Holomorphic bundles on $\mathcal{O}(-k)$ are algebraic
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Abstract

We show that holomorphic bundles on $\mathcal{O}(-k)$ for $k > 0$ are algebraic. We also show holomorphic bundles on $\mathcal{O}(-1)$ are trivial outside the zero section.

1 Preliminaires

The line bundle on $\mathbb{P}^1$ given by transition function $z^k$ is usually denoted $\mathcal{O}(-k)$. Since we will be studying bundles over this space, we will denote $\mathcal{O}(-k)$ by $M_k$ when we want to view this space as the base of a bundle. We give $M_k$ the following charts $M_k = U \cup V$, for

$$U = \mathbb{C}^2 = \{(z, u)\}$$
$$V = \mathbb{C}^2 = \{ (\xi, v) \}$$
$$U \cap V = (\mathbb{C} - \{0\}) \times \mathbb{C}$$

with change of coordinates

$$(\xi, v) = (z^{-1}, z^k u).$$

Since $H^1(\mathcal{O}(-k), \mathcal{O}) = 0$, using the exponential sheaf sequence it follows that $Pic(\mathcal{O}(-k)) = \mathbb{Z}$, and holomorphic line bundles on $M_k$ are classified by their Chern classes. Therefore it is clear that holomorphic line bundles over $M_k$ are algebraic. We will denote by $\mathcal{O}^l(j)$ the line bundle on $M_k$ given by transition function $z^{-j}$.

If $E$ is a rank $n$ bundle over $M_k$, then over the zero section (which is a $\mathbb{P}^1$) $E$ splits as a sum of line bundles by Grothendieck’s theorem. Denoting the zero section by $\ell$ it follows that for some integers $j_i$ uniquely determined up to order $E_\ell \simeq \bigoplus_{i=1}^n \mathcal{O}(j_i)$. We will show that such $E$ is an algebraic extension of the line bundles $\mathcal{O}^\ell(j_i)$. 
2 Bundles on $\mathcal{O}(-k)$ are algebraic

Lemma 2.1: Holomorphic bundles on $M_k$ are extensions of line bundles.

Proof: We give the proof for rank two for simplicity. The case for rank $n$ is proved by induction on $n$ using similar calculations. Suppose rank $E = 2$ and $E \cong \mathcal{O}(-j_1) \oplus \mathcal{O}(-j_2)$ which we may assume to satisfy $j_1 \geq j_2$. A transition matrix for $E$ from $U$ to $V$ therefore takes the form

$$T = \begin{pmatrix} z^{j_1} + ua & uc \\ ud & z^{j_2} + ub \end{pmatrix}$$

where $a, b, c, \text{ and } d$ are holomorphic functions in $U \cap V$. We will change coordinates to obtain an upper triangular transition matrix

$$\begin{pmatrix} z^{j_1} & uc \\ 0 & z^{j_2} \end{pmatrix},$$

which is equivalent to an extension

$$0 \to \mathcal{O}^l(-j_1) \to E \to \mathcal{O}^l(-j_2) \to 0.$$

Our required change of coordinates will be

$$\begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \begin{pmatrix} z^{j_1} + ua & uc \\ ud & z^{j_2} + ub \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}$$

where $\xi$ is a holomorphic function on $U$ and $\eta$ is a holomorphic function on $V$ whose values will be determined in the following calculations.

After performing this multiplication, the entry $e(2, 1)$ of the resulting matrix is

$$e(2, 1) = \eta(z^{j_1} + ua) + ud + [\eta uc + (z^{j_2} + ub)] \xi.$$

We will choose $\xi$ and $\eta$ to make $e(2, 1) = 0$. We write the power series expansions for $\xi$ and $\eta$ as $\xi = \sum_{i=0}^{\infty} \xi_i(z) u^i$ and $\eta = \sum_{i=0}^{\infty} \eta_i(z^{-1}) (z^k u)^i$, and plug into the expression for $e(2, 1)$. The term independent of $u$ in $e(2, 1)$ is

$$\eta_0(z^{-1}) z^{j_1} + \xi_0(z) z^{j_2}.$$

Since $j_2 - j_1 \leq 0$ we may choose $\eta(z^{-1}) = z^{j_2 - j_1}$ and $\xi(z) = 1$. After these choices $e(2, 1)$ is now a multiple of $u$. Suppose that the coefficients of $\eta$ and
\[ \xi \] have been chosen up to power \( u^{n-1} \) so that \( e(2, 1) \) becomes a multiple of \( u^n \). Then the coefficient of \( u^n \) in the expression for \( e(2, 1) \) is

\[ \eta_n z^{j_1+kn} + \xi_n z^{j_2} + \Phi \]

where

\[ \Phi = \sum_{s+i=n} \eta_s a_i d_n z^{sk} + \sum_{s+i+m=n} \eta_s c_i \xi_m z^{sk} \sum_{m+i=n} \xi_m b_i. \]

We separate \( \Phi \) into two parts

\[ \Phi = \Phi_{>j_2} + \Phi_{\leq j_2} \]

where \( \Phi_{>j_2} \) is the part of \( \Phi \) containing the powers \( z^i \) for \( i > j_2 \) and \( \Phi_{\leq j_2} \) is the part of \( \Phi \) containing powers \( z^i \) for \( i \leq j_2 \). We then choose the values of \( \eta_n \) and \( \xi_n \) as

\[ \eta_n = z^{-j_1-nk} \Phi_{\leq j_2} \]

and

\[ \xi_n = z^{-j_2} \Phi_{>j_2}. \]

These choices cancel the coefficient of \( u^n \) in \( e(2, 1) \). Induction on \( n \) gives \( e(2, 1) = 0 \) And provides a transition matrix of the form

\[ T = \begin{pmatrix} z^{j_1} + ua & uc \\ 0 & z^{j_2} + ub \end{pmatrix}. \]

Now do a similar trick using the change of coordinates

\[ \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \begin{pmatrix} z^{j_1} + ua & uc \\ 0 & z^{j_2} + ub \end{pmatrix} \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix} \]

and choose \( \xi_1, \xi_2, \eta_1 \) and \( \eta_2 \) appropriately to obtain a transition matrix of the form

\[ T = \begin{pmatrix} z^{j_1} & uc \\ 0 & z^{j_2} \end{pmatrix}. \]

\[ \blacksquare \]

**Theorem 2.2** : Holomorphic bundles over \( M_k, k > 0 \) are algebraic.
**Proof:** Let $E$ be a holomorphic bundle over $M_k$ whose restriction to the exceptional divisor is $E_t \simeq \oplus_{i=1}^n \mathcal{O}(j_i)$, then $E$ has a transition matrix of the form

\[
\begin{pmatrix}
  z^{j_1} & p_{12} & p_{13} & \cdots \\
  0 & z^{j_2} & p_{23} & p_{24} & \cdots \\
  \vdots & \vdots & & \ddots & \ddots \\
  0 & \cdots & 0 & z^{j_{n-1}} & p_{n-1,n} \\
  0 & \cdots & 0 & 0 & z^{j_n}
\end{pmatrix}
\]

from $U$ to $V$, where $p_{ij}$ are polynomials defined on $U \cap V$.

Once again we will give the detailed proof for the case $n = 2$. The general proof is by induction on $n$ and is essentially the same as for $n = 2$ only notationally uglier.

For the case $n = 2$ we restate the theorem giving the specific form of the polynomial.

**Theorem 2.3** : Let $E$ be a holomorphic rank two vector bundle on $M_k$ whose restriction to the exceptional divisor is $E_t \simeq \mathcal{O}(j_1) \oplus \mathcal{O}(j_2)$, with $j_1 \geq j_2$. Then $E$ has a transition matrix of the form

\[
\begin{pmatrix}
  z^{j_1} & p \\
  0 & z^{j_2}
\end{pmatrix}
\]

from $U$ to $V$, where the polynomial $p$ is given by

\[
p = \sum_{i=1}^{\lfloor (j_1-j_2-2)/k \rfloor} \sum_{l=ki+j_2+1}^{j_1-1} p_{il} z^l u^i
\]

and $p = 0$ if $j_1 < j_2 + 2$.

**Proof:** Based on the proof of Theorem 2.1 we know that $E$ has a transition matrix of the form

\[
\begin{pmatrix}
  z^{j_1} & uc \\
  0 & z^{j_2}
\end{pmatrix}
\]

We are left with obtaining the form of the polynomial $p$, for which we perform another set of coordinate changes as follows.

\[
\begin{pmatrix}
  1 & \eta \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  z^{j_1} & uc \\
  0 & z^{j_2}
\end{pmatrix}
\begin{pmatrix}
  1 & \xi \\
  0 & 1
\end{pmatrix}
\]
where the coefficients of $\xi = \sum_{i=0}^{\infty} \xi_i(z) u^i$ and $\eta = \sum_{i=0}^{\infty} \eta_i(z^{-1}) (z^k u)^i$, will be chosen appropriately in the following steps. After performing this multiplication, the entry $e(1, 2)$ of the resulting matrix is

$$e(1, 2) = z^{j_1} \xi + uc + z^{j_2} \eta.$$ 

The term independent of $u$ in the expression for $e(1, 2)$ is $z^{j_1} \xi_0(z) + z^{j_2} \eta_0(z^{-1})$. However, we know from the expression for our matrix $T$ (proof of lemma 2.1), that $e(1, 2)$ must be a multiple of $u$; accordingly we choose $\xi_0(z) = \eta_0(z^{-1}) = 0$. Placing this information into the above equation, we obtain

$$e(1, 2) = \sum_{n=1}^{\infty} (\xi_n(z) z^{j_1} + c_n(z, z^{-1}) + \eta_n(z^{-1}) z^{j_2+kn}) u^n.$$ 

Proceeding as we did in the proof of Lemma 2.1, we choose values of $\xi_n$ and $\eta_n$ to cancel as many coefficients of $z$ and $z^{-1}$ as possible. However, here $\xi_n$ appears multiplied by $z^{j_1}$ (and $\eta_n$ multiplied by $z^{j_2+kn}$), therefore the optimal choice of coefficients cancels only powers of $z^i$ with $i \geq j_1$ (resp. $z^i$ with $i \leq j_2 + kn$). Consequently, $e(1, 2)$ is left only with terms in $z^l$ for $j_2 + nk < l < j_1$, and we have the expression

$$e(1, 2) = \sum_{i=1}^{j_1-1} \sum_{l=nk+j_2+1}^{j_1-1} c_{il} z^l u^i.$$ 

But $i$ may only vary up to the point where $nk + j_2 + 1 \leq j_1 - 1$ and the polynomial $p$ is given by

$$p = \sum_{i=1}^{\left\lfloor (j_1-j_2-2)/k \right\rfloor} \sum_{l=ik+j_2+1}^{j_1-1} p_{il} z^l u^i.$$ 

### 3 Triviality outside the zero section

From the previous section we know that bundles on $M_k$ are extensions of line bundles. First we have the following lemma.

**Lemma 3.1**: Line bundles on $M_k$ are trivial outside the zero section.
Proof: A line bundle on $M_k$ can be given by a transition function $z^j$ for some integer $j$. Then the function given by $z^{k-j}u$ on $U$ and $z^ku$ on $V$ is a global holomorphic section which trivializes the bundle outside the zero section.

We now show that the extensions given in Section 2 are trivial outside the zero section.

**Theorem 3.2** Holomorphic vector bundles on $\mathcal{O}(-1)$ are trivial outside the zero section.

**Proof:** Let $E$ be a holomorphic bundle on $\mathcal{O}(-1)$. According to the previous section we know that $E$ is algebraic. Call $F$ the restriction of $E$ to the complement of the zero section, i.e. $F = E|_{\mathcal{O}(-1)}$. Let $\pi: \mathcal{O}(-1) \to \mathbb{C}^2$ be the blow up map. Then $\pi_*(F)$ is an algebraic bundle over $\mathbb{C}^2 - 0$ and therefore it extends to a coherent sheaf $\mathcal{F}$ over $\mathbb{C}^2$. Then $\mathcal{F}^{**}$ is a reflexive sheaf and as such has singularity set of codimension 3 or more, hence in this case $\mathcal{F}^{**}$ is locally free. Moreover, as a bundle on $\mathbb{C}^2$ it must be holomorphically trivial. But $\mathcal{F}^{**}$ restricts to $\pi_*(F)$ on $\mathbb{C}^2 - 0$, hence $\pi_*(F)$ is trivial and so is $F$.

**Corollary 3.3** Holomorphic bundles on the blow up of a surface are trivial on a neighborhood of the exceptional divisor minus the exceptional divisor.

**Proof:** Apply Theorem 3.2 to $\tilde{\mathbb{C}}^2 = \mathcal{O}(-1)$.

**References**

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