Phase transition for black holes with scalar hair and topological black holes

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Abstract

We study phase transitions between black holes with scalar hair and topological black holes in asymptotically anti-de Sitter spacetimes. As the ground state solutions, we introduce the non-rotating BTZ black hole in three dimensions and topological black hole with hyperbolic horizon in four dimensions. For the temperature matching only, we show that the phase transition between black hole with scalar hair (Martinez-Troncoso-Zanelli black hole) and topological black hole is second-order by using differences between two free energies. However, we do not identify what order of the phase transition between scalar and non-rotating BTZ black holes occurs in three dimensions, although there exists a possible decay of scalar black hole to non-rotating BTZ black hole.

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1 Introduction

Topological black holes in asymptotically anti-de Sitter spacetimes were first found in three and four dimensions [1]. Their black hole horizons are Einstein spaces of spherical ($k = 1$), hyperbolic ($k = -1$), and flat ($k = 0$) curvature for higher dimensions more than three [2,3]. The standard equilibrium and off-equilibrium thermodynamic analysis is possible to show that they are treated as the extended thermodynamic system even though their horizons are not spherical.

The Schwarzschild black hole with spherical horizon is in an unstable equilibrium with the heat reservoir of the temperature $T$ [4]. Its fate under small fluctuations will be either to decay to hot flat space by Hawking radiation or to grow without limit by absorbing thermal radiation in the heat reservoir [5]. This means that an isolated black hole is never in thermal equilibrium. There exists a way to achieve a stable black hole in an equilibrium with the heat reservoir. A black hole could be rendered thermodynamically stable by placing it in four-dimensional anti-de Sitter (AdS$_4$) spacetimes. An important point to understand is how a stable black hole with positive specific heat could emerge from thermal radiation through a phase transition. This is known as the Hawking-Page phase transition between thermal AdS space and Schwarzschild-AdS black hole [6,7,8], which is a typical example of the first-order phase transition.

It was proposed that a phase transition between AdS black hole with hyperbolic horizon and AdS massless black hole with degenerate horizon is unlikely to occur [9]. Moreover, the phase transition between AdS black hole with Ricci-flat horizon and AdS soliton has been proposed for a candidate of phase transition in topological black holes [10,11].

Recently, a black hole solution with a minimally coupled self-interacting scalar was found in AdS$_4$ spacetimes [12]. This is called the Martinez-Troncoso-Zanelli black hole (MTZ). Its horizon geometry is a surface of hyperbolically constant curvature. This solution dressed by a scalar reminds us of the topological black hole with hyperbolic horizon (TBH). It was conjectured that there is a second-order phase transition between MTZ and TBH [13,14]. Furthermore, it was argued that quasinormal modes show a signal to this phase transition.

On the other hand, there exists a black hole solution to three-dimensional gravity with a minimally coupled self-interacting scalar, called the scalar black hole (SBH) [15,16]. It was observed that the SBH can decay into the non-rotating Banados-Teitelboim-Zanelli black hole (NBTZ) [17]. We note that in three dimensions, there is no distinction between AdS black hole with spherical horizon (NBTZ) and those with hyperbolic and flat horizons [11]. Hence we expect to have a phase transition between SBH and NBTZ.
In this Letter, we show that the phase transition between MTZ and TBH is second-order by using temperature matching and off-shell free energies. However, we do not identify what order of the phase transition between SBH and NBTZ occurs in three dimensions, although there exists a possible decay of SBH into NBTZ.

Our study is based on the on-shell observations of temperature, mass, heat capacity and free energy as well as the off-shell observations of generalized (off-shell) free energy. In general, the on-shell thermodynamics implies equilibrium thermodynamics and thus the first-law of thermodynamics holds for this case. Hence it describes relationships among thermal equilibria, but not the transitions between equilibria. On the other hand, the off-shell thermodynamics is designed for the description of off-equilibrium configurations [18, 19]. We note that the first-law of thermodynamics does not hold for off-shell thermodynamics. This approach is suitable for the description of phase transitions between thermal equilibria. Especially, the off-shell free energy shows the phase transition characteristics more manifestly than on-shell free energy. Introducing this quantity leads to that the temperature dependence of phase transition is clearly understood. In this work, we focus on how the second-order phase transition do occur between MTZ and TBH.

The organization of this work is as follows. In section 2, we first review thermodynamics of SBH and NBTZ in three-dimensional AdS spacetimes. We discuss the phase transition between them by introducing off-shell free energies. In section 3, we review thermodynamics of MTZ and TBH in four-dimensional AdS spacetimes by introducing coordinate and temperature matchings. Then we study the phase transition between them by considering the difference of their off-shell free energies. We discuss a possible connection between quasinormal modes and phase transitions in section 4. Finally, we summarize our results in section 5.

2 Transition between SBH and NBTZ

We wish to study the scalar black hole in three-dimensional AdS (AdS$_3$) spacetimes. For this purpose, we introduce the action in three-dimensional spacetimes [15]

$$I_3[g, \phi] = \frac{1}{\pi G_3} \int d^3 x \sqrt{-g} \left[ \frac{R}{16} - \frac{1}{2} (\nabla \phi)^2 - V_\nu(\phi) \right]$$

(1)

where the potential $V_\nu(\phi)$ is given by

$$V_\nu(\phi) = -\frac{1}{8\ell^2} \left( \cosh^6 \phi + \nu \sinh^6 \phi \right).$$

(2)
Here $\nu$ is a parameter and $l$ is the curvature radius of AdS$_3$ spacetimes. In the case of $\phi = 0$, we have the BTZ black hole solution. For $\nu \geq -1$, there is a circularly symmetric black hole solution dressed with the scalar field
\[
\tilde{\phi}(r) = \tanh^{-1} \sqrt{\frac{B}{H(r) + B}}.
\] (3)
Here $B$ is a non-negative constant and $H(r) = (r + \sqrt{r^2 + 4Br})/2$. Then the line element of the scalar black hole is given by
\[
dS^2 = -\left(\frac{H}{H + B}\right)^2 F(r) dt^2 + \left(\frac{H + B}{H + 2B}\right)^2 \frac{d r^2}{F(r)} + r^2 d\varphi^2
\] (4)
with the metric function
\[
F(r) = \frac{H^2}{l^2} - \frac{1 + \nu}{l^2} \left(3B^2 + \frac{2B^3}{H}\right).
\] (5)
The causal structure of this geometry is the same as for the NBTZ [17]. The event horizon is located at
\[
r_+ = B \theta_\nu \geq 0
\] (6)
where $\theta_\nu$ is the first-zero of the Schuster function of order $\nu$ defined by
\[
\theta_\nu = 2 \left(\frac{z \bar{z}}{2/3} - \frac{2/3}{z - \bar{z}}\right)
\] (7)
with $z = 1 + i\sqrt{\nu}$ and its complex conjugate $\bar{z}$. As is shown in Fig. 1, $\theta_\nu$ is a monotonically increasing function of $\nu$ and grows as $\sqrt{\nu}$ asymptotically. Importantly, we observe that $\theta_{-1} = 0$ and $\theta_0 = 4/3$. From Eq.(5), we have $F(r) = \frac{r^2}{l^2} \to 0$ as $B \to 0(H(r) \to r)$, where the massless degenerate black hole is found.

Thermodynamic quantities of the SBH are Hawking temperature $T_S$, mass $M_S$, heat capacity $C_S$, entropy $S_S$, on-shell free energy $F_S$, and off-shell free energy $F_S^{\text{off}}$
\[
T_S = \frac{r_+}{2\pi l^2} \frac{3(1 + \nu)}{\theta_\nu^2}, \quad M_S = \frac{r_+^2}{8G_3 l^2} \frac{3(1 + \nu)}{\theta_\nu^2}, \quad C_S = \frac{\pi r_+}{2G_3} = \tilde{S}_S,
\] (8)
\[
F_S(r_+) = M_S - T_S S_S = -\frac{r_+^2}{8G_3 l^2} \frac{3(1 + \nu)}{\theta_\nu^2}, \quad F_S^{\text{off}}(r_+, T) = M_S - T S_S.
\] (9)
We note that the first-law of thermodynamics $dM_S = T_S dS_S$ holds for the SBH. Thanks to this, we check that for $dF_S^{\text{off}} = 0$ (saddle point), one has thermal equilibrium point of $T = T_S$ and thus $F_S^{\text{off}}(r_+, T) = F_S(r_+)$. Here we derive the free energy as a function of $T_S$
\[
F_S(T_S) = -\frac{(2\pi l^2 T_S)^2}{8G_3} \frac{\theta_\nu^2}{3(1 + \nu)}.
\] (10)
We observe an inequality of $\frac{3(1+\nu)}{\theta^2} > 1$ for $\nu \geq -1$ from Fig. 1.

On the other hand, for $\phi = 0$, we have the NBTZ whose thermodynamic quantities as Hawking temperature $T_B$, mass $M_B$, heat capacity $C_B$, entropy $S_B$, on-shell free energy $F_B$, and off-shell free energy $F_{\text{off}}$.

$$T_B = \frac{\rho_+}{2\pi l^2}, \quad M_B = \frac{\rho_+^2}{8G_3l^2}, \quad C_B = \frac{\pi \rho_+}{2G_3} = S_B,$$

$$F_B(\rho_+) = M_B - T_B S_B = -\frac{\rho_+^2}{8G_3l^2}, \quad F_{\text{off}}^B(\rho_+, T) = M_B - TS_B$$  \hspace{1cm} (11)

with $\rho_+ \geq 0$. It is obvious that the first-law holds for the NBTZ. Also we find the free energy as a function of $T_B$

$$F_B(T_B) = -\frac{(2\pi l^2 T_B)^2}{8G_3}.$$

(13)

The coordinate matching of $r_+ = \rho_+$ implies the heat capacity (entropy) matching of $C_S = C_B(S_S = S_B)$. Choosing $G_3 = l/4$ with $l = 1$ and $\frac{3(1+\nu)}{\theta^2} = 1.1$ for $\nu = 56.12$, the behaviors of thermodynamic quantities are shown in Fig. 2. We note that there is an inconsistency between $F_B(r_+) \geq F_S(r_+)$ and $F_B(T_B) \leq F_S(T_S)$. Furthermore, we evaluate differences between free energies

$$\Delta F(r_+) = F_B(r_+) - F_S(r_+) = -\frac{r_+^2}{8G_3l^2} \left[ 1 - \frac{3(1+\nu)}{\theta^2} \right],$$

$$\Delta F_{\text{off}}(r_+, T) = F_{\text{off}}^B(r_+, T) - F_{\text{off}}^S(r_+, T) = -\Delta F(r_+),$$

(14)

where the latter is independent of the external temperature $T$. Thus we cannot describe any phase transition between them under the coordinate matching.
Figure 2: Coordinate matching of $\rho_+ = r_+$. Hawking temperature $T(r_+)$ with external temperatures $T = 0.4, 0.2, 0.1$, mass $M(r_+)$, heat capacity $C(r_+)$, free energy $F(r_+)$ and $F(T_{S/B})$ for SBH (solid curves) and NBTZ (dashed curves). The right graph of the bottom panel shows the difference of the on-shell (solid curve) free energies $\Delta F(r_+)$ and that of off-shell (dashed curve) free energies of $\Delta F^{off}(r_+, T)$.

In order to resolve this situation, we need another matching for temperatures: $T_S = T_B$, which means that $\rho_+ = r_+ \frac{3(1+\nu)}{\theta^2}$ with $\rho_+ \geq r_+$. Then we find consistent inequalities for free energy

$$F_B(r_+) \leq F_S(r_+) \quad \text{and} \quad F_B(T_B) \leq F_S(T_B).$$

Actually, both of SBH and NBTZ have the singe phase of the black hole because of their positive heat capacities $C_{S/B} \geq 0$. A phase transition from the massless black hole at $r_+ = 0(\rho_+ = 0)$ to the black hole at $r_+ \neq 0(\rho_+ \neq 0)$ is always possible to occur in the SBH (NBTZ) \cite{27}. Now we study a phase transition between two black hole for $r_+ > 0$. As is shown in Fig. 3, there is a nonvanishing probability for decay of SBH into NBTZ because of Eq.(16). In order to express it more clearly, we consider the difference between free energies. For $r_+ > 0$, the ground state is the NBTZ. Hence we have to consider the free energy difference $\Delta F$ to show how a phase transition occurs between them. We note that there is no critical temperature for $T > 0$. Hence, the graph of $\Delta F$ shows that it is difficult to identify the order of phase transition. This arises because there exists a forbidden region of $r_+ < 0$. 
Figure 3: Temperature matching $\rho_+ = r_+^{\frac{\alpha(1+\nu)}{6}}$. Hawking temperature $T(r_+)$ with external temperatures $T = 0.4, 0.2, 0.1$, mass $M(r_+)$, heat capacity $C(r_+)$, free energy $F(r_+)$ and $F(T_{S/B})$ for SBH (solid curves) and NBTZ (dashed curves). The right graph of the bottom panel shows the difference of the on-shell (solid curve) free energies $\Delta F(r_+)$ and that of off-shell (dashed curves) free energies of $\Delta F^{\text{off}}(r_+, T)$ with $T = 0.1, 0.2, 0.4$ from top to bottom.

### 3 Transition between MTZ and TBH

We wish to study the four-dimensional black hole with scalar hair. For this purpose, we introduce the similar action [12]

$$I_4[g, \phi] = \int d^4x \sqrt{-g} \left[ \frac{R + 6/l^2}{16\pi G_4} - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right]$$

(17)

where the potential $V(\phi)$ is given by

$$V(\phi) = -\frac{3}{4\pi G_4 l^2} \sinh^2 \left[ \sqrt{\frac{4\pi G_4}{3}} \phi \right].$$

(18)

Here $l$ is the curvature radius of AdS$_4$ spacetimes. In the case of $\phi = 0$, we have the topological black hole solution with topology $\mathbb{R}^2 \times \Sigma$, where $\Sigma$ is a two-dimensional hyperbolic manifold of negative constant curvature ($k = -1$). The field equations are

$$G_{\mu\nu} - \frac{3}{l^2} g_{\mu\nu} = 8\pi T_{\mu\nu},$$

(19)

$$\nabla^2 \phi = V'(\phi)$$

(20)

with the stress-energy tensor

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \left( \partial_\eta \phi \partial^\eta \phi - 2V(\phi) \right).$$

(21)
Then the solution of the MTZ black hole is given by

$$ds^2_M = \frac{r(r + 2G_4\mu)}{(r + G_4\mu)^2} \left[ -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\sigma^2 \right]$$  \hspace{1cm} (22)

where the metric function \( f(r) \) is given by

$$f(r) = \frac{r^2}{l^2} - \left( 1 + \frac{G_4\mu}{r} \right)^2$$  \hspace{1cm} (23)

and the scalar field has the configuration

$$\tilde{\phi}(r) = \sqrt{\frac{3}{4\pi G_4}} \tanh^{-1} \left( \frac{G_4\mu}{r + G_4\mu} \right).$$  \hspace{1cm} (24)

Here \( \mu \) is a parameter, playing the role of the reduced mass. The event (cosmological) horizons are determined by solving \( f(r) = 0 \) as

$$r_\pm = \frac{l}{2} \left[ 1 \pm \sqrt{1 + \frac{4G_4\mu}{l}} \right]$$  \hspace{1cm} (25)

provided the reduced mass is bounded from below by \( \mu \geq -l/4G_4 \). For \( \mu = -l/4G_4 \), we have the degenerate horizon at \( r_+ = r_- = l/2 \). As is shown in Fig. 4, we have the range of \( 0 \leq r_- \leq l/2 \) and \( r_+ \geq l/2 \). Hence the causal structure is not the Schwarzschild-AdS black hole replacing \( S^2 \) by \( \Sigma \) but the BTZ black hole \( S \) by \( \Sigma \). For the massless case of \( \mu = 0(\tilde{\phi} = 0) \), we have the metric

$$ds^2 = -\left( \frac{r^2}{l^2} - 1 \right)dt^2 + \left( \frac{r^2}{l^2} - 1 \right)^{-1}dr^2 + r^2d\sigma^2$$  \hspace{1cm} (26)

which is a locally AdS_4 spacetime but has a topological horizon at \( r_+ = l \). For the case of \( \phi = 0(V(\phi) = 0) \), we have the vacuum solution of topological black hole with hyperbolic horizon (TBH)

$$ds^2_T = -\left( \frac{\rho^2}{l^2} - 1 - \frac{2G_4\mu_0}{\rho} \right)dt^2 + \left( \frac{\rho^2}{l^2} - 1 - \frac{2G_4\mu_0}{\rho} \right)^{-1}d\rho^2 + \rho^2d\sigma^2.$$  \hspace{1cm} (27)

Thermodynamic quantities of the MTZ are given by Hawking temperature \( T_M \), mass \( M_M \), heat capacity \( C_M \), entropy \( S_M \), and on-shell free energy \( F_M \) by

$$T_M = \frac{1}{2\pi l} \left( \frac{2r_+}{l} - 1 \right), \quad M_M = \frac{\sigma r_+}{4\pi G_4} \left( \frac{r_+}{l} - 1 \right),$$  \hspace{1cm} (28)

$$C_M = \frac{\sigma l}{4G_4} \left( 2r_+ - l \right) = S_M, \quad F_M(r_+) = -\frac{\sigma}{8\pi G_4} \left( \frac{2r_+^2}{l} - 2r_+ + l \right).$$  \hspace{1cm} (29)
Figure 4: Left panel: plot of the event horizon $r_+$ (solid curve) and the cosmological horizon $r_-$ (dashed curve) as functions of reduced mass $\mu$. Right panel: its inverse function $\mu(r_+)$. $\mu(r_+)$ (solid curve) represents the mass $M_M$ of MTZ for $\sigma/4\pi G_4 = 1$. Here we choose $G_4 = l/4$ with $l = 1$ for numerical computations.

where $r_+ \geq l/2$ and $\sigma$ denotes the area of $\Sigma$. We note that the first-law of thermodynamics $dM_M = T_M dS_M$ holds for the MTZ. The free energy can be rewritten as a function of $T_M$ as

$$F_M(T_M) = -\frac{\sigma l}{8\pi G_4} \left(1 - 2\pi(T_M + T_c) + 2\pi^2(T_M + T_c)^2\right)$$

(30)

with $T_c = 1/2\pi l$. Here we define the off-shell free energy as a function of $r_+$ and $T$

$$F_M^{\text{off}}(r_+, T) = M_M - T S_M.$$  

(31)

Here the horizon radius $r_+$ is the order parameter and the external temperature $T$ is the control parameter for the description of the phase transition. On the other hand, the TBH provides thermodynamic quantities as Hawking temperature $T_T$, mass $M_T$, heat capacity $C_T$, entropy $S_T$, and on-shell free energy $F_T$

$$T_T = \frac{1}{4\pi\rho_+} \left(\frac{3\rho_+^2}{l^2} - 1\right), \quad M_T = \frac{\sigma\rho_+}{8\pi G_4} \left(\frac{\rho_+^2}{l^2} - 1\right),$$

(32)

$$C_T = \frac{\sigma\rho_+^2 (3\rho_+^2 - l^2)}{2G_4(3\rho_+^2 + l^2)}, \quad S_T = \frac{\sigma\rho_+^2}{4G_4}, \quad F_T(\rho_+) = -\frac{\sigma\rho_+}{16\pi G_4} \left(\frac{\rho_+^2}{l^2} + 1\right)$$

(33)

with $\rho_+ \geq l/\sqrt{3}$. Solving the equation of $3\rho_+^2 - 4\pi l^2 T_T \rho_+ - l^2 = 0$, one finds $\rho_+ = \rho_+(T_T)$. Plugging this into $F_T(\rho_+)$ leads to

$$F_T(T_T) = -\frac{\sigma}{16\pi G_4} \left[ \left(\frac{2\pi l^2 T_T + \sqrt{3l^2 + (2\pi l^2 T_T)^2}}{3l^{2/3}}\right)^3 + \frac{2\pi l^2 T_T + \sqrt{3l^2 + (2\pi l^2 T_T)^2}}{3} \right].$$

(34)

Here we define the off-shell free energy as a function of $\rho_+$ and $T$

$$F_T^{\text{off}}(\rho_+, T) = M_T - T S_T.$$  

(35)
Figure 5: Coordinate matching of $\rho_+ = r_+$ with $\sigma/4\pi G_4 = 1$. Hawking temperature $T(r_+)$ with external temperatures $T = 0.25, 0.2, 0.16, 0.1$ from top to bottom, heat capacity $C(r_+)$, free energy $F(r_+)$ and $F(T_{M/T})$ for MTZ (solid curves) and TBH (dashed curves).

For coordinate matching of $r_+ = \rho_+$ with $l = 1$, we observe from Fig. 5 that inconsistent equalities of free energy appear for $r_+ > 1$

\[
F_T(r_+) > F_M(r_+) \quad \text{for} \quad r_+ < 1 \quad \text{and} \quad F_T(T) > F_M(T) \quad \text{for} \quad T < T_c, \tag{36}
\]

\[
F_T(r_+) > F_M(r_+) \quad \text{for} \quad r_+ > 1 \quad \text{and} \quad F_T(T) < F_M(T) \quad \text{for} \quad T > T_c. \tag{37}
\]

Hence we do not make a further progress on what kind of the phase transition happens for the coordinate matching.

In order to resolve this situation, we need to introduce the temperature matching: $T_S = T_B$, which means that $r_+ = \frac{3\rho_+}{4} - \frac{1}{4\rho_+} + \frac{1}{2}$ with $\rho_+ \geq r_+$. Then we find from Fig. 6 that consistent equalities are found for free energy

\[
F_T(\rho_+) > F_M(\rho_+) \quad \text{for} \quad \rho_+ < 1 \quad \text{and} \quad F_T(T) > F_M(T) \quad \text{for} \quad T < T_c, \tag{38}
\]

\[
F_T(\rho_+) < F_M(\rho_+) \quad \text{for} \quad \rho_+ > 1 \quad \text{and} \quad F_T(T) < F_M(T) \quad \text{for} \quad T > T_c. \tag{39}
\]

As is shown in Fig. 6, we could find the critical temperature $T_c = 1/2\pi = 0.16$ where $F_M(T_c) = F_T(T_c)$ at $\rho_+ = 1$. Hence, we separate the whole region into the left region
Figure 6: Temperature matching of $r_+ = 3\rho_+ + \frac{1}{4\rho_+} + \frac{1}{2}$ with $\sigma/4\pi G = 1$. Hawking temperature $T(\rho_+)$ with external temperatures $T = 0.25, 0.2, 0.16, 0.1$, heat capacity $C(\rho_+)$, free energy $F(\rho_+)$ and $F(T_{M/T})$ for MTZ (solid curves) and TBH (dashed curves).

of $1/\sqrt{3} \leq \rho_+ < 1$ and the right region of $\rho_+ > 1$ for the temperature matching. For $\rho_+ < 1$ the MTZ configuration is more favorable than the TBH, while for $\rho_+ > 1$, the TBH configuration is more favorable than the MTZ. This means that for $\rho_+ < 1$, the ground state is the MTZ, whereas for $\rho_+ > 1$, the ground state is chosen to be the TBH. Actually, both of MTZ and TBH have the single phase of the black hole because of their positive heat capacities $C_{S/B} \geq 0$. Also we note that the heat capacities are nothing special at the critical point $\rho_+ = 1$. Hence phase transitions from the extremal black hole at $\rho_+ = 1/\sqrt{3}(r_+ = 1/2)$ to the non-extremal black hole at $\rho_+ > 1/\sqrt{3}(r_+ > 1/2)$ are always possible to occur in the TBH (MTZ), respectively.

Now we are in a position to identify what kind of the phase transitions is going on for the temperature matching. We remind the reader that in this work, $r_+(\rho_+)$ are the order parameters and $T$ is the control parameter for the description of the phase transition in the MTZ-TBH system.\footnote{We do not use $\lambda_\phi = |\tanh \sqrt{4\pi G/3}\phi(r_+)| = |(r_+ - 1)/r_+|$ as the order parameter for $T < T_c$ \cite{12,13} because it vanishes for $T > T_c$.}
Figure 7: The left panel of coordinate matching shows the difference of the on-shell (solid curve) free energies $\Delta F_{L/R}(r_+)$ and that of off-shell (dashed curve) free energies of $\Delta F^\text{off}_{L/R}(r_+, T)$. The right panel denotes $\Delta F_{L/R}(\rho_+)$ and $\Delta F^\text{off}_{L/R}(\rho_+, T)$ for the temperature matching. Here $T = 0.25, 0.2, 0.16, 0.1$ for $r_+(\rho_+) < 1$ and $T = 0.1, 0.16, 0.2, 0.25$ for $r_+(\rho_+) > 1$ from top to bottom.

It was conjectured that this is a second-order phase transition as occurred in the ferromagnetic system. In order to prove it, we define the difference between on-shell (off-shell) free energies on both sides

$$\Delta F_{L/R}(r) = \pm \left[ F_M(r) - F_T(r) \right],$$

$$\Delta F^\text{off}_{L/R}(r, T) = \pm \left[ F^\text{off}_M(r, T) - F^\text{off}_T(r, T) \right]$$

with $r = r_+(\rho_+)$ for coordinate (temperature) matchings, respectively. These are depicted in Fig. 7. For the coordinate matching, there is no saddle point of thermal equilibrium as the crossing point between $\Delta F_{L/R}(r_+)$ and $\Delta F^\text{off}_{L/R}(r_+, T)$. However, for the temperature matching, there exist saddle points of thermal equilibria as the crossing points between $\Delta F_{L/R}(\rho_+)$ and $\Delta F^\text{off}_{L/R}(\rho_+, T)$.

Firstly, we consider the phase transition between two black hole on the left region of $1/\sqrt{3} \leq \rho_+ < 1$. As is shown in Fig. 6, there is a nonvanishing probability for decay of TBH into MTZ. In order to express it more precisely, we have to consider the difference between free energies. The graph of $\Delta F^\text{off}_{L}$ in Fig. 7 shows that there is a transition from TBH to MTZ configuration. For $T = 0.1 < T_c$, we find that the dominance of system is the MTZ, while for $T = 0.2, 0.25 > T_c$ the dominance is changed to be the TBH.

Secondly, we consider the phase transition between two black hole on the right region of $\rho_+ > 1$. As is shown in Fig. 6, there is a nonvanishing probability for decay of MTZ into TBH. The graph of $\Delta F^\text{off}_{R}$ in Fig. 7 indicates that there is a transition from MTZ to TBH configuration. There is a change of dominance at the critical temperature.
$T = T_c$. For $T = 0.1 < T_c$, we find that the dominance of system is the MTZ, while for $T = 0.2, 0.25 > T_c$ the dominance is given by the TBH. As the whole picture, for $T < T_c$ the MTZ is ground state while for $T > T_c$, the TBH is the ground state. However, as $T \rightarrow 0$, the MTZ configuration may lead to the extremal MTZ with $T_M = 0$ at $\rho_+ = 1/\sqrt{3}$ by evaporating process. Hence the non-extremal MTZ may not be a truly ground state near $T = 0$.

A symmetric configuration of $\Delta F_{L/R}$ around the critical point $\rho_+ = 1$ is expected to exist for a typical feature of the second-order phase transition in condensed matter physics [21]. Even though it is asymmetric in the MTZ-TBH system of black hole physics, it shows still the nature of second-order phase transition. To compare this with the first-order phase transition, one simply refers the Hawking-Page phase transition in the Schwarzschild-AdS black hole [9, 22].

4 Quasinormal modes and phase transitions

Recently, black holes’ quasinormal modes (QNMs) can reflect the black hole phase transition. It was claimed that the behavior of QNMs around $r_+ = 1 (\rho_+ = 1)$ shows a signal for the second-order phase transition by calculating the electromagnetic perturbations on the background of the MTZ and TBH [13]. Furthermore, there was a change in the slope in the $w_R - w_I$ diagrams: positive for $r_+ = 0.97$, infinity at $r_+ = 1$, and negative for $r_+ = 1.03$ by calculating the scalar perturbations on the background of the MTZ [14]. Also the same thing happened: positive for $\rho_+ = 0.97$, infinity at $\rho_+ = 1$, and negative for $\rho_+ = 1.03$ on the background of the TBH. So it is very important to justify whether the QNMs plays the role of an effective tool to disclose a phase transition in the general black hole background. However, there is no change in the slope on the $k = 0$ AdS black hole background which was introduced to explain the phase transition between the $k = 0$ AdS black hole and AdS soliton. Of course, there is no QNMs on the AdS soliton background because of the absence of event horizon.

Without the temperature matching, the points of $r_+ = 1$ in the MTZ and $\rho_+ = 1$ in the TBH are nothing special thermodynamically. That is, their positive heat capacities are continuous across at these points. These are not the point of the minimum temperature in the Schwarzschild-AdS black hole [9] and Davies’ point in the Reissner-Nordström black hole [23], where heat capacities blow up [24 25 26]. However, as is shown Fig. 5, two black holes have the same thermodynamic quantities at $r_+ = \rho_+ = 1$. An interesting point is that two black holes have the same free energy as $F_M(r_+ = 1) = F_T(\rho_+ = 1) = -0.5 \ (F_M(T_M = T_c) = F_T(T_T = T_c) = -0.5)$. Also this free energy corresponds to that of
topological black hole with zero mass. It is turned out that this black hole is stable \cite{20}, which can be confirmed by the positive heat capacity. Hence $r_+ = 1 (\rho_+ = 1)$ have nothing to do with the thermodynamic instability.

As was explained in the previous section, the second-order phase transition between MTZ and TBH at $\rho_+ = 1 (T = T_c)$ is shown to occur by using temperature matching between MTZ and TBH and the difference of their free energies. Even though the QNMs may give a signal to the phase transition, this is not a deterministic evidence for the second-order phase transition between MTZ and TBH.

5 Discussions

We start with discussion on three-dimensional black holes. We note that for three dimensions, the $k = 0, \pm 1$ AdS$_3$ black holes are the same and AdS$_3$ soliton is diffeomorphic to thermal AdS$_3$ spacetimes \cite{11}. Hence we could not distinguish topological black holes in three dimensions. Introducing a single self-interacting scalar field minimally coupled to gravity, then one finds the SBH. However, there is no non-zero critical temperature of $T_S = T_B$ for coordinate matching. Hence the transition between massless black hole at $r_+ = 0 (\rho_+ = 0)$ and SBH(NBTZ) is possible to occur \cite{27}. Two temperatures become the same only when $T_S = T_B = 0$. This point contrasts to the four-dimensional critical temperature with $T = T_c = 1/2\pi l$.

Requiring the temperature matching, there is a nonvanishing probability for decay of the SBH into the NBTZ for $r_+ > 0$. However, we could not identify its order because there is no critical temperature. Concerning the QNMs, we have the results: AdS soliton (thermal AdS) and massless black hole at $r_+ = 0 \rightarrow$ No QNMs and NBTZ with $r_+ \neq 0 \rightarrow$ QNMs \cite{28}. We expect that the same thing happens for the SBH: massless black hole at $\rho_+ = 0 \rightarrow$ No QNMs and scalar black hole with $\rho_+ \neq 0 \rightarrow$ QNMs. This implies that the QNMs may not show a direct evidence for the phase transition between SBH and NBTZ.

We summarize all possible phase transitions in AdS$_4$ spacetimes in Table 1. First of all, for $k = 1$ spherical horizon, we have the Hawking-Page phase transition between thermal AdS$_4$ and large black hole in the Schwarzschild-AdS black hole. This is the first-order phase transition. In the case of $k = 0$ Ricci-flat horizon, it is conjectured that there may exist a phase transition between AdS$_4$ soliton and large black hole with Ricci-flat horizon. This is similar to the Hawking-Page phase transition but there is no unstable small black hole as a mediator of the phase transition. Hence it is unclear to define its order of phase transition. For the case of $k = -1$ hyperbolic horizon, we expect that there exists a phase transition between extremal and non-extremal black holes. This happens for MTZ and
starting configuration | ending configuration | remarks
--- | --- | ---
k = 1 | thermal AdS$_4$ space | large black hole | first-order (HPT)
k = 0 | AdS$_4$ soliton | large black hole | *(HPT)*
k = −1 | EBH | TBH | *(HPT)*
k = −1 | EBH | MTZ | *(HPT)*
k = −1 | MTZ | TBH | second-order

Table 1: Summary of phase transitions in asymptotically AdS$_4$ spacetimes. EBH represents the extremal black hole with degenerate horizon and HPT denotes the Hawking-Page phase transition.

Finally, we observe a second-order phase transition between MTZ and TBH at the critical temperature $T = T_c$.

In conclusion, we explicitly show that the phase transition between MTZ and TBH is second-order by using the temperature matching and difference of free energies. Without temperature matching, we do not confirm that the phase transition is second-order.

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