Some Remarks on an Eigenvalue Problem for an Anisotropic Elliptic Equation with Indefinite Weight

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Abstract. In this paper, we consider an eigenvalue problem for an anisotropic elliptic equation with indefinite weight, in which the differential operator involves partial derivatives with different variable exponents. Under some suitable conditions on the growth rates of the anisotropic coefficients involved in the problem, we prove some results on the existence and non-existence of a continuous family of eigenvalues by using variational methods.

1. Introduction

In this paper, we are interested in the eigenvalue problem for the following anisotropic elliptic equation

\[
\begin{cases}
    - \sum_{i=1}^{N} \partial_{x_i} \left( |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \right) = \lambda V(x) |u|^{q(x)-2} u \\
    u = 0, \quad x \in \partial \Omega,
\end{cases}
\]

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial \Omega$, and $p_i, i = 1, 2, \ldots, N$ are continuous functions on $\Omega$ such that $2 \leq p_i(x) < N, q(x) > 1$ for all $x \in \Omega$, $V: \Omega \to \mathbb{R}$ is an indefinite weight function in the sense that $V$ can change sign in $\Omega$, $\lambda$ is a positive parameter.

In the particular case when $p_i(x) = p(x)$ for each $i$ the differential operator involved in (1) becomes

\[
\sum_{i=1}^{N} \partial_{x_i} \left( |\partial_{x_i} u|^{p(x)-2} \partial_{x_i} u \right)
\]

and has similar properties with the $p(x)$-Laplace operator $\Delta_{p(x)} u = \text{div}(\nabla u |\nabla u|^{p(x)-2})$. We know that the $p(x)$-Laplace operator helps us to model some materials involving nonhomogenities, such as electrorheological fluids, see [18], but if we want to seek for the model of an inhomogeneous material which has a different behavior on each direction we note it is not adequate. For this case, we need to use the differential operator of anisotropic type as in (1) and we then call (1) an anisotropic partial differential equation with variable exponent. Problems of this type were firstly studied by Mihailescu et al. in the celebrated paper [13], in which the authors developed some previous results on anisotropic elliptic
equations with constant exponents of Fraga\'la et al. [8] by considering eigenvalue problem (1) in the case when the weight function $V \equiv 1$ and obtained some interesting results. After that, many authors studied the existence of solutions for anisotropic partial differential equations by various methods, we refer the readers to some papers [1–4, 6, 9, 17, 20].

In [10], Kefi studied the nonlinear eigenvalue problem for the $p(x)$-Laplace operator of the form

$$
\begin{align*}
-\Delta_{p(x)} u &= \lambda V(x)|u|^{p(x)-2} u \\
u &= 0, \quad x \in \partial \Omega,
\end{align*}
$$

(2)

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial \Omega$, $\lambda > 0$ is a parameter, $V : \Omega \to \mathbb{R}$ is an indefinite weight function belonging to $L^{s(x)}(\Omega)$ in the sense that $V$ may change sign in $\Omega$ and $p, q, s \in C(\overline{\Omega})$. Using variational arguments based on the Ekeland variational principle [7], the author proved that any $\lambda > 0$ sufficient small is an eigenvalue of problem (2) under the following sublinear condition at infinity

$$
1 < q(x) < p(x) \leq N < s(x), \quad \forall x \in \overline{\Omega}.
$$

In spired by the ideas introduced by Kefi [10] and some results on anisotropic elliptic equations mentioned above, in this paper we study a class of nonlinear eigenvalue problems for anisotropic elliptic equations with indefinite weight of the form (1). Our goal is to complement and extend the previous ones in [13] and [10] in the sense that Mihailescu et al. [13] considered problem (1) in the special case $V \equiv 1$ while Kefi [10] considered $p(x)$-Laplacian problem (2) only in the sublinear case. We also find that our results are better than those presented in [15, 16] since we consider the problem in the anisotropic case. By considering two different situations concerning the growth rates involved in the problem, we prove the existence of a continuous family of eigenvalues by using variational methods. It should be noticed that our situations are different from ones considered in [6, 14, 17].

The remainder of the paper is organized as follows. In Section 2, we will recall the definitions and some properties of generalized Lebesgue-Sobolev spaces and anisotropic variable exponent Sobolev spaces. The readers can consult the papers [5, 13] for details on this class of functional spaces. In Section 3 we will state and prove the main results of the paper.

2. Preliminaries

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ where $\Omega$ is an open subset of $\mathbb{R}^N$. In that context, we refer to the books [5, 18] and the papers of Kov\'a\'cik et al. [11] and Mihailescu et al. [13]. Set

$$
C_+(\overline{\Omega}) := \{ h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega} \}.
$$

For any $h \in C_+(\overline{\Omega})$ we define

$$
h^+ = \sup_{x \in \Omega} h(x) \text{ and } h^- = \inf_{x \in \Omega} h(x).
$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$
L^{p(x)}(\Omega) = \left\{ u : \text{a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)} \, dx < \infty \right\}.
$$

We recall the following so-called Luxemburg norm on this space defined by the formula

$$
|u|_{p(x)} = \inf \left\{ \mu > 0 : \int_\Omega \frac{|u(x)|^{p(x)}}{\mu} \, dx \leq 1 \right\}.
$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the H\ölder inequality holds, they are reflexive if and only if $1 < p^- \leq p^+ < \infty$ and continuous
functions are dense if \( p^+ < \infty \). The inclusion between Lebesgue spaces also generalizes naturally: if \( 0 < |\Omega| < \infty \) and \( p_1, p_2 \) are variable exponents so that \( p_1(x) \leq p_2(x) \) a.e. \( x \in \Omega \) then there exists the continuous embedding \( L^{p_1(x)}(\Omega) \hookrightarrow L^{p_2(x)}(\Omega) \). We denote by \( L^{p(x)}(\Omega) \) the conjugate space of \( L^{p(x)}(\Omega) \), where \( \frac{1}{p_1} + \frac{1}{p_2} = 1 \). For any \( u \in L^{p(x)}(\Omega) \) and \( v \in L^{p(x)}(\Omega) \) the Hölder inequality

\[
\left| \int_{\Omega} u v dx \right| \leq \left( \frac{1}{p_1} + \frac{1}{p_2'} \right) |u|_{p(x)} |v|_{p(x)}
\]

holds true. Moreover, if \( p_1, p_2, p_3 \in C(\overline{\Omega}) \) and \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \), then for any \( u \in L^{p_1(x)}(\Omega), v \in L^{p_2(x)}(\Omega) \) and \( w \in L^{p_3(x)}(\Omega) \) we have

\[
\left| \int_{\Omega} u v w dx \right| \leq \left( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \right) |u|_{p_1(x)} |v|_{p_2(x)} |w|_{p_3(x)}.
\]

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the modular of the \( L^{p(x)}(\Omega) \) space, which is the mapping \( \rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R} \) defined by

\[
\rho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} dx.
\]

If \( u \in L^{p(x)}(\Omega) \) and \( p^+ < \infty \) then the following relations hold

\[
|u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-},
\]

provided \( |u|_{p(x)} > 1 \) while

\[
|u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}
\]

provided \( |u|_{p(x)} < 1 \) and

\[
|u_n - u|_{q(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u_n - u) \to 0.
\]

**Proposition 2.1.** Let \( p(x) \) and \( q(x) \) be measurable functions such that \( p \in L^{\infty}(\Omega) \) and \( 1 \leq p(x)q(x) \leq +\infty \) for a.e. \( x \in \Omega \). Let \( u \in L^{p(x)}(\Omega) \) and \( u \neq 0 \). Then we have

\[
|u|_{p(x)q(x)} \leq 1 \Leftrightarrow |u|^{p^-}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)},
\]

\[
|u|_{p(x)q(x)} \geq 1 \Leftrightarrow |u|^{p^-}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}.
\]

In particular, if \( p(x) = p \) is a constant then \( ||u||_{q(x)} = |u|^{p}_{p(x)} \).

If \( p \in C(\overline{\Omega}) \) the variable exponent Sobolev space \( W^{1,p(x)}(\Omega) \), consisting of functions \( u \in L^{p(x)}(\Omega) \) whose distributional gradient \( \nabla u \) exists almost everywhere and belongs to \( [L^{p(x)}(\Omega)]^N \), endowed with the norm

\[
\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)},
\]

is a separable and reflexive Banach space. The space of smooth functions are in general not dense in \( W^{1,p(x)}(\Omega) \), but if the exponent \( p \in C(\overline{\Omega}) \) is logarithmic Hölder continuous, that is,

\[
|p(x) - p(y)| \leq \frac{M}{\log(|x - y|)}, \quad \forall x, y \in \Omega, \quad |x - y| \leq \frac{1}{2},
\]
then the smooth functions are dense in $W^{1,p(x)}(\Omega)$ and so we can define the space $W^{1,p(x)}_0(\Omega)$ as the closure of $\mathcal{C}^\infty(\Omega)$ under the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}$$

by the $p(x)$-Poincaré inequality. We point out that the above norm is equivalent with the following norm

$$\|u\|_{p(x)} = \sum_{i=1}^N |\partial_i u|_{p_i(x)},$$

provied that $p(x) \geq 2$ for all $x \in \overline{\Omega}$. The space $\left(W^{1,p(x)}_0(\Omega), \|\cdot\|_{p(x)}\right)$ is a separable and Banach space.

**Proposition 2.2.** If $s \in C_+(\overline{\Omega})$ and $s(x) < p'(x)$ for all $x \in \overline{\Omega}$ then the embedding

$$W^{1,p(x)}_0(\Omega) \hookrightarrow L^{p'(x)}(\Omega)$$

is compact and continuous, where $p'(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p'(x) = \infty$ if $p(x) > N$.

We introduce a natural generalization of the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ that will enable us to study problem (1) with sufficient accuracy. Define $\overrightarrow{p} : \overline{\Omega} \rightarrow \mathbb{R}^N$ the vectorial function $\overrightarrow{p}(x) = (p_1(x), p_2(x), \ldots, p_N(x))$, the components $p_i \in C(\overline{\Omega})$, $i \in \{1, 2, \ldots, N\}$ are logarithmic Holder continuous, that is, there exists $M > 0$ such that $|p_i(x) - p_i(y)| \leq \frac{M}{\log(|x - y|)}$ for any $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$ and $i \in \{1, 2, \ldots, N\}$. We introduce the anisotropic variable exponent Sobolev space, $W^{1,\overrightarrow{p}(x)}_0(\Omega)$, as the closure of $\mathcal{C}^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{p}(x)} = \sum_{i=1}^N |\partial_i u|_{p_i(x)}.$$ 

Then $W^{1,\overrightarrow{p}(x)}_0(\Omega)$ is a reflexive and separable Banach space, see [13]. In the case when $p_i$ are all constant functions the resulting anisotropic space is denoted by $W^{1,\overrightarrow{p}}_0(\Omega)$, where $\overrightarrow{p}$ is the constant vector $(p_1, p_2, \ldots, p_N)$. The theory of such spaces has been developed in [8]. Let us introduce $\overrightarrow{p}_+, \overrightarrow{p}_- \in \mathbb{R}^N$ and $p^*_+, p^*_-, p^+_, p^-_+ \in \mathbb{R}^+$ as

$$\overrightarrow{p}_+ = (p^*_1, p^*_2, \ldots, p^*_N), \quad \overrightarrow{p}_- = (p^-_1, p^-_2, \ldots, p^-_N)$$

and

$$p^*_+ = \max\{p^*_1, p^*_2, \ldots, p^*_N\}, \quad p^+_+ = \max\{p^-_1, p^-_2, \ldots, p^-_N\},$$

$$p^-_- = \min\{p^-_1, p^-_2, \ldots, p^-_N\}, \quad p^-- = \min\{p^*_1, p^*_2, \ldots, p^*_N\}.$$ 

Throughout this paper we assume that

$$\sum_{i=1}^N \frac{1}{F_i} > 1 \quad (8)$$

and define $P^*_- \in \mathbb{R}$ and $P^*_{-\infty} \in \mathbb{R}^+$ by

$$P^*_- = \frac{N}{\sum_{i=1}^N \frac{1}{F_i} - 1}, \quad P^*_{-\infty} = \max\{P^*_-, P^*_{-\infty}\}.$$ 

**Proposition 2.3 (see [13, Theorem 1]).** If $s \in C_+ (\overline{\Omega})$ satisfies $1 < s(x) < P^-_{-\infty}$ for all $x \in \overline{\Omega}$ then the embedding

$$W^{1,\overrightarrow{p}(x)}_0(\Omega) \hookrightarrow L^{p'(x)}(\Omega)$$

is compact and continuous.
3. Main results

In this section, we will state and prove the main results of the paper. Let us denote by $X$ the anisotropic variable exponent Sobolev space $W^{1,\overline{r}_{\alpha}}(\Omega)$ and denote by $c_1$ a general positive constant whose value may change from line to line. Problem (1) will be considered in two different situations when the nonlinearity is sublinear or superlinear at infinity in the sense that $q^+ < P_{\alpha}$ or $P_{\alpha}^+ < q^+\,$, respectively.

**Definition 3.1.** We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1) if there exists $u \in W^{1,\overline{r}_{\alpha}}(\Omega)\setminus\{0\}$ such that

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_x u|^q(x) \partial_x u \partial_y v \, dx - \lambda \int_{\Omega} V(x) |u|^{q(x)-2} u v \, dx = 0$$

for all $v \in W^{1,\overline{r}_{\alpha}}(\Omega)$. If $\lambda$ is an eigenvalue of problem (1) then the corresponding eigenfunction $u \in W^{1,\overline{r}_{\alpha}}(\Omega)\setminus\{0\}$ is a weak solution of (1).

Our first result concerns the existence of a continuous family of eigenvalues for problem (1) in a neighborhood of the origin.

**Theorem 3.2.** Assume that the following conditions hold:

(H1) $1 < q(x) < P_{\alpha}^+ \leq \frac{N}{p_{i}(x)} < N < s(x)$ for all $x \in \overline{\Omega}$;

(H2) $V \in L^{s(x)}(\Omega)$ and there exists a measurable set $\Omega_0 \subset \Omega$, $|\Omega_0| > 0$ such that $V(x) > 0$ for all $x \in \overline{\Omega}_0$.

Then there exists $\overline{\lambda} > 0$ such that any $\lambda \in (0, \overline{\lambda})$ is an eigenvalue of problem (1).

**Proof.** Let us consider the energy functional $J_{\lambda} : X \to \mathbb{R}$ given by the formula

$$J_{\lambda}(u) = \Phi(u) - \lambda \Psi(u),$$

where

$$\Phi(u) = \sum_{i=1}^{N} \int_{\Omega} \frac{1}{p_{i}(x)} |\partial_x u|^{p_{i}(x)} \, dx, \quad \Psi(u) = \int_{\Omega} V(x) \frac{|u|^{s(x)} \, dx}{q(x)}.$$  \hfill (9)

From (H1)-(H2), using Proposition 2.1, it is clear that for all $u \in X$,

$$|\Psi(u)| \leq \frac{2}{q(x)} |V|_{s(x)} \frac{u^{p(x)}}{p_{i}(x)} \left\{ \begin{array}{ll} \frac{2}{q} |V|_{s(x)} u^{p(x)} & \text{if } |u|_{s(x)} \leq 1, \\
\frac{2}{q} |V|_{s(x)} u^{p(x)} & \text{if } |u|_{s(x)} \geq 1. \end{array} \right.$$  

On the other hand, by (H1)-(H2), we have $\alpha(x) = \frac{s(x)(q(x))}{q(x)-1} < P_{\alpha}$ and $\beta(x) = \frac{s(x)(q(x))}{q(x)-q(x)} < P_{\alpha}^+$ for all $x \in \overline{\Omega}$ and by Proposition 2.3, the embeddings $X \hookrightarrow L^{s(x)}(\Omega)$ and $X \hookrightarrow L^{q(x)}(\Omega)$ are continuous and compact. Thus, the functional $J_{\lambda}$ is well-defined on $X$. The proof is divided into the following four steps.

**Step 1.** We prove that $J_{\lambda} \in C^{1}(X, \mathbb{R})$ and its derivative is

$$J'_{\lambda}(u)(v) = \sum_{i=1}^{N} \int_{\Omega} |\partial_x u|^{p(x)-2} \partial_x u \cdot \partial_x v \, dx - \lambda \int_{\Omega} V(x) |u|^{q(x)-2} u v \, dx.$$
for all $u, v \in X$. Hence, we can find weak solutions of problem (1) as the critical points of the functional $J_\lambda$ in the space $X$.

We first need to prove that $\Psi \in C^2(X, \mathbb{R})$, that is, for given $u \in X$ we show that for all $v \in X$,

$$\lim_{t \to 0} \frac{\Psi(u + tv) - \Psi(u)}{t} = \Psi'(u)(v),$$

and $\Psi' : X \to X'$ is continuous, where we denote by $X'$ the dual space of $X$.

Indeed, by conditions (H1)-(H2), for $|t| < 1$ using inequality (4) and Proposition 2.1, it implies that

$$\int_\Omega |V(x)|u + tv|^{p(x)-2}(u + tv)dx \leq \int_\Omega |V(x)||u + tv|^{p(x)-1}dx \leq 3\int V(x)\beta_{\beta} \|u + tv\|_{\beta}^{p(x)-1}dx \leq 3\int V_{\|\cdot\|_{\beta}} \|u + tv\|_{\beta}^{p(x)-1}dx \leq +\infty,$$

where $\tau = +$ if $\|u\| + \|v\|_{\beta} > 1$ and $\tau = -$ if $\|u\| + \|v\|_{\beta} \leq 1$ since $X \hookrightarrow L^{p(x)}(\Omega), X \hookrightarrow L^{p(x)}(\Omega)$ and $V \in L^{s(x)}(\Omega).

For all $v \in X$, using the Lebesgue theorem we have

$$\lim_{t \to 0} \frac{\Psi(u + tv) - \Psi(u)}{t} = \frac{d}{dt} \Psi(u + tv)\bigg|_{t=0} = \frac{d}{dt} \left( \int_\Omega \frac{V(x)}{q(x)}|u + tv|^{p(x)}dx \right)\bigg|_{t=0} = \int_\Omega \frac{d}{dt} \left( \int_\Omega \frac{V(x)}{q(x)}|u + tv|^{p(x)}dx \right)dx = \int_\Omega V(x)|u + tv|^{p(x)-2}(u + tv)dx = |\Psi'(u)(v)|,$$

Since the embedding $X \hookrightarrow L^{p(x)}(\Omega)$ is continuous, there exists $c_1 > 0$ such that $|v|_{\beta} \leq c_1\|v\|_{\beta}^{p(x)}$ for all $v \in X$ and by condition (H1)-(H2),

$$|\Psi'(u)(v)| = \left| \int_\Omega V(x)|u|^{p(x)-2}uvdx \right| \leq \int_\Omega |V(x)||u|^{p(x)-1}uvdx \leq 3c_1\int V(x)|u|^{p(x)-1}uvdx \leq 3c_1\int V_{\|\cdot\|_{\beta}}|u|_{\beta}^{p(x)-1}\|v\|_{\beta}dx$$

for any $v \in X$, where $\tau = +$ if $|u|_{\beta} > 1$ and $\tau = -$ if $|u|_{\beta} \leq 1$. Combining this with the linearity of $\Psi'$ we deduce that $\Psi' \in X'$. Note that the map $u \mapsto |u|^{p(x)-2}u$ from $L^{p(x)}(\Omega)$ into $L^{\frac{p(x)}{p(x)-1}}(\Omega)$ is continuous. For the Fréchet differentiability, we conclude that $\Psi$ is Fréchet differentiable and

$$\Psi'(u)(v) = \int_\Omega V(x)|u|^{p(x)-2}uvdx.$$
for all \( u, v \in X \). Similarly, we can show that \( \Phi \in C^1(X, \mathbb{R}) \) and

\[
\Phi'(u)(v) = \sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u|^{p_i} \partial_{x_i} u \cdot \partial_{x_i} v \, dx
\]

for all \( u, v \in X \). The step 1 is completed.

**Step 2.** We prove that there exists \( \lambda > 0 \) such that for any \( \lambda \in (0, \lambda) \), there exist constants \( r, \rho > 0 \) such that \( J_1(u) \geq r \) for all \( u \in X \) with \( ||u||_{\overline{p}(\Omega)} = \rho \).

Indeed, since \( \alpha(x) = \frac{s \alpha(\Omega)}{\alpha(\Omega)-1} < P_\ast \), the embedding \( X \hookrightarrow L^{\alpha(x)}(\Omega) \) is continuous, there exists \( c_2 > 0 \) such that

\[
||u||_{\overline{p}(\Omega)} \leq c_2 ||u||_{\overline{p}(\Omega)}, \quad \forall u \in X.
\]

Now, let us assume that \( ||u||_{\overline{p}(\Omega)} = \rho < \min \left\{ 1, \frac{1}{c_2} \right\} \) sufficiently small, where \( c_2 \) is given in the above inequality. Then we have \( ||u||_{\overline{p}(\Omega)} < 1 \). For such an element \( u \) we get \( ||\partial_{x_i} u||_{p_i(x)} < 1 \) for all \( i = 1, 2, ..., N \). Using (6) and some simple computations, we obtain

\[
\sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u|^{p_i(x)} \, dx \geq \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)}
\]

\[
\geq \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)}
\]

\[
\geq N \left( \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)} \right)^{p_i} = \frac{||u||_{\overline{p}(\Omega)}}{N^{p_i-1}}.
\]

Thus, by (10) for any \( u \in X \) with \( ||u||_{\overline{p}(\Omega)} = \rho \) small enough,

\[
J_1(u) = \sum_{i=1}^{N} \int_{\Omega} \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)} \, dx - \lambda \int_{\Omega} \frac{V(x)}{q(x)} ||u||_{\overline{p}(\Omega)} \, dx
\]

\[
\geq \frac{||u||_{\overline{p}(\Omega)}^{p_i}}{P_\ast N^{p_i-1} - \lambda 2^{\frac{\gamma}{q}} |V|_{b(x)} ||u||_{\overline{p}(\Omega)}^{p_i}}
\]

\[
= \frac{1}{P_\ast N^{p_i-1} - \lambda 2^{\frac{\gamma}{q}} |V|_{b(x)}}
\]

\[
= \rho^\gamma \left( \frac{1}{P_\ast N^{p_i-1} - \lambda 2^{\frac{\gamma}{q}} |V|_{b(x)}} \right).
\]

Putting

\[
\overline{\lambda} = \frac{2^{\frac{\gamma}{q}} |V|_{b(x)}}{P_\ast N^{p_i-1}},
\]

then for any \( \lambda \in (0, \overline{\lambda}) \) and \( u \in X \) with \( ||u||_{\overline{p}(\Omega)} = \rho \), there exists \( r = \frac{2^{\frac{\gamma}{q}} |V|_{b(x)}}{P_\ast N^{p_i-1}} \) such that \( J_1(u) \geq r > 0 \).

**Step 3.** We prove that there exists \( \varphi \in X \) such that \( \varphi \geq 0, \varphi \neq 0 \) and \( J_1(t \varphi) < 0 \) for all \( t > 0 \) small enough.

Indeed, condition (H1) implies that \( q(x) < P_- \) for all \( x \in \overline{\Omega_0} \). In the sequel, we use the notation \( q_0^- = \inf_{x \in \Omega_0} q(x). \) Let \( \epsilon_0 > 0 \) be such that \( q_0^- + \epsilon_0 < P_- \). We also have since \( q \in C(\overline{\Omega_0}) \) that there exists an open subset \( \Omega_1 \subset \Omega_0 \) such that

\[
|q(x) - q_0^-| < \epsilon_0, \quad \forall x \in \Omega_1
\]
and thus 
\[ q(x) \leq q_0^- + \epsilon_0 < P_\Omega^-, \quad \forall x \in \Omega_1. \]

Let \( \varphi \in C_0^\infty(\Omega_0) \) such that \( \Omega_1 \subset \text{supp}(\varphi) \), \( \varphi(x) = 1 \) for all \( x \in \Omega_1 \) and \( 0 \leq \varphi \leq 1 \) in \( \Omega_0 \). Then, using the above information, for any \( t \in (0, 1) \) we have

\[
J_\lambda(t\varphi) = \sum_{i=1}^{N} \int_{\Omega} \frac{1}{p_i(x)} |\partial_{x_i} t \varphi|^p(x) \, dx - \lambda \int_{\Omega} V(x) \frac{1}{q(x)} |t \varphi|^q(x) \, dx
\]

\[
\leq \frac{P_\Omega^-}{P_\Omega^-} \sum_{i=1}^{N} \int_{\Omega_0} |\partial_{x_i} \varphi|^p(x) \, dx - \lambda \int_{\Omega} V(x) \frac{1}{q(x)} |\varphi|^q(x) \, dx
\]

\[
\leq \frac{P_\Omega^-}{P_\Omega^-} \sum_{i=1}^{N} \int_{\Omega_0} |\partial_{x_i} \varphi|^p(x) \, dx - \frac{\lambda \inf \{ \int_{\Omega} V(x) \varphi^q(x) \, dx \} }{q_0^+} \int_{\Omega} V(x) |\varphi|^q(x) \, dx.
\]

Therefore, \( J_\lambda(t\varphi) < 0 \) for \( 0 < t < \delta \leq \frac{1}{t} \) with

\[
0 < \delta < \min \left\{ 1, \frac{1}{q_0^+} \frac{\int_{\Omega} V(x) \varphi^q(x) \, dx}{\sum_{i=1}^{N} \int_{\Omega_0} |\partial_{x_i} \varphi|^p(x) \, dx} \right\}.
\]

The above fraction is meaningful if we can show that \( \sum_{i=1}^{N} \int_{\Omega_0} |\partial_{x_i} \varphi|^p(x) \, dx > 0 \). Indeed, it is clear that

\[
\int_{\Omega} |\varphi|^q(x) \, dx \leq \int_{\Omega} |\varphi|^q(x) \, dx \leq \int_{\Omega} |\varphi|^{q_0^+} \, dx.
\]

On the other hand, the space \( X \) is continuously embedded in \( L^{q_0^+} (\Omega) \) and thus, there exists \( c_3 > 0 \) such that \( |\varphi|^q \leq c_3 \| \varphi \|_{L^{q_0^+}} \), which implies that \( \| \varphi \|_{L^{q_0^+}} > 0 \). Combining this with (5) or (6) the claim follows at once.

By Step 2, we have

\[
\inf_{u \in \partial B_\rho(0)} J_\lambda(u) > 0.
\]

We also have from Step 2 again, the functional \( J_\lambda \) is bounded from below on \( B_\rho(0) \). Moreover, by Step 3, there exists \( \varphi \in X \) such that \( J_\lambda(t\varphi) < 0 \) for all \( t > 0 \) small enough. It follows from Step 2 that

\[
J_\lambda(u) \geq \frac{\| u \|^p_{L^p(\Omega)}}{P_\Omega^-} - \frac{\lambda c_3^q}{q_0^+} \| V(x) \|_{L^{q_0^+}} \| u \|^q \| \varphi \|_{L^{q_0^+}}
\]

which yields

\[
-\infty < c_\lambda = \inf_{u \in \partial B_\rho(0)} J_\lambda(u) < 0.
\]

Let us choose \( \epsilon > 0 \) such that \( 0 < \epsilon < \inf_{u \in \partial B_\rho(0)} J_\lambda(u) \). Applying the Ekeland variational principle in [7] to the functional \( L_\lambda : \overline{B}_\rho(0) \to \mathbb{R} \), it follows that there exists \( u_\epsilon \in \overline{B}_\rho(0) \) such that

\[
J_\lambda(u_\epsilon) < \inf_{u \in \partial B_\rho(0)} J_\lambda(u) + \epsilon,
\]

\[
J_\lambda(u_\epsilon) < J_\lambda(u) + \epsilon \| u - u_\epsilon \|_{L^p(\Omega)}, \quad u \neq u_\epsilon,
\]

then, we have \( J_\lambda(u_\epsilon) < \inf_{u \in \partial B_\rho(0)} J_\lambda(u) \) and thus, \( u_\epsilon \in B_\rho(0) \).

Now, we define the functional \( I_\lambda : \overline{B}_\rho(0) \to \mathbb{R} \) by \( I_\lambda(u) = J_\lambda(u) + \epsilon \| u - u_\epsilon \|_{L^p(\Omega)} \). It is clear that \( u_\epsilon \) is a minimum point of \( I_\lambda \) and thus

\[
\lim_{t \to 0} \frac{I_\lambda(u_\epsilon + t \varphi) - I_\lambda(u_\epsilon)}{t} \geq 0
\]
for all \( \tau > 0 \) small enough and all \( v \in B_r(0) \). The above information shows that

\[
\frac{J_\lambda(u_c + \tau v) - J_\lambda(u_c)}{\tau} + \epsilon \|v\|_{\mathcal{F}(\lambda)} \geq 0.
\]

Letting \( \tau \to 0^+ \), we deduce that

\[
\langle J'_\lambda(u_c), v \rangle \geq -\epsilon \|v\|_{\mathcal{F}(\lambda)}.
\]

It should be noticed that \( -v \) also belongs to \( B_r(0) \), so replacing \( v \) by \( -v \), we get

\[
\langle J'_\lambda(u_c), -v \rangle \geq -\epsilon \|v\|_{\mathcal{F}(\lambda)}
\]

or

\[
\langle J'_\lambda(u_c), v \rangle \leq \epsilon \|v\|_{\mathcal{F}(\lambda)}
\]

which helps us to deduce that \( \|J'_\lambda(u_c)\|_{X^*} \leq \epsilon \). Therefore, there exists a sequence \( \{u_n\} \subset B_r(0) \) such that

\[
J_\lambda(u_n) \to \xi_1 = \inf_{u \in B_r(0)} J_\lambda(u) < 0 \quad \text{and} \quad J'_\lambda(u_n) \to 0 \quad \text{in} \quad X^* \quad \text{as} \quad n \to \infty. \tag{11}
\]

It is clear that the sequence \( \{u_n\} \) is bounded in \( X \). Now, since the Banach space \( X \) is reflexive, there exists \( u \in X \) such that passing to a subsequence, still denoted by \( \{u_n\} \), it converges weakly to \( u \) in \( X \).

**Step 4.** We prove that \( \{u_n\} \) which is given by (11) converges strongly to \( u \) in \( X \), i.e. \( \lim_{n \to \infty} \|u_n - u\|_{\mathcal{F}(\lambda)} = 0 \).

By conditions (H1)-(H2), using (4) and Proposition 2.1 we have

\[
\left| \int_\Omega V(x)|u_n|^{\beta(x)} - 2u_n(u_n - u) \right| \leq 2\|V\|_{\mathcal{B}(\lambda)} \|u_n\|_{L^p(\lambda)}^\beta \|u_n - u\|_{L^p(\lambda)}^2
\]

\[
\leq 4\|V\|_{\mathcal{B}(\lambda)} \|u_n\|_{L^p(\lambda)}^{\beta - 1} \|u_n - u\|_{L^p(\lambda)}
\]

\[
\to 0 \quad \text{as} \quad n \to \infty,
\]

since \( X \) is continuously and compactly embedded in \( L^{\beta(x)}(\Omega) \) with \( \beta(x) = \frac{s(x)}{s(x) - q(x)} \). Moreover, by (11) we have \( \lim_{n \to \infty} J_\lambda(u_n)(u_n - u) = 0 \), i.e.

\[
\lim_{n \to \infty} \sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p(x)} - 2\partial_i u_n(\partial_i u_n - \partial_i u) dx - \lambda \int_\Omega V(x)|u_n|^{p(x)} - 2u_n(u_n - u) dx = 0,
\]

which yields

\[
\lim_{n \to \infty} \sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p(x)} - 2\partial_i u_n(\partial_i u_n - \partial_i u) dx = 0.
\]

Combining this with the fact that \( \{u_n\} \) converges weakly to \( u \) in \( X \), we get

\[
\lim_{n \to \infty} \sum_{i=1}^N \int_\Omega \left( |\partial_i u_m|^{p(x)} - 2\partial_i u_m - |\partial_i u|^{p(x)} - 2\partial_i u \right) (\partial_i u_m - \partial_i u) dx = 0. \tag{12}
\]

Next, we apply the following inequality (see [19])

\[
(\|\xi\|^{r-2} - |\eta|^{r-2}) \cdot (\xi - \eta) \geq 2^{-r} |\xi - \eta|^r, \quad \xi, \eta \in \mathbb{R}^N,
\]

valid for all \( r \geq 2 \). Relations (12) and (13) show actually the sequence \( \{u_n\} \) converges strongly to \( u \) in \( X \). Thus, in view of (11), we obtain \( J_\lambda(u) = \xi_1 < 0 \) and \( J'_\lambda(u) = 0 \). This means that \( u \) is a non-trivial weak solution of (1), i.e. any \( \lambda \in (0, \lambda) \) is an eigenvalue of problem (1). Theorem 3.2 is completely proved. \( \square \)
Now we will prove the second main result for problem (1) regarding the superlinear case. For the case of a non-constant sign weight $V$ we define the following:

$$W^+ = \left\{ u \in X : \int_\Omega V(x)|u|^{q(x)} \, dx > 0 \right\},$$

$$W^- = \left\{ u \in X : \int_\Omega V(x)|u|^{q(x)} \, dx < 0 \right\},$$

(14)

$$\lambda^* = \inf_{u \in W^+} \frac{\Phi(u)}{\Psi(u)^{1/q}}, \quad \lambda_* = \inf_{u \in W^-} \frac{\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)} \, dx}{\int_\Omega V(x)|u|^{q(x)} \, dx},$$

(15)

$$\mu^* = \sup_{u \in W^+} \frac{\Phi(u)}{\Psi(u)^{1/q}}, \quad \mu_* = \sup_{u \in W^-} \frac{\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)} \, dx}{\int_\Omega V(x)|u|^{q(x)} \, dx}.$$

Theorem 3.3. Assume that (8) and the following conditions hold:

(H3) $P_+ < q^- \leq q^+ < P_-$ and $q^+ - \frac{1}{2} < q^-$, where $P_* = \frac{N}{\sum_{i=1}^N \frac{1}{p_i}}$;

(H4) $V \in L^{s(x)}(\Omega)$ is a sign-changing function such that $s \in C(\Omega)$ and

$$s(x) > \max \left\{ 1, \frac{P_-}{P_+ - q(x)} \right\}$$

for all $x \in \Omega$.

Then we have the following assertions:

(i) The numbers $\lambda^*$ and $\mu^*$ are the positive and negative eigenvalues of problem (1) respectively, satisfying $\mu^* \leq \mu_* < 0 < \mu_* \leq \lambda^*$;

(ii) Any $\lambda \in (-\mu^*) \cup (\lambda^*, +\infty)$ is an eigenvalue of problem (1) while any $\lambda \in (\mu_*, \lambda_*)$ is not an eigenvalue of (1).

Proof. It is clear that if $\lambda$ is an eigenvalue of problem (1) with weight $V$ then $-\lambda$ is an eigenvalue of problem (1) with weight $-V$. For this reason, it is sufficient to prove Theorem 3.3 only for $\lambda > 0$ and we will consider problem (1) only in the set $W^+$ defined as before. For this case, the proof of Theorem 3.3 is divided into the following four steps.

Step 1. We prove that $\lambda_*>0$.

By relation (14), it follows that

$$\frac{q^-}{P_+} \lambda_* \leq \lambda^* \leq \frac{q^+}{P_-} \lambda_*,$$

(16)

and $0 \leq \lambda_* \leq \lambda^*$ since $P_+ < q^-$. We will prove that

$$\lim_{\|u\|_{p(x)} \to 0, \ u \in W^+} \frac{\Phi(u)}{\Psi(u)} = +\infty,$$

(17)

$$\lim_{\|u\|_{p(x)} \to +\infty, \ u \in W^+} \frac{\Phi(u)}{\Psi(u)} = +\infty.$$
Indeed, using (3) and Proposition 2.1, for all \( u \in X \) we have

\[
|\Psi(u)| \leq \frac{2}{q^-} |V| |u|_{q'(x)}^{q'(x)} \leq \frac{2}{q^-} |V| |u|_{\alpha(x)'}^{q'}.
\]

where \( \tau = -\) if \( |u|_{\alpha(x)} \leq 1 \) and \( \tau = + \) if \( |u|_{\alpha(x)} \geq 1 \) and \( \alpha(x) = \frac{\alpha(x)}{\alpha(x)'} \). By condition (H4), we have \( 1 < \alpha(x) < P_- \) for all \( x \in \Omega \), that is, \( X \) is continuously embedded in \( L^{\alpha(x)}(\Omega) \), so there exists \( c_4 > 0 \) such that

\[
|\Psi(u)| \leq \frac{2c_4}{q^-} |V| |u|_{\alpha(x)'}^{q'}.
\]

For \( u \in W^+ \) with \( ||u||_{(2)} \leq 1 \) by relations (10) and (19) we infer that

\[
\Phi(u) = \frac{\sum_{i=1}^{N} \int_{\Omega} \frac{1}{q(x)} |\partial_{x_i} u|_{p(x)}^{p(x)} dx}{\int_{\Omega} \frac{V(x)}{q(x)} |u|_{p(x)}^{p(x)} dx} \geq \frac{q^-}{2c_4 P^+ N^{p(x)-1}} |V| |u|_{p(x)}^{q(x)}.\]

which implies that relation (17) holds since \( P^+ < q^- \).

On the other hand, since \( q^+ - \frac{1}{2} < q^- \), it follows that there exists \( \theta > 0 \) such that \( q^+ - \frac{1}{2} < \theta < q^- \), which gives us \( q^+ - 1 < q^- - \frac{1}{2} < \theta \) and

\[
1 + \theta - q^+ > 0, \quad 2(q^- - \theta) \leq 2(q^+ - \theta) < 1.
\]

Take \( r(x) \) be any measurable function satisfying

\[
\max \left\{ \frac{s(x)}{1 + \theta s(x)}, \frac{P^-}{P^- + \theta - q(x)} \right\} \leq r(x) < \min \left\{ \frac{P^-}{P^- + \theta s(x)}, \frac{1}{1 + \theta - q(x)} \right\}
\]

for all \( x \in \Omega \) and

\[
\theta \left( \frac{r^+}{r^-} + 1 \right) \leq q^-.
\]

From relations (21)-(23), it implies that \( r \in L^{\alpha(x)}(\Omega) \) and \( 1 < r(x) < s(x) \) for any \( x \in \Omega \). Moreover, we have

\[
1 < \frac{\partial r(x) s(x)}{s(x) - r(x)} < P_-,
\]

for all \( x \in \Omega \), so there exists \( c_5 > 0 \) such that

\[
\frac{|u|_{(\alpha_0, \alpha_0 - \alpha_0)}}{\alpha_0} \leq c_5 |u|_{(\alpha_0, \alpha_0 - \alpha_0)} \leq c_5 |u|_{(\alpha_0, \alpha_0 - \alpha_0)} \quad \forall u \in X.
\]

For \( u \in W^+ \), by (3) and (5) we have

\[
|\Psi(u)| \leq \int_{\Omega} |V| |u|_{(\alpha, \alpha)}^{\alpha(x)-\theta} dx \leq 2 |V| |u|_{(\alpha, \alpha)}^{\alpha(x)-\theta} \left| |u|_{(\alpha, \alpha)}^{\alpha(x)-\theta} \right|_{(\alpha, \alpha)}.
\]

Using (3) and (5) we have

\[
|V| |u|_{(\alpha, \alpha)}^{\alpha(x)-\theta} \leq \left( \int_{\Omega} |V| |u|_{(\alpha, \alpha)}^{\alpha(x)-\theta} dx \right)^{\frac{\alpha}{\alpha(x)-\theta}} \leq 2 |V| |u|_{(\alpha, \alpha)}^{\alpha(x)-\theta} \left| |u|_{(\alpha, \alpha)}^{\alpha(x)-\theta} \right|_{(\alpha, \alpha)}.
\]
for any $u \in W^+$ with $|V|u^0|\Omega| > 1$ and we deduce from (24) that

$$|V|u^0|\Omega| \leq 1 + 2|V|^1\langle u^0|\Omega| > 1 + 2\left(1 + |V|^1\langle \right) \left(1 + \frac{|u|^\theta}{\theta} \right)$$

$$\leq c_6 \left(1 + \|u\|^\theta_p\right)$$

for any $u \in W^+$. Similarly,

$$\|u\|^{(\sigma-\theta)}|\Omega| \leq 1 + \|u\|^\theta_p\|, \quad u \in W^+.$$

From above information, it follows that

$$|\Psi(u)| \leq c_6 \left(1 + \|u\|^\theta_p\right) \left(1 + c_7\|u\|^\theta_p\right)$$

$$= c_6 + c_6 c_7\|u\|^\theta_p + c_6 c_7\|u\|^\theta_p + c_6 c_7\|u\|^\theta_p = c_8 \left(1 + \|u\|^\theta_p + \|u\|^\theta_p\right)$$

$$\leq c_8 \left(1 + \|u\|^\theta_p + \|u\|^\theta_p\right)$$

for all $u \in W^+$ with $\|u\|^\theta_p > 1$.

For each $i \in \{1, 2, ..., N\}$, we define

$$\alpha_i = \begin{cases} P^+, & \text{if } |\partial_{x_i}u|_{\Omega} < 1, \\ P^-, & \text{if } |\partial_{x_i}u|_{\Omega} > 1. \end{cases}$$

Then, for all $u \in W^+$ with $\|u\|_{\Omega} > 1$, it holds that

$$\sum_{i=1}^N \int_\Omega |\partial_{x_i}u|^p|\Omega| dx = \sum_{i=1}^N |\partial_{x_i}u|^p|\Omega|$$

$$\geq \sum_{i=1}^N |\partial_{x_i}u|^p|\Omega| - \sum_{i=1}^N \left(|\partial_{x_i}u|^p|\Omega| - |\partial_{x_i}u|^p|\Omega|\right)$$

$$\geq \frac{1}{N^p_{\Omega}}\|u\|^p|\Omega| - N.$$

Hence,

$$\Phi(u) = \frac{\sum_{i=1}^N \int_{\Omega} |\partial_{x_i}u|^p|\Omega| dx}{\int_{\Omega} |\partial_{x_i}u|^p|\Omega| dx}$$

$$\geq \frac{1}{N^p_{\Omega}}\|u\|^p|\Omega| - N$$

$$\rightarrow +\infty$$

as $\|u\|^p|\Omega| \rightarrow +\infty$ since $P^- > 1 > 2(q^+ - \theta) \geq 2(q^- - \theta) \geq 2\theta^p > 2\theta$. Thus relation (18) holds.
Now, we are in the position to prove that $\lambda_0 > 0$. Assume by contradiction that $\lambda_0 = 0$, from (16) we get $\lambda^* = 0$. Then, there exists a sequence $\{u_n\} \subset W^+\setminus\{0\}$ such that

$$
\lim_{n \to \infty} \frac{\Phi(u_n)}{\Psi(u_n)} = 0.
$$

We also obtain from (20) that

$$
\frac{\Phi(u_n)}{\Psi(u_n)} \geq \begin{cases} 
\frac{\alpha^-}{q^-} \|u_n\|_{P^-}^q, & \text{if } \|u_n\|_{P^-} \geq 1, \\
\frac{\alpha^-}{q^-} \|u_n\|_{P^-}^q, & \text{if } \|u_n\|_{P^-} < 1.
\end{cases}
$$

By (H3), $P^- - q^- < 0$ and $P^+ - q^- < 0$, so (29) implies that $\|u_n\|_{P^+} \to +\infty$ as $n \to \infty$, as $n \to \infty$. Using again (29), we get

$$
\lim_{n \to \infty} \frac{\Phi(u_n)}{\Psi(u_n)} = +\infty,
$$

which contradicts with (28) and thus, we conclude that $\lambda_0 > 0$.

**Step 2.** We prove that $\lambda_0$ is an eigenvalue of problem (1).

Indeed, let $\{u_n\} \subset W^+\setminus\{0\}$ be a minimizing sequence for the number $\lambda^*$, that is,

$$
\lim_{n \to \infty} \frac{\Phi(u_n)}{\Psi(u_n)} = \lambda^* > 0.
$$

By (18), we have $\{u_n\}$ is a bounded sequence in $X$. Since $X$ is reflexive, there exists $u^* \in X$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that $\{u_n\}$ converges weakly to $u^*$ as $n \to \infty$. Since $\Phi$ is a convex function, it is weakly lower semi-continuous, so we get

$$
\lim_{n \to \infty} \Phi(u_n) \geq \Phi(u^*).
$$

On the other hand, since the embedding $X \hookrightarrow L^{\alpha}(\Omega)$ is compact, the sequence $\{u_n\}$ converges strongly to $u^*$ in $L^{\alpha}(\Omega)$, where $\alpha(x) = \frac{\alpha(x)}{q(x)-1}$. It is noticed that $\|u_n\|_{\alpha(x)} \to \|u^*\|_{\alpha(x)}$, $\|u_n\|_{P^+}^{\alpha(x)} \to \|u^*\|_{P^+}^{\alpha(x)}$, $\|u_n\|_{P^-}^{\alpha(x)} \to \|u^*\|_{P^-}^{\alpha(x)}$, the sequence $\|u_n\|_{P^+}^{\alpha(x)} \to \|u^*\|_{P^+}^{\alpha(x)}$ in $L^{\frac{\alpha(x)}{q(x)}}(\Omega)$, and $\|u_n\|_{P^-}^{\alpha(x)} \to \|u^*\|_{P^-}^{\alpha(x)}$ in $L^{\frac{\alpha(x)}{q(x)}}(\Omega)$. From the above information, it implies that

$$
|\Phi(u_n) - \Phi(u)| \leq \int_{\Omega} \frac{|V(x)|}{q(x)} \left( \|u_n\|_{P^+}^{\alpha(x)} - \|u^*\|_{P^+}^{\alpha(x)} \right) dx
$$

$$
\leq \frac{2}{q} \|V(x)\|_{\alpha(x)} \|u_n\|_{P^+}^{\alpha(x)} - \|u^*\|_{P^+}^{\alpha(x)} \to 0,
$$

as $n \to \infty$, that is,

$$
\lim_{n \to \infty} \Psi(u_n) = \Psi(u^*) = \int_{\Omega} \frac{V(x)}{q(x)} \|u^*\|_{P^+}^{\alpha(x)} dx \geq 0.
$$

In view of (31) and (32) we obtain $\frac{\Phi(u^*)}{\Psi(u^*)} = \lambda^*$ if $\Psi(u^*) > 0$, i.e. $u^* \in W^+$. We need to show that $\Psi(u^*) > 0$. Assume by contradiction that $\Psi(u^*) = 0$ or

$$
\lim_{n \to \infty} \Psi(u_n) = 0.
$$

Now, taking $\varepsilon \in (0, \lambda^*)$ be fixed, by (9), for $n$ large enough,

$$
\left| \frac{\Phi(u_n)}{\Psi(u_n)} - \lambda^* \right| < \varepsilon.
$$
or

\[(\lambda^* - c)\Psi(u_n) < \Phi(u_n) < (\lambda^* + c)\Psi(u_n),\]

which follows from (33) that \(\lim_{n \to \infty} \Phi(u_n) = 0\). This means that \(u_n \to 0\) in \(X\), that is, \(\|u_n\|_{\mathcal{F}(\varepsilon)} \to 0\) as \(n \to \infty\) and thus,

\[
\lim_{n \to \infty} \Phi(u_n) = +\infty
\]

which is a contradiction. Therefore, the proof of Theorem 3.3 is now complete.

**Step 3.** We prove that any \(\lambda \in (\lambda^*, +\infty)\) is an eigenvalue of problem (1).

Let \(\lambda \in (\lambda^*, +\infty)\) be arbitrary but fixed. We know that \(\lambda\) is an eigenvalue of problem (1) if and only if there exists \(u_\lambda \in W^+\setminus \{0\}\) a critical point of \(J_\lambda\). From relations (26) and (27), it implies that

\[
J_\lambda(u) = \sum_{i=1}^{N} \int_{\Omega} \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)} dx - \lambda \int_{\Omega} \frac{V(x)}{q(x)} |u|^{q(x)} dx
\]

\[
\geq \frac{1}{P_+ N p_+} \|u\|_{\mathcal{F}(\varepsilon)}^{p_+} \left( 1 + \|u\|_{\mathcal{F}(\varepsilon)}^{2q^+} + \|u\|_{\mathcal{F}(\varepsilon)}^{2q^+ - \theta} \right)
\]

\[
\to +\infty,
\]

as \(\|u\|_{\mathcal{F}(\varepsilon)} \to +\infty\) since \(P_+ > 1 > 2(q^+ - \theta) \geq 2(q^+ - \theta) \geq 2\theta^{q^+} > 2\theta\). This follows that \(J_\lambda\) is coercive in \(W^+\).

By the proof of the step 2, the functional \(\Psi\) is weakly-strongly continuous in \(W^+\). We also know that \(\Phi\) is weakly lower semi-continuous, so by Weierstrass theorem, there exists \(u_\lambda \in W^+\) a global minimum point of \(J_\lambda\). We need to prove that \(u_\lambda\) is non-trivial critical point of \(J_\lambda\). Indeed, since \(\lambda = \inf_{u \in W^+} \frac{\Phi(u)}{\Psi(u)}\) and \(\lambda > \lambda^*\), it follows that there exists \(\bar{u}_\lambda \in W^+\) such that \(\frac{\Phi(\bar{u}_\lambda)}{\Psi(\bar{u}_\lambda)} < \lambda\), that is,

\[
J_\lambda(\bar{u}_\lambda) = \Phi(\bar{u}_\lambda) - \lambda \Psi(\bar{u}_\lambda) < 0.
\]

This means that \(\inf_{u \in W^+} J_\lambda(u) < 0\) and thus, \(u_\lambda\) is a non-trivial critical point of \(J_\lambda\) or \(\lambda\) is an eigenvalue of problem (1). The proof of step 3 is completed.

**Step 4.** We prove that any \(\lambda \in (0, \lambda_1)\) is not an eigenvalue of problem (1).

Indeed, assume by contradiction that there exists \(\lambda \in (0, \lambda^*)\) an eigenvalue of problem (1), that is, there exists \(u_\lambda \in W^+\) such that

\[
\Phi(u_\lambda)(\lambda) = \lambda \Psi(u_\lambda)(\lambda), \quad \forall \lambda \in W^+.
\]

Thus, for \(v = u_\lambda \in W^+\) we have

\[
\sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u_\lambda|^{p_i(x)} dx = \lambda \int_{\Omega} V(x) |u_\lambda|^{q(x)} dx.
\]  (34)

By the definition of the set \(W^+\), we have \(\int_{\Omega} V(x) |u_\lambda|^{q(x)} dx > 0\). By (34) and the definition of \(\lambda\), and the fact that \(\lambda < \lambda_1\), we deduce that

\[
\sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u_\lambda|^{p_i(x)} dx \geq \lambda \int_{\Omega} V(x) |u_\lambda|^{q(x)} dx
\]

\[
> \lambda \int_{\Omega} V(x) |u_\lambda|^{q(x)} dx
\]

\[
= \sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u_\lambda|^{p_i(x)} dx,
\]

which is a contradiction. Therefore, the proof of Theorem 3.3 is now complete. \(\square\)
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