Symmetries of the Renormalization Group Equations

S R Juárez W\textsuperscript{1}, P Kielanowski\textsuperscript{2} and L Vázquez M\textsuperscript{3}

\textsuperscript{1} Departamento de Física, Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, U.P. “Adolfo Lópezm Mateos”, C.P. 07738 Ciudad de México, Mexico

\textsuperscript{2} Departamento de Física, Centro de Investigación y de Estudios Avanzados, C.P. 07000 Ciudad de México, Mexico

\textsuperscript{3} Departamento de Física, Centro Universitario de Ciencias Exactas e Ingenierías. Universidad de Guadalajara Av. Revolución 1500, Colonia Olimpia C.P. 44430, Guadalajara, Jalisco, Mexico.

E-mail: rebeca@esfm.ipn.mx, kiel@fis.cinvestav.mx, liliana.vmercado@academicos.udg.mx

Abstract. The determination of the renormalization group equations in quantum field theory is a very laborious task. For example in the Standard Model the full set of these equations is known only up to two loops, while only some partial results are obtained at higher orders. We argue that one can simplify the calculation of the renormalization group equations by using the symmetry of a system. The origin of the simplification lies in the use of the invariant polynomials of the symmetry group of the theory. We consider a quantum field theory with three scalar fields that is invariant under the action of the permutation group $S_3$. We show, using the theorem of Molien, that for such a model there is a significant reduction of the amount of work needed for the derivation of the renormalization group equations.

1. Introduction

Symmetries \cite{1} have always played a very important role in the description of physical phenomena. The presence of a symmetry in a system leads to observable effects, like conservation laws and frequently introduces significant simplifications in the mathematical derivations. The group theory lies at the root of these symmetry considerations.

In quantum field theory the symmetry requires that the action be a scalar. This requires specific transformation properties of the fields under the group transformations. Let us take as an example the quantum electrodynamics \cite{2,3,4}, which describes the interactions of electrons (and positrons) with the electromagnetic field, and let us concentrate on the Poincaré invariance. The electromagnetic field transforms as a 4-vector and the electron field transforms as a bi-spinor. The QED action is equal

$$S = \int \mathcal{L} d^4x = \int \left( \bar{\psi} (i \gamma^\mu D_\mu - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) d^4x$$

and it is a Poincaré scalar. The theory is thus Poincaré invariant. In the lowest order the electron-electron scattering is described by the two Feynman diagrams shown in Figure 1. The amplitude corresponding to those diagrams is equal

$$iM = i(M_t - M_u) = -i(-e^2) \left( \frac{1}{t} \bar{u}(p_3) \gamma^\mu u(p_1) \bar{u}(p_4) \gamma_\mu u(p_2) - \frac{1}{u} \bar{u}(p_3) \gamma^\mu u(p_2) \bar{u}(p_4) \gamma_\mu u(p_1) \right),$$

where $M_t$ and $M_u$ are the tree-level and one-loop amplitudes, respectively.
Figure 1. Feynman diagrams in the lowest order for the electron-electron scattering.

from which one obtains

\[
\frac{d\sigma}{4} \sum_{\text{spins}} |M|^2 = 2e^4 \left( \frac{1}{t^2} (s^2 + u^2 - 8m^2(s + u) + 24m^4) \right. \\
+ \left. \frac{1}{u^2} (s^2 + t^2 - 8m^2(s + t) + 24m^4) + 24m^4 \frac{2}{tu}(s^2 - 8m^2s + 12m^4) \right), \\
\]

\( s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2. \)

The cross section is a Poincaré scalar and one obtains that it is only a function of the scalars \( s, t \) and \( u \). This result is obvious and could have been predicted without any calculations, due to the Poincaré invariance.

We will apply a similar line of reasoning to study the form of the renormalization group equations.

The Renormalization Group (RG) is a method in quantum field theory and statistical physics that allows the study of physical systems, at different scales, of the parameters of the theory. In quantum field theory the RG equations relate the parameters of the Lagrangian at different values of energy of the renormalization point. The convergence of the three gauge coupling constants of the Standard Model (or Minimal Supersymmetric Standard Model) \(^5\) obtained by the RG analysis is a strong argument for the Grand Unified Theories \(^6\).

The Renormalization Group Equations (RGE), in the quantum field theory, are a set of ordinary differential equations for the parameters of the Lagrangian (masses and coupling constants) and the independent variable is the renormalization point energy. They are obtained from the condition that the observables should not depend on the renormalization point energy. If \( F \) is an observable then we have the condition:

\[
F(x) = F(x, g(\mu)) = F(x, g(\mu')).
\]

Here \( g(\mu) \) is the set of the parameters of the Lagrangian, \( \mu \) and \( \mu' \) are two different renormalization point energies and \( x \) are other variables (e.g. kinematical variables).

2. Renormalization group equations

Renormalization group equations obtained from condition \(^1\) have the following generic form

\[
\mu \frac{d h_i}{d \mu} = \beta_i(g_1, \ldots, g_k), \quad i = 1, \ldots, k,
\]
where \( h_i \) are the parameters of the theory, contained in the Lagrangian (running coupling constants and masses), \( g_i \) are the coupling constants and \( \mu \) is the renormalization point energy. The functions \( \beta_i \) are specific for each theory and are determined from the Lagrangian as a perturbation series with respect to the number of loops.

Let us consider now the Standard Model (SM) as an example. The interaction Lagrangian of the SM is equal

\[
-L_{\text{int}} = \bar{e} F_L \phi^+ l + \bar{d} F_D \phi^+ q + \bar{u} H \phi^+ q + \text{h.c.} + m^2 \phi^+ \phi + \frac{\lambda}{2} (\phi^+ \phi)^2
\]

and the one loop RGEs have the following form [7]:

\[
\begin{align*}
H^{-1} \beta_H^{(1)} &= \frac{3}{2} (H^\dagger H - F_D^\dagger F_D) + Y_2(S) - \left( \frac{17}{20} g_1^2 + \frac{9}{4} g_2^2 + 8 g_3^2 \right), \\
F_D^{-1} \beta_D^{(1)} &= \frac{3}{2} (F_D^\dagger F_D - H^\dagger H) + Y_2(S) - \left( \frac{1}{4} g_1^2 + \frac{9}{4} g_2^2 + 8 g_3^2 \right), \\
F_L^{-1} \beta_L^{(1)} &= \frac{3}{2} F_L^\dagger F_L + Y_2(S) - \frac{9}{4} (g_1^2 + g_2^2), \\
Y_2(S) &= \text{Tr} \left[ 3 H^\dagger H + 3 F_D^\dagger F_D + F_L^\dagger F_L \right].
\end{align*}
\]

From Eqs. (2) and (4) one can make the following observations

(i) The RGEs depend only on the coupling constants — they do not contain kinematic variables.
(ii) At a given order the functions \( \beta_i \) are homogeneous polynomials of the coupling constants.
(iii) If the Lagrangian has a symmetry, then the RGEs are invariant polynomials that depend on the coupling constants.

The implications of the symmetry are especially important, because the properties of the invariant polynomials can reduce the number of the Feynman diagrams, which have to be calculated.

3. Invariant polynomials and the theorem of Molien

The theory of the algebraic invariants was started by A. Cayley [8] and later was extensively developed by D. Hilbert [9] and it studies the properties of the algebraic expressions, which are invariant under linear coordinate changes. Later developments are discussed in Ref. [10].

Here we will concentrate on a theorem of Molien [11], which determines the number of homogeneous polynomials under the action of a linear representation \( \rho \) of a finite group \( G \).

Definition. Let us have a linear representation \( \rho \) of a group \( G \) on a finite \( k \)-dimensional vector space \( V \) and let \( n_d \) be a number of the homogeneous, linearly independent invariant polynomials of degree \( d \) of \( k \) variables, then the Molien series is the following formal series

\[
M(t) = \sum_d n_d t^d.
\]

The Molien theorem gives an explicit formula for the function \( M(t) \).

Theorem (Molien).

\[
M(t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I - t \cdot \rho(g))}.
\]
4. Symmetric polynomials

Let us consider the set of symmetric polynomials of \( n \) variables. Then one introduces the notion of elementary symmetric polynomials \( e_i(x_1, x_2, \ldots, x_n) \) with the help of the equation

\[
\prod_{i=1}^{n} (\lambda - x_i) = \lambda^n - e_1(x_1, x_2, \ldots, x_n) \lambda^{n-1} + e_2(x_1, x_2, \ldots, x_n) \lambda^{n-2} + \cdots + (-1)^n e_n(x_1, x_2, \ldots, x_n)
\]

from which one obtains

\[
e_1(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n
\]
\[
e_2(x_1, x_2, \ldots, x_n) = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n
\]
\[
\cdots
\]
\[
e_n(x_1, x_2, \ldots, x_n) = x_1x_2\cdots x_n
\]

The following theorem holds

**Theorem** (The fundamental theorem of symmetric polynomials). Any symmetric polynomial of the variables \((x_1, x_2, \ldots, x_n)\) can be written as a polynomial of \( e_1(x_1, x_2, \ldots, x_n) \), \( e_2(x_1, x_2, \ldots, x_n) \), \ldots, \( e_n(x_1, x_2, \ldots, x_n) \) and this polynomial is unique.

This theorem facilitates the construction of the symmetric polynomials, but it does not determine the number of symmetric polynomials of a given order.

5. Field theory with three scalar fields

Let us consider a model of three quantum fields, with three coupling constants \( g_1 \), \( g_2 \) and \( g_3 \) with the permutation symmetry \( S_3 \). We will compare the structure of the RGEs for this model with that of the same model without any symmetry. As we discussed earlier the beta functions of the RGEs are expanded with respect to the number of loops and at each order they are homogeneous polynomials in the coupling constants. It means that for a full determination of the RGEs one has to calculate the coefficient for each monomial, which enters into the RGE. In the case of no symmetry we have to determine the coefficient of each monomial and in the case with symmetry we have to determine the coefficient of the invariant polynomial. This means that the presence of symmetry will reduce the number of coefficients that have to be fixed.

The order of the polynomials in the RGEs for the scalar field is equal to the number of loops+1, i.e. at one loop it is a polynomial of second order, at two loops it is of third order, etc. Thus, the number of coefficients that has to be calculated in our theory with three coupling constants at \( n \) loops is equal to the number of the monomials of type \( g_1^i g_2^j g_3^k \) with the conditions \( i + j + k = n + 1 \), \( 0 \leq i \leq n + 1 \), \( 0 \leq j \leq n + 1 \), \( 0 \leq k \leq n + 1 \) which is equal to

\[
\frac{1}{2} (n + 2)(n + 3).
\]

Now let us consider the case with the \( S_3 \) symmetry. The polynomials in the RGEs are now symmetric polynomials of \( g_1 \), \( g_2 \) and \( g_3 \) and the number of the independent polynomials of a given order can be obtained from the theorem of Molien and Eq. [6]. The matrices of the
representation of the group $S_3$ needed in Eq. (6) are equal
\[
\rho_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \rho_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \rho_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
\rho_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \rho_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \rho_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

Applying Eq. (6) one obtains
\[
M(t) = \frac{1}{6} \left( \frac{1}{(1-t)^3} + \frac{3}{(1-t^2)(1-t)} + \frac{2}{(1-t^3)} \right) = \frac{1}{(1-t)(1-t^2)(1-t^3)} = 1 + t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + 7t^6 + \cdots \quad (10)
\]

The coefficient at power $t^k$ in Eq. (10) gives the number of symmetric polynomials of $g_1$, $g_2$ and $g_3$ with a total power $k$.

Another approach to find the number of the symmetric polynomials of order $k$ of $g_1$, $g_2$ and $g_3$ is by the direct construction from three elementary symmetric polynomials
\[
e_0 = 1 \\
e_1 = g_1 + g_2 + g_3 \\
e_2 = g_1g_2 + g_1g_3 + g_2g_3 \\
e_3 = g_1g_2g_3
\]
and the number of the independent symmetric polynomials of order $k$ that can be constructed from the elementary symmetric polynomials (11) is equal
\[
1 + \left\lfloor \frac{k^2 + 6k}{12} \right\rfloor.
\]

Here $\lfloor x \rfloor$ denotes the integer part of $x$. Note that the number of symmetric polynomials of a given order obtained from Eq. (10) coincides with that obtained from (12). One should, however, stress that the method given by the theorem of Molien is more general and can be applied for any symmetry described by a final group.

In the Table 1 we compare the numbers of coefficients in the RGEs that have to be calculated in the case without symmetry, Eq. (9) and with symmetry $S_3$, Eq. (10) or (12). From this table

| number of loops | 1 | 2 | 3 | 4 | 5 |
|-----------------|---|---|---|---|---|
| power of the polynomial | 2 | 3 | 4 | 5 | 6 |
| number of monomials (no symmetry) | 6 | 10 | 15 | 21 | 28 |
| number of symmetric polynomials (with symmetry) | 2 | 3 | 4 | 5 | 7 |

we see that, at a given order, the ratio of the number of coefficients with no symmetry to the case...
with the $S_3$ symmetry is more than 3. It is increasing and asymptotically it tends to 6. It means that we have to calculate at least three time more coefficients if we do not take into account the symmetry of the theory. We thus see that the symmetry considerations can significantly simplify the calculation of the RGEs at a higher order, e.g. the number of the coefficients of the RGEs with symmetry that have to be calculated at four loops is smaller than the number of coefficients needed at one loop.

6. Conclusions and outlook

We discussed the renormalization group equations of quantum field theory with a symmetry described by a finite group. Application of a symmetry leads to a significant reduction of the necessary calculations of these equations.

In the Standard Model of elementary particles the symmetry is not discrete, but is a continuous non-Abelian gauge symmetry. The generalization of our results to the continuous case would extend the range of calculations that are technically possible. Some information about the generalization of our results to the continuous symmetry is given by the Hilbert-Weyl theorem:

Theorem (Hilbert-Weyl). Let $\Gamma$ be a compact Lie group acting on $V$. Then there exists a finite Hilbert basis for the ring $P(\Gamma)$ of the set of invariant polynomials.

The existence of a finite basis shows that the fundamental condition for a generalization of our method for the continuous symmetry is fulfilled, but the construction of such a basis is a formidable task.

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