THE GRAM MATRIX OF A TEMPERLEY-LIEB ALGEBRA IS SIMILAR TO THE MATRIX OF CHROMATIC JOINS

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Dedicated to the memory of Xiao-Song Lin (1957-2007)

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1. Introduction

Rodica Simion noticed experimentally that matrices of chromatic joins (introduced by W. Tutte in [Tu2]) and the Gram matrix of the Temperley-Lieb algebra, have the same determinant, up to renormalization. In the type A case, she was able to prove this by comparing the known formulas: by Tutte and R. Dahab [Tu2, Dah], in the case of chromatic joins, and by P. Di Francesco, and B. Webby [DiF, Wei] (based on the work by K. H. Ko and L. Smolinsky [KS]) in the Temperley-Lieb case; see [CSS]. She then asked for a direct proof of this fact [CSS, Sch], Problem 7.

The type B analogue was an open problem central to the work of Simion [Sch]. She demonstrated strong evidence that the type B Gram determinant of the Temperley-Lieb algebra is equal to the determinant of the matrix of type B chromatic joins, after a substitution similar to that in type A, cf. [Sch].

In this paper we show that the matrix $J_n$ of chromatic joins and the Gram matrix $G_n$ of the Temperley-Lieb algebra are similar (after rescaling), with the change of basis given by diagonal matrices. More precisely we prove the following two results:

**Theorem A.** We have $J_n^A(\delta^2) = PG_n^A(\delta)P$, where $P = (p_{ij})$ is a diagonal matrix with $p_{ii}(\delta) = \delta^{bk(\pi_i) - n/2}$; here $bk(\pi_i)$ denotes the number of blocks in the type A non-crossing $n$-partition $\pi_i$; see 2.1 of Section 2 for precise definitions.

**Theorem B.** We have $J_n^B(\delta^2) = P^BG_n^B(1, \delta)P^B$ where $P^B = (p_{ij}^B)$ is a diagonal matrix with $p_{ii}^B(\delta) = \delta^{nzbk(\pi_i) - n/2}$; here $nzbk(\pi_i)$ denotes half of the number of non-zero blocks in the type B non-crossing $n$-partition $\pi_i$; see 2.2 of Section 2 for precise definitions.
2. Definitions and notation

2.1. The type A case. An \( n \)-partition of type A is a partition \( \pi \) of the \( n \) element set \( \{1, 2, \ldots, n\} \) into blocks. The number of blocks is denoted by \( bk(\pi) \). To represent \( \pi \) pictorially, we place the numbers 1, 2, \ldots, \( n \) anti-clockwise around the boundary circle of the unit disk and draw a chord, called a connection chord, in the disk between two numbers \( i < j \) if they are in the same block of \( \pi \) and there is no \( k \) in the same block with \( i < k < j \). We say that \( \pi \) is non-crossing if all connection chords can be drawn without crossing each other. Notice that each block is represented by a tree. Denote the set of all non-crossing \( n \)-partitions of type A by \( \Pi_A^n \). On the other hand if \( n = 2m \) is even, we have bipartitions of \( 2m \) points of type A, those \( 2m \)-partitions of type A with every block containing exactly 2 numbers. Denote the set of all non-crossing \( 2m \)-bipartitions of type A by \( \Gamma_A^m \). We have a bijection \( \varphi_A : \Pi_A^n \rightarrow \Gamma_A^n \) realized by considering the boundary arcs of a regular neighborhood of the connection chords (see Fig. 1 and compare Fig. 2, Fig. 3).

2.2. The type B case. An \( n \)-partition of type B is a partition \( \pi \) of the \( 2n \) element set \( \{+1, +2, \ldots, +n, -1, -2, \ldots, -n\} \) into blocks with the property that for any block \( K \) of \( \pi \), its opposite \( -K \) is also a block of \( \pi \), and that there is at most one invariant block (called the zero block) for which \( K = -K \). Since all non-zero blocks occur in pairs \( \pm K \) one defines \( \text{nzbk}(\pi) \) as half of the number of all non-zero blocks. To represent \( \pi \) pictorially, we place the numbers \( +1, +2, \ldots, +n, -1, -2, \ldots, -n \) anti-clockwise around the boundary circle of a disk and draw a connection chord in the disk between two numbers \( i < j \) if they are in the same block of \( \pi \) and there is no \( k \) in the same block with \( i < k < j \). Then \( \pi \) is said to be non-crossing if all connection chords can be drawn without crossing each other. Denote the set of all non-crossing \( n \)-partitions of type B by \( \Pi_B^n \). The set \( \Pi_B^n \) is illustrated in Fig. 2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The bijection \( \varphi_A : \Pi_A^3 \rightarrow \Gamma_A^3 \).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The set \( \Pi_B^n \).}
\end{figure}

\footnote{For topologists the term invariant block is more natural than zero block so we use this names interchangeably in the paper.}

\footnote{Here we use the order \(+1 < +2 < \cdots < n < -1 < -2 < \cdots < -n\).}

\footnote{The non-crossing condition forces a partition to have at most one zero block.}
On the other hand if \( n = 2m \) is even, we have bipartitions of \( 4m \) points of type \( B \), those \( 2m \)-partitions of type \( B \) with every block containing exactly 2 numbers. Denote the set of all non-crossing \( 2m \)-bipartitions of type \( B \) by \( \Gamma_B^m \). Similar to the type \( A \) case we have a bijection \( \varphi_B : \Pi_B^n \to \Gamma_B^n \) realized by considering the boundary arcs of a regular neighborhood of the connection chords (see Fig. 3).

2.3. The matrices. For any \( n \)-partitions \( \pi \) and \( \pi' \) of type \( A \) (resp. \( B \)), denote by \( \pi \lor \pi' \) the finest \( n \)-partition (not necessarily non-crossing) of type \( A \) (resp. \( B \)) that is coarser than both \( \pi \) and \( \pi' \). The matrix of chromatic joins of type \( A \) and \( B \) are respectively:

\[
(J_A^n(\delta))_{\pi,\pi' \in \Pi_A^n} = \delta^{\text{bik}(\pi \lor \pi')} \quad \text{and} \quad (J_B^n(\delta))_{\pi,\pi' \in \Pi_B^n} = \delta^{\text{nabk}(\pi \lor \pi')}.
\]

For any \( 2n \)-bipartitions \( \pi \) and \( \pi' \) of type \( A \) (resp. \( B \)), one can glue them along the boundary circles respecting the labels. The result, denoted \( \pi \lor \pi' \), is a collection of disjoint circles on a 2-sphere. The Gram matrix of Temperley-Lieb algebra of type \( A \) and \( B \) are respectively:

\[
(G_A^n(\delta))_{\pi,\pi' \in \Gamma_A^n} = \delta^{\text{e}(\pi \lor \pi')} \quad \text{and} \quad (G_B^n(\alpha, \delta))_{\pi,\pi' \in \Gamma_B^n} = \alpha^{c_0(\pi \lor \pi')} \delta^{\text{c}(\pi \lor \pi')}.
\]

where \( c(\pi \lor \pi') \) is the number of circles, \( c_0(\pi \lor \pi') \) is the number of zero (i.e. invariant) circles \( C \) with \( C = -C \), and \( c_d(\pi \lor \pi') \) is the number of pairs of non-zero circles \( C, -C \) with \( C \neq -C \) in \( \pi \lor \pi' \).

\[\footnote{The matrix \( G_A^n(\delta) \) was first used by H. Morton and P. Traczyk to find a basis of the Kauffman bracket skein module of a tangle [MT], and played an important role in Lickorish's approach to Witten-Reshetikhin-Turaev invariants of 3-manifolds [Li]. The matrix \( G_B^n(1, \delta) \) was first considered by Rodica Simion in 1998; compare [Sch].}\]
3. PROOF OF THEOREMS [A] AND [B]

Proof of Theorem [A]. For \( \pi_i \in \Pi^A_n \) let \( b_i := \varphi_A(\pi_i) \in \Gamma^A_n \). Since \( c(b_i \lor b_j) \) is also equal to the number of boundary components of the regular neighborhood of the pictorial representation of \( \pi_i \lor \pi_j \) we have \( 2b\pi(\pi_i \lor \pi_j) = c(b_i \lor b_j) + b\pi(\pi_i) + b\pi(\pi_j) - n = b\pi(\pi_i) - \frac{c}{2} + c(b_i \lor b_j) + b\pi(\pi_j) - \frac{c}{2} \). The formula can be obtained from the expression for the Euler characteristic of a plane graph: Let \( G_{\pi} \) be a graph corresponding to the non-crossing partition \( \pi \). \( G_{\pi} \) is a forest of \( n \) vertices and \( n - b\pi(\pi) \) edges. Similarly, let \( G_{\pi_i \lor \pi_j} \) be the graph corresponding to \( \pi_i \lor \pi_j \). We should stress that \( \pi_i \lor \pi_j \) does not have to be a noncrossing partition and that the graph \( G_{\pi_i \lor \pi_j} \) is a plane graph obtained by putting \( G_{\pi_i} \) inside a disk and \( G_{\pi_j} \) outside the disk with \( \partial K_{\pi_i \lor \pi_j} \) composed of the \( n \) points on the unit circle (e.g.: \( \bigcirc \bigcirc \bigcirc - \bigcirc \), or \( \bigcirc \bigcirc \bigcirc - \bigcirc \bigcirc \bigcirc \)).

By construction, \( G_{\pi_i \lor \pi_j} \) is a plane graph of \( n \) vertices and \( b\pi(\pi_i \lor \pi_j) \) components. It has \( E(G_{\pi_i \lor \pi_j}) = E(G_{\pi_i}) + E(G_{\pi_j}) = n - b\pi(\pi_i) + n - b\pi(\pi_j) \) edges. Furthermore, if we embed \( G_{\pi_i \lor \pi_j} \) in a disjoint union of \( b\pi(\pi_i \lor \pi_j) \) 2-spheres (each component of \( G_{\pi_i \lor \pi_j} \) in a different sphere) we can identify \( c(b_i \lor b_j) \) with the number of regions of the embedded graph. The Euler characteristic is on the one hand equal to \( 2b\pi(\pi_i \lor \pi_j) \) and on the other hand equal to \( n - E(G_{\pi_i \lor \pi_j}) + c(b_i \lor b_j) = c(b_i \lor b_j) + b\pi(\pi_i) + b\pi(\pi_j) - n \), as needed. Theorem [A] follows directly from the formula.

Proof of Theorem [B]. For \( \pi_i \in \Pi^B_n \) let \( b_i := \varphi_B(\pi_i) \in \Gamma^B_n \) and \( b_0(\pi_i) \) be the number of zero blocks of \( \pi_i \). Recall that \( c_0(b_i \lor b_j) \) is the number of zero (i.e. invariant) components of \( b_i \lor b_j \). As in the type A case we have \( 2b\pi(\pi_i \lor \pi_j) = c(b_i \lor b_j) + b\pi(\pi_i) + b\pi(\pi_j) - 2n \). Furthermore, we have (see Lemma 1):

\[
2b\pi(\pi_i \lor \pi_j) = c_0(b_i \lor b_j) + b\pi(\pi_i) + b\pi(\pi_j).
\]

(Notice that \( b\pi(\pi_i \lor \pi_j) \) can be equal to 2, 1, or 0.) From these we conclude that:

\[
2n b\pi(\pi_i \lor \pi_j) = c_d(b_i \lor b_j) + n b\pi(\pi_i) + n b\pi(\pi_j) - n.
\]

Thus Theorem [B] follows.

Lemma 1. The zero blocks and zero components satisfy the following identity:

\[
2b\pi(\pi_i \lor \pi_j) = c_0(b_i \lor b_j) + b\pi(\pi_i) + b\pi(\pi_j),
\]

where \( b_i = \varphi_B(\pi_i) \) and \( b_j = \varphi_B(\pi_j) \).

Proof. The lemma reflects the basic properties of a 2-sphere with an involution fixing two points and its compact invariant submanifolds.

To demonstrate the formula we consider all cases of blocks of \( \pi_i; \pi_j \) and \( \pi_i \lor \pi_j \) divided into four classes:

1. If \( K \) is a non-zero (i.e. non-invariant) block of \( \pi_i \lor \pi_j \), then all its constituent blocks in \( \pi_i \) and \( \pi_j \) are non-zero blocks and the boundary components of a regular neighborhood of the geometric realization of \( K \) (denoted \( \partial K \)), that is circles in \( b_i \lor b_j \) corresponding to \( K \), are non-invariant (non-zero) curves, i.e. not in \( c_0(b_i \lor b_j) \).
(2) If \( K \) is a zero block of \( \pi_i \vee \pi_j \) but all its constituent blocks in \( \pi_i \) and \( \pi_j \) are non-zero blocks, then exactly two components of \( \partial K \) are invariant curves.

(3) If \( K \) is a zero block of \( \pi_i \vee \pi_j \) and exactly one constituent block is a zero block then exactly one component of \( \partial K \) is an invariant curve.

(4) If \( K \) is a zero block of \( \pi_i \vee \pi_j \) and exactly two constituent blocks are zero-blocks (necessarily one in \( \pi_i \) and one in \( \pi_j \)) then no component of \( \partial K \) is an invariant curve.

These conditions taken together prove the formula in Lemma 1. □

4. Corollaries

Theorem B and the results of [MS, CP] allow us to answer Problems 1 and 2 of [Sch] about a formula for the determinant of the type-\( B \) matrix of chromatic joins:

**Corollary 2.**

\[
\det(J_B^n(\delta^2)) = \prod_{i=1}^{n} \left(T_i(\delta)^2 - 1\right)^\binom{2n}{n-i}
\]

where \( T_i(\delta) \) is the Chebyshev polynomial of the first kind:

\[
T_0 = 2, \quad T_1 = \delta, \quad T_i = \delta T_{i-1} - T_{i-2}.
\]

The matrix \( J_B^n(\delta) \) can be generalized to a matrix of two variables as follows:

\[
(J_B^n(\alpha, \delta))_{\pi, \pi' \in \Pi_B^n} = \alpha^{\text{bk}_0(\pi \vee \pi')} \delta^{\text{nzbk}(\pi \vee \pi')}.
\]

It follows from Lemma 11 that

\[
J_B^n(\alpha^2, \delta^2) = P_B^n(\alpha, \delta)G_B^n(\alpha, \delta)P_B^n(\alpha, \delta),
\]

where \( P_B^n(\alpha, \delta) = (p_{ij}) \) is a diagonal matrix with \( p_{ii}(\alpha, \delta) = \alpha^{\text{bk}_0(\pi_i)} \delta^{\text{nzbk}(\pi_i)} \).

Furthermore, \( \det P_B^n(\alpha, \delta) = \alpha^{\frac{n}{2}} \) and thus we have:

**Corollary 3.**

\[
\det J_B^n(\alpha^2, \delta^2) = \alpha^{\binom{2n}{n} G_B^n(\alpha, \delta)} = \alpha^{\binom{2n}{n}} \prod_{i=1}^{n} \left(T_i(\delta)^2 - \alpha^2\right)^{\binom{2n}{n-i}}.
\]

**Remark 4.** Consider the Gram matrix of type \( B \) based on non-crossing connections in an annulus. This matrix is the same as the one considered before in Theorem 12 via the branched cover described in Fig. 4.

\*[This follows from Proposition 3 of [Rei], which asserts that there exists a fixed-point free involution \( \gamma \) on \( \Pi_B^n \) such that \( \text{bk}_0(\pi) + \text{bk}_0(\gamma(\pi)) = 1 \) and \( \text{nzbk}(\pi) + \text{nzbk}(\gamma(\pi)) = n \).]

\[††\]This interpretation of the Gram matrix of type B Temperley-Lieb algebra is mentioned in [Sch] as an annular skein matrix and utilized in [MS] and [CP].
Figure 4. The double branch cover $pr : D^2 \rightarrow D^2$ with the “cutting” arc $S$.

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