Quantum effects in five-dimensional brane-world: creation of deSitter branes and particles and stabilization of induced cosmological constant

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Abstract. The role of quantum effects in brane-world cosmology is investigated. It is shown in time-independent formulation that quantum creation of deSitter branes in five-dimensional (A)dS bulk occurs with also account of brane quantum CFT contribution. The surface action is chosen to include cosmological constant and curvature term. (The time-dependent formulation of quantum-corrected brane FRW equations is shown to be convenient for comparison with Supernovae data). The particles creation on deSitter brane is estimated and is shown to be increased due to KK modes. The deSitter brane effective potential due to bulk quantum matter on 5d AdS space is found. It may be used to get the observable cosmological constant in the minimum of the potential (stabilization). The appearence of the entropy bounds from bulk field equation is also mentioned.

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1. Introduction

In the brane-world scenario, our four-dimensional universe represents the brane (boundary) embedded into the higher dimensional space. Unlike the original Kaluza-Klein proposal, such a picture may be quite consistent as it predicts that brane gravity is trapped on the brane\cite{1} even if extra dimensions are relatively large. Among the other positive aspects of brane-world scenario, one may count: the natural solution of hierarchy problem \cite{2} (where Planck scale appears) and the connection with the AdS/CFT correspondence \cite{3} (and, hence, with string theory). Moreover, such a scenario is a successful manifestation of holographic principle. It is also interesting, that bulk contributions (like dark radiation) to brane gravitational equations may be important only at very early universe (at least, for a number of models). They play no role at the late epochs of the universe evolution (even at the remote past epoch with very high redshift observed in the case of supernovae Ia).

Taking into account that brane-world should be the consequence (ground state?) of some quantum gravity which is not constructed yet as consistent theory, the study of quantum effects in such models is of great importance. Indeed, quantum effects are expected to be important in the construction of non-singular and (or) stable brane-world cosmologies, in the resolution of the cosmological constant problem and in drawing more relations with string theory (AdS/CFT set-up). Moreover, such investigation may teach us the important lessons about the structure of the future quantum gravity. Unfortunately, even in semiclassical approximation it is not so easy to consider the quantum brane-world theory. Eventually, at first step one should investigate the particular aspects of quantum brane-world. For example, the good starting point is quantum matter theory on the classical brane-world background. Even such approach turns out to be too complicated and restrictive, as the very few, quite simple backgrounds (where one-loop calculations are possible to do) may be actually discussed. In such situation, one is forced to consider separately the bulk and the brane quantum effects and estimate their role in various aspects of brane-world evolution.

The purpose of this paper is to study the quantum (bulk and brane) matter effects and their influence to brane-world cosmology. In the next section we start from the general five-dimensional action where four-dimensional brane action includes cosmological constant and curvature. With the correspondent choice of sign for bulk cosmological constant one may consider AdS or dS five-dimensional background. Supposing that brane is constant curvature (deSitter or hyperbolic) space and working in time-independent setting, the quantum effects of conformal brane matter (via the conformal anomaly) are included into the brane gravitational equation. This is fourth-order algebraic equation which roots describe the quantum creation of deSitter or hyperbolic brane. Without brane curvature term and with the choice of brane cosmological constant as required by the cancellation of the leading divergence of AdS bulk, the equation simplifies and gives the inflationary deSitter brane (Brane New World) suggested some time ago in refs.\cite{4, 5, 6}. We also describe the connection with
the AdS/CFT correspondence (the specific choice of surface terms to cancell the leading and next-to-leading divergence of AdS space). The quantum creation of deSitter and hyperbolic branes and their stability in such a case depends on the content of brane matter. In particular, there is no creation when brane matter is $\mathcal{N} = 4$ super Yang-Mills theory as required by AdS/CFT. Our formulation is quite simple and general. Changing the sign of the bulk cosmological constant in the brane gravitational equation one arrives at the case with deSitter bulk.

In section three we consider another phenomenon related with the effect of brane gravity to quantum matter fields. The bulk scalar is seen as collection of massive scalars (Kaluza-Klein modes) on deSitter brane which is embedded to AdS space. Gravitational field leads to particle creation which is summed over KK modes. As the result the total creation probability is significally increased if compare with the classical four-dimensional consideration. The created KK particles decay into the light particles which indicates that particles creation in the brane inflationary universe should be more intensive than usually expected.

In the fourth section we evaluate the deSitter brane effective potential due to quantum bulk scalars and spinors in AdS bulk. It is shown that it contains the minimum where it may be identified with the observable cosmological constant. Simple estimation indicates that it may stabilize the present cosmological constant. Several remarks on the relation of the field equation for AdS space with entropy of the correspondent AdS black hole are given in section five, using Verlinde formulation.

Time-dependent formulation of quantum-corrected FRW brane equation is given in Discussion.

2. The creation of deSitter branes from (A)dS bulk with account of quantum effects

In this section we will derive the equations of motion for deSitter (FRW) brane when bulk is five-dimensional AdS or dS space. It is supposed that there are conformal fields on the brane (for recent discussion of bulk conformal fields in brane-world scenario and list of relevant references, see [7]). These conformally invariant fields are quantum fields and their quantum effects (via the account of the correspondent brane conformal anomaly) are included into the dynamical equation of motion. The number of solutions of brane gravitational equations describing the creation of deSitter (FRW) branes is presented.

We follow the approach developed by Shiromizu, Maeda and Sasaki [8] where it has been shown how the 4d effective Einstein equation appears from the 5d bulk Einstein equation. In this section, we consider the case that the brane action contains the 4d scalar curvature with arbitrary coefficient (for its cosmological applications, see [9]).

Let the 3-brane is embedded into the 5d bulk space as in [8]. Let $g_{\mu\nu}$ be the metric tensor of the bulk space and $n_\mu$ be the unit vector normal to the 3-brane. Then the
metric $q_{\mu\nu}$ induced on the brane has the following form:

$$q_{\mu\nu} = g_{\mu\nu} - n_{\mu}n_{\nu}. \tag{1}$$

The initial action is

$$S = \int d^5x \sqrt{-g} \left\{ \frac{1}{\kappa_5^2} R^{(5)} - 2\Lambda + \cdots \right\} + S_{\text{brane}}(q). \tag{2}$$

In this section, the 5d quantities are denoted by the suffix $(5)$ and 4d ones by $(4)$. In $(2)$, $\cdots$ expresses the matter fields contribution and $S_{\text{brane}}$ is the action on the brane, which will be specified later. The bulk Einstein equation is given by

$$\frac{1}{\kappa_5^2} \left( R^{(5)}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{(5)} \right) = T_{\mu\nu}. \tag{3}$$

If one chooses the metric near the brane as:

$$ds^2 = d\chi^2 + q_{\mu\nu} dx^\mu dx^\nu, \tag{4}$$

the energy-momentum tensor $T_{\mu\nu}$ has the following form:

$$T_{\mu\nu} = T_{\mu\nu}^{\text{bulk matter}} - \Lambda g_{\mu\nu} + \delta(\chi) \left( -\lambda q_{\mu\nu} + \tau_{\mu\nu} \right). \tag{5}$$

Here $T_{\mu\nu}^{\text{bulk matter}}$ is the energy-momentum tensor of the bulk matter, $\Lambda$ is the bulk cosmological constant, $\lambda$ is the tension of the brane, and $\tau_{\mu\nu}$ expresses the contribution due to brane matter. Without the bulk matter ($T_{\mu\nu}^{\text{bulk matter}} = 0$), following the procedure in [8], the bulk Einstein equation can be mapped into the equation on the brane:

$$\frac{1}{\kappa_5^2} \left( R^{(4)}_{\mu\nu} - \frac{1}{2} q_{\mu\nu} R^{(4)} \right) = -\frac{1}{2} \left( \Lambda + \frac{\kappa_5^2 \lambda^2}{6} \right) q_{\mu\nu} + \frac{\kappa_5^2 \lambda}{6} \tau_{\mu\nu} + \kappa_5^2 \pi_{\mu\nu} - \frac{1}{\kappa_5^2} E_{\mu\nu}. \tag{6}$$

Here $\pi_{\mu\nu}$ is given by

$$\pi_{\mu\nu} = -\frac{1}{4} \tau_{\mu\alpha} \tau_{\nu}^{\alpha} + \frac{1}{12} \tau^{\mu\nu} + \frac{1}{8} g_{\mu\nu} \tau_{\alpha\beta} \tau^{\alpha\beta} - \frac{1}{24} q_{\mu\nu} \tau^2. \tag{7}$$

On the other hand, $E_{\mu\nu}$ is defined by the bulk Weyl tensor $C^{(5)}_{\mu\nu\rho\sigma}$:

$$E_{\mu\nu} = C^{(5)}_{\alpha\beta\gamma\delta} n^{\alpha} q_{\beta} q^{\gamma} q_{\delta}. \tag{8}$$

In general $D$-dimensional spacetime, the Weyl tensor is

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} - \frac{1}{D-2} \left( g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} + g_{\mu\kappa} R_{\lambda\nu} - g_{\mu\nu} R_{\lambda\kappa} \right) + \frac{1}{(D-1)(D-2)} R \left( g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu} \right). \tag{9}$$

The brane action is taken as following

$$S_{\text{brane}} = \int \sqrt{-q} \left( -\alpha R^{(4)}(q) - 2\lambda \right). \tag{10}$$

where the coefficient of brane curvature term is an arbitrary parameter. It is not difficult to show that $\tau_{\mu\nu}$ is given by the 4d Einstein tensor:

$$\tau_{\mu\nu} = \alpha \left( R^{(4)}_{\mu\nu} - \frac{1}{2} q_{\mu\nu} R^{(4)} \right). \tag{11}$$
Therefore one has

\[
\frac{\pi_{\mu\nu}}{\alpha^2} = -\frac{1}{4} R_{\mu \alpha}^{(4)} R_{\nu \alpha}^{(4)} + \frac{1}{6} R^{(4)} R_{\mu \nu}^{(4)} + q_{\mu \nu} \left( - \frac{1}{16} R^{(4)^2} + \frac{1}{8} R_{\alpha \beta}^{(4)} R_{\alpha \beta}^{(4)} \right). \tag{12}
\]

Then Eq. (10) can be rewritten as

\[
\frac{1}{\kappa^5} \left( 1 - \frac{\kappa^2 \lambda \alpha}{6} \right) \left( R_{\mu \nu}^{(4)} - \frac{1}{2} q_{\mu \nu} R^{(4)} \right) = -\frac{1}{2} \left( \Lambda + \frac{\kappa^2 \lambda^2}{6} \right) q_{\mu \nu} \\
+ \alpha^2 \kappa^5 \left\{ -\frac{1}{4} R_{\mu \alpha}^{(4)} R_{\nu \alpha}^{(4)} + \frac{1}{6} R^{(4)} R_{\mu \nu}^{(4)} + q_{\mu \nu} \left( - \frac{1}{16} R^{(4)^2} + \frac{1}{8} R_{\alpha \beta}^{(4)} R_{\alpha \beta}^{(4)} \right) \right\} \\
- \frac{1}{\kappa^5} R_{\alpha \beta \gamma \delta} n^\alpha n^\gamma q^\mu q^\nu q_\delta. \tag{13}
\]

Note that one may identify the effective 4d gravitational constant \(\kappa_4\) and 4d cosmological constant \(\Lambda_4\) with

\[
\frac{1}{\kappa_4^2} = \frac{6}{\lambda \kappa_5^4} \left( 1 - \frac{\kappa_5^4 \lambda \alpha}{6} \right), \quad \Lambda_4 = \frac{\kappa_5^2}{2} \left( \Lambda + \frac{\kappa_5^2 \lambda^2}{6} \right) \left( 1 - \frac{\kappa_5^4 \lambda \alpha}{6} \right)^{-1}. \tag{14}
\]

The next step is to include the quantum effects from the conformal brane matter. As usually, the simplest way to do so is to consider the conformal anomaly\(^\dagger\): \(\tau^A = b \left( F^{(4)} + \frac{2}{3} \Box R^{(4)} \right) + b' G^{(4)} + b'' \Box R^{(4)}\), \(\tag{15}\)

where \(F^{(4)}\) is the square of 4d Weyl tensor, \(G^{(4)}\) is Gauss-Bonnet invariant, which are given as

\[
F^{(4)} = \frac{1}{3} R^{(4)^2} - 2 R_{ij}^{(4)} R^{(4)ij} + R_{ijkl}^{(4)} R^{(4)ijkl} \\
G^{(4)} = R^{(4)^2} - 4 R_{ij}^{(4)} R^{(4)ij} + R_{ijkl}^{(4)} R^{(4)ijkl}. \tag{16}\]

In general, with \(N\) scalar, \(N_{1/2}\) spinor, \(N_1\) vector fields, \(N_2\) (\(= 0\) or \(1\)) gravitons and \(N_{\text{HD}}\) higher derivative conformal scalars, \(b\), \(b'\) and \(b''\) are given by

\[
b = \frac{N + 6 N_{1/2} + 12 N_1 + 611 N_2 - 8 N_{\text{HD}}}{120 (4\pi)^2}, \\
b' = -\frac{N + 11 N_{1/2} + 62 N_1 + 1411 N_2 - 28 N_{\text{HD}}}{360 (4\pi)^2}, \quad b'' = 0. \tag{17}\]

For typical examples motivated by AdS/CFT correspondence\(^\bullet\) one has:

a) \(\mathcal{N} = 4\) \(SU(N)\) SYM theory

\[
b = -b' = \frac{N^2 - 1}{4 (4\pi)^2}, \tag{18}\]

b) \(\mathcal{N} = 2\) \(Sp(N)\) theory

\[
b = \frac{12 N^2 + 18 N - 2}{24 (4\pi)^2}, \quad b' = -\frac{12 N^2 + 12 N - 1}{24 (4\pi)^2}. \tag{19}\]

\(^\dagger\) For recent discussion of conformal anomaly as applied to brane black holes, see\(^\ddagger\).

\(^\bullet\) For recent discussion of conformal anomaly as applied to brane black holes, see\(^\ddagger\).
Note that \( b' \) is negative in the above cases. It is important to note that even brane quantum gravity may be taken into account via the contribution to correspondent parameters \( b, b' \).

Having in mind that observable universe was in past (or currently is) in deSitter phase, the natural assumption is that brane is the Einstein manifold defined by

\[
R^{(4)}_{\mu\nu} = \frac{k}{L^2} q_{\mu\nu} .
\]  

(20)

Here \( L \) is the length parameter and \( k = 0, \pm 3 \). For positive \( k \) the brane universe is deSitter space (FRW brane in Minkowski signature). Then

\[
R^{(4)}_{\mu\nu} - \frac{1}{2} q_{\mu\nu} R^{(4)} = - \frac{k}{L^2} q_{\mu\nu} .
\]  

(21)

Taking into account the energy-momentum tensor \( \tau^A_{\mu\nu} \) caused by the one-loop quantum effects, \( \tau_{\mu\nu} \) is modified as

\[
\tau_{\mu\nu} = - \frac{k_\alpha}{L^2} q_{\mu\nu} + \tau^A_{\mu\nu} .
\]  

(22)

Then \( \pi_{\mu\nu} \) has the following form:

\[
\pi_{\mu\nu} = - \frac{k_\alpha^2}{12L^4} q_{\mu\nu} + \frac{k_\alpha}{6L^2} \tau^A_{\mu\nu} - \frac{1}{4} \tau^A_{\mu\omega} \tau^{A\omega}_{\mu\nu} + \frac{1}{12} \tau^{A\mu} \tau^{A\nu} + \frac{1}{8} q_{\mu\nu} \tau^{A\alpha} \tau^{A\alpha} - \frac{1}{24} q_{\mu\nu} \tau^{A^2} .
\]  

(23)

Then the brane equation corresponding (13) is given by

\[
0 = \frac{k}{k_5^2 L^2} \left( 1 - \frac{k_5^2 \lambda \alpha}{6} \right) q_{\mu\nu} - \frac{1}{2} \left( \Lambda + \frac{k_5^2 \lambda^2}{6} \right) q_{\mu\nu} + \frac{k_5^2 \lambda}{6} \tau^A_{\mu\nu}
\]

\[
- \frac{k_5^2 \alpha^2}{12L^4} q_{\mu\nu} + \frac{k_5 \alpha}{6L^2} \tau^A_{\mu\nu}
\]

\[
+ k_5^2 \left( \frac{1}{4} \tau^A_{\mu\nu} \tau^{A\nu} + \frac{1}{12} \tau^{A\mu} \tau^{A\nu} + \frac{1}{8} q_{\mu\nu} \tau^{A\alpha} \tau^{A\alpha} - \frac{1}{24} q_{\mu\nu} \tau^{A^2} \right)
\]

\[
- \frac{1}{k_5^2} C^{(5)}_{\beta\gamma\delta} n^\alpha n^\gamma q_{\mu}^\beta q_{\nu}^\delta .
\]  

(24)

It is quite general gravitational brane equation with account of brane quantum effects.

It is right time now to specify the structure of bulk space. One can imagine that it is 5d AdS space, where

\[
R^{(5)}_{\mu\nu\rho\sigma} = - \frac{1}{l^2} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) .
\]  

(25)

Here

\[
\Lambda = - \frac{6}{k_5^2 l^2} .
\]  

(26)

Then

\[
C^{(5)}_{\alpha\beta\gamma\delta} n^\alpha n^\gamma q_{\mu}^\beta q_{\nu}^\delta = 0 .
\]  

(27)

We can make an assumption on the brane structure stronger than (20)

\[
R^{(4)}_{\mu\nu\rho\sigma} = \frac{k}{3L^2} (g_{\mu\rho} q_{\nu\sigma} - q_{\mu\sigma} q_{\nu\rho}) .
\]  

(28)
Then from Eq. (15), one gets

\[ \tau^A = \frac{8k^2}{3L^4}b'. \]  

(29)

Furthermore, we assume \( \tau^A_{\mu\nu} \) is proportional to \( q_{\mu\nu} \), which may be consistent with (24):

\[ \tau^A_{\mu\nu} = \frac{1}{4} \tau^A q_{\mu\nu} = \frac{2k^2b'}{3L^4}q_{\mu\nu}. \]  

(30)

Then Eq. (24) has the following form:

\[ 0 = \frac{k}{\kappa_5^2L^2} \left( 1 - \frac{\kappa_5^2 \lambda \alpha}{6} \right) - \frac{1}{2} \left( -\frac{6}{\kappa_5^2 l^2} + \frac{\kappa_5^2 \lambda^2}{6} \right) + \frac{\kappa_5^2 \lambda k^2 b'}{9L^4} \]

\[ - \frac{k^2 \alpha^2 \kappa_5^2}{12L^4} + \frac{k^3 \alpha \kappa_5^2}{9L^6} - \frac{\kappa_5^2 k^4 b'^2}{27L^8} \]

\[ = \left( \frac{3}{\kappa_5^2 l^2} - \frac{\kappa_5^2 \lambda^2}{12} \right) + \left( \frac{1}{\kappa_5^2} - \frac{\kappa_5^2 \lambda \alpha}{6} \right) \frac{k}{L^2} \]

\[ + \left( \frac{\kappa_5^2 \lambda b'}{9} - \frac{\alpha \kappa_5^2}{12} \right) \frac{k^2}{L^4} + \frac{k^3 \alpha \kappa_5^2}{9L^6} - \frac{\kappa_5^2 k^4 b'^2}{27L^8}. \]  

(31)

By solving the above equation, \( L^2 \) may be determined. In other words, the solution of above equation defines the creation of deSitter or hyperbolic branes with the account of quantum effects. In principle, there are four solutions of eq.(31) which should define created deSitter (or flat or hyperbolic) branes and their relative stability.

When \( k = 3 > 0 \), the 4d brane universe can be regarded as deSitter spacetime and \( \frac{1}{L} \) can be identified with the ratio of the expansion. Let assume there are two solutions of Eq.(20), one could be parametrized by the small \( L \) and another one-by large one. At the early epoch, the universe might expand rapidly (the small \( L \) solution). At some time, there occurs a jump (transition) to large \( L \) solution caused by some (quantum or thermal) fluctuations. As a result, the present universe might expand slowly (large \( L \) solution).

The interesting example is provided by the following choice

\[ \alpha = 0, \quad k = 3, \quad \lambda = \frac{6}{l\kappa_5^2}, \]  

(32)

This means that brane cosmological constant is predicted by AdS/CFT correspondence as the surface counterterm\[12\] which is necessary in order to cancel the leading divergence of AdS space.

Eq.(31) has the following form:

\[ \left( \frac{1}{l} - \frac{\kappa_5^2 b'}{L^4} \right)^2 = \frac{1}{L^2} \left( 1 + \frac{L^2}{l^2} \right) \quad \text{or} \quad \pm \frac{1}{L} \sqrt{1 + \frac{L^2}{l^2} - 1} = -\frac{\kappa_5^2 b'}{L^4}, \]  

(33)

which (plus sign) reproduces the result \[6\] (see also, \[5, 4\]). In other words, we demonstrated that for the particular choice of the boundary terms, our equation

\[^+\] Generally speaking, the situation is more complicated here as quantum-corrected energy-momentum tensor depends also on the vacuum state chosen. However, as we actually use only trace of brane gravitational equation below, such choice turns out to be enough for our purposes.
describes the quantum creation of deSitter (inflationary) brane which glues two AdS spaces. Such inflationary brane-world scenario is sometimes called Brane New World [5].

Let us define a new variable $X$ by

$$X = \frac{k}{l^2}.$$  \hspace{1cm} (34)

Eq. (31) can be rewritten as

$$0 = \left( \frac{3}{\kappa_5^2 l^2} - \frac{\kappa_5^2 \lambda^2}{12} \right) X + \left( \frac{1}{\kappa_5^2} - \frac{\kappa_5^2 \lambda \alpha}{6} \right) X^{12} + \left( \frac{\kappa_5^2 \lambda b'}{9} - \frac{\alpha^2 \kappa_5^2}{12} \right) X^2 + \frac{\alpha \kappa_5^2}{9} X^3 - \frac{\kappa_5^2 b'^2}{27} X^4.$$

(35)

The equation (20) might be obtained from an effective potential

$$V(X) = -C \left\{ \left( \frac{3}{\kappa_5^2 l^2} - \frac{\kappa_5^2 \lambda^2}{12} \right) X + \left( \frac{1}{\kappa_5^2} - \frac{\kappa_5^2 \lambda \alpha}{6} \right) \frac{X^2}{2} + \left( \frac{\kappa_5^2 \lambda b'}{9} - \frac{\alpha^2 \kappa_5^2}{12} \right) \frac{X^3}{3} + \frac{\alpha \kappa_5^2}{36} X^4 - \frac{\kappa_5^2 b'^2}{135} X^5 \right\}.$$  \hspace{1cm} (36)

Here $C$ is a constant*. Especially in case of (32), one gets

$$V(X) = -C \left\{ \frac{X^2}{2\kappa_5^2} + \frac{\kappa_5^2 \lambda b' X^3}{27} - \frac{\kappa_5^2 b'^2 X^5}{135} \right\}.$$  \hspace{1cm} (37)

Since

$$V'(X) = -C \frac{X}{\kappa_5^2} f(X), \quad f(X) \equiv 1 + \frac{\kappa_5^2 b'}{54 l} X - \frac{\kappa_5^2 b'^2}{27} X^3,$$

(38)

and

$$f'(X) = \frac{\kappa_5^2 b'}{54 l} - \frac{\kappa_5^2 b'^2 X^2}{9} < 0,$$

(39)

$f(X)$ is monotonically decreasing function and $V(X)$ (36) has two extremal values with respect to $X$, one is for $X = 0$ and another is for positive $X$. The latter corresponds to the solution of (33). As we know that the solution of (33) is stable, we should choose $C > 0$ in order that the corresponding solution could be stable. From the effective potential $V(X)$ (36), one finds that the largest solution of (35) is stable but the smallest solution is unstable. Then if there are two solutions, one being positive and another being negative, the positive one is more stable than the negative one. Since positive (negative) $X$ means $k = 3$ ($k = -3$), the spherical (deSitter) brane is more stable than the hyperbolic (anti-deSitter) brane.

In case of the AdS/CFT correspondence [3], the surface terms (the parameters of the brane action) are derived [12]

$$\lambda = \frac{6}{\kappa_5^2 l}, \quad \alpha = \frac{l}{\kappa_5^2}.$$  \hspace{1cm} (40)

* The effective potential whose stationary condition is given by (35) has, of course, some ambiguity but the potential (36) is sufficient when we discuss the local stability.
Furthermore in case of AdS/CFT correspondence, one uses
\[ \frac{l^3}{\kappa_5^2} = \frac{N^2}{8\pi^2}. \quad (41) \]

Using (40) and (41), Eq.(31) is simplified:
\[ 0 = \left( b' - \frac{N^2}{64\pi^2} \right) \frac{k^2}{L^4} + \frac{l^2k^3}{6L^6} - \frac{4\pi^2b'^2l^4k^4}{9N^2L^8}. \quad (42) \]

Without the quantum correction, that is, with \( b' = 0 \), there are trivial (flat brane) solution \( kL^2 = 0 \) and non-trivial solution \( \frac{k}{L^2} = \frac{3N^2}{64\pi^2} \), which corresponds to the brane of the sphere or deSitter space since \( k > 0 \). By including the quantum correction and using (18) in the large \( N \), Eq.(20) becomes
\[ 0 = \frac{k^2}{L^4} \left( -\frac{N^2}{32\pi^2} + \frac{l^2k^3}{6L^2} - \frac{N^2l^2k^2}{9216\pi^2L^4} \right). \quad (43) \]

Since the quantity inside \( ( ) \) is negative for the sufficiently large \( N \), there will be only trivial solution \( kL^2 = 0 \). In other words, when brane QFT is super Yang-Mills theory required by the duality with AdS space, there is no creation of dS brane. For general \( b' \), the nontrivial solutions of (42) are given by
\[ kL^2 = \frac{9N^2}{4\pi^2b'^2l^2} \left\{ \frac{1}{12} \pm \sqrt{\frac{1}{144} + \left( b' - \frac{N^2}{64\pi^2} \right) \frac{4\pi^2b'^2}{9N^2}} \right\}. \quad (44) \]

In order that the solution (44) is real, the following condition should be satisfied,
\[ \frac{4\pi^2}{9N^2} G(b') \equiv \frac{1}{144} + \left( b' - \frac{N^2}{64\pi^2} \right) \frac{4\pi^2b'^2}{9N^2} \geq 0. \quad (45) \]

The function \( G(b') \) can be factored as
\[ G(b') = (b' - x_0)(b' - x_+)(b' - x_+), \]
\[ x_0 \equiv \frac{N^2}{192\pi^2} + \beta_+ + \beta_- , \quad x_+ \equiv \frac{N^2}{192\pi^2} + \beta_+\xi + \beta_-\xi^2 , \]
\[ x_0 \equiv \frac{N^2}{192\pi^2} + \beta_+\xi^2 + \beta_-\xi , \quad \xi \equiv e^{2\pi i} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} , \]
\[ \beta_\pm \equiv \left( \frac{N}{8\pi} \right)^\frac{2}{3} \left\{ \frac{1}{27} \left( \frac{N}{8\pi} \right)^4 - \frac{1}{2} \right\} \pm \frac{1}{2} \sqrt{\left( 1 - \frac{8}{27} \left( \frac{N}{8\pi} \right)^4 \right) \left( 1 + \frac{4}{27} \left( \frac{N}{8\pi} \right)^4 \right)} \right\}^{\frac{1}{3}} . \quad (46) \]

If
\[ \left( \frac{N}{8\pi} \right)^4 > \frac{27}{8} , \quad (47) \]
\( \beta_\pm \) is complex and \( \beta_+ \) is the complex conjugate of \( \beta_- \). In this case, all of \( x_0 \) and \( x_\pm \) are real. If we reorder them and write as \( x_1, x_2, x_3 \) so that \( x_1 < x_2 < x_3 \), in order that the solution (44) is real, one arrives at the condition
\[ x_1 \leq b' \leq x_2 , \quad \text{or} \quad b' \geq x_3 . \quad (48) \]
On the other hand, if
\[
\left( \frac{N}{8\pi} \right)^4 < \frac{27}{8},
\] (49)
only \(x_0\) is real and \(x_\pm\) are complex. Then in order that the solution (44) is real, the condition appears
\[
b' \geq x_0.\] (50)
Thus, we demonstrated that if brane matter is different from super YM theory, there occurs quantum creation of deSitter brane. Since for usual matter \(b' < 0\) is negative, two of the solutions in (44) are positive, then the corresponding branes are deSitter. With \(b'\) being small, Eq. (44) has the following form:
\[
k \Lambda^2 \sim \frac{9N^2}{24\pi^2 b'^2 l^2} \frac{3N^2}{32\pi^2 l^2}.\] (51)
If \(b'\) is large, \(\frac{1}{L}\), which corresponds to the rate of the expansion of the universe, becomes very large. Here we assumed by (40) that the brane gravity vanishes but even if the brane gravity is non-trivial, the situation is not so changed at least if \(\alpha \sim \frac{1}{\kappa^2 l^2}\).

We now consider the case that the bulk is not AdS but deSitter. This can be obtained by changing the sign of the bulk cosmological constant in (6) by \(\Lambda = \frac{6}{\kappa^2 l^2}\). Furthermore as in [13], if the following situation is discussed
\[
\lambda = \frac{6}{\kappa^2 l}, \quad \alpha = 0, \quad b' = 0, \quad k = 3\] (52)
the equation corresponding to (31) looks as:
\[
0 = -\frac{6}{\kappa^2 l^2} + \frac{3}{\kappa^2 L^2}.\] (53)
Then the solution is
\[
L = \frac{l}{\sqrt{2}},\] (54)
which reproduces the result in [13]. (One can also search for deSitter branes in dS bulk using time-dependent setting [14].) This solution describes the classical creation of deSitter brane from deSitter bulk. With \(X\) defined as in (34), the effective potential corresponding to (53) is
\[
V(X) = C \left( \frac{6}{\kappa^2 l^2} X - \frac{1}{2\kappa^2} X^2 \right).\] (55)
With \(C > 0\) as in (38), the solution (54) becomes instable.

One may consider the case with quantum corrections as in (40) and (41) but with \(b'\) being arbitrary. The equation analogous to (31) is given by
\[
0 = -\frac{6}{\kappa^2 l^2} + \left( \frac{2b'}{3l} - \frac{N^2}{96\pi^2 l} \right) \frac{k^2}{L^4} + \frac{k^3 l}{9L^6} - \frac{8\pi^4 k^4 l^4 b'^2}{27N^2 L^8},\] (56)
and introducing \(X\) (34), the effective potential is given by
\[
V(X) = -C \left\{ -\frac{6}{\kappa^2 l^2} X + \left( \frac{2b'}{3l} - \frac{N^2}{96\pi^2 l} \right) \frac{X^3}{3} + \frac{l}{36} X^4 - \frac{8\pi^2 l^4 b'^2}{135N^2} X^5 \right\},\] (57)
Eq. (56) has maximally four solutions. First, Eq. (56) does not have flat solution \( k L = 0 \).

Since \( V''(X) = -CX \left\{ 2 \left( \frac{2b'}{3l} - \frac{N^2}{96\pi^2l} \right) + \frac{l}{3} X - \frac{32\pi^2b'^2l^2}{27N^2} X^2 \right\} \), the equation \( V''(X) = 0 \) has three solutions

\[
X = 0, \quad X = X_\pm \equiv \left( \frac{3N}{8\pi lb'} \right)^2 \pm \sqrt{ \left( \frac{3N}{8\pi lb'} \right)^4 + \frac{27N^2}{4\pi^2b'^4} \left( \frac{2b'}{3} - \frac{N^2}{96\pi^2} \right) }. \tag{59}
\]

For large \( X \), \( V'(X) \) behaves as \( \frac{V'(X)}{C} \sim \frac{8\pi^2b'^2}{27N^2} X^4 > 0 \). Since \( \frac{V'(0)}{C} = \frac{6}{\pi^2b'^2} > 0 \), if \( V'(X_+) \leq 0 \) or \( V'(X_-) \leq 0 \), there is non-trivial solution in (56). If \( b' < 0 \), we have \( X_+ \geq 0 \) and all the solutions in (59) express dS (not AdS) brane. Thus, the successful demonstration of dS brane creation from dS bulk is also made.

3. Particles creation on the dS brane

In the present section the role of the brane quantum effects to bulk matter is described. Specifically, we consider bulk scalar which appears on the brane as collection of massive scalars (KK modes). As in previous section, the brane is supposed to be deSitter one. Then, brane gravitational field leads to particles creation effect which should be summed over the KK modes.

We start from the following expression of the metric of the Euclidean five-dimensional anti-deSitter space (AdS5):

\[
ds^2 = dy^2 + l^2 \cosh^2 \frac{y}{l} \left( d\xi^2 + \sin^2 \xi d\Omega_3^2 \right). \tag{60}
\]

Here \( d\Omega_3^2 \) is the metric of the three-dimensional sphere (S3) with unit radius. Let assume that there is a brane at \( y = y_0 \) and the region of the bulk space is given by \( 0 \leq y \leq y_0 \). Then the brane is a four dimensional sphere (S4) with the radius \( R_b = l \cosh \frac{y_0}{l} \). We now Wick-rotate the coordinate \( \xi \) by \( \xi \rightarrow \frac{\pi}{2} + it \). The following metric is obtained

\[
ds^2 = \sum_{\mu, \nu=0}^{4} g_{\mu \nu} dx^\mu dx^\nu = dy^2 + \cosh^2 \frac{y}{l} \left( -dt^2 + \cosh^2 t d\Omega_3^2 \right). \tag{61}
\]

Then the brane becomes four-dimensional deSitter (dS4) space, whose rate of the expansion is \( \frac{1}{R_b} \). If we define a new time coordinate \( \tau \) by \( \tau = l \left( \cosh \frac{y_0}{l} \right) t \), \( \tau \) expresses the cosmological time on the brane.

The Klein-Gordon equation for the scalar field \( \phi \) with mass \( M_0^2 \) looks as:

\[
0 = -\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu \nu} \partial_\nu \phi \right) + M_0^2 \phi
= -\frac{\cosh^4 \frac{y}{l}}{\tanh \frac{y}{l}} \partial_y \left( \cosh^4 \frac{y}{l} \partial_y \phi \right) - \frac{1}{l^2 \cosh^2 \frac{y}{l}} \Delta_4 \phi + M_0^2 \phi, \tag{62}
\]

\[
\Delta_4 \phi = -\frac{1}{\cosh^3 \frac{y}{l}} \partial_t \left( \cosh^3 t \partial_t \phi \right) + \frac{1}{\cosh^2 t} \Delta_3 \phi. \tag{63}
\]
Here $\triangle_3$ is the Laplacian on $S_3$ with unit radius. Then $\triangle_4$ is the Laplacian on $dS_4$ with unit length parameter. The eigen-functions and eigen-values of $\triangle_3$ are known:

$$\triangle_3 Y_{klm} = -k(k+2)Y_{klm}. \quad (64)$$

Here $k = 0, 1, 2, \cdots$, $l = 0, 1, 2, \cdots$, and $m = -l, -l+1, \cdots, l$. Then the eigen-functions of $\triangle_4$ are given by [16]:

$$\varphi_{klm}^{(\pm)\gamma} = Y_{klm} \cosh^k t e^{(\pm)\frac{3}{2}it} F \left( k + \frac{3}{2}, k + \frac{3}{2} \pm i\gamma, 1 \pm i\gamma; -e^{2t} \right),$$

$$\varphi_{klm(\pm)}^{\gamma} = Y_{klm} \cosh^k t e^{(k + \frac{3}{2})it} F \left( k + \frac{3}{2}, k + \frac{3}{2} \mp i\gamma, 1 \mp i\gamma; -e^{2t} \right),$$

$$\triangle_4 \varphi = -M_4^2 \phi, \quad \varphi = \varphi_{klm}^{(\pm)\gamma}, \varphi_{klm(\pm)}^{\gamma}.$$  \quad (65)

Here $F$ is Gauss’ hypergeometric functions. $M_4$ can be identified with the mass of the Kaluza-Klein modes. When $t \to +\infty$, one gets

$$\varphi_{klm}^{(\pm)\gamma} \sim e^{(\mp)\frac{3}{2}it} e^{\pm \frac{3}{2}i\gamma}. \quad (66)$$

On the other hand, when $t \to -\infty$ the asymptotics looks as

$$\varphi_{klm(\pm)}^{\gamma} \sim e^{(\mp)\frac{3}{2}it} e^{\mp \frac{3}{2}i\gamma}. \quad (67)$$

Then $\varphi_{klm}^{(+)\gamma}$ ($\varphi_{klm}^{(-)\gamma}$) corresponds to the positive (negative) frequency mode at $t \to +\infty$. On the other hand, $\varphi_{klm(+)\gamma}$ ($\varphi_{klm(-)\gamma}$) corresponds to the positive (negative) frequency mode at $t \to -\infty$. In the 5d Klein-Gordon equation, by assuming

$$\phi = \varphi_{klm}^{(\pm)M_5\gamma} = \eta_{M_5\gamma}(y) \varphi_{klm}^{(\pm)\gamma}, \quad (68)$$

or

$$\phi = \varphi_{klm(\pm)}^{M_5\gamma} = \eta^{M_5\gamma}(y) \varphi_{klm(\pm)}^{\gamma}, \quad (69)$$

we obtain

$$0 = -\frac{l^2}{\cosh^4 \frac{y}{l}} \partial_y \left( \cosh^4 \frac{y}{l} \partial_y \eta^{M_5\gamma} \right) - \frac{M_4^2}{\cosh^2 y} \eta^{M_5\gamma} + M_5^2 l^2 \eta^{M_5\gamma}. \quad (70)$$

Replacing

$$z = \cosh^2 \frac{y}{l}, \quad \eta^{M_5\gamma} = z^\alpha \zeta^{M_5\gamma}(z), \quad \alpha \equiv \frac{1}{2} \sqrt{1 + \frac{M_4^2}{M_5^2 l^2}}, \quad (71)$$

one gets

$$0 = z(1 - z) \frac{d^2 \zeta}{dz^2} + \left( 2\alpha + \frac{5}{2} \right) \frac{d\zeta}{dz} - \left( \frac{M_4^2}{4} - M_5^2 l^2 \right) \zeta. \quad (72)$$

Assuming that the scalar field is not singular in the bulk ($0 \leq y \leq y_0$ or $1 \leq z \leq \cosh^2 \frac{y}{l}$), the solution of (72) is given by Gauss' hypergeometric function:

$$\zeta^{M_5\gamma} = \frac{1}{z^{\alpha+1 \pm \sqrt{3\alpha + M_5^2 l^2} + 1}} \times F \left( \alpha + 1 \pm \sqrt{3\alpha + M_5^2 l^2} + 1, -\alpha - \frac{1}{2} \mp \sqrt{3\alpha + M_5^2 l^2} + 1, \frac{1}{2}; \frac{z}{z_1} \right)$$
\[
= \left( \cosh \frac{y}{l} \right)^{-\left( \alpha + 1 \pm \sqrt{3\alpha + M^2_{4}l^2} + 1 \right)} \times F \left( \alpha + 1 \pm \sqrt{3\alpha + M^2_{4}l^2} + 1, -\alpha - \frac{1}{2} \mp \sqrt{3\alpha + M^2_{4}l^2} + 1; \tanh^2 \frac{y}{l} \right)
\]
(73)

Now \( M_{KK}^2 = \frac{M^2_{4}l^2}{l^2 \cosh \frac{y_0}{l}} \) can be regarded as the mass of the Kaluza-Klein modes. Imposing some boundary condition to the scalar field \( \phi \) on the brane, the Kaluza-Klein modes take the discrete values. It is technically very difficult to find the exact values of the Kaluza-Klein modes. Imagine the situation when \( M_{4} \) is large. Then from (71), one finds that \( \alpha \) is large:
\[
\alpha \sim \frac{M_{4}}{2}.
\]
(74)

Then we can approximate \( \zeta \) in (73) as follows,
\[
\zeta^{M_{5}\gamma}(y) \sim \left( \cosh \frac{y}{l} \right)^{-\alpha} F \left( \alpha, -\alpha, \frac{1}{2}; \tanh^2 \frac{y}{l} \right) = \cos^\alpha \theta \cos (2\alpha \theta).
\]
(75)

Here a new coordinate \( \theta \) is introduced
\[
\sin \theta = \tanh \frac{y}{l}, \quad 0 \leq \theta < \frac{\pi}{2}.
\]
(76)

By using (71), one finds
\[
\phi \propto \eta^{M_{5}\gamma} \sim \cos^{-\alpha} \theta \cos (2\alpha \theta).
\]
(77)

Then imposing the Neumann-type boundary condition \( \partial_y \phi \big|_{y=y_0} = 0 \) (\( \partial_y \eta \big|_{y=y_0} = 0 \)) or Dirichlet-type boundary condition \( \phi \big|_{y=y_0} = 0 \) (\( \eta \big|_{y=y_0} = 0 \)), the Kaluza-Klein masses take the discrete values:
\[
M_{KK}^2 \sim \frac{2\alpha}{l \cosh \frac{y_0}{l}} \sim \frac{n\pi}{l \theta_0 \cosh \frac{y_0}{l}} = \frac{n\pi \cos \theta_0}{l \theta_0}.
\]
(78)

Here \( n \) is a large integer and \( \theta = \theta_0 \) corresponds to \( y = y_0 \), where the brane exists. More exactly if we impose the Neumann-type boundary condition, we obtain \( 2\alpha \theta = n\pi \) with an integer \( n \). On the other hand, if we impose Dirichlet-type boundary condition \( [15] \), we have \( 2\alpha \theta = \left( n + \frac{1}{2} \right) \pi \). The difference \( \frac{\pi}{2} \) could be, however, negligible for the Kaluza-Klein modes with large mass since \( n \) becomes large.

We can now expand \( \phi \) by \( \phi_{klm}^{(+\cdot)M_{5}\gamma} \) or \( \phi_{klm}^{M_{5}\gamma} \):
\[
\phi = \sum_{\gamma,k,l,m} \left( a_{klm}^{M_{5}\gamma} \phi_{klm}^{(+\cdot)M_{5}\gamma} + a_{klm}^{M_{5}\gamma\dagger} \phi_{klm}^{(-\cdot)M_{5}\gamma} \right)
\]
\[
= \sum_{\gamma,k,l,m} \left( b_{klm}^{M_{5}\gamma} \phi_{klm}^{M_{5}\gamma} + b_{klm}^{M_{5}\gamma\dagger} \phi_{klm}^{M_{5}\gamma\dagger} \right).
\]
(79)

The creation operators \( a_{klm}^{M_{5}\gamma\dagger} \) and \( b_{klm}^{M_{5}\gamma\dagger} \) and/or annihilation operators \( a_{klm}^{M_{5}\gamma} \) and \( b_{klm}^{M_{5}\gamma} \) are related by the unitary transformation as in [16]:
\[
a_{klm}^{M_{5}\gamma\dagger} = \alpha^{\gamma k} b_{klm}^{M_{5}\gamma\dagger} - \beta^{\gamma k} b_{klm}^{M_{5}\gamma},
\]
\[
a_{klm}^{M_{5}\gamma} = - \beta^{\gamma k^*} b_{klm}^{M_{5}\gamma\dagger} + \alpha^{\gamma k^*} b_{klm}^{M_{5}\gamma}.
\]
\[ \alpha^k = i(-)^k \sinh \Theta \equiv \frac{i(-)^k}{\sinh \pi \gamma} \]
\[ \beta^k = e^{-2i \delta_k} \cosh \Theta \equiv \frac{\Gamma(1-i\gamma)\Gamma(-i\gamma)}{\Gamma(k+\frac{3}{2}-i\gamma)\Gamma(-k-\frac{1}{2}-i\gamma)} . \] (80)

Then the creation probability \( \Gamma \) per unit volume and unit time for the particle modes with \( \gamma \) is, as in [16] (for earlier discussion, see [17]), given by
\[ \Gamma = \frac{8}{\pi^2 l^4 \cosh^4 \frac{\theta_0}{T}} \ln \coth \pi \gamma . \] (81)

For the Kaluza-Klein modes with large mass (large \( M_4 \)), from [65], \( \gamma \) is large:
\[ \gamma \sim M_4 . \] (82)

Then the creation probability of the Kaluza-Klein modes with large mass is exponentially suppressed:
\[ \Gamma \sim \frac{16e^{-2\pi M_4}}{\pi^2 l^4 \cosh^4 \frac{\theta_0}{T}} . \] (83)

The result (81) or (83) is valid if the masses of the Kaluza-Klein modes are smaller than the Planck mass scale. If not, one cannot neglect the backreaction due to particles creation. Probably then one should consider quantum gravity effects. Now we have considered the case that \( M_4 \) is large but this does not always mean that the real masses of the Kaluza-Klein modes are large. The real masses of the Kaluza-Klein modes are given by \( M_{KK}^2 = \frac{M_4^2}{l^2 \cosh^2 \frac{\theta_0}{T}} \). The length parameter \( l \) of the bulk AdS5 space could be of the order of the Planck length. If \( M_4^2 \ll \cosh^2 \frac{\theta_0}{T} \), that is, the radius or the length parameter of the dS brane is large, one may neglect the backreaction to the gravity and Eqs. (81) and (83) are valid. Since for large \( n \) in (78)
\[ M_4 \sim \frac{n\pi}{\theta_0} \] (84)
by using (83), we can sum up the Kaluza-Klein modes to obtain the total creation probability:
\[ \Gamma_{\text{total}} = \sum_{\text{KK modes}} \Gamma \sim \sum_n \frac{16e^{-2\pi^2 n}}{\pi^2 l^4 \cosh^4 \frac{\theta_0}{T}} \sim \frac{16}{\pi^2 l^4 \cosh^4 \frac{\theta_0}{T}} \frac{C}{1-e^{-2\pi^2}} . \] (85)

The coefficient \( C \) could be determined by the contribution from the Kaluza-Klein modes with relatively small masses. Eventually, \( C \) is the order of the unity. The constant \( C \) may depend on the boundary condition of \( \phi \) but the difference could be given by a factor of order unity. Eq. (85) shows that the creation of the Kaluza-Klein modes could not be neglected in the inflationary universe. The created Kaluza-Klein particles decay into the light particles although the decay rate etc. depend on the details of the interactions of the bulk scalar \( \phi \) with the light particles. Then, the particle creation at the early Universe with orbifolded extra dimensions should be much larger than that expected from the naive Standard Model.
4. Stabilization of the brane cosmological constant

It would be of great interest to consider the role of both: brane and bulk quantum effects to brane-world cosmology. Unfortunately, technically it is not so easy to make such a study. Hence, one should discuss the role of such effects separately. In this section the quantum bulk scalar is considered and its contribution to brane effective potential is found. This may suggest the mechanism to stabilize the induced brane cosmological constant.

We start with the action for a conformally invariant massless scalar

$$S = \frac{1}{2} \int d^5x \sqrt{\mathcal{g}} \left[ -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \xi_5 R^{(5)} \phi^2 \right], \quad (86)$$

where $\xi_5 = -3/16$, $R^{(5)}$ being the five-dimensional scalar curvature.

Let us recall the expression for the Euclidean metric of the five-dimensional AdS bulk:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{l^2}{\sinh^2 z} \left( dz^2 + d\Omega_4^2 \right), \quad (87)$$

$$d\Omega_4^2 = dz^2 + \sin^2 z d\Omega_3^2, \quad (88)$$

where $l$ is the AdS radius which is related to the cosmological constant $\Lambda$ of the AdS bulk by $\Lambda = -\frac{l^2}{12}$, and $d\Omega_3$ is the metric on the 3-sphere. Two dS branes, which are four-dimensional spheres, are placed in the AdS background. If we put one brane at $z_1$, which is fixed, and the other brane at $z_2$, the distance between the branes is given by $L = |z_1 - z_2|$.

The action, Eq. (86), is conformally invariant under the conformal transformations for the metric Eq. (87) and the scalar field, which are given by

$$g_{\mu\nu} = \sinh^{-2} z \, l^2 \hat{g}_{\mu\nu}, \quad \phi = \sinh^{3/2} z \, l^{-3/2} \hat{\phi}. \quad (89)$$

The corresponding transformed Lagrangian looks like

$$\mathcal{L} = \phi \left( \partial_z^2 + \Delta^{(4)} + \xi_5 R^{(4)} \right) \phi. \quad (90)$$

where $R^{(4)} = 12$. This Lagrangian was used in ref. [18] in order to calculate the Casimir effect (for related study of bulk Casimir effect in brane-world with applications to radion stabilization, see [19]). We apply the result of this calculation in the study of brane cosmological constant induced by bulk quantum effects.

As shown in Ref. [18], the one-loop effective potential can be written as

$$V = \frac{1}{2 L V^4} \ln \det(L_5/\mu^2), \quad (91)$$

where $L_5 = -\partial_z^2 - \Delta^{(4)} - \xi_5 R^{(4)} = L_1 + L_4$ and $V_4$ is the volume of the four dimensional sphere with a unit radius. The explicit calculation gives [18]:

$$- \ln \det(L_5/\mu^2) = \zeta'(0)L_5 = \frac{\zeta'(-4)}{6} \frac{\pi^4}{L^4} + \frac{\zeta'(-2)}{12} \frac{\pi^2}{L^2}$$

$$\simeq \frac{0.129652}{L^4} - \frac{0.025039}{L^2} + \cdots \quad (92)$$
The expression of the zeta-function has been given in terms of an expansion on the brane distance \( L \), valid for \( L \leq 1 \), which complements the one for large brane distance obtained above. In \( \mathcal{S}^7 \), since \( z \) is dimensionless, \( L = |z_1 - z_2| \) is dimensionless.

Since the effective potential \( \mathcal{S}^9 \) is evaluated in the conformally transformed metric \( \mathcal{S}^9 \), the Casimir energy density \( \rho_0 \) in the real space is given by
\[
\rho_0 = \frac{\sinh^5 z}{l^5} V = -\frac{\sinh^5 z}{2l^5 LV_4^4} \zeta'(0|L_5). \quad (93)
\]

Then the effective potential \( V_{\text{eff}} \) per unit volume on the brane at \( z = z_1 \) is given by
\[
V_{\text{eff}} = \frac{\sinh^4 z_1}{l^4 V_4} \int_{z_2}^{z_1} dz \int d\Omega_4 \frac{l^5}{\sinh^3 z} \rho_0 = -\frac{1}{2V_4 R_1^4} \zeta'(0|L_5). \quad (94)
\]

Here \( R_1 \equiv \frac{l}{\sinh z_1} \) is the radius of the brane at \( z = z_1 \). Then when we Wick-rotate the metric into the Minkowski signature, the rate of the expansion of the universe, that is the Hubble parameter, is given by \( \frac{1}{R_1} \). In the leading order one has
\[
V_{\text{eff}} \sim -\frac{1}{2V_4 R_1^4} \left( \frac{\zeta'(-4)}{6} \frac{\pi^4}{L^4} + \frac{\zeta'(-2)}{12} \frac{\pi^2}{L^2} \cdot \cdot \cdot \right). \quad (95)
\]

As we will see later, the higher order terms do not contribute for large brane. We should also note that there can appear the (surface) terms corresponding to the tensions or classical cosmological constants of the branes. (This may be also considered as finite renormalization). The brane at \( z = z_1 \) gives a constant term. Since the radius \( R_2 \) of the brane at \( z = z_2 \) is given by \( R_2 \equiv \frac{l}{\sinh z_2} \), the ratio of the scales on the two branes is given by \( R_2 R_1 \). Then the tension of the brane at \( z = z_2 \) gives a term proportional to \( \left( \frac{R_2}{R_1} \right)^4 \) as the contribution into the effective potential. The total effective potential is given by
\[
V_{\text{eff}} = -\frac{1}{2V_4 R_1^4} \zeta'(0|L_5) + \lambda_1 + \lambda_2 \left( \frac{R_2}{R_1} \right)^4. \quad (96)
\]

This is the quantity which should be identified with observable cosmological constant including bulk quantum effects.

Note that the scalar curvatures on the branes are given by \( \frac{6}{R_1^2} \) and \( \frac{6}{R_2^2} \). Let assume \( R_1 > R_2 \), then \( z_2 < z_1 \) and therefore \( L = z_2 - z_1 \), which gives
\[
\frac{R_2}{R_1} = \frac{\sinh z_1}{\sinh (L + z_1)}. \quad (97)
\]

As \( L \) is assumed to be small, the effective potential \( \mathcal{S}^9 \) has the following form:
\[
V_{\text{eff}} \sim -\frac{1}{2V_4 R_1^4} \left( \frac{\zeta'(-4)}{6} \frac{\pi^4}{L^4} + \frac{\zeta'(-2)}{12} \frac{\pi^2}{L^2} \cdot \cdot \cdot \right)
+ \lambda_1 + \lambda_2 (1 - L \coth z_1)^4
\sim -\frac{1}{2V_4 R_1^4} \left( \frac{\zeta'(-4)}{6} \frac{\pi^4}{L^4} + \frac{\zeta'(-2)}{12} \frac{\pi^2}{L^2} \cdot \cdot \cdot \right)
+ \lambda_1 + \lambda_2 - \frac{4\lambda_2 L \sqrt{R_1^2 + l^2}}{R_1}. \quad (98)
\]
We should note that the “distance” $L$ is not the real distance between the two branes. From (87), one can find the real distance $L_R$ is given by

$$L_R = \int_{z_1}^{z_2} dz \frac{l}{\sinh z} = 2l \ln \left| \frac{\tanh \frac{z_2}{2}}{\tanh \frac{z_1}{2}} \right|. \quad (99)$$

In the limit where the radii $R_1$ and $R_2$ of the two branes become very large, which is the limit of flat branes, we have $z_1, z_2 \to 0$. Then in the limit, Eq. (99) reduces as

$$L_R \sim 2l \ln \frac{z_2}{z_1}. \quad (100)$$

Since

$$L = z_2 - z_1 \sim \left( e^{\frac{L_R}{2}} - 1 \right) z_1, \quad R_1 = \frac{l}{\sinh z_1} \sim \frac{l}{z_1},$$

$$R_2 \sim \frac{\sinh z_1}{\sinh z_2} \sim e^{-\frac{L_R}{2}},$$

Eq. (96) has the following form:

$$V_{\text{eff}} \sim -\frac{1}{2V_4 l^4} \zeta'(-4) \frac{\pi^4}{6} \left( e^{\frac{L_R}{2}} - 1 \right)^4 + \lambda_1 + \lambda_2 e^{\frac{2L_R}{l}}, \quad (101)$$

which coincides with the result in [20] where it was used to study the radion stabilization. Note that the higher order terms in the expansion (92) vanish in the limit of the large brane. Since $\frac{1}{R_1}$ corresponds to the Hubble parameter when we Wick-rotate the metric into the Lorentzian signature, the large $R_b$ corresponds to the slow expansion of the universe. In such a case, (102) is valid. Since $\frac{R_2}{R_1}$ gives the ratio of the scales between the two branes, Eq. (101) shows that the hierarchy is given by $e^{-\frac{L_R}{2}}$ for the small Hubble parameter case.

We now consider the case that $R_1$ and $R_2$ are small, that is, the Hubble parameter is large. In this case, $z_1$ and $z_2$ becomes large. Since $\tanh \frac{z}{2} \to 1 - 2e^{-z}$ when $z \to +\infty$, the distance $L_R$ (99) has the following form:

$$L_R = 2l \left\{ 1 + 2e^{-2z_1} \left( 1 - e^{-2L} \right) \right\}. \quad (103)$$

We should note that there is a minimum $2l$ in $L_R$. In the limit that $R_1$ and $R_2$ go to infinity, $L_R \to 2l$. By using Eq. (97), one finds

$$\frac{R_2}{R_1} \to e^{-L}. \quad (104)$$

Therefore if $L \sim 50$, the hierarchy between the weak scale and the gravity can be consistent with the present hierarchy.

The value of the parameter $L$ is defined by the minimum of the effective potential $V_{\text{eff}}$ (98) with respect to $L$. Assuming that $L$ is small, one may keep only the first term and neglect the other terms in ( ) of (98). Since we did not specify the value of $\lambda_2$, we cannot drop the last term. Then from the variation of $V_{\text{eff}}$ with respect to $L$, one finds that the minimum is given when

$$L^5 \sim \frac{\pi^4 \zeta'(-4)}{12\lambda_2 V_4 R_1^3 \sqrt{R_1^2 + l^2}}. \quad (105)$$
Since $\zeta'(-4) > 0$, $\lambda_2$ should be positive in order that $L^5$ is positive. Then if $R_1$ is not large, in order that $L$ is small, from the consistency, $\lambda_2$ must be large. An interesting point is that the parameter $L$ should depend on $R_1$ or the Hubble parameter $\frac{1}{R_1}$. When $R_1$ is large, Eq. (100) and (111) show

$$L_R \sim 2l \ln \left\{ 1 + \frac{LR_1}{l} \right\}.$$  \hspace{1cm} (106)

Since $LR_1 \propto R_1^\frac{2}{3}$ for large $R_1$, $L_R$ and therefore the hierarchy slowly depend on the $R_1$: $L_R \propto \ln R_1$.

At the minimum (105), the effective potential (108) has the following value

$$V_{\text{eff}} = \lambda_1 + \lambda_2 \left\{ 1 - \frac{\sqrt{R_1^2 + l^2}}{R_1} \left( \frac{\pi^4 \zeta'(-4)}{12\lambda_2 V_4 R_1^3 \sqrt{R_1^2 + l^2}} \right)^\frac{1}{2} \right\}.$$  \hspace{1cm} (107)

Thus, even if $\lambda_2$ is large, we can fine-tune $\lambda_1$ to make $V_{\text{eff}}$ be the value of the observable brane cosmological constant. Note, however, the minimum (105) is unstable, which can be found from the fact that the effective potential (105) is unbounded below for small $L$ since $\zeta'(-4) > 0$. Since the sign of the leading term in the contribution to $V_{\text{eff}}$ from the spinors is different from the one of the scalar fields, the stability of the effective potential depends on the field content. With $N$ scalars and $M$ spinors, the effective potential (103) looks like

$$V_{\text{eff}} = -\frac{N - M}{2V_4 R_1^4} \zeta'(0) L_5 + \lambda_1 + \lambda_2 \left( \frac{R_2}{R_1} \right)^4.$$  \hspace{1cm} (108)

Then with small $L$, the minimum of the effective potential is given, instead of (105) by

$$L^5 \sim \frac{(N - M)\pi^4 \zeta'(-4)}{12\lambda_2 V_4 R_1^3 \sqrt{R_1^2 + l^2}}.$$  \hspace{1cm} (109)

If the contribution from the spinor fields is dominant, that is $N - M < 0$ and also if $\lambda_2 < 0$, there exists a minimum. In case $N - M, \lambda_2 < 0$, the effective potential becomes stable at least for small $L$. At the minimum, the effective potential has the following value.

$$V_{\text{eff}} = \lambda_1 + \lambda_2 \left\{ 1 - \frac{\sqrt{R_1^2 + l^2}}{R_1} \left( \frac{\pi^4 (N - M) \zeta'(-4)}{12\lambda_2 V_4 R_1^3 \sqrt{R_1^2 + l^2}} \right)^\frac{1}{2} \right\}. \hspace{1cm} (110)$$

In the present universe, the Hubble parameter $H_0$ could be $H_0 \sim 60\text{km s}^{-1} \text{Mpc}^{-1} \sim 2 \times 10^{-18}\text{s}^{-1}$ (for a recent review of early Universe with positive cosmological constant, see[21]). We may identify $\frac{1}{R_1}$ with $\frac{H_0}{c} \sim 10^{-26}\text{m}^{-1}$ (c is the light velocity). On the other hand, the length parameter $l$ could be a Planck length $\sim 10^{-35}\text{m}$, which is much smaller than $R_1$. We may also identify $\kappa_4^2 V_{\text{eff}}$ ($\kappa_4$ is the four dimensional gravitational coupling constant) with $\kappa_4^2 \Lambda \sim 10^{-120}$ ($\Lambda$ is the cosmological constant). Then Eq. (110) may give

$$10^{-120} \sim \kappa_4^4 L_1 + \kappa_4^4 \lambda_2 \left\{ 1 - 10^{-49} \left( -\kappa_4^4 \lambda_2 \right)^{-\frac{1}{2}} \right\}. \hspace{1cm} (111)$$
If one assumes \( \kappa_4^4 \lambda_1, \kappa_4^4 \lambda_2 \sim 10^{-120} \), the second term (bulk quantum effects) in \( \{\} \) of (111) can be neglected. If \( \lambda_1 = -\lambda_2 \), we have \( 10^{-120} \sim 10^{-49} \left(-\kappa_4^4 \lambda_2\right)^{-\frac{4}{5}} \) or \( \kappa_4^4 \lambda_2 \sim 10^{-89} \). On the other hand, Eq. (109) gives \( L \sim 10^{-49} \left(-\kappa_4^4 \lambda_2\right)^{-\frac{4}{5}} \), then even if \( \kappa_4^4 \lambda_2 \sim 10^{-120} \) or \( \kappa_4^4 \lambda_2 \sim 10^{-89} \), \( L \) is small. Then the hierarchy is hard to explain in the same way as in [1], without more fine-tuning \( \lambda_1 \) and \( \lambda_2 \). Another possibility is to choose \( \lambda_2 \) so that the terms inside \( \{\} \) in (111) cancelled with each other. Then \( \kappa_4^4 \lambda_1 \sim 10^{-120}, \kappa_4^4 \lambda_2 \sim 10^{-244} \). In this case, as \( L \sim 10^{-49} \left(-\kappa_4^4 \lambda_2\right)^{-\frac{4}{5}} \sim 1 \), there is a possibility to solve the problem of the hierarchy between the weak scale and Planck scale.

As \( R_1 \) could be large, one may assume \( R_2 \) is also large. Then we may use the effective potential (102), modified by including \( N \) scalars and \( M \) spinors:

\[
V_{\text{eff}} \sim -\frac{(N-M)\pi^4 \zeta’(-4)}{2V_4 l^4} \frac{e^{\frac{4\pi R}{l}}}{6} + \lambda_1 + \lambda_2 e^{\frac{4\pi R}{l}},
\]

(112)

Here it is taken \( e^{\frac{4\pi R}{l}} \gg 1 \) since \( e^{\frac{4\pi R}{l}} \) corresponds to the ratio of the weak scale and Planck scale \( (e^{\frac{4\pi R}{l}} \sim 10^{17}) \). For \( N - M > 0 \) and \( \lambda_2 > 0 \), the effective potential (112) has a minimum at

\[
e^{\frac{4\pi R}{l}} = -\frac{(N-M)\pi^4 \zeta’(-4)}{2V_4 l^4} \frac{1}{6\lambda_2},
\]

(113)

and the value of the effective potential at the minimum is given by

\[
V_{\text{eff}} \sim \frac{(N-M)\pi^4 \zeta’(-4)\lambda_2}{3V_4 l^4} + \lambda_1.
\]

(114)

Taking \( \kappa_4^4 V_{\text{eff}} \sim l^4 V_{\text{eff}} \sim 10^{-120} \) and \( \lambda_1 = 0 \), one has \( \kappa_4^4 \lambda_2 \sim 10^{-240} \) and \( e^{\frac{4\pi R}{l}} \sim 10^{30} \), which is still much larger than \( 10^{17} \). Thus, the observable cosmological constant is induced by the only bulk quantum effects but the natural solution of hierarchy problem does not occur. On the other hand, if we require \( e^{\frac{4\pi R}{l}} \sim 10^{17} \), we have \( \kappa_4^4 \lambda_2 \sim 10^{-136} \) and \( \kappa_4^4 \sqrt{\frac{(N-M)\pi^4 \zeta’(-4)\lambda_2}{3V_4 l^4}} \sim 10^{-68} \). Then if \( \kappa_4^4 \lambda_1 \sim 10^{-68} \), one may fine-tune \( \lambda_1 \) for \( V_{\text{eff}} \) to vanish.

Thus, we demonstrated that the effective potential may be stable at the minimum where it coincides with the observable value of the 4d cosmological constant. The quantum bulk effects (of spinors) stabilize the effective potential which minimum defines the cosmological constant. However, the fine-tuning of brane tension is still necessary to recover the observable value of the brane cosmological constant. Moreover, if necessary cosmological constant is induced it is hard to get the natural solution for hierarchy.

5. Field equation as entropy bound

Let us make now several remarks on the form of field equation with (or without) quantum corrections.

If we write the metric (87) in the warped form

\[
ds^2 = dy^2 + l^2 e^{2\Lambda(y)} d\Omega_4^2,
\]

(115)
the Einstein equation has the following form:
\[
4A'' + 4 (A')^2 = \frac{4}{l^2} + \kappa^2 \left( -\frac{1}{3} T + T_{yy} \right)
\]
(116)
\[
A'' + 4 (A')^2 + \frac{3}{l^2} e^{-2A} = \frac{4}{l^2} - \kappa^2 \left( \frac{1}{12} T + \frac{1}{4} T_{yy} \right)
\]
(117)
Here \( T \) is the trace of the energy momentum tensor \( T_{\mu\nu} \). In (116) and (117), the derivative with respect to \( y \) is denoted by \( ' \) (\( \partial_y = ' \)). Combining (116) and (117), one obtains
\[
(A')^2 = \frac{\kappa^2}{6} T_{yy} - \frac{e^{-2A}}{l^2} + \frac{1}{l^2}.
\]
(118)
This has the form very similar to the FRW equation. One may identify \( T_{yy} \) with the Casimir energy density \( \rho_0 \). On the branes, from the matching condition, we have
\[
A' = \frac{|\sigma_{1,2}|}{12}.
\]
(119)
Here \( \sigma_{1,2} \) is the tension of the brane. Combining (118) and (119), one gets
\[
\frac{|\sigma_{1,2}|}{12} = \frac{\kappa^2}{6} \frac{\sinh^5 z_{1,2} - \zeta'(0)}{2l^5 L V_4} - \frac{\sinh z_{1,2}}{l^2} + \frac{1}{l^2}.
\]
(120)
It is very interesting that one can develop FRW-like interpretation of above field equation via corresponding entropy bounds. As in [22], one defines the “Hubble entropy” \( S_H \), the “Bekenstein-Hawking entropy” \( S_{BH} \), “Bekenstein entropy” \( S_B \) by
\[
S_H \equiv \frac{4\pi l^5}{\kappa^2} A' V_4 e^{4A}, \quad S_{BH} \equiv \frac{4\pi l^4}{\kappa^2} V_4 e^{3A},
\]
\[
S_B \equiv -2\pi l^6 V_4 e^{5A} \left( \frac{T_{yy}}{6} - \frac{1}{\kappa^2 l^2} \right).
\]
(121)
Then above field equation is rewritten as
\[
S_H^2 = 2S_B S_{BH} - S_{BH}^2.
\]
(122)
For the pure AdS case (\( T_{\mu\nu} = 0 \))
\[
A = \ln \cosh \frac{y}{l}.
\]
(123)
If the hypersurface with \( z \to +\infty \) is considered as a horizon, \( S_H \) gives the usual Bekenstein-Hawking entropy since \( A' \to \frac{1}{l} \):
\[
S_H \to \frac{4\pi A_{A}^{\infty}}{\kappa^2}, \quad A_{A}^{\infty} \equiv l^4 V_4 e^{4A}.
\]
(124)
On the other hand, \( S_{BH} \) and \( 2S_B \) correspond to the Bekenstein-Hawking entropy when the hypersurface with \( A = 0 \) is regarded as a horizon:
\[
S_{BH}, \quad 2S_B \to \frac{4\pi A_{A}^{0}}{\kappa^2}, \quad A_{A}^{0} \equiv l^4 V_4.
\]
(125)
Eq. (122) may be rewritten as
\[
S_H^2 + (S_B - S_{BH})^2 = S_B^2.
\]
(126)
Remarkably, the entropy bounds a la Verlinde follow
\[ S_H, \quad |S_B - S_{BH}| \leq S_B \] (127)
Thus, it is demonstrated the universality of Verlinde representation for field equation. In similar way, other spaces may be discussed. Even for pure AdS, one can impose the periodic boundary condition for the Euclidean time coordinate and introduce the temperature. Then we may consider the entropy, or other thermodynamical quantities even for the pure AdS case. The corresponding entropy is related with that of the dual CFT as in [23].

6. Discussion
In summary, the role of bulk and brane quantum effects in brane-world cosmology is considered. In particular, the quantum creation of dS branes from constant curvature five-dimensional bulk is discussed, the way to stabilize the observable cosmological constant due to bulk quantum effects is suggested.

It is interesting that some modification of our formulation may be done so that the possibility to compare with fitting coming from Supernovae observations appears. Indeed, let the metric of the 3-brane has the warped form:
\[ ds^2 = -dt^2 + L^2 e^{2A} \sum_{i,j=1}^{3} \tilde{g}_{ij} dx^i dx^j . \] (128)
Here \( \tilde{g}_{ij} \) is the metric which satisfies \( \tilde{R}_{ij} = k \tilde{g}_{ij} \) with the Ricci curvature \( \tilde{R}_{ij} \) given by \( \tilde{g}_{ij} \) and \( k = 0, \pm 2 \) is chosen. The energy density \( \rho \) and pressure \( p \) may be defined by
\[ \tau_{tt} = \rho , \quad \tau_{ij} = pg_{ij} = pe^{2A} \tilde{g}_{ij} (i,j = 1,2,3) , \] (129)
Then \( \pi_{\mu\nu} \) in (7) has the following form:
\[ \pi_{tt} = \frac{1}{12} \rho^2 , \quad \pi_{ij} = \left( \frac{1}{4} \rho p + \frac{1}{12} \rho^2 \right) e^{2A} \tilde{g}_{ij} . \] (130)
Especially, the contributions from the conformal anomaly are [24]:
\[ \rho_A = - \frac{1}{a^4} \left[ b' \left( 6a^4 H^4 + 12a^2 H^2 \right) \right. \\
+ \left( \frac{2}{3} b + b'' \right) \left\{ a^4 \left( -6H \dot{H} - 18H^2 \dot{H} + 3H^2 \right) + 6a^2 H^2 \right\} \\
- 2b + 6b' - 3b'' \right] , \] (131)
\[ p_A = b' \left\{ 6H^4 + 8H^2 \dot{H} + \frac{1}{a^2} \left( 4H^2 + 8H \dot{H} \right) \right\} \\
+ \left( \frac{2}{3} b + b'' \right) \left\{ -2H \ddot{H} - 12H \dot{H} - 18H^2 \dot{H} - 9H \right\} \\
+ \frac{1}{a^2} \left( 2H^2 + 4H \dot{H} \right) = -2b + 6b' - 3b'' . \] (132)
Here, the “radius” of the universe \( a \) and the Hubble parameter \( H \) are
\[ a \equiv Le^A , \quad H = \frac{1}{a} \frac{da}{dt} = \frac{dA}{dt} . \] (133)
Since the 4d curvature has the following forms:

\[
R^{(4)}_{tt} = -3H_{,t} - 3H^2, \quad R^{(4)}_{ij} = \left(-H_{,t} - 3H^2 + \frac{k}{a^2}\right)a^2 \tilde{g}_{ij},
\]  

(134)

if we choose the action on the brane as with matter, the \((tt)\) component of Eq. (6) has the FRW like form

\[
H^2 = -\frac{k}{2a^2} - \frac{\Lambda_4}{3} + \kappa_4^2 \left[\frac{1}{3} \rho_{\text{matter}} + \frac{2}{\lambda} \left\{-\alpha \left(3H^2 + \frac{3k}{2a^2}\right) + \rho_{\text{matter}} + \rho_A \right\}^2 - \frac{6}{\lambda \kappa_4^2} E_{tt}\right].
\]  

(135)

Here the 4d effective gravitational coupling constant and the cosmological constant are given in (14).

The last term including \(E_{tt}\) becomes non-trivial if there is a black hole in the 5d bulk. The term is called dark radiation and we may assume as in [9]

\[
- \frac{6\kappa_4^2}{\lambda \kappa_4^2} E_{tt} = \frac{C}{a^4},
\]  

(136)

with a constant \(C\). For the late universe, the matter can be regarded as a dust then

\[
\rho_{\text{matter}} = \frac{\rho_0}{a^3}.
\]  

(137)

Furthermore if we neglect the term including the derivative of the Hubble parameter \(H\), the contribution (131) from the conformal anomaly has the following form

\[
\rho_A \sim -6b' H^4 - \frac{(16b + b'') H^2}{a^2} - \frac{2b + 6b' - 3b''}{a^4}.
\]  

(138)

Then Eq. (135) can be rewritten as

\[
H^2 = -\frac{\Lambda_4}{3} + \frac{2\kappa_4^2}{\lambda} \left(3\alpha H^2 + 6b' H^4\right)^2 + \left\{-\frac{k}{2a^2} - 2\kappa_4^2 \left(3\alpha H^2 + 6b' H^4\right) \frac{2 \left(16b + b''\right) H^2 + 3k\alpha}{\lambda}\right\} \frac{1}{a^2} + \left\{\frac{1}{3} - \frac{4 \left(3\alpha H^2 + 6b' H^4\right)}{\lambda}\right\} \frac{\kappa_4^2 \rho_0}{a^3} + \left\{C + \frac{\kappa_4^2 \left(2 \left(16b + b''\right) H^2 + 3k\alpha\right)^2}{2\lambda}\right\} \frac{1}{a^4} + \frac{4 \left(3\alpha H^2 + 6b' H^4\right) \left(-2b + 6b' - 3b''\right)}{\lambda} \frac{1}{a^6} - \frac{2\kappa_4^2 \left(2 \left(16b + b''\right) H^2 + 3k\alpha\right) \rho_0}{\lambda} \frac{1}{a^5} + \frac{\left\{2\kappa_4^2 \rho_0 - 2 \left(2 \left(16b + b''\right) H^2 + 3k\alpha\right) \left(-2b + 6b' - 3b''\right)\right\}}{\lambda} \frac{1}{a^6} - \frac{4 \rho_0 \left(-2b + 6b' - 3b''\right) \frac{1}{a^7} + 2 \left(-2b + 6b' - 3b''\right)^2}{\lambda} \frac{1}{a^8}.  
\]  

(139)

\footnote{For recent review of FRW cosmology from AdS bulk black holes, see [24]. It is interesting that bulk AdS black hole may help to prevent QG era of FRW cosmology, for recent discussion, see [25].}
In order to compare the above expression with the Supernovae data, as in last reference from [9], we rewrite (139) in the following form:

\[
H^2(z) = H_0^2 \left[ \Omega_0^0 + \Omega_2^0 (1 + z)^2 + \Omega_3^0 (1 + z)^3 + \Omega_4^0 (1 + z)^4 \\
+ \Omega_5^0 (1 + z)^5 + \Omega_6^0 (1 + z)^6 + \Omega_7^0 (1 + z)^7 + \Omega_8^0 (1 + z)^8 \right]
\]

\[
\Omega_0 \equiv -\frac{\Lambda_4}{3} + \frac{2\kappa_4^2}{\lambda} \left( 3\alpha H^2 + 6b'H^4 \right)^2
\]

\[
\Omega_2 \equiv -\frac{k}{2} + \frac{2\kappa_4^2 \left( 3\alpha H^2 + 6b'H^4 \right)}{\lambda} \left\{ 2 \left( 16b + b'' \right) H^2 + 3k\alpha \right\}
\]

\[
\Omega_3 \equiv \left\{ \frac{1}{3} - \frac{4 \left( 3\alpha H^2 + 6b'H^4 \right)}{\lambda} \right\} \kappa_4^2 \rho_0
\]

\[
\Omega_4 \equiv C + \frac{\kappa_4^2 \left\{ 2 \left( 16b + b'' \right) H^2 + 3k\alpha \right\}^2}{2\lambda}
\]

\[
+ \frac{4 \left( 3\alpha H^2 + 6b'H^4 \right) \left( -2b + 6b' - 3b'' \right)}{\lambda}
\]

\[
\Omega_5 \equiv -\frac{2\kappa_4^2 \left\{ 2 \left( 16b + b'' \right) H^2 + 3k\alpha \right\} \rho_0}{\lambda}
\]

\[
\Omega_6 \equiv \frac{2\kappa_4^2 \rho_0^2}{\lambda} + \frac{2 \left\{ 2 \left( 16b + b'' \right) H^2 + 3k\alpha \right\} \left( -2b + 6b' - 3b'' \right)}{\lambda}
\]

\[
\Omega_7 \equiv -\frac{4\rho_0 \left( -2b + 6b' - 3b'' \right)}{\lambda}
\]

\[
\Omega_8 \equiv \frac{2 \left( -2b + 6b' - 3b'' \right)^2}{\lambda}
\]

(140)

Here \( z \equiv \frac{a^0}{a} \) is a redshift factop and \( a^0 \) is the length parameter in the present universe. In (139), the quantities in the present universe are expressed by the superscript “0”. It follows that parameters receive the quantum correction from the conformal anomaly. As quantum correction may be chosen to be non-dominant, the number of parameters choice to fit the Supernovae data exists.

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24

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