NILPOTENT SPACELIKE JORDEN OSSERMAN
PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. Pseudo-Riemannian manifolds of balanced signature which are both spacelike and timelike Jordan Osserman nilpotent of order 2 and of order 3 have been constructed previously. In this short note, we shall construct pseudo-Riemannian manifolds of signature $(2s, s)$ for any $s \geq 2$ which are spacelike Jordan Osserman nilpotent of order 3 but which are not timelike Jordan Osserman. Our example and techniques are quite different from known previously both in that they are not in neutral signature and that the manifolds constructed will be spacelike but not timelike Jordan Osserman.

1. Introduction

Let $(M, g)$ be a pseudo-Riemannian manifold of signature $(p, q)$. Let

$$S^\pm(M, g) := \{x \in TM : (x, x) = \pm 1\}$$

be the bundles of unit spacelike and unit timelike vectors, respectively. Let $R$ be the associated Riemann curvature tensor. If $x \in T_p M$, then the Jacobi operator $J(x)$ is the self-adjoint linear map of $T_p M$ which is characterized by the identity:

$$(1.a) \quad g(J(x)y, z) = R(y, x, x, z).$$

One says that $(M, g)$ is spacelike Osserman or timelike Osserman if the eigenvalues of $J$ are constant on $S^+(M, g)$ or on $S^-(M, g)$, respectively. These are equivalent notions if $p \geq 1$ and $q \geq 1$ [12] so such manifolds are simply said to be Osserman.

If $p = 0$, and similarly if $q = 0$, then one is in the Riemannian setting. If $(M, g)$ is a rank 1 symmetric space or if $(M, g)$ is flat, then the local isometries of $(M, g)$ act transitively on $S^+(M, g)$ so the eigenvalues of $J$ are constant on $S^+(M, g)$. Osserman [19] wondered if the converse held. Work of Chi [7] and of Nikolayevsky [17] has shown this to be the case if the dimension is different from 8 and 16.

If $p = 1$, and similarly if $q = 1$, then one is in the Lorentzian setting. Blažič, Bokan and Gilkey [6] and García–Río, Kupeli and Vázquez-Abal [9] have shown that Lorentzian Osserman manifolds have constant sectional curvature.

The situation is quite different in the higher signature setting where $p \geq 2$ and $q \geq 2$. There exist Osserman pseudo-Riemannian manifolds which are not symmetric spaces [2, 3, 4, 5, 11]; we refer to [10] for an excellent and quite comprehensive treatment of the subject.

In the higher signature setting, it is natural to impose a more restrictive hypothesis and study the Jordan normal form of the Jacobi operator. We say that $(M, g)$ is spacelike Jordan Osserman or is timelike Jordan Osserman if the Jordan normal form of $J(x)$ is constant on $S^+(M, g)$ or on $S^-(M, g)$, respectively. Relatively few examples of such manifolds are known.

The eigenvalue 0 is distinguished. One says that $(M, g)$ is nilpotent Osserman if $J(x)^{p+q} = 0$ or equivalently if 0 is the only eigenvalue of $J(x)$ for any $x \in TM$. 

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Lemma 2.1. Let $\mathfrak{g}$ be a finite dimensional real vector space which is equipped with a non-degenerate symmetric bilinear form $g(\cdot, \cdot)$ of signature $(p, q)$. Let $R \in \otimes^4 V^\ast$. We say that $R$ is an algebraic curvature tensor if $R$ satisfies the symmetries of the Riemann curvature tensor:

$$R(x, y, z, w) = -R(y, x, z, w),$$
$$R(x, y, z, w) = R(z, w, x, y),$$
$$R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0.$$  

The associated Jacobi operator is then defined using equation (2.a) and the notions spacelike Jordan Osserman and so forth are defined analogously.

Fiedler and Gilkey [8] gave examples of $m$ dimensional pseudo-Riemannian manifolds for any $m \geq 4$ where $n(M) = m - 2$; thus $n(M)$ can be arbitrarily large. However for these examples, $n(x)$ was constant neither on $S^+(M, g)$ or on $S^-(M, g)$ so these manifolds were neither spacelike nor timelike Jordan Osserman.

Results of Gilkey and Ivanova [13] show that if $(M, g)$ is spacelike Jordan Osserman of signature $(p, q)$ where $p < q$, then the Jacobi operator is diagonalizable and hence $(M, g)$ can not be not nilpotent. Thus we suppose $p \geq q$ henceforth. Examples of spacelike and timelike Jordan Osserman manifolds of neutral signature $(s, s)$ which are nilpotent of order 2 have been constructed Gilkey, Ivanova, and Zhang [14] for any $s \geq 2$. Examples of spacelike and timelike Jordan Osserman manifolds of signature $(2, 2)$ which are nilpotent of order 3 have been constructed by García-Riό, Vázquez-Abal and Vázquez-Lorenzo [14]. This brief note is devoted to the proof of the following result:

**Theorem 1.1.** If $s \geq 2$, then there exist pseudo-Riemannian manifolds of signature $(2s, s)$ which are spacelike Jordan Osserman nilpotent of order 3 and which are not timelike Jordan Osserman.

Our examples is quite different in flavor from those described in [11, 14] in several respects. The primary feature is that we are not in the balanced setting where $p = q$; the extra timelike directions play a central role in our construction. Additionally, the examples of [11, 14] are also timelike Jordan Osserman; this is not the case for our examples.

To prove Theorem 1.1 it is convenient to work first in a purely algebraic context. In Section 2 we shall construct a family of algebraic curvature tensors $R$ on a vector space $V$ of signature $(2s, s)$ which are spacelike Jordan Osserman nilpotent of order 3 and which are not timelike Jordan Osserman. We complete the discussion in Section 3 by realizing this family geometrically. Our construction will show that in fact there are many such examples; although we shall use quadratic polynomials to define the metric in question, this is an inessential feature.

## 2. Algebraic curvature tensors

Let $V$ be a finite dimensional real vector space which is equipped with a non-degenerate symmetric bilinear form $g(\cdot, \cdot)$ of signature $(p, q)$. We say that $R$ is an algebraic curvature tensor if $R$ satisfies the symmetries of the Riemann curvature tensor:

$$R(x, y, z, w) = -R(y, x, z, w),$$
$$R(x, y, z, w) = R(z, w, x, y),$$
$$R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0.$$  

The associated Jacobi operator is then defined using equation (2.a) and the notions spacelike Jordan Osserman and so forth are defined analogously.

Let $s \geq 2$. Let $U := \{U_1, \ldots, U_s\}$, $V := \{V_1, \ldots, V_s\}$, and $T := \{T_1, \ldots, T_s\}$ comprise a basis for $\mathbb{R}^{3s}$. We let indices $a, b, c, d$ range from 1 through $s$.

**Lemma 2.1.** Let $g_{ab} = g_{ba}$ be an arbitrary symmetric matrix. Define a metric $g$ of signature $(2s, s)$ on $\mathbb{R}^{3s}$ whose non-zero components are:

$$g(U_a, U_b) = g_{ab}, \quad g(U_a, V_b) = g(V_b, U_a) = \delta_{ab}, \quad g(T_a, T_b) = -\delta_{ab}.$$
Let $R^{(1)}$ and $R^{(2)}$ be algebraic curvature tensors on $\text{Span}\{U_a\}$. Define a 4 tensor $R = R(R^{(1)}, R^{(2)})$ on $\mathbb{R}^3$ whose non-zero entries are

\[
\begin{align*}
R(U_a, U_b, U_c, U_d) & := R^{(1)}(U_a, U_b, U_c, U_d), \\
R(U_a, U_b, U_c, T_d) & = R(U_a, U_b, T_d, U_d) = R(U_a, T_b, U_c, U_d) \\
& = R(T_a, U_b, U_c, U_d) := R^{(2)}(U_a, U_b, U_c, U_d).
\end{align*}
\]

1. $R$ is an algebraic curvature tensor on $\mathbb{R}^3$.
2. If $R^{(2)}(U_a, U_b, U_c, U_d) := \delta_{ab} \delta_{cd} - \delta_{ad} \delta_{bc}$, then $R$ is spacelike Jordan Osserman nilpotent of order 3 and not timelike Jordan Osserman.

**Proof.** The sum of algebraic curvature tensors is again an algebraic curvature tensor. If $R^{(2)} = 0$, then clearly $R$ is an algebraic curvature tensor since we may assume $x, y, z, w \in \mathcal{U}$ in establishing the relations of display (2.a). We may therefore set $R^{(1)} = 0$ and consider only the effect of $R^{(2)}$ in proving assertion (1). In that case, exactly one of the vectors $x, y, z, w$ must be taken from $\mathcal{T}$ and the remaining 3 vectors must be taken from $\mathcal{U}$. Suppose, for example, $x \in \mathcal{T}$ while $y, z, w \in \mathcal{U}$. Then replacing $x$ by the corresponding element $\bar{x} \in \mathcal{U}$ replaces $R$ by $R^{(2)}$ and thus the relations of display (2.a) follow for $R$ because of the corresponding relations for $R^{(2)}$. This proves assertion (1).

The tensor $R^{(2)}$ of assertion (2) is the algebraic curvature tensor of constant sectional curvature +1 with respect to the standard metric $(U_a, U_b) = \delta_{ab}$. Consequently, it is invariant under the action of the orthogonal group $O(s)$.

Expand a spacelike vector $X \in \mathbb{R}^3$ in the form $X = u_a U_a + v_a V_a + t_a T_a$ where we adopt the Einstein convention and sum over repeated indices. Then

\[
g(X, X) = g_{ab} u_a u_b + 2 \delta_{ab} u_a v_b - \delta_{ab} t_a t_b.
\]

If $\bar{u} = 0$, then $g(X, X) \leq 0$. Consequently $\bar{u} \neq 0$. By making an orthogonal rotation in the $U$ vectors and the same orthogonal rotation in the $V$ and in the $T$ vectors and by rescaling $X$, we may suppose without loss of generality that the bases $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{T}$ have been chosen so that the general form of $g$ and $R$ is the same, so that $u_1 = 1$, and so that $u_a = 0$ for $a > 1$. For $1 \leq a, b, c, d \leq s$, define $R^{(2)}_{abcd} := R^{(2)}(U_a, U_b, U_c, U_d)$. Then:

\[
\begin{align*}
(J(X)U_a, U_b) & = C_{ab}, \\
(J(X)T_a, U_b) & = R^{(2)}_{a11b}, \quad (J(X)U_a, T_b) = R^{(2)}_{a11b}, \\
(J(X)U_a, V_b) & = 0, \quad (J(X)U_a, V_b) = 0,
\end{align*}
\]

where $C_{ab} = C_{ba}$ is an appropriately chosen matrix. We then have:

\[
J(X)U_a = C_{ab} V_b - R^{(2)}_{a11b} T_b, \quad J(X)T_a = R^{(2)}_{a11b} V_b, \quad J(X)V_a = 0.
\]

It is now clear that $J(X)^3 = 0$. We have $J(X)X = 0$ and $J(X)V_a = 0$. Since $R^{(2)}_{a11b} = 0$ if $a = 1$ or $b = 1$, $J(X)T_1 = 0$. Set $R^{(2)}_{a11b} = \delta_{ab}$ for $a \geq 2$. Since $u_1 = 1$, \{ $X, U_2, ..., U_s, T_1, ..., T_s, V_1, ..., V_s$ \} is a basis for $V$. Consequently:

\[
\text{Range}(J(X)) = \text{Span}\{J(X)X, J(X)U_2, ..., J(X)U_s, J(X)T_1, ..., J(X)T_s, J(X)V_1, ..., J(X)V_s\}
\]

\[
= \text{Span}\{J(X)U_2, ..., J(X)U_s, J(X)T_2, ..., J(X)T_s\}
\]

\[
= \text{Span}\{C_{2b} V_b - T_2, ..., C_{sb} V_b - T_s, V_2, ..., V_s\}.
\]

The set \{ $C_{2b} V_b - T_2, ..., C_{sb} V_b - T_s, V_2, ..., V_s$ \} is linearly independent. Furthermore:

\[
\begin{align*}
\text{Range}(J(X)) \cap \ker(J(X)) & = \text{Span}\{V_2, ..., V_s\}, \\
\text{Range}(J(X)^2) & = \text{Span}\{V_2, ..., V_s\}.
\end{align*}
\]

It is now clear that $R$ is spacelike Jordan Osserman nilpotent of order 3. Since $J(T_1) = 0$ while $J(U_1 - V_1) = J(U_1) \neq 0$, $R$ is not timelike Jordan Osserman. \(\square\)
3. GEOMETRIC REALIZATIONS

We complete the proof of Theorem 1.1 by showing that the structures of Lemma 2.1 are geometrically realizable. The metrics we shall consider are similar those described in different contexts in [6, 15, 18]. We take coordinates of the form 

\[(u_1, ..., u_s, v_1, ..., v_s, t_1, ..., t_a)\]

on \(\mathbb{R}^{3s}\). Let 

\[U_a := \frac{\partial}{\partial u_a}, \quad V_a := \frac{\partial}{\partial v_a}, \quad \text{and} \quad T_a := \frac{\partial}{\partial t_a}\]

be the associated coordinate frame for the tangent bundle. We let the index \(r\) range from 1 to 3s and index the full coordinate frame 

\[\{e_1, ..., e_{3s}\} := \{U_1, ..., U_s, V_1, ..., V_s, T_1, ..., T_a\}.

Theorem 1.1 will follow from Lemma 2.1 and from the following Lemma:

**Lemma 3.1.** Let \(R^{(2)}\) be a fixed algebraic curvature tensor on \(\mathbb{R}^n\). Define a metric \(g\) of signature \((2s, s)\) on \(\mathbb{R}^{2s,s}\) whose non-zero inner products are given by:

\[g(U_a, U_b) = \psi_{abcd} U_d, \quad g(U_a, V_b) = g(V_b, U_a) = \delta_{ab}, \quad \text{and} \quad g(T_a, T_b) = -\delta_{ab}.

Let \(R^{(1)}_{abc}(u, t) := R(U_a, U_b, T_c, U_d)(u, t)\). Then \(R(u, t) = R(R^{(1)}(u, t), R^{(2)}).

**Proof.** At this point, we change our indexing convention slightly for the remainder of the proof. We shall let indices \(a, b, c\) index elements of \(\mathcal{U}\), indices \(\alpha, \beta, \gamma\) index elements of \(\mathcal{V}\), and indices \(i, j, k\) index elements of \(\mathcal{T}\). Indices \(r, s\) will index the full coordinate basis. By an abuse of notation, we shall set \(\Gamma_{abc} = g(\nabla U_a U_b U_c), \Gamma_{abi} = g(\nabla U_a U_b U_c, T_i)\), etc. We replace an element of \(\mathcal{T}\) by the corresponding element of \(\mathcal{U}\) to define \(\tilde{\psi}_{abc}, \tilde{R}_{abc}, \tilde{R}_{abc}, \tilde{R}_{abc}\), and \(\tilde{R}_{abc}\). The non-zero Christoffel symbols of the metric are:

\[\Gamma_{abc} = \frac{1}{2}(\tilde{\psi}_{bac} + \tilde{\psi}_{acb} - \tilde{\psi}_{bca})t_i.

We raise indices to see:

\[\Gamma_{r_1 r_2}^2 = 0, \quad \Gamma_{r_1 r_2}^i = -\Gamma_{r_1 r_2 i}, \quad \text{and} \quad \Gamma_{r_1 r_2}^\alpha = \Gamma_{r_1 r_2}^\alpha.

The curvature tensor is given by:

\[R_{r_1 r_2 r_3 r_4} = e_{r_1} \Gamma_{r_2 r_3 r_4} - e_{r_2} \Gamma_{r_1 r_3 r_4} + \Gamma_{r_1 r_2 r_4} \Gamma_{r_3 r_4} - \Gamma_{r_2 r_3 r_4},\]

If \(r_5\) indexes an element of \(\mathcal{V}\), then \(\Gamma_{r_5 r_5}^r = 0\) by equation (3.2), while if \(r_5\) indexes an element of \(\mathcal{U}\), then \(\Gamma_{r_5}^r = 0\) by equation (3.3). Thus \(r_5\) must index an element of \(\mathcal{T}\) and consequently, we may express:

\[R_{r_1 r_2 r_3 r_4} = e_{r_1} \Gamma_{r_2 r_3 r_4} - e_{r_2} \Gamma_{r_1 r_3 r_4} + \Gamma_{r_1 r_2 r_4} \Gamma_{r_3 r_4} - \Gamma_{r_2 r_3 r_4} \Gamma_{r_1 r_3}.

Thus by equation (3.4), quadratic terms in \(\Gamma\) can only appear in equation (3.4) if \(r_1, r_2, r_3, \) and \(r_4\) index elements of \(\mathcal{U}\). The only other non-zero curvatures occur when exactly one of \(r_5\) indexes an element of \(\mathcal{T}\) and the remaining \(r_5\) index elements of \(\mathcal{U}\). We may therefore compute the proof by computing:

\[R(U_a, U_b, U_c, T_i) = U_a \Gamma_{bci} - U_b \Gamma_{aci} = \frac{1}{2}(\tilde{\psi}_{abc} - \tilde{\psi}_{bca})

Thus by equation (3.3), quadratic terms in \(\Gamma\) can only appear in equation (3.4) if \(r_1, r_2, r_3, \) and \(r_4\) index elements of \(\mathcal{U}\). We may therefore compute the proof by computing:

\[R(U_a, U_b, U_c, T_i) = U_a \Gamma_{bci} - U_b \Gamma_{aci} = \frac{1}{2}(\tilde{\psi}_{abc} - \tilde{\psi}_{bca})

**Remark 3.2.** It is worth giving a very specific example. Define an inner product \(g\) on \(\mathbb{R}^6\) whose non-zero components are, up to the usual \(\mathbb{Z}_2\) symmetries given by:

\[g(U_1, U_1) = -2u_2 t_2, \quad g(U_2, U_2) = -2u_1 t_1, \quad g(U_1, U_2) = u_1 u_2, \quad g(U_1, V_1) = g(U_2, V_2) = g(T_1, T_1) = -g(T_2, T_2) = 1.

\[4 P. GILKEY AND S. NIKČEVIĆ\]
This manifold has signature $(4, 2)$. It is spacelike Jordan Osserman nilpotent of order 3. It is not timelike Jordan Osserman. Furthermore, it is curvature homogeneous up to order 0 as defined by Kowalski, Tricerri, and Vanhecke [10].

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