On a class of polynomials connected to Bell polynomials

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Abstract. In this paper, we study a class of sequences of polynomials linked to the sequence of Bell polynomials. Some sequences of this class have applications on the theory of hyperbolic differential equations and other sequences generalize Laguerre polynomials and associated Lah polynomials. We discuss, for these polynomials, their explicit expressions, relations to the successive derivatives of a given function, real zeros and recurrence relations. Some known results are significantly simplified.

Keywords. New class of polynomials, recurrence relations, real zeros, Bell polynomials, Laguerre polynomials, associated Lah polynomials.

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1 Introduction

In [10], there are many polynomials having applications to the hyperbolic partial differential equations
\[ Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0 \] with \( AD > BC \),
for which the following two sequences of polynomials \((U_n (x))\) and \((V_n (x))\) defined by
\[
\sum_{n \geq 0} U_n (x) \frac{t^n}{n!} = (1 - t)^{-1/2} \exp \left( x \left( (1 - t)^{-1/2} - 1 \right) \right),
\]
\[
\sum_{n \geq 0} V_n (x) \frac{t^n}{n!} = (1 - t)^{-3/2} \exp \left( x \left( (1 - t)^{-1/2} - 1 \right) \right)
\]
are considered. These polynomials have applications to the theory of hyperbolic partial differential equations, see [8, pp. 391–398]. They can be expressed as follows [10, pp. 257–258]:
\[
U_n (x) = x e^{-x} \left( \frac{d}{d(x^2)} \right)^n (x^{2n} e^{-x}),
\]
\[
V_n (x) = \frac{e^{-x}}{x} \left( \frac{d}{d(x^2)} \right)^n (x^{2n+1} e^{-x}).
\]

Recently, two studies of the sequence of polynomials \((U_n (x))\) are given in [15, 25].
Motivated by these works, to give more properties of these polynomials, we prefer to consider their generalized sequence of polynomials \(P_n^{(\alpha, \beta)} (x)\) defined by
\[
\sum_{n \geq 0} P_n^{(\alpha, \beta)} (x) \frac{t^n}{n!} = (1 - t)^{\alpha} \exp \left( x \left( (1 - t)^{\beta} - 1 \right) \right), \ \alpha, \beta \in \mathbb{R}, \ \beta \neq 0.
\]

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The first few values of the sequence \( P^{(\alpha,\beta)}(\frac{x}{\beta}) ; n \geq 0 \) are to be

\[
P_0^{(\alpha,\beta)} \left( \frac{x}{\beta} \right) = 1,
\]
\[
P_1^{(\alpha,\beta)} \left( \frac{x}{\beta} \right) = x - \alpha,
\]
\[
P_2^{(\alpha,\beta)} \left( \frac{x}{\beta} \right) = x^2 - (2\alpha + \beta - 1) x + (\alpha)_2,
\]
\[
P_3^{(\alpha,\beta)} \left( \frac{x}{\beta} \right) = x^3 - 3(\alpha + \beta - 1) x^2 + (3\alpha^2 + 3\alpha \beta - 6\alpha + \beta^2 - 3\beta + 2) x - (\alpha)_3,
\]

where \( (\alpha)_n := \alpha (\alpha - 1) \cdots (\alpha - n + 1) \) if \( n \geq 1 \) and \( (\alpha)_0 := 1 \).

We use also the notation \( \langle \alpha \rangle_n := \alpha (\alpha + 1) \cdots (\alpha + n - 1) \) if \( n \geq 1 \) and \( \langle \alpha \rangle_0 := 1 \).

The paper is organized as follows. In the next section we give different expressions for \( P^{(\alpha,\beta)}_n(x) \).

In the third section we give special expressions for \( P^{(\alpha,\beta)}_n(x) \) and we show that it has only real zeros under certain conditions on \( \alpha \) and \( \beta \). In the fourth section, we give differential equations arising from the polynomials \( P^{(\alpha,\beta)}_n \) and some recurrence relations, and, in the last section apply the obtained results to some particular polynomials.

## 2 Explicit expressions for the polynomials \( P^{(\alpha,\beta)}_n \)

In this section, we give some explicit expressions for \( P^{(\alpha,\beta)}_n(x) \). Two expressions of \( P^{(\alpha,\beta)}_n(x) \) related to Dobinski’s formula and generalized Stirling numbers are given by the following proposition.

**Proposition 1** There hold

\[
P^{(\alpha,\beta)}_n(x) = e^{-x} \sum_{k \geq 0} (-\alpha - \beta k)_n \frac{x^k}{k!}
\]
\[
P^{(\alpha,\beta)}_n(x) = \sum_{k=0}^{n} S_{\alpha,\beta}(n, k) x^k,
\]

where

\[
S_{\alpha,\beta}(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (-\alpha - \beta j)_n.
\]

**Proof.** The first identity follows from the expansion

\[
\sum_{n \geq 0} P^{(\alpha,\beta)}_n(x) \frac{t^n}{n!} = e^{-x} (1 - t)^\alpha \exp \left( x (1 - t)^\beta \right)
\]
\[
= e^{-x} \sum_{k \geq 0} x^k \frac{(1 - t)^{\alpha + \beta k}}{k!}
\]
\[
= e^{-x} \sum_{k \geq 0} \frac{x^k}{k!} \sum_{n \geq 0} (-\alpha - \beta k)_n \frac{t^n}{n!}
\]
\[
= \sum_{n \geq 0} \left( e^{-x} \sum_{k \geq 0} (-\alpha - \beta k)_n \frac{x^k}{k!} \right) \frac{t^n}{n!},
\]

and, the second identity follows from the first by expansion \( e^{-x} \) in power series. \( \square \)
**Proposition 2** There holds
\[ x^n = \sum_{k=0}^{n} \tilde{S}_{\alpha, \beta} (n, k) P_k^{(\alpha, \beta)} (x) \quad \text{with} \quad \tilde{S}_{\alpha, \beta} (n, k) = (-1)^{n-k} S_{n - \frac{\alpha}{\beta}, \frac{1}{\beta}} (n, k). \]

**Proof.** Since
\[
\sum_{n \geq k} S_{\alpha, \beta} (n, k) \frac{t^n}{n!} = \frac{1}{k!} \sum_{n \geq k} \left( \frac{d}{dx} \right)^k P_n^{(\alpha, \beta)} (0) \frac{t^n}{n!} = \frac{1}{k!} \left( (1-t)^{\beta} - 1 \right)^k (1-t)^\alpha,
\]
it results
\[
\sum_{n \geq 0} \left( \sum_{k=0}^{n} \tilde{S}_{\alpha, \beta} (n, k) P_k^{(\alpha, \beta)} (x) \right) \frac{t^n}{n!} = \sum_{k \geq 0} (-1)^k P_k^{(\alpha, \beta)} (x) \sum_{n \geq k} S_{n - \frac{\alpha}{\beta}, \frac{1}{\beta}} (n, k) \frac{(-t)^n}{n!}
\]
\[
= \sum_{k \geq 0} (-1)^k \frac{P_k^{(\alpha, \beta)} (x)}{k!} (1+t)^{\frac{k}{\beta}} - 1)^k (1+t)^{-\frac{k}{\beta}}
\]
\[
= (1+t)^{-\frac{\alpha}{\beta}} \sum_{k \geq 0} P_k^{(\alpha, \beta)} (x) \left( 1 - (1+t)^{\frac{k}{\beta}} \right)^k
\]
\[
= \exp (xt),
\]
which shows the desired identity. \[\square\]

**Corollary 3** There holds
\[ \langle -\alpha - \beta x \rangle_n = \sum_{j=0}^{n} S_{\alpha, \beta} (n, j) (x)_j. \]

**Proof.** Let \( \langle -\alpha - \beta x \rangle_n = \sum_{j=0}^{n} \delta (n, j) (x)_j. \) Then, from Proposition 1 we get
\[ P_n^{(\alpha, \beta)} (x) = e^{-x} \sum_{k=0}^{n} \langle -\alpha - \beta k \rangle_n \frac{x^k}{k!} = \sum_{j=0}^{n} \delta (n, j) \left( e^{-x} \sum_{k \geq j} (k)_j \frac{x^k}{k!} \right) = \sum_{j=0}^{n} \delta (n, j) x^j,
\]
which gives \( \delta (n, j) = S_{\alpha, \beta} (n, j). \) \[\square\]

If \( B_n \) denote the \( n \)-th Bell polynomial, then when we replace \( t \) by \( 1-e^t \) in the generating function of the sequence \( \left( P_n^{(\alpha, \beta)} (x) \right) \), then \( P_n^{(\alpha, \beta)} (x) \) can be written in the basis \( \{ 1, B_1 (x), \ldots, B_n (x) \} \) as follows:

**Proposition 4** There holds
\[
\sum_{k=0}^{n} (-1)^k S (n, k) P_k^{(\alpha, \beta)} (x) = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \beta^k B_k (x),
\]
or equivalently
\[ P_n^{(\alpha, \beta)} (x) = \sum_{j=0}^{n} \beta^j \left( \sum_{k=j}^{n} (-1)^k |s (n, k)| \alpha^{k-j} \right) B_j (x),
\]
where \( s (n, k) \) and \( S (n, k) \) are, respectively, the Stirling numbers of the first and second kind, see for instance [7, 33].
Let $B_{n+r,k+r}^{(r)}((a_i, i \geq 1); (b_i, i \geq 1))$ are the partial $r$-Bell polynomials [6, 21, 28] defined by
\[ \sum_{n \geq k} B_{n+r,k+r}^{(r)}(a_i; b_i) \frac{t^n}{n!} = \frac{1}{k!} \left( \sum_{j \geq 1} a_j \frac{t^j}{j!} \right)^r \left( \sum_{j \geq 0} b_{j+r} \frac{t^j}{j!} \right)^r. \]
and $B_{n,k}((a_i, i \geq 1)) = B_{n,k}^{(0)}((a_i, i \geq 1); (b_i, i \geq 1))$ are the partial Bell polynomials [1, 7, 18, 19]. An expression of $P^{(\alpha,\beta)}_n(x)$ in terms of the partial $r$-Bell polynomials is as follows.

**Proposition 5** For any non-negative integer $r$, there hold
\[ S_{r\alpha,\beta}(n, k) = B_{n+r,k+r}^{(r)}\left(\langle -\beta \rangle_j; \langle -\alpha \rangle_{j-1} \right), \]
which imply
\[ P^{(r\alpha,\beta)}_n(x) = \sum_{k=0}^n B_{n+r,k+r}^{(r)}\left(\langle -\beta \rangle_j; \langle -\alpha \rangle_{j-1} \right) x^k. \]

**Proof.** From definition we may state
\[ \sum_{n \geq 0} \left( \frac{d}{dx} \right)^k P^{(r\alpha,\beta)}_n(0) \frac{t^n}{n!} = \left( (1-t)^\beta - 1 \right)^k (1-t)^r \]
\[ = \left( \sum_{n \geq 1} \langle -\beta \rangle_n \frac{t^n}{n!} \right)^k \left( \sum_{n \geq 0} \langle -\alpha \rangle_n \frac{t^n}{n!} \right)^r \]
\[ = k! \sum_{n \geq k} B_{n+r,k+r}^{(r)}\left(\langle -\beta \rangle_j; \langle -\alpha \rangle_{j-1} \right) \frac{t^n}{n!}, \]
and by connection to $S_{\alpha,\beta}(n, k)$ defined above, this last expansion gives
\[ S_{r\alpha,\beta}(n, k) = \frac{1}{k!} \left( \frac{d}{dx} \right)^k P^{(r\alpha,\beta)}_n(0) = B_{n+r,k+r}^{(r)}\left(\langle -\beta \rangle_j; \langle -\alpha \rangle_{j-1} \right). \]

3. **Some properties of the polynomials $P^{(\alpha,\beta)}_n$**

In this section, we show the link of the sequence of polynomials $\left( P^{(\alpha,\beta)}_n(x) \right)$ to the successive derivatives of a given function and we give sufficient conditions on $\alpha$ and $\beta$ for which the polynomial $P^{(\alpha,\beta)}_n$ has only real zeros.

**Lemma 6** There holds
\[ P^{(\alpha,\beta)}_{n+1}(x) = (n - \alpha - \beta x) P^{(\alpha,\beta)}_n(x) - \beta x \frac{d}{dx} P^{(\alpha,\beta)}_n(x). \]

**Proof.** One can verify that the function
\[ F_{\alpha,\beta}(t, x) = (1-t)^\alpha \exp \left( x \left( (1-t)^\beta - 1 \right) \right) \]
is a solution of the partial differential equation
\[ (1-t) \frac{d}{dt} Y + \beta x \frac{d}{dx} Y + (\alpha + \beta x) Y = 0 \]
from which it results the desired identity. \[ \square \]
Theorem 7. For \( x > 0 \) and \( n \geq 0 \) we have

\[
P_n^{(\alpha, \beta)}(x^\beta) = (-1)^n x^{n-\alpha} e^{-x^\beta} \left( \frac{d}{dx} \right)^n \left( x^\alpha e^{x^\beta} \right),
\]

\[
P_n^{(\alpha, \beta)}(x^{-\beta}) = x^{\alpha+1} e^{-x^{-\beta}} \left( \frac{d}{dx} \right)^n \left( x^{\alpha-1} e^{x^{-\beta}} \right).
\]

Proof. For the first identity, Lemma 6 gives

\[
P_n^{(\alpha, \beta)}(x) = (n - 1 - \alpha - \beta x) P_n^{(\alpha, \beta)}(x) - \beta x \frac{d}{dx} P_n^{(\alpha, \beta)}(x),
\]

and if we set \( f_n^{(\alpha, \beta)}(x) := (-1)^n x^{\frac{n-\alpha}{\beta}} e^x P_n^{(\alpha, \beta)}(x) \), the last identity can also be written as

\[
f_n^{(\alpha, \beta)}(x) = \beta x^{1-\frac{1}{\beta}} \frac{d}{dx} f_{n-1}^{(\alpha, \beta)}(x) = \frac{d}{d\left(\frac{x}{x^{1/\beta}}\right)} f_{n-1}^{(\alpha, \beta)}(x)
\]

which implies \( f_n^{(\alpha, \beta)}(x) = \left( \frac{d}{d\left(\frac{x}{x^{1/\beta}}\right)} \right)^n f_0^{(\alpha, \beta)}(x) = \left( \frac{d}{d\left(\frac{x}{x^{1/\beta}}\right)} \right)^n \left( x^{\frac{n}{\beta}} e^x \right) \). So, we get

\[
P_n^{(\alpha, \beta)}(x) = (-1)^n x^{\frac{n-\alpha}{\beta}} e^{-x} \left( \frac{d}{d\left(\frac{x}{x^{1/\beta}}\right)} \right)^n \left( x^{\frac{n}{\beta}} e^x \right)
\]

or equivalently \( P_n^{(\alpha, \beta)}(y^\beta) = (-1)^n y^{n-\alpha} e^{-y^\beta} \left( \frac{d}{dy} \right)^n \left( y^\alpha e^{y^\beta} \right) \).

For the second identity, we proceed as follows

\[
\left( \frac{d}{d\left(\frac{x}{x^{1/\beta}}\right)} \right)^n \left( x^{\frac{\alpha+1-n}{\beta}} e^x \right) = \left( \frac{d}{dy} \right)^n \left. \frac{y^{n-\alpha-1} e^{-y^\beta}}{y=x^{1/\beta}} \right|_{y=x^{1/\beta}}
\]

\[
= \sum_{k \geq 0} (n - 1 - \alpha - k\beta) \frac{1}{k!} \frac{y^{-1-\alpha-k\beta}}{y=x^{1/\beta}}
\]

\[
= x^{\frac{\alpha+1}{\beta}} \sum_{k \geq 0} (-\alpha - k\beta) \frac{x^k}{k!}
\]

\[
= x^{\frac{\alpha+1}{\beta}} e^x P_n^{(\alpha, \beta)}(x),
\]

i.e. \( P_n^{(\alpha, \beta)}(x) = x^{\frac{\alpha+1}{\beta}} e^{-x} \left( \frac{d}{d\left(\frac{x}{x^{1/\beta}}\right)} \right)^n \left( x^{\frac{\alpha-\alpha+n}{\beta}} e^x \right) \) which is equivalent to the desired identity. □

Remark 8. The first hand of Theorem 7 can also be proved directly using Proposition 7 as follows:

\[
P_n^{(\alpha, \beta)}(x^\beta) = e^{-x^\beta} \sum_{k \geq 0} \frac{(-\alpha - \beta k)n x^{\alpha+\beta k-n}}{k!}
\]

\[
= (-1)^n \sum_{k \geq 0} (\alpha + \beta k) \frac{x^{\alpha+\beta k-n}}{k!}
\]

\[
= (-1)^n x^{n-\alpha} e^{-x^\beta} \left( \frac{d}{dx} \right)^n \left( \sum_{k \geq 0} \frac{x^{\alpha+\beta k}}{k!} \right)
\]

\[
= (-1)^n x^{n-\alpha} e^{-x^\beta} \left( \frac{d}{dx} \right)^n \left( x^\alpha e^{x^\beta} \right).
\]
which implies by using Theorem 7

\[ \text{Proof. From Proposition 4 we get} \]

\[ P(\beta) = \frac{(\beta)^k}{\alpha^k} B_k(x), \]

which implies by using Theorem 7

\[ \left( \frac{\beta}{\alpha} \right)^n B_n(x^\beta) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{1}{\alpha^k} \sum_{j=0}^{k} (-1)^j S(k, j) P_n^{(\alpha, \beta)}(x^\beta) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{1}{\alpha^k} \sum_{j=0}^{k} (-1)^j S(k, j) (-1)^j x^{j-\alpha}e^{-x^\beta} \left( \frac{d}{dx} \right)^j (x^\alpha e^{x^\beta}) \]

\[ = (-1)^{n} x^{-\alpha} e^{-x^\beta} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{-1}{\alpha} \right)^k \sum_{j=0}^{k} S(k, j) x^j \left( \frac{d}{dx} \right)^j (x^\alpha e^{x^\beta}). \]

On using the identity \( \sum_{j=0}^{k} S(k, j) x^j \left( \frac{d}{dx} \right)^j = (x \left( \frac{d}{dx} \right) )^k \), it follows

\[ B_n(x^\beta) = \left( \frac{-\alpha}{\beta} \right)^n x^{-\alpha} e^{-x^\beta} \sum_{k=0}^{n} \binom{n}{k} \left( -x \frac{d}{dx} \right)^k (x^\alpha e^{x^\beta}) = \left( \frac{-\alpha}{\beta} \right)^n x^{-\alpha} e^{-x^\beta} \left( 1 - x \frac{d}{dx} \right)^n (x^\alpha e^{x^\beta}) = x^{-\alpha} e^{-x^\beta} \left( x \frac{d}{dx} - \frac{\alpha}{\beta} \right)^n (x^\alpha e^{x^\beta}), \]

which remains true for \( \alpha = 0 \).

\[ \square \]

To study the real zeros of \( P_n^{(\alpha, \beta)} \), we use the following known theorem. Indeed, let \( P_1 \) and \( P_2 \) be two polynomials having only real zeros and let \( x_0 \leq \cdots \leq x_1 \) and \( y_m \leq \cdots \leq y_1 \) be the zeros of \( P_1 \) and \( P_2 \), respectively. Following [30], we say that \( P_2 \) interlaces \( P_1 \) if \( m = n - 1 \) and

\[ x_0 \leq y_{n-1} \leq x_{n-1} \leq \cdots \leq y_1 \leq x_1 \]

and that \( P_2 \) alternates left of \( P_1 \) if \( m = n \) and

\[ y_n \leq x_n \leq y_{n-1} \leq x_{n-1} \leq \cdots \leq y_1 \leq x_1. \]

**Theorem 10** [31] Th. 1] Let \( a_1, a_2, b_1, b_2 \) be real numbers, let \( P_1, P_2 \) be two polynomials whose leading coefficients have the same sign and let \( P(x) = (a_1 x + b_1) P_1(x) + (a_2 x + b_2) P_2(x) \). Suppose that \( P_1, P_2 \) have only real zeros and \( P_2 \) interlaces \( P_1 \) or \( P_2 \) alternates left of \( P_1 \). Then, if \( a_1 b_2 \geq b_1 a_2 \), \( P(x) \) has only real zeros.

**Theorem 11** Let

\[ A = \left\{ (\alpha, \beta) \in \mathbb{R}^2 : (\beta - 1)^2 + 4\alpha \beta \geq 0, \ \beta < 0, \ \alpha \leq 2 \right\}, \]

\[ \tilde{A} = \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \beta > 0, \ \alpha \geq 1 \right\}. \]
Then, for \((\alpha, \beta) \in A\), the polynomial \(P_n^{(\alpha, \beta)}\) has only real zeros, \(n \geq 1\), and, for \((\alpha, \beta) \in A\), the polynomials \(P_1^{(\alpha, \beta)}, \ldots, P_{\lceil \alpha \rceil}^{(\alpha, \beta)}\) has only real zeros, where \([\alpha]\) is the smallest integer \(\geq \alpha\).

**Proof.** We proceed by induction on \(n \geq 1\). For \(n = 1\), the polynomial \(P_1^{(\alpha)}(x) = -\beta x - \alpha\) has a real zero, and for \(n = 2\), the polynomial

\[
P_2^{(\alpha)}(x) = \beta^2 x^2 + \beta(2\alpha + \beta - 1)x + \alpha(\alpha - 1)
\]

has only real zeros when \((\beta - 1)^2 + 4\alpha \beta \geq 0\) and \(\beta < 0\).

Assume that \(P_n^{(\alpha, \beta)}(x)\) has \(n \geq 2\) real zeros different from zero, since the leading coefficient of \(P_n^{(\alpha, \beta)}(x)\) is \(S_{\alpha, \beta}(n, n) = (-\beta)^n\) and then leading coefficient of \(\frac{d}{dx}P_n^{(\alpha, \beta)}(x)\) is \(nS_{\alpha, \beta}(n, n) = n(-\beta)^n\), then they are of the same sign. Also, since \(\frac{d}{dx}P_n^{(\alpha, \beta)}(x)\) interlaces \(P_n^{(\alpha, \beta)}(x)\) it follows from Theorem 10 that if \(-\beta(n - \alpha) \geq 0\), \(P_{n+1}^{(\alpha, \beta)}\) has only real zeros. The condition \(-\beta(n - \alpha) \geq 0\) is satisfied when \((\alpha, \beta) \in A\) because \(-\beta(n - \alpha) \geq -\beta(2 - \alpha) \geq 0\). It is also satisfied when \(n \in [1, \lceil \alpha \rceil - 1]\) and \((\alpha, \beta) \in A\) because \(-\beta(n - \alpha) \geq -\beta(\lceil \alpha \rceil - 1 - \alpha) \geq 0\).

**Corollary 12** For \(\alpha \leq 0\) and \(\beta < 0\) the sequence \((S_{\alpha, \beta}(n, k); 0 \leq k \leq n)\) is strictly log-concave, more precisely

\[
(S_{\alpha, \beta}(n, k))^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n - k}\right) S_{\alpha, \beta}(n, k + 1) S_{\alpha, \beta}(n, k - 1), \quad 1 \leq k \leq n - 1.
\]

**Proof.** For \(\alpha \leq 0\) and \(\beta < 0\) the polynomial \(P_n^{(\alpha, \beta)}(x)\) has only real zeros and its coefficients \(S_{\alpha, \beta}(n, k)\) are non-negative, so Newton’s inequality [13, pp. 52] completes the proof.

**4** Differential equations and recurrence relations

In [15, Sec. 2] (see also [24]), the authors give a differential equation having as solution the function

\[
(1 - t)^{-1/2} \exp \left(x \left((1 - t)^{-1/2} - 1\right)\right)
\]

from which they conclude a generalized recurrence relation for the sequence \((U_n(x))\). The results of this section simplify and generalize these results.

**Theorem 13** Let \(m\) be a positive integer. The function

\[
F_{\alpha, \beta}(t, x) := (1 - t)^{\alpha} \exp \left(x \left((1 - t)^{\beta} - 1\right)\right)
\]

satisfies

\[
\left(\frac{d}{dt}\right)^m F_{\alpha, \beta}(t, x) = F_{\alpha, \beta}(t, x) (1 - t)^{-m} P_m^{(\alpha, \beta)}\left(x (1 - t)^{\beta}\right).
\]
Proof. From the definition of $F_{\alpha,\beta}(t, x)$ and Corollary 3 we obtain

$$\left( \frac{d}{dt} \right)^m F_{\alpha,\beta}(t, x) = \left( \frac{d}{dt} \right)^m \left( (1-t)^\alpha \exp \left( x \left( (1-t)^\beta - 1 \right) \right) \right)$$

$$= e^{-x} \sum_{k \geq 0} \frac{x^k}{k!} \left( \frac{d}{dt} \right)^m (1-t)^{\alpha + k\beta - m}$$

$$= e^{-x} (1-t)^{\alpha-m} \sum_{j=0}^{m} S_{\alpha,\beta}(m, j) \sum_{k \geq 0} \frac{(k)}{k!} \frac{x^k (1-t)^{k\beta}}{k!}$$

$$= (1-t)^{\alpha-m} \exp \left( x \left( (1-t)^\beta - 1 \right) \right) \sum_{j=0}^{m} S_{\alpha,\beta}(m, j) x^j (1-t)^{j\beta}$$

$$= F_{\alpha,\beta}(t, x) (1-t)^{-m} \sum_{j=0}^{m} S_{\alpha,\beta}(m, j) x^j (1-t)^{j\beta}$$

$$= F_{\alpha,\beta}(t, x) \left( 1-t \right)^{-m} P_m^{(\alpha,\beta)} \left( x (1-t)^\beta \right).$$

□

The next corollary gives an expression of $P_{n+m}^{(\alpha,\beta)}(x)$ in terms of the family $\left( x^k P_j^{(\alpha,\beta)}(x) \right)$. The obtained expression is similar to the expression of the Bell number $B_{n+m}$ given in [29], Bell polynomial $B_{n+m}$ given in [3, 12] and several generalizations given later, see [14, 16, 17, 20, 32].

Corollary 14 For $n, m = 0, 1, 2, \ldots$, we have

$$P_{n+m}^{(\alpha,\beta)}(x) = \sum_{j=0}^{m} \sum_{k=0}^{m} \binom{n}{j} (m - \beta k)_{n-j} S_{\alpha,\beta}(m, k) x^k P_j^{(\alpha,\beta)}(x).$$

In particular, for $m = 1$, we obtain

$$P_{n+1}^{(\alpha,\beta)}(x) = -\sum_{j=0}^{n} \binom{n}{j} \left( \alpha (n-j)! + \beta x (1-\beta)_{n-j} \right) P_j^{(\alpha,\beta)}(x).$$

Proof. On using Theorem 13 we have

$$\sum_{n \geq 0} P_{n+m}^{(\alpha,\beta)}(x) \frac{t^n}{n!} = \left( \frac{d}{dt} \right)^m F_{\alpha,\beta}(t, x)$$

$$= F_{\alpha,\beta}(t, x) (1-t)^{-m} P_m^{(\alpha,\beta)} \left( x (1-t)^\beta \right)$$

$$= \left( \sum_{i \geq 0} P_i^{(\alpha,\beta)}(x) \frac{t^i}{i!} \right) \left( \sum_{k=0}^{m} S_{\alpha,\beta}(m, k) x^k (1-t)^{-m+\beta k} \right).$$
Proposition 16

There holds

\[ P_n^{(\alpha, \beta)} (x) = \sum_{k=0}^{m} S_{\alpha, \beta} (m, k) x^k \left( \sum_{i \geq 0} P_i^{(\alpha, \beta)} (x) \frac{t^i}{i!} \right) \left( \sum_{j \geq 0} (m - \beta j) \frac{t^j}{j!} \right) \]

\[ = \sum_{k=0}^{m} S_{\alpha, \beta} (m, k) x^k \sum_{n \geq 0} \left( \sum_{j=0}^{n} \binom{n}{j} (m - \beta k) P_j^{(\alpha, \beta)} (x) \right) \frac{t^n}{n!} \]

\[ = \sum_{n \geq 0} \left( \sum_{j=0}^{m} \sum_{k=0}^{m} \binom{n}{j} (m - \beta k) P_j^{(\alpha, \beta)} (x) \right) \frac{t^n}{n!} \]

which follows gives the desired identity. \( \square \)

Remark 15 For \( n = 1 \) in Corollary 14 we get

\[ P_{m+1}^{(\alpha, \beta)} (x) = \sum_{j=0}^{m+1} ((m - \alpha - \beta j) S_{\alpha, \beta} (m, j) - \beta S_{\alpha, \beta} (m, j - 1)) x^j. \]

So, since from Proposition 4 we have \( P_{m+1}^{(\alpha, \beta)} (x) = \sum_{j=0}^{m+1} S_{\alpha, \beta} (m + 1, j) x^j \), it results

\[ S_{\alpha, \beta} (m + 1, j) = (m - \alpha - \beta j) S_{\alpha, \beta} (m, j) - \beta S_{\alpha, \beta} (m, j - 1), \]

with \( S_{\alpha, \beta} (m + 1, j) = 0 \) if \( j < 0 \) or \( j > m + 1 \).

Proposition 16 There holds

\[ P_n^{(\alpha, \beta)} (x) = \sum_{k=0}^{n} (-1)^k S_{\alpha - \frac{\alpha'}{\beta'}, \beta} (n, k) P_k^{(\alpha', \beta')} (x). \]

In particular, for \((\alpha', \beta') = (\alpha, \beta)\), we get

\[ P_n^{(\alpha, \lambda \beta)} (x) = \sum_{k=0}^{n} (-1)^k B_{n,k} \left( \langle -\lambda \rangle_j \right) P_k^{(\alpha, \beta)} (x), \ \lambda \in \mathbb{R}, \]

Proof. From Proposition 2 we have \( x^k = \sum_{j=0}^{k} (-1)^{k-j} S_{\frac{\alpha'-1}{\beta'}, \frac{1}{\beta'}} (k, j) P_j^{(\alpha', \beta')} (x) \).

So, use the identity \( P_n^{(\alpha, \beta)} (x) = \sum_{k=0}^{n} S_{\alpha, \beta} (n, k) x^k \) of Proposition 1 to obtain

\[ P_n^{(\alpha, \beta)} (x) = \sum_{k=0}^{n} S_{\alpha, \beta} (n, k) \left( \sum_{j=0}^{k} (-1)^{k-j} S_{\frac{\alpha'-1}{\beta'}, \frac{1}{\beta'}} (k, j) P_j^{(\alpha', \beta')} (x) \right) \]

\[ = \sum_{j=0}^{n} \left( \sum_{k=j}^{n} (-1)^{k-j} S_{\alpha, \beta} (n, k) S_{\frac{\alpha'-1}{\beta'}, \frac{1}{\beta'}} (k, j) \right) P_j^{(\alpha', \beta')} (x). \]

Now, since from the proof of Proposition 2 we have

\[ \sum_{n \geq k} S_{\alpha, \beta} (n, k) \frac{t^n}{n!} = \frac{1}{k!} \left( (1-t)^{\beta} - 1 \right)^k (1-t)^{\alpha}, \]
it follows that the exponential generating function of the sequence \((M(n,j) ; n \geq j)\) defined by

\[
M(n,j) := \sum_{k=j}^{n} (-1)^{k-j} S_{\alpha,\beta}(n,k) S_{\frac{\alpha'}{\beta'},\frac{\alpha'}{\beta'}}(k,j)
\]

is to be

\[
\sum_{n\geq j} M(n,j) \frac{t^n}{n!} = \frac{(-1)^{j}S_{\alpha'}{\beta'}(j)}{j!} \left((1-t)^{\frac{\alpha}{\beta}} -1\right)^j
\]

which shows that \(M(n,j) = (-1)^j S_{\alpha',\beta'}(n,j)\).

\[\Box\]

As a consequence of Proposition 16, by combining it with Propositions 1, it results:

**Corollary 17** For any real numbers \(\alpha, \alpha', \beta, \beta'\) such that \(\beta' \neq 0\), there hold

\[
\langle -\alpha - \beta x \rangle_n = \sum_{j=0}^{n} (-1)^j S_{\alpha',\beta'}(n,j) \langle -\alpha' - \beta' x \rangle_j,
\]

\[
S_{\alpha,\beta}(n,k) = \sum_{j=k}^{n} (-1)^j S_{\alpha',\beta'}(n,j) S_{\alpha',\beta'}(j,k).
\]

## 5 Application to particular polynomials

### 5.1 Application to the polynomials \(U_n\) and \(V_n\)

For \(n \geq 1\), the polynomials \(U_n = P_n^{(-1/2,-1/2)}\) and \(V_n = P_n^{(-3/2,-1/2)}\) defined above, Propositions 1, 4 and 5 give

\[
U_n(x) = e^{-x} \sum_{k=0}^{n} \left(\frac{k+1}{2}\right)_n \frac{x^k}{k!}, \quad V_n(x) = e^{-x} \sum_{k=0}^{n} \left(\frac{k+3}{2}\right)_n \frac{x^k}{k!},
\]

\[
U_n(x) = \sum_{k=0}^{n} S_{-1/2,-1/2}(n,k) x^k, \quad V_n(x) = \sum_{k=0}^{n} S_{-3/2,-1/2}(n,k) x^k,
\]

\[
U_n(x) = \sum_{j=0}^{n} \left(\sum_{k=j}^{n} \frac{\left|s(n,k)\right|^{(k,j)}}{2^k} \right) B_j(x), \quad V_n(x) = \sum_{j=0}^{n} \left(\sum_{k=j}^{n} \left|s(n,k)\right| \left(\frac{1}{2}\right)^k \right) B_j(x),
\]

\[
U_n(x) = \sum_{k=0}^{n} B_{n+1,k+1}^{(1)} \left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_{j-1} x^k, \quad V_n(x) = \sum_{k=0}^{n} B_{n+1,k+1}^{(1)} \left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_{j-1} x^k.
\]

Theorem 4 proves that the polynomials \(U_n\) and \(V_n\), \(n \geq 1\), have only real zeros and Theorem 7 shows that, for \(x > 0\), there hold

\[
U_n\left(\frac{1}{\sqrt{x}}\right) = (-1)^n x^n \sqrt{x} e^{-\frac{1}{\sqrt{x}}} \left(\frac{d}{dx}\right)^n \left(\frac{1}{\sqrt{x}} e^{\frac{1}{\sqrt{x}}}\right),
\]

\[
V_n\left(\frac{1}{\sqrt{x}}\right) = (-1)^n x^{n+1} \sqrt{x} e^{-\frac{1}{\sqrt{x}}} \left(\frac{d}{dx}\right)^n \left(\frac{1}{x\sqrt{x}} e^{\frac{1}{\sqrt{x}}}\right)
\]

and

\[
U_n\left(\sqrt{x}\right) = \sqrt{x} e^{-\sqrt{x}} \left(\frac{d}{dx}\right)^n \left(x^{n-1} \sqrt{x} e^{\sqrt{x}}\right),
\]

\[
V_n\left(\sqrt{x}\right) = \sqrt{x} e^{-\sqrt{x}} \left(\frac{d}{dx}\right)^n \left(x^{n-1} \sqrt{x} e^{\sqrt{x}}\right).
\]
5.2 Application to the generalized Laguerre polynomials

We note here that the sequence of generalized Laguerre polynomials \( L_n^{(\lambda)}(x) \) (see for example [4] [8] [27]) defined by
\[
\sum_{n \geq 0} L_n^{(\lambda)}(x) t^n = (1 - t)^{-\lambda - 1} \exp \left( -\frac{xt}{1 - t} \right)
\]
presents a particular case of the sequence \( P_n^{(\alpha, \beta)}(x) \), i.e. \( L_n^{(\lambda)}(x) = \frac{1}{n!} P_n^{(\lambda-1,-1)}(x) \).

Propositions [4] [4] and [5] give
\[
\begin{align*}
L_n^{(\lambda)}(x) &= \frac{e^{-x}}{n!} \sum_{k \geq 0} (\lambda + 1 + k)_n x^k, \\
L_n^{(\lambda)}(x) &= \frac{1}{n!} \sum_{k=0}^n S_{\lambda-1,-1}(n,k) x^k, \\
L_n^{(\lambda)}(x) &= \frac{1}{n!} \sum_{j=0}^n \left( \sum_{k=j}^n |s(n,k)| \right) (\lambda + 1)^{k-j} E_j(x), \\
L_n^{(\lambda)}(x) &= \frac{1}{n!} \sum_{k=0}^n P^{(1)}_{n+1,k+1} \langle \lambda+1 \rangle_j \langle \lambda+1 \rangle_{j-1} x^k.
\end{align*}
\]

To write \( P_n^{(\alpha, \beta)}(x) \) in the basis \( \{1, L_1^{(\lambda)}(x), \ldots, L_n^{(\lambda)}(x)\} \), set \((\alpha', \beta') = (-\lambda - 1, -1)\) in Proposition [10] to obtain
\[
P_n^{(\alpha, \beta)}(x) = \sum_{j=0}^n (-1)^j j! S_{\alpha-\lambda+1,0,\beta-\beta}(n,j) L_j^{(\lambda)}(x).
\]

Theorem [11] proves the known property on the generalized Laguerre polynomials \( L_n^{(\lambda)}(x) \), \( n \geq 1 \), that have only real zeros (here for \( \lambda \geq -2 \)), for more information about the real zeros of Laguerre polynomials see for example [5]. Theorem [7] shows that, for \( x > 0 \), there hold
\[
\begin{align*}
L_n^{(\lambda)}(\frac{1}{x}) &= \frac{(-1)^n}{n!} x^{n+1+\lambda} e^{-\frac{x}{2}} \left( \frac{d}{dx} \right)^n \left( x^{\lambda-1} e^{\frac{x}{2}} \right), \\
L_n^{(\lambda)}(x) &= \frac{x^{-\lambda} e^{-x}}{n!} \left( \frac{d}{dx} \right)^n \left( x^{n+\lambda} e^{x} \right).
\end{align*}
\]

We remark that for \( \lambda = 2r - 1 \) be a positive odd integer, we obtain
\[
L_n^{(2r-1)}(\frac{1}{x}) = \frac{(-1)^n}{n!} x^{n+2r} e^{-\frac{x}{2}} \left( \frac{d}{dx} \right)^n \left( \frac{1}{x^{2r} e^{\frac{x}{2}}} \right) = \frac{1}{n!} \sum_{k=0}^n L_r(n+r,k+r) x^k,
\]
where \( L_r(n,k) \) is the \((n,k)\)-th \( r \)-Lah number, see [4] [22] [24].

5.3 Application to the associated Lah polynomials

Let \( m \) be a positive integer. The sequence of the associated Lah polynomials \( \mathcal{L}_n^{(m)}(x) \) are studied in [2] [23] and are defined by
\[
\sum_{n \geq 0} \mathcal{L}_n^{(m)}(x) \frac{t^n}{n!} = \exp \left( x \left( (1 - t)^{-m} - 1 \right) \right).
\]
This shows that $\mathcal{L}_n^{(m)}(x) = P_n^{(0,-m)}(x)$. Propositions 1, 4 and 5 give

$$
\mathcal{L}_n^{(m)}(x) = e^{-x} \sum_{k \geq 0} \langle mk \rangle_n \frac{x^k}{k!}, \\
\mathcal{L}_n^{(m)}(x) = \sum_{k=0}^{n} S_{0,-m}(n, k) x^k,
$$

To write $P_n^{(\alpha,\beta)}(x)$ in the basis $\{1, \mathcal{L}_1^{(m)}(x), \ldots, \mathcal{L}_n^{(m)}(x)\}$, set $(\alpha', \beta') = (0, -m)$ in Proposition 16 to obtain

$$
P_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n} (-1)^j S_{\alpha,-\frac{\beta}{m}}(n, j) \mathcal{L}_j^{(m)}(x).
$$

Theorem 11 proves a known property of the associated Lah polynomials $\mathcal{L}_n^{(m)}$, $n \geq 1$, that have only real zeros and Theorem 7 shows that, for $x > 0$, there hold

$$
\mathcal{L}_n^{(m)}\left(\frac{1}{x^m}\right) = (-1)^n x^n e^{-1/x^m} \left(\frac{d}{dx}\right)^n \left(e^{1/x^m}\right), \\
\mathcal{L}_n^{(m)}(x^m) = x e^{-x^n} \left(\frac{d}{dx}\right)^n \left(x^n e^{x^m}\right).
$$

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