Solving the EPR paradox: an explicit statistical model for the singlet quantum state

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We present an explicitly local and realistic statistical model of pseudo-classical hidden variables that reproduces the expectation values and correlations of all quantum observables in Bohm’s system of two photons in their singlet polarization state and, hence, might solve the Einstein-Podolski-Rosen paradox. The model also offers a new insight on the so-called weak values of quantum observables and may become a useful tool for performing numerical simulations of the quantum system.

1. Quantum Mechanics is believed to be the ultimate theoretical framework within which all fundamental laws of Nature are to be formulated. Its postulates have been extensively and accurately tested in a very broad class of physical systems, including optics, atomic and molecular physics, condensed matter physics and high-energy particle physics. Indeed, one of the greatest challenges in physics at present is to formulate Einstein’s general relativity theory of gravitation within this framework.

Nevertheless, there remain crucial questions about the interpretation of Quantum Mechanics that have not been properly understood yet. In particular, it has been known since long ago that the current interpretation of the quantum formalism cannot accommodate together two fundamental principles of modern physics usually taken for granted, namely, the principle of locality and the principle of physical realism. The clash between these two principles was first noticed by Einstein, Podolsky and Rosen [1] and it has been known since then as the EPR paradox. The paradox is commonly formulated as follows [2]:

Consider a massive particle that decays into two distinguishable photons, which travel in opposite directions along the Z axis. The two photons are assumed to be emitted in their singlet polarization state:

$$|\Psi> = \frac{1}{\sqrt{2}}(|\uparrow \downarrow > - |\downarrow \uparrow >),$$  \hspace{1cm} (1)

where \{ |\uparrow >, |\downarrow > \} is an orthonormal linear basis of eigenstates of the \(\sigma_Z\) operator in the single particle polarization Hilbert space. If the photons travel freely, their polarization state remains entangled once they have travelled far away from each other. In this state the polarizations of the photons are perfectly anti-correlated when they are measured along parallel directions: if we would perform a measurement of the polarization of photon A, we would know with certainty also the polarization of photon B along that direction. Hence, accepting the principle of locality implies that we can gain certainty on the polarization of photon B along any direction without perturbing it in any sense. Furthermore, accepting the principle of physical realism implies then that the polarization properties of photon B were set at emission. Obviously, the same could be said about the polarization of photon A. Nevertheless, according to the current interpretation of Quantum Mechanics nothing can be said with certainty about the polarization of each of the photons from their wavefunction at emission [1]. Therefore, it is claimed that the description of the physical system provided by the wavefunction is not complete.

The description of the quantum system would be complete if we could interpret the wavefunction as a statistical mixture of classical states defined in some more fundamental hidden variables phase space, so that a measurement of the polarization of any of the two photons would imply nothing but a mere update of our knowledge about the actual state of the system. Unfortunately, Bell’s theorem [3] rules out the possibility to build such a statistical classical interpretation of the wavefunction based on the currently accepted notions of physical realism and locality.

Formally, Bell’s theorem can be stated as follows. Let us assume that Bohm’s two photons system could be described as a statistical mixture of classical states \(\{\lambda\} \in \mathcal{S}\) in a hidden variables phase space \(\mathcal{S}\), such that: 1) each hidden state has a well defined probability \(\rho(\lambda)\) and 2) in each hidden state the spin polarization of each of the photons \(s_{\Omega_0}^{(A)}(\lambda), s_{\Omega_0}^{(B)}(\lambda)\) have well defined values, either +1 or -1, along any directions \(\Omega_0, \Omega_0'\) perpendicular to their direction of flight. Furthermore, the perfect anti-correlation between the polarization of the photons when they are measured along the same direction demands \(s_{\Omega_0}^{(B)}(\lambda) = -s_{\Omega_0}^{(A)}(\lambda)\). Therefore, the expected correlation between the polarization of the photons along any two arbitrary directions is given in these models by:

$$E(s_{\Omega_0}^{(A)} s_{\Omega_0'}^{(B)}) = - \int d\lambda \rho(\lambda) s_{\Omega_0}^{(A)}(\lambda) s_{\Omega_0'}^{(A)}(\lambda).$$  \hspace{1cm} (2)

Bell’s theorem states that for any three arbitrary directions \(\Omega_0, \Omega_0'\) and \(\Omega_0''\) the following inequality holds:

$$|E(\Omega_0, \Omega_0') - E(\Omega_0, \Omega_0'')| \leq 1 + E(\Omega_0', \Omega_0'').$$  \hspace{1cm} (3)

The proof of this statement proceeds as follows:
The importance of Bell’s theorem lies on the fact that the correlations between the spin polarizations of the two photons predicted by Quantum Mechanics $E(s_0^{(A)}|s_0^{(B)}) = \langle \Psi | \sigma_0^{(A)} \cdot \sigma_0^{(B)} | \Psi \rangle = -\cos(\Omega_0 - \Omega_0')$ are not constrained by this inequality. For example, for $\Omega_0' - \Omega_0 = \pi/4$ and $\Omega_0'' - \Omega_0 = 3\pi/4$ we find that $|E(\Omega_0, \Omega_0') - E(\Omega_0, \Omega_0'')| = |\cos(\pi/4) - \cos(3\pi/4)| = \sqrt{2}$, while $1 + E(\Omega_0', \Omega_0'') = 1 - \cos(\pi/2) = 1$.

Bell’s theorem is widely accepted as a strict proof of the impossibility to integrate together the principles of locality and physical realism within the quantum mechanical framework. In this paper we notice, nonetheless, that the proof of the theorem relies on certain implicit assumptions that are not necessary consequences of these two basic principles. In particular, the proof implicitly assumes that the physical properties of the photons can be defined independently of the frame of reference in which they are described. We follow this observation and present an explicitly local and realistic statistical model of pseudo-classical hidden variables that avoids the constraints imposed by Bell’s theorem and successfully reproduces the average values, two-points correlations and weak values of all quantum observables in Bohm’s system of two photons in their singlet polarization state $|1\rangle$.

2. We consider a statistical system whose phase space consists of an infinitely large number of equally probable states distributed over the unit circle $S_1$. An observer that strongly measures the polarization of either one of the photons, say photon $A$, along certain arbitrary direction $\Omega_0$ fixes a reference on this circle and we assume that the number density distribution of states over the circle is given by

$$g(\omega_A) = \frac{1}{4} |\sin(\omega_A)|,$$

where $\omega_A \in [-\pi, \pi]$ is an angular coordinate defined with respect to the chosen reference direction $\Omega_0$.

By symmetry considerations we must demand that this number density distribution of states remains invariant under a rotation by an angle $\Delta \Omega_0$ of the reference frame set by the observer, $\Omega_0 \rightarrow \Omega_0' = \Omega_0 + \Delta \Omega_0$. Therefore, the angular coordinates of these hidden states must transform $\omega_A \rightarrow \omega_A'$ as follows (see Fig. 1):

- If $\Delta \Omega_0 \in [0, \pi)$,

$$\omega_A' = \begin{cases} 
q(\omega_A) \cdot \cos(-\cos(\Delta \Omega_0) - \cos(\omega_A)) + \cos(\omega_A), 
& \text{if } -\pi \leq \omega_A \leq \Delta \Omega_0 < \pi, \\
q(\omega_A) \cdot \cos(+\cos(\Delta \Omega_0) + \cos(\omega_A)), 
& \text{if } \Delta \Omega_0 - \pi \leq \omega_A < 0, \\
q(\omega_A) \cdot \cos(+\cos(\Delta \Omega_0) - \cos(\omega_A)) + \cos(\omega_A), 
& \text{if } 0 \leq \omega_A < \Delta \Omega_0, \\
q(\omega_A) \cdot \cos(-\cos(\Delta \Omega_0) + \cos(\omega_A)) + \cos(\omega_A), 
& \text{if } \Delta \Omega_0 \leq \omega_A < +\pi,
\end{cases}$$

where

$$q(\omega_A) = \frac{1}{2}(\omega_A - \Delta \Omega_0 \mod(-[\pi], \pi)),$$

and the function $y = \cos(x)$ is defined in his main branch, such that $y \in [0, \pi]$ while $x \in [-1, +1]$.

It is straightforward to prove that the transformation fulfills the following boundary conditions: $\omega_A = 0 \rightarrow \omega_A' = -\Delta \Omega_0$, $\omega_A = \Delta \Omega_0 \rightarrow \omega_A' = 0$, $\omega_A = \pm\pi \rightarrow \omega_A' = (\Delta \Omega_0 + \pi) \mod([\pi, \pi])$, $\omega_A = (\Delta \Omega_0 - \pi) \mod([-\pi, \pi]) \rightarrow \omega_A' = \pm\pi$ and leaves the number density distribution of states $|4\rangle$ invariant, as

$$d|\omega_A'\rangle |\sin(\omega_A')| = |d\cos(\omega_A')| = \frac{|d\cos(\omega_A)|}{|\sin(\omega_A)|}.$$
rotation of the reference direction chosen by the observer of photon B by an angle $\Delta \Omega_0$, then $\omega_0'$ transform into $\omega_A$ by a rotation of the reference direction chosen by the observer of photon A by an angle $-\Delta \Omega_0$.

We can now properly define a strong measurement of the polarization of photon A along the reference direction $\Omega_0$ as a test of the sign of the angular coordinate $\omega_A$ of the *hidden* state of the system in the fixed frame. That is,

$$S^{(A)}_{\Omega_0}(\omega_A) = \begin{cases} +1, & \text{if } \omega_A \in [0, \pi), \\ -1, & \text{if } \omega_A \in [-\pi, 0). \end{cases}$$  \hspace{1cm} (7)

Obviously, each of one of the two possible outputs happen with probability 1/2.

Similarly, a strong measurement of the polarization of photon B along the reference direction $\Omega'_0$ tests the sign of the angular coordinate $\omega'_B$ of the *hidden* state. That is,

$$S^{(B)}_{\Omega'_0}(\omega'_B) = \begin{cases} +1, & \text{if } \omega'_B \in [0, \pi), \\ -1, & \text{if } \omega'_B \in [-\pi, 0). \end{cases}$$  \hspace{1cm} (8)

Using the transformation laws (5), (6) it is straightforward to check that,

$$S^{(B)}_{\Omega'_0}(\omega_A) = \begin{cases} +1, & \text{if } \omega_A \in [\Delta \Omega_0 - \pi, \Delta \Omega_0), \\ -1, & \text{if } \omega_A \in [-\pi, \Delta \Omega_0 - \pi) \cup [\Delta \Omega_0, \pi), \end{cases}$$  \hspace{1cm} (9)

where we have assumed, without any loss of generality, that $\Delta \Omega_0 \in [0, \pi)$. Therefore, the four possible outputs of the two strong measurements define a partition of the *hidden* phase space into four coarse subsets,

$$(S^{(A)}_{\Omega_0} = +1; S^{(B)}_{\Omega'_0} = +1) \iff \omega_A \in [0, \Delta \Omega_0)$$

$$(S^{(A)}_{\Omega_0} = +1; S^{(B)}_{\Omega'_0} = -1) \iff \omega_A \in [\Delta \Omega_0, \pi)$$

$$(S^{(A)}_{\Omega_0} = -1; S^{(B)}_{\Omega'_0} = +1) \iff \omega_A \in [\Delta \Omega_0 - \pi, 0)$$

$$(S^{(A)}_{\Omega_0} = -1; S^{(B)}_{\Omega'_0} = -1) \iff \omega_A \in [-\pi, \Delta \Omega_0 - \pi),$$

whose probabilities to happen are, therefore:

$$p(+1,+1) = \int_0^{\Delta \Omega_0} g(\omega_A) \, d\omega_A = \frac{1}{4} (1 - \cos(\Delta \Omega_0)),$$

$$p(+1,-1) = \int_0^{\pi} g(\omega_A) \, d\omega_A = \frac{1}{4} (1 + \cos(\Delta \Omega_0)),$$

$$p(-1,+1) = \int_{\Delta \Omega_0 - \pi}^{\pi} g(\omega_A) \, d\omega_A = \frac{1}{4} (1 + \cos(\Delta \Omega_0)),$$

$$p(-1,-1) = \int_{-\pi}^{\Delta \Omega_0 - \pi} g(\omega_A) \, d\omega_A = \frac{1}{4} (1 - \cos(\Delta \Omega_0)).$$

Thus, this model reproduces the probabilities and the correlations for Bohm’s two photons system in their singlet polarization state as predicted by Quantum Mechanics,

$$E(S_{\Omega_0}, S_{\Omega'_0}) = 2 \int_0^{\Omega'_0 - \Omega_0} g(\omega_A) \, d\omega_A - 2 \int_{\Omega'_0 - \Omega_0}^{\pi} g(\omega_A) \, d\omega_A$$

$$= 4 \int_0^{\Omega'_0 - \Omega_0} g(\omega_A) \, d\omega_A - 1 = -\cos (\Omega'_0 - \Omega_0).$$

In order to clarify how our model avoids the constraints set up by Bell’s theorem we try to rewrite the general proof of the theorem that we provided in section 1. First we write

$$|E(S_{\Omega_0}, S_{\Omega'_0}) - E(S_{\Omega_0}, S_{\Omega''_0})| = 4 \int_{\Omega'_0 - \Omega_0}^{\Omega''_0 - \Omega_0} \, d\omega_A g(\omega_A) =$$

$$= 1 + 4 \int_{\Omega'_0 - \Omega_0}^{\Omega''_0 - \Omega_0} \, d\omega_A g(\omega_A) - 1,$$

where we have assumed without any loss of generality that $\Omega'_0 - \Omega_0, \Omega''_0 - \Omega_0 \geq 0$ and $\Omega'_0 - \Omega_0 \geq \Omega''_0 - \Omega_0$. Then, we notice that

$$4 \int_0^{\Omega'_0 - \Omega_0} \, d\omega_A g(\omega_A) - 1 =$$

$$= 4 \int_0^{\int_{\Omega'_0 - \Omega_0}^{\Omega''_0 - \Omega_0} \, d\omega_A g(\omega_A) - 1}$$

is not necessarily equal to

$$E(S_{\Omega''_0}, S_{\Omega'_0}) = 4 \int_0^{\Omega''_0 - \Omega'_0} \, d\omega_A g(\omega_A) - 1$$

and, therefore, Bell’s inequality does not necessarily hold. In other words, the number of states $4 \int_{\Omega'_0 - \Omega_0}^{\Omega''_0 - \Omega_0} \, d\omega_A g(\omega_A)$ between the two fixed external directions $\Omega'_0$ and $\Omega''_0$ is a function of the third reference direction $\Omega_0$ in which we choose to describe the system. This number of states can be directly related to the correlation $E(S_{\Omega''_0}, S_{\Omega'_0})$ only in
the particular case in which this third reference direction coincides with either one of the former two, \( \Omega_0 = \Omega_0' \) or \( \Omega_0 = \Omega_0'' \).

3. In this hidden phase space of finely resolved states we can now define detailed polarization properties for each one of the two photons. The definition is explicitly local, although it depends on the reference direction chosen by the observer. For example, we define the polarization properties of photon A in the frame of an observer that chooses the direction \( \Omega_0 \) as reference, as follows:

\[
\begin{align*}
  s^{(A)}_{\Omega_0}(\omega_A) &= \begin{cases} +1, & \omega_A \in [0, \pi), \\ -1, & \omega_A \in [-\pi, 0). \end{cases} \\
  s^{(A)}_{\Omega_0'}(\omega_A) &= \begin{cases} -\frac{1}{\tan \omega_A}, & \omega_A \in [0, \pi), \\ +\frac{1}{\tan \omega_A}, & \omega_A \in [-\pi, 0). \end{cases} \\
  s^{(A)}_Z(\omega_A) &= \begin{cases} +\frac{\sin(\omega_A - \Delta \Omega_0)}{-\sin \omega_A}, & \omega_A \in [0, \pi), \\ -\frac{\sin(\omega_A - \Delta \Omega_0)}{\sin \omega_A}, & \omega_A \in [-\pi, 0). \end{cases}
\end{align*}
\]

where \( s^{(A)}_{\Omega_0} \) denotes the polarization component along the reference direction set by the observer, \( s^{(A)}_Z \) denotes the polarization component along the photon’s travelling direction and \( s^{(A)}_{\Omega_0'} \) denotes the polarization component along their orthogonal, right-oriented direction. The polarization of the photon along any other direction is defined linearly with respect to these three. For example, the polarization along direction \( \Omega_0'' = \Omega_0 + \Delta \Omega \) is given by:

\[
s^{(A)}_{\Omega_0''}(\omega_A) = \cos(\Delta \Omega_0) \cdot s^{(A)}_{\Omega_0}(\omega_A) + \sin(\Delta \Omega_0) \cdot s^{(A)}_{\Omega_0'}(\omega_A) = \begin{cases} +\frac{\sin(\omega_A - \Delta \Omega_0)}{-\sin \omega_A}, & \omega_A \in [0, \pi), \\ -\frac{\sin(\omega_A - \Delta \Omega_0)}{\sin \omega_A}, & \omega_A \in [-\pi, 0). \end{cases}
\]

We can now define weak measurements as low precision tests of the polarization components of the photons in this finely resolved phase space, followed by strong measurements of any complete set of commuting observables \( \{\sigma^{(A)}_{\Omega_0}, \sigma^{(B)}_{\Omega_0'}\} \). We state by convention that these properties cannot be experimentally accessed with high precision in a single measurement. Nonetheless, we can obtain a high precision measurement of these properties by repeating the weak measurement on many identically prepared systems. The price to be paid for this gained precision is that these properties are necessarily measured only on average over each one of the four different coarse subsets of the phase space \([10]\) defined by these outputs \( S^{(A)}_{\Omega_0} = \pm 1, S^{(B)}_{\Omega_0'} = \pm 1 \) of the two strong measurements. It is straightforward to test that these average values reproduce the weak values of quantum observables as defined in Quantum Mechanics for each of the four possible options for the post-selection conditions:

\[
\begin{align*}
  \int_0^{\Delta \Omega_0} d\omega_A \quad & g(\omega_A) \quad S^{(A)}_{\Omega_0, \Omega_0'} Z(\omega_A) \\
  \int_0^{\Delta \Omega_0} d\omega_A \quad & g(\omega_A) \quad S^{(A)}_{\Omega_0, \Omega_0'} Z(\omega_A) \\
  \int_0^{\Delta \Omega_0} d\omega_A \quad & g(\omega_A) \quad S^{(A)}_{\Omega_0, \Omega_0'} Z(\omega_A) \\
  \int_0^{\Delta \Omega_0} d\omega_A \quad & g(\omega_A) \quad S^{(A)}_{\Omega_0, \Omega_0'} Z(\omega_A)
\end{align*}
\]

where \( \sigma^{(A)}_{\Omega_0}, \sigma^{(A)}_{\Omega_0'}, \sigma^{(A)}_Z \) are the quantum operators that represent the polarization components of photon A along the corresponding directions.

Finally, we need to define the dynamics of the polarization properties of the photons in these hidden states in the general case in which either one of them, say photon A, interacts with the external world. We do so in terms of the Heisenberg representation of their corresponding operators, \( \sigma^{(A)}(t) = e^{+iH_A t}\sigma^{(A)}e^{-iH_A t} \) where \( H_A \) is the hamiltonian of the interaction, by analogy with the pseudo-classical paths that we introduced in \([4, 5]\). This definition is explicitly local as the polarization properties of the other photon do not get modified by this interaction, \( \sigma^{(B)}(t) = e^{+iH_A t}\sigma^{(B)}e^{-iH_A t} = \sigma^{(B)} \).

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