Investigation of the Critical Behavior of the Critical Point of the Z2 Gauge Lattice

Y. Blum*, P.K. Coyle, S. Elitzur, E. Rabinovici, S. Solomon
Racah Institute of Physics, Hebrew University of Jerusalem, Jerusalem 91904, Israel.

H. Rubinstein
Institutionen för Teoretisk Fysik, Box 803, S-751 08 Uppsala, Sweden
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We investigate, through Monte-Carlo simulations, the nature of the second order point in a Z2
(Bosonic) + Z2 gauge theory in four dimensions. Detailed analysis of the critical exponents point
to the Ising universality class. Relevancy to extended models and possible Non-Gaussian behavior
is discussed.

In the present paper we investigate the character of the
critical point C (Fig 1) and show that the segment AC is
mapped on the segment \( H = 0, \) \( T \in [0, T_c] \) of the usual
Ising model. We find that the Ising critical exponents
and the Ising universal function are reproduced to a very
high precision.

I. INTRODUCTION

The Z2 gauge system has been investigated using
mean-field method [1], Monte Carlo simulations [2,3] and
exact solutions [4,5]. The objective of these studies was
to find the phase diagram from the action (equation 1.1),
and to classify the order of the phase transitions.

A. The Model

The action of the Z2 gauge lattice can be written as
\[
S = -\beta_P \sum_{ij} U_{ij} U_{jk} U_{kl} U_{li} - \beta_L \sum_{ij} \sigma_i U_{ij} \sigma_j \tag{1.1}
\]
where \( U_{ij} \) is the value of the link connecting the sites \( i \)
and \( j \) and \( \sigma_i \) is the value of site \( i \). The first sum is over
all sets of four neighboring links (plaquettes) and the
second sum is over all nearest neighbor sites. Both \( U_{ij} \)
and \( \sigma_i \) can obtain the values \( \pm 1 \) with \( U_{ij} = U_{ji} \).

The free energy is then
\[
F = \frac{1}{N} \ln Z \tag{1.2}
\]
with the partition function \( Z \) given by
\[
Z = \sum_{U_{ij},\sigma_i} e^{-S} \tag{1.3}
\]
and \( N \) is the number of sites in the lattice.

From the action Eq. (1.1) we look at two gauge invariant
observables \( L \) and \( P \) define as
\[
P = \langle U_{ij} U_{jk} U_{kl} U_{li} \rangle = -\frac{1}{4} \frac{\partial}{\partial \beta_P} F(\beta_L, \beta_P) \tag{1.4}
\]
\[
L = \langle \sigma_i U_{ij} \sigma_j \rangle = -\frac{1}{6} \frac{\partial}{\partial \beta_L} F(\beta_L, \beta_P) \tag{1.5}
\]
where the averaging is over all sites and links of the lattice. The factors \( \frac{1}{4} \) and \( \frac{1}{6} \) are the ratios between the
number of sites to the number of links and plaquettes in
a four-dimensional lattice.

In this system, \( L \) and \( P \) can be used as order param-
eters replacing the standard magnetization which is not
gauge invariant. They are order parameters in the sense
that they exhibit singularities of the bulk thermodynamics.
However they lack the property of a magnetization
in that they never vanish (except at temperature zero).

The phase diagram is shown schematically in Fig. 1. The three transition lines experienced by \( L \) and \( P \) meet at the
triple point (A) and split the phase space into three
regions. This phase space includes two familiar limits.
In the limit \( \beta_P \rightarrow \infty \) all plaquettes are forced to one.
The gauge degrees of freedom therefore disappear and
the system reduces to that of a normal four-dimension
Ising model. Here $L$ experiences a second order transition while $P$ remains smooth. It has been shown that the second order line is of Ising type all the way to the triple point.

On the $\beta L = 0$ axis the action reduces to a pure gauge theory and $P$ experiences a first order transition. This behavior continues all the way up to the triple point $A$ with $L$ remaining smooth. A line of first order in both $L$ and $P$ is then formed which ends with the critical point $C$.

II. NUMERICAL SIMULATION ON THE LATTICE

We implemented a standard Monte Carlo procedure using a heat bath algorithm to generate the configurations of the lattice. In this paper we shall use the term iteration to represent an update of all links and sites in the lattice.

Even using this method, the simulations require significant computational resources restricting us to modest lattice sizes for our investigation. Therefore we used lattice sizes ranging from $4^4$ sites (which other studies have shown to give good results) up to $14^4$, and at the critical point up to $18^4$. In order to minimize surface effects a toroidal geometry was adopted.

Each simulation consisted of measurements of the observables along a line of points in the phase diagram. Many such lines were made in order to cover the parameter space and measure variations along different directions. At each point the system was allowed to evolve for at least 10,000 iterations, with mean values of the observables calculated after discarding the first 1000. In order to minimize surface effects a toroidal geometry was adopted.

Along the first order line $\overline{AC}$, the behavior of $L$ is like a magnetization with two metastable states $L^+$ and $L^-$. However, unlike the Ising model, these two states are not symmetrical about the origin. By evaluating the mid point $L_0$ between $L^+$ and $L^-$, we can consider $L - L_0$ which is then symmetrical about zero. However $L_0$ can only be found along the line of first order where there are two stable states present.

Direct measurements of $L$ are further complicated by tunneling between $L^+$ and $L^-$. This has the effect of blurring the first order nature of the transition making measurements of $L_0$ difficult. In order to reduce the effects of tunneling we work with $L_{rms} = \sqrt{\langle (L - L_0)^2 \rangle}$. Due to fluctuations in $L$, $L_{rms}$ will never reduce to zero even though $(L - L_0)$ may become zero, however these fluctuations decrease as the lattice size increases.

In order to measure the critical exponents, the relevant parameterization of the phase space must be identified.

We therefore require an expression for the temperature $T$ and external field $H$ like directions in terms of $\beta_P$ and $\beta_L$.

In a magnetic system, a coexistence line occurs along the temperature axis. We therefore define the temperature axis $T$ for this system along the line of first order $\overline{AC}$.

Along this line the reduced temperature $t$ is given by

$$ t = \frac{\beta_P}{\beta_P - \beta_L} $$

The definition of the external field $H$ is less clear as there is no obvious preferred direction. In this work we used the definition used by Brézin and Drouffe in their mean field analysis of the critical exponents.

The measurements of the critical exponents were carried out in two stages. Firstly we made a direct fit for $L$ to the exponents. This method was applied for $\beta$ and $\delta$.

Next we used finite size scaling which gave more reliable measurements of $\beta$ and $\delta$ and also provided a value for $\nu$ and $\gamma$.

III. NUMERICAL RESULTS AND DISCUSSION

1. Determination of $\beta$ and $\nu$

Figure 2 shows how $L_{rms}$ changes with $t$ for the different lattices. After discarding points with small $t$ (where $\langle L_{rms} \rangle$ differs significantly from $\langle L - L_0 \rangle$), the best fit gives $\beta = 0.41 \pm 0.15$. The solid line is the Ising value of $\beta = 0.5$.

![FIG. 2. $L_{rms}$ versus the temperature](image-url)
A more accurate measurement of $\beta$ was obtained from finite size scaling. Figure 3 shows these results and compares the data with simulations run in the Ising limit, $\beta_g \to \infty$ (solid line). The values for the critical exponents used in this graph are those of a four dimensional Ising model, $\beta = 0.50$ and $\nu = 0.50$. Deviation from a single curve was observed if $\nu$ or $\beta$ were varied by more than 0.02.

FIG. 3. Finite size scaling for $\beta$. $L$ is the order parameter $L_{rms}$, $l$ is the lattice size and $t$ is the reduced temperature. The solid line represents the results from the Ising limit.

2. Determination of $\gamma$

Direct fits with $\gamma$ were unreliable and depended strongly on the dimensions of the lattices we used.

Results from finite size scaling proved to be much more consistent as shown in Fig. 4. Using the value of $\nu = 0.5$ found earlier gives $\gamma = 1.0 \pm 0.1$.

3. Determination of $\delta$

The measurement of $\delta$ is less straightforward. It proved to be sensitive both to the location of the critical point and to the lattice size (see Fig. 5). The main difficulty arises from the fact that $L$ is not symmetric under the operation $H \to -H$ generating different values either side of the temperature axis are $\delta^+ = 4.4 \pm 0.4$ for $H > 0$, and $\delta = 3.0 \pm 0.3$ for $H < 0$. Measurements of lattices with different sizes showed that the shape of $L$ vs. $H$ becomes more symmetric as the lattice size increases.

FIG. 4. Finite size scaling for $\gamma$. $\chi$ is the susceptibility, $l$ and $t$ are the lattice size and reduced temperature respectively.
The calculation of the range of $\delta$ values took place separately for each side around the zero of $L$. The combined result is $\delta = 3.7 \pm 1.1$.

The asymmetry also affects finite size scaling. It proved impossible to collapse both $H > 0$ and $H < 0$ data onto a single curve simultaneously as shown in Fig 6. Treating each side independently gave $\nu = 0.5$ and $\delta^+ = 2.5$ and $\nu = 0.5$ and $\delta^- = 4.7$.

4. Universality

The measured values of the critical exponents show that the critical point in this model most likely behaves like a four-dimensional Ising model. The values of $\beta$ and $\nu$ fit the Ising values with very high accuracy, $\gamma$ has the Ising value too, but the measurement is less accurate.

The only critical exponent that could be significant different from the Ising value is $\delta$. We ascribe our difficulties in measuring $\delta$ to our definition of the external field direction $H$. It is not obvious from the model which direction is relevant and we were unable to isolate it from our data. The direction we adopted may therefore be a combination of $T$ and $H$, thus our $\delta$ measurement is probably contaminated with effects from $\beta$.

IV. SUMMARY

This work is a numerically investigation of the behavior of the critical point in the $Z_2$ gauge lattice. The critical exponents were measured using Monte Carlo simulations, and were calculated directly as well as through finite size scaling. Other measurements were made in the Ising limit of the system.

The calculated exponents had the four-dimensional Ising model values. The numerical measurements gave the same values as the theoretically calculated exponents of $\beta$ and $\delta$. Two other critical exponents, $\nu$ and $\gamma$, that were measured, gave also the Ising values. The finite size scaling for $\delta$ created the same shape of curve as the one obtained from Ising data. The present high precision runs set a high precision standard for the estimation of the character of other interesting critical points, notably the vortex/monopole condensation points in mixed action gauge theories and analogue points in supersymmetric theories.

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* Current address: School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel.

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