Moduli spaces of 1-dimensional semi-stable sheaves and Strange duality on \( \mathbb{P}^2 \).

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Abstract. We study Le Potier’s strange duality conjecture on \( \mathbb{P}^2 \). We show the conjecture is true for the pair \((M(2,0,2), M(d,0))\) with \(d > 0\), where \(M(2,0,2)\) is the moduli space of semistable sheaves of rank 2, zero first Chern class and second Chern class 2, and \(M(d,0)\) is the moduli space of 1-dimensional semistable sheaves of first Chern class \(dH\) and Euler characteristic 0.

1 Introduction.

The strange duality conjecture is a very interesting and famous problem in the theory of moduli spaces of sheaves. It was first formulated for moduli spaces of vector bundles on curves by Beauville, Donagi and Tu in 1990s. Two groups of people proved this conjecture around 2007 ([2], [3] and [9]). Those are very remarkable works.

So far, there is no general extension of strange duality conjecture to moduli spaces of sheaves over surfaces. But under some conditions, this conjecture can be formulated, such as Le Potier’s formulation on the projective plan (see [5]) and Marian-Oprea’s formulation for K3 and Abelian surfaces (see [10]). In this article, we study the former.

Let us briefly review the set-up for strange duality conjecture. More details can be found in [5] and [10].

Let \(X\) be any smooth projective scheme. Let \(u\) and \(c\) be two elements in the Grothendieck group of coherent sheaves on \(X\), assume moreover \(u\) is orthogonal to \(c\) with respect to the Euler characteristic, i.e. the flat tensor \(F_u \otimes^L F_c\) is of Euler characteristic zero for any \(F_u\) (\(F_c\), resp.) a sheaf in class \(u\) (\(c\), resp.). Denote by \(M_u\) (\(M_c\), resp.) the moduli space of semistable sheaves
of class \( u \) (resp.), then there is a well-defined determinant line bundle \( \lambda_c \) (resp.) associated to \( c \) (resp.) on \( M(u) \) (resp.). Actually if there are strictly semistable sheaves, we will require a slightly stronger condition to define \( \lambda_u \) and \( \lambda_c \). We refer to Section 2 in [13] or Chapter 8 in [6] for the explicit definition of this “determinant line bundle”. Notice that the definition in [6] is dual to what we use in this paper.

The locus \( \mathcal{D} := \{ (F_u, F_c) \in M_u \times M_c | H^0(F_u \otimes F_c) \neq 0 \} \) is closed in \( M_u \times M_c \). If \( \mathcal{D} \) is a divisor of the line bundle \( \lambda_c \otimes \lambda_u \) (not always the case on surfaces), then the section induced by \( \mathcal{D} \) defines the following morphism up to scalars.

\[
SD : H^0(M_u, \lambda_c)^\vee \rightarrow H^0(M_c, \lambda_u).
\]

Strange duality conjecture then says \( SD \) is an isomorphism.

In the Le Potier’s version of strange duality, \( X = \mathbb{P}^2 \) with the hyperplane class \( H, u = (0, dH, 0) \) is a class of 1-dimensional sheaves with first Chern class \( dH \) and Euler characteristic 0, and \( c = (r, 0, n) \) is a class of rank \( r \) sheaves of first Chern class zero and second Chern class \( n \).

Very few is known in general about this conjecture for surfaces, even at numerical level, i.e. whether we have \( h^0(M_u, \lambda_c) = h^0(M_c, \lambda_u) \)? There are some results for special cases, for instance Danila proves that Le Potier’s strange duality holds for \( u = (0, dH, 0), c = (2, 0, n) \) for small \( n \) and \( d = 1, 2, 3 \) (see [4] and [5]); Abe shows that it holds for \( u = (0, dH, 0), c = (2, 0, n) \) for all \( n \) and \( d = 1, 2 \) (see [1]); the author shows that it holds for \( u = (0, dH, 0), c = (1, 0, n) \) for all \( n \) and \( d \) (see Section 4.3 in [13]). Marian and Oprea build a version of strange duality and prove that it holds in a large number of cases for generic K3 surfaces (see [11]).

In this article we prove the following theorem.

**Theorem 1.1** (Theorem 4.18). Let \( u = (0, dH, 0) \) with \( d > 0 \) and \( c = (2, 0, 2) \), then Le Potier’s strange duality conjecture is true for the pair \( (u, c) \), i.e. the map

\[
SD : H^0(M_c, \lambda_u)^\vee \rightarrow H^0(M_u, \lambda_c)
\]

is an isomorphism.

The proof of Theorem 1.1 is quite tricky: a priori we don’t have any numerical evidence of the conjecture at this case, and in fact we still don’t know how to compute directly the dimension of space at the right hand side. But the dimension of the space at the left hand side is relatively easy to compute, and we somehow get the injectivity of the map \( SD \) by using Fourier transform. Then we only need to show that \( \dim H^0(M_u, \lambda_c) \leq \dim H^0(M_c, \lambda_u) \).
In Section 2 we introduce some notations and also some basic properties of the moduli space $M_u$. In Section 3 we show the injectivity of $SD$ and finally in Section 4 we show the surjectivity.

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2 Notations and Preliminaries.

1. We are always on $\mathbb{P}^2$ and the base field is $\mathbb{C}$. $H$ is the hyperplane class in $\mathbb{P}^2$. $|dH|$ is the linear system of the divisor class $dH$.

2. Let $M(d,0)$ be the moduli space of 1-dimensional semi-stable sheaves with first Chern class $dH$ and Euler characteristic 0. $M(d,0)$ is a good quotient of a smooth quasi-projective variety, hence it is normal and Cohen-Macaulay. $M(d,0)$ is irreducible (Theorem 3.1 in [7]).

3. Let $c^r_n$ be a class in the Grothendieck group of coherent sheaves, such that $c^r_n = r[O_{\mathbb{P}^2}] - n[O_x]$ with $x$ a single point on $\mathbb{P}^2$.

4. Let $W(r,0,n)$ be the moduli space of semi-stable sheaves of class $c^r_n$, i.e. semi-stable sheaves of rank $r$, first Chern class 0 and second Chern class $n$.

5. We denote by $\Theta_d$ the determinant line bundle associated to $[O_{\mathbb{P}^2}]$ on $M(d,0)$. Since $\dim H^0(\Theta_d) = 1$ (see [5] or Theorem 4.3.1 in [13]), the line bundle $\Theta_d$ defines a unique divisor $D_{\Theta_d}$ which consists of sheaves with non trivial global sections.

6. By Proposition 2.8 in [8], $\Theta^r_{d}(n) := \Theta^r_d \otimes \pi^*O_{|dH|}(n)$ is the determinant line bundle associated to $c^r_n$ on $M(d,0)$ for any $r \geq 1$, $n \geq 0$, where $\pi : M(d,0) \rightarrow |dH|$ sends every sheaf to its support.

7. Let $\lambda_{c^r_n}(d)$ be the determinant line bundle associated to $u_d$ on $W(r,0,n)$, where $u_d$ is the class of sheaves in $M(d,0)$. We denote simply by $\lambda_r(d)$ if $r = n$.

3 Injectivity of the strange duality map.

In this section we prove the following proposition.
Proposition 3.1. The strange duality map

\[ SD : H^0(W(2,0,2),\lambda_2(d))^\vee \to H^0(M(d,0),\Theta_d^2(2)) \]  

is injective for all \( d > 0 \).

We first recall the Fourier transform on \( \mathbb{P}^2 \) (see also Section 4 in [7] or [12]). Let \( D \) be the universal curve in \( \mathbb{P}^2 \times |H| \) as follows.

\[
\mathbb{P}^2 \times |H| \supset D \overset{p}{\longrightarrow} \mathbb{P}^2. 
\]  

(3.2)

Let \( F \) be a pure 1-dimensional sheaf with Euler characteristic 0, then its Fourier transform is defined to be \( G_F := q_*(p^*(F \otimes \mathcal{O}_{\mathbb{P}^2}(2))) \otimes \mathcal{O}_{|H|}(-1) \).

Let \( G \) be a torsion free sheaf on \( |H| \) of rank \( r \), first Chern class 0 and Euler characteristic 0. Then its Fourier transform is defined to be \( F_G := R^1p_*(q^*(G \otimes \mathcal{O}_{|H|}(-1))) \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \). In general, these two Fourier transform need not be the inverse to each other, but the situation is quite better if we add semi-stability condition. Namely, identify \( |H| \) with \( \mathbb{P}^2 \) and Fourier transform gives a birational correspondence

\[ \Phi : M(d,0) \dashrightarrow W(d,0,d). \]  

(3.3)

We have the following theorem due to Le Potier (see Lemma 4.2 and Corollary 4.3 in [7])

**Theorem 3.2** (Le Potier). \( \Phi \) is defined over the complement of the divisor \( D_{\Theta_d} \subset M(d,0) \) and induces an isomorphism onto the open subset in \( W(d,0,d) \) corresponding to polystable sheaves whose restriction on a generic line \( \mathbb{P}^1 \cong l \in |H| \) is isomorphic to \( \mathcal{O}_l^{\oplus d} \). In particular, \( \Phi \) is an isomorphism for \( d = 1,2 \).

We have the following property for Fourier transform.

**Proposition 3.3.** Let \( G_1, G_2 \) be two torsion free sheaves with first Chern class zero and Euler characteristic zero. Let \( F_i \) be the Fourier transform of \( G_i \). Assume for a generic line \( l \in |H| \), \( H^0(G_i(-1)|_l) = 0 \) for \( i = 1,2 \). Then \( H^0(G_1 \otimes F_2) \neq 0 \Leftrightarrow H^0(G_2 \otimes F_1) \neq 0 \).

**Proof.** We view \( G_1 \) a sheaf on \( \mathbb{P}^2 \) and \( G_2 \) a sheaf on \( |H| \). It is enough to show

\[ H^0(G_1 \otimes F_2) \cong H^1(p^*(G_1(-1)) \otimes q^*(G_2(-1))) \cong H^0(G_2 \otimes F_1). \]  

(3.4)
Let \( \tilde{G} := p^*(G_1(-1)) \otimes q^*(G_2(-1)) \). Since \( H^0(G_i(-1)_l) = 0 \) for a generic line \( l \), we have \( p_*\tilde{G} = 0 = q_*\tilde{G} \) and then by spectral sequence we have

\[
H^0(R^1p_*(\tilde{G})) \cong H^1(\tilde{G}) \cong H^0(R^1q_*(\tilde{G})).
\]

To have (3.4), it is enough to have

\[
G_1 \otimes F_2 \cong R^1p_*(\tilde{G}) \quad \text{and} \quad G_2 \otimes F_1 \cong R^1q_*(\tilde{G}). \tag{3.5}
\]

Notice that \( G_2 = R^1p_*(q^*(G_2(-1)))(-1) \) and \( F_1 = R^1q_*(p^*(G_1(-1)))(-1) \). (3.5) is obvious if \( G_i \) are locally free. If \( G_i \) are not locally free, we then take the locally free resolution of \( G_i \) and it is easy to see that both of the two functors \( R^1p_*(q^*(G_2(-1)) \otimes -) \) and \( R^1p_*(q^*(G_2(-1)) \otimes p^*(-)) \) map exact complexes of torsion free sheaves into exact complexes of 1-dimensional sheaves, hence we have (3.5) and hence the proposition.

Analogously, we have the following proposition.

**Proposition 3.4.** Let \( F_1, F_2 \) be two pure 1-dimensional sheaves with Euler characteristic zero. Let \( G_i \) be the Fourier transform of \( F_i \). Assume that \( H^1(F_i(2)_l) = 0 \) for all lines \( l \in |H| \) and \( i = 1, 2 \). Then \( H^0(G_1 \otimes F_2) \neq 0 \iff H^0(G_2 \otimes F_1) \neq 0 \).

Proposition 3.3 and Proposition 3.4 implies the following corollary.

**Corollary 3.5.** For \( d = 1, 2 \), the isomorphism \( \Phi \) in (3.3) identifies the determinant line bundle \( \Theta^*_d(r) \) on \( M(d, 0) \) with \( \lambda_d(r) \) on \( M(d, 0, d) \) for any \( r > 0 \).

There is a strange duality map

\[
D: H^0(M(d, 0), \Theta^*_d(n))^\vee \to H^0(W(r, 0, n), \lambda_{c_n}(d)). \tag{3.6}
\]

Notice that \( D \) is the dual of \( SD \) in (3.1) for \( r = n = 2 \).

**Proposition 3.6.** Let \( d = 1, 2 \). Then the strange duality map \( D \) in (3.6) is injective for any \( r > 0 \) and \( n = ar \) with \( a \in \mathbb{Z}_{\geq 1} \).

**Proof.** Let \( G \) be a sheaf in \( W(r, 0, n) \). Then we can associate to \( G \) a section \( s_G \) of \( \Theta^*_d(n) \) given by the divisor \( D_G := \{ F \in M(d, 0) | H^0(F \otimes G) \neq 0 \} \). To show that \( D \) is injective, it is enough to show that we can find a collection of finitely many sheaves \( \{ G_i \} \) in \( W(r, 0, ar) \) such that \( \{ s_{G_i} \} \) spans \( H^0(M(d, 0), \Theta^*_d(ar)) \).

For \( d = 1, 2 \), \( M(d, 0) \cong |dH| \) and \( \Theta^*_d(n) \cong O_{dH}(n) \) for all \( r > 0 \) (see [3] or Proposition 4.1.1 in [13]). We choose a finite collection of distinct
points \( \{x_j\}_{j \in J} \), and associate each point \( x_j \) a divisor consisting of curves passing through \( x_j \), which gives a section \( t_j \) of \( \mathcal{O}_{dH}(1) \). Let \( N = \dim |dH| \), then it is possible to choose \( N + 1 \) distinct points \( x_j \) such that \( \{t_j\}_{j=1}^{N+1} \) spans \( H^0(\mathcal{O}_{dH}(1)) \). Hence we can choose \( n(N + 1) \) distinct points \( x_j \) with \( 1 \leq j \leq N + 1, 1 \leq k \leq n \) such that \( \{t_{j_1, \ldots, j_n}\} \) spans \( H^0(\mathcal{O}_{dH}(n)) \), where \( t_{j_1, \ldots, j_n} \) is defined as follows.

\[
t_{j_1, \ldots, j_n} := \prod_{k=1}^{n} t_{j_k}^{k}, \text{ with } t_{j_k}^{k} \text{ the section associated to } x_{j_k}^{k}.
\]

Let \( n = ar \), then we define a collection of semistable sheaves \( \{G_i\} \) consisting of all the sheaves of the form \( \bigoplus_{k=1}^{n} I_k \) with \( I_k \) the idea sheaf of a distinct points in \( \{x_k^{k}\} \), which we term the cosupport \( \text{Cosupp}(I_k) \) of \( I_k \). Let \( F \) be any semistable sheaf with first Chern class \( dH, d = 1, 2 \), Euler characteristic 0. It is easy to see that \( H^0(F \otimes G) \neq 0 \) for \( G = \bigoplus_{k=1}^{n} I_k \) if and only if \( \text{Supp}(F) \cap (\bigcup_{k=1}^{n} \text{Cosupp}(I_k)) \neq \emptyset \). Hence \( \{s_{G_i}\} \) spans \( H^0(\mathcal{O}_{dH}(ar)) = H^0(M(d, 0), \Theta^2_d(ar)) \). Hence the proposition. 

**Proof of Proposition 3.1.** According to Corollary 3.5 and Proposition 3.6, the injectivity of \( SD \) follows if we know that all \( G_i \) in \( W(r, 0, r) \) we used in the proof of Proposition 3.6 can be realized as Fourier transforms of sheaves \( F_i \) in \( M(r, 0) \). But this is true by Theorem 3.2 and the fact that the restrictions of \( G_i \) to a generic line are trivial. Hence the proposition.

**4 Surjectivity of the strange duality map.**

We already know that the map \( SD \) is injective, to prove the surjectivity, it is enough to show that

\[
dim H^0(M(d, 0), \Theta^2_d(2)) \leq \dim H^0(W(2, 0, 2), \lambda_2(d)). \tag{4.1}
\]

We also know that

\[
dim H^0(W(2, 0, 2), \lambda_2(d)) = \dim H^0(M(2, 0), \Theta^2_d(d)) = \binom{5 + d}{d}.
\]

**Lemma 4.1.** For \( d \leq 3 \), \( \dim H^0(M(d, 0), \Theta^2_d(2)) = \binom{5 + d}{d} \).

**Proof.** We can find this result in [5] or use Proposition 4.1.1 and Theorem 4.4.1 in [13] to compute directly.
Let \(M(d,0)^{int}\) be the open subscheme of \(M(d,0)\) parameterizing sheaves with integral supports. The following lemma follows from Proposition 3.8, Proposition 3.16 and Remark 4.11 in [14].

**Lemma 4.2.** The codimension of \(M(d,0) - M(d,0)^{int}\) is \(\geq \min\{d - 1, 7\}\), for all \(d\).

Now we assume \(d \geq 4\). Then S-equivalent classes of sheaves with non-integral supports form a subset of codimension \(\geq 3\) in \(M(d,0)\).

Recall that there is a unique divisor \(D_{\Theta_d}\) associated to the line bundle \(\Theta_d\). We have the following exact sequence.

\[
0 \rightarrow \Theta_d^{-1}(n) \rightarrow \Theta_d(n) \rightarrow \Theta_d(n)|_{D_{\Theta_d}} \rightarrow 0, \text{ for all } n, r. \tag{4.2}
\]

Recall that we have a projection \(\pi : M(d,0) \rightarrow |dH|\) sending every sheaf to its support. By Theorem 4.3.1 in [13], we have \(\pi_* \Theta_d \cong \mathcal{O}_{|dH|}\).

**Proposition 4.3.** \(R^i \pi_* \Theta^r_d = 0\) for all \(i > 0, r > 0\).

**Proof.** By Proposition 3.0.3 and Proposition 4.2.11 in [13], we know that \(\Theta^r_d(n)\) with \(r > 0\) has no higher cohomology as \(n \gg 0\). We may choose \(n\) very large such that \(R^i \pi_* \Theta^r_d(n)\) has no higher cohomology for all \(i\). Then we get a surjection \(H^i(\Theta^r_d(n)) \twoheadrightarrow H^0(R^i \pi_* \Theta^r_d(n))\) which implies for \(i > 0\), \(H^0(R^i \pi_* \Theta^r_d(n)) = 0\) as \(n \rightarrow +\infty\). Therefore \(R^i \pi_* \Theta^r_d\) has to be zero for \(i > 0\). \(\square\)

Proposition 4.3 together with the fact that \(\pi_* \Theta_d \cong \mathcal{O}_{|dH|}\) imply that \(H^i(\Theta_d(n)) = 0\) for all \(n \geq 0\). So for \(n \geq 0\) we have

\[
\dim H^0(\Theta^2_d(n)) = \dim H^0(\Theta_d(n)) + \dim H^0(\Theta^2_d(n)|_{D_{\Theta_d}}). \tag{4.3}
\]

In general we have

\[
\dim H^0(\Theta^r_d(n)) \leq \dim H^0(\Theta^{r-1}_d(n)) + \dim H^0(\Theta^r_d(n)|_{D_{\Theta_d}}). \tag{4.4}
\]

The crucial theorem is as follows.

**Theorem 4.4.** For \(d \geq 4, r > 0\) and \(n \geq 0\), we have

1. \(H^0(M(d,0), \Theta^r_d(n)|_{D_{\Theta_d}}) = 0\) if \(r > n\); 
2. \(\dim H^0(M(d,0), \Theta^r_d(n)|_{D_{\Theta_d}}) \leq \dim H^0(M(d-3,0), \Theta^{r-3}_{d-3}(r))\).
Actually, Proposition 4.3 is not necessary to our proof of the surjectivity. We will see that given Theorem 4.4, the equation in (4.4) will suffice to get the equation in (4.1).

**Corollary 4.5.** (1) \( H^0(M(d,0), \Theta_d^r(n)) \cong H^0(M(d,0), \Theta_d^r(n)) \) for \( r > n \).

(2) \( \dim H^0(M(d,0), \Theta_d^r(n)) \leq \sum_{k=0}^{\lfloor \frac{d}{3} \rfloor} \dim H^0(M(d-3k,0), \Theta_d^{r-1}(r)) + \dim H^0(M(d_0,0), \Theta_d^{r_0}(r)), \) with \( d_0 \equiv d \pmod{3} \) and \( 1 \leq d_0 \leq 3 \).

In order to prove Theorem 4.4, we first need to construct some birational maps relating \( D_{\Theta_d} \) and \( M(d-3,0) \) with the Hilbert scheme \( Hilb^{d(d-3) \over 2}(\mathbb{P}^2) \) of \( d(d-3) \) points on \( \mathbb{P}^2 \). The strategy is very similar to [14].

Let \( e := \frac{d(d-3)}{2} \) and \( H_e := Hilb^e(\mathbb{P}^2) \). Denote by \( \mathcal{I}_e \) the universal ideal sheaf over \( \mathbb{P}^2 \times H_e \).

Let \( Q_1 := Quot_{\mathbb{P}^2 	imes H_e/\mathcal{I}_e} (\mathcal{I}_e, dn) \) and \( Q_2 := Quot_{\mathbb{P}^2 	imes H_e/\mathcal{I}_e} (\mathcal{I}_e, (d-3)n) \) be the two relative Quot-schemes over \( H_e \) parametrizing quotients with Hilbert polynomial \( P(n) = dn \) and \( P(n) = (d-3)n \) respectively. Let \( \rho_1 : Q_1 \to H_e \) be the projection. Each point \([f_1 : I_e \to F_d] \in Q_1\) (\([f_2 : I_e \to F_{d-3}] \in Q_2\), resp.) over \([\mathcal{I}_e] \in H_e\) must have the kernel \( \mathcal{O}_{\mathbb{P}^2}(-3) \) \( (\mathcal{O}_{\mathbb{P}^2}, \text{resp.}) \). This is because \( Ker(f_i) \) are torsion free of rank 1 and second Chern class zero.

For any ideal sheaf \( I_e \) with colength \( e \), we have \( H^0(I_e(d-3)) \neq 0 \) and \( H^0(I_e(d)) \neq 0 \). Hence \( \rho_1 \) are surjective. We write down the following two exact sequences.

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to I_e(d-3) \to F_d \to 0; \tag{4.5}
\]
\[
0 \to \mathcal{O}_{\mathbb{P}^2} \to I_e(d-3) \to F_{d-3} \to 0. \tag{4.6}
\]

Notice that if both \( F_d \) and \( F_{d-3} \) are semi-stable, then (the class of) \( F_d \) is contained in \( D_{\Theta_d} \) and (the class of) \( F_{d-3} \) is contained in \( M(d-3,0) \).

We will construct rational maps \( g_1 : Q_1 \to D_{\Theta_d} \) and \( g_2 : Q_2 \to M(d-3,0) \). We then will use these two maps to relate \( H^0(M(d,0), \Theta_d^r(n)|_{D_{\Theta_d}}) \) with \( H^0(M(d-3,0), \Theta_d^{r-1}(r)). \)

**Convention.** If we have a product \( \mathbb{P}^2 \times M \) with \( M \) some moduli space (e.g. \( M(d,0), W(r,0,n), D_{\Theta_d}, H_e, Q_i \)), then we usually denote by \( p \) the projection \( \mathbb{P}^2 \times M \to \mathbb{P}^2 \), and \( q \) the projection \( \mathbb{P}^2 \times M \to M \). Most of the time, we use \( p \) and \( q \) without clarifying that they are maps from the product of \( \mathbb{P}^2 \) with the moduli space.
★ A birational map from $Q_1$ to $D_{\Theta_d}$.

Choose $m$ large enough. Let $\Omega_d$ be the smallest open subset of the Quot-scheme $\text{Quot}_{P^2}(O_{P^2}(-m)^\oplus dm, dn)$ containing all $GL(dm)$-orbits of semistable sheaves and sheaves appearing in $Q_1$.

Let $H^a_b$ with $a, b \in \mathbb{Z}_{\geq 0}$ be the locally closed subscheme of $H$ parametrizing ideal sheaves $I$ such that $h^0(I(e(d - 3)) = a, h^1(I_e(d)) = b$. Let $Q^a_b$ be the preimage of $H^a_b$ via $\rho_1$. $H^a_b$ is empty unless $a \geq 1$. $\mathcal{H}om_p(q^*O_{P^2}(-3), I_e(d - 3)|_{P^2 \times H^a_b})$ is a locally free sheaf of rank $3d + 1 + b$ on $H^a_b$ and the following lemma is trivial.

**Lemma 4.6.** $Q^a_b \cong \mathbb{P}(\mathcal{H}om_p(q^*O_{P^2}(-3), I_e(d - 3)|_{P^2 \times H^a_b}))$.

Let $\Omega^a_b$ parametrizing quotients $F$ such that $h^0(F) = a$ and $h^1(F(3)) = b$. We have a universal quotient $Q_d$ on $P^2 \times \Omega_d$. $\mathcal{V}^{a,b} := \mathcal{E}xt^1(Q_d|_{\Omega^a_b}, q^*O_{P^2}(-3))$ is locally free of rank $a$ on $\Omega^a_b$. $Q_d$ is naturally $GL(dm)$-linearized, hence so is $\mathcal{V}^{a,b}$. The projective bundle $\mathbb{P}(\mathcal{V}^{a,b})$ has a natural $PGL(dm)$-action, and the projection $\mathbb{P}(\mathcal{V}^{a,b}) \to \Omega^{a,b}$ is $PGL(dm)$-equivariant. In particular if $a = 1$, $\mathbb{P}(\mathcal{V}^{a,b}) = \Omega^{a,b}$. There is an open subscheme $P^0_1 \subset \mathbb{P}(\mathcal{V}^{a,b})$ parametrizing torsion free extensions of $F_s$ by $O_{P^2}(-3)$ for all $s \in \Omega^a_b$. Then we have a classifying map $f^{a,b}_1 : P^0_1 \to Q^a_1$. Notice that $H^0(I_e(d - 3)) \cong H^0(F_d)$ and $H^1(I_e(d)) \cong H^1(F_d(3))$.

**Lemma 4.7.** $f^{a,b}_1 : P^0_1 \to Q^a_1$ is a principal $PGL(dm)$-bundle.

*Proof.* $PGL(dm)$ acts freely on $\mathbb{P}(\mathcal{V}^{a,b})$ and the map $f^{a,b}_1$ is $PGL(dm)$-invariant with fiber isomorphic to $PGL(dm)$. \qed

$\Omega_d$ is a smooth atlas of the moduli stack $\mathcal{M}_d$ parametrizing pure 1-dimensional $m$-regular sheaves with first Chern class $dH$ and Euler characteristic 0, with fiber $GL(dm)$. Denote by $\xi$ the dimension of $GL(dm)$. Then $\text{dim} \ \Omega_d = \xi + d^2$ by Corollary 2.6 in [14]. Let $\Omega_{d,\text{int}}$ parametrizing quotients in $\Omega_d$ with integral support. Then by Proposition 3.8 and Proposition 3.16, the complement of $\Omega_{d,\text{int}}$ in $\Omega_d$ is of codimension $\geq \min\{d - 1, 7\}$. It is easy to see that $\Omega_{d,\text{int}}$ is smooth and connected, hence $\Omega_d$ is irreducible.

Let $D_{\Omega_d} = \cup_{a \geq 1} \Omega^a_d$, then $D_{\Omega_d}$ is actually a divisor associated to the determinant line bundle $(\text{det} \ R^*p_1^*\Omega_d)^\vee$ on $\Omega_d$. By the semicontinuity of the dimension of the cohomology groups, we know that $\Omega^0_d$ is of dimension $\xi + d^2 - 1$. Let $\Omega^a_d = \Omega^a_d \cap \Omega_{d,\text{int}}$, then the complement of $\Omega^a_d$ in $\Omega^a_d$ is of codimension
≥ \min\{d - 2, 6\} ≥ 2 since \(d ≥ 4\). Analogously we have \(P^o_1 ⊂ P^{1,0}_1\) and \(Q^o_1 ⊂ Q^{1,0}_1\), and the complement of \(P^o_1\) (\(Q^o_1\), resp.) in \(P^{1,0}_1\) (\(Q^{1,0}_1\)) is of codimension ≥ 2. In particular since \(a = 1\) and all quotients \(F\) are of integral support, \(P^o_1 = \Omega^o_d\). Since sheaves with integral supports are all stable, the universal family on \(Q^o_1\) induces a morphism \(g_1 : Q^o_1 → M(d, 0)\) with image contained in the divisor \(D_{Θ_o}\).

Let \(D^{\text{int}}_{Θ_o} := D_{Θ_o} ∩ (M(d, 0))^{\text{int}}\). The complement of \(D^{\text{int}}_{Θ_o}\) in \(D_{Θ_o}\) is of codimension ≥ 2, because \(d ≥ 4\). \((M(d, 0))\) is Cohen-Macaulay, hence so is \(D_{Θ_o}\) and \(D^{\text{int}}_{Θ_o}\). \(D_{Θ_o}\) is integral because \(D^{\text{int}}_{Θ_o}\) is. Let \(D^{\text{int}}_{Θ_o} ⊂ D^{\text{int}}_{Θ_o}\) parametrizing sheaves \(F\) such that \(\dim H^0(F) = 1\) and \(\dim H^1(F(3)) = 0\). The image of \(Q^o_1\) is obviously contained in \(D^{\text{int}}_{Θ_o}\) and also it is surjective.

**Lemma 4.8.** The map \(g_1 : Q^o_1 → D^{\text{int}}_{Θ_o}\) is an isomorphism.

**Proof.** This is because both \(Q^o_1\) and \(D^{\text{int}}_{Θ_o}\) are the geometric \(PGL(dm)\)-quotient of \(Ω^o_d\). \(\square\)

**Remark 4.9.** We know that there is no universal sheaf over any open subset of \((M(d, 0))^{\text{int}}\). But Lemma 4.8 implies that there is a universal sheaf over the locally closed subset \(D^{\text{int}}_{Θ_o}\) in \((M(d, 0))^{\text{int}}\).

**Remark 4.10.** Our argument for the birationality between \(Q^1\) and \(D_{Θ_o}\) can be simplified, if we use “stack language” as we did in [14]. But we stick to schemes here because we don’t want to talk about line bundles over stacks.

By Lemma 4.8 we have

\[
H^0(Θ^o_d(n)|D^{\text{int}}_{Θ_o}) \cong H^0(Q^o_1, g^*1(Θ^o_d(n))). \tag{4.7}
\]

We will see that \(g^*1(Θ^o_d(n))\) can be extended to a line bundle \(Λ^r_n\) on \(Q^{1,0}_1\).

By deformation theory, the relative obstruction space of \(Q^1\) over \(H\) is \(\text{Ext}^1(Ω^{−2}_2(−3), F) = H^1(F(3))\) at each \([l_e → F]\). Hence \(ρ\) restricted on \(Q^{a,0}_1\) is smooth for any \(a ≥ 1\). \(H^{1,0}_e\) is open in \(H_e\) hence smooth and hence so is \(Q^{1,0}_1\).

There is a universal quotient \(F^{1,0}_1\) on \(P^2 × Q^{1,0}_1\). By Lemma 4.6 we have an exact sequence as follows.

\[
0 → q^*Ω^{−2}_2(−3) ⊗ p^*Ω^1_{ρ1} → (id_{P^2} × ρ1)^*_Lc ⊗ q^*Ω^{−2}_2(d − 3) → F^{1,0}_1 → 0, \tag{4.8}
\]

where \(Ω^1_{ρ1} (−1)\) is the relative tautological line bundle of the projective bundle \(P(H\text{om}_p(Ω^{−2}_2(−3), Lc|_{P^2 × H^{1,0}_e}))\). Let \(G^r_n\) be a torsion free sheaf of class \(c^r_n\) on \(P^2\). Define

\[
Λ^r_n := (\text{det}(R^*p_*(F^{1,0}_1 ⊗ q^*G^r_n)))^\vee.
\]
Then by the universal property of the determinant line bundle, \( \Lambda^r_n \mid Q^r_1 \cong g^r_1 \Theta_{\delta}^r (n) \). Since \( Q^{1,0}_1 \) is smooth and the complement of \( Q^r_1 \) in \( Q^{1,0}_1 \) is of codimension \( \geq 2 \), we have

\[
H^0(Q^{1,0}_1; g^r_1 \Theta_{\delta}^r (n)) = H^0(Q^{1,0}_1, \Lambda^r_n). \tag{4.9}
\]

On the other hand by (4.8), we have

\[
\Lambda^r_n \cong \text{det}(R^*p_*(q^*G^r_n(-3) \otimes p^*O_{\rho_1}(-1))) \otimes (\text{det}((id_{\mathbb{P}^2} \times \rho_1)^*\mathcal{I}_e \otimes q^*G^r_n(d - 3)))^\vee
\cong O_{\rho_1}(-1) \otimes \chi(G^r_n(-3)) \otimes \rho_1^*(\text{det}(R^*p_*(\mathcal{I}_e \otimes q^*G^r_n(d - 3))))^\vee \tag{4.10}
\]

Notice that the maps \( p \) and \( q \) at the first line of (4.10) are from \( \mathbb{P}^2 \times Q^{1,0}_1 \) to \( Q^{1,0}_1 \) and \( \mathbb{P}^2 \) respectively, while \( p, q \) at the second line are from \( \mathbb{P}^2 \times H^{1,0}_e \) to \( H^{1,0}_e \) and \( \mathbb{P}^2 \). As is said in the convention before, we don’t change the letters although they are different maps.

**Proof of Statement (1) in Theorem 4.4.** By (4.10), \( \rho^1_1 \Lambda^r_n = 0 \) if \( \chi(G^r_n(-3)) > 0 \Leftrightarrow \chi(G^r_n) = r - n > 0 \). Since \( D_{\Theta_d} \) is integral and \( D^\circ_{\Theta_d} \) is open in \( D_{\Theta_d} \). We have an injection \( H^0(D_{\Theta_d}, \Theta^{r}_{d}(n)) \hookrightarrow H^0(D^\circ_{\Theta_d}, \Theta^{r}_{d}(n)) \). Then Statement (1) follows from (1.7) and (4.9).

Let \( r = n \), then \( \rho^1_1 \Lambda^r_n \cong (\text{det}(R^*p_*(\mathcal{I}_e \otimes q^*G^r_n(d - 3))))^\vee =: L^r_{1,0} \). There is an obvious extension of \( L^r_{1,0} \) to a line bundle \( L^r \) on the whole \( H^{1,0}_e \).

**Lemma 4.11.** The complement of \( D^\circ_{\Theta_d} \) in \( D_{\Theta_d} \) is of codimension \( \geq 2 \). In particular since \( D^\circ_{\Theta_d} \) is smooth, \( D_{\Theta_d} \) is normal.

**Proof.** It is enough to show the statement for \( D_{\Theta_d} \) replaced by \( D^\text{int.}_{\Theta_d} \). Denote by \( M(d, 0)^\text{int.}_{(a,b)} \) the analogous meaning to before. Then we want to prove that \( \cup_{a>1} \cup_{b>0} M(d, 0)^\text{int.}_{(a,b)} \) is of codimension \( \geq 3 \) in \( M(d, 0) \). By Proposition 2.14 in \[4\], we only need to show \( M(d, 0)^\text{int.}_{(2,0)} \) is of codimension \( \geq 3 \) in \( M(d, 0) \). Hence it is enough to show that \( \dim \Omega^{2,0}_{d, \text{int}} \leq \xi + d^2 - 3 \), where \( \xi = \dim \text{GL}(dm) \).

By Lemma 4.6 and Lemma 4.7, we have

\[
\dim \Omega^{2,0}_{d, \text{int}} = \dim H^{2,0}_{e} + 3d + \xi - 1 - 1 = \dim H^{2,0}_{e} + 3d + \xi - 2.
\]

\( H^{2,0}_{e} \) is locally closed in \( H_{e} \) and \( \dim H^{2,0}_{e} \leq \dim H_{e} - 1 = d(d - 3) - 1 \) because \( H_{e} \) is irreducible. Hence the lemma.

**Lemma 4.12.** The complement of \( H^{1,0}_{e} \) in \( H_{e} \) is of codimension \( \geq 2 \).
We will prove Lemma 4.12 in next subsection. Because $M(d,0)$ is irreducible, normal and Cohen-Macaulay, Lemma 4.11 and Lemma 4.12 together with (4.7) and (4.9) implies the following proposition.

**Proposition 4.13.** \( \dim H^0(D_{\Theta_d}, \Theta^*(d)\vert_{D_{\Theta_d}}) = \dim H^0(H_e, L^r) \), for all \( r > 0 \).

\[ \star \text{ A rational map from } Q_2 \text{ to } M(d - 3, 0). \]

Again we choose \( m \) large enough and let \( \Omega_{d-3} \) be the smallest open subset of the Quot-scheme \( \text{Quot}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(-m)^{(d-3)m}, (d - 3)n) \) containing all \( GL((d - 3)m) \)-orbits of semistable sheaves and sheaves appearing in \( Q_2 \). Let \( Q_{d-3} \) be the universal quotient over \( \mathbb{P}^2 \times \Omega_{d-3} \).

Define \( \Omega_{d-3, \text{int}} \) analogously. Define \( \Omega_{d-3}^{a,c} \) to be the locally closed subscheme of \( \Omega_{d-3} \) parametrizing quotients \( F \) such that \( h^0(F) = a \) and \( h^0(F(-3)) = c \). Let
\[
\Omega_{d-3}^A \coloneqq \sqcup_{a \leq 3} \Omega_{d-3}^{a,0} \quad \text{and} \quad \Omega_{d-3}^o \coloneqq \Omega_{d-3,\text{int}} \cap \Omega_{d-3}^1.
\]

**Lemma 4.14.** (1) The complement of \( \Omega_{d-3}^0 \) is of codimension \( \geq 1 \).

(2) The complement of \( \Omega_{d-3}^o \) in \( \Omega_{d-3} \) is of codimension \( \geq 2 \), if \( d - 3 \neq 2 \).

**Proof.** Directly follows from results in [14] (see Proposition 2.14, Remark 2.15, Proposition 3.8 and Proposition 3.16 in [14]). \( \square \)

\( \mathcal{E}x_{t_p}(Q_{d-3}, q^*\mathcal{O}_{\mathbb{P}^2}) \) is locally free of rank \( 3(d - 3) \) on \( \Omega_{d-3}^A \). The projective bundle \( P^A_2 \coloneqq \mathbb{P}(\mathcal{E}x_{t_p}(Q_{d-3}\vert_{\mathbb{P}^2 \times \Omega_{d-3}^A}, q^*\mathcal{O}_{\mathbb{P}^2})) \) has a natural \( PGL((d - 3)m) \)-action, and the projection \( P^A_2 \to \Omega_{d-3}^A \) is \( PGL((d - 3)m) \)-equivariant. Let \( P^A_2 \) be the open subset of \( P^A_2 \) parametrizing torsion free extensions. Denote \( P^o_2 \) the restriction of \( P^A_2 \) to \( \Omega^o_{d-3} \). \( P^o_2 \) is a projective bundles over \( \Omega^o_{d-3} \). We have a morphism \( f^A_2 : P^A_2 \to Q_2 \) induced by the universal extension on \( P^A_2 \). Denote by \( Q^A_2 \) (\( Q^o_2 \), resp.) the image of \( P^A_2 \) (\( P^o_2 \), resp.), and \( f^o_2 \) the restriction of \( f^A_2 \) to \( P^o_2 \). Then we have the following lemma analogous to Lemma 4.7.

**Lemma 4.15.** \( \forall A \geq 0, f^A_2 : P^A_2 \to Q^A_2 \) is a principal \( PGL((d - 3)m) \)-bundle.

We have a commutative diagram

\[
\begin{array}{ccc}
P^o_2 & \xrightarrow{\sigma^o_2} & \Omega^o_{d-3} \\
f^o_2 \downarrow & & \downarrow f^o_{M} \\
Q^o_2 & \xrightarrow{g^o_2} & M(d - 3, 0)^o
\end{array}
\]
In (4.11), \( f_2^o \) and \( f_M^o \) are principal \( PGL((d-3)m) \)-bundles. \( \sigma_2^o \) is \( PGL((d-3)m) \)-equivariant, hence it descends and gives the map \( g_2 \). Moreover, \( P_2^o \) is a projective bundle over \( \Omega_{d-3} \), hence \( \sigma_2^o \mathcal{O}_{P_2^o} \cong \mathcal{O}_{\Omega_{d-3}} \) and hence \( g_2 \mathcal{O}_{Q_2^o} \cong \mathcal{O}_{M(d-3,0)^{o}} \). Hence

\[
H^0(M(d-3,0)^{o}, \Theta_{d-3}^o(n)) \cong H^0(Q_2^o, g_2^o \Theta_{d-3}^o(n)). \tag{4.12}
\]

Recall the projective map \( \rho_2 : Q_2 \to H_e \). By deformation theory and the exact sequence (4.6), \( \rho_2 \) is smooth at the locus \( H_e^{sm} \) where \( h^0(I_e(d-3)) = 1 \). Actually it is easy to see that \( \rho_2 \) induces an isomorphism between \( H_e^{sm} \) and its preimage. Hence \( Q_2 \) and \( H_e \) are birational.

Denote by \( H_e^\leq A \) the open subscheme of \( H_e \) parametrizing ideal sheaves \( I_e \) such that \( h^0(I_e(d-3)) \leq A+1 \) and \( h^0(I_e(d-6)) = 0 \). By (4.9), \( h^0(I_e(d-3)) = h^0(F_{d-3}) + 1 \) and \( h^0(I_e(d-6)) = h^0(F_{d-3}(-3)) \). Hence the preimage of \( H_e^\leq A \) is exactly \( Q_2^e^{\leq A} \).

**Lemma 4.16.** The complement of \( H_e^\leq 0 \) in \( H_e \) is of codimension \( \leq 2 \). In particular, \( Q_2^e^{\leq A} \) is irreducible containing \( Q_2^e \) as an dense open subset for all \( A \geq 0 \).

**Proof.** By Lemma 4.2 in \[14\], points \([I_e]\) such that \( h^0(I_e(d-6)) \neq 0 \) forms a subset of codimension \( \geq 3(d-3) \geq 3 \). Hence to show the lemma, it is enough to show that \( H_e^\leq A \) is of dimension \( \leq d(d-3) - 2 \) for all \( A \geq 0 \). The relative dimension of \( \rho_2 \) is \( \geq 1 \) away from \( H_e^{sm} \), hence it is enough to show that \( Q_2^e^{\leq A} \) is of dimension \( \leq d(d-3) - 1 = \dim Q_2^{e^{\leq 0}} - 1 \), which follows from Statement (1) of Lemma 4.14 and Lemma 4.15.

Finally \( Q_2^e^{\leq A} \) is irreducible because the dense open subset \( Q_2^{e^{\leq 0}} \) is isomorphic to an open subset of \( H_e \), hence irreducible. Hence the lemma.

**Proof of Lemma 4.12** By Lemma 4.3 in \[14\], we know that points \([I_e]\) such that \( H^1(I_e(d)) \neq 0 \) forms a subset of codimension \( \geq 2 \) in \( H_e^{\leq 0} \), hence Lemma 4.12 follows directly from Lemma 4.16.

**Lemma 4.17.** \( \rho_2^* \mathcal{O}_{Q_2^e^{\leq A}} \cong \mathcal{O}_{H_e^{\leq A}} \) for all \( A \geq 0 \).

**Proof.** Since \( \rho_2 \) is a birational projective morphism and both \( Q_2^e^{\leq A} \) and \( H_e^{\leq A} \) are integral, the lemma follows from Zariski main theorem.

On \( \mathbb{P}^2 \times Q_2^{e^{\leq A}} \) we have an exact sequence given by the universal family with \( \mathcal{F}_2 \) the universal quotient.

\[
0 \to \mathcal{R} \to (id_{\mathbb{P}^2} \times \rho_2)^* \mathcal{I}_e \otimes q^* \mathcal{O}_{\mathbb{P}^2}(d-3) \to \mathcal{F}_2 \to 0. \tag{4.13}
\]
The kernel $\mathcal{R}$ is a line bundle on $\mathbb{P}^2 \times Q_2^{\leq A}$. Let $R := p_2^* \mathcal{R}$. Since $\mathcal{R}$ restricted to the fiber over each point in $Q_2^{\leq A}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}$, $R$ is a line bundle. There is a natural map $\mathcal{R} \rightarrow p^* \mathcal{R}$, which is injective since $\mathcal{R}$ is of rank one, and surjective since it is surjective when restricted to the fiber over any point of $Q_2^{\leq A}$. Hence $\mathcal{R} \cong p^* R = p^*(p, \mathcal{R})$.

Let $G_n^r$ be a torsion free sheaf of class $e_n^r$ on $\mathbb{P}^2$. Define
\[
\Sigma_n^r := (\det(R^* p_*(\mathcal{F}_2 \otimes q^* G_n^r)))^\vee.
\]
Then by the universal property of the determinant line bundle, $\Sigma_n^r |_{Q_2^r} \cong g_2^* \Theta_{d-3}^r(n)$. Since $Q_2^{\leq A}$ is irreducible and $Q_2^r$ is open in $Q_2^{\leq A}$, we have
\[
dim H^0(Q_2^r, g_2^* \Theta_{d-3}^r(n)) \leq \dim H^0(Q_1^{\leq A}, \Sigma_n^r).
\]
(4.14)

On the other hand by (4.13), we have
\[
\Sigma_n^r \cong \det(R^* p_*(q^* G_n^r \otimes p^* R)) \otimes (\det((id_{\mathbb{P}^2} \times \rho_2)^* \mathcal{I}_e \otimes q^* G_n^r(d-3)))^\vee \\
\cong R^\otimes (G_n^r) \otimes \rho_2^*(\det(R^* p_*(\mathcal{I}_e \otimes q^* G_n^r(d-3)))^\vee)
\]
(4.15)

Let $r = n$, then by Lemma 4.17 we have $\rho_2^* \Sigma_n^r \cong (\det(R^* p_*(\mathcal{I}_e \otimes q^* G_n^r(d-3)))^\vee) \otimes L^r |_{H^{\leq A}}$. Combine (4.12) (4.14) and Lemma 4.16 we have
\[
dim H^0(H_e, L^r) \leq \dim H^0(M(d-3,0)^o, \Theta_{d-3}^r(r)), \forall r > 0.
\]
(4.16)

**Proof of Statement (2) in Theorem 4.4.** For $d - 3 \neq 2$, by Statement (2) in Lemma 4.14, the complement of $M(d-3,0)^o$ in $M(d-3,0)$ is of codimension $\geq 2$. Hence by Proposition 4.13 and (4.16) we are done.

Let $d-3 = 2$. Notice that in this case any sheaf in $O_2$ is semistable if and only if $h^0(F) = 0$. Let $O_2'$ parametrizing sheaves $F$ such that $h^0(F) = 0$ and $\text{Supp}(F)$ is reduced, i.e. not a double line in $|2H|$. We then have the following diagram analogous to (4.11)
\[
P_2' \xrightarrow{\sigma'_2} \Omega'_2 \xrightarrow{f'_2} \Omega'_2.
\]
(4.17)

The complement of $M(2,0)'$ is of codimension $\geq 2$ in $M(2,0)$. Now in (4.17) we still have $f_2'$ a principal $\text{PGL}(2m)$-bundle, but $f_M'$ only a good quotient. $\sigma'_2$ is $\text{PGL}(2m)$-equivariant, hence it descends and gives the map $g_2$. $P_2' \subseteq P_2$ with $P_2'$ the corresponding projective bundle over $\Omega'_2$. What we want is
\( g_2\mathcal{O}_{Q_2} \cong \mathcal{O}_{M(2,0)} \), and once we have this condition the rest of our argument for \( d - 3 \neq 2 \) applies and then we are done.

In order to show \( g_2\mathcal{O}_{Q_2} \cong \mathcal{O}_{M(d-3,0)} \), we need to show that \( \sigma'_2, \mathcal{O}_{P_2} \cong \mathcal{O}_{\Omega'} \). We have that \( \sigma'_2, \mathcal{O}_{\tilde{P}_2} \cong \mathcal{O}_{\Omega'} \). \( \Omega' \) is smooth and irreducible and hence so is \( \tilde{P}_2 \). By a direct observation we see that the complement of \( P_2 \) in \( \tilde{P}_2 \) is of codimension 2 and hence \( j_* \mathcal{O}_{P_2} \cong \mathcal{O}_{\tilde{P}_2} \) with \( j : P_2 \hookrightarrow \tilde{P}_2 \). On the other hand \( \sigma'_2 = \tilde{\sigma}_2 \circ j \), hence \( \sigma'_2, \mathcal{O}_{P_2} \cong \sigma'_2, (j_* \mathcal{O}_{P_2}) \cong \sigma'_2, \mathcal{O}_{\tilde{P}_2} \cong \mathcal{O}_{\Omega'} \).

Hence we have proven Theorem 4.4. \( \square \)

**Theorem 4.18.** The strange duality map

\[
SD : H^0(W(2,0,2), \lambda_2(d))^{\vee} \rightarrow H^0(M(d,0), \Theta_2^2(d)) \quad (4.18)
\]

is an isomorphism for all \( d > 0 \).

**Proof.** By Theorem 4.3.1 in [13], we know that \( \pi_* \Theta_d \cong \mathcal{O}_{|dH|} \) for all \( d > 0 \) with \( \pi : M(d,0) \rightarrow |dH| \). Hence \( \dim H^0(M(d,0), \Theta_d(n)) = \binom{n + \frac{d(d+3)}{2}}{n} \). By Corollary 4.3 and Lemma 4.1 we get the following equation

\[
\dim H^0(M(d,0), \Theta_2^2(d)) \leq \binom{5 + d}{d} = \dim H^0(W(2,0,2), \lambda_2(d)).
\]

Hence \( SD \) is an isomorphism because it is injective by Proposition 3.1. \( \square \)

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