COMMON FIXED POINTS VIA $\lambda$-SEQUENCES IN $G$-METRIC SPACES.

YAÉ OLATOUNDJI GABA$^{1,2,*}$

Abstract. In this article, we use $\lambda$-sequences to derive common fixed points for a family of self-mappings defined on a complete $G$-metric space. We imitate some existing techniques in our proofs and show that the tools employed can be used at a larger scale. These results generalize well known results in the literature.

1. Introduction and preliminaries

The generalization of the Banach contraction mapping principle has been a heavily investigated branch of research. In recent years, several authors have obtained fixed and common fixed point results for various classes of mappings in the setting of many generalized metric spaces. One of them, the $G$-metric space, is our focus in this paper and fixed point results, in this setting, presented by authors like Abbas$^1$, Gaba$^{2,4}$, Mustafa$^7$, Vetro$^8$ and many more, are enlightening on the subject. Moreover, in $^3$, we introduced the concept of $\lambda$-sequence which extended the idea of $\alpha$-series proposed by Vetro et al. in $^8$. The present article exclusively presents natural extensions of some results already given by Abbas$^1$ and Vetro$^8$, and therefore generalizes some recent results regarding fixed point theory in $G$-metric spaces. We also show how the idea of $\lambda$-sequence are used in proving some of these results. The method builds on the convergence of an appropriate series of coefficients. We also make use of a special class of homogeneous functions. Recent and similar work can also be read in $^2, 3, 4, 5$.

We recall here some key results that will be useful in the rest of this manuscript. The basic concepts and notations attached to the idea of $G$-metric spaces can be read extensively in $^7$ but for the convenience of the reader, we discuss the most important ones.

Definition 1.1. (Compare $^7$, Definition 3) Let $X$ be a nonempty set, and let the function $G : X \times X \times X \to [0, \infty)$ satisfy the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$ whenever $x, y, z \in X$;
(G2) $G(x, x, y) > 0$ whenever $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$ whenever $x, y, z \in X$ with $z \neq y$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, (symmetry in all three variables);
(G5)

$$G(x, y, z) \leq [G(x, a, a) + G(a, y, z)]$$

for any points $x, y, z, a \in X$.

Then $(X, G)$ is called a $G$-metric space.

The property (G3) is crucial and shall play a key role in our proofs.

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Proposition 1.2. (Compare [7, Proposition 6]) Let \((X, G)\) be a \(G\)-metric space. Then for a sequence \((x_n) \subseteq X\), the following are equivalent

(i) \((x_n)\) is \(G\)-convergent to \(x \in X\).
(ii) \(\lim_{n,m \to \infty} G(x, x_n, x_m) = 0\).
(iii) \(\lim_{n \to \infty} G(x, x_n, x_n) = 0\).
(iv) \(\lim_{n \to \infty} G(x_n, x, x) = 0\).

Proposition 1.3. (Compare [7, Proposition 9]) In a \(G\)-metric space \((X, G)\), the following are equivalent

(i) The sequence \((x_n) \subseteq X\) is \(G\)-Cauchy.
(ii) For each \(\varepsilon > 0\) there exists \(N \in \mathbb{N}\) such that \(G(x_n, x_m, x_m) < \varepsilon\) for all \(m, n \geq N\).

Definition 1.4. (Compare [7, Definition 9]) A \(G\)-metric space \((X, G)\) is said to be complete if every \(G\)-Cauchy sequence in \((X, G)\) is \(G\)-convergent in \((X, G)\).

Definition 1.5. (Compare [7, Definition 4]) A \(G\)-metric space \((X, G)\) is said to be symmetric if

\[ G(x, y, y) = G(x, x, y), \] for all \(x, y \in X\).

Definition 1.6. (Compare [4, Definition 2.1]) A sequence \((x_n)_{n \geq 1}\) in a metric space \((X, d)\) is a \(\lambda\)-sequence if there exist \(0 < \lambda < 1\) and \(n(\lambda) \in \mathbb{N}\) such that

\[ \sum_{i=1}^{L-1} d(x_i, x_{i+1}) \leq \lambda L \] for each \(L \geq n(\lambda) + 1\).

Definition 1.7. (Compare [2, Definition 6]) A sequence \((x_n)_{n \geq 1}\) in a \(G\)-metric space \((X, G)\) is a \(\lambda\)-sequence if there exist \(0 < \lambda < 1\) and \(n(\lambda) \in \mathbb{N}\) such that

\[ \sum_{i=1}^{L-1} G(x_i, x_{i+1}, x_{i+1}) \leq \lambda L \] for each \(L \geq n(\lambda) + 1\).

Definition 1.8. (Compare [8, Definition 2.1]) For a sequence \((a_n)_{n \geq 1}\) of nonnegative real numbers, the series \(\sum_{n=1}^{\infty} a_n\) is an \(\alpha\)-series if there exist \(0 < \lambda < 1\) and \(n(\lambda) \in \mathbb{N}\) such that

\[ \sum_{i=1}^{L} a_i \leq \lambda L \] for each \(L \geq n(\lambda)\).

Remark 1.9. For a given \(\lambda\)-sequence \((x_n)_{n \geq 1}\) in a \(G\)-metric space \((X, d)\), the sequence \((\beta_n)_{n \geq 1}\) of nonnegative real numbers defined by

\[ \beta_i = d(x_i, x_{i+1}, x_{i+1}), \]

is an \(\alpha\)-series. Moreover, any non-increasing \(\lambda\)-sequence of elements of \(\mathbb{R}^+\) endowed with the max\(^1\) metric is also an \(\alpha\)-series. Therefore, \(\lambda\)-sequences generalise \(\alpha\)-sequences but to ease computations, we shall consider, throughout the paper, \(\alpha\)-series\(^2\).

\(^1\)The max metric \(m\) refers to \(m(x, y) = \max\{x, y\}\).
\(^2\)However, the reader can convince himself that using \(\lambda\)-sequences do not add to the complexity of the problem.
2. First generalizations results

We begin with the following generalisation of [8, Theorem 2.1], the main result of Vetro et al.

Let $\Phi$ be the class of continuous, non-decreasing, sub-additive and homogeneous functions $F : [0, \infty) \to [0, \infty)$ such that $F^{-1}(0) = \{0\}$.

**Theorem 2.1.** Let $(X, G)$ be a complete $G$-metric space and $\{T_n\}$ be a family of self mappings on $X$ such that

\[
F(G(T_ix, T_jy, T_kz)) \leq F \left( (\Theta_{i,j})^s \left[ G(x, T_ix, T_ix) + \frac{1}{2} \left[ G(y, T_jy, T_jy) + G(z, T_kz, T_kz) \right] \right] \right) \\
+ F((\Delta_{i,j}) G(x, y, z)) \tag{2.1}
\]

for all $x, y, z \in X$ with $x \neq y$, $0 \leq \Theta_{i,j}, \Delta_{i,j} < 1; i, j, k = 1, 2, \ldots$, and some $F \in \Phi$, homogeneous with degree $s$. If

\[
\sum_{i=1}^{\infty} \left[ \frac{(i+2\Theta_{i,i+1})^s + (i+2\Delta_{i,i+1})^s}{1 - (i+2\Theta_{i,i+1})^s} \right]
\]

is an $\alpha$-series, then $\{T_n\}$ have a unique common fixed point in $X$.

**Proof.** We will proceed in two main steps.

**Claim 1:** $\{T_n\}_{n \geq 1}$ have a common fixed point in $X$.

For any $x_0 \in X$, we construct the sequence $(x_n)$ by setting $x_n = T_n(x_{n-1})$, $n = 1, 2, \ldots$. We assume without loss of generality that $x_m \neq x_n$ for all $n \neq m \in \mathbb{N}$. Using (2.1), we obtain, for the triplet $(x_0, x_1, x_2)$,

\[
F(G(x_1, x_2, x_3)) = F(G(T_1x_0, T_2x_1, T_3x_2)) \\
\leq (3\Theta_{1,2})^s F \left( \left[ G(x_0, x_1, x_1) + \frac{1}{2} G(x_1, x_2, x_2) + \frac{1}{2} G(x_2, x_3, x_3) \right] \right) \\
+ (3\Delta_{1,2})^s F(G(x_0, x_1, x_2)).
\]

By property (G3) of $G$, one knows that

\[
G(x_i, x_{i+1}, x_{i+1}) \leq G(x_{i-1}, x_i, x_{i+1}) \quad \text{and} \quad G(x_i, x_i, x_{i+1}) \leq G(x_i, x_{i+1}, x_{i+2}).
\]
Hence,

\[ F(G(x_1, x_2, x_3)) = F(G(T_1x_0, T_2x_1, T_3x_2)) \]

\[
\leq (\Theta_{1,2})^s F \left( \left[ G(x_0, x_1, x_2) + \frac{1}{2} G(x_1, x_2, x_3) + \frac{1}{2} G(x_1, x_2, x_3) \right] \right) \\
+ (\Delta_{1,2})^s F(G(x_0, x_1, x_2)) \\
= (\Theta_{1,2})^s F(G(x_1, x_2, x_3)) + [(\Theta_{1,2})^s + (\Delta_{1,2})^s] F(G(x_1, x_2, x_0))
\]

i.e.

\[ F(G(x_1, x_2, x_3)) \leq \frac{[(\Theta_{1,2})^s + (\Delta_{1,2})^s]}{1 - (\Theta_{1,2})^s} F(G(x_0, x_1, x_2)). \]

Also we get

\[ F(G(x_2, x_3, x_4)) \leq \frac{[(\Theta_{2,3})^s + (\Delta_{2,3})^s]}{1 - (\Theta_{2,3})^s} F(G(x_1, x_2, x_3)) \]

\[
\leq \left[ \frac{[(\Theta_{2,3})^s + (\Delta_{2,3})^s]}{1 - (\Theta_{2,3})^s} \right] \left[ \frac{[(\Theta_{1,2})^s + (\Delta_{1,2})^s]}{1 - (\Theta_{1,2})^s} \right] F(G(x_0, x_1, x_2)).
\]

Repeating the above reasoning, we obtain

\[ F(G(x_n, x_{n+1}, x_{n+2})) \leq \prod_{i=1}^{n} \left[ \frac{[(\Theta_{i,i+1})^s + (\Delta_{i,i+1})^s]}{1 - (\Theta_{i,i+1})^s} \right] F(G(x_0, x_1, x_2)). \]

If we set

\[ r_i = \left[ \frac{[(\Theta_{i,i+1})^s + (\Delta_{i,i+1})^s]}{1 - (\Theta_{i,i+1})^s} \right], \]

we have that

\[ F(G(x_n, x_{n+1}, x_{n+2})) \leq \prod_{i=1}^{n} r_i \ F(G(x_0, x_1, x_2)). \]

Therefore, for all \( l > m > n > 2 \)

\[
G(x_n, x_m, x_l) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\
+ \cdots + G(x_{l-1}, x_{l-1}, x_l) \\
\leq G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) \\
+ \cdots + G(x_{l-2}, x_{l-1}, x_l).
\]
Using the fact that $F$ is sub-additive, we write

$$F(G(x_n, x_m, x_l)) \leq \left( \left[ \prod_{i=1}^{n} r_i \right] + \left[ \prod_{i=1}^{n+1} r_i \right] + \cdots + \left[ \prod_{i=1}^{l-2} r_i \right] \right) F(G(x_0, x_1, x_2))$$

$$= \sum_{k=0}^{l-n-2} \left[ \prod_{i=1}^{n+k} r_i \right] F(G(x_0, x_1, x_2))$$

$$= \sum_{k=n}^{l-2} \left[ \prod_{i=1}^{k} r_i \right] F(G(x_0, x_1, x_2)).$$

Now, let $\lambda$ and $n(\lambda)$ as in Definition 1.8, then for $n \geq n(\lambda)$ and using the fact that the geometric mean of non-negative real numbers is at most their arithmetic mean, it follows that

$$F(G(x_n, x_m, x_l)) \leq \sum_{k=n}^{l-2} \frac{1}{k} \left( \sum_{i=1}^{k} r_i \right)^k F(G(x_0, x_1, x_2))$$

$$\leq \left( \sum_{k=n}^{l-2} \alpha^k \right) F(G(x_0, x_1, x_2))$$

$$\leq \frac{\alpha^n}{1-\alpha} F(G(x_0, x_1, x_2)).$$

As $n \to \infty$, we deduce that $G(x_n, x_m, x_l) \to 0$. Thus $(x_n)$ is a $G$-Cauchy sequence. and since $X$ is complete there exists $u \in X$ such that $(x_n)$ $G$-converges to $u$.

Moreover, for any positive integers $k, l$, we have

$$F(G(x_n, T_k u, T_l u)) = F(G(T_n x_{n-1}, T_k u, T_l u))$$

$$\leq F \left( (\Theta_{n,k}) \left[ G(x_{n-1}, T_n x_{n-1}, T_n x_{n-1}) + \frac{1}{2} \left[ G(u, T_k u, T_k u) + G(u, T_l u, T_l u) \right] \right] \right)$$

$$+ F((\Delta_{n,k})G(x_{n-1}, u, u)).$$

Letting $n \to \infty$, and using property (G3) we obtain

$$F(G(u, T_k u, T_l u)) \leq (\Theta_{n,k})^* F(G(u, T_k u, T_l u)),$$

and this is a contradiction, unless $u = T_k u = T_l u$, since $\Theta_{n,k} < 1$. Then $u$ is a common fixed point of $\{T_n\}$.

Claim 2: $u$ is the unique common fixed point of $\{T_m\}$.

Finally, we prove the uniqueness of the common fixed point $u$. To this aim, let us suppose that $v$ is another common fixed point of $\{T_m\}$, that is, $T_m(v) = v$, $\forall m \geq 1$. Then, using (2.1) again, we have

$$F(G(u, v, v)) = F(G(T_m u, T_m v, T_m v)) \leq (\Delta_{m,m})^* F(G(u, v, v)),$$

which yields $u = v$, since $\Delta_{m,m} < 1$. So, $u$ is the unique common fixed point of $\{T_m\}$. \qed
Theorem 2.2. Let \((X, G)\) be a complete \(G\)-metric space and \(\{T_n\}\) be a family of self mappings on \(X\) such that

\[
F(G(T^p_i x, T^p_j y, T^p_k z)) \leq F \left( (k\Theta_{i,j}) \left[ G(x, T^p_i x, T^p_i x) + \frac{1}{2} (G(y, T^p_j y, T^p_j y) + G(z, T^p_k z, T^p_k z)) \right] 
+ F((k\Delta_{i,j}) G(x, y, z)) \right)
\]  

for all \(x, y, z \in X\) with \(x \neq y\), \(0 \leq k\Theta_{i,j}, k\Delta_{i,j} < 1\); \(i, j, k = 1, 2, \ldots\), some positive integer \(p\), and some \(F \in \Phi\), homogeneous with degree \(s\). If

\[
\sum_{i=1}^{\infty} \left[ \frac{(i+2\Theta_{i,i+1})^s + (i+2\Delta_{i,i+1})^s}{1 - (i+2\Theta_{i,i+1})^s} \right] \]

is an \(\alpha\)-series, then \(\{T_n\}\) have a unique common fixed point in \(X\).

Proof. It follows form Theorem 2.1, that the family \(\{T^p_n\}\) have a unique common fixed point \(x^*\). Now for any positive integers \(i, j, i \neq j\),

\[
T_i(x^*) = T_i^p T_i(x^*) = T^p_i T_i(x^*) \quad \text{and} \quad T_j(x^*) = T^p_j T_j(x^*) = T^p_j T_j(x^*)
\]

i.e. \(T_i(x^*)\) and \(T_j(x^*)\) are also fixed points for \(T^p_i\) and \(T^p_j\). Since the common fixed point of \(\{T^p_n\}\) is unique, we deduce that

\[
x^* = T_i(x^*) = T_j(x^*) \quad \text{for all } i.
\]

The next result, corollary of Theorem 2.1, corresponds to the result presented by Vetro [8, Theorem 2.1].

Corollary 2.3. (Compare [8, Theorem 2.1]) Let \((X, G)\) be a complete \(G\)-metric space and \(\{T_n\}\) be a family of self mappings on \(X\) such that

\[
G(T_i x, T_j y, T_k z) \leq (k\Theta_{i,j}) [G(x, T_i x, T_i x) + (G(y, T_j y, T_j y)) + (k\Delta_{i,j}) G(x, y, z)]
\]

for all \(x, y, z \in X\) with \(x \neq y\), \(0 \leq k\Theta_{i,j}, k\Delta_{i,j} < 1\); \(i, j, k = 1, 2, \ldots\). If

\[
\sum_{i=1}^{\infty} \left[ \frac{(i+2\Theta_{i,i+1}) + (i+2\Delta_{i,i+1})}{1 - (i+2\Theta_{i,i+1})} \right] \]

is an \(\alpha\)-series, then \(\{T_n\}\) have a unique common fixed point in \(X\).

Proof. In Theorem 2.1, take \(F = Id_{[0,\infty)}\), \(j = k\) and \(y = z\). □

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\(^3\)Remember that any fixed point of \(T^p_i\) is a fixed point of \(T^p_j\) for \(i \neq j\), Cf. Theorem 2.1.

\(^4\)The identity map on \([0,\infty)\)
3. Second generalizations results

The next generalisation is that of [1, Theorem 2.1], the main result of Abbas at al. Instead of considering three maps, we consider a family of maps like in the previous case. Moreover, to show the reader that λ-sequences do not add to the complexity of the problem, we shall use them in the next statement.

**Theorem 3.1.** Let X be a complete G-metric space (X, G) and \{T_n\} be a sequence of self mappings on X. Assume that there exist three sequences (a_n), (b_n) and (c_n) of elements of X such that

\[
G(T_i x, T_j y, T_k z) \leq (k \Delta_{i,j}) G(x, y, z) + (k \Theta_{i,j}) [G(T_i x, x, x) + G(y, T_j y, y) + G(z, z, T_k z)]
\]

\[
+ (k \Lambda_{i,j}) [G(T_i x, y, z) + G(x, T_j y, z) + G(x, y, T_k z)],
\]

(3.1)

for all \(x, y, z \in X\) with \(0 \leq k \Delta_{i,j} + 3(k \Theta_{i,j}) + 4(k \Lambda_{i,j}) < 1/2\), \(i, j, k = 1, 2, \ldots\), where \(k \Delta_{i,j} = G(a_i, a_j, a_k)\), \(k \Theta_{i,j} = G(b_i, b_j, b_k)\) and \(k \Lambda_{i,j} = G(c_i, c_j, c_k)\). If the sequence \((r_i)\) where

\[
r_i = \frac{[(i+2) \Delta_{i,i+1} + 2(i+2) \Theta_{i,i+1} + 3(i+2) \Lambda_{i,i+1}]}{1 - (i+2) \Theta_{i,i+1} - (i+2) \Lambda_{i,i+1}}
\]

is a non-increasing λ-sequence of \(\mathbb{R}^+\) endowed with the max \(^5\) metric, then \{T_n\} have a unique common fixed point in X. Moreover, any fixed point of \(T_i\) is a fixed point of \(T_j\) for \(i \neq j\).

**Proof.** We will proceed in two main steps.

**Claim 1:** Any fixed point of \(T_i\) is also a fixed point of \(T_j\) and \(T_k\) for \(i \neq j \neq k \neq i\).

Assume that \(x^*\) is a fixed point of \(T_i\) and suppose that \(T_j x^* \neq x^*\) and \(T_k x^* \neq x^*\). Then

\[
G(x^*, T_j x^*, T_k x^*) = G(T_i x^*, T_j x^*, T_k x^*)
\]

\[
\leq (k \Delta_{i,j}) G(x^*, x^*, x^*) + (k \Theta_{i,j}) [G(T_i x^*, x^*, x^*) + G(x^*, T_j x^*, x^*) + G(x^*, x^*, T_k x^*)]
\]

\[
+ (k \Lambda_{i,j}) [G(T_i x^*, x^*, x^*) + G(x^*, T_j x^*, x^*) + G(x^*, x^*, T_k x^*)]
\]

\[
\leq [(k \Theta_{i,j}) + (k \Lambda_{i,j})] [G(x^*, T_j x^*, T_k x^*) + G(x^*, T_j x^*, T_k x^*)]
\]

\[
\leq [(2k \Theta_{i,j}) + (2k \Lambda_{i,j})] [G(x^*, T_j x^*, T_k x^*)],
\]

which is a contradiction unless \(T_i x^* = x^* = T_j x^* = T_k x^*\).

**Claim 2:**

For any \(x_0 \in X\), we construct the sequence \((x_n)\) by setting \(x_n = T_n(x_{n-1})\), \(n = 1, 2, \ldots\). We assume without loss of generality that \(x_n \neq x_m\) for all \(n \neq m\). Using (3.1), we obtain

\[
G(x_1, x_2, x_3) = G(T_1 x_0, T_2 x_1, T_3 x_2)
\]

\[
\leq (3 \Delta_{1,2}) G(x_0, x_1, x_2) + (3 \Theta_{1,2}) [G(x_1, x_0, x_0) + G(x_1, x_2, x_1) + G(x_2, x_2, x_3)]
\]

\[
+ (3 \Lambda_{1,2}) [G(x_1, x_1, x_2) + G(x_0, x_2, x_2) + G(x_0, x_1, x_3)].
\]

By property (G3), one can write

\[^5\text{The max metric } m \text{ refers to } m(x, y) = \max\{x, y\}\]
\[
G(x_1, x_2, x_3) = G(T_1x_0, T_2x_1, T_3x_2)
\leq (3\Delta_{12})G(x_0, x_1, x_2) + (3\Theta_{12})[G(x_1, x_0, x_2) + G(x_1, x_2, x_0) + G(x_1, x_2, x_3)]
\]
\[
+ (3\Lambda_{12})[G(x_1, x_0, x_2) + G(x_0, x_1, x_2) + G(x_0, x_1, x_3)]
\]
Again since
\[
G(x_0, x_1, x_3) \leq G(x_0, x_2, x_2) + G(x_2, x_1, x_3) \leq G(x_0, x_1, x_2) + G(x_2, x_1, x_3),
\]
we obtain,
\[
G(x_1, x_2, x_3) = G(T_1x_0, T_2x_1, T_3x_2)
\leq (3\Delta_{12})G(x_0, x_1, x_2) + (3\Theta_{12})[G(x_1, x_0, x_2) + G(x_1, x_2, x_0) + G(x_1, x_2, x_3)]
\]
\[
+ (3\Lambda_{12})[G(x_1, x_0, x_2) + G(x_0, x_1, x_2) + G(x_0, x_1, x_3)],
\]
that is
\[
[1 - (3\Theta_{12}) - (3\Lambda_{12})]G(x_1, x_2, x_3) \leq [(3\Delta_{12}) + 2(3\Theta_{12}) + 3(3\Lambda_{12})]G(x_0, x_1, x_2).
\]
Hence
\[
G(x_1, x_2, x_3) \leq \frac{[(3\Delta_{12}) + 2(3\Theta_{12}) + 3(3\Lambda_{12})]}{1 - (3\Theta_{12}) - (3\Lambda_{12})}G(x_0, x_1, x_2).
\]
Also we get
\[
G(x_2, x_3, x_4) \leq \frac{[(4\Delta_{23}) + 2(4\Theta_{23}) + 3(4\Lambda_{23})]}{1 - (4\Theta_{23}) - (4\Lambda_{23})}G(x_1, x_2, x_3)
\]
\[
\leq \left[\frac{[(4\Delta_{23}) + 2(4\Theta_{23}) + 3(4\Lambda_{23})]}{1 - (4\Theta_{23}) - (4\Lambda_{23})}\right] \left[\frac{[(3\Delta_{12}) + 2(3\Theta_{12}) + 3(3\Lambda_{12})]}{1 - (3\Theta_{12}) - (3\Lambda_{12})}\right]G(x_0, x_1, x_2).
\]
Repeating the above reasoning, we obtain
\[
G(x_n, x_{n+1}, x_{n+2}) \leq \prod_{i=1}^{n} \left[\frac{[(i+2\Delta_{i,i+1}) + 2(i+2\Theta_{i,i+1}) + 3(i+2\Lambda_{i,i+1})]}{1 - (i+2\Theta_{i,i+1}) - (i+2\Lambda_{i,i+1})}\right]G(x_0, x_1, x_2)
\]
If we set
\[
r_i = \left[\frac{[(i+2\Delta_{i,i+1}) + 2(i+2\Theta_{i,i+1}) + 3(i+2\Lambda_{i,i+1})]}{1 - (i+2\Theta_{i,i+1}) - (i+2\Lambda_{i,i+1})}\right],
\]
we have that
\[
G(x_n, x_{n+1}, x_{n+2}) \leq \prod_{i=1}^{n} r_i G(x_0, x_1, x_2).
\]
Therefore, for all \( l > m > n > 2 \)
\[ G(n, m, x_l) \leq G(n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{l-1}, x_{l-1}, x_l) \]
\[
\leq G(n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots + G(x_{l-2}, x_{l-1}, x_l),
\]

and
\[
G(n, m, x_l) \leq \left( \prod_{i=1}^{n} r_i \right) + \left( \prod_{i=1}^{n+1} r_i \right) + \cdots + \left( \prod_{i=1}^{l-2} r_i \right) G(x_0, x_1, x_2)
\]
\[
= \sum_{k=0}^{l-n-2} \left( \prod_{i=1}^{n+k} r_i \right) G(x_0, x_1, x_2)
\]
\[
= \sum_{k=n}^{l-2} \left( \prod_{i=1}^{k} r_i \right) G(x_0, x_1, x_2).
\]

Now, let \( \lambda \) and \( n(\lambda) \) as in Definition 1.8, then for \( n \geq n(\lambda) \) and using the fact that the geometric mean of non-negative real numbers is at most their arithmetic mean, it follows that
\[
G(n, m, x_l) \leq \sum_{k=n}^{l-2} \left( \frac{1}{k} \left( \sum_{i=1}^{k} r_i \right) \right)^k G(x_0, x_1, x_2)
\]
\[
= \left( \sum_{k=n}^{l-2} \alpha^k \right) G(x_0, x_1, x_2)
\]
\[
\leq \frac{\alpha^n}{1 - \alpha} G(x_0, x_1, x_2).
\]

As \( n \to \infty \), we deduce that \( G(n, m, x_l) \to 0 \). Thus \( (x_n) \) is a \( G \)-Cauchy sequence. Moreover, since \( X \) is complete there exists \( u \in X \) such that \( (x_n) \) \( G \)-converges to \( u \).

If there exists \( n_0 \) such that \( T_{n_0} u = u \), then by the claim 1, the proof of existence is complete. Otherwise for any positive integers \( k, l \), we have
\[
G(n, k u, T_l u) = G(T_n x_{n-1}, k u, T_l u)
\]
\[
\leq (t \Delta_{n,k}) G(x_{n-1}, k u, T_l u) + (t \Theta_{n,k}) [G(T_n x_{n-1}, x_{n-1}, x_{n-1}) + G(u, k u, u) + G(u, u, T_l u)]
\]
\[
+ (t \Lambda_{n,k}) [G(T_n x_{n-1}, u, u) + G(x_{n-1}, k u, u) + G(x_{n-1}, u, T_l u)]
\]

Letting \( n \to \infty \), and using property (G3) we obtain
\[
G(u, k u, T_l u) \leq (t \Theta_{n,k}) [G(u, k u, u) + G(u, u, T_l u)]
\]
\[
+ (t \Lambda_{n,k}) [G(u, k u, u) + G(u, u, T_l u)]
\]
\[
\leq ([2_k \Theta_{ij}] + (2_k \Lambda_{ij}) [G(u, k u, T_l u) + G(u, k u, T_l u)]
\]
and this is a contradiction, unless \( u = T_k u = T_l u \).

Finally, we prove the uniqueness of the common fixed point \( u \). To this aim, let us suppose that \( v \) is another common fixed point of \( T_m \), that is, \( T_m(v) = v, \forall m \geq 1 \). Then, using 3.1, we have

\[
G(u, v, v) = G(T_n u, T_k v, T_l v) \leq (\Delta_{n,k}) G(u, v, v) + 3(\Lambda_{n,k}) G(u, v, v),
\]

which yields \( u = v \). So, \( u \) is the unique common fixed point of \( \{T_m\} \). \( \Box \)

Following the same lines of the proof of Theorem 2.2, one can prove the next theorem.

**Theorem 3.2.** Let \( X \) be a complete \( G \)-metric space \( (X, G) \) and \( \{T_n\} \) be a sequence of self mappings on \( X \). Assume that there exist three sequences \((a_n), (b_n)\) and \((c_n)\) of elements of \( X \) such that

\[
G(T^p_i x, T^p_j y, T^p_k z) \leq (k\Delta_{i,j}) G(x, y, z) + (k\Theta_{i,j})[G(T^p_i x, x, x) + G(y, T^p_j y, y) + G(z, z, T^p_k z)]
\]

\[
+ (k\Lambda_{i,j})[G(T^p_i x, y, z) + G(x, T^p_j y, z) + G(x, T^p_j x, T^p_k z)],
\]

(3.2)

for all \( x, y, z \in X \) with \( 0 \leq k\Delta_{i,j} + 3(k\Theta_{i,j}) + 4(k\Lambda_{i,j}) < 1/2 \), \( i, j, k = 1, 2, \ldots \), some positive integer \( p \), where \( k\Delta_{i,j} = G(a_i, a_j, a_k) \), \( k\Theta_{i,j} = G(b_i, b_j, b_k) \) and \( k\Lambda_{i,j} = G(c_i, c_j, c_k) \). If the sequence \( (r_i) \) where

\[
r_i = \left[ \frac{(i+2)\Delta_{i,i+1}}{1 - (i+2)\Theta_{i,i+1} - (i+2)\Lambda_{i,i+1}} \right]
\]

is a non-increasing \( \lambda \)-sequence of \( \mathbb{R}^+ \) endowed with the max \(^6 \) metric, then \( \{T_n\} \) have a unique common fixed point in \( X \). Moreover, any fixed point of \( T_i \) is a fixed point of \( T_j \) for \( i \neq j \).

The next result, corollary of Theorem 3.1, corresponds to the result presented by Abbas [1, Theorem 2.1].

**Corollary 3.3.** Let \( X \) be a complete \( G \)-metric space \( (X, G) \), \( f, g, h \) mappings on \( X \). Assume that there exist three positive reals \( a, b, c \) such that

\[
G(fx, gy, hz) \leq aG(x, y, z) + b[G(fx, x, x) + G(y, gy, y) + G(z, z, hz)]
\]

\[
+ c[G(fx, y, z) + G(x, gy, y) + G(x, y, hz)],
\]

(3.3)

for all \( x, y, z \in X \) with \( 0 \leq a + 3b + 4c < 1 \). Then \( f, g, h \) have a unique common fixed point in \( X \). Moreover, any fixed point of \( f \) is a fixed point of \( g \) and \( h \) and conversely.

**Proof.** In Theorem 3.1, take \( T_1 = f, T_2 = g, T_3 = h \). Also set

\[
3\Delta_{1,2} = a, \quad 3\Theta_{1,2} = b, \quad 3\Lambda_{1,2} = c.
\]

Hence, we have:

\(^6\)The max metric \( m \) refers to \( m(x, y) = \max\{x, y\} \)
\[ 0 \leq a + 3b + 4c < 1/2 \implies 0 \leq a + 3b + 4c < 1 \]

\[ \iff 0 \leq r = \frac{a + 2b + 3c}{1 - b - c} < 1. \]

The sequence \( r_i = r \) is constant, so in Definition 1.8, if we choose \( \lambda = \frac{1}{2} \) and \( n(\lambda) = 1 \), it is clear that \( \sum_{i=1}^{\infty} r_i \) is an \( \alpha \)-series. Indeed, since

\[ \frac{a + 2b + 3c}{1 - b - c} < a + 3b + 4c < \frac{1}{2}, \]

therefore, for any \( L \geq n(\lambda) + 1 = 1 + 1 = 2 \),

\[ \sum_{i=1}^{L-1} r_i = \sum_{i=1}^{L-1} r < \frac{1}{2}(L - 1) \leq \frac{1}{2}L. \]

\[ \square \]

We conclude this manuscript with the following result, whose proof is straightforward, following the steps of the proofs of the earliest results.

**Theorem 3.4.** Let \( X \) be a complete \( G \)-metric space \((X,G)\) and \( \{T_n\} \) be a sequence of self mappings on \( X \). Assume that there exist three sequences \((a_n),(b_n)\) and \((c_n)\) of elements of \( X \) such that

\[ F[G(T_i^p x, T_j^p y, T_k^p z)] \leq F[(k\Delta_{i,j})G(x, y, z) + (k\Theta_{i,j})[G(T_i^p x, x, x) + G(y, T_j^p y, y) + G(z, z, T_k^p z)] + (k\Lambda_{i,j})[G(T_i^p x, y, z) + G(x, T_j^p y, z) + G(x, y, T_k^p z)], \]

for all \( x, y, z \in X \) with \( 0 \leq (k\Delta_{i,j})^s + 3(k\Theta_{i,j})^s + 4(k\Lambda_{i,j})^s < 1/2 \), \( i, j, k = 1, 2, \cdots \), some positive integer \( p \) and some \( F \in \Phi \), homogeneous with degree \( s \), where \( k\Delta_{i,j} = G(a_i, a_j, a_k) \), \( k\Theta_{i,j} = G(b_i, b_j, b_k) \) and \( k\Lambda_{i,j} = G(c_i, c_j, c_k) \). If the sequence \((r_i)\) where

\[ r_i = \frac{[(i+2\Delta_{i,i+1})^s + 2(i+2\Theta_{i,i+1})^s + 3(i+2\Lambda_{i,i+1})^s]}{1 - (i+2\Theta_{i,i+1})^s - (i+2\Lambda_{i,i+1})^s} \]

is a non-increasing \( \lambda \)-sequence of \( \mathbb{R}^+ \) endowed with the max\(^7\) metric, then \( \{T_n\} \) have a unique common fixed point in \( X \). Moreover, any fixed point of \( T_i \) is a fixed point of \( T_j \) for \( i \neq j \).

In addition to the examples provided by Abbas and Vetro, illustrations of all the above results can be read in [2, Example 2.5] and [3, Example 2.8].

**Conflict of interests**

The author declares that there is no conflict of interests regarding the publication of this article.

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\(^7\)The max metric \( m \) refers to \( m(x, y) = \max\{x, y\} \)
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1École Normale Supérieure de Natitingou, Université de Parakou, Bénin.
2Institut de Mathématiques et de Sciences Physiques (IMSP)/UAC, Porto-Novo, Bénin.

*Corresponding author.
E-mail address: gabayae2@gmail.com