THE $L_p$-MINKOWSKI PROBLEM WITH SUPER-CRITICAL EXPONENTS

QIANG GUANG, QI-RUI LI, AND XU-JIA WANG

Abstract. The $L_p$-Minkowski problem deals with the existence of closed convex hypersurfaces in $\mathbb{R}^{n+1}$ with prescribed $p$-area measures. It extends the classical Minkowski problem and embraces several important geometric and physical applications. Existence of solutions has been obtained in the sub-critical case $p > -n - 1$, but the problem remains widely open in the super-critical case $p < -n - 1$. In this paper we introduce new ideas to solve the problem for all the super-critical exponents. A crucial ingredient in our proof is a topological method based on the calculation of the homology of a topological space of ellipsoids.

1. Introduction

A central problem in convex geometry is the characterisation of geometric measures for convex bodies in the Euclidean space $\mathbb{R}^{n+1}$. The best-known example is the classical Minkowski problem, which was a major impetus for the development of fully nonlinear PDEs. In the last three decades, a focus of research in convex geometry is the $L_p$-Minkowski problem introduced by Lutwak [35]. It includes the classical Minkowski problem ($p = 1$), the logarithmic Minkowski problem ($p = 0$), and the centro-affine Minkowski problem and elliptic affine spheres ($p = -n - 1$) as special cases [7, 16]. The $L_p$-Minkowski problem was derived from the Brunn-Minkowski theory, and research of the problem paved the way for further development of this theory [18, 30, 37]. The $L_p$-Minkowski problem also plays a significant role in other applications. Of particular interest is that it describes self-similar solutions to Gauss curvature flows [3, 4, 9, 17], and its projective invariance in the case $p = -n - 1$ makes it fundamental in image processing [2, 5].

Let $\mathcal{K}_o$ denote the set of closed convex bodies in $\mathbb{R}^{n+1}$ with the origin in the interior. For any $\Omega \in \mathcal{K}_o$ and $p \in \mathbb{R}$, its $p$-area measure is defined as $dS_p = u^{1-p}dS$ [35], where $u$ is the support function of $\Omega$ and $S$ is the classical surface area measure of $\Omega$. Given a finite non-negative Borel measure $\mu$ on the unit sphere $\mathbb{S}^n$, the $L_p$-Minkowski problem asks for the existence of solutions $\Omega \in \mathcal{K}_o$ such that its $p$-area measure coincides with the given measure $\mu$. If $d\mu = f d\sigma_{\mathbb{S}^n}$ for a density function $f$ on $\mathbb{S}^n$, then the $L_p$-Minkowski problem can be

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formulated as finding solutions to the Monge-Ampère equation

$$\det(\nabla^2 u + uI) = fu^{p-1}$$
on $S^n$,

where $\nabla$ denotes the covariant derivative with respect to an orthonormal frame on $S^n$, and $I$ is the identity matrix.

The last three decades have witnessed a great progress in the study of the $L_p$-Minkowski problem. The problem can be divided into three cases.

- In the sub-critical case $p > -n - 1$ (with respect to the Blaschke-Santaló inequality), the existence of solutions was obtained in [16]. However, there is no uniform estimate for equation (1.1) when $p < 0$ [26], and there may exist infinitely many solutions when $p < -n$ [22, 32]. When $p = 0$, (1.1) is called the logarithm Minkowski problem; necessary and sufficient conditions for the existence of solutions were obtained in [7] when the prescribed measure is an even Borel measure. For $p \geq 1$, the existence and regularity of solutions were obtained in [16, 36].

- In the critical case $p = -n - 1$, equation (1.1) is called the centro-affine Minkowski problem [16]. The quantity $u^{n+2} \det(\nabla^2 u + uI)$ is invariant under projective transforms and plays a key role in affine geometry. For example when $f = 1$, (1.1) is the equation for affine elliptic spheres [11]. The projective invariance also makes it of great interest to image processing [2]. A Kazdan-Warner type condition [16] implies that (1.1) admits no solutions for a general positive function $f$. There are many works dealing with the critical case [1, 25, 28, 32, 34]. However, results on the existence and multiplicity of solutions in this case are far from being satisfactory. The main difficulty is that the normalisation of blow-up sequences does not lead to a unique limit model.

- In the super-critical case $p < -n - 1$, there are some results in the one dimensional case. In [19, 40] the authors obtained the existence of $\frac{2}{k}$-periodic ($k$-fold symmetry, $k \geq 2$) convex solutions. In [1], Andrews proved that when $f = 1$ and $p \in [-7, -2)$, a convex solution to (1.1) must be a circle; when $p < -7$, a convex solution to (1.1) is either the circle, or a curve with $k$-fold symmetry. In high dimensions, Zhu [42] proved the existence of solutions when $f$ is a discrete measure with no essential subspaces, but we are unaware of any existence results when $f$ is a function.

There are many related research works on the $L_p$-Minkowski problem [6, 13, 14, 23, 43]. It is interesting to compare equation (1.1) with the semi-linear elliptic equation

$$-\Delta_{g_0} u + c_n R_{g_0} u = f(x) u^\gamma$$
on $M$, 

where $\Delta$ denotes the Laplace-Beltrami operator on a Riemannian manifold $(M, g_0)$, and $c_n$ is the $n$-th eigenvalue of the Laplace-Beltrami operator on $M$.
where \((M, g_0)\) is an \(n\)-dimensional Riemannian manifold, \(c_n\) is a constant depending only on \(n\), and \(R_{g_0}\) is the scalar curvature. There is a vast body of literature on equation (1.2). In the sub-critical case \(1 < \gamma < \frac{n+2}{n-2}\), there is a uniform estimate for solutions to (1.2), and one can obtain the existence of non-trivial solutions under suitable conditions. In the critical case \(\gamma = \frac{n+2}{n-2}\), (1.2) is the prescribing scalar curvature equation. In particular, it is Nirenberg’s problem when \(n = 2\) and \(M = S^2\), and the Yamabe problem when \(f = 1\). In this case, there is a very rich phenomena on the existence and multiplicity of solutions, and one can find many significant results [8, 10, 29, 39]. In the super-critical case \(\gamma > \frac{n+2}{n-2}\), numerous attempts have been made for the existence of non-trivial solutions to (1.2) but the solution was obtained only in some special cases.

Comparing with (1.2), we find that equation (1.1) is more complicated. There is no uniform estimate for (1.1) in the sub-critical case. There is no solutions in general in the critical case by the Kazdan-Warner type condition, and much less is known about sufficient conditions for the existence of solutions. Therefore, one would not expect a complete resolution for the existence of solutions to (1.1) in the super-critical case. Surprisingly, we find that the \(L_p\)-Minkowski problem (1.1) admits a solution for all \(p\) in the super-critical range, without any additional conditions, and thus completely resolve the existence problem.

**Theorem 1.1.** Suppose that \(p < -n - 1\). Let \(f\) be a positive and \(C^{1,1}\)-smooth function on \(S^n\). Then there is a uniformly convex, \(C^{3,\alpha}\)-smooth and positive solution to (1.1), where \(\alpha \in (0, 1)\).

By approximation, we also obtain the existence of solutions when \(f\) is a non-smooth function.

**Corollary 1.2.** Suppose that \(p < -n - 1\) and \(f\) is a function on \(S^n\) such that \(\frac{1}{c_0} \leq f \leq c_0\) for some constant \(c_0 > 1\). Then there is a strictly convex, \(C^{1,\alpha}\)-smooth and positive weak solution to (1.1) for some \(\alpha \in (0, 1)\).

We point out that the condition \(f > 0\) in Theorem 1.1 and Corollary 1.2 cannot be relaxed to \(f \geq 0\). Indeed, there exist functions \(f\) which are positive except at the north and south poles, such that equation (1.1) admits no solutions [20, Theorem 1.4].

It is well-known that the Monge-Ampère equation is of divergence form and equation (1.1) is the Euler equation of the following functional (for \(p \neq 0\) for convex bodies \(\Omega \in K_o\)

\[
J(\Omega) = \text{Vol}(\Omega) - \frac{1}{p} \int_{S^n} f u^p d\sigma_{S^n}.
\]
Therefore, a natural approach to the \( L_p \)-Minkowski problem is to combine the variational method with the Gauss curvature flow \([12, 15, 24, 33]\). In this paper we will employ the following Gauss curvature flow:

\[
\frac{\partial X}{\partial t}(x,t) = -f(\nu)K(x,t)\langle X,\nu \rangle^p \nu + X(x,t),
\]

where \( X(\cdot,t) \) is a parametrisation of the evolving convex hypersurfaces \( M_t \), \( \nu \) and \( K \) are respectively the unit outward normal and Gauss curvature of \( M_t \). We will show that the functional (1.3) is non-increasing under the flow (1.4) (Lemma 2.1).

The main difficulty is the lack of uniform estimate for the problem. The uniform estimate is the key estimate for many geometric problems such as the Yamabe problem \([39]\) or Calabi’s conjecture \([44]\). The \( L_p \)-Minkowski problem has been extensively studied in the past three decades, and various techniques have been developed to establish the uniform estimate for the \( L_p \)-Minkowski problem and the associated Gauss curvature flow, but none of them applies to the super-critical case.

To overcome the difficulty, our strategy is to use a topological method to find a special initial condition such that the evolving hypersurfaces \( M_t = \partial \Omega_t \) satisfies

\[
B_r(0) \subset \Omega_t \subset B_R(0),
\]

for positive constants \( R \geq r > 0 \) independent of \( t \), where \( B_r(x) \) denotes a closed ball of radius \( r \) centred at \( x \). Once the solution satisfies such a \( C^0 \)-estimates, one can establish the second derivative estimates, and higher regularity follows from Krylov’s regularity theory. Hence by the monotonicity of the functional (1.3), the flow converges to a solution of (1.1).

Therefore, the key point in the argument is to find the special initial hypersurface. A crucial ingredient in achieving this goal is to compute the homology for a class of ellipsoids centred at the origin. Let us outline the main ideas of the proof below.

For any convex body \( \Omega \) in \( \mathbb{R}^{n+1} \), it is well known that there is a unique ellipsoid \( E(\Omega) \), called John’s minimum ellipsoid \([41]\), which achieves the minimal volume among all ellipsoids containing \( \Omega \), such that

\[
\frac{1}{n+1} E(\Omega) \subset \Omega \subset E(\Omega).
\]

Let \( r_1(\Omega) \leq r_2(\Omega) \leq \cdots \leq r_{n+1}(\Omega) \) be the lengths of semi-axes of \( E(\Omega) \). Denote \( e_M = e_\Omega = \frac{r_{n+1}(\Omega)}{r_1(\Omega)} \) the eccentricity of \( M := \partial \Omega \) (or the eccentricity of \( \Omega \)). We will first prove the following property.

(P): For any given constant \( A > J(B_1(0)) \), if one of the quantities \( e_\Omega, \text{Vol}(\Omega), [\text{Vol}(\Omega)]^{-1} \), and \([\text{dist}(O,\partial \Omega)]^{-1}\) is sufficiently large, we have \( J(\Omega) \geq A \) (Lemmas 2.2, 2.4).
Denote by $\mathcal{A}_I$ the set of ellipsoids $E$ such that the origin $O \in E$, $e_E \in [1, \bar{e}]$, and $\bar{v} \leq \text{Vol}(E) \leq 1/\bar{v}$, where $\bar{e}$ is a large constant and $\bar{v}$ is a small constant. $\mathcal{A}_I$ is a metric space under the Hausdorff distance. For any ellipsoid $E \in \mathcal{A}_I$, let $\mathcal{M}_E(t)$ be the solution to the flow (1.4) with initial condition $E$. By the above property (P), $\mathcal{M}_E(t)$ has uniformly bounded eccentricity and volume, and $\text{dist}(O, \mathcal{M}_E(t))$ is uniformly bounded from zero if

$$J(\mathcal{M}_E(t)) \leq A \quad \forall \ t \geq 0.$$  

Now, our focus is to prove that at any given time $t_0 > 0$, there exists an initial $E_0 \in \mathcal{A}_I$ such that the minimum ellipsoid of $\mathcal{M}_{E_0}(t_0)$ is the unit ball centred at the origin (Lemma 3.8), thus validating the condition (1.5) (as a result of Lemma 3.11).

If to the contrary there is no such an initial $E_0$, we will construct a continuous map $T : \mathcal{A}_I \to \mathcal{P}$ which is the identity map on $\mathcal{P}$, where $\mathcal{P}$ is the boundary of $\mathcal{A}_I$ in the topological space of all ellipsoids. This implies the existence of an injection from the homology group of $\mathcal{P}$ to that of $\mathcal{A}_I$. As a consequence, $\mathcal{P}$ has trivial homology since $\mathcal{A}_I$ is contractible (Lemma 3.4). By Proposition 3.6 this leads to $H_k(\mathcal{E} \times S^n) = H_k(\mathcal{E}) \oplus H_k(S^n)$, where $\mathcal{E}$ is introduced in (3.3). We thus reach a contradiction by the Künneth formula and Theorem 3.7 if we take $k = \frac{n(n+1)}{2} + 2n - 1$. This topological fixed-point argument is the main novelty in this paper. A crucial ingredient in the argument is the computation of the homology of the class $\mathcal{E}$ of ellipsoids.

To complete the proof, we choose a sequence $t_k \to \infty$ and let $E_k \in \mathcal{A}_I$ be the initial condition such that the minimum ellipsoid of $\mathcal{M}_{E_k}(t_k)$ is the unit ball. By the Blaschke selection theorem, $E_k$ sub-converges to $E_* \in \mathcal{A}_I$. It follows by the above property (P) that the Gauss curvature flow (1.4) with initial condition $E_*$ satisfies (1.5). Hence, the flow (1.4) starting from $E_*$ converges to a solution of (1.1) as $t \to \infty$.

The paper is organised as follows. In Section 2 we derive some a priori estimates for the functional (1.3) and the Gauss curvature flow (1.4). In Section 3 we prove the main results (Theorem 1.1 and Corollary 1.2), assuming Proposition 3.6, Theorem 3.7, and Theorem 2.5 temporarily. The proofs of Proposition 3.6 and Theorem 3.7 will be given in Sections 4 and the proof of Theorem 2.5 will be given in Section 5.

The topological method introduced in this paper enables us to find an initial condition such that the solution has uniform estimate for all time $t$. The uniform estimate is the most difficult part for many geometric and analysis problems. This method can be adapted to other geometric problems, such as the $L_p$ dual Minkowski problem [23, 38], and the dual centro-affine Minkowski problem [27] and more general prescribing curvature problems. We will study these problems separately.
2. A priori estimates

Let $\mathcal{M}$ be a smooth, closed, and uniformly convex hypersurface in $\mathbb{R}^{n+1}$. The support function of $\mathcal{M}$ is given by
\[ u(x) = \langle x, \nu^{-1}_M(x) \rangle, \quad \forall \ x \in \mathbb{S}^n, \]
where $\nu_M : \mathcal{M} \to \mathbb{S}^n$ is the Gauss map and $\nu^{-1}_M$ is its inverse, namely, $\nu^{-1}_M(x)$ is the point $z(x) \in \mathcal{M}$ such that the unit outer normal of $\mathcal{M}$ at $z(x)$ is equal to $x$. It is well known that $\nu^{-1}_M(x) = u(x)x + \nabla u(x)$ and the Gauss curvature of $\mathcal{M}$ at $\nu^{-1}_M(x)$ is given by
\[ K = \frac{1}{\det(u_{ij} + u\delta_{ij})}, \tag{2.1} \]
where $u_{ij} = \nabla^2 u$. This implies that the $p$-area measure of $\mathcal{M}$ is given by
\[ dS_p = u^{1-p} \det(\nabla^2 u + uI) d\sigma_{\mathbb{S}^n}. \]
Hence, the $L_p$-Minkowski problem is equivalent to solving equation (1.1).

Denote by $\text{Cl}(\mathcal{M})$ the convex body enclosed by $\mathcal{M}$. When no confusion arises we may abuse the notation $\mathcal{M}$ for $\text{Cl}(\mathcal{M})$, such as writing the functional $J(\text{Cl}(\mathcal{M}))$ as $J(\mathcal{M})$. Assume that $\text{Cl}(\mathcal{M}) \in \mathcal{K}_{\mathbb{R}}$. Let $r$ be the radial function of $\mathcal{M}$, which is given by
\[ r(\xi) = \max\{\lambda : \lambda\xi \in \text{Cl}(\mathcal{M})\} \quad \forall \ \xi \in \mathbb{S}^n. \tag{2.2} \]
Then
\[ \text{Vol}(\text{Cl}(\mathcal{M})) = \frac{1}{n+1} \int_{\mathbb{S}^n} r^{n+1} d\sigma_{\mathbb{S}^n}. \tag{2.3} \]
Denote $\bar{r}(\xi) = r(\xi)\xi$. We also define the radial Gauss mapping by
\[ \nu_{\mathcal{M}}(\xi) = \nu_M(\bar{r}(\xi)) \quad \forall \ \xi \in \mathbb{S}^n. \]

Let $\mathcal{M}_t$ be a solution to the flow (1.4) and $X(\cdot, t)$ be its parametrisation. Consider the new parametrisation
\[ \overline{X}(x, t) = X(\nu_{\mathcal{M}_t}^{-1}(x), t). \]
It is straightforward to compute
\[ \frac{\partial \overline{X}}{\partial t} = \sum_i \frac{\partial X}{\partial z^i} \frac{\partial (\nu_{\mathcal{M}_t}^{-1})_i}{\partial t} + \frac{\partial X}{\partial t}. \]
Since the first term on the right hand side is tangential, taking inner product with the unit outer normal of $\mathcal{M}_t$ gives that
\[ \partial_t u(x, t) = \langle x, \partial_t \overline{X}(x, t) \rangle = \langle x, \partial_t X(\nu_{\mathcal{M}_t}^{-1}(x), t) \rangle. \]
Hence by (2.1), the flow (1.4) can be expressed as
\begin{equation}
\partial_t u(x, t) = - \frac{f(x)u^p(x)}{\det(\nabla^2 u + uI)} + u(x, t).
\end{equation}

We next show the monotonicity of the functional (1.3) under the flow (1.4).

**Lemma 2.1.** Suppose $\mathcal{M}_t, t \in [0, T)$, is a solution to the flow (1.4) in $\mathcal{K}_0$. Then
\[ \frac{d}{dt} J(\Omega_t) \geq 0, \]
where $\Omega_t = \text{Cl}(\mathcal{M}_t)$. Moreover, the equality holds if and only if $\mathcal{M}_t$ satisfies (1.1).

**Proof.** The following formulas can be found in [33]:
\begin{equation}
\frac{\partial r}{r}(\xi, t) = \frac{\partial u}{u}(\mathcal{A}_{\mathcal{M}_t}(\xi), t),
\end{equation}
\begin{equation}
|\text{Jac}\mathcal{A}|(\xi) = \frac{r^{n+1}K(\cdot, t)}{u(\mathcal{A}_{\mathcal{M}_t}(\xi))},
\end{equation}
where $\text{Jac}\mathcal{A}$ is the Jacobian of the radial Gauss mapping.

By virtue of (2.1)–(2.5), we obtain
\begin{equation}
\frac{d}{dt} J(\Omega_t) = - \int_{\mathbb{S}^n} fu^{p-1} \partial_t u(x) d\sigma_{\mathbb{S}^n}(x) + \int_{\mathbb{S}^n} r^n \partial_t r(\xi) d\sigma_{\mathbb{S}^n}(\xi)
= - \int_{\mathbb{S}^n} \left( \frac{1}{K} - fu^{p-1} \right) \partial_t u(x) d\sigma_{\mathbb{S}^n}(x)
= \int_{\mathbb{S}^n} \left( \frac{1}{K} - fu^{p-1} \right)^2 uK d\sigma_{\mathbb{S}^n} \geq 0.
\end{equation}
Clearly, the equality $\frac{d}{dt} J(\Omega_t) = 0$ holds if and only if $u(\cdot, t)$ satisfies (1.1). \hfill \Box

The proof of Lemma 2.1 also verifies that (1.1) is the Euler-Lagrangian equation of the functional (1.3).

### 2.1. Properties of the functional (1.3).

Next, we prove the property (P) stated in the introduction.

**Lemma 2.2.** Suppose that $p < -n - 1$ and $1/c_0 \leq f \leq c_0$ for some $c_0 \geq 1$. For any given constant $A > 0$, there exists a small constant $d_0 > 0$ depending only on $n$, $p$, $c_0$ and $A$ such that if $\Omega \in \mathcal{K}_0$ satisfies $\text{dist}(O, \partial \Omega) \in (0, d_0)$, then $J(\Omega) > A$.

**Proof.** Denote by $d = \text{dist}(O, \partial \Omega) > 0$. Take $x_0 \in \mathbb{S}^n$ such that
\[ u(x_0) = \min_{\mathbb{S}^n} u = d, \]
where \( u \) is the support function of \( \Omega \). Let \( E \) be the minimum ellipsoid of \( \Omega \). We choose the coordinates such that

\[
E - \zeta_E = \left\{ z \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} z_i^2 a_i^2 \leq 1 \right\},
\]

and

\[
x_0 \cdot e_{n+1} = \max \{|x_0 \cdot e_i| : 1 \leq i \leq n+1\},
\]

where \( \zeta_E \) is the center of \( E \). This implies that \( x_0 \cdot e_{n+1} \geq c_n \). We use \( c_n \) to denote a constant which depends only on \( n \), but it may change from line to line.

Let \( w(x) = u(x) + u(-x), \ x \in \mathbb{S}^n \), be the width function of \( \Omega \). Since the ball \( B_d(0) \) is contained in \( \Omega \) and \( \frac{1}{n+1} E \subset \Omega \subset E \), we have

\[
d \leq \min_{\mathbb{S}^n} w \leq c_n a_{n+1} \quad \text{and} \quad w(e_i) \leq c_n a_i.
\]

This yields that

\[
J(\Omega) > \text{Vol}(\Omega) \geq c_n \prod_{i=1}^{n+1} a_i \geq c_n d \prod_{i=1}^n w(e_i) \geq c_n d \prod_{i=1}^n u(e_i).
\]

Next, we consider the set \( \Omega_{n+1}^* = \Omega^* \cap L \), where \( \Omega^* \) is the polar dual of \( \Omega \) and \( L = \left\{ z \in \mathbb{R}^{n+1} : z \cdot e_{n+1} = 0 \right\} \). Let \( r^* \) be the radial function of \( \Omega^* \). Since the origin \( O \) and points \( r^*(e_i)e_i, i = 1, \ldots, n \), are contained in \( \Omega_{n+1}^* \), their convex hull is an \( n \)-dimensional convex set in \( \Omega_{n+1}^* \), namely,

\[
\mathcal{C} = \text{convex hull of} \ \{O, r^*(e_1)e_1, \ldots, r^*(e_n)e_n\} \subset \Omega_{n+1}^*.
\]

Let \( V \) be the cone in \( \mathbb{R}^{n+1} \) with base \( \mathcal{C} \) and vertex \( p_0 = r^*(x_0)x_0 \). Since \( r^*(x) = \frac{1}{u(x)} \) and \( u(x_0) = d \), the height of the cone \( V \) (in the direction of \( e_{n+1} \)) satisfies

\[
r^*(x_0)x_0 \cdot e_{n+1} = \frac{x_0 \cdot e_{n+1}}{d} \geq \frac{c_n}{d}.
\]

Consider the following subset of \( V \)

\[
V' = \left\{ z \in V : z_{n+1} \geq \frac{r^*(x_0)}{2} x_0 \cdot e_{n+1} \right\}.
\]

By (2.7), we have

\[
|z| \geq \frac{c_n}{2d} \quad \forall \ z \in V'.
\]

In view of \( V' \subset \Omega^* \), \( f \geq 1/c_0 \) and (2.8), we have

\[
J(\Omega) \geq -\frac{1}{p} \int_{\mathbb{S}^n} u^p f \geq \frac{1}{c_0} \int_{\Omega^*} |z|^{-p-n-1}dz \geq \frac{1}{c_0} \int_{V'} |z|^{-p-n-1}dz \geq \frac{c_n \text{Vol}(V')}{c_0 d^{-p-n-1}}.
\]
Since
\[ \text{Vol}(V') \geq \frac{c_n}{d} \text{Vol}(C), \]
\[ \text{Vol}(C) \geq c_n \prod_{i=1}^{n} r^*(e_i) = c_n \left[ \prod_{i=1}^{n} u(e_i) \right]^{-1}, \]
we further obtain that
\[ (2.9) \quad \mathcal{J}(\Omega) \geq \frac{c_n}{c_0 d^{n+p-n}} \left[ \prod_{i=1}^{n} u(e_i) \right]^{-1}. \]
Combining (2.6) and (2.9), we have
\[ \left[ \mathcal{J}(\Omega) \right]^2 \geq \frac{c_n}{c_0 d^{n+p-n}}. \]
Since \(-p - n - 1 > 0\), we see that \( \mathcal{J}(\Omega) > A \) if \( d \) is sufficiently small. \( \square \)

**Lemma 2.3.** Suppose that \( p < -n - 1 \) and \( 1/c_0 \leq f \leq c_0 \) for some \( c_0 \geq 1 \). For any given constant \( A > 1 \), there exists a small constant \( v > 0 \) depending only on \( n, p, c_0 \), and \( A \), such that if \( \Omega \in \mathcal{K}_0 \) satisfies either \( \text{Vol}(\Omega) \leq v \) or \( \text{Vol}(\Omega) \geq v^{-1} \), then \( \mathcal{J}(\Omega) > A \).

**Proof.** If \( \text{Vol}(\Omega) \geq v^{-1} \), by definition we have \( \mathcal{J}(\Omega) > \text{Vol}(\Omega) \geq v^{-1} > A \) by taking \( v \) small.

If \( \text{Vol}(\Omega) \leq v \). Denote \( d = \text{dist}(O, \partial \Omega) \). Since the ball \( B_d(0) \subset \Omega \), we have
\[ v \geq \text{Vol}(\Omega) \geq \text{Vol}(B_d) = \text{Vol}(B_1) d^{n+1}. \]
Hence, if \( v \) is sufficiently small, then \( d < d_0 \), where \( d_0 > 0 \) is the constant given by Lemma 2.2. Therefore, we have \( \mathcal{J}(\Omega) > A \) by Lemma 2.2. \( \square \)

**Lemma 2.4.** Suppose that \( p < -n - 1 \) and \( 1/c_0 \leq f \leq c_0 \) for some \( c_0 \geq 1 \). For any given constant \( A > 0 \), there exists a large constant \( e > 1 \) depending only on \( n, p, c_0 \) and \( A \), such that if \( \Omega \in \mathcal{K}_0 \) satisfies \( e_{\Omega} \geq e \), we have \( \mathcal{J}(\Omega) > A \).

**Proof.** For the given constant \( A \), let \( d_0 \) be the constant determined by Lemma 2.2. We assume that \( d = \text{dist}(O, \partial \Omega) \geq d_0 \); otherwise, we are done by Lemma 2.2.

Let \( E \) be the minimum ellipsoid of \( \Omega \) with semi-axes \( a_1 \leq \cdots \leq a_{n+1} \). Note that \( \frac{a_{n+1}}{a_1} = e_{\Omega} \). Since \( B_{d_0} \subset \Omega \) and \( \Omega \subset E \), we obtain that
\[ (2.10) \quad d_0 \leq d \leq a_1. \]
Noting also that \( \frac{1}{n+1} E \subset \Omega \), we have
\[ \mathcal{J}(\Omega) > \text{Vol}(\Omega) \geq c_n \text{Vol}(E) = c_n \prod_{i=1}^{n+1} a_i \geq c_n e_{\Omega} d_0^{n+1}. \]
Clearly, we have \( \mathcal{J}(\Omega) > A \) if \( e_{\Omega} \) is sufficiently large. \( \square \)
2.2. A priori estimates for the parabolic equation \((2.4)\).

In this subsection, we state the a priori estimates for the solution \(u\), assuming the uniform estimate for \(u\).

**Theorem 2.5.** Let \(f\) be a positive and \(C^{1,1}\)-smooth function on \(\mathbb{S}^n\). Let \(u(\cdot, t)\) be a positive, smooth and uniformly convex solution to \((2.4)\), \(t \in [0, T]\). Assume that
\[
1/C_0 \leq u(x, t) \leq C_0, \\
|\nabla u|(x, t) \leq C_0, \\
\]
for all \((x, t) \in \mathbb{S}^n \times [0, T]\). Then
\[
C^{-1}I \leq (\nabla^2 u + u I)(x, t) \leq CI \forall (x, t) \in \mathbb{S}^n \times [0, T),
\]
where \(C\) is a positive constant depending only on \(n, p, C_0, \min_{\mathbb{S}^n} f, \|f\|_{C^{1,1}(\mathbb{S}^n)}\), and the initial condition \(u(\cdot, 0)\).

The proof of Theorem 2.5 is based on proper choice of auxiliary functions, and will be given in Section 3.

By the second derivative estimates \((2.12)\), equation \((2.4)\) becomes uniformly parabolic. Hence, by Krylov’s regularity theory, we have the \(C^{3,\alpha}\) estimates for the solution \(u\). Namely
\[
\|u(\cdot, t)\|_{C^{3,\alpha}(\mathbb{S}^n)} \leq C \forall (x, t) \in \mathbb{S}^n \times [0, T),
\]
for any given \(\alpha \in (0, 1)\), where the constant \(C\) depends only on \(\alpha, n, p, C_0, \min_{\mathbb{S}^n} f, \|f\|_{C^{1,1}(\mathbb{S}^n)}\), and the initial condition \(u(\cdot, 0)\). By the a priori estimates \((2.13)\), we have the longtime existence of solutions to the flow \((1.4)\), provided that \(u\) satisfies \((2.11)\).

**Theorem 2.6.** Let \(f\) be a positive and \(C^{1,1}\)-smooth function on \(\mathbb{S}^n\). Let \(T_{\text{max}}\) be the maximal time such that \(u(\cdot, t)\) is a positive, \(C^{3,\alpha}\)-smooth, and uniformly convex solution to \((2.4)\) on \([0, T_{\text{max}})\). If \((2.11)\) holds for all the time \(t \in [0, T_{\text{max}})\), then \(T_{\text{max}} = \infty\) and \(u\) satisfies the estimates \((2.12)\) and \((2.13)\).

**Remark 2.7.** Let \(\mathcal{M}(t)_{t \in [0, T_{\text{max}}]}\) be a solution to \((1.4)\). By Lemmas 2.2, 2.3 and 2.4 if \(\mathcal{J}(\mathcal{M}(t)) < A\) for some constant \(A\) independent of \(t\), then there exists positive constants \(e, v, d\) depending on \(A\), but independent of \(t\), such that
\[
e_{\mathcal{M}(t)} \leq e, \quad v \leq \text{Vol}(\Omega(t)) \leq v^{-1}, \quad B_d(0) \subset \Omega(t),
\]
where \(\Omega(t)\) is the convex body enclosed by \(\mathcal{M}(t)\).

From \((2.14)\) one infers that \((2.11)\) holds. Hence the a priori estimates \((2.12)\) and \((2.13)\) hold, and one has the long-time existence of solution (Theorem 2.6). Therefore, for the a
priori estimates (2.12), (2.13) and the long-time existence of solution, all we need is that
the condition \( J(M(t)) < A \) holds for some constant \( A \).

3. Proof of Theorem 1.1

In this section, we show how to select an initial hypersurface \( N_0 \), such that the flow (1.4)
deforms \( N_0 \) to a solution of (1.1).

The initial hypersurface \( N_0 \) is an ellipsoid and will be chosen by a topological method.
In the proof of the existence of \( N_0 \), a key step is the computation of the homology groups
of a special class of ellipsoids. The homology groups will be given in Proposition 3.6 and
Theorem 3.7, whose proofs are postponed to the next section.

Denote
\[
A_0 = 2 \left( -\frac{\|f\|_{L^1(S^n)}}{p[2(n+1)]^p} + 2^{n+1} \text{Vol}(B_1) \right).
\]
where \( B_1 = B_1(0) \) is the unit ball in \( \mathbb{R}^{n+1} \) centred at the origin. Recall that \( p < -n-1 \).
Hence for any \( \Omega \in \mathcal{K}_o \) with \( \frac{1}{2(n+1)} B_1 \subseteq \Omega \subseteq 2 B_1 \), we have
\[
J(\Omega) \leq \frac{1}{2} A_0.
\]
In particular, if the minimum ellipsoid of \( \Omega \) is \( B_1 \), then \( \frac{1}{n+1} B_1 \subset \Omega \subset B_1 \) and hence
\[
J(\Omega) \leq -\frac{1}{p(n+1)^p} \int_{S^n} f + \text{Vol}(B_1) \leq \frac{1}{2} A_0.
\]

3.1. A modified flow of (1.4).

We introduce a modified flow of (1.4) such that for any initial condition, the solution exists
for all time \( t \geq 0 \). The purpose of introducing this modified flow is for the convenience of
later discussion.

For a closed, smooth and uniformly convex hypersurface \( \mathcal{N} \) such that \( \Omega_0 = \text{Cl}(\mathcal{N}) \in \mathcal{K}_o \), we
define a family of time-depending hypersurfaces \( \mathcal{M}_\mathcal{N}(t) \) with initial condition \( \mathcal{N} \) as follows:

- If \( J(M(t)) < A_0 \) for all time \( t \geq 0 \), let \( \mathcal{M}_\mathcal{N}(t) = M_\mathcal{N}(t) \) for all \( t \geq 0 \), where \( M_\mathcal{N}(t) \)
is the solution to (1.4). We point out that, by Remark 2.7, the solution \( M_\mathcal{N}(t) \) exists
as long as \( J(M_\mathcal{N}(t)) \) is finite.
- If \( J(\mathcal{N}) < A_0 \), and \( J(M_\mathcal{N}(t)) \) reaches \( A_0 \) at the first time \( t_0 > 0 \), we define
\[
\mathcal{M}_\mathcal{N}(t) = \begin{cases} 
M_\mathcal{N}(t), & \text{if } 0 \leq t < t_0, \\
M_\mathcal{N}(t_0), & \text{if } t \geq t_0.
\end{cases}
\]
\begin{itemize}
\item If $J(\mathcal{N}) \geq A_0$, we let $\mathcal{M}_N(t) \equiv \mathcal{N}$ for all $t \geq 0$. That is, the solution is stationary.
\end{itemize}

For convenience, we call $\mathcal{M}_N(t)$ a modified flow of (1.4).

\textbf{Remark 3.1.} Apparently $\text{Cl}(\mathcal{M}_N(t)) \in \mathcal{K}_o$ and it is easy to verify the following properties.

\begin{itemize}
\item $\mathcal{M}_N(t)$ is defined for all time $t \geq 0$, and by Lemma 2.1, $J(\mathcal{M}_N(t))$ is non-decreasing.
\item By Lemma 2.2 if dist$(O, \mathcal{N})$ is very small, then $\mathcal{M}_N(t) \equiv \mathcal{N} \ \forall \ t \geq 0$.
\item By Lemma 2.3 if Vol$(\Omega_0)$ is sufficiently large or small, then $\mathcal{M}_N(t) \equiv \mathcal{N} \ \forall \ t \geq 0$.
\item By Lemma 2.4 if $e_\Omega$ is sufficiently large, then $\mathcal{M}_N(t) \equiv \mathcal{N} \ \forall \ t \geq 0$.
\item We have $J(\mathcal{M}_N(t)) \leq \max\{A_0, J(\mathcal{N})\} \ \forall \ t \geq 0$.
\item By the a priori estimates, $\mathcal{M}_N(t)$ is smooth for any fixed time $t$, and Lipschitz continuous in time $t$.
\end{itemize}

3.2. A special class of ellipsoids $\mathcal{A}_I$.

\textbf{Lemma 3.2.} For the constant $A_0$ given by (3.1), there exist sufficiently small constants $\bar{v}$ and $\bar{d}$, and a sufficiently large constant $\bar{e}$, such that for any $\Omega \in \mathcal{K}_o$,

\begin{itemize}
\item (i) if dist$(O, \partial \Omega) \leq \bar{d}$, then $J(\Omega) > A_0$;
\item (ii) if $e_\Omega \geq \bar{e}$, then $J(\Omega) > A_0$;
\item (iii) if Vol$(\Omega) \leq \bar{v}$ or Vol$(\Omega) \geq [(n + 1)^{n+1} \bar{v}]^{-1}$, then $J(\Omega) > A_0$.
\end{itemize}

\textit{Proof.} This is an immediate consequence of Lemmas 2.2, 2.3 and 2.4. See also Remark 2.7. \hfill $\blacksquare$

Fix the constants $\bar{d}, \bar{v}$ and $\bar{e}$ as in Lemma 3.2. We introduce the following notations:

\begin{itemize}
\item $\mathcal{K}$ is the collection of all non-empty, compact and convex sets in $\mathbb{R}^{n+1}$ equipped with the Hausdorff distance, such that $\mathcal{K}$ is a metric space.
\item $\bar{\mathcal{K}}_o$ is the closure of $\mathcal{K}_o$ in $\mathcal{K}$.
\item $\mathcal{K}_e$ is the subset of $\mathcal{K}_o$ which consists of all origin-symmetric convex bodies.
\item Denote $\mathcal{A}_I = \{ E \in \bar{\mathcal{K}}_o : \text{an ellipsoid in } \mathbb{R}^{n+1} : \bar{v} \leq \text{Vol}(E) \leq 1/\bar{v} \text{ and } e_E \leq \bar{e} \}$,
\item $\tilde{\mathcal{A}} = \{ E \in \mathcal{A}_I : \text{Vol}(E) = \omega_n \text{ and } e_E \in [1, \bar{e}] \}$, where $\omega_n = \text{Vol}(B_1)$,
\item $\mathcal{A} = \{ E \in \tilde{\mathcal{A}} : \text{either } e_E = \bar{e} \text{ or dist}(O, \partial E) = 0 \}$.
\end{itemize}
To calculate the homology of $A_I$, we also introduce
\begin{align}
\mathcal{E}_I &= A_I \cap K_e, \\
\hat{\mathcal{E}} &= \hat{A} \cap K_e, \\
\mathcal{E} &= A \cap K_e.
\end{align}

The sets $A_I, \hat{A}, A$ and $\mathcal{E}_I, \hat{\mathcal{E}}, \mathcal{E}$ are all closed. In particular, $\mathcal{E}$ (resp. $A$) is the boundary of $\hat{\mathcal{E}}$ (resp. $\hat{A}$) in the space of all ellipsoids in $K_e$ (resp. $K$) with unit ball volume.

For any $E \in \mathcal{E}$, let $R$ be a rotation such that
\begin{align}
R(E) = \left\{ (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \frac{x_i^2}{a_i^2} \leq 1 \right\}.
\end{align}
Since $\text{Vol}(E) = \omega_n$, we have $\prod_{i=1}^{n+1} a_i = 1$. For $s \in [0,1]$, denote
\begin{align}
b_i(s) &= (1-s) + sa_i \quad \text{and} \quad \hat{a}_i(s) = \frac{b_i(s)}{\prod_{i=1}^{n+1} b_i(s)}^{\frac{1}{n+1}}.
\end{align}

We then obtain an ellipsoid $\hat{E}(s)$ such that
\begin{align}
R(\hat{E}(s)) = \left\{ (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \frac{x_i^2}{(\hat{a}_i(s))^2} \leq 1 \right\}.
\end{align}

In particular, $\hat{E}(0) = B_1$, $\hat{E}(1) = E$. The set $\{\hat{E}(s)\}_{s \in [0,1]}$ is a path in $K_e$ connecting $B_1$ to $E$, and satisfies $\text{Vol}(\hat{E}(s)) = \omega_n$ for all $s \in [0,1]$. As a result,
\begin{align}
\hat{\mathcal{E}} = \{ \hat{E}(s) : E \in \mathcal{E}, s \in [0,1] \},
\end{align}

namely, $\hat{\mathcal{E}}$ is the union of all such paths, and
\begin{align}
\mathcal{E}_I = \left\{ \left[ \tau/\omega_n \right]^\frac{1}{n+1} \hat{E} : \hat{E} \in \hat{\mathcal{E}}, \tau \in [\bar{v}, 1/\bar{v}] \right\}.
\end{align}

**Lemma 3.3.** Both $\mathcal{E}_I$ and $\hat{\mathcal{E}}$ are contractible. Hence the homology
\begin{align}
H_k(\mathcal{E}_I) = H_k(\hat{\mathcal{E}}) = 0 \quad \forall \ k \geq 1.
\end{align}

**Proof.** To see that $\hat{\mathcal{E}}$ is contractible, by (3.5) it suffices to notice that $(\hat{E}(s), t) \to \hat{E}((1-t)s)$, where $t \in [0,1]$, is a deformation retraction from $\hat{\mathcal{E}} \to \{B_1\}$. Hence $\hat{\mathcal{E}}$ are contractible.

Similarly, by (3.6),
\begin{align}
\left( \left[ \frac{\tau}{\omega_n} \right]^\frac{1}{n+1} \hat{E}, t \right) &\to \left[ \frac{(1-t)\tau + t\omega_n}{\omega_n} \right]^\frac{1}{n+1} \hat{E}
\end{align}
is a deformation retraction from $\mathcal{E}_I$ to $\hat{\mathcal{E}}$. As $\hat{\mathcal{E}}$ is contractible, one sees that $\mathcal{E}_I$ is also contractible. \qed
We can combine the two deformation retractions in the above proof and obtain a new one from \( E \) to \( \{ B_1 \} \) as follows

\[(\eta) : \left( \left[ \frac{\tau}{\omega_n} \right]_{\pi+1} E(s), t \right) \mapsto \left[ \frac{(1-t)\tau + t\omega_n}{\omega_n} \right]_{\pi+1} E((1-t)s), \]

where \( s \in [0,1], \tau \in [\bar{v}, 1/\bar{v}] \).

In the following, the notation \( H \simeq H' \) means two metric spaces \( H, H' \) are homeomorphic.

**Lemma 3.4.** We have \( A_I \simeq E_I \times B_1 \), and \( \hat{A} \simeq \hat{E} \times B_1 \) (with the product topology). It follows that both \( A_I \) and \( \hat{A} \) are contractible, and hence the homology

\[(3.9) \quad H_k(A_I) = H_k(\hat{A}) = 0 \quad \forall \ k \geq 1.\]

**Proof.** For any given ellipsoid \( E \in K_e \), the map

\[\tilde{\varphi}_E(y) = \begin{cases} y/r_E(y/|y|), & \text{if } y \in E \setminus \{0\}, \\ 0, & \text{if } y = 0, \end{cases}\]

defines a homeomorphism between \( E \) and \( B_1 \), where \( r_E \) is the radial function of \( E \) (see (2.2)). Denote by \( \tilde{\varphi}_E^{-1} \) the inverse of \( \tilde{\varphi}_E \).

We show \( A_I \simeq E_I \times B_1 \). For \( E \in A_I \), let \( \zeta_E \) be its centre. Then \( E_o := E - \zeta_E \in E_I \).

We define a map \( \phi : A_I \to E_I \times B_1 \) by

\[(3.10) \quad \phi(E) = (E - \zeta_E, \tilde{\varphi}_{E_o}(\zeta_E)).\]

Its inverse \( \phi^* : E_I \times B_1 \to A_I \) is given by

\[(3.11) \quad \phi^*(E, y) = E + \tilde{\varphi}_E^{-1}(y).\]

It is easy to verify that \( \phi^* \circ \phi = id_{A_I}, \phi \circ \phi^* = id_{E_I \times B_1} \) and both \( \phi \) and \( \phi^* \) are continuous.

Restricting \( \phi \) and \( \phi^* \) to \( \hat{A} \) and \( \hat{E} \times B_1 \) respectively, we see that \( \hat{A} \simeq \hat{E} \times B_1 \). \( \square \)

Denote

\[(3.12) \quad \mathcal{P} = \{ E \in A_I : \text{either } Vol(E) = \bar{v}, \text{ or } Vol(E) = 1/\bar{v}, \text{ or } e_E = \bar{e}, \text{ or } O \in \partial E \}.\]

Note that \( \mathcal{P} \) is the boundary of \( A_I \) if we regard \( A_I \) as a set in the topological space of all ellipsoids.

**Lemma 3.5.** There is a retraction \( \Psi \) from \( A_I \setminus \{B_1\} \) to \( \mathcal{P} \). Namely, \( \Psi : A_I \setminus \{B_1\} \to \mathcal{P} \) is continuous and \( \Psi|_{\mathcal{P}} = id \).
Proof. The retraction $\Psi$ can be constructed as follows.

1. By the map $\phi$ in (3.10), we have $A_I \setminus \{B_1\} \simeq E_I \times B_1 \setminus (\{B_1\} \times \{0\})$.

2. Let $\partial E_I$ be the topological boundary of $E_I$, regarding $E_I$ as a set in the space of all ellipsoids in $K_e$. Then $\partial E_I = \{E \in E_I : \text{either Vol}(E) = \bar{v}, \text{or Vol}(E) = 1/\bar{v}, \text{or } e_E = \bar{v}\}$. In this step we define a continuous map $\psi : (E_I \times B_1) \setminus (\{B_1\} \times \{0\}) \to (\partial E_I \times B_1) \cup (E_I \times \partial B_1)$.

   Recall the deformation retraction $\eta$ from $E_I$ to $\{B_1\}$ in (3.8). For any $t \neq 1$, $\eta(\cdot, t)$ defines a homeomorphism between $E_I$ and $\eta(E_I, t)$. For any $E \in E_I \setminus \{B_1\}$, let

   $$t_E = \sup\{t \in [0, 1] : \text{there exists } E' \in E_I \text{ such that } \eta(E', t) = E\}.$$

   We have $t_E < 1$. Since $E_I$ is closed, there exists $\tilde{E} \in \partial E_I \subset E_I$ such that $\eta(\tilde{E}, t_E) = E$. For any given $t \in (0, 1)$, $\eta$ also defines a homeomorphism between $\partial E_I$ and $\eta(\partial E_I, t)$, and $\eta(\cdot, 0)$ is the identity map on $\partial E_I$. Define a map $\psi_1 : E_I \setminus \{B_1\} \to \partial E_I$ by letting $\psi_1(E) = \tilde{E}$, where $\tilde{E}$ satisfies $\eta(\tilde{E}, t_E) = E$. Then $\psi_1$ is a retraction from $E_I \setminus \{B_1\}$ to $\partial E_I$ such that $\psi_1 = \text{id}$ on $\partial E_I$.

   Let $\psi_2(x) = \frac{x}{|x|}$. Then $\psi_2$ is a retraction $B_1 \setminus \{0\} \to \partial B_1$ such that $\psi_2 = \text{id}$ on $\partial B_1$.

   Combining $\psi_1$ and $\psi_2$ we can define the map $\psi$ by letting

   $$\psi : (E, x) \to \begin{cases} 
   (\psi_1(E), \frac{x}{1 - t_E}) & \text{if } 1 - t_E \geq |x|, \\
   (\eta(\psi_1(E), 1 - \frac{1 - t_E}{|x|}), \psi_2(x)) & \text{if } 1 - t_E < |x|. 
   \end{cases}$$

3. By the map $\phi^*$ in (3.11), we have $\partial(E_I \times B_1) \simeq \mathcal{P}$.

4. Let $\Psi = \phi^* \circ \psi \circ \phi : A_I \setminus \{B_1\} \to \mathcal{P}$. Then $\Psi$ is a retraction from $A_I \setminus \{B_1\}$ to $\mathcal{P}$.

The following two results are crucial for our later argument, whose proofs are postponed to Section 4.

**Proposition 3.6.** We have the following results.

(i) $H_{k+1}(\mathcal{P}) = H_k(A)$ for all $k \geq 1$.

(ii) There is a long exact sequence

$$\cdots \to H_{k+1}(A) \to H_k(\mathcal{E} \times S^n) \to H_k(\mathcal{E}) \oplus H_k(S^n) \to H_k(A) \to \cdots.$$ 

**Theorem 3.7.** Let $n^* = \frac{n(n+1)}{2}$. The homology group $H_{n^*+n-1}(\mathcal{E}) = \mathbb{Z}$. 

3.3. Selection of a good initial condition.

As mentioned in the introduction, we use a topological method to prove the existence of a special initial condition such that the solution to the Gauss curvature flow (1.4) satisfies the uniform estimate (1.5).

We will employ the modified flow with initial data in $A_I$. For any ellipsoid $N$ such that $\text{Cl}(N) \in A_I$, let $\tilde{M}_N(t)$ be the solution to the modified flow, with the constant $A_0$ given in (3.1). We have the following properties

(1) If $\text{Cl}(N)$ is close to $P$ in Hausdorff distance or on $P$, we have $J(N) \geq A_0$ and so $\tilde{M}_N(t) \equiv N$ for all $t$ (see Lemma 3.2).

(2) If $\text{Cl}(N)$ is close to $B_1(0)$ in Hausdorff distance, then $J(N) < A_0$.

(3) By our definition of the modified flow, if $J(\tilde{M}_N(t)) < A_0$ for all $t \geq 0$, then by Remark 2.7 we have

(3.13) $e_{\tilde{M}_N(t)} \leq \bar{e}, \quad \bar{v} \leq \text{Vol}(\bar{\Omega}_N(t)) \leq 1/\bar{v}, \quad \text{and} \quad B_\bar{d}(0) \subset \bar{\Omega}_N(t) \forall t \geq 0,$

where $\bar{\Omega}_N(t) = \text{Cl}(\tilde{M}_N(t))$, the convex body enclosed by $\tilde{M}_N(t)$. Here the bar over $\Omega$ means that $\bar{\Omega}_N(t)$ is the convex body enclosed by the modified flow $\tilde{M}_N(t)$, not the closure of $\Omega_N(t)$. In our notation, $\Omega_N$ is a closed convex body.

With these properties, we can prove the following key lemma.

**Lemma 3.8.** For every $t > 0$, there exists $N = N_t$ with $\text{Cl}(N) \in A_I$, such that the minimum ellipsoid of $\tilde{M}_N(t)$ is the unit ball $B_1(0)$.

**Proof.** Suppose by contradiction that there exists $t' > 0$ such that, for any $\Omega \in A_I$, $E_N(t') \neq B_1$, where $N = \partial \Omega$ and $E_N(t')$ is the minimum ellipsoid of $\Omega_N(t') := \text{Cl}(\tilde{M}_N(t'))$. We have

(3.14) $E_N(t') \in A_I$.

This is obvious when $\Omega_N(t) \equiv \Omega$ for all $t$. If $\Omega_N(t)$ is not identical to $\Omega$, then (3.14) follows from $\frac{1}{n+1} E_N(t') \subset \Omega_N(t') \subset E_N(t')$, (3.13) and Lemma 3.2.

Hence, we can define a continuous map $T : A_I \to \mathcal{P}$ by

$$\Omega \in A_I \mapsto E_N(t') \in A_I \setminus \{B_1\} \mapsto \Psi(E_N(t')) \in \mathcal{P},$$

where $\Psi$ is given in Lemma 3.5. Note that when $\Omega \in \mathcal{P}$, we have $J(\Omega) \geq A_0$ and thus $E_N(t') = E_N(0) = \Omega$. This implies that $T|_\mathcal{P} = \text{id}_\mathcal{P}$. Hence $T$ is a retraction from $A_I$ to $\mathcal{P}$, and so there is an injection from $H_*(\mathcal{P})$ to $H_*(A_I)$. By (3.9) we then have

$H_k(\mathcal{P}) = 0$ for all $k \geq 1$. 

It follows from Proposition 3.6 (ii) that
\[ H_k(\mathcal{E} \times \mathbb{S}^n) = H_k(\mathcal{E}) \oplus H_k(\mathbb{S}^n) \quad \forall k \geq 1. \]

Computing the left-hand side by the Künneth formula and using the fact \( H_k(\mathbb{S}^n) = \mathbb{Z} \) if \( k = 0 \) or \( k = n \), and \( H_k(\mathbb{S}^n) = 0 \) otherwise, we further obtain
\[ H_k(\mathcal{E}) \oplus H_{k-n}(\mathcal{E}) = H_k(\mathcal{E}) \oplus H_k(\mathbb{S}^n) \quad \forall k \geq 1. \]

However, this contradicts Theorem 3.7 by taking \( k = n^* + 2n - 1 \) in the above.

**Remark 3.9.** Lemma 3.8 asserts that for any \( t > 0 \), there is an initial hypersurface \( \mathcal{N} = \mathcal{N}_t \) such that the minimum ellipsoid \( E_{\mathcal{N}}(t) \) is the unit ball \( B_1(0) \), even in the case when the \( L_p \)-Minkowski problem (1.1) has a unique solution \( \partial B_R(z) \) for \( R \neq 1 \) and \( z \neq 0 \). In fact, our topological argument implies that for any ellipsoid \( E \in \mathcal{A}_I \setminus \mathcal{P} \), and any given time \( t > 0 \), there is an initial hypersurface \( \mathcal{N} = \mathcal{N}_t \) such that \( E_{\mathcal{N}}(t) = E \).

**Remark 3.10.** To derive a contradiction from (3.15), we only need to show that there exists one nontrivial homology group among \( H_k(\mathcal{E}) \) for all \( k \geq 1 \). Theorem 3.7 asserts that \( H_{n^*+n-1}(\mathcal{E}) = \mathbb{Z} \), which suffices for the proof of the key Lemma 3.8. By (4.33) below, \( k = n^* + n - 1 \) is the largest integer such that \( H_k(\mathcal{E}) \neq 0 \). We will not compute other homology groups of \( \mathcal{E} \) in this paper, as they are not needed in our proof.

In the following we prove the convergence of the flow (1.4) with a specially chosen initial condition. Take a sequence \( t_k \to \infty \) and let \( \mathcal{N}_k = \mathcal{N}_{t_k} \) be the initial data from Lemma 3.8. By our choice of \( A_0 \) (see (3.1) and (3.2)), Lemma 3.8 implies that
\[ J(\mathcal{M}_{\mathcal{N}_k}(t_k)) \leq \frac{1}{2} A_0. \]

Hence, by the monotonicity of the functional \( J \), we have
\[ \mathcal{M}_{\mathcal{N}_i}(t) = \mathcal{M}_{\mathcal{N}_i}(t) \quad \forall t \leq t_k. \]

Since \( \text{Cl}(\mathcal{N}_k) \in \mathcal{A}_I \) and \( B_\delta(0) \subset \text{Cl}(\mathcal{N}_k) \), by Blaschke’s selection theorem, there is a subsequence of \( \mathcal{N}_k \) which converges in Hausdorff distance to a limit \( \mathcal{N}_* \) such that \( \text{Cl}(\mathcal{N}_*) \in \mathcal{A}_I \) and \( B_\delta \subset \text{Cl}(\mathcal{N}_*) \).

Next, we show that the flow (1.4) starting from \( \mathcal{N}_* \) satisfying \( J(\mathcal{M}_{\mathcal{N}_*}(t)) < A_0 \) for all \( t \).

**Lemma 3.11.** For any \( t \geq 0 \), we have
\[ J(\mathcal{M}_{\mathcal{N}_*}(t)) \leq \frac{3}{4} A_0. \]

Hence
\[ \mathcal{M}_{\mathcal{N}_*}(t) = \mathcal{M}_{\mathcal{N}_*}(t) \quad \forall t > 0. \]
Proof. For any given \( t > 0 \), since \( N_k \to N^* \) and \( t_k \to \infty \), when \( k \) is sufficiently large such that \( t_k > t \), we have

\[
J(\bar{M}_{N^*}(t)) - J(M_{N_k}(t)) \leq \frac{1}{4}A_0.
\]

By the monotonicity of the functional \( J \),

\[
J(M_{N_k}(t_k)) \leq J(M_{N_k}(t_k)).
\]

Combining above two inequalities with (3.16), we obtain that

\[
J(\bar{M}_{N*}(t)) = J(\bar{M}_{N*}(t)) - J(M_{N_k}(t)) + J(M_{N_k}(t_k)) \leq J(\bar{M}_{N*}(t)) - J(M_{N_k}(t)) + J(M_{N_k}(t_k)) \leq \frac{1}{4}A_0 + \frac{1}{2}A_0 = \frac{3}{4}A_0.
\]

This completes the proof. \( \square \)

3.4. Convergence of the flow and existence of solutions to (1.1).

Let \( \Omega_{N^*}(t) = \text{Cl}(M_{N^*}(t)) \) and \( u(\cdot, t) \) be its support function. By Lemma 3.11 \( M_{N^*}(t) \) satisfies (3.13). Hence by (3.13) we infer that

\[
d \leq u(x, t) \leq C \ \forall \ (x, t) \in S^n \times [0, \infty).
\]

where \( C = (n + 1)/(\bar{v}\omega_{n-1}\bar{a}^n) \). By the convexity,

\[
|\nabla u(x, t)| \leq \max_{S^n} u(\cdot, t) \leq C \ \forall \ (x, t) \in S^n \times [0, \infty).
\]

Namely condition (2.11) holds. By Subsection 2.2, we obtain the existence of solutions to (1.1) as follows.

Proof of Theorem 1.1. Denote \( \mathcal{M}(t) = M_{N^*}(t) \) and \( J(t) = J(M(t)) \). By Lemma 2.1 and Lemma 3.11

\[
J(t) < A_0 \text{ and } J'(t) \geq 0 \ \forall \ t \geq 0.
\]

Therefore,

\[
\int_0^\infty J'(t) dt \leq \limsup_{T \to \infty} J(T) - J(0) \leq A_0.
\]

This implies that there exists a sequence \( t_i \to \infty \) such that

\[
J'(t_i) = \int_{S^n} \left[ \left( \frac{1}{K} - f u^{p-1} \right)^2 uK \right]_{t=t_i} d\sigma_{S^n} \to 0.
\]

Passing to a subsequence, we obtain by the a priori estimates (2.13) that \( u(\cdot, t_i) \to u_\infty \) in \( C^{3, \alpha}(S^n) \)-topology and \( u_\infty \) satisfies (1.1). \( \square \)
Corollary 1.2 follows from Theorem 1.1 by an approximation argument. To prove Corollary 1.2, we first point out that all arguments in Subsections 2.1 and 3.1–3.3 depend on \( \inf f \) and \( \sup f \) but are independent of the smoothness of \( f \). Therefore, the constants \( \bar{e}, \bar{v}, \bar{d} \) in (3.13) are independent of the smoothness of \( f \).

**Proof of Corollary 1.2.** Choose a sequence of functions \( f_j \in C^\infty(S^n) \) such that \( \inf_{S^n} f \leq f_j \leq \sup_{S^n} f \) and \( f_j \to f \) a.e. (such as the mollifications of \( f \)). By our proof of Theorem 1.1, there exists an initial convex hypersurface \( N_j \in \mathcal{A}_I \) such that the solution \( M_j(t) := M_{N_j}(t) \) to (1.4) converges to a solution \( M_j \) of (1.1) with \( f = f_j \), and \( M_j(t) \) satisfies (3.13), uniformly for all \( j \) and \( t \). Hence \( M_j \) satisfies the estimates in (3.13). Passing to a subsequence, we may assume that \( M_j \) converges in Hausdorff distance to a limit \( M \). Then \( M \) satisfies (3.13). By the weak convergence of the Monge-Ampère equation, \( M \) is a weak solution of (1.1). By the regularity theory of the Monge-Ampère equation, \( M \) is strictly convex and \( C^{1,\alpha} \) smooth for some \( \alpha \in (0, 1) \). □

4. Proofs of Proposition 3.6 and Theorem 3.7

In this section, we prove Proposition 3.6 and Theorem 3.7.

4.1. Proof of Proposition 3.6.

**Part (i).** Note that for any \( \tau \in [\bar{v}, \bar{v}^{-1}] \), we have

\[
\{ E \in \mathcal{A}_I : \text{Vol}(E) = \tau \} \simeq \check{A}, \\
\{ E \in \mathcal{A}_I : \text{Vol}(E) = \tau; \ e_E = \bar{e}, \text{ or } O \in \partial E \} \simeq \check{A}.
\]

Hence, \( \mathcal{P} \) consists of three components (up to homeomorphism)

\[ \mathcal{A} \times [\bar{v}, \bar{v}^{-1}], \ \check{A} \times \{ \bar{v} \}, \ \text{and} \ \check{A} \times \{ \bar{v}^{-1} \}. \]

Since \( \mathcal{A} \times [\bar{v}, \bar{v}^{-1}] \) is topologically a cylinder with base \( \mathcal{A} \), and \( \mathcal{A} \) can be viewed as the boundary of \( \check{A} \), we see that \( \mathcal{P} \) can be viewed as attaching two copies of \( \check{A} \) along the boundary of \( \mathcal{A} \times [\bar{v}, \bar{v}^{-1}] \). As \( \check{A} \) is contractable, we conclude that \( \mathcal{P} \) is homotopy-equivalent to \( \text{SA} \) (the suspension of \( \mathcal{A} \)), which is the quotient of \( \mathcal{A} \times [\bar{v}, \bar{v}^{-1}] \) obtained by collapsing \( \mathcal{A} \times \{ \bar{v} \} \) to one point and \( \mathcal{A} \times \{ \bar{v}^{-1} \} \) to another point. Hence the two spaces have the same homology \( H_*(\mathcal{P}) = H_*(SA) \). It is known from [21] that

\[ H_{k+1}(SA) = H_k(\mathcal{A}), \text{ for all } k \geq 1. \]

This completes the proof. □
Part (ii). Let $\mathcal{B}$ be the set of unit balls such that the origin lies on the boundary of the ball. Consider the following subspaces of $\mathcal{A}$:

$$\mathcal{A}_1 = \mathcal{A} \setminus \mathcal{B}, \quad \mathcal{A}_2 = \mathcal{A} \setminus \mathcal{E}, \quad \text{and} \quad \mathcal{A}_3 = \mathcal{A}_1 \cap \mathcal{A}_2.$$ 

We have the Mayer-Vietories sequence for the decomposition $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$:

(4.1) 

$$\cdots \to H_{k+1}(\mathcal{A}) \to H_k(\mathcal{A}_3) \to H_k(\mathcal{A}_1) \oplus H_k(\mathcal{A}_2) \to H_k(\mathcal{A}) \to \cdots$$

Let $L = (0, 1] \times \{1\} \cup (\{1\} \times [0, 1]) \subset \mathbb{R}^2$. Denote

$$L_1 = \{(s, \rho) \in L : s > 0\}, \quad L_2 = \{(s, \rho) \in L : \rho > 0\}, \quad \text{and} \quad L_3 = L_1 \cap L_2.$$ 

Let $G_1 : L_1 \times [0, 1] \to L_1$ be a strong deformation retraction from $L_1$ onto the point $(1, 0)$; $G_2 : L_2 \times [0, 1] \to L_2$ be a strong deformation retraction from $L_2$ onto $(0, 1)$; and $G_3 : L_3 \times [0, 1] \to L_3$ be a strong deformation retraction from $L_3$ onto $(1, 1)$. Denote by $G_{i,1}$ and $G_{i,2}$ the components of the map $G_i$ such that $G_i(s, \rho, t) = (G_{i,1}(s, \rho, t), G_{i,2}(s, \rho, t))$, where $1 \leq i \leq 3$.

Given $E \in \mathcal{A}$, we take $(E', \xi, s, \rho) \in \mathcal{E} \times S^n \times L$ such that $\phi(E) = (\hat{E}'(s), \rho \xi)$ with $\phi$ being the map (3.10). Define $\mathcal{G}_i : \mathcal{A}_i \times [0, 1] \to \mathcal{A}_i$ by

$$\mathcal{G}_i(E, t) = \phi^*(\hat{E}'(G_{i,1}(s, \rho, t)), G_{i,2}(s, \rho, t)\xi).$$

Then $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ are deformation retractions from $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ onto $\mathcal{E}$, $\mathcal{B}$ and $\mathcal{A}'$ respectively, where

$$\mathcal{A}' = \{E \in \mathcal{A} : e_E = \bar{e} \text{ and } O \in \partial E\}.$$ 

Therefore,

$$H_*(\mathcal{A}_1) = H_*(\mathcal{E}), \quad H_*(\mathcal{A}_2) = H_*(\mathcal{B}), \quad \text{and} \quad H_*(\mathcal{A}_3) = H_*(\mathcal{A}').$$

Inserting these identities into (4.1), we obtain the following long exact sequence

(4.2) 

$$\cdots \to H_{k+1}(\mathcal{A}) \to H_k(\mathcal{A}') \to H_k(\mathcal{E}) \oplus H_k(\mathcal{B}) \to H_k(\mathcal{A}) \to \cdots.$$ 

On the other hand, the maps $\phi$ and $\phi^*$ (see (3.10) and (3.11)) yield the homeomorphisms

(4.3) 

$$\mathcal{B} \simeq S^n \text{ and } \mathcal{A}' \simeq \mathcal{E} \times S^n.$$ 

Our conclusion follows by combining (4.2) and (4.3). \[\square\]

The remaining of this section is devoted to the proof of Theorem 3.7.
4.2. **Proof of Theorem [3.7]**

In the following, we always take \( n^* = \frac{n(n+1)}{2} \), which is the dimension of the Lie group \( SO(n+1) \).

For \( n = 1 \), the proof of Theorem [3.7] is straightforward. In this case, the lengths of semi-axes \( r_1 = r_1(E) \) and \( r_2 = r_2(E) \) of \( E \in \mathcal{E} \) satisfy \( r_1r_2 = 1 \) and \( r_2 = \bar{e}r_1 \). Hence,

\[
    r_1(E) = \frac{1}{\sqrt{\bar{e}}} \quad \text{and} \quad r_2(E) = \sqrt{\bar{e}}, \quad \forall E \in \mathcal{E}.
\]

Therefore, each element \( E \in \mathcal{E} \) is determined by the major axis of \( E \). It follows that \( \mathcal{E} \) is homeomorphic to \( RP^1 \). As a result,

\[
    H_1(\mathcal{E}) = H_1(RP^1) = \mathbb{Z}.
\]

This proves Theorem [3.7] when \( n = 1 \), since \( n^* = \frac{n(n+1)}{2} = 1 \).

In the following we deal with the case \( n \geq 2 \). First we introduce some notations. Let

\[
    H_1 = \{(x_2, \cdots, x_n) \in \mathbb{R}^{n-1} : x_2 \geq 1\},
\]

\[
    H_i = \{(x_2, \cdots, x_n) \in \mathbb{R}^{n-1} : x_{i+1} \geq x_i\}, \quad i = 2, \cdots, n-1,
\]

\[
    H_n = \{(x_2, \cdots, x_n) \in \mathbb{R}^{n-1} : \bar{e} \geq x_n\}.
\]

Then \( H_i \) are closed half spaces of \( \mathbb{R}^{n-1} \), \( i = 1, 2, \cdots, n \). Denote

\[
    (4.4) \quad \Delta_{n-1} = \bigcap_{i=1}^{n} H_i.
\]

If \( n = 2 \), then \( \Delta_{n-1} = \{1 \leq x_2 \leq \bar{e}\} \subset \mathbb{R} \) is an interval. If \( n = 3 \), then \( \Delta_{n-1} = \{(x_2, x_3) \in \mathbb{R}^2 : 1 \leq x_2 \leq x_3 \leq \bar{e}\} \) is a triangle. For \( n \geq 3 \), \( \Delta_{n-1} \) is an \((n-1)\)-simplex in \( \mathbb{R}^{n-1} \).

Denote by

\[
    F_i = \partial H_i \cap \Delta_{n-1},
\]

a face of \( \Delta_{n-1} \), \( i = 1, \cdots, n \). We also denote by \( \Delta_{n-1}^{(i)} \) the subset of \( \Delta_{n-1} \), obtained by removing the face \( F_i \) from \( \Delta_{n-1} \), namely

\[
    (4.5) \quad \Delta_{n-1}^{(i)} = \Delta_{n-1} \setminus F_i.
\]

There is a natural projection \( \pi : \mathcal{E} \to \Delta_{n-1} \), given by

\[
    E \mapsto \pi(E) = (\tilde{r}_2(E), \cdots, \tilde{r}_n(E)),
\]

where \( r_i(E) \) are lengths of semi-axes of \( E \) satisfying \( r_1(E) \leq r_2(E) \leq \cdots \leq r_{n+1}(E) \) and

\[
    \tilde{r}_i(E) = \frac{r_i(E)}{r_1(E)}, \quad i = 2, \cdots, n.
\]
Note that \( r_{n+1}(E)/r_1(E) = \bar{c} \) is a fixed constant for all \( E \in \mathcal{E} \).

The mapping \( \pi \) can be written as a composition of two mappings \( \pi_1 \) and \( \pi_2 \). Namely,
\[
\pi_1 : \mathcal{E} \to \mathbb{L}_{n+1}, \quad \text{given by } E \mapsto (r_1(E), \ldots, r_{n+1}(E)),
\]
\[
\pi_2 : \mathbb{L}_{n+1} \to \Delta_{n-1}, \quad \text{given by } (x_1, \ldots, x_{n+1}) \mapsto \left( \frac{x_2}{x_1}, \ldots, \frac{x_{n+1}}{x_1} \right),
\]
where
\[
\mathbb{L}_{n+1} = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : 0 < x_1 \leq \cdots \leq x_{n+1}, \prod_{i=1}^{n+1} x_i = 1, \ x_{n+1} = \bar{c} x_1 \right\}.
\]
It is readily seen that \( \pi_2 \) is a bijection and \( \pi = \pi_2 \circ \pi_1 \) is surjective.

**Remark 4.1.** For any given ellipsoid \( E \in \mathcal{E} \), there is a unique positive definite, unimodular matrix \( A \) such that \( E = \{ x \in \mathbb{R}^{n+1} : x' \cdot Ax = 1 \} \). Let \( \lambda_1 \geq \cdots \geq \lambda_{n+1} \) be the eigenvalues of \( A \). Then \( \pi_1(E) = (\lambda_1^{-\frac{1}{2}}, \ldots, \lambda_{n+1}^{-\frac{1}{2}}) \). Hence, for any point (vector) \( \bar{r} = (r_1, \ldots, r_{n+1}) \in \mathbb{R}^{n+1} \) such that \( 0 < r_1 \leq \cdots \leq r_{n+1} \) and \( \prod_{i=1}^{n+1} r_i = 1 \), we can define \( \pi_1^{-1}(\bar{r}) \) as the set of ellipsoids \( E \in \mathcal{E} \) such that the lengths of the semi-axes of \( E \) are equal to \( r_1, \ldots, r_{n+1} \), namely, the eigenvalues of the ellipsoid matrix \( A \) are equal to \( r_1^{-2}, \ldots, r_{n+1}^{-2} \). For convenience, we say that a component \( r_i \) of the vector \( \bar{r} \) is single if \( r_{i-1} < r_i < r_{i+1} \), and that a component \( r_i \) has multiplicity \( k \) (\( k \geq 2 \)) if \( r_{i-1} < r_i = \cdots = r_{i+k-1} < r_{i+k} \).

For \( 1 \leq i \leq n \), consider the following subsets of \( \mathcal{E} \):
\[
\mathcal{E}_i = \pi^{-1}(\Delta_{n-1}^{(i)}).
\]
By our notation \( \Delta_{n-1}^{(i)} \) in (4.5), \( \mathcal{E}_i \) is a subset of \( \mathcal{E} \) and can be written as
\[
\mathcal{E}_i = \{ E \in \mathcal{E} : r_i(E) \neq r_{i+1}(E) \}.
\]
For any \( 1 \leq j_1 < j_2 \cdots < j_l \leq n \), we denote
\[
\mathcal{E}_{j_1;j_2;\cdots;j_l} = \bigcup_{s=1}^{l} \mathcal{E}_{j_s} \quad \text{and} \quad \mathcal{W}_{j_1;j_2;\cdots;j_l} = \bigcap_{s=1}^{l} \mathcal{E}_{j_s}.
\]
For brevity we write \( \mathcal{E}_{j_1;j_2;\cdots;j_l} = \mathcal{E}_{j_1;j_2;\cdots;j_l} \) and \( \mathcal{W}_{j_1;j_2;\cdots;j_l} = \mathcal{W}_{j_1;j_2;\cdots;j_l} \). We see that
\[
\mathcal{E}_{j_1;j_2;\cdots;j_l} = \{ E \in \mathcal{E} : r_i(E) \neq r_{i+1}(E) \text{ for some } i = j_1, \cdots, j_l \},
\]
\[
\mathcal{W}_{j_1;j_2;\cdots;j_l} = \{ E \in \mathcal{E} : r_i(E) \neq r_{i+1}(E) \text{ for all } i = j_1, \cdots, j_l \}.
\]

For convenience of the reader, we first prove Theorem 3.7 for lower dimensions \( n = 2 \) and \( n = 3 \), and then for higher dimensions. One may also skip subsections 4.3 and 4.4 and go through subsection 4.5 for general case directly. Our method is based on dividing \( \mathcal{E} \) into suitable parts and employing the Mayer-Vietoris sequences [21].
4.3. Dimension \( n = 2 \).

For \( n = 2 \), the simplex (4.4) is \( \Delta_1 = [1, \bar{e}] \). Recall that
\[
\Delta_1^{(1)} = (1, \bar{e}], \quad \Delta_1^{(2)} = [1, \bar{e}), \quad F_1 = \{1\}, \quad F_2 = \{\bar{e}\}.
\]

The subsets \( E_1 = \pi^{-1}(\Delta_1^{(1)}) \) and \( E_2 = \pi^{-1}(\Delta_1^{(2)}) \) of \( E \) are given by
\[
E_1 = \{ E \in E : r_1(E) < r_2(E)\}, \quad E_2 = \{ E \in E : r_2(E) < r_3(E)\}.
\]

We have the Mayer-Vietoris sequence for the decomposition \( E = E_1 \cup E_2 \):
\[
\cdots \rightarrow H_k(E_1) \oplus H_k(E_2) \rightarrow H_k(E) \rightarrow H_{k-1}(E_1 \cap E_2) \rightarrow H_{k-1}(E_1) \oplus H_{k-1}(E_2) \rightarrow \cdots
\]

In order to prove Theorem 3.7 by using (4.8), we compute the homology groups of \( E_1, E_2 \) and \( E_1 \cap E_2 \).

The following lemma helps us to simplify the computation of the homology groups of \( E_1, E_2 \) and \( W_{12} = E_1 \cap E_2 \).

**Lemma 4.2.** The following statements hold:

1. For \( i = 1, 2 \), \( \pi^{-1}(F_{3-i}) \) is a deformation retract of \( E_i \).
2. For any given point \( P \in \Delta_1^{(1)} \cap \Delta_1^{(2)} \), \( \pi^{-1}(P) \) is a deformation retract of \( E_1 \cap E_2 \).

As a direct consequence, we have for all \( k \geq 1 \),
\[
H_k(E_i) = H_k(\pi^{-1}(F_{3-i})), \quad i = 1, 2,
\]
\[
H_k(E_1 \cap E_2) = H_k(\pi^{-1}(P)), \quad P \in \Delta_1^{(1)} \cap \Delta_1^{(2)}.
\]

**Proof.** For part (1), it suffices to consider the case when \( i = 1 \). Recall that \( \Delta_1^{(1)} = (1, \bar{e}] \) and \( F_2 = \{\bar{e}\} \). Let \( G : \Delta_1^{(1)} \times [0, 1] \rightarrow \Delta_1^{(1)} \) be a strong deformation retraction of \( \Delta_1^{(1)} \) onto \( F_2 \) (such deformation clearly exists). We then define \( G : E_1 \times [0, 1] \rightarrow E_1 \) as follows. For any \( E \in E_1 \), let \( G(E, t) \) be the ellipsoid such that its axial directions are all the same with \( E \), and its axial lengths are determined by
\[
(r_1(t), r_2(t), r_3(t)) = \pi_2^{-1} \circ G(\pi(E), t).
\]

Namely, we continuously deform the axial lengths of \( E \) while keeping the directions of its axes so that the resulting ellipsoid belongs to \( \pi^{-1}(F_2) \). It is easy to check that \( G \) is a deformation retraction from \( E_1 \) onto \( \pi^{-1}(F_2) \).

For part (2), the argument is similar. \( \square \)

Since \( \pi^{-1}(F_1) = \{ E \in E : r_1(E) = r_2(E)\} \) and \( \pi^{-1}(F_2) = \{ E \in E : r_2(E) = r_3(E)\} \), we see that \( \pi^{-1}(F_1) \) and \( \pi^{-1}(F_2) \) are both homeomorphic to \( RP^2 \). Using Lemma 4.2 and the
homology of $RP^2$, we have
\begin{equation}
H_k(\mathcal{E}_i) = 0, \ \forall \ k \geq 3.
\end{equation}

Next, we study the homology groups of $W_{12} = \mathcal{E}_1 \cap \mathcal{E}_2$.

**Lemma 4.3.** For any given $\vec{r} = (r_1, r_2, r_3) \in \mathbb{L}_3$ with $r_1 < r_2 < r_3$, denote $\mathcal{E}_{\vec{r}} = \pi_1^{-1}(\vec{r})$, i.e.,
\[ \mathcal{E}_{\vec{r}} = \{ E \in \mathcal{E} : r_i(E) = r_i, 1 \leq i \leq 3 \}. \]

Then $\mathcal{E}_{\vec{r}} \simeq SO(3)/\Gamma$, where $\Gamma = \{ A \in SO(3) : A = \text{diag}\{\pm 1, \pm 1, \pm 1\} \}$ is a finite discrete subgroup of $SO(3)$. It follows that $\mathcal{E}_{\vec{r}}$ is orientable and so
\[ H_3(\mathcal{E}_{\vec{r}}) = \mathbb{Z}. \]

**Proof.** We consider the following Lie group action on $\mathcal{E}_{\vec{r}}$:
\[ SO(3) \times \mathcal{E}_{\vec{r}} \to \mathcal{E}_{\vec{r}}, \ (g, E) \mapsto T_g(E), \]
where $T_g$ is the linear transformation of $\mathbb{R}^3$ associated to $g$. This groups action is clearly transitive and the stabiliser of $E_{\vec{r}} = \{ x \in \mathbb{R}^3 : \sum_{i=1}^{3} x_i^2 / r_i^2 \leq 1 \}$ is given by $\Gamma$. The subgroup $\Gamma$ is isomorphic to the dihedral group $D_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. It follows that $\mathcal{E}_{\vec{r}}$ is homeomorphic to $SO(3)/\Gamma [31]$. As $\Gamma$ is finite, we find that $SO(3)/\Gamma$ is an orientable manifold. This together with $\dim(SO(3)/\Gamma) = 3$ yields $H_3(\mathcal{E}_{\vec{r}}) = H_3(SO(3)/\Gamma) = \mathbb{Z}$. \qed

For any $P \in \Delta_1^{(1)} \cap \Delta_1^{(2)}$, let $\pi_2^{-1}(P) = (r_1, r_2, r_3) \in \mathbb{L}_3$ such that $r_1 < r_2 < r_3$. By Lemma 4.2 and Lemma 4.3, we obtain that
\begin{equation}
H_3(\mathcal{E}_1 \cap \mathcal{E}_2) = \mathbb{Z}.
\end{equation}

Now, we are ready to prove Theorem 3.7 for $n = 2$.

Taking $k = 4$ in (4.8) and using (4.10), we obtain the following short exact sequence
\[ 0 \to H_4(\mathcal{E}) \to H_3(\mathcal{E}_1 \cap \mathcal{E}_2) \to 0. \]

As a result,
\[ H_4(\mathcal{E}) = H_3(\mathcal{E}_1 \cap \mathcal{E}_2). \]

This together with (4.11) gives $H_4(\mathcal{E}) = \mathbb{Z}$, and thus proves Theorem 3.7 for $n = 2$. 

4.4. Dimension $n = 3$.

When $n = 3$, the simplex $\Delta_2 = \{(x_2, x_3) \in \mathbb{R}^2 : 1 \leq x_2 \leq x_3 \leq \bar{e}\}$ is a triangle, and $F_i$, $1 \leq i \leq 3$, are the sides of this triangle $\Delta_2$. Recall that $\mathcal{E}_i = \pi^{-1}(\Delta_2^{(i)})$, $1 \leq i \leq 3$, is given by

$$\mathcal{E}_i = \{ E \in \mathcal{E} : r_i(E) < r_{i+1}(E) \}.$$

Arguing similarly as in Lemma 4.2, we have the following result.

**Lemma 4.4.** The following statements hold:

1. For $i = 1, 2, 3$, $\pi^{-1}(P)$ is a deformation retract of $\mathcal{E}_i$, where $P = F_{j_1} \cap F_{j_2}$ and $j_1$ and $j_2$ are distinct such that $\{j_1, j_2\} = \{1, 2, 3\} \setminus \{i\}$.
2. For any point $P \in \bigcap_{k=1}^3 \Delta_2^{(k)}$, $\pi^{-1}(P)$ is a deformation retract of $\mathcal{W}_{123} = \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$.

Suppose that $P = F_1 \cap F_2$ and $\pi_2^{-1}(P) = (r_1, r_2, r_3, r_4) \in \mathbb{R}_4$. We then have $r_1 = r_2 = r_3$. Hence, $\pi^{-1}(P)$ is homeomorphic to $RP^3$. By Lemma 4.4, we conclude that $\mathcal{E}_3$ is homotopy equivalent to $RP^3$. Similarly, we obtain that $\mathcal{E}_1$ is homotopy equivalent to $RP^3$ and $\mathcal{E}_2$ is homotopy equivalent to the Grassmannian $G(2, 4)$. Using the homology of $RP^3$ and $G(2, 4)$, we have for $1 \leq i \leq 3$

$$H_k(\mathcal{E}_i) = 0, \quad \text{if } k \geq 5.$$  \hspace{1cm} (4.12)

By part (2) of Lemma 4.4 and an analog of Lemma 4.3 for $n = 3$ (see Lemma 4.7 for the general case), we conclude that $\mathcal{W}_{123}$ is homotopy equivalent to $SO(4)/\Gamma$, where $\Gamma = \{ A \in SO(4) : A = \text{diag}\{\pm 1, \pm 1, \pm 1, \pm 1\} \}$ is a finite subgroup of $SO(4)$. Since $SO(4)/\Gamma$ is orientable and has dimension 6, we have

$$H_6(\mathcal{W}_{123}) = H_6(SO(4)/\Gamma) = \mathbb{Z}. \hspace{1cm} (4.13)$$

Next, we consider the homology groups of $\mathcal{W}_{12} = \mathcal{E}_1 \cap \mathcal{E}_2$, $\mathcal{W}_{13} = \mathcal{E}_1 \cap \mathcal{E}_3$ and $\mathcal{W}_{23} = \mathcal{E}_2 \cap \mathcal{E}_3$. Recall that

$$\mathcal{W}_{12} = \{ E \in \mathcal{E} : r_1(E) < r_2(E) < r_3(E) \}.$$  \hspace{1cm (4.12)}

Take any point $P \in \Delta_2^{(1)} \cap \Delta_2^{(2)} \cap F_3$. Clearly, $\pi^{-1}(P) \in \mathcal{W}_{12}$. Arguing as in Lemma 4.2, we see that $\pi^{-1}(P)$ is a deformation retract of $\mathcal{W}_{12}$. Assume that $\pi_2^{-1}(P) = (r_1, r_2, r_3, r_4) := \vec{r}$. We then have $r_1 < r_2 < r_3 = r_4$. Denote $\mathcal{E}_{\vec{r}} = \pi^{-1}(P)$. We consider the Lie group action on $\mathcal{E}_{\vec{r}}$ as in Lemma 4.3. $SO(4) \times \mathcal{E}_{\vec{r}} \to \mathcal{E}_{\vec{r}}$. The stabiliser of this group action is given by $S(O(1) \times O(1) \times O(2))$, i.e., the set of matrices in $O(1) \times O(1) \times O(2)$ with determinant 1. Therefore, we conclude that

$$H_k(\mathcal{W}_{12}) = H_k(SO(4)/S(O(1) \times O(1) \times O(2))), \forall k.$$
As the space $SO(4)/S(O(1) \times O(1) \times O(2))$ has dimension 5, we get
$$H_k(W_{12}) = 0, \forall k \geq 6.$$ 

The above discussion yields the following result.

**Lemma 4.5.** Let $1 \leq j_1 < j_2 \leq 3$. Then

1. $\pi^{-1}(P)$ is a deformation retract of $W_{j_1j_2}$, where $P \in \Delta_2^{(j_1)} \cap \Delta_2^{(j_2)} \cap F_i$ with $\{i\} = \{1, 2, 3\} \setminus \{j_1, j_2\}$.
2. $H_k(W_{j_1j_2}) = 0$ for all $k \geq 6$.

The Mayer-Vietoris sequence for the decomposition $E = E_1 \cup (E_2 \cup E_3)$ gives
$$\cdots \rightarrow H_k(E_1) \oplus H_k(E_2 \cup E_3) \rightarrow H_k(E) \rightarrow H_{k-1}(W_{12} \cup W_{13}) \rightarrow H_{k-1}(E_1) \oplus H_{k-1}(E_2 \cup E_3) \cdots.$$ 

For the decomposition $E_{23} = E_2 \cup E_3$, we have
$$\cdots \rightarrow H_k(E_2) \oplus H_k(E_3) \rightarrow H_k(E_2 \cup E_3) \rightarrow H_{k-1}(W_{23}) \rightarrow H_{k-1}(E_2) \oplus H_{k-1}(E_3) \rightarrow \cdots.$$ 

Taking $k \geq 7$ in (4.15) and using (4.12) and the part (2) in Lemma 4.5, we see that
$$H_k(E_2 \cup E_3) = 0, \quad \forall k \geq 7.$$ 

Letting $k = 8$ in (4.14) and inserting (4.12) and (4.16) in the exact sequence, we obtain that
$$H_8(E) = H_7(W_{12} \cup W_{13}).$$ 

The proof reduces to the computation of the right hand side of (4.17), i.e., $H_7(W_{12} \cup W_{13})$. The Mayer-Vietoris sequence
$$\cdots \rightarrow H_7(W_{12}) \oplus H_7(W_{13}) \rightarrow H_7(W_{12} \cup W_{13}) \rightarrow H_6(W_{123}) \rightarrow H_6(W_{12}) \oplus H_6(W_{13}) \rightarrow \cdots$$
together with the part (2) in Lemma 4.5 yields that
$$H_7(W_{12} \cup W_{13}) = H_6(W_{123}).$$

Combining (4.17), (4.18) and (4.13), we complete the proof of Theorem 3.7 for $n = 3$.

### 4.5. General dimensions.

Now, we consider the general dimensions. Before we use the Mayer-Vietoris sequence and the induction arguments, we first prove several lemmas concerning the homology groups of $E_i$, $E_{j_1j_2\ldots j_l}$ and $W_{j_1j_2\ldots j_l}$ (see notations in (4.6) and (4.7)). Recall the following notations:

1. If $P \in \text{Int} \Delta_{n-1}$, then
$$\pi^{-1}(P) \in W_{12\ldots n} = \{E \in E : r_i(E) \neq r_j(E) \text{ whenever } i \neq j\}.$$
(b) If \( P \in (\cap_{s=1}^{l} \Delta_{n-1}^{(j_s)}) \cap (\cap_{i \neq j_1, \ldots, j_l} F_i) \), where \( 1 \leq l < n \) and \( 1 \leq j_1 < \cdots < j_l \leq n \), then
\[
\pi^{-1}(P) \in \mathcal{W}_{j_1 j_2 \cdots j_l} \cap \{ E \in \mathcal{E} : r_i(E) = r_{i+1}(E) \text{ for all } i \neq j_s, \ s = 1, \ldots, l \}.
\]

The following lemma shows that \( \pi^{-1}(P) \) in cases (a) and (b) above are deformation retracts of \( \mathcal{W}_{12 \cdots n} \) and \( \mathcal{W}_{j_1 j_2 \cdots j_l} \), respectively. It is the generalisation of Lemmas 4.3, 4.4 and 4.5 for high dimensions.

**Lemma 4.6.** The two statements below hold.

(i) For any given \( P \in \text{Int} \Delta_{n-1} \), \( \pi^{-1}(P) \) is a deformation retract of \( \mathcal{W}_{12 \cdots n} \).

(ii) For \( 1 \leq l < n \) and \( 1 \leq j_1 < \cdots < j_l \leq n \), if \( P \in (\cap_{s=1}^{l} \Delta_{n-1}^{(j_s)}) \cap (\cap_{i \neq j_1, \ldots, j_l} F_i) \).

Then \( \pi^{-1}(P) \) is a deformation retract of \( \mathcal{W}_{j_1 j_2 \cdots j_l} \).

**Proof.** For (i), let
\[
G : \text{Int} \Delta_{n-1} \times [0, 1] \to \text{Int} \Delta_{n-1}
\]
be a deformation retraction of \( \text{Int} \Delta_{n-1} \) onto \( P \). Define \( \mathcal{G} : \mathcal{W}_{12 \cdots n} \times [0, 1] \to \mathcal{W}_{12 \cdots n} \) as follows. For any \( E \in \mathcal{W}_{12 \cdots n} \), let \( \mathcal{G}(E, t) \) be the ellipsoid such that its axial directions are all the same as \( E \), and its axis lengths \( r_i(t) \) are determined by
\[
(4.19) \quad (r_1(t), \ldots, r_{n+1}(t)) = \pi_2^{-1} \circ G(\pi(E), t).
\]
It can be verified that \( \mathcal{G} \) is a deformation retraction from \( \mathcal{W}_{12 \cdots n} \) onto \( \pi^{-1}(P) \) that we want.

For (ii), the argument is similar. Denote \( W = \cap_{s=1}^{l} \Delta_{n-1}^{(j_s)} \). Now let \( G : W \times [0, 1] \to W \) be a deformation retraction form \( W \) onto \( P \) such that
- \( G(W \cap (\cap_{i \neq j_1, \ldots, j_l} F_i), t) \subset W \cap (\cap_{i \neq j_1, \ldots, j_l} F_i) \) for all \( t \in [0, 1] \);
- \( G(Q, t) \in W \setminus (\cap_{i \neq j_1, \ldots, j_l} F_i) \) for all \( t \in [0, 1] \) and \( Q \in W \setminus (\cap_{i \neq j_1, \ldots, j_l} F_i) \).

We then define the deformation retraction \( \mathcal{G} : \mathcal{W}_{j_1 j_2 \cdots j_l} \times [0, 1] \to \mathcal{W}_{j_1 j_2 \cdots j_l} \) as follows: \( \mathcal{G}(E, t) \) keeps all the axis-directions of \( E \) unchanged but its axis-lengths \( r_i(t) \) are again given by \( (4.19) \). \( \square \)

The next lemma gives the general case of Lemma 4.3

**Lemma 4.7.** Suppose that \( r_i, i = 1, 2, \cdots, n+1 \), are distinct positive constants such that \( \vec{r} = (r_1, \cdots, r_{n+1}) \in \mathbb{L}_{n+1} \). Denote \( \mathcal{E}_{\vec{r}} = \pi_1^{-1}(\vec{r}) \). Then \( \mathcal{E}_{\vec{r}} \) is homeomorphic to \( \text{SO}(n+1)/\Gamma \), where \( \Gamma = \{ A \in \text{SO}(n+1) : A = \text{diag}\{\pm 1, \cdots, \pm 1\} \} \) is a finite discrete subgroup of \( \text{SO}(n+1) \). As a result \( \mathcal{E}_{\vec{r}} \) is orientable and has dimension \( n^* \), and
\[
H_n^*(\mathcal{E}_{\vec{r}}) = \mathbb{Z}.
\]
**Proof.** Consider the Lie group action on $E_{\vec{r}}$:

\begin{equation}
SO(n+1) \times E_{\vec{r}} \to E_{\vec{r}}, \quad (g, E) \mapsto T_g(E),
\end{equation}

where $T_g$ represents the linear transformation of $\mathbb{R}^{n+1}$ associated to $g$. Then the stabiliser of

\begin{equation}
E_{\vec{r}} = \left\{ x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2/r_i^2 \leq 1 \right\}
\end{equation}

is given by $\Gamma = \{ A \in SO(n+1) : A = \text{diag}\{\pm 1, \cdots, \pm 1\}\}$, the set of diagonal $(n+1) \times (n+1)$ matrices with determinant 1. Since the group action is transitive and $\Gamma$ is closed in $SO(n+1)$, we conclude that $E_{\vec{r}}$ has a smooth manifold structure such that

\begin{equation}
\mathcal{F} : SO(n+1)/\Gamma \to E_{\vec{r}}, \quad \mathcal{F}(g\Gamma) \mapsto T_g(E_{\vec{r}}),
\end{equation}

is a diffeomorphism. As such manifold structure yields the same topology of $E_{\vec{r}}$ induced by the Hausdorff metric, we see that $E_{\vec{r}} \simeq SO(n+1)/\Gamma$ and these two spaces have the same homology.

Since $\Gamma$ is a finite discrete group and the orientation of $SO(n+1)$ is preserved by all the diffeomorphisms of $\Gamma$, we conclude that $SO(n+1)/\Gamma$ is an orientable closed manifold. As $\dim(SO(n+1)/\Gamma) = n^*$, we see that

\begin{equation}
H_{n^*}(SO(n+1)/\Gamma) = \mathbb{Z}.
\end{equation}

This completes the proof. Indeed, one can also show that $SO(n+1)/\Gamma$ is diffeomorphic to the complete flag variety in $\mathbb{R}^{n+1}$ (see [11]).

The following result is corresponding to the general case of Lemma 4.5.

**Lemma 4.8.** Suppose that $\vec{r} = (r_1, \cdots, r_{n+1}) \in \mathbb{L}_{n+1}$ such that $r_{i_k+1}$ has multiplicity $m_k$ for $k = 1, \cdots, l$ and all other components $r_i$ are single, where

\[ 0 \leq i_1 < i_2 < \cdots < i_l \leq n - 1, \quad i_j + m_j \leq i_{j+1}, \quad i_l + m_l \leq n + 1, \quad \sum_{j=1}^l m_j \leq n + 1. \]

Let $E_{\vec{r}} = \pi_1^{-1}(\vec{r})$. Then

\begin{equation}
H_k(E_{\vec{r}}) = 0, \quad \text{if } k \geq n^* + 1 - \sum_{j=1}^l m_j(m_j - 1)/2.
\end{equation}

**Proof.** The components of $\vec{r}$ can be divided into two groups:

\[ \{r_{s_1}, \cdots, r_{s_p}\} \quad \text{and} \quad \{r_{i_{l+1}}, \cdots, r_{i_l+m_l}\}. \]

Components in the first group are single ones and components in the second are multiple ones (see Remark 4.11). We have $p + \sum_{j=1}^l m_j = n + 1$. 

For simplicity, we may assume that (after a proper permutation) \( \vec{r} \) can be written as
\[
(4.23) \quad (r_{s_1}, \ldots, r_{s_p} | r_{i_1+1}, \ldots, r_{i_1+m_1}, \ldots, r_{i_l+1}, \ldots, r_{i_l+m_l}),
\]
where the components before the symbol \( | \) are single ones and the components after the symbol \( | \) are multiple ones, and the components are in the ascending order \( r_{s_1} < \cdots < r_{s_p} \), \( r_{i_1+1} < r_{i_2+1} < \cdots < r_{i_l+1} \).

Consider the Lie group action on \( \mathcal{E}_\vec{r} \) as in (4.20). This group action is transitive and the stabiliser of \( \mathcal{E}_\vec{r} \) (given by (4.21)) is the collection of matrices in the form
\[
S = \text{diag} \{ \pm 1, \ldots, \pm 1 | O_{m_1}, \ldots, O_{m_l} \}
\]
with the property \( \det S = 1 \), where \( O_{m_k} \in O(m_k) \), the set of \( m_k \times m_k \) orthogonal matrices.

By the same argument as in Lemma 4.7, we obtain
\[
\mathcal{E}_\vec{r} \cong SO(n+1)/S(1) \times \cdots \times O(m_1) \times \cdots \times O(m_l).
\]
It is known that the space on the right-hand side has dimension \( n^* - \sum_{j=1}^l m_j(m_j - 1)/2 \).
We then deduce (4.22) as desired.

By using Lemma 4.6 and Lemma 4.7, we have the following conclusion.

**Lemma 4.9.** We have \( H_{n^*}(\mathcal{W}_{12 \ldots n}) = \mathbb{Z} \).

**Proof.** Let \( P \) be a point of \( \cap_{i=1}^n \Delta_{n-1}^{(i)} \). By Lemma 4.6,
\[
H_*(\mathcal{W}_{12 \ldots n}) = H_*(\pi^{-1}(P)).
\]
Since \( \pi_2^{-1}(P) = \vec{r} \in \mathbb{L}_{n+1} \) satisfies \( r_1 < r_2 < \cdots < r_{n+1} \), it follows from Lemma 4.7 that
\[
H_{n^*}(\pi^{-1}(P)) = \mathbb{Z}.
\]
This completes the proof.

**Lemma 4.10.** For any \( 1 \leq j_1 < j_2 \cdots < j_l \leq n \), we have
\[
H_k(\mathcal{W}_{j_1 j_2 \ldots j_l}) = 0, \quad \forall k \geq n^* + 1 - \sum_{s=1}^{l+1} \frac{m_s(m_s - 1)}{2},
\]
where \( m_s = j_s - j_{s-1} \) and \( j_0 = 0, j_{l+1} = n + 1 \).

**Proof.** Let \( P \) be a point in \( \cap_{s=1}^l \Delta_{n-1}^{(j_s)} \cap (\cap_{i \neq j_1, j_2, \ldots, j_l} F_i) \). By Lemma 4.6, we find
\[
H_*(\mathcal{W}_{j_1 j_2 \ldots j_l}) = H_*(\pi^{-1}(P)).
\]
As \( \pi_2^{-1}(P) = \vec{r} \in \mathbb{L}_{n+1} \) satisfies
\[
r_1 = \cdots = r_{j_1} < r_{j_1+1} = \cdots = r_{j_2} < r_{j_2+1} = \cdots \]
\[
\cdots = r_{j_{l-1}} < r_{j_{l-1}+1} = \cdots = r_{j_l} < r_{j_l+1} = \cdots = r_{n+1},
\]
we obtain the conclusion by Lemma 4.8 and (4.24). □

Propositions 4.11 and 4.12 below are consequences of Lemmas 4.9 and 4.10 which can be viewed as a generalisation of these two lemmas.

**Proposition 4.11.** Suppose $1 \leq p_1 < \cdots < p_r < j \leq n$, we have

$$(4.25) \quad H_k(\cup_{l=j}^{n} W_{p_1 \cdots p_r l}) = 0, \quad \text{if } k \geq n^* + n + 1 - j - \sum_{s=1}^{r+1} m_s(m_s - 1) \cdot$$

where $m_s = p_s - p_{s-1}$, $p_0 = 0$ and $p_{r+1} = j$. Furthermore, if $k \geq n^* + n + 1 - j - \frac{j(j-1)}{2}$, then

$$(4.26) \quad H_k(\mathcal{E}_{j(j+1) \cdots n}) = 0.$$

**Proof.** For $j = n$, (4.25) follows from Lemma 4.10.

We now verify (4.25) by the induction argument on $j$. For this purpose, let us assume that (4.25) holds when $j = n - m$ for some $m \geq 0$. We next show that (4.25) holds for $j = n - (m + 1)$. The Mayer-Vietoris sequence for the decomposition

$$\cup_{l=n-m}^{n} W_{p_1 \cdots p_r l} = W_{p_1 \cdots p_r (n-m)} \cup (\cup_{l=n-m}^{n} W_{p_1 \cdots p_r l})$$

yields

$$\cdots \rightarrow H_k(\cup_{l=n-m}^{n} W_{p_1 \cdots p_r l}) \oplus H_k(\mathcal{W}_{p_1 \cdots p_r (n-m-1)})$$

(4.27)

$$\rightarrow H_k(\cup_{l=n-m-1}^{n} W_{p_1 \cdots p_r l}) \rightarrow H_{k-1}(\cup_{l=n-m}^{n} W_{p_1 \cdots p_r (n-m-1)})$$

$$\rightarrow H_{k-1}(\cup_{l=n-m}^{n} W_{p_1 \cdots p_r l}) \oplus H_{k-1}(\mathcal{W}_{p_1 \cdots p_r (n-m-1)}) \rightarrow \cdots.$$

By our induction assumption, (4.25) holds when $j = n - m$. That is

$$(4.28) \quad H_k(\cup_{l=n-m}^{n} W_{p_1 \cdots p_r l}) = 0, \quad \text{if } k \geq k(m),$$

where $k(m)$ is an integer function of $m$ given by

$$k(m) := n^* + m + 1 - \sum_{s=1}^{r} \frac{m_s(m_s - 1)}{2} - \frac{(n - m - p_r)(n - m - p_r - 1)}{2}.$$

By Lemma 4.10

$$(4.29) \quad H_k(\mathcal{W}_{p_1 \cdots p_r (n-m-1)}) = 0, \quad \text{if } k \geq k'(m),$$

where $k'(m)$ is another integer function of $m$ given by

$$k'(m) := n^* + 1 - \sum_{s=1}^{r} \frac{m_s(m_s - 1)}{2} - \frac{(n - m - p_r - 1)(n - m - p_r - 2)}{2} - \frac{(m + 2)(m + 1)}{2}.$$

It can be verified that $k(m + 1) \geq \max\{k(m), k'(m)\} + 1$. Now inserting (4.28) and (4.29) in the long exact sequence (4.27), we obtain then

$$(4.30) \quad H_k(\cup_{l=n-m}^{n} W_{p_1 \cdots p_r l}) = H_{k-1}(\cup_{l=n-m}^{n} W_{p_1 \cdots p_r (n-m-1)}), \quad \text{if } k \geq k(m + 1).$$
By our induction assumption again, the right hand side above

\[ H_{k-1}(\bigcup_{l=n-m}^n W_{p_1\ldots p_r(n-m-1)l}) = 0, \quad \text{if } k \geq k(m + 1). \]

Hence, by (4.30), we conclude that (4.25) holds when \( j = n - (m + 1). \)

Note that \( W_i = E_i \) and \( E_{j(j+1)\ldots n} = \bigcup_{l=j}^n W_l. \) By the same discussion as above but deleting \( p_i's, \) we obtain (4.26). \( \square \)

**Proposition 4.12.** For any \( 2 \leq j \leq n, \) we have

\[ H_{n^*+n-j}(\bigcup_{l=j}^n W_{12\ldots(j-1)l}) = \mathbb{Z}. \]

In particular, if \( j = 2, \) then \( H_{n^*+n-2}(\bigcup_{l=2}^n W_l) = \mathbb{Z}. \)

**Proof.** For \( j = n, \) (4.31) is the conclusion of Lemma 4.9. Suppose by induction argument that (4.31) holds for \( j = n - m \) for some \( m \geq 0. \) Applying the Mayer-Vietoris sequence to the pair \( \bigcup_{l=n-m}^n W_{12\ldots(n-m-2)l} \) and \( W_{12\ldots(n-m-2)(n-m-1)}, \) we obtain

\[
\cdots \to H_{k}(\bigcup_{l=n-m}^n W_{12\ldots(n-m-2)l}) \oplus H_{k}(W_{12\ldots(n-m-2)(n-m-1)}) \\
\to H_{k-1}(\bigcup_{l=n-m}^n W_{12\ldots(n-m-2)l}) \to H_{k-1}(\bigcup_{l=n-m}^n W_{12\ldots(n-m-1)l}) \\
\to H_{k-1}(\bigcup_{l=n-m}^n W_{12\ldots(n-m-2)l}) \oplus H_{k-1}(W_{12\ldots(n-m-2)(n-m-1)}) \to \cdots.
\]

It follows from (4.25) in Proposition 4.11 and Lemma 4.10 that

\[ H_k(\bigcup_{l=n-m}^n W_{12\ldots(n-m-2)l}) = H_k(W_{12\ldots(n-m-2)(n-m-1)}) = 0, \quad \text{if } k \geq n^* + m, \]

and therefore the long exact sequence above implies that

\[ H_{n^*+m+1}(\bigcup_{l=n-m-1}^n W_{12\ldots(n-m-2)l}) = H_{n^*+m}(\bigcup_{l=n-m}^n W_{12\ldots(n-m-1)l}). \]

Hence, (4.31) follows when \( j = n - m - 1 \) by our induction assumption. This completes the proof. \( \square \)

Now, we are ready to give the proof of Theorem 3.7 for general dimensions.

**Proof of Theorem 3.7.** The Mayer-Vietoris sequence for the decomposition \( E = E_1 \cup E_{23\ldots n} \) implies

\[ \cdots \to H_{n^*+n-1}(E_1) \oplus H_{n^*+n-1}(E_{23\ldots n}) \to H_{n^*+n-1}(E) \\
\to H_{n^*+n-2}(E_1 \cap E_{23\ldots n}) \to H_{n^*+n-2}(E_1) \oplus H_{n^*+n-2}(E_{23\ldots n}) \to \cdots \]

By virtue of (4.26) in Proposition 4.11 and Lemma 4.10 (for \( l = 1 \) and \( j_1 = 1), \)

\[ H_{n^*+n-1}(E_1) = H_{n^*+n-1}(E_{23\ldots n}) = H_{n^*+n-2}(E_1) = H_{n^*+n-2}(E_{23\ldots n}) = 0. \]

Hence, by (4.32),

\[ H_{n^*+n-1}(E) = H_{n^*+n-2}(E_1 \cap E_{23\ldots n}). \]
Since \( \mathcal{E}_1 \cap \mathcal{E}_{23 \cdots n} = \bigcup_{i=2}^{n} \mathcal{W}_{1i} \), we complete the proof by Proposition 4.12.

\[ \square \]

**Remark 4.13.** For any given \( k \geq n^* + n \), by Proposition 4.11 and Lemma 4.10 we have

\[ H_i(\mathcal{E}_1) = H_i(\mathcal{E}_{23 \cdots n}) = 0 \text{ for } i = k - 1 \text{ or } k. \]

Using (4.32) with \( n^* + n - 1 \) replaced by \( k \), we then obtain

\[ H_k(\mathcal{E}) = H_{k-1}(\mathcal{E}_1 \cap \mathcal{E}_{23 \cdots n}) = H_{k-1}(\bigcup_{i=1}^{n} \mathcal{W}_{1i}). \]

By (4.25) (with \( j = 2 \)), the right hand side above \( H_{k-1}(\bigcup_{i=1}^{n} \mathcal{W}_{1i}) = 0. \) Therefore

(4.33) \[ H_k(\mathcal{E}) = 0 \text{ for all } k \geq n^* + n. \]

**5. Proof of Theorem 2.5**

In this section, we prove Theorem 2.5 by showing

(i) the Gauss curvature of \( \mathcal{M}_t \) is bounded from above,

(ii) the principal curvatures of \( \mathcal{M}_t \) have a positive lower bound.

By approximation, we may assume directly that \( f \) is \( C^2 \)-smooth. The Gauss curvature flow has been extensively studied. The technique and calculation presented here are similar to those in [33].

Let \( X(\cdot, t) \) be the solution of the flow (1.4). Recall that the Gauss curvature of \( X(\cdot, t) \) is given by (2.1), and the principal radii of curvature of \( X(\cdot, t) \) are eigenvalues of the matrix \( \{b_{ij}\} \), where

\[ b_{ij} = u_{ij} + u_{ij}, \]

where \( u \) is the support function of \( X(\cdot, t) \).

First, we derive an upper bound for the Gauss curvature.

**Lemma 5.1.** Let \( X(\cdot, t) \) be a uniformly convex solution to the flow (1.4) for \( t \in [0, T) \). Suppose that the support function \( u \) satisfies (2.11). Then there exists a constant \( C \) depending on \( n, p, \min_{\mathbb{S}^n} f, \max_{\mathbb{S}^n} f \), the initial condition \( \mathcal{M}_0 \), and the constant \( C_0 \) in (2.11), such that

\[ K(\cdot, t) \leq C, \quad \forall t \in [0, T). \]

**Proof.** We introduce the auxiliary function

\[ Q = -\frac{u_t}{u - \varepsilon_0} = \frac{Ku^p f - u}{u - \varepsilon_0}, \]

where \( \varepsilon_0 = \frac{1}{2} \min_{\mathbb{S}^n \times [0, T]} u > 0. \) It suffices to show that \( Q(x, t) \leq C \) \( \forall (x, t) \in \mathbb{S}^n \times [0, T). \)
For any given $T' \in (0, T)$, we assume that $Q$ attains its maximum over $\mathbb{S}^n \times [0, T']$ at $(x_0, t_0)$. If $t_0 = 0$, then $\max_{\mathbb{S}^n \times [0, T']} Q = \max_{\mathbb{S}^n} Q(\cdot, 0)$ and we are through. If $t_0 > 0$, then at the point $(x_0, t_0)$, we have

\begin{equation}
0 = \nabla_i Q = -\frac{u_{ti}}{u - \varepsilon_0} + \frac{u_t u_i}{(u - \varepsilon_0)^2}.
\end{equation}

Hence $u_{ti} = -Q u_i$ and we have

\begin{equation}
0 \geq \nabla^2_{ij} Q = -\frac{u_{tij}}{u - \varepsilon_0} + \frac{u_t u_j}{(u - \varepsilon_0)^2} + \frac{2 u_t u_i u_j}{(u - \varepsilon_0)^3} - \frac{u_{tij} u_j}{(u - \varepsilon_0)^2}.
\end{equation}

It follows that

\begin{equation}
-b_{ijt} = -u_{ijt} - u_t \delta_{ij} \leq (b_{ij} - \varepsilon_0 \delta_{ij})Q.
\end{equation}

Let $\{h^{ij}\}$ be the inverse matrix of $\{b_{ij}\}$. Then

\[\sum h^{ii} \geq n(\prod h^{ii})^{1/n} = nK^{1/n}.\]

This, together with (5.4), yields

\begin{equation}
\partial_t K = -K \sum h^{ij} b_{ijt} \leq (n - \varepsilon_0 \sum h^{ii}) K Q \leq CQ^2 - \frac{\varepsilon_0}{C} Q^{2+1/n}.
\end{equation}

We next compute, at $(x_0, t_0)$,

\begin{equation}
0 \leq \partial_t Q = -\frac{u_{tt}}{u - \varepsilon_0} + Q^2 = \frac{1}{u - \varepsilon_0} \frac{\partial}{\partial t} (K u^p f) + Q + Q^2 \leq \frac{1}{u - \varepsilon_0} (f u^p \partial_t K) + CQ^2,
\end{equation}

where we assume without loss of generality that $K \approx Q \gg 1$.

Combining (5.5) and (5.6), we obtain, at $(x_0, t_0)$,

\[0 \leq (C - \varepsilon_0 Q^{1/n}) Q^2.\]

This implies that $\max_{\mathbb{S}^n \times [0, T']} Q$ is bounded from above. As this bound is independent of $T'$, by sending $T' \to T$, we complete the proof.

Next, we derive a lower bound on the principal curvatures.
Lemma 5.2. Let \( X(\cdot, t) \) be a uniformly convex solution to the flow (1.4) for \( t \in [0, T) \). Assume the support function \( u \) satisfies (2.11). Then there exists a constant \( \bar{\kappa} \) depending on \( n, p, C_0, \min_{S^n} f, \|f\|_{C^1(S^n)} \), and the initial condition \( M_0 \), such that
\[
\kappa_i(\cdot, t) \geq \bar{\kappa} \quad \forall t \in [0, T), \quad 1 \leq i \leq n,
\]
where \( \kappa_i \)'s are the principal curvatures of \( X(\cdot, t) \).

Proof. Consider the following auxiliary function
\[
\tilde{w}(x, t) = \log \lambda_{\text{max}}(\{b_{ij}\}) - A \log u + B|\nabla u|^2,
\]
where \( A \) and \( B \) are large constants to be determined, and \( \lambda_{\text{max}}(\{b_{ij}\}) \) denotes the maximal eigenvalue of \( \{b_{ij}\} \). Our purpose is to show that \( \tilde{w} \) is bounded from above.

For any given \( T' \in (0, T) \), assume that \( \tilde{w}(x, t) \) achieves its maximum over \( S^n \times [0, T'] \) at some point \((x_0, t_0)\). We also suppose \( t_0 > 0 \), otherwise estimate (5.7) follows from the initial condition. By a proper rotation, we may assume that \( \{b_{ij}\} \) is diagonal at \((x_0, t_0)\) and \( \lambda_{\text{max}}(\{b_{ij}\})(x_0, t_0) = b_{11}(x_0, t_0) \).

Then the function
\[
w(x, t) = \log b_{11} - A \log u + B|\nabla u|^2
\]
attains its maximum at \((x_0, t_0)\). We may assume \( b_{11} \gg 1 \), otherwise we are through. Denote by \( \{h^{ij}\} \) the inverse matrix of \( \{b_{ij}\} \). At \((x_0, t_0)\), we have
\[
0 = \nabla_i w = h^{11} \nabla_i b_{11} - A u_i u + B \sum_k u_k u_{ki} = h^{11}(u_{i11} + u_i \delta_{i1}) - A u_i u + 2B u_i u_{ii},
\]
and
\[
0 \geq \nabla_{ii} w = h^{11} \nabla_i^2 b_{11} - (h^{11})^2(\nabla_i b_{11})^2 - A \left( \frac{u_{ii}}{u} - \frac{u_{i}^2}{u^2} \right) + 2B \left( u_{ii}^2 + \sum_k u_k u_{kii} \right).
\]

In the above, we have used the properties that \( \nabla_k b_{ij} \) are symmetric in all indices and that \( \nabla_k b_{ij} = -h^{li} h^{jp} \nabla_k b_{lp} \).

We also have
\[
\partial_t w = h^{11}(u_{11t} + u_t) - A u_t u + 2B \sum_k u_k u_{kt}.
\]
Next, we estimate the term \( b_{11} u_{11t} \). Recall that
\[
\log(u - u_t) = \log K + \log(f u^p).
\]
Set
\[
\phi(x, u) = \log(f u^p).
\]
Differentiating (5.10) gives

\[
\frac{u_k - u_{kt}}{u - u_t} = -\sum h^{ij} \nabla_k b_{ij} + \nabla_k \phi
\]

(5.11)

\[
= -\sum h^{ii} (u_{ki} + u_i \delta_k) + \nabla_k \phi,
\]

and

\[
\frac{u_{11} - u_{11t}}{u - u_t} - \frac{(u_1 - u_{1t})^2}{(u - u_t)^2} = -\sum h^{ii} \nabla_{11}^2 b_{ii} + \sum h^{ii} h^{jj} (\nabla_1 b_{ij})^2 + \nabla_1^2 \phi.
\]

(5.12)

By (5.12) and the Ricci identity \(\nabla_{11}^2 b_{ii} = \nabla_{ii}^2 b_{11} - b_{11} + b_{ii}\), we have

\[
\frac{\partial_t w}{u - u_t} = h^{11} \left[ \frac{u_{111} - u_{11}}{u - u_t} + \frac{u_{11} + u - u + u_t}{u - u_t} \right] - \frac{A}{u} \frac{u_t - u + u}{u - u_t} + 2B \frac{\sum u_k u_{kt}}{u - u_t}
\]

\[
\leq h^{11} \left[ \sum h^{ii} \nabla_{11}^2 b_{ii} - \sum h^{ii} h^{jj} (\nabla_1 b_{ij})^2 - \nabla_1^2 \phi \right]
\]

\[
+ \frac{1}{u - u_t} + \frac{A}{u} - \frac{A}{u - u_t} + 2B \frac{\sum u_k u_{kt}}{u - u_t}
\]

(5.13)

\[
\leq h^{11} \left[ \sum h^{ii} (\nabla_{ii}^2 b_{11} - b_{11} + b_{ii}) - \sum h^{ii} h^{jj} (\nabla_1 b_{ij})^2 \right] - h^{11} \nabla_1^2 \phi
\]

\[
+ \frac{1 - A}{u - u_t} + \frac{A}{u} + 2B \frac{\sum u_k u_{kt}}{u - u_t}.
\]

Inserting (5.8) and (5.9) into (5.13), we obtain, at \((x_0, t_0)\),

\[
\frac{\partial_t w}{u - u_t} \leq \sum h^{ii} \left[ (h^{11})^2 (\nabla_i b_{11})^2 + A \left( \frac{u_{ii}}{u} - \frac{u_{ii}^2}{u^2} \right) - 2B \left( u_{ii}^2 + \sum u_k u_{ki} \right) \right]
\]

\[
+ h^{11} \sum h^{ii} (b_{ii} - b_{11}) - h^{11} \sum h^{ii} h^{jj} (\nabla_1 b_{ij})^2 - h^{11} \nabla_1^2 \phi
\]

\[
+ \frac{1}{u - u_t} + \frac{A}{u} + 2B \frac{\sum u_k u_{kt}}{u - u_t}
\]

\[
\leq \sum h^{ii} \left[ A \left( \frac{u_{ii}}{u} - \frac{u_{ii}^2}{u^2} \right) \right]
\]

\[
- h^{11} \nabla_1^2 \phi + 2B \frac{\sum u_k u_{kt}}{u - u_t} + \frac{1 - A}{u - u_t} + CA
\]

\[
\leq -A \sum h^{ii} - 2B \sum b_{ii} - 2B \sum h^{ii} u_{ki} u_{ki}
\]

\[
- h^{11} \nabla_1^2 \phi + 2B \frac{\sum u_k u_{kt}}{u - u_t} + \frac{1 - A}{u - u_t} + CA + CB,
\]
where $\sum h^i h^{11}(\nabla, b_{11})^2 \leq \sum h^i h^{jj}(\nabla, b_{ij})$ is used in the second inequality. By (5.11),

$$\frac{\partial_t w}{u - u_t} \leq (2B|\nabla u|^2 - A)\sum h^i - 2B\sum b_{ii} - h^{11}\nabla^2 \phi$$

$$-2B\sum u_k \nabla_k \phi + \frac{2B|\nabla u|^2 + 1 - A}{u - u_t} + CA + CB$$

$$\leq (2B|\nabla u|^2 - A)\sum h^i - 2B\sum b_{ii} + Cb_{11}$$

$$+ \frac{2B|\nabla u|^2 + 1 - A}{u - u_t} + CA + CB.$$  

(5.14)

Choose $B$ large such that $B \sum b_{ii} \geq Cb_{11}$, and let $A = 2B \max_{S^* \times [0,T']} |\nabla u|^2 + 1$. Since $\partial_t w \geq 0$ at $(x_0, t_0)$, we obtain by (5.14) that

$$0 \leq \frac{\partial_t w}{u - u_t} \leq -B\sum b_{ii} + CA + CB.$$  

Hence, $\lambda_{\max}(\{b_{ij}\})(x_0, t_0)$ is bounded and so $\max_{S^* \times [0,T']} \lambda_{\max}(b_{ij}) \leq C$. Since this upper bound is independent of $T'$, we then let $T' \to T$ and finish the proof.

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Mathematical Sciences Institute, The Australian National University, Canberra, ACT 2601, Australia.

Email address: qiang.guang@anu.edu.au

School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China

Email address: qi-rui.li@zju.edu.cn

Mathematical Sciences Institute, The Australian National University, Canberra, ACT 2601, Australia.

Email address: xu-jia.wang@anu.edu.au