Associativity Equations in Effective SUSY Quantum Field Theories

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The role of associativity or WDVV equations in effective supersymmetric quantum theories is discussed and it is demonstrated that for wide class of their solutions when residue formulas are valid the proof of associativity equations can be reduced to solving the system of ordinary linear equations and depends only upon corresponding matching and nondegeneracy conditions. The covariance of WDVV equations upon generic duality transformations and the role of associativity equations in general context of quasiclassical integrable systems is also discussed.

1 Introduction: effective nonperturbative SUSY theories

More than 30 years have passed after pioneering work of Golfand and Likhtman [1] but supersymmetry (SUSY) is still among both most hot and intriguing topics of modern theoretical physics. Looking around one may distinguish two main directions of investigations of supersymmetry, the first is devoted to the search of the (signs of) SUSY in real nature and the second mostly concerns supersymmetry as a very effective theoretical laboratory for study mainly nonperturbative effects in quantum field theories and string theory. While the first direction is still supported only by belief that superpartners will be found sometimes on colliders, the second direction is based on more solid ground, since on pure theoretical level SUSY has become already a part of reality in physics. Indeed, for example, it is enough to require world-line (or world-sheet in string theory) supersymmetry in the first-quantized theory to get the space-time fermions from the space-time bosons.

The main topic of this talk is to discuss some strong (though very indirect) outcomes of SUSY for the effective nonperturbative quantum field and string theories. Our starting point is that, speaking in sigma-model terms, SUSY determines or strongly restricts the geometry of corresponding effective action, which (dependently on the number of SUSY generators) necessarily possesses complex (Kähler, special Kähler, hyper-Kähler etc) structure. This is pure kinematic fact and it is very important since it is preserved if one goes beyond perturbation theory where conventional quantum field theory does not work. Beyond weak coupling one may rely only upon some heuristic tools, like duality, which should be, of course, consistent with the properties of corresponding target-space complex geometry.

Hence, in what follows we almost forget about fermions and Lagrangians of SUSY gauge theories and will not distinguish between SUSY and corresponding geometry it induces on moduli space. A possible way to formulate any nonperturbative statements comes from the relation between complex geometry and theory of integrable systems – nonlinear integrable differential equations. We will discuss one of the most intriguing examples of these equations which arises in the context of SUSY effective theories – the associativity or WDVV equations [2] and demonstrate in particular that they are indeed consistent with properties of corresponding geometry underlying SUSY gauge theories.

For moduli spaces of vector multiplets in $N = 2$ SUSY gauge theories this geometry is (a rigid analog of) special Kähler [3, 4]. It means that metric on moduli space is determined by a single holomorphic function $F(\mathbf{a})$ of several complex arguments (VEV of complex matrix scalar field). This function can be determined even non-perturbatively [4], i.e. beyond the scope of standard quantum gauge theory. It means that there should exist some other way to define nonperturbative effective theory and one of the possibilities to do this is to use

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the above relation between SUSY and geometry and connections between geometry of complex manifolds and theory of nonlinear integrable differential equations.

From this point of view it is very important that function $F$ satisfies some well-known nonlinear integrable differential equations [3] and, in particular, the set of associativity or WDVV equations which can be written in the form [4]

$$ F_i \cdot F_j^{-1} \cdot F_k = F_k \cdot F_j^{-1} \cdot F_i \quad \forall i, j, k , $$

for the matrices $\|F_i\|_{jk} \equiv F_{ijk}$ whose matrix elements are the third derivatives of the function $F(a)$,

$$ F_{ijk} = \frac{\partial^3 F}{\partial a_i \partial a_j \partial a_k} $$

(2)

Originally WDVV equations were found [2] in the context of two-dimensional topological ($\mathcal{N} = 2$ twisted superconformal) theory. However, it was shown later [4] that the WDVV equations are much more universal and can be extended at least to the $\mathcal{N} = 2$ SUSY gauge theories in four dimensions. Despite simplicity of their compact matrix form they form indeed an overdetermined system of highly nontrivial nonlinear differential equations satisfied by function $F$, apart from the case of functions of one or two variables, when equations [4] are empty, i.e. are satisfied by any function.

2 WDVV equations: associative algebra and residue formulas

In two-dimensional topological theories, for example $\mathcal{N} = 2$ SUSY Landau-Ginzburg models, which are defined by superpotential $W$ so that vacua are identified with $dW = 0$, the WDVV equations arose [2] as consequence of the crossing relations

$$ \sum_k C_{ij}^k C_{kl}^n = \sum_k C_{il}^k C_{kj}^n $$

(3)

for the structure constants of the operator algebra of primary or vacuum operators in topological (say, Landau-Ginzburg) model

$$ \phi_i \cdot \phi_j = \sum_k C_{ij}^k \phi_k $$

(4)

Equations [3] are algebraic relations and they turn into the system of nonlinear differential WDVV equations only upon identification of three-point functions of the operators $\{\phi_i\}$ with the third derivatives of some function $F(a_1, \ldots, a_n)$

$$ \langle \phi_i \phi_j \phi_k \rangle = \frac{\partial^3 F}{\partial a_i \partial a_j \partial a_k} $$

(5)

In the Landau-Ginzburg models algebra [4] is realized as ring of all polynomials modulo $dW = 0$ and formula (3) acquires the form of the residue formula [4, 8]

$$ F_{ijk} = \sum_{\alpha} \text{res}_{\lambda_\alpha} \frac{\phi_i(\lambda) \phi_j(\lambda) \phi_k(\lambda)}{W(\lambda)Q'(\lambda)} = \sum_{\alpha} \frac{\phi_i(\lambda_\alpha) \phi_j(\lambda_\alpha) \phi_k(\lambda_\alpha)}{W(\lambda_\alpha)Q'(\lambda_\alpha)} = \sum_{\alpha} \text{res}_{\lambda_\alpha} \frac{dH_i dH_j dH_k}{dW dQ} $$

(6)

For simplicity we have considered the case when superpotential is function of a single variable and denoted $\phi_i(\lambda) = \frac{dH_i}{d\lambda}, W'(\lambda) = \frac{dW}{d\lambda}$ and $Q'(\lambda) = \frac{dQ}{d\lambda}$. However, the properties of the formula (3) we are going to use below do not really depend on the number of variables, and universality of formula (3) goes far beyond the Landau-Ginzburg case.

It is necessary to stress that both algebra [4] and residue formula (3) are necessary for the validity of the WDVV equations [3]. In sect. 3 we demonstrate that imposing these two conditions is actually enough to prove (3) and almost nothing extra (except for nondegeneracy) should be added. In various models algebra [4] can be realized in different ways, but it can be always presented as algebra of functions (of course, not necessarily polynomials) modulo vanishing on some submanifold. If this submanifold as realized as a set of zeroes of some differential $dW = 0$ this algebra is finite and leads together with residue formula (3) (certainly with the same $dW$ in denominator) to the finite system of WDVV equations on function $F$ of finite number of variables. We also demonstrate in sect. 3 that when residue formula (3) is valid the proof of validity of the WDVV equations is reduced to the problem of solving the system of ordinary linear equations and for that all extra ingredients, common in the context of two-dimensional topological theories [3] (constancy of ”metric”, existence of unity operator etc) are absolutely inessential.

Finally in this section, let us make few remarks concerning appearence of the WDVV equations in general context of quasiclassical integrable hierarchies, i.e. beyond SUSY topological and Seiberg-Witten theories. It
turns out that at least in some cases they can be considered as a direct consequence of (infinite) dispersionless Hirota relations. On one hand from the point of view of associativity algebras this is a rather trivial statement since for infinite number of operators it is much more easy to construct a closed algebra. Nevertheless (see for details) it is possible to show, say in the known Landau-Ginzburg case, that the finite WDVV equations are consequence of these infinite ones and, moreover, some new solutions of the associativity equations can be obtained in this way.

3 WDVV as solving system of linear equations

Require now the differential \(dW(\lambda)\) in (10) to be 
meromorphic
with finite number of zeroes at some points \(\{\lambda_\alpha\}\)

\[
W'(\lambda_\alpha) = 0
\] (7)

and \(Q'(\lambda_\alpha) \neq 0\). The only extra condition one should impose below is that matrix \(\|\phi_i(\lambda_\alpha)\|\) is non-degenerate

\[
\det_{\alpha\beta} \|\phi_i(\lambda_\alpha)\| \neq 0
\] (8)

In particular, (8) requires "matching" \#(i) = \#(\alpha), i.e. the number of "hamiltonians" \(\{dH_i\}\) or "fields" \(\{\phi_i\}\) should be exactly equal to the number of zeroes \(\{\lambda_\alpha\}\). One may define now the structure constants \(C_{ij}^k\) of this finite-dimensional algebra (1) (where the sum is finite) from the system of linear equations

\[
\phi_i(\lambda_\alpha)\phi_j(\lambda_\alpha) = \sum_k C_{ij}^k \phi_k(\lambda_\alpha), \quad \forall \lambda_\alpha
\] (9)

which hold for all zeroes \(\{\lambda_\alpha\}\) of \(dW\). Formula (1) gives a realization of the finite-dimensional associative algebra (4) defined by any meromorphic differential \(dW\). Using matching and nondegeneracy conditions (8), one can simply solve the system (9) and write

\[
C_{ij}^k = \sum_\alpha \phi_i(\lambda_\alpha)\phi_j(\lambda_\alpha) (\phi_k(\lambda_\alpha))^{-1}
\] (10)

where the last factor means matrix inverse to \(\|\phi_i(\lambda_\alpha)\|\).

The situation does not change at all, if instead of (1) we consider an isomorphic algebra (1)

\[
\phi_i(\lambda_\alpha)\phi_j(\lambda_\alpha) = \sum_k C_{ij}^k(\xi)\phi_k(\lambda_\alpha) \cdot \xi(\lambda_\alpha), \quad \forall \lambda_\alpha
\] (11)

with the only requirement \(\xi(\lambda_\alpha) \neq 0\), for \(\forall \alpha\), and (11) is a particular case of (11) with \(\xi(\lambda) \equiv 1\). Then, instead of (10), one immediately gets

\[
C_{ij}^k(\xi) = \sum_\alpha \frac{\phi_i(\lambda_\alpha)\phi_j(\lambda_\alpha)}{\xi(\lambda_\alpha)} (\phi_k(\lambda_\alpha))^{-1}
\] (12)

When the algebra (11) leads to finite WDVV equations (1)? In order to get the answer one should check consistency between formulas (12) and (11), which has the form

\[
\mathcal{F}_{ijk} = \sum_l C_{ij}^l(\xi)\eta_{kl}(\xi)
\] (13)

and "metric" \(\eta_{kl}(\xi)\) (which depends upon \(\xi\) in order to cancel dependence of the structure constants) is non degenerate and satisfies

\[
\eta_{kl}(\xi) = \sum_\alpha \xi_\alpha\mathcal{F}_{k\lambda a}
\] (14)

where the third derivatives \(\mathcal{F}_{k\lambda a}\) are given by residue formula (11) and \(\{\xi_\alpha\}\) are some coefficients, which can even depend on times. Now, substituting (11) into (11) one gets

\[
\eta_{kl}(\xi) = \sum_\alpha \phi_k(\lambda_\alpha)\phi_l(\lambda_\alpha)\xi(\lambda_\alpha) W'(\lambda_\alpha) W'(\lambda_\alpha) = \sum_\alpha \phi_k(\lambda_\alpha)\phi_l(\lambda_\alpha)\xi(\lambda_\alpha) W''(\lambda_\alpha) Q'(\lambda_\alpha)
\] (15)

\footnote{The situation here is very similar to considered in (11) in the context of algebra of 1-differentials on Riemann surfaces. However, in contrast to algebra of forms, algebra of functions (11) is always associative.}
if $\xi(\lambda) = \sum_a \xi_a \phi_a(\lambda)$

Since we already required $\xi(\lambda)$ not to have zeros in the points $\{\lambda_a\}$, using condition (8) one can always find the corresponding $\xi_a$, solving again the system of linear equations.

The rest is simple matrix algebra, requiring again only matching condition $\#(\alpha) = \#(i)$. Write

$$\sum_k C^k_{ij}(\xi) \eta_{kl}(\xi) = \sum_{\alpha} \sum_k \sum_{\beta} \frac{\phi_i(\lambda_{\alpha}) \phi_j(\lambda_{\alpha})}{\xi(\lambda_{\alpha})} \cdot (\phi_k(\lambda_{\alpha}))^{-1} \cdot \phi_k(\lambda_{\beta}) \cdot \frac{\phi_l(\lambda_{\beta}) \xi(\lambda_{\beta})}{W'(\lambda_{\beta}) Q'(\lambda_{\beta})}$$

and consider it as a product of four matrices. Two mutually inverse factors in the middle cancel each other and one finally gets

$$\sum C^k_{ij}(\xi) \eta_{kl}(\xi) = \sum \frac{\phi_i(\lambda_{\alpha}) \phi_j(\lambda_{\alpha})}{\xi(\lambda_{\alpha})} \frac{\phi_l(\lambda_{\beta}) \xi(\lambda_{\beta})}{W'(\lambda_{\beta}) Q'(\lambda_{\beta})} = \mathcal{F}_{ijl}$$

and it means that algebra (14) leads to the WDVV equations (1). Note that derivation is valid for any function $\xi(\lambda)$ with the only restriction that $\xi(\lambda_{\alpha}) \neq 0$ and, thus, constancy metric is absolutely inessential. When all time dependence is hidden into differential $dW$ the matching condition is satisfied automatically, at least if $W(\lambda)$ is a polynomial (the Landau-Ginzburg case). Below we also consider two other examples when, in contrast to the Landau-Ginzburg case, the matching condition is at least naively violated into one or another direction. In the first case (e.g. Seiberg-Witten prepotential for softly broken $\mathcal{N} = 4$ Yang-Mills theory) one should necessarily add extra variables to the Seiberg-Witten periods, in the second case (one of the examples is given by tau-functions of conformal maps (9)) the situation is even more striking: $\mathcal{F} = \log \tau$ satisfies the WDVV equations as a function of only part of its variables, when the rest of the variables is fixed.

4 Consistency in the examples of Seiberg-Witten prepotentials and tau-functions of curves

In the Seiberg-Witten theory (the pure $\mathcal{N} = 2$ Yang-Mills or Toda chain case; on correspondence between Seiberg-Witten theories and integrable systems see (15)) the residue formula has the form (6)

$$\mathcal{F}_{ijk} = \int_{w=0} d\omega_i d\omega_j d\omega_k = \sum \text{res}_{\lambda_{\alpha}} d\omega_i d\omega_j d\omega_k =$$

$$= \sum \text{res}_{\lambda_{\alpha}} \frac{\phi_i(\lambda) \phi_j(\lambda) \phi_k(\lambda)}{y^2 P^i_N(\lambda)} = \sum \text{res}_{\lambda_{\alpha}} \frac{\phi_i(\lambda) \phi_j(\lambda) \phi_k(\lambda)}{(P_N(\lambda)^2 - 4) P^i_N(\lambda)}$$

where the Seiberg-Witten curve is (16)

$$w + \frac{1}{w} = P_N(\lambda)$$

or

$$y^2 = P_N(\lambda)^2 - 4$$

with $P_N(\lambda)$ being a polynomial of degree $N$ and $y = w - \frac{1}{w}$. In (11) the role of hamiltonians $\{dH_i\}$ from (6) is played by the set of canonical holomorphic differentials on Riemann surface (17), (18)

$$d\omega_i = \frac{\phi_i(\lambda) d\lambda}{y}$$

where $\{\phi_i(\lambda)\}$ are certain polynomials of degree not exceeding $(N-2)$. The set of these polynomials is supposed to be nondegenerate in the points $\{\lambda_{\alpha}\}$, which are the zeroes of $d\omega = \frac{dP_N(\lambda)}{y}$, i.e. solutions to $P^i_N(\lambda_{\alpha}) = 0$.

We know, that despite the difference between formulas (8) and (19) the WDVV equations (1) do hold for the Seiberg-Witten theory (6). The reason is the same as in sect. (8): the matching condition between the number of holomorphic differentials $d\omega_i$ and the zeroes of $P_N(\lambda)$ holds exactly since both numbers are equal to the genus of Riemann surface (20), (21) which is $g = N - 1$. And this is all (together with (6)) we need for derivation of (11) from (13), the proof literally repeats that of sect. (8).
Now, why this may cause difficulties in more general situation in Seiberg-Witten theory \[1\]? The reason is that, by derivation, formula \[19\] is accidental, since what was really derived in \[1\] is

\[
\mathcal{F}_{ijk} = -\oint_{d\lambda=0} \frac{d\omega_i d\omega_j d\omega_k}{d\lambda dw} = \oint_{dz=0} \frac{d\omega_i d\omega_j d\omega_k}{d\lambda dz}
\]

(23)

where the second integral (used in \[13\]) to be rewritten in the form of \[3\] is a consequence of the first due to holomorphic properties of the integrand. The first equality in \[23\] was obtained by differentiating the period matrix \(T_{ij} = \mathcal{F}_{ij}\) of the Riemann surface \(20\), \(21\) and using some relations for the values of canonical holomorphic differentials \(22\) at the branch points of hyperelliptic curve, defined by \(d\lambda = 0\) (and not by \(\frac{d\omega}{dw} = 0\)). However, the number of hyperelliptic branch points \(d\lambda = 0\), as follows from \(21\), equals to \(2N\) and the matching condition naively would fail! What saves the situation is that, using that integrand in \(23\) is holomorphic, one can rewrite the same contour integral around the zeroes \(\frac{d\omega}{dw} = \frac{dP_N(\lambda)}{g}\) = 0, which is still not enough, since the number of zeroes of \(\frac{d\omega}{dw}\) is \(2g = 2(N - 1)\), but due to hyperelliptic \(\mathbb{Z}_2\)-symmetry of exchanging \(\lambda\)-sheets one can finally bring the residue formula to the form of \(13\), i.e. as a sum over \(g = N - 1\) zeroes of the polynomial \(P_N(\lambda)\) (each of them corresponds in fact to the pair of points on curve \(20\), \(21\)). It means that the matching condition for the Seiberg-Witten Toda chain case finally holds!

Now, it becomes clear that for other prepotentials on nontrivial Riemann surfaces the matching conditions may fail \[1\]. For example, in the case of elliptic Calogero-Moser or broken \(N = 4\) Seiberg-Witten theory the generating differential, instead of \(\lambda \frac{d\omega}{dw}\) is \(\lambda dz\), where \(dz\) is canonical holomorphic differential on base torus and function \(\lambda\) satisfies the Lax equation

\[
\det (\lambda - L(z)) = 0
\]

(24)

with the Lax operator \(L(z)\) introduced in \[12\]. The number of zeroes of \(d\lambda\) and \(dz\) can be calculated from the Riemann-Roch theorem, saying, in particular, that for any meromorphic differential

\[
\# \text{(zeroes)} - \# \text{(poles)} = 2g - 2 = 2N - 2
\]

(25)

since the genus of the curve \(24\) is \(g = N\). The differential \(dz\) is holomorphic (it is holomorphic on base torus and does not acquire poles on the cover), so one gets

\[
\# \text{(zeroes } dz\text{)} = 2N - 2
\]

(26)

quite similar to its analog \(\frac{d\omega}{dw}\) in Toda case \(20\), \(21\). However, we do not have anymore the hyperelliptic symmetry, which allows to "reduce factor 2" and, say, rewrite \(19\) as a sum over only \((N - 1)\) points (half of \((2N - 2)\)). As for the second differential \(d\lambda\), since it follows from \(24\) and the properties of the elliptic Calogero-Moser Lax operator \(12\) that

\[
d\lambda \sim \frac{dz}{z^2}
\]

(27)

it has \(N\) second-order poles, hence

\[
\# \text{(zeroes } d\lambda\text{)} - \# \text{(poles } d\lambda\text{)} = \# \text{(zeroes } d\lambda\text{)} - 2N = 2N - 2
\]

(28)

or

\[
\# \text{(zeroes } d\lambda\text{)} = 4N - 2
\]

(29)

i.e. it has even more zeroes than \(26\). It means that restricting \(1\) to the case of holomorphic differentials only

\[
\mathcal{F}_{ijk} = -\oint_{d\lambda=0} \frac{d\omega_i d\omega_j d\omega_k}{d\lambda dz} = \oint_{dz=0} \frac{d\omega_i d\omega_j d\omega_k}{d\lambda dz}
\]

(30)

one would get \(#(\alpha) > #(i)\) and in order to close the algebra, corresponding to \(30\) one needs to add to the set of \(N\) holomorphic differentials at least \((N - 2)\) extra "hamiltonians". Naively there are two direct options to do that – to add either meromorphic differentials with higher order poles or non single-valued holomorphic differentials in spirit of \(6\). One needs then, however, to check the (extended) residue formula \(30\) with added new meromorphic or non single-valued differentials. From the point of view of SUSY quantum theory the main problem is the physical sense of corresponding extra time variables which should play the role of "hidden" moduli parameters in corresponding Seiberg-Witten theory.

The opposite example (which has no yet known direct relation to SUSY apart of being presented as "generalized" topological theory) is given by dispersionless tau-functions, in particular by tau-functions of analytic
curves or conformal maps \([13]\). These tau-functions satisfy well-known algebraic Hirota relations for the second derivatives

\[
F_{ij} \equiv \frac{\partial^2 F}{\partial a_i \partial a_j} \tag{31}
\]

(dispersionless limit of the bilinear Hirota relations) and (infinite) WDVV equations can be derived taking times-derivatives of algebraic Hirota relations \([9]\). The residue formula \([6]\) in this case can be also thought of as a consequence of Hirota relations.

One may consider tau-functions corresponding to rational conformal maps \([13]\) (tau-function of curves restricted to the finite number of time variables). It is easy to see that in this case matching condition will be violated into the opposite (compare to Seiberg-Witten examples) direction, so that \(#(\alpha) < #(i)\). It means that (logarithm of) tau-function of rational conformal maps satisfies the WDVV equation \([1]\) as a function of any (in the case of corresponding non-degeneracy) part of its variable whose total number is equal to \(#(\alpha)\) or to the number of zeroes of differential of rational map. Unfortunately in the only explicit case of ellipse there cannot be any reasonable WDVV since algebra \([1]\) is two-dimensional so that equations \([1]\) are empty, but if one manages to find explicit formulas for the other tau-functions of rational maps, their logarithms would give examples of functions, satisfying equations \([1]\) as functions of part of their variables.

5 Conclusions and outlook

We have tried to demonstrate in this talk that when conventional quantum field theory fails to describe non-perturbative effective theories one can still use the relation between SUSY and geometry and even encode the corresponding geometric information in the solutions to some very specific nonlinear integrable differential equations. As an example of such equations we have considered the WDVV equations \([1]\) and demonstrate that their existence is based on two very universal facts – associative algebra and residue formula. The corresponding proof, presented above, is extremely simple and does not at all involve any additional requirements like constancy of ”topological metric”, existence of unity operator, (see \([8]\)) etc. However, there is still no direct link between the WDVV equations and first principles of non-perturbative physics and finally we would like to make few comments on the possible connection.

• Duality

From the point of view of relation between SUSY and geometry it is very important that WDVV equations \([1]\) are consistent with generic duality transformations \([14]\)

\[
F^S(a^S) = F(a) + \frac{1}{2} a^t \cdot U^t \cdot W \cdot a + a^t \cdot W^t \cdot Z \cdot \left( \frac{\partial F}{\partial a} \right) + \frac{1}{2} \left( \frac{\partial F}{\partial a} \right)^t \cdot Z^t \cdot V \cdot \left( \frac{\partial F}{\partial a} \right). \tag{32}
\]

with matrices \(U, V, W\) and \(Z\) forming symplectic group or obeying relations

\[
U^t \cdot V - Z^t \cdot W = V \cdot U^t - Z \cdot W^t = 1,
\]

\[
U^t \cdot W = W^t \cdot U, \quad Z^t \cdot V = V^t \cdot Z. \tag{33}
\]

The proof of this fact is very simple and requires only trivial matrix algebra. However, the outcomes are very important: the WDVV equations are internally consistent with properties of special Kähler geometry and their form \([1]\), where only third derivatives of \(F\), but not (special Kähler) metric appear, is covariant under transformations which should be necessarily respected by nonperturbative physics. It is also very interesting to understand the sense of transformations \([12]\) in general context of quasiclassical integrable hierarchies and (generalized) dispersionless Hirota equations.

• From WDVV to algebraic relations on second derivatives

In the case of dispersionless hierarchies, the infinite WDVV equations follow from the dispersionless Hirota relations \([1]\) for the second derivatives \([11]\), which can be written in the form

\[
F_{ij} = T_{ij}(\varphi) \tag{34}
\]

where \(T_{ij}\) are some known functions and \(\{\varphi_i\}\) denote some restricted set of the second derivatives, say

\[
\varphi_i = F_{1i}(a) \tag{35}
\]
The structure constants (4) for the WDVV equations can be defined as

\[ C^k_{ij} = \partial T_{ij} / \partial \varphi_k \]  (36)

Indeed, then

\[ F_{ijk} = \partial F_{ij} / \partial a_k = \sum_l C^l_{ij} F_{kl} \]  (37)

which repeats (3) with some special ”topological metric” \( \eta_{ij} = F_{1ij} \). In dispersionless hierarchies crossing relations (3) for the structure constants (36) follow from bilinear Hirota identities (34), but naively this is true only for infinite number of variables \( a = (a_1, a_2, \ldots) \) and equations.

In the essentially finite dimensional case (say, prepotentials \( N = 2 \) SUSY gauge theories in four dimensions) the second derivatives of \( F \) (31) form symmetric \( N \times N \) matrix with \( 1/2 N(N+1) \) generally different elements \( F_{ij}(a) \) and in fact we have \( 1/2 N(N+1) - N = 1/2 N(N-1) \) nontrivial functions \( T_{ij}(\varphi) \). The WDVV equations (3), (1) become now the set of first-order differential equations for these functions

\[ \sum_l \partial T_{ik} / \partial \varphi_l \partial T_{lj} / \partial \varphi_n = \sum_l \partial T_{ik} / \partial \varphi_l \partial T_{lj} / \partial \varphi_n \]  (38)

The question we are going to address is whether (38) may be rewritten in some sensible form – for example as algebraic relations on \( T_{ij} \) similar to dispersionless Hirota equations. It is not clear even in the simplest nontrivial \( N = 3 \) case (\( N = 1 \) and \( N = 2 \) are trivial as they should be). If such algebraic relations exist they may play the role of RG relations for the theory with many couplings and/or some analogs of the Schottkey relations for the matrix elements of the (Seiberg-Witten) period matrices.

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