The conditional Gaussian multiplicative chaos structure underlying a critical continuum random polymer model on a diamond fractal

Jeremy Thane Clark*
University of Mississippi, Department of Mathematics

Abstract

We discuss a Gaussian multiplicative chaos (GMC) structure underlying a family of random measures $M_r$, indexed by $r \in \mathbb{R}$, on a space $\Gamma$ of directed pathways crossing a diamond fractal with Hausdorff dimension two. The laws of these random continuum path measures arise in a critical weak-disorder limiting regime for discrete directed polymers on disordered hierarchical graphs. For the analogous subcritical continuum polymer model in which the diamond fractal has Hausdorff dimension less than two, the random path measures can be constructed as subcritical GMCs through couplings to a spatial Gaussian white noise. This construction fails in the critical dimension two where, formally, an infinite coupling strength to the environmental noise would be required to generate the disorder. We prove, however, that there is a conditional GMC interrelationship between the random measures $(M_r)_{r \in \mathbb{R}}$ such that the law of $M_r$ can be constructed as a subcritical GMC with random reference measure $M_R$ for any choice of $R \in (-\infty, r)$. A similar GMC structure plausibly would hold for a critical continuum (2+1)-dimensional directed polymer model.

1 Introduction

A Gaussian field $W$ on a measure space $(\Gamma, \mu)$ and defined over a probability space $(\Omega, \mathcal{F}, P)$ is a bounded linear map $W : L^2(\Gamma, \mu) \to L^2(\Omega, \mathcal{F}, P)$ in which the range of $W$ is a Gaussian subspace of $L^2(\Omega, \mathcal{F}, P)$. For $\psi \in L^2(\Gamma, \mu)$ we have the alternative notations

$$W(\psi) \equiv \langle W, \psi \rangle \equiv \int_\Gamma W(p)\psi(p)\mu(dp),$$

(1.1)

where the field “variables” $\{W(p)\}_{p \in \Gamma}$, in nontrivial cases, can only be understood in a distributional sense as defining random variables when integrated against an appropriate test function $\psi$. A Gaussian multiplicative chaos (GMC) on $(\Gamma, \mu)$ generated by the field $W \equiv \{W(p)\}_{p \in \Gamma}$ for a coupling strength $\beta \geq 0$ is a random measure $M_\beta$ on $\Gamma$ that is formally expressed as

$$M_\beta(dp) = e^{\beta W(p) - \frac{\beta^2}{2} \mathbb{E}[W^2(p)]} \mu(dp).$$

(1.2)

In general, questions of existence and uniqueness for random measures of this type require an indirect technical interpretation since the field variables $W(p)$ are not actual random variables, and thus the factor $\exp\{\beta W(p) - \frac{\beta^2}{2} \mathbb{E}[W^2(p)]\}$ is not simply a random Radon-Nikodym derivative, $dM_\beta/d\mu$. In nontrivial cases, $M_\beta$ is a.s. mutually singular to $\mu$ even though the expectation $\mathbb{E}[M_\beta]$ is absolutely continuous with respect to $\mu$.

The first rigorous approach to defining GMC measures was introduced by Kahane [10] in 1985. Background for understanding the motivation for Kahane’s work and a discussion of some of the important directions that GMC theory has taken since then can be found in the review [12] by Rhodes

jeremy@olemiss.edu
and Vargas; see also Junnila’s dissertation \[^5\] for a summary of GMC theory that includes discussion of some additional recent contributions. In this article we will use the definitional framework for GMC proposed by Shamov in \[^3\].

A GMC \(?M_β\) is said to be subcritical if the expectation measure, \(?E[M_β]\), is σ-finite and critical otherwise. In the subcritical case, it can be assumed that \(?E[M_β] = μ\) and that the correlation operator \(?T\) of the field \(?W\) has a kernel \(?T(p, q) = E[W(p)W(q)]\) related to the correlations of \(?M_β\) as follows:

\[
E[M_β(dp)M_β(dq)] = e^{β^2T(p,q)μ(dp)μ(dq)}.
\]

(1.3)

In particular, the measure \(?ν_β := E[M_β × M_β]\) on \(?Γ × Γ\) is absolutely continuous with respect to the product measure \(?μ × μ = E[M_β] × E[M_β]\). The GMC formalism \((1.2)\) potentially defines a family of laws \((M_β)_β≥α\) for random measures on the space \(?Γ\), and for \(0 ≤ α < β\) the law of \(?M_β\) can be formally constructed from \(?M_α\) as

\[
M_β(dp) = e^{ν_βM_α(p) - \frac{β^2-α^2}{2}E[W^2_α(p)|M_α]M_α(dp)},
\]

(1.4)

where the field \(\{W_α(p)\}_{p ∈ Γ}\) is Gaussian with kernel \(?T(p, q) = E[W_α(p)W_α(q)|M_α]\) when conditioned on \(?M_α\). This conditional form for the mean fields that \(?W_α : L^2(Γ, M_α) → \mathcal{G}\) is a random bounded linear map adapted to \(?M_α\), where \(?\mathcal{G}\) is a Gaussian subspace of \(?L^2(Ω, F, P)\) whose variables are independent of \(?M_α\).

In this article we will show that a family of random measures, \((M_r)_{r ∈ R}\), introduced in \[^2\] satisfies a constructive GMC interrelationship analogous to \((1.4)\) even though the random measures \(?M_r\) are not GMCs with respect to a deterministic “pure” measure \(?μ\) as in \((1.2)\). The measures \(?M_r\) are defined on a space \(?Γ\) of directed pathways through a compact diamond fractal \(?D\) having Hausdorff dimension two, and their laws derive from a continuum/weak-disorder limiting regime for models of directed polymers on disordered hierarchical graphs \[^4\]. The family of random measure laws \((M_r)_{r ∈ R}\) satisfies properties (I)-(V) below for any fixed \(r, R ∈ R\) with \(R ≤ r\), where proving (IV) \& (V) is the focus of coming sections.

(I) \(?E[M_r] = μ\), where \(?μ\) is a probability measure on \(?Γ\). Moreover, the law of \(?M_r\) converges to the deterministic measure \(?μ\) as \(?r \searrow −∞\).

(II) Unlike \((1.3)\), the correlation measure \(?ν_r := E[M_r × M_r]\) on \(?Γ × Γ\) is not absolutely continuous with respect to the product measure \(?μ × μ\).

(III) The Radon-Nikodym derivative of \(?ν_r\) with respect to \(?ν_R\) is \(\exp\{(r−R)T(p, q)\}\) for a nonnegative kernel \(?T(p, q)\), and the product measure \(?μ × μ\) is supported on the set of pairs \((p, q) ∈ Γ \times Γ\) with \(?T(p, q) = 0\) (essentially non-intersecting paths). In contrast, the random product measure \(?M_r × M_r\) a.s. assigns positive weight to the set of \((p, q)\) such that \(?T(p, q) > 0\).

(IV) For a.e. realization of the random measure \((Γ, M_R)\) there is a field \(\{W_κ_R(p)\}_{p ∈ Γ}\) that is Gaussian with correlation kernel \(?T(p, q)\) when conditioned on \(?M_R\) and for which the following GMC, \(?M_{R,r−R}\), is well-defined:

\[
M_{R,r−R}(dp) = e^{ν_κRW_R(p) - \frac{r-R}{2}E[W^2_R(p)|M_R]M_R(dp)}.
\]

(V) The random measure \((Γ, M_{R,r−R})\) is equal in law to \((Γ, M_r)\).

Property (I) implies that \(?M_r\) is not a critical GMC since \(?E[M_r]\) is finite, and (II) implies that \(?M_r\) is not a subcritical GMC since it does not satisfy \(?E[M_r × M_r] ≤ E[M_r] × E[M_r]\). Intuitively, since \(?M_R\) approaches \(?μ\) as \(?R → −∞\) and the coupling strength \(?β = √r−R\) in (IV) blows up with \(?−R \gg 1\),

\[^1\]See Lemma 34 and the discussion following it in \[^3\].
the measures \(M_r\) are too heavily disordered to admit a subcritical GMC form with respect to \(\mu\). The assertion in (III) suggests a strong form of locality for the continuum disordered polymer model defined by \(M_r\) in which the “disordered environment” effectively confines the polymers to a measure zero portion of the space where independently chosen paths tend to have richer intersection sets.

Analogous continuum directed polymer models on disordered diamond fractals with Hausdorff dimension less than two have a canonical subcritical GMC construction [3], so dimension two is critical in this family of models. As is the case for the continuum (1+1)-dimensional directed polymer model, the random measures defining subcritical continuum polymer models on diamond fractals can be constructed through well-behaved Wiener chaos expansions; see [3, Theorem 1.25]. We conjecture that a similar conditional GMC structure to continuum polymer models on diamond fractals can be constructed through well-behaved Wiener chaos expansions; see [3, Theorem 1.25].

2 Statement of the result

Before defining the continuum polymer model considered in this article, we will state Theorem 2.7, which is a more precise formulation of the interrelational GMC structure of the family \((M_r)_{r \in \mathbb{R}}\) summarized in points (IV) & (V) above. To do this we first turn to the definition of subcritical which is a more precise formulation of the interrelational GMC structure of the family \((M_r)_{r \in \mathbb{R}}\)

In the discussion below, \((\Gamma, \mu)\) denotes a finite standard Borel measure space. The approach in [13] to addressing questions of existence and uniqueness of a GMC on \((\Gamma, \mu)\) over a field \(W \equiv \{W(p)\}_{p \in \Gamma}\) begins by reformulating the field \(W\) through a pair of linear operators \((W, Y)\) on a Hilbert space \(H\) in which \(W : H \to L^2(\Omega, \mathcal{F}, \mathbb{P})\) is a Gaussian field on \(H\), i.e., an isometric map into a Gaussian subspace of \(L^2(\Omega, \mathcal{F}, \mathbb{P})\), and the map \(Y : H \to L^0(\Gamma, \mu)\) is continuous, where the codomain \(L^0(\Gamma, \mu)\) denotes the space of measurable functions on \((\Gamma, \mu)\) equipped with the topology of convergence in measure. The linear operator \(Y\) determines a “generalized \(H\)-valued function” that sends elements \(p \in \Gamma\) to (typically unnormalizable) “vectors” \(Y(p)\) having inner product \(\langle Y(p), \phi \rangle := (Y\phi)(p)\) for \(\phi \in H\) and a.e. \(p \in \Gamma\). Similarly, \(W\) determines a generalized \(H\)-valued function such that \(\omega \in \Omega\) is mapped to \(W(\omega)\) with \(\langle W(\omega), \phi \rangle := (W\phi)(\omega)\). The “values” \(Y(p)\) and \(W(\omega)\) can be viewed as elements of a Frechet space containing \(H\) as a subspace, for instance, by choosing an orthonormal basis \((\phi_n)_{n \in \mathbb{N}}\) of \(H\) and identifying \(Y(p) \equiv \{(Y(p), \phi_n)\}_{n \in \mathbb{N}}\) and \(W(\omega) \equiv \{(W(\omega), \phi_n)\}_{n \in \mathbb{N}}\), i.e., as elements of \(\mathbb{R}^N\). The Gaussian field \(W\) is then formally defined as \(W(p) = \langle W, Y(p) \rangle\) through

\[
\int_{\Gamma} W(p)\psi(p)\mu(dp) := \left\langle W, \int_{\Gamma} Y(p)\psi(p)\mu(dp) \right\rangle
\]

for a suitable class of test functions \(\psi \in L^0(\Gamma, \mu)\) such that \(\int_{\Gamma} Y(p)\psi(p)\mu(dp) \in H\). The correlation operator of \(W\) is determined by the quadratic form sending \(\psi\) to \(\|\int_{\Gamma} Y(p)\psi(p)\mu(dp)\|_H^2\) and has kernel formally given by

\[
T(p, q) = \mathbb{E}[W(p)W(q)] = \langle Y(p), Y(q) \rangle \quad \text{for} \quad p, q \in \Gamma.
\]

The equivalence between defining the Gaussian field \(W\) through the pair \((W, Y)\) and through integration against \(L^2\) test functions as in [1.1] is explained in [13, Appendix A].
The following definition of GMC specializes [13] to the relevant subcritical case. The space of finite measures on $\Gamma$ is given the weak topology.

**Definition 2.1.** Let $W$ be a Gaussian field on $\mathcal{H}$ and $Y : \mathcal{H} \to L^0(\Gamma, \mu)$ be linear and continuous. A subcritical Gaussian multiplicative chaos $M$ on $(\Gamma, \mu)$ over the Gaussian field $(W, Y)$ is a random finite measure on $\Gamma$ satisfying (I)-(III) below.

(I) $E[M] = \mu$, i.e., $E[M(A)] = \mu(A)$ for any measurable set $A \subset \Gamma$.

(II) $M$ is adapted to the field $W$, i.e., $M \equiv M(W)$ is a measurable function of $W$.

(III) For $\phi \in \mathcal{H}$ and a.e. realization of the field $W$,

$$M(W + \phi, dp) = e^{(Y\phi)(p)}M(W, dp).$$

The formula in property (III) determines how the GMC measure is changed by a Cameron-Martin shift of the Gaussian field; see [7, Chapter 14] for a discussion of Cameron-Martin shifts. The relationship between GMC and shifts of the field $W$ is taken further in the theorem below.

**Definition 2.2.** Let $P$ denote the law of a Gaussian field $W$ on $\mathcal{H}$. A continuous linear map $Y : \mathcal{H} \to L^0(\Gamma, \mu)$ is said to be a randomized shift if the measure $\tilde{P}$ determined by $\mathcal{L}_{\tilde{P}}[W] := \mathcal{L}_{P \times \mu}[W + Y(p)]$ is absolutely continuous with respect to $P$.

**Remark 2.3.** By definition of the measure $\tilde{P}$, if $f$ is a functional of the field $W$, then $\int f(W)\mathcal{P}(dW) = \int f(W + Y(p))\mathcal{P}(dW)\mu(dp)$.

**Theorem 2.4** (Theorem 17 of [13]). There is unique subcritical GMC on $(\Gamma, \mu)$ over the field $(W, Y)$ iff $Y$ is a randomized shift.

2.2 Extending the GMC definition to a random reference measure

We can generalize Definition 2.5 to the case where $\mu$ is replaced by a random finite measure $M$ and the coupling $Y \equiv Y_M$ is a function of $M$. As before, $\Gamma$ denotes a Polish space.

**Definition 2.5.** Let $(\Gamma, M)$ be a random finite Borel measure with underlying probability space $(\Omega, \mathcal{F}, P)$ and $W : \mathcal{H} \to \mathfrak{G}$ be a Gaussian field on $\mathcal{H}$, where $\mathfrak{G}$ is a Gaussian subspace of $L^2(\Omega, \mathcal{F}, P)$ whose variables are independent of $M$. Also, let $Y_M : \mathcal{H} \to L^0(\Gamma, M)$ be a random linear map adapted to $M$ that is a.s. continuous. A conditional GMC on $(\Gamma, M)$ over the field $(W, Y_M)$ is defined as a random finite measure $M$ on $\Gamma$ satisfying (I)-(III) below.

(I) $E[M \mid M] = M$

(II) $M \equiv M(M, W)$ is adapted to the $\sigma$-algebra generated by the random measure $M$ and the Gaussian field $W$.

(III) For $\phi \in \mathcal{H}$ and a.e. realization of the pair $(M, W)$,

$$M(M, W + \phi, dp) = e^{(Y_M\phi)(p)}M(M, W, dp).$$

The following is a corollary of Theorem 2.4.

**Corollary 2.6.** If $Y_M : \mathcal{H} \to L^0(\Gamma, M)$ defines a randomized shift for a.e. realization of $M$, then there is a unique conditional GMC on $(\Gamma, M)$ over the field $(W, Y_M)$. 

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2.3 Main theorem

In the theorem statement below, $\Gamma$ denotes the space of directed paths on a diamond fractal defined in the next section, and $(\Gamma, M_r)$ is a random measure with underlying probability space $(\Omega, \mathcal{F}, P)$, whose law arises as a continuum limit of disordered Gibbs measures on discrete models for random polymers.

**Theorem 2.7.** Let the one-parameter family of laws for random measures $(M_r)_{r \in \mathbb{R}}$ on the space $\Gamma$ be defined as in [5, Theorem 2.11] (restated below in Theorem 3.1). For a fixed $r \in \mathbb{R}$, let $\mathcal{M}$ be an infinite-dimensional Gaussian subspace of $L^2(\Omega, \mathcal{F}, P)$ whose variables are independent of $M_r$. Moreover, let $W : \mathcal{H} \to \mathcal{M}$ be a Gaussian field on a Hilbert space $\mathcal{H}$. There is a random compact operator $Y_{M_r} : \mathcal{H} \to L^2(\Gamma, M_r)$ adapted to $M_r$ such that (i)-(iii) below hold for any $a \in \mathbb{R}_+$.

(i) The operator $\sqrt{a} Y_{M_r}$ is a.s. a randomized shift. The operator $T_{M_r} : L^2(\Gamma, M_r) \to L^2(\Gamma, M_r)$ defined by $T_{M_r} := Y_{M_r} Y_{M_r}^*$ is a.s. Hilbert-Schmidt but not trace class and has kernel $T(p, q)$ independent of $M_r$.

(ii) There is a unique conditional GMC $M_{r,a}$ on $(\Gamma, M_r)$ over the field $(W, \sqrt{a} Y_{M_r})$.

(iii) The random measure $(\Gamma, M_{r,a})$ is equal in law to $(\Gamma, M_{r+a})$.

3 A diamond fractal, its path space, and a critical continuum model

Sections 3.1-3.3 review the construction from [3] of a diamond fractal, which we refer to as the diamond hierarchical lattice (DHL), along with its space of directed paths. Section 3.4 outlines the properties of the continuum random polymer measures $(M_r)_{r \in \mathbb{R}}$ referred to in Theorem 2.7. The presentation in points (A)-(V) below is intended to be scannable and readily referred back to, and a reader familiar with [5] can skip to Section 4.

The DHL $D^{b,s}$, which depends on a branching number $b \in \{2, 3, \ldots\}$ and a segmenting number $s \in \{2, 3, \ldots\}$, defines a space $\Gamma^{b,s}$ of directed pathways between opposing nodes $A$ and $B$; see the figure below for a depiction of the diamond fractal’s self-similarity in the case of $(b, s) = (2, 3)$. A directed pathway is an isometrically embedded copy of the unit interval $[0, 1]$ with $0 \equiv A$ and $1 \equiv B$.

![Figure 1: The diamond fractal $D^{2,3}$ embeds shrunken copies $D^{2,3}_{i,j}$ of itself corresponding to each $(i, j) \in \{1, 2\} \times \{1, 2, 3\}$. The path space $\Gamma^{2,3}$ is canonically soluble as $\bigcup_{i=1}^{2} \times_{j=1}^{3} \Gamma^{2,3}$ through three-fold concatenation of paths crossing the subcopies of $D^{2,3}$.

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3.1 The DHL and its space of directed paths

(A) Sequences: Given \( b, s \in \{2, 3, \ldots \} \), define
\[
D^{b,s} := \left( \{1, \ldots, b\} \times \{1, \ldots, s\} \right) \infty,
\]
i.e., the set of sequences of ordered pairs \( x = \{(b_k, s_k)\}_{k \in \mathbb{N}} \), where \( b_k \in \{1, \ldots, b\} \) and \( s_k \in \{1, \ldots, s\} \). The DHL, \( D^{b,s} \), is defined as an equivalence relation on \( D^{b,s} \),
\[
D^{b,s} := D^{b,s} / (x, y \in D^{b,s} \text{ with } d_D(x, y) = 0),
\]
for a semi-metric \( d_D : D^{b,s} \times D^{b,s} \rightarrow [0, 1] \) to be defined below.

(B) The semi-metric: Define the map \( \tilde{\pi} : D^{b,s} \rightarrow [0, 1] \) such that a sequence \( x = \{(b_k^x, s_k^x)\}_{k \in \mathbb{N}} \) is assigned
\[
\tilde{\pi}(x) := \sum_{k=1}^{\infty} \frac{s_k^x - 1}{s_k^x},
\]
in other terms, the number with base-\( s \) decimal expansion having \( k^{th} \) digit \( s_k^x - 1 \in \{0, \ldots, s-1\} \). Define the extremal sets
\[
A := \{ x \in D^{b,s} \mid \tilde{\pi}(x) = 0 \} \quad \text{and} \quad B := \{ x \in D^{b,s} \mid \tilde{\pi}(x) = 1 \}.
\]
For \( x, y \in D^{b,s} \) we write \( x \downarrow y \) if \( x \) or \( y \) belongs to one of the sets \( A, B \) or if the sequences of pairs \( \{(b_k^x, s_k^x)\}_{k \in \mathbb{N}} \) and \( \{(b_k^y, s_k^y)\}_{k \in \mathbb{N}} \) defining \( x \) and \( y \), respectively, have their first disagreement at an \( s \)-component value, i.e., there exists an \( n \in \mathbb{N} \) such that \( b_k^x = b_k^y \) for all \( 1 \leq k \leq n \) and \( s_n^x \neq s_n^y \). We define the semi-metric \( d_D \) in terms of \( \tilde{\pi} \) as
\[
d_D(x, y) := \begin{cases} 
|\tilde{\pi}(x) - \tilde{\pi}(y)| & \text{if } x \downarrow y, \\
\inf_{z \in D^{b,s}, z \downarrow x, z \downarrow y} (d_D(x, z) + d_D(z, y)) & \text{otherwise}.
\end{cases}
\]
The semi-metric \( d_D(x, y) \) takes values \( \leq 1 \) since, by definition, \( z \downarrow x \) and \( z \downarrow y \) for any \( z \in A \) or \( z \in B \), and thus \( d_D(x, y) \leq \min(\tilde{\pi}(x) + \tilde{\pi}(y), 2 - \tilde{\pi}(x) - \tilde{\pi}(y)) \).

(C) Vertex set: Let \( E^{b,s} \) denote the set of points \( x \in D^{b,s} \) that correspond through \( (3.1) \) to a single-element equivalence class of \( D^{b,s} \). The complement \( V^{b,s} = D^{b,s} \setminus E^{b,s} \) is a countable, dense set.

(D) Directed paths: A directed path on \( D^{b,s} \) is a continuous function \( p : [0, 1] \rightarrow D^{b,s} \) such that \( \tilde{\pi}(p(r)) = r \) for all \( r \in [0, 1] \). Thus the path moves at a constant speed from \( A \) to \( B \). We can measure the distance between paths using the uniform metric:
\[
d_U(p_1, p_2) = \max_{0 \leq r \leq 1} d_D(p_1(r), p_2(r)) \quad \text{for} \quad p_1, p_2 \in \Gamma^{b,s}.
\]
Paths cross over \( V^{b,s} \) at the countable set \( V \subset [0, 1] \) of times \( t \) of the form \( t = \frac{k}{\pi^n} \) for \( k, n \in \mathbb{N}_0 \).

3.2 Cylinder sets and uniform measures

(E) Shift maps: Define the shift maps \( S_{i,j} : D^{b,s} \rightarrow D^{b,s} \) for \( (i, j) \in \{1, \ldots, b\} \times \{1, \ldots, s\} \) that send a sequence \( x \in D^{b,s} \) to a shifted sequence \( y = S_{i,j}(x) \) having initial term \( (i, j) \), i.e., \( \{(b_k^y, s_k^y)\}_{k \in \mathbb{N}} \) is mapped to \( \{(b_k^z, s_k^z)\}_{k \in \mathbb{N}} \) for \( (b_1^z, s_1^z) = (i, j) \) and \( (b_k^z, s_k^z) = (b_{k-1}^z, s_{k-1}^z) \) for \( k \geq 2 \). The maps \( S_{i,j} \) are well-defined on \( D^{b,s} \) and have the contractive property
\[
d_D(S_{i,j}(x), S_{i,j}(y)) = \frac{1}{s} d_D(x, y) \quad \text{for} \quad x, y \in D^{b,s}.
\]
Next we will outline the defining properties for the family of random measures \((M_{b,b})_I\) form of (ii) occurs in the sense of branching than segmenting. If the paths set of intersection times: \(I\). For any collection of cylinder sets \((H)\) Cylinder sets for directed paths: For any \(\langle H \rangle\) Cylinder subsets of the DHL: \(C(b,s)\) to shrunken, embedded copies \(D^{b,s}\) take \(D^{b,s}\) for which \(I_p,q\) is finite. Thus (i) above occurs with probability one, and there is no chance that a non trivial form of (ii) is a fixed point for the function \(M(x) = \frac{1}{b} \left[ 1 - (1-x)^s \right]\).

3.4 Continuum random polymer measures in the critical case of the DHL

Next we will outline the defining properties for the family of random measures \((M_{r})_r \in \mathbb{R}\) introduced in \([5]\). For the remainder of the article we will focus only on the critical case \(b = s\) in the sense of the DHL and maintain \(b \in \{2,3,\ldots\}\) as an underlying parameter that will be removed as a superscript from all DHL-related notations: \(D \equiv D^{b,b}, \Gamma \equiv \Gamma^{b,b}, E \equiv E^{b,b}, V \equiv V^{b,b}\). The expectation symbol \(\mathbb{E}\) will always refer to the underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on which \(M_{r}\) is defined.
(M) **Disordered measure on the path space:** The following theorem is from [5]. The uniqueness of the family of laws with properties (I)-(IV) will be used to prove part (iii) of Theorem 2.7.

**Theorem 3.1** (Theorem 2.11 of [5]). There is a unique family of laws for random measures \( (M_r)_{r \in \mathbb{R}} \) on the path space, \( \Gamma \), of \( D \) satisfying the properties (I)-(IV) below.

(I) The expectation of the measure \( M_r \) with respect to the underlying probability space is the uniform measure on paths, i.e., \( \mathbb{E}[M_r] = \mu \).

(II) For a correlation measure \((\Gamma \times \Gamma, \nu_r)\) discussed below, we have that \( \mathbb{E}[M_r \times M_r] = \nu_r \).

(III) For \( m \in \{2, 3, \ldots\} \), the \( m \)th centered moment of the total mass, \( M_r(\Gamma) \), is given by \( R^{(m)}(r) \) for an increasing function \( R^{(m)}: \mathbb{R} \to \mathbb{R}_+ \) that decays in proportion to \( (-r)^{-[m/2]} \) as \( r \to -\infty \) and grows without bound as \( r \to \infty \).

(IV) Let \( (\Gamma, M_r^{(i,j)}) \) be independent copies of \((\Gamma, M_r)\) corresponding to the first-generation embedded copies, \( D_{i,j} \), of \( D \). There is equality in distribution of the random measures \( M_{r+1} \) under the identification \( \Gamma \equiv \bigcup_{i=1}^b \times \bigcup_{j=1}^b \Gamma \).

(N) **Basic properties of \( M_r \):** The random measures \( M_r \) are a.s. mutually singular with respect to \( \mu \) although they a.s. assign positive measure \( M_r(A) > 0 \) to every open set \( A \subset \Gamma \). As \( r \to -\infty \), \( M_r \) converges to \( \mu \) in the sense that for any \( F \in L^2(\Gamma, \mu) \) the random variable \( \int_{\Gamma} F(p) M_r(dp) \) converges in \( L^2 \) to the constant \( \int_{\Gamma} F(p) \mu(dp) \).

(O) **Total mass variance function:** The variance \( R(r) := \text{Var}(M_r(\Gamma)) \) is a continuous increasing function \( R: \mathbb{R} \to \mathbb{R}_+ \) satisfying the recursive relation: \( \frac{1}{b} \left[ (1 + R(r))^b - 1 \right] = R(r+1) \) for all \( r \in \mathbb{R} \) and having the following vanishing asymptotics as \( r \to -\infty \):

\[
R(r) = -\frac{\kappa^2}{r} + \frac{\kappa^2 \eta \log(-r)}{r^2} + O\left(\frac{\log^{2}(-r)}{r^3}\right) \quad \text{for constants } \kappa = \sqrt{\frac{2}{b-1}} \quad \text{and } \eta = \frac{b+1}{3(b-1)}.
\]

(P) **Correlation measure:** The correlation measure \((\Gamma \times \Gamma, \nu_r)\) is determined by assigning products \( p \times q \) of cylinder sets \( p, q \in \Gamma_n \) the weight \( \nu_r(p \times q) = \frac{1}{|\Gamma_n|^2} (1 + R(r-n))^N_{n,(p,q)} \), where \( N_{n,(p,q)} \) is the number of \( k \in \{1, \ldots, n^a\} \) such that \( p(k) = q(k) \). The marginals of \( \nu_r \) are both \((1 + R(r))\mu \) although \( \nu_r \) is not absolutely continuous with respect to \( \mu \times \mu \); see (S) below.

(Q) **Intersection-time kernel:** The kernel \( T(p, q) \) in the definition below is \( \nu_r \)-a.e. finite by [5, Theorem 2.27] and effectively measures the set of intersection times \( I_{p,q} = \{t \in [0,1] | p(t) = q(t)\} \).

**Definition 3.2.** For \( p, q \in \Gamma \) define \( N_n(p, q) \) as \( N_n(p, q) \) for \( (p, q) = ([p]_n, [q]_n) \). We define \( T(p, q) := \lim_{n \to \infty} \frac{2}{n} N_n(p, q) \) when the limit exists and \( T(p, q) := \infty \) otherwise.

(R) **Exponential moments of the intersection-time kernel:** For any \( a, r \in \mathbb{R} \), the Radon-Nikodym derivative of \( \nu_{r+a} \) with respect to \( \nu_r \) is \( \exp\{a \Gamma(p, q)\} \). The exponential moments of \( T(p, q) \) with respect to \( \nu_r \) are thus finite with \( \int_{\Gamma \times \Gamma} e^{a \Gamma(p, q)} \nu_{r}(dp, dq) = \nu_{r+a}(\Gamma \times \Gamma) = 1 + R(r+a) \). Moreover, since \( \mathbb{E}[M_r \times M_r] = \nu_r \), the exponential moments \( \int_{\Gamma \times \Gamma} e^{a \Gamma(p, q)} M_r(dp) M_r(dq) \) are also a.s. finite.

(S) **Lebesgue decomposition of the correlation measure:** The Lebesgue decomposition of \( \nu_r \) with respect to \( \mu \times \mu \) has the form \( \nu_r = \mu \times \mu + \rho_r \), where \( \rho_r \) is a probability measure supported on the set of pairs \( (p, q) \in \Gamma \times \Gamma \) with \( 0 < T(p, q) < \infty \), and the product \( \mu \times \mu \) is supported on the set of pairs with \( T(p, q) = 0 \) and, in fact, for which \( N_n(p, q) \in \mathbb{N} \) in Definition 3.2 is zero for all large enough \( n \in \mathbb{N} \).

(T) **The product \( M_r \times M_r \):** For any \( r \in \mathbb{R} \), the product \( M_r \times M_r \) is a.s. supported on the set of pairs \( (p, q) \in \Gamma \times \Gamma \) such that \( T(p, q) < \infty \), i.e., for which the limit in Definition 3.2 exists. The
following lemma implies that $M_r$-a.e. path has nontrivial intersection set with a $M_r$-nonnegligible portion of the path space $\Gamma$:

**Lemma 3.3** (Theorem 2.27 of [5]). Given $p \in \Gamma$ define $\widehat{s}_p$ as the set of $q \in \Gamma$ such that $T(p, q) > 0$. The random measure $M_r$ a.s. satisfies that $M_r(\widehat{s}_p) > 0$ for $M_r$-a.e. $p \in \Gamma$.

(U) A Hilbert-Schmidt operator defined by the intersection-time kernel: The following theorem characterizes the operator on $L^2(\Gamma, M_r)$ defined by integrating against the kernel $T(p, q)$.

**Theorem 3.4** (Theorem 2.41 in [5]). For a.e. realization of $M_r$, the linear map, $T_{M_r}$, on $L^2(\Gamma, M_r)$ defined by $(T_{M_r}, \psi)(p) = \int_{\Gamma} T(p, q) \psi(q) M_r(dq)$ has the properties below.

(i) $T_{M_r}$ is Hilbert-Schmidt but not traceclass.

(ii) $T_{M_r} = \check{Y}_{M_r} \check{Y}_{M_r}^*$ for a compact operator $\check{Y}_{M_r} : L^2(D, \vartheta_{M_r}) \to L^2(\Gamma, M_r)$, where $(D, \vartheta_{M_r})$ is a random Borel measure adapted to $M_r$ that has total mass, $\vartheta_{M_r}(D)$, equal to $\int_{\Gamma \times \Gamma} T(p, q) M_r(dp)M_r(dq)$ and expectation $E[\vartheta_{M_r}] = R'(r)\nu$.

(iii) The operator $\check{Y}_{M_r}$ sends the constant function $1_D$ to $M_r(p) = \int_Y T(p, q) M_r(dq)$, which is $M_r$-a.e. positive for a.e. realization of $M_r$ by Lemma 3.3.

(V) A remark on renormalization symmetry: The following proposition describes the hierarchical relationship of the family of random measures $(\vartheta_{M_r})_{r \in \mathbb{R}}$ and compact operators $(\hat{Y}_{M_r})_{r \in \mathbb{R}}$.

**Proposition 3.5** (Sections 2.8 & 2.9 of [5]). Let $\{M_r(i,j)\}_{i,j \in \{1, \ldots, b\}}$ be a family of independent copies of $(\Gamma, M_r)$ such that $M_{r+1} = \sum_{i=1}^{b} \prod_{j=1}^{b} M_r(i,j)$ in the sense of property (IV) of Theorem 3.1. The measure $(D, \vartheta_{M_{r+1}})$ has decomposition

$$\vartheta_{M_{r+1}} = \frac{1}{b^2} \bigoplus_{1 \leq i,j \leq b} \left( \prod_{\ell \neq j} M_r(i,\ell)(\Gamma) \right)^2 \vartheta_{M_r(i,j)}$$

by identifying $D \equiv \bigcup_{1 \leq i,j \leq b} D_{i,j},$

where $D_{i,j}$ is a copy of $D$ corresponding to the first-generation sub-copy of the DHL situated at the $j^{th}$ segment along the $i^{th}$ branch. In the same vein, $\hat{Y}_{M_{r+1}}$ decomposes as

$$(\hat{Y}_{M_{r+1}}\hat{\phi})(p) = \sum_{j=1}^{b} \left( \hat{Y}_{M_r(i,j)}\hat{\phi}(i,j) \right)(p_j),$$

where $\hat{\phi}(i,j) \in L^2(D_{i,j}, \vartheta_{M_r(i,j)})$ are the components of $\hat{\phi} \in L^2(D, \vartheta_{M_{r+1}})$, and we identify $p \in \Gamma$ with the $(b+1)$-tuple $(i;p_1, \ldots, p_b) \in \{1, \ldots, b\} \times \Gamma^b$.\footnote{Note that the identification $\Gamma \equiv \{1, \ldots, b\} \times \Gamma^b$ is equivalent to $\Gamma \equiv \bigcup_{i=1}^{b} \Gamma^b$ in (IV) of Theorem 3.1.}

4 Proof of Theorem 2.7

Once the relevant definitions are formed, parts (i) and (ii) of Theorem 2.7 (restated in Corollary 4.5 and Corollary 4.6 below) follow easily from results in Section 3.4 and [13]. The proof of part (iii) of Theorem 2.7 will depend on the uniqueness of the family of laws $(M_r)_{r \in \mathbb{R}}$ satisfying properties (I)-(IV) of Theorem 3.1 and on the uniqueness of subcritical GMC.
4.1 Constructing the Gaussian field and the conditional GMC

Definition 4.1. Fix some $r \in \mathbb{R}$, and let the random measure $(D, \vartheta_{M_r})$ and the random linear map $\hat{Y}_{M_r} : L^2(D, \vartheta_{M_r}) \to L^2(\Gamma, \vartheta_{M_r})$ be defined as in Theorem 3.4. Let $\hat{U}_{M_r} : L^2(D, \vartheta_{M_r}) \to \mathcal{H}$ be an isometric embedding adapted to $M_r$, where $\mathcal{H}$ is an infinite-dimensional, separable, real Hilbert space.

(i) Let $W : \mathcal{H} \to \mathcal{F}$ be a Gaussian field on $\mathcal{H}$, where $\mathcal{F}$ is Gaussian subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ whose variables are independent of $M_r$.

(ii) Define the random linear map $Y_{M_r} : \mathcal{H} \to L^2(\Gamma, \vartheta_{M_r})$ as $Y_{M_r} := \hat{Y}_{M_r} \hat{U}_{M_r}^*$.

Definition 4.2. Let $W, \hat{U}_{M_r}$, and $Y_{M_r}$ be defined as in Definition 4.1. For a given realization of $(\Gamma, \vartheta_{M_r})$, define the random linear maps

(i) $\hat{W}_{M_r} : L^2(D, \vartheta_{M_r}) \to \mathcal{F}$ as $\hat{W}_{M_r} := W \hat{U}_{M_r}$, and

(ii) $W_{M_r} : L^2(\Gamma, \vartheta_{M_r}) \to \mathcal{F}$ as $W_{M_r} = W Y_{M_r}^*$.

Remark 4.3. For a.e. realization of $M_r$, the operator $\hat{W}_{M_r}$ defines a white-noise field on $D$ with variance measure $\vartheta_{M_r}$ for which we can formally write

$$\hat{W}_{M_r}(\hat{\phi}) \equiv \langle \hat{W}_{M_r}, \hat{\phi} \rangle \equiv \int_D \hat{W}_{M_r}(x) \hat{\phi}(x) \vartheta_{M_r}(dx) \quad \text{for} \quad \hat{\phi} \in L^2(D, \vartheta_{M_r}).$$

Remark 4.4. For a.e. realization of $M_r$, the operator $W_{M_r} \equiv \{W_{M_r}(p)\}_{p \in \Gamma}$ defines a Gaussian field over $(\Gamma, \vartheta_{M_r})$ with correlation operator $T_{M_r} = Y_{M_r} Y_{M_r}^* = \hat{Y}_{M_r} \hat{Y}_{M_r}^*$ having kernel $T(p, q)$. We formally write

$$W_{M_r}(\psi) \equiv \langle W, Y_{M_r}^* \psi \rangle \equiv \int_{\Gamma} W_{M_r}(p) \psi(p) \vartheta_{M_r}(dp) \quad \text{for} \quad \psi \in L^2(\Gamma, \vartheta_{M_r}).$$

Corollary 4.5. Fix $r \in \mathbb{R}$, and let $Y_{M_r} : \mathcal{H} \to L^2(\Gamma, \vartheta_{M_r})$ be defined as in Definition 4.1. For any $a \in \mathbb{R}_+$, the operator $\sqrt{a}Y_{M_r}$ defines a randomized shift for a.e. realization of $M_r$.

Proof. The correlation operator $Y_{M_r} Y_{M_r}^* = \hat{Y}_{M_r} \hat{Y}_{M_r}^* = T_{M_r}$ is Hilbert-Schmidt by Theorem 3.4 with kernel $T(p, q)$ having finite exponential moments $\int_{\Gamma \times \Gamma} \exp\{aT(p, q)\} \vartheta_{M_r}(dp) \vartheta_{M_r}(dq)$ for any $a \in \mathbb{R}_+$ by the remarks in part (R) of Section 3.4. This suffices for $Y_{M_r}$ to define a randomized shift on $\mathcal{H}$; see, for instance, [13, Theorem 25] and the simple approximation scheme for $Y_{M_r}$ by finite-dimensional operators in [5, Lemma 2.44].

Corollary 4.6. Fix $r \in \mathbb{R}$, and let $Y_{M_r} : \mathcal{H} \to L^2(\Gamma, \vartheta_{M_r})$ and $W : \mathcal{H} \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ be defined as in Definition 4.1. For any $a \in \mathbb{R}_+$, there is a unique conditional GMC $M_{r,a}$ on $(\Gamma, \vartheta_{M_r})$ over the field $(W, \sqrt{a}Y_{M_r})$.

Proof. The linear operator $\sqrt{a}Y_{M_r}$ is adapted to $M_r$ and a.s. defines a randomized shift by Corollary 4.5. Hence, by Theorem 2.4, for a.e. realization of $M_r$ there is a unique subcritical GMC $M_{r,a} \equiv M_{r,a}(M_r, W)$ over the field $(W, \sqrt{a}Y_{M_r})$ with (conditional) expectation $M_r$ when the field is integrated out. The random measure $M_{r,a}$ satisfies the properties of a conditional GMC and does so uniquely.

---

Footnote: Examples of the random isometry $\hat{U}_{M_r}$ are trivial to construct since $\vartheta_{M_r}$ is a function of $M_r$; see Appendix A.
The following proposition is a remark about the random Gaussian fields $W_{M_r}$ and $\hat{W}_{M_r}$ defining bounded linear operators on $L^2(\Gamma, \mu)$ and $L^2(D, \nu)$, respectively.

**Proposition 4.7.** If $\psi \in L^2(\Gamma, \mu)$ and $\hat{\phi} \in L^2(D, \nu)$, then $\psi \in L^2(\Gamma, M_r)$ and $\hat{\phi} \in L^2(D, \vartheta_{M_r})$ for a.e. realization of $M_r$ since $E[M_r] = \mu$ and $E[\vartheta_{M_r}] = R'(r)\nu$. Moreover, the linear maps $\hat{W} : L^2(D, \nu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $W : L^2(\Gamma, \mu) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ defined through $\hat{W} \equiv W_{M_r}$ and $W \equiv W_{M_r}$ have operator norms bounded by $\sqrt{R'(r)}$.

**Proof.** For $\psi \in L^2(\Gamma, \mu)$ notice that by integrating out the field we get the first equality below

$$
E\left[(W_{M_r}(\psi))^2\right] = E\left[\int_{\Gamma \times \Gamma} \psi(p)\psi(q)T(p, q)M_r(dp)M_r(dp)\right] = \int_{\Gamma \times \Gamma} \psi(p)\psi(q)T(p, q)\nu_r(dp, dq),
$$

where the second equality holds by (II) of Theorem 3.1 Since $\psi(p)\psi(q) \leq \frac{1}{2}(|\psi(p)|^2 + |\psi(q)|^2)$ and the marginals of $T(p, q)\nu_r(dp, dq)$ are equal to $R'(r)\mu$ as a consequence of the remarks in part (R) of Section 3.4 the above is bounded by $R'(r)\|\psi\|_{L^2(\Gamma, \mu)}^2$. Since $E[\vartheta_{M_r}] = R'(r)\nu$, the operator $\hat{W}_{M_r}$ is a multiple $\sqrt{R'(r)}$ of an isometry since $\hat{\phi} \in L^2(D, \nu)$

$$
E\left[(\hat{W}_{M_r}(\hat{\phi}))^2\right] = E\left[\int_D |\hat{\phi}(x)|^2 \vartheta_{M_r}(dx)\right] = R'(r)\int_D |\hat{\phi}(x)|^2 \nu(dx) = R'(r)\|\hat{\phi}\|_{L^2(D, \nu)}^2.
$$

\[\square\]

### 4.2 Renormalization transforms and the conditional GMC

The proof of part (iii) of Theorem 2.7 will involve verifying properties (I)-(IV) in Theorem 3.1 for the conditional GMC $M_{r-a,a}$. In particular property (IV) requires us to develop some notation for working with the renormalization transforms.

**Definition 4.8.** Let $\overline{M} := \{M^{(i,j)}\}_{i,j \in \{1, \ldots, b\}}$ be a family of measures on $\Gamma$. Define $\mathcal{T}\overline{M}$ to be the measure on $\Gamma$ such that

$$
\mathcal{T}\overline{M} := \frac{1}{b} \sum_{i=1}^b \prod_{j=1}^b M^{(i,j)} \quad \text{through the identification} \quad \Gamma \equiv \bigcup_{i=1}^b \times \Gamma.
$$

**Remark 4.9.** If $\overline{M}_r := \{M^{(i,j)}_r\}_{i,j \in \{1, \ldots, b\}}$ is a family of i.i.d. copies of $(\Gamma, M_r)$, then property (IV) of Theorem 3.1 implies that $M_{r+1}$ is equal in law to $\mathcal{T}\overline{M}_r$. Similarly $\vartheta_{\mathcal{T}\overline{M}_r}$ can be decomposed in terms of the measures $(D, \vartheta_{M^{(i,j)}_r})$ as in Proposition 3.5.

**Definition 4.10.** Fix $r \in \mathbb{R}$ and let $\overline{M}_r = \{M^{(i,j)}_r\}_{i,j \in \{1, \ldots, b\}}$ be a family of independent copies of the random measure $(\Gamma, M_r)$ and $Y_{M^{(i,j)}_r} : \mathcal{H} \to L^2(\Gamma, M^{(i,j)}_r)$ be the corresponding copies of the operator $Y_{M_r}$ defined as in (ii) of Definition 4.1. Similarly, let $W_{i,j} : \mathcal{H} \to \mathfrak{G}_{i,j}$ be independent copies of the Gaussian field defined in (i) of Definition 4.1 i.e., where $\mathfrak{G}_{i,j}$ are independent Gaussian linear spaces embedded in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

(i) For $\mathcal{H} := \bigoplus_{1 \leq i,j \leq b} \mathcal{H}$ and $\mathfrak{G} := \bigoplus_{1 \leq i,j \leq b} \mathfrak{G}_{i,j}$, define the Gaussian field $\overline{W} : \mathcal{H} \to \mathfrak{G}$ as

$$
\overline{W} := \bigoplus_{1 \leq i,j \leq b} W_{i,j}.
$$
(ii) Define the linear operator $Y_{M_r} : \mathcal{H} \to L^2(\Gamma, Y_M)$ to act on $\phi = \oplus_{1 \leq i,j \leq b} \phi^{(i,j)}$ for $\phi^{(i,j)} \in \mathcal{H}$ as

$$
(Y_{M_r} \phi)(p) := \sum_{j=1}^{b} (Y_{M_r}^{(i,j)} \phi^{(i,j)})(p_j),
$$

where $p \in \Gamma$ is identified with the $(b+1)$-tuple $p = (i;p_1, \ldots, p_b) \in \{1, \ldots, b\} \times_{j=1}^{b} \Gamma$.

**Lemma 4.11.** Fix $a \in \mathbb{R}_+$ and $r \in \mathbb{R}$.

(i) For a.e. realization of $M_r$, the operator $\sqrt{a}Y_{M_r} : \mathcal{H} \to L^2(\Gamma, Y_M)$ is a randomized shift.

(ii) The conditional GMC $M_{r,a}^T$ on $(\Gamma, Y_M)$ over the field $(W, \sqrt{a}Y_{M_r})$ is equal in law to the conditional GMC on $(\Gamma, M_{r+1})$ over the field $(W, \sqrt{a}Y_{M_{r+1}})$.

**Proof.** Part (i) follows as a corollary of (ii). To see (ii), first recall that the random measure $(\Gamma, Y_{M_r})$ is equal in law to $(\Gamma, M_{r+1})$ by Remark 4.9. Thus we need to argue that the fields defined by $(W, \sqrt{a}Y_{M_r})$ and $(W, \sqrt{a}Y_{M_{r+1}})$ are equal in law for a.e. realization of $Y_{M_r} \equiv M_{r+1}$. A pair $(W, Y)$ of generalized $\mathcal{H}$-valued functions defines the same field as the pair $(WU, YU)$ of generalized $\tilde{H}$-valued functions provided that $U : \tilde{H} \to \mathcal{H}$ is a linear isometry with $(\text{Range}(U))^\perp \subset \text{Null}(Y)$. Applying this with the isometry $\hat{U}_{M_{r+1}}$ in Definition 4.1, we get that $(W, \sqrt{a}Y_{M_{r+1}})$ defines the same field as $(\hat{W}_{M_{r+1}}, \sqrt{a}\hat{Y}_{M_{r+1}})$. Similarly, $(W, \sqrt{a}Y_{M_r})$ defines the same field as $(\hat{W}_{M_r}, \sqrt{a}\hat{Y}_{M_r})$ where $\hat{W}_{M_r} := \mathcal{W}\hat{U}_{M_r}$ and $\hat{Y}_{M_r} := \hat{Y}_{M_r}\hat{U}_{M_r}$ for the isometry $\hat{U}_{M_r}$ defined as the direct sum of isometric maps $\hat{U}_{M_r}^{(i,j)} : L^2(D, \vartheta^{M_r}) \to \mathcal{H}$ (i.e., copies of $\hat{U}_{M_r}$):

$$
\hat{U}_{M_r} := \bigoplus_{1 \leq i,j \leq b} \hat{U}_{M_r}^{(i,j)}.
$$

The operator $\hat{W}_{M_r}$ defines a white noise field on $D := \bigcup_{1 \leq i,j \leq b} D_{i,j}$ with variance measure $\vartheta_{M_r} := \oplus_{1 \leq i,j \leq b} \vartheta^{M_r^{(i,j)}}$ and $\hat{Y}_{M_r}$ acts on $\hat{\phi} \in L^2(D, \vartheta_{M_r})$ as

$$
(\hat{Y}_{M_r} \hat{\phi})(p) = \sum_{j=1}^{b} \left(\hat{Y}_{M_r}^{(i,j)} \hat{\phi}^{(i,j)}\right)(p_j)
$$

for $p \equiv (i;p_1, \ldots, p_b)$ and $\hat{\phi} = \oplus_{1 \leq i,j \leq b} \hat{\phi}^{(i,j)}$ with $\hat{\phi}^{(i,j)} \in L^2(D_{i,j}, \vartheta^{M_r^{(i,j)}})$. Since $\hat{W}_{M+1}$ is a white noise field on $D$ with variance measure $\vartheta_{M+1}$, the identities in Proposition 3.5 imply that the field $(\hat{W}_{M_r}, \sqrt{a}\hat{Y}_{M_r})$ is equal in law to $(\hat{W}_{M_{r+1}}, \sqrt{a}\hat{Y}_{M_{r+1}})$, which completes the proof.

\[\square\]

### 4.3 Proof of part (iii) of Theorem 2.7

**Proof (iii) of Theorem 2.7** Let the family of laws for random measures $(M_r)_{r \in \mathbb{R}}$ on $\Gamma$ be defined as in Theorem 3.1. For $r \in \mathbb{R}$ and $a \in \mathbb{R}_+$, define $M_{r-a}$ as the conditional GMC on $(\Gamma, M_{r-a})$ over the field $(W, \sqrt{a}Y_{M_{r-a}})$, which exists uniquely by Corollary 4.6. By the uniqueness of the family of laws $(M_r)_{r \in \mathbb{R}}$ satisfying properties (I)-(IV) in Theorem 3.1 it suffices for us to verify that (I)-(IV) hold for the family of laws $(M_{r-a})_{r \in \mathbb{R}}$.

**Property (I):** Since $M_{r-a}$ is a conditional GMC on $(\Gamma, M_{r-a})$,

$$
E[M_{r-a}] = E[E[M_{r-a} | M_{r-a}]] = E[M_{r-a} | M_{r-a}] = \mu,
$$

where $p \in \Gamma$ is identified with the $(b+1)$-tuple $p = (i;p_1, \ldots, p_b) \in \{1, \ldots, b\} \times_{j=1}^{b} \Gamma$. 

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where the third equality is from property (I) of Theorem 3.1.

**Property (II):** For measurable $g : \Gamma \times \Gamma \to [0, \infty)$, we can insert a conditional expectation with respect to $M_{r-a}$

$$
E \left[ \int_{\Gamma \times \Gamma} g(p,q)M_{r-a,a}(dp)M_{r-a,a}(dq) \right] = E \left[ E \left[ \int_{\Gamma \times \Gamma} g(p,q)M_{r-a,a}(dp)M_{r-a,a}(dq) \mid M_{r-a} \right] \right].
$$

Since the Gaussian field $W_{M_{r-a}} \equiv (W,Y_{M_{r-a}})$ generating the conditional GMC $M_{r-a,a}$ has kernel $T(p,q)$ when conditioned on $M_{r-a}$, the above is equal to

$$
= E \left[ \int_{\Gamma \times \Gamma} g(p,q) e^{aT(p,q)}M_{r-a}(dp)M_{r-a}(dq) \right].
$$

By property (II) of Theorem 2.7,

$$
= \int_{\Gamma \times \Gamma} g(p,q) e^{aT(p,q)}v_{r-a}(dp,dq).
$$

By the remark in (R) of Section 3.4, $\exp\{aT(p,q)\}$ is the Radon-Nikodym derivative of $v_r$ with respect to $v_{r-a}$, and thus the above is equal to $\int_{\Gamma \times \Gamma} g(p,q)v_r(dp,dq)$. Since $g$ is an arbitrary nonnegative measurable function, $E[M_{r-a,a} \times M_{r-a,a}] = v_r$.

**Property (III):** As before, we begin with a conditional expectation with respect to $M_{r-a}$ to write

$$
E \left[ \left( M_{r-a,a}(\Gamma) \right)^m \right] = E \left[ \left( M_{r-a,a}(\Gamma) \right)^m \mid M_{r-a} \right]
$$

$$
= E \left[ \int_{\Gamma^m} \exp \left\{ a \sum_{1 \leq i < j \leq m} T(p_i,p_j) \right\} M_{r-a}(dp_1) \cdots M_{r-a}(dp_m) \right],
$$

where we have applied Kahane’s moment formula [10, part (d) of Theorem 6], which holds without the $\sigma$-positive assumption on the kernel if the exponential moments of $T(p,q)$ are finite; see Appendix B.

With the inequality $x_1 \cdots x_n \leq \frac{1}{n} (x_1^2 + \cdots + x_n^2)$ for nonnegative $x_j$, the above can be bounded by

$$
\leq \frac{2}{m(m-1)} E \left[ \int_{\Gamma^m} \sum_{1 \leq i < j \leq m} \exp \left\{ a \frac{m(m-1)}{2} T(p_i,p_j) \right\} M_{r-a}(dp_1) \cdots M_{r-a}(dp_m) \right].
$$

Expanding the sum and integrating out the $m - 2$ variables absent from the integrand we get

$$
= E \left[ \int_{\Gamma \times \Gamma} \left( M_{r-a}(\Gamma) \right)^{m-2} \exp \left\{ a \frac{m(m-1)}{2} T(p,q) \right\} M_{r-a}(dp)M_{r-a}(dq) \right].
$$

Next by applying the inequality $xy \leq \frac{1}{2} x^2 + \frac{1}{2} y^2$ for $xy \geq 0$,\n
$$
\leq E \left[ \int_{\Gamma \times \Gamma} \left( \frac{1}{2} \left( M_{r-a}(\Gamma) \right)^{2m-4} + \frac{1}{2} e^{am(m-1)T(p,q)} \right) M_{r-a}(dp)M_{r-a}(dq) \right]
$$

$$
= \frac{1}{2} E \left[ \left( M_{r-a}(\Gamma) \right)^{2m-2} \right] + \frac{1}{2} E \left[ \int_{\Gamma \times \Gamma} e^{am(m-1)T(p,q)} M_{r-a}(dp)M_{r-a}(dq) \right].
$$
The left term above is finite since the moments of $M_{r-a}(\Gamma)$ are finite by property (III) of Theorem 3.1. By property (II) of Theorem 3.1, the right term above is equal to

$$\int_{\Gamma \times \Gamma} e^{am(m-1)/2} u_{r-a}(dp, dq) = \int_{\Gamma \times \Gamma} u_{r-a+am(m-1)}(dp, dq) = 1 + R(r - a + am(m - 1)), $$

where the first equality uses that $u_{t+u}$ has Radon-Nikodym derivative $\exp\{u T(p, q)\}$ with respect to $v_t$, and the second equality holds because $u_t$ has total mass $1 + R(t)$. Therefore, the $m^{th}$ moment of $M_{r-a}(\Gamma)$ is finite.

**Property (IV):** Let $\mathbb{M}_{r-a} := \{M_{r-a}^{(i,j)}\}_{i,j \in \{1, \ldots, b\}}$ be a family of independent copies of the random measure $M_{r-a}$. We must show that

$$M_{r+1-a} \leq \Upsilon \mathbb{M}_{r-a}, \tag{4.1}$$

where $\Upsilon$ is defined as in Definition 4.8. To construct the family $\mathbb{M}_{r-a}$,

- let $\mathbb{M}_{r-a} := \{M_{r-a}^{(i,j)}\}_{i,j \in \{1, \ldots, b\}}$ be a family of i.i.d. copies of $(\Gamma, M_{r-a})$,
- define the fields $(W_{i,j}, \sqrt{a} Y_{M_{r-a}^{(i,j)}})$ for $(i, j) \in \{1, \ldots, b\}$

Then we define $M_{r-a}^{(i,j)} = M_{r-a}^{(i,j)}(M_{r-a}^{(i,j)}, W_{i,j})$ as the conditional GMC on $(\Gamma, M_{r-a})$ generated by the field $(W_{i,j}, \sqrt{a} Y_{M_{r-a}^{(i,j)}})$ in the sense of Corollary 4.6.

Let $\Upsilon \mathbb{M}_{r-a}$ and $\bar{W}$ be defined as in Definition 4.10. Recall from Lemma 4.11 that the conditional GMC $M_{r-a}^{(i,j)}$ on $(\Gamma, \Upsilon \mathbb{M}_{r-a})$ over the field $(\bar{W}, \sqrt{a} Y_{\Upsilon \mathbb{M}_{r-a}})$ is equal in law to the conditional GMC $M_{r+1-a}^{(i,j)}$ on $(\Gamma, M_{r+1-a})$ over the field $(\bar{W}, \sqrt{a} Y_{\mathbb{M}_{r+1-a}})$. Therefore, to prove (4.1), it suffices to show that

$$M_{r-a}^{(i,j)} = \Upsilon \mathbb{M}_{r-a}^{(i,j)} \tag{4.2}$$

holds a.s. By uniqueness of the conditional GMC $M_{r-a}^{(i,j)}$, we can deduce (4.2) by verifying that $\Upsilon \mathbb{M}_{r-a}$ fulfills conditions (I)-(III) in Definition 2.5 for being a conditional GMC on $(\Gamma, \Upsilon \mathbb{M}_{r-a})$ over the field $(\bar{W}, \sqrt{a} Y_{\Upsilon \mathbb{M}_{r-a}})$. We check these conditions below.

Condition (I): Notice that taking the conditional expectation of $\Upsilon \mathbb{M}_{r-a}$ with respect to $\mathbb{M}_{r-a}$ yields

$$E[\Upsilon \mathbb{M}_{r-a} | \mathbb{M}_{r-a}] = \frac{1}{b^2} \sum_{i=1}^b \sum_{j=1}^b E[M_{r-a}^{(i,j)} | \mathbb{M}_{r-a}] = \frac{1}{b} \sum_{i=1}^b \sum_{j=1}^b E[M_{r-a}^{(i,j)} | M_{r-a}^{(i,j)}] = \frac{1}{b} \sum_{i=1}^b \sum_{j=1}^b M_{r-a}^{(i,j)} = \Upsilon \mathbb{M}_{r-a},$$

where the first and last equalities are understood through the canonical identification $\Gamma \equiv \bigcup_{i=1}^b \times_{j=1}^b \Gamma$.

The second equality above uses that the pairs $(M_{r-a}^{(i,j)}, W_{i,j})$ are independent. Since $\Upsilon \mathbb{M}_{r-a}$ is a function of $\mathbb{M}_{r-a}$, taking the conditional expectation of $\Upsilon \mathbb{M}_{r-a}$ with respect to $\Upsilon \mathbb{M}_{r-a}$ would also yield $\Upsilon \mathbb{M}_{r-a}$.

Condition (II): By definition, the random measure $\Upsilon \mathbb{M}_{r-a}$ is a function, $\Upsilon$, of the family $\mathbb{M}_{r-a} := \{M_{r-a}^{(i,j)}\}_{i,j \in \{1, \ldots, b\}}$, whose components $M_{r-a}^{(i,j)}$ are functions of $(M_{r-a}, W_{i,j})$ by (II) of Definition 2.5. Thus $\Upsilon \mathbb{M}_{r-a}$ is a function of $(M_{r-a}, W)$. To see that $\Upsilon \mathbb{M}_{r-a}$ is also a function of $(\Upsilon \mathbb{M}_{r-a}, W)$,
notice that the information lost about a family of measures \( \{ \lambda_i \}_{i,j \in \{1,...,b\}} \) through the operation \( \Upsilon \) can be characterized as follows: for any positive scalars \( \{ \lambda_i \}_{i,j \in \{1,...,b\}} \) with \( \prod_{j=1}^b \lambda_i = 1 \) for each \( i \), we get

\[
\Upsilon \{ \lambda_i M_{i,j} \}_{i,j \in \{1,...,b\}} = \Upsilon \{ M_{i,j} \}_{i,j \in \{1,...,b\}},
\]

and the family of measures \( \{ M_{i,j} \}_{i,j \in \{1,...,b\}} \) can be recovered from \( \Upsilon \{ M_{i,j} \}_{i,j \in \{1,...,b\}} \) up to such a family scalar multiples. However, since \( \lambda_i \) \( \{ M_{i,j} \}_{i,j \in \{1,...,b\}} \) \( \equiv \{ (i;j) \} \equiv \{ i ; p_1, \ldots, p_b \} \) under the identification \( \Gamma \equiv \{ 1, \ldots, b \} \times \times_{j=1}^b \). Since the random measures \( \lambda_i M_{i,j} \) are conditional GMCs over the field \( (W_i,j, \sqrt{a Y M_{i,j}}) \), we can apply property (iii) of Definition 2.5 to write the above as

\[
\Upsilon M_{r-a,a} (\Upsilon M_{r-a,a}, W + \phi, dp) = \frac{1}{b} \prod_{j=1}^b M_{r-a,a}^{(i,j)} \left( M_{r-a,a}^{(i,j)}, W_{i,j} + \phi^{(i,j)}, dp_j \right),
\]

where \( \phi = \oplus 1_{i,j \leq b} \phi^{(i,j)}(i) \) is the decomposition of \( \phi \) in terms of \( \phi^{(i,j)} \in \mathcal{H} \) and \( p \equiv (i; p_1, \ldots, p_b) \) under the identification \( \Gamma \equiv \{ 1, \ldots, b \} \times \times_{j=1}^b \). Since the random measures \( \lambda_i M_{i,j} \) are conditional GMCs over the field \( (W_i,j, \sqrt{a Y M_{i,j}}) \), we can apply property (iii) of Definition 2.5 to write the above as

\[
\Upsilon M_{r-a,a} (\Upsilon M_{r-a,a}, W + \phi, dp) = \frac{1}{b} \exp \left\{ \sqrt{a} \sum_{j=1}^b \left( Y M_{r-a,a}^{(i,j)} \phi^{(i,j)}(i) \right) (p_j) \right\} \prod_{j=1}^b M_{r-a,a}^{(i,j)} \left( M_{r-a,a}^{(i,j)}, W_{i,j}, dp_j \right).
\]

Since \( \left( \overline{M}_{r-a,a} \phi \right)(p) := \sum_{j=1}^b \left( Y M_{r-a,a}^{(i,j)} \phi^{(i,j)}(i) \right) (p_j) \), the definition of \( \Upsilon M_{r-a,a} \) implies that the above can be written as

\[
\Upsilon M_{r-a,a} (\Upsilon M_{r-a,a}, W, dp) = \exp \left\{ \sqrt{a} \left( \overline{M}_{r-a,a} \phi \right)(p) \right\} \Upsilon M_{r-a,a} (\Upsilon M_{r-a,a}, W, dp).
\]

This verifies condition (iii) and establishes that \( \Upsilon M_{r-a,a} \) is the conditional GMC on \( (\Gamma, M_{r-a,a}^Y) \) generated by the field \( (W, \sqrt{a Y M_{r-a,a}}) \). By the observations above and below \( \ref{equation:4.2} \), the proof is complete.

\[
\square
\]

5. An application of the GMC structure to strong disorder analysis

The proof of the following proposition is a modification of the proof of \( \ref{theorem:3.1} \) Theorem 1.15, which adapted an argument in \( \square \) for discrete polymers.

**Proposition 5.1.** Let the family of random measures \( \{ M_r \}_{r \in \mathbb{R}} \) be defined as in Theorem 3.1. As \( r \to \infty \) the total mass \( M_r(\Gamma) \) converges in probability to 0.

**Proof of Proposition 5.1.** Since we are characterizing the limiting behavior of the random measures \( M_r \) as \( r \to \infty \), it suffices to assume that \( r > 0 \). By part (iii) of Theorem 2.7, we can construct \( M_r \) as a conditional GMC on \( (\Gamma, M_0) \) over the field \( (W, \sqrt{a Y M_0}) \), where \( W \) and \( Y M_0 \) are defined as in Definition 4.1. The random measure \( M_r(dp) \equiv M_r(M_0, W, dp) \) is a function of \( (M_0, W) \) with

\[
E[M_r | M_0] = M_0 \quad \text{and} \quad M_r(M_0, W + \phi, dp) = e^{(Y M_0 \phi)(p)} M_r(M_0, W, dp)
\]

for \( \phi \in \mathcal{H} \). To prove that \( M_r(\Gamma) \) converges in probability to zero, it suffices to show that the fractional moment \( F(r, M_0) := E\left[ \left( M_r(M_0, W, \Gamma) \right)^{1/2} \right] \) converges to zero as \( r \to \infty \) for a.e. realization of \( M_0 \).
Let \( h \in \mathcal{H} \) be defined as \( \phi := -\hat{U}_{M_0}1_D \) where \( \hat{U}_{M_0} : L^2(D, \vartheta_{M_0}) \to \mathcal{H} \) is the linear isometry in Definition 4.1. Notice that
\[
(Y_{M_0}(p))(p) = (\hat{Y}_{M_0}1_D)(p) = -\int_{\Gamma} T(p,q)M_0(dq) =: -t_{M_0}(p),
\]
where the first equality is from the definition of \( Y_{M_0} \) in Definition 4.1 and the second equality is from (iii) of Theorem 3.4. For a.e. realization of \( M_0 \), the function \( t_{M_0} : \Gamma \to [0, \infty) \) is positive for \( M_0 \)-a.e. \( p \in \Gamma \) as a consequence of Lemma 3.3. In particular, this implies that we a.s. have
\[
\int_{\Gamma} e^{-\sqrt{r}t_{M_0}(p)}M_0(dp) \xrightarrow{r \to \infty} 0.
\]

Define \( \hat{\mathbb{P}} \) as the change of measure \( \frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = e^{(W,\phi) - \frac{1}{2}\|\phi\|^2} \). Let \( \hat{\mathbb{E}} \) denote the expectation with respect to \( \hat{\mathbb{P}} \). The Cauchy-Schwarz inequality yields that
\[
\mathbb{E}\left[\left(\int_{\Gamma} e^{-\sqrt{r}M_0(p)}M_0(dp)\right)^\frac{1}{2}|M_0\right] \leq \mathbb{E}\left[\left(\int_{\Gamma} e^{-\sqrt{r}M_0(p)}M_0(dp)\right)|M_0\right]^\frac{1}{2} \mathbb{E}\left[\left(e^{-(W,\phi) + \frac{1}{2}\|\phi\|^2}\right)^2\right]^{\frac{1}{2}}.
\]

Since \( \mathbb{E}[F(M_0, W) | M_0] = \mathbb{E}[F(M_0, W + \vartheta) | M_0] \) for any integrable function \( F \) of \( (M_0, W) \) and \( M_r(M_0, W + \vartheta, dp) = e^{\langle Y_{M_0}(\vartheta)p \rangle}M_r(M_0, W, dp) \) for \( (Y_{M_0}(\vartheta))(p) = -t_{M_0}(p) \), the above is equal to
\[
\mathbb{E}\left[\left(\int_{\Gamma} e^{-\sqrt{r}M_0(p)}M_r(dp)\right)^\frac{1}{2}|M_0\right] \leq \mathbb{E}\left[\left(e^{-(W,\phi) + \frac{1}{2}\|\phi\|^2}\right)^\frac{1}{2}\right] \mathbb{E}[\left(\int_{\Gamma} e^{-\sqrt{r}M_0(p)}M_0(dp)\right)^\frac{1}{2} e^{\frac{1}{2}\|\phi\|^2}].
\]

This expression a.s. converges to zero as \( r \to \infty \) by the remark above.

\[\square\]

### A Construction of the isometry \( \hat{U}_{M_r} \)

The construction of \( Y_{M_r} \) in Definition 4.1 requires a linear, isometric embedding \( \hat{U}_{M_r} : L^2(D, \vartheta_{M_r}) \to \mathcal{H} \) adapted to \( M_r \), where \( \mathcal{H} \) is a Hilbert space. Given a realization of the measure \( (D, \vartheta_{M_r}) \), we can construct such a map \( \hat{U}_{M_r} \) using, for instance, the recipe below.

- Let \( C(D) \) be the space of real-valued continuous functions on \( D \) equipped with the uniform norm: \( \|f\|_{\infty} = \max_{x \in D}|f(x)| \). Let \( \mathcal{D} \) be a countable, dense subset of the unit shell \( \{f \in C(D) \mid \|f\|_{\infty} = 1\} \) and \( \{h_n\}_{n \in \mathbb{N}} \) be an enumeration of \( \mathcal{D} \).

- Apply Gram-Schmidt to \( \{h_n\}_{n \in \mathbb{N}} \) to generate an orthonormal sequence \( \{h_n^{(r)}\}_{n \in \mathbb{N}} \) under the inner product

\[
\langle \phi_1, \phi_2 \rangle_{L^2(D, \vartheta_{M_r})} = \int_D \phi_1(x)\phi_2(x)\vartheta_{M_r}(dx), \quad \phi_1, \phi_2 \in L^2(D, \vartheta_{M_r}).
\]

Let \( c_{k,n}^{(r)} \in \mathbb{R} \) denote the Gram-Schmidt coefficients, i.e., the values satisfying
\[
h_n = \sum_{k=1}^n c_{k,n}^{(r)} h_k^{(r)}.
\]
• Let \( \{e_n\}_{n \in \mathbb{N}} \) denote an orthonormal basis of \( \mathcal{H} \). Define an isometric map \( \hat{U}_{M_r} : \text{Span} \{ h^{(r)}_n \}_{n \in \mathbb{N}} \to \mathcal{H} \) by sending \( h^{(r)}_n \mapsto e_n \) for all \( n \in \mathbb{N} \). The algebraic span of \( \{ h^{(r)}_n \}_{n \in \mathbb{N}} \) must be dense in \( L^2(D, \vartheta_{M_r}) \) since \( \|\phi\|_{L^2(D, \vartheta_{M_r})} \leq \sqrt{\vartheta_{M_r}(D)} \|\phi\|_\infty \) and \( \vartheta_{M_r} \) is a.s. finite. Therefore \( \hat{U}_{M_r} \) extends to an isometry from \( L^2(D, \vartheta_{M_r}) \) into \( \mathcal{H} \).

The map \( \hat{U}_{M_r} \) is adapted to \( M_r \) since the measure \( \vartheta_{M_r} \) is adapted to \( M_r \).

## B Moments of the GMC total mass

As in Section 2.1, let \((\Gamma, \mu)\) be a finite standard Borel measure space and \( \mathcal{H} \) is a Hilbert space. The following is a statement of Kahane’s GMC moment formula [10 Theorem 6] with the assumption that the kernel \( T(p, q) \) has finite exponential moments and not necessarily of \( \sigma \)-positive type.

**Lemma B.1.** Let \( M \) be a GMC on \((\Gamma, \mu)\) over the field \((W, Y)\) defined on the Hilbert space \( \mathcal{H} \). Suppose that \( Y \) defines a compact map from \( \mathcal{H} \) to \( L^2(\Gamma, \mu) \) and that \( T := YY^* \) is Hilbert-Schmidt with kernel \( T(p, q) \) having finite exponential moments

\[
\int_{\Gamma \times \Gamma} e^{aT(p_1, p_2)} \mu(dp_1)\mu(dp_2) < \infty \quad \text{for} \quad a \in \mathbb{R}_+.
\]

The positive integer moments of the total mass \( M(\Gamma) \) are finite and have the form

\[
\mathbb{E} \left[ (M(\Gamma))^m \right] = \int_{\Gamma^m} \exp \left\{ \sum_{1 \leq i < j \leq m} T(p_i, p_j) \right\} \mu(dp_1) \cdots \mu(dp_m). \tag{B.1}
\]

**Proof of Lemma B.1.** Since \((\Gamma, \mu)\) is a standard Borel measure space, the \( \sigma \)-algebra \( \mathcal{B}_\Gamma \) is countably generated, and there exists an increasing sequence of finite \( \sigma \)-algebras \( \mathcal{F}_n \subset \mathcal{B}_\Gamma \) such that \( \mathcal{B}_\Gamma \) is generated by the algebra \( \mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{F}_n \). Define the orthogonal projection \( P_n : L^2(\Gamma, \mu) \to L^2(\Gamma, \mu) \) through conditional averaging with respect to \( \mathcal{F}_n \), and define the finite rank operators \( Y_n := P_n Y \) and \( T_n := Y_n Y_n^* \). Since the projections \( P_n \) converge strongly to the identity operator on \( L^2(\Gamma, \mu) \) and \( Y \) is compact, \( Y_n \) converges in operator norm to \( Y \). The kernels of \( T_n \) satisfy \( T_n(p, q) = \mathbb{E}[T(p, q) | \mathcal{F}_n \otimes \mathcal{F}_n] \) and form a martingale with respect to the filtration \((\mathcal{F}_n \otimes \mathcal{F}_n)_{n \in \mathbb{N}}\). Thus \( T_n(p, q) \) converges \( \mu \times \mu \)-a.e. to \( T(p, q) \) as \( n \to \infty \).

Let \( M_n \) be the GMC on \((\Gamma, \mu)\) over the field \((W, Y_n)\). The \( m \)th moment of the total mass of \( M_n \) is

\[
\mathbb{E} \left[ (M_n(\Gamma))^m \right] = \mathbb{E} \left[ \left( \int_{\Gamma} \exp \left\{ \sum_{1 \leq i < j \leq m} (W, Y_n(p)) - \frac{1}{2} \mathbb{E} \left[ (W, Y_n(p))^2 \right] \right\} \mu(dp) \right)^m \right]
\]

\[
= \int_{\Gamma^m} \mathbb{E} \left[ \prod_{j=1}^{m} \exp \left\{ \sum_{1 \leq i < j \leq m} (W, Y_n(p_j)) - \frac{1}{2} \mathbb{E} \left[ (W, Y_n(p_j))^2 \right] \right\} \right] \mu(dp_1) \cdots \mu(dp_m).
\]

The random variables in the product are log-normal, and hence the above is equal to

\[
= \int_{\Gamma^m} \exp \left\{ \sum_{1 \leq i < j \leq m} \mathbb{E} \left[ (W, Y_n(p_i)) (W, Y_n(p_j)) \right] \right\} \mu(dp_1) \cdots \mu(dp_m),
\]

and since \( \mathbb{E}[(W, Y_n(p)) (W, Y_n(q))] = (Y_n(p), Y_n(q)) = T_n(p, q) \), we have

\[
= \int_{\Gamma^m} \exp \left\{ \sum_{1 \leq i < j \leq m} T_n(p_i, p_j) \right\} \mu(dp_1) \cdots \mu(dp_m). \tag{B.2}
\]
Since $T_n(p, q) = \mathbb{E}[T(p, q) \mid \mathcal{F}_n \otimes \mathcal{F}_n]$, Jensen’s inequality implies that the expression \[\text{(B.2)}\] is bounded by the right side of \[\text{(B.1)}\], which is finite by our assumption on the exponential moments of $T(p, q)$. Since $T_n(p, q)$ converges $\mu \times \mu$-a.e. to $T(p, q)$, the moment $\mathbb{E} \left[ (M_n(\Gamma))^m \right]$ converges as $n \to \infty$ to the right side of \[\text{(B.1)}\] by Fatou’s lemma. Since $\sup_{n \in \mathbb{N}} \mathbb{E} \left[ (M_n(\Gamma))^m \right]$ is finite for each $m$ and the random variables $M_n(\Gamma)$ converge in probability to $M(\Gamma)$ as a consequence of \[\text{[13, Theorem 25]}\], the moments $\mathbb{E} \left[ (M_n(\Gamma))^m \right]$ converge to $\mathbb{E} \left[ (M(\Gamma))^m \right]$ as $n \to \infty$ for each $m \in \mathbb{N}$.

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