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TRANSITIVITY IN THE THEORY OF THE LORENTZ GROUP
AND THE STOKES – MUeller FORMALISM IN OPTICS

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Summary

Group-theoretical analysis of arbitrary polarization devices is performed, based on the theory of the Lorentz group. In effective "non-relativistic" Mueller case, described by 3-dimensional orthogonal matrices, results of the one polarization measurement \( \mathbf{S} \xrightarrow{L} \mathbf{S}' \) determine group theoretical parameters within the accuracy of an arbitrary numerical variable. There are derived formulas, defining Muller parameter of the non-relativistic Mueller device uniquely and in explicit form by the results of two independent polarization measurements.

Analysis is extended to Lorentzian optical devices, described by 4-dimensional Mueller matrices. In this case, any single polarization measurement \((S_0, S) \xrightarrow{L_a} (S'_0, S')\) fixes parameters of the corresponding Mueller matrix up to 3 arbitrary variables. Formulas, defining Muller parameter of any relativistic Mueller device uniquely can be found from results of four independent polarization measurements.

Analytical expressions for parameters of any Mueller device can be given the most simple form when using the results of 6 independent measurements, the corresponding formulas are written down in explicit form.

1. The transitivity problem in the theory of the Lorentz group

It is known that in describing (fully or partly) polarized light noticeable role may given to the group of 3 + 1-pseudoorthogonal transformations consisting of a group \( SO(3,1) \) isomorphic to the Lorentz group. Therefore, techniques developed in the frames of the Lorentz group, in particular within relativistic kinematics, may play heuristic role in exploring optical problems (see big list of references in the end; a previous consideration of one of the authors is given in [101].

In the paper, when working with the Lorentz group we use technique developed in [102] and [103] and partly updated in [104]. This approach had been started many years ago by Einstein and Mayer in [105].

Let us recall the known transitivity problem in relativistic kinematics: in Stokes – Mueller approach it reads

\[
L_b^a(k, \bar{k}^*) \ S_a = +S'_b. \tag{1}
\]

From the very beginning, one peculiarity shout be noted: due to existence of the concept of little Lorentz group initial and final Stokes 4-vectors \( S \) and \( S' \), one can write down the transitivity condition in the form \( L (L_{\text{little}} S) = L'_{\text{little}} S' \), so that

\[
[ (L'_{\text{little}})^{-1} L \ L_{\text{little}} ] \ S = S'. \tag{2}
\]
This means that the transitive matrix $L$ cannot be defined uniquely in terms of $S$ and $S'$.

Let us use the factorized representation for Lorentzian matrices (we adhere notation given in [101,104]), eq. (1) gives

$$A^* S = A^{-1} S', \quad \text{and} \quad A S = (A^*)^{-1} S',$$

or in more detailed form (conjugate equation is written down too)

\[
\begin{pmatrix}
  k_0^* & -k_1^* & -k_2^* & -k_3^* \\
  -k_1^* & k_0^* & ik_3^* & -ik_2^* \\
  -k_2^* & -ik_3^* & k_0^* & ik_1^* \\
  -k_3^* & ik_2^* & -ik_1^* & k_0^*
\end{pmatrix}
\begin{pmatrix}
  S_0 \\
  S_1 \\
  S_2 \\
  S_3
\end{pmatrix}
= 
\begin{pmatrix}
  k_0 & k_1 & k_2 & k_3 \\
  k_1 & k_0 & ik_3 & -ik_2 \\
  k_2 & -ik_3 & k_0 & ik_1 \\
  k_3 & ik_2 & -ik_1 & k_0
\end{pmatrix}
\begin{pmatrix}
  S'_0 \\
  S'_1 \\
  S'_2 \\
  S'_3
\end{pmatrix},
\]

Below, the notation will be used

$$k_0 = n_0 + im_0, \quad k_j = -in_j + m_j, \quad k_0 - k^2 = 1.$$   

Summing and subtracting eqs we get

\[
\begin{pmatrix}
  n_0 & -m_1 & -m_2 & -m_3 \\
  -m_1 & n_0 & -n_3 & n_2 \\
  -m_2 & n_3 & n_0 & -n_1 \\
  -m_3 & -n_2 & n_1 & n_0
\end{pmatrix}
\begin{pmatrix}
  S_0 \\
  S_1 \\
  S_2 \\
  S_3
\end{pmatrix}
= 
\begin{pmatrix}
  n_0 & m_1 & m_2 & m_3 \\
  m_1 & n_0 & n_3 & -n_2 \\
  m_2 & -n_3 & n_0 & n_1 \\
  m_3 & n_2 & -n_1 & n_0
\end{pmatrix}
\begin{pmatrix}
  S'_0 \\
  S'_1 \\
  S'_2 \\
  S'_3
\end{pmatrix},
\]

So, we arrive at two homogeneous linear systems under 8 varianes

\[
\begin{align*}
  & n_0 \left(S_0 - S'_0\right) - m_1 \left(S_1 + S'_1\right) - m_2 \left(S_2 + S'_2\right) - m_3 \left(S_3 + S'_3\right) = 0, \\
  & -m_1 \left(S_0 + S'_0\right) + n_0 \left(S_1 - S'_1\right) + n_2 \left(S_3 + S'_3\right) - n_3 \left(S_2 + S'_2\right) = 0, \\
  & -m_2 \left(S_0 + S'_0\right) + n_0 \left(S_2 - S'_2\right) + n_3 \left(S_1 + S'_1\right) - n_1 \left(S_3 + S'_3\right) = 0, \\
  & -m_3 \left(S_0 + S'_0\right) + n_0 \left(S_3 - S'_3\right) + n_1 \left(S_2 + S'_2\right) - n_2 \left(S_1 + S'_1\right) = 0, \\
  & - m_0 \left(S_0 + S'_0\right) - n_1 \left(S_1 - S'_1\right) - n_2 \left(S_2 - S'_2\right) - n_3 \left(S_3 - S'_3\right) = 0, \\
  & -n_1 \left(S_0 - S'_0\right) - m_0 \left(S_1 + S'_1\right) - m_2 \left(S_3 - S'_3\right) + m_3 \left(S_2 - S'_2\right) = 0, \\
  & -n_2 \left(S_0 - S'_0\right) - m_0 \left(S_2 + S'_2\right) - m_3 \left(S_1 - S'_1\right) + m_1 \left(S_3 - S'_3\right) = 0, \\
  & -n_3 \left(S_0 - S'_0\right) - m_0 \left(S_3 + S'_3\right) - m_1 \left(S_2 - S'_2\right) + m_2 \left(S_1 - S'_1\right) = 0.
\end{align*}
\]
2. "Non-relativistic" 3-dimensional Mueller matrices

First, let us consider more simple (non-relativistic) case when \( S'_0 = S_0 = I = \text{inv.} \). Eqs. (5) takes the form (because we search solutions in 3-dimensional rotations, we require \( m_0 = 0, \ m_j = 0 \):

\[
\begin{align*}
&n_0 (S_1 - S'_1) + n_2 (S_3 + S'_3) - n_3 (S_2 + S'_2) = 0 , \\
&n_0 (S_2 - S'_2) + n_3 (S_1 + S'_1) - n_1 (S_3 + S'_3) = 0 , \\
&n_0 (S_3 - S'_3) + n_1 (S_2 + S'_2) - n_2 (S_1 + S'_1) = 0 , \\
&-n_1 (S_1 - S'_1) - n_2 (S_2 - S'_2) - n_3 (S_3 - S'_3) = 0 .
\end{align*}
\]

(6)

The fourth equation is not independent of three remaining – it follows from them. Therefore we have the system of 3 independent ones

\[
\begin{align*}
&n_2 (S_3 + S'_3) - n_3 (S_2 + S'_2) = -n_0 (S_1 - S'_1) , \\
&n_3 (S_1 + S'_1) - n_1 (S_3 + S'_3) = -n_0 (S_2 - S'_2) , \\
&n_1 (S_2 + S'_2) - n_2 (S_1 + S'_1) = -n_0 (S_3 - S'_3) .
\end{align*}
\]

(7)

They may be written in 3-vector form

\[
n \times (S + S') = -n_0 (S - S') .
\]

(8)

General solutions for \( n \) can be searched with the aid of substitution

\[
n = \alpha S + \rho S' + \beta S \times S' ,
\]

then eq. (8) leads to (below note \( S^2 = SS \))

\[
(\alpha - \rho) S \times S' + \beta \left[ S' S^2 + S' (SS') - S S^2 - S (SS') \right] = -n_0 S + n_0 S' ,
\]

from whence it follow \( \rho = \alpha , \ \alpha \) is arbitrary, and

\[
n_0 = \beta (S^2 + S S') , \quad n = \alpha (S + S') + \beta S \times S' .
\]

(9)

One must to take into account additional restriction for parameters of rotation matrices

\[
n_0^2 + n^2 = 1 ,
\]

(10)

which results in

\[
\beta^2 (S^2 + SS')^2 + [\alpha (S + S') + \beta S \times S']^2 = 1 ,
\]

or

\[
\beta^2 \left[ S^4 + 2S^2 (S S') + (S S')^2 \right] + \beta^2 [S^4 - (SS')^2] + 2\alpha^2 (S^2 + SS') = 1 ;
\]

and ultimately eq. (10) gives

\[
\beta^2 S^2 + \alpha^2 = \frac{1}{2(S^2 + SS')} .
\]
General solution of eq. (11) can be presented in terms of sin- and cos-functions of an angular variable

$$\alpha = \frac{\sin \Gamma}{\sqrt{2(S^2 + SS')}} , \quad \beta = \frac{\cos \Gamma}{S \sqrt{2(S^2 + SS')}} , \quad \Gamma \in [0, 2\pi] . \quad (12)$$

Thus, relations (9) read (here $\Gamma \in [0, 2\pi]$ stands for arbitrary parameter)

$$n_0^2 + n^2 = 1 , \quad n_0 = \frac{\cos \Gamma}{S \sqrt{2(S^2 + SS')}} (S^2 + SS') ,$$

$$n = \frac{\sin \Gamma}{\sqrt{2(S^2 + SS')}} (S + SS') + \frac{\cos \Gamma}{S \sqrt{2(S^2 + SS')}} S \times S' . \quad (13)$$

Note that when $S' = S$, relations (13) describe the case of little rotation group

$$n_0^2 + n^2 = 1 , \quad n_0 = \cos \Gamma , \quad n = \sin \Gamma \frac{S}{S} . \quad (14)$$

When $\Gamma = 0$, solution (13) becomes of the most simple form

$$n_0 = \frac{S^2 + SS'}{S \sqrt{2(S^2 + SS')}} , \quad n = \frac{S \times S'}{S \sqrt{2(S^2 + SS')}} . \quad (15)$$

Note, that we may transform all the relations to a Gibbs 3-vector parameter in the rotation group (the full treatment of the theory in this parametrization see in [102])

$$c = \frac{n}{n_0} , \quad (16)$$

then eqs. (13) give

$$c = \tan \Gamma \frac{S}{S^2 + SS'} (S + SS') + \frac{S \times S'}{S^2 + SS'} . \quad (17)$$

Note that in the non-relativistic case, for Stokes vectors one can use the following parametrization ($I$ is intensity of the light beam, $p$ is a polarization degree)

$$S_0 = I , \quad S = IpN , \quad I = \text{inv} , \quad N^2 = 1 ; \quad (18)$$

at this (13) and (15) change to

$$n_0^2 + n^2 = 1 , \quad n_0 = \cos \Gamma \frac{1 + N N'}{\sqrt{2(1 + NN')}} ,$$

$$n = \sin \Gamma \frac{N + N'}{\sqrt{2(1 + NN')}} + \cos \Gamma \frac{N \times N'}{\sqrt{2(1 + NN')}} , \quad (19)$$

and

$$c = \tan \Gamma \frac{N + N'}{1 + NN'} + \frac{N \times N'}{1 + NN'} . \quad (20)$$
3. On defining Mueller 3-matrices from the results of polarization measurements

Because a single polarization measurement relating \( S \xrightarrow{L} S' \) cannot fix Mueller 3-matrix uniquely, to obtain result values for parameters of the Mueller 3-matrix, one need to perform two independent measurements \( S_1 \xrightarrow{L} S'_1, \ S_2 \xrightarrow{L} S'_2 \). Mathematically, the problem of finding a definite Mueller 3-matrix can be formulated as a system to solve, describing two polarization measurement with one the same Mueller matrix.

First, let us consider this task with the aid of Gibbs 3-paramere

\[
c = \tan \Gamma \frac{N_1 + N'_1}{1 + N_1N'_1} + \frac{N_1 \times N'_1}{1 + N_1N'_1}, \quad c = \tan \Gamma \frac{N_2 + N'_2}{1 + N_2N'_2} + \frac{N_2 \times N'_2}{1 + N_2N'_2}; \tag{21}
\]

so we have a vector equation

\[
\tan \Gamma \left[ \frac{N_1 + N'_1}{1 + N_1N'_1} - \frac{N_2 + N'_2}{1 + N_2N'_2} \right] + \frac{N_1 \times N'_1}{1 + N_1N'_1} - \frac{N_2 \times N'_2}{1 + N_2N'_2} = 0. \tag{22}
\]

Multiplying it by \( N_1, N'_1, N_2, N'_2 \), we obtain four scalar equations

\[
\tan \Gamma \left[ 1 - \frac{N_1 (N_2 + N'_2)}{1 + N_2N'_2} \right] - \frac{N_1 (N_2 \times N'_2)}{1 + N_2N'_2} = 0, \tag{23}_1
\]

\[
\tan \Gamma \left[ 1 - \frac{N'_1 (N_2 + N'_2)}{1 + N_2N'_2} \right] - \frac{N'_1 (N_2 \times N'_2)}{1 + N_2N'_2} = 0, \tag{23}_2
\]

\[
\tan \Gamma \left[ \frac{N_2 (N_1 + N'_1)}{1 + N_1N'_1} - 1 \right] + \frac{N_2 (N_1 \times N'_1)}{1 + N_1N'_1} = 0, \tag{23}_3
\]

\[
\tan \Gamma \left[ \frac{N'_2 (N_1 + N'_1)}{1 + N_1N'_1} - 1 \right] + \frac{N'_2 (N_1 \times N'_1)}{1 + N_1N'_1} = 0. \tag{23}_4
\]

From whence it follow

\[
\tan \Gamma = \frac{N_1 (N_2 \times N'_2)}{(N_2 - N_1)(N_2 + N'_2)}, \quad \tan \Gamma = -\frac{N'_1 (N_2 \times N_2)}{(N'_2 - N'_1)(N'_2 + N_2)}, \tag{24}_1
\]

\[
\tan \Gamma = \frac{N_2 (N_1 \times N'_1)}{(N_1 - N_2)(N_1 + N'_1)}, \quad \tan \Gamma = -\frac{N'_2 (N_1 \times N_1)}{(N'_1 - N'_2)(N'_1 + N_1)}. \tag{24}_2
\]

Thus, we have a simple expression for \( \tan \Gamma \), together with four additional constraints, which determine the whole aggregate of all possible couples of Stokes 3-vectors related by one the same Mueller matrices.

Now let us detail considering of the task in the frames of unitary group \( SU(2) \) – evidently, two solutions cannot contradict each other. Here we have

\[
n_0 = \beta_1 S_1 (S_1 + S'_1), \quad n = \alpha_1 (S_1 + S'_1) + \beta_1 S_1 \times S'_1, \tag{25}_1
\]

\[
n_0 = \beta_2 S_2 (S_2 + S'_2), \quad n = \alpha_2 (S_2 + S'_2) + \beta_2 S_2 \times S'_2. \tag{25}_2
\]

what is equivalent to

\[
n_0 = \cos \Gamma \frac{1 + N_1N'_1}{\sqrt{2(1 + N_1N'_1)}}, \quad n = \sin \Gamma \frac{N_1 + N'_1}{\sqrt{2(1 + N_1N'_1)}} + \cos \Gamma \frac{N_1 \times N'_1}{\sqrt{2(1 + N_1N'_1)}}, \tag{26}_1
\]

\[
n_0 = \cos \Gamma \frac{1 + N_2N'_2}{\sqrt{2(1 + N_2N'_2)}}, \quad n = \sin \Gamma \frac{N_2 + N'_2}{\sqrt{2(1 + N_2N'_2)}} + \cos \Gamma \frac{N_2 \times N'_2}{\sqrt{2(1 + N_2N'_2)}}. \tag{26}_2
\]

5
From two different expressions for \( n_0 \), it follows
\[
N_1 N_1' = N_2 N_2'.
\] (27)
Taking this into account, from two different expressions for \( n \) we derive
\[
\sin \Gamma \left[ (N_1 + N_1') - (N_2 + N_2') \right] + \cos \Gamma \left[ (N_1 \times N_1') - (N_2 \times N_2') \right] = 0
\] (28)
It should be noted that due to (27), relation (22) becomes much more simpler
\[
tg \Gamma \left[ (N_1 + N_1') - (N_2 + N_2') \right] + N_1 \times N_1' - N_2 \times N_2 = 0 .
\] (29)
In fact, (28) and (29) coincide, difference consist in the following: (28) cannot distinguish between two solutions: \((+ \cos \Gamma, + \sin \Gamma)\) and \((- \cos \Gamma, - \sin \Gamma)\).

4. Relativistic Mueller matrices relating two Stokes 4-vectors
Let us turn back to general (relativistic) case of Mueller matrices (5):
\[
m_1 \left( S_1 + S_1' \right) + m_2 \left( S_2 + S_2' \right) + m_3 \left( S_3 + S_3' \right) = n_0 \left( S_0 - S_0' \right),
\]
\[
m_1 \left( S_0 + S_0' \right) - n_2 \left( S_3 + S_3' \right) + n_3 \left( S_2 + S_2' \right) = n_0 \left( S_1 - S_1' \right),
\]
\[
m_2 \left( S_0 + S_0' \right) - n_3 \left( S_1 + S_1' \right) + n_1 \left( S_3 + S_3' \right) = n_0 \left( S_2 - S_2' \right),
\]
\[
m_3 \left( S_0 + S_0' \right) - n_1 \left( S_2 + S_2' \right) + n_2 \left( S_1 + S_1' \right) = n_0 \left( S_3 - S_3' \right),
\]
\[
-n_1 \left( S_1 - S_1' \right) - n_2 \left( S_2 - S_2' \right) - n_3 \left( S_3 - S_3' \right) = m_0 \left( S_0 + S_0' \right),
\]
\[
-n_1 \left( S_0 - S_0' \right) - m_2 \left( S_3 - S_3' \right) - m_3 \left( S_2 - S_2' \right) = m_0 \left( S_1 + S_1' \right),
\]
\[
-n_2 \left( S_0 - S_0' \right) - m_3 \left( S_1 - S_1' \right) + m_1 \left( S_3 - S_3' \right) = m_0 \left( S_2 + S_2' \right),
\]
\[
-n_3 \left( S_0 - S_0' \right) - m_1 \left( S_2 - S_2' \right) + m_2 \left( S_1 - S_1' \right) = m_0 \left( S_3 + S_3' \right).
\] (30)
Because we search solutions among proper orthochronous Lorentzian transformations, unknown parameters must obey additional relations
\[
n_0^2 + n^2 - m_0^2 - m^2 = 1, \quad n_0 m_0 + n m = 0;
\] (31)
by this reason, the trivial solution \( n_a = 0, m_a = 0 \) for (30) is of no interest. Eqs. (30) can be rewritten in 3-vector form
\[
m \left( S + S' \right) = n_0 \left( S_0 - S_0' \right),
\]
\[
n \left( S - S' \right) = -m_0 \left( S_0 + S_0' \right),
\]
\[
m \left( S_0 + S_0' \right) + \left( S + S' \right) \times n = n_0 \left( S - S' \right),
\]
\[
n \left( S_0 - S_0' \right) - \left( S - S' \right) \times m = -m_0 \left( S + S' \right).
\] (32)
Note that the (non-relativity) requirement \( S_0 - S_0' = 0 \) immediately leads us to additional relations \( m = 0 \) and \( m_0 = 0 \), and we get eqs. (7)–(5).
Let us introduce notation
\[
S_0 + S_0' = A, \quad S_0 - S_0' = B, \quad S + S' = A, \quad S - S' = B, \quad N_+ = \nu, \quad M_- = \mu;
\] (33)
The complete system of equations to solve is

\[ n_0^2 + n^2 - m_0^2 - m^2 = 1 \,, \quad n_0 m_0 + nm = 0 \,; \quad (34) \]
\[ \mathbf{m} \mathbf{A} = n_0 \mathbf{B} \,, \quad \mathbf{n} \mathbf{B} = -m_0 \mathbf{A} \,; \quad (35) \]
\[ \mathbf{m} \mathbf{A} + \mathbf{A} \times \mathbf{n} = n_0 \mathbf{B} \,, \quad \mathbf{n} \mathbf{B} - \mathbf{B} \times \mathbf{m} = -m_0 \mathbf{A} \,; \quad (36) \]

In is convenient to use linear expansions for both 3-vectors

\[ \mathbf{n} = N_+ \mathbf{A} + N_- \mathbf{B} + N \mathbf{A} \times \mathbf{B} \,, \quad \mathbf{m} = M_+ \mathbf{A} + M_- \mathbf{B} + M \mathbf{A} \times \mathbf{B} \,; \quad (37) \]

From the first equation in (36) it follows

\[ A(M_+ \mathbf{A} + M_- \mathbf{B} + M \mathbf{A} \times \mathbf{B}) + A \times (N_+ \mathbf{B} + N \mathbf{A} \times \mathbf{B}) = n_0 \mathbf{B} \,; \]

which gives three equations

\[ AM_+ + AB N = 0 \,, \quad AM_- - A^2 N = n_0 \,, \quad AM + N_+ = 0 \,; \quad (38) \]

In the same manner, from the second equation in (36) we get

\[ B(N_+ \mathbf{A} + N_- \mathbf{B} + N \mathbf{A} \times \mathbf{B}) - B \times (M_+ \mathbf{A} + M \mathbf{A} \times \mathbf{B}) = -m_0 \mathbf{A} \,; \]

and further

\[ BN_- + ABM = 0 \,, \quad BN_+ - B^2 M = -m_0 \,, \quad BN + M_+ = 0 \,; \quad (39) \]

Thus, two vector equations (36) provide us with the system for six parameters

\[ AM_+ + AB N = 0 \,, \quad AM_- - A^2 N = n_0 \,, \quad AM + N_+ = 0 \,; \quad (36) \]

\[ BN_- + ABM = 0 \,, \quad BN_+ - B^2 M = -m_0 \,, \quad BN + M_+ = 0 \,; \quad (40) \]

After excluding the variables \( N_- \,, M_+ \):

\[ N_- = -AM \,, \quad M_+ = -BN \,; \quad (41) \]

eqs. (40) read

\[ -ABN + AB N = 0 \,, \quad AM_- - A^2 N = n_0 \,, \quad -ABM + ABM = 0 \,, \quad BN_+ - B^2 M = -m_0 \,; \quad (42) \]

Note that equations 1 and 3 are identities. In fact, eqs. (42) are equivalent to two equations only

\[ AM_- - A^2 N = n_0 \,, \quad BN_+ - B^2 M = -m_0 \,; \quad (43) \]

Substituting expressions

\[ \mathbf{n} = N_+ \mathbf{A} - M \mathbf{A} \mathbf{B} + N \mathbf{A} \times \mathbf{B} \,, \quad \mathbf{m} = M_- \mathbf{B} - N \mathbf{B} \mathbf{A} + M \mathbf{A} \times \mathbf{B} \,; \quad (44) \]
into (35), we arrive at
\[ M - N B A^2 = n_0 B \quad \Rightarrow \quad M - N A^2 = n_0 , \]
\[ N_+ A B - M A B^2 = -m_0 A \quad \Rightarrow \quad N_+ B - M B^2 = -m_0 ; \]
which coincide with (43). This means that eqs. (35) can be removed. The above substitutions for two vectors (44) are to be allowed in the conditions
\[ n^2_0 - m^2_0 = 1 + m^2 - n^2 , \quad n_0 m_0 = -nm = 0 . \]

Let us simplify notation
\[ M = x , \quad N = y , \quad N_+ = z , \quad M = w \]
In these variables, the main equations to solve read
\[ n_0 = A x - A^2 y , \quad n = z A - w AB + y A \times B ; \]
\[ m_0 = -B z + B^2 w , \quad m = x B - y B A + w A \times B ; \]
\[ n_0 m_0 = -nm , \quad n^2_0 - m^2_0 = 1 + m^2 - n^2 . \] (45)

First, let us detail \( n_0 m_0 = -nm \). Taking into account
\[ n_0 m_0 = -xz AB + w x A B^2 + yz BA^2 - wy A^2 B^2 , \]
\[ -nm = -(z A - w AB + y A \times B) (x B - y B A + w A \times B) = \]
\[ = -xz AB + yz BA^2 + w x A B^2 - yw ABAB - yw A^2 B^2 + yw (AB)^2 . \]
we arrive at
\[ 0 = xz (AB - AB) - yw ABAB + yw (AB)^2 . \] (46)
Because
\[ AB - AB = (S_0^2 - S^2) - (S'_0^2 - S'^2) = 0 , \] (47)
eq. (46) takes the form of an identity \( 0 = 0 \), subsequently, this equation can be excluded from (45). Remaining and independent relations are
\[ n^2_0 - m^2_0 = 1 + m^2 - n^2 , \]
\[ n_0 = A x - A^2 y , \quad n = z A - w AB + y A \times B , \]
\[ m_0 = -B z + B^2 w , \quad m = x B - y B A + w A \times B . \] (48)

Each of vector equation in (48) can be changed into three scalar ones; those are obtained through multiplying them by \( A, B, A \times B \):
\[ A n = z A^2 - w A^2 B , \]
\[ B n = z A B - w A B^2 , \]
\[ (A \times B) n = +y A^2 B^2 - y A^2 B^2 , \]
\[ A m = x A B - y B A^2 \]
\[ B m = x B^2 - y B^2 A , \]
\[ (A \times B) m = +w A^2 B^2 - w A^2 B^2 . \] (49)
These equations are easy to solve

\[
y = \frac{(A \times B)n}{A^2B^2 - A'^2B'^2}, \quad z = \frac{(Bn)AB - (An)B^2}{A^2B^2 - A'^2B'^2}, \quad w = \frac{1}{A} \left( \frac{(Bn)A^2 - (An)AB}{A^2B^2 - A'^2B'^2} \right),
\]

\[
w = \frac{(A \times B)m}{A^2B^2 - A'^2B'^2}, \quad x = \frac{-(Am)AB + (Bm)A^2}{A^2B^2 - A'^2B'^2}, \quad y = \frac{1}{B} \left( \frac{(Bm)AB - (Am)B^2}{A^2B^2 - A'^2B'^2} \right).
\] (50)

Taking (48), we may turn back to a starting complex parameter \(k_0\):

\[
k_0 = n_0 + im_0 = (xA - izB) - (yA^2 - iwB^2),
\]

\[
k = m - in = -(y B + i z) A + (x + iwA) B + (w - iy)A \times B.
\] (51)

Note that one can derive a more simple 3-vector, parameter for Lorentz group [...],

\[
q = \frac{k}{k_0} = \frac{- (y B + i z) A + (x + iwA) B + (w - iy)A \times B}{(xA - izB) - (yA^2 - iwB^2)}
\] (52)

It may be formally simplified

\[
q = \alpha A + \beta B + \gamma A \times B,
\]

\[
\alpha = \frac{-(y B + i z)}{(xA - izB) - (yA^2 - iwB^2)},
\]

\[
\beta = \frac{x + iwA}{(xA - izB) - (yA^2 - iwB^2)},
\]

\[
\gamma = \frac{w - iy}{(xA - izB) - (yA^2 - iwB^2)}.
\] (53)

The formulas allow transition to a more simple non-relativistic case \((x \equiv 0, w \equiv 0, B = 0)\)

\[
c = i q = i\alpha A + i\beta B + i\gamma A \times B,
\]

\[
i\alpha = -\frac{1}{A^2} \frac{z}{y}, \quad i\beta = 0, \quad i\gamma = -\frac{1}{A^2};
\] (54)

these relations describe 1-parametric set of 3-rotations. In relations [48], the non-relativistic case is reached as follow

\[
n_0^2 + n^2 = 1, \quad n_0 = yA^2, \quad n = zA + yA \times B.
\] (55)

let us obtain an explicit form of the relationship \(n_0^2 - m_0^2 = 1 + m^2 - n^2\) in (48). We have

\[
n_0^2 - m_0^2 = (A x - A^2 y)^2 - (B z + B^2 w)^2 = A^2 x^2 - B^2 z^2 - 2 A A^2 x y + 2 B B^2 z w + (A^2)^2 y^2 - (B^2)^2 w^2,
\]

and further

\[
m^2 = (xB - yB A + w A \times B)(xB - yB A + w A \times B) = x^2 B^2 - xy B(BA) - xy B(BA) + y^2 B^2 A^2 + w^2 A^2 B^2 - w^2(AB)^2,
\]

9
that is
\[ m^2 = x^2 B^2 - 2xy AB^2 + y^2 B^2 A^2 + w^2 A^2 B^2 - w^2 A^2 B^2. \]

In the same manner, we derive
\[ n^2 = (zA - wAB + yA \times B) (zA - wAB + yA \times B) = z^2 A^2 - 2zw BA^2 + w^2 A^2 B^2 + y^2 A^2 B^2 - y^2 A^2 B^2, \]
and further
\[ 1 + m^2 - n^2 = 1 + x^2 B^2 - 2xy AB^2 + y^2 B^2 A^2 + w^2 A^2 B^2 - w^2 A^2 B^2 - z^2 A^2 + 2zw BA^2 - w^2 A^2 B^2 - y^2 A^2 B^2 + y^2 A^2 B^2, \]
that is
\[ 1 + m^2 - n^2 = 1 + x^2 B^2 - z^2 A^2 - 2xy AB^2 + 2zw BA^2 + y^2 (B^2 - B^2)A^2 + w^2 A^2 B^2 - w^2 (A^2 - A^2)B^2 + A^2 B^2 \]
\[ + 1. \]

The quadratic equation for parameters of the Mueller matrix takes the form
\[ x^2 (A^2 - B^2) + 2xy A(B^2 - A^2) + y^2 [ (A^2 + B^2 - B^2)A^2 - A^2 B^2 ] = z^2 (B^2 - A^2) + 2zw B(A^2 - B^2) + w^2 [ (A^2 + B^2 - A^2)B^2 - A^2 B^2 ] + 1. \]

(56)

5. On defining 4-dimensional Mueller matrix from polarization measurements

As shown above, each polarization measurement
\[ S_a \xrightarrow{L} S'_a \quad \text{or} \quad (A_a, B_a) \xrightarrow{L} (A'_a, B'_a) \]
allows to obtain the quadratic constraint on Mueller’s characteristics of a polarization device
\[ x^2 (A^2 - B^2) + 2xy A(B^2 - A^2) + y^2 [ (A^2 + B^2 - B^2)A^2 - A^2 B^2 ] = z^2 (B^2 - A^2) + 2zw B(A^2 - B^2) + w^2 [ (A^2 + B^2 - A^2)B^2 - A^2 B^2 ] + 1; \]
\[ \text{the later has a 3-parametric set of solutions which describe all the possible Mueller matrices of} \]
\[ \text{the given optical device} \]
\[ n_0 = x A - y A^2, \quad n = z A - w AB + y A \times B, \]
\[ m_0 = -z B + w B^2, \quad m = x B - y B A + w A \times B. \]

(58)

It is evident, that to fix Mueller matrix uniquely, one should perform several polarization tests. Let start with four ones – the problem to solve is formulate as a system of 4 equations
\[ x^2 (A_1^2 - B_1^2) + 2xy A_1 (B_2^2 - A_2^2) + y^2 \left( (A_2^2 + B_2^2 - B_1^2)A_2^2 - A_1^2B_1^2 \right) = \\
= z^2 (B_1^2 - A_1^2) + 2zw B_1 (A_2^2 - B_2^2) + w^2 \left( (A_1^2 + B_1^2 - A_2^2)B_2^2 - A_2^2B_2^2 \right) + 1. \]

\[ x^2 (A_2^2 - B_2^2) + 2xy A_2 (B_3^2 - A_3^2) + y^2 \left( (A_3^2 + B_3^2 - B_2^2)A_2^2 - A_2^2B_2^2 \right) = \\
= z^2 (B_2^2 - A_2^2) + 2zw B_2 (A_3^2 - B_3^2) + w^2 \left( (A_2^2 + B_2^2 - A_3^2)B_3^2 - A_2^2B_2^2 \right) + 1. \]

\[ x^2 (A_3^2 - B_3^2) + 2xy A_3 (B_4^2 - A_4^2) + y^2 \left( (A_4^2 + B_4^2 - B_3^2)A_3^2 - A_3^2B_3^2 \right) = \\
= z^2 (B_3^2 - A_3^2) + 2zw B_3 (A_4^2 - B_4^2) + w^2 \left( (A_3^2 + B_3^2 - A_4^2)B_4^2 - A_3^2B_3^2 \right) + 1. \]

\[ x^2 (A_4^2 - B_4^2) + 2xy A_4 (B_5^2 - A_5^2) + y^2 \left( (A_5^2 + B_5^2 - B_4^2)A_4^2 - A_4^2B_4^2 \right) = \\
= z^2 (B_4^2 - A_4^2) + 2zw B_4 (A_5^2 - B_5^2) + w^2 \left( (A_4^2 + B_4^2 - A_5^2)B_5^2 - A_4^2B_4^2 \right) + 1. \]

It may be presented in a symbolical form as

\[ a_1 x^2 + 2b_1 xy + c_1 y^2 = \alpha_1 z^2 + 2\beta_1 zw + \sigma_1 w^2 + 1, \]

\[ a_2 x^2 + 2b_2 xy + c_2 y^2 = \alpha_2 z^2 + 2\beta_2 zw + \sigma_2 w^2 + 1, \]

\[ a_3 x^2 + 2b_3 xy + c_3 y^2 = \alpha_3 z^2 + 2\beta_3 zw + \sigma_3 w^2 + 1, \]

\[ a_4 x^2 + 2b_4 xy + c_4 y^2 = \alpha_4 z^2 + 2\beta_4 zw + \sigma_4 w^2 + 1. \]

In general, this mathematical task should have a definite solution, though rather cumbersome one. Indeed, we could successively exclude the variables as follows

\[ (1) \implies x = x(y, z, w), \]

\[ (2) \implies y = y(z, w), \quad x = x(y(z, w), z, w) = \bar{x}(z, w), \]

\[ (3) \implies z = z(w), \quad (4) \implies w = w(\ldots), \quad z = z(w(\ldots)). \]

However, there exist another and more beautiful way to solve the problem. Indeed, let us consider 6 independent polarization measurements – they provide us with 6 linear equations under 6 variables

\[ x^2, \quad y^2, \quad 2xy, \quad z^2, \quad w^2, \quad 2zw; \]

\[ a_1 x^2 + 2b_1 xy + c_1 y^2 - \alpha_1 z^2 - 2\beta_1 zw - \sigma_1 w^2 = +1, \]

\[ a_2 x^2 + 2b_2 xy + c_2 y^2 - \alpha_2 z^2 - 2\beta_2 zw - \sigma_2 w^2 = +1, \]

\[ a_3 x^2 + 2b_3 xy + c_3 y^2 - \alpha_3 z^2 - 2\beta_3 zw - \sigma_3 w^2 = +1, \]

\[ a_4 x^2 + 2b_4 xy + c_4 y^2 - \alpha_4 z^2 - 2\beta_4 zw - \sigma_4 w^2 = +1, \]

\[ a_5 x^2 + 2b_5 xy + c_5 y^2 - \alpha_5 z^2 - 2\beta_5 zw - \sigma_5 w^2 = +1, \]

\[ a_6 x^2 + 2b_6 xy + c_6 y^2 - \alpha_6 z^2 - 2\beta_6 zw - \sigma_6 w^2 = +1. \]

By physical reasons, we can presuppose existence of a unique solution of the task. This is given by Kramer’s rule

\[ x^2 = \frac{\Delta x^2}{\Delta}, \quad y^2 = \frac{\Delta y^2}{\Delta}, \quad 2xy = \frac{\Delta 2xy}{\Delta}, \]

\[ z^2 = \frac{\Delta z^2}{\Delta}, \quad w^2 = \frac{\Delta w^2}{\Delta}, \quad 2zw = \frac{\Delta 2zw}{\Delta}, \]

\[ (62) \]
from whence it follows (evidently, arising subtleties with ± should be examined additionally)

\[
x + y = \sqrt{\frac{\Delta x^2 + \Delta y^2 + \Delta 2xy}{\Delta}}, \quad x - y = \sqrt{\frac{\Delta x^2 + \Delta y^2 - \Delta 2xy}{\Delta}},
\]

\[
z + w = \sqrt{\frac{\Delta z^2 + \Delta w^2 + \Delta 2zw}{\Delta}}, \quad z - w = \sqrt{\frac{\Delta z^2 + \Delta w^2 - \Delta 2zw}{\Delta}},
\]

Recall (see (51) that Muller’s matrices are defined by \(k\)-parameter

\[
k_0 = (xA - izB) - (yA^2 - iwB^2),
\]

\[
k = -(yB + izA) + (x + iwA)B + (w-iy)A \times B;
\]

evidently, any orthogonal Lorentz matrix cannot distinguish between \((+k_0, +k)\) and \((-k_0, -k)\).

We may employ the same method in non-relativistic case as well. See (55); with the notation \(z = \nu, y = N\) we have

\[
n_0^2 + n^2 = 1, \quad n_0 = yA^2, \quad n = zA + yA \times B.
\]  

(64)

Note that because

\[
A^2 = (S + S')^2 = S^2 + S'^2 + 2SS = 2(S^2 + SS), \quad A \times B = 2S \times S',
\]
eqs. (64) are equivalent to

\[
n_0 = 2y(S^2 + SS), \quad n = zA + 2yS \times S'.
\]  

(65)

and thereby coincide with (9)

\[
n_0 = \beta(S^2 + SS'), \quad n = \alpha(S + S') + \beta S \times S'.
\]  

(66)

In this notation two independent polarization test provide us with a linear system

\[
y^2[A_1^2(A_1^2 + B_1^2) - (A_1B_1)^2] + z^2A_1^2 = 1, \\
y^2[A_2^2(A_2^2 + B_2^2) - (A_2B_2)^2] + z^2A_2^2 = 1,
\]  

(67)

its solution is

\[
y^2 = \frac{(A_1B_1)^2 - (A_1B_2)^2}{[A_1^2(A_1^2 + B_1^2) - (A_1B_1)^2][A_2^2(A_2^2 + B_2^2) - (A_2B_2)^2][A_1^2]}, \\
z^2 = \frac{[A_1^2(A_1^2 + B_1^2) - (A_1B_1)^2][A_2^2(A_2^2 + B_2^2) - (A_2B_2)^2] - [A_2^2(A_2^2 + B_2^2) - (A_2B_2)^2][A_1^2]}{[A_1^2(A_1^2 + B_1^2) - (A_1B_1)^2][A_2^2(A_2^2 + B_2^2) - (A_2B_2)^2][A_1^2]}.
\]  

(68)

6. On diagonalizing the transitivity equation

The transitivity equation \(LS = S'\) led us to a 3-surface in 4-parametric space

\[
x^2 (A^2 - B^2) + 2xy A(B^2 - A^2) + y^2 [(A^2 + B^2 - B^2)A^2 - A^2B^2] - \\
z^2 (B^2 - A^2) - 2zw B(A^2 - B^2) - w^2 [(A^2 + B^2 - A^2)B^2 - A^2B^2] = 1,
\]  

(69)
or in symbolical form
\[ ax^2 + 2bxy + cy^2 - \alpha z^2 - 2\beta zw - \sigma w^2 = +1. \] (70)

Let us examine the possibility to transform an elementary quadratic form to a diagonal form by means of 3-rotation in 2-plane
\[ ax^2 + 2bxy + cy^2 = FX^2 + GY^2, \]
\[ x = \cos \phi X + \sin \phi Y, \quad y = -\sin \phi X + \cos \phi Y. \] (71)

Eqs. (71) yield
\[ a(\cos \phi X + \sin \phi Y)^2 + 2b(\cos \phi X + \sin \phi Y)(-\sin \phi X + \cos \phi Y) + 
+ c(-\sin \phi X + \cos \phi Y)^2 = FX^2 + GY^2 \implies 
\]
\[ a(2XY \sin \phi \cos \phi + X^2 \cos^2 \phi + Y^2 \sin^2 \phi) + 
+ 2b[(Y^2 - X^2) \sin \phi \cos \phi + XY(\cos^2 \phi - \sin^2 \phi)] + 
+ c(-2XY \sin \phi \cos \phi + X^2 \sin^2 \phi + Y^2 \cos^2 \phi) = FX^2 + GY^2. \] (72)

So we have three equations
\[ X^2 : \quad a \cos^2 \phi - 2b \sin \phi \cos \phi + c \sin^2 \phi = F, \]
\[ Y^2 : \quad a \sin^2 \phi + 2b \sin \phi \cos \phi + c \cos^2 \phi = G, \]
\[ 2XY : \quad a \sin \phi \cos \phi + b(\cos^2 \phi - \sin^2 \phi) - c \sin \phi \cos \phi = 0. \]

With the help of the variable \(2\phi\), these are written as
\[ a \frac{\cos 2\phi + 1}{2} - b \sin 2\phi + c \frac{1 - \cos 2\phi}{2} = F, \]
\[ a \frac{1 - \cos 2\phi}{2} + b \sin 2\phi + c \frac{\cos 2\phi + 1}{2} = G, \]
\[ \frac{a - c}{2} \sin 2\phi + b \cos 2\phi = 0. \] (73)

This results in
\[ \sin 2\phi = \frac{2b}{\sqrt{(c-a)^2 + 4b^2}}, \quad \cos 2\phi = \frac{c-a}{\sqrt{(c-a)^2 + 4b^2}}; \] (74)

and
\[ F = \frac{a + c}{2} + \frac{a - c}{2} \cos 2\phi - b \sin 2\phi = \frac{a + c}{2} - \frac{\sqrt{(a-c)^2 + 4b^2}}{2}, \]
\[ G = \frac{a + c}{2} - \frac{a - c}{2} \cos 2\phi + b \sin 2\phi = \frac{a + c}{2} + \frac{\sqrt{(a-c)^2 + 4b^2}}{2}. \] (75)

In the same manner, the second quadratic form is considered
\[ -\alpha z^2 - 2\beta zw - \sigma w^2 = \Delta Z^2 + \Gamma W^2 \]
\[ z = \cos \rho Z + \sin \rho W, \quad w = -\sin \rho Z + \cos \rho W. \] (76)
For $2\rho$ we get

$$\sin 2\rho = \frac{2\beta}{\sqrt{(\sigma - \alpha)^2 + 4\beta^2}}, \quad \cos 2\rho = \frac{\sigma - \alpha}{\sqrt{(\sigma - \alpha)^2 + 4\beta^2}}; \quad (77)$$

$$\Delta = \frac{\alpha + \sigma}{2} - \frac{\sqrt{(\alpha - \sigma)^2 + 4\beta^2}}{2};$$

$$\Gamma = \frac{\alpha + \sigma}{2} + \frac{\sqrt{(\alpha - \sigma)^2 + 4\beta^2}}{2}. \quad (78)$$

For instance, conditions at which $F$ and $G$ are positive, and $\Delta, \Gamma$ are negative, are formulated in the form

$$(F, G, \Delta, \Gamma) \sim (+, +, -, -),$$

$$a > 0, \ c > 0, \quad a + c > +\sqrt{(a - c)^2 + 4b^2} > 0 \quad \Rightarrow \quad ac > b^2. $$

$$\alpha < 0, \ \sigma < 0, \quad \alpha + \sigma < -\sqrt{(\alpha - \sigma)^2 + 4\beta^2} \quad \Rightarrow \quad \alpha\sigma > \beta^2. \quad (79)$$

When specifying expressions for $a, b, c, \alpha, \beta, \sigma$ we should distinguish between a partly and completely polarized light. In the case of a partly polarized and completely polarized light we have respectively

$$S_0^2 - S^2 = S_0'^2 - S'^2 = 0, \quad S_0 = + |S|,$$

$$S_0^2 - S^2 = S_0'^2 - S'^2 > 0, \quad S_0 > |S|.$$

For the main invariant let us use the notation $S_0^2 - S^2 = S_0'^2 - S'^2 = \Sigma^2$.

Expression for $a, b, \alpha, \beta$ are given by

$$a = (S_0 + S_0')^2 - (S - S')^2 = 2\Sigma^2 + 2(S_0S_0' + SS') ,$$

$$\frac{b}{A} = (S_0 - S_0')^2 - (S + S')^2 = 2\Sigma^2 - 2(S_0S_0' + SS') ,$$

$$\alpha = (S_0 - S_0')^2 - (S + S')^2 = 2\Sigma^2 - 2(S_0S_0' + SS') ,$$

$$\frac{\beta}{B} = (S_0 + S_0')^2 - (S - S')^2 = 2\Sigma^2 + 2(S_0S_0' + SS') . \quad (80)$$

they become simpler for a completely polarized light

$$a_{polar} = + 2(S_0S_0' + SS') > 0, \quad \frac{b_{polar}}{A} = - 2(S_0S_0' + SS') < 0,$$

$$\alpha_{polar} = - 2(S_0S_0' + SS') < 0, \quad \frac{\beta_{polar}}{B} = + 2(S_0S_0' + SS') > 0. \quad (81)$$

Let us specify $c = (A^2 + B^2 - B^2)A^2 - A^2B^2$; accounting for

$$A^2 + B^2 - B^2 = (S + S')^2 + (S - S')^2 - (S_0 - S_0')^2 = 4\Sigma^2 + (S_0 + S_0')^2,$$

$$A^2 = (S + S')^2, \quad A^2B^2 = (S_0 + S_0')^2(S_0 - S_0')^2.$$
we get
\[
c = [-4\Sigma^2 + (S_0 + S'_0)^2](S + S')^2 - (S_0 + S'_0)^2(S_0 - S'_0)^2,
\]
\[
e_{\text{polar}} = 2(S_0 + S'_0)^2 \left(S_0S_0 + SS'\right).
\] (82)

In the same matter, for \(\sigma = \sigma = (B^2 + A^2 - A^2)B^2 - B^2A^2\) with relations
\[
B^2 + A^2 - A^2 = (S - S')^2 + (S + S')^2 - (S_0 + S'_0)^2 = -4\Sigma^2 + (S_0 - S'_0)^2,
\]
\[
B^2 = (S - S')^2, \quad B^2A^2 = (S_0 - S'_0)^2(S_0 + S'_0)^2
\]
we obtain
\[
\sigma = [-4\Sigma^2 + (S_0 - S'_0)^2](S - S')^2 - (S_0 - S'_0)^2(S_0 + S'_0)^2,
\]
\[
\sigma_{\text{polar}} = -2(S_0 - S'_0)^2 \left(S_0S_0 + SS'\right)
\] (83)

7. On the Lorentz little group for a partly polarized light

In the context of polarization optics, some interest may have the known problem of the little Lorentz group. What is the majority of Mueller matrices leaving invariant a given Stokes 4-vector. The problem is reduced to
\[
L_b^a (k, k^*) S_a = +S_b, \quad S^aS_a = \text{inv} > 0;
\] (84)

with the use of a factorized form \(L = A A^* = A^* A\), the previous equations are
\[
A = \begin{vmatrix} k_0 & -k_1 & -k_2 & -k_3 \\ -k_1 & k_0 & -i k_3 & i k_2 \\ -k_2 & i k_3 & k_0 & -i k_1 \\ -k_3 & -i k_2 & i k_1 & k_0 \end{vmatrix}, \quad \left(A^*\right)^{-1} = \begin{vmatrix} k_0^* & k_1^* & k_2^* & k_3^* \\ k_1^* & k_0^* & -i k_3^* & i k_2^* \\ k_2^* & i k_3^* & k_0^* & -i k_1^* \\ k_3^* & -i k_2^* & i k_1^* & k_0^* \end{vmatrix}
\] (85)

So we arrive at
\[
\begin{vmatrix} (k_0 - k_0^*) & -(k_1 + k_1^*) & -(k_2 + k_2^*) & -(k_3 + k_3^*) \\ -(k_1 + k_1^*) & (k_0 - k_0^*) & -i(k_3 - k_3^*) & i(k_2 - k_2^*) \\ -(k_2 + k_2^*) & i(k_3 - k_3^*) & (k_0 - k_0^*) & -i(k_1 - k_1^*) \\ -(k_3 + k_3^*) & -i(k_2 - k_3^*) & i(k_1 - k_1^*) & (k_0 - k_0^*) \end{vmatrix} \begin{vmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{vmatrix} = 0
\] (86)

which with notation \(k_0 = n_0 + im_0\), \(k_j = -in_j + m_j\) reads
\[
\begin{vmatrix} im_0 & -m_1 & -m_2 & -m_3 \\ -m_1 & im_0 & -n_3 & n_2 \\ -m_2 & n_3 & im_0 & -n_1 \\ -m_3 & -n_2 & n_1 & im_0 \end{vmatrix} \begin{vmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{vmatrix} = 0
\] (87)

Note that imposing restrictions \(m_0 = 0, m_j = 0\), we obtain a more simple equation
\[
\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -n_3 & n_2 \\ 0 & n_3 & 0 & -n_1 \\ 0 & -n_2 & n_1 & 0 \end{vmatrix} \begin{vmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{vmatrix} = 0 \quad \Rightarrow \quad n = \frac{S}{S}
\] (88)
which describes a 1-parametric group of 3-rotations \( O(\phi, n) \) about the axis \( S = Sn \). In general case, eq. (87) can be presented in the vector form

\[
im_0 S_0 - mS = 0, \quad -mS_0 + im_0 S + n \times S = 0.
\]  

To have solutions in real variables, we must require \( m_0 = 0 \). Therefore, an expression for \( m \) is

\[
m = \frac{n \times S}{S_0} = n \times p .
\]  

Thus, solution for the problem of little Lorentz group is (first it was obtained by Wigner [...])

\[
L_b^a(k, \bar{k}^*) S_a = +S_b, \quad S^a S_a = \text{inv} > 0 ;
\]

\[
k_0 = n_0 + i0, \quad k = -i n + n \times p .
\]  

Explicitly, additional condition for parameters looks

\[
k_0^2 - k^2 = 1 \quad \Rightarrow \quad n_0^2 + n^2(1 - p^2) + (np)^2 = 1 .
\]  

This relationship determines a 3-parametric majority of Mueller matrices leaving invariant the polarization vector \( S_a = (S_0, S_0p_i) \) of the partly polarized light. As known, this set of transformations consists of a group isomorphic to \( SU(2) \).

8. On the Lorentz little group for a completely polarized light

Analogous problem for a completely polarized light looks much the same

\[
L_b^a(k, \bar{k}^*) S_a = +S_b, \quad S^a S_a = 0 ;
\]

we again have equations

\[
im_0 S_0 - mS = 0, \quad -mS_0 + im_0 S + n \times S = 0 ,
\]

in which restriction \( m_0 = 0 \) must hold. Solution looks as follows

\[
L_b^a(k, \bar{k}^*) S_a = +S_b, \quad S^a S_a = 0 ;
\]

\[
k_0 = n_0 + i0, \quad k = -i n + n \times p , \quad p^2 = 1 .
\]  

The difference arises due to the relation \( p^2 = 1 \),

\[
k_0^2 - k^2 = 1 \quad \Rightarrow \quad n_0^2 + (np)^2 = 1 .
\]  

This relationship determines a 3-parametric majority of Mueller matrices leaving invariant a given isotropic Stokes 4-vector \( S_a = (S_0, S_0p_i) \), \( p^2 = 1 \).

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