Multistep shell model description of spin-aligned neutron-proton pair coupling: The formalism

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Abstract

The multistep shell model was extended recently to incorporate both neutron and proton degrees of freedom and applied to study the structure of $N = Z$ systems with four, six and eight particles [arXiv:1108.0269]. In this work we give a brief introduction to the formalism thus developed. A more detailed explanation with applications will be updated later.

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I. INTRODUCTION

In the multistep shell model method (MSM) one solves the shell-model equation in several steps. In the first step one constructs the two-particle states. In the second step one proceed by solving the three- or four-particle states in terms of the two-particle states calculated in the first step. In our case we will solve the two-neutron plus two-proton system within a non-orthogonal overcomplete basis in terms of the \((\nu\pi) \otimes (\nu\pi)\) excitations at the same time as the \((\nu\nu) \otimes (\pi\pi)\) ones. With the four-particle system thus evaluated, we will proceed to the next step and evaluate the six-particle system in terms of the four-particle states times the two-particle states. For the eight-particle system one can choose the MSM basis such that it consists of the four-particle states in the form \((\nu\pi) \otimes (\nu\pi)\) times themselves. Systems with more pairs can be described in the same fashion in successive steps.

The Hamiltonian is given as

\[
H = \sum_i \epsilon_p p_i^\dagger p_i + \sum_i \epsilon_n n_i^\dagger n_i + \sum_{ijkl} \langle ij| V_{pm}| kl \rangle p_i^\dagger n_j^\dagger n_k p_l + \frac{1}{2} \sum_{ijkl} \langle ij| V_{pp}| kl \rangle p_i^\dagger p_j^\dagger p_k p_l + \frac{1}{2} \sum_{ijkl} \langle ij| V_{nn}| kl \rangle n_i^\dagger n_j^\dagger n_k n_l, \tag{1}
\]

where \(p_i\) and \(n_i\) denote proton and neutron operators, respectively, \(\epsilon\) denote the single-particle energies, and \(V\) is the two-body interaction. The indices \(i, j, k, l\) label single-particle states. We will use the Greek letter \(\gamma_n\) to label the \(n\)-particle \(np\) states. Since we will only consider cases with equal number of neutrons and protons outside a closed shell, \(n\) will be an even number such that the number of neutrons \((n/2)\) is the same as the number of protons. Therefore the \(np\) states will be \(|\gamma_2\rangle = P^\dagger(\gamma_2)|0\rangle\) where the \(np\) creation operator is \(P^\dagger(\gamma_2) = \sum_{ij} X(ij; \gamma_2) p_i^\dagger n_j^\dagger\). In the same fashion the two-proton (two-neutron) creation operator is \(P^\dagger(\alpha_2) (P^\dagger(\beta_2))\). The amplitude \(X\) of the two-body wave function is determined by solving the corresponding TDA equations,

\[
(W(\alpha_2) - \epsilon_{p_1} - \epsilon_{p_2}) \langle \alpha_2| p_1^\dagger p_2^\dagger |0\rangle = \frac{1}{2} \sum_{p_1 p_2} \langle p_1 p_2| V_{pp}| p_1 p_2 \rangle \langle \alpha_2| p_1^\dagger p_2^\dagger |0\rangle, \]

\[
(W(\beta_2) - \epsilon_{n_1} - \epsilon_{n_2}) \langle \beta_2| n_1^\dagger n_2^\dagger |0\rangle = \frac{1}{2} \sum_{n_1 n_2} \langle n_1 n_2| V_{nn}| n_1 n_2 \rangle \langle \beta_2| n_1^\dagger n_2^\dagger |0\rangle, \]

\[
(W(\gamma_2) - \epsilon_{p_1} - \epsilon_{n_2}) \langle \gamma_2| p_1^\dagger n_2^\dagger |0\rangle = \sum_{p_1 n_2} \langle p_1 n_2| V_{pn}| p_1 n_2 \rangle \langle \gamma_2| p_1^\dagger n_2^\dagger |0\rangle. \]

We have \(X = 1\) for systems within a single-\(j\) shell. These correlated two-particle states are the basic building blocks in the construction of the MSM many-body basis vectors.
II. SYSTEM WITH TWO NEUTRONS AND TWO PROTONS

The four-particle state is $|\gamma_4\rangle = P^\dagger(\gamma_4)|0\rangle$ with

$$P^\dagger(\gamma_4) = \sum_{\alpha_2\beta_2} X(\alpha_2\beta_2; \gamma_4) P^\dagger(\alpha_2) P^\dagger(\beta_2) + \sum_{\gamma_2 \leq \gamma_2'} X(\gamma_2\gamma_2'; \gamma_4) P^\dagger(\gamma_2) P^\dagger(\gamma_2')$$

(2)

where all possible like-particle and $np$ pairs are taken into account. The physical meaningful quantities are the projections of the basis vectors upon the physical vector, which we denote as

$$F(\alpha_2\beta_2; \gamma_4) = \langle \gamma_4 | P^\dagger(\alpha_2) P^\dagger(\beta_2) | 0 \rangle,$$

$$F(\gamma_2\gamma_2'; \gamma_4) = \langle \gamma_4 | P^\dagger(\gamma_2) P^\dagger(\gamma_2') | 0 \rangle.$$  (3)

The orthonormality condition now reads

$$\delta_{\gamma_4\gamma_4'} = \sum_{\alpha_2\beta_2} X(\alpha_2\beta_2; \gamma_4) F(\alpha_2\beta_2; \gamma_4') + \sum_{\gamma_2 \leq \gamma_2'} X(\gamma_2\gamma_2'; \gamma_4) F(\gamma_2\gamma_2'; \gamma_4).$$  (4)

The norm of the MSM basis $|\gamma_2\gamma_2\rangle = P^\dagger(\gamma_2) P^\dagger(\gamma_2') | 0 \rangle$, i.e., $N(\gamma_2\gamma_2'; \gamma_4) = \sqrt{\langle \gamma_2\gamma_2 | \gamma_2\gamma_2 \rangle}$, may not be unity.

II.1. TDA Equation

The dynamic matrix of the two-neutron two-proton system is given as

$$(W(\gamma_4) - W(\gamma_2) - W(\gamma_2'))\langle \gamma_4 | (P^\dagger(\gamma_2) P^\dagger(\gamma_2')) | \gamma_4 \rangle | 0 \rangle =$$

$$\sum_{\gamma_2'' \leq \gamma_2'''} \left\{ \sum_{p_1p_2n_1n_2} (-1) \frac{W(\gamma_2'') + W(\gamma_2''') - \varepsilon_{p_1} - \varepsilon_{p_2} - \varepsilon_{n_1} - \varepsilon_{n_2}}{1 + \delta_{\gamma_2''\gamma_2'''} \langle \gamma_4 | (P^\dagger(\gamma_2'') P^\dagger(\gamma_2''')) | \gamma_4 \rangle | 0 \rangle} \right\} \times (A_1 + A_2)$$

$$+ \sum_{\alpha_2\beta_2} \left\{ \sum_{p_1p_2n_1n_2} (W(\alpha_2) + W(\beta_2) - \varepsilon_{p_1} - \varepsilon_{p_2} - \varepsilon_{n_1} - \varepsilon_{n_2}) \times B \right\} \langle \gamma_4 | (P^\dagger(\alpha_2) P^\dagger(\beta_2)) | \gamma_4 \rangle | 0 \rangle,$$

and

$$(W(\gamma_4) - W(\alpha_2) - W(\beta_2))\langle \gamma_4 | (P^\dagger(\alpha_2) P^\dagger(\beta_2)) | \gamma_4 \rangle | 0 \rangle =$$

$$\sum_{\gamma_2'' \leq \gamma_2'''} \left\{ \sum_{p_1p_2n_1n_2} \frac{W(\gamma_2'') + W(\gamma_2''') - \varepsilon_{p_1} - \varepsilon_{p_2} - \varepsilon_{n_1} - \varepsilon_{n_2}}{1 + \delta_{\gamma_2''\gamma_2'''} \langle \gamma_4 | (P^\dagger(\gamma_2'') P^\dagger(\gamma_2''')) | \gamma_4 \rangle | 0 \rangle} \right\} \times C$$

$$\langle \gamma_4 | (P^\dagger(\gamma_2'') P^\dagger(\gamma_2''')) | \gamma_4 \rangle | 0 \rangle.$$
where $W$ denotes the corresponding $n$-particle energy. To obtain above equation we have assumed that the four-particle system was decomposed into two different blocks in terms of $(\pi\pi) \otimes (\nu\nu)$ and $(\nu\pi) \otimes (\nu\pi)$.

The $A$, $B$ and $C$ matrix elements are defined as,

$$A_1 = (-1)^{2p_1+n_1+n_2+\gamma_2+\gamma_2'} \hat{\gamma}_1 \hat{\gamma}_2 \hat{\gamma}_2' \hat{\gamma}_2'' \hat{\gamma}_2''' \begin{pmatrix} p_1 & n_1 & \gamma_2 \\ n_2 & p_2 & \gamma_2' \\ \gamma_2'' & \gamma_2''' & \gamma_4 \end{pmatrix} \times X(p_1n_1; \gamma_2)X(p_2n_2; \gamma_2')X(p_1n_2; \gamma_2'')X(p_2n_1; \gamma_2''')$$

$$A_2 = (-1)^{2p_1+n_1+n_2+\gamma_2+\gamma_2'} \hat{\gamma}_1 \hat{\gamma}_2 \hat{\gamma}_2' \hat{\gamma}_2'' \hat{\gamma}_2''' \begin{pmatrix} p_1 & n_1 & \gamma_2 \\ n_2 & p_2 & \gamma_2' \\ \gamma_2'' & \gamma_2''' & \gamma_4 \end{pmatrix} \times X(p_1n_1; \gamma_2)X(p_2n_2; \gamma_2')X(p_1n_2; \gamma_2'')X(p_2n_1; \gamma_2''')$$

$$B = \hat{\gamma}_2 \hat{\gamma}_2' \hat{\gamma}_2'' \hat{\gamma}_2''' \hat{\gamma}_2'''' \begin{pmatrix} p_1 & n_1 & \gamma_2 \\ n_2 & p_2 & \gamma_2' \\ \gamma_2'' & \gamma_2''' & \gamma_4 \end{pmatrix} \times X(p_1n_1; \gamma_2)X(p_2n_2; \gamma_2')Y(p_1p_2; \alpha_2)Y(n_1n_2; \beta_2)$$

$$C = \hat{\alpha}_2 \hat{\beta}_2 \hat{\gamma}_2'' \hat{\gamma}_2''' Y(p_1p_2; \alpha_2)Y(n_1n_2; \beta_2)X(p_1n_1; \gamma_2')X(p_2n_2; \gamma_2'')$$

In all cases we use the same symbols to label states as well as the corresponding angular momenta. The coefficient $Y$ is related to $X$ by $Y(ij; \alpha_2) = (1 + \delta_{ij})^{1/2}X(ij; \alpha_2)$.

### II.2. Overlap Matrix

The overlap matrix is defined as follows,

$$\langle 0 | (P^\dagger(\gamma_2)P^\dagger(\gamma_2'))_{\gamma_4} (P^\dagger(\gamma_2')P^\dagger(\gamma_2))_{\gamma_4} | 0 \rangle = \delta_{\gamma_2\gamma_2'} \delta_{\gamma_2\gamma_2'} + (-1)^{\gamma_2+\gamma_2'} \delta_{\gamma_2\gamma_2'} \delta_{\gamma_2\gamma_2'}$$

$$- \sum_{p_1p_2n_1n_2} (A_1 + A_2)$$

$$\langle 0 | (P^\dagger(\gamma_2)P^\dagger(\gamma_2'))_{\gamma_4} (P^\dagger(\alpha_2)P^\dagger(\beta_2))_{\gamma_4} | 0 \rangle = \sum_{p_1p_2n_1n_2} B$$

$$\langle 0 | (P^\dagger(\alpha_2)P^\dagger(\beta_2'))_{\gamma_4} (P^\dagger(\alpha_2')P^\dagger(\beta_2'))_{\gamma_4} | 0 \rangle = \delta_{\alpha_2\alpha_2'} \delta_{\beta_2\beta_2'}$$
which correspond to the overlap between states of the forms \( \langle \nu \pi \otimes \nu \pi | \nu \pi \otimes \nu \pi \rangle \), \( \langle \nu \pi \otimes \nu \pi | \nu \nu \otimes \pi \pi \rangle \) and \( \langle \nu \nu \otimes \pi \pi | \nu \nu \otimes \pi \pi \rangle \), respectively.

### III. SYSTEM WITH THREE NEUTRONS AND THREE PROTONS

For the six-particle case we will use the MSM partition of two- times four-particles. Thus the corresponding wave function will be \( |\gamma_6\rangle = P^\dagger(\gamma_6)|0\rangle \), where

\[
P^\dagger(\gamma_6) = \sum_{\gamma_2\gamma_4} X(\gamma_2\gamma_4; \gamma_6) P^\dagger(\gamma_2) P^\dagger(\gamma_4).
\]

and

\[
|\gamma_6\rangle = \sum_{\gamma_2\gamma_4} X(\gamma_2\gamma_4; \gamma_6) \sum_{\alpha_2\beta_2} X(\alpha_2\beta_2; \gamma_4) \sum_{p_1n_1} X(p_1n_1; \gamma_2) \langle p_1n_1|\gamma_2\rangle \times \frac{1}{2} \sum_{p_2p_3} Y(p_2p_3; \alpha_2) \langle p_2p_3|\alpha_2\rangle \times \frac{1}{2} \sum_{n_2n_3} Y(n_2n_3; \beta_2) \langle n_2n_3|\beta_2\rangle p_1^\dagger n_1^\dagger p_2^\dagger n_2^\dagger p_3^\dagger n_3^\dagger |0\rangle.
\]

As before, we will evaluate the projection of the basis vectors upon the physical vectors, i.e., \( F(\gamma_2\gamma_4; \gamma_6) \). In this six-particle case one can also view the MSM basis elements as the direct tensorial product of three pairs which takes the forms \( \nu \pi \otimes \nu \pi \otimes \nu \pi \) and \( \nu \pi \otimes \nu \nu \otimes \pi \pi \).

#### III.1. TDA Equation of the 2 \times 4 Block

For the partition of one \( np \) pair times the 4-particle system, the dynamic matrix is given as

\[
(W(\gamma_6) - W(\gamma_2) - W(\gamma_4)) \langle \gamma_6| (\gamma_2^\dagger \gamma_4^\dagger) \gamma_6 |0\rangle = \sum_{\gamma_2'\gamma_4'} \left\{ \sum_{p_1n_1n_2n_3} \sum_{\alpha_2\beta_2} \sum_{\beta_4} (W(\gamma_2') + W(\beta_4') - \varepsilon_{p_1} - \varepsilon_{n_1} - \varepsilon_{n_2} - \varepsilon_{n_3}) \times A_1 \right. \\
+ \left. \sum_{p_1p_2p_3n_1} \sum_{\alpha_2\beta_2} \sum_{\alpha_2'\beta_4'} (W(\gamma_2) + W(\alpha_2') - \varepsilon_{p_1} - \varepsilon_{p_2} - \varepsilon_{p_3} - \varepsilon_{n_1}) \times A_2 \right\} \langle \gamma_6| (\gamma_2^\dagger \gamma_4^\dagger) \gamma_6 |0\rangle,
\]

where

\[
A_1 = (-1)^{n_2+n_3+\gamma_2+\gamma_4+\gamma_2'\beta_2'\beta_4'\gamma_2'\gamma_4'} \times X(p_1n_1; \gamma_2) Y(n_2n_3; \beta_2) X(p_1n_1; \gamma_2) Y(n_1n_2; \beta_2') \times \left\{ \begin{array}{c} p_1 n_1 \gamma_2 \\ n_3 n_2 \beta_2 \\ \gamma_2' \beta_2' \theta_4 \end{array} \right\} \left\{ \begin{array}{c} \gamma_2 \beta_2 \theta_4 \\ \alpha_2 \gamma_6 \gamma_4 \end{array} \right\} \left\{ \begin{array}{c} \gamma_2' \beta_2' \theta_4 \\ \alpha_2 \gamma_6 \gamma_4' \end{array} \right\},
\]

\[
A_2 = (-1)^{n_2+n_3+\gamma_2+\gamma_4+\gamma_2'\beta_2'\beta_4'\gamma_2'\gamma_4'} \times X(p_1n_1; \gamma_2) Y(n_2n_3; \beta_2) X(p_1n_1; \gamma_2) Y(n_1n_2; \beta_2') \times \left\{ \begin{array}{c} p_1 n_1 \gamma_2 \\ n_3 n_2 \beta_2 \\ \gamma_2' \beta_2' \theta_4 \end{array} \right\} \left\{ \begin{array}{c} \gamma_2 \beta_2 \theta_4 \\ \alpha_2 \gamma_6 \gamma_4 \end{array} \right\} \left\{ \begin{array}{c} \gamma_2' \beta_2' \theta_4 \\ \alpha_2 \gamma_6 \gamma_4' \end{array} \right\},
\]

\[
5
\]
\[ A_2 = (-1)^{n_1+p_1+\gamma_2+\alpha_2+\gamma_4+\phi_4} \hat{\gamma}_2 \hat{\alpha}_2 \hat{\gamma}_4 \hat{\gamma}_4' \hat{\phi}_4' \]
\[ \times X(p_1n_1; \gamma_2)Y(p_2p_3; \alpha_2)X(p_3n_1; \gamma_2')Y(p_1p_2; \alpha_2')X(\alpha_2 \beta_2; \gamma_4)F(\alpha_2' \beta_2; \gamma_4') \]
\[ \times \left\{ \begin{array}{c}
\begin{array}{c}
\{ p_1 \ n_1 \ \gamma_2 \\
\{ p_2 \ p_3 \ \alpha_2 \\
\{ \alpha_2' \ \gamma_4' \ \phi_4'
\end{array}
\end{array} \right\} \left\{ \begin{array}{c}
\begin{array}{c}
\{ \gamma_2 \ \alpha_2 \ \phi_4 \\
\{ \beta_2 \ \gamma_4 \ \gamma_4' \\
\{ \beta_2 \ \gamma_6 \ \gamma_4'
\end{array}
\end{array} \right\}. \quad (8) \]

The overlap matrix of a \(2 \times 4\) block is given as
\[ \langle 0 | (\gamma_2^t \gamma_4')^t \gamma_6^t (\gamma_2^t \gamma_4')^t | 0 \rangle = \delta_{\gamma_2 \gamma_2} \delta_{\gamma_4 \gamma_4'} + \sum_{p_1m_1} \sum_{p_2m_2} \sum_{\alpha_2 \beta_2 \phi_4} A_1 + \sum_{p_1m_1} \sum_{p_2m_2} \sum_{\alpha_2 \beta_2 \phi_4} A_2 \]
\[ + \sum_{p_1m_1} \sum_{p_2m_2} B \]
, where
\[ B = (-1)^{\gamma_2+\gamma_4'} \hat{\gamma}_2 \hat{\beta}_2 \hat{\gamma}_4 \hat{\beta}_4 \hat{\gamma}_4' \hat{\phi}_4' \hat{\phi}_4 \]
\[ \times X(p_1n_1; \gamma_2)Y(p_2p_3; \alpha_2)Y(n_2n_3; \beta_2)X(p_3n_3; \gamma_2')Y(p_1p_2; \alpha_2')Y(n_1n_2; \beta_2') \]
\[ \times X(\alpha_2 \beta_2; \gamma_4)X(\alpha_2' \beta_2'; \gamma_4') \]
\[ \times \left\{ \begin{array}{c}
\begin{array}{c}
\{ p_1 \ n_1 \ \gamma_2 \\
\{ p_2 \ n_2 \ \psi_2 \\
\{ \alpha_2' \ \beta_2' \ \gamma_4'
\end{array}
\end{array} \right\} \left\{ \begin{array}{c}
\begin{array}{c}
\{ \gamma_2 \ \psi_2 \ \gamma_4' \\
\{ \gamma_2 \ \gamma_6 \ \gamma_4' \\
\{ \gamma_2 \ \gamma_6 \ \gamma_4'
\end{array}
\end{array} \right\}. \quad (10) \]

### III.2. Transformation to the \(2 \times 2 \times 2\) block

The transformation from the \(2 \times 4\) block to the \(2 \times 2 \times 2\) block is given as
\[ \langle \gamma_6 | (\gamma_2^t \alpha_2^t \beta_2^t) \gamma_6 | 0 \rangle = \sum_{\gamma_4} \langle \gamma_6 | (\gamma_2^t \gamma_4') \gamma_6 | 0 \rangle \langle \gamma_4 | (\alpha_2^t \beta_2^t) \gamma_4 | 0 \rangle, \]
\[ \langle \gamma_6 | (\gamma_2^t \gamma_2^t \gamma_2') \gamma_6 | 0 \rangle = \sum_{\gamma_4} \langle \gamma_6 | (\gamma_2^t \gamma_4') \gamma_6 | 0 \rangle \langle \gamma_4 | (\gamma_2^t \gamma_2') \gamma_4 | 0 \rangle. \]

The overlap for the \(2 \times 2 \times 2\) coupling is
\[ \langle 0 | (\gamma_2^t A_2^t B_2^t) \gamma_6 | 0 \rangle = \sum_{\gamma_4 \gamma_4'} \langle \gamma_4 | (A_2^t B_2^t) \gamma_4 | 0 \rangle \langle 0 | (\gamma_2^t \gamma_4') \gamma_6 | 0 \rangle \langle \gamma_4' | (C_2^t D_2^t) \gamma_4' | 0 \rangle. \]

### IV. FOUR-PROTON FOUR-NEUTRON SYSTEM

We will describe the eight-particle states as \(| \gamma_8 \rangle = P^t(\gamma_8) | 0 \rangle\), where
\[ P^t(\gamma_8) = \sum_{\alpha_4 \leq \beta_4} X(\alpha_4 \beta_4; \gamma_8) P^t(\alpha_4) P^t(\beta_4). \quad (11) \]
and

\[ |\gamma_8\rangle = \sum_{\gamma_4'} X(\gamma_4' \gamma_8) (\gamma_4' \gamma_8) \sum_{\alpha_2 \beta_2} X(\alpha_2 \beta_2; \gamma_4) (\alpha_2 \beta_2 \gamma_4) \sum_{\alpha_2' \beta_2'} X(\alpha_2' \beta_2'; \gamma_4') (\alpha_2' \beta_2' \gamma_4) \]

\[ \times \frac{1}{2} \sum_{p_1 p_2} Y(p_1 p_2; \alpha_2) (p_1 p_2 \alpha_2) \times \frac{1}{2} \sum_{n_1 n_2} Y(n_1 n_2; \beta_2) (n_1 n_2 \beta_2) \]

\[ \times \frac{1}{2} \sum_{p_3 p_4} Y(p_3 p_4; \alpha_2') (p_3 p_4 \alpha_2') \times \frac{1}{2} \sum_{n_3 n_4} Y(n_3 n_4; \beta_2') (n_3 n_4 \beta_2') \]

\[ (W(\gamma_8) - W(\gamma_4) - W(\gamma_4')) |\gamma_8\rangle (\gamma_4' \gamma_4' \gamma_8) |0\rangle = \sum_{\gamma_4''} \frac{1}{1 + \delta_{\gamma_4'} \gamma_4''} \sum_{\alpha_2 \beta_2 \alpha_2' \beta_2'} 
\]

\[ \left\{ (W(\gamma_4') + W(\gamma_4'')) - W(\alpha_2) - W(\beta_2) - W(\alpha_2') - W(\beta_2') \right\} \times (A_1 + A_2) \]

\[ + \sum_{\alpha_2 \alpha_2'} \sum_{p_1 p_2 p_3 p_4} (W(\alpha_2') + W(\alpha_2'')) \varepsilon_{p_1} - \varepsilon_{p_2} - \varepsilon_{p_3} - \varepsilon_{p_4} (B_1 + B_2) \]

\[ + \sum_{\beta_2 \beta_2'} \sum_{n_1 n_2 n_3 n_4} (W(\beta_2') + W(\beta_2'')) \varepsilon_{n_1} - \varepsilon_{n_2} - \varepsilon_{n_3} - \varepsilon_{n_4} (C_1 + C_2) \}

\[ |\gamma_8\rangle (\gamma_4' \gamma_4' \gamma_8) |0\rangle \]

where

\[ A_1 = (-1)^{\beta_2 + \beta_2' + \gamma_4 + \gamma_4'} \times \hat{\gamma}_4 \hat{\gamma}_4' \hat{\gamma}_4' \]

\[ \times X(\alpha_2 \beta_2; \gamma_4) X(\alpha_2 \beta_2'; \gamma_4') F(\alpha_2 \beta_2; \gamma_4') F(\alpha_2 \beta_2'; \gamma_4') \]

\[ \left\{ \begin{array}{c} \alpha_2 \beta_2 \gamma_4 \\ \beta_2' \alpha_2' \gamma_4' \\ \gamma_4' \gamma_4'' \gamma_8 \end{array} \right\} \]

\[ A_2 = (-1)^{\beta_2 + \beta_2' + \gamma_4 + \gamma_4'' + \gamma_8} \times \hat{\gamma}_4 \hat{\gamma}_4' \hat{\gamma}_4' \hat{\gamma}_4'' \]

\[ \times X(\alpha_2 \beta_2; \gamma_4) X(\alpha_2 \beta_2'; \gamma_4') F(\alpha_2 \beta_2; \gamma_4') F(\alpha_2 \beta_2'; \gamma_4'') \]

\[ \left\{ \begin{array}{c} \alpha_2 \beta_2 \gamma_4 \\ \beta_2' \alpha_2' \gamma_4' \\ \gamma_4'' \gamma_4'' \gamma_8 \end{array} \right\} \]
\[ \mathbb{B}_1 = \sum_{\theta_4 \phi_4} (-1) \hat{\alpha}_2 \hat{\alpha}_2' \hat{\alpha}_2'' \hat{\gamma}_4 \hat{\gamma}_4' \hat{\gamma}_4'' \hat{\theta}_4' \hat{\phi}_4' \\
\times X(\alpha_2 \beta_2; \gamma_4) X(\alpha_2' \beta_2'; \gamma_4') F(\alpha_2'' \beta_2''; \gamma_4'') F(\alpha_2''' \beta_2'''; \gamma_4''') Y(p_1 p_2; \alpha_2) Y(p_3 p_4; \alpha_2') \]
\[ \times Y(p_1 p_3; \alpha_2'') Y(p_2 p_4; \alpha_2'') \]
\[ \alpha_2' \alpha_2'' \alpha_2''' \alpha_2''' \theta_4 \phi_4 \gamma_8 \]
\[ \alpha_2'' \alpha_2''' \theta_4 \phi_4 \gamma_8 \]

\[ \mathbb{B}_2 = \sum_{\theta_4 \phi_4} (-1)^{1+\beta_2+\beta_2'+\phi_4} \hat{\alpha}_2 \hat{\alpha}_2' \hat{\alpha}_2'' \hat{\gamma}_4 \hat{\gamma}_4' \hat{\gamma}_4'' \hat{\theta}_4' \hat{\phi}_4' \\
\times X(\alpha_2 \beta_2; \gamma_4) X(\alpha_2' \beta_2'; \gamma_4') F(\alpha_2'' \beta_2''; \gamma_4'') F(\alpha_2''' \beta_2'''; \gamma_4''') Y(p_1 p_2; \alpha_2) Y(p_3 p_4; \alpha_2') \]
\[ \times Y(p_1 p_3; \alpha_2'') Y(p_2 p_4; \alpha_2'') \]
\[ \alpha_2' \alpha_2'' \alpha_2''' \alpha_2''' \theta_4 \phi_4 \gamma_8 \]
\[ \alpha_2'' \alpha_2''' \theta_4 \phi_4 \gamma_8 \]

\[ \mathbb{C}_1 = \sum_{\theta_4 \phi_4} (-1) \hat{\beta}_2 \hat{\beta}_2' \hat{\beta}_2'' \hat{\beta}_2''' \hat{\gamma}_4 \hat{\gamma}_4' \hat{\gamma}_4'' \hat{\gamma}_4''' \hat{\theta}_4' \hat{\phi}_4' \\
\times X(\alpha_2 \beta_2; \gamma_4) X(\alpha_2' \beta_2'; \gamma_4') F(\alpha_2'' \beta_2''; \gamma_4'') F(\alpha_2''' \beta_2'''; \gamma_4''') Y(n_1 n_2; \beta_2) Y(n_3 n_4; \beta_2') \]
\[ \times Y(n_1 n_3; \beta_2'') Y(n_2 n_4; \beta_2'') \]
\[ \beta_2' \beta_2'' \gamma_4 \phi_4 \gamma_8 \]
\[ \beta_2'' \beta_2''' \gamma_4 \phi_4 \gamma_8 \]

\[ \mathbb{C}_2 = \sum_{\theta_4 \phi_4} (-1)^{1+\alpha_2+\alpha_2'+\phi_4} \hat{\beta}_2 \hat{\beta}_2' \hat{\beta}_2'' \hat{\beta}_2''' \hat{\gamma}_4 \hat{\gamma}_4' \hat{\gamma}_4'' \hat{\gamma}_4''' \hat{\theta}_4' \hat{\phi}_4' \\
\times X(\alpha_2 \beta_2; \gamma_4) X(\alpha_2' \beta_2'; \gamma_4') F(\alpha_2'' \beta_2''; \gamma_4'') F(\alpha_2''' \beta_2'''; \gamma_4''') Y(n_1 n_2; \beta_2) Y(n_3 n_4; \beta_2') \]
\[ \times Y(n_1 n_3; \beta_2'') Y(n_2 n_4; \beta_2'') \]
\[ \beta_2' \beta_2'' \gamma_4 \phi_4 \gamma_8 \]
\[ \beta_2'' \beta_2''' \gamma_4 \phi_4 \gamma_8 \]
IV.1. Overlap Matrix

The overlap matrix of $4 \times 4$ block is given by

$$\langle 0| (\gamma_4^t \gamma_4^t)_{\alpha_8} (\gamma_4^t \gamma_4^t)_{\alpha_8} |0 \rangle = \delta_{\gamma_4^t \gamma_4^t} \delta_{\gamma_4^t \gamma_4^t} + (-1)^{\gamma_4^t + \gamma_4^t} \alpha_8 \delta_{\gamma_4^t \gamma_4^t} \delta_{\gamma_4^t \gamma_4^t} + \sum_{\alpha_2 \beta_2 \alpha_2' \beta_2'} (A_1 + A_2)$$

$$+ \sum_{\alpha_2 \beta_2 \alpha_2' \beta_2'} \sum_{p_1 p_2 p_3 p_4} (B_1 + B_2) + \sum_{\alpha_2 \beta_2 \alpha_2' \beta_2'} \sum_{n_1 n_2 n_3 n_4} (C_1 + C_2)$$

$$+ \sum_{\alpha_2 \beta_2 \alpha_2' \beta_2'} \sum_{p_1 p_2 p_3 p_4} \sum_{n_1 n_2 n_3 n_4} D,$$

where

$$D = \sum_{\theta_3 \phi_4} \alpha_2 \beta_2 \alpha_2' \beta_2' \alpha_2'' \beta_2'' \alpha_2''' \beta_2''' \gamma_4^t \gamma_4^t \gamma_4^t \gamma_4^t \theta_3 \phi_4$$

$$\times X(\alpha_2 \beta_2; \gamma_4) X(\alpha_2' \beta_2'; \gamma_4') X(\alpha_2'' \beta_2''; \gamma_4'') X(\alpha_2''' \beta_2'''; \gamma_4''') Y(p_1 p_2; \alpha_2) Y(p_3 p_4; \alpha_2')$$

$$Y(p_1 p_2; \alpha_2''') Y(p_2 p_3; \alpha_2''') Y(n_1 n_2; \beta_2) Y(n_3 n_4; \beta_2')$$

$$\times \left\{ \begin{array}{c} p_1 & p_2 & \alpha_2 \\ p_3 & p_4 & \alpha_2' \\ \alpha_2'' & \alpha_2''' & \theta_4 \end{array} \right\} \left\{ \begin{array}{c} n_1 & n_2 & \beta_2 \\ n_3 & n_4 & \beta_2' \\ \beta_2'' & \beta_2''' & \phi_4 \end{array} \right\} \left\{ \begin{array}{c} \alpha_2 & \beta_2 & \gamma_4 \\ \alpha_2' & \beta_2' & \gamma_4' \\ \alpha_2''' & \beta_2''' & \gamma_4''' \end{array} \right\} .$$

IV.2. Transformation to the $2 \times 2 \times 2 \times 2$ coupling

The transformation from the $4 \times 4$ block to the $2 \times 2 \times 2 \times 2$ block is given as,

$$\langle \gamma_8 | (\alpha_2 \beta_2 \alpha_2' \beta_2')_{\gamma_8} |0 \rangle = \sum_{\gamma_4^t \gamma_4^t} \langle \gamma_8 | (\gamma_4^t \gamma_4^t)_{\gamma_8} |0 \rangle \langle \gamma_4 | (\alpha_2 \beta_2; \gamma_4) |0 \rangle \langle \gamma_4' | (\alpha_2' \beta_2'; \gamma_4') |0 \rangle.$$

$$\langle \gamma_8 | (\alpha_2 \beta_2 \gamma_2^t \gamma_2')_{\gamma_8} |0 \rangle = \sum_{\gamma_4^t \gamma_4^t} \langle \gamma_8 | (\gamma_4^t \gamma_4^t)_{\gamma_8} |0 \rangle \langle \gamma_4 | (\alpha_2 \beta_2; \gamma_4) |0 \rangle \langle \gamma_4' | (\gamma_2^t \gamma_2')_{\gamma_4} |0 \rangle.$$

$$\langle \gamma_8 | (\gamma_2^t \gamma_2' \gamma_2'' \gamma_2'''_{\gamma_8}) |0 \rangle = \sum_{\gamma_4^t \gamma_4^t} \langle \gamma_8 | (\gamma_4^t \gamma_4^t)_{\gamma_8} |0 \rangle \langle \gamma_4 | (\gamma_2^t \gamma_2')_{\gamma_4} |0 \rangle \langle \gamma_4' | (\gamma_2'' \gamma_2''')_{\gamma_4'} |0 \rangle.$$

The overlap matrix is as follows,

$$\langle 0 | (A_2 B_2 C_2 D_2)_{\gamma_8}^t (E_2 F_2 G_2 H_2)_{\gamma_8} |0 \rangle = \sum_{\gamma_4^t \gamma_4^t \gamma_4^t \gamma_4^t} \langle \gamma_4 | (A_2 B_2; \gamma_4) |0 \rangle \langle \gamma_4' | (B_2 D_2; \gamma_4') |0 \rangle \langle \gamma_4'' | (C_2 D_2; \gamma_4''') |0 \rangle \langle \gamma_4''' | (G_2 H_2; \gamma_4''''') |0 \rangle.$$
V. SUMMARY

Summarizing, in this work we introduced the formalism to study the structure of the wave function of \( N = Z \) systems with two (Section II), three (Section III) and four (Section IV) \( np \) pairs. Systems with more pairs can be described in the same fashion in successive steps.

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[1] Z.X. Xu, C. Qi, J. Blomqvist, R.J. Liotta, R. Wyss, arXiv: 1108.0269.

[2] R. J. Liotta and C. Pomar, Nucl. Phys. A 362 (1981) 137.

[3] C. Qi, T. Bäck, J. Blomqvist, B. Cederwall, R. J. Liotta and R. Wyss, arXiv: 1101.4046, Phys. Rev. C (R), in press.

[4] S. Zerguine and P. Van Isacker, Phys. Rev. C 83 (2011) 064314.