AN ALGEBRAIC CHARACTERIZATION OF A DEHN TWIST FOR NONORIENTABLE SURFACES

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Abstract. Let $N_{g}^{k}$ be a nonorientable surface of genus $g \geq 5$ with $k$-punctures. In this note, we will give an algebraic characterization of a Dehn twist about a simple closed curve on $N_{g}^{k}$. Along the way, we will fill some little gaps in the proofs of some theorems in [1] and [2] giving algebraic characterizations of Dehn twists about separating simple closed curves. Indeed, our results will give an algebraic characterization for the topological type of Dehn twists about separating simple closed curves.

1. Introduction

In this note, $N_{g,r}^{k}$ will denote the nonorientable surface of genus $g$ with $r$ boundary components and $k$ punctures (or distinguished points). The mapping class group of $N_{g,r}^{k}$, the group of isotopy classes of all diffeomorphisms of $N_{g,r}^{k}$, where diffeomorphisms and isotopies fix each point on the boundary is denoted by Mod$(N_{g,r}^{k})$. If we restrict ourselves to the diffeomorphisms and isotopies to those which do not permute the punctures then we obtain the pure mapping class group $P\text{Mod}(N_{g,r}^{k})$. The subgroup $P\text{Mod}^{+}(N_{g,r}^{k})$ of the pure mapping class group consists of the pure mapping classes that preserve the local orientation around each puncture. Also the twist subgroup of $\text{Mod}(N_{g,r}^{k})$, generated by Dehn twists about two-sided simple closed curves is denoted by $T$.

An algebraic characterization of a Dehn twist plays important role in the computation of the outer automorphism group of the mapping class group of a surface (orientable or nonorientable, see [1], [2] and [4]). Moreover, it is one of the main tools in the proof of the fact that any injective endomorphism of the mapping class group of an orientable surface must be an isomorphism proved by Ivanov and McCarthy ([6]). We note that it is also used in the proof of the fact that any isomorphism between two finite index subgroups of the extended mapping class group of an orientable surface is the restriction of an inner automorphism of this group ([5], [8]). Using an
algebraic characterization of Dehn twists, we show that any automorphism of the mapping class group of a surface takes Dehn twists to Dehn twists.

For orientable surfaces, N. V. Ivanov gave an algebraic characterization of Dehn twists in [4]. December in 2012, E. Irmak reported that the proofs of Ivanov’s theorem on algebraic characterization of Dehn twists about separating simple closed curves have some gaps (see counter example in Section 2). For closed nonorientable surfaces, we gave an algebraic characterization of Dehn twists in [1], closely following Ivanov’s work [4]. Therefore, the algebraic characterization in [1] has also gaps. In this paper, we will not only give an algebraic characterization of a Dehn twist about a simple closed curve on \(N_g^k\) but also an algebraic characterization of the topological type of the curve the Dehn twist is about. In particular, this paper will fill the gaps, mentioned above, in both [1] and [4].

The organization of the paper is as follows: In Section 2 we will state and prove an algebraic characterization for Dehn twists about nonseparating curves with nonorientable complements (Theorem 2.1). The statement and its proof are slightly different than the ones in [1, 2] and its proof also provides a proof for Corollary 2.3. In Section 3, after proving some preliminary results we will first characterize the Dehn twists about characteristic curves on a nonorientable surface of even genus. Then Lemma 3.9 will lead to an algebraic characterization of Dehn twists about separating curves (Theorem 4.1). Moreover, this algebraic characterization will encode the topological type of the separating curve the Dehn twist is about.

### 2. Preliminaries

Let \(S\) denote the surface \(N_g^k\) and let \(a\) be a simple closed curve on \(S\). If a regular neighborhood of \(a\) is an annulus or a Möbius strip, then we call \(a\) a two-sided or a one-sided simple closed curve, respectively. The curve \(a\) will be called trivial, if it bounds a disc with at most one puncture or a Möbius band on \(S\) (or if it is isotopic to a boundary component). Otherwise, it is called nontrivial.

We will denote by \(S^a\) the result of cutting of \(S\) along the simple closed curve \(a\). The simple closed curve \(a\) is called nonseparating if \(S^a\) is connected. Otherwise, it is called separating.

Let \(H\) be a group. If \(G \leq H\) is a subgroup and \(h \in H\) is an element of \(H\), then the center of \(H\), the centralizer of \(G\) in \(H\) and the centralizer of \(h\) in \(G\) will be denoted by \(C(H)\), \(C_H(G)\) and \(C_G(h)\), respectively.

Let \(P : \Sigma_{g-1}^{2k} \to N_g^k\) be the orientation double covering of \(N_g^k\) and \(\tau : \Sigma_{g-1}^{2k} \to \Sigma_{g-1}^{2k}\) the Deck transformation, which is an involution. It is well known that any diffeomorphism \(f : N_g^k \to N_g^k\) has exactly two lifts to the orientation double covering and exactly one of them is orientation preserving. Moreover, we can regard \(\text{Mod}(N_g^k)\) as the subgroup \(\text{Mod}(\Sigma_{g-1}^{2k})^\tau\), the subgroup of mapping classes which are invariant under the action of the deck transformation (see also [1]).
Let $\Gamma(m)$, where $m \in \mathbb{Z}$, $m > 1$, be the kernel of the natural homomorphism

$$\text{Mod}(\Sigma_{g-1}^{2k}) \to \text{Aut}(H_1(\Sigma_{g-1}^{2k}, \mathbb{Z}/m\mathbb{Z})).$$

Then $\Gamma(m)$ is a subgroup of finite index in $\text{Mod}(\Sigma_{g-1}^{2k})$. Let $\Gamma'(m) = \Gamma(m) \cap \text{Mod}(N^g_{h,n})$, regarding $\text{Mod}(N^g_{h,n})$ as a subgroup of $\text{Mod}(\Sigma_{g-1}^{2k})$ as described above.

It is well known that $\Gamma(m)$ and hence $\Gamma'(m)$ consist of pure elements only provided that $m \geq 3$, (I). In this paper, we will always assume that $\Gamma(m)$ for only $m \geq 3$.

Throughout the paper, $\Gamma'$ will denote a subgroup of finite index in $\Gamma'(m)$.

First, we give the following algebraic characterization of Dehn twists about nonseparating curves with nonorientable complement.

**Theorem 2.1.** Let $S = N^g_{g}$ be a connected nonorientable surface of genus $g \geq 5$ with $k$ punctures. Let $M(S)$ be any of three groups $\text{PMod}^+(S)$, $\text{PMod}(S)$ and $\text{Mod}(S)$ and let $\Gamma'$ be as above. An element $f \in M(S)$ is a Dehn twist about a nonseparating simple closed curve with nonorientable complement if and only if the following conditions are satisfied:

i) $C(\Gamma'(f^n)) \cong \mathbb{Z}$, for any integer $n \neq 0$ such that $f^n \in \Gamma'$.

ii) If $g$ is odd (even) there is an abelian subgroup $K$ of rank $\frac{3g-7}{2} + k$ (respectively, of maximal rank $\frac{3g-8}{2} + k$) generated by $f$ and its conjugates freely.

iii) $f$ is a primitive element of $C_{M(S)}(K)$.

**Proof.** We will prove the odd genus case mainly following the proof of Ivanov in [I]. The even genus case is basically the same with minor changes, which we will emphasize whenever needed.

Assume that the above conditions are satisfied, then we have to show that $f$ is a Dehn twist about a nonseparating circle.

Choose any integer $n \neq 0$ such that $f^n \in \Gamma'$. If $f^n$ is equal to the identity, then clearly $C_{\Gamma'}(f^n) = \Gamma'$ and hence the first condition, $C(\Gamma')$ is isomorphic to $\mathbb{Z}$. However, this is a contradiction to the condition (ii). Therefore, $f^n$ is not the identity element.

Let $C'$ be the minimal reduction system for $f^n$. Let $G$ denote the subgroup generated by the twists about the two-sided circles in $C'$. Set $G' = G \cap \Gamma'$. Then $G$ and $G'$ are free abelian groups. Firstly, we will show that $G' \subseteq C(\Gamma'(f^n))$. Let $g \in C_{\Gamma'}(f^n)$. Since $g$ commutes with $f^n$, it preserves the canonical reduction system $C'$. Because $g$ is pure, it fixes each circle of $C'$ and also preserves orientation of a regular neighbourhood of each two-sided circle of $C'$. It follows that $g$ commutes with each generator $G$, hence $G \subseteq C_{M(S)}(C_{\Gamma'}(f^n)).$ So, $G' \subseteq C_{\Gamma'}(C_{\Gamma'}(f^n)) = C(\Gamma'(f^n))$. For the last equality, we note that the inclusion $C(\Gamma'(f^n)) \subseteq C_{\Gamma'}(C_{\Gamma'}(f^n))$ is clear. For the other inclusion, let $g \in C_{\Gamma'}(C_{\Gamma'}(f^n))$. Thus, $[g, h] = 1$, for all $h \in C_{\Gamma'}(f^n)$. Since
\( f^n \in \Gamma' \) we have \( f^n \in C_{\Gamma'}(f^n) \) and hence \([g, f^n] = 1\). This yields that \( g \in C_{\Gamma'}(f^n) \). Now it follows that \( g \in C_{\Gamma'}(C_{\Gamma'}(f^n)) \cap C_{\Gamma'}(f^n) = C(C_{\Gamma'}(f^n)) \).

The assumption that \( C(C_{\Gamma'}(f^n)) = \mathbb{Z} \) yields that \( C' \) has at most one two-sided circle. Now assume that \( C' \) has no two-sided circle, so that \( C' = \{c_1, \ldots, c_l\} \), where each \( c_i \) is a one-sided circle. Then \( S' \) is connected and the restriction \( f^m_{|S'} \) is either the identity or pseudo-Anosov.

Let us denote the number \( \frac{3g - 7}{2} + k \left( \frac{3g - 8}{2} + k, \text{ in the even genus case} \right) \) by \( s \). If \( f^m_{|S'} \) is the identity, then \( f^n \) must be a product of Dehn twists about some circles in \( \mathcal{C}' \) which is not possible since each \( c_i \) is one-sided. Therefore, \( f^m_{|S'} \) and hence \( f^m_{|S'} \) is a pseudo-Anosov diffeomorphism. However, in this case, the maximal abelian group containing \( f \) has rank one. By the assumption \( g \geq 5 \) and thus \( s \geq 4 > 1 \). This is clearly a contradiction. (Similarly, in the even genus case, we have \( s \geq 5 > 1 \).) Hence, \( C' \) has exactly one two-sided circle and \( C' = \{c_1, \ldots, c_l, a\} \), where \( a \) is a two-sided circle and each \( c_i \) is one-sided.

Let \( D \) be the subgroup generated by \( f^n \) and the twist about \( a \) and denote the intersection \( D \cap \Gamma' \) by \( \Gamma' \). Hence, \( D' \subseteq C(C_{\Gamma'}(f^n)) \) and hence \( D' \) is isomorphic to \( \mathbb{Z} \). Now it follows that \( f^n \) is a power of the Dehn twist \( t_a \). In other words, \( f^n = t^m_a \) for some integer \( m \).

Next we will show that the circle \( a \) is nonseparating. Assume on the contrary that \( a \) is indeed a separating circle. Then, \( S^a = S_1 \cup S_2 \) such that \( \chi_j = \chi(S_j) \) and \( \chi_1 + \chi_2 = \chi(S) \). Since \( a \) is nontrivial we see that \( \chi_j < 0 \) for \( j = 1, 2 \). Without loss of generality, we can assume that \( \chi_1 \geq \chi_2 \).

Let \( f_1, \ldots, f_s \) denote the elements that are conjugate to \( f \) generating the abelian subgroup \( K \) in the statement of the theorem. In particular, each \( f_i^m = t_{a_i}^m \), where \( a_i \) is a two-sided circle. Since \( f_i \) and \( f \) are conjugate, each \( a_i \) separates the surface into two components, say \( S_1^i \) and \( S_2^i \), which are diffeomorphic to \( S_1 \) and \( S_2 \), respectively. By the structure of abelian groups, we know that the circles \( a_i \) can be chosen to be disjoint and pairwise nonisotopic. Furthermore, each \( S_1^i \) is diffeomorphic to \( S_1^i \) and \( a_i \) is not isotopic to \( a_j \), for all \( i \neq j \). Hence, \( S_1 \) cannot be contained inside the surface \( S_1^i \). Therefore, for any \( i \neq j \), we have \( S_1^i \cap S_1^j = \emptyset \); in other words, the surfaces \( S_1^i \) are all disjoint. Let \( S^0 = Cl(S \setminus (S_1^1 \cup \cdots \cup S_1^s)) \). The fact that \( a_i \) are being pairwise nonisotopic yields that \( S^0 \) cannot be an annulus. Furthermore, \( S^0 \) is not a sphere, a torus, a Klein bottle, a projective plane, a disk or a Möbius band. Thus, \( \chi(S^0) < 0 \).

Now, \( 2 - g - k = \chi(S) = \chi(S^0) + \chi(S_1^1) + \cdots + \chi(S_1^s) \) and hence, \( 2 - g - k < s \chi(S_1^1) \). Then, \( g - 2 + k > s \chi(S_1^1) \). Hence, we get \( 2g + 2k - 4 > 3g + 2k - 7 \), which implies \( 3 > g \). This is a clear contradiction. Thus, \( a \) must be a nonseparating circle. (For even genus, in a similar fashion, we would get \( 2g + 2k - 4 > 3g + 2k - 8 \), which would yield \( 4 > g \), a contradiction to the assumption that \( g \geq 6 \).)
Next, we take an element $\psi \in C_{M(S)}(K)$ and choose a diffeomorphism $\Psi : S \to S$ in the isotopy class $\psi$. The group $K$ is generated by $f_1, \ldots, f_s$, which are all conjugate to $f$ and hence $f_1^n = t_{a_1}^m$ as above. Since $\psi \in C_{M(S)}(K)$, $\psi f_1 \psi^{-1} = f_1$ and thus $f_1^n \psi^{-1} = f_1^n$. In other words, $\psi t_{a_1}^m \psi^{-1} = t_{a_1}^m$ and thus $t_{a_i}^m \psi^{-1} = t_{a_i}^m$. Hence, replacing $\Psi$ by an isotopy we may assume that $\Psi(a_i) = a_i$. Let $\bar{C} = \{a_1, \ldots, a_s\}$.

Since $s$ is the maximal number of pairwise nonisotopic disjoint two-sided circles on $S$, we see that every component of the surface $S^c$ is either a once-punctured annulus or a once-punctured Möbius band. The total number of the boundary components of the components of $S^c$ is $2s$ and hence all the components of $S^c$ except one must be pair of pants.

In the even genus case, some extra care is needed. One can easily see that in the even genus case, there are abelian subgroups of rank \(\frac{3g-8}{2} + k\) and \(\frac{3g-6}{2} + k\), which are maximal in the sense that they are not contained in abelian subgroups of higher ranks (compare with Lemma 2.7 of \cite{[1]}). By requiring that the subgroup $K$ of rank \(\frac{3g-8}{2} + k\) to be maximal in this sense we ensure that all but two of the components of $S^c$ are pair of pants and the remaining two components are one-punctured Möbius bands. This can be seen easily using an Euler characteristic argument (see also Remark \cite{22}).

Now suppose that $\Psi$ sends one of these components to another one. Since $\Psi$ fixes the circles (boundary components) of $S^c$, we see that $S = \bar{P} \cup \bar{R}$ where $\Psi(P) = R$. However, this is a contradiction to the assumption that $S$ is a nonorientable surface of odd genus. Hence, $\Psi$ does not permute the components. It follows that, $\Psi$ is a composition of twists about the circles $a_i$. (Similarly, in the even genus case, $\Psi$ cannot interchange the components since by assumption $g \geq 6$.)

Note that $f \in C_{M(S)}(K)$ because $f \in K$ and $K$ is abelian. Without loss of generality, assume that $f = f_1$. Then, $f^n = f_1^n = t_{a_1}^m$. Now by the above paragraph, we deduce that $f$ is a power of $t_{a_1}^l$, $f = t_{a_1}^l$, for some $l$, because $K$ is free abelian. Therefore, $f_1 = t_{a_1}^l$ and $K$ is generated by $t_{a_1}^l$. In particular, $t_{a_1}$ is in $C_{M(S)}(K)$. Finally, by the last condition, $f$ is primitive and hence, $f = t_{a_1} = t_a$. This finishes the proof for surfaces of odd genera.

In the even genus case, one has to prove also that the complement of $a$ is nonorientable. Assume on the contrary that the complement of $a$ is orientable. Since each $t_{a_i}$ is conjugate to $t_a$ the complement of each $a_i$ is also orientable. It follows that each $a_i$ represents the $\mathbb{Z}_2$-homology cycle Poincaré dual to the first Stiefel-Whitney class $\omega_1$ of the surface $S$. In particular, for any $1 \leq i, j \leq s$, $i \neq j$, the circles $a_i$ and $a_j$ form the boundary of a subsurface $S_{i,j}$ of $S$. Moreover, since the complement of each $a_i$ is orientable, $S_{i,j}$ is an orientable surface of genus at least one with at least two boundary components. By relabeling $a_i$’s if necessary, we may assume that the interior of each subsurface $S_{i,i+1}$ (where we set $S_{s,s+1} = S_{s,1}$) does
not intersect any \( a_k \). Hence, \( S_{i,i+1} \cap S_{j,j+1} = \emptyset \), if and only if \( |i-j| > 1 \). Now, \( S = \bigcup_{1 \leq i \leq s} S_{i,i+1} \), and thus, computing the Euler characteristics of both sides, we see that

\[
2 - g - k = \chi(S) = \sum_{i=1}^{s} \chi(S_{i,i+1}) \leq -2s = -3g + 8 - 2k,
\]

which yields the inequality \( 6 \geq 2g + k \), a contradiction since \( g \geq 6 \) by the assumption. Hence, we conclude that the complement of \( a \) must be nonorientable.

The other direction of the theorem is straightforward and left to the reader (see also [1]).

**Remark 2.2.** In the mapping class groups of even genus nonorientable surfaces there are abelian subgroups of different maximal ranks. This with one example is illustrated in [2].

Before we go on characterizing other Dehn twists algebraically, we state a corollary of the proof of the above theorem.

**Corollary 2.3.** Let \( S \) be the surface in the above theorem and \( f \in \text{M}(S) \) an element. Then there are integers \( m > 0 \), \( N \) and a simple closed curve \( c \) so that \( f^N = t_c^m \) if and only if the following conditions are satisfied:

i) \( C(C_f(f^n)) \cong \mathbb{Z} \), for any integer \( n \neq 0 \) such that \( f^n \in \Gamma' \).

ii) There exists a free abelian subgroup \( K \) of \( \text{M}(S) \) generated by \( f \) and \( t_c \), \( \frac{3g-9}{2} + k \), when \( g \) is odd, (respectively, \( \frac{3g-10}{2} + k \), when \( g \) is even), twists about two-sided nonseparating simple closed curves such that

\[
\text{rank } (K) = \frac{3g-7}{2} + k, \text{ when } g \text{ is odd}, \text{ (respectively, } \frac{3g-8}{2} + k, \text{ when } g \text{ is even}).
\]

Hence by the above theorem and the corollary, we have algebraically characterized, in all the three groups \( \text{PMod}^+(S) \), \( \text{PMod}(S) \) and \( \text{Mod}(S) \) of odd genus surfaces, the Dehn twists about nonseparating simple closed curves (respectively in the even genus case, the Dehn twists about nonseparating simple closed curves with nonorientable complements) and those elements \( f \in \text{M}(S) \), whose some power is a positive power of a Dehn twist about a nontrivial two-sided simple closed curve.

2.1. A Counter Example. The situation for separating curves is more involved. First we remark that Theorem 3.2 and Theorem 3.3 of [1] are not correct as they are stated. E. Irmak informed us about the following counter example constructed by L. Paris: Let \( c \) be a separating curve in a surface \( S \) so that one of the two components of \( S^c \) is a torus with one boundary component \( \Sigma_{1,1} \) as in the Figure [1]. Consider the mapping class \( \tau = (t_a t_b)^3 \), where \( a \) and \( b \) are the curves given in the same figure. Let \( K \) be an abelian subgroup as in the statement of Theorem 3.2 or Theorem 3.3 of [1] for the Dehn twist \( t_c \). Since \( \tau \) commutes with both \( t_a \) and \( t_b \), \( \tau \) is in \( C_{\text{M}(S)}(K) \).
However, \( t_c = \tau^2 \) and hence \( t_c \) is not a primitive element in \( C_{M(S)}(K) \). Note that this example exists since the center of the mapping class group of \( \Sigma_{1,1} \) is not trivial. Indeed it is isomorphic to the infinite cyclic group generated by \( \tau \). There are two more cases that might cause similar problem. The first one is \( \Sigma_{1,1} \), whose mapping class group is isomorphic to that of \( \Sigma_{1,1} \). The final problematic case is the surface \( N^1 \) whose mapping class group is the infinite cyclic group generated by the class of \( v \), the class of the puncture slide diffeomorphism (Figure 1). In this case, \( v^2 \) is the Dehn twist about the boundary curve \( c \).

Remark 2.4. The example we have just described above, also provides a counter example for Theorem 2.2 of [4], a result for algebraic characterization for Dehn twists (mainly, about separating curves) in orientable surfaces. The methods we will provide here, which work for both orientable and nonorientable surfaces, not only provide an algebraic characterization for Dehn twists about separating curves but also for the topological type of the separating curve the Dehn twist is about (see Remark 4.2).

**Figure 1.**

**3. Preparation for the algebraic characterization of Dehn twist about separating curves**

We start with the following technical result which we will use later.

**Proposition 3.1.** Let \( S = N^k \) be a nonorientable surface of even genus. Then for any integer \( s = 0, \cdots, \frac{g-2}{2} \), the group \( M(S) \) has an abelian subgroup of rank \( \frac{3g - 6 - 2s}{2} + k \), which is freely generated by Dehn twists about pairwise nonisotopic nonseparating circles, and so that no abelian subgroup containing this subgroup has bigger rank. Moreover, when we cut the surface along these circles, the resulting surface is a disjoint union of \( g + k - 2s - 2 \) many pair of pants and \( 2s \) many two holed real projective planes.

**Proof.** Let \( r = \frac{3g - 6 - 2s}{2} + k \). By the maximality condition of the proposition, all the components of the surface cut along these \( r \) circles have Euler characteristic \(-1\). In other words, each component is either a pair of pants or a two holed real projective plane. Assume that there are \( l \) many pair
of pants and \( m \) many two holed real projective planes. Hence, considering Euler characteristics we obtain the equation
\[
2 - g - k = \chi(S) = \chi\left( \bigcup_l N_0^3 \cup \bigcup_m N_1^2 \right) = -l - m .
\]
On the other hand, counting the number of punctures, we obtain
\[
3g - 6 - 2s + 3k = 3l + 2m .
\]
These two equations yield \( m = 2s \) and \( l = g + k - 2s - 2 \) as desired. Finally, the existence of such subgroups is readily seen by inspection.

A sequence of Dehn twists \( t_{a_1}, \cdots, t_{a_n} \), is called a chain if the following geometric intersection \( i(a_i, a_{i+1}) = 1 \), for \( i = 1, \cdots, n-1 \). The integer \( n \geq 1 \) is called the length of the chain. Note that if a chain has length more than one then each \( a_i \) must be nonseparating (and has nonorientable complement, if \( S \) is nonorientable of even genus). For a tree or chain of Dehn twists we always fix an orientation for a tubular neighborhood of the union of simple closed curves (which is always orientable) and consider Dehn twists using this orientation. It is known that two Dehn twists \( t_a, t_b \) satisfy the braid relation \( t_a t_b t_a = t_b t_a t_b \) if and only if \( i(a, b) = 1 \) on nonorientable surfaces (see [9]). Hence, by the above results any automorphism \( \Psi : M(S) \to M(S) \) maps a chain of Dehn twists of length at least two to another chain of Dehn twists of the same length. In this note, unless we state otherwise a chain or a tree in a nonorientable surface \( S \) will mean a chain or a tree of Dehn twists about nonseparating two-sided simple closed curves with nonorientable complement.

Below we will give a generalization of Lemma 3.7 of [1] to punctured surfaces. We will include the proof since the one in [1] has a gap, indicated by B. Szepietowski.

Lemma 3.2. Let \( S = N_g^k \) be a nonorientable surface of genus \( g \geq 5 \). Then the image of a disc separating chain under an automorphism of \( M(S) \) is again a chain which separates a disc.

Proof. If the genus of the surface is odd then a chain is separating if and only if it is maximal in \( M(S) \). However, being maximal is clearly preserved under an automorphism. Now Lemma 3.5 of [1] finishes the proof.

Now assume that the genus \( g \geq 6 \) is an even integer. Further assume that \( c_1, \cdots, c_{2l+1} \) is a disc separating chain in \( M(S) \). Hence, when we delete a tubular neighborhood of the chain from the surface we obtain a disjoint union of a disc and a nonorientable surface, call \( S_0 \), of genus \( g - 2l \) with \( k \) punctures and one boundary component. By Euler characteristic calculation and inspection we see that the group \( M(S_0) \) has an abelian subgroup \( K \) of rank \( \frac{3(g - 2l) - 6}{2} + k + 2 \), contained in each \( C_{M(S)}(t_{c_i}) \), which is freely generated by Dehn twists about pairwise nonisotopic circles.
Now suppose on the contrary that the image $d_1, \cdots, d_{2l+1}$ of the chain $c_1, \cdots, c_{2l+1}$ under an automorphism is not separating. So by Lemma 3.5 of [1] the complement of a tubular neighborhood of the chain $d_1, \cdots, d_{2l+1}$ in $S$ is an orientable surface of genus $g - 2l - 1$ with $k$ punctures and two boundary components, say $c_1$ and $c_2$. Let us call this surface $S_1$. (Since the surface $S$ is nonorientable the circles $c_1$ and $c_2$ are both characteristic in the surface $S$.) The image of the abelian subgroup $K$ under the same automorphism is again a maximal subgroup in $M(S)$ and it lies in each $C_{M(S)}(t_{d_i}), i = 1, \cdots, 2l + 1$. However, the orientable subsurface $S_1$ can support an abelian subgroup $K_0$ in $M(S)$, which lies in each $C_{M(S)}(t_{d_i})$, of rank at most $\text{rank}(K) - 1$. Some of the generators of both groups are Dehn twists about characteristic or separating curves. However, by Corollary 2.3 some powers of these generators are preserved under automorphisms. This finishes the proof. \[\square\]

Now we will characterize algebraically a separating pair of Dehn twists about some two-sided simple closed curves, each of which is nonseparating with nonorientable complement (so that together they separate the surface).

**Lemma 3.3.** Let $S = N^g_k$ be a nonorientable surface of genus $g \geq 5$ and $a_1$ and $a_2$ be disjoint, nonisotopic, nonseparating two-sided simple closed curves with nonorientable complements. Then $a_1$ and $a_2$ together separate the surface if and only if the following conditions are satisfied:

1. For any Dehn twist $t_b$ satisfying the braid relation $t_{a_1} t_b t_{a_1} = t_b t_{a_1} t_b$ we have $t_b \notin C_{M(S)}(t_{a_2})$;
2. In the even genus case, the twists $t_{a_1}$ and $t_{a_2}$ are contained in a free generating set, whose elements are all nonseparating circles with nonorientable complement, of a maximal abelian subgroup $K$ in $M(S)$, of rank $r = \frac{3g - 6 - 2s}{2} + k$, where $s = 1$ or $s = 2$.

Moreover, if the conditions of lemma are satisfied, then $s = 2$ if both components of the surface $S$ cut along the curves $a_1$ and $a_2$ are nonorientable of even genus, and $s = 1$ otherwise.

**Proof.** Suppose first that $a_1$ and $a_2$ separate the surface. In particular, the homology classes $[a_1]$ and $[a_2]$ are equal in $H_1(S, \mathbb{Z}_2)$ and thus the first condition is trivially satisfied. It is easy to construct the required free abelian subgroup $K$, in case the genus is even.

For the other direction assume that the conditions of the lemma are satisfied, but on the contrary suppose that the surface cut along the curves $a_1$ and $a_2$ is connected. Hence, the surface $S$ cut along only $a_2$, say $S_0$, is connected.

First we will treat the case where the genus $g$ is an odd integer. Hence, the surface $S_0$ is a connected nonorientable surface of genus at least 3. Moreover, the curve $a_1$ is still nonseparating in $S_0$. Hence, there is two-sided curve $b$ in $S_0$, whose geometric intersection with $a_1$ is one. This is a contradiction to the first condition of the assumption.
Now let us consider the even genus case. By Proposition 3.1 the surface cut along the \( r \) many curves, about which the Dehn twists generate the subgroup \( K \), has nonorientable components. Thus the surface \( S_0 \) is a connected nonorientable surface of genus at least four, and the curve \( a_1 \) is neither separating nor characteristic in \( S_0 \). Hence, as above there is a two-sided curve \( b \) in \( S_0 \), whose geometric intersection with \( a_1 \) is one. This finishes the proof for the even genus case.

The final part of the lemma is an immediate consequence of Proposition 3.1. □

Let \( a_1, a_2, a_3 \) be distinct, nonisotopic, nonseparating two-sided simple closed curves with nonorientable complements. We say that they form a triangle if each geometric intersection \( i(a_i, a_j) = 1 \), for all \( i \neq j \) (see [2]). A triangle is called orientation preserving if there are Dehn twists about these circles, naturally denoted by \( t_{a_1}, t_{a_2} \) and \( t_{a_3} \), so that any two satisfy the braid relation. (Note that on a nonorientable surface we have exactly two Dehn twists about any two-sided circle \( a \), which we may denote by \( t_a \) and \( t_a^{-1} \).) Otherwise, we call the triangle orientation reversing.

The following result is proved in [2].

**Lemma 3.4.** The above triangle of Dehn twists is orientation preserving if and only if the union of these curves has an orientable regular neighborhood. Moreover, a regular neighborhood of a nonorientable triangle is \( N_{4,1} \), a genus four nonorientable surface with one boundary component.

We will call a separating pair of Dehn twists \( \{t_{a_1}, t_{a_2}\} \) is of type I if there is no orientation reversing triangle in the intersection of centralizers

\[
C_{M(S)}(t_{a_1}) \cap C_{M(S)}(t_{a_2})
\]

Otherwise, we call it a type II pair. Now we prove as a consequence of what we have proved so far, an algebraic characterization of type I pairs of Dehn twists on nonorientable surfaces.

**Corollary 3.5.** Let \( S \) be a nonorientable surface of genus at least five and \( \{t_{a_1}, t_{a_2}\} \) be a separating pair of Dehn twists on \( S \) about disjoint, nonisotopic, nonseparating two-sided simple closed curves with nonorientable complements and \( \Psi : M(S) \to M(S) \) be any automorphism. If this pair is of type I then so is \( \{\Psi(t_{a_1}), \Psi(t_{a_2})\} \).

**Proof.** Assume on the contrary that the pair \( \{\Psi(t_{a_1}), \Psi(t_{a_2})\} \) is of type II. Hence, there is orientation reversing triangle contained in \( C_{M(S)}(t_{a_1}) \cap C_{M(S)}(t_{a_2}) \) so that its image under \( \Psi \) is also orientation reversing. It follows that there are two Dehn twists about two-sided nonseparating simple closed curves (necessarily meeting transversally at one point), say \( t_x \) and \( t_y \), so that we have both \( t_x t_y t_x = t_y t_x t_y \) and \( t_x^{-1} t_y t_x^{-1} = t_y t_x^{-1} t_y \). However, this is not possible. Indeed, multiplying these two relations side by side we get \( t_z t_y t_x^{-1} t_x = t_y t_x t_y t_x^{-1} t_y \). Letting \( t_z = t_x t_y t_x^{-1} = t_x(y) \), we obtain \( t_z^2 = t_y t_x^2 t_y \), where the curve \( z = t_x(y) \) intersects \( y \) transversally at one point. We may...
as well assume that the curves $y$ and $z$ are lying in a torus. Therefore, we have the relation $(t_z^2 t_y)^4 = 1$ (cf. page 82 of [3]). Combining this with the relation $t_z^2 = t_y t_z t_y$ we get,

$$1 = (t_z^2 t_y)^4 = t_z^2 (t_y t_z t_y) t_z^2 (t_y t_z t_y) = t_z^8,$$

a clear contradiction. This finishes the proof. □

Let $t_a$ be a Dehn twist about a separating simple closed curve $a$ in a nonorientable surface $S$. Maximal chains contained in the centralizer $C_{M(S)}(t_a)$ correspond to maximal chains of Dehn twists about two-sided simple closed curves (nonseparating with nonorientable complement) lying in different components of $S^a$. The lengths of these maximal chains determine the topological type the curve $a$ up to a great extend, however fail to characterize its topological type completely. A more powerful tool is to use maximal trees contained in the centralizers. By a maximal tree of Dehn twists in a surface nonorientable $S$ we will mean a connected maximal tree of Dehn twists about pairwise nonisotopic nonseparating two-sided simple closed curves with nonorientable complements. If $S$ is orientable we will only require that the curves in the tree to be nonseparating.

**Remark 3.6.** As it is seen in the figure below a maximal chain in a maximal tree need not to be a maximal chain in the surface. Note that the chain of circles $1, 2, \cdots, 7$ is maximal in the tree but not in the surface, which contains the longer chain $1, 2, \cdots, 7, 8, 9.$

![Figure 2.](image)

The trees below will be useful for the rest of the paper: $T_{2g+1,1}^k$, $T_{2g+2,1}^k$ and $OT_{g,1}^k$.

We may endow an embedding of one of these trees into the group $M(S)$, where $S$ is a nonorientable surface of genus at least five, with a coloring of its vertices. For example, the tree $T_{2g+1,1}^k$ in Figure 2 has the vertices $a_{4g-3}$ and $a_{4g}$ colored. This will mean that any orientation reversing triangle in $M(S)$, whose vertices commute with the colored vertices also commute with all the vertices of the tree. Equivalently, any orientation reversing triangle which lies in

$$C_{M(S)}(t_{a_{4g-3}}) \cap C_{M(S)}(t_{a_{4g}}),$$

also lies in $C_{M(S)}(t_a)$, for all vertices $a$ of the tree.
For nonorientable surfaces of even genera, first we present an algebraic characterization of a Dehn twist about a two-sided simple closed curve, whose complement is orientable.

**Lemma 3.7.** Let \( g \geq 2, \ k \geq 0 \) be integers and \( c \) be a nontrivial two-sided simple closed curve in \( S = N_{2g+2}^k \). Then \( S^c \) is orientable if and only if the colored tree \( NT_{2g+2}^k \) (see Figure 3) can be embedded in the centralizer \( C_{M(S)}(t_c) \) as a maximal tree, where

1) Each maximal chain in the tree is a maximal chain in \( M(S) \);
2) The tree \( T \) has a chain with length larger than or equal to any chain in the centralizer \( C_{M(S)}(t_c) \);
3) Any two vertices connected to \( a_{4g-3}, \) both different than \( a_{4g-4} \), form a separating pair.
Proof. One direction is clear. Now assume that $NT_{2g+2}^k$ lies in the centralizer $C_{M(S)}(t_c)$. We claim that the surface $S$ cut along the curves $a_1$, $a_2$ and $a_3$ has two components one of which is an orientable surface of genus $g - 1$ with $k$ punctures and one boundary component: To see this, consider a tubular neighborhood of the tree $NT_{2g+2}^k$ with the vertex $a_0$ deleted. This is an orientable surface of genus $g$ with $1 + k + 2(g - 1)$ boundary components. By the maximality of the tree, $2g - 2$ many of these components must bound discs. To illustrate this, consider, for example, the maximal chain $a_1, a_2, a_4, a_6, a_5$. The boundary component corresponding to this chain should bound a disc, because otherwise the tree would not be maximal (we could attach another two-sided simple closed curve to $a_2$). Note also that by the condition (3) of the hypothesis of the lemma the $k$ pairs $(t_{a_{4g-2}}, t_{a_{4g-1}}), (t_{a_{4g-1}}, t_{a_{4g}}), \cdots, (t_{a_{4g+k-3}}, t_{a_{4g+k-2}})$ on the right corner of the tree are all bounding. By maximality and the coloring they must all bound punctured annuli. This finishes the proof of the claim.

Now, if we attach a tubular neighborhood of the circle $a_0$ to this subsurface we obtain another subsurface, call $S_0$, of genus $g$ with $k$ punctures and two boundary components. Note that the two boundary components of $S_0$ should be glued so that the resulting surface would be nonorientable of genus $2g + 2$ with $k$ punctures. Finally, $c$ being disjoint from each vertex of the tree implies that $c$ lies in $S - \cup_{i \geq 1} a_i$, which is a disjoint union of $2g$ discs, $k$ punctured annuli and a one holed Klein bottle. In particular, up to isotopy, $c$ is the unique nontrivial two-sided simple closed curve in this punctured Klein bottle. This finishes the proof.
We note that the above lemma, Corollary 2.3 and the ideas in Theorem 2.1 yield the following algebraic characterization for Dehn twists about nonseparating simple closed curves with orientable complements on nonorientable surfaces of even genus.

**Corollary 3.8.** Let \( g \geq 2 \), \( k \geq 0 \) be integers \( f \) an mapping class in \( \text{M(S)} \) such that \( f^N = t^m_c \) for some integers \( m > 0 \), \( N \) and a nontrivial simple closed curve \( c \) on \( S = N_{2g+2} \). Then \( c \) is a characteristic curve (i.e., its complement \( S^c \) is orientable) and \( f = t_c \) if and only if \( f \) is a primitive element (as in the statement of Theorem 2.1) and the colored tree \( NT_{2g+2} \) (see Figure 6) can be embedded in the centralizer \( C_{\text{M(S)}}(t_c) \) as a maximal tree, where

1) Each maximal chain in the tree is a maximal chain in \( \text{M(S)} \);
2) The tree \( T \) has a chain with length larger than or equal to any chain in the centralizer \( C_{\text{M(S)}}(t_c) \);
3) Any two vertices connected to \( a_{4g-3} \), both different than \( a_{4g-4} \), form a separating pair.

For separating Dehn twists characterization we will make use of the following lemma.

**Lemma 3.9.** Let \( g \) be a positive integer and \( T \) be one of the colored trees \( T_{2g+1,1}^k \), \( T_{2g+2,1}^k \) or \( OT_{g,1}^k \), embedded in the group \( \text{M(S)} \), where \( S \) is a nonorientable surface of genus at least five. Suppose that \( c \) is a nontrivial separating simple closed curve in \( S \) and the tree \( T \) lies in the centralizer \( C_{\text{M(S)}}(t_c) \) as a maximal tree. Moreover, assume the followings:

1) Each maximal chain in the tree is a maximal chain in \( \text{M(S)} \);
2) If the tree \( T \) is \( T_{2g+1,1}^k \) or \( T_{2g+2,1}^k \), then it has a chain with length larger than or equal to any chain in the centralizer \( C_{M(S)}(t_c) \).

3) Any two vertices connected to \( a_{4g-2} \), except \( a_{4g-4} \), form a separating pair.

4a) If \( T = T_{2g+1,1}^k \) then \( g \geq 2 \), and if \( T = T_{2g+2,1}^k \) then \( g \geq 1 \). Moreover, in both cases, there is an orientation reversing triangle in \( C_{M(S)}(t_c) \), which is not contained in \( \bigcap_{a_i \in V(T)} C_{M(S)}(t_{a_i}) \), where \( V(T) \) is the set of vertices of \( T \).

4b) If \( T = OT_{g,1}^k \) then \( g \geq 2 \), and any orientation reversing triangle in \( M(S) \) also lies in \( C_{M(S)}(t_{a_{4g-3}}) \cap C_{M(S)}(t_{a_{4g-1}}) \).

Then \( S^c \) has a component homeomorphic to \( N_{2g+1,1}^k \), \( N_{2g+2,1}^k \) or \( \Sigma_{g,1}^k \), respectively.

**Proof.** Case 1: \( T = T_{2g+1,1}^k \). First we assume that \( S \) is of odd genus. For each Dehn twist belonging the tree choose a two-sided nonseparating simple closed curve \( a_i \) with nonorientable complement. Let \( S_0 \) be the closure of a tubular neighborhood of the tree of the curves \( a_i \). Then \( S_0 \) is an orientable subsurface of \( S \) with Euler characteristic \( \chi(S_0) = 2 - 4g - k \) and with \( 2g + k \) boundary components. So \( S_0 \) is an orientable surface of genus \( g \) with \( 2g + k \) boundary components. By the definition of maximal tree each maximal chain in \( T \) is a maximal chain in the surface \( S \). Moreover, \( S \) has odd genus and thus each maximal chain contained in \( T \) separates the surface (this is the only place we use the assumption that \( S \) is of odd genus). Hence, each boundary component bounds either a disc, a once punctured disc, an annulus or a Möbius band. Again by maximality of the tree, \( 2g - 2 \) many of these components must bound discs. To illustrate this, consider for example, the maximal chain \( a_1, a_2, a_4, a_6, a_5 \). The boundary component corresponding to this chain should bound a disc, because otherwise the tree would not be maximal (we could attach another two-sided simple closed curve to \( a_2 \)).

Note that by the condition (3) of the hypothesis of the lemma the \( k + 1 \) pairs \( (t_{a_{4g-3}}, t_{a_{4g}}), (t_{a_{4g}}, t_{a_{4g+1}}), \ldots, (t_{a_{4g+k-1}}, t_{a_{4g-1}}) \) on the right corner of the tree are all bounding.

The condition (2) of the hypothesis of the lemma implies that at most two of these pairs may bound a projective plane with two boundary components. On the other hand, the condition (4a) implies that at least one of them bounds a projective plane with two boundary components. Finally, the coloring of the vertices implies that boundary components corresponding to the chain \( a_{4g-3}, a_{4g} \) is the only pair that bounds a projective plane with two boundary components. Hence, the other \( k \) bounding pairs must bound punctured annuli.
By attaching $2g - 2$ discs and $k$ punctured discs and a Möbius band to $S_0$ we get a nonorientable surface, say $S_1$, of genus $2g + 1$ with one boundary component. $S_1$ is contained as a subsurface in one of the two components of $S^c = S_2 \cup S_3$, say in $S_2$. Since $S_2$ has only one boundary component, the boundary component corresponding to the chain $a_1, a_2, a_3$ must be parallel to the boundary of $S_2$. Hence, we are done in the odd genus case.

Now let us consider the case where $S$ has even genus. If each maximal chain in $T$ is separating in $S$ then the above proof works in this case as well. Now we will show that any maximal chain in $T$ is indeed separating in $S$. To prove this, assume that there is a maximal chain $t_{c_1}, \cdots, t_{c_2l+1}$ in $T$, and thus in $M(S)$, so that the chain of two-sided simple closed curves $c_1, \cdots, c_{2l+1}$ is not separating in $S$. A tubular neighborhood $\nu$ of the union of these curves is an orientable surface of genus $l$ with two boundary components. Since the chain is maximal in $M(S)$ we see that the surface $S - \text{Int}(\nu)$ is an annulus, possibly with punctures, so that when the orientable surfaces $\nu$ and $S - \text{Int}(\nu)$ are glued along the two boundary components, the resulting surface $S$ is nonorientable (see Figure 7) with genus $2l + 2$. Since the separating curve $c$ is disjoint from the tree and thus from the chain, we see that $c$ lies in the annulus and bounds a (at least twice) punctured disc.

![Figure 7](image)

So the topological type of $c$ is determined up to the number of punctures of $S$ contained in either sides of $c$. On the other hand, since the surface $S$ has even genus, exactly two of the bounding pairs $(t_{a_{4g-3}}, t_{a_{4g}}), (t_{a_{4g}}, t_{a_{4g+1}}), \cdots, (t_{a_{4g+k-1}}, t_{a_{4g-1}})$ bound a projective plane with two boundary components. On the other hand, the coloring of the vertices implies that $a_{4g-3}, a_{4g}$ is the only pair bound a projective plane with two boundary components. This gives the desired contradiction. Hence the proof finishes in the case where $T = T_{k}^{2g+1,1}$. 
Case 2: $T = T_{2g+2,1}^k$. Now first assume that the surface $S$ has odd genus. We proceed analogous to the previous case and arrive at the point, where the $k + 2$ pairs $(t_{a_{4g-3}}, t_{a_{4g}}), (t_{a_{4g}}, t_{a_{4g+1}}), \ldots, (t_{a_{4g+k}}, t_{a_{4g-1}})$ on the right corner of the tree are all bounding. Again the condition (2) of the hypothesis of the lemma implies that at most two of these pairs bound a projective plane with two boundary components and the others will bound punctured annuli. Then condition (4a) and the coloring of the vertices imply that exactly the two pairs $(t_{a_{4g-3}}, t_{a_{4g}})$ and $(t_{a_{4g}}, t_{a_{4g+1}})$ bound projective plane with two boundary components and all other pairs bound punctured annuli. This finishes the proof for the odd genus case.

The even genus case is again analogous to that in Case 1. We just need to show that any maximal chain in the tree is separating. We proceed as in case $T = T_{2g+1,1}^k$. Without loss of generality we may assume that $c_{2t+1}$ belongs to the set

$$\{a_{4g-3}, a_{4g}, a_{4g+1}, \ldots, a_{4g+k}, a_{4g-1}\}.$$

The condition (2) of the hypothesis of the lemma implies that the $k + 2$ pairs $(t_{a_{4g-3}}, t_{a_{4g}}), (t_{a_{4g}}, t_{a_{4g+1}}), \ldots, (t_{a_{4g+k}}, t_{a_{4g-1}})$ on the right corner of the tree are all bounding punctured annuli. However, this contradicts to the maximality of the tree, since in this case we may add two more vertices to the tree as in Figure 4. So we are done in this case too.

Case 3: $T = OT_{g,1}^k$. This case is easy now, because by condition (4b) all the bounding pairs on the right side of the tree will bound punctured annuli (see the paragraph below Remark 3.10). It follows that the surface $S_1$, in this case, will be an orientable surface of genus $g$ with one boundary component and with $k$ punctures (compare with the surface $S_1$ we obtained in the case $T = T_{2g+1,1}^k$ above). \hfill \Box

Remark 3.10. 1) One needs to be careful when using the above lemma to characterize the topological type of the curve (or the Dehn twist $t_c$) algebraically: Namely, if one of the components of $S^c$, the surface $S$ cut along the curve $c$, is orientable of genus at least two then we can embed $T = OT_{g,1}^k$ into the centralizer $C_{M(S)}(t_c)$ satisfying the conditions of the lemma so that the topological type of the curve (or the Dehn twist $t_c$) is algebraically characterized. If the orientable component is of genus one, then the intersection of $C_{M(S)}(t_c)$ with the centralizer of the tree (the intersection of all the centralizers of the vertices of $T$) contains a pair of Dehn twists $t_a$ and $t_b$, about nonseparating circles with nonorientable components so that $t_a$ and $t_b$ satisfy the Braid relation.

On the other hand, if both components of $S^c$ are nonorientable, then we have to choose the component of $S^c$ so that Condition (2) of the lemma is satisfied, which is always possible if $g \geq 7$ (see also the next theorem). In other words, we need to choose the component which contains the longer chain.
2) For odd genus surfaces $S$ the condition (3) of the hypothesis of the lemma is void. For example consider the pair $(t_{a_{4g-3}}, t_{a_{4g}})$ in the first tree $T_{2g+1,1}$. The maximal chain $t_{a_{4g-3}}, t_{a_{4g-2}}, t_{a_{4g}}$ in $T$ is maximal in the surface and thus is separating. Hence, a tubular neighborhood of the chain $t_{a_{4g-3}}, t_{a_{4g-2}}, t_{a_{4g}}$ is a torus with two boundary components, each of which is a separating curve. This implies that the pair $(t_{a_{4g-3}}, t_{a_{4g}})$ is separating.

4. Completing the proof of the algebraic characterization of Dehn twist about separating curves

If $c$ is a separating circle on a nonorientable surface $S = \Sigma^k_g$ of genus $g \geq 7$ then one of the two components of $S^c$ is either a nonorientable surface of genus at least four or an orientable surface of genus at least two. Hence, the above lemma, the remark following it, together with Corollary 2.3 and Corollary 3.8 give an algebraic characterization of Dehn twists about separating curves on nonorientable surfaces provided that $g \geq 7$.

Theorem 4.1. Let $g \geq 5, k \geq 0$ be integers $f$ an mapping class in $M(S)$ such that $f^N = t_c^m$ for some integers $m > 0$, $N$ and a nontrivial separating simple closed curve $c$ on $S = \Sigma^k_g$. If Lemma 3.9 is applicable, which is always the case if $g \geq 7$, then $f = t_c$ if and only if

a) $f$ is a primitive element of $C_{M(S^c)}(K)$ if $c$ does not separate $\Sigma_{1,1}$

b) $f = t_c = (t_a t_b)^6$ if $c$ separates $\Sigma_{1,1}$ or $\Sigma_{1,1}^1$ (see Figure 7);

c) $f = t_c = m^2$ if $c$ separates $\Sigma_{1,1}^1$ (see Figure 7).

Moreover, the topological type of $c$ is determined completely via Lemma 3.9.

If Lemma 3.9 is not applicable then $f = t_c$ if and only if $f$ primitive. If further $g = 6$ then each component of $S^c$ is a nonorientable surface of genus three and the number punctures is $r - 2$, where $r$ is the biggest integer such that the tree below can be embedded in $C_{M(S)}(t_c)$. Finally, if $g = 5$ then one of the components is again a nonorientable surface of genus three. The other component is orientable if and only if $C_{M(S)}(t_c)$ contains a Dehn twist about a characteristic curve. The number punctures in each component can be determined the same way as in genus six.

Proof. As we mentioned in the paragraph above the theorem, if $g \geq 7$ we are done. If $g = 5$ or 6 and the hypothesis of Lemma 3.9 is still satisfied then again there is nothing to do. Hence, we are left with the cases $g = 5$ or 6 and one of the components of $S^c$ is nonorientable of genus three. In other words, we need to characterize these two cases algebraically. In case $g = 6$, via Euler characteristic considerations, the maximal trees below contained in $C_{M(S)}(t_c)$ determines the number of punctures in each component. Indeed, it is $r - 2$ if $r$ is as in the theorem.

If $g = 5$ and $C_{M(S)}(t_c)$ does not contain a Dehn twist about a characteristic curve then the second component is a punctured Klein bottle. We know that any two two-sided circles in the second component intersect geometrically
in at least two points. In other words, this component does not contain any chain of length larger than one. However, the genus three component clearly contains a chain of length three. Hence, we can distinguish the topological components of $S^c$ algebraically. The number of punctures in the genus three component is determined exactly as in the genus six case (so it is $r - 2$). Finally, the number of punctures in the other component is then $k - r + 2$. □

Remark 4.2. Theorem 2.1 of [4] characterizes algebraically the Dehn twists about nonseparating curves in orientable surfaces of genus $g > 0$ (except closed surfaces of genus two). Clearly, some appropriate versions of Lemma 3.9 and Theorem 4.1 for orientable surfaces provide an algebraic characterization we mentioned in Remark 2.4. In fact, the versions for orientable surfaces will be even easier since a large portion of our efforts has been spent to distinguish algebraically an annulus with one puncture from a Möbius band.

5. Some Applications

The subgroup $T$ of the mapping class group generated by all Dehn twists about two-sided simple closed curves, is called the twist subgroup. It is known that this subgroup is of index $2^{k+1}k!$ in $\text{Mod}(N^k_g)$, provided that $g \geq 3$ ([7], [10]). Now, we state the following corollary of Theorem 4.1.

**Corollary 5.1.** For $g \geq 5$, let $\Phi: \text{M}(N^k_g) \to \text{M}(N^k_g)$ be an automorphism and $T \leq \text{M}(N^k_g)$ be the twist subgroup. If $t_c \in T$ is a Dehn twist then so is $\Phi(t_c)$. Moreover, the Dehn twists $t_c$ and $\Phi(t_c)$ are topologically equivalent. In other words, there is a homeomorphism $f: N^k_g \to N^k_g$ such that $\Phi(t_c) = t_{f(c)}$. In particular, the twist subgroup is a characteristic subgroup of $\text{M}(N^k_g)$.

The subgroup $\text{PMod}^+(N^k_g)$ has index $2^k$ in $\text{PMod}(N^k_g)$ and contains the twist subgroup $T$ as a subgroup of index two ([10]) provided that $g \geq 3$.

**Lemma 5.2.** The subgroup $\text{PMod}^+(N^k_g)$ is characteristic in $\text{Mod}(N^k_g)$ and $\text{PMod}(N^k_g)$, provided that $g \geq 5$.

**Proof.** We know that the twist subgroup is characteristic in all the three groups in the statement of the lemma. On the other hand, $\text{PMod}^+(N^k_g)$ is generated by the twist subgroup and the set of all $Y$-homeomorphisms,
each of which is supported inside Klein bottles with one boundary component, so that both components of the surface cut along this boundary curve are nonorientable. It is easy to see that this set is also characteristic (cf. Theorem 3.9 and Theorem 3.10 of [1]). Indeed, one can see this directly as follows: Note that, in the Klein bottle with the boundary circle \( e \), a \( Y \)-homeomorphism represents a mapping class, say \( \tau \), which is characterized as a mapping class that is not a Dehn twist but \( \tau^2 = t_e \). It follows that \( \text{PMod}^+(N_g^k) \) is characteristic in \( \text{Mod}(N_g^k) \) and \( \text{PMod}(N_g^k) \).

□

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