On Toeplitz operators between Fock spaces

Tesfa Mengestie

Abstract. We study mapping properties of Toeplitz operators $T_\mu$ associated to nonnegative Borel measure $\mu$ on the complex space $\mathbb{C}^n$. We, in particular, describe the bounded and compact operators $T_\mu$ acting between Fock spaces in terms of the objects $t$-Berezin transforms, averaging functions, and averaging sequences of their inducing measures $\mu$. An asymptotic estimate for the norms of the operators has been also obtained. The results obtained extend a recent work of Z. Hu and X. Lv and fills the remaining gap when both the smallest and largest Banach–Fock spaces are taken into account.

Mathematics Subject Classification (2010). 31B05, 39A12, 31C20.

Keywords. Fock space, Toeplitz operator, Fock–Carleson measures, Berezin transforms, Bounded, Compact, averaging sequences and functions.

1. Introduction

For a positive parameter $\alpha$, the classical Fock space $\mathcal{F}_\alpha^p$ consists of entire functions $f$ for which

$$
\|f\|_{(p,\alpha)}^p = \left(\frac{\alpha p}{2\pi}\right)^n \int_{\mathbb{C}^n} |f(z)|^p e^{-\frac{\alpha p}{2}|z|^2} dV(z) < \infty
$$

where $1 \leq p < \infty$ and $dV(z)$ denotes the usual Lebesgue measure on $\mathbb{C}^n$. For $p = \infty$, the corresponding growth type Fock space contains those entire functions $f$ for which

$$
\|f\|_{(\infty,\alpha)} = \sup_{z \in \mathbb{C}^n} |f(z)| e^{-\frac{\alpha}{2}|z|^2} < \infty.
$$

(1.1)

The space $\mathcal{F}_\alpha^2$ is a reproducing kernel Hilbert space with kernel and normalized kernel functions respectively $K_w(z) = e^{\alpha \langle z,w \rangle}$ and $k_w(z) = e^{\alpha \langle z,w \rangle - \alpha |w|^2/2}$ where

$$
\langle z,w \rangle = \sum_{j=1}^n z_j \overline{w_j}, \quad |z| = \sqrt{\langle z,z \rangle}, \text{and } w = (w_j), z = (z_j) \in \mathbb{C}^n.
$$
For each $\delta > 0$, we denote by $D(z, \delta)$ the ball $\{w \in \mathbb{C}^n : |z - w| < \delta\}$, and consider a sequence of points $z_k$ such that for $r > 0$, the sequences of the balls $D(z_k, r/2)$ covers the entire space $\mathbb{C}^n$ while the balls $D(z_k, r/4)$ are mutually disjoint. Because of this covering property, it has been known that for any $\delta > 0$, there exists a positive integer $N_{\text{max}} = N_{\text{max}}(\delta, r)$ such that every point in $\mathbb{C}^n$ belongs to at most $N_{\text{max}}$ of the balls $D(z_k, \delta)$. The sequence $z_k$ will refer to such fixed $r/2$-lattice throughout the paper.

For a nonnegative measure $\mu$ on $\mathbb{C}^n$, its average on $D(z, r)$ is the quantity $\mu(D(z, r))/|D(z, r)|$ where $|D(z, r)|$ refers to the volume of the ball $D(z, r)$ which depends only on $r$ for all $z$ in $\mathbb{C}^n$. Thus, we call $\mu(D(z, r))$ simply an averaging function of $\mu$, and $\mu(D(z_k, r))$ its averaging sequence.

Recent years have seen a lot of work on Toeplitz operators acting on different spaces of holomorphic functions since introduced by O. Toeplitz [12] in the year 1911. The characterization of symbols $\psi$ in $L^\infty$ which induce bounded or compact Toeplitz operators $T_\psi$ on the classical Fock space $\mathcal{F}_\alpha^2$ have been studied by several authors; for example see [1, 2, 3, 8, 10, 11]. In [5], J. Isralowitz and K. Zhu took the case further and studied conditions on positive Borel measure $\mu$ on the complex plane $\mathbb{C}$ under which the induced map $T_\mu$, \[ T_\mu f(z) = \int_\mathbb{C} K_w(z) f(w) e^{-\alpha |w|^2} d\mu(w), \] becomes bounded and compact on $\mathcal{F}_\alpha^2$ whenever the measure $\mu$ satisfies an admissibility condition \[ \int_\mathbb{C} |K_w(z)|^2 e^{-\alpha |w|^2} d\mu(w) < \infty. \]\[ (1.3) \]

Inspired by this work, Z. Hu and X. Lv [4] recently identified the bounded and compact $T_\mu$ when it acts between the Fock spaces $\mathcal{F}_\alpha^p$ and $\mathcal{F}_\alpha^q$ in terms of Fock–Carleson measures whenever the exponents $p$ and $q$ are limited in the range $1 < p, q < \infty$ and $\mu$ still satisfies condition (1.3) on $\mathbb{C}^n$. We mention that this condition along with density of the kernel functions or the atomic decomposition result [6, Theorem 8.2] and Cauchy–Schwarz inequality ensure that $T_\mu$ is densely defined on Fock spaces.

The main purpose of this note is to extend the results in [4] for the two extreme exponents 1 and $\infty$ assuming the same admissibility condition (1.3) with $\mathbb{C}$ replaced by $\mathbb{C}^n$ throughout the paper. It may be noted that the corresponding Fock spaces $\mathcal{F}_\alpha^1$ and $\mathcal{F}_\alpha^\infty$ are respectively the smallest and largest Banach–Fock spaces in the sense of inclusion [6, Theorem 7.2]. The proofs of some of Hu and Lv results rely on duality and complex interpolation arguments in Fock spaces. Because of the complicated nature and size of the dual space of $\mathcal{F}_\alpha^\infty$, carrying over those arguments to these extreme exponent cases seems more difficult especially when the growth space $\mathcal{F}_\alpha^\infty$ is mapped into smaller spaces $\mathcal{F}_\alpha^p$. Thus, we will take here a different approach in dealing with these two cases.
The notion of Carleson measures has been one of the basic tools in studying properties of several operators on spaces of holomorphic functions. Since the notion plays a role in our case too, we close this section by recalling a few words about it. For $0 < p \leq \infty$ and $0 < q < \infty$, we call a nonnegative measure $\mu$ on $\mathbb{C}^n$ a $(p,q)$ Fock–Carleson measure if the inequality

$$\int_{\mathbb{C}^n} |f(z)|^q e^{-\frac{q\alpha}{2}|z|^2} d\mu(z) \lesssim \|f\|_{q}^{p,\alpha}(1.4)$$

holds for all $f$ in $F^p_{\alpha}$. Thus $\mu$ is a $(p,q)$ Fock–Carleson measure if and only if the embedding map $I_{\mu} : F^p_{\alpha} \rightarrow L^q(\sigma_q)$ is bounded where $d\sigma_q(z) = e^{-q\alpha|z|^2/2}d\mu(z)$. We call $\mu$ a $(p,q)$ vanishing Fock–Carleson measure if such an embedding map is compact. For finite exponents $p$ and $q$ such measures were described in [4]. On the other hand, the $(\infty,q)$ Fock–Carleson measures were recently identified by the author in [7]. The description of such measure $\mu$ involves its $t$-Berezin transform defined by

$$\tilde{\mu}_t(w) = \int_{\mathbb{C}^n} |k_w(z)|^t e^{-\frac{4t\alpha}{2}|z|^2} d\mu(z), \quad t > 0.$$  

2. Main results

In this section we formulate our results on bounded and compact Toeplitz operators $T_{\mu}$ in terms of Fock–Carleson measures. Then the corollaries to follow give comprehensive lists of the different characterizations of these maps in terms of $t$-Berezin transforms, averaging functions, and averaging sequences of the symbols inducing the maps.

**Theorem 2.1.** Let $1 \leq p < \infty$ and $\mu$ be a nonnegative measure on $\mathbb{C}^n$. Then $T_{\mu} : F^p_{\alpha} \rightarrow F^\infty_{\alpha}$ is

(i) bounded if and only $\mu$ is a $(1,q)$ Fock–Carleson measure for some (or any) finite $q \geq p$. Furthermore, we have

$$\|T_{\mu}\| \simeq \|I_{\mu}\| \quad (2.1)$$

where $I_{\mu}$ is the embedding map $I_{\mu} : F^1_{\alpha} \rightarrow L^1(\sigma_1)$.

(ii) compact if and only if $\mu$ is a $(1,q)$ vanishing Fock–Carleson measure for some (or any) finite $q \geq p$.

While Theorem 2.1 gives conditions under which $T_{\mu}$ maps smaller spaces into the largest space $F^\infty_{\alpha}$, the following result gives a stronger condition on $\mu$ when it maps $F^\infty_{\alpha}$ into smaller Fock spaces $F^p_{\alpha}$ for finite $p$.

**Theorem 2.2.** Let $1 \leq p < \infty$ and $\mu$ be a nonnegative measure on $\mathbb{C}^n$. Then the following statements are equivalent.

(i) $T_{\mu} : F^\infty_{\alpha} \rightarrow F^p_{\alpha}$ is bounded;

---

1A word on notation: The notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant $C$ such that $U(z) \leq CV(z)$ holds for all $z$ in the set in question, which may be a Hilbert space or a set of complex numbers. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$. We also write $L^p$ to mean $L^p(\mathbb{C}^n,dV)$. 

(ii) $T_\mu : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\alpha^p$ is compact;
(iii) $\mu$ is an $(\infty, q)$ vanishing Fock–Carleson measure for some (or any) finite $q \geq 1$. Furthermore, we have
\[
\|T_\mu\| \simeq \|I_\mu\| \number{2.2}
\]
where $I_\mu$ is the embedding map $I_\mu : \mathcal{F}_\alpha^\infty \to L^p(\sigma_p)$.

The phenomena with condition (iii) is, in particular, interesting. It ensures that if there exists a finite $q \geq 1$ for which $T_\mu : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\alpha^q$ is bounded (compact), then the same conclusion holds for $T_\mu$ when we replace the target space by $\mathcal{F}_\alpha^p$ for all finite $p \geq 1$. This phenomena does not necessarily happen when $T_\mu$ maps $\mathcal{F}_\alpha^p$ into $\mathcal{F}_\alpha^q$ for finite exponents $p$ and $q$ even if the target exponent $q$ is smaller than $p$.

As consequences of the characterizations of the Fock–Carleson measures in Theorems 3.1 and 3.2 of [4], and from a particular case of Theorem 2.4 of [7], we obtain several other characterizations for the bounded and compact operators $T_\mu$ in terms of t-Berezin transform, averaging functions, and averaging sequences of $\mu$. For readers convenient, we populate these conditions in the next two corollaries.

**Corollary 2.3.** Let $1 \leq p < \infty$ and $\mu$ be a nonnegative measure on $\mathbb{C}^n$. Then the following statements are equivalent.

(i) $T_\mu : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^\infty$ is bounded;
(ii) $\mu$ is a $(1, q)$ Fock–Carleson measure for some (or any) finite $q \geq p$;
(iii) $\mu(D(., \delta)) \in L^\infty$ for some (or any) $\delta > 0$;
(iv) $\widetilde{\mu}_t \in L^\infty$ for some (or any) $t > 0$;
(v) $\mu(D(z_k, r)) \in \ell^\infty$ for some (or any) $r > 0$.

Furthermore, we have
\[
\|T_\mu\| \simeq \|I_\mu\| \simeq \|\widetilde{\mu}_t\|_{L^\infty} \simeq \|\mu(D(., \delta))\|_{L^\infty} \simeq \|\mu(D(z_k, r))\|_{\ell^\infty} \number{2.3}
\]
where $I_\mu$ is the embedding map $I_\mu : \mathcal{F}_\alpha^1 \to L^1(\sigma_1)$.

The last four conditions in the corollary above are proved in [4] and [7] where in the latter a more general setting is considered. The last three conditions are in addition independent of the exponents $p$ and $\infty$ under consideration, and remain valid if we replace the target space $\mathcal{F}_\alpha^\infty$ by a smaller space $\mathcal{F}_\alpha^q$ for any $q \geq p$. Thus, if need be any one of these conditions might be used to define the $(\infty, \infty)$ (vanishing) Fock–Carleson measures since an inequality of the form in (1.4) makes no sense as it stands. Taking the "little oh" counter part of the above conditions gives a list of different characterizations of the compact Toeplitz operator $T_\mu$ acting between $\mathcal{F}_\alpha^p$ and $\mathcal{F}_\alpha^\infty$.

**Corollary 2.4.** Let $1 \leq p < \infty$ and $\mu$ be a nonnegative measure on $\mathbb{C}^n$. Then the following statements are equivalent.

(i) $T_\mu : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\alpha^p$ is bounded;
(ii) $T_\mu : \mathcal{F}_\alpha^\infty \to \mathcal{F}_\alpha^p$ is compact;
(iii) $\mu$ is an $(\infty, q)$ Fock–Carleson measure for some (or any) finite $q \geq 1$;
(iv) \( \mu \) is an \((\infty, q)\) vanishing Fock–Carleson measure for some (or any) finite \( q \geq 1 \);
(v) \( \tilde{\mu}_t \in L^1 \) for some (or any) \( t > 0 \);
(vi) \( \mu \) is a finite measure on \( \mathbb{C}^n \);
(vii) \( T_\mu : \mathcal{F}_\alpha^2 \rightarrow \mathcal{F}_\alpha^2 \) belongs to the trace class;
(viii) \( \mu(D(., \delta)) \in L^1 \) for some (or any) \( \delta > 0 \);
(ix) \( \mu(D(z_k, r)) \in \ell^1 \) for some (or any) \( r > 0 \).

Furthermore, we have
\[
\| T_\mu \|^p \simeq \| I_\mu \|^p \simeq \| \tilde{\mu}_t \|_{L^1} \simeq \| \mu(D(., \delta)) \|_{L^1} \simeq \| \mu(D(z_k, r)) \|_{\ell^1} \simeq \mu(\mathbb{C}^n) \quad (2.4)
\]

where \( I_\mu \) is the embedding map \( I_\mu : \mathcal{F}_\alpha^\infty \rightarrow L^p(\sigma_p) \).

We note in passing that the equivalence of conditions (iii), (vi), (v), (vi), (viii) and (ix) has been proved in [7]. On the other hand the equivalencies of (vi) and (vii) comes from Proposition 10 in [5]. These two conditions follow from condition (v) when we in particular set \( t = 1 \).

We now return to the case when the exponent \( p = 1 \) and study properties of \( T_\mu \) acting between \( \mathcal{F}_\alpha^1 \) and \( \mathcal{F}_\alpha^p \) and conversely.

**Theorem 2.5.** Let \( \mu \) be a nonnegative measure on \( \mathbb{C}^n \). Then
(i) \( T_\mu : \mathcal{F}_\alpha^1 \rightarrow \mathcal{F}_\alpha^p, \ 1 \leq p \leq \infty, \) is
a) bounded if and only if \( \mu \) is a \((1, q)\) Fock–Carleson measure for some (or any) \( q \geq p \). Furthermore, we have
\[
\| T_\mu \| \simeq \| I_\mu \| \quad (2.5)
\]
where \( I_\mu \) is the embedding map \( I_\mu : \mathcal{F}_\alpha^1 \rightarrow L^p(\sigma_p) \).

b) compact if and only if \( \mu \) is a \((1, q)\) vanishing Fock–Carleson measure for some (or any) \( q \geq p \).
(ii) \( T_\mu : \mathcal{F}_\alpha^p \rightarrow \mathcal{F}_\alpha^1, \ 1 < p \leq \infty, \) is bounded (compact) if and only if \( \mu \) is a \((1, p)\) Fock–Carleson measure. We also have the norm estimate
\[
\| T_\mu \| \simeq \| I_\mu \| \quad (2.6)
\]
where \( I_\mu \) is the embedding map \( I_\mu : \mathcal{F}_\alpha^p \rightarrow L^1(\sigma_1) \).

As in the previous corollaries, the conditions in Theorem 2.5 can be also equivalently expressed in terms of the \( t \)-Berezin transform of \( \mu \), the averaging functions \( \mu(D(., r)) \), and averaging sequences \( \mu(D(z_j, r)) \). The reformulation in terms of these notions again follows easily from the characterizations of the Fock–Carleson measures in Theorems 3.1 and 3.2 of [4], and Theorem 2.4 of [7] as refereed above.

### 3. Proof of the results

**Proof of Theorem 2.1.** (i) We may first assume that \( T_\mu : \mathcal{F}_\alpha^p \rightarrow \mathcal{F}_\alpha^\infty \) is bounded and proceed to show the necessity of the condition. By the reproducing property of the kernel, we have
\[
K_w(z) = \left( \frac{\alpha}{\pi} \right)^n \int_{\mathbb{C}^n} K_w(\eta)K_\eta(z)e^{-\alpha|\eta|^2}dV(\eta). \quad (3.1)
\]
By condition (1.3), we observe that the 2-Berezin transform
\[ \tilde{\mu}_2(z) = \int_{\mathbb{C}^n} |K_w(z)|^2 e^{-\alpha(|z|^2 + |w|^2)} d\mu(w) \]
\[ = e^{-\alpha|z|^2} \int_{\mathbb{C}^n} |K_w(z)|^2 e^{-|w|^2} d\mu(w) \lesssim e^{-\alpha|z|^2} < \infty. \]

On the other hand, our assumption ensures
\[ \left| \int_{\mathbb{C}^n} T_\mu k_2(\eta) \overline{k_2(\eta)} e^{-\alpha|\eta|^2} dV(\eta) \right| \leq \|T_\mu k_2\|_{(\infty, \alpha)} \int_{\mathbb{C}^n} e^{-\frac{\alpha}{2} |\eta|^2} dV(\eta) \]
\[ \lesssim \int_{\mathbb{C}^n} |k_2(\eta)| e^{-\frac{\alpha}{2} |\eta|^2} dV(\eta) \]
\[ \simeq \|k_2\|_{(1, \alpha)} = 1 < \infty \]

for all \( z \in \mathbb{C}^n \).

Now, we may write the 2-Berezin transform as
\[ \tilde{\mu}_2(z) = \int_{\mathbb{C}^n} |K_w(z)|^2 e^{-\alpha(|z|^2 + |w|^2)} d\mu(w) \]
\[ = \int_{\mathbb{C}^n} K_w(z) e^{-\frac{\alpha}{2}(|z|^2 + |w|^2)} K_z(w) e^{-\frac{\alpha}{2}(|z|^2 + |w|^2)} d\mu(w). \]

Applying the kernel property in (3.1), we find that the integral above is equal to
\[ \int_{\mathbb{C}^n} K_z(w) e^{-\frac{\alpha}{2}(|z|^2 + |w|^2)} \left( \frac{\alpha}{\pi} \right)^n \int_{\mathbb{C}^n} \frac{K_w(\eta) K_z(\eta)}{e^{\alpha|\eta|^2}} dV(\eta) e^{-\frac{\alpha}{2}(|z|^2 + |w|^2)} d\mu(w) \]
\[ = \left( \frac{\alpha}{\pi} \right)^n \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \left( \frac{K_z(w) K_w(\eta)}{e^{\alpha|w|^2 + \alpha|z|^2}} \right) d\mu(w) \overline{K_z(\eta)} e^{-\alpha|\eta|^2} dV(\eta) \]
\[ = \left( \frac{\alpha}{\pi} \right)^n \int_{\mathbb{C}^n} T_\mu k_z(\eta) \overline{k_z(\eta)} e^{-\alpha|\eta|^2} dV(\eta). \quad (3.2) \]

Note that for each \( f \in \mathcal{F}_\alpha^\infty \), (1.1) implies
\[ |f(z)| \leq \| f \|_{(\infty, \alpha)} e^{\frac{\alpha}{2} |z|^2} \quad (3.3) \]
from which it follows
\[ \tilde{\mu}_2(z) \simeq \int_{\mathbb{C}^n} T_\mu k_z(\eta) \overline{k_z(\eta)} e^{-\alpha|\eta|^2} dV(\eta) \]
\[ \leq \|T_\mu k_z\|_{(\infty, \alpha)} \int_{\mathbb{C}^n} |k_z(\eta)| e^{-\frac{\alpha}{2} |\eta|^2} dV(\eta) \lesssim \| T_\mu \|, \quad (3.4) \]
where the last inequality follows because \( \|k_z\|_{(1, \alpha)} = \|k_z\|_{(\infty, \alpha)} = 1 \) for all \( z \) in \( \mathbb{C}^n \). This proves that \( \tilde{\mu}_2 \) is uniformly bounded on \( \mathbb{C}^n \). By Theorem 3.1 of [4], this happens if and only if \( \mu \) is a \((p, q)\) Fock–Carleson measure for any \( p \).
and $q$ in the range $0 < p \leq q < \infty$, from which the desired conclusion follows. In particular when $p = q = 1$ the result there and (3.4) imply

$$\sup_{z \in \mathbb{C}^n} \tilde{\mu}_2(z) \simeq \|I_\mu\| \lesssim \|T_\mu\|$$

(3.5)

where $I_\mu$ is the embedding map $I_\mu : \mathcal{F}_\alpha^1 \to L^1(\sigma_1)$. On the other hand, if $\mu$ is a $(1, q)$ Fock–Carleson measure for some finite $q \geq p \geq 1$, then it is a $(1, 1)$ Fock–Carleson measure; see Theorem 3.1 of [4] again. Thus,

$$|T_\mu f(z)| e^{-\frac{\alpha}{2}|z|^2} \leq \int_{\mathbb{C}^n} |k_z(w)||f(w)| e^{-\alpha |w|^2} d\mu(w)$$

$$\leq \|f\|_{(\infty, \alpha)} \int_{\mathbb{C}^n} |k_z(w)| e^{-\frac{\alpha}{2}|w|^2} d\mu(w)$$

$$\lesssim \|I_\mu\| \|f\|_{(\infty, \alpha)} \|k_z\|_{(1, \alpha)} = \|I_\mu\| \|f\|_{(\infty, \alpha)}.$$ (3.6)

Here the second inequality follows by (3.3). From this it follows that

$$\sup_{z \in \mathbb{C}^n} |T_\mu f(z)| e^{-\frac{\alpha}{2}|z|^2} \lesssim \|I_\mu\| \|f\|_{(\infty, \alpha)} \lesssim \|I_\mu\| \|f\|_{(p, \alpha)}$$

where we use the inclusion $\mathcal{F}_\alpha^p \subseteq \mathcal{F}_\alpha^\infty$, and hence $T_\mu$ is bounded and

$$\|T_\mu\| \lesssim \|I_\mu\|.$$ (3.8)

From (3.5) and (3.8), the asymptotic norm estimate in (2.1) follows.

(ii) We may first verify the sufficiency of the condition. Let $f_m$ be a sequence of functions in $\mathcal{F}_\alpha^p$ such that $\sup_{m \geq 1} \|f_m\|_{(p, \alpha)} < \infty$ and $f_m$ converges uniformly to zero on compact subsets of $\mathbb{C}^n$ as $m \to \infty$. An application of Lemma 1 of [9] gives

$$|T_\mu f_m(z)| e^{-\frac{\alpha}{2}|z|^2} \leq \sup_{m \geq 1} \|f_m\|_{(p, \alpha)} \int_{\mathbb{C}^n} |k_z(w)| e^{-\frac{\alpha}{2}|w|^2} d\mu(w)$$

$$\lesssim \int_{\mathbb{C}^n} |k_z(w)| e^{-\frac{\alpha}{2}|w|^2} d\mu(w).$$

Since $\mu$ is a $(1, q)$ vanishing Fock–Carleson measure for some finite $q \geq p \geq 1$, by Theorem 3.2 of [4], it is also a $(1, 1)$ vanishing Fock–Carleson measure. This along with the fact that $k_z$ converges uniformly to zero on compact subsets of $\mathbb{C}^n$ when $|z| \to \infty$ yields

$$\int_{\mathbb{C}^n} |k_z(w)| e^{-\alpha |w|^2} d\mu(w) \to 0$$

as $|z| \to \infty$ from which we conclude that $T_\mu$ is compact.

Conversely, suppose that $T_\mu : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^\infty$ is compact. By arguing with (3.2) and (3.3), we have

$$\tilde{\mu}_2(z) = \int_{\mathbb{C}^n} |k_z(w)|^2 e^{-\alpha |w|^2} d\mu(w) \leq \|T_\mu k_z\|_{(\infty, \alpha)} \|k_z\|_{(1, \alpha)}$$

$$= \|T_\mu k_z\|_{(\infty, \alpha)} \to 0$$

as $|z| \to \infty$ since $k_z$ is a sequence of unit norm functions which converges uniformly to zero on compact subsets of $\mathbb{C}^n$ when $|z| \to \infty$. Then, by a
particular case of Theorem 3.2 of [4], it follows that \( \mu \) is a \((1, q)\) vanishing Fock–Carleson measure for any finite \( q \geq p \).

Proof of Theorem 2.2. Since (ii) implies (i) is trivial, we only need to verify (iii) implies (ii) and (i) implies (iii). Suppose \( \mu \) is an \((\infty, q)\) vanishing Fock–Carleson measure for some finite \( q \geq 1 \). Then for each \( f \) in \( F_\alpha^\infty \),

\[
\|T_\mu f\|_{(p, \alpha)}^p \lesssim \int_{\mathbb{C}^n} |T_\mu f(z)e^{-\frac{\alpha}{2}|z|^2}|^p dV(z) \\
\leq \int_{\mathbb{C}^n} \left( \int_{\mathbb{C}^n} |K_w(z)| f(w) \right)^p dV(z) = I.
\]

(3.9)

We may now first assume that \( p > 1 \). Then applying Hölder’s inequality on the right-hand side of (3.9) and subsequently Fubini’s theorem give

\[
I \leq \mu(\mathbb{C}^n)^{\frac{p}{p'}} \int_{\mathbb{C}^n} |f(w) e^{-\frac{\alpha}{2}|w|^2}|^p \int_{\mathbb{C}^n} |K_w(z)| f(w) e^{-\frac{\alpha}{2}(|w|^2+|z|^2)} dV(z) d\mu(w) \\
\lesssim \mu(\mathbb{C}^n)^{\frac{p}{p'}} \int_{\mathbb{C}^n} |f(w) e^{-\frac{\alpha}{2}|w|^2}|^p d\mu(w)
\]

(3.10)

where \( p \) and \( p' \) are conjugate exponents, and by the fact that \( \|k_w\|_{(p, \alpha)} = 1 \) for each \( p \) and \( w \) in \( \mathbb{C}^n \). On the other hand for \( p = 1 \), it follows that

\[
I = \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |K_w(z)| e^{\frac{\alpha}{2}|w|^2} |f(w)| d\mu(w) dV(z) \\
\lesssim \int_{\mathbb{C}^n} |f(w)| d\mu(w)
\]

(3.11)

after interchanging the integrals. Since \( \mu \) is assumed to be an \((\infty, q)\) vanishing Fock–Carleson measure, a particular case of Theorem 2.4 of [7] ensures that \( \mu(\mathbb{C}^n) \) is finite. To this end, we combine (3.9), (3.10) and (3.11) to conclude the norm estimate

\[
\|T_\mu f\|_{(p, \alpha)}^p \lesssim \int_{\mathbb{C}^n} |f(z) e^{-\frac{\alpha}{2}|z|^2}| d\mu(z).
\]

(3.12)

We may now consider a sequence of functions \( f_m \) in \( F_\alpha^\infty \) such that \( \sup_m \|f_m\|_{(\infty, \alpha)} < \infty \) and \( f_m \) uniformly converges to zero on compact subsets of \( \mathbb{C}^n \) as \( m \to \infty \). Applying the estimate in (3.12), we have

\[
\|T_\mu f_m\|_{(p, \alpha)}^p \lesssim \int_{\mathbb{C}^n} |f_m(z) e^{-\alpha|z|^2/2}| d\mu(z) \to 0
\]

when \( m \to \infty \). This holds as \( \mu \) is an \((\infty, q)\) vanishing Fock–Carleson measure, by Theorem 2.4 of [7], it is also an \((\infty, p)\) vanishing Fock–Carleson measure. Thus, we conclude that \( T_\mu \) is compact.

Furthermore, observe that since \( \mu \) is an \((\infty, p)\) Fock–Carleson measure, inequality (3.12) gives the estimate

\[
\|T_\mu f\|_{(p, \alpha)}^p \lesssim \int_{\mathbb{C}^n} |f(z) e^{-\alpha|z|^2/2}| d\mu(z) \leq \|I_\mu\| \|f\|_{(\infty, \alpha)}^p
\]

for each \( f \) in \( F_\alpha^\infty \) from which we find

\[
\|T_\mu\| \lesssim \|I_\mu\|
\]

(3.13)
where $I_\mu$ is the embedding map $I_\mu : F^{\infty}_\alpha \to L^p(\sigma_p)$.

It remains to prove (i) implies (iii). We first consider the case $p = 1$. Suppose $T_\mu : F^{\infty}_\alpha \to F^1_\alpha$ is bounded. Then by Lemma 4.1 of [4]

$$\int_{\mathbb{C}^n} K_w(\eta)\overline{K_z(\eta)}e^{-\alpha|\eta|^2}d\mu(\eta) = \langle T_\mu K_w, K_z \rangle$$

for each kernel functions $K_w$ and $K_z$. From this and [6, Theorem 8.2], we easily see that for each $f$ in $F^\infty_\alpha$

$$\int_{\mathbb{C}^n} |f(z)|^2e^{-\alpha|z|^2}d\mu(z) \leq \|T_\mu f\|_{(1,\alpha)}\|f\|_{(\infty,\alpha)} \leq \|T_\mu\||f\|^2_{(\infty,\alpha)}. \quad (3.14)$$

This means that $\mu$ is an $(\infty, 2)$ Fock–Carleson measure. Then by a particular case of Theorem 2.4 of [7], $\tilde{\mu}_t \in L^1$ and hence $\mu$ is an $(\infty, q)$ Fock–Carleson measures for any $q \geq 1$ from which the desired conclusion follows. From (3.14), we also have

$$\|I_\mu\| \leq \|T_\mu\|. \quad (3.15)$$

We now consider the case $p > 1$. For this, we may assume that $T_\mu : F^{\infty}_\alpha \to F^p_\alpha$ is bounded and aim to show that $\mu$ is a finite measure on $\mathbb{C}^n$. Then the desired conclusion will follow once from a particular case of Theorem 2.4 of [7] again. To proceed further, we pick a suitable test function defined by

$$f_0 = \sum_{j=1}^{\infty} k_{z_j}.$$ 

Observe that $f_0$ belongs to $F^{\infty}_\alpha$ since it could be written as sum of $c_j k_{z_j}$ where $c_j = 1$ for all $j$ and $\|f_0\|_{(\infty,\alpha)} \simeq 1$; see the atomic decomposition result [6, Theorem 8.2]. Applying $T_\mu$ to this function gives

$$\|T_\mu\|^p \geq \int_{\mathbb{C}^n} \left| T_\mu \sum_{j=1}^{\infty} k_{z_j}(z) \right|^p e^{-\frac{\alpha}{4}|z|^2}dV(z)$$

$$= \int_{\mathbb{C}^n} \left\{ \int_{\mathbb{C}^n} \sum_{j=1}^{\infty} e^{-\alpha|z|^2} + \alpha\langle w, z_j \rangle + \alpha\langle z, w \rangle - \alpha|w|^2 d\mu(w) \right\} e^{-\frac{\alpha}{2}|z|^2}dV(z)$$

$$= \int_{\mathbb{C}^n} \sum_{j=1}^{\infty} \int_{\mathbb{C}^n} e^{i\alpha \Re \langle w, z_j \rangle + \langle z, w \rangle} e^{-\alpha|z|^2 - \frac{\alpha|z|^2}{2}} d\mu(w) \left\| e^{-\frac{\alpha}{2}|z|^2} dV(z) \right\|^p$$

$$\simeq \int_{\mathbb{C}^n} \left( \sum_{j=1}^{\infty} \int_{\mathbb{C}^n} e^{-\frac{\alpha}{2}|z|^2 - \frac{\alpha|z|^2}{2}} d\mu(w) \right)^p dV(z) = J$$
where \( \Im \eta \) refers to the imaginary part of a complex number \( \eta \). Since all the terms in the last sum are positive and \( p > 1 \), we also have the estimate

\[
J \geq \int_{\mathbb{C}^n} \sum_{j=1}^{\infty} \left( \int_{\mathbb{C}^n} e^{-\frac{\alpha |z_j-w|^2}{2} - \frac{\alpha |z-w|^2}{2}} d\mu(w) \right)^p dV(z)
\]

\[
\geq \int_{\mathbb{C}^n} \sum_{j=1}^{\infty} \int_{\mathbb{C}^n} e^{-\frac{\alpha |z_j-w|^2}{2} - \frac{\alpha |z-w|^2}{4}} d\mu(w) dV(z)
\]

\[
= \int_{\mathbb{C}^n} \left( \int_{\mathbb{C}^n} \sum_{j=1}^{\infty} e^{-\frac{\alpha |z_j-w|^2}{2} - \frac{\alpha |z-w|^2}{4}} dV(z) \right) d\mu(w)
\]

\[
\simeq \int_{\mathbb{C}^n} \sum_{j=1}^{\infty} e^{-\frac{\alpha |z_j-w|^2}{2}} d\mu(w) \simeq \int_{\mathbb{C}^n} d\mu(w) = \mu(\mathbb{C}^n)
\]

where the first equality is due to Fubini’s Theorem. From the above series of estimations, we deduce

\[
\mu(\mathbb{C}^n) \lesssim \|T_\mu\|^p
\]

from which, \((3.13)\), \((3.15)\) and the estimates in Theorem 2. of \([7]\), the asymptotic norm relation in \((2.4)\) follows and completes the proof.

**Proof of Theorem 2.5.** The proof of part (i) of the theorem follows from a simple variant of the arguments used to prove Theorem 2.1. Thus, we will omit it. To prove part (ii) of the result, it suffices to verify the statement: \( T_\mu : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^1 \) is bounded if and only if \( \mu(D(z_j, r)) \) belongs to \( \ell^{p/(p-1)} \) for some positive \( r \). Then the desired conclusion will follow once from Theorem 3.3 of \([4]\). We may first assuming that \( T_\mu : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^1 \) is bounded. Note that for \( p = \infty \), the result follows from Corollary 2.4 above. Thus, we only look at the case when \( p > 1 \) is finite. By Theorem 9.3 of \([6]\), our assumption holds if and only if \( T_\mu : \mathcal{F}_\alpha^{2p} \to \mathcal{F}_\alpha^{2p-1} \) is bounded. Following similar arguments as those leading to \((3.14)\), for each \( f \) in \( \mathcal{F}_\alpha^{2p} \) we have

\[
\int_{\mathbb{C}^n} |f(z)|^2 e^{-\alpha |z|^2} d\mu(z) \leq \|T_\mu f\|_{(2p-1, \alpha)} \|f\|_{(2p, \alpha)} \leq \|T_\mu\| \|f\|_{(2p, \alpha)}^2.
\]

This proves that \( \mu \) is an \((2p, 2)\) Fock–Carleson measure. This holds if and only if \( \mu(D(z_j, r)) \) belongs to \( \ell^{2p/(2p-2)} = \ell^{p/(p-1)} \) by special case of Theorem 2.3 of \([7]\). It follows from Theorem 4.4 of \([4]\) that \( T_\mu : \mathcal{F}_\alpha^{p} \to \mathcal{F}_\alpha^{1} \) is bounded and this along with the series of estimations in Theorem 2.3 of \([7]\) give

\[
\|I_\mu\| \simeq \|\mu(D(z_j, r))\|_{\ell^{p/(p-1)}} \lesssim \|T_\mu\|
\]

where \( I_\mu \) is here the embedding map \( I_\mu : \mathcal{F}_\alpha^{p} \to L^1(\sigma_1) \). Conversely, we assume that \( \mu(D(z_j, r)) \) belongs to \( \ell^{p/(p-1)} \) for some positive \( r \). Then for
each $f$ in $\mathcal{F}_\alpha^p$, we have
\[
\|T_\mu f\|_{(1, \alpha)} \leq \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |f(w)| e^{-\alpha|w|^2 - \frac{\alpha}{2}|z|^2} |K_w(z)| d\mu(w) dV(z)
\]
\[
\simeq \int_{\mathbb{C}^n} |f(w)| e^{-\frac{\alpha}{2}|w|^2} \left( \int_{\mathbb{C}^n} |K_w(z)| e^{-\frac{\alpha}{2}(|z|^2 + |w|^2)} dV(z) \right) d\mu(w)
\]
\[
\simeq \int_{\mathbb{C}^n} |f(w)| e^{-\frac{\alpha}{2}|w|^2} d\mu(w)
\]
by Fubini’s theorem and the fact that $\|k_w\|_{(p, \alpha)} = 1$ for each $p$ and $w$ in $\mathbb{C}^n$. By triangle inequality, it follows that $D(z, r) \subset D(z_j, 2r)$ whenever $z \in D(z_j, r)$. For each $z$, Lemma 1 of [5] gives
\[
|f(z)| e^{-\frac{\alpha}{2}|z|^2} \lesssim \int_{D(z_j, 2r)} |f(w)| e^{-\frac{\alpha}{2}|w|^2} dV(w).
\]
Applying this estimation to the right hand of (3.19) followed by Fubini’s theorem gives
\[
\int_{\mathbb{C}^n} |f(w)| e^{-\frac{\alpha}{2}|w|^2} d\mu(w) \lesssim \sum_{j=1}^\infty \mu(D(z_j, r)) \int_{D(z_j, 2r)} |f(z)| e^{-\frac{\alpha}{2}|z|^2} dV(z).
\]
Applying Hölder’s inequality, we see that the right hand side quantity is bounded by
\[
\left( \sum_{j=1}^\infty \mu(D(z_j, r))^\frac{p}{p-1} \right)^{\frac{p-1}{p}} \left( \sum_{j=1}^\infty \left( \int_{D(z_j, 2r)} |f(z)| e^{-\frac{\alpha}{2}|z|^2} dV \right)^p \right)^{\frac{1}{p}}
\]
\[
\lesssim \|\mu(D(z_j, r))\|_{\ell^{\frac{p}{p-1}}} \left( \sum_{j=1}^\infty \int_{D(z_j, 2r)} |f(z)|^p e^{-\frac{\alpha}{2}|z|^2} dV(z) \right)^{\frac{1}{p}}
\]
\[
\lesssim \|\mu(D(z_j, r/2))\|_{\ell^{\frac{p}{p-1}}} \|f\|_{(p, \alpha)}.
\]
where the first inequality follows by Jensen’s inequality as the Lebesgue measure is a finite measure on balls and $x^p$ is convex for all $p \geq 1$. From (3.19) and (3.20), we conclude
\[
\|T_\mu\| \lesssim \|\mu(D(z_j, r/2))\|_{\ell^{\frac{p}{p-1}}}.
\]
We combine this with (3.18) and the series of norm estimates in Theorem 3.3 of [4] to arrive at (2.6) and finish the proof.

References

[1] Berger, C., Coburn, L.: Toeplitz operators and quantum mechanics, J. Funct. Anal., 68, 273–299 (1986)

[2] Berger, C., Coburn, L.: Toeplitz operators on the Segal–Bargmann space, Trans. Am. Math. Soc., 301(2), 813–829 (1987)
[3] Coburn, L., Isralowitz, J., Li, B.: Toeplitz operators with BMO symbols on the Segal–Bargmann space, Trans. Am. Math. Soc., 363, 3015–3030 (2011)

[4] Hu, Z., Lv, X.: Toeplitz operators from one Fock space to another, Integr. Equ. Oper. Theory, 70, 541–559 (2011)

[5] Isralowitz, J., Zhu, K.: Toeplitz operators on the Fock space, Integr. Equ. Oper. Theory, 66(4), 593–611 (2010)

[6] Janson, S., Peetre, J., Rochberg, R.: Hankel forms and the Fock space, Rev. Mat. Iberoamericana, 3, 61–138 (1987)

[7] Mengestie, T.: Carleson type measures for Fock–Sobolev spaces, to appear at Complex Analysis and Operator Theory, DOI: 10.1007/s11785-013-0321-7

[8] Ramirez De Arellano, E., Vasilevski, E.: Toeplitz operators on the Fock space with pre symbols discontinuous on a thick set, Math. Nachr., 180, 299–315 (1996)

[9] Stević, S.: Weighted composition operators on Fock-type spaces in \( \mathbb{C}^N \), Applied Mathematics and Computation, 215, 2750–2760 (2009)

[10] Stroethoff, S.: Compact Toeplitz operators on generalized Fock spaces, Acta Sci. Math. (Szeged), 64, 657–669 (1998)

[11] Stroethoff, K.: Hankel and Toeplitz operators on the Fock space, Mich. Math. J., 39(1), 3–16 (1992)

[12] Toeplitz, O.: Zur Theorie der quadratischen und bilinearen Formen von unendlichvielen, Veränderlichen. Math. Ann., 70, 351–376 (1911)

Tesfa Mengestie
Department of Mathematical Sciences
Norwegian University of Science and Technology (NTNU)
NO-7491 Trondheim, Norway

e-mail: tesfantnu@gmail.com