A CLASS OF SYMPLECTIC INTEGRATORS WITH ADAPTIVE TIME STEP FOR SEPARABLE HAMILTONIAN SYSTEMS

MIGUEL PRETO AND SCOTT TREMAINE
Princeton University Observatory, Peyton Hall, Princeton, NJ 08544
Received 1999 June 22; accepted 1999 July 15

ABSTRACT

Symplectic integration algorithms are well suited for long-term integrations of Hamiltonian systems, because they preserve the geometric structure of the Hamiltonian flow. Although this desirable property is generally lost when adaptive time step control is added to a symplectic integrator. We describe an adaptive time step, symplectic integrator that can be used if the Hamiltonian is the sum of kinetic and potential energy components and the required time step depends only on the potential energy (e.g., test-particle integrations in fixed potentials). In particular, we describe an explicit, reversible, symplectic, leapfrog integrator for a test particle in a near-Keplerian potential; this integrator has a time step proportional to distance from the attracting mass and has the remarkable property of integrating orbits in an inverse-square force field with only “along-track” errors; i.e., the phase-space shape of a Keplerian orbit is reproduced exactly, but the orbital period is in error by $O(N^{-2})$, where $N$ is the number of steps per period.

Key words: celestial mechanics, stellar dynamics — methods: numerical — solar system: general

1. INTRODUCTION

During the last decade, a great deal of effort has been devoted to the development of symplectic integration algorithms (SIAs) for Hamiltonian systems (Channell & Scovel 1990; Yoshida 1993; Marsden, Patrick, & Shadwick 1996). An SIA is a symplectic mapping of phase space $z = (q, p)$ and time $M_h(z, t) \rightarrow (z', t' = t + h)$ that approximates the Hamiltonian flow over a small interval $h$. SIAs preserve much of the geometric structure of the Hamiltonian flow; as a result, they usually have only oscillatory and not secular errors in the integrals of motion and are useful when the main goal is minimizing long-term qualitative errors, rather than achieving the highest possible short-term precision.

The most popular SIA is the leapfrog or Verlet method, which can be applied to separable Hamiltonians of the form

$$H(q, p, t) = T(p) + U(q, t)$$

(usually $T$ and $U$ are the kinetic and potential energy). We define drift ($D$) and kick ($K$) operators as

$$D_h(q, p, t) = \left( q + h \frac{\partial T}{\partial p}, p, t + h \right),$$

$$K_h(q, p, t) = \left( q, p - h \frac{\partial U}{\partial q}, t \right),$$

and the leapfrog ($L$) operator is then

$$L_h = D_{h/2} K_h D_{h/2} \quad \text{or} \quad L_h = K_{h/2} D_h K_{h/2},$$

(3a)

(3b)

either of which defines a second-order SIA (drift-kick-drift [DKD] leapfrog and kick-drift-kick [KDK] leapfrog, respectively). Higher order SIAs can be derived by concatenating leapfrog operators; for example, $L_{x_3} L_{x_1} L_{x_0}$ is a fourth-order SIA if $x_0 = -2^{1/3}/(2 - 2^{1/3})$ and $x_1 = 1/(2 - 2^{1/3})$ (Yoshida 1990). The leapfrog and its higher order generalizations have several appealing features: (1) only a small number of force evaluations are required per step (one for a second-order integrator, three for a fourth-order integrator), (2) no auxiliary variables are required, thus minimizing memory requirements, and (3) the integrators are explicit and time reversible.

One limitation of SIAs is that they are usually restricted to a fixed time step. When a standard adaptive time step prescription is applied to an SIA, its performance is no better than that of nonsymplectic integrators; the reason for this is that the mapping $M_{h(t)}(z, t)$ is not symplectic, even when $M_h(z, t)$ is. This is a serious limitation, since in most applications a fixed time step is inefficient.

There have been many attempts to construct adaptive time step SIAs. In the context of molecular dynamics, Skeel & Biesiadecki (1994) split the interaction potential into a short-range component (rapidly varying, cheap to calculate, zero outside a limited range) and a long-range component (slowly varying, expensive to calculate); they then evaluate the effects of the short-range potential every time step, using a symplectic integrator, adding in $N$ times the long-range potential at every $N$th time step. This procedure retains symplecticity and can be generalized by splitting the potential into any number of parts. Duncan, Levison, & Lee (1998) describe a variant of the Skeel & Biesiadecki (1994) method for long-term integrations of solar system orbits.

In some situations, an adaptive time step integrator can be replaced by an integrator that uses different—but constant—time steps for different subsystems. In the context of solar system integrations, Saha & Tremaine (1994) describe an SIA that uses a different time step for each planet, which works well as long as the planetary orbits are well separated.

An alternative to using adaptive time step SIAs is to abandon symplecticity but demand that the integrator be time reversible. Formally, an integrator is reversible if $M_h T M_h = T$, where $T$ is the time-reversal operator. Reversible maps have many of the same geometric properties as symplectic maps (Arnold 1984 calls the similarities “astonishing”); hence we expect reversible integration algorithms to have virtues similar to those of SIAs when they are applied to reversible Hamiltonian systems. In particu-
lar, we expect that reversible methods should not exhibit secular errors in the integrals of motion.

Reversible integration algorithms with adaptive time step are relatively easy to generate. Any integration algorithm \( N_h \) can be converted to many reversible algorithms of the same order (Hut et al. 1997); one example is \( M_h \equiv N_{h/2} T N_{h/3} T \). Moreover, any reversible integration algorithm remains reversible with variable time step if the time step depends symmetrically on the initial and final phase-space coordinates (Hut, Makino, & McMillan 1995); unfortunately, such integrators are usually implicit and therefore are slower than explicit integrators. Various explicit reversible integrators with adaptive time step are described by Huang & Leimkuhler (1997), Quin et al. (1997), Calvo, López-Marcos, & Sanz-Serna (1998), and Evans & Tremaine (1999).

One problem that requires adaptive time step is the integration of highly eccentric, near-Keplerian orbits (e.g., long-period comets, which often have eccentricities \( e > 0.99995 \)). Here the standard approach is to regularize the equations of motion; in particular, the Kustaanheimo-Stiefel (KS) regularization converts the Keplerian Hamiltonian to a harmonic oscillator Hamiltonian, which is easier to integrate numerically. KS regularization can be extended to handle few-body problems but is restricted to inverse-square interparticle forces. This technique is also useful, in simulations of star clusters, to deal with the delicate problem of the formation and dynamical evolution of tightly bound binary stars over very long times (e.g., Mikkola & Aarseth 1993). Mikkola (1997, see also Rauch & Holman 1999) has combined KS regularization with an efficient SIA designed for nearly integrable problems by Wisdom & Holman (1991) to provide a sophisticated integrator for eccentric, near-Keplerian orbits.

A closely related problem in numerical celestial mechanics is the long-term integration of moderate-eccentricity, planet-crossing orbits (e.g., Earth-crossing asteroids, Centaurs, Jupiter-family comets), which are nearly Keplerian for millions of years, yet occasionally suffer strong perturbations from close planetary encounters that may last only a few hours (Duncan et al. 1998).

The aim of this paper is to discuss a class of explicit adaptive time step SIs that can be used to follow Hamiltonian systems of the form of equation (1) in the important special case where the time step depends only on the potential energy \( U(q, t) \). We provide a brief review of SIs in § 2 and discuss the use of extended phase space to derive adaptive time step SIs in § 3. A class of explicit adaptive time step SIs that are particularly suitable for following orbits in nearly Keplerian potentials is presented in § 4, and numerical tests are described in § 5. Section 6 contains a brief summary. As this work neared completion, we learned of a paper by Mikkola & Tanikawa (1999) that contains many similar conclusions.

2. REVIEW OF SYMPLECTIC INTEGRATION

An SIA is a mapping of the form \( M_h \) out of the phase-space trajectory of \( z \) over time \( h \), which is given by

\[
\dot{z} = \{z, H\},
\]

where \( H(z, t) \) is the Hamiltonian and the braces stand for the Poisson bracket. The SIA can be defined implicitly by a generating function \( W = W(q, p', t) \), with the equations of transformation

\[
\begin{align*}
\dot{p} &= \frac{\partial W}{\partial q}, \\
\dot{q}' &= \frac{\partial W}{\partial p'},
\end{align*}
\]

In the simplest SIA, the generating function is chosen to be

\[
W = q \cdot p' + hH(q, p', t),
\]

which implies

\[
\begin{align*}
\dot{q}' &= q + h \frac{\partial H(q, p', t)}{\partial p'}, \\
\dot{p}' &= p - h \frac{\partial H(q, p', t)}{\partial q}.
\end{align*}
\]

These equations define an implicit first-order SIA. Higher order schemes can be derived from more complicated generating functions (see, e.g., Channell & Scovel 1990).

We can go further if the Hamiltonian is separable and autonomous, that is, if

\[
H = H_A + H_B,
\]

where \( H_A \) and \( H_B \) are time-independent and separately integrable. For a system of this type, the equations of motion can be written as

\[
\dot{z} = \{z, H_A + H_B\}.
\]

We introduce the differential operators \( A = \{t, H_A\} \) and \( B = \{t, H_B\} \) and write the formal solution of equation (9) as

\[
z(t) = \exp[t(A + B)]z(0).
\]

By assumption, we know how to calculate \( \exp(tA) \) and \( \exp(tB) \). In general, these operators are noncommutative, so \( \exp[t(A + B)] \neq \exp(tA) \exp(tB) \). The correct expression is given by the Baker-Campbell-Hausdorff (BCH) identity (Yoshida 1993)

\[
\exp(X) \exp(Y) = \exp \left( Z \right),
\]

where \( Z = X + Y + [X, Y]/2 + [X - Y, [X, Y]]/12 + \cdots \). To construct an explicit symplectic integrator of order \( n \), we use the BCH identity to find a set of real numbers \( (c_i, d_i) \), such that

\[
\exp[h(A + B)] = \prod_{i=1}^{k} \exp(c_i hA) \exp(d_i hB) + O(h^{n+1}).
\]

The integrator is then

\[
z(t = h) = \prod_{i=1}^{k} \exp(c_i hA) \exp(d_i hB) z(0),
\]

where the operators are applied in the order \( (c_1, d_1, \ldots, c_s, d_s) \); this is an example of the general technique of operator splitting. See Yoshida (1990) for a systematic strategy for finding these numerical coefficients. In the important case of a Hamiltonian of the form of equation (1),
this map takes the simple form (cf. eqs. [2a] and [2b])

\[
q_{i+1} = q_i + h\frac{\partial \Gamma}{\partial p_i}, \\
p_{i+1} = p_i - h\frac{\partial U}{\partial q_{i+1}},
\]

(14a, 14b)

where \( z(0) = z_1 \) and \( z(t) = z_{k+1} \). The usual second-order leapfrog corresponds to the choice \( k = 2 \), \( c_1 = c_2 = \frac{1}{2} \), \( d_1 = 1 \), and \( d_2 = 0 \) for the DKD leapfrog (eq. [3a]) and \( c_1 = 0 \), \( c_2 = 1 \), and \( d_1 = d_2 = 1/2 \) for the KDK leapfrog (eq. [3b]). Using the BCH identity, one can show that

\[
\exp \left( \frac{1}{2} hA \right) \exp (hB) \exp \left( \frac{1}{2} hA \right) = \exp [h(A + B + C)],
\]

(15)

where \( C = \{\cdot, H_{\text{err}}\} \) and

\[
H_{\text{err}} = \frac{h^2}{12} \left\{ [H_A, H_B], H_B + \frac{1}{2} H_A \right\} + O(h^4). \tag{16}
\]

Thus, the DKD leapfrog describes the equations of motion in a surrogate Hamiltonian

\[
\hat{H} = H + H_{\text{err}}, \tag{17}
\]

more precisely, the numerical trajectory lies exponentially close to the exact trajectory of the surrogate Hamiltonian. The good properties of symplectic integrators, such as the absence of secular errors in the energy, are a consequence of the existence of this Hamiltonian.

3. VARIABLE TIME STEP AND EXTENDED PHASE SPACE

We want to construct an explicit adaptive time step SIA to integrate the Hamiltonian of equation (1). Following Mikkola (1997), we extend phase space by introducing a fictitious time variable \( \tau \) through the relation

\[
d \tau = g(q, p, t) dt \tag{18}
\]

and take \( t \equiv q_0 \) as a new coordinate together with the corresponding conjugate momentum \( p_0 = -H \). Thus, an extended phase space is defined by

\[
Q = (q_0, q), \quad P = (p_0, p), \tag{19}
\]

and the equations of motion in the extended phase space are

\[
\frac{dQ}{d\tau} = g(q, p, t) \frac{\partial H}{\partial P} = \frac{\partial \Gamma}{\partial P}, \tag{20a}
\]

\[
\frac{dP}{d\tau} = -g(q, p, t) \frac{\partial H}{\partial Q} = -\frac{\partial \Gamma}{\partial Q}, \tag{20b}
\]

where

\[
\Gamma(Q, P) = g(q, p, q_0)[H(q, p, q_0) + p_0] \tag{21}
\]

and only trajectories on the hypersurface \( \Gamma = 0 \) in the extended phase space correspond to solutions of the equations of motion in the original phase space. The equations of motion (eqs. [20a] and [20b]) are Hamiltonian in the extended phase space if \( \Gamma \) is chosen as the Hamiltonian. We can now integrate the equations of motion with an SIA having constant, fictitious time step \( \Delta \tau \) in the extended phase space, which is equivalent to a variable time step \( \Delta t = g(q, p, t) \Delta \tau \) in the reduced phase space.

The Hamiltonian equation (21) is not generally separable, and hence operator-splitting techniques cannot be used to derive explicit SIAs with arbitrary time steps. Nevertheless, this approach can yield useful SIAs for specific choices of the time step function \( g(q, p, t) \).

3.1. Separable Hamiltonian in Extended Phase Space

We choose the time step function to be

\[
g(Q, P) = f(T(P)) - f(-U(Q)), \tag{22}
\]

where \( T(P) = T(p) + p_0 \) and \( U(Q) = U(q, q_0) \). The Hamiltonian (21) becomes

\[
\Gamma(Q, P) = f(T(P)) - f(-U(Q)), \tag{23}
\]

which is separable. The equations of motion are

\[
\frac{dq_i}{dT} = \frac{\partial \Gamma}{\partial p_i} = f'(T(p) + p_0) \frac{\partial T(p)}{\partial p_i}, \tag{24a}
\]

\[
\frac{dt}{dT} = \frac{\partial \Gamma}{\partial p_0} = f'(T(p) + p_0), \tag{24b}
\]

\[
\frac{dp_i}{dT} = -\frac{\partial \Gamma}{\partial q_i} = -f'(-U(q, t)) \frac{\partial U(q, t)}{\partial q_i}, \tag{24c}
\]

\[
\frac{dp_0}{dT} = -\frac{\partial \Gamma}{\partial q_i} = -f'(-U(q, t)) \frac{\partial U(q, t)}{\partial q_0}. \tag{24d}
\]

To choose the function \( f \), we recall that \( H(q, p, q_0) + p_0 = T(P) + U(Q) = 0 \) for the Hamiltonian flow, i.e., \( T(P) \approx -U(Q) \) during the numerical integration. Consequently, \( f(T) - f(-U) \approx 0 \), and we can Taylor expand the function \( f \) around \( T(P) \) to obtain

\[
f(T(P)) = f(-U(Q)) + [T(P) + U(Q)] \times f'(-U(Q)) + O(T^2 + U^2). \tag{25}
\]

Therefore, equation (22) yields

\[
g(Q, P) \approx f'(-U) \tag{26}
\]

along the integration path. Thus, the time step can be chosen to be an arbitrary function of the potential energy \( g = g(-U) \) and a suitable \( f(-U) \) is determined by integrating \( g(-U) \).

The choice of the time step function is crucial to the success of an integrator, and the restriction that this function depends only on the potential energy \( U \) is severe. Nevertheless, time step functions of this form can be useful for a variety of dynamical problems.

3.2. Error Analysis

We now illustrate how to analyze the integration errors that arise when fixed time step SIAs are used to integrate the equations of motion (eqs. [24a]–[24d]) in extended phase space. For simplicity, we shall assume that the potential is stationary, \( U(Q) = U(q) \), and restrict our attention to the DKD leapfrog. The error Hamiltonian for the DKD leapfrog applied to the Hamiltonian equation (23) is

\[
\Gamma_{\text{err}}(Q, P) = \frac{1}{2} (\Delta \tau)^2 \left[ f'(-U)^2 \right] U_{i,j} \times \left[ f''(T) p_j \right] - \frac{1}{2} (\Delta \tau)^2 \times \left[ f''(T) p_j \right] \left[ -f''(-U) U_{i,j} \right] + f'(-U) U_{i,j} + O(\Delta \tau^4), \tag{27}
\]

where...
where \( U_i = \partial U/\partial q_i \) and summation over the indices \( i, j \in \{1, 2, 3\} \) is assumed.

Once we have the error Hamiltonian, the numerical error in the energy in the original phase space is easy to derive. The integrator accurately follows the trajectory of the surrogate Hamiltonian

\[
\hat{\Gamma} = \Gamma + \Gamma_{err} = f(T) - f(-U) + \Gamma_{err} ;
\]

thus, \( \hat{\Gamma} \) is conserved along the numerical trajectory. At the starting point \((Q_0, P_0)\), \( \Gamma = 0 \), so \( \hat{\Gamma} = \Gamma_{err}(Q_0, P_0) \equiv \Gamma_i \) throughout the integration. Since \( \hat{\Gamma} \) is independent of the coordinate \( q_0 \), the momentum \( p_0 \) is conserved throughout the integration and is therefore equal to minus the initial energy \( E_i \). Thus, \( T_\gamma = T(P) + p_0 = \Delta E - U(q) \), where \( \Delta E = E - E_i \) is the energy error. Equation (28) can now be rewritten as

\[
\Gamma_i = f(\Delta E - U(q)) - f(-U(q)) + \Gamma_{err}(Q, P) ;
\]

since the energy error is small, we can expand in a Taylor series to obtain

\[
\Delta E = \frac{\Gamma_i - \Gamma_{err}(Q, P)}{f'(-U(q))}.
\]

4. KEPLERIAN TWO-BODY PROBLEM

The long-term integration of nearly Keplerian orbits is central to the study of solar system dynamics, and the Keplerian two-body problem provides a natural laboratory for testing integration algorithms. For simplicity, we work in two dimensions, and to provide more general formulae we add an extra potential \( V(q, t) \) to the point-mass potential that defines the Kepler problem. The Hamiltonian for a test particle is thus

\[
H(q, p, t) = \frac{1}{2} p^2 - \frac{\mu}{r} + V(q, t),
\]

where \( q = (x, y), p = v = (v_x, v_y) \), \( r^2 = x^2 + y^2 \), and \( \mu \) is the mass. The equations of motion are

\[
\frac{d^2x}{dt^2} = \frac{dv_x}{dt} = -\frac{\mu}{r^3} \frac{\partial V}{\partial x},
\]

\[
\frac{dy}{dt} = \frac{dv_y}{dt} = -\frac{\mu}{r^3} \frac{\partial V}{\partial y}.
\]

There are two natural choices for the time step function \( g(q, p) \) when integrating bound Keplerian orbits: (1) \( g \propto r^{3/2} \) ensures that the time step is a constant fraction of the local free-fall time \( t_{ff} \sim r^{1/2} \mu^{-1/2} \) so that all phases of highly eccentric orbits are followed with the same relative accuracy; (2) \( g \propto r \) ensures that the coordinate trajectory as a function of the fictitious time is that of a harmonic oscillator, so that there are no high-frequency harmonics that are difficult for numerical integrators to follow.

We can accommodate both of these choices by assuming that the time step function is a power law in radius

\[
g(r) = \epsilon r^\gamma \mu^{1-\gamma},
\]

where \( \epsilon \) is a constant that parametrizes the size of the time step; having introduced \( \epsilon \), we can henceforth set \( \Delta \tau = 1 \) without loss of generality. Since \( U \approx -\mu/r \) for nearly Keplerian orbits, equation (26) then suggests that we choose

\[
f(x) = \begin{cases} 
\epsilon \mu x^{-\gamma+1} & \gamma \neq 1 \, , \\
\epsilon \mu \log x & \gamma = 1 \, .
\end{cases}
\]

The corresponding Hamiltonian is

\[
\Gamma(Q, P) = \begin{cases} 
\frac{\epsilon \mu}{1-\gamma} (\mu r - V)^{-\gamma+1} & \gamma \neq 1 \\
\epsilon \mu \log (\mu r - V) - \log (-U(Q)) & \gamma = 1 \, .
\end{cases}
\]

For the problem we consider here, \( U(Q) = -\mu/(x^2 + y^2)^{1/2} + V(x, y, t) \) and \( T(P) = (v_x^2 + v_y^2)/2 + p_0 \). The equations of motion (eqs. [24a]–[24d]) in the fictitious time read

\[
\frac{dx}{d\tau} = \epsilon \mu \frac{v_x}{(v_x^2/2 + v_y^2/2 + p_0)^{\gamma}},
\]

\[
\frac{dy}{d\tau} = \epsilon \mu \frac{v_y}{(v_x^2/2 + v_y^2/2 + p_0)^{\gamma}},
\]

\[
\frac{dt}{d\tau} = \epsilon \mu \frac{1}{(v_x^2/2 + v_y^2/2 + p_0)^{\gamma}},
\]

\[
\frac{dv_x}{d\tau} = -\frac{\epsilon \mu}{(\mu r - V)^{\gamma}} \left( \frac{\mu x}{r^3} + \frac{\partial V}{\partial x} \right),
\]

\[
\frac{dv_y}{d\tau} = -\frac{\epsilon \mu}{(\mu r - V)^{\gamma}} \left( \frac{\mu y}{r^3} + \frac{\partial V}{\partial y} \right),
\]

\[
\frac{dp_0}{d\tau} = -\frac{\epsilon \mu}{(\mu r - V)^{\gamma}} \frac{\partial V}{\partial t}.
\]

These equations can be integrated using leapfrog, since the right-hand sides of the first three depend only on momenta in the extended phase space and the latter three depend only on coordinates. For example, if the extra potential \( V = 0 \), then the DKD leapfrog for equations (36a)–(36f) with \( \gamma = 1 \) can be written

\[
r_{1/2} = r + \frac{\epsilon \mu v}{v^2 + 2p_0},
\]

\[
t_{1/2} = t + \frac{\epsilon \mu}{v^2 + 2p_0},
\]

\[
v' = v - \frac{\epsilon \mu r_{1/2}}{r_{1/2}^2},
\]

\[
r' = r_{1/2} + \frac{\epsilon \mu v}{(v')^2 + 2p_0},
\]

\[
t' = t_{1/2} + \frac{\epsilon \mu}{(v')^2 + 2p_0},
\]

where \( r = |r| \) and \( p_0 \) is an integral of motion equal to minus the initial energy. The fictitious time \( \tau \) is equal to the eccentric anomaly \( u \) to within a linear transformation; each
step of the integration corresponds to $\Delta \mu = e(\mu/a)^{1/2}$, where $a = -\mu/2E$ is the semimajor axis.

More generally, if the attracting mass has a trajectory $r_\alpha(t)$, the DKD leapfrog with $\gamma = 1$ reads

\begin{align*}
r_{1/2}^\alpha &= r + \frac{e\mu}{v^2 + 2p_0}, \\
t_{1/2}^\alpha &= t + \frac{e\mu}{v^2 + 2p_0},
\end{align*}

(38a)

(38b)

\begin{align*}
v' &= v - \frac{e\mu [r_{1/2}^\alpha - r_\alpha(t_{1/2}^\alpha)]}{|r_{1/2}^\alpha - r_\alpha(t_{1/2}^\alpha)|^2}, \\
p_0' &= p_0 + \frac{e\mu [r_{1/2}^\alpha - r_\alpha(t_{1/2}^\alpha)] \cdot dr_\alpha(t_{1/2}^\alpha)/dt}{|r_{1/2}^\alpha - r_\alpha(t_{1/2}^\alpha)|^2},
\end{align*}

(38c)

(38d)

\begin{align*}
r' &= r_{1/2} + \frac{e\mu v}{(v^2 + 2p_0)}, \\
t' &= t_{1/2} + \frac{e\mu v}{(v^2 + 2p_0)}.
\end{align*}

(38e)

(38f)

An appealing feature of equations (38a)-(38f) is that they do not contain a square root, which is the most time-consuming operation in integrating Keplerian orbits by conventional methods; however, square root evaluations do become necessary with this integrator as soon as the non-Keplerian potential $V(r, t)$ is nonzero.

### 4.1. Error Analysis for the Kepler Problem

To analyze the numerical error of the DKD leapfrog with $\gamma = 3/2$, we take $f(x)$ from equation (34) and set $\Delta \tau = 1$ in equation (27). The leading term of the error Hamiltonian becomes

\[
\Gamma_{\text{err}}(Q, P) = \frac{e^3 \mu^3}{24[-U(q)T_e(P)]^3} \left\{ 2T_e^{2/3} |VU|^2 \right. \\
\left. - p_i p_j (U)^{3/2} U_{ij} \right. \\
\left. - \left[ \frac{3}{2} (U)^{1/2} + 3T_e^{1/2} \right] (p \cdot \nabla U)^2 \right\}.
\]

(39)

For the Kepler problem, $U = -\mu/r$, and the error Hamiltonian simplifies to

\[
\Gamma_{\text{err}} = \frac{e^3 \mu^2}{24 T_e^{2/3}} \left[ 2 T_e^{2/3} \frac{3 (p \cdot r)^2}{r^3} + \frac{3 \mu^{1/2} (p \cdot r)^2}{2r^{1/2}} - \frac{\mu^{1/2} \rho^2}{r^{3/2}} \right],
\]

(40)

where $T_e = v^2/2 + p_0$.

Similarly for $\gamma = 1$, the leading term of the error Hamiltonian is

\[
\Gamma_{\text{err}} = \frac{e^3 \mu^3}{24 [-U(q)T_e(P)]^3} \times [2T_e |VU|^2 + p_i p_j U_{ij} - 3 (p \cdot \nabla U)^2];
\]

(41)

for the Kepler problem, this simplifies to

\[
\Gamma_{\text{err}} = \frac{e^3 \mu^3 P_0}{12r^2 (v^2/2 + p_0)^2}.
\]

(42)

This formula leads to a remarkable conclusion, specific to this potential and integrator. The original phase-space variables $(q, p)$ enter $\Gamma_{\text{err}}$ in the same combination that they enter the Hamiltonian $\Gamma$; in other words, the surrogate Hamiltonian may be written

\[
\Gamma(Q, P) = \Gamma(Q, P) + \Gamma_{\text{err}}(Q, P)
\]

\[
= \epsilon \mu \log W(Q, P) + \frac{e^3 \mu p_0}{12 W^2(Q, P)},
\]

(43)

where $W(Q, P) = r(p^2/2 + p_0)/\mu$. Thus, the equations of motion (eq. [4]) for $\Gamma$ read

\[
\dot{z} = (z, \Gamma) = \left( \frac{e \mu}{W} - \frac{e^3 \mu p_0}{6 W^3} \right) \{z, W\},
\]

(44a)

\[
\frac{e \mu}{W} \{z, W\},
\]

(44b)

while the equations of motion for $\Gamma$ read

\[
\dot{z} = (z, \Gamma) = \left( \frac{e \mu}{W} - \frac{e^3 \mu p_0}{6 W^3} \right) \{z, W\},
\]

(45a)

\[
\frac{e \mu}{W} \{z, W\},
\]

(45b)

Thus, the exact trajectory $\gamma(t)$ (Hamiltonian $\Gamma$) is the same as the numerical trajectory $\gamma(t)$ (Hamiltonian $\Gamma$) at a slightly different fictitious time, where the two timescales are related by $dt = d\tau [1 - e \mu p_0/(6 W^3)]$, or $\tau = \gamma(1 - e \mu p_0)$ since $W = 1$ on the trajectory. In other words, the algorithm (eqs. [38a]–[38f]) follows the Keplerian trajectory exactly: the position and velocity are precisely those of the Keplerian orbit, and the only error is in the time of arrival at a given location (i.e., the only error is “along track”).

Although we have only established this result for the leading order in the error Hamiltonian, we show in the Appendix that it holds at all orders, i.e., for arbitrarily large time steps. This result also applies in the more general case where the attracting mass is in uniform motion, rather than stationary at the origin (cf. eqs. [38a]–[38f]).

We also show in the Appendix that the timing error arising from a step $\Delta \mu$ in eccentric anomaly is $(\Delta \mu)^2/12n + O(\mu/\mu)^3$, where $n = (\mu/a^3/2)$ is the mean motion and the error is independent of eccentricity. The fractional error in the orbital period is then $\pi^2/3N^2$, where $N$ is the number of steps per period (eq. [A12]).

### 4.2. Error Analysis for the Perturbed Kepler Problem

Because the DKD leapfrog with $\gamma = 1$ follows a Keplerian trajectory exactly, this method is of particular interest for the perturbed Kepler problem, where the extra potential $V(r, t)$ in equations (36a)-(36f) is small but nonzero. To investigate the errors in this case, we take the error Hamiltonian equation (41), set $U = -\mu/r + V$ (for simplicity we assume that $V$ is stationary) and expand to first order in $V$:  

\[
\Gamma_{\text{err}}(Q, P) = \frac{e^3 \mu^2}{24 (v^2/2 + p_0)^2} \left[ 2 \mu p_0 + \frac{4 p_0 V}{r} \\
+ \frac{4}{2} \left( \frac{v^2 + p_0}{r} \right) \nabla V - r^2 \partial_i v d V_{ij} \\
+ \frac{v^2 V}{r} - \frac{3 (e \cdot r)^2 V}{r^3} - 6 \frac{e \cdot r}{r} v \cdot \nabla V \right].
\]

(46)

\[1\] This result holds only for the DKD leapfrog, not for the KDK leapfrog.
When this is evaluated on the trajectory, we have
\[
\Gamma_{err}(Q, P) = -\frac{1}{12} e^3 \mu E + \frac{e^3}{24} \left[ -8EvV \\ + 4\mu r \cdot VV - r^3 v_i v_j V_{ij} + rv^2 V \\
- \frac{3(v \cdot r)^2 V}{r} - 6r(v \cdot r)v \cdot VV \right].
\] (47)

The energy error (eq. [30]) is then
\[
\Delta E = \frac{\Gamma_i - \Gamma_{err}(Q, P)}{er}.
\] (48)

As \( r \to 0 \), we have \( v \sim r^{-1/2} \). Thus, if \( V \sim r^k \) as \( r \to 0 \), \( \Gamma_{err} \sim r^k \) as well. Then if \( k > 0 \), the energy error at close encounters with the attracting mass is
\[
\Delta E = \frac{\Gamma_i}{er}, \quad r \ll r_i;
\] (49)
in other words, the energy error at close encounters is determined by the initial conditions and varies as \( r^{-1} \), independent of the form of the perturbing potential at small radii, as long as \( V \to 0 \) as \( r \to 0 \).

The divergence of \( \Delta E \) as \( r \to 0 \), even when the perturbing potential \( V \to 0 \) as \( r \to 0 \), appears to contradict our proof that the integrator tracks Keplerian orbits exactly; the resolution is that the integrator only tracks Keplerian orbits on the hypersurface \( \Gamma = 0 \) in the extended phase space, and numerical errors at larger radii perturb the trajectory to the neighboring hypersurface \( \Gamma = \Gamma_i \). In fact, it can be shown that in this case the integrator is following a Kepler orbit exactly, but for an attracting mass \( \mu \exp (\Gamma_i/\epsilon \mu) \). Thus, even large energy errors at close encounters do not signal a catastrophic failure of the integrator, as long as \( |\Gamma_i/\epsilon \mu| \ll 1 \).

Moreover, there is a simple way to correct these errors. Normally, the initial value of \( p_0 \) is set equal to \( -E \), so that \( \Gamma = 0 \); instead, we modify the initial value of \( p_0 \) such that \( \tilde{\Gamma} = \Gamma + \Gamma_{err} \) is zero. This requires
\[
p_0 = -E + \frac{\mu}{r} \left[ \exp \left( -\frac{\Gamma_i}{\epsilon \mu} \right) - 1 \right].
\] (50)

5. NUMERICAL TESTS

5.1. Keplerian Two-Body Problem

We have tested these integration methods by following Keplerian orbits with eccentricities \( e = 0.9, 0.99, 0.999, 0.9999, 0.99999 \), and 0.999999. Each orbit is started at the pericenter and followed for \( 2 \times 10^4 \) orbital periods (although a shorter integration would have been sufficient, since the energy errors are oscillatory rather than growing). We characterize the performance of the integrator by the maximum energy error \( |\Delta E_{\text{max}}/E_0| = \max |(E - E_0)/E_0| \), as a function of the number of steps per orbital period.

Figure 1 shows the energy error that arises from integrating equations (36a)–(36f) using the DKD leapfrog with \( \gamma = 3/2 \) (i.e., time step \( \propto r^{3/2} \)). For comparison, we have also shown as open circles the energy error for the leapfrog with fixed time step \( \gamma = 0 \) at eccentricity \( e = 0.9 \) and 0.99. Clearly, \( \gamma = 3/2 \) provides far more accurate integrations than those provided by \( \gamma = 0 \).

We can compare these energy errors with those in the analysis of § 4.1. When evaluated on the trajectory \( (T_e = -U) \), the error Hamiltonian equation (40) is
\[
\Gamma_{err} = -\frac{e^3 r^{3/2}}{24\mu^{1/2}} \left[ \frac{3(r \cdot v)^2}{2r^2} + 2E \right]
\] 
\[
+ \frac{e^4 n a^2}{24} \left[ (1 - e \cos u)^{3/2} - \frac{3e^2 \sin^2 u}{2(1 - e \cos u)^{1/2}} \right],
\] (51)

where \( n, a, \) and \( u \) are the mean motion, semimajor axis, and eccentric anomaly, respectively. The energy error of equation (30) is then given by
\[
\Delta E = \frac{e^2 n^2 a^2}{24} \left[ \frac{(1 - e \cos u)^{3/2}}{1 - e \cos u} \right]
\] 
\[
- \frac{3e^2 \sin^2 u}{2(1 - e \cos u)^{1/2}(1 - e \cos u)^{1/2}}
\] 
\[
- 1 + \frac{3e^2 \sin^2 u}{2(1 - e \cos u)^2}. \]
\] (52)

For high-eccentricity orbits started at the pericenter \( (u_t = 0) \), the maximum error \( |\Delta E| \) occurs at \( u \sim \cos^{-1} e \) and is given by
\[
|\Delta E|_{\text{max}} = \frac{e^2}{16(1 - e)} + O(e^2(1 - e)^0). \] (53)
For high-eccentricity orbits started at the apocenter \((u_i = \pi)\), the maximum energy error occurs at the pericenter,

\[
\left| \frac{\Delta E}{E} \right|_{\text{max}} = \frac{e^2}{3-2^{1/2}(1-e)^{3/2}} + \mathcal{O}\left(\frac{e^2}{1-e}\right).
\]  

(54)

These formulæ show that pericenter starts lead to smaller energy errors than do apocenter starts, although in the latter case the errors can be reduced by the use of corrected initial conditions (cf. § 4.2).

The number of steps per orbit is defined as \(N \approx \int_0^P \frac{df}{\epsilon/g(r)}\), where \(g(r)\) is the time step function (eq. 33) and \(P\) is the orbital period. For \(\gamma = 3/2\),

\[
N = \frac{2}{\epsilon} \int_0^\pi \frac{df}{(1 + e \cos f)^{1/2}} = \frac{4}{\epsilon(1 + e)^{1/2}} K\left(\frac{2e}{1+e}\right),
\]  

(55)

where \(f\) is the true anomaly and \(K\) is an elliptic integral. Plotting equations (53) and (55) as a parametric function of \(\epsilon\), we obtain the dashed lines in Figure 1, which agree well with the energy errors from the numerical orbit integrations.

As we discussed in § 4, integrating the Keplerian equations of motion using the DKD leapfrog with \(\gamma = 1\) (eqs. [37a]–[37c]) yields even better behavior than that with \(\gamma = 3/2\), in which case the energy error is zero; in fact, there is zero error in all of the phase-space functions that are constants of motion in a point-mass potential (energy, angular momentum, and Runge-Lenz vector). Therefore, to test the practical value of this algorithm, we must turn to more general Hamiltonians, which we now do.

5.2. Stark Problem

The Stark problem is to follow the motion of a test particle subject to an inverse-square force plus a constant force; the Stark Hamiltonian is

\[
H = \frac{1}{2} p^2 - \frac{\mu}{r} - S \cdot r,
\]  

(56)

where the Stark vector \(S\) is a constant. The Stark Hamiltonian has three constants of motion and thus is integrable: these are the energy \(E\), the angular momentum component along \(S\), and a third analytic integral, which there is no reason to write out here (Pars 1965; Landau & Lifshitz 1976). We restrict ourselves to the planar case, which is particularly challenging because all orbits oscillate between retrograde and prograde and hence pass arbitrarily close to the attracting mass. We shall examine only a single integrator, the DKD leapfrog, applied to the equations of motion for \(\gamma = 1\) (eqs. [36a]–[36f]), since in this case the trajectory is followed exactly when \(S = 0\).

Rauch & Holman (1999) have recently tested several integrators on the Stark problem, and we shall usually use their initial conditions: the initial eccentricity is \(e = 0.9\), the Stark vector is oriented \(45^\circ\) to the initial line of apsides, and the orbit is started at the apocenter. The strength of the Stark perturbation is written as \(S = \eta E^2/\mu\), where \(E\) is the energy and \(\eta \ll 1\) for nearly Keplerian motion.

The error Hamiltonian and the expected energy error are given by equations (47) and (48), with \(V = -S \cdot r\). Figure 2 verifies the functional form \(\Delta E \propto r^{-1}\) predicted by equation (48) and demonstrates the improvement during close encounters that results from using the corrected initial condition (eq. [50]).

![Figure 2](image-url)

**Fig. 2.** Fractional energy error as a function of distance from the attracting mass, for a numerical integration of the Stark problem using the DKD leapfrog with \(\gamma = 1\), which integrates Keplerian orbits with zero energy error. The integration lasts for 1000 Keplerian periods of the initial orbit, and the error is plotted every 100 time steps. The integration parameters are \(\mu = 1\) and \(e = 0.1\). The initial eccentricity is \(e = 0.9\). The Stark vector is \(45^\circ\) from the initial line of apsides, and its magnitude is \(S = \eta E^2/\mu\), where \(\eta = 4 \times 10^{-3}\). For \(r \leq 1\), the points lie on a straight line, consistent with the prediction of equation (49) marked by solid line segments. The open circles show the much smaller energy errors when the initial conditions are corrected using equation (50).

![Figure 3](image-url)

**Fig. 3.** Fractional energy error for the Stark problem, as a function of number of force evaluations per orbit. The initial eccentricity is \(e = 0.9\); the Stark vector is \(45^\circ\) from the initial line of apsides; and the orbit starts at apocenter and is followed for \(10^6\) periods. The magnitude of the Stark vector is \(S = \eta E^2/\mu\), where \(\eta = 0.001\) (solid lines), 0.005 (dashed lines), and 0.02 (dash-dotted lines). The lower curves (filled circles) represent the average of the absolute value of the energy error, and the upper curves represent the maximum error. The integrator is the DKD leapfrog with \(\gamma = 1\) and initial conditions corrected using equation (50); the single solid line with open circles represents the average error for \(\eta = 0.001\) if no corrector is applied.
Figure 3 shows the energy error as a function of steps per orbit \( N \) and the strength of the Stark parameter for integrations lasting \( n = 10^4 \) orbital periods, using the Rauch & Holman (1999) initial conditions corrected by equation (50). We have plotted both the maximum energy error, which is dominated by very close encounters, and the average of the absolute value of the energy error, which provides a better estimate of the typical error. The average error exhibits the \( N^{-2} \) dependence expected for a second-order integrator; the maximum error is much larger and more irregular, reflecting its dependence on rare and rather unphysical close encounters (the typical maximum eccentricity in an integration of this length is given by \( 1 - e_{\text{max}} \approx n^{-2} = 10^{-6} \)). We have also plotted one error curve for uncorrected initial conditions; we see that the correction reduces the errors by about one order of magnitude in this case.

Rauch & Holman (1999) tested the Wisdom & Holman (1991) integrator on the Stark problem with \( \eta = 4 \times 10^{-3} \). They found that the Wisdom & Holman (1991) integrator—which works very well for low-eccentricity orbits—was generally unstable, in that the energy error grew by a random walk, until the orbit escaped to infinity. The instability arose through numerical chaos caused by the overlap of resonances in the error Hamiltonian and could only be evaded if the (constant) time step was in resonance with the orbital period, or the instability is small enough that the pericenter passage is well resolved—even though the Wisdom & Holman (1991) integrator follows Keplerian orbits exactly for any time step. Our integrator is evidently not subject to these limitations.

Rauch & Holman (1999) also tested several other methods. In particular, Mikkola’s (1997) regularized version of the Wisdom & Holman (1991) mapping was completely stable and gave energy errors comparable to those shown for our integrator in Figure 3. However, we expect that our method is faster in practice, because it requires fewer calculations per integration step.

6. SUMMARY

We have constructed adaptive time step, reversible, explicit SIAs for separable Hamiltonians of the form of equation (1), using extended phase space (Mikkola 1997); the principal restriction is that the time step must be a function of the potential energy alone.

Integrators of this kind would require modifications for problems with many degrees of freedom, since the total potential energy of the system is insensitive to local conditions that may demand a short time step (e.g., a close encounter between two bodies). However, for test-particle integrations in fixed, smooth potentials or few-body systems with similar masses, these integrators can provide both adaptive time step control and the excellent long-term error control associated with symplectic and reversible integration algorithms.

For close encounters or eccentric orbits in few-body gravitating systems, these adaptive time step SIAs provide an attractive alternative to fixed time step SIAs in regularized coordinates; moreover, unlike regularized integrators, adaptive time step SIAs can also be used to follow orbits in non-Keplerian potentials (e.g., galaxies).

Although we have discussed only second-order leapfrog integrators, higher order, adaptive time step SIAs can be derived by concatenating leapfrog steps of different lengths (Yoshida 1990). A particularly interesting example of these integrators is offered by equations (37a)–(37e), which follow a Keplerian trajectory exactly, with only “along-track” errors.

This research was supported in part by NASA grant NAG 5–7310. We thank Seppo Mikkola for sending us a copy of his paper (Mikkola & Tanikawa 1999), which also shows that the integrator (eqs. [38a]–[38f]) is exact for Keplerian orbits. We also thank Kevin Rauch for thoughtful comments.

APPENDIX

EXACT INTEGRATOR FOR THE KEPLER PROBLEM

We prove that the \( g = 1 \) leapfrog integrator (eqs. [38a]–[38f]) follows a Keplerian orbit exactly, except for errors in the time. For simplicity, we shall assume that the attracting mass is at rest at the origin, as in equations (37a)–(37e); the extension to an attracting mass in uniform motion is straightforward.

Let \((r, v)\) and \((r', v')\) be the position and velocity at two points on a bound Keplerian orbit with eccentric anomalies \( u \) and \( u' \), respectively. Then, \((r', v')\) satisfies the relation

\[
\begin{align*}
\dot{r'} &= f(u, u')r + g(u, u')v, \quad (A1a) \\
\dot{v'} &= f_i(u, u')r + g_i(u, u')v, \quad (A1b)
\end{align*}
\]

where

\[
\begin{align*}
f(u, u') &= \frac{\cos (u' - u) - e \cos u}{1 - e \cos u}, \quad (A2a) \\
g(u, u') &= \frac{1}{n} \left[ \sin (u' - u) - e \sin u' + e \sin u \right], \quad (A2b) \\
f_i(u, u') &= -\frac{n \sin (u' - u)}{(1 - e \cos u')(1 - e \cos u)}, \quad (A2c) \\
g_i(u, u') &= \frac{\cos (u' - u) - e \cos u'}{1 - e \cos u'} \quad (A2d)
\end{align*}
\]
are Gauss's $f$ and $g$ functions (e.g., Danby 1988); here $e$ is the eccentricity, $n = (\mu/a^3)^{1/2}$ is the mean motion, and $a$ is the semimajor axis, which is related to the radius by

$$r = a(1 - e \cos u).$$

These equations can be rewritten as

$$r_{1/2} = r + s(u', u)u' , \quad v' = v + z(u', u)r_{1/2} , \quad r' = r_{1/2} - s(u, u')v' , \quad \text{(A3a)}$$

$$s(u', u) = \frac{[1 - \cos (u' - u)][1 - e \cos u]}{n \sin (u' - u)} , \quad \text{(A4a)}$$

$$z(u', u) = - \frac{n \sin (u' - u)}{(1 - e \cos u')(1 - e \cos u)} . \quad \text{(A4b)}$$

Comparison with the $\gamma = 1$ leapfrog integrator (eqs. [A5a]–[A5f]) shows that the two maps $(r, v) \rightarrow (r', v')$ are the same if

$$\epsilon \mu \quad \frac{p^2 + 2p_0}{[1 - \cos (u' - u)][1 - e \cos u]} = \frac{n \sin (u' - u)}{[1 - e' \cos u'](1 - e \cos u)} , \quad \text{(A5a)}$$

$$\epsilon \mu \quad \frac{q_{1/2}^2}{[1 - \cos (u' - u)][1 - e \cos u]} = \frac{n \sin (u' - u)}{(1 - e' \cos u')(1 - e \cos u)} , \quad \text{(A5b)}$$

$$\epsilon \mu \quad \frac{(p')^2 + 2p_0}{[1 - \cos (u' - u)][1 - e \cos u']} = \frac{n \sin (u' - u)}{[1 - e' \cos u'](1 - e \cos u')} . \quad \text{(A5c)}$$

We use the relations $p^2 + 2p_0 = 2\mu[a(1 - e \cos u)]$ and $(p')^2 + 2p_0 = 2\mu[a(1 - e' \cos u')]$ and square equation (A3a) to eliminate $q_{1/2}^2$. After some algebra, we find that all of the relations (eq. [A5a]–[A5c]) are satisfied if

$$\epsilon = 2 \frac{1 - \cos \Delta u}{na \sin \Delta u} , \quad \text{(A6)}$$

where $\Delta u = u' - u$. Thus, we have proved that our mapping follows the Keplerian two-body problem exactly in the original phase space. Although our proof is for bound orbits, it is straightforward to show that unbound Keplerian orbits ($a < 0$) are also integrated exactly, with

$$\epsilon = 2 \frac{\cosh \Delta u - 1}{a \sinh \Delta u} ; \quad \text{(A7)}$$

here $n_a = (-\mu/a^3)^{1/2}$ and $r = -a(e \cosh u - 1)$.

We must still establish the relation between the time step given by equations (38b) and (38e) and the actual time $\Delta t_K$ required to travel from $r$ to $r'$. The time step is

$$t' = t + \Delta t = t + \epsilon \mu \left[ \frac{1}{p^2 + 2p_0} + \frac{1}{(p')^2 + 2p_0} \right] = t + \frac{1}{2} \epsilon a(2 + e \cos u + e \cos u') . \quad \text{(A8)}$$

The relation (A6) then implies that

$$n \Delta t = \frac{[1 - \cos (u' - u)][2 - e \cos u - e \cos u']}{\sin (u' - u)}$$

$$= 2 \frac{1 - \cos (u' - u)}{\sin (u' - u)} - e \sin u' + e \sin u . \quad \text{(A9)}$$

On the other hand, Kepler's equation states that the actual time step is given by

$$n \Delta t_K = u' - u - e \sin u' + e \sin u . \quad \text{(A10)}$$

Thus the time error is given by

$$n(\Delta t - \Delta t_K) = 2 \frac{1 - \cos \Delta u}{\sin \Delta u} - \Delta u = \frac{1}{12} (\Delta u)^3 + \frac{1}{120} (\Delta u)^5 + O(\Delta u)^7 , \quad \text{(A11)}$$

independent of eccentricity. If we take $N = 2\pi/\Delta u$ steps per orbit, the timing error per orbit is

$$\frac{\delta t}{P} = \frac{\pi^2}{3N^2} + O(N^{-4}) , \quad \text{(A12)}$$

independent of eccentricity.
REFERENCES

Arnold, V. I. 1984, in Nonlinear and Turbulent Processes in Physics, Vol. 3, ed. R. Z. Sagdeev (Chur: Harwood), 1161
Calvo, M. P., López-Marcos, M. A., & Sanz-Serna, J. M. 1998, Appl. Numer. Math., 28, 1
Channell, P. J., & Scovel, J. C. 1990, Nonlinearity, 3, 231
Danby, J. A. M. 1988, Fundamentals of Celestial Mechanics (Richmond: Willmann-Bell)
Duncan, M. J., Levison, H. F., & Lee, M. H. 1998, AJ, 116, 2067
Evans, N. W., & Tremaine, S. 1999, AJ, 118, 1888
Huang, W., & Leimkuhler, B. 1997, SIAM J. Sci. Stat. Comput., 18, 239
Hut, P., Funato, Y., Kokubo, E., Makino, J., & McMillan, S. 1997, in ASP Conf. Ser. 123, Computational Astrophysics, ed. D. A. Clarke & M. J. West (San Francisco: ASP), 26
Hut, P., Makino, J., & McMillan, S. 1995, ApJ, 443, L93
Landau, L., & Lifshitz, L. 1976, Mechanics (3d ed.; Oxford: Pergamon)

Marsden, J., Patrick, G., & Shadwick, W. F., eds. 1996, Integration Algorithms for Classical Mechanics (Fields Inst. Commun., Vol. 10) (Providence: Am. Math. Soc.), 217
Mikkola, S. 1997, Celest. Mech. Dyn. Astron., 67, 145
Mikkola, S., & Aarseth, S. J. 1993, Celest. Mech. Dyn. Astron., 57, 439
Mikkola, S., & Tanikawa, K. 1999, Celest. Mech. Dyn. Astron., 74, 287
Pars, L. A. 1965, A Treatise on Analytical Dynamics (London: Heinemann)
Quinn, T. R., Katz, N., Stadel, J., & Lake, G. 1997, preprint (astro-ph/9710043)
Rauch, K. P., & Holman, M. 1999, AJ, 117, 1087
Saha, P., & Tremaine, S. 1994, AJ, 108, 1962
Skeel, R. D., & Biesiadecki, J. J. 1994, Ann. Numer. Math., 1, 191
Wisdom, J., & Holman, M. 1991, AJ, 102, 1528
Yoshida, H. 1990, Phys. Lett. A, 150, 262
———. 1993, Celest. Mech. Dyn. Astron., 56, 27