Interpolation of Hilbert and Sobolev spaces: Quantitative estimates and counterexamples
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INTRODUCTION

Since we published the paper [4] in 2015, the quantitative results we derive therein, and the summary we provide of results in the literature on interpolation spaces, have been of use in our own work (for example, [5]) and elsewhere (for example, [13]). But the paper as published is marred by inaccuracies which we correct in this note, including the inaccuracy flagged in [13, p. 1768]. We use throughout the notations of [4]. As in [4], we intend primarily that vector space, Banach space, and Hilbert space should be read as their complex versions. But, except where we deal with complex interpolation, the results below apply equally in the real case, with minor changes to the statements and proofs.

CORRECTIONS TO SECTION 2

(1) The main inaccuracy in this section is in the first sentence, which should read: “Suppose that $X_0$ and $X_1$ are Banach spaces that are linear subspaces continuously embedded in some larger Hausdorff topological vector space $V$.” With this correction, the definition of a compatible pair in the second sentence coincides with the standard definition, as, for example, in [1, 3, 14]. The requirement that $X_0$ and $X_1$ are continuously embedded in a larger TVS, rather than just being linear subspaces of a larger linear space $V$ (as in [4, § 2] or [12, Appendix B]), is needed to establish that $|| \cdot ||_{\Sigma}$ is a norm (and not just a semi-norm) on $\Sigma(X)$. That this requirement

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is sufficient to establish that $\| \cdot \|_\Sigma$ is a norm is shown, for example, in [14, Lemma 1.2.1], [3, Proposition 2.1.7]. That $\| \cdot \|_\Sigma$ need not be a norm if the Banach spaces $X_0$ and $X_1$ are just linear subspaces of some larger vector space can be shown by Hamel basis-type constructions.

Following this strengthening of the meaning of compatible pair, one needs to check that every use of this term in the rest of the paper is accurate with compatible pair understood in this stronger sense. Where some additional justification is needed we have noted this below. Helpfully (and this is the case in all the applications considered in [4, §4]) it is immediate that $(X_0, X_1)$ is a compatible pair in this stronger sense if $X_1 \subset X_0$ with continuous embedding, for then we may take $X_0$ as the larger TVS. Similarly if $X_0 \subset X_1$ with continuous embedding.

(2) “$X_1 \subset X_0$” should read “$X_1 \subset X_0$ with continuous embedding” in the penultimate line of the first paragraph of Section 2 and in item (iii) of Theorem 2.2.

### 3 | CORRECTIONS TO SECTION 3

#### 3.1 | Corrections relating to Corollary 3.2

We provided in [4] no proof of Corollary 3.2, which we viewed at the time as a straightforward extension of the proof of Theorem 3.1. There is a typo; $|w_0 A \phi|^2$ should read $w_0 |A \phi|^2$ in the definition of $H^\theta$. But, even with this typo fixed, the corollary is false as stated. We will state the corrected corollary with a proof, and provide a counterexample to the corollary as stated in [4] (with the above typo fixed). The correction, in addition to fixing the above typo, is just to add the word “injective” before “linear” in the statement of the corollary. The corrected corollary is:

**Corollary 3.2′.** Let $\overline{H} = (H_0, H_1)$ be a compatible pair of Hilbert spaces, $(\mathcal{X}, M, \mu)$ be a measure space, and let $\mathcal{Y}$ denote the set of measurable functions $\mathcal{X} \to \mathbb{C}$. Suppose that there exists an injective linear map $A : \Sigma(\overline{H}) \to \mathcal{Y}$ and, for $j = 0, 1$, functions $w_j \in \mathcal{Y}$, with $w_j > 0$ almost everywhere, such that the mappings $A : H_j \to L^2(\mathcal{X}, M, w_\mu j)$ are unitary isomorphisms. For $0 < \theta < 1$, define intermediate spaces $H^\theta$, with $\Delta(\overline{H}) \subset H^\theta \subset \Sigma(\overline{H})$, by

$$H^\theta := \left\{ \phi \in \Sigma(\overline{H}) : \| \phi \|_{H^\theta} := \left( \int_{\mathcal{X}} |w_\theta| |A \phi|^2 \, d\mu \right)^{1/2} < \infty \right\},$$

where $w_\theta := w_0^{1-\theta} w_1^\theta$. Then, for $0 < \theta < 1$, $H^\theta = K_{\theta,2}(\overline{H}) = J_{\theta,2}(\overline{H})$, with equality of norms.

**Remark 1.** In the case that $H_1 \subset H_0$ (with $H_1$ a linear subspace of $H_0$), it holds that $\Sigma(\overline{H}) = H_0$ so that the injectivity of $A$ in the above corollary follows from the assumption that $A : H_0 \to L^2(\mathcal{X}, M, w_0 \mu)$ is an isomorphism. Similarly if $H_0 \subset H_1$.

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† Let $X_1$ be a Banach space with norm $\| \cdot \|_{X_1}$ and construct (using a Hamel basis and Zorn’s lemma, see, for example, [2, Example A, p. 249]) an unbounded linear functional $f$ on $X_1$, and let $X_0$ be the completion of $X_1$ with respect to the norm $\| \cdot \|_{X_0}$ given by $\| \phi \|_{X_0} := \| \phi \|_{X_1} + |f(\phi)|$, $\phi \in X_1$. Then $X_0$ and $X_1$ are Banach spaces with $X_1$ a linear subspace of $X_0$ (so $X_0$ and $X_1$ are linear subspaces of a larger linear space $V$, namely $V = X_0$), but the inclusion map is not continuous. This implies, by the closed graph theorem, that the inclusion map is not closed, that is, there exists a sequence $(\phi_n) \subset X_1$ which is convergent in $X_1$ to $x_1 \in X_1$, $i = 0, 1$, with $x_0 \neq x_1$. This in turn implies that $x_0 - x_1 \neq 0$ but $\| x_0 - x_1 \|_2 \leq \lim \inf_{n \to \infty} (\| x_0 - \phi_n \|_{X_0} + \| \phi_n - x_1 \|_{X_1}) = 0$, so that $\| \cdot \|_2$ is not a norm.
That Corollary 3.2’ holds is most easily seen as a corollary of the following more general result. We use in the statement and proof of this lemma the notations from the start of [4, § 2].

**Lemma 2.** Suppose that \( \overline{X} = (X_0, X_1) \) and \( \overline{Y} = (Y_0, Y_1) \) are compatible pairs of Banach spaces, that \( A : \Sigma(\overline{X}) \to \Sigma(\overline{Y}) \) is an injective linear map, and that \( A_j \), the restriction of \( A \) to \( X_j \), is an isometric isomorphism from \( X_j \) to \( Y_j \), \( j = 0, 1 \). Suppose also that \( (X, Y) \) is a pair of exact interpolation spaces relative to \( (\overline{X}, \overline{Y}) \) and that \( (Y, X) \) is a pair of exact interpolation spaces relative to \( (\overline{Y}, \overline{X}) \). Then \( A(X) = Y \) and \( A : X \to Y \) is an isometric isomorphism.

**Proof.** Note first that, since \( A(X_j) = Y_j \), \( j = 0, 1 \), \( A : \Sigma(\overline{X}) \to \Sigma(\overline{Y}) \) is surjective, and so bijective and a linear isomorphism\(^1\). Thus \( A \) has a linear inverse \( B : \Sigma(\overline{Y}) \to \Sigma(\overline{X}) \), which is a couple map with \( B_j = A_j^{-1} \), where \( B_j \) denotes the restriction of \( B \) to \( Y_j \). Since \( (X, Y) \) is a pair of exact interpolation spaces relative to \( (\overline{X}, \overline{Y}) \), and \( A \) is a couple map, \( A(\overline{X}) \subset \overline{Y} \), \( A : X \to Y \) is bounded, and \( \|A\|_{X,Y} \leq \max(\|A\|_{X_0,Y_0},\|A\|_{X_1,Y_1}) = 1 \). Similarly, since \( (Y, X) \) is a pair of exact interpolation spaces relative to \( (\overline{Y}, \overline{X}) \), \( B(\overline{Y}) \subset \overline{X} \) and \( \|B\|_{Y,X} \leq \max(\|B\|_{Y_0,X_0},\|B\|_{Y_1,X_1}) = 1 \). Thus \( A(X) = Y \) and \( A \) is an isometric isomorphism. \( \square \)

**Remark 3.** If, in the statement of the above lemma, we require only that each \( A_j \) is an isomorphism (not necessarily isometric), and that \( (X, Y) \) and \( (Y, X) \) are pairs of interpolation spaces (not necessarily exact), then the argument of the above proof shows that \( A(X) = Y \) and that \( A : X \to Y \) is an isomorphism (cf. [1, Theorem 6.4.2]).

**Remark 4.** As a concrete application of the above lemma we can take \( X \) and \( Y \) to be complex interpolation spaces, that is, \( X = (X_0, X_1)[\theta] \) and \( Y = (Y_0, Y_1)[\theta] \), for some \( 0 < \theta < 1 \), in the notation of [1], or we can take \( X = K_{\theta, q}(\overline{X}) \) and \( Y = K_{\theta, q}(\overline{Y}) \), for any \( 0 < \theta < 1 \) and \( 1 \leq q \leq \infty \). With either of these choices \( (X, Y) \) and \( (Y, X) \) are pairs of interpolation spaces that are exact of exponent \( \theta \) (see, for example, [1, Theorem 4.1.2] for the complex case, [12, Theorem B.2] or [4, Theorem 2.2(i)] for the \( K \)-interpolation spaces).

To see that the same is true for the choice \( X = J_{\theta, q}(\overline{X}) \) and \( Y = J_{\theta, q}(\overline{Y}) \), we just need to check that the standard argument (for example, [1, Theorem 3.2.2]) carries over to the not-quite-standard definition (suited to the Hilbert space case) we make for the \( J \)-functional in [4, § 2.2], following [12, Appendix B]. That \( (X, Y) \) and \( (Y, X) \) are pairs of interpolation spaces is clear since \( K_{\theta, q}(\overline{X}) = J_{\theta, q}(\overline{X}) \) and \( K_{\theta, q}(\overline{Y}) = J_{\theta, q}(\overline{Y}) \), with equivalence of norms [4, Theorem 2.3]. It remains to check that these pairs are exact of exponent \( \theta \). So suppose that \( \overline{X} = (X_0, X_1) \) and \( \overline{Y} = (Y_0, Y_1) \) are compatible pairs of Banach spaces, that \( A : \Sigma(\overline{X}) \to \Sigma(\overline{Y}) \) is a linear map, and that \( A_j \), the restriction of \( A \) to \( X_j \), is bounded from \( X_j \) to \( Y_j \), \( j = 0, 1 \), and set \( M_j := \|A_j\|_{X_j,Y_j} \), \( j = 0, 1 \), \( X := J_{\theta, q}(\overline{X}) \), and \( Y := J_{\theta, q}(\overline{Y}) \). Since \( (X, Y) \) is a pair of interpolation spaces, it follows that \( A : X \to Y \) and is bounded. Further, if \( \phi \in X \), \( \phi \) can be written as the Bochner integral

\[
\phi = \int_0^\infty f(t) \frac{dt}{t},
\]

\(^1\)Indeed, also a topological isomorphism, by the Banach bounded inverse theorem, since \( A \) is bounded as each \( A_j \) is bounded.
for some $\Delta(\overline{X})$-strongly measurable function $f : (0, \infty) \to \Delta(\overline{X})$, with the integral convergent in $\Sigma(\overline{X})$, and in $\Delta(\overline{X})$ when the interval of integration is reduced to $(a, b)$ with $0 < a < b < \infty$ (see [4, §2.2]; for the definition of the Bochner integral see, for example, [12, Appendix B] or [6, Appendix E]). Arguing as in the proof of [1, Theorem 3.2.2] it follows, by linearity, and since our assumptions imply that $A : \Delta(\overline{X}) \to \Delta(\overline{Y})$ and $A : \Sigma(\overline{X}) \to \Sigma(\overline{Y})$ are bounded, that the mapping $t \mapsto Af(t)$ is $\Delta(\overline{Y})$-strongly measurable and that

$$A\phi = \int_0^\infty Af(t) \frac{dt}{t},$$

with the integral (2) convergent in $\Sigma(\overline{Y})$ and in $\Delta(\overline{Y})$ on $(a, b)$ with $0 < a < b < \infty$. Moreover, for $t > 0$,

$$J(t, Af(t), \overline{Y}) = \left( \|Af(t)\|_{\overline{Y}_0}^2 + t^2\|Af(t)\|_{\overline{Y}_1}^2 \right)^{1/2} \leq M_0 J(tM_1/M_0, f(t), \overline{X}),$$

so that

$$\|A\phi\|_Y \leq \|J(\cdot, Af(\cdot), \overline{Y})\|_{\beta,q} \leq M_0 \|J(M_1/M_0, f(\cdot), \overline{X})\|_{\beta,q} = M_0^{1-\theta} M_1^\theta \|J(\cdot, f_{M_1/M_0}(\cdot), \overline{X})\|_{q,\theta},$$

where, for $s > 0$, $f_s(t) := f(st)$, $t > 0$. Noting that, for every $s > 0$, (1) holds if and only if $\phi = \int_0^\infty f_s(t) \frac{dt}{t}$, it follows, taking the infimum over all $f$ for which (1) holds, that $\|A\phi\|_Y \leq M_0^{1-\theta} M_1^\theta \|\phi\|_X$. Thus $(X, Y)$ is exact of exponent $\theta$; clearly the same holds true for $(Y, X)$.

**Proof of Corollary 3.2’.** Suppose that the conditions of Corollary 3.2’ are satisfied, let $Y_j := L^2(\mathcal{X}, M, w_j \mu)$, $j = 0, 1$, and note that $\overline{Y} := (Y_0, Y_1)$ is a compatible pair, since each $Y_j$ is continuously embedded in $Y^* := L^2(\mathcal{X}, M, w^* \mu)$ (for example), where $w^* := \min(w_0, w_1)$. It is clear from the conditions of the corollary that $A(\Sigma(H)) \subset \Sigma(\overline{Y})$, that is, $A : \Sigma(H) \to \Sigma(\overline{Y})$. Set $Y^\theta := L^2(\mathcal{X}, M, w_\theta \mu) \subset \Sigma(\overline{Y})$, for $0 < \theta < 1$. For each fixed $0 < \theta < 1$, where $H$ denotes either $K_{\theta,2}(\overline{H})$ or $J_{\theta,2}(\overline{H})$, and $Y$ denotes, correspondingly, $K_{\theta,2}(\overline{Y})$ or $J_{\theta,2}(\overline{Y})$, Lemma 2 and Remark 4 imply that $A(H) = Y$ and that $A : H \to Y$ is an isometric isomorphism. But also $Y = Y^\theta$, with equality of norms, by [4, Theorem 3.1]. It follows that $H^\theta = H$ and that, for $\phi \in H$, $\|\phi\|_H = \|A\phi\|_{Y^\theta} = \|\phi\|_{H^\theta}$.

**Remark 5 (Counterexample to Corollary 3.2 as stated in [4] (with the typo fixed)).** To see that Corollary 3.2’ is false if the word “injective” is deleted, let $\mathcal{X} = (0, \infty)$, $M$ be the Lebesgue measurable subsets of $\mathcal{X}$, and take $d\mu = dt/t$, where $dt$ is Lebesgue measure. For some $\alpha > 0$ and $2 < \beta < 0$, let $w_0(t) := t^{-\alpha}, w_1(t) := t^{-\beta}, t > 0$, let $Y_j := L^2(\mathcal{X}, M, w_j \mu)$, $j = 0, 1$, and define $A : \Sigma(\overline{Y}) \to \mathcal{Y}$ by

$$A\phi(s) = \phi(s) - \frac{1}{s} \int_0^s \phi(t) dt, \ s > 0, \ \phi \in \Sigma(\overline{Y}).$$

Then, see [10, Remark 3], $A : Y_j \to Y_j$ is an isomorphism, for $j = 0, 1$, but $A : \Sigma(\overline{Y}) \to \mathcal{Y}$ is not injective; explicitly, the constant $1 \in \Sigma(\overline{Y})$ and $A1 = 0$. Further, taking $\theta = \alpha/(\alpha - \beta)$ we
have that $Y^\vartheta := L^2(\mathcal{X}, M, w_0 \mu) = L^2(\mathcal{X}, M, \mu)$, and that $K_{\vartheta,2}(\bar{Y}) = J_{\vartheta,2}(\bar{Y}) = Y^\vartheta$, with equality of norms, by [4, Theorem 3.1], so that $A : Y^\vartheta \to Y^\vartheta$ is bounded by [4, Theorem 2.2(i)]. But, see [10, Remark 2], $A : Y^\vartheta \to Y^\vartheta$ is not an isomorphism; precisely, there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset Y^\vartheta$ with $\|\phi_n\|_{Y^\vartheta} = 1$ and $\|A\phi_n\|_{Y^\vartheta} \to 0$ as $n \to \infty$. For $j = 0, 1$ let $H_j$ denote the set $Y_j$ equipped with the norm $\| \cdot \|_{H_j}$ defined by $\|\phi\|_{H_j} := \|A^{-1}\phi\|_{Y_j}$, equivalent to the norm $\| \cdot \|_{Y_j}$. It is easy to see that this is a Hilbert-space norm, so that $H_j$ is a Hilbert space, and clearly $A : H_j \to Y_j$ is an isometric isomorphism, so a unitary isomorphism. Further, $K_{\vartheta,2}(H) = J_{\vartheta,2}(H) = Y^\vartheta$, with equivalence of norms. But, if it holds that $H^\vartheta = Y^\vartheta$, it does not hold that the norms are equivalent, for $\|\phi_n\|_{H^\vartheta} / \|\phi_n\|_{Y^\vartheta} = \|\phi_n\|_{Y^\vartheta} = \|A\phi_n\|_{Y^\vartheta} \to 0$ as $n \to \infty$. Thus it is not true that $H^\vartheta = K_{\vartheta,2}(H) = J_{\vartheta,2}(H)$ with equivalence of norms. Thus Corollary 3.2′ is false if the word “injective” is deleted.

### 3.2 | Corrections to Theorem 3.3

As pointed out in [13, p. 1768], there is an inaccuracy in the definition of the domain of the unbounded operator $T$ in [4, Theorem 3.3], and consequent inaccuracies in the proof. Additionally, the corrections made above to the statement of Corollary 3.2 necessitate modification where we apply this corollary in the proof of Theorem 3.3. Here is the amended statement of Theorem 3.3 together with an amended version of the second half of the proof. As in [4], we use in this theorem and its proof properties of non-negative, closed symmetric forms and their associated self-adjoint operators, see, for example, [9, VI §1] or [7, Chapter IV].

**Theorem 3.3′.** Suppose that $H = (H_0, H_1)$ is a compatible pair of Hilbert spaces. Then, for $0 < \vartheta < 1$, $\|\phi\|_{K_{\vartheta,2}(H)} = \|\phi\|_{J_{\vartheta,2}(H)}$ for $\phi \in (H_0, H_1)_{\vartheta,2}$. Further, where $H^\circ$ denotes the closure in $H_1$ of $\Delta(H)$, $(\cdot, \cdot)_{H_0}$, with domain $\Delta(H)$, is a closed, densely defined, non-negative symmetric form on $H^\circ_1$, with an associated unbounded, non-negative, self-adjoint, injective operator $T : H^\circ_1 \to H^\circ_1$, which satisfies

$$
(T\phi, \psi)_{H_1} = (\phi, \psi)_{H_0}, \quad \phi, \psi \in D(T),
$$

where $D(T)$, the domain of $T$, is a dense linear subspace of the Hilbert space $\Delta(H)$. Moreover, where $S : H^\circ_1 \to H^\circ_1$ is the unique non-negative, self-adjoint square root of $T$, which has domain $\Delta(H)$, it holds that

$$
\|\phi\|_{H_0} = \|S\phi\|_{H_1} \text{ and } \|\phi\|_{K_{\vartheta,2}(H)} = \|\phi\|_{J_{\vartheta,2}(H)} = \|S^{1-\vartheta}\phi\|_{H_1}, \quad \text{for } \phi \in \Delta(H), \quad 0 < \vartheta < 1,
$$

so that $K_{\vartheta,2}(H)$ is the closure of $\Delta(H)$ in $\Sigma(H)$ with respect to the norm defined by $\|\phi\|_{\vartheta} := \|S^{1-\vartheta}\phi\|_{H_1}$.

**Proof.** The proof that $\|\phi\|_{K_{\vartheta,2}(H)} = \|\phi\|_{J_{\vartheta,2}(H)}$ proceeds as the proof of [4, Theorem 3.3]. In particular, where $A_j : \Delta(H) \to \Delta(H)$, for $j = 0, 1$, is the bounded, non-negative, self-adjoint, injective operator defined by $(A_j\phi, \psi)_{\Delta(H)} = (\phi, \psi)_{H_j}$, for $\phi, \psi \in \Delta(H)$, it holds by the spectral theorem that there exists a measure space $(\mathcal{X}, M, \mu)$, bounded $\mu$-measurable functions $w_j$, with $w_j > 0$ almost everywhere and $w_0 + w_1 = 1$, and a unitary isomorphism $U : \Delta(H) \to L^2(\mathcal{X}, M, \mu)$, such that

$$
A_j\phi = U^{-1}w_jU\phi, \quad \text{for } \phi \in \Delta(H), \quad j = 0, 1.
$$
As we note in the proof in [4], where $H^j_\circ$ denotes the closure of $\Delta(H)$ in $H_j$, $U$ extends to a unitary isomorphism $U : H^j_\circ \rightarrow L^2(\mathcal{X}, M, w_j \mu)$, $j = 0, 1$, and extends further to a linear operator $U : \Sigma(H_\circ) \rightarrow \mathcal{Y}$, where $H_\circ = (H_0^\circ, H_1^\circ)$ and $\mathcal{Y}$ is the space of $\mu$-measurable functions defined on $\mathcal{X}$. Moreover, if $\phi \in \Sigma(H_\circ)$ and $U\phi = 0$, in which case $\phi = \phi_0 + \phi_1$ with $\phi_j \in H^j_\circ$ for $j = 0, 1$, then, defining $y := U\phi_0 = -U\phi_1$, we see that $y \in L^2(\mathcal{X}, M, w_j \mu)$, $j = 0, 1$, so that $y \in L^2(\mathcal{X}, M, \mu)$ and, since $U : \Delta(H) \rightarrow L^2(\mathcal{X}, M, \mu)$ is surjective, $y = U\psi$ for some $\psi \in \Delta(H)$. Since $U : H_j \rightarrow L^2(\mathcal{X}, M, w_j \mu)$ is injective, $j = 0, 1$, it follows that $\phi_0 = \psi = -\phi_1$, so that $\phi = 0$, that is, $U : \Sigma(H_\circ) \rightarrow \mathcal{Y}$ is injective.

This observation was missing in the original proof.

3.3 Other corrections to Section 3

(1) As a consequence of the correction above to Corollary 3.2 (see Corollary 3.2'), $w_1(j) = \lambda_j^{-1/2}$ in the last line of the proof of Theorem 3.4 should read $w_1(j) = \lambda_j^{-1}$. In the same sentence, “$w_0$ and $w_j$” should read “$w_0$ and $w_1$”, and, in the displayed equation three lines above, the two instances of $\lambda_i$ should read $\lambda_j$.

(2) In Section 3.2, the words “a Hilbert space” are missing from the first sentence of the last paragraph on page 428, which should start: “More briefly but equivalently, in the language introduced in §2, a Hilbert space $H$ is a geometric interpolation space of exponent $\theta$ ...”.

(3) The proof of Theorem 3.5 is arguably not quite complete because we do not say explicitly that $H_\theta$ is a Hilbert space. The first sentence of the proof should be replaced with the following sentences: “As we have noted at the beginning of §3, $H_\theta := K_{\theta, 2}(H)$ is a Hilbert space due to the particular definition we have made for $K_{\theta, 2}(H)$ and since $H_0$ and $H_1$ are Hilbert spaces (this follows, in particular, from Theorem 3.3, since $\|\phi\|_{K_{\theta, 2}(H)} = \|S^{1-\theta} \phi\|_{H_1}$, and the latter is clearly a Hilbert-space norm). That $H_\theta$ is moreover a geometric interpolation space of exponent $\theta$ follows from Lemma 2.1(iii) and Theorem 2.2(i) and (iv)”.

(4) In Remark 3.6 our justification that “the complex interpolation method is also an instance of that functor” is not quite complete, because we do not justify that complex interpolation of
Hilbert spaces produces a Hilbert space. This must be well known, for example, it is implicit in the last sentence of [11, § 1]. One way of seeing this is to note that a version of [4, Theorem 3.2] holds for complex interpolation (see [1, Theorem 5.5.3]) so that, noting the comments on complex interpolation in Remark 4, a version of Corollary 3.2′ holds for complex interpolation, so that Theorem 3.3′ can be strengthened to conclude that \( K_{\theta,2}(H) = J_{\theta,2}(H) = (H_0, H_1)_{\theta} \), with equal norms (the relationship \( \|\phi\|_{\theta,2} = \|S^{1-\theta}\phi\|_{H_1} \) makes clear that this norm is a Hilbert-space norm). Of course this argument also shows directly the claim in the last sentence of Remark 3.6.

(5) In Remark 3.8, the phrase “\((H_s, H_t)\) is a compatible pair and” is missing. The end of the second sentence of the remark should read: ... a collection of Hilbert spaces \( \{H_s : s \in I\} \), indexed by \( I \), is an interpolation scale if, for all \( s, t \in I \) and \( 0 < \eta < 1 \), \((H_s, H_t)\) is a compatible pair and

\[
(H_s, H_t)_{\eta,2} = H_{\theta}, \quad \text{for } \theta = (1 - \eta)s + \eta t.
\]

4 CORRECTIONS TO SECTION 4

(1) As a consequence of the correction above to Corollary 3.2 (see Corollary 3.2′), \( w_j(\xi) = (1 + |\xi|^2)^{s_j/2} \) in the proof of Theorem 4.1 should read \( w_j(\xi) = (1 + |\xi|^2)^{-s_j} \).

(2) As noted in [8, Appendix B], there is a sign error in [4, Equation (26)], which should read

\[
\|\phi\|^2_{H^2(\Omega)} = |\phi(0)|^2 + |\phi'(0)|^2 + |\phi(0) - \phi'(0)|^2 + |\phi(a)|^2 + |\phi'(a)|^2 + |\phi(a) + \phi'(a)|^2
+ \int_0^a (|\phi|^2 + 2|\phi'|^2 + |\phi''|^2) \, dx.
\]

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