Applications of the potential algebras of the two-dimensional Dirac-like operators

Vít Jakubský
Nuclear Physics Institute, Řež near Prague, 25068, Czech Republic
E-mail: v.jakubsky@gmail.com

Abstract

Potential algebras can be used effectively in the analysis of the quantum systems. In the article, we focus on the systems described by a separable, $2 \times 2$ matrix Hamiltonian of the first order in derivatives. We find integrals of motion of the Hamiltonian that close centrally extended $so(3)$, $so(2, 1)$ or oscillator algebra. The algebraic framework is used in construction of physically interesting solvable models described by the $(2 + 1)$ dimensional Dirac equation. It is applied in description of open-cage fullerenes where the energies and wave functions of low-energy charge-carriers are computed. The potential algebras are also used in construction of shape-invariant, one-dimensional Dirac operators. We show that shape-invariance of the first-order operators is associated with the $N = 4$ nonlinear supersymmetry which is represented by both local and nonlocal supercharges. The relation to the shape-invariant non-relativistic systems is discussed as well.

1 Introduction

Exactly solvable models play an exceptional role in physics. They are simple enough to be solvable, yet sufficiently complicated to grab the essence of physical reality. In quantum mechanics, exact solvability usually means that the eigenfunctions and eigenvalues of the Hamiltonian can be found explicitly. They can be computed analytically by solving directly corresponding differential equations.

Alternatively, they can be obtained in algebraic manner with the use of the integrals of motion; when the symmetries of the quantum system form Lie algebra, the spectrum and eigenstates of the energy operator can be deduced from the analysis of the admissible unitary representations [1], [2]. The symmetry operators preserve domain of the Hamiltonian by transforming one physical state into another. When they are time-dependent, they do not commute with the energy operator and, hence, they do not preserve energy. The associated algebra of the integrals of motion is called dynamical or spectrum generating; see [3], [4]. When the integrals of motion commute with the Hamiltonian, they rather reflect spectral degeneracy of the system. The algebraic structure, where the Hamiltonian plays the role of central element, is denoted as potential algebra. In the current article, we will focus on the analysis and applications of the potential algebras of the relativistic quantum systems described by the $(2 + 1)$ dimensional Dirac equation.

The low-dimensional Dirac equation appears in surprising variety of physical settings that are studied both in quantum field theory and in the condensed matter physics. Let us mention the $(1 + 1)$ dimensional variant of the famous Nambu-Jona-Lasinio (chiral Gross-Neveu) model [5], [6], [7], analysis of fractionally charged solitons [8], [9] as well as description of inhomogeneous superconductors [10], carbon nanotubes [11], [12], or linear molecules like polyacetylene [13]. The $(2 + 1)$ dimensional massless Dirac Hamiltonian appears in the low-energy approximation of dynamics of charge carriers in graphene and in related carbon nanostructures [14], [15], [16], [17].
The work is organized as follows: in the next section, the potential algebras are constructed for a separable Dirac-like operator with the use of a specific ansatz for both the ladder operators and for the structure of the algebra. Relevant aspects of the representations based on the lowest/highest weight vectors are discussed in more detail, as well as the application of the algebraic structure in description of Dirac fermions in the curved space.

The results are used directly in the third section, where solvable models describing low-energy charge carriers on the surface of open-cage fullerenes are analyzed. Two configurations are considered, with one and two holes in the surface of the crystal.

In the fourth section, we apply the potential algebras in construction of one-dimensional shape-invariant systems. First, we focus on the shape-invariance of one-dimensional Dirac Hamiltonians and of such system which possesses shape-invariance. We make a nonrelativistic shape-invariant systems whose supercharges are shape-invariant themselves. We consider superalgebraic structure associated with the supercharges. We also discuss how is the relation of this framework with the known, non-relativistic shape-invariant Hamiltonians. In this context, we consider superalgebraic structure associated with the nonrelativistic shape-invariant systems whose supercharges are shape-invariant themselves. We make a comment on the nonrelativistic systems with position dependent mass and present an illustrative example of such system which possesses shape-invariance.

In the last section, we discuss briefly two specific two-dimensional systems; we make few comments on Dirac oscillator and its dynamical symmetries. Additionally, we show that the so(2, 1) potential algebra can be used for analysis of (1 + 1)-dimensional quantum system whose metric can be identified with the (restricted) BTZ black hole metric. We conclude with short discussion of the results and outlook to possible future research.

2 Potential algebras of two-dimensional Dirac operator

Let us consider the operator $h_D$ which is given in terms of the Pauli matrices $\sigma_a$ and real functions $g_2$ and $g_3$,

$$h_D = i\sigma_1 \partial_{x_1} + \sigma_2 (-ig_2 \partial_{x_2} + g_3) + M\sigma_3, \quad g_a = g_a(x_1), \quad a = 2, 3.$$ \(M\) is a real constant. We leave the domain of $h_D$ as well as the range of the real variables $x_1$ and $x_2$ unspecified at the moment. The function $g_2$ should not be identically zero. If not stated otherwise, the coordinates $x_1$ and $x_2$ are considered to be space-like. The operator (1) can play the role of the Hamiltonian of the (2 + 1)-dimensional Dirac equation that governs dynamics of many physically interesting models mentioned in the previous section. The vector potential $g_3 = g_3(x_1)$ with real coupling constant $c_3$ is associated with an external magnetic field. The operator is manifestly Hermitian with respect to the standard scalar product.

The system represented by (1) has integral of motion $J_3$,

$$J_3 = -i\partial_{x_2}, \quad [h_D, J_3] = 0.$$ \(J_3\) = \pm J_{\pm}, \quad [J_-, J_+] = 2c_1 J_3 + c_2, \quad [J_\pm, h_D] = 0,$$ where $c_1$ and $c_2$ are real numbers the value of which will be specified later in the text. Let us notice that for $c_1 \neq 0$ the non-vanishing $c_2$ just implies a constant shift in definition of the angular operator.

Let us make the ansatz in the form $J_+ = e^{ix_2} \left( \sum_{k=1}^2 (A_{k,1} + \sigma_3 A_{k,2}) \partial_{x_k} + C_1 + \sigma_3 C_2 \right)$ where the coefficients $A_{k,a}$ and $C_a$ ($a, k = 1, 2$) depend on $x_1$ only. The operator $J_-$ is defined as $J_- = J_1^\dagger$. Inserting the ansatz into (2) and comparing the coefficients of corresponding derivatives, we find that the coefficients of $J_\pm$ can be expressed in terms of $g_2$ and $g_3$ as

$$J_\pm = i e^{\pm ix_2} \left( \partial_{x_1} + \frac{g_2 (\pm J_3 + \frac{1}{2})}{g_2} \pm \frac{g_3}{2g_2} \partial_{x_2} \right).$$
It suggests that the function $g_2$ should be node-less to avoid singularities in definition of these operators. Additionally, the functions $g_2$ and $g_3$ have to solve the following set of differential equations,

$$
\begin{align*}
  g_4^2 + c_1 g_2^2 &= g_2^2, \\
  g_3^2 + g_2^2 &= g_2 g_2', \\
  g_2 g_3 + \frac{c_2}{2} g_2^3 &= g_3 g_2'.
\end{align*}
$$

Quick inspection suggests that the system can be reduced to two equations as long as $g_3 = g_2$. However, we will omit this solution; looking at (1), it would just shift the operator $J_3$ by a constant factor. The equation (3) eliminates constant (nonzero) solution for $g_2$.

We can find the following nontrivial solutions of the system,

$$
\begin{align*}
  g_2 &= \begin{cases} 
  \sinh^{-1} x_1, & \text{for } c_1 = 1, c_2 = 0, \\
  \cos^{-1} x_1, & \text{and for } c_1 = -1, c_2 = 0, \\
  x_1^{-1}, & \text{for } c_1 = 0, c_2 = -4c_3,
\end{cases}
\end{align*}
$$

and

$$
\begin{align*}
  g_3 &= \begin{cases} 
  c_3 \coth x_1, & \text{for } c_1 = 1, c_2 = 0, \\
  c_3 \tan x_1, & \text{for } c_1 = -1, c_2 = 0, \\
  c_3 x_1, & \text{for } c_1 = 0, c_2 = -4c_3,
\end{cases}
\end{align*}
$$

where $c_3$ is a real coupling constant. To avoid singularities, the variable $x_1$ can acquire only nonzero values for $c_1 \in \{1, 0\}$ and $x_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for $c_1 = -1$. The equations (4)-(6) are invariant with respect to reflections $x_1 \rightarrow -x_1$ as well as they possess translational invariance. It means that the functions (7) solve the system even after a constant shift and reflection of the coordinate $x_1$. It is possible to write the potential term $g_3$ as the function of $g_2$ and $c_1$,

$$
\begin{align*}
  g_3^2 &= c_3^2 (g_2^2 + c_1), & \text{for } c_1 \neq 0, \\
  g_3^2 &= c_3^2 g_2^{-2}, & \text{for } c_1 = 0.
\end{align*}
$$

The constants $c_1$ and $c_3$ fix the structure of the algebra (2). For $c_1 \neq 0$, it can be identified either as $so(3)$ ($c_1 = -1$) or $so(2,1)$ ($c_1 = 1$), while the oscillator algebra is restored for $c_1 = 0$. $h_D$ commutes with both $J_3$ and $J_\pm$, and, hence, it is the central element of the following algebra

$$
\begin{align*}
  [J_3, J_\pm] &= \pm J_\pm, \\
  [J_-, J_+] &= \begin{cases} 
  2c_1 J_3, & \text{for } c_1 \neq 0, \\
  -4c_3, & \text{for } c_1 = 0,
\end{cases}
\end{align*}
$$

and

$$
[h_D, J_\pm] = [h_D, J_3] = 0.
$$

It is worth noticing that nonrelativistic systems with similar algebraic background were classified as superintegrable in [18]. The representation space of (5)-(9) is spanned by the eigenvectors of $h_D$. It can be decomposed into subspaces where $h_D$ acquires constant value and where the algebra (5)-(9) is irreducible. These subspaces can have either finite or infinite dimension, dependently on the type of the algebra.

The square of (1) can be written as a second-order polynomial in the generators of (3). When $c_1 \neq 0$, it coincides up to an additive constant with the standard Casimir operator $C = J_+ J_- - c_1 J_3 (J_3 - 1) = J_- J_+ - c_1 J_3 (J_3 + 1)$ of the algebra $so(3)$ or $so(2,1)$. When $c_1 = 0$, the algebra is not semi-simple (Killing form is degenerate) and the standard (quadratic) Casimir operator does not exist. There holds

$$
\begin{align*}
  h_D^2 - M^2 &= \begin{cases} 
  J_+ J_- - c_1 \left( (J_3 - \frac{1}{2})^2 - c_3^2 \right) = J_- J_+ - c_1 \left( (J_3 + \frac{1}{2})^2 - c_3^2 \right), & \text{for } c_1 \neq 0, \\
  J_+ J_- + 4c_3 (J_3 - \frac{1}{2}) = J_- J_+ + 4c_3 (J_3 + \frac{1}{2}), & \text{for } c_1 = 0.
\end{cases}
\end{align*}
$$

For any of the solutions (7), the operator $h_D$ commutes with the linear operator $P = \sigma_3 R_{x_1}$ where $R_{x_1}$ is defined as $R_{x_1} x_1 R_{x_1} = -x_1$ and $R_{x_1} x_2 R_{x_1} = x_2$. The ladder operators $J_\pm$ together with $J_3$ satisfy the following relations,

$$
\begin{align*}
  P J_\pm &= -J_\pm, \\
  [h_D, P] &= [J_3, P] = 0.
\end{align*}
$$

3
We will utilize these relations in the fourth section in the context of shape-invariance. When $M = 0$, the Hamiltonian $h_D$ also commutes with $\mathbf{R} = R_x R_y \sigma_1$. The reflection operators $R_x$ and $R_y$ are defined as $R_x f(x_1, x_2, c_3) = f(x_1, -x_2, c_3)$ and $R_y f(x_1, x_2, c_3) = f(x_1, x_2, -c_3)$. For the generators of the potential algebra, there holds

$$\mathbf{R} J_- \mathbf{R} = J_+, \quad \mathbf{R} J_3 \mathbf{R} = -J_3.$$  

In our analysis, we will focus on the representations that can be constructed from the lowest (highest) weight vectors, the zero-modes of ladder operator $J_- \ (J_+).$ These vectors, denoted as $\psi_m^\pm \ (\psi_m^-)$, can be fixed as the mutual eigenstates of $h_D$ and $J_3$,

$$J_\pm \psi_m^\pm = 0, \quad J_3 \psi_m^\pm = m \psi_m^\pm, \quad h_D \psi_m^\pm = E_m^\pm \psi_m^\pm. \quad (12)$$

The energies $E_m^\pm$ can be found with the use of the relations (10),

$$(E_m^\pm)^2 \begin{cases} -c_1 \left( (m \pm \frac{1}{2})^2 - c_3^2 \right) + M^2 & \text{for } c_1 \neq 0, \\ 4c_3 \left( m \pm \frac{1}{2} \right) + M^2 & \text{for } c_1 = 0. \end{cases} \quad (13)$$

To keep the energies real, there must hold $|m \pm \frac{1}{2}| \geq c_3$ for $c_1 = -1$. For $c_1 = 1$, the values of $m$ are constrained by the strength of the coupling parameter, $|m \pm \frac{1}{2}| \leq c_3$.

The other eigenstates of $h_D$ corresponding to the same energy can be found by repeated action of the ladder operator $J_+ \ (J_-)$ on the vectors $\psi_m^\pm \ (\psi_m^-)$. These eigenstates then establish irreducible representations of $\mathfrak{so}(3)$ that are specified by the energies $E_m^\pm$ of the corresponding lowest (highest) weight vector. These representations are usually associated with the bound states of the quantum system, see e.g. [20].

The unitary representations of $\mathfrak{so}(3)$ are finite dimensional. Consequently, the lowest weight vector, let us fix it as $\psi^-_{-m}$, has to be annihilated by specific power of the ladder operator $J_-$. There holds $(J_+)^d \psi^-_{-m} = 0$ where $d$ denotes dimension of the corresponding representation. The quantum number $m$ has to be either integer or semi-integer; we have $\psi^\pm_{-m} \sim (J_+)^{d-1} \psi^-_{-m}$ so that $J_3 J_+^{d-1} \psi^-_{-m} = m J_+^{d-1} \psi^-_{-m}$. Additionally, we can use [8] and write $J_3 J_+^{d-1} \psi^-_{-m} = (-m + d - 1) J_+^{d-1} \psi^-_{-m}$. Combining these relations together, there must hold $-2m + d - 1 = 0$. Since $d$ is a positive integer, we get that $m$ has to be non-negative integer or semi-integer.

The admissible values of the coupling constant $c_3$ depend on $m$; when $m$ acquires integer (half-integer) values, $c_3$ has to be half-integer (integer). We refer to Appendix A for details of the proof. Let us notice in this context that the representations of odd dimension for two different half-integer values of $c_3$ were considered in [21]. Representations of $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$ were also discussed in [22] where they were obtained analytically with the use of the master equation.

For the purposes of the forthcoming section, let us discuss shortly how the results can be used in description of the spin–1/2 particle living in the two-dimensional curved space. Let us consider the space where the metric tensor acquires diagonal form and the non-vanishing components depend on one coordinate only,

$$g_{\mu\nu} = \begin{pmatrix} g_{11}(x_1) & 0 \\ 0 & g_{22}(x_2) \end{pmatrix}. \quad (14)$$

In the most of the text, we will deal with the systems where both $x_1$ and $x_2$ are considered to be space-like $(g_{11} > 0, g_{22} > 0)$, i.e. the metric tensor corresponds to a curved surface embedded into three-dimensional euclidean space. Quantum settings of this kind appear frequently in the analysis of carbon nanostructures where deformations of graphene crystal are considered [23]. The Dirac Hamiltonian can be written in the following non-covariant but manifestly Hermitian form (see Appendix B for details),

$$h_D = i\sigma_1 \frac{1}{g_{11}^{1/4}} \frac{1}{g_{11}^{1/4}} \partial_x + \sigma_2 \frac{J_3}{\sqrt{g_{22}}} + g_3 \sigma_2 + M \sigma_3.$$\footnote{Similar operator that switches the sign of coupling constant was employed in [19] in the analysis of quantum systems with nonlinear supersymmetry.}
This operator is manifestly Hermitian with respect to the standard scalar product.

When we compare (15) with (1), it differs by the term containing $\partial_x$. However, we can reduced (15) into (1) by an appropriate change of coordinates. Let us set $z = z(x_1)$ such that $z'(x_1) = \sqrt{g_{11}}$. Rewriting (15) in the new coordinate $z$, the multiplicative factor $g_{11}^{1/2}$ in front of the derivative is eliminated. Then we make the similarity transformation $g_{11}^{-1/4}(x_1(z)) h_D(\partial_z, z, \partial_x) g_{11}^{1/4}(x_1(z))$ which brings the operator back to the manifestly Hermitian form. In this way, the term $\frac{1}{\sqrt{g_{11}}} \partial_x, \frac{1}{\sqrt{g_{11}}} h_D$ in (15) is effectively replaced by $\partial_z$. This transformed operator can be then identified with (1) for $g_2 = g_{22}^{-1/2}(x_1(z))$ and $g_3 = g_3(x_1(z))$.

Using the inverse transformation, we can derive integrals of motion of (15) from (3) that close the Lie algebra \([3, 9]\)\(^2\). They are of the following form

$$J_\pm = i e^{\pm i x_2} \left( \frac{1}{g_{11}} \partial_{x_1} + g_{12}^{-1} \right) - \frac{g_{12}'}{2g_{22}\sqrt{g_{11}}} \left( \pm J_3 + \frac{1}{2} \right) \pm \frac{\sqrt{g_{22}} g_3'}{\sqrt{g_{11}}} \mp \frac{1}{2g_{22}} \sigma_3, \quad J_3 = -i \partial_{x_2},$$

provided that $g_{11}, g_{22}$ and the external potential $g_3$ satisfy the following equations

$$g_{11} = \frac{1}{4} \frac{(g_{22}')^2}{g_{22}(1+c_1 g_{22})}, \quad g_3 = \begin{cases} c_3 \sqrt{\frac{1}{g_{22}} + c_1}, & \text{for } c_1 \neq 0, \\ c_3 g_{22}^{1/2}, & \text{for } c_1 = 0. \end{cases}$$

We will use the formulas (15), (17) and (17) extensively in the forthcoming text.

Concluding the section, let us notice that the operator (15) can describe low-energy charge carriers in two-dimensional crystals with hexagonal lattice where the (nontrivial) geometry of the crystal surface is encoded in $g_{\mu
u}$. In particular, we refer to graphene and boron-nitride crystals for $M = 0$ and $M \neq 0$ respectively, see (15). In this context, the generally non-constant coefficients of $\partial_{x_1}$ and $\partial_{x_2}$ in (15) can be interpreted as the space-dependent modulation of Fermi velocity of massless Dirac particle in graphene induced by the strains and ripples of the crystal [24]. The non-vanishing vector potential represented by $g_3$ can be induced by external magnetic fields or by defects of the crystal lattice [21, 17].

## 3 Solvable models of open-cage fullerenes

Fullerenes are spherical molecules made of carbon atoms. The spherical shape is due to twelve pentagons that are inserted into hexagonal carbon lattice. The most famous example is the $C_{60}$, “Buckminsterfullerene” [25], which can be classified as a truncated icosahedron with 60 vertices and 32 faces. Fixing the number of pentagons in the lattice while increasing the number of hexagons, we can get not only bigger fullerenes like $C_{240}$ or $C_{540}$, but also objects of more complicated shapes, e.g. capped nanotubes, elliptic fullerenes etc.. We will be interested just in the systems where the surface of the crystal can be approximated by the spherical geometry.

\(^2\)Generic form of the operators that can be obtained from (1) and (3) by the described transformation is

$$h_D = i \sigma_1 \left( g_1 \partial_{x_1} + \frac{g_1'}{2} \right) + \sigma_2 (-ig_2 \partial_{x_2} + g_1) + M \sigma_3, \quad g_a = g_a(x_1), \quad a = 1, 2, 3,$$

and

$$J_\pm = i e^{\pm i x_2} \left( g_1 \partial_{x_1} + \frac{g_1'}{2} + \frac{g_1'g_3(\pm J_3 + \frac{1}{2})}{g_2} + \frac{g_1 g_3' + g_2 g_3}{2} \right).$$

The functions $g_1, g_2$ and $g_3$ satisfy

$$g_2^2 = \frac{g_1^2 + c_1 g_3^2}{(g_2')^2}, \quad g_3^2 = \begin{cases} c_3^3 (g_2^2 + c_1), & \text{for } c_1 \neq 0, \\ c_3^3 g_2^2, & \text{for } c_1 = 0. \end{cases}$$

Then these operators close \([3, 9]\).
In quantum chemistry, the situation is considered where small enough atoms or molecules (e.g. of hydrogen) are inserted inside the fullerenes. During the process called “chemical surgery”, some bonds between the carbon atoms of the fullerene are broken, making the hole into the surface. The alien atoms are then inserted into the opened cage and the hole is closed again [26]. In this section, we will focus on the electronic properties of the open-cage fullerenes with one or two holes in the surface. We will construct exactly solvable models that can serve as the low-energy approximation of these systems.

3.1 Spectrum of fullerenes

First, let us go briefly through the analysis of the electronic properties of the (unopened) fullerenes. The low-energy excitations of charge carriers in the crystal can be investigated efficiently within the framework of the Dirac equation on the spherical surface [21]. The surface embedded into three-dimensional space can be parametrized in spherical coordinates as \(x = \sin x_1 \cos x_2, \ y = \sin x_1 \sin x_2, \ z = \cos x_1\) where \(x_1 \in (0, \pi)\) and \(x_2 = (0, 2\pi)\). The associated metric tensor is explicitly

\[g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 x_1 \end{pmatrix}.\]

The entries of the metric tensor satisfy (17) for \(c_1 = -1\). Employing the formula (15), we can write down immediately the Dirac Hamiltonian corresponding to the massless, spin–1/2 particle on the sphere,

\[h_D = \imath \sigma_1 \partial_{x_1} + \sigma_2 \frac{J_3}{\sin x_1} + c_3 \sigma_2 \cot x_1, \ \ x_1 \in (0, \pi).\]  

(18)

The potential term corresponds to Dirac monopole that is situated in the center of the sphere. It occurs due to the topological defects, twelve pentagons, that have to be inserted into the hexagonal lattice of graphene to close the spherical surface [16], [17], [21]. The actual value of the coupling constant is \(c_3 = \frac{\pi}{3}\) where \(n\) corresponds to the number of conical defects in the crystal. As \(n = 12\) for fullerenes, we get \(c_3 = \frac{3}{2},\) see [16], [21].

We will not write down the explicit form of \(J_\pm\), it can be extracted easily from (17). The lowest weight vectors can be found explicitly by solving (12). They are

\[
\psi_m^- = e^{i m x_2} w_F^{-c_3} (\sin x_1)^{-m+\frac{1}{2}} \left( \begin{array}{c} \sqrt{m - \frac{1}{2} + c_3 w_F^{-\frac{1}{2}}} \\ -\sqrt{-m + \frac{1}{2} + c_3 w_F^{-\frac{1}{2}}} \end{array} \right), \quad w_F = \tan \frac{x_1}{2},
\]

(19)

\[
h_D \psi_m^- = \sqrt{\left( m - \frac{1}{2} \right)^2 - c_3^2} \psi_m^-.
\]

(20)

We require the eigenfunctions of (15) to be square integrable and vanishing at \(x_1 = 0, \pi\). The vectors (19) are physically acceptable as long as \(m \leq -|c_3| + \frac{1}{2}\). In that case, they are regular and correspond to real energies, see the restriction below (13). Degeneracy of the zero-energy level is equal to the dimension of the representation of \(so(3)\) spanned by zero modes of the Hamiltonian. For \(c_3 = \frac{1}{2}\), the zero-energy has triple degeneracy.

For purposes of the forthcoming analysis, it is convenient to define transformation \(G_F\),

\[G_F = w_F^{c_3} \sqrt{\sin x_1} \begin{pmatrix} w_F^{1/2} & 0 \\ 0 & w_F^{-1/2} \end{pmatrix} \sigma_1,
\]

and rewrite the lowest weight vector in the factorized form

\[
\psi_m^- = e^{i m x_2} y^{-m} G_F \left( \begin{array}{c} \sqrt{-m + \frac{1}{2} + c_3 w_F^{k_a}} \\ \sqrt{m - \frac{1}{2} + c_3 w_F^{k_b}} \end{array} \right), \quad y = \sin x_1, \quad w_F(y) = \begin{cases} 1 - \sqrt{1 - y^2}, & x_1 < \pi/2, \\
\frac{1 + \sqrt{1 - y^2}}{y}, & x_1 \geq \pi/2, \end{cases}
\]

(21)
With the use of this notation, the ladder operator can be rewritten as
\[
G^{-1} J_+ G = i e^{ix^2} (\partial_{x_1} - \cot x_1 J_3) = i e^{ix^2} (1 - y \, w_F(y)) \left( \partial_y - \frac{J_3}{y} \right). 
\]  

It would be necessary to check that a multiple action of $J_+$ on $\psi_m^-$ does not violate regularity of the wave functions. However, we can skip this task as the considered wave functions are known to be regular and based on the Jacobi polynomials, see [22, 27]. It means that the action of (23) on (21) preserves the required boundary conditions. We will use this fact in the analysis of the open-cage fullerenes.

### 3.2 Open-cage fullerene with one hole

Let us consider the fullerene molecule with one hole carved into its surface (see Fig.1). We suppose that the twelve pentagons allowing the crystal to close the spherical shell are still present in the lattice. To meet this requirement, we can suppose that the model describes larger molecules like $C_{240}$ where one or few hexagons can be extracted from the crystal without affecting the pentagons. The considered spherical surface can be parametrized as
\[
x = \sin (\pi \, dn(x_1, k)) \cos x_2, \quad y = \sin (\pi \, dn(x_1, k)) \sin x_2, \quad z = \cos (\pi \, dn(x_1, k)) ,
\]
where $x_1 \in (0, K)$, $x_2 = (-\pi, \pi)$ and $k \in (0, 1)$. The components of the associated metric tensor
\[
g_{\mu \nu} = \begin{pmatrix} (\pi k^2 \, cn(x_1,k) \, sn(x_1,k))^2 & 0 \\ 0 & 0 \\ \sin^2(\pi \, dn(x_1,k)) \end{pmatrix}
\]
satisfy (17) for $c_1 = -1$.

The functions $dn(x_1, k)$, $cn(x_1, k)$ and $sn(x_1, k)$ are Jacobi elliptic functions and $K = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{-1} d\theta$ is the complete elliptic integral of the first kind with $0 \leq k \leq 1$. Let us briefly refresh relevant properties of these functions. $dn(x_1, k)$ is periodic with periodicity $2K$ while $cn(x_1, k)$ and $sn(x_1, k)$ have periodicity $4K$. There holds $0 < dn(x_1, k) \leq 1$ for all real $x_1$ and all values of the modular parameter $k \in (0, 1)$. The functions $cn(x_1, k)$ and $sn(x_1, k)$ can be understood as a generalization of trigonometric functions; they satisfy the relation $cn(x_1, k)^2 + sn(x_1, k)^2 = 1$. There holds $cn(x_1, 0) = \cos x_1$ and $sn(x_1, 0) = \sin x_1$, $dn(x_1, 0) = 1$. When $k \to 1$, the period of the Jacobi elliptic functions tends to infinity. There holds $sn(x_1, 1) = \tan h x_1$, whereas $sn(x_1, 1) = dn(x_1, 1) = \sech x_1$. In the cases where the actual value of the modular parameter is not informative or it is clear from the context, we will use the shortened notation $sn(x_1) \equiv sn(x_1, k)$, $cn(x_1) \equiv cn(x_1, k)$ and $dn(x_1) \equiv dn(x_1, k)$.

Dirac Hamiltonian on the considered surface acquires the following (non-covariant but manifestly Hermitian) explicit form
\[
h_D = i \sigma_1 \frac{1}{k^2 \, \pi \, cn x_1 \, sn x_1} \left( \partial_{x_1} + \frac{dn x_1 (sn^2 x_1 - cn^2 x_1)}{2 \, cn x_1 \, sn x_1} \right) + \sigma_2 \frac{J_3}{\sin (\pi \, dn x_1)} + c_3 \sigma_2 \cot (\pi \, dn x_1),
\]
where $x_1 \in (0, K)$. Likewise in (18), the potential term corresponds to the field generated by Dirac monopole. We will require the eigenstates of $h_D$ to be regular and vanishing at the edge of the hole (i.e. at $x_1 = K$). The lowest weight vectors $\psi_m^-$ annihilated by $J_-$ are
\[
\psi_m^- = e^{imx_2} \frac{\sqrt{cn x_1 \, sn x_1}}{w_f^{3/2}} (\sin (\pi \, dn x_1))^{-m + \frac{1}{2}} \left( \frac{\sqrt{c_3 + \frac{1}{2} - m \, w_f^{1/2}}}{\sqrt{c_3 - \frac{1}{2} + m \, w_f^{-1/2}}} \right),
\]
where $w_f = \tan \frac{\pi \, dn x_1}{2}$ and $h_D \psi_m^- = \sqrt{(m - 1/2)^2 - c_3^2} \psi_m^-$. 

7
The wave functions (25) comply with the required boundary condition \((cn(K) = 0)\). However, it is not clear whether the multiple action of \(J_+\) on (25) does not violate the boundary conditions. We can analyze this point using the close analogy with the model of fullerenes. Let us define the transformation \(G_I\) as
\[
G_I = \sqrt{cn x_1 sn x_1 \sin(\pi dn(x_1))} w_I^{c_3} \begin{pmatrix} w_I^{-1/2} & 0 \\ 0 & w_I^{1/2} \end{pmatrix}.
\]
The lowest weight vector can be then written as
\[
\psi_m^- = e^{imx_2 y^{-m}G_I} \left( \sqrt{c_3 + \frac{1}{2} - mw_I^{k_w}} \right), \quad y = \sin(\pi dn x_1), \quad w_I(y) = \begin{cases} 1 - \sqrt{1 - \frac{y^2}{c_3}}, & dn x_1 \leq 1/2, \\ 1 + \sqrt{1 - \frac{y^2}{c_3}}, & dn x_1 > 1/2, \end{cases}
\]
where the constants \(k_u\) and \(k_d\) are those in (22). Performing the “gauge” transformation of \(J_+\), we get
\[
G_I^{-1} J_+ G_I = i e^{ix_2} \left( \frac{1}{k^2 \pi cn x_1 sn x_1} \partial_{x_1} + \cot \pi dn x_1 J_3 \right) = -i e^{ix_2} (1 - y w_I(y)) \left( \partial_y - \frac{J_3}{y} \right).
\]
The latter operator is identical with (23). The vector (25) coincides with (21) up to the factors \(G_F\) and \(G_I\) and up to the sign of the upper component. Hence, likewise in the case of fullerenes, the multiple action of \(J_+\) on \(\psi_m^-\) will keep regularity of the wave functions, i.e. only positive powers of \(y\) will emerge.

### 3.3 Open-cage fullerene with two holes

Similar results can be obtained for spin-1/2 particle on the spherical surface with two holes centered in the poles, see Fig.1. The surface can be parametrized as \(x = dn x_1 \cos x_2\), \(y = dn x_1 \sin x_2\), \(z = k sn x_1\), where \(x_1 \in (-K, K)\), \(x_2 \in (-\pi, \pi)\) and \(k \in (0, 1)\). The metric tensor reads this time
\[
g_{\mu \nu} = \begin{pmatrix} (kn x_1)^2 & 0 \\ 0 & dn^2 x_1 \end{pmatrix}.
\]
The explicit form of the Hamiltonian can be written as follows
\[
h_D = i \sigma_1 \frac{d x_1 + dn x_1 sn x_1}{2 cn x_1} + \sigma_2 \frac{J_3}{dn x_1} + c_3 \sigma_2 \frac{kn x_1}{dn x_1}, \quad x_1 \in (-K, K).
\]
It complies with the required boundary conditions since \( cn(K) = cn(-K) = 0 \). In order to show that application of \( J_+ \) on these vectors does not violate the required properties, let us define the matrix

\[
G_{II} = \sqrt{cn(x_1)dn(x_1)} \frac{w_{II}^{3}}{w_{II}^{1/2}} \left( \begin{array}{cc} w_{II}^{-1/2} & 0 \\ 0 & w_{II}^{1/2} \end{array} \right).
\]

The lowest weight vector can be written in the following form

\[
\psi_m^- = e^{imx} y^{-m} G_{II} \left( \frac{\sqrt{c_3 + \frac{1}{2} - m w_{II}^{1/2}}}{\sqrt{c_3 - \frac{1}{2} + m w_{II}^{1/2}}} \right), \quad y = dn x_1, \quad w_{II}(y) = \begin{cases} \frac{1 - \sqrt{1 - y^2}}{y}, & sn x_1 > 0, \\ \frac{1 + \sqrt{1 - y^2}}{y}, & sn x_1 \leq 0. \end{cases}
\]

while the ladder operator is transformed into

\[
G_{II}^{-1} J_+ G_{II} = ie^{ix} \left( \frac{1}{k c n x_1} \partial_{x_1} + \frac{J_3 k s n x_1}{d n x_1} \right) = -ie^{ix} (1 - y w_{II}(y)) \left( \partial_y - \frac{J_3}{y} \right).
\]

Comparing (30) with (24) and (31) with (23), we can conclude that the repeated action of \( J_+ \) on \( \psi_m^- \) will preserve regularity of the wave functions.

The formulas (26), (27), (30) and (31) allow to find the required eigenstates of the corresponding Hamiltonian as well as energies which are given by (13) for \( c_1 = -1 \). Each of the systems has distinct topological genus as the number of the holes in the spherical surface varies. Despite this distinct topological nature, the models are spectrally identical and possess the same algebraic background. This follows from the fact that they can be related by the coordinate transformation discussed in the preceding section.

In the next section, we will discuss a different application of the potential algebras. We will consider one-dimensional, both relativistic and non-relativistic supersymmetric systems that possess shape invariance.

4 Shape invariant Dirac-like Hamiltonians and associated nonrelativistic systems

The concept of shape invariance was proposed by Gendenshtein [28]. It originated from the supersymmetric quantum mechanics dominated by Witten’s model [29, 30]. It attracted a lot of attention for being extremely useful in construction and analysis of exactly solvable models. Instead of going into the details of the concept, let us sketch briefly the main idea that will be relevant for our forthcoming discussion.

Let us suppose that the Hamiltonian \( H_m \), describing a (non-relativistic) quantum system, contains a potential term that depends explicitly on the coupling constant \( m \). The ground state \( \psi_m^{(0)} \) of \( H_m \) is known for any value of the coupling constant \( m \) and corresponds to the ground state energy \( E_m^{(0)} \).

\[
H_m \psi_m^{(0)} = E_m^{(0)} \psi_m^{(0)}.
\]

Suppose that the ground state energy is a decreasing function of the coupling constant, i.e. \( E_n^{(0)} > E_m^{(0)} \) for \( n < m \). Finally, let us take for granted that there exist operators \( A_m \) and \( A_m^\dagger \) which intertwine \( H_m \) with another Hamiltonian \( \tilde{H} \) that differs from \( H_m \) just by the value of the coupling constant, i.e. \( \tilde{H} = H_{m-\delta} \).

There holds

\[
A_m H_m = H_{m-\delta} A_m, \quad H_m A_m^\dagger = A_m^\dagger H_{m-\delta}.
\]

Then we say that the \( H_m \) is shape-invariant. For such setting, we can easily obtain excited states with energy \( E_m^{(n)} \). Indeed, it is sufficient to apply appropriate intertwining operators on the ground state \( \psi_m^{(0)} \) of \( H_{m-n\delta} \). We get the following formula,

\[
(H_m - E_m^{(n)}) A_m A_{m-\delta} \cdots A_{m-(n-1)\delta} \psi_m^{(0)} = 0, \quad E_m^{(n)} = E_m^{(0)}.
\]
In should be kept in mind that when a specific system is considered, one has to check that the intertwining operators do not violate boundary conditions prescribed for the eigenfunctions, that they preserve domains of the Hamiltonians etc..

As an example of non-relativistic shape-invariant setting, let us mention Poschl-Teller system described by the Hamiltonian \( H_m = -\partial_x^2 - \frac{m(m+1)}{\cosh^2 x_1} \). It has the ground state energy \( E_m^{(0)} = -m^2 \) and the ground state \( \psi_m^{(0)} = \cosh^{-m} x_1 \). There holds \( A_m H_m = H_{m-1} A_m \) where \( A_m = \partial_{x_1} + m \tanh x_1 \). It can be identified with \( (32) \) for \( \delta = 1 \). The explicit form and the energies of the excited bound state can be found directly by the formula \( (33) \). We have \( E_m^{(n)} = -(m-n)^2, n = 0, 1, \ldots, m-1 \).

The non-relativistic systems with shape-invariant potentials were studied extensively in the literature, see e.g. \( (31), (32) \). The systematic analysis was possible as the (second-order) Hamiltonians could be factorized in terms of the first order differential operators. The situation gets more complicated in case of Dirac operators which are itself of the first order, and, hence, cannot be factorized in the similar way.

Usually, the shape-invariance of Dirac Hamiltonian is treated such that the square of the operator is identified with a known, nonrelativistic, shape-invariant operator after a series of transformations \( (32) \). In this manner, exact eigenstates and eigenvalues of the original relativistic system can be found. Instead of following this way, we will consider directly the shape-invariance of the first-order Dirac-like Hamiltonians. In our approach, we will be inspired by Balantekin and Gangopadhyaya \( (34), (35) \). They found that the shape-invariance of (nonrelativistic) systems can be understood in terms of a higher dimensional Hamiltonian that possesses Lie algebra of integrals of motion. In this framework, the Hamiltonian \( H_m \) as well as the \( A_m \) (and \( A^*_m \)) correspond to an appropriate restriction of the higher-dimensional Hamiltonian and of the ladder operators to the subspaces with fixed value of angular momentum.

### 4.1 Shape-invariance of Dirac operators via \( N = 4 \) nonlinear supersymmetry

Let us consider restriction of the Hamiltonian \( (1) \) and of the ladder operators \( (43) \) to the subspaces where \( J_3 \) acquires fixed value. The operator \( h_D \) reduces to the one-dimensional Dirac Hamiltonian

\[
h_m = h_D|_{J_3=m} = e^{-imx_2} h_D e^{imx_2} = i\sigma_1 \partial_{x_1} + \sigma_2 g_2 m + g_3 \sigma_2 + M \sigma_3. \tag{34}\]

The ladder operators \( J_\pm \) exchange the subspaces where \( J_3 = m \) and \( J_3 = m+1 \). The restricted operators \( j_{\pm,m} \) read explicitly

\[
j^+_m = e^{-i(m+1)x_2} J_+ e^{imx_2} = i \left( \partial_{x_1} + \frac{g'_2(m + \frac{1}{2}) + g'_3}{g_2} - \frac{g_2}{2} \sigma_3 \right),
\]

\[
j^-_m = e^{-imx_2} J_- e^{i(m+1)x_2} = i \left( \partial_{x_1} - \frac{g'_2(m + \frac{1}{2}) + g'_3}{g_2} + \frac{g_2}{2} \sigma_3 \right). \tag{35}\]

Keeping in mind that \( [h_D, J_\pm] = 0 \), we can write down the following intertwining relations between the operators \( h_m \) and \( h_{m+1} \),

\[
h_m j^-_m = j^-_m h_{m+1}, \quad h_{m+1} j^+_m = j^+_m h_m. \tag{36}\]

Comparing with \( (32) \), they establish the shape-invariance of the one-dimensional Dirac operator \( h_m \) mediated by the operators \( j^\pm_m \). The operators \( j^\pm_m \) can be identified as the generalized matrix Darboux transformation. General properties of these transformations were discussed in \( (36) \) in detail. In particular, it was proved that the product \( j^-_m j^+_m \) or \( j^+_m j^-_m \) is proportional to the second-order polynomials in \( h_m \) or \( h_{m+1} \) respectively,

\[
j^-_m j^+_m = h^2_m - \mathcal{E}_m, \quad j^+_m j^-_m = h^2_{m+1} - \mathcal{E}_m, \tag{37}\]

where \( \mathcal{E}_m = (E^{(n)}_m)^2 \), see \( (13) \). Let us notice that similar approach to the shape-invariance of Dirac operators based on the intertwining relations \( (36) \) also appeared recently in \( (37) \).
We can understand the relations (39) and (57) as the manifestation of a supersymmetry where \( \tilde{j}_m^+ \) are components of the supercharges while \( h_m \) and \( h_{m+1} \) compose the superextended Hamiltonian. Picking up \( \tau_3 \) as the grading operator, the operators

\[
\begin{align*}
\mathcal{H}_m &= \begin{pmatrix} h_m & 0 \\ 0 & h_{m+1} \end{pmatrix}, & \mathcal{J}^{(1)}_m &= \begin{pmatrix} 0 & j_m^- \\ j_m^+ & 0 \end{pmatrix}, & \mathcal{J}^{(2)}_m &= i \tau_3 \mathcal{J}^{(1)}_m,
\end{align*}
\]

form the \( N = 2 \) supersymmetry,

\[
[\mathcal{H}_m, \mathcal{J}^{(a)}_m] = 0, \quad \{ \mathcal{J}^{(a)}_m, \mathcal{J}^{(b)}_m \} = 2\delta_{ab} (h_m^2 - \varepsilon_m), \quad a, b = 1, 2.
\]

(39)

The supersymmetry is nonlinear, see [38], as the anticommutator of the supercharges is a quadratic polynomial in the (extended) Hamiltonian \( h_m \). The operators (38) act on the bispinors with the upper spinor from the subspace where \( J_3 = m \) and with the lower one from the subspace where \( J_3 = m + 1 \). After the restriction, the operator \( J_3 \) reduces to \( \mathcal{J}^{(3)}_m = \text{diag}(m, m + 1) = (m + \frac{1}{2}) (1 \otimes 1) - \frac{m}{2} \) where \( \tau_3 = \sigma_3 \otimes 1 \). The intertwining relations (36) of the relativistic Hamiltonian (34) are encoded in the first commutator of (39).

The choice the grading operator is not unique; we could equally well accept the parity \( \mathcal{P} \) (see (11)) in this role, \( [\mathcal{P}, \mathcal{H}_m] = \{ \mathcal{J}^{(a)}_m, \mathcal{P} \} = 0 \), and define \( \mathcal{J}^{(2)}_m = i \mathcal{P} \mathcal{J}^{(1)}_m \). Then exactly the same superalgebra \( \mathfrak{g}(3) \) would emerge (with distinct realization of the supercharges). To treat these two parallel algebraic structures in a unified framework, let us consider the following fermionic operators

\[
\mathcal{J}^{(1, 1)}_m = \mathcal{J}^{(1)}_m, \quad \mathcal{J}^{(2, 1)}_m = i \mathcal{P} \mathcal{J}^{(1)}_m, \quad \mathcal{J}^{(1, 2)}_m = \tau_3 \mathcal{P} \mathcal{J}^{(1)}_m, \quad \mathcal{J}^{(2, 2)}_m = i \tau_3 \mathcal{J}^{(1)}_m.
\]

(40)

They close the following \( N = 4 \) nonlinear superalgebra where the bosonic operators \( \mathcal{P} \) and \( \tau_3 \) are included,

\[
[\mathcal{H}_m, \mathcal{J}^{(a)}_m] = 0, \quad \{ \mathcal{J}^{(a, c)}_m, \mathcal{J}^{(b, d)}_m \} = 2\delta_{ab} (\delta_{cd} + (1 - \delta_{cd}) \tau_3 \mathcal{P}) (h_m^2 - \varepsilon_m),
\]

\[
[\mathcal{P}, \mathcal{J}^{(a, b)}_m] = -2i \varepsilon_{ac} \mathcal{J}^{(c, b)}_m, \quad [\tau_3, \mathcal{J}^{(a, b)}_m] = -2i \left( \delta_{ab} \varepsilon_{ac} \mathcal{J}^{(c, b)}_m - \varepsilon_{ab} \varepsilon_{ac} \varepsilon_{bd} \mathcal{J}^{(c, d)}_m \right).
\]

Here \( \varepsilon_{ab} \) is completely antisymmetric in indices and \( \varepsilon_{12} = 1 \). Hence, the shape invariance of \( h_m \) is associated with nonlinear \( N = 4 \) superalgebra.

For \( M = 0 \), the operator \( H_m = h_m^2 \), coincides with the Hamiltonian of one of the nonrelativistic shape-invariant models that can be, dependently on the actual choice of \( g_2 \) and \( g_3 \) (see (39)), classified as the trigonometric Scarf I (for \( c_1 = -1 \)), Harmonic oscillator (\( c_1 = 0 \)) or Rosen-Morse II system (\( c_1 = 1 \)), see (39). In the regime of zero mass, the operator \( \tau_3 \) anticommutes with \( h_m \) and, hence, it can be regarded as the grading operator of the standard \( N = 2 \) supersymmetry that underlies shape-invariance of \( H_m \). It reads explicitly \( [H_m, Q^{(a)}_m] = 0, \{ Q^{(a)}_m, Q^{(b)}_m \} = 2\delta_{ab} H_m, \) where \( Q^{(1)}_m = h_m, Q^{(2)}_m = i\tau_3 h_m \) and \( a, b = 1, 2 \).

Is it possible to extend this structure with the fermionic operators \( \mathcal{J}^{(a, b)}_m \)? In order to do so, we define the following extended operators

\[
H_m = h_m^2 = \text{diag}(H_m, H_{m+1}), \quad Q^{(1)}_m = h_m, \quad Q^{(2)}_m = i1 \otimes \tau_3 Q^{(1)}_m.
\]

(43)

that close reducible \( N = 2 \) superalgebra graded by \( 1 \otimes \tau_3 \). In order to treat both \( Q^{(a)}_m \) and \( \mathcal{J}^{(a, b)}_m \) as fermionic operators, we have to fix \( \tau_3 \otimes \tau_3 \) as the grading operator. There are sixteen fermionic operators in the extended superalgebra; half of them are local and half are nonlocal operators. They are \( \mathcal{J}^{(a)}_m \) and \( i\tau_3 \otimes \tau_3 \mathcal{J}^{(a, b)}_m \) together with \( Q^{(a)}_m, \tau_3 \mathcal{P} Q^{(a)}_m, i(1 \otimes \tau_3) Q^{(a)}_m \) and \( i(\tau_3 \otimes \tau_3) Q^{(a)}_m \). Additionally, it is necessary to introduce bosonic operators \( \mathcal{J}^{(a, b)}_m = \mathcal{J}^{(a)}_m \mathcal{Q}^{(b)}_m \) and \( \mathcal{P} \mathcal{J}^{(a, b)}_m \), \( a, b = 1, 2 \). Instead of writing down all the relation of the superalgebra, let us focus on its part generated by the local operators. Then we have just eight fermionic operators which we denote in the following manner,

\[
Q^{(a, 1)} = Q^{(a)}, \quad Q^{(a, 2)} = \tau_3 Q^{(a)}, \quad \mathcal{J}^{(a, 1)} = \mathcal{J}^{(a)}, \quad \mathcal{J}^{(a, 2)} = \tau_3 \mathcal{J}^{(a)}, \quad \sigma_3 = 1 \otimes \tau_3.
\]
The sub-superalgebra generated by local operators then reads\(^3\)

\[
\begin{align*}
[H_m, Q_{m}^{(a,b)}] &= 0, \quad \{Q_{m}^{(a,c)}, Q_{m}^{(b,d)}\} = 2\delta_{ab}(\delta_{cd} + (1 - \delta_{cd})\tau_3)H_m, \quad (44) \\
[J_m, \tilde{f}_{m}^{(a,b)}] &= 0, \quad \{\tilde{f}_{m}^{(a,c)}, \tilde{f}_{m}^{(b,d)}\} = 2\delta_{ab}(\delta_{cd} + (1 - \delta_{cd})\tau_3)(H_m - \mathcal{E}_m), \quad (45) \\
[\tilde{\delta}_3, Q_{m}^{(a,c)}] &= -2i\varepsilon_{ab}\tilde{Q}_{m}^{(b,c)}, \quad [\tau_3, \tilde{f}_{m}^{(a,c)}] = -2i\varepsilon_{ab}\tilde{f}_{m}^{(b,c)}, \quad (46) \\
\{\tilde{f}_{m}^{(a,c)}, Q_{m}^{(b,d)}\} &= 2\delta_{a1}\delta_{d1}\tilde{X}_{m}^{(a,b)} + 2\delta_{b2}\delta_{c2}\varepsilon_{bc}\tilde{X}_{m}^{(a,c)}, \quad (47) \\
[X_m^{(a,b)}, X_m^{(c,d)}] &= -2i(\varepsilon_{bd}\delta_{ac}\tilde{\delta}_3 + \varepsilon_{ac}\delta_{bd}\tau_3)H_m(H_m - \mathcal{E}_m), \quad (48) \\
[X_m^{(a,b)}, Q_m^{(c,d)}] &= -2i\left(\varepsilon_{bc}\delta_{d1}\tilde{f}_{m}^{(a,2)} - \delta_{bc}\delta_{d2}\varepsilon_{ar}\tilde{f}_{m}^{(r,1)}\right)H_m, \quad (49) \\
[X_m^{(a,b)}, \tilde{f}_{m}^{(c,d)}] &= -2i\left(\varepsilon_{ac}\delta_{d1}\tilde{Q}_{m}^{(b,2)} - \delta_{ac}\delta_{d2}\varepsilon_{br}\tilde{Q}_{m}^{(r,1)}\right)(H_m - \mathcal{E}_m), \quad (50) \\
[\tilde{\delta}_3, X_m^{(a,b)}] &= -2i\varepsilon_{bc}\tilde{X}_{m}^{(a,c)}, \quad [\tau_3, X_m^{(a,b)}] = -2i\varepsilon_{ac}\tilde{X}_{m}^{(c,b)}, \quad a, b, c, d, r, s = 1, 2. \quad (51)
\end{align*}
\]

The superalgebra (44)-(51) is nonlinear again. Its linearity is restored in subspaces of fixed energy where \(H_m\) acquires constant value.

The complete set of the fermionic supercharges can be written in the current notation as \(\tilde{f}_{m}^{(a,b)}\), \(i\mathcal{P}\tilde{f}_{m}^{(a,b)}, Q_{m}^{(a,b)}\), and \(\mathcal{P}Q_{m}^{(a,b)}\). Having in mind that the parity commutes with \(Q_{m}^{(a,b)}\) while it anticommutes with \(\tilde{f}_{m}^{(a,b)}\), the structure of the complete \(N = 16\) superalgebra that would include both local and nonlocal fermionic operators can be found directly from the relations (44)-(51). In the complete superalgebra, there would also appear bosonic operators \(\mathcal{P}\tau_3\) and \(\mathcal{P}\tilde{\delta}_3\). It is worth noticing that supersymmetric structure of nonrelativistic systems with two distinct sets of supercharges (both local and nonlocal) was discussed e.g. in [40] and [41]. \(N = 8\) extended supersymmetry was considered for three-dimensional Schrödinger-Pauli equation in [42]. In [43], superalgebraic structure based on nonlocal supercharges was considered for one-dimensional nonrelativistic systems.

### 4.2 The non-relativistic systems with position-dependent mass by coordinate transformation

The same superalgebra (44)-(51) can be obtained when (44) and (47), corresponding to a generic (diagonal) metric tensor (13), are considered and reduced to the fixed subspaces of \(J_3\). We remind that (44) and (47) can be obtained from (11) and (33) by the change of coordinates. In this context, let us mention that there appears an interesting class of physical systems where the operators (15) and (17) can appear rather naturally. The Schrödinger operators with the position-dependent mass \(\Sigma(x_1)\),

\[
-\frac{1}{\Sigma(x_1)^{1/4}}\frac{\partial}{\partial x_1}\frac{1}{\sqrt{\Sigma(x_1)}}\frac{\partial}{\partial x_1}\frac{1}{\Sigma(x_1)^{1/4}} + V(x_1), \quad (52)
\]

emerge in condensed matter physics, e.g. in description of the semiconductors or heterostructures [44], [45]. These systems have attracted lots of attention in the literature, see [46]-[47] and references therein.

Let us pick up Dirac Hamiltonian (15). The square of the operator in the subspace of fixed angular momentum \(J_3 = m\) can be written as

\[
H_m = \frac{1}{g_{11}^{1/4}}\frac{\partial}{\partial x_1}\frac{1}{\sqrt{g_{11}}\frac{1}{g_{11}}}\frac{1}{\sqrt{g_{11}}} + \left(\frac{m}{\sqrt{g_{22}}} + c_3g_3\right)^2 - \sigma_3\frac{1}{g_{11}}\left(\frac{m}{\sqrt{g_{22}}} + c_3g_3\right)'. \quad (53)
\]

We can see that the diagonal entries of the matrix operator are of the type (52) provided that we fix \(\Sigma(x_1) = g_{11}(x_1)\). Let us present an example of a solvable shape-invariant system with position dependent

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\(^3\)In the computation of the commutators and anticommutators, we used \(\{J_{m}^{(a)}, Q_{m}^{(b)}\} = 0, \{J_{m}^{(a)}, J_{m}^{(b)}\} = -2i\varepsilon_{ab}\tau_3(H_m - \mathcal{E}_m), \{\tilde{Q}_{m}^{(a)}, Q_{m}^{(b)}\} = -2i\varepsilon_{ab}\tilde{\delta}_3H_m\).
mass. Since the effective mass in (52) is required to be bounded and positive for all $x_1$, we fix the coefficients in (15) as
g_{22} = sn(x_1)^2, \quad g_{11} = dn(x_1)^2, \quad g_3 = c_3 \frac{cn(x_1)}{sn(x_1)}, \quad x_1 \in (0, 2K).

It complies with (17) for $c_1 = -1$. Hence, we deal with quantum system which possesses $so(3)$ potential algebra. The explicit form of the supercharge $Q_m^{(1)}$ reads

$$Q_m^{(1)} = i \sigma_1 \left( \frac{1}{\sqrt{dn(x_1)}} \frac{\partial}{\partial x_1} x_1 - \frac{1}{\sqrt{dn(x_1)}} + \frac{c_3}{sn(x_1)} \frac{cn(x_1)}{sn(x_1)} \right).$$ (54)

The nonrelativistic Hamiltonian $H_m$ acquires the following form

$$H_m = - \frac{1}{dn(x_1)} \frac{\partial}{\partial x_1} x_1 \left( \frac{1}{dn(x_1)} \frac{\partial}{\partial x_1} x_1 - \frac{1}{dn(x_1)} + \frac{m + c_3 cn(x_1)}{sn(x_1)} \right)^2 + \frac{c_3}{sn(x_1)^2}.$$ (55)

The effective mass of the nonrelativistic particle is equal to $dn^2(x_1)$ and is strictly positive. Notice that it coincides with the mass term used in (17) for $k = 1, \Sigma(x_1) = dn^2(x_1) = \text{sech}^2 x_1$. The particle lives in the finite interval in presence of the external potential. We require the wave functions of $H_m$ to be regular and vanishing at the borders of the interval.

The explicit form of the operators $j_{m, \pm}$ reads

$$j_{m, \pm} = i \left( \frac{1}{\sqrt{dn(x_1)}} \frac{\partial}{\partial x_1} x_1 \right) \mp \left( \frac{2m + 1}{2sn(x_1)} m + 2c_3 + \sigma_3 \right).$$ (56)

We will consider $m - 1 > c_3 > 0$. For these values of $m$, the potential in $H_m$ is confining, i.e. it diverges to infinity at the boundaries. Energy levels of $H_m$ are doubly degenerate including the non-zero ground state energy $E_m = (m + \frac{1}{2})^2 - c_3^2$. Doublet of ground states is formed by $\psi_m^+$ and $\sigma_3 \psi_m^+$ where explicitly

$$\psi_m^+ = \sqrt{dn(x_1)sn(x_1)} \frac{1}{1 + cn(x_1)} \left( \frac{1 - cn(x_1)}{1 + cn(x_1)} \right)^{\frac{m}{2}} \left( \frac{1 - cn(x_1)}{1 + cn(x_1)} \right)^{\frac{1}{2}} \left( \frac{1 - cn(x_1)}{1 + cn(x_1)} \right)^{\frac{1}{2}}.$$(57)

The vector complies with the required boundary conditions. The action of $j_{m, \pm}$ preserves regularity of wave function provided that $m$ is an integer (semi-integer) and $c_3$ is a semi-integer (integer). The $N = 2$ supersymmetry of $H_m$ is spontaneously broken; the supercharges $Q_m^{(1)}$ and $i \sigma_3 Q_m^{(1)}$ do not annihilate ground states, but interchange them mutually.

The superpartner Hamiltonians $H_m$ and $H_{m+1}$ are spectrally almost identical up to the energy level $\mathcal{E}_m$ which is missing in the spectrum of $H_{m+1}$. Hence, the extended operator $\mathcal{H}_m$ in (18) has four-fold degeneracy of the energy levels up to the ground state which is doubly degenerate. The subspaces with fixed $m$ are invariant with respect to the action of the operators $Q_m^{(a)}$. The operators $j_{m}^{(a)}$ annihilate the ground states of $\mathcal{H}_m$, see in Fig. 2 for illustration.

5 Discussion

5.1 Dirac oscillator

The quantum settings presented in section 8 were based on the potential algebra $so(3)$. Let us make few comments on the quantum systems whose integrals of motion form either the oscillator algebra or
so(2,1). When $c_1 = 0$, we have $g_2 = \frac{1}{x_1}$ and $g_3 = c_3 x_1$. Substituting these coefficients into (1), we get the Hamiltonian

$$h_D = i\sigma_1 \partial_{x_1} + \sigma_2 \frac{J_3}{x_1} + c_3 \sigma_2 x_1, \quad x_1 > 0.$$  (58)

which coincides with the radial part of the well known Dirac oscillator Hamiltonian introduced by Moshinsky and Szczepaniak [48]. It has integrals of motion in the following form,

$$J_{\pm} = i e^{\pm i x_2} \left( \partial_{x_1} - \frac{(\pm J_3 + \frac{1}{2})}{x_1} \pm c_3 x_1 \pm \frac{\sigma_3}{2 x_1} \right).$$  (59)

The eigenfunctions of (58) are required to be vanishing at $x_1 = 0$. The system has bound states as long as $c_3 \neq 0$. They can be generated from the lowest weight vectors $\psi_m$,

$$\psi_m = e^{im x_2} e^{\frac{c_3}{2} x_1^2} x_1^{-m} \left( -i \sqrt{c_3 \left( -\frac{1}{2} + m \right)} \right)^T,$$  (60)

and comply with the required boundary condition provided that $c_3 < 0$ and $m$ is a semi-integer lower or equal to $\frac{1}{2}$.

In general, the solution of the stationary equation with (58) are not determined uniquely by the requirement of square integrability. The singularity of the potential at $x_1 = 0$ causes that the square integrability is not sufficient to determine the wave functions. Our requirement of regularity of the wave functions at the origin removes the ambiguity by fixing one of the self-adjoint extensions of $h_D$. In [49], different way of dealing with singular potentials was discussed. The potential of Calogero Hamiltonian was regularized by the complex shift of the coordinate. Its hermicity was violated, however, the complex shift of coordinate allowed for additional set of solutions and spectrum remained real (see also [50] for a recent reference where the technique was used). This approach should be applicable in the context of Dirac Hamiltonian (58) as well as in other quantum systems described by (1) with singular coefficients $g_2$ and $g_3$.

5.2 so(2,1) and BTZ black hole -like system

Finally, let us discuss the case where the quantum system possesses integrals of motion that form so(2,1). We shall consider $x_2$ as the time-like coordinate. In order to do so, we perform the substitution $x_2 \rightarrow -it$. It makes the replacement $dx_2^2 \rightarrow -dt^2$ in the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ and, hence, the sign of the
lower component of $g_{\mu\nu}$ gets changed accordingly. This substitution transforms Dirac Hamiltonian (70) into (71) (the flat space metric is now $\eta_{\mu\nu} = \text{diag}(1, -1)$) while keeping the condition (17) unchanged. As an example, let us consider the space with the metric tensor in the following form

$$g_{\mu\nu} = \text{diag}(N^{-2}, -N^2), \quad N = N(x_1),$$

where we leave the function $N$ unspecified at the moment. Let us comment that metric tensors of similar type appear in description of black holes \[51\]. The Dirac Hamiltonian (71) associated with (61) reads

$$h_D = iN\sigma_1 \partial_{x_1} + \sigma_2 \frac{\partial_t}{N} + i\sigma_1 \frac{N'}{2} + c_3\sigma_2 N'.$$

It has integrals of motion (17),

$$J_{\pm} = ie^{\pm t} \left( N\partial_{x_1} \mp N'\partial_t \mp \frac{c_3}{N} \mp \frac{\sigma_3}{2N} \right).$$

They, together with $J_3 = \partial_t$, form the $so(2, 1)$ algebra provided that the components of the metric tensor (61) solve (17). We find very interesting that there is a solution of (17) where the function $N$ acquires the following form

$$N^2 = -1 + x_1^2.$$

The metric (61) then coincides with the one of BTZ black hole \[52\] as long as the black hole has unit mass and zero angular momentum. The particle lives on the straight line with fixed angular coordinate which terminates in the singularity. This coincidence suggests that appropriate (higher-dimensional) modification of our current algebraic approach could be particularly helpful in the analysis of Dirac fermions in the gravitational background of BTZ black hole.

There appeared remarkable results recently where the geometrical properties of the settings with horizon were encoded into the spatial part of the metric tensor \[53\]. This observation made it possible to consider black hole simulations in the strained graphene. Analysis of these systems in the context of the potential algebras could be very interesting.

5.3 Outlook

In the current work, we considered potential algebras of the $(2 + 1)$-dimensional Dirac-like operator (1). The explicit form the operator was found that possessed Lie algebra $so(3)$, $so(2, 1)$ or oscillator algebra of integrals of motion. It should be mentioned that we do not make any statements about generality of the found representations; the work was rather focused on the applications of the potential algebras. The results can be understood in the context of classification of realizations of the low-dimensional Lie algebras. Here, let us mention \[54\] where representations in terms of differential operators without matrix degree of freedom were analyzed. The analysis of the generic form of the ladder operators $J_{\pm}$ that would satisfy the algebra (1)-(2) would be desirable. It would be a step towards the extension of the general results obtained in \[54\] to the operators with matrix coefficients.

The structure of the potential algebra, its rank in particular, was fixed by the ansatz (2) and by the explicit form of the Dirac Hamiltonian (1). It would be possible to extend the current approach to higher-dimensional settings by considering potential algebras of higher rank. Such algebraic structures could be used effectively in the analysis of quantum systems that live in spaces with non-trivial geometry. It might be applied in construction of the new, shape-invariant, two-dimensional Dirac Hamiltonians. As it was suggested above, it could be also useful in the analysis of Dirac particle in the gravitational background of BTZ black hole.

In the considered framework, the operators $J_{\pm}$ were required to commute with the Hamiltonian, i.e. they reflected degeneracy of its spectrum. The concept of dynamical algebras (or spectrum generating algebras) represents a natural extension of the current approach. Relaxing the condition $[J_{\pm}, h_D] = 0$, we
could consider a broader class of quantum systems. A possible hint on how to modify the commutator is provided by the shape-invariance in the nonrelativistic quantum mechanics, where \( H_{m+1} q_m - q_m H_m = r(m) q_m \) with \( r(m) \) being a constant. Having in mind the reduction (34) and (35), the commutator is suggested to be \( [h_D, J_\pm] \sim J_\pm \). The ansatz for the operators \( J_\pm \) should be extended appropriately to the matrix operator with generally non-vanishing anti-diagonal components.

It is worth mentioning in this context that a different approach to dynamical symmetries of non-relativistic systems was introduced in [55]. Dynamical algebra of Poschl-Teller system was treated as a potential algebra of a suitably modified Hamiltonian. Implementation of this approach to Dirac operators could be fruitful as well.

**Appendix A**

We shall show that for the representations of \( so(3) \) generated by \( X \) from the lowest weight vectors, the coupling constant \( c_3 \) has to be integer (half-integer) as long as \( m \) is half-integer (integer). Only in this case, the representations can be finite-dimensional.

We set \( g_2 = \sin^{-1} x_1 \) and \( g_3 = c_3 \cot x_1 \) in (3). The lowest weight vector, the kernel of \( J_- \), acquires the form

\[
\psi_m = e^{imx_2} \sin^{\frac{1}{2}+m} x_1 \left( \beta_1 \tan^{\frac{1}{2}+c_3} \frac{x_1}{2}, \beta_2 \tan^{-\frac{1}{2}+c_3} \frac{x_1}{2} \right)^T,
\]

where \( T \) denotes transposition. The actual value of the constants \( \beta_1 \) and \( \beta_2 \) is not important at the moment. Next we define a diagonal matrix \( G \),

\[
G = \sin^{\frac{1}{2}} x_1 \text{diag} \left( \tan^{\frac{1}{2}+c_3} \frac{x_1}{2}, \tan^{-\frac{1}{2}+c_3} \frac{x_1}{2} \right)
\]

We use this matrix to transform the ladder operator \( J_+ \) and the lowest-weight vector to the form that will be convenient for the forthcoming calculation,

\[
G^{-1} J_+ G = i e^{ix_2} g_2 \sqrt{-1 + \frac{g_2^2}{g_2^2} \left( \frac{1}{g_2^2} \partial_{x_1} + \frac{J_3}{g_2} \right), \quad \psi_m = e^{imx_2} G \left( \frac{\alpha - g_2^{m-1+2c_3} w^{2c_3-1}}{\beta - g_2^{m+1+2c_3} w^{2c_3+1}} \right), \quad (65)
\]

where \( w = 1 + \sqrt{1 - \frac{1}{g_2^2}} \). Let us compute how does \( J_+ \) act on the generic wave function \( \xi = e^{imx_2} G \left( g_2^{s} \right) \).

Using (65), we can calculate immediately

\[
J_+ e^{ix_2} G \left( g_2^{s} \right) = i e^{i(n+1)x_2} G \left( \begin{array}{c} -(n+r-s)w^{s-1}g_2^{r-1} + g_2^{r+1}w^{s}(n+r) \\ 0 \end{array} \right).
\]

(66)

This formula helps understand qualitatively the structure of the \((J_+)^k \xi \) (with \( k \) being a positive integer): we can find coefficients at the highest and the lowest power of \( g_2 \). The coefficient of the term \( g_2^{r-k} w^{s-k} \) is \((-1)^k \prod_{l=0}^{k-1} (n+l-1+r-s) \) while the coefficient of the term \( g_2^{r+k} w^s \) is \( \prod_{l=0}^{k-1} (n+r+2l) \).

Now, let us identify the nonzero element of \( \xi \) with the upper component of \( \psi_m \) in (65), i.e. we fix \( n = m, r = m+1 + 2c_3 \) and \( s = 2c_3 - 1 \). The requirement that \( \psi_m \) is annihilated by a specific power of \( J_+ \) implies that the above derived coefficients have to vanish for a specific value of \( k \). This happens provided that

\[
2m - 1 \in \mathbb{Z} \leq 0, \quad -(m+c_3) + \frac{1}{2} \in \mathbb{Z} \geq 0.
\]

The first relation restricts \( m \) to be integer or half-integer while the second relation tells that \( c_3 \) has to be half-integer or integer, respectively.
Appendix B

Let us review briefly how the Dirac Hamiltonian for the mass-less particle in the curved space can be constructed. We will consider two distinct scenarios that are distinguished by the sign $\varepsilon$ in the metric tensor $g_{\mu\nu}$,

$$g_{\mu\nu} = \begin{pmatrix} g_{11}(x_1) & 0 \\ 0 & \varepsilon g_{22}(x_1) \end{pmatrix}.$$  

We suppose that $g_{11}(x_1) \geq 0$ and $g_{22}(x_1) \geq 0$. For $\varepsilon = -1$, we deal with $(1+1)$ dimensional space-time, considering the coordinate $x_2$ to be time-like. When $\varepsilon = 1$, both coordinates are space-like and the metric describes curved space. In both cases, Dirac equation can be written in the following form \[56\],

$$i\gamma^\mu (\partial_\mu + \Omega_\mu) \psi = \lambda \psi, \quad \mu = 1, 2,$$  

(67)

where $\lambda$ corresponds to the mass of the particle for $\varepsilon = -1$ whereas it represents eigenvalues of Dirac Hamiltonian for $\varepsilon = 1$. In the latter case, \[67\] can be identified with the stationary equation. $\Omega_\mu$ is spin connection and $\gamma^\mu$ are generalized gamma matrices which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. These quantities can be computed with the use of tetrad formalism. The tetrads $e^a_{x\mu}$ are related to the metric tensor in the following manner,

$$e^a_{x\mu} e^b_{x\nu} \eta_{ab} = g_{\mu\nu}.$$  

The quantities $e^a_{x\mu}$ and $e^a_{x\nu}$ relate locally flat coordinates $y_a$ with the curvilinear coordinates $x\mu$ at a given point $X$, $e^a_{x\mu} = (\partial x^a/X, y_a) (X)$ and $e^a_{x\nu} = (\partial y_a/X, x\mu) (X)$. There holds $e^a_{x\mu} e^b_{x\nu} = \delta^a_b$. Flat-space metric is identified as $\eta_{ab} = \text{diag}(1, \varepsilon)$. The matrices $\gamma^\mu$ are defined as

$$\gamma^\mu = e^a \gamma^a \quad \text{where} \quad \{\gamma^a, \gamma^b\} = 2\eta^{ab}. \quad (68)$$

The spin connection $\Omega_\mu$ is defined as

$$\Omega_\mu = \frac{1}{4} \omega^{ab}_\mu [\gamma_a, \gamma_b],$$  

where

$$\omega^{ab}_\mu = e^a_{x\lambda} g^{\lambda\tau} (\partial_\mu e^b_{x\tau} - \Gamma^\kappa\mu\tau e^b_{x\kappa}) \quad \text{and} \quad \Gamma^{\lambda\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}). \quad (69)$$

In our present case, the tetrads can be fixed as

$$e^a_{x\mu} = \begin{pmatrix} \sqrt{g_{11}} & 0 \\ 0 & \sqrt{g_{22}} \end{pmatrix}, \quad e^a_{x\nu} = \begin{pmatrix} \frac{1}{\sqrt{g_{11}}} & 0 \\ 0 & \frac{1}{\sqrt{g_{22}}} \end{pmatrix}.$$  

Here, the upper index denotes row and the lower index the column of the matrix. Non-zero elements of the affine connection are $\Gamma^{1}_{11} = \frac{g'_{11}}{g_{11}^2}, \quad \Gamma^{1}_{22} = -\frac{g'_{22}}{g_{11}}, \quad \Gamma^{1}_{12} = \Gamma^{2}_{21} = \frac{g'_{22}}{g_{22}}$ and

$$\omega^{12} = -\omega^{21} = -\frac{g'_{22}}{2\sqrt{g_{11}g_{22}}}.$$  

The matrices $\gamma_a$ are defined as $\gamma_1 = \sigma_1, \quad \gamma_2 = i \frac{1-\varepsilon}{2} \sigma_2$ and satisfy $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$. The generalized gamma matrices are then

$$\gamma^{x_1} = \sqrt{g_{11}}^{-1} \sigma_1, \quad \gamma^{x_2} = \varepsilon i \frac{1-\varepsilon}{2} \sqrt{g_{22}}^{-1} \sigma_2.$$  

Finally, the spin connection is

$$\Omega_{x_1} = 0, \quad \Omega_{x_2} = -\frac{i \left(1-\frac{1-\varepsilon}{2}\right) g'_{22}}{4\sqrt{g_{11}g_{22}}} \sigma_3.$$  

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Dirac Hamiltonian for $\varepsilon = 1$ acquires the form

$$h_{D}^{\varepsilon=1} = i\sigma_1 \left( \frac{1}{\sqrt{g_{11}}} \partial_{x_1} + \frac{g_{22}^\prime}{4g_{22} \sqrt{g_{11}}} \right) + \sigma_2 \frac{i\partial_{x_2}}{\sqrt{g_{22}}}$$

(70)

while it has the following explicit form for $\varepsilon = -1$

$$h_{D}^{\varepsilon=-1} = i\sigma_1 \left( \frac{1}{\sqrt{g_{11}}} \partial_{x_1} + \frac{g_{22}^\prime}{4g_{22} \sqrt{g_{11}}} \right) - \sigma_2 \frac{i\partial_{x_2}}{\sqrt{g_{22}}}.$$  

(71)

We transform the operator $h_{D}^{\varepsilon=1}$ to the form which is manifestly hermitian with respect to the standard scalar product in the following manner

$$h_{D} = (\det g_{\mu\nu})^{\frac{1}{4}} h_{D}^{\varepsilon=1} (\det g_{\mu\nu})^{-\frac{1}{4}} = \frac{i\sigma_1}{\sqrt{g_{11}}} \partial_{x_1} + \frac{i\sigma_2}{\sqrt{g_{22}}} \partial_{x_2} - \frac{i\sigma_1 g_{11}^\prime}{4g_{11}^{3/2}}.$$  

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