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Abstract—In this paper, we study the influence of ill-conditioned regression matrix on two hyper-parameter estimation methods for the kernel-based regularization method: the empirical Bayes (EB) and the Stein’s unbiased risk estimator (SURE). First, we consider the convergence rate of the cost functions of EB and SURE, and we find that they have the same convergence rate but the influence of the ill-conditioned regression matrix on the scale factor are different: the scale factor for SURE contains one more factor \(\text{cond}(\Phi^T\Phi)\) than that of EB, where \(\Phi\) is the regression matrix and \(\text{cond}(\cdot)\) denotes the condition number of a matrix. This finding indicates that when \(\Phi\) is ill-conditioned, i.e., \(\text{cond}(\Phi^T\Phi)\) is large, the cost function of SURE converges slower than that of EB. Then we consider the convergence rate of the optimal hyper-parameters of EB and SURE, and we find that they are both asymptotically normally distributed and have the same convergence rate, but the influence of the ill-conditioned regression matrix on the scale factor are different. In particular, for the ridge regression case, we show that the optimal hyper-parameter of SURE converges slower than that of EB with a factor of \(1/n^2\), as \(\text{cond}(\Phi^T\Phi)\) goes to \(\infty\), where \(n\) is the FIR model order.

I. INTRODUCTION

In the past decade, kernel-based regularization method (KRM) has gained increasing attention and become a hot research topic in the system identification community. It has been shown to be a complement [1] to the classical system identification paradigm based on top of the maximum likelihood/prediction error methods (ML/PEM) [2].

There are two fundamental issues for KRM. The first one is the parameterization of kernel structure with hyper-parameters based on the prior knowledge of the system to be identified, for which several kernels have been invented, e.g. [3] and [4]. The other one is the tuning of hyper-parameters based on the given data to achieve balance in the bias-variance trade-off. Common methods for the hyper-parameter estimation include the cross-validation (CV), empirical Bayes (EB), \(C_p\) statistics, Stein’s unbiased estimator (SURE) and so on. Among them, SURE has two variants: \(\text{SURE}_g\) corresponds to the mean square error (MSE) related with the impulse response reconstruction (MSE\(_g\)), while \(\text{SURE}_y\) corresponds to the MSE with respect to output prediction (MSE\(_y\)) [5], [6].

An interesting phenomena we observed from Monte Carlo simulations in [1], [5], [6] is that when the regression matrix is ill-conditioned, \(\text{SURE}_g\) and \(\text{SURE}_y\) often converges slower than EB, which motivates us to consider the following two questions:

1) what is the impact of the condition number of \(\Phi^T\Phi\) on the convergence properties of the cost functions of EB and \(\text{SURE}_y\)?

2) what is the impact of the condition number of \(\Phi^T\Phi\) on the convergence properties of the optimal hyper-parameters of EB and \(\text{SURE}_y\)?

In this paper, we use asymptotic analysis tools from statistics to address these questions. In particular, we first consider the convergence rate of the cost functions of EB and SURE, and we find that they have the same convergence rate but the influence of the ill-conditioned regression matrix on the scale factor are different: the scale factor for SURE contains one more factor \(\text{cond}(\Phi^T\Phi)\) than that of EB, where \(\Phi\) is the regression matrix and \(\text{cond}(\cdot)\) denotes the condition number of a matrix. This finding indicates that when \(\Phi\) is ill-conditioned, i.e., \(\text{cond}(\Phi^T\Phi)\) is large, the cost function of SURE converges slower than that of EB. Then we consider the convergence rate of the optimal hyper-parameters of EB and SURE, and we find that they are both asymptotically normally distributed and have the same convergence rate, but the influence of the ill-conditioned regression matrix on the scale factor are different. In particular, for the ridge regression case, we show that the optimal hyper-parameter of SURE converges slower than that of EB with a factor of \(1/n^2\), as \(\text{cond}(\Phi^T\Phi)\) goes to \(\infty\), where \(n\) is the FIR model order.

The remaining parts of the paper is organized as follows. In Section II, we introduce the LS method and the RLS method for the linear regression model. In Section III, we show several common kernel structures and hyper-parameter estimation methods, including EB and SURE\(_y\). In Section IV, we compute the upper bounds of the convergence rates of EB and SURE\(_y\) cost functions and compare them in terms of the greatest power of the condition number of \(\Phi^T\Phi\). In Section V, we derive the asymptotic normality of EB and SURE\(_y\) hyper-parameter estimators. In Section VI, we illustrate our experiment results with the Monte Carlo simulation method. Our conclusion is given in Section VII. All proofs of the

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II. REGULARIZED LEAST SQUARES ESTIMATION FOR THE LINEAR REGRESSION MODEL

We focus on the linear regression model:

$$y(t) = \Phi^T(t) \theta + v(t), \quad t = 1, \cdots, N,$$

where $t$ denotes the time index, $y(t) \in \mathbb{R}$, $\Phi(t) \in \mathbb{R}^n$, $v(t) \in \mathbb{R}$ represent the output, the regressor and the disturbance at time $t$, and $\theta \in \mathbb{R}^n$ is the unknown parameter to be estimated. In addition, $v(t)$ is assumed to be independent and identically distributed (i.i.d.) white noise with zero mean and constant variance $\sigma^2 > 0$.

The model (1) can also be rewritten in matrix form as

$$Y = \Phi \theta + V,$$

where

$$Y = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix}, \quad \Phi = \begin{bmatrix} \Phi^T(1) \\ \vdots \\ \Phi^T(N) \end{bmatrix}, \quad V = \begin{bmatrix} v(1) \\ \vdots \\ v(N) \end{bmatrix}.$$  (3)

Our goal is to estimate the unknown $\theta$ as "good" as possible based on the historical data sets $\{y(t), \phi(t)\}_{t=1}^N$. Two types of mean square error (MSE) (4), (5) can be used to evaluate how "good" an estimator $\hat{\theta} \in \mathbb{R}^n$ of the true parameter $\theta_0 \in \mathbb{R}^n$ performs, which are defined as follows.

$$\text{MSE}_g(\hat{\theta}) = \mathbb{E}(\|\hat{\theta} - \theta_0\|_2^2),$$  (4a)

$$\text{MSE}_r(\hat{\theta}) = \mathbb{E}(\|\Phi \theta_0 + V - \Phi \hat{\theta}\|_2^2),$$  (4b)

where $\mathbb{E}(\cdot)$ denotes the mathematical expectation, $\| \cdot \|_2$ denotes the Euclidean norm, and $V^*$ is an independent copy of $V$. The smaller MSE indicates the better performance of $\hat{\theta}$. At the same time, MSE$_g$ and MSE$_r$ are closely connected with each other, which is stated in [6].

Assume that $\Phi \in \mathbb{R}^{N \times n}$ is full column rank with $N > n$, i.e. rank$(\Phi) = n$. One classic estimation method is the Least Squares (LS):

$$\hat{\theta}_{LS} = \arg \min_{\theta \in \mathbb{R}^n} \|Y - \Phi \theta\|_2^2$$  

$$= (\Phi^T \Phi)^{-1} \Phi^T Y,$$  

Although the LS estimator $\hat{\theta}_{LS}$ is unbiased, it may have large variance, which still results in large MSE$_g$.

$$\mathbb{E}(\hat{\theta}_{LS}) = \theta_0,$$

$$\text{Var}(\hat{\theta}_{LS}) = \sigma^2 \text{Tr}(\Phi^T \Phi)^{-1},$$

$$\text{MSE}_g(\hat{\theta}_{LS}) = \text{Var}(\hat{\theta}_{LS}) + \|\mathbb{E}(\hat{\theta}_{LS}) - \theta_0\|_2^2$$

$$= \sigma^2 \text{Tr}(\Phi^T \Phi)^{-1},$$

where $\text{Var}(\cdot)$ is the mathematical variance and $\text{Tr}(\cdot)$ denotes the trace of a square matrix.

Remark 1: When $\Phi^T \Phi$ is very ill-conditioned, the performance of $\hat{\theta}_{LS}$ will always be poor by the measure of MSE$_g$.

Define that eigenvalues of an $n$-by-$n$ positive definite matrix with $n \geq 2$ are $\lambda_1(\cdot) \geq \cdots \geq \lambda_n(\cdot)$ and the condition number of this matrix can be represented as $\text{cond}(\cdot) = \lambda_1(\cdot)/\lambda_n(\cdot)$. Then we can rewrite (4b) as

$$\text{MSE}_r(\hat{\theta}_{LS}) = \frac{\sigma^2}{\lambda_1(\Phi^T \Phi)} \left[ 1 + \sum_{i=2}^n \frac{\lambda_i(\Phi^T \Phi)}{\lambda_1(\Phi^T \Phi)} \right],$$  (7)

which means that

$$\frac{\sigma^2}{\lambda_1(\Phi^T \Phi)} \text{cond}(\Phi^T \Phi) < \text{MSE}_r(\hat{\theta}_{LS}) \leq \frac{n \sigma^2}{\lambda_1(\Phi^T \Phi)} \text{cond}(\Phi^T \Phi).$$  (8)

There are two factors influencing the lower bound of $\text{MSE}_r(\hat{\theta}_{LS})$: $\sigma^2/\lambda_1(\Phi^T \Phi)$ and $\text{cond}(\Phi^T \Phi)$.

- For the first factor $\sigma^2/\lambda_1(\Phi^T \Phi)$, if $\lambda_1(\Phi^T \Phi)$ is close to zero, then $\|\Phi\|_F$, where $\| \cdot \|_F$ denotes the Frobenius norm of a matrix, also becomes zero and we could not get enough valid information from outputs. Correspondingly, the estimation of $\theta$ would be very hard even if $\Phi^T \Phi$ is well-conditioned.

- For a fixed $\lambda_1(\Phi^T \Phi)$, as the second factor $\text{cond}(\Phi^T \Phi)$ becomes larger, i.e. $\Phi^T \Phi$ becomes more ill-conditioned, the $\text{MSE}_r(\hat{\theta}_{LS})$ will also increase, indicating the worse performance of $\hat{\theta}_{LS}$.

In the following part, we use the concept of almost sure convergence. We define that the random sequence $\{\xi_N\}$ converges almost surely to a random variable $\xi$ if and only if $\forall \varepsilon > 0$, $\lim_{N \to \infty} \mathbb{P}(|\xi_N - \xi| > \varepsilon$ for all $i \geq N) = 0$, which can be written as $\xi_N \overset{a.s.}{\to} \xi$ as $N \to \infty$.

Remark 2: Moreover, $\lambda_1(\Phi^T \Phi)/\sigma^2$ can conservatively act as the signal-to-noise ratio (snr), which can be defined as the ratio of variances of the noise-free output and the noise:

$$\text{snr} = \frac{\lambda_1(\Phi^T \Phi)}{\sigma^2}.\frac{\sum_{i=1}^N \lambda_i(\Phi^T \Phi) \theta_0^2}{\sum_{i=1}^N \theta_0^2}$$

If we assume that $\{\phi_i\}_{i=1}^N$ are independent and normally distributed with zero mean and constant covariance $\Sigma \in \mathbb{R}^{n \times n}$, it follows that $\phi_i^T \theta_0 \sim \mathcal{N}(0, \theta_0 \Sigma \theta_0)$ for $i = 1, \cdots, N$. Define that the eigenvector of $\Sigma$ is $u_i \in \mathbb{R}^n$ corresponding to $\lambda_i(\Sigma)$ with $i = 1, \cdots, n$. According to Corollary 2 in Appendix, we know that as $N \to \infty$,

$$\text{snr} \overset{a.s.}{\to} \frac{\sum_{i=1}^n \lambda_i(\Sigma) \theta_0^2 u_i^T \theta_0}{\sigma^2} \leq \frac{\lambda_1(\Sigma) \theta_0^2 u_1^T \theta_0}{\sigma^2},$$

Since $\lambda_1(\Phi^T \Phi)/\sigma^2 \overset{a.s.}{\to} \lambda_1(\Sigma)$ as $N \to \infty$, small $\lambda_1(\Phi^T \Phi)/\sigma^2$ always gives a smaller snr. When the snr is very small, even if the condition number of $\Phi^T \Phi$ is equal to one, $\hat{\theta}_{LS}$ still performs badly. We usually set $\text{snr} \geq 1$ in simulation experiments.

Remark 3: Interestingly, the MSE$_g$ of the LS estimator

$$\text{MSE}_g(\hat{\theta}_{LS}) = \mathbb{E}(\|\Phi \theta_0 + V^* - \Phi \hat{\theta}_{LS}\|_2^2)$$

$$= (N + n) \sigma^2$$

is irrespective of $\text{cond}(\Phi^T \Phi)$.  (11)
To handle this problem, we can introduce one regularization term in (5a) to obtain the regularized least squares (RLS) estimator:

\[
\hat{\theta}^R = \arg\min_{\theta \in \mathbb{R}^n} \| Y - \Phi \Phi^T \|^2 + \sigma^2 \theta^T P^{-1} \theta
\]

\[
= (\Phi^T \Phi + \sigma^2 P^{-1})^{-1} \Phi^T Y
\]

\[
= P \Phi^T Q^{-1} Y,
\]

where

\[
Q = \Phi P \Phi^T + \sigma^2 I_N,
\]

\[
P \in \mathbb{R}^{n \times n}
\]

is positive semidefinite and often known as the kernel matrix, and \(I_N\) denotes the \(N\)-dimensional identity matrix.

### III. Kernel Design and Hyper-parameter Estimation

For the regularization method, our main concerns are the kernel design and the hyper-parameter estimation.

#### A. Kernel Design

The structure of the kernel matrix \(P\) should be designed based on the prior knowledge about the true system by parameterizing it with the hyper-parameter \(\eta \in \mathbb{R}^p\), which can be tuned in the set \(\Omega \subset \mathbb{R}^p\). Several popular positive semidefinite kernels have been invented before,

- **SS**: \(P_{ii}(\eta) = c \left( \frac{\alpha^{i+j \max(i,j)}}{2} - \frac{\alpha^{3 \max(i,j)}}{6} \right)\), \(\eta = [c, \alpha] \in \Omega = \{ c \geq 0, \alpha \in [0, 1] \}\),

- **DC**: \(P_{ii}(\eta) = c \alpha^{i+j/2} P_{ii-j}, \eta = [c, \alpha, \rho] \in \Omega = \{ c \geq 0, \alpha \in [0, 1], |\rho| \leq 1 \}\),

- **TC**: \(P_{ii}(\eta) = c \alpha^{\max(i,j)}, \eta = [c, \alpha] \in \Omega = \{ c \geq 0, \alpha \in [0, 1] \}\),

where the spline kernel (SS) [14a] is firstly introduced in [3], the diagonal correlated (DC) kernel [14b] and the tuned-correlated (TC) kernel [14c] (also named as the first order stable spline kernel) are introduced in [4].

#### B. Hyper-parameter Estimation

If the structure of \(P(\eta)\) has been fixed, our next step is to estimate the hyper-parameter \(\eta\) using the historical data. There are many estimation approaches for the tuning of \(\eta\), such as the empirical Bayes (EB) method [1], the Stein’s unbiased estimation (SURE) method of MSE, and the generalized marginal likelihood method, the generalization cross validation (GCV) method [7] and so on.

Here we mainly investigate two hyper-parameter estimation methods: EB and SURE. The EB method assumes that \(\theta\) is Gaussian distributed with zero mean and covariance \(P\), and \(V\) is also normally distributed, i.e.,

\[
\theta \sim \mathcal{N}(0, P), V \sim \mathcal{N}(0, \sigma^2 I_N),
\]

\[
\Rightarrow Y \sim \mathcal{N}(0, \Phi P \Phi^T + \sigma^2 I_N).
\]

By maximizing the likelihood function of \(Y\), EB can be represented as

\[
EB: \hat{\eta}_{EB} = \arg\min_{\eta \in \Omega} \mathcal{F}_{EB}(\eta)
\]

\[
= \mathcal{F}_{EB} = \frac{1}{2} \| Y - \Phi \hat{\theta}^R(\eta) \|^2 + 2 \sigma^2 \text{Tr}(\Phi P \Phi^T Q^{-1}).
\]

The SURE can be written as

\[
SURE_y : \hat{\eta}_{Sy} = \arg\min_{\eta \in \Omega} \mathcal{F}_{Sy}(\eta)
\]

\[
= \mathcal{F}_{Sy} = \| Y - \Phi \hat{\theta}^R(\eta) \|^2 + 2 \sigma^2 \text{Tr}(\Phi P \Phi^T Q^{-1}).
\]

Before the further analysis, we firstly make some definitions and assumptions, which are consistent with [6]. Define the corresponding Oracle counterparts of EB and SURE as follows,

\[
EB^* : \hat{\eta}_{EB} = \arg\min_{\eta \in \Omega} \mathcal{F}_{EB}^*(\eta)
\]

\[
= \mathcal{F}_{EB}^* = \frac{1}{2} \| Y - \Phi \hat{\theta}^R(\eta) \|^2 + 2 \sigma^2 \text{Tr}(\Phi P \Phi^T Q^{-1})
\]

\[
MSE_y : \hat{\eta}_{MSE} = \arg\min_{\eta \in \Omega} \mathcal{F}_{MSE}(\eta)
\]

\[
= \mathcal{F}_{MSE} = \sigma^4 \theta^T \sigma^2 Q^{-2} \theta + \sigma^6 \text{Tr}(Q^{-2}) - 2 \sigma^4 \text{Tr}(Q^{-1}) + 2 N \sigma^2.
\]

**Assumption 1**: The optimal hyper-parameter estimates \(\hat{\eta}_{EB}, \hat{\eta}_{Sy}, \hat{\eta}_{MSE}\) are interior points of \(\Omega\).

**Assumption 2**: \(P\) is positive definite and as \(N \to \infty\), \((\Phi^T \Phi)/N\) converges to the positive definite \(\Sigma \in \mathbb{R}^{n \times n}\) almost surely, i.e. \((\Phi^T \Phi)/N \xrightarrow{a.s.} \Sigma > 0\).

Under Assumption 1 and 2, we can define the limit functions of EB, EEB, and SURE, respectively,

\[
\eta^*_b = \arg\min_{\eta \in \Omega} \mathcal{W}(P, \theta_0)
\]

\[
\mathcal{W}(P, \theta_0) = \theta_0^T \Phi^T Q^{-1} \Phi \theta_0 + \log \text{det}(Q),
\]

\[
\eta^*_c = \arg\min_{\eta \in \Omega} \mathcal{W}(P, \Sigma, \theta_0)
\]

\[
\mathcal{W}(P, \Sigma, \theta_0) = \sigma^2 \theta_0^T P^{-1} \Sigma^{-1} P^{-1} \theta_0 - 2 \sigma^4 \text{Tr}(\Sigma^{-1} P^{-1}).
\]

**Assumption 3**: The sets \(\eta^*_b\) and \(\eta^*_c\) are made of isolated points, respectively.

In the following assumption, we apply the concept of the boundedness in probability. Let \(\xi_N = O_P(a_N)\) denote that \(\{\xi_N/a_N\}\) is bounded in probability, which means that \(\forall \varepsilon > 0, \exists L > 0\) such that \(P(\|\xi_N/a_N\| > L) < \varepsilon\) for any \(N\).

**Assumption 4**: \(\|\Phi^T \Phi\|/N \xrightarrow{P} O_P(\delta_N)\) and as \(N \to \infty\), \(\delta_N \to 0\).

Under the Assumption 1, 2, 3, and 4, it has been shown in [6] that:

- \(\hat{\eta}_{Sy}\) is asymptotically optimal, while \(\hat{\eta}_{EB}\) is not, which means that as \(N \to \infty\),

\[
\hat{\eta}_{EB} \xrightarrow{a.s.} \eta^*_b, \ \hat{\eta}_{EB} \xrightarrow{a.s.} \eta^*_c,
\]

\[
\hat{\eta}_{Sy} \xrightarrow{a.s.} \eta^*_c, \ \hat{\eta}_{MSE} \xrightarrow{a.s.} \eta^*_c.
\]
• the convergence rate of \( \hat{\eta}_y \) to \( \eta^*_y \) is related with the convergence rate of \( (\Phi^T \Phi) N / \Sigma \) to \( \Sigma \), while that of \( \hat{\eta}_y \) to \( \eta^*_y \) is not, which means that

\[
\| \hat{\eta}_y - \eta^*_y \|_2 = O_p(1/\sqrt{N}) \tag{31a}
\]

\[
\| \hat{\eta}_y - \eta^*_y \|_2 = O_p(\mu_N), \tag{31b}
\]

\[
\mu_N = \max(\{O_p(\delta_N), O_p(1/\sqrt{N})\}). \tag{31c}
\]

Remark 4: For the SURE\(_y\), it will not encounter the ill-conditionedness for practical computation using the cholesky decomposition [8], but the ill-conditioned issue will appear if we analyze the property of its hyper-parameter estimator.

According to the findings and simulation experiments in [6], although \( \hat{\eta}_y \) is not asymptotically optimal, we can still observe better performance of \( \hat{\eta}_y \) than that of \( \hat{\eta}_y \) in the sense of MSE\(_y\), when \( \Phi^T \Phi \) is ill-conditioned and the sample size is small. Thus we draw attention to the influence of \( \text{cond}(\Phi^T \Phi) \) on the convergence rates of the cost functions and hyper-parameter estimators of EB and SURE\(_y\), respectively.

IV. Effects of \( \text{cond}(\Phi^T \Phi) \) on the Convergence Rates of Cost Functions of EB and SURE\(_y\)

Let

\[
\overline{\mathcal{F}}_{EB} = \mathcal{F}_{EB} + Y^T \Phi (\Phi^T \Phi)^{-1} \Phi^T Y / \sigma^2 - Y^T Y / \sigma^2 - (N - n) \log \sigma^2 - \log \det(\Phi^T \Phi) + (\delta^2 - \delta LS) S^{-1} - \log \det(S) \tag{32}
\]

\[
\mathcal{F}_{SY} = N [\mathcal{F}_{SY} + Y^T \Phi (\Phi^T \Phi)^{-1} \Phi^T Y - Y^T Y - 2n \sigma^2] \tag{33}
\]

\[
\mathcal{F}_{EB} = F_{EB} + Y^T \Phi (\Phi^T \Phi)^{-1} \Phi^T Y / \sigma^2 - Y^T Y / \sigma^2
\]

\[
= N \left[ (\delta^2 - \delta LS) S^{-1} - \log \det(S) \right] \tag{34}
\]

\[
\mathcal{F}_{SY} = N \left[ \sigma^2 (\delta^2 - \delta LS) S^{-1} - \sigma^2 \right] \tag{35}
\]

Under Assumption \([1, 2]\) and \([3]\) it has been proved in [6] that as \( N \to \infty \),

\[
\overline{\mathcal{F}}_{EB} \overset{a.s.}\to W_b, \quad \overline{\mathcal{F}}_{SY} \overset{a.s.}\to W_y. \tag{37}
\]

In fact, we can also investigate the influence of \( \text{cond}(\Phi^T \Phi) \) on the convergence rates of cost function by computing the upper bounds of \( |\overline{\mathcal{F}}_{EB} - W_b| \) and \( |\overline{\mathcal{F}}_{SY} - W_y| \).

Remark 5: To be clear; the first part of each upper bound in Theorem 7 and 8 indicates its boundedness in probability. For example, as shown in (183) of Corollary 8 in Appendix, \( \|A_N \|_F = O_p(1/\sqrt{N}) \). If one term is \( O_p(1) \), we omit this part for the convenience.

Applying Corollary 8 the upper bounds of \( |\overline{\mathcal{F}}_{EB} - W_b| \) and \( |\overline{\mathcal{F}}_{SY} - W_y| \) can be represented in the following Theorem 1 and 2.

Theorem 1: Under Assumption \([7, 2, 3]\) and \([3]\) we have

\[
|\overline{\mathcal{F}}_{EB} - W_b| \leq E_{1,b} + E_{2,b} + E_{3,b}, \tag{38}
\]

where

\[
E_{1,b} = \| \theta_0 \|_2 \| \Phi^T V \|_2 \| (\Phi^T \Phi)^{-1} \|_F \| S^{-1} \|_F + \| P^{-1} \|_F \tag{39}
\]

\[
E_{2,b} = \| (\Phi^T \Phi)^{-1} \|_F \left\| S^{-1} \right\|_F \left\| (\Phi^T \Phi)^{-1} \|_F + \sigma^2 \right\| \theta_0 \|_2 \| P^{-1} \|_F \right|_F \tag{40}
\]

\[
E_{3,b} = \| \theta_0 \|_2 \| (\Phi^T \Phi)^{-1} \|_F \| S^{-1} \|_F + \| P^{-1} \|_F \tag{41}
\]

\[
r_1 = \text{rank}(I_n - P^{1/2} S^{-1} P^{1/2}). \tag{42}
\]

Upper bounds of terms \([39], [40] \) and \([41] \) are shown as follows, respectively,

\[
E_{1,b} \leq \frac{1}{\sqrt{N}} n \| \theta_0 \|_2 \frac{N}{\lambda_1(\Phi^T \Phi)} \text{cond}(\Phi^T \Phi) \left( \frac{\| \Phi^T V \|_2^2}{N} \right) \tag{43}
\]

\[
E_{2,b} \leq \frac{1}{N} \frac{\| \Phi^T \|_2^2 N}{\lambda_1(\Phi^T \Phi)} \text{cond}(\Phi^T \Phi) \left( \frac{\| \Phi^T V \|_2^2}{N} \right) \tag{44}
\]

\[
E_{3,b} \leq \frac{1}{N} \frac{\| \theta_0 \|_2 \| (\Phi^T \Phi)^{-1} \|_F \| S^{-1} \|_F + \| P^{-1} \|_F}{\lambda_1(\Phi^T \Phi)} \text{cond}(\Phi^T \Phi) \left( \frac{\| \Phi^T V \|_2^2}{N} \right) \tag{45}
\]

Theorem 2: Under Assumption \([7, 2, 3]\) and \([3]\) we have

\[
|\overline{\mathcal{F}}_{SY} - W_y| \leq E_{1,y} + E_{2,y} + E_{3,y} + E_{4,y} + E_{5,y}, \tag{46}
\]

where

\[
E_{1,y} = \| \theta_0 \|_2 \| \Phi^T V \|_2 \| (\Phi^T \Phi)^{-1} \|_F \| S^{-1} \|_F + \| P^{-1} \|_F \tag{47}
\]

\[
E_{2,y} = \| \theta_0 \|_2 \| \Phi^T V \|_2 \| (\Phi^T \Phi)^{-1} \|_F \| S^{-1} \|_F + \| P^{-1} \|_F \tag{48}
\]

\[
E_{3,y} = \| \theta_0 \|_2 \| (\Phi^T \Phi)^{-1} \|_F \| S^{-1} \|_F \tag{49}
\]

\[
E_{4,y} = \| \theta_0 \|_2 \| \Phi^T V \|_2 \| (\Phi^T \Phi)^{-1} \|_F \| S^{-1} \|_F + \| P^{-1} \|_F \tag{50}
\]

\[
E_{5,y} = \| \theta_0 \|_2 \| \Phi^T V \|_2 \| (\Phi^T \Phi)^{-1} \|_F \| S^{-1} \|_F + \| P^{-1} \|_F \tag{51}
\]
Upper bounds of \((47), (48), (49), (50)\) and \((51)\) are shown as follows, respectively,

\[
E_{1,Y} \leq \frac{1}{\sqrt{N}} N^2 \sigma^4 \| \theta_0 \|^2 \left( \frac{N}{\lambda_1(\Phi^T \Phi)} \right) \text{cond}(\Phi^T \Phi) \frac{\| \Phi V \|_2}{\sqrt{N}} + \frac{1}{\lambda_1(\Sigma)} \text{cond}^2(S) \tag{53}
\]

\[
E_{2,Y} \leq \delta N n \sigma^4 \left( \frac{N}{\lambda_1(\Phi^T \Phi)} \right) \text{cond}(\Phi^T \Phi) \frac{\| \Phi V \|_2}{\sqrt{N}} + \frac{1}{\lambda_1(\Sigma)} \text{cond}^2(S) \tag{54}
\]

\[
E_{3,Y} \leq \frac{1}{\sqrt{N}} \frac{N}{\lambda_1(\Phi^T \Phi)} \frac{1}{\lambda_1(S)} \text{cond}^2(\Phi^T \Phi) \text{cond}(S) \left[ \frac{N}{\lambda_1(\Phi^T \Phi)} \right] \left( \frac{N}{\lambda_1(\Phi^T \Phi)} \right) \frac{\| \Phi V \|_2}{\sqrt{N}} + \frac{1}{\lambda_1(\Sigma)} \text{cond}^2(S) \tag{55}
\]

\[
E_{4,Y} \leq \frac{1}{N \sqrt{N}} \frac{N}{\lambda_1(\Phi^T \Phi)} \frac{1}{\lambda_1(S)} \text{cond}^2(\Phi^T \Phi) \text{cond}(S) \text{cond}(P) \frac{\| \Phi V \|_2}{\sqrt{N}} \tag{56}
\]

Remark 6: If \(\{ \phi(t) \}_{t=1}^N\) are assumed to be independent and normally distributed with zero mean, covariance matrix \(\Sigma\) and finite fourth moment, we can derive that \(\delta N = 1/\sqrt{N}\) using the Central Limit Theorem (CLT).

The comparison between upper bounds of \(|F_{EB} - W|\) and \(|F_{SY} - W|\) with respect to the powers of \(\text{cond}(\Phi^T \Phi)\) and \(\text{cond}(P)\) is summarized in the following table.

| \ \ | \(|F_{EB} - W|\) | \(|F_{SY} - W|\) |
|---|---|---|
| boundedness in probability | term of \(|F_{EB} - W|\) | maximum power of \(\text{cond}(\Phi^T \Phi)\) | maximum power of \(\text{cond}(P)\) |
| \(1/\sqrt{N}\) | \(E_{1,b}\) | 1 | 1 |
| \(1/N\) | \(E_{2,b}\) | 2 | 1 |
| \(1/N^{3/2}\) | \(E_{3,b}\) | 2 | 1 |
| boundedness in probability | term of \(|F_{SY} - W|\) | maximum power of \(\text{cond}(\Phi^T \Phi)\) | maximum power of \(\text{cond}(P)\) |
| \(1/\sqrt{N}\) | \(E_{1,y}\) | 2 | 2 |
| \(1/N\) | \(E_{2,y}\) | 3 | 2 |
| \(1/N^{3/2}\) | \(E_{3,y}\) | 3 | 1 |

Remark 7: Since \(N \to \infty\), \((\Phi^T \Phi)/N \xrightarrow{a.s.} \Sigma\), it can be seen that

\[
\text{cond}(\Phi^T \Phi) = \text{cond} \left( \frac{\Phi^T \Phi}{N} \right) \xrightarrow{a.s.} \text{cond}(\Sigma),
\]

which means that for any \(\varepsilon > 0\), \(\exists N > 0\), then for all \(N > N\)

\[
|\text{cond}(\Phi^T \Phi) - \text{cond}(\Sigma)| < \varepsilon \text{ almost surely.}
\]

Then in Table [7] for the term like \(\text{cond}(\Phi^T \Phi)\text{cond}(\Sigma)\), its greatest power of \(\text{cond}(\Phi^T \Phi)\) is regarded as 2.

As shown in Table [8], comparing the upper bounds of \(|F_{EB} - W|\) and \(|F_{SY} - W|\), the greatest power of \(\text{cond}(\Phi^T \Phi)\) of \(|F_{SY} - W|\) is always one larger than that of \(|F_{EB} - W|\), with regard to each term with the same boundedness in probability. At the same time, ill-conditioned \(\Phi^T \Phi\) may usually result in large \(\text{cond}(P)\). Table [8] also shows that the greatest power of \(\text{cond}(P)\) in each term of \(|F_{SY} - W|\) upper bound is one larger than that of \(|F_{EB} - W|\) upper bound, correspondingly. Thus, the large \(\text{cond}(\Phi^T \Phi)\) may lead to far slower convergence rate of \(F_{SY}\) to \(W\) than that of \(F_{EB}\) to \(W\). It also inspires us to continue the study of the effects of large \(\text{cond}(\Phi^T \Phi)\) on the comparison between the convergence rate of \(\tilde{W}_{EB}\) to \(\eta_{EB}^*\) and that of \(\tilde{W}_{SY}\) to \(\eta_{SY}^*\).

V. EFFECTS OF \(\text{cond}(\Phi^T \Phi)\) ON THE CONVERGENCE RATES OF HYPER-PARAMETER ESTIMATORS OF EB AND SURE

In this section, we show the asymptotic normality of \(\tilde{W}_{EB} - \eta_{EB}^*\) and \(\tilde{W}_{SY} - \eta_{SY}^*\). Here we define that the random sequence \(\{\xi_N\}\) converges in distribution to a random variable \(\xi\) with cumulative density function (CDF) \(F(\xi)\) if \(\lim_{N \to \infty} [F_N(\xi_N) - F(\xi)] = 0\), which can be written as \(\xi_N \xrightarrow{d} \xi\).

Assumption 5: Let \(\Omega\) be an open subset of the Euclidean p-space, which means that \(\eta_{EB}^*\) and \(\eta_{SY}^*\) are interior points of \(\Omega\).

Theorem 3: Assume that the noise is Gaussian distributed, i.e. \(V \sim N(0, \sigma^2 I_N)\). Under Assumption [2] [2] [3]
where the \((k,l)\)th elements of \(A_y(\eta^*_y)\) and \(B_y(\eta^*_y)\) can be represented as follows, respectively,

\[
A_y(\eta^*_y)_{k,l} = \left\{ \theta_0^T \frac{\partial P^{-1} - 1}{\partial \eta_k} \theta_0 + \text{Tr} \left( \frac{\partial P^{-1} \partial P}{\partial \eta_l} \right) \right\}_{\eta^*_y},
\]

\[
B_y(\eta^*_y)_{k,l} = 4\sigma^2 \left\{ \theta_0^T \frac{\partial P^{-1} - 1}{\partial \eta_k} \Sigma^{-1} \frac{\partial P^{-1}}{\partial \eta_l} \theta_0 - \text{Tr} \left( \Sigma^{-1} \frac{\partial P^{-1} - 1}{\partial \eta_l} \theta_0 \right) \right\}_{\eta^*_y}.
\]

\[\text{Theorem 4: In addition to Assumption 7, 8, 9, and the Gaussian noise assumption, we further suppose that} \delta_N = o(1/\sqrt{N}), \text{which means that} \delta_{\eta} \text{is an infinitesimal of}\]

\[\text{and as} \ N \to \infty, \text{we have}\]

\[
\sqrt{N}(\hat{\eta}_y - \eta^*_y) \xrightarrow{d} N\left(0, \frac{4\sigma^2}{n} \theta_0^T \Sigma^{-1} \theta_0 \right),
\]

\[
\sqrt{N}(\hat{\theta}_y - \theta^*_y) \xrightarrow{d} N\left(0, \frac{4\sigma^2}{\text{Tr}(\Sigma^{-1})} \theta_0^T \Sigma^{-3} \theta_0 \right).
\]

\[\text{As} \ \lambda_1(\Sigma) \to 0 \text{and other eigenvalues} \ \lambda_i(\Sigma) \text{with} i = 1, \ldots, n - 1 \text{are fixed, which leads to} \ \text{cond}(\Sigma) \to \infty, \text{the ratio of two limiting variances in (60) and (67) tends to be} 1/n^2, \ \text{i.e.}\]

\[
\frac{\theta_0^T \Sigma^{-1} \theta_0 / n^2}{\theta_0^T \Sigma^{-3} \theta_0 / \text{Tr}(\Sigma^{-1})} \to \frac{1}{n^2}.
\]

It implies that even for the ridge regression case, when \(\text{cond}(\Sigma) \to \infty\) and \(n \geq 2\), the asymptotic variance of \(\hat{\eta}_y - \eta^*_y\) still tends to \(n^2\) times larger than that of \(\hat{\eta}_y - \eta^*_y\). Then in this case, the convergence rate of \(\hat{\eta}_y - \eta^*_y\) is slower than that of \(\hat{\eta}_y - \eta^*_y\).

VI. NUMERICAL SIMULATION

To generate data sets, we construct \(\{\phi(i)\}_{i=1}^N\) as independent and Gaussian distributed vectors with zero mean and fixed covariance \(\Sigma\). Then it satisfies \((\Phi^T \Phi)/N \xrightarrow{d} \Sigma\) as \(N \to \infty\), which can be proved by Corollary 2. It is worth to note that under our simulation settings, \(\delta_N = 1/\sqrt{N}\), which is worse than the assumption in Theorem 4.

In our simulation experiments, we consider the ridge regression case and set \(n = 50\), \(\text{cond}(\Sigma) = 1 \times 10^5\) and \(\text{snr} = 5\). The number of Monte Carlo simulations is selected as \(1 \times 10^3\). Define that \(\theta_0 \triangleq \left[ g_1 \ldots g_n \right]^T\) and \(V^* \triangleq \left[ v(1)^* \ldots v(N)^* \right]^T\). The performance of \(\hat{\theta}_y^R\) in (12) can be evaluated by relative criteria [9] as follows,

\[
\text{Fit}_{\eta_y}(\hat{\theta}_y^R, \theta_0) = 100 \times \left(1 - \frac{\|\hat{\theta}_y^R - \theta_0\|_2}{\|\theta_0 - \theta_0\|_2} \right),
\]

\[
\text{Fit}_{\text{MSE}}(\hat{\theta}_y^R, \theta_0) = 100 \times \left(1 - \frac{\|\Phi \hat{\theta}_y^R - \Phi \theta_0 - V^*\|^2}{\|\Phi \theta_0 + V^* - \Phi \theta_0\|^2} \right),
\]

where

\[
\theta_0 = \frac{1}{n} \sum_{i=1}^n g_i, \quad V^* = \frac{1}{N} \sum_{i=1}^N (\phi(i)^T \theta_0 + v(i))^*. \]

In fact, \(\text{Fit}_{\eta_y}\) evaluates the performance of \(\hat{\theta}_y^R\) in the sense of \(\text{MSE}\), and \(\text{Fit}_{\text{MSE}}\) measures in the sense of \(\text{MSE}\). The convergences of \(\Phi^T \Phi/N\) to \(\Sigma\), \(\hat{\theta}_y^R\) to \(\theta_0\), \(\hat{\eta}_y\) to \(\eta^*_y\), \(\hat{\eta}_y\) to \(\eta^*_y\), \(\hat{\theta}_y\) to \(W_y\) and \(\hat{\eta}_y\) to \(W_y\) are also evaluated by the measure of fit similarly.

![Fig. 1: Convergence of \((\Phi^T \Phi)/N\) to \(\Sigma\)](image-url)
Fig. 2: Average Fit\(_{\chi}\) of $\hat{\theta}^R(\hat{\eta}_{EB})$ and $\hat{\theta}^R(\hat{\eta}_{Sy})$

Fig. 3: Average Fit\(_{\chi}\) of $\hat{\theta}^R(\hat{\eta}_{EB})$ and $\hat{\theta}^R(\hat{\eta}_{Sy})$

Fig. 4: Average fits of cost functions

Fig. 5: Logarithm of absolute cost function differences

Fig. 6: Average fits of hyper-parameter estimates

Fig. 7: Logarithm of absolute hyper-parameter estimate differences
Σ. It verifies the consistency of our simulation settings and Assumption 2.

Secondly, Fig 2 shows that the performance of \( \hat{\theta}^R(T) \) is better than that of \( \hat{\theta}^R(\eta) \) in the sense of MSE\(_F\). But in Fig 3 the overall Fit\(_T\) of \( \hat{\theta}^R(\eta) \) and \( \hat{\theta}^R(\eta) \) are almost identical for all sample size, indicating that cond(\( \Phi^T \Phi \)) may exert few influence on MSE\(_F\) of the RLS estimator.

Thirdly, according to Fig 4, it can be observed that when cond(\( \Phi^T \Phi \)) is very close to 10\(^3\), \( \mathcal{F}_{EB} \) converges to \( W_b \) much faster than \( \mathcal{F}_{SE} \) to \( W_y \). At the same time, the intercept of the vertical axis in Fig 5 is around 0.75, which indicates that \( |\mathcal{F}_{EB} - W_b| \) converges to zero faster than that of \( |\mathcal{F}_{SE} - W_y| \).

Lastly, according to Fig 6, it can be observed that when cond(\( \Phi^T \Phi \)) is very close to 10\(^5\), the convergence rate of \( \hat{\eta}_{EB} \) to \( \eta^* \) is much faster than that of \( \hat{\eta}_{SY} \) to \( \eta^* \). The vertical intercept of Fig 7 is about 2.5, which also reflects faster convergence rate of \( \|\hat{\eta}_{EB} - \eta^*\|_2 \) than \( \|\hat{\eta}_{SY} - \eta^*\|_2 \) to zero.

VII. CONCLUSIONS

In this paper, we focus on the comparison between two hyper-parameter estimation methods: EB and SURE\(_F\) with an emphasis on the influence of cond(\( \Phi^T \Phi \)), where cond(\( \cdot \)) denotes the condition number and \( \Phi \) is the regression matrix. Our major results are about the contrast between convergence rates of two pairs, \( \mathcal{F}_{EB} \) to \( W_b \) and \( \mathcal{F}_{SE} \) to \( W_y \), and \( \hat{\eta}_{EB} \) to \( \eta^* \) and \( \hat{\eta}_{SY} \) to \( \eta^*, \) respectively.

1) Comparing terms with the same boundedness in probability, the greatest power of cond(\( \Phi^T \Phi \)) of the upper bound of \( |\mathcal{F}_{SE} - W_y| \) is always one larger than that of the upper bound of \( |\mathcal{F}_{EB} - W_b| \). It indicates that the ill-conditioned \( \Phi^T \Phi \) may result in far slower convergence rate of \( \mathcal{F}_{SE} \) to \( W_y \) than that of \( \mathcal{F}_{EB} \) to \( W_b \).

2) As the sample size \( N \to \infty \), under the assumption of \( \delta_N = o(1/\sqrt{N}) \) and Gaussian distributed noise, we prove the asymptotic normality of \( \hat{\eta}_{EB} - \eta^* \) and \( \hat{\eta}_{SY} - \eta^* \). In this case, \( \|\hat{\eta}_{EB} - \eta^*\|_2 \) and \( \|\hat{\eta}_{SY} - \eta^*\|_2 \) are both bounded in probability with \( 1/\sqrt{N} \) but may have different scaling coefficients, which are connected with their asymptotic covariance matrices. For the ridge regression case, we derive that, as cond(\( \Phi^T \Phi \)) tends to infinity, the asymptotic variance of \( \hat{\eta}_{SY} - \eta^* \) tends to be \( n^2 \) times larger than that of \( \hat{\eta}_{EB} - \eta^* \), where \( n \) is the number of parameters to be estimated.

These findings provide more insights into the influence of cond(\( \Phi^T \Phi \)) upon different performances of EB and SURE\(_F\). Our next step is to relax the restriction \( \delta_N = o(1/\sqrt{N}) \) as \( N \to \infty \) to further explore the convergence rates of EB and SURE\(_F\) estimators with different boundedness in probability.

APPENDIX A

Proofs of Theorems 1, 2, 3 and 4 and Corollary 1 are shown in Appendix A.

A. Proof of Theorem 7

The difference of \( \mathcal{F}_{EB} \) and \( W_b \) can be represented as

\[
\mathcal{F}_{EB} - W_b = D_{1,b} + D_{2,b},
\]

where

\[
D_{1,b} = (\hat{\theta}^LS)^T S^{-1} \hat{\theta}^LS - \theta_0^T P^{-1} \theta_0
\]
\[
= (\hat{\theta}^LS - \theta_0)^T S^{-1} (\hat{\theta}^LS - \theta_0) + 2\theta_0^T (S^{-1} - P^{-1}) \hat{\theta}^LS
\]
\[
+ \theta_0^T P^{-1}(\hat{\theta}^LS - \theta_0),
\]

\[
(73)
\]
\[
D_{2,b} = \log \det(S) - \log \det(P)
\]
\[
= \log \det(SP^{-1})
\]
\[
= \log \det(P^{-1/2}SP^{-1/2}),
\]

\[
(74)
\]

• Computation of \( \|\hat{\theta}^LS - \theta_0\|_2 \)

Using Lemma 1, it can be known that

\[
\|\hat{\theta}^LS - \theta_0\|_2 = \|(\Phi^T \Phi)^{-1} \Phi^T V\|_2
\]
\[
\leq \|(\Phi^T \Phi)^{-1}\|_F \|\Phi^T V\|_2.
\]

(75)

The upper bound of \( \|(\Phi^T \Phi)^{-1}\|_F \) can be derived by applying Corollary 3. We can also derive that \( \|\Phi^T V\|_2 = O_p(1/\sqrt{N}) \) using CLT. It can be seen that

\[
\|\hat{\theta}^LS - \theta_0\|_2 \leq \frac{1}{\sqrt{N}} \sqrt{\lambda_1(\Phi^T \Phi)} \|\Phi^T V\|_2 \sqrt{N}.
\]

(76)

• Computation of \( \|\hat{\theta}^LS\|_2 \)

Since

\[
\|\hat{\theta}^LS\|_2 = \|\hat{\theta}^LS - \theta_0 + \theta_0\|_2
\]
\[
\leq \|\hat{\theta}^LS - \theta_0\|_2 + \|\theta_0\|_2,
\]

(77)

it leads to that

\[
\|\hat{\theta}^LS\|_2 \leq \|\theta_0\|_2 + \frac{1}{\sqrt{N}} \frac{N}{\lambda_1(\Phi^T \Phi)} \|\Phi^T V\|_2 \sqrt{N}.
\]

(78)

• Computation of \( \|S^{-1} - P^{-1}\|_F \)

It can be derived that

\[
\|S^{-1} - P^{-1}\|_F = \|S^{-1}(P - S)P^{-1}\|_F
\]
\[
= \|\sigma^2 S^{-1}(\Phi^T \Phi)^{-1}P^{-1}\|_F
\]
\[
= \|\sigma^2 S^{-1}(\Phi^T \Phi)^{-1}P^{-1}\|_F
\]
\[
\leq \sigma^2 \|S^{-1}\|_F \|\Phi^T \Phi\|^{-1}_F \|P^{-1}\|_F
\]
\[
\leq \frac{\sigma^2}{\sqrt{\lambda_1(\Phi^T \Phi)}} \frac{1}{\lambda_1(S)} \frac{1}{\lambda_1(P)} \frac{1}{\lambda_1(\Phi^T \Phi)} \text{cond}(\Phi^T \Phi) \text{cond}(S) \text{cond}(P).
\]

(79)

For \( D_{1,b} \) in (73), we can apply the inequalities above directly

\[
|D_{1,b}| \leq \|\theta_0\|_2 \|\Phi^T V\|_2 \|\Phi^T \Phi\|^{-1}_F \|S^{-1}\|_F \|P^{-1}\|_F
\]
\[
+ \|\Phi^T \Phi\|^{-1}_F \|S^{-1}\|_F \|\Phi^T V\|_2 \|\Phi^T \Phi\|^{-1}_F \|F
\]
\[
+ \sigma^2 \|\theta_0\|_2 \|\Phi^T V\|_2 \|\Phi^T \Phi\|^{-1}_F \|S^{-1}\|_F \|P^{-1}\|_F.
\]

(80)

Since \( P^{-1/2}SP^{-1/2} \) is positive definite, we can see that

\[
\text{Tr}(I_n - P^{-1/2}S^{-1}P^{1/2})
\]
\[
\leq \log \det(P^{-1/2}SP^{-1/2}) \leq \text{Tr}(P^{-1/2}SP^{-1/2} - I_n),
\]

(81)
which yields
\[ |D_{2,b}| = |\log(\det(P^{-1/2}SP^{-1/2}))| \leq \max\{ |\text{Tr}(I_n - P^{-1/2}S^{-1}P^{1/2})|, |\text{Tr}(P^{-1/2}SP^{-1/2} - I_n)| \}. \] (82)

Define
\[ \text{rank}(I_n - P^{-1/2}S^{-1}P^{1/2}) = r_1. \] (83)

It follows that
\[ |D_{2,b}| \leq \max\{ \sqrt{T_1} |I_n - P^{-1/2}S^{-1}P^{1/2}|, \sqrt{T_1} |P^{-1/2}SP^{-1/2} - I_n| \} \]
\[ = \max\{ \sqrt{T_1} |P^{1/2}(P^{-1/2} - S)P^{1/2}|, \sqrt{T_1} |P^{-1/2}(S - P)P^{-1/2}| \} \]
\[ \leq \max\{ \sqrt{T_1} \sigma^2 \|\Phi^T\Phi\|^{-1} \|\Phi\| \|\Sigma^{-1}\|F \|P^{-1/2}\|F, \sqrt{T_1} \sigma^2 \|\Phi^T\Phi\|^{-1} \|\Phi\| \|P^{-1/2}\|F \}. \] (84)

Combining (80) with (84), it can be known that
\[ |\overline{\mathcal{F}}_{EB} - W_y| \leq |D_{1,b}| + |D_{2,b}| \]
\[ \leq \alpha_{1,b} E_{1,b} + E_{2,b} + E_{3,b}, \] (85)

where
\[ E_{1,b} = \|\theta_0\|_2 \|\Phi^T V\|_2 \|\Phi^T F - \Phi\|^{-1} \|\Phi\| \|\Sigma^{-1}\|F \|P^{-1}\|F \] (86)
\[ E_{2,b} = \|\Phi^T \|F \]
\[ + \sqrt{T_1} \sigma^2 \max\{ \|\Sigma^{-1}\|F \|\Phi^T\|F \|P^{-1/2}\|F, \|\Phi\| \|P^{-1/2}\|F \} \] (87)
\[ E_{3,b} = \sigma^2 \|\theta_0\|_2 \|\Phi^T V\|_2 \|\Phi^T F - \Phi\|^{-1} \|\Phi\| \|\Sigma^{-1}\|F \|P^{-1}\|F. \] (88)
The upper bounds of \( E_{1,b}, E_{2,b} \) and \( E_{3,b} \) can be derived using Corollary 3.

B. Proof of Theorem 2

The difference of \( \overline{\mathcal{F}}_{SB} \) and \( W_y \) can be represented as
\[ \overline{\mathcal{F}}_{SB} - W_y = D_{1,y} + \text{Tr}(D_{2,y}), \] (89)

where
\[ D_{1,y} = \alpha^T (\hat{\theta}_L S) S^{-T} N(\Phi^T \Phi)^{-1} - S^{-1} \hat{\theta}_L \]
\[ = \alpha^T (\hat{\theta}_L S - \theta_0) S^{-T} N(\Phi^T \Phi)^{-1} - S^{-1} \hat{\theta}_L \]
\[ + \alpha^T \theta_0 (S^{-1} - P^{-1}) S^{-T} N(\Phi^T \Phi)^{-1} - S^{-1} \hat{\theta}_L \]
\[ + \alpha^T \theta_0 (S^{-1} - P^{-1}) S^{-T} N(\Phi^T \Phi)^{-1} - S^{-1} \hat{\theta}_L \]
\[ + \alpha^T \theta_0 (S^{-1} - P^{-1}) S^{-T} N(\Phi^T \Phi)^{-1} - S^{-1} \hat{\theta}_L \]
\[ + \alpha^T \theta_0 (S^{-1} - P^{-1}) S^{-T} N(\Phi^T \Phi)^{-1} - S^{-1} \hat{\theta}_L \]
\[ D_{2,y} = 2\alpha^T (\Sigma^{-1} P^{-1} - N(\Phi^T \Phi)^{-1} S^{-1}) \]
\[ = 2\alpha^T (\Sigma^{-1} P^{-1} - N(\Phi^T \Phi)^{-1} S^{-1}) \]
\[ + 2\alpha^T (\Sigma^{-1} P^{-1} - N(\Phi^T \Phi)^{-1} S^{-1}). \] (90)

• Computation of \( |\Sigma^{-1}|_F \)

\[ |\Sigma^{-1}|_F \leq \frac{\sqrt{n}}{\lambda_1(\Sigma)} \text{cond}(\Sigma). \] (92)

• Computation of \( |\Phi^T F|^{-1} - \Sigma^{-1}|_F \)

Since \( |(\Phi^T F)|/N - \Sigma^{-1}|_F = O_p(\delta_N) \), we can know that
\[ \|\Phi^T F - \Sigma\|_F \leq \delta_N \sqrt{n} \lambda_n(\Phi^T F/N - \Sigma) \text{cond}(\Phi^T F - \Sigma). \] (93)

Furthermore,
\[ |N(\Phi^T F - \Sigma)|_F \leq |\Phi^T F - \Sigma|_F \leq |N(\Phi^T F - \Sigma)|_F \]
\[ \leq \|N(\Phi^T F - \Sigma)|_F \leq \|\Phi^T F - \Sigma\|_F \|\Sigma^{-1}\|_F \]
\[ \leq \delta_N \frac{n^{3/2}}{\lambda_1(\Phi^T F)} \frac{\lambda_n(\Phi^T F/N - \Sigma)}{\delta_N} \frac{1}{\lambda_1(\Sigma)} \text{cond}(\Phi^T F) \text{cond}(\Phi^T F - \Sigma) \text{cond}(\Sigma). \] (94)

Suppose that
\[ r_2 = \text{rank}(D_{2,y}) = \text{rank}(\Sigma^{-1} P^{-1} - N(\Phi^T \Phi)^{-1} S^{-1}). \] (95)

Thus, the absolute difference term can be rewritten as
\[ |\overline{\mathcal{F}}_{SB} - W_y| = |D_{1,y} + \text{Tr}(D_{2,y})| \]
\[ \leq |D_{1,y}| + \sqrt{T_2} \|D_{2,y}\|_F. \] (96)

Its upper bound can similarly be obtained with the computation of building blocks above.

C. Proof of Theorem 3

We firstly derive the asymptotic normality of \( \hat{\eta}_{EB} - \eta_b \) using Lemma 5 with \( M_N = N\overline{\mathcal{F}}_{EB} \) and \( \eta^* = \eta_b \).

• assumptions 1, 2 and 3 in Lemma 5

The first task is to show that \( N\overline{\mathcal{F}}_{EB}(Y, \eta) \) is a measurable function of \( Y \) for all \( \eta \in \Omega \). Recall that
\[ N\overline{\mathcal{F}}_{EB}(\eta) = (\hat{\theta}_L S)^T S^{-1} \hat{\theta}_L S + \log(\det(S)) = \gamma^T Q^{-1} (\Phi^T F - \Phi^T Y) + \log(\det(\Phi^T F - \Phi^T F)^{-1}). \] (97)

We can observe that \( N\overline{\mathcal{F}}_{EB}(Y, \eta) \) is a continuous function of \( Y \), which leads to that \( N\overline{\mathcal{F}}_{EB}(Y, \eta) \) is also a measurable function of \( Y \), \( \forall \eta \in \Omega \).

Then show that \( \frac{\partial N\overline{\mathcal{F}}_{EB}}{\partial \eta} \) exists and is continuous in an open neighbourhood of \( \eta_b \), and \( \frac{\partial^2 N\overline{\mathcal{F}}_{EB}}{\partial \eta^2} \) exists and is continuous in an open and convex neighbourhood of \( \eta_b \).

For common kernel structures, like SS \( 15a \), DC \( 14b \) and TC \( 14c \), \( P(\eta) \) is a continuous, differentiable and second-order differentiable function of \( \eta \) in \( \Omega \). Meanwhile, the first-order derivative and the second-order derivative of \( P(\eta) \) with respect to \( \eta \) are both continuous for all \( \eta \in \Omega \). Then under Assumption 5 it can be derived that there exists an open neighbourhood of \( \eta_b \) such that \( \frac{\partial N\overline{\mathcal{F}}_{EB}}{\partial \eta} \) exists and is continuous. There also exists an open and convex neighbourhood of \( \eta_b \), in which \( \frac{\partial^2 N\overline{\mathcal{F}}_{EB}}{\partial \eta^2} \) exists and is continuous.
• assumption 5 in Lemma 5
In this part, we prove that $\mathcal{F}_{EB}(\eta)$ converges to $W_b(\eta)$ in probability and uniformly in one neighbourhood of $\eta^*_b$.
As shown in (72), (73) and (74), $\mathcal{F}_{EB} - W_b$ can be divided into two parts: $D_{1,b}$ and $\text{Tr}(\Sigma_{D,2,b})$. Recall that
\[ D_{1,b} = (\hat{\theta}^{LS} - \theta_0)^T S^{-1} \hat{\theta}^{LS} + \theta_0^0 (S^{-1} - P^{-1}) \hat{\theta}^{LS} + \theta_0^0 (S^{-1} - P^{-1}) \theta_0 + \hat{\theta}^{LS} - \theta_0 \] (98)
\[ D_{2,b} = \log(\det(S)) - \log(\det(P)) \] (99)
Under Assumption 5 there exists $\Omega_1 \subset \Omega$ containing $\eta^*_b$ such that $0 < d_1 \leq ||P(\eta)||_F \leq d_2 < \infty$ and $||S^{-1}||_F \leq ||P^{-1}||_F < 1/d_1$ for all $\eta \in \Omega_1$. Noting that as $N \to \infty$, $(\Phi^T \Phi)/N^{a_k} \to \Sigma$, it gives that $\hat{\theta}^{LS} \xrightarrow{a_k} \theta_0$, $S^{-1} \xrightarrow{a_k} P^{-1}$ as $N \to \infty$, which can be derived as follows,
\[ S^{-1} - P^{-1} = -\sigma^2 S^{-1} (\Phi^T \Phi)^{-1} P^{-1} \xrightarrow{\text{a.s.}} 0 \] (100)
\[ \hat{\theta}^{LS} - \theta_0 = N (\Phi^T \Phi)^{-1} P \to 0 \] (101)
It also follows that $||\hat{\theta}^{LS} - \theta_0||_2 = O_p(1/\sqrt{N})$, $||\hat{\theta}^{LS}||_2 = O_p(1)$ and $||S^{-1} - P^{-1}||_F = O_p(1/N)$. Then we can see that each term of $D_{1,b}$ and $D_{2,b}$ converge to zero almost surely and uniformly for all $\eta \in \Omega_1$.
In addition, the almost sure convergence can lead to the convergence in probability. Thus, $\mathcal{F}_{EB}(\eta)$ converges to $W_b(\eta)$ in probability and uniformly for all $\eta \in \Omega_1$.
We can also show that $W_b(\eta)$ attains a strict local minimum at $\eta^*_b$. If $\eta^*_b$ in (25) is an interior point in $\Omega$, it should satisfy the first order optimality condition of $W_b(\eta)$, i.e.
\[ \frac{\partial W_b(\eta)}{\partial \eta} \bigg|_{\eta^*_b} = 0. \] (102)
Combining with Assumption 3 $\eta^*_b$ can strictly and locally minimize $W_b(\eta)$.

• assumption 5 in Lemma 5
Our aim in this part is to prove that $\frac{\partial^2 \mathcal{F}_{EB}}{\partial \eta \partial \eta^T}$ converges to $A_b(\eta^*_b)$ in probability for any sequence $\eta^*_N$ such that $\lim_{N \to \infty} \eta^*_N = \eta^*_b$ in probability, where
\[ A_b(\eta^*_b) \triangleq \lim_{N \to \infty} \mathbb{E} \left[ \frac{\partial^2 \mathcal{F}_{EB}}{\partial \eta \partial \eta^T} \right]_{\eta^*_b} \] (103)
Here $\text{plim}$ denotes the limiting in probability.
Our proof consists of two steps.
The first is to show that $\frac{\partial^2 \mathcal{F}_{EB}}{\partial \eta \partial \eta^T} \bigg|_{\eta^*_b}$ converges to $\frac{\partial^2 W_b}{\partial \eta \partial \eta^T} \bigg|_{\eta^*_b}$ in probability for any sequence $\eta^*_N$ such that $\text{plim}_{N \to \infty} \eta^*_N = \eta^*_b$.
The $(k,l)$th elements of the Hessian matrices of $\mathcal{F}_{EB}$ and $W_b$ are shown as follows, respectively,
\[ \frac{\partial^2 \mathcal{F}_{EB}}{\partial \eta_k \partial \eta_l} = (\hat{\theta}^{LS})^T \frac{\partial^2 S^{-1}}{\partial \eta_k \partial \eta_l} (\hat{\theta}^{LS}) + \text{Tr} \left( \frac{\partial S^{-1}}{\partial \eta_k} \frac{\partial P}{\partial \eta_l} \right) + \text{Tr} \left( \frac{\partial S^{-1}}{\partial \eta_k} \frac{\partial P}{\partial \eta_l} \right) \] (104)
\[ \frac{\partial^2 W_b}{\partial \eta_k \partial \eta_l} = \theta_0^T \frac{\partial^2 P}{\partial \eta_k \partial \eta_l} \theta_0 + \text{Tr} \left( \frac{\partial P}{\partial \eta_k} \frac{\partial P}{\partial \eta_l} \right) + \text{Tr} \left( \frac{\partial P}{\partial \eta_k} \frac{\partial P}{\partial \eta_l} \right) \] (105)
Then the $(k,l)$th element of the difference between $\frac{\partial^2 \mathcal{F}_{EB}}{\partial \eta \partial \eta^T}$ and $\frac{\partial^2 W_b}{\partial \eta \partial \eta^T}$ can be represented as
\[ \frac{\partial^2 \mathcal{F}_{EB}}{\partial \eta_k \partial \eta_l} - \frac{\partial^2 W_b}{\partial \eta_k \partial \eta_l} = \Psi_{1,b} + \text{Tr}(\Psi_{2,b}), \] (106)
where
\[ \Psi_{1,b} = (\hat{\theta}^{LS} - \theta_0)^T \frac{\partial^2 S^{-1}}{\partial \eta_k \partial \eta_l} \hat{\theta}^{LS} + \theta_0^T \left( \frac{\partial^2 S^{-1}}{\partial \eta_k \partial \eta_l} - \frac{\partial^2 P}{\partial \eta_k \partial \eta_l} \right) \hat{\theta}^{LS} + \theta_0^T \frac{\partial^2 P}{\partial \eta_k \partial \eta_l} (\hat{\theta}^{LS} - \theta_0) \] (107)
\[ \Psi_{2,b} = \left( \frac{\partial S^{-1}}{\partial \eta_k} - \frac{\partial P}{\partial \eta_k} \right) \frac{\partial P}{\partial \eta_l} + (S^{-1} - P^{-1}) \frac{\partial P}{\partial \eta_k} \] (108)
Under Assumption 5 there exists a neighborhood $\Omega_2 \subset \Omega$ of $\eta^*_b$ such that for any $k = 1, \ldots, p$, $l = 1, \ldots, p$, $\frac{\partial^2 P}{\partial \eta_k \partial \eta_l}$ and $P$ are all bounded. Since $||S^{-1}||_F \leq ||P^{-1}||_F$ and
\[ \frac{\partial^2 P}{\partial \eta_k \partial \eta_l} = p^{-1} \frac{\partial P}{\partial \eta_k} \frac{\partial P}{\partial \eta_l} - p^{-1} \frac{\partial^2 P}{\partial \eta_k \partial \eta_l} + p^{-1} \frac{\partial P}{\partial \eta_k} \frac{\partial P}{\partial \eta_l} \] (109)
\[ \frac{\partial^2 S^{-1}}{\partial \eta_k \partial \eta_l} = S^{-1} \frac{\partial S^{-1}}{\partial \eta_k} \frac{\partial S^{-1}}{\partial \eta_l} - S^{-1} \frac{\partial^2 P}{\partial \eta_k \partial \eta_l} S^{-1} + S^{-1} \frac{\partial^2 P}{\partial \eta_k \partial \eta_l} S^{-1} \] (110)
it follows that $\frac{\partial^2 P}{\partial \eta_k \partial \eta_l}$ and $\frac{\partial^2 S^{-1}}{\partial \eta_k \partial \eta_l}$ are both bounded $\forall \eta \in \Omega_2$ with $k = 1, \ldots, p$, $l = 1, \ldots, p$. As $N \to \infty$, since $(\Phi^T \Phi)/N^{a_k} \to \Sigma$, we have $\hat{\theta}^{LS} \xrightarrow{a_k} \theta_0$, $S^{-1} \xrightarrow{a_k} P^{-1}$, $S^{-1} \xrightarrow{a_k} P^{-1}$ and $\frac{\partial^2 S^{-1}}{\partial \eta_k \partial \eta_l} \xrightarrow{a_k} \frac{\partial^2 P}{\partial \eta_k \partial \eta_l}$. The last two can be proved as follows,
\[ \frac{\partial S^{-1}}{\partial \eta_k} = -S^{-1} \frac{\partial P}{\partial \eta_k} S^{-1} + P^{-1} \frac{\partial P}{\partial \eta_k} P^{-1} \] (111)
Under the assumption that as $N \to \infty$, 
\begin{equation}
\sqrt{N} \frac{\partial \mathcal{F}_{EB}}{\partial \eta} \mid_{\eta^*_b} \to \mathcal{N}(0, B_b(\eta^*_b)),
\end{equation}
where 
\begin{equation}
B_b(\eta^*_b) \triangleq \plim_{N \to \infty} \left( \frac{\partial \mathcal{F}_{EB}}{\partial \eta} \right)_{\eta^*_b} \times \frac{\partial \mathcal{F}_{EB}}{\partial \eta^T} \mid_{\eta^*_b}.
\end{equation}

Our proof is made up of two steps. The first step is to show that \( \sqrt{N} \frac{\partial \mathcal{F}_{EB}}{\partial \eta} \mid_{\eta^*_b} \) converges in distribution to a Gaussian distributed random vector with zero mean and the (k,l)th element of the limiting covariance matrix is 
\begin{equation}
4\sigma^2 \theta_0^T \frac{\partial P^{-1}}{\partial \eta_k} \Sigma^{-1} \frac{\partial P^{-1}}{\partial \eta_l} \theta_0 \mid_{\eta^*_b}.
\end{equation}

The kth elements of \( \frac{\partial \mathcal{F}_{EB}}{\partial \eta} \) and \( \frac{\partial W_b}{\partial \eta} \) can be written as 
\begin{equation}
\frac{\partial \mathcal{F}_{EB}}{\partial \eta_k} = (\hat{\theta}^{LS})^T \frac{\partial S^{-1}}{\partial \eta_k} \hat{\theta}^{LS} + \text{Tr} \left( S^{-1} \frac{\partial P}{\partial \eta_k} \right),
\end{equation}
\begin{equation}
\frac{\partial W_b}{\partial \eta_k} = \theta_0^T \frac{\partial P^{-1}}{\partial \eta_k} \theta_0 + \text{Tr} \left( P^{-1} \frac{\partial P}{\partial \eta_k} \right).
\end{equation}

Since \( \frac{\partial W_b}{\partial \eta_k} \mid_{\eta^*_b} = 0 \), it leads to 
\begin{equation}
\sqrt{N} \frac{\partial \mathcal{F}_{EB}}{\partial \eta_k} \mid_{\eta^*_b} = \sqrt{N} \left( \frac{\partial \mathcal{F}_{EB}}{\partial \eta_k} - \frac{\partial W_b}{\partial \eta_k} \right) \mid_{\eta^*_b}.
\end{equation}

\begin{equation}
\mathcal{Y}_{1,b} = \langle \hat{\theta}^{LS} - \theta_0 \rangle^T \frac{\partial S^{-1}}{\partial \eta_k} \hat{\theta}^{LS} + \theta_0^T \frac{\partial P^{-1}}{\partial \eta_k} (\hat{\theta}^{LS} - \theta_0).
\end{equation}

1) For \( \sqrt{N} \mathcal{Y}_{1,b} \mid_{\eta^*_b} \), applying Lemma 3 with \( X_N = S^{-1} - P^{-1}, a_N = \frac{1}{N} \) and \( w_N = \sqrt{N} \), we have 
\begin{equation}
\sqrt{N} (S^{-1} - P^{-1}) \mid_{\eta^*_b} \Rightarrow 0
\end{equation}
as \( N \to \infty \). Similarly, we can prove that as \( N \to \infty \), 
\begin{equation}
\sqrt{N} \left( \frac{\partial S^{-1}}{\partial \eta_k} - \frac{\partial P^{-1}}{\partial \eta_k} \right) \mid_{\eta^*_b} \Rightarrow 0
\end{equation}
with \( X_N = \frac{\partial S^{-1}}{\partial \eta_k} - \frac{\partial P^{-1}}{\partial \eta_k}, a_N = \frac{1}{N} \) and \( w_N = \sqrt{N} \) by Lemma 1. Note that as \( N \to \infty \), \( \frac{\partial S^{-1}}{\partial \eta_k} \mid_{\eta^*_b} \Rightarrow \hat{\theta}^{LS}, \hat{\theta}^{LS} \mid_{\eta^*_b} \Rightarrow 0 \), (125) and (126) will not change with the value of \( \eta \). Thus, according to the sum and product rules of convergence in probability, it can be seen that as \( N \to \infty \), 
\begin{equation}
\sqrt{N} \mathcal{Y}_{1,b} \mid_{\eta^*_b} \Rightarrow 0.
\end{equation}
2) For $\sqrt{N}Y_{2,b}|_{\eta_b^*}$, let us investigate the limiting in distribution of $\sqrt{N}(\hat{\theta}^{LS} - \theta_0)$ firstly, which can be rewritten as

$$\sqrt{N}(\hat{\theta}^{LS} - \theta_0) = \sqrt{N}(\Phi^T \Phi)^{-1} \Phi^T \sqrt{N} F \rightarrow \mathcal{N}(0, \Sigma^{-1}).$$

Then we come to

$$\sqrt{N}(\hat{\theta}^{LS} - \theta_0) \rightarrow \mathcal{N}(0, \Sigma^{-1}).$$

(129)

Note that as $N \rightarrow \infty$, $N(\Phi^T \Phi)^{-1} \rightarrow \Sigma^{-1}$ and $\sqrt{N}(\Phi^T \Phi)^{-1} \rightarrow \mathcal{N}(0, \Sigma^{-1})$, which can be proved by CLT, then it follows that

$$\sqrt{N}(\hat{\theta}^{LS} - \theta_0) \rightarrow \mathcal{N}(0, \Sigma^{-1}).$$

(129)

Based on (125) and (126), we can prove that

$$\plim_{N \to \infty} \sqrt{N} \left[ \theta_0^T \frac{\partial S^{-1}}{\partial \eta_k} \theta_0 + \text{Tr} \left( S^{-1} \frac{\partial P}{\partial \eta_l} \right) \right]_{\eta_b^*} = 0.$$  

(135)

Then we have

$$B_b(\eta_b^*)_{k,l} = 4\sigma^2 \theta_0^T \frac{\partial P^{-1}}{\partial \eta} \Sigma^{-1} \frac{\partial P^{-1}}{\partial \eta} \theta_0 |_{\eta_b^*}. $$

(136)

Finally, we apply Lemma 3 with $M_N = N\mathcal{F}_{\mathcal{F}_S}$ and $\eta^* = \eta_b^*$ to prove the asymptotic normality of $\hat{\eta}_{EB} - \eta_b^*$.

**D. Proof of Theorem 2**

The asymptotic normality of $\hat{\eta}_{SY} - \eta_b^*$ can be shown through similar thoughts with $M_N = N\mathcal{F}_{\mathcal{F}_S}$ and $\eta^* = \eta_b^*$.

- **assumption 12 and 3 in Lemma 5**

First of all, we show that $N\mathcal{F}_{\mathcal{F}_S}(Y, \eta)$ is a measurable function of $Y$ for all $\eta \in \Omega$. Recall that

$$\mathcal{F}_{\mathcal{F}_S} = N(\sigma^4 Y^T \Phi^T \Phi^{-1} S^{-1} \theta^{LS} - 2\text{Tr}(\Phi^T \Phi^{-1} S^{-1})) = N(\sigma^4 Y^T Q^{-1} \Phi^T \Phi^{-1} \theta^{LS} S^{-1} Y + 2\sigma^2 \text{Tr}(\Phi^T \Phi + \sigma^2 P^{-1}) \Phi^T \Phi - l_n)).$$

(137)

It can be noticed that $N\mathcal{F}_{\mathcal{F}_S}(Y, \eta)$ is a continuous function of $Y$ for all $\eta \in \Omega$, which indicates that $\forall \eta \in \Omega, N\mathcal{F}_{\mathcal{F}_S}(Y, \eta)$ is a measurable function of $Y$.

Then we show that $\frac{\partial N\mathcal{F}_{\mathcal{F}_S}}{\partial \eta}$ exists and is continuous in an open and convex neighbourhood of $\eta_b^*$, and $\frac{\partial^2 N\mathcal{F}_{\mathcal{F}_S}}{\partial \eta \partial \eta}$ exists and is continuous in an open and convex neighbourhood of $\eta_b^*$. For common kernel structures, like SS (13a), DC (13b) and TC (13c), $P(\eta)$ is a continuous, differentiable and second-order differentiable function of $\eta$ in $\Omega$. Meanwhile, the first-order derivative and the second-order derivative of $P(\eta)$ with respect to $\eta$ are both continuous for all $\eta \in \Omega$. Then under Assumption 5, it can be derived that there exists an open neighbourhood of $\eta_b^*$ such that $\frac{\partial N\mathcal{F}_{\mathcal{F}_S}}{\partial \eta}$ exists and is continuous. There also exists an open and convex neighbourhood of $\eta_b^*$, in which $\frac{\partial^2 N\mathcal{F}_{\mathcal{F}_S}}{\partial \eta \partial \eta}$ exists and is continuous.

- **assumption 12 in Lemma 5**

In this part, we prove that $\mathcal{F}_{\mathcal{F}_S}(\eta)$ converges to $W(\eta)$ in probability and uniformly in one neighbourhood of $\eta_b^*$. 

$$\plim_{N \to \infty} \sqrt{N} \left[ \theta_0^T \frac{\partial S^{-1}}{\partial \eta_k} \theta_0 + \text{Tr} \left( S^{-1} \frac{\partial P}{\partial \eta_l} \right) \right]_{\eta_b^*} = 0.$$  

(135)
As mentioned in (89), (90) and (91), \( \mathcal{F}_N - W_N \) is computed with two parts: \( D_{1,y} \) and \( \text{Tr}(D_{2,y}) \). Recall that

\[
D_{1,y} = \sigma^4(\partial^2 S - \partial_0^2)T S^{-T} N(\Phi^T \Phi)^{-1} S^{-1} \partial^2 S
+ \sigma^4 \theta_0^T (S^{-1} - P^{-1}) T N(\Phi^T \Phi)^{-1} S^{-1} \partial^2 S
+ \sigma^4 \theta_0^T P^{-T} (N(\Phi^T \Phi)^{-1} - \Sigma^{-1}) S^{-1} \partial^2 S
+ \sigma^4 \theta_0^T P^{-T} \Sigma^{-1} (S^{-1} - P^{-1}) \partial^2 S
+ \sigma^4 \theta_0^T P^{-T} \Sigma^{-1} (S^{-1} - P^{-1}) \partial^2 S
\]

\[
D_{2,y} = 2\sigma^4 (\Sigma^{-1} - N(\Phi^T \Phi)^{-1}) P^{-1}
+ 2\sigma^4 N(\Phi^T \Phi)^{-1} (P^{-1} - S^{-1}).
\]

Based on Assumption [5] there exists \( \bar{\Omega}_3(\eta^*_y) \subset \Omega \) such that \( 0 < d_3 \leq \|P(\eta)||_F \leq d_4 < \infty \) and \( \|S^{-1}\|_F \leq \|P^{-1}\|_F \leq 1/d_3 \) for all \( \eta \in \bar{\Omega}_3 \). Noting that \( N(\Phi^T \Phi)^{-1} = \frac{1}{\Sigma} \), \( \bar{\Theta}LS \bar{\Theta} \), \( S^{-1} - P^{-1} \) as \( N \to \infty \), and \( \|\tilde{\Theta}LS - \tilde{\Theta}\|_2 = O_p(1/\sqrt{N}) \), \( \|\tilde{\Theta}LS\|_2 = O_p(1) \), \( \|S^{-1} - P^{-1}\|_F = O_p(1/N) \) and \( \|N(\Phi^T \Phi)^{-1} - \Sigma^{-1}\|_F = O_p(\delta_0) \), we can show that each term of \( D_{1,y} \) and \( D_{2,y} \) converges to zero in probability and uniformly for any \( \eta \) in \( \bar{\Omega}_3 \). Thus, as \( N \to \infty \), \( \mathcal{F}_N \) converges to \( W_N \) in probability and uniformly \( \forall \eta \in \bar{\Omega}_3 \).

Under Assumption [3] and [5] we can show that \( W_N \) attains a strict local minimum at \( \eta^*_y \).

**assumption [5] in Lemma 5**

Our goal in this part is to prove that \( \frac{\partial^2 \mathcal{F}_N}{\partial \eta \partial \eta^T} \bigg|_{\eta^*_N} \) converges to \( C_y(\eta^*_y) \) in probability for any sequence \( \Pi_N \) such that \( \lim_{N \to \infty} \Pi_N = \eta^*_y \) in probability, where

\[
C_y(\eta^*_y) \triangleq \lim_{N \to \infty} \mathbb{E} \left[ \left( \frac{\partial^2 \mathcal{F}_N}{\partial \eta \partial \eta^T} \right) \bigg|_{\eta^*_N} \right].
\]

Detailed procedure consists of two steps.

The first step is to prove that \( \frac{\partial^2 \mathcal{F}_N}{\partial \eta \partial \eta^T} \bigg|_{\eta^*_N} \) converges to \( \frac{\partial^2 W_N}{\partial \eta \partial \eta^T} \bigg|_{\eta^*_N} \) in probability for any sequence \( \Pi_N \) such that \( \lim_{N \to \infty} \Pi_N = \eta^*_N \) in probability.

The \( (k,l) \)th elements of Hessian matrices of \( \mathcal{F}_N \) and \( W_N \) are shown as follows, respectively,

\[
\frac{\partial^2 \mathcal{F}_N}{\partial \eta_k \partial \eta_l} = 2\sigma^4 (\partial^2 S - \partial_0^2) T S^{-T} N(\Phi^T \Phi)^{-1} S^{-1} \partial^2 S
+ 2\sigma^4 (\partial^2 S - \partial_0^2) T S^{-T} N(\Phi^T \Phi)^{-1} \partial^2 S
+ 2\sigma^4 (\partial^2 S - \partial_0^2) T S^{-T} \Sigma^{-1} S^{-1} \partial^2 S
+ 2\sigma^4 (\partial^2 S - \partial_0^2) T S^{-T} \Sigma^{-1} \partial^2 S
\]

\[
\frac{\partial^2 W_N}{\partial \eta_k \partial \eta_l} = 2\sigma^4 \theta_0^T P^{-T} \Sigma^{-1} \partial^2 P \partial_0^T \Sigma^{-1} \partial_0^T \theta_0
+ 2\sigma^4 \theta_0^T P^{-T} \Sigma^{-1} \partial^2 P \partial_0^T \theta_0
+ 2\sigma^4 \theta_0^T P^{-T} \Sigma^{-1} \theta_0
\]

Then the \((k,l)\)th element of the difference between \( \frac{\partial^2 \mathcal{F}_N}{\partial \eta_k \partial \eta_l} \) and \( \frac{\partial^2 W_N}{\partial \eta_k \partial \eta_l} \) can be represented as

\[
\frac{\partial^2 \mathcal{F}_N}{\partial \eta_k \partial \eta_l} - \frac{\partial^2 W_N}{\partial \eta_k \partial \eta_l} = Y_{1,y} + \text{Tr}(Y_{2,y}),
\]

where

\[
Y_{1,y} = 2\sigma^4 (\partial^2 S - \partial_0^2) T S^{-T} N(\Phi^T \Phi)^{-1} S^{-1} \partial^2 S
+ 2\sigma^4 (\partial^2 S - \partial_0^2) T S^{-T} N(\Phi^T \Phi)^{-1} S^{-1} \partial^2 S
+ 2\sigma^4 (\partial^2 S - \partial_0^2) T S^{-T} N(\Phi^T \Phi)^{-1} S^{-1} \partial^2 S
\]

Under Assumption [5] there exists \( \bar{\Omega}_4(\eta^*_y) \subset \Omega \) such that for any \( k = 1, \ldots, p \) and \( l = 1, \ldots, p \), \( \frac{\partial^2 \mathcal{F}_N}{\partial \eta_k \partial \eta_l} \) and \( P \) are all bounded, which leads to that \( \frac{\partial^2 \mathcal{F}_N}{\partial \eta_k \partial \eta_l} \) and \( \partial^2 S \) are both bounded \( \forall \eta \in \bar{\Omega}_4 \) with \( k = 1, \ldots, p \) and \( l = 1, \ldots, p \). As \( N \to \infty \), we have \( N(\Phi^T \Phi)^{-1} \to \Sigma^{-1} \), \( \tilde{\Theta}LS \to \tilde{\Theta}_0 \), \( S^{-1} - P^{-1} \to \tilde{\Theta}LS \to \tilde{\Theta}_0 \), \( S^{-1} - P^{-1} \to \tilde{\Theta}LS \to \tilde{\Theta}_0 \), \( \frac{\partial^2 S}{\partial \eta_k \partial \eta_l} \), and \( \frac{\partial^2 S}{\partial \eta_k \partial \eta_l} \) are all bounded.

Also note that \( \|N(\Phi^T \Phi)^{-1} - \Sigma^{-1}\|_F = O_p(\delta_0) \), \( \|\tilde{\Theta}LS - \tilde{\Theta}LS\|_2 = O_p(1/\sqrt{N}) \), \( \|\tilde{\Theta}LS - \tilde{\Theta}LS\|_F = O_p(1/N) \), \( \frac{\partial \mathcal{F}_N}{\partial \eta_k} \) and \( \frac{\partial \mathcal{F}_N}{\partial \eta_l} \) are all bounded. So each term in \( Y_{1,y} \) and \( Y_{2,y} \) converges to zero in probability and uniformly, \( \forall \eta \in \bar{\Omega}_4(\eta^*_y) \).

Therefore, \( \frac{\partial^2 \mathcal{F}_N}{\partial \eta_k \partial \eta_l} \) converges to \( \frac{\partial^2 W_N}{\partial \eta_k \partial \eta_l} \) in probability and uniformly in \( \bar{\Omega}_4(\eta^*_y) \). From Lemma 6 \( \frac{\partial^2 \mathcal{F}_N}{\partial \eta_k \partial \eta_l} \) converges to \( \frac{\partial^2 W_N}{\partial \eta_k \partial \eta_l} \) in probability for any sequence \( \Pi_N \) such that \( \lim_{N \to \infty} \Pi_N = \eta^*_y \) in probability.
The second step is to show that

\[
C_y(\eta^*_y) \equiv \lim_{N \to \infty} \mathbb{E} \left[ \frac{\partial^2 F_y}{\partial \eta \partial \eta^T} \right]_{\eta^*_y} = \left[ \frac{\partial^2 W_y}{\partial \eta \partial \eta^T} \right]_{\eta^*_y}.
\] (146)

Since \( \hat{\theta}^{LS} \sim N(\Phi \theta_0, \sigma^2 (\Phi^T \Phi)^{-1}) \), we can apply Lemma 5 to obtain the \((k,l)\)th element of \(C(\eta^*_y)\) as

\[
C_y(\eta^*_y)_{k,l} = \lim_{N \to \infty} 2\sigma^6 \text{Tr} \left[ \frac{\partial S^{-T}}{\partial \eta} N(\Phi^T \Phi)^{-1} \frac{\partial S^{-1}}{\partial \eta_k} (\Phi^T \Phi)^{-1} \right]
+ 2\sigma^4 \theta_0^T \frac{\partial S^{-T}}{\partial \eta} N(\Phi^T \Phi)^{-1} \frac{\partial S^{-1}}{\partial \eta_k} \theta_0
+ 2\sigma^4 \theta_0^T S^{-T} N(\Phi^T \Phi)^{-1} \frac{\partial S^{-1}}{\partial \eta_k} \theta_0
+ 2\sigma^4 \theta_0^T S^{-T} N(\Phi^T \Phi)^{-1} \frac{\partial^2 S^{-1}}{\partial \eta_k \partial \eta_l} \theta_0
- 2\sigma^4 \text{Tr} \left( N(\Phi^T \Phi)^{-1} \frac{\partial^2 S^{-1}}{\partial \eta_k \partial \eta_l} \right)
\] (147)

which is exactly equal to (142) at \( \eta^*_y \).

\* assumption 9 in Lemma 5

In this part, we aim to prove that \( N \to \infty \),

\[
\sqrt{N} \left[ \frac{\partial F_y}{\partial \eta} \right]_{\eta^*_y} \xrightarrow{d} N(0, D_y(\eta^*_y)),
\] (149)

where

\[
D_y(\eta^*_y) \equiv \lim_{N \to \infty} \mathbb{E} \left[ \frac{\partial^2 F_y}{\partial \eta \partial \eta^T} \right]_{\eta^*_y} \times \left[ \frac{\partial^2 F_y}{\partial \eta \partial \eta^T} \right]_{\eta^*_y}.
\] (150)

Our proof consists of two steps.

The first step is to show that \( \sqrt{N} \left[ \frac{\partial F_y}{\partial \eta} \right]_{\eta^*_y} \) converges in distribution to a Gaussian distributed random vector with zero mean and the \((k,l)\)th element of the limiting covariance matrix is

\[
4\sigma^4 \left\{ \theta_0^T \left[ \frac{\partial^2 -1}{\partial \eta_k \partial \eta_l} + \frac{\partial^2}{\partial \eta_l} \Sigma^{-1} P^{-1} \right] \Sigma^{-1} \right\} \left[ \frac{\partial^2 -1}{\partial \eta_k \partial \eta_l} + \frac{\partial^2}{\partial \eta_l} \Sigma^{-1} P^{-1} \right] \theta_0.
\] (151)

The \(k\)th elements of \( \frac{\partial F_y}{\partial \eta_k} \) and \( \frac{\partial W_y}{\partial \eta_k} \) can be written as

\[
\frac{\partial F_y}{\partial \eta_k} = 2\sigma^4 (\hat{\theta}^{LS})^T S^{-T} N(\Phi^T \Phi)^{-1} \frac{\partial S^{-1}}{\partial \eta_k} \hat{\theta}^{LS}
- 2\sigma^4 \text{Tr} \left( N(\Phi^T \Phi)^{-1} \frac{\partial S^{-1}}{\partial \eta_k} \right)
\] (152)

\[
\frac{\partial W_y}{\partial \eta_k} = 2\sigma^4 \theta_0^T P^{-T} \Sigma^{-1} \frac{\partial P^{-1}}{\partial \eta_k} \theta_0
- 2\sigma^4 \text{Tr} \left( \Sigma^{-1} \frac{\partial P^{-1}}{\partial \eta_k} \right).
\] (153)

Since \( \frac{\partial W_y}{\partial \eta_k} = 0 \), we have

\[
\sqrt{N} \left[ \frac{\partial F_y}{\partial \eta_k} \right]_{\eta^*_y} = \sqrt{N} \left[ \frac{\partial F_y}{\partial \eta_k} - \frac{\partial W_y}{\partial \eta_k} \right]_{\eta^*_y} = \sqrt{N}(Y_{1,y} | \eta^*_y + Y_{2,y} | \eta^*_y).
\] (154)

where

\[
Y_{1,y} = 2\sigma^4 \theta_0^T (S^{-1} - P^{-1})^T N(\Phi^T \Phi)^{-1} \frac{\partial S^{-1}}{\partial \eta_k} \hat{\theta}^{LS}
+ 2\sigma^4 \theta_0^T P^{-T} (N(\Phi^T \Phi)^{-1} - \Sigma^{-1}) \frac{\partial S^{-1}}{\partial \eta_k} \hat{\theta}^{LS}
+ 2\sigma^4 \theta_0^T P^{-T} \Sigma^{-1} \left( \frac{\partial S^{-1}}{\partial \eta_k} - \frac{\partial P^{-1}}{\partial \eta_k} \right) \hat{\theta}^{LS}
+ 2\sigma^4 \text{Tr} \left[ \Sigma^{-1} N(\Phi^T \Phi)^{-1} \frac{\partial P^{-1}}{\partial \eta_k} \right]
+ 2\sigma^4 \text{Tr} \left[ N(\Phi^T \Phi)^{-1} \left( \frac{\partial P^{-1}}{\partial \eta_k} - \frac{\partial S^{-1}}{\partial \eta_k} \right) \right]
\] (155)

\[
Y_{2,y} = 2\sigma^4 \hat{\theta}^{LS} \left( \theta_0 - \theta_0 \right) S^{-1} \theta_0 - \theta_0
\] (156)

1) For \( \sqrt{N}Y_{1,y} | \eta^*_y \), if we define that \( w_N = \frac{1}{\delta_N \sqrt{N}} \), from \( \delta_N = o(1/\sqrt{N}) \) as \( N \to \infty \), it can be derived that

\[
\lim_{N \to \infty} \frac{1}{w_N} = \lim_{N \to \infty} \frac{\delta_N}{\sqrt{N}} = 0,
\] (157)

namely \( w_N \to 0 \) as \( N \to \infty \). Thus we can use Lemma 4 with \( X_N = N(\Phi^T \Phi)^{-1} - \Sigma^{-1} \), \( a_N = \delta_N \) and \( w_N = \frac{1}{\delta_N \sqrt{N}} \) to obtain

\[
\sqrt{N}N(\Phi^T \Phi)^{-1} - \Sigma^{-1} \xrightarrow{P} 0.
\] (158)

As \( N \to \infty \), \( N(\Phi^T \Phi)^{-1} \xrightarrow{P} \Sigma^{-1} \), \( S^{-1} \xrightarrow{P} P^{-1} \), \( \frac{\partial S^{-1}}{\partial \eta_k} \xrightarrow{P} \frac{\partial P^{-1}}{\partial \eta_k} \), \( \hat{\theta}^{LS} \xrightarrow{P} \theta_0 \). (125), (126) and (158) do not change with the value of \( \eta \). It gives that as \( N \to \infty \),

\[
\sqrt{N}Y_{1,y} | \eta^*_y \xrightarrow{P} 0.
\] (159)

2) For \( \sqrt{N}Y_{2,y} | \eta^*_y \), as \( N \to \infty \), since \( N(\Phi^T \Phi)^{-1} \xrightarrow{P} \Sigma^{-1} \), \( S^{-1} \xrightarrow{P} P^{-1} \), \( \frac{\partial S^{-1}}{\partial \eta_k} \xrightarrow{P} \frac{\partial P^{-1}}{\partial \eta_k} \), \( \hat{\theta}^{LS} \xrightarrow{P} \theta_0 \) and
Applying Slutsky’s theorem, \( \sqrt{N} \frac{\partial F_{xy}}{\partial \eta} \) converges in distribution to a Gaussian distributed random variable with zero mean and the limiting variance is

\[
4 \sigma^2 \left\{ \Theta_0 \left[ p^{-1} \Sigma^{-1} \frac{\partial P^{-1}}{\partial \eta} + \frac{\partial P^{-1}}{\partial \eta} \Sigma^{-1} p^{-1} \right] \Sigma^{-1} \theta_0 \right\}_{\eta_*}.
\]

(160)

Thus

\[
D_y(\eta^*_k)_{k,l} \triangleq \lim_{N \to \infty} \left\{ \left( \frac{\partial F_{xy}}{\partial \eta} \right)_{\eta_*} \times \left( \frac{\partial F_{xy}}{\partial \eta'} \right)_{\eta_*} \right\}
\]

(162)

equals (161). Using Lemma 3 we have

\[
D_y(\eta^*_k)_{k,l} = 4 \sigma^2 \left\{ \Theta_0 \left[ p^{-1} \Sigma^{-1} \frac{\partial P^{-1}}{\partial \eta} + \frac{\partial P^{-1}}{\partial \eta} \Sigma^{-1} p^{-1} \right] \Sigma^{-1} \theta_0 \right\}_{\eta_*}.
\]

(163)

Based on (126) and (158), it can be derived that

\[
\lim_{N \to \infty} \sqrt{N} \left[ \left( \theta_0 \right)^T S^{-1} N(\Phi^T \Phi)^{-1} \frac{\partial S^{-1}}{\partial \eta_0} \right] \theta_0
\]

\[
= \lim_{N \to \infty} \sqrt{N} \left[ \left( \theta_0 \right)^T S^{-1} N(\Phi^T \Phi)^{-1} \frac{\partial S^{-1}}{\partial \eta_0} \right] \theta_0
\]

\[
= \lim_{N \to \infty} \sqrt{N} \left[ \left( \theta_0 \right)^T (S^{-T} - p^{-T}) N(\Phi^T \Phi)^{-1} \frac{\partial S^{-1}}{\partial \eta_0} \right] \theta_0
\]

\[
+ \theta_0^T p^{-T} \left[ \frac{\partial S^{-1}}{\partial \eta_0} \right] \theta_0
\]

\[
+ \theta_0^T p^{-T} \left[ \frac{\partial S^{-1}}{\partial \eta_0} \right] \theta_0
\]

\[
- \text{Tr} \left( N(\Phi^T \Phi)^{-1} \frac{\partial S^{-1}}{\partial \eta_0} \right)
\]

(164)

Thus

\[
D_y(\eta^*_k)_{k,l} = 4 \sigma^2 \left\{ \Theta_0 \left[ p^{-1} \Sigma^{-1} \frac{\partial P^{-1}}{\partial \eta} + \frac{\partial P^{-1}}{\partial \eta} \Sigma^{-1} p^{-1} \right] \Sigma^{-1} \theta_0 \right\}_{\eta_*}.
\]

(165)

Finally, we apply Lemma 3 with \( M_N = N \frac{\partial F_{xy}}{\partial \eta} \) and \( \eta^* = \eta_* \)

and the proof of the asymptotic normality of \( \eta_{sy} - \eta_0^* \) is complete.

E. Proof of Corollary 7

Inserting \( P = \eta I_n \) into the first order optimality conditions of \( W_b \) and \( W_y \), we can derive that

\[
\eta^*_k = \frac{\theta_0^T \theta_0}{n}, \quad \eta^*_l = \frac{\theta_0^T \Sigma^{-1} \theta_0}{\text{Tr}(\Sigma^{-1})}.
\]

(166)

Inserting them into (61), (62), (64) and (65), it gives that

\[
A_b(\eta^*_k) = \frac{n^3}{(\theta_0^T \theta_0)};
\]

(167)

\[
B_b(\eta^*_k) = 4 \sigma^2 n^4 \theta_0^T \Sigma^{-1} \theta_0;
\]

(168)

\[
C_b(\eta^*_k) = \frac{2 \sigma^4 \text{Tr}(\Sigma^{-1})}{(\theta_0^T \Sigma^{-1} \theta_0)^3};
\]

(169)

\[
D_b(\eta^*_k) = 16 \sigma^4 n^6 \Sigma^{-1} \theta_0^T \Sigma^{-1} \theta_0
\]

(170)

which leads to

\[
\frac{A_b^{-1}(\eta^*_k) B_b(\eta^*_k) C_b^{-1}(\eta^*_k)}{C_b^{-1}(\eta^*_k) D_b(\eta^*_k) C_b^{-1}(\eta^*_k)} = \frac{\text{Tr}(\Sigma^{-1}) \theta_0^T \Sigma^{-1} \theta_0}{n^2 \theta_0^T \Sigma^{-1} \theta_0}.
\]
Apply the singular value decomposition (SVD) in $\Sigma$ as
\[
\Sigma = U_S S U_s^T,
\]
(172)

where $U_s \in \mathbb{R}^{n \times n}$ is orthogonal and $S \in \mathbb{R}^{n \times n}$ is diagonal with, eigenvalues of $\Sigma$, $\lambda_i(\Sigma) \geq \cdots \geq \lambda_n(\Sigma)$. Set $U_S^T \theta_0 = \begin{bmatrix} \hat{g}_1 & \cdots & \hat{g}_n \end{bmatrix}^T$. As $\lambda_0(\Sigma) \to 0$ and the other eigenvalues are fixed, namely that cond$(\Sigma)$ = $\lambda_1(\Sigma) / \lambda_n(\Sigma)$ = $\infty$, we have
\[
\frac{\lambda_i^{-1}(\eta_k) B_{\lambda_i}(\eta_k^*) A_{\lambda_i}^{-1}(\eta_k^*)}{C_{\lambda_i}^{-1}(\eta_k^*) D_{\lambda_i}(\eta_k^*) C_{\lambda_i}^{-1}(\eta_k^*)} = \frac{1}{n^2} \left[ \frac{1}{\lambda_i^2} + \sum_{j=1}^{n-1} \frac{1}{\lambda_j^2} \right] \lambda_i^2 \sum_{j=1}^{n-1} \frac{1}{\lambda_j^2} \lambda_i^2
g_i^2 + \sum_{j=1}^{n-1} \frac{1}{\lambda_j^2} \lambda_i^2
g_i^2 + \sum_{j=1}^{n-1} \frac{1}{\lambda_j^2} \lambda_i^2
g_i^2 \to \frac{1}{n^2}.
\]
(173)

**APPENDIX B**

A. Matrix Norm Inequalities

**Lemma 1:** ( [10] Chapter 10.3 Page 61 – 62) For the symmetric $B \in \mathbb{R}^{m \times m}$ with its rank $r$ and $C \in \mathbb{R}^{m \times m}$, we have
\[
\|BC\|_F \leq \|B\|_F \|C\|_F
\]
(174)
\[
\|B + C\|_F \leq \|B\|_F + \|C\|_F
\]
(175)
\[
|\text{Tr}(B)| \leq \|B\|_F, \leq \sqrt{r} \|B\|_F,
\]
(176)

where $\|\cdot\|_F$ denotes the nuclear norm. $|\text{Tr}(B)| \leq \|B\|_F$ can be proved by $|\sum_{i=1}^{n} \lambda_i(B)| \leq \sum_{i=1}^{n} |\lambda_i(B)|$.

B. Strong Law of Large Numbers

**Lemma 2:** Kolmogorov’s Strong Law of Large Numbers( [11] Page 1166 Appendix D.7) If $x_i, i = 1, \cdots, N$ is a sequence of independent random variables such that $E(x_i) = \mu_i < \infty$ and $\text{Var}(x_i) = \sigma_i^2 < \infty$ such that $\sum_{i=1}^{\infty} \sigma_i^2 / i^2 < \infty$ as $N \to \infty$ then
\[
\frac{1}{N} \sum_{i=1}^{N} x_i - \frac{1}{N} \sum_{i=1}^{N} \mu_i \overset{a.s.}{\to} 0.
\]
(177)

C. Almost Sure Convergence of Sample Covariance Matrix

**Corollary 2:** Let $X_1, X_2, \cdots X_N$ be independent, identically distributed random vectors with mean $\mu$ and covariance matrix $\Sigma$, where $X_i \in \mathbb{R}^n$ for each $i = 1, \cdots, N$. $\mu_i$, $\mu_i \in \mathbb{R}^n$ with $\|\mu_i\|_2 < \infty$, and $\Sigma \in \mathbb{R}^{n \times n}$ with $\|\Sigma\|_F < \infty$. Then it can be seen that as $N \to \infty$,
\[
\frac{1}{N} \sum_{i=1}^{N} X_i \overset{a.s.}{\to} \mu
\]
(178)
\[
\frac{1}{N} \sum_{i=1}^{N} (X_i - \bar{X}) (X_i - \bar{X})^T \overset{a.s.}{\to} \Sigma
\]
(179)
\[
\frac{1}{N} \sum_{i=1}^{N} X_i \overset{a.s.}{\to} \mu, \frac{1}{N} \sum_{i=1}^{N} (X_i - \bar{X}) (X_i - \bar{X})^T \overset{a.s.}{\to} \Sigma,
\]
(180)

where $\bar{X} = \sum_{i=1}^{N} X_i / N$.

**Proof:** Define that the $i$th element of $\mu_i$ is $\mu_i$, the $(i,j)$th element of $\Sigma_i$ is $\sigma_{ij}$ and the $(i,j)$th $(i \neq j)$ element of $\Sigma$ is $c_{ij}$. Let $X_{i,j}$ represent the $i,j$th element of $X_i$.

According to Lemma 2 since $(X_{i,j})_{i=1}^{N}$ are i.i.d. with $\mu_j < \infty$ and $\sigma^2_j < \infty$, then it is clear that as $N \to \infty$, $\bar{X}_j \overset{a.s.}{\to} \mu_j$. Meanwhile, as $N \to \infty$, $\frac{1}{N} \sum_{i=1}^{N} (X_{i,j} - \bar{X}_j)^2 = \frac{1}{N} \sum_{i=1}^{N} X_{i,j}^2 - \left( \frac{1}{N} \sum_{i=1}^{N} X_{i,j} \right)^2$\]$$\overset{a.s.}{\to} \left( \sigma^2_j + \mu_j^2 - \sigma^2_j \right) = \sigma^2_j.$$\]
(181)

In addition, for each pair $(j,k)$ $(j \neq k)$, we can see that $\frac{1}{N} \sum_{i=1}^{N} X_{i,j} X_{i,k} = \left( \frac{1}{N} \sum_{i=1}^{N} X_{i,j} \right) \left( \frac{1}{N} \sum_{i=1}^{N} X_{i,k} \right) \overset{a.s.}{\to} (c_{jk} + \mu_j \mu_k - \mu_j \mu_k = c_{j,k}$.

Applying the results to respective elements of $X_i, i = 1, \cdots, N$. $\mu_i$ and $\Sigma_i$, then the results (178) and (179) can be obtained.

**D. Upper Bound of the Frobenius Norm of A Random Matrix**

**Corollary 3:** For a positive definite random matrix $X_\Sigma \in \mathbb{R}^{n \times n}$, if $A_\Sigma = O_p(a_\Sigma)$, the upper bound of $A_\Sigma^{-1} = O_p(1/a_\Sigma)$ can be written as
\[
\|A_\Sigma^{-1}\|_F \leq \frac{1}{a_\Sigma \lambda_1(A_\Sigma^*)} \text{cond}(A_\Sigma).
\]
(183)

**Proof:** According to the definition of Frobenius norm, we can know that
\[
\|A_\Sigma^{-1}\|_F = \sqrt{\sum_{i=1}^{n} \lambda_i^2(A_\Sigma^*)} \leq \frac{1}{a_\Sigma \lambda_1(A_\Sigma^*)} \text{cond}(A_\Sigma).
\]
(184)

**E. Expectation and Covariance of Gaussian Quadratic Forms**

**Lemma 3:** ( [12] Chapter 5.2 Page 107 – 110, [10] Chapter 8.2 Page 43) Assume that $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$. If $a \in \mathbb{R}^n$ follows the normal distribution with mean $\mu_a \in \mathbb{R}^n$ and the covariance matrix $\Sigma_a \in \mathbb{R}^{n \times n}$, i.e. $a \sim \mathcal{N}(\mu_a, \Sigma_a)$, then
\[
\mathbb{E}(a^T A a) = \text{Tr}(A^* A \Sigma_a) + \mu_a^T A \mu_a
\]
(185)
\[
\mathbb{E}(a^T A a u^T B a) = \text{Tr}(A^* A B + B^T \Sigma_a) + \mu_a^T (A + A^T) \Sigma_a (B + B^T) \mu_a + \left[ \text{Tr}(A^* A \Sigma_a) + \mu_a^T A \mu_a \right] \text{Tr}(B^* \Sigma_a) + \mu_B^T B \mu_a
\]
(186)

**F. Bounded in Probability and Convergence in Probability**

**Lemma 4:** ( [13], Page 5, Lemma 3) If $X_N = O_p(a_N)$ with $a_N$ to be positive number sequence, then for any positive number sequence $w_N$, which satisfies that as $N \to \infty$, $w_N \to \infty$, we have $X_N / w_N a_N \overset{P}{\to} 0$. Here $\overset{P}{\to}$ denotes the convergence in probability.
Proof: $X_N = O_p(a_N)$ means that $\forall \varepsilon_1 > 0, \exists L > 0$, such that

$$
P(|X_N| > a_N L) < \varepsilon_1, \quad (187)$$

$$
\lim_{N \to \infty} P(|X_N| > a_N L) \leq \varepsilon_1, \quad (188)$$

where $\lim_{N \to \infty} X_N = \lim_{k \to \infty} \sup_{n \geq k} X_N$ denotes the superior limit of the sequence. Since as $N \to \infty$, we have $\omega_N \to \infty$, which also leads to $\omega_2 \omega_N \to \infty$ for any $\omega_2 > 0$. Then for sufficiently large $N$, we always have $\omega_2 \omega_N > L$, i.e.

$$
\lim_{N \to \infty} P(|X_N| > \omega_2 \omega_N a_N) \leq \lim_{N \to \infty} P(|X_N| > L a_N) \leq \varepsilon_1. \quad (189)
$$

Thus for any fixed $\omega_2 > 0, \forall \varepsilon_1 > 0$, we have

$$
\lim_{N \to \infty} P(|X_N| / \omega_2 \omega_N a_N > \omega_2) \leq \varepsilon_1. \quad (190)
$$

If follows that $\forall \omega_2 > 0$, we have

$$
\lim_{N \to \infty} P(|X_N| / \omega_2 \omega_N a_N > \omega_2) = 0. \quad (191)
$$

Therefore, $X_N / \omega_2 \omega_N a_N \xrightarrow{p} 0$ as $N \to \infty$.

G. Asymptotic Normality of A Consistent Root

Lemma 5: ([14] Theorem 4.1.3 Page 111–112) Make the assumptions:

1. Let $\Omega$ be an open subset of the Euclidean $p$-space. (Thus the true value $\eta^*$ is an interior point of $\Omega$.)
2. $M_N(Y, \eta)$ is a measurable function of $Y$ for all $\eta \in \Omega$, and $\partial M_N / \partial \eta$ exists and is continuous in an open neighborhood $\Omega_1(\eta^*)$ of $\eta^*$.
3. There exists an open neighborhood $\Omega_2(\eta^*)$ of $\eta^*$ such that $N^{-1} M_N(\eta)$ converges to a nonstochastic function $M(\eta)$ in probability and uniformly in $\eta$ in $\Omega_2(\eta^*)$, and $M(\eta)$ attains a strict local minimum at $\eta^*$.  
4. $\partial^2 M_N / \partial \eta \partial \eta^T$ exists and is continuous in an open, convex neighborhood of $\eta^*$. 
5. $N^{-1} (\partial^2 M_N / \partial \eta \partial \eta^T)_{\eta^*}^{-1}$ converges to finite invertible $A(\eta^*) = \lim_{N \to \infty} E[N^{-1} (\partial^2 M_N / \partial \eta \partial \eta^T)_{\eta^*}^{-1}]$ in probability for any sequence $\hat{\eta}_N$ such that $\lim_{N \to \infty} \hat{\eta}_N = \eta^*$ in probability.
6. $N^{-1}(\partial M_N / \partial \eta)_{\eta^*} \xrightarrow{d} \mathcal{N}(0, B(\eta^*))$, where $B(\eta^*) = \lim_{N \to \infty} E[(\partial M_N / \partial \eta)_{\eta^*} \times (\partial M_N / \partial \eta^T)_{\eta^*}]$ in probability.

Let $\eta_N$ be the set of roots of the equation

$$
\frac{\partial M_N}{\partial \eta} = 0 \quad (192)
$$

corresponding to the local minima. Let $\{\hat{\eta}_N\}$ be a sequence obtained by choosing one element from $\eta_N$ such that $\lim \hat{\eta}_N = \eta^*$, where $\hat{\eta}_N$ can be called a consistent root. Then as $N \to \infty$,

$$
\sqrt{N}(\hat{\eta}_N - \eta^*) \xrightarrow{d} \mathcal{N}(0, A(\eta^*)^{-1} B(\eta^*) A(\eta^*)^{-1}). \quad (193)
$$

H. Convergence in Probability

Lemma 6: ([14] Theorem 4.1.5 Page 113) Suppose $M_N(\eta)$ converges in probability to a nonstochastic function $M(\eta)$ uniformly in $\eta$ in an open neighborhood of $\eta^*$. Then $\lim_{N \to \infty} M_N(\hat{\eta}) = M(\eta^*)$ if $\lim_{N \to \infty} \hat{\eta} = \eta^*$ and $M(\eta)$ is continuous at $\eta^*$.

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