Rings and Fields from Semigroups

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We introduce a ring and a field, generated by a semigroup, and we investigate some of their properties.

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We start with a semigroup \((\mathbb{H}, \cdot)\), where ‘\(\cdot\)’ means the operation in the semigroup \(\mathbb{H}\).

Definition 1.1. Let \((\mathbb{H}, \cdot)\) be a semigroup, \(\mathbb{H} \neq \emptyset\). We define the set \(\text{sums}(\mathbb{H})\).

\[
\text{sums}(\mathbb{H}) := \{ (\pm_1 x_1 \pm_2 x_2 \pm_3 x_3 \ldots \pm_{N-1} x_{N-1} \pm_N x_N) \mid N \in \mathbb{N}_0, \pm_i \in \{+, -\} \ (\text{the sum is only formal}), \ x_1, x_2, x_3, \ldots, x_{N-1}, x_N \ \text{are elements (not necessarily distinct) of the semigroup} \ \mathbb{H} \}.
\]

The set \(\text{sums}(\mathbb{H})\) contains at least three elements \(\langle \rangle\), \(\langle +h \rangle\), \(\langle -h \rangle\) for \(h \in \mathbb{H}\). It even has infinite many elements, it contains the subset \(\{(+h), (+h + h), (+h + h + h), \ldots\}\).

We call the number \(N\) the length of \(x\), and we call the symbols \(\pm_i x_i\) the entries of \(x\), for \(x = (\pm_1 x_1 \pm_2 x_2 \pm_3 x_3 \ldots \pm_{N-1} x_{N-1} \pm_N x_N) \in \text{sums}(\mathbb{H})\).

We define an equivalence relation on \(\text{sums}(\mathbb{H})\). Let \(x, y\) be elements of \(\text{sums}(\mathbb{H})\). We introduce the relation ‘\(\approx_1\)’ to create a zero in the ring \(R(\mathbb{H})\) we will construct below.

We say that the elements \(x\) and \(y\) are equivalent if \(x\) has length \(N\), \(y\) has length \(N - 2\), and \(y\) has the same entries as \(x\) at the same positions, except \(y\) lacks a pair of entries \(\{+h, -h\}, h \in \mathbb{H}\).

In symbols we write \(x \approx_1 y\). We get \(\langle +h - h \rangle \approx_1 \langle \rangle\) and \(\langle -h + h \rangle \approx_1 \langle \rangle\) for each \(h \in \mathbb{H}\).

We add a further relation ‘\(\approx_2\)’ to enforce the commutativity of the addition ‘\(+_{\mathbb{H}}\)’ in the ring \(R(\mathbb{H})\).

We say that two elements \(x\) and \(z\) fulfill the relation ‘\(\approx_2\)’ if

\[
x = \left(\sum_{i=1}^{N} \pm_i x_i\right) \quad \text{and} \quad z := \left(\sum_{i=1}^{N} \pm_{\nu(i)} x_{\nu(i)}\right) \text{where} \ \nu \ \text{is any permutation of the set} \ \{1, 2, \ldots, N\}.
\]

(1.1)
Let ‘\(\equiv\)’ be the smallest equivalence relation that includes the relations ‘\(\approx_1\)’ and ‘\(\approx_2\)’. We name the equivalent classes by \([\cdot]_\equiv\), i.e. here we get \([x]_\equiv = [y]_\equiv\) and \([x]_\equiv = [z]_\equiv\). We get a set of equivalent classes \(\text{sums}(\mathbb{N})/\equiv\). Every equivalent class \([x]_\equiv\) has an infinite number of elements. Every equivalent class contains an element \(p\) of minimal length, this means that \(p\) contains no pair \(\{+h,-h\}\) of entries. In the case of an element \([x]_\equiv \in \text{sums}(\mathbb{N})/\equiv\) with an element \(p\) of minimal length 0, i.e. \(p = \langle \rangle \in \text{sums}(\mathbb{H})\), we will write \([p]_\equiv =: 0_{\mathbb{R}(\mathbb{H})}\).

In the following we will not distinguish between the equivalent class \([x]_\equiv\) and its representative \(x\). This can be made since all the operations which are constructed in the following are independent from the picked representatives of the equivalent classes.

We define the sets \(\mathbb{R}(\mathbb{H})\) and Quot.

**Definition 1.2.** \(\mathbb{R}(\mathbb{H}) := \text{sums}(\mathbb{H})/\equiv\)

\[\text{Quot} := \{ p/q \mid p,q \in \mathbb{R}(\mathbb{H}) \mid q \neq 0_{\mathbb{R}(\mathbb{H})}\}\]

We have that \(\mathbb{R}(\mathbb{H})\) contains at least some elements, one is \(0_{\mathbb{R}(\mathbb{H})}\).

Note that the quotients in Quot are only formal. We will write \(0_{\text{Quot}}\) instead of \(0_{\mathbb{R}(\mathbb{H})}/q\) for all \(q \in \text{sums}(\mathbb{H}), q \neq 0_{\mathbb{R}(\mathbb{H})}\). Note that we will construct a field \(\mathbb{F}(\mathbb{H})\), and we assume that in this case the semigroup \((\mathbb{H}, \cdot)\) is commutative.

We find informations on rings, fields and semigroups in [1] and [2].

**Proposition 1.3.** The triple \(\mathbb{R}(\mathbb{H}), +_{\mathbb{H}}, \ast\) is a ring, where \(+_{\mathbb{H}}, \ast : \mathbb{R}(\mathbb{H}) \times \mathbb{R}(\mathbb{H}) \rightarrow \mathbb{R}(\mathbb{H})\).

Furthermore, we will define two operations \(+_{\text{Quot}}, \ast_{\text{Quot}} : \text{Quot} \times \text{Quot} \rightarrow \text{Quot}\). Note that we omit mostly the symbol ‘\([\cdot]_\equiv\)’ of equivalent classes in the following.

**Proof.** We define \(x \ast 0_{\mathbb{R}(\mathbb{H})} := 0_{\mathbb{R}(\mathbb{H})} =: 0_{\mathbb{R}(\mathbb{H})} \ast x\) for each \(x \in \mathbb{R}(\mathbb{H})\). For positive natural numbers \(N, K\) for \(x = \langle \sum_{i=1}^{N} \pm_{i,1} x_i \rangle\), \(y = \langle \sum_{j=1}^{K} \pm_{j,2} y_j \rangle\), \(x, y, p, q, r, s \in \text{sums}(\mathbb{H})\) we define the multiplications for \(q, s \neq 0_{\mathbb{R}(\mathbb{H})}\)

\[
\frac{p}{q} \cdot_{\text{Quot}} \frac{r}{s} := \frac{p \ast r}{q \ast s} \quad \text{in Quot}, \quad \text{while in } \mathbb{R}(\mathbb{H}) \text{ we multiply } (x, y \text{ as above }):
\]

\[
x \ast y := \left[ \sum_{i=1}^{N} \sum_{j=1}^{K} \pm_{i,j,\ast} x_i \cdot y_j \right]_{\equiv} \quad (1.2)
\]

where in the last term the signs \(\pm_{i,j,\ast} \in \{+, -\}\) are determined by the rules \(\pm_{i,j,\ast} := +\) if \(\pm_{i,1} \pm_{j,2} = ++\) or \(\pm_{i,1} \pm_{j,2} = -+\) and \(\pm_{i,j,\ast} := -\) if \(\pm_{i,1} \pm_{j,2} = +-\) or \(\pm_{i,1} \pm_{j,2} = --\). The summation in the last term is made by a formal sum of multiplications in the semigroup \((\mathbb{H}, \cdot)\), provided with signs \(\pm_{i,j,\ast}\).

**Remark 1.4.** The associativity of \((\mathbb{R}(\mathbb{H}), \ast)\) relies on the associativity of the semigroup \((\mathbb{H}, \cdot)\).

The construct \((\mathbb{R}(\mathbb{H}), \ast)\) is commutative if and only if the generating semigroup \((\mathbb{H}, \cdot)\) is commutative.

For length \(N = 0\) we have the empty sum \(\langle \rangle := \langle \sum_{i=1}^{0} \pm_{i,1} x_i \rangle\). For positive natural numbers \(N, K\) we define the sum

\[
x +_{\mathbb{H}} y := \left[ \sum_{l=1}^{\max(N,K)} \pm_{l,1} x_l \pm_{l,2} y_l \right]_{\equiv} \quad \text{where} \quad (1.3)
\]
for the left distributivity in $(\mathbb{R}(\mathbb{H}), +_{\mathbb{H}})$ we define $\pm_{l,1} x_l \pm_{l,2} y_l := \pm_{l,1} x_l$ for $K < l \leq N$,  
if $N < K$ we define $\pm_{l,1} x_l \pm_{l,2} y_l := \pm_{l,2} y_l$ for $N < l \leq K$.\n
(1.4)\n
Hence, with relation ‘$\approx_2$’ the addition ‘$+_{\mathbb{H}}$’ in $(\mathbb{R}(\mathbb{H}), +_{\mathbb{H}})$ is commutative. For $x = \langle \sum_{i=1}^{N} \pm_{i} x_i \rangle$ 
we define $-x := \langle \sum_{i=1}^{N} \mp_{i} x_i \rangle$, where if $\pm_{i}$ is ‘$+$’ then $\mp_{i}$ is ‘$-$’, and vice versa. By the relation ‘$\approx_1$’ we get $[x]_{/\mathbb{H}} +_{\mathbb{H}} [-x]_{/\mathbb{H}} = 0_{\mathbb{R}(\mathbb{H})}$.

It holds for $x, y, z \in \mathbb{R}(\mathbb{H})$:

$(x +_{\mathbb{H}} y) +_{\mathbb{H}} z = x +_{\mathbb{H}} (y +_{\mathbb{H}} z)$ and $(x * y) * z = x * (y * z)$.

We have the associativity for both operations. The pair $(\mathbb{R}(\mathbb{H}), +_{\mathbb{H}})$ is an Abelian group.

We are defining two operations $+_\text{Quot}, *_\text{Quot} : \text{Quot} \times \text{Quot} \rightarrow \text{Quot}$. For $p, q, x, y \in \mathbb{R}(\mathbb{H})$ and $q, y \not\in 0_{\mathbb{R}(\mathbb{H})}$ we define the addition ‘$+_\text{Quot}$’ in Quot,

\[ \frac{p}{q} +_\text{Quot} \frac{x}{y} := \frac{p * y +_{\mathbb{H}} q * x}{q * y} \]  

(1.6)

It holds for $a, b, c \in \text{Quot}$:

$(a +_\text{Quot} b) +_\text{Quot} c = a +_\text{Quot} (b +_\text{Quot} c)$ and $(a *_\text{Quot} b) *_\text{Quot} c = a *_\text{Quot} (b *_\text{Quot} c)$.

We have the associativity for both operations.

In the case we can construct a field $\mathbb{F}(\mathbb{H})$, the pair $(\mathbb{F}(\mathbb{H}), +_{\mathbb{H}})$ is an Abelian group, hence the semigroup $(\mathbb{H}, \cdot)$ and also the ring $(\mathbb{R}(\mathbb{H}), +_{\mathbb{H}}, *)$ have to be commutative.

Furthermore we have inverse elements and two neutral elements $0_{\mathbb{R}(\mathbb{H})}$ and $0_{\text{Quot}}$ in the pairs $(\mathbb{R}(\mathbb{H}), +_{\mathbb{H}})$ or $(\text{Quot}, +_\text{Quot})$, respectively, and we define the neutral element $1_{\text{Quot}}$. Note that we will omit the symbol ‘$/_{/\mathbb{H}}$’ of equivalent classes.

\[ x +_{\mathbb{H}} 0_{\mathbb{R}(\mathbb{H})} = x \text{ and } x +_{\mathbb{H}} -x = 0_{\mathbb{R}(\mathbb{H})}, \]

(1.7)

\[ 1_{\text{Quot}} *_{\text{Quot}} (x/y) := (x/y) *_{\text{Quot}} 1_{\text{Quot}} := x/y =: (x/y)/1_{\text{Quot}}, \]

(1.8)

\[ (r/s) *_{\text{Quot}} (x/1_{\text{Quot}}) := \frac{r * x}{s} =: (x/1_{\text{Quot}}) *_{\text{Quot}} (r/s), \]

(1.9)

\[ (x/y) *_{\text{Quot}} 0_{\text{Quot}} := 0_{\text{Quot}} =: 0_{\text{Quot}} *_{\text{Quot}} (x/y) \]

(1.10)

for all $r, s, x, y \in \mathbb{R}(\mathbb{H})$, for $s, y \not\in 0_{\mathbb{R}(\mathbb{H})}$. We get that the pair $(\mathbb{R}(\mathbb{H}), +_{\mathbb{H}})$ is an Abelian group.

For the left distributivity in $(\mathbb{R}(\mathbb{H}), +_{\mathbb{H}}, *)$ we need to show

\[ a * (x +_{\mathbb{H}} y) = a * x +_{\mathbb{H}} a * y \]

(1.11)

We use the definition of the sum in line (1.3). Additionally we define $a := \langle \sum_{j=1}^{A} \pm_{j,3} a_j \rangle \in \text{sums}(\mathbb{H})$. It holds

\[ a * (x +_{\mathbb{H}} y) = \langle \sum_{j=1}^{A} \pm_{j,3} a_j \rangle * (x +_{\mathbb{H}} y) \]

(1.12)

\[ = \langle \sum_{j=1}^{A} \pm_{j,3} a_j \rangle * \left( \sum_{l=1}^{\text{max}(N,K)} \pm_{l,1} x_l \pm_{l,2} y_l \right) \]

(1.13)

\[ = \langle \sum_{j=1}^{A} \sum_{l=1}^{\text{max}(N,K)} \pm_{j,l,4} a_j \cdot x_l \pm_{j,l,5} a_j \cdot y_l \rangle \]

(1.14)

\[ = \langle \sum_{j=1}^{A} \sum_{l=1}^{N} \pm_{j,l,4} a_j \cdot x_l \rangle +_{\mathbb{H}} \langle \sum_{j=1}^{A} \sum_{l=1}^{K} \pm_{j,l,5} a_j \cdot y_l \rangle = a * x +_{\mathbb{H}} a * y \]

(1.15)
The signs \( \pm_{j,l,4} \in \{+,-\} \) are determined by the following rules: If \( \pm_{j,3}\pm_{l,1} = ++ \) or if \( \pm_{j,3}\pm_{l,1} = -- \) we set \( \pm_{j,l,4} = + \). In the case \( \pm_{j,3}\pm_{l,1} = +- \) or if \( \pm_{j,3}\pm_{l,1} = -+ \) we define \( \pm_{j,l,4} = - \). For \( \pm_{j,15} \) also a corresponding rule holds. The signs \( \pm_{j,15} \in \{+,-\} \) are conditioned by the pair of signs \( \pm_{j,3}\pm_{l,2} \).

Please see the explanation after line (1.2). If \( N \neq K \) we use the rule (1.4) and the following one: If \( K < N \) we define \( \pm_{j,l,4} a_j \cdot x_l \pm_{j,l,5} a_j \cdot y_l := \pm_{j,l,4} a_j \cdot x_l \) for \( K < l \leq N \), or if \( K > N \) we define \( \pm_{j,l,4} a_j \cdot x_l \pm_{j,l,5} a_j \cdot y_l := \pm_{j,l,5} a_j \cdot y_l \) for \( N < l \leq K \).

This ensures the left distributivity of \( (\mathcal{R}(\mathbb{H}), +_{\mathbb{H}}, \cdot^*) \). The right distributivity works in the same manner.

Therefore it holds in \( (\mathcal{R}(\mathbb{H}), +_{\mathbb{H}}, \cdot^*) \) the laws of distributivity, i.e. \( (\mathcal{R}(\mathbb{H}), +_{\mathbb{H}}, \cdot^*) \) is a ring.

**Remark 1.5.** This ring \( \mathcal{R}(\mathbb{H}) \) is free of zero divisors and it has an unit if and only if the generating semigroup \( (\mathbb{H}, \cdot^\flat) \) has an unit \( 1_{\mathbb{H}} \), i.e. it holds \( 1_{\mathbb{H}} \cdot a = a = a \cdot 1_{\mathbb{H}} \) for all \( a \in \mathbb{H} \). The unit in the ring \( \mathcal{R}(\mathbb{H}) \) in this case is \( [(+1_\mathbb{H})]_{/\sim} \).

**Lemma 1.6.** In the case that the semigroup \( (\mathbb{H}, \cdot^\flat) \) is commutative, we get that both operations \( +_{\text{Quot}} \) and \( \cdot_{\text{Quot}} \) from the triple \( (\text{Quot}, +_{\text{Quot}}, \cdot_{\text{Quot}}) \) are commutative. Furthermore, from the equations \( x \cdot_{\text{Quot}} y = x \cdot_{\text{Quot}} z \) or \( y \cdot_{\text{Quot}} x = z \cdot_{\text{Quot}} x \) the equation \( y = z \) follows, for \( x, y, z \in \text{Quot} \) and \( x \neq 0_{\text{Quot}} \).

**Proof.** The commutativity and the associativity of the semigroup \( (\mathbb{H}, \cdot^\flat) \) causes the commutativity and the associativity of both operations of the ring \( (\mathcal{R}(\mathbb{H}), +_{\mathbb{H}}, \cdot^*_{\mathbb{H}}) \) and as a consequence the commutativity and associativity of the operations in \( (\text{Quot}, +_{\text{Quot}}, \cdot_{\text{Quot}}) \), too.

We prove the last claim of the lemma: Let \( x = p/q \) and \( y = r/s, z = v/w \) for \( p, q, r, s, v, w \in \mathcal{R}(\mathbb{H}) \) and \( p, q, s, w \neq 0_{\mathcal{R}(\mathbb{H})} \). We multiply the equation \( x \cdot_{\mathcal{F}} y = x \cdot_{\mathcal{F}} z \) with the inverse \( q/p \) of \( x \) and we get \( y = z \). □

We introduce a third relation \( \approx_{3} \) to cancel common factors in Quot to ensure the distributivity. With this relation and if the multiplication \( \cdot^* \) is commutative, the ordinary relation \( a/b \approx c/d \) if and only if \( ad = bc \) is fulfilled.

Let \( r, x, y \in \mathcal{R}(\mathbb{H}), r, y \neq 0_{\mathcal{R}(\mathbb{H})} \). We say that two elements have the relation \( \approx_{3} \),

\[
\frac{x}{y} \approx_{3} \frac{r \cdot x}{r \cdot y}.
\] (1.16)

Let \( \approx_{3} \) be the smallest equivalence relation on Quot that includes the relation \( \approx_{3} \). We name the equivalent classes by \( [\cdot]_{/\approx_{3}} \).

**Definition 1.7.** Let \( \mathcal{F}(\mathbb{H}) := \text{Quot}_{/\approx_{3}} \).

Let \( +_{\mathcal{F}}, \cdot_{\mathcal{F}} \) be the canonical continuations on \( \mathcal{F}(\mathbb{H}) \) of the operations \( +_{\text{Quot}}, \cdot_{\text{Quot}} \) for equivalent classes. For instance, we define the addition \( +_{\mathcal{F}} \) by

\[
\left[ \frac{x}{y} \right]_{/\approx_{3}} +_{\mathcal{F}} \left[ \frac{v}{w} \right]_{/\approx_{3}} := \left[ \frac{x \cdot w +_{\mathcal{F}} y \cdot v}{y \cdot w} \right]_{/\approx_{3}}.
\] (1.17)

**Lemma 1.8.** In the case of a commutative semigroup \( (\mathbb{H}, \cdot^\flat) \) the laws of distributivity hold in \( (\mathcal{F}(\mathbb{H}), +_{\mathcal{F}}, \cdot_{\mathcal{F}}) \).

**Proof.** The distributivity of \( (\mathcal{F}(\mathbb{H}), +_{\mathcal{F}}, \cdot_{\mathcal{F}}) \) relies on the distributivity and commutativity of the ring \( (\mathcal{R}(\mathbb{H}), +_{\mathbb{H}}, \cdot^*_{\mathbb{H}}) \). For instance, to prove the left distributivity in \( (\mathcal{F}(\mathbb{H}), +_{\mathcal{F}}, \cdot_{\mathcal{F}}) \) we need to confirm the equation

\[
\frac{r \cdot a}{s \cdot a} \cdot_{\mathcal{F}} \left( \frac{x \cdot b}{y \cdot b} +_{\mathcal{F}} \frac{v \cdot c}{w \cdot c} \right) = \left( \frac{r \cdot a}{s \cdot a} \cdot_{\mathcal{F}} \frac{x \cdot b}{y \cdot b} \right) +_{\mathcal{F}} \left( \frac{r \cdot a}{s \cdot a} \cdot_{\mathcal{F}} \frac{v \cdot c}{w \cdot c} \right),
\] (1.18)

\[
a, b, c, r, s, x, y, v, w \in \mathcal{R}(\mathbb{H}), \quad a, b, c, s, y, w \notin 0_{\mathcal{R}(\mathbb{H})}.
\] (1.19)
We omitted the symbols ‘\([\cdot]/_{\equiv 3}\)’.

**Proposition 1.9.** We can construct the set \( F(\mathbb{H}) \), and if the generating semigroup \((\mathbb{H}, \cdot)\) is commutative, we have that the triple \((F(\mathbb{H}), +_{\mathbb{H}}, \cdot_{\mathbb{H}})\) is a field, where

\[ +_{\mathbb{H}}, \cdot_{\mathbb{H}} : F(\mathbb{H}) \times F(\mathbb{H}) \rightarrow F(\mathbb{H}). \]

**Proof.** Most of the constructions are made already above. The operations are independent of the representatives of the equivalent classes. The division ‘\( \frac{r}{s} \)’ is defined by

\[
\left[ \frac{r}{s} \right]_{\equiv 3} / \left[ \frac{h}{g} \right]_{\equiv 3} := \left[ \frac{r}{s} \cdot \frac{g}{h} \right]_{\equiv 3} := \left[ \frac{r \cdot g}{s \cdot h} \right]_{\equiv 3} \quad \text{for } s, g, h \neq 0_{F(\mathbb{H})}.
\]

We use the symbols ‘\(+_{\mathbb{H}}\)’ and ‘\(+_{\mathbb{H}}\)’ for two different operations on \( R(\mathbb{H}) \) and \( F(\mathbb{H}) \), respectively. The neutral elements in \((F(\mathbb{H}), +_{\mathbb{H}})\) and \((F(\mathbb{H}), \cdot_{\mathbb{H}})\) are \(0_{F(\mathbb{H})} := [0_{\mathbb{H}}]_{\equiv 3}\) and \(1_{F(\mathbb{H})} := [1_{\mathbb{H}}]_{\equiv 3}\), respectively. The inverse element of \([r/s]_{\equiv 3}\) in \(F(\mathbb{H})\) is \([s/r]_{\equiv 3}\), of course.

We consider \(\mathbb{H}\) as a part of \(R(\mathbb{H})\) by the embedding \(e_1\),

\[
e_1 : \mathbb{H} \rightarrow R(\mathbb{H}), \quad a \mapsto [(+a)]_{\equiv 3}.
\]

Note that \(e_1 : (\mathbb{H}, \cdot) \rightarrow (R(\mathbb{H}), \cdot)\) is a semigroup homomorphism, i.e. \(e_1(a \cdot b) = e_1(a) \cdot e_1(b)\). We constructed the set \(F(\mathbb{H})\), and in the special case of a commutative semigroup \((\mathbb{H}, \cdot)\) we get that \((F(\mathbb{H}), +_{\mathbb{H}}, \cdot_{\mathbb{H}})\) is a field.

In this case of a commutative semigroup \((\mathbb{H}, \cdot)\) we regard \(R(\mathbb{H})\) as a part of \(F(\mathbb{H})\) by the embedding \(e_2\),

\[
e_2 : (R(\mathbb{H}), +_{\mathbb{H}}, \cdot_{\mathbb{H}}) \rightarrow (F(\mathbb{H}), +_{\mathbb{H}}, \cdot_{\mathbb{H}}) \quad \text{where } e_2(0_{R(\mathbb{H})}) := 0_{F(\mathbb{H})}, \text{ and for } [(\pm x_1 \pm x_2 \pm x_3 \ldots \pm x_{N-1} x_N)]_{\equiv 3} \in R(\mathbb{H}) \text{ we define }
\]

\[
e_2([(\pm x_1 \pm x_2 \pm x_3 \ldots \pm x_{N-1} x_N)]_{\equiv 3}) := [(\pm x_1 \pm x_2 \pm x_3 \ldots \pm x_{N-1} x_N)]_{\equiv 3} \text{ in } F(\mathbb{H}).
\]

We get that the map \(e_2\) is a ring homomorphism, i.e. it holds \(e_2(x +_{\mathbb{H}} y) = e_2(x) +_{\mathbb{H}} e_2(y)\) and \(e_2(x \cdot_{\mathbb{H}} y) = e_2(x) \cdot_{\mathbb{H}} e_2(y)\) for all \(x, y \in R(\mathbb{H})\). Also it holds that \(e_2 \circ e_1 : (\mathbb{H}, \cdot) \rightarrow (F(\mathbb{H}), \cdot_{\mathbb{H}})\) is a semigroup homomorphism, i.e. \(e_2 \circ e_1(a \cdot b) = e_2 \circ e_1(a) \cdot_{\mathbb{H}} e_2 \circ e_1(b)\), for \(a, b \in \mathbb{H}\).

**Theorem 1.10.** Let \( f : (\mathbb{H}, \cdot) \rightarrow (R, \cdot_{\mathbb{H}}) \) be a semigroup homomorphism, i.e. \(f(a \cdot b) = f(a) \cdot_{\mathbb{H}} f(b)\) for \(a, b \in \mathbb{H}\), where the triple \((R, +_{\mathbb{H}}, \cdot_{\mathbb{H}})\) is a ring.

Then there exists a unique ring homomorphism \(f^2 : (R(\mathbb{H}), +_{\mathbb{H}}, \cdot_{\mathbb{H}}) \rightarrow (R, +_{\mathbb{H}}, \cdot_{\mathbb{H}})\) such that the following diagram commutes, i.e. \(f = f^2 \circ e_1\).

**Proof.** We call \(0_{\mathbb{H}}\) the zero element in \((R, +_{\mathbb{H}}, \cdot_{\mathbb{H}})\), i.e. \(r +_{\mathbb{H}} 0_{\mathbb{H}} = r\), for all \(r \in R\). We abbreviate the element \(x \in \text{sums}(\mathbb{H}), \; x \neq 0_{\mathbb{H}}\) by

\[
x := (\pm x_1 \pm x_2 \pm x_3 \ldots \pm x_{N-1} x_N).
\]

We define the map \(f^2\) by

\[
f^2 : (R(\mathbb{H}), +_{\mathbb{H}}, \cdot_{\mathbb{H}}) \rightarrow (R, +_{\mathbb{H}}, \cdot_{\mathbb{H}}), \text{ we set } f^2(0_{R(\mathbb{H})}) := 0_{R} \text{ and }
\]

\[
f^2([(\pm a)]_{\equiv 3}) = +f(a) \text{ for each } a \in \mathbb{H}. \text{ Therefore the values of } f^2
\]
are determined by the values of \( f \) on \( \mathbb{H} \).

The map \( f^t \) is a ring homomorphism, i.e. \( f^t(x +_{\mathbb{H}} y) = f^t(x) +_{\text{Ring}} f^t(y) \) and \( f^t(x \cdot y) = f^t(x) \cdot_{\text{Ring}} f^t(y) \), where \( x, y \in \mathbb{R}(\mathbb{H}) \). The proof that \( f^t \) is truly a ring homomorphism is straightforward. The map \( f^t \) is independent of representatives.

\[
\begin{array}{ccc}
(\mathbb{H}, \cdot) & \xrightarrow{e_1} & (\mathbb{R}(\mathbb{H}), +_{\mathbb{H}}, \ast) \\
\downarrow f & & \downarrow f^t \\
(\mathbb{R}, +_{\text{Ring}}, \cdot_{\text{Ring}}) & &
\end{array}
\]

\[\square\]

**Theorem 1.11.** Let \( g : (\mathbb{H}, \cdot) \to (F, \cdot_{\text{Field}}) \) be another semigroup homomorphism, i.e. \( g(a \cdot b) = g(a) \cdot_{\text{Field}} g(b) \), where the triple \((F, +_{\text{Field}}, \cdot_{\text{Field}})\) is a field. Since it is a field, it is also a ring, too. By the above Theorem 1.10, there exists a unique ring homomorphism \( g^t \) such that \( g = g^t \circ e_1 \). Additionally we assume that this map \( g^t \) is injective.

Then it exists a unique field homomorphism \( g^\nabla : (F(\mathbb{H}), +_{\mathbb{H}}, \ast) \to (F, +_{\text{Field}}, \cdot_{\text{Field}}) \) such that the diagram after next commutes, i.e. \( g = g^\nabla \circ e_2 \circ e_1 \) and \( g^t = g^\nabla \circ e_2 \).

The following lemma is needed to define an embedding from the ring \((\mathbb{R}(\mathbb{H}), +_{\mathbb{H}}, \ast)\) into the field \((F(\mathbb{H}), +_{\mathbb{H}}, \ast)\). Note that we just have built the field of fractions \( F(\mathbb{H}) \) of the ring \( \mathbb{R}(\mathbb{H}) \).

**Lemma 1.12.** With the above conditions in this theorem we get that the semigroup \((\mathbb{H}, \cdot)\) is commutative, and it follows that the operations in the pairs \((\mathbb{R}(\mathbb{H}), \ast)\) and \((F(\mathbb{H}), \cdot)\) are also commutative. As a consequence we are able to construct the field \((F(\mathbb{H}), +_{\mathbb{H}}, \cdot_{\mathbb{H}})\).

**Proof.** of Theorem 1.11

Since the triple \((F, +_{\text{Field}}, \cdot_{\text{Field}})\) is a field, it is also a ring. By Theorem 1.10 there exists a unique ring homomorphism \( g^t \) such that the following diagram commutes, i.e. \( g = g^t \circ e_1 \).

\[
\begin{array}{ccc}
(\mathbb{H}, \cdot) & \xrightarrow{e_1} & (\mathbb{R}(\mathbb{H}), +_{\mathbb{H}}, \ast) \\
\downarrow g & & \downarrow g^t \\
(F, +_{\text{Field}}, \cdot_{\text{Field}}) & &
\end{array}
\]

Let \( 0_{\text{Field}}, 1_{\text{Field}} \) be the neutral elements in \((F, +_{\text{Field}})\) and \((F, \cdot_{\text{Field}})\), respectively. The map \( g^t \) is injective, i.e. the kernel of \( g^t \) is \( \{0_{\text{Field}}\} \).

We define \( g^\nabla : (F(\mathbb{H}), +_{\mathbb{H}}, \cdot_{\mathbb{H}}) \to (F, +_{\text{Field}}, \cdot_{\text{Field}}) \).

For arbitrary \( p := \langle \pm_{1,1} p_1 \pm_{2,1} p_2 \pm_{3,1} p_3 \ldots \pm_{N-1,1} p_{N-1} \pm_{N,1} p_N \rangle \) and \( y := \langle \pm_{1,2} y_1 \pm_{2,2} y_2 \pm_{3,2} y_3 \ldots \pm_{M-1,2} y_{M-1} \pm_{M,2} y_M \rangle \), \( p, y \in \text{sums}(\mathbb{H}) \), \( [y]_{\mathbb{H}} \neq 0_{\mathbb{R}(\mathbb{H})} \) we define

\[
\begin{align*}
g^\nabla(0_{F(\mathbb{H})}) & := 0_{\text{Field}} \quad \text{and} \quad g^\nabla(1_{F(\mathbb{H})}) := 1_{\text{Field}} \\
\left[ p \right]_{\mathbb{H}} & := \frac{g^\nabla\left( [p]_{\mathbb{H}} \right)}{1_{F(\mathbb{H})}} := g^\nabla\left( [p]_{\mathbb{H}} \right) := g^\nabla\left( [p]_{\mathbb{H}} \right) := (1.25) \\
\pm_{1,1} g(p_1) & \pm_{2,1} g(p_2) \pm_{3,1} g(p_3) \ldots \pm_{N-1,1} g(p_{N-1}) \pm_{N,1} g(p_N) \\
& \frac{1}{1_{\text{Field}}} \quad \text{for} \left[ p \right]_{\mathbb{H}} \neq 0_{\mathbb{R}(\mathbb{H})} \quad \text{and} \quad (1.26)
\end{align*}
\]
as well as for \( w := \frac{p}{y} \) we define

\[
g^\nabla (\frac{[w]}{\approx_3}) := \frac{g^\sharp([p]/\approx)}{g^\sharp([y]/\approx)} = \frac{\pm 1,1 g(p_1) \pm 2,1 g(p_2) \pm 3,1 g(p_3) \ldots \pm N-1,1 g(p_{N-1}) \pm N,1 g(p_N)}{\pm 1,2 g(y_1) \pm 2,2 g(y_2) \pm 3,2 g(y_3) \ldots \pm M-1,2 g(y_{M-1}) \pm M,2 g(y_M)}.
\]

The condition that \( g^\sharp \) is injective ensures that the denominator \( g^\sharp([y]/\approx) \) of \( g^\nabla (\frac{[w]}{\approx_3}) \) is not zero. Note that if \( \frac{p}{y} = \frac{r \ast a}{r \ast b} \), we have

\[
g^\nabla (\frac{[w]}{\approx_3}) = \frac{g^\sharp([p]/\approx)}{g^\sharp([y]/\approx)} = \frac{g^\sharp([r \ast a]/\approx \cdot_F \text{Field} \ g^\sharp([a]/\approx)}{g^\sharp([r \ast b]/\approx \cdot_F \text{Field} \ g^\sharp([b]/\approx)} = \frac{g^\sharp([a]/\approx)}{g^\sharp([b]/\approx)}.
\]

The values of \( g^\nabla \) are determined uniquely by the values of \( g \) on \( \mathbb{H} \). The uniqueness of \( g^\nabla \) is clear since \( g(a) = g^\nabla \circ e_2 \circ e_1(a) = g^\nabla \circ e_2 \left( \frac{[+a]}{\approx} \right) = g^\nabla \left( \frac{[\langle +a \rangle]}{\approx} \right) = +g(a) \), for \( a \in \mathbb{H} \).

Please see the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{H}, & \xrightarrow{e_1} & \mathbb{R}(\mathbb{H}), (+_\mathbb{H}, \ast) \xrightarrow{e_2} \mathbb{F}(\mathbb{H}), (+_F, \cdot_F) \\
g & \downarrow & g^\nabla \\
(F, +_\text{Field}, \cdot_F) & \xrightarrow{id} & (F, +_\text{Field}, \cdot_F)
\end{array}
\]

\[\square\]

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**References**

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