THE DYNAMICS OF SEMILATTICE NETWORKS

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Abstract. Time-discrete dynamical systems on a finite state space have been used with great success to model natural and engineered systems such as biological networks, social networks, and engineered control systems. They have the advantage of being intuitive and models can be easily simulated on a computer in most cases; however, few analytical tools beyond simulation are available. The motivation for this paper is to develop such tools for the analysis of models in biology. In this paper we have identified a broad class of discrete dynamical systems with a finite phase space for which one can derive strong results about their long-term dynamics in terms of properties of their dependency graphs. We classify completely the limit cycles of semilattice networks with strongly connected dependency graph and provide polynomial upper and lower bounds in the general case.

1. Introduction and Background

Time-discrete dynamical systems on a finite state space play an important role in several different contexts. Examples of such systems include Boolean networks, cellular automata, agent-based models, and finite state machines, to name a few. This modeling paradigm has been used with great success to model natural and engineered systems such as biological networks, social networks, and engineered control systems. It has the advantage of being intuitive and models can be easily simulated on a computer in most cases. A disadvantage of discrete models of this type is that few analytical tools beyond simulation are available. The motivation for this paper is to develop such tools for the analysis of models in biology, but the results are of independent mathematical interest. Some theoretical results have been proven for Boolean networks and are reviewed in [7]. In that paper the authors study conjunctive Boolean networks, that is, Boolean networks for which the future state of each node is computed using the Boolean AND operator. It is shown that the dynamics of such networks is controlled strongly by the topology of the network. The current paper shows that the results in [7] are valid much more broadly. We briefly describe the main results in [7].

A Boolean network \( f \) on \( n \) nodes \( x_1, \ldots, x_n \) can be viewed as a time discrete dynamical system over the Boolean field \( \mathbb{F}_2 \):

\[
    f = (f_1, \ldots, f_n) : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n,
\]

where the coordinate functions \( f_i \) are the Boolean functions assigned to the nodes of the network. We can associate two directed graphs to \( f \). The dependency graph (or wiring diagram) \( \mathcal{D}(f) \) of \( f \) has \( n \) vertices corresponding to the variables \( x_1, \ldots, x_n \) of \( f \). There is a directed edge \( i \rightarrow j \) if the function \( f_j \) depends on \( x_i \) (i.e. there is an instance where changing the value of \( x_i \) changes the value of \( f_j \)). That is, \( \mathcal{D}(f) \) encodes the variable dependencies in \( f \). The dynamics of \( f \) is encoded by its phase space, denoted by \( \mathcal{S}(f) \). It is the directed graph with vertex set \( \mathbb{F}_2^n \) and a directed edge from \( u \) to \( v \) if \( f(u) = v \). Thus, the graph \( \mathcal{D}(f) \) encodes part of the structure of \( f \) and the graph \( \mathcal{S}(f) \) encodes its dynamics. The results in [7] relate these two graphs, deriving information about \( \mathcal{S}(f) \) from \( \mathcal{D}(f) \), in the case where each \( f_i \) is a conjunction of the variables \( x_j \) for which there is an edge \( j \rightarrow i \) in \( \mathcal{D}(f) \). These networks were called conjunctive networks in [7].

It is easy to see that the graph \( \mathcal{S}(f) \) has the following structure. Its connected components consist of a unique directed cycle, a limit cycle, with directed “trees” feeding into the nodes of the cycle, representing the transient states of the network, each of which eventually maps to a periodic point. If the graph \( \mathcal{D}(f) \) is strongly connected, that is, there is a directed path from any vertex to any other vertex, then it is shown in [7] that there is a precise closed formula for the number of limit cycles of any given length. This formula depends on a numerical invariant of \( \mathcal{D}(f) \), its loop number. If \( \mathcal{S}(f) \) is not strongly connected, then there is a sharp lower bound and an upper bound on the number of limit cycles, in terms of the loop numbers of the strongly connected components of \( \mathcal{S}(f) \) and the antichains in the poset of strongly connected components.
These two bounds agree for the number of fixed points of $f$, so that one corollary is a closed formula for the number of fixed points of a general conjunctive network.

The current paper shows that these results hold in much broader generality, namely for time discrete dynamical systems over any finite set $X$:

$$f = (f_1, \ldots, f_n) : X^n \rightarrow X^n,$$

such that the $f_i$ are constructed from an operator $\land : X^2 \rightarrow X$, which has the property that it endows $X$ with the structure of a semilattice, that is, $\land$ is commutative, associative, and idempotent. We will call such functions $f_i$ semilattice functions. That is, any semilattice $X$ gives rise to a dynamical system to which the formulas and bounds for limit cycles hold. It is shown furthermore that they hold precisely for the class of semilattice networks. It is important to mention that since semilattice networks are not linear (see Example 6.1), our results complement those for linear systems in [3, 5].

2. Semilattice networks

Let $X$ be a set with $M$ elements and consider a dynamical system with $n$ variables over $X$:

$$f = (f_1, \ldots, f_n) : X^n \rightarrow X^n.$$

**Definition 2.1.** A function $\land : X^2 \rightarrow X$ is called a semilattice operator if it satisfies the following (with notation $x \land y := \land(x, y)$):

- $x \land y = y \land x$, $\land$ is commutative
- $x \land (y \land z) = (x \land y) \land z$, $\land$ is associative
- $x \land x = x$, $\land$ is idempotent

Note that semilattice operators induce the structure of a semilattice on $X$. Conversely, the meet or join operations on a semilattice are semilattice operators in the above sense.

**Example 2.2.** Some examples of semilattice operators are:

- $\land = \text{AND over } [0, 1]^2$
- $\land = \text{OR over } [0, 1]^2$
- $\land = \text{MIN over } [0, m]^2$
- $\land = \text{MAX over } [0, m]^2$

That is, semilattice operators are a generalization of the conjunctive and disjunctive operators used in [7].

The domain of a semilattice operator $\land : X^2 \rightarrow X$ can be extended to $\land : X^k \rightarrow X$ with $k \geq 1$ by

$$x_1 \land x_2 \land \ldots \land x_k = x_1 \land (x_2 \land (\ldots \land (x_{k-1} \land x_k)))$$

if $k > 1$ and

$$\land x = x \land x = x$$

We will call such an extended operator a semilattice function.

**Definition 2.3.** A dynamical system $f = (f_1, \ldots, f_n) : X^n \rightarrow X^n$ is called a semilattice network if there exists a semilattice operator, $\land$, such that $f_j = \land |_{X^k}$ is a semilattice function for all $j = 1, \ldots, n$ (where $f_j$ depends only on $x_{i_1}, \ldots, x_{i_k}$).

Let $f : X^n \rightarrow X^n$ be a semilattice network, $G = \mathcal{D}(f)$, and $A$ the adjacency matrix of $G$. We will assume here and in the remainder of the paper that none of the coordinate functions of $f$ are constant, that is, all vertices of $G$ have positive in-degree.

2.1. Structure of the Dependency Graph. Define the following relation on the vertices of $G$: $a \sim b$ if and only if there is a directed path from $a$ to $b$ and a directed path from $b$ to $a$. It is easy to check that $\sim$ is an equivalence relation. Suppose there are $t$ equivalence classes $V_1, \ldots, V_t$. For each equivalence class $V_i$, the subgraph $G_i$ with the vertex set $V_i$ is called a strongly connected component of $G$. The graph $G$ is called strongly connected if $G$ consists of a unique strongly connected component. A trivial strongly connected component is a graph on one vertex and no self loop. Since such components do not influence the cycle structure of the network, we assume that all strongly connected components are non-trivial.
Example 2.4. Our running example for this paper will be the semilattice network $f : \{0, 1, 2\}^6 \to \{0, 1, 2\}^6$ with dependency graph given by Figure 1 and semilattice function $\land = \text{MIN}$.

Let $G_i$ be a strongly connected component, and let $h_i$ be the semilattice network with dependency graph $\mathcal{D}(h_i) = G_i$. Let $h : X^n \to X^n$ be the semilattice network defined by $h = (h_1, \ldots, h_t)$. That is, the dependency graph of $h$ is the disjoint union of the strongly connected graphs $G_1, \ldots, G_t$, and $h$ is obtained from $g$ by deleting all edges between strongly connected components.

Now define the following order relation on the strongly connected components $G_1, \ldots, G_t$ of the dependency graph $\mathcal{D}(f)$ of the network $f$.

$G_i \preceq G_j$ if there is at least one edge from a vertex in $G_i$ to a vertex in $G_j$.

In this way we obtain a partially ordered set $\mathcal{P}$. In this paper, we relate the dynamics of $f$ to the dynamics of its strongly connected components and the poset $\mathcal{P}$.

Example 1 (Cont.). The dependency graph of $f$ has four strongly connected components, $G_1$, $G_2$, $G_3$, and $G_4$ (bottom, left, right and top, resp.). The poset is given by $G_1 \preceq G_2, G_3 \preceq G_4$.

**Figure 1.** Dependency graph of the semilattice network in Example 2.4. The labels have been omitted for simplicity.

**Figure 2.** The strongly connected components of $f$ (left) and their poset (right).
2.2. The Loop Number.

Definition 2.5. The loop number of a strongly connected graph is the greatest common divisor of the lengths of its simple (no repeated vertices) directed cycles. The loop number of any directed graph $G$ is the least common multiple of the loop numbers of its non-trivial strongly connected components.

Example 2.5 (Cont.). In our running example, the loop numbers are $\text{loop}(G_1) = 1$, $\text{loop}(G_2) = 2$, $\text{loop}(G_3) = 3$ and $\text{loop}(G_4) = 1$.

3. Semilattice Networks with Strongly Connected Dependency Graph

In this section we give an exact formula for the cycle structure of semilattice networks with strongly connected dependency graphs. In the next section we will also consider networks with general dependency graphs and give upper and lower bounds for the cycle structure.

The formula for conjunctive networks was given and proven in [2]. It is not difficult to show that the proofs are also valid for general semilattice networks. Let $f : X^n \rightarrow X^n$ be a semilattice network with semilattice operator $\land$. Assume that the dependency graph $\mathcal{D}(f)$ of $f$ is strongly connected with loop number $c$.

Lemma 3.1. The set of vertices of $\mathcal{D}(f)$ can be partitioned into $c$ non-empty sets $W_1, \ldots, W_c$ such that each edge of $\mathcal{D}(f)$ is an edge from a vertex in $W_i$ to a vertex in $W_{i+1}$ for some $i$ with $1 \leq i \leq c$ and $W_{c+1} = W_1$.

Proof. For a proof of this fact see [1] Lemma 3.4.1(iii) or [2] Lemma 4.7.

Let $s_i$ be the number of elements of $W_i$. Without loss of generality we assume for the rest of the section that $W_1 = \{1, \ldots, s_1\}$, $W_2 = \{s_1 + 1, \ldots, s_1 + s_2\}$, $\ldots$, $W_c = \{s_1 + \ldots + s_{n-1} + 1, \ldots, s_1 + \ldots + s_c = n\}$.

Proposition 3.2. Let $c$ be the loop number of $f$; then there exists $k_0$ such that for all $k \geq k_0$ we have

\[
\begin{align*}
f^{ck}(x) &= (y_1, \ldots, y_1, \ldots, y_c) \\
f^{ck+1}(x) &= (y_1, \ldots, y_c, y_1, \ldots, y_1, \ldots, y_c) \\
\vdots & \\
\vdots & \\
f^{ck+j}(x) &= (y_{c+1-j}, \ldots, y_{c+1-j}, \ldots, y_c, y_c, y_1, \ldots, y_1) \\
\vdots & \\
\vdots & \\
f^{ck+c}(x) &= f^c(x)
\end{align*}
\]

where $y_j = \bigwedge_{i \in W_j} x_i$.

Proof. The proof is analogous to [2] Theorem 4.10).

The following corollary states that the period of $f$ can be obtained from the topology of its dependency graph.

Corollary 3.3. The period of $f$ is equal to the loop number of $\mathcal{D}(f)$.

The following corollary states that the long-term dynamics of a semilattice network can be reduced to the dynamics of a rotation over $X^c$, with $c$ as in Lemma 3.1.

Corollary 3.4. The cycle structure of $f$ is equal to the cycle structure of $R : X^c \rightarrow X^c$ where $R(y_1, \ldots, y_c) = (y_c, y_1, \ldots, y_{c-1})$.

Proof. Let $\Gamma : X^c \rightarrow X^n$ and $\Phi : X^n \rightarrow X^c$ be defined by $\Gamma(y_1, y_2, \ldots, y_c) = (y_1, y_2, \ldots, y_2, y_2, \ldots, y_c, \ldots, y_c)$ and $\Phi(x_1, \ldots, x_n) = (x_{s_1}, x_{s_1+s_2}, \ldots, x_n)$. The proof now follows from the equalities $\Phi \circ f \circ \Gamma = \tau$ and $\Gamma \circ \tau \circ \Phi = f|\{\text{periodic points of } f\}$.

\]
For any positive integers \( p, k \) that divide \( c \), let \( A(p) \) be the set of periodic points of period \( p \) and let \( D(k) := \bigcup_{p \mid k} A(p) \).

**Proposition 3.5.** The cardinality of the set \( D(k) \) is \( |D(k)| = M^k \).

**Proof.** It follows from the fact that if \( k \mid c \), then a rotation in \( M \) colors and \( c \) variables has \( M^k \) colorings of periods that divide \( k \).

**Corollary 3.6.** If \( p \) is a prime number and \( p^k \) divides \( c \) for some \( k \geq 1 \), then
\[
|A(p^k)| = M^{p^k} - M^{p^k-1}.
\]

**Proof.** It is clear that \( |D(1)| = M \) (there are \( M \) constant colorings). Now if \( p \) is prime and \( k \geq 1 \), then the proof follows from the fact that \( D(p^k) = D(p^{k-1}) \cup A(p^k) \), where \( \cup \) is the disjoint union.

Next we prove Theorem 3.7 which gives the exact number of periodic points of any possible length.

**Theorem 3.7.** Let \( f \) be a semilattice network whose dependency graph is strongly connected and has loop number \( c \). If \( c = 1 \), then \( C(f) = MC^1 \). If \( c > 1 \) and \( k = p_1^{s_1} \cdots p_r^{s_r} \) is a divisor of \( c \), then the number of periodic states of period \( k \) is \( |A(k)| \)
\[
|A(k)| = \sum_{s=0}^{1} \cdots \sum_{s=0}^{1} (-1)^{t_1+t_2+\cdots+i_r} M^{p_1^{s_1-t_1} p_2^{s_2-t_2} \cdots p_r^{s_r-t_r}}.
\]

**Proof.** The statement for \( c = 1 \) is part of the previous corollary. Now suppose that \( c > 1 \). For \( 1 \leq j \leq r \), let \( k_j = p_j^{s_j-1} \prod_{i=1, i \neq j}^{r} p_i^{s_i} \). Then \( D(k) = A(k) \cup (\bigcup_{j=1}^{r} D(k_j)) \), where \( \cup \) is a disjoint union, in particular,
\[
A(k) = D(k) \setminus \bigcup_{j=1}^{r} D(k_j).
\]

The formula (3.1) follows from the inclusion-exclusion principle and the disjoint union above.

**Corollary 3.8.** If \( k \) divides \( c \), then the number of cycles of length \( k \) in the phase space of \( f \) is \( C(f)_k = \frac{|A(k)|}{k} \). Hence the cycle structure of \( f \) is
\[
C(f) = \sum_{k \mid c} \frac{|A(k)|}{k} c^k.
\]

**Example (Cont.).** In our running example we obtain:
\[
C(h_1) = 3C_1, \quad C(h_2) = 3C_1 + 3C_2, \quad C(h_3) = 3C_1 + 8C_3, \quad C(h_4) = 3C_1
\]

**Remark 3.9.** Notice that the cycle structure of \( f \) depends on the loop number and \( |X| = M \) only. In particular, a semilattice network with loop number 1 on a strongly connected dependency graph only has as limit cycles the \( M \) fixed points \((w, w, \ldots, w)\) where \( w \in X \), regardless of how many vertices its dependency graph has and how large \( n \) is.

4. **Networks with General Dependency Graph**

Let \( f : S \rightarrow S \) be a semilattice network with dependency graph \( \mathcal{D}(f) \). Let \( G_1, \ldots, G_t \) be the strongly connected components of \( \mathcal{D}(f) \). Furthermore, suppose that none of the \( G_i \) is trivial. For \( 1 \leq i \leq t \), let \( h_i \) be the semilattice network that has \( G_i \) as its dependency graph and suppose that the loop number of \( h_i \) is \( c_i \). In particular, the loop number of \( f \) is \( c := \text{lcm}(c_1, \ldots, c_t) \).

First we study the effect of deleting an edge in the dependency graph between two strongly connected components. Let \( G_1 \) and \( G_2 \) be two strongly connected components in \( \mathcal{D}(f) \) and suppose \( G_1 \preceq G_2 \). Let \( x \rightarrow y \) be a directed edge in \( \mathcal{D}(f) \) between a vertex \( x \) in \( G_1 \) and a vertex \( y \) in \( G_2 \). Let \( \mathcal{D}' \) be the graph \( \mathcal{D}(f) \) after deleting this edge, and let \( y \) be the semilattice network such that \( \mathcal{D}(y) = \mathcal{D}' \).

**Theorem 4.1.** Any cycle in the phase space of \( f \) is a cycle in the phase space of \( g \). In particular \( C(f) \leq C(g) \) componentwise.
Proof. First notice that if \( x \) appears in the expression \( f^k_y \) if and only if there is a path from \( x \) to \( y \) of length \( k \). Let \( C := \{ u, f(u), \ldots, f^{m−1}(u) \} \) be a cycle of length \( m \) in \( S(f) \). To show that \( C \) is a cycle in \( S(g) \), it is enough to show that, \( u_x \wedge u_{g^2} = u_{g^2} \). Thus, the value of \( y \) is determined already by the value of \( y' \) and the edge \((x,y)\) does not contribute anything new and hence \( C \) is a cycle in \( S(g) \). This is equivalent to show that \( u_x \wedge u_{g^2} = u_{g^2} \) for all \( y' \in G_y \) such that there is an edge from \( y' \) to \( y \).

Suppose the loop number of \( G_1 \) (resp. \( G_2 \)) is \( a \) (resp. \( b \)). Now, any path from \( x \) (resp. \( y \)) to itself is of length \( pa \) (resp. \( qb \)) where \( q \geq T \) and \( T \) is large enough, see [2 Corollary 4.4]. Thus there is a path from \( x \) to \( y \) of length \( qa + 1 \) for any \( q \geq T \). Also, there is a directed path from \( y \) to \( y' \) of length \( qb - 1 \) for any \( q \geq T \). This implies the existence of a path from \( x \) to \( y' \) of length \( q(a+b) \) for all \( q \geq T \), then \( x \) appears in \( f^{m}y' \).

Now \( u = f^{m}(u) = f^{mk}(u) \), for all \( k \geq 1 \). Choose \( q, k \) large enough such that \( qa + b = km \geq T \). Then, \( u_y = f^{mk}(u) = u_x \wedge \ldots \) and \( u_x \wedge u_y = u_x \wedge u_y = \ldots \). Therefore \( u_x \wedge u_{G_2} = \wedge u_{G_2} \). \( \square \)

Let \( h : S \rightarrow S \) be the semilattice network with the disjoint union of \( G_1, \ldots, G_t \) as its dependency graph. That is, \( h = (h_1, \ldots, h_t) \). For \( h \), there are no edges between any two strongly connected components of the dependency graph of the network. Its cycle structure can be completely determined from the cycles structures of the \( h_i \) alone.

**Theorem 4.2.** Let \( C(h_i) = \sum_{j} a_{i,j} C^j \) be the cycle structure of \( h_i \). Then the cycle structure of \( h \) is \( C(h) = \prod_{i=1}^{t} C(h_i) \) (where \( C^C := r_{m}^{s_{\lcm(r,s)}} C_{\lcm(r,s)} \)) and the number of cycles of length \( m \) (where \( m \lvert l \)) in the phase space of \( h \) is

\[
C(h)_m = \sum_{\substack{j \mid m \vdot j_i \mid \lcm(j_1, \ldots, j_t) = m}} \frac{j_1 \cdot \cdots \cdot j_t}{m} \prod_{i=1}^{t} a_{i,j_i}.
\]

**Proof.** This follows from the fact that if \( u \) is a periodic point of \( h_i \) of period \( k_i \) and \( v \) is a periodic point of \( h_j \) of period \( k_j \), then \( (u, v) \) is a periodic point of \( (h_i, h_j) \) of period \( \lcm(k_i, k_j) \). \( \square \)

**Corollary 4.3.** Let \( f \) and \( h \) be as above. The number of cycles of any length in the phase space of \( f \) is less than or equal to the number of cycles of that length in the phase space of \( h \). That is \( C(f) \leq C(h) \) componentwise. In particular, the period of \( f \) is a divisor of the loop number of its dependency graph.

In [7], it was shown that the poset structure of \( \mathcal{P} \) gives an algebraic way to combine the cycle structure of \( h_i \) to obtain lower and upper bounds for the cycle structure of \( f \), where \( f \) was a conjunctive Boolean network. It is not difficult to see that the proofs of these results still also hold for general semilattice networks if the corresponding semilattice operator has a “neutral” and an “absorbent” element (analogous to the identities \( 1 \wedge x = x \) and \( 0 \wedge x = 0 \)). In order to state the theorem about lower and upper bounds of semilattice networks, we need the following definitions.

**Definition 4.4.** Let \( \wedge : X^2 \rightarrow X \) be a semilattice operator. An element \( \lambda \in X \) is called a neutral element if \( \lambda \wedge x = \wedge x \) for all \( x \in X \). An element \( \theta \in X \) is called an absorbent element if \( x \wedge \theta = \wedge \theta \) for all \( x \in X \).

**Example 4.5.** All the functions in Example 2.2 have a neutral and absorbent element; they are:

\[
\begin{align*}
\lambda = 1, & \quad \theta = 0 \\
\lambda = 0, & \quad \theta = 1 \\
\lambda = m, & \quad \theta = 0 \\
\lambda = 0, & \quad \theta = m
\end{align*}
\]

**Remark 4.6.** Since \( X \) is finite, every semilattice operator has an absorbent element (\( \theta = \wedge_{x \in X} x \)).

**Remark 4.7.** Any semilattice operator \( \wedge \) on \( X \) can be extended to a set with a neutral element by defining \( \wedge' : (X \cup \{\lambda\})^2 \rightarrow X \cup \{\lambda\} \) as \( x \wedge' y = x \wedge y \) if \( x, y \in X \) and \( x \wedge' \lambda = \lambda \wedge' x = x \) otherwise.

Let \( f, h, G_1, \ldots, G_t \) be as above and let \( \ell_i \) be the loop number of \( G_i \). Furthermore, let \( \mathcal{P} \) be the poset of the strongly connected components. Let \( \Omega \) be the set of all maximal antichains in \( \mathcal{P} \). For \( J \subseteq [\ell] \), let \( x_J := \prod_{j \in J} x_j \) and let \( J := [\ell] \setminus J \). In the remainder of the paper we assume that \( f : X^n \rightarrow X^n \) is a
semilattice network such that its semilattice operator, \( \land \), has idempotent neutral and absorbent elements, \( \lambda \) and \( \theta \), resp.

**Definition 4.8.** For any subset \( J \subseteq [t] \), let

\[
J^{\geq} := \{ k : G_j \preceq G_k \text{ for some } j \in J \},
\]

\[
J^{=} := \{ k : G_j \simeq G_k \text{ for some } j \in J \},
\]

\[
J^{<} := \{ k : G_j \prec G_k \text{ for some } j \in J \}, \quad \text{and} \quad J^{>} := \{ k : G_j \succ G_k \text{ for some } j \in J \}.
\]

A limit cycle \( C \) in the phase space of \( f \) is \( J_\theta \) (resp. \( J_\lambda \)) if the \( G_j \) component of \( C \) is \( \theta \) (resp. \( \lambda \)) for all \( j \in J \).

Denote \( \langle K, L \rangle = \begin{cases} 0, & \text{if } K \cap L \neq \emptyset, \\ 1, & \text{if } K \cap L = \emptyset. \end{cases} \) and \( I_N = I^{\geq} \cup N, J^M = J^{\geq} \cup M. \)

**Definition 4.9.** The \( L \)- and \( U \)-polynomial associated to \( P \) are defined as follows:

\[
L(z_1, \ldots, z_t) = \sum_{J \subseteq \Omega} (-1)^{|J|+1} \prod_{k \in \cap_{I \in J} J} z_k
\]

\[
U(z_1, \ldots, z_t) = \sum_{I \subseteq N \subseteq [t]} (-1)^{|N|+|M|+|J|} \prod_{k \in I_N \cup J^M} z_k
\]

**Example 1 (Cont.)**. For the poset in Figure 2 we obtain:

\[
L(z_1, z_2, z_3, z_4) = -2 + z_1 + z_2 z_3 + z_4
\]

\[
U(z_1, z_2, z_3, z_4) = 14 - 7z_1 + 3z_1 z_2 - 4 z_2 + 3z_1 z_3 - z_1 z_2 z_3 + z_2 z_3 - 4 z_3 + 4z_1 z_4 - 2z_1 z_2 z_4 + 3z_2 z_4 - 2z_1 z_3 z_4 + z_1 z_2 z_3 z_4 - z_2 z_3 z_4 + 3z_3 z_4 - 7z_4
\]

**Theorem 4.10.** With the notation above we have the following coefficient-wise inequalities

\[
L(C(h_1), \ldots, C(h_t)) \leq C(f) \leq U(C(h_1), \ldots, C(h_t)).
\]

Here, the polynomials \( L \) and \( U \) are evaluated using the “multiplication” described in Theorem 4.2 and coefficient-wise addition.

**Proof.** The proof is analogous to the proofs of Theorems 6.2 and 7.4 in [7]. \( \square \)

Note that the left and right sides of the inequalities (4.10) are polynomial functions in the variables \( C(h_i) \), with integer coefficients. That is, the lower and upper bounds are polynomial functions depending exclusively on measures of the network topology.

**Example 1 (Cont.)**. In our running example we obtain:

\[
L(C(h_1), C(h_2), C(h_3), C(h_4)) = 13C_1 + 9C_2 + 24C_3 + 24C_6
\]

\[
U(C(h_1), C(h_2), C(h_3), C(h_4)) = 20C_1 + 24C_2 + 64C_3 + 96C_6
\]

and therefore \( 13C_1 + 9C_2 + 24C_3 + 24C_6 \leq C(f) \leq 20C_1 + 24C_2 + 64C_3 + 96C_6 \). It is important to mention that the phase space of \( f \) has \( 3^{63} \approx 10^{20} \) nodes so it is not feasible to obtain information about the cycle structure from exhaustive enumeration. Also, although the bounds agree on fixed points for Boolean semilattice networks [7], our example shows that they do not agree for general lattice networks.

5. **Characterization of Semilattice Networks**

In this section we characterize semilattice networks; in order to do this, it is enough to characterize semilattice operators. Since semilattice operators are semilattice operations, the number of semilattice operators over a set with \( m \) elements (up to permutation) is the number of semilattices with \( m \) elements. According to the next proposition, the number of semilattice operators over a set with \( m \) elements is the number of lattices of size \( m + 1 \). Although there is no closed formula for the number of lattices of a given size, algorithms for counting them have been developed [1].

**Proposition 5.1.** There is a one-to-one correspondence between semilattices with \( m \) elements and lattices with \( m + 1 \) elements.
Proof. If \((X, \wedge)\) is a semilattice with \(m\) elements, consider \((X \cup \{\lambda\}, \wedge, \vee)\), by defining \(x \wedge' y = x \wedge y\) if \(x, y \in X\) and \(x \wedge' \lambda = \lambda \wedge' x = x\) otherwise; also, \(x \vee y = \wedge\{z : z \wedge x = x, z \wedge y = y\}\). It follows that \((X \cup \{\lambda\}, \wedge', \vee)\) is a lattice with \(m + 1\) elements. On the other hand, if \((Z, \wedge, \vee)\) is a lattice with \(m + 1\) elements, let \(\lambda = \bigwedge Z\); then it follows that \((Z \setminus \{\lambda\}, \wedge|_{Z \setminus \{\lambda\}})\) is a semilattice with \(m\) elements. \(\Box\)

6. Infinite Semilattice Networks

We present some results on infinite semilattice networks, both networks on an infinite set \(X\) and networks on an infinite Cartesian product of a set \(X\).

6.1. Semilattice networks on infinite sets. If \(X\) is a set with infinitely many elements (that is, \(M = \infty\)), some of the theorems remain valid. Notice that if \(S \subseteq X\) is finite, there exists a finite set \(Z \supseteq S\) that is closed under \(\wedge\); that is, \(\wedge|_Z : Z^2 \to Z\) is a semilattice function.

Suppose that \(\mathcal{D}(f)\) is strongly connected with loop number \(c\), and let \(u = (u_1, \ldots, u_n)\) be a periodic point of \(f\). Let \(Z \supseteq \{u_1, \ldots, u_n\}\) be a finite subset of \(X\) that it is closed under \(\wedge\); then we can consider \(u\) as a periodic point of \(f|_Z\). Therefore, the results on finite semilattice networks apply, and the period of \(u\) must divide \(c\). That is, Corollary 3.3 is valid for \(M = \infty\). Now, consider a divisor \(k\) of \(c\) and consider \(Z \subseteq X\) finite with at least \(m\) elements such that \(\wedge|_Z\) is a semilattice operator. Then, the number of periodic states of period \(k\) of \(f|_Z\) is at least \(|A(k)|\) (see Theorem 3.7 and notice that \(|A(k)|\) is increasing with respect to \(M\)). Since \(\lim_{m \to \infty} |A(k)| = \infty\), it follows that \(f\) has infinitely many periodic points and limit cycles of length \(k\).

Then Theorem 4.1 (and the corresponding corollary) holds for \(M = \infty\).

If \(\mathcal{D}(f)\) is not necessarily strongly connected, suppose \(\wedge\) has a neutral and an absorbent element. Let \(h_1, \ldots, h_n\) correspond to \(f|_Z\). Then, by using the argument in the paragraph above, it follows that if at least a limit cycle of length \(k\) appears in \(\mathcal{C}(\Xi(h_1), \ldots, \Xi(h_n))\), then \(f\) has infinitely many limit cycles of length \(k\).

6.2. Infinite-dimensional semilattice networks. If the dimension of \(f\) is infinite, that is, \(n = \infty\), then \(\wedge\) needs to satisfy an additional condition to be properly defined: Every (possibly infinite) subset \(S \subseteq X\) has a largest lower bound. This means that there exists \(x \in X\) such that \(x \wedge s = x\) for all \(s \in S\) (in lattice terminology: \(x\) is a lower bound); and if there is another such \(x'\), then \(x \wedge x' = x'\) (in lattice terminology: \(x\) is the largest lower bound). This additional condition allows for a function \(f_1\) to have infinitely many inputs.

Suppose that \(\mathcal{D}(f)\) is strongly connected with loop number \(c\), and let \(u = (u_1, \ldots, u_n)\) be a periodic point of \(f\) of period \(d\). Consider \(x, y \in W_i\) (see Lemma 3.1); then there exists \(k_0\) such that for all \(k \geq k_0\) there is a path of length \(ck\) from \(x\) to \(y\) and from \(y\) to \(x\). Then, \(f^{\text{ck}}_y = x \wedge \ldots, f^{\text{ck}}_x = y \wedge \ldots\) for all \(k \geq k_0\); in particular, \(u_y = f^{\text{ck}}_{y\text{cod}}(u) = u_x \wedge w\) and \(u_w = f^{\text{ck}}_{x\text{cocd}}(u) = u_y \vee v\) for some \(v, w \in X\). Then, \(u_x \wedge u_y = u_x \wedge u_x \wedge w = u_x \wedge w = u_y\), similarly \(u_x \wedge u_y = u_x\); therefore \(u_x = u_y\). It follows that Corollary 3.3 and Theorem 5.7 remain valid for \(n = \infty\) (and \(M \in \mathbb{Z}^+ \cup \{\infty\}\)).

If \(\mathcal{D}(f)\) is not necessarily strongly connected, suppose \(\wedge\) has a neutral and an absorbent element. It is not difficult to show that Theorem 4.1 remains valid for \(n = \infty\) (and \(M \in \mathbb{Z}^+ \cup \{\infty\}\)).

Finally, we show with counterexamples that we cannot omit any of the properties of \(\wedge\) in Definition 2.1. That is, the formulas and bounds derived in this paper are valid exactly for semilattice networks.

Example 6.1. Consider \(\wedge = +\), then \(\wedge\) is commutative and associative, but not idempotent. Consider \(f : \mathbb{F}_2^2 \to \mathbb{F}_2^2\), defined by \(f(x_1, x_2) = (x_1 + x_2, x_1 + x_2)\). It is not difficult to see that \(f\) has a unique limit cycle (a fixed point), \(\{(0, 0)\}\); that is, \(\mathcal{C}(f) = C^1\). On the other hand, the loop number of its dependency graph is 1, so from Theorem 5.7 we would obtain \(2C^1 = \mathcal{C}(f)\).

Example 6.2. Consider \(x \wedge y = x^2y\) defined over \(\mathbb{F}_3\). It is easy to show that \(\wedge\) is associative and idempotent, but not commutative. Consider \(f : \mathbb{F}_3^2 \to \mathbb{F}_3^2\), defined by \(f(x_1, x_2) = (x_1^2 + x_2, x_1)\). It is not difficult to show that \(f\) has the limit cycle of length 2, \(\{(1, 2), (2, 1)\}\) (it also has 3 fixed points). On the other hand, the loop number of its dependency graph is 1, so from Theorem 5.7 we would obtain \(3C^1 = \mathcal{C}(f)\).

Example 6.3. Consider \(x \wedge y = 2x + 2y\) defined over \(\mathbb{F}_3\). It is easy to show that \(\wedge\) is commutative and idempotent, but not associative. Consider \(f : \mathbb{F}_3^2 \to \mathbb{F}_3^2\), defined by \(f(x_1, x_2) = (2x_1 + 2x_2, x_1)\). It is not difficult to show that \(f\) has the limit cycle of length 3, \(\{(0, 1), (2, 0), (1, 2)\}\) (it also has 3 fixed points and another limit cycle of length 3). On the other hand, the loop number of its dependency graph is 1, so from Theorem 5.7 we would obtain \(3C^1 \neq \mathcal{C}(f)\).
7. Discussion

In this paper we have identified a broad class of discrete dynamical systems with a finite phase space for which one can derive strong results about their long-term dynamics in terms of properties of their dependency graphs. We classify completely the limit cycles of semilattice networks with strongly connected dependency graph and provide polynomial upper and lower bounds in the general case. It is our hope that the formulas in this paper are related to general properties of semilattices, which is a subject for future investigation. As mentioned in the Introduction, the motivation for this investigation was the need for theoretical tools to analyze discrete models in biology. An example of such an application is given in [6], where it is shown that the results about conjunctive Boolean networks can be applied to determining the limiting behavior of certain types of epidemiological models.

The results in this paper apply to certain types of Boolean networks and cellular automata, which in many cases have the property that the update functions are of the same type for all nodes. Another model type to which the results in this paper apply in some cases is that of so-called logical models, developed by René Thomas for the purpose of modeling gene regulatory networks. It is shown in [8] that logical models can be translated into the framework of polynomial dynamical systems. If the dynamical system arises from a semilattice function, then the results of this paper give information about the steady states and limit cycles of the model under synchronous update.

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