Time Evolution and Thermodynamics for the Nonequilibrium System in Phase-Space

Chen-Huan Wu *
College of Physics and Electronic Engineering, Northwest Normal University, Lanzhou 730070, China

November 10, 2017

The integrable system is constrained strictly by the conservation law during the time evolution, and the nearly integrable system or nonintegrable system is also constrained by the conserved parameters (like the constants of motion) with corresponding generalized Gibbs ensemble (GGE) which is indubitably a powerful tool in the prediction of the relaxation dynamics. For stochastic evolution dynamic with considerable noise, the obviously quantum or thermal correlations which don’t exhibit the thermal behavior, (like the density of kinks or transverse magnetization correlators), display a asymptotic nonthermalization, and in fact it’s a asymptotic quasisteady state with an infinite temperature, therefore the required distance to the nonthermal steady state is in an infinite time average. In this paper, we unambiguously investigate the relaxation of a nonequilibrium system in a canonical ensemble for integrable system or nonintegrable system, and the temporal behavior of many-body quantum system and the macroscopic system, as well as the corresponding linear-coupling between harmonic oscillators. Matrix-method in entropy ensemble is also utilized to discuss the boundary and the important diagonalization, the approximation by the perturbation theory is also obtained.

1 Introduction

The investigation of evolution of nonequilibrium system is important to the particle physics or condensed matter physics and even the cosmology (like the entropy of Bekenstein-Hawking black hole\(^1\)), especially in the many-body theory prediction which by, e.g., the trapped ultracold atomic gases which have weak energy interaction with the environment and therefore allow the observation of unitary time evolution\(^2\). For nonequilibrium system, the usual form of glass can be blocked by the pinning field\(^3\) and produce a glass transition like the process of ergodic to non-ergodic. In replica theory, since the homogeneous liquid given by replica symmetry have a inhibitory effect for entropy production, whereas the replica symmetry broken result in the increase of overlap of replicas. With the increase of degree of overlap which can be realized by enlarge the system size, the number of metastable states (or the hidden one) is grows exponentially, and furthermore, the entropy is grows logarithmic.

We already know that the observable chaotic classical system require the processing resource which increase exponentially with time and Kolmogorov entropy \(h^\parallel\) due to it’s exponential sensitivity in initial state\(^5\), while the integrable one, which is solvable by the Bethe ansatz\(^6\), is increase polynomially. The time evolution of quantum entangled state may cause decoherence effect which is widely found in condensate system and it take a important role in quantum information processing, quantum computation and metrology, quantum teleporation, quantum

*chenhuanwu1@gmail.com

\(^1\)arXiv:1711.00547v2 [cond-mat.stat-mech] 9 Nov 2017
key agreement\textsuperscript{7,8}, and even the decoherence in neural network\textsuperscript{9}. The entanglement is mostly produced by the dynamical evolution with nonlinear interaction\textsuperscript{10} and the non-destructive measurement. like the Dzyaloshinskii-Moriya interaction\textsuperscript{11,12}, and in extreme case, e.g., through the axion field\textsuperscript{13,14}. Usually the quantum entanglement is studied by the two-qubit or qutrit\textsuperscript{11,15} system, in some case the tripartite system\textsuperscript{12,16} or even more one is consider. In nonequilibrium and nonstationary open system, the coarse graining which connecting numerous subsystems’ degrees of freedom make more possible to realize this process\textsuperscript{17}, and the thermal entropy is a good measurement for the effect of coarse graining. The quantum spaces’ dimension increases exponentially with particle number due to the tensor-productor\textsuperscript{5}, similarly, the number of metastable which as the subsystems of the spin glasses system is increase exponentially with size in high temperature\textsuperscript{18}, phase transition and critical fluctuation occur when it from one kind of subsystem into another and the broken and restoration of symmetry is also affect the properties of materials\textsuperscript{19}, like the dielectric constant, etc.

In solid-state quantum system, the spin is the best candidate among various microscopic atom intrinsic degrees of freedom in thermal entanglement which has higher stability compared to other entanglements due to the spins’ relatively long decoherence time\textsuperscript{20} and it’s in close connection with the local free energy. The long coherence time in many-body systems is useful to detecting the unitary dynamics, e.g., the Hubbard-type model, and it’s important to the coherent nonequilibrium dynamics for the multiple phases transition. Since the models that can be mapped to a spinless free fermions through Wigner-Jordan transformation and show a in-phase fermion liquid state\textsuperscript{21}, have show a stationary behavior in such a equilibrium integrability model which consider as a powerful tool to obtain the exactly solution of model\textsuperscript{22}. A numerical method as time-dependent density matrix renormalization group (t-DMRG) have show that the matrix produce operator $D(t)$ is simulation-inefficiently for nonintegrable model which is similar to the tensor-productor, but it’s efficient for integrable and local disordered case\textsuperscript{23}. Except that, the method of matrix produce wave function is also a good tool to deal with this time-evolving one-dimension quantum system\textsuperscript{24}. The time evolution on free fermions or bosons, when the time scale to infinity the thermal average of $z$-component spin $S_z$ is zero and the spin states is half-filled\textsuperscript{21}, in this case the interaction between particles is strongest due to the zero-polarization\textsuperscript{23}, and the entanglement entropy is also increase and becomes more extensive\textsuperscript{25}. The first implementation of using the density matrices in prediction of many-body system (equilibrium or nonequilibrium) is the Ref.\textsuperscript{26}. It discuss the situation similar to the quantum irreversible process in a energy- and information-lossy system.

A fact that the many-body quantum system will tend to equilibrium has been verified by many recently experiments, like the trapped ultracold atoms in optical lattices or the interactions with optical resonance. Whereas for the nonequilibrium system, the relaxation and thermal entanglement and the stochastic force also attract a lot of attention\textsuperscript{27,28}. Furthermore, the system may relax to analogue of thermal state if the initial state is ground state\textsuperscript{27}. The method of fluctuation-dissipation relation (FDR) and quantum state diffusion (QSD) is utilized for the evolution to steady states in integrable system whose final states are constrainted by the conserved law (indeed, it’s the scattering process of particles which constrainted by conserved law) and with a finite speed of algebraically relaxation and information transfer under the thermodynamic limit (the large-N limit). Note that the speed here will not bounded by the speed of light like the relativistic quantum theory, but bounded by a well known Lieb-Robinson group velocity\textsuperscript{29}. The integrable system of quantum Newton’s cradle with groundbreaking is a example\textsuperscript{30}. The classical system also have found the same result, like the Fermi-Pasta-Ulam (FPU) theorem\textsuperscript{31} and Kolmogorov-Arnold-Moser (KAM) theorem\textsuperscript{32}. While for some nonintegrable system, the constant of motion can be expressed by second quantized operator\textsuperscript{33} (see below).

The collection variables are applied to investigate the evolution in studied system, except
this, we also applied the method of density matrix and complex tensor grid to make this paper self-contained. For local observable system the stationary and linear value may exist (like thermal state), but for integrable system whose time evolution found no thermalization and it may tends to a distribution of GGE with a important fundamental hypothesis for statistical ensemble that has maximized entropy which is constrained by local conservation law(34), (e.g., the conservation quantity of momentum occupation number), hence restrict the ergodicity and can’t reach the thermal state. For a framework of macroscopic system in finite dimension is important to introduce the quantum field theory for both the equilibrium and nonequilibrium state in open system(17) to investigate its time-dependent nature and coupling (or interaction) in local and nonlocal case as well as the dynamical fluctuation in short distance. It’s also necessary to consider a quantum field when the Hilbert space is too large to implement a well numerical simulation(35). While the importance of entangled states for quantum computation is well understand, to reduce the confusion from decoherence, there is a topology way that storing the quantum information non-localized(36) or through the non-Abelian braiding statistics which support the Majorana fermions(37,38) by Majorana modes in finite wire(39), and it can better solve the problem of infinity dilution of the stored information in local area(27).

Since for nearly integrable system, the behavior of relaxation is under the crossover effect of prethermalization and thermalization, which is associate with the thermal correlation and the speed of information transfer, and the prethermalized state can be well described by the GGE(40), i.e., may be view as a integrable system. Like the Ref.41 which also using the method of t-DMRG and show the nonthermalization in soft-bosons model, have perform the off-diagonal correlation in the two-dimension square model, and the relaxation with some fluctuation is presented in short time evolution. The suppressed thermalization can be freed by enough perturbation to break the integrability. This crossover effect affect both the nonintegrable system and open system. Through the study of this paper, we know that the recurrence will appear for large time evolution. In the configuration which considered in this paper, part of mixed system which is of interest is coupling with the environment (not isolated), and hence the degrees of freedom of environment system (i.e., the counterpart of the target one) can be traced out in the canonical ensemble(32), i.e., tracing over the variables outside the target region. This provide the support on the matrix method in Sect.10. A large number of degrees of freedom is also a important precondition to implement global relaxation with the thermodynamic limit(42). For nonlocal operators in equilibrium state, the dynamical parameters display a effective asymptotic thermal behavior (follow the Gibbs disturbance) during equilibrium time evolution with determined temperature and decay with a asymptotic exponent law, while the model what we focus on is towards the asymptotic quasisteady state with a infinite temperature, which decay with a asymptotic power law(9) acted by a diffusion term (see Sect.11). The prethermalization will shares the same properties of nonthermal steady state due to the dynamical parameters, which makes the model after quench close to the integrable points (or superintegrable point). But in fact, for integrable quenches, the stationary behavior for both the local and nonlocal observables can be well described by the corresponding GGE, and the particles scattering which constrained by conserved law is purely diagonal(44,45).

This paper is organized as follows. We introduce the model of two-coupled subsystem in Sect.2, and the bare coupling is further discussed in Appendix.A. The evolutions in non-dissipation system is discussed in Sect.3, and the quenching for many-body system is discussed in Sect.4. A system-environment partition is mentioned in these two section. In Sect.5, we discuss the dissipation for nonlocal model. In Sect.6, the time evolution and thermal entanglement of Heisenberg XXZ model is investigated. In Sect.7, the correlation and transfer speed of information in quantum system is discuss where we take the one-dimension chain model as the explicit example. The relations between thermal behavior and the integrability is also discussed in this Section. We discuss the nonequilibrium dynamics with strong and weak in-
teraction in Hubbard model in Sect. 8. In this section, we investigate the phase transition of nonintegrable Hubbard model, and the relaxation of double occupation and the kinetic energy. We also use the method of nonequilibrium dynamical mean-field theory (DMFT) to detect the evolution by mapping the lattice model to the self-consistent single-site problem which can be solved numerically. In Sect. 9, we discuss the relaxation to a Gaussian state. In Sect. 10, we resort to the matrix method, and the propertice of the boundary and the transfer speed are also discussed. In Sect. 11, we discuss the relaxation of nonequilibrium system with stochastic dynamical variables in a free energy surface, the quantum dissipation in the damp-out process is also discussed. The diagonal contribution to symplectic spectrum of covariance matrix is further detected in Appendix B. The bulk-edge-coupling type materials which is related to the spectrum gap is presented in Sect. 10 and Appendix C, and the perturbation theory and diagonalized Hamiltonian is also discussed in Appendix C.

2 Model Introduction and the Coupling in Field Theory

We begin with the perturbation theory with space-time dimension, which is important to consider in the strong coupling case \( \text{eq} \), weak-perturbation limit of nonintegrable system, and even the breaking of ergodicity \( \text{eq} \). In dimension of \((d+1)\) in space-time, since the particles obtain mass from the broken of non-Abelian gauged symmetry, the coupling constant \( g \) is dimension-dependent, except the bare coupling \( g^0 \) which vanish in \( d + 1 = 4 \) limit \( \text{eq} \). The broken translational symmetry also make the spin liquid state rapidly solidified and turn into the crystalline structure \( \text{eq} \). Then we define two \( d \)-dimension state \( \psi_i \) and \( \psi_j \) with potential \( \phi_i \) and \( \phi_j \), respectively. In weak coupling condition which suitable for the perturbative calculation \( \text{eq} \), there exist a spin density wave (SDW) which in a Fourier expression is \( \psi_i = L^{-d} \sum e^{-ir \cdot \phi(x - r_i)} \), and \( \psi_j \) is as the same form. Although the \( L \) here is constrained by the model dimension \( d \), but \( L \) itself could be dimensionless and with dimensionless length scale and time scale (see Ref. \( \text{eq} \)). The \( \phi \) here describe the fluctuation as a function in arbitrary dimension, and it’s also useful for quantum fluctuation or even the vacuum fluctuation. The dimension of \( \phi \) may even up to ten according to D-branes of string theory \( \text{eq} \). In the space dimension of \( d = 3 \), the kitaev model despit a triangular parameter space with different degrees of coupling in three direction \( x, y, \) and \( z \), and the small triangular area which connecting the three midpoints of three side is gapless phase region \( \text{eq} \). In this model, I set coupling in these three direction in a range of 0 to \( n \), for which the top value is \( n = 2^{d/2} \) in \( SO(d) \times SU(N) \) system \( \text{eq} \). So a continuous phase transition with weak coupling pertubative RG under the time evolution can be expressed by \( S = \int d^d x \mathcal{L} \) which is a exponent appear in the imaginary-time path integral \( Z = \int D\psi_1 D\psi_1 D\psi_2 e^{-S} \).

The nonrelativistic Lagrange \( \mathcal{L} \) is \( \text{eq} \):

\[
\mathcal{L} = \int_\tau^{\tau'} d\tau [\{i\psi_i^\dagger \partial_\tau \psi_i + \frac{1}{2\mu} \psi_i^\dagger \nabla \psi_i - \mu \psi_i^\dagger \psi_i\} + \{i\psi_j^\dagger \partial_\tau \psi_j + \frac{1}{2\mu} \psi_j^\dagger \nabla \psi_j - \eta \psi_j^\dagger \psi_j\}],
\]

where \( \tau \) and \( \tau' \) is initial and final time, \( \nabla \) is the Laplace operator, and \( \eta \) is the chemical potential. This time evolution Lagrange ignore the interactions, e.g., the impurity induced long range order \( \text{eq} \). Since the half integer spin corresponding to the gapless area which mentioned above, the fermion system in this area can be written as \( H = \sum_{a=x,y,z} J_a \sum_{(i,j)} a^i \psi_i^a \psi_j^a \) \( (a = x, y, z) \), with \( \psi_i^x = s_i / i \) and \( \psi_j^y = s_j / i \), then we have \( H = -\sum_{a=x,y,z} J_a \sum_{(i,j)} s_i^a s_j^a \). In Eq.(1) we take the imaginary time approach which the quantum Monte Carlo (QMC) method is utilized \( \text{eq} \), the differential symbol \( \partial_\tau \) has the below relation according to the definition of Bernoulli number \( \text{eq} \).
and the differential symbol for mass is as the same form

\[ n \partial_{\tau} \left( \frac{\tau^{-1}-z}{1-z} \right) = \frac{\tau^{-1}-z}{1-z} \sum_{n=0}^{\infty} B_n \frac{(-\partial_{\tau})^n}{n!}, \]  
(2)

and the action of Landau-Ginzburg-Wilson (LGW) Hamiltonian with N-component O(N) symmetry and noncollinear order is

\[ \beta \]

\[ n \partial_{\mu} \left( \frac{\mu^{-1}-z}{1-z} \right) = \frac{\mu^{-1}-z}{1-z} \sum_{n=0}^{\infty} B_n \frac{(-\partial_{\mu})^n}{n!}. \]  
(3)

The Gardner transition which the critical dimension \( d_c = 3 \) is a important object in study of properties of amorphous solids. In (3+1) space-time dimension using the renormalized coupling, since the bare coupling is absent in the dimension \( d+1 = 4 \), the resulting dimensionless bare action with unbroken Quantum electrodynamics (QED) symmetry is

\[ S = \int dx \left\{ \frac{1}{2} \sum_{x,y=0}^{n} [(\partial_{\mu} \phi_{xy})^2 + r \phi_{xy}^2] - \frac{1}{3!} \left( g^b_i \sum_{x,y=0}^{n} \phi_{xy}^3 + g^b_j \sum_{x,y,z=0}^{n} \phi_{xy} \phi_{xz} \phi_{yz} \right) \right\}, \]  
(4)

and the action of Landau-Ginzburg-Wilson (LGW) Hamiltonian with N-component O(N) symmetry and noncollinear order is

\[ S = \int d^4x \int_{\tau}^{\tau'} d\tau \left\{ \frac{1}{2} \sum_{x,y=0}^{n} [(\partial_{\mu} \phi_{xy})^2 + r \phi_{xy}^2] \right. \]

\[ + \frac{1}{4!} \left[ g_i \left( \sum_{x,y=0}^{n} \phi_{xy}^2 \right)^2 + g_j \sum_{x,y,z=0}^{n} \left( \phi_{xy} \phi_{xz} \phi_{yz} \right) \right] \} \}

The summation index \( xyz \) range from zero to \( n-1 \) corresponding to the parameter space setted above, and the average term \( \sum_{x,y,z=0}^{n} \left( \phi_{xy} \phi_{xz} \phi_{yz} \right) \) exhibit the correlation between these two fluctuation functions. Using the method of time dependent density matrix RG which have been proved valid for particles at a fixed evolution time, this fermion system shown as \( T_{ij} \delta_{ij} = \text{Tr} \{ \sigma^i \sigma^j \} \) where \( T_{ij} \) is the interaction tensor, the \( \sigma^i \) and \( \sigma^j \) are the matrices of \( \psi_i \) and \( \psi_j \) respectively and \( \delta_{ij} = \{ c_i, c_j \} \). This expression is indeed take the diagonal part of \( T_{ij} \). Ref. put forward a valuable view that connecting the bare coupling to the renormalized coupling by a infinite cutoff, and then the mass-independent bare coupling can be shown as

\[ g^b = \mu^{3-d} \left\{ g + \delta_{11} \frac{g^3}{3-d} \right. \]

\[ + \delta_{21} \frac{g^5}{3-d} + \delta_{22} \frac{g^5}{(3-d)^2} \]

\[ + \delta_{31} \frac{g^7}{3-d} + \delta_{32} \frac{g^7}{(3-d)^2} + \delta_{33} \frac{g^7}{(3-d)^3} + O(g^9) \} \}

which is satisfactory consistent with the series expansion of \( \beta \) function given in Ref.

\[ \beta(g) = -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{(16\pi^2)^2} - \beta_2 \frac{g^7}{(16\pi^2)^3} + O(g^9). \]  
(7)

This \( \beta \)-function is series expand to the seven-order of coupling, i.e., the three-loop level for the gauge field. The specific quantitative analysis of \( \beta \)-function is presented in the Appendix A. Fig.1 show the \( \beta(g) \) as a function of \( g \) in SU(3) system (i.e. \( C_{ij}^{(2)} = 3 \) (see Appendix.A) with different number of fermion multiplets \( m \), We set the \( m \) from 0 to 20. It’s obvious to see that the curves shows a drastic non-linear change, and the \( m \)-dependent interaction tensor \( T_{ij} \) also played a decisive role in the relation between \( \beta(g) \) and \( g \).
3 Evolution Behavior in Non-Dissipation System

Since the long time scales exist in the metastable states which the quantity grows exponentially with system size\(^{20}\), e.g., the single positive charge state in p-type material\(^{20}\) or the p-spin model\(^{20}\), the imaginary-time path integral can be expressed by the trace of time evolution operator \(Z = \text{Tr}(e^{-\beta H})\) with the evolution propagator \(U = e^{-\beta H} = \text{Tr}(\sigma_i^1 \sigma_j^1 \cdots \sigma_i^n \sigma_j^n)\), where \(\beta\) is inverse temperature \(1/T\) which we use the unit of Boltzmann constant \(k = 1\). Note that the spin pauli matrix here is contain all the component in finite dimension of Hilbert space and \(H_{ij}\) is the nearest neighbor Hamiltonian and can be decomposed using the Trotter-Suzuki method which mapping the one dimension quantum system into two dimension\(^{61}\) and the path integral becomes \(Z = \text{Tr}(\Pi_i e^{-\beta H_i})\). In this way, the long range interaction can be treated locally as a nearest-neighbor pair in this spin isotropic system through a single two-qubit exchange gate \(U_{i,i+1} = e^{-\delta H_{i,i+1}}\). Due to the iterative nature and acting on two adjacent site with single time step \(\delta \tau\) evolution, it is also meets with the realignment criterion\(^{15}\), that is, the local field effect. Then we have

\[
e^{-\beta H_{i,i+1}} = \prod_i U_{i,i+1}. \tag{8}\]

Except the Anderson localization, the local length may strongly increase obey the logarithmic law\(^{62}\). The Hamiltonian here was divided by the partition function \(Z\) through the temperature interval or the external magnetic field \(h\).\(^{20}\) By investigate the asymptotic behavior of \(Z\), when \(\beta \to \infty\), i.e., the temperature decrease with the imaginary time evolution, the \(Z \to 0\), and the system tends to the ground state which is \(|\psi(0)\rangle = |\psi_1^1 \otimes |\psi_2^1 \otimes \cdots \otimes |\psi_1^n \otimes |\psi_2^n \otimes \cdots \otimes |\psi_n^n\rangle\), and denote the \(\varepsilon_{n'}\) is the energy of the \(n'\)th level \((n' < n)\) in this system above the ground state, \(\varepsilon_{n'} = E_{n'} - E_{n'-1}\). Then the pauli operator \(\sigma_{n'}^{i,j}\) within the evolution propagator \(U\) is \(\sigma_{n'}^{i,j} = \sigma_0^{0 \otimes n'} \otimes \sigma_{n'}^{i,j} \times \sigma_0^{0 \otimes (n-n')}\). Before that happen, the entanglement between particles which depending on time rapidly reaches the maximum value, which make the method of time dependent density matrix RG invalid due to the too large growth speed of entanglement entropy. The evolution by the evolution propagator \(U\) is

\[
|\psi(\beta)\rangle = U|\psi(0)\rangle, \tag{9}\]

and specifically, in the form with imaginary-time analogue \(e^{\pm H(\tau)}\) it has\(^{24,63}\)

\[
|\psi(\tau)\rangle = \frac{e^{\pm H(\tau)}|\psi(0)\rangle}{\|e^{\pm H(\tau)}|\psi(0)\rangle\|}, \tag{10}\]

where we define the imaginary-time as \(\tau = t + i0^+\), while for the evolution Hamiltonian is \(H(\tau) = e^{\alpha H} H e^{-\alpha H}\) where \(\alpha = \beta + i0^+\). Since \(\partial_\beta \psi(\beta) = H \psi(\beta)\), we have \(\beta \propto (\partial_\tau)^n\), which is also shown in the Eq.(2). For thermal average of a imaginary-time-dependent quantity \(F\), its expectation value which describe the ensemble average can be written as

\[
\langle F_\tau \rangle = \frac{\langle \psi(\tau)|F|\psi(\tau)\rangle}{\langle \psi(\tau)|\psi(\tau)\rangle}, \tag{11}\]

where \(\langle \psi(\tau)|\psi(\tau)\rangle\) is the partition function here, and the accurate value of \(\langle \psi(\tau)|\psi(\tau)\rangle\) and \(\langle \psi(\tau)|F|\psi(\tau)\rangle\) can be determined by the method of tensor RG. The cumulative effect is efficiently in this averaging process\(^{27}\) and often do a cumulant expansion at the expectation value for simplifield result whose truncation depends on the detail of dissipation\(^{64}\). Through this, a world-line tensor grid RG can be formed by taking coordinate as the horizontal axis, and the time (or temperature) as vertical axis, i.e., form a tensor network. The tensor network separated by the inverse temperature \(\beta\) have the spacing \(\zeta = \beta/M\) where \(M\) is the total number of lattices
in the network (also called the Trotter number \( \text{number} \)). Such a method which utilizes the evolution of time and phase also called Trotterization \( \text{trotterization} \). Through the theory of t-DMRG, the \( F \) can be treated as a matrix product operator which depends on the time-evolution, \( F(\tau) = U(\tau)F U^\dagger(\tau) \), here \( F_\tau \) and \( F \) base on different basis. With the nonequilibrium time evolution, the integrable system which has the important feature of localization will relax to the stationary state after quantum quenches, i.e., the suddenly change of interaction strength \( \text{strength} \), and the density matrices which constraint by the expectation value will leads to a maximum entropy ensemble \( \text{ensemble} \). Usually we model the integrable (superintegrable) model by choosing the special initial state, typically, like the XY spin model, and it can be affected deeply by the constants of motion in the integrable (superintegrable) points like reach the nonthermal steady state and so on. The density matrices here is depends exponentially on conserved quantity and the Hamiltonians which related to the initial state. For the matrix-product operators which describe the quantum states, the minimal rank \( D \) is required to the maximal one of the the reduced density matrix of bipartition system \( \text{system} \) (i.e., bipartition of the target one and its environment) and it needed to truncated by the method of singular values decomposing to keep the size of \( D \) polynomial increase which is local and time-computable, and we keep only the largest singular values after the truncation, i.e., only keep the basis states \( \text{states} \). In fact, for dissipation system, the linear or nonlinear dissipation coupling accompanied by the phase noise \( \text{noise} \) (like the Wiener noise (see Sect. 11)) as well as the white noise or colored ones \( \text{ones} \) also have inhibition on the exponential increase.

In Schrödinger picture, the observables of thermal states are achieved by carry the integrable system into the nonintegrable one (by perturbations) and in the mean time the energy-level spacing distribution is evolves from the Poisson distribution with diagonal matrices to the Gaussian one (i.e., the wigner-Dyson type one) with level repulsion and random symmetric matrices \( \text{matrices} \) (there are also symmetrically ordered operators in quantum dynamics by Wigner representation \( \text{representation} \)). It’s possible to back to the Possion distribution by applying a series of single gate which prevent the exponential increase of rank \( D \) but introduces the norm error \( \text{error} \),

\[
\eta = \sum_{i=0}^{n-1} (1 - \sum_{j=0}^{D-1} \lambda_j^2(U_i)),
\]

(12)

where \( U_i \) is a single gate and \( \lambda_j(U_i) \) is the decreasing ordered singular values after removing the maximum one, and the maximum entropy is accessible through the local relaxation and the same as the entanglement. Although for nonintegrable system the growth of \( D \) is founded to be exponential, there exist methods like the diagonalization which keep the size of matrix always proportional to the time (or the system size), like Bogoliubov rotation (see Appendix.C). The procedure of eliminating the small singular values result in a low-rank matrix, and this is also to keep the local free energy

\[
E_{\text{free}} = -\frac{1}{\beta} \ln(\sum \lambda_i^2)
\]

(13)

smallest \( (\lambda_i \text{is the singular values}) \), and also to enhance the equilibrium characteristics which treated as a thermodynamics anomaly in glass system \( \text{system} \). This equation also explicitly show the measurement of energies in units of (inverse) temperature. To solve the problem of density matrix in the t-DMRG, one introduce a way to solve the rank minimization problem which make this method valid even for the low rank matrices (see Ref. \( \text{Ref} \)), and it’s help to reducing the error and keep computational cost low at the same time. On the other hand, that also provide the convenience that make the matrix nondecreasing and so that the maximum rank is always appear in the final step of the algorithm.

Since we have implement the system-environment partition, in a full quantum dynamics, we can yield a well approximation in the weak-couping regime by the low-order truncation, e.g.,
the Wigner truncation approximation which truncate in the power of one-order \(48\). In such a phase space, the coupled two subsystem have the relation 
\[
\sum_{k_i,k_j}(-k_i!/A^{k_i})(-k_j!/A^{k_j})g^{k_i}g^{k_j} = \sum_k(-k!/A^k)g^{2k}\]
where \(k\) is the number of powers of truncation in phase space (e.g., \(k = 1\) when truncate in the first-order) and \(A\) is the angles which dominate the series expansion of the dimensionless coupling \(g\) (see Sect.2).

From the discussion on this section, we can see that the imaginary-time propagation has the similar behavior with the real-time one, it will provide us another way to detect the decaying progress including the die out of excitations, and it’s available for similar real-time setups\(54\), or application to the nonequilibrium problem with stochastic series expansion in integrable system without the constraint of local conservation law. Therefore it’s more feasible to detect the asymptotics phenomenon in time evolution, especially for the low-order perturbation theory with extended potential.

4 Quenching in Many-Body Local System

For integrable open system, we imagine the bipartion of the Hilbert space and into the two formulated finite-dimension linear space (two associated configuration) \(V_i\) and \(V_j\) which assumed have same spectrum and their reduced density matrices are

\[
J_i = \sum_{R=1}^{R} \lambda_R |\psi_R^i\rangle \langle \psi_R^i|,
\]
\[
J_j = \sum_{R=1}^{R} \lambda_R |\psi_R^j\rangle \langle \psi_R^j|,
\]

where \(\lambda_R\) is the Schmidt coefficients (the decreasing singular values). The bipartite state \(|\psi\rangle \in \mathbb{C}^{d_i} \otimes \mathbb{C}^{d_j}\) which realized through the Schmidt decomposition via singular value decomposition, and the Schmidt rank is \(\min[d_i, d_j]\). For inseparable case, the reduced density matrix \(J'_i\) (if it’s pure state density matrix with feature of unitarily invariant) can be obtained by tracing over the the pure state in its extended subsystem (i.e., \(\mathbb{C}^{d_j}\)), and the product space which form by two subsystem is \(V_i \otimes V_j\). This bipartition can be used in most of the quantum many-body model, like the Ising transverse field model, XXZ model, and kitaev model, etc.

Integrability is usually relies on the localization, especially the superintegrable one (like the XY spin model) which are fully relies on the localization\(34\). For a concrete example, we consider a XY spin two-chain model without the magnetic field, which the bulk Hamiltonian is

\[
H_{i,i+1} = \sum_{i=0}^{N-1} \frac{1}{2}(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) \cdot \exp[iJ \sum_{j=0}^{N-1} (\sigma_i^{2x} + \Theta(j-i) + \sigma_i^{2y} + \Theta(j-i))],
\]

where \(J\) is the coupling, \(i, j\) stands for the different chains, and \(\Theta(j-i)\) is a step function. The correlation in such a system is

\[
\langle s_i, s_j \rangle = \frac{1}{N-1} \sum_{i,j} s_i s_j = \frac{1}{2} q_{ij}(N-1),
\]

where \(q_{ij}\) is the overlap between these two spin configuration. The local quantum integrability in the bounded bulk model can be deriving by the explicit form of the quantum R-matrix as well as the boundary transfer matrices, e.g, see Ref.\(34,73,74\).

For quench behavior due to the perturbation from local operators, which for the out-of-equilibrium protocol is striking, the amplitude from initial state to instantaneous \(n\) state is\(75\)
\[ A_n(t) = -\int_{t_i}^{t_f} dt \langle n | \partial_t | 0 \rangle \exp[i(\varphi_n(t) - \varphi_0(t))] \quad (0 \leq i \leq n), \]  

where \( \varphi(t) \) is the dynamics phase. Such a amplitude is also the eightvalue of density matrices in entropy ensemble with the specific heat \( \sum_n [E_n - E_0] A_n(t)^2 \). The sum of square of amplitudes is the excitation probability \( P_{ex} = \sum_n |A_n(t)|^2 \) for electrons, particles, or holes, i.e., quenched away from initial (ground state) to new state. Here we suppose the quench is very fast that the initial state \( \psi_0 \) and the quenched state \( \psi_n \) are amlost exist at the same time \( t_i \). Then using the evolution propagator \( U(t) \), we obtain the amplitude\(^{76}\)

\[
\langle n | U(t) | 0 \rangle = -i \langle n | \int_{t_i}^{t_f} dt H(t) | 0 \rangle \\
= -i \langle i | H_{int} | 0 \rangle \int_{t_i}^{t_f} dt \exp[i(E_n - E_0)t'] \\
= -\langle i | H_{int} | 0 \rangle \frac{\exp[i(E_n - E_0)t] - 1}{E_n - E_0},
\]

where \( E_0 \) is the energy in the initial state \( \psi_0 \). Through the fermi golden rule, where \( H_{int} \) is the interaction Hamiltonian with scattering amplitude \( A_i \), which is

\[
H_{int} = \frac{U(t)(E_n - E_0)}{\sqrt{2 - 2\cos[(E_n - E_0)t]}},
\]

For further detect the perturbation from local operators, we present in Fig.2 (a) the energy difference between the excited state and initial one with different staggered magnetic field \( h_s \) in different dimension \( D \) of a quantum lattice model, and (b) the excitation probability as a function of the temperature, it’s clearly that the probability distribution obey a Gaussian form. Since the quantum noise comes from the random initial state, we define a Gaussian in the initial state white noise which have a zero mean and therefore the initial probability distribution is Gaussian. Then the probability distribution in the process of relaxation is\(^{77}\)

\[
P = \sum_{m,n} \delta[\Delta E - (E_m(t') - E_n(t_i))]|\langle \psi_{m}(t') | U(t) | \psi_{n}(t_i) \rangle|^2|\langle \psi_{n}(t_i) | \psi_{0}(t_i) \rangle|,
\]

where \( \delta \) is the amplitude of the Gaussian (see Sect.11) and \( \Delta E \) is the energy-difference between the initial and final state of relaxation. In phase space, such a relaxation can be expressed by the density matrix

\[
\mathcal{J}(t) = \sum_{k,k+q} \exp[-\phi(k)t] \mathcal{J}(0) = \sum_{k,k+q} \exp[-(E_{k+q} - E_k)t] \mathcal{J}(0),
\]

where \( k \) and \( k + q \) are two spectral parameters. For slow quench which the time scale to infinity, the non-diagonal contribution to \( \mathcal{J}(t) \) (i.e., the part of \( q \neq 0 \)) is vanish due to the fast oscillation of Fourier kernel \( \exp[-(E_{k+q} - E_k)t] \).

In fact, the non-diagonal contribution to the mean-field-representation (or the second moments of the distribution of momentum\(^{27}\)) \( \langle c_i c_{i+1}^\dagger \rangle = \int d^n k f(k) \cos(\varphi(k)t) \) is asymptotically to a fixed value with the time evolution\(^{14}\). When a external perturbing field is considered in the free energy landscape, a perturbing term should be added to the local free energy, and since the perturbation is bad for the conservation of energy, the quantum system under the influence of noise variables will not completely isolated even for the closed quantum system. The coupling between this perturbing field and the Hamiltonian is beneficial to enhance the system ergodicity.
by increase the coupling of metastates. For closed system which have total energy conservation, the ergodicity for observables under the long-time limit can be large enough to expect the time average to the thermal average\cite{28}, but there are restriction on the observables like the bound of the von Neumann entropy, and hence prevent it closing the thermal state. (Note that here the correlation between each distinguishable particle and the environment is still localized.) The entropy of pinning field is increase with the overlap in a metastate, can associate with the hidden glass states, and it’s confirmed equal to the mean field potential of glass system\cite{78}.

Both the entropy $S_{\text{hidden}}$ (not the diagonal one) and its free energy as well as the non-diagonal contribution vanish in the final of the process of relaxation to steady equilibrium state, e.g., the commensurate superfluid state.

Since for the integrable system, most solvable Hamiltonian can be mapped to the effective noninteracting Hamiltonian\cite{32}

$$H_{\text{eff}} = \sum_{i} \epsilon_i P_i$$  \hspace{1cm} (23)

with the eigenenergy $\epsilon_i$ and conserved quantity $P_i$, and the maximum entropy ensemble after quench with local conserve-law can be written using the density matrix as

$$J_{\text{quenched}} = \frac{1}{Z} \exp(-\sum_{i} P_i Y_i)$$ \hspace{1cm} (24)

where the conserved observable quantity $P_i$ has the form $P_i = a_i^\dagger a_i$ where $a_i$ is the annihilation operator of bosons or fermions and has commute relation $[H, P_i] = [P_i, P_i'] = 0$, the $Y_i$ is a initial state-dependent quantity. The partition function $Z = \text{Tr}[(\exp(-\sum_i P_i Y_i))]$. This is in fact only a local steady state but not canonical steady states for the full system\cite{25}. For integrable system begin with the maximal entropy in GGE, the $Y_i$ here can be replaced by a Lagrange multiplier set \{$\lambda_i$\}\cite{32,34,79,80}, (which is\cite{81} $\lambda_i = \ln[(1 - \langle \psi(0)|P_i|\psi(0)\rangle)/\langle \psi(0)|P_i|\psi(0)\rangle]$ and constrained by $\langle n \rangle_{\text{GGE}} = \langle \psi(0)|c^\dagger c|\psi(0)\rangle = \text{Tr}(\rho n)$ where $n$ is the conserved number of particles). For integrable systems which are exact solvable (i.e., all the eigenvalues and eigenfunctions can be obtained), since the $\epsilon_i$ is linear eigenenergy, for a simplest conserved quantity, the number of particles $n_i$, the number eigenstate can be treated as the energy eigenstate $E = \sum_i \epsilon_i n_i$ which on the eightbasis of \{$n_i$\}\cite{48}.

Within the scheme of adiabatic perturbative $(k \cdot p)$ theory, the asymptotic behavior be manipulated by the velocity and acceleration of tuning parameter in quench dynamic\cite{75}. The tuning-dependent Hamiltonian $\psi(\lambda(t))$ ($\lambda(t)$ is the time-dependent tuning parameter) can also take effect in the adiabatic excitation of system in ground state which is similar to $\psi(t)$, and recover due to the asymptotic effect of time evolution\cite{54}. The asymptotic freedom of system will preserved until the number of fermion species is too large\cite{30}, so this asymptotic state with the scaling theory is depend only on the configuration, e.g., the fluctuation of system\cite{18,54,82}, and scales show a collection of the effects from fluctuation and tend to Gibbs value when the momentum vector $q \rightarrow 0$\cite{82}. One of the reflection is the equilibrium Gibbs free energy as below\cite{3} (without restrictions)

$$E_{\text{Gibbs}} = -\frac{1}{\beta} \ln \int dt e^{-\beta H(t)}$$ \hspace{1cm} (25)

and since the Hamiltonian here is often the potential field-characterized, the free energy also treated as a potential function with determined weigh (probability distribution).
5 Dissipation in Nonlocal Model

For nonlocal model, there is a large different compare to the local one. The nonequilibrium long-range force is also usually unobservable in localiaed interaction models\cite{2020}. Consider the Yang-Mills theory, the action of field can be expressed as

\[ S = \frac{1}{4} \int d^d x \int_0^\tau d\tau \hat{F}_i^{\mu \nu} F_{i i}^{\mu \nu}, \]

where \( F_{\mu \nu} \) is the field strength tensor (see Appendix.A), \( F_{i i}^{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g C_{iab} A_\mu^a A_\nu^b \), where \( A_\mu^a \) and \( A_\nu^b \) are the vector potential of the field and here \( C_{iab} \) is for introduce a SU(3) structure factor which is \( C_{iab} = \gamma_{iab} F^a F^b \), where \( \gamma_{iab} \) is the SU(3) structure constant and \( F^a \) is the group generator. The relation between the Lie group structure constant \( C \) and quadratic Casimir operator is \( \sum_{ij} C_{iab} C_{jab} = C_{ij} (2) \delta_{ij} \).

The dissipative effect which derived from the macroscopic entangled system give arise the reservoir problem and accompanied by a process of coarse-graining by the isometry that integrating the degrees of freedom of subsystems\cite{2020} and with a dimension smaller than the maximum dimension of Hilbert space\cite{2020}. The nonlocal correlation between the nearest neighbors can be treated locally by using the matrix product operator with determined rank and the unitary transformation with time-evolution oparator (see below). For the localized interaction with nearest neighbor spin accompanied by the local field effect, since there is for large coupling constant and long time configuration, it’s priority to use the nonperturbative method\cite{2020}, but the quantum dissipation which is nolinear is more acceptable to use the perturbative RG, and the reservoir interaction is also perturbed, The Gaussian probability distribution which exist in the linear case is not exist in the nolinear case anymore, and the dimension of density matrices is also grows non-linearly with time\cite{2020}. But there still exist some linear relation, e.g., the entanglement entropy is change linearly with the evolution of time with a staggered magnetic field in the disordered case\cite{2020}.

In a open quantum system, the thermal average of observable \( \mathcal{F} \) can be written as (here \( \tau \) is the complex-time for propagators)

\[ \langle \mathcal{F} \rangle_\tau = \frac{\text{Tr}(e^{-\beta H} e^{-H\tau} \mathcal{F} e^{H\tau})}{\text{Tr}(e^{-\beta H})}. \]

For integrable system, this equation which describe the thermal average in Gibbs ensemble\cite{2020} is equal to the energy of initial state of relaxation process after quench which evolution with time \( \tau \). Such thermal average is also meaningful in thermodynamics description for quasi-equilibrium state\cite{2020}. Base on the Eq.(8) and using the second order Trotte-Suzuki formation, the evolution propagator can be decomposed as \( e^{-\beta H} = e^{-\beta H_x} e^{-\beta H_y} + O(\tau^2) \), and the Eq.(8) can be rewritten as

\[ e^{-\beta H_{ij}} = \prod_{i=0,j=0}^{n} U_{a_i+a_j,j+a_j} \quad (a = x, y, z). \]

To study the dissipation of the remaining degrees of freedom in subsystems after coarse granulation in such a no-spacing-interaction macroscopic model, the reservoir is very important. To introducing FDR to the steady state, we rewrite the Eq.(27) by the method of path integral as

\[ \langle \mathcal{F} \rangle_\tau = \int D\psi(\tau) e^{\tau H} \frac{\langle \psi(0 + \varepsilon^+) | \mathcal{F}(0) | \psi(0 + \varepsilon^-) \rangle}{\langle \psi(\tau + \varepsilon^+) | \mathcal{F}(\tau) | \psi(\tau + \varepsilon^-) \rangle}, \]

with \( \varepsilon \to 0 \), and \( \psi(\tau) = U \psi(0) \) where \( U \) the time-evolution oparator \( U = T_\varepsilon \exp(- \int_0^\tau d\tau H(\tau)) \). For statistical linear dissipation system, the correlation between reservoirs \( \langle R_i R_j \rangle \neq 0 \), the
method of unperturbed linear dissipation is also suitable for perturbed macroscopic model if the perturbation Hamiltonian is linear with reservoir $H_p = \sum_i f(i) R_i$ where $f(i)$ is a linear term and therefore the collective response to perturbation is mostly linear. This form of $H_p$ is suitable for all the integrable or nonintegrable linear dissipation model. While for the non-linear dissipation case, since the reservoirs in different subsystems is independent with each other, so we constraint the reservoir states in the Liouville spaces, and have $\langle R(0)|H_{SR}|R(\tau)\rangle = 0$, where $H_{SR}$ is the interaction term between system and reservoirs and there exist shared influence function for all constituent.

For non-dissipation system, the propagation along time scale can be expressed by the initial Hamiltonian and the observable conserved quantity (i.e., Eq.(24)), whereas for linear dissipation, since it need a stochastic term to compensate the lost energy, and it has a history-independent potential term $\partial_{\tau} \psi(\tau) = H_0(\tau) \psi(\tau) - \sum_i f(i) q(i) \psi(\tau)$, where $q(i)$ is the stochastic force or the noise. For nonlinear-dissipation system, the state of reservoir variables is span only in the Liouville space. Both the linear-dissipation and nonlinear-dissipation contain a friction force term but the nonlinear-dissipation have a complex memory which it’s obvious from the feature of history-dependent evolution while the linear one haven’t.

6 Time Evolution and Thermal Entanglement in Integrable Heisenberg XXZ Model

We already know that for non-dissipation system the antiferromagnetic Ising chain and the XY spin chain and the bulk model is integrable and can be exactly solved. The Heisenberg XXZ model is also suggested integrable and own the local conserved quantity, e.g., the observable microscopic quantity like the $S^z$ or the observable macroscopic one like energy or number of particles. To investigate the imaginary-time evolution in Heisenberg XXZ model, we firstly need to use a c-number representation which depict a shift of $J_{x,y,z}$ transformation which turns the regular integrable system terms into the chaotic one. In this case, this

$$H = \frac{J_z}{J} = \left\{ \begin{array}{ll} \cos \gamma, & J_z \leq J \\ \cosh \mu, & J_z > J \end{array} \right. \quad (31)$$

where the tilted angle $\gamma$ and $\mu$ is enlarge with the increase of degrees of anisotropy. We focus on the $J_z/J = \cos \gamma$ case. In the case of $J_z = 0$, i.e., becomes the noninteracting spinless fermion system with strongly correlated electronic characteristics under the Wigner-Jordan (WJ) transformation which turns the regular integrable system terms into the chaotic one. In this case, the fermion representation of the gapless bilinear fermionic system is

$$H_{bf} = \sum_i (c_i c_{i+1}^\dagger + c_i^\dagger c_{i+1} + h_i n_i), \quad (32)$$

with $\Delta_i = \langle c_i c_{i+1}^\dagger \rangle$ stands for a mean-field and also represent the covalent bonding of WJ fermions, and this is also the tight-binding fermionic model with dispersion relation $\kappa = \pm 2\cos k$ in $\pi$-phase (the phase difference between neighbor site is $\pi$). In this case, this
Heisenberg Hamiltonian becomes a strongly correlated electronic system with a finite entropy (will saturation). The operator of number of the spinless particles is \( n_i = c_i^\dagger c_i \), and the electron correlation is \( J_z n_i n_{i+1} \). To investigate the nonlinear-dissipation in this spinless fermions chain model, we need to introduce the master equation with system density matrix \( \mathcal{J} \):

\[
\partial_t \mathcal{J} = -i[H, \mathcal{J}] + \mathcal{K} \sum_i (O_i \mathcal{J} O_i^\dagger - \frac{1}{2} (O_i^\dagger O_i \mathcal{J} + \mathcal{J} O_i^\dagger O_i)) \equiv \mathcal{L} \mathcal{J},
\]

where \( \mathcal{J} \) corresponds to the pure state or mixed state and \( O_i \) is the Lindblad operator describing the bath coupling. The right-hand side of this equation contain two terms, the first one is the unitary part of the Liouvillean, while the second one is the dissipative term and \( \mathcal{K} \) is the coupling strengths within the dissipation scenario. We consider the damping here due to the nonlinear-dissipation. The Gaussian area arrived in time evolution have \( \partial_t \mathcal{J} = 0 \), in this case \( \mathcal{K} \) is almost vanish and produce a zero dissipative area, that suggest that the observables exponential fast approach to the steady state, while the entries of the density matrix is close to the main diagonal.

To introduce the thermal entanglement in the evolution, we define the generate and annihilate operator for \( i \) sites as

\[
c_i^\dagger = e^{i\varphi_i} S_i^+, \quad c_i = e^{-i\varphi_i} S_i^-.
\]

The operators obey commutation relation \([c_i, c_j^\dagger] = \delta_{ij} \) (boson operator and fermion operator for \( \alpha = 1 \) and \(-1 \), respectively), and \( c_i^\dagger c_j + c_j c_i^\dagger = \delta_{ij} (\alpha = -1) \) under the WJ transformation that treat \( c_i \) as operator field. The time-involve phase \( \varphi_i \) have

\[
\varphi_{i+1} - \varphi_i = cn_i,
\]

where \( c \) is a c-number-correlated factor which defined as the imaginary part of \( \text{In} (\tau' - \tau) \), i.e., the scale of imaginary-time, and the phase function \( \varphi_i = \sum_i c_i^\dagger c_i c \). Then the Hamiltonian (Eq.(30)) can be represented as

\[
H = \begin{bmatrix}
J_z/2 + h & 0 & 0 & 0 \\
0 & -J_z/2 & J & 0 \\
0 & J & -J_z/2 & 0 \\
0 & 0 & 0 & J_z/2 - h
\end{bmatrix},
\]

when \( |J| < h - J_z \), the ground state is disentangled state \(|0, 0\rangle \) which have the eightvalue \( J_z/2 - h \); when \( |J| > h - J_z \), the ground state is entangled state \( \frac{1}{\sqrt{2}}(|0, 1\rangle - |1, 0\rangle) \) for \( J > 0 \) or \( \frac{1}{\sqrt{2}}(|0, 1\rangle + |1, 0\rangle) \) for \( J < 0 \) which have the eightvalue \( -J_z/2 - |J| \), and this entangled state will goes to maximal with time-evolution. Thus, the entanglement increase with the enhancement of coupling \( J \) and \( J_z \) no matter they are both greater than zero (ferromagnetic) or both less than zero (antiferromagnetic), but it’s always symmetry compare to the case of inhomogeneous magnetic field. We can obtain the relaxations in long-time scale after the sudden quench of \( J \) and \( J_z \), and regulate the entanglement by the quench of magnetic field \( h \). In equilibrium case, the density matrix of this thermal state can be written as

\[
\mathcal{J} = \frac{1}{Z} \exp(-\beta H) = \frac{1}{Z} \begin{bmatrix}
e^{-(J_z/2+h)/T} & 0 & 0 & 0 \\
0 & e^{J_z/2T} \cosh(|J|/T) & -s & 0 \\
0 & -s & e^{J_z/2T} \cosh(|J|/T) & 0 \\
0 & 0 & 0 & e^{-(J_z/2-h)/T}
\end{bmatrix},
\]

where \( Z = e^{-(J_z/2+h)/T} (1 + e^{2h/T}) + 2e^{(J_z+h)/T} \cosh(|J|/T) \) and \( s = J e^{J_z/2T} \sinh(|J|/T)/|J| \). Usually, we can creating strong entanglement by raising the ratio of \( J_z/J \), or raising the degree
of inhomogeneity of magnetic field $h$, or properly lower the temperature through the previous study\cite{20,89,90}. Sometimes the lower temperature which can be implemented by increase the system size\cite{90} can decrease the eigenvalue of density matrix (Eq.(37)).

7 Correlation and Transfer Speed in One-dimension Chain Model

In this section, we focus on the two-point spin correlation in $S = 1/2$ Heisenberg chain and $S = 1$ Ising chain, and define that the $J_1$ ad $J_2$ as the nearest neighbor coupling and next-nearest neighbor coupling in the chain, respectively. The $\beta$ (inverse temperature)-dependent magnetic susceptibility can be written as $\chi(\beta, t, i) = \beta 2^{-n} \sum_{i=1}^{n-1} \langle S_i^z S_i^z \rangle$ for a n-qubit chain, the latter term in this expression is the spin-spin correlation function for the Heisenberg model\cite{23}. The Fig.3 shows the spin correlation $C$ and inverse correlation length $\xi^{-1}$ for (a) $S = 1$ Ising spin chain and (b) $S = 1/2$ Heisenberg chain with different $J_2$ at different site $i$. We show that the nonlocal order parameter decay exponentially due to the perturbations from long-range spin-spin interaction which breakig the integrability and therefore exhibit a effectively asymptotic thermal behavior, though the latter one is exactly solvable (i.e., all eigenvalues can be obtained by the method of Bethe ansatz in thermodynamic Bethe ansatz (TBA)\cite{91}) before the perturbation. Such an exponentially decay for the nonlocal operators in nonintegrable model has been widely observed, e.g., the order parameter in transverse field Ising chain for ferromagnetic state or paramagnetic state\cite{92} or the number of quasiparticles in the time evolution in a quantum spin chain\cite{93}, etc. We also can see that the $\xi^{-1}$ is tends to saturated with the increase of distance which is obey the equilibrium law, and in fact it’s equivalent to the coherent state with coherent amplitude in terms of a exponential form, and therefore the phase coherence rate will display a similar behavior with the correlation length. Fig.4 shows the spin correlation for $S = 1/2$ Heisenberg chain as a function of temperature with different $J_2$. We can see that, with the increase of $J_2$, the spin correlation is increase. We also make the comparision for the spin correlation $C$ at different temperature for $S = 1$ Ising chain and $S = 1/2$ Heisenberg chain in the Fig.5. It’s obviously that the $S = 1/2$ Heisenberg chain is earlier becoming saturated compare to the Ising one. Furthermore, we present the correlation (a) for $S = 1/2$ Heisenberg chain which is obtained by the by the method of Bethe ansantz and make a comparison on the results of correlation in low-temperature for $S = 1/2$ Heisenberg chain between the methods of Bethe ansatz and renormalization group (b) in Fig.6.

Since the equal time spin correlation $C$ have the relation

$$C(r, t) = \langle S(0, t) S(r, t) \rangle \propto \exp(-r/\xi),$$

which is consistent with the expression of correlation length $\xi^{-1} = - \lim_{L \to \infty} \ln \langle S_i S_{i+L} \rangle$ in Ref.\cite{41}, here the distance $r$ can be quantified as $i$ which stands for number of position in spin chains and $\xi$ is the correlation length. Note that this expression for equal time two-point correlation is well conform the ordered phase in the long-time limit, while for disordered phase the $\xi$ has more complicated form\cite{22}. Now that this spin correlation function display a effective asymptotic thermal behavior as introduced in Sect.1, and correlation length $\xi$ is related to the quantum quench protocol\cite{40}, the thermal behavior for a nondissipation system after quench can also have the relation which mentioned above (Eq.(38)), but note that although this spin correlation here is in a exponential form, the correlation length is not follow the thermal distribution but a nonthermal distribution\cite{39} and guided by GGE. This is because the correlation length is local quantity which behave nonthermally. Similar behavior appear in the correlators like the transverse magnetization and so on. We still need to note that though for infinite system which follow the effective thermal distribution is mostly nonintegrable, but the initial state of integrable system which dictated by the noninteracting Hamiltonian may still follow the thermal disturbance since without the affect of interactional quench Hamiltonian.
Further, if we mapping to the Fourier space, the equal time correlation (Eq.(38)) for the spin-1/2 square lattice model has a more specific form:

\[ \langle S_i(0)S_j(r) \rangle e^{ikr} \propto \frac{e^{-r/\xi}}{r^4} (1 + \frac{r}{\xi}) \delta_{ij}, \]  

(39)

which follow the power law decay when \( r \ll \xi \) and exponentially decay when \( r \gg \xi \).

Since the pinning field play an important role in the process of ergodic to non-ergodic transition which plug the correlation between subsystems and even the velocity of spin wave \( v_s \), which associate with the slope of the dispersion relations in momentum space. For the case of \( J_z/J = \cos \gamma \), \( v_s \) can be written as:

\[ v_s = \frac{J \pi \sin \gamma}{2} \]  

(40)

which is consistent with slope of dispersion relation \( \partial_k \kappa = \pm \sin k \). Then a question is arisen that if the speed of information transfer which govern the relaxation time of a post-quench state relate to the speed of spin wave in a spin system? The answer is yes. A direct evidence is the Lieb-Robinson type boundary (the details in a Bose-Hubbard model is presented in the next section). In fact the spin wave is also related to the momentum transfer and even the damping of oscillation of superfluid regime (see Sect.8 and Ref. 101). We know that the missing of symmetry is related to the influence of initial states, and the collapse of physical phenomena like the interference pattern or the collective excitation by inhomogeneous oscillation in condensate with a density wave order which act like a single phase wave or standing wave, is revives in the latter time of relaxation. the transfer of correlation with a finite velocity also construct a line-cone which well describe the relaxation behavior.

8 Double Occupation and The Interaction Quench in Nonintegrable Hubbard Model Near The Phase Transition Point

Since the time evolution operator is dependents on the Hamiltonian (like Eq.(8)), we next construct the Bose-Hubbard lattice model as a explicit example:

\[ H = -P \sum_{i=0}^{n-2} (b_i^\dagger b_{i+1} + H.c.) + U \sum_{i=0}^{n-1} n_i(n_i - 1) - \mu_i \sum_{i=0}^{n-1} n_i \]  

(41)

where \( P \) is the hopping constant, \( U \) is the chemical potential and \( \mu_i \) is the local potential of each particles. The interaction between the next-nearest neighbor is assumed zero in this model, and so that this model is integrable, i.e., the second term of above equation can be replaced as \( U \sum_{i=0}^{N-1} n_i n_{i+1} \). A dimensionless reduced coupling is defined as:

\[ g_{\text{red}} = \frac{U N}{P} \]  

(42)

where \( N \) is the number of interactional particles. We can implement the phase transition from Mott-insulator to the condensed state or the superfluid by modulating \( g_{\text{red}} \), and it has been implemented experimentally. Even for systems which without hopping at all (i.e., \( P = 0 \)), the phase transition of metallic state and the Mott insulator are also realizable by the interaction quench of \( U \), and in this case the oscillations with the collapse-and revival are periodic with period \( 2\pi U/\hbar \) (The Table.A shows the time scale of relaxation and the period of collapse and revival for several models). In fact, most mang-body system can exhibit different quantum phase with different entanglement structure in the complex mixed dynamical, and it’s
usually realizable by tuning the strength of this competing interaction. The fluctuation of correlation amplitude due to the fast oscillation of phase factor are related to the distribution of initial state, and the short-range correlation also shows distinguishable differences for different configuration of initial states.

In this model we next define the hopping-determined operator \( R := itP \), this periodic-time-dependent evolution operator for a single-site can be expanded as

\[
e^R := e^{itP} = \sum_{k \geq d_r} \frac{(itP)^k}{k!} \leq \sum_{k \geq d_r} \frac{(6Pt)^k}{k^k}
\]

where \( k \) denotes the unit vector in phase space and \( d_r \) is the distance between site \( i \) and \( i + r \). There exist a upper bound for \( d_r \) as \( d_r < 6Pt/e \) where \( e \) is the natural constant, since it’s a insurmountable maximum speed for information transfer in this model. The summation of all the other places which beyond the distance \( d_r \) have the above relation. Thus we also have

\[
e^R \leq \frac{(6Pt)^{d_c}}{d_c^c - 6Pt \cdot d_c^{d-1}},
\]

which requires \( d_r > 6Pt \) while the critical distance \( d_c \) which corresponds to the upper bound is nearly equals to \( 6Pt \). If we relate the conserved particles-number \( P \) to a matrix, then it has operator norm \( \|PP^*\|_{op} = 1 \) and \( PP^* = PP^t = I \) where \( I \) is a identity operator. This is related to the case mentioned in the Ref. that \( n_i \) only have the two eightvalues 0 and 1, and here the maximal eightvalue 1 is nondegenerate for our scenario, while other eightvalues approaches to 1 smoothly in the long-time limit.

Since the in long-time limit the relaxation will removing the non-diagonal part of the density matrix, the difference between the density matrices and its diagonal one is \( \Delta J = J(t) - J_G \), thus for the hopping matrix which mentioned above, its trace norm has

\[
\Delta J \leq \frac{(6Pt)^{d_c}}{d_c^c - 6Pt \cdot d_c^{d-1}},
\]

Note that here the critical value \( d_c \) is independent of the size of system.

We have present the upper bound of of speed of information transfer by a form of suppressed exponent. Since the nondiagonal contribution won’t vanish until \( t \to \infty \) (which corresponds to \( \Delta J = 0 \)), and it’s decay in a time scale as \( 1/t^{(d+\alpha)} \), i.e., the dephasing process, (note the for large-size system, the inequality of Eq.(45) will becomes more obvious, and the vanished nondiagonal contribution will reappear if the size is large enough, which called “rephasing”), the phase can be expressed as \( \varphi(k) = \varphi(0) + q^\ell + O(q^\ell+1) \) where \( \ell \) is a tunable parameter in phase space. The contribution in such a dephasing with scale \( 1/t \) in phase space is

\[
k^\ell = \int dk^\ell e^{i\varphi(k)} \frac{k^{1-\ell}}{\ell} \int d^{d-1}kf(k),
\]

where \( \varphi(k) = \varphi_0 + k^\ell \).

Next we form the the Bessel formula to show the reducing property of the evolution operator \( e^{itP} \) which with large size \( N \) and can be viewed as the Riemann sum approximation of the following function with phase number \( \alpha \).

\[
J_\alpha(x) = \frac{1}{2\pi i^\alpha} \int_0^{2\pi} \exp[i(\alpha \varphi + x \cos \varphi)]d\varphi
= \frac{1}{2\pi} \int_0^{2\pi} \exp[i(\alpha \varphi - x \sin \varphi)]d\varphi
\]
which is shown in the Fig.7. Through this, the maximum rate for the system to relaxation to the Guassian state is obtained as $(2\mathcal{P}t)^{-N/3}$ for a $N$-site system.

For this one dimension bosonic system what we are discussing, the Mott gap $U - U_c$ is allowed to exist during the relaxation process, for a experiment, see Ref.109. For coupled bose-lattice model, one forms the time-dependent continuous variable $n(t)$ to describe the quasiperiodic decaying, the semiclassical motion equation which in a continuum bath of harmonic potential and additively applied a confining parabolic potential, is

$$
\frac{d^2n}{dt^2} + 4n + 4g_{red}n \left[ \cos(\varphi(0)) + \frac{g_{red}n^2}{2} \right] = 0,
$$

where $\varphi(0)$ is the initial phase. Thus the double occupation $\langle n^2(t) \rangle$ (also the double momenta occupation number in momentum space) under the quenches from different Mott insulator initial state (with different initial phase) to weak interaction one (with weak $g_{red}$) is

$$
\langle n^2(t) \rangle \approx n^2(0) - \frac{1}{2\pi} \int_0^\pi \sin^2 \varphi(0) \cos[4t\sqrt{1 + g_{red}\cos \varphi(0)}] d\varphi(0),
$$

where $n^2(0) = 1/4$ here as a effective approximation for two uncouping system in semiclassical theory. The $n^2(t)$ with weak $g_{red}(< 1)$ according to Eq.(49) is shown in the Fig.8, note that since the critical value of interaction for superfluid-to-Mott insulator phase transition in the Bose-Hubbard lattice model requires $U/P \approx 16.7^{107}$, and the reduced coupling $g_{red} \sim N^2$, so the ground state of this system will keep this superfluid regime in a large range of $g_{red}$ if without excitation like the quench behavior. But this expression doesn’t works for the region of $g_{red} > 1$, e.g., see (d) and (e) in Fig.8. The long-time behavior with very weak $g_{red}$, the asymptotic behavior of Eq.(49) is

$$
\langle n^2(t) \rangle \approx n^2(0) - \frac{1}{\sqrt{16\pi g_{red}}} [\cos(4t\sqrt{g_{red}} + 1 + \frac{\pi}{4}) + \cos(4t\sqrt{1 - g_{red}} - \frac{\pi}{4})],
$$

which is presented in Fig.9. We can see that the amplitude fluctuation is increse with the reduction $g_{red}$, and in long-time limit the undulate of oscillation becomes more flat but no completely governed by the time-independent Hamiltonian. This corresponds to the superfluid regime with obvious amplitude fluctuation and the recurrences and interference pattern will occur (not shown). For the case of initial $g_{red} = N$, when the quenched $g_{red} \gtrsim 7N$, this nonequilibrium system will into the nonthermal steady state though it’s a nonintegrable system according to the results shown in Ref.110.

For one-dimension nonintegrable case of hard-core bosons (which generalized eigenstate thermalization occurs), a typical model of $1/r$ Hubbard chain also have the feature of collapse-and-revival oscillations like the nonintegrable one, but it’s dispersion-linear, i.e., it can be effectively solved by Eq.(23) while the nonintegrable one can not. Now we consider the large $g_{red}$ into strong-couping perturbation in a two-dimension version of $1/r$ Hubbard model, the lattice fermions Hubbard model, the double occupation $d(t) = \langle n_\uparrow n_\downarrow \rangle/N$ can be written as

$$
d(t) = d(0) + \sum_{i=0}^{N-1} \frac{1}{g_{red}} \langle c_i^\dagger c_{i+1}(n_i - n_{i+1})^2 \rangle + O(\frac{V^2}{U^2}).
$$

whose graphs have been presented in the Fig.2 of Ref.109. This is corresponds to the state of Mott insulator with strong interaction and have

$$
\mathcal{P} \langle c_i^\dagger c_{i+1}(n_i(0) - n_{i+1}(U))^2 \rangle = 2 \sum_i \left[ c_i(n_i(0) - n_{i+1}(U)) \right],
$$

17
where $\kappa_i$ is the dispersion relation related to the kinetic energy $T_{\text{kin}}$. The prethermalization regime is also exist in this case for one-dimension or two-dimension Bose-Hubbard model\textsuperscript{[41]}, but this prethermalization regime as well as the general collapse-and-revival oscillations vanish in a little range before the critical value $U_c$ which origin from the discontinuity momentum distribution in Fermi surface under the quenching.

We show the bandwidth-dependent kinetic energy of $1/r$ Hubbard chain in Fig.10. with different bandwidth: $W = 1$, $W = 4$, and $W = 1/2$ which have been obtained by the method of local density approximation (LDA)\textsuperscript{[40]}. It’s obviously to see that the amplitude of hopping is increases as the bandwidth $W$ increases (inset), and the $T_{\text{kin}}$ decay rapidly with the increase of distance along the chain. When quenches to large $U$, the oscillations of Eq.(51) makes a difference\textsuperscript{[40]} $\Delta d = P\pi (1 - 2n/3)/U$ which is halved for Falicov-Kimball model in nonequilibrium dynamical mean-field theory (DMFT) due to the vanishing of $P$ for one of its two spin species and therefore only one spin specie contributes to kinetic energy. In DMFT, this kinetic function due to the considerable noise (see Sect.10, Appendix.C) yields a single-site Green’s function

$$G(t,t') = i\langle c(t)c^\dagger(t') \rangle,$$  \hspace{1cm} (53)

where the contour-order correlation $\langle c(t)c^\dagger(t') \rangle$ has

$$\langle c(t)c^\dagger(t') \rangle = \frac{\text{Tr}[e^{i\beta T_c}c(t)c^\dagger(t)\gamma_\sigma \cdot S]}{\text{Tr}[e^{i\beta T_c}c^\dagger(t)]},$$  \hspace{1cm} (54)

where $T_c$ is the contour-order temperature, and the single-site action\textsuperscript{[41]}

$$S = \int_c dt dt' c^\dagger(t)A(t,t')c(t') + \int_c dt V(t),$$  \hspace{1cm} (55)

where $A(t,t')$ is a hybridization of site with fermion operators and the rest of the lattice.

By the nonequilibrium DMFT, which well describe the time evolution of an interacting many-body system (fermions lattice Hubbard model here), we can map the lattice model to the single-site impurity model as shown in above. Unlike the Eq.(51), the method of DMFT is nonperturbative, but since we consider the perturbation from noise into the Green’s function, the resulting Green’s function is

$$G(t,t') = G_0(t,t') + G_0(t,t_i)\Sigma_v G(t_i,t'),$$  \hspace{1cm} (56)

where $G_0$ is the unperturbed Green’s function, and it has\textsuperscript{[41]}

$$\frac{e^V - 1}{e^V - e^V_0(e^V_0 - 1)} * G_0(t,t') = \Sigma * G(t,t'),$$  \hspace{1cm} (57)

where $V = H - H_0$ is the non-Gaussian part of the Hamiltonian, i.e., the interaction term $U(t)n_\uparrow n_\downarrow$ which is noncommuting\textsuperscript{[41]}. So to linearize the rest part of the Hamiltonian, we need to tend the partial function which is the denominator of Eq.(54) into interaction representation with decomposed Boltzmann operator using the method of Hubbard-Stratanovich transformation which require the convergence of the gaussian integrals\textsuperscript{[41]}. Since this partial function select all the possible configuration of single-site along the contour $C$, which make it possible to be decoupled by a auxiliary-field quantum Monte Carlo methods\textsuperscript{[41,41]} (Note that the integrable lattice model for soft-core bosons, the nonGaussian distortion is origin from the off-site hopping\textsuperscript{[41]} term unlike the case what we are talking). the single-energy variables $s_i$ along the contour $C$ have\textsuperscript{[41]} $e^V_\sigma = \text{diag}(e^{\gamma s_1}, e^{\gamma s_2}, \ldots, e^{\gamma s_\sigma})$ where $\sigma$ denote the spin order $\sigma = \pm 1$ and $\gamma$ here is a temperature- and interaction-dependent parameter. This equation means that eighvalues (which can be specificized as the band energy $\epsilon_k$ in Hubbard model) of
hopping matrix $V$ can be diagonalized by the diagonal matrices which shown in the bracket of this equation.

Since the total Hamiltonian must be conserved in the evolution, the kinetic energy of $1/r$ Hubbard chain is suppressed by the term $E_{pot} = Ud(t)$. For half-filling Hubbard Hamiltonian $(n_\uparrow = n_\downarrow = 1/2)$ with a semielliptic density of state $\rho_{hf} = \sqrt{4P^2 - \epsilon_k^2}/(2\pi P^2)$, the kinetic energy per lattice site is $T_{kin} = 2 \int d\epsilon_k \rho_{hf}(\epsilon_k)n(\epsilon_k, t)\epsilon_k$, where the band energy $\epsilon_k$ here which obey the Dyson equation in lattice model with Green’s function $G_k(t, t')$

$$G_k(t, t')(i\partial_t + \mu - \epsilon_k - \Sigma) = 1, \quad t = t'$$

(58)

where the convolution product of local self-energy $\Sigma$ with $G_k$ yields the equal time double occupation in the homogeneity phase and the self-consistency local Green function has

$$G_k(t, t') = \int d\epsilon_k \rho(\epsilon_k)G_k(t, t'),$$

where $G_k(t, t')$ is diagonal. The approximation of Hartree-Fock which works well for the single-particle problem, affect the chemical potential $\mu$ which have a zero mean, by the particle number in canonical ensemble

$$\langle n_\uparrow n_\downarrow \rangle = \frac{1}{N^2} \sum_{k,k'} \langle n_{k\uparrow} n_{k'\downarrow} \rangle = \frac{n^2}{4},$$

(59)

and it contribute to the self-energy by the diagonalized Hartree-Fock Hamiltonian and provide the precise result in half-filling case, but since the Hartree-Fock is sensitive to the spin-correlations, it fails when the spin degrees of freedom disappear. In this case, one gives the second-order contribution to the self-energy by the form of

$$\Sigma(t, t') = -U(t)U(t')G_{0\sigma}(t, t')G_{0\sigma}(t', t)G_{0\sigma}(t, t'),$$

(60)

where the unperturbed Green’s function $G_{0\sigma}$ can be replaced by the full interacting one $G_\sigma$, and the interaction $U$ can be viewed as a evolution propagator here.

Since the fact that the phase transition of metal-to-insulator in half-filling $1/r$ Hubbard chain occurs when the $U = W$, which we set the bandwidth $W = 4$ here, i.e., $U_c = 4$. Note that the band energy $\epsilon_k$ is closely related to the continuity of momentum distribution, e.g., it’s discontinuity when $\epsilon_k = 0^−$ and $0^+$ in the each side of critical value $U_c$. When quench approaches to critical value $U_c$, $d(0) = 1/8$, and since we set the $n = 1$ and the critical value is $U_c = 4$, the one-dimension half-filling $1/r$ Hubbard model have the double occupation as

$$d_{hf}(t) = \frac{1}{8U} + \frac{(4 - U)^2}{16U^2} + \frac{(16 - U^2)^2}{16U^2} \ln \left| \frac{4 - U}{4 + U} \right| + \frac{\cos(Ut)\cos(4t)}{2Ut^2}, \quad \text{for quench from } 0 \text{ to } U;$$

$$d_{hf}(t) = \frac{1}{8U} - \frac{(4 - U)^2}{16U^2} + \frac{(16 - U^2)^2}{16U^2} \ln \left| \frac{4 - U}{4 + U} \right| + \frac{\cos(Ut)\cos(4t)}{2Ut^2}, \quad \text{for quench from } \infty \text{ to } U,$$

(61)

while for the quench to reach $U_c$, the behavior of double occupation is described by

$$d_c(t) = \frac{1}{8} - \frac{1}{512} \left[ \frac{48\sin(8t)}{t^2} + \left( \frac{6 - 32t^2}{t^4} \right)(\cos(8t) - 1) \right] - \frac{3}{32t^2}, \quad \text{for quench from } 0 \text{ to } U;$$

$$d_c(t) = \frac{1}{32} + \frac{1}{2048} \left[ \frac{48\sin(8t)}{t^2} + \left( \frac{6 - 32t^2}{t^4} \right)(\cos(8t) - 1) \right] + \frac{3}{128t^2}, \quad \text{for quench from } \infty \text{ to } U.$$

(62)

Fig.11 shows the graphs of $d_{hf}(t)$ of Mott insulator for quenches from 0 to $U$ and from $\infty$ to $U$ (according to Eq.(61)), we can see that the later one is roughly the inverse version of the former one, and a significant features is the fast-saturation. The larger the interaction $U$, the faster the curve tends to saturated. Note that the double occupation here is indeed related to the realistic physical quantity of global correlation for bosons system, and the discussion above is for
a prediction for the behavior of long-time limit, i.e., the stationary result, which consistent with the thermal values\cite{42}: 1/4 for interaction quenches from 0 to \( \infty \), 1/6 for interaction quenches from \( \infty \) to \( 0 \), 1/8 for interaction quenches from 0 or \( \infty \) to \( U_c \), (we set \( n = 1 \) here). The collapse of oscillations are scale as 1/\( \sqrt{g_{\text{red}}} \), i.e., the amplitude are continually decaying along the long-time scale limit cover the phase transition, and \( d(t) \) will shows strictly periodic behavior in the noninteracting regime with \( g_{\text{red}} = 0 \) (not shown in the Fig.11). For quenches from 0 to finite \( U \), the prethermalization regime also shows large agreement with the stationary values of \( d(t) \) in long-time limit. The effect of damping on the amplitude of collapse-and-revival oscillations is always exist in the long-time scale, and has important influence on the relaxation. It produce the “overdamp” in the regime of sufficiently large \( U \), which nearly reduce the amplitude to 0 after instantly tends to saturate. The process of damping is related to the velocity of spin wave in Goldstone model that for zero frequency Goldstone mode is followed by a additional standing wave modes.\cite{97,106} By setting a list of interaction in Fig.11, we found that, for quench from 0 to a infinite interaction \( U_c \), the closer the quenches to critical value \( U_c \), the closer the \( d_{\text{hf}}(t) \) to quasistationary value which is obtained from the Fig.12 as 0.125 (see the bottom inset of Fig.12(a)): the \( U \) which close to \( U_c \) in Fig.11(a) is setted as 3.299, and the long-time result for quench to this \( U \) is 0.12499, which is very close to the stationary prediction 1/8, and it’s reasonably differ from the thermal prediction of 0.098 by the equilibrium result\cite{118}. While for the quench from \( \infty \) to \( U \), we obtain the same conclusion: the result of quench to \( U = 3.299 \) is 0.032 which is very close to the stationary value 0.0312 which is shown in the bottom inset of Fig.12(b). That is the long-time behavior of nonequilibrium system which show agreement with the result of quasistationary value in phase transition point (this conclusion will always exist in the time scale of 1/|\( P \)| \( \ll t \ll U/P^2 \)).

While for the anharmonicity case, the coupling \( g_{\text{red}} \) is still usable by the form of a symmetrical anharmonic term (see Sect.10), the bare action of quantum system with \( N \)-component bosonic field \( \phi_a \) in \( \phi^4 \) field theory, when the \( g_{\text{red}} \) close to the critical value with \( U_c \), is\cite{106,119}

\[
S = \int d^4r \int d\tau \frac{1}{2} \left[ (\nabla \phi_a)^2 + \left( \frac{\partial \phi_a}{\partial \tau} \right)^2 - (r_c + r)\phi_a^2 + \frac{\lambda r^4}{N} \phi_a^4 \right],
\]

(63)

where \( \alpha = 1 \cdots N \), \( c \) is the velocity, \( \lambda r^4 \) is the quartic nonlinear coupling term, and the critical \( r_c \) is reach in the \( r = 0 \). For the case of quenches from large \( U \) to a small one which is close to zero, i.e., from the Mott insulator initial state to the superfluid or metallic state, we introduce the vectors \( k_1 = 2\pi n_1/N \) and \( k_2 = 2\pi n_2/N \) which obey periodic boundary condition (see Appendix.C) and have \( n_1 \neq n_2 < N \), then when the coupling is close to zero, the time-dependent nearest-neighbor correlation in the bath with harmonic potential is given as\cite{106}

\[
\langle n_r(t)n_{r+1}(t) \rangle = \frac{2g_{\text{red}}}{N} \sum_{r}^{N-1} \sin^2Gt, \quad (64)
\]

where the periodic correlator \( G = 1 + \cos k_1 - \cos k_2 - \cos(k_1 - k_2) \). This utilize the periodicity of harmonic oscillators in superfluid regime and exclude the high-frequency part due to the periodic boundary condition, i.e., keep the stable low-frequency only.

For many-body system, the dispersion relation \( \kappa \) of this bosonic model is oscillate as a function of \( k \) with the period \( \pi \) (see Fig.13). From Fig.14, it’s obvious to see that the periodic dispersion relation resulting in the degeneracy of energy. In the process of relaxation of correlations, the relevant parameter is assumed change linearly. By setting the dispersion relations \( \kappa \) before and after quench, the corresponding relaxation of correlations between the bosons is shown in the Fig.14, we see that the oscillations approach to quasisteady state with small (non-zero) frequency, and with the increasing of dispersion relation, the amplitude of correlation is decreased and the required-relaxation time is shorter. In fact this conclusion is always exist for all the many-body system in phase-space.
9 Investigation of Relaxation of Chain Model to Gaussian State By the Transfer Matrices

We then define the transfer matrix

\[ t(x) = \text{Tr}\left( \prod_{i=1}^{a} T_i(x) \right), \] (65)

where \( T_i(x) = R_{n-1}^l(x)R_{n-2}^l(x) \cdots R_0^l(x) \) is the monodromy matrix with \( n \)-site R-matrices and \( x \) is the spectral parameter. Employing this transfer matrix representation, the initial state can be written as

\[ F_0(x) = \lim_{n \to \infty} \frac{1}{n-1} \partial_x \langle \psi(0)|t(x)t^\dagger(x)|\psi(0) \rangle \] (66)

where the total number of particles \( N \) is a integer multiple of number of transfer matrices \( \text{num}(t_1(x)) \). Based on this, the localized free energy of per spin (or grid point in the network) is

\[ E_{\text{free}} = -\frac{\text{num}(t_1(x))}{N} \frac{1}{\beta} \lim_{M \to \infty} \ln \lambda_{\text{max}} \] (67)

where \( M \) is the number depends on how many parts temperature divided into (i.e., the Trotter number), and \( \lambda_{\text{max}} \) is the maximum eigevalue of transfer matrix and in the limit of \( N \to \infty \), it has

\[ \lambda_{\text{max}}^N = \lim_{M \to \infty} \text{Tr} t_{\text{num}}(t_1(x))(x), \] (68)

i.e., in the case of infinity-system-size the maximum eigevale is equal to the trace of transfer matrices. Further, we deduce that

\[ \lim_{N \to \infty} \frac{\ln(\lambda_{\text{max}}^N)}{N} = \lim_{M \to \infty} \frac{\ln\lambda_{\text{max}}}{N} \cdot \text{num}(t_1(x)), \] (69)

which can be easily confirmed by numerical methods. In the framework of auxiliary space which established in above, one can define the matrix \( A_i \) which acting on the auxiliary space \( 22 \), then the wave function of ground state can be redefined as

\[ |\psi(0)\rangle = \sum_{s_i} \text{Tr}\left( \prod_{i=0}^{n-1} A_i \right) \prod_{i=0}^{n-1} s_i \] (70)

where \( \prod_{i=0}^{n-1} s_i \) denotes a normalized computational basis state \( 23 \), while the set of unnormalized part form a projective space \( \mathbb{P} \) with dimension \( d_i d_j - 1 \).20

Since in normalization case the expectation value of initial state is \( \langle \psi(0)|J_i|\psi(0) \rangle \) with \( \langle \psi(0)|\psi(0) \rangle = 1 \), the transfer matrices in two subspaces can be obtained by the algebraic Bethe ansatz20:

\[ t(i + R) = \text{Tr}(A_{n-1}(\mathcal{R})A_{n-2}(\mathcal{R}) \cdots A_0(\mathcal{R})), \]
\[ t^\dagger(i + R) = \text{Tr}(A_{n-1}^\dagger(\mathcal{R})A_{n-2}^\dagger(\mathcal{R}) \cdots A_0^\dagger(\mathcal{R})), \] (71)

where \( \mathcal{R} \) is a constants of motion and the matrices \( A \) and \( A^\dagger \) are isomorphic with the bipartite space of \( \mathbb{C}^d_i \otimes \mathbb{C}^d_j \). In convex hull construction for nuclear norm, a direction of subgradient is consist of the orthogonal set \( \{ s_i \} \) and \( \{ s_i \} \),21 and it’s well know that the Schmidt rank \( R \) is invariant by local operations and classical communication (LOCC) when the but variable when bipartite state is mixed.22,21 For localized quantum communication, Eqs.(43,44) give the exponential suppression for transfer which reflected as the the exponentially fast quantum propagation in branched tree graph and the exponentially slow down of latter-time motion.
in the quantum graph\textsuperscript{122} for which the information flow toward the random path in local relaxation process.

In the above Bose-Hubbard model, using the Wigner representation which is generally negative definite\textsuperscript{48} we also have the characteristic function of density matrix $\mathcal{J}$ as\textsuperscript{27}

$$\text{Tr}[\mathcal{J}e^{ab^\dagger - a^*b^\dagger}] = e^{-|\alpha|^2/2} \prod_{d_r} L_m(|\alpha|^2 e^{2itP(d_r)}),$$

where the translation operator $e^{ab^\dagger - a^*b^\dagger} = e^{ab^\dagger} e^{-a^*b^\dagger} e^{-|\alpha|^2/2}$ where the state of $c$-number variable $|\alpha\rangle = e^{-|\alpha|^2/2}(\alpha b^\dagger - \alpha^* b^\dagger)\textsuperscript{20}$, and $L_m$ is the Laguerre polynomial, which is noniterative and utilized to express the boundary conditions of parameter space. Here the density matrix $\mathcal{J}_i = \text{Tr}(|\psi\rangle\langle\psi|)$ and $b^\dagger_i b_i = -(\frac{\partial}{\partial \alpha} + \frac{\alpha^2}{2})(\frac{\partial}{\partial \alpha^*} + \frac{\alpha^*}{2})$, $b^\dagger_i b_i = (\frac{\alpha}{2} - \frac{\partial}{\partial \alpha})(\frac{\alpha}{2} - \frac{\partial}{\partial \alpha^*})$. After the local relaxation (dephasing) to a steady state ensemble with stationary state $\overline{\rho}_i$, the Eq.(72) tends to the Gaussian form with $e^{-(\overline{\rho}_i + 1/2)\alpha^2} \overline{\rho}_i$ where $\overline{\rho}_i$ is the average of initial states for finite system and reach the maximum entanglement related to the second moments. The Hamiltonian has $\lim_{t \to \infty} (\langle \psi \rangle | e^{iHt} e^{-iH} | \psi \rangle) = \text{Tr}(\overline{\rho}_i H)$. For integrable homogeneous system (like the one we present in the Sect.6), the translation invariance in transition states and it’s also meaningful in the investigation of relaxation of degrees of freedom, the small displacement of coordinates due to the local potential produce a negative Hessian eigenvalue\textsuperscript{123}, and since the site-shift invariance has been broken by the local conservation law\textsuperscript{41}, The result of Ref.\textsuperscript{27} shows that the local relaxation is always preserves the full information of initial state, which shows that the information of initial state is not or at least not only be recorded by the factors of Lagrange multipliers\textsuperscript{24}, and this is consistent with the above result in Gaussian form which contain the term about initial states. While for inhomogeneous case (like most of the damped or polarized model), since the translation invariance is broken, the thermal behaviors and scattering is very different compare to the homogeneous one, and the prediction of GGE to the final state is also inadequate\textsuperscript{124}. Further, the relaxed result for nonequilibrium system can be constructed as the sum of Gaussians which is associated to the related collective variables\textsuperscript{125} or canonical variables which can be utilized to diagonalize the inhomogeneous model\textsuperscript{30}. Note that this Gaussian state is quasifree and contains only second moments, i.e., the redistribution by the scattering. We will further represent this process by matrix method in the next section. When the system have already relax to the equilibrium distribution, the dynamic is well described by a stochastic partial differential equation, e.g., the quantum Langevin equation\textsuperscript{127}. For this equilibrium state under large time evolution, the diffusion have a non-negligible influence to system and produce the recurrences which occur in a time scale larger than the relaxation time (i.e., the diffusion time is larger than the relaxation time), and the recurrences period is also depends on the transfer velocity of information.

10 Matrices Processing

The density matrices of Eqs.(14,15) can be represented by the Schmidt decomposition of bipartite state

$$|\psi\rangle = \sum_{R}^{R} \lambda_{R} |\mathcal{J}_{1R}\rangle \otimes |\mathcal{J}_{jR}\rangle$$

where $\lambda_{R}$ is the maximum eightvalues of density matrix for each $R$. If we set the the maximum rank is $R$, then it have $\sum_{R}^{R} \lambda_{R} = 1$ and $(\sum_{R}^{R} \sqrt{\lambda_{R}})^2 \leq R$. Definition\textsuperscript{121} shows that the Schmidt rank is just $R$ under the condition $R - 1 < (\sum_{R}^{R} \sqrt{\lambda_{R}})^2 \leq R$. In the case of $(\sum_{R}^{R} \sqrt{\lambda_{R}})^2 < R$, only the eigenvector which has maximum rank $R$ is needed, that also explain why the singular values decomposition reserved only the largest singular value (Eq.(12)). The set spaces S
with convex construction always have \( S_R \subset S_R \). In the zero-entanglement case, the square root of eightvalue of \( J \) have \( \sqrt{\lambda_R} = (VJV^\dagger)_{ij} \) with another index \( j \) when \( (VJV^\dagger)_{ij} \) is diagonal, and in another exposition is \( \langle A_i | \sigma_y A_j^\ast \rangle = \lambda_R \delta_{ij} \), where \( \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) and \( A \) is the matrix-product state. Here we consider the spin flip in the term \( \sigma_y A_j^\ast \), and it also have \( \langle A_i | \sigma_y A_j^\ast \rangle = \text{Tr}[\sigma_y A_j^\ast A_i] = \text{Tr}[(A_j^T \sigma_y)A_i] \) (not the scalar product). In such a flip in tilted state scheme\(^{22}\) we let the eightvalue \( \lambda_i = e^{S_i} \), and \( A = e^{\theta \sum_i S_i} \), i.e., spin flip when the \( \theta = \pi \).

It’s found that \( \sum_i e^{2i\theta} \lambda_i = 0 \) in the zero-entanglement case\(^{23}\).

A prerequisite to satisfy this formula is zero-entanglement, i.e., the two subsystem \( i \) and \( j \) is separable (or distillable). The density matrix \( J \) matrix here is assume have the eightvalue \( \lambda_R \) and it diagonalized by matrix \( V \) when \( J \) is symmetry, and in this case, the eightvalues of \( J \) is non-negative. Assuming \( V \) is a \( m \times n \) matrix with \( n \) orthonormal columns and \( m < n \), thus \( V \) acts periodic with period of square of number of column \( n \). Let \( \Sigma \) is the \( m \times m \) diagonal matrix which made up of singular values of \( J \), then its nuclear norm can be expressed as \( ||J||_* = \text{Tr}(V \Sigma V^T) \).

If here \( V \) is the matrix \( A \) which appear in Eq.(71), then the trace norm of \( A \) which equals the sum of square root of eightvalues \( \sum_i \sqrt{\lambda_i} \) have \( ||A||_\text{tr} = \text{Tr} \sqrt{AA^\dagger} \) and \( ||A \otimes A||_\text{tr} = ||A||_\text{tr} \cdot ||A||_\text{tr} = ||A||_\text{tr}^2 \). In normalized case with \( \langle \psi | \psi \rangle = 1 \), the operator norm of \( A \) has the similar property with Hermitian conjugate matrices: \( ||A^\dagger \otimes A||_\text{op} = ||A||_\text{op}^2 = 1 \). This corresponds to a absolute value of the maximal eightvalue which is normalized and it’s found nondegenerate for \( S = 1/2 \) Heisenberg model\(^{24}\). For separable case, \( \sum_R \lambda_R \leq 1 \) due to the convertibility and the decomposition of entangled state into unentangled pure states in the case that the maximum eightvalue is smaller than the sum of rest eightvalues\(^{25}\), i.e., \( \lambda_1 < \lambda_2 + \lambda_3 + \cdots \lambda_n \) (here set the \( i = 1, 2, \cdots, n \)). For pure state we have

\[
\frac{n-1}{n} \geq 1 - n \sum_i \lambda_i^2 \geq \frac{4}{n(n-1)} \left( \sum_{i<j} \sqrt{\lambda_i \lambda_j} \right).
\]

(74)

A general bound of dimension of subspace is that the largest dimension of space is almost \( d_i \times d_j \) and the smallest one is \( (d_i - R + 1)(d_j - R + 1) \), and these subspaces which dimension within this range, i.e., the rank \( R < R \) can be represented by the affine variety\(^{26}\). Since a precondition of increase of the Schmidt rank is incresing the dimension of subspace, and the degree of entanglement is also reaches maximally when it grows into the largest subspace, we can obtain that in most case the largest subsystem which almost full rank have the almost maximal entanglement, except somecase for the pure state which is unmixed\(^{27}\). The largest subspace form the largest-probability set with the constants of motion which proportional to the dimension of corresponding Hilbert space or projector onto its eightvalues, or its integer powers of Hamiltonian\(^{28}\).

Without losing general, for distillable state, the upper bound of entanglement entropy formed by the logarithmic negativity\(^{29}\) \( S_N = \ln ||J^\Gamma||_\text{tr} \), where \( J^\Gamma \) is the partial transpose of density matrix \( J \) and the corresponds covariance matric is \( \gamma^\Gamma = P \gamma P \) where \( \gamma \) is covariance matric and the diagonal matrix \( P = (-I_i) \oplus I_j \) with the diagonal identities matric \( I \). Let \( V \) is the nonsingular and skew-symmetric column vector, and it’s real. Then we have \( V^T J V = J \), (i.e., \( J \) is diagonal) so the nonincreasing ordered symplectic eightvalues \( \lambda^\Gamma \) with symplectic matrix \( \Omega = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \) which describe the reduced Gaussian state\(^{29,30}\) have

\[
\ln ||J^\Gamma||_\text{tr} = \sum_i \ln(\max[1, (\lambda^\Gamma_i)^2]) \leq \sum_i ((\lambda^\Gamma_i)^2 - 1),
\]

(75)

while the normal eightvalue of \( J \) is \( \lambda_i \) which equal to \( (\lambda^\Gamma_i)^2 \) (see Appendix.B for the detail).

23
11 Relaxation of Nonequilibrium System With Stochastic Dynamical Variables

Since for mixed system, if the initial state is homogeneous, the second moments is conserved and it prevent the system to relax to the thermal state, so the effective disentanglement is impossible in this case, and therefore some microstates are inaccessible since the final state is constrained by the conserved constants of motion no matter the system is integrable or not. Like the integrable system which guided by the corresponding GGE with maximal entropy $S_{ij} = -\text{Tr}(\rho_{ij} \ln \rho_{ij})$, will reach nonthermal steady states and share the similar property with the prethermalization plateaus in the long time limit, which also called the prerelaxation in the time evolution of GGE, and this has been founded in the isolated or open quantum system, while for the nonintegrable system it’s thermalize directly. In the inhomogeneous case like the most damping model, the conserved law is no more exist and then the thermal state is achievable directly. The local minima free energy which separated by barriers in free energy surface is connected by along the steepest descent path in the scenario of discretized evolution, and thereforce update the collective coordinates. This is a powerful way to obtain the symmetric tensor in the flattened space macroscopically, and even the supersymmetry system with global minimal potential energy. In these special points, the gradient of free energy as well as the potential energy vanish, and the energy is rised by the little displacement of coordinates.

Defining $Z$ as a collective variable with the coordinate $x$, then for harmonic oscillators with mass $m$, in the free energy surface, the distribution of Gaussians can be described by the biasing potential which is guided by the difference of free energy $E(Z) - E_G(Z, \tau)$, and in the limit of $w \to 0$, it have $\int dZ e^{-\beta E(Z)} = e^{-\beta E_G(Z, \tau)}$. Such a biasing potential is indeed a history-dependent term which is appear in the non-Markovian dynamics equation and as a biased estimator for the free energy, and while the unbiased estimate require a Markovian one. An experiment done recently of one-dimension Tomonage-Luttinger liquid model that the Gaussians propagation, which are adjusted by microwave, are along the one-dimension trajectories (“tubes”) and accompanied by a negative perturbation in the time evolution of $w$ and $\delta Z$ which shows stability in the chaotic scenario. This biasing potential is indeed a bias estimator for the quantum states with multiple phases, and we can see that it follows the Gaussian decay. Here the summation symbols is used due to the discretized evolution. Note that this expression is for harmonic oscillators, i.e., the system is linear response. While for anharmonicity oscillators, which produced by, e.g., the detuned Gaussian laser (blue-shift or red-shift) or the (two-photon) Raman detuning, this potential need to modified by adding some variational parameter describing the asymmetry (three-order term) or symmetry (quartic term) anharmonic to the exponent part of Eq.(76). The coupling in this case is nonlinear, like the scenario in FPU theorem. The free energy $E_{\text{free}} = -V_{\text{bias}}$, and it’s govern by the force $F = -\partial G E_{\text{free}}$. After the flatting process on free energy surface (for a intuitive schematic view, see, e.g., the Ref.), the change of the distribution makes the new Gaussians which given by new Hamiltonians, and hence the new equilibrium states, that can only happen in the inhomogeneous situation. After the local minimums of difference of free energy were mostly eliminated, then the probability distribution is nearly uniform, and the remaining corrugations are independents of the $E(Z) - E_G(Z, \tau)$. The action describing this dynamic of evolution in complex time scale is ($\tau$ is the complex time here)
\[ S(Z) = \frac{m}{2} \int_{\tau}^{\tau'} d\tau [Z'(\tau)]^2 - \mu^2 Z^2(\tau), \]  

(77)

where \( \mu \) here is the natural frequency. Note that for macroscopic model, the actions of harmonic oscillators which are viewed as matter fields coupled with the reservoir or the external electric field is not stationary and therefore belongs to the nonequilibrium dynamic, and the corresponding kernel functions are also in a nonequilibrium form, (see Ref.111).

The correlation matrix \( \Gamma_G \) which obey the Gaussian distribution is

\[ \Gamma_G(\tau) = \langle Z(\tau')Z(\tau) \rangle_G = \frac{\delta Z}{2} \langle R_i(\tau')R_j(\tau) \rangle, \]  

(78)

where \( R \) is the coupling operators between the states with dissipation scenario (e.g., the reservoir), and the evolution is \( \Gamma_G(\tau) = e^{-\tau H_G} \Gamma(0) \). The coupling is fadeout in damping system through this evolution. Then we have the action function

\[ S(Z) = \int_{\tau}^{\tau'} d\tau g(Z, Z'(\tau)), \]  

(79)

which contain the non-Markovian kernel \( g(Z, Z'(\tau)) \). In the classical limit approximately, the harmonic motion can be described by

\[ M \ddot{Z} + s \dot{Z} = -\frac{d}{d\tau} V(Z) + F_n(\tau), \]  

(80)

where \( s \) is a friction parameter and \( s = M \int_{\tau}^{\tau'} d\tau S(\tau' - \tau) \) where \( S \) is the friction kernel, \( F_n \) is the noise force. Since for Markovian noise which obey the Markovian evolution and can be well fitted to the master equation Eq.(33), we then need to replace the history-independent potential term which mentioned above by the form of Eq.(78), i.e., taking the bath coupling \( R \) as the noise source which are real and Gaussian, and then it have \( \langle R_i(t')R_j(t) \rangle = \delta_{ij}\delta_{t'-t} \). Here is because that for the harmonic oscillator, using Wick theorem, the density matrix can be diagonalized with a quadratic Gaussian potential (see Ref.6), and then the Green's function with infinite imaginary-time becomes\(138\)

\[ G(Z_i, Z_j; \tau) = \int_{\tau}^{\tau'} Ds(t) \exp(-S_{\text{eff}}(Z(t))/\hbar), \]  

(81)

where the Euclidean effective action

\[ S_{\text{eff}}(Z(t)) = \int_{\tau}^{\tau'} \left( \frac{1}{2} M \ddot{Z}(t) + V(Z) \right) dt - \int_{\tau}^{\tau'} dt \delta(Z - Z(t)) + V_0 \]  

(82)

where \( V_0 \) is the time-independent potential. In this expression, the state in next time step is only depends on the state in this time, i.e., variables satisfy the Markovian evolution, and more important, the contributions of noise in the imaginary axis is vanish, that’s also match the real noise source, so we only need to consider the noise in real part. Then the time derivative of \( Z \) has the form

\[ \frac{d}{dt} Z = A(t) + B(t) F_n(t), \]  

(83)

with the \( 2d \times 2d \) positive definite diffusion matrix \( D = BB^T \) which is symmetry in Wigner representation and both \( A \) and \( B \) are positive and real matrix. By the way, in this case, the quantum Fisher information matrix satisfy its saturation condition\(139\). This Markovian
stochastic evolution can be expressed by the second-order Fokker-Planck equation in a stochastic description

$$\frac{\partial}{\partial t} E = \left[ -\sum_i \frac{\partial}{\partial Z} A(t) + \frac{1}{2} \sum_{ij} \frac{\partial}{\partial Z_i} \frac{\partial}{\partial Z_j} D_{ij} \right] E$$

where $E$ is the free energy of the system influenced by the noise variables. Indeed this expression for the anharmonic case is due to the truncation which discard the asymmetry or symmetry anharmonic terms (see above). While in a probabilistic description, a Laplacian operator equal to the second time-derivative of non-Markovian kernel which is negative definite is contained in the Markovian form Fokker-Planck equation (see Ref.\textsuperscript{129}).

Now that in macroscopic system the observables are usually represented by thermal states directly since the error of statistical prediction is negligible\textsuperscript{33}. We then investigate the rate of variance of the statistical prediction of observable $P$, which belong to the canonical ensemble, i.e., the relate to the decay rate of Liouvillean relaxation\textsuperscript{140}. Writing its statistical prediction as $\text{Tr}(\rho P_i)$ where $\rho$ is the canonical ensemble. As we discussed above, the damp-out process is associate with the decoupling with the dissipation, and therefore we also can define the Hamiltonian here as the damping spectrum of the observable, which classified discussion here for bosons and fermions, i.e., decompose the $P_i$ into real part and imaginary part. Consider a bath with space $\mathbb{C}^{2d} \otimes \mathbb{C}^{2d}$, then for bosons, the communication relation is $[b_i, b_j^\dagger] = \delta_{ij}$ and for linear bath Hamiltonian which is in a quadratic form (even sector) is $H = u^T H_b u$ where $H_b$ is symmetry, and for fermions $[f_i, f_j^\dagger]_{-1} = \{f_i, f_j\} = \delta_{ij}$ with Hamiltonian $H = u^T H_f u$ where $H_f$ is antisymmetry, where $u$ and $w$ are real vectors. Since the real part of prediction can be represented by the covariance matrix\textsuperscript{130} $(\gamma_b)_{ij} = \frac{1}{2} \text{Tr} \rho P_b$ where $P_b = \{u_i, u_j\}$ and the imaginary part $(\gamma_f)_{ij} = \frac{1}{2} \text{Tr} \rho P_f$ where $P_f = \{w_i, w_j\}$, and here always have $\gamma_b \geq \sigma_y$. Writing the bath matrix as $M = \sum_i l_i \otimes l_i^\dagger$ with $l_i$ the vector with dimension $2d$ describing the bath coupling, then we have\textsuperscript{39,129}

$$\partial_t \gamma = X^T \gamma + \gamma X - Y$$

where for fermions $X = 2\text{Re} M$ and $Y = 4\text{Im} M$ while for bosons $X = 2\text{Im} M$ and $Y = 4\text{Re} M$. This Sylvester matrix equation also clarify the FDR.

Through this, the materials of bulk-edge-coupling type like the topological insulators or topological superconductors with the quantum spin Hall effect, have the full pairing gap inside the bulk and the gapless state which protected by the time-reversal invariance in the edge\textsuperscript{37} can decoupling with the bulk part, i.e., without dissipation at the sample boundary\textsuperscript{141} and the subspaces of edge and bulk will separate through the long-enough time evolution (The closing of gap is due to the effect of off-diagonal term here and often leads to the phase transition, e.g., which follow the power law decay with system size $N$ in the a spinor condensate system\textsuperscript{142}). For example, the chiral superconductor with $d + id'$ pairing phase\textsuperscript{143} which break the time-reversal symmetry (it’s realizable by, e.g., applying a strong magnetic field\textsuperscript{144}), or the non-Abelian statistical in the Majorana zero model\textsuperscript{145}. For most the bulk-edge-coupling type model which is the spinless fermions model, the time evolution is presented in the Appendix.C.

We already know that the integrable system in the homogeneous phase can only relaxes to the nonthermal steady state, but there are some models which can’t find the thermalization (e.g., can only to the generalized canonical), like the soft-core bosons model (e.g., the Mott insulator\textsuperscript{44}), spinless fermions model, integrable Luttinger model\textsuperscript{150}, etc. This kind of model can’t be effectively predicted by the form of Eq.(24). While for the models which nearly integrable (like the Hubbard model) or nonintegrable, the expectation will relax to thermal equilibrium finally, the resulting quasistationary state of this kinds of model is nonthermal\textsuperscript{33}. The final state which not be thermalized is quasisteady due to the off-diagonal contribution.
But there are still some integrable system which have the features of thermalization for some specific variables whose final state is described by the Gibbs ensemble, like the hard-core bosons system\cite{33,146}, so the integrability is not the only criterion of the thermodynamic behavior, the varied or conservative observables which have nonnegligible effect and their off-diagonal contribution as well as the integrability broken (broken of integrals of motion\cite{147})(see Appendix.C) are also important to consider. The required distance to the nonthermal steady state is in a infinite time average, and the required distance away from integrable point for thermalization occur is infinitesimal\cite{32}, while for a nonintegrable system, the thermalization will gradually (“smoothly”) broken when approaches to an integrable point\cite{148} with a infinite time scale.

12 Conclusion

This work mainly investigate time evolution of quantum many-body system as well as the thermodynamics of macroscopical system with the non-Markovian processes in the free-energy surface for which the steepest descent is used to find the minimal coupling (similar to the method of covariant derivatives). The condition of the presence of thermalization in a relaxation process of quantum many-body system is discussed in this work as well as the entropy and entanglement in the harmonic and anharmonic system. The main model of our investigation is the nonisolated system and so that the degrees of freedom can be traced out from the discussed canonical ensembles (or the microcanonical one), and therefore the ergodic is suppressed which the detial investigation is presented in the above. Although the integrable system which governed by the corresponding GGE keeps the expectation value of observables in initial state while the chaotic one keeps the initial memory little and it helps to understand the quenches towards the stationary state in the ordered phase or disordered phase in thermodynamic limit or scaling limit respectively, the required numberical computation is more demanding and the eigenstate thermalization hypothesis is failure\cite{148}. We also obtain that, the integrability is not only affected by the constants of motion, but some other important considerable factors which constitute the integrability breaking term (see Appendix.C).

To investigate the approaching to Gaussian state with maximum local entropy within the relaxation, a estimator in terms of trace norm is presented in the Sect.8 which related to the matrix method. The open quantum system is discussed in depth in the above sections, while for a closed quantum system which begin with a pure state with Tr$\rho^2 = 1$ ($\rho$ is the square root of eigenvalue of the density matrix), will never relax to the thermal state with Tr$\rho^2 < 1$ which corresponds to $\sum_R \lambda_R \leq 1$ which is discussed in Sect.9. For the diagonal Hamiltonian which make the observables tend to diagonal form with the infinite time average can be implemented by the methods like Bogoliubov transformation and a fast relaxation to diagonal ensemble (reach a quasisteady state) required the system spectrum is nondegenerate\cite{148} where we exclude the accidental degeneracies of diagonal ensemble. In this case, the eightenergies is linear like the one which mentioned in Sect.4, and the globally observable follows the relation Eq.\((126)\) in long-time limit. Note that such a nondegenerate will not long-live since the irregular dispersion in boundaries or the nonlinear waveguide will generate the degeneracies.

13 Appendix A : Deduction of $\beta$-function and the coupling in perturbed system

The $\beta$-function can be defined as $\beta = \mu \frac{\partial}{\partial \mu} g = \frac{d}{d \ln \lambda} g$, where $\ln \lambda = \frac{1}{2} \mu^2$. When $\lambda \rightarrow +\infty$, the $g \rightarrow 0$\cite{149}. Since the bare coupling $g^b$ is independent of the mass, so $\frac{d}{d \mu} g^b = 0$, according to the relation given in the Ref.\cite{46}.
\[
\mu \frac{d}{d\mu} g^b = (\mu \frac{\partial}{\partial \mu} + \mu \frac{d}{d\mu} \frac{\partial}{\partial g}) g^b.
\]

(86)

We can deduce that \(-\partial_\mu g^b = \frac{d}{d\mu} g \frac{\partial}{\partial g} g^b \neq 0\), according to the asymptotic series expansion which given in the Ref.\(^{46}\)

\[
\mu \frac{d}{d\mu} g^b = \varepsilon g^b - \varepsilon g \frac{\partial}{\partial g} g^b + (b_5 g^2 + b_5 g^4 + b_7 g^6 + O(g^8)) g \frac{\partial}{\partial g} g^b,
\]

(87)

when the \(\varepsilon \rightarrow 0\), i.e., dimension \(n \rightarrow n_c\).

\[
\mu \frac{d}{d\mu} g^b \rightarrow \mu \frac{d}{d\mu} \frac{\partial}{\partial g} g^b.
\]

(88)

The coefficient of Eq.(7) is\(^{58,149}\)

\[
\beta_0 = \frac{11}{3} C_{ij}^{(2)} - \frac{4}{3} T_{ij}
\]

\[
\beta_1 = \frac{34}{3} (C_{ij}^{(2)})^2 - \frac{20}{3} C_{ij}^{(2)} T_{ij} - 4 C_F^{(2)} T_{ij}
\]

\[
\beta_2 = \frac{2857}{54} (C_{ij}^{(2)})^3 - \frac{5033}{162} C_{ij}^{(2)} T_{ij} + \frac{2925}{864} C_F^{(2)} T_{ij}^2,
\]

(89)

where \(C_{ij}^{(2)}\) is the quadratic Casimir operator acting on the adjacent nodes, which equal to \(N\) for SU(N) system\(^{149}\), \(C_F^{(2)}\) is the quadratic Casimir operator acting on fermions, and has the relation with mass as \(\frac{1}{4} C_{ij}^{(2)} \text{dim}(T_{ij}) = m\)\(^{58,149}\), where \(m\) is the number of fermion multiplets\(^{84}\).

With the increase of \(m\), there will be a lot of novel nature in fermion stand model which we don’t discuss here, for a reference can see the Ref.\(^{150}\).

According to the supersymmetry SU(3) Yang-Mills theory in Ref.\(^{151,152}\), the quadratic Casimir operator which have \(C_{ij}^{(2)} = F^{\mu\nu} F_{\mu\nu}\) where \(F^{\mu\nu}\) is the field strength tensor or the SU(N) generate meta (here is the group generator of SU(3)) which have the below relation with the coupling \(g\)

\[
\frac{\beta(g)}{g} F \tilde{F} = -\frac{11}{4} \partial_\mu (\psi^\dagger(x) \gamma^\mu \gamma_5 \psi(x)),
\]

(90)

where \(F \tilde{F} = \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}\), \(\varepsilon_{\mu\nu\rho\sigma}\) is the Levi-Civita symbol. Note that this relation is correct for \(l\)-loop order where \(l \geq 2\) since it’s gauge-independent for \(\beta(g)\) in one-loop order. It’s easy to obtain that

\[
F \tilde{F} = -\frac{11}{4} \frac{g}{\beta(g)} \partial_\mu (\psi^\dagger(x) \gamma^\mu \gamma_5 \psi(x)),
\]

\[
C_{ij}^{(2)} = \frac{16 \pi^2}{g} \left[ -\frac{8}{33} \left( \frac{\beta(g)}{g} \right)^2 - \frac{1}{3} \frac{\beta(g)}{g} \right],
\]

(91)

where \(\frac{\beta(g)}{g}\) in SU(3) system obeys\(^{151}\).

\[
\frac{\beta(g)}{g} = \frac{-3 C_{ij}^{(2)}}{16 \pi^2 - 2 C_{ij}^{(2)}} = \frac{-9}{16 \pi^2 - 6},
\]

(92)

here utilize the virtue of invariance of \(\gamma_5\) as \(\Lambda_\frac{1}{2} \gamma_5 \Lambda^{-1}_\frac{1}{2} = \gamma_5\).
14 Appendix B : The Supplement of Covariance Matrix

Firstly we consider the Minkowski space function

\[ Z = \text{Tr} e^{-iHt} = \int \text{DP} e^{-iS(t)}, \]  

(93)

where \( A \) is the vector potential, and the partition function \( Z(\beta) = Z(-i\beta) = \text{Tr} e^{-\beta H} \). Consider the canonical ensemble

\[ \rho(\beta) = \frac{e^{-\beta H}}{Z(\beta)}. \]  

(94)

We take the Hamiltonian of components of decomposed covariance matrix \( \gamma = (H_1 \oplus H_2)/2 \), where \( H_1 = V^{-1/2} \) and \( H_2 = V^{1/2} \) and \( V \) is the potential matrix, into the blocks of \( 1/\beta \). Then the free energy in entropy ensemble is

\[ E(\beta) = \text{Tr} H_2(\beta) = \sum \ln \gamma(\beta) = \sum \ln \frac{H_1(\beta) \oplus H_2(\beta)}{2} \]  

(95)

where \( H_1(\beta) = V^{-1/2}[I_d + 2(\exp(\beta H_2) - I_d)^{-1}] \) and \( H_2(\beta) = V^{1/2}[I_d + 2(\exp(\beta H_2) - I_d)^{-1}] \). Then the Eq.(75) can be represented as

\[ \ln ||J^F||_\text{Tr} = \sum_i \ln(\max[1, \lambda_i^{-1}]) \leq ||\lambda_i^{-1} - 1||_\text{Tr} \leq 2(e^{\beta H_2} - I_d)^{-1} \]  

(96)

here \( e^{\beta H_2} = -\Omega^{-1/2} \gamma F \sigma_y \gamma F (-\Omega)^{1/2} \) is the blocks of \( H_2 \) and indeed it play a key role in the coupling between the target region with the rest. The maximal \( l_1 \)-norm\(\text{155)}\) of \( (e^{\beta H_2} - I_d) \) is linear bounded\(\text{129)}\) by the size of target region, (linear with the number of degrees of freedom of boundary of \( \rho \)), but it’s independent of size of the total size (contain the nontarget-region).

Then we take the equation of stochastic-description dynamics (Eq.(83)) into consider and let the \( A, B, F \) be the matrices. For quantitative analysis, we form a new potential matrix \( Q = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \). Through the mathematical method, we have\(\text{154)}\)

\[ S^{-1} \begin{pmatrix} A & B \\ B & A \end{pmatrix} S = (A + BF) \oplus (A - BF) \]  

(97)

where \( S = (P + F)/\sqrt{2} \) and \( S^{-1} = S \), and we have \( A + BF = (A - BF)^{-1} \). The Hamiltonian which describe the conserved observable becomes

\[ H = \text{Tr}(F \text{In} Q) = \text{Tr}(\text{In} \frac{A + BF}{A - BF}), \]  

(98)

then the determinant \( \det[A + BF] = \exp(-p\text{Tr}(F \text{In} Q)) \), where \( p \) is the probability within the canonical ensemble \( \rho = \sum p|\psi\rangle \langle \psi| \). Through Jacobi’s formula, we have

\[ \partial_t \det[A + BF] = \text{Tr}(\text{adj}[A + BF] \cdot \partial_t[A + BF]) = \exp(-p\text{Tr}(F \text{In} Q)) \cdot \partial_t Q, \]  

(99)

where \( \text{adj}[:] \) denotes the adjoint matrix.
Appendix C: The Perturbation Theory Applied to Diagonalized Ising Chain Hamiltonian and The Discuss of Off-Diagonal Contribution Term

We next taking the Ising chain model \( H = -J \sum_{i=0}^{N-2} \sigma_i^x \sigma_{i+1}^x - gJ \sum_{i=0}^{N-2} \sigma_i^z \) with \( g < 1 \) as an example to detect the effect of perturbation theory in diagonalization. For fermion quasiparticles with quasimomentum \( \frac{2\pi}{N} \) which have even parity, have even fermion number \( N_{\text{even}} \) and obey the antiperiodic boundary conditions \( \psi(r + N) = -\psi(r) \) with essential vectors \( k = \pi(2n - 1)/N \) \( (n \) is a integer), while for the odd parity one which have a odd fermion number \( N_{\text{odd}} \) is obey the periodic boundary conditions \( \psi(r + N) = \psi(r) \) with essential vectors \( k' = 2\pi n/N \). Note that these two sectors can well describe the stationary phase-space probability distribution by the WKB spectrum \( \text{Ref. 106, 157} \). Then the WJ fermions \( c_r^\dagger \) satisfy

\[
\sigma_r^+ = \frac{\sigma_r^x + i\sigma_r^y}{2} = c_r^\dagger e^{i\pi N},
\]

\[
\sigma_r^- = \frac{\sigma_r^x - i\sigma_r^y}{2} = c_r^\dagger e^{i\pi N},
\]

with \( r \) and \( r' \) satisfy the anticommutate relation \( \{c_r, c_{r'}^\dagger\} = \delta_{r,r'} \). We introduce the Gaussian white noise to this in this model, then the conserved observables follow the Gaussian distribution after the quench, which with the Gaussian amplitude \( \omega = 1/(\delta Z \sqrt{2\pi}) \) (see Eq.\((76))

For the currents which is proportional to the diagonalization \( \text{Ref. 126} \), the antiperiodic boundary conditions which also called the Neveu-Schwarz sector \( \text{Ref. 92} \) corresponds to the left current \( J^L_L \), and the periodic boundary conditions corresponds to the right current \( J^R_R \), which are

\[
J^L_L(k) = \sum_k \psi^\dagger_R(k + k') \psi_R(k') + \psi^\dagger_R(k + k') \psi_R(k') + \text{H.c.}
\]

\[
J^R_R(k') = \sum_{k'} \psi^\dagger_L(k + k') \psi_L(k) + \psi^\dagger_L(k + k') \psi_L(k) + \text{H.c.}
\]

The largerest current is appear in the ground state, i.e., the \( J^c(0) \), and the net current \( J_{\text{net}} = N_R - N_L \) which is conserved. The observable \( A \) in long-time limit has

\[
\lim_{t \to \infty} \langle \psi(t)|A|\psi(t)\rangle = \lim_{t \to \infty} \frac{\langle \psi_R(t)|A|\psi_R(t)\rangle + \langle \psi_L(t)|A|\psi_L(t)\rangle}{2},
\]

and

\[
\frac{\langle \psi_R(t)|\psi_R(t)\rangle}{\langle \psi_L(t)|\psi_L(t)\rangle} = 1 + O(e^{-nt}),
\]

where \( n \) is a constant associate with the \( J_{\text{net}} \), i.e., the wave function in the pictures of left current and right current are nearly equivalence if \( J_{\text{net}} \) is small enough.

Mapping the fermi field into the Fourier space for simplicity through the transformation \( \sigma_r^x = 1 - 2c_r^\dagger c_r \) and \( \sigma_r^z = -\prod_{r'=0}^{r-1}(1 - 2c_{r'}^\dagger c_{r'}) \), we have \( \psi_r(k) = \frac{1}{\sqrt{N}} \sum_{k'} \psi_{kr} e^{ikr} \) for even parity, and \( \psi_{r'}(k') = \frac{1}{\sqrt{N}} \sum_{k'} \psi_{r'k'} e^{ik'r} \) for odd parity, we obtain the quadratic Hamiltonian (but no diagonalized)

\[
H = 2 \sum_{k>0} c_k^\dagger H_k c_k,
\]

where Nambu vector \( c_k^\dagger = \begin{pmatrix} e_k^\dagger \\ c_{-k} \end{pmatrix} \), and \( H_k = H_0 + R(t,k)\sigma_z \) where \( R(t,k)\sigma_z \) is the term associate to the noise and \( H_0 \) is the Hamiltonian without the noise which is

\[
H_0 = \begin{pmatrix} 2J(g - \cos k) & -2J\sin k \\ 2J\sin k & -2J(g - \cos k) \end{pmatrix}
\]
To make the Hamiltonian diagonal in a nonperturbative treatment, we use the Bogoliubov transformation (rotation) to obtain the expression of Bogoliubov quasiparticles with Bogoliubov angle \( \theta(k) \) (assuming the lattice spacing \( \tilde{a} = 1 \))

\[
c(k) = \cos \theta(k)c_0(k) + i\sin \theta(k)c_0^\dagger(-k),
\]

\[
c^\dagger(k) = i\sin \theta(k)c_0(-k) + \cos \theta(k)c_0^\dagger(k),
\]

with the gap is \( \Delta = \epsilon_0 = 2J|1 - g| \) which vanish in the phase transition point (quantum critical point \( k_c = 1 \)) where the interactions of quasiparticle become more effective. The excitation probability of quasiparticles becomes \( \langle \psi(0)|c^\dagger(k)c(k)|\psi(0)\rangle = \tan^2[(\theta(k) - \theta(0))/2] \) and obey the nonthermal distribution. When \( g \gg 1 \), the ground state is strictly a paramagnetic, while when \( g \ll 1 \), the ground states are two degenerate ferromagnetic. If we ignore the noise term, the diagonalized Hamiltonian after the transformation is

\[
H_0 = 2 \sum_k \epsilon_k(c_k^\dagger c_k - c_0(-k)c_0^\dagger(-k) - 1),
\]

where the linear dispersion \( \epsilon_k \) which depends on \( H_0 \), and \( \epsilon_k = \sqrt{|H_0|} = 2J\sqrt{g^2 - 2g\cos k} + 1 \).

This a noninteracting Hamiltonian and has the accidental degeneracies due to the periodic dispersion which has mentioned above. This procedure is also available for the phonon field operators, which the Hamiltonian can be exactly diagonalized in harmonic-oscillator. If we consider the noise term, the density matrix of diagonalized Hamiltonian which satisfy the master equation (Eq.(33)) can be written as

\[
J(k) = \begin{pmatrix}
c_0^\dagger c_0 & c_0^\dagger c_0^\dagger
\end{pmatrix},
\]

where the two elements in the main diagonal stands for the number of levels in momentum space which is invariant under the time evolution, and the two elements in the vice diagonal describe the coherence which will decay exponentially under time evolution and finally lead the system to the mixed state with decoherence superposition. For example, we denote the element \( c_0(-k)c_0(k) \) as \( c_{10} \), then \( c_{10}(t) = e^{-\kappa t}c_{10}(0) \), i.e., it vanish when \( t \gg 1/\kappa \), this result is obey the thermal Glauber dynamics. So it has \( \partial_t J(k) \neq 0 \). Base on the Bogoliubov transformation introduced above, the initial state before the quench can be written as

\[
|\psi(g_0)\rangle = N \prod_{k,k' > 0} [1 + i\tan \Delta \theta \ c^\dagger(k)c^\dagger(-k)]|\psi(g)\rangle,
\]

where the difference of Bogoliubov angle \( \Delta \theta(k) = \theta(k; g) - \theta(k; g_0) \) for the left current regime or \( \Delta \theta(k) = \theta(k'; g) - \theta(k'; g_0) \) for right current regime, and \( N = \exp[-1/2 \sum_{k,k' > 0} \ln(1 + \Delta \theta(k))] \).

More parameterized, the difference of Bogoliubov angle \( \Delta \theta(k) \) has

\[
\cos \Delta \theta(k) = \frac{\epsilon_k^2(g_0)}{\epsilon_k(g_0)\epsilon_k(g)},
\]

where \( \epsilon_k(g_0) = 2J\sqrt{g_0g - (g_0 + g)\cos k + 1} \).

Since the WJ fermions is exist here, it’s spinless and therefore the thermalization can’t be found in this model, which it’s similar to the one mentioned in the Ref.\textsuperscript{320}. For the setups of model mentioned in Sect.10 which have a damping model with damping spectrum, the result is different with what discussed above. In integrable case for this Majorana fermions setup, the Hamiltonian can be simplified as \( H = -i\mathcal{P}_f(i\gamma_L + \gamma_R) \) where \( \gamma \) are the Majorana models and

\[
\text{(33)}
\]
\( P_f \) is the hopping of nearest-neighbor fermions. The Majorana model in the edge of sample is nonlocal and decoherence, the total edge localized model is

\[
e_M(k) = \frac{1}{2}(i\gamma_L(k) + \gamma_R(k)),
\]

i.e., the conserved currents coupling to the Majorana models. This combination process cost energy \( 2P_f \) and form a dissipative gap with the bulk (this gap requires that the on-site interaction \( U < 2P_f \)). Since the damping feature, the bulk part of density matrix (not the Eq.\,(111)) is decay with time evolution, and its time derivative have the same form with Eq.\,(85), while the edge part is not, i.e., the both the main diagonal and vice diagonal are decay with time exponentially, so the final state is become a pure state \( (|\psi\rangle \langle \psi|) \) with coherence superposition (in a similar way to Eq.\,(112)).

In perturbation theory, with the variables driven by time-dependent white noise, the correlation matrix becomes \( \Gamma(t) = -\frac{i}{2} (R_i(t') R_j(t)) = \frac{K}{2} \delta_{ij} \delta_{t-t'} \), i.e., the coupling strength \( K \) is associate with the dephasing effect of noise which accelerate the relaxation in a time scale of order \( 1/K \), while the diverging length scale is \( 1/\Delta \). We take the approximation \( H = H_0 + gH_1 \), where \( H_0 = \sum_k \epsilon_k c^\dagger(k)c(k) \) and \( H_1 = \sum_k \frac{g}{2} c^\dagger(k)ckc(k) \) where \( H_1 \) is second quantized and \( \delta_k \) is a nonlinear two-body interaction potential unlike the linear eigenenergy \( \epsilon_k \). Then we introduce the anti-Hermitian operator \( s \) as \( s = g_1 + \frac{1}{2} g^2 s_2 + O(g^3) \) where \( g \) is time-dependent parameter and diagonalize the Hamiltonian through canonical transformation have been presented in the Ref.\,32

\[
H_d = H_0 + gH_d^{(1)} + g^2 H_d^{(2)} + O(g^3)
\]

\[
= H_0 + g(H_1 + [s_1, H_0]) + g^2(\frac{1}{2}[s_2, H_0] + [s_1, H_1]) + \frac{1}{2}(s_1, s_1, H_0)] + O(g^3)
\]

then the conserved observable \( P_i \) have \([H_d, P_i] = O(g^3)\). In this way, the diagonalized quasi-particles are \( c^\dagger(k, t) = e^{iH_0 t} c^\dagger(k) e^{-iH_0 t} \) and \( c(k, t) = e^{iH_0 t} c(k) e^{-iH_0 t} \). In the range of \( 1/|g| \ll \) time scale \( \ll 1/g^2 \), the pure state have the same expectation value with the mixed state, i.e., the main diagonal and vice diagonal of diagonalized Hamiltonians’ density matrix have the same degree of decaying.

In the case of \( g^2 \ll 1 \), the \( s \) can be viewed as \( g s_1 \), then since \( H_d(t) = e^{gs_1 H} e^{-gs_1} \), we obtain

\[
\frac{d}{dg} H_d(t) = e^{gs_1} [s_1, H] e^{-gs_1},
\]

\[
\frac{d^2}{dg^2} H_d(t) = e^{gs_1} [s_1, [s_1, H]] e^{-gs_1},
\]

\[\ldots\]

, then we further obtain

\[
\frac{d}{dg} H_d(t) = e^{\frac{s}{g}} [\frac{s}{g}, H] e^{-s},
\]

\[
\frac{d^2}{dg^2} H_d(t) = e^{\frac{s}{g}} [\frac{s}{g}, [\frac{s}{g}, H]] e^{-s},
\]

\[\ldots\]

For a globally conserved observabe \( A = \prod_i P_{\alpha_i} \), apply \( H_d \) to it with the GGE average, we have\,23

\[
\langle A \rangle_{\text{GGE}} = \sum_{\alpha_1 \ldots \alpha_n} \tilde{A}_{\alpha_1 \ldots \alpha_n} \prod_{i=1}^n \langle P_{\alpha_i} \rangle_{\text{GGE}},
\]

32
where $\tilde{A}_{\alpha_1 \cdots \alpha_n}$ is the perturbation-averaged matrix elements which is utilized to diagonalize the $P_{\alpha_i}$ here and it have the property of

$$
\langle A \rangle_{\text{GGE}} = \langle \prod_{i=1}^{n} P_{\alpha_i} \rangle_{\text{GGE}} = \langle \prod_{i=1}^{n} P_{\alpha_i} \rangle_{\text{GGE}} = \langle \prod_{i=1}^{n} P_{\alpha_i} \rangle_0 = \langle \prod_{i=1}^{n} P_{\alpha_i} \rangle_0 + O(g^3) \tag{121}
$$

we have\(^{32}\)

$$
\langle A(t) \rangle = \langle \psi(0) | e^{i H t} F e^{-i H t} | \psi(0) \rangle
= \langle \psi(0) | e^{-s} e^{i H d t} e^s F e^{-s} e^{-i H d t} e^s | \psi(0) \rangle,
$$

which is diagonalized, and with $s(t) = e^{i H d t} e^{-i H d t}$. This transformation use the formula $e^{i H t} = e^{-s} e^{i H d t} e^s$, we define the $e^{-s} e^{i H d t} e^s = e^{R - s} e^{i H d t}$ where the real linear map $R_{-s} := ad_{-s}$\(^{160}\), and have\(^{161}\)

$$
-s \cdot (i H d t) = -s + \frac{R_{-s}(i H d t)}{1 - e^{R_{-s}}},
$$

then it’s easy to obtain

$$
\ln(e^{-s} e^{i H d t}) \approx -s + \frac{s^{-1} e^{R_{-s}}}{e^{R_{-s}} - 1}.
$$

A estimator for the integrability breaking is given by the Ref.\(^{147}\) that add the integrability broken term to the expression of observable

$$
A(t) \approx \mu A_{\text{initial}} + (1 - \mu) A_{\text{thermal}},
$$

for which the system in a completely integrable case when $\mu = 1$, the system expectation value is the same as the initial one in this case, and it’s different from the thermal expectation value of microcanonical ensemble in the completely chaotic case (nonintegrable) which can be well described by the standard statistical mechanics\(^{148}\). The later case appear in the case $\mu \ll 1$ and average over the initial states which equal to the thermal one, as $\langle \psi(0) | A_{\text{thermal}} | \psi(0) \rangle = \langle \psi(t) | A_{\text{thermal}} | \psi(t) \rangle$, and all these eight state are within the relevant energy windows with different weights\(^{81}\). That allow the precise prediction for thermal state in long-time limit with the energy close to the initial one. So the thermalization require a large number of coarse-grained observables\(^{32}\). As predicted in the classical system by KAM theorem, it’s a crossover of regular and chaotic regime\(^{147}\), and the achievement of thermalization require enough integrability breaking (otherwise the ergodicity is ineffective and the thermalization is suppressed) and a long-time process ($\sim 1/g^3$ in our limit) or as a infinite time average to the diagonal ensemble and fluctuate around it in the latter time\(^{149}\), which shown as (not consider the possible degeneracies here)

$$
\langle A(t) \rangle = \lim_{t \to \infty} \frac{1}{t} \int_0^t dt \text{Tr}(A \rho(t)) = \langle \psi(t) | A | \psi(t) \rangle_{\text{diag}}
= \sum_\alpha |\langle \alpha | \psi(0) \rangle|^2 \langle \alpha | A | \alpha \rangle,
$$

where $|\alpha\rangle = \sum_\beta (|b\rangle \langle b| g H_1 |\alpha\rangle)/(E_a - E_b)$. This equation gives the long-time average, and keeps the diagonal term only. This long-time average will equal to the GGE expectation value or not which dominated by the conserved $P_z$. For Eq.\((122)\), when the state $\rho$ which can be described by the Hamiltonian $H = H_0 + g H_1$ is nondiagonal while the observable $A$ is diagonal (i.e., $[A, H_d] = 0$), it becomes\(^{32}\)

$$
\langle A(t) \rangle = -\langle \psi(0) | (s(t) - s) A(s(t) - s) | \psi(0) \rangle + O(g^3)
= -2\langle \psi(0) | s A | \psi(0) \rangle - \text{Re} \langle \psi(0) | s A s(t) | \psi(0) \rangle,
$$

33
where the term $-\text{Re}\langle \psi(0) | sA s(t) | \psi(0) \rangle$ is due to the off-diagonal contribution as
\begin{equation}
-\text{Re}\langle a | sA s(t) | a \rangle = \text{Re} \sum_b \frac{|\langle a | gH_1 | b \rangle|^2}{(E_a - E_b)^2} (b | A | b) e^{-i(E_a - E_b)t} + O(g^3),
\end{equation}
where we simplify the initial state $\psi(0)$ as $a$ and the quenched state $\psi(t)$ ($t > 0$) as $b$. But in the case of both $\rho$ and $A$ are off-diagonal, this off-diagonal contribution term becomes
\begin{equation}
-2\text{Re} \sum_b \frac{|\langle a | gH_1 | b \rangle|^2 - |\langle a | gH_1 | a \rangle|^2}{(E_a - E_b)^2} (a | A | b) e^{-i(E_a - E_b)t} + O(g^3).
\end{equation}
While the diagonalized state is
\begin{equation}
\rho_{\text{diag}}(|b\rangle) = \sum_a P_a \rho_0 P_a,
\end{equation}
where the projector $P_a = |a\rangle \langle a|$ which project onto the subspace of initial state $|a\rangle$.

References

[1] Eling C, Guedens R, Jacobson T. Phys. Rev. Lett. 96(12): 121301(2006)
[2] Calabrese P, Essler F H L, Fagotti M. Phys. Rev. Lett. 106(22): 227203(2011)
[3] Monasson R. Phys. Rev. Lett. 1995, 75(15): 2847.
[4] For quantum chaotic non-integrable system, the Kolmogorov entropy $S_k = 1.1$ and it’s independent of $n$ and error$^5$. Since the Kolmogorov entropy can be written as
\begin{equation}
S_k = -\sum_{i+1} \lambda_{i+1} \ln \lambda_{i+1} + \sum_i \lambda_i \ln \lambda_i
\end{equation}
So we can obtain that for the non-integrable chaotic system, the eigenvalue $\lambda_i = 1.82632$ with $i = 0, 1, \cdots, n - 1$ here.
[5] Prosen T, Žnidarič M. Phys. Rev. E,75(1): 015202(2007)
[6] Karrasch C, Rentrop J, Schuricht D, et al. Phys. Rev. Lett. 2012, 109(12): 126406.
[7] Zhou N, Zeng G, Xiong J. Electronics Letters, 2004, 40(18): 1149-1150.
[8] Liu B, Xiao D, Jia H Y, et al. Quantum Information Processing, 2016, 15(5): 2113-2124.
[9] Tegmark M. Phys. Rev. E. 2000, 61(4): 4194.
[10] Luo X Y, Zou Y Q, Wu L N, et al. Science, 2017, 355(6325): 620-623.
[11] Sharma K K, Pandey S N. Quantum Information Processing, 2016, 15(4): 1539-1551.
[12] Qiang Z, Qi-Jun Z, Xiao-Ping Z, et al. Chinese physics C, 2011, 35(2): 135.
[13] Li R, Wang J, Qi X, et al. arXiv preprint arXiv:0908.1537, 2009.
[14] Qi X L, Li R, Zang J, et al. Science, 2009, 323(5918): 1184-1187.
[15] Zhang C J, Zhang Y S, Zhang S, et al. Phys. Rev. A. 2008, 77(6): 060301.
[16] Horodecki M, Horodecki P, Horodecki R. Open Systems & Information Dynamics, 2006, 13(01): 103-111.
[17] Arimitsu T, Umezawa H, Progress of theoretical physics, 74(2): 429-432(1985)
[18] Franz S, Parisi G. Journal de Physique I, 1995, 5(11): 1401-1415.
[19] Charbonneau P, Yaida S. Phys. Rev. Lett. 118(21): 215701(2017)
[20] Liu J, Du X H. International Journal of Theoretical Physics, 2013, 52(8): 2623-2630.
[21] Wang Y R. Phys. Rev. B. 1992, 46(1): 151.
[22] Fagotti M, Collura M, Essler F H L, et al. Phys. Rev. B. 2014, 89(12): 125101.
[23] Žnidarič M, Prosen T, Prelovšek P. Phys. Rev. B. 2008, 77(6): 064426.
[24] Vidal G. Phys. Rev. Lett. 2007, 98(7): 070201.
[25] Barthel T, Schollwöck U. Phys. Rev. Lett. 2008, 100(10): 100601.
[26] Jaynes E T. Physical review, 1957, 106(4): 620.
[27] Cramer M, Dawson C M, Eisert J, et al. Phys. Rev. Lett. 2008, 100(3): 030602.
[28] Behunin R O, Hu B L. Phys. Rev. A. 2011, 84(1): 012902.
[29] Lieb E H, Robinson D W. Statistical Mechanics. Springer Berlin Heidelberg, 1972: 425-431.
[30] Kollath C, Läuchli A M, Altman E. Phys. Rev. Lett. 2007, 98(18): 180601.
[31] Kollar M, Eckstein M. Phys. Rev. B. 2011, 84(5): 054304.
[32] Manmana S R, Wessel S, Noack R M, et al. Phys. Rev. Lett. 2007, 98(21): 210405.
[33] Fagotti M. Journal of Statistical Mechanics: Theory and Experiment, 2014, 2014(3): P03016.
[34] Diehl S, Rico E, Baranov M A, et al. Nature Physics, 2011, 7(12).
[35] Rossini D, Suzuki S, Mussardo G, et al. Phys. Rev. B. 2010, 82(14): 144302.
[36] Calabrese P, Essler F H L, Fagotti M. Phys. Rev. Lett. 2011, 106(22): 227203.
[37] De Grandi C, Gritsev V, Polkovnikov A. Phys. Rev. B. 2010, 81(22): 224301.
[38] Elias V, McKeon D G C. International Journal of Modern Physics A, 2003, 18(13): 2395-2401.
[39] Elcoro L, Etxebarria I, Perez-Mato J M. Journal of Physics: Condensed Matter, 2000, 12(6): 841.
[40] Drummond P D, Opanchuk B. Phys. Rev. A. 2017, 96(4): 043616.
[41] Hořava P. Phys. Rev. Lett. 2005, 95(1): 016405.
[42] Baskaran G, Mandal S, Shankar R. Phys. Rev. Lett. 2007, 98(24): 247201.
[43] Pelissetto A, Rossi P, Vicari E. Phys. Rev. B. 2001, 65(2): 020403.
[44] Machida M, Koyama T. Phys. Rev. A. 2006, 74(3): 033603.
[45] Matsumoto M, Yasuda C, Todo S, et al. Phys. Rev. B. 2001, 65(1): 014407.
[46] De Grandi C, Polkovnikov A, Sandvik A W. Phys. Rev. B. 2011, 84(22): 224303.
The quadratic Casimir operator $C(2)_{ij}$ here is the first-order one, i.e., the bilinear relation and fit for the two-body model. For the three- or more-body model we need to introduce the second Casimir operator $C(3)_{ijk} = \gamma_{abc} F^a F^b F^c$ (see, for example, Ref.83).
For finite dimension symplectic space, the searching of orthogonal set of state $|s_i\rangle$ often use the Darboux basis $\omega_n = |\partial_n|S_i\rangle + |s_i\rangle \langle \partial_n s_i|s_i\rangle$ which have $\langle s_i|\omega_n\rangle = \partial_n \langle s_i|s_i\rangle = 0$ in the Gram-Schmidt process which presented in the Ref. 139.

[121] Terhal B M, Horodecki P. Phys. Rev. A. 2000, 61(4): 040301.
[122] Keating J P, Linden N, Matthews J C F, et al. Phys. Rev. A. 2007, 76(1): 012315.
[123] Morgan J W R, Mehta D, Wales D J. Physical Chemistry Chemical Physics, 2017.
[124] Lancaster J, Mitra A. Phys. Rev. E. 2010, 81(6): 061134.
[125] Bussi G, Laio A, Parrinello M. Phys. Rev. Lett. 2006, 96(9): 090601.
[126] Suttorp L G, Van Wonderen A J. EPL (Europhysics Letters), 2004, 67(5): 766.
[127] Gardiner C W, Collett M J. Phys. Rev. A. 1985, 31(6): 3761.
[128] Wootters W K. Phys. Rev. Lett. 1998, 80(10): 2245.
[129] Plenio M B, Eisert J, Dreissig J, et al. Phys. Rev. Lett. 2005, 94(6): 060503.
[130] Eisert J, Prosen T. arXiv preprint arXiv:1012.5013, 2010.
[131] Laio A, Rodriguez-Fortea A, Gervasio F L, et al. The journal of physical chemistry B, 2005, 109(14): 6714-6721.
[132] Yang B, Chen Y Y, Zheng Y G, et al. Phys. Rev. Lett. 2017, 119(16): 165701.
[133] Arlt J, Hitomi T, Dholakia K. Applied Physics B: Lasers and Optics, 2000, 71(4): 549-554.
[134] Bretin V, Stock S, Seurin Y, et al. Phys. Rev. Lett. 2004, 92(5): 050403.
[135] Zhang J Y, Ji S C, Chen Z, et al. Phys. Rev. Lett. 2012, 109(11): 115301.
[136] Hou J X. Physics Letters A, 2007, 368(5): 366-370.
[137] Caldeira A O, Leggett A J. Phys. Rev. Lett. 1981, 46(4): 211.
[138] Pezzè L, Ciampini M A, Spagnolo N, et al. arXiv preprint arXiv:1705.03687, 2017.
[139] Picek I, Radovčič B. Physics Letters B, 2010, 687(4): 338-341.
[140] Elias V. Journal of Physics G: Nuclear and Particle Physics, 2001, 27(2): 217.
[141] Jones D R T. Physics Letters B, 1983, 123(1-2): 45-46.
[142] Rahi S J, Emig T, Graham N, et al. Phys. Rev. D. 2009, 80(8): 085021.
[154] Audenaert K, Eisert J, Plenio M B, et al. Phys. Rev. A. 2002, 66(4): 042327.
[155] Yu L W, Ge M L. Physics Letters A, 2017, 381(11): 958-963.
[156] Gebhard F, Ruckenstein A E. Phys. Rev. Lett. 1992, 68(2): 244.
[157] Bloch I, Dalibard J, Zwerger W. Rev. Mod. Phys. 2008, 80(3): 885.
[158] Dziarmaga J. Phys. Rev. Lett. 2005, 95(24): 245701.
[159] Mattis D C, Lieb E H. Journal of Mathematical Physics, 1965, 6(2): 304-312.
[160] Hall B. Lie groups, Lie algebras, and representations: an elementary introduction[M]. Springer, 2015. P. 64.
[161] Hall B. Lie groups, Lie algebras, and representations: an elementary introduction[M]. Springer, 2015. P. 148.
[162] Lundgren L, Svedlindh P, Nordblad P, et al. Phys. Rev. Lett. 1983, 51(10): 911.
[163] Binder K, Young A P. Reviews of Modern physics, 1986, 58(4): 801.
[164] Moeckel M, Kehrein S. Phys. Rev. Lett. 2008, 100(17): 175702.
16 Figure captions

Fig.1:(Color online) $\beta(g)$ as a function of $g$ in SU(3) system (i.e. $C^{(2)}_{ij} = 3$ (see Appendix.A) with the number of fermion multiplets $m = 0, 1, 2, 3, 5, 8, 10, 15, 20$, i.e., the 0-plet, 1-plet, · · ·, 20-plet fermion configuration.

Fig.2:(Color online) (a) Energy difference between the excited state and initial state as a function of staggered magnetic field $h_s$ for different dimension of matrices. (b) Probability of excitation $P_{ex}$ as a function of temperature for different dimension.

Fig.3:(Color online)(a)Inverse spin correlation length (square) and spin correlation (triangle) for $S = 1$ Ising spin chain at different site $i$. (b)Inverse spin correlation length and spin correlation for $S = 1/2$ Heisenberg spin chain at different site $i$ for different $J_2$. The $J_1$ here is setted as 0.7.

Fig.4:(Color online)Spin correlation for $S = 1/2$ spin chain as a function of temperature for different next-nearest neighbor coupling $J_2$.

Fig.5:Spin correlation for $S = 1$ Ising spin chain and $S = 1/2$ Heisenberg chain as a function of temperature.

Fig.6:(left) Spin correlation as a function of temperature by the method of Bethe ansantz; (right) Comparison of the results of spin correlation under low temperature between Bethe ansatz and renormalization group (RG).

Fig.7:(Color online)Graph of Eq.(47) with phase $\alpha = 1, 2, 3$. It’s obviously to see that the contours is bounded by a power function.

Fig.8:The graphs of $\langle n^2(t) \rangle$ as a function of $t$ (Eq.(49)). The reduced coupling $g_{red} = 0.01, 0.1, 1, 1.5, 2$ from (a) to (e), respectively.

Fig.9:The large time behavior of $\langle n^2(t) \rangle$ with coupling $g_{red} = 0.1, 0.05, 0.01, 0.001$ from left to right (Eq.50).

Fig.10:(Color online) Double occupation for half-filling Mott insulator $d_{hf}(t)$ quenches from $U = 0$ to $U$ (a) and from $\infty$ to $U$ (b). The insets show the enlarged views of the $d_{hf}(t)$ for quenches to $U = 1$.

Fig.11:Double occupation for quenches from 0 to critical value $U_c$ (a) and from $\infty$ to $U_c$. The top insets show the enlarge views on short-time scale, and the bottom insets show the large-time behavior in more detial.

Fig.12:Kinetic energy of $1/r$ Hubbard chain as a function of $U$ with different $n$ and bandwidth $W = 1/2, 1, 4$. The bandwidth-dependent hopping constants of $1/r$ Hubbard chain as a function of distance is shown in the inset.

Fig.13:(left)The dispersion relation in $k$ space with different regulatory paramater (0 to 1 from bottom of graph to the upper); (right)The upper, lowest, and ground state energy in a same space according to Ref.61).

Fig.14:(Color online) Correlations as a function of distance $r$ for different quench of dispersion relation are shown, the curves with different colors from outside to inside corresponds to $\kappa_1$ to $\kappa_2$, $\kappa_2$ to $\kappa_3$, $\kappa_4$ to $\kappa_5$, $\kappa_6$ to $\kappa_7$, and $\kappa_7$ to $\kappa_8$, respectively. The dispersion relations are $\kappa_1 = 0.191820018, \kappa_2 = 0.331662479, \kappa_3 = 0.45825757, \kappa_4 = 0.5, \kappa_5 = 0.619656837, \kappa_6 = 0.866025404, \kappa_7 = 1.118033989, \kappa_8 = 1.322875656$. 
### Tables

Table A:

| Model                        | Time scale of relaxation | Period of collapse and revival | Ref(s) |
|------------------------------|--------------------------|--------------------------------|--------|
| Falicov-Kimball              | $\hbar$/bandwidth        | $\hbar/U^3$                    | 116    |
| Bose-Hubbard                 | $1/P$                    | $\hbar/U$                      | 41     |
| Spin glasses                 | macroscopical and with a very broad range | - | 162, 163 |
| Tomonaga-Luttinger           | 2–3 orders of time       | $\hbar/J$ ($J$ is the coupling of nearest-neighbor) | 53      |
| Hubbard                      | $\rho_F^{-1}U^{-2} \sim \rho_F^{-3}U^{-4}$ | - | 164     |
| One-dimension hard core bosons | $1/P_F^3$                | $\hbar/U$                      | 148, 33 |

* Here taking the decaying of time derivative of initial Hamiltonian as the criterion of relaxation.

** $\hbar$ is the Planck constant and $U$ is the strength of nearest interaction (The belows are also follow this).

† $\rho_F$ is the density of states at the Fermi level.

‡ $P_F$ is the hopping of final state after quench.
18 Figures

Fig. 1

![Graph showing the function $\beta(g)$ with different values of $m$.](image)
Fig. 4

![Figure 4](image1.png)

Fig. 5

![Figure 5](image2.png)
