A Combinatorial Formula for Principal Minors of a Matrix with Tree-metric Exponents and Its Applications

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Abstract

Let $T$ be a tree with a vertex set $\{1, \ldots, N\}$. Denote by $d_{ij}$ the distance between vertices $i$ and $j$. In this paper, we present an explicit combinatorial formula of principal minors of the matrix $(t^{d_{ij}})$, and its applications to tropical geometry, study of multivariate stable polynomials, and representation of valuated matroids. We also give an analogous formula for a skew-symmetric matrix associated with $T$.

1 Introduction

Let $T = (V,E)$ be a tree, where $V = \{1, 2, \ldots, N\}$. For $i, j \in V$, denote by $d_{ij}$ the number of edges of the unique path between $i$ and $j$ in $T$. With an indeterminate $t$, define the matrix $A = (a_{ij})$ by

$$a_{ij} = t^{d_{ij}} \quad (i, j \in V).$$

Yan and Yeh [26] considered the matrix $A$ and showed that \det $A$ is given by the following simple formula:

**Theorem 1.1** (Yan–Yeh [26]). \det $A = (1 - t^2)^{N-1}$.

Our main result can be understood as an extension of Yan–Yeh’s formula to principal minors of $A$. The motivation of our investigation, however, comes from study of multivariate stable polynomials [5, 6, 8], tropical geometry [10, 24], and representation of valuated matroids [12, 14]. To state our result, let us introduce some notions. For $X \subseteq V$, denote by $A[X]$ the principal submatrix of $A$ consisting of $a_{ij}$ for $i, j \in X$. We say that a forest $F = (V_F, E_F)$ is spanned by $X$ if $X \subseteq V_F$ and all leaves of $F$ are contained in $X$. Note that the subtree of $T$ spanned by $X$ is the unique minimal subtree including $X$, which is denoted by $T_X = (V_X, E_X)$. Define $c(F)$ as the number of connected components of $F$. Denote by $\deg_F(v)$ the degree of a vertex $v$ in $F$. Then our main result is the following:

**Theorem 1.2.**

$$\det A[X] = \sum_F (-1)^{|X| + c(F)} t^{2|E_F|} \prod_{v \in V_F \setminus X} (\deg_F(v) - 1),$$

where the sum is taken over all subgraphs $F$ of $T$ spanned by $X$. In particular, the leading term is given by

$$(-1)^{|X| + 1} t^{2|E_X|} \prod_{v \in V_X \setminus X} (\deg_{T_X}(v) - 1).$$

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In the case $X = V$, the formula (1.1) coincides with the binomial expansion of Yan-Yah’s formula.

Our formula brings a strong consequence on the signature of $A[X]$. Recall that the signature of a symmetric matrix is a pair $(p, q)$ of the number $p$ of positive eigenvalues and the number $q$ of negative eigenvalues. When we substitute a sufficiently large value for $t$, the sign of $\det A[X]$ is determined by the leading term. By (1.2), $\det A[X] > 0$ if $|X|$ is odd, and $\det A[X] < 0$ if $|X|$ is even. From Sylvester’s law of inertia, the number of sign changes of leading principal minors is equal to the number of negative eigenvalues (see [16, Theorem 2 in Chapter X]). Therefore the signature of $A[X]$ is $(1, |X| - 1)$. This argument works on the field $\mathbb{R}\{t\}$ of Puiseux series (defined in Section 2). Thus we have the following.

**Corollary 1.3.** The signature of $A[X]$ is $(1, |X| - 1)$.

In particular, $A[X]$ is nonsingular and defines the Minkowski inner product, i.e., a nondegenerate bilinear form with exactly one positive eigenvalue.

We also consider a skew-symmetric matrix $B = (b_{ij})$ defined by

$$b_{ij} = -b_{ji} = t^{\delta_{ij}} \quad (i < j).$$

Denote by $B[X]$ the principal submatrix of $B$ as above. In contrast with the symmetric case, the Phaffian $\text{Pf} B[X]$ depends on the ordering of $X$. We give a simple formula for the case where $X$ has a special ordering, though we do not know such a formula for general case. A vertex subset $X = \{i_1, i_2, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$ is said to be nicely-ordered (with respect to a given tree $T$) if the tour $i_1 \to i_2 \to \cdots \to i_k \to i_1$ in $T$ passes through each edge in $T$ at most twice. An edge $e$ of $T$ is said to be odd (with respect to $X$) if both two components obtained by the removal of $e$ from $T$ include an odd number of vertices in $X$. Let $O_X \subseteq E$ be the set of odd edges.

**Theorem 1.4.** If $X$ is nicely-ordered and $|X|$ is even, then

$$\text{Pf} B[X] = t^{|O_X|}. \quad (1.3)$$

The both formulas are easily generalized to a tree metric, that is, a dissimilarity matrix that can be embeddable to an edge-weighted tree. More precisely a dissimilarity matrix is a nonnegative symmetric matrix $D \in \mathbb{Q}^{n \times n}$ with zeros on diagonal, and a tree metric is a dissimilarity matrix $D$ such that there are a tree $T = (V, E)$, a positive edge weight $l$ on $E$, and a map $\varphi : \{n\} \to V$ such that $D_{ij}$ is equal to the sum of weight of edges of the unique path between $\varphi(i)$ and $\varphi(j)$. In this case, if $\varphi$ is injective, then we can regard $\{n\} \subseteq V$, and the formula of $\det(t^{D_{ij}})$ is obtained by replacing $|E_F|$ and $|O_X|$ by weighted sums $\sum_{e \in E_F} l(e)$ and $\sum_{e \in O_X} l(e)$, respectively. (If $\varphi$ is not injective, then $\det(t^{D_{ij}}) = 0$). The well-known tree metric theorem [7] says that a dissimilarity matrix $D = (D_{ij})$ is a tree metric if and only if $D$ satisfies

$$[4\text{PC}] \quad D_{ij} + D_{kl} \leq \max\{D_{ik} + D_{jl}, D_{il} + D_{jk}\} \quad (i, j, k, l \in \{n\}).$$

This condition is called the four-point condition [4PC]. A symmetric matrix $W = (w_{ij}) \in \mathbb{Q}^{n \times n}$ satisfying [4PC] (not necessarily a dissimilarity matrix) can be represented with a tree metric $D = (D_{ij})$ and a vector $p = (p_i) \in \mathbb{Q}^n$ (defined by $p_i := w_{ii}/2$) as

$$w_{ij} = D_{ij} + p_i + p_j. \quad (1.4)$$

Then we have

$$\det(t^{w_{ij}}) = t^2 \sum_{k=1}^n p_k \det(t^{D_{ij}}). \quad (1.5)$$
Therefore our formulas are applicable to matrices with exponents satisfying [4PC].

The organization of this paper is as follows. In Section 2 we present applications of the formulas. The space of tree metrics (called the space of phylogenetic trees in [3]), and related spaces arise ubiquitously from the literature of tropical geometry: examples include the tropical grassmannian of rank 2 [24], the Bergman fan of the matroid of a complete graph [1], and the space of matrices with tropical rank 2 [10]. In Section 2.2 we present yet another appearance of the space of tree metrics from tropicalization of the space of Hermite matrices of signature (1, n − 1). This type of matrices also has interest from theory of multivariate stable polynomials [8] Theorem 5.3. Recent studies [5, 6, 8] explored an interesting link between stable polynomials, matroids, and related discrete concave functions. In Section 2.2 utilizing the formula (1.1), we establish a correspondence between tree metrics (D_{ij}) and quadratic stable polynomials z^\top D_{ij} z in R\{t\}. Our formula also sheds a new insight on the dissimilarity map X \mapsto \left| E_X \right| of a tree T = (V, E) [22]. The dissimilarity map of a tree has a significance in phylogenetic analysis as well as has interests from tropical geometry and representation of valuated matroids [9, 17, 19]. Observe that our leading term formula (1.2) gives a new type of representation of the dissimilarity map by the degree of principal minors of a symmetric matrix. In Section 2.3 we address this subject. In Section 3 we prove Theorems 1.2 1.3

2 Applications

To describe applications of our formulas, let us recall the notion of Puiseux series. A Puiseux series in the indeterminate t and a field K(= R, C) is a formal series of the form \( \sum_{i=0}^{\infty} a_i t^{i/k} \), where \( i_0 \) and \( k > 0 \) are some integers and each coefficient \( a_i \) is an element in \( K \). Let \( K\{t\} \) denote the fields of all Puiseux series in the indeterminate \( t \) and a field \( K \). Define a binary relation \( > \) on \( K\{t\} \) by \( x > y \) if the leading coefficient of \( x - y \) is positive. By this relation, \( K\{t\} \) becomes an ordered field. Any statement in \( R \) is naturally formulated in \( R\{t\} \). From Tarski’s principle, any true (first order) statement in \( R \) is also true in \( R\{t\} \); see Appendix A. Hence Corollary 1.3 is true in \( R\{t\} \). Let \( \bar{Q} := Q \cup \{-\infty\} \). The valuation \( \text{val} : K\{t\} \rightarrow \bar{Q} \) is defined by

\[
\text{val}(x) := \max\{j/k \mid a_j \neq 0\} \quad \left(x = \sum a_i t^{i/k} \in K\{t\}\right),
\]

where \( \text{val}(0) := -\infty \). Namely \( \text{val}(x) \) is the degree of the leading term of \( x \).

Define \( \text{val} : K\{t\}^n \rightarrow \bar{Q}^n \) as \( \text{val}(z) := (\text{val}(z_1), \ldots, \text{val}(z_n)) \) for \( z \in K\{t\}^n \). Through this map, an algebraic object \( \mathcal{V} \) in \( K\{t\}^n \) is transformed to a polyhedral object \( \text{val}(\mathcal{V}) \) in \( \bar{Q}^n \), and an algebraic condition \( c_0 z^{b_0} = \sum_i c_i z^{b_i} \) satisfied by \( \mathcal{V} \) is transformed to a max-plus condition \( \text{val}(c_0) + \langle b_0, v \rangle \leq \max_i \{\text{val}(c_i) + \langle b_i, v \rangle\} \) satisfied by \( \text{val}(\mathcal{V}) \), which is obtained by replacing \((+, \times)\) with \((\max, +)\) in the original condition. We will refer to this process as a tropicalization. This is a basic idea in tropical geometry; see [24].

2.1 Tropicalizing Hermite matrices with nonnegative diagonals and signature \((1, n - 1)\)

Let \( \mathcal{M}_n \) be the set of \( n \times n \) Hermite matrices on \( C\{t\} \) having signature \((1, n - 1)\) and nonnegative diagonal entries. Let \( \overline{\mathcal{M}}_n \) be the closure of \( \mathcal{M}_n \), that is, the set of Hermite matrices having nonnegative diagonal entries and at most one positive eigenvalue. We regard a symmetric matrix \( M = (m_{ij}) \) of size \( n \) as a vector of dimension \( n(n + 1)/2 \). Then the tropicalization of \( \overline{\mathcal{M}}_n \) is essentially the space of tree metrics as follows.

3
Theorem 2.1. For a symmetric matrix \( W = (w_{ij}) \in \mathbb{Q}^{n(n+1)/2} \), the following conditions are equivalent:

1. \( W \) belongs to \( \text{val}(\mathcal{M}_n) \).
2. \( W \) satisfies [4PC].

In particular, \( (w'_{ij}) \in \text{val}(\mathcal{M}_n) \) if and only if \( W \) satisfies [4PC].

Proof. (1) \( \Rightarrow \) (2). Since \( W \in \text{val}(\mathcal{M}_n) \), there is a matrix \( M = (m_{ij}) \in \mathcal{M}_n \) such that \( w_{ij} = \text{val}(m_{ij}) \) for \( i, j = 1, \ldots, n \). Every principal submatrix \( M[X] \) has at most one positive eigenvalue, and if \( M[X] \) has no positive eigenvalue, then \( M[X] \) must be a zero matrix (since all diagonal entries of \( M \) must be zero, and all \( 2 \times 2 \) principal minors of \( M \) must be nonnegative). From this we have the following:

\( \ast \) \( \det M[X] \geq 0 \) if \( |X| \) is odd, and \( \det M[X] \leq 0 \) if \( |X| \) is even.

We show that \( W \) satisfies [4PC]:

(i) \( w_{ii} + w_{kk} \leq 2w_{ik} \) for distinct \( i, k \).
(ii) \( w_{ii} + w_{kk} \leq w_{ik} + w_{il} \) for distinct \( i, k, l \).
(iii) \( w_{ij} + w_{kl} \leq \max\{w_{ik} + w_{jl}, w_{il} + w_{jk}\} \) for distinct \( i, j, k, l \).

(i): By \( \ast \) we have \( \det M[{i, k}] = m_{ii}m_{kk} - |m_{ik}|^2 \leq 0 \). Since \( m_{ii} \) and \( m_{kk} \) are nonnegative, it must hold that \( w_{ii} + w_{kk} = \text{val}(m_{ii}m_{kk}) \leq \text{val}(|m_{ik}|^2) = 2w_{ik} \).

(ii): \( \det M[{i, k, l}] = m_{ii}m_{kk}m_{ll} + \left(m_{ik}m_{il}m_{kl} + \frac{m_{ik}m_{il}m_{kl}}{m_{ii}m_{il}m_{kl}} \right) - m_{ii}|m_{kl}|^2 - m_{kk}|m_{il}|^2 - m_{ll}|m_{ik}|^2 \). (2.1)

From (i), \( \text{val}(m_{ii}m_{kk}m_{ll}) \leq \text{val}(m_{ik}m_{il}m_{kl}) = \text{val}(\frac{m_{ik}m_{il}m_{kl}}{m_{ii}m_{il}m_{kl}}) \). Since \( \det M[{i, k, l}] \geq 0 \) by \( \ast \), and the last three terms in (2.1) are nonpositive, it must hold that

\( \text{val}(m_{ii}|m_{kl}|^2) \leq \max\{\text{val}(m_{ik}m_{kk}m_{ll}), \text{val}(m_{ik}m_{il}m_{kl} + \frac{m_{ik}m_{il}m_{kl}}{m_{ii}m_{il}m_{kl}}) \} \leq \text{val}(m_{ik}m_{il}m_{kl}) \).

Therefore we obtain (ii).

(iii): Consider the expansion of \( \det M[{i, j, k, l}] \). For a term containing \( m_{i'j'}m_{j'k'} \) in the expansion, the term obtained by replacing \( m_{i'j'}m_{j'k'} \) with \( m_{ij}m_{jk} \) also appears in the expansion and has degree at least the original by (i) and (ii). From this we see that the degree of a term including a diagonal element \( a_{i'j'} \) is no more than the degree of \( m_{i'j'}m_{j'k'}m_{k'l'}m_{l'i'} \) for some different \( i', j', k', l' \). Observe \( \text{val}(m_{i'j'}m_{j'k'}m_{k'l'}m_{l'i'}) \) is equal to \( \text{val}(|m_{i'j'}|^2|m_{j'k'}|^2) + \text{val}(|m_{i'j'}|^2|m_{j'k'}|^2) / 2 \). Therefore, if [4PC] is violated, say, \( w_{ij} + w_{kl} > \max\{w_{ik} + w_{jl}, w_{il} + w_{jk}\} \), then \( |m_{ij}|^2|m_{kl}|^2 \) becomes the unique leading term in \( \det M[{i, j, k, l}] \). This implies that \( \det M[{i, j, k, l}] > 0 \), which contradicts \( \ast \). Thus \( W \) satisfies [4PC].

(2) \( \Rightarrow \) (1). It suffices to show that \( M := (w_{ij}) \) belongs to \( \mathcal{M}_n \). We use the induction on \( n \). If \( n = 1 \), then the statement obviously holds. Suppose \( n > 1 \). If \( M \) is singular, then some \( i \)-th column (row) can be represented as a linear combination of other column (row). Hence the signature of \( M \) is equal to that of the matrix \( M' \) obtained by deleting \( i \)-th column and row; we can apply the induction. We can assume that \( M \) is nonsingular. If \( w_{ij} = -\infty \) for distinct \( i, j \), then [4PC] implies \( w_{ik} + w_{jl} \leq w_{il} + w_{jk} \). Exchanging the role of \( k \) and \( l \), we have \( w_{ik} + w_{jl} = w_{il} + w_{jk} \). This means \( m_{ik}m_{jl} = m_{il}m_{jk} \). Hence the \( i \)-th row and the \( j \)-th row are linearly dependent, and this contradicts the nonsingularity assumption of \( M \). Thus \( W \) has \(-\infty \) only on diagonals (if it exists). By
If $P$ has no root $z = (z_1, z_2, \ldots, z_n)$ with $\text{Re}(z_i) > 0$ ($i = 1, 2, \ldots, n$). Such a polynomial is also called an HPP polynomial. Choe, Oxley, Sokal and Wagner [8] and Brändén [5, 6] explored an interesting link between the half-plane property and matroidal convexity. A set $B$ of integer vectors in $\mathbb{Z}^*_+$ is called $M$-convex [21] if $B$ satisfies the following property:

[EXC] For $u, v \in B$ and $i \in [n]$ with $u_i < v_i$, there exists $j \in [n]$ such that $u_j > v_j$ and

$$u + e_i - e_j, v + e_j - e_i \in B.$$  

An M-convex set is nothing but the base set of an integral polymatroid [15]. In addition, if $B$ belongs to $\{0, 1\}^n$, then $B$ is the set of characteristic vectors of bases of a matroid. [3] An M-convex set $B$ lies on a hyperplane $\{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = k\}$ for some $k \in \mathbb{Z}_+$, and this $k$ is called the rank of $B$. The support of a polynomial $P(z) = \sum a_u z^n$ is the set of vectors $u \in \mathbb{Z}_+^n$ such that $a_u \neq 0$, where $z^n := z_1^{n_1} \cdots z_n^{n_n}$.

**Theorem 2.2** (Choe–Oxley–Sokal–Wagner [8, Theorem 7.2]). For every homogeneous HPP polynomial $P$, the support of $P$ is an $M$-convex set.

The converse of this theorem is not true in general: there is no HPP polynomial having Fano matroid support [5]. In rank-2 case, however, the following is known.

**Theorem 2.3** (Choe–Oxley–Sokal–Wagner [8, Corollary 5.4]). For every $M$-convex set $B$ of rank 2, the polynomial $P_B(z) = \sum_{i,j \in [n], e_i + e_j \in B} z_i z_j$ has the half-plane property.

A key ingredient of their proof is the following.

**Theorem 2.4** (Choe–Oxley–Sokal–Wagner [8, Theorem 5.3]). For a nonnegative real symmetric matrix $A$, the following conditions are equivalent:

1. $z^\top Az$ has the half-plane property.
2. $A$ has at most one positive eigenvalue.

Brändén [6] considered HPP polynomials on the field of Puiseux series. Since $\mathbb{R}\{t\}$ is an ordered field, the half-plane property is also defined on $\mathbb{C}\{t\}$. Namely, $P \in \mathbb{R}\{t\}[z_1, z_2, \ldots, z_n]$ is said to have the half-plane property if $P$ has no root $z$ with $\text{Re}(z_i) > 0$ ($i = 1, 2, \ldots, n$). For a polynomial $P = \sum a_u z^n$, define a function $\text{val}_P$ on $\mathbb{Z}_+^n$ by

$$\text{val}_P(u) := \text{val}(a_u) \quad (u \in \mathbb{Z}_+^n).$$

Again $\text{val}_P$ has a matroidal concavity. A function $f : \mathbb{Z}_+^n \to \mathbb{Q}$ is called $M$-concave [21] if

\[\text{val}_P(u) \leq \text{val}(a_u) \quad (u \in \mathbb{Z}_+^n).\]
[M-EXC] for \( u, v \in \mathbb{Z}_n^+ \) and \( i \in [n] \) with \( u_i < v_i \), there exists \( j \in [n] \) such that \( u_j > v_j \) and
\[
f(u) + f(v) \leq f(u + e_i - e_j) + f(v_j + e_j - e_i).
\]
Note that if \( f \) is an M-concave function, then the domain of \( f \) is the M-convex set \([24]
Proposition 6.1\), where the domain is the set of elements \( u \) such that \( f(u) > -\infty \).
Therefore we define the rank of an M-concave function as the rank of the domain. If the
domain of \( f \) is contained by \((0,1)^n\), then \( f \) is a valuated matroid \([14]\); see Section 2.3.

**Theorem 2.5** (A corollary of Brändén \([6]\) Theorem 4). For every homogeneous HPP
polynomial \( P \), \( \text{val}_P \) is an M-concave function.

We consider the rank-2 case. A function \( f \) on \( \{ e_i + e_j \mid i, j \in [n] \} \) is identified with
a symmetric matrix \( (f_{ij}) \) by the correspondence
\[
f(e_i + e_j) \leftrightarrow f_{ij}.
\]
By this correspondence, the condition [M-EXC] for \( f \) is equivalent to [4PC] for \( (f_{ij}) \). This fact was observed by Dress and Tarhella \([12]\), Hirai and Murata \([18]\). Thus Theorem
2.4 implies that \( A := (t^{f_{ij}}) \) has at most one positive eigenvalue. Theorem 2.4 is true in
\( \mathbb{R}[t] \) by Tarski’s principle (Appendix A). Therefore we have the following.

**Corollary 2.6.** For every M-concave function \( f \) of rank 2, the polynomial
\( P_f(z) = \sum_{i,j} (t^{f(e_i + e_j)}) z_i z_j \) has the half-plane property.

**Remark 2.7.** For a valuated matroid \( f \) of rank 2, the existence of an HPP polynomial \( P \)
with \( \text{val}_P = f \) can also be derived from a combination of the following two facts: (i) every
valuated matroid of rank 2 is representable in \( \mathbb{R}[t] \) \([24]\), and (ii) for a representable
valuated matroid \( f \) represented by a \( k \times n \) matrix \( M \), the polynomial
\( Q(z) = \det M Z M^T \) is an HPP polynomial with \( \text{val}_Q = f \), where \( Z = \text{diag}(z_1, z_2, \ldots, z_n) \) \([8]\) Theorem 8.2];
see Section 2.3 for a valuated matroid and its representability.

### 2.3 Valuated matroid and dissimilarity map on tree

Our formulas shed some insight on valuated (\( \Delta \)-)matroids arising from weights of sub-
trees in a tree. Denote by \( \binom{V}{k} \) the set of all subsets of \( V \) with cardinality \( k \). For a matrix
\( M \), denote by \( M_X \) the submatrix of \( M \) consisting of the \( i \)-th columns over \( i \in X \), and
by \( M_{X,Z} \) the submatrix consisting of the \( i \)-th rows and the \( j \)-th column over \( i \in X \) and
\( j \in Z \).

A valuated matroid of rank \( k \) is a map \( \omega : \binom{V}{k} \rightarrow \mathbb{Q} \) satisfying
\[
\omega(X) + \omega(Y) \leq \max_{j \in Y \setminus X} \{ \omega(X \setminus \{ i \} \cup \{ j \}) + \omega(Y \setminus \{ j \} \cup \{ i \}) \} \quad (X, Y \in \binom{V}{k}, \ i \in X \setminus Y).
\]
This condition is the tropicalization of the Grassmann-Plücker relation of the Plücker
coordinate \( u_X := \text{val}(\det M_X) \) for a \( k \times n \) matrix \( M \):
\[
u_X \cdot \nu_Y = \sum_{j \in Y \setminus X} \sigma_{ij} \cdot u_{X \setminus \{ i \} \cup \{ j \}} \cdot u_{Y \setminus \{ j \} \cup \{ i \}} \quad (X, Y \in \binom{V}{k}, \ i \in X \setminus Y),
\]
where \( \sigma_{ij} \in \{1, -1\} \) depends on the ordering of the elements \( i, j \). In particular for any
\( k \times n \) matrix \( M \), the map \( X \mapsto \text{val}(\det M_X) \) is a valuated matroid. Such a valuated
matroid is called representable. In tropical geometry, a representable valuated matroid
is a point of the tropical Grassmannian \([21]\).
In study on phylogenetic trees, Pachter and Speyer \[22\] found that weight of subtrees in a tree yields a class of valuated matroids. Let \( T = (V, E) \) be a tree with a positive edge weight \( l(e) \). For a vertex set \( Y \), define the dissimilarity \( D(Y) \) of \( Y \) by the sum of edge weights \( l(e) \) over edges \( e \) in the minimal subtree in \( T \) containing \( Y \). Let \( X = \{1, 2, \ldots, n\} \) be the set of leaves of \( T \). The \( k \)-dissimilarity map \( D^k \) is a function on the \( k \)-leaf set \( \binom{X}{k} \) defined by \( D^k(Y) := D(Y) \).

**Theorem 2.8** (Pachter–Speyer \[22\]). The \( k \)-dissimilarity map is a valuated matroid.

Pachter and Speyer \[22\] asked whether a \( k \)-dissimilarity map is in the tropical grassmannian, or equivalently, is a representable valuated matroid (in our terminology). Recently this problem was affirmatively solved:

**Theorem 2.9** (Cools \[9\], Giraldo \[17\], Manon \[19\]). The \( k \)-dissimilarity map is a representable valuated matroid.

Compared with this theorem, our formula (1.1) gives another type of a representation of the dissimilarity map \( D \) — a representation by the degree of principal minors of a symmetric matrix. Combinatorial properties of the map \( X \mapsto \text{val}(\det A[X]) \) for a symmetric matrix \( A \) are not well-studied and not well-understood, though it is known that the nonzero support \( \{ X \mid \det A[X] \neq 0 \} \) forms a \( \Delta \)-matroid \[8, 11\]. A natural question is: does the map \( X \mapsto \text{val}(\det A[X]) \) have a kind of a matroidal concavity? We hope that our new representation of dissimilarity maps will stimulate this line of research.

Giraldo \[17\] proved Theorem 2.9 by showing that the total length of a tree is represented as the degree of the determinant of a certain matrix associated with the tree. His formula is somewhat similar to our formula, although we could not find a relationship between them.

**Representation of rooted \( k \)-dissimilarity map.** Nevertheless our formula gives a linear representation for a special class of dissimilarity maps. Fix a root vertex \( 0 \in V \setminus X \). The **rooted \( k \)-dissimilarity map** \( D^k_0 \) is a function on \( \binom{X}{k} \) defined by \( D^k_0(Y) := D(Y \cup \{0\}) \). A linear representation of \( D^k_0 \) is constructed as follows.

Define an \( n \times n \) matrix \( M = (m_{ij}) \) by \( m_{ij} := t^{d_{ij}} - t^{d_{0i} + d_{0j}} \). Namely \( M \) is the Schur complement of the 0-th diagonal element in \( A[X \cup \{0\}] = (t^{d_{ij}}) \). Hence we have

\[
\text{(1) } \det M[Y] = \det A[Y \cup \{0\}] \text{ for } Y \subseteq X, \text{ and} \\
\text{(2) } M \text{ is negative definite.}
\]

We see (2) from the sign pattern of \( \det M[[1, 2, \ldots, k]] = \det A[[0, 1, 2, \ldots, k]] \). By (2) and the Cholesky factorization, there is an \( n \times n \) matrix \( Q \) with \( -M = Q^\top Q \). Take an arbitrary \( k \times n \) matrix \( J \) in \( \mathbb{R} \) whose entries have no algebraic dependence. By the Binet–Cauchy formula we have

\[
2 \text{val}(\det(JQ)_{YZ}) = 2 \text{val}(\sum_Z \det J_Z \det Q_{Z,Y}) = 2 \max_Z \text{val}(\det Q_{Z,Y})
\]

\[
= \max_Z \text{val}(\det Q_{Z,Y})^2 = \text{val}(\sum_Z (\det Q_{Z,Y})^2) = \text{val}(\det Q_Y)^\top Q_Y
\]

\[
= \text{val}(|M[Y]|) = \text{val}(\det A[Y \cup \{0\}]) = D^k_0(Y),
\]

where \( Z \) ranges all elements in \( \binom{X}{k} \), and the second equality follows from the algebraic independence of elements in \( J \). Hence let \( R := JQ \) and replace \( t \) by \( t^{1/2} \) in \( R \). Then \( D^k_0(Y) = \text{val}(\det R_Y) \), and we obtain a linear representation of \( D^k_0 \).
Valuated $\Delta$-matroid and odd-dissimilarity map. A valuated $\Delta$-matroid is a function $\omega : 2^V \rightarrow \overline{Q}$ satisfying

$$\omega(X) + \omega(Y) \leq \max_{j \in (X \triangle Y) \setminus i} \{\omega(X \triangle \{i, j\}) + \omega(Y \triangle \{i, j\})\} \quad (X, Y \subseteq V, i \in X \triangle Y).$$

This is the tropicalization of the Wick relation of principal minors $b_X := \text{Pf} B[X]$ ($X \subseteq V$) of a skew-symmetric matrix $B$:

$$b_X \cdot b_Y = \sum_{j \in (X \triangle Y) \setminus i} \sigma_{ij}' \cdot b_{X \triangle \{i, j\}} \cdot b_{Y \triangle \{i, j\}} \quad (X, Y \subseteq V, i \in X \triangle Y),$$

where $\sigma_{ij}' \in \{1, -1\}$ depends on the ordering of the elements $i, j$. Hence the map $X \mapsto \text{val}(b_X)$ is a valuated $\Delta$-matroid [25]; see also [20, Section 7.3]. Such a valuated $\Delta$-matroid is called representable.

Let $T = (V, E)$ be a tree where $V = \{1, 2, \ldots, N\}$. For any tree $T$, the odd-dissimilarity map $D^o \in 2^V$ is defined as follows.

$$D^o(X) := \begin{cases} |O_X| & \text{if } |X| \text{ is even}, \\ -\infty & \text{if } |X| \text{ is odd}, \end{cases} \quad (X \subseteq V),$$

where $O_X$ is the set of odd edges with respect to $X$, defined in Section 1. After reordering, we suppose that $V$ is nicely-ordered. One can easily see that any subset $X \subseteq V$ is also nicely-ordered. By [13], we have

$$D^o(X) = \text{val}(\text{Pf} B[X]) \quad (X \subseteq V).$$

Moreover, let $B^\nu$ be the matrix obtained by replacing $t$ by $t^{-1}$ in $B$. Then we have

$$-D^o(X) = \text{val}(\text{Pf} B^\nu[X]) \quad (X \subseteq V).$$

Therefore we obtain the following.

Corollary 2.10. The odd-dissimilarity map and its negative are both representable valuated $\Delta$-matroids.

This theorem implies that the odd-dissimilarity map is a nontrivial example of a valuated $\Delta$-matroid whose negative is also a valuated $\Delta$-matroid.

An algebraic variety determined by the Wick relation is called the spinor variety. The spior variety parametrizes maximal isotropic vector subspaces in a vector space with an antisymmetric bilinear form, analogous to the grassmannian that parametrizes vector subspaces. Rincón [24] considered the tropical spinor variety (a tropicalization of the spinor variety). A representable valuated $\Delta$-matroid is nothing but a point of the tropical spinor variety in his sense. Corollary 2.10 is therefore an isotropic analogue of Theorem 2.9.

3 Proof

3.1 Proof of Theorem 1.2

Let $T = (V, E)$ be a tree, and $X \subseteq V$. Let us recall the formula for the determinant of $A[X]$. Without loss of generality, we can assume that $X = \{1, \ldots, n\}$.

$$\det A[X] = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^n a_{\sigma(i)},$$

8
where $S_n$ is the symmetric group of degree $n$. Our first step is to identify each permutation of this formula with a path on the corresponding tree. Let us define following terminology.

- A cycle of $X$ is a cyclic sequence $(x_1, x_2, \ldots, x_k)$ ($k \geq 1$) of a subset $\{x_1, x_2, \ldots, x_k\} \subseteq X$, where we do not distinguish $(x_1, x_2, \ldots, x_k)$ and $(x_k, x_1, x_2, \ldots, x_{k-1})$, and indices are considered in modulo $k$.

- A cycle partition $W$ of $X$ is a set of cycles of $X$ such that every element of $X$ belongs to exactly one cycle.

- The support $\text{supp}(C)$ of a cycle $C = (i_1, i_2, \ldots, i_k)$ is a function on $E$ defined by:
  \[
  \text{supp}(C)(e) \text{ is the number of indices } i \text{ such that } e \text{ belongs to the unique path between } x_i \text{ and } x_{i+1} \text{ in } T.\]
  The support $\text{supp}(W)$ of a cycle partition $W$ is defined as $\sum_{C \in W} \text{supp}(C)$. Note that the support is even-valued.

- $\text{sign}(W) := \prod_{C \in W} (-1)^{\|C\|+1}$.

- $\|W\| := \sum_{e \in E} \text{supp}(W)(e)$. For a cycle $C = (i_1, i_2, \ldots, i_k)$, this definition means that
  \[
  \sum_{j=1}^{k} d_{i_j, i_{j+1}} = \sum_{e \in E} \text{supp}(C)(e) = \|C\|. \tag{3.1}
  \]

By using these notions, the formula of the determinant can be rephrased as follows.

**Lemma 3.1.**

\[
\det A[X] = \sum \left\{ \text{sign}(W) t^\|W\| \mid W: \text{cycle partition of } X \right\}. \tag{3.2}
\]

**Proof.** Observe that there is a one-to-one correspondence between permutations and cycle partitions: a permutation is uniquely represented as the product of (disjoint) cyclic permutations, and each cyclic permutation $(i_1, i_2, \ldots, i_k)$ is naturally identified with a cycle in our sense. In this correspondence, the sign of a permutation $\sigma$ is equal to $\text{sign}(W)$ of the corresponding $W$, and by the equation (3.1), we have

\[
\prod_{i \in X} a_{\sigma(i)} = t^{\sum_{i \in X} d_{\sigma(i)}} = t^\|W\|.
\]

Hence we have

\[
\det A[X] = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i \in X} a_{\sigma(i)} = \sum_{W} \text{sign}(W) t^\|W\|.
\]

\[\square\]

A cycle partition $W$ of $X$ is said to be tight if the support of $W$ is $\{0, 2\}$-valued. In fact, non-tight cycle partitions vanish in (3.2).

**Lemma 3.2.**

\[
\det A[X] = \sum \left\{ \text{sign}(W) t^\|W\| \mid W: \text{tight cycle partition of } X \right\}.
\]
Proof. Let us first introduce an operation on cycle partitions, called a flip. Let \( W \) be a cycle partition of \( X \), and let \( e = xy \) be an edge of \( T \). Suppose that \( \text{supp}(W)(xy) \geq 4 \). Then (i) there are two cycles \( C, C' \) passing through \( e \) in order \( x \to y \), or (ii) there is a single cycle \( C'' \) passing through in order \( x \to y \) twice. For the case (i), suppose that \( C = (v_1, \ldots, v_k), C' = (u_1, \ldots, u_l) \), the unique path from \( v_i \) to \( v_{i+1} \) passes through \( xy \) in order \( v_i \to x \to y \to v_{i+1} \), and the unique path from \( u_j \) to \( u_{j+1} \) passes through \( xy \) in order \( u_j \to x \to y \to u_{j+1} \). Replace \( C \) and \( C' \) by

\[
C'' = (v_1, \ldots, v_i, u_{j+1}, \ldots, u_l, u_1, \ldots, u_j, v_{i+1}, \ldots, v_k).
\]  

(3.3)

Then we obtain a cycle partition \( W' = W \setminus \{C, C'\} \cup \{C''\} \). Similarly, for the case (ii), there is a single cycle \( C'' \) as in (3.3). By reversing the operation above, we obtain two cycles \( C, C' \). Replacing \( C'' \) by \( C \) and \( C' \), we obtain a cycle partition \( W' = W \setminus \{C''\} \cup \{C, C'\} \). In this way, we obtain an operation \( W \mapsto W' \) on cycle partitions, which we call a flip.

If a cycle partition \( W' \) is obtained by applying a flip to another cycle partition \( W \), then \( \text{supp}(W') = \text{supp}(W) \) and \( \text{sign}(W') = -\text{sign}(W) \). The former equation is obvious from the definition of a flip. The latter equation holds since \(|W| - |W'| \in \{1, -1\}\) and

\[
\text{sign}(W) = \prod_{C \in W} (-1)^{|C|+1} = (-1)^{|X|+|W|},
\]

For \( l : E \to 2\mathbb{Z}_+ \), let \( W_l \) be the set of all cycle partitions \( W \) with \( \text{supp}(W) = l \). Let \( W_l^+ \) denote the set of cycle partitions \( W \in W_l \) with \( \text{sign}(W) = 1 \), and let \( W_l^- := W_l \setminus W_l^+ \). Then, from (3.2), we have

\[
\det A[X] = \sum_{l : E \to 2\mathbb{Z}_+} (|W_l^+| - |W_l^-|)l\|l\|,
\]  

(3.4)

where \( \|l\| := \sum_{e \in E} l(e) \). It suffices to show that if there is an edge \( e \in E \) with \( l(e) \geq 4 \), then \( |W_l^+| = |W_l^-| \).

Let \( \Gamma \) be the graph on \( W_l \) such that two vertices \( W, W' \in W_l \) are adjacent if and only if \( W \) can be obtained from \( W' \) by a single flip on \( e \). The graph \( \Gamma \) is a bipartite graph of bipartition \( \{W_l^+, W_l^-\} \), since a flip operation changes the sign, and \( \Gamma \) cannot have an edge joining vertices of the same sign. Moreover \( \Gamma \) is a regular graph, since the number of different flips on \( e \) is determined only by \( l(e) \) (which is equal to \( 2(l(e)/2) \)), and different flips yield different cycle partitions. Thus \( \Gamma \) is a regular bipartite graph with bipartition \( \{W_l^+, W_l^-\} \), which implies \( |W_l^+| = |W_l^-| \).

For any set \( W \) of cycle partitions, define the number \( \langle W \rangle \) by

\[
\langle W \rangle := \sum_{W \in W} \text{sign}(W).
\]

For a forest \( F \) (not necessarily a subgraph of \( T \)) spanned by \( X \), define \( W_{X,F} \) by the set of cycle partitions \( W \in X \) such that each cycle \( C \in W \) belongs to some connected component of \( F \), and each edge in \( F \) is traced by cycles in \( W \) exactly twice. By using this notation, we have the following.

Lemma 3.3.

\[
\det A[X] = \sum_F \langle W_{X,F} \rangle t^{2|E_F|},
\]  

(3.5)

where \( F \) ranges all subgraphs in \( T \) spanned by \( X \).
Proof. This immediately follows from Lemma 3.2 and the fact that for any tight cycle partition $W$ the forest formed by edges $e$ with $\text{supp}(W)(e) = 2$ is spanned by $X$. \(\square\)

Thus it suffices to show the following:

**Lemma 3.4.**

\[
\langle W_{F,X} \rangle = (-1)^{|X| + c(F)} \prod_{v \in V_F \setminus X} (\text{deg}_F(v) - 1).
\]

This lemma is an easy corollary of the following properties of $\langle W_{X,F} \rangle$.

**Lemma 3.5.**

(i) Suppose that $F$ is the vertex-disjoint union of two forests $H, H'$. Then we have

\[
\langle W_{F,X} \rangle = \langle W_{H,X \cap V_H} \rangle \langle W_{H',X \cap V_{H'}} \rangle.
\]

(ii) For $e = xy \in E_F$, let $F'$ be the forest obtained from $F$ by adding new vertices $x', y'$ and by replacing $e$ by new edges $xy', x'y$. Then $F'$ is spanned by $X \cup \{x', y'\}$, and we have

\[
\langle W_{F,X} \rangle = -\langle W_{F',X \cup \{x', y'\}} \rangle.
\]

(iii) If $F$ is a star with the center vertex $v$, then

\[
\langle W_{F,X} \rangle = \begin{cases} (-1)^{|X| + 1} & \text{if } v \in X, \\ (-1)^{|X| + 1}(\text{deg}_F(v) - 1) & \text{otherwise}. \end{cases}
\]

**Proof of Lemma 3.4.** By Lemma 3.3 (i), it suffices to prove the formula for the case where $F$ is connected. We use the induction on the number $k$ of inner vertices. If $k = 0, 1$, then $F$ is a star, and the corresponding formula follows from (iii). Let $k > 1$. Since $F$ is connected, there exists an edge $e$ joining two inner vertices. Applying (ii) for $e$, we have $\langle W_{F,X} \rangle = -\langle W_{F',X \cup \{x', y'\}} \rangle$, where $F'$ has two connected components $H, H'$, each of which has less inner vertices than $F$ has. Let $Y := (X \cup \{x', y'\}) \cap V_H$, and $Y' := (X \cup \{x', y'\}) \cap V_{H'}$. From (i) and inductive hypothesis, we get

\[
\langle W_{F,X} \rangle = -\langle W_{F',X \cup \{x', y'\}} \rangle = -\langle W_{H,Y} \rangle \langle W_{H',Y'} \rangle
\]

\[
= -(-1)^{|X \cup \{x', y'\}| + |V_H| + |V_{H'}| + |E_H| + |E_{H'}|} \prod_{v \in V_F \setminus (X \cup \{x', y'\})} (\text{deg}_F(v) - 1)
\]

where $|V_H| + |V_{H'}| = |V_F| + 2$ and $|E_H| + |E_{H'}| = |E_F| + 2$. Since $c(F) = |V_F| - |E_F|$, we have the desired equation. \(\square\)

**Proof of Lemma 3.5.** (i) Since every cycle partition $W \in W_{F,X}$ is uniquely decomposed into cycle partitions $Z \in W_{H,X \cap V_H}$ and $Z' \in W_{H',X \cap V_{H'}}$, with $W = Z \cup Z'$, and vice versa, we have

\[
\langle W_{F,X} \rangle = \sum_{W \in W_{F,X}} \text{sign}(W) = \sum_{Z \in W_{H,X \cap V_H}} \sum_{Z' \in W_{H',X \cap V_{H'}}} (\text{sign}(Z))(\text{sign}(Z'))
\]

\[
= \langle W_{H,X \cap V_H} \rangle \langle W_{H',X \cap V_{H'}} \rangle.
\]

(ii) For every cycle partition $W \in W_{F,X}$, there is the unique cycle

\[
C = (u, v, \alpha_1, \ldots, \alpha_i, u', \beta_1, \ldots, \beta_j) \in W
\]
such that the path between \(u, v\) and the path between \(v', u'\) include \(xy\) in order \(u \to x, y \to v\) and \(v' \to y, x \to u'\), respectively. Define two cycles \(C', C''\) by

\[
C' := (u, y', u', \beta_1, \ldots, \beta_j), \quad C'' := (v', x', v, \alpha_1, \ldots, \alpha_i). \tag{3.6}
\]

Let \(W' := W \setminus \{C\} \cup \{C', C''\}\). Then \(W'\) is a cycle partition in \(W_{F', X \cup \{x', y'\}}\) with \(\text{sign}(W) = -\text{sign}(W')\). Thus we obtain a map from \(W_{F, X}\) to \(W_{F', X \cup \{x', y'\}}\) such that \(W \mapsto W'\). Observe that this map is a bijection: any cycle partition \(W' \in W_{F', X \cup \{x', y'\}}\) includes cycles \(C, C'\) with property (3.6), and the reverse operation is valid on every cycle partition. Hence we obtain

\[
\langle W_{F, X \cap V_F} \rangle = \sum_{W \in W_{F, X}} \text{sign}(W) = -\sum_{W' \in W_{F', X \cup \{x', y'\}}} \text{sign}(W') = -\langle W_{F', X \cup \{x', y'\}} \rangle.
\]

(iii) Let \(k := |X|\). In the both cases, \(\langle W_{F, X} \rangle\) depends only on the cardinality of \(X\). We may denote \(W_{F, X}\) by \(A_k\) if \(v \not\in X\), and denote \(W_{F, X}\) by \(B_k\) if \(v \in X\). We will prove the following two claims.

(*1) \(\langle A_k \rangle = -(k-1)(\langle A_{k-1} \rangle + \langle A_{k-2} \rangle)\), \((k > 3)\).

(*2) \(\langle B_k \rangle = \langle A_k \rangle + \langle A_{k-1} \rangle\), \((k > 2)\).

Since \(\langle A_2 \rangle = -1\) and \(\langle A_3 \rangle = 2\) from the recursion (*1) we have

\[
\langle A_k \rangle = -(k-1)(\langle A_{k-1} \rangle + \langle A_{k-2} \rangle) = (-1)^{k+1}(k - 1) = (-1)^{|X|+1}(\deg_F(v) - 1).
\]

Also we have \(\langle B_1 \rangle = 1\), \(\langle B_2 \rangle = -1\), and from (*2), \(\langle B_k \rangle = \langle A_k \rangle + \langle A_{k-1} \rangle = (-1)^{|X|+1}\).

For (*1), fix an arbitrary vertex \(u \in X\). Let \(A'_{k}\) denote the set of cycle partitions \(W\) in \(A_k\) such that the unique cycle in \(W\) containing \(u\) has length at least three. We will show that

\[
\langle A'_{k} \rangle = -(k-1)\langle A_{k-1} \rangle, \tag{3.7}
\]

\[
\langle A_k \rangle - \langle A'_{k} \rangle = \langle A_k \setminus A'_{k} \rangle = -(k-1)\langle A_{k-2} \rangle. \tag{3.8}
\]

To see (3.7), for a cycle partition \(W \in A_{k-1}\), take a consecutive pair \(x, y\) in some cycle \(C\) in \(W\). Replace \(x, y\) by \(x, u, y\) in \(C\). Then we get a cycle partition \(W'\) in \(A'_{k}\) with sign change. There are \(k - 1\) ways of choosing a consecutive pair in each cycle partition. Also every cycle partition in \(A'_{k}\) is obtained in this way. Hence we have (3.7).

To see (3.8), observe that \(A_k \setminus A'_{k}\) is the disjoint union, over \(x \in X \setminus u\), of the sets \(W_x\) of cycle partitions including \((u, x)\). Delete \((u, x)\) from each cycle partition of \(W_x\). Then we get a cycle partition in \(A_{k-2}\) with sign change. Also, every cycle partition in \(W_x\) is obtained by adding the cycle \((u, x)\) to cycle partitions in \(A_{k-2}\). Hence \(\langle A_k \setminus A'_{k} \rangle = \sum_{x \in X \setminus u} \langle W_x \rangle = -(k-1)\langle A_{k-2} \rangle\), and we have (3.8).

Consider (*2). Let \(B'_k\) denote the set of cycle partitions \(W\) in \(B_k\) such that \(W\) includes the singleton cycle \((v)\). It suffice to show that \(\langle B'_k \rangle = \langle A_{k-1} \rangle\) and \(\langle B_k \setminus B'_k \rangle = \langle A_k \rangle\).

The first equation follows from the observation that the deletion of \((u)\) from cycle partitions in \(B'_k\) makes a one-to-one correspondence between \(B'_k\) and \(A_{k-1}\). For the latter equation, add a new leaf \(v'\) to \(F\), and replace \(v\) by \(v'\) in each cycle partition in \(B_k \setminus B'_k\). This procedure maps cycle partitions in \(B_k \setminus B'_k\) to ones in \(A_k\) bijectively, and thus we have the latter equation.
3.2 Proof of Theorem 1.4

Suppose that \( X = \{1, 2, \ldots, 2n\} \) and \( X \) is nicely-ordered with respect to \( T \). We denote \( \text{Pf} B[X] \) by \( \text{Pf}[X] \) for simplicity. Let us recall the recursive definition of Pfaffian:

\[
\text{Pf}[X] = \sum_{j \in X} (-1)^{i+j+1} b_{ij} \text{Pf}[X \setminus \{i, j\}] \quad (i \in X).
\] (3.9)

Since the deletion of an element in \( X \) only omits paths of the corresponding tour, we have the following lemma.

**Lemma 3.6.** If \( X \) is nicely-ordered, then every subset \( Y \) of \( X \) is nicely-ordered.

In the following, we tacitly use this lemma. For distinct \( i, j \in X \), define \( P_{ij} \subseteq E \) as the set of edges which belong to the unique path from \( i \) to \( j \).

**Lemma 3.7.** \( O_{X \setminus \{i, j\}} = O_X \triangle P_{ij} \).

**Proof.** Let \( e \in E \), and let \( T', T'' \) be the two components obtained by the removal of \( e \). If \( e \notin P_{ij} \), then either \( T' \) or \( T'' \) must include both \( i \) and \( j \), and hence \( e \in O_{X \setminus \{i, j\}} \iff e \in O_X \). If \( e \in P_{ij} \), then \( T' \) must include just one of \( i \) and \( j \), and hence \( e \in O_{X \setminus \{i, j\}} \iff e \notin O_X \). These imply the statement.

The following lemma gives a pairing of elements of \( X \) via odd edges.

**Lemma 3.8.** There is a partition \( \{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_n, j_n\}\} \) of \( X \) such that

(i) \( i_k + j_k \) is odd for all \( k = 1, \ldots, n \), and

(ii) \( O_X \) is the disjoint union of \( P_{i_1j_1}, \ldots, P_{i_nj_n} \). In particular, it follows that

\[
P_{uik} \setminus O_X = P_{ujk} \setminus O_X \quad (u \in X, \ k = 1, 2, \ldots, n).
\]

**Proof.** We show that there exists \( i \) with \( P_{i(i+1)} \subseteq O_X \), where the indices are considered in modulo \( 2n \). Suppose that this is true. Take \( \{i, i + 1\} \) as \( \{i_1, j_1\} \). Then \( O_X \) is the disjoint union of \( P_{i(i+1)} \) and \( O_{X \setminus \{i, i+1\}} \), and \( i_1 + j_1 \) is odd. We can renumber \( X \setminus \{i, i+1\} \) with keeping the parity of the indices. Apply the induction to \( O_{X \setminus \{i, i+1\}} \).

We may assume that all leaves of \( T \) belong to \( X \). Consider the subgraph \( H \) of \( T \) formed by \( O_X \). There exists a connected component \( T' \) of \( H \) incident to (at most) one edge \( e \in E \setminus O_X \) in \( T \). Necessarily \( T' \) contains at least two vertices \( i, j \) in \( X \). Then \( i - 1 \) or \( i + 1 \) also belongs to \( T' \); otherwise the edge \( e \) is traced at least four times by the tour \( 1 \to 2 \to \cdots \to 2n \to 1 \); contradiction to the fact that \( X \) is nicely-ordered. This implies \( P_{(i-1)i} \subseteq O_X \) or \( P_{(i+1)i} \subseteq O_X \).

We are ready to prove Theorem 1.4.

**Proof of Theorem 1.4** We prove the statement by the induction on the cardinality of \( X \). If \( X = \{1, 2\} \), then \( \text{Pf}[\{1, 2\}] = b_{12} = t^{d_{12}} = t^{O(1, 2)} \). Suppose \( |X| > 2 \). Fix a partition \( \{\{i_1, j_1\}, \ldots, \{i_n, j_n\}\} \) of \( X \) satisfying the condition in Lemma 3.8. We can assume that \( i_1 = 1 \). Since \( j_1 \) is even and \( i_k + j_k \) is odd for all \( k \), from the formula (3.9) we have

\[
\text{Pf}[X] = b_{1j_1} \text{Pf}[X \setminus \{1, j_1\}]
+ \sum_{k=2}^{n} (-1)^k \left( b_{1i_k} \text{Pf}[X \setminus \{1, i_k\}] - b_{1j_k} \text{Pf}[X \setminus \{1, j_k\}] \right).
\] (3.10)
Since $O_X$ is the disjoint union of $P_{1j_i}$ and $O_{X\backslash \{1,j_i\}}$, by inductive hypothesis we have
\[b_{1j_i} Pf[X \backslash \{1, j_i\}] = t|P_{1j_i}| + |O_{X\backslash \{1, j_i\}}| = t|O_X|,\]
\[b_{1i_k} Pf[X \backslash \{1, i_k\}] = t|P_{1i_k}| + |O_{X\backslash \{1, i_k\}}|,\]
\[b_{1j_k} Pf[X \backslash \{1, j_k\}] = t|P_{1j_k}| + |O_{X\backslash \{1, j_k\}}|.
\]
From Lemma \[3.7\] (ii), we have
\[|P_{1i_k}| + |O_{X\backslash \{1, i_k\}}| = |P_{1i_k}| + |O_X \triangle P_{1i_k}| = |O_X| + 2|P_{1i_k} \backslash O_X| = |O_X| + 2|P_{1j_k} \backslash O_X| = |P_{1j_k}| + |O_{X\backslash \{1, j_k\}}|.
\]
Hence the sum of the equation \[3.10\] vanishes, and we have \[1.3\].

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\section*{A Tarski’s principle for real closed fields}

A field $K$ is a real closed field if $K$ is an ordered field such that every positive element is a sum of squares in $K$, and every polynomial on $K$ of odd degree has at least one root in $K$ (see [2, p. 34]). It is known that $\mathbb{R}\{t\}$ is a real closed field (see [2 Theorem 2.91]). An important fact in a real closed field is the following:

\textbf{Theorem A.1} (Tarski’s principle (see [2 Theorems 2.80, 2.81])). A first-order statement is true in a real closed field if and only if it is true in every real closed field.
Here a first-order statement is a predicate constructed from addition, multiplication, equality, inequality, and the standard logical connectives and quantifiers. Hence any true first order statement in \( \mathbb{R} \) is also true in \( \mathbb{R}\{t\} \). For example, the statement “a polynomial \( P(z) \) in \( \mathbb{R}\{t\} \) is HPP” can be written as a first-order statement in \( \mathbb{R}\{t\} \) as follows. Substitute \( u + iv \) to \( z \in P(z) \), and represent \( P \) as \( P(u + iv) = Q(u, v) + iR(u, v) \), where \( Q, R \) are polynomials in \( \mathbb{R}\{t\} \). Then the HPP statement is equivalent to

\[
\forall u \forall v (Q(u, v) = 0 \land R(u, v) = 0 \rightarrow \neg(u \geq 0)).
\]

In this way, any polynomial relation in \( \mathbb{C}\{t\} \) can be written in polynomial relations in \( \mathbb{R}\{t\} \). Therefore the statement “an Hermite matrix has real eigenvalues only” and Sylvester’s law hold in \( \mathbb{C}\{t\} \), which were used in Section 2.1. Also Theorem 2.4 in Section 2.2 holds in \( \mathbb{R}\{t\} \).