TRIVIALITY PROPERTIES OF PRINCIPAL BUNDLES ON SINGULAR CURVES-II

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ABSTRACT. For $G$ a split semi-simple group scheme and $P$ a principal $G$-bundle on a relative curve $X \to S$, we study a natural obstruction for the triviality of $P$ on the complement of a relatively ample Cartier divisor $D \subset X$. We show, by constructing explicit examples, that the obstruction is nontrivial if $G$ is not simply connected but it can be made to vanish, if $S$ is the spectrum of a dvr (and some other hypotheses), by a faithfully flat base change. The vanishing of this obstruction is shown to be a sufficient condition for etale local triviality if $S$ is a smooth curve, and the singular locus of $X - D$ is finite over $S$.

1. INTRODUCTION

Let $f : X \to S$ be a proper, flat and finitely presented curve over an arbitrary scheme (over Spec($\mathbb{Z}$)) $S$. Let $G$ be a split reductive group scheme over Spec($\mathbb{Z}$), base changed to $S$, and $B$ a Borel subgroup of $G$. Let $D \subset X$ be a relatively ample effective Cartier divisor which is flat over $S$, and set $U = X \setminus D$. Generalizing results of Drinfeld and Simpson [DS95] for the case of smooth $f$, the following result was proved in [BF, Theorem 1.4] without any conditions on the singularities of $f$:

Theorem 1.1. Let $P$ be a principal $G$-bundle on $X$ with $G$ semisimple and simply connected. Then, after a surjective etale base change $S' \to S$, $P$ is trivial on $U_{S'}$.

Now suppose $G$ is semisimple, but not necessarily simply connected. and. Triviality statements similar to the above are proved in [BF, Theorem 1.5] but with stronger hypotheses: for example, in characteristic zero, the Cartier divisor $D$ is not allowed to pass through the singular locus of $f$, and $D$ is also assumed to be set theoretically a union of sections of $f$ (and some other mild conditions).

In this note, motivated in part by the article [Sol], we study the analogue of Theorem 1.1 for non-simply connected $G$. In this case there is a natural obstruction to local triviality constructed as follows:

Let $\tilde{G}$ be the simply connected cover of $G$ and denote by $\pi_1(G)$ the scheme theoretic kernel o the covering map $\tilde{G} \to G$. The central exact sequence of sheaves of groups (on the fppf site of $X$),

$$1 \to \pi_1(G)_X \to \tilde{G}_X \to G_X \to 1$$

(1.1)

gives rise to a boundary map in fppf cohomology

$$H^1_{\text{fppf}}(X, G) \to H^2_{\text{fppf}}(X, \pi_1(G)).$$

Therefore, from $P \in H^1_{\text{fppf}}(X, G)$, we get an element $\alpha_P \in H^2_{\text{fppf}}(X, \pi_1(G))^1$. It is clear that if $P$ is trivial on $U$, then $\alpha_P$ maps to zero in $H^2_{\text{fppf}}(U, \pi_1(G))$. Thus, for the generalization of Theorem 1.1 to hold for $P$, the following property (L) must hold:

(L) There exists a surjective etale morphism $S' \to S$ such that $\alpha_P$ maps to zero in $H^2_{\text{fppf}}(U_{S'}, \pi_1(G))$.

We show that this property is nontrivial: For $G = \text{PGL}(m)$, we construct principal $G$-bundles on families of curves $X \to S$ with nodal singularities, and $D$ passing through the singularities of $f$, where property (L), so also the direct generalization of Theorem 1.1, fails (Proposition 3.1). These examples include cases when $S$ is a smooth curve, and $X \to S$ is a family of smooth curves degenerating to curve with a single nodal singularity and the divisor $D$ passes through the node. Examples for other classical groups $G$ can be constructed using similar methods.

\footnote{We note that for a smooth group scheme, fppf cohomology is the same as etale cohomology. In particular, if $|\pi_1(G)|$ is invertible in $\mathcal{O}_S$, we may replace fppf cohomology by etale cohomology throughout this paper.}
Even though condition (L) is not always satisfied, we show in Lemma 2.2 that the weaker condition (L') below often holds, e.g., when $S$ is a smooth curve and $U$ is smooth over $S$:

(L') There exists a faithfully flat morphism $S' \to S$ such that $\alpha_P$ maps to zero in $H^2_{fl}(U_{S'}, \pi_1(G))$.

We are thus faced with:

**Question 1.2.** Does condition (L') always hold? If so, does there always exist a faithfully flat morphism $S' \to S$ such that $P$ becomes trivial on $U_{S'}$?

We do not know the answer to this in full generality. However, we prove (Theorem 2.5):

**Theorem 1.3.** Let $f : X \to S$ be flat projective curve, $D \subset X$ a relatively ample Cartier divisor which is flat over $S$ and set $U = X \setminus D$. Let $G$ be a semisimple group and let $P$ be a principal $G$-bundle on $X$. Assume further that

1. $S$ is an excellent regular (purely) one dimensional scheme and
2. $U$ is smooth over $S$.

Then there is a faithfully flat morphism $S' \to S$ such that $P$ becomes trivial on $U_{S'}$.

We also show in Proposition 2.6 that for $G = \text{PGL}(m)$, and $P$ satisfying a condition weaker than (L), and $S$ smooth, $P$ lifts to a principal $\text{GL}(m)$-bundle on $X$ (after an étale base change in $S$). This result generalizes to arbitrary groups, see Remark 2.7.

2. **Consequences of condition (L)**

2.1. **The case $S$ is regular of dimension one.**

**Lemma 2.1.** Let $U$ be a regular Noetherian scheme and $D \subset U$ a closed subscheme. Let $V = U \setminus D$ and let $\{D_i\}$ be the irreducible components of $D$ of codimension one. Then for any integer $n > 0$, the kernel of the restriction map $H^2_{fl}(U, \mu_n) \to H^2_{fl}(V, \mu_n)$ is the subgroup generated by the first Chern classes of all $\mathcal{O}_U(D_i)$.

The lemma is well-known for étale cohomology, but we do not know a reference for fppf cohomology so we give a proof.

**Proof.** We first note that for any scheme $U$, $H^1_{fl}(U, \mathbb{G}_m) = \text{Pic}(U)$, and if $U$ is regular then $H^2_{fl}(U, \mathbb{G}_m) = \text{Br}(U)$.

Consider the commutative diagram

\[
\begin{array}{ccccccccc}
H^0_{fl}(U, \mathbb{G}_m) & \longrightarrow & H^0_{fl}(V, \mathbb{G}_m) & \longrightarrow & H^1_{D,fl}(U, \mathbb{G}_m) & \xrightarrow{g_1} & H^1_{fl}(U, \mathbb{G}_m) & \xrightarrow{g_2} & H^1_{fl}(V, \mathbb{G}_m) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1_{fl}(U, \mu_n) & \longrightarrow & H^1_{fl}(V, \mu_n) & \longrightarrow & H^2_{D,fl}(U, \mu_n) & \xrightarrow{c} & H^2_{fl}(U, \mu_n) & \longrightarrow & H^2_{fl}(V, \mu_n) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1_{fl}(U, \mathbb{G}_m) & \xrightarrow{f_1} & H^1_{fl}(V, \mathbb{G}_m) & \longrightarrow & H^2_{D,fl}(U, \mathbb{G}_m) & \longrightarrow & H^2_{fl}(U, \mathbb{G}_m) & \xrightarrow{f_2} & H^2_{fl}(V, \mathbb{G}_m) \\
\end{array}
\]

where the rows come from the long exact sequence of cohomology with supports and the columns from the Kummer sequence.

The map $f_1$ is surjective because $U$ is regular and the map $f_2$ is injective by [Gro68, Corollaire 1.10]. This implies that $H^2_{D,fl}(U, \mathbb{G}_m) = 0$ so the map $c$ is surjective. The claim then follows by a simple diagram chase, noting that the map $c_U$ gives, by definition, the first Chern class of a line bundle on $U$ and the kernel of $g_2$ (equal to the image of $g_1$) is precisely the subgroup of $\text{Pic}(U)$ generated by the $\mathcal{O}_U(D_i)$.

□
Lemma 2.2. Let $f : U \to S$ be a smooth morphism of relative dimension one with $S$ the spectrum of a henselian dvr $R$ with quotient field $K$ and residue field $k$. Given any element $\alpha \in H^2_{fl}(U, \mu_n)$ there exists a faithfully flat morphism $S' \to S$, with $S'$ also the spectrum of a dvr, such that the pullback of $\alpha$ in $U_{S'}$ is $0$.

If $R$ is excellent, or $n$ is invertible in $O_S$, the proof shows that $S' \to S$ can be chosen to be finite.

Proof. Let $U_0$ be the closed fibre of $f$ and set $V = U \setminus U_0$. Since $V$ is an affine curve over $K$, $H^2_{fl}(V_K, \mu_n) = 0$. This follows from the Kummer sequence, and the following facts for smooth curves $Y$ over an algebraically closed field: The vanishing of $\text{Br}(Y)$ (Tsen’s theorem) and, if $Y$ is affine, the surjectivity of multiplication by $n$ map on Pic($Y$) for $n > 0$. It follows that there exists a finite extension $K_1$ of $K$ so that the image of $\alpha$ in $H^2_{fl}(V_{K_1}, \mu_n)$ becomes $0$ in $H^2_{fl}(V_K, \mu_n)$. By replacing $R$ by its integral closure $R_1$ in $K_1$ (which is still a dvr), functoriality implies that we may assume $\alpha$ is in the kernel of the restriction map $r : H^2_{fl}(U, \mu_n) \to H^2_{fl}(V, \mu_n)$.

Since $f$ is smooth, $U$ is regular, so by Lemma 2.1 the kernel of $r$ is spanned by the fundamental classes of the irreducible components of $U_0$. Let $R \to R'$ be a finite map, with $R'$ a dvr, such that the ramification degree is divisible by $n$ and set $S' = \text{Spec}(R')$. The pullbacks of the Chern classes of all the components of $U_0$ become divisible by $n$, hence are all $0$ in $H^2_{fl}(U_{S'}, \mu_n)$. We conclude that $\alpha$ also becomes $0$ in $H^2_{fl}(U_{S'}, \mu_n)$. □

Proposition 2.3. Assume

(1) $S$ is an excellent regular (purely) one dimensional scheme (e.g., a smooth curve over a field).
(2) $X$ is reduced.
(3) The closure of the non-regular locus of $U = X - D$ does not intersect $D$; this is equivalent to assuming that the non-regular locus of $U$ is finite over $S$, e.g., $U$ has isolated singularities.
(4) $\alpha_P$ is zero when restricted to $U$.

Then there is a surjective étale morphism $S' \to S$ such that $P$ is trivial on $U_{S'}$.

Proof. For the sake of clarity we first deal with the case $U$ is regular. Let $g : \tilde{X} \to X$ be a resolution of singularities of $X$ [Lip78]. By our assumptions $U$ is regular so we can assume $U \subset \tilde{X}$. Note that $\tilde{X} \to S$ is flat since $S$ is regular and one-dimensional.

We first show that $\tilde{X} - U$ supports a relatively ample Cartier divisor $\tilde{D}$ (possibly non-effective). By assumption $D$ supports a relatively ample divisor $D'$. A resolution of singularities for $X$ can be obtained by iterating the process of normalization and then blowing up the singular locus (cf. Lip78, and $X$ is excellent). Let the resulting schemes be denoted by $X_0 = X, X_1, \ldots, X_s = \tilde{X}$. We build relatively ample Cartier divisors $D_r$ at each step of this resolution $X_r, D_0 = D'$, and finally set $\tilde{D} = D_s$. For the normalization step, we just pull back the Cartier divisor from the previous step. For a resolution step $g : X_{r+1} \to X_r$, let $E_{r+1}$ be the exceptional divisor of the blow up $g$. It is easy to see that then $D_{r+1} = g^*(nD_r) - E_{r+1}$ is relatively ample for $n$ sufficiently large (use Har77, Proposition II.7.10): We may assume that $D'$, and hence each $D_r$, is actually ample since $S$ is affine).

Therefore, $\tilde{X} - U$ supports a relatively ample Cartier divisor $\tilde{D}$ (possibly non-effective). Let $L = O_{\tilde{X}}(\tilde{D})$ be the corresponding relatively ample line bundle on $\tilde{X}$; it is trivial on $U$.

Assume that $S$ is affine. By replacing $S$ by an étale cover we may assume that $P$ has a $B$-reduction, where $B$ is a Borel subgroup of $G$, and then we may also assume (as in §3.2.2 of [BF]) that $P$ is induced from an $H$-bundle $E$, where $H$ is a maximal torus of $B$. 
Let \( \tilde{G} \) be the simply connected cover of \( G \) and let \( Z \cong \pi_1(G) \) be the kernel of the covering map \( \tilde{G} \to G \). Let \( \tilde{H} \) be the maximal torus in \( \tilde{G} \) mapping onto \( H \), so \( Z \subset \tilde{H} \). We have a commutative diagram

\[
\begin{array}{ccc}
1 & \rightarrow & Z \\
\downarrow & & \downarrow \\
1 & \rightarrow & \tilde{G} \\
\downarrow & & \downarrow \\
1 & \rightarrow & \tilde{H} \\
\downarrow & & \downarrow \\
1 & \rightarrow & H \\
\end{array}
\]

whose rows are exact sequences of group schemes. Since we have assumed that \( \alpha_P \) becomes 0 on \( U \), by the commutativity of the diagram it follows that \( E|_U \) lifts to a \( \tilde{H} \)-bundle \( \tilde{E}_U \) on \( U \). Since \( \tilde{H} \) is a torus and \( \tilde{X} \) is regular, \( \tilde{E}_U \) extends to a \( \tilde{H} \)-bundle \( \tilde{E} \) on \( \tilde{X} \) (this follows from the fact that line bundles on \( U \) extend to \( \tilde{X} \)).

Let \( \tilde{P} \) be the induced \( \tilde{G} \)-bundle on \( \tilde{X} \). Since \( \tilde{G} \) is simply connected, and \( \tilde{X} - U \) supports a relatively ample Cartier divisor \( \tilde{D} \), by (almost) the same argument as in \( \S 3.2.2 \) of [BF] (see Remark 2.4 below) we see that \( \tilde{P}|_U \) is trivial, hence \( P|_U \) is also trivial.

If \( U \) is not regular, we employ the following strategy: We choose a partial desingularization \( \tilde{X} \to X \) which is an isomorphism over \( U \), such that \( \tilde{X} \) is regular in a neighborhood of \( \tilde{X} - U \). To see that such a partial desingularization exists, first consider a full desingularization \( \tilde{Q} \to X \). We just carry out only those blow ups with support over \( X - U \), and normalize only in neighbourhoods of inverse images of \( X - U \), and obtain \( \tilde{X} \to X \) which is an isomorphism over \( U \), with \( \tilde{X} \) regular on the complement of \( U \).

**Remark 2.4.** [BF, Theorem 1.4] can be generalized as follows: Let \( S \) be an arbitrary scheme over \( \text{Spec}(\mathbb{Z}) \) and let \( f : X \to S \) be a proper, flat and finitely presented curve over \( S \). Let \( E \) be a principal \( G \)-bundle on \( X \) with \( G \) semisimple and simply connected. Let \( U \subset X \) be an open subset, affine over \( S \), such that \( X - U \) supports a relatively ample Cartier divisor \( D \) for \( X \to S \) (possibly non-effective, whose components need not be flat over \( S \)). Then, after a surjective étale base change \( S' \to S \), \( E \) is trivial on \( U_{S'} \).

To prove this we need to slightly modify the proof of [BF, Proposition 3.2] as follows: Assume \( S \) is affine. We twist by tensor powers of \( L = \mathcal{O}(D) \) to find subbundles \( \mathcal{O} \subset E_i \otimes \mathcal{O}^r \) with corresponding quotients \( T_i \), \( i = 1, 2 \). On the affine open subset \( U \subset X \), these extensions of vector bundles split, and \( L \) is trivial. Therefore, restricted to \( U \) we get \( E_i = \mathcal{O} \oplus T_i \), with \( T_i \) a line bundle. But \( T_i \) is the determinant of \( E_i \), hence \( T_1 \) and \( T_2 \) are isomorphic on \( U \).

By combining Lemma 2.2 and Proposition 2.3 we obtain:

**Theorem 2.5.** Let \( f : X \to S \) be flat projective curve, \( D \subset X \) a relatively ample Cartier divisor which is flat over \( S \) and set \( U = X \setminus D \). Let \( G \) be a semisimple group and let \( P \) be a principal \( G \)-bundle on \( X \).

Assume further that

1. \( S \) is an excellent regular (purely) one dimensional scheme and
2. \( U \) is smooth over \( S \).

Then there is a faithfully flat morphism \( S' \to S \) such that \( P \) becomes trivial on \( U_{S'} \).

**Proof.** Since \( \pi_1(G) \) is a finite group scheme of multiplicative type, by applying Lemma 2.2 we may find a faithfully flat finite type cover \( S_1 \to S \), with \( S_1 \) also excellent regular and one dimensional, such that \( \alpha_P \) becomes 0 on \( U_{S_1} \). We then apply Proposition 2.3 to the induced morphism \( X_1^\text{red} \to S_1 \), where \( X_1 := X \times_S S_1 \), to get an étale cover \( S' \to S_1 \) so that \( P \) becomes trivial on \( U_{S'} \). The composition of the maps \( S' \to S_1 \to S \) is the desired faithfully flat map.

\[ \square \]

### 2.2. Lifting to vector bundles

A principal \( \text{PGL}(m) \)-bundle \( P \) on \( X \) gives rise to a cohomology class \( \alpha_P \in \text{H}^2_{\text{ff}}(X, \mu_m) \) as well a cohomology class \( \beta_P \in \text{H}^2_{\text{et}}(X, \mathbb{G}_m) (= \text{H}^2_{\text{ff}}(X, \mathbb{G}_m)) \), by considering the exact sequence of group schemes

\[ 1 \to \mathbb{G}_m \to \text{GL}(m) \to \text{PGL}(m) \to 1 \]
Clearly $\alpha_P$ maps to $\beta_P$ under the natural map $H^1_f(X, \mu_m) \to H^2_{et}(X, \mathbb{G}_m)$. It is easy to see that $\beta_P$ represents the obstruction to lifting $P$ to a principal $\text{GL}(m)$-bundle, i.e., a vector bundle on $X$.

Condition (L) implies that $\beta_P$ maps to zero in $H^2_{et}(U, \mathbb{G}_m)$, i.e., $P$ can be lifted to a vector bundle on $U$. In fact, under somewhat mild conditions, $P$ can be lifted to a vector bundle on $X$ after a surjective étale base change of $S$:

**Proposition 2.6.** Assume $S$ is smooth, and the smooth locus of $X \to S$ is dense in every fiber. If $\beta_P$ maps to zero in $H^2_{et}(U, \mathbb{G}_m)$, then after an étale base change in $S$, $P$ comes from a vector bundle on $X$ and hence $\beta_P \in H^2_{et}(X, \mathbb{G}_m)$ becomes zero.

**Proof.** After an étale base change in $S$, we can find sections of $X \to S$ such that their union is disjoint from $D$ and contained in the smooth locus of $X \to S$. We can also assume that the union of these sections is relatively ample. Let $U' \subset X$ be the complement of these sections. Using [BF, Theorem 1.5], we may assume that $P$ is trivial on $U'$ after an étale base change in $S$. Lift this trivial $\text{PGL}(m)$-bundle to a vector bundle $W$ on $U'$.

By assumption, $P$ comes from a vector bundle $V$ on $U$. Thus, we have two $\text{GL}(m)$-bundles $W$ and $V$ on $U' \cap U$ which coincide as $\text{PGL}(m)$-bundles. Let $L^*$ be the sheaf of isomorphisms $V \to W$ which induce identity on the underlying $\text{PGL}(m)$-bundle. Clearly $L^*$ is a $\mathbb{G}_m$-bundle, let $L$ be the corresponding line bundle. Hence $W$ is isomorphic to $V \otimes L$ on $U' \cap U$. Extend $L$ to a line bundle on $U$ ($U$ is smooth along $U - U' \cap U$ since $S$ is smooth and the sections have images in the smooth locus of $X \to S$). Now glue the vector bundle $V \otimes L$ (a vector bundle on $U'$) with $W$ (a vector bundle on $U'$) over $U' \cap U$, to get a vector bundle $A$ on $X$. The $\text{PGL}(m)$-bundle induced from $A$ equals $P$ which completes the proof. \qed

The examples in Section 3 all had $\beta_P = 0$, i.e., came from vector bundles on $X$. So $\beta_P = 0$ is a lot weaker than $\alpha_P = 0$.

**Remark 2.7.** The proof of Prop 2.6 works more generally: For $G$ any semisimple group (replacing $\text{PGL}(m)$) let $G'$ be as in Lemma 2.1 below. For $P$ a principal $G$-bundle on $X$ we get, as above, elements $\alpha_P \in H^1_f(X, \pi_1(G))$ and $\beta_P \in H^2_{et}(X, K)$. Suppose $\beta_P$ maps to zero in $H^2_{et}(U, K)$, which would be the case if $\alpha_P$ maps to zero in $H^1_1(U, \pi_1(G))$ (i.e., if condition (L) holds). Then, after an étale base change in $S$, $P$ comes from a principal $G'$-bundle on $X$ and hence $\beta_P \in H^2_{et}(X, K)$ becomes zero.

The following lemma is well-known, we give a proof for the convenience of the reader.

**Lemma 2.1.** For any semisimple group $G$ there exists a reductive group $G'$ mapping surjectively to $G$ with kernel a central torus $K$, and such that the derived group of $G'$ is simply connected.

**Proof.** Let $\tilde{G}$ be the simply connected cover of $G$ and let $\tilde{T}$ be any torus in $\tilde{G}$ containing the kernel $Z \cong \pi_1(G)$ of the covering map $\tilde{G} \to G$. Then we may take $G'$ to be $(\tilde{G} \times \tilde{T}/Z)$, where $Z$ is embedded diagonally. There is a natural map $G' \to G$ induced by projection to the first factor and the kernel of this is $\tilde{T}$ (embedded in $G'$ via the second factor). Moreover, the derived group of $G'$ is equal to $\tilde{G}$ (embedded via the first factor). One gets a somewhat canonical construction by taking $\tilde{T}$ to be a maximal torus. \qed

**Remark 2.8.** There are many choices for $G'$, e.g., for $G = \text{PGL}(m)$ (resp. $\text{PGSp}(2m)$) one may also take $G'$ to be $\text{GL}(m)$ (resp. $\text{GSp}(2m)$).

3. The examples

3.1. By constructing examples where property (L) does not hold, we show that Theorem 1.1 fails if $G$ is not assumed to be simply connected.

Let $S$ be a smooth curve over a field of characteristic zero (for simplicity) and let $X \to S$ a family of projective curves with a unique singular fibre over the point $s_0 \in S$ having an ordinary double point at the point $x_0$ over $s_0$. Locally in the étale topology at $x_0$, the family looks like the surface with equation $xy - z^{n+1} = 0$, for some $n > 0$, with the map given by $(x, y, z) \mapsto z$.

We assume that the family has a section $\sigma : S \to X$ not passing through $x_0$ and we also assume that there is a section $\tau : S \to X$ with $\tau(s_0) = x_0$. Such families with sections can be constructed by base
changing any family as above by a suitable map \( S' \to S \), with \( S' \) also a smooth curve, factoring through \( X \). Note that the local class group at \( x_0 \) is \( \mathbb{Z}/(n+1)\mathbb{Z} \) (use the method of proof of [Har77, Example II.6.5.2]), so \( D = (n+1)\tau(S) \) is a Cartier divisor which is flat and, as is easily seen, relatively ample over \( S \). Set \( U = X - D \).

Let \( L = \mathcal{O}_X(\sigma(S)) \); for any positive integer \( m \), the first Chern class of \( L \) gives a cohomology class \( c_1(L) \in H^2_{et}(X, \mu_m) \). Let \( P \) be the \( \text{PGL}(m) \)-bundle on \( X \) induced from the \( \text{GL}(m) \)-bundle \( L \oplus \mathcal{O}^{\oplus(m-1)} \).

Using the identification \( \pi_1(G) = \mu_m \), it is easy to see that \( \alpha_P = c_1(L) \in H^2_{et}(X, \mu_m) \).

**Proposition 3.1.** For suitable \( m \) and \( n \), \( P \) is not trivial on \( U_{S'} \) for any \( \text{étale} \) neighborhood \( S' \) of \( s_0 \).

Let \( S' \) be an \( \text{étale} \) neighbourhood of \( s_0 \in S \). By functoriality of the boundary map (1.1) Proposition 3.1 follows from

**Claim 3.2.** For suitable \( m \) and \( n \), the class \( c_1(L) \) restricts to a non-zero class in \( H^2_{et}(X_{S'} - \tau(S'), \mu_m) \).

It suffices to prove this for \( S = S' \), since \( \text{étale} \) base change does not alter any of the properties of the family \( X \to S \).

### 3.2. The singularity of \( X \) at \( x_0 \) is \( \text{étale} \) locally equivalent to \( xy - zn+1 = 0 \), so of type \( A_n \). The exceptional divisor of the minimal resolution \( Q \) consists of a chain of \( n \) smooth rational curves \( E_1, E_2, \ldots, E_n \), each with self-intersection \(-2\) and with \( E_i \) intersecting \( E_{i+1} \) transversely, \( i = 1, \ldots, n-1 \); see for example, [BHPVdV04, III.7]. The fibre \( F \) over \( s_0 \), hence its strict transform \( \tilde{F} \), may or may not be irreducible; in the latter case we write \( \tilde{F}_1 \) and \( \tilde{F}_2 \) for the components. Locally the fibre corresponds to the curve \( xy = 0, z = 0 \), so its strict transform intersects \( E_1 \) and \( E_n \) transversely (in a single point each). The strict transform \( \tilde{D} \) of \( D = \tau(S) \) intersects the exceptional divisor in a single point which must be smooth (since \( Q \) is smooth), so lies on a unique exceptional divisor \( E_i \). Note that \( t \) can be arbitrary: if \( S' \) is a smooth curve in \( Q \) which intersects \( E_1 \) transversely at a point \( s_0' \) then \( X' = S' \times_S X \) has a singular fibre over \( s_0' \) and a section which on the corresponding desingularization \( Q' \) passes through \( E_i \).

By considering tangent directions, one sees that \( \tilde{D} \) does not intersect \( \tilde{F}_1 \); \( F \) and \( \tilde{D} \) become disjoint after a single blowup.

Let \( C = \sigma(S) \) and \( \tilde{C} \) its strict transform in \( Q \). Let \( E = \cup E_i \), so we have \( X - D = Q - \tilde{D} \cup E \).

Note that \( H^2_{D\cup E, et}(Q, \mu_m) \) is freely generated by the classes of \( \tilde{D} \) and all the \( E_i \). Thus, using the Gysin sequence, it suffices to show that the class of \( \tilde{C} \) in \( H^2_{et}(Q, \mu_m) \) is not in the span of the classes of \( \tilde{D} \) and the \( E_i \).

Assume that we have an equation

\[
[\tilde{C}] = a[\tilde{D}] + \sum_i b_i[E_i] \in H^2_{et}(Q, \mu_m)
\]

We will get a contradiction, in certain cases, by using elementary intersection theory. Each irreducible component \( G \) of \( E \cup \tilde{F} \) is a proper curve, so we have maps

\[
H^2_{et}(Q, \mu_m) \to H^2_{et}(G, \mu_m) \to \mathbb{Z}/m\mathbb{Z},
\]

where the first map is pullback and the second is the degree map (which is an isomorphism). Moreover, the pullback map is compatible with the restriction of line bundles or, equivalently, intersections of divisors.

#### 3.2.1. Suppose \( \tilde{F} \) is reducible. We may assume that \( \tilde{C} \) passes through \( \tilde{F}_1 \), \( \tilde{F}_1 \) intersects \( E_1 \) and \( \tilde{F}_2 \) intersects \( E_n \). Restricting both sides of (3.1) to each of \( \tilde{F}_1, \tilde{F}_2 \) and all the \( E_i \) and using the degree isomorphisms, we get \( n+2 \) equations in \( n+1 \) unknowns. We show below that this leads to a contradiction if \( m \) does not divide \( t \), which will certainly be the case if \( m > n \).

- Intersecting (3.1) with \( \tilde{F}_1 \), we get \( b_1 = 1 \). Intersecting with \( \tilde{F}_2 \) gives \( b_n = 0 \).
- By induction, we may prove that if \( i \leq t \), \( b_i = ib_1 \). These equations are obtained by intersecting the two sides of (3.1) by \( E_1, \ldots, E_{t-1} \). Therefore \( b_t = tb_1 \).
- By descending induction from \( i = n \), we can prove that \( b_i = 0 \) if \( i \geq t \): The case \( i = n \) is already known. Intersecting with \( E_n \) (if \( t < n \)) gives \( b_{n-1} = 2b_n \). Intersect (3.1) with \( E_i \) (if \( i > t \)) to get \( b_{i-1} - 2b_i + b_{i+1} = 0 \), and hence \( b_{i-1} = 0 \).

Therefore we have \( b_t = tb_1 = 0 \in \mathbb{Z}/m\mathbb{Z} \), and hence \( t = 0 \in \mathbb{Z}/m\mathbb{Z} \).
3.2.2. Now suppose $\hat{F}$ is irreducible.

We now intersect both sides of (3.1) with the classes of $\hat{F}$ and all the $E_i$ and then apply the degree isomorphisms, getting $n + 1$ equations in $n + 1$ unknowns. As we show below these equations imply that $t$ is a linear combination of $n + 1$ and $m$. Therefore, if the gcd of $n + 1$ and $m$ does not divide $t$ (for example, if $n + 1$ divides $m$), we reach a contradiction.

- Intersecting (3.1) with $\hat{F}$, we get $b_1 + b_n = 1$.
- By induction, we may prove that if $i \leq t$, $b_i = ib_1$. These equations are obtained by intersecting the two sides of (3.1) by $E_1, \ldots, E_{t-1}$. Therefore $b_t = t b_1$.
- By descending induction from $i = n$, we can prove that that $b_i = (n - i + 1)b_n$ if $i \geq t$: The case $i = n$ is already known. Intersecting with $E_n$ (if $t < n$) gives $b_{n-1} = 2b_n$. Intersect (3.1) with $E_i$ (if $i > t$) to get $b_{i-1} - 2b_i + b_{i+1} = 0$, and hence $b_{i-1} = (n - (i - 1) - 1)b_n$.

Therefore $b_t = t b_1 = (n - t + 1)b_n$, so $t (1 - b_n) = (n - t + 1)b_n$, hence $b_n(n + 1) = t$. We thus see that $t$ is a linear combination of $n + 1$ and $m$.

3.3. Let $X \to S$ be a family of curves as at the beginning of this section and let $f : Q \to S$ be the minimal desingularization of $X$. Let $S^{sh}$ be the strict henselisation of $S$ at $s_0$ and $Q^{sh} = Q \times_S S^{sh}$. By the proper base change theorem, the restriction map $H^2_{et}(Q^{sh}, \mu_m) \to H^2_{et}(X_0, \mu_m)$ is an isomorphism, where $X_0$ is the (geometric) fibre of $f$ over $s_0$. Now $H^2_{et}(X_0, \mu_m)$ is a free $\mathbb{Z}/m\mathbb{Z}$-module with basis the irreducible components of $X_0$, i.e., the $E_i$ and $\hat{F}$ (or $\hat{F}_1$ and $\hat{F}_2$). It follows from this that for any $\alpha \in H^2_{et}(Q, \mu_m)$, if the numerical equations obtained by intersecting both sides of an equality

$$\alpha = a[D] + \sum_i b_i [E_i] \in H^2_{et}(Q, \mu_m)$$

with the $E_i$ and $\hat{F}$ (or $\hat{F}_1$ and $\hat{F}_2$) can be solved, then in fact (3.2) itself can be solved (for the pullbacks) in $H^2_{et}(Q^{sh}, \mu_m)$, so also when pulled back to some étale neighbourhood $S'$ of $s_0 \in S$. In particular, this applies to $[\hat{C}]$ as above.

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