ON CONSTRUCTION OF A NEW INTERPOLATION TOOL:
CUBIC \( q \)-SPLINE

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Abstract. This work presents a new interpolation tool, namely, cubic \( q \)-spline. Our new analogue generalizes a well-known classical cubic spline. This analogue, based on the Jackson \( q \)-derivative, replaces an interpolating piecewise cubic polynomial function by \( q \)-polynomials of degree three at most. The parameter \( q \) provides a solution flexibility.

1. INTRODUCTION

The interpolation problem of an unknown function \( f(x) \) when only the values of \( f(x_i) \) at some point \( x_i \) are given arises in different areas. One of widely used methods is a spline interpolation, and, particularly, cubic spline interpolation. That means that the function \( f(x) \) is interpolated between two adjacent points \( x_i \) and \( x_{i+1} \) by a polynomial of degree three at most. Such interpolation is very suitable for smooth functions which do not have oscillating behaviour (cf. [1]). Another advantage of the cubic polynomial interpolation is that it leads to a system of linear equations which is described by a tridiagonal matrix. This linear system has a unique solution which can be fast obtained. A generalization of cubic spline interpolation was done by Marsden (see [2]). He chose a \( q \)-representation for knots and normalized \( B \)-splines as interpolating functions. \( B \)-splines are the generalization of the Bezier curves which are built with the help of Bernstein polynomials and they play an important role in the theory of polynomial interpolation. Their \( q \)-analogues were defined and studied in [3, 4]. For further information and other details related to \( q \)-analogues of Bernstein polynomials, Bezier curves and splines, see the recent works [3, 6, 7, 8, 9, 10]. Another \( q \)-generalization of polynomial interpolation was studied in [11]. Despite the popularity of \( B \)-spline, new generalizations of spline continue to appear. Some of them concerns about preserving of convexity [12], some of them about smoothness of interpolation [13], and others about degrees of freedom [14]. Classical cubic spline already proved itself in data and curve fitting problems. Our new \( q \)-generalization gives it a new interesting twist. The matrix describing the linear system is tridiagonal like in the classic case, and a solution of this linear system can be obtained by simple recursive algorithm (see for example [15]). The \( q \)-parameter provides a flexibility of the solution.

We start from a short review of definitions coming from the quantum calculus (cf. [16]). The \( q \)-derivative is given by

\[
D_q f(x) = \frac{f(qx) - f(x)}{qx - x}.
\]

For any complex number \( c \), its \( q \)-analogue is defined as \([c]_q = \frac{q^c - 1}{q-1}\). For natural \( n \), a \( q \)-factorial is defined as \([n]_q! = \prod_{k=1}^{n} [k]_q \), with \([0]_q! = 1\).
The $q$-analogue of the polynomial is $(x-c)^n_q = \prod_{k=1}^n (x - cq^{k-1})$ by assuming as usually that $(x-c)^0_q = 1$. It is easy to show that $D_q(x-c)^n_q = [n]_q(x-c)^{n-1}_q$. We will denote by $D_q^k$ the $k$-th $q$-derivative. The Jackson $q$-integral of $f(x)$ is defined as (cf. [16, 17])

$$\int f(x)d_qx = (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x),$$

and $D_q \int f(x)d_qx = f(x)$. Note that $q$ is usually considered to be $0 < q < 1$.

In the next section we describe a process of building of a cubic $q$-spline.

2. BUILDING OF A $q$-ANALOGUE OF CUBIC SPLINE

Let a function $f(x)$ is given by its values $f(x_i) = f_i$, $0 \leq i \leq n$, at $n + 1$ nodes $a = x_0 < x_1 < \ldots < x_n = b$. We define the cubic $q$-spline function $S(x; q)$ as a function of a variable $x$ with parameter $q$ as following:

$$S(x; q) = \begin{cases} S_i(x_1; q) & x_0 \leq x \leq x_1, \\ S_i(x_2; q) & x_1 < x \leq x_2, \\ \ldots \ldots \\ S_n(x; q) & x_{n-1} < x \leq x_n, \end{cases}$$

where each $S_i(x_1; q)$, $1 \leq i \leq n$, is a $q$-polynomial in variable $x$ of degree at most three, and $S(x; q)$, $D_qS(x; q)$, $D_q^2S(x; q)$ are continuous on $[a, b]$. To provide these properties we demand

$$\begin{cases} S_i(x_{i-1}; q) = f(x_{i-1}), & i = 1, \ldots, n, \\ S_i(x_i; q) = f(x_i), & i = 1, \ldots, n, \end{cases}$$

$$D_qS_i(x_i; q) = D_qS_{i+1}(x_i; q), \quad i = 1, \ldots, n-1,$$

$$D_q^2S_i(x_i; q) = D_q^2S_{i+1}(x_i; q), \quad i = 1, \ldots, n-1.$$  

The boundary conditions for the clamped cubic $q$-spline are

$$\begin{cases} D_qS_0(x_0; q) = D_qf(x_0), \\ D_qS_n(x_n; q) = D_qf(x_n). \end{cases}$$

Let us denote by $\mu_i(q)$, $i = 0, \ldots, n$ the value of the second $q$-derivative of the spline $S(x; q)$ at the node $x_i$, that is $\mu_i(q) = D_q^2S(x; q)$. Since the spline $S(x; q)$ is a polynomial of degree at most three, its second derivative is a polynomial of degree at most one. Therefore it can be written as

$$D_q^2S_i(x; q) = \mu_{i-1}(q) \frac{x_i - x}{h_i} + \mu_i(q) \frac{x - x_{i-1}}{h_i},$$

where $h_i = x_i - x_{i-1}$ and $x_{i-1} \leq x \leq x_i$ for $1 \leq i \leq n$. By performing $q$-integration we obtain

$$D_qS_i(x; q) = \int D_q^2S_i(x; q)d_qx = \frac{\mu_i(q)}{2_q h_i} (x - x_{i-1})^2 - \frac{\mu_{i-1}(q)}{2_q h_i} (x - x_i)^2 + A_i(q).$$

Let us denote by $f[x_{i-1}, x_i] = \frac{f(x) - f(x_{i-1})}{x_i - x_{i-1}}$ the first order divided difference of a function $f(x)$ at nodes $x_{i-1}, x_i$. By integrating (6), we obtain

$$S_i(x; q) = \int D_qS_i(x; q)d_qx = \frac{\mu_i(q)}{3_q h_i} (x - x_{i-1})^3 - \frac{\mu_{i-1}(q)}{3_q h_i} (x - x_i)^3 + A_i(q)(x - x_{i-1}) + B_i(q),$$
where \( A_i(q) \) and \( B_i(q) \) depend on parameter \( q \) only and can be found by substituting \( x = x_i \) and \( x = x_{i-1} \) in (7) respectively and using the conditions (2) as following

\[
(8) \quad B_i(q) = f(x_{i-1}) + \frac{\mu_{i-1}(q)}{[3]_q h_i} (x_{i-1} - x_i)^3,
\]

\[
(9) \quad A_i(q) = \frac{f(x_i) - f(x_{i-1})}{h_i} - \frac{\mu_i(q)}{[3]_q h_i^2} (x_i - x_{i-1})^3 - \frac{\mu_{i-1}(q)}{[3]_q h_i^2} (x_{i-1} - x_i)^3.
\]

By substituting the detailed expressions (8–9) for functions \( A_i(q) \) and \( B_i(q) \), we obtain the \( i \)th spline function as following

\[
S_i(x; q) = \frac{\mu_i(q)}{[3]_q h_i} (x - x_{i-1})^3 - \frac{\mu_{i-1}(q)}{[3]_q h_i} (x - x_i)^3
\]

\[
+ \left( f[x_{i-1}, x_i] - \frac{\mu_i(q)}{[3]_q h_i^2} (x_i - x_{i-1})^3 - \frac{\mu_{i-1}(q)}{[3]_q h_i^2} (x_{i-1} - x_i)^3 \right) (x - x_{i-1})
\]

\[
+ f(x_{i-1}) + \frac{\mu_{i-1}(q)}{[3]_q h_i} (x_{i-1} - x_i)^3.
\]

In order to obtain the unknown moments \( \mu_i(q) \), \( i = 0, \ldots, n \) we use the conditions (3) for the first derivative of the \( q \)-spline (6). Hence, for \( i = 1, \ldots, n - 1 \), we have

\[
\frac{\mu_i(q)}{[2]_q h_i} (x_i - x_{i-1})^2 + f[x_{i-1}, x_i] - \frac{\mu_i(q)}{[3]_q h_i^2} (x_i - x_{i-1})^3 - \frac{\mu_{i-1}(q)}{[3]_q h_i^2} (x_{i-1} - x_i)^3
\]

\[
= - \frac{\mu_i(q)}{[2]_q h_i+1} (x_i - x_{i+1})^2 + f[x_i, x_{i+1}] - \frac{\mu_{i+1}(q)}{[3]_q h_i+1} (x_{i+1} - x_i)^3
\]

\[
- \frac{\mu_i(q)}{[3]_q h_i+1} (x_i - x_{i+1})^3.
\]

Moreover, from the boundary conditions (5) for the clamped \( q \)-spline we obtain

\[
D_q S_0(x_0; q) = -\frac{\mu_0(q)}{[2]_q h_1} (x_0 - x_1)^2 + f[x_0, x_1]
\]

\[
+ \frac{\mu_1(q)}{[3]_q h_1^2} (x_1 - x_0)^3 - \frac{\mu_0(q)}{[3]_q h_1^2} (x_0 - x_1)^3 = D_q f(x_0),
\]

\[
D_q S_n(x_n; q) = \frac{\mu_n(q)}{[2]_q h_n} (x_n - x_{n-1})^2 + f[x_{n-1}, x_n]
\]

\[
- \frac{\mu_n(q)}{[3]_q h_n^2} (x_{n-1} - x_n)^3 - \frac{\mu_{n-1}(q)}{[3]_q h_n^2} (x_{n-1} - x_n)^3 = D_q f(x_n).
\]

The equations (11)–(13) form a system of \( n + 1 \) linear equations with respect to the moments \( \mu_i(q) \) that can be written shortly as \( A \mu = b \). With the notation \( H_i(q) = \frac{(x_{i-1} - x_i)(x_i - x_{i-1})}{x_{i-1} - x_i} \), we have

\[
(14) \quad A = \begin{pmatrix}
\frac{[2]_q H_1(q)}{(x_0 - x_{-1})^2} & \frac{(x_1 - x_0)^2}{x_1 - x_0} & 0 & 0 & \cdots & 0 \\
\frac{[2]_q H_1(q) + H_2(q)}{(x_1 - x_0)^2} & \frac{(x_2 - x_1)^2}{x_2 - x_1} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \frac{[2]_q H_n(q)}{(x_n - x_{n-1})^2}
\end{pmatrix}.
\]
\[ b = [3] q^1 \begin{pmatrix} f[x_0, x_1] - D_q f(x_0) \\ f[x_1, x_2] - f[x_0, x_1] \\ \vdots \\ f[x_{n-1}, x_n] - f[x_{n-2}, x_{n-1}] \\ D_q f(x_n) - f[x_{n-1}, x_n] \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_0(q) \\ \vdots \\ \mu_n(q) \end{pmatrix}. \]

The unknown moments \( \mu = \mu(q) \) (15) are the solution of the matrix equation \( A\mu = b \), where \( A \) is given by (14) and \( b \) is given by (15). In view of the fact that \( A \) is a tridiagonal matrix, the method proposed in [15] may be used for evaluating the moments \( \mu_i(q) \), \( i = 0, \ldots, n \).

The moments \( \mu_i(q) \), \( i = 0, \ldots, n \) are the solution of the matrix equation \( A\mu = b \), therefore they are the unique functions of parameter \( q \). Thus we can state the following result.

**Theorem 1.** Let the piecewise function \( S(x; q) \) be defined by (11) and (10), where \( \mu = \mu(q) \) is a unique solution of the matrix equation \( A\mu = b \), with \( A \) given by (14) and \( b \) given by (15). Then \( S(x; q) \) is a \( q \)-analogue of the cubic spline interpolation of a function \( f(x) \).

**Example 2.** Let us consider a \( q \)-spline interpolation of a function \( f(x) = x^4 \) on the interval \( x \in [-1, 1] \) at the knots \( x_0 = -1 \), \( x_1 = 0 \), \( x_2 = 1 \). We assume that the values of the function itself and its first derivation at the knots are given. The classical cubic spline solution is given in Example 1 on page 824–825 of [18]. By using the above proposed method, one can obtain the cubic \( q \)-spline solution:

\[ S(x; q) = \begin{cases} \frac{a^3 - 3a^2 - 2a - 2}{2a^2 + q} x^3 + \frac{a^3 - a^2 - 2a}{2a^2 + q} x^2, & -1 \leq x \leq 0, \\ -\frac{a^3 - 3a^2 - 2a - 2}{2a^2 + q} x^3 + \frac{a^3 - a^2 - 2a}{2a^2 + q} x^2, & 0 \leq x \leq 1. \end{cases} \]

It is easy to obtain the classical cubic spline solution corresponding to \( q \to 1 \). One can see that there exist a slight oscillation of the spline regarding to the original function. Decreasing the parameter \( q \) leads to more intensive oscillation. However, increasing the parameter \( q \) overcomes the oscillation effect and interpolates the original function with significant improvement.

![Figure 1. Interpolation of \( f(x) = x^4 \) by cubic \( q \)-spline.](image)
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