UNCONDITIONAL UNIQUENESS FOR THE MODIFIED KORTEWEG-DE VRIES EQUATION ON THE LINE

LUC MOLINET*, DIDIER PILOD† AND STÉPHANE VENTO*

Abstract. We prove that the modified Korteweg-de Vries equation (mKdV) equation is unconditionally well-posed in $H^s(\mathbb{R})$ for $s > \frac{1}{4}$. Our method of proof combines the improvement of the energy method introduced recently by the first and third authors with the construction of a modified energy. Our approach also yields a priori estimates for the solutions of mKdV in $H^s(\mathbb{R})$, for $s > \frac{1}{10}$.

1. Introduction

We consider the initial value problem (IVP) associated to the modified Korteweg-de Vries (mKdV) equation

\begin{equation}
\begin{cases}
\partial_t u + \partial_x^3 u + \kappa \partial_x (u^3) = 0, \\
u(\cdot, 0) = u_0,
\end{cases}
\end{equation}

where $u = u(x, t)$ is a real function, $\kappa = 1$ or $-1$, $x \in \mathbb{R}$, $t \in \mathbb{R}$.

In the seminal paper [10], Kenig, Ponce and Vega proved the well-posedness of (1.1) in $H^s(\mathbb{R})$ for $s \geq \frac{1}{4}$. This result is sharp in the sense that the flow map associated to mKdV fails to be uniformly continuous in $H^s(\mathbb{R})$ if $s < \frac{1}{4}$ in both the focusing case $\kappa = 1$ (cf. Kenig, Ponce and Vega [11]) and the defocusing case $\kappa = -1$ (cf. Christ, Colliander and Tao [2]). Global well-posedness for mKdV was proved in $H^s(\mathbb{R})$ for $s > \frac{1}{3}$ by Colliander, Keel, Staffilani, Takaoka and Tao [4] by using the $I$-method. We also mention that another proof of the local well-posedness result for $s \geq \frac{1}{4}$ was given by Tao by using the Fourier restriction norm method [19].

The proof of the well-posedness result in [10] relies on the dispersive estimates associated with the linear group of (1.1), namely the Strichartz estimates, the local smoothing effect and the maximal function estimate. A normed function space is constructed based on those estimates and allows to solve (1.1) via a fixed point theorem on the associated integral equation. Of course the solutions obtained in this way are unique in this resolution space. The same occurs for the solutions constructed by Tao which are unique in the space $X^{s, \frac{1}{2}+}_{T}$.

Date: November 24, 2014.

2010 Mathematics Subject Classification. Primary: 35A02, 35E15, 35Q53; Secondary: 35B45, 35D30.

Key words and phrases. Modified Korteweg-de Vries equation, Unconditional uniqueness, Well-posedness, Modified energy.

* Partially supported by the french ANR project GEODISP.
† Partially supported by CNPq/Brazil, grants 302632/2013-1 and 481715/2012-6.
The question to know whether uniqueness holds for solutions which do not belong to these resolution spaces turns out to be far from trivial at this level of regularity. This kind of question was first raised by Kato [6] in the Schrödinger equation context. We refer to such uniqueness in $C([0, T]: H^s(\mathbb{R}))$, or more generally in $L^\infty([0, T]: H^s(\mathbb{R}))$, without intersecting with any auxiliary function space as unconditional uniqueness. This ensures the uniqueness of the weak solutions to the equation at the $H^s$-regularity. This is useful, for instance, to pass to the limit on perturbations of the equation as the perturbative coefficient tends to zero (see for instance [13] for such an application).

Unconditional uniqueness was proved for the KdV equation to hold in $L^2(\mathbb{R})$ [20] and in $L^2(\mathbb{T})$ [1] and for the mKdV in $H^s_\mathbb{T}(\mathbb{T})$ [13].

The aim of this paper is to prove the unconditional uniqueness of the mKdV equation in $H^s(\mathbb{R})$ for $s > \frac{1}{4}$. By doing so, we also provide a different proof of the existence result. Next, we state our main result.

**Theorem 1.1.** Let $s > 1/4$ be given.

**Existence:** For all $u_0 \in H^s(\mathbb{R})$, there exists $T = T(\|u_0\|_{H^s}) > 0$ and a solution $u$ of the IVP (1.1) such that

\[ u \in C([0, T]: H^s(\mathbb{R})) \cap L^4_T L^\infty_x \cap X^{s-1,1}_T \cap \tilde{L}^\infty_T H^s_x. \]

**Uniqueness:** The solution is unique in the class

\[ u \in L^\infty([0, T]: H^s(\mathbb{R})). \]

Moreover, the flow map data-solution $u_0 \mapsto u$ is Lipschitz from $H^s(\mathbb{R})$ into $C([0, T]: H^s(\mathbb{R}))$.

**Remark 1.2.** We refer to Section 2.2 for the definition of the norms $\|u\|_{\tilde{L}^\infty_T H^s_x}$ and $\|u\|_{X^{s-1,1}_T}$.

**Remark 1.3.** According to the equation, the time derivative of a weak solution satisfying (1.3) belongs to $L^\infty([0, T]: H^{s-3}(\mathbb{R}))$. Thus, such a solution has to belong to $C([-T, T]: H^{s-3}(\mathbb{R}))$, so that the initial value condition in (1.1) still makes sense.

Our technique of proof also yields a priori estimates for the solutions of mKdV in $H^s(\mathbb{R})$ for $s > \frac{1}{10}$. It is worth noting that a priori estimates in $H^s(\mathbb{R})$ were already proved by Christ, Holmer and Tataru for $-\frac{1}{8} < s < \frac{1}{4}$ in [3]. Their proof relies on short time Fourier restriction norm method in the context of the atomic spaces $U$, $V$ and the $I$-method. Although our result is not as strong as Christ, Holmer and Tataru’s one, we hope that it still may be of interest due to the simplicity of his proof.

**Theorem 1.4.** Assume that $s > \frac{1}{10}$. For any $M > 0$, there exist a positive time $T = T(M) > 0$ and a positive constant $C$ such that for any initial data $u_0 \in H^\infty(\mathbb{R})$ such that $\|u_0\|_{H^s} \leq M$, the smooth solution of (1.1) satisfies

\[ \|u\|_{Z_{T}^s} := \|u\|_{\tilde{L}^\infty_T H^s_x} + \|u\|_{X^{s-1,1}_T} + \|u\|_{L^4_T L^\infty_x} \leq C \|u_0\|_{H^s}. \]
Remark 1.5. By passing to the limit on a sequence of smooth solutions, the above \textit{a priori} estimate ensures the existence of a $L^\infty_T H^s_x$- weak solution of (1.1) for $s > 1/10$. Note that, since $s > 0$, there is no difficulty to pass to the limit on the nonlinear term by a compactness argument.

To prove Theorems 1.1 and 1.4, we derive energy estimates on the dyadic blocks $\|P_N u\|_{L^2_T H^s_x}$ by using the norms $\|u\|_{L^\infty_T H^s_x}$ and $\|u\|_{X^{s-1,1}}$. This technique has been introduced by the first and the third authors in [16]. Note however that in addition to use the fractional Leibniz rule to control the $X^{s-1,1}$-norm as in [16], we also introduce the norm $\| \cdot \|_{L^4_T L^\infty_x}$. This norm is in turn controlled by using a refined Strichartz estimate derived by chopping the time interval in small pieces whose length depends on the spatial frequency. It is this estimate which provides the restriction $s > 1/10$ in Theorem 1.4. Note that it was first established by Koch and Tzvetkov [12] (see also Kenig and Koenig [7] for an improved version) in the Benjamin-Ono context.

The main difficulty to estimate $\frac{d}{dt}\|P_N u\|_{L^2_T H^s_x}$ is to handle the resonant term $R_N$, typical of the cubic nonlinearity $\partial_x(u^3)$. When $u$ is the solution of mKdV, $R_N$ writes $R_N = \int \partial_x(P_{+N} u P_{+N} u P_{-N} u)P_{-N} u dx$. Actually, it turns out that we can always put the derivative appearing in $R_N$ on a low frequency product by integrating by parts, as it was done in [5] for quadratic nonlinearities. This allows us to derive the \textit{a priori} estimate of Theorem 1.4 in $H^s(\mathbb{R})$ for $s > 1/10$. Unfortunately, this is not the case anymore for the difference of two solutions of mKdV due to the lack of symmetry of the corresponding equation. To overcome this difficulty we modify the $H^s$-norm by higher order terms up to order 6. These higher order terms are constructed so that the contribution of their time derivatives coming from the linear part of the equation will cancel out the resonant term $R_N$. The use of a modified energy is well-known to be a quite powerful tool in PDE’s (see for instance [14] and [8]). Note however that, in our case, we need to define the modified energy in Fourier variables due to the resonance relation associated to the cubic nonlinearity. This way to construct the modified energy has much in common with the way to construct the modified energy in the I-method (cf. [4]).

Finally let us mention that the tools developed in this paper together with some ideas of [15] and [17] will enable us, in a forthcoming paper, to get the unconditional well-posedness of the periodic mKdV equation in $H^s(\mathbb{T})$ for $s > 1/3$. We also hope that the techniques introduced here could be useful in the study of the Cauchy problem at low regularity of other cubic nonlinear dispersive equations such as the modified Benjamin-Ono equation and the derivative nonlinear Schrödinger equation.

The rest of the paper is organized as follows. In Section 2 we introduce the notations, define the function spaces and state some preliminary estimates. The multilinear estimates at the $L^2$-level are proved in Section 3. Those estimates are used to derive the energy estimates in Section 4. Finally, we give the proofs of Theorems 1.1 and 1.4 respectively in Sections 5 and 6.

\footnote{For technical reason we perform this integration by parts in Fourier variables.}
2. Notation, Function spaces and preliminary estimates

2.1. Notation. For any positive numbers $a$ and $b$, the notation $a \lesssim b$ means that there exists a positive constant $c$ such that $a \leq cb$. We also denote $a \sim b$ when $a \lesssim b$ and $b \lesssim a$. Moreover, if $\alpha \in \mathbb{R}$, $\alpha_{+}$, respectively $\alpha_{-}$, will denote a number slightly greater, respectively lesser, than $\alpha$.

Let us denote by $\mathbb{D} = \{N > 0 : N = 2^n \text{ for some } n \in \mathbb{Z}\}$ the dyadic numbers. Usually, we use $n_i, j_i, m_i$ to denote integers and $N_i = 2^{n_i}, L_i = 2^{j_i}$ and $M_i = 2^{m_i}$ to denote dyadic numbers.

For $N_1, N_2 \in \mathbb{D}$, we use the notation $N_1 \lor N_2 = \max\{N_1, N_2\}$ and $N_1 \land N_2 = \min\{N_1, N_2\}$. Moreover, if $N_1, N_2, N_3 \in \mathbb{D}$, we also denote by $N_{\text{max}} \geq N_{\text{med}} \geq N_{\text{min}}$ the maximum, sub-maximum and minimum of $\{N_1, N_2, N_3\}$.

For $u = u(x,t) \in S'(\mathbb{R}^2),$ $\mathcal{F}u$ will denote its space-time Fourier transform, whereas $\mathcal{F}_x u = \hat{u}$, respectively $\mathcal{F}_t u$, will denote its Fourier transform in space, respectively in time. For $s \in \mathbb{R}$, we define the Bessel and Riesz potentials of order $-s$, $J^s_x$ and $D^s_x$, by

$$J^s_x u = \mathcal{F}_x^{-1}((1 + |\xi|^2)^{\frac{s}{2}}\mathcal{F}_x u) \quad \text{and} \quad D^s_x u = \mathcal{F}_x^{-1}(|\xi|^s\mathcal{F}_x u).$$

We also denote by $U(t) = e^{-it\partial_x^2}$ the unitary group associated to the linear part of (1.1), i.e.,

$$U(t)u_0 = e^{-it\partial_x^2}u_0 = \mathcal{F}_x^{-1}(e^{it\xi^3}\mathcal{F}_x(u_0)(\xi)).$$

Throughout the paper, we fix a smooth cutoff function $\chi$ such that $\chi \in C^\infty_0(\mathbb{R})$, $0 \leq \chi \leq 1$, $\chi|_{[-1,1]} = 1$ and $\text{supp}(\chi) \subset [-2,2]$.

We set $\phi(\xi) := \chi(\xi) - \chi(2\xi)$. For $l \in \mathbb{Z}$, we define

$$\phi_{2^l}(\xi) := \phi(2^{-l}\xi),$$

and, for $l \in \mathbb{N}^*$,

$$\psi_{2^l}(\xi, \tau) = \phi_{2^l}(\tau - \xi^3).$$

By convention, we also denote

$$\phi_0(\xi) = \chi(2\xi) \quad \text{and} \quad \psi_0(\xi, \tau) := \chi(2(\tau - \xi^3)).$$

Any summations over capitalized variables such as $N, L, K$ or $M$ are presumed to be dyadic. Unless stated otherwise, we will work with non-homogeneous dyadic decompositions in $N$, $L$ and $K$, i.e. these variables range over numbers of the form $\mathbb{D}_{nh} = \{2^k : k \in \mathbb{N} \} \cup \{0\}$, whereas we will work with homogeneous dyadic decomposition in $M$, i.e. these variables range over $\mathbb{D}$. We call the numbers in $\mathbb{D}_{nh}$ nonhomogeneous dyadic numbers. Then, we have that $\sum N \phi_N(\xi) = 1,$

$$\text{supp}(\phi_N) \subset I_N := \{\frac{N}{2} \leq |\xi| \leq 2N\}, \quad N \geq 1, \quad \text{and} \quad \text{supp}(\phi_0) \subset I_0 := \{|\xi| \leq 1\}.$$

Finally, let us define the Littlewood-Paley multipliers $P_N, R_K$ and $Q_L$ by

$$P_N u = \mathcal{F}_x^{-1}(\phi_N \mathcal{F}_x u), \quad R_K u = \mathcal{F}_x^{-1}(\phi_K \mathcal{F}_x u) \quad \text{and} \quad Q_L u = \mathcal{F}_x^{-1}(\psi_L \mathcal{F}_x u),$$

$$P_{\geq N} := \sum_{K\geq N} P_K, \quad P_{< N} := \sum_{K\leq N} P_K, \quad Q_{\geq L} := \sum_{K\geq L} Q_K \quad \text{and} \quad Q_{< L} := \sum_{K< L} Q_K.$$
2.2. Function spaces. For $1 \leq p \leq \infty$, $L^p(\mathbb{R})$ is the usual Lebesgue space with the norm $\| \cdot \|_{L^p}$. For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R})$ denotes the space of all distributions of $S'(\mathbb{R})$ whose usual norm $\|u\|_{H^s} = \|J_s^x u\|_{L^2}$ is finite.

If $B$ is one of the spaces defined above, $1 \leq p \leq \infty$ and $T > 0$, we define the space-time spaces $L^p_t B_x$, $L^p_t B_x$, $L^p_t B_x$ and $L^p_t B_x$ equipped with the norms

$$\|u\|_{L^p_t B_x} = \left(\int_0^T \|f(\cdot, t)\|_{L^p_x}^p dt\right)^{\frac{1}{p}}, \quad \|u\|_{L^p_t B_x} = \left(\int_0^T \|f(\cdot, t)\|_{L^p_x}^p dt\right)^{\frac{1}{p}}$$

with obvious modifications for $p = \infty$, and

$$\|u\|_{L^p_t B_x} = \left(\sum_N \|P_N u\|_{L^p_{T} B_x}^2\right)^{\frac{1}{2}}, \quad \|u\|_{L^p_t B_x} = \left(\sum_N \|P_N u\|_{L^p_{T} B_x}^2\right)^{\frac{1}{2}}.$$  

For $s, b \in \mathbb{R}$, we introduce the Bourgain spaces $X^{s,b}$ related to the linear part of \((1.1)\) as the completion of the Schwartz space $S(\mathbb{R}^2)$ under the norm

$$\|u\|_{X^{s,b}} := \left(\int_{\mathbb{R}^2} \langle x - \xi^3 \rangle^{2b} \langle \xi \rangle^{2s} |\mathcal{F}(u)(\xi, \tau)|^2 d\xi d\tau\right)^{\frac{1}{2}},$$

where $\langle x \rangle := 1 + |x|$. By using the definition of $U$, it is easy to see that

$$\|u\|_{X^{s,b}} \sim \|U(-t)u\|_{H^{s,b}} \quad \text{where} \quad \|u\|_{H^{s,b}} = \|J_s^x J_t^b u\|_{L^2_{s,b}}.$$  

We will also use restriction in time versions of these spaces. Let $T > 0$ be a positive time. The restriction space $X^{s,b}_T$ will be the space of functions $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ or $\mathbb{C}$ satisfying

$$\|u\|_{X^{s,b}_T} := \inf \{ \|\tilde{u}\|_{X^{s,b}} \mid \tilde{u} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ or } \mathbb{C}, \quad \tilde{u}|_{[0, T]} = u \} < \infty.$$  

Finally, we define our resolution spaces $Y^s = X^{s-1,1}_T \cap \tilde{L}^\infty_T H^s_x$, $Y^s = X^{s-1,1}_T \cap \tilde{L}^\infty_T H^s_x$ and $Z^s_T = Y^s_T \cap L^1_t L^\infty_x$ with the associated norms

$$\|u\|_{Y^s} = \|u\|_{X^{s-1,1}_T} + \|u\|_{\tilde{L}^\infty_T H^s_x}, \quad \|u\|_{Y^s_T} = \|u\|_{X^{s-1,1}_T} + \|u\|_{\tilde{L}^\infty_T H^s_x}$$

and

$$\|u\|_{Z^s_T} = \|u\|_{Y^s_T} + \|u\|_{L^1_t L^\infty_x}.$$  

It is clear from the definition that $\tilde{L}^\infty_T H^s_x \hookrightarrow L^\infty_T H^s_x$, i.e.

$$\|u\|_{L^\infty_T H^s_x} \lesssim \|u\|_{L^\infty_T H^s_x}, \quad \forall u \in L^\infty_T H^s_x.$$  

2.3. Extension operator. In this subsection, we introduce an extension operator $\rho_T$ which is a bounded operator from $X^{s-1,1}_T \cap L^\infty_T H^s_x$ into $X^{s-1,1} \cap L^\infty_T H^s_x \cap L^1_T H^s_x$ for any $s \in \mathbb{R}$.

**Definition 2.1.** Let $0 < T \leq 1$ and $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ or $\mathbb{C}$ be given. Let us first define

$$v(x, t) = U(-t)u(\cdot, t)(x), \quad \forall (x, t) \in \mathbb{R} \times [0, T].$$

Then, we extend $v$ on $[-2, 2]$ by setting $\partial_t v = 0$ on $[-2, 2] \setminus [0, T]$. We define the extension operator $\rho_T$ by

$$\rho_T(u)(x, t) := \chi(t)U(t)v(\cdot, t)(x), \quad \forall (x, t) \in \mathbb{R}^2,$$

which extends functions defined on $\mathbb{R} \times [0, T]$ to $\mathbb{R}^2$. 

UNCONDITIONAL UNIQUENESS FOR MKDV 5
It is clear from the definition that \( p_T(u)(x,t) = u(x,t) \) for \((x,t) \in \mathbb{R} \times [0,T]\), \( p_T(P_N u) = P_N(p_T(u)) \) and supp \( p_T(u) \subset [-2,2] \).

Lemma 2.2. Let \( 0 < T \leq 1 \) and \( s \in \mathbb{R} \). Then,

\[
\rho_T : X^{s-1,1}_T \cap L^{\infty}_T H^s_x \to X^{s-1,1}_T \cap L^{\infty}_T H^s_x, \quad u \mapsto \rho_T(u)
\]

is a bounded linear operator, i.e. there exists \( C_1 > 0 \) independent of \( T \in (0,1] \) such that

\[
\|\rho_T(u)\|_{X^{s-1,1}_T} + \|\rho_T(u)\|_{L^{\infty}_T H^s_x} \leq C_1 (\|u\|_{X^{s-1,1}_T} + \|u\|_{L^{\infty}_T H^s_x}),
\]

for all \( u \in X^{s-1,1}_T \cap L^{\infty}_T H^s_x \).

Proof. First, it is clear from the construction of \( \rho_T(u) \) that

\[
\|\rho_T(u)\|_{L^{\infty}_T H^s_x} \lesssim \|u\|_{L^{\infty}_T H^s_x},
\]

since \( U \) is a unitary group in \( H^s \).

Next, we explain how to bound \( \|\rho_T(u)\|_{X^{s-1,1}_T} \). From the definition of \( X^{s-1,1}_T \), there exists an extension \( \tilde{u} \) of \( u \) on \( \mathbb{R}^2 \) such that

\[
\|\tilde{u}\|_{X^{s-1,1}_T} \leq 2\|u\|_{X^{s-1,1}_T}.
\]

Now, by using (2.2), we have that

\[
\|\rho_T(u)\|_{X^{s-1,1}_T} = \|\chi J_x^{s-1}v\|_{H^1_x L^2_T} \lesssim \|J_x^{s-1}v\|_{L^{2,2}_x L^2_T} + \|J_x^{s-1} \partial_t v\|_{L^{2,2}_x L^2_T}.
\]

Since \( \tilde{u} = u \) on \([0,T] \), we deduce from the definition of \( v \) that

\[
\|J_x^{s-1}v\|_{L^{2,2}_x L^2_T} \lesssim \|J_x^{s-1}u(\cdot,0)\|_{L^{2,2}_x L^2_T} + \|J_x^{s-1}U(\cdot)\tilde{u}(\cdot,t)\|_{L^{2,2}_x L^2_T}.
\]

and

\[
\|J_x^{s-1} \partial_t v\|_{L^{2,2}_x L^2_T} = \|J_x^{s-1} \partial_t U(\cdot)\tilde{u}(\cdot,t)\|_{L^{2,2}_x L^2_T},
\]

It follows then gathering (2.11) - (2.13) and using (2.2) that

\[
\|\rho_T(u)\|_{X^{s-1,1}_T} \lesssim \|u\|_{L^{\infty}_T H^s_x} + \|\tilde{u}\|_{X^{s-1,1}_T},
\]

which implies (2.8) in view of (2.10). \( \square \)

2.4. Refined Strichartz estimates. First, we recall the Strichartz estimate associated to the unitary Airy group derived in [9]. Then

\[
\|e^{-it\partial^3_x} D_x^\frac{1}{2} u_0\|_{L^1 L^\infty_T} \lesssim \|u_0\|_{L^2},
\]

for all \( u_0 \in L^2(\mathbb{R}) \).

Following the arguments in [7] and [12], we derive a refined Strichartz estimate for the solutions of the linear problem

\[
\partial_t u + \partial^3_x u = F.
\]

Proposition 2.3. Assume that \( T > 0 \) and \( \delta \geq 0 \). Let \( u \) be a smooth solution to (2.16) defined on the time interval \([0,T]\). Then,

\[
\|u\|_{L^2_T L^\infty_x} \lesssim \|J_x^{\frac{1}{4}+\theta} u\|_{L^2_T L^2_x} + \|J_x^{\frac{3\delta+1}{4}+\theta} F\|_{L^2_T L^2_x},
\]

for any \( \theta > 0 \).
Proof: Let \( u \) be solution to (2.16) defined on a time interval \([0, T]\). We use a nonhomogeneous Littlewood-Paley decomposition, \( u = \sum_N u_N \) where \( u_N = \mathcal{P}_N u, \) \( N \) is a nonhomogeneous dyadic number and and also denote \( F_N = \mathcal{P}_N F \). Then, we get from the Minkowski inequality that
\[
\| u \|_{L^4_t L^\infty_x} \leq \sum_N \| u_N \|_{L^4_t L^\infty_x} \lesssim \sup_N N^\delta \| u \|_{L^4_t L^\infty_x},
\]
for any \( \delta > 0 \). Recall that \( \mathcal{P}_0 \) corresponds to the projection in low frequencies, so that we set \( \mathcal{P}_0 = 1 \) by convention. Therefore, it is enough to prove that
\[
(2.18) \quad \| u_N \|_{L^4_t L^\infty_x} \lesssim \| D_x^{\frac{1}{2}} u_N \|_{L^\infty_x} + \| D_x^{\frac{1}{4}} F_N \|_{L^4_t L^\infty_x},
\]
for any \( \delta \geq 0 \) and any dyadic number \( N \in \{2^k : k \in \mathbb{N}\} \cup \{0\} \).

Let \( \delta \) be a nonnegative number to be fixed later. we chop out the interval in small intervals of \( N^{-\delta} \). In other words, we have that \([0, T] = \bigcup_{j \in J} I_j\) where \( I_j = [a_j, b_j]\), \(|I_j| \sim N^{-\delta}\) and \(|J| \sim N^\delta\). Since \( u_N \) is a solution to the integral equation
\[
\begin{aligned}
  u_N(t) &= e^{-(t-a_j)D_x} u_N(a_j) + \int_{a_j}^t e^{-(t-t')D_x} F_N(t') dt' \\
  &= e^{-(t-a_j)D_x} u_N(a_j) + \int_{a_j}^t e^{-(t-t')D_x} F(t') dt',
\end{aligned}
\]
for \( t \in I_j \), we deduce from (2.15) that
\[
\begin{aligned}
  \| u_N \|_{L^4_t L^\infty_x} &\lesssim \left( \sum_j \| D_x^{\frac{1}{2}} u_N(a_j) \|_{L^\infty_x}^4 \right)^{\frac{1}{4}} + \left( \sum_j \left( \int_{I_j} \| D_x^{\frac{1}{4}} F_N(t') \|_{L^4_t L^\infty_x}^4 dt' \right)^{\frac{1}{4}} \right) \leq N^{\frac{\delta}{2}} \| D_x^{\frac{1}{2}} u_N \|_{L^\infty_x} + \left( \int_{I_j} \| D_x^{\frac{1}{4}} F_N(t') \|_{L^4_t L^\infty_x}^4 dt' \right)^{\frac{1}{4}} \lesssim \| D_x^{\frac{1}{2}} u_N \|_{L^\infty_x} + \| D_x^{\frac{1}{4}} F_N \|_{L^4_t L^\infty_x},
\end{aligned}
\]
which concludes the proof of (2.18). \( \square \)

3. \( L^2 \) Multilinear Estimates

In this section we follow some notations of [19]. For \( k \in \mathbb{Z}_+ \) and \( \xi \in \mathbb{R} \), let \( \Gamma^k(\xi) \) denote the \( k \)-dimensional “affine hyperplane” of \( \mathbb{R}^{k+1} \) defined by
\[
\Gamma^k(\xi) = \{ (\xi_1, \ldots, \xi_{k+1}) \in \mathbb{R}^{k+1} : \xi_1 + \cdots + \xi_{k+1} = \xi \},
\]
and endowed with the obvious measure
\[
\int_{\Gamma^k(\xi)} F = \int_{\Gamma^k(\xi)} F(\xi_1, \ldots, \xi_{k+1}) := \int_{\mathbb{R}^k} F(\xi_1, \ldots, \xi_k, \xi - (\xi_1 + \cdots + \xi_k)) d\xi_1 \cdots d\xi_k,
\]
for any function \( F : \Gamma^k(\xi) \to \mathbb{C} \). When \( \xi = 0 \), we simply denote \( \Gamma^k = \Gamma^k(0) \) with the obvious modifications.

Moreover, given \( T > 0 \), we also define \( \mathbb{R}_T = \mathbb{R} \times [0, T] \) and \( \Gamma^k_T = \Gamma^k \times [0, T] \) with the obvious measures
\[
\int_{\mathbb{R}_T} u := \int_{\mathbb{R} \times [0, T]} u(x, t) dx dt
\]
and
\[
\int_{\Gamma^k_T} F := \int_{\mathbb{R}^k \times [0, T]} F(\xi_1, \ldots, \xi_k, \xi - (\xi_1 + \cdots + \xi_k), t) d\xi_1 \cdots d\xi_k dt.
\]
3.1. 3-linear estimates.

Lemma 3.1. Let $f_j \in L^2(\mathbb{R})$, $j = 1, \ldots, 4$ and $M \in \mathbb{D}$. Then it holds that

\begin{equation}
\int_{\mathbb{R}^3} \phi_M(\xi_1 + \xi_2) \prod_{j=1}^{4} |f_j(\xi_j)| \lesssim M \prod_{j=1}^{4} \|f_j\|_{L^2}.
\end{equation}

Proof. Let us denote by $J^3_j(f_1, f_2, f_3, f_4)$ the integral on the left-hand side of (3.1).

We can assume without loss of generality that $f_i \geq 0$ for $i = 1, \ldots, 4$. Then, we have that

\begin{equation}
J^3_j(f_1, f_2, f_3, f_4) \leq A_M(f_1, f_2) \times \sup_{\xi_1, \xi_2} \int_{\mathbb{R}} f_3(\xi_3) f_4(-\xi_1 + \xi_2 + \xi_3) d\xi_3,
\end{equation}

where

\begin{equation}
A_M(f_1, f_2) = \int_{\mathbb{R}^2} \phi_M(\xi_1 + \xi_2) f_1(\xi_1) f_2(\xi_2) d\xi_1 d\xi_2.
\end{equation}

Hölder’s inequality yields

\begin{equation}
A_M(f_1, f_2) \leq \int_{\mathbb{R}} \phi_M(\xi_1)(f_1 * f_2)(\xi_1) d\xi_1 \lesssim M\|f_1 * f_2\|_{L^\infty} \lesssim M\|f_1\|_{L^2} \|f_2\|_{L^2}.
\end{equation}

Moreover, the Cauchy-Schwarz inequality yields

\begin{equation}
\int_{\mathbb{R}} f_3(\xi_3) f_4(-\xi_1 + \xi_2 + \xi_3) d\xi_3 \leq \|f_3\|_{L^2} \|f_4\|_{L^2}.
\end{equation}

Therefore, estimate (3.1) follows from (3.2)–(3.5). \qed

For a fixed $N \geq 1$ dyadic, we introduce the following disjoint subsets of $\mathbb{D}^3$:

$\mathcal{M}^{low}_{3} = \{(M_1, M_2, M_3) \in \mathbb{D}^3 : M_{min} \leq N^{-\frac{1}{4}}$ and $M_{med} \leq 2^{-9}N\}$,

$\mathcal{M}^{med}_{3} = \{(M_1, M_2, M_3) \in \mathbb{D}^3 : N^{-\frac{1}{4}} < M_{min} \leq M_{med} \leq 2^{-9}N\}$,

$\mathcal{M}^{high,1}_{3} = \{(M_1, M_2, M_3) \in \mathbb{D}^3 : M_{min} \leq N^{-1}$ and $2^{-9}N < M_{med} \leq M_{max}\}$,

$\mathcal{M}^{high,2}_{3} = \{(M_1, M_2, M_3) \in \mathbb{D}^3 : N^{-1} < M_{min} \leq 1$ and $2^{-9}N < M_{med} \leq M_{max}\}$,

$\mathcal{M}^{high,3}_{3} = \{(M_1, M_2, M_3) \in \mathbb{D}^3 : 1 < M_{min}$ and $2^{-9}N < M_{med} \leq M_{max}\}$,

where $M_{min} \leq M_{med} \leq M_{max}$ denote respectively the minimum, sub-maximum and maximum of $\{M_1, M_2, M_3\}$. Moreover, we also denote $\mathcal{M}^{high}_{3} = \mathcal{M}^{high,1}_{3} \cup \mathcal{M}^{high,2}_{3} \cup \mathcal{M}^{high,3}_{3}$.

We will denote by $\phi_{M_1, M_2, M_3}$ the function

$\phi_{M_1, M_2, M_3}(\xi_1, \xi_2, \xi_3) = \phi_{M_1}(\xi_2 + \xi_3) \phi_{M_2}(\xi_1 + \xi_3) \phi_{M_3}(\xi_1 + \xi_2)$.

Next, we state a useful technical lemma.

Lemma 3.2. Let $(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ that satisfies $|\xi_j| \sim N_j$ for $j = 1, 2, 3$ and $|\xi_1 + \xi_2 + \xi_3| \sim N$. Let $(M_1, M_2, M_3) \in \mathcal{M}^{low}_{3} \cup \mathcal{M}^{med}_{3}$. Then it holds that

$N_1 \sim N_2 \sim N_3 \sim M_{max} \sim N$ if $(\xi_1, \xi_2, \xi_3) \in \text{supp } \phi_{M_1, M_2, M_3}$,

where $M_{max}$ denote the maximum of $\{M_1, M_2, M_3\}$.
Proof. Without loss of generality, we can assume that $M_1 \leq M_2 \leq M_3$. Let $(\xi_1, \xi_2, \xi_3) \in \text{supp} \phi_{M_1, M_2, M_3}$. Then, we have $|\xi_2 + \xi_3| \ll N$ and $|\xi_1 + \xi_3| \ll N$, so that $N_1 \sim N_2 \sim N$ since $|\xi_1 + \xi_2 + \xi_3| \sim N$.

On one hand $N_3 \ll N$ would imply that $M_1 \sim M_2 \sim N$ which is a contradiction. On the other hand, $N_3 \gg N$ would imply that $|\xi_1 + \xi_2 + \xi_3| \gg N$ which is also a contradiction. Therefore, we must have $N_3 \sim N$.

Finally, $M_1 \ll N$ imply that $\xi_2 \cdot \xi_3 < 0$ and $M_2 \ll N$ imply $\xi_1 \cdot \xi_3 < 0$. Thus, $\xi_1 \cdot \xi_2 > 0$, so that $M_3 \sim N$.

For $\eta \in L^\infty$, let us define the trilinear pseudo-product operator $\Pi^3_{\eta, M_1, M_2, M_3}$ in Fourier variables by

$$
\mathcal{F}_\epsilon (\Pi^3_{\eta, M_1, M_2, M_3}(u_1, u_2, u_3)) (\xi) = \int_{\Gamma^2(\xi)} (\eta \phi_{M_1, M_2, M_3})(\xi_1, \xi_2, \xi_3) \prod_{j=1}^{3} \hat{u}_j (\xi_j).
$$

It is worth noticing that when the functions $u_j$ are real-valued, the Plancherel identity yields

$$
\int_R \Pi^3_{\eta, M_1, M_2, M_3}(u_1, u_2, u_3) u_4 \, dx = \int_{\Gamma^3} (\eta \phi_{M_1, M_2, M_3})(\xi_1, \xi_2, \xi_3) \prod_{j=1}^{4} \hat{u}_j (\xi_j).
$$

Finally, we define the resonance function of order 3 by

$$
\Omega^3(\xi_1, \xi_2, \xi_3) = \xi_1^3 + \xi_2^3 + \xi_3^3 - (\xi_1 + \xi_2 + \xi_3)^3
$$

$$
= -3(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_2 + \xi_3).
$$

The following proposition gives suitable estimates for the pseudo-product $\Pi^3_{M_1, M_2, M_3}$ when $(M_1, M_2, M_3) \in \mathcal{M}_3^{\text{high}}$

**Proposition 3.3.** Assume that $0 < T \leq 1$, $\eta$ is a bounded function and $u_i$ are real-valued functions in $Y^0 = X^{-1,1} \cap L^\infty_t L^2_x$ with time support in $[0, 2]$ and spatial Fourier support in $I_N^i$, for $i = 1, \cdots, 4$. Here, $N_i$ denote nonhomogeneous dyadic numbers. Assume also that $N_{\max} \geq N_4 = N \gg 1$, and $(M_1, M_2, M_3) \in \mathcal{M}_3^{\text{high}}$. Then

$$
\int_{R^4 \times [0, T]} \Pi^3_{\eta, M_1, M_2, M_3}(u_1, u_2, u_3) u_4 \, dx \, dt \lesssim N_{\max}^{-1} \prod_{i=1}^{4} \|u_i\|_{Y^0}.
$$

Moreover, the implicit constant in estimate (3.9) only depends on the $L^\infty$-norm of the function $\eta$.

Before giving the proof of Proposition 3.3, we derive some important technical lemmas.

**Lemma 3.4.** Let $L$ be a nonhomogeneous dyadic number. Then the operator $Q_{\leq L}$ is bounded in $L^\infty_t L^2_x$ uniformly in $L$. In other words,

$$
\|Q_{\leq L} u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{L^\infty_t L^2_x},
$$

for all $u \in L^\infty_t L^2_x$ and the implicit constant appearing in (3.10) does not depend on $L$.

**Proof.** A direct computation shows that

$$
Q_{\leq L} u = e^{itw(D)} R_{\leq L} e^{-itw(D)} u.
$$
Since $e^{itw(D)}$ is a unitary group in $L^2$, it follows from (3.11), Minkowski and Hölder’s inequalities that

$$\|Q_{\leq L} u(\cdot, t)\|_{L^2} = \|R_{\leq L} e^{-itw(D)} u(\cdot, t)\|_{L^2} \leq \int_R \|((\chi_L)') e^{-i(t-t_1)w(D)} u(\cdot, t-t_1)\|_{L^2} dt_1 \leq \|((\chi)') L^1 u\|_{L^\infty L^2},$$

which implies estimate (3.10). □

For any $0 < T \leq 1$, let us denote by $1_T$ the characteristic function of the interval $[0, T]$. One of the main difficulties in the proof of Proposition 3.3 is that the operator $1_T$ does not commute with $Q_L$. To handle this situation, we follow the arguments introduced in [16] and use the decomposition

$$(3.12) \quad 1_T = 1_{T,R}^\text{low} + 1_{T,R}^\text{high}, \quad \text{with} \quad \mathcal{F}_t(1_{T,R}^\text{low})(\tau) = \chi(\tau/R)\mathcal{F}_t(1_T)(\tau),$$

for some $R > 0$ to be fixed later. The following lemmas were derived in [16]. For the sake of completeness, we will give their proof here.

**Lemma 3.5.** For any $R > 0$ and $T > 0$ it holds

$$(3.13) \quad \|1_{T,R}^\text{high}\|_{L^1} \lesssim T \wedge R^{-1},$$

and

$$(3.14) \quad \|1_{T,R}^\text{low}\|_{L^\infty} \lesssim 1.$$ 

**Proof.** It follows from the definition of $1_{T,R}^\text{high}$ in (3.12) that

$$\|1_{T,R}^\text{high}\|_{L^1} = \int_R \int_R (1_T(t) - 1_T(t-s/R)) \mathcal{F}_t^{-1}(\chi)(s) ds \, dt \leq \int_R \int_{[0,T)\setminus[s/R,T+s/R]} \mathcal{F}_t^{-1}(\chi)(s) ds \, dt \lesssim T \wedge R^{-1}.$$ 

Finally it is easy check that $\|1_{T,R}^\text{low}\|_{L^\infty} \lesssim \|\chi\|_{L^1} \|1_T\|_{L^\infty} \lesssim 1$. □

**Lemma 3.6.** Assume that $T > 0$, $R > 0$ and $L \gg R$. Then, it holds

$$(3.15) \quad \|Q_L(1_{T,R}^\text{low} u)\|_{L^2} \lesssim \|Q_{\sim L} u\|_{L^2},$$

for all $u \in L^2(\mathbb{R}^2)$.

**Proof.** By Plancherel we get

$$A_L = \|Q_L(1_{T,R}^\text{low} u)\|_{L^2} = \|\phi_L(\tau - \omega(\xi)) \mathcal{F}_\tau(1_{T,R}^\text{low} u)(\tau, \xi)\|_{L^2} \lesssim \|\sum_{L_1} \phi_L(\tau - \omega(\xi)) \int_R \phi_{L_1}(\tau' - \omega(\xi)) \mathcal{F}(u)(\tau', \xi) \chi((\tau - \tau')/R) e^{-i\tau' - \tau'} \chi((\tau - \tau')/R) \frac{e^{-i\tau(\tau - \tau')}}{\tau - \tau'} \, dt'\|_{L^2}.$$
In the region where $L_1 \ll L$ or $L_1 \gg L$, we have $|\tau - \tau'| \sim L \lor L_1 \gg R$, thus $A_L$ vanishes. On the other hand, for $L \sim L_1$, we get

$$A_L \lesssim \sum_{L_1 \sim L} \|Q_L(1_{T,R}^{-1} Q_{L_1} u)\|_{L^2} \lesssim \|Q_{\sim L} u\|_{L^2}.$$ 

\[ \square \]

**Proof of Proposition 3.3.** Given $u_i$, $1 \leq i \leq 4$, satisfying the hypotheses of Proposition 3.3 let $G_{M_1,M_2,M_3}^3 = G_{M_1,M_2,M_3}^3(u_1, u_2, u_3, u_4)$ denote the left-hand side of (3.9). We use the decomposition in (3.12) and obtain that

$$G_{M_1,M_2,M_3}^3 = G_{M_1,M_2,M_3,R}^{3,\text{low}} + G_{M_1,M_2,M_3,R}^{3,\text{high}},$$

where

$$G_{M_1,M_2,M_3,R}^{3,\text{low}} = \int_{R^2} 1_{T,R}^{-1} \Pi_{\eta,M_1,M_2,M_3}^3 (u_1, u_2, u_3) u_4 \, dx \, dt$$

and

$$G_{M_1,M_2,M_3,R}^{3,\text{high}} = \int_{R^2} 1_{T,R}^{-1} \Pi_{\eta,M_1,M_2,M_3}^3 (u_1, u_2, u_3) u_4 \, dx \, dt.$$

We deduce from Hölder’s inequality in time, (3.1), (3.7), (3.13) and (3.18) that

$$\|G_{M_1,M_2,M_3,R}^{3,\text{high}}\|_{L^1} \lesssim \|1_{T,R}^{-1}\|_{L^1} \int_{R} \Pi_{\eta,M_1,M_2,M_3}^3 (u_1, u_2, u_3) u_4 \, dx \|_{L^\infty} \lesssim R^{-1} M_{\min} \prod_{i=1}^{4} \|u_i\|_{L_t^{\infty}L_x^2},$$

which implies that

$$\|G_{M_1,M_2,M_3,R}^{3,\text{high}}\|_{L^1} \lesssim N_{\max}^{-1} \prod_{i=1}^{4} \|u_i\|_{L_t^{\infty}L_x^2}$$

if we choose $R = M_{\min} N_{\max}$.

To deal with the term $G_{M_1,M_2,M_3,R}^{3,\text{low}}$, we decompose with respect to the modulation variables. Thus,

$$G_{M_1,M_2,M_3,R}^{3,\text{low}} = \sum_{L_1,L_2,L_3} \int_{R^2} \Pi_{\eta,M_1,M_2,M_3}^3 (Q_{L_1} (1_{T,R}^{-1} Q_{L_1} u_1), Q_{L_2} u_2, Q_{L_3} u_3) Q_{L_4} u_4 \, dx \, dt.$$

Moreover, observe from the resonance relation in (3.8) and the hypothesis $(M_1, M_2, M_3) \in \mathcal{N}_{\text{high}}^{N_{\max}}$ that

$$L_{\max} \gtrsim M_{\min} N_{\max}^2.$$ 

Indeed, in the case where $N_{\max} \sim N$, (3.13) is clear from the definition of $\mathcal{N}_{\text{high}}$. In the case where $N_{\max} \gg N$, then we have $N_{\max} \sim N_{\med} \gg N_{\min}$ since $\xi_1 + \xi_2 + \xi_3 \sim N$. This implies that $M_{\max} \sim M_{\med} \gtrsim N_{\max}$.

In particular, (3.12) implies that $L_{\max} \gtrsim R = M_{\min} N_{\max}$, since $N_{\max} \gg 1$.

In the case where $L_{\max} = L_1$, we deduce from (3.1), (3.7), (3.10) and (3.11) that

$$\|G_{M_1,M_2,M_3,R}^{3,\text{low}}\|_{L^1} \lesssim \sum_{L_1 \gtrsim M_{\min} N_{\max}^2} M_{\min} L_1^{-1} L_1 \|Q_{L_1} (1_{T,R}^{-1} Q_{L_1} u_1)\|_{L^2} \prod_{i=1}^{4} \|Q_{\leq L_1} u_i\|_{L_t^{\infty}L_x^2} \lesssim N_{\max}^{-1} \|u_1\|_{X^{-1,1}} \prod_{i=2}^{4} \|u_i\|_{L_t^{\infty}L_x^2},$$
which implies that

(3.19) \[ |G_{M_1, M_2, M_3, R}^{3, \text{low}}| \lesssim N_{\max}^{-1} \prod_{i=1}^{4} \|u_i\|_Y^0. \]

We can prove arguing similarly that (3.19) still holds true in all the other cases, i.e. \( L_{\max} = L_2, L_3 \) or \( L \). Note that for those cases we do not have to use (3.13) but we only need (3.14). Therefore, we conclude the proof of estimate (3.9) gathering (3.16), (3.17) and (3.19). \( \square \)

3.2. 5-linear estimates.

**Lemma 3.7.** Let \( f_j \in L^2(\mathbb{R}) \), \( j = 1, \ldots, 6 \) and \( M_1, M_4 \in \mathbb{D} \). Then it holds that

(3.20) \[ \int_{\Gamma^5} \phi_{M_1}(\xi_2 + \xi_3)\phi_{M_4}(\xi_5 + \xi_6) \prod_{j=1}^{6} |f_j(\xi_j)| \lesssim M_1 M_4 \prod_{j=1}^{6} \|f_j\|_{L^2}. \]

If moreover \( f_j \) are localized in an annulus \( \{|\xi| \sim N_j\} \) for \( j = 5, 6 \), then

(3.21) \[ \int_{\Gamma^5} \phi_{M_1}(\xi_2 + \xi_3)\phi_{M_4}(\xi_5 + \xi_6) \prod_{i=1}^{6} |f_i(\xi_i)| \lesssim M_1 M_4^{1/2} N_5^{1/2} N_6^{1/2} \prod_{i=1}^{6} \|f_i\|_{L^2}. \]

**Proof.** Let us denote by \( J^5 = J_5(f_1, \ldots, f_6) \) the integral on the right-hand side of (3.20). We can assume without loss of generality that \( f_j \geq 0, j = 1, \ldots, 6 \). We have by using the notation in (3.4) that

(3.22) \[ J^5 \leq J_{M_1}(f_2, f_3) \times J_{M_4}(f_5, f_6) \times \sup_{\xi_2, \xi_3, \xi_5, \xi_6} \int_{\mathbb{R}} f_1(\xi_1)f_4(-\sum_{j=1, j \neq 4}^{6} \xi_j) d\xi_1. \]

Thus, estimate (3.20) follows applying (3.22) and the Cauchy-Schwarz inequality to (3.22).

Assuming furthermore that \( f_j \) are localized in an annulus \( \{|\xi| \sim N_j\} \) for \( j = 5, 6 \), then we get arguing as above that

(3.23) \[ J^5 \leq M_1 \times J_{M_4}(f_5, f_6) \times \prod_{j=1}^{4} \|f_j\|_{L^2}. \]

From the Cauchy-Schwarz inequality

\[ J_{M_4}(f_5, f_6) \leq \left( \int_{\mathbb{R}} f_5(\xi_5)d\xi_5 \right) \times \left( \int_{\mathbb{R}} f_6(\xi_6)d\xi_6 \right) \lesssim N_5^{1/2} N_6^{1/2} \|f_5\|_{L^2}\|f_6\|_{L^2}, \]

which together with (3.23) yields

(3.24) \[ J^5 \lesssim M_1 N_5^{1/2} N_6^{1/2} \prod_{i=1}^{6} \|f_i\|_{L^2}. \]

Therefore, we conclude the proof of (3.21) interpolating (3.20) and (3.24). \( \square \)
For a fixed $N \geq 1$ dyadic, we introduce the following subsets of $\mathbb{D}^6$:

\[
M_5^{low} = \{(M_1, \ldots, M_6) \in \mathbb{D}^6 : (M_1, M_2, M_3) \in M_3^{med}, M_{min(5)} \leq 2^9 M_{med(3)} \text{ and } M_{med(5)} \leq 2^{-9} N\},
\]

\[
M_5^{med} = \{(M_1, \ldots, M_6) \in \mathbb{D}^6 : (M_1, M_2, M_3) \in M_3^{med} \text{ and } 2^9 M_{med(3)} < M_{min(5)} \leq M_{med(5)} \leq 2^{-9} N\},
\]

\[
M_5^{high,1} = \{(M_1, \ldots, M_6) \in \mathbb{D}^6 : (M_1, M_2, M_3) \in M_3^{med}, M_{min(5)} \leq N^{-1} \text{ and } 2^{-9} N < M_{med(5)} \leq M_{max(5)}\},
\]

\[
M_5^{high,2} = \{(M_1, \ldots, M_6) \in \mathbb{D}^6 : (M_1, M_2, M_3) \in M_3^{med}, N^{-1} < M_{min(5)} \text{ and } 2^{-9} N < M_{med(5)} \leq M_{max(5)}\},
\]

and

\[
\tilde{\mathcal{N}}_5^{low} = \{(M_1, \ldots, M_6) \in \mathbb{D}^6 : (M_1, M_2, M_3) \in M_3^{high,2}, M_{min(5)} \leq 2^9 M_{min(3)}\},
\]

\[
\tilde{\mathcal{N}}_5^{med,1} = \{(M_1, \ldots, M_6) \in \mathbb{D}^6 : (M_1, M_2, M_3) \in M_3^{high,2}, 2^9 M_{min(3)} < M_{min(5)} \leq M_{med(5)} \leq 2^{-9} N \text{ and } M_{min(5)} \leq 2^9 N^4\},
\]

\[
\tilde{\mathcal{N}}_5^{med,2} = \{(M_1, \ldots, M_6) \in \mathbb{D}^6 : (M_1, M_2, M_3) \in M_3^{high,2}, 2^9 M_{min(3)} < M_{min(5)} \leq M_{med(5)} \leq 2^{-9} N \text{ and } M_{min(5)} > 2^9 N^4\},
\]

\[
\tilde{\mathcal{N}}_5^{high} = \{(M_1, \ldots, M_6) \in \mathbb{D}^6 : (M_1, M_2, M_3) \in M_3^{high,2}, 2^9 M_{min(3)} < M_{min(5)} \text{ and } 2^{-9} N < M_{med(5)} \leq M_{max(5)}\},
\]

where $M_{max(3)} \geq M_{med(3)} \geq M_{min(3)}$, respectively $M_{max(5)} \geq M_{med(5)} \geq M_{min(5)}$, denote the maximum, sub-maximum and minimum of $\{M_1, M_2, M_3\}$, respectively $\{M_4, M_5, M_6\}$. We will also denote by $\phi_{M_1, \ldots, M_6}$ the function defined on $\mathbb{R}^6$ by

\[
\phi_{M_1, \ldots, M_6}(\xi_1, \ldots, \xi_6) = \phi_{M_1, M_2, M_3}(\xi_1, \xi_2, \xi_3) \phi_{M_4, M_5, M_6}(\xi_4, \xi_5, \xi_6).
\]

For $\eta \in L^\infty$, let us define the operator $\Pi_{\eta, M_1, \ldots, M_6}^5$ in Fourier variables by

\[
\mathcal{F}_x (\Pi_{\eta, M_1, \ldots, M_6}^5(u_1, \ldots, u_5))(\xi) = \int_{\Gamma^5(\xi)} (\eta \phi_{M_1, \ldots, M_6})(\xi_1, \ldots, \xi_5) - \sum_{j=1}^5 \xi_j \prod_{j=1}^5 \hat{u}_j(\xi_j).
\]

Observe that, if the functions $u_j$ are real valued, the Plancherel identity yields

\[
\int_{\mathbb{R}^5} \Pi_{\eta, M_1, \ldots, M_6}^5(u_1, \ldots, u_5) \, u_6 \, dx = \int_{\mathbb{R}^5} (\eta \phi_{M_1, \ldots, M_6}) \prod_{j=1}^6 \hat{u}_j(\xi_j).
\]

Finally, we define the resonance function of order 5 for $\xi_{(5)} = (\xi_1, \ldots, \xi_6) \in \Gamma^5$ by

\[
\Omega^5(\xi_{(5)}) = \xi_1^3 + \xi_2^3 + \xi_3^3 + \xi_4^3 + \xi_5^3 + \xi_6^3.
\]

It is worth noticing that a direct calculus leads to

\[
\Omega^5(\xi_{(5)}) = \Omega^3(\xi_1, \xi_2, \xi_3) + \Omega^3(\xi_4, \xi_5, \xi_6).
\]
The following proposition gives suitable estimates for the pseudo-product $\Pi_{M_1,\ldots,M_6}^5$ when $(M_1, \ldots, M_6) \in M_5^{\text{high}}$ in the non resonant case $M_1 M_2 M_3 \not\sim M_4 M_5 M_6$, when $(M_1, \ldots, M_6) \in M_5^{\text{high}}$ and when $(M_1, \ldots, M_6) \in \tilde{N}_5^{\text{med}}$.

**Proposition 3.8.** Assume that $0 < T \leq 1$, $\eta$ is a bounded function and $u_i$ are functions in $Y^0 = X^{-1,1} \cap L^2_t L^2_x$ with time support in $[0,2]$ and spatial Fourier support in $I_N$ for $i = 1, \ldots, 6$. Here, $N_i$ denote nonhomogeneous dyadic numbers. Assume also that $\max \{N_1, \ldots, N_6\} \geq N \gg 1$.

1. If $(M_1, \ldots, M_6) \in M_5^{\text{high}}$ satisfies the non resonance assumption $M_1 M_2 M_3 \not\sim M_4 M_5 M_6$, then

$$
\left| \int_{\mathbb{R} \times [0,T]} \Pi_{\eta,M_1,\ldots,M_6}^5(u_1, \ldots, u_5) u_6 \, dx \, dt \right| \lesssim M_{\min(3)} N^{-1} \prod_{i=1}^6 \left\| u_i \right\|_{Y^0},
$$

where $N_{\max(5)} = \max \{N_1, N_5, N_6\}$.

2. If $(M_1, \ldots, M_6) \in \tilde{N}_5^{\text{high}}$, $\max \{N_1, N_2, N_3\} \sim N$ and $\text{med} \{N_1, N_2, N_3\} \ll N$, then

$$
\left| \int_{\mathbb{R} \times [0,T]} \Pi_{\eta,M_1,\ldots,M_6}^5(u_1, \ldots, u_5) u_6 \, dx \, dt \right| \lesssim M_{\min(3)} N^{-1} \prod_{i=1}^6 \left\| u_i \right\|_{Y^0}.
$$

3. If $(M_1, \ldots, M_6) \in \tilde{N}_5^{\text{med}}$, $\max \{N_1, N_2, N_3\} \sim N$ and $\text{med} \{N_1, N_2, N_3\} \ll N$, then

$$
\left| \int_{\mathbb{R} \times [0,T]} \Pi_{\eta,M_1,\ldots,M_6}^5(u_1, \ldots, u_5) u_6 \, dx \, dt \right| \lesssim M_{\min(3)} N^{-1} \prod_{i=1}^6 \left\| u_i \right\|_{Y^0}.
$$

Here, we used the notations $M_{\min(3)} = \min(M_1, M_2, M_3)$. Moreover, the implicit constant in estimate (3.30) only depends on the $L^{\infty}$-norm of the function $\eta$.

**Proof.** The proof is similar to the proof of Proposition 3.3. We may always assume $M_1 \leq M_2 \leq M_3$ and $M_4 \leq M_5 \leq M_6$.

In Case 1, we deduce from identities (3.28) and (3.8), the non resonance assumption and the fact that $(M_1, ..., M_6) \in M_5^{\text{high}}$ that

$$
L_{\max} \gtrsim \max(M_1 M_2 M_3, M_4 M_5 M_6) \gtrsim M_4 N_{\max(5)}^2.
$$

Estimate (3.30) follows from this and estimate (3.29).

In Case 2, we get from the assumptions on $M_i$ and $N_i$ that

$$
M_4 M_5 M_6 \geq 2^9 M_1 2^{-9} N N_{\max(5)} \gtrsim M_1 N^2 \gg M_1 M_2 M_3
$$

so that $L_{\max} \gtrsim M_3 M_5 M_6 \gtrsim N_{\max(5)}^2 M_4$.

The proof of estimate (3.31) follows then exactly as above.

In Case 3, we get from the assumptions on $M_i$ and $N_i$

$$
M_4 M_5 M_6 \gg N^2 \geq N^2 M_1, \quad \text{so that} \quad L_{\max} \gtrsim N^\frac{3}{2} M_4.
$$

Moreover, observe from Lemma 3.2 that $\max \{N_1, \ldots, N_6\} \sim N$. The proof of estimate (3.32) follows then exactly as above. □
3.3. 7-linear estimates.

Lemma 3.9. Let \( f_i \in L^2(\mathbb{R}) \), \( i = 1, \ldots, 8 \) and \( M_1, M_2, M_4, M_5 \) and \( M_7 \in \mathbb{D} \). Then it holds that

\[
\int_{\Gamma^7} \phi_{M_1}(\xi_1 + \xi_3)\phi_{M_3}(\xi_5 + \xi_6)\phi_{M_7}(\xi_7 + \xi_8) \prod_{i=1}^{8} |f_i(\xi_i)| \lesssim M_1 M_4 M_7 \prod_{i=1}^{8} ||f_i||_{L^2} .
\]

and

\[
\int_{\Gamma^7} \phi_{M_1}(\xi_2 + \xi_5)\phi_{M_2}(\xi_1 + \xi_5)\phi_{M_4}(\xi_5 + \xi_6)\phi_{M_6}(\xi_4 + \xi_6) \prod_{i=1}^{8} |f_i(\xi_i)| \lesssim M_1 M_2 M_4 M_5 \prod_{i=1}^{8} ||f_i||_{L^2} .
\]

If moreover \( f_j \) is localized in an annulus \( \{ |\xi| \sim N_j \} \) for \( j = 7, 8 \), then

\[
\int_{\Gamma^7} \phi_{M_1}(\xi_2 + \xi_5)\phi_{M_4}(\xi_5 + \xi_6)\phi_{M_7}(\xi_7 + \xi_8) \prod_{i=1}^{8} |f_i(\xi_i)| \lesssim M_1 M_4 M_7^2 N_7^2 N_8^2 \prod_{i=1}^{8} ||f_i||_{L^2} .
\]

Proof. Let us denote by \( J^7 = J^7(f_1, \ldots, f_8) \) the integral on the right-hand side of (3.36). We can assume without loss of generality that \( f_j \geq 0 \), \( j = 1, \ldots, 8 \). We have by using the notation in (3.4) that

\[
J^7 \leq \partial_{M_1}(f_2, f_3) \times \partial_{M_4}(f_5, f_6) \times \sup_{\xi_2, \xi_5, \xi_6, \xi_7, \xi_8} \left( \int_{\mathbb{R}} f_1(\xi_1) f_4(-\sum_{j \neq 4}^{8} \xi_j) d\xi_1 \right).
\]

Thus, estimate (3.36) follows applying (3.4) and the Cauchy-Schwarz inequality to (3.36).

Assuming furthermore that \( f_j \) are localized in an annulus \( \{ |\xi| \sim N_j \} \) for \( j = 7, 8 \), then we get arguing as above that

\[
J^7 \leq M_1 M_4 \times \partial_{M_7}(f_7, f_8) \times \prod_{j=1}^{6} ||f_j||_{L^2} .
\]

From the Cauchy-Schwarz inequality

\[
\partial_{M_7}(f_7, f_8) \leq \left( \int_{\mathbb{R}} f_7(\xi_7) d\xi_7 \right) \times \left( \int_{\mathbb{R}} f_8(\xi_8) d\xi_8 \right) \lesssim N_7^2 N_8^2 ||f_7||_{L^2} ||f_8||_{L^2} ,
\]

which together with (3.37) yields

\[
J^7 \lesssim M_1 M_4 N_7^2 N_8^2 \prod_{j=1}^{8} ||f_j||_{L^2} .
\]

Therefore, we conclude the proof of (3.35) interpolating (3.38) and (3.39).

Now, we prove (3.34). Let denote by \( \widetilde{J}^7 = \widetilde{J}^7(f_1, \ldots, f_8) \) the left-hand side of (3.34). Then,

\[
\widetilde{J}^7 \leq \partial_{M_1}(f_1, f_2) \times \partial_{M_4}(f_5, f_6) \times \sup_{\xi_2, \xi_3, \xi_5, \xi_6} \mathcal{K}(\xi_2, \xi_3, \xi_5, \xi_6) \]

\[\cdots\]
where $\mathcal{K} = \mathcal{K}(\xi_2, \xi_3, \xi_5, \xi_6)$ is defined by
\[
\mathcal{K}(\xi_2, \xi_3, \xi_5, \xi_6) = \int_{\mathbb{R}^2} \phi M_2 (\xi_1 + \xi_3) \phi M_6 (\xi_4 + \xi_6) f_1 (\xi_1) f_4 (\xi_4) f_7 (\xi_7) f_8 (\xi_8) \frac{1}{7} \sum_{k=1}^{7} \xi_k d\xi_1 d\xi_4 d\xi_7.
\]

We deduce from Cauchy-Schwarz in $(\xi_1, \xi_4)$ that
\[
\mathcal{K} = \int_{\mathbb{R}^2} \phi M_2 (\xi_1 + \xi_3) \phi M_6 (\xi_4 + \xi_6) f_1 (\xi_1) f_4 (\xi_4) (f_7 \ast f_8) (- (\xi_1 + \cdots + \xi_6)) d\xi_1 d\xi_4
\]
\[
\leq \| \phi M_2 (\xi_1 + \xi_3) \phi M_6 (\xi_4 + \xi_6) \|_{L^2_{\xi_1, \xi_4}} \| f_1 (\xi_1) f_4 (\xi_4) \|_{L^2_{\xi_1, \xi_4}} \| f_7 \ast f_8 \|_{L^\infty}
\]
\[
\lesssim (M_2 M_6)^{1/2} \| f_1 \|_{L^2} \| f_4 \|_{L^2} \| f_7 \|_{L^2} \| f_8 \|_{L^2},
\]
which together with (3.41) and (3.40) concludes the proof of (3.34).

For a fixed $N \geq 1$ dyadic, we introduce the following subsets of $\mathbb{D}^9$: $\mathcal{M}^{\text{low}}_7 = \{(M_1, \ldots, M_9) \in \mathbb{D}^9 : (M_1, \ldots, M_6) \in \mathcal{M}^{\text{med}}_5, \ M_{\text{min}(7)} \leq M_{\text{med}(7)} \leq 2^{-9} N\}$, $\mathcal{M}^{\text{high}}_7 = \{(M_1, \ldots, M_9) \in \mathbb{D}^9 : (M_1, \ldots, M_6) \in \mathcal{M}^{\text{med}}_5, \ 2^{-9} N < M_{\text{med}(7)} \leq M_{\text{max}(7)}\}$ where $M_{\text{max}(7)} \geq M_{\text{med}(7)} \geq M_{\text{min}(7)}$ denote respectively the maximum, sub-maximum and minimum of $\{M_7, M_8, M_9\}$.

We will denote by $\phi_{M_1, \ldots, M_9}$ the function defined on $\Gamma_7$ by
\[
\phi_{M_1, \ldots, M_9} (\xi_1, \ldots, \xi_7, \xi_8) = \phi_{M_1, \ldots, M_9} (\xi_1, \ldots, \xi_5, - \sum_{j=1}^{5} \xi_j) \phi_{M_7, M_8, M_9} (\xi_6, \xi_7, \xi_8).
\]

For $\eta \in L^\infty$, let us define the operator $\Pi_7^{\eta, M_1, \ldots, M_9}$ in Fourier variables by
\[
\mathcal{F}_x (\Pi_7^{\eta, M_1, \ldots, M_9} (u_1, \ldots, u_7)) (\xi) = \int_{\Gamma^7 (\xi)} (\eta \phi_{M_1, \ldots, M_9}) (\xi_1, \ldots, \xi_7) \prod_{j=1}^{7} \hat{u}_j (\xi_j).
\]

Observe that, if the functions $u_j$ are real valued, the Plancherel identity yields
\[
\int_{\mathbb{R}} \Pi_7^{\eta, M_1, \ldots, M_9} (u_1, \ldots, u_7) u_8 dx = \int_{\mathbb{R}} (\eta \phi_{M_1, \ldots, M_9}) \prod_{j=1}^{8} \hat{u}_j (\xi_j).
\]

We define the resonance function of order 7 for $\xi(7) = (\xi_1, \cdots, \xi_8) \in \Gamma^7$ by
\[
\Omega^7 (\xi(7)) = \sum_{j=1}^{8} \xi_j^3.
\]

Again it is direct to check that
\[
\Omega^7 (\xi(7)) = \Omega^5 (\xi_1, \ldots, \xi_5, - \sum_{i=1}^{5} \xi_i) + \Omega^3 (\xi_6, \xi_7, \xi_8).
\]

The following proposition gives suitable estimates for the pseudo-product $\Pi_{M_1, \ldots, M_9}^7$ when $(M_1, \cdots, M_9) \in \mathcal{M}^{\text{high}}_7$ in the nonresonant case $M_2 M_8 M_6 \not\sim M_7 M_8 M_9$.

**Proposition 3.10.** Assume that $0 < T \leq 1$, $\eta$ is a bounded function and $u_j$ are functions in $Y^0 = X^{-1,1}_t \cap L^\infty_\sigma L^2_\sigma$ with time support in $[0, 2]$ and spatial Fourier support in $I_{N_j}$ for $j = 1, \cdots, 8$. Here, $N_j$ denote nonhomogeneous dyadic numbers.
Assume also that $N_{\text{max}} \geq N \gg 1$, and $(M_1,\ldots,M_9) \in \mathcal{M}_{T}^{\text{high}}$ satisfies the non resonance assumption $M_4 M_5 M_6 \not\sim M_7 M_8 M_9$. Then

\begin{equation}
\left| \int_{\mathbb{R} \times [0,T]} \Pi_{\eta,M_1,\ldots,M_9}(u_1,\ldots,u_7) u_8 \, dxdt \right| \lesssim M_{\text{min}(3)} M_{\text{min}(5)} N^{-1} \prod_{j=1}^{8} \|u_j\|_{Y^0},
\end{equation}

where $M_{\text{min}(3)} = \min(M_1, M_2, M_3)$ and $M_{\text{min}(5)} = \min(M_4, M_5, M_6)$. Moreover, the implicit constant in estimate (3.45) only depends on the $L^\infty$-norm of the function $\eta$.

**Proof.** The proof is similar to the proof of Proposition 3.3. From identities (3.29) and (3.44), the non resonance assumption and the fact that $(M_1,\ldots,M_9) \in \mathcal{M}_{T}^{\text{high}}$ we get

$L_{\text{max}} \gtrsim \max(M_4 M_5 M_6, M_7 M_8 M_9) \gtrsim \min(M_7, M_8, M_9) N_{\text{max}}^2$.

The claim follows from this and estimate (3.33). □

4. **Energy estimates**

The aim of this section is to derive energy estimates for the solutions of (1.1) and the solutions of the equation satisfied by the difference of two solutions of (1.1) (see equation (5.3) below).

In order to simplify the notations in the proofs below, we will instead derive energy estimates on the solutions $u$ of the more general equation

\begin{equation}
\partial_t u + \partial_x^3 u = c_i \partial_x(u_{i,1} u_{i,2} u_{i,3}),
\end{equation}

where for any $i \in \{1, 2, 3\}$, $u_i$ solves

\begin{equation}
\partial_t u_i + \partial_x^3 u_i = c_i \partial_x(u_{i,1} u_{i,2} u_{i,3}).
\end{equation}

Finally we also assume that each $u_{i,j}$ solves

\begin{equation}
\partial_t u_{i,j} + \partial_x^3 u_{i,j} = c_{i,j} \partial_x(u_{i,j,1} u_{i,j,2} u_{i,j,3}),
\end{equation}

for any $(i,j) \in \{1, 2, 3\}^2$. We will sometimes use $u_4$, $u_{4,1}$, $u_{4,2}$, $u_{4,3}$ to denote respectively $u$, $u_1$, $u_2$, $u_3$. Here $c_j, j \in \{1, \cdots, 4\}$ and $c_{i,j}, (i,j) \in \{1,2,3\}^2$ denote real constant. Moreover, we assume that all the functions appearing in (4.1)-(4.3) are real-valued.

Also, we will use the notations defined at the beginning of Section 3.

The main obstruction to estimate $\frac{d}{dt} \|P_N u\|_{L^2}^2$ at this level of regularity is the resonant term $\int \partial_x(P_{+N} u_1 P_{+N} u_2 P_{-N} u_3) P_{-N} u \, dx$ for which the resonance relation (3.8) is not strong enough. In this section we modify the energy by a fourth order term, whose part of the time derivative coming from the linear contribution of (4.1) will cancel out this resonant term. Note however, that we need to add a second modification to the energy to control the part of the time derivative of the first modification coming from the resonant nonlinear contribution of (4.1).
4.1. Definition of the modified energy. Let $N_0 = 2^9$ and $N$ be a nonhomogeneous dyadic number. For $t \geq 0$, we define the modified energy at the dyadic frequency $N$ by

$$(4.4)\quad \mathcal{E}_N(t) = \begin{cases} \frac{1}{2} \| P_N u(\cdot,t) \|_{L^2_2}^2 & \text{for } N \leq N_0 \\ \frac{1}{2} \| P_N u(\cdot,t) \|_{L^2_2}^2 + \alpha \mathcal{E}_{N,med}^3(t) + \gamma \mathcal{E}_N^{3,\text{high}}(t) + \beta \mathcal{E}_N^b(t) & \text{for } N > N_0, \end{cases}$$

where $\alpha$, $\gamma$ and $\beta$ are real constants to be determined later,

$\mathcal{E}_{N,med}^3(t) = \sum_{(M_1,M_2,M_3) \in \mathcal{M}^{med}_{3}} \int_{\Gamma^3} \phi_{M_1,M_2,M_3}(\tilde{\xi}_{(3)}) \phi_2(\xi_4) \frac{\xi_4}{\Omega^3(\tilde{\xi}_{(3)})} \prod_{j=1}^4 \tilde{u}_j(\xi_j),$

$\mathcal{E}_N^{3,\text{high}}(t) = \sum_{N_j, N_{med} \leq N^{1/2}} \sum_{(M_1,M_2,M_3) \in \mathcal{M}^3_{N_j} \setminus N_{med}^3} \int_{\Gamma^3} \phi_{M_1,M_2,M_3}(\tilde{\xi}_{(3)}) \phi_2(\xi_4) \frac{\xi_4}{\Omega^3(\tilde{\xi}_{(3)})} \times \prod_{j=1}^3 (P_{N_j} u_j)^\wedge(\xi_j) (P_N u_4)^\wedge(\xi_4),$

where $\tilde{\xi}_{(3)} = (\xi_1, \xi_2, \xi_3)$ and the dyadic decompositions in $N_j$ are nonhomogeneous,

$\mathcal{E}_N^b(t) = \sum_{(M_1,\ldots,M_6) \in \mathcal{M}^6} \sum_{j=1}^4 c_j \int_{\Gamma^3} \phi_{M_1,\ldots,M_6}(\tilde{\xi}_{(5)}) \phi_2(\xi_4) \frac{\xi_4 \xi_j}{\Omega^3(\tilde{\xi}_{(3)}) \Omega^5(\tilde{\xi}_{(5)})} \times \prod_{k=1 \atop k \neq j}^4 \tilde{u}_k(\xi_k) \prod_{l=1}^3 \tilde{u}_{j,l}(\xi_{j,l}),$

with the convention $\xi_j = - \sum_{k=1 \atop k \neq j}^4 \xi_k = \sum_{l=1}^3 \xi_{j,l}$ and the notation

$\tilde{\xi}_{(5)} = (\tilde{\xi}_{(3)}, \xi_{j,1}, \xi_{j,2}, \xi_{j,3}) \in \Gamma^5$

where $\tilde{\xi}_{(3)}$ is defined by

$\tilde{\xi}_{(3)} = (\xi_2, \xi_3, \xi_4), \; \tilde{\xi}_{(2)} = (\xi_1, \xi_3, \xi_4), \; \tilde{\xi}_{(2)} = (\xi_1, \xi_2, \xi_4), \; \tilde{\xi}_{(3)} = (\xi_1, \xi_2, \xi_3).$

For $T > 0$, we define the modified energy by using a nonhomogeneous dyadic decomposition in spatial frequency

$$(4.5)\quad E_T^s(u) = \sum_N N^{2s} \sup_{t \in [0,T]} \| \mathcal{E}_N(t) \|.$$

By convention, we also set $E_T^0(u) = \sum_N N^{2s} | \mathcal{E}_N(0) |$.

Next, we show that if $s > \frac{1}{4}$, the energy $E_T^s(u)$ is coercive.

**Lemma 4.1** (Coercivity of the modified energy). Let $s > 1/4$, $0 < T \leq 1$ and $u, u_i, u_{i,j} \in L^{\infty}_T H^s_x$ be solutions of (4.1)-(4.4) on $[0,T]$. Then it holds that

$$\|u\|_{L^{\infty}_T H^s_x} \lesssim E_T^s(u) + \sum_{j=1}^4 \| u_j \|_{L^{\infty}_T H^s_x} + \sum_{k=1 \atop k \neq j}^4 \| u_k \|_{L^{\infty}_T H^s_x} \prod_{l=1}^3 \| u_{j,l} \|_{L^{\infty}_T H^s_x}. \quad (4.6)$$
Proof. We infer from (4.5) and the triangle inequality that
\[ \|u\|_{L^8_tH^4_x}^2 \lesssim E_1^4(u) + \sum_{N \geq N_0} N^{2s} \sup_{t \in [0,T]} |E_N^{4,med}(t)| + \sum_{N \geq N_0} N^{2s} \sup_{t \in [0,T]} |E_N^{5,high}(t)| \]
(4.7) \[ + \sum_{N \geq N_0} N^{2s} \sup_{t \in [0,T]} |E_N^5(t)|. \]

We first estimate the contribution of \( E_N^{4,med} \). By symmetry, we can always assume that \( M_1 \leq M_2 \leq M_3 \), so that we have \( N^{-1/2} < M_1 \leq M_2 \ll N \) and \( M_3 \sim N \), since \( (M_1, M_2, M_3) \in M^{med}_3 \). Then, we have thanks to Lemma 3.1
\[ N^{2s} |E_N^{4,med}(t)| \lesssim \sum_{N^{-1/2} < M_1, M_2 \ll N} \frac{N^{2s+1}}{M_1 M_2 M_3} M_1 \prod_{j=1}^4 \|P_{\sim N} u_j(t)\|_{L^2_x} \]
(4.8) \[ \lesssim \prod_{j=1}^4 \|P_{\sim N} u_j(t)\|_{H^s_t}. \]

Now, we deal with the contribution of \( E_N^{5,high} \). Observe from the frequency localization, that we have \( N^{-1} < M_{min} \leq 1 \), \( M_{med} \sim M_{max} \sim N_{max} \sim N \) and \( N_{min} \leq N_{med} \leq N^{3/2} \). Without loss of generality, assume moreover that \( N_1 \leq N_2 \leq N_3 \). Thus it follows from estimate (5.1) that
\[ N^{2s} |E_N^{5,high}(t)| \]
(4.9) \[ \lesssim \sum_{N^{-1/2} < M_{min} \leq 1} \sum_{N_{max} \sim N} \frac{N^{2s+1}}{M_{min} N^2} M_{min} \prod_{j=1}^3 \|P_{N_j} u_j(t)\|_{L^2_x} \|P_{N_3} u_3(t)\|_{H^s_t} \|P_{N_2} u_2(t)\|_{H^s_t}. \]

To estimate the contribution of \( E_N^3(t) \), we notice that for \((M_1, ..., M_6) \in M^{med}_6 \), the integrand in the definition of \( E_N^3 \) vanishes unless \( |\xi_1| \sim ... \sim |\xi_4| \sim N \) and \( |\xi_{j,1}| \sim |\xi_{j,2}| \sim |\xi_{j,3}| \sim N \). Moreover, we assume without loss of generality \( M_1 \leq M_2 \leq M_3 \) and \( M_4 \leq M_5 \leq M_6 \), so that
\[ \frac{\xi_1 \xi_j}{\Omega^3(\xi_j(3)) \Omega^5(\xi_j(6))} \sim \frac{N^2}{M_1 M_2 N M_4 M_5 N} \sim \frac{1}{M_1 M_2 M_4 M_5}. \]

Thus we infer from (4.20) that
\[ N^{2s} |E_N^1(t)| \lesssim \sum_{j=1}^4 \sum_{M_2 > N^{-1/2}} \sum_{M_5 \geq M_2} \frac{N^{2s}}{M_2 M_5} \prod_{k=1}^4 \|P_{\sim N} u_k(t)\|_{L^2_x} \prod_{l=1}^3 \|P_{\sim N} u_{j,l}(t)\|_{L^2_x} \]
(4.10) \[ \lesssim \sum_{j=1}^4 \sum_{k=1}^4 \|P_{\sim N} u_k(t)\|_{H^s_t} \prod_{l=1}^3 \|P_{\sim N} u_{j,l}(t)\|_{H^s_t}, \]
since \( 2s + 1 < 6s \).

Finally, we conclude the proof of (4.10) gathering (4.7)–(4.10) and using the Cauchy-Schwarz inequality. \( \square \)
4.2. Estimates for the modified energy.

**Proposition 4.2.** Let $s > 1/4$, $0 < T \leq 1$ and $u$, $u_i$, $u_{i,j} \in Y_T^s$ be solutions of \( f(t) \) on \([0,T]\). Then we have

\[
E_T^E(u) \leq E_0^E(u) + \sum_{j=1}^4 \|u_j\|_{Y_T^s} + \sum_{j=1}^4 \prod_{k \neq j}^4 \|u_k\|_{Y_T^s} \prod_{l=1}^3 \|u_{j,l}\|_{Y_T^s}
\]

(4.11)

\[\sum_{j=1}^4 \sum_{m=1}^3 \prod_{k \neq j}^4 \|u_k\|_{Y_T^s} \prod_{l=1}^3 \|u_{j,l}\|_{Y_T^s} \prod_{n=1}^3 \|u_{j,m,n}\|_{Y_T^s} .\]

**Proof.** Let $0 < t \leq T \leq 1$. First, assume that $N \leq N_0 = 2^9$. By using the definition of $E_N$ in (4.4), we have

\[
\frac{d}{dt} E_N(t) = c_4 \int_{\mathbb{R}} P_N \partial_x (u_1 u_2 u_3) P_N u \, dx ,
\]

which yields after integrating between 0 and $t$ and applying Hölder’s inequality that

\[
E_N(t) \leq E_N(0) + |c_4| \int_{\mathbb{R}_t} P_N \partial_x (u_1 u_2 u_3) P_N u \bigg| \leq E_N(0) + \prod_{i=1}^4 \|u_i\|_{L_T^\infty L_x^4} \lesssim E_N(0) + \prod_{i=1}^4 \|u_i\|_{L_T^\infty H_x^4/4}
\]

where the notation $\mathbb{R}_t = \mathbb{R} \times [0,t]$ defined at the beginning of Section 3 has been used. Thus, we deduce after taking the supreme over $t \in [0,T]$ and summing over $N \leq N_0$ that

(4.12) \[
\sum_{N \leq N_0} N^{2s} \sup_{t \in [0,T]} |E_N(t)| \lesssim \sum_{N \leq N_0} N^{2s} |E_N(0)| + \prod_{j=1}^4 \|u_j\|_{Y_T^{1/4}} .
\]

Next, we turn to the case where $N \geq N_0$. As above, we differentiate $E_N$ with respect to time and then integrate between 0 and $t$ to get

\[
N^{2s} E_N(t) = N^{2s} E_N(0) + c_4 N^{2s} \int_{\mathbb{R}} P_N \partial_x (u_1 u_2 u_3) P_N u + \alpha N^{2s} \int_0^t \frac{d}{dt} E_N^{\text{med}}(t') dt' + \gamma N^{2s} \int_0^t \frac{d}{dt} E_N^{\text{high}}(t') dt' + \beta N^{2s} \int_0^t \frac{d}{dt} E_N(t') dt'
\]

(4.13) \[
=: N^{2s} E_N(0) + c_4 I_N + \alpha J_N + \gamma L_N + \beta K_N .
\]

We rewrite $I_N$ in Fourier variable and get

\[
I_N = N^{2s} \int_{\Gamma_3^1} (-i \xi \cdot \phi_N^2 (\xi) \tilde{u}_1 (\xi) \tilde{u}_2 (\xi) \tilde{u}_3 (\xi) \tilde{u}_4 (\xi) = \sum_{(M_1, M_2, M_3) \in D^3} N^{2s} \int_{\Gamma_3^1} (-i \xi \cdot \phi_{M_1, M_2, M_3} (\xi) \phi_N^2 (\xi) \prod_{j=1}^4 \tilde{u}_j (\xi) .
\]
Next we decompose $I_N$ as

$$I_N = N^{2s} \left( \sum_{M^\text{low}_3} + \sum_{M^\text{med}_3} + \sum_{k=1}^{3} \sum_{M^\text{high,k}_3} \right) \int_{\mathbb{R}^3} (-i\xi) \phi_{M_1,M_2,M_3}^{\xi_3} \tilde{\phi}_N^{\xi_4} \prod_{j=1}^{4} \tilde{u}_j(\xi_j)$$

(4.14)

$$=: I_N^{\text{low}} + I_N^{\text{med}} + \sum_{k=1}^{3} I_N^{\text{high,k}},$$

by using the notations in Section 3.

**Estimate for $I_N^{\text{low}}$**. Thanks to Lemma 3.2 the integral in $I_N^{\text{low}}$ is non trivial for $|\xi_1| \sim |\xi_2| \sim |\xi_3| \sim |\xi_4| \sim N$ and $M_{\text{min}} \leq N^{-1/2}$. Therefore we get from Lemma 3.1 that

$$|I_N^{\text{low}}| \lesssim \sum_{\substack{M_{\text{min}} \leq N^{-\frac{1}{2}} \\text{in N}}} N^{2s+1} M_{\text{min}} \prod_{j=1}^{4} \|P_{\sim N} u_j\|_{L^2_T L^2_x} \lesssim \prod_{j=1}^{4} \|P_{\sim N} u_j\|_{L^2_T H^s_x},$$

since $(2s + \frac{1}{2}) < 4s$. This leads to

(4.15)

$$\sum_{N \geq N_0} |I_N^{\text{low}}| \lesssim \prod_{j=1}^{4} \|u_j\|_{Y^s_T}.$$

**Estimate for $I_N^{\text{high,1}}$**. We perform nonhomogeneous dyadic decompositions $u_j = \sum_{N_j} P_{N_j} u_j$ for $j = 1, 2, 3$. We assume without loss of generality that $N_1 = \max(N_1, N_2, N_3)$. Recall that this ensures that $M_{\text{max}} \sim N_1$. Then we apply Lemma 3.1 on the sum over $M_{\text{med}}$ and use the discrete Young’s inequality to get

$$|I_N^{\text{high,1}}| \lesssim \sum_{M_{\text{min}} \leq N^{-1}} N^{2s+1} M_{\text{min}} \prod_{N_1 \geq N, N_2, N_3} \prod_{j=2}^{3} \|P_{N_j} u_j\|_{L^2_T L^2_x} \|P_{N_1} u_1\|_{L^2_T H^s_x} \|P_{N} u_4\|_{L^2_T H^s_x}$$

$$\lesssim \sum_{N_1 \geq N} \left(\frac{N}{N_1}\right)^s \|P_{N_1} u_1\|_{L^2_T H^s_x} \|P_{N} u_4\|_{L^2_T H^s_x} \|u_2\|_{L^2_T H^0_x} \|u_4\|_{L^2_T H^0_x}$$

(4.16)

$$\lesssim \delta_N \|P_{N} u_4\|_{L^2_T H^s_x} \prod_{i=1}^{3} \|u_i\|_{L^2_T H^s_x},$$

with $\{\delta_N\} \in l^2(\mathbb{N})$. Summing over $N$ this leads to

(4.17)

$$\sum_{N \geq N_0} |I_N^{\text{high,1}}| \lesssim \prod_{j=1}^{4} \|u_j\|_{Y^s_T}.$$

**Estimate for $I_N^{\text{high,3}}$**. For $j = 1, \cdots, 4$, let $\tilde{u}_j = \rho_T(u_j)$ be the extension of $u_j$ to $\mathbb{R}^2$ defined in (4.1). Now, we define $u_{N_j} = P_{N_j} \tilde{u}_j$ and perform nonhomogeneous dyadic decompositions in $N_j$, so that $I_N^{\text{high,3}}$ can be rewritten as

$$I_N^{\text{high,3}} = N^{2s+1} \sum_{N_j, N_4 \sim N} \sum_{(M_1,M_2,M_3) \in N_3^{\text{high,3}}} \int_{\mathbb{R}^3} \Pi_{\eta,M_1,M_2,M_3}^{3} (u_{N_1},u_{N_2},u_{N_3}) u_{N_4},$$

where $\Pi_{\eta,M_1,M_2,M_3}^{3}$ are the usual projections for $M_1, M_2, M_3$. Now we estimate $I_N^{\text{high,3}}$.
with \( \eta(\xi_1, \xi_2, \xi_3) = \phi_{M_1 M_2 M_3}(\xi_1, \xi_2, \xi_3)\phi_N^3(\xi_4) \frac{d\gamma}{\gamma} \in L^\infty(\Gamma^3) \). Thus, it follows from (3.9) that

\[
|I_N^{\text{high}, 3}| \lesssim N^{2s} \sum_{N_j, N_4 \sim N} \frac{N}{N_{\text{max}}} \sum_{1 < M_{\text{min}} \leq M_{\text{med}} \leq M_{\text{max}} \leq N_{\text{max}}} \|u_{N_1}\|_{Y^0}\|u_{N_2}\|_{Y^0}\|u_{N_3}\|_{Y^0}\|u_{N_4}\|_{Y^0}.
\]

Proceeding as in (4.16) (here we sum over \( M_{\text{min}} \) by using that \( M_{\text{min}} \leq N_{\text{med}} \)) we get

\[
(4.18) \quad \sum_{N \geq N_0} |I_N^{\text{high}, 3}| \lesssim \prod_{j=1}^4 \|u_j\|_{Y^2}.
\]

**Estimate for** \( c_4 I_N^{\text{high}, 2} + \gamma L_N \). As above, we perform the nonhomogeneous dyadic decompositions \( u_j = \sum_{N_j} P_{N_j} u_j \) for \( j = 1, 2, 3 \) and use the notation \( N_4 = N \) to rewrite \( I_N^{\text{high}, 2} \) as

\[
N^{2s} \sum_{N_j} \sum_{M_{\text{high}}, 2} \int_{\Gamma^3} (-i\xi_4)\phi_{M_1 M_2 M_3}(\xi_3)\phi_N(\xi_4) \prod_{j=1}^4 \left( P_{N_j} u_j \right)^\gamma(\xi_j).
\]

First, we deal with the case \( N_{\text{med}} > N^{1/4} \). Then arguing as for the contribution \( I_N^{\text{high}, 3} \), we conclude that

\[
(4.19) \quad \sum_{N \geq N_0} |I_N^{\text{high}, 2}| \lesssim \prod_{j=1}^4 \|u_j\|_{Y^2}.
\]

Now, we treat the case \( N_{\text{med}} \leq N^{1/4} \). Using (4.11), (4.12), we can rewrite \( \frac{d}{dt} c_N^{3, \text{high}} \) as the sum of the linear contribution

\[
\sum_{N_j, N_{\text{med}} \leq N^{1/4}} \sum_{M_{\text{high}}, 3} \int_{\Gamma^3} \phi_{M_1 M_2 M_3}(\xi_3)\phi_N(\xi_4) \frac{i\xi_4(\xi_4^3 + \xi_2^3 + \xi_3^3 + \xi_4^3)}{\Omega^3(\xi_3)} \prod_{j=1}^4 \left( P_{N_j} u_j \right)^\gamma(\xi_j)
\]

and the nonlinear contribution

\[
(4.20) \quad \sum_{j=1}^4 c_j \sum_{N_j, N_{\text{med}} \leq N^{1/4}} \sum_{M_{\text{high}}, 3} \int_{\Gamma^3} \phi_{M_1 M_2 M_3}(\xi_3)\phi_N(\xi_4) \frac{\xi_4}{\Omega^3(\xi_3)} \times \prod_{k=1, k \neq j}^4 \left( P_{N_k} u_k \right)^\gamma(\xi_k) \partial_x \partial_y P_{N_j}(u_{j,1} u_{j,2} u_{j,3})(\xi_j).
\]

Using the resonance relation (3.8), we see by choosing \( \gamma = c_4 \) that \( I_N^{\text{high}, 2} \) is canceled out by the linear contribution of \( \frac{d}{dt} c_N^{3, \text{high}} \). Hence,

\[
(4.21) \quad c_4 I_N^{\text{high}, 2} + \gamma L_N = c_4 \sum_{j=1}^4 A_N^j,
\]
where
\[ A_N^1 = iN^{2\sigma} \sum_{N_j, N_k, N_{med} \leq N^3} \sum_{M_3^{high, 2}} \int_{I_1} \phi_{M_1, M_2, M_3}(\xi_j) \phi_N(\xi_k) \frac{\xi_l \xi_j}{\Omega^3(\xi_j)} \times \prod_{k=1}^{3} (P_{N_k} u_k)^{3} \prod_{l=1}^{3} \hat{u}_{j,l} \left( \xi_{j,l} \right). \]

It thus remains to treat the terms \( A_N^1 \) corresponding to the nonlinear contribution \( \frac{d}{dt} \xi_{3, high}^N \). Observe that
\[ (4.22) \quad N^{-1} \leq M_{min} \leq 1, \quad M_{med} \sim M_{max} \sim N \quad \text{and} \quad N_{min} \leq N_{med} \leq N^{\frac{1}{2}} \]
in this case. Moreover, without loss of generality, we can always assume that \( N_1 \leq N_2 \leq N_3 \).

**Estimates for \( A_N^1 \) and \( A_N^2 \).** We only estimate \( A_N^2 \), since the estimate for \( A_N^1 \) follows similarly. We get from (3.11) that
\[ |A_N^2| \leq \sum_{N_1 \leq N_2 \leq N^{\frac{1}{2}}} \sum_{N_3^3 \leq \min \leq 1} \frac{N^{2\sigma + 1}}{M_{min}^N N^2} \times \| \left( P_{N_1} u_1 \right) \|_{L^p_t L^q_x} \| \left( P_{N_2} \left( u_{2,1} u_{2,2} u_{2,3} \right) \right) \|_{L^p_t L^q_x} \| \left( P_{N_3} u_3 \right) \|_{L^p_t L^q_x} \| \left( P_{N} u_4 \right) \|_{L^p_t L^q_x}. \]

Moreover thanks to Berstein and Hölder’s inequalities
\[ \| \left( P_{N_2} \left( u_{2,1} u_{2,2} u_{2,3} \right) \right) \|_{L^p_t L^q_x} \lesssim N^{\frac{1}{4}} \| u_{2,1} u_{2,2} u_{2,3} \|_{L^p_t L^q_x} \lesssim N^{\frac{3}{10}} \prod_{l=1}^{3} \| u_{2,l} \|_{L^p_t L^q_x}, \]
which implies after summing over \( N \) and using the Sobolev embedding \( H^{\frac{1}{4}}(\mathbb{R}) \hookrightarrow L^3(\mathbb{R}) \) that
\[ (4.23) \quad \sum_{N \geq N_0} |A_N^2| \leq \prod_{k=1}^{4} \| u_k \|_{Y^2} \prod_{l=1}^{3} \| u_{2,l} \|_{Y^2}. \]

**Estimates for \( A_N^3 \) and \( A_N^4 \).** We only estimate \( A_N^3 \), since the estimate for \( A_N^1 \) follows similarly. By using the notations in (3.25), we decompose \( A_N^3 \) as
\[ A_N^3 = iN^{2\sigma} \left( \sum_{M_3^{low}} + \sum_{M_3^{med, 1}} + \sum_{M_3^{med, 2}} + \sum_{M_3^{high}} \right) \sum_{N_1 \leq N_2 \leq N^{3}} \sum_{N_3, N_4} \int_{I_1} \phi_{M_1, \cdots, M_6}(\xi_j) \phi_N(\xi_k) \frac{\xi_l \xi_j}{\Omega^3(\xi_j)} \times \prod_{k=1}^{4} (P_{N_k} u_k)^{3} \prod_{l=1}^{3} (P_{N_3} u_{3,l})^{3} \left( \xi_{j,l} \right). \]

We observe that \( N_{max}(5) = \max \{ N_3, N_2, N_3, 3 \} \gtrsim N \) since \( N_3 \sim N \). Without loss of generality, we can assume \( M_4 \leq M_5 \leq M_6 \). Note that this forces \( M_6 \sim N_{max}(5) \).
Estimates for \( A_{N}^{3,\text{low}} \). We apply (3.20) on the sum over \( M_{j} \). On account of (4.22), we obtain that

\[
|A_{N}^{3,\text{low}}| \lesssim \sum_{N^{-1} < M_{min} \leq 1} \sum_{N_{1} \leq N_{2} \leq N} \sum_{N_{3} \leq N} \frac{N^{2s+2}M_{min}M_{4}}{M_{min}N^{2M_{min}}}M_{min}M_{4}
\]

\[\times \prod_{k=1}^{4} \|P_{N_{k}}u_{k}\|_{L_{t}^{\infty}L_{x}^{2}} \prod_{l=1}^{3} \|P_{N_{3,l}}u_{3,l}\|_{L_{t}^{\infty}L_{x}^{2}}.\]  

(4.25)

Therefore, we deduce after proceeding as in (4.16) and summing over \( N \) that

\[
\sum_{N \geq N_{0}} \sum_{N_{1} \leq N_{2} \leq N} \sum_{N_{3} \leq N} \|u_{3,l}\|_{Y_{Lt}^{2}} \|u_{3,l}\|_{Y_{Lt}^{2}}.\]  

(4.26)

Estimates for \( A_{N}^{3,\text{med},1} \). Let \( N_{max}(5) \geq N_{med}(5) \geq N_{min}(5) \) denote the maximum, sub-maximum and minimum of \( \{N_{3,1}, N_{3,2}, N_{3,3}\} \). Thanks to Lemma 3.2, we know that \( N_{max}(5) \sim N_{med}(5) \sim N_{min}(5) \sim N \) and \( M_{6} \sim N \). Thus, we deduce from (3.20) and (4.22) that

\[
|A_{N}^{3,\text{med},1}| \lesssim \sum_{N^{-1} < M_{min} \leq 1} \sum_{N_{1} \leq N_{2} \leq N} \sum_{N_{3} \leq N} \frac{N^{2}M_{min}M_{4}}{M_{min}N^{2M_{min}}}M_{min}M_{4}N^{-2s}
\]

\[\times 2 \prod_{k=1}^{2} \|P_{N_{k}}u_{k}\|_{L_{t}^{\infty}L_{x}^{2}} \|P_{N}u_{4}\|_{L_{t}^{\infty}H_{x}^{2}} \prod_{l=1}^{3} \|P_{\sim N}u_{3,l}\|_{L_{t}^{\infty}H_{x}^{2}},\]  

which implies that

\[
\sum_{N \geq N_{0}} |A_{N}^{3,\text{med},1}| \lesssim \prod_{k=1}^{4} \|u_{k}\|_{Y_{Lt}^{2}} \prod_{l=1}^{3} \|u_{3,l}\|_{Y_{Lt}^{2}},\]  

(4.27)

since \( s > \frac{1}{4} \).

Estimates for \( A_{N}^{3,\text{med},2} \). Again, Lemma 3.2 implies that \( N_{max}(5) \sim N_{med}(5) \sim N_{min}(5) \sim M_{6} \sim N \). For \( 1 \leq k \leq 4, k \neq 3, 1 \leq l \leq 3 \) let \( \tilde{u}_{k} = \rho_{T}(u_{k}) \) and \( \tilde{u}_{3,l} \) be the extensions of \( u_{k} \) and \( u_{3,l} = \rho_{T}(u_{3,l}) \) to \( \mathbb{R}^{3} \) defined in (2.7). We define \( u_{N_{k}} = P_{N_{k}} \tilde{u}_{k}, u_{N_{j,l}} = P_{N_{j,l}} \tilde{u}_{j,l} \). We deduce from (3.32) that

\[
|A_{N}^{3,\text{med},2}| \lesssim \sum_{N^{-1} < M_{min} \leq 1} \sum_{N_{1} \leq N_{2} \leq N} \sum_{N_{3,1} \sim N} \frac{N^{2s+2}M_{min}^{1}M_{min}M_{4}}{M_{min}N^{2M_{min}}}M_{min}M_{4}N^{-\frac{1}{4}}
\]

\[\times 2 \prod_{k=1}^{2} \|u_{N_{k}}\|_{Y^{0}} \|u_{N_{4}}\|_{Y^{0}} \prod_{l=1}^{3} \|u_{N_{3,l}}\|_{Y^{0}},\]  

(4.28)

Therefore, we conclude from (2.8) that

\[
\sum_{N \geq N_{0}} |A_{N}^{3,\text{med},2}| \lesssim \prod_{k=1}^{4} \|u_{k}\|_{Y_{Lt}^{2}} \prod_{l=1}^{3} \|u_{3,l}\|_{Y_{Lt}^{2}}.\]
Estimates for $A_N^{3,\text{high}}$. We argue as above by using (3.31) instead of (3.32). We obtain that
\[
|A_N^{3,\text{high}}| \lesssim \sum_{3 \leq M_4 \leq N} \sum_{N \leq M_5 \leq N_{\max(5)}} \sum_{N_1 \leq M_6 \leq N_{\min(5)}} \frac{N^{2s+2}}{M_{\min} N^2} M_{\min} N^{-1} \times \prod_{k=1}^{2} \|u_{N_k}\|_{Y^0} \|u_{N_4}\|_{Y^0} \prod_{l=1}^{3} \|u_{N_{3,l}}\|_{Y^0},
\]
which leads to
\[
(4.29) \quad \sum_{N \geq N_0} |A_N^{3,\text{high}}| \lesssim \prod_{k=1}^{4} \|u_k\|_{Y^2} \prod_{l=1}^{3} \|u_{3,l}\|_{Y^2},
\]
in this case. Note we can use the factor $N^{-s}_{\max}$ to sum over $M_{\min}$, $M_4$, $M_5$ and $M_6$ here.

Estimate for $c_4 I_N^{\text{med}} + \alpha J_N + \beta K_N$. Using (4.11)-4.12, we can rewrite $\frac{d}{dt} \xi_N^{3,\text{med}}$ as
\[
\sum_{M_2^{\text{med}}} \int \phi_{M_1,M_2,M_3}(\xi_3) \frac{\xi_3^2 \xi_4^2}{\Omega^3(\xi_3)} \prod_{j=1}^{4} \tilde{u}_k(\xi_j)
\]
\[
+ \sum_{j=1}^{4} c_j \sum_{M_2^{\text{med}}} \int \phi_{M_1,M_2,M_3}(\xi_3) \frac{\xi_3^2 \xi_4^2}{\Omega^3(\xi_3)} \prod_{k=1}^{4} \tilde{u}_k(\xi_k) \partial_x (u_{j,1} u_{j,2} u_{j,3})(\xi_j).
\]

Using (3.3), we see by choosing $\alpha = c_4$ that $I_N^{\text{med}}$ is canceled out by the first term of the above expression. Hence,
\[
(4.30) \quad c_4 I_N^{\text{med}} + \alpha J_N = c_4 \sum_{j=1}^{4} c_j J_N^j,
\]
where, for $j = 1, \cdots, 4$,
\[
J_N^j = i N^{2s} \sum_{M_2^{\text{med}}} \int \phi_{M_1,M_2,M_3}(\xi_3) \frac{\xi_4 \xi_j}{\Omega^3(\xi_3)} \prod_{k=1}^{4} \tilde{u}_k(\xi_k) \prod_{l=1}^{3} \tilde{u}_{j,l}(\xi_{j,l}),
\]
with the convention $\xi_j = -\sum_{k=1}^{4} \xi_k = \sum_{k \neq j} \xi_{j,l}$ and the notation $\xi_3 = (\xi_1, \xi_2, \xi_3)$.

Now, we define $\xi_j(3)$, for $j = 1, 2, 3, 4$ as follows:
\[
\xi_1(3) = (\xi_2, \xi_3, \xi_4), \quad \xi_2(3) = (\xi_1, \xi_3, \xi_4), \quad \xi_3(3) = (\xi_1, \xi_2, \xi_4), \quad \xi_4(3) = (\xi_1, \xi_2, \xi_3).
\]

With this notation in hand and by using the symmetries of the functions $\sum_{M_2^{\text{med}}} \phi_{M_1,M_2,M_3}$ and $\Omega^3$, we obtain that
\[
J_N^j = i N^{2s} \sum_{M_2^{\text{med}}} \int \phi_{M_1,M_2,M_3}(\xi_3) \frac{\xi_4 \xi_j}{\Omega^3(\xi_3)} \prod_{k=1}^{4} \tilde{u}_k(\xi_k) \prod_{l=1}^{3} \tilde{u}_{j,l}(\xi_{j,l}).
\]
Moreover, observe from the definition of $M_{med}^{med}$ in Section 3 that

$$|\xi_1| \sim |\xi_2| \sim |\xi_3| \sim |\xi_4| \sim N \quad \text{and} \quad \left| \frac{\xi_j \xi_4}{\Omega^3(\tilde{\xi}_j(3))} \right| \sim \frac{N}{M_{min(3)} M_{med(3)}}$$

on the integration domain of $J_N^j$. Here $M_{max(3)} \geq M_{med(3)} \geq M_{min(3)}$ denote the maximum, sub-maximum and minimum of $\{M_1, M_2, M_3\}$.

Since $\max(\xi_{j,1} + \xi_{j,2}, |\xi_{j,1} + \xi_{j,3}|, |\xi_{j,2} + \xi_{j,3}|) \gtrsim N$ on the integration domain of $J_N^j$, we may decompose $\sum_j c_j J_N^j$ as

$$\sum_{j=1}^4 c_j J_N^j = i N^{2s} \left( \sum_{M_{low}^{med}} + \sum_{M_{high,1}^{med}} + \sum_{M_{high,2}^{med}} \right) \sum_{j=1}^4 c_j$$

$$\times \int \prod_{l=1}^s \phi_{M_1, \ldots, M_6}(\xi_j(5)) \theta_N^2(\xi_4) \prod_{k \neq j} \hat{u}_k(\xi_k) \prod_{l=1}^3 \hat{u}_{j,l}(\xi_{j,l})$$

(4.31)

$$:= J_N^{low} + J_N^{med} + J_N^{high,1} + J_N^{high,2},$$

where $\bar{\xi}_j(5) = (\xi_{j,1}, \xi_{j,2}, \xi_{j,3}) \in \Gamma^5$.

Moreover, we may assume by symmetry that $M_1 \leq M_2 \leq M_3$ and $M_4 \leq M_5 \leq M_6$.

**Estimate for $J_N^{low}$.** In the region $M_{low}^{med}$, we have that $M_4 \leq M_2$. Moreover thanks to Lemma 3.2, the integral in $J_N^{low}$ is non trivial for $|\xi_1| \sim \cdots \sim |\xi_4| \sim N$, $|\xi_{j,1}| \sim |\xi_{j,2}| \sim |\xi_{j,3}| \sim N$ and $M_3 \sim M_6 \sim N$. Therefore by using (3.20), we can bound $|J_N^{low}|$ by

$$\sum_{j=1}^4 \sum_{N_{\geq N_0}^{\geq N_0}} \sum_{M_{low}^{med}} \sum_{M_{4,5}^{med}} N^{2s} M_1 M_2 N \prod_{k=1}^{N_{\geq N_0}} \prod_{k \neq j} \left\| P_{\sim N} u_k \right\|_{L_2^2 L_2^2} \prod_{l=1}^3 \left\| P_{\sim N} u_{j,l} \right\|_{L_2^2 H_2^s},$$

$$\lesssim \sum_{j=1}^4 \prod_{k=1}^{N_{\geq N_0}} \prod_{k \neq j} \left\| P_{\sim N} u_k \right\|_{L_2^2 H_2^s} \prod_{l=1}^3 \left\| u_{j,l} \right\|_{Y_{2s}}.$$

(4.32)

since $s > 1/4$. Thus, we deduce that

**Estimate for $J_N^{high,1}$.** From Lemma 3.2, the integral in $J_N^{high,1}$ is non trivial for $|\xi_1| \sim \cdots \sim |\xi_4| \sim N$, $M_3 \sim N$, $M_{max(5)} = \max\{N_{j,1}, N_{j,2}, N_{j,3}\} \gtrsim N$, $M_4 \leq N^{-1}$ and $M_5 \sim M_6 \sim N_{max(5)}$. Therefore by using (3.20), we can bound $|J_N^{high,1}|$ by

$$\sum_{j=1}^4 \sum_{N_{\geq N_0}^{\geq N_0}} \sum_{M_{low}^{med}} \sum_{M_{4,5}^{med}} N^{2s+1} M_1 M_2 \prod_{k=1}^{N_{\geq N_0}} \prod_{k \neq j} \left\| P_{\sim N} u_k \right\|_{L_2^2 L_2^2} \prod_{l=1}^3 \left\| P_{\sim N} u_{j,l} \right\|_{L_2^2 L_2^2},$$

$$\lesssim \sum_{j=1}^4 \prod_{k=1}^{N_{\geq N_0}} \prod_{k \neq j} \left\| P_{\sim N} u_k \right\|_{L_2^2 H_2^s} \prod_{l=1}^3 \left\| u_{j,l} \right\|_{Y_{2s}}.$$
since $s > 1/4$. This leads to

$$
\sum_{N \geq N_0} |J_N^{\text{high,1}}| \lesssim \sum_{j=1}^{4} \prod_{k=1}^{4} \|u_k\|_{Y^+_T} \prod_{l=1}^{3} \|u_{j,l}\|_{Y^+_T}.
$$

**Estimate for $J_N^{\text{high,2}}$.** For $1 \leq k \leq 4$, and $1 \leq l \leq 3$ let $\tilde{u}_k = \rho_T(u_k)$ and $\tilde{u}_{j,l}$ be the extensions of $u_k$ and $u_{j,l} = \rho_T(u_{j,l})$ to $\mathbb{R}^2$ defined in (2.7). We define $u_{N_k} = P_{N_k} \tilde{u}_k$, $u_{N,l} = P_{N,l} \tilde{u}_{j,l}$ and perform nonhomogeneous dyadic decompositions in $N_k$ and $N_{j,l}$.

We first estimate $J_N^{\text{high,2}}$ in the resonant case $M_1 M_2 M_3 \sim M_4 M_5 M_6$. We assume to simplify the notations that $M_1 \leq M_2 \leq M_3$ and $M_4 \leq M_5 \leq M_6$. Since we are in $M_0^{\text{high}}$, we have that $M_5, M_6 \gg N$ and $M_1, M_2 \ll N$ which yields

$$\quad M_3 \sim N \quad \text{and} \quad M_4 \sim \frac{M_1 M_2 N}{M_5 M_6} \ll N.
$$

This forces $N_{j,l} \sim N$ and it follows from (3.21) that

$$
|J_N^{\text{high,2}}| \lesssim \sum_{j=1}^{4} \sum_{N_{j,l}^{\text{high}}} N^{2s+1} M_1 M_2 M_4^{\frac{1}{2}} N_{j,l}^{\frac{1}{2}} \prod_{k=1}^{4} \|P_{\sim N} \tilde{u}_k\|_{L^\infty_T L^2_x} \prod_{l=1}^{3} \|u_{N,l}\|_{L^\infty_T L^2_x}
$$

$$
\lesssim \sum_{j=1}^{4} \sum_{N^{-\frac{1}{2}} \leq M_1 \leq M_2 \ll N} N^{s+\frac{1}{2}} \left(\frac{M_1 M_2}{M_2}\right)^{\frac{1}{2}} \prod_{k=1}^{4} \|P_{\sim N} \tilde{u}_k\|_{L^\infty_T L^2_x} \prod_{l=1}^{3} \|u_{j,l}\|_{L^\infty_T H^s}.
$$

Summing over $N^{-1/2} \leq M_1, M_2 \ll N$ and $N \geq N_0$ and using the assumption $s > \frac{1}{4}$, we get

$$
\sum_{N \geq N_0} |J_N^{\text{high,2}}| \lesssim \sum_{j=1}^{4} \prod_{k=1}^{4} \|u_k\|_{Y^+_T} \prod_{l=1}^{3} \|u_{j,l}\|_{Y^+_T},
$$

in the resonant case.

Thanks to (3.30), we easily estimate $J_N^{\text{high,2}}$ in the non resonant case $M_1 M_2 M_3 \not\sim M_4 M_5 M_6$ by

$$
|J_N^{\text{high,2}}| \lesssim \sum_{j=1}^{4} \sum_{N^{-\frac{1}{2}} \leq M_1 \leq M_2 \ll N} \sum_{N = M_4 \leq N} \sum_{N_{j,l}} \sum_{N_{j,l}^{\text{max}}} N^{2s+1} M_1 M_2^{\frac{1}{2}} M_4^{\frac{1}{2}} \prod_{k=1}^{4} \|P_{\sim N} \tilde{u}_k\|_{Y^0} \prod_{l=1}^{3} \|u_{N,l}\|_{Y^0}.
$$

Recalling that $\max\{N_{j,1}, N_{j,2}, N_{j,3}\} \gg N$, we conclude after summing over $N$ that (4.34) also holds, for $s > 1/4$, in the non resonant case.
Estimate for $\alpha J_{N}^{med} + \beta K_{N}$. Using equations \([1.1] - [1.3]\) and the resonance relation \([3.28]\), we can rewrite $N^{2s} \int_{0}^{t} e^{\xi N} dt$ as
\[
N^{2s} \sum_{N_{0}^{med}} \sum_{j=1}^{4} c_{j} \int_{\Gamma_{j}^{2}} \phi_{M_{1}, \ldots, M_{6}}(\tilde{\xi}(3)) \phi_{N}^{2}(\xi) \frac{i \xi \xi j}{\Omega_{3}(\xi(3))} \prod_{k=1}^{3} \hat{u}_{k}(\xi_{k}) \prod_{l=1}^{3} \hat{u}_{j,l}(\xi_{j,l})
\]
\[
+ N^{2s} \sum_{N_{0}^{med}} \sum_{j=1}^{4} c_{j} \sum_{m=1}^{4} c_{m} \int_{\Gamma_{j}^{2}} \phi_{M_{1}, \ldots, M_{6}}(\tilde{\xi}(3)) \phi_{N}^{2}(\xi) \frac{\xi \xi j}{\Omega_{3}(\xi(3))} \Omega^{3}(\xi(5)) \frac{\xi \xi j}{\Omega^{3}(\xi(3))} \Omega^{3}(\xi(5))
\]
\[
\times \prod_{k=1}^{4} \hat{u}_{k}(\xi_{k}) \prod_{k \neq j} \hat{u}_{j,l}(\xi_{j,l}) \prod_{l=1}^{3} \hat{u}_{j,l}(\xi_{j,l}) \prod_{m=1}^{3} \hat{u}_{j,m}(\xi_{j,m}) (\xi_{j,m})
\]
\[
:= K_{N}^{2} + K_{N}^{3} + K_{N}^{4}.
\]
By choosing $\beta = -\alpha$, we have that
\[
\alpha J_{N}^{med} + \beta K_{N} = \beta(K_{N}^{2} + K_{N}^{3}) .
\]
For the sake of simplicity, we will only consider the contribution of $K_{N}^{3}$ corresponding to a fixed $(j, m) \in \{1, 2, 3, 4\} \times \{1, 2, 3\}$, since the other contributions on the right-hand side of \([4.35]\) can be treated similarly.

Thus, for $(j, m)$ fixed, we need to bound
\[
\tilde{K}_{N} := i N^{2s} \sum_{N_{0}^{med}} \int_{\Gamma_{j}^{2}} \sigma(\tilde{\xi}(3)) \prod_{k=1}^{4} \hat{u}_{k}(\xi_{k}) \prod_{l=1}^{3} \hat{u}_{j,l}(\xi_{j,l}) \prod_{n=1}^{3} \hat{u}_{j,n}(\xi_{j,m,n})(\xi_{j,m,n}),
\]
with the conventions $\xi_{j} = -\sum_{k=1}^{4} \xi_{k}$ and $\xi_{j,m} = \sum_{n=1}^{3} \xi_{j,m,n}$ and where
\[
\sigma(\tilde{\xi}(3)) = \phi_{M_{1}, \ldots, M_{6}}(\tilde{\xi}(3)) \phi_{N}^{2}(\xi(3)) \frac{\xi \xi j}{\Omega_{3}(\xi(3))} \Omega^{3}(\xi(5)).
\]
Now, we define $\tilde{\xi}_{j,m,n} \in \Gamma_{j}^{2}$ as follows:
\[
\tilde{\xi}_{j,1,n} = (\tilde{\xi}_{j,1,n}, \tilde{\xi}_{j,2,n}, \tilde{\xi}_{j,3,n}, \tilde{\xi}_{j,1,1,n}, \tilde{\xi}_{j,1,2,n}, \tilde{\xi}_{j,1,3,n}) ,
\]
\[
\tilde{\xi}_{j,2,n} = (\tilde{\xi}_{j,2,n}, \tilde{\xi}_{j,1,n}, \tilde{\xi}_{j,3,n}, \tilde{\xi}_{j,2,1,n}, \tilde{\xi}_{j,2,2,n}, \tilde{\xi}_{j,2,3,n}) ,
\]
\[
\tilde{\xi}_{j,3,n} = (\tilde{\xi}_{j,3,n}, \tilde{\xi}_{j,1,n}, \tilde{\xi}_{j,2,n}, \tilde{\xi}_{j,3,1,n}, \tilde{\xi}_{j,3,2,n}, \tilde{\xi}_{j,3,3,n}) .
\]
Observe from Lemma \([3.2]\) that the integrand is non trivial for
\[
|\xi_{1}| \sim \cdots \sim |\xi_{4}| \sim |\xi_{j,1}| \sim |\xi_{j,2}| \sim |\xi_{j,3}| \sim |\xi_{j,m,1} + \xi_{j,m,2} + \xi_{j,m,3}| \sim N .
\]
Hence,
\[ |\sigma(\tilde{\xi}_j(5))| \lesssim \frac{N^3}{M_1 M_2 N \cdot M_4 M_5 N} \sim \frac{N}{M_1 M_2 M_4 M_5}. \]

Now we decompose \( \tilde{K}_N \) as
\[ \tilde{K}_N = i N^{2s} \left( \sum_{M_i^{low}} + \sum_{M_i^{high}} \right) \]
\[ \times \int \tilde{\sigma}(\tilde{\xi}_{j,m(7)}) \prod_{k=1}^{4} \tilde{u}_k(\xi_k) \prod_{l=1}^{3} \tilde{u}_{j,l}(\xi_{j,l}) \prod_{n=1}^{3} \tilde{u}_{j,n}(\xi_{j,m,n})(\xi_{j,m,n}) \]
(4.36)
\[ = \tilde{K}_N^{low} + \tilde{K}_N^{high}, \]
where
\[ \tilde{\sigma}(\tilde{\xi}_{j,m(7)}) = \phi_{M_1,...,M_6}(\tilde{\xi}_{j,m(7)}) \sigma(\tilde{\xi}_j(5)). \]

Moreover, we may assume without loss of generality that \( M_1 \leq M_2 \leq M_3, M_4 \leq M_5 \leq M_6 \) and \( M_7 \leq M_8 \leq M_9 \). This forces \( M_2 \ll M_4 \) and \( M_3 \sim M_6 \sim N \) since \((M_1, \cdots, M_6) \in M^{med}_5 \).

**Estimate for \( \tilde{K}_N^{low} \).** In the integration domain of \( \tilde{K}_N^{low} \) we have from Lemma 3.32 that \(|\xi_{j,m,1}| \sim |\xi_{j,m,2}| \sim |\xi_{j,m,3}| \sim N\). Then, applying (3.33) on the sum over \((M_7, M_8, M_9)\) we get
\[ |\tilde{K}_N^{low}| \lesssim \sum_{N^{-1/2} \leq M_1 \leq M_2 \leq N} \frac{N^{2s+1}}{M_1 M_2 M_4 M_5 M_7^2 M_8^2} \]
\[ \times \prod_{k=1}^{4} \|P_{\sim N} u_k\|_{L_T^{\infty} L_x^2} \prod_{l=1}^{3} \|P_{\sim N} u_{j,l}\|_{L_T^{\infty} L_x^2} \prod_{n=1}^{3} \|P_{\sim N} u_{j,m,n}\|_{L_T^{\infty} L_x^2}. \]

This implies that
(4.37)
\[ \sum_{N \geq N_0} |\tilde{K}_N^{low}| \lesssim \prod_{k=1}^{4} \|u_k\|_{L_T^{\infty} H_x^s} \prod_{l=1}^{3} \|u_{j,l}\|_{L_T^{\infty} H_x^s} \prod_{n=1}^{3} \|u_{j,m,n}\|_{L_T^{\infty} H_x^s}, \]
since \(2s + \frac{5}{2} < 8s\).

**Estimate for \( \tilde{K}_N^{high} \).** We first estimate \( \tilde{K}_N^{high} \) in the resonant case \( M_4 M_5 M_6 \sim M_7 M_8 M_9 \). Since we are in \( M^{high}_7 \), we have that \( M_9 \geq M_8 \gtrsim N \) and \( M_4 \leq M_5 \ll N \). It follows that \( M_6 \sim N \) and
\[ M_7 \sim \frac{M_4 M_5 N}{M_8 M_9} \ll N. \]

This forces \( N_{j,m,1} \sim N \) and we deduce from (3.35) that
\[ |\tilde{K}_N^{high}| \lesssim \sum_{N^{-1/2} \leq M_1 \leq M_2 \leq N} \sum_{M_6 \geq M_7 \leq M_9 \leq N} \sum_{N_{j,m,1} \sim N} \frac{N^{2s+\frac{5}{2}}}{M_2} \]
\[ \times \prod_{k=1}^{4} \|P_{\sim N} u_k\|_{L_T^{\infty} L_x^2} \prod_{l=1}^{3} \|P_{\sim N} u_{j,l}\|_{L_T^{\infty} L_x^2} \prod_{n=1}^{3} \|P_{N_{j,m,1}} u_{j,m,n}\|_{L_T^{\infty} L_x^2}. \]
which yields summing over $N \geq N_0$ and using the assumption $s > \frac{1}{4}$ that

\begin{equation}
\sum_{N \geq N_0} |\tilde{K}^{high}_N| \lesssim \prod_{k=1}^4 \|u_k\|_{Y^0_x} \prod_{l=1}^3 \|u_{j,l}\|_{Y^0_x} \prod_{n=1}^3 \|u_{j,m,n}\|_{Y^0_x}.
\end{equation}

Now, in the non resonant case we separate the contributions of the region $M_T \leq N^{-1}$ and $M_T > N^{-1}$. In the first region, applying (3.33) on the sum over $(M_8, M_9)$, we get

\begin{equation}
|\tilde{K}^{high}_N| \lesssim \sum_{N \geq N_0} \sum_{0 \leq M_1 \leq M_2 \leq N} \sum_{M_T \leq N^{-1}} \sum_{N_{j,m,n}} \frac{N^{2s+1}M_T}{M_2M_5} \times \prod_{k=1}^4 \|P_{\sim N}u_k\|_{L^\infty_x L^2} \prod_{l=1}^3 \|P_{\sim N}u_{j,l}\|_{L^\infty_x L^2} \prod_{n=1}^3 \|P_{N_{j,m,n}}u_{j,m,n}\|_{L^\infty_x L^2}.
\end{equation}

Observing that $\max\{N_{j,m,1}, N_{j,m,2}, N_{j,m,3}\} \geq N$, we conclude after summing over $N \geq N_0$ that

\begin{equation}
\sum_{N \geq N_0} |\tilde{K}^{high}_N| \lesssim \prod_{k=1}^4 \|u_k\|_{L^\infty_x H^1} \prod_{l=1}^3 \|u_{j,l}\|_{L^\infty_x H^1} \prod_{n=1}^3 \|u_{j,m,n}\|_{L^\infty_x H^1},
\end{equation}

since $2s + 1 < 6s$.

Finally we treat contribution of the region $M_T > N^{-1}$. For $1 \leq j \leq 4$, $1 \leq l \leq 3$ and $1 \leq n \leq 3$ let $\tilde{u}_j = \rho_T(u_j)$, $\tilde{u}_{j,l} = \rho_T(u_{j,l})$ and $\tilde{u}_{j,l,n} = \rho_T(u_{j,l,n})$ be the extensions of $u_j$, $u_{j,l}$ and $u_{j,l,n}$ to $\mathbb{R}^d$ defined in (2.7). We define $u_{N_k} = P_{N_k}\tilde{u}_k$, $u_{N_{j,l}} = P_{N_{j,l}}\tilde{u}_{j,l}$, $u_{N_{j,l,n}} = P_{N_{j,l,n}}\tilde{u}_{j,l,n}$ and perform nonhomogeneous dyadic decompositions in $N_k$, $N_{j,l}$ and $N_{j,m,n}$. Thanks to Proposition 3.10 we easily estimate $\tilde{K}^{high}_N$ on this region by

\begin{equation}
|\tilde{K}^{high}_N| \lesssim \sum_{N \geq N_0} \sum_{0 \leq M_1 \leq M_2 \leq N} \sum_{M_T \leq N^{-1}} \sum_{N_{j,m,n}} \sum_{N_{j,m,n}} \frac{N^{2s+1}M_T}{M_2M_5} N^{-1} \times \prod_{k=1}^4 \|u_{N_k}\|_{Y^0} \prod_{l=1}^3 \|u_{N_{j,l}}\|_{Y^0} \prod_{n=1}^3 \|u_{N_{j,m,n}}\|_{Y^0},
\end{equation}

where $N_{max(T)} = \max\{N_{j,m,1}, N_{j,m,2}, N_{j,m,3}\}$. Therefore, we deduce from (2.8) that (4.35) also holds, for $s > 1/4$, in this region.

Finally, we conclude the proof of Proposition 4.2 gathering (4.12)-(4.38). □

4.3. Estimates of the $X_T^{-1,1}$ norm. In this subsection, we explain how to control the $X_T^{-1,1}$ norm that we used in the energy estimates.

**Proposition 4.3.** Assume that $0 < T \leq 1$ and $s \geq 0$. Let $u \in L^\infty(0,T; H^s(\mathbb{R})) \cap L^4(0,T; L^\infty(\mathbb{R}))$ be a solution to (4.1). Then,

\begin{equation}
\|u\|_{X_T^{-1,1}} \lesssim \|u_0\|_{H^s} + \sum_{i=1}^3 \prod_{j=1}^3 \|u_j\|_{L^\infty_x L^4_x} \|J_x^s u_i\|_{L^\infty_x L^2_x}.
\end{equation}
Proof. By using the Duhamel formula associated to (1.1), the standard linear estimates in Bourgain’s spaces and the fractional Leibniz rule (c.f. Theorem A.12 in [10]), we have that

\[
\|u\|_{X^{s-1,1}_T} \lesssim \|u_0\|_{H^s} + \|\partial_x(u_1u_2u_3)\|_{X^{s-1,0}_T} \\
\lesssim \|u_0\|_{H^s} + \|J_x^4(u_1u_2u_3)\|_{L^2_T X^1}
\]

(4.41)

Therefore the existence and uniqueness of a solution of (1.1) on the time interval \([0, T]\) follows gathering estimates (2.5), (4.6), (4.11), (4.40) and (4.42) that

\[
\|u\|_{L^4_T L^\infty_x} \lesssim \|u\|_{L^\infty_T H^s} + \prod_{j=1}^3 \|u_j\|_{L^\infty_T H^s}.
\]

Proposition 4.4. Assume that \(0 < T \leq 1\) and \(s > \frac{1}{4}\). Let \(u \in L^\infty(0, T; H^s(\mathbb{R}))\) be a solution to (4.1). Then,

\[
\|u\|_{L^4_T L^\infty_x} \lesssim \|u\|_{L^\infty_T H^s} + \prod_{j=1}^3 \|u_j\|_{L^\infty_T H^s}.
\]

Proof. Since \(u\) is a solution to (4.1), we use estimate (2.17) with \(F = \partial_x(u_1u_2u_3)\) and \(\delta = 2\) and the Sobolev embedding to obtain

\[
\|u\|_{L^4_T L^\infty_x} \lesssim \|u\|_{L^\infty_T H^s} + \|u_1u_2u_3\|_{L^4_T L^1_x} \lesssim \|u\|_{L^\infty_T H^s} + \prod_{j=1}^3 \|u_j\|_{L^\infty_T H^s}.
\]

□

5. Proof of Theorem 1.1

Fix \(s > \frac{1}{4}\). First it is worth noticing that we can always assume that we deal with data that have small \(H^s\)-norm. Indeed, if \(u\) is a solution to the IVP (1.1) on the time interval \([0, T]\) then, for every \(0 < \lambda < \infty\), \(u_\lambda(x, t) = \lambda u(\lambda x, \lambda^4 t)\) is also a solution to the equation in (1.1) on the time interval \([0, \lambda^{-3} T]\) with initial data \(u_{0,\lambda} = \lambda u_0(\lambda)\). For \(\varepsilon > 0\) let us denote by \(B^s(\varepsilon)\) the ball of \(H^s(\mathbb{R})\), centered at the origin with radius \(\varepsilon\). Since

\[
\|u_\lambda(\cdot, 0)\|_{H^s} \lesssim \lambda^\frac{s}{4}(1 + \lambda^s)\|u_0\|_{H^s},
\]

we see that we can force \(u_{0,\lambda}\) to belong to \(B^s(\varepsilon)\) by choosing \(\lambda \sim \min(\varepsilon^2\|u_0\|_{H^s}, 1)\). Therefore the existence and uniqueness of a solution of (1.1) on the time interval \([0, 1]\) for small \(H^s\)-initial data will ensure the existence of a unique solution \(u\) to (1.1) for arbitrary large \(H^s\)-initial data on the time interval \(T \sim \lambda^3 \sim \min(\|u_0\|_{H^s}, 1)\).

5.1. Existence. First, we begin by deriving a priori estimates on smooth solutions associated to initial data \(u_0 \in H^\infty(\mathbb{R})\) that is small in \(H^s(\mathbb{R})\). In other words, we assume that \(u_0 \in B^s(\varepsilon)\). It is known from the classical well-posedness theory that such an initial data gives rise to a global solution \(u \in C(\mathbb{R}; H^\infty(\mathbb{R}))\) to the Cauchy problem (1.1).

Then, we deduce gathering estimates (2.5), (4.6), (4.11), (4.40) and (4.42) that

\[
\|u\|_{L^\infty_T H^s} \lesssim \|u_0\|_{H^s}^2 (1 + \|u_0\|_{H^s}^2)^2 + \|u\|_{L^\infty_T H^s}^4 (1 + \|u\|_{L^\infty_T H^s}^2)^4,
\]
for any $0 < T \leq 1$. Moreover, observe that $\lim_{T \to 0} \|u\|_{\tilde{L}^p_T H^s_x} = c\|u_0\|_{H^s}$. Therefore, it follows by using a continuity argument that there exists $\epsilon_0 > 0$ and $C_0 > 0$ such that

$$\|u\|_{\tilde{L}^p_T H^s_x} \leq C_0 \|u_0\|_{H^s} \quad \text{provided} \quad \|u_0\|_{H^s} = \epsilon \leq \epsilon_0.$$ 

Now, let $u_1$ and $u_2$ be two solutions of the equation in (1.1) in $\tilde{L}^\infty_T H^s_x$ for some $0 < T \leq 1$ emanating respectively from $u_1(\cdot, 0) = \varphi_1$ and $u_2(\cdot, 0) = \varphi_2$. We also assume that

$$\|u_i\|_{\tilde{L}^p_T H^s_x} \leq C_0 \epsilon_0, \quad \text{for } i = 1, 2.$$ 

Let us define $w = u_1 - u_2$ and $z = u_1 + u_2$. Then $(w, z)$ solves

$$\begin{cases}
\partial_t w + \partial^2_x w + \frac{\lambda}{2} \partial_x (z^2 w) + \frac{\lambda}{2} \partial_x (w^3) = 0, \\
\partial_t z + \partial^2_x z + \frac{\lambda}{2} \partial_x (z^3) + \frac{\lambda}{2} \partial_x (zw^2) = 0.
\end{cases}$$

Therefore, it follows from (2.3), (3.6), (4.11), (4.40) and (4.42) that

$$\|u_1 - u_2\|_{\tilde{L}^p_T H^s_x} \leq \|\varphi_1 - \varphi_2\|_{H^s}, \quad \text{provided } u_1 \text{ and } u_2 \text{ satisfy } (5.1) \text{ with } 0 < \epsilon < \epsilon_1, \text{ for some } 0 \leq \epsilon_1 \leq \epsilon_0.$$ 

We are going to apply (6.3) to construct our solutions. Let $u_0 \in H^s$ with $s > 1/4$ satisfying $\|u_0\|_{H^s} \leq \epsilon_1$. We denote by $u_N$ the solution of (1.1) emanating from $P_{\leq N} u_0$ for any dyadic integer $N \geq 1$. Since $P_{\leq N} u_0 \in H^\infty(\mathbb{R})$, there exists a solution $u_N$ of (1.1) satisfying

$$u_N \in C(\mathbb{R} : H^\infty(\mathbb{R})) \quad \text{and} \quad u_N(\cdot, 0) = P_{\leq N} u_0.$$ 

We observe that $\|u_{0,N}\|_{H^s} \leq \|u_0\|_{H^s} \leq \epsilon_1$. Thus, it follows from (6.1)-(6.4) that for any couple of dyadic integers $(N, M)$ with $M < N$,

$$\|u_N - u_M\|_{\tilde{L}^p_{\infty} H^s_x} \lesssim \|(P_{\leq N} - P_{\leq M}) u_0\|_{H^s} \underset{M \to +\infty}{\longrightarrow} 0.$$ 

Therefore $\{u_N\}$ is a Cauchy sequence in $C([0,1]; H^s(\mathbb{R}))$ which converges to a solution $u \in C([0,1]; H^s(\mathbb{R}))$ of (1.1). Moreover, it is clear from Propositions 4.13 and 4.22 that $u$ belongs to the class $\tilde{H}^\infty$.

5.2. Uniqueness. Next, we state our uniqueness result.

**Lemma 5.1.** Let $u_1$ and $u_2$ be two solutions of the equation in (1.1) in $\tilde{L}^\infty_T H^s_x$ for some $T > 0$ and satisfying $u_1(\cdot, 0) = u_2(\cdot, 0) = \varphi$. Then $u_1 = u_2$ on $[-T, T]$.

**Proof.** Let us define $K = \max\{\|u_1\|_{L^p_T H^s_x}, \|u_2\|_{L^p_T H^s_x}\}$. Let $s'$ be a real number satisfying $\frac{1}{2} < s' < s$. We get from the Cauchy-Schwarz inequality that

$$\|u_i\|_{\tilde{L}^\infty_T H^{s'}_x} = \left(\sum_N \|P_N u_i\|_{L^p_T H^{s'}_x}^2\right)^{\frac{1}{2}} \lesssim \|u_i\|_{L^p_T H^{s'}_x}, \quad \text{for } i = 1, 2.$$ 

As explained above, we use the scaling property of (1.1) and define $u_{i,\lambda}(x,t) = \lambda u_i(\lambda x, \lambda^2 t)$. Then, $u_{i,\lambda}$ are solutions to the equation in (1.1) on the time interval $[-S, S]$ with $S = \lambda^{-3}T$ and with the same initial data $\varphi_\lambda = \lambda \varphi(\lambda)$. Thus, we deduce from (5.5) that

$$\|u_{i,\lambda}\|_{\tilde{L}^\infty_T H^{s'}_x} \lesssim \lambda^{\frac{s}{2}}(1 + \lambda^{s'}) \|u_i\|_{\tilde{L}^\infty_T H^{s'}_x} \lesssim \lambda^{\frac{s}{2}}(1 + \lambda^{s'}) K, \quad \text{for } i = 1, 2.$$
Thus, we can always choose \( \lambda = \lambda > 0 \) small enough such that \( \|u_{i, \lambda}\|_{L_\infty^\infty H_{x}^s} \leq C_0 \epsilon \) with \( 0 < \epsilon \leq \epsilon_1 \). Therefore, it follows from (6.4) that \( u_{\lambda, 1} = u_{\lambda, 2} \) on \([0, \delta]\). This concludes the proof of Lemma 5.1 by reverting the change of variable. \( \square \)

Finally, the Lipschitz bound on the flow is a consequence of estimate (6.4).

6. A priori estimates in \( H^s \) for \( s > \frac{1}{10} \)

Let \( u \) be a smooth solution of (1.1) defined in the time interval \([0, T]\) with \( 0 < T \leq 1 \). Fix \( \frac{1}{10} < s < \frac{1}{8} \). The aim of this section is to derive estimates for \( u \) in the norms \( \| \cdot \|_{L_\infty^\infty H_{x}^s}, \| \cdot \|_{L_4^s L_\infty^\infty} \) and \( \| \cdot \|_{X_T^{-1, 1}} \).

6.1. Estimate for the \( X_T^{-1, 1} \) and \( L_4^1 L_\infty^\infty \) norms.

**Proposition 6.1.** Assume that \( 0 < T \leq 1 \) and \( s \geq 0 \). Let \( u \) be a solution to (1.1) defined in the time interval \([0, T]\). Then,

\[
\|u\|_{X_T^{-1, 1}} \lesssim \|u_0\|_{H^s} + \|u\|_{L_4^1 L_\infty^\infty}^2 \|u\|_{L_\infty^\infty H_{x}^s}.
\]

**Proof.** The proof is exactly the same as the one of Proposition 4.3. \( \square \)

**Proposition 6.2.** Assume that \( 0 < T \leq 1 \) and \( s > \frac{1}{10} \). Let \( u \) be a solution to (1.1) defined in the time interval \([0, T]\). Then,

\[
\|u\|_{L_4^1 L_\infty^\infty} \lesssim \|u\|_{L_\infty^\infty H_{x}^s} + \|u\|_{L_4^1 L_\infty^\infty} \|u\|_{L_\infty^\infty H_{x}^s}^2.
\]

**Proof.** Since \( u \) is a solution to (1.1) we use estimate (2.17) with \( F = \partial_x (u^3) \). Then, it follows from the Sobolev inequality and the fractional Leibniz rule that

\[
\|J_x \frac{3\delta + 1}{4} F\|_{L_4^1 L_\infty^\infty} \lesssim \|J_x \frac{3\delta + 1}{4} + \frac{1}{p} \partial_x (u^3)\|_{L_4^1 L_\infty^\infty}
\lesssim \|u\|_{L_4^1 L_\infty^\infty} \|u\|_{L_\infty^\infty H_{x}^s} \|D_x u\|_{L_\infty^\infty L_4^1},
\]

for all \( 1 \leq p \leq 2 \) and \( 2 \leq q_1 \leq \infty \) satisfying \( \frac{1}{q_1} + \frac{1}{2} = \frac{1}{p} \) and \( \kappa = -\frac{3\delta}{4} + \frac{1}{4} + \frac{1}{p} \). Thus, the Sobolev inequality yields

\[
\|J_x \frac{3\delta + 1}{4} F\|_{L_4^1 L_\infty^\infty} \lesssim \|u\|_{L_4^1 L_\infty^\infty}^2 \|u\|_{L_\infty^\infty H_{x}^s}^2,
\]

if we choose \( \kappa \) satisfying \( \kappa = \frac{1}{2} - \frac{1}{q_1} = 1 - \frac{1}{p} \). This implies that

\[
\kappa = -\frac{3\delta}{4} + \frac{1}{4} + \frac{1}{p} = -\frac{3\delta}{4} + \frac{5}{4} - \kappa \quad \Rightarrow \quad \kappa = -\frac{3\delta}{8} + \frac{5}{8}.
\]

Then, we choose \( \delta \) such that \( \frac{\delta + 1}{4} = -\frac{3\delta}{8} + \frac{5}{8} \), which gives

\[
\delta = \frac{7}{5}, \quad \kappa = \frac{1}{10}, \quad p = \frac{10}{9} \quad \text{and} \quad q_1 = \frac{5}{2}.
\]

Therefore, we conclude estimate (6.2) by using (2.17) with \( \delta = \frac{7}{5} \) and arguing as in (6.3)-(6.4). \( \square \)
6.2. **Integration by parts.** In this Section, we will use the notations of Section 3. We also denote \( m = \min_{1 \leq i \neq j \leq 3} |\xi_i + \xi_j| \) and

\[
A_j = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\sum_{k=1 \atop k \neq j}^3 \xi_k| = m\}, \quad \text{for } j = 1, 2, 3.
\]

Then, it is clear from the definition that

\[
\sum_{j=1}^3 \chi_{A_j}(\xi_1, \xi_2, \xi_3) = 1, \quad \text{a.e. } (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.
\]

For \( \eta \in L^\infty \), let us define the trilinear pseudo-product operator \( \tilde{\Pi}^{(j)}_{\eta, M} \) in Fourier variables by

\[
\mathcal{F}\left(\tilde{\Pi}^{(j)}_{\eta, M}(u_1, u_2, u_3)\right)(\xi) = \int_{\mathbb{R}^3(\xi)} (\chi_{A_j}(\eta)(\xi_1, \xi_2, \xi_3)\phi_M(\sum_{k=1 \atop k \neq j}^3 \xi_k)) \prod_{j=1}^3 \hat{u}_j(\xi_j).
\]

Moreover, if the functions \( u_j \) are real-valued, the Plancherel identity yields

\[
\int_\mathbb{R} \tilde{\Pi}^{(j)}_{\eta, M}(u_1, u_2, u_3) u_4 \, dx = \int_\mathbb{R} (\chi_{A_j}(\eta)(\xi_1, \xi_2, \xi_3)\phi_M(\sum_{k=1 \atop k \neq j}^3 \xi_k)) \prod_{j=1}^4 \hat{u}_j(\xi_j).
\]

Next, we derive a technical lemma involving the pseudo-products which will be useful in the derivation of the energy estimates.

**Lemma 6.3.** Let \( N \) and \( M \) be two homogeneous dyadic numbers satisfying \( N \gg 1 \). Then, for \( M \ll N \), it holds

\[
\int_\mathbb{R} P_N \tilde{\Pi}^{(3)}_{1, M}(f_1, f_2, g) P_N \partial_x g = M \sum_{N_3 \sim N} \int_\mathbb{R} \tilde{\Pi}^{(3)}_{\eta_3, M}(f_1, f_2, P_{N_3}g) P_N \partial_x g,
\]

for any real-valued functions \( f_1, f_2, f_3, g \in L^2(\mathbb{R}) \) and where \( \eta_3 \) is a function of \( (\xi_1, \xi_2, \xi_3) \) whose \( L^\infty \)-norm is uniformly bounded in \( N \) and \( M \).

**Proof.** Let us denote by \( T_{M,N}(f_1, f_2, g, g) \) the right-hand side of (6.10). From Plancherel’s identity we have

\[
T_{M,N}(f_1, f_2, g, g) = \int_{\mathbb{R}^3} \chi_{A_3}(\xi_1, \xi_2, \xi_3)\phi_M(\xi_1 + \xi_2)\phi_M(\xi_1 + \xi_3)\phi_M(\xi_1 + \xi_2 + \xi_3) \overline{\tilde{f}_1(\xi_1)} \tilde{f}_2(\xi_2) g(\xi_3) \overline{\xi(\xi_3)} \, d^3\xi,
\]

where \( \xi = \xi_1 + \xi_2 + \xi_3 \) and \( d^3\xi = d\xi_1 d\xi_2 d\xi_3 \). We use that \( \xi = \xi_1 + \xi_2 + \xi_3 \) to decompose \( T_{M,N}(f_1, f_2, g, g) \) as follows.

\[
T_{M,N}(f_1, f_2, g, g) = M \sum_{N_2 \leq N_3 \leq 2N} \int_\mathbb{R} \tilde{\Pi}^{(3)}_{\eta_3, M}(f_1, f_2, P_{N_3}g) P_N g \, dx
\]

\[
+ M \sum_{N_2 \leq N_3 \leq 2N} \int_\mathbb{R} \tilde{\Pi}^{(3)}_{\eta_2, M}(f_1, f_2, P_{N_3}g) P_N g \, dx
\]

\[
+ \tilde{T}_{M,N}(f_1, f_2, g, g),
\]
where

\[ \tilde{\eta}_1(\xi_1, \xi_2, \xi_3) = \phi_N(\xi) \frac{\xi_1 + \xi_2}{M} \chi_{\text{supp} \phi_M}(\xi_1 + \xi_2), \]

\[ \tilde{\eta}_2(\xi_1, \xi_2, \xi_3) = \phi_N(\xi) - \phi_N(\xi_3) \frac{\xi_3 \chi_{\text{supp} \phi_M}(\xi_1 + \xi_2)}{M}, \]

and

\[ \tilde{T}_{M,N}(f_1, f_2, g, g) = \int_{\mathbb{R}^3} \chi_{A_3}(\xi_1, \xi_2, \xi_3) \phi_M(\xi_1 + \xi_2) \xi_3 \tilde{f}_1(\xi_1) \tilde{f}_2(\xi_2) \tilde{g}_N(\xi_3) \tilde{g}_N(\xi) d\xi \]

with the notation \( g_N = P_N g \).

First, observe from the mean value theorem and the frequency localization that \( \tilde{\eta}_1 \) and \( \tilde{\eta}_2 \) are uniformly bounded in \( M \) and \( N \).

Next, we deal with \( \tilde{T}_{M,N}(f_1, f_2, g, g) \). By using that \( \xi_3 = \xi - (\xi_1 + \xi_2) \) observe that

\[ \tilde{T}_{M,N}(f_1, f_2, g, g) = -\int_{\mathbb{R}^3} \chi_{A_3}(\xi_1, \xi_2, \xi_3) \phi_M(\xi_1 + \xi_2) (\xi_1 + \xi_2) \tilde{f}_1(\xi_1) \tilde{f}_2(\xi_2) \tilde{g}_N(\xi_3) \tilde{g}_N(\xi) d\xi \]

\[ + S_{M,N}(f_1, f_2, g, g) \]

with

\[ S_{M,N}(f_1, f_2, g, g) = \int_{\mathbb{R}^3} \chi_{A_3}(\xi_1, \xi_2, \xi_3) \phi_M(\xi_1 + \xi_2) \tilde{f}_1(\xi_1) \tilde{f}_2(\xi_2) \tilde{g}_N(\xi_3) \tilde{g}_N(\xi) d\xi. \]

Since \( g \) is real-valued, we have \( \tilde{g}_N(\xi) = \tilde{g}_N(-\xi) \), so that

\[ S_{M,N}(f_1, f_2, g, g) = \int_{\mathbb{R}^3} \chi_{A_3}(\xi_1, \xi_2, \xi_3) \phi_M(\xi_1 + \xi_2) \tilde{f}_1(\xi_1) \tilde{f}_2(\xi_2) \tilde{g}_N(\xi_3) \tilde{g}_N(-\xi) d\xi. \]

We change variable \( \xi_3 = -\xi = -(\xi_1 + \xi_2 + \xi_3) \), so that \( -\xi_3 = \xi_1 + \xi_2 + \xi_3 \). Thus, \( S_{M,N}(f_1, f_2, g, g) \) can be rewritten as

\[ -\int_{\mathbb{R}^3} \chi_{A_3}(\xi_1, \xi_2, -\xi_1 - \xi_2 - \xi_3) \phi_M(\xi_1 + \xi_2) \tilde{f}_1(\xi_1) \tilde{f}_2(\xi_2) \xi_3 \tilde{g}_N(\xi_3) \tilde{g}_N(\xi_1 + \xi_2 + \xi_3) d\xi, \]

where \( d\xi = d\xi_1 d\xi_2 d\xi_3 \). Now, observe that \(|\xi_1 + (-\xi_1 - \xi_2 - \xi_3)| = |\xi_2 + \xi_3| \) and \(|\xi_2 + (-\xi_1 - \xi_2 - \xi_3)| = |\xi_1 + \xi_3| \). Thus \( \chi_{A_3}(\xi_1, \xi_2, -\xi_1 - \xi_2 - \xi_3) = \chi_{A_3}(\xi_1, \xi_2, \xi_3) \) and we obtain

\[ S_{M,N}(f_1, f_2, g, g) = -\tilde{T}_{M,N}(f_1, f_2, g, g), \]

so that

\[ (6.12) \quad \tilde{T}_{M,N}(f_1, f_2, g, g) = M \int_{\mathbb{R}} \Pi_{\eta_2, M}^1(f_1, f_2, P_N g) P_N g dx \]

where

\[ \eta_2(\xi_1, \xi_2, \xi_3) = -\frac{1}{2} \frac{\xi_1 + \xi_2}{M} \chi_{\text{supp} \phi_M}(\xi_1 + \xi_2) \]

is also uniformly bounded function in \( M \) and \( N \).

Finally, we define \( \eta_1 = \tilde{\eta}_1 + \tilde{\eta}_2 \) and \( \eta_3 = \eta_1 + \eta_2 \). Therefore the proof of (6.10) follows gathering (6.11) and (6.12). \( \Box \)

Finally, we state a \( L^2 \)-trilinear estimate involving the \( X^{-1,1} \)-norm and whose proof is similar to the one of Proposition 3.5.
Proof. Observe from the definition that
\[
\|u\|_{L_t^\infty L_x^2} \lesssim \sum_{N} \|P_N u\|_{L_t^\infty L_x^2}.
\]
Moreover, by using (1.1), we have
\[
\frac{1}{2} \frac{d}{dt} \|P_N u(\cdot, t)\|_{L_x^2}^2 = \int_R (P_N \partial_x (u^3) P_N u)(x, t) dx.
\]
which yields after integration in time between 0 and t and summation over N
\[
\|u\|_{L_t^\infty L_x^2}^2 \lesssim \|u_0\|_{H^s}^2 + \sum_{i \in [0, T]} \sup_{i \in [0, T]} |L_N(u)|,
\]
where
\[
L_N(u) = N^{2s} \int_{R \times [0, t]} P_N \partial_x (u^3) P_N u dx ds.
\]
In the case where $N \lesssim 1$, Hölder’s inequality and (2.5) imply that
\[
\sum_{N \lesssim 1} |L_N(u)| \lesssim \|u\|_{L_t^1 L_x^\infty}^2 \|u\|_{L_t^\infty L_x^2}^2 \lesssim \|u\|_{Z_T^s}^4.
\]
In the following, we can then assume that $N \gg 1$. By using the decomposition in (6.7), we get that $L_N(u) = \sum_{j=1}^{3} L_N^{(j)}(u)$ with
\[
L_N^{(j)}(u) = N^{2s} \sum_{M} \int_{R \times [0, t]} P_N \tilde{\Pi}_{1, M}^{(j)}(u, u, u) P_N \partial_x u dx ds,
\]
where we performed a homogeneous dyadic decomposition in $m \sim M$. Thus, by symmetry, it is enough to estimate $L_N^{(3)}(u)$, that still will be denoted $L_N(u)$ for the sake of simplicity.
We decompose $J_N(u)$ depending on whether $M < 1$, $1 \leq M \ll N$ and $M \gtrsim N$. Thus
\[ L_N(u) = N^{2s} \left( \sum_{M \geq N} + \sum_{1 \leq M < N} + \sum_{M < 1} \right) \int_{\mathbb{R} \times [0,t]} P_N \tilde{\Pi}_{1,M}^{(3)}(u, u, u) P_N \partial_x u \, dx \, ds \]
(6.19) \[ =: L_{N}^{\text{high}}(u) + L_{N}^{\text{med}}(u) + L_{N}^{\text{low}}(u). \]

Estimate for $L_{N}^{\text{high}}(u)$. Let $\tilde{u} = \rho_t(u)$ be the extension of $u$ to $\mathbb{R}^2$ defined in (2.7). Now we define $u_N = P_N \tilde{u}$, for $i = 1, 2, 3$, $u_N = P_N \tilde{u}$ and perform dyadic decompositions in $N_i$, $i = 1, 2, 3$, so that
\[ L_{N}^{\text{high}}(u) = N^{2s} \sum_{M \geq N} \sum_{N_1, N_2, N_3} \int_{\mathbb{R} \times [0,t]} P_N \tilde{\Pi}_{1,M}^{(3)}(u_{N_1}, u_{N_2}, u_{N_3}) P_N \partial_x u \, dx \, ds. \]

Define
\[ \eta_{\text{high}}(\xi_1, \xi_2, \xi_3) = \frac{\xi}{N} \phi_N(\xi). \]

It is clear that $\eta_{\text{high}}$ is uniformly bounded in $M$ and $N$. Thus, by using estimate \[ \eta \] , we have that
\[ \left| L_{N}^{\text{high}}(u) \right| \lesssim N^{2s} \sum_{M \geq N} \sum_{N_1, N_2, N_3} N \int_{\mathbb{R} \times [0,t]} P_N \tilde{\Pi}_{1,M}^{(3)}(u_{N_1}, u_{N_2}, u_{N_3}) P_N \partial_x u \, dx \, ds \]
(6.20) \[ \lesssim N^{2s} \left\| u_N \right\|_{Y^s} \sum_{N_1, N_2, N_3} \prod_{i=1}^3 \left\| u_{N_i} \right\|_{Y^s}, \]

since $\sum_{M \geq N} N/M \lesssim 1$. Let us denote $N_{\text{max}}$, $N_{\text{med}}$ and $N_{\text{min}}$ the maximum, sub-maximum and minimum of $N_1$, $N_2$, $N_3$. It follows then from the frequency localization that $N \lesssim N_{\text{med}} \sim N_{\text{max}}$. Thus, we deduce summing (6.20) over $N$, using the Cauchy-Schwarz inequality in $N_1$, $N_2$, $N_3$ and $N$ and estimate (2.8) that
\[ \sum_{N \gg 1} \left| L_{N}^{\text{high}}(u) \right| \lesssim \left\| \tilde{u} \right\|_{Y^s}^4 \lesssim \left\| u \right\|_{Y^s}^4, \]
(6.21) since $s > 0$.

Estimate for $L_{N}^{\text{med}}(u)$. To estimate $L_{N}^{\text{med}}(u)$, we decompose $\int_{\mathbb{R}} P_N \tilde{\Pi}_{1,M}^{(3)}(u, u, u) P_N \partial_x u$ as in (6.10), since we are in the case $1 \leq M \ll N$ and $N \gg 1$.

Once again, let $\tilde{u} = \rho_t(u)$ be the extension of $u$ to $\mathbb{R}^2$ defined in (2.7) and $u_N = P_N \tilde{u}$, for $i = 1, 2, 3$, $u_N = P_N \tilde{u}$. Observe from the frequency localization that $N_3 \sim N$. We perform dyadic decompositions in $N_i$, $i = 1, 2, 3$ and deduce from (6.10) that
\[ \left| L_{N}^{\text{med}}(u) \right| \lesssim N^{2s} \sum_{1 \leq M \ll N} \sum_{N_1, N_2} \sum_{N_3 \sim N} \int_{\mathbb{R} \times [0,t]} P_N \tilde{\Pi}_{n_1,M}^{(3)}(u_{N_1}, u_{N_2}, u_{N_3}) P_N \partial_x u \, dx \, ds, \]
where $\eta_n^3$ is uniformly bounded in the range of summation of $M$, $N$, $N_1$, $N_2$ and $N_3$. Then, we deduce from (6.13) that
\[ \left| L_{N}^{\text{med}}(u) \right| \lesssim \sum_{1 \leq M \ll N} \sum_{N_1, N_2} \sum_{N_3 \sim N} \left\| u_{N_1} \right\|_{Y^s} \left\| u_{N_2} \right\|_{Y^s} \left\| u_{N_3} \right\|_{Y^s} \left\| u_N \right\|_{Y^s}. \]

\[ \text{see the proof of Lemma (6.13) for a definition of } \eta_n. \]
Observe that \( \max\{N_1, N_2\} \geq M \). Therefore, we deduce after summing over \( N \sim N_3 \gg 1 \), \( N_1 \), \( N_2 \) and \( M \) that

\[
\sum_{N \gg 1} |L_N^{med}(u)| \lesssim \|\tilde{u}\|_{\mathcal{Y}_T}^4 \lesssim \|u\|_{\mathcal{Y}_T}^4,
\]

since \( s > 0 \).

**Estimate for \( L_N^{low} \).** In this case, we also have \( N \gg 1 \) and \( M \ll N \). Thus the decomposition in (6.10) yields

\[
L_N^{low}(u) = N^{2s} \sum_{M \leq \frac{1}{N}} M \sum_{N_3 \sim N} \int_{R \times [0, t]} \tilde{\Pi}^{(3)}_{\eta_3,M}(u, u, P_{N_3}u)P_Nu \, dx \, ds,
\]

where \( \eta_3 \) is defined in the proof of Lemma 6.3. Since \( \eta_3 \) is uniformly bounded in \( N \) and \( M \), we deduce from (6.4) and Hölder’s inequality in time (recall here that \( 0 < t \leq T \leq 1 \)) that

\[
|L_N^{low}(u)| \lesssim N^{2s} \sum_{M \leq 1/2} M^2 \|u\|_{L_T^\infty L_x^2}^2 \sum_{N_3 \sim N} \|P_{N_3}u\|_{L_T^\infty L_x^2} \|P_Nu\|_{L_T^\infty L_x^2}.
\]

Thus, we infer that

\[
\sum_{N \gg 1} |L_N^{low}(u)| \lesssim \|u\|_{L_T^\infty L_x^\infty}^2 \|u\|_{L_T^\infty H_x^\infty}^2 \|u\|_{L_T^\infty L_x^\infty}^4 \lesssim \|u\|_{\mathcal{Y}_T}^4.
\]

Finally, we conclude the proof of estimate (6.14) gathering (6.10), (6.18), (6.19), (6.21), (6.23) and (6.24).

**6.4. Proof of Theorem 1.4.** By using a scaling argument as in Section 5 it suffices to prove Theorem 1.4 in the case where the initial datum \( u_0 \) belongs to the ball \( \mathcal{B}^\epsilon(\epsilon) \) of \( H^s \) centered at the origin and of radius \( \epsilon \), where \( 0 < \epsilon < \epsilon_0 \) and \( \epsilon_0 \) is a small number to be determined later, and where the solution \( u \) is defined on a time interval \([0, T]\) with \( 0 < T \leq 1 \).

Let us define \( \Gamma_T^{\epsilon}(u) = \|u\|_{L_T^\infty H_x^\epsilon} + \|u\|_{L_T^4 L_x^\infty} \). Then it follows gathering (6.11), (6.2) and (6.14) that

\[
\Gamma_T^{\epsilon}(u) \lesssim \|u_0\|_{H^s} + \Gamma_T^{\epsilon}(u)^2 + \Gamma_T^{\epsilon}(u)^3,
\]

if \( \epsilon_0 \) is chosen small enough. Moreover, observe that \( \lim_{T \to 0} \Gamma_T^{\epsilon}(u) = c \|u_0\|_{H^s} \). Therefore, it follows by using a continuity argument that there exists \( \epsilon_0 > 0 \) such that

\[
\Gamma_T^{\epsilon}(u) \lesssim \|u_0\|_{H^s} \quad \text{provided} \quad \|u_0\|_{H^s} \leq \epsilon \leq \epsilon_0.
\]

This concludes the proof of Theorem 1.4 by using (6.1).

**Acknowledgments.** L.M. and S.V. were partially supported by the ANR project GEO-DISP. D.P. would like to thank the L.M.P.T. at Université François Rabelais for the kind hospitality during the elaboration of this work. He is also grateful to Gustavo Ponce for pointing out the reference 3.

**References**

1. A. Babin, A. Ilyin and E. Titi, *On the regularization mechanism for the periodic Korteweg-de Vries equation*, Comm. Pure Appl. Math., 64 (2011), no. 5, 591–648.

2. M. Christ, J. Colliander and T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*, Amer. J. Math., 125 (2003), 1235–1293.

3. M. Christ, J. Holmer and D. Tataru, *Low regularity a priori bounds for the modified Korteweg-de Vries equation*, Lib. Math. (N.S.), 32 (2012), no.1, 51–75.
4. J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$, J. Amer. Math. Soc., 16 (2003), 705–749.
5. A. Ionescu, C. E. Kenig and D. Tataru, Global well-posedness of the KP-I initial value problem in the energy space, Invent. Math., 173 (2008), 265–304.
6. T. Kato, On nonlinear Schrödinger equations II. $H^s$-solutions and unconditional well-posedness, J. Anal. Math., 67 (1995), 281–306.
7. C. E. Kenig and D. Koenig, On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations, Math. Res. Let., 10 (2003), 879–895.
8. C. E. Kenig and D. Pilod, Well-posedness for the fifth-order KdV equation in the energy space, preprint (2012), arXiv:1205.0169, to appear in Trans. Amer. Math. Soc.
9. C. E. Kenig, G. Ponce and L. Vega, Oscillatory integrals and regularity of dispersive equations, Indiana Univ. Math. J., 40 (1991), 33–69.
10. C. E. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, Comm. Pure Appl. Math., 46 (1993), 527–620.
11. C. E. Kenig, G. Ponce and L. Vega, On the ill-posedness of some canonical dispersive equations, Duke Math. J., 106 (2001), 617–633.
12. H. Koch and N. Tzvetkov, Local well-posedness of the Benjamin-Ono equation in $H^s(\mathbb{R})$, Int. Math. Res. Not., 14 (2003), 1449–1464.
13. S. Kwon and T. Oh, On unconditional well-posedness of modified KdV, Internat. Math. Res. Not., 15 (2012), 3509–3534.
14. N. Masmoudi and K. Nakanishi, From the Klein-Gordon-Zakharov system to the nonlinear Schrödinger equation , J. Hyperbolic Differ. Equ. 2 (2005), 975–1008.
15. L. Molinet, A note on the inviscid limit of the Benjamin-Ono-Burgers equation in the energy space, Proc. Amer. Math. Soc., 14 (2013), 2793–2798.
16. L. Molinet and S. Vento, Improvement of the energy method for strongly non resonant dispersive equations and applications, preprint (2014).
17. K. Nakanishi, H. Takaoka and Y. Tsutsumi, Local well-posedness in low regularity of the mKdV equation with periodic boundary condition, Disc. Cont. Dyn. Systems 28 (2010), no. 4, 1635–1654.
18. H. Takaoka and Y. Tsutsumi, Well-posedness of the Cauchy problem for the modified KdV equation with periodic boundary condition, Int. Math. Res. Not. (2004), 3009–3040.
19. T. Tao, Multilinear weighted convolution of $L^2$ functions and applications to nonlinear dispersive equations, Amer. J. Math., 123 (2001), 839–908.
20. Y. Zhou, Uniqueness of weak solution of the KdV equation, Int. Math. Res. Not., 6 (1997), 271–283.

Luc Molinet, Laboratoire de Mathématiques et Physique Théorique, Université François Rabelais Tours, Fédération Denis Poisson-CNRS, Parc Grandmont, 37200 Tours, France.
E-mail address: Luc.Molinet@lmpt.univ-tours.fr

Didier Pilod, Instituto de Matemática, Universidade Federal do Rio de Janeiro, Caixa Postal 68530, CEP: 21945-970, Rio de Janeiro, RJ, Brasil.
E-mail address: didier@im.ufrj.br

Stéphane Vento, Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS ( UMR 7539), 99, avenue Jean-Baptiste Clément, F-93 430 Villetaneuse, France.
E-mail address: vento@math.univ-paris13.fr