Soliton–potential interaction in the nonlinear Klein–Gordon model

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Abstract
The interaction of solitons with external potentials in the nonlinear Klein–Gordon field theory is investigated using an improved model. The presented model has been constructed with a better approximation for adding the potential to the Lagrangian through the metric of background space–time. The results of the model are compared with another model and the differences are discussed.

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1. Introduction
Nonlinear evolution equations in mathematical physics are mostly studied because of their important role in most of the branches of science. The Klein–Gordon (KG) equation is a relativistic version of Schrödinger’s equation which describes the field equation for scalar particles (spin-0). The KG equation has been the most frequently studied equation for describing the particle dynamics in quantum field theory. This equation with various types of potentials has appeared in field theories where it is called the nonlinear Klein–Gordon (NKG) model. The pion form factor [1], hadronic atoms [2], Josephson junction [3], superfluid current disruption [4], shallow water [5] and Hawking radiation from a black hole [6] are some examples of the NKG’s applications.

In recent years, several solitonic solutions for the NKG equations have been proposed using different methods. These solutions have appeared in a homogeneous and well-behaved medium, while in the real world the medium of propagation contains disorders and impurities which add local space-dependent potentials to the problem. The behaviour of the NKG solitons during interaction with these local potentials and their stability after the interaction are important questions. Therefore, the investigation of these situations is very interesting. It is a very important subject because of its applications and from the mathematical point of view. Motivated by this situation, we have studied the interaction of the NKG solitons with defects by using different methods and the results are presented in this paper. Hereafter, a brief description of the NKG model and its solitons will be given in section 2. The motivation of the present work and some applications of the model in the presence of local potentials will be proposed in this section, too. Methods of adding the potential to the soliton equation of motion will be described in section 3. The interaction of solitons with the potential walls and potential wells will be investigated in section 4. Conclusions and remarks will be presented in the final section.

2. The nonlinear Klein–Gordon equation
Different types of NKG equations have been proposed. They differ in their nonlinear terms which arise from different applications. Some of them are as follows:

\[
\frac{\partial^2 \phi}{\partial t^2} - a^2 \frac{\partial^2 \phi}{\partial x^2} + \alpha \phi - \beta \phi^3 = 0,
\]

and

\[
\frac{\partial^2 \phi}{\partial t^2} - a^2 \frac{\partial^2 \phi}{\partial x^2} + \alpha \phi - \beta \phi^3 + \gamma \phi^5 = 0,
\]

in which \(\alpha\) and \(\beta\) are arbitrary constants and \(\gamma = \frac{3\epsilon^2}{16\sigma}\). Several localized solutions such as solitons, compactons and solitary waves have been found for these equations by using different methods [7–10]. In this paper, we will focus on the soliton solutions of equation (2.2). The results are valid for the other equations, too.

Wazwaz has found several localized solutions for equation (2.2) using the ‘tanh’ method. The one-soliton solution for this equation is [7]

\[
\phi(x, t) = \sqrt{\frac{2\alpha}{\beta}} \left[ 1 \pm \tanh \left( \frac{\alpha}{a^2 - u^2} (x - x_0 - ut) \right) \right],
\]

(2.3)
where $x_0$ is the soliton initial position and $u$ is its velocity. The signs ‘+’ and ‘−’ in equation (2.3) denote the kink and antikink solutions, respectively. Figure 1 presents kink and antikink solutions with $u = 0.5$, $a = 1$, $x_0 = 0$ and $\alpha = \beta = 1$ at $t = 0$.

As mentioned before, the KG-type field theories have been applied in many investigations. The quantum aspects of the black hole have been studied considering a time-dependent classical black hole solution induced by the KG soliton [6]. The Hawking radiation and the Bondi reflected energy of the back reaction of the metric were calculated by adding a lump of energy through the KG field as

$$U(\phi) = \frac{\beta}{4} \left( \phi^2 - \frac{\mu^2}{\beta} \right)^2. \quad (2.4)$$

The parameters $\beta$ and $\mu$ are constant if the soliton is sufficiently narrow. In this situation, the effects of gravity on the characteristics of the soliton are neglected. For broader energy lumps, we cannot take constant parameters for the soliton and its characteristics which are spatial functions [11, 12]. Therefore, a better situation can be investigated if we consider a potential for the KG part of the theory with space-dependent parameters such as $\beta(x)$ and $\mu(x)$.

The behaviour of transition regions in one-dimensional (1D) multistable continuous media is described by the NKG equation with dissipation (NKGD) [13]

$$\mu \phi_{tt} + \gamma \phi_t - D \phi_{xx} + P(\phi) = 0, \quad (2.5)$$

where $P(\phi)$ is a function of the field $\phi$. Generally, the parameters of the model are functions of the medium’s characteristics such as temperature, viscosity and so on; however, such a system has not been studied with constant parameters so far. It is clear that we need a suitable model containing variable (at least slowly varying) parameters.

Much work has been devoted to the dynamics of the 1D chain of coupled double–well oscillators and its applications in condensed matter [14–16]. This model as well as the Frenkel–Kontorova chain became a very convenient and popular tool for the theoretical study of dynamical and statistical properties of a number of materials with strong quasi-1D anisotropy [17]. One of the interesting systems of this class is a chain of hydrogen bonds where the proton transfers in adjacent hydrogen bonds are correlated because of the cooperative nature of hydrogen bonding. The proton dynamics in an ice crystal can be described using the discrete NKG equation [18, 19]

$$\phi_{tt} - c_0^2 \phi_{xx} + \omega_0^2 V(\phi) = 0. \quad (2.6)$$

It is clear that a more realistic model contains disorders and dislocations in the hydrogen chain, which can be modelled using space-dependent parameters in the NKG field theory.

### 3. Two models for ‘the nonlinear Klein–Gordon soliton–potential’ system

**Model 1:** Consider a scalar field with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) \quad (3.7)$$

and the following potential:

$$U(\phi) = \lambda(x) \left( \frac{1}{2} \alpha \phi^2 - \frac{1}{6} \beta \phi^4 + \frac{1}{6} \gamma \phi^6 \right). \quad (3.8)$$

The equation of motion for the field becomes

$$\partial_\mu \partial^\mu \phi + \lambda(x) \left( \alpha \phi^2 - \beta \phi^4 + \gamma \phi^6 \right) = 0. \quad (3.9)$$

The effects of the external potential can be added to the equation of motion by using a suitable definition for $\lambda(x)$, such as $\lambda(x) = 1 + v(x)$ [20–22]. For a constant value of $\lambda$, $\lambda(x) = 1$, equation (3.9) reduces to (2.2) and therefore (2.3) is its exact solution. Thus (2.3) is used as an initial condition for solving (3.9) with a space-dependent $\lambda(x)$ when the potential $v(x)$ is small.

The Hamiltonian density of the Lagrangian (3.7) is

$$\mathcal{H}_1 = \frac{1}{2} \phi^2 + \frac{1}{2} \dot{\phi}^2 + \lambda(x) \left( \frac{1}{2} \alpha \phi^2 - \frac{1}{6} \beta \phi^4 + \frac{1}{6} \gamma \phi^6 \right). \quad (3.10)$$

**Model 2:** The potential can also be added to the Lagrangian of the system, through the metric of background space–time. So the metric carries the medium’s characteristics. The general form of the action in an arbitrary metric is

$$I = \int \mathcal{L}(\phi, \partial_\mu \phi) \sqrt{-g} \, d^4x \, dt, \quad (3.11)$$

where $g$ is the determinant of the metric $g^{\mu \nu}(x)$. A suitable metric in the presence of a smooth and slowly varying weak potential $v(x)$ is [23, 24]

$$g^{\mu \nu}(x) \equiv \begin{pmatrix} 1 + v(x) & 0 \\ 0 & -\frac{1}{1 + v(x)} \end{pmatrix}. \quad (3.12)$$

The equation of motion for the field $\phi$ which is described by the Lagrangian (3.7) in the action (3.11) is [24, 25]

$$\frac{1}{\sqrt{-g}} \left( \sqrt{-g} \partial_\mu \partial^\mu \phi + \partial_\mu \phi \partial^\mu \sqrt{-g} \right) + \frac{\partial U(\phi)}{\partial \phi} = 0. \quad (3.13)$$
This equation of motion in the background space–time (3.12) becomes [26]

\[
(1 + v(x)) \frac{\partial^2 \phi}{\partial t^2} = \frac{1}{1 + v(x)} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial U(\phi)}{\partial \phi} = 0.
\]

(3.14)

The field energy density is

\[
\mathcal{H}_2 = \left(g^00(x) \right)^2 \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \dot{\phi}^2 + g^00(x) U(\phi).
\]

(3.15)

The energy density is calculated by varying both the field and the metric (see equation (11.81) page 643 of [25]). Solution (2.3) can be used as an initial condition for solving (3.14) when the potential \(v(x)\) is small.

4. Numerical simulations

Several simulations with different types of external potential \(v(x)\) have been performed using the two presented models. The smooth and slowly varying potential \(v(x) = a e^{-b(x-c)^2}\) has been used in the simulations that are reported below. Parameter \(a\) controls the strength of the potential, \(b\) represents its width and \(c\) adjusts the centre of the potential. If \(a > 0\), the potential shows a barrier, and for \(a < 0\) we have a potential well.

Simulations have been performed using the fourth-order Runge–Kutta method for time derivatives. Space derivatives were expanded using the finite difference method. Grid spacing has been taken as \(h = 0.01, 0.02\) and sometimes \(h = 0.001\). Time steps have been chosen as \(\frac{1}{2}\) of the space step \(h\) because of numerical stability considerations. Simulations have been set up with fixed boundary conditions and solitons have been kept far from the boundaries. We have controlled the results of simulations by checking the total energy as a conserved quantity during the simulation.

A moving soliton has kinetic and potential energies. Consider a static (zero-velocity) soliton; in this situation the soliton has only potential energy. The difference between the energies of the two static solitons in different positions comes from the difference between the potentials in those places. This means that one can find the potential as a function of the collective coordinate \(X\) using the energy of a static soliton in a different position \(X\). The shape of the potential (as a function of \(X\)) has been found by placing a static soliton at different positions and calculating its energy. It is clear that its energy is equal to the potential in that position. Figure 2 shows the shape of the potential barrier \(v(x) = 0.5 e^{-4x^2}\) in different models, which has been calculated using this method. This figure shows that the static parts of the potential are almost the same in both models. But we will find that the dynamical behaviour of the solitons is very different in the above models.

Suppose that a soliton is placed far away from the potential. It goes toward the barrier in order to interact with the potential. There exist two different kinds of trajectories for the soliton during the interaction with the barrier depending on its initial velocity, which are separated by a critical velocity \(u_c\). At low velocities \(u_i < u_c\), the soliton reflects back and reaches its initial place with the final velocity of \(u_f \approx u_i\). Figure 3 presents trajectories of a soliton with different values of initial velocity for model 2. The critical velocity can be found by sending a soliton toward the potential with different initial velocities.

Figure 4 presents the critical velocity as a function of potential height in the two models. The figure shows that the general behaviour of the soliton in the two models is the same. However, there is a small difference between the results of models 1 and 2. This comes from the difference in the rest mass of the soliton in the two models. Let us compare the soliton energy in model 1 (equation (3.10)) and in model 2 (equation (3.15)). The static parts of the energy in the two models are almost the same, while the kinetic energies in the two models are different. The effective mass of the soliton in model 1 is \(M_{\text{eff1}} = 1\), while the effective mass in model 2 is \(M_{\text{eff2}} = (1 + v(x))^2\). The soliton rest mass of model 2 is greater than the soliton rest mass of model 1. The difference in the soliton rest mass of the two models increases as the potential height grows. Thus the difference between the soliton critical velocities in the two models becomes larger, as clearly shown in figure 4.

Simulations show that the scattering of a soliton with a potential barrier is nearly elastic. The soliton radiates a small
amount of energy during the interaction. The energy radiated during the interaction in models 1 and 2 is almost the same.

The scattering of topological solitons on a potential well is more interesting. Unlike a classical point particle that always passes through a potential well, a soliton may be trapped in a potential well with a specific depth \[26, 27\]. It is shown that the \(\phi^4\) soliton does not have a fixed mass during the interaction with a potential well \[28\] and it works for the NKG soliton too. So we cannot look at the soliton as a point particle in some cases. Several simulations have been done for the NKG soliton–well system using the two models. Like in the case of the potential barrier, the general behaviour of the system is almost the same in both models. However, there are some differences in the details of the interactions. Figure 5 presents a comparison of the shapes of the potential well \(v(x) = -0.5e^{-4x^2}\), as seen by an NKG soliton in the two models. There is not a critical velocity for a soliton–well system, but we can define an escape velocity. Consider a soliton which is located at the initial position \(x_0\) near the centre of the potential \(v(x)\). As figure 5 clearly shows, the soliton energy is a function of its initial position. This soliton can escape to infinity if its initial speed becomes greater than the escape velocity. The escape velocity for this situation is defined as the one for which the asymptotic speed is null at infinity. A soliton with initial velocity above the escape velocity passes through the well, whereas a soliton with initial velocity lower than the escape velocity falls into the well and is trapped by the potential.

The rest mass can be calculated with integration of the Hamiltonian density (equation (3.10) for model 1; equation (3.15) for model 2) with respect to the position \(x\). The calculated rest masses using models 1 and 2 are very close to each other. The rest masses in the two models are exactly the same because of the same static part of the energy in the two models. Integration of the Hamiltonian density is calculated by the Runberg method for different values of the potential height.

As figure 5 shows, the effective potentials for these two models are the same. Therefore it is expected that the escape velocity of the soliton in models 1 and 2 finds the same values for different values of the potential height. Figure 6 presents
the escape velocity of a soliton in the potential well as a function of the potential depth. However, the potentials in the models are the same, but the calculated escape velocities using these models are not equal. The reason for this difference can be explained using figure 7 as follows.

Figure 7 presents the trajectory of a soliton with an initial velocity \( u = 0.35 \) during the interaction with the potential \( v(x) = -0.5e^{-4x^2} \). The final velocity of the soliton in model 2 is less than the final velocity of the soliton in model 1. This means that the energy loss due to radiation in model 2 is greater than in model 1. This is the main reason for the differences between the critical velocities observed in figure 6.

The most interesting behaviour of a soliton during the process of scattering on a potential well is seen in some very narrow windows of initial velocities. At some velocities less than \( u_c \) the soliton may reflect back or pass over the potential while one would expect that the soliton should be trapped in the potential, well. These narrow windows can be found by scanning the soliton initial velocity with small steps. Figure 8 presents this phenomenon in models 1 and 2. Figure 8(a) shows that a soliton with an initial velocity within the window \( 0.062 \leq u_i \leq 0.065 \) reflects back after interaction with the potential \( v(x) = -0.4e^{-4x^2} \) simulated using model 1. Figure 8(b) presents the same situation simulated using model 2. We could not find a soliton reflection in model 2 for the potential \( v(x) = -0.4e^{-4x^2} \). Some simulations have been performed for different values of the potential height \( a = -0.2, -0.3, -0.5, -0.6, -0.7, -0.8 \) in model 2, but such a soliton reflection from the potential well has not been observed at all. Model 2 is built by varying the Lagrangian density with respect to both ‘field’ and ‘metric’; therefore an energy exchange between the field and the potential is possible in this model. Some authors have related such a soliton reflection to the energy exchange between the field and the potential [28, 29]. It seems that this point is not true for the NKG solitons. Our simulations for the NKG solitons did not confirm this situation for model 2. This phenomenon needs to be subjected to more investigations.

5. Conclusion

Two models have been used to study the interaction of the NKG solitons with defects. The general behaviour of a soliton during the interaction with external potentials in the two models is almost the same. Model 1 adds the potential to the equation of motion by a different method than the one used in model 2. Model 1 is easy to simulate, while model 2 is more analytic. The two models confirm that the interaction of a soliton with a potential barrier is nearly elastic. At low velocities it reflects back, but at a high velocity it climbs the barrier and passes over the potential. The soliton radiates some amount of energy during the interaction with the potential. There exists a critical velocity that separates these two kinds of trajectories. The interaction of a soliton with the potential well is more inelastic. It is possible that a soliton scatters on a potential well and reflects back from the potential. This phenomenon depends on the model. Model 1 predicts this behaviour, but this is contrary to what can be seen for the sine-Gordon and \( \phi^4 \) models. These models show this behaviour for model 2, while such a behaviour is observed in model 1 for the NKG solitons.

It is interesting to investigate the scattering of solitons of other models on defects using model 2. These studies help us to gain more comprehensive knowledge of the general behaviour of solitons.

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