ROBUST OPTIMAL CONSUMPTION-INVESTMENT STRATEGY WITH NON-EXPONENTIAL DISCOUNTING

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Abstract. This paper extends the existing dynamic consumption-investment problem to the case with more general discount functions under the robust framework. The decision-maker is ambiguity-averse and invests her wealth in a risk-free asset and a risky asset. Since non-exponential discounting is considered in our model, our optimization problem is time inconsistent. By solving the extended Hamilton-Jacobi-Bellman equations, the corresponding optimal consumption-investment strategies for sophisticated and naive investors under power and logarithmic utility functions are derived explicitly. Our model and results extend some existing ones and derive some interesting phenomena.

1. Introduction. The continuous-time optimal consumption-investment problem was first studied by [31, 32]. Therein the goal of the investor is to seek an optimal consumption-investment strategy to maximize the expected utility from consumption and terminal wealth. Since then, there has been an increasing interest in consumption-investment problems and the Merton’s problem has been extended in many directions, such as investigations with stochastic volatilities ([17], [41]), transaction costs ([22], [1]), models with random coefficients ([7], [13], [4]), incorporating life insurance ([9], [37]), and so on.

Although the optimal consumption-investment problem has been widely investigated by many scholars, two aspects ought to be explored further. On the one hand, an optimization problem with non-exponential discounting is worthy to be

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considered. Most of the above-mentioned literature uses exponential discounting that the rate of time preference is constant. However, there is growing evidence to demonstrate that the exponential discounting disagree with actual behavior of decision-makers, see [19]. Moreover, [26] presents four drawbacks of exponential discounting and proposes a model which accounts for them. The authors analyze the implications for savings behavior and estimation of discount rates. Therefore, it is necessary to take non-exponential discounting into consideration. If the non-exponential discounting is considered, the dynamic optimal control problem becomes time inconsistent, in the sense that Bellman’s principle of optimality does not hold. Hence, the dynamic programming approach cannot be used directly in the time inconsistent problem. [38] first studies the time inconsistent problem and proposes three basic ways to handle it: (i) The decision-maker recognizes the time inconsistency of her preferences and finds the most effective precommitment strategy, which means that the decision-maker view a precommitment strategy as an optimal strategy derived at the initial time \( t_0 = 0 \) even if this strategy is not optimal at the later time; (ii) the decision-maker is naive and does not recognize the time inconsistency of the problem she considers, she constantly revises her strategy according to her current preferences without considering her future preferences may change; (iii) the decision-maker is sophisticated and recognizes that her future preferences change over time in a temporally inconsistent way; she views the time inconsistent problem as a non-cooperative game, i.e., at each time \( t \), there is a player \( t \), representing the future incarnation of the decision-maker at time \( t \), and the decision-maker seeks a subgame perfect Nash equilibrium strategy, which is time-consistent\(^1\). In past several years, some scholars are increasingly interested in the optimal consumption-investment problem with time-inconsistent preference, such as [23], [15], [14], [16], [44], [45], and so on. Besides, there are also a great deal of literature about other time-inconsistent problems in recent years, such as [21], [12], [6], [10], [11], and so on.

On the other hand, an optimization problem with model uncertainty is another important direction worthy to be investigated, while a fundamental assumption in most above-mentioned works is absence of model uncertainty. It is a notorious fact that the return of risky assets is difficult to be estimated with precision, and more and more researchers are sceptical about the reliability of standard historical estimates, which implies that it becomes more and more significant to incorporate model uncertainty into portfolio selection problems. In past ten years, many scholars hope for a systematic and quantitative way to take such model uncertainty into account. For example, [27, 28] investigate the effect of ambiguity on the intertemporal portfolio choice in a setting with constant investment opportunities and in a setting with a mean-reverting equity risk premium, respectively. A number of other papers are built on [27] to address the effects of ambiguity on portfolio choice. [36] considers a robust optimal control for a consumption-investment problem under HARA utility. [24] examines the robust consumption and portfolio choice for time-varying investment opportunities. [25] discusses a continuous-time intertemporal consumption and portfolio choice problem under ambiguity, where expected

\(^1\)As the statement in [8], for a dynamic optimization problem, if the strategy \( \pi_{t_1} \) is optimal for the decision-maker at some time \( t_1 \), and for any later time \( t_2 > t_1 \), she will follow the strategy \( \pi_{t_1} \) if it is still optimal at time \( t_2 \), i.e., \( \pi_{t_1}(t) = \pi_{t_2}(t) \) for all \( t > t_2 \), then it is called a time-consistent strategy, see [5], [6], and so on. If a strategy is optimal at a time \( t_1 \), while is not optimal at some later time \( t_2 \), i.e., \( \pi_{t_1}(t) \neq \pi_{t_2}(t) \) for some \( t > t_2 \), then \( \pi_{t_1}(t) \) is a time-inconsistent strategy.
returns of a risky asset follow a hidden Markov chain. [18] determines the optimal investment strategy for an ambiguity-averse investor with a stochastic interest rate. [33] introduces a stochastic interest rate and inflation into a portfolio management problem for an ambiguity-averse investor. Moreover, some other literature about model uncertainty we refer to [3], [29], [20], [42], [39], [43], and so on.

As far as we know, there is no literature considering both non-exponential discounting and model uncertainty in the optimal consumption-investment problem, while the two elements are quite important. In this paper, we try to incorporate both the two elements into the optimal consumption-investment problem. Suppose that the financial market consists of a risk-free asset and a risky asset, and the decision-maker is ambiguity-averse about the risk of the risky asset. The consideration of non-exponential discounting results in that the corresponding dynamic optimization problem becomes time-inconsistent and Bellman’s principle of optimality does not hold. By solving the extended Hamilton-Jacobi-Bellman (HJB) equation (see [40]), we derive the corresponding optimal consumption-investment strategies for sophisticated and naive investors under power and logarithmic utility functions explicitly. From our results, we find that: (i) the robust optimal investment strategies for sophisticated and naive investors are the same, which are independent of the discount rate; (ii) the robust optimal consumption strategies for the two investors are quite different; (iii) the results for logarithmic utility are indeed the limits (as $\gamma \to 1$) for the power utility. The main contributions of this paper are as follows: (i) A new continuous time optimal consumption-investment model incorporating non-exponential discounting and model uncertainty is established; (ii) the robust optimal consumption-investment strategies for sophisticated and naive investors are derived explicitly, and the two strategies are compared; and (iii) our model and results are more general, which can reduce to models only considering time-inconsistent preference (e.g. [30]) or model uncertainty (e.g. [27]).

This paper proceeds as follows. Section 2 describes the formulation of the model. Section 3 obtains the sophisticated consumption-investment strategy for power and logarithmic utilities. Section 4 provides the corresponding naive consumption-investment strategy. Section 5 presents some numerical illustrations and Section 6 concludes this paper.

2. The model. Let $W(\cdot)$ be a one-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and the filtration $\mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the augmentation under $\mathbb{P}$ of $\sigma(W(s), 0 \leq s \leq t), \ t \in [0, T]$, where $\mathbb{P}$ is a reference measure and $T > 0$ is a finite constant representing the investment time horizon. All stochastic process and variables below are assumed to be well-defined on this probability space. Moreover, it is assumed that there are no transaction costs or taxes in the financial market and trading can be continuous. In this section, we first describe the financial market and model uncertainty, and then establish our robust time-inconsistent consumption-investment problem with non-exponential discount function and ambiguity.

2.1. The financial market. We consider a market which consists of two assets, a bond and a stock. The price of the bond evolves according to the differential equation

$$
\begin{align*}
\begin{cases}
dB(s) = r(s)B(s)ds, & s \in [0, T], \\
B(0) = 1,
\end{cases}
\end{align*}
$$

where $r(\cdot) > 0$ is the interest rate. The price of the stock is modelled by the stochastic differential equation
\[
\begin{aligned}
\begin{cases}
\mathrm{d}S(s) &= \mu(s)S(s)\mathrm{d}s + \sigma(s)S(s)\mathrm{d}W(s), \quad s \in [0,T], \\
S(0) &= s_0,
\end{cases}
\end{aligned}
\]

where \( \mu(\cdot) > r(\cdot) \) is the appreciation rate and \( \sigma(\cdot) \geq \bar{\sigma} > 0 \) is the volatility. We assume that \( r(\cdot), \mu(\cdot) \) and \( \sigma(\cdot) \) are bounded deterministic functions.

A consumption-investment strategy is a bivariate process \( (c(\cdot), u(\cdot)) \in \mathbb{R}^+ \times \mathbb{R} \), where \( c(\cdot) \) is the consumption at time \( t \) and \( u(\cdot) \) is the amount of wealth invested in the stock at time \( t \).

For any initial time \( t \in [0, T] \) and initial wealth \( x_t > 0 \), applying a consumption-investment policy \( (c(\cdot), u(\cdot)) \), the wealth process of the investor, denoted by \( X(\cdot) \), is governed by

\[
\begin{aligned}
\begin{cases}
\mathrm{d}X(s) &= [r(s)X(s) - c(s) + u(s)\sigma(s)\theta(s)]\mathrm{d}s + u(s)\sigma(s)\mathrm{d}W(s), \quad s \in [t,T], \\
X(t) &= x_t,
\end{cases}
\end{aligned}
\]

where \( \theta(s) := \frac{\mu(s) - r(s)}{\sigma(s)} \) is the Sharp ratio of the risky asset.

In the following, to derive the optimal strategy, we restrict ourselves to Markov controls. This is not too restrictive, as under proper conditions one can always obtain as good as performance with a Markov control as with an arbitrary \( \mathbb{F} \)-adapted control (see [34], pp.250). By abuse of notation, we write \( c(t) = c(t, X(t)), u(t) = u(t, X(t)) \), where the map \( (c, u) : [0, T] \times (0, \infty) \to [0, \infty) \times \mathbb{R} \) is a Borel measurable function.

### 2.2. Model uncertainty

In this paper, we work under the framework of [2] and [27] which study the portfolio selection problems involving uncertainty about the return process for stocks. In this framework, the alternative models can be obtain from the reference model by using Girsanov's theorem.

We define a density processes for the Brownian motion. Let \( \xi(\cdot) \) be an \( \mathbb{F} \)-adapted process satisfying Novikov condition

\[
\mathbb{E}\left[ \exp\left\{ \frac{1}{2} \int_0^T \xi^2(s)\mathrm{d}s \right\} \right] < \infty.
\]

Denote by \( \Xi \) the set of all such processes. For each process \( \xi(\cdot) \in \Xi \), the density process for the Brownian motion associated with \( \xi(\cdot) \) is defined as an \( \mathbb{F} \)-adapted process \( \Lambda^\xi(\cdot) \) given by as below:

\[
\Lambda^\xi(t) := \exp\left\{ -\int_0^t \xi(s)\mathrm{d}W(s) - \frac{1}{2} \int_0^t \xi^2(s)\mathrm{d}s \right\}.
\]

Then we have

\[
\begin{aligned}
\begin{cases}
\mathrm{d}\Lambda^\xi(t) &= -\Lambda^\xi(t)\xi(t)\mathrm{d}W(t), \quad t \in [0,T], \\
\Lambda^\xi(0) &= 1.
\end{cases}
\end{aligned}
\]

Since the Novikov condition is satisfied, we know \( \Lambda^\xi(\cdot) \) is an \( (\mathbb{F}, \mathbb{P}) \)-martingale.

Now, we can define a measure \( Q^\xi \sim \mathbb{P} \) on \( \mathcal{F}_T \) by

\[
\frac{\mathrm{d}Q^\xi}{\mathrm{d}\mathbb{P}} := \Lambda^\xi(T).
\]

In the following, we denote by \( \mathbb{E}^\xi[\cdot] \) the expectation under the measure \( Q^\xi \). Let

\[
W^\xi(\cdot) = W(\cdot) + \int_0^\cdot \xi(s)\mathrm{d}s.
\]
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By Girsanov's theorem, \( W^\xi(\cdot) \) is a standard Brownian motion under \( Q^\xi \).

Then, we can rewrite the wealth process under \( Q^\xi \) as follows:

\[
\begin{align*}
\{ & dX(t) = \left[ r(s)X(t) - c(s) + u(s)\sigma(s)(\theta(s) - \xi(s)) \right] ds + u(s)\sigma(s)dW^\xi(s), \\
& s \in [t, T], \\
& X(t) = x_t.
\end{align*}
\]

(4)

Similar to the Subsection 2.1, by abuse of notation, we write \( \xi(t) = \xi(t, X(t)) \), where the map \( \xi : [0, T] \times (0, \infty) \rightarrow \mathbb{R} \) is a Borel measurable function. Let \( \pi(\cdot, \cdot) := (c(\cdot, \cdot), u(\cdot, \cdot), \xi(\cdot, \cdot)) \). To emphasize the dependence of the wealth process on the initial state and the policy, we also write the solution of (4) as \( X(\cdot; t, x_t, \pi) \).

2.3. Robust time-inconsistent consumption-investment problem. In this subsection, we introduce the optimization problem of the investor. At any time \( t \) with initial wealth \( x_t \), the performance functional under probability measure \( Q^\xi \), i.e., the expected discounted utility from the consumption and terminal wealth plus the penalty incurred when moving away from the reference model, is given by

\[
J(t, x_t; c, u, \xi) = \mathbb{E}^\xi \left[ \int_t^T h(s - t)U(c(s, X(s)))ds + \alpha h(T - t)U(X(T)) + \int_t^T h(s - t) \frac{1}{2\phi(t, s, X(s))}\xi^2(s, X(s))ds \right],
\]

(5)

where \( \mathbb{E}^\xi[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t] \), \( \alpha \) is a nonnegative constant, \( \phi(\cdot, \cdot, \cdot) \) is usually referred as ambiguity aversion, \( h(\cdot) \) is a general discount function and \( U(\cdot) \) is the utility function.

In this paper we assume that the discount function satisfies \( h(0) = 1, h(\cdot) > 0, h'(\cdot) \leq 0 \) and the discount rate function \( -\frac{h'(\cdot)}{h(\cdot)} \) is bounded above by a constant \( \bar{\theta} > 0 \). For any given \( (t, x) \), the function \( \phi(\cdot, s, x) \) is continuous and we shall focus on the power and logarithmic utility, i.e., \( U(x) = x^{1-\gamma} \) where \( \gamma \in (0, 1) \) and \( U(x) = \ln x, x > 0 \).

Remark 1. The last term in (5) is the entropy penalty. In the case with \( \phi(\cdot, \cdot, \cdot) \equiv 0 \), the investor is extremely convinced that the true model is the reference model \( P \), any deviation from \( P \) will be penalized heavily by \( h(s - t) \frac{1}{2\phi(t, s, x)} \xi^2 \). Thus, \( \xi \equiv 0 \) must be satisfied to guarantee \( h(s - t) \frac{1}{2\phi(t, s, x)} \xi^2 \equiv 0 \), where no model uncertainty is allowed. At the other extreme, if \( \phi(\cdot, \cdot, \cdot) = \infty \), the investor has no information about the true model, and the term \( h(s - t) \frac{1}{2\phi(t, s, x)} \xi^2 \equiv 0 \) vanishes.

Remark 2. If \( \alpha = 0 \), the discount function \( h(\cdot) \) is exponential and \( \phi(t, s, x) \equiv \phi(s, x) \), our model reduces to the one studied in [27]. If \( \alpha = 1 \) and without model uncertainty (i.e., \( \phi(\cdot, \cdot, \cdot) \equiv 0 \)), then the problem is studied in [30].

It is well-known that the robust portfolio selection problem can be regarded as a game between the investor and the market, in which the investor chooses the consumption-investment strategy \( (c(\cdot, \cdot), u(\cdot, \cdot)) \), while the market chooses the probability measure (i.e. \( \xi(\cdot, \cdot) \)). Thus, we can say \( \pi(\cdot, \cdot) = (c(\cdot, \cdot), u(\cdot, \cdot), \xi(\cdot, \cdot)) \) is a strategy (of the two-person differential game). Before we state the optimization problem of the investor, we give the following definition of admissible strategies.
Definition 2.1. A measurable function \( \pi : [0, T] \times (0, \infty) \to [0, \infty) \times \mathbb{R} \times \mathbb{R} \) is called an admissible strategy if the following hold:

1. For all \( x_0 \geq 0 \), \( X(T; 0, x_0, \pi) \geq 0 \), \( c(\cdot, X(\cdot)) \geq 0 \) and \( \xi(\cdot, X(\cdot)) \in \Xi \), a.s.;
2. for all \( t \in [0, T] \) and \( x \geq 0 \),

\[
E^\xi \left[ \int_t^T \left( |U(c(s, X(s)))| + \left| \frac{1}{\phi(t, s, X(s))} \xi^2(s, X(s)) \right| \right] < \infty, \quad \text{a.s.,}
\]

\[
E^\xi \left[ |U(X(T))| \right] < \infty, \quad \text{a.s.,}
\]
where \( X(\cdot) = X(\cdot; t, x_t, \pi) \).

Denote by \( \Pi \) the set of all admissible strategies.

The robust optimal consumption-investment problem of the investor is the following:

Problem 1. For any initial state \( (t, x) \), find an admissible strategy \( \hat{\pi} := (\hat{c}, \hat{u}, \hat{\xi}) \in \Pi \) such that

\[
J(t, x; \hat{c}, \hat{u}, \hat{\xi}) = \sup_{c, u} \inf_{\xi} J(t, x; c, u, \xi). \quad (6)
\]

Note that both the ambiguity aversion and discount function depends on the initial time \( t \) in a general way. This makes Problem 1 time-inconsistent in the sense that the classical dynamic programming principle is no longer valid, and thus cannot be solved by some classical stochastic control methods.

As we mentioned in Introduction, there are three methods dealing with the time-inconsistent problems. With the first two approaches, the optimal strategies are time-inconsistent since an optimal strategy determined at a particular moment is not necessarily optimal at a later moment. However, the third approach imposes that the manager should take into account her future actions induced by her changing preferences. The third approach generates a time-consistent strategy and is implemented by taking the game theory point of view and considering the so-called subgame perfect Nash equilibrium strategies. Since the pre-commitment strategy can be obtained by studying the naive investor at the initial time, in this paper, we shall derive the solution for both sophisticated and naive investors.

3. Sophisticated investor. In this section, we derive an equilibrium strategy for a sophisticated investor who can be modeled as a sequence of autonomous selves. Each self decides the consumption-investment strategy of the present and is concerned about-but does not control-the strategy of the future. Furthermore, the sophisticated manager’s current self correctly foresees that her future selves will act according to their own preferences, and will reject her optimal strategy. As a result, the current self must select her strategy in light of her future selves’ actions, although this strategy may not be optimal at that time. The optimal control problem under this case is an intra-personal game between successive selves.

As we have mentioned in Subsection 2.3, Problem 1 can be viewed as a two-person game. More specifically, the investor chooses the strategy \( (\hat{c}, \hat{u}) \) such that the performance functional

\[
J_1(t, x; c, u) := \inf_{\xi} J(t, x; c, u, \xi)
\]

is maximized, while the market chooses \( \hat{\xi} \) such that
is minimized. The following definition of time-consistent equilibrium strategy is a natural extension of the one in [15] and [5] to the two-person differential game. Intuitively, the following time-consistent equilibrium strategies are the strategies such that, given that they will be implemented by the market (resp. investor) in the whole planning horizon and will be implemented by the investor (resp. market) in the future, it is optimal for the investor (resp. market) to implement them right now.

**Definition 3.1.** Given a control \( \hat{\pi} = (\hat{c}, \hat{u}, \hat{\xi}) \in \Pi \), for any \( t \in [0, T] \) and fixed real number \( \epsilon > 0 \), define

\[
(c'(s, y), u'(s, y), \xi'(s, y)) = \begin{cases} 
(c_0, u_0, \xi_0), & \text{for } s \in [t, t + \epsilon), y \in [0, \infty), \\
(\hat{c}(s, y), \hat{u}(s, y), \hat{\xi}(s, y)), & \text{for } s \in [t + \epsilon, t], y \in [0, \infty),
\end{cases}
\]

(7)

where \((c_0, u_0, \xi_0) \in [0, \infty) \times \mathbb{R} \times \mathbb{R}\) such that \(\pi^1_\epsilon := (c', u', \xi')\) and \(\pi^2_\epsilon := (\hat{c}, \hat{u}, \hat{\xi}')\) are in \(\Pi\). Let \(\hat{X}(\cdot)\) be the state process corresponding to \(\hat{\pi}\). We say that \(\hat{\pi}\) is a subgame equilibrium strategy for Problem 1 if

\[
\liminf_{\epsilon \to 0} \frac{J(t, \hat{X}(t); \hat{c}, \hat{u}, \hat{\xi}) - J(t, \hat{X}(t); c', u', \xi')}{\epsilon} \geq 0,
\]

(8)

and

\[
\limsup_{\epsilon \to 0} \frac{J(t, \hat{X}(t); \hat{c}, \hat{u}, \hat{\xi}) - J(t, \hat{X}(t); \hat{c}, \hat{u}, \hat{\xi}')}{\epsilon} \leq 0,
\]

(9)

for any \( t \in [0, T] \).

The following subsections are devoted to deriving the solutions for a sophisticated investor for our optimization problem. We first provide the equilibrium HJB equation, and then consider the solutions for the cases of power and logarithmic utility functions, respectively.

### 3.1. Time-consistent equilibrium HJB equation

In this section, we present the equilibrium HJB equation. First, we introduce some notations. Define

\[
H(\tau, t, x, c, u, \xi, p) := r(t)x - c + u\sigma(t)(\theta(t) - \xi) + \frac{1}{2}u^2\sigma^2(t)P + h(t - \tau)\left[U(c) + \frac{1}{2\phi(t, t, x)}\xi^2\right].
\]

Let

\[
\varphi(\tau, t, x, p) := \left(\varphi_1(\tau, t, x, p), \varphi_2(\tau, t, x, p), \varphi_3(\tau, t, x, p, P)\right)
\]

be the \((c, u, \xi)\) such that the \(\sup_{c, u, \xi} H(\tau, t, x, c, u, \xi, p)\) is attained.

If \(P < 0\) and \(p \geq 0\), then

\[
\begin{cases}
\varphi_1(\tau, t, x, p) = I \left(\frac{p}{h(t - \tau)\theta(t)p}\right), \\
\varphi_2(\tau, t, x, p) = -\frac{\sigma(t)\theta(t)p}{h(t - \tau)\phi(t, t, x)P}, \\
\varphi_3(\tau, t, x, p) = -\frac{\theta(t)p}{h(t - \tau)P - \phi(t, t, x)P},
\end{cases}
\]

(10)
where $I(\cdot) = (U')^{-1}(\cdot)$ is the inverse function of the marginal utility function.

Motivated by (4.77) in [40], we consider the equilibrium HJB equation:

\[
\begin{align*}
\Theta_t(\tau, t, x) + H(\tau, t, x, \varphi(t, t, x, \Theta_x(t, t, x), \Theta_{xx}(t, t, x)), \Theta_x(\tau, t, x), \Theta_{xx}(\tau, t, x)) &= 0, \\
(\tau, t, x) &\in \mathbb{D}[0, T] \times (0, \infty), \\
\Theta(\tau, T, x) &= \alpha h(T - \tau) U(x), \\
(\tau, x) &\in [0, T] \times (0, \infty),
\end{align*}
\]

(11)

where $\mathbb{D}[0, T] = \{(\tau, t) : 0 \leq \tau \leq t \leq T\}$.

**Theorem 3.2.** Suppose that there exists a function $\Theta(\cdot, \cdot, \cdot) \in C^{0,1,2}(\mathbb{D}[0, T] \times (0, \infty))$ (i.e., $\Theta(\tau, t, x)$ is continuous with respect to $\tau$, first-order continuously differentiable with respect to $t$ and second-order continuously differentiable with respect to $x$) with $\Theta_{xx}(\cdot, \cdot, \cdot) < 0$ and $\Theta_x(\cdot, \cdot, \cdot) \geq 0$. Define the strategy $\hat{\pi}(\cdot)$ by

\[
\hat{\pi}(t, x) = \left(\hat{c}(t, x), \hat{u}(t, x), \hat{\xi}(t, x)\right) := \varphi(t, t, x, \Theta_x(t, t, x), \Theta_{xx}(t, t, x)),
\]

(12)

where $\varphi(\cdot)$ is given by (10). If $\Theta(\cdot, \cdot, \cdot)$ satisfies the equilibrium HJB equation (11) and $\hat{\pi}(\cdot)$ given by (12) is admissible, then $\hat{\pi}(\cdot)$ is a subgame equilibrium strategy to Problem 1 and $V(t, x_t) = \Theta(t, t, x_t)$ is the corresponding value function.

**Proof.** We provide the proof in two steps: 1. Showing that $V(\cdot, \cdot)$ is the value function corresponding to $\hat{\pi}(\cdot)$, i.e., $V(t, x_t) = J(t, x_t; \hat{\pi}(\cdot))$; 2. proving that $\hat{\pi}(\cdot)$ is indeed the equilibrium control defined by Definition 3.1.

**Step 1.** For any $\tau \in [0, T]$, applying Dynkin’s formula to the function $\Theta(\tau, \cdot, \cdot)$, by (11), we have

\[
\Theta(\tau, t, x_t)
\]

\[
= E_t^x \left\{ \Theta(\tau, T, \hat{X}(T)) - \int_t^T \left\{ \Theta_t(\tau, s, \hat{X}(s)) + \frac{1}{2} \hat{u}^2(s, \hat{X}(s)) \Theta_{xx}(\tau, s, \hat{X}(s)) \\
+ \left( r(s) \hat{X}(s) - \hat{c}(s, \hat{X}(s)) + \hat{u}(s, \hat{X}(s)) \sigma(s) \left( \theta(s) - \hat{\xi}(s, \hat{X}(s)) \right) \right) \Theta_x(\tau, s, \hat{X}(s)) \right\} ds \right\} \\
= E_t^x \left\{ \int_t^T h(s - \tau) \left( \hat{c}(s, \hat{X}(s)) + \frac{1}{2} \frac{\sigma^2(s)}{\phi(r, s, \hat{X}(s))} \hat{\xi}^2(s, \hat{X}(s)) \right) ds \right\} \\
+ \alpha h(T - \tau) U(\hat{X}(T)) .
\]

(13)

Obviously, we have

\[
V(t, x_t) := \Theta(t, t, x_t) = J(t, x_t; \hat{c}, \hat{u}, \hat{\xi}).
\]

**Step 2.** Denote by $X_1^t(\cdot)$ and $X_2^t(\cdot)$ the paths under the controls $\pi_1^t(\cdot)$ and $\pi_2^t(\cdot)$, respectively. We first prove (8). Applying Itô’s formula to $\Theta(t, \cdot, \cdot)$, we have

\[
\Theta(t, t + \epsilon, \hat{X}(t + \epsilon)) - \Theta(t, t, x_t)
\]

\[
= \int_t^{t + \epsilon} \left\{ \Theta_t(t, s, \hat{X}(s)) + \frac{1}{2} \Theta_{xx}(t, s, \hat{X}(s)) \hat{u}^2(s, \hat{X}(s)) \sigma^2(s) \\
+ \Theta_x(t, s, \hat{X}(s)) \left[ r(s) \hat{X}(s) - \hat{c}(s, \hat{X}(s)) + \hat{u}(s, \hat{X}(s)) \sigma(s) \left( \theta(s) - \hat{\xi}(s, \hat{X}(s)) \right) \right] \right\} ds \\
+ \int_t^{t + \epsilon} \Theta_x(t, s, \hat{X}(s)) \hat{u}(s, \hat{X}(s)) \sigma(s) dW^Z(s)
\]
and
\[
\Theta(t, t + \epsilon, X^*_t(t + \epsilon)) - \Theta(t, t, x_i)
\]
\[
= \int_t^{t+\epsilon} \left\{ \Theta_t(t, s, X^*_t(s)) + \frac{1}{2} \Theta_{xx}(t, s, X^*_t(s)) u_0^2 \sigma^2(s)
+ \Theta_x(t, s, X^*_t(s)) \left[ r(s)X^*_t(s) - c_0 + u_0 \sigma(s)(\theta(s) - \xi(s, X^*_t(s))) \right] \right\} ds
+ \int_t^{t+\epsilon} \Theta_x(t, s, X^*_t(s)) u_0 \sigma(s) dW^\xi(s).
\]
Then
\[
J(t, x_i; \hat{\pi})
\]
\[
= \mathbb{E}^\hat{\pi} \left[ \int_t^T h(s - t) \left( U \left( \hat{c}(s, \hat{X}(s)) \right) + \frac{1}{2\phi(t, s, \hat{X}(s))} \hat{\xi}^2(s, \hat{X}(s)) \right) ds
+ \alpha h(T - t) U(\hat{X}(T)) \right]
\]
\[
= \mathbb{E}^\hat{\pi} \left[ \int_t^{t+\epsilon} h(s - t) \left( U \left( \hat{c}(s, \hat{X}(s)) \right) + \frac{1}{2\phi(t, s, \hat{X}(s))} \hat{\xi}^2(s, \hat{X}(s)) \right) ds
+ \Theta(t, t + \epsilon, \hat{X}(t + \epsilon)) \right]
\]
\[
= \mathbb{E}^\hat{\pi} \left\{ \int_t^{t+\epsilon} \left[ \Theta_t(t, s, \hat{X}(s)) + \frac{1}{2} \Theta_{xx}(t, s, \hat{X}(s)) \hat{u}^2(s, \hat{X}(s)) \sigma^2(s)
+ \Theta_x(t, s, \hat{X}(s)) \left( r(s)\hat{X}(s) - \hat{c}(s, \hat{X}(s)) + \hat{u}(s, \hat{X}(s)) \sigma(s)(\theta(s) - \hat{\xi}(s, \hat{X}(s))) \right) \right] ds
\]
\[
+h(s - t) \left( U \left( \hat{c}(s, \hat{X}(s)) \right) + \frac{1}{2\phi(t, s, \hat{X}(s))} \hat{\xi}^2(s, \hat{X}(s)) \right) \right\} + \Theta(t, t, x_i),
\]
and
\[
J(t, x_i; \pi^*_1)
\]
\[
= \mathbb{E}^{\pi^*_1} \left[ \int_t^T h(s - t) \left( U \left( c^*(s, X^*_t(s)) \right) + \frac{1}{2\phi(t, s, X^*_t(s))} \hat{\xi}^2(s, X^*_t(s)) \right) ds
+ \alpha h(T - t) U(X^*_t(T)) \right]
\]
\[
= \mathbb{E}^{\pi^*_1} \left[ \int_t^{t+\epsilon} h(s - t) \left( U \left( c^*(s, X^*_t(s)) \right) + \frac{1}{2\phi(t, s, X^*_t(s))} \hat{\xi}^2(s, X^*_t(s)) \right) ds
+ \Theta(t, t + \epsilon, X^*_t(t + \epsilon)) \right]
\]
\[
= \mathbb{E}^{\pi^*_1} \left\{ \int_t^{t+\epsilon} \left[ \Theta_t(t, s, X^*_t(s)) + \frac{1}{2} \Theta_{xx}(t, s, X^*_t(s)) u_0^2 \sigma^2(s)
+ \Theta_x(t, s, X^*_t(s)) \left( r(s)X^*_t(s) - c_0 + u_0 \sigma(s)(\theta(s) - \hat{\xi}(s, X^*_t(s))) \right) \right] ds
\]
\[
+h(s - t) \left( U \left( c^*(s, X^*_t(s)) \right) + \frac{1}{2\phi(t, s, X^*_t(s))} \hat{\xi}^2(s, X^*_t(s)) \right) \right\} + \Theta(t, t, x_i).
\]
Therefore, we have
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( J(t, x_i; \hat{\pi}) - J(t, x_i; \pi^*_1) \right)
\]
\[ \begin{align*}
&= \Theta_x(t, t, x_t) \left( r(t)x_t - \dot{c}(t, x_t) + \dot{u}(t, x_t)\sigma(t)(\theta(t) - \dot{\xi}(t, x_t)) \right) \\
&\quad + \frac{1}{2} \Theta_{xx}(t, t, x_t)u^2(t, x_t)\sigma^2(t) + \left( U(\dot{c}(t, x_t)) + \frac{1}{2\phi(t, t, x_t)}\dot{\xi}^2(t, x_t) \right) \\
&\quad - \left[ \Theta_x(t, t, x_t) \left( r(t)x_t - c_0 + u_0\sigma(t)(\theta(t) - \dot{\xi}(t)) \right) \\
&\quad + \frac{1}{2} \Theta_{xx}(t, t, x_t)u^2(t, x_t)\sigma^2(t) + \left( U(c_0) + \frac{1}{2\phi(t, t, x_t)}\dot{\xi}^2(t, x_t) \right) \right] \geq 0,
\end{align*} \]
where the inequality follows from the definition of \( \dot{\pi}(\cdot) \).

Now, we are going to verify (9). Noting that \( \dot{X}(\cdot; t, x_t) \) and \( X^*_2(\cdot; t, x_t) \) have the same distribution under probability measure \( P \) for any initial state \( (t, x_t) \in [0, T] \times (0, \infty) \), it holds that

\[ \begin{align*}
\Theta(t, t + \epsilon, X^*_2(t + \epsilon)) \\
= & E_{t+\epsilon} \left\{ \int_{t+\epsilon}^T h(s - t) \left[ U \left( \dot{c}(s, \dot{X}(s; t + \epsilon, X^*_2(t + \epsilon))) \right) \right. \\
&\quad + \frac{1}{2\phi(t, s, \dot{X}(s; t + \epsilon, X^*_2(t + \epsilon)))}\dot{\xi}^2(s, \dot{X}(s; t + \epsilon, X^*_2(t + \epsilon))) \\
&\left. + \alpha h(T - t)U(\dot{X}(T; t + \epsilon, X^*_2(t + \epsilon))) \right] \right. \\
&\left. + \alpha h(T - t)U(\dot{X}(T; t + \epsilon, X^*_2(t + \epsilon))) \right\}
\end{align*} \]

Then

\[ \begin{align*}
J(t, x_t; \pi^*_2) \\
= & E^C_t \left\{ \int_t^T h(s - t) \left[ U \left( \dot{c}(s, X^*_2(s)) \right) + \frac{1}{2\phi(t, s, X^*_2(s))} (\dot{\xi}^*(s, X^*_2(s)))^2 \right] \right. \\
&\left. + \alpha h(T - t)U(X^*_2(T)) \right\}
\end{align*} \]
Applying Itô's formula to $\Theta(t, \cdot, \cdot)$, we have

$$\Theta(t, t + \epsilon, X^*_2(t + \epsilon)) - \Theta(t, t, x_t)$$

$$= \int_t^{t+\epsilon} \left\{ \Theta_t(t, s, X^*_2(s)) + \frac{1}{2} \Theta_{xx}(t, s, X^*_2(s)) \hat{u}^2(s, X^*_2(s)) \sigma^2(s) 
\quad + \Theta_x(t, s, X^*_2(s)) \left[ r(s) X^*_2(s) - \hat{c}(s, X^*_2(s)) + \hat{u}(s, X^*_2(s)) \sigma(s)(\theta(s) - \xi_0) \right] \right\} ds$$

$$+ \int_t^{t+\epsilon} \Theta_x(t, s, X^*_2(s)) \hat{u}(s, X^*_2(s)) \sigma(s) dW^c(s).$$

Hence,

$$J(t, x_t; \hat{\pi}) - J(t, x_t; \pi^*_2)$$

$$= \mathbb{E}_{t, x} \left\{ \int_t^{t+\epsilon} \left[ \Theta_t(t, s, \hat{X}(s)) + \frac{1}{2} \Theta_{xx}(t, s, \hat{X}(s)) \hat{u}^2(s, \hat{X}(s)) \sigma^2(s) 
\quad + \Theta_x(t, s, \hat{X}(s)) \left[ r(s) \hat{X}(s) - \hat{c}(s, \hat{X}(s)) + \hat{u}(s, \hat{X}(s)) \sigma(s)(\theta(s) - \hat{\xi}(s, \hat{X}(s))) \right] 
\quad + h(s - t) \left( U(\hat{c}(s, \hat{X}(s))) + \frac{1}{2 \phi(t, s, \hat{X}(s))} \hat{\xi}^2(s, \hat{X}(s)) \right) \right] ds \right\}$$

$$- \mathbb{E}_{t, x} \left\{ \int_t^{t+\epsilon} \left[ \Theta_t(t, s, X^*_2(s)) + \frac{1}{2} \Theta_{xx}(t, s, X^*_2(s)) \hat{u}^2(s, X^*_2(s)) \sigma^2(s) 
\quad + \Theta_x(t, s, X^*_2(s)) \left[ r(s) X^*_2(s) - \hat{c}(s, X^*_2(s)) + \hat{u}(s, X^*_2(s)) \sigma(s)(\theta(s) - \xi_0) \right] 
\quad + h(s - t) \left( U(\hat{c}(s)) + \frac{1}{2 \phi(t, s, X^*_2(s))} \hat{\xi}_0^2 \right) \right] ds \right\}. $$

Consequently,

$$\limsup_{\epsilon \to 0} \frac{1}{\epsilon} (J(t, x_t; \hat{\pi}) - J(t, x_t; \pi^*_2))$$

$$= \Theta_x(t, t, x_t) \left( r(t)x_t - \hat{c}(t, x_t) + \hat{u}(t, x_t) \sigma(t)(\theta(t) - \hat{\xi}(t)) \right)$$

$$+ \frac{1}{2} \Theta_{xx}(t, t, x_t) \hat{u}^2(t, x_t) \sigma^2(t) + \left( U(\hat{c}(t, x_t)) + \frac{1}{2 \phi(t, t, x_t)} \hat{\xi}^2(t, x_t) \right)$$

$$- \left[ \Theta_x(t, t, x_t) \left( r(t)x_t - \hat{c}(t, x_t) + \hat{u}(t, x_t) \sigma(t)(\theta(t) - \xi_0) \right) \right]$$

$$+ \frac{1}{2} \Theta_{xx}(t, t, x_t) \hat{u}^2(t, x_t) \sigma^2(s) + \left( U(\hat{c}(t, x_t)) + \frac{1}{2 \phi(t, t, \xi_0^2)} \right) \leq 0,$$

where the inequality follows from the definition of $\hat{\pi}(\cdot)$.

3.2. Power utility. In [2], the ambiguity aversion is a constant $\phi(\tau, t, x) \equiv \eta > 0$. To explicitly solve the model, we shall follow [27, 28] to impose the homothetic robustness specification, i.e., suppose that

$$\phi(\tau, t, x) = \frac{\eta(\gamma)}{(1 - \gamma) \Theta(t, t, x)},$$

where $\eta(\cdot) > 0$ is a bounded deterministic function, which scales $\eta$ by the value function.
Theorem 3.3. If \( U(x) = \frac{x^1}{1-\gamma}, x > 0, \) with \( \gamma \in (0, 1), \) then the equilibrium strategy is given by
\[
\hat{c}(t, x) = f^{-\frac{1}{\gamma}}(t)x, \quad \hat{u}(t, x) = \frac{\theta(t)}{\sigma(t)(\gamma + \eta(t))}x, \quad \hat{\xi}(t, x) = \frac{\theta(t)\eta(t)}{\gamma + \eta(t)},
\]
and the corresponding equilibrium value function is
\[
\Theta(t, x) = f(t) \frac{x^{1-\gamma}}{1-\gamma},
\]
where the function \( f(\cdot) \) is the unique solution to the following integral equation
\[
f(t) = \alpha h(T-t) \exp \left\{ \int_t^T (1-\gamma) \left[ r(s) + \frac{1}{2} \frac{\gamma \theta^2(s)}{(\gamma + \eta(s))^2} - f^{-\frac{1}{\gamma}}(s) \right] ds \right\} \\
+ \int_t^T \exp \left\{ \int_t^s (1-\gamma) \left[ r(s) + \frac{1}{2} \frac{\gamma \theta^2(s)}{(\gamma + \eta(s))^2} - f^{-\frac{1}{\gamma}}(s) \right] ds \right\} \\
\times h(s-t) \left[ f^{-\frac{1}{\gamma}}(z) + (1-\gamma) \frac{1}{2\eta(t)} \left( \frac{\theta(z)\eta(z)}{\gamma + \eta(z)} \right)^2 f(z) \right] dz.
\]

\[ (14) \]

Proof. Obviously, \( I(x) = x^{-\frac{1}{\gamma}}. \) Then the equilibrium HJB equation (11) becomes
\[
\left\{ \Theta_\tau(t, x) + \left[ r(t) x - \Theta_x^{-\frac{1}{\gamma}}(t, x) - \frac{\theta^2(t)\phi(t,t,x)\phi_x(t,t,x)}{\eta(t)} \frac{2}{(\gamma + \eta(t))^2} x \right] \Theta_x(t, x) \right. \\
+ \frac{1}{2} \frac{\theta^2(t)\phi_x^2(t,t,x)}{\eta(t)} \Theta_{xx}(t, x) \\
+ h(t-\tau) \left[ \Theta_x^{-\frac{1}{\gamma}}(t, x) + (1-\gamma) \frac{1}{2\eta(t)} \left( \frac{\theta(t)\phi(t,t,x)\phi_x(t,t,x)}{\eta(t)} \right)^2 \right]^2 \right. \\
\left. - (\tau, t, x) \in \mathbb{D}[0, T] \times (0, \infty), \right. \\
\left. \Theta(\tau, T, x) = \alpha h(T-\tau) \frac{x^{1-\gamma}}{1-\gamma}, \quad (\tau, x) \in [0, T] \times (0, \infty). \right.
\]

\[ (15) \]

Consider the following Ansatz:
\[
\Theta(\tau, t, x) = F(\tau, t) \frac{x^{1-\gamma}}{1-\gamma},
\]
where \( F(\cdot, \cdot, \cdot) \) is a function to be determined. Inserting (16) into (15) and comparing the coefficients of \( x^{1-\gamma}, \) we have
\[
\left\{ F_\tau(t, t) + (1-\gamma) \left[ r(t) + \frac{1}{2} \frac{\gamma \theta^2(t)}{(\gamma + \eta(t))^2} - F^{-\frac{1}{\gamma}}(t, t) \right] F(t, t) \\
+ h(t-\tau) \left[ F^{-\frac{1}{\gamma}}(t, t) + (1-\gamma) \frac{1}{2\eta(t)} \left( \frac{\theta(t)\eta(t)}{\gamma + \eta(t)} \right)^2 \right] F(t, t) \right. \\
\left. - (\tau, t, x) \in \mathbb{D}[0, T] \times (0, \infty), \right. \\
F(\tau, T) = \alpha h(T-\tau), \quad (\tau, x) \in [0, T] \times (0, \infty).
\]

Thus,
\[
F(\tau, t) = \alpha h(T-\tau) \exp \left\{ \int_t^T (1-\gamma) \left[ r(s) + \frac{1}{2} \frac{\gamma \theta^2(s)}{(\gamma + \eta(s))^2} - F^{-\frac{1}{\gamma}}(s, s) \right] ds \right\} \\
+ \int_t^T \exp \left\{ \int_t^s (1-\gamma) \left[ r(s) + \frac{1}{2} \frac{\gamma \theta^2(s)}{(\gamma + \eta(s))^2} - F^{-\frac{1}{\gamma}}(s, s) \right] ds \right\}
\]
\[ \times h(z - \tau) \left[ F^{-\frac{1}{2\eta}}(z, z) + (1 - \gamma) \frac{1}{2\eta(\tau)} \left( \frac{\theta(z) \eta(z)}{\gamma + \eta(z)} \right)^2 F(z, z) \right] \, dz. \] (17)

By letting \( t = \tau \) in (17) and denoting \( f(t) = F(t, t) \), we have (14).

In the following, we consider the existence and uniqueness of the solution to (14). Similar to Section 6 in [40], we only need to show the uniform lower and upper boundedness of \( f(\cdot) \).

Let
\[ \tilde{f}(t) = \exp \left\{ - \int_t^T (1 - \gamma) \left[ r(s) + \frac{1}{2} \frac{\gamma \theta^2(s)}{(\gamma + \eta(s))^2} - f^{-\frac{1}{2}}(s) \right] \, ds \right\} f(t). \] (18)

We first show the lower boundedness. Recalling that the discount rate function \(-h(\cdot)/h(\cdot)\) is bounded above by a constant \( \bar{\delta} > 0 \), we have \( h(t) \geq e^{-\bar{\delta} t} \). Thus,
\[ \tilde{f}(t) e^{-\bar{\delta} t} \geq \alpha e^{-\bar{\delta} T} + \int_t^T e^{-\bar{\delta} z} \tilde{f}(z) f^{-\frac{1}{2}}(z) \, dz \]
\[ \geq \alpha e^{-\bar{\delta} T} + \int_t^T (1 - \gamma) e^{-\bar{\delta} z} \tilde{f}(z) f^{-\frac{1}{2}}(z) \, dz := \zeta(t). \]

Thus,
\[ \zeta'(t) = -(1 - \gamma) e^{-\bar{\delta} t} \tilde{f}(t) f^{-\frac{1}{2}}(t) \leq -(1 - \gamma) f^{-\frac{1}{2}}(t) \zeta(t), \]
which implies that \( \left[ \zeta(t) e^{-\int_t^T (1 - \gamma) f^{-\frac{1}{2}}(s) \, ds} \right]' \leq 0 \). Thus,
\[ \zeta(t) e^{-\int_t^T (1 - \gamma) f^{-\frac{1}{2}}(s) \, ds} \geq \alpha e^{-\bar{\delta} T}. \]

Consequently,
\[ f(t) = \tilde{f}(t) \exp \left\{ \int_t^T (1 - \gamma) \left[ r(s) + \frac{1}{2} \frac{\gamma \theta^2(s)}{(\gamma + \eta(s))^2} - f^{-\frac{1}{2}}(s) \right] \, ds \right\} \]
\[ \geq \zeta(t) \exp \left\{ \bar{\delta} t + \int_t^T (1 - \gamma) \left[ r(s) + \frac{1}{2} \frac{\gamma \theta^2(s)}{(\gamma + \eta(s))^2} - f^{-\frac{1}{2}}(s) \right] \, ds \right\} \]
\[ \geq \alpha \exp \left\{ \bar{\delta}(t - T) + \int_t^T (1 - \gamma) \left[ r(s) + \frac{1}{2} \frac{\gamma \theta^2(s)}{(\gamma + \eta(s))^2} \right] \, ds \right\} \geq \bar{\kappa}, \]
for some constant \( \bar{\kappa} > 0 \).

Now, we look at the upper boundedness. By (14), we have
\[ \tilde{f}(t) = a h(T - t) + \int_t^T h(z - t) \tilde{f}(z) \left[ f^{-\frac{1}{2}}(z) + (1 - \gamma) \frac{1}{2\eta(t)} \left( \frac{\theta(z) \eta(z)}{\gamma + \eta(z)} \right)^2 \right] \, dz \]
\[ \leq \kappa_1 + \int_t^T \tilde{f}(z) \left[ f^{-\frac{1}{2}}(z) + \kappa_2 \right] \, dz, \]
for some constants \( \kappa_1, \kappa_2 > 0 \). It follows from Gronwall’s inequality that
\[ \tilde{f}(t) \leq \kappa_1 e^{\int_t^T \left[ f^{-\frac{1}{2}}(z) + \kappa_2 \right] \, dz}, \]
\[ 2 \text{Given the boundedness, the existence and uniqueness of the solution can be obtained by the standard method (and we omit the details of this step): first show the well-posedness by using contraction mapping theorem on some small interval } [T - \bar{\delta}, T] \text{; then show the well-posedness on } [T - 2\bar{\delta}, T - \bar{\delta}], \ldots, [0, \bar{\delta}], \text{ backw
which together with (18) implies that \( f(t) \leq \kappa e^{\gamma \int_t^T f^{-\frac{1}{\gamma}}(z)dz} \), for some constant \( \kappa > 0 \). By the lower bound of \( f(\cdot) \), we know that \( f(\cdot) \) is bounded from above uniformly.

**Remark 3.** We find that the equilibrium investment strategy is the Merton solution adjusted by the ambiguity aversion and it is independent of the discount rate. The equilibrium consumption rate now has more complicated structure due to the non-exponential discounting.

**Remark 4.** If \( \eta(\cdot) \equiv \eta, r(\cdot) \equiv r, \theta(\cdot) \equiv \theta \) and \( h(t) = e^{-\delta t} \) for some \( \delta > 0 \), then (14) becomes

\[
\begin{align*}
    f(t) &= \alpha e^{-\delta(T-t)} \exp \left\{ \int_t^T (1-\gamma) \left[ r + \frac{1}{2} \frac{\gamma \theta^2}{(\gamma+\eta)^2} - f^{-\frac{1}{\gamma}}(s) \right] ds \right. \\
    &\quad + \int_t^T \exp \left\{ \int_t^s (1-\gamma) \left[ r + \frac{1}{2} \frac{\gamma \theta^2}{(\gamma+\eta)^2} - f^{-\frac{1}{\gamma}}(s) \right] ds \right. \\
    &\quad \left. \times e^{-\delta(z-s)} \left[ f^{-\frac{1}{\gamma}}(z) + (1-\gamma) \frac{\theta \eta}{\gamma+\eta} f(z) \right] dz. \right\} f(t).
\end{align*}
\]

Thus,

\[
    f_t(t) = -\gamma f^{-\frac{1}{\gamma}}(t) + \left\{ \delta - (1-\gamma) \left[ r + \frac{\theta^2}{2(\eta+\gamma)} \right] \right\} f(t). \tag{19}
\]

Let \( w(t) = f^{\frac{1}{\gamma}}(t) \). Then

\[
    w_t(t) = \frac{1}{\gamma} f^{\frac{1}{\gamma}-1}(t) \left\{ -\gamma f^{-\frac{1}{\gamma}}(t) + \left\{ \delta - (1-\gamma) \left[ r + \frac{\theta^2}{2(\eta+\gamma)} \right] \right\} f(t) \right\}
\]

\[
    = -1 + \frac{1}{\gamma} \left\{ \delta - (1-\gamma) \left[ r + \frac{\theta^2}{2(\eta+\gamma)} \right] \right\} w(t),
\]

whose solution is given by

\[
    w(t) = \alpha^{\frac{1}{\gamma}} e^{-a(T-t)} + \frac{1}{a} \left( 1 - e^{-a(T-t)} \right),
\]

where

\[
    a = \frac{1}{\gamma} \left[ \delta - (1-\gamma)r - \frac{1-\gamma}{2(\gamma+\eta)} \theta^2 \right].
\]

Therefore, we can derive the solution to (14) as

\[
    f(t) = \left[ \alpha^{\frac{1}{\gamma}} e^{-a(T-t)} + \frac{1}{a} \left( 1 - e^{-a(T-t)} \right) \right]^\gamma. \tag{20}
\]

Moreover, if \( \alpha = 0 \), the equilibrium strategy for a sophisticated investor is the optimal strategy obtained by [27].

### 3.3 Logarithmic utility.

Although homotheticity obtains for logarithmic utility even if \( \phi(\tau, t, x) \) is a constant, similar to [27], to show that the homothetic results for power utility converge to the solution for logarithmic utility as \( \gamma \) tends to one, we impose the following homothetic robustness specification:

\[
    \phi(\tau, t, x) = \lim_{\gamma \to 1} \frac{\eta(\tau)}{(1-\gamma)\Theta(t, t, x)}, \tag{21}
\]

where \( \eta(\cdot) > 0 \) is a bounded deterministic function and \( \Theta(t, t, x) \) is the equilibrium value function of the investor with power utility.
Thus
\[
\phi(\tau, t, x) = \lim_{\gamma \to 1} \frac{\eta(\tau)}{(1 - \gamma)\Theta(t, t, x)} = \frac{\eta(\tau)}{H(t)},
\]
where \( H(\cdot) \) is given by
\[
H(t) = \int_t^T h(s - t)ds + ah(T - t).
\]

**Theorem 3.4.** If \( U(x) = \ln x, x > 0 \), then the equilibrium strategy is given by
\[
\hat{c}(t, x) = \tilde{f}^{-1}(t, t)x, \quad \hat{u}(t, x) = \frac{\theta(t)}{\sigma(t)(1 + \eta(t))}x, \quad \hat{\xi}(t, x) = \frac{\theta(t)\eta(t)}{1 + \eta(t)},
\]
and the corresponding equilibrium value function is \( \tilde{\Theta}(t, t, x) = \tilde{f}(t, t)\ln x + \tilde{g}(t, t) \), where
\[
\tilde{f}(\tau, t) = \int_t^T h(s - \tau)ds + ah(T - \tau) \tag{22}
\]
and
\[
\tilde{g}(\tau, t) = - \int_t^T \left\{ r(s) - \frac{1}{H(s)} + \frac{1}{2} \left( \frac{\theta^2(s)}{1 + \eta(s)} \right)^2 \right\} \tilde{f}(\tau, s)
\]
\[
+ h(s - t) \left[ - \ln H(s) + \frac{H(s)}{2\eta(s)} \left( \frac{\theta(s)\eta(s)}{1 + \eta(s)} \right)^2 \right] \right\} ds. \tag{23}
\]

**Proof.** In this case the equilibrium HJB equation (11) becomes
\[
\begin{align*}
\Theta_t(t, t, x) + \Theta_x(t, t, x) + & \left\{ r(t) \rho - \frac{1}{\Theta_x(t, t, x)} - \frac{\sigma^2(t, t, x)\Theta_{xx}(t, t, x)}{\Theta_x(t, t, x)\Theta_x(t, t, x)} \right\} \Theta_x(t, t, x) \\
+ & \frac{1}{2} \left( \Theta_{xx}(t, t, x) - \rho(t, t, x)\Theta_x(t, t, x) \right)^2 \Theta_{xx}(t, t, x) \\
+ & h(t - \tau) \left[ \ln \left( \frac{1}{\Theta_x(t, t, x)} \right) + \frac{H(t)}{2\eta(t)} \left( \frac{\theta(t)\eta(t)}{H(t) + \eta(t)f(t, t)} \right)^2 \right] = 0,
\end{align*}
\]
\[
(\tau, t, x) \in \mathbb{D}[0, T] \times (0, \infty),
\]
\[
\Theta(\tau, T, x) = ah(T - \tau)\ln x, \quad (\tau, x) \in [0, T] \times (0, \infty).
\]
Consider the following Ansatz:
\[
\tilde{\Theta}(\tau, t, x) = \tilde{f}(\tau, t)\ln x + \tilde{g}(\tau, t), \tag{24}
\]
where \( \tilde{f}(\cdot, \cdot) \) and \( \tilde{g}(\cdot, \cdot) \) are functions to be determined. Putting (24) into the equilibrium HJB equation and comparing the coefficients of \( \ln x \), we have
\[
\begin{align*}
\tilde{f}_{\tau}(\tau, t) + h(t - \tau) = 0, \quad 0 \leq \tau \leq t \leq T, \\
\tilde{f}(\tau, T) = ah(T - \tau),
\end{align*}
\]
and
\[
\begin{align*}
\tilde{g}_{\tau}(\tau, t) + & \left\{ r(t) - \frac{1}{\tilde{f}(t, t)} + \frac{1}{2} \left( \frac{\theta^2(t, t, x)}{1 + \frac{\eta(t)f(t, t)}{H(t)}} \right)^2 \right\} \tilde{f}(\tau, t) \\
+ & h(t - \tau) \left[ - \ln \tilde{f}(t, t) + \frac{H(t)}{2\eta(t)} \left( \frac{\theta(t)\eta(t)\tilde{f}(t, t)}{H(t) + \eta(t)\tilde{f}(t, t)} \right)^2 \right] = 0,
\end{align*}
\]
\[
\tilde{g}(\tau, T) = 0.
\]
It is easy to see
\[
\tilde{f}(\tau, t) = \int_t^T h(s - \tau)ds + ah(T - \tau).
\]
Thus, \( \bar{f}(t, t) = H(t) \) and
\[
\begin{align*}
\bar{g}(\tau, t) + \left[ r(t) - \frac{1}{H(t)} + \frac{1}{2} \frac{\sigma^2(t)}{(1+\theta(t))^2} \right] \bar{f}(\tau, t) \\
+ h(t - \tau) \left[ -\ln H(t) + \frac{H(t)}{2\theta(t)} \left( \frac{\theta(t)\eta(t)}{T} \right)^2 \right] = 0,
\end{align*}
\]
which implies (23).

We find that under the homothetic robustness assumption (21), the results for logarithmic utility are indeed the limits (as \( \gamma \to 1 \)) for the power utility. Furthermore, in this case, the equilibrium consumption rate for a sophisticated investor is independent of the ambiguity aversion. This is consistent with the results in [27].

4. **Naive investor.** In this section, we aim to derive the optimal strategies for the naive investor for the optimization problem with power and logarithmic utility functions, respectively. Naive investors take decisions without taking into account their calculated choices for the future (see [30]). We add a superscript na for the value function, optimal strategy and other functions for a naive investor.

At time \( \tau \), the naive agent (called naive agent-\( \tau \)) makes optimal decision based on the discount function \( h(\cdot - \tau) \) and ambiguity aversion \( \phi(\tau, \cdot, \cdot) \), where \( \tau \) is regarded as a fixed parameter. Then Problem 1 becomes a standard optimal control problem for the naive agent-\( \tau \), and the HJB equation is given by
\[
\begin{align*}
\sup_{c, u} \inf_{\xi} \{ \Theta^{\text{na}}(t, x; \tau) + [r(t)x - c + u\sigma(t)(\theta(t) - \xi)] \Theta^{\text{na}}(t, x; \tau) \\
+ \frac{1}{2} u^2 \sigma^2(t) \Theta_{x\tau}^{\text{na}}(t, x; \tau) + h(t - \tau) \left[ U(c) + \frac{1}{2\phi(\tau, t, x)} \xi^2 \right] \} = 0, \quad \tau \leq t \leq T, \quad x > 0,
\end{align*}
\]
where \( \Theta^{\text{na}}(\cdot, \cdot; \tau) \) is the value function for the naive agent-\( \tau \).

If \( \Theta^{\text{na}}_x(t, x; \tau) \geq 0 \) and \( \Theta^{\text{na}}_{x^2}(t, x; \tau) < 0 \) for all \((t, x) \in [0, T] \times (0, \infty) \), then by the first-order condition, (25) becomes
\[
\begin{align*}
\Theta^{\text{na}}_t(t, x; \tau) + [r_x - I \left( \frac{\Theta^{\text{na}}_{x^2}(t, x; \tau)}{h(t-\tau)} \right) \Theta^{\text{na}}(t, x; \tau) + h(t - \tau)U \left( 1 \left( \frac{\Theta^{\text{na}}_{x}(t, x; \tau)}{h(t-\tau)} \right) \right) ] \\
- \frac{1}{2} h(t-\tau)(\Theta^{\text{na}}_{x^2}(t, x; \tau))^2 - \phi(\tau, t, x) \Theta^{\text{na}}_{x}(t, x; \tau) \Theta^{\text{na}}_{x}(t, x; \tau) = 0, \\
\tau \leq t \leq T, \quad x \in (0, \infty),
\end{align*}
\]
and the the supremum and infimum in the HJB equation (25) is attained at
\[
\begin{align*}
c &= I \left( \frac{\Theta^{\text{na}}_x(t, x; \tau)}{h(t-\tau)} \right), \\
u &= -\sigma(t)h(t-\tau)\Theta^{\text{na}}_{x^2}(t, x; \tau) - \phi(\tau, t, x)^\prime \left( \Theta^{\text{na}}_x(t, x; \tau) \right)^\prime, \\
\xi &= \frac{-\theta(t)\eta(t)(\Theta^{\text{na}}_{x^2}(t, x; \tau))^2}{h(t-\tau)\Theta^{\text{na}}_{x^2}(t, x; \tau) - \phi(\tau, t, x)(\Theta^{\text{na}}_x(t, x; \tau))^2},
\end{align*}
\]
for \( t \in [\tau, T] \).

Since the naive agent constantly updates her strategy and implements the optimal strategy of the naive agent-\( \tau \) whenever she is at time \( \tau \), the strategy of the naive agent is given by
\[
\begin{align*}
\dot{\eta}_{\text{na}}(t) &= I \left( \Theta_{x}^{na}(t, \hat{X}_{\text{na}}(t); t) \right), \\
\ddot{\nu}_{\text{na}}(t) &= -\frac{\sigma(t) \left[ \Theta_{x}^{na}(t, \hat{X}_{\text{na}}(t); t) - \dot{\phi}(t, t, x) \right]}{\left( \Theta_{x}^{na}(t, \hat{X}_{\text{na}}(t); t) \right)^2}, \\
\ddot{\xi}_{\text{na}}(t) &= -\frac{\nu(t) \left[ \Theta_{x}^{na}(t, \hat{X}_{\text{na}}(t); t) - \dot{\phi}(t, t, x) \right]}{\left( \Theta_{x}^{na}(t, \hat{X}_{\text{na}}(t); t) \right)^2},
\end{align*}
\]

for \( t \in [0, T] \), where \( \hat{X}_{\text{na}}(\cdot) \) is the wealth process for the naive agent that satisfies

\[
\left\{ \begin{array}{l}
\mathrm{d}\hat{X}_{\text{na}}(s) = \left[ r(s)\hat{X}_{\text{na}}(s) - I \left( \Theta_{x}^{na}(s, \hat{X}_{\text{na}}(s); s) \right) \\
- \frac{\sigma(s) \Theta_{x}^{na}(s, \hat{X}_{\text{na}}(s); s) \Theta_{xx}^{na}(s, \hat{X}_{\text{na}}(s); s)}{\Theta_{x}^{na}(s, \hat{X}_{\text{na}}(s); s)} \right] \mathrm{d}s \\
- \frac{\nu(s) \Theta_{x}^{na}(s, \hat{X}_{\text{na}}(s); s) \Theta_{xx}^{na}(s, \hat{X}_{\text{na}}(s); s)}{\Theta_{x}^{na}(s, \hat{X}_{\text{na}}(s); s)} \right] \mathrm{d}W_{\hat{X}_{\text{na}}}(s), & s \in [0, T], \\
\hat{X}_{\text{na}}(0) = x_0.
\end{array} \right.
\]

Note that \( \Theta^a(t, x; t) \) is not the actual value function that the naive agent obtains, though he/she pursues this value function at the state \((t, x)\). To derive the actual value function of the naive agent, we introduce the following function:

\[
\bar{\Theta}^{na}(t, x; \tau) = \mathbb{E}_t^{\hat{X}_{\text{na}}}[\int_t^T h(s - \tau)U(\hat{X}_{\text{na}}(s))\mathrm{d}s + \alpha h(T - \tau)U(\hat{X}_{\text{na}}(T)) + \int_t^T h(s - \tau) \frac{1}{2\sigma^2(\tau, s, x)} \left( \hat{\xi}^{na}(s) \right)^2 \mathrm{d}s],
\]

which is the expectation of the discounted utility conditioned on the initial state \((t, x)\) for the agent who uses discount function \(h(- \tau)\), the ambiguity aversion \(\phi(\tau, \cdot, \cdot)\) and implements the strategy \(\left(\hat{\xi}^{na}(\cdot), \hat{\nu}^{na}(\cdot), \hat{\chi}^{na}(\cdot)\right)\) on \([t, T]\). Obviously, \(\bar{\Theta}^{na}(t, x; t)\) is the actual value function of the naive agent.

Similar to (4.25) in [40], we know that for any \(\tau, \bar{\Theta}^{na}(\cdot, \cdot, \tau)\) satisfies

\[
\left\{ \begin{array}{l}
\bar{\Theta}^{na}(t, x; \tau) + \left[ r x - I \left( \Theta_{x}^{na}(t, x; t) \right) - \frac{\sigma^2(t) \Theta_{x}^{na}(t, x; t)}{\Theta_{xx}^{na}(t, x; t) - \dot{\phi}(t, t, x) \Theta_{x}^{na}(t, x; t)} \right] \bar{\Theta}^{na}(x, x; t) \\
+ h(t - \tau) \left[ U \left( \frac{\Theta_{x}^{na}(t, x; t)}{\Theta_{xx}^{na}(t, x; t) - \dot{\phi}(t, t, x) \Theta_{x}^{na}(t, x; t)} \right) \right] + \frac{\nu(t) \Theta_{x}^{na}(t, x; t) \Theta_{xx}^{na}(t, x; t)}{\Theta_{x}^{na}(t, x; t)} \right] \right) \bar{\Theta}^{na}(x, x; t) \\
+ \frac{1}{2} \left[ \frac{\theta(t) \Theta_{x}^{na}(t, x; t)}{\Theta_{xx}^{na}(t, x; t) - \dot{\phi}(t, t, x) \Theta_{x}^{na}(t, x; t)} \right] \bar{\Theta}^{na}(x, x; t) = 0, & t \leq T, x > 0,
\end{array} \right.
\]

(27)

Note that (27) can be obtained from (11) by replacing \(\Theta(t, t, x)\) with \(\Theta^a(t, x; t)\).

This is because if we change \((\hat{\xi}(\cdot), \hat{\nu}(\cdot), \hat{\chi}(\cdot))\) to \((\hat{\xi}^{na}(\cdot), \hat{\nu}^{na}(\cdot), \hat{\chi}^{na}(\cdot))\) in (13), then \(\bar{\Theta}^{na}(t, x; \tau) = \Theta(t, t, x)\).

4.1. **Power utility.** Similar to the sophisticated case, we impose the following homothetic robustness specification:

\[
\phi(\tau, t, x) = \frac{\eta(\tau)}{1 - \gamma(\tau)} \Theta^a(t, x; \tau),
\]

where \(\eta(\cdot) > 0\) is a bounded deterministic function.
Theorem 4.1. If \( U(x) = \frac{x^{1-\gamma}}{1-\gamma}, x > 0, \) with \( \gamma \in (0, 1) \), then the optimal strategy for a naive investor is given by

\[
\hat{c}^{na}(t, x) = (v^{na}(t; t))^{-1} x, \quad \hat{u}^{na}(t, x) = \frac{\theta(t)}{(\theta(t) + \gamma)\sigma(t)} x, \quad \hat{\xi}^{na}(t, x) = \frac{\eta(t)\theta(t)}{\eta(t) + \gamma},
\]

where

\[
v^{na}(t; \tau) = \alpha \frac{\hat{\Theta}}{\hat{\Theta}} (T - \tau) \exp \left\{ \int_t^T (1 - \gamma) \left[ r(s) - \frac{1}{2} \theta^2(s) h(s - \tau) \right] \, ds \right\} + \int_t^T \exp \left\{ \int_t^\tau (1 - \gamma) \left[ r(s) - \frac{1}{2} \theta^2(s) h(s - \tau) \right] \, ds \right\} \gamma h(\tau - \tau) \frac{1}{\tau} \, d\tau.
\]

The value function is given by

\[
\bar{\Theta}^{na}(t, x; t) = f^{na}(t; t) \frac{x^{1-\gamma}}{1-\gamma},
\]

where

\[
f^{na}(t; \tau) = ah(T - \tau) \exp \left\{ \int_t^T (1 - \gamma) \left[ r(s) - (f^{na}(s; s))^{-\frac{1}{\gamma}} + \frac{1}{2} \theta^2(s) \left( 1 - \frac{\eta(s)}{\theta(s) + \gamma} \right)^2 \right] \, ds \right\} + \int_t^T \exp \left\{ \int_t^\tau (1 - \gamma) \left[ r(s) - (f^{na}(s; s))^{-\frac{1}{\gamma}} + \frac{1}{2} \theta^2(s) \left( 1 - \frac{\eta(s)}{\theta(s) + \gamma} \right)^2 \right] \, ds \right\} \times h(\tau - \tau) \left[ (f^{na}(\tau; \tau))^{-\frac{1}{\gamma}} + (1 - \gamma) f^{na}(\tau; \tau) \frac{\theta(\tau)\eta(\tau)}{\theta(\tau) + \gamma} \right] \, d\tau,
\]

and \( f^{na}(t; \tau) = [v^{na}(t; \tau)]^\gamma \).

Proof. In this case, (26) becomes

\[
\begin{cases}
\Theta_t^{na}(t, x; \tau) + \left[ r(t) x - \left( \frac{\Theta^{na}(t, x; \tau)}{h(t - \tau)} \right)^{-\frac{1}{\gamma}} \right] \Theta_x^{na}(t, x; \tau) + h(t - \tau) \left( \frac{\Theta^{na}(t, x; \tau)}{h(t - \tau)} \right)^{-\frac{1}{\gamma}} - \frac{1}{2} \theta^2(t) \left( \frac{\Theta^{na}(t, x; \tau)}{h(t - \tau)} \right)^{-\frac{1}{\gamma}} - \frac{1}{2} \theta^2(t) \left( \frac{\Theta^{na}(t, x; \tau)}{h(t - \tau)} \right)^{-\frac{1}{\gamma}} = 0, \\
\Theta^{na}(t, x; \tau) = ah(T - \tau) \frac{x^{1-\gamma}}{1-\gamma}, \quad x \in (0, \infty).
\end{cases}
\]

Consider the Ansatz:

\[
\Theta^{na}(t, x; \tau) = f^{na}(t; \tau) \frac{x^{1-\gamma}}{1-\gamma}.
\]

Inserting (31) into (30) and comparing coefficients, it yields that

\[
\begin{cases}
f^{na}(t; \tau) + (1 - \gamma) \left[ r(t) - \frac{1}{2} \frac{\theta^2(t - \tau)}{h(t - \tau)} \right] f^{na}(t; \tau), \\
+ \gamma h(t - \tau) \frac{1}{\gamma} \left( f^{na}(t; \tau) \right)^{-\frac{1}{\gamma}} = 0, \quad \tau \leq t \leq T, \\
f^{na}(T; \tau) = ah(T - \tau), \quad x \in (0, \infty).
\end{cases}
\]
Let 
\[ v^{na}(t; \tau) = (f^{na}(t; \tau))^\frac{1}{\gamma}. \]

Then
\[
\begin{cases}
\gamma v_t^{na}(t; \tau) + (1 - \gamma) \left[ r - \frac{1}{2} \frac{\partial^2 h(t;\tau)}{\partial \tau^2} \right] v^{na}(t; \tau) + \gamma h(t; \tau)^\frac{1}{\gamma} = 0, \\
v^{na}(T; \tau) = \alpha \frac{1}{\gamma} h^\gamma(T - \tau), \quad x \in (0, \infty),
\end{cases}
\]

which implies \((28)\).

Similarly, we guess the value function has the following form
\[ \tilde{\Theta}^{na}(t, x; \tau) = \bar{f}^{na}(t; \tau) \frac{x^{1-\gamma}}{1-\gamma}. \]

It follows from \((27)\) that
\[
\begin{cases}
\bar{f}_t^{na}(t; \tau) + (1 - \gamma) \left[ r - (f^{na}(t; t)) \right]^{-\frac{1}{\gamma}} + \frac{1}{2} \frac{\partial^2}{\partial \tau^2} \left( 1 - \frac{\eta(t) \tau}{\eta(t) + \tau} \right)^2 \bar{f}^{na}(t; \tau) \\
+ h(t; \tau) \left[ f^{na}(t; t) \right]^{-\frac{1}{\gamma}} + (1 - \gamma) f^{na}(t; \tau) \left( \frac{\theta(t) \tau}{\theta(t) + \tau} \right)^2 = 0, \quad \tau \leq t \leq T, \ t > 0, \\
\bar{f}^{na}(T; \tau) = \alpha h(T - \tau), \quad \tau \in [0, T], \ x > 0,
\end{cases}
\]

which implies \((29)\).

Since the optimal investment strategy is independent of the discount rate, it is the same as that for a sophisticated investor. However, the optimal consumption rate for a naive investor is different from that for a sophisticated investor.

4.2. Logarithmic utility. Similarly, we use the homothetic robustness specification
\[ \phi(\tau, t, x) = \lim_{\gamma \to 1} \frac{\eta(\tau)}{(1 - \gamma) \Theta^{na}(t, x; \tau)}, \]
where \(\eta(\cdot) > 0\) is a bounded deterministic function and \(\Theta^{na}(t, x; \tau)\) is the value function of the naive investor-\(\tau\) with power utility. Thus
\[ \phi(\tau, t, x) = \lim_{\gamma \to 1} \eta(\tau) = \frac{\eta(\tau)}{H(t)}. \]

Theorem 4.2. If \(U(x) = \ln x, x > 0\), then the optimal strategy for naive investor is given by
\[ \epsilon^{na}(t, x) = H(t)^{-1} x, \quad \hat{u}^{na}(t, x) = \frac{\theta(t)}{(\eta(t) + 1)\sigma(t)} x, \quad \hat{\epsilon}^{na}(t, x) = \frac{\eta(t) \theta(t)}{\eta(t) + 1}. \] (32)

The value function is
\[ \tilde{\Theta}^{na}(t, x; \tau) = \bar{f}(t, x) = \bar{f}(t) \ln x + \bar{g}(t, t). \] (33)

Proof. If \(U(x) = \ln x\), then \(I(x) = x^{-1}\). The HJB equation \((25)\) becomes
\[
\begin{cases}
\Theta_t^{na}(t, x; \tau) + \left[ r x - \frac{h(t; \tau)}{\Theta^{na}(t, x; \tau)} \right] \Theta_x^{na}(t, x; \tau) + h(t; \tau) \ln \left( \frac{h(t; \tau)}{\Theta^{na}(t, x; \tau)} \right) \\
\quad - \frac{1}{2} \frac{\partial^2}{\partial \tau^2} \Theta^{na}(t, x; \tau) \left( \frac{\theta(t)}{\eta(t) + 1} \right)^2 \Theta^{na}(t, x; \tau) \left( \frac{\theta(t)}{\eta(t) + 1} \right)^2 = 0, \quad \tau \leq t \leq T, \ x \in (0, \infty), \\
\Theta^{na}(T, x; \tau) = \alpha h(T - \tau) \ln x, \quad x \in (0, \infty). \end{cases}
\] (34)
Let us consider the ansatz:
\[
\Theta^{na}(t, x; \tau) = \hat{f}^{na}(t; \tau) \ln x + \hat{g}^{na}(t; \tau).
\] (35)

Putting (35) back into (34) and comparing the coefficients, it yields that
\[
\begin{cases}
\hat{f}^{na}(t; \tau) + h(t - \tau) = 0, & \tau \leq t \leq T, \\
\hat{f}^{na}(T; \tau) = \alpha h(T - \tau).
\end{cases}
\]

Obviously, we have
\[
\hat{f}^{na}(t; \tau) = \alpha h(T - \tau) + \int_t^T h(s - \tau) ds
\]
and \(\hat{f}^{na}(t; t) = H(t)\). Therefore, it is easy to derive (32).

Note that, in the logarithmic utility case, the optimal strategies for naive and sophisticated investors are the same (this property is consistent with [35] and [30]). Thus, we have (33).

5. Numerical illustrations. Since the optimal strategies for naive and sophisticated investors of the logarithmic utility case are the same, we take the optimal strategies of the power utility case as an example in the numerical illustrations, and take the discount function as
\[
h(t) = \omega_1 \exp(-\delta_1 t) + \omega_2 \exp(-\delta_2 t).
\]
Throughout numerical analysis, unless otherwise stated, the basic parameters are given by \(\alpha = 1; \omega_1 = 0.3; \omega_2 = 0.7; \delta_1 = 0.1; \delta_2 = 0.2; \gamma = 0.8; \eta = 1; \theta = 2; r = 0.05; T = 5\) and \(x = 1\). The value of parameters is referred to the existing studies such as Ekeland and Pirva (2008).

To simplify the analysis, we consider the optimal proportion strategy of consumption and investment, i.e., \(\hat{c}(t, x)/x\) and \(\hat{u}(t, x)/x\). From Theorems 2 and 4, we find that the optimal proportion strategy of investment for naive and sophisticated investors are the same which is given by
\[
\frac{\hat{u}(t, x)}{x} = \frac{\hat{u}^{na}(t, x)}{x} = \frac{\theta(t)}{\sigma(t)(\gamma + \eta(t))}.
\]
Furthermore, if the investor is more ambiguity to the financial market, both the naive and sophisticated investors will become more conservative, and invest less proportion of wealth in the risky asset.

Because the expressions of optimal consumption strategies and the corresponding value functions are complex, we show the effects of the ambiguity parameter \(\eta\) on the optimal consumption strategies and the optimal value functions through numerical results. From the left figure of Figure 5.1, we can find that: (i) The proportion of wealth to consume for the sophisticated investor is lower than that of the naive investor. One possible reason is that the sophisticated investor recognizes that her future preferences change and become smaller over time, then she need to consume more in the future to make the discounted utility smooth. Then she need consume less at the beginning period to accumulate more wealth for future consumption. But the naive investor does not recognizes her future preferences change over time, so she does not accumulate more wealth for future consumption and then consumes more at the beginning period. (ii) For the sophisticated investor, at the beginning period, if the investor is more ambiguity averse about the financial market, she will consume less wealth, and as time goes by, the consumption strategy with ambiguity may exceed that without ambiguity; for the naive investor, she will always consume less if she is ambiguity averse. This can be attributed to the fact that if \(\eta\) increases,
the investor is more ambiguity averse about the financial market and invest less in the risky asset, then the expected return from the financial market will become less. Thus, the investor with ambiguity averse will consume less to accumulate more wealth for future consumption. However, for the sophisticated investor, if the time is large enough, since the investor with ambiguity averse accumulates more wealth, then the consumption strategy with ambiguity is higher than that without ambiguity. (iii) We can also find that the effect of ambiguity on the naive investor’s consumption is more significant than it on the sophisticated investor’s.

From the right figure of Figure 5.1, we find that: (i) The value function for the naive investor is smaller than that of the sophisticated investor. We think the reason is that the naive investor consumes more than the sophisticated investor and then she has less wealth to invest in the financial market to get more return for future consumption. (ii) The value function with ambiguity is smaller than that without ambiguity. We think the reason is that the investor with ambiguity averse invest less in the risky asset and derive less expected return from the financial market.

6. Conclusion. In this paper, we consider an optimal consumption-investment problem with more general discount rates under the robust framework. In our model, the decision-maker is ambiguity-averse and invest her wealth in a risk-free asset and a risky asset. With consideration of non-exponential discounting, the dynamic optimization problem becomes time-inconsistent and Bellman’s principle of optimality does not hold. By solving the equilibrium HJB equation, we derive the equilibrium strategies for a sophisticated investor with power and logarithmic utility functions. Furthermore, we also obtain the corresponding optimal consumption-investment strategy for a naive investor. From this paper, we find that: (i) the robust optimal investment strategies for sophisticated and naive investors are the same, which are independent of the discount rate; (ii) the robust optimal consumption strategies for the two investors are quite different, the sophisticated investor will consume more than the naive investor, and for the sophisticated investor, at initial time, if the investor is more ambiguity to the financial market, she will consume more wealth, and as time goes by, the consumption strategy without considering ambiguity may exceed that with considering ambiguity, and for the naive investor, she will always consume more if she is ambiguity averse; (iii) the results
for logarithmic utility are indeed the limits (as $\gamma \to 1$) for the power utility. The main contributions of this paper are as follows: (i) A new continuous time optimal consumption-investment model incorporating non-exponential discounting and model uncertainty is established; (ii) the robust optimal consumption-investment strategies for sophisticated and naive investors are derived explicitly, and the two strategies are compared; and (iii) our model and results are more general, which can reduce to models only considering time-inconsistent preference (e.g. [30]) or model uncertainty (e.g. [27]).

In future research, more complicated models, such as uncertain lifetime and mortality risk can be taken into account, although doing so may make it difficult to obtain a closed-form solution. Thus, other methods, such as asymptotic or other practical methods, may be introduced to deal with the robust optimal consumption-investment problem with non-exponential discounting.

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