HOW CAN NEWLY BORN RAPIDLY ROTATING NEUTRON STARS BECOME MAGNETARS?

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ABSTRACT

In a newly born (high-temperature and Keplerian rotating) neutron star, \( r \)-mode instability can lead to stellar differential rotation, which winds the seed poloidal magnetic field (\( \sim 10^{11} \) G) to generate an ultra-high (\( \sim 10^{17} \) G) toroidal field component. Subsequently, by succumbing to the Tayler instability, the toroidal field could be partially transformed into a new poloidal field. Through such dynamo processes, the newly born neutron star with sufficiently rapid rotation could become a magnetar on a timescale of \( \sim 10^{-2} \) s, with a surface dipolar magnetic field of \( \sim 10^{15} \) G. Accompanying the field amplification, the star could spin down to a period of \( \sim 5 \) ms through gravitational wave radiation due to the \( r \)-mode instability and, in particular, the non-axisymmetric stellar deformation caused by the toroidal field. This scenario provides a possible explanation for why the remnant neutron stars formed in gamma-ray bursts and superluminous supernovae could be millisecond magnetars.

Key words: gamma-ray burst: general – stars: neutron

Online-only material: color figures

1. INTRODUCTION

Since the operation of the Swift satellite, it has been widely suggested that the remnant compact objects formed in some gamma-ray bursts (GRBs) could be rapidly rotating, highly magnetized neutron stars (NSs). Such a millisecond magnetar scenario is helpful for understanding observations such as X-ray shallow decay and, in particular, plateau afterglows on timescales of \( \sim 10^{-2} \) s (e.g., Dai & Lu 1998; Zhang & Mészáros 2001; Fan & Xu 2006; Yu et al. 2010; Zhang 2013; Rowlinson et al. 2013; Gompertz et al. 2013). The growth of the gamma-ray emission on timescales of minutes of short GRBs (Gao & Fan 2006; Metzger et al. 2008; Bucciantini et al. 2012). Recently, a similar energy source scenario was employed to interpret the high luminosity of some superluminous supernovae (e.g., Kasen & Bildsten 2010) and was also suggested to power bright merger nova emission during the merger of a double NS system (Yu et al. 2013). In all of these cases, the high magnetic field of the NS is required to ensure that most of the rotational energy of the star can be released into the stellar wind in a sufficiently short time.

Magnetars can generally be defined as special types of NSs with surface (though sometimes only interior) magnetic fields that are as high as \( \sim 10^{14} \) G at least. The dissipation of the magnetic fields could power some high-energy electromagnetic emission, e.g., the GRB X-ray flares (Dai et al. 2006) and the bursts of soft gamma-ray repeaters (SGRs) and anomalous X-ray pulsars (AXPs; Thompson & Duncan 1993). The strength of the surface, dipolar magnetic fields of most Galactic SGRs and AXPs is indeed inferred to be on the order of \( 10^{14} \)–\( 10^{15} \) G by \( B_p = 3.2 \times 10^{19} \sqrt{\dot{P}P} \), where \( P \) is the pulse period and \( \dot{P} \) is the period derivative (Olausen & Kaspi 2013). However, the recently observed SGR 0418+5729 and Swift J1822.3–1606 both indicate much lower dipolar fields (\( 6 \times 10^{12} \) G and \( 3.8 \times 10^{13} \) G, respectively; Rea et al. 2013), which are typical for normal pulsars rather than magnetars. This “contradiction” hints that the high magnetic fields of some magnetars could be dominated by multipolar (e.g., toroidal) components, which are probably hidden in the stellar interior. The interior fields could be much stronger than those on the surface.

The most straightforward consideration one might suggest is that the high magnetic field of a newly born magnetar may originate from the fossil magnetic fluxes in the progenitor core via the magnetic flux conservation, where the progenitor should be highly magnetized (Ferrario & Wickramasinghe 2006). However, considering the possible existence of interior multipolar magnetic fields in SGRs/AXPs and the extremely rapid rotation of the GRB magnetars, it is believed that the origin of the high magnetic field is more likely to be associated with a dynamo process located deep in the stellar interior. Duncan & Thompson (1992) proposed that an \( \alpha - \omega \) dynamo could be supported by neutrino-driven turbulent convection and initially existing differential rotation in newly born millisecond NSs. Alternatively, in this Letter we suggest that the extremely rapid rotation of newly born NSs can spontaneously initiate a dynamo process via \( r \)-mode instability and the magnetic Tayler instability.

In a rotating NS, \( r \)-modes arise due to the action of the Coriolis force with positive feedback (Andersson 1998; Friedman & Morsink 1998). The growth of the \( r \)-modes can be suppressed by viscous damping and, in particular, by some non-linear effects. Specifically, by expanding the \( r \)-modes up to the second order of amplitude, the differential rotation induced by the modes can be found to determine a saturation state of the instability (Sá & Tomé 2005, 2006; Yu et al. 2009). As a result, the toroidal magnetic field component can be formed and amplified by winding up the seed poloidal field (Rezzolla et al. 2000; Rezzolla et al. 2001a, 2001b). As it increases in a stably stratified stellar interior, the toroidal field could enter into the Tayler instability and therefore can be partly transformed into a new poloidal component. Finally, a stable poloidal–toroidal twisted torus configuration appears in the stellar interior, which is connected by an enhanced dipolar field on the stellar surface (Braithwaite & Spruit 2004). Such a dynamo mechanism has been previously investigated in the framework of accreting NS binaries (Cuofano & Drago 2010; Cuofano et al. 2012), where the solid crust of the NSs can provide an extra effective suppression on the \( r \)-mode instability. In contrast, for a newly born NS, the crust cannot form initially due to the high stellar temperature. Moreover, the rotation of the newly born NS could be very close to the Keplerian limit, which is much more rapid.
than what an accreting binary NS can reach (Hessels et al. 2006).

The chief purpose of this Letter is to report on the formation of the remnant magnetars harbored in GRBs. In Section 2, we describe the model for the evolutions of r-modes, magnetic fields, and stellar rotations. Calculated results are presented in Section 3. Conclusion and discussions are given in Section 4.

2. AN EVOLUTIONARY MODEL OF NEWLY BORN NEUTRON STARS

2.1. r-mode Evolution

A phenomenological model for the evolution of the \( l = m = 2 \) mode of primary importance was first developed by Owen et al. (1998) and subsequently improved by Ho & Lai (2000), who demonstrated that the r-mode evolution is guided by the conservation of angular momentum. The saturation amplitude of the r-mode is determined by some nonlinear effects. Specifically, the second-order solution of the r-mode gives a saturation amplitude of \( \alpha_{\text{sat}} \approx (\delta + 2)^{-1/2} \) (Sá & Tomé 2005), where \(-5/4 \leq \delta \leq 10^{13}\) is a free model parameter representing the initial amount of differential rotation. In the same model framework, the physical angular momentum of the r-mode can be calculated by \( J_r = (1/2)(4\delta + 5)\alpha^2 l^2/\Omega \), where \( l^* = 1.635 \times 10^{-2} M R^2 \) is an effective moment of inertia for the mode, \( \Omega \) is the angular spin frequency, and \( M \) and \( R \) are the mass and radius of the NS, respectively.

The r-mode angular momentum can increase through a gravitational wave (GW) radiation back-reaction and, meanwhile, be decreased by viscous damping and by winding the seed poloidal magnetic field to form a toroidal component. Therefore, the evolution equation for the r-mode angular momentum can be written as

\[
\frac{dJ_r}{dt} = 2J_r \left( \frac{1}{\tau_{\text{gr}}} - \frac{1}{\tau_v} - \frac{1}{\tau_l} \right),
\]

where the timescales \( \tau_{\text{gr}}, \tau_v, \) and \( \tau_l \) correspond to the GW radiation induced by the r-mode, the viscous damping, and the formation of the toroidal magnetic field, respectively. The expressions of the two former timescales can be found in Yu et al. (2009), while the last one is defined as \( \tau_l = 2E_r/\dot{E}_r \), where \( E_r = (1/2)(4\delta + 9)\alpha^2 l^2/\Omega^2 \) is the energy of the r-mode and \( \dot{E}_r \) is the change rate of the toroidal field’s energy. For simplicity, some magnetic back reactions are not taken into account in Equation (1). When the toroidal field’s energy becomes comparable to the kinetic energy of the differential rotation, the Lorentz force exerted on the plasma would approach to reverse the differential rotation (Braithwaite 2006a). Then a dynamic equilibrium could be built at which point the toroidal field formation can no longer be regarded as a dissipation process. Such an effect could suppress the peak strength of the magnetic fields that are presented in Section 3 by a factor of a few. In more detail, a sufficiently strong Lorentz force could affect the drift velocity of the given fluid element, the azimuthal displacement, and the rate of energy transfer (e.g., Morsink & Rezania 2002).

2.2. Magnetic Field evolution

Secular azimuthal drifts on the isobaric surfaces due to the differential rotation gradually generate a large scale azimuthal magnetic field. Following Rezzolla et al. (2000), Sá (2004), Sá & Tomé (2005), and Cuofano & Drago (2010), the strength of the azimuthal field at coordinate, \( \mathbf{r} \), in the star at a given time, \( t \), can be calculated by

\[
B^\theta(\mathbf{r}, t) = B_d \left( \frac{R}{r} \right)^3 (4 \cos^2 \theta + \sin^2 \theta)|\xi^\theta(\mathbf{r}, t)|,
\]

where an internal dipolar magnetic field with a surface strength, \( B_d \), is assumed and

\[
\xi^\theta(\mathbf{r}, t) = \frac{15}{32\pi} \left( \frac{r}{R} \right)^2 \sin^2 \theta(2\delta + 3) \int_0^t \alpha^2 \Omega dt.
\]

is the total azimuthal displacement from the onset of the r-mode instability to time, \( t \). The increase rate of the total energy of the toroidal field can be calculated by integrating over the whole stellar volume

\[
\frac{dE_t}{dt} = \frac{d}{dt} \int \frac{B^2(\mathbf{r}, t)}{8\pi} dV \\
\approx \frac{15}{56\pi^2} B_d^2 R^3(2\delta + 3)^2 \alpha^2 \Omega \int_0^t \alpha^2 \Omega dt.
\]

Here, we further define a volume-averaged strength of the toroidal field by using \( E_t = (4\pi R^3/3)(B_t^2/8\pi) \), and then the evolution of \( B_t \) can be determined by

\[
\frac{d\bar{B}_t}{dt} \approx \frac{3}{2\pi} \left( \frac{5}{14} \right)^{1/2} B_d(2\delta + 3)\alpha^2 \Omega.
\]

A quadrupolar deformation of the NS may appear with an ellipticity of \( \epsilon = -5B_t^2 R^4/6GM^2 \), an important consequence of the toroidal field’s formation. Such a deformation will cause the NS to produce additional GW radiation that is much stronger than that induced by the r-mode itself.

It is further expected that the ultra-high toroidal field will succumb to the Tayler instability that closes the dynamo loop by generating a new poloidal field. Here, it is required that the spin frequency is lower than the Alfvén frequency \( \omega_{\text{Alfv}} = \dot{\Omega}/(\sqrt{4\pi\rho}) \) of the stellar material (Braithwaite 2006b). In other words, the timescale of the Tayler instability, \( \tau_{\text{TI}} \approx 2\pi/\omega_{\text{Alfv}} = 2.2 \times 10^{-7} M R^3 G^{-1} \), would be shorter than the spin period of the star. Such a condition could be satisfied in a short time due to the increase of \( \dot{B} \) and the decrease of \( \Omega \) by GW radiation. As a result, a poloidal–toroidal twisted torus shape can be built in the stellar interior. Nevertheless, some previous studies for a stable magnetic configuration suggested that the poloidal component is probably overwhelmingly subordinate to the toroidal one (e.g., Mastrano et al. 2011). Moreover, since some poloidal field lines could be closed in the stellar interior, the field that extends to the stellar surface to connect with the outer dipolar field could be much weaker than the internal toroidal one. Therefore, in the following calculations, we will adopt

\[
B_d = \xi \bar{B}_t \text{ for } \Omega < \omega_{\text{Alfv}}
\]

with a reference value of \( \xi = 0.01 \). This assumption could be supported by the fact that the surface dipolar magnetic field of some SGRs is inferred to be lower than the internal field (Stella et al. 2005; Dall’Osso et al. 2009).
2.3. Spin Evolution

A rapidly rotating, newly born NS could be spun down by GW radiation and magnetic dipole radiation, whereas the former can be due to both the r-mode oscillation and the magnetic deformation of the star. Therefore, the decrease of the total stellar angular momentum can be written as

$$\frac{dJ}{dt} = -\frac{3\alpha^2 I^*\Omega}{\tau_{g,r}} - \frac{I\Omega}{\tau_{g,i}} - \frac{I\Omega}{\tau_d},$$

(7)

where $J = I\Omega + J_r$ with $I = 0.261M R^2$ being the star’s moment of inertia. The timescales corresponding to the magnetic dipole radiation read $\tau_d = 6t_c^2/(R_i^2 R^4 \Omega^2 \sin^2 \chi)$ and to the GW radiation due to magnetic deformation reads $\tau_{g,i} = 5e^5/[2G\ell c^2 \Omega^4 \sin^2 \chi (1 + 15 \sin^2 \chi)]$ (Cutler & Jones 2001), where $\chi$ is the inclined angle between the magnetic and spin axes. The initial value of $\chi$ could be close to zero, but a deviation between the two axes is also expected to happen quickly though the details of the processes are uncertain (e.g., Dall’Osso et al. 2009; Cutler 2002).

Combining Equations (1) and (7), we can obtain the coupled evolution equations for the r-mode amplitude and spin frequency as

$$\frac{d\alpha}{dt} = \left[ 1 + \frac{2\alpha^2}{15} (\delta + 2) \right] \frac{\alpha}{\tau_{g,r}} - \left[ 1 + \frac{2\alpha^2}{30} (4\delta + 5) \right] \left( \frac{\alpha}{\tau_v} + \frac{\alpha}{\tau_1} \right)$$

$$+ \frac{\alpha}{2\tau_{g,i}} + \frac{\alpha}{2\tau_d},$$

(8)

$$\frac{d\Omega}{dt} = -\frac{4\alpha^2}{15} (\delta + 2) \frac{\Omega}{\tau_{g,r}} - \frac{\Omega}{\tau_{g,i}} - \frac{\Omega}{\tau_d},$$

(9)

where $I^*/I \approx 1/15$ is taken and the timescale, $\tau$, reads

$$\tau = \left[ \frac{1}{\tau_{g,r}} - \frac{(4\delta + 5)}{4(\delta + 2)} \left( \frac{1}{\tau_v} + \frac{1}{\tau_1} \right) \right]^{-1}.$$  

(10)

3. RESULTS

From Equations (5), (6), (8), and (9), we can calculate the strengths of the magnetic fields, the spin frequency, and the r-mode amplitude as functions of time since the birth of an NS, where an analytical cooling history dominated by a modified Urca process is adopted as $T = T_0 (1 + t/\tau_c)^{-1/6}$ with $\tau_c = 20(T_i/10^{10})^{-1}$ s and $T_0$ represents the stellar temperature. Our results reveal that the secular evolutions of these quantities are very insensitive to the initial values of $T_i$, $\alpha_i$, and $B_{d,i}$ within a wide parameter range, whereas the initial spin frequency could significantly influence the evolutions. By taking $T_i = 10^{10}$ K, $\alpha_i = 10^{-10}$, and $B_{d,i} = 10^{11}$ G, which is typical for normal pulsars, we plot the evolution curves of $B_t$, $B_d$, $v (= \Omega/2\pi r)$, and $\alpha$ in Figure 1 for three different initial spin frequencies as $\Omega_i = \Omega_K$, $(3/4)\Omega_K$, and $(1/2)\Omega_K$, where $\Omega_K = 2\sqrt{\pi G\rho}/3$ is the Keplerian spin frequency.

The solid red and green lines in the top panel of Figure 1 show that, for $\Omega_i \geq (3/4)\Omega_K$, the toroidal magnetic field can be increased to as high as a few times $10^{17}$ G on a timescale of $\sim 10^{-3}$ s. Subsequently, the high quadrupolar ellipticity $\epsilon \sim 0.01$ of the NS gives rise to a strong GW radiation, which leads the NS to spin down with a temporal behavior of $\propto \tau^{-1/4}$, as shown in the middle panel of Figure 1 (for $\chi = \pi/2$). Strictly speaking, the start time of this GW braking phase is determined by the uncertain deviation of the magnetic axis from the spin axis. Nevertheless, for $\Omega_i \geq (3/4)\Omega_K$, such an uncertainty may not influence the stellar magnetic evolution, because the ultra-high toroidal field can make the timescale of the Tayler instability shorter than the spin period even though the NS is only slightly spun down by the r-mode-induced GW radiation. So, the surface dipolar magnetic field of the NS can be amplified to $\sim 10^{15}$ G on the same timescale of $\sim 10^{-3}$ s. However, for lower initial spin frequencies (e.g., $\Omega_i = (1/2)\Omega_K$; the blue lines in Figure 1), the consequent lower toroidal field ($\sim 10^{16}$ G) would determine a Tayler instability timescale much longer than the initial spin period ($P_i = 2\pi/\Omega_i$). Hence, the occurrence of the Tayler instability requires a remarkable spin-down of the NS, which is beyond the ability of the r-mode-induced GW radiation. Therefore, a long-time braking by the GW radiation, due to the magnetic deformation, becomes necessary. The amplification of the surface field is delayed until it is too late (e.g., $\sim 10^8$ s) to be consistent with the GRB timescales. The magnetar’s formation should occur much earlier than the GRB-associated supernova on a timescale of $\sim 10^5$ s. More calculations will reveal a critical initial spin period of $P_{i,\text{crit}} = 1.7$ ms. When $P_i \leq P_{i,\text{crit}}$, the Tayler instability can happen with only r-mode-induced GW radiation, and the surface field can be simultaneously amplified to the generation of the toroidal field.

For the r-mode evolution, the bottom panel of Figure 1 shows that the maximum amplitude of the r-mode is restricted to $\lesssim 10^{-5}$ for the adopted parameter $\delta = 10^{10}$, which is taken

1 On such a long timescale, more complexity will be involved. For $\tau \gtrsim 10^7$ s, the temperature of the NS dropped to $T \lesssim 10^9$ K, at which a stellar crust and a superconducting layer could form. Therefore, if the field amplification happens later than $10^7$ s, the emergence of the amplified field from the stellar surface will be seriously suppressed by the solid crust for an extremely long time.
to be consistent with the saturation amplitude determined by some other possible nonlinear effects. For example, Bondarescu et al. (2007) revealed a saturation amplitude on the order of $\sim 10^{-5}$ by coupling the r-modes with other two inertial modes. In fact, a higher saturation amplitude would not significantly change the toroidal magnetic field’s order of magnitude because the r-mode energy, $E_r \propto \alpha_i^2 \delta_i$, is weakly dependent on the parameter. On the other hand, the duration of the instability is restricted to $\sim 10^{-2} - 3$ s for $\Omega_i \gtrsim (1/2)\Omega_k$, which is drastically shorter than that obtained without the consideration of the magnetic field evolution (Yu et al. 2009). For a more general understanding, in Figure 2, we display the temporal-dependent r-mode instability windows, the boundaries of which (solid lines) are determined by the equation $dF_r/dt = 0$. As shown, the instability window shrinks to the high-$\nu$ region very quickly and becomes temperature-independent. Such a window evolution is caused by the magnetic field evolution, because the dissipation of the r-mode is primarily through the energy transfer from the r-mode to the toroidal field. As a result, the r-mode instability could be switched off at a very early time, even though the star has only slightly spun down. The stellar temperature at which the r-mode instability ends can be found to be around $5.5 \times 10^9$ K. Therefore, the solid crust of the NS could not have been formed during the action of the r-mode instability (Chamel & Haensel 2008; Dall’Osso et al. 2009), therefore, the damping effects arising from the boundary of the crust (Mendell 2001) can be ignored. Finally, the instability window for $\tau = 0$ s shows that, for an initial spin period longer than $\sim 3$ ms, the r-mode instability can be effectively suppressed by the viscous damping.

4. CONCLUSION AND DISCUSSIONS

By considering the differential rotation caused by r-mode instability in a newly born, rapidly rotating NS, we calculate the evolution of the stellar magnetic fields, where an ultrastrong toroidal magnetic field is generated. Succumbing to the Tayler instability, the toroidal field is partially transformed into a new poloidal field. Through such dynamo processes, the NS could become a magnetar with a surface dipolar field of a strength $\sim 10^{15}$ G on timescales $\sim 10^{2-3}$ s, the precondition of which is that the NS should rotate initially with a nearly Keplerian period, $P_i \lesssim 1.7$ ms. Such a condition could easily be satisfied in the situation of GRBs. For somewhat longer periods, $1.7$ ms $< P_i \lesssim 3$ ms, this dynamo could work in principle, but the strengths of the fields become much lower. Moreover, the amplification of the surface field is delayed to a very late time, at which more complexity (e.g., the formation of a crust) is involved. In any case, the long time delay could make the model inapplicable for GRB magnetars. Finally, for $P_i$ $> 3$ ms, the dynamo processes would never happen and a normal magnetic field keeps in the NS, because the r-mode instability is suppressed by viscosities.

Due to the magnetic dissipation, the r-mode-induced GW radiation becomes very weak. Alternatively, another strong GW radiation is produced due to the high deformation of the NS by the toroidal magnetic field, which could cause the star to be a promising target for GW detection. As a result, accompanying the magnetic field amplification, the spin periods of GRB magnetars would be increased to $\sim 5$ ms. In other words, the “initial” spin periods derived from GRB afterglow observations should be basically consistent with such a value. Furthermore, due to the GW radiation, a remarkable amount of the rotational energy of the NS can be released into the GW. Therefore, the supernova remnant around the magnetar cannot be as highly energized as usually considered. In observation, analysis of the X-ray spectra of some supernova remnants associated with magnetar candidates Vink & Kuiper (2006) revealed that the total energy in these supernova remnants is almost nothing, which may favor our model. In other words, some Galactic magnetars may share the same origin mechanism presented here.

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Figure 2. R-mode instability windows (regions above the solid lines) at different times, as labeled, which evolve with time because of the magnetic evolution. The dash–dotted line represents the evolution trajectory of an NS for $R_{41} = 10^{11}$ G, $\Omega_i = \Omega_k$, and $T_i = 10^{10}$ K in the $\nu$–$T$ plane. The open circle on the evolution curve indicates the time of 611 s at which the evolution curve crosses the boundary of the instability window.

(A color version of this figure is available in the online journal.)
