0} W_{g^{-1}(\tau)}, \]
remember that the Lie commutator can be defined as
\[ [A; B] \equiv \frac{d}{d\tau} |_{\tau=0} g(\tau) B g^{-1}(\tau). \]
Consider the derivatives of sides of eq.(A.6) at \(\tau = 0\). We obtain property (A.5). Lemma A.3 is proved.

Consider now the infinitesimal properties of the transformation \(U\). Suppose that on some dense subset \(D\) of \(\mathcal{F}\) the vector functions \(U_{g[X]} \Psi (\Psi \in D)\) are strongly continuously differentiable with respect to \(g\) and smooth with respect to \(X\). Define operators
\[ H(A : X) \Psi = i \frac{d}{d\tau} |_{\tau=0} U_{g(\tau)} [X] \Psi, \]  \tag{A.7}
where \(g(\tau)\) is a curve on the group \(G\) with the tangent vector \(A\) at \(g(0) = e\).

Lemma A.4. 1. The operator \(H(A : X)\) does not depend on the choice of the curve \(g(\tau)\) with the tangent vector \(A\).
2. The following property is satisfied:
\[ H(A_1 + \alpha A_2 : X) = H(A_1 : X) + \alpha H(A_2 : X). \]

The proof is analogous to lemma A.1.

Let \(h(\tau)\) be a curve with the tangent vector \(B\) at \(h(0) = e\). Eq.(A.1) implies:
\[ U_{g[u_{h(\tau)}X]} u_{h(\tau)} [X] U_{g^{-1}[X]} \Psi = U_{g h(\tau) u_g^{-1}} [u_{g}X] \Psi, \quad \Psi \in D \]
Differentiating this identity by \(\tau\) at \(\tau = 0\), we obtain:

Lemma A.5. For \(\Psi \in D\)
\[ -i U_{g[X]} H(B : X) U_{g^{-1}[X]} \Psi + (\delta[B]U_{g[X]} U_{g^{-1}[X]} \Psi = -i H(B g^{-1} ; u_g X) \Psi. \]  \tag{A.8}

Let \(g = g(t)\) be a curve with the tangent vector \(A\) at \(g(0) = e\). Differentiating eq.(A.8) by \(t\) in a weak sense, we obtain:

Lemma A.6. On the subset \(\mathcal{D}\) the following bilinear form vanishes:
\[ -[H(A; X); H(B; X)] - i\delta[B] H(A; X) + i\delta[A] H(B; X) + i H([A; B]; X) = 0. \]  \tag{A.9}

Remark. Introduce the operator
\[ \hat{H}(A : X) = H(A : X) - i\delta[A] \]  \tag{A.10}
on the space of smooth sections of the bundle \(\mathcal{X} \times \mathcal{D} \to \mathcal{X}\). The property (A.9) can be rewritten as
\[ [\hat{H}(A; X); \hat{H}(B; X)] = i\hat{H}([A, B]; X). \]  \tag{A.11}
Note that the operator (A.10) is an analog of the covariant differentiation operator in the theory of bundles (see, for example, [42]).

Thus, the group property (A.1) is reformulated in terms of Lie algebras.

Investigate now the problem of reconstructing the group representation if the algebra representation is known. Our purpose is to prove some lemmas which are useful in constructing the representation of the Poincare group.

Impose the following conditions on the operators $H(A : X)$, $A \in T_e \mathcal{G}$, $X \in \mathcal{X}$.

H1. Hermitian operators $H(A : X)$ are defined on a common domain $\mathcal{D}$ which is dense in $\mathcal{F}$.

H2. For each smooth curve $h(\alpha)$ on $\mathcal{G}$ and each $\Psi \in \mathcal{D}$ the vector function $H(A : u_{h(\alpha)}X)\Psi$ is strongly continuously differentiable with respect to $\alpha$.

H3. The bilinear form (A.9) vanishes on $\mathcal{H}_1$. Impose the following conditions on the operators $H$.

H4. Let $B \in \mathcal{Z}$. If $\Psi_0 \in \mathcal{D}$, there exists a solution of the Cauchy problem for eq.(A.12).

Therefore, it is invertible and unitary.

Impose also the following conditions.

H5. Let $B \in \mathcal{Z}$. For each smooth curve $h(\alpha)$ on $\mathcal{G}$ and each $\Psi \in \mathcal{D}$ the quantity $||\frac{\partial}{\partial \alpha} H(A : u_{h(\alpha)}X)\Psi_t||$ is bounded uniformly with respect to $\alpha, t \in [\alpha_1, \alpha_2] \times [t_1, t_2]$ for any finite $\alpha_1, \alpha_2, t_1, t_2$.

H6. For $\psi \in \mathcal{D}$, $B \in \mathcal{Z}$, $A \in T_e \mathcal{G}$, the following property is satisfied:

$$||H(g_B(\tau)A g_B^{-1}(\tau) : u_{g_B(\tau)}X)[U_B^t(X)\Psi - \Psi]|| \rightarrow_{\tau \rightarrow 0} 0.$$ 

Under these conditions, we obtain:

**Lemma A.7.** Let $B \in \mathcal{Z}$, $A \in T_e \mathcal{G}$, The following property is satisfied on the domain $\mathcal{D}$:

$$H(A : X) + i(U_B^t(X))^{-1}(\delta[A]U_B^t(X) - (U_B^t(X))^{-1}H(g_B(t)A g_B(t)^{-1} : u_{g_B(t)}X)U_B^t(X) = 0. \quad (A.13)$$

**Proof.** Let us check that under these conditions the operator $(\delta[A]U_B^t(X)$ is correctly defined, i.e. the strong derivative

$$(\delta[A]U_B^t(X)\Psi = \frac{d}{d\alpha}|_{\alpha=0}U_B^t(u_{h(\alpha)}X)\Psi \quad (A.14)$$

exists for all $\Psi \in \mathcal{D}$, where $h(\alpha)$ is a curve on $\mathcal{G}$ with tangent vector $A$.

Denote

$$\Psi_{\alpha,t} = U_B^t(u_{h(\alpha)}X)\Psi.$$

This vector obeys the equation

$$i\frac{\partial}{\partial t} \Psi_{\alpha,t} = H(B : u_{g_B(t)h(\alpha)}X)\Psi_{\alpha,t}.$$
so that
\[ i\frac{\partial}{\partial t}(\Psi_{\alpha+\delta\alpha,t} - \Psi_{\alpha,t}) = H(B : u_{gb(t)}h(\alpha+\delta\alpha)X)(\Psi_{\alpha+\delta\alpha,t} - \Psi_{\alpha,t}) \\
+ (H(B : u_{gb(t)}h(\alpha+\delta\alpha)X) - H(B : u_{gb(t)}X))\Psi_{\alpha,t}. \]

Since \( \Psi_{\alpha,0} = \Psi_{\alpha+\delta\alpha,0} = \Psi \), we have
\[ \Psi_{\alpha+\delta\alpha,t} - \Psi_{\alpha,t} = -i \int_0^t d\tau U_B^{t-\tau}(u_{gb(\tau)}h(\alpha+\delta\alpha)X)(H(B : u_{gb(\tau)}h(\alpha+\delta\alpha)X) - H(B : u_{gb(\tau)}X))\Psi_{\alpha,\tau}. \]

Because of unitarity of the operators \( U_B^t \), the following estimation takes place:
\[ ||\Psi_{\alpha+\delta\alpha,t} - \Psi_{\alpha,t}|| \leq \int_0^t d\tau ||(H(B : u_{gb(\tau)}h(\alpha+\delta\alpha)X) - H(B : u_{gb(\tau)}X))\Psi_{\alpha,\tau}||. \]

Making use of the Lesbegue theorem \( [12] \) and condition H5, we find that \( ||\Psi_{\alpha+\delta\alpha,t} - \Psi_{\alpha,t}|| \to \delta\alpha \to 0 \), so that the operator \( U_B^t(u_{h(\alpha)}X) \) is strongly continuous with respect to \( \alpha \).

Furthermore,
\[ \frac{\Psi_{\alpha+\delta\alpha,t} - \Psi_{\alpha,t}}{\delta\alpha} = -i \int_0^\tau \int_0^1 d\gamma U_B^{t-\gamma}(u_{gb(\tau)}h(\alpha+\delta\alpha)X) \left( \frac{\partial}{\partial \alpha} H(B : u_{gb(\tau)}h(\alpha+\gamma\delta\alpha)X) \right) \Psi_{\alpha,\tau}. \]

Denote
\[ \frac{\partial \Psi_{\alpha,t}}{\partial \alpha} \equiv -i \int_0^t d\tau U_B^{t-\tau}(u_{gb(\tau)}h(\alpha+\delta\alpha)X) \left( \frac{\partial}{\partial \alpha} H(B : u_{gb(\tau)}h(\alpha+h)) \right) \Psi_{\alpha,\tau}. \]  

(A.15)

The following estimation takes place:
\[ \frac{||\Psi_{\alpha+\delta\alpha,t} - \Psi_{\alpha,t}|| - \frac{\partial \Psi_{\alpha,t}}{\partial \alpha}}{\delta\alpha} \leq \int_0^t d\tau \int_0^1 d\gamma \left[ \frac{\partial}{\partial \alpha} H(B : u_{gb(\tau)}h(\alpha+\gamma\delta\alpha)X) \right] \Psi_{\alpha,\tau} \right| \]  

+ \[ \left| \left| \left( U_B^{t-\tau}(u_{gb(\tau)}h(\alpha+\delta\alpha)X) - U_B^{t-\tau}(u_{gb(\tau)}h(\alpha)X) \right) \frac{\partial}{\partial \alpha} H(B : u_{gb(\tau)}h(\alpha)X) \right| \Psi_{\alpha,\tau} \right| \]  

(A.16)

Making use of the Lesbegue theorem, conditions H2,H5, we find that the quantity \( (A.16) \) tends to zero as \( \delta\alpha \to 0 \). Thus, the vector \( (A.14) \) is correctly defined.

It follows from the expression \( (A.13) \) that
\[ \frac{\partial}{\partial \tau} \frac{\partial}{\partial \alpha} U_B^t(u_{h(\alpha)}X) - iH(B : u_{gb(t)}h(\alpha)X) \frac{\partial}{\partial \alpha} U_B^t(u_{h(\alpha)}X) \]  
in a strong sense.

Let us prove now property \( (A.13) \). At \( t = 0 \) the property \( (A.13) \) is satisfied. The derivative with respect to \( t \) of any matrix element of the operator \( (A.13) \) under conditions H1-H6 vanishes. Therefore, equality \( (A.13) \) viewed in terms of bilinear forms is satisfied on \( D \). Since the left-hand side of eq.\( (A.13) \) is defined on \( D \), it also vanishes on \( D \).

To construct the representation of the Poincare group, the following statement is used in this paper.

Let the property
\[ g_{B_n}(t_n(\alpha))...g_{B_1}(t_1(\alpha)) = e, \]  

be satisfy for \( \alpha \in [0,\alpha_0] \) and \( B_1,...,B_n \in \mathcal{Z} \). Here \( t_k(\alpha) \) are smooth functions. Denote \( h_k(\alpha) = g_{B_k}(t_k(\alpha)), s_k(\alpha) = h_k(\alpha)...h_1(\alpha) \).

**Lemma A.8.** Under condition \( (A.17) \) the operator
\[ U_{B_n}^{t_n(\alpha)}(u_{s_{n-1}(\alpha)}X) ... U_{B_1}^{t_1(\alpha)}(X) \]  
(A.18)
is \( \alpha \)-independent.

To prove lemma, denote \( U_k \equiv U_k(u_{s_k(\alpha)}X) \equiv U_{B_k}^{t_k(\alpha)}(u_{s_k(\alpha)}X) \). Let us use the following lemma.
Lemma A.9. Let \( s(\alpha) \) be a smooth curve on the group \( \mathcal{G} \), \( t(\alpha) \) is a smooth real function, \( B \in T_c \mathcal{G} \). Then the operator function \( U_B^{t(\alpha)}(u_{s(\alpha)}X) \) is strongly differentiable with respect to \( \alpha \) on \( \mathcal{D} \) and

\[
\frac{\partial}{\partial \alpha} U_B^{t(\alpha)}(u_{s(\alpha)}X) = -i \frac{dt}{d\alpha} H(B : u_{g_B(t(\alpha))s(\alpha)}X) U_B^{t(\alpha)}(u_{s(\alpha)}X) + (\delta \frac{ds}{d\alpha} s^{-1}) U_B^{t(\alpha)}(u_{s(\alpha)}X), \tag{A.19}
\]

where \( \frac{ds}{d\alpha} s^{-1} \) is a tangent vector to the curve \( s(\alpha + \tau)s^{-1}(\alpha) \) at \( \tau = 0 \).

Proof. Let \( \Psi \in \mathcal{D} \). One has

\[
\frac{1}{\delta \alpha} (U_B^{t(\alpha + \delta \alpha)}(u_{s(\alpha + \delta \alpha)}X) - U_B^{t(\alpha)}(u_{s(\alpha)}X)) \Psi = \frac{1}{\delta \alpha} (U_B^{t(\alpha + \delta \alpha)}(u_{s(\alpha + \delta \alpha)}X) - U_B^{t(\alpha)}(u_{s(\alpha)}X)) \Psi + \frac{1}{\delta \alpha} (U_B^{t(\alpha + \delta \alpha)}(u_{s(\alpha + \delta \alpha)}X) - U_B^{t(\alpha)}(u_{s(\alpha)}X)) \Psi. \tag{A.20}
\]

The first term in the right-hand side of eq. (A.20) tends to

\[-i \frac{dt}{d\alpha} H(B : u_{g_B(t(\alpha))s(\alpha)}X) \Psi \]

by definition of the operator \( U_B^t(X) \). Consider the second term. It can be represented as

\[
\int_0^1 d\tau \frac{\partial}{\partial \alpha} U_B^{t(\alpha + \gamma \delta \alpha)}(u_{s(\alpha + \gamma \delta \alpha)}X)|_{\tau = \alpha + \delta \alpha} \Psi.
\]

Making use of eq. (A.13), we take this term to the form

\[-i \int_0^1 d\tau \int_0^{t(\alpha + \delta \alpha) - \tau} U_B^{t(\alpha + \delta \alpha) - \tau}(u_{g_B(\tau)s(\alpha + \gamma \delta \alpha)}X) \frac{\partial}{\partial \alpha} H(B : u_{g_B(\tau)s(\alpha + \gamma \delta \alpha)}X) U_B^{t(\alpha + \gamma \delta \alpha)}(u_{s(\alpha)}X) \Psi. \tag{A.21}
\]

Making use of the Lesbegue theorem and property H6, we see that the vector (A.21) is strongly continuous as \( \delta \alpha \to 0 \), so that it is equal to

\[
\frac{\partial}{\partial \alpha} U_B^{t(\alpha)}(u_{s(\alpha)}X)|_{\beta = 0} = ((\delta \frac{ds}{d\alpha} s^{-1}) U_B^{t(\alpha)}(u_{s(\alpha)}X) \Psi.
\]

We obtain formula (A.19).

Let us return to proof of lemma A.8. To check formula (A.18), let us obtain that

\[
\frac{d}{d\alpha}(U_n...U_1)(U_n...U_1)^{-1} = 0 \tag{A.22}
\]

on \( \mathcal{D} \) (the derivative is viewed in the strong sense). The property (A.22) is equivalent to the following relation:

\[
\sum_{k=1}^n U_n...U_{k+1}\frac{\partial U_k}{\partial \alpha} U_k^{-1}...U_n^{-1} = 0. \tag{A.23}
\]

Making use of eq. (A.19), we take eq. (A.23) to the form

\[
\sum_{k=1}^n U_n...U_{k+1} H(-i \frac{dt}{d\alpha} B_k : u_{s_k}X) U_k^{-1}...U_n^{-1} + \sum_{k=1}^n U_n...U_{k+1} (\delta \frac{ds}{d\alpha} s^{-1}) U_k(U_{s_k-1}X) U_k^{-1}...U_n^{-1} = 0. \tag{A.24}
\]

Applying properties (A.13) \( n - k \) times, we obtain

\[
-i H(\sum_{k=1}^n \frac{dt}{d\alpha} h_n...h_{k+1} B_k h_{k+1}^{-1}...h_n^{-1} : X) + \sum_{l=1}^n U_n...U_{l+1} (\delta \frac{ds}{d\alpha} s^{-1}) U_l(U_{s_l-1}X) U_l^{-1}...U_n^{-1} - \sum_{k=1}^n \sum_{l=k+1}^n U_n...U_{l+1} \delta [h_{l-1}...h_{k+1} B_k h_{k+1}^{-1}...h_{l-1}] U_l(U_{s_l-1}X) U_l^{-1}...U_n^{-1}. \tag{A.25}
\]

Eq. (A.17) implies that

\[
\sum_{k=1}^n \frac{dt}{d\alpha} h_n...h_{k+1} B_k h_{k+1}^{-1}...h_n^{-1} = 0;
\]

\[
\frac{ds}{d\alpha} s^{-1} - \sum_{k=1}^{l-1} h_{l-1}...h_{k+1} B_k h_{k+1}^{-1}...h_{l-1} = 0.
\]

Lemma A.8 is proved.

Corollary. Let \( t_k(0) = e \). Then

\[
U_B^{t_n(\alpha)}(u_{s_n(\alpha)}X)...U^{t_1(\alpha)}(X) = 1
\]

under conditions of lemma A.8.
B Some properties of quadratic Hamiltonians in the Fock space

The purpose of this appendix is to introduce some notations and check some properties of operators in Fock space.

Remind that the Fock space \( \mathcal{F}(L^2(\mathbb{R}^d)) \) is defined as a space of sets

\[
\Psi = (\Psi_0, \Psi_1(x_1), \ldots, \Psi_n(x_1, \ldots, x_n), \ldots)
\]

of symmetric with respect to \( x_1, \ldots, x_n \) symmetric functions \( \Psi_n \) such that \( ||\Psi||^2 = \sum_{n=0}^{\infty} ||\Psi_n||^2 < \infty \). By \( A^\pm(x) \) we denote, as usual, the creation and annihilation operator distributions:

\[
\begin{align*}
(f \, dx A^+(x)f(x)\psi)_n(x_1, \ldots, x_n) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j)\psi_{n-1}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n); \\
(f \, dx A^-(x)f^*(x)\psi)_{n-1}(x_1, \ldots, x_{n-1}) &= \sqrt{n} \int dx f^*(x)\psi_n(x, x_1, \ldots, x_{n-1}).
\end{align*}
\]

By \( |0\rangle \) we denote, as usual, the vacuum vector of the form \((1, 0, 0, \ldots)\). Arbitrary vector of the Fock space can be presented via the creation operators and vacuum vector as follows \([1]\)

\[
\Psi = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int dx_1 \ldots dx_n \Psi_n(x_1, \ldots, x_n) A^+(x_1) \ldots A^+(x_n) |0\rangle.
\]

Introduce the operator of number of particles \( \hat{n} \) as \( \langle \hat{n} \psi \rangle = n\psi_n \). Let \( T \) be a nonbounded self-adjoint operator in \( L^2(\mathbb{R}^d) \) such that \( T - 1 \geq 0 \). By \( A^+T A^- \) we denote the operator of the form

\[
(A^+T A^-)^n = \sum_{j=1}^{n} 1^{\otimes j-1} \otimes T \otimes 1^{\otimes n-j} \psi_n.
\]

Introduce also the following norms in the Fock space:

\[
||\psi||_m = ||(\hat{n} + 1)^m \psi||, \quad ||\psi||_T^m = ||(A^+T A^- + 1)^m \psi||.
\]

**Lemma B.1.** Let \( ||\psi||_T^m < \infty \). Then \( ||\psi||_m \leq ||\psi||_T^m \).

**Proof.** It is sufficient to check that

\[
(\psi_n, (\sum_{j=1}^{n} 1^{\otimes j-1} \otimes T \otimes 1^{\otimes n-j} + 1)^{2m} \psi_n) \geq (\psi_n, (\sum_{j=1}^{n} 1 + 1)^{2m} \psi_n).
\]

This relation is a corollary of the formula

\[
(\psi_n, T^{2l_1} \otimes \cdots \otimes T^{2l_n} \psi_n) \geq (\psi_n, \psi_n)
\]

for all \( l_1, \ldots, l_n \geq 0 \). The latter formula is obtained from the relation \( ||T^{-l_1} \otimes \cdots \otimes T^{-l_n}|| \leq 1 \). Lemma B.1 is proved.

Let \( \mathcal{H}^{++} \) be a nonbounded operator in \( L^2(\mathbb{R}^d) \) such that operators \( T^{-1/2} \mathcal{H}^{++} T^{-1/2} \) and \( \mathcal{H}^{+-} T^{-1} \) are bounded.

**Lemma B.2.** The following estimation is satisfied:

\[
||A^+ \mathcal{H}^{+-} A^- \psi|| \leq C ||\psi||_T^1
\]

with \( C = \max(||T^{-1/2} \mathcal{H}^{++} T^{-1/2}||, ||\mathcal{H}^{+-} T^{-1}||) \).

**Proof.** One should check

\[
(\psi_n, \mathcal{H}^{++}_{i^-} \mathcal{H}^{++}_{j^-} \psi_n) \leq C(\psi_n, T_i T_j \psi_n) \quad \text{(B.1)}
\]

with \( \mathcal{H}^{++}_{i^-} = 1^{i^-} \otimes \mathcal{H}^{-+} \otimes 1^{n-i}, T_i = 1^{i^-} \otimes T \otimes 1^{n-i} \). Denote \( T_i^{1/2} T_j^{1/2} \psi_n = \phi_n \). Inequality (B.1) takes the form

\[
(\phi_n, T_i^{1/2} T_j^{1/2} \mathcal{H}^{++}_{i^-} \mathcal{H}^{++}_{j^-} T_i^{-1/2} T_j^{-1/2} \phi_n) \leq C^2(\phi_n, \phi_n). \quad \text{(B.2)}
\]
For $i \neq j$, property (B.2) is satisfied if $C = \|T^{-1/2} \mathcal{H}^\mp T^{-1/2}\|$ as a corollary of the Cauchy-Bunyakovski-Schwartz inequality. For $i = j$, property (B.2) is satisfied if $C = \|\mathcal{H}^\pm T^{-1}\|$. Lemma B.2 is proved.

**Lemma B.3.** Consider the operator

$$\hat{\varphi} = \int dx_1...dx_m dy_1...dy_k \varphi(x_1,...,x_n,y_1,...,y_k)A^+(x_1)...A^+(x_m)A^-(y_1)...A^-(y_k)$$

with $\varphi \in L^2(\mathbb{R}^{d(m+k)})$. Let $\Psi \in \mathcal{F}$ and $\|\Psi\|_{l^2} < \infty$. Then

$$\|\hat{\varphi}\psi\| \leq C\|\varphi\|_{L^2}\|\Psi\|_{l^2},$$

where $C^2 = \max\{1, (m-k)!(m-k)^2\}$.

**Proof.** One has

$$(\hat{\varphi}\Psi)_n(z_1,...,z_n) = \frac{(n-m+k)!}{(n-m)!}\sqrt{\frac{n!}{(n-m)!}}\text{Sym} \int dy_1...dy_k \varphi(z_1,...,z_m,y_1,...,y_k)$$

$$\times \Psi_{n+m+k}(y_1,...,y_k,z_{m+1},...,z_n)$$

where $\text{Sym}$ is a symmetrization operator. Since $\|\text{Sym}\Phi_n\| \leq ||\Phi_n||$ and

$$\|\int dy \varphi(z,y)\Psi(y,z')\| \leq ||\varphi||\|\Psi||,$$

one has

$$\|\hat{\varphi}\Psi\| \leq \sqrt{\frac{(n-m+k)!}{(n-m)!}\frac{n!}{(n-m)!}}\|\varphi\||\Psi_{n+m+k}||.$$

Therefore,

$$\|\hat{\varphi}\Psi\|^2 = \sum_{s=0}^\infty (n+1)^{s} \frac{(n-m+k)!}{(n-m)!} \frac{n!}{(n-m)!} \|\varphi\|^2 \|\Psi_{n+m+k}\|^2$$

$$= \sum_{s=0}^\infty (s+m-k)^{2(s-k+1)}(s-k+1)^{s-k+1} \frac{(n-m+k)!}{(n-m)!} \frac{n!}{(n-m)!} \|\varphi\|^2 \|\Psi_{s}\|^2 \leq C^2 \|\varphi\|^2 \|\Psi\|^2_{l^2},$$

where $s = m - n + k$. Lemma is proved.

**Corollary.**

$$\|\frac{1}{2}A^\pm \mathcal{H}^\pm A^\pm \psi\| \leq \frac{1}{\sqrt{2}} \|\mathcal{H}^\pm\|_2 \|\psi\|_1,$$

Here $\| \cdot \|_2$ is a Hilbert-Schmidt norm $\|A\|_2 = \sqrt{TrA^+A}$.

Consider the quadratic Hamiltonian

$$H = \frac{1}{2}A^+\mathcal{H}^+ A^- + A^+\mathcal{H}^- A^- + \frac{1}{2}A^-\mathcal{H}^- A^- + \mathcal{H}.$$ 

Let $H, \mathcal{H}^+, \mathcal{H}^-, \mathcal{H}^{-}, \mathcal{H}^{++}, \mathcal{H}^{+-}, \mathcal{H}^{-+}, \mathcal{H}^{--}$ be $\alpha$-dependent, $\alpha \in \mathbb{R}$, and $(\mathcal{H}^{++})^+ = \mathcal{H}^{-}, (\mathcal{H}^{+-})^+ = \mathcal{H}^+.$

**Lemma B.4.** Let $\mathcal{H}_\alpha$ be a continuously differentiable function, $\mathcal{H}^{++}_\alpha$ be a continuously differentiable Hilbert-Schmidt operator in the norm $\| \cdot \|_2$, while $T^{-1/2}\mathcal{H}^{++}_\alpha T^{-1/2}$ and $\mathcal{H}^{++}_\alpha T^{-1}$ are operator functions being continuously differentiable in the operator norm. Let $\psi \in \mathcal{F}$, $\|\psi\|^2 < \infty$. Then the vector function $H_\alpha \psi$ is continuously differentiable in the strong topology.

**Proof.** One has from lemmas B.2 and B.3 that

$$\max \left( \|T^{-1/2} \left[ \frac{H_{\alpha+\delta\alpha} \mathcal{H}^{++}_{\alpha+\delta\alpha}}{\delta\alpha} - \frac{H_{\alpha+\delta\alpha} \mathcal{H}^{++}_{\alpha}}{\delta\alpha} \right] T^{-1/2} \|, \| \left[ \frac{H_{\alpha+\delta\alpha} \mathcal{H}^{++}_{\alpha+\delta\alpha}}{\delta\alpha} - \frac{H_{\alpha+\delta\alpha} \mathcal{H}^{++}_{\alpha}}{\delta\alpha} \right] T^{-1} \right) \right) \|\psi\|^2 +$$

$$- \left| \frac{\mathcal{H}^{++}_{\alpha+\delta\alpha} T_{\alpha+\delta\alpha}}{\delta\alpha} - \frac{\mathcal{H}^{++}_{\alpha} T_{\alpha}}{\delta\alpha} \right| \rightarrow 0 \delta \alpha \rightarrow 0.$$

The fact that $\frac{d}{d\alpha} H_\alpha \psi$ is continuous is checked analogously. Lemma B.4 is proved.
Consider now the Cauchy problem for the equation

\[ H_t = \frac{1}{2} A^+ \mathcal{H}_t^{++} A^- + A^+ \mathcal{H}_t^{+-} A^- + \frac{1}{2} A^- \mathcal{H}_t^{-+} A^- + \mathcal{H}_t. \] (B.3)
onumber

on the Fock vector \( \Psi_t \); the strong derivative enters to eq.(B.3).

Formally, the solution for the initial condition

\[ \Psi_0 = \sum_{n=0}^\infty \frac{1}{\sqrt{n!}} \int dx_1 \ldots dx_n A^+ (x_1) \ldots A^+ (x_n) \Psi_{0,n}(x_1, \ldots, x_n) |0> \] (B.4)

is looked for in the following form \[ \text{[41, 50]} \]

\[ \Psi_t = \sum_{n=0}^\infty \frac{1}{\sqrt{n!}} \int dx_1 \ldots dx_n A^+_t (x_1) \ldots A^+_t (x_n) \Psi_{0,n}(x_1, \ldots, x_n) |0>_t \] (B.5)

with

\[ |0>_t = c^t \exp \left[ \frac{i}{2} \int dxdy M(t) A^+(x) A^+(y) \right] |0> . \] (B.6)

while operators \( A^+_t (x) \) are chosen to be

\[ A^+_t (x) = \int dy [A^+ (y) G^*_t (y, x) - A^- (y) A^*_t (y, x)]. \]

Namely, the Gaussian ansatz \[ \text{[B.6]} \] formally satisfies eq.(B.3) if

\[ i \frac{dA^+}{dt} = \frac{1}{2} Tr H^- + M^t c^t + \mathcal{H}_t c^t, \]

\[ i \frac{dM^t}{dt} = \mathcal{H}_t^{++} + \mathcal{H}_t^{+-} M_t + M_t \mathcal{H}_t^{+-} + M_t \mathcal{H}_t^{-+} M_t. \] (B.7)

Here \( M_t \) is the operator with kernel \( M(t, x, y), \mathcal{H}_t^{+-} = (\mathcal{H}_t^{-+})^* \). The operators \( A^+_t (x) \) commute with

\[ i \frac{dA^+}{dt} - H_t \]

if

\[ i \frac{dF_t}{dt} = \mathcal{H}_t^{-+} F_t + \mathcal{H}_t^{++} G_t, \quad -i \frac{dG_t}{dt} = \mathcal{H}_t^{--} G_t + \mathcal{H}_t^{+-} F_t. \] (B.8)

Here \( F_t, G_t \) are operators with kernels \( F_t(x, y) \) and \( G_t(x, y) \). Note that the operator \( M_t = F_t G_t^{-1} \)

formally satisfies eq.(B.7). Initial conditions (B.4) are satisfied if \( F_0 = 0, G_0 = 1 \).

Let us check that eq.(B.3) is satisfied in a strong sense.

First of all, let us present some auxiliary lemmas.

**Lemma B.5.** Let \( M \) be a Hilbert-Schmidt operator and \( ||M|| < 1 \). Then

\[ \exp \left[ \frac{1}{2} A^+ M A^+ \right] |0> \] (B.9)

The proof is presented in \[ \text{[41]} \].

**Corollary.** For the state \( \text{[B.9]} \), the estimation

\[ ||\psi_n|| \leq Ae^{-\alpha n} \] (B.10)

is satisfied under conditions of lemma B.5 for some \( A > 0 \) and \( 0 < \alpha < -\frac{1}{2} \log ||M|| \).

**Proof.** Since \( ||M|| < 1, ||e^{2\alpha M}|| < 1 \). Since expression \( \tilde{\psi} = \exp \left[ \frac{1}{2} e^{2\alpha A^+ M A^+} \right] |0> \) specifies a Fock space vector, \( ||\tilde{\psi}_{2n}|| = ||e^{2\alpha n} \psi_{2n}|| \leq A \). Corollary is proved.

**Lemma B.6.** Let \( M, \delta M \) be Hilbert-Schmidt operators, \( ||M|| \leq 1, ||M + \delta M|| \leq 1 \) and

\[ ||\delta M||_2 \leq \frac{1}{4} \log ||M||^{-1} ||M||^{-3/8} \]
Then
\[ \exp\left[ \frac{1}{2} A^+ (M + \delta M) A^+ \right]|0 > = \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \frac{1}{2} A^+ \delta M A^+ \right]^k \exp\left[ \frac{1}{2} A^+ M A^+ \right]|0 > \]

**Proof.** One should check that
\[
s - \lim_{N \to \infty} \sum_{k,l,k+l \leq N} \frac{1}{2 k!}(A^+ \delta MA^+)^k \frac{1}{2 l!}(A^+ MA^+)^l|0 > =
\]
\[
s - \lim_{N \to \infty} \sum_{k=0}^{N} \frac{1}{2 k!}(A^+ \delta MA^+)^k e^\frac{1}{2} A^+ MA^+|0 > \tag{B.11}
\]
Since the strong limit in the left-hand side of equality \((B.11)\) exists, eq. \((B.11)\) can be presented as
\[
\sum_{k=0}^{N} \sum_{l=N-k+1}^{\infty} \psi_{k,l} \to_{N \to \infty} 0 \tag{B.12}
\]
with
\[
\psi_{k,l} = \frac{1}{2 k!}(A^+ \delta MA^+)^k \frac{1}{2 l!}(A^+ MA^+)^l|0 > .
\]
Since
\[
([A^+ \delta MA^+] \psi)_n(x_1, \ldots, x_n) = \text{Sym}\sqrt{n(n-1)}\delta M(x_1, x_2) \psi_{n-2}(x_3, \ldots, x_n),
\]
one has
\[
||([A^+ \delta MA^+] \psi)_n|| \leq \sqrt{n(n-1)} ||\delta M||_2 ||\psi||_{n-2}.
\]
By induction, one obtains:
\[
||[A^+ \delta MA^+]^k \psi_{n-2k}|| \leq \sqrt{\frac{n!}{(n-2k)!}} ||\delta M||^k_2 ||\psi_{n-2k}||.
\]
It follows from the estimation \((B.10)\) that
\[
||\psi_{k,l}|| \leq \sqrt{\frac{(l+2k)!}{2k!}} ||\delta M||^k_2 e^{-\alpha l/2} e^{-\alpha l/2} \leq \max(l + 2k)! e^{-\alpha l/2} ||\delta M||^k_2 e^{\alpha l} = e^{-\alpha l/2} \frac{k^k}{e^{\alpha l}} \left( \frac{||\delta M||_2 e^{\alpha l}}{\alpha} \right)^k.
\]
Since \(k! \sim (k/e)^k \sqrt{2\pi k}\) as \(k \to \infty\), one has \(e^{-k} k^k/k! \leq A_1\). Therefore,
\[
||\psi_{k,l}|| \leq A A_1 e^{-\alpha l/2} b^k
\]
with \(b = ||\delta M||_2 e^{\alpha l}/\alpha\). Therefore,
\[
\sum_{k=0}^{N} \sum_{l=N-k+1}^{\infty} ||\psi_{k,l}|| = \sum_{k=0}^{N} A A_1 b^k e^{-\frac{\alpha}{2} (N-k+1)} \frac{1}{1 - e^{-\alpha/2}} \leq A A_1 e^{-\alpha (N+1)/2} \left( \frac{e^{\alpha}}{1 - e^{-\alpha/2}} \right)^k (1 - e^{-\alpha/2}) (1 - e^{-\alpha/2}).
\]
Therefore, for \(||\delta M||_2 e^{3\alpha/2} \leq \alpha\) property \((B.11)\) is satisfied. Choosing \(\alpha = -\frac{1}{3} \log ||M||\), we obtain statement of lemma.

**Lemma B.7.** Let \(M_t, t \in [t_1, t_2]\) be a differentiable operator function, \(||M_t||_2 < \infty\),
\[
||\frac{M_{t+\delta t} - M_t}{\delta t} - \frac{dM_t}{dt}||_2 \to_{\delta t \to 0} 0. \tag{B.14}
\]
Then
\[
||e^{\frac{1}{2} A^+ M_{t+\delta t} A^+}||_2 > \left( -e^{\frac{1}{2} A^+ M_t A^+} \right)\frac{dM_t}{dt} A^+ e^{\frac{1}{2} A^+ M_{t+\delta t} A^+} ||_m \to_{\delta t \to 0} 0. \tag{B.15}
\]
**Proof.** Denote \(\delta M \equiv \delta M_{t, \delta t} = M_{t+\delta t} - M_t\). It is sufficient to check the following formulas:
\[
||\frac{e^{\frac{1}{2} A^+ M_{t+\delta t} A^+} - 1 - \frac{1}{2} A^+ M_t A^+}{\delta t} \to_{\delta t \to 0} 0; \tag{B.16}
\]
\[ \left\| A^+ \frac{\delta M}{\delta t} - \frac{dM}{dt} \right\| A^+ e^{\frac{t}{2} A^+ M_t A^+} \right\| 0 > \left\| m \to \delta t \to 0. \quad (B.17) \]

The latter formula is a direct corollary of lemma B.3, property \( \left\| e^{\frac{t}{2} A^+ M_t A^+} \right\| 0 > \left\| m+1 < \infty \) following from formula (B.10) and relation \( \left\| \frac{\delta M}{\delta t} - \frac{dM}{dt} \right\| 2 \to \delta t \to 0. \) Formula (B.30) is a corollary of the relation
\[ \sum_{k=2}^{\infty} \sum_{l=0}^{k-1} \frac{1}{\delta t} (2k + 1 + l)^m \left\| \psi_{k,l} \right\| \to \delta t \to 0. \quad (B.18) \]

Making use of the estimation (B.13) and formula \( \left\| \delta M \right\| ^2 / \delta t \to \delta t \to 0, \) we prove relation (B.18). Lemma B.7 is proved.

**Lemma B.8.** Let \( T \) be such nonbounded self-adjoint operator in \( L^2(\mathbb{R}^d) \) that \( T - 1 > 0, \) \( D(T) \subset D(\mathcal{H}^+), \) \( \mathcal{H}^+ T^{-1} \) be uniformly bounded operator. Let the initial condition for eq. (B.3) be of the form (B.4), where \( \Psi_{0,n} = 0 \) as \( n \geq N_0, \)
\[ \Psi_{0,n}(x_1, \ldots, x_n) = \sum_{j=1}^{J} f_j(x_1) \ldots f_n(x_n), \quad f_j^* \in D(T). \quad (B.19) \]

Let Hilbert-Schmidt operator \( M_t \) satisfy eq. (B.4) (the derivative is defined in the Hilbert-Schmidt sense (B.14)) and initial condition \( M_0 = 0, \) \( c_t \) obey eq. (B.17). \( F_t \) and \( G_t \) be uniformly bounded operators \( F_t : D(T) \to D(T), G_t : D(T) \to D(T) \) satisfying eq. (B.8) in the strong sense on \( D(T), F_0 = 0, G_0 = 1. \) Then the Fock vector (B.3) obeys eq. (B.3) in the strong sense.

**Proof.** It is sufficient to prove lemma for the initial condition
\[ \Psi_0 = \frac{1}{\sqrt{n!}} A^+[f^1] \ldots A^+[f^n] \right\| 0 > \]
where \( A^+[f] = \int dxf(x) A^+(x). \) Let us show that the Fock vector
\[ \Psi_t = \frac{1}{\sqrt{n!}} A^+_t [f^1] \ldots A^+_t [f^n] \right\| 0 > t \]
with
\[ A^+_t [f] = \int dy A^+(y)(G_t^* f)(y) - A^-(y)(F_t^* f)(y) \]
satisfies eq. (B.3). Let
\[ \Psi_t = \frac{1}{\sqrt{n!}} (\sum_{j=1}^{n} A^+_t [f^1] \ldots A^+_t [f^j] \ldots A^+_t [f^n] \right\| 0 > t + A^+_t [f^1] \ldots A^+_t [f^j] \ldots A^+_t [f^n] \frac{d}{dt} \right\| 0 > t \]
with
\[ \dot{A}^+_t [f] = \int dy A^+(y) \frac{d}{dt}(G_t^* f)(y) - A^-(y) \frac{d}{dt}(F_t^* f)(y), \]
\[ \frac{d}{dt} \right\| 0 > t = \frac{d}{dt} e^{\frac{t}{2} A^+ M_t A^+} \right\| 0 > + e^{\frac{t}{2} A^+ M_t A^+} \frac{d}{dt} \right\| 0 > t. \]
One has
\[ \frac{\Psi_{t+\delta t} - \Psi_t}{\delta t} - \dot{\Psi}_t = \frac{1}{\sqrt{n!}} A^+_t [f^1] \ldots A^+_t [f^n] \right\| 0 > t + \frac{\Psi_{t+\delta t} - \Psi_t}{\delta t} - \frac{d}{dt} \right\| 0 > t \]
\[ \sum_{j=1}^{n} \frac{A^+_t [f^1] \ldots A^+_t [f^j] \ldots A^+_t [f^n] \right\| 0 > t + \frac{\Psi_{t+\delta t} - \Psi_t}{\delta t} - \frac{d}{dt} \right\| 0 > t + \sum_{j=1}^{n} \frac{A^+_t [f^1] \ldots A^+_t [f^j] \ldots A^+_t [f^n] \right\| 0 > t + \sum_{j=1}^{n} \frac{A^+_t [f^1] \ldots A^+_t [f^j] \ldots A^+_t [f^n] \right\| 0 > t. \]
It follows from lemmas B.3, B.7 and conditions of lemma B.8 that
\[ \left\| \frac{\Psi_{t+\delta t} - \Psi_t}{\delta t} - \dot{\Psi}_t \right\| \to \delta t \to 0. \]
Eqs. (B.7), (B.8) imply that $\dot{\Psi}_t = -iH_t\Psi_t$. Lemma B.8 is proved.

Denote by $D_1 \subset F$ the set of all Fock vectors $\Psi \in F$ such that $\Psi_n$ vanish at $n \geq N_0$ and have the form (B.19) as $n < N_0$. Lemma B.8 allows us to construct the mapping $U_t : D_1 \to F$ of the form $U_t\Psi_0 = \Psi_t$. Note that the domain $D_1$ is dense in $F$.

Denote

$$A_t^-[f] \equiv (A_t^+[f])^+ \equiv \int dy[A^-(y)(G_tf)(y) - A^+(y)(F_tf)(y)].$$

**Lemma B.9.** 1. The operators $A_t^+[f]$ obey the commutation relations

$$[A_t^-[f], A_t^+[g]] = (f, g), \quad [A_t^+[f], A_t^+[g]] = 0.$$  

2. The following property is satisfied:

$$A_t^-[f] - t = 0.$$  

3. The operator $U_t$ is isometric.

**Proof.** The commutation relations (B.20) are rewritten as

$$\begin{align*}
(G_tf, G_tg) - (F_tf, F_tg) &= (f, g); \\
(F_t^-f, G_tg) - (G_t^-f, F_tg) &= 0.
\end{align*}$$

(B.22)

They are satisfied at $t = 0$. The time derivatives of the left-hand sides of eqs. (B.22) vanish because of eqs. (B.8). Statement 1 is proved.

The fact that $U_t$ is an isometric operator is a corollary of the property $\frac{d}{dt}(\Psi_t, \Psi_t) = 0$.

Analogously to lemma B.8, we find that the vector $\tilde{\Psi}_t = A_t^-[f]0 > t$, obeys eq. (B.8) in the strong sense. Since $\Psi_0 = 0$ and $||\Psi_t|| = ||\Psi_0||$, one has $\Psi_t = 0$. Property (B.21) is proved. Note that it means that

$$M_t G_t = F_t.$$  

(B.23)

Lemma B.9 is proved.

Therefore, the operator $U_t$ can be extended to the whole space $F$, $U_t : F \to F$.

**Lemma B.10.** Let the operator

$$\begin{pmatrix}
G^+ & -F^+
\end{pmatrix}
\begin{pmatrix}
F & G^*
\end{pmatrix}$$

be invertible. Then the following relation is satisfied on $D_1$:

$$U_t^{-1}A^+TA^-U_t\Psi_0 = (A^+G_t^T + A^-F^+)T(FA^+ + G^*A^-)\Psi_0$$

(B.24)

**Proof.** It follows from lemma B.9 that

$$\begin{pmatrix}
G^+ & -F^+
\end{pmatrix}
\begin{pmatrix}
F & G^*
\end{pmatrix}^{-1} = 1
$$

Therefore,

$$\begin{pmatrix}
G^+ & -F^+
\end{pmatrix}
\begin{pmatrix}
G^T & F^*
\end{pmatrix} = 1
$$

and

$$A^-(y) = \int dz(F_t(y, z)A_t^-(z) + G_t^*(y, z)A_t^+(z)),
A^+(y) = \int dz(F_t^*(y, z)A_t^-+(z) + G_t(y, z)A_t^+(z)).$$

Identity (B.24) is then a corollary of definition of the operator $U_t$.

By $D \in F$ we denote set of such Fock vectors $\Psi$ that $||\Psi||^2 < \infty$.

**Lemma B.11.** Let $\Psi_0 \in D$. Suppose that $TF_t$ and $G^{++}$ are continuous operator functions in the $\|\cdot\|_2$-norm, $G_t$, $T^{1/2}G_tT^{-1/2}$, $TG_tT^{-1}$, $T^{-1/2}G^{++}T^{-1/2}$, $G^{++}T^{-1}$ are continous operator functions in the $\|\cdot\|$-norm. Then the following statements are satisfied.
1. $\Psi_t \equiv U_t \Psi_0 \in \mathcal{D}$.
2. $\Psi_t$ obeys eq. (B.3) in the strong sense.
3. 

$$||\Psi_t - \Psi_0||_1^T \rightarrow_{t \to 0} 0.$$  \hfill (B.25)

Proof. Let $\Psi_0 \in \mathcal{D}_1$. For $||U_t \Psi_0||_1^T$, one has the following estimation:

$$||U_t \Psi_0||_1^T = ||U_t^{-1}(\hat{T} + 1)U_t \Psi_0|| \leq ||\Psi_0|| + ||(A^T G + A^{-1} A^T)T(FA^T + G^* A)\Psi_0|| \leq$$

$$(1 + ||F||_2 ||T F||_2 ||\Psi_0|| + (2||G^T T F||_2 + ||F^T T F|| + ||F + T G||_2)||\Psi_0||$$

$$+ (||T^{-1/2} G^T T g^* T^{-1/2}|| + ||A^T A^* T^{-1}||)||\Psi_0|| \leq const ||\Psi||_1^T$$

at $t \in [0, t_1]$. Therefore, the operator $U_t$ is bounded in norm $|| \cdot ||_1^T$. The extension of the operator $U_t$ to $\mathcal{D}$ is then also a bounded operator in $|| \cdot ||_1^T$ norm. One therefore has $\Psi_t \in \mathcal{D}$.

The fact that $||U_t \Psi_0 - \Psi_0||_1^T \rightarrow_{t \to 0} 0$ if $\Psi_0 \in \mathcal{D}_1$ is justified analogously to lemma B.8. Since the operator $U_t : \mathcal{D} \to \mathcal{D}$ is uniformly bounded at $t \in [0, t_1]$ in $|| \cdot ||_1^T$-norm, the Banach-Steinhaus theorem (see, for example, [51]) implies relation (B.25).

To check the second statement, note that lemma B.8 imply that

$$\frac{U_{t+\delta t} - U_t}{\delta t} \rightarrow_{\delta t \to 0} \frac{dU_t}{dt}$$  \hfill (B.26)

in the strong sense on $\mathcal{D}_1$. For showing that relation (B.26) is satisfied in the strong sense on $\mathcal{D}$, it is sufficient to show that the operator

$$\delta U : \mathcal{D} \to \mathcal{F}$$

is uniformly bounded,

$$||\frac{\delta U}{\delta t} \Psi || \leq C||\Psi||_1^T.$$  

One has

$$||\frac{\delta U}{\delta t} \Psi || = || \int_0^1 ds U_{t+\delta st} \Psi || = || \int_0^1 ds H_{t+s\delta t} U_{t+s\delta t} \Psi || \leq$$

$$\max_{s \in [0,1]} ||\sqrt{2}||H_{t+s\delta t}^{+}||_2 + ||T^{-1/2} H_{t+s\delta t}^{-} T^{-1/2}|| + ||H_{t+s\delta t}^{-} T^{-1}|| ||U_{t+s\delta t} \Psi ||_1^T.$$  

Lemma B.11 is proved.

Let us now check properties of operators $F_t$, $F_t$, $M_t$.

First of all, consider the Cauchy problem

\begin{align*}
    i\dot{f}_t &= Y_t f_t + Z_t g_t, \\
    -i\dot{g}_t &= Z_t^* f_t + Y_t^* g_t, \\
    f_0 &= 0, g_0 = 1, 
\end{align*}  \hfill (B.27)

where $g_t$ is a bounded operator functions, $f_t$ is a Hilbert-Schmidt operator function. The derivatives in (B.27) are understood as

$$||\left(\frac{g_{t+\delta t} - g_t}{\delta t} - \dot{g}_t\right)\varphi|| \rightarrow_{\delta t \to 0} 0, \quad ||\left(\frac{f_{t+\delta t} - f_t}{\delta t} - \dot{f}_t\right)||_2 \rightarrow_{\delta t \to 0} 0.$$  \hfill (B.28)

Lemma B.12. Let $Y_t$ be a strongly continous operator function, while $||Z_{t+r} - Z_t|| \rightarrow_{r \to 0} 0$, $||TZ_t|| \leq a_1$, $||T Y_t^{-1}|| \leq a_2$, $||T^{1/2} Y_t^{-1}|| \leq a_3$ for smooth functions $a_k$. Then there exist a solution to the Cauchy problem (B.24) such that

$$||T f_t|| \leq a_4, \quad ||T^{1/2} g_t T^{-1/2}|| \leq a_5^t, \quad ||T g_t T^{-1}|| \leq a_6^t, \quad ||g_t|| \leq a_7^t$$  \hfill (B.29)

for smooth functions $a_k^t$. 

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Proof (cf. [50]). Let us look for the solution to the Cauchy problem in the following form:

\[ f_t = \sum_{n=0}^{\infty} f_t^n, \quad g_t = \sum_{n=0}^{\infty} g_t^n. \]  

(B.30)

where \( f_t^0 = 0, g_t^0 = 1, \)

\[ f_t^{n+1} = -i f_t^0 d\tau (Y_{r} f_t^n + Z_{r} g_t^n), \]

\[ g_t^{n+1} = -i f_t^0 d\tau (Y_{r}^* g_t^n + Z_{r}^* f_t^n). \]  

(B.31)

By induction we find that \( \|f_t^n\|_2 \leq C_1 t^n/n!, \|g_t^n\|_2 \leq C_1 t^n/n! \) for \( t \in [0, t_1] \). Here \( C_1 \) is a constant.

Therefore, the series (B.30) converge. \( f_t \) is a Hilbert-Schmidt operator, while \( g_t \) is a bounded operator. Analogously, we show

\[ \|T f_t^n\|_2 \leq \frac{C_2 t^n}{n!}, \quad \|T g_t^n T^{-1}\| \leq \frac{C_2 t^n}{n!}, \quad \|T^{1/2} g_t^n T^{-1/2}\| \leq \frac{C_2 t^n}{n!}, \]

where \( t \in [0, t_1] \). Therefore, properties (B.23) are satisfied.

To check relations (B.28), note that

\[ f_t = -i \int_0^t d\tau (Y_{\tau} f_{\tau} + Z_{\tau} g_{\tau}), \]

\[ g_t = i \int_0^t d\tau (Z_{\tau}^* f_{\tau} + Y_{\tau}^* g_{\tau}). \]  

(B.32)

Eqs. (B.32) imply that the operator functions \( f_t, g_t \) obey properties

\[ \|T(f_{t+\delta t} - f_t)\|_2 \to_{\delta t \to 0} 0, \quad \|(g_{t+\delta t} - g_t)\| \to_{\delta t \to 0} 0, \]

Therefore,

\[ \|i \frac{f_{t+\delta t} - f_t}{\delta t} - Y_t f_t - Z_t g_t\|_2 \leq \int_0^1 ds \|Y_{t+\delta t} f_{t+\delta t} + Z_{t+\delta t} g_{t+\delta t} - Y_t f_t - Z_t g_t\|_2, \]

\[ \|(-i \frac{g_{t+\delta t} - g_t}{\delta t} - Z_t^* f_t - Y_t^* g_t)\| \leq \int_0^1 ds \|(Z_{t+\delta t} f_{t+\delta t} + Y_{t+\delta t} g_{t+\delta t} - Z_t^* f_t - Y_t^* g_t)\| \]

Since the integrands are uniformly bounded functions, the Lesbegue theorem (see, for example, [52]) tells us that it is sufficient to check that

\[ \|Y_{t+\tau} f_{t+\tau} - Y_t f_t\|_2 \to_{\delta t \to 0} 0, \quad \|\frac{s}{\tau} - \lim_{\tau \to 0} Z_{t+\tau}^* f_{t+\tau} - Z_t^* f_t, \]

\[ \|Y_{t+\tau} g_{t+\tau} - Y_t g_t\| \to_{\delta t \to 0} 0, \]

These relations are corollaries of conditions of lemma B.12 and formulas (B.31).

Lemma B.13. Let \( \mathcal{H}_{t+\delta t} = L + \mathcal{H}_{t+\delta t}, \mathcal{H}_{t}, T^{1/2} \mathcal{H} T^{-1/2}, T \mathcal{H} T^{-1} \) be strongly continuous operator functions, \( \|\mathcal{H}_{t+\delta t} - \mathcal{H}_{t\delta t}\|_2 \to_{\delta t \to 0} 0, L \) be a t-independent (maybe nonbounded) self-adjoint operator, such that \( \|LT^{-1}\|_2 \leq C_1 t^n/n!, \|T^{1/2}\|_2 \leq C_2 t^n/n! \) for \( t \in [0, t_1] \). Then there exists a solution to the Cauchy problem for system (B.13) for the initial condition \( F_0 = 0, G_0 = 1 \):

\[ \frac{i}{\delta t} \frac{F_{t+\delta t} - F_t}{\delta t} - \mathcal{H}_+ F_t - \mathcal{H}_t^+ G_t \|_2 \to_{\delta t \to 0} 0, \]

\[ \|(-i \frac{G_{t+\delta t} - G_t}{\delta t} - \mathcal{H}_t G_t - \mathcal{H}_t^+ F_t)\| \to_{\delta t \to 0} 0, \quad \varphi \in D(T). \]  

(B.33)

Moreover,

\[ \|T F_t\|_2 \leq b(t), \quad \|T G_t T^{-1}\| \leq b(t), \quad \|T^{1/2} G_t T^{-1/2}\| \leq b(t), \quad \|G_t\| \leq b(t) \]  

(B.34)

for some smooth function \( b(t) \) on \( t \in [0, t_1] \). The properties (B.23) are also satisfied.

Proof. Consider the operator functions

\[ F_t = e^{-iL_t f_t}, \quad G_t = e^{iL_t g_t}, \]

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where \((f_t, g_t)\) is a solution to the Cauchy problem \((\text{B.27})\) with \(Y_t = e^{iLt}H_t e^{-iLt}, \ Z_t = e^{iLt}H_t^+ e^{iLt},\ f_0 = 0, \ g_0 = 1\). Check of properties \((\text{B.34})\) is straightforward. Let us prove relations \((\text{B.33})\). One has

\[
\begin{align*}
&\frac{d}{dt}F(t) \quad (L + \frac{d}{dt}H_t)\frac{d}{dt}G(t) = (\frac{d}{dt} e^{-iLt} - LT^{-1})TF_t + i e^{-iLt} (\frac{d}{dt}F_t - \frac{d}{dt}G_t), \\
&-\frac{d}{dt}G(t) \quad (L + \frac{d}{dt}H_t)G(t) - \frac{d}{dt}F(t) = \left(-\frac{d}{dt} e^{-iLt} - LT^{-1}\right)TF_t + i e^{-iLt} (\frac{d}{dt}G_t - \frac{d}{dt}F_t).
\end{align*}
\]

Since

\[
||(e^{-iLt} - \frac{d}{dt}T^{-1} - LT^{-1})|\| \leq \int_0^1 ds||(e^{-iLt} - \frac{d}{dt}T^{-1} - LT^{-1}|| \to_{\tau \to 0} 0,
\]

we obtain relations \((\text{B.33})\).

Property \((\text{B.23})\) is proved analogously to \((\text{B.10})\): one should consider the convergent in \(||\cdot||\)-norm series

\[
\begin{pmatrix} G & F^* \\ F & G^* \end{pmatrix}^{(n)} = \sum_{n=0}^{\infty} \begin{pmatrix} G^{(-n)} & F^{(-n)*} \\ F^{(-n)} & G^{(-n)*} \end{pmatrix} \begin{pmatrix} e^{iLt} & 0 \\ 0 & e^{-iLt} \end{pmatrix}
\]

with

\[
\begin{pmatrix} G^{(-n)} & F^{(-n)*} \\ F^{(-n)} & G^{(-n)*} \end{pmatrix} = i \int_0^t d\tau \begin{pmatrix} G^{(-n-1)} & F^{(-n-1)*} \\ F^{(-n-1)} & G^{(-n-1)*} \end{pmatrix} \begin{pmatrix} Y^* & Z^* \\ -Z^* & Y^* \end{pmatrix}
\]

Lemma B.13 is proved.

**Lemma B.14.** Under conditions of lemma B.13 there exists a solution to the Cauchy problem for eq. \((\text{B.7})\) with the initial condition \(M_0 = 0\).

**Proof.** It follows from \((\text{B.1})\) that the matrix \(G\) is invertible and \(||G^{-1}|| < 1\). Consider the operator \(M_t = F_t G_t^{-1}\). Note that \(||TM_t|| \leq \infty, ||LM_t|| \leq \infty\). One has

\[
M_{t+\delta t} - M_t = M_{t+\delta t}(G_t - G_{t+\delta t})G_t^{-1} + (F_{t+\delta t} - F_t)G_t^{-1},
\]

so that \(||TM_{t+\delta t} - M_t|| \to_{\delta t \to 0} 0\). Therefore,

\[
M_{t+\delta t} TT^{-1}(\frac{d}{dt}G_t^{-1}) - \frac{d}{dt}F_t G_t^{-1} + F_t G_t^{-1} \frac{d}{dt}G_t^{-1} = \frac{d}{dt}G_t^{-1}.
\]

Analogously to lemmas B.12, B.13, one finds

\[
\left\|(\frac{G^{+}_{t+\delta t} - G^{+}_t}{\delta t} - G^{+}_t)T^{-1} \varphi\right\| \to_{\delta t \to 0} 0.
\]

Therefore,

\[
\left\|\frac{M_{t+\delta t} - M_t}{\delta t} - M_t\right\| \to_{\delta t \to 0} 0.
\]

Lemma B.14 is proved.

Therefore, we have proved the following theorem.

**Theorem B.15.** Let \(T, L\) be self-adjoint operators in \(L^2(\mathbb{R}^d)\) such that

\[
||T^{-1/2}LT^{-1/2}|| < \infty, \quad ||LT^{-1}|| < \infty, \quad ||T^{1/2}e^{-iLt}T^{-1/2}|| \leq C, \quad ||Te^{-iLt}T^{-1}|| \leq C, \quad t \in [0, t_1].
\]

Let \(T - c\) be positively definite for some positive constant \(c\), \(H_t^+ = L + H_t, H_t^{++} \) be operator-valued functions such that \(||T(H_t^{++} - H_t^+)|| \to_{\delta t \to 0} 0\), \(H_t, TH_tT^{-1}, T^{1/2}H_tT^{-1/2}\) are strongly continuous operator functions, \(\overline{H_t}\) be a continuous function. Then there exists a unique solution \(\Psi_t\) to the Cauchy problem \((\text{B.3})\), provided that \(\Psi_0 \in D \equiv \{\Psi \in F||\Psi||_1^2 < \infty\}\) It satisfies the properties \(\Psi_t \in D\) and \(||\Psi_t - \Psi_0||_1^2 \to_{\delta t \to 0} 0\).
C Some properties of the Weyl symbol

The purpose of this appendix is to investigate some properties of Weyl symbols of operators which are useful in justification of properties H1-H6 of Appendix A.

C.1 Definition of Weyl symbol

First of all, remind the definition of Weyl symbol of operator (see, for example, \cite{34,53}). Let \( A(x,k), x,k \in \mathbb{R}^d \) be a classical observable depending on coordinates \( x = (x_1, ..., x_d) \) and momenta \( k = (k_1, ..., k_d) \). To specify the corresponding quantum observable \( \hat{A} \) (to "quantize" the observable \( A \)), one should substitute the coordinates \( x_i \) by operators \( \hat{x}_i \) of multiplication by \( x_i \), while the momenta \( k_j \) should be substituted by the operators \( \hat{k}_j = -i\partial/\partial k_j \). However, it is not easy to determine the operator \( \hat{A}(\hat{x}, \hat{k}) \) for arbitrary function \( A \), since the coordinate and momenta operators do not commute. Different ways of ordering operators \( \hat{x} \) and \( \hat{k} \) are known. In the Weyl approach, one first considers the partial case

\[
A = e^{i\alpha k + i\beta x} \quad (C.1)
\]

and sets

\[
\hat{A} = e^{i\alpha \hat{k} + i\beta \hat{x}} \quad (C.2)
\]

The operator (C.2) can be defined as a transformation taking the initial condition \( f^0(x) \) for the Cauchy problem for the equation

\[
-i\frac{\partial f^t}{\partial t} = (\alpha \hat{k} + \beta \hat{x}) f^t(x) \quad (C.3)
\]

to the solution \( f^t(x) \) to the Cauchy problem at \( t = 1 \). Eq.(C.3) is exactly solvable:

\[
f^t(x) = e^{i\beta \hat{x}} e^{\frac{t}{2}\alpha \beta} f^0(x + \alpha t),
\]

so that

\[
(e^{i\alpha k + i\beta \hat{x}} f)(x) = e^{i\beta \hat{x}} e^{\frac{t}{2}\alpha \beta} f(x + \alpha). \quad (C.4)
\]

One finds

\[
e^{i\alpha k + i\beta \hat{x}} = e^{i\beta \hat{x}} e^{\frac{t}{2}\alpha \beta}.
\]

For an arbitrary function \( A \), one presents it as a superposition of exponents (C.1),

\[
A(x,k) = \int d\alpha d\beta \bar{A}(\alpha, \beta) e^{i\alpha k + i\beta x}
\]

one sets

\[
\hat{A} = \int d\alpha d\beta \bar{A}(\alpha, \beta) e^{i\alpha \hat{k} + i\beta \hat{x}}
\]

Applying the formula for inverse Fourier transformation and making use of formula (C.4), we find

\[
(\hat{A}f)(x) = \int \frac{d\alpha dp}{(2\pi)^d} A(x + \frac{\alpha}{2}; p) e^{-iap} f(x + \alpha). \quad (C.5)
\]

We denote the operator \( \hat{A} \) of the form (C.5) as \( \hat{A} = \mathcal{W}(A) \). We will also write \( A = \mathcal{W}(\hat{A}) \) if \( \hat{A} = \mathcal{W}(A) \).

Definition C.1. The operator \( \mathcal{W}(A) \) is called a Weyl quantization of the function \( A \). The function \( \mathcal{W}(\hat{A}) \) is called as a Weyl symbol of the operator \( \hat{A} \).
C.2 Some classes of Weyl symbols

C.2.1 Classes $A_N$ and $B_N$

For investigations of QFT ultraviolet divergences, we are interested in behavior of Weyl symbols of operators at large values of momenta. Let us introduce some important spaces. Let $\omega_k = \sqrt{k^2 + m^2}$ for some $m$.

**Definition C.2.** 1. We say that a smooth function $A(x, k)$ is of the class $B_N$ if and only if the functions

$$\omega_k^{N+i} \frac{\partial^s A}{\partial k^{i_1} \cdots \partial k^{i_s}} \quad (C.6)$$

are bounded for all $s, i_1, \ldots, i_s$.

2. Let $A_n \in B_N$, $n = 1, \infty$, $A \in B_N$. We say that $B_N - \lim_{n \to \infty} A_n = A$ if and only if

$$\lim_{n \to \infty} \max_{k, x} \omega_k^{N+i} \frac{\partial^s (A_n - A)}{\partial k^{i_1} \cdots \partial k^{i_s}} = 0$$

for all $s, i_1, \ldots, i_s$.

3. We say that a function $A \in B_N$ is of the class $A_N$ if and only if

$$(x_{j_1}, \ldots, x_{j_R}) \frac{\partial}{\partial x_{s_1}} \cdots \frac{\partial}{\partial x_{s_p}} A \in B_N$$

for all $R, P, j_1, \ldots, j_R, s_1, \ldots, s_p$.

4. Let $A_n \in A_N$, $A \in A_N$. We say that $A_N - \lim_{n \to \infty} A_n = A$ if and only if

$$B_N - \lim_{n \to \infty} x_{j_1}, \ldots, x_{j_R} \frac{\partial}{\partial x_{s_1}} \cdots \frac{\partial}{\partial x_{s_p}} (A_n - A) = 0$$

for all $R, P, j_1, \ldots, j_R, s_1, \ldots, s_p$.

Let us investigate some properties of introduced classes $A_N$ and $B_N$.

**Lemma C.1.** 1. $A_{N+R} \subseteq A, B_{N+R} \subseteq B$ for $R \geq 0$.

2. Let $A_{N+R} - \lim_{n \to \infty} A_n = A$ and $R \geq 0$. Then $A_N - \lim_{n \to \infty} A_n = A$.

3. Let $B_{N+R} - \lim_{n \to \infty} A_n = A$ and $R \geq 0$. Then $B_N - \lim_{n \to \infty} A_n = A$.

The proof is obvious: it is sufficient to notice that $\omega_k^{-R}$ is a bounded function.

**Lemma C.2.** 1. Let $A \in B_N$. Then $\frac{\partial A}{\partial x_i} \in B_{N+1}$.

2. Let $A \in A_N$. Then $x_i A \in A_N$, $\frac{\partial A}{\partial x_i} \in A_{N+1}$, $f(x) A \in A_N$ for smooth bounded function $f(x)$.

3. Let $B_N - \lim_{n \to \infty} A_n = A$. Then $B_{N+1} - \lim_{n \to \infty} \frac{\partial A_n}{\partial x_i} = \frac{\partial A}{\partial x_i}$.

4. Let $A_N - \lim_{n \to \infty} A_n = A$. Then $A_N - \lim_{n \to \infty} x_i A_n = x_i A$, $A_N - \lim_{n \to \infty} \frac{\partial A_n}{\partial x_i} = \frac{\partial A}{\partial x_i}$.

The proof is also obvious.

**Lemma C.3.** Let $A_1 \in B_{N_1}, A_2 \in B_{N_2}$. Then $A_1 A_2 \in B_{N_1+N_2}$.

**Proof.** It is sufficient to check that the expression

$$\omega_k^{N_1} \omega_k^{N_2} \frac{\partial^s}{\partial k^{i_1} \cdots \partial k^{i_s}} (A_1 A_2)$$

is bounded. This statement is a corollary of properties $A_1 \in B_{N_1}, A_2 \in B_{N_2}$ and formula

$$\frac{\partial}{\partial k_i} (f g) = \frac{\partial}{\partial k_i} f \cdot g + f \cdot \frac{\partial}{\partial k_i} g.$$ 

Lemma C.3 is proved.
Lemma C.4. The following properties are satisfied: \( k_i \in \mathcal{B}_{-1}, \omega_k^\alpha \in \mathcal{B}_{-\alpha} \).

Proof. Since \(|k_i/\omega_k| < 1\), we obtain the property \( k_i \in \mathcal{B}_{-1} \). For the function \( \omega_k^\alpha \), one has

\[
\frac{\partial}{\partial k_{i_1}} \ldots \frac{\partial}{\partial k_{i_s}} \omega_k^\alpha = \omega_k^\alpha \mathcal{P}(k_i/\omega_k)
\]

(C.7)

where \( \mathcal{P} \) is a polynomial in \( k_i/\omega_k \). Property (C.7) is checked by induction. Therefore, functions (C.6) are bounded for \( N = 1 \). Lemma C.4 is proved.

Lemma C.5. 1. Let \( A \in \mathcal{B}_N \). Then

\[ k_i A \in \mathcal{B}_{N-1}, \quad \omega_k \omega_k^\alpha = \omega_k^\alpha A \in \mathcal{B}_{N+\alpha}, \quad \frac{\partial A}{\partial k_i} \in \mathcal{B}_{N+1}. \]

2. Let \( A \in \mathcal{A}_N \). Then

\[ k_i A \in \mathcal{A}_{N-1}, \quad \omega_k \omega_k^\alpha = \omega_k^\alpha A \in \mathcal{A}_{N+\alpha}, \quad \frac{\partial A}{\partial k_i} \in \mathcal{A}_{N+1}. \]

Proof. Property 1 is a corollary of lemmas C.2 and C.4. Property 1 implies property 2. Lemma is proved.

Lemma C.6. 1. Let \( \mathcal{B}_N - \lim_{n \to \infty} A_n = A \). Then

\[ \mathcal{B}_{N-1} - \lim_{n \to \infty} k_i A_n = k_i A, \quad \mathcal{B}_{N+\alpha} - \lim_{n \to \infty} \omega_k \omega_k^\alpha A_n = \omega_k \omega_k^\alpha A, \quad \mathcal{B}_{N+1} - \lim_{n \to \infty} \frac{\partial A_n}{\partial k_i} = \frac{\partial A}{\partial k_i}. \]

2. Let \( \mathcal{A}_N - \lim_{n \to \infty} A_n = A \). Then

\[ \mathcal{A}_{N-1} - \lim_{n \to \infty} k_i A_n = k_i A, \quad \mathcal{A}_{N+\alpha} - \lim_{n \to \infty} \omega_k \omega_k^\alpha A_n = \omega_k \omega_k^\alpha A, \quad \mathcal{A}_{N+1} - \lim_{n \to \infty} \frac{\partial A_n}{\partial k_i} = \frac{\partial A}{\partial k_i}. \]

The proof is analogous to the proof of lemma C.3.

Lemma C.7. 1. Let \( A_1 \in \mathcal{A}_{N_1}, A_2 \in \mathcal{A}_{N_2} \). Then \( A_1 A_2 \in \mathcal{A}_{N_1+N_2} \).

2. Let \( \mathcal{A}_{N_1} - \lim_{n \to \infty} A_{1,n} = A_1, \quad \mathcal{A}_{N_2} - \lim_{n \to \infty} A_{2,n} = A_2 \). Then \( \mathcal{A}_{N_1+N_2} - \lim_{n \to \infty} A_{1,n} A_{2,n} = A_1 A_2 \).

The proof is analogous to lemma C.3.

C.2.2 Properties of operators and symbols

Lemma C.8. 1. Let \( A \in \mathcal{A}_0 \). Then the operator \( \mathcal{W}(A) \) (C.5) is bounded.

2. Let \( \mathcal{A}_0 - \lim_{n \to \infty} A_n = 0 \). Then \( \lim_{n \to \infty} ||\mathcal{W}(A_n)|| = 0 \).

Proof (cf. [36]). Let us obtain an estimation for the norm ||\( \hat{A} \)||. One has

\[ \hat{A} = \int d\beta e^{i\beta \hat{x}} \int d\alpha e^{i\alpha(k+\beta/2)} \tilde{A}(\alpha, \beta). \]

The estimation ||\( \int d\beta \hat{F}(\beta) || \leq \int d\beta ||\hat{F}(\beta)|| ||\hat{\alpha}|| \) implies

\[ ||\hat{A}|| \leq \int d\beta ||\int d\alpha e^{i\alpha(k+\beta/2)} \tilde{A}(\alpha, \beta)||. \]

However, for operator \( F(k) \) one has ||\( F(k) || = \sup_k ||F(k)|| \), since in the momentum representation \( F(\hat{k}) \) is the operator of multiplication be \( F(k) \). Therefore,

\[ ||\int d\alpha e^{i\alpha(k+\beta/2)} \tilde{A}(\alpha, \beta)|| = \max_k ||\int d\alpha e^{i\alpha(k+\beta/2)} \tilde{A}(\alpha, \beta)|| = \max_k ||\int d\alpha e^{ik\beta} \tilde{A}(\alpha, \beta)|| = \max_k \int \frac{dx}{(2\pi)^d} A(x, k) e^{-i\beta x} = \frac{1}{(\beta^2 + 1)^{\frac{d}{2}}} \max_k \int \frac{dx}{(2\pi)^d} e^{-i\beta x} (x^2 + 1)^N (-\Delta_x + 1)^N A(x,k). \]
Here \( N \) is an arbitrary number such that \( N > d/2 \). Thus,

\[
\|\hat{A}\| \leq \frac{1}{(2\pi)^d} \int \frac{d\beta dx}{(\beta^2 + 1)^N(x^2 + 1)^N} \max_{kx} |(x^2 + 1)^N(-\Delta_x + 1)^N A(x, k)|.
\]

The first statement is justified. Proof of the second statement is analogous. Lemma C.8 is proved.

**Lemma C.9.1.** Let \( A \in \mathcal{A}_N, \ N > d/2 \). Then \( \mathcal{W}(A) \) is a Hilbert-Schmidt operator.

2. Let \( \mathcal{A}_N = \lim_{n \to \infty} A_n = 0, \ N > d/2 \). Then \( \lim_{n \to \infty} \|\mathcal{W}(A_n)\| = 0 \).

**Proof.** Let us use the property [51, 53]

\[
\|\hat{A}\|_2^2 = \int \frac{dx}{(2\pi)^d} |A(x, k)|^2
\]

which can be obtained from definition (C.5). One has

\[
\|\hat{A}\|_2^2 \leq \int \frac{dx}{(2\pi)^d} \frac{dk}{\omega_k^N} \max_{xk} |(x^2 + 1)^N/2 \omega_k^N A(x, k)|^2.
\]

The first statement is justified. Proof of the second statement is analogous. Lemma C.9 is proved.

### C.3 Properties of *-product

Remind that the Weyl symbol of the product of operators

\[
A \ast B = \overline{\mathcal{W}(\mathcal{W}(A)\mathcal{W}(B))}
\]

can be presented as [51, 53]

\[
(A \ast B)(x, k) = \int \frac{d\beta_1 d\beta_2 d\xi_1 d\xi_2}{(2\pi)^{2d}} A(x + \xi_1, k + \frac{\beta_1}{2}) B(x + \xi_2, k - \frac{\beta_2}{2}) e^{-i\beta_1 \xi_1 - i\beta_2 \xi_2} \quad (C.8)
\]

Formula (C.8) can be obtained from definition (C.7).

Let us investigate some properties of formula (C.8). Let us find an expansion of formula (C.8) as \(|k| \to \infty\). Formally, one has

\[
A(x + \xi_1, k + \frac{\beta_1}{2}) = \sum_{n_2 = 0}^{\infty} \frac{1}{2^n n_2!} \frac{\partial^{n_2} A(x + \xi_1, k)}{\partial k_1^{n_2}} \beta_2^{n_2},
\]

\[
B(x + \xi_2, k - \frac{\beta_2}{2}) = \sum_{n_1 = 0}^{\infty} \frac{(-1)^{n_1}}{2^{n_1} n_1!} \frac{\partial^{n_1} B(x + \xi_2, k)}{\partial k_1^{n_1}} \beta_1^{n_1}.
\]

Therefore,

\[
(A \ast B)(x, k) = \sum_{n_1 n_2 = 0}^{\infty} \frac{(-1)^{n_1}}{2^{n_1 + n_2} n_1! n_2!} \int \frac{d\beta_1 d\beta_2 d\xi_1 d\xi_2}{(2\pi)^{2d}} e^{-i\beta_1 \xi_1 - i\beta_2 \xi_2} \frac{\partial^{n_1 + n_2} A(x, k)}{\partial k_1^{n_1} \partial k_2^{n_2}} \beta_1^{n_1} \beta_2^{n_2} \quad (C.9)
\]

Denote

\[
(A \ast^K B)(x, k) = \sum_{n_1 n_2 = 0, n_1 + n_2 \leq K} \frac{\partial^{n_1 + n_2} A(x, k)}{\partial x^{i_1} \partial x^{j_1} \partial x^{i_2} \partial x^{j_1} \partial x^{i_2} \partial x^{j_2} \partial x^{i_2} \partial x^{j_2} \partial x^{i_2} \partial x^{j_2} \partial x^{i_2} \partial x^{j_2}} \frac{\partial^{n_1 + n_2} B(x, k)}{\partial x^{i_1} \partial x^{j_1} \partial x^{i_2} \partial x^{j_1} \partial x^{i_2} \partial x^{j_2} \partial x^{i_2} \partial x^{j_2} \partial x^{i_2} \partial x^{j_2} \partial x^{i_2} \partial x^{j_2}}
\]

This is an asymptotic expansion in \(1/|k|\) as \(|k| \to \infty\). Let us estimate an accuracy of the asymptotic series.

Making use of the relation

\[
A(x + \xi_1, k + \frac{\beta_2}{2}) - A(x + \xi_1, k) = \int_0^1 d(\alpha_2 - 1) \frac{\partial}{\partial \alpha_2} A(x + \xi_1, k + \alpha_2 \frac{\beta_2}{2})
\]

This is...
and integrating by parts $N_2$ times, we find

$$A(x + \xi_1, k + \frac{\partial^2}{2}) = \sum_{n_2=0}^{N_2} \frac{1}{2^{n_2} n_2!} \frac{\partial^{n_2} A(x + \xi_1, k)}{\partial k_1^{n_2} \partial k_{n_2+2}^{n_2+2}} \beta_2^{i_2} \ldots \beta_2^{i_{n_2+2}} + \int_0^1 d\alpha_2 \frac{1}{2^{n_2+1} N_2!} \frac{\partial^{n_2+1} A(x + \xi_1, k + \frac{\partial^2}{2})}{\partial k_1^{n_2+1} \partial k_{n_2+2}^{n_2+2}} \beta_2^{i_2} \ldots \beta_2^{i_{n_2+2+1}}.$$ 

Analogously,

$$B(x + \xi_2, k - \frac{\partial^2}{2}) = \sum_{n_1=0}^{N_1} \frac{(-1)^{n_1}}{2^{n_1} n_1!} \frac{\partial^{n_1} B(x + \xi_2, k)}{\partial k_1^{n_1} \partial k_{n_1}^{n_1}} \beta_1^{i_1} \ldots \beta_1^{i_{n_1+2}} + \int_0^1 d\alpha_1 \frac{1}{2^{n_1+1} N_1!} \frac{\partial^{n_1+1} B(x + \xi_2, k - \frac{\partial^2}{2})}{\partial k_1^{n_1+1} \partial k_{n_1+2}^{n_1+2}} \beta_1^{i_1} \ldots \beta_1^{i_{n_1+2+1}}.$$ 

Therefore,

$$(A \ast B)(x, k) = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \frac{1}{2^{n_1+n_2} n_1! n_2!} \frac{\partial^{n_1+n_2} A(x, k)}{\partial k_1^{n_1} \partial k_2^{n_2} \partial k_{n_1+n_2}^{n_1+n_2}} \frac{\partial^{n_1+n_2} B(x, k)}{\partial k_1^{n_1} \partial k_2^{n_2} \partial k_{n_1+n_2}^{n_1+n_2}} \beta_1^{i_1} \ldots \beta_1^{i_{n_1+1}} \beta_2^{i_2} \ldots \beta_2^{i_{n_2+1}}$$ 

with the following remaining terms,

$$r^{(1)}_{N_1 N_2} = \int_0^1 d\alpha_1 \frac{1}{2^{n_1+n_2} n_1! n_2!} \frac{\partial^{n_1+n_2} A(x, k)}{\partial k_1^{n_1} \partial k_2^{n_2} \partial k_{n_1+n_2}^{n_1+n_2}} \frac{\partial^{n_1+n_2} B(x, k)}{\partial k_1^{n_1} \partial k_2^{n_2} \partial k_{n_1+n_2}^{n_1+n_2}} \beta_1^{i_1} \ldots \beta_1^{i_{n_1+1}} \beta_2^{i_2} \ldots \beta_2^{i_{n_2+1}}$$ 

$$r^{(2)}_{N_1 N_2} = \int_0^1 d\alpha_2 \frac{1}{2^{n_1+n_2} n_1! n_2!} \frac{\partial^{n_1+n_2} A(x, k)}{\partial k_1^{n_1} \partial k_2^{n_2} \partial k_{n_1+n_2}^{n_1+n_2}} \frac{\partial^{n_1+n_2} B(x, k)}{\partial k_1^{n_1} \partial k_2^{n_2} \partial k_{n_1+n_2}^{n_1+n_2}} \beta_1^{i_1} \ldots \beta_1^{i_{n_1+1}} \beta_2^{i_2} \ldots \beta_2^{i_{n_2+1}}$$

Let us investigate the remaining terms.

### C.3.1 The $k$-independent case

**Definition C.3.** We say that the function $f(x), x \in \mathbb{R}^d$ is of the class $C$ if $f$ is a smooth function such that for each set $(i_1, \ldots, i_l)$ there exists $m > 0$ such that the function

$$(x^2 + 1)^{-m} \frac{\partial^l}{\partial x^{i_1} \ldots \partial x^{i_l}} f$$

is bounded.

Let $A = f(x), f \in C$. Then the only nontrivial term is $r^{(2)}_{N_10}$ which is taken by integrating by parts to the form

$$r^{(2)}_{N_10} = \int \frac{d\beta_1 d\beta_2 d\xi_1 d\xi_2}{(2\pi)^d} e^{-i\beta_1 \xi_1 - i\beta_2 \xi_2} \int_0^1 d\alpha_1 \frac{1}{2^{n_1+n_2} n_1! n_2!} \frac{\partial^{n_1+n_2} A(x + \xi_1, k + \frac{\partial^2}{2})}{\partial k_1^{n_1} \partial k_2^{n_2} \partial k_{n_1+n_2}^{n_1+n_2}} \frac{\partial^{n_1+n_2} B(x + \xi_2, k - \frac{\partial^2}{2})}{\partial k_1^{n_1} \partial k_2^{n_2} \partial k_{n_1+n_2}^{n_1+n_2}} \beta_1^{i_1} \ldots \beta_1^{i_{n_1+1}} \beta_2^{i_2} \ldots \beta_2^{i_{n_2+1}}.$$ 

Let us prove some auxiliary statements.

**Lemma C.10.** For some constant $A_1$ the estimation

$$\omega_k \leq A_1 \omega_{p_1} \omega_{k-p}$$

is satisfied.

**Proof.** Let $p = (\frac{1}{2} + \alpha)k + p_\perp, \alpha \in \mathbb{R}, p_\perp \perp k$. Then

$$\frac{\omega_k}{\omega_{p_1} \omega_{k-p}} \leq \frac{\omega_k}{\omega_{(1/2+\alpha)k} \omega_{(1/2-\alpha)k}} \equiv f(\alpha, k),$$
so that it is sufficient to check estimation \((C.11)\) for \(p = \alpha k\) only. For the function \(1/f^2\), one has

\[
\frac{1}{f^2(\alpha, k)} = \begin{cases} 
\frac{1}{k^2 + m^2 \left[ \left( \frac{1}{2} + \alpha \right)^2 k^2 + m^2 \right]} & \text{if } k^2 < 4m^2, \quad \alpha = 0, \quad \frac{k^2}{k^2 + m^2} > \frac{m^2}{k^2}, \\
\frac{1}{k^2 m^2} & \text{if } k^2 > 4m^2, \quad \alpha = \sqrt{\frac{m^2}{k^2} - 1}.
\end{cases}
\]

It has the following minimal value

\[
\min_{\alpha} \frac{1}{f^2(\alpha, k)} = \begin{cases} 
\frac{(k^2/4 + m^2)^2}{k^2 + m^2} & \text{if } k^2 < 4m^2, \\
k^2 m^2 & \text{if } k^2 > 4m^2, \quad \alpha = \sqrt{\frac{m^2}{k^2} - 1}.
\end{cases}
\]  \((C.12)\)

The quantity \((C.12)\) is bounded below. Thus, lemma is proved.

**Corollary.** For \(0 < \gamma < 1\),

\[
\frac{\omega_k}{\omega_{k - \gamma p}} \leq A_1.
\]

**Lemma C.11.** Let \(C \in \mathcal{A}_N\), \(\chi \in \mathcal{C}\), \(\varphi \in \mathcal{C}[0, 1]\). Then for

\[
F(x, k) = \int_0^1 d\alpha \varphi(\alpha) \int \frac{d\beta d\xi}{(2\pi)^d} e^{-i\beta \xi} \chi(x + \xi) C(x, k - \frac{\alpha \beta}{2})
\]  \((C.13)\)

the function \(\omega_k^N F\) is bounded.

**Proof.** Inserting the identity

\[
e^{-i\beta \xi} = (\xi^2 + 1)^{-L_1} (-\frac{\partial^2}{\partial \beta^2} + 1)^{L_1} e^{-i\beta \xi}
\]  \((C.14)\)

and integrating by parts, we obtain that

\[
F(x, k) = \int_0^1 d\alpha \varphi(\alpha) \int \frac{d\beta d\xi}{(2\pi)^d} \frac{1}{(\xi^2 + 1)^{L_1}} e^{-i\beta \xi} \chi(x + \xi) \left(1 - \frac{\alpha^2}{4 k^2}\right)^{L_1} C(x; k - \frac{\alpha \beta}{2}).
\]

For the function \(\omega_k^N F\), one has

\[
\omega_k^N F = \int_0^1 d\alpha \varphi(\alpha) \int \frac{d\beta d\xi}{(2\pi)^d} \frac{1}{(\xi^2 + 1)^{L_1} \omega_{\beta/2}} \chi(x + \xi) \left(-\frac{1}{4} \frac{\partial^2}{\partial \xi^2} + m^2\right)^{\frac{L_2 + N}{2}} e^{-i\beta \xi} \frac{\omega_k^N}{\omega_{\beta/2}} \chi(x; k - \frac{\alpha \beta}{2}).
\]  \((C.15)\)

Choose \(L_2\) to be such a number that \(\frac{L_2 + N}{2}\) is integer, \(L_2 > d\). The property \(\chi \in \mathcal{C}\) implies that there exists such \(K\) that

\[
\frac{\partial^m}{\partial \xi_{i_1} \cdots \partial \xi_{i_m}} \chi(x + \xi) = ((x + \xi)^2 + 1)^K f_{m, i_1 \cdots i_m}(x + \xi), \quad m = 0, \frac{L_2 + N}{2},
\]

where \(f_{m, i_1 \cdots i_m}\) are bounded functions. Choose \(L_1\) to be integer and \(L_1 > \frac{K + d}{2}\). Integrating expression \((C.13)\) by parts, making use of corollary of lemma C.10 and property \(C \in \mathcal{A}_N\), we obtain that \(\omega_k^N F\) is a bounded function. Lemma C.11 is proved.

**Lemma C.12.** Under conditions of lemma C.11 \(F \in \mathcal{A}_N\).

**Proof.** It is sufficient to consider the functions

\[
\omega_k^{N+1} \frac{\partial^l}{\partial k_{i_1} \cdots \partial k_{i_l}} x_{j_1} \cdots x_{j_k} \frac{\partial}{\partial x_{s_1}} \cdots \frac{\partial}{\partial x_{s_p}} F
\]  \((C.16)\)

which are expressed via linear combinations of integrals of the type \((C.13)\). Lemma C.12 is a corollary of lemma C.11.

**Lemma C.13.** Let \(\mathcal{A}_N = \lim_{n \to \infty} C_n = C\), \(\chi \in \mathcal{C}\), \(\varphi \in \mathcal{C}[0,1]\). Then \(\mathcal{A}_N = \lim_{n \to \infty} F_n = F\).
The proof is analogous to lemmas C.11 and C.12.
We obtain therefore the following theorem.

**Theorem C.14.** 1. Let \( f \in \mathcal{C} \), \( B \in \mathcal{A}_N \). Then

\[
f \ast B = f \ast K \ast B + R_K
\]

with \( R_K \in \mathcal{A}_{N+K+1} \).

2. Let \( f \in \mathcal{C} \), \( \mathcal{A}_N - \lim_{n \to \infty} B_n = 0 \). Then \( \mathcal{A}_{N+K+1} - \lim_{n \to \infty} (f \ast B_n - f \ast K \ast B_n) = 0 \).

C.3.2  The \( x \)-independent case

Let \( A = A(k), A \in \mathcal{B}_{M_1}, B \in \mathcal{A}_{M_2} \). The only nontrivial term is taken to the form:

\[
r_{0,N}(x,k) = \int_0^1 d\alpha_2 \left( -\frac{i}{2} \right)^{N_2+1} \frac{(1 - \alpha)^N_2}{N_2!} \int \frac{d\beta_2 d\xi_2}{(2\pi)^d} e^{-i\beta_2 \xi_2} \frac{\partial^{N_2+1} A(k + \alpha \beta_2)}{\partial k^{i_1} \cdots \partial k^{i_{N_2+1}}} \frac{\partial^{N_2+1} B(x + \xi_2; k)}{\partial x^{i_1} \cdots \partial x^{i_{N_2+1}}}. 
\]

**Lemma C.15.** \( C = C(k), C \in \mathcal{B}_{K_1}, K_1 > 0, D \in \mathcal{A}_{K_2}, \varphi \in C[0,1] \). Then for

\[
F(x,k) = \int_0^1 d\alpha \varphi(\alpha) \int \frac{d\beta d\xi}{(2\pi)^d} e^{-i\beta \xi} C(k + \alpha \beta) D(x + \xi, k) \xi_1 \cdots \xi_m
\]

the function \( \omega_K^{K_1+K_2} F \) is bounded.

**Proof.** Inserting the identity \( (\mathcal{C.14}) \) and integrating by parts, we obtain that

\[
F(x,k) = \int_0^1 d\alpha \varphi(\alpha) \int \frac{d\beta d\xi}{(2\pi)^d} \left( -\frac{\alpha}{2} \right)^m \frac{\partial}{\partial k_{j_1}} \cdots \frac{\partial}{\partial k_{j_m}} C(k + \alpha \beta) D(x + \xi, k).
\]

For the function \( \omega_k^{K_1+K_2} F \), one has

\[
\omega_k^{K_1+K_2} F(x,k) = \int_0^1 d\alpha \varphi(\alpha) \int \frac{d\beta d\xi}{(2\pi)^d} \left( \frac{1}{(\alpha + 1)^{L_1}} \right) \omega_k^{K_1} D(x + \xi, k) \left( -\frac{\alpha}{2} \right)^m \frac{\partial}{\partial k_{j_1}} \cdots \frac{\partial}{\partial k_{j_m}} C(k + \alpha \beta) \right)^{L_2+K_1}
\]

Integrating by parts for sufficiently large \( L_1, L_2 \), making use of lemmas C.10, we check proposition of lemma C.15.

**Lemma C.16.** Under conditions of lemma C.15 \( F \in \mathcal{A}_{K_1+K_2} \).

**Lemma C.17.** Let \( \mathcal{A}_{K_2} - \lim_{n \to \infty} D_n = D, C = C(k), C \in \mathcal{B}_{K_1}, K_1 > 0, \varphi \in C[0,1] \). Then

\[
\mathcal{A}_{K_1+K_2} - \lim_{n \to \infty} F_n = F.
\]

The proof is analogous to lemmas C.12 and C.13. We obtain then the following theorem.

**Theorem C.18.** 1. Let \( A = A(k), A \in \mathcal{B}_{M_1}, B \in \mathcal{A}_{M_2} \). Then

\[
A \ast B = A \ast K \ast B + R_K
\]

with \( R_K \in \mathcal{A}_{M_1+M_2+K+1} \), provided that \( K + M_1 + 1 > 0 \).

2. Let \( A = A(k), A \in \mathcal{B}_{M_1}, \mathcal{A}_{M_2} - \lim_{n \to \infty} B_n = B \). Then

\[
\mathcal{A}_{M_1+M_2+K+1} - \lim_{n \to \infty} (A \ast B_n - A \ast K \ast B_n) = 0,
\]

provided that \( K + M_1 + 1 > 0 \).

**Remark.** If the proposition of theorem C.18 is satisfied for \( K = K_0 \), it is satisfied for all \( K \leq K_0 \).

Therefore, the condition \( K + M_1 + 1 > 0 \) can be omitted.

The following lemma is a corollary of theorem C.18.
Lemma C.19.1. Let $A \in \mathcal{A}_N$, $N > d$. Then $\mathcal{W}(A)$ is of the trace class.
2. Let $\mathcal{A}_N - \lim_{n \to \infty} A_n = 0$, $N > d$. Then $\lim_{n \to \infty} \text{Tr} \mathcal{W}(A_n) = 0$.

Proof. Consider the operator

$$\hat{B} = \mathcal{W}(B) = \hat{\omega}^{N/2}(x^2 + 1)^{N/2} \mathcal{W}(A)$$

with

$$B = \omega_k^{N/2} (x^2 + 1)^{N/2} A$$

Since $B \in \mathcal{A}_{N/2}$, $\mathcal{W}(B)$ is a Hilbert-Schmidt operator according to lemma C.9. Therefore, $\mathcal{W}(A)$ is a product of two Hilbert-Schmidt operators $(x^2 + 1)^{-N/2} \hat{\omega}^{-N/2}$ and $\mathcal{W}(B)$. Thus, $\mathcal{W}(A)$ is of the trace class.

One also has:

$$|\text{Tr} \mathcal{W}(A_n)| = |\text{Tr} (x^2 + 1)^{-N/2} \hat{\omega}^{-N/2} \mathcal{W}(B_n)| \leq \|(x^2 + 1)^{-N/2} \hat{\omega}^{-N/2}\|_2 \|\mathcal{W}(B_n)\|_2.$$

Making use of lemma C.9, we prove lemma C.19.

C.3.3 The $\mathcal{A}_N$-case

Let $A \in \mathcal{A}_{M_1}$, $B \in \mathcal{A}_{M_2}$. The $r$-terms can be investigated as follows.

1. We substitute $\beta_{1,2}^r e^{-i \beta_{1,2} \xi_{1,2}} \equiv i \frac{\partial}{\partial \xi_{1,2}} e^{-i \beta_{1,2} \xi_{1,2}}$ and integrate the expressions for $r^{(1)}, r^{(2)}$, $R$ by parts with respect to $\xi_1, \xi_2$.

2. We consider the quantities like

$$\omega_k^{N_1+N_2+M_1+M_2+1+L} \frac{\partial^L}{\partial k^{i_1} \cdots \partial k^{i_L}} x_{j_1} \cdots x_{j_L} \frac{\partial}{\partial x_{s_1}} \cdots \frac{\partial}{\partial x_{s_p}} r$$

for $r = r^{(1)}, r^{(2)}, R$ and show them to be bounded. We use the following statement.

Lemma C.20. Let $F \in \mathcal{A}_{K_1}$, $G \in \mathcal{A}_{K_2}$, $K_1, K_2 > 0$. Then the function

$$\int \frac{d\beta_1 d\beta_2 d\xi_1 d\xi_2}{(2\pi)^{2d}} e^{-i \beta_1 \xi_1 - i \beta_2 \xi_2} \omega_k^{K_1+K_2} F(x + \xi_1, k + \alpha \frac{\beta_2}{2}) G(x + \xi_2, k - \alpha \frac{\beta_1}{2}) \xi_1^{j_1} \cdots \xi_1^{j_m}$$

is uniformly bounded with respect to $\alpha_1, \alpha_2 \in [0, 1]$.

This lemma is proved analogously to lemmas C.11 and C.15.

3. Analogously to previous subsections, we prove the following theorem.

Theorem C.21. 1. Let $A \in \mathcal{A}_{M_1}$, $B \in \mathcal{A}_{M_2}$. Then

$$A \ast B = A^K \ast B + R_K$$

with $R_K \in \mathcal{A}_{M_1+M_2+K+1}$.

2. Let $A_n \in \mathcal{A}_{M_1}$, $B_n \in \mathcal{A}_{M_2}$. Then

$$\mathcal{A}_{M_1+M_2+K+1} - \lim_{n \to \infty} (A_n \ast B_n - A_n^K \ast B_n) = A \ast B - A^K \ast B.$$

C.4 Properties of the exponent

Let us investigate now the properties of the exponent of the operator $\exp \mathcal{W}(A) \equiv \mathcal{W}(\ast \exp A)$. It is convenient to consider the Fourier transformations of Weyl symbols,

$$\tilde{A}(\gamma, k) = \int \frac{dx}{(2\pi)^d} e^{-ix} A(x, k).$$
Introduce the following norms for Weyl symbols,

$$||A||_{I,K} = \max_{J+M+N \leq K} \max_{\gamma,k} |\omega_k^J \frac{\partial^J}{\partial k_1 \cdots \partial k_J} \gamma_{m_1} \cdots \gamma_{m_M} \frac{\partial^N A}{\partial \gamma_{m_1} \cdots \partial \gamma_{m_N}}|.$$  \hfill (C.17)

**Lemma C.22.** \(A \in \mathcal{A}_I\) if and only if \(||A||_{I,K} < \infty\) for all \(k = 0, \infty\).

The proof is obvious.

Let \(C = A \ast B\). Then the Fourier transformation \(\tilde{C}\) can be expressed via \(\tilde{A}\) and \(\tilde{B}\) as follows,

$$\tilde{C}(\gamma, k) = \int d\alpha \tilde{A}(\alpha, k + \frac{\gamma - \alpha}{2}) \tilde{B}(\gamma - \alpha, k - \frac{\alpha}{2}).$$  \hfill (C.18)

The following estimation is satisfied.

**Lemma C.23.** For arbitrary integer numbers \(K, L > d/2\) there exists such a constant \(b_K\) that

$$||A \ast B||_{0,K} \leq b_K ||A||_{0,K+2L} ||B||_{0,K}.$$  \hfill (C.19)

To prove estimation (C.19), one should use definition (C.17) and formula (C.18):

(i) the derivatives \(\partial/\partial \gamma_n\) are applied as

$$\frac{\partial}{\partial \gamma_n}(\tilde{A}(\alpha, k + \frac{\gamma - \alpha}{2}) \tilde{B}(\gamma - \alpha, k - \frac{\alpha}{2})) = \frac{1}{2} \frac{\partial \tilde{A}}{\partial \alpha_n}(\alpha, k + \frac{\gamma - \alpha}{2}) \tilde{B}(\gamma - \alpha, k - \frac{\alpha}{2}) + \tilde{A}(\alpha, k + \frac{\gamma - \alpha}{2}) \frac{\partial}{\partial \alpha_{n}} \tilde{B}(\gamma - \alpha, k - \frac{\alpha}{2});$$

(ii) the derivatives \(\partial/\partial k_j\) are applied analogously;

(iii) the multiplicators \(\gamma_m\) are written as \(\alpha_m + (\gamma_m - \alpha_m)\);

(iv) the estimations

$$\omega_k \leq C \omega_{\alpha/2} \omega_{k-\alpha/2}, \quad \omega_k \leq C \omega_{\alpha/2} \omega_{k+\alpha/2};$$

(lemma C.10) are taken into account.

(v) the integrating measure is written as

$$d\alpha = \frac{d\alpha}{(\alpha^2 + 1)^L}.$$ 

We obtain the estimation (C.19).

Consider the Weyl symbol of the exponent

$$\ast \exp At - 1 = \sum_{n=1}^{\infty} A^{n} t^n n!$$  \hfill (C.20)

with \(A^n = A \ast ... \ast A\).

**Lemma C.24.** Let \(A \in \mathcal{A}_M, M > 0\). Then the estimation (C.20) is convergent in the \(||\cdot||_{0,K}\)-norm. The estimation \(||\ast \exp At - 1||_{0,K} \leq C_K\) is satisfied for \(t \in [0, T]\).

**Proof.** One has

$$||A^n||_{0,K} \leq b_K^{n-1} ||A||_{0,K+2L}^{n-1} ||A||_{0,K} \leq b_K^{n-1} ||A||_{0,K+2L}^n.$$

Therefore,

$$||\ast \exp At - 1||_{0,K} \leq \sum_{n=1}^{\infty} \frac{1}{b_K} \frac{(t ||A||_{0,K+2L} b_K)^n}{n!} \leq \frac{e^{t ||A||_{0,K+2L} b_K} - 1}{b_K} \leq C_K$$

on \(t \in [0, T]\). Lemma C.24 is proved.

**Lemma C.25.** Let \(A \in \mathcal{A}_M, M > 0\). Then

$$\sum_{m=N}^{\infty} \frac{A^{m}}{m!} \in \mathcal{A}_{MN}.$$
\[ \sum_{m=N}^{\infty} \frac{A^{*m}}{m!} = A^{*N} \left( \frac{1}{N!} + \int_0^1 d\tau \frac{(1 - \tau)^{N-1}}{(N-1)!} (\exp A\tau - 1) \right) \]  
(C.21)

Lemma C.24 implies that
\[ \int_0^1 d\tau \frac{(1 - \tau)^{N-1}}{(N-1)!} (\exp A\tau - 1) \in A_0. \]

It follows from theorem C.21 that the symbol (C.21) is of the \( A_{NM} \)-class. Lemma C.25 is proved.

**Lemma C.26.** Let \( A_n \in A_M, M > 0 \) and \( A_M - \lim_{n \to \infty} A_n = A \). Then
\[ \mathcal{A}_{MN} - \lim_{n \to \infty} \sum_{m=N}^{\infty} \frac{A^{*m}}{m!} = \sum_{m=N}^{\infty} \frac{A^{*m}}{m!}. \]

**Proof.** Because relation (C.21) and theorem C.21 it is sufficient to prove that
\[ \mathcal{A}_0 - \lim_{n \to \infty} \int_0^1 dt \frac{(1 - t)^{N-1}}{(N-1)!} (\exp A_n t - \exp A t) = 0. \]  
(C.22)

One has
\[ \exp A_n t - \exp A t = \int_0^t d\tau \exp A(t - \tau) * (A_n - A) * \exp A_n \tau. \]

Making use of lemma C.23, we obtain then estimation (C.22).

### C.5 Estimations for the commutator

Let \( \hat{A} = f(\hat{x}), \hat{B} = g(\hat{k}) \). To investigate the properties of the commutator \( \hat{K} = [\hat{A}; \hat{B}] \), it is convenient to introduce the notion of \( \hat{x}\hat{k}\)-symbol of the operator instead of Weyl symbol. For the \( \hat{x}\hat{k}\)-quantization, the operator \( e^{i\hat{x}\hat{k}} e^{iak} \) corresponds to the function (C.1). Therefore, the function
\[ A(x, k) = \int d\alpha d\beta \hat{A}(\alpha, \beta) e^{i\alpha k + i\beta x} \]
corresponds to the operator
\[ \hat{A} = \int d\alpha d\beta \hat{A}(\alpha, \beta) e^{i\alpha k} e^{iak} \]

For \( \hat{x}\hat{k}\)-quantization, the *-product defined from the relations \( \hat{C} = \hat{A} \hat{B}, C = A * B \) has the form [14, 53]
\[ (A * B)(x, k) = A(x, k - i \frac{\partial}{\partial y}) B(y, k)|_{y=x}. \]

**Lemma C.27.** 1. Let \( A(x, k) = \varphi_1(x) \varphi_2(k) \) with bounded functions \( \varphi_1, \varphi_2 \). Then \( ||\hat{A}|| < \infty \).
2. Let \( A \in L^2(\mathbb{R}^{2d}) \). Then \( \hat{A} \) is a Hilbert-Schmidt operator.

**Proof.** 1. One has \( \hat{A} = \varphi_1(\hat{x}) \varphi_2(\hat{k}), ||\hat{A}|| \leq ||\varphi_1(\hat{x})|| ||\varphi_2(\hat{k})|| = \max |\varphi_1| \max |\varphi_2| < \infty. \)
2. One has
\[ \text{Tr} A^+ A = \frac{1}{(2\pi)^{2d}} \int dxdk |A(x, k)|^2 < \infty. \]

The commutator \( \hat{K} = [f(\hat{x}), g(\hat{k})] \) has the following \( \hat{x}\hat{k}\)-symbol:
\[ K(x, k) = [g(k) - g(k - i \frac{\partial}{\partial x})] f(x) = \sum_{n=0}^{L} \frac{\partial^n g}{\partial k^n} (-i)^n \frac{\partial^n f}{\partial x^n} + \int_0^1 \frac{d\alpha}{L!} (-i)^{L+1} \frac{\partial^{L+1} g(k - i\alpha \frac{\partial}{\partial x})}{\partial x^{L+1}} \frac{\partial^{L+1} f}{\partial x^{L+1}}. \]
Lemma C.28. Let $C(x, k) = A(k - i\alpha \partial / \partial x)B(x)$. Then $||C||_{L^2} = ||A||_{L^2}||B||_{L^2}$.

Proof. Consider the Fourier transformation of the function $A$:

$$A(k) = \int d\gamma \tilde{A}(\gamma) e^{i\gamma k}.$$ 

One has $||A||_{L^2} = (2\pi)^{d/2} ||\tilde{A}||_{L^2}$ and

$$C(x, k) = \int d\gamma \tilde{A}(\gamma) e^{i\gamma k} e^{\gamma^2 \partial / \partial x} B(x).$$

Since $e^{\gamma^2 \partial / \partial x} B(x) = B(x + \gamma \alpha)$, one has

$$||C||_{L^2}^2 = \int d\gamma d\gamma' \tilde{A}(\gamma) \tilde{A}(\gamma') B(x + \gamma \alpha) B(x + \gamma' \alpha) = (2\pi)^d \int d\gamma |\tilde{A}(\gamma)|^2 \int dx |B(x + \gamma \alpha)|^2 = ||A||_{L^2}^2 ||B||_{L^2}^2.$$

Lemma C.28 is proved.

We have obtained the following important statement.

Lemma C.29. Let $\frac{\partial^n f}{\partial x^{j_1} \ldots \partial x^{j_n}}$, $\frac{\partial^m g}{\partial k^{l_1} \ldots \partial k^{l_m}}$ be bounded functions, $m, n = 1, L$, while

$$\frac{\partial L+1 f}{\partial x^{j_1} \ldots \partial x^{j_{L+1}}} \in L^2, \quad \frac{\partial L+1 g}{\partial k^{l_1} \ldots \partial k^{l_{L+1}}} \in L^2.$$

Then $[f(\hat{x}), g(\hat{k})]$ is a bounded operator.

C.6 Asymptotic expansions of Weyl symbol

To check the property of Poincare invariance, it is important to investigate the large-$k$ expansion of the Weyl symbols. Introduce the corresponding definitions.

Definition C.4. 1. We say that a smooth function $A(x, n)$, $x, n \in \mathbb{R}^d$, $|n| < 1$, is of the class $\mathcal{L}$ if the functions

$$\frac{\partial^l}{\partial n_{i_1} \ldots \partial n_{i_l}} x_{j_1} \ldots x_{j_l} \frac{\partial^m}{\partial x_{m_1} \ldots \partial x_{m_m}} A$$

are bounded.

2. Let $A_s \in \mathcal{L}$, $s = \overline{1, \infty}$. We say that $\mathcal{L} = \lim_{s \to \infty} A_s$ if

$$\sup_{|n| \leq 1} \lim_{s \to \infty} \left| \frac{\partial^l}{\partial n_{i_1} \ldots \partial n_{i_l}} x_{j_1} \ldots x_{j_l} \frac{\partial^m}{\partial x_{m_1} \ldots \partial x_{m_m}} A \right| = 0.$$

Definitions C.2 and C.4 imply the following statement.

Lemma C.30. 1. Let $A \in \mathcal{L}$. Then the function $B(x, k) = A(x, k/\omega_k)$ is of the class $\mathcal{B}_0$.

2. Let $\mathcal{L} = \lim_{s \to \infty} A_s = 0$. Then $\mathcal{B}_0 = \lim_{s \to \infty} A_s(x, k/\omega_k) = 0$.

Making use of definition C.2 and lemma C.25, we obtain the following corollary.

Corollary. 1. Let $A \in \mathcal{L}$. Then the function $\omega_k^{-\alpha} A(x, k/\omega_k)$ is of the class $\mathcal{A}_{\alpha}$.

2. Let $\mathcal{L} = \lim_{s \to \infty} A_s = 0$. Then $\mathcal{A}_{\alpha} = \lim_{s \to \infty} \omega_k^{-\alpha} A_s(x, k/\omega_k) = 0$.

Definition C.5. 1. A formal asymptotic expansion is a set $\tilde{A}$ of $\alpha \in \mathbb{R}$ and functions $A_0, A_1, \ldots \in \mathcal{L}$. We say that the formal asymptotic expansions $\tilde{A} = (\alpha, A_0, A_1, \ldots)$ and $\tilde{B} = (\beta, B_0, B_1, \ldots)$ are equivalent if $\alpha - \beta$ is an integer number and $A_{l-\alpha + \beta} = B_l$ for all $l = -\infty, +\infty$ (we assume $A_l = 0$ and $B_l = 0$ for $l < 0$. We denote formal asymptotic expansions of Weyl symbols as

$$\tilde{A} \equiv \sum_{n=0}^{\infty} \omega_k^{-n-\alpha} A_n(x, k/\omega_k).$$

If $A_0 = 0$, $\ldots$, $A_{l-1} = 0$, $A_l \neq 0$, the quantity $\deg \tilde{A} \equiv \alpha + n$ is called as a degree of a formal asymptotic expansion $\tilde{A}$.
2. Let $\hat{A}_s$, $s = 1, \infty$ and $\hat{A}$ be formal asymptotic expansions of Weyl symbols. We say that $F.E - \lim_{s \to \infty} \hat{A}_s = A$ if $\alpha_s = \alpha$ and $\mathcal{L} - \lim_{s \to \infty}(A_{s,n} - A_s) = 0$.

The summation and multiplication by numbers are obviously defined:

$$\hat{A} + \lambda \hat{B} = \sum_{n=0}^{\infty} \omega_k^{-n-\alpha}(A_n(x, k\omega_k) + \lambda B_n(x, k\omega_k)).$$

The product of formal asymptotic expansions of Weyl symbols

$$\hat{A} \equiv \sum_{n=0}^{\infty} \omega_k^{-n-\alpha} A_n(x, k\omega_k), \quad \hat{B} \equiv \sum_{n=0}^{\infty} \omega_k^{-n-\beta} B_n(x, k\omega_k)$$

is defined as

$$\hat{A} \hat{B} \equiv \sum_{n=0}^{\infty} \omega_k^{-n-\alpha-\beta} \sum_{s,l \geq 0, s+l=n} A_s(x, k\omega_k) B_l(x, k\omega_k).$$

Let $f = f(x)$, $f \in \mathcal{C}$. Then

$$f(x) \hat{A} \equiv \sum_{n=0}^{\infty} \omega_k^{-n-\alpha} f(x) A_n(x, k\omega_k).$$

One also defines

$$\omega_k^{-\beta} \hat{A} \equiv \sum_{n=0}^{\infty} \omega_k^{-n-\alpha-\beta} A_n(x, k\omega_k)$$

and

$$\frac{\partial \hat{A}}{\partial k_s} = \sum_{l=0}^{\infty} \omega_k^{-l-\alpha-1} \left[ -(l+\alpha) A_l(x, n) + \frac{\partial A_l}{\partial n_p}(x, n) (\delta_{ps} - n_p n_s) \right] |_{n=k/\omega_k}$$

The *-product of formal asymptotic expansions is introduced as

$$\hat{A} \ast \hat{B} \equiv \sum_{K=0}^{\infty} \sum_{n_1 n_2 \geq 0, n_1 + n_2 = K} \frac{\omega_k^{-n_1-\alpha_1}}{\partial^{n_1+\alpha_1}} \frac{\omega_k^{-n_2-\alpha_2}}{\partial^{n_2+\alpha_2}} \frac{\omega_k^{-l_1-\alpha_1}}{\partial^{l_1+\alpha_1}} \frac{\omega_k^{-l_2-\alpha_2}}{\partial^{l_2+\alpha_2}} \sum_{l_1=0}^{\infty} \omega_k^{-l_1} A_{l_1}(x, k\omega_k) \sum_{l_2=0}^{\infty} \omega_k^{-l_2} A_{l_2}(x, k\omega_k)$$

The formal asymptotic expansions $\hat{A} \ast \omega_k^\alpha$, $\hat{A} \ast f(x)$ are defined analogously. The *-exponent of a formal asymptotic expansion $\hat{A}$ is defined as

$$* \exp \hat{A} - 1 = \sum_{n=1}^{\infty} \frac{\hat{A}^n}{n!}$$

provided that $\text{deg} A$ is a positive integer number.

**Definition C.6.** 1. An asymptotic expansion of the Weyl symbol is a set $A \equiv (A, \hat{A})$ of the Weyl symbol $A$ and a formal asymptotic expansion $\hat{A}$ such that

$$A(x, k) - \sum_{l=0}^{n-1} \frac{A_l(x, k/\omega_k)}{\omega_k^{l+\alpha}} \in A_{n+\alpha}$$

for all $n = 0, \infty$.

2. We say that $E - \lim_{s \to \infty} A_s = A$ if $F.E - \lim_{s \to \infty} \hat{A}_s = \hat{A}$ and

$$A_{n+\alpha} - \lim_{s \to \infty} (A_s(x, k) - \sum_{l=0}^{n-1} \frac{A_{s,l}(x, k/\omega_k)}{\omega_k^{l+\alpha}}) = A(x, k) - \sum_{l=0}^{n-1} \frac{A_l(x, k/\omega_k)}{\omega_k^{l+\alpha}}$$

for all $n = 0, \infty$.

**Remark.** For given Weyl symbol $A$, the asymptotic expansion is not unique. For example, let

$$A(x, k) = m^2 f(x)/\omega_k.$$
One can choose α = 2, A_0(x, n) = m^2 f(x) and find A(x, k) = \omega_k^{-2} A_0(x, k/\omega_k). On the other hand, one can set α = 0, A_0(x, n) = f(x)(1 - n, n_1) and obtain A(x, k) = A_0(x, k/\omega_k) since \omega_k^2 - k_i k = m^2. We see that a degree is a characteristic feature of an expansion rather than of a symbol.

Let \( \mathbf{A} = (A, \dot{A}), \mathbf{B} = (B, \dot{B}). \) Denote \( \mathbf{A} \ast \mathbf{B} \equiv (A \ast B, \dot{A} \ast \dot{B}), \)

\[
\omega_k^\alpha \ast \mathbf{A} \equiv (\omega_k^\alpha \ast A, \omega_k^\alpha \ast \dot{A}),
\]

\[
f(x) \ast \mathbf{A} \equiv (f(x) \ast A, f(x) \ast \dot{A}),
\]

\[
\exp \mathbf{A} - 1 \equiv (\exp A - 1, \exp \dot{A} - 1).
\]

Theorems C.14, C.18, C.21 and lemmas C.25 and C.26 imply the following statement.

**Theorem C.31.** 1. Let \( \mathbf{A} \) be an asymptotic expansion of a Weyl symbol. Then \( \omega_k^\alpha \ast \mathbf{A} \) and \( f(x) \ast \mathbf{A} \) are asymptotic expansions of Weyl symbols under conditions of theorem C.14, while \( \exp \mathbf{A} - 1 \) is an asymptotic expansion of a Weyl symbol, provided that \( \deg \mathbf{A} \) is a positive integer number.

2. Let \( \mathbf{A} \) and \( \mathbf{B} \) be asymptotic expansions of Weyl symbols. Then \( \mathbf{A} \ast \mathbf{B} \) is an asymptotic expansion of a Weyl symbol.

**Theorem C.32.** 1. Let \( E \ast n \rightarrow \infty \mathbf{A}_n = \mathbf{A}. \) Then:

(a) \( E \ast \lim_{n \rightarrow \infty} \omega_k^\alpha \ast \mathbf{A}_n \omega_k^\alpha \ast \mathbf{A}; \)

(b) \( E \ast \lim_{n \rightarrow \infty} f(x) \ast \mathbf{A}_n f(x) \ast \mathbf{A} \) under conditions of theorem C.14;

(c) \( E \ast \lim_{n \rightarrow \infty} (\ast \exp \mathbf{A}_n - 1) = \ast \exp \mathbf{A} - 1 \) if \( \deg \mathbf{A}_n, \deg \dot{A} \) are positive integer numbers.

2. Let \( E \ast \lim_{n \rightarrow \infty} \mathbf{A}_n = \mathbf{A} \) and \( E \ast \lim_{n \rightarrow \infty} \mathbf{B}_n = \mathbf{B}. \) Then \( E \ast \lim_{n \rightarrow \infty} \mathbf{A}_n \ast \mathbf{B}_n = \mathbf{A} \ast \mathbf{B}. \)

The time derivative of the asymptotic expansion \( \mathbf{A}(t) \) with respect to \( t \) is defined in a standard way

\[
E \ast \lim_{\delta t \rightarrow 0} \frac{\mathbf{A}(t + \delta t) - \mathbf{A}(t)}{\delta t} = \frac{d \mathbf{A}(t)}{dt}.
\]

The integral \( \int_0^1 t \mathbf{A}(t) dt \) is also defined in a standard way.

**Theorem C.33.** 1. Let \( \mathbf{A}(t) \) be a continuously differentiable asymptotic expansion of a Weyl symbol. Then

(a) \( \frac{d}{dt}(\omega_k^\alpha \ast \mathbf{A}) = \omega_k^\alpha \ast \frac{d\mathbf{A}}{dt}; \)

(b) \( \frac{d}{dt}(f(x) \ast \mathbf{A}) = f(x) \ast \frac{d\mathbf{A}}{dt} \) under conditions of theorem C.14.

(c) \( \frac{d}{dt}(\ast \exp \mathbf{A} - 1) = \int_0^1 d\tau e^{\mathbf{A}(t-\tau)} \ast \frac{d\mathbf{A}}{dt} \ast e^{\mathbf{A}\tau}; \)

(d) \( \frac{d}{dt}(\mathbf{A} \ast \mathbf{B}) = \frac{d}{dt} \mathbf{A} \ast \mathbf{B} + \mathbf{A} \ast \frac{d}{dt} \mathbf{B}. \)

The only nontrivial statement is (c). It is proved by using the identity \( \exp \mathbf{A}_1 - \ast \exp \mathbf{A}_2 = \int_0^1 d\tau \ast \exp(\mathbf{A}_1(1 - \tau)) \ast (\mathbf{A}_1 - \mathbf{A}_2) \ast \exp(\mathbf{A}_2\tau). \)
References

[1] R. Dashen, B. Hasslacher, A. Neveu, *Phys. Rev.* **D10** (1974), 4114

[2] R. Rajaraman, ”Solitons and Instantons. An Introduction to solitons and instantons in quantum field theory”, North-Holland, Amsterdam, Netherlands, 1982.

[3] J. Coldstone, R. Jackiw, *Phys. Rev.** D11** (1975), 1486

[4] R. Jackiw, *Rev. Mod. Phys.* **49** (1977), 681.

[5] L. D. Faddeev, V. E. Korepin, *Phys. Rep.* **42** (1978) 1.

[6] A. A. Grib, S. G. Mamaev, V. M. Mostepanenko, ”Vacuum Quantum Effects in Strong Fields”, Atomizdat, Moscow, 1988; Friedmann Laboratory Publishing, St. Petersburg 1994.

[7] N. D. Birrell, P. C. W. Davies, ”Quantum Fields in Curved Space”, Cambridge, UK: Univ. Pr., 1982.

[8] D. Boyanovsky, H. J. de Vega and R. Holman, *Phys. Rev.* **D49** (1994), 2769.

[9] D. Boyanovsky, H. J. de Vega, R. Holman, D. S. Lee and A. Singh, *Phys. Rev.* **D51** (1995), 4419.

[10] Ju. Baacke, K. Heitmann and C. Pätzold, *Phys. Rev.* **D55** (1997), 2320.

[11] Ju. Baacke, K. Heitmann and C. Pätzold, *Phys. Rev.* **D56** (1997), 6556.

[12] F. Cooper, E. Mottola, *Phys. Rev.* **D36** (1987), 3114

[13] S.-Y. Pi, M. Samiullah, *Phys. Rev.* **D36** (1987), 3128

[14] R. Jackiw and A. Kerman, *Phys. Lett.* **A71** (1979), 158.

[15] F. Cooper, S.-Y. Pi and P. Stancioff, *Phys. Rev.* **D34** (1986), 3831.

[16] O. Eboli, R. Jackiw and S.-Y. Pi, *Phys. Rev.* **D37** (1988), 3557.

[17] O. Eboli, S.-Y. Pi, M. Samiullah, *Ann. Phys.* **193** (1989), 102.

[18] A. S. Wightman, *Phys. Rev.* **101** (1956), 860.

[19] R. F. Streater, A. S. Wightman ”PCT, spin and statistics and all that”, N. Y., Benjamin, 1964.

[20] N. N. Bogoliubov, A. A. Logunov, A. I. Oksak, I. T. Todorov, ”General principles of Quantum Field Theory”, Moscow, Nauka, 1987; Kluwer Academic Publishers, 1990.

[21] K. Hepp, ”Theorie de la renormalisation”, Springer-Verlag, 1969.

[22] J. Glimm and A. Jaffe, ”Boson quantum field models”. In ”London 1971, Mathematics Of Contemporary Physics”, London 1972, pp. 77-143.

[23] I. Ya. Arefieva, *Teor. Mat. Fiz.* **14** (1973) 3.

[24] I. Ya. Arefieva, *Teor. Mat. Fiz.* **15** (1973) 207.

[25] N. N. Bogoliubov, D. V. Shirkov, ”Introduction to the Theory of Quantized Fields”, N.-Y., Interscience Publishers, 1959.
[26] A.A.Slavnov, L.D.Faddeev, "Introduction to the quantum theory of gauge fields", Moscow, Nauka, 1988.

[27] E.C.G. Stueckelberg Phys.Rev. 81 (1951) 130.

[28] R.Haag, Kgl. Danske Videnskab. Selsk. Mat.-Fiz. Medd. 29 (1955) N12.

[29] L.D.Faddeev, Doklady Akademii Nauk SSSR 152 (1963) 573.

[30] V.P.Maslov, O.Yu.Shvedov, Trudy MIAN 226 (1999) 112.

[31] O.Yu.Shvedov, Teor. Mat. Fiz. 125 (2000) 91.

[32] O.Yu.Shvedov, hep-th/0002108, Ann. Phys. 287 (2001) 260.

[33] V.P.Maslov, O.Yu.Shvedov Teor. Mat.Fiz. 114 (1988) 233.

[34] V.P.Maslov, ”Operational Methods”, Moscow, Nauka, 1973; English translation: Moscow, Mir Publishers, 1976.

[35] V.P.Maslov, ”The Complex-WKB Method for Nonlinear Equations”, Moscow, Nauka, 1977.

[36] V.P.Maslov, O.Yu.Shvedov, ”The Complex Germ Method in Many-Particle Problem and Quantum Field Theory”, Moscow, Editorial URSS, 2000.

[37] A.A.Grib, S.G.Mamaev, Yadernaya Fizika, 10 (1969), 1276.

[38] M.I.Shirokov, Yadernaya Fizika, 7 (1968) 672.

[39] O.Yu.Shvedov, Matematicheskie zametki 65 (1999) 437

[40] O.Yu.Shvedov, Matematisheski ii sborik 190 (1999) N10, 123.

[41] F.A.Berezin, ”The Method of Second Quantization”, Moscow, Nauka, 1965; N.Y.1996.

[42] M.M.Postnikov, ”Differential geometry”, Moscow, Nauka, 1988.

[43] K.Wilson, Phys. Rev. D7 (1974) 2911

[44] J.Collins, ”Renormalization. An introduction to renormalization, the renormalization group and the operator-product expansion”. Cambridge, Cambridge University Press, 1984.

[45] M.M.Popov, Zapiski Nauchnogo Seminara LOMI,104 (1981) 195.

[46] M.V.Karasev, Zapiski Nauchnogo Seminara LOMI,172 (1989) 41.

[47] M.V.Karasev, Yu.M.Vorobiev, preprint ITP-90-85E, Kiev, 1990.

[48] V.P.Maslov, O.Yu.Shvedov Teor. Mat.Fiz. 104 (1995) 479.

[49] M.Henneaux Phys.Rep. 126 (1985) 1

[50] V.P.Maslov, O.Yu.Shvedov, Russian J. Math Phys. 4 (1996) 173.

[51] L.V.Kantorovich, G.P.Akilov, ”Functional Analysis”, Moscow, Nauka, 1984.

[52] A.N.Kolmogorov, S.V.Fomin, ”Elements of Functions Theory and Functional Analysis”, Moscow, Nauka, 1989.

[53] M.V.Karasev, V.P.Maslov, ”Nonlinear Poisson Bracket. Geometry and Quantization”, Moscow, Nauka, 1991.