Gravitational reheating in quintessential inflation

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We provide a detailed study of gravitational reheating in quintessential inflation generalizing previous analyses only available for the standard case when inflation is followed by an era dominated by the energy density of radiation. Quintessential inflation assumes a common origin for inflation and the dark energy of the Universe. In this scenario reheating can occur through gravitational particle production during the inflation-kination transition. We calculate numerically the amount of the radiation energy density, and determine the temperature $T^\ast$ at which radiation starts dominating over kination. The value of $T^\ast$ is controlled by the Hubble parameter $H_0$ during inflation and the transition time $\Delta t$, scaling as $H_0^2 \left[ \ln(1/H_0 \Delta t) \right]^{3/4}$ for $H_0 \Delta t \ll 1$ and $H_0^2 (H_0 \Delta t)^{-c}$ for $H_0 \Delta t \gg 1$. The model-dependent parameter $c$ is found to be around 0.5 in two different parametrizations for the transition between inflation and kination.

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I. INTRODUCTION

Over the last few years strong evidence has been found allowing to conclude that the Universe underwent an accelerated expansion with negative pressure for at least twice in its history: in its very early infancy during inflation, and at the present day. While inflation was introduced as an elegant solution to the horizon and flatness problem and to explain the anisotropy of the cosmic microwave background radiation \cite{1}, today’s accelerated expansion was an unexpected discovery which strongly favors the existence of a dark energy component contributing a fraction $\Omega_{DE} \simeq 0.7$ to the closure density. It is somehow intriguing that both periods of accelerated expansion can be explained by scalar fields whose potential energy dominates the energy density at some time of its evolution. In fact, while for inflation an “inflaton” field is introduced, the dark energy component in the present Universe can be explained in a dynamical way by modifying the standard cosmology with the introduction of a slowly evolving scalar field called “quintessence” \cite{2}. Compared to a cosmological constant, the latter approach has the nice feature of explaining in a natural way why radiation and dark energy provide comparable contributions to the energy budget of the present Universe, in spite of having very different time evolutions (the coincidence problem), through “tracking solutions” \cite{2} of the quintessence field. It is then natural to imagine whether it is possible to unify the two scenarios, identifying inflaton and quintessence with the same field $\phi$ \cite{4,5}. As a consequence of this, various models of quintessential inflation have been discussed in the literature \cite{6}.

In the quintessential inflation scenario two main qualitative properties emerge as follows.

i) The potential $V(\phi)$ needs to account for the large mismatch between the inflationary plateau at the beginning of the $\phi$ field evolution (whose natural value $V_0$ is expected to be within a few orders of magnitude of the Plank scale $m_P = 2.4 \times 10^{18}$ GeV) and the tiny scale of the quintessential tail $V_F$ eventually accounting for the cosmological constant today, about $10^{-121}$ times smaller. As a consequence of this, a crucial requirement of $V(\phi)$ is to have a rapid fall at the end of inflation, so that at the end of its slow–roll phase the field $\phi$ experiences a strong acceleration as it “deep-dives” from $V_0$ toward $V_F$. Thus the Universe undergoes a period of “kination” expansion when its energy density is dominated by the kinetic energy of $\phi$, whose potential energy eventually provides the dark energy at later time.

ii) In this scenario the standard reheating mechanism which is usually assumed to create the initial plasma by the decay of $\phi$ is not at work. However, the mechanism of gravitational particle production is able to reheat the Universe \cite{7,8} without introducing extra ingredients in the scenario of quintessential inflation. Although it is much less efficient than the usual reheating mechanism, there is no difficulty to accommodate the required cosmology of radiation domination. Initially at the gravitational reheating temperature, the tiny amount of radiation produced by this mechanism is largely sub-dominant compared to the energy contribution from kination. However, since the
energy density of kination is red-shifted away by the cosmological expansion much faster than the radiation density, radiation eventually dominates at some lower temperature \( T_\ast \) which, in viable models, must be above the MeV scale in order to preserve the successes of Big Bang Nucleosynthesis. Here we remark that other reheating mechanisms like preheating \cite{9} or curvaton reheating \cite{10} could be realized typically by introducing additional scalar fields in quintessential inflation.

The purpose of this work is to provide a quantitative analysis for the amount of particle production and determine the temperature \( T_\ast \) assuming the gravitational reheating mechanism. The actual value of \( T_\ast \) has important phenomenological consequences. During kination the expansion rate of the Universe is larger compared to the usual radiation domination case, and this can lead to various non-standard cosmological scenarios. For instance, if \( T_\ast \) is at the GeV scale the relic abundance of a thermal cold dark matter candidate can be significantly enhanced compared to the canonical prediction \cite{11}, because its decoupling time from the plasma is anticipated, providing the correct amount of Dark Matter for values of the cross sections between the DM particle and ordinary matter sizeably higher compared to the standard case. This has interesting phenomenological implications not only for the LHC and other future collider experiments but also for astrophysical observations like direct or indirect dark matter detection. Indeed, the enhanced dark matter annihilation rate which is hinted by the recent PAMELA \cite{12} or ATIC/PPB-BETS \cite{13,14} data, can be made compatible with the thermal dark matter production in the context of quintessential inflation.

The presence of a kination era may also have an impact on the properties of the electroweak baryogenesis \cite{15} or the thermal leptogenesis induced by the CP-violating decay of a right-handed neutrino \cite{16}.

Due to its interesting phenomenological implications, we aim to discuss \( T_\ast \) in detail within the quintessential inflation scenario. The original calculation of gravitational reheating has been performed in Ref. \cite{7} in the case of the inflation-radiation transition, and its results have been widely used also in papers discussing quintessential inflation \cite{8}. In our discussion we will generalize the analysis of Ref. \cite{7} to the case of the inflation-kination transition by performing full numerical calculations of the Bogolubov coefficients as is reviewed in Section 2. In Section 3 we introduce two ways of parametrizing the transitions between inflation and radiation or kination, in order to discuss how the particular way of modeling them may affect our conclusions. The results of our numerical calculation of the radiation energy density are presented in Section 4 and the kination-radiation equality temperature \( T_\ast \) is discussed in Section 5, showing its dependence on the inflationary scale and the transition time. We give our conclusions in Section 6.

II. GRAVITATIONAL PARTICLE CREATION: A SETUP

We briefly review here the mechanism of gravitational particle creation \cite{17,18} in order to set up the relevant equations in a suitable form for numerical integration. The main results of this section are Eqs. \eqref{eq:1} and \eqref{eq:2}. In a spatially flat background the Robertson-Walker metric is written as:

\[
\text{\( ds^2 = a^2(\eta)(d\eta^2 - dx^i dx^i) \),}
\]

where \( a(\eta) \) is the scale factor as a function of the conformal time \( \eta \). Recall that the conformal time \( \eta \) is related to the usual time \( t \) by \( d\eta = dt/a \). Let us consider a massless scalar field \( \Phi \) minimally coupled to gravity. Spatial translation symmetry allows to separate spatial and time dependence in the Fourier-transform of the field: \( \phi_k(\vec{x}, t) = \rho_k(t)e^{i\vec{k} \cdot \vec{x}} \).

Factoring out the scale factor: \( \chi_k \equiv \rho_k/a \), one finds the equation of motion of the (conformal) time-dependent harmonic oscillator:

\[
\chi_k'' + \omega_k^2(\eta)\chi_k = 0
\]

where

\[
\omega_k^2(\eta) = k^2 + V = k^2 - \frac{a''}{a}.
\]

Here the prime indicates a derivative with respect to conformal time and \( k \) is the comoving momentum. Assuming the adiabatic condition \( V \ll k^2 \) in the early past and in the late future (so that \( \omega(\eta \to \pm \infty) = \text{constant} \)) implies that the field operator can be expanded in terms of a complete set of positive–frequency solutions. Denoting the solutions asymptotically free in the past and in the future by \( f_j \) and \( F_j \), respectively, we have

\[
\phi = \sum_j \left( a_j f_j + a_j^* f_j^* \right) = \sum_j \left( b_j F_j + b_j^* F_j^* \right) .
\]
In the above equation \( a_j \) and \( b_j \) are operators that annihilate the in and out vacua, and are connected by the Bogolubov transformations:

\[
\begin{align*}
a_j &= \sum_k \left( \alpha_{jk}^* b_k - \beta_{jk}^* b_k^* \right) \\
b_k &= \sum_j \left( \alpha_{jk} a_j + \beta_{jk}^2 a_j^* \right).
\end{align*}
\]

Assuming that at \( \eta = -\infty \) no particles are present, the initial vacuum \( |0\rangle_{in} \), which in the Heisenberg picture is the state of the system for all the time, is annihilated by the \( a_j \) operators: \( a_j |0\rangle_{in} = 0 \). However the physical number operator that counts particles in the out-region is \( N_k = b_k^* b_k \), so that the mean number of particles created into mode \( k \) is \( \langle N_k \rangle_{in} = \langle 0 | b_k^* b_k | 0 \rangle_{in} = \sum_j | \beta_{jk} |^2 \), which implies that particle creation is proportional to the negative–frequency content of out states with the boundary condition that the in states contain only negative frequencies. This is tantamount to solving Eq. (2) with the boundary condition that the in states contain only negative frequencies. This is Eq. (2) with the boundary conditions:

\[
\begin{align*}
\chi_k (\eta \to -\infty) &= \frac{e^{-ik\eta}}{\sqrt{2k}} \\
\chi_k (\eta \to +\infty) &= \frac{1}{\sqrt{2k}} (\alpha_k e^{-ik\eta} + \beta_k e^{ik\eta})
\end{align*}
\]

with \( k \equiv |\vec{k}| \). The Wronskian condition: \( \chi_k \chi_k' - \chi_k^* \chi_k' = i \) keeps the correct normalization of states and implies the additional relation:

\[
|\alpha_k|^2 - |\beta_k|^2 = 1. \tag{7}
\]

Once the function \( \omega(\eta) \) is given, setting \( \chi_k = r_k e^{\omega_k \eta} \) the real and imaginary part of Eq. (2) can be separated:

\[
\begin{align*}
R_k'' + (\omega_k^2 - \Omega_k^2) R_k &= 0 \\
R_k' \Omega_k + 2R_k' \Omega_k &= 0
\end{align*}
\]

where \( \Omega_k \equiv \phi_k' \). The corresponding boundary conditions are

\[
\begin{align*}
r(\eta \to -\infty) &= \frac{1}{\sqrt{2k}} \\
r'(\eta \to -\infty) &= 0 \\
\Omega_k (\eta \to -\infty) &= -k.
\end{align*}
\]

The last equation of (3) is separable and readily solved, yielding \( \Omega_k = -\frac{1}{2\pi \sqrt{k}} \). Then the problem reduces to solving the single second-order differential equation:

\[
R_k'' + \left( \omega_k^2(\eta) - \frac{1}{4r_k'^2} \right) R_k = 0, \tag{10}
\]

whose solutions can be found by numerical integration given the function \( \omega_k(\eta) \) from Eq. (3). Once the values of \( R_k(\eta \to \infty) \) and \( R_k'(\eta \to \infty) \) are found, the Bogolubov coefficient can be extracted using a suitable combination of \( \chi_k \) and its derivative \( \chi_k' \). Making use of the asymptotic conditions (4) and of Eq. (7), one finds

\[
|\beta_k|^2 = \frac{k^2 |\chi_k|^2 + |\chi_k'|^2 - k}{2k^2 r_k'^2 + (r_k')^2 + \frac{1}{4r_k'^2} - k}. \tag{11}
\]

In the above equation for \( \eta \to -\infty \) the boundary conditions \( r = 1/\sqrt{2k}, r' = 0 \) consistently imply \( |\beta_k|^2 = 0 \). Note that whenever \( V \ll k^2 \) (i.e. close to the adiabatic regime) a perturbative approach is possible in the calculation of the Bogolubov coefficient \( \beta_k \), which can be approximated by the Fourier transform of the function \( V \) (9):

\[
\beta_k \simeq -\frac{i}{2k} \int_{-\infty}^{+\infty} e^{-2ik\eta V(\eta)} d\eta. \tag{12}
\]
Although we will mainly rely on the numerical solutions of Eq. (10) for our discussions, we will also give some examples where the numerical solution is compared to Eq. (12). Given the Bogolubov coefficient $\beta_k$, one obtains directly the final energy density produced:

$$\rho = \frac{1}{(2\pi a)^3} \int d^3 \tilde{k} |\beta_k|^2.$$  (13)

### III. GRAVITATIONAL REHEATING FOR INFLATION–RADIATION/KINATION TRANSITION

The amount of gravitational particle production can be calculated by solving numerically Eq. (10) given the potential $V = -a''/a = -a^2 R/6$ in $\omega_k(\eta)$ of Eq. (3) (here $R$ is the Ricci scalar). Its functional forms are uniquely defined during the inflation and radiation/kination periods, while a certain degree of arbitrariness arises from the modelization of the transition period. For this reason in our analysis we will adopt two different parametrizations for the transition. One is a polynomial parametrization for $a^2(\eta)$ following the original paper [2], and the other is a parametrization of the equation state $w \equiv p/\rho$ smoothly connecting the inflation ($w = -1$) and the radiation ($w = 1/3$) or kination ($w = 1$) regimes with an hyperbolic tangent.

Let us now apply such a parametrization to the different case when inflation is followed by a period of kination.

$$f(x) = \begin{cases} 
1/x^2 & \text{for } x < -1 \text{ (inflation)} \\
(1/x^2, a_0 + a_1 x + a_2 x^2 + a_3 x^3 & \text{for } -1 < x < x_0 - 1 \text{ (transition)} \\
b_0 (x + b_1 x)^2 & \text{for } x > x_0 - 1 \text{ (radiation)}, 
\end{cases}$$  (15)

where the 6 parameters $a_i$ and $b_i$ can be fixed by imposing continuity of $f(x)$, $f(x)'$ and $f(x)''$ in the two points $x = -1$ and $x = x_0 - 1$. Since the corresponding expressions we find are quite involved and they differ from the ones given in [2], we give them in the Appendix.

Let us now apply such a parametrization to the different case when inflation is followed by a period of kination. In this case one has $\rho \propto 1/a^6$ and $H \propto 1/a^3$, so that $R \propto 1/a^6$ and the adiabatic condition, $R \rightarrow 0$ is verified only asymptotically for $\eta \rightarrow \infty$ and $a(\eta) \rightarrow \infty$. Recalling $H = a''/a^2$ one finds that $a^2(\eta)$ is a linear function of $\eta$ during kination. The equivalent of Eq. (15) for a transition between inflation and kination is then given by

$$f(x) = \begin{cases} 
1/x^2 & \text{for } x < -1 \text{ (inflation)} \\
1/x^2, a_0 + a_1 x + a_2 x^2 + a_3 x^3 & \text{for } -1 < x < x_0 - 1 \text{ (transition)} \\
b_0 + b_1 x & \text{for } x > x_0 - 1 \text{ (kination)}, 
\end{cases}$$  (16)

in which the 6 parameters $a_i$ and $b_i$ can be fixed in the same way as in Eq. (15). In this case the corresponding expressions are particularly simple, and given by:

$$\begin{cases} 
a_0 = -\frac{1}{x_0} + 6, \quad a_1 = -\frac{3}{x_0} + 8, \quad a_2 = -\frac{3}{x_0} + 3, \quad a_3 = -\frac{1}{x_0} \\
b_0 = 3 + 3x_0 - x_0^2, \quad b_1 = 2 + 3x_0. 
\end{cases}$$  (17)
In Fig. 1 we plot the equation of state $w = p/\rho$ (upper panel) and the function $V = -a''/a$ from Eq. (3) (lower panel) for both cases of radiation domination (dashed line) and kination domination (solid line) for $x_0 = 0.1$ (left), $x_0 = 1$ (center), and $x_0 = 2$ (right), using the parametrizations of Eqs. (15,16). Taking into account the approximate expression of the Bogolubov coefficient given in Eq. (12) it is possible from this figure to anticipate that an enhancement of the process of particle creation is expected whenever $x_0 \ll 1$, and especially for kination compared to radiation. On the other hand, at large $x_0$ the process is generally suppressed compared to the previous case and is almost independent on $x_0$. Notice however that the parametrization (15,16) can be considered as a reasonable choice capturing model-independent properties of a generic transition only when $x_0 \ll 1$. When $x_0 \gtrsim 1$ the actual behavior of the functions $w(x)$ and $V(x)$ will depend on the details of the model-dependent transition. For instance, the peculiar behavior of the function $w(x)$ for the inflation–kination transition in the third panel of Fig. 1 is to be ascribed to an artifact of the parametrization used rather than to the physical properties of a realistic system.

In order to estimate the model dependence of our results, in the following analysis we will also adopt a smoother transition applied to the equation of state:

$$w(y) = \frac{w_f - 1}{2} + \frac{w_f + 1}{2} \tanh \left( \frac{2y}{y_0} \right),$$

(18)
where \( y = \log(a) + \text{const} \), and \( y_0 \) parametrizes the duration of the transition in e-foldings, while \( w_f = 1/3, 1 \) is the asymptotic equation of state at late times for radiation and kination, respectively. With the above parametrization, the relation

\[
\frac{d \log H}{dy} = \frac{d \log H}{d \log a} = -\frac{3}{2} (1 + w),
\]

(19)
can be integrated analytically to yield:

\[
\tilde{H} \equiv H/H_0 = e^{-\frac{3}{2}(1+w_f)} \left[ y + \log \left( \frac{\log \cosh \frac{y}{y_0} + \log 2}{\cosh \frac{y}{y_0}} \right) \right],
\]

(20)
with \( H_0 = H(y = -\infty) \). The relation between \( y \) and \( x \) is given by \( dx/dy = 1/(a\tilde{H}) \). Identifying the transition period with the interval \(-y_0 < y < +y_0 \) \((w(y_0) = \pm 0.96)\), we have:

\[
x_0 = \int_{-y_0}^{y_0} \frac{dy'}{H(y') a(y')}.
\]

(21)
The quantities \( w = p/\rho \) (upper panels) and \( V(x) \) (lower panels) in this parametrization are shown in Fig. 2 for \( x_0 = 0.1, 1, 2 \), where \( V(x) \) is obtained from the general relation:

\[
V(x) = -\frac{a''}{a} = -\frac{a(y)^2 \tilde{H}(y)^2}{2} (1 - 3w(y)).
\]

(22)
In Fig. 2 we have chosen the (arbitrary) normalization of the scale factor \( a \propto e^y \) and the origin of the conformal time variable \( x \) in such a way that the minimum of \( V(x) \) is for \( x = -1 \), and \( V(x = -1) = -2 \), in order to have a direct comparison with the previous parametrization discussed in Fig. 1. That is, our normalization in terms of the variable \( y \) is set to satisfy \( V(y_{min}) = -2 \) and \( x(y_{min}) = -1 \) where \( y_{min} \) can be calculated by solving \( dV/dy \bigg|_{y=y_{min}} = 0 \). From Eq. (20) the asymptotic behavior of the Hubble constant follows in a straightforward way:

\[
\tilde{H}(y > +y_0) \simeq e^{-\frac{3}{2}(1+w_f)} \left[ y + \log \left( \frac{\log \cosh \frac{y}{y_0} + \log 2}{\cosh \frac{y}{y_0}} \right) \right].
\]

(23)
which gives, for the case of kination,

\[
\tilde{H}(y > +y_0) \simeq \frac{b_1}{2 a^3} \quad \text{with} \quad b_1 = 2 a(y_{min})^3 e^{-3 y_{min}}.
\]

(24)
All the quantities discussed above can be expressed explicitly in terms of \( y_0 \). In the case of a transition from inflation to kination \((w_f = 1)\), we get:

\[
y_{min} = -\frac{y_0}{4} \log \left( \frac{(9y_0 + 6)^{\frac{2}{3}} + (y_0 + 6)^{\frac{2}{3}}}{(9y_0 + 6)^{\frac{2}{3}} - (y_0 + 6)^{\frac{2}{3}}} \right),
\]

(25)
from which explicit forms of other variables can be derived.

Note that in the limit \( y_0 \to 0 \) a power expansion allows to recover the parametrization of Eq. (16). In this case \( a(y_{min}), \tilde{H}(y_{min}) \to 1 \), but when making a direct comparison with the parametrization of Eq. (16) the correspondence between \( y_0 \) and \( x_0 \) is somewhat arbitrary since \( w \equiv \pm 1 \) only when \( y \to \pm \infty \). However, by choosing the boundaries of the transition at \( y = \pm \bar{y}_0 \equiv \pm cy_0 \) in such a way that \( y_{min} \to -\bar{y}_0 \simeq -x_0/2 \) (so that in both parametrizations the beginning of the transition coincides asymptotically to the minimum of the potential), we find

\[
b_1 \to 2 e^{\frac{3}{x_0}} \simeq 2 + 3x_0,
\]

(26)
which corresponds to the expression for \( b_1 \) found in the last line of Eq. (17) (The above procedure corresponds to setting \( c = \log(y_0/4) \)). With such normalizations Figs. 1 and 2 can be directly compared. As expected they differ when \( x_0 \gtrsim 1 \) while they coincide when \( x_0 \ll 1 \). Notice however that the definition of the conformal time depends on the arbitrary normalization of the scale factor \( a \). For this reason in the following section we will discuss our results expressing the duration of the transition \( x_0 \) in term of the physical time \( t \). Of course the physical results will not depend on these conventions.
As we will see later, $b_1$ is one of the important parameters determining the temperature of the radiation component in quintessential inflation models. In the limit of an abrupt transition ($\Delta t \to 0$), we have $b_1 \approx 2$ as discussed above. In the limit of a slow transition, $b_1$ behaves like:

$$b_1 \propto (H_0 \Delta t)^c,$$

where $\Delta t$ is the transition time and the exponent $c$ is a model-dependent parameter. We find $c = 0.5$ and $c \approx 0.48$ in the case of the polynomial and $tanh$ parametrization, respectively. For the latter, we define $\Delta t$ by $\Delta t \equiv \int_{-y_0}^{y_0} dy/H$.

**IV. NUMERICAL RESULTS**

In this section we discuss our results for the amount of radiation (13) with the Bogolubov coefficient $|\beta_k|^2$ given by the asymptotic value of Eq. (11) at $\eta \to \infty$. The latter is obtained in a straightforward way by solving numerically the second-order differential equation (10) with the boundary conditions given in (9). In Fig. 3 we present the quantity radiation transition (thin lines) and inflation–kinetation transition (bold lines). In the left-hand panel the transition is parametrized according to Eqs. (15,16), while in the right-hand one Eq. (18) is used. In both figures solid lines are the result of numerical integration of Eq. (10), while dashed lines are obtained by making use of the perturbative approximation given in Eq. (12). A couple of comments are in order here.

Fig. 3 shows an infrared divergence for the inflation-radiation transition, but not for the inflation-kinetation transition. Such a property can be studied analytically in the limit of an instantaneous transition for which we can obtain simple solutions for particle creation. We assume that the inflation-radiation transition occurs at $x = H_0 \eta = -1$ ($x_0 \equiv 0$). Apparently natural choices of the in-modes are:

$$\chi^in_k = \left(1 - \frac{i}{k\eta} \right) e^{-ik\eta} \sqrt{2k},$$

which are the solutions of Eq. (2) during inflation. Now using the out-modes during radiation in Eq. (6), it is easy to find

$$|\beta_k|^2 = \frac{1}{4k^3},$$

which results in the infrared divergence of the energy density (13) for the modes $k \to 0$. This implies that the in-modes for $k \to 0$ correspond to unphysical states since otherwise this large infrared contribution to the energy density cannot represent a self-consistent solution to the Einstein’s equations. One way forward to solve this problem is to find a suitable choice of the initial vacuum state such that the two-point function for the field does not diverge in the infrared limit (19). Once such choices are made to obtain finite results for $k \to 0$, their contribution to the integration of the total energy density becomes negligible. In this paper, we take a practical approach to cut off the $k < H_0$ modes to calculate the energy density. A possible alternative approach to solve this problem would be to calculate the radiation energy density by using the renormalized energy momentum tensor with a proper regularization scheme (17), which would not display this infrared behavior during the quasi de Sitter stage (20).

It is interesting to note that the situation is different for the inflation-kinetation transition. Again assuming an instantaneous transition at $x = -1$, one gets $V = H_0^2/(3+2\kappa)^2$. While we can take the in-modes as in Eq. (28) during inflation, the out-modes during kination can be written as

$$\chi^out_k = \frac{\sqrt{\pi}}{2} \sqrt{\frac{2}{H_0}} \left[ \alpha_k H_0^{(2)}(\kappa z) + \beta_k H_0^{(1)}(\kappa z) \right],$$

where $\kappa \equiv k/H_0$, $z \equiv x + 3/2$ and $H_0^{(1,2)}$ are the Hankel functions. Matching the in-modes and out-modes at $x = -1$ one readily finds for $k \to 0$,

$$|\beta_k|^2 \sim \frac{1}{\pi k^3},$$

which is consistent with our numerical results in Fig. 3. Therefore, we can conclude that the radiation-kinetation transition may be free from infrared divergences in reasonable models for the transition.
FIG. 3: $k^3 \times |\delta_k|^2$ as a function of $k$ normalized to $H_0$ in the case of inflation–radiation transition (thin lines) and inflation–kination transition (bold lines), and for $x_0 = 0.1$. Solid curves are the result of numerical integration of Eq. (10), while dashed lines are obtained by making use of the perturbative approximation given in Eq. (12). In the left–hand panel the transition is modeled according to Eqs. (15,16), while in the right–hand one the parametrization of Eq. (18) is used.

Another interesting observation is that the epoch of particle production is confined to a short interval close to the minimum of the potential $V(x)$, which, by our convention, is for $x = -1$ or $y = y_{min}$. This can be seen by considering the adiabaticity parameter $\omega'/\omega^2$ which becomes much smaller than 1 for the adiabatic regime. In the case of a slow transition ($x_0 \gg 1$) the adiabaticity parameter is suppressed at all times except for $x \simeq -1$, so particle creation is a quasi–instantaneous process even in this case, as in the situation of an abrupt transition with $x_0 \ll 1$. For this reason when the transition time is large the amount of particle creation becomes a very slowly varying function of $x_0$ or $H_0 \Delta t$, as will be shown below.

The computation of the energy density is shown in Fig. 4, where the quantity $a^4 \rho$ normalized to $H_0^4$, is plotted as a function of $H_0 \Delta t$ in both cases of a transition between inflation and radiation (dashed lines) and inflation and kination (solid lines). Notice how, as previously anticipated, the comoving energy density $a^4 \rho$ shows a plateau at large $\Delta t$, while an enhancement is present at smaller $\Delta t$. In the case of kination, due to the fact that the function $V$ has a second peak at positive $\eta$ for small $\Delta t$ (see the lower part of the left panel of Fig. 1), which is not present for radiation due to a vanishing Ricci scalar $R$, the quantity $a^4 \rho$ is larger by about a factor of two compared to the case of radiation, and in both cases, we find the behavior of ultra-violet divergence: $a^4 \rho \propto \ln(H_0 \Delta t)$ for $\Delta t \to 0$ as discussed previously. However, the amount of radiation produced by gravitational reheating for large $\Delta t$ is similar in the two cases, since in that case the potential $V$ for kination is strongly suppressed for $\eta > 0$ (see the lower part of the right–hand panel in Fig. 1) and thus very similar to the case of radiation. From Fig. 4 we obtain the typical amount of particle creation parameterized by $I \equiv a^4 \rho/H_0^4$:

$$0.03 \lesssim I \lesssim 0.08 \quad \text{Inflation} \to \text{Radiation}$$
$$0.03 \lesssim I \lesssim 0.19 \quad \text{Inflation} \to \text{Kination}$$

restricting ourselves to $H_0 \Delta t \gtrsim 10^{-3}$.

V. RADIATION–KINATION EQUALITY

A crucial element for the phenomenology of quintessential inflation is the epoch when the standard cosmology with radiation domination starts. The amount of radiation produced by gravitational reheating is very small, so that at
the end of inflation the energy of the Universe is dominated by the kinetic energy of the quintessence field. However, the energy density of kination $\rho_K$ is red-shifted away much faster than that of radiation $\rho_R$ as

$$\rho_K \propto a^{-6} \quad \text{and} \quad \rho_R \propto a^{-4},$$

and thus the latter can eventually dominate at some temperature much below the initial (reheat) temperature $T_I$. Let us define the kination–radiation equality temperature $T_\ast$ by

$$\rho_K(T_\ast) = \rho_R(T_\ast),$$

which should follow the constraint: $T_\ast > \sim 1 \text{ MeV}$, in order not to spoil nucleosynthesis.

Consider thermalization of the gravitationally produced particles occurring at $a_I$ with the corresponding (reheat) temperature $T_I$. At this moment, the radiation energy density can be expressed as

$$\rho_R^I = \frac{g_I}{a_I^4} H_0^4 = \frac{\pi^2}{30} g_I T_I^4,$$

from which one determines $a_I T_I$ given $I$ and $H_0$. The specific value of $T_I$ can be explicitly calculated if the interaction rates of particles are given in a particular model [21]. For instance, assuming a typical interaction rate of $\Gamma \sim a^2 T$ with a certain coupling strength $\alpha$, and equating it with the Hubble parameter $H = b_I/2a^3$ during kination at $a_I$ or $T_I$, we find

$$T_I = \alpha \left( \frac{30}{\pi^2} \right)^{\frac{1}{6}} \left( \frac{2I^4}{b_I} \right)^{\frac{1}{2}} H_0,$$

which can be much smaller than the Gibbons-Hawking temperature $H_0/2\pi$ [22]. Irrespective of the specific values of $a_I$ or $T_I$, we can determine the temperature $T_\ast$ when the radiation domination starts. Using Eq. (35) and the relation of iso-entropic expansion, $g_I a_I^3 T_I^3 = g_* a_*^3 T_*^3$, one obtains

$$a_\ast T_\ast = \left( \frac{g_I}{g_*} \right)^{\frac{1}{6}} \left( \frac{30 I}{\pi^2} \right)^{\frac{1}{2}} H_0.$$
FIG. 5: The combination $I_{3/4}/b_1$, entering into the kination–radiation equality temperature $T_*$ given in Eq. (39), as a function of the transition time $\Delta t/H_0^{-1}$ expressed in absolute time. Solid lines are for the transition given in Eq. (18), and dashed lines are for the transition in Eq. (16). In both cases the upper line is the result of a numerical integration of Eq. (10) and the lower one is obtained by making use of the approximation of Eq. (12).

Since the energy densities of radiation $\rho^*_R$ and kination $\rho^*_K$ at $a_*$ or $T_*$ are

$$\rho^*_R = \frac{\pi^3}{30} g_* T_*^4,$$

$$\rho^*_K = 3 m_P^2 H_0^2 \frac{b_1^2}{4 a_*^6},$$

the conditions (34) and (37) enable us to get

$$T_* = \frac{2 \times 30^{3/4}}{\sqrt{3\pi}} \frac{g_*}{g_s} \frac{I_{3/4}}{b_1} \frac{H_0^2}{m_P}.$$  \hspace{1cm} (39)

In the above equation $b_1$ is given by Eq. (17) or (24), depending on which parametrization for the transition is adopted. Moreover, notice that both quantities $I$ and $b_1$ depend on the arbitrary normalization of the scale factor, while, as expected, in Eq. (39) the combination $I_{3/4}/b_1$ entering the physical quantity $T_*$ or $T_I$ does not.

In Fig. 5 the combination $I_{3/4}/b_1$ is plotted in terms of the transition time $\Delta t/H_0^{-1}$. In the figure the solid lines show $I_{3/4}/b_1$ when the transition is modeled according to Eq. (18), while in the dashed lines the transition is modeled according to Eqs. (15,16). In both cases the upper line is the result of a numerical integration of Eq. (10) and the lower one is obtained by making use of the approximation of Eq. (12). Note that the quantity $I_{3/4}/b_1$ shows the scaling behavior:

$$\frac{I_{3/4}}{b_1} \sim \begin{cases} \frac{0.06}{\ln \left( \frac{1}{H_0 \Delta t} \right)^{1/4}} & \text{for } \Delta t \rightarrow 0 \\ \frac{0.02}{0.026 (H_0 \Delta t)^{-0.48}} & \text{for } \Delta t \rightarrow \infty \end{cases}$$  \hspace{1cm} (40)

where the two values in the second line correspond to the transitions given in Eqs. (16) and (18), respectively. From this, we find the following numerical value of $T_*$:

$$T_* \sim \frac{g_*}{\sqrt{g_s}} \frac{H_0^2}{m_P} \begin{cases} \frac{0.09}{\ln \left( \frac{1}{H_0 \Delta t} \right)^{1/4}} & \text{for } \Delta t \rightarrow 0 \\ \frac{0.03}{0.039 (H_0 \Delta t)^{-0.48}} & \text{for } \Delta t \rightarrow \infty \end{cases}$$  \hspace{1cm} (41)

which depends two important model parameters: the inflation scale $H_0$ and the transition time $\Delta t$. 


VI. CONCLUSIONS

In this paper we have provided a detailed study of gravitational reheating in quintessential inflation generalizing previous analyses only available for the standard case when inflation is followed by an era dominated by the energy density of radiation. In the quintessential inflation scenario both inflation and dark energy are caused by the same scalar field: initially the inflaton/quintessence potential energy drives inflation, which is then followed by a kination period when the energy density of the Universe is dominated by the kinetic energy of the field; this kination period is eventually ended by radiation domination, while later on the potential energy of the quintessence field prevails again, providing the dark energy observed today. In quintessential inflation, among several ways to produce radiation and thus reheat the Universe, gravitational particle creation is the minimal scenario being able to provide a sufficient amount of radiation in spite of a much lower efficiency compared to other mechanisms, thanks to the fact that the kination energy density is red–shifted at a much faster rate compared to radiation. In this scenario, the kination-radiation equality temperature $T_*$ is an important quantity which has implications for various cosmological events like baryogenesis, dark matter decoupling and nucleosynthesis, etc.

Motivated by this, we performed a numerical analysis of Bobolubov transformations to calculate the radiation energy density produced gravitationally, and determine the dependence of $T_*$ on the inflation scale $H_0$ and the transition time $\Delta t$ of inflation to kination. To study how our conclusions depend on the details in modelling the transition period we considered two different parameterizations: a polynomial expansion of $a^2(\eta)$ and a parametrization of the equation of state making use of a tangent hyperbolic function. Our main results are summarized in Eq. (41). In both cases we obtain numerically similar results, both in the small and in the large $\Delta t$ limit which may hold for generic models with a reasonable transition behavior. It is also interesting to observe that the inflation-kination transition does not show an infrared divergence for natural choices of the in-modes which represented unphysical vacuum states for small $k$ in the case of the inflation-radiation transition. Nevertheless the problem of such an infrared divergence would be removed in a consistent way by using the renormalized energy momentum tensor, which could provide an independent study of the gravitational reheating process to confirm our results. We leave this issue to a future investigation.

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APPENDIX A

We give here the explicit expressions for the parameters $a_i$ and $b_i$ introduced in Eq. (15) for the case of the inflation-radiation transition:

$$\begin{align*}
a_1 &= 3a_0 - 10 \\
a_2 &= 3a_0 - 15 \\
a_3 &= a_0 - 6 \\
b_0 &= 3(a_0(1 + t) - 5 - 6t) \\
b_1 &= \frac{a_1 + 2a_2 t + 3a_3 t^2}{2b_0} - t,
\end{align*}$$  \(A1\)

where $a_0$ is the physical solution of the quadratic equation:

$$C_1 a_0^2 + C_2 a_0 + C_3 = 0, \quad (A2)$$

chosen in such a way that $f(x) = a^2(x)$ is always positive. In (A2) the constants $C_i$ are given by

$$\begin{align*}
C_1 &= K_1K_7 - K_3K_5 \\
C_2 &= K_1K_8 + K_2K_7 - K_4K_5 - K_3K_6 \\
C_3 &= K_2K_8 - K_6K_4,
\end{align*}$$  \(A3\)
where

\[
\begin{align*}
K_1 &= 3 + 6t + 3t^2 \\
K_2 &= -10 - 30t - 18t^2 \\
K_3 &= 6 + 6t \\
K_4 &= -30 - 15t \\
K_5 &= 2 + 6t + 6t^2 + 2t^3 \\
K_6 &= -20t - 30t^2 - 12t^3 \\
K_7 &= 3 + 6t + 3t^2 \\
K_8 &= -10 - 30t - 18t^2,
\end{align*}
\]

(A4)

and \( t = x_0 - 1 \). The above equations differ from Eqs. (18,19) in Ref. [7], which do not yield continuity of \( f'(x) \) and \( f''(x) \) in \( x = x_0 - 1 \), as we checked by numerical inspection.

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