Quantum Solution to Scalar Field Theory Models

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Abstract

Amplitudes $A_n$ in $d$-dimensional scalar field theory are generated, to all orders in the coupling constant and at $n$-point. The amplitudes are expressed as a series in the mass $m$ and coupling $\lambda$. The inputs are the classical scattering, and these generate, after the integrals are performed, the series expansion in the couplings $\lambda_i$. The group theory of the scalar field theory leads to an additional permutation on the $L$ loop trace structures. Any scalar field theory, including those with higher dimension operators and in any dimension, are amenable.
Introduction

The quantum scalar $\phi^3$ theory has been studied for many years and is a textbook quantum field theory. The interactions in this theory are typically examined to lowest order in perturbation theory, or to higher orders in the ultraviolet so as to find the critical exponents and scaling. Large order studies in perturbation theory were performed over twenty years ago, without detailed knowledge of the diagrams. The diagrams in the usual perturbation theory are complicated to evaluate in general, which has slowed progress.

The derivative expansion has been pursued for several years [1]-[10]. This formulation has been placed in the context of many theories, including scalar, gauge and supersymmetric models. The derivation of the quantum scattering has been simplified in [1]; this approach is used here to find the amplitudes of all scalar field theories in any dimension, to all orders in the couplings. All coefficients of the following $n$-point amplitude expansion can be determined,

$$A_n = \sum c(i, m) \lambda^i (k^2)^m,$$  \hspace{1cm} (1)

with $\lambda$ the coupling constant and $k^{2m}$ representing the generic product of the $n(n - 1)/2$ momentum invariant at $n$-point.

The tree amplitudes in $\phi^3$ theory are given in [11]. The amplitudes for any scalar field theory follow from this result by pinching propagators.

In general the classical amplitudes in any quantum field theory, including massless ones, are required to recursively construct, in this formulation, the solution to the amplitudes and effective action. The recently appeared tree amplitudes of scalar, and gauge and gravity theory, are based on a simple number theoretic parameterization [11], [12]. These scalar amplitudes and their coefficients are used here in the quantum scalar solution.

The classical Lagrangian that generates the amplitudes are those of massive scalar field theory, and includes the possible interactions. These pertain to $\phi^n$, scaled with an appropriate coupling, and the derivatives $\partial^{n_1} \phi \ldots \partial^{n_m} \phi$. Group theory, the inclusion of additional modes, and mixed interactions are also included. The group theory adds a complication associated with permutations of external lines.

$n$-Point Amplitudes

The genus zero amplitudes are first presented. Then the formulae describing the quantum amplitudes are given and used to find the full amplitudes.
The tree amplitudes in $\phi^3$ at all $n$-point have recently been described in the literature [10]: a set of numbers $\phi_n(i)$ are required to specify individual diagrams. These $n - 2$ numbers label the vertices and range from $n$ to 1. In a color ordered tree, they occur at most $n - 2$ times for the greatest number to none in the case of the lowest number in an incremental manner. The set $\phi_n$ generates the momentum routing of the propagators and describe the diagram.

The $\phi_n$ numbers can be changed to the set of numbers $i, p$ which describe the poles in the diagram through the invariants

$$i_t^{[p]} = (k_i + \ldots + k_{i+p-1})^2 .$$  \hspace{1cm} (2)

These invariants are defined for a fixed ordering of the external legs and the numbers are cyclic around the final number. A second set of numbers, besides the $\sigma(i, n)$ are required when the mass expansion is performed. Due to the series $1/(m^2 - p^2) = m^2 \sum (p^2/m^2)^k$, the coefficients $\tilde{\sigma}(i, p)$ are numbers from 0 to $\infty$ and label the exponent in the series for each propagator.

The numbers

$$\begin{align*}
\sigma(i, p) & \quad \tilde{\sigma}(i, p) ,
\end{align*}$$  \hspace{1cm} (3)

describe the individual diagrams in the mass expansion.

The fundamental iteration is accomplished via the sewing procedure as described in [4]-[10]. The integrals are simple free-field ones in $x$-space, and generate an infinite series of relations between the parameters of the coupling expansion $\alpha_{n,q}$.

$$\sum_{q} \alpha_{n,q}^{p_{ij}} \lambda^{n-2+q} = \sum_{i,j,p,l_{ij},n_{ij},m_{ij}} \alpha_{n+p,i}^{l_{ij}} \alpha_{n+p,j}^{m_{ij}} \lambda^{2n+2p+i+j-4} I^{p_{ij}}_{l_{ij},n_{ij}} .$$  \hspace{1cm} (4)

The indices $i, j$ are exemplified below. The $\alpha$ parameters are quantum corrected vertex parameters, and they take into account propagator corrections. The coefficients $I^{p_{ij}}_{l_{ij},n_{ij}}$ are defined by the momentum expansion of the 'rainbow' integrals

$$J^{\sigma}_{l_{ij},m_{ij}} = \int \prod_{a=1}^{p} d^d q_a \left( \frac{1}{(q_{\rho(a)} - k_{\sigma(a)})^2 + m^2} \right)^{l_{ij} + m_{ij}} |_{p_{ij}} ,$$  \hspace{1cm} (5)
with $\tilde{l}_{ij}$ and $\tilde{m}_{ij}$ parameterize a subset of the vertex lines which are contracted inside the loop. The indices $\rho$ label the linear combination of the loop momenta in the internal lines.

The integrals (5) are symmetrized over the external lines in the formula (4); there are $n_1$ and $n_2$ external lines on each side of the graph and $b$ parameterizes a subset of these numbers (e.g. $n_1 = 1, 2, 3, 4, n_2 = 5, 6, 7, 8$ and $b = 3, 4, 5, 6$; the $l_{ij}$ and $m_{ij}$ parameterize the kinematics associated with the external and internal lines. The expansion of the integral in (5) in the momenta generate the coefficients $p_{ij}$. The set of numbers $\sigma(a)$ parameterize the subset of numbers of the two vertices (forming an integral with $n$ external lines. The numbers $\sigma(a)$ label numbers beyond the external lines $1, \ldots, n_1$ and $n_1 + 1, \ldots, n$) and are irrelevant because the integral is a function of their sum; this property lends to a group theory interpretation of the final result in terms of the coefficients

$$I_{\tilde{l}_{ij}, \tilde{m}_{ij}}^{\sigma, \rho_{ij}} ,$$

after summing the permutations. The numbers $i, j$ in $l_{ij}$ and $n_{ij}$ span 1 to $m$ (including internal lines) and those in $p_{ij}$ span 1 to $n$:

$$l_{ij} = (l_{ij}, 0, \ldots, 0, l_{ij}) \quad n_{ij} = (0, 0, \ldots, n_{ij}, \ldots, n_{ij}, \ldots, n_{ij}) ,$$

and

$$p_{ij} = (p_{ij}, \ldots, p_{ij}) .$$

This notation of $l_{ij}, m_{ij}$, and $p_{ij}$ is used to setup a (pseudo-conformal) group theory interpretation of the scattering.

The details of the expansion of the integrals in (5) depend on the selection of the internal lines found via the momenta of the vertices

$$\lambda_{n_1}^{(p_{11}, p_{12}, \ldots, p_{nn})}$$

on either side of the double vertex graph. Although the $\tilde{l}_{ij}, \tilde{m}_{ij}$, and $\tilde{p}_{ij}$ depend on the details of the contractions and sums of the lines of the individual vertices, the actual coefficients of the iteration, i.e. $I_{\tilde{l}_{ij}, \tilde{m}_{ij}}^{\rho_{ij}}$, are functions only of the vertex parameters. The details of the expansion and the contractions of the tensors in the integrals (5) are parameterized by $p_{ij}$, which label the momentum expansion of the integrals. The
coefficients \( p_{ij} \) range from 0 to \( \infty \), in accordance with the momentum expansion of the massive theory.

Although the coefficients \( \Gamma_{l_{ij},n_{ij}}^{p_{ij}} \) arise from the integral expansion, they also have a group theory description. The dynamics of the expansion are dictated via these coefficients for an arbitrary initial condition of the bare Lagrangian.

The iteration of the coefficients results in the simple expression,

\[
\alpha_{m_{ij}} = \sum_{i,j,l_{ij},n_{ij}} \alpha_{n+p,i}^{l_{ij}} \alpha_{n+p,j}^{n_{ij}} \Gamma_{l_{ij},n_{ij}}^{m_{ij}} .
\]  

(10)

The sums are on the number of internal lines \( p \) and the powers of the shared couplings \( i \) and \( j \),

\[
n - 2 + q = 2(n + p) - 4 + i + j \quad q = n + 2p - 2 + i + j ,
\]  

(11)

for the example of \( \phi^3 \). The numbers of momenta \( l_{ij} \) and \( n_{ij} \) are accorded to \( s_{ij} \) (some of which are within the integral). The parameters \( m_{ij} \) label the external momenta, interpreted group theoretically through the coefficient \( I \).

The integrals and the iteration in (10) have to be performed. The initial condition on the sum is the form of the classical \( n \)-point amplitudes, i.e. \( \alpha^{p_{ij}}_{n,cl} \). The integral complication is that there are invariants that: 1) contain both external and loop momenta, and 2) contain only loop momenta. The sum must have attention to both types of invariants as the integrals are different with differing numbers of loop momenta.

**Integrals**

The simplest integral is when all there is no tensor numerator,

\[
\Gamma_{0,0}^{m_{ij}} = \int d^d x \prod_{a=1}^{b} \Delta(m, x) e^{ik \cdot x} .
\]  

(12)

The tensor integrals are computed via the identity,

\[
\Gamma_{l_{ij},n_{ij}}^{m_{ij}} = \mathcal{O}^{l_{ij}}[s_{ij}] \mathcal{O}^{n_{ij}}[s_{ij}] \Gamma_{0,0}^{m_{ij}} |_{m_{ij}} ,
\]  

(13)
which expresses all integrals via a derivative iteration on the integral (12). In this expression (13) the internal momenta in $s_{ij}$, with the invariants numbered by $l_{ij}$ and $n_{ij}$, are replaced with a differential $\partial_k$.

The integrals are evaluated via transforming to $x$-space and using the Bessel form of the massive propagators. The propagator is in $d$ dimensions,

$$\Delta(m, x) = \lambda_d(x^2)^{-d/2+1} K_{d/2}(mx). \quad (14)$$

The parameters are left as variables to span unusual propagation, such as $\phi e^{-\hat{\Lambda} \mathbf{\Box} \phi}$ or $\phi \mathbf{\Box}^2 \phi$, in view of $\phi^4$ theory, and the quantization of perturbatively nonrenormalizable theories. The integral in (12) evaluates to,

$$I_{0,0} = (k^2)^{d-(L+1)(d/2-1)} \sum \beta_{a,b} \left( \frac{k^2}{m^2} ight)^a \left( \frac{k^2}{\Lambda^2} \right)^b. \quad (15)$$

A momentum regulator is used, and dimensional reduction is also possible. The coefficients in (13) are,

$$\Delta(m, x)^N = (x^2)^{N\beta_1-N\beta_2/2}(m^2)^{-N\beta_2/2} \sum_{n=0}^{\infty} \sum_{a=1}^{m} \frac{1}{n!} \sum_{n_a, n_b} \left| \sum_{n=1}^{m} b_{a,n_a} n \right| n! \prod_{a=1}^{m} (n_a) ! \quad (16)$$

$$x \times \frac{\Gamma(N+1)}{\Gamma(N-m)} c_{\{n_a, n_b\}} \ (x^2 m^2)^n \quad (17)$$

with

$$c_{\{n_a, n_b\}} = \prod_{a=1}^{m} \partial_u^{(b_a),n_a} \ (x^2 m^2)^{\beta_2/2} K_{\beta_2}(u) \big|_{u=m^2 x^2}, \quad \partial_u^{(b_a),n_a} f = (\partial_u^{(b_a)} f)^{n_a} \quad (18)$$

The coefficients in (18) are derived via the Bessel function series,

$$K_{\beta_2}(u) = e^{-i\pi \beta_2/2} \left( -\frac{u^2}{4} \right)^{\beta_2/2} \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m+1-\beta_2)} \left( -\frac{u^2}{4} \right)^m + \beta_2 \to -\beta_2, \quad (19)$$

which is the contour rotated Euclidean version. The scalar expansion in even powers of $x^2 m^2$ is,

$$e^{-i\pi \beta_2/2} 2^{-\beta_2} \frac{\Gamma(m+1)}{\Gamma(m+1-\beta_2)} \left( \frac{1}{4} \right)^m \quad (20)$$
\[-e^{i\pi \beta_2/2} \frac{\Gamma(m + \beta_2)}{\Gamma(m + 1 + \beta_2)} \left(\frac{1}{4}\right)^{m + \beta_2} \quad m \to m - \beta_2 .\]  

(21)

Due to the dimension $d$, there is a Taylor series for $d$ even; the propagator should be $\Delta_+$ and $\Delta_-$, which is expanded in $\Delta^N$. Then a Taylor series expansion can be defined again for general $d$; else the coefficients could be used as a variant of dimensional reduction.

The parameters in (18) are,

$$
\Delta(m, x)^N = \sum_{a=0}^{\delta} \delta_{N,a} m^{2(a-\beta_2/2)} (x^2)^{a+N(\beta_1-\beta_2/2)} .
$$

(22)

The integrals are,

$$
\int d^d x e^{ik \cdot x} (x^2)^{a+N(\beta_1-\beta_2/2)}
$$

(23)

$$
= (-\partial_k \cdot \partial_k)^{-a} \frac{\Gamma(-N(\beta_1 - \beta_2/2) + 1 - a)}{\Gamma(-N(\beta_1 - \beta_2/2) + 1)} \int d^d x e^{ik \cdot x} (x^2)^{N(\beta_1-\beta_2/2)}
$$

(24)

$$
= \left(\frac{\Gamma}{\Gamma}\right) (\partial_k \cdot \partial_k)^{-a} \rho(\beta_1, \beta_2) (k^2)^{-d/2-N(\beta_1-\beta_2/2)}
$$

(25)

$$
= \rho(\beta_1, \beta_2) \frac{\Gamma(-N(\beta_1 - \beta_2/2) + 1 - a)}{\Gamma(-N(\beta_1 - \beta_2/2) + 1)}
$$

(26)

$$
\times \frac{\Gamma(-d/2 - N(\beta_1 - \beta_2/2) + 1)}{\Gamma(-d/2 - N(\beta_1 - \beta_2/2) + 1 - a)} (k^2)^{-d/2-N(\beta_1-\beta_2/2)-a} ,
$$

(27)

with

$$
\rho(\beta_1, \beta_2) = (-1)^a \int d^d x e^{ik \cdot x} (x^2)^{N\beta_1-N\beta_2/2} .
$$

(28)

The coefficients are defined dimensional reduction and momentum (string-inspired) cutoff and to find the theory in multiple dimensions.

The tensor integrals have been examined in [6]. Momentum of plane waves in $x$-space, within the effective action have the form of a derivative, as in quantum mechanics: $k_i \leftrightarrow i\partial_i$ via the Fourier transform. The integrals with the internal derivatives could be evaluated directly.
Those in (5) have the form (13), with the internal derivatives (or momenta) extracted from the internal lines; these momenta have an action on the integral in (12) as,

\[ \partial^\mu \Delta(m, x)^N = [(2 - d) + m^2 \partial_m^2] \times \left( \frac{x^\mu}{x^2} \right) \Delta(m, x)^N. \]  

The number \( n \) counts the \( \partial_x \)'s. The \( x \) factors are removable via,

\[ (\partial_{k^\mu})^{-1} = -\frac{x^\mu}{x^2}, \]

so that the general tensor integral requires only the scalar evaluation, followed by tensor derivatives as in (13).

The derivatives in (29) have the effect of changing \( \delta \) to

\[ \delta_{N,a} \rightarrow \delta_{N,a} \rho \sum_{l=0}^{n!} \frac{n!}{l! (n - l)!} (2 - d)^{n-l} \frac{\Gamma(a - N\beta_2/2 + 1)}{\Gamma(a - N\beta_2/2 + 1 - l)}. \]  

The \( \delta \) changes, due to the \( m^2 \partial_m^2 \) differential operators, if \( l \geq a - N\beta_2/2 \). If \( l < a + N\beta_2/2 \) then \( \delta = 0 \).

The action of the inverse derivatives in (30) on (31) further modifies the \( \delta \) to,

\[ \delta_{N,a}(k^2)^{-d/2-N(\beta_1-\beta_2/2)-a} \rightarrow \delta_{N,a} \frac{\Gamma(d/2 + N(\beta_1 - \beta_2/2) + a + 1 - n)}{\Gamma(d/2 + N(\beta_1 - \beta/2) + a + 1)} \]

\[ \times \prod_{j=1}^{n} \partial_{k^\mu_j} (k^2)^{-d/2-N(\beta_1-\beta_2/2)-a-n}. \]

The latter tensor is,

\[ \prod_{j} k_{\mu_j} 2^n \frac{\Gamma(-d/2 - N(\beta_1 - \beta_2/2) - a - n + 1)}{\Gamma(-d/2 - N(\beta_1 - \beta_2/2) - a - 2n + 1)} (k^2)^{-d/2-N(\beta_1-\beta_2/2)-a-2n} \]

\[ \sum_{\text{perms}} \eta_{\mu_1\mu_2} \prod_{j} k_{\mu_j} 2^{n-1} \frac{\Gamma(-d/2 - N(\beta_1 - \beta_2/2) - a - n + 1)}{\Gamma(-d/2 - N(\beta_1 - \beta_2) - a - 2n + 2)} \]

8
$$\times (k^2)^{-d/2-N(\beta_1-\beta_2/2)-a-2n+1}$$ (36)

and on, via incrementing the factorial and multiplying the number of metrics when
the number of derivatives is even. The general form is,

$$\sum_{\sigma_w, \tilde{\sigma}_w} \prod_{\mu, \tilde{\mu}} \eta_{\sigma(\mu)} \mu_{\tilde{\sigma}(\tilde{\mu})} \prod_{n=1}^{n-w} k_{\mu(\tilde{\mu})} \frac{2^{n-w}}{\Gamma(-d/2-N(\beta_1-\beta_2/2)-a-n+1)} \frac{\Gamma(-d/2-N(\beta_1-\beta_2/2)-a-2n+1+w/2)}{\Gamma(-d/2-N(\beta_1-\beta_2/2)-a-n+1+w/2)}. \tag{37}$$

with a factor $(k^2)^{-d/2-N(\beta_1-\beta_2/2)-a-2n+w/2}$ The $\sigma$ and $\tilde{\sigma}$ are vectors with $w$ components, and there is a summation over all combinations. The $\rho$ set is the complement of these two vectors in the space of the $n$ components.

The net result for the tensor integrals is,

$$\int d^d x e^{ik \cdot x} \prod_{\mu} \partial_{\mu} \Delta(m, x)^N = T^n_{\mu} \sum_{\alpha=1}^{\infty} \delta(N, a)(m^2)^{a-N\beta_2/2} (k^2)^{-d/2-N(\beta_1-\beta_2/2)-a} \tag{38}$$

$$\times \Gamma(\beta_1, \beta_2) \frac{\Gamma(-N(\beta_1-\beta_2/2)+1-p)}{\Gamma(-N(\beta_1-\beta_2/2)+1)} \frac{\Gamma(-d/2-N(\beta_1-\beta_2/2)+1)}{\Gamma(-d/2-N(\beta_1-\beta_2/2)+1-p)} \tag{39}$$

$$\rho(\beta_1, \beta_2) \frac{\Gamma(p-N\beta_2/2+1)}{\Gamma(p-N\beta_2/2+1-l)} \frac{\Gamma(p-N\beta_2/2+1-l)}{\Gamma(p-N\beta_2/2+1)} (m^2)^{p-N\beta_2/2} (k^2)^{-d/2-N(\beta_1-\beta_2/2)-p} \tag{40}$$

$$\times \frac{\Gamma(d/2+N(\beta_1-\beta_2/2)+p+1-n)}{\Gamma(d/2+N(\beta_1-\beta_2/2)+p+1)} \tag{41}$$

$$\sum_{\sigma_w, \tilde{\sigma}_w} \prod_{\mu, \tilde{\mu}} \eta_{\sigma(\mu)} \mu_{\tilde{\sigma}(\tilde{\mu})} \prod_{n=1}^{n-w} k_{\mu(\tilde{\mu})} \frac{2^{n-w}}{\Gamma(-d/2-N(\beta_1-\beta_2/2)-p-n+1)} \frac{\Gamma(-d/2-N(\beta_1-\beta_2/2)-p-2n+1+w/2)}{\Gamma(-d/2-N(\beta_1-\beta_2/2)-p-2n+1+w/2)}. \tag{42}$$

The sums on $l$ and $a$ should be performed, in order to have a simplified expression at fixed tensor structure. The momentum $k$ is the sum of the momenta on the exterior of the integral, i.e. $k = \sum_{j=1}^{q} k_j$. The number $n$ refers to the number of derivatives $\partial_{\mu}$ on the internal lines of the integral. $m$ refers to the maximum number $a$ in $b_a$ and $p_a$, except where the mass term is obvious.
Figure 1: The diagram showing the iteration to the recursive formula. This figure shows how tree diagrams are used to construct loop amplitudes, when both are momentum expanded. The recursion is not required from this point of view. The internal and external lines within the loop are to be an indefinite number.

Solution to Coefficients

The iterative formula in (10) can be expanded into a product form. The substitution of the prior $\alpha$ terms into the expression will continue until the $\alpha^{q ij}$ represents the classical scattering. In $\phi^3$ theory this occurs at $q - 2 = i$ coupling order ($q - 2$ vertices); more general scalar theories have more than one coupling constant and the count is more complex.

The expansion of the iterative formula is represented in Figure 1; the sum of nodes from 2 to $n_{\text{max}}$ is required. The number of an individual propagator can be any integer, in conjunction with the expansion of a tree diagram. The external legs are permuted at the nodes appropriate to the color structure.

The tree level initial conditions are required to solve the recursion; a scalar field theory possessing higher derivative terms can model any initial condition. The $\phi^3$ and $\phi^4$ initial conditions are described in [11], with a bootstrap condition $q - 2 = i$ and $q - 2 = 2i$; $q$ is the external leg number and $i$ counts either the 3- or 4-point vertices (coupling constants).

The recursion solution is,
\[
\alpha_{n,q}^{n_{ij}} = \sum_{a_{\text{nodes}}=1}^{N_{\text{nodes}}} \sum_{a_{\text{nodes}}=1}^{n_{(c)}} \prod_{a=1}^{n_{(a)}} \alpha_{a_{\text{nodes}}}^{n_{ij}(a)} \prod_{b=1}^{n_{(b)}+n_{(b+1)}} I_{n_{ij},n_{ij}}^{n_{(b)},n_{(b+1)}},
\] (44)

with \( b_{\text{nodes}} = a_{\text{nodes}} - 1 \). The number of nodes is to be summed; the maximum is set by the initial conditions. External lines may exit from any of the nodes. The numbers of propagators have to be summed at each of the nodes, when \( a_{\text{nodes}} \geq 2 \). The \( \alpha_{n_{ij},q(a)}^{n_{(a)}} \) parameters are classical (loop zero); different boundary conditions could iterate from non-classical data without altering the form of (44). When there is more than one coupling constant, \( q = \sum q^{(a)} \). The classical coefficients in \( \phi^3 \) theory are

\[
\alpha_{n_{ij},q(a)}^{n_{(a)}} = (m^2)^{n^{(a)}-3} \alpha^{n^{(a)}-2} \prod (m^2)^{-n_{ij}^{(a)}},
\] (45)

with the latter factor representing the mass expansion of the propagators.

The parameters \( n_{ij}^{(a)} \) project the form in (5) at a fixed tensor. At each node there are \( n_{b-1} \) lines to the left and \( n_{b} \) lines to the right. The tensor structure is denoted by \( n_{ij} \), and all of the kinematics of the \( s_{ij} \) in the integrals add to form the tensor of \( \alpha_{n_{ij}} \).

In order to find the product of the integrals at fixed parameters, in (5), the kinematics at the vertex are expanded as,

\[
s_{ij} = (k_i + i\partial_x)^2 = 2ik_i \cdot \partial_x - \partial_x^2,
\] (46)

and

\[
s_{ij} = -4\partial_x^2.
\] (47)

The first example is the situation when \( i \) is an external leg and the other line is an internal leg; the latter has both internal legs. The derivatives are those in (29). Starting at the left node, the number of momenta which are internal are counted so as to define the tensor in (5). The numbers \( \phi_n \) are used for this count [11, 12]. These numbers are a (symmetric) set theoretic foundation to build any \( \phi^3 \) tree diagram. Given the set \( \phi_n \), a function has has to be made that counts the number of \( s_{ij} \) belonging to the internal-external (ie) class. Else the explicit tree diagram labeled by the numbers of \( n_{ij} \) has to be used, without the simple set theoretic definition.

At the node, there is a tensor from the expansion of the invariants in (46) and (47). The number of spatial derivatives ranges from \( \Gamma_1(\phi_n, n_{ij}, u) \) to \( \Gamma_2(\phi_n, n_{ij}, u) \).
These numbers depend on the external leg set $v$ and the set of $s_{ij}$ as defined by $\phi_n$ and $n_{ij}$. The difference between the two numbers is due to the counts $s_{ii}$ and $s_{ie}+s_{ii}$, i.e. the number of invariants with shared legs. The sets $\phi_n$ are not required if the input $n_{ij}$ is given independently. To each of these counts is a tensor $W^{(b)}_{\nu,j}$.

Each of the integrals has $p_{(b)}$ internal lines. The tensors $W_{\nu;n}$ contract with the momentum of the external line as the integral is a function only of $k$; the explicit form is in (43). If there are external legs attached to the node, as illustrated on node 2 in Figure 2, then $k = \sum_{\text{node}_b} k_j = \sum_{\text{node}_i} p_i$; these momenta contract with the node tensors $W_{\nu;n}$.

The expansion of the invariants

$$(-4\partial^2)^{s_{ii}} \prod_{i,j \in \text{ie}} (2ik_i \cdot \partial_x - \partial^2)^{n_{ij}} = T^{(b)}_{\mu_i,w,k_j} \prod \partial^{\mu_i},$$

defines the tensors $T_{\mu_i,w,k_j}$ for the variable $w$ and node $b$. The node momentum $p = \sum_b p_a$ is a sum of the previous on-shell momenta $k_{\sigma(m_i-m_0)}$ to $k_{\sigma(m_f-\tilde{m}_0)}$. As a result these invariants are expressed in terms of the two-particle invariants through

$$p^2 = \sum_{i<j} s_{ij};$$

these $s_{ij}$ variables are used to define the tree and the adjacent loop integrations.

The tensor in (48) is

$$T^{(b)}_{\mu_i,w,k_j} = C(q_1)(-4)^{q_2}(2i)^{q_1} \prod_{i=1}^{q_1} k_{\alpha_\sigma(i)} \prod_{i=1}^{q_2} \eta_{\alpha_\nu(i+1/2)} \prod_{i=1}^{q_1} \eta_{\alpha_\sigma(i)\nu_i}$$

with the prefactor $C(q_1)$ defined from (48)

$$C(q_1) = \prod_{ij \in \text{ie}} \frac{n_{ij}!}{(n_{ij}-\tilde{n}_{ij})!(\tilde{n}_{ij})!} (-1)^{(n_{ij}-\tilde{n}_{ij})} q_1 = \sum \tilde{n}_{ij}.$$ 

The remaining derivatives contract with the external momenta set $\sigma(i)$. The loop tensor from (45) is

$$\sum_{\sigma_w,\tilde{\sigma}_w} \prod_{i=1}^{w} \eta_{\sigma(i)\mu_\rho(i)} \prod_{i=1}^{n-w} k_{\mu_\rho(i)}.$$ 

The contraction of the two tensors results in
\[ T(k_i) = \sum_{\beta, \tilde{\beta}, \alpha} B_{d_b} \left( P^2_b \right)^c \prod s_{\beta(i)} \tilde{\beta}(i) \prod i \cdot k_{\alpha(i)} , \] (52)

with \( \beta(i) \) and \( \tilde{\beta}(i) \) denoting labels in the set of indices \( \sigma(m_i - m_b) \) to \( \sigma(m_f + m_b) \).

The \( (P^2_b)^c \) is expanded to the terms,

\[ \left( \sum s_{ij} \right)^c = \sum \prod s_{\rho(k)} \tilde{\rho}(k) , \] (53)

with \( \sigma \) and \( \tilde{\sigma} \) specifying the permutations in the product from the power \( c \); \( k \) range from 1 to \( c \). The numbers \( i \) and \( j \) are pairs of numbers between \( \sigma(m_i - m_b) \) and \( \sigma(m_f + m_b) \), which represent the indices of the external momenta at node \( b \). The permutation sets are all combinations of the pairs of numbers \( i, j \) including repeating pairs, which is the same as all sets of numbers \( i \) and \( j \) including repeating ones.

The last term in (52) in terms of two-particle invariants is,

\[ \sum_{\tilde{\alpha}} \prod_{i,j} s_{\tilde{\alpha}(j)k_{\alpha(i)}} . \] (54)

The \( \tilde{\alpha} \) is summed over all combinations of the momentum labels in the set of \( P_b \), the momentum flowing into the loop at node \( b \). The expansion is then all pairs of numbers \( \tilde{\alpha} = \sigma(m_i - m_b), \ldots, \sigma(m_f + m_b) \) and \( \alpha \) with the first set repeating in all possible ways.

The net result for the tensor at level \( w \) is a collection of \( s_{ij}^{n_{ij}} \). These two-particle invariants are all external lines to the loop system at node \( b + 1 \). The number of these invariants is denoted \( m_{ij}^{n_{ij}-1} \), which is a function of the preceeding node \( b - 1 \) and the number of propagators.

The integral factors in the formula (44) multiply the tensor produc ts in (48). These functions are the product

\[ \alpha_{n_{ij}}^{m_{ij}} = \sum_{\{n_b\}} \prod_{b=1}^{b_{\text{max}}} I_{n(b), w_b} \ T(k_i) , \] (55)

\[ T(k_i) = \prod_{b=1}^{\text{nodes}} \prod_{i,j} s_{ij}^{m_{ij}, n_{b-1}} \prod_{i,j \in \text{ee}, b_{\text{max}}} s_{ij}^{m_{ij}} \] (56)
The integral product is found from multiplying the scalar integrals in \((5)\); these depend on the number of propagators and the index \(w\). The summation over the propagators is independent at each integral, which does change the initial condition \(\alpha_{n_{ij}}^{n_{b-1}}\). The \(n_{b-1}\) is the propagator number at node \(b\), counting lines to the left; at \(b = 1\), \(m_{ij}\) counts the external-external invariants.

Having found the scalar product and the tensor product, the bounds on the sum require to be defined. The coupling order \(q\) can partition into the nodal orders via \(q = \sum_b q_b\); the minimum and maximum for \(\phi^3\) theory is \(q_b = 2\) and \(q_b = q - 2\).

The number of partitions is

\[
\sum_{b=2}^{b_{\text{max}}} \sum_{\{q_m\}; q_{b_{\text{nodes}}}} \frac{q!}{q_1!q_2! \ldots q_b!} = \sum_{b=2}^{\max} \frac{q!}{b!} b^b - \text{boundary terms} \tag{57}
\]

with the boundary conditions appropriate to the theory; the latter remove the \(q_b = 1, 0\) and \(q_b = q - 1, q\).

The permutation sum on the external lines has to be performed. The \(n\) external lines are to be placed in all possible ways located at the \(b\) nodes. The trace structures of a non-abelian theory require the permutation subsets of the external lines; the color flow appears simpler due to the topology of the rainbow graphs.

The iterative formulae has to include the propagator corrections. It appears that these corrections were not included in the product form of the amplitudes. However, the quantum vertices in \((10)\) take into account these quantum corrections, and so should the latter form. Tree diagrams in the mass expansion allow the generation of propagator corrections.

(If there is a formal reason to examine the amplitudes without the propagator corrections, or \(m\)-point corrections, then it is possible to extract them. This is done by modifying the external lines of the tree amplitudes with the mass expansion of the full quantum two-point function, i.e. \(\sum T_p m^{2p} \Box^p \Delta\). Each of the \(\Box\)s on the internal lines is included in the iteration and the \(T_p\) coefficient modifies the vertex. The summation on the derivatives in the classical vertex takes into account the propagator correction by altering the limits on the classical vertex and multiplying the \(T_p\); there is a \(p_j\) on each internal line that modifies the count of \(n\) by \(n \to n + \sum 2p_j\). The vertex gets a factor \(\prod T_{p_j}\). To eliminate the propagator correction, at each vertex divide by the \(T_{p_j}\) numbers and alter the sum by \(n \to n - \sum 2p_j\). The \(m\)-point corrections are eliminated in a similar fashion.)

Concluding remarks
The quantum theory to any scalar field theory is generated in a direct manner; \( n \)-point scattering amplitudes are composed through a product of tree amplitudes of varying coupling orders and with varying numbers of legs. The conservation of both is used to find the coefficient of the scattering at any loop order \( L \) and \( n \)-point. Formulae in this paper demonstrate a simple sum of products which arise from a breaking of these orders into partitions; an example is \( \phi^3 \) theory in which the coupling order is \( m - 2 \) with \( m \) being the number of legs. All scalar field theories, including those models with any number of higher dimensional operators, are quantized with the initial condition of the classical scattering. The solutions could lead to better formulations of the quantum scalar models and the possible geometries, or potential conformal models, generating them; this includes the large coupling regime.

The requirements for any of these massive theories to be solved at any order are the tree amplitudes, which are expanded in low energy (e.g. see [11]). The formulae involving the \( \Gamma \) summations and the tensors can be simplified, and this will lead to a more compact representation, less than this the half page of algebra given in this work.

The unitarity has been made indiscreet due to the large mass or low momentum expansion. This can be found by resumming the momentum modes at a specific order. This has been examined in [1], [5], [6], [8], [10] in the this expansion.

This work shows that the solution to a quantum field theory can be obtained, in the case of an arbitrary scalar field theory in any \( d \)-dimension. The same is true for gauge and gravity theories (for which the tree amplitudes have been found in a number context in [12], and also in the standard model. The formulae are similar but with more complicated tensor algebra.

The complete scalar classical scattering in generic non-linear \( N = 2 \) sigma models also follows number theoretically from the classical \( \phi^3 \) theory, and this includes all toric Calabi-Yau quotients [13]; the complete quantum solution of these scalar models can be found.

\textit{Note added}: There are further simplifications of the tensor algebra. The index and integral forms are suitable for a computer implementation of the scattering derivation. Revision to text is notation and a gamma function in equations (38)-(43).

The form of amplitude is,

\[ A_n = g^L \prod s_{ij}^{n_{ij}} \sum \prod B(n_i, m_i, w_i), \]

\textbf{(58)}
with $n_i$ and $m_i$ parameterizing the number of internal lines and derivatives acting on them, and $w_i$ the number of metric tensors. The notation follows equation (43). The sum on these numbers depends on the number of loops $L$ and the kinematics $n_{ij}$. The functions $B(n, m, w)$ are the 'building blocks', the sums given in (43).
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