The Lie $h$-Invariant Conformal Field Theories and the Lie $h$-Invariant Graphs

M. B. Halpern, E. B. Kiritsis, N. A. Obers

Department of Physics, University of California and Theoretical Physics Group, Lawrence Berkeley Laboratory Berkeley, CA 94720, USA

and

Physikalisches Institut der Universität Bonn Nussallee 12, D-5300 Bonn 1, GERMANY

Abstract

We use the Virasoro master equation to study the space of Lie $h$-invariant conformal field theories, which includes the standard rational conformal field theories as a small subspace. In a detailed example, we apply the general theory to characterize and study the Lie $h$-invariant graphs, which classify the Lie $h$-invariant conformal field theories of the diagonal ansatz on $SO(n)$. The Lie characterization of these graphs is another aspect of the recently observed Lie group-theoretic structure of graph theory.
Table of Contents

1. Introduction
2. The Virasoro master equation
3. Group Invariant Ansätze
4. Lie $h$-Invariant Ansätze
5. (0,0) and (1,0) Operators in Lie $h$-Invariant CFTs
6. The Lie $h$-Invariant Conformal Multiplets
7. Lie Symmetry in Graph Theory
   7.1 Strategy
   7.2 A Review of $SO(n)_{\text{diag}}$
   7.3 Lie $h$-Invariant Subansätze in $SO(n)_{\text{diag}}$
   7.4 The Lie $h$-Invariant Graphs
   7.5 On the Number of Lie $h$-Invariant Graphs
8. Graphical Identification of (0,0) and (1,0) Operators
9. The Lie $h$-Invariant Graph Multiplets
10. Examples of Lie $h$-Invariant Graphs
    10.1. Counting on Small Manifolds
    10.2. Self $K_{g/h}$-Complementary Graphs
11. Exact Solutions on Lie $h$-Invariant Graphs
    11.1. Small Subansätze with New Level-Families
    11.2. An $(SO(2) \times SO(2))$-Invariant Octet in $SO(6)_{\text{diag}}$
    11.3. An $SO(2)$-Invariant Quartet in $SO(6)_{\text{diag}}$
    11.4. The Self $K_{g/h}$-Conjugate Level-Families of $SO(6)_{\text{diag}}$
Appendix A. Counting (0,0) and (1,0) Operators

1. Introduction

Affine Lie algebra, or current algebra on $S^1$, was discovered independently in mathematics [1] and physics [2]. The first representations [2] were obtained with world-sheet fermions [2, 3] in the construction of current-algebraic spin and internal symmetry on the string [2]. Examples of affine-Sugawara constructions [2, 5, 10] and coset constructions [2, 5] were also given in the first string era, as well as the vertex operator construction of fermions and $SU(n)_1$ from compactified spatial dimensions [3, 4]. The generalization of these constructions [3, 4, 11] and their application to the heterotic string [11].

*The Sugawara-Sommerfield model [4] was in four dimensions on the algebra of fields. The first affine-Sugawara constructions, on affine Lie algebra, were given by Bardakçi and Halpern [3, 5] in 1971.
mark the beginning of the present era. See [12, 13, 14] for further historical remarks on affine-Virasoro constructions.

The general Virasoro construction on the currents \( J_a \) of affine \( g \) [15, 16]

\[
T(L) = L^{ab} J_a J_b^* \tag{1.1}
\]
systematizes the algebraic approach used by Bardakci and Halpern [2, 5] to obtain the original affine-Sugawara and coset constructions. The general construction is summarized by the Virasoro master equation [15, 16] for the inverse inertia tensor \( L^{ab} = L^{ba} \), which contains the affine-Sugawara nests and many new affine-Virasoro constructions \( g^\# \) on the currents of affine \( g \).

In particular, broad classes of exact unitary solutions with irrational central charge [20] have recently been obtained on affine compact \( g \). The growing list presently includes the unitary irrational constructions [20, 22-28]

\[
((\text{simply-laced } g_x)^q)^\#_M
\]

\[
SU(3)^\#_{\text{BASIC}} = \left\{ \begin{array}{l}
SU(3)^\#_{D(1)}, \text{ } SU(3)^\#_{D(2)}, \text{ } SU(3)^\#_{D(3)} \\
SU(3)^\#_{A(1)}, \text{ } SU(3)^\#_{A(2)} \\
SO(2n)^\#[d, 4], \text{ } n \geq 3 \\
SO(5)^\#[d, 6]_2 \\
SO(2n + 1)^\#[d, 6]_{1,2}, \text{ } n \geq 3 \\
SU(5)^\#[m, 2] \\
SU(n)^\#[m(N = 1), rs] \\
SU(\Pi^s n_i)^\#[m(N = 1); \{r\}\{t\}] \end{array} \right. \tag{1.2}
\]

which are called conformal level-families because they are defined for all levels of affine \( g \). As in general relativity, these level-families were obtained in various ansätze and subansätze (BASIC\( \supset \)Dynkin\( \supset \)Maximal, diagonal, metric, etc.) of the Virasoro master equation and its companion, the superconformal master equation [29].

As an example, the value at level 5 of \( SU(3) \) [24]

\[
c \left( (SU(3)_5)^\#_{D(1)} \right) = 2 \left( 1 - \frac{1}{\sqrt{61}} \right) \approx 1.7439 \tag{1.3}
\]
is the lowest unitary irrational central charge yet obtained. The simplest exact unitary irrational level-families yet obtained are the \(rs\)-superconformal set in (1.2) with central charge \([27]\)

\[
c(SU(n)\# [m(N = 1), rs]) = \frac{6nx}{nx + 8 \sin^2(rsp/n)}
\]

where \(r, s \in \mathbb{N}\) and \(x\) is the level of affine \(SU(n)\).

More generally, the solution space of the master equation is immense, with a very large number \([20]\)

\[
N(g) = 2^{\dim g(\dim g - 1)/2}
\]

of level-families expected generically on any \(g\). The level-families have generically irrational central charge and it is a puzzle that the level-families that have been observed on compact \(g\), either exactly or by high-level expansion \([24]\), are generically unitary.

In order to classify their conformal field theories, the Virasoro master equation and the superconformal master equation are generating generalized graph theories on Lie \(g\) \([25-31]\), including conventional graph theory as a special case on the orthogonal groups. In this development, the graphs and generalized graphs not only classify large sets of generically unitary and irrational conformal level-families, but an underlying Lie group-theoretic structure is seen in each of the graph theories. We mention in particular the Lie-algebraic form of the edge-adjacency matrix and isomorphism groups of the graph theories, and we refer the interested reader to Ref. \([28]\), which axiomatizes the subject. See also Ref. \([32]\) for a review of this and other developments in the Virasoro and superconformal master equations.

In this paper, we study the \(Lie h\)-invariant conformal field theories, that is, the solutions of the Virasoro master equation which possess a Lie symmetry, with associated \((0,0)\) and \((1,0)\) operators. Known examples in this class are the affine-Sugawara nests and the exact \(U(1)\)-invariant level-families \([23, 24]\)

\[
SU(3)_{D(1)}^\#, \; SU(3)_{D(2)}^\#, \; SU(3)_{A(1)}^\#, \; SU(3)_{A(2)}^\#
\]

which are generically unitary with generically irrational central charge. In fact, the set of Lie \(h\)-invariant conformal field theories has generically irrational central charge, and is much larger than the set of affine-Sugawara nests, although Lie symmetry is not generic in the space of conformal field theories.

The space of Lie \(h\)-invariant conformal field theories is organized by K-conjugation \([2, 3, 10, 13]\) and Lie symmetry into sets of \(Lie h\)-invariant conformal multiplets, which exhibit a large variety of generalized K-conjugations through coset constructions and general affine-Sugawara nests. We mention in particular the \textit{self K}_{g/h}-conjugate
constructions, whose central charges

\[ c = \frac{1}{2} \frac{c_{g/h}}{c_{g/h}} \]  

(1.6)

are half that of the corresponding \(g/h\) coset constructions. These constructions generalize the self K-conjugate constructions [25, 26, 33] which have half affine-Sugawara central charge.

As an example, we apply the general theory of Lie \(h\)-invariant conformal constructions to the original graph-theory ansatz \(SO(n)_{\text{diag}}\) [25], whose generically unitary and irrational level-families are classified by the conventional graphs of order \(n\). The Lie \(h\)-invariant level-families of this ansatz live on the Lie \(h\)-invariant graphs, whose Lie group-theoretic characterization is obtained in Section 7.4. The Lie characterization of these graphs is another facet of the underlying Lie group structure of graph theory.

In parallel with the conformal field-theoretic development, we also characterize:

a) The Lie \(h\)-invariant graph multiplets.

b) \(K_{g/h}\)-complementarity through the \(g/h\) coset graphs, and other generalized graph complementarities which correspond to the generalized K-conjugations.

c) The self \(K_{g/h}\)-complementary graphs.

Examples and some enumeration of these new graph categories are given in Sections 7.5 and 10, and some exact solutions on Lie \(h\)-invariant graphs are given in Section 11.

It would be interesting to apply the general theory of Lie \(h\)-invariant conformal constructions to the graph theory of superconformal level-families [29, 30], and to the other generalized graph theories of the Virasoro and superconformal master equations.

2. The Virasoro Master Equation

The general affine-Virasoro construction begins with the currents \(J_a\) of untwisted affine Lie \(g\) [1, 2]

\[ [J_a^{(m)}, J_b^{(n)}] = i f_{ab}^\ c J_c^{(m+n)} + m G_{ab} \delta_{m+n,0} \]  

(2.1a)

\[ a, b = 1, \cdots, \dim g \ , \ m, n \in \mathbb{Z} \]  

(2.1b)

where \(f_{ab}^\ c\) and \(G_{ab}\) are respectively the structure constants and general Killing metric of \(g = \oplus_I g_I\). To obtain level \(x_I = 2k_I/\psi_I^2\) of \(g_I\) with dual Coxeter number \(\tilde{h}_I = Q_I/\psi_I^2\), take

\[ G_{ab} = \oplus_I k_I \eta_{ab}^I \ , \ f_{ac}^d f_{bd}^e = - \oplus_I Q_I \eta_{ab}^I \]  

(2.2)
where $\eta^I_{ab}$ and $\psi^I_*$ are respectively the Killing metric and highest root of $g_I$. The class of operators

$$T(L) \equiv L^{ab} J^*_a J^*_b \equiv \sum_{m \in \mathbb{Z}} L^{(m)} z^{-m-2}$$

is defined with symmetric normal ordering, $T_{ab} \equiv J^*_a J^*_b = T_{ba}$, and $L^{ab} = L^{ba}$ is called the inverse inertia tensor, in analogy with the spinning top. In order that $T(L)$ be a Virasoro operator

$$[L^{(m)}, L^{(n)}] = (m - n)L^{(m+n)} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0}$$

the inverse inertia tensor must satisfy the Virasoro master equation (VME) \cite{15, 16}

$$L^{ab} = R^{ab}(L)$$

$$R^{ab}(L) = 2L^{ac}G_{cd}L^{db} - L^{cd}L^{ef}f_{de}^a f_{df}^b - L^{cd}L^{ef}f_{de}^f (a L)^{bc}$$

$$c(L) = 2G_{ab}L^{ab}$$

where we have defined $A(a B_b) \equiv A_a B_b + A_b B_a$. The VME has been identified \cite{34} as an Einstein-like system on the group manifold with central charge $c = \text{dim} g - 4R$, where $R$ is the Einstein curvature scalar.

We remark on some general properties of the master equation which will be useful below.

1. The affine-Sugawara construction \cite{2, 5, 9} $L_g$ is

$$L^{ab}_g = \bigoplus_I \frac{\eta^{ab}_I}{2k_I + Q_I}, \quad c_g = \sum_I \frac{x_I \dim g_I}{x_I + h_I}$$

for arbitrary levels of affine $g$, and similarly for $L_h$ when $h \subset g$.

2. K-conjugation covariance \cite{2, 5, 10, 15}. When $L$ is a solution of the master equation on $g$, then so is the K-conjugate partner $\check{L}$ of $L$

$$\check{L}^{ab} = L^{ab}_g - L^{ab}_h, \quad c(\check{L}) = c_g - c(\check{L})$$

while the corresponding stress tensors $T(L)$ and $T(\check{L})$ form a commuting pair of Virasoro operators. The coset constructions \cite{2, 5, 10} $L_{g/h} = L_g - L_h$ are the K-conjugate partners of $L_h$ on $g$.

3. Affine-Sugawara nests \cite{20}. Repeated K-conjugation on the embedded subgroup sequence $g \supset h_1 \supset \cdots \supset h_d$ generates an affine-Sugawara nest of depth $d + 1$

$$L_{g/h_1/h_2/\cdots/h_d} = L_g - L_{h_1} + L_{h_2} - \cdots + (-1)^d L_{h_d}$$

(2.8a)
\[ c_{g/h_1/h_2\cdots/h_d} = c_g - c_{h_1} + c_{h_2} - \cdots + (-1)^d c_{h_d} \]  

so that the affine-Sugawara and coset constructions are affine-Sugawara nests of depth 1 and 2 respectively. The more general affine-Virasoro nests \[ \text{[20]} \] are formed from (2.8) by the replacement \( L_{h_d} \rightarrow L(h_d) \), where \( L(h_d) \) is any construction on \( h_d \).

4. **Unitarity** \[ \text{[10, 20]} \]. Unitary constructions on positive integer level of affine compact \( g \) are recognized when \( L^{ab} = \text{real} \) in any Cartesian basis, and corresponding forms in other bases.

5. **Automorphisms** \[ \text{[34, 24, 25]} \]. The elements \( \omega \in \text{Aut} g \) of the automorphism group of \( g \) satisfy

\[
\begin{align*}
  f_{ab}^c &= \omega_a^d \omega_b^e (\omega^{-1})_f^c f_{de}^f \\
  G_{ab} &= \omega_a^c \omega_b^d G_{cd}
\end{align*}
\]

and (2.9b) may be written as

\[
(\omega^T)^b_a \equiv G^{bc} \omega_b^d G_{da} = (\omega^{-1})_a^b
\]

so that \( \omega \) is a (pseudo) orthogonal matrix which is an element of \( SO(p,q) \), \( p + q = \text{dim} g \) with metric \( G_{ab} \). The automorphism group includes the outer automorphisms of \( g \) and the inner automorphisms

\[
g(\omega) T_a g^{-1}(\omega) = \omega_a^b T_b , \quad g(\omega) \in G
\]

where \( T \) is any representation of \( G \).

The transformation

\[
J_a^{(m)} = \omega_a^b J_b^{(m)} , \quad \omega \in \text{Aut} g
\]

is an automorphism of affine Lie \( g \) in (2.1), and \( (L')^{ab} \) defined by

\[
(L')^{ab} = L^{cd} (\omega^{-1})_c^a (\omega^{-1})_d^b
\]

\[
R(L')^{ab} = R^{cd} (L) (\omega^{-1})_c^a (\omega^{-1})_d^b , \quad \omega \in \text{Aut} g
\]

is an automorphically equivalent solution of the VME when \( L^{ab} \) is a solution. The corresponding matrix form of the automorphism (2.13)

\[
L' = \omega L \omega^{-1} , \quad L_a^b \equiv G_{ac} L^c_b
\]

\[
R(L') = \omega R(L) \omega^{-1} , \quad R_a^b \equiv G_{ac} R^c_b , \quad \omega \in \text{Aut} g
\]

is preferred below.
6. Self K-conjugate constructions [25, 26, 33]. These constructions satisfy
\[ \bar{L} = \omega L \omega^{-1} \quad \text{for some } \omega \in \text{Aut} \, g \]  
(2.15a)
\[ c = \frac{c_g}{2} \]  
(2.15b)
on Lie group manifolds of even dimension. The first examples were observed on \( SO(4n) \) and \( SO(4n+1) \), where they correspond to the self-complementary graphs of graph theory, and they have also been observed on \( SU(3) \) and \( SU(5) \).

3. Group-Invariant Ansätze

As in general relativity, consistent ansätze and subansätze have played a central role in obtaining exact solutions of the VME on affine \( g \). In this section, we study the broad class of consistent ansätze associated to symmetry under subgroups of \( \text{Aut} \, g \).

In the VME on affine \( g \), the \( H \)-invariant ansatz \( A_g(H) \) is
\[ A_g(H) : L = \omega L \omega^{-1} , \quad \forall \omega \in H \subset \text{Aut} \, g \]  
(3.1)
where \( H \) may be any finite subgroup or Lie subgroup of \( \text{Aut} \, g \), and may involve inner or outer automorphisms of \( g \). The ansatz \( A_g(H) \) is a set of linear relations on \( L^{ab} \) which requires that all the conformal field theories (CFTs) of the ansatz are invariant under \( H \). The ansatz is consistent because, according to (2.13), the same linear relations
\[ R(L) = R(\omega L \omega^{-1}) = \omega R(L) \omega^{-1} , \quad \forall \omega \in H \subset \text{Aut} \, g \]  
(3.2)
are obtained for the right-hand side of the VME.

As a set, the \( H \)-invariant ansätze on \( g \) follow the subgroup embeddings in \( \text{Aut} \, g \), so that
\[ A_g(\text{Aut} \, g) \subset A_g(H_1) \subset A_g(H_2) \cdots \subset A_g(H_n) \subset A_g(1) \]  
(3.3a)
\[ \text{Aut} \, g \supset H_1 \supset H_2 \cdots \supset H_n \supset 1 \]  
(3.3b)
where 1 is the trivial subgroup. Subgroup embedding in \( \text{Aut} \, g \) is a formidable problem, but we may discuss many examples.

The largest \( H \)-invariant ansatz \( A_g(1) \) is the VME itself on affine \( g \). The smallest \( H \)-invariant ansatz \( A_g(\text{Aut} \, g) \) contains no new constructions because it resides in the smallest inner automorphic ansatz
\[ A_g(G) : L = \omega L \omega^{-1} , \quad \forall \omega \in G \]  
(3.4)
which itself contains no new constructions: The explicit form of $A_g(G)$ on $G = \otimes I G_I$ is

$$A_g(G) : L^{ab} = \oplus I L_I \eta_I^{ab}$$

(3.5)

and the solutions of $A_g(G)$, which are all possible $G$-invariant CFTs on affine $g$, are $L = 0$ and the affine-Sugawara constructions on all subsets of $g_I$.

Here are some less trivial examples.

1. The graph-symmetry subansätze in the graph-theory ansatz $SO(n)_{diag}$ [25]. These H-invariant subansätze are reviewed in Section 7.2.

2. Outer automorphism. The Chevalley involution on simple $g$,

$$H'_A = -H_A , \quad E'_\alpha = -E_{-\alpha}$$

(3.6a)

$$\omega_a^b = -\eta_{ab} = -\begin{pmatrix} \delta_{AB} & 0 \\ 0 & \delta_{a+\beta,0} \end{pmatrix} , \quad \forall A \in CSA , \alpha \in \Phi(g)$$

(3.6b)

is an involutive automorphism of $g$ which is an outer automorphism for $SU(n \geq 3), SO(2n \geq 6)$ and $E_6$. The corresponding consistent “complex conjugation”-invariant ansatz

$$L^{Aa} = L^{A,-\alpha} , \quad L^{a\beta} = L^{-\alpha,-\beta} , \quad \text{no restriction on} \ L^{AB}$$

(3.7)

follows from (3.1) and (3.6).

3. Inner automorphism. The “grade automorphism” of simple $g$

$$H'_A = H_A , \quad E'_\alpha = e^{\frac{2\pi}{h} G(\alpha)} E_{\alpha}$$

(3.8a)

$$\omega_a^b = \begin{pmatrix} \delta_{AB} & 0 \\ 0 & e^{\frac{2\pi}{h} G(\alpha) \delta_{a+\beta}} \end{pmatrix}$$

(3.8b)

generates a cyclic subgroup of $\text{Aut} g$, where $h$ is the Coxeter number of $g$ and $G(\alpha)$ is the grade of $\alpha \in \Phi(g)$ (grade=number of simple roots in $\alpha$). The corresponding consistent invariant ansatz has non-zero components

$$L^{AB} , \quad L^{a\beta} \text{ when } G(\alpha) + G(\beta) = 0.$$  

(3.9)

4. A metric ansatz. Consider $SU(n)$ in the Pauli-like basis $[33]$

$$\eta_{\vec{p},\vec{q}} = \sigma(\vec{p}) \delta_{\vec{p},(-\vec{q}) (\text{mod} \ n)}$$

(3.10a)

$$f_{\vec{p},\vec{q}}^{\vec{r}} = -\sqrt{\frac{2 \sigma^2}{n}} \sigma(\vec{p},\vec{q}) \sin\left(\frac{\pi}{n} (\vec{p} \times \vec{q})\right) \delta_{\vec{p}+\vec{q} (\text{mod} \ n),\vec{r}}$$

(3.10b)
where the adjoint index is \( a = \vec{p} = (p_1, p_2) \) with \( p_1, p_2 \) integers between 0 and \( n - 1 \), excluding \( \vec{p} = \vec{0} \). The phases \( \sigma(\vec{p}) \) and \( \sigma(\vec{p}, \vec{q}) \) are given in Ref. [35]. It is not difficult to check that
\[
\omega_{\vec{p} \vec{q}}^{(1)} = e^{i2\pi p_1/n} \delta_{\vec{p}, \vec{q}}, \quad \omega_{\vec{p} \vec{q}}^{(2)} = e^{i2\pi p_2/n} \delta_{\vec{p}, \vec{q}}
\]
are automorphisms which generate the finite subgroup \( Z_n \times Z_n \) of \( \text{Aut} SU(n) \). The corresponding consistent invariant ansatz
\[
L_{\vec{p} \vec{q}} = L_{\vec{p}} L_{\vec{q}}
\]
is the metric ansatz \( SU(n)_{\text{metric}} \) on \( SU(n) \), whose unitary irrational level-families are classified by the sine-area graphs [26, 27, 28].

4. Lie \( h \)-Invariant Ansätze

When \( H \subset G \) is a connected Lie subgroup, the ansatz \( A_g(H) \) in (3.1) is equivalently described by its infinitesimal form
\[
A_g(\text{Lie } h) : \delta L^{ab}(\psi) = L^{c(a f_{cd})\psi^d} = 0
\]
where \( \psi \) parametrizes \( H \) in the neighborhood of the origin
\[
(\omega)^b_a = \delta^b_a + \psi^c f_{ca}^b + \mathcal{O}(\psi^2), \quad \forall \omega \in H
\]
and \( \delta L = \omega L \omega^{-1} - L + \mathcal{O}(\psi^2) \) is an infinitesimal transformation of \( L \) in \( H \). The ansätze \( \{A_g(\text{Lie } h)\} \) will be called the Lie \( h \)-invariant ansätze because all the CFTs of \( A_g(\text{Lie } h) \) have at least the Lie \( h \) symmetry (4.1). The generic Lie \( h \)-invariant ansatz is, like the VME itself, a large set of coupled quadratic equations, so we expect that the generic Lie \( h \)-invariant construction has irrational central charge.

Since the affine-Sugawara construction \( L_g \) in (2.6) is Lie \( h \)-invariant for all \( h \subset g \), it follows from (4.1) that the K-conjugate partner \( \tilde{L} = L_g - L \) of a Lie \( h \)-invariant construction \( L \) is also Lie \( h \)-invariant. Put another way, every Lie \( h \)-invariant ansatz is K-conjugation covariant.

The Lie \( h \)-invariant ansätze follow the Lie subalgebra embeddings of \( g \), as in Eq. (3.3). Some examples of \( A_g(\text{Lie } h) \) are:

1. \( A_g(\text{CSA}) \). The non-vanishing components of the Cartan subalgebra-invariant ansatz
\[
A_g(\text{CSA}) : L^{AB}, L^{a, -\alpha}, \quad A \in \text{CSA} , \quad a \in \Phi(g)
\]
are obtained from (4.1) by setting \( \psi^a = 0 \) and \( \psi^A \) arbitrary.
2. $A_{SU(3)}(\text{Lie } h)$. Up to automorphisms, the complete list of Lie $h$-invariant ansätze on $SU(3)$ is

$$A_{SU(3)}(U(1)) : \begin{cases} L^{11} = L^{22}, & L^{44} = L^{55}, & L^{66} = L^{77} \\ L^{46} = -L^{57}, & L^{47} = L^{56} \\ L^{33}, & L^{88}, & L^{38} \end{cases} \quad (4.4)$$

$$A_{SU(3)}(U(1) \times U(1)) : \begin{cases} L^{11} = L^{22}, & L^{44} = L^{55}, & L^{66} = L^{77} \\ L^{33}, & L^{88}, & L^{38} \end{cases} \quad (4.5)$$

$$A_{SU(3)}(SU(2)_{\text{reg}}) = A_{SU(3)}(SU(2)_{\text{reg}} \times U(1)) : \begin{cases} L^{11} = L^{22} = L^{33}, & L^{88} \\ L^{44} = L^{55} = L^{66} = L^{77} \end{cases} \quad (4.6)$$

$$A_{SU(3)}(SU(2)_{\text{irreg}}) : \begin{cases} L^{11} = L^{33} = L^{44} = L^{66} = L^{88} \\ L^{22} = L^{55} = L^{77} \end{cases} \quad (4.7)$$

This set of ansätze was obtained by choosing

$$U(1) \sim J_3 \subset U(1) \times U(1) \sim J_3, J_8 \subset SU(2)_{\text{reg}} \times U(1) \sim J_1, J_2, J_3, J_8 \quad (4.8a)$$

$$SU(2)_{\text{irreg}} \sim J_2, J_5, J_7 \quad (4.8b)$$

in the Gell-Mann basis, and automorphic copies of these ansätze are obtained by choosing other representatives of the subgroups

Here is what we know about these ansätze:

a) $A_{SU(3)}(SU(2)_{\text{irreg}})$. All 4 level-families of this ansatz are affine-Sugawara nests: $L = 0$, $SU(2)_{4x}$ and their K-conjugates on $SU(3)_{x}$.

b) $A_{SU(3)}(SU(2)_{\text{reg}})$. All 8 level-families of this ansatz are affine-Sugawara nests: $L = 0$, $SU(2)_x$, $U(1)$, $SU(2)_x \times U(1)$ and their K-conjugates on $SU(3)_{x}$.

c) $A_{SU(3)}(U(1) \times U(1))$. This is a subansatz of the ansatz $SU(3)_{\text{BASIC}}$, which has been completely solved \cite{24}. We have checked that $SU(3)_{\text{BASIC}}$, and hence $A_{SU(3)}(U(1) \times U(1))$, contains no $(U(1) \times U(1))$-invariant level-families beyond the relevant affine-Sugawara nests, Cartan $SU(3)^\#$ and $SU(3)/\text{Cartan } SU(3)^\#$ \cite{20}.

d) $A_{SU(3)}(U(1))$. We have not obtained all 256 level-families of this ansatz, but it contains at least one automorphic copy of each of the 8 known $U(1)$-invariant* level-families with unitary irrational central charge \cite{23, 24}.

$$SU(3)_{\text{BASIC}} \supset SU(3)^\#_{D(1)}, SU(3)/SU(3)^\#_{D(1)}, SU(3)^\#_{D(2)}, SU(3)/SU(3)^\#_{D(2)} \quad (4.9a)$$

$$SU(3)^\#_{A(1)}, SU(3)/SU(3)^\#_{A(1)}, SU(3)^\#_{A(2)}, SU(3)/SU(3)^\#_{A(2)} \quad (4.9b)$$

*The representatives of (4.9a) and (4.9b) in Ref. \cite{24} have $U(1) \sim J_5$ and $U(1) \sim (\sqrt{3}J_3 - J_8/2)$ respectively.
3. $A_{SO(n)}(SO(m))$. According to (1.5), the generic number of conformal level-families in the VME on affine $SO(n)$ is

$$N(SO(n)) = 2^{n(n-2)(n^2-1)/8} = \mathcal{O}(2^{n^4/8}).$$

(4.10)

Among these, we estimate the number of $SO(m)$-invariant level-families as follows. The standard Cartesian basis of $SO(n)$ has adjoint index

$$a = ij, \quad 1 \leq i < j \leq n$$

(4.11)

where $i$ and $j$ are vector indices of $SO(n)$, and the inverse inertia tensor is labeled $L^{ij,kl}$ in this basis. Let

$$i = (\mu, I), \quad \mu = 1, 2, \ldots, m, \quad I = m+1 \ldots, n$$

(4.12)

so that $\mu$ is a vector index of $SO(m)$. The $SO(m)$-invariant components of the inverse inertia tensor

$$\sum_{\mu} L^{\mu \nu,\mu \nu}, \sum_{\mu} L^{I \mu, I \mu}, \ L^{I I, K L}$$

(4.13)

may be taken as the non-zero unknowns of the ansatz $A_{SO(n)}(SO(m))$, so we count

$$|A_{SO(n)}(SO(m))| = 2^{1+ \frac{(n-m)(n-m+1)(n-m)(n-m-1)+6}{8}} = \mathcal{O}(2^{n^4/8}) \text{ at fixed } m$$

(4.14)

$SO(m)$-invariant level-families in the VME on $SO(n)$, where the exponent in (4.14) is the number of unknowns in (4.13). The number (4.14) has not been corrected for residual continuous or discrete automorphisms of the ansatz, but this estimate and the corresponding Lie $h$-invariant fraction

$$\frac{|A_{SO(n)}(SO(m))|}{N(SO(n))} = \mathcal{O}(2^{-n^3 m/2}) \text{ at fixed } m$$

(4.15)

provide a strong indication that Lie $h$-invariant constructions, although not generic, are copious in the space of CFTs. A more precise statement of these conclusions, including a comparison with the number of affine-Sugawara nests, is obtained with graph theory in Section 7.5.

5. (0,0) and (1,0) Operators in Lie $h$-Invariant CFTs

The simplest examples of Lie $h$-invariant CFTs on $g$ are the affine-Sugawara nests, which include the affine-Sugawara constructions, the coset construction and the coset nests. Each of these contains certain numbers $N_0$ and $N_1$ of (0,0) and (1,0) operators among the currents of affine $g$. As examples, consider

$$L_g \subset A_g(Lie h) : \quad N_0 = 0, \quad N_1 = \dim g$$

$$L_h \subset A_g(Lie h) : \quad N_0 = \dim h', \quad N_1 = \dim h$$

$$L_{g/h} \subset A_g(Lie h) : \quad N_0 = \dim h, \quad N_1 = \dim h'$$

(5.1)
where \( h' \subset g \) is the centralizer of \( h \) in \( g/h \) (see Ref. [12]). In the case of non-trivial \( h' \), the subgroup and coset constructions have the higher Lie symmetry \( h \oplus h' \), and also appear in \( A_g(h \oplus h') \).

The general affine-Sugawara nest of depth \( d + 1 \)

\[
L_{g/h_1/\cdots/h_d} \subset A_g(\text{Lie } h_d)
\]

(5.2)

has at least a Lie \( h_d \)-symmetry, which is generated by the current subalgebra \( \{J_a, a \in h_d\} \). This set of currents is \( (0,0) \) or \( (1,0) \) of the nest when \( d \) is odd or even respectively.

Consistent with this data, Appendix A contains the proof of the following general results.

**THEOREM.** Let \( L(\text{Lie } h) \in A_g(\text{Lie } h) \) be a unitary conformal construction whose Lie symmetry is exactly \( h \). Then

\[
N_0 + N_1 = \dim h
\]

(5.3)

where \( N_0, N_1 \) is the number of \( (0,0) \) and \( (1,0) \) currents in the construction, and the Lie \( h \) symmetry of \( L(\text{Lie } h) \) is generated by this set of currents.

**REMARK.** This theorem was stated without proof in Ref. [33].

**PROPOSITION.** The symmetry algebra \( \text{Lie } h \) of \( L(\text{Lie } h) \) has the form

\[
h = h_0 \oplus h_1, \quad \dim h_0 = N_0, \quad \dim h_1 = N_1.
\]

(5.4)

**REMARK.** Since

\[
[L^{(m)}(\text{Lie } h), J_A^{(n)}] = 0, \quad \forall n, \quad A \in h_0 \quad (5.5a)
\]

\[
[L^{(m)}(\text{Lie } h), J_I^{(0)}] = 0, \quad I \in h_1 \quad (5.5b)
\]

we will call \( h_0 \) and \( h_1 \) the affine (or local) and global components of the Lie \( h \) symmetry respectively. It also follows from (5.5) that K-conjugation interchanges the global and local components of Lie \( h \)

\[
h_0(\tilde{L}) = h_1(L), \quad h_1(\tilde{L}) = h_0(L)
\]

(5.6)

where \( \tilde{L}(\text{Lie } h) \) is the K-conjugate partner of \( L(\text{Lie } h) \).

**6. The Lie \( h \)-Invariant Conformal Multiplets**
In what follows, we refer to ordinary K-conjugation on affine g as K$_g$-conjugation, in order to distinguish this operation from the generalized K-conjugations discussed below. We also visualize the K$_g$-conjugate partners L and ˜L as a K$_g$-doublet

\[
\begin{array}{c}
L \\
\uparrow_{K_g}
\end{array} 
\begin{array}{c}
\tilde{L} = L_g - L
\end{array}
\]

(6.1)

which is closed under K$_g$-conjugation because K$_g^2 = 1$.

When a conformal construction L(˚h on g) on affine g has an affine (or ˚h) invariance

\[
[L^{(m)}(˚h on g), J_A^{(n)}] = 0, \ A \in h
\]

(6.2)

then it resides in the Lie h-invariant quartet

\[
\begin{array}{c}
L(˚h on g) \ \xrightarrow{+L_h} \ L(˚h on g) + L_h \\
\uparrow_{K_g}
\end{array} 
\begin{array}{c}
L_g - L(˚h on g) \ \xrightarrow{+L_h} \ L_{g/h} - L(˚h on g)
\end{array}
\]

(6.3a)

\[
\begin{array}{c}
c \ \xrightarrow{+L_h} \ c + c_h \\
\uparrow_{K_g}
\end{array} 
\begin{array}{c}
c_g - c \ \xleftarrow{+L_h} \ c_{g/h} - c
\end{array}
\]

(6.3b)

where L$_h$ is the affine-Sugawara construction on h and L$_{g/h} = L_g - L_h$ is the g/h coset construction. All 4 members of the quartet are Lie h-invariant conformal constructions because T(L$_h$) commutes with T(L(˚h on g)), and all 4 members of the quartet occur in the ansatz A$_g$(Lie h). Note, however, that only two of the constructions in the quartet

\[
L(˚h on g), \ L_{g/h} - L(˚h on g)
\]

(6.4)

are ˚h-invariant, while the other two constructions

\[
L(˚h on g) + L_h , \ L_g - L(˚h on g)
\]

(6.5)

are globally h-invariant.

Looking more closely, we see that two generalized K-conjugations are defined in the quartet. In the first place, there is a K$_{g/h}$-conjugation through the g/h coset construction

\[
\begin{array}{c}
L(˚h on g) \ \xrightarrow{K_{g/h}} \ L(˚h on g) + L_h
\end{array} 
\begin{array}{c}
L_g - L(˚h on g) \ \xrightarrow{K_{g/h}} \ L_{g/h} - L(˚h on g)
\end{array}
\]

(6.6)
which acts with $K_{g/h}^2 = 1$ in the space of $\hat{h}$-invariant CFTs. Algebraically, we may write that the $K_{g/h}$-conjugate partner $(\tilde{L})_{g/h}$ of an $\hat{h}$-invariant conformal construction $L$

$$K_{g/h} : (\tilde{L})_{g/h} \equiv L_{g/h} - L \quad , \quad (\tilde{c})_{g/h} = c_{g/h} - c \quad (6.7)$$

is also an $\hat{h}$-invariant conformal construction. We also see a $K_{g+h}$-conjugation (≡ first $K_g$ and then $+L_h$

$$L(\hat{h} \text{ on } g) \quad L(\hat{h} \text{ on } g) + L_h$$

which acts with $K_{g+h}^2 = 1$ in the space of globally $h$-invariant CFTs. In this case, the $K_{g+h}$-conjugate partner $(\tilde{L})_{g+h}$ of a globally $h$-invariant conformal construction $L$

$$K_{g+h} : (\tilde{L})_{g+h} \equiv (L_g + L_h) - L \quad , \quad (\tilde{c})_{g+h} = (c_g + c_h) - c \quad (6.9)$$

is also a globally $h$-invariant conformal construction, although $L_g + L_h$ is not a conformal construction.

The simplest example of a Lie $h$-invariant quartet is the familiar set

$$L = 0 \quad \overset{+L_h}{\xrightarrow{K_g}} \quad L_h \quad \overset{+L_h}{\xrightarrow{K_g}} \quad L_g$$

obtained by choosing $L(\hat{h} \text{ on } g) = 0$. Moreover, the 8 known unitary $U(1)$-invariant level-families with irrational central charge $[23, 24]$

$$SU(3)^A_{D(1)} \overset{+L_{U(1)}}{\xrightarrow{K_{SU(3)}}} SU(3)^A_{D(2)} \quad \overset{+L_{U(1)}}{\xrightarrow{K_{SU(3)}}} SU(3)^A_{D(1)} \quad (6.11a)$$

$$SU(3)/SU(3)^A_{D(1)} \overset{+L_{U(1)}}{\xrightarrow{K_{SU(3)}}} L_{SU(3)/U(1)} - SU(3)^A_{D(1)}$$

obtain 2 $U(1)$-invariant quartets, as shown.

In analogy with the self $K_g$-conjugate constructions $[25, 26, 33]$ in (2.15), it is natural to expect the occurrence of self $K_{g/h}$-conjugate constructions

$$(\tilde{L})_{g/h} = L_{g/h} - L(\hat{h} \text{ on } g) = \omega L(\hat{h} \text{ on } g)\omega^{-1} \quad , \quad \omega \in \text{ Aut } g \quad (6.12a)$$
in which $K_{g/h}$-conjugate partners are automorphically equivalent. Note that self $K_{g/h}$-conjugate constructions will occur with half coset central charge because $c_{g/h} - c = c$.

The existence of these constructions is established with graph theory in Section 10.2, and exact self $K_{g/h}$-conjugate level-families are obtained on $SO(6)$ in Section 11.4.

Moreover, self $K_{g/h}$-conjugate constructions occur in conjunction with self $K_{g+h}$-conjugate constructions

$$\tilde{L}(\hat{h} \text{ on } g) = L_g - L(\hat{h} \text{ on } g) = \omega^{-1}(L(\hat{h} \text{ on } g) + L_h)\omega, \quad \omega \in \text{Aut } g$$

because (6.13a) follows from (6.12a).

The Lie $h$-invariant quartets are only the first glimpse into a web of Lie $h$-invariant conformal multiplets, associated with multiple subgroup embeddings. As an example, the embedding $g \supset h \supset h'$ underlies the Lie $h'$-invariant conformal octet shown in Fig. 1, which is constructed from $L(\hat{h} \text{ on } g)$ by the moves $K_g$, $+L_h$ and $+L_{h'}$. A number of new $K$-conjugations are observed in the octet, including $K_{g/h/h'}$-conjugation through the affine-Sugawara nest of depth 3,

$$K_{g/h/h'} : (L')_{g/h/h'} \equiv L_{g/h/h'} - L(\hat{h} \text{ on } g).$$

Examples of these octets are given in Section 10.1.

More generally, a $2^{d+1}$-plet of Lie $h_d$-invariant CFTs is associated to the general subgroup nest $g \supset h_1 \cdots \supset h_d$ of depth $d + 1$. These $(d + 1)$-dimensional cubes may be generated from $L(\hat{h}_1 \text{ on } g)$ by the moves $K_g$ and $+L_{h_i}$, $i = 1, \cdots, d$, and the multiplets show a broad variety of generalized $K$-conjugations, including $K_{g/h_1/\cdots/h_d}$-conjugation through the general affine-Sugawara nest. We conjecture that generalized self $K$-conjugate constructions exist for all the generalized $K$-conjugations, including self $K_{g/h_1/\cdots/h_d}$-conjugate constructions with central charge

$$c = \frac{1}{2} c_{g/h_1/\cdots/h_d}$$

but we will not investigate these higher multiplets in this paper.

We finally remark that the Lie $h$-invariant multiplets are special cases of a more general class of conformal multiplets associated to affine-Virasoro nesting \cite{20}. Consider,

\footnote{The $h$-invariant quartet of $L(\hat{h} \text{ on } g)$ is the top face of the cube, while the lower members of the multiplet are only $h'$-invariant generically. The entire octet is $h$-invariant in the special case where $h = h_1 \oplus h_2$ and $h' = h_1$ or $h_2$.}
for example, the conformal quartet
\[
\begin{align*}
L(\hat{h} \text{ on } g) & \xrightarrow{+L(h^\#)} L(\hat{h} \text{ on } g) + L(h^\#) \\
L - L(\hat{h} \text{ on } g) & \xleftarrow{+L(h^\#)} L_{g/h^\#} - L(\hat{h} \text{ on } g)
\end{align*}
\]

(6.16)

where \(L(h^\#)\) is any construction on affine \(h\). These quartets show \(K_{g/h^\#}\)-conjugation through the simple affine-Virasoro nest \(L_{g/h^\#} = L_g - L(h^\#)\), but the constructions on the right of (6.16) carry only the Lie symmetry of \(L(h^\#)\), which is generically asymmetric.

7. Lie Symmetry in Graph Theory

7.1 Strategy

The generically unitary and irrational conformal level-families of the ansatz \(SO(n)_{\text{diag}}\) are isomorphic to the graphs of order \(n\), and the Lie group-theoretic structure of graph theory and generalized graph theory has been studied in Refs. [25-31]. In what follows, we apply the general theory of Lie \(h\)-invariant CFTs to characterize and study the \(Lie h\)-invariant graphs, which classify the Lie \(h\)-invariant level-families of \(SO(n)_{\text{diag}}\).

7.2 A Review of \(SO(n)_{\text{diag}}\)

The standard Cartesian basis of \(SO(n)\) has the adjoint index
\[
a = ij \ , \quad 1 \leq i < j \leq n
\]

(7.2.1)

where \(i\) and \(j\) are vector indices of \(SO(n)\), and the basic non-zero structure constants of Cartesian \(SO(n)\) are
\[
f_{ij,il,jl} = -\sqrt{\frac{\tau_n \psi_n^2}{2}} \ , \quad \tau_n = \begin{cases} 1 & , \ n \neq 3 \\ 2 & , \ n = 3 \end{cases}
\]

(7.2.2)

where \(\psi_n\) is the highest root of \(SO(n)\) and \(\tau_n\) is the embedding index of Cartesian \(SO(n)\) in Cartesian \(SO(p > n)\). The other non-zero structure constants of this basis may be obtained from (7.2.2) by permutation of the adjoint indices \(ij\), \(il\) and \(jl\).

The diagonal ansatz \(SO(n)_{\text{diag}}\) on \(SO(n)\) has the form
\[
L^{ab} = L^{ij,kl} = L_{ij} \psi_n^{-2} \delta_{ik} \delta_{jl}
\]

(7.2.3)

and the reduced master equation of \(SO(n)_{\text{diag}}\)
\[
L_{ij}(1 - xL_{ij}) + \tau_n \sum_{l \neq i,j} [L_l L_{ij} - L_{ij}(L_{il} + L_{jl})] = 0
\]

(7.2.4a)
follows from (2.5), (7.2.2) and (7.2.3), where $x$ is the level of affine $SO(n)$. The symmetry conditions $L_{ij} = L_{ji}, L_{ii} = 0$ are assumed in (7.2.4a).

The $2^{2n}$ level-families $L_{ij}(G_n, x)$ of $SO(n)_{\text{diag}}$ are in one to one correspondence with the labeled graphs $G_n$ of order $n$, and, in particular, the high-level form of each level-family

$$L_{ij}(G_n, x) = \frac{1}{x} \theta_{ij}(G_n) + \mathcal{O}(x^{-2}) , \ \theta_{ij}(G_n) \in \{0, 1\}$$

shows the adjacency matrix $\theta_{ij}(G_n)$ of its graph $G_n$. Here are some basic facts about the correspondence, which will be useful below.

1. The values $i$ or $j = 1, \cdots, n$ of the $SO(n)$ vector index are the graph points of $G_n$ and the values $ij$ of the $SO(n)$ adjoint index are the possible graph edges of $G_n$.

2. The graph isomorphisms

$$G'_n \sim G_n \text{ when } \theta_{ij}(G'_n) = \theta_{\pi(i)\pi(j)}(G_n) , \ \pi \in S_n \subset AutSO(n)$$

are the residual isomorphisms of $SO(n)_{\text{diag}}$

$$L_{ij}(G'_n, x) = L_{\pi(i)\pi(j)}(G_n, x) = \sum_{k<l} L_{kl}((\omega^{-1}_\pi)_{kl,ij})^2 , \ \omega_\pi \in S_n .$$

It follows that the automorphically-inequivalent level-families of $SO(n)_{\text{diag}}$ live on the unlabeled graphs of order $n$.

3. The level-family of a graph has the symmetry of its graph,

$$L_{\pi(i)\pi(j)}(G_n, x) = L_{ij}(G_n, x) , \ \forall \pi \in \text{auto}G_n .$$

For each possible graph symmetry $H \subset S_n \subset AutSO(n)$, the linear relations (7.2.8) define the consistent graph-symmetry subansatz which collects the $H$-invariant level-families of $SO(n)_{\text{diag}}$. Historically, these were the first examples of the $H$-invariant ansätze (3.1), and, as we shall see, they include the Lie $h$-invariant subansätze of $SO(n)_{\text{diag}}$ as special cases (see Section 7.4).

4. The affine-Sugawara construction on $SO(n)$ lives on the complete graph $K_n$,

$$L_{SO(n)} = L(K_n, x) .$$

5. $SO(n)_{\text{diag}}$ contains only the subgroup constructions

$$L_{h(SO(n)_{\text{diag}})} = L(G_h(SO(n)_{\text{diag}}), x)$$

(7.2.10a)
\[ G_{h(SO(n)_{\text{diag}})} = \bigcup_{i=1}^{N} K_{m_i} \quad (7.2.10b) \]

\[ h(SO(n)_{\text{diag}}) = \times_{i=1}^{N} SO(m_i), \quad \sum_{i=1}^{N} m_i = n \quad (7.2.10c) \]

on those subgroups \( h(SO(n)_{\text{diag}}) \) whose generators are a subset of the generators \( J_{ij} \) of cartesian \( SO(n) \).

6. K-conjugate level-families live on complementary graphs,

\[ \tilde{L}(G_n, x) = L(\tilde{G}_n, x), \quad \tilde{G}_n = K_n - G_n \quad (7.2.11) \]

so that, e.g., \( \tilde{L}(K_n, x) = L(\tilde{K}_n, x) = 0 \), where \( \tilde{K}_n \) is the completely disconnected graph of order \( n \).

7. The coset constructions of \( SO(n)_{\text{diag}} \) live on the \( g/h \) coset graphs

\[ L_{SO(n)/h(SO(n)_{\text{diag}})} = L(G_{SO(n)/h(SO(n)_{\text{diag}})}, x) \quad (7.2.12a) \]

\[ G_{SO(n)/h(SO(n)_{\text{diag}})} = K_n - \bigcup_{i=1}^{N} K_{m_i}, \quad \sum_{i=1}^{N} m_i = n \quad (7.2.12b) \]

which are the complete \( N \)-partite graphs of order \( n \).

8. The affine-Sugawara nests (2.8) live on the affine-Sugawara nested graphs, for example,

\[ L_{SO(n)/SO(m_1)/.../SO(m_d)} = L(G_{SO(n)/SO(m_1)/.../SO(m_d)}, x) \quad (7.2.13a) \]

\[ G_{SO(n)/SO(m_1)/.../SO(m_d)} = K_n - (K_{m_1} - (K_{m_2} - (\cdots - K_{m_d})) \cdots) . \quad (7.2.13b) \]

These are the graphs of the “old” or standard rational conformal field theories in \( SO(n)_{\text{diag}} \), and the more general affine-Virasoro nested graphs are formed from these by the replacement \( K_{m_d} \rightarrow G_{m_d} \), where \( G_{m_d} \) is any graph of order \( m_d \).

9. The self K-conjugate level-families

\[ \tilde{L}(G_n, x) = L(\tilde{G}_n, x) = \omega L(G_n, x)\omega^{-1}, \quad \omega \in S_n \quad (7.2.14) \]

live on the self-complementary graphs \( \tilde{G}_n \sim G_n \).

See Ref. [28] for further discussion of the Lie group-theoretic structure of graph theory including the Lie-algebraic form of the edge-adjacency matrix and the isomorphism groups of graph theory and the generalized graph theories on Lie \( g \).

7.3 Lie ℎ-Invariant Subansätze in \( SO(n)_{\text{diag}} \)
According to (4.1) and (7.2.3), the $h(SO(n)_{diag})$-invariant subansatz of $SO(n)_{diag}$ is

$$(L_{ij} - L_{kl}) \sum_{r<s} f_{ij,kl,rs} \psi_{rs} = 0$$

(7.3.1)

where $\psi_{rs}$ parametrizes $h(SO(n)_{diag})$ in the vicinity of the origin. In fact, the sum in (7.3.1) may be dropped, and we may write

$A_{SO(n)_{diag}}(h(SO(n)_{diag})) : (L_{ij} - L_{kl}) f_{ij,kl,rs} \psi_{rs} = 0 , \forall rs \in h(SO(n)_{diag})$ (7.3.2)

because the generators of $h(SO(n)_{diag})$ are a subset of the generators of Cartesian $SO(n)$.

As an example, we consider the $SO(m)$-invariant subansatz in $SO(n)_{diag}$. Decompose the vector indices of $SO(n)$ as

$$i = (\mu, I) , \; i = 1, \cdots , m , \; I = m + 1, \cdots , n$$

(7.3.3)

so that Greek letters are vector indices of $SO(m)$. Then the non-zero coefficients of the ansatz

$A_{SO(n)_{diag}}(SO(m)) : L_{\mu\nu} = L , \; L_{\mu I} = L_I , \; L_{IJ}$

(7.3.4)

are obtained from (7.3.2) with the structure constants (7.2.2). The reduced VME of the $SO(m)$-invariant subansatz

$$\tau_n^{-1} L(1 - xL) - (m - 2)L^2 + \sum_{I} L_I (L_I - 2L) = 0$$

(7.3.5a)

$$\tau_n^{-1} L_I(1 - xL_I) - (m - 1)L^2_I + \sum_{J \neq I} [L_{IJ}(L_J - L_I) - L_I L_J] = 0$$

(7.3.5b)

$$\tau_n^{-1} L_{IJ}(1 - xL_{IJ}) + m(L_I L_J - L_{IJ}(L_I + L_J))$$

$$+ \sum_{K \neq I,J} [L_{IK}L_{KJ} - L_{IJ}(L_{IK} + L_{JK})] = 0$$

(7.3.5c)

follows from (7.2.4). Any other decomposition $i = (\mu, I)$ with $\dim \mu = m$ gives an automorphically equivalent copy of $A_{SO(n)_{diag}}(SO(m))$.

Similarly, the $(SO(m_1) \times SO(m_2))$-invariant subansatz of $SO(n)_{diag}$

$$L_{\mu_1\nu_1} = L_1 , \; L_{\mu_2\nu_2} = L_2 , \; L_{\mu_1\mu_2} = L$$

(7.3.6a)

$$L_{\mu_1 I} = L_{I,1} , \; L_{\mu_2 I} = L_{I,2} , \; L_{IJ}$$

(7.3.6b)

is obtained with $i = (\mu_1, \mu_2, I)$, where $\mu_1$ (or $\nu_1$) and $\mu_2$ (or $\nu_2$) are vector indices of $SO(m_1)$ and $SO(m_2)$ respectively.

### 7.4 The Lie $h$-Invariant Graphs
The Lie $h$-invariant level-families of $SO(n)_{\text{diag}}$ are in one to one correspondence with the Lie $h$-invariant graphs, whose characterization may be obtained by taking the high-level limit (7.2.5) of the Lie $h$-invariant level-families (7.3.2):

A graph $G_n$ of order $n$, with adjacency matrix $\theta_{ij}(G_n)$, is (at least) Lie $h(SO(n)_{\text{diag}})$-invariant when

$$(\theta_{ij}(G_n) - \theta_{kl}(G_n)) f_{ij,kl,rs} = 0, \quad \forall rs \in h(SO(n)_{\text{diag}}) \quad (7.4.1)$$

where the subgroups $h(SO(n)_{\text{diag}})$ are characterized in (7.2.10) and $f_{ij,kl,rs}$ are the structure constants (7.2.2) of Cartesian $SO(n)$. The equivalent Lie group characterization

$$\Theta(G_n) = \omega \Theta(G_n) \omega^{-1} \quad (7.4.2a)$$

$$\Theta_{ij,kl} \equiv \delta_{ik}\delta_{jl} \theta_{ij}(G_n), \quad \forall \omega_{ij,kl} \in h(SO(n)_{\text{diag}}) \quad (7.4.2b)$$

also follows with (7.2.3) as the high-level limit of (3.1).

The forms (7.4.1) and (7.4.2) are the Lie-algebraic and Lie group-theoretic characterization of the Lie $h$-invariant graphs. The Lie characterization of these graphs is another facet of the underlying Lie group-theoretic structure of graph theory and generalized graph theory on Lie $g$ [28].

To obtain a visual characterization of the Lie $h$-invariant graphs, we must solve (7.4.1) or (7.4.2). Consider first the $SO(m)$-invariant graphs of order $n$, and let $i = (\mu, I)$ where $\mu$ (the vector index of $SO(m)$) runs over any set of $m$ graph points while $I$ runs over the remaining $n - m$ points. The solution of (7.4.1) for $rs = \mu \nu \in SO(m)$

$$SO(m)\text{-invariant graphs of order } n \quad \begin{cases} \theta_{\mu\nu}(G_n) \text{ is independent of } \mu, \nu \\ \theta_{\mu I}(G_n) \text{ is independent of } \mu \\ \theta_{IJ}(G_n) \text{ arbitrary} \end{cases} \quad (7.4.3)$$

follows easily with the structure constants (7.2.2), or as the high-level limit (7.2.5) of the $SO(m)$-invariant conformal level-families in (7.3.4).

The $SO(m)$-invariant graphs of order $n$ are shown schematically in Fig. 2, which distinguishes two cases,

$$G_{n-m} \oplus_p \tilde{K}_m \quad (\theta_{\mu\nu} = 0) \quad (7.4.4a)$$

$$G_{n-m} \oplus_p K_m \quad (\theta_{\mu\nu} = 1). \quad (7.4.4b)$$

Here, $G_{n-m}$ is any graph of order $n - m$, and the partial join $\oplus_p$ is defined to connect $p \leq n - m$ points of $G_{n-m}$ to all points in $\tilde{K}_m$ or $K_m$.

It is also clear from Fig. 2 that the $SO(m)$-invariant graphs are those with a graph-local discrete symmetry $S_m$, which, by definition, acts only on a fixed set of $m$ points,
without permutation of the remaining \(n - m\) points of the graph. Conversely, any graph with a graph-local \(S_m\) symmetry is also \(SO(m)\)-invariant.

Similarly, to characterize the \((SO(m_1) \times SO(m_2))\)-invariant graphs of order \(n\), decompose the \(SO(n)\) vector index as \(i = (\mu_1, \mu_2, I)\), where \(\mu_1\) (or \(\nu_1\)) and \(\mu_2\) (or \(\nu_2\)) run over distinct sets of \(m_1\) and \(m_2\) graph points respectively and \(I\) runs over the remaining \(n - m_1 - m_2\) points. Then, the characterization of the \((SO(m_1) \times SO(m_2))\)-invariant graphs

\[
\theta_{\mu_1 \nu_1}(G_n), \theta_{\mu_2 \nu_2}(G_n), \theta_{\mu_1 \mu_2}(G_n)
\]

are independent of their labels \(7.4.5a\)

\[
\theta_{\mu_1 I}(G_n), \theta_{\mu_2 I}(G_n)
\]

are independent of \(\mu_1, \mu_2\) \(7.4.5b\)

\[
\theta_{I J}(G_n)
\]

arbitrary \(7.4.5c\)

is obtained as the high-level limit of the \((SO(m_1) \times SO(m_2))\)-invariant level-families in \(7.3.6\). These graphs are shown schematically in Fig. 3, where \(G_{n - m_1 - m_2}\) is a general graph of order \(n - m_1 - m_2\). As above, the subgroup components \((K_{m_1}\) or \(\tilde{K}_{m_1}\)) and \((K_{m_2}\) or \(\tilde{K}_{m_2}\)) are connected to \(G_{n - m_1 - m_2}\) by the independent partial joins \(\oplus_{p_1}\) and \(\oplus_{p_2}\); the two cases shown in the figure correspond to \(\theta_{\mu_1 \mu_2} = 0\) or 1, that is, the connection of all or none of the points of \((K_{m_1}\) or \(\tilde{K}_{m_1}\)) to the points of \((K_{m_2}\) or \(\tilde{K}_{m_2}\)).

The general \(h(SO(n)_{\text{diag}})\)-invariant graph is constructed as follows:

1. Choose the subgroup components \(h_i = (\tilde{K}_{m_i}\) or \(K_{m_i}\)) for each \(SO(m_i)\) in \(h(SO(n)_{\text{diag}}) = \cup_{i=1}^N SO(m_i), m_i \geq 2, \sum_{i=1}^N m_i \leq n\).

2. Choose a set of independent partial joins \(\{\oplus_{p_i}, p_i \leq n - \sum_{i=1}^N m_i\}\) to connect each subgroup component \(h_i\) to any graph of order \(n - \sum_{i=1}^N m_i\).

3. Choose to join or not to join each pair of subgroup components \(h_i\) and \(h_j\).

Moreover, a graph-local discrete symmetry \(S_{m_i}\) is associated to each \(\tilde{K}_{m_i}\) or \(K_{m_i}\) in the general Lie \(h\)-invariant graph. In particular, it was noted in Ref. [25] that every affine-Sugawara nested graph has at least a \(Z_2\) symmetry, and we have checked that this \(Z_2\) is the graph-local \(S_2\) associated to the minimal Lie symmetry \(SO(2)\) of any affine-Sugawara nest. The conformal field-theoretic interpretation of \(\tilde{K}_m\) or \(K_m\) is given in Section 8.

### 7.5 On the Number of Lie \(h\)-Invariant Graphs

In this section, we discuss the (unlabeled or labelled) graph hierarchy:

\[
\text{graphs}
\]

\[
\gg\gg\text{ graphs with symmetry}
\]
Lie $h$–invariant graphs

affine – Sugawara nested graphs

which tells us, as in Section 4, that the Lie $h$-invariant level-families, though copious in $SO(n)_{\text{diag}}$, are not generic. The last inequality also shows that the generic Lie $h$-invariant level-family is a set of new (generically unitary and irrational) CFTs.

It is well known that the generic graph has no symmetry, and it is clear that the Lie $h$-invariant graphs, having a special symmetry, are not generic among the graphs with symmetry.

Bounds on the number of Lie $h$-invariant graphs are obtained as follows. The number of labelled $SO(m)$-invariant graphs in the subansatz $A_{SO(n)_{\text{diag}}}(SO(m))$ is

$$|A_{SO(n)_{\text{diag}}}(SO(m))| = 2^{1 + \frac{(n-m)(n-m+1)}{2}}$$ (7.5.2)

where the exponent is the number of unknowns in the subansatz (see Eq. (7.3.4)). The number (7.5.2) is a lower bound on the number of labelled $SO(m)$-invariant graphs of order $n$ because the subansatz corresponds to the choice of a fixed $SO(m)$, that is, a fixed set of graph points, say $\mu = 1, \ldots, m$. Moreover, (7.5.2) is an upper bound on the number of unlabeled $SO(m)$-invariant graphs of order $n$ because the subansatz contains at least one labelled representative of the equivalence class of each $SO(m)$-invariant graph. We conclude that the number of inequivalent $SO(m)$-invariant level-families of $SO(n)_{\text{diag}}$ satisfies

$$\frac{1}{n!} 2^{1 + \frac{(n-m)(n-m+1)}{2}} \leq N \left( \text{unlabelled } SO(m) \text{-invariant graphs of order } n \right) \leq 2^{1 + \frac{(n-m)(n-m+1)}{2}}$$ (7.5.3)

because no graph of order $n$ is isomorphic to more than $n!$ other graphs. Moreover, the total number of inequivalent Lie $h$-invariant constructions in $SO(n)_{\text{diag}}$ satisfies

$$\frac{1}{n!} 2^{\frac{n^2 - 2n + 4}{2}} \leq N \left( \text{unlabelled Lie } h \text{-invariant graphs of order } n \right) \leq 2^{\frac{n^2 - 2n + 4}{2}}$$ (7.5.4)

because every Lie $h$-invariant graph has at least an $SO(2)$ symmetry.

Comparison of these inequalities to the total number of “old” Lie $h$-invariant level-families in $SO(n)_{\text{diag}}$

$$N \left( \text{unlabelled affine-Sugawara nested graphs of order } n \right) \leq \frac{1}{2n - 1} \binom{2n}{n} = \mathcal{O}(e^{2n \log 2}).$$ (7.5.5)

shows that the generic Lie $h$-invariant level-family in $SO(n)_{\text{diag}}$ is a set of new CFTs.

Some precise counting of Lie $h$-invariant graphs on small manifolds is given in Section 10.1.
8. Graphical Identification of (0,0) and (1,0) Operators

Recall from Section 5 that the symmetry of Lie $h$-invariant CFTs is $h = h_0 \oplus h_1$, where $h_0$ and $h_1$ are the affine and global components of $h$ respectively.

To understand this in the graphs, consider the conformal weights $\Delta_{ij}(G_n, x)$ of the one-current states $J_{ij}^{(-1)}|0\rangle$ of the level-family $L(G_n, x)$ of each graph (see Appendix A): In $SO(n)_{\text{diag}}$, the matrix $M_{ij}^{kl}(G_n, x)$ is diagonal

$$M_{ij}^{kl}(G_n, x) = \Delta_{ij}(G_n, x) \delta_{ik} \delta_{jl}$$

so that a conformal weight is associated to each pair of points $ij$ in $G_n$, and we obtain

$$\Delta_{ij}(G_n, x) = x L_{ij}(G_n, x) + \frac{1}{2} \sum_{l \neq i, j} (L_{il}(G_n, x) + L_{jl}(G_n, x)) \quad (8.2a)$$

$$= \theta_{ij}(G_n) + O(x^{-2}) \quad (8.2b)$$

For most $ij$ in most graphs, the conformal weights in (8.2a) will be irrational at finite level. For the Lie $h$-invariant level-families, on the other hand, we recall the exact result of Section 5,

$$\Delta_{ij}(G_n, x) = \begin{cases} 
0, & ij \in h_0 \\
1, & ij \in h_1 
\end{cases} \quad (8.3)$$

which shows that the higher-order corrections in (8.2) will vanish when $ij \in h$. Comparing (8.2) and (8.3) with (7.4.4), we learn that

$$J_{ij} \text{ is a (0,0) operator of } L(G_{n-m} \oplus_p \tilde{K}_m) \text{ when } i, j \in \tilde{K}_m \quad (8.4a)$$

$$J_{ij} \text{ is a (1,0) operator of } L(G_{n-m} \oplus_p K_m) \text{ when } i, j \in K_m \quad (8.4b)$$

More generally, it follows that the Lie symmetry $SO(m_i) \subset h(SO(n)_{\text{diag}})$ is affine for all subgroup components $\tilde{K}_{m_i}$ and global for all subgroup components $K_{m_i}$. In what follows, we refer to the set of graphs with any affine $h$-invariance as the $h$-invariant graphs.

9. The Lie $h$-Invariant Graph Multiplets

The Lie $h$-invariant graph multiplets are the Lie $h$-invariant conformal multiplets of Section 6, restricted to the ansatz $SO(n)_{\text{diag}}$.

As an example, begin with the $\tilde{SO}(m)$-invariant graph $G_{n-m} \oplus_p \tilde{K}_m$, and consider the $SO(m)$-invariant graph quartet on $SO(n)$

$$G_{n-m} \oplus_p \tilde{K}_m \quad \xrightarrow{+SO(m)} \quad G_{n-m} \oplus_p K_m$$

$$\tilde{G}_{n-m} \oplus_{n-m-p} \tilde{K}_m \quad \xleftarrow{+SO(m)} \quad \tilde{G}_{n-m} \oplus_{n-m-p} K_m$$

(9.1)
which corresponds to (6.3a). The graph moves here are

\[ +SO(m) \equiv \text{add the edges of } K_m = +L_{SO(m)} \quad (9.2a) \]

\[ K_{SO(n)} = \text{complementarity in the graphs of order } n . \quad (9.2b) \]

The \( K_{g/h} \)-conjugation in (6.6)

\[ G_{n-m} \oplus_p K_m \quad \overset{K_{g/h}}{\longrightarrow} \quad G_{n-m} \oplus_p K_m \]

\[ \tilde{G}_{n-m} \oplus_{n-m-p} K_m \quad \overset{K_{g/h}}{\longrightarrow} \quad \tilde{G}_{n-m} \oplus_{n-m-p} K_m \]

is realized in these graphs as the \( K_{SO(n)/SO(m)} \)-complementarity

\[ K_{SO(n)/SO(m)} : \quad \tilde{G}_{n-m} \oplus_{n-m-p} K_m = G_{SO(n)/SO(m)} - G_{n-m} \oplus_p \tilde{K}_m \quad (9.4a) \]

\[ G_{SO(n)/SO(m)} = K_n - K_m = K_{n-m} + \tilde{K}_m \quad (9.4b) \]

through the coset graph \( G_{SO(n)/SO(m)} \). The \( K_{g+h} \)-complementarity along the other diagonal of (9.1) is the composite move \( K_{SO(n)} \) followed by \( +SO(m) \).

More generally, \( K_{g/h} \)-complementarity through the general \( g/h \) coset graph

\[ K_{SO(n)/h(SO(n)_{diag})} : \quad (\tilde{G}_n)_{SO(n)/(h(SO(n)_{diag}))} \equiv G_{SO(n)/(h(SO(n)_{diag}))} - G_n \quad (9.5a) \]

\[ G_{SO(n)/(h(SO(n)_{diag}))} = K_n - \bigcup_{i=1}^N K_{m_i} , \quad \sum_{i=1}^N m_i = n \quad (9.5b) \]

satisfies \( K_{g/h}^2 = 1 \) on the space of \( h(SO(n)_{diag}) \)-invariant graphs.

The self \( K_{g/h} \)-conjugate level-families of \( SO(n)_{diag} \) live with \( c = \frac{1}{2} c_{g/h} \) on the self \( K_{g/h} \)-complementary graphs, which satisfy

\[ (\tilde{G}_n)_{SO(n)/(h(SO(n)_{diag}))} \sim G_n . \quad (9.6) \]

Examples of these graphs are given in Section 10, and the exact self \( K_{g/h} \)-conjugate level-families in \( SO(6)_{diag} \) are obtained in Section 11.4. When a Lie \( h \)-invariant quartet contains a self \( K_{g/h} \)-complementary graph, we know from Section 6 that the graphs along the other diagonal are also isomorphic (self \( K_{g+h} \)-complementary), with level-families whose central charge is \( c = (c_g + c_h)/2. \)

The higher Lie \( h \)-invariant graph multiplets can be studied from their corresponding conformal multiplets, viz. Fig. 1: In particular, a \( 2^{d+1} \)-plet of Lie \( h_d \)-invariant graphs is associated to the general subgroup nest \( SO(n) \supset h_1 \supset \cdots \supset h_d, \; h_i \in h(SO(n)_{diag}) \). These \( (d+1) \)-dimensional cubes may be generated from any \( h_1 \)-invariant graph by the
graph moves $K_{SO(n)}$ and $+h_i$, $i = 1, \cdots, d$, and the multiplets show a broad variety of generalized complementarities, including $K_{SO(n)/h_1/\cdots/h_d}$-complementarity through the general affine-Sugawara nested graph $G_{SO(n)/h_1/\cdots/h_d}$. Because self-complementary and self $K_{g/h}$-complementary graphs exist, we conjecture that generalized self-complementary graphs exist for all the generalized complementarities. Examples of Lie $h$-invariant graph octets are given in the next section.

10. Examples of Lie $h$-Invariant Graphs

10.1. Counting on Small Manifolds

On $SO(2)$ and $SO(3)$, all graphs are affine-Sugawara nested graphs, and hence at least $SO(2)$-invariant.

On $SO(4)$, 10 of the 11 unlabeled graphs are Lie $h$-invariant. These 10 graphs are arranged in Fig. 4 as an $SO(2)$-invariant octet on $SO(4) \supset SO(3) \supset SO(2)$ and an $(SO(2) \times SO(2))$-invariant quartet on $SO(4) \supset SO(2) \times SO(2)$. The graphs on the top face of the cube form an $SO(3)$-invariant quartet. The 11th graph of $SO(4)$, with no Lie $h$-symmetry, is the self-complementary path graph of order 4. More generally, we have found no self-complementary graphs which are also Lie $h$-invariant.

On $SO(5)$, we find that 28 of the 34 unlabeled graphs are Lie $h$-invariant. These include 24 affine-Sugawara nested graphs and an $SO(2)$-invariant quartet, shown in Fig. 5, whose conformal level-families are new. The coset graph $G_{SO(5)/SO(2)} = K_5 - K_2$, through which $K_{g/h}$-complementarity acts in this case, is the sum of the edges of the $K_{g/h}$-complementary graphs in the quartet.

Among the 156 unlabeled graphs of $SO(6)$, we find 120 Lie $h$-invariant graphs, including 66 affine-Sugawara nested graphs and 54 Lie $h$-invariant graphs whose level-families are new. Among the graphs of the new level-families, 42 are new irreducible graphs, which can be arranged as follows:

a) 1 $SO(3)$-invariant quartet in an $SO(2)$-invariant octet on $SO(6) \supset SO(3) \supset SO(2)$.

b) 2 $(SO(2) \times SO(2))$-invariant octets on $SO(6) \supset SO(2) \times SO(2) \supset SO(2)$. One of these octets has 2 pairs of isomorphic graphs (see Section 11.2).

c) 2 $SO(2)$-invariant quartets on $SO(6) \supset SO(2)$ which contain self $K_{SO(6)/SO(2)}$-complementary graphs (see Section 10.2 and 11.4).

d) 4 other $SO(2)$-invariant quartets on $SO(6) \supset SO(2)$ (see Section 11.3).

\*\*The new irreducible graphs \[25\] are those which cannot be constructed by affine-Virasoro nesting from smaller manifolds.
As an example on $SO(n)$, we mention the $SO(n - 3)$-invariant graphs of Fig. 6, which are the new irreducible $\hat{h}$-invariant graphs with the largest Lie symmetry on each manifold.

10.2. Self $K_{g/h}$-Complementary Graphs

Fig. 7 shows the 2 $SO(2)$-invariant quartets on $SO(6)$ which contain self $K_{g/h}$-complementary graphs, and the graphical form of $K_{g/h}$ is shown under each quartet. According to the general theory, both of these self $K_{g/h}$-conjugate level-families have central charge

$$c = \frac{1}{2} \frac{c_{SO(6)}/SO(2)}{x + 4} = \frac{7x - 2}{x + 4}$$

(10.2.1)

while the self $K_{g+h}$-conjugate level-families on the other diagonal of the quartets have

$$c = \frac{(c_{SO(6)} + c_{SO(2)})}{2} = \frac{2(4x + 1)}{(x + 4)}.$$ 

We consider self $K_{g/h}$-complementary graphs in further detail for the case $g/h = SO(n)/SO(m)$. According to (9.3) and (9.6), the self $K_{SO(n)/SO(m)}$-complementary graphs satisfy

$$G_{n-m} \oplus_p \tilde{K}_m \sim \tilde{G}_{n-m} \oplus_{n-m-p} K_m$$

(10.2.2)

so the generic self $K_{SO(n)/SO(m)}$-complementary graph has the structure

$$G_{4q} \oplus_{2q} \tilde{K}_{m=n-4q} , \quad \tilde{G}_{4q} \sim G_{4q}$$

(10.2.3)

which is a half-partial join of $\tilde{K}_m$ to a self-complementary graph of order $4q$. The structure (10.2.3) predicts self $K_{g/h}$-complementary graphs on $SO(n) \supset SO(n - 4q)$, $n \geq 6$, and the central charges of the corresponding self $K_{SO(n)/SO(n-4q)}$-conjugate level-families of these graphs are

$$c = \frac{1}{2} \frac{c_{SO(n)}/SO(n-4q)}{n \geq 6}.$$ 

(10.2.4)

The examples in Fig. 7 on $SO(6) \supset SO(2)$ are constructed with $q = 1$ on the single self-complementary graph $G_4 = P_4$ of order 4, and these examples are the only self $K_{g/h}$-complementary graphs on $SO(6)$. Fig. 8 shows the entire $q = 1$ series of self $K_{SO(n)/SO(n-4)}$-complementary graphs (constructed on $P_4$), and, following (10.2.3), we have also constructed 36 distinct series of self $K_{SO(n)/SO(n-8)}$-complementary graphs on the 10 self-complementary graphs of $SO(8)$.

11. Exact Solutions on Lie $h$-Invariant Graphs

11.1. Small Subansätze with New Level-Families
Smaller ansätze are generally more amenable to exact solution, and these occur with higher symmetry on smaller manifolds. The Lie $h$-invariant subansätze in $SO(5)_{\text{diag}}$ and $SO(6)_{\text{diag}}$ which contain new level-families are

\[
SO(5)[d(SO(2)), 7] \equiv A_{SO(5)_{\text{diag}}}(SO(2)) \\
SO(6)[d(SO(3)), 7] \equiv A_{SO(6)_{\text{diag}}}(SO(3)) \\
SO(6)[d(SO(2) \times SO(2)), 8] \equiv A_{SO(6)_{\text{diag}}}(SO(2) \times SO(2)) \\
SO(6)[d(SO(2)), 11] \equiv A_{SO(6)_{\text{diag}}}(SO(2))
\]  

(11.1.1)

where we have introduced the nomenclature of Ref. [25] to show the number of unknowns in the subansatz.

\subsection*{11.2. An (SO(2)×SO(2))-Invariant Octet in SO(6)_{\text{diag}}}

Because it has a subansatz of higher symmetry, we focus first on the 8-dimensional $(SO(2) \times SO(2))$-invariant subansatz of $SO(6)_{\text{diag}}$,

\[
L_{15} = L_{25}, \quad L_{16} = L_{26}, \quad L_{35} = L_{45}, \quad L_{36} = L_{46} \\
L_{13} = L_{23} = L_{14} = L_{24} \\
L_{12}, \quad L_{34}, \quad L_{56}
\]  

(11.2.1)

where we have chosen $1, 2 \in SO(2)$ and $3, 4 \in SO(2)'$. This subansatz contains 14 new irreducible $(SO(2) \times SO(2))$-invariant level-families, whose graphs form the 2 $(SO(2) \times SO(2))$-invariant octets shown in Fig. 9.

All 256 labelled graphs of the subansatz have the graph-local discrete symmetry

\[
S_2 \times S_2 : \quad 1 \leftrightarrow 2 \quad \text{and/or} \quad 3 \leftrightarrow 4
\]  

(11.2.2)

associated to their $(SO(2) \times SO(2))$ invariance, but those graphs for which both $SO(2)$s are affine (or both global) have the higher symmetry $SO(2) \times SO(2) \times Z_2$, where the discrete symmetry

\[
Z_2 : \quad 1 \leftrightarrow 3, \quad 2 \leftrightarrow 4, \quad 5 \leftrightarrow 6
\]  

(11.2.3)

is the exchange symmetry of the two $SO(2)$ subgroups. Among the 14 new irreducible graphs, only 4 of the graphs in the right octet of Fig. 9 have this symmetry, and we have redrawn two of them in Fig. 10 to show the $Z_2$ in (11.2.3) as a left-right symmetry. The remaining 2 new irreducible $(SO(2) \times SO(2) \times Z_2)$-invariant graphs are the complements of those in the figure.

27
The \((SO(2) \times SO(2) \times Z_2)\)-invariant graphs of order 6 are collected in the 5-dimensional subansatz

\[
L_{15} = L_{25} = L_{36} = L_{46} \\
L_{16} = L_{26} = L_{35} = L_{45} \\
L_{13} = L_{23} = L_{14} = L_{24} \\
L_{12} = L_{34}, L_{56}
\]

which follows from (11.2.1) by imposing the symmetry (11.2.3) in the form \(L_{\pi(i)\pi(j)}=L_{ij}\). In what follows, we refer to this subansatz as \(SO(6)[d(SO(2) \times SO(2)), 5]\), and we have named the level-families of the first two graphs in Fig. 10 accordingly. The reduced master equation of \(SO(6)[d(SO(2) \times SO(2)), 5]\)

\[
L_{15}(1 - xL_{15}) = L_{15}(L_{15} + 3L_{16} + 2L_{13} + L_{12} + L_{56}) - L_{12}L_{15} - L_{16}L_{56} - 2L_{16}L_{13} \\
L_{16}(1 - xL_{16}) = L_{16}(L_{16} + 3L_{15} + 2L_{13} + L_{12} + L_{56}) - L_{12}L_{16} - L_{15}L_{56} - 2L_{15}L_{13} \\
L_{13}(1 - xL_{13}) = 2L_{13}(L_{13} + L_{12} + L_{15} + L_{16}) - 2L_{12}L_{13} - 2L_{15}L_{16} \\
L_{12}(1 - xL_{12}) = 2L_{12}(L_{15} + L_{16} + 2L_{13}) - L_{15}^2 - L_{16}^2 - 2L_{13}^2 \\
L_{56}(1 - xL_{56}) = 4L_{56}(L_{15} + L_{16}) - 4L_{15}L_{16} \\
c = x(4L_{13} + L_{15} + L_{16} + 2L_{12} + L_{56})
\]

follows from (7.2.4) and (11.2.4).

Except for possible sporadic solutions, we have solved the 5-dimensional subansatz completely. The new \((SO(2) \times SO(2) \times Z_2)\)-invariant level-families are

\[
L_{15} = \frac{(1 + \eta R)}{2(x + 4)} + \frac{\eta \sigma}{2} \frac{\sqrt{x^2 - 4x - 4}}{x - 2} R \\
L_{16} = \frac{(1 + \eta R)}{2(x + 4)} - \frac{\eta \sigma}{2} \frac{\sqrt{x^2 - 4x - 4}}{x - 2} R \\
L_{13} = \frac{1}{2(x + 4)} \left( 1 - \eta \frac{x^2 - x - 14}{x - 2} R \right) \\
L_{12} = \frac{\eta \xi}{2x} + \frac{1}{2(x + 4)} \left( 1 + 2\eta \frac{x^2 - 2x - 12}{x(x - 2)} \right) \\
L_{56} = \frac{1}{2(x + 4)} \left( 1 + \eta \frac{x^2 - x - 26}{x - 2} R \right) \\
c = \frac{1}{2(x + 4)} \left( 15x - \eta (3x^2 - 9x - 24) R \right) + \xi \eta
\]
\[ R \equiv \sqrt{\frac{(x - 2)}{(x + 1)(x^2 - x - 18)}} \]  

(11.2.6)

where \( \eta = \pm 1 \) is K-conjugation on \( SO(6) \), \( \sigma = \pm 1 \) labels automorphically equivalent copies and \( \xi = \pm 1 \) are inequivalent level-families.

The high-level limit of the level-families (11.2.6) establishes the correspondence

\[ SO(6)^\#[d(SO(2) \times SO(2)), 5]_1 : \eta = -\xi = 1 \]  

(11.2.7a)

\[ SO(6)^\#[d(SO(2) \times SO(2)), 5]_2 : \eta = \xi = 1 \]  

(11.2.7b)

with the first two graphs in Fig. 10, where \( \sigma = 1 \) is required to obtain the labelling shown. In particular, the level-family \( SO(6)^\#[d(SO(2) \times SO(2)), 5]_1 \) is \( (\hat{SO}(2) \times \hat{SO}(2) \times Z_2) \)-invariant (with \((0,0)\) operators \( J_{12} \) and \( J_{34} \) and the globally-invariant level-family \( SO(6)^\#[d(SO(2) \times SO(2)), 5]_2 \) can be obtained from it by the move \((SO(2) \times SO(2)) = +L_{SO(2) \times SO(2)}\), as in Fig. 9.

As a bonus, we obtain the \( (\hat{SO}(2) \times SO(2)) \)-invariant conformal level-family

\[ L(SO(6)^\#[d(SO(2) \times SO(2)), 8]_1) = L(SO(6)^\#[d(SO(2) \times SO(2)), 5]_1 + L_{SO(2)} \]  

(11.2.8)

where \( SO(2) \) is either 1, 2 or 3, 4. This level-family is the third graph in Fig. 10, and, with K-conjugation, we have obtained all the level-families of the right \( (SO(2) \times SO(2)) \)-invariant octet in Fig. 9.

The level-families of this octet have generically irrational central charge and they are unitary for integer level \( x \geq 5 \). The lowest unitary irrational central charge in the octet is found at level \( 6 \)

\[ c(SO(6)_{6}[d, 5]^\#_1) = \frac{7}{2} \left( 1 - \frac{3}{7\sqrt{21}} \right) \approx 3.1727 \]  

(11.2.9)

and, more generally, all the central charges in the octet increase monotonically with the level toward the integers \( c_0 = \text{dim} E(G) \).

11.3. An \( SO(2) \)-Invariant Quartet in \( SO(6)_{\text{diag}} \)

The 11-dimensional subansatz \( SO(6)[d(SO(2)), 11] \) in \( SO(6)_{\text{diag}} \) is

\[ L_{1i} = L_{2i}, \quad 3 \leq i \leq 6 \]

\[ L_{12}, \quad L_{ij}, \quad 3 \leq i < j \leq 6 \]  

(11.3.1)

where we have chosen \( 1, 2 \in SO(2) \). This subansatz contains representatives of all the Lie \( h \)-invariant level-families of \( SO(6)_{\text{diag}} \).
All $2^{11}$ labelled graphs of the subansatz have the graph-local discrete symmetry

\[ S_2 : \quad 1 \leftrightarrow 2 \]  

(11.3.2)

associated to their $SO(2)$-invariance, but we will consider only those graphs with the higher symmetry $SO(2) \times Z_2$, where $Z_2$ is the discrete symmetry

\[ Z_2 : \quad 3 \leftrightarrow 4 \, , \, 5 \leftrightarrow 6 \, . \]  

(11.3.3)

The new irreducible graphs which have this symmetry are the 2 self K$_{SO(6)/SO(2)}$-complementary graphs (and the 2 self K$_{SO(6)+SO(2)}$-conjugate graphs) in the 2 quartets of Fig. 7 and the $SO(2)$-invariant graph quartet in Fig. 11.

The $SO(2) \times Z_2$-invariant graphs of order 6 are collected in the 7-dimensional subansatz of (11.3.1)

\[
L_{13} = L_{23} = L_{14} = L_{24} \\
L_{15} = L_{25} = L_{16} = L_{26} \\
L_{35} = L_{46} \, , \, L_{45} = L_{36} \\
L_{12} \, , \, L_{34} \, , \, L_{56}
\]  

(11.3.4)

which is called $SO(6)[d(SO(2)), 7']$ in the nomenclature of Ref. [25]. The reduced master equation of $SO(6)[d(SO(2)), 7']$

\[
L_{13}(1-xL_{13}) = L_{13}(2L_{13} + 2L_{15} + L_{12} + L_{34} + L_{35} + L_{45}) \\
- L_{13}(L_{12} + L_{34}) - L_{15}(L_{35} + L_{45}) \\
L_{15}(1-xL_{15}) = L_{15}(2L_{13} + 2L_{15} + L_{12} + L_{56} + L_{35} + L_{45}) \\
- L_{13}(L_{35} + L_{45}) - L_{15}(L_{12} + L_{56}) \\
L_{35}(1-xL_{35}) = L_{35}(2L_{13} + 2L_{15} + 2L_{45} + L_{34} + L_{56}) \\
- 2L_{13}L_{15} - L_{45}(L_{34} + L_{56}) \\
L_{45}(1-xL_{45}) = L_{45}(2L_{13} + 2L_{15} + 2L_{35} + L_{34} + L_{56}) \\
- 2L_{13}L_{15} - L_{35}(L_{34} + L_{56}) \\
L_{12}(1-xL_{12}) = 4L_{12}(L_{13} + L_{15}) - 2(L_{13}^2 + L_{15}^2) \\
L_{34}(1-xL_{34}) = 2L_{34}(2L_{13} + L_{35} + L_{45}) - 2(L_{13}^2 + L_{35}L_{45}) \\
L_{56}(1-xL_{56}) = 2L_{56}(2L_{15} + L_{35} + L_{45}) - 2(L_{15}^2 + L_{35}L_{45})
\]  

(11.3.5)

follows as above from Eqs (7.2.4) and (11.3.4).
Except for possible sporadic solutions, we have solved the 7-dimensional subansatz completely and we present the new solutions in 2 groups.

The \((SO(2) \times Z_2)\)-invariant level-families which form the quartet in Fig. 11 are

\[
L_{13} = \frac{1}{2(x+4)} \left( 1 - \eta R \left( 4 - \epsilon(x+4) \sqrt{x(x-4)} \right) \right)
\]

\[
L_{15} = \frac{1}{2(x+4)} \left( 1 - \eta R \left( 4 + \epsilon(x+4) \sqrt{x(x-4)} \right) \right)
\]

\[
L_{35} = \frac{1}{2(x+4)} (1 + 2\eta(x+2)R) + \frac{\sigma\eta}{2} \sqrt{\frac{x^2 - 4x - 4}{x^4 - 16x^2 + 16}}
\]

\[
L_{45} = \frac{1}{2(x+4)} (1 + 2\eta(x+2)R) - \frac{\sigma\eta}{2} \sqrt{\frac{x^2 - 4x - 4}{x^4 - 16x^2 + 16}}
\]

\[
L_{12} = \frac{1}{2(x+4)} \left( 1 + \eta \left( 16R - \xi \frac{x+4}{x} \right) \right)
\]

\[
L_{34} = \frac{1}{2(x+4)} \left( 1 - \eta R \left( x^2 + 2x - 4 + 2\epsilon(x+4) \sqrt{(x-4)/x} \right) \right)
\]

\[
L_{56} = \frac{1}{2(x+4)} \left( 1 - \eta R \left( x^2 + 2x - 4 - 2\epsilon(x+4) \sqrt{(x-4)/x} \right) \right)
\]

\[
c = \frac{1}{2(x+4)} \left( 15x - \eta(x-2)(x^2 + 4)R \right) - \frac{(1 + \xi)\eta}{2}
\]

\[
R \equiv (x^4 - 16x^2 + 16)^{-1/2}
\]

where \(\eta = \pm 1\) is K-conjugation on \(SO(6)\), \(\xi = \pm 1\) labels the left and right sides of the quartet and \(\sigma,\epsilon = \pm 1\) label 4 automorphic copies of the quartet. More precisely, the high-level limit of the level-families establishes the correspondence

\[
SO(6)^\#[d(SO(2)), 7^1_1]: \quad \eta = \xi = 1 \quad (11.3.7a)
\]

\[
SO(6)^\#[d(SO(2)), 7^1_2]: \quad \eta = -\xi = 1 \quad (11.3.7b)
\]

with the graphs in Fig. 11, where \(\sigma = \epsilon = 1\) is required to obtain the labelling shown. The rest of the quartet is obtained by K-conjugation of these two level-families.

The level-families of this quartet have generically irrational central charge, and they are unitary for integer level \(x \geq 5\). The value at level 5

\[
c \left( SO(6)^\#_5[d(SO(2)), 7^1_1] \right) = \frac{1}{18} \left( 57 - \frac{87}{\sqrt{241}} \right) \approx 2.8553 \quad (11.3.8)
\]

is the lowest unitary irrational central charge in the quartet, and all the central charges of the quartet increase monotonically with the level toward the integers \(c_0 = \text{dim}E(G)\).
11.4. The Self $K_{g/h}$-Conjugate Level-Families of $SO(6)_{\text{diag}}$

The $SO(2)$-invariant quartets of Fig. 7 contain the only 2 self $K_{g/h}$-complementary graphs of order 6, where $g/h = SO(6)/SO(2)$. The corresponding level-families of these quartets

\[
L_{13} = \frac{1}{2(x+4)} \left( 1 + \eta \epsilon \frac{\sqrt{x+4}}{x} \right)
\]

\[
L_{15} = \frac{1}{2(x+4)} \left( 1 - \eta \epsilon \frac{\sqrt{x+4}}{x} \right)
\]

\[
L_{35} = \frac{1}{2(x+4)} + \frac{\sigma \eta}{2 \sqrt{x(x+4)}}
\]

\[
L_{45} = \frac{1}{2(x+4)} - \frac{\sigma \eta}{2 \sqrt{x(x+4)}}
\]

\[
L_{12} = \frac{1}{2(x+4)} \left( 1 - \eta \frac{x+4}{x} \right)
\]

\[
L_{34} = \frac{1}{2(x+4)} \left( 1 - \frac{2 \eta \epsilon}{x} \frac{\sqrt{x+4}}{x} \frac{(x+4)(x^2+4)}{x} \right)
\]

\[
L_{56} = \frac{1}{2(x+4)} \left( 1 + \frac{2 \eta \epsilon}{x} \frac{\sqrt{x+4}}{x} \frac{(x+4)(x^2+4)}{x} \right)
\]

\[
c = \frac{15x}{2(x+4)} - \frac{\eta}{2}
\]  

are also solutions of (11.3.5), where $\eta = \pm 1$ is K-conjugation, and $\sigma, \epsilon = \pm 1$ label automorphic copies. The values $\xi = 1$ or $\xi = -1$ correspond to solutions in the left and right quartets of Fig. 7 respectively. The level-families of both quartets are unitary on all integer levels $x \geq 1$.

More precisely, the high-level limit of the level-families (11.4.1) establishes the correspondence

\[
SO(6)^\#[d(SO(2)), 7']_3 : \quad \eta = \xi = 1
\]  \hspace{1cm} \text{(11.4.2a)}

\[
SO(6)^\#[d(SO(2)), 7']_4 : \quad \eta = -\xi = 1
\]  \hspace{1cm} \text{(11.4.2b)}

where 3 and 4 are the level-families of the left and right self $K_{SO(6)/SO(2)}$-complementary graphs respectively in Fig. 7. Both of these self $K_{SO(6)/SO(2)}$-conjugate level-families are $SO(2)$-invariant with $SO(2) = J_{12}$, and, in agreement with Eq. (10.2.1), they have central charges $c = c_{SO(6)/SO(2)}/2$. Their K-conjugate level-families, which are self $K_{SO(6)+SO(2)}$-conjugate, have $c = (c_{SO(6)} + c_{SO(2)})/2$. 

32
It is clear from the square roots in (11.4.1) that these self $K_{g/h}$-conjugate constructions have generically irrational conformal weights. Since this situation was also observed for self $K$-conjugate constructions [25, 26, 33], we expect that irrational conformal weights will occur generically in all the generalized self $K$-conjugate constructions.

Acknowledgements

We thank F. Harary for a helpful discussion.

Two of the authors (M.B.H. and E.K.) are grateful to T. Eguchi, T. Miwa and the members of RIMS for their kind invitation to participate in the “Infinite Analysis” workshop, and for their financial support and hospitality. E.K. would also like to thank Prof. T. Inami for hospitality at the Yukawa Institute for Fundamental Physics.

Appendix A. Counting (0,0) and (1,0) Operators

To study (0,0) and (1,0) operators of Lie $g$-invariant constructions, recall the general $TJ$ algebra

\[ [L^{(m)}, J_a^{(n)}] = -n M_a^b(L) J_b^{(m+n)} + N_a^{bc}(L) T_{bc}^{(m+n)} \]  

\[ M_a^b(L) = 2 G_{ac} L^{cb} + f_{ad}^c L^{dc} f_{ce}^b \]  

\[ N_a^{bc}(L) = -i f_{ad}^c (b L^{cd}) \]

of any conformal construction $L^{ab}$ on affine $g$. The matrix $M_a^b(L)$ controls the spectrum of the level-one current states in the affine vacuum module,

\[ L^{(m)} \psi_i^a J_a^{(-1)} |0\rangle = \delta_{m,0} \Delta_i \psi_i^a J_a^{(-1)} |0\rangle , \quad m \geq 0 \]

\[ \psi_i^b M_a^b(L) = \Delta_i \psi_i^a , \quad i = 1, \cdots, \dim g \]

where $\psi_i^a$ are the left eigenvectors of $M_a^b(L)$ and $\Delta_i$ are the corresponding conformal weights. It is also known that $M_{ab}(L) = M_a^c(L) G_{cb}$ is a symmetric matrix.

Consequences of unitarity on integer levels of affine compact $g$ are obtained as follows. Work for simplicity in any Cartesian frame of $g$, where $f_{ab}^c$, $G_{ab}$ and all unitary constructions $L^{ab}$ are real. It follows that $M_a^b(L)$, $M_{ab}(L)$ and $\Delta_i$ are real, and we may choose orthonormality in the form

\[ \psi_i^a G_{ab} \psi_j^b = \delta_{ij} \]  

The stronger result

\[ 0 \leq \Delta_i \leq 1 \]  

\[ \psi_i^a G_{ab} \psi_j^b = \delta_{ij} \]  

\[ 0 \leq \Delta_i \leq 1 \]
follows by the standard operator unitarity argument on K-conjugate partners, which satisfy
\( \Delta(L) + \Delta(\bar{L}) = \Delta(L_g) \) for all conformal weights.

Restrict attention now to any conformal construction whose Lie symmetry is exactly \( h \). In this case, the construction also satisfies Eq. (4.1)
\[
\delta L^{ab}(\psi_h) = -i \psi_h^c N_c^{ab}(L) = 0 \quad \text{(A.5)}
\]
where \( \psi_h \) parametrizes \( H \) in the vicinity of the origin, so it follows from (A.1) that
\[
[L^{(m)}, \psi_h^a J_a^{(n)}] = -n \psi_h^a M_a^{b}(L) J_b^{(m+n)} \quad \text{(A.6)}
\]
Next, consider the state
\[
L^{(-1)} \psi_h^a J_a^{(-1)}|0\rangle = \psi_h^a M_a^{b}(L) J_b^{(-2)}|0\rangle \quad \text{(A.7)}
\]
in a unitary Lie \( h \)-invariant theory and compute its norm from each of the two forms (A.7) to obtain
\[
\psi_h^d M_a^{b}(L) G_{ba} (\psi_h^a - \psi_h^c M_a^{c}(L))^* = 0 \quad \text{(A.8)}
\]
A more transparent form of this identity
\[
\sum_{i=1}^{\dim g} \Delta_i (1 - \Delta_i) |\psi_i^* G_{ba} \psi_h^a|^2 = 0 \quad \text{(A.9)}
\]
is obtained with (A.3) by expanding \( \psi_h^a = \sum_i c_i(\psi_h) \psi_i^a \) in the left eigenbasis of \( M_a^{b}(L) \).

It follows from (A.9) that
\[
[L^{(m)}, \psi_i^a J_a^{(n)}] = -n \Delta_i \psi_i^a J_a^{(m+n)} \quad \text{(A.10a)}
\]
\[
\Delta_i = 0 \text{ or } 1 \text{ when } \psi_i^* G_{ab} \psi_h^b \neq 0 \quad \text{(A.10b)}
\]
so all the currents of \( h \) are \((0,0)\) or \((1,0)\) operators of a unitary Lie \( h \)-invariant construction.

The converse of this result was obtained in Ref. 33: When \( L \) is unitary and \( \psi^a \) is a left eigenvector of \( M_a^{b}(L) \) with \( \Delta_i = 0 \text{ or } 1 \), then \( \delta L^{ab}(\psi) = -i \psi^c N_c^{ab}(L) = 0 \). It follows that the set of all \((0,0)\) and \((1,0)\) currents of a unitary theory generate a Lie symmetry, so that a unitary theory with a Lie symmetry which is exactly \( h \) contains
\[
\dim h = N_0 + N_1 \quad \text{(A.11)}
\]
\((0,0)\) and \((1,0)\) currents, where \( N_0 \) and \( N_1 \) are the numbers of each type respectively.

This completes the proof of the theorem in Section 5.

\( \square \)
We can also show that

\[ h = h_0 \oplus h_1 \]  (A.12)

where \( h_0 \) and \( h_1 \) are the commuting affine subalgebras generated by the \((0,0)\) currents \( J_A \) and the \((1,0)\) currents \( J_I \) respectively of \( L^{ab} \),

\[ [L^{(m)}, J^{(n)}_A] = 0 \, , \, [L^{(m)}, J^{(n)}_I] = -nJ^{(m+n)}_I. \]  (A.13)

To see this, note first that the commutator of two \((0,0)\) currents commutes with the stress tensor, so that the set of \((0,0)\) currents of \( L^{ab} \) is closed under commutation. Similarly, the set of \((1,0)\) currents of \( L^{ab} \) is closed under commutation because these currents and their commutators are \((0,0)\) currents of the K-conjugate construction \( \tilde{L}^{ab} \), and hence \((1,0)\) currents of \( L^{ab} \). Finally, the Jacobi identity and current algebra

\[ \left[ L^{(m)}, [J^{(p)}_A, J^{(q)}_I] \right] = -q[J^{(p)}_A, J^{(m+q)}_I] \]  (A.14a)

\[ [J^{(p)}_A, J^{(q)}_I] = iF_A^{\ a}J^{(p+q)}_a + C_{AIp}d_{p+q,0} \]  (A.14b)

imply that \( F_{AI}^\ a \) and \( C_{AI} \) are zero, so that the \((0,0)\) currents commute with the \((1,0)\) currents. This completes the proof of the proposition in Section 5. \[ \square \]
References

[1] V.G. Kač, Funct. Anal. Appl. 1, (1967) 328; R.V. Moody, Bull. Am. Math. Soc. 73 (1967) 217.

[2] K. Bardakçì and M.B. Halpern, Phys. Rev. D3 (1971) 2493.

[3] P. Ramond, Phys. Rev. D3 (1971) 2415.

[4] H. Sugawara, Phys. Rev. 170 (1968) 1659; C. Sommerfield, Phys. Rev. 176 (1968) 2019.

[5] M.B. Halpern, Phys. Rev. D4 (1971) 2398.

[6] M. B. Halpern, Phys. Rev. D12 (1975) 1684; Phys. Rev. D13 (1976) 337.

[7] T. Banks, D. Horn, H. Neuberger, Nucl. Phys. B108 (1976) 119.

[8] J. Lepowsky and R. Wilson, Comm. Math. Phys. 62 (1978) 43; I. B. Frenkel and V. G. Kač, Inv. Math. 62 (1980) 23; G. Segal, Comm. Math. Phys. 80 (1981) 301.

[9] E. Witten, Comm. Math. Phys. 92 (1984) 455; G. Segal, unpublished; V.G. Knizhnik and A. M. Zamolodchikov, Nucl. Phys. B247 (1984) 83.

[10] P. Goddard, A. Kent and D. Olive, Phys. Lett. B152 (1985) 88.

[11] D. Gross, J.A. Harvey, E. Martinec, R. Rohm, Phys. Rev. Lett. 54 (1985) 502.

[12] J.K. Freericks and M.B. Halpern, Ann. of Phys. 188 (1988) 258; Erratum, ibid. 190 (1989) 212.

[13] M.B. Halpern Ann. of Phys. 194 (1989) 247.

[14] P. Goddard and D. Olive, Int. J. Mod. Phys. A1 (1986) 303.

[15] M.B. Halpern and E. Kiritsis, Mod. Phys. Lett. A4 (1989) 1373; Erratum ibid. A4 (1989) 1797.

[16] A. Yu. Morozov, A.M. Perelomov, A.A. Rosly, M.A. Shifman and A.V. Turbiner, Int. J. Mod. Phys. A5 (1990) 803.

[17] E. Kiritsis, Mod. Phys. Lett. A4 (1989) 437.

[18] G. V. Dunne, I. G. Halliday and P. Suranyi, Phys. Lett. B213 (1988) 139.
[19] Y. Kazama and H. Suzuki, Mod. Phys. Lett. A4 (1989) 235; R. Cohen and D. Gepner, Mod. Phys. Lett. A6 (1991) 2249.

[20] M.B. Halpern, E. Kiritsis, N.A. Obers, M. Porrati and J.P. Yamron, Int. J. Mod. Phys. A5 (1990) 2275.

[21] E. Witten in Memorial Volume for V. Knizhnik, eds. L. Brink et al., World Scientific, 1990.

[22] M.B. Halpern and N.A. Obers, Int. J. Mod. Phys. A6 (1991) 1835.

[23] S. Schrans and W. Troost, Nucl. Phys. B345 (1990) 584.

[24] M.B. Halpern and N.A. Obers, Nucl. Phys. B345 (1990) 607.

[25] M.B. Halpern and N.A. Obers, Comm. Math. Phys. 138 (1991) 63.

[26] M.B. Halpern and N.A. Obers, “Magic Bases, Metric Ansätze and Generalized Graph Theories in the Virasoro Master Equation”, Berkeley preprint, UCB-PTH-91/12 (1991). To appear in Ann. of Phys.

[27] M.B. Halpern and N.A. Obers, “Generalized Graphs and Unitary Irrational Central Charge in the Superconformal Master Equation”, Berkeley preprint, UCB-PTH-91/30 (1991). To appear in J. Math. Phys.

[28] M.B. Halpern and N.A. Obers, “Generalized Graph Theory on Lie g and Applications in Conformal Field Theory”, Berkeley preprint, UCB-PTH-91/36 (1991).

[29] A. Giveon, M.B. Halpern, E.B. Kiritsis and N.A. Obers, “The Superconformal Master Equation”, Berkeley preprint, UCB-PTH-91/2 (1991). To appear in Int. J. Mod. Phys. A.

[30] M.B. Halpern and N.A. Obers, “New Superconformal Constructions on Triangle-Free Graphs”, Berkeley preprint, UCB-PTH-91/18 (1991).

[31] M.B. Halpern and N.A. Obers, “Superconformal Constructions on Two-Dimensional Simplicial Complexes”, Berkeley preprint, UCB-PTH-91/22 (1991). To appear in Int. J. Mod. Phys. A.

[32] M. B. Halpern, “Recent Developments in the Virasoro Master Equation”, Berkeley preprint, UCB-PTH-91/43 (1991), invited talk at the conference “Strings and Symmetries 1991”, Stony Brook 1991.
[33] A. Giveon, M.B. Halpern, E.B. Kiritsis and N.A. Obers, Nucl. Phys. B357 (1991) 655.

[34] M.B. Halpern and J.P. Yamron, Nucl. Phys. B332 (1990) 411.

[35] J. Patera and H. Zassenhaus, J. Math. Phys. 29 (1988) 665; D. B. Fairlie, P. Fletcher and C. K. Zachos, Phys. Lett. B218 (1989) 203; I. Bars, Phys. Lett. B245 (1990) 35.