MODULAR FORMS AND ALMOST LINEAR DEPENDENCE OF GRADED DIMENSIONS

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Abstract. For every positive integral level $k$ we study arithmetic properties of certain holomorphic modular forms associated to modular invariant spaces spanned by graded dimensions of $L_{\hat{sl}_2}(k\Lambda_0)$-modules. We found a necessary and sufficient condition for their vanishing and showed that these modular forms resemble classical Eisenstein series $E_{2k+2}(\tau)$. Furthermore, we derived similar results for $M(p,p')$ Virasoro minimal models, thus generalizing some results of Mortenson, Ono and the author.

1. Introduction

Automorphic and modular forms are omnipresent in rational conformal field theories (RCFTs). The most distinguished automorphic functions that arise from a RCFT are graded dimensions (or simply, characters) of irreducible modules of the underlying vertex operator algebra $V$. Let us recall here that the graded dimension of a $V$-module $M$ is defined as the graded trace

\begin{equation}
\text{ch}_M(q) = \text{tr}|_M q^{L(0)} - c/24,
\end{equation}

where $L(0)$ is a Virasoro generator and $c$ is the central charge ($q = e^{2\pi i \tau}$, for $\tau$ in the upper half-plane).

In [M2], [M3] and [MMO] we introduced and studied certain remarkable modular forms associated to modular invariant vector spaces coming from rational vertex operator algebras. In fact, under some mild conditions, most of our results from [M2] hold for general modular invariant spaces. We briefly recall this construction here. Let $V$ be a rational vertex operator algebra such that modular invariance holds for $V$-modules (see [Zh] for some sufficient conditions). Consider the modular invariant vector space $V$ spanned by characters of $V$-modules. Fix an ordered basis of $V$ and let

\begin{equation}
F_V(\tau) = \frac{W'_V(\tau)}{W_V(\tau)},
\end{equation}

where $W'_V$ and $W_V$ are certain Wronskians associated to the same ordered basis (see Section 2 for details). If $F_V$ is nonzero we will write $F_V(\tau)$ for the normalization of $F_V$ with leading coefficient 1. In [MMO] we proved that $F_{L(c_2,2k+1)}(\tau)$, $k \neq 6i^2 - 6i + 1$, resembles the classical Eisenstein series

\begin{equation}
E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \frac{n^{2k-1}q^n}{1-q^n}, \quad k \geq 2.
\end{equation}

More precisely we showed (resp. conjectured) that $F_{L(c_2,2k+1,0)}(\tau)$ satisfy (i) and (ii) (resp. (iii)) from the following list of properties which are known to hold for Eisenstein series:

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(i) (Holomorphicity) For every $k \geq 2$, $E_{2k}(\tau)$ is a holomorphic modular form of weight $2k$.

(ii) (Clausen–von Staudt congruences) For prime $p = 2k + 1 \geq 5$, $E_{2k}(q)$ is $p$-integral. More importantly

$$E_{2k}(q) \equiv 1 \pmod{p}.$$ 

(iii) (Rankin-Swinnerton-Dyer [RSD]) All zeros of $E_{2k}(\tau)$ in the fundamental region lie on boundary arc from $e^{2\pi i/3}$ to $e^{\pi i/2}$ (i.e., on the $j$-line the zeros are inside the interval $[0, 1728]$).

Our purpose in this short note is to study arithmetic properties of $F_V$ for two important families of rational vertex operator algebras: those associated to affine Lie algebra $\widehat{sl}_2$, denoted by $L_{\widehat{sl}_2}(k\Lambda_0)$, $k \in \mathbb{N}$ and vertex operator algebras associated to $\mathcal{M}(p, p')$ Virasoro minimal models, denoted by $L(c_{p,p'},0)$, where

$$c_{p,p'} = 1 - \frac{6(p - p')^2}{pp'}, \quad (p, p') = 1, \quad p \geq 2, \quad p' \geq 2.$$ 

For more about these vertex operator algebras see [LL]. To simplify the notation we shall write $L(k\Lambda_0)$ instead of $L_{\widehat{sl}_2}(k\Lambda_0)$.

As we already noticed in [MMO] and [M3], a nontrivial task is to determine precisely the vanishing condition for $F_V(\tau)$. Our first result establishes this for $F_{L(k\Lambda_0)}$.

**Theorem 1.1.** For every $k \geq 1$, $F_{L(k\Lambda_0)}$ is a holomorphic modular form. Moreover, $F_{L(k\Lambda_0)}(\tau) = 0$ if and only if $k = 2i^2 - 2$ for some $i \geq 1$.

Similarly,

**Theorem 1.2.** For every $p$ and $p'$, $F_{L(c_{p,p'},0)}$ is a holomorphic modular form. We have $F_{L(c_{p,p'},0)}(\tau) = 0$ if and only if $(p, p') = (2p^2, 3p'^2)$.

The previous theorem has been established for $L(c_{2,2k+1},0)$, $k \geq 2$ in [MMO]. In the process of proving Theorems [1.1] and [1.2] an important ingredient is an application of $q$-series identities among irreducible characters of the form

$$\sum_{i=1}^{s} m_i c_{M_i}(\tau) = C \neq 0,$$ 

where $M_i$ are irreducible $V$-modules, $C$ is a constant and $m_i \in \mathbb{Z}$. The equation (1.2) is what we refer to as an "almost linear dependence" relation among graded dimensions $c_{M_i}$.

In order to prove (ii) for $F_V$, we have to prove the $p$-integrality first. This is a nontrivial problem because in some cases the leading term of $F_V$ is divisible by $p$.

**Theorem 1.3.** For every prime $p = 2k + 3 \geq 5$, $F_{L(k\Lambda_0)}(\tau)$ is $p$-integral.

This result is far from being obvious, simply because the leading coefficient of $F_{L(k\Lambda_0)}$ is divisible by $p = 2k + 3$ for every $p$. Based on overwhelming numerical evidence, Theorem [1.1] and Theorem [1.3] we conjecture:

**Conjecture 1.4.** For $k \neq 2i^2 - 2$, $i \geq 1$ and prime $p = 2k + 3 \geq 5$ we have

$$F_{L(k\Lambda_0)}(q) \equiv 1 \pmod{p}.$$ 

(1.3)
Moreover, the zeros of $F_{L(k,\Lambda_0)}$, in the fundamental domain, lie on boundary arc from $e^{2\pi i/3}$ to $e^{\pi i/2}$.

For $M(p, p')$ minimal models the property (iii) fails to hold in general (e.g., $F_{L(c_3,\tau, 0)}$). Because of complexity of computation we do not have enough numerical evidence which would support a precise conjecture for $F_{L(c_3,\tau, 0)}$. Nevertheless, we did observe that properties (ii) and (iii) seem to hold for unitary minimal models $M(m+2, m+3)$.

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## 2. Irreducible characters of $L(k,\Lambda_0)$–modules and $L(c_p, p', 0)$-modules.

In this section we are following [K]. Let $k$ be a positive integer (level). For every $i = 1, \ldots, k+1$, let $L((k-i+1)\Lambda_0 + (i-1)\Lambda_1)$ be the integrable highest weight $\hat{sl}_2$-module of level $k$ with highest weight $(k-i+1)\Lambda_0 + (i-1)\Lambda_1$. For simplicity we shall write $L((k-i+1)\Lambda_0 + (i-1)\Lambda_1)$ instead of $L((k-i+1)\Lambda_0 + (i-1)\Lambda_1)$. It is known that $L(k,\Lambda_0)$ has a natural vertex operator algebra structure, but we will not use this fact in the rest of the text. The conformal vector $\omega_k \in L(k,\Lambda_0)$ is given by

$$\omega_k = \frac{1}{2(k+2)} \left( \frac{h(-1)^2}{2} + e(-1)f(-1) + f(-1)e(-1) \right),$$

where $1$ is the vacuum vector and $\{e, f, h\}$ is the standard basis of the finite-dimensional simple Lie algebra $\mathfrak{sl}_2$. The central charge of $L(k,\Lambda_0)$ is given by

$$c_k = \frac{3k}{k+2},$$

and the lowest conformal weight of $L(k, i-1)$ is

$$h_{k,i} = \frac{i^2 - 1}{4(k+2)},$$

where $i = 1, \ldots, k+1$. Thus,

$$h_{k,i} - c_k/24 = \frac{i^2}{4(k+2)} - \frac{1}{8}.$$ (2.1)

The homogeneous specialization in the Weyl-Kac character formula yields (cf. [K]): For every positive integer $k$, and $i = 1, \ldots, k+1$, we have

$$\sum_{n \in \mathbb{Z}} nq^{n^2/24(k+2)}$$

$$ch_{k,i}(q) := \text{tr}|_{L(k, i-1)} q^{L(0) - c/24} = \sum_{n \in \mathbb{Z}} \frac{nq^{n^2/24(k+2)}}{\eta(\tau)^3},$$ (2.2)

where $\eta(\tau) = q^{1/24} \prod_{i=1}^{\infty} (1 - q^i)$ is the Dedekind eta-functions.
Let \( p \geq 2 \) and \( p' \geq 2 \) be two relatively prime positive integers. We have the following well-known formula for the graded dimensions of \( L(c_{p,p'},0) \)-modules (cf. [RC]).

\[
\text{ch}_{p,p'}^{r,s}(q) := \text{tr}|_{L(c_{p,p'},0)}q^{L(0)-c_{p,p'}/24} = \sum_{n \in \mathbb{Z}} q^{\frac{(2npp'+p'r-ps)^2}{4pp'}} - q^{\frac{(2npp'+p'r+ps)^2}{4pp'}},
\]

where \( 1 \leq r \leq p-1 \) and \( 1 \leq s \leq p'-1 \).

3. Modular forms and \( SL(2,\mathbb{Z}) \)-modules

Form now on we will be using the Ramanujan’s derivative \((\frac{d}{dq})\). Let \( \{f_1, ..., f_m\} \) be a basis of the modular invariant space spanned by characters of a rational vertex operator algebra \( V \). Let also \( W_V(\tau) = W(\frac{d}{dq})(f_1, ..., f_m) \), where \( W(\frac{d}{dq})(\cdot) \) denotes the Wronskian determinant with respect to \( \frac{d}{dq} \). We will denote by \( W_V \) a multiple of \( W(\frac{d}{dq})(f_1, ..., f_m) \) with the leading coefficient 1. This normalization is important in the theory of mod \( p \) modular forms [Se]. Similarly, we will denote by \( W'_V = W(\frac{d}{dq})(f'_1, ..., f'_m) \) the Wronskian of derivatives of \( f_i \), and by \( W'_V \) its normalization (if nonzero) with the leading coefficient being 1. Now, \( W_V \) and \( W'_V \) are determined only up to a nonzero constant, while \( W'_V \), \( W'_V \),

\[
F_V(\tau) := \frac{W'_V(\tau)}{W_V(\tau)},
\]

and \( F_V := \frac{W'_V}{W} \) do not depend on the choice of a basis of \( V \). If \( W(\frac{d}{dq})(f_1, ..., f_m) \) has no zeros in the upper half-plane, the quotient \( F_V(\tau) \) is a holomorphic modular form of weight \( 2m \) (see [M2], [M3]).

The following result is from [M2]

**Theorem 3.1.** For every \( k \geq 1 \),

\[
W_{L(k\Lambda_0)}(q) = \eta(q)^{2k(k+1)}.
\]

Consequently, \( F_{L(k\Lambda_0)} \) is a holomorphic modular form of weight \( 2k+2 \) (possibly zero).

Similarly, we have a result from [M1] (see also [M2] for a different proof).

**Theorem 3.2.** For every \( p \) and \( p' \) as above

\[
W_{L(c_{p,p'},0)}(q) = \eta(q)^{\frac{(p-1)(p'-1)(pp'-p-p'-1)}{2}}.
\]

Thus, \( F_{L(c_{p,p'},0)} \) is a holomorphic modular form of weight \( \frac{(p-1)(p'-1)(pp'-p-p'-1)}{4} \) (possibly zero).

4. Vanishing results for \( F_{L(k\Lambda_0)} \) and triangular numbers

Let us recall a classical Jacobi’s \( q \)-series identity

\[
\prod_{n=1}^{\infty} (1-q^n)^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1)q^{n(n+1)/2}.
\]

This \( q \)-series identity has a nice representation theoretic interpretation. We will need the following lemma.
Lemma 4.1. For every positive integer \( i \geq 2 \),
\[
\prod_{n=1}^{\infty} (1 - q^n)^3 = \sum_{j=0}^{i-1} \sum_{m=0}^{\infty} (4mi + 2j + 1)(-1)^j q^{(2mi+j)(2mi+j+1)/2}
\]
\[
+ \sum_{j=0}^{\infty} \sum_{m=1}^{\infty} (4mi - 2j - 1)(-1)^{j+1} q^{(2mi-j)(2mi-j-1)/2}.
\]
(4.2)

Proof. For every \( n \in \mathbb{N} \) let \( n \equiv j \pmod{2i} \), where \( j \in \{-i, ...,-1,0,1,...,i-1\} \). Now, we apply (4.1) and rewrite the sum \( \sum_{n=0}^{\infty} (-1)^n (2n+1)q^{n(n+1)/2} \) modulo \( 2i \) and sum over \( j \) as above. \( \square \)

Firstly, we classify all \( k \), such that for some \( i \), \( ch_{k,i}(q) \) has a nonzero leading constant term and consequently only positive integer powers. It is easy to see that this is equivalent to
\[
h_{k,i} = \frac{c_k}{24},
\]
which holds if and only if \( 2i^2 = (k+2) \), or equivalently \( k = 2i^2 - 2 \), for some \( i \geq 2 \). Moreover, for \( k = 2i^2 - 2 \)
\[
h_{2i^2-2,m} - \frac{c_{2i^2-2}}{24} \in \mathbb{N},
\]
if and only if \( m = i(2j+1), \ j = 0,...,i \).

Theorem 4.2. We have \( F_{L(k\Lambda_0)}(q) = 0 \) if and only if \( k = 2i^2 - 2 \) for some \( i \geq 1 \).

Proof. We rewrite the irreducible characters (2.2) in the following form:
\[
ch_{k,i}(q) = \sum_{m \in \mathbb{Z}} \frac{(i+2m(k+2))q^{(i+2m(k+2))^2/4(k+2)}}{\eta(\tau)^3}.
\]
(4.3)

Now, let us use the fact that
\[
ch_{k,i}(q) \in a_0 + q\mathbb{Z}_{\geq 0}[[q]], \ a_0 \neq 0
\]
if and only if \( k = 2i^2 - 2 \) and for such \( k \), for \( j = 0,...,i-1 \), we have
\[
ch_{(2i^2-2),(2j+1)i}(q) \in \mathbb{Z}_{\geq 0}[[q]].
\]

Now, for \( j = 0,...,i-1 \)
\[
ch_{(2i^2-2),(i(2j+1))}(q) = \sum_{m \in \mathbb{Z}} \frac{i(4mi + 2j + 1)q^{(j+2m)(j+2m+1)/2}}{(q;q)_\infty^3}.
\]

Let us rewrite the numerator. Clearly,
\[
\sum_{m \in \mathbb{Z}} i(4mi + 2j + 1)q^{(j+2m)(j+2m+1)/2} = i \sum_{m=0}^{\infty} (4mi + 2j + 1)q^{(j+2m)(j+2m+1)/2}
\]
\[
+i \sum_{m=0}^{\infty} (-1)(4mi + 4i - 2j - 1)q^{(2m+2i-2)(2m+2i-2j-1)/2}.
\]
(4.4)
Now,
\[
\sum_{j=0}^{i-1} (-1)^j \sum_{m \in \mathbb{Z}} i(4mi + 2j + 1)q^{\frac{(i+2m)(j+2mi+1)}{2}} = \sum_{j=0}^{i-1} \sum_{m=0}^{\infty} (-1)^j (4mi + 2j + 1)q^{\frac{(i+2m)(j+2mi+1)}{2}} + \sum_{j=0}^{i-1} \sum_{m=0}^{\infty} (-1)^j (4mi + 4i - 2j - 1)q^{\frac{(2mi+2i-1)(2mi+2i+1)}{2}}.
\]

By Lemma 4.1 and formula (4.4) we have
\[
\sum_{j=0}^{i-1} (-1)^j ch_{2i^2-2j(2i+1)}(q) = i.
\]

Therefore \(W_L((2i^2-2)\Lambda_0)(q)\) is zero.

Now, we prove the converse. If \(W_L'(k\Lambda_0)\) is zero, then there exist \(ch_{k,i}(q), i \in I \subseteq \{0, 1, \ldots, k\}\) such that
\[
\sum_{i \in I} m_i ch_{k,i}(\tau) = 0.
\]

On the other hand, \(ch_{k,i}(q), i \in I\) are linearly independent, so
\[
(4.5) \quad \sum_{i \in I} m_i ch_{k,i}(\tau) = C \neq 0,
\]
for some constant \(C\). The previous equation implies that there exists a subset \(J \subseteq I\), such that \(ch_{k,j}(q), j \in J\) admits only positive integer powers of \(q\). From (4.5) it follows that
\[
\text{ord}_{\infty}(ch_{k,j}) = 0,
\]
for some \(j\). Therefore, \(k = 2i^2 - 2\) for some \(i\).

5. VANISHING RESULTS FOR \(F_L(c_{p,p'},0)\) AND PENTAGONAL NUMBERS

In this section we will give necessary and sufficient conditions for vanishing of \(F_L(c_{p,p'},0)\). We will use approach from the previous section and [MMO]. The following lemma gives a necessary condition for the vanishing.

Lemma 5.1. If \(F_L(c_{p,p'},0)(q) = 0\), then \(pp' = 6m^2\), for some \(m \in \mathbb{N}\).

Proof. As in the previous section, it is not hard to see that the vanishing of \(F_L(c_{p,p'},0)\) implies that there exists a pair \((r, s)\), such that \(\text{ord}_{\infty}(ch_{p,p'}^{r,s}) = 0\). Now the equation \(h_{p,p'}^{r,s} - \frac{c_{p,p'}}{24} = 0\) implies the wanted condition. \(\square\)
The previous lemma gives the following four possibilities on \( p \) and \( p' \):

\[
\begin{align*}
  p &= \tilde{p}^2, & p' &= 6\tilde{p}^2, \\
  p &= 6\tilde{p}^2, & p' &= \tilde{p}^2, \\
  p &= 3\tilde{p}^2, & p' &= 2\tilde{p}^2, \\
  p &= 2\tilde{p}^2, & p' &= 3\tilde{p}^2.
\end{align*}
\]

We will rule out the first case a little bit later. Now, let \( p = 2\tilde{p}^2 \) and \( p' = 3\tilde{p}^2 \). We shall classify all \( \text{ch}^{r,s}_{p,p'}(q) \) with positive integer powers of \( q \). A simple computation shows that \( \text{ch}^{r,s}_{p,p'}(q) \) has positive integer powers of \( q \) if and only if

\[
6(p'r - sp)^2 = pp'(24k + 1), \quad \text{for some } k \geq 0.
\]

Now, in Lemma 5.1 we already established that \( pp' = 6m^2 \). Thus, \((p'r - sp)^2 = m^2(24k + 1)\) implies that \( 24k + 1 \) is a perfect square. Equivalently, \( k = \frac{3l^2 + l}{2} \) for some \( l \in \mathbb{Z} \). In this case

\[
\text{ord}_{\infty}(\text{ch}^{r,s}_{p,p'}(q)) = \frac{3l^2 + l}{2}.
\]

Now we have to classify all \((r,s)\)-pairs satisfying (5.1). The equation

\[
(p'r - sp)^2 = p^2p'^2(24k + 1)
\]

adds an additional constraint on \( r \) and \( s \). Let us consider the case \( p = 2\tilde{p}^2, p' = 3\tilde{p}^2 \) for \( \tilde{p} \) and \( \tilde{p}' \) relatively prime. It follows then that \( \tilde{p}|r \) and \( \tilde{p}'|s \), so that \( s = \tilde{p}'s' \) and \( r = \tilde{p}'r' \), for some \( r' \) and \( s' \) such that \( 1 \leq r' \leq 2\tilde{p} - 1 \) and \( 1 \leq s' \leq 3\tilde{p}' - 1 \). If we substitute everything in (5.2) and factor \( \tilde{p}\tilde{p}' \), we obtain \((3r'\tilde{p}' - 2s'\tilde{p})^2 = (24k + 1)\), which holds if and only if \((3r'\tilde{p}' - 2s'\tilde{p}) \equiv \pm 1 \pmod{6}\). Clearly, \( \tilde{p} \equiv \pm 1 \pmod{3} \) and \( \tilde{p}' \equiv 1 \pmod{2} \). Now if \( s' \equiv -1 \pmod{3} \), then because of the symmetry \( \text{ch}^{\tilde{p}r',\tilde{p}'s'}_{2\tilde{p}^2,3\tilde{p}'^2}(q) = \text{ch}^{\tilde{p}'r,\tilde{p}s}_{2\tilde{p}^2,3\tilde{p}'^2}(q) \), it suffices to consider only \( r' \) and \( s' \) such that \( r' \) is odd and \( s' \equiv 1 \pmod{3} \). This completes the classification of \( \text{ch}^{r,s}_{2\tilde{p}^2,3\tilde{p}'^2}(q) \) with positive integer powers.

The following result is a generalization of a series of identities for \( \mathcal{M}(2, 3(2k - 1)^2) \) Virasoro minimal model characters from [MMO]. Essentially the same formula was obtained recently by Mukhin [Mu] by using similar methods, but apparently motivated by a different circle of ideas (cf. [BE]).

**Proposition 5.2.** The following \( q \)-series identity holds among characters of \( \mathcal{M}(2\tilde{p}^2, 3\tilde{p}'^2) \) minimal models

\[
\sum_{r'=1, r' \equiv 1 \pmod{2}} \sum_{s'=1, s' \equiv 1 \pmod{3}} (-1)^{3r'\tilde{p}' - 2s'\tilde{p}' + 1} \text{ch}^{r',s'}_{2\tilde{p}^2,3\tilde{p}'^2}(q) = 1.
\]

**Proof.** The idea is the same as in the previous chapter and [MMO] so we will omit some details. Let us recall Euler’s Pentagonal Number Theorem [A]:

\[
\eta(\tau) = \sum_{k \in \mathbb{Z}} (-1)^{k} q^{\frac{(6k+1)^2}{24}}.
\]

\(1\). \(\mathcal{M}(p, p') \) and \(\mathcal{M}(p', p) \) are the same minimal models, so there are actually only two cases to consider.
Proof. It suffices to consider the case (5.5) \[ \chi_{2p^2, 3p^2}(q) = \eta^{-1}(\tau) \sum_{m \in \mathbb{Z}} \left( q^{(12p^2 m + 3p^2 - 2p^2)2} - q^{(12p^2 m + 3p^2 + 2p^2)2} \right). \]

Notice that all powers of \( q \) in the numerator of (5.5) are pentagonal numbers shifted by \( \frac{1}{24} \). Now, for \( r' \equiv 1 \pmod{2} \) and \( s' \equiv 1 \pmod{3} \), the numbers \( 3r'p' - 2s'p \) and \( 3r'p' + 2s'p \) are all distinct (2p\( p' \) values in total) and congruent to \( \pm 1 \) modulo 6. Thus

\[
\left\{ \begin{array}{l}
(6(2p^2 m + 3r'p' - 2s'p)^2)/24 : m \in \mathbb{Z}, 1 \leq r' \leq 2p - 1, r' \equiv 1 \pmod{2}; 1 \leq s' \leq 3p - 1, s' \equiv 1 \pmod{3} \\
(6(2p^2 m + 3r'p' + 2s'p)^2)/24 : m \in \mathbb{Z}, 1 \leq r' \leq 2p - 1, r' \equiv 1 \pmod{2}; 1 \leq s' \leq 3p - 1, s' \equiv 1 \pmod{3} \\
\end{array} \right.
\]

\[ = \left\{ (6(p^2 + i)^2)/24 : -pp' < i \leq pp', m \in \mathbb{Z} \right\} = \left\{ (6k + 1)^2/24, k \in \mathbb{Z} \right\}. \]

Finally, the formula (5.5) gives

\[
\sum_{r' = 1 \atop r' \equiv 1 \pmod{2}}^{2p - 1} \sum_{s' = 1 \atop s' \equiv 1 \pmod{3}}^{3p - 1} (-1)^{3r'p' - 2s'p + 1} \chi_{2p^2, 3p^2}(q) = \frac{\tilde{p}p'}{\eta(\tau)} \sum_{t = -\tilde{p}p' + 1 \atop m \in \mathbb{Z}} (-1)^{t} q^{(6(2p^2 m + i + 1)^2)/24}.
\]

The last expression equals 1 by the Euler’s identity (5.4).

\[ \square \]

Lemma 5.3. If \( p = \tilde{p}^2 \) and \( p' = 6\tilde{p}^2 \), then \( F_{L(c, p', 0)} \neq 0 \).

Proof. It suffices to consider the case \( p = \tilde{p}^2 \) and \( p' = 6\tilde{p}^2 \). Then \( \tilde{p} \equiv \pm 1 \pmod{6} \). Here the q-series \( \chi_{p^2, 6p2}(q) \) with integer powers are given by

\[
\chi_{p^2, 6p2}(q) = \eta^{-1}(\tau) \sum_{m \in \mathbb{Z}} \left( q^{(12p^2 m + 6p^2 - 6p^2 + p')2}/24 - q^{(12p^2 m + 6p^2 + 2s'p)^2}/24 \right),
\]

where \( 1 \leq r' \leq \tilde{p} - 1 \) and \( 1 \leq s' \leq 6\tilde{p} - 1 \). Now, we have to show that no linear combination of numerators in (5.6) equals \( \eta(\tau) \). Thus, it suffices to show that there exists \( k \in \mathbb{Z} \), such that \( q^{(6k + 1)^2}/24 \) does not appear in the numerator of (5.6) for any admissible choices of \( r' \) and \( s' \). Because of \( (p, p') = 1 \), we have \( \tilde{p} \equiv \pm 1 \pmod{6} \). But equations \( 12\tilde{p}^2 m + r'p' - 6s'\tilde{p} = \tilde{p} \) and \( 12\tilde{p}^2 m + r'p' + 6s'\tilde{p} = \tilde{p} \) have no solutions in the given range for \( r' \) and \( s' \), thus it is impossible to find a linear combination of characters \( \chi_{p^2, 6p2} \) which equals to 1.

\[ \square \]

Theorem 5.4. The modular form \( F_{L(c, p', 0)} \) is zero if and only if \( p = 2\tilde{p}^2 \), \( p' = 3\tilde{p}^2 \).

Proof. If \( p = 2\tilde{p}^2 \) and \( p' = 3\tilde{p}^2 \) the result follows from Lemma 5.1, the discussion after Lemma 5.2 and Lemma 5.3.

\[ \square \]

Remark 5.5. For fixed \( p \) and \( p' \), the q-series identities obtained in Proposition 5.3 is the only almost-linear dependence relation of the form

\[
\sum_{i \in \mathfrak{f}} m_i \chi_{p, p'}(q) = 1, \ m_i \neq 0.
\]
Otherwise the irreducible characters would be linearly dependent, which is false.

6. \textit{p-integrality of } $F_{L(k\Lambda_0)}$, $p = 2k + 3$

As usual, we say that $a \in \mathbb{Q}$ is $p$-integral if $v_p(a) \geq 0$. For two formal $q$-series

\begin{align*}
A(q) &= \sum_{n \in \mathbb{C}} a(n)q^n \in \mathbb{Q}\{q\}, \\
B(q) &= \sum_{n \in \mathbb{C}} b(n)q^n \in \mathbb{Q}\{q\}
\end{align*}

with $p$-integral coefficients we will write

\[ A(q) \equiv B(q) \pmod{p}, \]

if

\[ a(n) \equiv b(n) \pmod{p}, \quad \text{for every } n. \]

\textbf{Lemma 6.1.} For $k \geq 2$ and $p = 2k + 3$ prime, the $q$-series $W_{L(k\Lambda_0)}(q)$, $W_{L(k\Lambda_0)}(q)$ and $W'_{L(k\Lambda_0)}(q)$ are $p$-integral.

\textbf{Proof.} The infinite product $W_{L(k\Lambda_0)}(\tau)$ (cf. Theorem 3.1) is clearly $p$-integral. Furthermore $4(k+2)$ is not divisible by $p$, so it follows that $W_{L(k\Lambda_0)}(\tau)$ and $W'_{L(k\Lambda_0)}$ are $p$-integral as well. \qed

It is not clear at all that the $q$-series $W'_{L(k\Lambda_0)}(q)$ is $p$-integral.

For a formal series $F(y) \in \mathbb{Q}\{y, q\}$, let

\[ [y^k]F(y) := \text{Coeff}_{y^k}F(y) \in \mathbb{Q}\{q\}. \]

\textbf{Definition 6.2.} Let $F(q)$ be a (formal) $q$-series. Then the $s$-th moment of $F(q)$ is defined as

\[ \left( \frac{q^{d_2}}{y} \right)^s F(q). \]

\textbf{Proposition 6.3.} (a) For every $k \geq 2$, and $p = 2m + 1$ prime we have the following congruence

\[ m! [y^m] \left( \frac{\eta(qey)}{\eta(q)} \right)^3 \equiv 2^{-3m} \pmod{p}. \]  

(b) For every $m$ the left hand side in (6.1) is a quasimodular form.

\textbf{Proof.} By using Jacobi’s formula we have

\[ \eta^3(qey) = \sum_{n=0}^{\infty} (-1)^n (2n + 1) e^{y(2n+1)^2/8} q^{(2n+1)^2/8}. \]

Thus,

\[ m! [y^m] \eta(qey)^3 = \sum_{n=0}^{\infty} (-1)^{n} \frac{(2n + 1)^{2n+1}}{8^m} q^{(2n+1)^2/8} \]

\begin{equation}
\equiv 2^{-3m} \sum_{n=0}^{\infty} (-1)^{n} (2n + 1) q^{(2n+1)^2/8} \pmod{p}.
\end{equation}
Now,
\[ m! |y^m_\eta| \left( \frac{\eta(yq)}{\eta(q)} \right)^3 \equiv 2^{-3m} \pmod{p}. \]

Part (b) follows from the fact that the logarithmic derivative of \( \eta(\tau) \) is (up to a non-zero constant) the quasimodular form \( E_2(\tau) \) and the ring of quasimodular forms is closed under the differentiation \( \left( q \frac{d}{dq} \right) \) (cf. [KZ2]). \( \square \)

**Example.** Let \( E_{2m,3}(\tau) \) denote the normalization of the series on the left hand side of (6.2) such that the leading coefficient in the \( q \)-expansion is one, then
\[
E_{2,3}(\tau) = E_2(\tau),
E_{4,3}(\tau) = \frac{5}{3} E_2(\tau) - \frac{2}{3} E_4(\tau),
E_{6,3}(\tau) = \frac{35}{9} E_2(\tau)^3 - \frac{14}{3} E_2(\tau) E_4(\tau) + \frac{16}{9} E_6(\tau).
\]

Let us denote the numerator in (4.3) by \( \theta_{k,i}(\tau) \), so that
\[(6.3) \quad \theta_{k,i}(\tau) = \sum_{m \in \mathbb{Z}} (i + 2m(k + 2)) q^{(i+2m(k+2))^2/4(k+2)}.
\]
Let \( p = 2k + 3 \) be prime. Clearly, \( \text{GCD}(4(k+2), p) = 1 \).

**Lemma 6.4.** For every \( i = 1, \ldots, k + 1 \), we have
\[(6.4) \quad \theta_{k,i}^{(k+1)}(\tau) \equiv (4(k+2))^{-k-1} \theta_{k,i}(\tau) \pmod{p}.
\]

Now we are ready to prove

**Theorem 6.5.** For every \( k \geq 1 \) such that \( p = 2k + 3 \) is prime, \( W_{L(k\Lambda_0)}' \) (and henceforth \( F_{L(k\Lambda_0)} \)) is \( p \)-integral.

**Proof.** It suffices to prove
\[(6.5) \quad \frac{W_k'(\tau)}{W_k(\tau)} \equiv 0 \pmod{p}
\]
and
\[(6.6) \quad v_p(a_0) = 1, \quad \text{where} \quad \frac{W_k'}{W_k} = a_0 + a_1 q + \cdots.
\]
We first prove (6.6). Notice that
\[ p \not| h_{k,i} - \frac{c_i}{24}, \]
for \( i = 1, \ldots, k \), but \( p \) divides
\[ h_{k,k+1} - \frac{c_k}{24} = \frac{(2k+3)k}{8(k+2)}. \]
Thus, the leading coefficient \( a_0 \) in the \( q \)-expansion of \( W_{L(k\Lambda_0)}' \) is divisible by \( p \). In fact,
\[ a_0 = \prod_{i=1}^{k+1} \left( h_{k,i} - \frac{c_i}{24} \right), \]
so that $v_p(a_0) = 1$. It is easy to see that the leading coefficient in the $q$-expansion of $W_{L(k\Lambda_0)}$ is (up to a sign) $\prod_{1 \leq m < n \leq k+1} \frac{m^2 - n^2}{4(k + 2)}$, which is not divisible by $p = 2k + 3$. This proves (6.6).

To prove (6.5) we will use some row operations on the determinant $W'_{L(k\Lambda_0)}$. Firstly, by the Leibnitz rule

$$\text{ch}_{k,i}^{(r)}(\tau) = \sum_{j=0}^{r} \binom{r}{j} \theta_{k,i}^{(j)}(\tau) \left( \frac{1}{(\eta(\tau))^3} \right)^{(r-j)}.$$ 

By applying a sequence of row operations, we first rewrite the determinant $W'_{L(k\Lambda_0)}(\tau)$ so that in the $j$-th row and $i$-th column we have

$$\frac{\theta_{k,i}^{(j)}(\tau)}{(\eta(\tau))^3} + a_j(\tau) \theta_{k,i}(\tau),$$

where $a_j(\tau)$ does not depend on $i$. For instance, in the first row the entries are now

$$\frac{\theta_{k,i}(\tau)'}{(\eta(\tau))^3} + \frac{1}{(\eta(\tau))^3} \theta_{k,i}(\tau)' \left( \frac{1}{(\eta(\tau))^3} \right)' \left( \frac{1}{(\eta(\tau))^3} \right)' \left( \frac{1}{(\eta(\tau))^3} \right)', \quad i = 1, \ldots, k + 1,$$

so

$$a_1(\tau) = \left( \frac{1}{(\eta(\tau))^3} \right)'.$$

In the second row the entries are

$$\frac{\theta_{k,i}(\tau)''}{(\eta(\tau))^3} + \left( \left( \frac{1}{(\eta(\tau))^3} \right)'' - 2\eta(\tau)^3 \left( \left( \frac{1}{(\eta(\tau))^3} \right)' \right)^2 \right) \theta_{k,i}(\tau), \quad i = 1, \ldots, k + 1,$$

thus

$$a_2(\tau) = \left( \frac{1}{(\eta(\tau))^3} \right)'' - 2\eta(\tau)^3 \left( \left( \frac{1}{(\eta(\tau))^3} \right)' \right)^2.$$

Clearly, $a_j(\tau)$ can be defined recursively. Let

$$A_j(q) = (\eta(\tau))^3 a_j(\tau).$$

Consider an exponential generating function

$$A(q) = \sum_{n=0}^{\infty} A_n(q) \frac{y^n}{n!}.$$ 

We claim that

(6.7) $\sum_{n=0}^{\infty} A_n(q) \frac{y^n}{n!} = - \left( \frac{\eta(q)}{\eta(e^y q)} \right)^3.$

To see this notice that the exponential generating function for moments of $\frac{1}{(\eta(q))^3}$ is given by

(6.8) $\left( \frac{\eta(q)}{\eta(e^y q)} \right)^3.$
where the \( j \)-th moment of \( \frac{1}{\eta(q)} \) is
\[
\eta(q)^3 \left( \frac{1}{\eta(q)^3} \right)^{(j)}.
\]
Now, the generating function \( A(q) \) is just \((-1)^j\) times the reciprocal of the generating function for the moments (6.8). This can be seen from the recursion formula for \( A_j(q) \)'s. Now we have a closed expression for \( a_j(\tau) \), so in the \( j \)-th row the entries are
\[
\theta^{(j)}_{k,i}(\tau) = 1728 \frac{E_3(\tau)^{3}}{E_3(\tau) - E_6(\tau)} \quad \text{for} \quad i=1,\ldots,k+1.
\]
Let us focus at the \((k+1)\)-st row. The formula (6.9) together with Proposition 6.3 and Lemma 6.4 imply
\[
\theta^{(k+1)}_{k,i}(\tau) + a_{k+1}(\tau)\theta^{(k)}_{k,i}(\tau) \equiv 4(k+2)^{-k-1}\frac{\theta^{(k)}_{k,i}(\tau)}{\eta^3(\tau)} - 2^{-3(k+1)}\frac{\theta^{(k)}_{k,i}(\tau)}{\eta^3(\tau)} \pmod{p}.
\]
Finally, the formula
\[
(4(k+2))^{-k-1} - 2^{-3(k+1)} \equiv 0 \pmod{p}
\]
implies
\[
W_k' \equiv \frac{W_k'}{W_k} \equiv 0 \pmod{p}.
\]
The proof now follows.

\[\square\]

**Conjecture 6.6.** For every \( k \in \mathbb{N} \), and \( p = 2k + 3 \geq 5 \) prime,
\[
\mathcal{F}_{L(k\Lambda_0)}(q) \equiv 1 \pmod{p}.
\]

7. **The zeros of \( G(\mathcal{F}_{L(k\Lambda_0)}, j) \)**

Let us recall that every holomorphic modular form \( f(\tau) \) can be uniquely expressed as
\[
\begin{align*}
    f(\tau) &= \Delta(t)E_4^3(\tau)E_6^3(\tau)G(f,j(\tau)),
\end{align*}
\]
where
\[
    j(\tau) = \frac{1728E_4^3(\tau)}{E_4^3(\tau) - E_6^3(\tau)},
\]
\( G(f,j) \) is a polynomial of degree \( \leq t \) and
\[
k = 12t + 4\delta + 6\epsilon,
\]
where \( 0 \leq \delta \leq 2 \) and \( 0 \leq \epsilon \leq 1 \). It is known that the \( j \)-function defines a one-to-one map from the arc \( [\theta^{2\pi i/3}, \theta^{\pi i/2}] \) onto the interval \([0,1728] \). As in [MMO], based on extensive computations we conjecture

**Conjecture 7.1.** For every \( k \neq 2t^2 - 2, \ i \geq 2 \), the zeros of \( G(\mathcal{F}_{L(k\Lambda_0)}, j) \) are simple and inside the interval \([0,1728] \).

Here is a sample of \( j \)-zeros for \( 1 \leq k \leq 11 \) which clearly supports our conjecture.
Notice that (1.3) would follow from the congruence

\[ W'_{L(k\Lambda_0)}(q) \equiv h W_{L(k\Lambda_0)}(q) \pmod{p^2}, \]

for some \( h \).

The method in [RSD] is peculiar to Eisenstein series and it does not apply directly to \( F_{L(k\Lambda_0)} \). So in order to probe Conjecture [M3] E. Mortenson computed the \( j \)-zeros of \( F_{L(k\Lambda_0)} \) for all \( 1 \leq k \leq 22 \). From his data we observed certain interlacing properties of zeros, which indicates a possibility that \( G(F_{L(k\Lambda_0)}, j) \), \( k \in \mathbb{N} \) forms an orthogonal polynomial sequence (cf. [KZ1]).

An unpleasant feature of \( F_{L(k\Lambda_0)} \) is rather irregular pattern of vanishing which occurs for \( k = 2i^2 - 2 \). This, and some other clues, makes it hard to believe that a simple recursive formula for \( F_{L(k\Lambda_0)} \) will settle down the conjecture. We should mention here that recursion formulas do arise naturally in the context of \( L(\Lambda_0)^{\otimes k} \)-modules (see [M3]). But as we know \( L(k\Lambda_0) \)-modules and \( L(\Lambda_0)^{\otimes k} \)-modules are related in a nontrivial way via the unitary minimal models.

8. Final Remarks

| \( k \) | \( \epsilon \) | \( \delta \) | \( G(F_{L(k\Lambda_0)}, j) \) |
|---|---|---|---|
| 1 | 1 | 0 | 1 |
| 2 | 0 | 1 | 1 |
| 3 | 2 | 0 | 1 |
| 4 | 1 | 1 | 1 |
| 5 | 0 | 0 | \( j - \frac{1302528}{1073} \) |
| 6 | 2 | 1 | \( - \) |
| 7 | 1 | 0 | \( j - \frac{187021824}{587349} \) |
| 8 | 0 | 1 | \( j - \frac{3896490}{201119} \) |
| 9 | 2 | 0 | \( j - \frac{1381580800}{10776887} \) |
| 10 | 1 | 1 | \( j - \frac{956352}{2021} \) |
| 11 | 0 | 0 | \( j^2 - \frac{20462710947840}{13928908741} j + \frac{1908473415598080}{13928908741} \) |

100.0843760, 1368.997756

\[ 100.0843760, 1368.997756 \]

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