Performance Equations for DC Commutatorless Motors Using Salient-Pole Synchronous-Type Machines

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Abstract—A large volume of research has been done to analyze commutatorless dc motors using salient-pole synchronous-type machines. These studies have not resulted in clear design-oriented expressions relating performance to machine geometry. The development of design-oriented algebraic expressions of machine performance measures in terms of machine inductances is described. Speed independent equations for commutation angle, commutating current, critical angle, safety angle, average and instantaneous torques, and instantaneous damper currents are developed. Shunt- and series-type machines are analyzed, and numerical examples of both are included.

I. INTRODUCTION

COMMUTATORLESS dc motors using synchronous-type machines have been studied over 40 years [1]. The system uses a machine that is built like a conventional polyphase synchronous motor, and it is interfaced to a dc power supply by a variable-frequency static inverter which switches the power to the appropriate stator windings of the machine. Inverter switching is controlled by signals from a rotor position sensor or an induced-voltage sensor. When a rotor position sensor is used, the combination of sensor and inverter replaces the commutator of conventional dc machines. Inverter switching is essentially controlled by rotor position, and the time the rotor is at a particular position and the inverter frequency are functions of the motor speed. Thus the machine itself is the controller, much the same as a conventional dc motor.

The development of silicon-controlled rectifiers (SCR's) produced renewed interest in synchronous-type machines used as commutatorless dc motors during the early 1960's [2], [3]. In 1964 Sato [4] published his work, and since that time considerable advancements have been made in the theory, development, and application of commutatorless dc motors. An important consideration for system performance is the permissible conduction period of the main SCR's. This conduction period can be fixed at any interval from less than 90 to 180 electrical degrees, but it is particularly advantageous to fix the interval at 120 or 180 electrical degree intervals [5].

The commutation zones of conventional dc motors are fixed by machine geometry. However, inverter-fed motors are electrically commutated by forcing the current to zero in the conducting SCR's. This can be accomplished by the use of auxiliary commutation devices in the inverter circuit, or by the back electromotive force (EMF) of the synchronous-type motor. The former method requires special logic to control the auxiliary devices, but the commutation zone can be held to acceptable limits. The latter method, called natural commutation, has a load-dependent commutation zone since the motor back EMF is a function of the load current.

The synchronous-type machine, when used as a commutatorless dc motor, can be of either the series (field current proportional to line current) or shunt (field current independent of line current) type. Series-type machines using natural commutation have been shown to have constant commutation zones for varying load conditions [6]. Shunt-type machines using natural commutation exhibit increasing commutation zones for increasing loads. This increase continues to the point where commutation is no longer possible, and the machine exhibits commutation failure [4]. The shunt-type commutatorless motor has limited overload capacity when compared to the series-type machine.

A large volume of research has been done to analyze the synchronous-type commutatorless dc motors; however, the major part of this work has emphasized the electronics of the inverter circuits. These studies have not developed clear design-oriented relationships of machine performance in terms of the winding parameters that can be used by the machine designer.

This paper presents the development of semi-explicit design-oriented algebraic relationships in terms of the machine winding inductances. The method developed by Franklin [7] is extended to the commutatorless dc motor using a salient-pole synchronous-type machine and a 120 electrical degree square-wave inverter. Inductance, rather than the reactance concept, is used to develop speed independent equations for commutation angle, commutating current, critical angle, instantaneous torques, and the safety angle. Equations are developed for shunt- and series-type machines with damper windings. The commutating inductance is identified. A numerical example for both shunt- and series-type machines is included to illustrate the performance characteristics described by the equations.

II. DESCRIPTION OF SYSTEM

Fig. 1 shows the schematic diagram for a separately excited naturally commutated commutatorless dc motor. The mechanical configuration is that of a salient-pole synchronous-type machine with wye-connected stator windings. The battery symbol represents the power supply. \( V_L \) is the source voltage, and \( I_L \) is the line current. Currents are defined...
as negative when flowing into the dotted end of a coil. The current labeled $i_k$ is a circulating current in two stator coils during commutation and is henceforth called the commutating current.

The smoothing choke is used to eliminate the ripple from the current supplied to the conventional SCR bridge inverter. Auxiliary commutation devices which may be required for starting are not shown; nor is the logic which provides gating signals for the SCR's every 120 electrical degrees.

Angular displacement of the rotor is indicated by the angle $\theta$ which has a dimension of electrical degrees. The fixed space reference, $\theta = 0$, is that rotor position where the positive field pole is aligned with the magnetic centerline of the $A$-phase winding.

The dc field current consists of a controlled value $I_f$ and a fluctuating value $i_f$ caused by voltages induced in the field circuit through mutual inductances. Direct-axis and quadrature-axis damper currents are represented by $i_d$ and $i_q$, respectively.

### III. DEFINITION OF OPERATING INTERVALS

Typical voltage and current waveshapes are essentially the same for constant field and proportional field machines. It can be seen that electromagnetic events are repeated every 60 electrical degrees of rotor travel, and that two distinct operating intervals occur within each 60 degree region. The conduction interval is defined as that interval during which only two stator phases carry current. Switching of the line current from one stator phase to another stator phase occurs during the commutation interval. Both intervals are described in detail in this section.

Referring to Fig. 2, the internal voltage $e$ is that voltage induced in the stator windings by the field magnetomotive force (MMF). The phase voltages are the total induced voltages in each of the three stator phases. Terminal voltage is the sum of the induced voltages of all current-carrying stator windings, exclusive of the smoothing choke, and is the instantaneous voltage measured from the load side of the smoothing choke to the negative side of the battery.

Phase current is the actual current carried by each stator phase. The phases that carry current and the rotor position where switching occurs are determined by the gating logic. Since the operation is periodic with 60 electrical degree regions, only one region needs to be analyzed. The specific region used in the derivations begins where the positive $C$-phase SCR (Fig. 1) has just been commutated or shut off, and the positive $A$-phase SCR has just achieved full conduction. This instant corresponds to the angle $\beta + \mu - \pi/3$ in Fig. 2. The positive $A$-phase and negative $B$-phase SCR's are now the only ones conducting, and a dc current of magnitude $I_k$ is present in the $A$- and $B$-phase stator windings. This is the conduction interval, and it extends from the angle $\beta + \mu - \pi/3$ to the angle $\beta$.

The commutation interval starts at the angle $\beta$ where a gating signal is applied to the negative $C$-phase SCR. The SCR's are assumed ideal switches, and the $C$-phase voltage at angle $\beta$ is positive with respect to the negative side of the battery, the SCR begins to conduct instantaneously, and forces the $B$- and $C$-phase terminal voltages to be equal. The $C$-phase current, which is the commutating current $i_k$ in this particular interval, now starts to increase toward the value $I_k$. Current in the $B$-phase decreases in proportion to the increase of $C$-phase current. The rate of change of the $B$- and $C$-phase currents induces a voltage in phase $B$, which reverse biases the negative $B$-phase SCR. This SCR can now be commutated when the $B$-phase current becomes zero. The angular distance the rotor travels while the $B$-phase current decreases to zero is the commutation interval measured by the commutation angle $\mu$ in electrical degrees. Therefore, the negative $B$-phase SCR is commutated at angle $\beta + \mu$ by the induced phase voltage, or back EMF, of the machine, and natural commutation has been completed. The $C$-phase current is now the line current $I_L$.

There are two criteria that must be satisfied for successful commutation. Restricting analysis to the commutation interval shown in Fig. 2, the first criterion is that commutation must be achieved before the induced $B$-phase voltage becomes more positive than the induced $C$-phase voltage. If this criterion is not met, the commutating current $i_k$ will reverse when the potential reverses, and the $B$-phase current will not go to zero. Thus the negative $B$-phase SCR will not be commutated. The next SCR to be gated, at angle $\beta + \pi/3$, is the positive $B$-phase SCR. Since the negative $B$-phase SCR has not been commutated, gating of the positive $B$-phase SCR creates a short circuit across the stator terminals, and commutation failure results. The second criterion for successful commutation is that the commutation angle $\mu$ cannot exceed 60 electrical degrees. From Fig. 2 it is seen that the positive $B$-phase SCR is gated at angle $\beta + \pi/3$. Therefore, if angle $\mu$ exceeds 60 electrical degrees commutation failure results.

The angle where the $B$- and $C$-phase voltages cross is defined as the critical angle $\phi$ in Fig. 2. The safety angle $\gamma$ is the margin of safety between the rotor position where commutation was actually achieved, $\beta + \mu$, and the rotor position beyond which natural commutation cannot be achieved: $\phi$ or $\beta + \pi/3$, whichever applies. Since the phase voltage leads the internal voltage $e$, the critical angle $\phi$ must be less than 90 degrees. Therefore, referring to Fig. 2, the angle between $\beta$ and $\phi$ can only exceed 60 degrees if $\beta$ is less than 30 degrees.
Commutation failure occurs when the safety angle $\gamma$ becomes zero or negative.

Commutatorless dc motors using synchronous-type machines and square-wave inverters do not operate in the conventional phase balanced synchronous manner. A synchronous motor operated from a symmetrical three-phase source develops a synchronously rotating MMF in the stator if only the first harmonic is considered. The 120 degree inverter-fed machine develops a discontinuous MMF [8]. During the conduction interval, between $\beta + \mu - \pi/3$ and $\beta$ in Fig. 2, the machine operates essentially as a single-phase motor, and the stator MMF is stationary in space with respect to the stator. During commutation, between $\beta$ and $\beta + \mu$, the line current switches from phase $B$ to phase $C$ causing the stator MMF to "jump" 60 electrical degrees while the rotor travels only $\mu$ degrees [7]. Assuming a constant rotor speed, the stator MMF travels at variable speed with respect to the rotor. The damper and field windings must alternately absorb and supply the MMF required to smooth the discontinuities caused by the MMF "jump." This smoothing action results in large instantaneous currents in the field and damper windings at a frequency six times the rotor frequency. Therefore, the instantaneous rotor currents are periodic with 60 electrical degree periods.

IV. ASSUMPTIONS

The following assumptions are made to simplify the derivations.

1) All circuit resistances, brush drops, and SCR drops are neglected.

2) The dc input current $I_L$ is assumed to be ripple free. This requires the use of a smoothing choke.

3) Only the first stator and rotor MMF harmonics and the first two saliency permeance harmonics are considered.

4) The magnetic circuit is linear and all inductions are independent of saturation of the main flux path, leakage flux path, or load current.

5) The damper winding is represented as two asymmetrical windings [7].

6) The mechanical moment of inertia is assumed to be so large that the motor speed is constant at a given operating point.

7) Mutual coupling inductance between stator phases is 50 percent of the single-phase leakage inductance and is independent of the rotor position.

8) Eddy current effects are neglected.

9) The external inductance of the field current source is included in the field leakage inductance.

10) Iron losses are neglected.

11) A gating signal for the SCR's is automatically available at the angle $\beta$.

12) The source voltage $V_L$ is constant.

V. DERIVATION OF EQUATIONS

Derivation of the performance equations is based on known values of dc source voltage $V_L$, controlled gating angle $\beta$, and ripple-free dc field current $I_F$. The ripple-free line current $I_L$ is the independent variable. The symbol $\theta$ represents angular displacement of the rotor in electrical degrees unless specifically defined by subscripts. Actual derivations begin with the definition of matrices and basic relationships followed by development of equations for the instantaneous rotor currents, rotor speed, commutating current, commutation angle, critical angle, and instantaneous torque.

A. Definition of Matrices and Basic Relationships

The inductance matrix shown in Fig. 3 considers the first MMF harmonic and the first two permeance harmonics. Equations for the inductances are in the literature [9], [10], [11].

Referring to Figs. 1 and 2, it is seen that the commutating current $i_k$ is zero during the conduction interval $\beta + \mu - \pi/3 \leq \theta < \beta$; therefore, the current matrices are defined separately for the conduction and commutation intervals.

Conduction ($\beta + \mu - \pi/3 \leq \theta < \beta$):

$$[1] = \begin{bmatrix} -I_L & I_L & 0 & I_F + i_f & i_d & i_q \end{bmatrix}$$

(phase A)

(phase B)

(field)

(direct damper)

(quad damper).
Commutation ($\beta < \theta < \beta + \mu$):

$$[i] = \begin{bmatrix} -I_L \\ I_L - i_k \\ i_k \\ I_f + i_f \\ i_d \\ i_q \end{bmatrix}$$

The flux interlinkage matrix is defined for the entire 60 degree region $\beta + \mu - \pi/3 < \theta < \beta + \mu$:

$$[\lambda] = \begin{bmatrix} \lambda_A \\ \lambda_B \\ \lambda_C \\ \lambda_f \\ \lambda_d \\ \lambda_q \end{bmatrix} = [L][i].$$

Induced voltages across each winding are described by the voltage matrix. The voltages $v_f$, $v_d$, and $v_q$ are zero since all resistances are neglected and the windings are considered short-circuited:

$$[v] = \begin{bmatrix} v_{A-N} \\ v_{B-N} \\ v_{C-N} \\ v_f = 0 \\ v_d = 0 \\ v_q = 0 \end{bmatrix} = \frac{d[\lambda]}{dt} = \frac{d[\lambda]}{d\theta} \frac{d\theta}{dt} = -\omega_e \frac{d[\lambda]}{d\theta}.$$

The instantaneous terminal voltage ($v_t$) is the sum of the instantaneous induced voltages of all series-connected stator windings measured on the load side of the smoothing choke. Considering winding polarities, during the conduction interval $\beta + \mu - \pi/3 < \theta < \beta$,

$$v_t = (v_{A-N} - v_{B-N}).$$

During the commutation period $\beta < \theta < \beta + \mu$, the $B$- and $C$-phase terminal voltages are equal, and

$$u_t = (v_{A-N} - v_{B-N}) = (v_{A-N} - v_{C-N})$$

(2) Substituting (4) into (6) yields, for the entire 60 degree region,

$$u_t = -\omega_e \frac{d}{d\theta} (\lambda_A - \lambda_B).$$

(7)

From Fig. 2 it is seen that the average terminal voltage on the load side of the smoothing choke is the source voltage $V_L$. Therefore, the average terminal voltage over the region of interest is

$$V_L = (3/\pi) \int_{\beta + \mu - \pi/3}^{\beta + \mu} v_t \, d\theta$$

$$= (3/\pi) \int_{\beta + \mu - \pi/3}^{\beta + \mu} [v_{A-N} - v_{B-N}] \, d\theta$$

$$= -(3\omega_e/\pi) \int_{\beta + \mu - \pi/3}^{\beta + \mu} \left[ \frac{d\lambda_A}{d\theta} - \frac{d\lambda_B}{d\theta} \right] \, d\theta$$

$$= -(3\omega_e/\pi) \left[ (\lambda_A - \lambda_B)(\beta + \mu) - (\lambda_A - \lambda_B)(\beta + \mu - \pi/3) \right]$$

(8)

Equation (8) is important for the derivations that follow since it contains $\omega_e$. Another important relationship follows from the assumptions of steady state, zero resistance, short-circuited field and damper windings, and constant flux interlinkages. The short-circuited winding currents are periodic, have the same value at the beginning and end of each period, and average zero; therefore,

$$\int_{\beta + \mu - \pi/3}^{\beta + \mu} i_d \, d\theta = \int_{\beta + \mu - \pi/3}^{\beta + \mu} i_d \, d\theta = \int_{\beta + \mu - \pi/3}^{\beta + \mu} i_f \, d\theta = 0.$$  

(9) The final basic relationship is that the shaft power output
must equal the electrical input power:

\[ T_{AVG} = V_L I_L/\omega_m \text{ N·m.} \] (10)

**B. Instantaneous Currents of Short-Circuited Rotor Windings**

After expanding (3) and applying some elementary trigonometric identities the following flux linkage relationships are obtained:

\[ \lambda_q = \sqrt{3} I_L M_q \sin (\theta + \pi/6) - \sqrt{3} i_q M_q \]
\[ \times \cos (\theta) + i_q (L_q + l_q) \] (11)

\[ \lambda_d = -\sqrt{3} I_L M_d \cos (\theta + \pi/6) - \sqrt{3} i_k M_d \cos (\theta - \pi/2) \]
\[ + (L_f + i_f) M_{fd} + i_d (L_d + I_f) \] (12)

\[ \lambda_f = -\sqrt{3} I_L M_f \cos (\theta + \pi/6) - \sqrt{3} i_k M_d \cos (\theta - \pi/2) \]
\[ + (L_f + i_f) (L_f + I_f) + i_d M_{fd} \] (13)

Since the instantaneous voltages of the short-circuited windings are zero,

\[ v_q = v_d = v_f = \frac{-d\lambda_q}{dt} = \frac{-d\lambda_d}{dt} = \frac{-d\lambda_f}{dt} = 0, \] (14)

and \( \lambda_q, \lambda_d, \) and \( \lambda_f \) are constant with time, supporting the constant flux interlinkage concept.

Equation (11) can be evaluated at \( \beta + \mu - \pi/3 \) and set equal to (11) at \( \theta \). Remembering that \( i_k \) is zero at \( \beta + \mu - \pi/3 \), and letting \( i_{q0} \) be the current in the quadrature-axis damper winding at the beginning of the region,

\[ \lambda_q (\beta + \mu - \pi/3) = 3I_L M_q \sin (\beta + \mu - \pi/6) + i_{q0} (L_q + I_q). \] (15)

Setting (15) equal to (11) and solving for the current over the entire 60 electrical degree region yields

\[ i_q = \sqrt{3} I_L K_q \left[ \sin (\beta + \mu - \pi/6) - \sin (\theta + \pi/6) \right] \]
\[ + \sqrt{3} i_k K_q \cos (\theta) + i_{q0} \] (16)

with

\[ K_q = M_q/(L + I_q). \] (17)

The same approach is used to obtain expressions for the instantaneous field current and the direct-axis damper current:

\[ i_d = \sqrt{3} I_L K_d \left[ \cos (\theta + \pi/6) - \cos (\beta + \mu - \pi/6) \right] \]
\[ + \sqrt{3} i_k K_d \sin (\theta) + i_{d0} \] (18)

\[ i_f = \sqrt{3} I_L K_f \left[ \cos (\theta + \pi/6) - \cos (\beta + \mu - \pi/6) \right] \]
\[ + \sqrt{3} i_k K_f \sin (\theta) + i_{f0} \] (19)

with

\[ K_d = M_d (L + I_f) / M_{fd} \]
\[ \left[ (L_d + l_d) (L_f + l_f) - (M_{fd})^2 \right] \] (20)

and

\[ K_f = M_d (L_d + I_d) / M_{fd} \]
\[ \left[ (L_d + l_d) (L_f + l_f) - (M_{fd})^2 \right]. \] (21)

The initial conditions \( i_{q0}, i_{d0}, \) and \( i_{f0} \) are evaluated by substituting (16), (18), and (19) into (9) and recalling that the commutating current \( i_k \) is defined only for the region \( \beta \leq \theta \leq \beta + \mu \). Substituting (19) into (9) yields

\[ \sqrt{3} I_L K_f \int_{\beta + \mu - \pi/3}^{(\beta + \mu)} \cos (\theta + \pi/6) d\theta \]
\[ - \sqrt{3} I_L K_f \cos (\beta + \mu - \pi/6) \int_{\beta + \mu - \pi/3}^{(\beta + \mu)} d\theta \]
\[ + \sqrt{3} K_f \int_{\beta + \mu - \pi/3}^{(\beta + \mu)} i_k \sin (\theta) d\theta + i_{f0} \int_{\beta + \mu - \pi/3}^{(\beta + \mu)} \]
\[ \times \cos (\theta) d\theta = 0. \] (22)

Since the nature of \( i_k \) is not known, the following simplifying terms are defined:

\[ W = \int_{\beta}^{(\beta + \mu)} i_k \sin (\theta) d\theta \] (23)

and

\[ V = \int_{\beta}^{(\beta + \mu)} i_k \cos (\theta) d\theta. \] (24)

After integrating, (22) is solved for \( i_{f0} \)

\[ i_{f0} = \sqrt{3} I_L K_f \left[ \cos (\beta + \mu - \pi/6) \right] \]
\[ - (3/\pi) \cos (\beta + \mu) \right] \]
\[ - 3 \sqrt{3} W K_f / \pi. \] (25)

The terms \( i_{d0} \) and \( i_{q0} \) are found in the same manner as \( i_{f0} \). These results are substituted in (16), (18), and (19) to find the expressions for the instantaneous currents of the short-circuited windings. These equations apply to both the conduction and commutation regions:

\[ i_d = \sqrt{3} I_L K_d \left[ \cos (\theta + \pi/6) - (3/\pi) \cos (\beta + \mu) \right] \]
\[ + \sqrt{3} K_d \left[ i_k \sin (\theta) - 3W / \pi \right] \] (26)

\[ i_q = \sqrt{3} I_L K_q \left[ (3/\pi) \sin (\beta + \mu) - \sin (\theta + \pi/6) \right] \]
\[ + \sqrt{3} K_q \left[ i_k \cos (\theta) - 3V / \pi \right] \] (27)
\[ i_f = \sqrt{3} I_L K_f [\cos (\theta + \pi/6) - (3\pi/2) \cos (\beta + \mu)] \]
\[ + \sqrt{3} K_f [i_k \sin (\theta) - W/3] \tag{28} \]

Rotational speed is derived from (8). Expansion of (3), after some trigonometric transformations, yields
\[ \lambda_d = -(3/2) I_L (L_0 + l_0) - \sqrt{3} I_L L_{00} \cos (2\theta + \pi/6) \]
\[ - \sqrt{3} i_k L_{00} \cos (2\theta - \pi/2) + [(I_f + i_f)M_f + i_d M_d] \cdot \cos (\theta - i_q M_q \sin (\theta)) \tag{29} \]
\[ \lambda_B = (3/2) I_L (L_0 + l_0) - \sqrt{3} I_L L_{00} \cos (2\theta - \pi/2) \]
\[ - (3/2)i_k (L_0 + l_0) - \sqrt{3} i_k L_{00} \cos (2\theta + 5\pi/6) \]
\[ + [(I_f + i_f)M_f + i_d M_d] \cdot \cos (\theta - 2\pi/3) - i_q M_q \sin (\theta - 2\pi/3) \tag{30} \]
\[ \lambda_C = -\sqrt{3} I_L L_{00} (2\theta + 5\pi/6) + (3/2) i_k (L_0 + l_0) \]
\[ - \sqrt{3} i_k L_{00} \cos (2\theta + \pi/6) + [(I_f + i_f)M_f + i_d M_d] \cdot \cos (\theta - 4\pi/3) - i_q M_q \sin (\theta - 4\pi/3). \tag{31} \]

After making the following simplifying reductions
\[ M_0 = K_f M_f + K_d M_d \]
\[ = M_f^2 (L_d + l_d) + M_d^2 (L_f + l_f) - 2M_d M_f M_{fd} \]
\[ (L_d + l_d)(L_f + l_f) - (M_{fd})^2 \]
\[ M_{00} = K_q M_q = \frac{(M_q)^2}{(L_q + l_q)} \tag{32} \]
\[ \Lambda_f = (M_0 + M_{00}) \tag{33} \]
\[ \Lambda_d = (M_0 - M_{00}) \tag{34} \]
\[ \Delta_0 = [2(L_0 + l_0) - \Lambda_f] \tag{35} \]
\[ \Delta_{00} = [2L_{00} - \Lambda_d]. \tag{36} \]

Equations (29) and (30) are substituted into (8) to yield
\[ V_L = (3\omega_{e N L} \pi) [\sqrt{3} I_f M_f \sin (\beta + \mu) + (3/2) I_f L_{00} \cos (\beta + \mu)] \cdot \cos (\theta - \pi/2 - \beta/2) \]
\[ + I_f L_\Lambda_d \sin (\beta + \mu) \tag{37} \]
\[ + [M_0 W \sin (\beta + \mu) + M_{00} V \cos (\beta + \mu)] \], \tag{38} \]

which is solved for the rotational velocity
\[ \omega_e = V_L \pi / [3\sqrt{3} I_f M_f \sin (\beta + \mu) - (3/4) \Lambda_d L_\Delta \]
\[ + (3/2) I_f L_{00} \cos (\beta + \mu) - (9/2) I_f L_\Lambda_d \cdot \sin (\beta + \mu) \cdot (9/\pi) [M_0 W \sin (\beta + \mu) + M_{00} V \cos (\beta + \mu)] \] \tag{39} \]

The terms \( \Lambda_f, \Lambda_d, \Delta_0, \) and \( \Delta_{00} \) were shown by Franklin [7] to be functions of the individual machine leakage inductances. The term \( \Lambda_f \) is the sum of the rotor leakage inductances, and \( \Lambda_d \) is the difference of the rotor leakage inductances. First permeance harmonic stator inductances are combined with the rotor inductances by the term \( \Delta_0, \) and \( \Delta_{00} \) combines the second permeance harmonic inductances of the stator with the rotor inductances.

Equating the line current \( I_L \) to zero in (39) also causes the terms \( W, V, \) and \( \mu \) to become zero. The resulting equation defines the no-load speed:
\[ \omega_{e N L} = \frac{V_L \pi}{3\sqrt{3} I_f M_f \sin (\beta)}. \tag{40} \]

Equation (40) clearly shows the effect of the field current \( I_f \), the gating angle \( \beta \), and the source voltage \( V_L \) on the no-load speed. The relationship is very similar to a conventional shunt dc motor, particularly if the angle \( \beta \) is compared to the displacement of the brushes from the main pole axis.

C. Commutating Current and Commutation Angle

During the commutation interval the terminals of phases \( B \) and \( C \) are forced to the same potential. Therefore, phase voltages \( v_B \) and \( v_C \) are equal. This does not imply that \( \lambda_B \) and \( \lambda_C \) have equal values, but their derivatives are equal. The difference between \( \lambda_B \) and \( \lambda_C \) is constant throughout the commutation interval; hence
\[ \Delta_{B C} = (\lambda_B - \lambda_C)(\beta = 0 < \beta + \mu) = \text{constant}. \tag{41} \]

Substituting (30) and (31) into (41) and solving for the commutating current \( i_k \), recalling that the value of \( i_k \) at \( \theta = \beta \) is zero, yields
\[ i_k = \frac{1}{[\Delta_0 - \Delta_{00} \cos (2\theta)]}
\[ + (6/\pi)(M_0 w \sin (\beta) - \sin (\theta)) + M_{00} V \cos (\beta) - \sin (\theta + \beta + \mu) \cdot \Lambda_f \sin (\mu) + \sin (\theta - \beta - \mu)] \]
\[ + I_f L_{00} [\sin (\theta + \beta + \mu) - \Lambda_f \sin (\mu)] \cdot [\sin (\theta + \beta + \mu)] \cdot [\sin (\theta - \beta - \mu)]. \tag{42} \]

The commutation angle \( \mu \) is found by evaluating (42) at the angle \( \beta + \mu \). By definition, the commutating current \( i_k \) equals the line current \( I_L \) at the angle at which commutation is achieved. After some trigonometric transformations and rearranging of terms
\[ \sqrt{3} (I_f / I_L) M_f \sin (\beta) - (3/2) I_f L_{00} \sin (\theta + \pi/6) + 3 \Delta_0 / 2 \]
\[ = \sqrt{3} (I_f / I_L) M_f \sin (\beta + \mu) + (3 \sqrt{3}/2) \Delta_{00} \cdot \cos (\beta + \mu) - \sin (\beta + \mu) - \Lambda_f \sin (\mu) / \pi - (9/2) I_f L_{00} \sin (\beta + \mu) - \sin (\beta + \mu) + M_{00} V \cos (\beta + \mu) / \pi. \tag{43} \]
This equation shows the commutation angle to be independent of speed and voltage. The left side of (43) is constant for a given gating angle \( \beta \), excitation \( I_f \), and load \( I_L \). Commutation is influenced by the ratio of field current to line current \( I_f/I_L \), the gating angle \( \beta \), and the commutating inductance \( \Delta_0 \) which is independent of all operating variables.

### D. Critical and Safety Angles

Referring to Fig. 2, it can be seen that the critical angle \( \phi \) is in the region \( \beta + \mu < \phi < \beta + \mu + \pi/3 \). Since commutation from phase \( B \) to phase \( C \) has been completed, and commutation from phase \( A \) to phase \( B \) has not been initiated prior to angle \( \phi \), a new current matrix is defined:

\[
[i](\phi) = \begin{bmatrix}
-I_L \\
0 \\
I_L \\
I_f + i_f \\
i_d \\
i_q \\
\end{bmatrix}.
\] (44)

The critical angle \( \phi \) is defined as the rotor position where the \( B \) - and \( C \)-phase induced voltages are equal; therefore

\[
\frac{d}{d\theta} (\lambda_B - \lambda_C)(\phi) = 0.
\] (45)

The flux linkage matrix (3) is expanded with (44) inserted for the current matrix, and the new boundaries, to find an equation describing the critical angle \( \phi \):

\[
\frac{d}{d\theta} (\lambda_B - \lambda_C)(\phi) = 0 = -3I_L\Delta_0 \cos(2\phi - \pi/6) + \sqrt{3} I_f M_f \cos(\phi) + (3/\pi)(M_{10} \sin(\phi) [I_L \sin(\beta + \mu) + V] - M_0 \cos(\phi) [I_L \cos(\beta + \mu) + W]).
\] (46)

The terms \( W \) and \( V \) are exactly those defined by (23) and (24) due to the periodic nature of the instantaneous currents. The previous description of the safety angle \( \gamma \) leads to the following:

\[
\gamma = \phi - \beta - \mu, \quad (\phi < \beta + \pi/3)
\] (47)

\[
\gamma = (\pi/3) - \mu, \quad (\phi \geq \beta + \pi/3).
\] (48)

If the value of the safety angle \( \gamma \) becomes zero or negative, commutation failure results.

### E. Instantaneous Torque

Applying the principle of conservation of energy to the specific machine being investigated [12],

\[
\text{electrical energy supplied} = \left\{ \begin{array}{ll}
\text{change in stored magnetic energy of machine and choke} \\
\text{change in stored mechanical energy} \\
\text{mechanical output}
\end{array} \right\}
\] (49)

Recalling the initial assumptions, (49) can be expressed as

\[
dW_e = dW_f + dW_m,
\] (50)

and since the stored mechanical energy is constant at constant speed,

\[
V_L I_L dt = dW_f(\text{CHOKE}) + dW_f/d\theta_m d\theta_m + T_{\text{INST}} d\theta_m.
\] (51)

The average input energy over the region of interest is

\[
(3/\pi) \int_{\beta + \mu - \pi/3}^{\beta + \mu} V_L I_L dt d\theta
\] (52)

Since steady-state operation is assumed (51), (52), and (10) are combined as

\[
V_L I_L = \frac{dW_f(\text{CHOKE})}{dt} + \frac{dW_f}{d\theta_m} + T_{\text{INST}} \frac{d\theta_m}{dt} = T_{\text{AVG}} \omega_m
\] (53)

which is rearranged as

\[
T_{\text{INST}} = T_{\text{AVG}} - (1/\omega_m) \frac{dW_f(\text{CHOKE})}{dt} - \frac{dW_f}{d\theta_m}.
\] (54)

The magnetic field energies of the machine and smoothing choke are described as

\[
W_f = (1/2) [I]^T [L] [I] - (1/2) [I]^T [\lambda]
\] (55)

and

\[
W_f(\text{CHOKE}) = (1/2) I_L \lambda(\text{CHOKE}).
\] (56)

Since the line current \( I_L \) is assumed constant at a particular
operating point
\[
\frac{d}{dt} W_{(CHOK)} = (I_L/2) \frac{d}{dt} \lambda_{(CHOK)}.
\] (57)

Furthermore,
\[
\frac{d}{dt} \lambda_{(CHOK)} = \nu(t)_{(CHOK)} = V_L - (\nu_A - N - \nu_B - N)
\] (58)

and (54) becomes
\[
T_{INST} = T_{AVG} - (1/\omega_m)(I_L/2)[V_L - \nu_A + \nu_B] - \frac{dW_f}{d\theta_m}
\] (59)

Substituting (4) and (10) into (59) yields
\[
T_{INST} = (T_{AVG}/2) - (I_L/2) \frac{d}{d\theta_m} (\lambda_A - \lambda_B) - \frac{dW_f}{d\theta_m}
\] (60)

Using (2) and (3) and applying the boundary conditions to expand (60) yields
\[
T_{INST} = (V_L/2)(I_L/\omega_m) + (p/2) \left[ \lambda_B - \lambda_C \right] \frac{d\theta}{d\theta_m}
\] (61)

After doing the indicated operation on (61) the desired algebraic relationship is obtained:
\[
T_{INST} = (V_L/2)(I_L/\omega_m) + (p/2)
\]
\[
\cdot \left[ \sqrt{3} I_d A M_f \sin (\theta + \pi/6) - (9I_L/\pi) \right]
\]
\[
\cdot ((I_L/2)[L_d \sin (\theta - \beta + \mu + \pi/6)] + \lambda_d \sin (\theta + \beta + \mu/6) - M_0 W \sin (\theta + \pi/6)
\]
\[
+ M_0 V \cos (\theta + \pi/6)] + i_k(-\sqrt{3} I_d A M_f \cos (\theta)
\]
\[
+ (9/\pi)(I_L/2)[L_f \cos (\theta - \beta - \mu) + \lambda_f \cos (\theta + \beta + \mu)]
\]
\[
+ (9/\pi)(M_0 W \cos (\theta) - M_0 V \sin (\theta))
\]
\[
+ \frac{di_k}{d\theta_m}(I_L[(3\Delta_0/4) - (3\Delta_{00}/2) \sin (2\theta + \pi/6)]
\]
\[
+ (3i_k/2)[\Delta_{00} \cos (2\theta - \Delta_0)]
\] (62)

This is the final equation for instantaneous torque. It is valid for both conduction and commutation intervals.

The formal derivation of performance equations is now complete. If the machine inductances, the source voltage \(V_L\), line current \(I_L\), field current \(I_f\), and gating angle \(\beta\) are known, the following equations can be used to determine performance: (23), (24), (26), (27), (28), (39), (42), (43), (46), and (62). These equations are valid for the series-type machine as well as the shunt-type machine.

VI. SOLUTION METHOD

All of the derived equations contain the terms \(W\) and \(V\), (23) and (24), respectively, which can be evaluated iteratively with a computer using the known values for the machine inductances \(I_L\), \(I_f\), and \(\beta\). The first step is to iteratively solve (43) for \(\mu\), with \(W\) and \(V\) equal to zero. After substitution of (42), (23) and (24) are numerically integrated using the initial value of \(\mu\). Next, using the initial value for \(\mu\), and the new values for \(W\) and \(V\), (42) is evaluated at \(\theta = \beta + \mu\) and compared to \(I_L\). The process is repeated with updated values until \(i_k = I_L\) at \(\theta = \beta + \mu\). At this point \(\mu\), \(W\), and \(V\) are known, and the other equations are evaluated in a straightforward manner.

VII. EXAMPLE

A typical four-pole commercial synchronous-type motor of the 1000 hp class is used for this example. The calculated machine inductances, with the dimension of Henries, are
\[
\begin{align*}
L_q &= 9.334 \times 10^{-5} \\
L_q &= 9.09 \times 10^{-4} \\
l_0 &= 5.42 \times 10^{-6} \\
l_0 &= 6.171 \times 10^{-5} \\
L_{00} &= 3.536 \times 10^{-5} \\
M_f &= 1.684 \times 10^{-2} \\
M_f &= 3.29 \times 10^{-4} \\
I_f &= 1.20 \times 10^{-1} \\
M_q &= 2.24 \times 10^{-4} \\
L_d &= 8.842 \times 10^{-4} \\
M_{fd} &= 4.525 \times 10^{-2}
\end{align*}
\]

The following quantities, defined as reference values, are used as norms for the normalized display of the results;

- torque: \(T_R = 3259 \text{ N\cdot m}\),
- speed: \(N_R = 1760 \text{ r/min}\),
- line current: \(I_{LR} = 2400 \text{ A}\),
- field current: \(I_{FR} = 45.3 \text{ A}\),
- source voltage: \(V_R = 250 \text{ V}\).

These values are not necessarily the rated values for the machine being analyzed.

The machine was analyzed as both a series- and shunt-type motor. Source voltage was set at 250 V, and the line current was adjusted so the ratio \(I_L/I_{LR}\) changed in increments of 0.2. As a shunt-type motor, the field current was fixed at 45.3 A. The ratio \(I_f/I_L\) was held constant when analyzing the series-type motor.

Fig. 4 shows the commutation angle \(\mu\) as a function of the line current. The shunt-type motor was unable to achieve commutation when the current ratio exceeded 1.2. Since the
current ratio was increased in increments of 0.2, the curve shows that commutation failure occurred between 1.2 and 1.4. The critical angle $\phi$ is shown in Fig. 5. Of particular interest is the shift of $\phi$ with respect to the $A$-phase centerline $\theta = 0$ as the line current, or load, increases for the shunt-type motor. This is, of course, indicating the load-related phase shift between the terminal voltage and field generated EMF of the synchronous machine. A similar shift does not occur in the series-type motor as the net air gap MMF remains constant over the load range.

Rotor speed as a function of the line current is shown in Fig. 6. These curves clearly reflect the typical characteristics of dc motors. As expected from (40), removing the load from the series-type motor causes the speed to increase toward infinity.

Fig. 7 shows average torque as a function of line current. These curves also show a close resemblance to those of conventional dc motors.

**VIII. CONCLUSION**

Semi-explicit design-oriented algebraic equations that describe the performance characteristics of commutatorless dc motors in terms of the actual winding inductances have been developed. The use of actual inductances instead of the reactance concept has enabled the development of speed independent equations for some of the important performance characteristics. For specified performance characteristics, the designer can solve the equations for the required inductance values and subsequently determine the required winding geometry and magnetic circuit details. The final equations can be calculated on an inexpensive small computer with limited storage capability.

**NOMENCLATURE**

- $e_{A,B,C}$: Internal voltage, V.
- $I_f$: DC field current, A.
- $I_{fR}$: Reference field current, A.
- $I_L$: DC input line current, A.
- $I_{L,R}$: Reference line current, A.
- $i_d$: Instantaneous direct-axis damper current, A.
- $i_f$: Instantaneous fluctuating field current, A.
- $i_k$: Commutating current, A.
- $i_q$: Instantaneous quadrature-axis damper current, A.
- $i_{d0}$, $i_{f0}$, $i_{q0}$: Field and damper currents at $\theta = (\beta + \mu - \pi/3)$, A.
- $K_d$, $K_f$, $K_q$: Constants.
- $L_i$: Inductance, H.
- $M_l$: Mutual inductance, H.
- $M_{01}$, $M_{00}$: Inductance, H.
- $N$: Speed, r/min.
- $N_R$: Reference speed, r/min.
- $P$: Number of pole pairs.
- $T_{AVG}$: Average torque, N.m.
- $T_{INST}$: Instantaneous torque, N.m.
- $\tau$: Time, s.
- $V$: Constant.
- $V_L$: DC input voltage, V.
- $v_{(A,B,C)}$: Instantaneous phase voltages, V.
- $v_{f,d,q}$: Instantaneous winding voltages, V.
- $v_r$: Instantaneous terminal voltage, V.
- $W$: Constant.
- $W_e$, $W_f$, $W_m$: Energy, J.
- $\beta$: Angle where commutation starts, rad.
- $\gamma$: Safety angle, rad.
- $\Delta_0$, $\Delta_00$: Machine constants, H.
- $\theta_e$: Electrical displacement of rotor, rad.
- $\theta_m$: Mechanical displacement of rotor, rad.
- $\lambda_{A,B,C,d,f,q}$: Machine constants, H.
\[
\begin{align*}
\mu & \quad \text{Commutation angle, rad.} \\
\phi & \quad \text{Critical angle, rad.} \\
\omega_e & \quad \text{Electrical angular velocity rad/s,} \\
\omega_m & \quad \text{Mechanical angular velocity, rad/s.}
\end{align*}
\]

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