Exact Computation of Minimum Sample size for Estimation of Poisson Parameters *

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Abstract

In this paper, we develop an approach for the exact determination of the minimum sample size for the estimation of a Poisson parameter with prescribed margin of error and confidence level. The exact computation is made possible by reducing infinite many evaluations of coverage probability to finite many evaluations. Such reduction is based on our discovery that the minimum of coverage probability with respect to a Poisson parameter bounded in an interval is attained at a discrete set of finite many values.

1 Introduction

The estimation of a Poisson parameter finds numerous applications in various fields of sciences and engineering [3]. The problem is formulated as follows.

Let \( X \) be a Poisson random variable defined in a probability space \((\Omega, \mathcal{F}, \Pr)\) such that \( \Pr\{X = k\} = \frac{\lambda^k e^{-\lambda}}{k!}, \ k = 0, 1, \cdots \), where \( \lambda > 0 \) is referred to as a Poisson parameter. It is a frequent problem to estimate \( \lambda \) based on \( n \) identical and independent samples \( X_1, \cdots, X_n \) of \( X \).

An estimate of \( \lambda \) is conventionally taken as \( \hat{\lambda}_n = \frac{\sum_{i=1}^{n} X_i}{n} \). The nice property of such estimate is that it is of maximum likely-hood and possesses minimum variance among all unbiased estimates.

A crucial question in the estimation is as follows:

Given the knowledge that \( \lambda \) belongs to interval \([a, b]\), what is the minimum sample size \( n \) that guarantees the difference between \( \hat{\lambda}_n \) and \( \lambda \) be bounded within some prescribed margin of error with a confidence level higher than a prescribed value?

The main contribution of this paper is to provide exact answer to this important question. The paper is organized as follows. In Section 2, the techniques for computing the minimum sample size is developed with the margin of error taken as a bound of absolute error. In Section 3, we

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derive corresponding sample size method by using relative error bound as the margin of error. In Section 4, we develop techniques for computing minimum sample size with a mixed error criterion. Section 5 is the conclusion. The proofs are given in Appendices.

Throughout this paper, we shall use the following notations. The set of integers is denoted by $\mathbb{Z}$. The ceiling function and floor function are denoted respectively by $\lceil . \rceil$ and $\lfloor . \rfloor$ (i.e., $\lceil x \rceil$ represents the smallest integer no less than $x$; $\lfloor x \rfloor$ represents the largest integer no greater than $x$). The multivariate function $S(n,k,l,\lambda) = \sum_{i=k}^{l} \frac{\lambda^i e^{-\lambda}}{i!}$. The left limit as $\eta$ tends to 0 is denoted as $\lim_{\eta \to 0}$. The other notations will be made clear as we proceed.

2 Control of Absolute Error

Let $\varepsilon \in (0,1)$ be the margin of absolute error and $\delta \in (0,1)$ be the confidence parameter. In many applications, it is desirable to find the minimum sample size $n$ such that

$$\Pr\{ |\hat{\lambda}_n - \lambda| < \varepsilon \} > 1 - \delta$$

for any $\lambda \in [a,b]$. Here $\Pr\{ |\hat{\lambda}_n - \lambda| < \varepsilon \}$ is referred to as the coverage probability. The interval $[a,b]$ is introduced to take into account the knowledge of $\lambda$. The exact determination of minimum sample size is readily tractable with modern computational power by taking advantage of the behavior of the coverage probability characterized by Theorem 1 as follows.

**Theorem 1** Let $0 < \varepsilon < 1$ and $0 \leq a < b$. Let $X_1, \cdots, X_n$ be identical and independent Poisson random variables with mean $\lambda \in [a,b]$. Let $\hat{\lambda}_n = \frac{\sum_{i=1}^{n} X_i}{n}$. Then, the minimum of $\Pr\{ |\hat{\lambda}_n - \lambda| < \varepsilon \}$ with respect to $\lambda \in [a,b]$ is achieved at the finite set $\{a, b\} \cup \{k + \varepsilon \in (a,b) : k \in \mathbb{Z}\} \cup \{k - \varepsilon \in (a,b) : k \in \mathbb{Z}\}$, which has less than $2n(b-a) + 4$ elements.

See Appendix A for a proof. The application of Theorem 1 in the computation of minimum sample size is obvious. For a fixed sample size $n$, since the minimum of coverage probability with $\lambda \in [a,b]$ is attained at a finite set, it can be determined by a computer whether the sample size $n$ is large enough to ensure $\Pr\{ |\hat{\lambda}_n - \lambda| < \varepsilon \} > 1 - \delta$ for any $\lambda \in [a,b]$. Starting from $n = 2$, one can find the minimum sample size by gradually incrementing $n$ and checking whether $n$ is large enough.

3 Control of Relative Error

Let $\varepsilon \in (0,1)$ be the margin of relative error and $\delta \in (0,1)$ be the confidence parameter. It is interesting to determine the minimum sample size $n$ so that

$$\Pr\left\{ \left| \frac{\hat{\lambda}_n - \lambda}{\lambda} \right| < \varepsilon \right\} > 1 - \delta$$
for any $\lambda \in [a, b]$. As has been pointed out in Section 2, an essential machinery is to reduce infinite many evaluations of the coverage probability $\Pr\{|\hat{\lambda}_n - \lambda| < \varepsilon\}$ to finite many evaluations. Such reduction can be accomplished by making use of Theorem 2 as follows.

**Theorem 2** Let $0 < \varepsilon < 1$ and $0 < a < b$. Let $X_1, \ldots, X_n$ be identical and independent Poisson random variables with mean $\lambda \in [a, b]$. Let $\hat{\lambda}_n = \frac{\sum_{i=1}^{n} X_i}{n}$. Then, the minimum of $\Pr\left\{|\hat{\lambda}_n - \lambda| < \varepsilon\right\}$ with respect to $\lambda \in [a, b]$ is achieved at the finite set $\{a, b\} \cup \left\{\frac{\ell}{n} \in (a, b) : \ell \in \mathbb{Z}\right\}$, which has less than $2n(b - a) + 4$ elements.

See Appendix B for a proof.

### 4 Control of Absolute Error or Relative Error

Let $\varepsilon_a \in (0, 1)$ and $\varepsilon_r \in (0, 1)$ be respectively the margins of absolute error and relative error. Let $\delta \in (0, 1)$ be the confidence parameter. In many situations, it is desirable to find the smallest sample size $n$ such that

$$\Pr\left\{|\hat{\lambda}_n - \lambda| < \varepsilon_a \text{ or } \frac{|\hat{\lambda}_n - \lambda|}{\lambda} < \varepsilon_r\right\} > 1 - \delta$$

(1)

for any $\lambda \in [a, b]$. To make it possible to compute exactly the minimum sample size associated with (1), we have Theorem 3 as follows.

**Theorem 3** Let $0 < \varepsilon_a < 1$, $0 < \varepsilon_r < 1$ and $0 \leq a < \frac{\varepsilon_a}{\varepsilon_r} < b$. Let $X_1, \ldots, X_n$ be identical and independent Poisson random variables with mean $\lambda \in [a, b]$. Let $\hat{\lambda}_n = \frac{\sum_{i=1}^{n} X_i}{n}$. Then, the minimum of $\Pr\left\{|\hat{\lambda}_n - \lambda| < \varepsilon_a \text{ or } \frac{|\hat{\lambda}_n - \lambda|}{\lambda} < \varepsilon_r\right\}$ with respect to $\lambda \in [a, b]$ is achieved at the finite set $\{a, b\} \cup \left\{\frac{\ell}{n} + \varepsilon_a \in (a, \frac{\varepsilon_a}{\varepsilon_r}) : \ell \in \mathbb{Z}\right\} \cup \left\{\frac{\ell}{n} - \varepsilon_a \in (\frac{\varepsilon_a}{\varepsilon_r}, b) : \ell \in \mathbb{Z}\right\} \cup \left\{\frac{\ell}{n(1+\varepsilon_r)} \in (\frac{\varepsilon_a}{\varepsilon_r}, b) : \ell \in \mathbb{Z}\right\}$, which has less than $2n(b - a) + 7$ elements.

Theorem 3 can be shown by applying Theorem 1 and Theorem 2 with the observation that

$$\Pr\left\{|\hat{\lambda}_n - \lambda| < \varepsilon_a \text{ or } \frac{|\hat{\lambda}_n - \lambda|}{\lambda} < \varepsilon_r\right\} = \begin{cases} \Pr\left\{|\hat{\lambda}_n - \lambda| < \varepsilon_a\right\} & \text{for } \lambda \in \left[a, \frac{\varepsilon_a}{\varepsilon_r}\right], \\ \Pr\left\{\frac{|\hat{\lambda}_n - \lambda|}{\lambda} < \varepsilon_r\right\} & \text{for } \lambda \in \left(\frac{\varepsilon_a}{\varepsilon_r}, b\right]. \end{cases}$$

By virtue of Chernoff bounds, it can be shown that, for any $\varepsilon \in (0, 1)$,

$$\Pr\{\hat{\lambda}_n \leq (1 - \varepsilon)\lambda\} < \exp\left(-\frac{\lambda n \varepsilon^2}{2}\right),$$

$$\Pr\{\hat{\lambda}_n \geq (1 + \varepsilon)\lambda\} < \exp\left(-(2\ln 2 - 1)\lambda n \varepsilon^2\right).$$
As a result, \( \Pr\{ |\widehat{\lambda}_n - \lambda| > \varepsilon \lambda \} < \delta \) if
\[
\lambda > \frac{\ln \frac{2}{\delta}}{(2 \ln 2 - 1)n\varepsilon^2}.
\]

Therefore, to check whether (1) is satisfied for any \( \lambda \in [a, b] \), it suffices to check (1) for
\[
a \leq \lambda \leq \min \left\{ b, \frac{\ln \frac{2}{\delta}}{(2 \ln 2 - 1)n\varepsilon^2} \right\}.
\]

Finally, we would like to point out that similar characteristics of the coverage probability can be shown for the problem of estimating binomial parameter or the proportion of finite population, which allows for the exact computation of minimum sample size. For details, see our recent papers [1, 2].

5 Conclusion

We have developed an exact method for the computation of minimum sample size for the estimation of Poisson parameters, which only requires finite many evaluations of the coverage probability. Our sample size method permits rigorous control of statistical sampling error.

A Proof of Theorem 1

Define \( K = \sum_{i=1}^{n} X_i \) and
\[
C(\lambda) = \Pr \left\{ \left| \frac{K}{n} - \lambda \right| < \varepsilon \right\} = \Pr \{ g(\lambda) \leq K \leq h(\lambda) \}
\]
where
\[
g(\lambda) = \max(0, \lfloor n(\lambda - \varepsilon) \rfloor + 1), \quad h(\lambda) = \lceil n(\lambda + \varepsilon) \rceil - 1.
\]

It should be noted that \( C(\lambda) \), \( g(\lambda) \) and \( h(\lambda) \) are actually multivariate functions of \( \lambda \), \( \varepsilon \) and \( n \). For simplicity of notations, we drop the arguments \( n \) and \( \varepsilon \) throughout the proof of Theorem 1.

We need some preliminary results.

Lemma 1 Let \( \lambda_{\ell} = \frac{\ell}{n} - \varepsilon \) where \( \ell \in \mathbb{Z} \). Then, \( h(\lambda) = h(\lambda_{\ell+1}) = \ell \) for any \( \lambda \in (\lambda_{\ell}, \lambda_{\ell+1}) \).

Proof. For \( \lambda \in (\lambda_{\ell}, \lambda_{\ell+1}) \), we have \( 0 < n(\lambda - \lambda_{\ell}) < 1 \) and
\[
h(\lambda) = \lfloor n(\lambda + \varepsilon) \rfloor - 1
\]
\[
= \lfloor n(\lambda_{\ell} + \varepsilon + \lambda - \lambda_{\ell}) \rfloor - 1
\]
\[
= \lfloor n \left( \frac{\ell}{n} - \varepsilon + \varepsilon + \lambda - \lambda_{\ell} \right) \rfloor - 1
\]
\[
= \ell - 1 + \lfloor n(\lambda - \lambda_{\ell}) \rfloor
\]
\[
= \ell
\]
\[
= \lfloor n \left( \frac{\ell + 1}{n} - \varepsilon + \varepsilon \right) \rfloor - 1 = h(\lambda_{\ell+1}).
\]
Lemma 2 Let \( \lambda = \frac{\ell}{n} + \varepsilon \) where \( \ell \in \mathbb{Z} \). Then, \( g(\lambda) = g(\lambda \ell) = \max\{0, \ell + 1\} \) for any \( \lambda \in (\lambda \ell, \lambda \ell + 1) \).

Proof. For \( \lambda \in (\lambda \ell, \lambda \ell + 1) \), we have 
\[
g(\lambda) = \max(0, \lfloor n(\lambda - \varepsilon) \rfloor + 1) = \max(0, \lfloor n(\lambda - \varepsilon + \lambda - \lambda \ell + 1) \rfloor + 1) = \max(0, \lfloor n(\lambda - \lambda \ell + 1) \rfloor + 1 - 1 + 1) = \max(0, \ell + 1) = \max(0, \lfloor n(\frac{\ell}{n} + \varepsilon - \varepsilon) \rfloor + 1) = g(\lambda \ell).
\]

Lemma 3 Let \( \alpha < \beta \) be two consecutive elements of the ascending arrangement of all distinct elements of \( \{a, b\} \cup \{\frac{\ell}{n} + \varepsilon \in (a, b) : \ell \in \mathbb{Z}\} \cup \{\frac{\ell}{n} - \varepsilon \in (a, b) : \ell \in \mathbb{Z}\} \). Then, both \( g(\lambda) \) and \( h(\lambda) \) are constants for any \( \lambda \in (\alpha, \beta) \).

Proof. Since \( \alpha \) and \( \beta \) are two consecutive elements of the ascending arrangement of all distinct elements of the set, it must be true that there is no integer \( \ell \) such that \( \alpha < \frac{\ell}{n} + \varepsilon < \beta \) or \( \alpha < \frac{\ell}{n} - \varepsilon < \beta \). It follows that there exist two integers \( \ell \) and \( \ell' \) such that \( (\alpha, \beta) \subseteq \left(\frac{\ell}{n} + \varepsilon, \frac{\ell + 1}{n} + \varepsilon\right) \) and \( (\alpha, \beta) \subseteq \left(\frac{\ell'}{n} - \varepsilon, \frac{\ell' + 1}{n} - \varepsilon\right) \). Applying Lemma 1 and Lemma 2 we have \( g(\lambda) = g(\lambda) \) and \( h(\lambda) = h\left(\frac{\ell' + 1}{n} - \varepsilon\right) \) for any \( \lambda \in (\alpha, \beta) \).

Lemma 4 For any \( \lambda \in (0, 1) \), \( \lim_{\eta \downarrow 0} C(\lambda + \eta) \geq C(\lambda) \) and \( \lim_{\eta \downarrow 0} C(\lambda - \eta) \geq C(\lambda) \).

Proof. Observing that \( h(\lambda + \eta) \geq h(\lambda) \) for any \( \eta > 0 \) and that 
\[
g(\lambda + \eta) = \max(0, \lfloor n(\lambda + \eta - \varepsilon) \rfloor + 1) = \max(0, \lfloor n(\lambda - \varepsilon) \rfloor + 1 + \lfloor n(\lambda - \varepsilon) - \lfloor n(\lambda - \varepsilon) \rfloor + n\eta \rfloor) = \max(0, \lfloor n(\lambda - \varepsilon) \rfloor + 1) = g(\lambda)
\]
for \( 0 < \eta < \frac{1 + \lfloor n(\lambda - \varepsilon) \rfloor - n(\lambda - \varepsilon)}{n} \), we have
\[
S(n, g(\lambda + \eta), h(\lambda + \eta), \lambda + \eta) \geq S(n, g(\lambda), h(\lambda), \lambda + \eta) \tag{2}
\]
for $0 < \eta < \frac{1+|n(\lambda-\varepsilon)|-n(\lambda-\varepsilon)}{n}$. Since
\[
h(\lambda + \eta) = \lfloor n(\lambda + \eta + \varepsilon) \rfloor - 1 = \lfloor n(\lambda + \varepsilon) \rfloor - 1 + \lfloor n(\lambda + \varepsilon) - \lfloor n(\lambda + \varepsilon) \rfloor \rfloor + n\eta,
\]
we have
\[
h(\lambda + \eta) = \begin{cases} 
\lfloor n(\lambda + \varepsilon) \rfloor & \text{for } n(\lambda + \varepsilon) = \lfloor n(\lambda + \varepsilon) \rfloor \text{ and } 0 < \eta < \frac{1}{n}, \\
\lfloor n(\lambda + \varepsilon) \rfloor - 1 & \text{for } n(\lambda + \varepsilon) \neq \lfloor n(\lambda + \varepsilon) \rfloor \text{ and } 0 < \eta < \frac{1+|n(\lambda+\varepsilon)|-n(\lambda+\varepsilon)}{n}.
\end{cases}
\]
It follows that both $g(\lambda + \eta)$ and $h(\lambda + \eta)$ are independent of $\eta$ if $\eta > 0$ is small enough. Since $S(n, g, h, \lambda + \eta)$ is continuous with respect to $\eta$ for fixed $g$ and $h$, we have that $\lim_{\eta \to 0} S(n, g(\lambda + \eta), h(\lambda + \eta), \lambda + \eta)$ exists. As a result,
\[
\lim_{\eta \to 0} C(\lambda + \eta) = \lim_{\eta \to 0} S(n, g(\lambda + \eta), h(\lambda + \eta), \lambda + \eta) \geq \lim_{\eta \to 0} S(n, g(\lambda), h(\lambda), \lambda + \eta) = S(n, g(\lambda), h(\lambda), \lambda) = C(\lambda),
\]
where the inequality follows from (2).

Observing that $g(\lambda - \eta) \leq g(\lambda)$ for any $\eta > 0$ and that
\[
h(\lambda - \eta) = \lfloor n(\lambda - \eta + \varepsilon) \rfloor - 1 = \lfloor n(\lambda + \varepsilon) \rfloor - 1 + \lfloor n(\lambda + \varepsilon) - \lfloor n(\lambda + \varepsilon) \rfloor \rfloor + n\eta = \lfloor n(\lambda + \varepsilon) \rfloor - 1 = h(\lambda)
\]
for $0 < \eta < \frac{1+|n(\lambda+\varepsilon)|-\lfloor n(\lambda+\varepsilon) \rfloor}{n}$, we have
\[
S(n, g(\lambda - \eta), h(\lambda - \eta), \lambda - \eta) \geq S(n, g(\lambda), h(\lambda), \lambda - \eta)
\]
for $0 < \eta < \min \left\{ \lambda, \frac{1+|n(\lambda+\varepsilon)|-\lfloor n(\lambda+\varepsilon) \rfloor}{n} \right\}$. Since
\[
g(\lambda - \eta) = \max(0, \lfloor n(\lambda - \eta - \varepsilon) \rfloor + 1) = \max(0, \lfloor n(\lambda - \varepsilon) \rfloor + 1) = \max(0, \lfloor n(\lambda - \varepsilon) \rfloor + 1 + \lfloor n(\lambda - \varepsilon) - \lfloor n(\lambda - \varepsilon) \rfloor \rfloor - n\eta),
\]
we have
\[
g(\lambda - \eta) = \begin{cases} 
\max(0, \lfloor n(\lambda - \varepsilon) \rfloor) & \text{for } n(\lambda - \varepsilon) = \lfloor n(\lambda - \varepsilon) \rfloor \text{ and } 0 < \eta < \frac{1}{n}, \\
\max(0, \lfloor n(\lambda - \varepsilon) \rfloor + 1) & \text{for } n(\lambda - \varepsilon) \neq \lfloor n(\lambda - \varepsilon) \rfloor \text{ and } 0 < \eta < \frac{1+|n(\lambda-\varepsilon)|-\lfloor n(\lambda-\varepsilon) \rfloor}{n}.
\end{cases}
\]
It follows that both $g(\lambda - \eta)$ and $h(\lambda - \eta)$ are independent of $\eta$ if $\eta > 0$ is small enough. Since $S(n, g, h, \lambda - \eta)$ is continuous with respect to $\eta$ for fixed $g$ and $h$, we have that $\lim_{\eta \to 0} S(n, g(\lambda - \eta), h(\lambda - \eta), \lambda - \eta)$ exists. Hence,
\[
\lim_{\eta \to 0} C(\lambda - \eta) = \lim_{\eta \to 0} S(n, g(\lambda - \eta), h(\lambda - \eta), \lambda - \eta) \geq \lim_{\eta \to 0} S(n, g(\lambda), h(\lambda), \lambda - \eta) = S(n, g(\lambda), h(\lambda), \lambda) = C(\lambda),
\]
where the inequality follows from (3).
Lemma 5 Let $\alpha < \beta$ be two consecutive elements of the ascending arrangement of all distinct elements of $\{a, b\} \cup \{\frac{\ell}{n} + \varepsilon \in (a, b) : \ell \in \mathbb{Z}\} \cup \{\frac{\ell}{n} - \varepsilon \in (a, b) : \ell \in \mathbb{Z}\}$. Then, $C(\lambda) \geq \min\{C(\alpha), C(\beta)\}$ for any $\lambda \in (\alpha, \beta)$.

Proof. By Lemma 5 both $g(\lambda)$ and $h(\lambda)$ are constants for any $\lambda \in (\alpha, \beta)$. Hence, we can drop the argument and write $g(\lambda) = g$, $h(\lambda) = h$ and $C(\lambda) = S(n, g, h, \lambda)$.

For $\lambda \in (\alpha, \beta)$, define interval $[\alpha + \eta, \beta - \eta]$ with $0 < \eta < \min\{\lambda - \alpha, \beta - \lambda, \frac{\beta - \alpha}{2}\}$. Then, $C(\lambda) \geq \min_{\mu \in [\alpha + \eta, \beta - \eta]} C(\mu)$. Note that $\frac{\partial S(n, g, h, \lambda)}{\partial \lambda} = -\frac{\lambda e^{-\lambda}}{h}$ and thus, for $g > 0$,

$$\frac{\partial S(n, g, h, \lambda)}{\partial \lambda} = \frac{\partial S(n, 0, h, \lambda)}{\partial \lambda} - \frac{\partial S(n, 0, g - 1, \lambda)}{\partial \lambda} = \frac{\lambda^{g-1} e^{-\lambda}}{(g-1)!} - \frac{\lambda^{h} e^{-\lambda}}{h!} = \left[\frac{h!}{(g-1)!} - \lambda^{h-g+1}\right] \frac{\lambda^{g-1} e^{-\lambda}}{h!} > 0$$

if $\lambda < \left[\frac{h!}{(g-1)!}\right]^{-\frac{1}{g-1}}$. From such investigation of the derivative of $S(n, g, h, \lambda)$ with respective to $\lambda$, we can see that, for $0 < \eta < \min\{\lambda - \alpha, \beta - \lambda, \frac{\beta - \alpha}{2}\}$, one of the following three cases must be true: (1) $C(\mu)$ decreases monotonically for $\mu \in [\alpha + \eta, \beta - \eta]$; (2) $C(\mu)$ increases monotonically for $\mu \in [\alpha + \eta, \beta - \eta]$; (3) there exists a number $\theta \in (\alpha + \eta, \beta - \eta)$ such that $C(\mu)$ increases monotonically for $\mu \in [\alpha + \eta, \theta]$ and decreases monotonically for $\mu \in (\theta, \beta - \eta)$. It follows that

$$C(\lambda) \geq \min_{\mu \in [\alpha + \eta, \beta - \eta]} C(\mu) = \min\{C(\alpha + \eta), C(\beta - \eta)\}$$

for $0 < \eta < \min\{\lambda - \alpha, \beta - \lambda, \frac{\beta - \alpha}{2}\}$. By Lemma 4 both $\lim_{\eta \uparrow 0} C(\alpha + \eta)$ and $\lim_{\eta \uparrow 0} C(\beta - \eta)$ exist and

$$C(\lambda) \geq \lim_{\eta \uparrow 0} \min\{C(\alpha + \eta), C(\beta - \eta)\} = \min \left\{ \lim_{\eta \uparrow 0} C(\alpha + \eta), \lim_{\eta \uparrow 0} C(\beta - \eta) \right\} \geq \min\{C(\alpha), C(\beta)\}$$

for any $\lambda \in (\alpha, \beta)$.

Finally, to show Theorem 1 note that the statement about the coverage probability follows immediately from Lemma 5. The number of elements of the finite set can be calculated by using the property of the ceiling and floor functions.

B Proof of Theorem 2

Define

$$C(\lambda) = \Pr\left\{\left|\frac{K}{n} - \lambda\right| < \varepsilon\lambda\right\} = \Pr\{g(\lambda) \leq K \leq h(\lambda)\}$$
where
\[ g(\lambda) = \left\lfloor n\lambda(1 - \varepsilon) \right\rfloor + 1, \quad h(\lambda) = \left\lceil n\lambda(1 + \varepsilon) \right\rceil - 1. \]

It should be noted that \( C(\lambda), g(\lambda) \) and \( h(\lambda) \) are actually multivariate functions of \( \lambda, \varepsilon \) and \( n \). For simplicity of notations, we drop the arguments \( n \) and \( \varepsilon \) throughout the proof of Theorem 2.

We need some preliminary results.

**Lemma 6** Let \( \lambda_\ell = \frac{\ell}{n(1+\varepsilon)} \) where \( \ell \in \mathbb{Z} \). Then, \( h(\lambda) = h(\lambda_{\ell+1}) = \ell \) for any \( \lambda \in (\lambda_\ell, \lambda_{\ell+1}) \).

**Proof.** For \( \lambda \in (\lambda_\ell, \lambda_{\ell+1}) \), we have \( 0 < n(1+\varepsilon)(\lambda - \lambda_\ell) < 1 \) and
\[
\begin{align*}
h(\lambda) &= \left\lceil n\lambda(1 + \varepsilon) \right\rceil - 1 \\
&= \left\lceil n\lambda_\ell(1 + \varepsilon) + (1 + \varepsilon)(\lambda - \lambda_\ell) \right\rceil - 1 \\
&= \left\lceil \frac{\ell}{n} + (1 + \varepsilon)(\lambda - \lambda_\ell) \right\rceil - 1 \\
&= \ell - 1 + \left\lceil (n(1+\varepsilon)(\lambda - \lambda_\ell) \right\rceil \\
&= \ell = h(\lambda_{\ell+1}).
\end{align*}
\]

**Lemma 7** Let \( \lambda_\ell = \frac{\ell}{n(1-\varepsilon)} \) where \( \ell \in \mathbb{Z} \). Then, \( g(\lambda) = g(\lambda_\ell) = \ell + 1 \) for any \( \lambda \in (\lambda_\ell, \lambda_{\ell+1}) \).

**Proof.** For \( \lambda \in (\lambda_\ell, \lambda_{\ell+1}) \), we have \( -1 < n(1-\varepsilon)(\lambda - \lambda_{\ell+1}) < 0 \) and
\[
\begin{align*}
g(\lambda) &= \left\lfloor n\lambda(1 - \varepsilon) \right\rfloor + 1 \\
&= \left\lfloor n[\lambda_{\ell+1}(1 - \varepsilon) + (1 - \varepsilon)(\lambda - \lambda_{\ell+1})] \right\rfloor + 1 \\
&= \left\lceil n \times \frac{\ell + 1}{n(1 - \varepsilon)} \times (1 - \varepsilon) \right\rceil + [n(1 - \varepsilon)(\lambda - \lambda_{\ell+1})] + 1 \\
&= \left\lceil n \times \frac{\ell + 1}{n(1 - \varepsilon)} \times (1 - \varepsilon) \right\rceil - 1 + 1 \\
&= \ell + 1 = g(\lambda_\ell).
\end{align*}
\]

**Lemma 8** Let \( \alpha < \beta \) be two consecutive elements of the ascending arrangement of all distinct elements of \( \{a, b\} \cup \{\frac{\ell}{n(1-\varepsilon)} \in (a, b) : \ell \in \mathbb{Z}\} \cup \{\frac{\ell}{n(1+\varepsilon)} \in (a, b) : \ell \in \mathbb{Z}\} \). Then, both \( g(\lambda) \) and \( h(\lambda) \) are constants for any \( \lambda \in (\alpha, \beta) \).
Proof. Since $\alpha$ and $\beta$ are two consecutive elements of the ascending arrangement of all distinct elements of the set, it must be true that there is no integer $\ell$ such that $\alpha < \frac{\ell}{n(1+\varepsilon)} < \beta$ or $\alpha < \frac{\ell}{n(1+\varepsilon)} < \beta$. It follows that there exist two integers $\ell$ and $\ell'$ such that $(\alpha, \beta) \subseteq \left(\frac{\ell}{n(1+\varepsilon)}, \frac{\ell'+1}{n(1+\varepsilon)}\right)$ and $(\alpha, \beta) \subseteq \left(\frac{\ell'}{n(1+\varepsilon)}, \frac{\ell'+1}{n(1+\varepsilon)}\right)$. Applying Lemma 9 and Lemma 7 we have $g(\lambda) = g\left(\frac{\ell}{n(1+\varepsilon)}\right)$ and $h(\lambda) = h\left(\frac{\ell'+1}{n(1+\varepsilon)}\right)$ for any $\lambda \in (\alpha, \beta)$.

\[\square\]

Lemma 9 For any $\lambda \in (0, 1)$, $\lim_{\eta \downarrow 0} C(\lambda + \eta) \geq C(\lambda)$ and $\lim_{\eta \downarrow 0} C(\lambda - \eta) \geq C(\lambda)$.

Proof. Observing that $h(\lambda + \eta) \geq h(\lambda)$ for any $\eta > 0$ and that
\[
g(\lambda + \eta) = \lfloor n(\lambda + \eta)(1 - \varepsilon) \rfloor + 1 = \lfloor n\lambda(1 - \varepsilon) \rfloor + 1 + \lfloor n\lambda(1 - \varepsilon) - \lfloor n\lambda(1 - \varepsilon) \rfloor + \eta(1 - \varepsilon) \rfloor = \lfloor n\lambda(1 - \varepsilon) \rfloor + 1 = g(\lambda)
\]
for $0 < \eta < \frac{1 + [n\lambda(1 - \varepsilon)] - n\lambda(1 - \varepsilon)}{n(1 - \varepsilon)}$, we have
\[
S(n, g(\lambda + \eta), h(\lambda + \eta), \lambda + \eta) \geq S(n, g(\lambda), h(\lambda), \lambda + \eta)
\]
for $0 < \eta < \frac{1 + [n\lambda(1 - \varepsilon)] - n\lambda(1 - \varepsilon)}{n(1 - \varepsilon)}$. Since
\[
h(\lambda + \eta) = \begin{cases} 
\lfloor n\lambda(1 + \varepsilon) \rfloor & \text{for } n\lambda(1 + \varepsilon) = \lfloor n\lambda(1 + \varepsilon) \rfloor \text{ and } 0 < \eta < \frac{1}{n(1 + \varepsilon)}, \\
\lfloor n\lambda(1 + \varepsilon) \rfloor - 1 & \text{for } n\lambda(1 + \varepsilon) \neq \lfloor n\lambda(1 + \varepsilon) \rfloor \text{ and } 0 < \eta < \frac{[n\lambda(1 + \varepsilon)] - n\lambda(1 + \varepsilon)}{n(1 + \varepsilon)}.
\end{cases}
\]
It follows that both $g(\lambda + \eta)$ and $h(\lambda + \eta)$ are independent of $\eta$ if $\eta > 0$ is small enough. Since $S(n, g, h, \lambda + \eta)$ is continuous with respect to $\eta$ for fixed $g$ and $h$, we have that $\lim_{\eta \downarrow 0} S(n, g(\lambda + \eta), h(\lambda + \eta), \lambda + \eta)$ exists. As a result,
\[
\lim_{\eta \downarrow 0} C(\lambda + \eta) = \lim_{\eta \downarrow 0} S(n, g(\lambda + \eta), h(\lambda + \eta), \lambda + \eta)
\]
\[
\geq \lim_{\eta \downarrow 0} S(n, g(\lambda), h(\lambda), \lambda + \eta) = S(n, g(\lambda), h(\lambda), \lambda) = C(\lambda),
\]
where the inequality follows from (4).

Observing that $g(\lambda - \eta) \leq g(\lambda)$ for any $\eta > 0$ and that
\[
h(\lambda - \eta) = \lfloor n(\lambda - \eta)(1 + \varepsilon) \rfloor - 1 = \lfloor n\lambda(1 + \varepsilon) \rfloor - 1 + \lfloor n\lambda(1 + \varepsilon) - \lfloor n\lambda(1 + \varepsilon) \rfloor - \eta(1 + \varepsilon) \rfloor = \lfloor n\lambda(1 + \varepsilon) \rfloor - 1 = h(\lambda)
\]
for $0 < \eta < \frac{1 + n\lambda(1 + \varepsilon) - \lceil n\lambda(1 + \varepsilon) \rceil}{n(1 + \varepsilon)}$, we have

$$S(n, g(\lambda - \eta), h(\lambda - \eta), \lambda - \eta) \geq S(n, g(\lambda), h(\lambda), \lambda - \eta)$$ (5)

for $0 < \eta < \min \left\{ \lambda, \frac{1 + n\lambda(1 + \varepsilon) - \lceil n\lambda(1 + \varepsilon) \rceil}{n(1 + \varepsilon)} \right\}$. Since

$$g(\lambda - \eta) = \left\lfloor n\lambda(1 - \varepsilon) \right\rfloor + 1$$
$$= \left\lfloor n\lambda(1 - \varepsilon) \right\rfloor + 1 + \left\lfloor n\lambda(1 - \varepsilon) - \left\lfloor n\lambda(1 - \varepsilon) \right\rfloor \eta(1 - \varepsilon) \right\rfloor,$$

we have

$$g(\lambda - \eta) = \begin{cases} 
\left\lfloor n\lambda(1 - \varepsilon) \right\rfloor & \text{for } n\lambda(1 - \varepsilon) = \left\lfloor n\lambda(1 - \varepsilon) \right\rfloor \text{ and } 0 < \eta < \frac{1}{n(1 - \varepsilon)}, \\
\left\lfloor n\lambda(1 - \varepsilon) \right\rfloor + 1 & \text{for } n\lambda(1 - \varepsilon) \neq \left\lfloor n\lambda(1 - \varepsilon) \right\rfloor \text{ and } 0 < \eta < \frac{n\lambda(1 - \varepsilon) - \left\lfloor n\lambda(1 - \varepsilon) \right\rfloor}{n(1 - \varepsilon)}. 
\end{cases}$$

It follows that both $g(\lambda - \eta)$ and $h(\lambda - \eta)$ are independent of $\eta$ if $\eta > 0$ is small enough. Since $S(n, g, h, \lambda - \eta)$ is continuous with respect to $\eta$ for fixed $g$ and $h$, we have that $\lim_{\eta \downarrow 0} S(n, g(\lambda - \eta), h(\lambda - \eta), \lambda - \eta)$ exists. Hence,

$$\lim_{\eta \downarrow 0} C(\lambda - \eta) = \lim_{\eta \downarrow 0} S(n, g(\lambda - \eta), h(\lambda - \eta), \lambda - \eta) \geq \lim_{\eta \downarrow 0} S(n, g(\lambda), h(\lambda), \lambda - \eta) = S(n, g(\lambda), h(\lambda), \lambda) = C(\lambda),$$

where the inequality follows from (5).

By a similar argument as that of Lemma 6, we have

**Lemma 10** Let $\alpha < \beta$ be two consecutive elements of the ascending arrangement of all distinct elements of $\{a, b\} \cup \{\frac{\ell}{n(1 - \varepsilon)} \in (a, b) : \ell \in \mathbb{Z}\} \cup \{\frac{\ell}{n(1 + \varepsilon)} \in (a, b) : \ell \in \mathbb{Z}\}$. Then, $C(\lambda) \geq \min\{C(\alpha), C(\beta)\}$ for any $\lambda \in (\alpha, \beta)$.

Finally, to show Theorem 2 note that the statement about the coverage probability follows immediately from Lemma 10. The number of elements of the finite set can be calculated by using the property of the ceiling and floor functions.

**References**

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