Nonholomorphic N=2 terms in N=4 SYM:
1-Loop Calculation in N=2 superspace

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Abstract

The effective action of \( N = 2 \) gauge multiplets in general includes higher-dimension UV finite nonholomorphic corrections integrated with the full \( N = 2 \) superspace measure. By adding a hypermultiplet in the adjoint representation we study the effective action of \( N = 4 \) SYM. The nonanomalous \( SU(4) \) R-symmetry of the classical \( N = 4 \) theory must be also present in the on-shell effective action, and therefore we expect to find similar nonholomorphic terms for each of the scalars in the hypermultiplet. The \( N = 2 \) path integral quantization formalism developed in projective superspace allows us to compute these hypermultiplet nonholomorphic terms directly in \( N = 2 \) superspace. The corresponding gauge multiplet expression can be successfully compared with the result inferred from a \( N = 1 \) calculation in the abelian subsector.

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1 Introduction

The nonholomorphic $N = 2$ potential $\mathcal{H}(W, \bar{W})$ in the effective action of SYM theories has been an object of study for some time now [1]-[3]. This potential is integrated with the full $N = 2$ superspace superspace measure and therefore is a dimensionless real function of the $N = 2$ gauge field strengths $W$ and $\bar{W}$. In the nonabelian sector it contributes to the $N = 1$ Kähler potential $K(\phi, \bar{\phi})$ and in the abelian sector it can only contribute to $N = 1$ higher derivative terms [2]. Scale invariance and $U(1)_R$ invariance restricts the form of the $N = 2$ potential to be

$$\mathcal{H} = \mathcal{H}^0 + c \ln \frac{W^2}{\Lambda^2} \ln \frac{\bar{W}^2}{\Lambda^2},$$

(1)

where $\mathcal{H}^0$ depends on gauge invariant, scale independent combinations of the nonabelian $N = 2$ field strengths. The pure abelian piece is contained only in the second term. In $N = 4$ SYM theories the abelian nonholomorphic potential is believed to be generated only at 1-loop [3] since higher loop and nonperturbative contributions would break the scale and $U(1)_R$ invariance of $\mathcal{H}$. Nonperturbative contributions have been studied in [4] and they give vanishing results.

It is therefore possible to determine the exact form of $\mathcal{H}^{Abelian}$ by performing a 1-loop calculation. Recently this type of calculation has been done in $N = 1$ superspace [5] by considering the higher derivative operator

$$\int dxd^4\theta \left( i\mathcal{H}_{AB} \bar{W}^B \bar{\alpha}^A \nabla_{\bar{\alpha}} \right).$$

(2)

This is one of the $N = 1$ components of (1) [2]. Its contribution to the 1-loop abelian effective action was computed using $N = 1$ superspace quantization. The resulting coefficient $c$ was found to be nonvanishing. This has interesting implications for 3-branes in ten dimensions because they are believed to be effectively described by $N = 4$ SYM at low energies: the presence of a nonvanishing $N = 2$ nonholomorphic potential introduces acceleration dependent terms in the scattering of 3-branes in addition to the standard velocity dependent terms [5].

In this article we compute the nonholomorphic corrections to the $N = 4$ SYM effective action directly in $N = 2$ superspace using the $N = 2$ path integral quantization that we developed in [6], [7]. This quantization involves $N = 2$ superfields that contain the familiar $N = 1$ hypermultiplet and gauge degrees of freedom $1$.

First we calculate all finite 1-loop corrections to the $N = 2$ hypermultiplet effective action dropping terms with spinor or space-time derivatives on the external fields. The calculation is greatly simplified using $N = 2$ gauge propagators in the Landau gauge.

We then isolate the dependence of the effective action on the $N = 2$ hypermultiplet superfield whose $N = 1$ projection is part of the chiral hypermultiplet isodoublet $\Upsilon_0|_{\theta^2 = 0} = \bar{Q}$. This contribution to the $N = 2$ effective action is a nonholomorphic potential $\mathcal{H}(\Upsilon_0, \bar{\Upsilon}_0)$ whose $N = 1$ projection can be rotated by a $Z_2$ subgroup of the global $SU(4)_R$ of $N = 4$ SYM into the $N = 1$ projection of a corresponding pure gauge piece $\mathcal{H}(W, \bar{W})$. Symmetry

\footnote{Related but different $N = 2$ superfield Feynman rules have been developed in harmonic superspace [3]. Up to now they have not correctly reproduced the calculations we describe here [10].}
arguments therefore determine the form of the 1-loop $N = 2$ nonholomorphic potential in the low energy gauge effective action of $N = 4$ SYM.

The $N = 2$ potential we find for the abelian sector is of the form \( (1) \). The coefficient $c$ is exactly the same as that calculated in $N = 1$ superspace \([3]\). Due to the nonlinearity of the nonabelian superfield strengths it is not clear if the 1-loop nonabelian piece $H^o$ can be reproduced from the knowledge of the hypermultiplet effective action and we cannot test the proposal in \([3]\).

2 N=2 formalism

In this section we briefly review the superfield content of the $N = 4$ SYM in $N = 2$ superspace and we give the Feynman rules for quantization of these $N = 2$ multiplets. For a more detailed explanation we refer the reader to \([3]-[5]\) and references therein. The conventions we follow are those of ref. \([8]\).

Gauge multiplets and hypermultiplets can be described by off-shell representations of $N = 2$ supersymmetry using superfields that live in projective superspace. This is a subspace of $N = 2$ superspace whose anticommuting coordinates are the following linear combinations of the $N = 2$ Grassmann coordinates: $\Theta^\alpha = \theta^{2\alpha} - \zeta \theta^{1\alpha}$ and $\bar{\Theta}^{\dot{\alpha}} = \bar{\theta}^{\dot{1}\alpha} + \zeta \bar{\theta}^{\dot{2}\alpha}$ parameterized by a complex projective coordinate $\zeta$. Accordingly, projective superfields $\Omega$ obey the constraint

$$\nabla_\alpha \Omega(\Theta, \bar{\Theta}) = (D_{1\alpha} + \zeta D_{2\alpha}) \Omega = 0 = (\bar{D}_{\dot{1}\alpha} - \zeta \bar{D}_{\dot{2}\alpha}) \Omega = \nabla_{\dot{\alpha}} \bar{\Omega}$$ \hspace{1cm} (3)

Charged hypermultiplets can be described by an infinite power series in the projective coordinate $\zeta$. We refer to this multiplet as the polar multiplet

$$\Upsilon = \sum_{n=0}^{\infty} \Upsilon_n \zeta^n$$

$$\bar{\Upsilon} = \sum_{n=0}^{\infty} \bar{\Upsilon}_n (-\zeta)^n.$$ \hspace{1cm} (4)

As a consequence of the constraints (3) the highest order coefficient is a chiral superfield in $N = 1$ superspace $D_\dot{a} \Upsilon_0 = 0$ and the next order is a complex linear superfield $D^2 \Upsilon_1 = 0$. These two superfields contain the physical degrees of freedom of the hypermultiplet. The other coefficients are auxiliary superfields in $N = 1$ superspace.

Gauge vector multiplets are described by an infinite series with negative and positive powers of the projective complex coordinate that we call the tropical multiplet. This multiplet is real under conjugation and since there are no lowest order or highest order coefficients, all of them are unconstrained in $N = 1$ superspace

$$V = \sum_{n=-\infty}^{+\infty} v_n \zeta^n , \quad v_n = (-)^n \bar{v}_n.$$ \hspace{1cm} (5)

The coefficients $v_n$, $|n| > 1$ are gauge degrees of freedom, $v_0$ is related to the usual $N = 1$ real gauge prepotential $v = v_0 + \text{nonlinear corrections}$ of the covariantly chiral spinor field

\footnote{Under conjugation $\zeta \leftrightarrow -\zeta^{-1}$. See \([3]\).}
strength and \( v_{-1} \) is related to the prepotential \( \psi = iv_{-1} + \) nonlinear corrections of the ordinary chiral scalar \( \phi \) that appears in the \( N = 1 \) components of the classical gauge action

\[
S_{\text{gauge}} = \int dx^2 \theta d^2 \bar{\theta} \frac{Tr}{4g^2} e^{-v} e^v \phi + \frac{1}{2} \left( \int dx^2 \theta d^2 \bar{\theta} \frac{Tr}{2} W^a W_a + \int dx^2 \theta \frac{Tr}{2} W^a \bar{W}_a \right) + \frac{1}{2} W_a = i \bar{D}^2 e^{-v} D_a e^v, \quad \phi = \bar{D}^2 \psi . \tag{6}
\]

The manifestly \( N = 2 \) supersymmetric action describing the coupling of a polar hypermultiplet in the adjoint representation of the gauge group to the tropical gauge multiplet is the following

\[
S_T = \int dx^2 \theta d^2 \bar{\theta} \int \frac{d \zeta}{2\pi i \zeta} \frac{Tr}{4g^2} \left( e^{-V} e^V \hat{Y} \right) . \tag{7}
\]

This action can be dualized to the give the usual description of the hypermultiplet in terms of two chiral fields: the complex antilinear superfield \( \hat{Y}_1 \) is traded for a chiral Lagrange multiplier \( Q \), the chiral coefficient superfield \( \hat{Y}_0 \) is identified with its \( N = 2 \) partner \( Q \) and the auxiliary superfields decouple. The resulting interacting action is the usual one

\[
S_{\hat{Q}Q} = \int dx^3 \theta \frac{Tr}{4g^2} (e^{-v} \bar{Q} e^v \hat{Q} + e^{-v} Q e^v \hat{Q}) + \left( i \int dx^2 \theta \frac{Tr}{4g^2} \bar{Q} [\phi, Q] + \text{h.c.} \right) . \tag{8}
\]

The convention we adopt to define the path integral is such that the kinetic term of scalars is convergent in euclidean space

\[
Z = \int [dQ][d\hat{Q}] e^{-iS} = \int [dQ][d\hat{Q}] e^{S_E} , \quad S_E = \int d^4k_E Q(k_E) (-k_E^2) \bar{Q}(-k_E) + \ldots . \tag{9}
\]

For the gauge multiplet, the kinetic piece of (8) has a simple expression in terms of tropical multiplets

\[
S_{\hat{g}^\text{gauge}} = -\frac{Tr}{8g^2} \int dx^2 \theta d^2 \bar{\theta} \int \frac{d \zeta_1}{2\pi i} \frac{d \zeta_2}{2\pi i} \frac{V(\zeta_1)V(\zeta_2)}{(\zeta_1 - \zeta_2)^2} , \tag{10}
\]

while the interaction vertices are more complicated. Since we are only going to compute 1-loop amplitudes with external hypermultiplets coupling to internal gauge multiplets, all we need is the gauge propagator in projective superspace. The kinetic action (10) is therefore enough to use the path integral quantization of the model.

We recall examine the Feynman rules that we use to compute the set of diagrams proposed. The polar hypermultiplet propagator is \[ \int \frac{d \zeta_1}{2\pi i} \frac{d \zeta_2}{2\pi i} \frac{V(\zeta_1)V(\zeta_2)}{(\zeta_1 - \zeta_2)^2} \]

\[
\langle \hat{Y}_a(1) \hat{Y}_b(2) \rangle = -\frac{4g^2 \delta^{ab}}{C^2(A)} \sum_{n=0}^{\infty} \left( \frac{\zeta_2}{\zeta_1} \right)^n \frac{\nabla_1^4 \nabla_2^4}{\zeta_1^2 (\zeta_1 - \zeta_2)^2} \delta^8(\theta_1 - \theta_2) \delta(x_1 - x_2) , \tag{11}
\]

\[
\langle \hat{Y}_a(1) \hat{Y}_b(2) \rangle = -\frac{4g^2 \delta^{ab}}{C^2(A)} \sum_{n=0}^{\infty} \left( \frac{\zeta_1}{\zeta_2} \right)^n \frac{\nabla_1^4 \nabla_2^4}{\zeta_2^2 (\zeta_1 - \zeta_2)^2} \delta^8(\theta_1 - \theta_2) \delta(x_1 - x_2) .
\]
where $C_2(A)$ is the second Casimir in the adjoint representation of the gauge group $TrT_aT_b = C_2(A)\,\delta_{ab}$ and $\nabla_1^4 = \nabla^2(\zeta_1)\nabla^2(\zeta_1)$. The tropical multiplet propagator is

$$
\langle V^a(1)V^b(2) \rangle = \frac{4 g^2}{C_2(A)} \left( \alpha \sum_{n=-\infty}^{+\infty} \left( \frac{\zeta_2}{\zeta_1} \right)^n + (1 - \alpha) \right) \frac{\nabla_1^4 \nabla_2^4}{\zeta_1\zeta_2(\zeta_2 - \zeta_1)^2\Box^2} \delta^8(\theta_{12}) \delta^4(x_{12}) ,
$$

where $\alpha$ denotes the gauge fixing parameter (the factor $g^2$ in the propagators is the one consistent with the holomorphicity of the kinetic term as opposed to canonical normalization in the sense of [11]). The interaction vertices that contribute to the relevant graphs are obtained from the first two orders in the expansion of (11)

$$
- \int dxd^4\theta \oint \frac{d\zeta}{2\pi i\zeta} \frac{Tr}{4g^2} [V,\gamma] \bar{\gamma} + \int dxd^4\theta \oint \frac{d\zeta}{2\pi i\zeta} \frac{Tr}{4g^2} \frac{1}{2} [V, [V,\gamma]] \bar{\gamma} .
$$

Now we can use all the powerful tools of path integral quantization to calculate the 1-loop effective action of the polar multiplet. The only peculiarity of the $N = 2$ formalism is that we must complete the Grassmann measure of each vertex to have a full $N = 2$ superspace measure. This procedure eliminates four projective spinor derivatives in one of the propagators stemming from the vertex. For example in the vertex with an external arctic multiplet

$$
\int dx d^2\bar{D} \int \frac{d\zeta_1}{2\pi i\zeta_1} \frac{C_2(A)}{4g^2} (-i) f_{abc} Y^b(\zeta_1) \left[ 4g^2\delta^{ad} \frac{\nabla_1^4 \nabla_2^4 \delta_{12}}{C_2(A) \zeta_1\zeta_2(\zeta_2 - \zeta_1)^2\Box^2} \right] \left[ 4g^2\delta^{ce} \frac{\nabla_1^4 \nabla_3^4 \delta_{13}}{C_2(A) \zeta_1\zeta_2(\zeta_3 - \zeta_1)^2\Box^2} \right] \cdots
$$

$$
= \int dx d^2(\bar{D}^1)^2 d^2(\bar{D}^2)^2 \int \frac{d\zeta_1}{2\pi i\zeta_1} \gamma(\zeta_1) \left[ 4g^2\delta^{ad} \frac{\nabla_1^4 \nabla_2^4 \delta_{12}}{C_2(A) \zeta_1\zeta_2(\zeta_2 - \zeta_1)^2\Box^2} \right] \left[ \delta^{ce} \frac{\nabla_3^4 \delta_{13}}{(\zeta_3 - \zeta_1)^2\Box^2} \right] \cdots
$$

Once the measure has been completed in all the vertices, we do the “D”-algebra to reduce all propagators but one to bare Grassmann delta functions. These are the basic Feynman rules that we use in the next section to construct the 1-loop effective action of the polar multiplet.

### 3 1-loop nonholomorphic terms in the hypermultiplet effective action

Now that we have presented the rules to calculate Feynman diagrams in $N = 2$ superspace, we focus our attention on those amplitudes of interest to us. We want to consider graphs with any number of external polar multiplets at zero momentum. The calculation is greatly simplified working with the gauge propagator in the Landau gauge $\alpha = 0$. To illustrate the techniques used in this novel $N = 2$ quantization we present the simplest graphs in some detail. The more complicated ones only involve a larger amount of algebra.

In the $N = 2$ formalism tadpoles and seagulls vanish automatically [13], and at one loop we always have the same number of external arctic and antarctic polar fields. Therefore the first graph we study is the two point function. After completing the Grassmann measure
on both vertices we find the graph on Fig. 3 and a similar graph in which the external hypermultiplets are exchanged.

The "D"-algebra on this graph is trivial: we already have one bare propagator and the other one is acted upon by eight spinor derivatives. We just have to reduce

\[ \delta^8(\theta_1 - \theta_2) \nabla_1^4 \nabla_2^4 \delta^8(\theta_1 - \theta_2) = (\zeta_1 - \zeta_2)^4 \delta^8(\theta_1 - \theta_2). \]  

(15)

The resulting contribution to the effective action \(-i\Gamma(2)\) is UV finite (this is the well known nonrenormalization of hypermultiplets) and local in the \(N = 2\) Grassmann coordinates

\[ \frac{1}{2!} \int \frac{d\zeta_1}{2\pi i \zeta_1} \frac{d\zeta_2}{2\pi i \zeta_2} \int d^4\theta \frac{id^4p}{(2\pi)^4 (-p^2)^3} Tr \left( \sum_{n=1}^{\infty} \left( \frac{\zeta_2}{\zeta_1} \right)^n \Upsilon(\zeta_1) \bar{\Upsilon}(\zeta_2) + \sum_{n=1}^{\infty} \left( \frac{\zeta_1}{\zeta_2} \right)^n \bar{\Upsilon}(\zeta_1) \Upsilon(\zeta_2) \right). \]  

(16)

Next we consider the graphs with four external polar multiplets. After completing the superspace measure in the vertices we obtain the graphs in Fig. 4 and similar ones in which we exchange the external hypermultiplet of each cubic vertex by the internal hypermultiplet.

The "D" algebra is trivial only on the upper graph. In the other two we integrate by parts the spinorial derivatives of all propagators but one. As usual, it is easiest to do so on the graph. Since we are interested on nonholomorphic terms without external derivatives we keep only the contribution where all spinor derivatives end up acting on the same propagator.

Finally we can integrate the bare Grassmann delta functions and reduce the spinor derivatives acting on the last one. For example in the middle graph

\[ \delta^8(\theta_1 - \theta_2) \nabla_3^2 \nabla_3^2 \nabla_1^2 \nabla_2^2 \nabla_2^2 \delta^8(\theta_1 - \theta_2) = (\zeta_3 - \zeta_1)^2 \Box \delta^8(\theta_1 - \theta_2) \nabla_3^2 \nabla_1^2 \nabla_2^2 \nabla_2^2 \delta^8(\theta_1 - \theta_2) \]

\[ = (\zeta_3 - \zeta_1)^2 (\zeta_3 - \zeta_2)^2 (\zeta_1 - \zeta_2)^2 \Box \delta^8(\theta_1 - \theta_2). \]  

(17)

As a result all the graphs with four external hypermultiplets and no external derivatives
Figure 2: Hypermultiplet 4-point function.
have the same euclidean loop momentum integral \( \int d^4p/(-2\pi p^2)^4 \). The complex coordinate dependence is slightly different though. The first graph gives a contribution to \(-i\Gamma(4)\)

\[
\frac{1}{2!} \oint d\zeta_1 d\zeta_2 d\zeta_3 d\zeta_4 \quad \text{Tr} \quad \frac{1}{4} \left[ \left( Y(\zeta_1)\bar{Y}(\zeta_1) + Y(\zeta_1)Y(\zeta_1) \right) \frac{Y(\zeta_2)Y(\zeta_2) + \bar{Y}(\zeta_2)\bar{Y}(\zeta_2)}{2} \right]
\]

(18)

the second is

\[
-\frac{1}{2} \oint d\zeta_1 d\zeta_2 d\zeta_3 d\zeta_4 \quad \text{Tr} \quad \frac{1}{2} \left( \sum_{n=1}^{\infty} \left( \frac{\zeta_2}{\zeta_3} \right)^n Y(\zeta_3)Y(\zeta_3) + \sum_{n=1}^{\infty} \left( \frac{\zeta_2}{\zeta_3} \right)^n \bar{Y}(\zeta_2)\bar{Y}(\zeta_2) \right)
\]

(19)

and the last one

\[
\frac{1}{4} \oint d\zeta_1 d\zeta_2 d\zeta_3 d\zeta_4 \quad \text{Tr} \quad \left[ \sum_{n=1}^{\infty} \left( \frac{\zeta_4}{\zeta_1} \right)^n Y(\zeta_1)\bar{Y}(\zeta_1) + \sum_{n=1}^{\infty} \left( \frac{\zeta_4}{\zeta_1} \right)^n \bar{Y}(\zeta_4)Y(\zeta_4) \right]
\]

(20)

This simple result illustrates a few features that will be reproduced by higher n-point functions:

i) integrating the complex coordinates we can see that the coefficient superfields enter quadratically \(Y_i\bar{Y}_i\);

i) graphs containing one or more internal hypermultiplet propagators do not contribute terms that depend purely on powers of \(Y_0\bar{Y}_0\).

This simplifies our calculation considerably because we are interested in selecting terms that depend only on the \(N=2\) superfield containing \(\bar{Q} = Y_0|_{\theta^2=0}\). Terms mixing auxiliary superfields \(Y_i, i > 1\) and \(Y_0\) do not modify the pure \(\bar{Y}_0Y_0\) piece because they enter at least quadratically. We may set the auxiliary fields to zero using their algebraic field equations.

We focus our attention on \(Y_0\) because the \(N=1\) superfield \(\bar{Q}\) is rotated by \(Z_2\) subgroup of \(SU(4)_R\) into the \(N=1\) gauge scalar \(\bar{\phi}\). Since this symmetry is nonanomalous the nonholomorphic potential

\[
\int d^8\theta \, \mathcal{H}(Y_0, \bar{Y}_0)
\]

(21)

must be accompanied by a corresponding nonholomorphic function of \(N=2\) superfields whose \(N=1\) projection is precisely \(\bar{\phi}, \phi\). These are the \(N=2\) chiral gauge field strengths \(W, \bar{W}\).

Now that we know what kind of amplitudes to look for, we collect all the relevant graphs with any number of external hypermultiplets but no internal hypermultiplets and find
\[ \sum_{m \geq 4} \Gamma(m) = -\frac{1}{2} \int d^8 \theta \frac{dp^2}{(4\pi)^2 p^2} \sum_{n=2}^{+\infty} Tr \frac{1}{n} \left( \frac{\Upsilon_0 \bar{\Upsilon}_0 + \bar{\Upsilon}_0 \Upsilon_0}{-2p^2} \right)^n. \] (22)

This is almost the Taylor expansion of a logarithm but it is missing the first order term. To find this term let us go back for a moment to the result of the first graph (16). It does not seem to contain a piece depending on the chiral superfield we are interested in \( \Upsilon_0 \bar{\Upsilon}_0 \).

Notice however that it is possible to rewrite the contour integrals in (16) as follows

\[ \oint \frac{d\zeta_1}{2\pi i \zeta_1} \frac{d\zeta_2}{2\pi i \zeta_2} \sum_{n=1}^{\infty} \left( \frac{\zeta_1}{\zeta_2} \right)^n Tr \Upsilon(\zeta_1) \bar{\Upsilon}(\zeta_2) = \oint \frac{d\zeta}{2\pi i \zeta} Tr \left( \Upsilon(\zeta) \bar{\Upsilon}(\zeta) - \Upsilon_0 \bar{\Upsilon}_0 \right). \] (23)

The first term is a projective quantity and therefore it vanishes if we integrate it with the full \( N=2 \) superspace measure. Thus we obtain a contribution to the effective action that we can write in two equivalent ways

\[ \Gamma(2) = \int \frac{dp^2}{(4\pi)^2 p^2} Tr \sum_{i>1} \frac{\Upsilon_i \bar{\Upsilon}_i + \bar{\Upsilon}_i \Upsilon_i}{-2p^2} = -\int \frac{dp^2}{(4\pi)^2 p^2} Tr \frac{\Upsilon_0 \bar{\Upsilon}_0 + \bar{\Upsilon}_0 \Upsilon_0}{-2p^2}. \] (24)

To help us decide which form we use let us recall that the physical superfield \( \Upsilon_1 | \theta^2 = 0 \) is mapped by duality to one of the chiral fields \( Q \) of the hypermultiplet on-shell description and the superfield \( \bar{\Upsilon}_0 | \theta^2 = 0 \) is identified with its partner \( \tilde{Q} \). Since we expect the global \( SU(2)_R \) symmetry of these two chiral fields to be realized in the effective action it seems natural to choose an equally weighted combination

\[ -\frac{1}{4p^2} Tr \sum_{i>1} (\Upsilon_i \bar{\Upsilon}_i + \bar{\Upsilon}_i \Upsilon_i) + \frac{1}{4p^2} Tr (\Upsilon_0 \bar{\Upsilon}_0 + \bar{\Upsilon}_0 \Upsilon_0). \] (25)

This choice will prove to be correct when we compare the corresponding nonholomorphic gauge effective action with the result inferred from its \( N=1 \) components [5].

Adding (22) and the \( \Upsilon_0 \bar{\Upsilon}_0 \) piece in (23) we find the nonholomorphic contribution to the effective action

\[ \int d^8 \theta \mathcal{H}(\Upsilon_0, \bar{\Upsilon}_0) = \frac{1}{2} \int d^8 \theta \frac{dp^2}{(4\pi)^2 p^2} Tr \ln \left( 1 + \frac{\Upsilon_0 \bar{\Upsilon}_0 + \bar{\Upsilon}_0 \Upsilon_0}{2p^2} \right). \] (26)

The corresponding nonholomorphic potential for the \( N=2 \) gauge field strength \( W \) is therefore

\[ \mathcal{H}(W, \bar{W}) = \frac{1}{2} \int \frac{dp^2}{(4\pi)^2 p^2} Tr \ln \left( 1 + \frac{W \bar{W} + \bar{W} W}{2p^2} \right). \] (27)

To simplify our analysis let us consider the case of \( SU(2) \) SYM. The gauge operator in the argument of the logarithm can be diagonalized [2]

\[ U \left( W \bar{W} + \bar{W} W \right) U^\dagger = \begin{pmatrix} 2W \cdot \bar{W} & 0 & 0 \\ 0 & W \cdot \bar{W} + \sqrt{W^2 \bar{W}^2} & 0 \\ 0 & 0 & W \cdot \bar{W} - \sqrt{W^2 \bar{W}^2} \end{pmatrix}. \] (28)

\footnote{Actually in \( N=4 \) SYM this is just a subgroup of the larger \( SU(4)_R \) we mentioned before.}
This facilitates the evaluation of the trace in (27)

\[ \mathcal{H}(W, \bar{W}) = \frac{1}{2(4\pi)^2} \int \frac{dp^2}{p^2} \ln \left( 1 + \frac{W \cdot \bar{W}}{p^2} \right) + \ln \left( 1 + \frac{W \cdot \bar{W} + \sqrt{W^2 \bar{W}^2}}{2p^2} \right) \]

\[ + \ln \left( 1 + \frac{W \cdot \bar{W} - \sqrt{W^2 \bar{W}^2}}{2p^2} \right). \]  

(29)

4 Loop momentum integration and regularization issues

We have obtained the 1-loop N=2 nonholomorphic potential of N = 4 SYM as a loop momentum integral. Now we want to study the abelian sector of the theory by performing this momentum integral.

Then we can compare our result with the explicit form (1) consistent with scale and \( U(1)_R \) invariance. We can also compare the coefficient \( c \) obtained in \( N = 1 \) superspace [5] with the coefficient we obtain. Let us briefly review the result of the \( N = 1 \) calculation. The sum of all 1-loop amplitudes with two external abelian spinor field strengths, one space time derivative and arbitrary numbers of abelian gauge scalars is of the form (2) with

\[ \frac{\partial^2}{\partial W \partial \bar{W}} \mathcal{H}^{Abel}(W, \bar{W}) \bigg|_{\theta^2 = 0} = \frac{1}{(4\pi)^2} \frac{1}{\phi \bar{\phi}}. \]  

(30)

Promoting the \( N = 1 \) chiral field strengths to \( N = 2 \) chiral field strengths it is straightforward to integrate \( \mathcal{H}_{W \bar{W}} \) with respect to \( W \) and \( \bar{W} \) to obtain the postulated result

\[ \mathcal{H}^{Abel} = \frac{1}{(4\pi)^2} \ln W \ln \bar{W}. \]  

(31)

Let us consider now the abelian piece of (29). When \( W \) and \( \bar{W} \) commute the third term vanishes and the other two give equal contributions

\[ \mathcal{H}^{Abel}(W, \bar{W}) = 2 \times \frac{1}{2(4\pi)^2} \int \frac{dp^2}{p^2} \ln \left( 1 + \frac{W \bar{W}}{p^2} \right). \]  

(32)

Now we have to perform the loop momentum integral and verify that we reproduce the postulated form (1) of \( \mathcal{H}^{Abel} \). To show that the dependence on any mass scale is irrelevant we regulate the divergences of this integral in two different ways. First we introduce an IR cutoff \( \Lambda^2 \) and we rescale the loop momentum into a dimensionless variable

\[ \mathcal{H}^{Abel} = \frac{1}{(4\pi)^2} \int_{\Lambda^2}^{\infty} dy \frac{dy}{y} \ln (1 + y^{-1}) = \frac{1}{(4\pi)^2} \int_{0}^{\frac{W \bar{W}}{\Lambda^2}} \frac{dx}{x} \ln (1 + x). \]  

(33)

The lower limit of the integral does not give any contribution. To see this we split the integral in two pieces
\[ \int d^8 \theta \mathcal{H}_{\text{Abel}} = \frac{1}{(4\pi)^2} \int d^8 \theta \left( \int_0^\xi + \int_\xi^W \right) \frac{dx}{x} \ln(1 + x) . \] (34)

The first term is just a numerical constant and the integration over Grassmann coordinates cancels it. Since the splitting point is arbitrary we can choose \( \xi \gg 1 \) and the second term gives

\[ \int d^8 \theta \mathcal{H}_{\text{Abel}} = \frac{1}{(4\pi)^2} \int d^8 \theta \frac{1}{2} \left( \ln \frac{W\tilde{W}}{\Lambda^2} \right)^2 = \frac{1}{4(4\pi)^2} \int d^8 \theta \ln \frac{W^2}{\Lambda^2} \ln \frac{\tilde{W}^2}{\Lambda^2} . \] (35)

to an arbitrary degree of accuracy. In this form it is easy to see that the IR scale is irrelevant, since all the terms that depend on it are killed by the \( N = 2 \) superspace integral \[2, 3\]. Our calculation gives the correct answer for the 1-loop nonholomorphic abelian potential with a coefficient

\[ c = \frac{1}{4(4\pi)^2} . \] (36)

We can alternatively regularize the momentum integral using dimensional regularization

\[ \mathcal{H}_{\text{Abel}}(W, \tilde{W}) = \frac{1}{(4\pi)^2} \left( \mu^2 \right)^{-\epsilon} \int \frac{dp^2}{(p^2)^{1-\epsilon}} \ln \left( 1 + \frac{W\tilde{W}}{p^2} \right) . \] (37)

where \( 0 < \epsilon < 1 \). Rescaling the momentum variable we rewrite (37)

\[ \mathcal{H}_{\text{Abel}}(W, \tilde{W}) = \frac{1}{(4\pi)^2} \left( \frac{W\tilde{W}}{\mu^2} \right)^\epsilon \int_0^{+\infty} dy \frac{1}{y^{1-\epsilon}} \ln(1 + y^{-1}) . \] (38)

The divergence of the integral can be isolated by standard manipulation

\[ \int_0^1 \frac{dy}{y^{1-\epsilon}} \left( \ln(1 + y) - \ln y \right) + \int_1^{+\infty} \frac{dy}{y^{1-\epsilon}} \ln(1 + y^{-1}) \]

\[ = \int_0^1 \frac{dy}{y^{1-\epsilon}} \ln(1 + y) - \frac{y^\epsilon}{\epsilon} \left[ \ln y - \frac{1}{\epsilon} \right]_0^1 + \int_0^1 \frac{dx}{x^{1+\epsilon}} \ln(1 + x) \]

\[ = \int_0^1 \frac{dy}{y} \left( y^{-\epsilon} + y^\epsilon \right) \ln(1 + y) - \frac{y^\epsilon}{\epsilon} \left[ \ln y - \frac{1}{\epsilon} \right]_0^1 \]

\[ = C + \frac{1}{\epsilon^2} . \] (39)

The integral in (39) gives a finite constant and the upper limit of the last term contains the regulated divergence as we let \( \epsilon \to 0 \) . The resulting nonholomorphic potential is

\[ \mathcal{H}_{\text{Abel}}(W, \tilde{W}) = \frac{1}{(4\pi)^2} \lim_{\epsilon \to 0} \left( C + \frac{1}{\epsilon^2} \right) \exp \left( \epsilon \ln \frac{W\tilde{W}}{\mu^2} \right) \]

\[ = \frac{1}{(4\pi)^2} \lim_{\epsilon \to 0} \left( C + \frac{1}{\epsilon} \ln \frac{W\tilde{W}}{\mu^2} + \frac{1}{2} \left( \ln \frac{W\tilde{W}}{\mu^2} \right)^2 \right) . \] (40)
The constant $C$ and the chiral divergence $\ln W + \ln \bar{W}$ are killed by the $N = 2$ superspace measure and we are left with the same nonholomorphic potential we found using an IR cutoff.

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