Eigenfunction Statistics of Complex Systems: A Common Mathematical Formulation

Pragya Shukla
Department of Physics, Indian Institute of Technology, Kharagpur, India.
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We derive a common mathematical formulation for the eigenfunction statistics of Hermitian operators, represented by a multi-parametric probability density. The system-information in the formulation enters through two parameters only, namely, system size and the complexity parameter, a function of all system parameter including size. The behavior is contrary to the eigenvalue statistics which is sensitive to complexity parameter only and shows a single parametric scaling.

The existence of a mathematical formulation, of both eigenfunctions and eigenvalues, common to a wide range of complex systems indicates the possibility of a similar formulation for many physical properties. This also suggests the possibility to classify them in various universality classes defined by complexity parameter.

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I. INTRODUCTION

The eigenfunction correlations of various generators of dynamics contain a wealth of information about the system e.g. localized or delocalized nature of the dynamics, decay rate etc. Recently the correlations were shown to be relevant for description of fluctuations of physical properties e.g. conductance in mesoscopic systems, peak-height statistics in coulomb blockade regime of quantum dots [1, 2]. The correlations may vary from level to level or fluctuate in different realizations of a complex system. The strong fluctuations of eigenfunctions are already known to be the hallmark of many critical point studies e.g. metal-insulator transition in disorder systems [3], spin glass [4], and stock market fluctuations [5] etc. Recent studies have revealed existence of the fluctuations in a wider range of complex systems e.g. in the area of quantum information, nanotechnology [2] and complex networks etc. [6]. As a consequence, a detailed information about the eigenfunction statistics of complex systems is very important and desirable.

During recent years, many attempts have been made to statistically formulate the eigenfunction correlations of complex systems; see, for example, [1, 3, 5, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. One of the main tools used in this context is the random matrix approach which can briefly be described as follows (see [8] for details). The presence of complicated interactions (among its various sub-units) in the system under investigation often makes it impossible to exactly determine the relevant operator e.g. in a matrix representation. The elements (some or all) of the operator in the representation can then be best described by a probability distribution. This permits one to replace the operator by an ensemble of the operators which is supposed to describe the generic properties and is referred as random matrix model of the operator. (In this paper, we focus on systems where complicated interactions, of any origin, lead to a partial or full randomization, of the operator, thus, allowing one to use a random matrix approach).

The choice of an appropriate random matrix model for a system is sensitive to its physical conditions (i.e nature and degree of interactions in various sub-units, symmetry and topological conditions, dimensionality etc.). This is because the distribution parameters of each matrix element depend on the interactions between related basis states (or parts of the system) which in turn are governed by system-conditions. In past, this has motivated an introduction of a variety of random matrix ensembles as models for a wide range of complex systems e.g. nuclei, atoms, molecules, disordered and chaotic systems, quantum chromodynamics, elastomechanics, electrodynamics (see reviews [8, 9, 20, 22, 23, 24, 26, 27] and references therein for details), mathematical areas such as Riemann zeta function, enumeration problems in geometry and fluctuations in random permutations [27], biological systems [28], stock markets [5], atmospheric sciences [29] etc [30]. For example, the systems with delocalized wave dynamics (extended throughout the system) and antiunitary symmetries can be well modeled by Wigner-Dyson ensembles; the latter are the Hermitian ensembles with Gaussian distributed matrix elements, with ratio of diagonal to off-diagonal variance \( \alpha = 2 \), (originally introduced by Cartan [31], later developed by Wigner and Dyson to model compound nuclei and other systems) [8, 32]. The cases with partially violated anti-unitary symmetries can be well-modeled by Dyson’s Brownian ensembles (BE) [8, 32] (see section VI also). The ensembles with arbitrary \( \alpha \) (\( \neq 2 \)) [33], banded
matrices (elements with non-zero variance within a band around main diagonal) and sparse matrices (with many elements with zero variance) have been successfully used to model statistical properties of the energy levels and eigenfunctions of systems with localized wave dynamics (e.g. quasi one dimensional wires and disordered systems of higher dimensions, chiral systems). During last decade, many new ensembles were introduced to model the systems with unitary symmetries e.g. block form matrices for the cases with parity violation and pre-compound nuclei, chiral ensembles for systems with chiral symmetry in quantum chromodynamics, C and CI ensembles for cases with particle/ hole structures, superconductivity etc. The non-Hermitian operators e.g. scattering matrices, transfer matrices or correlation matrices (appearing in time-series analysis e.g. stock market, brain, atmospheric studies) can similarly be modeled by circular ensembles and their more generic forms. The breadth of the subject is such that it is not possible to give a detailed account of all ensembles or include all references here.

The applicability of random matrix ensembles to complex systems has been under investigation for past few decades. The validity of the models, however, has been extensively verified in context of the eigenvalue fluctuation only; see reviews and references therein. The validity in the domain of eigenfunction fluctuations is so far mostly studied either in ergodic regime of the wavefunctions (see for some original papers and reviews) or for quasi 1-d systems and specific cases e.g. disordered systems have made it imperative to seek the statistical information in higher dimensions and beyond ergodic regime. This motivates us to pursue the present study. It is also desirable to explore the possibilities of any connection among the critical point behavior of the eigenfunctions of different complex systems. One way to show the connection is by describing their various measures by a common mathematical formulation if possible. A recent study has shown the existence of a similar formulation for the case of level-statistics where system information enters through a single parameter, basically a function of all system parameters. The well-known connection between the statistics of eigenfunctions and eigenvalues in non-ergodic regime motivates us to seek a similar formulation for the eigenfunctions too. Such a formulation can also be useful in deriving the measure of one complex system from another.

The paper is organized as follows. The section II contains a brief revision of the single parametric formulation of the multi-parametric probability density of matrix elements for a wide range of complex systems (see for details). The section III describes the derivation of the complexity parameter governed diffusion equation for the eigenfunction components (of the same eigenfunction or different ones) which is used in section IV to study the distribution of some of the important fluctuation measures. The other measures can also be derived following the same route. Although, the diffusion approach seems to complicate the calculation by introducing a dependence on the initial conditions, however, as discussed in section IV, the statistics of the system can be recovered by integrating over all physically allowed initial conditions. The approach has an extra advantage: it provides a common analytical base for the systems which can be modeled by our ensemble (given below by eq.(1,2)) The section IV briefly discusses the role of complexity parameter in various transitions induced due to change in system-specifics. The section V contains details of the numerical verification of our analytical claims. We conclude in section VI by summarizing our main results and their potential applications.

II. SINGLE PARAMETRIC FORMULATION OF THE MATRIX ELEMENTS PROBABILITY DENSITY

The eigenvalues and eigenfunctions of an operator, say $H$, of a system can be obtained by solving the eigenvalue equation $H U_i = \lambda_i U_i$ (with $U_i$ and $\lambda_i$ as the eigenfunction and corresponding eigenvalue respectively) and any other physical information can then be deduced, in principle, from this knowledge. In the case of a complex system, however, the exact form of an operator e.g Hamiltonian may not be known or it may be far too complicated to solve. To deal with such a situation, one has to make a statistical hypotheses, known as maximum entropy hypothesis, for $H$: a sufficiently complicated system can be described by a matrix which is as random as possible under the conditions compatible with the nature of the dynamics as well as the symmetry requirements. Thus if the symmetries and the nature of the operation is approximately known in a basis space preserving the symmetries, it can be modeled by an ensemble of full or sparse random matrices in that basis. For example, an equal probability of dynamics in each region of a specific space suggests a uniform spread of the eigenfunctions in the entire associated basis space. The operator in such a basis will therefore be a full matrix, $\langle k | H | l \rangle = \sum_i \lambda_i U_{ki} U_{li}$ being of the same order for all combinations of
basis vectors $|k\rangle,|l\rangle$ (with $U_{kl}$ as the $k^{th}$ component of eigenvector $U_i$). On the other hand, the dynamics localized in a space leads to variation of the eigenfunction intensities in the associated basis and the operator will be a sparse matrix.

It is clear from the above that, contrary to eigenvalues, the eigenfunction statistics depends on the basis in which the matrix is represented. The knowledge however is still relevant because (i) it can provide important information about the system dynamics in a given basis-space of interest, (ii) it is also possible to define a relevant basis to represent an operator: it is the basis in which the constraints on the operator appear in a natural way. For example, for the time-reversal invariant systems with integer angular momentum, the relevant basis is the one in which their Hamiltonians are simultaneously expressed as real-symmetric matrices [24].

In this paper, we consider a prototype distribution which can model a wide range of complex systems, namely, an ensemble of Hermitian matrices $H$, each of size $N$, described by a Gaussian probability density

$$\rho(H,h,b) = C\exp[-\sum_{s=1}^{\beta}\sum_{k,l}^{N}\sum_{k,l}^{N} (1/2h_{kl,s}).(H_{kl,s} - b_{kl,s})^2]$$

(1)

Here $|k\rangle,|l\rangle$ are unit vectors of the arbitrary basis of size $N$, chosen to represent $H$ with $H_{kl} \equiv \langle k|H|l\rangle$. The subscript $"s"$ refers to the components of $H_{kl}$, $\beta$ as their total number ($\beta = 1$ for real variable, $\beta = 2$ for the complex one), $C$ as the normalization constant, $h$ as the variance matrix with $h_{kl,s} = \langle H_{kl,s}\rangle$ and $b$ as the mean value matrix with $\langle H_{kl,s}\rangle = b_{kl,s}$. Our choice of Hermitian nature of the ensemble restricts the present discussion to the class of systems with conservative dynamics. Following maximum entropy hypothesis, the above ensemble can well-describe the distribution of the operators for which the average behavior of the matrix elements and their variances is known. Based on the complexity of the system, the elements of the parametric matrices $h, b$ can have various functional forms e.g. exponential, power law etc. For example, the limit $h_{kl,s} \rightarrow 0$, corresponds to non-random nature of $H_{kl}$. The limit $h_{kl,s} \rightarrow \gamma, b_{kl,s} \rightarrow 0$ for all $\{k,l,s\}$ gives the density for a Wigner-Dyson ensemble [8]: $\rho(H) \propto e^{-TrH^2}$. The limit $h_{kl,s} \rightarrow [\sigma_{a}\delta_{kl} + \sigma_{b}(1 - \delta_{kl})], b_{kl,s} \rightarrow 0$ for all $\{k,l,s\}$ gives the density for a Rosenzweig-Porter ensemble [33]; (note Brownian ensembles have the same density-form too, see section VI). A band matrix ensemble [10, 20, 23, 34] with Gaussian distributed matrix elements and band length $t$ can be obtained by substituting $h_{kl,s} \rightarrow 0, b_{kl,s} \rightarrow 0$ if $|k-l| > t$ and $h_{kl,s} \propto a(1-t)$ for $|k-l| \leq t$ with various possible forms of function $a$ e.g. exponential, rectangular etc. Similarly other ensembles with uncorrelated matrix elements, some of them with Gaussian randomness and others non-random, can be represented by appropriate choice of $h$ and $b$ parameters [50].

The eq.(1) is applicable for the cases with mutually independent matrix elements, with no condition imposed on their moments higher than $2^{nd}$ order. Here we briefly mention only two such cases, namely, disordered systems and mixed dynamical systems; (the application to other cases e.g. algebraic or algorithmic complexity [55] will be discussed elsewhere). During recent past, specific cases of eq.(1) have been extensively used to model the energy level statistics of disordered system within independent electron approximation; (the latter results in independent matrix elements of the Hamiltonian). One such example is the power law random banded matrix (PRBM) ensemble (each $h_{kl}$ with a power law dependence on the distance from the diagonal) [34] which has been shown to be a good model for the level-statistics of Anderson Hamiltonian (AH) [2]. The ensemble (1) was also used recently to prove, analytically as well as numerically, the single parameter scaling of the level-statistics of Anderson Hamiltonian and its mapping to single parametric Brownian ensembles [50]. (Later, in this paper, AH, BE and PRBM cases are also used to verify our analytical predictions for the eigenfunction statistics).

Another potential application of eq.(1) is to systems with mixed dynamics where, similar to disorder, KAM tori lead to a localization of dynamics [48]. The connection of quantum systems in classically chaotic and integrable regime to Wigner-Dyson ensembles and Poisson ensembles, respectively, is already well-established [24, 41, 52, 53]. In past, it has been suggested that a mixed Hamiltonian (or time-evolution operator) in a relevant basis should appear as a block diagonal matrix, each block being associated with an isolated region of the classical phase space [24, 52, 53]. In cases, where a chaotic region can be decomposed in nearly, but not completely isolated subregions, blocks are expected to be connected through small but non-zero matrix elements. The average size of these matrix elements i.e the quantum constraints will be related to flux connecting different regions i.e to classical information. We further suggest that the regimes with stable islands can be modeled by the blocks with non-random elements (e.g. zero variance and non-zero mean). The chaotic regimes can be modeled by blocks with randomly distributed elements (e.g. same non-zero variance for all elements or only within a band). Based on nature of the dynamics, a chaotic block may further contain hierarchy of random and non-random sub-blocks; various diagonal blocks may also be correlated.
Eq.(1) can then be applied to model the Hamiltonian by choosing the matrix element variances appropriate to the block in which they appear.

Eq.(1) can not serve as a good model for the cases with correlated matrix elements. For example, particle-particle interactions in nuclei [7, 21, 22] and electron-electron interaction in disordered systems can lead to correlations among elements of the Hamiltonian [1, 51]: the correlation coefficients depend on various system parameters. In general, such cases can occur when the interaction (described by $H$) between any two basis states is influenced by the other states. In past, consideration of particle correlations in nuclei led to introduction of embedded ensembles [7, 21]; however no significant progress has been made so far in dealing analytically with these ensembles.

In general, an increase of constraints on the system-dynamics subjects higher moments of the matrix elements to certain specific conditions. This motivates us to consider an alternative ensemble, namely, maximum entropy ensemble with restricted higher moments. Within maximum entropy hypothesis, the probability density for such cases turns out to be non-Gaussian [51]: $\hat{\rho}(H) = C\rho(H)$ where

$$\rho(H) = \prod_{s=1}^{\beta} \prod_{r=1}^{n} \exp \left[ -\sum_{p(r)} b_{p(r)} \left( \prod_{r=p(r)}^{r} H_{i,j,p(r)} \right) \right]$$

with $C$ as a normalization constant. Here each $H_{jk}$ is expressed in terms of its $\beta$ components, ($\beta = 1$ for the real-symmetric matrices and $\beta = 2$ for the complex Hermitian case): $H_{jk} = \sum_{s=1}^{\beta} (i)^{s-1} H_{jk,s}$. Here symbol $p(r)$ refers to a combination of $r$ matrix elements chosen from a set of total $\hat{M} = N(N+1)/2$ of them; note the terms present in a given combination need not be all different. The $\prod_{r=p(r)}^{r}$ implies a product over $r$ terms present in the $p^\beta$ combination with coefficient $b_{p(r)}$ as a measure of their correlation: $(\prod_{r=p(r)}^{r} H_{i,j,p(r)}) \equiv \frac{\partial \log C}{\partial b_{p(r)}}$. The $\sum_{p(r)}$ is a sum over all possible combinations (total $(\hat{M})^\beta$) of $r$ elements chosen from a set of total $\hat{M} = N(N+1)/2$ of them.

The potential use of eq.(2) to disordered systems with e-e interaction is discussed in [51]. Here we briefly discuss few more examples. The systems with chiral symmetry can be modeled by Hermitian ensembles with block form matrices $H = \begin{pmatrix} 0 & W \\ W^+ & 0 \end{pmatrix}$, with $W$ as a matrix of size $N$. Here, due to $H_{k',l} = H_{l,k'}^*$ for $1 \leq k,l \leq N$ (with $k' \equiv k + N, l' \equiv l + N$), the correlations between these elements are subjected to condition:

$$\langle H_{k',l} H_{l,k'} \rangle = \langle |H_{k',l}|^2 \rangle = \langle |H_{l,k'}|^2 \rangle,$$

$$\langle H_{k,l} \rangle = \langle H_{k',l'} \rangle$$

(3)

However, due to $H_{k,l} = H_{k',l'} = 0$, all other matrix elements are uncorrelated. For a simple explanation, let us restrict to a case of real matrix $W$ with Gaussian distributed elements. The ensemble can then be represented by eq.(2) with $n = 2$ or equivalently by density,

$$\rho(H, a, b) = C \exp \left[ -\sum_{i \leq j, k \leq l} b_{i,j,k,l} H_{i,j} H_{k,l} - \sum_{k,l} a_{k,l} H_{k,l} \right],$$

(4)

with following conditions on $a$ and $b$:

$$b_{k',l,l,k'} = b_{k',l',k',l} = b_{l,k',l,k},$$

$$a_{k,l} = a_{k',l'} = 0,$$

$$a_{k',l} = a_{l,k'},$$

(5)

Note, $b$ parameters corresponding to other pairs of elements (both or at least one in diagonal blocks) diverge due to zero correlation between elements in such pairs. The cases with other types of correlated blocks can similarly be modeled by applying appropriate conditions on the $b$ parameters which correspond to combinations of matrix elements appearing in opposite blocks. For example, the ensemble $C$ describes the cases with particle-hole symmetry with a Hamiltonian $H = \begin{pmatrix} A & B \\ B^+ & -A^+ \end{pmatrix}$ (for details). Now the correlations between various elements must be subjected to conditions $H_{k',l} = H_{l,k'}^*, H_{k,l} = -H_{l',k'}$: this implies another set of non-zero correlations (besides those given by eq.(3)): $\langle H_{k,l} H_{l',k'} \rangle = -\langle |H_{k,l}|^2 \rangle = -\langle |H_{l',k'}|^2 \rangle$. For Gaussian distributed real matrices $A$ and $B$, the case can again
be modeled by eq.(4) however now $b$ parameters for other pairs (besides those given in eq.(5)) can also be finite and satisfy the equality: (i) $ob_{k,l,k',l'} = b_{k,l,k',l'} = b_{k',l',k',l'}$; (ii) $b_{k,l,k',l'} = a b_{k',l',k',l'}$ with $\alpha = -1$.

Recently it was shown [49, 51] that the distribution $\rho$ for both cases (eqs.(1,2)) appear as the non-equilibrium stages of a Brownian type diffusion process in the matrix-space, evolving with respect to a single parameter which is a function of the distribution parameters of the ensemble: ...

$$\frac{\partial \rho}{\partial Y} = L \rho$$  \hspace{1cm} (6)

with

$$L_{\pm} = \sum_{k,l,s} \frac{\partial}{\partial H_{kl,s}} \left[ \frac{g_{kl}}{2} \frac{\partial}{\partial H_{kl,s}} \pm \gamma H_{kl,s} \right]$$  \hspace{1cm} (7)

where $g_{kl} = 1 + \delta_{kl}$. The variable $Y$ is the parameter governing the evolution of matrix elements subjected to various system conditions. For case (1),

$$Y = -\frac{1}{2M\gamma} \ln \left[ \prod_{k \leq l} \prod_{s=1}^{\beta} |x_{kl,s}|^2 |b_{kl,s}|^2 \right] + C_0$$  \hspace{1cm} (8)

with $\prod$ implying a product over non-zero $b_{kl,s}$ and $x_{kl,s}$; $x_{kl,s} = 1 - (2 - \delta_{kl}) \gamma h_{kl,s}$, $C_0$ as a constant determined by the initial distribution, $M$ as the number of all non-zero parameters $x_{kl,s}$ and $b_{kl,s}$. The parameter $\gamma$ is arbitrary, giving the freedom to choose the end of the evolution; the $\lim h_{kl,s} \rightarrow \gamma, b_{kl,s} \rightarrow 0$ for all $k,l$ gives $Y \rightarrow \infty$ and the steady state (a Wigner-Dyson ensemble). The distribution parameters being indicators of the complexity of the system, $Y$ can be termed as the complexity parameter [49]. Some examples of the calculation of $Y$ from eq.(8) are discussed in [49] (for banded ensembles) and in [50] (for Anderson Hamiltonian). The $Y$ in case of a mixed system can similarly be calculated if one knows the details of the mixed dynamics.

In general, the form of parameter $Y$ for eq.(2) is quite complicated; its details can be found in [51]. However, for case (4), that is the Gaussian version of case (2), $Y$ can be given as

$$Y = \sum_{kl} \int d a_{kl} \ X + \sum_{ijkl} \int d b_{ijkl} \ X + \text{constant} \hspace{1cm} (9)$$

where summation is implied over the distribution parameters with finite values only, and, $X = [\sum_{kl} f_{kl} + \sum_{ijkl} f_{ijkl}]^{-1}$ with $f_{kl} = \gamma a_{kl} - 2[\sum_{mn} a_{mn} b_{klmn} + a_{mn} b_{klmn}], f_{ijkl} = [\gamma b_{ijkl} - 2 \sum_{mn} b_{ijkl} b_{klmn}]$. For further clarification we refer the reader to [51] where an example, namely, the modelling of lowest Landau level of a disordered quantum Hall system by eq.(4) and calculation of corresponding $Y$ is discussed.

It is easy to solve eq.(6) for arbitrary initial condition, say $H_0$ at $Y = Y_0$: $\rho(H, Y|H_0, Y_0) \propto \exp[-(\alpha/2)\text{Tr}(H - \eta H_0)^2]$ with $\alpha = \gamma (1 - \eta^2)^{-1}$ and $\eta = e^{-Y}$. The probability density of $H$ can now be extracted by integrating over an ensemble of initial conditions: $\rho(H, Y|Y_0) \propto \int \rho(H, Y|H_0, Y_0) \rho(H_0, Y_0) dH_0$. It is often useful to study the statistics of the perturbed Hamiltonian $H$ in the eigenfunction basis of unperturbed Hamiltonian $H_0$. Thus if the eigenfunctions of $H_0$ are chosen as the basis vectors $|k\rangle, |l\rangle$ etc, and, the initial distribution is given by $\rho(H_0) \propto \exp[-(1/2) \sum_{jj} H_{0,jj}^2]$, the eigenvalue equation $UH = \Lambda U$ can be used to transform $\rho(H)$ from matrix space to eigenvalue-eigenvector space $\{\lambda, U\}$:

$$\rho(H, Y) \propto \prod_{k,l < k'}^{N} |\lambda_k - \lambda_{l'}|^\beta \exp \left[ -\left( \gamma/2 \right) \sum_{j=1}^{N} \lambda_j^2 - \left( \gamma \mu/2 \right) \sum_{k < l} |\lambda_k - \lambda_l|^2 |U_{k,l}^2| \right]$$  \hspace{1cm} (10)

where $\mu = [e^{2Y}(Y - Y_0) - 1]^{-1}$.

As indicated by eqs.(6) and (10), the ensemble densities for various complex systems (i.e. different $h, b$ matrices) undergo a similar evolution as a function of $Y$. The $Y$-governed flow for the joint distribution of the desired eigenfunctions components and eigenvalues can be obtained, in principle, by integrating either eq.(6) or eq.(10) over all the undesired ones; however it is easier to integrate eq.(6). To explain the technique, we consider some of the important cases in this paper.
III. DIFFUSION EQUATION FOR EIGENFUNCTION COMPONENTS AND RELATED EIGENVALUES

The $k^{th}$ component $U_{kl}$ of an eigenstate $U_i$ is a measure of the contribution of $k^{th}$ basis state to the eigenstate. Experimental observations of complex systems indicate the level to level variations as well as sample to sample fluctuations of the contribution. As a result, the knowledge of average behavior of the components is not enough and one needs to study their distribution. In this section, we consider the joint probability distributions of a few relevant combinations of the components of the operator $H$. The basis chosen for the representation of the eigenfunctions is the one in which the matrix elements of $H$ have distribution (1) (or (2)). We use following notation in reference to various correlations: For a joint distribution $P_{rs}$, the subscripts $r$ and $s$ refer to the number of components of each eigenvector and the number of eigenvectors considered, respectively. For example, for a joint distribution of $n$ components of $m$ eigenvectors along with their eigenvalues, $r = n$ and $s = m$.

A. Joint Distribution of a Given Component of All Eigenfunctions and Eigenvalues

It is often relevant to know the influence of a particular basis state on the system dynamics at various energies and with varying complexity of the system. The information can be obtained by a knowledge of the distribution of the same component of various eigenfunctions and its $Y$ governed evolution. For example, let us calculate the joint distribution of a given component of all eigenvectors and the eigenvalues. Let $P_{1N}(Z,E,Y)$ be the probability, at a given $Y$, of finding the $j^{th}$ component $U_{j_n}$ of the eigenvectors $U_n$ of $H$ between $z_{jn}$ and $z_{jn} + dz_{jn}$ and the eigenvalues $\lambda_n$ between $e_n$ and $e_n + de_n$ for $n = 1 \rightarrow N$ (with $Z \equiv \{z_{jn}\}, E \equiv \{e_n\}$). It can be expressed as an average over entire ensemble $\rho$:

$$P_{1N}(Z,E,Y) = \int f_N(Z,E,U,\lambda) \rho(H,Y) \, dH$$  \hspace{1cm} (11)

with $f_s(Z,E,U,\lambda) = \prod_{n=1}^{N} \delta(z_{jn} - U_{jn}) \delta^{s-1}(\lambda_n - \lambda_n)$. The $Y$ dependent evolution equation for $P_{1N}$ can now be derived by connecting the parametric derivatives of $P_{1N}$ to its derivatives with respect to eigenvectors. The steps can briefly be described as follows: As $Y$-dependence of $P_{1N}$ comes only through $\rho$, one can write

$$\frac{\partial P_{1N}}{\partial Y} = \int dH f_N \rho L_+ + \frac{\gamma}{2} P_{1N}$$  \hspace{1cm} (12)

with $\gamma = \beta N(N+1)^2/2$. Eq.(12) is obtained, first, by differentiating eq.(11) with respect to $Y$, then using eq.(6), followed by partial integration. Due to $\delta$-function nature of $f_N$, its derivatives with respect to matrix elements can further be reduced to the derivatives with respect to $Z$ and $E$.

$$\frac{\partial f_N}{\partial H_{kl:s}} = -\sum_{n=1}^{N} \left[ \frac{\partial \lambda_n}{\partial H_{kl:s}} \frac{\partial f_N}{\partial e_n} + \frac{\partial U_{jn}}{\partial H_{kl:s}} \frac{\partial f_N}{\partial z_{jn}} + \frac{\partial \gamma}{\partial H_{kl:s}} \frac{\partial f_N}{\partial z_{jn}} \right]$$  \hspace{1cm} (13)

The 2nd derivative of $f_N$ can now be obtained from eq.(13) (see [49]). The substitution of eq.(13) in eq.(12) helps as the derivatives with respect to $z_{jn}$ and $e_n$ can be taken out of the integral. It can further be simplified by a knowledge of the effect of a small perturbation of $H$ on its eigenvalues and eigenvectors; the related results are given in Appendix B (see [49] for the details). Using the relations, eq.(12) can be rewritten as

$$\frac{\partial P}{\partial Y} = (L_Z + L_{Z^*}) P + L_E P$$  \hspace{1cm} (14)

where $P = C_1 P_{1N}, C_1 = e^{-\gamma Y}$, and
\[ L_Z = \frac{\beta^2}{4} \sum_{n,m=1,n \neq m}^N \frac{1}{(e_n - e_m)^2} \frac{\partial}{\partial z_{jm}} \left( \frac{\partial}{\partial z^{*}_{jm}} |z_{jm}|^2 - \frac{\partial}{\partial z_{jm}} z_{jm} z_{jm} + z_{jm} \right), \]

\[ L_E = \sum_n \frac{\partial}{\partial e_n} \left[ \gamma e_n + \sum_{m:m \neq n} \frac{\beta}{e_m - e_n} + \frac{\partial}{\partial e_n} \right]. \]  

where \( L_Z^* \) implies the complex conjugate of \( L_Z \); note \( L_Z = L_Z^* \) for \( \beta = 1 \) case. Eq.(14) describes the \( Y \)-governed diffusion of a given component of all eigenvectors and all eigenvalues. Its solution depends on the choice of initial condition \( H_0 \). In the diagonal representation of \( H_0 \) (taking \( \rho(H_0) \propto e^{-(1/2) \sum_j H_0 z_j^2} \), the solution can be given as

\[ P_{1N} \propto \prod_{k,l:k<l} |e_k - e_l|^\beta \exp \left[ -(1/2) \sum_{j=1}^N e_j^2 - (\mu/2) \sum_{m<n} |e_m - e_n|^2 |U_{jm}|^2 |U_{jm}|^2 \right] \]  

with \( \mu \) same as in eq.(10). (Note, the above result can directly be obtained from eq.(10) too).

### B. Joint Distribution of all Components of A given Eigenfunction and its Eigenvalue

The distribution of the components of a specific eigenstate contains information about various basis states contributing to the state which in turn determines its spread. Proceeding along the same lines as for \( P_{1N} \), the diffusion equation for the joint probability \( P_{N1} \) of the components \( U_{nk}, n = 1 \rightarrow N \), of an eigenvector \( U_k \) and the corresponding eigenvalue \( \lambda_k \) can also be obtained. The evolution of

\[ P_{N1}(Z_k, e_k, Y) = \int \tilde{f}_k \rho(H, Y) \, dH, \]  

with \( \tilde{f}_k = \delta(Z_k - U_k)\delta^{\beta-1}(Z_k^* - U_k^*)\delta(e_k - \lambda_k) \), can again be shown to be described by

\[ \frac{\partial P_{N1}}{\partial Y} = F_e + F_{e}^* + L_{e_k} P_{N1} \]  

where \( F_k = (\beta^2/4) \sum_{q=1}^2 L_{qk} \) with

\[ L_{1k} = \sum_{n=1}^N \frac{\partial}{\partial z_{nk}} \left[ z_{nk} Q^0_{mn,k} \right] \]

\[ L_{2k} = \sum_{n,m=1}^N \frac{\partial^2}{\partial z_{nk} \partial z_{mk}^*} Q^{12}_{mn,k}, \]

\[ L_{e_k} P_{N1} = \frac{\partial}{\partial e_k} \left[ \gamma e_k P + \frac{\partial P}{\partial e_k} + \beta Q^0_{mn,k} \right] \]  

and

\[ Q^r_{mn,k} = \sum_{j:j \neq k} \int \left( \frac{z_{nj} z_{mj}^*}{(e_k - e_j)} \right)^r P_{N2} \, d\tau_j. \]  

Here \( d\tau_j \equiv d\epsilon_j d^{\beta} Z_j \) with \( d^{\beta} Z \equiv dZ dZ^* \) and \( P_{N2} = P_{N2}(Z_k, Z_j, e_k, e_j) \) is the joint probability of all components of the two eigenvectors \( Z_j \equiv \{z_{nj}\} \) and \( Z_k \equiv \{z_{nk}\} \) \( (n = 1 \rightarrow N) \) alongwith their eigenvalues \( e_j \) and \( e_k \), respectively:

\[ P_{N2} = \int \tilde{f}_k \tilde{f}_j \rho(H, Y) \, dH. \]
The presence of eigenvalue-eigenfunction correlations in exponent of \( \rho(H) \) (e.g., eq.(10)) as well as the terms of type \((e_j - e_k)^{-2}\) in the denominator of eq.(20) makes it difficult to write \(F_k\) (in eq.(18)) as a function of \(P_{N1}(Z_k, e_k)\). To write eq.(18) in a closed form, it is necessary to approximate \(Q^{rs}_{mn;k}\) (Appendix A):

\[
Q^{rs}_{mn;k} \approx D^{-s}(N - 1)^{1-r} \chi^{s/2} [\delta_{mn} - z_{mk}^* z_{nk}]^r P_{N1}
\]

with \(D_k\) as the local mean level spacing at energy \(e_k\). Here \(\chi = 1\) for \(\mu < \zeta_k^d\), \(\chi \sim (\mu/\zeta_k^d)\) for \(\mu > \zeta_k^d\) with \(\zeta\) as the ensemble averaged localization length of the eigenfunction \(U_k\) and \(d\) as the system-dimension. The length \(\zeta\) enters in the formulation due to its relation with typical intensity of a wavefunction: \(|U_{nk}|^2 \approx |z_{nk}|^2 \sim \zeta_k^{-d}\).

The substitution of eq.(22) in eq.(19) helps to write \(F_k\) in terms of \(P_{N1}\), thus reducing the evolution equation (18) for \(P_{N1}\) in a closed form:

\[
F_k = \frac{\beta^2}{4D^2} \sum_{n=1}^{N} \partial_{z_{nk}} \left[ \sum_{m} \frac{\partial h_2}{\partial z_{mk}} + h_1 \right]
\]

with \(h_1 = (N - 1)\chi z_{nk} P_{N1}\), \(h_2 = \chi (\delta_{mn} - z_{nk}^* z_{mk}) P_{N1}\).

C. Joint Distribution of all components of \(q\) Eigenfunctions and their eigenvalues

For certain physical properties e.g susceptibility, a knowledge of the correlations among two (or more) eigenvectors at two different space points may be required. The fluctuations of such correlations can be determined by the joint probability density \(P_{Nq}\) of the components \(U_{nk}\) \((n = 1 \to N)\) of \(q\) eigenvectors \(U_k\) \((k = 1 \to q)\) where

\[
P_{Nq}(Z_1, Z_2, \ldots Z_q, Y) = \int \prod_{k=1}^{q} \tilde{f}_k \rho(H, Y) \, dH,
\]

Proceeding exactly as in previous two cases, the \(Y\)-governed diffusion of \(P_{Nq}\) can be shown to be described as

\[
\frac{\partial P_{Nq}}{\partial Y} = \sum_{k=1}^{q} \left[ \tilde{F}_k + \tilde{F}_k^* + L_{ec} P_{Nq} \right]
\]

where

\[
\tilde{F}_k = F_k + \frac{\beta^2}{4} \sum_{l=1}^{q} \sum_{m,n=1}^{N} \frac{\partial^2}{\partial z_{nk} \partial z_{ml}} \left[ \frac{z_{nk}^* z_{ml}}{(e_k - e_l)^2} \right] P_{Nq}.
\]

Note although \(F_k, L_{1k}, L_{2k}\) are still defined as in eqs.(18, 19) however the definition of \(Q\) is now slightly altered with \(P_{N(q+1)}\) replacing \(P_{N2}\) in eq.(20). Here \(P_{N(q+1)}\) is the joint probability density of \(q + 1\) eigenfunctions, namely, \(Z_1, Z_2, \ldots, Z_q\) along with \(Z_j\) (with \(j > q\)). Similar to previous case, the integral \(Q\) can again be approximated so as to express \(F_k\) in terms of \(P_{Nq}\): \(Q_{mn;k}^{rs} \approx D^{-s}(N - 1)^{1-r} \chi^{s/2} [\delta_{mn} - \sum_{l=1}^{q} z_{ml}^* z_{nl}]^r P_{Nq}\). Here again \(\chi = 1\) for \(\mu < \zeta^d_k\) and \(\chi \sim (\mu/\zeta^d_k)\) for \(\mu > \zeta^d_k\).

The above approximation for \(Q\) leaves the expression for \(F_k\) in the same form as in eq.(23) however now \(h_1 = \chi (N - 1) P_{Nq}\), and, \(h_2 = \chi (\delta_{mn} - \sum_{l=1}^{q} z_{ml}^* z_{nl}) P_{Nq}\). The substitution of \(F_k\) in \(F_k\) gives the latter as a function of \(P_{Nq}\) which in turn reduces eq.(26) in a closed form for \(P_{Nq}\). The equation can then be used, by integrating over undesired components, to obtain the distributions of various combinations of eigenfunction components.
IV. DIFFUSION EQUATION FOR FLUCTUATION MEASURES OF EIGENFUNCTIONS

The ensemble average of any measure of the eigenfunction correlations can be expressed in terms of $P$ ($P \propto P_{rq}$ for a correlation function of $r$ components of $q$ eigenstates). For example, the average of a measure, say $C$, describing the correlation among a set $X$ of eigenfunction components can be written as

$$
\langle C(X;Y) \rangle = \int_0^\infty C(X;Y)P(X;Y)\mathrm{d}X.
$$

(27)

where $\langle \rangle$ denotes an averaging over various realizations of the sample. However the strength of the reproducible fluctuations of the correlations in different realizations of same complex system is of the order of the averages. As a consequence, a knowledge of just the averages is not enough and it is necessary to know the distributions of correlations.

The $Y$-governed evolution of the distribution $P_C$ of a measure $C$ can be obtained by an integration of the undesired variables in eq.(14) (or eq.(18), eq.(25) as per requirement). As examples, we derive the evolution equations for few important measures in this section. The involved integrals are, however, quite tedious and analytical approximations seem necessary to reduce the equation in a closed form. As a check on our results, we study the $Y \to \infty$ limit of each measure. This limit corresponds to the flow of ensemble (1) (and ensemble (2)) to its steady state, that is, a Wigner-Dyson Ensemble. As a consequence, each measure is expected to evolve to its Wigner-Dyson limit as $Y \to \infty$. We verify our results numerically too; the details are given in section VI.

A. Distribution of Local Eigenfunction Intensity

The distribution function of the local eigenfunction intensity i.e. the eigenfunction intensity $u$ at a given basis state, say $n$ can be defined as $P_u(u,e) = \langle \sum_{k=1}^N \delta(u - N|z_{nk}|^2)\delta(e - e_k) \rangle$. The diffusion of $P_u$ as a function of $\Lambda$ can be obtained from either eq.(14) with $P \propto P_{1N}$ or eq.(18). For technical simplification, however, we choose the former and first study the evolution of the distribution $P_{11}(x,e)$ of an eigenfunction component $x = N^{1/2}z_{nk} = (u^{1/2})$ at an energy $e$, defined as

$$
P_{11}(x,x^*,e) = \langle \delta^2 \delta_x \rangle = \int \delta^2 \delta_x P_{1N}(Z,E,Y) \mathrm{d}E \mathrm{d}^2 Z
$$

(28)

where $\delta_x = \delta(x - \sqrt{N}z_{nk})\delta^{-1}(x^* - \sqrt{N}z_{nk}^*)$ and $\delta_x = \delta(e - e_k)$ and $\tau = \mathrm{d}E \mathrm{d}^2 Z$. The diffusion equation for $P_{11}(x,e)$ can be obtained by integrating eq.(14), with $P \propto P_{1N}$, over the variables $e_j$ and $z_{nj}$, $j = 1 \to N$,

$$
\frac{\partial P_{11}}{\partial Y} = \frac{\beta^2}{4} \left[ 2 \frac{\partial^2 G_1}{\partial x \partial x^*} + \frac{\partial(xG_0)}{\partial x} + \frac{\partial(x^*G_0)}{\partial x^*} \right] + L_r P_{11}
$$

(29)

where

$$
G_r(x,e) \equiv \sum_{j\neq k} \int \delta^2 \delta_x \frac{|z_{nj}|^{2r}}{(e_k - e_j)^2} P_{1N} \mathrm{d}\tau,
$$

(30)

with $r = 0,1$ and $\int \delta^2 \delta_x [L_E P_{1N}] \mathrm{d}E \mathrm{d}^2 Z = L_r P_{11}$.

Eq.(29) describes the sensitivity of the local intensity distribution to the energy scale $e$ as well as various system parameters. As discussed in appendix A (see eq.(A7)), $G_r$ can be approximated as

$$
G_r \approx \mu \chi_0(N - 1)^{1-r}(N - |x|^2)^r P_{11}(x)/D^2_k
$$

(31)

with $\chi_0 = \mu^{-1}$ for $\mu|x|^2 < 1$ and $\chi_0 \sim |x|^2$ for $\mu|x|^2 > 1$ where $\mu = [e^{2g(Y-Y_0)} - 1]^{-1}$ and $D_k$ as the local mean level spacing at energy $e_k$. A substitution of approximated $G_r$ in eq.(29) and an integration over $e$ gives the energy-averaged local intensity distribution $P_x(x) = \int P_{11}(x,e)\mathrm{d}e$:
\[
\frac{\partial P_x}{\partial \Lambda} = \frac{\beta^2}{4} \left[ 2 \frac{\partial^2 h_2(x) P_x}{\partial x \partial x^*} + \frac{\partial [h_1(x) P_x]}{\partial x} + \frac{\partial [h_1(x^*) P_x]}{\partial x^*} \right]
\]

with \( h_2(x) = \chi_0(N - x^2) \), \( h_1(x) = \chi_0(N - 1)x \). Here \( \Lambda = \mu \Lambda \) with \( \Lambda = (Y - Y_0)/D_\xi^2 \). Eq.(32) suggests that the evolution of \( P_x \) is governed by a rescaled parameter \( \Lambda \) instead of \( Y \).

For cases \( |x|^2 << N \) (thus \( \Lambda \approx \Lambda \)), the above equation can easily be solved: \( P_x(x, \Lambda|x_0) \propto e^{-\beta|x-x_0|^2/(1-\gamma^2)} \) with \( \gamma = e^{-\beta \Lambda N/2} \) and \( P_{x}(x_0) \) as the initial distribution. The steady state limit \( \frac{\partial P_x}{\partial \Lambda} \to 0 \) of eq.(32) occurs at \( \Lambda \to \infty \). The solution in this limit corresponds to Wigner-Dyson case i.e. \( P_x(x, \Lambda \to \infty) \propto e^{-\beta |x|^2/2} \) or, equivalently, Porter-Thomas distribution \( P_{\mu}(u, \Lambda \to \infty) \propto u^{(\beta-2)/2} e^{-\beta u/2} \) \( \tilde{[}8]\).

It is desirable to know the solution \( P_{\mu}(u) \) for finite, non-zero \( \Lambda \) and all ranges of \( u \). In the diagonal representation of \( H_{0} \), which corresponds to an initial distribution \( P_{\mu}(0) = N^{-1}[\delta(u-1) + (N-1)\delta(u)] \)\( \), eq.(32) gives following short range behavior of \( P_{\mu}: \)

\[
P_{\mu} = (\beta u/2)^{\beta/2-1} e^{-\beta u/2} \frac{1 + \kappa}{\Gamma(\beta/2)} \left[ \frac{1 + \kappa}{2} \left( (\beta + 2) - (\beta + 2)\sqrt{u} + \beta u/2 \right) + \ldots \right] \quad u \lesssim \kappa^{-1/2}
\]

\[
\approx (\beta u/2)^{\beta/2-1} \frac{1}{\Gamma(\beta/2)} \exp \left[ \left( \frac{\beta}{2} \right) \frac{u}{\kappa} \right] \quad \kappa^{-1/2} \lesssim u \lesssim \kappa^{-1}
\]

where \( \kappa = e^{-\beta \Lambda N} \) (note \( \kappa \approx \mu \) in large \( Y \)-limit and for \( \Omega_{\Lambda}^2 \sim (\Lambda N)^{-1} \)).

The tail behavior of a distribution has a significant influence on its moments and the related physical properties. The asymptotic analysis of eq.(32) shows \( P_{\mu}(u) \) to be a broad distribution:

\[
P_{\mu}(u) \simeq \exp \left[ -\alpha_{u0} u^{1/2} - \sum_{n=1}^{M} \alpha_{un} n^u \right] \quad u \gtrless \kappa^{-1}
\]

Here the coefficients are sensitive to system-specifics: \( \alpha_{u0} \approx 4N_{\beta}^{-1}(e^{\beta \Lambda N} - 1) \), \( \alpha_{u1} \approx -N/4, \alpha_{u2} \approx (N\beta/16)e^{\beta \Lambda N} \), \( \alpha_{un; n>2} \approx (1)^n (\nu_n \beta^2 N/4) e^{2\beta \Lambda N} \) with \( \nu_n \) decreasing as \( n \) increases. The decreasing coefficients alongwith alternate \( \pm \) signs lead to near-cancellation of higher order terms (with \( n > 2 \)) in the exponent. Consequently, the tail is dominated by a log-normal behavior for the systems with large, finite \( \Lambda \)-strengths and a weaker than exponential decay in \( \Lambda \to 0 \) limit.

Eq.(35) indicates the existence of a log-normal asymptotic tail for the local eigenfunction intensity of any complex system with finite, non-zero \( \Lambda \). A log-normal behavior of \( P(u) \) suggests a power-law behavior of its moments: \( \langle u^n \rangle \propto N^{-d_q} \) \( \tilde{[}17]\). Here \( d_q \) is an effective dimension which can be different from a spatial dimension \( d \). The form of \( P_{\mu}(u) \) at finite \( \Lambda \) is therefore fixed by a spectrum of scaling exponents (as the moments can be used to recreate the distribution); the situation is termed as multifractal scaling. Further, as shown later, a log-normal tail of \( P(u) \) results in the similar behavior of the distributions of other related correlations and physical properties. Such a behavior has already been indicated for the physical properties e.g. conductance, density of states, local density of states and relaxation time etc. of disordered systems \( \tilde{[}3]\).

The significance of above \( P(u) \)-formulation is that here system dependence (other than size) enters only through one parameter, namely, \( \Lambda \). This being valid for any complex system, modeled by eq.(1) (and eq.(2)), is thus applicable for disordered systems too. It is therefore relevant to compare our result with those obtained for disordered systems using other techniques (using renormalization group theory approach for dimension \( d = 2 + \epsilon, \epsilon < 1 \) \( \tilde{[}42]\), and, by using Berezinski and Abrikosov-Ryzkhin techniques for strictly \( d = 1 \) cases \( \tilde{[}8, 11, 57]\); the techniques predict a \( e^{-\alpha u^{1/2}} \) tail for \( d = 1 \), a log-normal tail for \( d = 2 \) and a log-cube tail for \( d = 3 \) case. However our technique predicts a log-polynomial behavior however dominated by log-normal term for all dimensions.

B. Inverse Participation Ratio (IPR)

The \( q^{th} \) order inverse participation ratio \( I_q \) of an eigenvector, say \( U_k \), is defined as \( I_q(k) = \sum_{j=1}^{N} |U_{jk}|^{2q} \). The physical meaning of \( I_q \) can be illustrated by two limiting cases: (i) an eigenvector with identical components
$U_{jk} = N^{-1/2}$ corresponds to $I_q(k) = N^{1-q}$, and, (ii) an eigenfunction with only one non-zero component (say $n^1k$) which gives $U_{jk} = \delta_{nk}$ and $I_q(k) = 1$. The case (i) corresponds to completely ergodic eigenfunctions covering randomly but uniformly the whole sample of volume $V$. The case (ii) corresponds to a wavefunction localized in the neighborhood of a single basis state. Thus $I_q$, in general, is related to reciprocal of the number of components significantly different from zero and contains information about spread of a wavefunction in the basis space. For example, for a d-dimensional exponentially localized state, $I_2 \sim (a/\zeta)^d$, where $a$ and $\zeta$ are the lattice constant and localization length, respectively. Consequently, the typical value of $I_2$ is a frequently used characteristic of the eigenfunction localization [3]: $I_2^{typ} = \exp(lnI_2) \approx N^{-D_2}$ with $D_2$ a system dependent scaling exponent (also known as correlation dimension).

The ensemble average of $I_q$ is related to $q$-th moment of the distribution $P_q(x)$: $\langle I_q \rangle = N^{1-q} \int_0^\infty |x|^q P_q(x) dx = \int \delta_{nk} P_{nk} dI_q$. The average inverse participation ratios can therefore provide information about the scaling exponents. As a consequence, it is useful to know the effect of changing system parameters on $\langle I_q \rangle$. Due to $P(u)$ decay for the ranges $\mu u \gtrsim 1$, major contribution to $\langle I_q \rangle$ comes from the region $\mu u \leq 1$. From eq.(32), it can be shown that

$$\frac{\partial \langle I_q \rangle}{\partial \Lambda} \approx q \alpha \langle I_q - 1 \rangle - qt \langle I_q \rangle. \quad (36)$$

where $\alpha = 2q + \beta - 2$, $t = N \beta + 2q - 2$. Eq.(36) depends on two parameters, namely $\Lambda$ and $t$ which results in a different power law behavior for each $\langle I_q \rangle$,

$$\langle I_q(\Lambda) \rangle = e^{-\alpha t \Lambda} \left[ \langle I_q(0) \rangle + \alpha \int_0^\Lambda \langle I_q - 1(r) \rangle e^{\alpha t r} dr \right]. \quad (37)$$

For $\Lambda \rightarrow \infty$, eq.(37) gives a correct steady state limit, namely, Wigner-Dyson behavior: $\langle I_q \rangle \rightarrow \frac{2}{q} \langle I_q - 1 \rangle$ or $\langle I_q \rangle = \frac{(2q)!}{2q^q} N^{1-q}$ for $\beta = 1$ and $\langle I_q \rangle = q! N^{1-q}$ for $\beta = 2$. For finite nonzero $\Lambda$, $\langle I_q \rangle$ can be determined if $\langle I_q - 1(\Lambda) \rangle$ as well as some past information about the system (to choose it as an initial state which will give $\langle I_q(0) \rangle$ is known. For example, for the systems where completely localized wavefunction dynamics is a valid physical possibility (e.g. disordered systems, mixed systems etc.), it can be chosen as the initial state which corresponds to $\langle I_q(0) \rangle = 1$; this gives $\langle I_q(\Lambda) \rangle = 1$, $\langle I_q(\Lambda) \rangle \approx e^{-q^2 \Lambda}$ for $q < N$.

In general, the IPR fluctuations reflect the level to level variations of the spatial structure of eigenfunctions. In a complex system e.g. nano-system, however, the sample to sample fluctuations of the eigenfunctions also manifest themselves through IPR fluctuations which makes a knowledge of the $I_q$-distribution over whole ensemble of samples relevant. The distribution $P_{nk}$ of $I_q$ of an eigenfunction, say $Z_k$, with the components $\{z_{nk}\}_{k=1}^{N}$ is related to $P \propto P_{N1}$: $P_{nk}(I_q) = \int \delta_{nk} P_{N1}(Z_k, \textbf{e}_k, \textbf{Y}) d\textbf{e}_k d\textbf{Y}$ with $\delta_{nk} = \delta(I_q - \sum_k |z_{nk}|^2)$ and the volume element $d\textbf{e}_k$ same as in eq.(20). The $Y$ governed evolution of $P_{nk}$ can therefore be obtained from Eq.(18) for $P \propto P_{N1}$:

$$\frac{\partial P_{nk}}{\partial Y} = \frac{\beta^2}{4} (X_1 + X_2) + X_3 \quad (38)$$

where $X_3 = \int \delta_{nk} |L_E P_{N1}| d\tau_k = 0$ and

$$X_1 = \int \delta_{nk} \left[ L_{1k} + L_{1k}^* \right] d\textbf{e}_k d\beta Z_k \quad (39)$$

$$X_2 = 2 \int \delta_{nk} L_{2k} d\tau_k \quad (40)$$

$$X_3 = \frac{8q^2}{\beta^2} \int \delta_{nk} F_2 d\tau_k - \frac{4q(2q + \beta - 2)}{\beta^2} \int \delta_{nk} F_3 d\tau_k \quad (42)$$
with \( F_1 = \sum_{n,k} Q^{02}_{nn;k} + Q^{02*}_{mn;k} \), \( F_2 = \sum_{m,n} |z_{mk}|^{2(q-1)}|z_{nk}|^{2(q-1)}z_{mk}^* z_{nk} Q^{12}_{mn;k} \) and \( F_3 = \sum_n |z_{nk}|^{2(q-1)}Q^{12}_{nn;k} \) where \( L_{1k}, L_{2k} \) and \( Q_{mn;k} \) are given by eqs.(19, 20). Using the approximate form (22) for \( Q^{12}_{mn;k} \), \( F_1 \)'s can further be reduced:

\[
F_1 \approx q \chi (N - 1)/D^2
\]

\[
F_2 \approx \frac{\chi}{D^2} \left[ \sum_n |z_{nk}|^{2(2q-1)} - \left( \sum_n |z_{nk}|^{2q} \right)^2 \right] P_{N1}
\]

\[
F_3 \approx \frac{\chi}{D^2} \sum_n [ |z_{nk}|^{2q-2} - |z_{nk}|^{2q} ] P_{N1}.
\]

where \( \chi = 1 \) for \( \mu < \zeta_k^d \) and \( \chi = \mu/\zeta_k^d \) for \( \mu > \zeta_k^d \).

In general, the fluctuations of different moments (or measures) of the eigenfunction intensity need not be mutually independent. We can therefore define the joint distribution of two measures, say, \( h_1(z), h_2(z) \):

\[
P_{h_1, h_2}(h_1, h_2) = \int \delta[h_1 - h_1(z)] \delta[h_2 - h_2(z)] P_{N1} d\alpha d\beta Z_k
\]

The above definition along with the equality \( \sum_n |z_{nk}|^{2(q-1)} = \sum_{m,n} |z_{mk}|^{2q}|z_{nk}|^{2(q-1)} - \sum_{m,n; m \neq n} |z_{mk}|^{2q}|z_{nk}|^{2(q-1)} \) gives:

\[
\int \delta I_q \left[ \sum_n |z_{nk}|^{2(2q-1)} \right] P_{N1} d\alpha d\beta Z_k \approx \int I_q I_{q-1} P_{I_{q-1}, I_{q}}(I_q, I_{q}) dI_{q-1} - \int W P_{I_q, W}(I_q, W) dW
\]

\[
\approx I_{q-1}^{typ} I_q P_{I_q} - W_{typ}^{typ} P_{I_q}
\]

where \( W_q \) is a measure of the correlation between the intensities localized at two different basis states: \( W_q = \sum_{m,n; m \neq n} |z_{mk}|^{2q}|z_{nk}|^{2(q-1)} \). The 2\( n \)th equality in eq.(45) is obtained from first by replacing \( I_{q-1} \) and \( W \) by their typical values; the superscript \( typ \) over a variable \( R \) indicates its typical value: \( R^{typ} = \exp(\ln R) \). Using eqs.(44,45), the terms \( X_1 \) and \( X_2 \) can be rewritten as the functions of \( I_q \) and \( P_{I_q} \) which in turn leads to:

\[
\frac{\partial P_{I_q}}{\partial \Lambda ip} \approx 2q \frac{\partial^2}{\partial I_q^2} [ I_{q-1}^{typ} I_q - W_q^{typ} - I_q^2 ] P_{I_q} - \frac{\partial}{\partial I_q} [ \alpha I_{q-1}^{typ} - t I_q ] P_{I_q}
\]

with \( \alpha, t \) same as in eq.(36) and

\[
\Lambda ip = q \chi \Lambda.
\]

Note, the above equation alongwith the definition \( \langle I_q \rangle = \int I_q P_{I_q} dI_q \) again leads to eq.(36).

The behavior of \( P_{I_q} \) in different \( I_q \) regimes can now be probed by analyzing eq.(46), using completely localized eigenstates as the initial state. The behavior varies from an exponential decay for small-\( I_q \) regime to log-power law decay for asymptotic tail regime of \( I_q \):

\[
P_{I_q} \approx \exp \left[ -\alpha_0 I_q \right] \quad I_q \lesssim e^{-I_q^{typ}}
\]

\[
\approx I_q^{-1} \exp \left[ -\sum_{n=1}^{M} \alpha_{in} \ln^n (I_q/I_q^{typ}) \right] \quad I_q \gtrsim e^{-I_q^{typ}}
\]

with \( e \approx 2.72, \alpha_0 \approx t(1 - e^{-tA^{typ}})^{-1}/2q \) and \( \alpha_{in} \sim (1)^n 2^2 I^{typ}_{q-1}/(2q I^{typ}_q)^n \) (valid for \( q < N \) and \( M \) as a large integer. Note the alternate \( \pm \) signs of terms with increasing powers lead to convergence of the series in the exponent. However the tail is dominated by increasingly higher powers of the logarithmic term as \( I_q \) increases above its typical value. For example, for \( e^{-I_q^{typ}} < I_q \lesssim I_q^{typ}, n = 1 \) dominates the exponent and \( P_{I_q} \) behaves as a power-law. Similarly, the tail shows a log-normal decay for regime \( I_q^{typ} < I_q \lesssim e^{-I_q^{typ}} \).
Eq. (46) depends on more than one parameter, namely, $\Lambda_{ip}$ as well as size-dependent parameters (appearing through $t$). This suggests an absence of single parameter scaling in IPR distributions. However, as suggested by eqs. (48, 49), it seems possible to define a single parameter locally (that is, different single parameters governing different IPR regimes). Further note that the asymptotic behavior of $P_{w}$ is sensitive to $\Lambda$-strength and is therefore system-specific. This result also agrees with the NLSM-result obtained for disordered systems [17].

C. Pair-Function $w(r,r')$

The measure contains important information about the spatial correlations between components of an eigenfunction $Z_{j}$ at two different basis points of the sample and at an energy $e$: $w(n,m) = \langle \delta_{|n|,|m|} | z_{|n|} z_{|m|} \rangle^{2}$ (equivalently $w(r,r') = \langle \delta_{r-r'} | z_{r} z_{r'} \rangle^{2}$ in a continuous basis e.g. coordinate space $r$). In the localized phase, the asymptotic behavior of $\ln(w(r,r'))$ at $|r-r'| \rightarrow \infty$ determines the rate of exponential decay of the eigenfunction amplitude. It is also useful for many physical applications e.g. in determination of the form factor of resonance scattering in the complex nuclei or the resonance conductance of the quantum dot with point contacts in the coulomb blockade regime [15].

The distribution $P_{w,e} = (\sum_{j} \delta(w - |z_{np}|^{2}))$ of the correlation between $n^{th}$ and $m^{th}$ components of an eigenfunction, at an energy $e$, is related to $P_{N1}$: $P_{w}(w,e) = \sum_{j} \int \delta(\delta_{e,j}) P_{N1}(Z_{j},e,j,Y) \, d\tau_{j}$ with $\delta_{w} \equiv \delta(w - |z_{np}|^{2})$ and $d\tau_{j}$ same as in eq.(20). Consequently, its rate of change with respect to $Y$ can be determined by eq.(18),

$$\frac{\partial P_{w}}{\partial Y} = N \Lambda_{w} + \frac{\beta^{2}}{4} (A_{1} + A_{2})$$

where

$$A_{1} = \frac{4}{\beta} \sum_{j} \frac{\partial}{\partial w} \int \delta_{w} \delta_{e,j} \left[ Q_{0}^{2n,m,j} + Q_{0}^{2m,n,j} \right] \, d\tau_{j}$$

$$A_{2} = \frac{8}{\beta^{2}} \sum_{j} \frac{\partial^{2}}{\partial w^{2}} \int \delta_{w} \delta_{e,j} \left[ F_{1} + F_{2} \right] \, d\tau_{j}$$

with $F_{1} = |z_{np}|^{2} Q_{0}^{2n,m,j} + |z_{np}|^{2} Q_{0}^{2m,n,j}$ and $F_{2} = z_{np} z_{nm} Q_{0}^{2n,m,j} z_{np} z_{nm} Q_{0}^{2m,n,j}$. Eq.(50) is derived by first using eq.(18), followed by repeated partial integration. Note $\int \delta_{w} \delta_{e,j} \left[ L_{\parallel} P_{N1} \right] \, d\tau_{j} = N \Lambda_{w}$. Within approximation (20) for $Q_{s}, A_{1}, A_{2}$ can further be simplified which on substitution in eq.(50) give the diffusion of $P_{w}$ in a closed form:

$$\frac{\partial P_{w}}{\partial \Lambda_{w}} = \frac{\beta^{2}}{2 \partial w^{2}} \left[ w(\Omega_{1} - 4w) \right] P_{w} - \frac{\partial}{\partial w} \left[ \Omega_{2} - bw \right] + N \Lambda_{w} P_{w}$$

where $\Omega_{1} = |z_{np}|^{2} \Omega_{2}^{2} + |z_{np}|^{2} \Omega_{2}^{2} |z_{np}|^{2} \delta_{nm} = 2(1 + \delta_{nm})u_{t}^{w}$, with $u_{t}^{w} = \exp[|\ln w|]$ as the typical local intensity of the $j^{th}$ eigenfunction, $\Omega_{2} = (\beta/2) |z_{np}|^{2} + (4/\beta) |z_{np}|^{2} \delta_{nm} = (\beta + 2\delta_{nm})u_{t}^{w}$, $\Lambda_{w} = 2\chi \Lambda$ and $b = \beta N + 2$.

The last term on the right of eq.(53) can be removed by an integration over energy $e$, leaving us with an evolution equation for energy-averaged $P_{w}$. Note the energy averaging of eq.(53) for case $n = m$ corresponds to eq.(46) for $P(\Omega_{2})$ (as $\sum_{n} P(w(n,n)) = \Omega_{2}$).

Exploiting the simplicity of the form of energy-averaged eq.(53) to eq.(46), the behavior of $P_{w}(w)$ in different regimes can again be given by eqs.(48,49) after following replacements (everywhere in the equations): $I_{q} \rightarrow w, \alpha_{\infty} \rightarrow \alpha_{wn}$ where $\alpha_{wn} \sim -1^{\alpha_{wn}} \Omega_{2}^{2-n} \Omega_{2}^{2-n} w^{-(n-2)}/n$. Thus $P_{w}(w)$ decays exponentially for small $w$ ranges ($w \lesssim w^{R}_{w}$): $P_{w} \approx \exp[-\alpha_{wn} w]$. It shows a power-law behavior for regimes $w^{R}_{w} < w < w^{L}_{w}$, a log-normal decay for regimes $w^{L}_{w} < w < w^{R}_{w}$. Such a behavior was predicted by non-linear sigma model studies of quasi 1-d disordered wires too [2].

Eq.(53) can be used to study the behavior of various moments of the distribution of pair-correlation. For example, for average behavior of $w$, that is, $\langle w \rangle = \int w P_{w}(w; \Lambda) \, dw$, eq.(53) gives its $\Lambda$ evolution. The evolution equation turns out to be of the same form as eq.(36) (with $q = 2$) with following replacements: $I_{q} \rightarrow w$, $\alpha \rightarrow \Omega_{2}$ and $2\Lambda \rightarrow \Lambda_{w}$ (note $I_{q-1} = 1$, $t \rightarrow b$ for $q = 2$). It can be solved to show that $\langle w(\Lambda_{w}) \rangle = e^{-k_{w}} \left( \langle w(0) \rangle + (\Omega_{2}/b)(e^{k_{w}} - 1) \right.$}. A
choice of \( \langle w(0) \rangle = \delta_{ml} \) (corresponding to localized regime) gives \( \langle w(\Lambda_w) \rangle = \delta_{ml} e^{-b\Lambda_w} + (\Omega_2/b)(1-e^{-b\Lambda_w} ) \approx (1-\kappa)/N \) which is analogous to the result obtained for disordered systems (by NLSM techniques); see [3].

\[ \text{D. Correlation Between Eigenfunctions At Two Different Energies} \]

Critical point studies of many systems indicate the presence of multifractal structures among eigenfunctions. The multifractality suggests that the wavefunction is effectively located in a vanishingly small fraction of the system volume. However such extremely sparse wavefunctions can exhibit strong correlations if they belong to neighboring energy levels; the correlations therefore preserve the level-repulsion despite the sparsity of the wavefunction. Thus, for a complete analysis of level-statistics and associated physical properties, knowledge of correlations among eigenfunction components is very important. The correlations are also used in the analysis of many other physical properties e.g. for the measurement of the linear response of the system, or, to determine the fluctuations of matrix elements of some operator in a given basis. This information is useful in studies of the effect of a particular interaction on the statistical properties of the system e.g. effect of electron-electron interaction on a single particle disordered system.

The correlations between components of two eigenfunctions at different energies can be described as \( \sigma(n, m, e_k, e_l) = |z_{nk}z_{ml}| \) (equivalently, in a continuous basis: \( \sigma(r, r', e, e') = |\psi_e(r)\psi_{e'}(r')|^2 \)). The distribution \( P_\sigma \) of the correlation \( \sigma = |z_{nk}z_{ml}|^2 \) between \( n \)th and \( m \)th components of the eigenfunctions \( Z_k \) and \( Z_l \), respectively, is related to \( P_{N2} \): \( P_\sigma(\sigma, e, \omega) = \sum_{k,l} \delta_\sigma \delta_{e,k} \delta_{e+l} * P_{N2}(Z_k, Z_l, e_k, e_l, Y) \left( \prod_{j=1}^{2} \delta_{r_j} \right) \) with \( \delta_\sigma = \delta(\sigma - |z_{nk}z_{ml}|^2) \), \( dr_j \) same as in eq.(20), and, \( \omega = \epsilon_k - \epsilon_l \) as the energy difference between two states. Using eq.(25) and proceeding as in the case of \( P'_\sigma(\omega) \), the diffusion of \( P'_\sigma(\sigma) \) can be shown to be described by the equation

\[
\frac{\partial P_\sigma}{\partial \Lambda_\sigma} = \left[ \frac{\partial^2}{\partial \sigma^2} \left[ \sigma(\tilde{\Omega}_1 - \sigma) - \frac{\partial}{\partial \sigma} \left( \tilde{\Omega}_2 - b\sigma \right) \right] + L_\epsilon + L_{e+\omega} \right] P_\sigma
\]

where \( \Lambda_\sigma = 2\chi \Lambda, \tilde{\Omega}_1 = (|z_{sl}|^2 \text{typ} + |z_{sr}|^2 \text{typ} + 2|z_{sr}|^2 \text{typ} \delta_{ld} \delta_{rs}, \tilde{\Omega}_2 = [\beta(|z_{sr}|^2 \text{typ} + |z_{sl}|^2 \text{typ}) + 4|z_{sr}|^2 \text{typ} \delta_{kl} \delta_{rs}]/2 \) and \( b = \beta N + 2 \). Note eq.(53) is a special case of the above equation (as \( P(w) \equiv P(\sigma) \) for \( k = l \)).

The dependence of average behavior of \( \langle \sigma \rangle \) now may be determined by multiplying eq.(54) by \( \sigma \) and then integrating over \( \sigma; \) the equation again turn out to be same as the \( q = 2 \) case of eq.(36) after following replacements: \( \langle I_q \rangle \rightarrow \langle \sigma \rangle \), \( \alpha \rightarrow \tilde{\Omega}_2 \) and \( 2\alpha \rightarrow \Lambda_\sigma \). Solving the so-obtained evolution equation gives \( \langle \sigma(\Lambda_\sigma) \rangle = e^{-b\Lambda_\sigma} \left( \langle \sigma(0) \rangle + (\tilde{\Omega}_2/b)(e^{b\Lambda_\sigma} - 1) \right) \).

The choice of a localized initial state (e.g. an insulator at \( \Lambda_\sigma = 0 \)) corresponds to \( \langle \sigma(0) \rangle = 0 \) which gives \( \langle \sigma(\Lambda) \rangle \approx d \right) N (1-e^{-b\Lambda}) \).

In this paper, we have considered only two point correlations. The other correlations e.g. \( \langle z_{pq}z_{rl} z_{sk}z_{dl} \rangle \) related to linear response of the system or, higher order ones e.g. \( \langle |z_{pq}|^4 |z_{sk}|^4 \rangle \) related to IPR fluctuations, can similarly be determined using eq.(25).

\[ \text{E. Local Density of States } \rho(e, j) \]

The local density of states or the spectral function, defined as \( \rho(e, j) = \sum_n |U_{jn}|^2 \delta(e-e_n) \), is an important measure of localization. This is because it counts the eigenstates \( U_n \) having appreciable overlap with (or equivalently, located close to) the site \( j \). Note this is distinct from the global density of states \( \rho(e) \) which counts all the eigenstates at the energy \( e \) irrespective of their location in space. The measure \( \rho(e, j) \) is of special interest as it gives information about the decay of a specific unperturbed state into other states due to interaction. The width of the LDOS defines the effective "life-time" of the unperturbed basis state. Its distribution is an experimentally accessible quantity related to the position and form of NMR line [3].
The probability density $P_{\rho}(\rho)$ of $\rho(e,j)$ is related to $P_{1N}$: $P_{\rho}(\rho) = \int \delta_{\rho} P_{1N}(E, Z, Y) \, dE \, d^3Z$ where $\delta_{\rho} = \delta(\rho - \sum_n |z_{jn}|^2 \delta(e - e_n))$ (note here $Z \equiv \{z_{jn}\}_{n=1...N}$). The diffusion of $P_{\rho}$ due to changing system parameters can again be studied with the help of eq.(14) for $P_{1N}$.

$$\frac{\partial P_{\rho}}{\partial Y} = L EP_{\rho} + B + B^*$$

with $B = B^* = \int \delta_{\rho} L_z P_{1N} \, d^3Z \, dE = \frac{\partial^2 F_{1N}}{\partial \rho^2} + \frac{\partial F_{1N}}{\partial \rho}$. Here the $2^{nd}$ form of $B$ is obtained from the $1^{st}$ by a substitution of three terms of $L_z$ (eq.(15)) in the integral, and, a subsequent partial integration which gives

$$F_1 = \int \delta_{\rho} \left[ \sum_{m,n,m\neq n} \frac{|z_{jm}|^2 |z_{jm}|^2}{(e_n - e_m)^2} \delta(e - e_n) (1 - \delta(e - e_m)) \right] P_{1N} \, d^3Z \, dE$$

$$F_2 = \int \delta_{\rho} \left[ \sum_{m,n,m\neq n} \frac{|z_{jn}|^2 - |z_{jm}|^2}{(e_n - e_m)^2} \delta(e - e_n) \right] P_{1N} \, d^3Z \, dE$$

As in the case of the integrals $Q$ and $G$ (see Appendix A), the dominant contribution to the integrals $F_1$ and $F_2$ comes from the regions where the exponent term in $P_{1N}$, that is $f = (\mu/2) \sum_{m<n} |e_n - e_m|^2 |U_{jm}|^2 |U_{jm}|^2 < 1$. This can occur under two conditions:

(1) $\mu(\sigma) < 1$: Here $\sigma = |U_{jm}|^2 |U_{jm}|^2$ describes the correlation between two different eigenfunction components in the same basis state. Under the condition, a neighborhood of the order of mean level spacing can contribute to the integral over e-variables i.e. $|e_n - e_m|^2 \sim D^2$.

(2) $\mu(\sigma) \geq 1$: In this case, $f \gg 1$ only for those regions where $|e_n - e_m|^2 \sim D^2/(\mu(\sigma))$. Note however in both cases, almost entire eigenfunction space can contribute to the integral.

Thus $F_1$ and $F_2$ can be approximated as $F_1 \approx \chi D^{-2} \rho(1 - \rho) P_{\rho}$ and $F_2 \approx N\chi D^{-2} \rho - \langle \rho \rangle P_{\rho}$ where $\chi = 1$ if $\mu(\sigma) \leq 1$ and $\chi \sim \mu(\sigma)$ for $\mu(\sigma) \geq 1$. The approximate forms of $F_1$ and $F_2$ can now be used to rewrite $B$ as a function of $P_{\rho}$ which on substitution in eq.(55) gives the diffusion equation for $P_{\rho}$:

$$\frac{\partial P_{\rho}}{\partial \Lambda_{\rho}} = L_{\epsilon} P_{\rho} + \frac{\partial^2}{\partial \rho^2} \rho (1 - \rho) P_{\rho} + \frac{\beta N}{2} \frac{\partial}{\partial \rho} \rho - \langle \rho \rangle P_{\rho}$$

where $\Lambda_{\rho} = \chi \Lambda$. Note that the above equation is analogous in form as the eqs.(46, 53, 54) of $P_{t_\epsilon}, P_{t_\omega}$ and $P_{t_\sigma}$, respectively. This similarity is reflected in both short as well as long-range behavior of $P_{\rho}$: (i) $\approx \exp[-\alpha_1(\rho)]$ for $\rho \lesssim \rho^{up}$, (ii) $\approx \rho^{-1} \exp[ - \alpha_1 \rho \ln(\rho) ]$ for $\rho^{-1} \gtrsim \rho \gtrsim \rho^{up}$, and, (iii) $\approx \rho^{-1} \exp[ - \alpha_3 \ln^2(\rho) ]$ for $\rho \gtrsim \kappa^{-1}$ where $\alpha_1 \gtrsim b(1 - e^{-\lambda_1 c})^{-1}/4$ and $\alpha_{up} \sim (1)^{n+1}(2^{n} \rho^{up} - n)$.

V. THE PARAMETER $\Lambda$

The set of Eqs.(14,18,25) provides a common mathematical formulation for the eigenfunction-statistics of various complex systems modeled by eqs.(1,2): here the information about the system enters only through $Y$. As shown explicitly in [49], the same $Y$ also enters in the common mathematical formulation of the eigenvalue-statistics of ensemble (1) and (2) [51]: this is implied by eqs.(14,18,25) too. However, as discussed in [49, 51], the evolution of the $n^{th}$ order eigenvalue correlations ($n > 1$) as a function of $Y$, is abrupt in large $N$-limit; a smooth crossover can only be seen in terms of a rescaled parameter $\Lambda_{\epsilon}$ where

$$\Lambda_{\epsilon}(e, Y) = \Lambda = \frac{Y - Y_0}{D_{\epsilon}^2}$$

with $D_{\epsilon}(e, Y) = D(\zeta/L)^d$ as the local mean level spacing, $D(e, Y)$ as the mean level spacing of the full spectrum and $\zeta$ as the correlation/localization length for a $d$-dimensional system of length $L$ ($N = L^d$), at an energy $e$ and parameter.
Y (with Y₀ as its initial value). Thus Λₓ, for various systems e.g disordered systems, mixed systems, systems with chiral or particle-hole symmetry etc., can be calculated by a prior knowledge of system parameters; (e.g. see [50] for the calculation for Anderson and Brownian ensembles). As Λₓ increases from zero to infinity, the level-statistics changes from its initial state (with \( Y = Y₀ \)) to that of Wigner-Dyson ensemble. For example, the initial state corresponds to the critical limit of disordered systems or integrable limit of mixed systems; both limits show Poisson level-statistics \([3, 9, 42, 52, 53]\). A variation of system parameters changes Λ from zero, causing diffusion of levels towards Wigner-Dyson steady state. According to Λ formulation, the level-statistics, for the system parameters resulting in finite Λ, is then an intermediate point of Poisson \( \rightarrow \) Wigner-Dyson transition. The prediction is in agreement with previous works on the two systems \([2, 20, 50, 52, 53]\); (note Wigner-Dyson statistics corresponds to metal and chaotic limits of disordered and mixed systems, respectively \([24, 42]\)).

For later reference, it is worth reviewing the role of Λₓ, that is, Λ in locating the critical point of level statistics. As both \( |Y − Y₀| \) as well as the local mean level density are functions of \( N \), the latter can affect Λ significantly. As a consequence, the size \( N \) plays a crucial role in determining the level statistics in the critical regime. For finite systems, the eigenvalue statistics smoothly approaches one of the two end points, namely, \( Λ → 0 \) or \( Λ → ∞ \), with increasing system size. The variation of Λ in infinite systems, however, may lead to an abrupt transition of the statistics, with its critical point given by the condition \( Λ = \text{size independent} \) (see ref. \([12]\) for the definition of a critical distribution).

The finite, non-zero Λ strength, say \( Λ_{\text{critical}} \), at the critical point, results in an eigenvalue-statistics different from the two end points. Note, however, that the existence of a critical point or its absence depends on the relative size-dependencies of \( |Y − Y₀| \) and the local mean level spacing. If the size-dependence of \( D^2 \) remains different from that of \( |Y − Y₀| \) under all complexity conditions, Λ will never achieve a finite non-zero value in infinite size limit. As a consequence, such a system will not show a critical behavior of eigenvalue-statistics. For example, as discussed in \([50]\) for a \( d \)-dimensional Anderson Hamiltonian of linear size \( L \), Λ turns out to be size-independent only for \( d > 2 \). The Λ-formulation, therefore, indicates the lack of metal-insulator transition for dimensions \( d ≤ 2 \) which is in agreement with several studies of previous years.

The connection of the eigenvalue fluctuations to those of eigenfunctions suggested Λ as the evolution parameter for the eigenfunctions correlations (of order \( n > 1 \)) too. As shown in section III, the evolution parameters \( Λ_{\text{measure}} \) of various eigenfunction fluctuation measures are indeed functions of Λ: \( Λ_{\text{measure}} = f(Λ) \). Here \( f(Λ) ∝ Λ \) on short length scales and \( f(Λ) \sim Λ^{−α Λ} \) in the tail regime.

The parameter Λ, being a function of the distribution parameters of the matrix elements, is sensitive to changes in the system parameters; this is due to latter’s influence on the uncertainties associated with system-interactions. Some examples of such system parameters are disorder, dimensionality, boundary and topological conditions, system size etc. For example, the presence of disorder randomizes the interactions, with degree of disorder affecting the distribution parameters \( h, b \) and consequently Λ. The dependence of Λ on the dimensionality and boundary conditions originates from their influence on the matrix element i.e degree of sparsity of the matrix which is reflected in the distribution parameters \( h, b \). For example, for nearest neighbor hopping and hard wall boundary conditions in \( d \)-dimensions, the matrix element \( H_{jk} ≠ 0 \) only if \( j = |k + L^{d−1}| \) (with \( L \) as linear size). The variance \( h_{jk} \) of the distribution ρ\( (H_{jk}) \) is therefore finite only for \( j = k \) or \( |j = k ± L^{d−1}| \) and is zero for all other \( j, k \). The information about dimensionality in Λ also enters through the local mean level spacing which depends on the correlation volume \( ζ^d \). (See also \([50]\) where the dependence of Λ on system parameters is explained by considering an example of Anderson Hamiltonian).

The system size \( N \) is another important parameter which affects the evolution of the measures significantly. As shown in section IV, it appears independently as well as through \( Λ_{\text{measure}} \) in the evolution equations which suggests a two parametric dependence, namely, \( Λ_{\text{measure}} \) and \( N \) (separately) of these measures. As a consequence, even at the critical point of level statistics, the eigenfunction statistics remains sensitive to size \( N \). This in turn results in a multifractal behavior of the eigenfunctions at the critical point of any complex system, modeled by eqs.\((1,2)\). The scaling exponents at the critical point, referred as critical exponents or multifractal dimensions, depend on system parameters. In finite size systems, changing system parameters can change Λ (and therefore \( Λ_{\text{measure}} \)) continuously between 0 and \( ∞ \) which may lead to intermediate stages of varying degree of multifractality. However, the physically interesting cases usually correspond to infinite sizes where Λ takes only three possible values, namely, \( Λ = 0, ∞, Λ_{\text{critical}} \); for these cases therefore only one multifractal stage, that is at the critical point corresponding to \( Λ_{\text{critical}} \) can exist. As \( Λ_{\text{critical}} \) is sensitive to system-specifics, the critical (multifractal) exponents can vary from system to system. Note, as already mentioned above, the occurrence of critical point and, therefore, a multifractal behavior of eigenstates is not a necessary feature of all infinite size complex systems.

The Λ-governed diffusion equations, derived in section IV, are valid for arbitrary initial conditions at \( Λ = 0 \) (which implies \( Λₓ = 0 \)) and their solutions \( P_x(X, Λₓ |X₀, 0) \) describe the probability of the measure, say \( X \), at \( Λ_x \) for a given
initial state of $X = X_0$. Thus $P$ is subjected to an initial constraint $\lim \Lambda \rightarrow 0 P(X, \Lambda_x | X_0, 0) = \delta(X - X_0)$. By integration of the solution over the distribution of initial values $P_0(X_0, 0)$, one can recover $P(X, \Lambda_x)$, that is, the distribution of measure $X$ for a system with complexity parameter strength $\Lambda$:

$$P(X, \Lambda_x) = \int P(X, \Lambda_x | X_0, 0)P_0(X_0, 0)\,dX_0.$$  

(60)

The eq.(60) implies that the statistics evolved in "time" $\Lambda$ is sensitive to the collective behavior of system parameters contributing to $\Lambda$ and the initial distribution only. The latter can always be chosen same for the systems operating in the matrix spaces of similar type e.g. same symmetry conditions; (the initial values of their $Y$ parameters need not be equal). Thus if both $A, B$ operate in the matrix spaces of same type, their behavior at the system parameter strengths which lead to $\Lambda_{x,A} = \Lambda_{x,B} = t$ would also be same (although they may show different behavior between $0 < \Lambda_{x,A}, \Lambda_{x,B} < t$). This implies a great deal of universality among systems of widely different origins of complexity. For example, consider the cases of a three dimensional disordered system, say $A$, and a clean, closed quantum dot, say $B$. In the first case, $\Lambda_{x,A} = \Lambda_{x,\text{disorder}}$ is a function of disorder, hopping strength, dimensionality, boundary condition etc. In the case of a dot, $\Lambda_{x,B} = \Lambda_{x,\text{dot}}$ is a function of shape as well as size. It is well-known that, in strong disorder limit and for circular shape, respectively, both systems show localized wavefunction dynamics and same statistical behavior of the eigenfunctions and eigenvalues of the Hamiltonians. Reducing the degree of disorder or change of shape of the dot from circle to stadium type results in a transition from localized to delocalized dynamics of the wavefunctions. The statistics in the intermediate stages during the transition for each case is governed by the respective $\Lambda$ strengths. If, however, the $\Lambda_{x,\text{dot}} = \Lambda_{x,\text{disorder}}$ at some shape parameter and disorder strength, respectively, our analysis predicts a same statistical behavior for both systems. The implication can also be extended to classical systems e.g. stock market fluctuations which are analyzed by statistical studies of the correlation matrix of stocks. (Note, the correlation matrices of classical systems are, in general, non-Hermitian; however, as shown in [40], the $\Lambda$ formulation remains valid for non-Hermitian version of eq.(1)). Here a very weak interaction among certain stocks due to various socio-economic conditions results in a localized dynamics of the eigenfunctions. The statistics in the intermediate stages during the transition for each case is governed by the respective $\Lambda$ strengths. If, however, the $\Lambda_{x,\text{stock}} = \Lambda_{x,\text{dot}}$ for some combinations of SEP and dot parameters, respectively, the spectral and strength fluctuations in correlation matrix of the stock market and the Hamiltonian of quantum dot will show same behavior. Note the analogy of statistical behavior of the eigenvalues and eigenfunctions among the three systems, mentioned above, has already been numerically verified in delocalized waves limit $\Lambda \rightarrow \infty$. The above universality makes $\Lambda$ formulation useful as it can be exploited to obtain the statistics of a complex system if the same information is available about another system (under same symmetry conditions) by another method. For example, for Anderson type disordered Hamiltonian, the distributions of many measures are known by non-linear sigma model techniques. The formulations can then be used for the complex systems e.g stock markets undergoing a transition from localized $\rightarrow$ delocalized wave dynamics. In this case, $\Lambda$ is a function of the socio-economic parameters (SEP). However if $\Lambda_{x,\text{stock}} = \Lambda_{x,\text{dot}}$ for some combinations of SEP and dot parameters, respectively, the spectral and strength fluctuations in correlation matrix of the stock market and the Hamiltonian of quantum dot will show same behavior.

The above universality makes $\Lambda$ formulation useful as it can be exploited to obtain the statistics of a complex system if the same information is available about another system (under same symmetry conditions) by another method. For example, for Anderson type disordered Hamiltonian, the distributions of many measures are known by non-linear sigma model techniques. The formulations can then be used for the complex systems e.g stock markets undergoing a transition from localized $\rightarrow$ delocalized wave dynamics; one just needs to replace $\Lambda($Anderson$)$ by that of the system.

The formulation can also be used to search for the system conditions leading to a critical state or multifractal wavefunctions of various complex systems. For example, the suggested modelling of mixed systems by eq.(1) would imply the possible existence of a critical point of level-statistics in the systems and multifractal eigenstates. The intuition suggests that the occurrence of such a point may correspond to breaking of the last KAM curve, thus allowing classical diffusion or delocalization of the dynamics above the critical point and localization below it; however it needs to be further explored. The critical $\Lambda$ can then given by the critical value of system parameter leading to last KAM curve breaking.

VI. NUMERICAL ANALYSIS

For numerical analysis, we choose three different ensembles (for both cases $\beta = 1, 2$); the choice is dictated by the reason (i) the ensembles are prototype models of many physical systems related to different areas [3, 6, 10, 35], (ii) a comparative study of the eigenvalue fluctuations of these systems has already been carried out, with their $\Lambda$ parameters and other results given in [49].

(i) Critical Anderson ensemble (AE):

(a) Time-Reversal case AE$_c$: we analyze cubic ($d = 3$) Anderson lattice of linear size $L$ ($N = L^d$) with a Gaussian site disorder (of variance $W^2/12$, $W = 4.05$ and mean zero), same for each site, an isotropic random hopping between
nearest neighbors with hard wall boundary conditions \[3, 50\]. The ensemble density in this case can be described by eq.(1) with \( h_{kk} = W^2/12, h_{kl} = f(kl)/12, b_{kl} = 0 \) where \( f(kl) = 1 \) for \( \{k, l\} \) pairs representing hopping, \( f(kl) \to 0 \) for all \( \{k, l\} \) values corresponding to disconnected sites. A substitution of above values in eq.(8) gives \( Y \) which subsequently gives \( \Lambda \) by eq.(59): (see eq.(19) of [49]),

\[
\Lambda_\alpha(E, Y) = |\alpha - \alpha_0| F^2 \gamma^{-d} L^{-d}\gamma^{-1}
\]

(61)

with \( \alpha - \alpha_0 = 1.36 \) and \( F(E) = 0.26e^{-E^2/5} \) (see section V of [49]). (Note, for later reference, \( F(E) \) is the mean level density: \( F(E) = (ND)^{-1} \)).

(b) Broken Time- Reversal case AE_{init}: we analyze cubic \( d = 3 \) Anderson lattice of linear size \( L (N = L^d) \) with a Gaussian site disorder (of variance \( W^2/12, W = 21.3 \) and mean zero), same for each site, an isotropic non-random hopping \( t = 1 \) between nearest neighbors with periodic boundary conditions \[3, 50\]. The time-reversal symmetry is broken by applying an Aharonov Bohm flux \( \phi \) which gives rise to a nearest neighbor hopping \( H_{kl} = \exp(\phi) \) for all \( k, l \) values related to the nearest-neighbor pairs \[3, 50\]. The flux \( \phi \) is chosen to be non-random in nature, that is, \( \langle \cos^2(\phi) \rangle = W_1 = 0, < \sin^2(\phi) >= W_2 = 0 \) and \( \langle \cos(\phi) \rangle = t_1 = 1, < \sin(\phi) >= t_2 = 1 \). The ensemble density in this case can again be described by eq.(1) with \( h_{kk} = W^2/12, h_{kl} = 0, h_{kl,s} = W_s = 0, b_{kl,s} = t_s f(kl;s) \) where \( f(kl;s) = 1 \) for all \( \{k, l\} \) pairs representing hopping, \( f(kl;s) \to 0 \) for all \( \{k, l\} \) values relating to disconnected sites. The \( \Lambda \) for this case is still given by eq.(61) (except for a factor \( \beta^{-1} \)); however now \( \alpha - \alpha_0 = 5.43, F(E) = 0.016e^{-E^2/400} \) (see section V of [49]).

(ii) Critical Power Law Random Banded Ensemble (critical PRBM or PE): As mentioned in section II, PRBM ensemble was introduced as a possible model for the level statistics of Anderson Hamiltonian \[3, 30\]. It is defined as the ensemble of random Hermitian matrices with matrix elements \( H_{ij} \) as independently distributed Gaussian variables with zero mean i.e \( < H_{ij} >= 0 \) and a power-law decay of the variances away from the diagonal \[3, 11, 31\]:

\[
\langle |H_{ij}|^2 \rangle = a(i - j)^\delta \quad \text{function of } a(r) \quad \text{decaying as } r \text{ increases.}
\]

The PRBM ensemble with specific choice \( \langle |H_{ij}|^2 \rangle = G_{ij}^{-1} \left[ 1 + ((i - j)/b)^2 \right]^{-1}, G_{ij} = \beta(2 - \delta_{ij}) \) and \( G_{ij} = 1/2 \) (referred as critical PRBM or PE in this paper) leads to a critical behavior of eigenfunction and eigenvalue statistics at arbitrary values of the parameter \( b \) and is believed to show all the key features of the Anderson critical point, including multifractality of eigenfunctions and the fractional spectral compressibility \[3, 52\]. The ensemble density in this case corresponds to eq.(1) with \( b_{kl} = 0 \), and, \( h_{kl,s} = G_{kl}^{-1} \left[ 1 + ((k - l)/p)^2 \right]^{-1} \). The corresponding \( \Lambda \) can be shown to be given by (see section VI of [49]),

\[
\Lambda_p(p, E) = \alpha_p^{-1} f(p) F^2(E) \gamma^2 N^{-1}.
\]

(62)

where \( \alpha_p = 2N(N + 2 - \beta), f(p) = \sum_{r=1}^{N} (N - r)ln[1 + (p/r)^2] \).

(iii) Critical Brownian Ensemble (BE): A Brownian ensemble can be described as a non-stationary state of the matrix elements undergoing a cross-over due to a random perturbation of a stationary ensemble by another one \[3, 11, 32, 49\]. For example, in the case of Hermitian operators, a Brownian ensemble \( H \) can be defined as \( H = \sqrt{f}(H_0 + \lambda V) \) (with \( f = (1 - X^2)^{-1} \); here \( V \) is a random perturbation of strength \( \lambda \), taken from a stationary ensemble \[3, 55\] e.g. Wigner-Dyson ensemble, and applied to an initial stationary state \( H_0 \) (see also \[11\]). Here we consider a specific class of BEs, namely, those appearing during a transition from Poisson \( \to \) Wigner-Dyson ensemble, caused by a perturbation of the former by the latter (that is, taking \( H_0, V \) as Poisson and Wigner-Dyson ensemble respectively). As in above two cases, this transition also results in a change of localized eigenstates to delocalized ones. The BEs related to the Poisson \( \to \) Wigner-Dyson transition can be described by a \( N \times N \) ensemble \( H \) represented by eq.(1) with mean \( \langle H_{kl} \rangle = b_{kl} = 0 \), the variance \( \langle H_{kk}^2 \rangle = h_{kk,s} = (2\gamma)^{-1} \) and \( \langle H_{kl}^2 \rangle = h_{kl,s} = |4\gamma(1 + \mu)^{-1} \rangle \) for \( k \neq l \) with \( (1 + \mu) = (\lambda^2 f)^{-1} \); here \( H = H_0 \) for \( \lambda \to 0 \) or \( \mu \to \infty \). As mentioned in section II, the ensemble density in this case has the same form as for Rosenzweig-Porter (RP) ensemble \[33\]; it can also describe an ensemble of Anderson Hamiltonians with very long range, isotropic, random hopping. Further, as discussed in \[49\], the special case \( \mu = cN^2 \) corresponds to the critical BEs; their mean level density is given as \( F(E) = (\pi)^{-1/2} e^{-E^2} \) and

\[
\Lambda_b(E) = (1/4\pi n^2) n^{-1/2} e^{-E^2}.
\]

(63)

Our aim is to show that the behavior of an eigenfunction fluctuation measure of AE, BE and PE is analogous at system parameters which lead to a same \( \Lambda_{\text{measure}} \) value for all the three cases. Using the latter as a condition, we can obtain the desired system parameters in each case (that is, \( p \) for PE and \( c \) for BE for a given AE). As \( \Lambda \)
for the three cases is energy dependent, the fluctuation measures should be compared at precisely a given value of energy. For numerical analysis, however, one needs to consider averages over an energy range \( \Delta E \) which should be sufficiently large in order to improve the statistics. On the other hand, choice of a very large \( \Delta E \) will lead to mixing of different statistics (in a range \( \Delta \Lambda \propto \delta E \)). As a consequence, one needs to consider an optimized range of \( \Delta E \). In our simulations, we analyze large ensembles of about 1400 matrices of size \( N = 2197 \). We choose \( \Delta E \) to be about 10% of the bandwidth, at the band center \( \bar{E} = 0 \) which gives approximately \( 3 \times 10^5 \) levels for each case. As the chosen \( \Delta E \) corresponds to a 1\% variation of the density of states, it avoids mixing of different statistics.

As discussed in previous section, the eigenfunction fluctuations are influenced by both \( \Lambda \) as well as system size \( N \). To compare \( \Lambda_{\text{measure}} \) dependence of a fluctuation measure (of eigenfunctions), therefore, same system size be taken for all systems under consideration. As examples, here we consider distributions of three measures, namely, local eigenfunction intensity \( P_u(u) \), inverse participation ratio \( P_I(I_2) \) and pair correlation function \( P_w(w) \) for the three systems under time-reversal symmetry (\( \beta = 1 \)) i.e. AE\(_t\), BE\(_t\), PE\(_t\). As, for \( P_u \),

\[
\Lambda_u = \frac{\mu(Y - \mu)}{N^2D^2} \approx \left( \frac{F}{I_{2u}^{\text{typ}}} \right)^2,
\]

(with \( \zeta^d \approx (I_{2u}^{\text{typ}})^{-1} \)), the BE and PE analogs for the intensity distribution of AE\(_t\) can be obtained by the condition \( I_{2u}^{\text{typ}}/F_0 = I_{2u}^{\text{typ}}/F_0 = I_{2u}^{\text{typ}}/F_0 \). This requires a prior information about \( I_{2u}^{\text{typ}} \). Our numerical study for various sizes of the three systems shows that, for each case, \( I_{2u}^{\text{typ}} \approx 1N^{-D_2} \) with \( I \) and \( D_2 \) system dependent. The numerical information about \( I_{2u} \) and \( F \) can now be used to obtain the parameters \( p \) and \( c \) for PE and BE analogs of AE\(_t\) for \( P_u \) case (i.e PE and BE with the ratio \( I_{2u}^{\text{typ}}/F \) same as for AE); we find \( p = 0.4 \), \( c = 0.02 \). The figure 1 shows the distribution for PE\(_t\) and BE\(_t\) along with AE\(_t\) case; the close agreement among the three cases confirms our theoretical prediction. This is also confirmed by the comparison of \( P_I(I_2) \) and \( P_w(w) \) for the three systems, shown in figure 3 and figure 5, respectively. Here again the parameters \( p \) and \( c \) for PE and BE analogs for both measures are obtained by the relation \( \Lambda_{I,u} = \Lambda_{I,b} = \Lambda_{I,p} \) (similarly for \( w \)).

The above numerical analysis is repeated also for the case of AE in a magnetic field and its BE and PE analog; the results for the three measures, shown in figures 2, 4 and 6, further support our claim: the eigenfunction fluctuations of different complex systems show same behavior if their complexity parameters and sizes are equal. It is worth reminding that behavior of the eigenvalue fluctuations is governed only by the related complexity parameter (that is, no independent influence of size). The details of analytical and numerical evidence about the eigenvalue statistics are already published in \[40, \]54, \]55]. However for the sake of completeness and to convince the reader, we include here the numerical analysis of an eigenvalue fluctuation measure, namely, nearest neighbor spacing distribution \( P(S) \) for the three system (for both \( \beta = 1, 2 \) cases) at parametric values leading to \( \Lambda_{I,a} = \Lambda_{I,b} = \Lambda_{I,c} \) (where \( \Lambda_e = \Lambda \)); the plots shown in figures 7,8 reconfirm the claim about eigenvalue statistics.

VII. CONCLUSION

In the end, we summarize our main results. Our analysis of the eigenfunction correlations of complex systems indicates a two parameter dependence, namely, complexity parameter \( \Lambda \) and system size \( N \), of the distributions of eigenfunction components. The independent appearance of size parameter (besides through \( \Lambda \)) seems to suggest a lack of finite size scaling in eigenfunction distributions and an absence of their critical limit. This is in contrast with the behavior of eigenvalue distribution which shows a single parametric scaling as well as a critical limit if the condition \( \lim N \rightarrow \infty \Lambda = \text{finite} \) is satisfied by the system. Note the above implies the size-dependence of eigenfunction correlations at the critical point of level-statistics too.

We have also studied the distribution of a few important measures of eigenfunction correlations e.g. local density of states, pair correlation function etc. We find that the form of complexity parameter governing an eigenfunction fluctuation measure is sensitive to its nature (e.g. \( \Lambda_u \) for local intensity distribution, \( \Lambda_I \) for inverse participation ratio distribution etc). This is again different from the eigenvalue fluctuations (except for level density) which are all governed by the same complexity parameter, namely, \( \Lambda_e = \Lambda \). Our analysis indicates a log-normal behavior of the asymptotic tails of the distributions at finite \( \Lambda \)-strength. In context of disordered systems, a similar behavior was predicted by other studies using different techniques e.g. Berezinski and Abrikosov-Ryzhkin technique (for
one dimension) and by non-linear sigma model (for higher dimensions). However the complexity parameter formulation suggests the existence of such a tail-behavior and multifractal eigenfunctions for almost any complex system, irrespective of the origin of complexity, if the parameter $N_{\text{measure}}$ is finite. A recent numerical study of the eigenfunction of the correlation matrix of stock prices confirms the suggestion in case of stock market. As finite $\Lambda$ corresponds to the critical point condition in infinite size systems, a log-normal tail-behavior seems to be associated with the existence of a critical point (and vice versa). The above study can thus be used to search and predict the critical stages of other complex systems, e.g., stock market, brain, etc.

In this paper, we have considered the cases modeled by generalized Gaussian ensembles with uncorrelated matrix elements as well as a wide range of non-Gaussian ensembles with correlated matrix elements. The latter are suitable models, for example, for disordered systems with varying degree of particle-particle interactions. In context of disordered systems, therefore, we expect a same statistical behavior of a measure for both the cases, namely, with or without particle interactions, if the strengths of their parameters $N_{\text{measure}}$ are equal. This suggests the sensitivity of the statistical behavior of a disordered system to degree of its complexity only (measured by complexity parameter), irrespective of the origin. The statement is expected to be valid for correlated and uncorrelated cases of other complex systems too. This in turn would indicate the existence of an infinite family of universality classes, parametrized by $\Lambda$, of statistical behavior among complex systems.

**APPENDIX A: CALCULATION OF INTEGRALS** $Q^{rs}_{mn;k}$ AND $G_r$

The integral $Q^{rs}_{mn;k}$ defined by eq. (20) can be rewritten in terms of $\rho(H)$ as

$$Q^{rs}_{mn;k} = \sum_{j:j\neq k} \int \frac{(U_{nj} U_{mj})^r}{(\lambda_k - \lambda_j)^s} \rho(H, Y) \, dH$$  \hspace{1cm} (A1)

To express $Q$ in terms of $P_{N1}$, it is necessary to write $\rho(H)$ in eigenvalue-eigenvector space i.e. $\{\lambda, U\}$ space. The steps can briefly be given as follows. The solution of eq.(6) for arbitrary initial condition, say $H_0$ at $Y = Y_0$ can be given as $\rho(H, Y|H_0, Y_0) \propto \exp[-(\alpha/2)\text{Tr}(H - \eta H_0)^2]$ with $\alpha = \gamma(1 - \eta^2)^{-1}$ and $\eta = e^{-\gamma Y}$. Without loss of generality, the basis space for $H$ can be chosen as the eigenvector space of $H_0$. The initial ensemble $H_0$ in this basis consists of diagonal matrices. For simplification, consider the initial distribution given by $\rho(H_0) \propto e^{-\sum_{j=1}^N H_{0,ij}^2}$. Using eigenvalue equation $UH = \Lambda U$, $\rho(H, Y|H_0, Y_0)$ can be transformed from matrix space to eigenvalue-eigenvector space $\{\lambda, U\}$ which followed by an integration over ensemble $H_0$ gives

$$\rho(H, Y) \propto \prod_{k,l,k<l}^{N} |\lambda_k - \lambda_l|^2 \exp \left[ -(1/2) \sum_{j=1}^{N} \lambda_j^2 - \mu/2 \sum_{k<l} [\lambda_k - \lambda_l]^2 U_{jk}^2 U_{jl}^2 \right]$$  \hspace{1cm} (A2)

where $\mu = (e^{2\eta(Y-Y_0)} - 1)^{-1}$.

A substitution of eq.(A2) for $\rho$ in eq.(A1) and using $dH = \prod_{j} \prod_{k<l} |\lambda_k - \lambda_l|^2 d\lambda_j dU_j$, gives $Q$ as a function of $\{U, \Lambda\}$ variables. As eq.(A2) indicates, the behavior of $Q^{rs}_{mn;k}$ is significantly influenced by the term $R \equiv \mu |\lambda_k - \lambda_j|^2 \sum_{n=1}^{N} |U_{nk}|^2 |U_{nj}|^2$ present in the exponent of $\rho$. Consequently, for a given $Y$, the dominant contribution to the integrals over the variables $U_j$ and $\lambda_j$ in eq.(A1) comes from those regions which lead to $R \to 0$. Also note that the eigenvalue-eigenfunction correlations appear in $\rho$ only through $R$. The limit $R \to 0$ therefore allows a mutually independent integration over $\lambda_j$ and $U_{nj}$ variables. As the typical local intensity $|U_{nk}|^2_{\text{typical}} \sim e^{-d}$ with $\zeta_k$ as the localization length of the eigenfunction $U_k$ (as system-dimension), this implies $R \sim \mu |\zeta_k|^d |\lambda_k - \lambda_j|^2$. Consequently, the regions of variable $\lambda_j$ and $U_{nj}$ which contribute to integral depend on mutual competition between $\mu$ and $\zeta_k^d$.

(i) for $\mu < \zeta_k^d$, almost entire region of $U_{nj}$ can contribute to integral (due to $0 < |U_{nj}|^2 < 1$). However only a small neighborhood of the order of local mean level spacing, i.e. $|\lambda_k - \lambda_j| \approx D_k$ around $\lambda_k$, contributes to $\lambda_j$ integration. Here $D_k$ is the local mean level spacing at eigenvalue $\lambda_k$. As a consequence, an approximation of repulsion term
\[ |\lambda_k - \lambda_j| \approx D_k \] along with the relation \( \sum_{k=1}^{N} U_{nk} U_{mk}^* = \delta_{mn} \) (due to unitary nature of \( U \)) gives

\[ \sum_{j=1; \neq k}^{N} \frac{(U_{nj} U_{mj}^*)^r}{(\lambda_k - \lambda_j)^s} = \frac{[\delta_{mn} - U_{nk} U_{mk}^*]^r}{(N-1)^{r-1} D_k^s} \] (A3)

Here \( r = 0, 1 \) only, \( \chi = 1 \), \( D_k \) is the local mean level spacing at eigenvalue \( \lambda_k \).

(ii) for \( \mu > \zeta_k^d \), the significant contribution comes from the regions of \( \lambda_j \) where \( |\lambda_j - \lambda_k| \sim D_k \zeta_k^d / \mu^{1/2} \). Here again, as a typical \( |U_{nj}|^2 \sim \zeta^{-d} \) < 1, the entire region of \( U_j \) can contribute to the integral. Consequently one can approximate

\[ \sum_{j=1; \neq k}^{N} \frac{(U_{nj} U_{mj}^*)^r}{(\lambda_k - \lambda_j)^s} = \left( \frac{\mu}{\zeta_k^d} \right)^{s/2} \frac{[\delta_{mn} - U_{nk} U_{mk}^*]^r}{(N-1)^{r-1} D_k^s} \] (A4)

(One may also consider the contribution from regions where \( |U_{nj}|^2 < (\mu |U_{nk}|^2 D_k^2)^{-1} \) however it is weaker than the above).

By substituting approximations (A3,A4) in eq.(A1), \( Q_{mn;k}^{rs} \) can be written as (for \( r = 0, 1 \) only):

\[ Q_{mn;k}^{rs} \approx \chi^{s/2} \frac{(\delta_{mn} - z_{nk}^* z_{mk})^r}{(N-1)^{r-1} D_k^s} \] (A5)

where \( \chi = 1 \) for \( \mu < \zeta_k^d \) and \( \chi = \mu / \zeta_k^d \) for \( \mu > \zeta_k^d \).

The integral \( G_r \) (see eq.(30) can also be rewritten in terms of \( \rho(H) \) and can similarly be approximated:

\[ G_r(x,e) \equiv \sum_{j: j \neq k} \int \delta e \frac{|U_{nj}|^{2r}}{(\lambda_k - \lambda_j)^s} dH \] (A6)

The dominant contribution in this case comes from those regions of integration over \( U_j \) and \( \lambda_j \) which lead to \( \tilde{R} \equiv \mu |U_{nk}|^2 \sum_j |\lambda_k - \lambda_j|^2 |U_{nj}|^2 \) present in the exponent of \( \rho \). (Note, unlike the dominating term \( R \) in \( Q_{mn;k}^{rs} \) case, \( \tilde{R} \) contains only a single component of the \( k^{th} \) eigenfunction, namely, \( U_{nk} \), and, the latter takes a fixed value \( x/\sqrt{N} \)) Consequently, for a given \( Y \), \( G_r \) depends on the mutual competition between \( \mu \) and \( x \). Reasoning as in the case of \( Q_{mn;k}^{rs} \), \( G_r \) can be approximated as

\[ G_r \approx \mu \chi_0 (N-1)^{-r} (N - |x|^2)^r P_{11}(x)/D^2 \] (A7)

with \( \chi_0 = \mu^{-1} \) for \( \mu |x|^2 < 1 \) and \( \chi_0 \sim |x|^2 \) for \( \mu |x|^2 > 1 \).

**APPENDIX B: EFFECT OF MATRIX ELEMENTS PERTURBATIONS ON EIGENVALUES AND EIGENFUNCTIONS**

Consider the perturbation of a Hermitian matrix \( H \) with matrix elements \( H_{kl} \equiv \sum_{s=1}^{2} (i)^{s-1} H_{kl;s} \), eigenvalues \( \lambda_n \) and eigenfunctions \( U_n \), \( n = 1, 2, \ldots N \). By using the eigenvalue equation \( \sum_m H_{nm} U_{mj} = \lambda_n U_{nj} \) along with the ortho-normal condition on eigenvectors i.e. \( \sum_j U_{nj} U_{mj}^* = \delta_{mn} \), it can be shown that

\[ \frac{\partial \lambda_n}{\partial H_{kl;s}} = 2 g_{kl}^{-1} U_{kn} U_{ln} \]
\[ \frac{\partial U_{nj}}{\partial H_{kl;s}} = \frac{i^{s-1}}{g_{kl}} \sum_{m \neq j} \frac{1}{\lambda_j - \lambda_m} U_{nm} (U_{kn}^* U_{ln} + (-1)^{s+1} U_{lm}^* U_{kj}) \] (B1)
The details of the steps used in derivation of eq.(B1) can be found in [40]. The set of equations (B1) can further be used to show following relations:

\[ \sum_{k,l,s:k \leq l} \frac{\partial \lambda_n}{\partial H_{kl;s}} H_{kl;s} = \lambda_n \]  \hspace{1cm} (B2)

\[ \sum_{k,l,s:k \leq l} \frac{\partial U_{nj}}{\partial H_{kl;s}} H_{kl;s} = 0, \]  \hspace{1cm} (B3)

and,

\[ \sum_{k,l,s:k \leq l} g_{kl} \frac{\partial^2 U_{nj}}{\partial H_{kl;s}^2} = - \sum_{m \neq j} \frac{U_{nj}}{(\lambda_j - \lambda_m)^2} \]  \hspace{1cm} (B5)

\[ \sum_{k,l,s:k \leq l} g_{kl} \frac{\partial \lambda_i}{\partial H_{kl;s}} \frac{\partial U_{nj}}{\partial H_{kl;s}} = 0 \]  \hspace{1cm} (B6)

\[ \sum_{k,l,s:k \leq l} g_{kl} \frac{\partial U_{ni}}{\partial H_{kl;s}} \frac{\partial U_{pj}}{\partial H_{kl;s}} = - \beta \sum_{m \neq j} \frac{U_{nm} U_{pm}^*}{(\lambda_j - \lambda_m)^2} \delta_{ij} \]  \hspace{1cm} (B7)

\[ \sum_{k,l,s:k \leq l} g_{kl} \frac{\partial U_{ni}}{\partial H_{kl;s}} \frac{\partial U_{pj}^*}{\partial H_{kl;s}} = \beta \sum_{m \neq j} \frac{U_{nm} U_{pm}^*}{(\lambda_j - \lambda_m)^2} \delta_{ij} \]  \hspace{1cm} (B8)
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III. FIGURE CAPTION

Fig. 1. Distribution $P_u(u')$ with $u' = [\ln u - \langle u \rangle]/(\ln^2 u)$ of the local intensity of an eigenfunction near band center for AE$_t$ (Cubic lattice of linear size $L = 13$, with hard wall boundary conditions, random hopping and time-reversal symmetry) and its BE and PE analog. The parts (a) and (b) of the figure show short and long range behavior of the distribution, respectively. The analogs are obtained by the relation $I^{\text{typical}}_{2, a}/F_a(0) = I^{\text{typical}}_{2, b}/F_b(0) = I^{\text{typical}}_{2, p}/F_p(0)$. For $N = 2197$, we find $I^{\text{typical}}_{2, a} = 0.018$, $I^{\text{typical}}_{2, b} = 0.02$, and, $I^{\text{typical}}_{2, p} = 0.02$, and, $F_a(0) = 0.26$, $F_b(0) = 0.26$, $F_p(0) = (\pi)^{-1/2}$.

Fig. 2 The local intensity distribution for AE$_{nt}$ (Cubic lattice of linear size $L = 13$, with periodic boundary conditions, non-random hopping and no time-reversal symmetry) and its BE and PE analogs. In this case, $I^{\text{typical}}_{2, a} = 0.013$, $I^{\text{typical}}_{2, b} = 0.03$, $I^{\text{typical}}_{2, p} = 0.4 = 0.00045$, and, $F_a(0) = 0.016$, $F_b(0) = 0.016$, $F_p(0) = (\pi)^{-1/2}$. Other details are same as in figure 1.

Fig. 3. Distribution $P(I_2')$ of the rescaled inverse participation ratio $I_2' = \ln[I_2/I_2^{\text{typ}}]$ for AE$_t$ (same as in Figure 1(a)) and its BE and PE analog: (a) short range behavior (lin-lin plot), (b) tail behavior (lin-log plot). Here the BE and PE analogs are obtained by the relation $\Lambda_{I, a} = \Lambda_{I, b} = \Lambda_{I, p}$. This gives a BE analog of AE different from the figure 1 although PE analog remains unaffected; the reason lies in almost similar mean level density behavior near band-center for the AE, PE cases.

Fig. 4. Distribution $P(I_2')$ of the rescaled inverse participation ratio $I_2'$ for AE$_{nt}$ (same as in Figure 1(b)) and its BE and PE analog. The other details are same as in figure 3. Note the BE analog of AE, PE in this case is different from the figure 2.

Fig. 5. Distribution $P(w')$ of the spatial correlation $w' = \ln w = \ln |z_{1n} - z_{Nn}|^2$ between two points belonging to opposite end of the sample: (a) short range behavior (lin-lin plot), (b) tail behavior (lin-log plot). The cases compared here are AE$_t$ (same as in Figure 1(a)) and its BE and PE analog (obtained by the relation $\Lambda_{w, a} = \Lambda_{w, b} = \Lambda_{w, p}$). Again the BE analog of AE, PE in this case turns out to be different from the figure 1 but same as in the figure 3.

Fig. 6. Distribution $P(w')$ of the spatial correlation $w'$ for AE$_{nt}$ (same as in figure 2) and its BE and PE analog; other details are same as in figure 5. Here the BE and PE analogs are obtained by the relation $\Lambda_{w, a} = \Lambda_{w, b} = \Lambda_{w, p}$. The BE analog of AE, PE in this case is different from the figure 2 but same as in the figure 4.

Fig. 7. Distribution $P(S)$ of the nearest-neighbor spacing distribution $S$ of the eigenvalues, with (a), (b) showing short and long range behavior, respectively, for AE$_t$ (same as in Figure 1) and its BE and PE analog. Here the BE and PE analogs are obtained by the relation $\Lambda_{e, a} = \Lambda_{e, b} = \Lambda_{e, p}$. Note the BE analog of AE, PE in this case is different from the figure 1 but same as in the figures 3 and 5.

Fig. 8. Distribution $P(S)$ of the nearest-neighbor spacing distribution $S$ for the case AE$_{nt}$ (same as in Figure 2) and its BE and PE analog (other details same as in figure 7).
Figure 1(a)

Figure 1(b)

$P(u')$

$u'$

$P(u)$

$u'$
Figure 2(a)

Figure 2(b)
AE, c=0.1
BE, c=0.1
PE, p=0.4

Figure 3(a)

Figure 3(b)
Figure 4(a) and Figure 4(b) illustrate the probability distribution of $P(I'_2)$ as a function of $I'_2$. The graphs show different scenarios with parameters $AE_{nt'}$, $BE_{nt'}$, and $PE_{nt'}$.

- $AE_{nt'}$: c=0.1
- $BE_{nt'}$: p=0.4

The figures highlight the distribution patterns under these conditions.
Figure 7(a) and Figure 7(b) show the probability distributions $P(S)$ for different values of $c$ and $p$. The graphs illustrate the behavior of $AE_t$, $BE_t$, and $PE_t$ with $c=0.1$ and $p=0.4$. The $S$ axis represents the parameter range, while the $P(S)$ axis shows the probability distribution for each value of $S$. The diagrams are used to visualize the statistical properties of the system under study.
Figure 8(a)

Figure 8(b)

$P(S)$ vs. $S$

$AE_{nt}$

$BE_{nt}$, $c=0.2$

$PE_{nt}$, $p=0.4$