Kink Oscillations of a Curved, Gravitationally Stratified Coronal Loop

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ABSTRACT

Loops of magnetic field in the corona are observed to oscillate and these oscillations have been posited to be the superposition of resonant kink waves, trapped between the loop’s footpoints in the photosphere. To date, most analyses of these oscillations have concentrated on calculating the frequency shifts that result from spatial variation in the kink wave speed (produced primarily by stratification in the density and magnetic field strength). Further, most have ignored gravity and treated the loop as a straight tube. Here we ignore spatial variation in the wave speed, but self-consistently include the effects of gravity and loop curvature in both the equilibrium loop model and in the wave equation that governs the propagation of the kink waves that live upon the loop. We model a coronal loop as an isolated, thin, magnetic fibril that is anchored at two points in the photosphere. The equilibrium shape of the loop is determined by a balance between magnetic buoyancy and magnetic tension, which is characterized by a Magnetic Bond Number $\epsilon$ that is typically small $|\epsilon| \ll 1$. This balance produces a loop that has a variable radius of curvature, with the legs being relatively straight and the apex of the loop the most curved. Further, a loop becomes unstable to buoyant rise if the footpoint separation becomes larger than $2\pi$ times the coronal pressure scale height. The resonant kink waves of such a loop come in two polarizations that are decoupled from each other: waves with motion completely within the plane of the loop (normal oscillations or “vertical modes”) and waves with motions that are completely horizontal, perpendicular to the plane of the loop (binormal oscillations or “horizontal modes”). We solve for the eigensolutions of both polarizations using perturbation theory for small Magnetic Bond
Number. For modes of the same order, normal oscillations have smaller eigen-frequencies than binormal oscillations. The additional forces of buoyancy and magnetic tension from the curvature of the loop increase and decrease the mode frequencies, respectively. The ratio of the frequencies of the first overtone to the fundamental mode—a common diagnostic used to assess the stratification of the wave speed along the loop—is modified by the inclusion of buoyancy and curvature. We find that the normal polarization possesses a frequency ratio that exceeds the canonical value of 2, whereas the binormal polarization has a ratio less than 2.

Subject headings: MHD — waves — Sun: Corona — Sun: magnetic fields — Sun: oscillations

1. Introduction

Initially through TRACE and EIT observations, and now ongoing with AIA imagery (e.g., White et al. 2012), the loops of magnetic field that often overlie magnetic active regions are seen to oscillate with frequencies of typically 2–4 mHz (e.g., Schrijver et al. 1999; Verwichte et al. 2004; van Doorsselaere et al. 2007). The attribution of these oscillations to the modes of a one-dimensional cavity, and the observational identification of overtone frequencies of this cavity, has given birth to the field of coronal seismology (see Nakariakov & Verwichte 2005 and the references therein). The promise offered by observable resonant oscillation frequencies has raised hopes for determining physical parameters of coronal loops which cannot otherwise be measured directly.

Although multiple claims have been made for the observation of sausage waves in post-flare coronal loops (e.g., Srivastava et al. 2008; Nakariakov & Verwichte 2005), the majority of the observed oscillations useful for seismological purposes are kink oscillations (Schrijver et al. 1999, 2002; Aschwanden et al. 1999, 2002; Verwichte et al. 2004). The dispersion in the observed overtones, compared to theoretical predictions, have been used mainly to estimate the mean magnetic field strength (e.g., Nakariakov & Ofman 2001), it’s longitudinal gradient (Verth & Erdélyi 2008; Ruderman et al. 2008), or the density stratification inside and outside the loop (Donnelly et al. 2006; Diaz et al. 2004, 2007; McEwan et al. 2008; Orza et al. 2012; van Doorsselaere et al. 2007; Goossens et al. 2006). Observational efforts are now focused on improving the measurement of oscillation eigenfrequencies by reducing the observational errors and by analyzing additional loops. A relevant overview of observational measurements can be found in Andries et al. (2009).
On the theoretical side, advancements are steadily being made by including additional physical effects with the aim of developing a general model of coronal loops and their oscillations. Most of this work is based on the pioneering calculations of Edwin & Roberts (1983), which were presciently performed prior to the first observations of coronal loop waves. This work revealed the MHD wave modes of a translationally invariant magnetic cylinder. Since then Diaz et al. (2004) and Andries et al. (2005) have investigated modifications to the eigenfrequencies that arise from density stratification along the loop. The expected dispersion has been compared to measured values with the goal of constraining coronal loop models (see also Goossens et al. 2006; Andries et al. 2005; McEwan et al. 2008). Further, the quantitative effects of loop geometry were investigated by Dymova et al. (2006) and Verth & Erdélyi (2008), with particular focus on the curvature of the loop axis. For example, van Doorsselaere et al. (2004) have shown that the frequency shift induced by curvature appears at second order in the ratio of tube radius to the loop length, which is believed to be quite small for the majority of loops. They and others have also suggested that since the eigenfrequencies for both horizontal and vertical (Wang & Solar 2004) polarizations are identical, there is no distinction between the oscillations with different polarization (e.g., Terradas et al. 2006), unless of course the tube is not locally axisymmetric (Ruderman 2003).

Here, our goal is to self-consistently include the effects of gravity and curvature, both in the establishment of the static equilibrium of a coronal loop and in the propagation of kink waves along the loop. We accomplish both of these goals by exploiting the thin-tube approximation, whereby lateral variation across the tube is ignored. Under this assumption the equilibrium shape of the coronal loop is established by a balance between the forces of magnetic buoyancy and magnetic tension. The eigenmode problem is reduced to a one-dimensional wave equation which we solve through perturbation theory. The paper is organized as follows: in Section 2 we present the equation of motion appropriate for a thin magnetic fibril. In Section 3 we derive the equilibrium height and shape of the loop, while Section 4 details the solution to the kink wave eigenmode problem. In Section 5, we discuss the significance of our findings and the implications of our base assumptions. Finally, we state our findings and conclusions in Section 6.

### 2. Equation of Motion

We will treat a coronal loop as a thin magnetic fibril embedded within a field-free coronal atmosphere. We recognize that both of these assumptions are suspect. Even though the thin tube approximation may only apply to a subset of coronal loops, it provides mathematical tractability. The assumption that the loop is an isolated magnetic structure is probably
never strictly valid, as the entirety of the solar corona is magnetically dominated and most loops are an integral part of larger magnetic structures. In fact, most loops are probably just selectively illuminated field lines amongst many that form magnetic arcades. Despite these objections, we adopt an isolated fibril model for convenience. The consequences of these assumptions are discussed in more detail in Section 5.

The forces acting on an isolated, thin magnetic fibril have been previously derived by Spruit (1981) through averaging the MHD momentum equation over the cross-sectional area of the tube. In terms of the acceleration of the tube, the resulting equation of motion is given by

$$\frac{dv}{dt} = \left[ \hat{s} \cdot \left( -\frac{1}{\rho} \nabla p + g \right) \right] \hat{s} + c^2 \mathbf{k} + D (\hat{s} \times \mathbf{g}) \times \hat{s},$$

(2.1)

where all fluid properties represent cross-sectional averages and \( \hat{s} \) and \( \mathbf{k} \) are the instantaneous tangent unit vector and curvature vector of the tube’s axis. The gravitational acceleration is \( \mathbf{g} \) and \( p \) is the gas pressure. The longitudinal forces (those in the \( \hat{s} \) direction) are just the pressure and gravitational forces that act parallel to the magnetic field. The last two terms on the right-hand side are the transverse forces of magnetic tension and buoyancy with \( c \) being the kink wave speed and \( D \) the tube’s fractional overdensity compared to its surroundings,

$$c^2 \equiv \frac{B^2}{4\pi (\rho + \rho_e)},$$

(2.2)

$$D \equiv \frac{\rho - \rho_e}{\rho + \rho_e}.$$

(2.3)

The densities \( \rho \) and \( \rho_e \) are, respectively, the mass density internal and external to the tube and \( B \) is the axial magnetic field strength of the fibril.

The transverse forces (tension and buoyancy) in equation (2.1) include the effects of “enhanced inertia,” which represents the backreaction caused by the external medium when transverse motions of the tube move surrounding fluid out of the way. The form of the enhanced inertia is the same as propounded by Spruit (1981), whereby the density in the inertial term is simply augmented, \( \rho \rightarrow \rho + \rho_e \). This form has inspired much controversy (e.g., Choudhuri 1990, Cheng 1992, Fan et al. 1994), but faithfully reproduces the proper kink wave speed and is consistent with all alternate formulations as long as (1) the background atmosphere lacks rotational shear and magnetic field, (2) flow along the tube is not permitted, and (3) correction terms that are quadratic in the fluid velocity are ignored.
3. The Equilibrium Shape of the Loop

We neglect the spherical geometry of the solar atmosphere and assume that the corona can be treated as plane-parallel with constant gravity \( \mathbf{g} \). We employ a Cartesian coordinate system, with the \( x-y \) plane corresponding to the photosphere and the \( z \) coordinate increasing upwards (i.e., \( \mathbf{g} = -g\hat{z} \)). The equilibrium position of the coronal loop is confined to the \( x-z \) plane with the endpoints of the loop anchored in the photosphere at the coordinates \((\pm X,0,0)\). The height of the loop above the photosphere is given by the function \( z_0(x) \), such that the loop’s axis is traced by the position vector \( \mathbf{r}_0(x) = x\hat{x} + z_0(x)\hat{z} \). Figure 1 provides an illustration of the geometry of the loop and the coordinate system.

In addition to this Cartesian coordinate system, we define a local set of Frenet coordinates for the equilibrium position of the loop. Let \( s \) denote the arclength along the loop measured from the footpoint located at \( x = -X \). The longitudinal unit vector, or the tangent vector that lies parallel to the loop’s axis pointing in the direction of increasing \( s \), is denoted \( \hat{s}_0 \). Within the \( x-z \) plane and everywhere perpendicular to \( \hat{s}_0 \) is the principle normal \( \hat{k}_0 \). We shall soon see that for the loop model we adopt, only convex loops are stable. Therefore, the curvature vector \( \mathbf{k}_0 \) always points inward and downward in the direction of the principle normal \( \hat{k}_0 \), with a magnitude equal to the reciprocal of the local radius of curvature \( R_0^{-1} \) of the loop’s axis. The third orthogonal direction is given by the binormal unit vector \( \hat{b}_0 = \hat{s}_0 \times \hat{k}_0 \), which is everywhere constant \( \hat{b}_0 = \hat{y} \). In what follows we will need the standard geometrical equations which describe the Frenet unit vectors, \( \hat{s}_0 \) and \( \hat{k}_0 \), and the radius of curvature \( R_0 \) in terms of the loop height,

\[
\hat{s}_0 \equiv \frac{\partial \mathbf{r}_0}{\partial s} = \hat{x} + z'_0(x)\hat{z} \quad s'(x),
\]

\[
\mathbf{k}_0 \equiv \frac{\partial \hat{s}_0}{\partial s} = \frac{1}{R_0} \frac{z'_0(x)\hat{x} - \hat{z}}{s'(x)},
\]

\[
R_0(x) \equiv \frac{1}{|\mathbf{k}_0|} = -\frac{[s'(x)]^3}{z'_0(x)},
\]

\[
s'(x) = \sqrt{1 + [z'_0(x)]^2},
\]

where primes denote differentiation with respect to the photospheric coordinate \( x \).

Since a loop in a state of static equilibrium is confined to the \( x-z \) plane, none of the forces on the fibril have a component in the binormal \( y \)-direction and we only need to consider two components of the equation of motion (2.1),
\[-\frac{1}{\rho_0} \frac{\partial \rho_0}{\partial s} - g \hat{z} \cdot \hat{s}_0 = 0, \tag{3.8}\]
\[\frac{c_0^2}{R_0} - g D_0 \hat{z} \cdot \hat{k}_0 = 0. \tag{3.9}\]

In these two equations the subscript 0 indicates a background, equilibrium quantity within the loop. The triple cross-product appearing in the buoyancy term in equation (2.1) has been reduced to a simpler form by making use of the identities \((\hat{s}_0 \times \hat{z}) \times \hat{s}_0 = \hat{z} - (\hat{z} \cdot \hat{s}_0)\hat{s}_0 = (\hat{z} \cdot \hat{k}_0)\hat{k}_0\).

Equation (3.8) expresses the balance of forces in the tangential or axial direction \(\hat{s}_0\), whereas equation (3.9) describes the balance in the transverse direction of the principle normal \(\hat{k}_0\). The axial equation is simply hydrostatic balance along magnetic field lines which we will satisfy trivially by assuming that the loop and the surrounding corona are both isothermal. Furthermore, since the tube is thin and thermal diffusion wipes out lateral variations, the temperatures inside and outside the tube are identical, \(T_0 = T_e\). All densities and pressures (both gas and magnetic) therefore vary exponentially with height \(z\) with a common scale height \(H = R_{\text{gas}} T_0 / g\). A consequence of this thermal uniformity is that the kink speed \(c_0\), the plasma-parameter \(\beta = 8\pi p_0 / B_0^2\), and the overdensity \(D_0\) are all constants.

The shape of the loop \(z_0(x)\) is constrained by the balance of the transverse forces of magnetic tension and magnetic buoyancy, which is quantified by equation (3.9). Since the temperatures inside and outside the loop are identical, and the external fluid is field free, the tube must be partially evacuated \((D_0 < 0)\) in order to ensure pressure continuity across the lateral boundary of the flux tube. Thus, magnetic buoyancy tries to lift the loop higher into the atmosphere and magnetic tension arising from the convex curvature of the loop tries to hold it down. Since the kink speed \(c_0\) is constant for our isothermal model, all variation along the loop in the magnitude of the magnetic tension force comes from changes in the radius of curvature \(R_0\). Similarly, the buoyancy only varies because the transverse direction \(\hat{k}_0\) rotates relative to the vertical \(\hat{z}\).

If we insert equations (3.5) and (3.6) into equation (3.9), we obtain a nonlinear ODE for the height of the loop \(z_0(x)\),

\[\frac{z''_0}{1 + (z'_0)^2} = \frac{\epsilon}{X}, \tag{3.10}\]

where we have defined a Magnetic Bond Number,
The Magnetic Bond Number can be written in a more recognizable form if we multiply the numerator and denominator by the enhanced inertia \( \rho + \rho_e \),

\[
\epsilon = \frac{gD_0 X}{c_0^2}.
\]  

(3.11)

In this expression the surface tension that normally appears in the denominator of the Bond Number has been replaced by magnetic tension. The Magnetic Bond Number \( \epsilon \) is a signed, dimensionless number that embodies the relative importance of the buoyancy and magnetic forces. The Magnetic Bond Number appears in calculations of the Rayleigh-Taylor instability when one of the fluid layers is filled with a horizontal field, providing the critical wavenumber below which instability ensues, \( k_c = -\epsilon/X \) (Chandrasekhar [1961]).

For our isothermal model, \( \epsilon \) is a constant and furthermore it can be expressed as a ratio of length scales, in particular the ratio of the footpoint separation to the pressure scale height \( \epsilon = -X/2H \). This property is easily derived by realizing that the combination of the perfect gas law and pressure continuity across the lateral surface of the tube allows one to express the magnetic pressure inside the tube as a ratio of the density difference to the fluid temperature. When this expression is inserted into equation (3.12), the density difference cancels in the numerator and denominator, leaving just the pressure scale height in the denominator (which is proportional to the temperature).

Since \( \epsilon \) is a constant, equation (3.10) can be integrated directly to obtain the derivative of the height function \( z'_0(x) \). Similarly, this expression for the derivative can be subsequently integrated to obtain the height function itself \( z_0(x) \). We choose the constants of integration such that \( z_0(\pm X) = 0 \) and \( z'_0(0) = 0 \). A summary of the resulting properties of the loop is provided below:

\[
z_0(x) = \frac{X}{\epsilon} \ln \left( \frac{\cos \epsilon}{\cos (\epsilon x/X)} \right),
\]  

(3.13)

\[
z'_0(x) = \tan (\epsilon x/X),
\]  

(3.14)

\[
z''_0(x) = \frac{\epsilon}{X} \sec^2 (\epsilon x/X),
\]  

(3.15)
\[ R_0(x) = -\frac{X}{\epsilon} \sec \left( \frac{\epsilon x}{X} \right) \text{,} \quad (3.16) \]
\[ s(x) = \frac{X}{2\epsilon} \ln \left( \frac{1 + \sin \epsilon}{1 - \sin \left( \frac{\epsilon x}{X} \right)} \frac{1 + \sin \left( \frac{\epsilon x}{X} \right)}{1 - \sin \epsilon} \right) \text{.} \quad (3.17) \]

The last of these equations was obtained by integrating equation (3.7), with the constant of integration chosen such that \( s(-X) = 0 \). Figure 2 displays solutions for several different values of \( \epsilon \). For reference, the dashed curve corresponds to a semi-circle, clearly illustrating the variable radius of curvature for the loops.

The length of the loop \( L \) is obtained by inserting the footpoint position \( x = X \) into equation (3.17),
\[ L = \frac{X}{\epsilon} \ln \left( \frac{1 + \sin \epsilon}{1 - \sin \epsilon} \right) \text{.} \quad (3.18) \]

Two interesting limits of this equation exist. As \( \epsilon \to 0 \) the length of the loop converges to the footpoint separation \( L \to 2X \). This arises because the loop becomes straight, flat, and confined to the photosphere. As \( \epsilon \to \pm \pi/2 \) the loop length diverges logarithmically because the height of the loop grows without bound. This can be seen by recognizing that the loop reaches its apex at its center \( (x = 0) \), therefore achieving a maximum height
\[ z_{\text{apex}} = z_{0}(0) = \frac{X}{\epsilon} \ln \left( \cos \epsilon \right) \text{.} \quad (3.19) \]

One can easily see that \( \epsilon = \pm \pi/2 \) corresponds to a logarithmic singularity in the height.

Clearly the height \( z_0(x) \) should be a positive function for the range \( x \in (-X, X) \). With a little thought, from equation (3.13) we can see that this is only possible if \( \epsilon \in (-\pi/2, 0) \). Therefore, for constant \( c_0 \) only underdense tubes \( (D_0 < 0) \) form stable loops and those loops are comprised of a single arch without inflection points \( (z''_0 < 0) \). Furthermore, for \( z_0 \) to remain positive, \( \epsilon \) must be bounded and an equilibrium is only possible if the magnetic tension exceeds a threshold,
\[ \frac{B_0^2}{4\pi} > \frac{B^2}{4\pi} = 2\pi^{-1}g(\rho_e - \rho)X \text{.} \quad (3.20) \]

This condition for stability can also be used to constrain the footpoint separation. Since, \( \epsilon = -X/2H \), stability requires that the footpoint separation be less than a critical value that depends on the corona’s pressure scale height.
Typically, we do not need to be overly concerned about the limit of large buoyancy forces. For most coronal loops the Magnetic Bond Number \( \epsilon \) is quite small. Using typical values of the footpoint separation \( 2X = 150 \) Mm, kink wave speed \( c_0 = 1 \) Mm s\(^{-1}\) (Tomczyk et al. 2007), overdensity \( D_0 = -1 \) (appropriate for very small plasma parameter \( \beta \)), and gravity \( g = 2.7 \times 10^{-4} \) Mm s\(^{-2}\), we find from equation (3.11) that \( \epsilon = -2 \times 10^{-2} \). For small values of \( \epsilon \), the loop is short and low-lying with \( z_{\text{apex}} = |\epsilon| X/2 \). This geometry is realized because the field lines exit the photosphere at an oblique angle—i.e., \( z'_0(\pm X) = \tan \epsilon \). As \( \epsilon \) becomes small, the angle between the fibril’s axis and the photospheric surface approaches zero (see Figure 2).

4. Kink Wave Oscillations

Kink waves cause the axis of the fibril to undulate; therefore, we need to consider wave-induced modulation of the local direction vectors, \( \hat{s} \) and \( \hat{k} \). Let the instantaneous position vector of the loop be described as a time-varying perturbation about a static equilibrium. We will consider two transverse displacements of the axis, one displacement \( \xi(s, t) \) polarized in the direction of the principle normal \( \hat{k}_0 \) (which is confined to the \( x-z \) plane) and a perpendicular displacement \( \zeta(s, t) \) polarized in the binormal \( \hat{y} \) direction (i.e., horizontal and parallel to the photospheric plane),

\[
\mathbf{r}(s, t) = \mathbf{r}_0(s) + \xi(s, t) \hat{k}_0(s) + \zeta(s, t) \hat{y} .
\]

In the equation above, as before, the coordinate \( s \) is the pathlength along the unperturbed position of the tube. If we carefully consider the perturbations to the unit vectors, we find to first order in the displacements

\[
\dot{s} = \dot{s}_0 + \frac{\partial \xi}{\partial s} \hat{k}_0 + \frac{\partial \zeta}{\partial s} \hat{y} ,
\]

\[
\mathbf{k} = \frac{\hat{k}_0}{R_0} - \frac{1}{R_0} \frac{\partial \xi}{\partial s} \dot{s}_0 + \left( \frac{\partial^2 \xi}{\partial s^2} + \frac{\xi}{R_0} \right) \hat{k}_0 + \frac{\partial^2 \zeta}{\partial s^2} \hat{y} ,
\]

\[
R^{-1} = R_0^{-1} + \left( \frac{\partial^2 \xi}{\partial s^2} + \frac{\xi}{R_0^2} \right) .
\]
Since, the equilibrium configuration of the loop lacks curvature in the \( \hat{y} \) direction, the radius of curvature does not depend on the \( \zeta \) displacement, at least to linear order. Because of this, the two polarizations decouple, and \( \xi \) and \( \zeta \) satisfy independent equations. These two wave equations are obtained by noting that transverse oscillations of the thin fibril are largely incompressive; therefore, the longitudinal component of the equation of motion, equation (2.1), can be ignored.

Differential equations describing the two polarizations of oscillation can be derived by inserting equations (4.23)–(4.25) into equation (2.1), Fourier transforming in time, and linearizing in the displacements,

\[
c_0^2 \left( \frac{d^2}{ds^2} + \frac{1}{R_0^2} \right) \xi - g\| D_0 \frac{d\xi}{ds} + \omega^2 \xi = 0 ,
\]

\[
c_0^2 \frac{d^2 \zeta}{ds^2} - g\| D_0 \frac{d\zeta}{ds} + \omega^2 \zeta = 0 .
\]

The influence of the geometry of the equilibrium loop model is felt only through the radius of curvature \( R_0 \) and through the parallel component of gravity \( g\| \equiv \mathbf{g} \cdot \hat{s}_0 \). The terms proportional to \( c_0^2 \) represent magnetic tension, with the radius of curvature corresponding to the contribution from the curvature of the equilibrium field lines. The terms with \( g\| \) are buoyancy forces and the inertial terms are those that depend on the temporal frequency \( \omega \). These are quite general equations that describe oscillations for any loop equilibrium—not just the isothermal model we consider here. In fact, except for the inclusion of curvature, these are the same equations used by Jain & Hindman (2012) to study the effects of buoyancy and wave speed variations on mode frequencies.

Let’s first examine the tension terms in detail. When these two equations are compared, we see that the curvature of the tube only appears in the equation for the normal oscillations. This additional tension term from curvature typically has the opposite sign from the tension arising from undulations. Thus, the two tension forces that appear in the equation for normal oscillations act in opposition and we should expect that curvature of the loop reduces the eigenfrequencies. Further, if we consider the limit of small Magnetic Bond Number \( \epsilon \), we can see from equation (3.16) that the radius of curvature is large \( R_0 = O(\epsilon^{-1}) \). Therefore, the curvature of the tube should generate a second-order frequency shift \( \Delta \omega / \omega = O(\epsilon^2) \).

If we insert \( g\| \equiv -g\hat{z} \cdot \hat{s}_0 \) into equation (4.27), and include the Magnetic Bond Number dependence directly, we get
\[
\frac{d^2 \zeta}{ds^2} + \frac{\epsilon}{X} \hat{z} \cdot \hat{s}_0 \frac{d \zeta}{ds} + \frac{\omega^2}{c_0^2} \zeta = 0 .
\] (4.28)

A cursory examination of this equation would seem to indicate that for small \( \epsilon \), the frequency shift from buoyancy forces should be a first-order quantity. However, this is incorrect. We have previously demonstrated that the equilibrium height and shape of the loop sensitively depend on the Magnetic Bond Number, and in particular as \( \epsilon \to 0 \) the loop becomes flat and horizontal. In fact, by using equations (3.4), (3.7), and (3.14) one finds

\[
\hat{z} \cdot \hat{s}_0 = \sin \left( \frac{\epsilon x}{X} \right) .
\] (4.29)

Since \( \sin \left( \frac{\epsilon x}{X} \right) = O(\epsilon) \) for small \( \epsilon \), it is now clear that buoyancy should cause frequency shifts that are second order in the Magnetic Bond Number. This makes sense, as curvature also enters at second order and buoyancy and curvature are balanced in the equilibrium configuration of the loop.

The quantities \( g_\parallel \) and \( R_0 \) depend on the equilibrium shape of the loop, which depends explicitly on the horizontal coordinate \( x \)—see equations (3.13)–(3.17). Therefore, it is quite natural to make a change of variable from the arclength variable \( s \) to the horizontal \( x \) coordinate. After inserting the properties of the equilibrium, we find that the resulting ODEs lack first-derivative terms,

\[
\frac{d^2 \xi}{dx^2} + \left[ \frac{\omega^2}{c_0^2} \frac{1}{\cos^2 \left( \frac{\epsilon x}{X} \right)} + \frac{\epsilon^2}{X^2} \right] \xi = 0 ,
\] (4.30)

\[
\frac{d^2 \zeta}{dx^2} + \frac{\omega^2}{c_0^2} \frac{1}{\cos^2 \left( \frac{\epsilon x}{X} \right)} \zeta = 0 .
\] (4.31)

The absence of the first derivatives occurs because the tension and buoyancy forces in the equilibrium are in exact balance.

Both of these equations have solutions in the form of Associated Legendre Functions,

\[
\xi(x) = (\cos \theta)^{1/2} \left[ A_1 P_{1/2}^\mu \left( \sin \theta \right) + A_2 Q_{1/2}^\mu \left( \sin \theta \right) \right] ,
\] (4.32)

\[
\zeta(x) = (\cos \theta)^{1/2} \left[ A_3 P_{-1/2}^\mu \left( \sin \theta \right) + A_4 Q_{-1/2}^\mu \left( \sin \theta \right) \right] ,
\] (4.33)
The dependence on the frequency $\omega$ is buried in the upper index $\mu$ of the Associated Legendre Functions. Therefore, these solutions are not particularly friendly, and applying boundary conditions to quantize the eigenfrequencies would require a nontrivial numerical computation. We avoid this difficulty by realizing that the Magnetic Bond Number $\epsilon$ is a small dimensionless parameter, thus allowing the use of perturbation theory.

The equations that describe oscillations in both the normal and binormal directions have very similar form. In fact, a brief inspection of equations (4.30) and (4.31) reveals that the eigenfunctions of the two polarizations should be identical up through second order in $\epsilon$. This means that the derivation of the solutions for the two polarizations are similar; therefore, we will demonstrate the perturbation analysis for only the normal component and simply state the result for the binormal component. Assume that the eigenfunction and eigenfrequency can be written as perturbation expansions,

$$\xi_n(x) = \Xi_n(x) + \epsilon^2 \delta\xi_n(x) + O(\epsilon^4) ,$$  \hspace{1cm} (4.36)$$

$$\omega_n^2 = \Omega_n^2 + \epsilon^2 \delta\omega_n^2 + O(\epsilon^4) .$$  \hspace{1cm} (4.37)$$

Insert these expansions into equation (4.30) and solve in the usual fashion for the zeroth-order eigenfunction and eigenfrequency, noting that the cosine appearing in the second term can be expanded for small argument,

$$\Xi_n(x) = X \sin [\kappa_n(X - x)] ,$$  \hspace{1cm} (4.38)$$

$$\Omega_n = \kappa_n c_0 = \frac{n\pi c_0}{2X} .$$  \hspace{1cm} (4.39)$$

The boundary condition of vanishing displacement at the footpoints $\xi(\pm X) = 0$ has been applied.
If we now collect terms of the next order, we find the following inhomogeneous equation for the perturbed eigenfunction,

\[
\left( \frac{d^2}{dx^2} + \frac{\Omega_n^2}{c_0^2} \right) \delta \xi_n = - \left( \frac{\Omega_n^2}{c_0^2} \frac{x^2}{X^2} + \frac{\delta \omega_n^2}{c_0^2} + \frac{1}{X^2} \right) \Xi_n .
\]  

(4.40)

The particular solution can be found by judicious guessing and by enforcing that the perturbed eigenfunction has the same symmetry about the center of the loop as the unperturbed eigenfunction,

\[
\delta \xi_n(x) = - \frac{x^2}{X^2} \Xi_n(x) + \left[ \frac{x^3}{6X^2} + \frac{c_0^2}{\Omega_n^2} \frac{x}{2} \left( \frac{\delta \omega_n^2}{c_0^2} + \frac{1}{2X^2} \right) \right] \Xi_n'(x) .
\]  

(4.41)

The perturbed eigenfrequency is fixed by reinforcing that the eigenfunction vanishes at the footpoints for all orders of expansion \( \delta \xi_n(\pm X) = 0 \),

\[
\delta \omega_n^2 = - \left( \frac{\Omega_n^2}{3} + \frac{c_0^2}{2X^2} \right) .
\]  

(4.42)

If we now put all the pieces together, to second order in the small parameter \( \epsilon \), the eigensolution for the normal displacement is given by

\[
\xi_n(x) = \Xi_n(x) - \epsilon^2 \left[ \frac{x^2}{X^2} \Xi_n(x) + \frac{1}{2} \left( 1 - \frac{x^2}{X^2} \right) \Xi_n'(x) \right] + O(\epsilon^4) ,
\]  

(4.43)

\[
\omega_n^2 = \kappa_n^2 c_0^2 \left[ 1 - \epsilon^2 \left( \frac{1}{3} + \frac{1}{2\kappa_n^2 X^2} \right) + O(\epsilon^4) \right] .
\]  

(4.44)

By the same procedure the eigensolution for the binormal polarization can be obtained,

\[
\zeta_n(x) = \xi_n(x) + O(\epsilon^4) ,
\]  

(4.45)

\[
\omega_n^2 = \kappa_n^2 c_0^2 \left[ 1 - \epsilon^2 \left( \frac{1}{3} - \frac{1}{2\kappa_n^2 X^2} \right) + O(\epsilon^4) \right] .
\]  

(4.46)

Figure 3 illustrates the resulting eigenfrequencies as a function of \( \epsilon \) for displacements in both the normal and binormal directions. Figure 4 shows sample eigenfunctions.
5. Discussion

In the previous sections we have modeled a coronal loop as an isolated, thin magnetic flux tube. The effects of gravity and curvature have been self-consistently included, leading to a static equilibrium determined by the balance of magnetic buoyancy and tension. We have computed the eigensolutions for transverse oscillations of both polarizations. In the following subsections we will discuss in detail the effects that gravity and curvature have on the structure of the eigenfrequencies, assess the differences between the two polarizations of wave motion, and explore the implications of relaxing several of our operating assumptions.

5.1. The Effects of Gravity and Curvature on the Eigenfrequencies

There are only two free parameters in our model, the footpoint separation $2X$ and the Magnetic Bond Number $\epsilon$. The footpoint separation just provides a physical scale for the problem and the structure of the solutions do not depend upon it. The solutions do depend intrinsically on the Magnetic Bond Number, because the equilibrium shape of the loop is sensitive to the strength of buoyancy and magnetic tension. As the magnitude of the parameter $\epsilon$ becomes larger (more negative), buoyancy forces increase in strength, which causes the loop to rise and a corresponding increase in the curvature of the loop is needed to allow magnetic tension to bring the loop back into equilibrium. This results in three ways by which the frequencies are modified: the loop lengthens, the restoring force of buoyancy increases, and the increased curvature modifies magnetic tension. We can identify the first of these effects on the eigenfrequencies by expanding the loop length $L$, equation (3.18), for small $\epsilon$,

$$L = 2X \left[ 1 + \frac{\epsilon^2}{6} + O(\epsilon^4) \right].$$  \hfill (5.47)

If the loop were straight and without gravitational effects, the eigenfrequencies would be

$$\omega_n^2 \sim \frac{n^2 \pi^2 c_0^2}{L^2} = \Omega_n^2 \left[ 1 - \frac{\epsilon^2}{3} + O(\epsilon^4) \right].$$  \hfill (5.48)

As expected, an increase in the loop length (increasing $\epsilon^2$) decreases the eigenfrequencies and the resulting fractional frequency shift is dispersionless and independent of mode order,

$$\frac{\Delta \omega_n^2}{\Omega_n^2} = -\frac{\epsilon^2}{3}.$$  \hfill (5.49)
This effect appears directly, in equations (4.44) and (4.46) as the first term in the parentheses on the right-hand sides.

Buoyancy appears in identical form in the equations describing each polarization of transverse oscillation; both equation (4.26) and (4.27) have the same first-derivative term. This common form, leads to identical perturbations to the eigenfrequencies due to the inclusion of buoyancy. From equations (4.44) and (4.46) we can directly see that buoyancy increases the eigenfrequencies by the fractional amount,

$$\frac{\Delta \omega_n^2}{\Omega_n^2} = \epsilon^2 \frac{2}{n^2 \pi^2}.$$  

(5.50)

This might be a somewhat surprising result. A simple dimensional analysis of the equation of motion would indicate that buoyancy causes frequency shifts that are first order in the Magnetic Bond Number \(\epsilon\), and indeed this is what was found in Jain & Hindman (2012). However, the buoyancy force in the wave equation is proportional to \(\epsilon\) times the geometrical factor \(\hat{z} \cdot \hat{s}_0\). This geometrical factor also depends on \(\epsilon\) because the shape of the coronal loop is sensitive to \(\epsilon\). Low Magnetic Bond Numbers correspond to loops that are low-lying and nearly horizontal, and because of this the geometrical factor is first order in the Magnetic Bond Number, and the buoyancy force in total is second order.

The last effect, curvature of the loop, modifies the magnetic tension than can act as a restoring force. It only appears in the equation for the normal polarization, and its sign indicates that it counteracts the local curvature induced by the wave motions (i.e., the second-derivative of the displacement). Therefore, we would expect curvature of the background loop model to decrease the eigenfrequencies. This does indeed occur, as we can see directly from the difference between the frequencies of the normal and binormal polarizations. The curvature of the loop results in a negative frequency shift that is twice as large in magnitude as the effect of buoyancy,

$$\frac{\Delta \omega_n^2}{\Omega_n^2} = -\epsilon^2 \frac{4}{n^2 \pi^2}.$$  

(5.51)

The frequency shifts induced by both buoyancy and curvature enter at second order in the Magnetic Bond Number. That both enter at the same order is a simple consequence of the balance between buoyancy and tension in the equilibrium. That they both enter at second order could easily be predicted. From dimensional arguments alone, the shift in the square frequency due to curvature must scale with the radius of curvature as \(\Delta \omega^2 \sim c_0^2 R_0^{-2}\). Since the radius of curvature is inversely proportional to \(\epsilon\), the frequency shift must be second order in \(\epsilon\).
A consistent result for the curvature was found by van Doorsselaere et al. (2004) where they solved for fast waves in a toroidal coordinate system in the absence of gravity. The equilibrium field was assumed to be purely toroidal and a coronal loop modeled by a partial section of the torus. With an extensive mathematical analysis they concluded that the curvature of their torus induces a frequency shift that vanishes at first order in the ratio of $a/R_0$, where $a$ is the minor radius of the torus (and $R_0$ corresponds to the major radius). Therefore, the first non-zero perturbation must occur at second or higher order. Here we have shown that for our thin tube model the first non-zero perturbation to the frequency does indeed occur at second order in the radius of curvature, as a simple dimensional analysis would indicate.

5.2. The Two Polarizations of Oscillation

Many of the observed transverse oscillations appear to be horizontal (e.g., Aschwanden et al. 2002), and therefore in the binormal direction. Unfortunately, our assumption of an isolated magnetic loop is particularly suspect for such oscillations. Most of what we call coronal loops are probably just a bundle of field lines that are part of a much larger magnetic structure like a magnetic arcade. We identify this particular bundle as a “coronal loop” solely because it is selectively illuminated by a local heating process. If the loop is part of a magnetic arcade, oscillations in the direction of the principle normal (sometimes called “vertical” oscillations) may be unaffected by the larger arcade structure, because the undulations do not impinge on neighboring arcade field lines. However, binormal (horizontal) oscillations must inherently impact nearby field lines and the anisotropic nature of the arcade structure must modify wave propagation. In particular, waves may no longer be confined to the loop. Spanwise propagation along the arcade is possible, and since the other loops in the arcade may not be illuminated, this propagation may go undetected. Such waves may still be trapped, but the problem is inherently two-dimensional and our one-dimensional analysis incorrect.

Despite these difficulties, there are distinct differences between the two polarizations of waves that may be observable. Unfortunately, the eigenfunctions are of little direct use because the two polarizations have identical eigenfunctions (to second order in the square of the Magnetic Bond Number). Therefore, observations of loop motion must be able to directly account for projection and occultation effects if they are to determine polarization. The frequencies of oscillation are not degenerate, however, and offer promise to winnow between polarization modes.

Many observational and theoretical studies have examined the diagnostic value of the ratio of eigenfrequencies (for a review see Andries et al. 2009), in particular the ratio of
the fundamental and first overtone. For a straight magnetic fibril in the absence of gravity and stratification, the ratio of the first overtone to the fundamental $\omega_2/\omega_1$ is 2. Observational differences from 2 have been interpreted as evidence for wave speed stratification (e.g., McEwan et al. 2008), and potential wave speed models have been rejected based on whether the model generates a ratio that is less than or greater than 2. As we shall soon see, curvature and buoyancy forces can also cause the ratio to shift from the canonical value of 2, without any spatial variation in the wave speed.

For our model, the frequency ratio for any given pair of neighboring modes is given by

$$\frac{\omega_{n+1}}{\omega_n} = \frac{n+1}{n} \left[ 1 \pm \frac{\epsilon^2}{\pi^2} \frac{2n+1}{n^2(n+1)^2} + O(\epsilon^4) \right],$$

where the upper sign (+) corresponds to oscillations in the normal direction and the lower sign (−) to those in the binormal direction. For the ratio of the first overtone to the fundamental, this becomes

$$\frac{\omega_2}{\omega_1} = 2 \pm \frac{3}{2} \frac{\epsilon^2}{\pi^2} + O(\epsilon^4).$$

Since the effects of the variation in the loop length are dispersionless, they have dropped out in these ratio expressions, leaving just the effects of buoyancy and curvature. Direct buoyant augmentation of the restoring force causes the ratio to decrease, while curvature of the loop causes the ratio to increase. Since the normal polarization senses the effects of loop curvature, whereas the binormal polarization does not, the ratio of the first overtone to the fundamental is greater than 2 for normal oscillations and less than 2 for binormal displacements. The diagnostic value of the frequency ratio is clearly predicated on the knowledge of which polarization one is observing.

5.3. The Effects of an External Magnetic Field

While we have assumed that the coronal loop is embedded in a field-free atmosphere, we can speculate what the effects of an external magnetic field might be. Let’s first consider the equilibrium. The inclusion of an external magnetic field only provides minor modifications to the steady state form of the equation of motion (2.1). The tangential component, equation (3.8), remains unchanged as it depends only on internal fluid properties. The same can be argued for the magnetic tension as long as the thin-tube approximation holds. The buoyancy force is modified, however, because the density difference between the internal and
external fluid is a function of the external magnetic field through equilibration of the total pressure across the tube. However, if the external field is force-free, the density scale height of the external fluid is unaffected and density changes only manifest as a different value of the Magnetic Bond Number \( \epsilon \)—see equation (3.11). Therefore, the equation that describes the equilibrium shape of the loop, equation (3.10), remains valid in the presence of external field, and of course the solutions of that equation also remain valid.

The existence of an external magnetic field is not as benign for the wave equations satisfied by the kink waves. The presence of the surrounding magnetic field can be felt by the waves through backreaction forces caused by the tube’s motion. The well-known effect of enhanced inertia, which accounts for the fact that a moving tube must move external fluid out of its way, has already been included in our equation of motion (2.1). However, when the external fluid is magnetized, there is an additional effect because bending of the tube requires bending of the external field as well. Therefore, the external field \( B_e \) provides additional stiffness that increases the kink wave speed,

\[
c^2_K = \frac{B^2 + B_e^2}{4\pi(\rho + \rho_e)} .
\]  

(5.54)

This well-known expression for the kink wave speed (e.g., Edwin & Roberts 1983) displays both backreaction forces explicitly. In the denominator the sum of the internal and external densities is the effect of the enhanced inertia, while in the numerator the appearance of \( B^2_e \) is the result of the increases stiffness. These forces are derived and identified explicitly in Appendix A for a thin tube in the absence of gravity.

When including the effects of an external magnetic field, one is tempted to simply replace the kink speed \( c_0 \) appearing in the wave equations (4.26) and (4.27) with the proper wave speed \( c_K \),

\[
\frac{d^2\xi}{ds^2} - \frac{g_\parallel D_0}{c^2_K} \frac{d\xi}{ds} + \left( \frac{\omega^2}{c^2_K} + \frac{1}{R_0^2} \right) \xi = 0 ,
\]

(5.55)

\[
\frac{d^2\zeta}{ds^2} - \frac{g_\parallel D_0}{c^2_K} \frac{d\zeta}{ds} + \frac{\omega^2}{c^2_K} \zeta = 0 .
\]

(5.56)

From the motivating calculation in Appendix A it is evident that such a substitution is justified in the absence of curvature of the loop. When curvature is present, the validity of the substitution seems physically appropriate, but remains mathematically unproven. While these two wave equations possess the same form as those solved previously, equations (4.26)
and (4.27), there is one subtle difference. The ratio appearing in front of the first derivative in each equation is no longer related to the parameter $\epsilon$ in the same fashion. Previously, the denominator depended on $c_0$ instead of $c_K$. Since, the magnetic tension in the equilibrium still depends on $c_0$ (and not $c_K$), after inserting $g_\parallel$ appropriate for the equilibrium, we find

\[
\frac{d^2 \xi}{ds^2} + \alpha \frac{\epsilon}{X} \sin(\epsilon x/X) \frac{d\xi}{ds} + \left( \frac{\omega^2}{c_K^2} + \frac{1}{R_0^2} \right) \xi = 0 ,
\]

(5.57)

\[
\frac{d^2 \zeta}{ds^2} + \alpha \frac{\epsilon}{X} \sin(\epsilon x/X) \frac{d\zeta}{ds} + \frac{\omega^2}{c_K^2} \zeta = 0 ,
\]

(5.58)

with $\alpha$ being a factor that depends on the plasma parameter within the tube, $\beta$, and outside the tube, $\beta_e$,

\[
\alpha \equiv \frac{B^2}{B^2 + B_e^2} = \frac{1 + \beta_e}{2 + \beta + \beta_e} .
\]

(5.59)

Since the corona is magnetically dominated (i.e., $\beta << 1$ and $\beta_e << 1$), $\alpha$ is generally quite close to a value of one-half. The form of our previous wave equations can be recovered only for $\alpha = 1$. While this seems like a minor difference, the change of variable that we performed to remove the first derivatives in equations (4.30) and (4.31) is predicated on $\alpha = 1$. So, while the basic form of the wave equations remains unchanged when we include an external magnetic field, the detailed perturbation analysis performed in $\S4$ is no longer valid. It would be relatively trivial to repeat the perturbation analysis for this new case, but we choose to leave such an effort to a future study.

6. Conclusions

We have developed a model of a coronal loop that self-consistently includes the effects of gravity and curvature. This has been accomplished for a loop that is treated as an isolated magnetic fibril embedded in an isothermal field-free corona. For such a model, the shape adopted by the loop when in a state of static equilibrium is completely determined by a balance between the forces of buoyancy and magnetic tension. The height of the loop depends on the Magnetic Bond Number $\epsilon$ which is usually small for coronal loops. For these small values, the loops are low-lying with weak curvature. In the other extreme, no stable equilibrium exists for $|\epsilon| > \pi/2$. This stability criterion is actually a restriction on the footpoint separation compared to the coronal density scale height. Loops with a footpoint separation greater than $2\pi H$ are unstable.
Since the static loop is confined within a vertical plane, and only possesses curvature within this plane, the two polarizations of transverse oscillations (those in the direction of the principle normal and those in the direction of the binormal) decouple. The wave equations that describe these oscillations both have restoring forces from buoyancy and from the magnetic tension induced by undulations. The equation that describes the normal polarization also possesses a tension term that arises from the curvature of the loop itself. The effects of the restoring force of buoyancy and of background curvature both enter the eigenfrequencies at second order in the Magnetic Bond Number. Buoyancy increases the frequency and curvature reduces it. The ratio of the frequencies of two eigenmodes of neighboring order $\omega_{n+1}/\omega_n$ differ from the canonical value of $(n + 1)/n$ that would be achieved for a straight loop in the absence of gravity with constant kink speed. Oscillations confined to the plane of the loop, or those polarized in the normal direction, have a frequency ratio that exceeds this canonical value, $\omega_{n+1}/\omega_n > (n + 1)/n$, whereas horizontal oscillations, or the binormal polarization, have a ratio that is less than the canonical value, $\omega_{n+1}/\omega_n < (n + 1)/n$. These conditions on the frequency ratios are achieved for a loop with a constant kink wave speed, independent of the pressure scale height.

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A. Backreaction Forces on a Magnetized Tube in a Magnetized Environment

In order to see how the backreaction forces might appear for a magnetic fibril moving through a magnetized external fluid, we perform a simple wave emission calculation for a field configuration amenable to analytical treatment and direct identification of the relevant forces. Following Edwin & Roberts (1983), consider a magnetized cylinder embedded in a magnetized atmosphere in the absence of gravity. The magnetic field within the cylinder is constant and points in the $z$-direction, $\mathbf{B} = B\hat{z}$. The magnetic field in the region exterior to the cylinder is also constant and points in the same direction $\mathbf{B}_e = B_e\hat{z}$. While the two fields are aligned, they may potentially differ in strength. The mass densities inside and outside the cylinder are $\rho$ and $\rho_e$, respectively.

Magnetosonic waves outside the cylinder have a simple analytic form when expressed in polar coordinates with the axis of the coordinate system coaligned with the axis of the cylinder. A general form for the radial fluid displacement $\xi_r$ and the total pressure perturbation $\delta\Pi$ are as follows (Edwin & Roberts 1983):
\[ \xi_r = \kappa^{-1} e^{im\phi} Q'_m(\lambda r) \, e^{i\kappa z} \, e^{-i\omega t}, \quad (A1) \]

\[ \delta \Pi = \rho_e (\lambda \kappa)^{-1} \left( \omega^2 - \kappa^2 V_e^2 \right) \, e^{im\phi} Q_m(\lambda r) \, e^{i\kappa z} \, e^{-i\omega t}, \quad (A2) \]

with the dispersion relationship

\[ \lambda^2 = \frac{\left( \omega^2 - \kappa^2 c_e^2 \right) \left( \omega^2 - \kappa^2 V_e^2 \right)}{\left( c_e^2 + V_e^2 \right) \left( \omega^2 - \kappa^2 U_e^2 \right)}. \quad (A3) \]

In these equations, \( c_e, V_e \) and \( U_e \) correspond to the external values of the sound, Alfvén, and slow wave speeds, respectively. The axial and radial wavenumbers are denoted \( \kappa \) and \( \lambda \), and the temporal frequency is \( \omega \). The function \( Q_m \) is a \( m \)th-order Bessel function or Hankel function. Primes denote differentiation with respect to the argument of the Bessel function.

If an axially propagating wave causes the tube to oscillate back and forth in the \( \hat{\mathbf{r}} \) direction with an amplitude \( \eta \),

\[ \xi_{\text{tube}} = \eta e^{i\kappa z} e^{-i\omega t} \hat{\mathbf{r}}, \quad (A4) \]

the radial displacement at the lateral surface of the tube \( (r = a) \) is given by

\[ \xi_{r,\text{tube}} = \eta \cos \phi \, e^{i\kappa z} \, e^{-i\omega t}. \quad (A5) \]

The radial displacement of the external fluid must match the normal motion of the tube’s outer surface. Therefore, the wavefield in the external medium must have the same \( \cos \phi \) azimuthal dependence as appears in equation (A5) and will be composed of only the dipole components \( (m = \pm 1) \). Further if \( \lambda^2 > 0 \), causality requires that the wavefield only possesses outward propagating waves, leading to the following form:

\[ \xi_r = A\kappa^{-1} \cos \phi \, H'_1(\lambda r) \, e^{i\kappa z} \, e^{-i\omega t}, \quad (A6) \]

\[ \delta \Pi = A\kappa^{-1} \rho_e \lambda^{-1} \left( \omega^2 - \kappa^2 V_e^2 \right) \cos \phi \, H_1(\lambda r) \, e^{i\kappa z} \, e^{-i\omega t}, \quad (A7) \]

with the function \( H_1 \) corresponding to an outward propagating Hankel function of the first kind, \( H_1(\lambda r) = H^{(1)}_1(\lambda r) \), and the wave amplitude \( A \) fixed by the displacement matching condition at the tube’s surface,
The backreaction forces acting on the tube due to the tube’s motion through the external fluid can be derived by integrating the forces acting on the tube over the tube’s cross-section. In the absence of gravity the only forces acting on the tube are magnetic tension and total pressure. If the tube is thin, we may ignore internal variations of the magnetic field strength $B$ and the cross-sectional mean of the tension force acting on the tube is given by

$$ F_{\text{tension}} = \frac{B^2}{4\pi} \frac{d^2 \xi_{\text{tube}}}{dz^2} = -\rho V^2 \kappa^2 \eta \ e^{i\kappa z} \ e^{-i\omega t} \ \hat{x} , $$

where $V$ is the Alfvén speed within the tube.

The cross-sectional mean of the total pressure force can be expressed as the integral of the total pressure perturbation around the circumference of the tube by using Stokes theorem to convert the area integral into a contour integral,

$$ F_{\text{pressure}} = -\frac{1}{\pi a^2} \int_0^{2\pi} \delta \Pi(a, \phi, z) \ a \ d\phi . $$

The $x$-component of the total pressure force is the only nonvanishing component, and the mean force in this direction is therefore given by the sum of the pressure and tension forces,

$$ F_x = -\frac{1}{\pi a} \int_0^{2\pi} \cos \phi \delta \Pi(a, \phi, z) \ d\phi - \rho V^2 \kappa^2 \eta \ e^{i\kappa z} \ e^{-i\omega t} . $$

The integral appearing in this equation is trivial because the total pressure field on the surface of the tube only has dipole dependence. Using equations (A7) and (A8) this reduces to

$$ F_x = -\left[ \kappa^2 B^2 + \rho \left( \omega^2 - \kappa^2 V^2 \right) \right] \frac{H_1(\lambda a)}{H_1^\prime(\lambda a)} \eta \ e^{i\kappa z} \ e^{-i\omega t} . $$

Since the tube is thin we can perform small argument expansions on the Hankel functions to obtain a simple form,

$$ F_x = \left[ -\kappa^2 \frac{B^2}{4\pi} \eta + \left( \rho \omega^2 - \kappa^2 \frac{B^2}{4\pi} \right) \eta \right] \ e^{i\kappa z} \ e^{-i\omega t} . $$
The equation of motion can therefore be written

\[- \rho \omega^2 \xi_{\text{tube}} = \frac{B^2 + B_e^2}{4\pi} \frac{d^2}{dz^2} \xi_{\text{tube}} + \rho_e \omega^2 \xi_{\text{tube}}.\]  \hspace{1cm} (A14)

We can immediately see that there are two backreaction forces. Both arise from the total pressure in the external fluid. The term proportional to \(\rho_e \omega^2 \xi_{\text{tube}}\) is the familiar effect of enhanced inertia and arises from the \(\omega^2\) factor appearing in equation \((A7)\). In the absence of external field, this would be the only term. However, when external field is present, the total pressure in the external fluid is reduced by the fractional amount \((\omega^2 - \kappa^2 V_e^2) / \omega^2\) compared to pressure in the unmagnetized fluid. The term proportional to the square of the Alfvén speed leads to an additional backreaction term \(-\kappa^2 (B_e^2 / 4\pi) \xi_{\text{tube}}\) that represents the force needed to bend the external field lines when the tube itself bends. The external field provides additional stiffness that increases the wave speed.

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Fig. 1.— The geometry of the coronal loop, with the loop shown in black. The photosphere corresponds to the $x$-$y$ plane, while the loop is confined to the $x$-$z$ plane. The local Frenet coordinates are indicated in red. The local tangent vector is $\hat{s}_0$ and the principle normal $\hat{k}_0$ lies in the direction of curvature. The binormal is everywhere constant and pointed in the $\hat{y}$ direction.

Fig. 2.— Equilibrium (a) height and (b) radius of curvature of coronal loop models with differing ratios of buoyancy to magnetic tension, i.e., differing values of the Magnetic Bond Number $\epsilon$. The value of $\epsilon$ associated with each color is indicated in the right panel. In both panels the dashed line corresponds to a semi-circle with constant radius of curvature $R_0 = X$. Loops with weak buoyancy ($|\epsilon| << 1$) are flat with large radius of curvature everywhere. Loops with substantial buoyancy ($\epsilon \approx -\pi / 2$) are tall with large radius of curvature in the legs and small radius of curvature at its apex. Loops with positive $\epsilon$ or with $\epsilon < -\pi / 2$ are unstable.
Fig. 3.— The eigenfrequencies of coronal loop waves as a function of the Magnetic Bond Number $\epsilon$. (a) Fractional perturbation in the square eigenfrequency for both polarizations of motion. Transverse oscillations in the normal direction are shown with the solid curves, while those in the binormal direction are illustrated with dashed curves. The different colors correspond to different mode orders: the fundamental ($n = 1$) shown in black and the overtones in red ($n = 2$), green ($n = 3$), blue ($n = 4$), and aqua ($n = 5$). The dotted curve corresponds to the fractional frequency shift (independent of mode order) that arises solely from changes in the length of the loop. (b) Frequency ratios of nearby mode orders, with the same line styles and colors as above (i.e., black corresponds to $\omega_2/\omega_1$). The dotted curves indicate the value appropriate for a straight loop in the absence of gravity $(n + 1)/n$. 

Fig. 4.— Eigenfunctions for the fundamental (black) and first two overtones (red and blue) as a function of arclength along the loop. The solid curves illustrate the waveform for a Magnetic Bond Number of $\epsilon = -\pi/4$, whereas the dashed curves correspond to those for the unperturbed eigenfunction with $\epsilon = 0$. Both displacement polarizations, $\xi$ and $\zeta$, possess the same eigenfunctions.