AN INTERESTING FAMILY OF CONFORMALLY INVARIANT
ONE-FORMS IN EVEN DIMENSIONS

JEFFREY S. CASE

Abstract. We construct a natural conformally invariant one-form of weight $-2k$ on any $2k$-dimensional pseudo-Riemannian manifold which is closely related to the Pfaffian of the Weyl tensor. On oriented manifolds, we also construct natural conformally invariant one-forms of weight $-4k$ on any $4k$-dimensional pseudo-Riemannian manifold which are closely related to top degree Pontrjagin forms. The weight of these forms implies that they define functionals on the space of conformal Killing fields. On Riemannian manifolds, we show that this functional is trivial for the former form but not for the latter forms. As a consequence, we obtain global obstructions to the existence of an Einstein metric in a given conformal class.

1. Introduction

Recent work [5,6,17] in CR geometry has identified an interesting family of natural CR invariant $(1,0)$-forms on all nondegenerate CR manifolds of dimension $2n+1$, $n \geq 2$. These $(1,0)$-forms can be regarded as CR invariant modifications of $\partial_b c_b(S)$, where $c_b(S)$ is the potential of a characteristic form of degree $2n$ determined by a homogeneous invariant polynomial $\Phi$ and the Chern tensor $S$. For strictly pseudo-convex CR manifolds, a result of Takeuchi [21] implies that these $(1,0)$-forms are all divergences. This fact leads to counterexamples to Hirachi’s conjecture on the generalization of the Deser–Schwimmer conjecture to CR geometry [13].

The purpose of this article is to construct the conformal analogues of the above CR invariant one-forms. The forms we construct retain three key properties of their CR analogues. First, they are natural; that is, they can be written as a linear combination of partial contractions of tensor products of the pseudo-Riemannian metric, its inverse, the Riemann curvature tensor, and its covariant derivatives; when restricted to oriented manifolds, we also allow these products to include factors of the pseudo-Riemannian volume form. Second, they can be regarded as conformally invariant modifications of the exterior derivative of the Pfaffian of the Weyl tensor or, in the oriented case, the potential of a top degree Pontrjagin form. Third, a result of Ferrand [15] and Obata [19] implies that, in Riemannian signature, the conformally invariant one-form related to the Pfaffian of the Weyl tensor is a divergence. The conformally invariant one-forms related to top degree Pontrjagin forms need not be divergences, and their failure to be a divergence obstructs the existence of an Einstein metric in a given conformal class.

To make these points explicit requires some notation. Let $(\mathbb{M}^n, g)$ be a pseudo-Riemannian manifold. Let $W_{ijpq}$ and $C_{ijp}$ denote the Weyl and Cotton tensors,
respectively, with the convention $\nabla^p W_{ijpq} = (n-3)C_{ijpq}$; here and throughout we use Penrose’s abstract index notation \cite{20}. Given $k \in \mathbb{N}$, define
\begin{equation}
(1.1) \quad \xi_i^{(k)} \ := \frac{1}{k!} \delta^{i j_1 \cdots j_{2k}}_{i_j \cdots i_{2k}} C_{j_1 j_2} \ i_2 W_{j_3 j_4} \ i_3 i_4 \ \cdots \ W_{j_{2k-1} j_{2k}} \ i_{2k-1} i_{2k} + \frac{1}{2nk} \nabla_i \Pr^{(k)}(W),
\end{equation}
where $\delta^{i j_1 \cdots j_{2k}}_{i_j \cdots i_{2k}}$ is the generalized Kronecker delta and
\begin{equation}
(1.2) \quad \Pr^{(k)}(W) := \frac{1}{k!} \delta^{i_1 \cdots i_{2k}}_{j_1 \cdots j_{2k}} W_{i_1 j_1} \ i_2 i_2 \ \cdots \ W_{i_{2k-1} j_{2k-1}} \ i_{2k-1} i_{2k}.
\end{equation}
In dimension $n = 2k$, it holds that $\Pr^{(k)}(W)$ is the Pfaffian of the Weyl tensor.

Suppose additionally that $(M^n, g)$ is an even-dimensional oriented manifold. Set $n = 2k$. Denote by $\epsilon_{i_1 \cdots i_n}$ the pseudo-Riemannian volume form. Let $\Phi$ be a homogeneous invariant polynomial of degree $k$; i.e. $\Phi$ is a linear combination of compositions of $\text{Id}^{\otimes k}$ with braiding maps such that
\begin{equation}
\Phi_{i_1 \cdots i_k}^{j_1 \cdots j_k} = \Phi_{i_1^{(1)} \cdots i_k^{(1)}}^{j_1^{(1)} \cdots j_k^{(1)}} \Phi_{i_1^{(2)} \cdots i_k^{(2)}}^{j_1^{(2)} \cdots j_k^{(2)}},
\end{equation}
for all elements $\sigma$ of $S_k$, the symmetric group on $k$ elements. Define
\begin{equation}
(1.3) \quad \rho_i^\Phi := \frac{1}{(2k-1)!} \epsilon_{i_1 \cdots i_{2k}}^{i_2 \cdots i_{2k}} \Phi_{s_1 \cdots s_k}^{t_1 \cdots t_k} C_{t_1 t_2} \ s_2 s_3 \ i_2 i_3 \ \cdots \ W_{t_{2k-1} t_{2k}} \ s_k s_{k-1} s_{k-2} + \frac{1}{2k} \nabla_i p_\Phi(W),
\end{equation}
where
\begin{equation}
(1.4) \quad p_\Phi(W) := \frac{1}{(2k)!} \epsilon_{i_1 \cdots i_{2k}}^{i_2 \cdots i_{2k}} \Phi_{s_1 \cdots s_k}^{t_1 \cdots t_k} W_{t_1 s_1} \ i_1 i_2 \ \cdots \ W_{t_{2k-1} s_{k-1}} \ i_{2k-1} i_{2k}.
\end{equation}
Note that $p_\Phi(W) = 0$ if $k$ is odd and that $p_\Phi(W) = p_\Phi(Rm)$ for all $k \in \mathbb{N}$, where $p_\Phi(Rm)$ is defined in terms of the Riemann curvature tensor $R_{ijkl}$ using Equation \cite{20}. The latter observation recovers the well-known fact \cite{4, 7} that the Pontrjagin form $\star p_\Phi(Rm)$ determined by $\Phi$ depends only on the Weyl tensor of $(M^n, g)$.

The one-form $\xi_i^{(k)}$ is conformally invariant in dimension $n = 2k$ and the one-forms $\rho_i^\Phi$ are conformally invariant in the dimensions where they are defined.

**Theorem 1.1.** Let $(M^{2k}, g)$ be a pseudo-Riemannian manifold and let $\Phi$ be a homogeneous invariant polynomial of degree $2k$. Then
\begin{equation}
e^{2kY} \xi_i^{(k)} = \xi_i^{(k)},
e^{2kY} \rho_i^\Phi = \rho_i^\Phi
\end{equation}
for all conformal metrics $\tilde{g} := e^{2Y} g$, where $\tilde{\xi}_i^{(k)}$ and $\tilde{\rho}_i^\Phi$ are defined in terms of $\tilde{g}$.

In terms of conformal density bundles, Theorem \cite{22} states that $\xi_i^{(k)}$ and $\rho_i^\Phi$ are natural conformally invariant elements of $\mathcal{E}_i[-2k]$ in dimension $2k$; see Section \cite{2} for definitions. In particular, $\xi_i^{(k)}$ defines a conformally invariant functional on the space of compactly-supported vector fields. More generally, let $(M^n, g)$ be a pseudo-Riemannian manifold. Given an element $\omega_i \in \mathcal{E}_i[-n]$, the formula
\begin{equation}
(1.5) \quad \Omega(X^i) := \int_M \omega_i X^i \ d\text{vol}
\end{equation}
defines a conformally invariant functional on the space of compactly-supported vector fields on $M$. More significantly, $\mathcal{E}_i[-n]$ is the codomain of the formal adjoint
$K^*: \mathcal{E}_{(ij)\sslash\blacksquare}[2-n] \to \mathcal{E}_i[-n]$ of the conformal Killing operator $K: \mathcal{E}_i[2] \to \mathcal{E}_{(ij)\sslash\blacksquare}[2]$, where $\mathcal{E}_{(ij)\sslash\blacksquare}[w]$ denotes the space of conformally invariant, trace-free symmetric $(0, 2)$-tensor fields with weight $w \in \mathbb{R}$. These operators are both conformally invariant, and the operator $K^*$ is a divergence: $K^*(T_{ij}) := -2\nabla^j T_{ij}$.

It is thus natural to ask whether $\xi_{i}^{(k)}$ or $\rho_{i}^{\Phi}$ are in the image of $K^*$. A necessary condition is that, on compact manifolds, the associated functional $\Xi^{(k)}$ or $P^{\Phi}$ annihilates conformal Killing fields. For Riemannian manifolds, the fact that $K^*$ has surjective principal symbol implies that this condition is also sufficient.

On closed Riemannian manifolds, $\xi_{i}^{(k)}$ is in the image of $K^*$.

**Theorem 1.2.** Let $(M^{2k}, g)$ be a closed Riemannian manifold. Then

$$\xi_{i}^{(k)} \in \text{im} \left( K^*: \mathcal{E}_{(ij)\sslash\blacksquare}[2-2k] \to \mathcal{E}_i[-2k] \right).$$

This result is remarkable due to the fact that $\xi_{i}^{(2)}$ is not the divergence of a natural trace-free symmetric $(0, 2)$-tensor field; see Section 6. To the best of our knowledge, this is the first example of a natural conformally invariant tensor field which is in the image of a natural conformally invariant differential operator, but is not the image of a natural tensor field. By contrast, in dimension four, the Bach tensor

$$B_{ij} := \nabla^s C_{sij} + W_{isjt} P_{st} \in \mathcal{E}_{(ij)\sslash\blacksquare}[-2]$$

is the image of the Weyl tensor under the natural conformally invariant differential operator $W_{ijkl} \mapsto (\nabla^s \nabla^t + P_{st})W_{isjt}$ (cf. [12]).

Our proof of Theorem 1.2 relies on the Ferrand–Obata Theorem [15, 19]. Taken together, Theorems 1.1 and 1.2 indicate that $\xi_{i}^{(k)}$ should be regarded as the conformal analogue of the aforementioned CR invariant $(1, 0)$-forms.

By contrast, the one-forms $\rho_{i}^{\Phi}$ need not be in the image of $K^*$. In fact, the failure of this to hold gives a global obstruction to the existence of an Einstein metric in the given conformal class.

**Theorem 1.3.** Let $\Phi$ be a homogeneous invariant polynomial of degree $2k$, $k \in \mathbb{N}$.

1. If $(M^{4k}, g)$ is a closed conformally Einstein manifold of Riemannian signature, then $\rho_{i}^{\Phi} \notin \text{im} \left( K^* \right)$.

2. There are examples of closed manifolds $(M^{4k}, g)$ for which $\rho_{i}^{\Phi} \notin \text{im} \left( K^* \right)$.

The proof of the first statement relies on the fact that, except on the round sphere, any conformal Killing field on a closed Einstein manifold of Riemannian signature is necessarily Killing [18]. In Section 5 we show that the product of $S^3$ and a non-round Berger three-sphere, as well as its products with copies of $\mathbb{C}P^2$, give examples with $\rho_{i}^{\Phi} \notin \text{im} \left( K^* \right)$. Our examples are not locally conformally Einstein. We are not aware of an example of a locally conformally Einstein manifold which can be proven via Theorem 1.3 to not be globally conformally Einstein.

Note that on locally conformally flat and obstruction flat even-dimensional $n$-manifolds, $K^*: \mathcal{E}_{(ij)\sslash\blacksquare} \to \mathcal{E}_i[-n]$ is the last nontrivial map in the conformal deformation complex [11, 12] and the conformal deformation detour complex [3], respectively. In particular, Theorems 1.2 and 1.3 indicate that there may be an interesting interpretation of the conformally invariant one-forms $\xi_{i}^{(k)}$ and $\rho_{i}^{\Phi}$ on even-dimensional obstruction flat manifolds.

As previously noted, $\xi_{i}^{(k)}$ is not the divergence of a natural trace-free symmetric $(0, 2)$-tensor field. However, one can express $\xi_{i}^{(k)}$ as the sum of the divergence of
a natural trace-free symmetric \((0,2)\)-tensor field and the exterior derivative of a natural scalar function.

**Theorem 1.4.** Let \((M^{2k}, g)\) be a pseudo-Riemannian manifold. Define \(\Omega^{(k)}_{ij} \in \Gamma(S^2T^*M)\) by

\[
(\Omega^{(k)})^j_i := \sum_{\ell=0}^{k-1} 4^{k-\ell} \frac{1}{\ell!(k-\ell)!} \delta_{i_{i_1\ldots i_{2\ell}}j_{j_1\ldots j_{2\ell}}} W_{j_1j_2}^{i_{i_1}i_{i_2} \ldots W_{j_2j_{2\ell}}^{i_{i_2}i_{i_3} \ldots j_{j_1\ldots j_{2\ell}}} P_{j_{2\ell+1}j_{2\ell+1}}^{i_{i_{2\ell+1}}} \ldots P_{j_{k+\ell}}^{i_{i_{k+\ell}}},
\]

where \(P_{ij}\) is the Schouten tensor of \(g\). Then

\[
(1.6) \quad 2k\xi^{(k)}_i = \nabla^j(\text{tr} \Omega^{(k)})_{ij} + \frac{1}{2k} \nabla_i \text{Pr}^{(k)}(Rm),
\]

where \((\text{tr} \Omega^{(k)})_{ij} := \Omega^{(k)}_{ij} - \frac{1}{2k} \text{tr} \Omega^{(k)} g_{ij}\) is the trace-free part of \(\Omega^{(k)}_{ij}\).

There is a nice heuristic based on Branson’s method of analytic continuation in the dimension \(\mathbb{R}^4\) which explains Theorems 1.1, 1.2 and 1.4. Let \((M^n, g)\) be a pseudo-Riemannian manifold and fix \(k \in \mathbb{N}\). Define

\[
T^{(k)}(W)^j_i := \frac{1}{k!} \delta_{i_{i_1\ldots i_{2k}}j_{j_1\ldots j_{2k}}} W_{j_1j_2}^{i_{i_1}i_{i_2} \ldots W_{j_2j_{2k}}^{i_{i_2}i_{i_3} \ldots j_{j_1\ldots j_{2k}}} 1_{2k-1}^{1_{2k}}.
\]

Observe that \(T^{(k)}(W)^i_j\) is conformally invariant and \(T^{(k)}(W)^i_j = 0\) if \(n \leq 2k\). Straightforward computations establish that

\[
(1.7) \quad \nabla^j(\text{tr} T^{(k)}(W))_{ij} = -2k(n-2k)\xi^{(k)}_i
\]

and

\[
(1.8) \quad e^{2k\gamma} \mathcal{V}^j(\text{tr} T^{(k)}(W))_{ij} = \nabla^j(\text{tr} T^{(k)}(W))_{ij} + (n-2k)\gamma^i(\text{tr} T^{(k)}(W))_{ij}
\]

for all \(\gamma := e^{2\gamma} g\). Combining Equations (1.7) and (1.8) yields

\[
e^{2k\gamma} \xi^{(k)}_i = \xi^{(k)}_i - \frac{1}{2k} \gamma^i(\text{tr} T^{(k)}(W))_{ij}
\]

when \(n > 2k\). Theorem 1.1 follows by taking the limit \(n \to 2k\). Equation (1.7) exhibits \(\xi^{(k)}_i\) in the image of the divergence on \(E_{(i)(j)\ell}^\ell\); dividing by \(n-2k\) and taking the limit \(n \to 2k\) yields Theorem 1.2, provided one can make sense of the limit

\[
(1.9) \quad \lim_{n \to 2k} \frac{1}{n-2k} \left(\text{tr} T^{(k)}(W)\right)_{ij}.
\]

Finally, the generalized Einstein tensor

\[
(E^{(k)})^{ij} := \frac{1}{k!} \delta_{i_{i_1\ldots i_{2k}}}^{j_{j_1\ldots j_{2k}}} R_{j_1j_2}^{i_{i_1}i_{i_2} \ldots R_{j_{2k-1}j_{2k}}^{i_{i_2}i_{i_3} \ldots j_{j_1\ldots j_{2k}}} 1_{2k-1}^{1_{2k}}
\]

is symmetric and divergence-free \(10\). Note that \(\text{tr} E^{(k)} = (n-2k)\text{Pr}^{(k)}(Rm)\) and

\[
E^{(k)}_{ij} = T^{(k)}(W)_{ij} + (n-2k)\Omega^{(k)}_{ij},
\]

where

\[
(\Omega^{(k)})^j_i := \sum_{\ell=0}^{k-1} 4^{k-\ell} \binom{k}{\ell} \frac{(n-k-\ell-1)!}{k!(n-2k)!} \delta_{i_{i_1\ldots i_{2\ell+1}}}^{j_{j_1\ldots j_{2\ell+1}}} W_{j_1j_2}^{i_{i_1}i_{i_2} \ldots W_{j_2j_{2\ell}}^{i_{i_2}i_{i_3} \ldots j_{j_1\ldots j_{2\ell}}} P_{j_{2\ell+1}j_{2\ell+1}}^{i_{i_{2\ell+1}}} \ldots P_{j_{k+\ell}}^{i_{i_{k+\ell}}}.
\]
In particular,
\[-\frac{n-2k}{n} \nabla_i \text{Pf}^{(k)}(Rm) = \nabla^j (\text{tf} T^{(k)}(W))_{ij} + (n-2k)\nabla^j (\text{tf} \Omega^{(k)})_{ij}.
\]
Combining this with Equation (1.7), dividing by \(n-2k\), and taking the limit \(n \to 2k\), yields Theorem 1.3.

We do not here attempt to make rigorous sense of the limit \(n \to 2k\). Indeed, the failure of \(\xi_i^{(2)}\) to be the divergence of a natural element of \(\mathcal{E}_{(ij)_0}[-2]\) in dimension four indicates that it is particularly difficult to make sense of Equation (1.9). Instead, we give direct proofs of Theorems 1.1 and 1.3 using elementary multilinear algebra and then deduce Theorem 1.2 from Theorem 1.3 and the Ferrand–Obata Theorem.

The above heuristic also illustrates the distinction between the one-forms \(\xi_i^{(k)}\) and \(\rho_i^\Phi\), namely through how they are naturally extended to other dimensions. In terms of the wedge product and Hodge star on double forms [14], the discussion above realizes \(\xi_i^{(k)}\) as the divergence of a dimensional multiple of \(*(W^\wedge k \wedge g^{(n-2k-1)})\) when \(n > 2k\). By contrast, the natural extension of \(\rho_i^\Phi\) to arbitrary dimension is in terms of (ordinary) differential forms. More precisely, define
\[
(*p_\Phi(W))_{i_1 \ldots i_{2k}} := \Phi_{s_1 \ldots s_k} W_{[t_1 t_2 | t_1]}^{s_1} \cdots W_{t_{2k-1} t_{2k}}^{s_k},
\]
(1.10)
\[
(*\rho^\Phi)_{i_2 \ldots i_{2k}} := (\Phi W^{k-1})_{i_2 \ldots i_{2k}} := \Phi_{s_1 \ldots s_k} C_{t_1}^{s_1} W_{[t_2 | t_1 t_3 \ldots t_{2k}]}^{s_2} \cdots W_{t_{2k} | t_2}^{s_k}
\]
\[
(*)_{i_2 \ldots i_{2k}} := (\Phi W^{k-1})_{i_2 \ldots i_{2k}} - \frac{1}{n-4k} \nabla^i (*p_\Phi(W))_{i_2 \ldots i_{2k}},
\]
where our notation in the first and second lines means that we skew symmetrize over the indices \(i_1, \ldots, i_{2k}\) and \(i_2, \ldots, i_{2k}\), respectively. Note that these objects are defined without reference to a given orientation. These normalizations are such that, in dimension \(n = 2k\), the definitions of \(\rho_i^\Phi\) by Equation (1.3) and the above display agree. Moreover, \((*\rho^\Phi)_{i_2 \ldots i_{2k}}\) is a conformally invariant \((2k-1)\)-form of weight \(-2\) in all dimensions; see Section 3.

This note is organized as follows. In Section 2 we recall some relevant facts from conformal geometry. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorems 1.2 and 1.3. In Section 5 we prove Theorem 1.3. In Section 6 we show that \(\xi_i^{(2)}\) is not the divergence of a natural element of \(\mathcal{E}_{(ij)_0}[-2]\).

2. Background

2.1. Abstract index notation. Let \((M^n, g)\) be a pseudo-Riemannian manifold. We denote by \(T^{(r,s)}M\) the tensor product of the bundles \(\otimes^r TM\) and \(\otimes^s \mathcal{L}TM\). We use abstract index notation [20] to denote sections of tensor bundles. Specifically, we denote a section of \(T^{(r,s)}M\) by \(T^{j_1 \cdots j_r}_{i_1 \cdots i_s}\); the \(r\) distinct superscripts denote contravariant indices and the \(s\) distinct subscripts denote covariant indices. Repeated indices denote contractions between the corresponding components. We use the metric \(g_{ij}\) to raise and lower indices in the usual way, and often offset subscripts and superscripts to clarify which components are raised or lowered. For example, as a section of \(T^{(1,3)}M\), the Riemann curvature tensor is defined by
\[
R_{ijk}^k \ell X^\ell := \nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k
\]
for all vector fields \(X^k\), where \(\nabla\) is the Levi-Civita connection. The Ricci curvature is \(R_{ij} = R_{k1}^k j\) and the scalar curvature is \(R = R_k^k\). The Schouten tensor of \((M^n, g)\)
where \( J = \frac{1}{2(n-1)} R \) is the trace of \( P_{ij} \). When clear from context, we write covariant derivatives of a scalar function using subscripts; e.g. given \( f \in C^\infty(M) \), we may write \( f_i \) for \( \nabla_i f \).

We use round and square brackets to denote symmetrization and skew symmetrization, respectively, over the enclosed indices. For example, if \( T_{ijk} \) is a section of \( T^{(0,3)} M \), then

\[
\begin{align*}
T_{(ijk)} &:= \frac{1}{3!} (T_{ijk} + T_{ikj} + T_{jki} + T_{jik} + T_{kij} + T_{kji}), \\
T_{[ijk]} &:= \frac{1}{3!} (T_{ijk} - T_{ikj} + T_{jki} - T_{jik} + T_{kij} - T_{kji})
\end{align*}
\]

denote the projections of \( T_{ijk} \) to its symmetric and antisymmetric parts, respectively. In this notation, the algebraic symmetries of the Weyl tensor \( W_{ijkl} \) are expressed as

\[
W_{ijkl} = W_{ij[kl]} = W_{kl[ij]}, \quad W_{ijkl}^k = 0, \quad W_{ijkl}^k = 0,
\]

which express that \( W_{ijkl} \) is a section of \( S^2 \Lambda^2 T^* M \), that \( W_{ijkl} \) satisfies the first Bianchi identity, and that \( W_{ijkl} \) is trace-free, respectively. The differential symmetries of the Weyl tensor \( W_{ijkl} \) and the Cotton tensor \( C_{ijk} \) are also succinctly expressed in abstract index notation:

**Lemma 2.1.** Let \((M^n, g)\), \( n \geq 3 \), be a Riemannian manifold. Then

\[
\begin{align*}
2 \nabla_{[i} P_{jkl]} &= C_{ijkl}, \\
\nabla_{[i} W_{jkl]}^{lm} &= -2 C_{ij[l} [g^m_{kl]}.
\end{align*}
\]

**Proof.** With our convention \( \nabla_s W_{ijsk} = (n-3) C_{ij}^{\phantom{ij}kl} \) from the introduction, the first equation is the customary definition of the Cotton tensor. The second equation follows from the second Bianchi identity \( \nabla_{[i} R_{jkl]}^{lm} = 0 \). \( \square \)

We use the symbol \( \mathcal{E} \) together with abstract indices to denote the spaces of sections of a given tensor bundle. For example, \( \mathcal{E}^i \) denotes the space of sections of \( TM \) and \( \mathcal{E}_{[i_1\ldots i_k]} \) denotes the space of \( k \)-forms. We denote by \( \mathcal{E}_{(ij)} \) the space of trace-free symmetric \((0,2)\)-tensor fields.

Suppose for the moment that \((M^n, g)\) is oriented. Denote by \( \epsilon_{i_1\ldots i_n} \) the pseudo-Riemannian volume form determined by \((M^n, g)\) and the orientation. Given an integer \( 0 \leq k \leq n \), the *Hodge star operator* \( \star : \mathcal{E}_{[i_1\ldots i_k]} \to \mathcal{E}_{[i_{k+1}\ldots i_n]} \) is defined by

\[
(\star \alpha)_{i_{k+1}\ldots i_n} := \frac{1}{k!} \epsilon^{i_1\ldots i_k}_{\phantom{i_1\ldots i_k}i_{k+1}\ldots i_n} \alpha_{i_1\ldots i_k}.
\]

A straightforward computation shows that

\[
\epsilon_{i_1\ldots i_k j_{k+1}\ldots j_n} \epsilon^{i_1\ldots i_k j_{k+1}\ldots j_n} = k! \delta^{j_{k+1}\ldots j_n}_{i_{k+1}\ldots i_n}.
\]

This implies the familiar identity

\[
(\star \star \alpha)_{i_1\ldots i_k} = (-1)^{k(n-k)} \alpha_{i_1\ldots i_k}.
\]
2.2. Conformal density bundles. Let \((M^n, c)\) be a conformal manifold (possibly of mixed signature). The conformal class \(c\) is naturally an \(\mathbb{R}_+\)-principle bundle with \(\mathbb{R}_+\)-action given by \(s \cdot g_x = s^2 g_x\) for all \(s \in \mathbb{R}_+, \) all \(g \in c,\) and all \(x \in M.\) Given \(w \in \mathbb{R},\) the \emph{conformal density bundle} of weight \(w\) is the line bundle associated to \(c\) via the representation \(s \mapsto s^{-w/2} \in \mathrm{End}(\mathbb{R})\) of \(\mathbb{R}_+\). We denote by \(\mathcal{E}[w]\) the space of smooth sections of this bundle; equivalently, an element of \(\mathcal{E}[w]\) is an equivalence class of pairs \((f, g) \in \mathcal{C}^\infty(M) \times c\) with respect to the equivalence relation \((f, g) \sim (e^{wT} f, e^{2T} g)\) for all \(T \in \mathcal{C}^\infty(M).\) Similarly, we denote by \(\mathcal{E}_1^{i_1 \cdots i_n}[w]\) the space of smooth sections of the tensor product of \(T^{(r,s)} M\) with the conformal density bundle of weight \(w.\)

Recall that a tensor field \(A^{i_1 \cdots i_n}_{i_1' \cdots i_n'}\) is \emph{natural} if it can be written as a linear combination of partial contractions of the Riemannian metric, its inverse, the Riemann curvature tensor, and its covariant derivatives; when restricted to oriented manifolds, we also allow these products to include factors of the Riemannian volume form. When \(M\) is fixed, we may regard \(A^{i_1 \cdots i_n}_{i_1' \cdots i_n'}\) as a map from \(\mathrm{Met}(M),\) the space of pseudo-Riemannian metrics on \(M,\) to \(\mathcal{E}^{i_1 \cdots i_n}_{i_1' \cdots i_n'}.\)

A \emph{natural element of} \(\mathcal{E}^{i_1 \cdots i_n}_{i_1' \cdots i_n'}[w]\) is an equivalence class \([A^{i_1 \cdots i_n}_{i_1' \cdots i_n'}(g), g],\) where \(A^{i_1 \cdots i_n}_{i_1' \cdots i_n'}\) is a natural tensor field. We say that \([A^{i_1 \cdots i_n}_{i_1' \cdots i_n'}(g), g]\) is \emph{conformally invariant} if it is independent of the choice of metric \(g \in c.\) For example, \(g_{ij}\) determines a natural conformally invariant element of \(\mathcal{E}(ij)[2];\) \(W_{ijkl}\) determines a natural conformally invariant element of \(\mathcal{E}_{ijkl}[2]\); and, if \((M, c)\) is oriented, then \(\epsilon_{i_1 \cdots i_n}\) determines a natural conformally invariant element of \(\mathcal{E}_{i_1 \cdots i_n}[n].\) In particular, we may use \(g_{ij}\) to raise and lower indices in conformal density bundles, and hence, for example, identify \(\mathcal{E}^i[0] \cong \mathcal{E}_i[2].\)

If \((M^n, c)\) is closed, then the total integral of any conformal density \(f \in \mathcal{E}[-n]\) is well-defined: simply pick \(g \in c,\) integrate against the Riemannian volume density of \(g,\) and observe that the result is independent of the choice of \(g.\) It follows that there is a conformally invariant pairing \(\mathcal{E}_i[w] \times \mathcal{E}_i[2 - n - w] \to \mathbb{R}\) given by

\[\langle \alpha_i, \beta_j \rangle := \int_M g^{ij} \alpha_i \beta_j.\]

These comments extend to general conformal manifolds by requiring \(f\) or one of \(\alpha_i, \beta_i\) to be compactly-supported.

The \emph{conformal Killing operator} \(K: \mathcal{E}_i[2] \to \mathcal{E}_{(ij)n}[2],\)

\[K(\alpha_i) := 2 \nabla_i \alpha_j - \frac{2}{n} \nabla^k \alpha_k g_{ij},\]

is conformally invariant. The kernel \(\mathcal{K} := \ker K \subset \mathcal{E}_i[2]\) of \(K\) is (after raising the index) the space of conformal Killing fields. The conformal invariance of Equation \(2.2\) and the analogous conformally invariant pairing of \(\mathcal{E}_{(ij)n}[w]\) and \(\mathcal{E}_{(ij)n}[4 - n - w]\) implies that the formal adjoint \(K^*: \mathcal{E}_{(ij)n}[2 - n] \to \mathcal{E}_i[-n],\)

\[K^*(A_{ij}) := -2 \nabla^k A_{ki},\]

of \(K\) is also conformally invariant.

2.3. \textbf{Infinitesimal conformal invariance.} Recall that a natural tensor field \(T^{i_1 \cdots i_n}_{i_1' \cdots i_n'}\) is \emph{homogeneous} of degree \(w \in \mathbb{R}\) if

\[T^{i_1 \cdots i_n}_{i_1' \cdots i_n'}(c^2 g) = c^w T^{i_1 \cdots i_n}_{i_1' \cdots i_n'}(g)\]
for all \( g \in \text{Met}(M) \) and all constants \( c > 0 \). Given such a tensor field, conformal invariance is equivalent to infinitesimal conformal invariance \([1]\). More precisely, given such a tensor field and a metric \( g \in \text{Met}(M) \), the conformal linearization of \( T_{i_1 \cdots i_s}^{j_1 \cdots j_r} \) at \( g \) is the map \( D_g T_{i_1 \cdots i_s}^{j_1 \cdots j_r} : C^\infty(M) \rightarrow \mathcal{E}_{i_1 \cdots i_s}^{j_1 \cdots j_r} \) defined by

\[
D_g T_{i_1 \cdots i_s}^{j_1 \cdots j_r}(\Upsilon) := \frac{\partial}{\partial t} \Bigg|_{t=0} e^{-wt}\Upsilon T_{i_1 \cdots i_s}^{j_1 \cdots j_r}(e^{2t\Upsilon} g).
\]

Observe that \( D_g T_{i_1 \cdots i_s}^{j_1 \cdots j_r} \) is linear and annihilates constants. One says that \( T_{i_1 \cdots i_s}^{j_1 \cdots j_r} \) is infinitesimally conformally invariant if and only if \( D_g T_{i_1 \cdots i_s}^{j_1 \cdots j_r} = 0 \) for all \( g \in M \). By integrating along paths in the conformal class \( c \), one observes that \( T_{i_1 \cdots i_s}^{j_1 \cdots j_r} \) is infinitesimally conformally invariant if and only if \( T_{i_1 \cdots i_s}^{j_1 \cdots j_r} \) determines a natural conformally invariant element of \( \mathcal{E}_{i_1 \cdots i_s}^{j_1 \cdots j_r} [\omega] \).

Our proof of the conformal invariance of \( \xi^{(k)}_i \) and \( \rho^\Phi_i \) relies on three ingredients. First are the well-known conformal linearizations of the Weyl and Cotton tensors.

**Lemma 2.2.** Let \( (M^n, g) \) be a pseudo-Riemannian manifold and let \( \Upsilon \in C^\infty(M) \). Then

\[
D_g W_{ijkl}(\Upsilon) = 0,
\]

\[
D_g C_{ijk}(\Upsilon) = W_{ij} \partial_k \Upsilon.
\]

Second is the conformal linearization of the exterior derivative of a natural homogeneous scalar function.

**Lemma 2.3.** Let \( (M^n, g) \) be a pseudo-Riemannian manifold and let \( \Upsilon \in C^\infty(M) \). For any natural homogeneous Riemannian scalar function \( f \) of degree \( w \), it holds that

\[
D_g \nabla_i f(\Upsilon) = w f \Upsilon_i + \nabla_i D_g f(\Upsilon).
\]

**Proof.** This follows directly from Equation \((2.3)\). \(\square\)

Third is the conformal linearization of the divergence of a natural homogeneous differential form.

**Lemma 2.4.** Let \( (M^n, g) \) be a pseudo-Riemannian manifold and let \( \Upsilon \in C^\infty(M) \). For any natural homogeneous Riemannian \( k \)-form \( \alpha_{i_1 \cdots i_k} \) of degree \( w \), it holds that

\[
D_g \nabla^i \alpha_{i_2 \cdots i_k}(\Upsilon) = (n + w - 2k) \Upsilon^i \alpha_{i_2 \cdots i_k} + \nabla^i D_g \alpha_{i_2 \cdots i_k}(\Upsilon).
\]

**Proof.** This follows directly from Equation \((2.3)\) and the fact that

\[
\tilde{\nabla}_i \alpha_j = \nabla_i \alpha_j - \Upsilon_i \alpha_j - \alpha_i \Upsilon_j + \Upsilon^s \alpha_s g_{ij}
\]

for all one-forms \( \alpha_i \) and all metrics \( g \) and \( \tilde{g} = e^{2\Upsilon} g \) on \( M \). \(\square\)

3. Conformal invariance

In this section we prove Theorem \([1]\). We separate the proof into two parts.

We begin by proving that \( \xi^{(k)}_i \) is conformally invariant on 2\( k \)-dimensional pseudo-Riemannian manifolds.
Proposition 3.1. Let \((M^{2k}, g)\) be a pseudo-Riemannian manifold and define \(\xi^{(k)}_i\) as in Equation \((1.1)\). For any \(\Upsilon \in C^\infty(M)\), it holds that
\[
e^{2k\Upsilon} \xi^{(k)}_i = \xi^{(k)}_i,
\]
where \(\xi^{(k)}_i\) is defined in terms of \(\hat{g} := e^{2\Upsilon} g\).

Proof. As discussed in Section 2, it suffices to show that the conformal linearization of \(\xi^{(k)}_i\) vanishes. A direct computation using Lemmas 2.2 and 2.3 yields
\[
D\xi^{(k)}_i (\Upsilon) = \frac{1}{k!} \delta^{ij_{2k}}_{i_1 \cdots i_{2k}} W_{j_{2k} j_{2k-1} \cdots j_{2k-1}} \Upsilon_{s} - \frac{1}{2k} \text{Pf}^{(k)}(W) \Upsilon_i.
\]
Since \(M\) is 2\(k\)-dimensional, we conclude that
\[
0 = \frac{1}{k!} \delta^{ij_{2k}}_{i_1 \cdots i_{2k}} W_{j_{2k} j_{2k-1} \cdots j_{2k-1}} \Upsilon_{s} - \frac{1}{2k} \text{Pf}^{(k)}(W) \Upsilon_i = -2k D\xi^{(k)}_i (\Upsilon).
\]

Let \(\Phi\) be a homogeneous invariant polynomial of degree \(k\). We now turn to the proof that \(\rho^{\Phi}_k\) is conformally invariant on oriented \(2k\)-dimensional pseudo-Riemannian manifolds. We in fact prove the stronger claim that the \((2k-1)\)-form \((\ast \rho^{\Phi}_k)_{i_2 \cdots i_{2k}}\) defined by Equation \((1.10)\) is conformally invariant on any pseudo-Riemannian manifold.

Proposition 3.2. Let \(\Phi\) be a homogeneous invariant polynomial of degree \(k\), let \((M^n, g)\) be a pseudo-Riemannian manifold, and let \((\ast \rho^{\Phi}_k)_{i_2 \cdots i_{2k}}\) be defined by Equation \((1.10)\). For any \(\Upsilon \in C^\infty(M)\), it holds that
\[
e^{2\Upsilon} (\ast \rho^{\Phi}_k)_{i_2 \cdots i_{2k}} = (\ast \rho^{\Phi}_k)_{i_2 \cdots i_{2k}},
\]
where \((\ast \rho^{\Phi}_k)_{i_2 \cdots i_{2k}}\) is defined in terms of \(\hat{g} := e^{2\Upsilon} g\).

Proof. As discussed in Section 2, it suffices to show that the conformal linearization of \((\ast \rho^{\Phi}_k)_{i_2 \cdots i_{2k}}\) vanishes. A direct computation using Lemma 2.2 yields
\[
D\rho (\Phi W^{k-1} C)_{i_2 \cdots i_{2k}} (\Upsilon) = \Phi^{t_1 \cdots t_k}_{s_1 \cdots s_k} W^{s_1 t_1}_{i_2} W^{s_2 t_2}_{i_3 \cdots i_2} \cdots W^{s_{2k-1} t_{2k-1}} W^{s_{2k} t_{2k}} \Upsilon_i.
\]
A direct computation using Lemma 2.3 yields
\[
D_g \nabla \ast (\ast \rho^{\Phi}(W))_{i_2 \cdots i_{2k}} (\Upsilon) = (n-4k) \Phi^{t_1 \cdots t_k}_{s_1 \cdots s_k} W^{s_1 t_1}_{i_2} W^{s_2 t_2}_{i_3 \cdots i_2} \cdots W^{s_{2k-1} t_{2k-1}} W^{s_{2k} t_{2k}} \Upsilon_i.
\]
Combining the previous two displays yields
\[
D_g (\ast \rho^{\Phi}_k)_{i_2 \cdots i_{2k}} = 0.
\]

Corollary 3.3. Let \(\Phi\) be a homogeneous invariant polynomial of degree \(k\), let \((M^{2k}, g)\) be an oriented pseudo-Riemannian manifold, and define \(\rho^{\Phi}_k\) as in Equation \((1.3)\). For any \(\Upsilon \in C^\infty(M)\), it holds that
\[
e^{2k\Upsilon} \hat{\rho}^{\Phi}_k = \rho^{\Phi}_k,
\]
where \(\hat{\rho}^{\Phi}_k\) is defined in terms of \(\hat{g} := e^{2\Upsilon} g\).

Proof. It follows directly from Equation \((2.1)\) that \(\rho^{\Phi}_k = - (\ast \ast \rho^{\Phi}_k)\). The conclusion now follows from Proposition 3.2 and the conformal invariance of the Hodge star operator \(\ast: \mathcal{E}_{i_1 \cdots i_{2k-1} [-2} \rightarrow \mathcal{E}_{i][-2k]}\).

Finally, combining Proposition 3.1 and Corollary 3.3 yields Theorem 1.1.
4. $\xi_i^{(k)}$ and the Image of $K^*$

There are two steps in our proof that $\xi_i^{(k)} \in \text{im } K^*$ on closed Riemannian 2k-manifolds. The first step is to write $\xi_i^{(k)}$ in a way that is manifestly orthogonal to the space of Killing fields. We accomplish this by proving Theorem 1.4.

Proof of Theorem 1.4. First observe that

$$\text{tr} \Omega^{(k)} = \sum_{\ell=0}^{k-1} \frac{4^{k-\ell}}{k!} \delta^{j_1\cdots j_{k-\ell}}_{i_1\cdots i_{k-\ell}} W_{j_1 j_2} W_{j_2 j_3} \cdots W_{j_{2\ell-1} j_{2\ell}} P^{i_1 i_2} P^{i_3 i_4} \cdots P^{i_{2\ell-1} i_{2\ell}}.$$

Since $R_{ijkl} = W_{ijkl} + P_{ik} g_{jl} - P_{il} g_{jk} + P_{jl} g_{ik} - P_{jk} g_{il}$, we compute that

$$\text{Pf}^{(k)}(Rm) = \sum_{\ell=0}^{k} \frac{4^{k-\ell}}{k!} \delta^{j_1\cdots j_{k-\ell}}_{i_1\cdots i_{k-\ell}} W_{j_1 j_2} W_{j_2 j_3} \cdots W_{j_{2\ell-1} j_{2\ell}} P^{i_1 i_2} P^{i_3 i_4} \cdots P^{i_{2\ell-1} i_{2\ell}}.$$

Combining these formulae yields

\[ \text{tr} \Omega^{(k)} = \text{Pf}^{(k)}(Rm) - \text{Pf}^{(k)}(W). \]

Next, a straightforward computation using Lemma 2.1 yields

\[ \nabla^i (\Omega^{(k)})_{ij} = \frac{2k}{k!} \delta^{j_{2\ell} \cdots j_{2k}}_{i_{2\ell} \cdots i_{2k}} C_{j_1 j_2} W_{j_3 j_4} \cdots W_{j_{2\ell-1} j_{2\ell}} X^i. \]

The desired conclusion follows from Equations (4.1) and (4.2). \[\square\]

The second step is to apply the Ferrand–Obata Theorem.

Proof of Theorem 1.2. Suppose first that $(M^{2k}, g)$ admits an essential conformal Killing field $X$; i.e. $\mathcal{L}_X \hat{g} \neq 0$ for all conformal metrics $\hat{g} \in [g]$. The Ferrand–Obata Theorem [9, 19] implies that $g$ is locally conformally flat. Hence $\xi_i^{(k)} = 0$.

Suppose instead that $(M^{2k}, g)$ does not admit an essential conformal Killing field. Let $X$ be a conformal Killing field. Then there is a conformally equivalent metric $\hat{g} \in [g]$ such that $\mathcal{L}_X \hat{g} = 0$. In particular, $\hat{\nabla} X^i = 0$. It follows from Theorems 1.1 and 1.2 that

$$\int_M \xi_i^{(k)} X^i \, \text{dvol} g = \int_M \mathcal{E}_i^{(k)} X^i \, \text{dvol} \hat{g} = -\frac{1}{4k^2} \int_M \text{Pf}^{(k)}(Rmg) \hat{\nabla} X^i \, \text{dvol} \hat{g} = 0.$$

Now, since $(M, g)$ is Riemannian, the divergence $K^*: \mathcal{E}_{(ij)} \to \mathcal{E}_i$ has surjective principal symbol. Therefore we have the $L^2$-orthogonal splitting

$$\mathcal{E}_i = \text{im } K^* \oplus \ker K.$$

The previous two paragraphs imply that $\xi_i^{(k)} \in \text{im } K^*$. The final conclusion follows from conformal covariance. \[\square\]

Remark 4.1. Our proof of Theorem 1.2 uses the fact that if $X^i \in K$ is essential, then $g$ is locally conformally flat [9, 19]. Frances [10] has constructed counterexamples to this statement for manifolds of signature $(p, q)$, $p, q \geq 2$, though it remains unknown whether this statement holds in Lorentzian signature. However, it is straightforward to check that $\xi_i^{(k)} = 0 \in \text{im } K^*$ for Frances’ even-dimensional counterexamples. In particular, it is not known if Theorem 1.2 is false in non-Riemannian signatures.
5. $\rho^\Phi_i$ and the Image of $K^*$

The purpose of this section is to prove Theorem \textbf{1.3}. We separate the proof into two pieces, corresponding to the two conclusions of Theorem \textbf{1.3}.

We first prove that the restriction of the induced functional $P^\Phi$ to the space $K$ of conformal Killing fields vanishes on any closed conformal manifold of Riemannian signature which admits an Einstein metric.

**Proposition 5.1.** Let $\Phi$ be a homogeneous invariant polynomial of degree $k \in \mathbb{N}$ and let $(M^{2k}, g)$ be a closed conformally Einstein manifold of Riemannian signature. Then

$$P^\Phi(X^i) := \int_M \rho^\Phi_i X^i \, d\text{vol} = 0$$

for all conformal Killing fields $X^i \in K$, where $\rho^\Phi_i$ is defined by Equation (\textbf{1.3}).

**Proof.** Since $P^\Phi(X^i) := \int_M \rho^\Phi_i X^i \, d\text{vol}$ is conformally invariant, we may assume that $(M^{2k}, g)$ is Einstein. Hence

$$\rho^\Phi_i = \frac{1}{2k} \nabla_i p^\Phi(W).$$

Let $X^i \in K$. Obata \textbf{[18]} proved that either $X^i$ is Killing or $(M^{2k}, g)$ is isometric to the round $2k$-sphere. In the former case,

$$P^\Phi(X^i) = -\frac{1}{2k} \int_M p^\Phi(W) \nabla^i X_i \, d\text{vol} = 0.$$ 

In the latter case, $\rho^\Phi_i = 0$, and hence $P^\Phi(X^i) = 0$. \hfill \Box

We now construct examples of closed Riemannian $4k$-manifolds and homogeneous invariant polynomials $\Phi$ of degree $2k$ for which $P^\Phi|_K \neq 0$. To that end, let $\mathbb{H}$ denote the space of quaternions and let $X, Y, Z$ be the frame of left-invariant vector fields on $S^3 \subset \mathbb{R}^4 \cong \mathbb{H}$ which restrict to $i, j, k$ at the identity. Let $\alpha, \beta, \gamma$ be the dual coframe. Given $t > 0$, the Berger sphere is the Riemannian manifold $(S^3, g_t)$, where

$$g_t := t \alpha \otimes \alpha + \beta \otimes \beta + \gamma \otimes \gamma.$$ 

We begin by finding an example in dimension four.

**Proposition 5.2.** Fix $\Phi^\gamma_{ij} = \frac{1}{2} \delta_i^\gamma \delta_j^\gamma$. Let $(S^3, g_t), t > 0$, be a Berger sphere and let $\theta$ be a nonvanishing left-invariant one-form on $S^1$. If $t \neq 1$, then the Riemannian product $(S^3 \times S^1, \pi_t) := g_t + \theta^2$ is such that

$$P^\Phi|_K \neq 0.$$ 

In particular, if $t \neq 1$, then $(S^3 \times S^1, \pi_t)$ is not conformal to an Einstein metric.

**Remark 5.3.** If $t \neq 1$, then $(S^3 \times S^1, \pi_t)$ is not Bach flat, and hence not even locally conformally Einstein. We are not aware of an example of a closed, locally conformally Einstein four-manifold which can be shown to not be conformally Einstein using Proposition \textbf{5.1}.

**Proof.** For clarity of the exposition, we write this proof in index-free notation.
It is well-known that
\[ \nabla^\omega \alpha = -\beta \otimes \gamma + \gamma \otimes \beta, \]
\[ \nabla^\beta \beta = -(t-2)\alpha \otimes \gamma - t \gamma \otimes \alpha, \]
\[ \nabla^\beta \gamma = (t-2)\alpha \otimes \beta + t \beta \otimes \alpha, \]
\[ \text{Ric}_{\gamma t} = 2t^2 \alpha \otimes \alpha + 2(2-t)\beta \otimes \beta + 2(2-t)\gamma \otimes \gamma. \]

From this it readily follows that
\[ W^{\gamma t} = \frac{2(t-1)}{3} \left[ t(\alpha \wedge \beta) \otimes (\alpha \wedge \beta) + t(\alpha \wedge \gamma) \otimes (\alpha \wedge \gamma) - 2(\beta \wedge \gamma) \otimes (\beta \wedge \gamma) \right.
\]
\[ - 2t(\alpha \wedge \theta) \otimes (\alpha \wedge \theta) + (\beta \wedge \theta) \otimes (\beta \wedge \theta) + (\gamma \wedge \theta) \otimes (\gamma \wedge \theta) \right] \]
\[ C^{\gamma t} = 2(t-1) \left[ (\alpha \wedge \beta) \otimes \gamma - (\alpha \wedge \gamma) \otimes \beta - 2(\beta \wedge \gamma) \otimes \alpha \right]. \]

We deduce that \( p_\Phi(W) = 0 \) and
\[ *p_\Phi = \Phi WC = -\frac{8t(t-1)^2}{3} \alpha \wedge \beta \wedge \gamma. \]

Let \( T \) be the vector field on \( S^1 \) dual to \( \theta \). Then \( T \) is a Killing field for \( (S^3 \times S^1, g_t) \).
We compute that
\[ \Phi(W) = \frac{8t(t-1)^2}{3} \int_{S^1 \times S^1} \alpha \wedge \beta \wedge \gamma \wedge \theta. \]
In particular, if \( t \neq 1 \), then \( \Phi(W)|_{\mathcal{C}} \neq 0 \). The final conclusion follows from Proposition \ref{prop:5.4}. \qed

Taking Riemannian products with \( k - 1 \) copies of \( \mathbb{C}P^2 \) yields examples in general dimension \( 4k \).

**Proposition 5.4.** Let \( \Phi \) be the homogeneous invariant polynomial of degree \( 2k \), \( k \in \mathbb{N} \), such that
\[ \Phi_{s_1 \ldots s_{2k}}^{t_1 \ldots t_{2k}} \omega_{i_1}^{s_1} \cdots \omega_{i_{2k}}^{s_{2k}} = \left( \omega^{s_1} \omega^{s_2} \right)^k \]
for all \( \omega_{ij} \in \mathcal{E}_{ij} \). Let \( t > 0 \) and consider the Riemannian product
\[ \left( S^3 \times S^1 \times \mathbb{C}P^2 \times \cdots \times \mathbb{C}P^2, G_t := \mathcal{F}_t + g_{FS} + \cdots + g_{FS} \right) \]
\( k - 1 \) times

of \( (S^3 \times S^1, g_t) \) with \( k - 1 \) copies of \( \mathbb{C}P^2 \) equipped with the Fubini–Study metric \( g_{FS} \). If \( t \neq 1 \), then \( \Phi|_{\mathcal{C}} \neq 0 \).

**Proof.** Let \( \tilde{\Phi} \) be the invariant polynomial of Proposition \ref{prop:5.2}

First observe that \( p_\Phi(W_{G_t}) \) is a nonzero multiple of \( p_1(\mathbb{C}P^2)^{k-1} \wedge p_\Phi(W_{\mathcal{F}_t}) \).
As noted in the proof of Proposition \ref{prop:5.2}, it holds that \( p_\Phi(W_{\mathcal{F}_t}) = 0 \). Therefore \( p_\Phi(W_{G_t}) = 0 \).

Next observe that \( \Phi W_{G_t}^{2k-1} C_{G_t} \) is a nonzero multiple of \( p_1(\mathbb{C}P^2)^{k-1} \wedge *p_{\tilde{\Phi}_{\gamma t}} \).
Since \( \int_{CPS} p_1(\mathbb{C}P^2) \neq 0 \), we conclude that \( \Phi_{G_t}(T) \) is a nonzero multiple of \( \Phi_{\gamma t} \Phi_{G_t}(T) \). Hence, by the proof of Proposition \ref{prop:5.2}, it holds that \( \Phi_{G_t}|_{\mathcal{C}} \neq 0 \). \qed

Finally, combining Propositions \ref{prop:5.1}, \ref{prop:5.2}, and \ref{prop:5.4} yields Theorem \ref{thm:1.3}.
6. $\xi^{(2)}_i$ AND THE DIVERGENCE OF INVARIANT TENSORS

We conclude by proving that, in dimension four, the natural conformal invariant

$$\xi^{(2)}_i = 2W_{istu}C^{stu} + \frac{1}{8}\nabla_i(W_{stu}W^{stu})$$

is not expressible as the divergence of a natural symmetric $(0,2)$-tensor field of weight $-2$. This follows from the classification of the natural elements of $\mathcal{E}_{(ij)0}[-2]$ in dimension four.

**Proposition 6.1.** In dimension four, the vector space of natural elements of $\mathcal{E}_{(ij)0}[-2]$ is spanned by the set

$$\{ B_{ij}, W_{isjt}P^{st}, tfP^s_j, tfJP_{ij}, tf\nabla^2_{ij}J \}.$$  

In particular, $\xi^{(2)}_i$ is not the divergence of a natural element of $\mathcal{E}_{(ij)0}[-2]$.

**Proof.** On a pseudo-Riemannian four-manifold, the space of natural symmetric $(0,2)$-tensor fields of weight $-2$ is spanned by partial contractions of $\nabla^2R^m\otimes g$ and $R^m\otimes R^m\otimes g$. Equivalently, it is spanned by $\Delta P, \nabla^2_{ij}J, W^2_{ij} := W_{istu}W^{stu}_j, W_{isjt}P^{st}, P^s_i P^s_j, JP_{ij}$, and products of their traces with $g_{ij}$. Using the facts that, in dimension four,

$$B_{ij} = \Delta P_{ij} - \nabla^2_{ij}J + 2W_{isjt}P^{st} - 4P^s_i P^s_j + |P|^2g_{ij}$$

and $tfW^2_{ij} = 0$, we conclude that the space of natural elements of $\mathcal{E}_{(ij)0}[-2]$ is spanned by Equation (6.1).

Next, it is known that the Bach tensor is divergence-free [8]. Direct calculation gives

$$\nabla^j(W_{isjt}P^{st}) = -C_{sIt}P^{st} + \frac{1}{2}W_{istu}C^{stu},$$

$$\nabla^j(tfP^s_j) = P_{is}\nabla^s J + \frac{1}{4}\nabla_i(P^s_j P^{st}) + C_{sIt}P^{st},$$

$$\nabla^j(tfJP_{ij}) = P_{is}\nabla^s J + \frac{1}{4}\nabla_i(J^2),$$

$$\nabla^j(tf\nabla^2_{ij}J) = \frac{3}{4}\nabla_i\Delta J + 2P_{is}\nabla^s J + \frac{1}{2}\nabla_i(J^2).$$

It readily follows that there is not a natural element of $\mathcal{E}_{(ij)0}[-2]$ with divergence equal to $\xi^{(2)}_i$. 

It is natural to conjecture that $\xi^{(k)}_i$, $k \geq 2$, cannot be expressed as the divergence of a natural element of $\mathcal{E}_{(ij)2k}[-2k]$ in dimension $2k$. However, an attempt to verify this by identifying a basis for $\mathcal{E}_{(ij)0}[-2k]$ is impractical for general $k$.

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References

[1] T. P. Branson, Differential operators canonically associated to a conformal structure, Math. Scand. 57 (1985), no. 2, 293–345. MR832360

[2] ———, Sharp inequalities, the functional determinant, and the complementary series, Trans. Amer. Math. Soc. 347 (1995), no. 10, 3671–3742. MR1316845

[3] T. P. Branson and A. R. Gover, The conformal deformation detour complex for the obstruction tensor, Proc. Amer. Math. Soc. 135 (2007), no. 9, 2961–2965. MR2317974

[4] ———, Pontrjagin forms and invariant objects related to the $Q$-curvature, Commun. Contemp. Math. 9 (2007), no. 3, 335–358. MR2336821

[5] J. S. Case and A. R. Gover, The $P'$-operator, the $Q'$-curvature, and the CR tractor calculus, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 20 (2020), no. 2, 565–618. MR4105911

[6] J. S. Case and Y. Takeuchi, $I'$-curvatures in higher dimensions and the Hirachi conjecture, preprint. arXiv:2003.08201.

[7] S. S. Chern and J. Simons, Characteristic forms and geometric invariants, Ann. of Math. (2) 99 (1974), 48–69. MR0353327

[8] C. Fefferman and C. R. Graham, The ambient metric, Annals of Mathematics Studies, vol. 178, Princeton University Press, Princeton, NJ, 2012. MR2858236

[9] J. Ferrand, The action of conformal transformations on a Riemannian manifold, Math. Ann. 304 (1996), no. 2, 277–291. MR1371767

[10] C. Frances, About pseudo-Riemannian Lichnerowicz conjecture, Transform. Groups 20 (2015), no. 4, 1015–1022. MR3416437

[11] J. Gasqui and H. Goldschmidt, Déformations infinitésimales des structures conformes plates, Progress in Mathematics, vol. 52, Birkhäuser Boston, Inc., Boston, MA, 1984. MR776970

[12] A. R. Gover and L. J. Peterson, The ambient obstruction tensor and the conformal deformation complex, Pacific J. Math. 226 (2006), no. 2, 309–351. MR2247867

[13] K. Hirachi, $Q$-prime curvature on CR manifolds, Differential Geom. Appl. 33 (2014), no. suppl., 213–245. MR3159959

[14] M.-L. Labbi, Double forms, curvature structures and the $(p,q)$-curvatures, Trans. Amer. Math. Soc. 357 (2005), no. 10, 3971–3992. MR2159696

[15] J. Lelong-Ferrand, Transformations conformes et quasi-conformes des variétés riemanniennes compactes (démonstration de la conjecture de A. Lichnerowicz), Acad. Roy. Belg. Cl. Sci. Mém. Coll. in-8° (2) 39 (1971), no. 5, 44. MR0322739

[16] D. Lovelock, The Einstein tensor and its generalizations, J. Mathematical Phys. 12 (1971), 498–501. MR0275835

[17] T. Marugame, Renormalized characteristic forms of the Cheng-Yau metric and global CR invariants, Adv. Math. 377 (2021), Paper No. 107468, 55. MR4186011

[18] M. Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere, J. Math. Soc. Japan 14 (1962), 333–340. MR0142086

[19] ———, The conjectures on conformal transformations of Riemannian manifolds, J. Differential Geometry 6 (1971/72), 247–258. MR0303464

[20] R. Penrose and W. Rindler, Spinors and space-time. Vol. I, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1984. Two-spinor calculus and relativistic fields. MR776784

[21] Y. Takeuchi, A constraint on Chern classes of strictly pseudoconvex CR manifolds, SIGMA Symmetry Integrability Geom. Methods Appl. 16 (2020), Paper No. 005, 5. MR4053867

109 McAllister Building, Penn State University, University Park, PA 16802

Email address: jscase@psu.edu