Asymptotic approximations of the solution
to a boundary-value problem in a thin aneurysm-type domain

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Abstract

A nonuniform Neumann boundary-value problem is considered for the Poisson equation in a thin 3D aneurysm-type domain that consists of thin curvilinear cylinders that are joined through an aneurysm of diameter $O(\varepsilon)$. A rigorous procedure is developed to construct the complete asymptotic expansion for the solution as the parameter $\varepsilon \to 0$.

The asymptotic expansion consists of a regular part that is located inside of each cylinder, a boundary-layer part near the base of each cylinder, and an inner part discovered in a neighborhood of the aneurysm. Terms of the inner part of the asymptotics are special solutions of boundary-value problems in an unbounded domain with different outlets at infinity. It turns out that they have polynomial growth at infinity. By matching these parts, we derive the limit problem ($\varepsilon = 0$) in the corresponding graph and a recurrence procedure to determine all terms of the asymptotic expansion.

Energetic and uniform pointwise estimates are proved. These estimates allow us to observe the impact of the aneurysm.

Key words: asymptotic expansion, multiscale analysis, thin aneurysm-type domains

MOS subject classification: 35B40, 74K30, 35C20, 35J05,

1 Introduction

In this paper we continue our investigation of boundary-value problems in thin multi-structures with a local geometric irregularity, which we have begun in [23, 24]. Namely, we modify and generalize our approach for more complicated structures that consist of thin curvilinear cylinders connected through a domain of small diameter.

Investigations of various physical and biological processes in channels, junctions and networks are urgent for numerous fields of natural sciences (see, e.g., [1, 2, 3, 4, 5, 7, 9, 10, 11, 12, 13, 25, 27, 31, 28, 35, 37, 39] and the references therein). A particular interest is the investigation of the influence of a local geometric heterogeneity in vessels on the blood flow. This is both an aneurysm (a pathological extension of an artery like a bulge) and a stenosis (a pathological restriction of an artery). In [21] the authors classified 12 different aneurysms and proposed a numerical approach for this study. The aneurysm models have been meshed with 800,000 – 1,200,000 tetrahedral cells containing three boundary layers. It was showed that the geometric aneurysm form essentially impacts on the haemodynamics of the blood flow. However, as was noted by the authors, the question how to model blood flow with sufficient accuracy is still open.
This question was the main motivation for us to begin the study of boundary-value problems in domains of such type and to detect the influence of their local geometric irregularity on properties of solutions. It is clear that such domains are prototypes of many other biological and engineering structures, but we prefer to call them *aneurysm-type domains* as comprehensive and concise.

In this paper we use the asymptotic approach for boundary-value problems in thin domains. The idea is as follows. Let us consider a boundary-value problem in a neighborhood of an aneurysm. After the nondimensionalization, we get a parameter $\varepsilon$ characterizing thickness of the domain. In many cases, e.g. for brain aneurysms, $\varepsilon$ is a small parameter. Thus, it is natural to study the behaviour of this problem as $\varepsilon$ tends to zero. As we can see from Fig. 1, the thin aneurysm-type domain is transformed into a graph and the aneurysm is transformed into the origin if $\varepsilon \to 0$.

![Figure 1: Transformation of a thin aneurysm-type domain into a graph](image)

So, the aim is to find the corresponding limit problem and detect the impact of the aneurysm. Obviously, this limit problem in the graph will be simpler, since it is one-dimensional problem. Then we can either analytically solve the limit problem or apply numerical methods.

There are several approaches to construct asymptotic approximations for solutions to boundary-value problems in thin rod structures. The method of the partial asymptotic domain decomposition (MPADD), proposed in [34], was applied in the book [35, Chapter 4] to the following problem in a finite thin rod structure $B_\varepsilon$:

$$
\Delta u_\varepsilon = \begin{cases} 
  f^e(\bar{x}_1), & \text{in } S_0, \\
  0, & \text{in } \Pi_{x_0} := B_\varepsilon \setminus S_0,
\end{cases}
$$

$$
u_\varepsilon|_{\partial_1 B_\varepsilon} = 0, \quad \frac{\partial u_\varepsilon}{\partial n}|_{\partial_2 B_\varepsilon} = 0.
$$

Here $S_0$ is the union of sections of the rod structure $B_\varepsilon$, $\Pi_{x_0}$ is the connected component of $B_\varepsilon \setminus S_0$ containing the node $x_0$. The main idea of this method is to reduce the problem to a simplified form on $S_0$, where the regular asymptotics of the solution is located. The initial formulation is kept on a small neighbourhood of the domain $\Pi_{x_0}$, where the asymptotic behavior is singular. Then these two models are coupled by some special interface conditions that are derived from some...
projection procedure. For this, the author proposed a method of redistribution of constants to have the boundary-layer solutions exponentially decaying at infinity. As a result, for the leading term of the asymptotic expansion the following estimate was obtained:

$$\|u_{\varepsilon} - v\|_{L^2(B_{\varepsilon})} \frac{\sqrt{\text{meas}(B_{\varepsilon})}}{\varepsilon} = O(\sqrt{\varepsilon}) \quad \text{as} \quad \varepsilon \to 0.$$  \hspace{1cm} (1)

Then this method has been applied to constructions of asymptotic expansions both for the solution of the wave equation on a thin rod structure [36], for the solution of non-steady Navier-Stokes equations with uniform boundary conditions in a thin tube structure $B_{\varepsilon}$ [37, 38, 6], for the uniform Dirichlet boundary value problem for the biharmonic equation in a thin T-like shaped plane domain [15] and for other problems [7, 8]. Thus, the method (MPADD) is used in the case of the uniform boundary conditions on the lateral rectilinear surfaces of thin rods (cylinders) and if the right-hand sides depend only on the longitudinal variable in the direction of the corresponding rod and they are constant in some neighbourhoods of the nodes and vertices.

We see that the main difficulty in such problems is the identification of the behaviour of solutions in neighbourhoods of the nodes (aneurysms). In [32, 33] the authors made the following assumptions:

- the first terms of the volume force $f$ and surface load $g$ on the rods satisfy special orthogonality conditions (see (3.5) and (3.6) in [33]) and the second term of the volume force $f$ has an identified form and depends only on the longitudinal variable,

- similar orthogonality conditions for the right-hand sides on the knots are satisfied (see (3.41)), and the second term is a piecewise constant vector-function (see (3.42)),

to overcome this difficulty and to construct the leading terms of the elastic-field asymptotics for solutions of the equations of anisotropic elasticity on junctions of thin beams (2D and 3D cases). Due to these assumptions the displacement field at each knot can be approximated by a rigid displacement. As a result, the approximation does not contain boundary-layer terms, i.e. the asymptotic expansion is not complete a priori [33, Remark 3.1].

Using the method of two-scale expansions, the complete asymptotic expansion in powers $\varepsilon$ and $\mu$ for a solution of a linear partial deferential equation in the simplest $s$-dimensional rectangular periodic carcass was constructed in [1] (here $\varepsilon$ is the period of the carcass and $(\varepsilon \mu)^{s-1}$ is the area of the cross-section of beams).

1.1 Novelty and Methods

A new feature of the present paper in comparison with the papers mentioned above is construction and justification of the complete asymptotic expansion for the solution to a nonuniform Neumann boundary-value problem for the Poisson equation in a thin 3D aneurysm-type domain and the proof both energetic and pointwise uniform estimates for the difference between the solution of the starting problem ($\varepsilon > 0$) and the solution of the corresponding limit problem ($\varepsilon = 0$) without any orthogonality conditions for the right-hand side in the equation and for the right-hand sides in the Neumann boundary conditions. In addition, the right-hand sides can depend both on longitudinal and transversal variables and the thin cylinders can be curvilinear.

To construct the asymptotic expansion in the whole domain, we use the method of matching asymptotic expansions (see [18]) with special cut-off functions. The asymptotic expansion consists of three parts, namely, the regular part of the asymptotics located inside of each thin cylinder,
the boundary-layer part near the base of each thin cylinder, and the inner part of the asymptotics discovered in a neighborhood of the aneurysm.

The terms of the inner part of the asymptotics are special solutions of boundary-value problems in an unbounded domain with different outlets at infinity. It turns out they have polynomial growth at infinity. Matching these parts, we derive the limit problem \((\varepsilon = 0)\) in the corresponding graph and the recurrence procedure to determine all terms of the asymptotic expansion.

We proved energetic estimates that allow us to identify more precisely the impact of the aneurysm on some properties of the whole structure. One of the main results obtained in this paper is the energetic estimate (see Corollary 4.1)

\[
\frac{\|u_\varepsilon - U^{(0)}\|_{H^1(\Omega_\varepsilon)}}{\sqrt{\text{meas}(\Omega_\varepsilon)}} = \mathcal{O}(\varepsilon^{\alpha}) \quad \text{as} \quad \varepsilon \to 0
\]

(2)
in the Sobolev space \(H^1(\Omega_\varepsilon)\) instead of \(L^2\)-space in [35, Chapter 4] (see (1)). Here \(\alpha\) is a fixed number from the interval \((\frac{3}{2}, 1)\). In addition, pointwise uniform estimates are deduced in the case of rectilinear cylinders.

It should be stressed that the error estimates and convergence rate are very important both for justification of adequacy of one- or two-dimensional models that aim at description of actual three-dimensional thin bodies and for the study of boundary effects and effects of local (internal) inhomogeneities in applied problems. Pointwise estimates are of particular importance for engineering practice, since large values of tearing stresses in a small region at first cause local material damage and then lead to destruction of the whole construction.

Thus, our approach makes it possible to take into account various factors (e.g. variable thickness of thin curvilinear cylinders, inhomogeneous boundary conditions, geometric characteristics of aneurysms, etc.) in statements of boundary-value problems on graphs. In addition, the transition from two-dimensional to three-dimensional problems, the variable thickness of thin cylinders and their arbitrary number, special behaviour of the inner terms, and the matching procedure are the major differences and difficulties that were overcome in contrast to our previous paper [24].

Also this approach has been applied to nonlinear monotone boundary-value problems with nonlinear boundary conditions in thin aneurysm-type domains to construct the leading terms of the asymptotics and these results were announced at the conference [22]. It should be mentioned here that in [23] we studied a boundary-value problem in the union of thin rectangles without any local geometric irregularity. In this case the inner part of the asymptotics is absent and we have to equate simply the regular parts to obtain transmission conditions in the respective limit problem.

### 1.2 Structure of the paper

In section 2, we describe the domain \(\Omega_\varepsilon\) and the statement of the problem. The formal asymptotic expansion for the solution to the problem (3) is constructed in section 3. In section 4, we justify the asymptotics (Theorem 4.1) and prove asymptotic estimates for the leading terms of the asymptotics (Corollaries 4.1, 4.2 and 4.3). In the Conclusions, we analyze results obtained in this paper and discuss possible generalizations.
2 Statement of the problem

The model thin aneurysm-type domain $\Omega_\varepsilon$ consists of three thin curvilinear cylinders

$$\Omega^{(i)}_\varepsilon = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \varepsilon \ell < x_i < 1, \sum_{j=1}^{3} (1 - \delta_{ij}) x_j^2 < \varepsilon^2 h_i^2(x_i) \right\}, \ i = 1, 2, 3,$$

that are joined through a domain $\Omega^{(0)}_\varepsilon$ (referred in the sequel "aneurysm"). Here $\varepsilon$ is a small parameter; $\ell \in (0, \frac{1}{3})$; the positive functions $\{h_i\}_{i=1}^{3}$ belong to the space $C^1([0, 1])$ and they are equal to some constants in neighborhoods at the points $x = 0$ and $x_i = 1$, $i = 1, 2, 3$; the symbol $\delta_{ij}$ is the Kroneker delta, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$.

The aneurysm $\Omega^{(0)}_\varepsilon$ (see Fig. 2) is formed by the homothetic transformation with coefficient $\varepsilon$ from a bounded domain $\Xi^{(0)} \in \mathbb{R}^3$, i.e., $\Omega^{(0)}_\varepsilon = \varepsilon \Xi^{(0)}$. In addition, we assume that its boundary contains the disks

$$\Upsilon^{(i)}_\varepsilon(\ell) = \left\{ x \in \mathbb{R}^3 : x_i = \varepsilon \ell, \sum_{j=1}^{3} (1 - \delta_{ij}) x_j^2 < \varepsilon^2 h_i^2(\ell) \right\}, \ i = 1, 2, 3,$$

and denote $\Gamma^{(0)}_\varepsilon := \partial \Omega^{(0)}_\varepsilon \setminus \left\{ \Upsilon^{(1)}_\varepsilon(\ell) \cup \Upsilon^{(2)}_\varepsilon(\ell) \cup \Upsilon^{(3)}_\varepsilon(\ell) \right\}$.

Thus the model thin aneurysm-type domain $\Omega_\varepsilon$ (see Fig. 3) is the interior of the union $\bigcup_{k=0}^{3} \Omega^{(k)}_\varepsilon$ and we assume that it has the Lipschitz boundary.

Remark 2.1. We can consider more general thin aneurysm-type domains with arbitrary orientation of thin cylinders (their number can be also arbitrary). But to avoid technical and huge calculations and to demonstrate the main steps of the proposed asymptotic approach we consider a such kind of the thin aneurysm-type domain, when the cylinders are placed on the coordinate axes.
In $\Omega_\varepsilon$, we consider the following mixed boundary-value problem:

$$
\begin{cases}
-\Delta u_\varepsilon(x) = f(x), & x \in \Omega_\varepsilon, \\
\partial_\nu u_\varepsilon(x) = 0, & x \in \Gamma^{(0)}_\varepsilon, \\
-\partial_\nu u_\varepsilon(x) = \varphi_\varepsilon(x), & x \in \Gamma^{(i)}_\varepsilon, \quad i = 1, 2, 3, \\
u_\varepsilon(x) = 0, & x \in \Upsilon^{(i)}_\varepsilon(1), \quad i = 1, 2, 3,
\end{cases}
$$

(3)

where $\Gamma^{(i)}_\varepsilon = \partial \Omega^{(i)}_\varepsilon \cap \{x \in \mathbb{R}^3 : \varepsilon \ell < x_i < 1\}$, $\partial_\nu$ is the outward normal derivative, the given functions $f$ and $\varphi_\varepsilon$ are smooth and

$$
\varphi_\varepsilon(x) = \varepsilon \varphi^{(i)}(x_i, \frac{x_i}{\varepsilon}), \quad x \in \Gamma^{(i)}_\varepsilon, \quad i = 1, 2, 3,
$$

where

$$
\bar{x}_i = \begin{cases}
(x_2, x_3), & i = 1, \\
(x_1, x_3), & i = 2, \\
(x_1, x_2), & i = 3.
\end{cases}
$$

It follows from the theory of linear boundary-value problems that, for any fixed value of $\varepsilon$, the problem (3) possesses a unique weak solution $u_\varepsilon$ from the Sobolev space $H^1(\Omega_\varepsilon)$ such that its traces on the ends $\Upsilon^{(i)}_\varepsilon(1), i = 1, 2, 3$, of the domain $\Omega_\varepsilon$ are equal to zero and the solution satisfies the integral identity

$$
\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla \psi \, dx = \int_{\Omega_\varepsilon} f \psi \, dx - \sum_{i=1}^{3} \int_{\Gamma^{(i)}_\varepsilon} \varphi_\varepsilon \psi \, d\sigma_x
$$

(4)

for any function $\psi \in H^1(\Omega_\varepsilon)$ such that $\psi|_{x_i=1} = 0, \ i = 1, 2, 3$.

The aim of the present paper is to

- develop a procedure to construct the complete asymptotic expansion for the solution to the problem (3) as the small parameter $\varepsilon \to 0$;
• justify this procedure and prove the corresponding asymptotic estimates;
• derive the corresponding limit problem \((\varepsilon = 0)\);
• prove energetic and uniform pointwise estimates for the difference between the solution of the problem (3) and the solution of the limit problem, from which the influence of the aneurysm will be observed.

### 3 Formal asymptotic expansions

We propose the following asymptotic ansatzes for the solution to the problem (3):

1) the regular part of the asymptotics

\[
\begin{align*}
    u_{\infty}^{(i)} &:= \omega^{(i)}_0(x_i) + \varepsilon \omega^{(i)}_1(x_i) + \sum_{k=2}^{+\infty} \varepsilon^k \left( u_k^{(i)}(x_i, \frac{\xi}{\varepsilon}) + \omega_k^{(i)}(x_i) \right)
\end{align*}
\]  

(5)

is located inside of each thin cylinder \(\Omega^{(i)}_{\varepsilon}\) (see Fig. 4) and their terms depend both on the corresponding longitudinal variable \(x_i\) and so-called ”quick variables” \(\frac{\xi}{\varepsilon}\) \((i = 1, 2, 3)\);

2) the boundary-layer part of the asymptotics

\[
\begin{align*}
    \Pi^{(i)}_{\infty} &:= \sum_{k=0}^{+\infty} \varepsilon^k \Pi_k^{(i)} \left( \frac{1 - x_i}{\varepsilon}, \frac{\xi}{\varepsilon} \right) \quad (i = 1, 2, 3)
\end{align*}
\]  

(6)

is located in a neighborhood of the base \(\Gamma^{(i)}_{\varepsilon}(1)\) of each thin cylinder \(\Omega^{(i)}_{\varepsilon}\);

3) the inner part of the asymptotics

\[
\begin{align*}
    N_{\infty} &:= \sum_{k=0}^{+\infty} \varepsilon^k N_k \left( \frac{x}{\varepsilon} \right)
\end{align*}
\]  

(7)

is located in a neighborhood of the aneurysm \(\Omega^{(0)}_{\varepsilon}\).

#### 3.1 Regular part of the asymptotics

Formally substituting the series (5) into the differential equation of the problem (3) and expanding function \(f\) in the Taylor series at the point \(\pi_i = (0, 0)\), we obtain

\[
\begin{align*}
    &- \sum_{k=2}^{+\infty} \varepsilon^k \frac{d^2 u_k^{(i)}}{dx_i^2}(x_i, \xi_i) - \sum_{k=0}^{+\infty} \varepsilon^k \Delta_{\xi_i} u_k^{(i)}(x_i, \xi_i) - \sum_{k=0}^{+\infty} \varepsilon^k \frac{d^2 \omega_k^{(i)}}{dx_i^2}(x_i) - \sum_{k=0}^{+\infty} \varepsilon^k f_k^{(i)}(x_i, \xi_i),
\end{align*}
\]

where \(\xi_i = \frac{x_i}{\varepsilon}, \ \xi_i = \frac{\xi}{\varepsilon}\)

and

\[
\begin{align*}
    f_k^{(i)}(x_i, \xi_i) &:= \frac{1}{k!} \left( \sum_{j=1}^{3} (1 - \delta_{ij}) \xi_j \frac{\partial}{\partial x_j} \right)^k f(x)|_{\pi_i=(0,0)}, \ k \in \mathbb{N}_0;
\end{align*}
\]  

(8)
e.g. at $i = 1$ the last symbol means as follows $f_k^{(1)}(x_1, \xi_2, \xi_3) = f(x_1, 0, 0)$ and $$f_k^{(1)}(x_1, \xi_2, \xi_3) = \frac{1}{k!} \left( \sum_{n=0}^{k} \binom{k}{n} \xi_2^n \frac{\partial^n f}{\partial x_2^n}(x_1, 0, 0) \xi_3^k \frac{\partial^{k-n} f}{\partial x_3^{k-n}}(x_1, 0, 0) \right), \quad k \in \mathbb{N}.$$ Then, taking into account the view of the outer normal to $\Gamma^{(i)}_\varepsilon$ $$\nu^{(i)}(x_i, \xi_i) = \frac{1}{\sqrt{1 + \varepsilon^2 |h_i'(x_i)|^2}} (-\varepsilon h_i'(x_i), \nu_i(\xi_i)) = \begin{cases} \left( -\varepsilon h_1'(x_1), \nu_2^{(1)}(\xi_1), \nu_3^{(1)}(\xi_1) \right), & i = 1, \\ \left( \nu_1^{(2)}(\xi_2), -\varepsilon h_2'(x_2), \nu_3^{(2)}(\xi_2) \right), & i = 2, \\ \left( \nu_1^{(3)}(\xi_3), \nu_2^{(3)}(\xi_3), -\varepsilon h_3'(x_3) \right), & i = 3, \end{cases}$$ where $\nu_i(\xi_i)$ is the outward normal for the disk $\Upsilon^{(i)}_\varepsilon(x_i)$ for each value of $x_i$, $i = 1, 2, 3$, we put the series (5) into the third relation of the problem (3) and derive $$h_i'(x_i) \sum_{k=3}^{+\infty} \varepsilon^k \frac{d u^{(i)}_{k-1}}{d x_i}(x_i, \xi_i) - \sum_{k=1}^{+\infty} \varepsilon^k \partial_{a(x_i)} u^{(i)}_{k+1}(x_i, \xi_i) + h_i'(x_i) \sum_{k=1}^{+\infty} \varepsilon^k \frac{d w^{(i)}_{k-1}}{d x_i}(x_i)$$ $$\approx \varepsilon \sqrt{1 + \varepsilon^2 |h_i'(x_i)|^2} \cdot \varphi^{(i)}(x_i, \xi_i) = \sum_{k=0}^{+\infty} \varepsilon^{2k+1} \frac{(-1)^k (2k)!}{(1-2k)(k!)^2 4^k} |h_i'(x_i)|^{2k} \varphi^{(i)}(x_i, \xi_i).$$
Equating the coefficients of the same powers of \( \varepsilon \), we deduce recurrent relations of the boundary-value problems for the determination of the expansion coefficients in (5). Let us consider the problem for \( u_2^{(i)} \):

\[
\begin{aligned}
&\left\{ \begin{array}{l}
-\Delta \xi u_2^{(i)}(x_i, \xi_i) = \frac{d^2 \omega_0^{(i)}}{dx_i^2}(x_i) + f_0^{(i)}(x_i, \xi_i), \quad \xi_i \in \mathcal{Y}_i(x_i), \\
-\partial_{\xi_i} u_2^{(i)}(x_i, \xi_i) = -h_i'(x_i) \frac{d \omega_0^{(i)}}{dx_i}(x_i) + \varphi^{(i)}(x_i, \xi_i), \quad \xi_i \in \partial \mathcal{Y}_i(x_i), \\
\langle u_2^{(i)}(x_i, \cdot) \rangle_{\mathcal{Y}_i(x_i)} = 0.
\end{array} \right.
\tag{9}
\end{aligned}
\]

Here the variable \( x_i \) is regarded as a parameter from the interval \( I^{(i)}_x = \{ x : x_i \in (\varepsilon, 1), \overline{x}_i = (0, 0) \} \), \( \mathcal{Y}_i(x_i) = \{ \overline{x}_i \in \mathbb{R}^2 : |\overline{x}_i| < h_i(x_i) \} \), \( \langle u(x_i, \cdot) \rangle_{\mathcal{Y}_i(x_i)} := \int_{\mathcal{Y}_i(x_i)} u(x_i, \overline{x}_i) d\overline{x}_i \), \( i = 1, 2, 3 \).

For each value of \( i \in \{1, 2, 3\} \), the problem (9) is the inhomogeneous Neumann problem for the Poisson equation in the disk \( \mathcal{Y}_i(x_i) \) with respect to the variable \( \overline{x}_i \in \mathcal{Y}_i(x_i) \). Writing down the necessary and sufficient conditions for the solvability of problem (9), we get the following differential equation for the function \( \omega_0^{(i)} \):

\[
-\pi \frac{d}{dx_i} \left( h_i^2(x_i) \frac{d \omega_0^{(i)}}{dx_i}(x_i) \right) = \int_{\mathcal{Y}_i(x_i)} f_0^{(i)}(x_i, \xi_i) d\xi_i - \int_{\partial \mathcal{Y}_i(x_i)} \varphi^{(i)}(x_i, \xi_i) d\xi_i, \quad x_i \in I^{(i)}_x.
\tag{10}
\]

Let \( \omega_0^{(i)} \) be a solution of the differential equation (10) (boundary conditions for this differential equation will be determined later). Then the solution of problem (9) exist and the third relation in (9) supplies the uniqueness of solution.

For determination of the coefficients \( u_3^{(i)}, i = 1, 2, 3 \), we obtain the following problems:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
-\Delta \xi u_3^{(i)}(x_i, \xi_i) = \frac{d^2 \omega_1^{(i)}}{dx_i^2}(x_i) + f_1^{(i)}(x_i, \xi_i), \quad \xi_i \in \mathcal{Y}_i(x_i), \\
-\partial_{\xi_i} u_3^{(i)}(x_i, \xi_i) = -h_i'(x_i) \frac{d \omega_1^{(i)}}{dx_i}(x_i), \quad \xi_i \in \partial \mathcal{Y}_i(x_i), \\
\langle u_3^{(i)}(x_i, \cdot) \rangle_{\mathcal{Y}_i(x_i)} = 0.
\end{array} \right.
\tag{11}
\end{aligned}
\]

Repeating the previous reasoning, we find

\[
-\pi \frac{d}{dx_i} \left( h_i^2(x_i) \frac{d \omega_1^{(i)}}{dx_i}(x_i) \right) = \int_{\mathcal{Y}_i(x_i)} f_1^{(i)}(x_i, \xi_i) d\xi_i, \quad x_i \in I^{(i)}_x, \quad i = 1, 2, 3.
\tag{12}
\]

Let us consider boundary-value problems for the functions \( u_k^{(i)}, k \geq 4, i = 1, 2, 3 \):

\[
\begin{aligned}
&\left\{ \begin{array}{l}
-\Delta \xi u_k^{(i)}(x_i, \xi_i) = \frac{d^2 \omega_k^{(i-2)}}{dx_i^2}(x_i) + \partial^2_{\xi_i} u_k^{(i-2)}(x_i, \xi_i) + f_k^{(i)}(x_i, \xi_i), \quad \xi_i \in \mathcal{Y}_i(x_i), \\
-\partial_{\xi_i} u_k^{(i)}(x_i, \xi_i) = -h_i'(x_i) \left( \frac{d \omega_k^{(i-2)}}{dx_i}(x_i) + \partial_{\xi_i} u_k^{(i-2)}(x_i, \xi_i) \right) \left. \right|_{
\xi_i = \partial \mathcal{Y}_i(x_i)}, \\
+ \eta_k^{(i-2)}(x_i, \xi_i) \varphi^{(i)}(x_i, \xi_i), \quad \xi_i \in \partial \mathcal{Y}_i(x_i), \\
\langle u_k^{(i)}(x_i, \cdot) \rangle_{\mathcal{Y}_i(x_i)} = 0.
\end{array} \right.
\tag{13}
\end{aligned}
\]
Assume that all coefficients \( u_2^{(i)}, \ldots, u_{k-1}^{(i)}, \omega_0^{(i)}, \ldots, \omega_{k-3}^{(i)} \) of the expansion (5) are determined. Then we can find \( u_k^{(i)} \) and \( \omega_{k-2}^{(i)} \) from problem (13). Indeed, it follows from the solvability condition of problem (13) that \( \omega_{k-2}^{(i)} \) must be a solution to the following ordinary differential equation:

\[
-\pi \frac{d}{dx_i} \left( h_i^{(2)}(x_i) \frac{d\omega_{k-2}^{(i)}}{dx_i}(x_i) \right) = \int_{\gamma_i(x_i)} f_{k-2}^{(i)}(x_i, \xi_i) d\xi_i - \eta_{k-2}^{(i)}(x_i) \int_{\partial\gamma_i(x_i)} \varphi^{(i)}(x_i, \xi_i) d\xi_i
\]

\[
+ h_i'(x_i) \int_{\partial\gamma_i(x_i)} \frac{\partial u_{k-2}^{(i)}}{\partial x_i}(x_i, \xi_i) d\xi_i, \quad x_i \in I_{\varepsilon}^{(i)}, \quad i = 1, 2, 3, \quad k \geq 4,
\]

where

\[
\eta_{k}^{(i)}(x_i) = \begin{cases} 
0, & \text{if } k \text{ is odd}, \\
\frac{(-1)^{k} k! |h_i^{(i)}(x_i)|^k}{(1-k)!((\frac{k}{2})!)^2 4^\frac{k}{2}}, & \text{if } k \text{ is even}, \quad x_i \in I_{\varepsilon}^{(i)}, \quad k \in \mathbb{N}.
\end{cases}
\]

**Remark 3.1.** Boundary conditions for the differential equations (10), (12) and (14) are unknown in advance. They will be determined in the process of construction of the asymptotics.

Thus, the solution of problem (13) is uniquely determined. Hence, the recursive procedure for the determination of the coefficients of series (5) is uniquely solvable.

### 3.2 Boundary-layer part of the asymptotics

In the previous subsection, we have considered the regular asymptotics taking into account the inhomogeneity of the right-hand side of the differential equation in (3) and the boundary conditions on the lateral surfaces the thin cylinders \( \Omega_{k}^{(i)}, \ i = 1, 2, 3 \). In what follows, we construct the boundary-layer part of the asymptotics compensating the residuals of the regular one at the base of \( \Omega_{k}^{(i)} \).

Substituting the series (6) into (3) and collecting coefficients with the same powers of \( \varepsilon \), we get the following mixed boundary-value problems:

\[
\begin{cases}
-\Delta_{\xi_i, \bar{\xi}_i} \Pi_{k}^{(i)}(\xi_i, \bar{\xi}_i) = 0, & \xi_i \in (0, +\infty), \quad \bar{\xi}_i \in \gamma_i(1), \\
-\partial_{\xi_i} \Pi_{k}^{(i)}(\xi_i, \bar{\xi}_i) = 0, & \xi_i \in (0, +\infty), \quad \bar{\xi}_i \in \partial\gamma_i(1), \\
\Pi_{k}^{(i)}(0, \bar{\xi}_i) = \Phi_{k}^{(i)}(\bar{\xi}_i), & \bar{\xi}_i \in \gamma_i(1), \\
\Pi_{k}^{(i)}(\xi_i, \bar{\xi}_i) \rightarrow 0, & \xi_i \rightarrow +\infty, \quad \bar{\xi}_i \in \gamma_i(1),
\end{cases}
\]

where \( \xi_i = \frac{1-x_i}{\varepsilon}, \quad \bar{\xi}_i = \frac{\pi_i}{\varepsilon} \),

\[
\Phi_{k}^{(i)} = -\omega_{k}^{(i)}(1), \quad k = 0, 1; \quad \Phi_{k}^{(i)}(\bar{\xi}_i) = -u_{k-1}^{(i)}(1, \bar{\xi}_i) - \omega_{k-2}^{(i)}(1), \quad k \geq 2, \quad k \in \mathbb{N}.
\]

Using the method of separation of variables, we determine the solution

\[
\Pi_{k}^{(i)}(\xi_i, \bar{\xi}_i) = a_{k,0}^{(i)} + \sum_{p=1}^{+\infty} a_{k,p}^{(i)} \Theta_p^{(i)}(\xi_i) \exp(-\lambda_p^{(i)} \xi_i)
\]
of problem (16) at a fixed index $k$, where

$$a_{k,0}^{(i)} = \frac{1}{|\mathcal{Y}_i(1)|} \int_{\mathcal{Y}_i(1)} \Phi_k^{(i)}(\xi) d\xi_i$$

$$a_{k,p}^{(i)} = \frac{1}{\|\Theta_p^{(i)}\|_{L^2(\mathcal{Y}_i(1))}^2} \int_{\mathcal{Y}_i(1)} \Phi_k^{(i)}(\xi) \Theta_p^{(i)}(\xi) d\xi_i,$$

Here $\Theta_0^{(i)} \equiv 1$, $\Theta_p^{(i)}(\xi)$ and $\{0 = \lambda_0^{(i)} < \lambda_1^{(i)} \leq \lambda_2^{(i)} \leq \cdots \leq \lambda_p^{(i)} \leq \cdots\}$ are the eigenfunctions and eigenvalues of the spectral problem

$$\begin{cases}
-\Delta_\xi \Theta^{(i)} = (\lambda^{(i)})^2 \Theta^{(i)} & \text{in } \mathcal{Y}_i(1), \\
\partial_{\nu_\xi} \Theta^{(i)} = 0 & \text{on } \partial\mathcal{Y}_i(1).
\end{cases} \tag{18}$$

It follows from the fourth condition in (16) that coefficient $a_{k,0}^{(i)}$ must be equal to 0. As a result, we arrive at the following boundary conditions for the functions $\{\omega_k^{(i)}\}$:

$$\omega_k^{(i)}(1) = 0, \quad k \in \mathbb{N}_0, \quad i = 1, 2, 3. \tag{19}$$

**Remark 3.2.** Since $\Phi_0^{(i)} \equiv \Phi_1^{(i)} \equiv 0$, we conclude that $\Pi_0^{(i)} \equiv \Pi_1^{(i)} \equiv 0$, $i = 1, 2, 3$. Moreover, from representation (17) it follows the following asymptotic relations

$$\Pi_k^{(i)}(\xi^*_i, \bar{\xi}_i) = \mathcal{O}(\exp(-\lambda_1^{(i)}\xi^*_i)) \quad \text{as} \quad \xi^*_i \to +\infty, \quad i = 1, 2, 3. \tag{20}$$

### 3.3 Inner part of the asymptotics

To obtain conditions for the functions $\{\omega_k^{(i)}\}$, $i = 1, 2, 3$ at the point 0, we introduce the inner part (7) of the asymptotics in a neighborhood of the aneurysm $\Omega^{(0)}$. For this we pass to the variables $\xi = \bar{\xi}$. Then forwarding the parameter $\varepsilon$ to 0, we see that the domain $\Omega_\varepsilon$ is transformed into the unbounded domain $\Xi$ that is the union of the domain $\Xi^{(0)}$ and three semibounded cylinders

$$\Xi^{(i)} = \{\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \ell < \xi_i < +\infty, \quad |\bar{\xi}_i| < h_i(0), \quad i = 1, 2, 3,$$

i.e., $\Xi$ is the interior of $\bigcup_{i=0}^3 \Xi^{(i)}$ (see Fig. 5).

Let us introduce the following notation for parts of the boundary of $\Xi$:

- $\Gamma_i = \{\xi \in \mathbb{R}^3 : \ell < \xi_i < +\infty, \quad |\bar{\xi}_i| = h_i(0), \quad i = 1, 2, 3,$

- $\Gamma_0 = \partial\Xi \setminus \left(\bigcup_{i=1}^3 \Gamma_i\right)$.

Substituting the series (7) into the problem (3) and equating coefficients at the same powers of $\varepsilon$, we derive the following relations for $\{N_k\}$:

$$\begin{cases}
-\Delta_\xi N_k(\xi) = F_k(\xi), & \xi \in \Xi, \\
\partial_{\nu_\xi} N_k(\xi) = 0, & \xi \in \Gamma_0, \\
-\partial_{\nu_{\bar{\xi}_i}} N_k(\xi) = B_k^{(i)}(\xi), & \xi \in \Gamma_i, \quad i = 1, 2, 3, \\
N_k(\xi) \sim \omega_k^{(i)}(0) + \Psi_k^{(i)}(\xi), & \xi_i \to +\infty, \quad \bar{\xi}_i \in \mathcal{Y}_i(0), \quad i = 1, 2, 3.
\end{cases} \tag{21}$$
Here
\[ F_0 \equiv F_1 \equiv 0, \quad F_k(\xi) = \frac{(\xi, \nabla_x)^{k-2} f(0)}{(k-2)!} := \frac{1}{(k-2)!} \left( \sum_{i=1}^{3} \xi_i \frac{\partial f(x)}{\partial x_i} \right)^{k-2} \bigg|_{x=0}, \quad \xi \in \Xi, \]
\[ B_0^{(i)} \equiv B_1^{(i)} \equiv 0, \quad B_k^{(i)}(\xi) = \frac{\xi_i^{k-2}}{(k-2)!} \frac{\partial^{k-2} \varphi^{(i)}}{\partial x_i^{k-2}}(0, \xi_i), \quad \xi \in \Gamma_i, \quad i = 1, 2, 3. \]

The right hand sides in the differential equation and boundary conditions on \{\Gamma_i\} of the problem (21) are obtained with the help of the Taylor decomposition of the functions \(f\) and \(\varphi^{(i)}\) at the points \(x = 0\) and \(x_i = 0\), \(i = 1, 2, 3\), respectively.

The fourth condition in (21) appears by matching the regular and inner asymptotics in a neighborhood of the aneurysm, namely the asymptotics of the terms \(\{N_k\}\) as \(\xi_i \to +\infty\) have to coincide with the corresponding asymptotics of terms of the regular expansions (5) as \(x_i = \varepsilon \xi_i \to +0, \quad i = 1, 2, 3\), respectively. Expanding each term of the regular asymptotics in the Taylor series at the points \(x_i = 0, \quad i = 1, 2, 3\), and collecting the coefficients of the same powers of \(\varepsilon\), we get
\[
\Psi_0^{(i)} \equiv 0, \quad \Psi_1^{(i)}(\xi) = \xi_i \frac{d \omega_0^{(i)}}{d x} (0), \quad i = 1, 2, 3, 
\]
\[
\Psi_k^{(i)}(\xi) = \sum_{j=1}^{k} \frac{\xi_i^j}{j!} \frac{d \omega_k^{(i)}}{d x_i} (0) + \sum_{j=0}^{k-2} \frac{\xi_i^j}{j!} \frac{\partial^{j} u_k^{(i)}}{\partial x_i^j} (0, \xi_i), \quad i = 1, 2, 3, \quad k \geq 2. \tag{22}
\]

A solution of the problem (21) at \(k \in \mathbb{N}\) is sought in the form
\[
N_k(\xi) = \sum_{i=1}^{3} \Psi_k^{(i)}(\xi) \chi_i(\xi_i) + \tilde{N}_k(\xi), \tag{23}
\]
where \(\chi_i \in C^\infty(\mathbb{R}_+), \quad 0 \leq \chi_i \leq 1\) and
\[
\chi_i(\xi_i) = \begin{cases} 
0, & \text{if } \xi_i \leq 1 + \ell, \\
1, & \text{if } \xi_i \geq 2 + \ell, \end{cases} \quad i = 1, 2, 3.
\]
Then $\tilde{N}_k$ has to be a solution of the problem

\[
\begin{cases}
-\Delta \xi \tilde{N}_k(\xi) &= \tilde{F}_k(\xi), \quad \xi \in \Xi, \\
\partial_{\nu_i} \tilde{N}_k(\xi) &= 0, \quad \xi \in \Gamma_0, \\
-\partial_{\nu_i} \tilde{N}_k(\xi) &= \tilde{B}^{(i)}_k(\xi), \quad \xi \in \Gamma_i, \quad i = 1, 2, 3,
\end{cases}
\]  

(24)
and has to satisfy the following conditions:

\[
\tilde{N}_k(\xi) \to \omega_k^{(i)}(0) \quad \text{as} \quad \xi_i \to +\infty, \quad \xi_i \in \mathcal{Y}_i(0), \quad i = 1, 2, 3,
\]

(25)

where

\[
\tilde{F}_1(\xi) = \sum_{i=1}^{3} \left( \xi_i \frac{d\omega_0^{(i)}}{dx_i}(0) \chi_i''(\xi_i) + 2 \frac{d\omega_0^{(i)}}{dx_i}(0) \chi_i'(\xi_i) \right),
\]

\[
\tilde{F}_k(\xi) = \sum_{i=1}^{3} \left[ \left( \sum_{j=1}^{k} \frac{\xi_i^{j}}{j!} \frac{d^j \omega_k^{(i-j)}}{dx_i^j}(0) + \sum_{j=0}^{k-2} \frac{\xi_i^{j}}{j!} \frac{\partial^j u_k^{(i-j)}}{\partial x_i^j}(0, \xi_i) \right) \chi_i''(\xi_i) \\
+ \sum_{j=1}^{k} \frac{\xi_i^{j}}{(j-1)!} \frac{d^j \omega_k^{(i-j)}}{dx_i^j}(0) + \sum_{j=0}^{k-2} \frac{\xi_i^{j}}{(j-1)!} \frac{\partial^j u_k^{(i-j)}}{\partial x_i^j}(0, \xi_i) \right) \chi_i'(\xi_i) \\
+ \left( 1 - \sum_{i=1}^{3} \chi_i(\xi_i) \right) \frac{1}{(k-2)!} (\xi, \nabla_x)^{k-2} f(0)
\]

and

\[
\tilde{B}_1^{(i)} \equiv 0, \quad \tilde{B}_k^{(i)}(\xi) = \frac{\xi_i^{k-2}}{(k-2)!} \frac{\partial^{k-2} \varphi^{(i)}}{\partial x_i^{k-2}}(0, \xi_i) (1 - \chi_i(\xi_i)), \quad i = 1, 2, 3, \quad k \geq 2.
\]

The existence of a solution to the problem (24) in the corresponding energetic space can be obtained from general results about the asymptotic behavior of solutions to elliptic problems in domains with different exits to infinity [19, 20, 29, 30]. We will use approach proposed in [30, 26].

Let $C^{\infty}_{0, \xi}(\Xi)$ be a space of functions infinitely differentiable in $\Xi$ and finite with respect to $\xi$, i.e.,

\[
\forall v \in C^{\infty}_{0, \xi}(\Xi) \quad \exists R > 0 \quad \forall \xi \in \Xi \quad \xi_i \geq R, \quad i = 1, 2, 3 : \quad v(\xi) = 0.
\]

We now define a space $\mathcal{H} := \left( C^{\infty}_{0, \xi}(\Xi), \|\cdot\|_{\mathcal{H}} \right)$, where

\[
\|v\|_{\mathcal{H}} = \sqrt{\int_{\Xi} |\nabla v(\xi)|^2 d\xi + \int_{\Xi} |v(\xi)|^2 |\rho(\xi)|^2 d\xi},
\]

and the weight function $\rho \in C^{\infty}(\mathbb{R}^3), \ 0 \leq \rho \leq 1$ and

\[
\rho(\xi) = \begin{cases} 
1, & \text{if} \quad \xi \in \Xi^{(0)}, \\
|\xi|^{-1}, & \text{if} \quad \xi_i \geq \ell + 1, \ \xi \in \Xi^{(i)}, \quad i = 1, 2, 3.
\end{cases}
\]
Definition 3.1. A function $\tilde{N}_k$ from the space $H$ is called a weak solution of the problem (24) if the identity

$$\int_{\Xi} \nabla \tilde{N}_k \cdot \nabla v \, d\xi = \int_{\Xi} \tilde{F}_k v \, d\xi - \sum_{i=1}^{3} \int_{\Gamma_i} \tilde{B}_k^{(i)} v \, d\sigma,$$

holds for all $v \in H$.

Proposition 3.1. Let $\rho^{-1}\tilde{F}_k \in L^2(\Xi)$, $\rho^{-1}\tilde{B}_k^{(i)} \in L^2(\Gamma_i)$, $i = 1, 2, 3$.

Then there exist a weak solution of problem (24) if and only if

$$\int_{\Xi} \tilde{F}_k \, d\xi = \sum_{i=1}^{3} \int_{\Gamma_i} \tilde{B}_k^{(i)} \, d\sigma.$$

This solution is defined up to an additive constant. The additive constant can be chosen to guarantee the existence and uniqueness of a weak solution of problem (24) with the following differentiable asymptotics:

$$\tilde{N}_k(\xi) = \begin{cases} 
\mathcal{O}(\exp(-\gamma_1 \xi_1)) & \text{as } \xi_1 \to +\infty, \\
\delta_k^{(2)} + \mathcal{O}(\exp(-\gamma_2 \xi_2)) & \text{as } \xi_2 \to +\infty, \\
\delta_k^{(3)} + \mathcal{O}(\exp(-\gamma_3 \xi_3)) & \text{as } \xi_3 \to +\infty,
\end{cases}$$

(28)

where $\gamma_i$, $i = 1, 2, 3$ are positive constants.

The constants $\delta_k^{(2)}$ and $\delta_k^{(3)}$ in (28) are defined as follows:

$$\delta_k^{(i)} = \int_{\Xi} \mathfrak{M}_i \tilde{F}_k(\xi) \, d\xi + \sum_{j=1}^{3} \int_{\Gamma_j} \mathfrak{M}_i \tilde{B}_k^{(j)}(\xi) \, d\sigma, \quad i = 2, 3, \quad k \in \mathbb{N}_0,$$

(29)

where $\mathfrak{M}_2$ and $\mathfrak{M}_3$ are special solutions to the corresponding homogeneous problem

$$-\Delta \mathfrak{M} = 0 \quad \text{in } \Xi, \quad \partial_{\nu} \mathfrak{M} = 0 \quad \text{on } \partial\Xi,$$

(30)

for the problem (24).

Proposition 3.2. The problem (30) has two linearly independent solutions $\mathfrak{M}_2$ and $\mathfrak{M}_3$ that do not belong to the space $H$ and they have the following differentiable asymptotics:

$$\mathfrak{M}_2(\xi) = \begin{cases} 
-\frac{\xi_1}{\pi h_1^2(0)} + \mathcal{O}(\exp(-\gamma_1 \xi_1)) & \text{as } \xi_1 \to +\infty, \\
C_2^{(2)} + \frac{\xi_2}{\pi h_2^2(0)} + \mathcal{O}(\exp(-\gamma_2 \xi_2)) & \text{as } \xi_2 \to +\infty, \\
C_2^{(3)} + \mathcal{O}(\exp(-\gamma_3 \xi_3)) & \text{as } \xi_3 \to +\infty,
\end{cases}$$

(31)

$$\mathfrak{M}_3(\xi) = \begin{cases} 
-\frac{\xi_1}{\pi h_1^2(0)} + \mathcal{O}(\exp(-\gamma_1 \xi_1)) & \text{as } \xi_1 \to +\infty, \\
C_3^{(2)} + \mathcal{O}(\exp(-\gamma_2 \xi_2)) & \text{as } \xi_2 \to +\infty, \\
C_3^{(3)} + \frac{\xi_3}{\pi h_3^2(0)} + \mathcal{O}(\exp(-\gamma_3 \xi_3)) & \text{as } \xi_3 \to +\infty,
\end{cases}$$

(32)

Any other solution to the homogeneous problem, which has polynomial growth at infinity, can be presented as a linear combination $\alpha_1 + \alpha_2 \mathfrak{M}_2 + \alpha_3 \mathfrak{M}_3$. 

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Proof. The solution $\mathfrak{M}_2$ is sought in the form of a sum

$$\mathfrak{M}_2(\xi) = -\frac{\xi_1}{\pi h_1^2(0)} \chi_1(\xi_1) + \frac{\xi_2}{\pi h_2^2(0)} \chi_2(\xi_2) + \tilde{\mathfrak{M}}_2(\xi),$$

where $\tilde{\mathfrak{M}}_2 \in \mathcal{H}$ and $\tilde{\mathfrak{M}}_2$ is the solution to the problem (24) with right-hand sides

$$\tilde{F}_2^i(\xi) = \begin{cases} \frac{1}{\pi h_1^2(0)} \left( (\xi_1 \chi_1'(\xi_1))' + \chi_1'(\xi_1) \right), & \xi \in \Xi^{(1)}, \\ \frac{1}{\pi h_2^2(0)} \left( (\xi_2 \chi_2'(\xi_2))' + \chi_2'(\xi_2) \right), & \xi \in \Xi^{(2)}, \\ 0, & \xi \in \Xi^{(0)} \cup \Xi^{(3)}, \end{cases}$$

and $\tilde{B}_2^i = 0$. It is easy to verify that the solvability condition (27) is satisfied. Thus, by virtue of Proposition 2.1 there exist a unique solution $\tilde{\mathfrak{M}}_2 \in \mathcal{H}$ that has the asymptotics

$$\tilde{\mathfrak{M}}_2(\xi) = (1 - \delta_{1j})C_2^{(j)} + O(\exp(-\gamma_j \xi_j)) \quad \text{as} \quad \xi_j \to +\infty, \quad j = 1, 2, 3.$$

Similar we can prove the existence of the solution $\mathfrak{M}_3$ with the asymptotics (32).

Obviously, that $\mathfrak{M}_2$ and $\mathfrak{M}_3$ are linearly independent and any other solution to the homogeneous problem, which has polynomial growth at infinity, can be presented as $\alpha_1 + \alpha_2 \mathfrak{M}_2 + \alpha_3 \mathfrak{M}_3$. \hfill $\Box$

Remark 3.3. To obtain formulas (29) for the constants $\delta_k^{(2)}$ and $\delta_k^{(3)}$, it is necessary to substitute the functions $\tilde{\mathcal{N}}_k, \mathfrak{F}_2$ and $\tilde{\mathcal{N}}_k, \mathfrak{F}_3$ in the second Green-Ostrogradsky formula

$$\int_{\Xi_R} (\tilde{\mathcal{N}} \Delta_\xi \mathfrak{M} - \mathfrak{M} \Delta_\xi \tilde{\mathcal{N}}) \, d\xi = \int_{\partial \Xi_R} (\tilde{\mathcal{N}} \partial_\nu \mathfrak{M} - \mathfrak{M} \partial_\nu \tilde{\mathcal{N}}) \, d\sigma_\xi$$

respectively, and then pass to the limit as $R \to +\infty$. Here $\Xi_R = \Xi \cap \{ \xi : |\xi_i| < R, \ i = 1, 2, 3 \}$.

3.3.1 Limit problem and problems for $\{\omega_k\}$

The problem (21) at $k = 0$ is as follows:

$$\begin{cases} -\Delta_\xi N_0(\xi) = 0, & \xi \in \Xi, \\ \partial_\nu_\xi N_0(\xi) = 0, & \xi \in \Gamma_0, \\ -\partial_\nu_{\xi_i} N_0(\xi) = 0, & \xi \in \Gamma_i, \ i = 1, 2, 3, \\ N_0(\xi) \sim \omega_0^{(i)}(0), & \xi_i \to +\infty, \ \xi_i \in \Upsilon_i(0), \ i = 1, 2, 3, \end{cases} \quad (33)$$

It is easy to verify that $\delta_0^{(2)} = \delta_0^{(3)} = 0$ and $\tilde{N}_0 \equiv 0$. Thus, this problem has a solution in $\mathcal{H}$ if and only if

$$\omega_0^{(1)}(0) = \omega_0^{(2)}(0) = \omega_0^{(3)}(0); \quad (34)$$

in this case $N_0 \equiv \tilde{N}_0 \equiv \omega_0^{(1)}(0)$.

In the problem (24) at $k = 1$, we have $\tilde{F}_1^{(i)} \equiv 0, \ i = 1, 2, 3$, and

$$\tilde{F}_1(\xi) = \sum_{i=1}^3 \left( \xi_i \frac{d\omega_0^{(i)}}{dx_i}(0) \chi_i''(\xi_i) + 2 \frac{d\omega_0^{(i)}}{dx_i}(0) \chi_i' (\xi_i) \right).$$
The solvability condition (27) reads in this case as follows:

$$\pi h^2_1(0) \frac{d\omega_1^{(1)}}{dx_1}(0) + \pi h^2_2(0) \frac{d\omega_2^{(2)}}{dx_2}(0) + \pi h^2_3(0) \frac{d\omega_3^{(3)}}{dx_3}(0) = 0. \quad (35)$$

Thus for the functions $$\{\omega_0^{(i)}\}_{i=1}^3$$ that are the first terms of the regular asymptotic expansion (5), we obtain the following problem:

$$\begin{cases}
-\pi \frac{d}{dx_i} \left(h^2_i(x_i) \frac{d\omega_0^{(i)}}{dx_i}(x_i)\right) = \widehat{F}_0^{(i)}(x_i), \quad x_i \in I_i, \quad i = 1, 2, 3, \\
\omega_0^{(i)}(1) = 0, \quad i = 1, 2, 3, \\
\omega_0^{(1)}(0) = \omega_0^{(2)}(0) = \omega_0^{(3)}(0), \\
\sum_{i=1}^3 \pi h^2_i(0) \frac{d\omega_0^{(i)}}{dx_i}(0) = 0,
\end{cases} \quad (36)$$

where $$I_i := \{x : x_i \in (0, 1), \; \xi_i = (0, 0)\}$$ and

$$\widehat{F}_0^{(i)}(x_i) := \pi h^2_i(x_i) f(x) \big|_{x_i=(0,0)} - \int_{\partial \Gamma_i(x_i)} \varphi^{(i)}(x_i, \xi_i) d\tau_i, \quad x_i \in I_i. \quad (37)$$

The problem (36) is called limit problem for problem (3).

Let us verify the solvability condition (27) for the problem (24) at any fixed $$k \in \mathbb{N}, \; k \geq 2.$$ Taking into account the third relation in problems (9), (11) and (13), the equality (27) can be re-written as follows:

$$\sum_{i=1}^3 \pi h^2_i(0) \int_{\ell+1}^{\ell+2} \int_{\varGamma_i(0)} (1 - \chi_i(\xi_i)) \int_{\varTheta_i(0)} (\xi, \nabla x)^{k-2} f(0) d\xi_i d\tau_i$$

$$+ \frac{1}{(k-2)!} \int_{\ell}^{\ell+2} (1 - \chi_i(\xi_i)) \int_{\varTheta_i(0)} \frac{\partial^{k-2} \varphi^{(i)}}{\partial x_i^{k-2}}(0, \xi_i) d\xi_i d\tau_i$$

$$- \frac{1}{\Xi(0)} \int_{\ell}^{\ell+2} (1 - \chi_i(\xi_i)) \int_{\varTheta_i(0)} \frac{\partial^{k-2} \varphi^{(i)}}{\partial x_i^{k-2}}(0, \xi_i) d\tau_i d\xi_i$$

$$+ \frac{1}{(k-2)!} \int_{\Xi(0)} (\xi, \nabla x)^{k-2} f(0) d\xi = 0, \quad k \in \mathbb{N}, \; k \geq 2.$$

Whence, integrating by parts in the first integrals with regard to (10), (12) and (14), we obtain the following relations for $$\{\omega_k^{(i)}\}$$:

$$\sum_{i=1}^3 \pi h^2_i(0) \frac{d\omega_k^{(i)}}{dx_i}(0) = d_{k-1}^{i}, \quad (38)$$
Remark 3.4. Due to (41), the functions
\[ f(x) \] where
\[ f(x) = \frac{1}{k!} \int f_{k-1}(0, \xi) d\xi \]
represent a weak solution of the problem (24) in the following form:
\[ \text{asymptotics (28).} \]

According to Proposition 3.1, it can be chosen in a unique way to guarantee the
and
\[ \text{Relations (40) and (38) are the first and second transmission conditions for the functions } \]
\[ \text{Let us denote by } G_k(x) := \omega_k^{(0)}(x) + \Psi_k(x), \quad x \in \Xi^{(i)}, \quad i = 1, 2, 3, \quad k \in \mathbb{N}. \]

Hence, if the functions \( \{\omega_{k-1}\}_{i=1}^3 \) satisfy (38), then there exist a weak solution \( \tilde{N}_k \) of the
problem (24). According to Proposition 3.1, it can be chosen in a unique way to guarantee the
asymptotics (28).

However till now, we do not take into account the condition (25). To satisfy this condition, we
represent a weak solution of the problem (24) in the following form:
\[ \tilde{N}_k = \omega_k^{(1)}(0) + \tilde{N}_k. \]

Taking into account the asymptotics (28), we have to put
\[ \omega_k^{(1)}(0) = \omega_k^{(2)}(0) - \delta_k^{(2)} = \omega_k^{(3)}(0) - \delta_k^{(3)}, \quad k \in \mathbb{N}. \]

As a result, we get the solution of the problem (21) with the following asymptotics:
\[ N_k(x) = \omega_k^{(0)}(x) + \Psi_k(x) + O(\exp(-\gamma_i x)) \quad \text{as } \xi_i \to +\infty, \quad i = 1, 2, 3. \]

Let us denote by
\[ G_k(x) := \omega_k^{(0)}(x) + \Psi_k(x), \quad x \in \Xi^{(i)}, \quad i = 1, 2, 3, \quad k \in \mathbb{N}. \]

Remark 3.4. Due to (41), the functions \( \{N_k - G_k\}_{k \in \mathbb{N}} \) are exponentially decrease as \( \xi_i \to +\infty, \)
\( i = 1, 2, 3. \)

Relations (40) and (38) are the first and second transmission conditions for the functions \( \{\omega_k^{(i)}\} \)
at \( x = 0. \) Thus, the functions \( \{\omega_k^{(1)}, \omega_k^{(2)}, \omega_k^{(3)}\} \) are determined from the problem
\[ \begin{aligned}
-\pi \frac{d}{dx_i} & \left( h_i^{2}(x_i) \frac{d\omega_k^{(i)}}{dx_i} \right) = \tilde{F}_k^{(i)}(x_i), \quad x_i \in I_i, \quad i = 1, 2, 3, \\
\omega_k^{(1)}(0) &= \omega_k^{(2)}(0) - \delta_k^{(2)} = \omega_k^{(3)}(0) - \delta_k^{(3)}, \\
3 & \sum_{i=1}^{3} \pi h_i^{2}(0) \frac{d\omega_k^{(i)}}{dx_i} = d_k^{*},
\end{aligned} \]
where
\[ \tilde{F}_k^{(i)}(x_i) := \int_{\Gamma_i(x_i)} f_k^{(i)}(x_i, \xi) d\xi_i - \eta_k^{(i)}(x_i) \int_{\partial\Gamma_i(x_i)} \varphi^{(i)}(x_i, \xi) d\xi_i \]
\[ + h_i^{*}(x_i) \int_{\partial\Gamma_i(x_i)} \frac{\partial f_k^{(i)}}{\partial x_i}(x_i, \xi) d\xi_i, \quad x \in I_i, \quad i = 1, 2, 3, \quad k \in \mathbb{N}. \]
From the limit problem (36) we uniquely determine the first terms of the regular asymptotic expansion (5). Next we uniquely determine the first term at any fixed index \( k \), in the form

\[
\hat{f}^{(i)}(x_i) = \omega_k^{(i)}(x_i), \quad \phi_k^{(2)}(x_2) = \omega_k^{(2)}(x_2) - \delta^{(2)}(1 - x_2), \quad \phi_k^{(3)}(x_3) = \omega_k^{(3)}(x_3) - \delta^{(3)}(1 - x_3).
\]

As a result for functions \( \{\phi_k^{(i)}\}_{i=1}^3 \), we get the problem

\[
\begin{align*}
-\pi \frac{d}{dx_i} \left( h_i^2(x_i) \frac{d\phi_k^{(i)}(x_i)}{dx_i} \right) &= \Phi_k^{(i)}(x_i), & x_i \in I_i, & i = 1, 2, 3, \\
\phi_k^{(1)}(0) &= \phi_k^{(2)}(0) = \phi_k^{(3)}(0), \\
3 \sum_{i=1}^{3} \pi h_i^2(0) \frac{d\phi_k^{(i)}(0)}{dx_i} &= \delta_k^* + \delta_k^{(2)} + \delta_k^{(3)},
\end{align*}
\]

where \( \Phi_k^{(1)}(x_1) = \tilde{F}_k^{(1)}(x_1), \Phi_k^{(i)}(x_i) = \tilde{F}_k^{(i)}(x_i) - 2\pi \delta_k^{(i)} h_i(x_i) h_i'(x_i), \ i = 2, 3. \)

Next for functions

\[
\tilde{\phi}(x) = \begin{cases} \phi^{(1)}(x_1), & \text{if } x_1 \in I_1, \\ \phi^{(2)}(x_2), & \text{if } x_2 \in I_2, \\ \phi^{(3)}(x_3), & \text{if } x_3 \in I_3,
\end{cases}
\]

defined on the graph \( I_1 \cup I_2 \cup I_3 \), we introduce the Sobolev space

\[
\mathcal{H}_0 := \left\{ \tilde{\phi} : \phi^{(i)} \in H^1(I_i), \phi^{(i)}(1) = 0, \ i = 1, 2, 3, \ \text{and} \ \phi^{(1)}(0) = \phi^{(3)}(0) = \phi^{(3)}(0) \right\}
\]

with the scalar product

\[
\langle \tilde{\phi}, \tilde{\psi} \rangle := \sum_{i=1}^{3} \pi \int_{0}^{1} h_i^2(x_i) \frac{d\phi^{(i)}(x_i)}{dx_i} \frac{d\psi^{(i)}(x_i)}{dx_i} dx_i, \quad \tilde{\phi}, \tilde{\psi} \in \mathcal{H}_0.
\]

A function \( \tilde{\phi}_k \in \mathcal{H}_0 \) is called a weak solution to problem (44) if it satisfies the integral identity

\[
\langle \tilde{\phi}_k, \tilde{\psi} \rangle = \sum_{i=1}^{3} \int_{0}^{1} \Phi_k^{(i)}(x_i) \psi_i(x_i) dx_i + (\delta_k^* + \delta_k^{(2)} + \delta_k^{(3)}) \psi_i(0) \quad \forall \tilde{\psi} \in \mathcal{H}_0.
\]

It follows from the Riesz representation theorem that problem (44) has a unique weak solution \( \tilde{\phi}_k \) at any fixed index \( k \in \mathbb{N} \).

### 4 Complete asymptotic expansion and its justification

**The first step.** From the limit problem (36) we uniquely determine the first terms \( \{\omega_0^{(i)}\}_{i=1}^3 \) of the regular asymptotic expansion (5). Next we uniquely determine the first term \( N_0 \) of the inner asymptotic expansion (7); it is a solution to the problem (33) and \( N_0 = \omega_0^{(1)}(0) \). Then we rewrite problem (9) for each index \( i = 1, 2, 3 \) and a fixed \( x_i \in I_i \) in the form

\[
\begin{align*}
-\Delta_{x_i} u_2^{(i)}(x_i, \xi_i) &= \frac{d^2 \omega_0^{(i)}}{dx_i^2}(x_i) + f(x) \big|_{x_i = (0,0)}, & \xi_i \in \mathcal{Y}_i(x_i), \\
-\partial_{\xi_i} u_2^{(i)}(x_i, \xi_i) &= - h_i'(x_i) \frac{d\omega_0^{(i)}}{dx_i}(x_i) + \varphi^{(i)}(x_i, \xi_i), & \xi_i \in \partial \mathcal{Y}_i(x_i), \\
\langle u_2^{(i)}(x_i, \cdot) \rangle_{\mathcal{T}_i(x_i)} &= 0.
\end{align*}
\]
It is easy to verify that the solvability condition for this problem is satisfied (see (10)). Therefore, thanks to the third relation in (45), there exists a unique solution to the problem (45).

Now with the help of formulas (17), we determine the first terms \( \Pi_2^{(i)} \), \( i = 1, 2, 3 \) of the boundary-layer expansions (6), as solutions of problems (16) that can be rewritten as follows:

\[
\begin{cases}
-\Delta_{\xi_i} \Pi_2^{(i)}(\xi^*_i, \bar{\xi}_i) = 0, & \xi^*_i \in (0, +\infty), \quad \bar{\xi}_i \in \mathcal{Y}_i(1), \\
-\partial_{\nu_{\xi_i}} \Pi_2^{(i)}(\xi^*_i, \bar{\xi}_i) = 0, & \xi^*_i \in (0, +\infty), \quad \bar{\xi}_i \in \partial\mathcal{Y}_i(1), \\
\Pi_2^{(i)}(0, \bar{\xi}_i) = -u_2^{(i)}(1, \bar{\xi}_i), & \bar{\xi}_i \in \mathcal{Y}_i(1), \\
\Pi_2^{(i)}(\xi^*_i, \bar{\xi}_i) \rightarrow 0, & \xi^*_i \rightarrow +\infty, \quad \bar{\xi}_i \in \mathcal{Y}_i(1).
\end{cases}
\] (46)

The second step. The second terms \( \{\omega_1^{(i)}\}_{i=1}^3 \) of the regular asymptotics (5) are found from the problem (42) that can be rewritten as follows:

\[
\begin{cases}
-\pi \frac{d}{dx_i} \left( h_i^2(x_i) \frac{d\omega_1^{(i)}}{dx_i}(x_i) \right) = \int_{\mathcal{Y}_i(x_i)} \sum_{j=1}^3 (1 - \delta_{ij}) \xi_j \frac{\partial f(x)}{\partial x_j} f(x)|_{x_i=0} d\xi_i, \\
\omega_1^{(i)}(1) = 0, & i = 1, 2, 3, \\
\omega_1^{(1)}(0) = \omega_1^{(2)}(0) - \delta_1^{(2)} = \omega_1^{(3)}(0) - \delta_1^{(3)}, \\
\sum_{i=1}^3 \pi h_i^2(0) \frac{d\omega_1^{(i)}}{dx_i}(0) = d_1^*.
\end{cases}
\] (47)

The constants \( \delta_1^{(2)} \) and \( \delta_1^{(3)} \) are uniquely determined (see Remark 3.3) by formula

\[
\delta_1^{(i)} = \int_{\Xi} \mathcal{N}_i \sum_{j=1}^3 \left( \xi_j \frac{d\omega_0^{(j)}}{dx_j}(0) \chi_j''(\xi_j) + 2 \frac{d\omega_0^{(j)}}{dx_j}(0) \chi_j'(\xi_j) \right) d\xi, \quad i = 2, 3.
\] (48)

and the constant \( d_1^* \) is determined from formula (39) and

\[
d_1^* = \ell \sum_{i=1}^3 \left( \pi h_i^2(0) f(0) - \int_{\partial\mathcal{Y}_i(0)} \varphi^{(i)}(0, \xi_i) d\xi_i \right) - |\Xi^{(0)}| f(0),
\] (49)

where \( |\Xi^{(0)}| \) is the volume of the aneurysm \( \Xi^{(0)} \) (see Section 2).

Knowing \( \{\omega_1^{(i)}\}_{i=1}^3 \), we can uniquely find the second terms of the regular asymptotics \( \{u_3^{(i)}\}_{i=1}^3 \) series (5) and boundary asymptotics \( \{\Pi_3^{(i)}\}_{i=1}^3 \) series (6) from the problems

\[
\begin{cases}
-\Delta_{\xi_i} u_3^{(i)}(x_i, \bar{\xi}_i) = \frac{d^2\omega_3^{(i)}}{dx_i^2}(x_i) + \sum_{j=1}^3 (1 - \delta_{ij}) \xi_j \frac{\partial f(x)}{\partial x_j} f(x)|_{x_i=0}, & \bar{\xi}_i \in \mathcal{Y}_i(x_i), \\
-\partial_{\nu_{\xi_i}} u_3^{(i)}(x_i, \bar{\xi}_i) = -h_i'(x_i) \frac{d\omega_3^{(i)}}{dx_i}(x_i), & \bar{\xi}_i \in \partial\mathcal{Y}_i(x_i), \\
\langle u_3^{(i)}(x_i, \cdot) \rangle_{\mathcal{Y}_i(x_i)} = 0,
\end{cases}
\] (50)
and
\[
\begin{align*}
-\Delta_{\xi_i^*} \Pi_3^{(i)}(\xi_i^*, \xi_i) &= 0, \quad \xi_i^* \in (0, +\infty), \quad \xi_i \in \Psi_i(1), \\
-\partial_{\xi_i} \Pi_3^{(i)}(\xi_i^*, \xi_i) &= 0, \quad \xi_i^* \in (0, +\infty), \quad \xi_i \in \partial \Psi_i(1), \\
\Pi_3^{(i)}(0, \xi_i) &= -u_3^{(i)}(1, \xi_i), \quad \xi_i \in \Psi_i(1), \\
\Pi_3^{(i)}(\xi_i^*, \xi_i) &\to 0, \quad \xi_i^* \to +\infty, \quad \xi_i \in \Psi_i(1),
\end{align*}
\]
(51)
respectively.

The second term $N_1$ of the inner asymptotic expansion (7) is the unique solution of the problem (21) that can now be rewritten in the form
\[
\begin{align*}
-\Delta_{\xi_i} N_1(\xi) &= 0, \quad \xi \in \Xi, \\
\partial_{\xi_i} N_1(\xi) &= 0, \quad \xi \in \Gamma_0, \\
-\partial_{\xi_i} N_1(\xi) &= 0, \quad \xi \in \Gamma_i, \quad i = 1, 2, 3, \\
N_1(\xi) &\sim \omega_1^{(i)}(0) + \xi_i \frac{d\omega_0^{(i)}}{dx}(0), \quad \xi_i \to +\infty, \quad \xi_i \in \Psi_i(0), \quad i = 1, 2, 3.
\end{align*}
\]
(52)
Thus we have uniquely determined the first terms of the expansions (5), (6) and (7).

The inductive step. Assume that we have determined the coefficients $u_0^{(i)}, \omega_1^{(i)}, \ldots, \omega_{n-3}^{(i)}$, $u_2^{(i)}, u_3^{(i)}, \ldots, u_{n-1}^{(i)}$, of the series (5), coefficients $\Pi_2^{(i)}, \Pi_3^{(i)}, \ldots, \Pi_{n-1}^{(i)}$ of the series (6), coefficients $N_1, \ldots, N_{n-3}$ of the series (7), constants $\delta_1^{(i)}, \ldots, \delta_{n-3}^{(i)}$ and $d_1^{(i)}, \ldots, d_{n-3}^{(i)}$.

Then we can find the solution $\{\omega_1^{(i)}\}_{i=1}^{n-3}$ of problem (42) with the constants $\delta_2^{(n-2)}, \delta_3^{(n-2)}$ (see (29)) in the first transmission condition and with the constant $d_n^{(n-2)}$ in the second transmission conditions. It should be noted that constants $\{d_k^{(n)}\}_{k=1}^{n}$ depend only on $f$ and $\{\varphi^{(i)}(\xi)\}_{i=1}^{3}$ and they are uniquely defined by formulas (39).

The coefficients $u_2^{(i)}$, $i = 1, 2, 3$, are determined as solutions of the following problems:
\[
\begin{align*}
-\Delta_{\xi_i} u_2^{(i)}(x, \xi_i) &= \frac{d^2 \omega_{n-2}^{(i)}(x)}{dx_i^2}(x) + \frac{\partial^2 u_{n-2}^{(i)}(x, \xi_i)}{\partial x_i^2}(x) + f_{n-2}^{(i)}(x, \xi_i), \quad \xi_i \in \Psi_i(x_i), \\
-\partial_{\xi_i} u_2^{(i)}(x, \xi_i) &= -h_i^{(i)}(x_i) \left( \frac{d\omega_{n-2}^{(i)}(x_i)}{dx_i} + \frac{\partial u_{n-2}^{(i)}(x_i, \xi_i)}{\partial x_i}(x_i) \right) \\
&\quad + \eta_{n-2}^{(i)}(x_i, \xi_i) \varphi^{(i)}(x_i, \xi_i), \quad \xi_i \in \partial \Psi_i(x_i), \\
\langle u_2^{(i)}(x, \cdot)\rangle_{\Psi_i(x_i)} &= 0,
\end{align*}
\]
(53)
where $f_{n-2}^{(i)}$ and $\eta_{n-2}^{(i)}$ are defined in (8) and (15) respectively. We note that solvability condition for problems (53) takes place, because $\langle u_{n-2}^{(i)}(x, \cdot)\rangle_{\Psi_i} = 0, \quad i = 1, 2, 3$. Here $x_i \in I_{\xi_i}^{(i)}$.

Further we find the coefficients $\Pi_2^{(i)}$, $i = 1, 2, 3$ of the boundary asymptotic expansions (6) as solutions of problems (16) that can be rewritten in the form
\[
\begin{align*}
-\Delta_{\xi_i} \Pi_2^{(i)}(\xi_i^*, \xi_i) &= 0, \quad \xi_i^* \in (0, +\infty), \quad \xi_i \in \Psi_i(1), \\
-\partial_{\xi_i} \Pi_2^{(i)}(\xi_i^*, \xi_i) &= 0, \quad \xi_i^* \in (0, +\infty), \quad \xi_i \in \partial \Psi_i(1), \\
\Pi_2^{(i)}(0, \xi_i) &= -u_2^{(i)}(1, \xi_i), \quad \xi_i \in \Psi_i(1), \\
\Pi_2^{(i)}(\xi_i^*, \xi_i) &\to 0, \quad \xi_i^* \to +\infty, \quad \xi_i \in \Psi_i(1),
\end{align*}
\]
(54)
Finally, we find the coefficient $N_{n-2}$ of the inner asymptotic expansion (7), which is the unique solution of the problem (21) that can now be rewritten in the form

\[
\begin{cases}
-\Delta_\xi N_{n-2}(\xi) &= \frac{(\xi, \nabla_\xi)^{n-4} f(0)}{(n-4)!}, & \xi \in \Xi, \\
\partial_{\xi\xi} N_{n-2}(\xi) &= 0, & \xi \in \Gamma_0, \\
-\partial_{\xi, i} N_{n-2}(\xi) &= \frac{\xi_i^{n-4}}{(n-4)!} \frac{\partial \phi^{(i)}}{\partial x_i^{n-4}}(0, \bar{\xi}_i), & \xi \in \Gamma_i, \\
N_{n-2}(\xi) &\sim \sum_{j=0}^{n-2} \frac{\xi_j^{n-4} \phi^{(i)}}{j!} \frac{\partial \phi^{(i)}}{\partial x_i^{n-4}}(0, \bar{\xi}_i), \quad \text{as} \quad \xi_i \to +\infty, \quad \bar{\xi}_i \in \Psi_i(0) \quad i = 1, 2, 3.
\end{cases}
\]

Thus we can successively determine all coefficients of series (5), (6) and (7).

4.1 Justification

With the help of the series (5), (6), (7) we construct the following series:

\[
\sum_{k=0}^{+\infty} \varepsilon^k \left( \overline{u}_k(x, \varepsilon, \alpha) + \Pi_k(x, \varepsilon) + \mathcal{N}_k(x, \varepsilon, \alpha) \right), \quad x \in \Omega_\varepsilon,
\]

where

\[
\overline{u}_k(x, \varepsilon, \alpha) := \sum_{i=1}^{3} \chi_{\ell}^{(i)} \left( \frac{x_i}{\varepsilon} \right) \left( u_{k}^{(i)} \left( \frac{x_i}{\varepsilon} \right) + \omega_{\ell}^{(i)}(x_i) \right), \quad (u_0 \equiv u_0 \equiv 0),
\]

\[
\Pi_k(x, \varepsilon) := \sum_{i=1}^{3} \chi_{\delta}^{(i)}(x_i) \Pi_k^{(i)} \left( \frac{1 - x_i}{\varepsilon}, \frac{\overline{x}_i}{\varepsilon} \right), \quad (\Pi_0 \equiv \Pi_1 \equiv 0),
\]

\[
\mathcal{N}_k(x, \varepsilon, \alpha) := \left( 1 - \sum_{i=1}^{3} \chi_{\ell}^{(i)} \left( \frac{x_i}{\varepsilon} \right) \right) \mathcal{N}_k \left( \frac{x}{\varepsilon} \right), \quad (N_0 \equiv \omega_0^{(i)}(0)),
\]

$\alpha$ is a fixed number from the interval $(\frac{2}{3}, 1)$, $\chi_\ell^{(i)}$, $\chi_\delta^{(i)}$ are smooth cut-off functions defined by formulas

\[
\chi_\ell^{(i)}(x_i) = \begin{cases} 
1, & \text{if } x_i \geq 3 \ell, \\
0, & \text{if } x_i \leq 2 \ell,
\end{cases} \quad \chi_\delta^{(i)}(x_i) = \begin{cases} 
1, & \text{if } x_i \geq 1 - \delta, \\
0, & \text{if } x_i \leq 1 - 2\delta,
\end{cases} \quad i = 1, 2, 3,
\]

and $\delta$ is a sufficiently small fixed positive number.

**Theorem 4.1.** Series (56) is the asymptotic expansion for the solution $u_\varepsilon$ to the boundary-value problem (3) in the Sobolev space $H^1(\Omega_\varepsilon)$, i.e., $\forall m \in \mathbb{N} \ (m \geq 2) \ \exists C_m > 0 \ \exists \varepsilon_0 > 0 \ \forall \varepsilon \in (0, \varepsilon_0)$:

\[
\| u_\varepsilon - U_\varepsilon^{(m)} \|_{H^1(\Omega_\varepsilon)} \leq C_m \varepsilon^\omega(m^{-\frac{1}{2}} + \frac{1}{2}),
\]

where

\[
U_\varepsilon^{(m)}(x) = \sum_{k=0}^{m} \varepsilon^k \left( \overline{u}_k(x, \varepsilon, \alpha) + \Pi_k(x, \varepsilon) + \mathcal{N}_k(x, \varepsilon, \alpha) \right), \quad x \in \Omega_\varepsilon,
\]

is the partial sum of (56).
Remark 4.1. Hereinafter, all constants in inequalities are independent of the parameter $\varepsilon$.

Proof. Take an arbitrary $m \geq 2$, $m \in \mathbb{N}$. Substituting the partial sum $U_{\varepsilon}^{(m)}$ in the equations and the boundary conditions of problem (3) and taking into account relations (36)–(54) for the coefficients of series (56), we find

$$
\Delta U_{\varepsilon}^{(m)}(x) + f(x) = \sum_{j=1}^{7} R_{\varepsilon,j}^{(m)}(x) =: R_{\varepsilon}^{(m)}(x), \quad x \in \Omega_{\varepsilon},
$$

where

$$
R_{\varepsilon,1}^{(m)}(x) = \varepsilon^{k} \sum_{k=m-1}^{3} \sum_{i=1}^{\alpha} \left( \frac{x_i}{\varepsilon} \right) \left( \frac{\partial^{2} u_{1}(i)}{\partial x_{1}^{2}} \left( \frac{x_i}{\varepsilon} \right) + \frac{\partial^{2} u_{2}(i)}{\partial x_{2}^{2}} \left( \frac{x_i}{\varepsilon} \right) \right),
$$

$$
R_{\varepsilon,2}^{(m)}(x) = \varepsilon^{k} \sum_{k=1}^{3} \sum_{i=1}^{\alpha} \left( -2 \varepsilon^{-1} - \alpha \frac{d\chi_{i}^{(i)}}{d\xi_{i}} \left( \frac{\partial N_{k}(\xi) - \partial G_{k}(\xi)}{\partial \xi_{i}} \right) - \varepsilon^{-2} \alpha \frac{d\chi_{i}^{(i)}}{d\xi_{i}} \left( N_{k}(\xi) - G_{k}(\xi) \right) \right) \bigg|_{\xi \in \frac{\varepsilon}{\varepsilon}, \xi \in \frac{\varepsilon}{\varepsilon}},
$$

$$
R_{\varepsilon,3}^{(m)}(x) = \varepsilon^{k} \sum_{k=2}^{3} \sum_{i=1}^{\alpha} \left( -2 \varepsilon^{-1} \frac{d\chi_{i}^{(i)}}{dx_{i}} \left( \frac{\partial N_{k}(\xi) - \partial G_{k}(\xi)}{\partial \xi_{i}} \right) \right)
$$

$$
+ \frac{d^{2} \chi_{i}^{(i)}}{dx_{i}^{2}} \left( \frac{\partial^{2} \Pi_{k}^{(i)}}{\partial \xi_{i}} \left( \xi^{*}, \xi \right) \right) \bigg|_{\xi^{*} = \frac{1-x}{\varepsilon}, \xi = \frac{x}{\varepsilon}},
$$

$$
R_{\varepsilon,4}^{(m)}(x) = \varepsilon^{m-1} \sum_{i=1}^{3} \chi_{i}^{(i)} \left( \frac{x_i}{\varepsilon} \right) \frac{\varepsilon^{-1}}{(m-2)!}
$$

$$
\times \left[ \sum_{p=1}^{3} (1 - \delta_{ip}) \int_{0}^{x_{p}} \left( \sum_{s=1}^{3} (1 - \delta_{is}) \frac{x_{s} - y_{s}}{\varepsilon} \frac{\partial}{\partial y_{s}} \right) \frac{m-2}{\partial f(\gamma)} \bigg|_{y_{s} = \gamma} \int_{y_{r} = \gamma} \frac{d\gamma}{\partial y_{r}} \right],
$$

$$
R_{\varepsilon,5}^{(m)}(x) = \varepsilon^{m-1} \left( 1 - \sum_{i=1}^{3} \chi_{i}^{(i)} \left( \frac{x_i}{\varepsilon} \right) \right) \frac{\varepsilon^{-1}}{(m-2)!}
$$

$$
\times \left[ \sum_{p=1}^{3} \int_{0}^{x_{p}} \left( \sum_{s=1}^{3} (1 - \delta_{is}) \frac{x_{s} - y_{s}}{\varepsilon} \frac{\partial}{\partial y_{s}} \right) \frac{m-2}{\partial f(\gamma)} \bigg|_{y_{s} = \gamma} \int_{y_{r} = \gamma} \frac{d\gamma}{\partial y_{r}} \right].
$$
\[ R_{\varepsilon,0}^{(m)}(x) = \varepsilon^{\alpha(m-1)} \sum_{i=1}^{3} \frac{3}{2} \frac{d^3 x_i^{(i)}}{d\xi_i^3} \left( \zeta_i^\dagger \right) \cdot \left[ \varepsilon^{(1-\alpha)m} \left( \frac{\partial u_m^{(i)}}{\partial x_i} \left( x_i, \frac{\varphi_i}{\varepsilon} \right) + \frac{d\omega_m^{(i)}}{dx_i} (x_i) \right) \right] \]

\[ + \sum_{k=0}^{m-1} \varepsilon^{(1-\alpha)k} \varepsilon^{-\alpha} \varepsilon + \varepsilon^{m-1} \sum_{k=0}^{m-1} \varepsilon^{(1-\alpha)k} \frac{\varepsilon^{-\alpha}}{(m-k-1)!} \int_{0}^{x_i} \left( x_i - y_i \right)^{m-k-1} \frac{\partial^{m-k+1} \left( u_k^{(i)} \left( y_i, \frac{\varphi_i}{\varepsilon} \right) + \omega_k^{(i)} (y_i) \right)}{d\xi_i^{m-k+1}} dy_i, \quad (66) \]

\[ R_{\varepsilon,7}^{(m)}(x) = \varepsilon^{\alpha(m-1)} \sum_{i=1}^{3} \frac{d^2 x_i^{(i)}}{d\xi_i^2} \left( \zeta_i^\dagger \right) \cdot \left[ \varepsilon^{(1-\alpha)m} \left( \frac{\partial u_m^{(i)}}{\partial x_i} \left( x_i, \frac{\varphi_i}{\varepsilon} \right) + \frac{d\omega_m^{(i)}}{dx_i} (x_i) \right) \right] \]

\[ \times \sum_{k=0}^{m} \varepsilon^{(1-\alpha)k} \frac{\varepsilon^{-\alpha}}{(m-k)!} \int_{0}^{x_i} \left( x_i - y_i \right)^{m-k} \frac{\partial^{m-k+1} \left( u_k^{(i)} \left( y_i, \frac{\varphi_i}{\varepsilon} \right) + \omega_k^{(i)} (y_i) \right)}{d\xi_i^{m-k+1}} dy_i. \quad (67) \]

From (61) we conclude that
\[ \exists \tilde{C}_m > 0 \quad \exists \varepsilon_0 > 0 \quad \forall \varepsilon \in (0, \varepsilon_0) : \quad \sup_{x \in \Omega_\varepsilon} \left| R_{\varepsilon,1}^{(m)}(x) \right| \leq \tilde{C}_m \varepsilon^{m-1}. \quad (68) \]

Due to the exponential decreasing of functions \( \{ N_k - G_k, \Pi_k^{(i)} \} \) (see Remark 3.4 and (20)) and the fact that the support of the derivatives of cut-off function \( \chi_i^{(i)} \) belongs to the set \( \{ x_i : 2\ell \varepsilon \leq x_i \leq 3\ell \varepsilon \} \), we arrive at
\[ \sup_{x \in \Omega_\varepsilon} \left| R_{\varepsilon,2}^{(m)}(x) \right| \leq \tilde{C}_m \varepsilon^{-1-\alpha} \exp \left( -\frac{2\ell}{\varepsilon^{1-\alpha}} \min_{i=1,2,3} \gamma_i \right), \quad (69) \]

similarly we obtain that
\[ \sup_{x \in \Omega_\varepsilon} \left| R_{\varepsilon,3}^{(m)}(x) \right| \leq \tilde{C}_m \varepsilon^{-1} \exp \left( -\frac{\delta}{\varepsilon} \min_{i=1,2,3} \lambda_i^{(i)} \right), \quad (70) \]

We calculate terms \( R_{\varepsilon,j}^{(m)}, \quad j = 4, 5, 6, 7 \) with the help of the Taylor formula with the integral remaining term for functions \( f, \{ \omega_k \} \) and \( \{ u_k \} \) at the point \( x_i = 0 \). It is easy to check that
\[ \sup_{x \in \Omega_\varepsilon} \left| R_{\varepsilon,4}^{(m)}(x) \right| \leq \tilde{C}_m \varepsilon^{m-1}, \quad \sup_{x \in \Omega_\varepsilon} \left| R_{\varepsilon,j}^{(m)}(x) \right| \leq \tilde{C}_m \varepsilon^{\alpha(m-1)}, \quad j = 5, 6, 7. \quad (71) \]

The partial sum leaves the following residuals on the boundary of \( \Omega_\varepsilon \):
\[ \partial_{\nu} U_{\varepsilon}^{(m)}(x) + \varepsilon \varphi_i^{(i)} \left( x_i, \frac{\varphi_i}{\varepsilon} \right) = \tilde{R}_{\varepsilon,(i)}^{(m)}(x), \quad x \in \Gamma_{\varepsilon}^{(i)}, \quad i = 1, 2, 3, \]
\[ U_{\varepsilon}^{(m)}(x) = 0, \quad x \in \Upsilon_{\varepsilon}^{(1)}, \quad i = 1, 2, 3, \]
\[ \partial_{\nu} U_{\varepsilon}^{(m)}(x) = 0, \quad x \in \Gamma_{\varepsilon}^{(0)}, \]

where \( \tilde{R}_{\varepsilon,(i)}^{(m)} := \sum_{j=8}^{9} \tilde{R}_{\varepsilon,j(i)}^{(m)} \)
\[ \tilde{R}_{\varepsilon,j(i)}^{(m)}(x) = \frac{\varepsilon}{\sqrt{1 + \varepsilon^2 |h_i^{(i)}(x_i)|^2}} \chi_i^{(i)} \left( \frac{x_i}{\varepsilon} \right) \cdot \left[ - \sum_{k=m-1}^{m-1} \varepsilon^{k} h_k^{(i)}(x_i) \left( \frac{\partial u_k^{(i)}}{\partial x_i} \left( x_i, \frac{\varphi_i}{\varepsilon} \right) + \frac{d\omega_k^{(i)}}{dx_i} (x_i) \right) \right. \]
\[ + \varepsilon^2 \left[ \varepsilon \left( \left[ \frac{\varepsilon}{\varepsilon} \right] + \left[ \frac{\varepsilon}{\varepsilon} \right] \right) \right] \frac{\varepsilon^{-2}}{1 - 2 \left[ \frac{\varepsilon}{\varepsilon} \right]} \left[ \frac{\varepsilon}{\varepsilon} \right] h_i^{(i)}(x_i) \varphi_i^{(i)} \left( x_i, \frac{\varphi_i}{\varepsilon} \right) \int_{0}^{x_i} \left( \frac{\varepsilon^2 |h_i^{(i)}(x_i)|^2 - t}{\varepsilon^2} \right) \left( 1 + t \right)^{1 - \left[ \frac{\varepsilon}{\varepsilon} \right]} dt, \quad (72) \]
\[
R^{(m)}_{\varepsilon,9,(i)}(x) = \varepsilon^{1+\alpha(m-1)} \left( 1 - \chi^{(i)} \left( \frac{x_i}{\varepsilon^{\alpha}} \right) \right) \frac{\varepsilon^{-\alpha}}{(m-2)!} \int_0^{x_i} \left( \frac{x_i - y_i}{\varepsilon^{\alpha}} \right)^{m-2} \frac{\partial^{m-1} \varphi^{(i)}}{\partial y_i^{m-1}}(y_i, \frac{\pi_i}{\varepsilon}) \, dy_i. \tag{73}
\]

In (72) the symbol \( \lceil \eta \rceil \) denotes the ceiling of the number \( \eta \). It follows from (72) and (73) that there exist positive constants \( \mathcal{C}_m \) and \( \varepsilon_0 \) such that for all \( i = 1, 2, 3 \) and

\[
\forall \varepsilon \in (0, \varepsilon_0) : \sup_{x \in \Gamma^{(i)}_{\varepsilon}} \left| R^{(m)}_{\varepsilon,8,(i)}(x) \right| \leq \mathcal{C}_m \varepsilon^m, \quad \sup_{x \in \Gamma^{(i)}_{\varepsilon}} \left| \mathcal{R}^{(m)}_{\varepsilon,9,(i)}(x) \right| \leq \mathcal{C}_m \varepsilon^{1+\alpha(m-1)}. \tag{74}
\]

Using (68) – (71) and (74), we obtain the following estimates:

\[
\left\| R^{(m)}_{\varepsilon,j} \right\|_{L^2(\Omega_{\varepsilon})} \leq \hat{C}_m \left( \frac{\pi}{\varepsilon} \right)^{\frac{3}{2}} \max_{i=1}^{\frac{3}{2}} h_i^2(x_i) \varepsilon^m, \quad j = 1, 4, \tag{75}
\]

\[
\left\| R^{(m)}_{\varepsilon,2} \right\|_{L^2(\Omega_{\varepsilon})} \leq \hat{C}_m \left( \frac{\pi}{\varepsilon} \right)^{\frac{3}{2}} \sum_{i=1}^{\frac{3}{2}} h_i^2(0) \varepsilon^{-\frac{2}{3}} \min_{i=1,2,3} \gamma_i, \tag{76}
\]

\[
\left\| R^{(m)}_{\varepsilon,3} \right\|_{L^2(\Omega_{\varepsilon})} \leq \hat{C}_m \left( \frac{\pi}{\varepsilon} \right)^{\frac{3}{2}} \sum_{i=1}^{\frac{3}{2}} h_i^2(1) \delta_i \varepsilon^{-\frac{4}{3}} \min_{i=1,2,3} \lambda_i, \tag{77}
\]

\[
\left\| R^{(m)}_{\varepsilon,5} \right\|_{L^2(\Omega_{\varepsilon})} \leq \hat{C}_m \left( \frac{\pi}{\varepsilon} \right)^{\frac{3}{2}} \sum_{i=1}^{\frac{3}{2}} h_i^2(0) \varepsilon^{\alpha(m-\frac{1}{2})+1}, \tag{78}
\]

\[
\left\| R^{(m)}_{\varepsilon,j} \right\|_{L^2(\Omega_{\varepsilon})} \leq \hat{C}_m \left( \frac{\pi}{\varepsilon} \right)^{\frac{3}{2}} \sum_{i=1}^{\frac{3}{2}} h_i^2(0) \varepsilon^{\alpha(m-\frac{1}{2})+1}, \quad j = 6, 7, \tag{79}
\]

\[
\left\| \mathcal{R}^{(m)}_{\varepsilon,8,(i)} \right\|_{L^2(\Gamma^{(i)}_{\varepsilon})} \leq \hat{C}_m \sqrt{2\pi \max_{x_i \in I_i} h_i(x_i) \varepsilon^{m+\frac{1}{2}}}, \quad i = 1, 2, 3, \tag{80}
\]

\[
\left\| \mathcal{R}^{(m)}_{\varepsilon,9,(i)} \right\|_{L^2(\Gamma^{(i)}_{\varepsilon})} \leq \hat{C}_m \sqrt{6\pi \varepsilon^{m+\frac{1}{2}}}, \quad i = 1, 2, 3. \tag{81}
\]

Thus, the difference \( W_{\varepsilon} := u_{\varepsilon} - U^{(m)}_{\varepsilon} \) satisfies the following relations:

\[
\begin{cases}
-\Delta W_{\varepsilon} = R^{(m)}_{\varepsilon} & \text{in } \Omega_{\varepsilon}, \\
-\partial_{\nu} W_{\varepsilon} = R^{(m)}_{\varepsilon,(i)} & \text{on } \Gamma^{(i)}_{\varepsilon}, \quad i = 1, 2, 3, \\
W_{\varepsilon} = 0 & \text{on } \Gamma^{(1)}_{\varepsilon}, \quad i = 1, 2, 3, \\
\partial_{\nu} W_{\varepsilon} = 0 & \text{on } \Gamma^{(0)}_{\varepsilon}.
\end{cases} \tag{82}
\]

From (82) we derive the following integral relation:

\[
\int_{\Omega_{\varepsilon}} |\nabla W_{\varepsilon}|^2 \, dx = \int_{\Omega_{\varepsilon}} R^{(m)}_{\varepsilon} W_{\varepsilon} \, dx - \sum_{i=1}^{3} \int_{\Gamma^{(i)}_{\varepsilon}} \mathcal{R}^{(m)}_{\varepsilon,(i)} W_{\varepsilon} \, d\sigma_x.
\]
In view of the Friedrichs inequality and estimates (75) – (81), this yields the following inequality:
\[
\int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^2 \, dx \leq C_m \varepsilon^{\alpha (m-\frac{1}{2}) + \frac{1}{2}} \| W_\varepsilon \|_{L^2(\Omega_\varepsilon)} + \varepsilon m \| W_\varepsilon \|_{L^2(T_\varepsilon^{(i)})} + \sum_{i=1}^{3} \| W_\varepsilon \|_{L^2(T_\varepsilon^{(i)})}
\]
\[
\leq C_m \varepsilon^{\alpha (m-\frac{1}{2}) + \frac{1}{2}} \| \nabla W_\varepsilon \|_{L^2(\Omega_\varepsilon)}.
\]

This, in turn, means the asymptotic estimate (58) and proves the theorem. \qed

**Corollary 4.1.** The differences between the solution \( u_\varepsilon \) of problem (3) and the partial sums \( U_\varepsilon^{(0)}, U_\varepsilon^{(1)} \) (see (59)) admit the following asymptotic estimates:
\[
\| u_\varepsilon - U_\varepsilon^{(0)} \|_{H^1(\Omega_\varepsilon)} \leq \widetilde{C}_0 \varepsilon^{1+\frac{2}{3}}, \quad \| u_\varepsilon - U_\varepsilon^{(0)} \|_{L^2(\Omega_\varepsilon)} \leq \widetilde{C}_0 \varepsilon^{\frac{3}{2} + \frac{1}{2}}, \quad (83)
\]
\[
\| u_\varepsilon - U_\varepsilon^{(1)} \|_{H^1(\Omega_\varepsilon)} \leq \tilde{C}_0 \varepsilon^{1+\alpha}, \quad (84)
\]

where \( \alpha \) is a fixed number from the interval \( \left( \frac{2}{3}, 1 \right) \).

In thin cylinders \( \Omega_{\varepsilon, \ell}^{(i)} := \Omega_\varepsilon^{(i)} \cap \{ x \in \mathbb{R}^3 : \ x_i \in I_{\varepsilon, \ell}^{(i)} := (3\ell \varepsilon^\alpha, 1) \} \), \( i = 1, 2, 3 \), the following estimates hold:
\[
\| u_\varepsilon - \omega_0^{(i)} \|_{H^1(\Omega_{\varepsilon, \ell}^{(i)})} \leq \bar{C}_1 \varepsilon^2, \quad i = 1, 2, 3, \quad (85)
\]

where \( \{ \omega_0^{(i)} \}_{i=1}^3 \) is the solution of the limit problem (36).

In the neighbourhood \( \Omega_{\varepsilon, \ell}^{(0)} := \Omega_\varepsilon \cap \{ x : \ x_i < 2\ell \varepsilon, \ i = 1, 2, 3 \} \) of the aneurysm \( \Omega_\varepsilon^{(0)} \), we get estimates:
\[
\| \nabla x u_\varepsilon - \nabla \xi N_1 \|_{L^2(\Omega_{\varepsilon, \ell}^{(0)})} \leq \| u_\varepsilon - \omega_0^{(i)} (0) - \varepsilon N_1 \|_{H^1(\Omega_{\varepsilon, \ell}^{(0)})} \leq \bar{C}_4 \varepsilon^\frac{5}{2}, \quad (86)
\]

**Proof.** Denote by \( \chi_{\ell, \alpha, \varepsilon}^{(i)}(\cdot) := \chi_{\ell}^{(i)}(\frac{\cdot}{\varepsilon \alpha}) \) (the function \( \chi_{\ell}^{(i)} \) is determined in (57)). Using the smoothness of the functions \( \{ \omega_k^{(i)} \} \) and the exponential decay of the functions \( \{ N_k - G_k \} \) and \( \{ \Pi_k^{(i)} \} \), \( i = 1, 2, 3 \), at infinity, we deduce the inequality (83) from estimate (58) at \( m = 2 \):
\[
\| u_\varepsilon - U_\varepsilon^{(0)} \|_{H^1(\Omega_\varepsilon)} \leq \| u_\varepsilon - U_\varepsilon^{(2)} \|_{H^1(\Omega_\varepsilon)} + \| \varepsilon \sum_{i=1}^{3} \left( \chi_{\ell, \alpha, \varepsilon}^{(i)} \omega_1^{(i)} + \left( 1 - \sum_{i=1}^{3} \chi_{\ell, \alpha, \varepsilon}^{(i)} \right) N_1 \right) \|_{H^1(\Omega_\varepsilon)}
\]
\[
+ \| N_1 \|_{H^1(\Omega_\varepsilon)} + \| N_2 \|_{H^1(\Omega_\varepsilon)} + \| \varepsilon \sum_{i=1}^{3} \left( \chi_{\ell, \alpha, \varepsilon}^{(i)} (u_2 + \omega_2) + \left( 1 - \sum_{i=1}^{3} \chi_{\ell, \alpha, \varepsilon}^{(i)} \right) N_2 \right) \|_{H^1(\Omega_\varepsilon)}
\]
\[
\leq C_2 \varepsilon^{\frac{3}{2} + \frac{1}{2}} + \| \varepsilon \sum_{i=1}^{3} \left( 1 - \chi_{\ell, \alpha, \varepsilon}^{(i)} \right) \left( \omega_1^{(i)} (0) + \frac{x_2^2 \omega_0^{(i)} (0)}{2} \right) \|_{H^1(\Omega_\varepsilon)}
\]
\[
+ \varepsilon \sum_{i=1}^{3} \left( 1 - \chi_{\ell, \alpha, \varepsilon}^{(i)} \right) \left( \omega_1^{(i)} (0) + \frac{x_2^2 \omega_0^{(i)} (0)}{2} \right) \|_{H^1(\Omega_\varepsilon)}.
\]
\[ + \varepsilon^2 \sum_{i=1}^{3} \left\| (1 - \chi^{(i)}_{t,\alpha,\varepsilon}) (u_2^{(i)}(0, \cdot) - u_2^{(i)}) \right\|_{H^1(\Omega_0^{(i)})} + \varepsilon^2 \sum_{i=1}^{3} \left\| (1 - \chi^{(i)}_{t,\alpha,\varepsilon}) (\omega_2^{(i)}(0) - \omega_2^{(i)}) \right\|_{H^1(\Omega_0^{(i)})} \]
\[ + \varepsilon^2 \sum_{i=1}^{3} \left( \varepsilon \left\| \omega_1^{(i)} \right\|_{H^1(\Omega_0^{(i)})} + \varepsilon^2 \left\| u_2^{(i)} + \omega_2^{(i)} \right\|_{H^1(\Omega_0^{(i)})} \right) \]
\[ + \varepsilon^2 \sum_{i=1}^{3} \left( \varepsilon \left\| (1 - \chi^{(i)}_{t,\alpha,\varepsilon}) (N_1 - G_1) \right\|_{H^1(\Omega_0^{(i)})} + \varepsilon^2 \left\| (1 - \chi^{(i)}_{t,\alpha,\varepsilon}) (N_2 - G_2) \right\|_{H^1(\Omega_0^{(i)})} \right) \]
\[ + \varepsilon^2 \left\| N_1 \right\|_{H^1(\Xi^{(0)})} + \varepsilon^2 \left\| N_2 \right\|_{H^1(\Xi^{(0)})} + \varepsilon^2 \sum_{i=1}^{3} \left\| \chi^{(i)}_3 \Pi_2^{(i)} \right\|_{H^1(\Omega_0^{(i)})} \leq \bar{C}_0 \varepsilon^{1 + \frac{\alpha}{2}}. \]

To prove the second estimate in (83), we need to calculate the \( L^2 \)-norm of terms in the right-hand side of the previous inequality.

The inequality (84) can be similarly obtained from the estimate (58) at \( m = 4 \).

Again with the help of estimate (58) at \( m = 4 \), we deduce
\[ \left\| u_\varepsilon - \omega_0^{(i)} \right\|_{H^1(\Omega_0^{(i)})} \leq \left\| u_\varepsilon - U_\varepsilon^{(4)} \right\|_{H^1(\Omega_0^{(i)})} + \varepsilon \left\| \omega_1^{(i)} \right\|_{H^1(\Omega_0^{(i)})} \]
\[ + \varepsilon^k \left\| u_k^{(i)} + \omega_k^{(i)} + \chi_3^{(i)} \Pi_k^{(i)} \right\|_{H^1(\Omega_0^{(i)})} \leq \bar{C}_2 \varepsilon^2, \]
whence we get (85).

From inequality
\[ \left\| u_\varepsilon - \omega_2^{(i)}(0) - \varepsilon N_1 \right\|_{H^1(\Omega_0^{(i)})} \leq \left\| u_\varepsilon - U_\varepsilon^{(4)} \right\|_{H^1(\Omega_0^{(i)})} + \sum_{k=2}^{4} \varepsilon^k \left\| N_k \right\|_{H^1(\Omega_0^{(i)})} \leq \bar{C}_4 \varepsilon^{\frac{3}{2}}, \]

it follows more better energetic estimate (86) in a neighborhood of the aneurysm \( \Omega_0^{(0)} \).

Using the Cauchy-Buniakovskii-Schwarz inequality and the continuously embedding of the space \( H^1(\Omega_0^{(i)}) \) in \( C(\Omega_0^{(i)}) \), it follows from (85) the following corollary.

**Corollary 4.2.** If \( h_i(x_i) \equiv h_i \equiv \text{const}, \) \( i = 1, 2, 3, \) then
\[ \left\| E_\varepsilon^{(i)}(u_\varepsilon) - \omega_0^{(i)} \right\|_{H^1(\Omega_0^{(i)})} \leq \bar{C}_2 \varepsilon, \] (87)
\[ \max_{x_i \in \Omega_0^{(i)}} \left| E_\varepsilon^{(i)}(u_\varepsilon)(x_i) - \omega_0^{(i)}(x_i) \right| \leq \bar{C}_3 \varepsilon, \] \( i = 1, 2, 3, \) (88)

where
\[ (E_\varepsilon^{(i)} u_\varepsilon)(x_i) = \frac{1}{\pi \varepsilon^2 h_i} \int_{\Omega_0^{(i)}} u_\varepsilon(x) \, d\tau_i, \] \( i = 1, 2, 3. \)

**Corollary 4.3.** If to the assumptions \( h_i(x_i) \equiv h_i \equiv \text{const}, \) \( i = 1, 2, 3, \) the function \( \varphi_\varepsilon \equiv 0 \) and \( f = f(x_i), \) \( x \in \Omega_0^{(i)} \) \( i = 1, 2, 3, \) then the asymptotic expansion for the solution \( u_\varepsilon \) has the following more simple form:
\[ \sum_{k=0}^{+\infty} \varepsilon^k \left( \sum_{i=1}^{3} \chi_k^{(i)} \left( \frac{x_i}{\varepsilon^{\alpha}} \right) \omega_k^{(i)}(x_i) + \left( 1 - \sum_{i=1}^{3} \chi_k^{(i)} \left( \frac{x_i}{\varepsilon^{\alpha}} \right) \right) N_k \left( \frac{x}{\varepsilon} \right) \right), \] \( x \in \Omega_\varepsilon, \) (89)
and the asymptotic estimates are improved:

\[ \| u_\varepsilon - U_\varepsilon^{(m)} \|_{H^1(\Omega_\varepsilon)} \leq C_m \varepsilon^{\alpha(m-\frac{1}{2})+1}. \]  

\[ \| u_\varepsilon - \omega_0^{(i)} - \varepsilon \omega_1^{(i)} \|_{H^1(\Omega_{\varepsilon,\alpha}^{(i)})} \leq C_1 \varepsilon^3, \quad i = 1, 2, 3; \]  

\[ \| E_\varepsilon^{(i)} u_\varepsilon - \omega_0^{(i)} - \varepsilon \omega_1^{(i)} \|_{H^1(I_{\varepsilon,\alpha}^{(i)})} \leq C_2 \varepsilon^2, \quad i = 1, 2, 3; \]  

\[ \max_{x \in I_{\varepsilon,\alpha}^{(i)}} \left| (E_\varepsilon^{(i)} u_\varepsilon)(x) - \omega_0^{(i)}(x) - \varepsilon \omega_1^{(i)}(x) \right| \leq C_3 \varepsilon^2, \quad i = 1, 2, 3. \]  

5 Conclusions

1. An important problem of existing multi-scale methods is their stability and accuracy. The proof of the error estimate between the constructed approximation and the exact solution is a general principle that has been applied to the analysis of the efficiency of a multi-scale method. In our paper, we have constructed and justified the asymptotic expansion for the solution to problem (3) and proved the corresponding estimates. It should be noted here that we do not assume any orthogonality conditions for the right-hand sides in the equation and in the Neumann boundary conditions.

The results showed the possibility to replace the complex boundary-value problem (3) with the corresponding 1-dimensional boundary-value problem (36) in the graph \( I = \bigcup_{i=1}^{3} I_i \) with sufficient accuracy measured by the parameter \( \varepsilon \) characterizing the thickness and the local geometrical irregularity. In this regard, the uniform pointwise estimates (88) and (93), that are very important for applied problems, also confirm this conclusion.

2. To construct the asymptotic expansion in the whole domain, we have used the method of matching asymptotic expansions with special cut-off functions. It is the natural approach for approximations of solutions to boundary-value problems in perturbed domains. In comparison to the method of the partial asymptotic domain decomposition [34], this method gives better estimate even for the first terms \( \{ \omega_0^{(i)} \}_{i=1}^{3} \) in the \( L^2 \)-norm (compare (1) and the second estimate in (83)) without any additional assumptions for the right-hand sides.

3. The energetic estimate (83) partly confirms the first formal result of [14] (see p. 296) that the local geometric irregularity of the analyzed structure does not significantly affect on the global-level properties of the framework, which are described by the limit problem (36) and its solution \( \{ \omega_0^{(i)} \}_{i=1}^{3} \) (the first terms of the asymptotics).

Therefore, convergence results, which were obtained for second-order problems in a thin T-like shaped domain (see [16, 17] and references therein) cannot show the influence of the aneurysm. But thanks to estimates (84) and (91) – (93) it became possible now to identify the impact of the geometric irregularity and material characteristics of the aneurysm on the global level through the second terms \( \{ \omega_1^{(i)} \}_{i=1}^{3} \) of the regular asymptotics (5). They depend on the constants \( d_1^{(i)}, \delta_1^{(2)} \) and \( \delta_1^{(3)} \) that take into account all those factors (see (48) and (49)). This conclusion does not coincide with the second main result of [14] (see p. 296) that “the joints of normal type manifest themselves on the local level only”.

In addition, in [14] the authors stated that the main idea of their approach “is to use a local perturbation corrector of the form \( \varepsilon N(x)/\varepsilon \frac{du_0}{d\varepsilon} \) with the condition that the function \( N(y) \) is localized near the joint”, i.e., \( N(y) \to 0 \) as \( |y| \to +\infty \), and the main assumption of this approach is that \( \nabla_y N \in L_1(Q_\infty) \).
As shown, the coefficients \( \{N_k\} \) of the inner asymptotics (7) behave as polynomials at infinity and do not decrease exponentially (see (41)). Therefore, they influence directly the terms of the regular asymptotics beginning with the second terms. Thus, the main assumption made in [14] is not correct.

4. From the first estimate in (83) it follows that the gradient \( \nabla u_\varepsilon \) is equivalent to \( \left\{ \frac{d\omega^{(i)}_0}{dx_i} \right\}_{i=1}^3 \) in the \( L^2 \)-norm over whole junction \( \Omega_\varepsilon \) as \( \varepsilon \to 0 \). Obviously, this estimate is not informative in the neighbourhood \( \Omega^{(0)}_{\varepsilon,l} \) of the aneurysm \( \Omega^{(0)}_0 \).

Thanks to estimates (84) and (86), we get the approximation of the gradient (flux) of the solution both in the curvilinear cylinders \( \Omega^{(i)}_{\varepsilon,\alpha} \), \( i = 1, 2, 3 \):

\[
\nabla u_\varepsilon(x) \sim \frac{d\omega^{(i)}_0}{dx_i}(x_i) + \varepsilon \frac{d\omega^{(i)}_1}{dx_i}(x_i) \quad \text{as} \quad \varepsilon \to 0
\]

and in the neighbourhood \( \Omega^{(0)}_{\varepsilon,l} \) of the aneurysm:

\[
\nabla u_\varepsilon(x) \sim \nabla N_1(\xi) \quad \text{as} \quad \varepsilon \to 0.
\]

Also using estimates (58), we can obtain better approximations for the solution and its gradient with preset accuracy \( \mathcal{O}(\varepsilon^{\alpha(m-\frac{1}{2})+\frac{1}{2}}) \), \( \forall m \in \mathbb{N} \).

5. We regard that the estimates (87) and (88) can be proved without the assumptions that \( h_i(x_i) \equiv h_i \equiv \text{const}, \ (i = 1, 2, 3) \). For this we should apply the following second energy inequality in \( \Omega_\varepsilon \):

\[
\|u_\varepsilon\|_{H^2(\Omega_\varepsilon)} \leq M\left(\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \|f\|_{L^2(\Omega_\varepsilon)} + \sum_{i=1}^{3} \|\varphi_\varepsilon\|_{H^{\frac{1}{2}}(\Gamma^{(i)}_\varepsilon)} \right).
\]

But the question how the constant \( M \) depends on the parameter \( \varepsilon \) remains open. The answer will allow to get better estimates both in the norms of energy spaces and in the uniform metric.

References

[1] N.S. Bakhvalov and G.P. Panasenko, *Homogenization: Averaging processes in periodic media*. Nauka, Moscow, 1984 (in Russian); English translation in Kluwer, Dordrecht/Boston/London,1989. 1, 1

[2] D. Blanchard, A. Gaudiello, Homogenization of highly oscillating boundaries and reduction of dimension for a monotone problem, *ESAIM Control. Optim. Calc. Var.* 9 (2003) 449–460. 1

[3] D. Blanchard, A. Gaudiello and T.A. Mel’nyk, Boundary homogenization and reduction of dimention in a Kirchhoff-Love plate, *SIAM J. Math. Anal.* 39 (2008) 1764–1787. 1

[4] A.O. Borisyuk, Experimental study of wall pressure fluctuations in rigid and elastic pipes behind an axisymmetric narrowing, *Journal of Fluids and Structures*, 26 (2010) 658–674. 1

[5] R. Buncio, G. Cardone and S.A. Nazarov, Scalar boundary value problems on junctions of thin rods and plates. I. Asymptotic analysis and error estimates, *ESAIM: Mathematical Modelling and Numerical Analysis*, 48 (2014) 1495–1528. 1
[6] G. Cardone, L. Carraro, R. Fares and G.P. Panasenko, Asymptotic analysis of the steady Stokes equation with randomly perturbed viscosity in a thin tube structure, *Journal of Mathematical Sciences*, **176** (2011) 797–817. 1

[7] G. Cardone, A. Corbo-Esposito and G. Panasenko, Asymptotic partial decomposition for diffusion with sorption in thin structures, *Nonlinear Analysis* **65** (2006) 79–106. 1, 1

[8] G. Cardone, G.P. Panasenko and Y. Sirakov, Asymptotic analysis and numerical modeling of mass transport in tubular structures, *Mathematical Models and Methods in Applied Sciences (M3AS)* **20** (2010) 1–25. 1

[9] D. Cioranescu, J. Saint Jean Paulin, *Homogenization of Reticulated Structures*, Appl. Math. Sci., Vol. 139, Springer-Verlag, New York (1999). 1

[10] G.A. Chechkin, V.V. Jikov, D. Lukkassen and A.L. Piatnitski, On homogenization of networks and junctions, *Asymptotic Analysis*, **30** (2002) 61–80. 1

[11] G.A. Chechkin and T.A. Mel’nyk, Spatial-skin effect for eigenvibrations of a thick cascade junction with “heavy” concentrated masses, *Mathematical Methods in Applied Sciences* **37** (2014) 56–74. 1

[12] U. De Maio, T. Durante and T.A. Mel’nyk, Asymptotic approximation for the solution to the Robin problem in a thick multi-level Junction’, *Mathematical Models and Methods in Applied Sciences (M3AS)* **15** (2005) 1897–1921. 1

[13] A. Gaudiello and E. Zappale, Junction in a thin multidomain for a fourth order problem, *Math. Models Methods Appl. Sci.* **16** (2006) 1887–1918. 1

[14] A. Gaudiello and A.G. Kolpakov, Influence of non degenerated joint on the global and local behavior of joined rods, *International Journal of Engineering Science* **49** (2010) 295–309. 5

[15] A. Gaudiello, G.P. Panasenko and A. Piatnitski, Asymptotic analysis and domain decomposition for a biharmonic problem in a thin multi-structure, *Communications in Contemporary Mathematics* **18** (2016) 1–27. 1

[16] A. Gaudiello, B. Gustafsson, C. Lefter and J. Mossino, Asymptotic analysis of a class of minimization problems in a thin multidomain, *Calc. Var. Partial Differ. Eqs.* **15**:2 (2002) 181–201. 5

[17] A. Gaudiello and A. Sili, Asymptotic analysis of the eigenvalues of an elliptic problem in an anisotropic thin multidomain, *Proc. Roy. Soc. Edinburgh Sect. A*, **141**:4 (2011) 739–754. 5

[18] A.M. Il’in, *Matching of asymptotic expansions of solutions of boundary value problems*. Translations of Mathematical Monographs, 102. American Mathematical Society, Providence, RI, 1992. 1.1

[19] V. A. Kondratiev and O. A. Oleinik, Boundary-value problems for partial differential equations in non-smooth domains, *Russian Mathematical Surveys*, **38**:2 (1983) 1–86. 3.3

[20] E. M. Landis and G. P. Panasenko, A variant of a theorem of Phragmen-Lindelof type for elliptic equations with coefficients that are periodic in all variables but one, *Trudy Seminara imeni I.G. Petrovskogo*, **5** (1979) 105–136 (in Russian). 3.3
[21] ØEvju, K. Valen-Sendstad and K.A. Mardal, A study of wall shear stress in 12 aneurysms with respect to different viscosity models and flow conditions, *Journal of Biomechanics*, **46** (2013) 2802–2808.

[22] A.V. Klevtovskiy, Asymptotic approximation for the solution to the nonlinear monotone boundary-value problem in the thin aneurysm-type domain. In: Buryachenko, K. (ed.) Book of Abstracts of the 5th International Conference for Young Scientists on Differential Equations and Applications Dedicated to Yaroslav Lopatynsky: 09-11 November, 2016, Kyiv, Ukraine, p. 78 (2016).

[23] A.V. Klevtovskiy and T.A. Mel’nyk, Asymptotic expansions of the solution of an elliptic boundary-value problem for a thin cascade domain, *Nonlinear Oscillations* **16** (2013) 214–237; English translation in *J. Math. Sci.* **198** (2014) 303–327.

[24] A.V. Klevtovskiy and T.A. Mel’nyk, Asymptotic expansion for the solution to a boundary-value problem in a thin cascade domain with a local joint, *Asymptotic Analysis* **97** (2016) 265–290.

[25] T.A. Mel’nyk and S.A. Nazarov, Asymptotic structure of the spectrum of the Neumann problem in a thin comb-like domain, *C.R. Acad. Sci., Paris* **319**, Serie 1 (1994), 1343–1348.

[26] T.A. Mel’nyk, Homogenization of the Poisson equation in a thick periodic junction, *Zeitschrift für Analysis und ihre Anwendungen* **18** (1999), 953–975.

[27] T.A. Mel’nyk, Asymptotic approximation for the solution to a semi-linear parabolic problem in a thick junction with the branched structure, *J. Math. Anal. Appl.* **424** (2015), 1237–1260.

[28] S.A. Nazarov and B.A. Plamenevskii, Asymptotics of the spectrum of the Neumann problem in a singularly degenerate thin domains. I, *Algebra i Analiz*, **2** (1990), 85–111.

[29] S. A. Nazarov and B. A. Plamenevskii, *Elliptic problems in domains with piecewise smooth boundaries*, Walter de Gruyter, Berlin, 1994.

[30] S.A. Nazarov, Junctions of singularly degenerating domains with different limit dimensions, *J. Math. Sci.* **80** (1996) 1989–2034.

[31] S.A. Nazarov, *Asymptotic Theory of Thin Plates and Rods. Dimension Reduction and Integral Bounds*, Nauchnaya Kniga, Novosibirsk (2002). (in Russian)

[32] S.A. Nazarov and A.S. Slutskii. Arbitrary plane systems of anisotropic beams, *Tr. Mat. Inst. Steklov.* **236** (2002), 234–261.

[33] S.A. Nazarov and A.S. Slutskii. Asymptotic analysis of an arbitrary spatial system of thin rods, *Proceedings of the St. Petersburg Mathematical Society*, **10** (2004), 59 – 109.

[34] G.P. Panasenko, Method of asymptotic partial decomposition of domain, *Mathematical Models and Methods in Applied Sciences* **8** (1998), 139–156.

[35] G.P. Panasenko, *Multi-Scale Modelling for Structures and Composites*, Springer, Dordrecht, 2005.
[36] G. Panasenko, Method of asymptotic partial domain decomposition for non-steady problems: wave equation on a thin structure. In book: Springer Proceedings in Mathematics & Statistics, Vol. 116 (2015), Analytic Methods in Interdisciplinary Applications, Editors: Vladimir V. Mityushev, Michael Ruzhansky, Springer, 2015.

[37] G. Panasenko, K. Pileckas, Asymptotic analysis of the non-steady Navier-Stokes equations in a tube structure. I. The case without boundary-layer in time. *Nonlinear Analysis* **122** (2015), 125–168.

[38] G. Panasenko, K. Pileckas, Asymptotic analysis of the non-steady Navier-Stokes equations in a tube structure. II. General case. *Nonlinear Analysis* **125** (2015), 582–607.

[39] V.V. Zhikov and S.E. Pastukhova, Homogenization of elasticity problems on periodic grids of critical thickness, *Matem. Sbornik* **194** (2003) 61–96.