Quantum switch of quantum switches

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Abstract

Recent results have shown that quantum theory is compatible with novel causal structures where events happen without a definite causal order. In particular, the quantum switch describes a process in which two quantum channels act in a coherent superposition of their two possible orders. Furthermore, the quantum switch can perform communication tasks that are impossible within the framework of the standard quantum Shannon theory. The present paper considers the scenario of one-shot heralded qubit communication using a higher-order quantum switch constructed from two quantum switches. Specifically, we show that two quantum switches put in a superposition of their alternative causal orders can transmit a qubit, without any error, with a probability strictly higher than that achievable with each quantum switch. We discuss three examples that demonstrate this communication advantage. Notably, a higher-order quantum switch not only can outperform useful quantum switches but also becomes useful as a resource even if the quantum switches making it up are useless.

1 Introduction

The causal structure embedded in a physical theory tells us about all possible causal relationships between events. It is generally assumed that events take place in a definite causal order. That means if we look at a sequence of events, then the order in which they appear is fixed. Recently, however, it was shown that quantum theory is compatible with situations where the order of the events is controlled by a quantum system, which, in turn, leads to events with no definite causal order. Specifically, the quantum switch [1] describes an operation in which two quantum channels, say $\mathcal{E}$ and $\mathcal{F}$ act on a quantum system in an order that is entangled with the state of an ancillary system—the order qubit. For example, if the order qubit is $|0\rangle$, the channels act in a particular

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order, say $\mathcal{EF}$, but if the order qubit is $|1\rangle$, then the order is $\mathcal{FE}$. However, if the order qubit is initially in a superposition of $|0\rangle$ and $|1\rangle$, then the channels act in a superposition of the two orders $\mathcal{EF}$ and $\mathcal{FE}$. Now that one cannot determine the definite order of their applications, the quantum switch is said to exhibit *indefinite causal order*, also known as causal nonseparability [2, 3, 4]. Note here that the quantum switch can neither be realized by a circuit with a definite causal order between $\mathcal{E}$ and $\mathcal{F}$ nor by a statistical mixture of circuits using $\mathcal{E}$ and $\mathcal{F}$ with definite causal orders [5].

Recent results have shown that indefinite causal order is a bonafide resource for some quantum information processing tasks, such as winning nonlocal games [2], testing properties of quantum channels [5, 6], reducing communication complexity [7], and quantum communication [7, 8, 9, 10, 11, 12, 13, 18]. Specific examples include: The quantum switch from two completely depolarizing channels can transmit classical information [8], even though the channels have zero capacities, and the quantum switch from two completely dephasing channels can transfer quantum information with nonzero probability although, individually, the dephasing channels cannot [9].

The quantum switch is, therefore, significant for two reasons. First, its ability to superpose different causal orders gives rise to a new kind of causal structure that is purely quantum mechanical. Second, it is a resource that exploits indefinite causal order for quantum information processing tasks. Our motivation for the present work is twofold. First, studies on the quantum switch, so far, have involved only the “elementary” quantum channels, such as depolarizing, dephasing, bit-flip, phase-flip, etc., even though the formalism allows for constructing quantum switch from any two quantum channels. So there is scope for novel quantum switches using quantum channels with relatively complex structures and studying their properties, especially from the resource point of view. Such an exercise, of course, will not lead to any new kind of causal structure, but, nevertheless, the possibility of outperforming quantum switches constructed from elementary quantum channels cannot be ruled out. Second, we would like to know whether it is possible to better a quantum switch in some well-defined communication task by another process that involves indefinite causal order. The purpose of the present paper is to investigate the paradigm of the quantum switch along these lines. We will show a natural way to define a higher-order quantum switch that is both simple and intuitively satisfying and outperforms quantum switches using elementary quantum channels.

Let us begin by noting that the quantum switch, in fact, is a higher-order quantum channel, a bilinear supermap [9, 14, 15]. So, in principle, one can construct a quantum switch from quantum switches of elementary channels. This is what we do in this paper. Specifically, we define a higher-order quantum switch of two quantum switches using an order qubit that controls the order of the quantum switches. Note that now there are two order qubits. The order of the quantum channels making up the individual quantum switches is controlled by one of them, and we will show that one qubit is sufficient for both. The order of the quantum switches, as noted earlier,
is determined by the other one, and choosing its state appropriately, one can place the quantum switches in a quantum superposition of their alternative causal orders.

So, one could now ask the following question: Does a higher-order quantum switch provide any advantage, if at all, over the individual quantum switches for some well-defined communication task? This question, in fact, makes a lot of sense because unless there is any kind of advantage, the whole exercise will not be of any significance. Fortunately, we can answer this question in the affirmative in the scenario of one-shot heralded qubit communication. We show that the quantum superposition of the possible causal orders of two quantum switches allows for noiseless transfer with a probability higher than that achievable using the individual quantum switches. We discuss three examples to demonstrate this communication advantage. The first two show this advantage over useful quantum switches. The third example shows that the higher-order quantum switch becomes useful even when the two quantum switches are both useless.

Before we proceed, we want to briefly remark on a couple of things. Since our configuration employs two order qubits, one has the freedom to choose any initial two-qubit state, which can even be entangled. Our examples, however, use specific product states, and we did not find any particular advantage using maximally entangled states. But we suspect this may not be the case always. So in situations (like ours or similar) where the control system is composite, one may try to see if an entangled control has an advantage over a product one. The other point we want to make here is that a higher-order quantum switch does not necessarily outperform the quantum switches from which it is constructed. In fact, in two of our examples, we observe outperformance only for subsets of parameter values.

## 2 Quantum switch

Quantum channels describe the evolution of quantum systems. Mathematically, a quantum channel is a completely positive, trace-preserving linear map that transforms quantum states into quantum states. Let us first briefly explain the mechanism of the quantum switch constructed from the quantum channels $\mathcal{E}$ and $\mathcal{F}$. First, the action of the channels on a quantum state $\rho$ can be expressed as:

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger,$$
$$\mathcal{F}(\rho) = \sum_j F_j \rho F_j^\dagger,$$  

where $\{E_i\}$ and $\{F_j\}$ are the Krauss operators satisfying $\sum_i E_i E_i^\dagger = \sum F_j F_j^\dagger = I$, $I$ being the identity operator.
Now suppose that the channels are applied sequentially. This gives rise to two possibilities:

\[ \mathcal{FE}(\rho) = \sum_{i,j} F_j E_i \rho E_i^\dagger F_j^\dagger, \]  
\[ \mathcal{EF}(\rho) = \sum_{i,j} E_i F_j \rho F_j^\dagger E_i^\dagger. \]

Note that, in each of the above scenarios the order in which the channels are applied to the target state \( \rho \) remains fixed. That is, in (3), \( \rho \) is first subjected to \( E \) followed by \( F \), whereas in (4), it is just the opposite.

The quantum switch is a higher-order quantum channel constructed from \( E, F \), and an ancilla \( a \)–the order qubit, which is accessible only to the receiver\(^1\). It is defined as [9, 14, 15]:

\[ S(\mathcal{E}, \mathcal{F}, a)(\rho) = \sum_{i,j} K_{ij} (\rho \otimes \omega) K_{ij}^\dagger, \]

where \( \omega = |\omega\rangle \langle \omega| \) is the state of the order qubit\(^2\) and \( \{K_{ij}\} \) are the Krauss operators

\[ K_{ij} = E_i F_j \otimes |0\rangle \langle 0| + F_j E_i \otimes |1\rangle \langle 1|, \]

where \( \{|0\rangle, |1\rangle\} \) is an orthonormal basis of \( \mathbb{C}^2 \). Note that (5) is independent of the Krauss operators \( \{E_i\} \) and \( \{F_j\} \)\(^1\).

From (6) it is clear that the order in which the channels \( \mathcal{E} \) and \( \mathcal{F} \) apply is determined by the state of the order qubit. In particular, if \( |\omega\rangle = |0\rangle \), first \( \mathcal{F} \) and then \( \mathcal{E} \) is applied to \( \rho \), whereas if \( |\omega\rangle = |1\rangle \) they apply in the reverse order. However, if the order qubit is initially in a superposition state \( |+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \), the effective channel is described by a superposition of the two orders, and now, we cannot determine which channel acts first definitely. Thus, a quantum switch exhibits indefinite causal order.

We now take the idea of a quantum switch forward and define a higher-order quantum switch constructed from two quantum switches.

\(^1\)The order qubit is assumed to be part of the communication channel.

\(^2\)We have assumed the order qubit is in a pure state.
3 Quantum switch of quantum switches

3.1 Definition

Consider the quantum channels $\mathcal{E}(x)$ with $\{E_i^{(x)}\}$ being the corresponding set of Krauss operators for $x = 1, 2, 3, 4$. Let $S(\mathcal{E}(1), \mathcal{E}(2), a) \equiv S_1$ and $S(\mathcal{E}(3), \mathcal{E}(4), a) \equiv S_2$ denote the quantum switches from the pairs $(\mathcal{E}(1), \mathcal{E}(2))$ and $(\mathcal{E}(3), \mathcal{E}(4))$ respectively. Following (5) they are defined as

$$S_1 (\rho) = \sum_{i,j} K_{ij}^{(1)} (\rho \otimes \omega) K_{ij}^{(1)\dagger},$$

$$S_2 (\rho) = \sum_{i,j} K_{ij}^{(2)} (\rho \otimes \omega) K_{ij}^{(2)\dagger},$$

where

$$K_{ij}^{(1)} = E_i^{(1)} E_j^{(2)} \otimes |0\rangle \langle 0| + E_j^{(2)} E_i^{(1)} \otimes |1\rangle \langle 1|,$$

$$K_{ij}^{(2)} = E_i^{(3)} E_j^{(4)} \otimes |0\rangle \langle 0| + E_j^{(4)} E_i^{(3)} \otimes |1\rangle \langle 1|.\tag{9}$$

The quantum switch constructed from the quantum switches $S_1$ and $S_2$ is now defined as:

$$S(S_1, S_2, a') (\rho) = \sum_{i,j,k,l} K_{ijkl} (\rho \otimes \Omega) K_{ijkl}^\dagger,\tag{11}$$

where $a'$ is the order qubit controlling the order of the switches, $\Omega = |\Omega\rangle \langle \Omega|$ is the joint state of $aa'$, $a$ being the order qubit associated with both $S_1$ and $S_2$, and $\{K_{ijkl}\}$ are the Krauss operators that are defined as

$$K_{ijkl} = K_{ij}^{(1)} K_{kl}^{(2)} \otimes |0\rangle \langle 0| + K_{kl}^{(2)} K_{ij}^{(1)} \otimes |1\rangle \langle 1|.\tag{12}$$

Observe that the order-ancilla $aa'$ is now a two-qubit system. These two qubits are accessible only to the receiver.

Note that, one can also construct a quantum switch from the four channels $\mathcal{E}(x), x = 1, 2, 3, 4$ (see, e.g., [12, 16]). In this case, if we allow all possible relative orders, then the dimension of the order-system will be equal to $4!$. Our definition (11) of the higher-order quantum switch $S$, however, can be viewed as a natural extension of the quantum switch, where the quantum channels are now replaced with quantum switches (see Figure 1).

In what follows, we discuss three examples in the context of one-shot error-free transfer of a qubit and show that in each example the higher-order quantum switch outperforms the constituent quantum switches.
Figure 1: The mechanism of the higher-order quantum switch constructed from two quantum switches. Observe that the order-ancilla consists of two qubits $a'$ and $a$. The first qubit controls the order of the switches whereas the second one controls the order of the channels making up the individual switches.

3.2 Examples

Example 1. Consider the Pauli channel $\mathcal{P}$ with the following Krauss operators:

$$P_0 = \sqrt{p_0}I, \quad P_1 = \sqrt{p_1}\sigma_y, \quad P_2 = \sqrt{p_2}\sigma_z,$$

where $(p_0, p_1, p_2)$ is a probability vector with $0 < p_0, p_1, p_2 < 1$ and $\sum_{i=0}^{2} p_i = 1$, and $\sigma_y$ and $\sigma_z$ are the Pauli $y$ and $z$ matrices respectively. The action of the Pauli channel on a qubit in state $\rho$ is given by

$$\mathcal{P}(\rho) = p_0\rho + p_1\sigma_y\rho\sigma_y + p_2\sigma_z\rho\sigma_z. \quad (14)$$

From (14), it is clear that if the channel $\mathcal{P}$ is used only once, error-free qubit communication is not possible.

Following (5), the quantum switch from two Pauli channels $\mathcal{P}$ is defined as

$$S(\mathcal{P}, \mathcal{P}, a) (\rho) = \sum_{i,j=0}^{2} M_{ij} (\rho \otimes \omega) M_{ij}^\dagger,$$

where the Krauss operators $\{M_{ij}\}$ are given by

$$M_{ij} = P_iP_j \otimes |0\rangle \langle 0| + P_jP_i \otimes |1\rangle \langle 1|,$$
and $\omega$ is the state of the order qubit $a$. After simplification, one finds that [10]

$$S(P,P,a)(\rho) = q_1\rho_1 \otimes \omega + q_2\rho_2 \otimes \sigma_z\omega\sigma_z,$$

(17)

where

$$q_1 = 1 - 2p_1p_2 \quad \rho_1 = \frac{1}{q_1} \left[ \left( p_0^2 + p_1^2 + p_2^2 \right) \rho + 2p_0p_1\sigma_y\rho\sigma_y + 2p_0p_2\sigma_z\rho\sigma_z \right];$$

$$q_2 = 2p_1p_2 \quad \rho_2 = \sigma_x\rho\sigma_x,$$

where $\sigma_x$ is the Pauli $x$ matrix. Note that, (17) holds for any $\omega$, so we have the freedom to choose $\omega$ appropriately.

In particular, let us choose $\omega = |+\rangle\langle+|$. Then, $\sigma_z\omega\sigma_z = |--\rangle\langle-|$. Consider now the output of the Pauli switch given by (17). If we measure the order qubit in the $\{|\pm\rangle\}$ basis, the $+$ outcome will herald the presence of $\rho_1$, whereas the $-$ outcome will herald the presence of $\rho_2$. The first outcome, which occurs with probability $q_1$, is not of any particular interest. But for the second outcome, which occurs with probability $q_2 = 2p_1p_2$, an application of $\sigma_x$ on the target qubit leads to an intact transmission of the input state. Thus in the one-shot scenario the quantum switch defined by (15) offers a clear advantage over the Pauli channel. It is, however, important to note that since (17) holds for any choice of $\omega$, the probability $q_2$ cannot be exceeded for any input state.

We now describe the quantum switch of two identical Pauli switches $S(P,P,a)$. Following (11) it is defined as:

$$S[S(P,P,a),S(P,P,a),a'] (\rho) = \sum_{i,j,k,l=0}^{2} K_{ijkl} (\rho \otimes \Omega) K_{ijkl}^\dagger,$$

(18)

where $\Omega = |\Omega\rangle\langle\Omega|$ is the joint state of the order qubits $a$ and $a'$, and $\{K_{ijkl}\}$ are the Krauss operators given by

$$K_{ijkl} = M_{ij}M_{kl} \otimes |0\rangle\langle0| + M_{kl}M_{ij} \otimes |1\rangle\langle1|.$$

(19)

Let us now choose $|\Omega\rangle = |++\rangle$. With this choice, one obtains

$$S[S(P,P,a),S(P,P,a),a'] (\rho) = \sum_{i=1}^{4} q_i \rho_i \otimes \Omega_i,$$

(20)

where
Since the input qubit with probability one achieves perfect transfer of the input qubit.

Now observe that:

\[ q_2 = 8p_0^2p_1p_2 \]
\[ q_3 = 4p_1p_2 (p_0^2 + p_1^2 + p_2^2) \]
\[ q_4 = 8p_0p_1p_2 (p_1 + p_2) \]

with

\[ u_1 = \frac{4p_0p_1 (p_0^2 + p_1^2 + p_2^2)}{(p_0 + p_1)^4 + 4p_0p_2 (p_0^2 + p_1^2 + p_2^2) + 2p_2^2 (3p_0^2 + 2p_0p_1 + 3p_1^2) + p_2^4}; \]  
\[ u_2 = \frac{4p_0p_2 (p_0^2 + p_1^2 + p_2^2)}{(p_0 + p_1)^4 + 4p_0p_2 (p_0^2 + p_1^2 + p_2^2) + 2p_2^2 (3p_0^2 + 2p_0p_1 + 3p_1^2) + p_2^4}; \]  

Now observe that: \( \Omega_2 = |+-\rangle \langle -+|, \) \( \Omega_3 = |--\rangle \langle +|, \) and \( \Omega_4 = |++\rangle \langle -| \). So, if we measure each order qubit in the \( \{|\pm\rangle\} \) basis, then either of the outcomes \(+-\) and \(-+\) will herald the presence of \( \sigma_x \rho \sigma_x \), in which case the receiver will be able to recover the input state by applying \( \sigma_x \). Therefore, with probability

\[ q_{23} = q_2 + q_3 = 4p_1p_2 (3p_0^2 + p_1^2 + p_2^2) \]  

one achieves perfect transfer of the input qubit.

Now recall that, using the Pauli-switch only once one could achieve error-free transmission of the input qubit with probability \( q_2 = 2p_1p_2 \). So the higher-order quantum switch obtained from two Pauli switches will do better than that if there exist nonzero \( p_0, p_1, p_2 \) satisfying

\[ q_{23} = 4p_1p_2 (3p_0^2 + p_1^2 + p_2^2) > q_2 = 2p_1p_2, \]  

such that \( p_0 + p_1 + p_2 = 1. \)  

Since \( 0 < p_0, p_1, p_2 < 1 \), we therefore need to search for \( p_0, p_1, p_2 \) that satisfy

\[ 3p_0^2 + p_1^2 + p_2^2 > \frac{1}{2}, \]  

such that \( p_0 + p_1 + p_2 = 1. \)  

The readers can easily convince themselves that solutions do exist. For example, suppose that
Figure 2: The shaded region depicts the possible values of $p_1$ and $p_2$ for which a single use of the quantum switch constructed from two Pauli switches enables the noiseless transfer of an arbitrary qubit with probability higher than the Pauli switch.

$3p_0^2 = \frac{1}{2}$. Then, (26) and (27) become

$$p_1^2 + p_2^2 > 0,$$

such that $p_1 + p_2 = 1 - \frac{1}{\sqrt{6}}$.  

It is now easy to see that there exist $p_1, p_2 > 0$ satisfying the second and these pairs automatically satisfy the first, and therefore, are legitimate solutions.

To find the complete set of solutions one may proceed as follows. Define

$$\Delta = q_{23} - 2p_1p_2$$

$$= 2p_1p_2 \left[ 2 \left( 3p_0^2 + p_1^2 + p_2^2 \right) - 1 \right]$$

$$= 2p_1p_2 \left[ 2 \left\{ 3 (1 - p_1 - p_2)^2 + p_1^2 + p_2^2 \right\} - 1 \right],$$

where to arrive at the last line we have used $p_0 = 1 - p_1 - p_2$. Therefore, all possible pairs $(p_1, p_2)$, where $0 < p_1, p_2 < 1$ and $p_1 + p_2 < 1$, satisfying $\Delta > 0$ are admissible solutions (see, Figure 2).

**Example 2.** Consider the bit flip channel $B$ with the Krauss operators

$$B_0 = \sqrt{1-r}I, \quad B_1 = \sqrt{r}\sigma_x,$$
where $0 < r < 1$, and the phase flip channel $\mathcal{G}$ with the Krauss operators

\[ G_0 = \sqrt{1-s}I, \quad G_1 = \sqrt{s}\sigma_z, \quad (32) \]

where $0 < s < 1$. The actions of the channels on a qubit state $\rho$ can be written as

\begin{align*}
\mathcal{B} (\rho) &= (1 - r) \rho + r\sigma_x \rho \sigma_x, \quad (33) \\
\mathcal{G} (\rho) &= (1 - s) \rho + s\sigma_z \rho \sigma_z. \quad (34)
\end{align*}

It is clear from (33) and (34) that for single use of the above channels error-free transfer of a qubit state is not possible.

Consider now the quantum switch constructed from $\mathcal{B}$ and $\mathcal{G}$, which is defined as

\[ S (\mathcal{B}, \mathcal{G}, a) (\rho) = \sum_{i,j=0}^1 T_{ij} (\rho \otimes \omega) T_{ij}^\dagger, \quad (35) \]

where

\[ T_{ij} = B_i G_j \otimes |0\rangle \langle 0 | + G_j B_i \otimes |1\rangle \langle 1 | \quad (36) \]

are the Krauss operators.

Simplifying (35) one obtains

\[ S (\mathcal{B}, \mathcal{G}, a) (\rho) = [(1 - r)(1 - s) \rho + r(1 - s) \sigma_x \rho \sigma_x + s(1 - r) \sigma_z \rho \sigma_z] \otimes \omega \]

\[ + r s (\sigma_y \rho \sigma_y) \otimes (\sigma_z \omega \sigma_z) \quad (37) \]

If we choose $\omega = |+\rangle \langle + |$, then we have $\sigma_z \omega \sigma_z = |-\rangle \langle - |$. From (37) then it follows that measuring the order qubit in the $\{ |\pm\rangle \}$ basis, the outcome $- \rangle$ is obtained with probability $rs$, in which case, the target qubit ends up in the state $\sigma_y \rho \sigma_y$ and can be subsequently recovered by applying $\sigma_y$. Thus error-free transmission of an arbitrary qubit is possible with probability $rs$ for single use of the quantum switch $S (\mathcal{B}, \mathcal{G}, a)$.

Let us now consider the quantum switch of two identical quantum switches $S (\mathcal{B}, \mathcal{G}, a)$. Following (11), it is defined as:

\[ S [S (\mathcal{B}, \mathcal{G}, a), S (\mathcal{B}, \mathcal{G}, a), a'] (\rho) = \sum_{i,j,k,l=0}^1 K_{ijkl} (\rho \otimes \Omega) K_{ijkl}^\dagger, \quad (38) \]

where $\Omega = |\Omega\rangle \langle \Omega |$ is the joint state of the order qubits $a$ and $a'$, and $\{K_{ijkl}\}$ are the Krauss
operators given by

$$K_{ijkl} = T_{ij} T_{kl} \otimes |0\rangle \langle 0| + T_{kl} T_{ij} \otimes |1\rangle \langle 1|.$$  \hfill (39)

Let us now choose $|\Omega\rangle = |++\rangle$. Then we get

$$S \left[ B, G, a \right], S \left( B, G, a \right), a' \left( \rho \right) = \sum_{i=1}^{4} q'_i \rho \otimes \Omega'_i,$$  \hfill (40)

where

| $q'_1$ | $q'_2$ | $q'_3$ | $q'_4$ |
|-------|-------|-------|-------|
| $1 - \sum_{i=2}^{4} q'_i$ | $(1 - v_1 - v_2) \rho + v_1 \sigma_x \rho \sigma_x + v_2 \sigma_z \rho \sigma_z$ | $\sigma_y \rho \sigma_y$ | $\sigma_y \rho \sigma_y$ |
| $2 rs (1 - r) (1 - s)$ | $\sigma_y \rho \sigma_y$ | $\sigma_y \rho \sigma_y$ | $\sigma_z \otimes I \Omega \left( \sigma_z \otimes I \right)$ |
| $2 rs (r + s - 2rs)$ | $\frac{1}{(r + s - 2rs)} \left[ s (1 - r) \sigma_x \rho \sigma_z + r (1 - s) \sigma_z \rho \sigma_z \right]$ | $\sigma_z \otimes \sigma_z \Omega \left( \sigma_z \otimes \sigma_z \right)$ | $\sigma_z \otimes \sigma_z \Omega \left( \sigma_z \otimes \sigma_z \right)$ |

with

$$v_1 = \frac{2 r (1 - r) (1 - s)^2}{1 - 2rs (2 - r - s)},$$  \hfill (41)

$$v_2 = \frac{2 s (1 - s) (1 - r)^2}{1 - 2rs (2 - r - s)}.$$  \hfill (42)

Now: $\Omega'_2 = |++\rangle \langle ++|$, $\Omega'_3 = |--\rangle \langle +--|$, and $\Omega'_4 = |--\rangle \langle --|$. Therefore, measuring each order qubit in the $\{|\pm\rangle\}$ basis will herald the presence of $\sigma_y \rho \sigma_y$ for each of the outcomes $+-$ and $-+$ and the receiver can now recover the input state by applying $\sigma_y$. Hence, with probability

$$q'_{23} = q'_2 + q'_3 = 4 rs (1 - r) (1 - s)$$  \hfill (43)

the higher-order quantum switch achieves error-free transfer of the input state.

Once again, this whole exercise will yield something meaningful provided we could find $0 < r, s < 1$ such that $q'_{23} > rs$, that is,

$$4 rs (1 - r) (1 - s) > rs$$  \hfill (44)
or, equivalently,

\[(1 - r)(1 - s) > \frac{1}{4} \tag{45}\]

So, whenever the above inequality is satisfied the higher-order quantum switch given by (38) will transfer an arbitrary qubit without any error with a probability higher than that achievable using the quantum switch (35). One can now easily see that there exist \(0 < r, s < 1\) so that the above inequality is indeed satisfied. For example, for all \(0 < r, s < \frac{1}{2}\), (45) will be satisfied. This is, of course, a subset of all possible solutions, the complete set of admissible solutions is depicted in Figure 3.

Example 3. Consider now two quantum switches, the first constructed from two identical bit flip channels \(\mathcal{B}\) defined in (31) and (33), and the second one from two identical phase flip channels \(\mathcal{G}\) defined in (32) and (34):

\[
\mathbb{S} (\mathcal{B}, \mathcal{B}, a) (\rho) = \sum_{i,j=0}^{1} V_{ij} (\rho \otimes \omega) V_{ij}^\dagger, \tag{46}\]

\[
\mathbb{S} (\mathcal{G}, \mathcal{G}, a) (\rho) = \sum_{i,j=0}^{1} W_{ij} (\rho \otimes \omega) W_{ij}^\dagger, \tag{47}\]
where

\[ V_{ij} = B_i B_j \otimes |0\rangle \langle 0| + B_j B_i \otimes |1\rangle \langle 1| \]  
\[ W_{ij} = G_i G_j \otimes |0\rangle \langle 0| + G_j G_i \otimes |1\rangle \langle 1| \]

are the corresponding Krauss operators.

Simplifying (48) and (49) one obtains

\[ S(B, B, a)(\rho) = [(1 - b) \rho + b \sigma_x \rho \sigma_x] \otimes \omega, \]  
\[ S(G, G, a)(\rho) = [(1 - g) \rho + g \sigma_z \rho \sigma_z] \otimes \omega, \]

where \( b = 2r(1 - r) \) and \( g = 2s(1 - s) \). Since (50) and (51) hold for an arbitrary \( \omega \), we conclude that neither switch when used only once can transfer a qubit without error.

Let us now consider the quantum switch of \( S(B, B, a) \) and \( S(G, G, a) \). Following (11) it is defined as:

\[ S[S(B, B, a), S(G, G, a), a'](\rho) = \sum_{i,j,k,l=0} K_{ijkl} (\rho \otimes \Omega) K_{ijkl}^*, \]

where \( \Omega = |\Omega\rangle \langle \Omega| \) is the joint state of the order qubits \( a \) and \( a' \), and \( \{K_{ijkl}\} \) are the Krauss operators given by

\[ K_{ijkl} = V_{ij} W_{kl} \otimes |0\rangle \langle 0| + W_{kl} V_{ij} \otimes |1\rangle \langle 1|. \]

Let us now choose \( |\Omega\rangle = |++\rangle \). Then, we get

\[ S[S(B, B, a), S(G, G, a), a'](\rho) = \sum_{i=1}^2 q_i'' q_i'' \otimes \Omega_i'', \]

where

| \( q_i'' \) | \( q_i'' \) | \( \Omega_i'' \) |
|---|---|---|
| \( q_1'' = 1 - q_2'' \) | \( q_1'' = (1 - w_1 - w_2) \rho + w_1 \sigma_x \rho \sigma_x + w_2 \sigma_z \rho \sigma_z \) | \( \Omega_1'' = \Omega \) |
| \( q_2'' = 4rs(1 - r)(1 - s) \) | \( q_2'' = \sigma_y \rho \sigma_y \) | \( \Omega_2'' = (I \otimes \sigma_z) \Omega (I \otimes \sigma_z) \) |
with

\[ w_1 = \frac{2[(1-s) s - 1]}{4rs(1-r)(1-s) - 1}, \quad (55) \]

\[ v_2 = \frac{2[(1-r) r - 1]}{4rs(1-r)(1-s) - 1}. \quad (56) \]

Since \( \Omega''_2 = |+\rangle \langle +| \), measuring each of the two order qubits in the \( \{|\pm\rangle\} \) basis will herald the presence of \( \sigma_y \rho \sigma_y \) whenever the outcome is \(+−\). This will happen with probability \( q''_2 = 4rs(1-r)(1-s) \) and when it does, the input state can be recovered completely by applying \( \sigma_y \). So with probability \( q''_2 \) one achieves error-free transfer of a qubit for single use of the higher-order switch.

Now, recall that the two switches \( S(B, B, a) \) and \( S(G, G, a) \) are completely useless for this particular task of one-shot error-free transfer of a qubit with some nonzero probability. But, as we have just shown, the higher-order switch can perform this task with a nonzero probability for all \( 0 < r, s < 1 \), so it is useful as a resource.

### 4 Conclusions

The quantum switch leads to a novel causal structure where two quantum channels act in a quantum superposition of their possible causal orders [1]. It has also been shown that the indefinite causal order manifested in a quantum switch is a resource for quantum communication [8, 9, 10, 11, 12]. In this paper, we discussed a higher-order quantum switch of two quantum switches. Here, a quantum state could pass through two quantum switches in a superposition of different causal orders, where the order is controlled by an order qubit. We presented instances in one-shot heralded quantum communication where this higher-order quantum switch can perform better than the individual quantum switches. In particular, two quantum switches placed in a quantum superposition of their alternative causal orders can transmit a qubit without any error with a probability higher than that of the individual quantum switches. We discussed three examples in detail. The first two showed this outperformance over useful quantum switches whereas, the last one showed that a higher-order quantum switch becomes useful even when constructed from two useless quantum switches.

There, however, are situations where the communication advantage using a quantum switch can also be obtained using coherently controlled quantum channels without requiring indefinite causal order [17, 18], although this is not always the case [11]. So it would be interesting to find out whether our results or similar ones can also be reproduced in a set-up of coherently controlled quantum switches without involving indefinite causal order. We do not know the answer either way and leave it for future considerations.
There is another problem one might consider. It has been shown there exist noisy quantum channels that can act as a perfect quantum communication channel when used to form a quantum switch [10]. However, these examples are unique up to unitary freedom. In a practical situation, it could be the case that particular quantum channels are not available. This stipulates the question: How to improve the efficacy of a quantum switch from two arbitrary noisy quantum channels? We answered this question partially by showing that the performances of certain quantum switches can be improved by a higher-order quantum switch. But there could be other ways to achieve the same, so other ideas also need to be explored. So, it would be interesting to know whether a higher-order quantum switch from two “noisy” quantum switches could behave as a perfect quantum communication channel.

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