RATIONAL POINTS ON CERTAIN HYPERELLIPTIC CURVES OVER FINITE FIELDS

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Abstract. Let $K$ be a field, $a, b \in K$ and $ab \neq 0$. Let us consider the polynomials $g_1(x) = x^n + ax + b$, $g_2(x) = x^n + ax^2 + bx$, where $n$ is a fixed positive integer. In this paper we show that for each $k \geq 2$ the hypersurface given by the equation

$$S_k : u^2 = \prod_{j=1}^{k} g_i(x_j), \quad i = 1, 2,$$

contains a rational curve. Using the above and Woestijne’s recent results we show how one can construct a rational point different from the point at infinity on the curves $C_i : y^2 = g_i(x)$, $(i = 1, 2)$ defined over a finite field, in polynomial time.

Dedicated to the memory of Andrzej Mąkowski

1. Introduction

R. Schoof in [4] showed how one can count rational points on the elliptic curve $E : y^2 = x^3 + ax + b$ defined over finite field $\mathbb{F}_p$, where $p > 3$ is a prime, in polynomial time. Surprisingly, this algorithm allows to compute the order of the group $E(\mathbb{F}_p)$ without providing any point (different from the point at infinity) on the curve $E$ explicitely. In this paper a problem was posed to construct an algorithm which would allow to determine a rational point $P \in E(\mathbb{F}_p) \setminus \{O\}$ in polynomial time.

To author’s best knowledge, the first work concerning this problem appeared in 2004. A. Schinzel and M. Skalba showed in [3] how to determine efficiently a rational point on the curves of the form $y^2 = x^n + a$, where $n = 3, 4, a \in \mathbb{F}_q$ and $q = p^m$. In case $n = 3$, the authors give explicit elements $y_1, y_2, y_3, y_4 \in \mathbb{F}_q$ with the property that for at least one $i \leq 4$ the equation $y_i^2 = x^3 + a$ has a solution in the field $\mathbb{F}_q$. In case $n = 4$, the authors give a construction of elements $y_1, y_2, y_3 \in \mathbb{F}_q$ with the property that for at least one $i \leq 3$, the equation $y_i^2 = x^4 + a$ has a solution in the field $\mathbb{F}_q$.

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A solution of the subproblem of finding the \(x\)-coordinate of a rational point on the elliptic curve

\[ E : y^2 = x^3 + ax + b =: f(x), \]

in case when \(a, b \in \mathbb{F}_q\), \(a \neq 0\), \(q = p^m\) and \(\mathbb{F}_q\) is a finite field with \(p > 3\), was provided in [7] (for many applications the \(y\)-coordinate is not needed). The key element of the proof was a construction of non-constant rational functions \(x_1, x_2, x_3, u \in K(t)\) which satisfy the equation

\[ u^2 = f(x_1)f(x_2)f(x_3). \]  

(1.1)

We know that multiplicative group \(\mathbb{F}_q^*\) is cyclic. This fact plus the obtained parametric solution prove that for at least one \(i \leq 3\), the element \(f(x_i)\) is a square in \(\mathbb{F}_q\). If now \(q = p\) or \(q = p^m\) and an element \(v \in \mathbb{F}_q \setminus \mathbb{F}_q^2\) is given, then using Schoof’s and Tonelli-Shanks’ algorithm (given in [6]) respectively, we can calculate a square root from \(f(x_i)\) in polynomial time.

In his PhD dissertation [8], Ch. van de Woestijne showed how in polynomial time, for given \(b_0, b_1, \ldots, b_n \in \mathbb{F}_q^*\) we can find integers \(i, j\) such that \(0 \leq i < j \leq n\), and an element \(b \in \mathbb{F}_q^*\) such that \(b_i/b_j = b^n\). We should pay attention that for calculating this \(n\)-th root, it is not necessary to have the element \(v \in \mathbb{F}_q \setminus \mathbb{F}_q^m\) and that the algorithm which computes this root is deterministic. It is easy to see that having elements \(x_1, x_2, x_3\) fulfilling the identities (1.1) for a certain \(u \in \mathbb{F}_q\) and using Woestijne’s result we can find a rational point on the curve \(y^2 = f(x)\) in polynomial time. This idea was used in [5]. The authors constructed a rational curve (different from the one constructed in [7]) on the hypersurface \(u^2 = f(x_1)f(x_2)f(x_3)\), where \(f\) is a given polynomial of degree three. However, we should note that in order to obtain an explicit form of the curve, it is necessary to solve the equation \(\alpha x^2 + \beta y^2 = \gamma\) in \(\mathbb{F}_q\) for certain \(\alpha, \beta, \gamma\) (this can be done in deterministic polynomial time, but of course this lengthens the time needed to compute rational point on the curve \(E\)). The authors also showed how to construct rational points on elliptic curves defined over finite fields of characteristic 2 and 3.

A natural question arises here concerning the existence of rational curves on the hypersurface of the form

\[ S_k : u^2 = \prod_{i=1}^{k} g(x_i), \]

(1.2)

where \(g \in \mathbb{Z}[x]\) and \(g\) is without multiple roots. It is worth noting that in this case \(S_k\) is smooth. It appears that the problem posed in this form has not been considered so far and seems to be interesting. Papers [7] and [5] also show that in case when \(k\) is odd, the ability to construct rational curves on the hypersurface \(S_k\) can be useful in finding rational points on hyperelliptic curves (defined over finite field) of the form

\[ C : y^2 = g(x). \]
It is also worth noting that in case when \( \deg g = 2, 3, 4 \) and \( k \) is an even number, it is easy to find rational curves on \( S_k \). Indeed, if \( k = 2 \), then on the surface \( S_2 \) we have a rational curve \((x_1, x_2, u) = (t, t, g(t))\). If now \( \deg g = 2 \), then using a standard procedure we parametrize rational solutions of equation \( u^2 = g(t)g(x) \). We act similarly in case when \( \deg g = 3 \) or \( \deg g = 4 \) with such a difference that this time we use an algorithm of adding points on curve with genus one with known rational points. In this way we obtain infinitely many rational curves on the surface \( S_2 \). As an immediate consequence of the above reasoning, we obtain curves on \( S_k \) in case when \( k > 2 \) is an even integer.

However, if \( \deg g > 4 \) or \( k \) is an odd integer, then the task seems to be much more difficult and the crucial question arises whether for a given \( g \in \mathbb{Z}[x] \) there is a \( k \) such that \( S_k \) contains a rational curve?

Let now \( a, b \in K, \ ab \neq 0 \) and let us consider polynomials

\[
\begin{align*}
g_1(x) &= x^n + ax + b, \\
g_2(x) &= x^n + ax^2 + bx,
\end{align*}
\]

where \( n \) is a fixed positive integer. In this paper we prove that if \( g = g_1 \) or \( g = g_2 \), then for each \( k \geq 2 \) there is a rational curve on the surface \( S_k \).

These and Woestijne’s results show that the construction of rational points on the curve \( C_i: y^2 = g_i(x) \) can be performed in polynomial time. Let us also note that if \( n \) is even, then \( g_i(-b/a) = (b/a)^n \) and the point \( P = (-b/a, \ (b/a)^{n/2}) \) lies on the curve \( y^2 = g_i(x) \) and is different from the point at infinity. One can see that in this case the problem of existence of rational points on \( C_i \) is easy. Certainly, it does not provide us with an answer to the question about constructing a rational curve on hypersurface \( S_k \) when \( n \) is even.

2. Rational curves on \( S_k^i \)

In this section we consider the hypersurface

\[
S_k^i: u^2 = \prod_{j=1}^{k} g_i(x_j),
\]

where \( i \in \{1, 2\} \) is fixed. As a direct examination of the existence of rational curves on the hypersurface \( S_k^i \) is difficult, let us reduce our problem to the examination of simpler objects.

Let \( a, b, c, d \in K \) fulfill the condition

\[(*) \quad a \neq 0 \text{ or } c \neq 0 \text{ and } b \neq 0 \text{ or } d \neq 0.\]
Let now $m, n$ be fixed positive integers and let us consider the surfaces

\[ S_1 : g_1(x)z^m = y^n + cy + d, \]
\[ S_2 : g_2(x)z^m = y^n + cy^2 + dy. \]

We will prove that a rational curve lies on each of these surfaces. Using constructed curves we will show how to construct curves on $S_i^1$ and $S_i^2$. Because each positive integer $\geq 2$ is of the form $2^k + 3l$, then as an immediate consequence we obtain the existence of rational curves on hypersurface $S_i^k$ for each $k \geq 2$.

We start with the following

**Lemma 2.1.** Let $n, m \in \mathbb{N}_+$ and $a, b, c, d \in K$ fulfill condition $(\ast)$. Then on each of the surfaces $S^1, S^2$ there is a rational curve.

**Proof.** Let $F_1(x, y, z) := g_1(x)z^m - (y^n + cy + d)$ and $F_2(x, y, z) := g_2(x)z^m - (y^n + cy^2 + dy)$. For the proof, let us put $x = T$, $y = tmT$, $z = t^n$. It is easy to see that for $x, y, z$ defined in this way, the equation $F_1(T, t^{m}T, t^n) = 0$ has the root

\[ T = -\frac{bt^{mn} - d}{at^{mn} - ct^m}, \]

which gives us a parametric curve $L_1$ on the surface $S^1$ given by the equations:

\[ L_1 : x(t) = -\frac{bt^{mn} - d}{at^{mn} - ct^m}, \quad y(t) = -\frac{bt^{mn} - d}{at^{mn(n-1)} - c}, \quad z(t) = t^n. \]

The same method can be applied to find a rational curve on the surface $S^2$. In this case the equation $F_2(T, t^{m}T, t^n) = 0$ has two roots, $T = 0$ and

\[ T = -\frac{bt^{m(n-1)} - d}{at^{m(n-1)} - ct^m}. \]

Rational curve $L_2$ on $S^2$ is given by the equations

\[ L_2 : x(t) = -\frac{bt^{m(n-1)} - d}{at^{m(n-1)} - ct^m}, \quad y(t) = -\frac{bt^{m(n-1)} - d}{at^{m(n-2)} - c}, \quad z(t) = t^n. \]

Note that the condition $(\ast)$ plays a crucial role in our reasoning in both cases. $\square$

**Remark 2.2.** The surface $S_1$ appeared in [2] with an additional assumption $a = c$, $b = d$, $m = 2$, $n = 3$. In this case the curve $L_1$ was used to show that for a given $j \neq 0$, 1728 there are infinitely many elliptic curves with $j$-invariant equal to $j$ and Mordell-Weil rank $\geq 2$.

It should be noted that a special case of the surface $S_1$, when $m = 2$, $n = 3$, was also considered in [1]. In this case the curve $L_1$ was used to show that on the surface $S_1$ the set of rational points is dense in the topology of $\mathbb{R}^3$. 

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We consider the surfaces $S^1$ and $S^2$ from the Lemma 2.1 with $m = 2$. For the proof of the first part of our theorem, let us make a change of variables $z = z_1/g_1(x)$ in the equation of the surface $S^1$. By this change of variables the surface $S^1$ is birational with the surface given by the equation

$$S' : z_1^2 = (x^n + ax + b)(y^n + cy + d).$$

Putting now $a = c$, $b = d$ and using equations of the curve $L_1$ from the proof of Lemma 2.1 we obtain the statement of our theorem.
Now we take the equations defining the curve $L_2$ from the proof of Lemma 2.1 and repeat the above reasoning in case of the surface $S^2$. This ends the proof of the first part of our theorem.

For the proof of the second part of our theorem let us go back to the surface $S'$ given by the equation (2.1). If we now put $c = a/g_1(u)^{n-1}$, $d = b/g_1(u)^n$ and perform a change of variables
\begin{equation}
(2.2) \quad u = X_1, \quad x = X_2, \quad y = \frac{X_3}{g_1(u)}, \quad z_1 = U_1 g_1(u)^{-\frac{n+1}{2}},
\end{equation}
then after elementary calculations the equation of the surface $S'$ is of the form

\[ U_1^2 = g_1(X_1)g_1(X_2)g_1(X_3). \]

If now $x$, $y$, $z$ are rational functions defining the curve $L_1$ on the surface $S^1$ for $c = a/g_1(u)^{n-1}$, $d = b/g_1(u)^n$, then calculating $X_1$, $X_2$, $X_3$ from equations (2.2), we obtain a two-parametric solution of the above equation given by the expressions from the statement of our theorem.

The proof of the second part of our theorem in case of the polynomial $g_2(x) = x^n + ax^2 + bx$ is similar, with one difference: we should substitute $a/g_2(u)^{n-2}$ and $b/g_2(u)^{n-1}$ in place of $c$, $d$ respectively. \hfill \Box

**Remark 2.4.** Let us note that in case when $K$ is a finite field with char $K > 3$, $a$, $b \in K$, $ab \neq 0$ and we are interested in construction of a rational point on the elliptic curve

\[ E : y^2 = x^3 + ax + b =: f(x), \]
then our rational curve lying on the hypersurface $S : u^2 = f(x_1)f(x_2)f(x_3)$ is much simpler than that obtained by Skalba. If $x_i = X_i(t)$, $i = 1$, $2$, $3$ are the equations defining the curve on $S$, then if $X_1X_2X_3 = N/D$ for certain relatively prime polynomials $N$, $D \in K[t]$, then $\deg N \leq 26$, $\deg D \leq 25$ in case of parametrization obtained by Skalba and $\deg N \leq 8$, $\deg D \leq 6$ in case of our parametrization (with $u \in K$ such that $f(u) \neq 0$) from Theorem 2.3. Multiplicative structure of functions $X_i$ is very simple in our case, too, which influences the speed of calculations.

Our parametrisation has also this advantage over the one obtained by Shallue and Woestijne that it is not necessary to solve the equation of the form $\alpha x^2 + \beta y^2 = \gamma$ in $K$ in order to obtain it.

As in case when $n$ is even we have $g_i(-a/b) = ((b/a)^{n/2})^2$, then from the above theorem we obtain

**Corollary 2.5.** Let $K$ be a field, $a$, $b \in K$, $ab \neq 0$. Then for each positive integer $k \geq 2$ there is a rational curve on the hypersurface

\[ S_k^1 : u^2 = \prod_{j=1}^{k} g_i(x_j), \quad i = 1, 2. \]
Because the case \( k = 3 \) and \( K = \mathbb{F}_q \), \( q = p^m \) is especially interesting for us, we have to decide about the assumptions permitting to calculate the values of function \( X_i(u, t) \) for \( i = 1, 2, 3 \) from the second part of Theorem 2.3. We can limit our considerations to examining the case of polynomial \( g_1 \), of odd degree \( n \). In case of the polynomial \( g_2 \) the reasoning will be similar.

Firstly, let us note that the functions \( X_i(t, u) \) for \( i = 1, 2, 3 \) are non-constant. Moreover, functions \( X_2 \) and \( X_3 \) have the same denominator which equals

\[
D(t, u) = g_1(u)t^2( t^2g_1(u) )^{n-1} - t^2
\]

We know that for each \( v \in \mathbb{F}_q \) we have \( v^{p^m-1} = 1 \). There are \( p^m - n \) elements \( u \in \mathbb{F}_q \) for which \( g_1(u) \neq 0 \). If we fix such an element now, then because \( \text{deg}_u D(t, u) = 2(n - 1) \) and \( t^2 \mid D(t, u) \), there are at least \( p^m - 2(n - 1) + 1 \) elements \( t \in \mathbb{F}_q \) for which \( D(t, u) \neq 0 \). Thus we can see that there are at least \( (p^m - n)(p^m - 2(n - 1) + 1) \) elements in \( \mathbb{F}_q \times \mathbb{F}_q \) for which \( D(t, u) \neq 0 \). From this observation we see that if \( p > 2(n - 1) - 1 \) then we can find \( t, u \in \mathbb{F}_q \) such that for at least one \( j \in \{1, 2, 3\} \) the \( g_1(X_j(t, u)) \) is a square.

3. Some remarks and questions

Let us define the set \( T \) which contains pairs \((t, u) \in \mathbb{F}_q \times \mathbb{F}_q \) for which we can compute \( X_i(t, u) \), \( i = 1, 2, 3 \) from the preceding section. Then, we can define the map \( \Phi \) from \( T \) to the curve \( C : y^2 = g_1(x) \) in the following way

\[
\Phi(t, u) = (X_j(t, u), \sqrt{g_1(X_j(t, u))}),
\]

where square root is taken in \( \mathbb{F}_q \) and \( j = \min \{i : g_1(X_i(t, u)) \text{ is a square} \} \).

Note that there are at most \( 2q \) rational points on \( C \) over \( \mathbb{F}_q \) but \( T \), as we have proved, contains at least \( (q - n)(q - 2(n - 1) + 1) \) elements. This suggest the following

**Question 3.1.** Is the map \( \Phi : T \ni (t, u) \mapsto \Phi(t, u) \in C \) a surjective map?

Another question which comes to mind is the following,

**Question 3.2.** Let us fix a polynomial \( g \in \mathbb{Z}[x] \) without multiple roots. Is there an integer \( k \geq 2 \) such that on the hypersurface

\[
S_k : u^2 = \prod_{j=1}^{k} g(x_j)
\]

there are infinitely many rational points with \( u \neq 0 \)? In this question we are interested in non-trivial points on \( S_k \), i.e. such points \((x_1, \ldots, x_k, u) \) that \( g(x_i) \neq g(x_j) \) for \( i \neq j \).
It would be also interesting to know whether if we treat hypersurface $S_k$ over $\mathbb{C}$ (instead $\mathbb{Q}$) then there are rational curves on $S_k$.

It seems that the following question is much more difficult.

**Question 3.3.** Let us fix a polynomial $g \in \mathbb{Z}[x]$ without multiple roots and a positive integer $k \geq 2$. Is there a non-trivial rational point with $u \neq 0$ on the hypersurface $S_k$?

If our fixed $k$ in Question 3.3 is odd we should also assume that for each $p \in \mathbb{P} \cup \{\infty\}$ the curve $y^2 = g(x)$ has a point over $\mathbb{Q}_p$ (as usual $\mathbb{Q}_\infty = \mathbb{R}$). It is clear that the assumption concerning local solubility is necessary. For example, consider the polynomial $g(x) = 3 - x^2$. There are no $\mathbb{Q}_3$-rational points on the curve $y^2 = g(x)$ and this immediately implies that there are no $\mathbb{Q}_3$-rational points on $S_k$.

Let us note that if the polynomial $g$ fulfills the condition $x^n g(1/x) = g(x)$ (this is so called reciprocal polynomial), then we have a rational curve $x_1 = t^2$, $x_2 = 1/t^2$, $u = t^n g(1/t^2)$ on the surface $S_2$. As an immediate consequence we conclude that if $k$ is even, then there is a rational curve on $S_k$. Additionally, if the degree of $g$ is odd, then on the surface $S_3$ we have a rational curve given by equations

$$x_1 = t, \quad x_2 = g(t), \quad x_3 = \frac{1}{g(t)}, \quad u = g(t)^{\frac{n+1}{2}} g\left(\frac{1}{g(t)}\right),$$

and immediately we have that for each $k \geq 2$ there is a rational curve on $S_k$.

Let us also note that if $g(x) = x^4 + 1$, then on the hypersurface $S_3$ we have a rational curve with $x_i = x_i(t), i = 1, 2, 3$ given by

$$x_1 = \frac{2t + 1}{3t^2 + 3t + 1}, \quad x_2 = \frac{3t^2 + 2t}{3t^2 + 3t + 1}, \quad x_3 = \frac{3t^2 + 4t + 1}{3t^2 + 3t + 1}$$

It would be very interesting to construct other families of polynomials of such a property that for each $k \geq 2$ there are rational curves (or infinitely many non-trivial rational points) on $S_k$.

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