SCHUR-WEYL DUALITY FOR QUANTUM TOROIDAL SUPERALGEBRAS

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Abstract. We establish the Schur-Weyl type duality between double affine Hecke algebras and quantum toroidal superalgebras, generalizing the well known result of Vasserot-Varagnolo [VV96] to the super case.

Keywords: Schur-Weyl duality, double affine Hecke algebra, quantum toroidal superalgebra

1. Introduction

In the last 30 years, quantum toroidal algebras [GKV95] and double affine Hecke algebras (DAHA for short) [Che92] are central objects in the area of representation theory. They have rich representation theory and also many important applications in algebra, combinatorics, geometry, and mathematical physics. In [VV96], it is shown that these two remarkable algebras are related via Schur-Weyl duality.

Recently, quantum toroidal superalgebras associated to $\mathfrak{sl}_{m|n}$ for arbitrary root systems were introduced in [BM21a]. A related geometric construction of the Drinfeld half of quantum toroidal superalgebras using the deformed K-theoretic Hall algebra of a quiver with potential is given in [VV22]. The present paper is devoted to establishing the Schur-Weyl duality between double affine Hecke algebras and quantum toroidal superalgebras, generalizing the well known result of Vasserot-Varagnolo [VV96] to the super case. We expect that this duality could be an important tool to study representations of quantum toroidal superalgebras, cf. e.g. [BL22], and to obtain results for super case from the (certain) known results in the even case, see e.g. [LM21, Section 4].

Schur-Weyl duality, being one of the most important and beautiful classical results in representation theory, is the equivalence between the category of modules over the symmetric group $S_\ell$ and the category of modules of level $\ell$ over the Lie algebra $\mathfrak{sl}_n$ for $\ell < n$. Since the introduction of quantum groups in the 1980s, it is interesting and important to generalize Schur-Weyl duality in the quantum setting. In fact, similar equivalences or related results have been established between finite Hecke algebras and quantum enveloping (super)algebras [Jim86, Moo03, Mit06], between degenerate affine Hecke algebras and (super) Yangians [Dri86, Ara99, LM21, Lu21], between affine Hecke algebras and quantum affine (super)algebras [Che87, GRV94, CP96, Fli20, KL22], between double affine Hecke algebras (also called elliptic Cherednik algebras) and quantum toroidal algebras [VV96], between trigonometric Cherednik algebras and affine Yangians [Gua05, Gua07], and between rational Cherednik algebras and deformed double current algebras [Gua05, Gua07]. These relations can be summarized by combining the table below and the table in [Rou05, Introduction].

| quantum (super)algebras | quantum affine (super)algebras | quantum toroidal (super)algebras |
|-------------------------|--------------------------------|---------------------------------|
| (super) Yangians         | affine (super) Yangians        | deformed double current (super)algebras |

More specifically, the (super)algebras in this table are the dual (superalgebras) for the corresponding algebras in the table of [Rou05, Introduction].

1This is borrowed from N. Guay’s talk in Representations and Lie Theory Seminar at Ohio State University.
It is also interesting to generalize the last two cases to the super setting. Note that the affine super Yangians (of type A associated to the standard root system) have been introduced in [Ued19] while deformed double current superalgebras are not discussed in the literature yet.

We almost follow the arguments used in [VV96] except [VV96, Theorem 3.3] which was deduced by using the braid group action on algebras and integrable modules. A similar description of [VV96, Theorem 3.3] using affine Hecke algebra does not seem to work in the super case. In order to generalize [VV96, Theorem 3.3] to the super case, a modification of the action of affine Hecke algebra on polynomial tensor representation is probably needed, see [GRV94, Section 4]. We obtain similar results by investigating the coproduct of the quantum affine superalgebra, see Propositions 2.7, 2.9.

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2. Preliminaries

We fix \( m, n \in \mathbb{Z}_{>0} \) such that \( m \neq n \) and set \( \kappa = m + n \). Set \( I = \{1, 2, \ldots, \kappa - 1\} \) and \( \hat{I} = \{0, 1, \ldots, \kappa - 1\} \). Fix \( q \in 
\mathbb{C}^* \) to be not a root of unity.

2.1. Double affine Hecke algebras. Let \( \zeta \in \mathbb{C}^* \) and \( \ell \in \mathbb{Z}_{>0} \).

**Definition 2.1 ([Che92]).** The double affine Hecke algebra (or elliptic Cherednik algebra) of type \( \mathfrak{gl}_\ell \), denoted by \( \mathcal{H}_\ell \), is the unital associative algebra with the generators \( T_i^{\pm 1}, X_j^{\pm 1}, Y_j^{\pm 1} \), \( 1 \leq i < \ell \), \( 1 \leq j \leq \ell \), and the relations:

\[
T_i T_j^{-1} = T_j^{-1} T_i = 1, \quad (T_i + 1)(T_i - q^2) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},
\]

\[
X_0 Y_1 = \zeta Y_1 X_0, \quad X_i X_j = X_j X_i, \quad Y_i Y_j = Y_j Y_i, \quad X_i X_j^{-1} = X_j^{-1} X_i = Y_i Y_j^{-1} = Y_j^{-1} Y_i = 1,
\]

\[
T_i X_i T_i = q X_i T_i, \quad T_i^{-1} Y_i T_i^{-1} = q^{-2} Y_i, \quad X_2 Y_1^{-1} X_2^{-1} Y_1 = q^{-2} T_2, \quad T_i T_j = T_j T_i \text{ if } |i - j| > 1, \quad X_j T_i = T_i X_j, \quad Y_j T_i = T_i Y_j \text{ if } j \neq i, i + 1,
\]

where \( X_0 = X_1 X_2 \cdots X_\ell \).

Here we set \( y = 1 \) in [VV96, Definition 1.1].

Let \( \mathcal{G}_\ell \) be the symmetric group permuting the set \( \{1, 2, \ldots, \ell\} \). Given an element \( w \in \mathcal{G}_\ell \), let \( T_w \in \mathcal{H}_\ell \) be the element defined in terms of a reduced expression of \( w \).

For a sequence of \( \ell \) integers \( r = (r_1, \ldots, r_\ell) \), set \( X^r := X_1^{r_1} \cdots X_\ell^{r_\ell} \) and \( Y^r := Y_1^{r_1} \cdots Y_\ell^{r_\ell} \). It is known that elements \( X^s Y^r T_w \) for all possible \( \ell \)-tuples \( s, r \) and \( w \in \mathcal{G}_\ell \) form a basis of \( \mathcal{H}_\ell \), see e.g. [Che92, Theorem 2.6 (a)].

Let \( \mathcal{H}_\ell^{(1)} \) and \( \mathcal{H}_\ell^{(2)} \) be the subalgebras of \( \mathcal{H}_\ell \) generated by \( T_i^{\pm 1}, Y_j^{\pm 1} \) and \( T_i^{\pm 1}, X_j^{\pm 1} \), \( 1 \leq i < \ell \), \( 1 \leq j \leq \ell \), respectively. Then \( \mathcal{H}_\ell^{(1)} \) and \( \mathcal{H}_\ell^{(2)} \) are isomorphic to the affine Hecke algebra of type \( \mathfrak{gl}_\ell \), which we denote it by \( \mathcal{H}_\ell \). Similarly, the subalgebra generated by \( T_i^{\pm 1}, 1 \leq i < \ell \), is isomorphic to the Hecke algebra of type \( \mathfrak{gl}_\ell \) and we denote it by \( \mathcal{H}_\ell \).

For \( 1 \leq i \leq j < \ell \), we use the convenient notation,

\[
T_{i,j} := T_i T_{i+1} \cdots T_j, \quad T_{j,i} := T_j T_{j-1} \cdots T_i.
\]  

The double affine Hecke algebra \( \mathcal{H}_\ell \) admits another well-known presentation as follows.
Proposition 2.2. The double affine Hecke algebra $\mathfrak{H}_\ell$ is the unital associative algebra with the generators $Q^\pm, T_{i_{i+1}}^T, Y_j^\pm, 1 \leq i < \ell, 1 \leq j \leq \ell$, and the relations:

\[
T_iT_i^{-1} = T_i^{-1}T_i = 1, \quad (T_i + 1)(T_i - q^2) = 0, \quad T_iT_{i+1} = T_{i+1}T_iT_{i+1},
\]

\[
QQ^{-1} = Q^{-1}Q = 1, \quad Y_iY_j = Y_jY_i, \quad T_i^{-1}Y_iT_i^{-1} = q^{-2}Y_{i+1}, \quad Y_iY_i^{-1} = Y_i^{-1}Y_i = 1,
\]

\[
T_jT_j = T_jT_j \quad \text{if } |i-j| > 1, \quad Y_jT_j = T_jY_j \quad \text{if } j \neq i, i+1,
\]

\[
QT_{i-1}Q^{-1} = T_i (1 < i < \ell - 1), \quad Q^2T_{i-1}Q^{-2} = T_i,
\]

\[
QY_iQ^{-1} = Y_{i+1} (1 \leq i \leq \ell - 1), \quad QY_iQ^{-1} = \zeta Y_i.
\]

Here $Q$ is identified with $X_1T_1T_{-1}$ in Definition 2.1.

For $1 \leq i < j < \ell$ and $r < \ell$, set

\[
Q_{i,j} := X_iT_{i,j} \in \mathfrak{H}_\ell, \quad P_r := Q_{\ell-r,\ell-1} \cdots Q_{2,1}Q_{1,r} \in \mathfrak{H}_\ell.
\] (2.2)

We shall need the following lemmas later.

Lemma 2.3 ([VV96]). If $i \leq a \leq j$ and $i < b < j$, then

\[
Q_{i,j}Y_aQ_{i,j}^{-1} = Y_{a+1}, \quad Q_{i,j}T_{b-1}Q_{i,j}^{-1} = T_b.
\]

Lemma 2.4 ([VV96]). If $r < a + 1$ and $r < b < \ell$, then

\[
P_rY_aP_r^{-1} = \zeta Y_{a-r+1}, \quad P_rP_r^{-1} = T_{b-r}.
\]

2.2. Quantum affine superalgebras. Let $s = (s_1, \ldots, s_n)$ where $s_i \in \{\pm 1\}$ and the occurrence of 1 is exactly $m$. We call such a sequence $s$ a parity sequence. Denote the set of all parity sequences by $S_{m|n}$. We call the parity $s = (1, \ldots, 1, -1, \ldots, -1)$ the standard parity sequence. For an $s \in S_{m|n}$, we extend it to $s = (s_i)_{i \in \mathbb{Z}}$ by enforcing periodicity, $s_{i+k} = s_i$ for all $i \in \mathbb{Z}$.

Given a parity sequence $s \in S_{m|n}$, we have the Cartan matrix $A^s = (a^s_{i,j})_{i,j \in I}$ and the affine Cartan matrix $\hat{A}^s = (a^s_{i,j})_{i,j \in I}$ given by

\[
a^s_{i,j} = (s_i + s_{i+1})\delta_{i,j} - s_i\delta_{i,j+1} - s_j\delta_{i+1,j}, \quad i, j \in \hat{I}.
\] (2.3)

Denote $\mathfrak{sl}_s$ and $\widehat{\mathfrak{sl}}_s$ be the Lie superalgebras corresponding to Cartan matrices $A^s$ and $\hat{A}^s$, respectively. Note that the Lie superalgebras $\mathfrak{sl}_s$ (resp. $\widehat{\mathfrak{sl}}_s$) are all isomorphic for all $s \in S_{m|n}$.

Let $P_s$ be the integral lattice spanned by the basis $e_i, 1 \leq i \leq \kappa$, with a bilinear form on it defined by $\langle e_i|e_j \rangle = s_i\delta_{i,j}$ for $1 \leq i, j \leq \kappa$. Set $\alpha_i := e_i - e_{i+1}$ for $i \in I$ and let $Q_s := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ be the root lattice of $\mathfrak{sl}_s$.

Let $\delta$ be the null root of $\widehat{\mathfrak{sl}}_s$ such that $\langle \delta|\delta \rangle = \langle \delta|\alpha_i \rangle = 0$ for $i \in I$. Let $\alpha_0 := \delta + \varepsilon_{\kappa} - \varepsilon_1$. Then $\langle \alpha_i|\alpha_j \rangle = a^s_{i,j}$ for $i, j \in \hat{I}$.

For two homogeneous elements $X, Y \in \mathfrak{C}$, set $[X, Y]_a = XY - (-1)^{|X||Y|}aYX$. We simply write $[X, Y]$ for $[X, Y]_1$.

Definition 2.5 ([Yam99] Drinfeld-Jimbo presentation). The quantum affine superalgebra $U_q(\widehat{\mathfrak{sl}}_s)$ is generated by the Chevalley generators $e_i, f_i, t_i^{\pm 1}, i \in \hat{I}$, whose parities are given by $|e_i| = |f_i| = |i| := (1 - s_is_{i+1})/2$, $|t_i^{\pm 1}| = 0$, with the defining relations given by

\[
t_i t_j = t_j t_i, \quad t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad t_i e_j t_i^{-1} = q^{a^s_{i,j}} e_j, \quad t_i f_j t_i^{-1} = q^{-a^s_{i,j}} f_j,
\]

\[
[e_i, f_j] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q - q^{-1}}.
\]
\[
[e_i, e_j] = [f_i, f_j] = 0 \quad \quad (a_{i,j}^s = 0),
\]
\[
[e_i, [e_i, e_{i+1}]] = [f_i, [f_i, f_{i+1}]] = 0 \quad \quad (a_{i,i}^s \neq 0),
\]
\[
[e_i, [e_{i+1}, [e_i, e_{i-1}]]] = [f_i, [f_{i+1}, [f_i, f_{i-1}]]] = 0 \quad \quad (mn \neq 2, a_{i,i}^s = 0),
\]
\[
[e_{i+1}, [e_{i-1}, [e_{i+1}, e_{i-1}]]] = [e_{i-1}, [e_{i+1}, [e_{i-1}, e_{i+1}]]] = [f_{i-1}, [f_{i+1}, [f_{i-1}, f_{i+1}]]] = 0 \quad \quad (mn = 2, a_{i,i}^s \neq 0),
\]
\[
[f_{i+1}, [f_{i-1}, [f_{i+1}, [f_{i-1}, f_{i+1}]]]] = 0 \quad \quad (mn = 2, a_{i,i}^s \neq 0),
\]

where \([X,Y] = [X,Y]_{q-(a,q)}\) if \(t_i X t_i^{-1} = q^{(a)}(\beta)\) and \(t_i Y t_i^{-1} = q^{(a)}(\gamma)\) for \(\beta, \gamma \in Q_s\) and \(i \in I\).

Note that the element \(t_0 t_1 \cdots t_{\kappa-1}\) is central and \(U_q(sl_n)\) for different \(s \in S_{m/\kappa}\) are isomorphic.

The superalgebra \(U_q(sl_n)\) is endowed with a coproduct \(\Delta\) given by
\[
\Delta(e_i) = e_i \otimes t_i + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + t_i^{-1} \otimes f_i, \quad \Delta(t_i) = t_i \otimes t_i.
\]

The subalgebra of \(U_q(sl_n)\) generated by \(e_i, f_i, t_i\), \(i \in I\), is isomorphic to \(U_q(sl_n)\) as a Hopf superalgebra.

The superalgebra \(U_q(sl_n)\) admits another presentation as follows.

Let \(\delta(z) = \sum_{r \in \mathbb{Z}} z^r\) be the formal delta function. For \(k \in \mathbb{Z}\), set \([k] = q^k - q^{-k}\).

**Definition 2.6** ([Yam99] New Drinfeld Presentation). The superalgebra \(U_q(sl_n)\) is generated by the *current generators* \(x^+_i, h_i, k_i, c_i, \) \(i \in I, r \in \mathbb{Z}'\). Here and below, we use the following convention: \(r \in \mathbb{Z}'\) means \(r \in \mathbb{Z}\) if \(r\) is an index of a non-Cartan current generator \(x^+_i,\) and \(r \in \mathbb{Z}'\) means \(r \in \mathbb{Z} \setminus \{0\}\) if \(r\) is an index of a Cartan current generator \(h_i\). The parity of current generators is given by \(|x^+_i| = |r| = (1 - s_is_{i+1})/2\) while all remaining generators have parity 0. The defining relations are as follows:

\(c\) is central, \(k_i k_j = k_j k_i, k_i k_i^{-1} = k_i^{-1} k_i = 1, k_i x^+_i(z) k_i^{-1} = q^{a_{i,i}^s} x^+_i(z),\)
\([h_i, h_j, h_k] = \delta_{r,s,0} \frac{[r a_{i,j}^s]}{r} c^r - c^{-r},\)
\([h_i, x^+_j(z)] = \pm \frac{[r a_{i,j}^s]}{r} c^{(r+|r|)/2} x^+_j(z),\)
\([x^+_i(z), x^-_j(w)] = \frac{\delta_{i,j}}{q - q^{-1}} \left( \delta \left( \frac{w}{z} \right) k^+_i(w) - \delta \left( \frac{z}{w} \right) k^-_i(z) \right),\)
\(z - q a_{i,j}^s w x^+_i(z) x^+_j(w) + (-1)^{|i||j|}(w - q a_{i,j}^s z)x^-_i(w) x^+_j(z) = 0 \quad (a_{i,j}^s \neq 0),\)
\([x^+_i(z), x^-_j(w)] = 0 \quad (a_{i,j}^s = 0),\)
\(\text{Sym}_{z_1, z_2} [x^+_i(z_1), [x^+_i(z_2), x^+_i(w_1)] = 0 \quad (a_{i,j}^s \neq 0, i \pm 1 \in I),\)
\(\text{Sym}_{z_1, z_2} [x^+_i(z_1), [x^+_i(w_1), x^+_i(z_2), x^+_i(w_2)] = 0 \quad (a_{i,j}^s = 0, i \pm 1 \in I),\)

where \(x^+_i(z) = \sum_{r \in \mathbb{Z}} x^+_i z^r\) and
\(k^+_i(z) = k^+_i \exp \left( \pm (q - q^{-1}) \sum_{r \geq 0} h_{i,\pm r} z^{\pm r} \right) = k^+_i \pm \sum_{r \geq 1} k^+_i \pm r z^{\pm r}.\)

Here and below, \(\text{Sym}_{z_1, z_2}\) stands for the symmetrization map on \(z_1, z_2\). For instance,
\(\text{Sym}_{z_1, z_2} [x^+_i(z_1), [x^+_i(z_2), x^+_i(w_1)] = [x^+_i(z_1), [x^+_i(z_2), x^+_i(w_1)] + [x^+_i(z_2), x^+_i(z_1), x^+_i(w_1)].\)

An isomorphism between Drinfeld-Jimbo and new Drinfeld presentations is given by
\(e_i \mapsto x^+_i, f_i \mapsto x^-_i, t_i \mapsto k_i, t_0 \mapsto c(k_1 k_2 \cdots k_{\kappa-1})^{-1},\)
\((i \in I),\)
\[
\begin{align*}
    \epsilon_0 & \mapsto (-1)^n s_k [x^{+}_{m+n-1,0}, \ldots, [x^{-}_{2,0}, x^{+}_{1,1}]_{q_2^{-1}} \cdots]_{q_{k-1}^{-1}} (k_1 k_2 \cdots k_{k-1})^{-1}, \\
    f_0 & \mapsto s_k k_1 k_2 \cdots k_{k-1} [\cdots [x^{+}_{1,1}, x^{-}_{2,0}]_{q_2}, \ldots, x^{+}_{-1,0}]_{q_{k-1}},
\end{align*}
\]
where \( q_i = q^s_i \) for \( i \in I \), see e.g. [Zha14, Theorem 5.2] and [LYZ22]. Note that \( t_0 t_1 \cdots t_{k-1} \mapsto c \).

### 2.3. Representations of quantum affine superalgebras.
For simplicity, let
\[
    \mathcal{U} := \mathcal{U}_q(\hat{\mathfrak{sl}}_s)/(c-1)
\]
be the quantum loop superalgebra. Here and below we use the same notation for the images of the generators in \( \mathcal{U}_q(\hat{\mathfrak{sl}}_s) \) under the quotient.

The quantum loop superalgebra \( \mathcal{U} \) has a Hopf superalgebra structure inherited from \( \mathcal{U}_q(\hat{\mathfrak{sl}}_s) \). We shall need the following coproduct formula.

We start with introducing necessary notations.

There is a natural \( \mathcal{Q}_s \)-grading on \( \mathcal{U} \) given by
\[
(\mathcal{U})_\alpha := \{ u \in \mathcal{U} \mid k_i u k_i^{-1} = q^{(\alpha_i, \alpha)} u, \text{ for } i \in I \}, \quad \alpha \in \mathcal{Q}_s.
\]
Let \( \mathcal{Q}_s^+ \) be the positive root lattice, \( \mathcal{Q}_s^+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \). Define the length function \( \iota : \mathcal{Q}_s^+ \to \mathbb{Z}_{\geq 0} \) by
\[
\iota(\sum_{i \in I} n_i \alpha_i) = \sum_{i \in I} n_i.
\]
Moreover, whenever \( \iota(\alpha) \) is used, we implicitly assume that \( \alpha \in \mathcal{Q}_s^+ \). Note that \( \iota \) depends on \( s \). However, we shall not write \( s \) explicitly. Finally, for \( i \in I \), let \( \mathcal{U}_i^\pm \) be the subalgebra of \( \mathcal{U} \) generated by \( x_{j,0}^\pm \) for \( j \in I \setminus \{i\} \).

#### Proposition 2.7
Let \( i \in I \) and \( r \in \mathbb{Z} \). We have the following properties for the coproduct of \( \mathcal{U} \),

1. **modulo** \( \sum_{\alpha \in \mathcal{Q}_s^+ \setminus \{0\}} (\mathcal{U})_\alpha \otimes (\mathcal{U})_{-\alpha} \),
   \[
   \Delta(k_i^{\pm}(z)) \equiv k_i^{\pm}(z) \otimes k_i^{\pm}(z),
   \]
2. **modulo** \( \sum_{\iota(\alpha)>1} (\mathcal{U}_i^+)_{\alpha} \otimes (\mathcal{U})_{\alpha-i} + \sum_{\iota(\alpha-i)>0} (\mathcal{U})_{\alpha} \otimes (\mathcal{U})_{\alpha-i} \),
   \[
   \Delta(x_{i,r}^+) \equiv x_{i,r}^+ \otimes k_i^+ + 1 \otimes x_{i,r}^+ + \sum_{j=1}^{r} x_{i,r-j}^+ \otimes k_{i,j}^+,
   \quad (r \geq 0),
   \]
   \[
   \Delta(x_{i,r}^-) \equiv x_{i,r}^- \otimes k_i^- + 1 \otimes x_{i,r}^- + \sum_{j=1}^{-r-1} x_{i,r+j}^- \otimes k_{i,-j}^-,
   \quad (r < 0),
   \]
3. **modulo** \( \sum_{\iota(\alpha)>1} (\mathcal{U})_{\alpha-i} \otimes (\mathcal{U})_{\alpha} + \sum_{\iota(\alpha-i)>0} (\mathcal{U})_{\alpha-i} \otimes (\mathcal{U})_{\alpha} \),
   \[
   \Delta(x_{i,r}^-) \equiv x_{i,r}^- \otimes 1 + k_i^- \otimes x_{i,r}^- + \sum_{j=1}^{r-1} k_{i,j}^+ \otimes x_{i,r-j}^-,
   \quad (r > 0),
   \]
   \[
   \Delta(x_{i,r}^-) \equiv x_{i,r}^- \otimes 1 + k_i^- \otimes x_{i,r}^- + \sum_{j=1}^{-r} k_{i,-j}^- \otimes x_{i,r+j}^-,
   \quad (r \leq 0).
   \]

**Proof.** The proof follows from that of [Zha14, Proposition 5.4] and [Zha16, Proposition 3.6], cf. also [CP91, Proposition 4.4]. We only sketch the key points.

First, one shows as in [Zha14, Lemma 5.3] and [Zha16, Lemma A.3] that for any \( s \in I \),
\[
\Delta(h_{i,1}) \equiv h_{i,1} \otimes 1 + 1 \otimes h_{i,1} + \sum_{j \in I, j=i-1}^{i+1} \nu_{i,j} x_{j,0}^+ \otimes x_{j,1}^-.
\]
modulo $\sum_{\alpha>1}(U^+_{\alpha})_\alpha \otimes (U)_{\alpha} + \sum_{\alpha>\alpha>0}(U^+_{\alpha})_\alpha \otimes (U)_{-\alpha}$, where $\nu_{i,j} \in \mathbb{C}^\times$ and $\nu_{i,j+1} = q - q^{-1}$. Then by induction, one proves the formula for $\Delta(x^+_{i,r})$, $r \geq 0$, as in [Zha16, Lemma A.4]. In this step, one needs to calculate explicitly certain coefficients which are obvious from the commutator relations, cf. [Zha14, Proposition 5.4]. Similarly, one obtains the other coproduct formulas for $\Delta(x^+_{i,r})$ with proper modifications. The proof for $\Delta(k^\pm_i(z))$ is parallel to that of [Zha16, Corollary A.5].

Given a parity sequence $s \in S_{m|n}$, define a map $\mu_s : \hat{I} \to \mathbb{Z}$ by

$$\mu_s(i) := \sum_{j=1}^i s_j, \quad i \in \hat{I}$$

where, by convention, $\mu_s(0) = 0$.

For $r \in \mathbb{Z}$, set

$$\psi_r(z) = \frac{q^r - q^{-r}z}{1 - z}.$$

Then $\psi_r(0)\psi_r(\infty) = 1$.

For a rational function $\phi(z)$ such that $\phi(0)\phi(\infty) = 1$, denote by $\phi^\pm(z)$ the expansions of $\phi(z)$ as power series in $z^{\mp 1}$, respectively.

**Example 2.8** (cf. [BM21b, Lemma 3.1]). Let $V_s \cong \mathbb{C}^{m|n}$ be the superspace with a basis $v_j$ for $1 \leq j \leq \kappa$ such that $|v_j| = (1 - s_j)/2$. Let $\xi$ be a formal variable and set $V_s(\xi) := \mathbb{C}[\xi^{\pm 1}] \otimes V_s$. Then $V_s(\xi)$ has a $\mathbb{U}_q(\hat{sl}_s)$-module structure as follows,

$$x^+_i(z)^\xi v_j = \delta_{i+1,j} \xi z (q^{\mu_s(i)} \xi) v_{j-1},$$

$$x^-_i(z)^\xi v_j = s_j \delta_{i,j} \xi z (q^{\mu_s(i)} \xi) v_{j-1},$$

$$k^\pm_i(z)^\xi v_j = \begin{cases} \psi^\pm_{s_j}(q^{\mu_s(i)} \xi) c v_j, & \text{if } i = j, \\ \psi^\pm_{s_{j+1}}(q^{\mu_s(i)} \xi) c v_j, & \text{if } i = j - 1, \\ c v_j, & \text{otherwise.} \end{cases}$$

Moreover, $c$ acts by identity.

One can also specialize $\xi$ to a nonzero complex number $a$. Then the same relations define a $\mathbb{U}_q(\hat{sl}_s)$-module structure on $V_s$ which is the evaluation vector representation at the evaluation parameter $a$. We denote it by $V_s(a)$.

The action of Chevalley generators on $V_s(\xi)$ is given by

$$e_i(v_j) = \delta_{i+1,j} v_{j-1}, \quad f_i(v_j) = s_j \delta_{i,j} v_{j+1}, \quad t_i(v_j) = q^{\delta_{i,j} - \delta_{i+1,j}} v_j, \quad (i \in I),$$

$$e_0(v_j) = \delta_{1,j} a v_\kappa, \quad f_0(v_j) = s_{\kappa,j} a^{-1} v_1, \quad t_0(v_j) = q^{\delta_{1,j} - \delta_{1,j}} v_j.$$

The restriction of $V_s(a)$ to a $\mathbb{U}_q(\hat{sl}_s)$-module is called the vector (or natural) representation of $\mathbb{U}_q(\hat{sl}_s)$.

A $\mathbb{U}_q(\hat{sl}_s)$-module (resp. $\mathbb{U}_q(\hat{sl}_s)$-module) is called integrable if it is a weight module over $\mathbb{U}_q(\hat{sl}_s)$ and $e_i$, $f_i$, for all $i \in \hat{I}$ (resp. $i \in I$), act locally nilpotent. Clearly, $V_s(\xi)$ and $V_s(a)$ are integrable.

Let $\ell \in \mathbb{Z}_{>0}$ and let $\xi = (\xi_1, \ldots, \xi_\ell)$ be a sequence of commuting formal variables. Denote by $V_s(\xi)$ the tensor product $V_s(\xi_1) \otimes V_s(\xi_2) \otimes \cdots \otimes V_s(\xi_\ell)$. Then $V_s(\xi)$ is a $\mathbb{U}_q(\hat{sl}_s)$-module induced by the coproduct (2.4). Note that the $\mathbb{U}_q(\hat{sl}_s)$-action on $V_s(\xi)$ commutes with multiplication by the elements of $\mathbb{C}[\xi^{\pm 1}], \ldots, \xi^{\pm 1}]$.

Let $e_\theta$, $f_\theta$, $k_\theta$ in $\text{End}(V_s)$ be defined by

$$e_\theta v_j = \delta_{j,k} v_1, \quad f_\theta v_j = \delta_{j,1} v_k, \quad k_\theta v_j = q^{\delta_{j,1} - \delta_{j,k}} v_j.$$
Define
\[ e_{\theta,j} = k_{\theta}^{j-1} \otimes e_{\theta} \otimes 1^{\otimes \ell-j}, \quad f_{\theta,j} = 1^{\otimes j-1} \otimes f_{\theta} \otimes (k_{\theta}^{-1})^{\otimes \ell-j} \]  
(2.7)
in \text{End}(V_{\theta}^{\otimes \ell}). Then for any \( v \in V_{\theta}^{\otimes \ell} \subset V_{\theta}(\xi) \), we have
\[ e_{0}v = \sum_{j=1}^{\ell} \xi_{j}f_{\theta,j}v, \quad f_{0}v = s_{\ell} \sum_{j=1}^{\ell} \xi_{j}^{-1}e_{\theta,j}v, \quad t_{0}v = (k_{\theta}^{-1})^{\otimes \ell}v, \]  
(2.8)
For any sequence of integers \( j = (j_{1}, j_{2}, \ldots, j_{\ell}) \), we say that \( j \) is non-decreasing if \( 1 \leq j_{1} \leq \cdots \leq j_{\ell} \leq \kappa \). For any non-decreasing \( j \) and \( 1 \leq r \leq \ell \), set
\[ v_{j} = v_{j_{1}} \otimes \cdots \otimes v_{j_{r}}, \quad j_{r}^{\pm} = (j_{1}, \ldots, j_{r-1}, j_{r} \pm 1, j_{r+1}, \ldots, j_{\ell}), \]  
and define
\[ |j_{r}| = \sum_{a=1}^{r-1} |v_{j_{a}}|, \quad t_{s}(i, r; j) = (-1)^{i||j_{r}|}. \]  
(2.9)
For nonnegative integers \( a < b \), we use the convenient notation \( (a, b) := \{ a+1, \ldots, b \} \). Regard \( j \) as a map from \( (0, \ell] \) to \( (0, \kappa] \).

For \( a \in \mathbb{C}^{\times} \) (or any formal variable \( \xi \)) and a rational function \( \phi(z) \) such that \( \phi(0)\phi(\infty) = 1 \), define the normal order products by
\[ :[\delta(a/z)\phi(z)]^{\pm} := \phi^{+}(z) \sum_{r \geq 0} a^{r}z^{-r} + \phi^{-}(z) \sum_{r > 0} a^{-r}z^{r}, \]  
and define
\[ |j_{r}| = \sum_{a=1}^{r-1} |v_{j_{a}}|, \quad t_{s}(i, r; j) = (-1)^{i||j_{r}|}. \]  
(2.9)

**Proposition 2.9.** If \( j \) is non-decreasing, we have the following in \( V_{\theta}(\xi) \),
\[ x_{i}^{\pm}(z)v_{j} = \sum_{r=a_{2}+1}^{a_{3}} t_{s}(i, r; j) : \left[ \delta(q^{\mu_{s}(i)}\xi_{r}/z) \prod_{p=r+1}^{a_{3}} \psi_{-s_{i+1}}(q^{\mu_{s}(i)}\xi_{p}/z) \right]^{\pm} : v_{j_{r}^{\pm}}; \]
\[ x_{i}^{-}(z)v_{j} = s_{i} \sum_{r=a_{1}+1}^{a_{2}} t_{s}(i, r; j) : \left[ \delta(q^{\mu_{s}(i)}\xi_{r}/z) \prod_{p=r+1}^{r-1} \psi_{s}(q^{\mu_{s}(i)}\xi_{p}/z) \right] : v_{j_{r}^{\pm}}; \]
\[ k_{i}^{\pm}(z)v_{j} = \prod_{j_{r}=i}^{a_{1}} \psi_{s}(q^{\mu_{s}(i)}\xi_{r}/z) \prod_{j_{r}=i+1}^{a_{2}} \psi_{-s_{i+1}}(q^{\mu_{s}(i)}\xi_{r}/z)v_{j}, \]
where \( (a_{1}, a_{2}) = j^{-1}(i) \) and \( (a_{2}, a_{3}) = j^{-1}(i+1) \).

**Proof.** Since \( j \) is non-decreasing, the statement follows from Proposition 2.7. We only show it for the last equality. The first and second equalities are similar.

Consider the coproduct \( \Delta(k_{i}^{\pm}(z)) \) where the first factor acts on the first factor of \( V_{\theta}(\xi) \) while the second factors acts on the tensor product of the rest of factors of \( V_{\theta}(\xi) \). Comparing the classical weights, it is clear that terms from \( \sum_{a_{s} \in Q_{\theta}^{+} \setminus \{0\}} (\mu)_{\alpha} \otimes (\mu)_{-\alpha} \) acting on \( v_{j} \) do not contribute to the final result of \( k_{i}^{\pm}(z)v_{j} \) as \( j \) is non-decreasing. Therefore, we have
\[ k_{i}^{\pm}(z)v_{j} = k_{i}^{\pm}(z)v_{j_{1}} \otimes k_{i}^{\pm}(z)(v_{j_{2}} \otimes \cdots \otimes v_{j_{\ell}}). \]
As the subsequence \( (j_{2}, \ldots, j_{\ell}) \) is also non-decreasing, we can repeat the procedure and obtain the last equality by formulas in Example 2.8.

Note that, in general, the proposition does not hold if \( j \) is not non-decreasing. \( \square \)
2.4. Quantum toroidal superalgebras. We recall the definition of quantum toroidal superalgebra from [BM21a]. Fix $d \in \mathbb{C}^\times$ and set

$$q_1 = dq^{-1}, \quad q_2 = q^2, \quad q_3 = d^{-1}q^{-1}. \quad (2.10)$$

Note that $q_1q_2q_3 = 1$. We always assume that $q_1$, $q_2$ are generic, namely $q_1^{n_1}q_2^{n_2}q_3^{n_3} = 1$, $n_1, n_2, n_3 \in \mathbb{Z}$, if and only if $n_1 = n_2 = n_3$. Also fix $d^{1/2}, q^{1/2} \in \mathbb{C}^\times$ such that $(d^{1/2})^2 = d$, $(q^{1/2})^2 = q$.

Define the matrix $M^s = (m^s_{i,j})_{i,j \in \hat{I}}$ by $m^s_{i+1,i} = -m^s_{i,i+1} = s_i$, and $m^s_{i,j} = 0$, $i \neq j \pm 1$. Recall the affine Cartan matrix $\hat{A}^s = (a^s_{i,j})_{i,j \in \hat{I}}$ from (2.3).

Definition 2.10 ([BM21a]). The quantum toroidal algebra associated with $\mathfrak{gl}_{m\mid n}$ and parity sequence $s$ is the unital associative superalgebra $E_s = E_s(q_1, q_2, q_3)$ generated by $E_{i,r}, F_{i,r}, H_{i,r}$, and invertible elements $K_i, C_i$, where $i \in \hat{I}$, $r \in \mathbb{Z}'$, subject to the defining relations (2.11)-(2.25) below. The parity of the generators is given by $|E_{i,r}| = |F_{i,r}| = |i| = (1 - s_is_{i+1})/2$, and all remaining generators have parity 0. We use generating series

$$E_i(z) = \sum_{r \in \mathbb{Z}} E_{i,r}z^{-r}, \quad F_i(z) = \sum_{r \in \mathbb{Z}} F_{i,r}z^{-r},$$

$$K_i^\pm(z) = K_i^{\pm 1}\exp\left(\pm(q - q^{-1}) \sum_{r \geq 1} H_{i,\pm r} z^{\mp r}\right) = K_i^{\pm 1} + \sum_{r \geq 1} K_{i,\pm r}^\pm z^{\mp r}.$$

Then the defining relations are as follows.

**C, K relations**

$$K_iK_j = K_jK_i, \quad K_iE_j(z)K_i^{-1} = q^{a^s_{i,j}}E_j(z), \quad K_iF_j(z)K_i^{-1} = q^{-a^s_{i,j}}F_j(z). \quad (2.11)$$

**K-K, K-E and K-F relations**

$$K_i^+(z)K_j^+(w) = K_j^+(w)K_i^+(z), \quad (2.12)$$

$$\frac{d^{m^s_{i,j}}C^{-1}z - q^{a^s_{i,j}}w}{d^{m^s_{i,j}}Cz - q^{a^s_{i,j}}w} K_i^-(z)K_j^+(w) = \frac{d^{m^s_{i,j}}q^{a^s_{i,j}}C^{-1}z - w}{d^{m^s_{i,j}}q^{a^s_{i,j}}Cz - w} K_j^-(w)K_i^-(z), \quad (2.13)$$

$$\left(\frac{d^{m^s_{i,j}}z - q^{a^s_{i,j}}w}{d^{m^s_{i,j}}z - q^{a^s_{i,j}}w}\right)K_i^\pm(\frac{C^{-1}+1}{2})E_j(z) = \left(\frac{d^{m^s_{i,j}}q^{a^s_{i,j}}z - w}{d^{m^s_{i,j}}q^{a^s_{i,j}}z - w}\right)E_j(z)K_i^\pm(\frac{C^{-1}+1}{2}), \quad (2.14)$$

$$\left(\frac{d^{m^s_{i,j}}z - q^{-a^s_{i,j}}w}{d^{m^s_{i,j}}z - q^{-a^s_{i,j}}w}\right)K_i^\pm(\frac{C^{-1}+1}{2})F_j(z) = \left(\frac{d^{m^s_{i,j}}q^{-a^s_{i,j}}z - w}{d^{m^s_{i,j}}q^{-a^s_{i,j}}z - w}\right)F_j(z)K_i^\pm(\frac{C^{-1}+1}{2}). \quad (2.15)$$

**E-F relations**

$$[E_i(z), F_j(w)] = \frac{\delta_{i,j}}{q - q^{-1}}(\delta(C\frac{w}{z})K_i^+(w) - \delta(C\frac{z}{w})K_i^-(z)). \quad (2.16)$$

**E-E and F-F relations**

$$[E_i(z), E_j(w)] = 0, \quad [F_i(z), F_j(w)] = 0 \quad (a^s_{i,j} = 0), \quad (2.17)$$

$$\left(\frac{d^{m^s_{i,j}}z - q^{a^s_{i,j}}w}{d^{m^s_{i,j}}z - q^{a^s_{i,j}}w}\right)E_j(z)E_j(w) = (-1)^{i|i|}\left(\frac{d^{m^s_{i,j}}q^{a^s_{i,j}}z - w}{d^{m^s_{i,j}}q^{a^s_{i,j}}z - w}\right)E_j(z)E_j(z) \quad (a^s_{i,j} \neq 0), \quad (2.18)$$

$$\left(\frac{d^{m^s_{i,j}}z - q^{-a^s_{i,j}}w}{d^{m^s_{i,j}}z - q^{-a^s_{i,j}}w}\right)F_i(z)F_j(w) = (-1)^{i|i|}\left(\frac{d^{m^s_{i,j}}q^{-a^s_{i,j}}z - w}{d^{m^s_{i,j}}q^{-a^s_{i,j}}z - w}\right)F_i(z)F_j(z) \quad (a^s_{i,j} \neq 0). \quad (2.19)$$

**Serre relations**

$$\text{Sym}_{z_1,z_2}[E_i(z_1), [E_i(z_2), E_{i+1}(w_1)]] = 0 \quad (a^s_{i,i} \neq 0), \quad (2.20)$$

$$\text{Sym}_{z_1,z_2}[F_i(z_1), [F_i(z_2), F_{i+1}(w_1)]] = 0 \quad (a^s_{i,i} \neq 0). \quad (2.21)$$

If $mn \neq 2$,

$$\text{Sym}_{z_1,z_2}[E_i(z_1), [E_{i+1}(w_1), [E_i(z_2), E_{i-1}(w_2)]]] = 0 \quad (a^s_{i,i} = 0). \quad (2.22)$$
over an isomorphism of superalgebras \( \hat{\tau} \). Basics about quantum toroidal superalgebras.

Namely, 

\[ \text{Sym}_z \left[ F_i(z_1), [F_i+1(w_1), [F_i(z_2), F_i-1(w_2)] \right] = 0 \quad (a^s_{i,i} = 0). \tag{2.23} \]

If \( mn = 2 \),

\[ \text{Sym}_z \left[ E_i-1(z_1), [E_i+1(w_1), [E_i-1(z_2), E_i(w_2)] \right] = (a^s_{i,i} \neq 0), \]

\begin{align*}
\text{Sym}_z \left[ F_i-1(z_1), [F_i+1(w_1), [F_i-1(z_2), F_i(w_2)] \right] = (a^s_{i,i} \neq 0). \tag{2.24} \\
\text{Sym}_z \left[ F_i-1(z_1), [F_i+1(w_1), [F_i-1(z_2), F_i(w_2)] \right] = (a^s_{i,i} \neq 0). \tag{2.25} \\
\end{align*}

Note that the element \( K_0 K_1 \cdots K_{n-1} \) is central and \( \mathcal{E}_s \) for different \( s \in S_{m|n} \) are isomorphic, see [BM21a].

We use the abbreviation \( E_i := E_{i,0} \) and \( F_i := F_{i,0} \) for \( i \in \hat{I} \).

2.5. Basics about quantum toroidal superalgebras. Let \( s \in S_{m|n} \). Define the \textit{vertical homomorphism} of superalgebras \( v_s : \mathbb{U}_q(\mathfrak{sl}_s) \to \mathcal{E}_s \) by

\[
v_s(x^+_i(z)) = E_i(d^{-\mu_s(i)}z), \quad v_s(x^-_i(z)) = F_i(d^{-\mu_s(i)}z),
\]

\[
v_s(k^+_i(z)) = K^+_i(d^{-\mu_s(i)}z), \quad v_s(c) = C, \quad (i \in I).
\]

The map \( v_s \) is injective for generic parameters. We call the image of \( v_s \) the \textit{vertical subalgebra} of \( \mathcal{E}_s \) and denote it by \( \mathbb{U}_q^v(\mathfrak{sl}_s) \).

We have an injective (for generic parameters) \textit{horizontal homomorphism} of superalgebras \( h_s : \mathbb{U}_q(\mathfrak{sl}_s) \to \mathcal{E}_s \) given by

\[
e_i \mapsto E_i, \quad f_i \mapsto F_i, \quad t_i \mapsto K_i, \quad (i \in \hat{I}).
\]

We call the image of \( h_s \) the \textit{horizontal subalgebra} of \( \mathcal{E}_s \) and denote it by \( \mathbb{U}_q^h(\mathfrak{sl}_s) \).

Note that \( \mathcal{E}_s \) is generated by \( \mathbb{U}_q^v(\mathfrak{sl}_s) \) and \( \mathbb{U}_q^h(\mathfrak{sl}_s) \).

For any \( u \in \mathbb{C}^\times \), denote by \( \gamma_{u,s} : \mathcal{E}_s \to \mathcal{E}_s \) the \textit{shift automorphism} by \( u \) defined by

\[
\gamma_{u,s}(C) = C, \quad \gamma_{u,s}(A_s(z)) = A_s(uz), \quad (i \in \hat{I}, \ A = K^\pm, E, F).
\]

Define a map \( \tau : S_{m|n} \to S_{m|n} \) which sends \( s = (s_1, \ldots, s_n) \) to \( \tau s := (s_\ell, s_1, \ldots, s_{\ell-1}) \). There exists an isomorphism of superalgebras \( \hat{\tau}_s : \mathcal{E}_s \to \mathcal{E}_{\tau s} \) given by

\[
\hat{\tau}_s(C) = C, \quad \hat{\tau}_s(A_s(z)) = A_{s+i}(q^{-s_n}z), \quad (i \in \hat{I}, \ A = K^\pm, E, F). \tag{2.26}
\]

An \( \mathcal{E}_s \)-module \( M \) has \textit{trivial central charge} if the restrictions of \( M \) to \( \mathbb{U}_q^v(\mathfrak{sl}_s) \) and \( \mathbb{U}_q^h(\mathfrak{sl}_s) \) also have trivial central charge. Namely, \( C = 1 \) and \( K_0 K_1 \cdots K_{n-1} = 1 \).

We say that a \( \mathbb{U}_q(\mathfrak{sl}_s) \)-module is \textit{of level} \( \ell \) if all its irreducible components are isomorphic some submodules of \( \mathcal{V}_s^{\otimes \ell} \). A \( \mathbb{U}_q(\mathfrak{sl}_s) \)-module or an \( \mathcal{E}_s \)-module is said to be \textit{of level} \( \ell \) if it is of level \( \ell \) as a \( \mathbb{U}_q(\mathfrak{sl}_s) \)-module.

Set \( P_\ell := \{ \lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}^\geq_0 \mid \lambda_1 + \lambda_2 + \cdots + \lambda_\ell = \ell \} \). We call \( \lambda \in P_\ell \) a \textit{polynomial weight}.

An \( \mathcal{E}_s \)-module \( M \) with trivial central charge and of level \( \ell \) is \textit{integrable} if \( M \) is integrable as modules over \( \mathbb{U}_q^v(\mathfrak{sl}_s) \) and \( \mathbb{U}_q^h(\mathfrak{sl}_s) \), and

\[
M = \bigoplus_{\lambda \in P_\ell} M_\lambda, \quad M_\lambda = \{ v \in M \mid K_i v = q^{s_i \lambda_i - s_{i+1} \lambda_{i+1}} v, \ i \in \hat{I} \}.
\]
3. Super Schur-Weyl duality

Since $E_s$ are all isomorphic for different $s \in S_{m|n}$, in the rest of this paper, we shall set $s$ to be the standard parity sequence or the images of the standard parity sequence under repeated application of $\tau$ for simplicity. However, our computations work for all parity sequences.

3.1. Super Schur-Weyl duality for finite and affine cases. We start with the super Schur-Weyl duality for finite case established in [Moo03, Mit06], cf. [Jim86].

Recall the vector representation $V_s$ of $U_q(sl_s)$ and consider the linear map $T: V_s \otimes V_s \to V_s \otimes V_s$ given by

$$T(v_i \otimes v_j) = \begin{cases} s_i q^{1+s_i} v_i \otimes v_i, & \text{if } i = j, \\ (-1)^{|v_i||v_j|} q v_j \otimes v_i, & \text{if } i < j, \\ (-1)^{|v_i||v_j|} q v_j \otimes v_i + (q^2 - 1)v_i \otimes v_j, & \text{if } i > j. \end{cases}$$

Fix $\ell > 1$. Let $T_\ell \in \text{End}(V_s^{\otimes \ell})$ be the map which acts on the $i$-th and $(i + 1)$-st factors as $T_i$ and the other factors as the identity.

Note that our choices of coproduct and $T$ follow that of [CP96, VV96] which are slightly different from that of [Moo03, Mit06].

**Theorem 3.1** ([Jim86, Moo03, Mit06]). There is a left $\mathbb{H}_\ell$-module structure on $V_s^{\otimes \ell}$ such that $T_i$ acts as $T_\ell$ for all $1 \leq i < \ell$. Moreover, the action of $\mathbb{H}_\ell$ commutes with the action of $U_q(sl_s)$ on $V_s^{\otimes \ell}$.

Let $M$ be a right $\mathbb{H}_\ell$-module. Define $J(M) := M \otimes_{\mathbb{H}_\ell} V_s^{\otimes \ell}$ with the $U_q(sl_s)$-module structure induced by that on $V_s^{\otimes \ell}$. If $\ell < mn + \kappa$, then the functor $J: M \to J(M)$ is an equivalence from the category of finite-dimensional $\mathbb{H}_\ell$-modules to the category of finite-dimensional $U_q(sl_s)$-modules of level $\ell$. □

The statement has been extended to the quantum affine superalgebra $U_q(\hat{sl}_s)$ in [Fli20, KL22], cf. [GRV94, CP96].

We identify $\mathbb{H}_\ell$ with $\mathbb{H}_\ell^{(1)}$. Recall (2.6), (2.7), and the generators $Y_j$ in $\mathbb{H}_\ell^{(1)}$.

**Theorem 3.2** ([GRV94, CP96, Fli20, KL22]). There exists a functor $F$ from the category of finite-dimensional right $\mathbb{H}_\ell$-modules to the category of finite-dimensional $U_q(sl_s)$-modules with trivial central charge and of level $\ell$, defined as follows. If $M$ is a right $\mathbb{H}_\ell$-module, then $F(M) = J(M)$ as a $U_q(sl_s)$-module and the action of $e_0$, $f_0$, $t_0$ is given by

$$e_0(w \otimes v) = \sum_{j=1}^{\ell} w Y_j^{-1} \otimes f_{\theta,j} v, \quad f_0(w \otimes v) = s_{\kappa} \sum_{j=1}^{\ell} w Y_j \otimes e_{\theta,j} v, \quad t_0(w \otimes v) = w \otimes (k_0^{-1})^{\otimes \ell} v, \quad (3.1)$$

where $w \in M$ and $v \in V_s^{\otimes \ell}$. Moreover, if $\ell < \kappa$, then the functor $F$ is an equivalence of categories. □

Note that the $U_q(\hat{sl}_s)$-module $F(M)$ can be understood as the tensor product of evaluation vector representations $V_s(Y)$, where $Y = (Y_1^{-1}, \ldots, Y_\ell^{-1})$, with values in $M$, see Example 2.8 and Proposition 2.9.

3.2. Super Schur-Weyl duality for toroidal case. Our main result is the Schur-Weyl duality between double affine Hecke algebra $\mathbb{H}_\ell$ and the quantum toroidal superalgebra $E_s$, extending the main result of [VV96] to the supersymmetric case.

Recall that $E_i$, $F_i$, $K_i$, $i \in \hat{I}$, are Chevalley generators of the horizontal subalgebra $\mathbb{U}_q^{\text{h}}(\hat{sl}_s)$. It is also convenient to introduce extra generators $E_0$, $F_0$, $K_0$ of $E_s$ so that combining with $E_i$, $F_i$, $K_i$, $i \in I$, they form Chevalley generators of the vertical subalgebra $\mathbb{U}_q^{\text{v}}(\hat{sl}_s)$. Note that $E_i$, $F_i$, $K_i$, $i \in I$, are Chevalley generators of $U_q(sl_s)$.
Let $M$ be a right $\mathfrak{H}_\ell$-module. From Theorem 3.2, $\mathcal{F}(M)$ is a $\mathbb{U}_q(\widehat{\mathfrak{sl}_\ell})$-module such that

$$E_0(w \otimes v) = \sum_{j=1}^\ell w Y_j^{-1} \otimes q_{\ell,j} v, \quad F_0(w \otimes v) = s_\kappa \sum_{j=1}^\ell w Y_j \otimes e_{\ell,j} v, \quad K_0(w \otimes v) = w \otimes (k_0^{-1})^{s_\ell} v,$$

where the action of $A_i$, for $i \in I$ and $A = E,F,K$, is as in Theorem 3.1.

Recall $\zeta$ from Definition 2.1 and $q_1 = dq^{-1}$ from (2.10). Our main result is the toroidal super Schur-Weyl duality.

**Theorem 3.3.** If $\zeta = q^{n-m}_1$ and $\kappa \geq 4$, then there exists a functor $\mathcal{F}$ from the category of right $\mathbb{H}_\ell$-modules to the category of integrable $\mathcal{E}_s$-modules with trivial central charge and of level $\ell$, defined as follows. If $M$ is a right $\mathbb{H}_\ell$-module, then $\mathcal{F}(M) = \mathcal{F}(M)$ as a $\mathbb{U}_q(\widehat{\mathfrak{sl}_\ell})$-module and the action of $E_0, F_0, K_0$ is given by

$$E_0(w \otimes v) = \sum_{j=1}^\ell w X_j \otimes q_{\ell,j} v, \quad F_0(w \otimes v) = s_\kappa \sum_{j=1}^\ell w X_j^{-1} \otimes e_{\ell,j} v, \quad K_0(w \otimes v) = w \otimes (k_0^{-1})^{s_\ell} v, \quad (3.2)$$

where $w \in M$ and $v \in V_s^{\otimes \ell}$. Moreover, if $\ell < \kappa - 2$, then the functor $\mathcal{F}$ is an equivalence of categories.

We shall prove the theorem in the next section. Before that, we make a few remarks which will be used later.

**Remark 3.4.** Since $q$ is not a root of unity, the $\mathbb{H}_\ell$-modules and integrable $\mathbb{U}_q(\mathfrak{sl}_\ell)$-modules are direct sums of finite-dimensional modules. (Note that in general the category of finite-dimensional $\mathbb{U}_q(\mathfrak{sl}_\ell)$-modules is not semisimple, however we restrict to the subcategory of polynomial modules only which is semisimple.) Therefore, if $\ell < mn + \kappa$, Theorem 3.1 implies indeed an equivalence between the category of $\mathbb{H}_\ell$-modules and the category of integrable $\mathbb{U}_q(\mathfrak{sl}_\ell)$-modules of level $\ell$. □

**Remark 3.5.** Similarly, if $q$ is generic and $\ell < \kappa$, then Theorem 3.2 gives an equivalence between the category of $\mathbb{H}_\ell$-modules and the category of integrable $\mathbb{U}_q(\mathfrak{sl}_\ell)$-modules with trivial central charge and of level $\ell$. □

## 4. Proof of the main result

In this section, we prove that the $\mathbb{U}_q(\widehat{\mathfrak{sl}_\ell})$-action and the action of $E_0, F_0, K_0$ on $\mathcal{F}(M)$ extend to an $\mathcal{E}_s$-module structure on $\mathcal{F}(M)$. Moreover, the resulting $\mathcal{E}_s$-module $\mathcal{F}(M)$ is integrable with trivial central charge and of level $\ell$. Finally, we show that the functor $\mathcal{F}$ is an equivalence of categories if $\ell < \kappa - 2$.

### 4.1. Explicit action of vertical subalgebra

Clearly, any vector $w \otimes v \in \mathcal{F}(M)$ can be written as $\sum_j w_j \otimes v_j$ summed over non-decreasing $j$, where $j$ is an $\ell$-tuple of integers from $\{0, \kappa\}$ and $w_j \in M$. Hence it suffices for us to concentrate on $v$ of the form $v_j$ for non-decreasing $j$.

We need the explicit action of Drinfeld currents of $\mathbb{U}_q(\widehat{\mathfrak{sl}_\ell})$ on $\mathcal{F}(M)$ which follows directly from Proposition 2.9.

**Corollary 4.1.** If $j$ is non-decreasing, we have the following in $\mathcal{F}(M)$,

$$E_i(z)(w \otimes v_j) = \sum_{r=a_2+1}^{a_3} i_\bullet(i,r,j \backslash z)w : \left[ \delta(q_1^{\mu_a(i)} Y_\ell z) \prod_{p=r+1}^{a_3} \psi_{s_{i+1}} (q_1^{\mu_a(i)} Y_p z) \right]^{+} \otimes v_{j^2}.$$
Proof. Comparing \((2.8)\) with \((3.1)\) and noting the shifts in the vertical homomorphism, the statement follows from Proposition 2.9. \(\square\)

4.2. An important proposition. We shall define the action of the series \(E_0(z), F_0(z), K_0^{±}(z)\) on \(\mathcal{F}(M)\) in Section 4.3. To this end, we need the following linear map and its properties.

Let \(\Psi_s : M \otimes_{\mathbb{H}_{\ell}} \mathcal{V}_{\tau_s}^{\otimes \ell} \to M \otimes_{\mathbb{H}_{\ell}} \mathcal{V}_{\tau_s}^{\otimes \ell}\) be the linear map defined by

\[
\Psi_s(w \otimes v_j) = w X_1^{-\delta_{j_1,\kappa}} X_2^{-\delta_{j_2,\kappa}} \cdots X_\ell^{-\delta_{j_\ell,\kappa}} \otimes v_{j_1+1} \otimes v_{j_2+1} \otimes \cdots \otimes v_{j_\ell+1},
\]

for any \(\ell\)-tuple \(j = (j_1, \ldots, j_\ell)\), \(w \in M\). Here by convention \(v_{k+1} = v_1\). Note that the tensor product of vector representations in the target space is for superalgebras associated to the parity sequence \(\tau_s\). In particular, we have \(|v_j|\) in \(\mathcal{V}_s\) coincides with \(|v_{j+1}|\) in \(\mathcal{V}_{\tau_s}\), and \(\Psi_s\) is an even linear map.

**Lemma 4.2.** The linear map \(\Psi_s\) is well-defined.

Proof. It reduces to show that

\[
\Psi_s(w \otimes T_i v_j) = \Psi_s(w T_i \otimes v_j)
\]

for all \(1 \leq i < \ell\). It suffices to show it for the case of \(\ell = 2\).

We have four situations.

1. If \(j_1 \neq \kappa\) and \(j_2 \neq \kappa\), this is obvious.
2. If \(j_1 = \kappa\) and \(j_2 \neq \kappa\), one uses

\[
T_1 X_1^{-1} = (q^2 - 1) X_1^{-1} + q^2 T_1^{-1} X_1^{-1} = (q^2 - 1) X_1^{-1} + X_2^{-1} T_1,
\]

which is obtained from \((T_1 + 1)(T_1 - q^2) = 0\) and \(T_1 X_1 T_1 = q^2 X_2\).
3. If \(j_1 \neq \kappa\) and \(j_2 = \kappa\), this is clear from \(T_1 X_1 T_1 = q^2 X_2\).
4. If \(j_1 = j_2 = \kappa\), it follows from the fact that \(T_1\) commutes with \(X_1^{-1} X_2^{-1}\). \(\square\)

We follow the main idea of [VV96]. Recall that for \(s = (s_1, \ldots, s_\kappa)\), we have \(\tau s = (s_\kappa, s_1, \ldots, s_{\kappa-1})\) and the superalgebra isomorphism \(\tilde{\tau}_s : \mathcal{E}_s \to \mathcal{E}_{\tau s}\), see (2.26).

For \(r \in \mathbb{Z}_{>0}\), define

\[
\Psi_s^r := \Psi_{s^{r-1}} \circ \Psi_{s^{r-2}} \circ \cdots \circ \Psi_s \circ \Psi_s, \quad \Psi_s^{-r} := (\Psi_s^r)^{-1}. \tag{4.1}
\]

We also use the superscript \(s\) to distinguish generators from \(\mathcal{E}_s\) (also other notations) for various \(s\). The following proposition is crucial in the proof of Theorem 3.3.

**Proposition 4.3.** For \(1 < i < \kappa\), we have the following identities in \(\text{End}(M \otimes_{\mathbb{H}_{\ell}} \mathcal{V}_{\tau_s}^{\otimes \ell})\),

\[
\begin{align*}
\Psi_{s}^{-1} &\circ E_{i}^{\tau s}(z) \circ \Psi_{s} = E_{i}^{s-1}(z q_1^{s_\kappa}) , & \Psi_{s}^{-2} &\circ E_{i}^{\tau s}(z) \circ \Psi_{s} = E_{i}^{s-2}(z q_1^{n-m+s_{\kappa-1}+s_\kappa}) , \\
\Psi_{s}^{-1} &\circ F_{i}^{\tau s}(z) \circ \Psi_{s} = F_{i}^{s-1}(z q_1^{s_\kappa}) , & \Psi_{s}^{-2} &\circ F_{i}^{\tau s}(z) \circ \Psi_{s} = F_{i}^{s-2}(z q_1^{n-m+s_{\kappa-1}+s_\kappa}) , \\
\Psi_{s}^{-1} &\circ K_{i}^{\pm,\tau s}(z) \circ \Psi_{s} = K_{i}^{\pm,s-1}(z q_1^{s_\kappa}) , & \Psi_{s}^{-2} &\circ K_{i}^{\pm,\tau s}(z) \circ \Psi_{s} = K_{i}^{\pm,s-2}(z q_1^{n-m+s_{\kappa-1}+s_\kappa}) ,
\end{align*}
\]
Proof. We only show identities in the first line. The rests are similar.

We start with the first one. If $1 < i < \kappa$, it suffices to show that the action of $E_{i}^{\tau s}(z) \circ \Psi_{s}$ and $\Psi_{s} \circ E_{i-1}^{\tau s}(zq_{1}^{s})$ on $w \otimes v_{j}^{s}$ coincides for $w \in M$ and non-decreasing $\ell$-tuple $j$. Put

$$j^{-1}(i-1) = (a_{1}, a_{2}), \quad j^{-1}(i) = (a_{2}, a_{3}), \quad j^{-1}(\kappa) = (b, \ell),$$

$$j_{1} = (j_{1} + 1, \ldots, j_{b} + 1), \quad j_{2} = (1, \ldots, 1, j_{1} + 1, \ldots, j_{b} + 1).$$

Then we have

$$j_{2}^{-1}(i) = (\ell - b + a_{1}, \ell - b + a_{2}), \quad j_{2}^{-1}(i + 1) = (\ell - b + a_{2}, \ell - b + a_{3}).$$

Recall $T_{j,i}$ from (2.1) and $|j_{i}|$ from (2.9). Set

$$R_{b} = (-1)^{(\ell - b + a_{3})(|j_{b+1}|q^{b}(b - \ell)T_{b+1,2} \cdots T_{\ell-1,\ell - b}.$$}

Here and below, the notation of parity is always the one induced from $s$.

On one hand, by Corollary 4.1, we have

$$E^{\tau s}_{i}(z) \circ \Psi_{s}(w \otimes v_{j}^{s}) = E^{\tau s}_{i}(z)(w X_{b}^{-1} X_{b+1}^{-1} \cdots X_{\ell}^{-1} \otimes v_{j}^{s}) = E^{\tau s}_{i}(z)(w X_{b}^{-1} X_{b+1}^{-1} \cdots X_{\ell}^{-1} R_{b} \otimes v_{j}^{s})$$

$$= \sum_{r = \ell - b + a_{3} + 1}^{\ell - b + a_{3}} \ell_{r_{s}}(i, r; j_{2}) w X_{b}^{-1} X_{b+1}^{-1} \cdots X_{\ell}^{-1} R_{b} : \left[ \delta(q_{1}^{\mu r_{s}(i)} Y_{r} z) \prod_{p = r+1}^{a_{1}} \psi_{s_{p}}(q_{1}^{\mu r_{s}(i)} Y_{p} z) \right] : \otimes v_{j_{2}}^{s}.$$

On the other hand, note that $\mu_{r_{s}(i)} = s_{\kappa} + \mu_{s}(i - 1)$, we have

$$\Psi_{s} \circ E^{\tau s}_{i-1}(zq_{1}^{s}) (w \otimes v_{j}^{s}) = \Psi_{s}(w \otimes v_{j}^{s}) = \Psi_{s}(s_{\kappa} + \mu_{s}(i - 1), w \otimes v_{j}^{s})$$

$$= \Psi_{s}(s_{\kappa} + \mu_{s}(i - 1), w \otimes v_{j}^{s}) : \left[ \delta(q_{1}^{\mu r_{s}(i)} Y_{r} z) \prod_{p = r+1}^{a_{1}} \psi_{s_{p}}(q_{1}^{\mu r_{s}(i)} Y_{p} z) \right] : \otimes v_{j_{2}}^{s}.$$

where in the last equality, we used

$$\ell_{r_{s}}(i, r + \ell - b; j_{2}) = (-1)^{(\ell - b)\nu_{i}^{s}(|j_{b+1}| - \nu_{i}^{s})} \ell_{s_{\kappa} + \mu_{s}(i - 1), r; j_{2}}$$

which follows from that the parity of $|i - 1|$ is the same as that of $|\nu_{i}^{s} - |\nu_{i}^{s}||.$

Recall $P_{b} = Q_{\ell-\beta, \ell-\gamma} \cdots Q_{2b+1, \ell} T_{b}$ from (2.2). It follows from Lemma 2.3 that

$$P_{b} : \left[ \delta(q_{1}^{\mu r_{s}(i)} Y_{r} z) \prod_{p = r+1}^{a_{1}} \psi_{s_{p}}(q_{1}^{\mu r_{s}(i)} Y_{p} z) \right] : = P_{b}^{-1} = : \left[ \delta(q_{1}^{\mu r_{s}(i)} Y_{-b+r+\beta} z) \prod_{p = r+1}^{a_{1}} \psi_{s_{p}}(q_{1}^{\mu r_{s}(i)} Y_{-b+p+\beta} z) \right] : .$$

Since

$$R_{b}^{-1} X_{b} \cdots X_{b+1} = (-1)^{(\ell - b)\nu_{i}^{s}(|j_{b+1}| - |\nu_{i}^{s}||)} P_{b},$$

we conclude from the above equations that $E^{\tau s}_{i}(z) \circ \Psi_{s}(w \otimes v_{j}^{s}) = \Psi_{s} \circ E^{\tau s}_{i-1}(zq_{1}^{s})(w \otimes v_{j}^{s})$ and hence

$$\Psi_{s} \circ E^{\tau s}_{i-1}(zq_{1}^{s}) = E^{\tau s}_{i-1}(zq_{1}^{s})$$

in $\text{End}(M \otimes_{H_{i}} V_{s}^{\otimes \ell})$ for $1 < i < \kappa.$
We then consider the second one. Set \( \mathbf{j}^{-1}(\kappa - 1) = (a, b), \mathbf{j}^{-1}(\kappa) = (b, \ell) \), and
\[
\mathbf{j}_1 = (1, \ldots, 1, 2, \ldots, 2, j_1 + 2, \ldots, j_a + 2), \quad \mathbf{j}_2 = (j_1 + 2, \ldots, j_a + 2, 1, \ldots, 1, 2, 2) 
\]
where
\[
\mathbf{j}_1^{-1}(1) = (0, b - a), \quad \mathbf{j}_1^{-1}(2) = (b - a, \ell - a), \quad \mathbf{j}_2^{-1}(1) = (a, b), \quad \mathbf{j}_2^{-1}(2) = (b, \ell). 
\]
Define \( R_a = (-1)^{((b-a)|\nu_s^a|+(\ell-b)|\nu_s^b|)|\mathbf{j}_a+1|}q^{α(a-\ell)}T_{a+1}T_a \cdots T_{\ell-1,\ell-α} \). We have
\[
E_1^{\tau_s}(\zeta) \circ \Psi_{s}^2(w \otimes v_j^s) = E_1^{\tau_s}(\zeta)(wX_{a+1} \cdots X_\ell^{-1} R_α \otimes v_j^s) 
\]
\[
= \sum_{r = b-a+1}^{\ell-a} \ell_{\tau_s}(1, r; \mathbf{j})wX_{a+1} \cdots X_\ell^{-1} R_α : \left[ \delta(q_1^{s-1-\zeta Y_r z}) \prod_{p = r+1}^{\ell-a} \psi_{s_n}(q_1^{s-1-\zeta Y_{p} z}) \right]^{\dagger} \otimes v_j^{\tau_s} 
\]
Note that \( \mu_s(\kappa - 1) + n - m + s_{\kappa-1} + s_\kappa = s_{\kappa-1} \), we also have
\[
\Psi_{s}^2 \circ E_{\kappa-1}(zq_1^{-m+s_{\kappa-1}+s_\kappa})(w \otimes v_j^s) 
\]
\[
= \Psi_{s}^2\left( \sum_{r = b-a+1}^{\ell-a} \ell_a(\kappa - 1, r; \mathbf{j})w : \left[ \delta(q_1^{s-1-\zeta Y_r z}) \prod_{p = r+1}^{\ell-a} \psi_{s_n}(q_1^{s-1-\zeta Y_{p} z}) \right]^{\dagger} \otimes v_j^{\tau_s} \right) 
\]
\[
= \sum_{r = b-a+1}^{\ell-a} \ell_{\tau_s}(1, r; \mathbf{j})w : \left[ \delta(q_1^{s-1-\zeta Y_r z}) \prod_{p = r+1}^{\ell-a} \psi_{s_n}(q_1^{s-1-\zeta Y_{p} z}) \right]^{\dagger} X_{a+1} \cdots X_\ell^{-1} \otimes v_j^{\tau_s} 
\]
where in the last equality we used that the parity of \( |\kappa - 1| \) is the same as that of \( |v_\kappa^{a-1}| - |v_\kappa^{b-1}| \). The rest is similar to the last case by using Lemma 2.4. 

### 4.3. Proof of part 1

Now we define the action of the series \( E_0(z), F_0(z), K_0^\pm(z) \) on \( \mathcal{F}(M) = M \otimes \mathbb{E}_d \mathcal{V}^\otimes_\ell \) by
\[
E_0^s(z) = \Psi_{s}^{-1} \circ E_1^{\tau_s}(q_1^{-s-\kappa} z) \circ \Psi_{s}, \\
F_0^s(z) = \Psi_{s}^{-1} \circ F_1^{\tau_s}(q_1^{-s-\kappa} z) \circ \Psi_{s}, \\
K_0^\pm(z) = \Psi_{s}^{-1} \circ K_1^{\pm,\tau_s}(q_1^{-s-\kappa} z) \circ \Psi_{s}.
\]
If \( \zeta = q_1^{n-m} \), then it follows from Proposition 4.3 that we have
\[
E_i^s(z) = \Psi_{s}^{-1} \circ E_{i+1}^{\tau_s}(q_1^{-s-\kappa} z) \circ \Psi_{s}, \\
F_i^s(z) = \Psi_{s}^{-1} \circ F_{i+1}^{\tau_s}(q_1^{-s-\kappa} z) \circ \Psi_{s}, \\
K_i^{\pm}(z) = \Psi_{s}^{-1} \circ K_{i+1}^{\pm,\tau_s}(q_1^{-s-\kappa} z) \circ \Psi_{s},
\]
for all \( i \in \hat{I} \) and any desired \( s \). Here we read indices modulo \( \kappa \). Recall the isomorphism \( \tau_s \) defined in (2.26), then we have \( \Psi_{s} \circ E_i^s(z) \circ \Psi_{s}^{-1} = \tau_{s}(E_i^s(z)) \) for all \( i \in \hat{I} \).

Under this action, it is straightforward that (3.2) is true. Thus, if these operators do define an \( \mathcal{E}_s \)-action on \( \mathcal{F}(M) \), then the \( \mathcal{E}_s \)-module structure defined in Theorem 3.3 is also well-defined. Moreover, these two \( \mathcal{E}_s \)-module structures coincide. In particular, it is straightforward to check that \( K_0K_1 \cdots K_{\kappa-1}(w \otimes v) = w \otimes v \) for all \( w \in M \) and \( v \in \mathcal{V}^\otimes_\ell \). 

\[\square\]
Similarly to (4.1), we use the convention
\[ \tilde{\tau}_s^r := \tilde{\tau}_{\tau^{-1}s} \circ \cdots \circ \tilde{\tau}_{\tau s} \circ \tilde{\tau}_s, \]
where \( r \in \mathbb{Z}_{>0} \). To simplify the notation, we drop the dependence of \( s \) in \( \tilde{\tau}_s \) and \( \Psi_s \) but keep \( s \) in generating series.

**Proof of Theorem 3.3, part 1.** By Corollary 4.1, the operators \( E_i^q(z), F_i^q(z), K_i^{\pm q}(z) \in \text{End}(M \otimes_{\mathbb{H}_q} \mathcal{V}_{\mathfrak{s}}^{\otimes \ell}) \) satisfy relations in Definition 2.10 of \( \mathcal{E}_s \) for \( i \in I \). To verify all the other relations, it suffices to check the relations involving \( \tilde{\tau}^r(E_i^{\tau^{-r}s}(z)), \tilde{\tau}^r(F_i^{\tau^{-r}s}(z)), \tilde{\tau}^r(K_i^{\pm \tau^{-r}s}(z)) \) for \( r = 1, \ldots, \kappa - 1 \) and \( i \in I \) which are also the relations of \( \mathcal{E}_{\tau^{-r}s} \) for \( r \in I \). By construction, these operators are equal to \( \Psi^r \circ E_i^{\tau^{-r}s}(z) \circ \Psi^{-r}, \Psi^r \circ F_i^{\tau^{-r}s}(z) \circ \Psi^{-r}, \Psi^r \circ K_i^{\pm \tau^{-r}s}(z) \circ \Psi^{-r} \), respectively. Since by Corollary 4.1, \( E_i^{\tau^{-r}s}(z), F_i^{\tau^{-r}s}(z), K_i^{\pm \tau^{-r}s}(z) \) satisfy the relations of \( \mathcal{E}_{\tau^{-r}s} \) for \( r \in I \), we are done.

The fact that the \( \mathcal{E}_s \)-module \( \mathcal{F}(M) \) has trivial central charge is clear from Theorem 3.2 and Corollary 4.1. The integrability of \( \mathcal{F}(M) \) follows from that of \( \mathcal{V}_{\mathfrak{s}}^{\otimes \ell} \). Moreover, by Theorem 3.1, \( \mathcal{F}(M) \) is of level \( \ell \), see Remark 3.4.

**4.4. Proof of part 2.** Assume for the remainder of the proof that \( \ell < \kappa - 2 \). We show that \( \mathcal{F} \) is an equivalence of categories, which means we must prove that

1. (Surjectivity) every integrable \( \mathcal{E}_s \)-module \( M \) with trivial central charge and of level \( \ell \) is isomorphic to \( \mathcal{F}(M) \) for some \( \mathbb{H}_q \)-module \( M \);
2. (Fully faithfulness) \( \mathcal{F} \) is bijective on sets of morphisms.

We need the following useful lemma.

**Lemma 4.4.** (1) If \( v \) is a generator of \( \mathcal{V}_{\mathfrak{s}}^{\otimes \ell} \) as a module over \( \mathcal{U}_q(\mathfrak{s}\mathfrak{l}_s) \), then \( w \otimes v \in M \otimes_{\mathbb{H}_s} \mathcal{V}_{\mathfrak{s}}^{\otimes \ell} \) is zero if and only if \( w = 0 \).

2. If \( j_1, \ldots, j_\ell \in (0, \kappa] \) are pairwise distinct, then \( v_j \) is a generator of \( \mathcal{V}_{\mathfrak{s}}^{\otimes \ell} \) over \( \mathcal{U}_q(\mathfrak{s}\mathfrak{l}_s) \).

**Proof.** The first statement follows directly from Theorem 3.1 and Remark 3.4 while the second one is clear.

**Proof of Theorem 3.3 part 2.** Let \( M \) be an integrable \( \mathcal{E}_s \)-module with trivial central charge and of level \( \ell \). Then the restriction of \( M \) to \( \mathcal{U}_q(\mathfrak{s}\mathfrak{l}_s) \) is integrable with trivial central charge and of level \( \ell \). Since \( \mathbb{H}_q(1) \) is isomorphic to \( \mathbb{H}_q \), it follows from Theorem 3.2 and Remark 3.5 that there exists an \( \mathbb{H}_q(1) \)-module \( M(1) \) such that \( M \cong M(1) \otimes_{\mathbb{H}_q} \mathcal{V}_{\mathfrak{s}}^{\otimes \ell} \) as \( \mathcal{U}_q(\mathfrak{s}\mathfrak{l}_s) \)-modules. Similarly, there exists an \( \mathbb{H}_q(2) \)-module \( M(2) \) such that \( M \cong M(2) \otimes_{\mathbb{H}_q} \mathcal{V}_{\mathfrak{s}}^{\otimes \ell} \) as \( \mathcal{U}_q(\mathfrak{s}\mathfrak{l}_s) \)-modules. Moreover, these two modules \( M(1) \) and \( M(2) \) are isomorphic as \( \mathbb{H}_q \)-modules. Hence we denote them by \( M \).

The action of \( \mathcal{U}_q(\mathfrak{s}\mathfrak{l}_s) \) on \( M \otimes_{\mathbb{H}_q} \mathcal{V}_{\mathfrak{s}}^{\otimes \ell} \) is as in Corollary 4.1 while the action of \( E_0, F_0, K_0^\pm \) is as in (3.2). Note that the action of \( X_i, Y_j \in \mathbb{H}_q \) on \( M \) is given by the \( \mathbb{H}_q(2) \)-module and the \( \mathbb{H}_q(1) \)-module structure of \( M \), respectively. We would like to show that these two actions extend to an \( \mathbb{H}_q \)-module structure on \( M \). By Proposition 2.2, it suffices to show that for any \( w \in M \), we have

\[ wQY_{i-1}Q^{-1} = wY_i \quad (1 < i \leq \ell), \quad wQY_iQ^{-1} = \zeta wY_1, \]  
(4.2)

where \( Q = X_1T_{\ell-1} = X_1T_1 \cdots T_{\ell-1} \).

We first show \( wQY_{i-1}Q^{-1} = wY_i \) for \( 1 < i \leq \ell \). Fix \( 1 < i \leq \ell \). Set

\[ v = v_1 \otimes \cdots \otimes v_i \otimes v_{i+2} \otimes \cdots \otimes v_{\ell+1}, \quad \tilde{v} = v_2 \otimes \cdots \otimes v_i \otimes v_{i+2} \otimes \cdots \otimes v_{\ell+1} \otimes v_\kappa. \]
Then it is clear that
\[ E_0(w \otimes v) = (-1)^{|v_0|(|v_1| + \cdots + |v_{i+1}|)} q^{1-\ell} wQ \otimes \tilde{v}. \]

We have
\[ E_0 K^+_i(z)(w \otimes v) = E_0(w \psi^+_{s_i}(q^{\mu_s(i)} Y_i z) \otimes v) \]
\[ = (-1)^{|v_0|(|v_1| + \cdots + |v_{i+1}|)} q^{1-\ell} w \psi^+_{s_i}(q^{\mu_s(i)} Y_i z) Q \otimes \tilde{v} \]
and
\[ K^+_i(z) E_0(w \otimes v) = (-1)^{|v_0|(|v_1| + \cdots + |v_{i+1}|)} q^{1-\ell} K^+_i(z) wQ \otimes \tilde{v} \]
\[ = (-1)^{|v_0|(|v_1| + \cdots + |v_{i+1}|)} q^{1-\ell} wQ \psi^+_{s_i}(q^{\mu_s(i)} Y_{i-1} z) \otimes \tilde{v}. \]

Note that \( E_0 K^+_i(z) = K^+_i(z) E_0 \) and \( \tilde{v} \) is a generator of \( \mathcal{V}^{\otimes \ell}_s \) over \( \mathcal{U}_q(\mathfrak{sl}_\ell) \). It follows from Lemma 4.4 that \( w \psi^+_{s_i}(q^{\mu_s(i)} Y_i z) Q = wQ \psi^+_{s_i}(q^{\mu_s(i)} Y_{i-1} z) \). In particular, \( wQ Y_i Q^{-1} = wY_i \) for \( 1 < i \leq \ell \).

Then we show \( wQY_i Q^{-1} = \zeta wY_1 \). By taking the coefficients of \( zw \) in (2.14), we have
\[ d^{-m^s_{i,j}}(E_j K_{i-1} - q^{a^s_{i,j}} K_{i-1} E_j) K_i = (q^{\alpha^s_{i,j}} - q^{-\alpha^s_{i,j}}) E_{j-1}. \]

Note that \( m^s_{1,0} = -a^s_{1,0} = s_1 \) and \( m^s_{\kappa-1,0} = -s_{\kappa} \), we have
\[ s_1 d^{-s_1}(E_0 K_{1-1} - q^{-s_1} K_{1-1} E_0) K_1 = s_{\kappa} d^{\kappa_s}(E_0 K_{\kappa-1-1} - q^{-s_{\kappa}} K_{\kappa-1} E_0) K_{\kappa-1}. \]

Set \( v = v_1 \otimes v_3 \otimes v_4 \otimes \cdots \otimes v_{\ell+1} \) and \( \tilde{v} = v_3 \otimes v_4 \otimes \cdots \otimes v_{\ell+1} \otimes v_{\kappa} \). We have
\[ E_0(w \otimes v) = (-1)^{|v_0|(|v_1| + \cdots + |v_{i+1}|)} q^{1-\ell} wQ \otimes \tilde{v}. \]

A direct computation implies that
\[ s_1 d^{-s_1}(E_0 K_{1-1} - q^{-s_1} K_{1-1} E_0) K_1(w \otimes v) \]
\[ = s_1 d^{-s_1} E_0 K_{1-1} K_1(w \otimes v) = (-1)^{|v_0|(|v_3| + \cdots + |v_{i+1}|)} q^{1-\ell} s_1(q^{-s_1} - q^{s_1}) wY_1 Q \otimes \tilde{v}. \]

Similarly, one has
\[ s_{\kappa} d^{\kappa_s}(E_0 K_{\kappa-1} - q^{-s_{\kappa}} K_{\kappa-1} E_0) K_{\kappa-1}(w \otimes v) \]
\[ = -s_{\kappa} q^{\kappa_s} K_{\kappa-1} E_0 K_{\kappa-1}(w \otimes v) = (-1)^{|v_0|(|v_3| + \cdots + |v_{i+1}|)} q^{1-\ell} s_{\kappa}(q^{-s_{\kappa}} - q^{s_{\kappa}}) q^{m-n} wQY_{\ell} \otimes \tilde{v}, \]
where we used \( \ell + 1 < \kappa - 1 \) and \( s_1 + s_2 + \cdots + s_{\kappa} = m - n \). Since \( \tilde{v} \) is a generator of \( \mathcal{V}^{\otimes \ell}_s \) over \( \mathcal{U}_q(\mathfrak{sl}_\ell) \), it follows from Lemma 4.4 that \( wY_1 Q = q^{m-n} wQY_{\ell} \). Note that \( \zeta = q^{a_{\kappa-m}} \), we conclude that \( wQY_1 Q^{-1} = \zeta wY_1 \).

It remains to show that the functor \( \mathcal{F} \) is fully faithful. The fact that \( \mathcal{F} \) is injective on morphisms is clear from Theorem 3.2. To show \( \mathcal{F} \) is surjective on morphisms, one only needs to use Theorem 3.2 and the fact that \( \mathbb{H}_\ell \) is generated by the subalgebras \( \mathbb{H}_\ell^{(1)} \) and \( \mathbb{H}_\ell^{(2)} \). \( \square \)

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