Breathing mode of a BEC immersed in a Fermi sea

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By analyzing breathing mode of a Bose-Einstein condensate repulsively interacting with a polarized fermionic cloud, we further the understanding of a Bose-Fermi mixture recently realized by Louden et al. [Phys. Rev. Lett. 120, 243403]. We show that a hydrodynamic description of a domain wall between bosonic and fermionic atoms reproduces experimental data of Huang et al. [Phys. Rev. A 99, 041602(R)]. Two different types of interaction renormalization are explored, based on lowest order constrained variational and perturbation techniques. In order to replicate nonmonotonic behavior of the oscillation frequency observed in the experiment, temperature effects have to be included. We find that the frequency down-shift is caused by the fermion-induced compression and rethermalization of the bosonic species as the system is quenched into the strongly interacting regime.

Introduction Quantum mixtures have proved to be a fruitful playground for both theoretical and experimental physicists [1, 2]. For a long time, most of the interest was aimed at various states of superfluid helium [3] and solid-state settings. However, introduction of experimental platform of ultracold gases opened a route towards increasingly better control and fine tuning of investigated systems [4–6]. Widely utilized Feshbach resonances allow to freely adjust interaction between components in a quantum mixture, providing a highly clean environment for precise measurements of many-body and collective excitations [7].

The latter are usually firstly investigated as an early probe of the properties of the system [8, 9]. Two types of collective modes are usually distinguished – surface and compression ones. An excitation is of surface type, when the volume of gas remains unchanged, in contrast to the other type. Surface modes have been utilized to study e.g. collisionless to collisional crossover in both Bose [10, 11] and Fermi [12, 13] gases. On the other hand, the response of the system under compression is usually used to study an equation of state of a given sample [14–18].

It has been shown that intercomponent interaction can cause plethora of effects on collective oscillations of the system, yielding e.g. damping and frequency shifts [19–35]. The particular role of phase separation in such problems has been extensively studied in Bose-Bose mixtures, in which description can be carried out accurately at the mean-field level [36–42]. The introduction of fermionic species provides a greater challenge, as much stronger interaction is needed to overcome the mean-field value of the kinetic energy in the system [43–45]. The involvement of beyond-mean-field corrections cannot be therefore neglected, as shown in the case of repulsive two-component Fermi gas that undergoes phase separation [46–59].

First experiments with phase-separated Bose-Fermi mixtures were performed with the mixtures of 4He and 3He [3]. However, only recent ultracold atomic experiments provide an environment to freely study physics at the thin layer between two interacting components [60–64]. In opposition to the fully overlapping mixture, in which bulk properties manifest, the presence of a domain wall makes it much harder to explore, both theoretically and experimentally. As such settings happen routinely in various physical systems including alloys, glasses, polymers, colloids etc., highly controllable ultracold system provides the hope of achieving and investigating new and interesting quantum phases [65–71].

In this Letter, we study a binary mixture of Bose-Einstein condensate (BEC) of potassium-41 and polarized Fermi sea of lithium-6, that was recently realized and studied by the Innsbruck group [62, 64]. Both components are optically trapped and their interaction is tuned by the means of Feshbach resonances. The boson-boson s-wave interaction, characterized by the scattering length a_{bb}, is kept constant, while interspecies interaction (a_{bf}) is finely tuned, allowing to access phase-separated regime.

The mixture is initially prepared as a very weakly interacting state (a_{bb} ≈ 60a_0 and a_{bf} ≈ 60a_0) in an elongated trap with the aspect ratio of λ = 7.6. The temperature is kept very low for fermions, staying at ca. 0.17T_f for each investigated setting. However, the condensed fraction of bosons is below 0.5, suggesting the need of inclusion of the thermal cloud to the analysis. After thermalization, to excite the breathing mode of the bosonic cloud, Bose-Fermi scattering length is alternately changed by the means of short radio-frequency pulses that switch the internal state of potassium atoms [72, 73]. The interaction is firstly quenched into value of a_{bf} = 700a_0 then after half the radial breathing mode period of unaccompanied bosons, it is switched back to base weak interaction, a_{bf} = 60a_0. Such a procedure is repeated once more and then the gas evolves in the presence of ultimate interaction strength, a_{bf}. Next, the frequency of a breathing mode is measured during couple of oscillation periods.

The theory presented in the original paper [64], based
on two approaches, adiabatic Fermi sea (AFS) and full phase separation (FPS) models, manages to qualitatively reproduce the up-shift of the frequency for weak-to-moderate interaction strengths. However, it quantitatively overestimates the frequency and does not provide an explanation for a nonmonotonic behavior of the curve for a very strong repulsion. By combining nonzero temperature classical fields description of cold Bose gas and hydrodynamics-derived pseudo-Schrödinger representation of fermions, we acquire an approach that allows us to study dynamics of the Bose-Fermi mixture in a fully quantum way. Our method provides a quantitative description of this previously unexplained behavior and suggests a way to further verify its predictions.

The Letter is organized as follows. First, we present the predictions for the breathing mode of fermions and nonzero temperature classical fields approximatively overestimates the frequency and does not provide a way to further verify its predictions. The Letter is closed by the recapitulations and final remarks.

Energy of uniform mixture

The starting point is the interaction energy of fermions, \( \epsilon_b = 6/5 \eta 2/3 = 0.473 \). The latter is necessary to describe \( a + b \). A \( \alpha \) is given in the Supplemental Materials (SM) [81]. The other approach not a perturbative one and in general can provide reliable results even in a strongly interacting regime. Lowest order constrained variational (LOCV) method assumes an explicit symmetric term for the relative two-body correlations of \( s \)-wave interacting particles in the form of Jastrow factor, \( \Psi \sim \Pi_i f_i (r_i - r_j) \) [82–86]. Such a trial function is also widely used in other variational approaches, mainly of quantum Monte Carlo type [70]. It was successfully applied in cold atoms systems, e.g., repulsive Fermi-Fermi and attractive Bose-Fermi mixtures [85, 86].

Within this approximation, the Bose-Fermi interaction energy can be then put in a simple form, \( \epsilon_f = \frac{2 \pi \hbar^2}{m_f} (\eta) \), where \( \eta \) is a numerically evaluated (see details in SM) function of dimensionless parameter \( \eta = (k_f a_f)^{-1} \), where \( k_f = (6 \pi^2 n_f)^{1/3} \). The boson-fermion interaction energy is also renormalized at the perturbative level, yielding \( \epsilon_f = \frac{2 \pi \hbar^2}{m_f} (1 + 4 \Delta(\eta)) \), where \( \Delta(\eta) \) can be numerically evaluated [86]. The details of computations are presented in SM.

The comparison between two approaches allows us to check how important going beyond the mean field is in the considered Bose-Fermi mixture and how well different renormalizations of energy spectrum fare in dealing with the experimental results. We use local density approximation (LDA) to describe the mixture in the harmonic trap, \( V_h = \frac{1}{2} m_b \omega^2 (\rho^2 + \lambda^2 z^2) \) and \( V_f = \frac{1}{2} m_f \omega_f^2 (\rho^2 + \lambda^2 z^2) \), for bosons and fermions, respectively. Then, the three energy spectra of a trapped Bose-Fermi mixture, mean field, LOCV, and VG read:

\[
E_{MF} = \int d^3 r \left( t_f(r) + t_b(r) + \epsilon_b^0(r) + \epsilon_f^0(r) + n_b V_b(r) + n_f V_f(r) \right)
\]

\[
E_{LOCV} = \int d^3 r \left( t_f(r) + t_b(r) + \epsilon_b^{LOCV}(r) + \epsilon_f^{LOCV}(r) + n_b V_b(r) + n_f V_f(r) \right)
\]

\[
E_{VG} = \int d^3 r \left( t_f(r) + t_b(r) + \epsilon_b^0(r) + \epsilon_f^0(r) + \epsilon_f^{VG}(r) + n_b V_b(r) + n_f V_f(r) \right)
\]
Coupled time evolution equations  We decide to represent bosons with the usual Bose field, however in the classical fields approximation (CFA) [87–89]. In the CFA, the quantum Bose field $\hat{\psi}_b(r,t) = \sum_p \hat{\psi}_b^p(r) \hat{a}_p(t)$ describing the bosonic system is replaced by the classical Bose field $\sqrt{N_b} \psi_b(r,t) = \sum_p \hat{a}_p(t) \psi_b(r,t)$. It constitutes an extension of the Bogolyubov approach, in which only $p = 0$ mode is macroscopically occupied. This mode is then described by a single complex amplitude $\alpha_0$, that is governed by the Gross-Pitaevskii equation (GPE). In the CFA, under the physically satisfied assumption of a finite system, each of the classical amplitudes $\alpha$ is macroscopically occupied and evolves under GPE. The $|\alpha_0|^2$, the largest eigenvalue of the averaged single particle density matrix, is interpreted as a condensed fraction of bosons and higher amplitudes are part of the thermal cloud. Such a description have been successfully employed for many problems in the past.

Nonzero temperature behavior is handled by a real time thermalization governed by nonlinear dynamics of GPE. The starting point is an excited bosonic state chosen conveniently for the following evolution. This state is then evolved in a real time to the point of thermalization – understood as a saturation of observables describing the system. The saturated, thermalized bosons undergo small fluctuations, continuously, but slightly changing their state. Then, one of these realizations is taken as a starting point for the analysis.

One of the ways to derive GPE is to start from the hydrodynamic description of Madelung [90]. Similar approach can also be readily utilized in the case of fermions, in the spirit of recent advances in the quantum continuum mechanics [91, 92]. The starting point is to represent fermions by a mean field pseudo-wave function, $\psi_f = \sqrt{N_f} \exp \left( i \frac{\mathbf{r}}{\sqrt{m_f}} \chi \right)$, where $\nabla \chi = v$ gives irrotational velocity field of the collective motion [55, 57]. By assuming that Hamiltonian is of the mean field form (1) and casting the time evolution equations in hydrodynamic (Euler-Lagrange) form, one can obtain coupled nonlinear pseudo-Schrödinger equations by the means of inverse Madelung transformation:

$$i\hbar \partial_t \psi_j = \left[ -\frac{\hbar^2}{2m_j} \nabla^2 + \frac{\hbar^2}{2m_j} \nabla^2 |\psi_j| - \frac{\delta E}{\delta n_j} \right] \psi_j, \quad (2)$$

where $j = \{ b, f \}$.

Such a representation seems to neglect a lot of information about fermionic many-body wave function. However, it can be rigorously shown that it can provide an exact quantum evolution given an appropriate effective Hamiltonian [91, 93]. Nonetheless, it was successfully employed to describe physics at the border between two quantum gases, yielding a quantitative agreement with the experiment [57]. Specifically, such a hydrodynamic-based description provides a very good description of an overlapping part of bosons and fermions at the domain wall, where interaction provides a fast thermalization of the system [59]. Eqs. (2) can be readily solved by split-step methods [94].

Results  We now proceed to describe our results. We study the breathing mode with parameters taken from the experiment and with the excitation procedure following the experimental one. To quantify the interaction strength, critical interaction parameter is introduced, $a_c = \sqrt{\frac{3\eta_{F0} M_{15}}{2(m_0 + m_f) \sqrt{\pi}}}$, where $k_f^0$ is the Fermi wavenumber evaluated at the center of the trap in the noninteracting case. $a_c$ describes the value of the interaction strength, at which phase separation happens in the mean-field calculation. Fig. 1 presents the results for the number of bosons in the condensate equaling $N_b^0 = 1.6 \cdot 10^4$ and number of fermions $N_f = 1 \cdot 10^5$. The total number of bosonic atoms was chosen in such a way that the initial thermalized state has an appropriate condensed fraction for a given temperature. The critical parameter in this case reads $a_c = 619 a_0$ and the breathing mode frequency is normalized to $\omega_b = 2\omega_0$.

In the experiment (stars in the plot) the frequency grows from $\omega / \omega_0 = 1$ for the weakly interacting mixture, $a_c / a_{bf} = 3.5$, starting to slightly diminish at ca. $a_c / a_{bf} = 0.6$. The theory of Ref. [64] (black lines) clearly overestimates the experimental results and does not reproduce nonmonotonicity for a strong interaction.

With our framework, we find what follows:

(i) The zero-temperature version of our method lies closer to the experimental results than AFS and FPS models, however it still overestimates the frequency and does not reproduce the nonmonotonicity. It fares better, because unlike in these two simpler approaches, it also accounts for the dynamical response of the fermionic cloud, allowing for its excitations.

(ii) There is only very small difference between mean-field, VG and LOCV frequencies for a weak-to-moderate interaction (also within finite temperature framework).

The only appreciable (however, still within 15%) mismatch is visible only very close to $a_c / a_{bf} = 0$, where there is no experimental data. It suggests that quantum corrections play much smaller role for Bose-Fermi mixture than for Fermi-Fermi one, for which strong renormalization dependence was predicted and observed even in a weakly interacting regime. We stress that such a lack of dependence may be specific for this particular experiment.

(iii) Introduction of a finite temperature shifts the theoretical curve much closer to the experimental one and manifests down-shift of the frequency for a very strong interaction. For a weak-to-moderate interaction, the thermal cloud frequency stays constant at $\omega / \omega_0 = 1$. As the number of noncondensed bosons is higher than condensed ones, the BEC is dragged and slowed by the thermal cloud, yielding decrease in frequency in comparison
1. The experimental data from Ref. [64] is denoted by stars. The theoretical data from Ref. [64] is indicated by black solid (full phase separation model) and black dashed (adiabatic Fermi sea model) lines. The rest of points comes from combined classical fields and pseudo-wave function model from this work. The blue color indicates zero temperature calculation, while the orange and red colors pertain to finite temperature ones. The black color signifies the oscillation frequency of the thermal cloud associated with the red markers. The circles denote bare mean field calculations, the squares – Viverit-Giorgini perturbation scheme, and the diamonds – lowest order constrained variational approach. For nonzero temperature calculation that involves temperature taken from the experiment [64] (red markers), the initial condensed fraction of bosons is 40% and the ratio of condensed bosons to fermions is $N_b^o/N_f = 0.16$. The orange markers denote lower temperature, with the same $N_b^o/N_f = 0.16$, but with the condensed fraction being 53%. The theory from Ref. [64] clearly overestimates the experimental curve and does not reproduce nonmonotonicity for a strong interaction. So does the zero temperature calculation, however to lesser extent. The full nonzero temperature calculation grasps the experimental data both qualitatively and quantitatively.

For a strong interaction, up-shift of thermal cloud’s frequency is observed. (Left, inset) The breathing mode frequency as a function of number of condensed bosons to number of fermions ratio, $N_b^o/N_f$. The interaction strength is kept constant, at $a_c/a_{bf} = 0.45$. Analogously, the blue markers pertain to the zero temperature model and the red ones to the nonzero one. Again, the former overestimates the experimental data, however to the lesser extent than FPS and APS models. Accounting for a finite temperature allows to reproduce the experimental results. (Right) The fraction of condensed bosons (associated with the red markers from the left figure) as a function of time and the interaction strength. For a weak interaction it stays at the initial value, while for a strong one it increases in time and saturates at some value.

Additionally, we have performed calculations for the other available experimental sets of data and for different values of initial condensed fractions. The obtained results agree with the above-mentioned observations and are presented in the SM.
Recapitulation and outlook. Summing up, we have created and presented a theoretical framework that allows us to effectively study quantum Bose-Fermi mixtures. It is based on classical fields approximation from the side of the bosonic part and on the quantum continuum mechanical or hydrodynamic approach from the fermionic side. It includes beyond mean field corrections and can be further utilized for studying physics of phase-separated states. We have used it to describe recent experimental setting in which the breathing mode of phase-separated Bose-Fermi mixture was analyzed. Not only did it agree with the experimental curve in the weakly coupled regime, but it also reproduced nonmonotonic behavior in a strongly interacting gas, providing an insight into its underlying mechanism. Further research in describing other quantum mixtures seems to be a natural way of continuing presented work.

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SUPPLEMENTAL MATERIALS

1. Energy of the uniform mixture

Here we present detailed schemes of calculation of the given energy spectrum – LOCV and VG.

1.1 Lowest order constrained variational approach

We follow the approach from Ref. [86], however with amendments due to the repulsive character of the mixture we consider. LOCV was firstly constructed and utilized to study dense quantum fluids, e.g. liquid helium, neutron stars etc. [82]. In the context of quantum Bose-Fermi mixtures, the starting point is the Hamiltonian describing the system:

\[ H = - \sum_{i=1}^{N_b} \frac{\hbar^2}{2m_b} \nabla_i^2 \sum_{j=1}^{N_f} \frac{\hbar^2}{2m_f} \nabla_j^2 + \sum_{i,j} U_{bf}(r_i - r_j) + \frac{1}{2} \sum_{i,i'} U_{bb}(r_i - r_{i'}), \tag{3} \]

where $U_{bf}$ and $U_{bb}$ are zero-range pseudopotentials that are characterized by the scattering lengths, $a_{bf}$ and $a_{bb}$, consecutively. The regime that we are interested in is Feshbach-tuned Bose-Fermi interaction that can be comparable to Fermi energy and therefore cannot be treated perturbatively. On the other hand, Bose-Bose interaction is small compared to other energy scales in the system and will be treated perturbatively.

The other base assumption is the Jastrow-Slater form of the trial wave function:

\[ \psi = \prod_{i,j} f(r_i - r_j) \left( \frac{1}{\sqrt{V}} \right)^{N_b} |FS\rangle, \tag{4} \]

where $|FS\rangle$ is the Fermi sea of $N_f$ fermions and Jastrow function $f(r)$ describes two-body correlations between fermions and bosons and is to be variationally minimized.

The main physical assumption is that contribution of far (further than the closest one) neighbors of a given particle to instantaneous potential felt by that particle should be included only into the average field. It implies a neglect of explicit correlations between far atoms, meaning $f$ should tend to value of 1 (no distortion in the wave function) in large distances. Such a criterion is usually handled with two boundary conditions that introduce the healing length $d$ that is to be calculated self-consistently:

\[ f(r = d) = 1 \tag{5} \]
\[ f'(r = d) = 0, \tag{6} \]

assuming that we deal with a spherically symmetric potential.

The energy is evaluated in the so-called LOCV approximation, that takes into account only two-body term in the linked cluster expansion, giving expression in the lowest order $O(f^2 - 1)$:

\[ \epsilon = n_b n_f \int d^3 r f(r) \left[ - \frac{\nabla^2}{2\mu} + U_{bf}(r) \right] f(r). \tag{7} \]

Before $\epsilon$ is minimized with respect to the variations in $f$, the part that comes from the average field and is not related to correlations has to be subtracted. We assume that the potential contributes only to the average field for $r > d$ and this contribution is denoted as $\lambda$. Then, the variational equation for $\epsilon$ can be written as:

\[ \delta \int d^3 r \left[ - f \frac{\nabla^2}{2\mu} + f U_{bf} - \lambda f^2 \right] = 0. \tag{8} \]

The two-body Schrödinger-like equations follows (already assuming spherical symmetry):

\[ \left[ - \frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + U_{bf}(r) \right] r f(r) = \lambda r f(r). \tag{9} \]
One more constraint is needed to have a self-consistent set of equations. We need to choose healing length $d$ in such a way that on average only one particle resides within it:

$$4\pi n_f \int_0^d dr r^2 f^2(r) = 1.$$  \hspace{1cm} (10)

These equations allow to evaluate Bose-Fermi interaction energy as:

$$\epsilon_{bf} = n_b n_f \int_0^d d^3r f(r) \left[ -\nabla^2 + U_{bf}(r) \right] f(r) = \lambda n_b.$$  \hspace{1cm} (11)

This equation can be rewritten as

$$\epsilon_{bf} = \lambda n_b = \frac{\hbar^2}{4\mu} \left( 6\pi^2/2n_f^{2/3} n_b B(\eta) \right),$$  \hspace{1cm} (12)

where $B(\eta)$ is a function of dimensional interaction parameter $\eta = (k_f a_{bf})^{-1}$. The details of numerical evaluation of the function $B(\eta)$ will be presented shortly.

The other part of energy spectrum is a perturbative treatment of the Bose-Bose interaction energy. Within LOCV, the correction to the mean-field energy reads:

$$\Delta \epsilon_{bb}^{LOCV} = \frac{8\pi \hbar^2}{m_{bb}} a_{bb} n_b^2 D(\eta),$$  \hspace{1cm} (13)

where

$$D(\eta) = n_f \int d^3r \left[ f(r) - 1 \right]^2.$$  \hspace{1cm} (14)

Then, the whole energy spectrum of the uniform mixture in LOCV approximation can be written as:

$$\epsilon^{LOCV} = \frac{6^{5/3} \hbar^{10/3} \pi^{4/3}}{20 m_f} n_f^{5/3} + \frac{(6\pi^2/2)^{2/3} \hbar^{2/3} n_f^{2/3} n_b B(\eta)}{4\mu} + \frac{2\pi \hbar^2}{m_b} a_{bb} n_b^2 (1 + 4D(\eta)).$$  \hspace{1cm} (15)

Now, we proceed to numerical evaluation of the function $B(\eta)$ and $D(\eta)$. Eq. 9 is just a usual Schrödinger type equation with a short-range pseudopotential. It yields one bound state and a continuum of scattering states. In Ref. [85] authors analyzed the bound solution, or in a language of polaron physics – the attractive branch of the energy spectrum. We are however interested in the repulsive mixture and therefore we shall investigate a lowest scattering state that satisfies boundary conditions coming from the LOCV approximation.

The effect of the short-range pseudopotential can be written as the Bethe-Peierls boundary condition,

$$\left( \frac{rf'}{rf} \right)'(r = 0) = -\frac{1}{a_{bf}}.$$  \hspace{1cm} (16)

A scattering state, satisfying this boundary condition behaves like $rf(r) \sim \sin(kr + \delta)$, where $k$ is a wavevector connected to the energy $\lambda$ by $\lambda = \hbar^2 k^2 / 2\mu$ and $\tan \delta/k = -a_{bf}$. Let’s introduce a parameter $b$, such that $kb = \arctan ka$. Then, the solution that satisfies $f(d) = 1$ reads:

$$f(r) = \frac{d \sin(k(r - b))}{r \sin(k(d - b))}.$$  \hspace{1cm} (17)

With conditions for a derivative and for a normalization, a set of three equations is to be solved:

$$kb = \arctan ka_{bf}$$  \hspace{1cm} (18)

$$4\pi n_f \int_0^d f^2(r)r^2 dr = 1$$  \hspace{1cm} (19)

$$\frac{a_{bf}}{d} = \frac{(1/kd) \tan kd - 1}{kd \tan kd + 1}.$$  \hspace{1cm} (20)
By introducing $k_1 = kd$ and $k_2 = ka_{bf}$ and bit of algebra, the solution for $k_1(\eta)$ can be obtained by solving single parameter function:

$$\eta^{-3}F_1(k_1) - F_2(k_1) = 0, \quad (21)$$

where

$$F_1(k_1) = \frac{1}{3\pi}k_1^2\left[k_1 - \sin k_1 \cos \left(k_1 - 2 \arctan \left(\frac{\tan k_1 - k_1}{k_1 \tan k_1 + 1}\right)\right)\right] \quad (22)$$

$$F_2(k_1) = \left(\frac{\tan k_1 - k_1}{k_1 \tan k_1 + 1}\right)^3 \sin^2 \left(k_1 - \arctan \left(\frac{\tan k_1 - k_1}{k_1 \tan k_1 + 1}\right)\right) \quad (23)$$

The functions $B(\eta)$ and $D(\eta)$ can then be written as:

$$B(\eta) = 2\eta^2 \left(\frac{\tan k_1(\eta) - k_1(\eta)}{k_1(\eta) \tan k_1(\eta) + 1}\right)^2 \quad (24)$$

$$D(\eta) = 1 + \frac{2}{9\pi}\eta^{-3} \left(\frac{k_1(\eta)}{k_2(\eta)}\right)^3 \left(1 - 6 \frac{k_2(\eta) - (k_1(\eta) + k_2(\eta)) \cos k_1(\eta) + (1 - k_1(\eta) k_2(\eta)) \sin k_1(\eta)}{k_1^2(\eta)(\sin k_1(\eta) - k_2(\eta) \cos k_1(\eta))}\right), \quad (25)$$

where

$$k_2(\eta) = \frac{\tan k_1(\eta) - k_1(\eta)}{k_1(\eta) \tan k_1(\eta) + 1}. \quad (26)$$

To compare how the result compares with another approaches, let us compare the function $B(\eta)$ with quantum Monte Carlo results from Ref. [48]. The comparison is direct, as $B(\eta)$ can be also interpreted as a repulsive polaron energy divided by the Fermi energy. In Fig. 2 the comparison is presented, showing remarkable agreement.

![Graph showing comparison of $B(\eta)$ with QMC results.]

**FIG. 2.** Comparison of the repulsive polaron energy calculated by Quantum Monte Carlo method and in lowest order constrained variational approximation.

### 1.2 Viverit-Giorgini perturbation scheme

Here we invoke results obtained by Ref. [79] and then generalized in Ref. [80]. The contribution to the Bose-Fermi interaction energy obtained in a frame of second-order perturbation theory leads to the following quantum correction to the energy:

$$\epsilon_{bf}(n_b, n_f) = \epsilon_f n_b (n_f a_{bf}^3)^{2/3} A(w, \alpha) = C_{bf} a_{bf}^2 n_b n_f^{4/3} A(w, \alpha), \quad (27)$$
with \( C_{bf} = (6\pi^2)^{2/3}\hbar^2/2m_f \) and dimensionless parameters \( w = m_b/m_f \) and \( \alpha = 16\pi n_b a_{bb}/(6\pi^2)^{2/3}n_f^{2/3} \). The function \( A(w, \alpha) \) is given in a form of integral:

\[
A(w, \alpha) = \frac{2(1 + w)}{3w} \left( \frac{6}{\pi} \right)^{2/3} \int_0^\infty dk \int_{-1}^{+1} d\Omega \left[ 1 - \frac{3k^2(1 + w)}{\sqrt{k^2 + \alpha}} \int_0^1 dq q^2 \frac{1 - \Theta(1 - \sqrt{q^2 + k^2 + 2kq\Omega})}{\sqrt{k^2 + \alpha + wk + 2qw\Omega}} \right],
\]

where \( \Theta(x) \) is the step theta-function. The above formula, Eq. (27), coincides with the results of Ref. [79] for \( \alpha \ll 1 \), i.e. in the limit when the Fermi energy is much larger than the chemical potential of bosons.

2. Coupled Schrödinger-like equations

We start with a hydrodynamic (Madelung) description of bosons and fermions. Each of the species can be described in the form of a classical pseudo-wavefunction:

\[
\psi = \begin{pmatrix} \psi_f \\ \psi_b \end{pmatrix} = \begin{pmatrix} \sqrt{n_f} e^{\frac{m_f}{2\hbar^2} \chi_f} \\ \sqrt{n_b} e^{\frac{m_b}{2\hbar^2} \chi_b} \end{pmatrix},
\]

where \( n_j = \psi_j^\dagger \psi_j, j = \{b, f\} \) are one-particle densities for fermions and bosons and \( \nabla \chi_j = \mathbf{v}_j \) are the velocity fields of the collective motion. The full system Hamiltonian is given by \( H = T_{\text{tot}} + E_{\text{int}} + E_{\text{pot}} \). The total kinetic energy \( T_{\text{tot}} = T + T_c \) consists of the intrinsic kinetic energy \( T \), which in case of fermions we approximate by the Thomas-Fermi-Weizsäcker functional and in case of bosons by \( \frac{k^2}{2m_b} (\nabla \sqrt{n_b})^2 \). The other part is the kinetic energy of the collective motion, \( T_c = \sum_{j=b,f} \int dr \, \frac{m}{2} \mathbf{v}_j^2 \). The interaction and potential energies are obtained by either mean-field, LOCV or VG approaches. One can then write four Euler-Lagrange equations for the system:

\[
\begin{aligned}

\partial_t n_j &= -\nabla (n_j \mathbf{v}_j), \\
m \partial_t \mathbf{v}_j &= -\nabla \left( \frac{\delta E}{\delta n_j} + \frac{m}{2} \mathbf{v}_j^2 \right),
\end{aligned}
\]

where the energy are evaluated with three different procedures:

\[
\begin{aligned}

E_{\text{MF}} &= \int d^3r \, (t_f(r) + t_b(r) + \epsilon_{bf}^0(r) + \epsilon_{bf}^1(r) + n_b V_b(r) + n_f V_f(r)) \\
E_{\text{LOCV}} &= \int d^3r \, (t_f(r) + t_b(r) + \epsilon_{bf}^{\text{LOCV}}(r) + \epsilon_{bf}^{\text{LOC}}(r) + n_b V_b(r) + n_f V_f(r)) \\
E_{\text{VG}} &= \int d^3r \, (t_f(r) + t_b(r) + \epsilon_{bf}^{\text{LOC}}(r) + \epsilon_{bf}^{\text{LOC}}(r) + n_b V_b(r) + n_f V_f(r))
\end{aligned}
\]

The hydrodynamic equations can be however recast into the form of nonlinear Schrödinger equation by the means of inverse Madelung transformation. In the case of the mean-field it yields

\[
\begin{aligned}

i\hbar \partial_t \psi_f &= \left[ -\frac{\hbar^2}{2m_f} \nabla^2 + \frac{4\hbar^2}{9m_f} \frac{\nabla^2 \psi_f}{|\psi_f|^2} + \frac{6^{5/3} \hbar^2 \pi^{4/3}}{12m_f} |\psi_f|^{4/3} + \frac{1}{2} m_f \omega_f^2 (\rho^2 + \lambda_f^2 z^2) + \frac{2\pi \hbar^2}{\mu} a_{bf} |\psi_b|^2 \right] \psi_f, \\
\end{aligned}
\]

\[
\begin{aligned}

i\hbar \partial_t \psi_b &= \left[ -\frac{\hbar^2}{2m_b} \nabla^2 + \frac{1}{2} m_b \omega_b^2 (\rho^2 + \lambda_b^2 z^2) + \frac{4\pi \hbar^2}{m_b} a_{bb} |\psi_b|^2 + \frac{2\pi \hbar^2}{\mu} a_{bf} |\psi_f|^2 \right] \psi_b,
\end{aligned}
\]
Finally we find the condensate frequency (38), we use split-step methods [94]. We use grid with dimensions $\Delta x$ and $\Delta y = \Delta z = 200 a_{bh} = 0.6445 \mu m$.

3. Numerical details and procedure

Here we present numerical details for zero temperature and nonzero temperature calculations. In numerical procedures we use the following units: length $a_{bh}$, energy $\hbar^2/(m_b a^2_{bh})$ and time $m_b a_{bh}^2/\hbar$. To solve equations (32), (33) and (38), we use split-step methods [94]. We use grid with dimensions $m_x = m_y = 128$ and $m_z = 1024$. Spatial steps are $\Delta x = \Delta y = \Delta z = 200 a_{bh} = 0.6445 \mu m$.

3.1 Zero temperature

First, we look for the ground state of the mixture at zero temperature prepared in a weakly interacting state $(a_{bh} \approx 60.9 a_0$ and $a_{bf} \approx 60 a_0)$ in an elongated trap with the aspect ratio of $\lambda = 7.6$. To do so, we solve equations (32), (33) and (38) using imaginary time technique. The time step is $\Delta t = 0.3349 \text{ ns}$. Then, we propagate these equations in real time applying the excitation scheme which exactly follows the experiment. The time step for real time evolution reads $\Delta t = 3.349 \text{ ns}$. After the excitation sequence, evolution still continues and we extract the condensate width according to the following formula:

$$w_b(t) = \sqrt{\int d^3r \left(x^2 + y^2\right) |\psi_b|^2 / \int d^3r |\psi_b|^2}.$$  

Finally we find the condensate frequency $\omega_b$ by fitting the expression given below:

$$w_b(t) = \alpha_b e^{-\gamma_b t} \sin(\omega_b t + \phi_b) + \beta_b + A_b \sin(\Omega_b t + \Phi_b).$$
3.2 Nonzero temperature

To prepare a thermal state we start with the ground state of the mixture initially prepared in a weakly interacting state \( \langle a_b b_b \rangle \approx 60.9 a_0 \) and \( \langle a_f b_f \rangle \approx 60 a_0 \) in an elongated trap with the aspect ratio of \( \lambda = 7.6 \). Then we put additional energy by randomizing the wave function of bosons at each lattice location \( r_j \) according to the following formula:

\[
\psi_b(r_j) \rightarrow \psi_p^b(r_j) = S_r [r_1 \Re(\psi_b(r_j)) + r_2 \Im(\psi_b(r_j))] \hat{t},
\]

where \( S_r \) is the strength of randomization, \( r_1 \) and \( r_2 \) are two random numbers chosen uniformly from \(-1\) to \(1\) interval, \( \psi_b \) is the ground state bosonic wave function and \( \psi_p^b \) is the perturbed wave function. The fermionic pseudo-wave function is unperturbed at the beginning. Then, both wave functions are evolved according to the nonlinear Schrödinger equations. Depending on the case we use the version with standard meanfield terms, with VG corrections or with LOCV terms. The time evolution is conducted until we reach the thermal state. We check that by monitoring the condensate fraction. At the beginning the condensate fraction changes rapidly, then starts to saturate and finally only fluctuates around some final mean value.

In CFA the condensate fraction is extracted from the classical field by solving the eigenvalue problem for the averaged one-particle density matrix. The one particle density matrix for bosons is given by:

\[
\rho_b(r, r', t) = \frac{1}{N_b} \psi_b^*(r, t) \psi_b(r', t).
\]

The eigenvalues give the occupation of modes. According to the Penrose-Onsager criterion the condensate fraction is given by the eigenvalue with the highest occupation and the condensate wave function is given by the eigenfunction corresponding to this mode. However, the one particle density matrix given by (42) is a pure state. The mixed state appears after an averaging procedure:

\[
\bar{\rho}_b = \langle \rho_b(r, r', t) \rangle,
\]

where \( \langle \cdot \rangle \) may involve averaging over space, time, an ensemble or be a combination of mentioned techniques. In our case, we integrate along axial direction of the trap:

\[
\bar{\rho}_b(x, y, x', y', t) = \int dz \psi_b^*(x, y, z, t) \psi_b(x', y', z, t).
\]

After solving the eigenvalue problem we get:

\[
\bar{\rho}_b(x, y, x', y', t) = \sum_k \frac{N_k^b}{N_b} \varphi_k^b(x, y, t) \varphi_k^b(x', y', t),
\]

where the eigenvectors \( \varphi_k^b(x, y, t) \) are functions corresponding to macroscopically occupied modes:

\[
\psi_k^b(x, y, t) = \sqrt{\frac{N_k^b}{N_b}} \varphi_k^b(x, y, t),
\]

and the eigenvalues \( N_k^b / N_b \) give the occupations of modes. The condensate wave function averaged over axial direction is given by:

\[
\psi_0^b(x, y, t) = \sqrt{\frac{N_0^b}{N_b}} \varphi_0^b(x, y, t)
\]

Subtracting the column condensate density \( \rho_0^b = |\psi_0^b(x, y, t)|^2 \) from the diagonal part of the one-particle density matrix one obtains the thermal cloud column density:

\[
\rho_T(x, y, t) = \bar{\rho}_b(x, y, t) - |\psi_0^b(x, y, t)|^2.
\]

Having the thermal state we follow a similar path like at zero temperature. We apply the excitation sequence and then monitor the width of the condensate fraction. What differs is that we also can measure the width of the thermal
cloud. So, we calculate both widths:

\[ w_0^b(t) = \sqrt{\int d^2r \ (x^2 + y^2) \rho_0^b / \int d^2r \ \rho_0^b}, \quad (45) \]

\[ w_T^b(t) = \sqrt{\int d^2r \ (x^2 + y^2) \rho_T / \int d^2r \ \rho_T}. \quad (46) \]

Finally we find the condensate frequency \( \omega_0^b \) and the thermal cloud frequency \( \omega_T \) by fitting expressions given below:

\[ w_0^b(t) = \alpha_0^b e^{-\gamma_0^b t} \sin (\omega_0^b t + \phi_0^b) + \beta_0^b + \Lambda_0^b \sin (\Omega_0^b t + \Phi_0^b), \quad (47) \]

\[ w_T(t) = \alpha_T e^{-\gamma_T t} \sin (\omega_T t + \phi_T) + \beta_T + \sigma_T t + \Lambda_T \sin (\Omega_T t + \Phi_T). \quad (48) \]

4. Additional results

In Fig. 3 we present more sets of theoretical data that pertain to the experimental results from Ref. [64]. The explanation of these results is contained within the caption to that figure.

![FIG. 3. (Left) The breathing mode frequency for a Bose-Einstein condensate immersed in a Fermi sea as a function of interaction parameter \( a_c/a_{bf} \). The experimental data from Ref. [64] is denoted by stars. The theoretical data from Ref. [64] is indicated by black solid (full phase separation model) and black dashed (adiabatic Fermi sea model) lines. The rest of points comes from combined classical fields and pseudo-wave function model from this work. The blue color indicates zero temperature calculation, while red and orange colors pertain to finite temperature ones. The circles denote bare mean field calculations, the squares – Viverit-Giorgini perturbation scheme, and the diamonds – lowest order constrained variational approach. For nonzero temperature calculations, the initial condensed fraction of bosons vary from 20\% (red markers) to 40\% (orange markers). The number of condensed bosons equals 8000 and the number of fermions equals 170000. It yields the ratio of condensed bosons to fermions, \( N_0^b/N_f = 0.047 \). One can see that also in this case, introduction of a finite temperature reproduces the experimental data.](image)