A BAER-KAPLANSKY THEOREM FOR MODULES OVER PRINCIPAL IDEAL DOMAINS

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Abstract. We will prove that if $G$ and $H$ are modules over a principal ideal domain $R$ such that the endomorphism rings $\text{End}_R(R \oplus G)$ and $\text{End}_R(R \oplus H)$ are isomorphic then $G \cong H$. Conversely, if $R$ is a Dedekind domain such that two $R$-modules $G$ and $H$ are isomorphic whenever the rings $\text{End}_R(R \oplus G)$ and $\text{End}_R(R \oplus H)$ are isomorphic then $R$ is a PID.

1. Introduction

The Baer-Kaplansky Theorem, [6, Theorem 108.1], states that two primary abelian groups with isomorphic endomorphism rings are necessarily isomorphic. This statement was extended to various classes of modules (abelian groups), e.g. in [8], [14], [16], [20], [21]. However straightforward examples show that in order to obtain such extensions we need to impose restrictions on these classes. For instance the endomorphism rings of the Prüfer group $\mathbb{Z}(p^\infty)$ and of the group of $p$-adic integers $\hat{\mathbb{Z}}_p$ are both isomorphic to the ring $J_p$ of $p$-adic integers. This fact suggests that we need to restrict to some good classes of modules in order to obtain a Baer-Kaplansky type theorem. Such a result (valid for torsion-free modules over valuation domains) was proved in [21]. It is well known that Baer-Kaplansky Theorem cannot be extended to torsion-free groups (of rank 1) since there are infinitely many pairwise non-isomorphic torsion-free groups of rank 1 whose endomorphism rings are isomorphic to $\mathbb{Z}$, [1]. However, similar results to Baer-Kaplansky Theorem hold for some special classes of torsion-free groups, see [2]. In the setting of modules over complete valuation domains W. May proved a theorem, [15, Theorem 1], for reduced modules which are neither torsion nor torsion-free and have a nice subgroup $B$ such that $M/B$ is totally projective: if $M$ is such a module and $N$ is an arbitrary module such that they have isomorphic endomorphism rings then $M \cong N$.

The main aim of this note is to prove a Baer-Kaplansky theorem for arbitrarily modules over principal ideal domains (Theorem 2): if $R$ is (commutative) principal ideal domain then the correspondence (from the class of $R$-modules to the class of rings)

$$\Phi : G \mapsto \text{End}_R(R \oplus G)$$

reflects isomorphisms of endomorphism rings. Moreover, this property characterizes principal ideal domains in the class of Dedekind domains: if $R$ is a Dedekind domain such that the correspondence $\Phi$ reflects isomorphisms then $R$ is a PID. The restriction to Dedekind domains is motivated by the fact that these domains have

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the cancellation property, i.e. the endofunctor $R \oplus - : \text{Mod-}R \rightarrow \text{Mod-}R$ on the category of all $R$-modules reflects isomorphisms:

**Theorem 1.** [212 Proposition 3.6] Let $R$ be a Dedekind domain. If $M$ and $N$ are two $R$-modules such that $R \oplus M \cong R \oplus N$ then $M \cong N$.

We need this property in order to obtain that $\Phi$ reflects isomorphisms (cf. Remark [3]). However, in order to obtain such a correspondence which reflects isomorphisms the cancellation property is not enough, as it is proved in Proposition [7] (in contrast with the similar problem for subgroup lattices, approached in [3] Lemma 2).

2. A Baer-Kaplansky theorem

The main result proved in this note is the following

**Theorem 2.** Let $R$ be a Dedekind domain. The following are equivalent:

1. The ring $R$ is a principal ideal domain;
2. If $G$ and $H$ are $R$-modules such that $G' = R \oplus G$ and $H' = R \oplus H$ have isomorphic endomorphism rings then $G$ and $H$ are isomorphic.

**Proof.** (1)$\Rightarrow$(2) Let $e$ and $f$ be the idempotents in $\text{End}_R(G')$ which are induced by the direct decomposition $G' = R \oplus G$. Using the version for principal ideal domains of [3, Theorem 106.1], we observe that there are isomorphisms

$$e \text{ End}_R(G')f \cong \text{Hom}_R(G, R)$$

and

$$f \text{ End}_R(G')e \cong \text{Hom}_R(R, G) \cong G.$$  

If $\varphi : \text{End}_R(G') \rightarrow \text{End}_R(H')$ is an isomorphism then the idempotents $\overline{e} = \varphi(e)$ and $\overline{f} = \varphi(f)$ induce a direct decomposition $H' = B \oplus K$, where $B = \overline{e}(H')$ and $K = \overline{f}(H')$. By [6, 106(d)] there is an isomorphism $\text{End}_R(B) \cong R$. Moreover, as before, we have the isomorphisms (of $R$-modules)

$$\text{Hom}_R(K, B) \cong \overline{e} \text{ End}_R(H')\overline{f} \cong \text{Hom}_R(G, R),$$

and

$$\text{Hom}_R(B, K) \cong \overline{f} \text{ End}_R(H')\overline{e} \cong \text{Hom}_R(R, G) \cong G.$$  

We claim that $B \cong R$. Using this and Theorem 1 we obtain $H \cong K$, and we have

$$H \cong \text{Hom}_R(R, K) \cong \text{Hom}_R(B, K) \cong \text{Hom}_R(R, G) \cong G.$$  

In order to prove our claim, suppose that $B \ncong R$. Let $\alpha : B \rightarrow R$ be an $R$-homomorphism. Since $R$ is a PID it follows that $\text{Im}(\alpha) \cong R$, hence $\text{Ker}(\alpha) \neq 0$. Moreover, $\text{Im}(\alpha)$ is a projective module, hence we have a direct decomposition $B \cong \text{Ker}(\alpha) \oplus \text{Im}(\alpha)$. But $\text{End}(B) \cong R$ has no non-trivial idempotents, hence $B$ is indecomposable. It follows that $\text{Im}(\alpha) = 0$, hence $\text{Hom}_R(B, R) = 0$.

If we consider the direct decomposition $H' = R \oplus H$ and the canonical projection $\pi_R : H' \rightarrow R$, it follows that $B$ is contained in $H$, the kernel of $\pi_R$. From $H' = B \oplus K$ we obtain $H = (H \cap K) \oplus B$. Using the equalities

$$K \oplus B = R \oplus H = R \oplus (H \cap K) \oplus B$$

we deduce that $K \cong R \oplus (H \cap K)$ (as complements for the direct summand $B$), hence $K$ has a direct summand isomorphic to $R$. Therefore $\text{Hom}_R(G, R) \cong \text{Hom}_R(K, B)$.
has a direct summand isomorphic to $B$. Since $R$ is commutative, $\text{Hom}_R(G, R)$ is an $R$-module which can be embedded as a submodule in the direct product $R^G$ of copies of $R$ (here we view $R^G$ as the set of all maps $G \to R$, endowed with pointwise addition and scalar multiplication; see [5 Exercise 43.1]). Therefore we can embed $B$ in $R^G$. Since $B \neq 0$ it follows that we can find a projection $\pi : R^G \to R$ such that $\pi(B) \neq 0$. This implies $\text{Hom}_R(B, R) \neq 0$, a contradiction, and it follows that $B \cong R$.

(2)⇒(1) Let $I$ be a non-zero ideal in $R$. Since $R$ is noetherian and integrally closed we can apply [7] Theorem 1.3.7 to conclude that $\text{End}_R(I) \cong R$. Moreover, since $I$ is invertible, we can use Steinitz isomorphism formula, [7] p.165. Therefore, for every positive integer $n$ we have an isomorphism $(\oplus_{k=1}^{n-1} R) \oplus I^n \cong \oplus_{k=1}^{n-1} I$, hence there are ring isomorphisms

$$\text{End}_R((\oplus_{k=1}^{n-1} R) \oplus I^n) \cong \text{End}_R(\oplus_{k=1}^{n-1} I) \cong M_n(R) \cong \text{End}_R(\oplus_{k=1}^{n-1} R).$$

If $n \geq 2$ we obtain, from (2), that $(\oplus_{k=1}^{n-2} R) \oplus I^n \cong \oplus_{k=1}^{n-1} R$. Using again the cancellation property of $R$, Theorem 1 we conclude that $I^n$ is principal for all $n \geq 2$. If $C(R)$ is the ideal class group associated to $R$ and $[I]$ is the class of $I$ in this group, it follows that $[I]^n = 1$ for all $n \geq 2$, hence $[I] = 1$. Then $I$ is principal and the proof is complete. □

**Remark 3.** From the above proof it follows that if $R$ is a principal ideal domain then every ring isomorphism $\varphi : \text{End}_R(R \oplus G) \to \text{End}_R(R \oplus H)$ induces a direct decomposition $R \oplus H = B \oplus K$ with $B = \varphi(e)(R \oplus H) \cong R$ and $(1 - \varphi(e))(R \oplus H) = K \cong G$, where $e$ is the idempotent such that $e(R \oplus G) = R$ and $(1 - e)(R \oplus G) = G$. Since $B \cong R$, it is not hard to see, using the same technique as in the proof for the bounded case of [9] Theorem 108.1, that $\varphi$ is induced by an isomorphism $\psi : R \oplus G \to R \oplus H$. Therefore the above theorem can be viewed as an improvement of [19] Theorem 2.1 for the case of principal ideal domains.

**Remark 4.** A class $C$ of modules is called Baer-Kaplansky if any two of its modules are isomorphic whenever their endomorphism rings are isomorphic as rings, [9] p. 1489. Therefore, Theorem 2 says that the class of modules over a Dedekind domain $R$ which have a direct summand isomorphic to $R$ is a Baer-Kaplansky class if and only if $R$ is a principal ideal domain. Similar results for other kind of rings were obtained in [8] Theorem 8 for a similar class, respectively in [9] Theorem 4 for a particular class of modules over FGC-rings.

As a consequence of Theorem 2 we obtain that locally free modules over principal ideal domains are determined by their endomorphism rings. This is also a consequence of [21] Theorem A].

**Corollary 5.** If two locally free modules over a principal ideal domain have isomorphic endomorphism rings then they are isomorphic.

**Remark 6.** In the proof of Theorem 2 we used the cancellation property of the regular module $R$. If $R$ has not this property (e.g. there are Dedekind-like domains without cancellation property, [11]) then there are two $R$-modules $G \cong H$ such that $R \oplus G \cong R \oplus H$, hence $\text{End}_R(R \oplus G) \cong \text{End}_R(R \oplus H)$. If we write these endomorphism rings as matrix rings

$$\text{End}_R(R \oplus G) = \left( \begin{array}{cc} \text{End}_R(R) & \text{Hom}_R(G, R) \\ \text{Hom}_R(R, G) & \text{End}_R(G) \end{array} \right) \cong \left( \begin{array}{cc} R & \text{Hom}_R(G, R) \\ G & \text{End}_R(G) \end{array} \right),$$

we can apply [7, Theorem I.3.7] to conclude that $\text{End}_R(R \oplus G) \cong \text{End}_R(R \oplus H)$.
respectively

\[ \text{End}_R(R \oplus H) \cong \begin{pmatrix} R & \text{Hom}_R(H, R) \\ H & \text{End}_R(H) \end{pmatrix}, \]

we observe that the (2,1)-blocks in these representations are isomorphic to \( G \), respectively to \( H \). These two blocks are not isomorphic even the corresponding matrix rings are isomorphic. It is obvious that in this case Theorem 2 is not valid.

We will prove that we cannot replace in the implication (1) ⇒ (2) of Theorem 2 the direct summand \( R \) by an arbitrary module which have the cancellation property. The following proposition shows that the property of the regular module \( R \) stated in Theorem 2 is more stronger than the usual cancellation property (see [18, Theorem B]).

**Proposition 7.** The following are equivalent for an indecomposable torsion-free abelian group \( F \neq 0 \) of finite rank:

1. If \( G \) and \( H \) are abelian groups such that \( \text{End}(F \oplus G) \cong \text{End}(F \oplus H) \) then \( G \cong H \);
2. \( F \cong \mathbb{Z} \).

**Proof.** (1)⇒(2) If \( F \) is not isomorphic to \( \mathbb{Z} \) then \( F \cong \mathbb{Q} \) or \( F \) is a reduced abelian group which has no free direct summands.

For the case \( F \cong \mathbb{Q} \), we can choose \( G \) and \( H \) two non-isomorphic subgroups of \( \mathbb{Q} \) such that \( \text{End}(G) = \text{End}(H) = \mathbb{Z} \). It is not hard to see that both endomorphism rings \( \text{End}(F \oplus G) \) and \( \text{End}(F \oplus H) \) are isomorphic to the matrix ring

\[ \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{pmatrix}, \]

so \( F \) does not verify the condition (1).

If \( F \) is a reduced abelian group which has no free direct summands then we can construct, using [18, Theorem], two (reduced) finite rank torsion-free groups \( G \) and \( H \) of the same rank such that

\[ \text{Hom}(F, G) = \text{Hom}(F, H) = \text{Hom}(G, F) = \text{Hom}(H, F) = 0 \]

and

\[ \text{End}(G) = \text{End}(H) = \mathbb{Z}. \]

In this case both endomorphism rings \( \text{End}(F \oplus G) \) and \( \text{End}(F \oplus H) \) are isomorphic to the ring \( \text{End}(F) \times \mathbb{Z} \), so \( F \) does not verify the condition (1).

(2)⇒(1) This is a consequence of Theorem 2. \( \square \)

**Remark 8.** There are also versions for the Baer-Kaplansky theorem proved for automorphism groups, Jacobson radicals or for ring anti-isomorphisms, [4], [10], [13], [17]. It would be nice to know if Theorem 2 is still true if we consider only automorphism groups or Jacobson radicals.

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A BAER-KAPLANSKY THEOREM

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