Gauge Systems with Finite Chemical Potential in 2+1 Dimensions by Bosonization

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We present a bosonization method to study generic low energy behavior of gauge systems with finite chemical potential in 2+1 dimensions. Benefit from the existence of gap (e.g. Gribov gap) in gauge systems at low energy, the fermion fields can be explicitly bosonized by new gauge fields. When chemical potential of the gauge systems is introduced, we find that topological terms (such as Chern-Simons term in 2+1D) as constraints inevitably emerge at low energy. The fermion sign problem at finite chemical potential and its deep connection to the Chern-Simons theories are discussed. The Wilson’s criteria of confinement in pure gauge theories is generalized to finite chemical potential case. The chemical potential dependence of physical quantities at strong coupling are explicitly calculated, including the expectation value of the Wilson loop, the confining potential and the confined/deconfined transition temperature. The bosonization puts discussions on chiral symmetry breaking and confined/deconfined transition on an equal footing, so it is suitable for the study of the subtle interplay between them. We find that the chiral symmetry breaking is a necessary (not sufficient) condition for the confinement in 2+1D, and argue that the confined/deconfined phases are not characterized by any local symmetries but distinguished by their non-local topologies. The low energy modes of the strongly coupled gauge systems in a non-symmetry breaking phase is also discussed. The results of the paper can be widely applied to real strongly coupled gauge systems, e.g. high-temperature superconductor and quantum chromodynamical systems in 2+1 dimensions.

I. INTRODUCTION

Gauge system is the most fundamental model of our world, not only describing the basic interactions but also emergent structures in many body systems in condensed matter. However, when the gauge system is strongly coupled, standard perturbative methods fails. At present, there are no systematic , non-perturbative and analytic approaches to study these systems. In many cases, we rely on numerical simulations, e.g. quantum Monte Carlo (QMC) on lattice. The QMC on lattice is the only systematic method with sufficient numerical accuracy in calculating many thermodynamics quantities of strongly coupled pure gauge system, such as the quantum chromodynamics (QCD) system without dynamical quarks. However, the numerical method has its own limitations, the QMC has the notorious fermion sign problem [1, 2] when the pure gauge system are doped by fermions, or equivalently, the gauge system is at finite chemical potential. The problem makes the studying of properties of gauge systems at finite chemical potential inaccessible.

On the other hand, Landau’s theory of phase transition is developed for system at finite temperature while quantum effects are ignored. The finite chemical potential induced new phenomenon and phase transition are in essential due to quantum effects, which are conjectured has richer structures [3, 4] than the thermal (classical) phase transition. It seems very likely that new states of matter we have not encountered before would emerge in finite chemical potential region. The phase structure of gauge systems such as the QCD system [5, 6] and certain condensed matter systems [7] (e.g. Hubbard model at half-filling and Heisenberg spin system) at finite chemical potential is one of the most important and practical issues of quantum matter. Generic phase structures are qualitative common in these gauge system, e.g. the confined/deconfined phase transition, chiral symmetry breaking/restoring, high temperature superconducting phase transition are expected occur in the finite chemical potential region. It is gradually become a general believe that a theory of gauge system at finite chemical potential holds the key to understand the exotic new states of matter observed at finite chemical potential, e.g. the quark-gluon plasma in the QCD system [11, 12], the strange metal [13] and/or pseudo-gap [14] in cuprates. The central question is how these phenomena take place as a function of chemical potential in these gauge systems.

We suggest that the strategy to overcome the difficulty accessing the finite chemical potential region is to reformulate the systems in a more proper language. True enough, the fermionic and bosonic languages are just mathematical machineries that invented independently for their special use and for convenient in different realms. The idea of the paper is very transparent, if we can map a bosonic equivalent of a theory of fermions, then the gauge system at finite chemical potential coupling with fermion matter is able to identify with a new pure gauge system. This line of thought leads to the idea of bosonization [15]. The problems arouse in the introducing fermions are hence translated to problems of the well studied theories of pure gauge system. In 1+1 dimensions, the bosonization has been shown as a powerful technique in studying low energy behaviors of many strong correlated condensed matter systems [16, 17]. A lot of efforts have been spent in generalizing this method to higher dimensions, the explicit results are seem available in 2+1 dimensions. The bosonization in 2+1 dimensions has many discussions in literature e.g. [18, 21]. However, these bosonization only applies to systems without chemical potential. In this paper, we present
a new bosonization rule through a direct variable replacement which relates the massless fermion theories with the Chern-Simons theories. The Chern-Simons term is relevant term controlling the low energy behavior and the higher order contributions from dynamical Maxwell/Yang-Mills terms are considered as perturbations. Although the idea of bosonization is not new, the deep implications and consequences of such recipe to the problem of finite chemical potential of gauge systems has not been fully studied and investigated. Through the bosonization recipe there are several focuses in the paper, (i) how the bosonization impacts on the fermion sign problem at finite chemical potential, (ii) what the bosonization in general tells us about the confinement of gauge systems at finite chemical potential, (iii) what is the relation between the chiral symmetry breaking and the confinement, and (iv) what is the low energy modes of gauge systems in a phase without symmetry breaking at finite chemical potential.

The rest of the paper is organized as follows. We begin by the explicit formulation of the bosonization recipe and give dictionaries to the translation of terminologies between fermions and bosons. The completely bosonized actions for the Abelian and non-Abelian gauge theories are obtained in the section II. In section III, we discuss the fermion sign problem by the new language of the bosonization instead of fermions, and show its deep connection to the induced Chern-Simons theories. Based on a large energy gap of the effective gauge fields at strong coupling, we introduce a factorization to alleviate the sign problem. In section IV, we generalize the Wilson’s criteria of confinement in pure gauge theories to finite chemical potential. The chemical potential dependence of some physical quantities, including the expectation value of the Wilson loop, the confining potential and the confined/deconfined transition temperature, are explicitly calculated at strong coupling limit. In section V, we discuss the interplay between the chiral symmetry breaking and the confinement, based on the bosonized formalism of the chiral order parameter. The low energy modes of gauge systems in a non-symmetry breaking phase at finite chemical potential is discussed in section VI. We finally draw conclusions of the paper in section VI. In the last part of the paper, several appendices are given to prove the translation of terminologies between the fermionic and its bosonized languages.

II. BOSONIZATION IN 2+1 DIMENSIONS

The non-locality nature of fermion is largely responsible for the fermion sign problem. The basic idea of bosonization is that a non-local bosonic string encodes all relevant information of a fermion. In 1+1 dimension, bosonic scalar field is enough for the bosonization. But in 2+1 dimensions, the description of the non-local bosonic string requires at least vector fields. Following the bosonic-string-end-point picture of fermion, a fermion located at point $\vec{x}$ at given time $\tau$ is identical to a phase string connecting a spatial fixed point $\vec{x}_f$ (it is assumed unobservable, so it must be moved to negative infinity) to its location $\vec{x}$ along certain path defined on a 2 dimensional space-like plane at given time $\tau$,

$$\psi(\vec{x}, \tau) = \lim_{\vec{x}_f \to -\infty} e^{-i \int_{\vec{x}_f}^{\vec{x}} dy^i a_i(\vec{y}, \tau)},$$

where $a_i$ is a vector field with subscript $i = 1, 2$. We can not see the string itself, only the end point of the string, we can imagine that a fermion always attaches an unobservable string, when the position of the fermion changes, it brings along the attached string. The string or the path in the formula is arbitrary, such phase arbitrariness of $\psi$ reflects the fact that there are redundant degree of freedoms for $\psi$ and $a_i$. This property indicates that the $a_i$ field is essentially a gauge field, one specific path gives a specific gauge. The temporal component of the gauge field $a_0$, which is not included in Eq. (1), can be introduced by a gauge condition. Then space-like plane mentioned above does not necessarily be space-like, it can be any Lorentz rotated hyper-plane from the spatial one in the 2+1 dimensions spacetime. And the starting fixed point $\vec{x}_f$ at spatial negative infinity is now Lorentz rotated to a new starting fixed point that is not necessarily space-like, but a fixed point defined on the hyper-plane and then be moved to the negative infinity of the hyper-plane. Thus now the gauge field can be formally generalized to a covariant form, i.e. the subscript now is $i = 0, 1, 2$. We could write the fields in a covariant form $\psi(x)$ and $a_i(x), x = (\vec{x}, \tau)$, and the gauge transformations of them is given by

$$\psi(x) \to e^{i \varphi(x)} \psi(x), \quad a_i(x) \to a_i(x) + \partial_j \varphi(x).$$

(2)

For massless fermions, there is another independent chiral fermionic variable $\tilde{\psi}$ constructed as

$$\tilde{\psi}(\vec{x}, \tau) = \lim_{\vec{x}_f \to -\infty} e^{-i \int_{\vec{x}_f}^{\vec{x}} dy^i \tilde{a}_i(\vec{y}, \tau)},$$

in which $\tilde{a}_i$ denotes $\tilde{a}(k) \equiv a(-k)$. After performing the replacement of $\psi$ by the new field $a_i$, then all operators appearing in the theory become bosonic, there will be no fermions in our discussions any more. It is worth stressing that all the paths in the definitions start
from a fixed point $x_f$ (although it is unobserved at negative infinity) is very important in reproducing the fermionic statistics. We will show the proof that the fermionic anti-commutation relation of $\psi$ on equal time is automatically ensured in Appendix I. Sometimes in our formulas we do not write explicitly the starting point $x_f$ in the path integral and then take the limit $x_f \to -\infty$, but just writing the starting point of the path integral instead as $-\infty$, we must keep in mind that all the paths integral from negative infinity start from a negative infinite fixed point.

The bosonization recipe gives a dictionary translating the fermionic terminologies to their bosonic counterparts as follows.

|                | fermionic                      | bosonic                        |
|----------------|-------------------------------|-------------------------------|
| current $J_i$  | $\psi_i \gamma_i \psi$      | $\pm \frac{i}{\hbar} \epsilon_{ijk} \partial_j a_k$ |
| kinetic energy | $i \bar{\psi} \gamma_i \partial_i \psi$ | $\pm \frac{i}{\hbar} \epsilon_{ijk} A_i a_k$ |
| chemical potential $\mu J_0$ | $\mu \bar{\psi} \gamma_0 \psi$ | $\pm \frac{i}{\hbar} \epsilon_{ij} A_i a_j$ |
| source $A_i J_i$ | $A_i \bar{\psi} \gamma_i \psi$ | $\pm \frac{i}{\hbar} \epsilon_{ijk} A_i a_k$ |
| chiral density $\mathcal{O}_\pm$ | $\bar{\psi} (1 \pm \gamma_5) \psi$ | $M^2 \exp (\pm i \Phi)$ |
| mass term      | $m \bar{\psi} \psi$         | $mM^2 \cos \Phi$              |

The bosonization recipe can be formally generalized to the non-Abelian fermion, by replacing the Abelian gauge field $a_i$ by the non-Abelian gauge field $a_i = a_i^J t^J$. Similarly, when the path defined on a spatial hyper-plane, the bosonization is given by

$$
\psi(x, \tau) = \exp \left( -i \int_{-\infty}^{\infty} \int dx dx' a_i^J (\bar{\psi}_i(x), \tau) \right), \quad \bar{\psi}(x, \tau) = \exp \left( i \int_{-\infty}^{\infty} \int dx dx' a_i^J (\bar{\psi}_i(x), \tau) \right)
$$

where $t^J$ with $J = (1, 2, \ldots, N^2 - 1)$ are the generators of the $SU(N)$ gauge group. The gauge transformations of the covariant form of the fields are given by

$$
\psi(x) \to e^{i t^J \psi^i(x)} \psi(x), \quad a_i^J (x) \to a_i^J (x) + \partial_i \psi^J (x) + \theta^{IJK} a_i^J \phi^K (x),
$$

in which $\theta^{IJK}$ is the structure constant of the gauge group satisfying $[t^I, t^J] = i \theta^{IJK} t^K$. Since there are $N^2 - 1$ gauge-invariant conserved currents, the number of chemical potentials $\mu^J$ is generalized to $N^2 - 1$ correspondingly. The dictionary for the non-Abelian fermions is as follows.

|                | fermionic                      | bosonic                        |
|----------------|-------------------------------|-------------------------------|
| current $J_i^I$ | $\psi \gamma_i \psi$      | $\pm \frac{i}{\hbar} \epsilon_{ijk} (\partial_j a_k + \theta^{IJK} a_j^I a_k^K)$ |
| kinetic energy | $i \bar{\psi} \gamma_i \partial_i \psi$ | $\pm \frac{i}{\hbar} \epsilon_{ijk} (a_i^J \partial_j a_k^K + \theta^{IJK} a_i^J a_j^I a_k^K)$ |
| chemical potential $\mu J_0^I$ | $\mu \bar{\psi} \gamma_0 \psi$ | $\pm \frac{i}{\hbar} \epsilon_{ij} \partial_j a_i^I a_k^K$ |
| source $A_i J_i^I$ | $A_i \bar{\psi} \gamma_i \psi$ | $\pm \frac{i}{\hbar} \epsilon_{ijk} A_i^J (\partial_j a_i^I + \theta^{IJK} a_j^I a_k^K)$ |
| chiral density $\mathcal{O}_\pm$ | $\bar{\psi} (1 \pm \gamma_5) \psi$ | $M^2 \exp (\pm i \Phi)$ |
| mass term      | $m \bar{\psi} \psi$         | $mM^2 \cos \Phi$              |

The detail proofs of the two dictionaries will be given in the appendix-II to -IV. We can see that the current is identically conserved, since the current is just $J = \pm \ast f$ and $d \ast f = 0$. The significant features of the above translations are mathematically transparent. The first fundamental feature is that the bosonized theory is very closely related to Chern-Simons and/or topological field theory. The kinetic energy term of fermions directly takes the Chern-Simons form, and the source term which describes the interaction between fermion and gauge field is of the mutual-Chern-Simons and/or topological field theory. The kinetic energy term of fermions directly takes the Chern-Simons form, and the source term which describes the interaction between fermion and gauge field is of the mutual-Chern-Simons and/or topological field theory. The kinetic energy term of fermions directly takes the Chern-Simons form, and the source term which describes the interaction between fermion and gauge field is of the mutual-Chern-Simons and/or topological field theory. The kinetic energy term of fermions directly takes the Chern-Simons form, and the source term which describes the interaction between fermion and gauge field is of the mutual-Chern-Simons and/or topological field theory.

We also observe in these dictionaries that the chiral density of fermions, from the bosonic perspective, measures the flux density of the $a$ field, where the $\Phi (x) = \lim_{\Sigma \to 0} \int_{\Sigma} dx dx' f_{ij}$ is the flux density (the non-Abelian case is just replacing $f_{ij}$ by $f_{ij}^0$). Here the energy scale $M^2 = \frac{1}{\lambda - \frac{\kappa}{\lambda}}$ is explicitly introduced to the system which sets the scale for the chiral symmetry breaking. This energy gap is related to the Gribov gap [22] at low energy which inevitably exists in quantum gauge theories especially in non-Abelian cases. As we will see that this energy gap $M$ is responsible for a ordered chiral symmetry breaking and protecting a low energy incompressible fluid-like state at disorder, both of which are common low energy features in a strongly coupled gauge system at finite chemical potential. This reformulation of the chiral density (and the corresponding chiral order parameter) provides a possibility to discuss the relation between the chiral symmetry breaking and confined/deconfined phase transition on an equal footing in section V.
For the practical interest, e.g. the QCD system and many strong correlated electron systems in 2+1 dimensions, our main focus in this paper is the massless fermions. By the construction, the action of a gauge theory with massless fermion matters can be straightforwardly rewritten as a pure bosonic action without any fermion fields,

\[
Z = \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left( -\int d^3x \bar{\psi} \left[ i\gamma_i \left( \partial_i - iA_i \right) + \mu \gamma_0 \right] \psi - S[A] \right) 
\]

\[
= \int \mathcal{D}A \mathcal{D}a \left( \prod_x \bar{\psi}_x \psi_x \right) \exp \left( -S_0[a, A] \right),
\]

\[
= M^2 \int \mathcal{D}A \mathcal{D}a \left( \prod_x \cos \Phi_x \right) \exp \left( -S_0[a, A] \right)
\]

\[
= M^2 \int \mathcal{D}A \mathcal{D}a \exp \left( -S_0[a, A] + \sum_x \ln \cos \Phi_x \right)
\]

\[
= M^2 \int \mathcal{D}A \mathcal{D}a \exp \left( -S_0[a, A] + \sum_x \ln \left( 1 - \frac{1}{2} \Phi^2 + ... \right) \right)
\]

\[
= M^2 \int \mathcal{D}A \mathcal{D}a \exp \left( -S_0[a, A] - \frac{1}{8\pi^2 M} \int d^3x f_{ij}f_{ij} + ... \right)
\]

\[
= M^2 \int \mathcal{D}A \mathcal{D}a \exp \left( -S_{eff}[a, A] \right)
\]

with

\[
S_{eff} = \pm \frac{1}{2\pi i} \int d^3x \left[ \epsilon_{ijk} \left( a_i \partial_j a_k + A_i \partial_j a_k + \mu \epsilon_{ij} \partial_a a_j \right) + S[A] + \frac{1}{8\pi^2 M} \int d^3x \left( \partial a_j \right)^2, \right.
\]

in which \( S[A] \) is the action of the original pure gauge fields \( S[A] = -\frac{1}{8\rho} F_{ij} (A) F_{ij} (A) \), \( \mu \) is the chemical potential of the fermion, and the last term comes from the change of the functional integral measure (i.e. \( \prod_x \cos \Phi_x \)) expanding in powers of the small parameter \( 1/M \). The mass gap \( M \) suppresses the Maxwell term of \( a \) field, thus leading to a protection of the topological behavior of the \( a \) field due to the Chern-Simons term. This gap originated from the requirement of a symmetry breaking of the system, however, it is striking that such requirement directly implies the existence of a topological order beyond the ordered symmetry breaking phase, in which the gap \( M \) will protect a fluid-like state as its low energy modes described by the Chern-Simons term. Therefore, the effective action Eq. (7) is at the vicinity of the IR fixed point far below the scale \( M \), thus describing the low energy long wavelength behavior of the gauge systems.

The non-Abelian version of Eq. (7) is

\[
S_{eff}^N = \pm \frac{1}{2\pi i} \text{tr} \left( d^3x \left[ \epsilon_{ijk} \left( a_i^l f_{jk}^l + \frac{2}{3} \theta^{IJK} a_i^I a_j^J a_k^K + A_i^l f_{jk}^l \right) + \epsilon_{ij} \mu \right] + S[A] + \frac{1}{8\pi^2 M} \text{tr} \left( d^3x f_{ij}^l f_{ij}^l \right), \right.
\]

where \( f_{ij}^l = \frac{1}{2} \left( \partial_i a_j^l + \theta^{IJK} a_i^I a_j^J a_k^K \right) \) is the curvature of \( a_i^l \).

If we consider that fermions are massive, it is equivalent to introduce the extra Lagrangian terms,

\[
m M^2 \text{tr} \cos \Phi = m M^2 \text{tr} \left( 1 - \frac{1}{2} \phi^2 + ... \right)
\]

\[
= m M^2 - \frac{m}{8\pi^2 M^2} \text{tr} f_{ij} f_{ij} + ...
\]

in which the first constant term is irrelevant, and the second term is the conventional Maxwell/Yang-Mills term of \( a_i \) fields.

The similar bosonized effective actions without chemical potential are also found in literature, however, what we focus in the paper is at their finite chemical potential regions. An important observation to these effective actions is that the chemical potential just formally shifts the temporal component of the gauge field at least in the Abelian case [23]. It is obvious that since there is no dynamics of the \( a_0 \) or \( a_0^l \), it plays the role of a Lagrange multiplier in the action which keeps the particle number \( J_0 \) or \( J_0^l \) conserved, similar with the chemical potential \( \mu \) or \( \mu^l \). Therefore, we conclude that when chemical potential is introduced in the gauge systems, the Chern-Simons term inevitably emerges as a constraint, when fermions are bosonized. Only when the fermions are reformulated by bosons this point can
be easily seen. This Chern-Simons term inevitably presenting at finite chemical potential is protected by the finite gap \( M \) and strong coupling of Yang-Mills term, thus leading to a low energy topological dominant theory at finite chemical potential, while the dynamical Maxwell/Yang-Mills part plays the role of perturbation.

One may caution that the Chern-Simons term although gauge invariant, usually breaks the parity and time reversal symmetry. It is worth emphasizing that there is an arbitrariness for prefactor \( \pm i \) of the Chern-Simons action, which comes from the arbitrariness of the phase in the anti-commutation relation \((ab - e^{\pm i\pi}ba) = 0\) of the fermions (see the appendix I) and the bosonized form of the current (see the appendix II). The arbitrariness makes the theory does not break the parity and time reversal symmetry as it should, when the Chern-Simons term changes sign under parity \( \epsilon_{ijk} a_i \partial_j a_k \rightarrow -\epsilon_{ijk} a_i \partial_j a_k \), and it \( \rightarrow -it \) under time reversal. This property indicates the particle/anti-particle symmetry \( (\mu \leftrightarrow -\mu) \) in the phase diagram of the system. By convention, we only pick up the plus sign in our following discussions. A salience change of the new actions concerns the center symmetry \( Z_N \) of the gauge symmetry \( SU(N) \). The bosonization makes the dynamic matter fields be adjoint representations, so all the fields are \( Z_N \)-invariant, thus the center symmetry does not break, which will discuss in Section IV. To summarize, the bosonized effective action preserves all symmetries of the origin action Eq.\( \text{(10)} \), and hence Eq.\( \text{(7)} \) and/or Eq.\( \text{(8)} \) serve a good starting point for studying the gauge system at finite chemical potential in \( 2+1 \) dimensions.

### III. SIGN STRUCTURE AT FINITE CHEMICAL POTENTIAL

In the quantum mechanical picture, the fermion sign problem arises due to the fact that the many particle wave-function changes sign when any two fermions are interchanged. As a consequence, considering a fermion in a fermionic density bath and travels around a closed spatial trajectory \( C \). In the path integral formalism, every time a fermion crossing (equivalent to exchange) an environment fermion gives an extra minus sign to the statistic weight of the path integral, beside the conventional part \( Z_0[C] \) describing the zero density contribution. So the partition function of the many fermions system is given by \( Z = \sum_C \epsilon \cdot (-1)^N[C] Z_0[C] \), where \( N[C] \) denotes the total number of the fermions on the trajectory \( C \) that needs to cross. The partition function sums over all possible closed loop \( C \) and takes an extra sign structure. Compared with the zero density partition function \( Z = \sum_C Z_0[C] \), the statistic weight \( (-1)^N[C] \) as a functional of the loop \( C \) in the summation is highly oscillatory. This property makes it hard to evaluate the partition function numerically, since the sign or the phase lead to dramatic cancellations which makes statistical errors scale exponentially when the system approaches to the thermodynamic limit.

In the field theoretical approach to the strongly coupled many particle system, the fermion density is characterized by the value of the fermion chemical potential or chemical potential \( \mu \). When we integrate out the fermions in the partition function, it gives a statistic weight \( \det D(\mu, A) \) to the path integral, where

\[
D(\mu, A) = i \gamma_\mu (\partial_\mu - i A_\mu) + \mu \gamma_0 \tag{10}
\]

is the Dirac operator. The notorious sign problem arises when the fermion determinant is not positive but in general be a complex number with non-trivial phase \( \det D \equiv |\det D| e^{i\theta} \) at finite chemical potential. Several proposals have been trying to at least partially solve the sign problem \( \text{[23] [27]} \).

The bosonized formulation provides us a new perspective to the fermion sign problem in \( 2+1 \) dimensions. Take the Abelian case as an example. In Eq.\( \text{(7)} \), we note that the action is quadratic in \( a_i \), so the bosonized matter fields \( a_i \) can be integrated out. It is equivalent to integrate out the matter fields, we obtain

\[
Z = \mathcal{N} \int DA \exp \left( \int d^3 x \frac{i}{8\pi} \epsilon_{ijk} (A_i + \mu \delta_{0i}) \partial_j A_k - \frac{1}{32\pi^2 M} F_{ij}(A)^2 \right) - S[A] \right). \tag{11}
\]

One find that there is a new emergent Chern-Simons term for \( A_i \) beside the original \( S[A] \), which plays the role of the fermion determinant,

\[
\det D(\mu, A) = \exp \left( -\frac{1}{32\pi^2 M} \int d^3 x F_{ij}(A)^2 \right) \exp \left( \frac{i}{8\pi} \int d^3 x \epsilon_{ijk} (A_i + \mu \delta_{0i}) \partial_j A_k \right). \tag{12}
\]

Note that under homotopically non-trivial large gauge transformation of \( A \), the phase of the fermion determinant changes as \( \det (\theta + A) \rightarrow (-1)^n \det (\theta + A) \) \( \text{[28] [29]} \) (\( n \) being an integer), which can be absorbed into the sign arbitrariness in front of the Chern-Simons action, and hence free from the anomaly. Therefore, the Chern-Simons term consistently identifies with the phase of the fermion determinant. While, the first exponent \( \exp \left( -\frac{1}{32\pi^2 M} \int d^3 x F_{ij}^2 \right) \) is real and it equals to the absolute value of the fermion determinant, which renormalizes the gauge fields term \( S[A] \), by \( \frac{1}{g_M^2} = \frac{1}{g^2} + \frac{1}{8\pi^2 M} \).
In contrast to the original gauge fields term \( S[A] \) which is metric dependent (it is real in Euclidean spacetime while imaginary in Minkowski spacetime), the Chern-Simons term is metric independent. This is because why Chern-Simons term is called topological. The consequence of this topological property is that the Euclidean continuation does not pick up any factor of \( i \), since when we perform the Wick rotation \( t \rightarrow it \), the time component of the gauge field changes as \( A_0 \rightarrow -iA_0 \). Therefore, the action always has an intrinsic imaginary part, no matter in Euclidean or Minkowski spacetime. We come to a crucial conclusion that the non-trivial phase capturing the essential of the sign problem is topological origin in \( 2+1 \) dimensions. This connection between the notion of the zero temperature finite chemical potential physics and the topology gives a further support to the idea of topological order/phase that beyond the Landau’s theories of finite temperature phase transition.

Only for special cases the theory could be free from the sign problem. When we choose the temporal-axial gauge \((A_0 = 0)\) which is trivial for the reason that it is equivalent to a pure gauge system without chemical potential. Or when \( \mu \) is very large (compared with \( A_0 \)) to be dominant, together with performing an analytic continuation of \( \mu \) from real to imaginary, since now \( \frac{1}{\mu} |\mu| \epsilon_{ij} \partial_i A_j \) is positive \([30,32]\). However, in general, this method has not led to any improvement for the sign problem, the non-trivial phase coming from fermionic statistic is still there, we just transform the non-trivial phase into a generally under-controlled topological gauge fields structure through bosonization. The non-trivial phase in the statistic weight of the path integral now has a complete new physical meaning, connecting to Chern-Simons term, so we could manipulate the integral according to the properties of the Chern-Simons theory, the sign problem in this sense is alleviated. Let us consider an index usually measuring the badness of the sign problem

\[
\langle S \rangle = \langle e^{i\theta} \rangle = Z^{-1} \int DAe^{iS_{CS}}e^{-S[A]} = \langle e^{iS_{CS}} \rangle ,
\]

where \( e^{i\theta} \) is defined as \( \det D \equiv |\det D| e^{i\theta} \). If it is much less than one then the sign problem is thought severe. At high temperature, the system is \( S[A] \) dominant, so the sign problem is not severe, since \( \langle S \rangle \sim Z^{-1} \int DAe^{-S} \approx 1 \). And since the Chern-Simons action is an infrared fixed point action and becomes dominant only at low temperature, the expectation value gives \( \langle S \rangle \sim Z^{-1} \int DAe^{iS_{CS}} \), which is a topological invariant only depending on the topological of the 3-manifold, not on the detail spatial size or temperature (temporal size) of the system. In particular, for certain 3-manifold this quantity may not be small, e.g. \( M = S^2 \times S^1 \), \( \langle S \rangle \sim Z^{-1} \int DAe^{iS_{CS}} \approx 1 \) for any gauge group. The sign problem related to the Chern-Simons term is irrelevant to the thermodynamics property we are interested in.

On the other hand, the low energy behavior governed by the Chern-Simons term is protected by an energy gap proportional to the square of the gauge coupling and \( M \), so it is well protected especially at strong coupling. Therefore, we can first safely integrate out the infrared part which eliminates most of the sign problem, and have a factorization into a topological (size and temperature independent) part and a dynamic (size and temperature dependent, free from sign problem) part,

\[
\langle O \rangle = Z^{-1} \int DAO(dA)e^{iS_{CS}[A]}e^{-S[dA]}
\]

\[
= Z^{-1} \int D(A_{IR} + B)O(dA_{IR} + dB)e^{iS_{CS}[A_{IR}+B]}e^{-S[dA_{IR}+dB]}
\]

\[
\approx Z^{-1}e^{iS_{CS}[A_{IR}]}, \int_{|k|<\Lambda} DB(k)e^{iS_{CS}[B]} \int_{|k|>\Lambda} DB(k)O(dB)e^{-S[dB]}
\]

\[
= \text{topological invariant} \times \int DAO(dA)e^{-S[dA]} \quad (14)
\]

where \( \Lambda = g^2/\pi \) is the energy gap of \( A \), \( A_{IR} \) are a set of flat gauge connections \((dA_{IR} = 0)\) dominating the infrared behaviors. \( O(dA) \) is a gauge invariant thermodynamic observable which is thought temperature and/or other thermodynamic variables dependent and low energy topology irrelevant, so we formally put it into the dynamic part of the integral. In principle, if the observable can also be factorized according to the energy gap, it could contribute to the low energy part of the integral, e.g. the Wilson loop, which we will discuss in the next Section. The factorization is based on the well separated scales of the topological part and the high energy dynamic part. The stronger the gauge coupling, the larger the gap between these two part, the more exact the factorization. This factorization gives an approximation to hivve off the Chern-Simons term, which is relevant to the sign problem but irrelevant to the thermodynamic property. If we are interested in the thermodynamical behaviors of \( \langle O \rangle \), this approximation may be useful.
IV. WILSON’S CONFINEMENT CRITERIA AT FINITE CHEMICAL POTENTIAL

The most interesting physics happens in the finite chemical potential region of gauge systems. It is known that in 2+1 dimensions the pure gauge field theories only have one phase, i.e., confinement \[32,33\]. The deconfined transition occurs in the finite chemical potential regime. However, a theoretical problem is, if the fermion matter fields \( \psi \) in the fundamental representations are included, the system will lose its center invariance. Only when all the dynamical fields are center invariant, the Wilson loop serves as an order parameter to distinguish the confined and deconfined phases. As a consequence the Wilson loop can not be used anymore to characterize these two phases in this situation. On the other hand, the sign problem makes the numerical method accessing the finite chemical potential region very difficult. So we confront a difficulty how to study the confined and deconfined phase at finite chemical potential.

The bosonization recipe is a promising approach, at least partially alleviating the sign problem, to investigate the physics in the finite chemical potential region. First of all, it bridges a gauge field theory at finite chemical potential to a pure gauge field theory, and hence, numbers of existing beautiful results and well developed techniques for pure gauge system could be straightforwardly generalized to finite chemical potential. Second, in contrast to the fact that the matter fields \( \psi \) in fundamental representation are not \( Z_N \) invariant, the bosonized matter fields \( a \) in adjoint representations are \( Z_N \) invariant. Therefore, all fields in the effective theory are \( Z_N \) invariant. This property lays the foundation to the validity of Wilson loop being an order parameter, even when the dynamical matters are included. This framework brings us a proper approach to discuss the issue of confined/deconfined transition at finite chemical potential which the fermionic framework can not do. Besides that, the Wilson loop is a suitable observable for applying the factorization Eq.(14), since the operator itself is factorisable.

Let us briefly recall the results from the pure gauge systems. It is well known that the expectation value of the Wilson loop satisfies the area law at strong gauge coupling limit \[36\],

\[
\langle W[C] \rangle = \int DA \exp \left( i \oint_C A \right) \exp (iS[A]) \sim \exp \left( -\frac{1}{2} \sigma A \right),
\]

where \( \sigma \) is the string tension at zero chemical potential, and \( A \) is the minimal area enclosed by a loop \( C \). The area law of the Wilson loop implies a confining linear potential in the static limit \( L_t \to \infty \), i.e. \( V = -\lim_{L_t \to \infty} \frac{1}{2L_t} \ln \langle W \rangle = \frac{1}{2} \sigma r \), in which we have set \( \mathcal{A} = rL_t \), where \( r \) and \( L_t \) are the spatial and temporal lengths of the loop. At finite chemical potential, the correction to such well known behavior of Wilson loop could be considered as follows. Starting from the action Eq.(11), we note that the Abelian action \( S[A] \) is \( A \) shift invariant, so the only physical effect of the chemical potential in Abelian gauge theories is just a shift of the time component of the gauge field \( A_0 \). The Eq.(11) becomes

\[
Z = \int DA \exp \left( i \frac{\pi}{8} \int d^3 x \epsilon_{ijk} \hat{A}_i \partial_j \hat{A}_k - S[A] \right),
\]

where \( \hat{A}_i = A_i + i \mu \delta_0i \). At strong coupling limit, the infrared behavior governed by the Chern-Simons term is well protected by a gap. In this case, the Chern-Simons term which is responsible for the sign problem can be first safely integrated out which gives a topological invariant (knot polynomial). Thanks to its topological nature, the topological invariant does not depend on the length or size of the loop, the contribution from the Chern-Simons term does not connect to the confining potential we are interested in. We can ignore the Chern-Simons term in Eq.(16) and perform the calculation of the Wilson loop as follows (in Minkowski background with real time \( t \in [0, L_i] \)),

\[
\langle W[C, \mu] \rangle = \int DA \exp \left( i \oint_C A \right) \exp (iS[\hat{A}] - \mu \delta_0),
\]

where \( \hat{A}_i = A_i - i \mu \delta_0i \). Because the path integral measure \( DA \) is shift invariant, we have

\[
\langle W[C, \mu] \rangle = \int D\hat{A} \exp \left( i \oint_C \hat{A} \right) \exp (iS[\hat{A}] - \mu \delta_0).
\]

By performing a variable replacement \( \hat{A} \to A \), then \( A_i \) becomes \( A_i + i \mu \delta_0i \), we obtain

\[
\langle W[C, \mu] \rangle = \int DA \exp \left( i \oint_C dx^j (A_j + i \mu \delta_0j) \right) \exp (iS[A])
\]

\[
= \exp \left( i \oint_C dx^j (A_j + i \mu \delta_0j) - \frac{1}{2} \oint_C dx^j \oint_C dy^j \bar{K}_{ij}(x - y) \right),
\]
For \( \langle A_j \rangle = 0 \), the first part in the exponent gives
\[
i \oint_C dx^i i \mu \delta_{0j} = -\mu L_t. \tag{20}\]

The \( \hat{K}_{ij}(x-y) \equiv \langle T (A_i(x) + i \mu \delta_{0i}) (A_j(y) + i \mu \delta_{0j}) \rangle \) is the correlator of the shift gauge field, which is given by
\[
\hat{K}_{ij}[\hat{A}_t] = \hat{K}_{ij}[\hat{A}_t + i \mu \delta_{0t}]
= K_{ij}[\hat{A}_t + i \mu \delta_{0t}] - i \mu \delta_{0t} \delta_{ij} \frac{\delta^2 K_{ij}}{\delta \hat{A}_t \delta \hat{A}_t}
= K_{ij}(x-y) - \mu^2 P_{ij}^L(x-y), \tag{21}\]
where \( P_{ij}^L \) is the longitudinal projector. The unshifted part correlator \( K_{ij} \) gives rise to the standard area law at strong coupling limit,
\[
-\frac{1}{2} \oint_C dx^i \oint_C dy^j K_{ij} = -\frac{1}{2} \sigma A = -\frac{1}{2} \sigma r L_t. \tag{22}\]
The shift part \( -\mu^2 P_{ij}^L \) gives a correction to the existing area law behavior. By using the Stokes’s theorem, we get
\[
-\frac{1}{2} \oint_C dx^i \oint_C dy^j (-\mu^2 P_{ij}^L) = \frac{1}{2} \mu^2 \int \Sigma d^2x \int \Sigma d^2y \epsilon_{ijk} \epsilon_{jkl} \partial_k \partial_l P_{ij}^L
= \frac{1}{2} \mu^2 \int \Sigma d^2x \int \Sigma d^2y (\delta_{ij} \partial^2 - \partial_i \partial_j) P_{ij}^L, \tag{23}\]
where \( \Sigma \) is the minimal surface enclosed by the loop \( C \). According to the Ward identity \( \partial_i P_{ij}^L = 0 \), and the Possion equation \( \partial^2 P_{ij}^L(x) = \delta_{ij} \sigma^2(x) \) in 2+1 dimensions, we obtain
\[
\frac{1}{2} \mu^2 \int \Sigma d^2x \int \Sigma d^2y (\delta_{ij} \partial^2 - \partial_i \partial_j) P_{ij}^L = \frac{1}{2} \mu^2 \int \Sigma d^2x \int \Sigma d^2y \delta^2(x-y)
= \frac{1}{2} \mu^2 A = \frac{1}{2} \mu^2 r L_t. \tag{24}\]

Put all Eq. (20,22,24) together, we finally have
\[
\langle W[C, \mu] \rangle \sim \exp \left[ -\mu L_t - \frac{1}{2} (\sigma - \mu^2) A \right]. \tag{25}\]

where “\( \sim \)” means, at strong gauge coupling limit, the result asymptotically behaves as the following form. The static confining potential at finite chemical potential is therefore obtained
\[
V = -\lim_{L_t \to \infty} \frac{1}{L_t} \ln \langle W \rangle = \mu + \frac{1}{2} (\sigma - \mu^2) r. \tag{26}\]

The first term of the static potential naturally comes from the chemical potential, the second term is a confining linear potential with a chemical potential dependent string tension \( \sigma \to \sigma - \mu^2 \). It is very physically importance for the minus sign, since it makes the chemical potential play the role of a competition against the confinement, i.e. deconfinement. The chemical potential weakens the string tension, and finally snaps the string at a critical value \( \mu_c = \sigma^{1/2} \).

At finite temperature, the string tension is in general a function of temperature. Consider the dimension of the string tension is of energy squared, so near the critical temperature we have the behavior,
\[
\sigma(T) \sim \sigma_0 T_c^2 \left( 1 - \frac{T^\eta}{T_c^2} \right), \tag{27}\]
in which \( T_c \) is the critical temperature at zero chemical potential \( (\mu = 0) \), the \( \eta \) is the critical exponent near \( T_c \), and \( \sigma_0 \) is a constant. Then the chemical potential and temperature dependence of the string tension is given by
\[
\sigma(T, \mu) = \sigma_0 T_c^2 \left( 1 - \frac{T^\eta}{\sigma_0 T_c^2} \right) = \sigma_0 T_c^2 \left( 1 - \frac{T^\eta}{T_c} \right), \tag{28}\]
where the chemical potential dependence of the critical temperature is found
\[ T_c(\mu) = T_c \left(1 - \frac{\mu^2}{\sigma_0 T_c^2}\right)^{\frac{1}{2}}. \]  
(29)

This relation depicts the confined/deconfined phase transition boundary in the $\mu - T$ phase diagram. The critical chemical potential is predicted at $\mu_c = \sigma_0^{1/2} T_c$, i.e. ratio between the critical chemical potential and the critical temperature in confined/deconfined transition is a constant.

The above result could be generalized to the non-Abelian case, which is not very trivial. Compared with the Abelian case, the similar aspects comes from the fact that the chemical potential term $\mu^I J^I_0$ is also able to absorbed into the source term $A^I_0 J^I_1$, and so we could treat the chemical potential as a shift to the time component of the non-Abelian gauge field, $A^I_0 \rightarrow A^I_0 + \mu^I \delta_{0I}$. However, the physical effects of chemical potential would no longer be just the formally shift of $A^I_0$, like the Abelian case. In other words, the shift will introduces extra terms due to its intrinsic non-linearity, which do not appear in the Abelian gauge theories,

\[ S_{YM}[A] + A^I_0 J^I_1 + \mu^I J^I_0 = S_{YM}[\tilde{A}] + \tilde{A}^I_0 J^I_1 \]
\[ = S_{YM}[\tilde{A}] + \tilde{A}^I_0 J^I_1 + \mathcal{O}(\mu^4) + \mathcal{O}(A\mu^3) + \mathcal{O}(A^2 \mu^2) + \mathcal{O}(A^3 \mu). \]  
(30)

The term $\mathcal{O}(\mu^4)$ is a constant, and the terms $\mathcal{O}(A\mu^3)$ do not give correction to the correlator of $A^I_0$, so both are irrelevant to the physics. The terms $\mathcal{O}(A^2 \mu^2)$ and $\mathcal{O}(A^3 \mu)$ are vanished, i.e.

\[ \theta^{IJK} A^I_\mu K^L_\delta_0 \theta^{LMN} A^L_\mu M^\delta_0 + \theta^{IJK} A^I_\mu K^L \theta^{LMN} A^L_\mu M^\delta_0 \delta_0 \equiv 0 \]  
(31)

Only $\mathcal{O}(A^3 \mu)$ terms are non-vanished. But note that this term always has $A^I_0$ due to the contraction between $A^I_0$ and $\mu^I \delta_{0I}$, therefore, if we choose the temporal-axial gauge to eliminate $A^I_0$, this term vanishes as well. To summarize, under the temporal-axial gauge, these extra terms in Eq. (30) are irrelevant to what we are interested in. So the reasoning in the Abelian case still holds, just replacing the ordinary shift of $A^I_0$, under the temporal-axial gauge, these extra terms in Eq.(30) are irrelevant to what we are interested in. So the

Thus, similar with what we have done in the Abelian case, we ignore the (non-Abelian) Chern-Simons term which gives rise to loop-size-irrelevant topological invariant, and the Wilson loop can be calculated as,

\[ \langle W[C, \mu] \rangle = \int \mathcal{D}A \left[ \frac{1}{N} \text{tr} \exp \left( i \oint_C dx^3 A^I_0 t^I \right) \right] \exp \left( i S[\tilde{A}] \right) \]
\[ = \int \mathcal{D}A \left[ \frac{1}{N} \text{tr} \exp \left( i \oint_C dx^3 (A^I_\mu + i \mu^I \delta_{0I}) t^I \right) \right] \exp \left( i S[A] \right) \]
\[ = \exp \left( \frac{i}{N} \text{tr} \oint_C dx^3 \left( A^I_\mu + i \mu^I \delta_{0I} \right) t^I - \frac{1}{2N} \text{tr} \oint_C dx^3 \oint_C dy^3 (K_{ij} t^I t^J) \right). \]
(32)

Unlike the Abelian case, the first term in the exponent is vanished for the tracelessness of the group generators,

\[ \frac{i}{N} \text{tr} \oint_C dx^3 \left( A^I_\mu + i \mu^I \delta_{0I} \right) t^I \equiv \frac{i}{N} \text{tr} (t^I) \oint_C dx^3 i \mu^I \delta_{0I} = 0. \]  
(33)

This result exhibits the expected that if there are several chemical potential components in the non-Abelian gauge theories, they will cancel each other in the density bath and do not contribute to the averaged static potential.

The second term in the exponent is given by

\[ -\frac{1}{2N} \text{tr} \oint_C dx^3 \oint_C dy^3 (K^{IJ} t^I t^J) = -\frac{1}{2N} \text{tr} (t^I t^J) \oint_C dx^3 \oint_C dy^3 (K^{IJ} - \mu^I \mu^J P_{ij}^{IJ}) \]
\[ = -\frac{1}{2} \left( \sigma - \frac{1}{N} \sum_I \mu^2_i \right) A. \]  
(34)

Put them together and we find the non-Abelian generalization of Eq. (25)

\[ \langle W[C, \mu] \rangle \sim \exp \left[ -\frac{1}{2} \left( \sigma - \frac{1}{N} \sum_I \mu^2_i \right) A \right]. \]  
(35)
It straightforwardly gives the static potential

\[ V = \frac{1}{2} \left( \sigma - \frac{1}{N} \sum_{i} \mu_{i}^{2} \right) r, \]

and the chemical potential dependence of the critical temperature, i.e. non-Abelian version of Eq. (29) is given by

\[ \tilde{T}_{c}(\mu) = T_{c} \left( 1 - \frac{1}{\sigma} \sum_{i} \mu_{i}^{2} \right)^{\frac{1}{2}}. \]

To summarize, up to a topological invariant, we calculate the chemical potential dependence of the Wilson loop at strong coupling limit for the Abelian and non-Abelian gauge theories in 2+1 dimensions. It should be emphasized two key points here. First, the chemical potential formally shifts the time-component gauge field. In the Abelian case, that is all, but in the non-Abelian case, beside the shift it introduces several extra terms due to its intrinsic non-linearity. However, these extra terms coming from the chemical potential shift are proportional to the “dynamic part” of \( A_{0}\)-component gauge field, so they could be gauged away and do not contribute to our final observables. Second, the chemical potential formally shifts the area law, or equivalently the string tension with a minus sign. The area law induced by the chemical potential comes from the fact that the chemical potential gives a non-trivial longitudinal part to the correlator which could not be gauged away and has crucial physical effect. It is intuitive reasonable that the gauge field gains its non-trivial longitudinal component in a density bath. The confinement or the area law is looked like that the gauge fields are gaped in a specific way so that they could no longer be excited, only the longitudinal part is responsible for the gap (note the massive photon propagator behaves like a longitudinal projector in the large momentum limit) and breaks the gauge symmetry in the ground-state (spontaneously). The minus sign signifies that the effect of the density bath is anti-gaped at strong coupling, which indicates the deconfinement. The anti-gaped effect of the density bath at strong-coupling may play a similar role of an instability that possibly develops (e.g. superconducting) phase transition near the confined/deconfined phase boundary.

V. RELATION BETWEEN CHIRAL SYMMETRY BREAKING AND CONFINEMENT

Besides the under-controlled sign problem, the generalization of Wilson’s confinement criteria at finite chemical potential mentioned in previous sections, we will refer to another advantage of the bosonization recipe in this section, this recipe provides us a suitable method to discuss the one of the long-standing puzzles, the relation between the chiral symmetry breaking and confinement.

As is discussed in the introduction, the chiral symmetry breaking phase is an old story through the Landau-Ginzburg’s framework, in which the phase is completely characterized by the local order parameter and classified by the breaking chain of symmetry group, and we have Goldstone bosons as the low energy excitations in the phase. The fermionic language (e.g. BCS-like NJL model) is good at such type of description and be our standard understanding of spontaneous chiral phase transition. However, the gauge (gluonic) degrees of freedom are integrated out and replaced by a local four-point interaction of fermions (quarks) currents. As a consequence, the local \( SU(N) \) gauge invariance is replaced by a global \( SU(N) \) symmetry in this type of models, so that the property of confinement is lost, this fermionic type of models are fail in describing the confined/deconfined phase transition. On the other hand, the confined/deconfined phases are thought beyond the Landau-Ginzburg’s framework, since, strictly speaking, there seems no local order parameter or symmetry breaking to distinguish these two phases, especially at finite chemical potential. In order to investigate the confined/deconfined phases deeply, the bosonic gauge theories pictures are inevitable. Thanks to the advantage of the bosonization, it helps us at least partially overcome the shortcomings at finite chemical potential and defining a local chiral order parameter, both of which are straightforward in the fermionic picture. Therefore, we propose that the bosonization recipe is a proper approach being able to treat the chiral symmetry breaking and the confinement on an equal footing.

From the dictionary, one can see that the chiral density of fermions in 2+1 dimensions, from the bosonic language, is just the flux vortex density of bosonized \( a \) fields. So the chiral symmetry breaking phase is characterized by the non-vanishing (local) chiral order parameter

\[ \langle \bar{\psi}(x)\psi(x) \rangle = M^{2} \langle \text{tr}\cos(\Phi(x)) \rangle \equiv M^{2} \langle \text{tr}U(x) \rangle \neq 0. \]

On the other hand, because the system Eq. (27) and/or Eq. (28) is a pure gauge system with two types of gauge fields \( a \) and \( A \), thus the system is confined only when both pure gauge fields are confined. By using the Wilson’s criteria of
confinement, we have,

\[ \langle W[C] \rangle \approx \langle W[A, C] \rangle \langle W[a, C] \rangle \sim \exp \left( -\frac{1}{2} \sigma_{AD} A \right) \exp \left( -\frac{1}{2} \sigma_{A} A C \right) \]

(39)

The relation qualitatively reproduces the behaviors of Eq. (28) and/or Eq. (34). The above two points are standard results we have known. In this section, one of the goals is to study the interplay between these two points.

In the flux vortex picture, suppose that we can subdivide any surface enclosing by a loop into smaller surfaces, such that if a loop \( C \) contains smaller loops \( C_1 \) and \( C_2 \), \( \langle W[C] \rangle \approx \langle W[C_1] \rangle \langle W[C_2] \rangle \). The less correlated between the loops the more exact the approximation. Then consider a surface of planar loop \( C \) with area \( A \). The surface \( C \) encloses a number \( \langle A/A_{\min} \rangle \) of surfaces that each is bounded by \( C_i \) with unit area \( A_{\min} = \epsilon^2 \). The flux vortex \( \langle \text{tr} U_i(x) \rangle \) pierces the surface at point \( x \) within loop \( C_i \). For the Stokes’s theorem, then the Wilson loop \( C \) area pierced by these flux vortices behaves like

\[ \langle W[a, C] \rangle = \langle \prod_i \text{tr} U_i \rangle \approx \langle \prod_i \text{tr} A_{\min} \rangle \langle \text{tr} U \rangle = \langle \text{tr} U \rangle \frac{\langle A \rangle}{\langle A_{\min} \rangle} . \]

(40)

From this formula, we can see clearly that only when \( \langle \text{tr} U \rangle \neq 0 \) the behavior of the Wilson loop possesses an area law. It indicates an important conclusion of the paper that the chiral symmetry breaking \( \langle \text{tr} U \rangle = M^{-2} \langle \bar{\psi}(x) \psi(x) \rangle \neq 0 \) is a necessary condition for the confinement in 2+1 dimensions, i.e. confinement implies chiral symmetry breaking. In a phase diagram for the 2+1 dimensional gauge systems at finite chemical potential, this result leads to the consequence that a confined phase is always contained inside a chiral symmetry breaking phase regime. There could also exists a deconfined phase in the chiral symmetry breaking phase, in which fermion pairs have been formed, a gap coming from the pairing is opened up, but the system is deconfined. The deconfinement phase could coexist with the chiral symmetry breaking phase. It qualitatively behaves like the pseudo-gap phenomenon in 2+1 dimensional doped cuprates [34]. It is worthwhile to mention that this property is in contrast to the counterpart in the 3+1 dimensional QCD deduced from large-N, in which the authors argued that there is a phase in QCD system the confined phase coexisting with the chiral restored phase, called quarkyonic phase [37].

In our argument, there could exist both confined and deconfined phases in the chiral symmetric broken phase, so let us discuss these two phases based on the non-vanishing chiral order parameter/flux vortices density. For the flux \( \Phi \) is real, so \( 0 < \langle \text{tr} U \rangle < 1 \), the string tension in Eq. (39)

\[ \sigma_{a} = -\frac{2 \ln \langle \text{tr} U \rangle}{\langle A_{\min} \rangle} = -4\pi M^2 \ln \langle \text{tr} U \rangle \]

(41)

is positive defined, which means that the Wilson loop contributed from the matter part possesses an area law decay and the matter field is confined. Although the chiral condensation breaks gauge symmetry, under a large gauge transformation, the system has a residue center symmetry for the chiral order parameter \( \langle \text{tr} \cos \Phi \rangle \rightarrow \langle \text{tr} \cos (\Phi + \frac{2\pi}{N} \cdot n) \rangle \), with \( n \)-integer. The gauge symmetry is broken down to its discrete center group, i.e. \( SU(N_f)_{L} \times SU(N_f)_{R} \rightarrow SU(N_f) \rightarrow \mathbb{Z}_N \). Therefore, we conclude that, although the 2+1 dimensional gauge theory alone is in a confined phase, the system can also be confined when matter is included, it is possible only when the matter fields are condensed in such a way that the gauge symmetry is spontaneously broken down to its discrete subgroup.

The system also has a deconfined phase in another situation. The flux vortices piercing the loop surface could wind in (Euclidean) time direction by virtue of the periodic boundary condition. This phenomenon frequently happens when the radius of the winding flux vortex loop is smaller than that of the Wilson loop \( C \), so that they will usually pierce the loop surface twice in opposite directions inside the Wilson loop, thus leading to a cancellation of these two contributions. Only those flux vortex loops near the boundary of the Wilson loop surface could pierce it once, which gives a perimeter law,

\[ \langle W[a, C] \rangle \sim \exp (-\alpha L_C) , \]

(42)

where \( L_C \) is the perimeter of the Wilson loop \( C \), \( \alpha \) is certain constant. The behavior of perimeter law signifies a deconfinement. Note that the center symmetry is not locally broken due to the non-vanished local center flux vortices \( \langle \text{tr} U_i \rangle \). The deconfined transition comes from the change of the global properties of the center vortices in a finite size regime, in particular, their size, shape and winding in the Wilson loop. In this sense, the confinement and deconfinement are not characterized by their local symmetries, but distinguished by their global topological properties, the confined/deconfined phase transition is beyond the Landau-Ginzburg type of phase transition.

Generally speaking, there are two ways “seeing” the flux vortices in the system. One way is seeing the local vacuum expectation value of the flux vortices, which characterizes the chiral symmetry breaking phase in our bosonization
picture. Another way is placing a finite size Wilson loop in the system, and seeing the behavior of the Wilson loop enclosing the flux vortices. The non-local behavior of it detect the size, shape and winding properties of these flux vortices contained inside the loop, which characterizing the confinement and deconfinement phase.

To summarize this section. In $2+1$ dimensions, the chiral symmetry breaking is a necessary condition for the confinement, but not sufficient. The confined phase is always contained inside the chiral symmetry breaking phase, and the deconfined phase could also coexist with the chiral symmetry breaking phase. The non-vanishing center vortices takes value of the center group. When the center vortices are long, they pierce the Wilson loop surface only once, it gives an area law and be confined. When the center vortices are short and winding to form small loops, they usually pierce the Wilson loop surface twice in opposite directions and cancel each other, it gives an perimeter law and indicates a deconfined phase. The confined/deconfined transition does not lead to any local symmetry breaking, they are distinguished by their non-local topologies.

VI. LOW ENERGY FLUID-LIKE STATE IN NON-SYMMETRY BREAKING PHASE

The main goal of condensed matter and many-body physics is to study the low energy behavior of the system, for the low energy modes responsible for its core properties. As it is well known that in an ordered symmetric breaking phase, the Goldstone theorem tells us that there must exist massless Goldstone modes dominating the low energy behavior. However, without an analog theorem in a non-symmetry breaking phase and local order parameter to characterize the phase, low energy modes of a strongly coupled gauge system is poorly understood and thought notoriously hard to solve. The observation that topological terms such as Chern-Simons term in $2+1$D are inevitable in a gaped gauge system at finite chemical potential provides us a route to the problem. The bosonized low energy effective action Eq.(7) and Eq.(8) shows the fact that the leading contribution to the partition function is the topological term, while the Yang-Mills term is perturbation due to its strong coupling and the gap $M$. It was first numerically found \cite{38,39} that the Laughlin wavefunction borrowed from the theory of Quantum Hall system effectively minimizes the groundstate energy of the system deviating from the half-filled large band gap Hubbard model which is exactly a gauge symmetric system. The bosonized effective theory of the paper is consistent with the numerical result, and it convinces us that the low energy modes of a gaped gauge system at finite chemical potential is dominated by a gauge symmetric system. The bosonized effective action of the paper is consistent with the numerical result, and it convinces us that the low energy modes of a gaped gauge system at finite chemical potential is dominated by a topological fluid describe by the Chern-Simons theory. The low energy fluid-like state emerges from the gauge system at finite chemical potential in many aspects similar with the quantum Hall fluid $L = i \frac{Ch}{4\pi} \epsilon_{\mu\nu\rho} a_\mu \partial_\nu a_\rho$ with Chern number $Ch = 2$ which is well-studied:

(1) The low energy modes is fluid-like and incompressible, when we only consider the topological Chern-Simons term. Here the terminology “incompressible” is tantamount to “topological” since there is no response to its volume change. When the gaped terms like the non-topological Yang-Mills term at order $O(1/M)$ and $O(1/g^2)$ are considered at finite temperature, the fluid gets its compressibility corrections, thus gradually deviating from incompressible. As a consequence, the quasi-particle picture fails, there is no sharp quasi-particle peak in experimental probe.

(2) The equation of state of the new fluid state of matter deviates from the idea gas especially at the critical point, since it is purely topological term dominant at the critical point, only the perturbative dynamical Yang-Mills term contributes to a quasi-particle/gas -like behavior.

(3) The fluid-like low energy modes are of conformal at the critical point due to the fact that the topological action is a fixed point action.

(4) The gauge charge of the excitation of the phase is fractionalized.

All these salient features of fluid-like low energy modes are expected shared by the non-Fermi-liquid normal state of high-temperature superconductor, and quark-gluon plasma of QCD system in finite chemical potential region beyond ordered phases, just because all these systems are generally Gribov gaped and strongly coupled gauge symmetric systems as well as the Chern-Simons term inevitably emerges at finite chemical potential. Although they are in a non-symmetry breaking phase, they are not completely disorder as we took it as a matter of course, they are actually topological ordered, since the system will develops robust groundstate degeneracy when we put it on base manifolds with different topologies.

VII. CONCLUSIONS

In this paper, due to the fact that gauge systems are gaped by $M$ which sets a scale for the chiral phase transition, we show that the fermions described by the Grassman numbers can be effectively simulated by the conventional numbers of bosons in $2+1$ dimensions. Based on the non-locality of fermions, we regard a fermion as an end point of a bosonic non-local string. This picture introduces an identification between theories of fermions and bosons. The new bosonization recipe is systematically developed in $2+1$ dimensions. For practical reasons, we only focus on the
massless fermions in this paper. In general, the new bosons required to bosonize fermions are of a \((d - 2)\)-form gauge fields, which explains why a conventional Maxwell/Yang-Mills type of 1-form gauge bosons is enough to bosonize the fermions in 2+1 dimensions. A more realistic QCD system in 3+1 dimensions requires a 2-form Kalb-Ramond type gauge fields [40], which is left for future discussions. The feasibility of this identification between fermions and bosons is reflected by the fact that the fermion statistic, or equivalently the sign structure of the fermions, can be totally attributed to the gauge phase controlled by the new gauge fields.

The bosonization recipe bridges the gap between a gauge system at finite chemical potential and a pure gauge system. We find that a Maxwell/Yang-Mills gauge system (with massless fermion) with finite chemical potential in 2+1 dimensions is equivalent to a gauge system with new gauge field governed by the action of Chern-Simons form up to certain gaped corrections. The new pure gauge theories preserves all symmetry of the original gauge theories at finite chemical potential. The induced Chern-Simons-Maxwell/Yang-Mills gauge system is well defined and be our starting point. The new theories provide us a new viewing angle to the well-known fermion sign problem. The non-trivial phase of fermion determinant equivalently comes from the Chern-Simons term of the theories, which could not be eliminated by conventional Wick rotation due to its metric independent nature. The sign problem can only be alleviated. Since the gauge fields of the system are gaped especially at strong coupling, a factorization can be introduced to hive off the contribution from the Chern-Simons part which is relevant to the sign problem but irrelevant to the size or temperature dependent thermodynamic properties.

Beyond the ordered Landau-Ginzburg type phase transition, we study the low energy modes of a non-symmetry breaking phase at finite chemical potential. Benefit from the emergence of topological term of a gaped gauge theories at finite chemical potential, the low energy modes could be studied analogous with the quantum Hall system. Similar with the Hall fluid, they are fluid-like without any quasi-particle sharp peak in spectrum, nearly incompressible, strongly deviate from gas behavior and gauge charge fractionalized.

The bosonization is a new attempt to study the quantum phase of a gauge system at finite chemical potential, with a new perspective to the long-standing fermion sign problem. The method can also be generalized to \((3+1)D\) [41, 42] based on two main reasons: (1) Non-dynamical components of gauge fields, e.g. \(a_0\) (1-form) and \(b_{ij} = -b_{ji}\) (2-form), play the roles of different Lagrange multipliers like different chemical potentials, which keep corresponding charges conserved. Thus they inevitably lead to the emergence of topological terms as constraints at finite chemical potential. (2) The quantum Yang-Mills gauge theory in \((3+1)D\) is generally believed to be gaped at low energy especially at strong coupling (even though the exact proof is still lacking). The gap strongly suppresses the dynamical infrared modes induced from Yang-Mills term and protects the emergent topological degrees of freedom at low energy, which will prohibits a hydrodynamic low energy behavior in gauge systems. It suggests that the fluid-like quark-gluon plasma found at Relativistic Heavy Ion Collider (RHIC) may has deep connection to the topological fluid governing by the topological terms (e.g. the BF-terms and theta-terms in \((3+1)D\)). We believe that bosonization method is worth pursuing and hopeful to give us new insights to the physics at finite chemical potential.
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Appendix I

In this appendix we proof that the bosonization recipe reproduces the fermion statistic. By using Eq. (1). The equal time anti-commutation relation for the fermions is given by

\[
\psi(\vec{x}_1, \tau)\psi(\vec{x}_2, \tau) = \lim_{\vec{x}_f \to -\infty} \exp \left( -i \int_{\vec{x}_f}^{\vec{x}_1} dy^i a_i(\vec{y}_1, \tau) \int_{\vec{x}_f}^{\vec{x}_2} dy^j a_i(\vec{y}_2, \tau) \right) \\
= \lim_{\vec{x}_f \to -\infty} e^{i\Phi(\vec{x}_1, \vec{x}_2)} \exp \left( -i \int_{\vec{x}_f}^{\vec{x}_1} dy^i a_i(\vec{y}_1, \tau) \right) \exp \left( -i \int_{\vec{x}_f}^{\vec{x}_2} dy^j a_i(\vec{y}_2, \tau) \right) \\
= e^{i\Phi(\vec{x}_1, \vec{x}_2)} \psi(\vec{x}_1, \tau) \psi(\vec{x}_2, \tau),
\]

(43)
in which we have used the identity

\[
e^A e^B = e^B e^A e^{-[A,B]}.
\]

(44)
The canonical commutation relation of \(a_i\) field is written as

\[
i\delta^2(\vec{x} - \vec{y}) = [a_i(\vec{x}, \tau), \Pi_i(\vec{y}, \tau)] = [a_i(\vec{x}, \tau), \frac{\delta S}{\delta \partial_\tau a_i(\vec{y}, \tau)}] \\
= \pm i\epsilon_{ij} [a_i(\vec{x}, \tau), a_j(\vec{y}, \tau)],
\]

(45)
in which the conjugate momentum is calculated by using the action Eq.(7) or Eq.(8) in which we have neglected the Maxwell/Yang-Mills terms suppressed by gap and strong coupling. So the equal time anti-commutation relation of fermions can be given as follows,

\[
i\Phi(\vec{x}_1, \vec{x}_2) = \lim_{\vec{x}_f \to -\infty} \left[ \int_{\vec{x}_f}^{\vec{x}_1} dy^i a_i(\vec{y}_1, \tau), \int_{\vec{x}_f}^{\vec{x}_2} dy^j a_j(\vec{y}_2, \tau) \right] \\
= \lim_{\vec{x}_f \to -\infty} \int_{\vec{x}_f}^{\vec{x}_1} dy^i \int_{\vec{x}_f}^{\vec{x}_2} dy^j [a_i(\vec{y}_1, \tau), a_j(\vec{y}_2, \tau)] \\
= \pm i\pi \lim_{\vec{x}_f \to -\infty} \int_{\vec{x}_f}^{\vec{x}_1} dy^i \int_{\vec{x}_f}^{\vec{x}_2} dy^j \epsilon_{ij} \delta^2(\vec{y}_1 - \vec{y}_2).
\]

(46)

Note that the integral of the delta function equals 1 when the coordinates of two paths meet at the fixed point \(\vec{x}_f\), so we reproduce the equal time anti-commutation relation of fermion \(i\Phi(\vec{x}_1, \vec{x}_2) = \pm i\pi\). It is worth mentioning that the \(\psi\) satisfies fermionic statistics on the equal time spatial hyper-plane, independent with any specific detail of the paths, so it is gauge independent. The arbitrariness of the plus and minus sign here is crucial for the parity conservation for the action. The deduction shows how the anti-commutation relation of fermions connects with the commutator of the \(a\) fields, and its relation with the coefficient of the Chern-Simons action.

Appendix II

In this appendix, we proof the bosonized form of the current in Abelian and non-Abelian case in the dictionaries. First, we proof the bosonized current for the Abelian case. The definition of current is

\[
J_i(x) =: \bar{\psi}(x) \gamma_i \psi(x) : = \lim_{\epsilon \to 0} : \bar{\psi}(x + \epsilon) \gamma_i \psi(x) :.
\]

(47)
in which $\therefore$ stands for the normal ordering, so we use the identity between the normal ordering and time ordering
\[
\phi(x)\chi(y) := T\phi(x)\chi(y) - \langle T\phi(x)\chi(y) \rangle,
\]
we get
\[
J_i(x) = \lim_{\epsilon \to 0} \left[ T\psi^\dagger(x + \epsilon)\gamma_0\gamma_i\psi(x) - \langle \psi^\dagger(x + \epsilon)\gamma_0\gamma_i\psi(x) \rangle \right].
\]
For the massless fermions, there are two chiral components $\psi_\pm$. We set $u_\pm$ the spinor bases with unit norm for different chirality. Then we write the current as
\[
J_i(x) = \lim_{\epsilon \to 0} \left[ T\psi^\dagger(x + \epsilon)\gamma_0\gamma_i\psi(x) - \langle \psi^\dagger(x + \epsilon)\gamma_0\gamma_i\psi(x) \rangle \right] = \lim_{\epsilon \to 0} \left[ i\int_{-\infty}^{x+\epsilon} dy^i a_j(y) - \int_{-\infty}^{x} dy^i a_j(y) \right] \left( u_\pm^\dagger \gamma_0 \gamma_i u_\pm \right),
\]
in which we have used the 2+1 dimensional correlator of fermions in the bases of $u_\pm$.
\[
\langle \psi^\dagger(x + \epsilon)\psi(x) \rangle = \int \frac{1}{(2\pi)^2} d^2k \int \frac{1}{(2\pi)^2} d^2q(0)\alpha^\dagger(k)\alpha(q)0 e^{i\vec{x}-i\vec{q}\cdot\vec{x}'} = \frac{1}{2\pi} \int_{k>0} dk e^{i\vec{k}\cdot\vec{x}-i\vec{k}\cdot\vec{x}'} = \frac{1}{2\pi(\vec{x}-\vec{x}')^2}.
\]
By using the identity of normal ordering
\[
e^{ia\phi(z)} : = e^{ib\phi(z')},
\]
we have
\[
J_i = \lim_{\epsilon \to 0} \left[ \exp \left( i\int_{-\infty}^{x+\epsilon} dy^i a_j(y) - i\int_{-\infty}^{x} dy^i a_j(y) \right) \exp \left( \int_{-\infty}^{x+\epsilon} dy^i a_j^\dagger(y) \int_{-\infty}^{x} dy^j a_k(y_1), a_k(y_2) \right) - \frac{1}{2\pi\epsilon^2} \right] \left( u_\pm^\dagger \gamma_0 \gamma_i u_\pm \right),
\]
in which we have used the correlator of $a$
\[
(a_i(x)a_j(x')) = \frac{1}{2\pi} \epsilon_{ijk} \frac{x_k - x'_k}{|x-x'|^2}.
\]
Then we use the Stokes's theorem,
\[
J_i = \lim_{\epsilon \to 0} \left[ \exp \left( i\sum_{j,k\neq i} \int_{x+\epsilon_j}^{x+\epsilon_k} \int_x^{x+\epsilon_k} dy^i dy^k \epsilon_{jk} \partial_j a_k \right) \frac{1}{2\pi\epsilon^2} - \frac{1}{2\pi\epsilon^2} \right] \left( u_\pm^\dagger \gamma_0 \gamma_i u_\pm \right) = \frac{1}{2\pi} \sum_{j,k\neq i} \epsilon_{ijk} \partial_j a_k + \frac{1}{2\pi\epsilon^2} \left( u_\pm^\dagger \gamma_0 \gamma_i u_\pm \right).
\]
in which we have used the identities
\[ \epsilon_{ij} \gamma_0 = \sum_{k \neq i,j} \epsilon_{ijk} \gamma_k. \] (56)

For the non-Abelian case, the current is defined as
\[ J_i^I(x) = \lim_{\epsilon \to 0} tr \left[ \psi(x + \epsilon) \gamma_i t^I \psi(x) \right] = tr \left[ T \psi(x + \epsilon) \gamma_i t^I \psi(x) - \langle T \psi(x + \epsilon) \gamma_0 \gamma_i t^I \psi(x) \rangle \right]. \] (57)

We write the current in its eigenstates of \( t^I \) and \( u_{\pm} \),
\[ J_i^I = \lim_{\epsilon \to 0} \left[ \exp \left( i \int_{-\infty}^{x + \epsilon} dy^j a^j_I t^j \right) \right] \left( \frac{1}{2\pi \epsilon^2} \right) \] (58)

where \( U \) is unit norm eigen-vector of \( t^I \). By using the identity
\[ e^{A} e^{B} = e^{A + B + \frac{1}{2}[A, B]} ; \quad e^{i\alpha \phi} (z) \equiv e^{i\alpha \phi (z)} ; \quad e^{-\alpha b (\phi (z))} ; \quad e^{-\alpha b (\phi (z))}, \] (59)
we get
\[ J_i^I = \lim_{\epsilon \to 0} \left[ \exp \left( i \int_{-\infty}^{x + \epsilon} dy^j a^j_I t^j \right) \right] \left( \frac{1}{2\pi \epsilon^2} \right) \] (60)

in which we have used the following relation
\[ \frac{1}{2} \left[ \int_{-\infty}^{x + \epsilon} dy^j a^j_I , \int_{-\infty}^{x + \epsilon} dy^k a^k_I \right] = \int_{-\infty}^{x + \epsilon} dy^j \int_{-\infty}^{x + \epsilon} dy^k [a^j_I , a^k_I]. \] (61)

Then using the Stokes’s theorem
\[ J_i^I = \lim_{\epsilon_j, \epsilon_k \to 0} \left[ \exp \left( i \sum \int_{x}^{x + \epsilon_j} \int_{x}^{x + \epsilon_k} dy^j dy^k \epsilon_{j k} \partial \partial_j a^j_k t^j + i a^j_k [t^j, t^k] \right) \left( \frac{1}{2\pi \epsilon^2} \right) \right] \] (62)

**Appendix III**

In this appendix we proof the bosonized form of kinetic term in the dictionaries. The massless fermion kinetic energy is identified to the Chern-Simons term at the Lagrangian (classical) level. Using the Eq. (11), the kinetic energy
is given by

\[ i\bar{\psi}_i \partial_i \psi = i\bar{\psi}_i \partial_i \exp \left( -i \int_{-\infty}^{x} dy' a_i \right) \]
\[ = i\bar{\psi}_i \exp \left( -i \int_{-\infty}^{x} dy' a_i \right) \partial_i \left( -i \int_{-\infty}^{x} dy' a_j \right) \]
\[ = \bar{\psi}_i \psi a_i. \quad (63) \]

By using the bosonized form of current deduced from the appendix II,

\[ J_i = \bar{\psi}_i \psi = \pm \frac{1}{2\pi} i\epsilon_{ijk} \partial_j a_k, \quad (64) \]

so

\[ i\bar{\psi}_i \partial_i \psi = J_i a_i = \pm \frac{1}{2\pi} i\epsilon_{ijk} a_i \partial_j a_k. \quad (65) \]

The non-Abelian counterpart is

\[ i\bar{\psi}_i \partial_i \psi = i\bar{\psi}_i \partial_i \exp \left( -i \int_{-\infty}^{x} dy' a_i^{t_l} \right) \]
\[ = i\bar{\psi}_i \exp \left( -i \int_{-\infty}^{x} dy' a_i^{t_l} \right) \partial_i \left( -i \int_{-\infty}^{x} dy' a_i^{t_l} \right) \]
\[ = \bar{\psi}_i \exp \left( -i \int_{-\infty}^{x} dy' a_i^{t_l} \right) \left( a_i^{t_l} \right). \quad (66) \]

By using the relation

\[ \exp (-i\alpha^l t^l) \left( a_i^{t_l} \right) \exp (i\alpha^l t^l) = a_i^{t_l} + i \left[ a_i^{t_l}, \alpha^l t^l \right] \]

we have

\[ i\bar{\psi}_i \partial_i \psi = \bar{\psi}_i \left( a_i^{t_l} + i \int_{-\infty}^{x} dy' \left[ a_i^{t_l}, a_i^{t_l} \right] \right) \exp \left( -i \int_{-\infty}^{x} dy' a_i^{t_l} \right) \]
\[ = \text{tra}^l J^l_i + \text{tr} \int_{-\infty}^{x} dy' \left( \bar{\psi}_i a_i^{t_l} \left[ t^l, t^l \right] \psi \right). \quad (68) \]

We use the non-Abelian form of the current

\[ J^l_i = \bar{\psi}_i t^l \psi = \pm \frac{1}{2\pi} i\epsilon_{ijk} \text{tr} \left( \partial_j a^l_k + \theta^{lJK} a^j K a^l_k \right), \quad (69) \]

then

\[ i\bar{\psi}_i \partial_i \psi = \text{tra}^l J^l_i - \theta^{lJK} \text{tr} \int_{-\infty}^{x} dy' \left[ a^l_i, a^l_j \right] J^K_j \]
\[ = \text{tra}^l J^l_i + \frac{1}{2\pi} i\epsilon_{ijk} \theta^{lJK} \text{tr} \int_{-\infty}^{x} dy' \left[ a^l_i, a^l_j \right] \left( \partial_j a^K_k + \theta^{lJK} a^j K a^l_k \right). \quad (70) \]

Since \text{tra}^4 = 0, so

\[ i\bar{\psi}_i \partial_i \psi = \text{tra}^l J^l_i + \frac{1}{2\pi} i\epsilon_{ijk} \theta^{lJK} \text{tr} \int_{-\infty}^{x} dy' \left[ a^l_i, a^l_j \right] \left( \partial_j a^K_k + \theta^{lJK} a^j K a^l_k \right) \]
\[ = \text{tra}^l J^l_i + \frac{1}{2\pi} i\epsilon_{ijk} \theta^{lJK} \frac{1}{3} \text{tr} \int_{-\infty}^{x} dy' \left[ \partial_i \left( a^l_j a^l_k \right) K \right] \]
\[ = \pm \frac{1}{2\pi} i\epsilon_{ijk} \text{tra}^l \left( \partial_j a^K_k + \frac{2}{3} \theta^{lJK} a^j K a^l_k \right) \]
\[ = \pm \frac{1}{2\pi} i\epsilon_{ijk} \text{tra}^l \left( \partial_j a^K_k + \frac{2}{3} \theta^{lJK} a^j K a^l_k \right). \quad (71) \]
Appendix IV

In this appendix we prove the bosonized form of the chiral densities. For the massless fermions, there are two chiral components, $\psi_{\pm}$. The definition of the chiral densities are given by

$$\bar{\psi}_+ \psi_+ = \bar{\psi} (1 + \gamma_5) \psi, \quad \bar{\psi}_- \psi_+ = \bar{\psi} (1 - \gamma_5) \psi, \quad (72)$$

where

$$\bar{\psi}_-(x) \psi_+(x) = \lim_{\epsilon \to 0} \exp \left( i \int_{-\infty}^{x+\epsilon} dy^i \dot{a}_i - i \int_{-\infty}^{x} dy^i a_i + \frac{1}{2} \left[ \int_{-\infty}^{x+\epsilon} dy^i \dot{a}_i, \int_{-\infty}^{x} dy^i a_i \right] \right) \exp \left( \int_{-\infty}^{x+\epsilon} \int_{-\infty}^{x} dy^i dy^j \langle \dot{a}_i a_j \rangle \right)$$

$$= \lim_{\epsilon \to 0} M^2 \text{tr} \exp \left( i \int_{-\infty}^{x+\epsilon} dy^i \dot{a}_i - i \int_{-\infty}^{x} dy^i a_i + \int_{x}^{x+\epsilon} \int_{-\infty}^{x} dy^i dy^j \dot{a}_i [a_j, a_j] \right)$$

$$= \lim_{\epsilon \to 0} M^2 \text{tr} \exp \left( -i \int_{x}^{x+\epsilon} \int_{x}^{x+\epsilon} dy^i dy^j \dot{a}_i \dot{a}_j + i [a_i, a_j] \right)$$

$$= \lim_{\Sigma \to 0} M^2 \text{tr} \exp \left( -i \int_{\Sigma = \epsilon \times \epsilon} dy^i dy^j f_{ij} \right)$$

$$= M^2 \text{tr} \exp (-i \Phi(x)), \quad (74)$$

where

$$M^2 (\epsilon) = \exp \left( \int_{-\infty}^{x+\epsilon} \int_{-\infty}^{x} dy^i dy^j \langle a_i (-k) a_j (k) \rangle \right) = \frac{1}{2 \pi \epsilon^2} \quad (75)$$

is a universal cut-off scale with dimensions of mass squared, the cut-off $\epsilon$ is the shortest distance between the fermions. Similarly, one finds

$$\bar{\psi}_+(x) \psi_-(x) = \lim_{\epsilon \to 0} \text{tr} \exp \left( i \int_{-\infty}^{x+\epsilon} dy^i a_i - i \int_{-\infty}^{x} dy^i \dot{a}_i + \frac{1}{2} \left[ \int_{-\infty}^{x+\epsilon} dy^i a_i, \int_{-\infty}^{x} dy^i \dot{a}_i \right] \right) \exp \left( \int_{-\infty}^{x+\epsilon} \int_{-\infty}^{x} dy^i dy^j \langle a_i \dot{a}_j \rangle \right)$$

$$= \lim_{\epsilon \to 0} M^2 \text{tr} \exp \left( i \int_{-\infty}^{x+\epsilon} dy^i a_i - i \int_{-\infty}^{x} dy^i \dot{a}_i + \int_{x}^{x+\epsilon} \int_{-\infty}^{x} dy^i dy^j [a_i, \dot{a}_j] \right)$$

$$= \lim_{\epsilon \to 0} M^2 \text{tr} \exp \left( i \int_{x}^{x+\epsilon} \int_{x}^{x+\epsilon} dy^i dy^j \dot{a}_i \dot{a}_j + i [a_i, a_j] \right)$$

$$= \lim_{\Sigma \to 0} M^2 \text{tr} \exp \left( i \int_{\Sigma = \epsilon \times \epsilon} dy^i dy^j f_{ij} \right)$$

$$= M^2 \text{tr} \exp (i \Phi(x)), \quad (76)$$

where $\Phi$ is the flux density at $x$. This result includes both the Abelian and non-Abelian case, the bracket $[a_i, a_j]$ automatically vanishes in the Abelian case. This result translates the (left/right) chiral density to the flux/anti-flux vortex density in the dictionaries

$$\bar{\psi} (1 \pm \gamma_5) \psi = M^2 \text{tr} \exp (\pm i \Phi(x)). \quad (77)$$
The result shows the scalar and axial vector is bosonized as
\[
\bar{\psi}(x)\psi(x) = \frac{1}{2} \left( \bar{\psi}_+ \psi_- + \bar{\psi}_- \psi_+ \right) = \frac{1}{2} M^2 \left( \text{tr} e^{i\Phi} + \text{tr} e^{-i\Phi} \right) = M^2 \text{tr} \cos(\Phi),
\]
\[
\bar{\psi}_i \gamma_5 \psi = M^2 \text{tr} \sin(\Phi).
\]
So the mass term is written as
\[
m\bar{\psi}\psi = m M^2 \text{tr} \cos(\Phi).
\]