RESEARCH ARTICLE

The number of the non-full-rank Steiner triple systems

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Abstract
The p-rank of a Steiner triple system (STS) B is the dimension of the linear span of the set of characteristic vectors of blocks of B, over GF(p). We derive a formula for the number of different STSs of order v and given 2-rank $r_2$, $r_2 < v$, and a formula for the number of STSs of order v and given 3-rank $r_3$, $r_3 < v - 1$. Also, we prove that there are no STSs of 2-rank smaller than $v$ and, at the same time, 3-rank smaller than $v - 1$. Our results extend previous study on enumerating STSs according to the rank of their codes, mainly by Tonchev, V.A. Zinoviev, and D.V. Zinoviev for the binary case and by Jungnickel and Tonchev for the ternary case.

KEYWORDS
2-rank, 3-rank, Steiner triple system

1 | INTRODUCTION

A Steiner triple system STS(v) is a finite set S of cardinality v, whose elements are called points, provided with a collection of 3-subsets, called blocks, such that every 2-subset of S is contained in one and only one block. We assume that $S = \{1,...,v\}$ and identify a block b with its characteristic vector, that is, the v-tuple of 0’s and 1’s having 1’s in the coordinates numbered by the elements of b. For example, $(0, 1, 0, 1, 1, 0, 0) = \{2, 4, 5\} (v = 7)$. The dimension of the space over GF(p) spanned by the blocks (to be exact, by their characteristic vectors) of an STS T is called the p-rank of T.
A series of papers was devoted to the study of STS(\(v\)) of deficient rank. In 1978, Doyen, Hubaut, and Vandensavel [4] characterized the classical examples—the point-line designs in binary projective and ternary affine spaces—as the unique STS with these parameters which have minimal 2- and 3-rank, respectively, and also proved that the \(p\)-rank of STS(\(v\)), \(v > 3\), equals \(v\) for every prime \(p\) larger than 3. Later, Assmus in his fundamental paper [1] gave an extensive theoretical study of the binary space spanned by the blocks of an STS. A formula for the exact number of different STS(\(2^m - 1\)) of 2-rank \(2^m - m\), which is one more than the minimum value, was derived by Tonchev [19]. V.A. Zinoviev and D.V. Zinoviev [27,28] proved the formulas for the number of different STS(\(2^m - 1\)) of rank \(2^m - m + 1\) and for the number of such STS with prescribed linear span. Four years later, D.V. Zinoviev [26] found the corresponding formulas for the next value of the rank, \(2^m - m + 2\). Recently, Jungnickel and Tonchev [8] described the possible structures of the linear span of the blocks of an STS over GF(2) or GF(3), in terms of parity check matrices. In the subsequent paper [7], Jungnickel and Tonchev proved characterization theorems for STSs whose linear span, binary or ternary, lies in a prescribed subspace; it occurs that such systems are in one-to-one correspondence with special collections of designs of smaller order (STS, transversal designs, and 1-factorizations), which yield a formula for the number of such systems, that is, of different systems whose linear span is upper bounded by a prescribed subspace. For the partial cases of STS(\(2^m - 1\)) of 2-rank \(2^m - m + 1\) and \(2^m - m + 2\), similar results were obtained earlier by V.A. Zinoviev and D.V. Zinoviev [27,29]. The goal of our paper is to make the next step for the general case and derive a formula for the number of all different STS(\(v\)) whose rank coincides with a given value (less than \(v\) for the 2-rank or less than \(v - 1\) for the 3-rank), but the linear span is not limited by any prescribed subspace.

Another very important direction in the enumeration of any kind of combinatorial configurations is evaluating the number of nonisomorphic configurations, and the STSs of prescribed rank are not an exception. Two STSs on the same point set \(S\) are isomorphic if there is a permutation of \(S\) (an isomorphism) that sends the blocks of one system to the blocks of the other. On the basis of the exact formula on the number of different STS (\(2^m - 1\)) of 2-rank \(2^m - m\), Tonchev [19] derived an exponential lower bound on the number of nonisomorphic systems with these parameters. Recently, Jungnickel and Tonchev [7] generalized that result and obtained estimates for the number of isomorphism classes of STS(\(2^m w - 1\)) of 2-rank \(2^m w - 1 - m\) and STS(\(3^m w\)) of 3-rank \(3^m w - m - 1\), for any positive integer \(w\) and \(m\) (formally, the statements are formulated for \(w = 2^i\) and \(3^i\), respectively, but the theory developed works for an arbitrary \(w\) as well).

There are several computational results related with the calculation of the exact number of nonisomorphic STSs of given 2- or 3-rank. We summarize all known numbers in Table 1. There are 80 nonisomorphic STS(15); their 2-ranks were studied by Tonchev and Weishaar [20]. Stinson and Seah [18] calculated the number 284 457 of isomorphism classes of STS(19) that have a sub-STS of order 9; these systems are exactly the STS(19) of 2-rank 18. Kaski and Östergård [9] classified all nonisomorphic STS(19); their number is 11 084 874 829, and hence the number of STS(19) of 2-rank 19 is also known. Osuna [14] found that there are 1239 nonisomorphic STS(31) of 2-rank 27. Kaski, Östergård, Topalova, and Zlatarski [10] counted the number 2 166 351 of nonisomorphic STS(21) of Wilson-type, which are essentially the STS(21) of 3-rank 19. Recently, Jungnickel, Magliveras, Tonchev, and Wassermann [6] calculated the number of STS(27) of 3-rank 24.

In the current paper, we derive formulas for the number of STS(\(v\)) of arbitrary 2-rank smaller than \(v\), see Theorem 4.6, or 3-rank smaller than \(v - 1\) [note that the 3-rank of STS cannot
exceed $v - 1$, as it is always orthogonal to the all-one vector $(1, 1, \ldots, 1)$ over $\mathbb{GF}(3)$], see Theorem 3.6. In particular, our result generalizes the formulas for 2-rank $2^m - m$ [19], $2^m - m + 1$ [28], and $2^m - m + 2$ [26], obtained before. The generalization is based on the Möbius transform, which makes possible to derive a common formula for different ranks and also to simplify some arguments. The formulas are tight but conditional: they involve the number of objects of smaller order (STSs, 1-factorizations of complete graph, and latin squares).

For partial cases where these numbers are known, we obtain explicit values. Namely, in addition to the results known before, we get the number of STS($3^k$) of 3-rank $v - k$ and $v - k + 1$, the number of STS($7 \cdot 3^k$) of 3-rank $7 \cdot 3^k - k - 1$, and the number of STS($10 \cdot 2^k - 1$) of 2-rank $10 \cdot 2^k - k - 1$, for every $k$ (Corollaries 3.7 and 4.7). In the other cases, our formulas can be combined with the asymptotic estimations of the number of STSs [11,24], 1-factorizations [3,11,25], and latin squares, see, for example, [21, Theorems 17.2 and 17.3], or with some bounds on the number of different objects of the required order. For example, from the formulas derived in the current paper (Corollary 3.7), we know that the number of STS(27) of 3-rank at most 25 is $124\,363\,532\,158\,160\,295\,774\,288\,076\,917\,115\,534\,405\,632\,000\,000$, which gives a lower bound on the number of all STS(27) (however, this bound is, hypothetically, incredibly small against the number of STS(27) of 3-rank 26).

We emphasize that the main difference of our result with the exact formulas of Jungnickel and Tonchev [7] is that the formula of [7] (concerning the 2- or 3-rank) counts the number of distinct STSs contained a given linear subspace, while the rank of the counted STS is less than or equal the dimension $d$ of the subspace. Our main attempt is focused on excluding from this formula the systems of rank smaller than $d$ and counting only the STS whose linear span is exactly the given subspace. After that, multiplying by a simple factor, we obtain the total number of STSs of rank $d$.

In Section 2, we define necessary concepts and mention related facts. In Section 3, we describe the structure of an STS($v$) of prescribed 3-rank $r_3$, $r_3 < v - 1$, and of its dual space and derive a formula for the number of such systems. In Section 4, similar results are obtained for STS($v$) of given 2-rank $r_2$, $r_2 < v$. In concluding Section 5, we show that there is no STS which is

### Table 1

| Order $v$ | Of 2-rank | Of 3-rank |
|-----------|-----------|-----------|
| $v$       | $v-1$     | $v-2$     | $v-3$     | $v-4$     | $v-5$     | $v-1$     | $v-2$     | $v-3$     | $v-4$     |
| 7         | 0         | 0         | 0         | 1         | ...       | 1         | ...       | ...       | ...       |
| 9         | 1         | ...       | ...       | ...       | ...       | ...       | 0         | 0         | 1         | ...       |
| 13        | 2         | ...       | ...       | ...       | ...       | ...       | 2         | ...       | ...       | ...       |
| 15        | 57        | 16        | 5         | 1         | 1         | ...       | 80        | ...       | ...       | ...       |
| 19        | 11,084,590,372 | 284,457   | ...       | ...       | ...       | ...       | ...       | 11,084,874,829 | ...       | ...       |
| 21        | ?         | ...       | ...       | ...       | ...       | ...       | ?         | 2,166,351 | ...       | ...       |
| 25        | ?         | ...       | ...       | ...       | ...       | ...       | ?         | ...       | ...       | ...       |
| 27        | ?         | ...       | ...       | ...       | ...       | ...       | ?         | ?         | 2,324 | 1       |
| 29        | ?         | ...       | ...       | ...       | ...       | ...       | ?         | ...       | ...       | ...       |
| 31        | ?         | ?         | ?         | ?         | 1,239     | 1         | ?         | ...       | ...       | ...       |

Abbreviation: STS, Steiner triple system.
both non-full-2-rank and non-full-3-rank, and briefly discuss the number of the isomorphism classes of STS of prescribed rank.

2 | DEFINITIONS AND NOTATIONS

2.1 | Orthogonality and duality

Two \( v \)-tuples \( x = (x_1, \ldots, x_v) \) and \( y = (y_1, \ldots, y_v) \) understood as vectors over \( \text{GF}(q) \) are said to be orthogonal, denoted \( x \perp y \), if \( x_1 y_1 + \cdots + x_v y_v = 0 \). Given a set \( B \) of vectors, the dual space \( B^\perp \) is the set of all vectors orthogonal to each element of \( B \). By \( \langle B \rangle \), we denote the linear span of the vector set \( B \).

To simplify the formulas, we will use the standard notation of \( q \)-factorial \( \prod_{s=1}^{n} \sum_{i=0}^{s-1} q^i \). Using this notation, the number \( \prod_{t=1}^{n} (q^n - q^{t-1}) \) of different bases in an \( n \)-dimensional space over \( \text{GF}(q) \) can be written as \( q^{n(n-1)/2}(q - 1)^{n} [n]_q ! \).

2.2 | Latin squares

A latin square of order \( n \) is a function \( f : \{1, \ldots, n\}^2 \rightarrow \{1, \ldots, n\} \) invertible in each argument. Traditionally, latin squares are represented by their value tables, whose rows and columns, by definition, contain all elements from 1 to \( n \). (A system from the set \( \{1, \ldots, n\} \) with a latin square operation \( f \) is known as a quasigroup of order \( n \).) A latin square \( f \) is called symmetric if \( f(x, y) = f(y, x) \) (ie, the corresponding quasigroup is commutative). A latin square \( f \) is called totally symmetric if \( f(x, y, z) = f(y, z, x) = f(z, y, x) = f(x, z, y) = f(z, x, y) \), and \( f(x, y) = y \). A latin square \( f \) is called idempotent if \( f(x, x) = x \). It is well known and obvious that the idempotent totally symmetric latin squares of order \( n \) are in one-to-one correspondence with the STSs of order \( n \).

**Proposition 2.1.** Let \( S = \{1, \ldots, n\} \). For every STS \( (S, B) \), the function \( f \) defined as the identity on \( S \), and \( f(x, y) = z \) for every \( \{x, y, z\} \in B \) is an idempotent totally symmetric latin square. Conversely, every idempotent totally symmetric latin square \( f : S^2 \rightarrow S \) induces the STS \( (S, B) \), \( B = \{\{x, y, z\} : x \neq y, f(x, y) = z\} \).

Slightly less obvious but also well known is the following bijection.

**Proposition 2.2.** For every odd \( n \), there is a one-to-one correspondence between symmetric latin squares of order \( n \) and symmetric latin squares \( f \) of order \( n + 1 \) such that \( f(x, x) = n + 1 \).

**Proof.** Given symmetric latin square \( f \) of order \( n + 1 \) such that \( f(x, x) = n + 1 \), the function \( g : \{1, \ldots, n\}^2 \rightarrow \{1, \ldots, n\} \) defined by \( g(x, x) = f(x, n + 1) \), \( g(x, y) = f(x, y) \) for every different \( x \) and \( y \) from \( \{1, \ldots, n\} \) is straightforwardly a symmetric latin square.

![Symmetric Latin Square Example]

\[
\begin{array}{ccc}
4 & 1 & 3 \\
1 & 4 & 2 \\
3 & 2 & 4 \\
2 & 3 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
2 & 1 & 3 \\
1 & 3 & 2 \\
3 & 2 & 1 \\
\end{array}
\]
To see the converse, it is important to note that for a symmetric latin square \( g \) of odd order \( n \), the set \( \{ g(x, x) : x \in \{1, \ldots, n\} \} \) coincides with \( \{1, \ldots, n\} \) (indeed, for every \( x \) the number of the pairs \( (y, z) \) such that \( g(y, z) = x \) is \( n \), which is odd, whereas the number of the pairs \( (y, z) \) such that \( g(y, z) = x \) and \( y \neq z \) is even, from the symmetry). Then we define the required \( f \) by the identities

\[
 f(x, x) = 1, \quad f(x, n) = f(n, x) = g(x, x), \quad \text{and} \quad f(x, y) = g(x, y) \text{ for every } x, y \in \{1, \ldots, n\}, x \neq y.
\]

\[\square\]

Remark 1. The symmetric latin squares \( f \) of even order \( n \) such that \( f(x, x) \equiv n \) are in a straightforward one-to-one correspondence with the ordered 1-factorizations of the complete graph on the vertex set \( \{1, \ldots, n\} \) (ie, the ordered partitions of the set of edges of this graph into \( n - 1 \) sets of mutually disjoint edges; the number of the ordered partitions equals \( (n - 1)! \) times the number of unordered partitions, see [30]), with the tournament schedules for \( n \) teams, see, for example, [33], and with the resolutions of the complete system of pairs from \( \{1, \ldots, n\} \), see, for example, [1]. Under these different names, but in the same context as in the current paper, the symmetric latin squares can be mentioned in the literature on combinatorial designs.

### 2.3 Möbius coefficients

For a prime power \( q \), define the numbers \( \mu_i^{(q)} \), \( i = 0, 1, 2, \ldots \) by the following recursion: \( \mu_0^{(q)} = 1 \), and for an \( i \)-dimensional space \( S \) over \( GF(q) \), \( i \geq 1 \), and the set \( S \) of its subspaces,

\[
 \mu_i^{(q)} = -\sum_{C \in S/\{S\}} \mu_{\dim(C)}^{(q)}, \quad \text{or, equivalently,} \quad \sum_{C \in S} \mu_{\dim(C)}^{(q)} = 0. \tag{1}
\]

Remark 2. The numbers \( \mu_i^{(q)} \) are related with the so-called Möbius function on the poset of spaces over \( GF(q) \). Namely, for two spaces \( U \) and \( V \), the Möbius function \( \mu_U(V) \) equals \( \mu_{\dim(U) - \dim(V)}^{(q)} \) if \( V \subseteq U \) and 0 otherwise.

**Lemma 2.3** (Stanley [17, Sect. 3.10]). For every prime power \( q \), one has \( \mu_i^{(q)} = (-1)^i q^{i \choose 2} \).

### 3 NON-FULL-3-RANK STS

Let \( v \equiv 1, 3 \mod 6 \); that is, there exist \( STS(v) \). By \( V^v \), we denote the vector space of all \( v \)-tuples over \( GF(3) \). Denote by \( \mathcal{D} \) the set of subspaces of \( V^v \), each including the all-one vector and being orthogonal to at least one \( STS(v) \); denote

\[
 \mathcal{D}_i = \{ D \in \mathcal{D} : \dim(D) = i + 1 \}.
\]

The following lemma can be considered as a treatment of the results of [4, Section 4] in terms of the structure of a basis for the dual space of \( STS \).

**Lemma 3.1** (Jungnickel and Tonchev [8, Theorem 5.1]). Let \( M \) be an \( (i + 1) \times v \) generator matrix for \( D \in \mathcal{D}_i \), and let the first row of \( M \) be the all-one vector. Then \( M \) consists of \( 3^i \) different columns, each occurring \( v/3^i \) times; for example,
The proof given in [8] includes a proof of more deep mathematical result, a variation of Bonisoli’s theorem. We give an independent simple proof.

Proof. Without loss of generality, we can assume that the first column of $M$ is $(1, 0,\ldots,0)\mathbf{T}$ (we can achieve this by subtracting the first row from some of the others).

Claim (*). If $a$ and $b$ are columns of $M$, then $a - b$ is also a column of $M$. Consider an STS$(v)$ orthogonal to the rows of $M$. Let $a$ and $b$ be the $j$th and $k$th columns of $M$, and let $[j, k, l]$ be the STS block containing $j$ and $k$. Since all rows of $M$ are orthogonal to the characteristic vector of this block, the $l$th column $c$ satisfies $a + b + c = 0$, that is, $c = -a - b$. This proves (*).

Claim (**). If $c$ and $d$ are columns of $M$, then $c + d - (1, 0,\ldots,0)\mathbf{T}$ is also a column of $M$. This is proved by applying (*) with $a = c$, $b = d$ first, and then with $a = -c - d$, $b = (1, 0,\ldots,0)\mathbf{T}$.

The last claim means that the set of columns of the matrix $M'$ obtained from $M$ by removing the first row is closed under addition. Since there are $i$ linearly independent columns, this set contains all possible $3^i$ columns of height $i$.

It remains to prove that different columns $a$ and $b$ occur the same number of times in $M$. Let $J$ and $K$ be the sets of positions in which $M$ has the columns $a$ and $b$, respectively. And let $l$ be a position of the column $a - b$. For each $j$ from $J$, there is $k$ from $K$ such that $[j, k, l]$ is a block of the STS. Moreover, different $j$’s correspond to different $k$’s. This shows that $|J| \leq |K|$. Similarly, $|K| \leq |J|$.

Lemma 3.2. Let $i$ and $j$ be nonnegative integers such that $i \leq j$. If $D_i$ is not empty, then every subspace from $D_i$ is contained in exactly $\Gamma_{v,i,j}$ subspaces from $D_j$, where

$$\Gamma_{v,i,j} = \left(\frac{v}{3!}\right)^j / 3^{(j+i+1)(j-i)/3} \left(\frac{v}{3!}\right)^{2j-i[j-i]}.$$

In particular, $|D_i| = V_{v,0,j}$.

Proof. First, let us fix some $D_i$ from $D_i$ and construct all $D_j$ from $D_j$ that satisfy $D_i \subseteq D_j$. Let $M_i$ be a generator matrix of $D_i$ whose first row contains only 1’s. According to Lemma 3.1, $M_i$ divides the coordinates into $3^i$ “cells” such that each cell contains the same columns in it. Since $D_i \subseteq D_j$, so $D_j$ has a generator matrix $M_j$ that starts with the $i + 1$ rows of $M_i$, and this matrix subdivides the cells into $3^j$ “subcells” of the same size. The number of such subdivisions is $A = (\frac{v}{3!})^j / (\frac{v}{3!})^{3^j}$.

Next, let us count the number of such matrices that generate the same code. Let $M'_j$ be another generator matrix of $D_j$ that also starts with $i + 1$ rows of $M_i$; what is more, its $t^{th}$ row, $i + 2 \leq t \leq j + 1$, is a linear combination of the rows of $M_j$, but is not a linear combination of the rows above it in the matrix $M'_j$. So, we can get there are $3^{j+1} - 3^i$ kinds of values for the row $i + 2$, $3^{j+1} - 3^{j+2}$ kinds of values for the row $i + 3$, and so on. Therefore, the number of such matrices $M'_j$ is
Lemma 3.3 (the structure of non-full-3-rank STS [7]). Given a subspace $D$ from $D_j$, the set of STS(v) orthogonal to $D$ is in one-to-one correspondence with the collections of $3^j$ STSs of order $v/3^j$ and $3^j(3^j - 1)/6$ latin squares of order $v/3^j$.

Proof: (a sketch). According to Lemma 3.1, a generator matrix $M$ of $D$ divides the coordinates into $3^j$ groups of size $v/3^j$. It can be seen that every STS(v) orthogonal to $D$ is divided into the $3^j + 3^j(3^j - 1)/6$ following subsets:

- For each of $3^j$ groups, the triples with all 3 points in these groups form STS($v/3^j$).
- For every 3 distinct groups $\alpha_1, \alpha_2, \alpha_3$, corresponding to columns $a, b, c$ with $a + b + c = 0$, the triples with one point in each of these 3 groups have the form $\{x, \beta_y, \gamma_f(x,y)\}$ for some latin square $f$ of order $v/3^j$.

Corollary 3.4 (Jungnickel and Tonchev [7]). Given a subspace $D$ from $D_j$, the number $\Phi(v)$ of STS(v) orthogonal to $D$ equals $\Phi(v)$, where

$$\Phi(v) = \Psi_{v/3^j} \Lambda_{v/3^j}^{3^j(3^j - 1)/6},$$

where $\Psi_u$ is the number of STS(u), and $\Lambda_u$ is the number of latin squares of order u.

Now, given a subspace $D$ from $D_j$, we know the number $\Phi(D)$ of STSs that are orthogonal to some subspace of $D$. To find the number of STSs that are dual to $D$, we should apply to the function $\Phi(D)$ the Möbius transform on the poset of subspaces of $D$. This is essentially done in the next lemma.

Lemma 3.5. Assume that $v$ is divided by $3^k$ and $k$ is the largest integer with this property. Let $i \in \{0, \ldots, k\}$, and let $D$ be in $D_i$. The number of STS(v) with dual space $D$ equals $Y(v, i)$, where

$$Y(v, i) = \sum_{j=i}^{k} \Gamma_{v,i,j} \Phi^{(3)}_{v,j},$$

where $\Gamma_{v,i,j}$ and $\Phi^{(3)}_{v,j}$ are from Lemma 3.2 and Corollary 3.4, respectively.

Proof. Utilizing the definition of $\Gamma_{v,i,j}$ and then expanding $\Phi(v,j)$, we have

$$\sum_{j=i}^{k} \Gamma_{v,i,j} \Phi^{(3)}_{v,j} = \sum_{j=1}^{k} \sum_{P \subseteq D_j} \mu^{(3)}_{j-1} \Phi(v,j) = \sum_{j=1}^{k} \sum_{P \subseteq D_j} \sum_{B \in P(D')} \mu^{(3)}_{j-1},$$
where \( P(D') \) is the set of \( \text{STS}(v) \) orthogonal to \( D' \). We continue

\[
\begin{align*}
&= \sum_{D \subseteq D'} \sum_{B \in P(D')} \mu^{(3)}_{\dim(D')-1-i} = \sum_{D \subseteq D'} \sum_{B \in P(D')} \mu^{(3)}_{\dim(D')-1-i} \\
&= \sum_{B \in P(D)} \sum_{D \subseteq D'} \mu^{(3)}_{\dim(D')-1-i} = \sum_{B \in P(D)} \sum_{D \subseteq D'} \mu^{(3)}_{\dim(D')-1-i} \\
D^* &= D'/D \sum_{B \in P(D)} \mu^{(3)}_{\dim(D')} = \sum_{B \in P(D)} (1 \text{ if } B^\perp = D; \ 0 \text{ otherwise}).
\end{align*}
\]

We see that the last formula meets the definition of \( Y_{v,i} \).

**Theorem 3.6.** Assume that \( v \) is divided by \( 3^k \) and \( k \) is the largest integer with this property. Let \( i \in \{0, \ldots, k\} \). The total number of different \( \text{STS}(v) \) of 3-rank \( v - i - 1 \) equals

\[
\Gamma_{v,0,i} \sum_{j=i}^{k} \Gamma_{v,i,j} \mu^{(3)}_{j-1} \Phi_{v,j}, \text{ where } \mu^{(3)}_{j} = (-1)^j 3^{j-2} / 2, \ \Phi_{v,j} = \Psi_{v/3}^{j/3} \Lambda_{v/3}^{3^{j-1}/3},
\]

\[
\Gamma_{v,i,j} = \left( \frac{v!}{3^j} \right)^{3j} / 3^{j(j+1)} / 2^{j-1} \left( \frac{v!}{3^j} \right)^{3j} \prod_{s=1}^{j-1} (3^s - 1),
\]

\( \Psi_u \) is the number of \( \text{STS}(u) \) (and also the number of idempotent totally symmetric latin squares of order \( u \)), and \( \Lambda_u \) is the number of latin squares of order \( u \).

**Proof.** The number of \( \text{STS}(v) \) of 3-rank \( v - i - 1 \) equals the number \( Y_{v,i} \) of \( \text{STS}(v) \) of 3-rank \( v - i - 1 \) orthogonal to a given subspace \( D \) from \( D_i \) multiplied by the number \( \Gamma_{v,0,i} \) of such subspaces. Utilizing the formulas from Lemma 3.5 and Corollary 3.4, we get the result.

**Corollary 3.7.** The number of \( \text{STS}(v) \), \( v = 3^k \), of 3-rank \( v - k - 1 \) is

\[
\frac{v!}{3^{k(k+1)/2} 2^{k-1}[k]_3!}.
\]

The number of \( \text{STS}(v) \), \( v = 3^k \), of 3-rank \( v - k \) is

\[
\frac{v! \left( 2^{v^2/27-4v/9+1} 3^{v^2/54-7v/18+k-1} - 2^{v^2/27-4v/9+3} 3^{v^2/54-7v/18+k-1} + 1 \right)}{2^k 3^{k(k+1)/2} [k - 1]_3!}.
\]

The number of \( \text{STS}(v) \), \( v = 3^k \), of 3-rank \( v - k + 1 \) is

\[
\frac{v!}{2^{k+2} 3^{(k(k+1)/2)-1} [k - 2]_3!} \times \left( \frac{2^{35} 3^8 5^2 7^2 5231 3824477}{2^{4v/9-4} 3^{v/3-2k+1}} - 2^{v^2/27-4v/9+3} 3^{v^2/54-7v/18+k-1} + 1 \right).
\]
The number of STS($v$), $v = 7 \cdot 3^k$, of 3-rank $v - k - 1$ is

$$\frac{v! \cdot 61479419904000^{3^k(3^k-1)/6}}{2^k \cdot 3^k(k+1)/2 \cdot 168 \cdot 3^k \cdot [k]!}.$$ 

Proof. According to [31], we have $Λ_1 = 1$, $Λ_3 = 12$, $Λ_7 = 61479419904000 = 2^{18} \cdot 3^5 \cdot 7 \cdot 1103$ [15,16], $Λ_9 = 5524751496156892842531225600 = 2^{35} \cdot 3^5 \cdot 7^2 \cdot 5231 \cdot 382447$ [2] (the last known value is $Λ_{11}$ [13]). According to [32], we have $Ψ_1 = Ψ_3 = 1$, $Ψ_7 = 30$, $Ψ_9 = 840$ (the last known value is $Ψ_{19}$ [9]). Applying the result of Theorem 3.6, we get the formulas.

A computer-aided classification of equivalence classes of STS(27) of 3-rank 24 is described in [6]. In particular, the total number of different systems with these parameters can be calculated from [6, table 1] as the sum $\sum N_i/|\text{Aut}(S)|$ over all rows of the table except the last one (corresponding to the 3-rank 23). This number coincides with the one given by our formula, $22\,300\,404\,167\,684\,260\,773\,163\,008\,000\,000$.

## 4 | NON-FULL-2-RANK STS

In this section, to simplify the formulas, we denote the order of STS by $w - 1$ instead of $v$. By $V^{w-1}$, we denote the vector space of all $(w - 1)$-tuples over GF(2). Denote by $\hat{D}_i$ the set of $i$-dimensional subspaces of $V^{w-1}$ orthogonal to at least one STS($w - 1$). The following lemma can be considered as a treatment of the results of [4, Section 3] in terms of the structure of a basis for the dual space of STS.

**Lemma 4.1** (Jungnickel and Tonchev [8, Theorem 4.1]). Let $M$ be an $i \times (w - 1)$ generator matrix for $D \in \hat{D}_i$. Then each of the $2^i - 1$ nonzero columns of height $i$ occurs $w/2^i$ times as a column of $M$, while the all-zero column occurs $w/2^i - 1$ times; for example,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ \end{bmatrix}.$$ 

Proof. Claim (*). If $a$ and $b$ are different nonzero columns of $M$, then $a + b$ is also a column of $M$. The proof is similar to that of Claim (*) in the proof of Lemma 4.1.

Since the rank of $M$ is $i$, it contains $i$ linearly independent columns. It follows from (*) that it contains all $2^i - 1$ different nonzero columns of height $i$.

It remains to show that every nonzero column $a$ occurs $|K| + 1$ times, where $K$ is the set of positions in which $M$ has the all-zero column. Let $J$ be the sets of positions in which $M$ has the column $a$, and let $l \in J$. For each $j$ from $J/|l|$, there is $k$ from $K$ such that $[j, k, l]$ is a block of the STS. Moreover, different $j$'s correspond to different $k$'s. This shows that $|J/|l|| \leq |K|$. Similarly, $|K| \leq |J/|l||.$

□
Lemma 4.2. Let \( i \) and \( j \) be nonnegative integers such that \( i \leq j \). If \( \mathcal{D}_j \) is not empty, then every subspace from \( \mathcal{D}_i \) is contained in exactly \( \Gamma_{w,i,j} \) subspaces from \( \mathcal{D}_j \), where

\[
\Gamma_{w,i,j} = \left( \frac{w}{2^i} \right)^{2j} \binom{2j}{2^i} \cdot \left( \frac{w}{2^j} \right)^{2^i} \cdot \left( \frac{w}{2^j} \right)^{2^j}.
\]  

(4)

In particular, \( |\mathcal{D}_j| = \Gamma_{w,0,j} \).

Proof. The proof is similar to that of Lemma 3.2. The difference is that the size of one cell, corresponding to the all-zero columns of the generator matrix, is one less than for each of the other cells; the same can be said for the subcells. So, totally we have

\[
\begin{align*}
A = \left( \frac{w}{2^i} \right)^{2j} \left( \frac{w}{2^j} \right)^{2^i} \left( \frac{w}{2^j} \right)^{2^j} = \left( \frac{w}{2^i} \right)^{2j - 1} \left( \frac{w}{2^j} \right)^{2^i - 1} \left( \frac{w}{2^j} \right) \left( \frac{w}{2^j} \right)^{2^j - 1}.
\end{align*}
\]

subdivisions. Dividing this number by the number \( B = 2^{(i-j)(j+i+1)/2} \prod_{s=1}^{j-i} (2^s - 1) \) of the matrices generating the same space, we get the result. \( \square \)

The following lemma describes the structure of an arbitrary non-full-2-rank STS. It was proved in [29] for the partial case of STS(2\(^k\)− 1) of rank 2\(^k\) + 2; the arguments, however, are applicable to the general case. It should be also noted that Theorem 4.1 in [1] is close to this result, but the structure of the part of the block set connected with latin squares is not described there (with the exception of one partial example in Remark 6).

Lemma 4.3 (the structure of non-full-2-rank STS [7]). Given a subspace \( D \) from \( \mathcal{D}_j \), the set of STS\((w - 1)\) orthogonal to \( D \) is in one-to-one correspondence with the collections of one STS\((w/2^j - 1)\), \( 2^j - 1 \) symmetric latin squares of order \( w/2^j - 1 \), and \((2^j - 1)(2^j - 2)/6 \) latin squares of order \( w/2^j \).

Proof (a sketch). According to Lemma 3.1, a generator matrix \( M \) of \( D \) divides the coordinates into \( 2^j - 1 \) groups of size \( w/2^j \) and one group of size \( w/2^j - 1 \) (the last group corresponds to the all-zero columns of \( M \)). It can be seen that the set of triples of every STS\((v)\) orthogonal to \( D \) is divided into the \( 2^j + (2^j - 1)(2^j - 2)/6 \) following subsets:

- The triples with all 3 points in the group of size \( w/2^j - 1 \) form STS\((w/2^j - 1)\).
- The triples with one points in the group \( \{y_1, y_2, y_3/2^j - 1\} \) of size \( w/2^j - 1 \) and two points in one of the \( 2^j - 1 \) groups \( \{x_1, \ldots, x_{w/2^j}\} \) of size \( w/2^j \). Such triples have the form \( \{x, y, f(x,y)\} \) for some symmetric latin square \( f \) satisfying \( f(x, x) \equiv w/2^j \). Proposition 2.2 relates \( f \) with a symmetric latin square of order \( w/2^j - 1 \).
- For every 3 distinct groups \( \{x_1, \ldots, x_{w/2}\} \), \( \{\beta_1, \ldots, \beta_{w/2}\} \), and \( \{y_1, \ldots, y_{w/2}\} \) corresponding to columns \( a, b, \) and \( c \) with \( a + b + c = 0 \), the triples with one point in each of these 3 groups have the form \( \{x, y, f(x,y)\} \) for some latin square \( f \) of order \( v/2^j \).

\( \square \)
Corollary 4.4 (Jungnickel and Tonchev [7]). Given a subspace \(D\) from \(\mathcal{D}_j\), the number \(\Phi(D)\) of STS\((w - 1)\) orthogonal to \(D\) equals \(\Phi_{w-1,j}\), where

\[
\Phi_{w-1,j} = \Psi_{w/2-1,j}^2 \Pi_{w/2-1}^{2/3} \Lambda_{w/2}\mathbb{Z},
\]

where \(\Psi_u\) is the number of STS\((u)\) (and the number of idempotent totally symmetric latin squares of order \(u\)), \(\Pi_u\) is the number of symmetric latin squares of order \(u\), and \(\Lambda_u\) is the number of latin squares of order \(u\).

Lemma 4.5. Assume that \(w\) is divided by \(2^k\) and \(k\) is the largest integer with this property. Let \(i \in \{0, \ldots, k\}\), and let \(D\) be in \(\mathcal{D}_i\). The number of STS\((w - 1)\) with dual space \(D\) equals \(\Upsilon_{w-1,i}\), where

\[
\Upsilon_{w-1,i} = \sum_{j=1}^{k} \Gamma_{w,i,j} \mu_{j-i}^{(2)} \Phi_{w-1,j},
\]

where \(\Gamma_{w,i,j}\) and \(\Phi_{w-1,j}\) are from Lemma 3.2 and Corollary 3.4, respectively.

The proofs of the lemma and the following theorem are the same as those of Lemma 3.5, and we omit them.

Theorem 4.6. Assume that \(w\) is divided by \(2^k\) and \(k\) is the largest integer with this property. Let \(i \in \{0, \ldots, k\}\). The total number of different STS\((w - 1)\) of \(2\)-rank \(w - i - 1\) equals

\[
\Gamma_{w,0,i} \sum_{j=1}^{k} \Gamma_{w,i,j} \mu_{j-i}^{(2)} \Phi_{w-1,j},
\]

where \(\mu_{j}^{(2)} = (-1)^{\binom{j}{2}}\), \(\Phi_{w-1,j} = \Psi_{w/2-1,j}^2 \Pi_{w/2-1}^{2/3} \Lambda_{w/2}\mathbb{Z}\), \(\Psi_u\) is the number of STS\((u)\) (and also the number of idempotent totally symmetric latin squares of order \(u\)), \(\Pi_u\) is the number of symmetric latin squares of order \(u\) (and also \(u!\) times the number of \(1\)-factorizations of the complete graph of order \(u + 1\)), \(\Lambda_u\) is the number of latin squares of order \(u\), and

\[
\Gamma_{w,i,j} = \left(\frac{w}{2^j}\right)^{2} / 2^{(j-i)(i+1)/2} \left(\frac{w}{2^j}\right)^{j-i} \prod_{x=1}^{j-i} (2^x - 1).
\]

Corollary 4.7. The number of STS\((w - 1)\), \(w = 2^k\), of \(2\)-rank \(w - k\) is

\[
w!(2^{w^2/24-3w/4+k+1/3} - 1) / 2^{k(k+1)/2}[k - 1]_2! \quad \text{(see [19]).}
\]

The number of STS\((w - 1)\), \(w = 2^k\), of \(2\)-rank \(w - k + 1\) is

\[
w!(3^{w^2/48-w/4+2/3} - 2^{w^2/16-5w/4+2k-1} - 3.2^{w^2/24-3w/4+k-2/3} + 1) / 3.2^{k(k+1)/2}[k - 2]_2! \quad \text{(see [28]).}
\]
The number of STS\((w - 1)\), \(w = 2^k\), of 2-rank \(w - k + 2\) is
\[
\frac{2^k 1}{21 \cdot 2^{k(k+1)/2-3} [k - 3]_2!} \times (780^{w/8-1} \cdot (2^{28}, 3^5, 5^2, 7^2, 1361291)^{w^2/384 - w/16 + 1/3, 2^{3(k-1)}} - 7 \cdot 2^{w^2/16 - 5w/4 + 2k - 3, 3^{w^2/48 - w/4 + 2/3} + 7 \cdot 2^{w^2/24 - 3w/4 - 5/3 + k - 1}) \text{ (see [26])}.
\]

The number of STS\((10w - 1)\), \(w = 2^k\), of 2-rank \(10w - 1 - k\) is
\[
\frac{(10w)! \cdot 122556672^{w-1} \cdot (2^{43}, 3^{10}, 5^4, 7^2, 31, 37, 547135293937)^{(w-1)(w-2)/6}}{2^{(k(k+1)/2)+5, 135} [k]_2!}.
\]

**Proof.** To apply the formula from Theorem 4.6, in addition to the values considered in the proof of Corollary 3.7, we need \(\Pi_1 = 1\), \(\Pi_2 = 6\), \(\Pi_3 = 31449600 = 7! \cdot 6240\) [22], \(\Pi_9 = 444733651353600 = 9! \cdot 1225566720\) [5], see also [33],

\[
\begin{align*}
\Lambda_2 & = 2, \quad \Lambda_4 = 576, \quad \Lambda_8 = 108776032459082956800 = 2^{28} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 1361291 \quad [23], \\
\Lambda_{10} & = 9982437658213039871725064756920320000 = 2^{43} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 31 \cdot 37 \\
& \quad \cdot 547135293937 [12],
\end{align*}
\]

see also [31]. We also formally need the trivial values \(\Pi_0 = 1\) and \(\Phi_0 = 1\). \(\square\)

**Remark 3.** Taking into account Propositions 2.1 and 2.2, we know that \(\Psi(u - 1)\) and \(\Pi(u - 1)\) are the numbers of latin squares of order \(u\) with certain restrictions. So, \(\Psi(u - 1) < \Pi(u - 1) < \Lambda(u)\). It can be then noted that if \(j \geq 3\), then the most valuable factor in the formulas for the number of STS\((v)\) of 2- or 3-rank at most \(v - j\) is connected with the number of unrestricted latin squares.

### 5 CONCLUDING REMARKS

As we see from the results of [7], the structure of the STSs of deficient rank, either 2- or 3-rank, with fixed orthogonal subspace, is well understood, meaning that it can be described in terms of latin squares and STSs of smaller order. The possibility to derive an explicit formula for the number of the non-full-rank STS (involving the number of latin squares and smaller STS) implies that this description is constructive even if we do not fix the orthogonal subspace of the systems. The following simple statement shows that the benefits given by the knowledge of the structure of an STS depending on the value of its 2-rank or the value of its 3-rank cannot be combined in the same system.

**Theorem 5.1.** There is no an STS of order \(v\) larger than 3 that is both non-full-2-rank and non-full-3-rank, that is, of 2-rank less than \(v\) and 3-rank less than \(v - 1\).
Proof. Assume that $S$ an STS($v$), $v > 3$, which is (a) of 3-rank at most $v - 2$ and (b) 2-rank at most $v - 1$. By Lemma 3.1, (a) means that there is a vector with $v/3$ zeros, $v/3$ ones, and $v/3$ twos that is orthogonal to $S$ over GF(3); in particular, $v \equiv 0 \mod 3$. Assumption (b) means that $S$ has a Steiner subsystem $S'$ of order $(v - 1)/2$, by Lemma 4.3. Since $v > 3$ implies $(v - 1)/2 > v/3$, the system $S'$ is orthogonal over GF(3) to a vector that is not all-0, all-1, or all-2. By Lemma 3.1, the order $(v - 1)/2$ is an integer divisible by 3, and we get $v \equiv 1 \mod 3$, a contradiction. □

Finally, we briefly discuss the number of isomorphism classes of STS of a prescribed 2- or 3-rank, which can be evaluated utilizing the following observation.

**Proposition 5.2** (see Jungnickel and Tonchev [7, equation (12)], where $C$ is the whole space). The number $N$ of isomorphism classes of STS($v$) from some family $S$ closed under isomorphism satisfies

$$\frac{|S|}{v!} \leq N \leq U \frac{|S|}{v!}$$

where $U$ is the maximum number of automorphisms of an STS from $S$.

So, the formulas given in Theorems 3.6 and 4.6 for the number of STSs of a prescribed 3- or 2-rank, respectively, immediately give the lower bounds on the number of isomorphism classes of such systems. (Note that these lower bounds are asymptotically the same as the bounds in [7, Theorems 4.4 and 4.12]; the difference is that using Theorems 3.6 and 4.6, we more accurately exclude from the counting the systems of smaller rank.) To obtain an upper bound, one can substitute the upper bounds on $U$ proposed in [7, Lemma 4.1] and [7, Lemma 4.9] for the cases of 3- and 2-rank, respectively. Clearly, the upper bound in Proposition 5.2 is far from the real value, as the most of systems have less than $U$ automorphisms. Obtaining asymptotically tight formulas for the number of STSs($v$) of rank $i$, for different behaviors of $i = i(v)$, remains an important problem in this topic.

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