A non-existence result for the $L_p$-Minkowski problem

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Abstract

We show that given a real number $p < 1$, a positive integer $n$ and a proper subspace $H$ of $\mathbb{R}^n$, the measure on the Euclidean sphere $S^{n-1}$, which is concentrated in $H$ and whose restriction to the class of Borel subsets of $S^{n-1} \cap H$ equals the spherical Lebesgue measure on $S^{n-1} \cap H$, is not the $L_p$-surface area measure of any convex body. This, in particular, disproves a conjecture from [Bianchi, B"orczyk, Colesanti, Yang, The $L_p$-Minkowski problem for $-n < p < 1$, Adv. Math. (2019)].

1 Introduction

Given a convex body $K$ in $\mathbb{R}^n$ and $p \in \mathbb{R}$, the $L_p$-surface area measure $S_p(K, \cdot)$ of $K$ is a Borel measure on the Euclidean sphere $S^{n-1}$, defined by

$$dS_p(K, \cdot) := h_K^{1-p}dS(K, \cdot),$$

where $h_K$ is the support function of $K$ and $S(K, \cdot)$ is the surface area measure of $K$ (see Section 2 for definitions). The $L_p$-Minkowski problem (as of the existence) asks the following.

**Problem 1.1.** Let $\mu$ be a given Borel measure on $S^{n-1}$. Find necessary and sufficient conditions for $\mu$ to be the $L_p$-surface area measure of some convex body in $\mathbb{R}^n$.

The case $p = 1$ is the classical Minkowski problem (see Section 2). The problem for general $p$ was initiated by Lutwak [15], where it was solved in the even case, for $p \in (1, n) \cup (n, \infty)$. Problem [11] was subsequently studied by several authors and it is still of current interest. The $L_p$-Minkowski problem for $p > 1$ in the non-symmetric case was treated in [14]. Chou and Wang [9] studied the problem for $p \geq -n - 1$, in the case where $\mu$ is absolutely continuous with respect to the spherical Lebesgue measure (and some extra regularity is assumed). A necessary and sufficient condition in the even case for $p = 0$ was found in [2], while a sufficient condition was obtained in [8] in the non-symmetric case, for $p = 0$. For existence results in the case $p < 0$, see [1], [18] and the references therein (see, also, [12], [13], [3] for other important variants of the Minkowski problem).

In the case $0 < p < 1$, it was shown in [7] that if $\mu$ is a finite Borel measure on $S^{n-1}$ that is not concentrated in a great subsphere, then there exists a convex body whose $L_p$-surface area measure equals $\mu$. Note that, as the example of a simplex with one vertex at the origin shows, $S_p(K, \cdot)$ (for $p < 1$) may be concentrated in a proper subspace of $\mathbb{R}^n$ and, therefore, the condition mentioned previously is not necessary. A more general result (still in the case $0 < p < 1$) appeared in [1], where the following conjecture was proposed.

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2020 Mathematics Subject Classification. Primary: 52A20; Secondary: 52A38, 52A39.

Key words and phrases. convex bodies, $L_p$-Minkowski problem, non-existence, surface area measure.
Conjecture 1.2. (Bianchi, Boroczky, Colesanti, Yang [12]) Let $0 < p < 1$ and $\mu$ be a finite Borel measure on $\mathbb{S}^{n-1}$ that is not concentrated in a pair of antipodal points. Then, there exists a convex body $K$ in $\mathbb{R}^n$, such that

$$S_p(K, \cdot) = \mu.$$

Conjecture 1.2 was confirmed in the plane independently by [4] and [7]. The main purpose of this note is to show that Conjecture 1.2 fails if the measure $\mu$ is additionally assumed to have finite support. Notice that by the result mentioned previously, the case $k = 1$ is already known, while for $k = n$ it is impossible to find such a measure). Moreover, our examples are valid for all $p < 1$; other non-existence results, for $p \leq -n$, appear for instance in [9] and [10].

Theorem 1.3. Let $p < 1$ and $H$ be a subspace of $\mathbb{R}^n$ of dimension $1 \leq k \leq n - 1$. If $\mu_H$ is the measure on $\mathbb{S}^{n-1}$ which is concentrated in $H$ and whose restriction to the family of Borel sets in $\mathbb{S}^{n-1} \cap H$ equals the $(k-1)$-dimensional Hausdorff measure on $\mathbb{S}^{n-1} \cap H$, then there exists no convex body $K$ in $\mathbb{R}^n$ satisfying

$$S_p(K, \cdot) = \mu_H.$$

We mention that it is well known that for $p \geq 1$, $S_p(K, \cdot)$ cannot be concentrated in a proper subspace of $\mathbb{R}^n$. Therefore, Theorem 1.3 trivially holds for $p \geq 1$ as well.

Although we are unable to formulate a general conjecture, it seems plausible to us that Conjecture 1.2 should hold if the measure $\mu$ is additionally assumed to have finite support.

The proof of Theorem 1.3 will be given in Section 3. In Section 2, we fix some notation and collect some facts concerning convex sets, that will be used in the proof.

2 Background and notation

We denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $\mathbb{R}^n$ and the origin by $o$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x|$. We set $B^n_2 = \{ x \in \mathbb{R}^n : |x| \leq 1 \}$ for the unit ball in $\mathbb{R}^n$ and $\mathbb{S}^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \}$ for its unit sphere. For $v \in \mathbb{S}^{n-1}$ and $t \in \mathbb{R}$, set

$$H^-(v, t) := \{ x \in \mathbb{R}^n : \langle x, v \rangle \leq t \}.$$

Let $A \subseteq \mathbb{R}^n$. The notation $A/E$ stands for the orthogonal projection of $A$ onto a subspace $E$. We write $\partial A$ for the boundary of $A$. Set, also, $A^\perp := \{ x \in \mathbb{R}^n : \langle x, y \rangle = 0, \forall y \in A \}$ for its orthogonal complement and $u^\perp := \{ u \}^\perp$, if $u \in \mathbb{R}^n$. Finally, for $a \geq 0$, the $a$-dimensional Hausdorff measure will be denoted by $H^a(\cdot)$.

Below, we collect some basic facts from convex geometry that will be needed in the proof of Theorem 1.3. For more information, we refer to the books of Schneider [17] and Gardner [11]. A convex set $C$ is called a convex cone, if $rx \in C$, for all $r \geq 0$ and for all $x \in C$. Clearly, a set $C$ is a convex cone if and only if $ax + \beta y \in C$, for all $a, \beta \geq 0$ and for all $x, y \in C$. Notice that according to this definition, any non-empty convex cone contains the origin.

We say that a vector $u \in \mathbb{S}^{n-1}$ supports a convex set $C \subseteq \mathbb{R}^n$ at $w \in \partial C$, if $\langle u, x \rangle \leq \langle u, w \rangle$, for all $x \in C$. If $C$ happens to be a convex cone and $u \in \mathbb{S}^{n-1}$ supports $C$ at some boundary point of $C$, then it is clear that $\langle u, x \rangle \leq 0$, for all $x \in C$. That is, $u$ supports $C$ at $o$ (and $o \in \partial C$). In this case, we will briefly say that $u$ supports $C$. 

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Let $K$ be a convex body in $\mathbb{R}^n$ (that is, convex, compact, with non-empty interior). The support function $h_K$ of $K$ is defined by

$$h_K(x) := \sup\{ \langle x, y \rangle : x \in K \}, \quad x \in \mathbb{R}^n.$$ 

It is clear by the definition that $h_K$ is convex and positively homogeneous, that is $h_K(tx) = th_K(x)$, for $t \geq 0$. Moreover, $h_K(0) = 0$ and $h_K$ is non-negative (resp. strictly positive) on $\mathbb{R}^n \setminus \{0\}$ if and only if $K$ contains the origin (resp. $K$ contains the origin in its interior). The support function $h_{K|E} : E \to \mathbb{R}$ of the orthogonal projection of $K$ onto a subspace $E$ can be computed by

$$h_{K|E} = (h_K)|_E.$$ 

The surface area measure $S(K, \cdot)$ of $K$ is a (necessarily finite) Borel measure on $\mathbb{S}^{n-1}$, given by

$$S(K, \omega) = \mathcal{H}^{n-1}(\{ x \in \partial K : \exists v \in \omega, \text{ such that } \langle x, v \rangle = h_K(v) \}), \quad \omega \text{ a Borel set in } \mathbb{S}^{n-1}.$$ 

Minkowski’s Existence and Uniqueness Theorem states that a finite Borel measure $\mu$ is the surface area measure of a convex body $K$ in $\mathbb{R}^n$ if and only if $\mu$ is not concentrated in a proper subspace of $\mathbb{R}^n$ and the barycentre of $\mu$ is at the origin. Moreover, $K$ is unique up to translation.

### 3 Proof of Theorem 1.3

We start the proof of Theorem 1.3 with the following observation.

**Lemma 3.1.** Let $K$ be a convex body in $\mathbb{R}^n$ that contains the origin and assume that for some $u \in \mathbb{S}^{n-1}$ it holds $h_K(u) = 0$. If for some $q < 0$, the function $v \mapsto \langle v, u \rangle h_K(v)^q$ is integrable on $\mathbb{S}^{n-1}$, then

$$\int_{\mathbb{S}^{n-1}} \langle v, u \rangle h_K(v)^q \, d\mathcal{H}^{n-1}(v) > 0.$$ 

**Proof.** For $t \geq 0$, define the hyperplane $H(t) := u^+ + tu$. Set, also, $H^+ := \{ x \in \mathbb{R}^n : \langle x, u \rangle \geq 0 \}$ and $H^- := \{ x \in \mathbb{R}^n : \langle x, u \rangle \leq 0 \}$.

Using the fact that the function $v \mapsto \langle v, u \rangle h_K(v)^q$ is integrable on $\mathbb{S}^{n-1}$ and integrating in polar coordinates, yields

$$\int_{B_2^n \cap H^\pm} \langle x, u \rangle h_K(x)^q e^{-1/|x|} \, dx = \int_{\mathbb{S}^{n-1} \cap H^\pm} \int_0^{1} r^{n-1} \langle rv, u \rangle h_K(rv)^q e^{-1/|rv|} \, dr \, d\mathcal{H}^{n-1}(v)$$

$$= \int_0^{1} r^{n+q} e^{-1/r} \, dr \cdot \int_{\mathbb{S}^{n-1} \cap H^\pm} \langle v, u \rangle h_K(v)^q \, d\mathcal{H}^{n-1}(v).$$

Thus, $\int_{B_2^n \cap H^\pm} \langle x, u \rangle h_K(x)^q e^{-1/|x|} \, dx < \infty$ and we need to prove that

$$\int_{B_2^n \cap H^+} \langle x, u \rangle h_K(x)^q e^{-1/|x|} \, dx > - \int_{B_2^n \cap H^-} \langle x, u \rangle h_K(x)^q e^{-1/|x|} \, dx. \quad (1)$$

Let $z \in u^\perp$ and $t > 0$. Observe that

$$h_K(z + tu) \leq h_K(z - tu) + 2th_K(u) = h_K(z - tu) \quad \text{(2)}$$
and, hence,
\[ h_K(z + tu)^q e^{-1/|z + tu|} \geq h_K(z - tu)e^{-1/|z - tu|}. \]

This shows that for \( 0 < t < 1 \), it holds
\[
\int_{[B_2^n \cap H(t)]-tu} h_K(z + tu)^q e^{-1/|z + tu|} \, dz \geq \int_{[B_2^n \cap H(t)]-tu} h_K(z - tu)^q e^{-1/|z - tu|} \, dz
\]
\[
= \int_{[B_2^n \cap H(t)]+tu} h_K(z - tu)^q e^{-1/|z - tu|} \, dx. \tag{3}
\]

Let \( t \in (0, 1) \) be such that the function \( z \mapsto h_K(z + tu)^q e^{-1/|z + tu|} \) is integrable on \([B_2^n \cap H(t)] - tu\). Then it is clear that equality holds in (3) if and only if equality holds in (2), for all \( z \in [B_2^n \cap H(t)] - tu \).

Notice that (3) can be equivalently written as
\[
\int_{B_2^n \cap H(t)} h_K(y)^q e^{-1/|y|} \, dy \geq \int_{B_2^n \cap H(-t)} h_K(y)^q e^{-1/|y|} \, dy, \tag{4}
\]
for all \( 0 < t < 1 \). Therefore, by Fubini’s Theorem, we obtain
\[
\int_{B_2^n \cap H^+} \langle x, u \rangle h_K(x)^q e^{-1/|x|} \, dx = \int_0^1 t \int_{B_2^n \cap H(t)} h_K(y)^q e^{-1/|y|} \, dy \, dt
\]
\[
\geq \int_0^1 t \int_{B_2^n \cap H(-t)} h_K(y)^q e^{-1/|y|} \, dy \, dt
\]
\[
= \int_{B_2^n \cap H} -\langle x, u \rangle h_K(x)^q e^{-1/|y|} \, dx.
\]

Assume, now, that equality holds in the previous inequality. Then equality holds in (4) (and therefore in (3)) for almost every \( t \in (0, 1) \). Consequently, equality must hold in (2) for almost every \( t \in (0, 1) \) and for all \( z \in [B_2^n \cap H(t)] - tu \). Given this and the continuity of \( h_K \), we conclude that \( h_K(-u) = h_K(u) = 0 \), which is a contradiction because \( K \) is assumed to have non-empty interior. The validity of (1) follows.

We will need the following statement concerning convex cones.

**Proposition 3.2.** Let \( R \) be closed convex cone in \( \mathbb{R}^n \), which is supported by at least one unit vector contained in the linear span of \( R \) (that is, \( R \) is not a subspace of \( \mathbb{R}^n \)). Then, there exists \( u_0 \in S^{n-1} \cap (-R) \), such that \( u_0 \) supports \( R \).

**Proof.** We may, clearly, assume that \( R \) is full dimensional. Furthermore, we may assume by a standard approximation argument that if \( x, y \) are boundary points of \( R \), then \( ax + \beta y \) is an interior point of \( R \), provided that \( a^2 + \beta^2 > 0 \). Set
\[
A := \sup_{(u, v) \in (S^{n-1})^2} \{ \langle -u, v \rangle : u \text{ supports } R \text{ and } v \in R \}.
\]

It follows by compactness that there exists a pair \((u_0, v_0) \in (S^{n-1})^2\), such that \( u_0 \) supports \( R \), \( v_0 \in R \) and \( \langle -u_0, v_0 \rangle = A \). We need to prove that \( A = 1 \). Let us assume that \( A < 1 \) instead.
Claim 1. There exists a unique vector \( v_1 \in u_0^+ \cap R \cap S^{n-1} \).

Proof. First assume that there exist two distinct vectors \( v_1, v_1' \in u_0^+ \cap R \cap S^{n-1} \). Then, \( v_1 + v_1' \in u_0^+ \cap R \). Since \( u_0^+ \) is a supporting hyperplane of \( R \), it follows that \( u_0^+ \cap R \subseteq \partial R \), hence all \( v_1, v_1' \) are contained in \( \partial R \), which contradicts the previous assumption. This shows that if there exists such \( v_1 \), then this must be unique.

Let us show the existence of \( v_1 \). If \( u_0^+ \cap R \cap S^{n-1} = \emptyset \), then by compactness, there exists \( 0 < \epsilon < 1 \), such that \( \langle -u_0, v \rangle > \epsilon \), for all \( v \in S^{n-1} \cap R \). Thus, there exists \( t_0 > 0 \), such that for all \( t \in (-t_0, t_0) \), for all \( v \in S^{n-1} \cap R \) and for all \( u \in S^{n-1} \), it holds

\[
\langle -(u_0 - tu), v \rangle > 0.
\]

Consequently, if \( u \in S^{n-1} \), then for all \( t \in (-t_0, t_0) \), the unit vector \( (u_0 - tu)/|u_0 - tu| \) supports \( R \). Thus,

\[
\left( \frac{\langle -u_0 + tu, v_0 \rangle}{| -u_0 + tu |} \right)'_{t=0} = 0.
\]

Since

\[
\left( \frac{\langle -u_0 + tu, v_0 \rangle}{| -u_0 + tu |} \right)'_{t=0} = \langle u, v_0 \rangle - \langle u_0, v_0 \rangle \langle u, u_0 \rangle
\]

(5)

(this holds even if we do not assume \( u \) to be a unit vector), we conclude that

\[
\langle u, v_0 \rangle = \langle u_0, v_0 \rangle \langle u, u_0 \rangle,
\]

for all \( u \in S^{n-1} \). This shows that \( v_0 = \pm u_0 \), which is a contradiction, because we assumed that \( A < 1 \) and because \( u_0 \notin R \). Our claim follows. \( \Box \)

Claim 2. \( \langle v_0, v_1 \rangle \geq 0 \).

Proof. For \( t > 0 \), \( v_0 + tv_1 \in R \), thus

\[
\left( \frac{\langle -u_0, v_0 + tv_1 \rangle}{|v_0 + tv_1|} \right)'_{t=0^+} \leq 0.
\]

Eqs (5), then, shows that

\[
\langle -u_0, v_1 \rangle - \langle -v_0, -u_0 \rangle \langle v_1, -v_0 \rangle \leq 0
\]

or equivalently (since \( \langle u_0, v_0 \rangle = 0 \) and \( \langle u_0, v_0 \rangle < 0 \)), \( \langle v_0, v_1 \rangle \geq 0 \). \( \Box \)

To finish with our proof set

\[
u_s := \frac{u_0 - s(v_0 + v_1)}{|u_0 - s(v_0 + v_1)|}.
\]

We claim that \( u_s \) supports \( R \), if \( s > 0 \) and \( s \) is sufficiently small. To see this, notice that since \( u_0 \notin R \), it holds \( u_0 - s(v_0 + v_1) \neq 0 \), for all \( s > 0 \). Moreover, since \( v_0 \neq v_1 \), we have \( \langle v_1, v_0 + v_1 \rangle > 0 \). By continuity, there exists \( t > 0 \), such that \( \langle v, v_0 + v_1 \rangle > 0 \), for all \( v \in S^{n-1} \cap R \) with \( |v - v_1| < t \). This shows that

\[
\langle -(u_0 - s(v_0 + v_1)), v \rangle \geq s(v, v_0 + v_1) > 0,
\]

for all \( s > 0 \) and for all \( v \in S^{n-1} \cap R \) with \( |v - v_1| < t \). On the other hand, since \( \langle -u_0, v \rangle > 0 \), for all \( u \in (S^{n-1} \cap R) \setminus \{v_1\} \), there exists \( s_0 > 0 \), such that

\[
\langle -u_0 + s(v_0 + v_1), v \rangle > 0,
\]
for all \( v \in S^{n-1} \cap R \) with \( |v - v_1| \geq t \) and for all \( 0 < s < s_0 \). Consequently, for \( 0 < s < s_0 \) and for \( v \in S^{n-1} \cap R \), it holds \( -(u_0 - s(v_0 + v_1)), v) \geq 0 \). That is, \( u_s \) supports \( R \).

As a consequence, we have

\[
((-u_s, v_0))_s=0' \leq 0,
\]

which by (5) gives

\[
\langle v_0 + v_1, v_0 \rangle - \langle u_0, v_0 \rangle \langle v_0 + v_1, u_0 \rangle \leq 0.
\]

Since \( \langle v_1, u_0 \rangle = 0 \) and since by Claim 2, \( \langle v_0 + v_1, v_0 \rangle \geq |v_0|^2 = 1 \), we find

\[
1 - \langle u_0, v_0 \rangle \langle v_0 + v_1, u_0 \rangle \leq 0,
\]

which is impossible by the assumption \( A < 1 \) and the fact that \( u_0 \not\in R \). This completes our proof. \( \square \)

The following well known fact will also be needed.

**Lemma 3.3.** For any convex body \( K \) in \( \mathbb{R}^n \), it holds

\[
K = \bigcap_{v \in \text{supp} S(K, \cdot)} H^{-}(v, h_k(v)).
\]

**Proof.** The proof follows (for instance) from [17, Theorem 2.2.6 and Lemma 4.5.2]. \( \square \)

Below, we state and prove a consequence of Lemma 3.3 and Proposition 3.2.

**Lemma 3.4.** Let \( p < 1 \), \( K \) be a convex body in \( \mathbb{R}^n \) that contains the origin and \( H \) be a proper subspace of \( \mathbb{R}^n \). Set \( S := \text{supp} S(K, \cdot) \setminus H \) and \( T := \{v \in S^{n-1} : h_K(v) = 0\} \). If \( S_p(K, \cdot) \) is concentrated in \( H \), then there exist \( u_0 \in S^{n-1} \cap H \) and \( y \in H^\perp \), satisfying the following.

(i) \( h_K|_H(u_0) = 0 \).

(ii) \( \langle u_0, v \rangle \geq 0 \), for all \( v \in T \cap H \).

(iii) \( \langle u_0 + y, v \rangle \geq 0 \), for all \( v \in S \).

**Proof.** Notice that \( S \neq \emptyset \), since \( S(K, \cdot) \) cannot be concentrated in \( H \). We first claim that \( S \subseteq T \). That is, \( h_K(v) = 0 \), for all \( v \in S \) (in particular, this shows that \( o \in \partial K \)). Indeed, if this is not the case, then \( h_K(v) > 0 \), for some \( v \in S \). By continuity, there exists an open set \( \Omega \) in \( S^{n-1} \), such that \( (h_K)|\Omega \) is bounded away from zero. In fact, we can choose \( \Omega \) to be contained in the open set \( S^{n-1} \setminus H \). Then, by the definition of the support of a measure, we have \( S(K, \Omega) > 0 \), thus

\[
S_p(K, \Omega) = \int_{\Omega} h_K(v)^{1-p} dS(K, v) > 0,
\]

which contradicts the fact that \( S_p(K, \cdot) \) is concentrated in \( H \).

Next, define the closed convex cone

\[
C := \bigcap_{v \in T} H^{-}(v, 0).
\]
Notice that by Lemma 3.3 it holds

\[
K = \left( \bigcap_{v \in S} H^-(v, h_K(v)) \right) \cap \left( \bigcap_{v \in S^n-1 \cap H} H^-(v, h_K(v)) \right)
\]

\[
\supseteq \left( \bigcap_{v \in T} H^-(v, 0) \right) \cap \left( \bigcap_{v \in S^n-1 \cap H} H^-(v, h_K(v)) \right)
\]

\[
= C \cap \left( \bigcap_{v \in S^n-1 \cap H} H^-(v, h_K(v)) \right) =: C \cap P.
\]

Since the reverse inclusion trivially holds true, we conclude that

\[
K = C \cap P. \quad \text{(6)}
\]

We claim that

\[
H^\perp \cap C = \{0\}. \quad \text{(7)}
\]

To see this, observe that since \(C\) is a convex cone, for \(y \in H^\perp \setminus \{0\}\), only the following cases are possible.

(a) \(\{ry : r \geq 0\} \subseteq C\).

(b) \(\{ry : r \leq 0\} \subseteq C\).

(c) \(\mathbb{R}y \cap C = \{0\}\).

If any of the two first cases occurs, since \(\mathbb{R}y\) is (trivially) contained in the (unbounded) cylinder \(P\), then by (6), either \(\{ry : r \geq 0\} \subseteq K\) or \(\{ry : r \leq 0\} \subseteq K\). This contradicts the fact that \(K\) is bounded. Thus, we are left with case (c) and (7) is proved.

Eqs (7) and the Hahn-Banach Theorem show easily that there exists a supporting hyperplane \(E\) (necessarily a subspace of \(\mathbb{R}^n\)) of \(C\), that contains \(H^\perp\). Notice that if \(E = u^\perp\) for some \(u \in \mathbb{R}^{n-1}\), then \(u\) must be contained in \(H\). Thus, we have found a unit vector \(u \in H\) that supports \(C\).

Since \(\langle u, x \rangle \leq 0\), for all \(x \in C\), it follows that \(\langle u, x \rangle \leq 0\), for all \(x \in C|H\). Furthermore, (6) shows that \(C|H \supseteq K|H\) and, therefore, \(C|H\) is a closed convex cone of dimension \(k = \dim H\). Thus, by Proposition 3.2, there exists a unit vector \(u_0 \in H\), such that \(u_0\) supports \(C|H\) and \(-u_0 \in C|H\). However, \(o \in K|H \subseteq C|H\), which immediately shows that \(h_{K|H}(u_0) = 0\). Thus, \(u_0\) satisfies assertion (i). Furthermore, observe that there exists \(y \in H^\perp\), such that if we set \(w := -u_0 - y\), then \(w \in C\). By the definition of \(C\), this is equivalent to

\[
\langle w, v \rangle \leq 0, \quad \forall v \in T \quad \text{(8)}
\]

and, since \(S \subseteq T\), \(u_0\) and \(y\) satisfy (iii). The fact that \(u_0\) satisfies (ii) follows trivially from (8). Our proof is complete. \(\blacksquare\)
Proof of Theorem 1.3. Assume that there exists such $K$. Let $S$, $T$, $u_0$, $y$ be as in the statement of Lemma 3.4 and set $z := u_0 + y$. Then, $\mathcal{H}^{k-1}(T \cap H) = 0$, otherwise the support of $\mu_H$ would be strictly contained in $\mathbb{S}^{n-1} \cap H$. By the assumption of Theorem 1.3, for any continuous function $\varphi : (\mathbb{S}^{n-1} \cap H) \setminus T \to \mathbb{R}$, it holds

$$
\int_{(\mathbb{S}^{n-1} \cap H) \setminus T} \varphi(v) dS(K, v) = \int_{\mathbb{S}^{n-1} \cap H} \varphi(v) h_K(v)^{(1-p)} d\mathcal{H}^{k-1}(v).
$$

In particular, since $S(K, \cdot)$ is finite, (9) shows that the function $h_K^{(1-p)}$ is integrable on $\mathbb{S}^{n-1} \cap H$ and, therefore, the function $v \mapsto \langle u_0, v \rangle h_K(v)^{(1-p)}$ is also integrable on $\mathbb{S}^{n-1} \cap H$.

Using (9), Lemma 3.1, Lemma 3.4 (i) and Lemma 3.4 (ii), we deduce

$$
\int_{\mathbb{S}^{n-1} \cap H} \langle u_0, v \rangle dS(K, v) = \int_T \langle u_0, v \rangle dS(K, v) + \int_{\mathbb{S}^{n-1} \cap H} \langle u_0, v \rangle dS(K, v)
\geq \int_{(\mathbb{S}^{n-1} \cap H) \setminus T} \langle u_0, v \rangle dS(K, v)
= \int_{\mathbb{S}^{n-1} \cap H} \langle u_0, v \rangle h_K(v)^{(1-p)} d\mathcal{H}^{k-1}(v)
= \int_{\mathbb{S}^{n-1} \cap H} \langle u_0, v \rangle h_{K|H}(v)^{(1-p)} d\mathcal{H}^{k-1}(v) > 0. \quad (10)
$$

Finally, Lemma 3.4 (iii) and (10) give

$$
\int_{\mathbb{S}^{n-1} \cap H} \langle z, v \rangle dS(K, v) = \int_S \langle z, v \rangle dS(K, v) + \int_{\mathbb{S}^{n-1} \cap H} \langle z, v \rangle dS(K, v)
\geq \int_{\mathbb{S}^{n-1} \cap H} \langle z, v \rangle dS(K, v)
= \int_{\mathbb{S}^{n-1} \cap H} \langle u_0, v \rangle dS(K, v) > 0.
$$

This violates Minkowski’s Existence and Uniqueness Theorem (in particular the part that states that the barycentre of $S(K, \cdot)$ is at the origin) and, therefore, there cannot be any convex body $K$ satisfying $S_p(K, \cdot) = \mu_H$. \hfill \Box

Remark 3.5. The strategy to prove Theorem 1.3 was to arrive at a contradiction as follows: If $K$ is a convex body such that $S_p(K, \cdot) = \mu_H$, then the barycentre of $S(K, \cdot)$ is not at the origin. The same reasoning can be applied to the equation

$$
h_K^{1-p} dS(K, \cdot) = dS_p(K, \cdot) = d\mathcal{H}^{n-1}(\cdot), \quad p \neq 1, \quad (11)
$$

to show that if a convex body $K$ that contains $0$ satisfies (11), then $K$ contains $0$ in its interior. This is, of course, an immediate consequence of Lemma 3.1 if $p < 1$ and of the analogue of Lemma 3.4 in the case $q > 0$, if $p > 1$. We mention that in \cite{C1}, the authors proved that a strictly convex body with $C^\infty$-smooth boundary (due to a result of Caffarelli \cite{C2}, this turns out to be equivalent to the fact that $K$ contains the origin in its interior) satisfying (11), for some $p > -n - 1$, must
be a ball. Clearly, the previous discussion shows that the regularity assumption in their result can be omitted. An extension of the result in [5] appears in [16], where \( h_{K}^{-p} \) in (11) is replaced with 

\[
1/G(h_K) \quad \text{and} \quad G : (0, \infty) \to (0, \infty)
\]

is an appropriate function. A similar argument as above shows that the assumption “\( K \) contains \( o \) in its interior” in the main result of [16] can be also omitted (after some obvious modifications), if \( G \) is assumed to be strictly monotone.

**Remark 3.6.** One can easily prove (without using Proposition 3.2; see the proof of Lemma 3.4) that if \( S_p(K,\cdot) \) is concentrated in a \( k \)-dimensional subspace \( H \) of \( \mathbb{R}^n \) (where \( p < 1 \) and \( 1 \leq k \leq n - 1 \)), then \( o \in \partial(K|H) \). Because of this and since the support function of any \( k \)-dimensional convex body that contains the origin in its boundary, raised to the power \(-k\), is never integrable on \( S^{k-1} \), the proof of Theorem 1.3 becomes much easier (and the result is much more expected) if we restrict ourselves to the case \( p \leq 1 - k \).

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