On Hurwitz stable polynomials with integer coefficients

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Let $H(N)$ denote the set of all polynomials with positive integer coefficients which have their zeros in the open left half-plane. We are looking for polynomials in $H(N)$ whose largest coefficients are as small as possible and also for polynomials in $H(N)$ with minimal sum of the coefficients. Let $h(N)$ and $s(N)$ denote these minimal values. Using Fekete's subadditive lemma we show that the $N$th square roots of $h(N)$ and $s(N)$ have a limit as $N$ goes to infinity and that these two limits coincide. We also derive tight bounds for the common value of the limits.

1 Introduction

A polynomial $p(z) = p_N z^N + \cdots + p_0$ with real coefficients is called Schur stable if all its zeros are in the open unit disk and is said to be Hurwitz stable if its zeros are all located in the left open half plane. Such polynomials appear as the result of Wiener-Hopf and spectral factorizations. To test numerical algorithms for these factorizations, it is desirable to have some supply of Schur and Hurwitz stable polynomials. For example, one starts with a Hurwitz stable polynomial $p(z)$, forms the product $f(z) = p(-z)p(z)$, applies the algorithm to get a factorization $f(z) \approx q(-z)q(z)$, and finally one measures the error $q(z) - p(z)$.

It is easy to produce nice Schur stable polynomials of arbitrary degree. For instance, Bini, Fiorentino, Gemignani, and Meini [1] introduced the beautiful polynomials

$$p(z) = 1 + z + \cdots + z^{N-1} + 2z^N.$$  

To reveal that $p(z)$ is Schur stable, we show that the reverse polynomial

$$z^N p(1/z) = 2 + z + \cdots + z^N$$

has no zeros in the open unit disk. And indeed, for $z \neq 1$ we have

$$2 + z + \cdots + z^N = 1 + \frac{1-z^{N+1}}{1-z} = \frac{2 - z - z^N}{1-z},$$

and this cannot be zero for $|z| < 1$ because then $|z + z^N| < 2$. In fact, the zeros of $p(z)$ cluster extremely close to the unit circle as $N$ increases. In addition, the coefficients of $p(z)$ are all small. (Note that the constant term $p_0$ of a monic Schur stable polynomial

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\( p(z) = z^N + \cdots + p_0 \) is always of modulus less than 1.) For these two reasons, these polynomials are excellent test polynomials for factorization algorithms.

Finding nice Hurwitz stable polynomials is a much harder task. The Wilkinson polynomials \( p(z) = (z + 1)(z + 2) \cdots (z + N) \) have astronomically large coefficients and are therefore not feasible. Well, one could take \( p(z) = (z + 1)^N \), but already for \( N = 20 \) the largest coefficient is \( \binom{20}{10} = 184756 \). The choice \( p(z) = (z + \mu)^N \) with \( 0 < \mu < 1 \) is also critical, since then the constant term \( p_0 = \mu^N \) may become the machine zero. Thus, I pose the following as a test: find a Hurwitz stable polynomial of degree 20 with positive integer coefficients such that the largest coefficient is about a hundred times better than 184756, that is, such that it does not exceed 2000.

2 Möbius transformation

A point \( z \) lies in the left open half-plane if and only if its distance to \(-1\) is smaller than that to 1, that is, if and only if \(|1 - z|/|1 + z| > 1\). Consequently, if \( u(z) \) has degree \( N \) and all zeros of \( u(z) \) are of modulus greater than 1, then \((1 + z)^N u\left(\frac{1-z}{1+z}\right)\) is a Hurwitz stable polynomial of degree \( N \).

Let \( u(z) = 2 + z + \cdots + z^N \) be the reverse of the polynomial we encountered in the introduction. Then, for \( z \neq 0 \),

\[
\ell_N(z) := (1 + z)^N u\left(\frac{1-z}{1+z}\right) = \frac{2(1+z)^{N+1} - (1-z)(1+z)^N - (1-z)^{N+1}}{2z} = \frac{(1+z)^{N+1} - (1-z)^{N+1}}{2z} + (1+z)^N.
\] (1)

Thus, \( \ell_N(z) \) is a Hurwitz stable polynomial of degree \( N \). In what follows we frequently represent polynomials by their coefficient vectors as in Matlab, that is, we write the polynomial \( p_N z^N + \cdots + p_0 \) as \([p_N \ldots p_0]\). For even \( N \), the coefficients of the polynomials \( [\text{I}] \) are all even, and hence we divide them by 2. The first polynomials are

\[
[1 3], \quad [1 1 2], \quad [1 7 3 5], \quad [1 2 8 2 3], \quad [1 3 18 10 25 3 4],
\]

and for the degrees 10, 16, 20 we obtain

\[
N = 10 : \quad [1 5 50 60 270 126 336 60 105 5 6], \\
N = 16 : \quad [1 8 128 280 2100 2184 10192 5720 18590 5720 13728 2184 5720 60 209950 38760 77520 268736 83980 83980 38760 77520 12597 570 760 10 11], \\
N = 20 : \quad [1 10 200 570 5415 7752 46512 38760 164730 83980 83980 209950 38760 77520 7752 12597 570 760 10 11].
\]

Figure 1 shows the zeros of \( \ell_{20}(z) \).
3 The inequality of Beauzamy

The following is a slight improvement of an inequality which, for $v = 1$, is stated (without proof) and attributed to Beauzamy in [5].

**Theorem 3.1** Let $p_N(z) = p_N z^N + \cdots + p_0$ be a Hurwitz stable polynomial of even degree with $p_N \geq 1$ and $p_0 \geq 1$. Then for every real number $v \geq 1$,

$$p_N(v) \geq (v^2 + 1)^{N/2}.$$  

**Proof.** Suppose $p_N(z)$ has exactly $n$ real zeros and exactly $m$ pairs of genuinely complex conjugate zeros, multiplicities taken into account. Then $N = n + 2m$ and we may write

$$p_N(z) = p_N \prod_{j=1}^{n} (z + \mu_j) \prod_{j=1}^{m} (z^2 + 2x_jz + |w_j|^2)$$  

(2)
with $\mu_1, \ldots, \mu_n > 0$ and with $w_j = -x_j + iy_j$, $x_j > 0$ $(j = 1, \ldots, m)$. The constant term of (2) is $p_0 = p_N \mu_1 \cdots \mu_n \lvert w_1 \rvert^2 \cdots \lvert w_m \rvert^2$. Clearly, all coefficients of (2) are positive. We have

$$p_N(v) = p_N \prod_{j=1}^{n} (v + \mu_j) \prod_{j=1}^{m} (v^2 + 2x_j v + \lvert w_j \rvert^2)$$

$$\geq \frac{p_N}{v^n} \prod_{j=1}^{n} (v^2 + \mu_j v) \prod_{j=1}^{m} (v^2 + \lvert w_j \rvert^2).$$

Put $\varphi_1 = \mu_1 v, \ldots, \varphi_n = \mu_n v, \varphi_{n+1} = \lvert w_1 \rvert^2, \ldots, \varphi_{n+m} = \lvert w_m \rvert^2$ and let $M = n + m$. Then

$$p_N(v) \geq \frac{p_N}{v^n} \prod_{j=1}^{M} (v^2 + \varphi_j), \prod_{j=1}^{M} \varphi_j = \frac{v^n}{p_N} p_0.$$

It follows that

$$p_N(v) \geq \frac{p_N}{v^n} (v^2)^M + (v^2)^{M-1} s_1 + (v^2)^{M-2} s_2 + \cdots + (v^2)^{M-M}s_M,$$

where

$$s_1 = \varphi_1 + \varphi_2 + \cdots + \varphi_M,$$

$$s_2 = \varphi_1 \varphi_2 + \varphi_1 \varphi_3 + \cdots + \varphi_{M-1} \varphi_M,$$

$$\ldots,$$

$$s_M = \varphi_1 \varphi_2 \cdots \varphi_M$$

are the symmetric functions of $\varphi_1, \ldots, \varphi_M$. The sum $s_k$ $(1 \leq k \leq M - 1)$ contains $\binom{M}{k}$ terms and each $\varphi_j$ occurs exactly $\binom{M-1}{k-1}$ times in $s_k$. The inequality between the arithmetic and geometric means therefore gives

$$s_k \geq \binom{M}{k} \left( \frac{1}{\binom{M}{k}} \right)^{1/\binom{M}{k}} = \binom{M}{k} \left( \frac{1}{\binom{M}{k}} \right)^{k/M} = \binom{M}{k} s_M^{k/M}.$$

Thus, using the binomial theorem and taking into account that $p_N \geq 1$, $p_0 \geq 1$, $v \geq 1$ we get

$$p_N(v) \geq \frac{p_N}{v^n} \sum_{k=0}^{M} \binom{M}{k} (v^2)^{M-k}s_M^{k/M} = \frac{p_N}{v^n} \left( v^2 + s_M^{1/M} \right)^M$$

$$= \frac{p_N}{v^n} \left( v^2 + \left( \frac{v^n p_0}{p_N} \right)^{1/M} \right)^M = \frac{1}{v^n} \left( \frac{p_N^{1/M} v^2 + v^{n/M} p_0^{1/M}}{p_N} \right)^M$$

$$\geq \frac{1}{v^n} (v^2 + v^{n/M})^M \geq \frac{1}{v^n} (v^2 + 1)^M = \left( \frac{v^2 + 1}{v^2} \right)^{n/2} (v^2 + 1)^{n/2+m}$$

$$\geq (v^2 + 1)^{n/2+m} = (v^2 + 1)^{N/2}. \quad \square$$
Corollary 3.2 Let \( p_N(z) = p_N z^N + \cdots + p_0 \) be a Hurwitz stable polynomial of even degree with \( p_N \geq 1 \) and \( p_0 \geq 1 \). Then the sum of the coefficients is greater than or equal to \( 2^{N/2} \) and at least one of the coefficients is greater than or equal to \( 2^{N/2}/(N+1) \).

Proof. The sum of the coefficients is \( p_N(1) \), and this is at least \( 2^{N/2} \) by Theorem 3.1 with \( v = 1 \). The polynomial has \( N + 1 \) coefficients, and denoting the maximum of the coefficients by \( p_{\text{max}} \), we have \( p_N(1) \leq (N+1)p_{\text{max}} \), which implies the asserted estimate \( p_{\text{max}} \geq 2^{N/2}/(N+1) \). \( \square \)

The previous corollary provides us with a very crude lower bound for the largest coefficient. I conjecture that the \( N + 1 \) can be replaced by its square root, possibly with a multiplicative constant. However, this is not the point for our purpose.

Example 3.3 Suppose \( p_{50}(z) = p_{50} z^{50} + \cdots + p_0 \) is a Hurwitz stable polynomial of degree 50. Since a polynomial \( p_N(z) \) of degree \( N \) is Hurwitz stable if and only if so is the reverse polynomial \( z^N p(1/z) \) (a property which is not shared by Schur stability), we may without loss of generality assume that \( p_{50} \leq p_0 \). Then we may write

\[
p_{50}(z) = p_{50} \left( z^{50} + \frac{p_{49}}{p_{50}} z^{49} + \cdots + \frac{p_0}{p_{50}} \right)
\]

and apply Corollary 3.2 to the polynomial in parentheses. We have \( 2^{25} = 33554432 \). Consequently, if \( p_{\text{max}} \) is the largest coefficient, then \( p_{\text{max}}/p_{50} \geq 2^{25}/51 > 650000 \). If the coefficients are required to be integers, this means that \( p_{\text{max}} > 650000 \).

The first even \( N \) for which \( 2^{N/2}/(N+1) > 10000 \) is \( N = 38 \). Thus, a Hurwitz stable polynomial of even degree with positive integer coefficients not exceeding 10000 must have a degree of at most 36. \( \square \)

Example 3.4 Let \( p_2(z) = z^2 + 2xz + 1 \). Take \( x = 0.1 \) and consider the polynomial \( p_{20}(z) := p_2(z)^{10} = (z^2 + 2xz + 1)^{10} \). The sum of the coefficients of \( p_{20}(z) \) is about 2656, which is comparable to \( 2^{N/2} = 2^{10} = 1024 \).

The polynomial \( q_{20}(z) \) resulting from \( p_{20}(z) \) by taking only the first 4 digits of the coefficients after the comma is

\[
\begin{bmatrix}
1.0000 & 2.0000 & 11.8000 & 18.9600 & 59.7360 & 78.8006 & 172.4294 & 188.5647 \\
315.8939 & 286.4110 & 384.8009 & 286.4110 & 315.8939 & 188.5647 & 172.4294 \\
78.8006 & 59.7360 & 18.9600 & 11.8000 & 2.0000 & 1.0000
\end{bmatrix}.
\]

Note that \( q_{20}(z) \) has moderately sized coefficients which, in contrast to those of \( p_{20}(z) \), are precisely given within the machine precision. Figure 2 shows the zeros of \( p_{20}(z) \) and \( q_{20}(z) \) as they are given by Matlab. Thus, the polynomial \( q_{20}(z) \) has six zeros in the right half-plane and is therefore not Hurwitz stable! \( \square \)
We consider Hurwitz stable polynomials $p(z)$ whose coefficients are positive integers. The degree is denoted by $N$, the maximal coefficient by $p_{\text{max}}$, the sum of the coefficients by $\sigma$, and the maximum of the real parts of the roots (= spectral abscissa) by $\alpha$. For each degree $N$, there are two kinds of optimal polynomials: the polynomials with minimal largest coefficient and the polynomials with minimal sum $\sigma$ of the coefficients. We call these polynomials $c$-optimal and $\sigma$-optimal. Small degrees $N$ are easy, because all possible cases can be checked by Matlab.

**4 Integer coefficients**

We consider Hurwitz stable polynomials $p(z)$ whose coefficients are positive integers. The degree is denoted by $N$, the maximal coefficient by $p_{\text{max}}$, the sum of the coefficients by $\sigma$, and the maximum of the real parts of the roots (= spectral abscissa) by $\alpha$. For each degree $N$, there are two kinds of optimal polynomials: the polynomials with minimal largest coefficient and the polynomials with minimal sum $\sigma$ of the coefficients. We call these polynomials $c$-optimal and $\sigma$-optimal. Small degrees $N$ are easy, because all possible cases can be checked by Matlab.

$\mathbf{N = 1.}$ The polynomial $a_1(z) = z + 1 = [1 \ 1]$ is the best in all respects.

$\mathbf{N = 2.}$ Here the polynomial $a_2(z) = z^2 + z + 1 = [1 \ 1 \ 1]$ is optimal on all accounts. Its spectral abscissa is $-1/2$. 

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*Figure 2: Zeros of $p_{20}(z)$ (blue) and of $q_{20}(z)$ (red) from Example 3.4 obtained by Matlab.*
**N = 3.** The five $c$-optimal polynomials are

\[
\begin{align*}
b_3(z) &= [1 \ 1 \ 2 \ 1], \quad \alpha = -0.2151, \quad \sigma = 5, \\
c_3(z) &= [1 \ 2 \ 1 \ 1], \quad \alpha = -0.1226, \quad \sigma = 5, \\
d_3(z) &= [1 \ 2 \ 2 \ 1], \quad \alpha = -0.5000, \quad \sigma = 6, \\
e_3(z) &= [1 \ 2 \ 2 \ 2], \quad \alpha = -0.2282, \quad \sigma = 7, \\
f_3(z) &= [2 \ 2 \ 2 \ 1], \quad \alpha = -0.1761, \quad \sigma = 7,
\end{align*}
\]

and the first two of them are the $\sigma$-optimal polynomials.

**N = 4.** The list of the nine $c$-optimal polynomials is

\[
\begin{align*}
a_4(z) &= [1 \ 1 \ 3 \ 1 \ 1], \quad \alpha = -0.1484, \quad \sigma = 7, \\
c_4(z) &= [1 \ 1 \ 3 \ 2 \ 1], \quad \alpha = -0.1049, \quad \sigma = 8, \\
d_4(z) &= [1 \ 2 \ 3 \ 1 \ 1], \quad \alpha = -0.0433, \quad \sigma = 8, \\
e_4(z) &= [1 \ 2 \ 3 \ 2 \ 1], \quad \alpha = -0.5000, \quad \sigma = 9, \\
f_4(z) &= [1 \ 2 \ 3 \ 3 \ 1], \quad \alpha = -0.2151, \quad \sigma = 10, \\
g_4(z) &= [1 \ 3 \ 3 \ 2 \ 1], \quad \alpha = -0.1226, \quad \sigma = 10, \\
h_4(z) &= [1 \ 2 \ 3 \ 3 \ 2], \quad \alpha = -0.0433, \quad \sigma = 11, \\
i_4(z) &= [1 \ 3 \ 3 \ 3 \ 1], \quad \alpha = -0.1910, \quad \sigma = 11, \\
j_4(z) &= [2 \ 3 \ 3 \ 2 \ 1], \quad \alpha = -0.0287, \quad \sigma = 11.
\end{align*}
\]

The only $\sigma$-optimal polynomial is $a_4(z)$.

**N = 5.** The nine $c$-optimal polynomials are

\[
\begin{align*}
b_5(z) &= [1 \ 1 \ 4 \ 3 \ 2 \ 1], \quad \alpha = -0.0835, \quad \sigma = 12, \\
c_5(z) &= [1 \ 2 \ 3 \ 4 \ 1 \ 1], \quad \alpha = -0.0320, \quad \sigma = 12, \\
d_5(z) &= [1 \ 1 \ 4 \ 3 \ 3 \ 1], \quad \alpha = -0.0582, \quad \sigma = 13, \\
e_5(z) &= [1 \ 2 \ 3 \ 4 \ 2 \ 1], \quad \alpha = -0.0354, \quad \sigma = 13, \\
f_5(z) &= [1 \ 2 \ 4 \ 3 \ 2 \ 1], \quad \alpha = -0.0204, \quad \sigma = 13, \\
g_5(z) &= [1 \ 3 \ 3 \ 4 \ 1 \ 1], \quad \alpha = -0.0203, \quad \sigma = 13, \\
h_5(z) &= [1 \ 2 \ 4 \ 4 \ 2 \ 1], \quad \alpha = -0.1484, \quad \sigma = 14, \\
i_5(z) &= [1 \ 2 \ 4 \ 3 \ 1 \ 1], \quad \alpha = -0.2151, \quad \sigma = 15, \\
j_5(z) &= [1 \ 3 \ 4 \ 4 \ 2 \ 1], \quad \alpha = -0.1226, \quad \sigma = 15,
\end{align*}
\]

and the $\sigma$-optimal polynomials are $b_5(z)$ and $c_5(z)$. Of course, it might be that there exist $\sigma$-optimal polynomials with $p_{\text{max}} \geq 5$ and $\sigma \leq 11$. However, the coefficients of such polynomials are either a permutation of 6, 1, 1, 1, 1, 1 or a permutation of 5, 2, 1, 1, 1, 1, and none of these 36 polynomials is Hurwitz stable.
\(N = 6\). We have the five \(c\)-optimal polynomials

\[
b_6(z) = [1 1 5 3 5 1 1], \quad \alpha = -0.0485, \quad \sigma = 17, \\
c_6(z) = [1 1 5 4 5 2 1], \quad \alpha = -0.0393, \quad \sigma = 19, \\
d_6(z) = [1 2 4 5 4 2 1], \quad \alpha = -0.0399, \quad \sigma = 19, \\
e_6(z) = [1 2 5 4 5 1 1], \quad \alpha = -0.0108, \quad \sigma = 19, \\
f_6(z) = [1 2 5 5 5 2 1], \quad \alpha = -0.1484, \quad \sigma = 21,
\]

and the first of them is the only \(\sigma\)-optimal polynomial.

\(N = 7\). Beginning with this degree things become challenging. Inspection of the \(7^8 = 5764801\) polynomials with \(p_{\text{max}} \leq 7\) shows that exactly two of them are Hurwitz stable, namely,

\[
b_7(z) = [1 2 5 7 7 6 2 1], \quad \alpha = -0.0175, \quad \sigma = 31, \\
c_7(z) = [1 2 6 7 7 5 2 1], \quad \alpha = -0.0077, \quad \sigma = 31.
\]

Consequently, these are the \(c\)-optimal polynomials of degree 7. Note that each of the two polynomials is the reverse of the other one. These two polynomials are not \(\sigma\)-optimal, because, for example, we also have the polynomials

\[
d_7(z) = [1 2 5 8 5 6 1 1], \quad \alpha = -0.0131, \quad \sigma = 29, \\
e_7(z) = [1 3 4 9 4 6 1 1], \quad \alpha = -0.0526, \quad \sigma = 29.
\]

I have not examined whether the last two polynomials are \(\sigma\)-optimal.

**Multiplication and doubling.** One way of getting Hurwitz stable polynomial of higher degrees is to multiply Hurwitz stable polynomials of lower degrees. Another way is as follows. Since \(z\) is in the open left half-plane if and only if so is \(z + 1/z\), it follows that \(q_N(z)\) is Hurwitz stable of degree \(N\) if and only if \(p_{2N}(z) = z^N q_N(z + 1/z)\) is a Hurwitz stable polynomial of degree \(2N\). We refer to the passage from \(q_N(z)\) to \(p_{2N}(z) = z^N q_N(z + 1/z)\) as doubling. The sum of the coefficients of a product is equal to the product of the sums of the coefficients, and if \(p_{2N}(z)\) results from \(q_N(z)\) by doubling, then the sum of the coefficients of \(p_{2N}(z)\) is \(p_{2N}(1) = q_N(2)\).

\(N = 8\). Multiplying \(a_4(z)\) by itself we obtain

\[
b_8(z) = [1 2 7 8 13 8 7 2 1], \quad \alpha = -0.1484, \quad \sigma = 7 \cdot 7 = 49.
\]

However, such products are usually far away from the optimal polynomials. Doubling of \(a_4(z)\) gives

\[
a_8(z) = [1 1 7 4 13 4 7 1 1], \quad \alpha = -0.0518, \quad \sigma = 39.
\]

I don’t know whether \(a_8(z)\) is \(\sigma\)-optimal. It is surely not \(c\)-optimal, because the polynomials

\[
c_8(z) = [1 2 6 9 11 10 7 2 1], \quad \alpha = -0.0171, \quad \sigma = 49, \\
d_8(z) = [1 2 6 9 11 11 7 3 1], \quad \alpha = -0.0075, \quad \sigma = 51, \\
e_8(z) = [1 2 7 11 11 11 6 3 1], \quad \alpha = -0.0135, \quad \sigma = 53
\]
are Hurwitz stable. Note that the reverses of these polynomials are Hurwitz stable, too. Clearly, these six polynomials are closer to the $c$-optimal polynomials.

$10 \leq N \leq 18$. Doubling of $c_5(z)$, $d_6(z)$, $d_7(z)$, $c_8(z)$ yields

$N = 10$, $b_{10}(z) = [1 \ 2 \ 8 \ 12 \ 20 \ 21 \ 20 \ 12 \ 8 \ 2 \ 1]$, $\alpha = -0.0117$, $\sigma = 107$,

$N = 12$, $b_{12}(z) = [1 \ 2 \ 10 \ 15 \ 35 \ 37 \ 53 \ 37 \ 35 \ 15 \ 10 \ 2 \ 1]$, $\alpha = -0.0134$, $\sigma = 253$,

$N = 14$, $b_{14}(z) = [1 \ 2 \ 12 \ 20 \ 51 \ 68 \ 101 \ 101 \ 68 \ 51 \ 20 \ 12 \ 2 \ 1]$, $\alpha = -0.0050$, $\sigma = 611$,

$N = 16$, $b_{16}(z) = [1 \ 2 \ 14 \ 23 \ 75 \ 97 \ 197 \ 192 \ 192 \ 197 \ 97 \ 75 \ 23 \ 14 \ 2 \ 1]$, $\alpha = -0.0042$, $\sigma = 1473$,

and for $N = 18$, I found

$b_{18}(z) = [1 \ 2 \ 16 \ 27 \ 98 \ 139 \ 303 \ 353 \ 523 \ 479 \ 523 \ 353 \ 303 \ 139 \ 98 \ 27 \ 16 \ 2 \ 1]$

with $\alpha = -0.0046$ and $\sigma = 3403$.

$N = 20$. We arrive at the test posed in the introduction. Multiplication of $b_{10}(z)$ by itself gives

$[1 \ 4 \ 20 \ 56 \ 152 \ 314 \ 588 \ 920 \ 1288 \ 1548 \ 1667 \ 1548 \ 1288 \ 920 \ 588 \ 314 \ 152 \ 56 \ 20 \ 4 \ 1]$

with $\alpha = -0.0117$ and $\sigma = 11449$. Multiplying other combinations yields similar results, the best being $a_2(z) \cdot b_{18}(z)$, which is

$[1 \ 3 \ 19 \ 45 \ 141 \ 264 \ 540 \ 795 \ 1179 \ 1355 \ 1525 \ 1355 \ 1179 \ 795 \ 540 \ 264 \ 141 \ 45 \ 19 \ 3 \ 1]$

with $\alpha = -0.0046$ and $\sigma = 10209$. Thus, eventually we easily passed the test and constructed a polynomial with $p_{\text{max}} = 1525$. However, notice that the success resulted from knowing the very good polynomials $b_{10}(z)$ and $b_{18}(z)$. In fact we can do it even better. Doubling $b_5(z) = [1 \ 1 \ 4 \ 3 \ 2 \ 1]$ we get

$c_{10}(z) = [1 \ 1 \ 9 \ 7 \ 24 \ 13 \ 24 \ 7 \ 9 \ 1 \ 1]$, $\sigma = 97$,

and doubling this again, we arrive at

$[1 \ 1 \ 19 \ 16 \ 141 \ 98 \ 540 \ 303 \ 1179 \ 523 \ 1525 \ 523 \ 1179 \ 303 \ 540 \ 98 \ 141 \ 16 \ 19 \ 1 \ 1]$, (3)

with $\alpha = -0.0067$ and $\sigma = 7167$, which has the smallest $\sigma$ we have found. This polynomial will be denoted by $c_{20}(z)$. Moreover, doubling of $b_{10}(z)$ yields the polynomial

$[1 \ 2 \ 18 \ 30 \ 129 \ 177 \ 484 \ 537 \ 1046 \ 920 \ 1349 \ 920 \ 1046 \ 537 \ 484 \ 177 \ 129 \ 30 \ 18 \ 2 \ 1]$

with $p_{\text{max}} = 1349$, $\alpha = -0.0038$, $\sigma = 8037$. We henceforth denote this polynomial by $b_{20}(z)$. I have not found a Hurwitz stable polynomial of degree 20 whose largest coefficient is smaller than 1349. Figures 3 and 4 show the zeros of $b_{18}(z)$ and $b_{20}(z)$.
Figure 3: Location of the zeros of the product of the polynomials $a_2(z)$ (two circles) and $b_{18}(z)$ (18 asterisks).

**Powers of 2.** Repeated doubling of $a_1(z) = [1 \ 1]$ yields the polynomials

$$a_2(z) = [1 \ 1 \ 1], \quad \sigma = 3,$$
$$a_4(z) = [1 \ 1 \ 3 \ 1 \ 1], \quad \sigma = 7,$$
$$a_8(z) = [1 \ 1 \ 7 \ 4 \ 13 \ 4 \ 7 \ 1 \ 1], \quad \sigma = 39,$$
$$a_{16}(z) = [1 \ 1 \ 15 \ 11 \ 83 \ 45 \ 220 \ 88 \ 303 \ 88 \ 220 \ 45 \ 83 \ 11 \ 15 \ 1 \ 1], \quad \sigma = 1231,$$

and finally

$$a_{32}(z) = [1 \ 1 \ 31 \ 26 \ 413 \ 293 \ 3141 \ 1896 \ 15261 \ 7866 \ 50187 \ 22122 \ 115410 \ 43488 \ 189036 \ 60753 \ 222621 \ 60753 \ 189036 \ 43488 \ 115410 \ 22122 \ 50187 \ 7866 \ 15261 \ 1896 \ 3141 \ 293 \ 413 \ 26 \ 31 \ 1 \ 1], \quad \sigma = 1242471.$$

The sum of the coefficients of $a_{64}(z)$ equals $a_{32}(2) \approx 1.2791 \cdot 10^{12}$. When comparing $a_{16}(z)$ and $b_{16}(z)$, we see that $a_{16}(z)$ has the smaller $\sigma$ and that $b_{16}(z)$ has the smaller $p_{\text{max}}$. 

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Figure 4: Zeros of the polynomial $b_{20}(z)$.

5 Higher degrees

Let $N$ be even. From Theorem 3.1, we infer that if $p_N(z) = p_N z^N + \cdots + p_0$ is Hurwitz stable with $p_N \geq 1$ and $p_0 \geq 1$, then the sum of the coefficients always satisfies $p_N(1) \geq 2^{N/2}$. The polynomials $p_N(z) = (z^2 + 2xz + 1)^{N/2}$ with sufficiently small $x > 0$ show that Theorem 3.1 is sharp. In particular, given any $\varepsilon > 0$, there is such a polynomial for which $p_N(1) < (2 + \varepsilon)^{N/2}$. But what happens if the coefficients are required to be integers?

Let $p_{\text{max}}(N)$ denote the minimum of the largest coefficients and let $\sigma(N)$ be the minimum of the sum of the coefficients of the Hurwitz stable polynomials of degree $N$ with positive integer coefficients. Equivalently, $p_{\text{max}}(N)$ is the largest coefficient of the $c$-optimal polynomials and $\sigma(N)$ is the sum of the coefficients of the $\sigma$-optimal polynomials. From the previous section we know the following.
| \( N \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 10 | 20 | 32 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( p_{\text{max}}(N) \) | 1 | 1 | 2 | 3 | 4 | 5 | 7 | 11 | \( \leq 21 \) | \( \leq 1349 \) | \( \leq 222621 \) |
| \( \sigma(N) \) | 2 | 3 | 5 | 7 | 12 | 17 | \( \leq 29 \) | \( \leq 39 \) | \( \leq 97 \) | \( \leq 7167 \) | \( \leq 1242471 \) |

**Theorem 5.1** The limits of \( \sigma(N)^{1/N} \) and \( p_{\text{max}}(N)^{1/N} \) exist and we have

\[
\lim_{N \to \infty} \frac{\sigma(N)}{N} = \inf_{N \geq 1} \frac{\sigma(N)}{N} = \lim_{N \to \infty} p_{\text{max}}(N)^{1/N} = \inf_{N \geq 1} p_{\text{max}}(N)^{1/N} =: \beta.
\]

**Proof.** Since the product of Hurwitz stable polynomials is again Hurwitz stable, it follows that

\[
\sigma(N + M) \leq \sigma(N) \sigma(M).
\]

Fekete’s subadditive lemma, for which see [2] or [3, p. 16], therefore implies that the limit of \( \sigma(N)^{1/N} \) exists and coincides with the infimum of \( \sigma(N)^{1/N} \) for \( N \geq 1 \). For every polynomial \( p_N(z) \) of degree \( N \) with positive coefficients, the inequalities

\[
\frac{\sigma}{N + 1} \leq p_{\text{max}} \leq \sigma
\]

hold, where \( \sigma \) is the sum and \( p_{\text{max}} \) is the maximum of the coefficients. If \( p_N(z) \) is \( \sigma \)-optimal, then \( \sigma = \sigma(N) \) and hence

\[
p_{\text{max}}(N) \leq p_{\text{max}} \leq \sigma = \sigma(N).
\]

In case \( p_N(z) \) is \( c \)-optimal, we have \( p_{\text{max}} = p_{\text{max}}(N) \) and consequently,

\[
\frac{\sigma(N)}{N + 1} \leq \frac{\sigma}{N + 1} \leq p_{\text{max}} = p_{\text{max}}(N).
\]

Thus, \( \sigma(N)/(N + 1) \leq p_{\text{max}}(N) \leq \sigma(N) \), which shows that the limit and the infimum of \( p_{\text{max}}(N)^{1/N} \) coincide with the limit and the infimum of \( \sigma(N)^{1/N} \): \( \square \)

**Theorem 3.1** with \( v = 1 \) shows that \( \beta \geq \sqrt{2} = 1.4142 \ldots \).

**Proposition 5.2** Let \( p_k(z) \) be any Hurwitz stable polynomial with positive integer coefficients. If \( N \) is divisible by \( k \), then \( \sigma(N) \leq (\sqrt[p_k(1)]{N})^N.\)

*Interestingly, in their equally titled papers [2], [4], Schur and Fekete considered the problem whether there are infinitely many polynomials with integer coefficients and given leading coefficient whose zeros are all simple and lie in a compact subset \( E \) of the plane. For example, in the case where \( E \) is a half-disk with diameter \( 2R < 3\sqrt{3}/2 = 2.5981 \ldots \), Fekete showed that the number of such polynomials must be finite. We here are concerned with the case where \( E \) is the open left half-plane, which is neither bounded nor closed. And indeed, the polynomials \( (z + j) \ldots (z + j + N) \) \((j = 1, 2, \ldots)\) constitute an infinite family of monic polynomials of even fixed degree with integer coefficients whose zeros are all simple and are located in \( E \).*
Proof. Let \( N = nk \) and consider \( p_N(z) = p_k(z)^n \). Then
\[
\sigma(N) \leq p_N(1) = p_k(1)^n = (p_k(1)^{1/k})^N.
\]
The best results from Proposition 5.2 are delivered by taking \( \sigma \)-optimal polynomials, in which case \( p_k(1) = \sigma(k) \). Here are the numbers.

| \( k \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 10 | 20 | 32 |
|-------|---|---|---|---|---|---|---|---|----|----|----|
| \( \sigma(k) \) | 2 | 3 | 5 | 7 | 12 | 17 | 29 | 39 | 97 | 7167 | 1242471 |
| \( \sqrt[k]{\sigma(k)} \) | 2 | 1.74 | 1.72 | 1.63 | 1.65 | 1.61 | 1.62 | 1.59 | 1.59 | 1.56 | 1.56 |

Since \( \sqrt{7167} = 1.5587 \ldots \) and \( \sqrt[32]{1242471} = 1.5504 \ldots \), we arrive at the following.

**Corollary 5.3** We have
\[
1.4142 \ldots = \sqrt{2} \leq \beta \leq \sqrt[32]{1242471} = 1.5504 \ldots.
\]

**Corollary 5.4** If \( N \) is divisible by 20 or 32, then \( 1.41^N < \sigma(N) < 1.56^N \).

Proof. We know from Theorem 3.1 that \( \sigma(N) \geq 2^{N/2} = (2^{1/2})^N > 1.41^N \), and Proposition 5.2 implies that \( \sigma(N) \leq (\sqrt{7167})^N < 1.56^N \) if \( N \) is divisible by 20 and that \( \sigma(N) \leq (\sqrt[32]{1242471})^N < 1.56^N \) if \( N \) is divisible by 32.

In what follows we need the sequence \( v_0, v_1, v_2, \ldots \) given by \( v_0 = 1 \) and \( v_{n+1} = v_n + 1/v_n \). The first terms are
\[
v_0 = 1, \quad v_1 = 2, \quad v_2 = \frac{5}{2} = 2.5, \quad v_3 = \frac{29}{10} = 2.9, \quad v_4 = \frac{941}{290} = 3.2448 \ldots.
\]

**Lemma 5.5** We have \( \sqrt{n+1} < v_n < 2\sqrt{n} \) for \( n \geq 2 \) and
\[
v_n = \sqrt{2n} \left( 1 + O\left( \frac{\log n}{n} \right) \right).
\]

Proof. We prove the inequalities \( \sqrt{n+1} < v_n < 2\sqrt{n} \) by induction on \( n \). They are obviously true for \( n = 2 \). So suppose they hold for \( n \). We then have
\[
v_{n+1} = v_n + \frac{1}{v_n} \leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+1}
\]
because
\[
2\sqrt{n+1} - 2\sqrt{n} = \frac{2}{\sqrt{n+1} + \sqrt{n}} > \frac{2}{\sqrt{n+1} + \sqrt{n+1}} = \frac{1}{\sqrt{n+1}}.
\]
In the same vein,
\[
v_{n+1} = v_n + \frac{1}{v_n} > \sqrt{n+1} + \frac{1}{2\sqrt{n}} > \sqrt{n+2}
\]
since
$$\sqrt{n+2} - \sqrt{n+1} = \frac{1}{\sqrt{n+2} + \sqrt{n+1}} < \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}}.$$ 

This completes the proof of the inequalities. To prove the asymptotics, note first that the numbers \(v_n\) satisfy \(v_n^2 = v_{n-1}^2 + 1/v_{n-1}^2 + 2\). Consequently,
$$v_n^2 > v_{n-1}^2 + 2 > v_{n-2}^2 + 2 \cdot 2 > \ldots > v_0^2 + n \cdot 2 = 2n + 1,$$
which implies that \(v_n > \sqrt{2n}\). On the other hand,
$$v_n^2 = v_{n-1}^2 + \frac{1}{v_{n-1}^2} + 2 = v_{n-2}^2 + \frac{1}{v_{n-2}^2} + \frac{1}{v_{n-2}^2} + 2 \cdot 2$$
and so on, which eventually gives
$$v_n^2 = v_1^2 + \frac{1}{v_1^2} + \ldots + \frac{1}{v_{n-1}^2} + (n-1) \cdot 2.$$

As \(v_k > \sqrt{2k}\), we conclude that
$$\frac{v_n^2}{2n} < \frac{1}{2n} \left(4 + \frac{1}{2 \cdot 1} + \ldots + \frac{1}{2(n-1)}\right) + \frac{n-1}{2n} \cdot 2 = 1 + O \left(\frac{\log n}{n}\right)$$
and hence
$$v_n \leq \sqrt{2n} \left(1 + O \left(\frac{\log n}{n}\right)\right).$$

This estimate in conjunction with the inequality \(v_n > \sqrt{2n}\) proves the lemma. \(\square\)

We define
$$\gamma_k = \sum_{j=0}^{k-1} \frac{\log v_j}{2^{j+1}}, \quad \gamma = \sum_{j=0}^{\infty} \frac{\log v_j}{2^{j+1}} = 0.4329 \ldots.$$ 

The first values are

| \(k\) | \(e^{\gamma_k}\) | \(\gamma_k\) |
|------|-----------------|----------|
| 1    | 1               | 0        |
| 2    | \(\sqrt{2}\) = 1.1892\ldots | 0.1733\ldots |
| 3    | \(\sqrt[4]{10}\) = 1.3335\ldots | 0.2878\ldots |
| 4    | \(\sqrt{290}\) = 1.4252\ldots | 0.3544\ldots |

Throughout what follows, if \(N = 2^n\) is a power of 2, we denote by \(a_N(z)\) the polynomials obtained from \(a_1(z) = z + 1\) by \(n\) doublings. The first of these polynomials are listed at the end of Section 4.

**Lemma 5.6** Let \(N = 2^n\). Then
$$a_N(1) = (v_n + 1)e^{\gamma_n} < (2\sqrt{n} + 1)e^{\gamma_N}$$
with \(e^{\gamma} = 1.5417\ldots\) and
$$a_N(1) = ((2n)^{1/4} + (2n)^{-1/4})e^{\gamma_N} \left(1 + O \left(\frac{\log n}{n}\right)\right).$$
Proof. We have

\[ a_N(1) = v_0^{N/2}a_{n/2}(v_1) = v_0^{N/2}v_1^{N/4}a_{N/4}(v_2) = \ldots = v_0^{N/2}v_1^{N/4}\ldots v_{n-1}a_1(v_n), \]

and since

\[ \log(v_0^{N/2}v_1^{N/4}\ldots v_{n-1}) = 2^n \left( \log \frac{v_0}{2} + \frac{\log v_1}{2} + \ldots + \frac{\log v_{n-1}}{2n} \right) = N \gamma_n, \]

we get \( a_N(1) = (v_n + 1)e^{\gamma N}. \) The upper bound \( (2\sqrt{n} + 1)e^{\gamma N} \) follows from Lemma 5.5.

To prove the asymptotics, we write

\[ \log a_N(1) = \log(v_n + 1) + 2^n \gamma_n = \log(v_n + 1) + 2^n \gamma - r_n, \quad r_n = \sum_{j=0}^{\infty} \frac{\log v_{n+j}}{2^{j+1}}. \]

By Lemma 5.5

\[ r_n = \sum_{j=0}^{\infty} \frac{\log 2}{2^{j+2}} + \sum_{j=0}^{\infty} \frac{\log(n+j)}{2^{j+2}} + O \left( \frac{\sum_{j=0}^{\infty} \log(n+j)}{2^{j+1}(n+j)} \right). \quad (4) \]

The first sum in (4) is \((1/4)\log 2.\) Using that \(\log(1+x) < x\) for \(x > 0,\) the second sum can be estimated as follows:

\[ \sum_{j=0}^{\infty} \frac{\log(n+j)}{2^{j+2}} = \sum_{j=0}^{\infty} \frac{\log n + \log(1 + j/n)}{2^{j+2}} = \frac{1}{4} \log n + O \left( \sum_{j=0}^{\infty} \frac{j}{2^{j+2}n} \right) = \frac{1}{4} \log n + O \left( \frac{1}{n} \right). \]

Multiplying the \(j\)th term in the third sum by \(n/\log n,\) it becomes

\[ \frac{\log(n+j)}{n} \cdot \frac{1}{n+j} \cdot \frac{1}{2^{j+1}} = \frac{\log n + \log(1+j/n)}{n} \cdot \frac{1}{n+j} < \left( 1 + \frac{j}{n \log n} \right) \cdot \frac{1}{2^{j+1}}, \]

and as this is smaller than \((1+j)/2^{j+1},\) we arrive at the conclusion that the third term in (4) is \(O((\log n)/n).\) Putting things together we obtain

\[ a_N(1) = (v_n + 1)e^{\gamma N}e^{-\left(1/4\right)\log n}e^{-\left(1/4\right)\log 2} \left( 1 + O \left( \frac{\log n}{n} \right) \right). \]

From Lemma 5.5 we infer that

\[ v_n + 1 = \sqrt{2n} \left( 1 + O \left( \frac{\log n}{n} \right) \right) + 1 = (\sqrt{2n} + 1) \left( 1 + O \left( \frac{\log n}{n} \right) \right). \]

What finally results is

\[ a_N(1) = \frac{\sqrt{2n} + 1}{(2n)^{1/4}}e^{\gamma N} \left( 1 + O \left( \frac{\log n}{n} \right) \right), \]

which is equivalent to the assertion. \( \Box \)

Here is a slight improvement of Corollary 5.3.
Corollary 5.7 We have

\[ 1.4142 \ldots = \sqrt{2} \leq \beta \leq e^\gamma = 1.5417 \ldots \]

Proof. Let \( \varepsilon > 0 \) be arbitrarily given. Choose \( K = 2^k \) so that \( (2\sqrt{k} + 1)^{1/K} < 1 + \varepsilon \). Lemma 5.6 then gives \( a_K(1)^{1/K} < (1 + \varepsilon)e^\gamma \). If \( N \) is divisible by \( 2^k \), Proposition 5.2 implies that \( \sigma(N)^{1/N} \leq a_K(1)^{1/K} < (1 + \varepsilon)e^\gamma \), whence \( \beta \leq (1 + \varepsilon)e^\gamma \). As \( \varepsilon > 0 \) was arbitrary, we conclude that \( \beta \leq e^\gamma \). □

A polynomial \( p_N(z) = p_Nz^N + \cdots + p_0 \) of even degree \( N \) is called symmetric if \( p_j = p_{N-j} \) for all \( j \). In that case there is a unique polynomial \( p_{N/2}(z) \) of degree \( N/2 \) such that \( p_N(z) = z^{N/2}p_{N/2}(z+1/z) \). The polynomial \( p_{N/2}(z) \) is Hurwitz stable if and only if so is \( p_N(z) \), and \( p_{N/2}(z) \) has integer coefficients if and only if \( p_N(z) \) has integer coefficients. If \( N/2 \) is also even and \( p_{N/2}(z) \) is symmetric, we call \( p_N(z) \) a 2-fold symmetric polynomial. We then have \( p_{N/2}(z) = z^{N/4}p_{N/4}(z) \). If \( N/4 \) is even and \( p_{N/4}(z) \) is symmetric, then \( p_N(z) \) is said to be 3-fold symmetric and so on. In other words, a polynomial is \( k \)-fold symmetric if and only if it results after \( k \) doubling procedures from another polynomial. Symmetry is 1-fold symmetry in this context.

We denote by \( \sigma_k(N) \) the minimum of the sum of the coefficients among all Hurwitz stable \( k \)-fold symmetric polynomials of degree \( N \) with positive integer coefficients. Clearly, \( \sigma(N) \leq \sigma_1(N) \leq \sigma_2(N) \leq \ldots \)

Theorem 5.8 Let \( N \) be divisible by \( 2^k \). Then

\[ \sigma(N) \leq \sigma_k(N) \leq (v_k + 1)^{N/2^k}e^\gamma N \]

and

\[ e^\gamma N \leq \left( (v_k^2 + 1)^{1/2^k+1} \right)^N e^\gamma N \leq \sigma_k(N). \]

Proof. Let \( N = 2^km \). The polynomial \( p_N(z) = a_{2^k}(z)^m \) is \( k \)-fold symmetric and hence \( \sigma_k(N) \leq p_N(1) \). From Lemma 5.6 we therefore obtain that

\[ \frac{\log \sigma_k(N)}{N} \leq \frac{m \log a_{2^k}(1)}{N} = \frac{m \log(v_k + 1) + m \cdot 2^k \gamma_k}{N} = \frac{\log(v_k + 1)}{2^k} + \gamma_k, \]

which proves the upper estimate for \( \sigma_k(N) \). To get the lower estimate, let \( p_N(z) \) be an arbitrary \( k \)-fold symmetric Hurwitz stable polynomial of degree \( N \) with positive integer coefficients. We then have

\[ p_N(1) = p_{N/2}(v_1) = v_1^{N/4}p_{N/4}(v_2) = v_1^{N/4}v_2^{N/8}p_{N/8}(v_3) \]

and so on, terminating with

\[ \log p_N(1) = \frac{N}{4} \log v_1 + \frac{N}{8} \log v_2 + \cdots + \frac{N}{2^k} \log v_k + \log p_{N/2^k}(v_k), \]
which is the same as \( \log p_N(1) = N \gamma_k + \log p_{N/2^k}(v_k) \). From Theorem 3.1 we now deduce that

\[
\log p_N(1) \geq N \gamma_k + \frac{N}{2^{k+1}} \log(v_k^2 + 1) \geq N \gamma_k.
\]

Taking the exponential we arrive at the asserted lower estimates. \( \square \)

For \( k = 1, 2, 3 \) the bounds provided by Theorem 5.8 read as follows.

| \( k \) | \((v_k^2 + 1)^{1/2^{k+1}} \) | \( \sigma_k(N)^{1/N} \) |
|---|---|---|
| 1 | \( 1.4953 \ldots = \sqrt{5} \leq \sigma_k(N)^{1/N} \leq \sqrt{3} = 1.7320 \ldots \) | \( 1.49 \leq \sigma_k(N)^{1/N} \leq \sqrt{3} \leq 1.7320 \ldots \) |
| 2 | \( 1.5233 \ldots = \sqrt[4]{29} \leq \sigma_k(N)^{1/N} \leq \sqrt{7} = 1.6265 \ldots \) | \( 1.52 \leq \sigma_k(N)^{1/N} \leq \sqrt[4]{29} \leq 1.6265 \ldots \) |
| 3 | \( 1.5340 \ldots = \sqrt[8]{941} \leq \sigma_k(N)^{1/N} \leq \sqrt{39} = 1.5808 \ldots \) | \( 1.53 \leq \sigma_k(N)^{1/N} \leq \sqrt[8]{941} \leq \sqrt{39} \leq 1.5808 \ldots \) |

Clearly, the \( k \)-fold sigmas \( \sigma_k(N) \) also satisfy the inequality \( \sigma_k(N+M) \leq \sigma_k(N)\sigma_k(M) \), and hence, by the argument of the proof of Theorem 5.1, the limits \( \beta_k \) of \( \sigma_k(N)^{1/N} \) exist as well.

**Corollary 5.9** We have

\[
1.41 < 1.4142 \ldots < \sqrt{2} \leq \beta \leq e^\gamma = 1.5417 \ldots < 1.55,
\]

\[
1.49 < 1.4953 \ldots < \sqrt{5} \leq \beta_1 \leq e^\gamma = 1.5417 \ldots < 1.55,
\]

\[
1.52 < 1.5233 \ldots < \sqrt[4]{29} \leq \beta_2 \leq e^\gamma = 1.5417 \ldots < 1.55,
\]

\[
1.53 < 1.5340 \ldots < \sqrt[8]{941} \leq \beta_3 \leq e^\gamma = 1.5417 \ldots < 1.55.
\]

**Proof.** The only thing we need to prove is the upper bound for \( \sigma_k(N) \). So fix \( k \) and take \( N = 2^{k+\ell} \) with \( \ell = 0, 1, 2, \ldots \). Then \( \sigma_k(N) \leq a_{2^{k+\ell}}(1) \), and Lemma 5.6 tells us that, given any \( \varepsilon > 0 \),

\[
\frac{\log \sigma_k(N)}{N} \leq \frac{\log(2^k + \ell + 1)}{2^{k+\ell}} + \frac{\gamma}{\varepsilon + \gamma}
\]

whenever \( \ell \) is large enough. This shows that the limit of \( \sigma_k(N)^{1/N} \) does not exceed \( e^\gamma \), as desired. \( \square \)

**Theorem 5.10** If \( N \) is divisible by 20, then

\[
\frac{1.41^N}{N} < p_{\text{max}}(N) < 1.56^N \left( \frac{0.68}{\sqrt{N}} + 0.97^N \right).
\]

**Proof.** To show the lower bound, let \( p_N(z) \) be \( c \)-optimal. Then the sum of the coefficients of \( p_N(z) \) is at most \( N p_{\text{max}} \), and since this sum is greater than \( 1.41^N \) by Theorem 5.4 we conclude that \( p_{\text{max}} > 1.41^N/N \).
The upper bound will follow once we have shown that the largest coefficient of the polynomial \( c_{20k}(z) := c_{20}(z)^k \) is smaller than this bound, where \( k = N/20 \) and \( c_{20}(z) \) is the polynomial \( f(z) \). The polynomial \( c_{20}(z) \) is symmetric and hence

\[
c_{20}(z) = z^{10} \left( c_{10} + c_9(z + z^{-1}) + \cdots + c_0(z^{10} + z^{-10}) \right).
\]

Thus,

\[
c_{20k}(z) = z^{10k} \left( c_0(k) + c_1(k)(z + z^{-1}) + \cdots + c_{10k}(k)(z^{10k} + z^{-10k}) \right),
\]

and the numbers \( c_j(k) \) are the coefficients of \( c_{20k}(z) \). For real \( x \), we define

\[
f(x) = c_{10} + c_9(e^{ix} + e^{-ix}) + \cdots + c_0(e^{10ix} + e^{-10ix}).
\]

Then \( c_j(k) \) is just the \( j \)th Fourier coefficient of \( f(x)^k \), that is,

\[
c_j(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)^k e^{-ix} dx.
\]

It follows that

\[
c_j(k) = |c_j(k)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^k dx.
\]

(Actually \( f(x) > 0 \) for all \( x \), but we don’t need this.) We have \( f(0) = 7167 =: \sigma \) and the function

\[
\frac{f(x)}{\sigma} = \frac{c_{10}}{\sigma} + \frac{2c_9}{\sigma} \cos(x) + \cdots + \frac{2c_0}{\sigma} \cos(10x)
\]

satisfies \( 0 < f(x)/\sigma \leq e^{-3.5x^2} \) for \( |x| < 1 \) and \( |f(x)/\sigma| < 1/2 \) for \( 1 \leq |x| < \pi \). Figure 5 shows the graphs of \( f(x)/\sigma \) and \( e^{-3.5x^2} \).

Consequently,

\[
\frac{1}{2\pi \sigma^k} \int_{-\pi}^{\pi} f(x)^k dx < \frac{1}{2\pi} \int_{|x|<1} e^{-3.5x^2} dx + \frac{1}{2\pi} \int_{1<|x|<\pi} 0.5^k dx
\]

\[
< \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-3.5x^2} dx + 0.5^k = \frac{1}{\sqrt{14\pi}} + 0.2^k.
\]

In summary we have

\[
p_{\text{max}}(N) \leq \sigma^k \left( \frac{1}{\sqrt{14\pi}} + 0.2^k \right).
\]

Inserting \( k = N/20 \) and \( \sigma = 7167 \) and taking into account that \( 7167^{1/20} < 1.56 \) and \( 0.5^{1/20} < 0.97 \), we arrive at the asserted upper bound. \( \square \)

**Remark 5.11** It is easy to find the asymptotics of the largest coefficient of the polynomials used in the preceding proof.
Let $c_{20}(z)$ be the polynomial (3) and put

$$
\sigma = c_{20}(1) = 7167, \quad \tau = \frac{26313}{7167} = 3.6714 \ldots .
$$

Then the maximum of the coefficients of $c_{20}(z)^k$ is

$$
\frac{\sigma^k}{\sqrt{4\pi \tau k}} (1 + o(1)).
$$

Indeed, we observed that the maximum in question is

$$
\frac{\sigma^k}{2\pi} \int_{-\pi}^{\pi} \left( \frac{f(x)}{\sigma} \right)^k \, dx = \frac{\sigma^k}{2\pi} \int_{-\pi}^{\pi} e^{k\sigma (x)} \, dx
$$

with

$$
\frac{f(x)}{\sigma} = \frac{c_{10}}{\sigma} + \frac{2c_9}{\sigma} \cos(x) + \cdots + \frac{2c_0}{\sigma} \cos(10x)
$$

$$
= 1 - \frac{1}{\sigma} (c_9 + 2^2 c_8 + 3^2 c_7 + \cdots + 10^2 c_0) x^2 + O(x^4) = 1 - \tau x^2 + O(x^4).
$$
and \( g(x) := \log(f(x)/\sigma) = -\tau x^2 + O(x^4) \). The function \( g(x) \) is twice differentiable and attains it maximum on \([-\pi, \pi]\) only at \( x = 0 \). A well known theorem by Laplace therefore implies that

\[
\frac{\sigma^k}{2\pi} \int_{-\pi}^{\pi} e^{kg(x)} \, dx = \frac{e^{kg(0)} \sigma^k}{\sqrt{2\pi k |g''(0)|}} (1 + o(1)) = \frac{\sigma^k}{\sqrt{2\pi k \cdot 2\tau}} (1 + o(1)).
\]

Writing \( k = N/20 \) we get for the maximal coefficient of \( c_N(z) = c_{20}(z)^k \) the asymptotics

\[
\left( \frac{20}{\sqrt{7167}} \right)^N \sqrt{\frac{5}{\pi \tau N}} (1 + o(1)) \approx 1.5587^N \frac{0.6584}{\sqrt{N}} (1 + o(1)),
\]

which is in accordance with Theorem 5.10. The number 3.5 we used in the proof of Theorem 5.10 comes from the estimate \( 3.5 < \tau = 3.6714 \ldots \) \( \square \)

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