Decoherence of two qubits in a non-Markovian reservoir without rotating-wave approximation

Fa-Qiang Wang, Zhi-Ming Zhang and Rui-Sheng Liang
Lab of Photonic Information Technology, School of Information and Photoelectric Science and Engineering, South China Normal University, Guangzhou 510066, China
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The decoherence of two initially entangled qubits in a non-Markovian reservoir has been investigated exactly without Born Markovian approximation and rotating-wave approximation(RWA). The non-perturbative quantum master equation is derived and its exact solution is obtained. The decoherence behaviors of two qubits, initially entangled in Bell states, has been investigated in three different cases of parameters. The results show that the counter-rotating wave terms have great influence on the decoherence behavior, and there are differences between the exact solution of the Hamiltonian with RWA and that of the exact Hamiltonian without RWA.

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I. INTRODUCTION

In recent years, the phenomenon, termed as “entanglement sudden death”(ESD), has been found theoretically[1, 2] and shown experimentally[3, 4]. It is shown that spontaneous disentanglement may take only a finite-time to be completed, while local decoherence (the normal single-atom transverse and longitudinal decay) takes an infinite time[2]. It’s quite different from the case of continuous variable two-atom model discussed by Dodd and Halliwell[5]. Thereafter, many works have been devoted to the related topics[6, 7, 8, 9, 10], and some authors extended the results in Markovian regime to non-Markovian case[11, 12].

The RWA, which neglecting counter rotating terms corresponding to the emission and absorption of virtual photon without energy conservation, is widely used in quantum optics. Generally, the RWA is justified for small detunings and small ratio of the atom-field coupling divided by the atomic transition frequency[13, 14, 15]. In atom-field cavity systems, this ratio is typically of the order $10^{-7} \sim 10^{-6}$. Recently, cavity systems with very strong couplings have been discussed[16]. The ratio may also become order of magnitudes larger in solid state systems, and the full Hamiltonian, including the virtual processes (counter-rotating terms), must be considered[17].

The neglect of counter-rotating wave terms in Hamiltonian strongly simplifies the mathematical treatment of the problem and usually give exact solution of the approximate Hamiltonian, while the perturbative approach to the systems beyond the RWA usually is complicated and gives approximate solution[14]. Otherwise, with the same parameters, the approximation in Hamiltonian might lead to different result from the approximation of exact solution with exact Hamiltonian. Fortunately, non-perturbative master equation could be obtained by path integral and it is beneficial to study, exactly, the systems beyond the RWA[18].

In this paper, we will focus on the influence of counter-rotating wave terms on the decoherence behavior of two qubits in a non-markovian reservoir from the exact solution of the system beyond the RWA and Born-Markovian approximation. In section III, the reduced non-perturbative non-Markovian quantum master equation of an atom in a non-Markovian reservoir is derived and its the exact solution is obtained by algebraic approach of Lie superoperator. In section IV, the decoherence of two initially entangled atoms coupled with two cavities separately has been discussed. The conclusion will be given in section V.

II. MODEL AND EXACT SOLUTION

A. Hamiltonian and non-perturbative master equation

Now we restrict our attention to two noninteracting two-level atoms A and B coupled individually to two environment reservoirs[12]. To this aim, we first consider the Hamiltonian of the subsystem of a single qubit coupled to its reservoir as

$$H = H_a + H_r + H_{ar}$$

where

$$H_a = \omega_0 \frac{\sigma_z}{2}$$

$$H_r = \sum_k \omega_k a_k^+ a_k$$

$$H_{ar} = (\sigma_+ + \sigma_-) \sum_k g_k (a_k^+ + a_k)$$

where $\omega_0$ is the atomic transition frequency between the ground state $|0\rangle$ and excited state $|1\rangle$. $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$, $\sigma_+ = |1\rangle\langle 0|$, and $\sigma_- = |0\rangle\langle 1|$ are pseudo-spin operators of atom. The index $k$ labels the field modes of the reservoir with frequency $\omega_k$, $a_k$ and $a_k^+$ are the

*Electronic address: zmzhang@scnu.edu.cn
modes’ creation and annihilation operators, and $g_k$ is the frequency-dependent coupling constant between the transition $|1\rangle - |0\rangle$ and the field mode $k$.

The reduced non-perturbative non-Markovian quantum master equation of atom could be obtained by path integral [18]

$$
\frac{\partial}{\partial t} \rho_a = -i \mathcal{L}_a \rho_a - \int_0^t ds \{ \mathcal{L}_{ar} e^{-i \mathcal{L}_a (t - s)} \mathcal{L}_{ar} e^{-i \mathcal{L}_a (s - t)} \rho_a 
$$

(5)

where $\mathcal{L}_0$, $\mathcal{L}_a$ and $\mathcal{L}_{ar}$ are Liouvillian operators defined as

$$
\mathcal{L}_0 \rho \equiv [H_a + H_r, \rho] \\
\mathcal{L}_a \rho \equiv [H_a, \rho] \\
\mathcal{L}_{ar} \rho \equiv [H_{ar}, \rho]
$$

and $\langle \ldots \rangle_r$ stands for partial trace of the reservoir.

Then we assume that the reservoir is initially in vacuum states and the spectral density of the reservoir is in Lorentzian form [12, 19]

$$
J(\omega) = \sum_k g_k^2 \{ \delta(\omega - \omega_k) + \delta(\omega + \omega_k) \}
$$

$$
= \frac{1}{2\pi} \frac{\lambda \gamma^2}{(\omega - \omega_0)^2 + \gamma^2}
$$

(6)

where $\gamma$ represents the width of the spectral distribution of the reservoir modes and is related to the correlation time of the noise induced by the reservoir, $\tau_r = 1/\gamma$. The parameter $\lambda$ is related to the subsystem-reservoir coupling strength. There are two correlation functions in this model [20]

$$
\alpha_1 = \int_{-\infty}^{\infty} J(\omega) e^{-i(\omega - \omega_0)t} = \frac{\gamma \lambda}{2} e^{-\gamma t}
$$

(7)

$$
\alpha_2 = \int_{-\infty}^{\infty} J(\omega) e^{i(\omega + \omega_0)t} = \frac{\gamma \lambda}{2} e^{(-\gamma + i2\omega_0)t}
$$

(8)

$\alpha_1$ comes from the rotating-wave interaction and $\alpha_2$ from the counter-rotating wave interaction.

So, we could obtain the non-perturbative master equation of the subsystem from Eq. (5)

$$
\frac{\partial}{\partial t} \rho_a = -\lambda \left( \gamma \alpha_R + f(t) \right) \rho_a - i (2\omega_0 - \lambda \gamma \alpha_I) J_0 \rho_a \\
+ \frac{\lambda}{2} \left( \gamma \alpha + f(t) \right) J_+ \rho_a + \frac{\lambda}{2} \left( \gamma \alpha^* + f(t) \right) J_- \rho_a \\
+ \lambda \left( \gamma \alpha_R - f(t) \right) K_0 \rho_a + \lambda \gamma \alpha_R K_+ \rho_a \\
+ \lambda f(t) K_- \rho_a
$$

(9)

where $J_0$, $J_+$, $J_-$, $K_0$, $K_+$ and $K_-$ are superoperators defined as

$$
J_0 \rho_a = \left[ \sigma_+ \rho_a \right] \\
J_+ \rho_a = \sigma_+ \rho_a \sigma_+ \\
J_- \rho_a = \sigma_- \rho_a \sigma_- \\
K_0 \rho_a = (\sigma_+ \sigma_- - \rho_a \sigma_+ \sigma_-) / 2 \\
K_+ \rho_a = \sigma_+ \rho_a \sigma_- \\
K_- \rho_a = \sigma_- \rho_a \sigma_+
$$

and

$$
\alpha = \frac{1 - e^{-(\gamma + 2\omega_0)t}}{\gamma + i2\omega_0}.
$$

$\alpha^R$, $\alpha^I$ and $\alpha^*$ are real part, image part and conjugate of $\alpha$, respectively. $f(t) = 1 - \exp(-\gamma t)$.

From Eq. (9), we find that there is a frequency shift caused by the interaction between the atom and the non-Markovian reservoir [21]. Eq. (9) reveals that the contribution of counter-rotating terms is in order of $\lambda \gamma^2 / \omega_0^2$ besides the rapidly oscillating terms, which is quite different from the result of Born and Markovian approximation of atom in vacuum [22].

B. Exact solution of master equation

The time evolution of density operator in Eq. (9) could be obtained with algebraic approach in Ref. [23] because the superoperators herein satisfy $SU(2)$ Lie algebraic communication relations, i.e.

$$
|J_-, J_+ \rangle \rho_a = -2J_0 \rho_a \\
|J_0, J_\pm \rangle \rho_a = \pm J_\pm \rho_a \\
|K_0, K_\pm \rangle \rho_a = -2K_0 \rho_a \\
|K_\pm, J_\pm \rangle = 0
$$

(10)

where $i, j = 0, \pm$. By directly integrating Eq. (9), the formal solution is obtained as [14]

$$
\rho_a(t) = e^{-\Gamma_k \tilde{T}_e \rho_a dt (c_0 J_0 + c_+ J_+ + c_- J_-)} \\
\times e^{\Gamma_0 \tilde{T}_e \rho_a dt (c_0 K_0 + c_+ K_+ + c_- K_-)} \rho_a(0)
$$

(11)

where $\tilde{T}$ is time time order operator, $\varepsilon_0 = -i (2\omega_0 - \lambda \gamma \alpha^I)$, $\varepsilon_+ = \lambda (\gamma \alpha + f(t)) / 2$, $\varepsilon_- = \lambda (\gamma \alpha^* + f(t)) / 2$, $\nu_0 = \lambda (\gamma \alpha_R - f(t))$, $\nu_+ = \lambda \gamma \alpha_R$, $\nu_- = \lambda f(t)$, $\Gamma_k = \lambda (\gamma \alpha^R F(t)) / 2$, $F(t) = t - [1 - \exp(-\gamma t)] / \gamma$ and

$$
\tilde{\alpha} = \int_0^t \alpha dt = \tilde{\alpha}^R + i \tilde{\alpha}^I \\
\tilde{\alpha}^* = \tilde{\alpha}^R - i \tilde{\alpha}^I
$$

(12)
\[
\hat{\alpha}^R = \frac{1}{4\omega_0^2 + \gamma^2} \left[ \gamma t + \frac{1}{4\omega_0^2 + \gamma^2} \left( (4\omega_0^2 - \gamma^2)(1 - e^{-\gamma t}\cos(2\omega_0 t)) - 4\omega_0 \gamma e^{-\gamma t}\sin(2\omega_0 t) \right) \right] \\
\hat{\alpha}^I = \frac{1}{4\omega_0^2 + \gamma^2} \left[ -2\omega_0 t + \frac{1}{4\omega_0^2 + \gamma^2} \left( 4\omega_0 \gamma (1 - e^{-\gamma t}\cos(2\omega_0 t)) + (4\omega_0^2 - \gamma^2)e^{-\gamma t}\sin(2\omega_0 t) \right) \right]
\]

The exponential functions of superoperators in Eq. (11) could be disentangled in form [14]

\[
\hat{T}_e^f \rho(t) \frac{dt}{\delta_0 \epsilon + e^{\sum J_+ + \epsilon J_-}} = e^{i_J + e^{j_0 \epsilon}} e^{j_J - J_+ - J_-} \\
\hat{T}_e^f \rho(t) \frac{dt}{\delta_0 = e^{k_+ k - \epsilon_0 \epsilon} e^{k_+ k - K_-}}
\]

where \( j_+ \), \( j_0 \), \( j_- \) and \( k_+ \), \( k_0 \), \( k_- \) satisfy the following equation

\[
\begin{align*}
\dot{X}_+ &= \mu_+ - \mu_+ X_+ \mu_0 X_+ \\
\dot{X}_0 &= \mu_0 - 2\mu_+ X_+ \\
\dot{X}_- &= \mu_0 - \exp(X_0)
\end{align*}
\]

\[\mu = \varepsilon \text{ for } X = \hat{J} \text{ and } \mu = \nu \text{ for } X = \hat{K}.\]

Using the results in Appendix, the exact solution of the master equation Eq. (10) is obtained

\[
\rho_\alpha(t) = e^{-\gamma_\alpha \hat{\rho}(t)}
\]

\[
\hat{\rho}(t) = \left( \begin{array}{cccc}
\rho_{11}^{(0)}(0) + m \rho_{00}^{(0)(0)} + y \rho_{01}^{(0)(0)} & x \rho_{00}^{(0)(0)} & y \rho_{01}^{(0)(0)} & 0 \\
q \rho_{00}^{(0)(0)} & n \rho_{00}^{(0)(0)} + m \rho_{11}^{(0)(0)} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right)
\]

\[l = e^{k_0/2} + e^{-k_0/2}k_+ k_- \quad m = e^{-k_0/2}k_+ \]

\[n = e^{-k_0/2} \quad p = e^{-k_0/2}k_- \]

\[q = e^{-j_0/2} \quad r = e^{-j_0/2}j_- \]

\[x = e^{j_0/2} + e^{-j_0/2}j_+ \quad y = e^{-j_0/2} j_+ \]

\[\]

### III. Numerical Results and Discussion

In order to study the effects of non-Markovian reservoir on the decoherence, we assume \( \lambda = 10 \gamma \) in Eq. (6), which could be realized in a high-Q cavity [12].

First, we focus on the decoherence of two qubits with an initial state \(| \Psi \rangle \). For the RWA model in Ref. [12], Fig. [11] shows that the concurrence periodically vanishes with a damping of its revival amplitude. For the non-RWA model in this paper, the decoherence of the system was categorized into three cases

\[\lambda > \omega_0 \gg \gamma \]

Fig. [12] reveals that the concurrence \( C_\Phi \) decreases exponentially to zero with very small amplitude oscillation and the decoherence time is about \( 1/\gamma \), which corresponds to the correlation time of the counter-rotating wave terms. As for the correlation function \( \alpha_2 \), it has no memory effect because it’s value averages to zero on time if \( \gamma / \omega_0 \ll 1 \). Compared with Fig. [11] for RWA model, there is no revival of entanglement for non-RWA model, which exhibits that the counter-rotating wave terms could not be neglected in Hamiltonian because the contribution of counter-rotating terms in master equation is in the order of \( \lambda^2 / \omega_0^2 \), which influences the decoherence behavior in long time scale. This characteristic will hold on for more weaker coupling constant.
(B) $\omega_0 = \lambda > \gamma$. From Fig. 3 we could find that the concurrence $C_\Phi$ first decreases to a finite value and maintain it for a period of time, then periodically vanishes with a damping of its revival amplitude, like that for RWA model. This is the result of coaction of rotating wave process and counter-rotating process of atom with a non-Markovian reservoir because the contribution of counter-rotating wave process to the system is bigger than the former case and its memory effect is nonzero.

(C) $\omega_0 = 3\gamma < \lambda$. Fig. 4 exhibits that the concurrence $C_\Phi$ decreases to zero, then periodically vanishes with a damping of its revival amplitude, which resulted from the strong interaction between atom and the non-Markovian reservoir through virtual photon process.

From the discussion above, we find that the decoherence behavior of $C_\Phi$ is symmetric with $\beta^2$ because of the symmetry of initial state $|\Phi\rangle$. And the revival of entanglement could be found only with big ratio of coupling constant to the atom transition frequency. For no-RWA model, the decoherence behaviors are richer than that for RWA model.

Then, we focus on the decoherence of two qubits with initial state of $|\Psi\rangle$. For the RWA model in Ref. [12], Fig. 1 shows that the entanglement represented by $C_\Psi$ has a similar behavior to $C_\Phi$ for $\beta^2 \geq 1/2$ in Fig. 1. In contrast, for $\beta^2 < 1/2$, there is ESD because $C_\Psi$ vanishes permanently after a finite time, similar to the Markovian case. Second, revival of entanglement appears after periods of times when disentanglement is complete. For the non-RWA model, the decoherence of the system were also categorized into three cases

(A) $\omega_0 > \lambda, \omega_0 \gg \gamma$. Fig. 2 reveals that the concurrence $C_\Phi$ decreases exponentially to zero for $\beta^2 \geq 1/2$, while $C_\Psi$ vanishes permanently after a finite time for $\beta^2 < 1/2$.

(B) $\omega_0 = \lambda > \gamma$. Fig. 3 shows that the entanglement represented by $C_\Phi$ has a similar behavior to $C_\Psi$ for $\beta^2 \geq 1/2$ in Fig. 1. In contrast, for $\beta^2 < 1/2$, the concurrence first decreases to zero and maintain it for a long period of time, then revives with very small amplitude before vanishes permanently.
(C) $\omega_0 = 3\gamma < \lambda$. The evolution dynamics of concurrence $C_{\Phi}$ is almost the same as that in Fig. IV. Unlike the two cases above, the evolution behavior of concurrence $C_{\Psi}$ becomes symmetric, like that of $C_{\Phi}$, because of the strong interaction of atom with reservoir through the emission and absorption of virtual photon.

The above characteristics of $C_{\Psi}$ also verify that the revival of entanglement could be found only with big ratio of coupling constant to the atom transition frequency.

IV. CONCLUSION

The reduced non-perturbative non-Markovian quantum master equation of atom in non-Markovian reservoir has been derived and its exact solution is obtained by algebraic approach of Lie superoperator. The decoherence of two initially entangled atoms, coupled with two vacuum cavities separately, has been discussed.

The results show that the decoherence behavior of two qubits in a non-markovian reservoir is dependent on the ratio of the coupling strength to atomic frequency. First, with the increasing of coupling strength, the decoherence behavior becomes more and more complicated, and there is revival of entanglement after a period of time of disentanglement, due to the strong interaction and the memory effect of the non-markovian reservoir. Second, the strong coupling and the counter-rotating wave interaction could make the decoherence of even parity Bell entanglement state become symmetrical. Third, with no-RWA model, one could find much more decoherence behaviors than that of RWA model under different conditions of system parameters. The case for finite temperature could also be obtained by the method in this paper and will be published elsewhere.

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APPENDIX A

Defining two superoperators as

$$J \equiv e^{j+J_+} e^{j+J_0} e^{j- J_-}$$

$$K \equiv e^{k+K_+} e^{k_0 K_0} e^{k- K_-}$$

and using the following relations

$$e^{j+J_+} \rho = \rho + j_+ \sigma_+ \rho \sigma_+$$

$$e^{j- J_-} \rho = \rho + j_- \sigma_- \rho \sigma_-$$

$$e^{k+K_+} \rho = \rho + k_+ \sigma_+ \rho \sigma_-$$

$$e^{k- K_-} \rho = \rho + k_- \sigma_- \rho \sigma_+$$
\[ e^{j_0 \rho} = (\text{ch} \, j_0^4 + \sigma_z \text{sh} \, j_0^4)\rho (\text{ch} \, j_0^4 - \sigma_z \text{sh} \, j_0^4) \]  
(A7)

\[ e^{k_0 K_0 \rho} = e^{k_0} [1 + (e^{k_0} - 1)\sigma_+ \sigma_-] \rho [1 + (e^{k_0} - 1)\sigma_+ \sigma_-], \]  
(A8)

one could get

\[ J \begin{pmatrix} \rho_{11} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{pmatrix} = \begin{pmatrix} \rho_{11} & (e^{j_0/2} + e^{-j_0/2} j_-) \rho_{10} + e^{-j_0/2} j_+ \rho_{01} \\ e^{-j_0/2} \rho_{01} + e^{-j_0/2} j_- \rho_{10} & \rho_{00} \end{pmatrix} \]  
(A9)

\[ K \begin{pmatrix} \rho_{11} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{pmatrix} = \begin{pmatrix} (e^{k_0/2} + e^{-k_0/2} k_-) \rho_{11} + e^{-k_0/2} k_+ \rho_{00} & \rho_{10} \\ e^{-k_0/2} \rho_{00} + e^{-k_0/2} k_- \rho_{11} & \rho_{01} \end{pmatrix} \]  
(A10)

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