Symmetric energy-momentum tensor:
the Abraham form and the explicitly covariant formula

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(Dated: April 21, 2015)
Abstract

We compare the explicitly covariant 4-dimensional formula, recently proposed by V.P. Makarov and S.A. Rukhadze [Phys. Usp. 52 937 (2011)] for symmetric energy-momentum tensor of electromagnetic field in a medium, and the energy-momentum tensor derived by Abraham in the 3-dimensional vector form. It is shown that these two objects coincide only on the physical configuration space $\Gamma$, formed by the field vectors and the velocity of the medium, which satisfy the constitutive relations. It should be emphasized that the 3-dimensional vector formulae for the components of the energy-momentum tensor were obtained by Abraham only on $\Gamma$, and the task of their extension to the whole unconditional configuration space $\Gamma$ was not posed. In order to accomplish the comparison noted above we derive the Makarov-Rukhadze formula a new by another method, namely, by generalizing the Abraham reasoning. The comparison conducted enables one to treat the Makarov-Rukhadze formula as a unique consistent extension of the Abraham formulae to the whole configuration space $\Gamma$. Thus the question concerning the relativistic covariance of the original 3-dimensional Abraham formulae defined on $\Gamma$ is solved positively. We discuss in detail the relativistic covariance of the 3-dimensional vector formulae for individual components of the 4-dimensional tensors in electrodynamics which is manifested in the form-invariance of these formulae under Lorentz transformations.

PACS numbers: 03.30.+p, 03.50.De, 41.20.-q

Keywords: Symmetric energy-momentum tensor, the Abraham tensor, the Minkowski energy-momentum tensor, Minkowski’s phenomenological electrodynamics, electrodynamics of continuous moving media, relativistic covariance and form-invariance.

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I. INTRODUCTION

In the electrodynamics of continuous media, it is generally accepted that the symmetric energy-momentum tensor of electromagnetic field was derived by Abraham (see, for example, the comprehensive survey by Pauli [1, Section 35]). However it is worth noting that Abraham has derived the components of his energy-momentum tensor only assuming that the Minkowski constitutive relations are satisfied, i.e. only on the physical configuration space \( \Gamma \) of changing the field vectors \( E, H, D, B \) and the velocity of the medium \( v, v < c \). The task to extend the Abraham formulae to the whole unconditional configuration space \( \Gamma \) was not posed. This peculiarity of the Abraham formulae completely concerns also the 4-dimensional presentation of these formulae proposed by Grammel [2]. This fact explains in particular the lack of uniqueness of the Grammel formulae (three versions) and their explicit dependence on \( \varepsilon \) and \( \mu \). Obviously the Abraham and Grammel formulae should be used only on \( \Gamma \), where they coincide and do not depend explicitly on \( \varepsilon \) and \( \mu \), but not on the whole configuration space \( \Gamma \). In the Pauli survey [1] and in subsequent papers dealing with the Abraham and Grammel formulae these peculiarities were not noted and not taken into account [3–6].

Let us make more precise the used terminology. Following Pauli, we define the symmetric energy-momentum tensor by its known values in co-moving frame [1, Section 35, Eqs. (298) and (300)]. By Abraham’s formulae, or simply by the Abraham tensor, we call, following again Pauli [1, Section 35, p.p. 666, 667], the set of formulae for all the components of the energy-momentum tensor, which were obtained by Abraham on \( \Gamma \) (see §§38, 39 in his book [7, 8] and also the articles [9, 10]). These formulae will also be brought about in the present paper in Section III. The formulae derived by Abraham are valid only on \( \Gamma \), therefore in advance it is not apparently that they define the components of a 4-dimensional relativistic vector. This point is also investigated in our paper.

The symmetric energy-momentum tensor, as it was specified by Pauli and defined on the whole configuration space \( \Gamma \), has been constructed only recently by Makarov and Rukhadze [3]. These authors proposed explicitly covariant 4-dimensional formula for this tensor, which is applicable both in the rest frame and for moving media. It is important that when deriving this formula the Minkowski constitutive equations were not used. Therefore this formula does not contain explicitly the material characteristics \( \varepsilon \) and \( \mu \). The authors of the paper
call their tensor by the Abraham tensor. However the comparison of these two objects was not conducted. In the present paper this gap will be filled up.

For this aim, we generalize the Abraham reasoning which he followed when deriving his energy-momentum tensor on \( \Gamma \). This enables us to construct the symmetric energy-momentum tensor on the whole configuration space \( \Gamma \) in another way different from the calculations in [3]. In our approach, the identity of the symmetric energy-momentum tensor and the Abraham tensor on \( \Gamma \) is proved easily. In addition we derive at the same time the explicit 3-dimensional vector formulae for all the components of the symmetric energy-momentum tensor. The latter is important due to the following.

In practical calculations, one uses, as a rule, the individual components of the energy-momentum tensor in 3-dimensional vector notation. In particular, such a situation takes place in the electrodynamics of the moving bodies. The point is the Lorentz vector describing the velocity of the medium

\[
\mathbf{u}_\alpha = \gamma \{ \mathbf{q}, \mathbf{i} \}, \quad \mathbf{q} = \mathbf{v}/c, \quad \gamma^{-1} = \sqrt{1 - \mathbf{q}^2}, \quad \mathbf{u}_\alpha \mathbf{u}_\alpha = -1, \quad \alpha = 1, 2, 3, 4 \quad (1.1)
\]

cannot be itself considered as a small parameter. Only its spatial part, \( \mathbf{q} \), may be such a parameter. The 3-dimensional vector formulae for the energy-momentum tensor in the Minkowski form and in the Abraham form are known [1, 4, 7, 8]. In the present paper, the analogous formulae will be derived for the symmetric energy-momentum tensor. A special attention will be paid to the discussion of the relativistic covariance of these formulae.

The layout of the paper is as follows. In Section II the formulae are brought which define the energy-momentum tensor in the Minkowski form and the Minkowski constitutive relations are presented. This material is substantially used in the following. In Section III the derivation of the energy-momentum tensor by Abraham is traced in detail. In Section IV the generalization of the Abraham reasoning enables us to construct the symmetric energy-momentum tensor on the whole configuration space \( \Gamma \). Here the identity of the symmetric energy-momentum tensor and the Abraham tensor on \( \Gamma \) is proved. At the same time, the explicit 3-dimensional vector formulae are derived for all the components of the symmetric energy-momentum tensor. These formulae are analogous to those derived by Abraham.

In Section V we discuss different methods of defining the tensors of the second rank in

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1 The method used in [3] is discussed in Section V B of the present paper.
2 To do the same proceeding from the formula obtained in [3] is a formidable task.
electrodynamics and the form-invariance of the pertinent 3-dimensional vector formulae. It
is shown in detail in what way this form-invariance become apparent in the case of the
energy-momentum tensor in the Minkowski form, in the Abraham form, and in the case of
the symmetric energy-momentum tensor. In Conclusion, Section VI the obtained results are
summarized briefly and some comments concerning the bibliography, in particular textbooks,
are made.

We do not touch the Abraham-Minkowski controversy and the problem of determining
the ‘correct energy-momentum’ tensor in macroscopic electrodynamics. There is a large
body of the literature on this subject, see, for example, the reviews and references therein
[3, 5, 11–15].

We use the following notations. The Greek indexes take values 1, 2, 3, 4, the Latin indexes
assume the values $x, y, z$. The unrationalized Gaussian units are used for the electromagnet
ic field and the notations generally accepted in macroscopic electrodynamics [16] are adopted.

II. THE MINKOWSKI ENERGY-MOMENTUM TENSOR

We will frequently refer to the energy-momentum tensor in the Minkowski form. Therefore we bring here the formulae needed. This tensor is a straightforward generalization of
the energy-momentum tensor of electromagnetic field in vacuum [17]

$$T_{\alpha\beta} = T_{\beta\alpha} = \frac{1}{4\pi} \left( F_{\alpha\gamma} F_{\gamma\beta} - \frac{\delta_{\alpha\beta}}{4} F_{\gamma\delta} F_{\delta\gamma} \right), \quad (2.1)$$

where $F_{\alpha\beta} = -F_{\beta\alpha}$ is electromagnetic field tensor in vacuum, $F_{ij} = \varepsilon_{ijk} H_k$, $F_{4j} = -F_{j4} = iE_j$.

Minkowski has formulated relativistic macroscopic electrodynamics in terms of two ten-
sors which are defined by the formulae [1, Section 33], [16, §76]

$$F_{ij} = \varepsilon_{ijk} B_k, \quad F_{4j} = -F_{j4} = iE_j; \quad (2.2)$$

$$H_{ij} = \varepsilon_{ijk} H_k, \quad H_{4j} = -H_{j4} = iD_j. \quad (2.3)$$

Generalizing (2.1), Minkowski proposed the following energy-momentum tensor of electro-
magnetic field in a material medium [1]

$$T^M_{\alpha\beta} = \frac{1}{4\pi} \left( F_{\alpha\gamma} H_{\gamma\beta} - \frac{\delta_{\alpha\beta}}{4} F_{\gamma\nu} H_{\nu\gamma} \right). \quad (2.4)$$
This definition holds both for the medium at rest and for the moving medium. According to construction, the velocity of the medium does not enter into Eq. (2.4). The Minkowski energy-momentum tensor is not symmetric \( T^M_{\alpha\beta} \neq T^M_{\beta\alpha} \) and its trace vanishes \( T^M_{\alpha\alpha} = 0 \).

By making use of the 3-dimensional vector notation we can present the components of the tensor \( T^M_{\alpha\beta} \) in the arbitrary inertial reference frame, in the form [1, 4]

\[
T^M_{\alpha\beta} = \left( \begin{array}{cc} \sigma^M_{ij} & -ic\, S^M \\ -\frac{1}{c}\, S^M & w^M \end{array} \right),
\]

(2.5)

where

\[
\sigma^M_{ij} = \frac{1}{4\pi} \left\{ E_i D_j + H_i B_j - \frac{\delta_{ij}}{2} (ED + HB) \right\},
\]

(2.6)

\[
-\frac{1}{c}\, S^M = \frac{1}{c} S = \frac{1}{4\pi} [EH], \quad c\, g^M = \frac{1}{4\pi} [DB],
\]

(2.7)

\[
w^M = \frac{1}{8\pi} (ED + HB).
\]

(2.8)

Equation (2.7) involves the Poynting vector

\[
S = \frac{C}{4\pi} [EH].
\]

(2.9)

In macroscopic electrodynamics, the important role is played by the Minkowski constitutive relations [1, Section 33], [16, §76]:

\[
D_\alpha = \varepsilon E_\alpha,
\]

(2.10)

\[
B_{\alpha\beta\gamma} = \mu H_{\alpha\beta\gamma}.
\]

(2.11)

Here we have used the notation from Ref. [18] which we will use in what follows

\[
E_\alpha \equiv F_{\alpha\beta} u_\beta = \gamma \{E + [qB], i(qE)\}, \quad D_\alpha \equiv H_{\alpha\beta} u_\beta + \gamma \{D + [qH], i(qD)\},
\]

(2.12)

\[
B_{\alpha\beta\gamma} = F_{\alpha\beta\gamma} u_\gamma + F_{\beta\gamma} u_\alpha + F_{\gamma\alpha} u_\beta, \quad H_{\alpha\beta\gamma} = H_{\alpha\beta} u_\gamma + H_{\beta\gamma} u_\alpha + H_{\gamma\alpha} u_\beta.
\]

(2.13)

In terms of the 3-dimensional vector notation Eqs. (2.10), (2.11) acquire the form [1, Section 33], [16, §76]

\[
D + [qH] = \varepsilon (E + [qB]),
\]

(2.14)

\[
B - [qE] = \mu (H - [qD]).
\]

(2.15)

Only due to the constitutive relations (2.10), (2.11) or (2.14), (2.15) the formal scheme of the macroscopic electrodynamics acquires the physical content [1, Section 33]. The values
of the vectors \( \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B} \) and the velocity of the medium \( \mathbf{v} \), \( v < c \), which satisfy the Minkowski constitutive relations (2.10), (2.11) will be referred to as the physical configuration space \( \Gamma \).

In the relativistic electrodynamics of moving media, since the Minkowski works, the 4-dimensional generalization of the Poynting vector (2.9) is used (the so-called Ruhstrahlvektor \([1\text{, Section 35}]\))

\[
\Omega_\alpha = H_{\alpha\beta\gamma}u_\beta E_\gamma.
\] (2.16)

For simplicity, we shall call \( \Omega_\alpha \) the Minkowski vector. In the co-moving reference frame \((K')\), Eq. (2.16) yields \([1\text{, Section 35, equation (304)}], [18\text{, equation (103a)}]\)

\[
\frac{1}{4\pi} (\Omega'_1, \Omega'_2, \Omega'_3) = S', \quad \Omega'_4 = 0.
\] (2.17)

Following Pauli \([1]\), we mark quantities in a co-moving frame by prime. From Eqs. (2.17) and (1.1) it follows

\[
u_\alpha \Omega_\alpha = 0.
\] (2.18)

### III. THE ABRAHAM ENERGY-MOMENTUM TENSOR

Abraham imposes the following conditions on the energy-momentum tensor to be found. This tensor should be symmetric; in the co-moving reference frame \((\mathbf{v} = 0)\) and under fulfillment of the constitutive relations \(3\)

\[
\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}
\] (3.1)

(isotropic medium) the components of this tensor should assume the values

\[
\sigma'_{ij}^A = \frac{1}{2}(\sigma'_{ij}^M + \sigma'_{ji}^M),
\] (3.2)

\[
\frac{1}{c} S'^A = c g'^A = \frac{1}{c} S'^C = \frac{1}{4\pi} [\mathbf{E}'\mathbf{H'}],
\] (3.3)

\[
w'^A = \frac{1}{8\pi} (\mathbf{E}'\mathbf{D'} + \mathbf{H}'\mathbf{B'}).
\] (3.4)

We have to stress here that all these requirements are not sufficient to uniquely define the energy-momentum tensor on the whole configuration space \( \Gamma \). Indeed, Abraham, having

\(3\) The account of the constitutive relations in Abraham’s derivation of the energy-momentum tensor is not noted in the Pauli review \([1]\) and in the subsequent papers, see, for example, Refs. \([3\text{–}5, 11\text{–}15]\)
been fixed the reference frame ($v = 0$), assigns the components of his tensor only on the physical configuration space $\Gamma$ (i.e. under fulfillment of the constitutive relations (3.1)), but not on the whole configuration space $\Gamma$, as the theory of relativity demands.

However Abraham did not use the theory of relativity in constructing his energy-momentum tensor. “He went along the way of extrapolations proceeding from the medium at rest, relying on rather arbitrary assumptions and very poor experimental data.” [11, p. 289, Russian edition]. Following this way, Abraham has derived the energy-momentum tensor first in the framework of his approach to the electrodynamics of moving bodies [7, §§38, 39], [9] and later on in application to the Minkowski electrodynamics [10]. In the paper [10], Abraham approached the most close to using the tensor formalism in the problem under consideration. The Abraham reasoning [10] was ‘translated’ to the standard tensor language by Grammel [10]; the pertinent detailed calculations were also conducted by Kafka [18]. We shall follow the papers [2, 10, 18].

By making use of the standard tensor notation the Abraham reasoning [10] can be reformulated like this. In the co-moving reference frame ($v = 0$) and when the constitutive relations (3.1) hold, the symmetric tensor

$$\frac{1}{2}(T^M_{\alpha \beta} + T^M_{\beta \alpha})$$

(3.5)
gives the following value for the density of the energy flux

$$c \frac{1}{4\pi} \frac{c}{2} (|DB| + |EH|)_{v=0} = \frac{\varepsilon \mu + 1}{2} c \frac{1}{4\pi} |EH|.$$  (3.6)

In order to get here the Poynting vector (2.9), in accordance with the requirement (3.3), one has to subtract from (3.6) the following quantity

$$\frac{\varepsilon \mu - 1}{2} c \frac{1}{4\pi} |EH| = \frac{c}{4\pi} \frac{1}{2} (|DB| - |EH|)_{v=0}.$$  (3.7)

In the co-moving frame, the spatial component of the 4-vector

$$\frac{\varepsilon \mu - 1}{2} c \frac{1}{4\pi} \Omega_\alpha,$$

(3.8)
where $\Omega_\alpha$ is the Minkowski vector (2.16), (2.17), reproduces the left-hand side of Eq. (3.7). Bearing this in mind, it is natural to add to the tensor (3.5) the symmetric tensor constructed from two 4-vectors $u_\alpha$ and ($\varepsilon \mu - 1$) $\Omega_\beta$:

$$T^{(1)}_{\alpha \beta} = \frac{1}{2}(T^M_{\alpha \beta} + T^M_{\beta \alpha}) + \frac{1}{2} \frac{\varepsilon \mu - 1}{2} (u_\alpha \Omega_\beta + u_\beta \Omega_\alpha).$$  (3.9)
Taking into account (1.1) and (2.17), one can easily verify that the tensor (3.9) in the rest frame \((v = 0)\) and under the fulfillment the constitutive relations (3.1) satisfies the conditions (3.2)–(3.4). The tensor (3.9) in view of (2.18) has vanishing trace.

Formally the tensor (3.9) is defined on the whole configuration space \(\Gamma\). However in order to carry out the needed subtraction at \(v = 0\) and fulfill condition (3.3), this tensor should be considered only on \(\Gamma\). Obviously this restriction must be also kept in the case of moving medium \((v \neq 0)\), in order to have a smooth dependence on \(v\) of the tensor looking for.

Abraham proceeds exactly in this way. He defines the energy-momentum tensor on \(\Gamma\) as the contraction of the tensor \(T^{(1)}_{\alpha\beta}\) on \(\Gamma\) [10, p. 42]:

\[
T^A_{\alpha\beta} \bigg|_\Gamma = T^{(1)}_{\alpha\beta} \bigg|_\Gamma.
\]

(3.10)

The task to define the energy-momentum tensor on the whole configuration space \(\Gamma\) was not posed by Abraham.

By making use of Eqs. (2.10), (2.11), we can represent the 4-vector \((\varepsilon \mu - 1) \Omega_\alpha\) on \(\Gamma\) in the form [18, p. 46 equation (106)]

\[
(\varepsilon \mu - 1) \Omega_\alpha = u_\beta (D_\gamma B_{\alpha\beta\gamma} - E_\gamma H_{\alpha\beta\gamma})|_\Gamma.
\]

(3.11)

Further it is convenient to express the 4-vector (3.11) in terms of one three dimensional vector \(\mathbf{w}\) [18, Eq. (107)]:

\[
(\varepsilon \mu - 1) \Omega = \frac{\mathbf{w}}{\gamma}, \quad (\varepsilon \mu - 1) \Omega_4 = \frac{i}{\gamma} (\mathbf{q} \mathbf{w}).
\]

(3.12)

In the case of isotropic medium moving with the velocity \(v\) all the components of the energy-momentum tensor (3.9), (3.10)

\[
T^A_{\alpha\beta} = \begin{pmatrix}
\sigma^A_{ij} & -ic g^A \\
-ic S^A & w^A
\end{pmatrix}
\]

were expressed by Abraham in terms of the field vectors \(\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}\), the velocity of the medium \(\mathbf{q}\) and the vector \(\mathbf{w}\) [10, Eqs. (24a,b,c)]

\[
\sigma^A_{ij} = \frac{1}{2} (\sigma^M_{ij} + \sigma^M_{ji}) + \frac{1}{8\pi} (q_i w_j + q_j w_i),
\]

(3.14)

\[
\frac{1}{c} S^A = \frac{1}{8\pi} \{[\mathbf{E}\mathbf{H}] + [\mathbf{D}\mathbf{B}] - 2\mathbf{w} - \mathbf{q}(\mathbf{q}\mathbf{w})\} = cg^A,
\]

(3.15)

\[
w^A = w^M - \frac{1}{4\pi} (\mathbf{q}\mathbf{w}).
\]

(3.16)

\footnote{The subtraction needed is brought about due to the sign minus in front of \(S^M\) and \(g^M\) in (2.5).}
In turn the vector $\mathbf{W}$, defined in (3.12), is also expressed in terms of the field vectors and the velocity of the medium. Abraham [10, p. 42] uses here the constitutive relations in the 3-dimensional vector form (2.14), (2.15). Kafka [18, pp. 48–51] applies here the 4-dimensional notation. As a preliminary Kafka [18, p. 51, Eq. (119)] proves the relation

$$
(\varepsilon\mu - 1)(u_\alpha\Omega_\beta - u_\beta\Omega_\alpha) |_{\Gamma} = F_{\alpha\gamma}H_{\gamma\beta} - F_{\beta\gamma}H_{\gamma\alpha} |_{\Gamma},
$$

(3.17)

which holds on $\Gamma$.

We shall not reproduce here a rather cumbersome proof of this formula (see work [18]), and only note the following. On the left-hand side in Eq. (3.17), the dependence on $\varepsilon\mu$ disappear as a result of using the constitutive relations (2.10), (2.11). When the pair of indices $\alpha \beta$ in Eq. (3.17) assumes the values 1 4, 2 4, 3 4, the following equality arises here

$$
- i\{\mathbf{W} - q(q\mathbf{W})\} = -i\{[\mathbf{DB}] - [\mathbf{EH}]\},
$$

(3.18)

or in another form

$$
\mathbf{W} = [\mathbf{DB}] - [\mathbf{EH}] + q(q\mathbf{W}).
$$

(3.19)

From (3.19) it follows

$$
(q\mathbf{W}) = (q, [\mathbf{DB}] - [\mathbf{EH}]) + q^2(q\mathbf{W}) = 
\gamma^2(q, [\mathbf{DB}] - [\mathbf{EH}]).
$$

(3.20)

Finally we arrive at the result

$$
\mathbf{W} = [\mathbf{DB}] - [\mathbf{EH}] + \gamma^2q(q, [\mathbf{DB}] - [\mathbf{EH}]).
$$

(3.21)

The substitution of (3.21) into (3.14) and (3.16) yields

$$
\frac{1}{c}S^A = \varepsilon g^A = \frac{1}{4\pi}[\mathbf{EH}] - \frac{q}{4\pi(1 - q^2)} (q, [\mathbf{DB}] - [\mathbf{EH}]),
$$

(3.22)

$$
w^A = \frac{1}{8\pi}(\mathbf{ED} + \mathbf{HB}) - \frac{1}{4\pi(1 - q^2)} (q, [\mathbf{DB}] - [\mathbf{EH}]).
$$

(3.23)

Here we shall not write out the bulky formulae which are obtained as a result of substituting Eq. (3.21) into (3.11). Their explicit form is obvious. The energy-momentum tensor constructed in this way is symmetric.

Let us note once more that Eqs. (3.14)–(3.16) and (3.21)–(3.23) determine the components of the Abraham energy-momentum only on $\Gamma$ and do not include the material characteristics
of the medium. The Minkowski constitutive relations (2.10), (2.11) or (2.14), (2.15) are used by Abraham specifically; namely, they are not used directly to express one pair of field vectors from the set \( \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B} \) in terms of the rest pair, but special formulae are used which contain all the vectors \( \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \mathbf{v} \) and the material characteristics \( \varepsilon, \mu \), these formulae being valid only under fulfillment of the Minkowski constitutive relations, i.e. on \( \Gamma \). An important example of such a formula is the equality (3.17). When the pair of indices \( \alpha \beta \) in Eq. (3.17) takes the values 23, 31, and 12, one obtains the equality \[ [\mathbf{qW}] = [\mathbf{ED}] + [\mathbf{HB}] \].

Taking into account (3.19), we deduce
\[ [\mathbf{q}, [\mathbf{DB}] - [\mathbf{EH}]] = [\mathbf{ED}] + [\mathbf{HB}] \]. (3.24)

It is obvious that this equality holds only under fulfillment of the Minkowski relations (2.14), (2.15).

Abraham deduced his formulae only on \( \Gamma \). Therefore it is not clear whether 10 quantities defined in this way can be definitely considered as the components of a 4-dimensional tensor. In order to verify this one has, first of all, to define the quantities under consideration on the whole configuration space \( \Gamma \). This will be done in the next Section.

Taking advantage of formula (3.17) we can represent the tensor \( T^{(1)}_{\alpha\beta} \) (3.9), defined on \( \Gamma \), in another form. Let us write the definition (3.9) and formula (3.17), multiplied by \( 1/2 \), on the neighbouring lines:
\[
4\pi T^{(1)}_{\alpha\beta} = \frac{1}{2} F_{\alpha\gamma} H_{\gamma\beta} + \frac{1}{2} F_{\beta\gamma} H_{\gamma\alpha} - \frac{1}{4} \delta_{\alpha\beta} F_{\gamma\delta} H_{\gamma\delta} + \frac{\varepsilon \mu - 1}{2} (u_{\alpha} \Omega_{\beta} + u_{\beta} \Omega_{\alpha}),
\]

\[ 0 = \frac{1}{2} F_{\alpha\gamma} H_{\gamma\beta} - \frac{1}{2} F_{\beta\gamma} H_{\gamma\alpha} - \frac{\varepsilon \mu - 1}{2} (u_{\alpha} \Omega_{\beta} - u_{\beta} \Omega_{\alpha}). \] (3.26)

Adding and subtracting the left-hand sides and the right-hand sides of these equations we obtain on \( \Gamma \):
\[ T^{(1)}_{\alpha\beta} = T^{(2)}_{\alpha\beta} = T^{(3)}_{\alpha\beta}, \] (3.27)

where
\[ T^{(2)}_{\alpha\beta} = T^{M}_{\alpha\beta} + \frac{\varepsilon \mu - 1}{4\pi} u_{\beta} \Omega_{\alpha}, \] (3.28)
\[ T^{(3)}_{\alpha\beta} = T^{M}_{\beta\alpha} + \frac{\varepsilon \mu - 1}{4\pi} u_{\alpha} \Omega_{\beta}. \] (3.29)

Appealing to the works [2, 10, 18], Abraham believed that the energy-momentum tensor, constructed by him, possesses correct properties under Lorentz transformations [8, p. 309].
If we use in (3.26) Eq. (3.17), multiplied by an arbitrary real number, then we obviously can continue to infinity the chain of equalities in (3.27), and consequently in the definition (3.11). However the tensors $T^{(i)}_{\alpha\beta}$, $i = 2, 3, \ldots$ constructed in this way give nothing new in comparison with $T^{(1)}_{\alpha\beta}$. On $\Gamma$ they are equal to $T^{(1)}_{\alpha\beta}$ and to the Abraham tensor according to (3.10), that is all these tensors are simply another form of representation for the Abraham tensor. In spite of the fact that these tensors represented in terms of vector $\mathbf{W}$ in accordance with (3.12), do not, in general case, coincide with Eqs. (3.14)–(3.16) and are not explicitly symmetric, but upon substituting $\mathbf{W}$ by (3.21), the components of these tensors are defined by (3.22)–(3.23) and Maxwell stress tensors also coincide.

Let us show this using the tensor $T^{(2)}_{\alpha\beta}$, which is, according to construction, the most close to the Minkowski energy-momentum tensor. The components of the tensor $T^{(2)}_{\alpha\beta}$ are defined by Eqs. (3.12) and (3.28):

\[
\sigma^{(2)}_{ij} = \sigma^M_{ij} + \frac{1}{4\pi} \mathbf{W}_i q_j, \tag{3.30}
\]

\[
\frac{1}{c} S^{(2)} = \frac{1}{4\pi} [\mathbf{E}\mathbf{H}] + \frac{1}{4\pi} q(\mathbf{q}\mathbf{W}), \tag{3.31}
\]

\[
cg^{(2)} = \frac{1}{4\pi} [\mathbf{D}\mathbf{B}] - \frac{1}{4\pi} \mathbf{W}, \tag{3.32}
\]

\[
w^{(2)} = w^M - \frac{1}{4\pi} (\mathbf{q}\mathbf{W}). \tag{3.33}
\]

It is this form that was used by Abraham for the representation of the components of his energy-momentum tensor in his book [7], [8] (see. $\S$ 39, Eqs. (199e), (199d), (201) (201a)). Substituting in (3.31) and (3.32) the vector $\mathbf{W}$ by (3.21), we can easily obtain (3.22) and (3.23). The identity of the 3-dimensional Maxwell stress tensor represented in the forms (3.30) and (3.11) is proved by Abraham in his article [10, p.p. 42, 43].

Pauli in his survey [1, p. 666, Eq. 303] do not note that the tensors $T^{(i)}_{\alpha\beta}$, $i = 1, 2, 3$ are equal to each other only on $\Gamma$ and that in the rest frame these tensors coincide only due to the constitutive relations (3.1). In view of this the status of Eq. (303) in the Pauli treatment (see [1, p. 666]) was left unclear.

The employment of the tensors $T^{(i)}_{\alpha\beta}$, $i = 1, 2, 3$ outside $\Gamma$ is not physically justified because outside $\Gamma$ these tensors have no relation to the Abraham energy-momentum tensor. Indeed, outside $\Gamma$ these tensors are different and already due to this reason one cannot consider them as an extension of the Abraham formulae (3.14)–(3.16), (3.21)–(3.23) on the whole configuration space $\Gamma$. The consistent extension of the Abraham formulae on $\Gamma$ will
be implemented in the next section of the present paper.

IV. THE SYMMETRIC ENERGY-MOMENTUM TENSOR

In the Introduction, it was explained that term ‘symmetric energy-momentum tensor’ implies the following. This tensor is defined by known values of its components in the rest frame (see Eqs. (298) and (300) in [5, Section 35] and is determined on the whole configuration space $\Gamma$. In the present Section we construct this tensor generalizing the Abraham reasoning considered in the preceding Section. This will enable us to prove, rather simple, the identity of the symmetric energy-momentum tensor and the Abraham tensor on $\Gamma$. Let us recall that the term ‘Abraham tensor’ we kept for Eqs. (3.13)–(3.16), (3.21)–(3.23) derived by Abraham himself only on $\Gamma$.

It is obvious that conditions (3.2)–(3.4) enable us to define the symmetric energy-momentum tensor on $\Gamma$ if we demand their fulfilment in the co-moving reference frame without using the constitutive relations (3.1). Below we demonstrate this.

In the generic inertial reference frame, tensor (3.5) yields the density of the energy flux

$$\frac{c}{4\pi} \left( [DB] + [EH] \right).$$

(4.1)

In order to get here the Poynting vector (2.9), in agreement with condition (3.3), one must subtract from (4.1) the following quantity

$$\frac{c}{4\pi} \left( [DB] - [EH] \right).$$

(4.2)

Let us consider the 4-vector $\tilde{\Omega}_\alpha$ which is obtained from the Minkowski vector $\Omega_\alpha$ (2.16) by the substitution

$$E \rightarrow D, \quad H \rightarrow B, \quad D \rightarrow E, \quad B \rightarrow H.$$  

(4.3)

Using Eq. (2.16), one can construct for $\tilde{\Omega}_\alpha$ the explicitly covariant formula

$$\tilde{\Omega}_\alpha = B_{\alpha\beta\gamma} u_\beta D_\gamma.$$  

(4.4)

In the co-moving reference frame, Eq. (4.4) gives:

$$\frac{1}{4\pi} (\tilde{\Omega}_1', \tilde{\Omega}_2', \tilde{\Omega}_3') = \frac{1}{4\pi} [D'B'], \quad \tilde{\Omega}_4' = 0$$

(4.5)

(compare with Eqs. (2.17), (2.9)).
Now it is clear that the 4-vector (3.11), which was used by Abraham in constructing the energy-momentum tensor only on $\Gamma$, should be substituted by the difference

$$\tilde{\Omega}_\alpha - \Omega_\alpha = u_\beta (D_\gamma B_\alpha B_\gamma - E_\gamma H_\alpha B_\gamma) =$$

$$= F_{\alpha\nu} H_{\nu\lambda} u_\lambda - H_{\alpha\nu} F_{\nu\lambda} u_\lambda = \tilde{\omega}_\alpha - \omega_\alpha,$$  

(4.6)

where

$$\omega_\alpha = H_{\alpha\nu} F_{\nu\lambda} u_\lambda; \quad \tilde{\omega}_\alpha = F_{\alpha\nu} H_{\nu\lambda} u_\lambda.$$  

(4.7)

In place of $T^{(1)}_{\alpha\beta}$ defined in (3.9), we have now, on the whole configuration space $\Gamma$, the tensor which we call, for definiteness, the symmetric energy-momentum tensor:

$$T^{\text{sym}}_{\alpha\beta} = \frac{1}{2} (T^{M}_{\alpha\beta} + T^{M}_{\beta\alpha}) + A_{\alpha\beta},$$  

(4.9)

where

$$A_{\alpha\beta} = A_{\beta\alpha} = \frac{1}{8\pi} \{u_\alpha (\tilde{\Omega}_\beta - \Omega_\beta) + u_\beta (\tilde{\Omega}_\alpha - \Omega_\alpha)\} =$$

$$= \frac{1}{8\pi} \{u_\alpha (F_{\beta\nu} H_{\nu\lambda} u_\lambda - H_{\beta\nu} F_{\nu\lambda} u_\lambda) + u_\beta (F_{\alpha\nu} H_{\nu\lambda} u_\lambda - H_{\alpha\nu} F_{\nu\lambda} u_\lambda)\}.$$  

(4.10)

Taking into account Eqs. (2.17) and (4.5), one can easily verify that in the rest frame ($v = 0$) the tensor $T^{\text{sym}}_{\alpha\beta}$ satisfies the conditions (3.2)–(3.4) without utilizing the constitutive relations (3.1).

In another way (see Section V B of the present paper), the symmetric energy-momentum tensor (4.9), (4.11) was constructed in the recent paper by V.P. Makarov and A.A. Rukhadze [3]. The authors called it by the Abraham tensor but juxtaposition of Eqs. (4.9), (4.11) and Eqs. (3.14)–(3.16), (3.21)–(3.23) was not implemented in [3]. In our approach, the coincidence of these tensors on $\Gamma$ follows immediately from the comparison of (4.6) and (3.11). In the analytical way, it is proved as follows. Taking advantage of relation (3.17), valid on $\Gamma$ and being read from right to left, and using property (2.18) of the Minkowski vector $\Omega_\alpha$, formula (4.11) can be transformed to the form

$$A_{\alpha\beta} |_{\Gamma} = \frac{1}{4\pi} \frac{\varepsilon \mu - 1}{2} (u_\alpha \Omega_\beta + u_\beta \Omega_\alpha) |_{\Gamma}.$$  

(4.12)

Keeping in mind Eqs. (4.9), (3.9) and (3.10), we obtain

$$T^{\text{sym}}_{\alpha\beta} |_{\Gamma} = T^{(1)}_{\alpha\beta} |_{\Gamma} = T^{A}_{\alpha\beta} |_{\Gamma}.$$  

(4.13)

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Below we prove equality \((4.13)\) once more by making use of the 3-dimensional vector formula \((3.24)\) in place of \((3.17)\).

When the energy-momentum tensor applies to practical calculations, the explicit form of its components is, as a rule, needed in terms of the 3-dimensional vector notation. We turn now to the construction of suchformulae for the symmetric energy-momentum tensor on \(\Gamma\).

First we derive the 3-dimensional vector representation for the auxiliary 4-vectors \(\omega_\alpha\) and \(\tilde{\omega}_\alpha\), introduced in Eqs. \((4.7)\) and \((4.8)\). From \((1.1)\) and \((2.12)\) it follows

\[
\omega_\alpha = \gamma\{ [EH] - [H[qB]] + D(qE); i(ED) - i(q[DB]) \} .
\] (4.14)

Obviously the 4-vector \(\tilde{\omega}_\alpha\) is derived from \(\omega_\alpha\) by the substitution \((4.3)\):

\[
\tilde{\omega}_\alpha = \gamma\{ [DB] - [B[qH]] + E(qD); i(ED) - i(q[EH]) \} .
\] (4.15)

The difference \(\tilde{\omega}_\alpha - \omega_\alpha\) becomes

\[
\tilde{\omega}_\alpha - \omega_\alpha = \gamma\{ [DB] + E(qD) + H(qB) - [EH] - D(qE) - B(qH); i(q[DB]) - i(q[EH]) \} .
\] (4.16)

By analogy with \((3.12)\) it is convenient to represent the difference \((4.6), \(4.7), \(4.16)\) in the following form

\[
\tilde{\Omega} - \Omega = \frac{W}{\gamma} , \quad \tilde{\Omega}_4 - \Omega_4 = \frac{i}{\gamma} (qW) ,
\] (4.17)

where the vector \(W\) is

\[
W = \gamma^2\{ [DB] - [EH] + E(qD) + H(qB) - B(qH) - D(qE) \} .
\] (4.18)

Equation \((4.18)\) yields in particular

\[
(qW) = \gamma^2\{ (q[DB]) - (q[EH]) \} .
\] (4.19)

Let us introduce the standard notation for the components of the tensor \(T^{sym}_{\alpha\beta}\) in a generic inertial reference frame \(K\), where the medium velocity is \(v\):

\[
T^{sym}_{\alpha\beta} = \begin{pmatrix}
\sigma^{sym}_{ij} & -ic g^{sym}_{ij} \\
-ic S^{sym}_{ij} & w^{sym}_{ij}
\end{pmatrix} .
\] (4.20)
Equations (4.9), (4.10), and (4.17) yield for the individual components of the symmetric tensor (4.20) the following 3-dimensional vector formulae

\[ \sigma_{ij}^{\text{sym}} = \frac{1}{2}(\sigma^{M}_{ij} + \sigma^{M}_{ji}) + \frac{1}{8\pi}(q_{i}W_{j} + q_{j}W_{i}) , \]  
\[ \frac{1}{c}S_{i}^{\text{sym}} = \frac{1}{8\pi}\{[EH] + [DB] - W - q(qW)\} = c\mathbf{g}_{i}^{\text{sym}} , \]  
\[ w_{i}^{\text{sym}} = w^{M} - \frac{1}{4\pi}\mathbf{q}(W) . \]  

These formulae are also obtained by the substitution \( 2\mathbf{w} \rightarrow \mathbf{W} \) in analogous equations (3.14)–(3.16) for the Abraham tensor. Equations (4.21)–(4.23) and (4.18) determine, in an explicit form, the symmetric energy-momentum tensor (4.20) on the whole configuration space \( \Gamma \).

Comparing Eqs. (3.20) and (4.19), we infer that the energy density given by the symmetric energy-momentum tensor (4.23) and by the Abraham tensor (3.16) are equal to each other on the whole configuration space \( \Gamma \) and are defined by (3.23)

\[ w^{\text{sym}} = w^{A} = \frac{1}{8\pi}(ED + HB) - \frac{1}{4\pi(1 - q^{2})}(\mathbf{q}[DB] - [EH]) . \]  

In the general case, the remaining components of these tensors, by virtue of (3.21) and (4.18), do not coincide beyond \( \Gamma \). However when the problem under consideration allows to direct one of the coordinate axes (for example, the \( x \)-axis) along the medium velocity \( \mathbf{v} \):

\[ \mathbf{q} = (q_{x}, 0, 0) , \]  

then some other components of these tensors will also equal on \( \Gamma \), namely, the diagonal components of the Maxwell stress tensors

\[ \sigma_{xx}^{\text{sym}} = \sigma_{xx}^{A} , \quad \sigma_{yy}^{\text{sym}} = \sigma_{yy}^{A} , \quad \sigma_{zz}^{\text{sym}} = \sigma_{zz}^{A} \]  

and the components

\[ \sigma_{yz}^{\text{sym}} = \sigma_{zy}^{\text{sym}} = \sigma_{yz}^{A} = \sigma_{zy}^{A} . \]  

The first equality in (4.26) is the consequence of the relation

\[ W_{x} = \gamma^{2}\{[DB]_{x} - [EH]_{x}\} = 2\mathbf{w}_{x} , \]  

which holds under condition (4.25) (see Eqs. (4.18) and (3.21)). The remaining equalities in (4.26) and equalities (4.27) follow from (4.25).
Now we prove that the 3-dimensional vectors $\mathbf{W}$ (4.18) and $\mathbf{W}$ (3.21) equal on $\Gamma$. For this purpose, we construct the vector product of $\mathbf{q}$ and the relation (3.24) valid on $\Gamma$:

$$[\mathbf{q}[\mathbf{q}[\mathbf{DB} - \mathbf{EH}]]] = \mathbf{q}(\mathbf{q}, [\mathbf{DB} - \mathbf{EH}]) - q^2([\mathbf{DB} - \mathbf{EH}]) =$$

$$= [\mathbf{q}, [\mathbf{ED}]] + [\mathbf{q}[\mathbf{HB}]] = E(qD) - D(qE) + H(qB) - B(qH). \quad (4.29)$$

The substitution of (4.29) into (4.18) yields

$$\mathbf{W} = \gamma^2 \{[\mathbf{DB} - \mathbf{EH}] - q^2([\mathbf{DB} - \mathbf{EH}] + \mathbf{q} \cdot [\mathbf{DB} - \mathbf{EH}])\} =$$

$$= [\mathbf{DB} - \mathbf{EH}] + \gamma^2 \mathbf{q}(\mathbf{q}, ([\mathbf{DB} - \mathbf{EH}]) = 2\mathbf{W}. \quad (4.30)$$

It implies that the symmetric energy-momentum tensor (4.20)–(4.23) and the Abraham tensor (3.13)–(3.16) are identical on $\Gamma$. In other words, this inference may be formulated like this: the correct extension of the Abraham formulae (3.13)–(3.16) onto the whole configuration space $\Gamma$ is accomplished by the symmetric energy-momentum tensor (4.20)–(4.23).

Closing this Section, we make an important remark. In practical calculations, aimed at obtaining the final physical result, it is obligatory assumed the use of the Minkowski constitutive relations (2.10), (2.11) or (2.14), (2.15). Therefore in such studies, it makes sense to utilize, from the very beginning, the energy-momentum tensor in the Abraham form as the respective formulae are substantially more compact in comparison with those given by the symmetric tensor. This concerns the density of the energy current, the linear momentum density (3.22), and especially the components of the 3-dimensional Maxwell stress tensor in the form (3.30).

V. FORM-INVARINACE OF THE VECTOR FORMULAE FOR THE ENERGY-MOMENTUM TENSOR

A. The methods of defining the Minkowski tensor and the form-invariance of the vector formulae involved

In this Subsection, we compare two methods of defining the 4-dimensional tensor: in the explicitly covariant form and in the 3-dimensional vector form. As an example, we shall use the Minkowski energy-momentum tensor. In this case one can clear show the covariance properties of the 3-dimensional vector formulae for the Minkowski tensor, which result in their form-invariance.
1. Definition of the Minkowski tensor by Eq. (2.4)

First we consider the definition of the Minkowski tensor by the 4-dimensional explicitly covariant expression (2.4), and in addition we postulate that the latter is valid both for the medium at rest and for the moving medium. It is worth emphasizing that the velocity of the medium does not enter Eq. (2.4). In terms of the 3-dimensional vector notation the components of the tensor $T^{M}_{\alpha\beta}$ (2.4), in the arbitrary inertial reference frame $K$, are given by Eqs. (2.5)–(2.8).

It is obvious that all the components of the Minkowski tensor (2.5), expressed in terms of the field vectors $E, H, D, B$ according to Eqs. (2.6)–(2.8), keep their functional dependence, specified by these formulae, in a generic reference frame. Indeed, when passing from Eq. (2.4) to Eqs. (2.5)–(2.8) the reference frame was not fixed or specified. This assertion is true both for the medium at rest and for the moving medium since the velocity of medium enter neither explicitly covariant formula (2.4) nor the consequent vector formulae (2.5)–(2.8).

Thus the transformation of the tensor $T^{M}_{\alpha\beta}$, defined in a generic reference frame $K$ originally by an explicitly covariant formula (2.4) and afterwards written in the vector form (2.5)–(2.8), to a new inertial reference frame, $K''$, is simply reduced to the substitution $F \to F''$ in Eqs. (2.6)–(2.8):

$$T''^{M}_{\alpha\beta}(F'') = T^{M}_{\alpha\beta}(F')$$

(5.1)

where $F$ means the set of the field vectors

$$F = \{ E, H, D, B \}.$$  

(5.2)

As before, the field vectors without prime belong to an arbitrary inertial reference frame $K$ and the vectors with two primes pertain to analogous frame $K''$. This form-invariance property of the 3-dimensional vector formulae (2.5)–(2.8) is a straightforward consequence of the relativistic covariance of definition (2.4).

Certainly, the numerical values of the individual components of the tensor, calculated by Eqs. (2.5)–(2.8) at respective points $r_{\alpha} = (x, y, z, ict)$ and $r''_{\alpha} = (x'', y'', z'', ict'')$, vary in accordance with the transformation law for the tensor

$$T''^{M}_{\alpha\beta}(F''(r''_{\gamma})) = \Lambda_{\alpha\delta} \Lambda_{\beta\rho} T^{M}_{\delta\rho}(F(r_{\sigma})),$$

(5.3)

where $\Lambda_{\alpha\beta}$ is the matrix of the Lorentz transformation connecting the points $r''_{\alpha}$ and $r_{\beta}$:

$$r''_{\alpha} = \Lambda_{\alpha\beta} r_{\beta}.$$  

(5.4)
The assertion above, concerning the Lorentz transformation of the components of the tensor $T^M_{\alpha\beta}$ in the vectorial form, holds obviously for an arbitrary tensor of any rank.

2. Definition of the Minkowski tensor by vector Eqs. (2.5)–(2.8)

The Minkowski energy-momentum tensor, as a generic tensor, can be uniquely defined by fixing its components in a specified reference frame. Let us assume that in the co-moving frame $K'$, where the medium is at rest, the components of the tensor $T^M_{\alpha\beta}$ are defined by vector formulae (2.5)–(2.8). All the vectors and their components in these formulae should be provided with the prime, in accord with our notation. In order to find the components of the tensor under consideration in arbitrary inertial reference frame we can proceed in two ways: i) to construct the 4-dimensional explicitly covariant formula, which reproduces Eqs. (2.5)–(2.8) in the reference frame $K'$, for example, generalizing the energy-momentum tensor in the vacuum \cite{4, §115}; ii) by making use of Eq. (5.3) to transform the 3-dimensional Eqs. (2.5)–(2.8) to the generic reference frame $K$. Here we cannot take advantage of the form-invariance (5.1) since Eqs. (2.5)–(2.8), according to the task posing, define the Minkowski tensor not in an arbitrary reference frame $K$, but in the co-moving frame $K'$.

The first way was, in fact, traced above (Section \text{V A 1}). In order to apply here the Lorentz transformations one has to resort to Eq. (5.2) which becomes in the case under consideration

$$T^M_{\alpha\beta}(F(r_\gamma)) = \Lambda_{\alpha\delta}\Lambda_{\beta\epsilon}T'^M_{\delta\epsilon}(F'(r'_{\sigma})). \tag{5.5}$$

According to the conditions of the task, the components of the tensor $T'^M_{\delta\epsilon}(F')$ in (5.5) are given by the vectorial equations (2.5)–(2.8). Further one has to express the field vectors $F'$ on the right-hand side in (5.5) in terms of $F$. Having in mind that for Eqs. (2.5)–(2.8) there is the 4-dimensional explicitly covariant representation (2.4), one can predict beforehand the result of these transformations

$$T^M_{\alpha\beta}(F(r_\gamma)) = T'^M_{\alpha\beta}(F(r_\gamma)). \tag{5.6}$$

\footnote{I.E. Tamm \cite[§115]{4} has postulated the form of the Minkowski tensor by Eqs. (2.5)–(2.8) (the present paper) both in the co-moving frame and in a generic inertial frame at once, in order to escape resort to employment of the relativity theory.}

\footnote{The employment of the Lorentz transformations in construction of the symmetric energy-momentum tensor is discussed in Section \text{V B}.}
Nevertheless, in a recent paper \[19\], these transformations were carried out for the component \( T_{\text{M}x} \), and it was shown its form-invariance. This property of the Minkowski tensor was treated in \[19\], not completely correct, as the manifestation of this tensor (see critical note in \[3\]). As was shown above, this form-invariance is in fact the consequence of the covariance property of individual components of generic tensor written in vector notation.

In pertinent literature (see, for example, \[17\]), the form-invariance of the vectorial formulae, determining the components of the 4-dimensional vectors and tensors in the theory of electromagnetic field is not considered, probably owing to clearness of this property. Nevertheless we believe that detailed discussion of this point presented above will be useful.

B. Form-invariance of the symmetric energy-momentum tensor and the Abraham tensor in the vector notation

For these tensors we have explicitly covariant 4-dimensional representations \( (4.9) \), \( (4.10) \) and \( (3.9) \), \( (3.10) \). Therefore the derived from them 3-dimensional vectorial Eqs. \( (4.20) \)–\( (4.23) \), \( (4.18) \) and \( (3.13) \)–\( (3.16) \), \( (3.21) \), \( (3.22) \), \( (3.23) \) preserve the form of functional dependence on field vectors \( \mathbf{F} \) \( (5.2) \) and velocity of medium \( \mathbf{v} \) when passing from one inertial reference frame to another. The transformation rule \( (5.1) \) generalized to this case reads

\[
T_{\alpha\beta}^{\prime\prime\text{sym}}(\mathbf{F}^{\prime\prime}, \mathbf{v}^{\prime\prime}) = T_{\alpha\beta}^{\text{sym}}(\mathbf{F}^{\prime\prime}, \mathbf{v}^{\prime\prime}), \quad (5.7)
\]

\[
T_{\alpha\beta}^{\prime\prime\text{A}}(\mathbf{F}^{\prime\prime}, \mathbf{v}^{\prime\prime}) = T_{\alpha\beta}^{\text{A}}(\mathbf{F}^{\prime\prime}, \mathbf{v}^{\prime\prime}). \quad (5.8)
\]

This rule enables one to find the tensors \( T_{\alpha\beta}^{\prime\prime\text{sym}}(\mathbf{F}^{\prime\prime}, \mathbf{v}^{\prime\prime}) \) and \( T_{\alpha\beta}^{\prime\prime\text{A}}(\mathbf{F}^{\prime\prime}, \mathbf{v}^{\prime\prime}) \) in an generic inertial reference frame \( K^{\prime\prime} \), provided the reference frame \( K \) with tensors \( T_{\alpha\beta}^{\text{sym}}(\mathbf{F}, \mathbf{v}) \) and \( T_{\alpha\beta}^{\text{A}}(\mathbf{F}, \mathbf{v}) \) is a reference frame of a generic type, in particular, \( \mathbf{v} \neq 0 \), i.e. \( K \) is not the co-moving frame. Hence the transformation rules \( (5.7) \) and \( (5.8) \) do not allow to find the symmetric energy-momentum tensor and the Abraham tensor in arbitrary reference frame \( K^{\prime\prime} \) proceeding from Eqs. \( (3.2) \)–\( (3.4) \) for these tensors in the rest frame \( K^{\prime} \) with \( \mathbf{v} = 0 \) (without allowance for the constitutive relations \( (3.1) \) and with this allowance, respectively). Here the method must be used, which was employed in Sections III and IV or one has to resort to the Lorentz

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\( ^8 \) In the work \[19\] the co-moving frame was denoted by \( K \) and generic inertial frame by \( K^{\prime} \). The component \( T_{\text{M}x}^{\text{M}} \), defined in \( K \) by Eq. \( (2.6) \) (present paper), was transformed into the reference frame \( K^{\prime} \). It was shown that \( T_{\text{M}x}^{\prime\prime}(\mathbf{F}^{\prime\prime}) = T_{\text{M}x}^{\prime}(\mathbf{F}^{\prime}) \).
transformations (5.5). The latter became in the case of the tensor $T_{\alpha\beta}^{\text{sym}}$

$$T_{\alpha\beta}^{\text{sym}}(F(r)) = \Lambda_{\alpha\delta}^{\beta\epsilon} T_{\delta\epsilon}^{\prime \text{sym}}(F'(r')) ,$$

(5.9)

where $\Lambda_{\alpha\beta}$ is the matrix of the Lorentz transformation connecting the points $r_{\alpha}$ and $r'_{\delta}$:

$$r_{\alpha} = \Lambda_{\alpha\delta} r'_{\delta} .$$

(5.10)

As before, the variables in an arbitrary inertial reference frame $K$ are not marked by prime, and primed variables are referred to the co-moving frame $K'$. Obviously the velocity of generic inertial reference frame $K$ with respect to the rest frame $K'$ is $-v$, where $v$ is the velocity of the medium in $K'$. Hence the matrix of the Lorentz transformation in (5.10) is defined by the velocity $-v$: $\Lambda_{\alpha\beta} = \Lambda_{\alpha\beta}(-v)$.

The tensor $T_{\alpha\beta}^{\prime \text{sym}}(F')$ in the rest frame $K'$ is specified by Eqs. (3.2)–(3.4). Further the field vectors $F'$ on the right-hand side in Eq. (5.9) must be expressed in terms of the field vectors $F$ in the generic inertial reference frame $K$. The latter is implemented by the transformation $\Lambda(+v)$.

The Lorentz transformations, considered above, were carried out by V.P. Makarov and A.A. Rukhadze [3] and, what is more, for all the components of the tensor $T_{\alpha\beta}^{\text{sym}}$ at once and for arbitrary direction of the medium velocity with respect to the coordinate axes.

For a single component of the symmetric energy-momentum tensor $T_{xx}^{\text{sym}} = \sigma_{xx}^{\text{sym}}$ the analogous transformations were in fact performed in the work [19], the $x$-axis being directed along the velocity of the medium $v$: $v = (v_{x}, 0, 0)$.

(5.11)

The authors of the paper [19] erroneously interpreted the obtained dependence of the component $T_{xx}^{\text{sym}}$ on $v$ as an indication that this energy-momentum tensor ‘is not relativistically invariant’. As a matter of fact, in this paper the exact expression was derived for the $xx$-component of the symmetric energy-momentum tensor in a generic inertial reference frame, where the medium has the velocity $v_{x}/c = -\beta$. One can be easily convinced of this by comparing equation (23) in the work [19], preliminarily changing the sign of $\beta$, with the $xx$-component of the Abraham tensor in the form $T_{\alpha\beta}^{(2)}$ (3.28), (3.30)–(3.33). This tensor, borrowed from the book [7], is presented in the Appendix 4 of the survey [11] under the condition (5.11). Here one has to bear in mind the following. The diagonal elements of
the tensor $T^{(2)}_{\alpha \beta}$, Eq. (3.28)), and of the tensor $T^A_{\alpha \beta}$, Eqs. (3.10), (3.14)–(3.16)), obviously coincide on the whole configuration space $\Gamma$. Further, when the condition (4.25) is valid, the equalities (4.26) and (4.27) hold. Thus, $\sigma^{(2)}_{xx} = \sigma_{xx}^{\text{sym}}$ on $\Gamma$. In the paper [3, p. 1361, Russian addition] it was explained in what way the setting of the problem in the work [19] should be supplemented in order to get the correct inference given above. However this inference was not formulated in [3].

VI. CONCLUSION

Let us summarize briefly the results obtained in our paper.

1. The Abraham energy-momentum tensor and the covariant Grammel formulae for this tensor are defined only on $\Gamma$, where they coincide and do not depend on the material characteristics of the medium $\varepsilon$ $\mu$.

2. Employment of the Abraham and Grammel formulae outside $\Gamma$ is not justified physically.

3. The generalization of the Abraham reasoning enables us to construct the symmetric energy-momentum tensor on the whole configuration space $\Gamma$.

4. The symmetric energy-momentum tensor coincides with the Abraham tensor on $\Gamma$ and provides the unique relativistically covariant extension of the Abraham tensor on the whole configuration space $\Gamma$.

5. In practical calculations, the energy-momentum tensor can be used in the Abraham form which is more compact in comparison with the symmetric tensor.

6. The vectorial 3-dimensional formulae for the components of the symmetric energy-momentum tensor have the same form (functional dependence) in the arbitrary inertial reference frame, i.e. these formulae are form-invariant. The same statement is also true for the Abraham tensor.

The physical basis of the Abraham tensor and the symmetric energy-momentum tensor is the definition of the linear momentum density of electromagnetic field in a medium, which is postulated, in the rest frame, in the form (3.3). Just this definition is the physical reason...
which distinguishes the Abraham approach from the Minkowski tensor and other versions of the electromagnetic energy-momentum tensor in the medium. The symmetry property of the Abraham tensor (3.13)–(3.16) and of the tensor (4.9)–(4.11), as well as their explicit dependence on the velocity of the medium, are, really, the consequences of the condition (3.3).

On the face of it, one may infer from (4.9) that these properties, symmetry and explicit dependence on the medium velocity, are implemented independently of each other. Indeed, the symmetric tensor is already produced by the first two terms in (4.9) and to take into account the dependence on the velocity of the medium, one has to add one more term $A_{\alpha\beta}(u) = A_{\beta\alpha}(u)$. As a matter of fact, it is not the case. Namely, the symmetrization alone, without introducing an explicit dependence on $u_\alpha$, does not enable one to meet on $\Gamma$ conditions (3.2)–(3.4) determining the tensor $T_{\alpha\beta}^{\text{sym}}$.

Not numerus references, concerning the Abraham tensor, are, practically all, enumerated in the present paper [1, 4, 7–10]. Unfortunately in the Russian edition of the Abraham book [22] the material devoted to the Abraham energy-momentum tensor is absent. In the educational literature, the Abraham energy-momentum tensor is, as a rule, not considered. The textbooks [4, 23] are rare exception. However the pertaining information supplied there is restricted and sometimes misleading. In particular, it is not noted that the Abraham and Grammel formulae are valid only on $\Gamma$. The problem to extend these formulae onto the whole configuration space $\Gamma$ is not posed. In the textbook [23], it is asserted erroneously that proceeding from Eqs. (3.2)–(3.4) and taking advantage of the Lorentz transformations for the energy flux density and for the density of the liner momentum one can derive Eq. (3.22) (see the solution of the problem 11.78 on page 438). A a matter of fact, doing in this way one arrives at Eq. (4.22) of the present paper with the vector $\mathbf{W}$ defined in Eq. (4.18).

The same erroneous assertion is brought in the review [5, p. 186, Russian edition] and in the book [24, p. 317, Rissian edition].

Until now, the problem of defining the energy-momentum tensor of electromagnetic field in the medium has been considered substantially for the medium at rest. However the complete solution of the problem in question demands also determining the dependence of this tensor on the medium velocity. One may hope that precise and comprehensive

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9 The symmetrized Minkowski energy-momentum tensor is used in practical calculations, see, for example, Refs. [20, 21].
consideration of the mathematical aspects relating to the Abraham approach will be helpful for these investigations.

ACKNOWLEDGMENTS

VVN thanks I. Brevik for useful advices on the subject under study and A.A. Rukhadze for providing the reference to the paper [3].

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