Large angular scale CMB anisotropy from an excited initial mode

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Abstract: According to inflationary cosmology, the CMB anisotropy gives an opportunity to test predictions of new physics hypotheses. The initial state of quantum fluctuations is one of the important options at high energy scale, as it can affect observables such as the CMB power spectrum. In this study a quasi-de Sitter inflationary background with approximate de Sitter mode function built over the Bunch-Davies mode is applied to investigate the scale-dependency of the CMB anisotropy. The recent Planck constraint on spectral index motivated us to examine the effect of a new excited mode function (instead of pure de Sitter mode) on the CMB anisotropy at large angular scales. In so doing, it is found that the angular scale-invariance in the CMB temperature fluctuations is broken and in the limit \(\ell < 200\) a tiny deviation appears. Also, it is shown that the power spectrum of CMB anisotropy is dependent on a free parameter with mass dimension \(H << M < M_p\) and on the slow-roll parameter \(\epsilon\).

Keywords: initial state, power spectrum, CMB anisotropy

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1 Introduction

Today’s cosmological observations give us a useful pattern of the Cosmic Microwave Background Anisotropies (CMB). This pattern gives us a good snapshot of the temperature fluctuations of photons on the last scattering surface (LSS) [1, 2]. Temperature anisotropies in the photons arise due to several physical effects. One of the interesting effects in which the CMB photons are gravitationally red-shifted on the LSS, as they decoupled from matter, is known as the Sachs-Wolfe effect. Moreover, it is the dominant effect in the CMB anisotropy at large angular scales [1–3]. Indeed, this anisotropy is seeded by the primordial perturbation in the early universe which manifests itself in the anisotropies of the CMB photons as well as the matter density perturbation today [2–6]. So, the large angular scale anisotropies of these photons are actually encoded by primordial perturbations.

It is well-known that a particular pattern of thermal anisotropies depends not only on the particular model of inflation, but also on the initial vacuum state of the quantum fluctuations [7–16]. It is usually considered that when these fluctuations are generated, they are initially in the minimum energy state so-called Bunch-Davies or de Sitter vacuum [13–17]. But according to new observational data about the scalar spectral index, it has been shown that the background geometry of inflation is not pure de Sitter [7, 8]. So, one can consider a non-Bunch-Davies state as an initial state of scalar field fluctuations of which the physical origin is unknown. According to some studies, it may be due to some initial condition arising from pre-inflationary evolution as calculated in Ref. [18]; or from a nonsingular bounce as studied in Refs. [19–26]; or from trans-Planckian physics in Refs. [27, 28]; or from the string theory effects in Refs. [29–31]; and so on. These motivate us to choose, from among the various possibilities for initial vacuum states of quantum fluctuations, an excited state which is constructed based on an excited de Sitter mode and which leads to the higher order trans-Planckian corrections [32]. Actually, these higher order corrections are produced because the applied non-trivial mode are non-linear (up to second order) with respect to \(\frac{1}{k\eta}\) [32–34]. On the other hand the free parameter with mass dimension as a cutoff scale in the corrections emphasises that the effect of these excited modes is bounded. In addition, as a result of using this excited initial state, it has been shown that we can have particle creation [35]. They have also been used to

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calculate the scale-dependency of the primordial power spectrum [36]. In this work, it is expected that employing an initial excited de Sitter mode for calculating the primordial perturbed gravitational potential leads to the breaking of angular-scale symmetry in the CMB power spectrum.

The layout of the paper is as follows. In Section 2, the excited de Sitter mode for nearly de Sitter inflationary background is briefly introduced and the power spectrum with this mode is calculated. In Section 3, the definition of large-scale inhomogeneity and anisotropy is reviewed and the scale-dependency of large-scale matter density perturbation and Large-Angular Scale CMB Anisotropy resulting from the excited de Sitter mode are calculated. Conclusions are given in the final section.

## 2 Non-trivial initial vacuum for inflationary background

In curved space-time, the quantization of a scalar field is similar to the quantization in flat space-time (i.e. Minkowskian), but the gravitational interaction due to the curvature of space-time can act as an external classical field on flat space-time, which could be generally non-homogeneous and non-stationary [37]. In general, due to the absence of Killing vector in a curved space-time, the notion of vacuum is ambiguous [38]. However, in de Sitter space with maximal symmetry, it is possible to define the vacuum state under the de Sitter symmetry group [17]. So, it is a logical choice that we consider de Sitter space as a background for our theory and use the following metric to describe the expanding inflationary universe with curved space-time

\[ ds^2 = dt^2 - a^2(t) dx^2 = a^2(\eta)(d\eta^2 - dx^2). \] (1)

For de Sitter space-time, the scale factor is given by \( a(t) = \exp(Ht) \), or equivalently in a conformal formalism \( a(\eta) = \frac{1}{H\eta} \). There are several models for the inflationary universe, but the simple and most popular one is a minimally coupled scalar field (inflaton) in an inflating background

\[ S = \frac{1}{2} \int d^4x \sqrt{-g} \left( R - (\nabla \phi)^2 - m^2 \phi^2 \right). \] (2)

The Fourier components corresponding to the inflaton field satisfy the equation of motion [5]

\[ \phi''_k - \frac{2}{\eta} \phi'_k + (k^2 + a^2m^2) \phi_k = 0, \] (3)

where the prime denotes the derivative with respect to conformal time \( \eta \). By considering the massless case and re-scaling of \( \phi_k \) as \( u_k = a\phi_k \), Equation (3) becomes

\[ u''_k + \omega_k^2(\eta)u_k = 0, \] (4)

where \( \omega_k^2(\eta) = k^2 - \frac{a''}{a} \). The solutions of this equation give the positive and negative frequency modes. So the general solutions of the equation of motion are given by

\[ u_k = A_k H^{(1)}_\mu(|k\eta|) + B_k H^{(2)}_\mu(|k\eta|), \] (5)

where \( H^{(1,2)}_\mu \) are the Hankel functions of the first and second kind. By imposing \( \{u(x, \eta), \pi(y, \eta)\} = i\delta(x - y) \), namely the equal-time commutation relations, and also implementing secondary quantization in Fock representation, the canonical quantization can be performed for the scalar field \( u \) and its canonically conjugate momentum \( \pi = u' \), [38].

### 2.1 Excited de Sitter mode for quasi-de Sitter space-time

In the pure de Sitter background, with \( \frac{a''}{a} = \frac{2}{\eta^2} \), the general solution of (4) becomes [5]

\[ u_k^S = \frac{A_k}{\sqrt{2k}} \left( 1 - \frac{i}{k\eta} \right) e^{-ik\eta} + \frac{B_k}{\sqrt{2k}} \left( 1 + \frac{i}{k\eta} \right) e^{ik\eta}, \] (6)

where \( A_k \) and \( B_k \) are Bogoliubov coefficients. By setting \( A_k = 1 \) and \( B_k = 0 \), this solution leads to the Bunch-Davies mode [17]

\[ u_k^{BD} = \frac{1}{\sqrt{2k}} \left( 1 - \frac{i}{k\eta} \right) e^{-ik\eta}. \] (7)

As previously mentioned, the observational data released by WMAP and Planck reveal that the inflationary universe can be described by nearly de Sitter space-time, where for the initial state any approximate de Sitter mode can be selected as an acceptable mode. Motivated by this fact, in Ref. [33] we suggested the following excited mode function as the fundamental mode during the inflationary era:

\[ u_k^{exc} \simeq \frac{1}{\sqrt{2k}} \left( 1 - \frac{i}{k\eta} - \frac{1}{2} \left( \frac{i}{k\eta} \right)^2 \right) e^{-ik\eta}. \] (8)

In the far past time limit, in a straightforward manner, if \( \frac{1}{k\eta} \neq 0 \) and \( \frac{1}{k^2\eta^2} \rightarrow 0 \), this mode function asymptotically approaches to the de Sitter mode function.

Our reason to introduce this excited-de Sitter solution are physical considerations, as noted in Section 3 of Ref. [32]. In Ref. [33], for the first time, we used this excited solution with the auxiliary fields to calculate the finite and renormalized power spectrum. Also, in Ref. [36], by the Planck results (2013) for scalar spectral index, we showed that the index of the Hankel function \( \mu \) lies in the range of \( 1.51 \leq \mu \leq 1.53 \), and this important
result stimulates us to move from the dS mode to excited dS mode. Therefore in approximate de Sitter space-time, according to (6), the general solutions of the equations of motion, including negative and positive frequency solutions, can be given by [32],

\[
u^\text{dS}_k \simeq \frac{A_k}{\sqrt{2k}} \left(1 - \frac{i}{k \eta} - \frac{i}{2} \left(\frac{1}{k \eta}\right)^2 \right) e^{-ik\eta} + \frac{B_k}{\sqrt{2k}} \left(1 + \frac{i}{k \eta} - \frac{i}{2} \left(-\frac{1}{k \eta}\right)^2 \right) e^{+ik\eta}.
\] (9)

It is necessary to emphasise that we consider the mode functions (6) for pure de Sitter space-time as the general exact solution of (4), while for approximate de Sitter space-time, we consider excited-de Sitter mode functions (9) as the general approximate solutions of (4).

2.2 Power Spectrum with excited-de Sitter mode

According to the usual definition used for matter density perturbations, the power spectrum of the scalar field has dimension $k^{-3}$, so we will have [5]

\[p_\phi(k) = \frac{|u_k(\eta)|^2}{a^2}.\] (10)

By inserting mode function (8) in (10) and doing some straightforward calculations, the power spectrum is obtained by [32]

\[p_\phi(k) = \left(\frac{H^2}{2k^3}\right) \left(2 + \frac{H^2}{4M_p^2}\right),\] (11)

where $M_* = -Hk\eta$, and $H \ll M_* < M_p$ is the scale of new physics or the cutoff scale.

3 Large-scale inhomogeneity and anisotropy

3.1 Large-scale matter density perturbation

So far, non-linear corrections due to applying an excited-de Sitter mode have been reviewed. We are interested in investigation of its effect on the large-angular CMB anisotropy or, equivalently, the Sachs-Wolfe effect. After inflation, when density perturbations re-enter the horizon, they do not grow appreciably before matter-domination [1–3]. At this time pressure is too large to allow increasing of density perturbations. In other words, the density of radiation acts as a repulsive force and suppresses the growth of inhomogeneities whose scales are smaller than the horizon, while large-scale inhomogeneities remain unaffected [1–3, 5]. So the horizon size at equality is an important scale for investigating structure formation.

After the equality epoch, in the matter domination era, density perturbations are related to the gravitational potential via the perturbed Poisson’s equation in the short wavelength limit: [2]

\[\Phi(k, a) = \frac{4\pi G \rho_m a^2}{k^2} \delta(k, a),\] (12)

where $4\pi G \rho_m = \left(\frac{3\Omega_m}{2a^2}\right) H_0^2$. So the density perturbation is

\[\delta(k, a) = \frac{2k^2}{3\Omega_m H_0^2} \Phi(k, a).\] (13)

The primordial potential formed during the inflationary epoch is related to the gravitational potential after equality. On the other hand, when the universe passes through equality, the potential on the large scale drops by a factor of $(9/10)$ [1, 2]. So in this case, the relation between the two of them takes the following form [2],

\[\Phi(k, a) = \frac{9}{10} \Phi_p(k) T(k) D_1(a) / a,\] (14)

where $\Phi_p(k)$ and $T(k)$ are the primordial potential and transfer function respectively. The transfer function determines the modification of the fluctuation amplitudes (due to their evolution through the horizon crossing) at different scales. $D_1(a)$ is called the growth function and describes the growth of the perturbation amplitudes after the epoch of equality. Considering (13) together with (14) yields

\[\delta(k, a) = \frac{3}{5} \frac{k^2}{\Omega_m H_0^2} \Phi_p(k) T(k) D_1(a).\] (15)

So, the power spectrum of density perturbations after equality is related to the power spectrum of the primordial potential by

\[P_\delta(k, a) = \frac{9}{25} \left(\frac{k}{H_0}\right)^4 P_{\Phi_p}(k) T^2(k) \left(\frac{D_1^2(a)}{\Omega_m^2}\right)^2.\] (16)

In the slow-roll approximation, by considering $P_{\Phi_p}(k) = \frac{16\pi G}{9\epsilon} p_\phi(k)$ and (11), we have

\[P_{\Phi_p} = \frac{32\pi G}{9\epsilon} \left(\frac{H^2}{2k^3}\right) \left(1 + \frac{H^2}{8M_p^2}\right).\] (17)

Now, this result can be used for finding the power spectrum of matter density perturbations in the matter domination era as

\[P_\delta(k, a) = 4\pi^2 \delta_H^2 T^2(k) \left(\frac{D_1(a)}{D_1(a = 1)}\right)^2 \left(\frac{k}{H_0}\right) \left(1 + \frac{H^2}{8M_p^2}\right).\] (18)

where $\delta_H$ is the Harrison-Zel’dovich-Peebles power spectrum on the horizon scale. In the limit of very large scale, the transfer function can be replaced by unity in (18), so
the matter perturbation power spectrum at present takes the following form

\[
P(k, \alpha) = 4\pi^2 \delta^2 H \left( \frac{k}{H_0} \right) \left[ 1 + \frac{H^2}{8M^2} \left( \frac{k}{H_0} \right)^{-\epsilon} \right].
\]  

(19)

Also, in the regime of slow-roll inflation one can assume the Hubble parameter is scale dependent as

\[
\frac{H}{H_*} \sim \left( \frac{k}{k_*} \right)^{-\epsilon},
\]

(20)

where \( H_* \) is the Hubble parameter evaluated when the perturbation with scale \( k \), leaves the horizon [39]. Substituting (20) into relation (19) yields

\[
P(k, \alpha) = 4\pi^2 \delta^2 H \left( \frac{k}{H_0} \right) \left[ 1 + \frac{H^2}{8M^2} \left( \frac{k}{k_*} \right)^{-\epsilon} \right].
\]

(21)

### 3.2 Large angular scale CMB anisotropy

As mentioned previously, the angular scale which corresponds to the horizon scale at recombination (\( \theta \sim 1^\circ \)) is an important scale as a dividing line between large-scale and small-scale inhomogeneities. The temperature fluctuations at large angular scales (\( \theta \gg 1^\circ \)) are induced by large-scale perturbations which are not affected by photon pressure before equality. So observation of the temperature fluctuations at these scales gives direct information about the primordial power spectrum of density perturbations.

In Section 3.1, we noticed that employment of the new spectrum for calculating the matter density spectrum leads to the scale dependent matter density power spectra. Now, we will use the result obtained in (21) to calculate the power spectrum of temperature fluctuations of CMB photons at large angular scale. If the CMB anisotropy at this scale is denoted by \( C_{\ell}^{SW} \), it can be read as [2]

\[
C_{\ell}^{SW} \approx \frac{\Omega_m H_0^4}{2\pi D_1(a=1)} \int_0^\infty \frac{dk}{k} P_{\delta}(k) j_\ell^2(k(\eta_0 - \eta_{rec})),
\]

(22)

where \( \eta_{rec} \) is the conformal time at recombination. By considering \( \eta_0 \gg \eta_{rec} \), one can neglect \( \eta_{rec} \) compared to \( \eta_0 \) in (22). Moreover, in the case of large-angular scale CMB anisotropy, the transfer function can be set to unity. So, by substituting (21) in (22), \( C_{\ell}^{SW} \) takes the following form

\[
C_{\ell}^{SW} \approx \frac{2\pi \delta^2 H \Omega_m}{D_1(a=1)} \int_0^\infty \frac{dk}{k} j_\ell^2(k\eta_0) \left[ 1 + \frac{H^2}{8M^2} \left( \frac{k}{k_*} \right)^{-\epsilon} \right].
\]

(23)

If we neglect the correction term, by considering \( X = k\eta_0 \) and with the help of the identity [40]

\[
\int_0^\infty dX X^2 j_\ell^2(X) = 2^{\ell+4} \pi \frac{\Gamma \left( \ell + \frac{n}{2} - \frac{1}{2} \right) \Gamma \left( 3 - n \right)}{\Gamma \left( \ell + \frac{5}{2} - \frac{n}{2} \right) \Gamma^2 \left( 2 - \frac{n}{2} \right)},
\]

(24)

the temperature fluctuation spectra for the CMB radiation at large angular scales would be read as

\[
C_{\ell}^{SW} = \frac{\pi \Omega_m^2}{2D_1(a=1)} \frac{\delta^2 H}{\ell(\ell + 1)}.
\]

(25)

So the quantity \( \ell(\ell + 1)C_{\ell}^{SW} \) is independent of \( \ell \). Indeed, this is the reason that the CMBA power spectrum is typically plotted in form of \( \ell(\ell + 1)C_{\ell}^{SW} \) versus \( \ell \). Now, we consider the excited dS mode instead of the pure dS mode and after straightforward calculations, we obtain the scale-dependent new result as

\[
C_{\ell}^{SW} = \frac{\Omega_m}{2D_1(a=1)} \delta^2 H \left( \frac{\ell}{\ell + 1} \right)^{-\epsilon} \left( \frac{\Gamma \left( \ell + \frac{5}{2} - \frac{n}{2} \right) \Gamma \left( 3 - \frac{\ell}{2} \right)}{\Gamma \left( \ell + 2 + \frac{\ell}{2} \right) \Gamma^2 \left( \frac{3}{2} - \frac{\ell}{2} \right)} \right).
\]

(26)

Since in most of the Sachs-Wolfe limit we have \( \frac{\epsilon}{2} \ll \ell \), we can ignore \( \epsilon \) in comparison to \( \ell \) and consequently the final answer can be written in the following form,

\[
\ell(\ell + 1)C_{\ell}^{SW} \approx \frac{\pi \delta^2 H}{2D_1(a=1)} \left( \frac{\Omega_m}{\ell(\ell + 1)} \right)^{\frac{1}{2}} \left[ 1 + \frac{\pi H^2}{2M^2} \left( \frac{\eta_0 k_*}{2} \right)^{\epsilon} \left( \frac{\Gamma \left( \ell + \frac{5}{2} - \frac{n}{2} \right) \Gamma \left( 3 - \frac{\ell}{2} \right)}{\Gamma \left( \ell + 2 + \frac{\ell}{2} \right) \Gamma^2 \left( \frac{3}{2} - \frac{\ell}{2} \right)} \right) \right].
\]

(27)

The correction term on the right hand side of (27), which results from the Sachs-Wolfe effect, shows that the quantity \( \ell(\ell + 1)C_{\ell}^{SW} \) depends on \( \epsilon \) and has a slight deviation from the standard scale-invariant result (25).

### 4 Conclusions

In this paper, we have calculated the scale-dependency of large angular scale CMB anisotropy resulting from an excited-dS mode as the fundamental mode function during inflation. The result indicates that considering the new excited mode as an initial quantum state for primordial fluctuations can affect the angular scale invariance of CMB anisotropy spectra. In addition, the appearance of a cutoff scale in the results shows that the effect of these modes should be suppressed by some unknown energy scale, which must be higher than the inflationary scale. This excited mode was prepared essentially by expanding the Hankel function in the
quantum mode in de Sitter space-time to quadratic order of $\frac{1}{k^2}$ before quantization. This approach is similar to performing the quantization at finite wavelength, rather than fully in the ultraviolet (i.e. Bunch-Davies) limit. In fact, taking into account the resent observational constraint together with the result obtained in [36] motivates us to use quasi-de Sitter curved space-time and excited mode functions. We have seen that slight deviation of the Bunch-Davies mode leads to corrections in the primordial gravitational potential spectra and in the large angular scales CMB anisotropy. The result also shows that the final $\epsilon$-dependent correction term is very tiny for low $\ell < 200$ and close to the plateau result, while for the limit $\ell \ll 200$, maybe due to dark energy fluid evolution, the size of the correction term can be significant. Also, from the Planck data, it is known that there might exist an anomaly of the power asymmetry at $\ell < 50$, as analyzed in Ref. [41, 42]. We plan to check the possible connection between this anomaly and the present study.

In our next study, we plan to examine the accuracy of our approach observationally.

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