RANDOM BAND MATRIX LOCALIZATION BY SCALAR FLUCTUATIONS

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ABSTRACT. We show the eigenvectors of a Gaussian random band matrix are localized when the band width is less than the 1/4 power of the matrix size. Our argument is essentially an optimized version of Schenker’s proof of the 1/8 exponent.

1. Introduction

1.1. Some context and prior work. A random band matrix is a random matrix whose entries vanish outside of a band around the diagonal. The standard random band matrix model uses independent Gaussian entries: Fix integers \( N \geq W \geq 1 \) and sample \( A \in \mathbb{R}^{N \times N} \) according to the probability density \( a \mapsto Z^{-1} e^{-\frac{W}{4} \text{tr} a^*a} \) on the space of matrices \( a \in \mathbb{R}^{N \times N} \) that satisfy \( a^* = a \) and \( a_{i,j} = 0 \) for \( |i-j| > W \). It is conjectured that, when \( \varepsilon > 0 \) small and \( N \geq C \varepsilon \) large, the eigenvectors of \( A \) are typically localized when \( W \leq N^{1/2-\varepsilon} \) and typically delocalized when \( W \geq N^{1/2+\varepsilon} \). See Casati, Molinari, and Izrailev [5].

At the time of writing, the rigorous state of the art for this model was localization when \( W \leq N^{1/7-\varepsilon} \) and delocalization when \( W \geq N^{3/4+\varepsilon} \). These were proved by Peled, Schenker, Shamis, and Sodin [7] and Bourgade, Yau, and Yin [3], respectively. In this article we advance the localization regime to \( W \leq N^{1/4-\varepsilon} \) by optimizing the proof of the 1/8 exponent in Schenker [8]. In a simultaneous and independent work, Cipolloni, Peled, Schenker, and Shapiro [6] also prove localization when \( W \leq N^{1/4-\varepsilon} \).

There is a related integrable band matrix model where the \( N^{1/2} \) threshold is known to be sharp. See Shcherbina [9] and Shcherbina and Shcherbina [10]. There is also a recent breakthrough on delocalization when \( W \geq N^{\varepsilon} \) for a \( d \geq 8 \) dimensional toroidal band model. See Yang, Yau, and Yin [12, 13].

1.2. Main result. For convenience we work with random block tridiagonal matrices. We expect our methods can be easily adapted to handle standard random band matrices. We sample a random block tridiagonal matrix

\[
A = \begin{pmatrix}
A_{1,1} & A_{1,2} & \cdots & \cdots & A_{1,N} \\
A_{2,1} & A_{2,2} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
A_{N-1,1} & \cdots & \cdots & \cdots & A_{N-1,N} \\
A_{N,1} & \cdots & \cdots & \cdots & A_{N,N}
\end{pmatrix} \in (\mathbb{R}^{M \times M})^{N \times N}
\]

according to the probability density

\[
(1.1) \quad a \mapsto Z^{-1} e^{-\frac{W}{4} \text{tr} a^*a}
\]
on the space of block matrices \( a \in (\mathbb{R}^{M \times M})^{N \times N} \) that satisfy \( a^* = a \) and \( a_{i,j} = 0 \) for \(|i - j| > 1\). We prove exponential off-diagonal decay of the resolvent:

**Theorem 1.2.** If \( \varepsilon > 0, |\lambda| < \varepsilon^{-1} \), and \( N, M \geq C_\varepsilon \), then

\[
\mathbf{E} \|(A - \lambda)^{-1}\|_{i,j}^{1 - \varepsilon} \leq e^{-M^{-3-\varepsilon}|i-j|}
\]

holds for all \( 1 \leq i, j \leq N \).

Here and throughout the paper, we let \( C > 1 > c > 0 \) denote positive universal constants that may differ in each instance. We use subscripts to denote dependence, so that \( C_\varepsilon > 1 > c_\varepsilon > 0 \) are positive constants that depend on \( \varepsilon \).

The resolvent bound \((1.3)\) is known to imply localization of the eigenvectors when \( N \geq M^{3+\varepsilon} \). See for example Aizenman, Friedrich, Hundertmark, and Schenker \([1]\). Since the flattened version of \( A \) is an \( NM \times NM \) matrix, this corresponds to localization when \( W \leq N^{1/4-\varepsilon} \) for the standard random band model.

By arguments of Schenker \([8]\) and Bourgain \([4]\), Theorem 1.2 can be deduced from the following lower bound on the logarithmic fluctuations of the corner block of the resolvent.

**Lemma 1.4.** If \( \varepsilon > 0, |\lambda| < \varepsilon^{-1} \), and \( N, M \geq C_\varepsilon \), then

\[
\mathbf{V} \log \|(A - \lambda)^{-1}\|_{1,N} \geq M^{-3-\varepsilon}N.
\]

In particular, the logarithmic fluctuation lower bound \((1.5)\) is the main contribution of our paper. The reader interested in the reduction of Theorem 1.2 to Lemma 1.4, which is arguably the most complicated part of the proof, is referred to Schenker \([8]\) and Bourgain \([4]\).

We remark that our only essential use of the Gaussian law \((1.1)\) of \( A \) is to compute the conditional law of \( S_k \) given \( D_{k-1}, D_k, D_{k+1}, B_{k-1}, B_k \) in Lemma 3.2. We conjecture that this usage can be removed, and that the same matrices treated in Schenker \([8]\) can also be handled using our methods.

### 1.3. Scalar fluctuations

We recall from Schenker \([8]\) the reformulation of the lower bound \((1.3)\) in terms of the scalar fluctuations of a cocycle of a Markov chain. Assume the hypotheses of Lemma 1.4. Almost surely, Gaussian elimination yields the explicit formula

\[
(A - \lambda)^{-1}_{1,N} = D_{1}^{-1} B_1 D_2^{-1} B_2 \cdots D_{N-1}^{-1} B_{N-1} D_N^{-1},
\]

where

\[
\begin{align*}
D_1 &= A_{1,1} - \lambda \\
B_k &= -A_{k,k+1} \\
D_{k+1} &= A_{k+1,k+1} - \lambda - B_k D_k^{-1} B_k
\end{align*}
\]

for \( k = 1, \ldots, N - 1 \). We bound the logarithmic fluctuations of the product \((1.6)\) from below by the sum of the conditional logarithmic fluctuations of the norms of its terms. First, we let

\[
S_k = \|D_k\|
\]

and

\[
\bar{D}_k = \|D_k\|^{-1} D_k,
\]

and then observe

\[
\mathbf{V} \log \|(A - \lambda)^{-1}\|_{1,N} \geq \mathbf{E}(\mathbf{V}(\log(S_1 \cdots S_N) | \bar{D}_1, \ldots, \bar{D}_N, B_1, \ldots, B_{N-1})).
\]
Second, we use independence to distribute the variance through the logarithm. Since $S_k$ and $(D_1, ..., D_{k-2}, D_{k+2}, ..., D_N, B_1, ..., B_{k-2}, B_{k+1}, ..., B_{N-1})$ are conditionally independent given $(D_{k-1}, \bar{D}_k, D_{k+1}, B_{k-1}, B_k)$, it follows that

$$\text{log } V((A - \lambda)^{-1})_{1,N} \geq \sum_{k=2}^{2N_{\mathbb{N}}} \mathbb{E}(V((\log S_k|D_{k-1}, \bar{D}_k, D_{k+1}, B_{k-1}, B_k))).$$

Here we dropped the odd terms in order to achieve conditional independence. The lower bound (1.8) follows from the inequality (1.8) and the lower bound (1.9)

$$\min_{1<k<N} \mathbb{E}(V((\log S_k|D_{k-1}, \bar{D}_k, D_{k+1}, B_{k-1}, B_k)) \geq M^{-3-\varepsilon}.$$

We prove the lower bound (1.9) by explicitly computing the conditional law of $S_k$ and estimating the log concavity of its density. See Lemma 3.6 below.

1.4. Discussion of optimality. We expect the $1/4$ exponent is the best possible for any argument that relies on the scalar fluctuations of the terms in the product (1.6). Indeed, for energies close to zero, we conjecture an upper bound on the logarithmic variance of the norms of the terms:

Conjecture 1.10. If $\varepsilon > 0$ is small, $|\lambda| < \varepsilon$, $M \geq C_\varepsilon$, and $1 < k < N$, then

$$\mathbb{E}(V((\log S_k|D_{k-1}, \bar{D}_k, D_{k+1}, B_{k-1}, B_k)) \leq M^{-3+\varepsilon}.$$  

Intuitively, the fluctuations of the norm $S_k$ should contain only an $O(M^{-2})$ fraction of the total randomness in the matrix $D_k$. In particular, any improvement beyond the exponent $1/4$ should give finer information about the Lyapunov exponents of the cocycle (1.6) of the Markov chain (1.7).

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2. Random full matrix bounds

We collect some known estimates for random full matrices. Recall that $\|A\| = \max_{\|x\|=1} \|Ax\|$ is the operator norm of $A$, $\|A\|_F = (\text{tr } A^* A)^{1/2}$ is the Frobenius norm of $A$, and $\text{tr } I_1(A)$ is the number of eigenvalues of $A$ in the interval $I \subseteq \mathbb{R}$.

Lemma 2.1. If $E \in \mathbb{R}^{M \times M}$ has independent $N(0, M^{-1})$ Gaussian entries, $G = (E + E^*)/\sqrt{2}$, and $H \in \mathbb{R}^{M \times M}$ is symmetric and deterministic, then

1. $\mathbb{E}\text{tr } I_1(G + H) \leq CM|I|$,  
2. $\mathbb{P}(\|G + H\|_F \geq t) \leq CM t^{-1}$,  
3. $\mathbb{E}(\text{tr } GH)^2 = 2M^{-1}\|H\|_F^2$,  
4. $\mathbb{P}(\|E\| \geq C) \leq e^{-cM}$,  
5. and $\mathbb{P}(\|E^* HE\|_F \leq c\|H\|_F) \leq e^{-cM}$

hold for all intervals $I \subseteq \mathbb{R}$ and $t > 0$.

Proof. The estimates (2.2) and (2.3) are part of the main theorem in Aizenman, Peled, Schenker, Shamis, and Sodin [2]. The estimate (2.4) follows from $\text{tr } GH = \sum_{ij} \sqrt{2} E_{ij} H_{ij}$ and the fact that the $E_{ij}$ are independent $N(0, M^{-1})$ Gaussians. The estimate (2.5) is a standard operator norm bound for Wigner matrices and can be found in Tao [11]. The final estimate (2.6) is somewhat non-standard, so
we give an ad hoc proof. Since the law of $E$ is invariant under left multiplication by an orthogonal matrix, we may assume $H$ is diagonal and therefore $(E^*HE)_{ij} = \sum_k E_{ik}H_{kk}E_{kj}$. For $i \neq j$, compute
\[
\mathbb{E}(E^*HE)_{ij}^2 = M^{-2}\|H\|_F^2
\]
and
\[
\mathbb{E}(E^*HE)_{ij}^4 = 3M^{-4}\|H\|_F^4.
\]
It follows that
\[
\mathbb{P}((E^*HE)_{ij}^2 \geq cM^{-2}\|H\|_F^2) \geq c.
\]
Since $(E^*HE)_{i_1,j_1}, \ldots, (E^*HE)_{i_m,j_m}$ are independent whenever $\{i_1, j_1\}, \ldots, \{i_m, j_m\}$ are disjoint, we can partition the entries $\{(E^*HE)_{ij} : i \neq j\}$ into at most $cM$ sets of at least $c$ independent entries. In particular, the event $\|E^*HE\|_F \leq c\|H\|_F$ is contained in a union of at most $cM$ events of probability at most $e^{-cM}$. □

The diagonal and off-diagonal blocks $A_{k,k}$ and $A_{k,k+1}$ have, respectively, the same law as $G$ and $E$ from Lemma 2.1. We can therefore use the random full matrix bounds in Lemma 2.1 to control the typical sizes of the matrices appearing in the Markov chain (1.7):

**Lemma 2.7.** If $\varepsilon > 0$, $|\lambda| < \varepsilon^{-1}$, $M \geq C\varepsilon$, and $1 \leq k < N$, then, with probability at least $1 - M^{-c\varepsilon}$,
\[
\begin{align*}
M^{1/2-\varepsilon} &\leq \|A_{k+1,k+1}\|_F \leq M^{1/2+\varepsilon}, \\
M^{1/2-\varepsilon} &\leq \|B_k^*D_k^{-1}B_k\|_F \leq M^{1+\varepsilon}, \\
|\text{tr} A_{k+1,k+1}(\lambda + B_k^*D_k^{-1}B_k)| &\leq M^{1/2+\varepsilon}, \\
\text{and} \quad |\text{tr} A_{k+1,k+1}B_k^*D_k^{-1}B_k| &\leq M^{1/2+\varepsilon}.
\end{align*}
\]

**Proof.** We prove a series of estimates valid for all $\varepsilon > 0$, $|\lambda| < \varepsilon^{-1}$, $M \geq C\varepsilon$, and $1 \leq k < N$. For brevity, we omit these quantifiers from the notation. However, it is important to note that an estimate for some $\varepsilon > 0$ may follow from a previous estimate for a different $\varepsilon > 0$. The estimate
\[
\mathbb{P}(\|A_{k+1,k+1}\|_F \geq M^{1/2+\varepsilon}) \leq M^{-c\varepsilon}
\]
follows from Markov’s inequality and $\mathbb{E}\|A_{k+1,k+1}\|_F^2 = M + 1$. The estimate
\[
\mathbb{P}(\text{tr} 1_{[-M^{-\varepsilon},M^{-\varepsilon}]}(A_{k+1,k+1}) \geq \varepsilon M) \leq M^{-c\varepsilon}
\]
follows from the Wegner estimate (2.2). The estimate
\[
\mathbb{P}(\|A_{k+1,k+1}\|_F \leq M^{1/2-\varepsilon}) \leq M^{-c\varepsilon}
\]
follows from the previous estimate and the fact that $A_{k+1,k+1}$ has $M$ eigenvalues. The estimate
\[
\mathbb{P}(\|D_k^{-1}\|_F \geq M^{1+\varepsilon}) \leq M^{-c\varepsilon}
\]
follows from the independence of $D_k - A_{k,k}$ and $A_{k,k}$ and the inverse Frobenius norm bound (2.6). The estimate
\[
\mathbb{P}(\|B_k^*D_k^{-1}B_k\|_F \geq M^{1+\varepsilon}) \leq M^{-c\varepsilon}
\]
follows from the previous estimate and the operator norm bound (2.6) applied to $B_k$. The estimate
\[
\mathbb{P}(\text{tr} 1_{[-M^{-\varepsilon},M^{-\varepsilon}]}(D_k) \geq \varepsilon M) \leq M^{-c\varepsilon}
\]
follows from the independence of $D_k - A_{k,k}$ and $A_{k,k}$ and the Wegner estimate \[2.2\]. The estimate

$$
P(\text{tr} 1_{[-M^\epsilon, M^\epsilon]}(B_k^* D_k^{-1} B_k) \leq (1 - \epsilon)M) \leq M^{-c_\epsilon}
$$

follows from the previous estimate and the operator norm bound \[2.5\] applied to $B_k$. The estimate

$$
P(1 \leq (1 - \epsilon)M) \leq M^{-c_\epsilon}
$$

follows from the previous estimate, $|\lambda| < \epsilon^{-1}$, and the operator norm bound \[2.5\] applied to $A_{k,k}$. The estimate

$$
P(1 \leq (1 - \epsilon)M) \leq M^{-c_\epsilon}
$$

follows from the previous estimate and the fact that $\|D_k^{-1}\|_F^2$ is the sum of the squares of the eigenvalues of $D_k^{-1}$. The estimate

$$
P(\|B_k^* D_k^{-1} B_k\|_F \leq M^{1/2 - \epsilon}) \leq M^{-c_\epsilon}
$$

follows from the previous estimate, the independence of $B_k$ and $D_k$, and the conjugated Frobenius norm bound \[2.6\]. Finally, the last two estimates in the statement of the lemma follow from the Frobenius inner product bound \[2.4\], Markov’s inequality, and the independence of $A_{k+1,k+1}$ and $B_k^* D_k^{-1} B_k$.

\[\square\]

3. Fluctuation lower bound

For an arbitrary random vector whose law has a continuous density, we recall the conditional law of its norm given its direction:

**Lemma 3.1.** If the random vector $X \in \mathbb{R}^n$ has continuous density $\phi$, $Y = \|X\|$, $\bar{X} = \|X\|^{-1}X$, and $f \in C_c((0, \infty))$, then

$$
E(f(Y) | \bar{X}) = Z^{-1} \int_0^\infty f(y)y^{n-1} \phi(y\bar{X}) \, dy,
$$

where $Z = \int_0^\infty y^{n-1} \phi(y\bar{X}) \, dy$. \[\square\]

The conditional law of $S_k$ can now be computed from the density \[1.1\] of the random band matrix.

**Lemma 3.2.** For $1 < k < N$ and $f \in C_c((0, \infty))$,

$$
E(f(S_k) | D_{k-1}, \bar{D}_k, D_{k+1}, B_{k-1}, B_k) = Z_k^{-1} \int_0^\infty f(s)e^{-\phi_k(s)} \, ds,
$$

where

$$
\phi_k(sS_k) = \frac{M}{4}\|sD_k + \lambda + B_k^* D_k^{-1} B_{k-1}\|_F^2
$$

$$
+ \frac{M}{4}\|D_{k+1} + \lambda + s^{-1} B_k^* D_k^{-1} B_k\|_F^2
$$

$$
- \frac{M^2 + M - 2}{2} \log(sS_k)
$$

(3.3)

and $Z_k = \int_0^\infty e^{-\phi_k(s)} \, ds$. 

Proof. Since the law of $A$ has density $f$ and the change of variables $A \mapsto (D, B)$ in (1.7) preserves Lebesgue measure, the law of $(D, B)$ has density
\[
\begin{align*}
(d, b) \mapsto Z_{N, M}^{-1} e^{-\frac{d}{4} \| d_b \|_p^p - \frac{d}{4} \sum_b \| d_{k+1} + \lambda + B_k^{-1} b \|_p^p - \frac{d}{4} \sum_k \| b_k \|_p^p}
\end{align*}
\]
on the space of pairs of sequences of matrices $(d, b) \in (\mathbb{R}^{M \times M})^N \times (\mathbb{R}^{M \times M})^{N-1}$ that satisfy $d_k^* = d_k$. The conditional law of $D_k$ given $(D_{k-1}, D_{k+1}, B_{k-1}, B_k)$ therefore has density
\[
\begin{align*}
d \mapsto Z^{-1} e^{-\frac{d}{4} \| d + \lambda + B_{k-1}^{-1} D_{k-1}^{-1} B_{k-1} \|_p^p - \frac{d}{4} \| D_{k+1} + \lambda + B_k^* d^{-1} B_k \|_p^p}
\end{align*}
\]
where $Z > 0$ is $(D_{k-1}, D_{k+1}, B_{k-1}, B_k)$-measurable. Conclude using Lemma 3.1 together with the fact that the space of $d \in \mathbb{R}^{M \times M}$ with $d^* = d$ has dimension $M(M+1)/2$. \hfill \Box

We estimate the typical growth of the logarithmic density $\phi_k$.

Lemma 3.4. If $\varepsilon > 0$, $|\lambda| < \varepsilon^{-1}$, $M \geq C_\varepsilon$, and $1 < k < N$, then, with probability at least $1 - M^{-\varepsilon r}$, the logarithmic density $\phi_k$ defined in (3.3) satisfies
\[
S_k \phi_k'(S_k s) \geq M^{2-\varepsilon}s \quad \text{for } s \geq M^\varepsilon,
\]
\[
S_k \phi_k'(S_k s) \leq -M^{2-\varepsilon}s^{-3} \quad \text{for } 0 < s \leq M^{-\varepsilon},
\]
and
\[
|S_k \phi_k''(S_k s)| \leq M^{3+\varepsilon}(1 + s^{-4}) \quad \text{for } s > 0.
\]

Proof. Using the recursion (1.7), compute
\[
s D_k + \lambda + B_{k-1}^* D_{k-1}^{-1} B_{k-1} = s A_{k, k} + (1 - s)(\lambda + B_{k-1}^* D_{k-1}^{-1} B_{k-1})
\]
and
\[
D_{k+1} + \lambda + s^{-1} B_k^* D_k^{-1} B_k = A_{k+1, k+1} + (s^{-1} - 1) B_k^* D_k^{-1} B_k.
\]
Inserting these into the definition of $\phi_k$ in (3.3), obtain
\[
\phi_k(S_k s) = \alpha_1 s^2 + \alpha_2 (s-1)^2 - 2\alpha_3 s(s-1) + \alpha_4 + \alpha_5 (s^{-1} - 1)^2 + 2\alpha_6 (s^{-1} - 1)^2 - \alpha_7 \log(S_k s),
\]
where
\[
\begin{align*}
\alpha_1 &= \frac{M}{4} \| A_{k, k} \|_F^2, \\
\alpha_2 &= \frac{M}{4} \| \lambda + B_{k-1}^* D_{k-1}^{-1} B_{k-1} \|_F^2, \\
\alpha_3 &= \frac{M}{4} \text{tr} A_{k, k} (\lambda + B_{k-1}^* D_{k-1}^{-1} B_{k-1}), \\
\alpha_4 &= \frac{M}{4} \| A_{k+1, k+1} \|_F^2, \\
\alpha_5 &= \frac{M}{4} \| B_k^* D_k^{-1} B_k \|_F^2, \\
\alpha_6 &= \frac{M}{4} \text{tr} A_{k+1, k+1} B_k^* D_k^{-1} B_k, \\
\text{and} \quad \alpha_7 &= \frac{1}{2} (M^2 + M - 2).
\end{align*}
\]
Using Lemma 2.7 the bounds

$$M^{2-\varepsilon} \leq \alpha_1 \leq M^{2+\varepsilon},$$
$$M^{2-\varepsilon} \leq \alpha_2 \leq M^{4+\varepsilon},$$
$$|\alpha_3| \leq M^{3/2+\varepsilon},$$
$$M^{2-\varepsilon} \leq \alpha_4 \leq M^{2+\varepsilon},$$
$$M^{2-\varepsilon} \leq \alpha_5 \leq M^{3+\varepsilon},$$
$$|\alpha_6| \leq M^{3/2+\varepsilon},$$
and $$M^{2-\varepsilon} \leq \alpha_7 \leq M^{2+\varepsilon},$$

hold with probability at least $$1 - M^{-c\varepsilon}$$. Without loss of generality, assume these bounds hold almost surely. Compute

$$S_k \phi'_k(S_k s) = 2\alpha_1 s + 2\alpha_2(s - 1) - 4\alpha_3 s + 2\alpha_5(s^2 - s^3) - 2\alpha_6 s^2 - \alpha_7 s^{-1},$$

and use the bounds on $$\alpha_k$$ to estimate

$$S_k \phi'_k(S_k s) \geq M^{2-\varepsilon}s - M^{2+\varepsilon} \quad \text{for } s > 1$$
and

$$S_k \phi'_k(S_k s) \leq M^{2+\varepsilon} + M^{2-\varepsilon}(s^2 - s^3) \quad \text{for } 0 < s < 1.$$

Conclude the first and second inequalities in the lemma statement. Compute

$$S_k^2 \phi''_k(S_k s) = 2\alpha_1 + 2\alpha_2 - 4\alpha_3 + 2\alpha_5(3s^4 - 2s^3) + 4\alpha_6 s^3 + \alpha_7 s^{-2}$$

and use the bounds on the $$\alpha_k$$ and $$s^2 + s^{-3} \leq 2 + s^{-4}$$ to conclude the third inequality in the lemma statement. \(\square\)

We prove an elementary logarithmic variance bound.

**Lemma 3.5.** If $$Y$$ is a positive random variable, the law of $$Y$$ has continuous density $$e^{-\psi}$$, and there are $$y_0 > 0, \alpha \geq 1, \text{ and } \beta \geq 1$$ such that

$$y_0 \psi'(y_0 y) \geq \beta y \quad \text{for } y \geq \beta,$$
$$y_0 \psi'(y_0 y) \leq -\beta y^{-3} \quad \text{for } 0 < y \leq \beta^{-1},$$

and

$$|y_0^2 \psi''(y_0 y)| \leq \alpha(1 + y^{-4}) \quad \text{for } y > 0,$$

then $$\mathbb{V} \log Y \geq c\alpha^{-1}\beta^{-6}$$.

**Proof.** Since $$\mathbb{V} \log(y_0^{-1}Y) = \mathbb{V} \log Y$$ and $$y_0^{-1}Y$$ has density $$y \mapsto y_0 e^{-\psi(y_0 y)}$$, we can rescale to make $$y_0 = 1$$. Using $$\psi'(y) \geq \beta y$$ for $$y \geq \beta$$, compute

$$\int_{2\beta}^{\infty} |\log y| e^{-\psi(y)} \, dy \leq \int_{2\beta}^{\infty} y e^{-\psi(y)} + \frac{1}{2} \psi'(2\beta)^2 - \frac{1}{2} \beta^2 y^2 \leq \beta^{-1} e^{-\psi(2\beta)} \leq \beta^{-2} \int_{\beta}^{2\beta} e^{-\psi(y)} \, dy \leq \beta^{-2}. $$
Similarly, using $\psi'(y) \leq -\beta y^{-3}$ for $0 < y \leq \beta^{-1}$, compute
\[
\int_0^{(2\beta)^{-1}} |\log y| e^{-\psi(y)} \, dy \\
\leq (2\beta)^{-2} \int_0^{(2\beta)^{-1}} y^{-3} e^{-\psi((2\beta)^{-1}) + \beta(2\beta)^2 \frac{1}{2} \beta y^{-2}} \\
= (2\beta)^{-2} \beta^{-1} e^{-\psi((2\beta)^{-1})} \\
\leq \frac{1}{2} \beta^{-2} \int_{(2\beta)^{-1}}^{\beta^{-1}} e^{-\psi(y)} \, dy \\
\leq \frac{1}{2} \beta^{-2}.
\]
Conclude $E|\log Y| \leq \log \beta + C$ and therefore $y_1 = \exp E \log Y$ satisfies $c\beta^{-1} \leq y_1 \leq C\beta$. Using $|\psi''(y)| \leq \alpha (1 + y^{-4})$, compute
\[
\int_{0}^{y_1 - t} e^{-\psi} + \int_{y_1 + t}^{\infty} e^{-\psi} \geq e^{-C\alpha \beta^3 t^2} \int_{y_1 - t}^{y_1 + t} e^{-\psi}
\]
for $0 < t < c\beta^{-1}$. Using Markov’s inequality, compute
\[
t^{-2} V \log Y \geq P(|\log Y - \log y_1| \geq t) \geq P(|Y - y_1| \geq C\beta t) \geq e^{-C\alpha \beta^3 t^2}
\]
for $0 < t < c\beta^{-1}$. Conclude by setting $t^{-2} = \alpha \beta^6$. \hfill \qed

Combining the previous two lemmas yields the main result.

**Lemma 3.6.** If $\varepsilon > 0$, $|\lambda| < \varepsilon^{-1}$, $M \geq C\varepsilon$, and $1 < k < N$, then, with probability at least $1 - M^{-\varepsilon}$,
\[
V(\log S_k|D_{k-1}, \hat{D}_k, D_{k+1}, B_{k-1}, B_k) \geq M^{-3-\varepsilon}.
\]
In particular, the lower bound (3.1) holds.

**Proof.** This is immediate from Lemma 3.4 and Lemma 3.5 \hfill \qed

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