Introduction to Duality

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ABSTRACT

We describe the duality between different geometries which arises by considering the classical and quantum harmonic map problem.

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1 Introduction and summary

Consider a manifold with certain geometric data: a metric, a compatible connection with torsion, and possibly a scalar field. Duality is a map between different geometries; in general, it changes not just the metric and connection on the manifold, but the topology as well. Though there are reasons to believe that it can be generalized beyond the cases described here, we understand how to construct the dual of a given geometry only for geometries with isometries.

The duality map is found by considering a generalized harmonic map problem. Geometries that are classically dual have the same harmonic maps. Geometries may also be dual in a stronger sense: as explained below, they give rise to the same quantum field theory. We call this map “quantum duality”\[1\].

We now briefly summarize our results. We begin with a manifold \( M \) with a metric \( G \) (which can be written in terms of frames \( e^a \) as \( G = e^a \otimes e^a \)), a closed 3-form \( T, dT = 0 \)\[2\] and a compatible connection \( \nabla = d + \omega \) with torsion \( T^a = e^a \otimes T = de^a + \omega^a_b \wedge e^b \). In a patch on \( M \), the torsion may be written as \( T = \frac{1}{2} dB \) for some two form \( B \). We suppose also that \( M \) has some Lie group of isometries preserving \( T \). Duality with respect to some subgroup \( G \) of the isometry group gives a new manifold \( \tilde{M} \) that is a certain \( \tilde{G} \) quotient of \( *g \times M \) (where \( *g \) is the dual of the Lie algebra of \( G \)). On \( *g \times M \) we find the metric \( \hat{G} \) and the two form \( \hat{B} \) in terms of the original metric plus two form \( E_{ij} = G_{ij} + B_{ij} \) and a basis \( \{ k_A \} \) of Killing vector fields generating the Lie algebra \( g \) of \( G \):

\[
\begin{pmatrix}
\hat{E}_{AB} & \hat{E}_{Aj} \\
\hat{E}_{iB} & \hat{E}_{ij}
\end{pmatrix}
= \begin{pmatrix}
[E_{AB}]^{-1} & \hat{E}^{AB} E_{Bj} \\
-E_{iA} \hat{E}^{AB} & E_{ij} - E_{iA} \hat{E}^{AB} E_{Bj}
\end{pmatrix}, \tag{1.1}
\]

where\[1\] When the geometry admits \( N = 2 \) superconformal symmetry, and if duality maps the left-moving \( N = 2 \) \( U(1) \) current \( J \) to \( -J \), it is also called a “mirror map”.\[2\] We assume that \( T \) represents an integral (possibly trivial) element of the cohomology.
Here $f^A_{BC}$ are the structure constants of $g$, $\lambda_A$ are coordinates on $\ast g$ and $X^i$ are coordinates on $M$. The dual metric $\tilde{G}$ and two form $\tilde{B}$ are the restriction of $\hat{G}$ and $\hat{B}$ to the $G$ orbits of $\ast g \times M$.

If the connection $\nabla$ has restricted holonomy, i.e., in some subgroup of the orthogonal group, then in general $M$ will carry some more structure. Typically, there will be a covariantly constant $p$-form $\omega_p$. If $\omega_p$ is preserved by $G$ as well, then duality leads to a covariantly constant $\tilde{\omega}_p$ on the dual manifold $\tilde{M}$. We give the explicit form of the dual form when we discuss the example $G = U(1)$; the general case has not been worked out.

This article is organized as follows: In the next section, we introduce $\sigma$-models. In section 3, we setup the duality transformation, and in section 4, we derive the (classical) dual transformation. In section 5 we discuss quantum duality and global issues. Finally, in section 6, we work out the example of $G = U(1)$ in greater detail.

## 2 Sigma-models

Recall that we consider a manifold $M$ with a metric $G$, a closed 3-form $T = \frac{3}{2} dB$, and a compatible connection $\nabla$ with torsion $T^a$.

The generalized harmonic map problem is defined by extremizing a functional $S$ (the “action”) for maps $X: \Sigma \to M$ from a Riemann surface $\Sigma$ (the “worldsheet”) to the manifold $M$ (the “target space”). The action $S$ is:

$$S = -\frac{1}{2\pi} \int_{\Sigma} \left( \|dX\|^2 \text{vol} + iX^*B \right),$$

where $X^*B$ is the pullback of $B$ from $M$ to $\Sigma$ by $X$. Explicitly (for future

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\footnote{Since in general $T$ is closed but need not be exact, $B$ is only defined locally, and $-\frac{1}{2\pi} \int X^*B$ is only defined modulo $2\pi n$, where $n \in \mathbb{Z}$.}
use, we label this action as $S_O$,

$$S_O[X, \gamma] = -\frac{1}{2\pi} \int_{\Sigma} d^2\sigma (\sqrt{\gamma} \gamma^{\alpha\beta} G_{ij}(X) + i e^{\alpha\beta} B_{ij}(X)) \partial_\alpha X^i \partial_\beta X^j \ ,$$

(2.2)

where $\gamma_{\alpha\beta}$ is the metric on $\Sigma$ ("the worldsheet metric"), $\gamma = \det \gamma_{\alpha\beta}$, $\alpha, \beta = 1, 2$, and $i, j = 1, \ldots, D = \dim M$.

In addition, one may couple a scalar field $\Phi$ (the "dilaton") to the curvature of the Riemann surface $\Sigma$:

$$S_{\text{dil}} = \frac{1}{8\pi} \int_{\Sigma} \Phi R^{(\Sigma)} \ .$$

(2.3)

Physicists call $S$ the action for a nonlinear $\sigma$-model; the generalized harmonic maps are called classical solutions\(^4\). As we shall describe in detail, classical duality preserves the harmonic maps when $\Sigma$ is a sphere. Quantum duality is stronger in two ways: it preserves the set of classical solutions on higher genus Riemann surfaces as well, and it preserves the quantum field theory defined by the functional integral $Z$ of $e^S$ with respect to $X$:

$$Z[\gamma] = \int [DX] e^{S[X, \gamma]} \ .$$

(2.4)

### 3 Isometries and the duality transformation

Isometries are generated by Killing vectors $k_A$ where $A = 1, \ldots, d$ labels the element of the Lie algebra $\mathfrak{g}$ that the Killing vectors generate:

$$[k_A, k_B] = f^C_{AB} k_C \ .$$

(3.1)

Here $f^C_{AB}$ are the structure constants of the Lie algebra $\mathfrak{g}$. The Killing vectors preserve the metric $G$, the torsion $T$, and the dilaton $\Phi$:

$$\mathcal{L}_{k_A} G_{ij} \equiv \nabla_i k_A j + \nabla_j k_{Ai} = 0 \ ,$$

$$\mathcal{L}_{k_A} T_{ijk} \equiv k_A^l \partial_l T_{ijk} + \partial_i k_A^l T_{jlk} + \partial_j k_A^l T_{ilk} + \partial_k k_A^l T_{ijl} = 0 \ ,$$

$$\mathcal{L}_{k_A} \Phi \equiv k_A^l \partial_l \Phi = 0 \ ,$$

(3.2)

\(^4\)When $B = \Phi = 0$, these are harmonic maps.
where $\mathcal{L}_{k_A}$ is the Lie derivative with respect to the vector $k_A$ (which has components $k^i_A$). When these conditions are satisfied, the action (2.2, 2.3) is invariant (modulo boundary terms) under the transformations

$$\delta X^i = -\epsilon^A k^i_A$$

where $\epsilon^A$ are constant parameters.

The duality transformation is found by a two step procedure [2, 3, 4]:

1. We “gauge” (a subgroup of) the symmetry (3.3), i.e., we introduce a $G$-connection $A$ on $\Sigma$, and use it to enhance the symmetry (3.3) to a “local” symmetry when the parameters become functions $\epsilon^A(\sigma)$ on $\Sigma$. When the action (2.2) is invariant without boundary terms, the procedure is called minimal coupling: We substitute

$$\partial_\alpha X^i \rightarrow \nabla_\alpha X^i \equiv \partial_\alpha X^i + A^B_k k^i_B .$$

More generally, one can follow the procedure of [5] to get a gauge-invariant action.

2. We constrain the connection $A$ to be trivial by adding a term to the gauged action that we constructed in step (1). For classical duality, when $\Sigma$ is a sphere, this term is simply

$$S_\lambda[A, \lambda] = \frac{i}{2\pi} \int_{\Sigma} Tr \lambda F(A) ,$$

where $F(A) = dA + A^2$ is the curvature of the connection $A$ and $\lambda$ is a Lagrange multiplier in the dual $*g$ of the Lie algebra. Extremizing the gauged action (including $S_\lambda$) with respect to the Lagrange multiplier $\lambda$ implies that the curvature vanishes: $F(A) = 0$. On a simply connected worldsheet $\Sigma$ such as $S^2$, this implies that the connection $A$ is pure gauge, and in particular,

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5 If a symmetry is anomalous, it is not possible to gauge it.

6 In certain cases, e.g., Abelian groups, it is possible to perform a duality transformation even when the gauging is anomalous; essentially, the anomaly can be cancelled by a transformation of $\lambda$. It is sometimes possible to compactify $\lambda$, e.g., in the Abelian case, for quantum duality, we take $\lambda$ in a toroidal subspace of $*g$ (see section 6).
can be chosen to vanish. Thus, classically, we are dealing with the original system. The duality transformation is found by extremizing the gauged action (including the Lagrange multiplier term) with respect to the connection $A$ and eliminating it in the classical case, or more carefully, functionally integrating $A$ out in the quantum case.

4 The dual action

For the case when we can use minimal coupling \((3.4)\), the gauged action is simply:

$$S[X, \gamma, A] = -\frac{1}{2\pi} \int_{\Sigma} d^{2}\sigma (\sqrt{\gamma} \gamma^{\alpha \beta} G_{ij}(X) + i\epsilon^{\alpha \beta} B_{ij}(X)) \nabla_{\alpha} X^{i} \nabla_{\beta} X^{j}. \quad (4.1)$$

At this stage it is convenient, though by no means essential, to choose complex coordinates on the surface $\Sigma$. After integrating by parts and collecting terms, the gauged action $S[X, \gamma, A]$ \((4.1)\) with the Lagrange multiplier term $S_{\lambda[A, \lambda]}$ \((3.5)\) added becomes:

$$S_{1}[X, A, \lambda] = -\frac{1}{2\pi} \int_{\Sigma} d^{2}z \left( E_{ij} \partial X^{i} \bar{\partial} X^{j} + (E_{iB} \partial X^{i} - \partial \lambda_{B}) \bar{\partial} A^{B} \right.$$

$$+ A^{B} (E_{Bi} \bar{\partial} X^{i} + \bar{\partial} \lambda_{B}) + E_{BC} A^{B} A^{C} \left. \right), \quad (4.2)$$

where the matrices $E$ are:

$$E \equiv \begin{pmatrix} E_{AB} & E_{Aj} \\ E_{iB} & E_{ij} \end{pmatrix} = \begin{pmatrix} k_{A}^{i} E_{ij} k_{B}^{j} + \lambda_{C} f_{AB}^{C} & k_{A}^{i} E_{ij} \\ E_{ij} & G_{ij} + B_{ij} \end{pmatrix}. \quad (4.3)$$

(Factors of $i$ that appear in covariant forms of various actions are absorbed by the coordinate change $\sigma \rightarrow z, \bar{z}$. ) Extremizing the action $S_{1}$ \((4.2)\) with respect to the connections $A, \bar{A}$, leads to a new action $S_{D}$ on what appears to be a larger space with coordinates $X^{i}, \lambda_{A}$. However, because the first order action is invariant under the gauge transformations

$$\delta X^{i} = -\epsilon^{A}(z) k_{A}^{i}, \quad \delta \lambda_{A} = \lambda_{C} f_{AB}^{C} \epsilon^{B}(z), \quad \delta A_{\alpha}^{C} = \partial_{\alpha} \epsilon^{C} + f_{AB}^{C} A_{\alpha}^{B} \epsilon^{A}(z),$$

$$\quad \delta A_{\alpha}^{B}.$$
the dual action $S_D$ is actually defined on the orbits of the symmetry (4.4), which is a manifold with the same dimension $D$ that we started with. Explicitly [6, 7, 8, 9]

$$S_D = -\frac{1}{2\pi} \int_{\Sigma} d^2z \left( \tilde{E}_{ij} \partial X^i \partial X^j + \tilde{E}_i^B \partial X^i \partial \lambda_B 
+ \tilde{E}^B_i \partial \lambda_B \partial X^i + \tilde{E}^{BC} \partial \lambda_B \partial \lambda_C \right), \quad (4.5)$$

where the dual geometry is specified by:

$$\hat{E} \equiv \begin{pmatrix} \tilde{E}_{AB} & \tilde{E}^A_{\ j} \\ \tilde{E}_i^B & \tilde{E}_{ij} \end{pmatrix} = \begin{pmatrix} [E_{AB}]^{-1} & \tilde{E}^{AB} E_{Bj} \\ -E_{iA} \tilde{E}^{AB} & E_{ij} - E_{iA} \tilde{E}^{AB} E_{Bj} \end{pmatrix}. \quad (4.6)$$

The defining feature of duality, which clearly follows by construction and may also be verified by explicit calculation [7] is that the extremal conditions that follow from $S_O$ (the “field equations”) and the obvious condition $d^2X^i = 0$ (the “Bianchi identities”) are rotated into each other by the duality transformation.

5 Quantum duality and global issues

In the previous section, we discussed classical duality, namely, a transformation between geometries that preserves their harmonic maps (from $S^2$ to the target geometry). We now turn to quantum duality; this is a more refined notion that preserves more structure. After a brief sketch of what a quantum field theory (QFT) is, we consider how to preserve maps from arbitrary surfaces $\Sigma$ to the target geometry. This leads us to new geometries which are orbifolds of the original geometry with respect to some subgroup (which may be infinite or even continuous) of the isometry group [4, 7, 9]. We then discuss the transformation of the functional measure, and show that duality, suitably defined, is an exact symmetry of a QFT.

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7See section 6.2, and ref. [3].
A quantum field theory can be defined by a set of correlation functions, associated to an action functional $S[\phi]$ that depends on some fields $\phi$. Formally, these may be computed from a functional integral; typically, for some set of operators $\{O_i(\phi)\}$, one computes

$$\langle \prod_i O_i \rangle = Z^{-1} \int [D\phi] e^{S[\phi]} \prod_i O_i(\phi), \quad (5.1)$$

where

$$Z \equiv \langle 1 \rangle = \int [D\phi] e^{S[\phi]} \quad (5.2)$$

is the “vacuum” functional integral or “partition function”, and $[D\phi] e^{S[\phi]}$ is the functional measure up to the normalization factor $Z^{-1}$. In general, this is not well defined, but for particular cases of interest one may make some kind of sense of these integrals.

The partition function of the system with action $S_1[X,A,\lambda]$ (4.2) is

$$Z = \int [DX][D\lambda][DA][D\bar{A}] e^{S_1[X,A,\lambda]} \quad (5.3)$$

We can proceed in two ways: integrating over the Lagrange multiplier $\lambda$, or instead integrating over the gauge field $A$. Although we will eventually be able to define these integrals in such a way that the order of integration will not matter, the issues in defining them are very different, and we first focus on the integration over $\lambda$.

### 5.1 Integration over the Lagrange multiplier $\lambda$

Since the Lagrange multiplier $\lambda$ enters the action $S_1$ (1.2) linearly, integrating over it gives a functional $\delta$-function of whatever it multiplies; this is exactly what one found in the classical case by extremizing with respect to $\lambda$. When the Lagrange multiplier is in the dual Lie algebra $\text{\dual g}$, integrating over it constrains the curvature $F$ to vanish, but does not force the connection $A$.

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8 Examples of these correlation functions are knot invariants in Chern-Simons theory [10], the Donaldson invariants on 4-manifolds [11], and intersection forms on Kähler manifolds [12]; needless to say, not all correlation functions are topological.
to be trivial on a multiply connected worldsheet \( \Sigma \). Therefore, the theory that one gets after integrating out \( \lambda \) (the “F-theory”) must be integrated over all flat connections, and is not equivalent to the original theory (the “O-theory”). The relation between the models is that the F-theory is the orbifold of the O-theory with respect to the isometry group \( G \). We can summarize this as:

\[
[DX] \int [D\lambda][DA][D\bar{A}]e^{S_{1}[X,A,\lambda]} = N[DX]e^{S_{F}[X]} = N_{G}[D_{\bar{G}}X]e^{S_{O}[X]} , \tag{5.4}
\]

where \( N, N_{G} \) are constant (\( X \)-independent) normalization factors, \( S_{O} \) is the action of the original model (2.2), \( S_{F} \) is the action of the F-theory (and formally is identical to \( S_{O}[X] \), but is a functional of \( X \) obeying different conditions), and \([D_{\bar{G}}X]\) indicates that the manifold coordinatized by \( X \) is the \( G \)-orbifold of the original manifold. If we restrict \( \lambda \), then we change the F-theory. For example, for \( G = U(1) \) (see section 6), if we choose \( \lambda \) to be a periodic coordinate on the group \( U(1) \) (and add a boundary term to the action \[4, 7\]) then the F-theory can be made exactly equivalent to the O-theory; if one changes the periodicity of \( \lambda \) by a factor \( k \), the F-theory becomes a \( Z_{k} \) orbifold of the O-theory.

### 5.2 Integration over the connection \( A \)

Integration over \( A, \bar{A} \) gives rise to the dual theory (the “D-theory”) with classical action \( S_{D} \). Though the integral is gaussian, it gives rise to factors that are not visible in the classical theory for two reasons: (1) Being a gauge field, \( A \) has a nontrivial measure, and (2) the gaussian integral itself involves the nontrivial quadratic form \( E_{AB} \) in (4.3). This gives rise to a functional determinant that must be carefully defined. This determinant has two factors: one is just a naive factor of \([det(E_{AB})]^{-1}\) and is the correct change in the target space volume element: \( det(\tilde{G})^{\frac{1}{2}} = det(E_{AB})^{-1} det(G)^{\frac{1}{2}} \) which implies

\[
[DX] = [det(G)^{\frac{1}{2}}dX] \rightarrow [D\tilde{X}] = [det(\tilde{G})^{\frac{1}{2}}d\tilde{X}] , \tag{5.5}
\]

where \( G \) is the metric on \( M \) (the target manifold of the F-theory), \( \tilde{G} \) is the metric on \( \tilde{M} \) (the target manifold of the D-theory), namely, the metric
on the $G$ orbits of $\ast g \times M$ that follows from the classical dual action \( (4.5) \), and \([dX],[d\tilde{X}]\) are the naive flat volume elements on the spaces $M$, $\tilde{M}$, respectively.

The second factor arises because of hidden dependence on the worldsheet metric $\gamma$ in $E_{AB}$ and in $[DA][D\tilde{A}]$. Suitably regularized, this gives rise to a shift in the dilaton \( (5.6) \)
\[
\tilde{\Phi} = \Phi + \ln(\det(E_{AB})) .
\]

We can summarize this as:
\[
[DX][D\lambda] \int [DA][D\tilde{A}] e^{S_1[X,A,\lambda]+S_{dil}[X,\gamma]} = N[D\tilde{X}]e^{S_D[\tilde{X}]+\tilde{S}_{dil}[\tilde{X},\gamma]} ,
\]
where the actions $S_1, S_{dil}, S_D$ are given above in \( (4.2, 2.3, 4.5) \), and $\tilde{S}_{dil}$ is simply $S_{dil}$ with $\Phi \to \tilde{\Phi}$ \( (5.6) \).

We emphasize that the F-theory and the D-theory are always equivalent as quantum field theories. The question of whether duality is a symmetry between the original O-theory and the D-theory is a matter of the global issues discussed in the previous subsection.

6 An example: Abelian duality

In this section, we briefly discuss the case when the gauge group is Abelian \( [3, 4, 13] \). Specifically, we consider a target space geometry with a $U(1)$ isometry.

6.1 The dual action

Without loss of generality, away from fixed points of the $U(1)$ action on the target space,\( \square \) we can choose coordinates $X = (\theta, x^i)$ on $M$ where the symmetry acts by shifts of a single periodic coordinate $\theta \equiv \theta + 2\pi$, and the

\(9\) In some cases, there may be further so-called higher order corrections. In the context of duality, they are not completely understood.

\(10\) Fixed points lead to singularities on the dual space.
remaining coordinates \( x^i \) are left inert. In these coordinates, the background is independent of \( \theta \). The action of the original model takes the form

\[
S_O[\theta, x] + S_{dil} = -\frac{1}{2\pi} \int_{\Sigma} d^2z \left( E_{00}(x) \partial \theta \bar{\partial} \theta + E_{0j}(x) \partial \theta \bar{\partial} x^j + E_{i0}(x) \partial x^i \bar{\partial} \theta \\
+ E_{ij}(x) \partial x^i \bar{\partial} x^j - \frac{1}{4} \Phi(x) R(\Sigma) \right),
\]

(6.1)

In the \( U(1) \) case, gauging by minimal coupling is performed by the substitution \( \partial \theta \rightarrow \partial \theta + A, \bar{\partial} \theta \rightarrow \bar{\partial} \theta + \bar{\partial} A \). We also add the Lagrange multiplier term (3.5), which here takes the form

\[
\frac{1}{2\pi} \int_{\Sigma} d^2z \left( \bar{A} \partial \lambda - \bar{A} \partial \lambda \right);
\]

(6.2)

up to an important total derivative, this is just \( \frac{i}{\pi} \int \lambda F \). When \( \lambda \) is chosen to have periodicity \( 2\pi \), as discussed in \([4, 7]\), the boundary term ensures that when one integrates over \( \lambda \), the winding modes of \( \lambda \) constrain the holonomy of the gauge field \( A, \bar{A} \) so that it is not just flat, but actually trivial, and we recover the original model.

An interesting special feature of the Abelian case is that the dual model has the same symmetry as the original model. It acts on \( \lambda \) by

\[
\lambda \rightarrow \lambda + \epsilon,
\]

(6.3)

for \( \epsilon \) constant. If we make a duality transformation with respect to this symmetry, we immediately see that the dual of the dual is the original model: We gauge (6.3) and add a second Lagrange multiplier \( \tilde{\lambda} \):

\[
S_1[\theta, x, A, \lambda] \rightarrow S[\theta, x, A, \tilde{A}, \lambda, \tilde{\lambda}]
\equiv S_1[\theta, x, A, \lambda] + \frac{1}{2\pi} \int_{\Sigma} d^2z \; tr \left( \tilde{A} A - \tilde{A} A + \tilde{A} \partial \tilde{\lambda} - \tilde{A} \partial \tilde{\lambda} \right). 
\]

(6.4)

Functionally integrating out \( \tilde{A}, \bar{A} \) gives the constraints:

\[
A = \partial \tilde{\lambda}, \quad \bar{A} = \bar{\partial} \tilde{\lambda},
\]

(6.5)
and hence, after the redefinition $\theta + \tilde{\lambda} \rightarrow \theta$, we recover the original model. This inverse transformation is a sign of an underlying group of duality transformations. For $d$ commuting $U(1)$ symmetries, one finds an $O(d, d, \mathbb{Z})$ group [13] (for a review, see, e.g., ref. [14]).

Returning to our example and choosing a representative on the $U(1)$ orbits (choosing “a gauge”) such that $\theta = 0$, we find the gauged action

$$S_1[x, A, \lambda] + S_{dil} = -\frac{1}{2\pi} \int \Sigma d^2 z \left( E_{00} A \bar{A} + E_{0i} A \partial x^i \bar{A} + E_{ij} \partial x^i \partial x^j \right. \left. - \frac{1}{4} \Phi R^{(\Sigma)} + (A \partial \lambda - \bar{A} \partial \lambda) \right). \quad (6.6)$$

Functionally integrating out the gauge field $A, \bar{A}$, we may replace the gauge field with

$$A(\lambda, x) \rightarrow (\partial \lambda - \partial x^i E_{i0})(E_{00})^{-1}, \quad \bar{A}(\lambda, x) \rightarrow -(E_{00})^{-1}(\bar{\partial} \lambda + E_{0i} \bar{\partial} x^i). \quad (6.7)$$

Substituting (6.7) into the gauged action (6.6) and recalling the shift in the dilaton that the integration over $A, \bar{A}$ gave, we find the dual action:

$$S_D[x, \lambda] + \tilde{S}_{dil} = -\frac{1}{2\pi} \int \Sigma d^2 z \left( (\partial \lambda - \partial x^i E_{i0})(E_{00})^{-1}(\bar{\partial} \lambda + E_{0i} \bar{\partial} x^i) \right. \left. + E_{ij} \partial x^i \partial x^j \right. \left. - \frac{1}{4}(\Phi + \ln E_{00})R^{(\Sigma)} \right). \quad (6.8)$$

From this we read off the dual geometry [3]:

$$\tilde{G} = \begin{pmatrix} (E_{00})^{-1} & (E_{00})^{-1} B_{0j} \\ (E_{00})^{-1} B_{i0} & G_{ij} - (E_{00})^{-1}(G_{i0} G_{0j} + B_{i0} B_{0j}) \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} 0 & (E_{00})^{-1} G_{0j} \\ -(E_{00})^{-1} G_{i0} & B_{ij} - (E_{00})^{-1}(G_{i0} B_{0j} + B_{i0} G_{0j}) \end{pmatrix},$$

$$\tilde{\Phi} = \Phi + \ln E_{00}, \quad (6.9)$$

where (recall) $E = G + B$. 

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6.2 Field Equations

A general feature of classical duality is that field equations and Bianchi identities are rotated into each other \[15\]. Here, they are simply interchanged.

In the original model (6.1), the field equation and Bianchi identity are:

Field Equation: \[ \bar{\partial} J + \partial \bar{J} = 0 \] (6.10)

for

\[ J = E_{00} \partial \theta + E_{i0} \partial x^i \], \[ \bar{J} = E_{00} \bar{\partial} \theta + E_{0i} \bar{\partial} x^i \], (6.11)

and

Bianchi Identity: \[ \bar{\partial}[(E_{00})^{-1}(J - E_{0i} \bar{\partial} x^i)] - \bar{\partial}[(J - \partial x^i E_{i0})(E_{00})^{-1}] = 0 \] (6.12)

(substituting the definition of \( J, \bar{J} \) (6.11) into (6.12), this becomes just the triviality \( \partial \bar{\partial} \theta - \bar{\partial} \partial \theta = 0 \)). We may write these as:

\[ d^* J = 0 \], \[ dA = 0 \], (6.13)

for \( A = d\theta \).

In the dual model, we have the obvious currents

\[ \tilde{J} = \partial \lambda \], \[ \tilde{\bar{J}} = -\bar{\partial} \lambda \]. (6.14)

These obey the dual Bianchi identity:

Dual Bianchi Identity: \[ \bar{\partial} \tilde{\bar{J}} + \partial \tilde{J} = 0 \]. (6.15)

By construction, the \( \lambda \) field equation is \( F(A, \bar{A}) = 0 \), with \( A, \bar{A} \) given in (6.7); in terms of the dual currents \( \tilde{J}, \tilde{\bar{J}} \), this takes the form:

Dual Field Equation: \[ \bar{\partial}[(E_{00})^{-1}(\tilde{J} - E_{0i} \bar{\partial} x^i)] - \bar{\partial}[(\tilde{J} - \partial x^i E_{i0})(E_{00})^{-1}] = 0 \] (6.16)

In the compact notation of (6.13), these become:

\[ d^* \tilde{J} = 0 \], \[ dA = 0 \], (6.17)

where now \( \tilde{J} = *d\lambda \) and \( A = A(\lambda, x) \). Thus we see that the field equation (6.11) becomes the dual Bianchi identity (6.15) and the Bianchi identity (6.12) becomes the dual field equation (6.16).
6.3 Further structures

When the original manifold has restricted holonomy, it can have a covariantly constant $p$-form $\omega_p$. Very recently, it was shown by B. B. Kim [1] that if the $p$-form is independent of $\theta$ (or, in invariant language, if $\mathcal{L}_\theta \omega_p = 0$), then one can find a dual covariantly constant form $\tilde{\omega}$ on the dual space $\tilde{M}$:

\[
\tilde{\omega}_{0k_1\ldots k_{p-1}} = (E_{00})^{-1} \omega_{0k_1\ldots k_{p-1}} \tag{6.18}
\]

\[
\tilde{\omega}_{jk_1\ldots k_{p-1}} = \omega_{jk_1\ldots k_{p-1}} - (E_{00})^{-1} W_{jk_1\ldots k_{p-1}} , \tag{6.19}
\]

where

\[
W_{jk_1\ldots k_{p-1}} = E_{j0} \omega_{0k_1\ldots k_{p-1}} - E_{k_10} \omega_{0jk_2\ldots k_{p-1}} + E_{k_20} \omega_{0k_1jk_3\ldots k_{p-1}} + \ldots + (-1)^{p-1} E_{k_{p-1}0} \omega_{0k_1\ldots k_{p-2}j} . \tag{6.20}
\]

Thus the dual connection (in general with torsion) also has restricted holonomy.

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