THE COMPLEXITY OF COMPUTING THE CYLINDRICAL
AND THE $t$-CIRCLE CROSSING NUMBER OF A GRAPH

FRANK DUQUE, HERNÁN GONZÁLEZ-AGUILAR, CÉSAR HERNÁNDEZ-VÉLEZ*, JESÚS LEAÑOS†, AND CAROLINA MEDINA§

Abstract. A plane drawing of a graph is cylindrical if there exist two concentric circles that contain all the vertices of the graph, and no edge intersects (other than at its endpoints) any of these circles. The cylindrical crossing number of a graph $G$ is the minimum number of crossings in a cylindrical drawing of $G$. In his influential survey on the variants of the definition of the crossing number of a graph, Schaefer lists the complexity of computing the cylindrical crossing number of a graph as an open question. In this paper we settle this by showing that this problem is NP-complete. Moreover, we show an analogous result for the natural generalization of the cylindrical crossing number, namely the $t$-circle crossing number.

Keywords: cylindrical crossing number; book crossing number; $t$-circle crossing number
Mathematics Subject Classifications 2010: 05C10; 68R10; 05C85

1. Introduction

This work is motivated by a question posed by Marcus Schaefer in his survey on the variants of the definition of the crossing number of a graph. In [10], Schaefer listed as open the problem of the complexity of computing the cylindrical crossing number of a graph. We recall that a cylindrical drawing of a graph $G$ is a plane drawing where all the vertices are in two concentric cycles, and no circle is intersected by the interior of an edge. The cylindrical crossing number $cr_\circ(G)$ of a graph $G$ is the minimum number of crossings in a cylindrical drawing of $G$.

The concept of a cylindrical drawing is motivated by a family of graph drawings of the complete graph $K_n$, originally conceived by the British artist Anthony Hill. As narrated in the lively account given in [4], Hill’s construction produces drawings of $K_n$ that are cylindrical, according to the definition above, and have exactly $Z(n) := \frac{1}{4} \left\lfloor \frac{n^2}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$ crossings. It is a long-standing conjecture that the crossing number of $K_n$ is $Z(n)$, for every $n \geq 3$ [8]. In [1], Ábrego et al. proved that $cr_\circ(K_n) = Z(n)$, for every $n \geq 3$.

Let $D$ be a plane drawing of a graph $G$. We say that a Jordan curve $\rho$ (that is, a simple closed curve) is clean (with respect to $D$) if the interior of no edge intersects $\rho$. Now suppose that there are two clean disjoint circles with respect to $D$, say $\rho_1$ and $\rho_2$, such that every vertex of $G$ is in $\rho_1 \cup \rho_2$. Note that not only concentricity is not assumed, but also it is not required that the disk bounded by one of these circles contains the other circle. It is a straightforward exercise in plane topology that there is a cylindrical drawing $D'$ with the same cellular structure as $D$; in particular, $D'$ has the same number of crossings as $D$. Thus for crossing number purposes it is totally valid to adopt the following definition of a cylindrical drawing.

**Definition 1** (Equivalent definition of cylindrical drawing). A plane drawing of a graph $G$ is cylindrical if there exists two disjoint clean circles $\rho_1, \rho_2$ such that every vertex of $G$ is in $\rho_1 \cup \rho_2$.

*Partially supported by FAPESP (Proc. 2013/03447-6) and CNPq (Proc. 456792/2014-7).
†Partially supported by CONACyT Grant 179867.
§Partially supported by CONACyT Grant 222667.
The advantage of adopting this definition of a cylindrical drawing is that it allows us to generalize this notion to an arbitrary number of circles, as follows. We should mention that the term “t-circle drawing” has been suggested by Éva Czabarka and Marcus Schaefer (private communication).

**Definition 2** (t-circle drawing and t-circle crossing number). Let \( t \geq 1 \) be an integer. A plane drawing of a graph \( G \) is a t-circle drawing if there exist \( t \) pairwise disjoint clean circles \( \rho_1, \ldots, \rho_t \) such that every vertex of \( G \) is in \( \rho_1 \cup \cdots \cup \rho_t \). The t-circle crossing number \( \text{cr}_t(G) \) of a graph \( G \) is the minimum number of crossings in a t-circle drawing of \( G \).

Thus a cylindrical drawing is simply a 2-circle drawing. Moreover, for \( t = 1 \) there is an immediate connection with 2-page drawings. We recall that a 2-page drawing of a graph is a drawing in which the vertices lie on the \( x \)-axis, and each edge is contained (except for its endpoints) either in the upper halfplane, or in the lower halfplane. A straightforward argument shows that a 1-circle drawing can be transformed into a 2-page drawing with the same cellular structure.

Thus the 1-circle crossing number of a graph coincides with its 2-page crossing number, and the 2-circle crossing number of a graph is the same as its cylindrical crossing number. The 3-circle crossing number is related to the pair of pants crossing number \([10]\), but these are different notions, since in the latter it is required that none of the disks bounded by the circles contains another circle, and that no edge intersects the interior of any of these disks.

For the arguments we will use in this paper, it will be useful to relax the condition that the clean Jordan curves in Definition 2 need to be circles:

**Definition 3** (t-curve drawing and t-curve crossing number). Let \( t \geq 1 \) be an integer. A plane drawing of a graph \( G \) is a t-curve drawing if there exist \( t \) pairwise disjoint clean Jordan curves \( \rho_1, \ldots, \rho_t \) such that every vertex of \( G \) is in \( \rho_1 \cup \cdots \cup \rho_t \). The t-curve crossing number of a graph \( G \) is the minimum number of crossings in a t-curve drawing of \( G \).

It follows from the Jordan-Schönflies theorem that if \( D \) is a t-curve drawing of a graph \( G \), then there is a self-homeomorphism of the plane that takes \( D \) to a t-circle drawing. In particular, for any graph \( G \), its t-circle crossing number and its t-curve crossing number are the same. Thus the difference between these notions is rather cosmetic. On the other hand, as we hinted above, the advantage of dealing with t-curve drawings instead of t-circle drawings is that being able to work with arbitrary Jordan curves, instead of exclusively with circles, makes our arguments simpler.

As we mentioned above, our motivation in this work is to settle the complexity of computing the cylindrical crossing number of a graph, that is, the complexity of the decision problem CYLINDRICAL CROSSING NUMBER: “given a graph \( G \) and an integer \( k \), is \( \text{cr}_2(G) \leq k \)?”

As it happens, with very little additional effort we can settle the complexity of the decision problem t-CURVE CROSSING NUMBER, that considers a fixed integer \( t \) and asks “given a graph \( G \) and an integer \( k \), is the t-curve crossing number of \( G \) at most \( k \)?”

**Theorem 1.** For each fixed integer \( t \geq 2 \), \( t \)-CURVE CROSSING NUMBER is NP-complete.

As we mentioned above, the t-circle crossing number of a graph and its t-curve crossing number are the same. Thus this settles the complexity of the decision problem t-CIRCLE CROSSING NUMBER, that considers a fixed integer \( t \) and asks “given a graph \( G \) and an integer \( k \), is \( \text{cr}_t(G) \leq k \)?”. Since the 2-circle crossing number of a graph is its cylindrical crossing number, this settles in particular the complexity of computing the cylindrical crossing number. For completeness, we state these observations formally:

**Corollary 2.** For each fixed integer \( t \geq 2 \), t-CIRCLE CROSSING NUMBER is NP-complete. In particular, CYLINDRICAL CROSSING NUMBER is NP-complete.

Before proceeding to the proof of Theorem 1 (Section 3), we establish in the next section a result on plane triangulations that are minimal with respect to having a t-curve embedding.
2. Minimal t-curve embeddings

An essential ingredient in the proof that \textsc{CurveCrossingNumber} is NP-hard is the existence of plane triangulations that are minimal with respect to having a t-curve embedding. Our aim in this section is to establish this result (Lemma 4 below). We will need the following statement.

**Proposition 3.** Let $G$ be a maximal planar graph, and let $t$ be a positive integer. Suppose that $G$ has a t-curve embedding. Then there is a collection $\{H_1, \ldots, H_t\}$ of pairwise disjoint subgraphs of $G$ with the following properties: (i) if $H_i$ has at least 3 vertices for some $i \in \{1, \ldots, t\}$, then $H_i$ is a cycle; and (ii) $\bigcup_{i=1}^t H_i$ contains all the vertices of $G$.

**Proof.** Let $E$ be a t-curve embedding of $G$, and let $\rho_1, \ldots, \rho_t$ be the underlying $t$ clean Jordan curves of $E$. Let $i \in \{1, \ldots, t\}$. If $\rho_i$ does not contain any vertex, then we let $H_i$ be the null graph. If $\rho_i$ contains at least one vertex, let $v_1, \ldots, v_{m_i}$ be the vertices on $\rho_i$, in the (cyclic) order in which they appear in $\rho_i$. If $m_i = 1$, then we let $H_i$ be the subgraph of $G$ that consists only of the vertex $v_1$. If $m_i \geq 2$, we proceed as follows.

For $j = 1, \ldots, m_i$, there is a subarc of $\rho_i$ whose endpoints are $v_j$ and $v_{j+1}$ (indices are taken modulo $m_i$), and that is otherwise disjoint from $G$. This implies that for $j = 1, \ldots, m_i$, there is a face incident with $v_j$ and $v_{j+1}$. Since $G$ is maximal planar, $E$ is a plane triangulation and it is the unique plane embedding of $G$ (up to homeomorphism). Therefore the existence of a face incident with $v_j$ and $v_{j+1}$ implies that $v_j$ and $v_{j+1}$ are adjacent.

If $m_i = 2$, then we let $H_i$ be the subgraph of $G$ that consists of the vertices $v_1$ and $v_2$, and the edge joining them. If $m_i \geq 3$, then $v_1v_2 \ldots v_{m_i}v_1$ is a cycle $C_i$ of $G$, and we let $H_i = C_i$.

Since each vertex of $G$ is contained in a curve in $\{\rho_1, \ldots, \rho_t\}$, and these curves are pairwise disjoint, it follows that the collection $\{H_1, \ldots, H_t\}$ satisfies the required conditions. \hfill $\square$

**Lemma 4.** For every $t \geq 2$ there is a 3-connected simple graph $G_t$ such that (i) $G_t$ triangulates the plane; (ii) $G_t$ has a t-curve embedding; and (iii) $G_t$ has no $(t-1)$-curve embedding.

**Proof.** The heart of the proof is the existence of triangulations whose longest cycles are relatively small. Following Chen and Yu [6], let $T_1, T_2, \ldots$ be the family of triangulations constructed as follows. First, $T_1$ is the plane triangulation induced by $K_4$. Now, $T_{i+1}$ is constructed from $T_i$, for $i = 1, 2, \ldots$, as follows: in each inner face of $T_i$, add one new vertex and join it to the vertices of $T_i$ incident with the face containing it. We refer the reader to Figure 1.

![Figure 1](https://example.com/figure1.png)

**Figure 1.** On the left hand side we have the triangulation $T_1$. For each inner face $T$ of $T_1$, we add a (white) vertex inside $T$ and join it with edges to the three vertices incident to $T$; the result is the middle triangulation $T_2$. We obtain $T_3$ (right hand side) similarly: for each inner face $T$ of $T_2$, we add a (grey) vertex inside $T$, and join it with edges to the three vertices incident to $T$.

In [6] it is proved that, for $i \geq 1$, the length of the longest cycle of $T_i$ is less than $\frac{7}{3}|V(T_i)|^{\log_3 2}$. Now let $j$ be an integer large enough such that $\frac{7}{2}|V(T_j)|^{\log_3 2} \cdot (t-1) < |V(T_j)|$. Toward a contradiction,
suppose that \( T_j \) has a \((t - 1)\)-curve embedding, and let \( \{H_1, \ldots, H_{t-1}\} \) be the subgraphs of \( T_j \)
guaranteed by Proposition 3. Since \( V(T_j) = \bigcup_{i=1}^{t-1} V(H_i) \), it follows that there is some \( H_i \) such that
\[ |V(H_i)| > \frac{7}{2} |V(T_j)|^\log_3 2. \]
Since \( \frac{7}{2} s^{\log_3 2} \geq 3 \) for every \( s \geq 1 \), it follows from Proposition 3 that \( H_i \)
must be a cycle, contradicting that the length of the longest cycle of \( T_j \) is less than \( \frac{7}{2} |V(T_j)|^\log_3 2. \)
Thus \( T_j \) has no \((t - 1)\)-curve embedding.

In the previous paragraph we have shown that the family of graphs which are no \((t - 1)\)-curve
embeddable is not empty. Now we will choose the required graph from such a family. Let \( m \) be the
least integer such that \( T_m \) has no \((t - 1)\)-curve embedding. Note that \( m \geq 3 \), since \( T_2 \) has a 1-cycle
embedding, and thus a \((t - 1)\)-curve embedding for every \( t \geq 2 \). By the minimality of \( m \), \( T_{m-1} \)
has a \((t - 1)\)-curve embedding. Let \( Q_1 := T_{m-1}, Q_2, \ldots, Q_k := T_m \) be a sequence of triangulations
(subtriangulations of \( T_m \)) such that \( Q_{i+1} \) is obtained from \( Q_i \) by adding a new vertex and its three
incident edges, for \( i \in \{1, \ldots, k - 1\} \). Let \( \ell \) be the largest integer such that \( Q_{\ell} \) is a \((t - 1)\)-curve
embedding. Let \( v \) be the vertex that gets added (together with its three incident edges) to \( Q_{\ell} \), in
order to get \( Q_{\ell + 1} \).

The maximality of \( \ell \) implies that \( Q_{\ell + 1} \) is not a \((t - 1)\)-curve embedding, and we claim that \( Q_{\ell + 1} \)
is a \( t \)-curve embedding. To see this, let \( x, y, z \) be the three vertices adjacent to \( v \) in \( Q_{\ell + 1} \). Thus
\( x, y, z \) form a 3-cycle, which bounds the face \( f \) in \( Q_{\ell} \) in which \( v \) is placed. Let \( \rho_1, \ldots, \rho_{t-1} \) be clean
Jordan curves that witness the \((t - 1)\)-curve embeddability of \( Q_{\ell} \). It is easy to see that if one of
these Jordan curves intersects \( f \) then we can slightly perturb it so that it also intersects \( v \). But
this is impossible, since then \( Q_{\ell + 1} \) would be a \((t - 1)\)-curve embedding. Thus none of \( \rho_1, \ldots, \rho_{t-1} \)
intersects \( v \) or its incident edges, and so they are also clean Jordan curves in \( Q_{\ell + 1} \). We now draw
in a small neighborhood of \( v \) a clean Jordan curve \( \rho_t \) that only contains \( v \), so that \( \rho_1, \ldots, \rho_t \) is a collection of pairwise disjoint clean Jordan curves
that contain all the vertices of \( Q_{\ell + 1} \). Therefore \( Q_{\ell + 1} \) is a \( t \)-curve embedding, as claimed.

Let \( G_t \) be the underlying graph of the triangulation \( Q_{\ell + 1} \). It is readily checked that \( G_t \) is
3-connected and simple, and \( Q_{\ell + 1} \) witnesses that \( G_t \) triangulates the plane, and that \( G_t \)
has a \( t \)-curve embedding. Since \( G_t \) is 3-connected it follows that \( Q_{\ell + 1} \) is its unique
embedding (up to isomorphism) in the plane. Since \( Q_{\ell + 1} \) is not a \((t - 1)\)-curve embedding, it follows that \( G_t \) does not have a \((t - 1)\)-curve embedding. \( \square \)

3. Proof of Theorem 1

First we prove membership in NP, and then we prove NP-hardness.

(A) \textsc{t-curveCrossingNumber} is in NP.

\textit{Proof.} A drawing \( D \) of a graph \( G \) with at most \( k \) crossings may be described combinatorially, with
an amount of information bounded by a polynomial function of \( |V(G)| + |E(G)| + k \), by giving the
cellular structure of \( D \). Now a collection \( R \) of \( t \) clean Jordan curves with respect to \( D \) can be also
described combinatorially, simply by regarding each of these curves as the edge set of a cycle that
gets added to \( D \). (We remark that for this purpose we regard a graph that consists of a pair of
vertices joined by two parallel edges, or of a vertex with a loop-edge, as a cycle.) We let \( D' \) denote
the drawing that is obtained from \( D \) by adding (the edges of) these \( t \) cycles, which we colour blue
to help comprehension.

Thus the fact that \( G \) has a \( t \)-curve drawing with at most \( k \) crossings can be attested by the
existence of such a drawing \( D' \), with the properties that the \( t \) blue cycles are pairwise disjoint,
and each vertex of \( G \) is contained in a blue cycle. We finally note that such a drawing \( D' \) can be
described combinatorially with an amount of information bounded by a polynomial function of
\( |V(G)| + |E(G)| + k + t \), and that it can be verified in polynomial time that \( D' \) satisfies the required
properties. \( \square \)
(B) \textsc{t-curveCrossingNumber} is NP-hard.

\textit{Proof.} Let $t \geq 2$ be fixed. Let $G$ be a graph and let $G'$ be the disjoint union of $G$, a graph $G_t$ that satisfies the conditions in Lemma 4 and $k$ disjoint copies of $K_{3,3}$. It was proved in [7] that testing if a graph has a 2-page embedding is NP-complete. Since the size of $G'$ is bounded by a polynomial function of $|V(G)| + |E(G)| + k$ (the size of $G_t$ is a constant, for each fixed $t$), it follows that to prove (B) it suffices to show that $G$ has a 2-page embedding if and only if $G'$ has a t-curve drawing with at most $k$ crossings.

Suppose that $G$ has page-number 2. Let $E$ be a t-curve embedding of $G_t$. Let $\rho$ be one of the $t$ clean Jordan curves that witness that $E$ is a t-curve embedding, and let $p$ be a point on $\rho$ that is not a vertex of $G$. Let $\delta$ be a disk with center $p$, small enough so that $\delta$ does not intersect any vertex or edge of $G_t$. Then we can embed $G$ in the interior of $\delta$, with the vertices lying on $\rho \cap \delta$. Moreover, since the 2-page crossing number of $K_{3,3}$ is 1, it follows that we can also draw the $k$ copies of $K_{3,3}$ in the interior of $\delta$, with one crossing per copy, so that all the vertices lie on $\rho \cap \delta$. This yields a t-curve drawing of $G'$ with at most (actually, exactly) $k$ crossings.

For the other direction, suppose that $D'$ is a t-curve drawing of $G'$ with at most $k$ crossings. We note that each copy of $K_{3,3}$ must contribute with exactly 1 crossing. Thus it follows that if we let $G''$ denote the disjoint union $G$ and $G_t$, then the restriction $E''$ of $D'$ to $G''$ is a t-curve embedding.

Let $R := \{\rho_1, \ldots, \rho_t\}$ be a set of clean Jordan curves that witness that $E''$ is a t-curve embedding. We let $E_t$ denote the restriction of $E''$ to $G_t$. Then obviously the collection $R$ witnesses that $E_t$ is a t-curve embedding.

\textbf{Claim.} Let $f$ be any face of $E_t$. Then there is at most one curve in $R$ that intersects $f$.

\textit{Proof.} Let $f$ be any face of $E_t$. By Lemma 4, every face in an embedding of $G_t$ is a triangle, and so $f$ is bounded by a 3-cycle $C$. Let $u, v, w$ be the vertices of $C$.

To prove the claim, first note that at most three curves in $R$ can intersect $f$; this follows simply because $C$ has exactly three vertices, and the curves in $R$ are pairwise disjoint and clean with respect to $E_t$. Suppose that exactly two curves $\rho_i, \rho_j$ in $R$ intersect $f$. Since the curves in $R$ are pairwise disjoint, then it is not possible that each of $\rho_i$ and $\rho_j$ intersects two vertices of $C$. Thus at least one of these curves, say $\rho_i$, must be a loop based on a vertex of $C$, say $u$. The other curve $\rho_j$ either contains both $v$ and $w$, or exactly one of them. Suppose first that $\rho_j$ contains both $v$ and $w$. Thus the scenario is as depicted on the left side of Figure 2.

We can then remove $\rho_i$, and reroute the part of $\rho_j$ inside $f$, so that the resulting curve $\rho'_j$ contains $v, u,$ and $w$, as illustrated on the right side of Figure 2. Hence $(R \setminus \{\rho_i, \rho_j\}) \cup \rho'_j$ is a set of $t-1$ pairwise disjoint clean Jordan curves whose union contains all the vertices of $G_t$. Therefore $G_t$ has a $(t-1)$-curve embedding, contradicting (iii) in Lemma 4. Now, if $\rho_j$ contains exactly one of $v$ and $w$ (say, $v$, without loss of generality), then the scenario is as shown on the left side of Figure 3. In this case we can replace $\rho_i$ and $\rho_j$ by a curve $\rho'_j$ that contains both $u$ and $v$ (as in the right side of Figure 3). Thus $(R \setminus \{\rho_i, \rho_j\}) \cup \rho'_j$ is a set of $t-1$ pairwise disjoint clean Jordan curves whose union contains all the vertices of $G_t$, again contradicting (iii) in Lemma 4.

In the remaining case, exactly three curves $\rho_i, \rho_j, \rho_k$ intersect $f$. In this case each of these curves must contain exactly one of $u, v,$ and $w$, as illustrated on the left side of Figure 4. We can then replace these three curves by a curve $\rho$ contained in $f$, as shown on the right side of Figure 4. Thus $(R \setminus \{\rho_i, \rho_j, \rho_k\}) \cup \rho$ is a set of $t-2$ pairwise disjoint clean Jordan curves whose union contains all the vertices of $G_t$. Hence $G_t$ has a $(t-2)$-curve embedding, (and therefore, a $(t-1)$-curve embedding), contradicting (iii) in Lemma 4. \hfill \Box
Figure 2. The curve $\rho_j$ intersects the (shaded) face $f$, and contains $v$ and $w$. Since the clean curve $\rho_i$ contains $u$, then $\rho_i \setminus \{u\}$ must be contained in $f$. In this case we can replace these two curves by a single curve $\rho'_j$ that contains $u, v,$ and $w$, as shown on the right-hand side.

Figure 3. The curve $\rho_i$ contains $u$, and is otherwise contained in $f$. The curve $\rho_j$ contains $v$, and is otherwise contained in $f$. In this case $\rho_j$ can be re-routed inside $f$, as illustrated on the right-hand side, so that the result is a clean Jordan curve $\rho'_j$ that contains both $u$ and $v$.

Figure 4. If the clean Jordan curves $\rho_i, \rho_j, \rho_\ell$ contain $u, v,$ and $w$, respectively, and each of these curves intersects $f$, then $\rho_i \setminus \{u\}, \rho_j \setminus \{v\},$ and $\rho_\ell \setminus \{w\}$ are contained in $f$, as shown in the left-hand side figure. These three curves can then be replaced by a single curve $\rho$ that contains $u, v,$ and $w$. 

6
Since $G$ and $G_t$ are disjoint, it follows that there is a face $f$ of $E_t$ such that, in $E''$, $G$ is drawn inside $f$. Thus it follows that some curve in $R$ must intersect $f$.

From the Claim, there is exactly one curve $\rho_m$ in $R$ that intersects $f$. Since $G$ is contained in $f$, it follows that all the vertices of $G$ are contained in $\rho_m$. Since $\rho_m$ is clean in $E''$, it follows that $\rho_m$ does not intersect any edge of $G$. Thus $\rho_m$ witnesses that the restriction of $E''$ to $G$ is a 1-curve embedding. Hence we are done, since $G$ has a 1-curve embedding if and only if it has a 2-page embedding. □

4. Concluding remarks

It follows from the proof of Theorem 1 that, for each fixed $t \geq 2$, even the problem of deciding whether a given graph admits a $t$-curve embedding, is already NP-complete. As we have observed, this is also true for $t = 1$, as testing if a graph has pagenumber 2 (which is equivalent to testing if it has a 1-curve embedding) is NP-complete.

We recall that a $p$-page book consists of $p$ halfplanes (the pages) whose boundaries lie on a common line (the spine). In a $p$-page drawing, all the vertices lie on the spine, and each edge (except for its endpoints) lies on a single page [5]. The $p$-page crossing number $b_{cr_p}(G)$ of a graph $G$ is the minimum number of crossings in a $p$-page drawing of $G$ [11].

In the Book Crossing Number entry in [10], Schaefer mentions that testing if a graph $G$ satisfies $b_{cr_p}(G) = 0$ is NP-complete, for every integer $p \geq 2, p \neq 3$ (the case $p = 3$ remains open). We note that analogous arguments to those we used in part (B) of the proof of Theorem 1 can be used to prove the following.

Observation 5. Let $p$ be a fixed integer such that $p \geq 2$ and $p \neq 3$. Then the decision problem “given a graph $G$ and an integer $k$, is $b_{cr_p}(G) \leq k$?” is NP-complete.

It is reasonable to argue that, alternatively to the definition of a $t$-circle drawing, we could obtain a generalization of the definition of a cylindrical drawing by asking that the vertices are contained in $t > 2$ clean concentric circles. To illustrate an issue with such a definition, let us consider drawings of the complete graph in which the vertices are placed on three clean concentric circles. Then there cannot be a vertex in the inner circle and a vertex in the outer circle, as then an edge joining these two vertices would necessarily cross the middle circle. Thus either all the vertices must lie in the union of the middle circle and the outer circle, or in the union of the middle circle and the inner circle. That is, any such drawing of the complete graph is necessarily cylindrical. Thus, for general graphs, such an alternative definition of a $t$-circle drawing is not really more general than the definition of a cylindrical embedding. On the other hand, if one slightly relaxes the condition that the interior of no edge intersects a circle, then we arrive to the radial crossing number [2,9,10] (see also the related notion of the cyclic level crossing number [3]).

References

[1] Bernardo M. Ábrego, Oswin Aichholzer, Silvia Fernández-Merchant, Pedro Ramos, and Gelasio Salazar, Shellable drawings and the cylindrical crossing number of $K_n$, Discrete Comput. Geom. 52 (2014), no. 4, 743–753.
[2] Christian Bachmaier, A radial adaptation of the Sugiyama framework for visualizing hierarchical information, IEEE Trans. Vis. Comput. Graph 13 (2007), no. 3, 583–594.
[3] Christian Bachmaier, Franz J. Brandenburg, Wolfgang Brunner, and Ferdinand Hübner, Global $k$-level crossing reduction, J. Graph Algorithms Appl. 15 (2011), no. 5, 631–659.
[4] Lowell Beineke and Robin Wilson, The early history of the brick factory problem, Math. Intelligencer 32 (2010), no. 2, 41–48.
[5] Frank Bernhart and Paul C. Kainen, The book thickness of a graph, J. Combin. Theory Ser. B 27 (1979), no. 3, 320–331.
[6] Guantao Chen and Xingxing Yu, Long cycles in 3-connected graphs, J. Combin. Theory Ser. B 86 (2002), no. 1, 80–99.
[7] Fan R. K. Chung, Frank Thomson Leighton, and Arnold L. Rosenberg, *Embedding graphs in books: a layout problem with applications to VLSI design*, SIAM J. Algebraic Discrete Methods 8 (1987), no. 1, 33–58.

[8] Frank Harary and Anthony Hill, *On the number of crossings in a complete graph*, Proc. Edinburgh Math. Soc. (2) 13 (1962/1963), 333–338.

[9] Mary L. Northway, *A method for depicting social relationships obtained by sociometric testing*, Sociometry 3 (1940), no. 2, 144–150.

[10] Marcus Schaefer, *The graph crossing number and its variants: A survey*, Electron. J. Combin. (2014), Dynamic Survey 21, 100.

[11] Farhad Shahrokhi, László A. Székely, Ondrej Sýkora, and Imrich Vrtú, *The book crossing number of a graph*, J. Graph Theory 21 (1996), no. 4, 413–424.

Instituto de Matemáticas, Universidad de Antioquia. Medellín, Colombia 050010
*E-mail address*: rodrigo.duque@udea.edu.co

Facultad de Ciencias, Universidad Autónoma de San Luis Potosí. San Luis Potosí, México 78290
*E-mail address*: hernan@fc.uaslp.mx

Facultad de Ciencias, Universidad Autónoma de San Luis Potosí. San Luis Potosí, México 78290
*E-mail address*: cesar.velez@uaslp.mx

Unidad Académica de Matemáticas, Universidad Autónoma de Zacatecas. Zacatecas, México 9800
*E-mail address*: jleanos@matematicas.reduaz.mx

Instituto de Física, Universidad Autónoma de San Luis Potosí. San Luis Potosí, México 78290
*E-mail address*: cmedina@ifisica.uaslp.mx