MACHOs in a Flattened Halo

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ABSTRACT

If massive compact halo objects (MACHOs) are detected in ongoing searches, then $\tau_{\text{SMC}}/\tau_{\text{LMC}}$, the ratio of the optical depth toward the Small and Large Magellanic Clouds, will be a robust indicator of the flattening of the Galactic dark matter halo. For a spherical halo, $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ is about 1.45, independent of details of the shape of the Galactic rotation curve, the assumed mass of the Galactic disk and spheroid, and the truncation distance (if any) of the dark halo. For an E6 halo (axis ratio $c/a = 0.4$), the ratio of optical depths is $\tau_{\text{SMC}}/\tau_{\text{LMC}} \sim 0.95$, again independent of assumptions about Galactic parameters. This ratio can be measured with a precision as good as $\sim 10\%$ depending on the typical mass of the MACHOs. If the halo is highly flattened (e.g., E6) and closely truncated (e.g., at twice the solar galactocentric radius), then the optical depth toward the LMC can be reduced by a factor of about two. For these extreme parameters, and the assumption of a heavy Galactic disk and spheroid, the upper limit of the MACHO mass range to which ongoing experiments are sensitive is reduced from $\mathcal{O}(10^6) M_\odot$ to $\mathcal{O}(10) M_\odot$.

Subject Headings: gravitational lensing, dark matter, Magellanic Clouds
1. Introduction

Three experiments are currently underway to detect Massive Compact Halo Objects (MACHOs) in the Galactic halo using a method originally suggested by Paczyński (1986). The MACHO Collaboration (Alcock et al. 1992) is now monitoring $\sim 10^7$ stars in the Magellanic Clouds to search for the characteristic microlensing light curve induced by the passing of a MACHO across the line of sight to the source. The program stars are mostly in the Large Magellanic Cloud (LMC), with some in the Small Magellanic Cloud (SMC). During the southern winter when the LMC is down, the MACHO Collaboration plans to observe several million stars in the Galactic bulge. The experiment is scheduled to last four years. The Warsaw-Carnegie-Princeton Optical Gravitational Lens Experiment (OGLE) is observing several million stars in the Galactic bulge over several years, and has completed one season of observations (Udalski 1992). A French collaboration is using Schmidt plates and CCD frames to look for microlensing in LMC fields (Aubourg et al. 1991).

Paczyński (1991) and Griest et al. (1991) have suggested that by observing the bulge one could measure the low-mass end of the disk luminosity function and any baryonic dark matter that may be present in the Galactic disk. The bulge observations will also be sensitive to MACHOs, although as noted by Griest et al., assuming standard halo parameters the optical depth to microlensing by MACHOs in this direction is only $\sim 1/3$ the optical depth toward the LMC and also only $\sim 1/3$ the optical depth toward the bulge due to known disk stars. However, as Griest et al. pointed out, the optical depth toward the bulge would be enhanced if the halo were flattened.

Here we consider the general effect of halo flattening on the detectability of MACHOs by addressing two questions: (1) If MACHOs are detected, is there any specific signature that they are distributed in a flattened rather than spherical halo? (2) If MACHOs are not detected, could this negative result be attributed to the MACHOs being in a flattened halo?
At first sight these questions appear difficult to answer because the core radius, asymptotic speed, and truncation radius of the “standard” spherical isothermal model for the dark halo are not well constrained, so that adding a fourth poorly-known parameter (halo flattening) would seem to render the problem nearly insoluble. We will show, however, that there are signatures along certain lines of sight for which the optical depth of a flattened versus spherical halo are largely independent of these considerations.

The standard spherical halo is given by

$$\rho(r) = \frac{v_\infty^2}{4\pi G} \left( \frac{1}{a^2 + r^2} \right) \theta(R_T - r),$$

where \(r\) is the Galactocentric radius, \(v_\infty\) is the asymptotic circular speed of the halo (assuming that it extends to infinity); \(a\) is the core radius of the halo; and \(R_T\) is the truncation radius. This form can be generalized to include a family of oblate halos of arbitrary axis ratio (Sackett & Sparke 1990, and §2).

As we discuss in §3, the circular speed and the core radius are partially constrained by observations, while the truncation radius is largely unknown. We will therefore consider separately the effect of flattening on truncated and untruncated halos (or halos truncated only beyond the Magellanic Clouds).

Untruncated halos described by equation (1.1) have only two parameters, \(a\) and \(v_\infty\), both of which are constrained by the observed rotation curve and the assumed masses and mass distributions of the Galactic disk and spheroid. The Galactic rotation curve measures the total gravitational force in the plane as a function of Galactocentric distance; the assumed disk and spheroid models tell us what fraction of this force is accounted for by the Galaxy’s observed components, and hence what fraction is contributed by the dark halo. The principal uncertainties are the slope of the rotation curve interior and exterior to the solar circle and the mass-to-light \((M/L)\) ratios of the disk and spheroid. The shape of dark matter halos is still an open question (for a review, see Ashman 1992), but substantially
flattened halos (E4-E6) have been indicated in some studies (Dubinski & Carlberg 1991; Katz 1991; Katz & Gunn 1991; and Sackett & Sparke 1990.)

Since we do not know the exact Galactic rotation curve, the shape of the Galactic halo, or precise disk and bulge normalizations, we choose arbitrary values for these quantities within the range allowed by observations. We then find the best fit values of $v_\infty$ and $a$ for a spherical halo and for an E6 (axis ratio $c/a = 0.4$ in density). For each of these halo models, we determine the optical depth to microlensing by MACHOs toward the LMC, the SMC, and the Galactic bulge (choosing Baade’s Window to be specific).

Although the optical depth toward individual lines of sight can vary by a factor of a few between models, we find that the ratio of optical depth toward the SMC over that to the LMC, $\tau_{\text{SMC}}/\tau_{\text{LMC}}$, varies by only a few percent over all Galactic parameters except halo flattening, to which it is much more sensitive. For a spherical halo, the ratio is $\tau_{\text{SMC}}/\tau_{\text{LMC}} \sim 1.45$; for an E6 halo, $\tau_{\text{SMC}}/\tau_{\text{LMC}} \sim 0.95$. If the halo is not truncated, these ratios vary by only about 3-5% over the whole range of models considered, while the individual optical depths to the LMC and SMC vary by as much as a factor of 3 to 4 between models. For a truncated halo, the $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ ratios vary a bit more, by as much as 8%. For the most extreme case that we examined, a highly flattened (E6) and closely truncated halo ($2R_0$), the optical depth toward the LMC decreases by a factor of $\sim 2$ relative to a standard model. For such a halo, the current searches are sensitive only to MACHOs of mass $M \lesssim \mathcal{O}(10) M_\odot$. By comparison, if one assumes a spherical untruncated halo, then a null result would rule out MACHOs to as high as $M \lesssim \mathcal{O}(10^6) M_\odot$ (Gould 1992).

In contrast to the optical depths toward the LMC and SMC, the optical depth toward the bulge seems to provide very little model-independent information. The principal value of the bulge observations to constraining the mass distribution of the Galaxy is that they will allow us to search for dark matter in the disk.

The remainder of the paper is organized as follows: In § 2, we derive expressions for the optical depth along arbitrary lines of sight to microlensing from within
an isothermal halo of arbitrary flattening, and discuss its dependence on halo parameters and the geometry of the line of sight relative to the halo. In § 3, we describe our procedure for determining the best fit halo parameters for various observationally constrained models of the Galactic rotation curve, disk, and bulge. In § 4 we compare, for a wide range of Galactic parameters, the optical depths of E0 versus E6 untruncated halos along lines of sight to the LMC, SMC, and the Galactic bulge; in § 5 a similar comparison is made for truncated halos. We argue from these results that if the Galactic dark halo is composed of MACHOs, the ratio $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ is an reliable measure of the ellipticity of the halo. In § 6, we discuss limitations imposed by statistical fluctuations and various lensing “backgrounds.” We summarize our conclusions in § 7, and in the Appendix we invert the $R^{1/4}$ law to derive analytic expressions for the volume density and corresponding force from a spherical de Vaucouleurs bulge.

2. Optical Depth to Microlensing by a Flattened Halo

To model the Galactic dark halo, we construct a four-parameter family of flattened density profiles $\rho(r)^*$,

$$\rho(r) = \frac{\tan \psi}{\psi} \frac{v_\infty^2}{4\pi G} \left( \frac{1}{a^2 + \zeta^2} \right) \theta(R_T - \zeta), \quad (2.1)$$

where

$$\zeta^2 \equiv r^2 + z^2 \tan^2 \psi, \quad (2.2)$$

and where $r$ is the Galactocentric radius and $z$ is the height above the Galactic plane. The flattening parameter, $\psi$, is defined such that $\cos \psi$ is the axis ratio

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* We note that this formulation for the halo truncation radius is not necessarily physical since it does not represent a solution to the collisionless Boltzmann equation. The optical depth to microlensing, however, is not sensitive to the precise shape of the truncation profile and, as we shall see, is only weakly dependent on the location of even a rather severe truncation. Similarly, we do not consider here the effect of anisotropic orbits of MACHOs in a flattened halo. The degree of anisotropy will depend on the amount of rotation support in the halo. Although any anisotropy in the orbits could alter estimates of the timescales of lensing events by as much as factor of two, the primary conclusions of this paper, which involve optical depths and their ratios, would not be affected.
\[ q = c/a \] of the halo density profile (which is assumed to be axisymmetric in the \( z = 0 \) plane). The remaining three parameters, \( v_\infty, a, \) and \( R_T \), in equation (2.1) are defined as in equation (1.1). Note that the halo is truncated on an isodensity contour that crosses the Galactic plane at \( R_T \); since the halo is assumed to be oblate, the truncation distance in other directions is smaller.

A MACHO of mass \( M \) in this halo will have an Einstein ring radius, \( r_\ast \), given by (e.g., , Blandford & Kochanek 1987)

\[
r_\ast^2 = \frac{4GML(D - L)}{c^2 D},
\]

(2.3)

where \( D \) is the distance from the observer to the source and \( L \) is the distance from the observer to the lensing MACHO. The cross section to lensing is defined by convention to be the probability that a given observed source lies within an angle \( r_\ast/L \) of a MACHO. If the halo is composed of MACHOs, the optical depth along a line of sight parameterized by \( L \), is therefore given by

\[
\tau = \int_0^D dL \frac{4\pi G \rho[r(L)]}{c^2} L \left(1 - \frac{L}{D}\right).
\]

(2.4)

Substituting equation (2.1) into equation (2.4) leads to the optical depth toward Galactic coordinates \( \ell \) and \( b \)

\[
\tau(\ell, b) = \frac{\tan \psi}{\psi} \frac{v_\infty^2}{c^2} \frac{1}{D} \int_0^{D_T} \frac{dL}{(a^2 + R_0^2)} - (2R_0 \cos \ell \cos b) L + (1 + \sin^2 b \tan^2 \psi) L^2
\]

(2.5)

where \( R_0 \) is the Galactocentric radius at the solar circle and \( D_T \) is the distance to the source or the truncation distance along the line of sight, whichever is smaller. By completing the square in the denominator, this integral can be evaluated to
yield

\[ \tau(\ell, b) = \frac{\tan \psi}{\psi} \frac{v_\infty^2}{c^2 p^2} \frac{1}{D} \left[ -s + \left( \frac{D}{2} - \Delta \right) \ln(s^2 + q^2) + \left( \frac{q^2 + D\Delta - \Delta^2}{q} \right) \arctan \frac{s}{q} \right]_{s=s_{\text{max}}}^{s=-\Delta} \]

where

\[ p^2 \equiv 1 + \tan^2 \psi \sin^2 b; \quad \Delta \equiv \frac{R_0 \cos \ell \cos b}{p^2}, \quad (2.7) \]

\[ q^2 \equiv \frac{R_0^2 + a^2}{p^2} - \Delta^2, \quad \text{and} \]

\[ s_{\text{max}} \equiv \min \left( D - \Delta, \sqrt{\frac{R_T^2 - R_0^2}{p^2} + \Delta^2} \right). \quad (2.9) \]

Once the parameters of a given model are established, it is straightforward to evaluate the optical depth in any direction using equation (2.5).

One can see from equation (2.5) that information about the shape of the dark halo is carried in the prefactor \((\tan \psi/\psi)\) and in the factor \((1 + \sin^2 b \tan^2 \psi)\) that multiplies \(L^2\) in the denominator of the integrand. The prefactor, which decouples from other halo parameters and from the source direction, is equal to unity for a spherical halo, and \(1.98\) for an E6 halo. This is the only dependence of \(\tau\) on the flattening of the halo for lines of sight that lie in the Galactic plane. In principle, it might seem that this would imply that searches for MACHOs toward the bulge would provide a strong discriminator for the shape of the dark halo, since the optical depth would be increased by a factor of two for a substantially flattened (E6) halo. However, as we shall discuss in § 3, in practice the poorly known halo parameters, \(a\) and \(v_\infty\), and microlensing contamination from disk stars render the bulge fields nearly useless as probes of the ellipticity of the dark halo.

High latitude sources are more effective at measuring halo shape because of decreased disk contamination and because of the \(L^2\) term in the denominator of the integrand of equation (2.5). Shown in Figure 1 are the optical depths toward
SMC ($\ell = 303^\circ, b = -44^\circ$) and LMC ($\ell = 280^\circ, b = -33^\circ$) fields for both E0 and E6 halos as a function of halo core radius. A somewhat arbitrary, but reasonable, halo asymptotic speed of $v_\infty = 220$ km s$^{-1}$ was used for the purposes of the calculation, but since from equation (2.5) it is clear that $\tau$ varies strictly as the square of $v_\infty$, these results can be rescaled to any other value. The decline of optical depths with halo core radius is easily understood from Eq. (2.5); the decline with halo truncation can be a factor of two in the most severe case. In general, $\tau_{\text{SMC}}$ is larger than $\tau_{\text{LMC}}$ because the line of sight to the SMC passes closer to the center of the halo; $\cos \ell \cos b = 0.39$ for the SMC and 0.15 for the LMC. The optical depth to the SMC is more sensitive to the shape of the halo because of its larger Galactic latitude.

Note that along the LMC line of sight, the effect of the $(\sin^2 b \tan^2 \psi)$ term in the denominator is nearly offset by the prefactor $(\tan \psi/\psi)$ in the numerator for an E6 halo; toward the SMC, the $(\sin^2 b \tan^2 \psi)$ term is substantially larger, and thus the gap between E0 and E6 curves in Figure 1 is wider for the SMC than the LMC.

Difficulty in determining the core radius and asymptotic speed of the halo translate into substantial uncertainties in the individual opticals depths toward the SMC and LMC. In the ratio of optical depths $\tau_{\text{SMC}}/\tau_{\text{LMC}}$, however, the asymptotic halo speed, $v_\infty$, drops out completely, and, as can be seen from Figure 2, the remaining dependence on the core radius, $a$, is a rather weak one. Furthermore, regardless of the truncation radius of the halo and the value of $a$ (up to more than 20 kpc), $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ is 20 - 40% lower for an E6 than an E0 halo. We thus propose $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ as an indicator of the ellipticity of the dark halo. What remains to be shown is that the range of core radii shown in Figures 1 and 2 are appropriate to the Milky Way, and that this difference is one that can be detected in on-going MACHO searches; this we do in subsequent sections.
3. Description of Models

We model the mass of the Galaxy in terms of three components, a disk, a spheroid, and a dark halo. In order to use the Galactic rotation curve as a constraint on the dark halo parameters $v_\infty$ and $a$ for a halo of given flattening, we must evaluate the contribution of the disk and spheroid to the rotational support as a function of radius.

The disk we take as a double exponential in density, 
$$
\rho(R, z) = \rho_0 e^{-|z|/z_0} e^{-R/h},
$$
with scale height, $z_0 = 350 \text{ pc}$ and scale length, $h = 3.5 \text{ kpc}$ (Binney & Tremaine 1987). The disk is then completely specified by a single parameter, the local column density, $\Sigma_0$, which is related to the central volume density, $\rho_0$, of the disk through 
$$
\Sigma_0 = 2 z_0 \rho_0 e^{-R_0/h}.
$$

Bahcall (1984) estimates the total mass of the visible components of the disk in the local neighborhood to be $\Sigma_0 \sim 50 M_\odot \text{ pc}^{-2}$, which places a lower limit on the surface mass density of the Galactic disk. The disk may also have dark components (Bahcall, Flynn, & Gould 1992); a so-called “maximal disk” (which is often used to model the mass distribution in other galaxies) provides a hard upper limit on $\Sigma_0$ of about twice that seen in visible components. The total column density of the disk is further constrained by the measurement of Kuijken & Gilmore (1989, KG), but not as tightly as those authors suggest. KG show a best fit to their data (prior to imposing the rotation-curve constraint) of $\Sigma_0 = 71 \pm 6 M_\odot \text{ pc}^{-2}$ and essentially no dark halo. After imposing the rotation constraint, they find a best fit of $\Sigma_0 = 46 M_\odot \text{ pc}^{-2}$. However, Gould (1990) showed that this estimate was statistically biased and that a better treatment yields $\Sigma_0 = 54 \pm 9 M_\odot \text{ pc}^{-2}$. Moreover, the best combined fit of the data and the rotation curve constraint is not particularly good, so that the confidence of the rotation constraint may have been overestimated. Hence, it seems conservative to allow a local column density as high as $\Sigma_0 = 75 M_\odot \text{ pc}^{-2}$. In our models, we examined $\Sigma_0 = 50$ and $75 M_\odot \text{ pc}^{-2}$.

* Although maximal disk models are widely and, in some respects, successfully used in fitting
The other major visible component of the Galaxy is the spheroid, which has a total luminosity of \(2.4 \times 10^9 L_\odot\) \(\text{V} \) (Bahcall & Soneira 1980), after correction to the current IAU standard of \(R_\odot = 8.5\) kpc for the solar Galactocentric distance. The total mass in the spheroid is uncertain because its mass-to-light ratio is not known. The spheroids of spiral galaxies are often thought to be similar to elliptical galaxies and the mass-to-light ratios of the inner parts of elliptical galaxies is estimated to be \(\sim 10 M_\odot/L_\odot \text{V} \) (cf. Lauer 1985; Peletier 1989). On the other hand, the mass-to-light ratios of the stellar disks of spirals are typically \(\lesssim 5 M_\odot/L_\odot \text{V} \) (cf. Begeman 1987; Kent 1987). We therefore consider models with total spheroid masses of \(1.2 \times 10^{10} M_\odot\) and \(2.4 \times 10^{10} M_\odot\). For the density profile of the spheroid, we adopt the spherical distribution that produces a de Vaucouleurs \(R^{1/4}\) law in projection. (A derivation of this volume density profile and the resulting expression for the gravitational acceleration it provides can be found in the Appendix.) We took the effective radius, \(R_e\), of the spheroid to be 2.87 kpc, which agrees with that from Bahcall and Soneira after rescaling to \(R_\odot = 8.5\) kpc.

The third element in our model of the Galaxy is the rotation curve. The Galactic rotation curve is approximately flat between 3 and 17 kpc, but is consistent with rising or falling by 14\% (or 30 km s\(^{-1}\)) over that range (Fich, Blitz & Stark 1989). Although the Galactic rotation curve is consistent with being flat where it has been measured, the slope is tied to local values of \(R\) and \(v\) (Schechter 1993). This means that because the the rotation curve is generally measured in fundamentally different ways inside and outside the solar circle, it is possible that the rotation curve rises inside the solar circle, but falls outside it, or vice versa. We were therefore led to examine a total of nine model rotation curves that span this range of possibilities. Each curve is pinned at the solar circle to have the IAU value of 220 km s\(^{-1}\) for the circular speed of the local standard of rest. This value

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the rotation curves of external galaxies, we do not include here models with a Galactic maximal disk of \(\Sigma_0 = 100 M_\odot \text{ pe}^{-2}\) because such models are inconsistent with the data of Kuijken & Gilmore (1989). In any event, the amount of disk dark matter will be directly measured by the MACHO and OGLE experiments themselves (Paczyński 1991; Griest et al. 1991).
has become somewhat controversial of late (for a discussion, see Merrifield 1992; Fich & Tremaine 1991). However, the optical depth of a halo having a smaller rotation speed than the standard value by $\eta$ can be found by incorporating a disk and spheroid that are larger by $\eta^{-2}$ and scaling the resulting optical depth down by $\eta^2$. The speeds at 3 and 17 kpc are allowed to vary up or down from this local speed and a linear extrapolation is taken to the local neighborhood; the total deviation from flatness is less than or equal to a 14% difference in rotation speed in this range. For example, the model rotation curve labeled “rise-rise” in our set of nine corresponds to a linear rise in rotation speed from 208.2 to 220 km s$^{-1}$ from 3 to 8.5 kpc, and then a linear rise to 238.2 km s$^{-1}$ to 17 kpc.

Once the disk, spheroid, and rotation curve are specified, we find the best fit spherical halo by setting the flattening parameter $\cos \psi = 1$ in equation (2.1), and minimizing $\chi^2$ with respect to $v_\infty$ and $a$. We then repeat the procedure for an E6 halo by setting $\cos \psi = 0.4$. The optical depths along various lines of sight are then calculated using equation (2.6), and assuming distances to the LMC and SMC of 50 and 63 kpc respectively (Binney & Tremaine 1987).

Within the constraints provided by observations, our choice for the parameterization of the Galactic rotation curve is arbitrary. In many cases the minima in $\chi^2$ were quite broad, so that the halo core radius, especially, was poorly constrained. Since the fits were not made to data points weighted by observational uncertainties, but to the model curves, the $\chi^2$ values associated with the fits are not meaningful for testing the absolute goodness-of-fit for a particular model, or the relative goodness-of-fit between models. We stress that our only aim was to explore a reasonable range of the halo parameters $a$ and $v_\infty$, constrained by our knowledge of the rotation curve of the Galaxy and the mass of its visible components. As we will show, the most important diagnostic for halo flattening is independent of the present uncertainty in these quantities.
4. Untruncated Halos

Our results for untruncated halos (that is, halos that extend to at least the Magellanic clouds) are illustrated in Table 1, where we display optical depths and ratios of optical depths for Galactic models consisting of a disk with no dark mass normalized locally at $\Sigma_0 = 50 M_\odot \text{pc}^{-2}$ and a relatively light spheroid with a mass-to-light ratio of $5M_\odot/L_\odot$. The nine columns of the table represent nine different rotation curves that span current observational constraints, as described above. The first two rows give the core radius, $a$ (in kpc), of the best fit models for a spherical and an E6 halo; rows 3 and 4 show the corresponding asymptotic circular speed, $v_\infty$, of the halo. Rows 5, 6, and 7 show the optical depth toward the LMC, the SMC, and Baade’s Window for a spherical halo. Row 8 gives the ratio of the optical depth toward the LMC for an E6 relative to a spherical halo. Rows 9 and 10 indicate the ratio of the optical depth toward the SMC compared to that toward the LMC for a spherical and E6 halo, respectively. Rows 11 and 12 give the ratios of the optical depth to Baade’s Window compared to the LMC for spherical and E6 halos.

Across the range of rotation curves considered, the best fit core radii and asymptotic circular halo speeds displayed in Table 1 vary significantly — from about 0 to 1.7 kpc, and 160 to 200 km s$^{-1}$, respectively — but these quantities vary much less between the E0 and E6 models for a given model rotation curve. This is especially true of asymptotic circular speed. The optical depth toward each of the three lines of sight varies by about a factor of 1.5 across the range of model rotation curves. Nevertheless, the ratio of the optical depths toward the SMC and the LMC is constant over the entire range of models to within 1%. This is true separately for the E0 and E6 halos, but the ratio is different in the two cases. For E0, $\tau_{\text{SMC}}/\tau_{\text{LMC}} = 1.47$, while for E6, $\tau_{\text{SMC}}/\tau_{\text{LMC}} = 0.96$. As anticipated in § 2, this indicates that for the range of models considered in Table 1, the ratio $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ is an excellent indicator of halo flatness. By contrast, the ratio toward Baade’s window in the Galactic bulge, $\tau_{\text{BW}}/\tau_{\text{LMC}}$, varies by more than a factor of nearly 2.
Finally, note from row 8 that $\tau_{\text{LMC}}$ is almost exactly the same for an E0 as an E6 halo for each rotation curve considered.

In Table 2 we present results for a model in which the observed components of the Galaxy are substantially heavier. The disk is increased in mass by 50% and the spheroid by 100%.

General comments made about Table 1 also apply to Table 2, except that the core radii and asymptotic circular speeds vary over a wider range (now, $0 \leq a \leq 12$ kpc and $110 \leq v_\infty \leq 300$ km s$^{-1}$). The full range in halo core radius and asymptotic speeds spanned by our models agrees favorably with that inferred for external Sb and Sc spirals (2.5 $\leq a \leq 15$ kpc and $116 \leq v_\infty \leq 307$ km s$^{-1}$, Begeman 1987), and with the values of $a = 3$ kpc and $v_\infty = 230$ km s$^{-1}$ advocated by Merrifield (1992) for the Milky Way. Note that the range of best-fit halo core radii lie comfortably within the range plotted in Figures 1 and 2. We should not be surprised, therefore, that the ratio of $\tau_{\text{LMC}}$ for E6 compared to that for E0 halos

| Rotation Curve | Fall- | Fall- | Fall- | Flat- | Flat- | Flat- | Rise- | Rise- | Rise- |
|----------------|------|------|------|------|------|------|------|------|------|
| $a$(kpc) (E0)  | 0.00 | 0.00 | 0.33 | 0.00 | 0.24 | 0.86 | 0.16 | 0.79 | 1.59 |
| $a$(kpc) (E6)  | 0.00 | 0.00 | 0.41 | 0.00 | 0.29 | 0.96 | 0.20 | 0.94 | 1.70 |
| $v_c$(km s$^{-1}$) (E0) | 162 | 170 | 184 | 158 | 170 | 190 | 158 | 176 | 200 |
| $v_c$(km s$^{-1}$) (E6) | 162 | 170 | 183 | 158 | 170 | 188 | 157 | 176 | 196 |
| $\tau_{\text{LMC}} \times 10^6$ (E0) | 0.339 | 0.373 | 0.436 | 0.325 | 0.375 | 0.463 | 0.321 | 0.401 | 0.508 |
| $\tau_{\text{SMC}} \times 10^6$ (E0) | 0.499 | 0.549 | 0.642 | 0.479 | 0.553 | 0.682 | 0.473 | 0.590 | 0.747 |
| $\tau_{\text{BW}} \times 10^6$ (E0) | 0.527 | 0.580 | 0.631 | 0.505 | 0.562 | 0.521 | 0.490 | 0.467 | 0.404 |
| $\tau_{\text{LMC}}$ (E6/E0) | 1.002 | 1.002 | 0.999 | 1.002 | 1.001 | 0.988 | 1.002 | 0.999 | 0.970 |
| $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ (E0) | 1.474 | 1.474 | 1.473 | 1.474 | 1.473 | 1.472 | 1.474 | 1.472 | 1.469 |
| $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ (E6) | 0.962 | 0.962 | 0.962 | 0.962 | 0.962 | 0.962 | 0.962 | 0.962 | 0.962 |
| $\tau_{\text{BW}}/\tau_{\text{LMC}}$ (E0) | 1.556 | 1.556 | 1.448 | 1.556 | 1.497 | 1.124 | 1.527 | 1.166 | 0.796 |
| $\tau_{\text{BW}}/\tau_{\text{LMC}}$ (E6) | 1.915 | 1.916 | 1.858 | 1.915 | 1.886 | 1.647 | 1.901 | 1.654 | 1.313 |
(row 8), and the ratio $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ for both E0 (row 9) and E6 (row 10) halos are nearly constant across the row, i.e., they are nearly independent of the assumptions made about the shape of the Galactic rotation curve between 3 and 17 kpc. Not only are the values constant within a row, but they are equal to the values in the corresponding row in Table 1. That is, we find that for untruncated halos, $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ depends only on the ellipticity of the halo and not on the details of the rotation curve or on the masses of the disk and the spheroid. Furthermore, for any of mass models that we explored that led to reasonable values of the halo parameters, we find that $\tau_{\text{LMC}} (E6)/\tau_{\text{LMC}} (E0) \sim 1$, that is: for untruncated halos, the optical depth toward the LMC is independent of the halo flattening regardless of the value of other Galactic parameters.

### TABLE 2
Optical Depths for Micro-Lensing (Heavy Disk and Spheroid)

\[ M_{\text{sph}} = 2.4 \times 10^{10} M_\odot \quad \Sigma_0 = 75 \ M_\odot \text{ pc}^{-2} \]

| Rotation Curve | Fall- | Fall- | Fall- | Flat- | Flat- | Flat- | Rise- | Rise- | Rise- | Rise- |
|----------------|------|------|------|-------|-------|-------|-------|-------|-------|-------|
| $a(\text{kpc})$ (E0) | 0.00 | 0.11 | 1.77 | 0.00 | 1.58 | 6.39 | 1.48 | 5.17 | 11.6  |
| $a(\text{kpc})$ (E6)  | 0.00 | 0.13 | 1.81 | 0.00 | 1.56 | 6.76 | 1.74 | 6.01 | 12.6  |
| $v_c(\text{km s}^{-1})$ (E0) | 117 | 130 | 163 | 113 | 144 | 224 | 124 | 184 | 300   |
| $v_c(\text{km s}^{-1})$ (E6)  | 117 | 130 | 160 | 113 | 140 | 216 | 124 | 184 | 292   |
| $\tau_{\text{LMC}} \times 10^6$ (E0) | 0.178 | 0.217 | 0.339 | 0.164 | 0.264 | 0.532 | 0.197 | 0.381 | 0.710 |
| $\tau_{\text{SMC}} \times 10^6$ (E0) | 0.263 | 0.320 | 0.498 | 0.242 | 0.387 | 0.760 | 0.289 | 0.549 | 0.997 |
| $\tau_{\text{BW}} \times 10^6$ (E0) | 0.277 | 0.335 | 0.250 | 0.256 | 0.211 | 0.114 | 0.165 | 0.104 | 0.078 |
| $\tau_{\text{LMC}}$ (E6/E0) | 1.002 | 1.000 | 0.960 | 1.002 | 0.956 | 0.961 | 0.997 | 1.001 | 1.015 |
| $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ (E0) | 1.474 | 1.474 | 1.468 | 1.474 | 1.469 | 1.430 | 1.470 | 1.440 | 1.404 |
| $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ (E6) | 0.962 | 0.962 | 0.963 | 0.962 | 0.963 | 0.971 | 0.963 | 0.969 | 0.996 |
| $\tau_{\text{BW}}/\tau_{\text{LMC}}$ (E0) | 1.556 | 1.541 | 0.739 | 1.556 | 0.801 | 0.215 | 0.837 | 0.272 | 0.109 |
| $\tau_{\text{BW}}/\tau_{\text{LMC}}$ (E6) | 1.915 | 1.909 | 1.271 | 1.915 | 1.374 | 0.374 | 1.299 | 0.430 | 0.174 |
5. Truncated Halos

By introducing a truncation radius, \( R_T \), for the halo, one introduces additional uncertainty into the problem; we describe here our motivation for considering this possibility. From the motion of distant globular clusters and satellite galaxies, it is believed that the dark halo must extend at least to \( \sim 3R_0 \) (for a review, see Ashman 1992 and reference therein). Zaritsky et al. (1992) have shown that the dark halo in other spiral galaxies extends to several hundred kpc. Thus, if the dark halo of the Milky Way is similar to that of other galaxies, it might seem unnecessary to explore the effect of halo truncation. MACHO searches, however, are not sensitive to dark matter per se, but only to the fraction of the dark matter in compact objects. It is quite possible there are two forms of dark matter, baryonic and non-baryonic. The baryonic dark matter, being dissipational, would form MACHOs that might lie in a relatively compact (and perhaps flattened) halo, while the non-baryonic matter might form a more extended halo. The observed orbits of satellite galaxies and other dynamical arguments would not be sensitive to such a distinction. The introduction of a truncation radius for MACHOs, therefore, allows us to explore the possibility that baryonic dark matter does not extend very far beyond the luminous disk of our Galaxy.

To examine this question, we computed optical depths along the same lines of sight for the same Galactic parameters used for untruncated halos, but truncated the dark halo on isodensity surfaces that cross the Galactic plane at twice or four times the solar circle. In Table 3, we compare truncated and untruncated halos for the class of models considered in Table 1, i.e., those with \( \Sigma_0 = 50 M_\odot \text{pc}^{-2} \) and \( M_{\text{sph}} = 1.2 \times 10^{10} M_\odot \). The subscripts ‘2’ and ‘4’ refer to halo truncation at \( R_T = 2R_0 \) and \( R_T = 4R_0 \) respectively. The subscript ‘\( \infty \)’ denotes an untruncated halo. The first four rows give the ratios of the optical depths of truncated to untruncated halos toward the LMC and the SMC for spherical halos of various truncation radii. Rows 5, 6, and 7 give the ratio of the optical depths toward the LMC in E6 compared to E0 halos. Rows 8, 9, and 10 give the ratios of the optical
depths toward the SMC compared to the LMC for spherical halos of differing \( R_T \); the final three rows give the same quantities for E6 halos.

**TABLE 3**

Optical Depths for Micro-Lensing by Truncated Halos (Light Disk and Spheroid)

\[
M_{\text{sph}} = 1.2 \times 10^{10} M_\odot \quad \Sigma_0 = 50 \, M_\odot \, \text{pc}^{-2}
\]

| Rotation Curve | Fall- | Fall- | Fall- | Flat- | Flat- | Flat- | Rise- | Rise- | Rise- |
|----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( \tau_{\text{LMC},4}/\tau_{\text{LMC},\infty} \) (E0) | 0.944 | 0.944 | 0.944 | 0.945 | 0.944 | 0.934 | 0.944 | 0.937 | 0.917 |
| \( \tau_{\text{LMC},2}/\tau_{\text{LMC},\infty} \) (E0) | 0.619 | 0.619 | 0.614 | 0.619 | 0.615 | 0.567 | 0.615 | 0.583 | 0.495 |
| \( \tau_{\text{SMC},4}/\tau_{\text{SMC},\infty} \) (E0) | 0.917 | 0.917 | 0.915 | 0.917 | 0.916 | 0.898 | 0.916 | 0.904 | 0.868 |
| \( \tau_{\text{SMC},2}/\tau_{\text{SMC},\infty} \) (E0) | 0.646 | 0.646 | 0.640 | 0.646 | 0.642 | 0.583 | 0.642 | 0.602 | 0.499 |
| \( \tau_{\text{LMC},\infty} \) (E6/E0) | 1.002 | 1.000 | 0.960 | 1.002 | 0.956 | 0.961 | 0.997 | 1.001 | 1.015 |
| \( \tau_{\text{LMC},4} \) (E6/E0) | 0.861 | 0.859 | 0.823 | 0.861 | 0.820 | 0.805 | 0.855 | 0.841 | 0.812 |
| \( \tau_{\text{LMC},2} \) (E6/E0) | 0.770 | 0.768 | 0.734 | 0.770 | 0.733 | 0.706 | 0.762 | 0.733 | 0.689 |
| \( \tau_{\text{SMC},\infty}/\tau_{\text{LMC},\infty} \) (E0) | 1.474 | 1.474 | 1.468 | 1.474 | 1.469 | 1.430 | 1.470 | 1.440 | 1.404 |
| \( \tau_{\text{SMC},4}/\tau_{\text{LMC},4} \) (E0) | 1.431 | 1.431 | 1.424 | 1.430 | 1.425 | 1.374 | 1.426 | 1.389 | 1.329 |
| \( \tau_{\text{SMC},2}/\tau_{\text{LMC},2} \) (E0) | 1.539 | 1.538 | 1.531 | 1.539 | 1.533 | 1.471 | 1.533 | 1.488 | 1.415 |
| \( \tau_{\text{SMC},\infty}/\tau_{\text{LMC},\infty} \) (E6) | 0.962 | 0.962 | 0.963 | 0.962 | 0.963 | 0.971 | 0.963 | 0.969 | 0.996 |
| \( \tau_{\text{SMC},4}/\tau_{\text{LMC},4} \) (E6) | 0.867 | 0.867 | 0.866 | 0.867 | 0.866 | 0.857 | 0.865 | 0.859 | 0.851 |
| \( \tau_{\text{SMC},2}/\tau_{\text{LMC},2} \) (E6) | 0.886 | 0.885 | 0.884 | 0.885 | 0.884 | 0.872 | 0.884 | 0.874 | 0.861 |

The most remarkable thing about Table 3 is that all 27 of the entries for the ratio \( \tau_{\text{SMC}}/\tau_{\text{LMC}} \) for truncated, spherical halos (rows 8, 9, and 10) and for truncated, flattened (E6) halos (rows 11, 12, and 13) are approximately equal. We found this to be true for all models that we explored that gave reasonable best-fit halo parameters.

In Figure 3, we present the ratio \( \tau_{\text{SMC}}/\tau_{\text{LMC}} \) as a function of the assumed Galactic rotation curve model for each of the four mass models (four combinations of light and heavy disk and bulge) at each of three halo truncations. It is clear from this figure that the optical depth ratio \( \tau_{\text{SMC}}/\tau_{\text{LMC}} \) is quite insensitive to assumptions about the rotation curve and \( M/L \) of the luminous components of the Galaxy. The ratio is sensitive to the halo truncation radius, but even allowing for the possibility
of strong truncation (at $2R_0$), the difference in $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ for E6 as opposed to E0 is unambiguous, and as we shall show in the next section, should be measurable. We conclude that, the ratio of optical depths, $\tau_{\text{SMC}}/\tau_{\text{LMC}}$, is a robust, model-independent indicator of halo flatness.

6. Detectability

If MACHOs are not detected by ongoing experiments, how would allowance for ellipticity and truncation of the halo affect the interpretation of this null result? If MACHOs are detected, to what accuracy can the ellipticity and truncation of the halo be measured? In order to address these questions, we must review the current MACHO search technique.

The characteristic time scale of a MACHO event is the time taken by the MACHO to cross the Einstein ring radius:

$$\omega^{-1} \equiv \frac{r_*}{v_t} = \frac{\sqrt{4GML(1-L/D)}}{vtc} \sim 70 \text{ days } (M/M_\odot)^{1/2}, \quad (6.1)$$

where $v_t$ is the transverse speed and the evaluation is for “typical” parameters of $L \sim 10$ kpc, $v_t \sim 200 \text{ km s}^{-1}$, and $L/D \sim 1/5$. The expected number of events underway at any given time is $N_* \tau$, where $N_*$ is the number of stars being observed and $\tau$ is the optical depth. Note that if $M \gtrsim 100M_\odot$, then the characteristic time of a MACHO event becomes several years, which is comparable to or longer than the planned four years of the MACHO Collaboration experiment. In this case, the full light curve would not be observed. For this reason, it was originally believed that the experiment would be insensitive to MACHOs in this high mass range. However, Gould (1992) showed that candidate events could be recognized from the data for MACHOs as massive as $10^6 M_\odot$ and that these candidates could be distinguished from backgrounds (such as variable stars) by a variety of techniques. For example, for masses $M \gtrsim 10^6 M_\odot$, the two lensed stellar images could be resolved by (an unrepaired) Hubble Space Telescope, and further confirmation could be obtained by measuring the proper motion of these images.
The total number of observed events depends on the characteristic mass of the MACHOs. If the events are longer than the duration of the experiment, then the approximately $N_* \tau$ events that are taking place on the first day of the observations will be the only events during the entire span of observations. If the events are shorter than the observation time (i.e., $M \lesssim 100 M_\odot$), then the expected total number of events is

$$N_{\text{events}} = \frac{2}{\pi} N_* \tau \omega T_{\text{eff}},$$  \hspace{1cm} (6.2)$$

where $T_{\text{eff}}$ is the effective time of the observations, which depends weakly on the mass scale of the MACHOs.

The LMC will be observed for about 8 months per year. If the MACHO events last more than a few months (i.e., $M \gtrsim M_\odot$), and therefore bridge the four month hiatus, then $T_{\text{eff}} \sim 4$ yrs, the length of the MACHO experiment. For smaller MACHOs, $T_{\text{eff}} = 4 \times 8$ months = 2.7 yrs. If the events last less than a few days (i.e., $M \lesssim 10^{-3} M_\odot$), then the light curves will be too poorly sampled to be recognized as lensing events. In this case, only “spikes” will be detected. Spikes are high-magnification single-observation events from a light curve that is too short and poorly sampled to be resolved temporally. Spikes can be distinguished from observational errors, however, because the stars are observed simultaneously in two bands. A genuine spike will appear equally magnified in both bands while an error (such as a cosmic ray event) will not. In order to be distinguished from background, however, the spike will need to be magnified by a factor of at least a few; we will adopt an average minimum detectable magnification of $A_{\min} = 2$. The number of such spike events is

$$N_{\text{spike}} = 2N_* \tau [ (1 - A_{\min}^2)^{-1/2} - 1 ] N_{\text{obs}}$$

$$\simeq N_* \tau A_{\min}^{-2} N_{\text{obs}} \sim 135 N_* \tau$$  \hspace{1cm} (6.3)$$

where $N_{\text{obs}}$ is the number of observations made for each of the $N_*$ program stars. In making this evaluation, we assumed 55% good weather over 8 months of observation per year, for a total of 4 years (K. Griest 1992, private communication). We
caution that this estimate of the number of spike events is rather uncertain due to difficulties in estimating $N_{\text{obs}}$; due to field crowding, the number of photometric observations is not, in general, a linear function of time. Furthermore, the value of $A_{\text{min}}$ varies from star to star with photometric errors.

While the statistical properties of the observed spikes would be a fairly convincing signature of lensing, one would want additional confirmation. This could be obtained by observing a subset of the fields, say 10%, more frequently, say 10 times per night. In this way, the individual light curves could be observed in detail. This altered observing schedule, however, would not substantially change the estimate of the number of recognizable events given by equation (6.3), but only the confidence with which spike events could be associated with microlensing events.

6.1. Null Result

Suppose that no candidate events were observed during the four years of observations, or that candidates were observed, but were shown to be non-lensing events. What classes of MACHO dark matter models could then be ruled out? The answer depends on the flattening of the halo and whether or not it is truncated.

Consider first untruncated spherical halos. From Table 1, we find that for models with a light disk and spheroid, the optical depth toward the LMC is $\tau_{\text{LMC}} \gtrsim 3.2 \times 10^{-7}$, and is somewhat greater toward the SMC. Since $N_* \gtrsim 10^7$ in both Magellanic Clouds combined (K. Griest 1993, private communication), we obtain $N_* \tau \gtrsim 3.5$. Thus, even for very massive MACHOs, for which $\omega T_{\text{eff}} \sim 1$, at least 3.5 events are expected. If none were observed, this would rule out this class of MACHO models with good confidence, that is, at the $1 - e^{-3.5} \sim 97\%$ level. From Table 2, we find that for models with a heavy disk and spheroid, the expected number of events may fall as low as 1.8. Very massive MACHOs could then be ruled out with only modest confidence, $\sim 83\%$. However, a more detailed analysis, taking into account the increased sampling of long duration events, shows that moderately massive $\lesssim 100M_\odot$ MACHOs could be ruled out with good confidence.
If the halo is assumed to be spherical and highly truncated \( R_T = 2R_0 \), then we see from Table 3 that the optical depth toward the LMC is reduced by a factor of about 0.6. If the halo is assumed to be highly flattened (E6) as well as highly truncated, then the optical depth is reduced by another factor of about 0.75. In this extreme case, the expected number of events would be about half the number in a standard spherical halo. If, in addition, the disk and spheroid were heavy, then the expected number of events would fall below unity, in which case no definite statement could be made about heavy MACHOs. From equations (6.1) and (6.2), we see that then only MACHOs of mass \( M \lesssim \mathcal{O}(10) M_\odot \) would be ruled out at a 97% confidence level.

In short, a null result for the MACHO Collaboration experiment would rule out a standard (spherical and untruncated) MACHO halo for masses \( M \lesssim 10^3 - 10^6 M_\odot \), depending on the mass of disk as measured by the MACHO experiment itself. However, a null result would rule out a highly truncated and highly flattened halo only for \( M \lesssim \mathcal{O}(10 - 100) M_\odot \). The higher limit of 100 \( M_\odot \) assumes that very long duration events will be so well sampled that events of lower amplification will contain enough information to discern the signature of microlensing.

### 6.2. Measurement of Halo Flattening

If MACHOs are detected then, as discussed in §§ 2, 4, and 5, the ratio of the optical depths toward the SMC and LMC will be a robust indicator of halo flatness. The precision with which this ratio can be measured depends on the Poisson fluctuations in the number of events detected, and the lensing “backgrounds” toward the two Magellanic Clouds. We address each of these in turn.

For the purposes of making our estimates of the statistical fluctuations, we assume that about 20% of the Magellanic program stars will be in the SMC, so that \( N_{*,\text{SMC}} \sim 2 \times 10^6 \). Most of the statistical fluctuation in the measurement of \( \tau_{\text{SMC}} / \tau_{\text{LMC}} \) will then from \( \tau_{\text{SMC}} \). The optical depth toward the SMC varies somewhat from model to model, but for these purposes we adopt \( \tau_{\text{SMC}} \sim 4 \times 10^{-7} \). Thus,
From equations (6.2) and (6.3), we therefore estimate the fractional precision of the ratio measurement as

$$\sigma \left( \ln \frac{\tau_{\text{SMC}}}{\tau_{\text{LMC}}} \right) \sim \left( \min \{ N_{\text{events,SMC}} \times N_{\text{spike,SMC}} \} \right)^{-1/2} \sim 10\% \times \max \{ 1, (M/0.004M_\odot)^{1/4} \}. \quad (6.4)$$

Since $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ varies by a factor $\sim 1.5$ between E0 and E6 halos, it should be possible to distinguish between these two cases, provided that the MACHOs are substellar.

The principal lensing “background” comes from events due to MACHOs in the Magellanic Clouds themselves. The LMC has a modest halo characterized by a rotation speed of about 80 km s$^{-1}$ (Schommer et al. 1992); this halo may or may not be composed of MACHOs. The optical depth of the SMC halo (if it exists) is an order of magnitude smaller than that of the LMC and can be ignored here. (See Gould 1993 for an extensive discussion of lensing by MACHOs in the LMC and SMC.) The optical depth due to MACHOs in a spherical LMC halo is $\tau_{\text{LMC, int}} \sim 10^{-7}$. For a flattened LMC halo, the optical depth is reduced by approximately the flatness ratio, $\cos \psi_{\text{LMC}}$. Hence, the fraction of all lensing events toward the LMC that are due to LMC MACHOs is not necessarily negligible, and the a priori uncertainty in this fraction may be as much as 10 or 15%.

It is possible to reduce this uncertainty in two ways. First, the difference in the optical depths toward the far and near sides of the LMC would allow one to measure $\tau_{\text{LMC, int}} \cos \psi_{\text{LMC}}$ (Gould 1993), which, unfortunately, is not quite the quantity of most interest, $\tau_{\text{LMC, int}}$. Moreover, the planned four years of observations by the MACHO Collaboration will not be sufficient to make a precise measurement of this quantity. Nevertheless, even the crude constraint obtained from this measurement would be useful. Second, by making several observations per day of all lensing events identified to be in progress, it would be possible to measure directly and with good precision the fraction of events due to LMC Machos (Gould, in preparation). These additional observations would allow one to pick out those lensing events in
which the MACHO passed directly over (or very near) the face of a lensed star. For these events, one could measure the angular speed of the MACHO from the deviation of the light curve from a standard “point-source” light curve. Since the angular speeds of LMC MACHOs are about 15 times slower than those of Galactic MACHOs, LMC MACHOs could be recognized unambiguously. In this way, the optical depth of LMC MACHOs could be measured accurately. Unfortunately, the organization of these additional observations would be a major undertaking.

In brief, there is a systematic uncertainty of $\lesssim 15\%$ in the measurement of the ratio $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ due to the “background” of lensing by LMC MACHOs. This uncertainty can be reduced somewhat by analyzing the spatial distribution of lensing events and can be greatly reduced by undertaking significant additional observations. The statistical uncertainty of the measurement of $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ is 10% provided that the MACHOs are substellar.

7. Conclusions

The primary conclusions of this work can be summarized in four main points:

- If the dark halo of the Galaxy is composed of MACHOs, the ratio of optical depths toward the SMC as compared to the LMC, $\tau_{\text{SMC}}/\tau_{\text{LMC}}$, is a robust measure of the ellipticity of the halo, independent of any current limitations on our knowledge of the the mass of the Galactic disk and spheroid, the slope of the Galactic rotation curve, and the truncation radius of the dark halo.

- If MACHOs are substellar and the SMC is sufficiently sampled, the ongoing MACHO Collaboration experiment will be sensitive enough to use $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ to measure the flattening of the Galactic dark halo.

- The optical depth to the Galactic bulge is not an effective probe of the shape of the dark halo because the expected rate of microlensing is sensitive to Galactic parameters (e.g., the mass of the disk and spheroid), and because contamination due to disk and bulge stars make any events due to MACHOs difficult to disentangle.
• For extreme assumptions about the shape of the Galactic dark halo (E6), the halo truncation radius (twice the solar circle), and the mass of the Galactic disk ($\Sigma_0 = 75 M_\odot \text{pc}^{-2}$) and spheroid ($2.4 \times 10^{10} M_\odot$), the optical depth toward the LMC is sufficiently reduced that the upper limit on the MACHO mass range to which ongoing experiments are sensitive would be lowered to $O(10) M_\odot$. This contrasts sharply with the upper limit of $O(10^6) M_\odot$ for more standard Galactic parameters.

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Appendix: Inversion of the $R^{1/4}$ Law to Obtain the Volume Density and Force: Spherical Case

A derivation of the spherical volume density distribution consistent with a de Vaucouleurs ($R^{1/4}$ law) surface mass density was given in integral form by Poveda et al. (1960). Young (1976) used Poveda’s expression to tabulate numerically the enclosed mass, mean density, force, potential, and escape velocity as a function of radius in the galaxy. A closed-form, analytic approximation to the integral form has been given by Mellier & Mathez (1987); two parameters of their analytic form must be varied with galactocentric radius in order to reproduce the $R^{1/4}$ law over a large radial range. Here, we use Young’s normalized form to derive an expression for the mass interior to $r$, which is needed to calculate the gravitational force due to the spheroid at any point. Since the original reference can be difficult to find, we first rederive the form of the density, $\rho(r)$, that gives an $R^{1/4}$ spheroid in projection, and show that it reduces to Young’s normalized form.

The projected surface density (assuming $M/L$ is constant) of a $R^{1/4}$ spheroid is given by

$$\Sigma(R) = \Sigma(0) \exp \left[ -b \left( \frac{R}{R_e} \right)^{1/4} \right], \quad (7.1)$$

where $R$ is the projected galactocentric radius on the sky, $R_e$ is the effective radius that encloses half the projected mass (or light), and $b = 7.66925$. We require that the spherical density distribution $\rho(r)$ give this form for the surface mass density in projection:

$$\Sigma(R) = \int_{-\infty}^{\infty} dz \, \rho(r) = 2 \int_{R}^{\infty} dr \, \frac{\rho(r)}{\sqrt{1 - (R/r)^2}}. \quad (7.2)$$

If we change to the normalized variables $y \equiv (r/R_e)^2$ and $x \equiv (R/R_e)^2$, then

$$\tilde{\Sigma}(x) = \Sigma(0) \exp (-bx^{1/8}) = R_e \int_{x}^{\infty} dy \, \tilde{\rho}(y) \frac{1}{\sqrt{y - x}}. \quad (7.3)$$
We can now use an Abel Transform (see e.g., Binney & Tremaine 1987) to derive an integral expression for $\tilde{\rho}(y)$.

For any functions $f(x)$ and $g(y)$ satisfying

$$f(x) = \int_x^\infty dy \frac{g(y)}{(y-x)^{1-\alpha}}, \quad \text{where} \quad 0 < \alpha < 1, \quad (7.4)$$

$g(y)$ is given by

$$g(y) = \frac{-\sin \pi \alpha}{\pi} \frac{d}{dy} \int_y^\infty dx \frac{f(x)}{(x-y)^{1-\alpha}}$$

$$= \frac{-\sin \pi \alpha}{\pi} \left[ \int_y^\infty \left( \frac{df(x)}{dx} \right) \frac{dx}{(x-y)^{1-\alpha}} - \lim_{x \to \infty} \frac{f(x)}{(x-y)^{1-\alpha}} \right] \quad (7.5)$$

Thus, applying the Abel Transform to equation (7.3) yields

$$R_e \tilde{\rho}(y) = \frac{b \Sigma(0)}{\pi} \int_y^\infty dx \frac{x^{-7/8} \exp \left[ -bx^{1/8} \right]}{8 \sqrt{x-y}}. \quad (7.6)$$

To retrieve the form given by Poveda et al. (as recast by Young), we must first change variables

$$s^2 \equiv (r/R_e)^2 \equiv j^8 = y, \quad x/y = t^8$$

so that

$$\rho(r) = \frac{b \Sigma(0)}{\pi R_e j^3} \int_1^\infty dt \frac{e^{-bjt}}{\sqrt{j^8 - 1}} \quad (7.7)$$

and then rewrite the normalization in terms of the total mass, $M_T$. Integrating
the surface mass density of an $R^{1/4}$ profile given by eq. (7.1), we find

$$\Sigma(0) = \frac{M_T}{8! \pi R^2_c} b^8,$$

which implies

$$\rho(r) = \frac{M_T}{8! \pi^2 R^3_c j^3} \int_1^\infty \frac{dt}{\sqrt{t^8 - 1}} e^{-bjt}$$  \hspace{1cm} (7.8)

This form should be compared to that of Young, whose dimensionless density is given as

$$\tilde{\rho}(s) = \frac{1}{2 j^3} \int_1^\infty \frac{dt}{\sqrt{t^8 - 1}} e^{-bjt}$$  \hspace{1cm} (7.9)

and whose the dimensionless total mass, $\tilde{M}$, is defined as $\tilde{M} \equiv \int_0^\infty ds \tilde{\rho}(s) 4\pi s^2$

From these definitions of $s$ and $\tilde{M}$, then,

$$\rho(r) = \left( \frac{M_T}{\tilde{M} R^3_c} \right) \tilde{\rho}(s)$$  \hspace{1cm} (7.10)

which requires, by comparison of equations (7.8) with (7.9), that

$$\tilde{M} = \frac{8! \pi^2}{2 b^9},$$  \hspace{1cm} (7.11)

This analytic result agrees with the numerical result of Young who found that $\tilde{M} = 2.1676 \times 10^{-3}$.

Our primary interest is to compute the gravitational force due to a spherical de Vaucouleurs spheroid at any distance $r_0$, so that we can compare the expected circular rotation speed for a given mass model with an assumed Galactic rotation
curve. To this end, we require an expression for the mass from $\rho(r)$ enclosed by a sphere of radius $r = r_0$. From equation (7.10) we have

$$M(< r_0) = \frac{4\pi M_T}{M} \int_0^{r_0/R_e} ds \, \bar{\rho}(s) \, s^2$$

By repeated integrations by parts, the integral over $j$ can be done analytically to obtain a polynomial in $(bt)^{-1}$ that multiplies an exponential factor,

$$M(< r_0) = \frac{8\pi M_T}{M} \int_1^{(r_0/R_e)^{1/4}} \frac{dt}{\sqrt{t^8 - 1}} \int_0^{(r_0/R_e)^{1/4}} dj \, j^8 e^{-btj}$$

One is then left with the integral over $dt$, which can be done numerically. Since $(t^8 - 1) = (t - 1)(t + 1)(t^2 + 1)(t^4 + 1)$, the integrand does have an integrable square root singularity at the lower limit, which requires some care. The radial acceleration at $r$ is then just given by $a = GM(< r)/r^2$. 

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FIGURE CAPTIONS

Fig. 1  The optical depth to microlensing as a function of halo core radius $a$ in kpc for a spherical (E0, left panels) and flattened (E6, right panels) Galactic dark halo, along lines of sight to the SMC (top panels) and LMC (bottom panels). The halo parameter $v_{\infty}$ (see text) is assumed to be 220 km s$^{-1}$ for the purposes of this figure; all optical depths scale strictly as $v_{\infty}^2$. Solid lines trace the optical depths through an untruncated Galactic halo (that is, a halo that extends to at least the Magellanic Clouds); dashed lines are for a halo truncated in the Galactic plane at four times the solar circle; dotted lines indicate a halo truncated at twice the solar circle.

Fig. 2  The ratio of optical depths along SMC and LMC lines of sight, $\tau_{\text{SMC}}/\tau_{\text{LMC}}$, as a function of halo core radius for a spherical (E0) and flattened (E6) Galactic dark halo for different halo truncations. Note the expanded vertical scale; the ratios vary only weakly with core radius, but are strong functions of halo flattening.

Fig. 3  The ratio of optical depths along SMC and LMC lines of sight, $\tau_{\text{SMC}}/\tau_{\text{LMC}}$, as a function of assumed Galactic rotation curve for four different mass models for the Galactic disk and spheroid, and three different halo truncation radii. (See the text for a detailed description of the models.) The rotation curve models are numbered as in Tables 1, 2 and 3, namely: 1 = fall – fall, 2 = fall – flat, 3 = fall – rise, 4 = flat – fall, 5 = flat – flat, 6 = flat – rise, 7 = rise – fall, 8 = rise – flat, 9 = rise – rise. Note that for a given halo flattening and truncation radius, all four mass models and all nine rotation curve models give nearly identical $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ ratios. Only the heaviest disk and spheroid model deviates somewhat for rotation curve models 6, 8, and 9. The ratio $\tau_{\text{SMC}}/\tau_{\text{LMC}}$ is more sensitive to halo flattening than any other Galactic parameter, including halo truncation.