ON DUAL F-SIGNATURE

AKIYOSHI SANNAI

Abstract. We define the dual F-signature of modules, which is equivalent to the F-signature if the module is the base ring. By using this invariant, we give characterizations of regular, F-regular, F-rational, and Gorenstein singularities.

1. Introduction

Let $R$ be a complete $d$-dimensional reduced Noetherian local ring with prime characteristic $p > 0$ and perfect residue field $k = k^p$. There is the Frobenius map $F: R \rightarrow R$ sending $r$ to $r^p$. For $e \in \mathbb{N}$ the inclusion $R \subseteq R^{1/p^e}$ into the corresponding ring of $p^e$-th roots of elements of $R$ is naturally identified with the $e$-th iterate of the Frobenius endomorphism. The $R$-module $R^{1/p^e}$ has important information about the singularity of $R$. Write $R^{1/p^e} = R^{p^{de}} \oplus M_e$ as $R$-modules where $M_e$ has no free direct summands. The number $a_e$ is called the $e$-th Frobenius splitting number of $R$. Kunz showed that $a_e = p^{de}$ holds for some $e$ if and only if $R$ is regular [Kun69]. This result also tells us that the ratio of the rank of the free direct summand $a_q$ to the rank of $R^{1/p^e} = p^{de}$ reflects the distance to the regularity. We define the $F$-signature by the asymptotic behavior of the sequence $\{a_e/p^{de}\}$, namely, $s(R) = \lim_{e \rightarrow \infty} \frac{a_e}{p^{de}}$. The $F$-signature first appeared implicitly in the work of K. Smith and M. Van den Bergh [SVdB97], and its formal study was started in [HL02]. Though the existence of the limit had been open for several years, K.Tucker showed the existence in general [Tuc].

C. Huneke and G. J. Leuschke showed that $s(R) \leq 1$ and the equality holds if and only if $R$ is regular [HL02]. In their paper, they believed that $s(R)$ characterizes $F$-rationality of $R$ by its positivity. But I. Aberbach and G. J. Leuschke showed $s(R) > 0$ if and only if $R$ is strongly $F$-regular. In this paper, we prove that what they believe...
is the right idea, namely, $F$-rationality is characterized by how the canonical module relates to iterated Frobenius powers. To do this, we extend this invariant $s(R)$ to modules.

**Definition 1.1.** Let $(R, m, k)$ be a reduced $F$-finite local ring of characteristic $p > 0$ and $M$ be an $R$-module. For each natural number $e$, put $q = p^e$, $\alpha = \log_p[k : k^p]$, and $b_q = \max\{ n \mid 3F_e^*M \twoheadrightarrow M^n \}$ and define

$$s(M) = \limsup_{e \to \infty} \frac{b_q}{q^{\dim R + \alpha}}$$

We call $b_q$ the $F$-surjective number of $M$ and call $s(M)$ the dual $F$-signature of $M$.

The point is that though the $F$-signature depends only on the $R$-module structure of $F_e^*R$, the dual $F$-signature depends not on the $R$-module structure of $F_e^*M$ but on the relative structure of $F_e^*M$ and $M$. It is easy to see the dual $F$-signature of $R$ is equivalent to $F$-signature. By using dual $F$-signature, we have the characterization of the singularities of $R$.

**Theorem 1.2.** Let $(R, m, k)$ be a reduced $F$-finite Cohen-Macaulay local ring of characteristic $p > 0$ and $\omega_R$ be the canonical module, then the following hold:

1. The following are equivalent.
   - (a) $R$ is regular.
   - (b) $s(R) = 1$
   - (c) $s(\omega_R) = 1$

2. $R$ is strongly $F$-regular if and only if $s(R) > 0$

3. Assume $k$ is infinite, then $R$ is $F$-rational if and only if $s(\omega_R) > 0$

4. $s(\omega_R) \geq s(R)$

5. Furthermore, assume $s(\omega_R) > 0$, then $R$ is Gorenstein if and only if $s(\omega_R) = s(R)$

**Remark 1.3.** The $F$-finiteness of $R$ implies the existence of the dualizing complex by Gabber [G04]. Therefore, there always exists a canonical module under the assumption in the theorem above.

**Remark 1.4.** Though the theorem is only for absolute version, we can also generalize $F$-signature of pair, $F$-splitting ratio, and $s$-dimension in the dual situation. See the section 4.

**Acknowledgments.** The author would like to express his gratitude to Professor Mitsuyasu Hashimoto for valuable discussions and helpful comments. The author would like to thank Professor Vasudevan Srinivas and Professor Kenichi Yoshida for examples about the dual
F-signature. The author also would like to thank Professor Yujiro Kawamata for his suggestion about pair of the dual F-signature. The author is partially supported by JSPS postdoctoral fellow.

2. Preliminary

We first review the theory of Hilbert-Kunz multiplicity. Let \((R, m, k)\) be a local Noetherian ring of prime characteristic \(p\), \(I\) an \(m\)-primary ideal and \(M\) an \(R\)-module. The Hilbert–Kunz function of \(M\) along \(I\) is the function taking an integer \(n\) to the length of \(R/I[p^n] \otimes M\), where \(I[p^n]\) is the ideal generated by all the \(p^n\)th powers of elements of \(I\). The Hilbert–Kunz multiplicity of \(M\) along \(I\), denoted \(e_{HK}(I, M)\), is \(\lim_{q=p^n \to \infty} l(R/I[p^q] \otimes M)\), where \(l(M)\) is the length of \(M\). The limit always exists by the following theorem.

**Theorem 2.1.** [Mon83] Suppose \((R, m, k)\) is a local ring of dimension \(d\) and characteristic \(p > 0\). If \(I\) is any \(m\)-primary ideal and \(M\) is a finitely generated \(R\)-module, then the limit

\[
e_{HK}(I, M) := \lim_{q \to \infty} \frac{1}{p^{d-1}} l(R/I[p^q] \otimes M)
\]

exists and is called the Hilbert-Kunz multiplicity of \(M\) along \(I\).

We need the following result which is special case of Theorem 8.17 of [HH90]:

**Theorem 2.2.** Let \((R, m, k)\) be a reduced, \(F\)-finite local ring of characteristic \(p > 0\). Let \(I \subseteq J\) be two \(m\)-primary ideals. Then \(I^* = J^*\) if and only if \(e_{HK}(I, R) = e_{HK}(J, R)\). (Here \(I^*\) denotes the tight closure of \(I\).)

We note that the assumption concerning test elements in [HH90, Theorem 8.17] is satisfied in this case since the ring is reduced, local and \(F\)-finiteness implies excellentness. See [HH94, Theorem 6.1]. We also need the following proposition in [Hun96].

**Proposition 2.3.** Let \((R, m, k)\) be a local Noetherian ring of characteristic \(p > 0\) and dimension \(d\). Let \(M, N\) be finitely generated \(R\)-modules, and let \(I\) be an \(m\)-primary ideal.

1. If \(\dim M < d\), then \(l(M/I[l]M) = O(q^{d-1})\) and thus \(e_{HK}(I, M) = 0\).
2. Let \(W\) be the complement of the set of minimal primes \(Q\) such that \(\dim(R/Q) = d\). If \(M_W \cong N_W\),

\[
|l(M/I[l]M) - l(N/I[l]N)| = O(q^{d-1})
\]

In particular, \(e_{HK}(I, M) = e_{HK}(I, N)\).
For the proofs, we need to consider about the minimal difference of Hilbert-Kunz multiplicity along some ideals. Though the minimal difference of Hilbert-Kunz multiplicity along $m$-primary ideals (called minimal relative Hilbert-Kunz multiplicity) is introduced by K.-i. Watanabe and K. Yoshida [WY04], we need the “parameter ideal” version of minimal relative Hilbert-Kunz multiplicity. The following definition and theorem are due to M. Hochster and Y. Yao [HY].

**Definition 2.4.** Let $(R, m, k)$ be a local ring of prime characteristic $p > 0$ and $M$ be a finitely generated $R$-module. Define (here s.o.p. stands for system of parameters)

$$r_{R(M)} = \inf\{e_{HK}((x), M) - e_{HK}((x, \Delta), M) \mid x \text{ is a s.o.p. and } ((x) : \Delta) = m\}$$

**Theorem 2.5.** Let $(R, m, k)$ be a Noetherian local ring of characteristic $p$. Suppose $R$ is excellent or there exists a common parameter (weak) test element for $R$ and $\hat{R}$. Then $R$ is $F$-rational if and only if $r_{R(R)} > 0$.

We review the $F$-signature of modules defined by Y. Yao [Yao06] and the result by K. Tucker about it [Tuc].

**Definition 2.6.** Let $(R, m, k)$ be an $F$-finite local ring and $M$ a finitely generated $R$-module. For each $e \in \mathbb{N}$, put $q = p^e$, $\alpha = \log_p[k : k^p]$, and write $F^e M \cong R^{a_e} \oplus M_e$ as left $R$-modules such that $M_e$ has no non-zero free direct summand. In other words, the number $a_e$ is the maximal rank of a free direct summand of the left $R$-module $F^e M$. We define

$$s'(M) = \lim_{e \to \infty} \frac{a_e}{q^{\dim R + \alpha}}$$

**Theorem 2.7.** Let $(R, m, k)$ be a d-dimensional $F$-finite characteristic $p > 0$ local domain and let $M$ be a finitely generated $R$-module. Denote by $a_e$ the maximal rank of a free $R$-module appearing in a direct sum decomposition of $F^e M$. Then

$$\lim_{q \to \infty} \frac{a_e}{q^{(d + \alpha)}} = s'(M) = \text{rank}(M) \cdot s(R).$$

### 3. The dual $F$-signature

We define the dual $F$-signature of modules.

**Definition 3.1.** Let $(R, m, k)$ be a reduced $F$-finite local ring of characteristic $p > 0$ and $M$ be an $R$-module. For each natural number $e$, Put $q = p^e$, $\alpha = \log_p[k : k^p]$, and $b_q = \max\{n \mid F^e M \to M'\}$ and define

$$s(M) = \limsup_{e \to \infty} \frac{b_q}{q^{\dim R + \alpha}}$$
We call $b_q$ $q$-th $F$-surjective number of $M$ and call $s(M)$ the dual $F$-signature of $M$.

**Remark 3.2.** Since any homomorphism $\phi : F^e R \to R^n$ splits, we can easily see that $s(R)$ coincides with the $F$-signature defined by G. J. Leuschke and C. Huneke.

Theorem 1.2 (2) follows from the following result due to I. M. Aberbach and G. J. Leuschke [AL03].

**Theorem 3.3.** Let $(R, m, k)$ be a reduced excellent $F$-finite local ring containing a field of characteristic $p$, let $d = \dim R$. Then $s(R)$ is positive if and only if $R$ is strongly $F$-regular.

**Proposition 3.4.** Let $(R, m, k)$ be a reduced $F$-finite local ring and $M$ be a finitely generated $R$-module. Assume $\dim M < \dim R$, then $s(M) = 0$.

**Proof.** Assume we have

$$F^e_1 M \to M^{b_q} \to 0$$

Tensoring $R/I$, we have

$$F^e_1(M/I^{[q]}M) \to (M/IM)^{b_q} \to 0$$

Therefore we have

$$l(F^e_1(M/I^{[q]}M) \geq b_q \cdot l(M/IM)$$

Dividing $q^{d+\alpha}$ and taking the limit, we have

$$e_{HK}(I, M) \leq s(M) \cdot l(M/IM)$$

The left hand side is zero by Theorem 2.3 (1).

**Proposition 3.5.** Let $(R, m, k)$ be a reduced $F$-finite local ring and $M$ be a finitely generated $R$-module. Then $0 \leq s(M) \leq 1$.

**Proof.** Assume $\dim M < \dim R$, then $s(M) = 0$ by Proposition 3.4. We may assume $\dim M = \dim R$. Therefore there is a minimal prime $\mathfrak{p}$ such that $M$ has rank at $\mathfrak{p}$. Since the rank of $(F^e M)_\mathfrak{p}$ is rank $M \cdot q^{d+\alpha}$ and the rank of $(M^{b_q})_\mathfrak{p}$ is rank $M \cdot b_q$, we obtain $b_q \leq q^{d+\alpha}$. This implies $s(M) \leq 1$.

**Lemma 3.6.** Let $(R, m, k)$ be a reduced $F$-finite Cohen-Macaulay local ring and $n$ be a non-negative integer. There is a one to one correspondence between non-isomorphic surjective homomorphisms from $F^e_1 \omega_R$ to $\omega^*_n$ and non-isomorphic injective homomorphisms from $R^n$ to $F^e_1 R$ such that the cokernel is a maximal Cohen-Macaulay module.
Proof. Assume we have

\[
0 \longrightarrow \ker(f) \longrightarrow F_\omega R \longrightarrow \omega_R^{b_i} \longrightarrow 0
\]

Since \(F_\omega R\) and \(\omega_R\) are maximal Cohen-Macaulay module, the kernel is maximal Cohen-Macaulay module. Taking \(\omega_R\)-dual, we obtain

\[
0 \longrightarrow R^{b_i} \longrightarrow \Hom(F_\omega R, \omega_R) \cong F_\omega R \longrightarrow K \longrightarrow 0
\]

where \(K\) is the \(\omega_R\)-dual of \(\ker(f)\). Therefore \(K\) is maximal Cohen-Macaulay module. Conversely, assume we have

\[
0 \longrightarrow R^{b_i} \longrightarrow F_\omega R \longrightarrow \coker(g) \longrightarrow 0
\]

such that \(\coker(g)\) is a maximal Cohen-Macaulay module. Taking \(\omega_R\)-dual, we obtain

\[
0 \longrightarrow L \longrightarrow F_\omega R \longrightarrow \omega_R^{b_i} \longrightarrow \text{Ext}^1_R(\coker(g), \omega_R) = 0
\]

where \(L\) is the \(\omega_R\)-dual of \(\coker(g)\). The last equality follows from Grothendieck duality.

\[
\square
\]

Remark 3.7. If there is a surjective homomorphism from \(\ker(f)\) to \(\omega_R\), then there is a surjective homomorphism from \(F_\omega R\) to \(\omega_R^{b_i+1}\).

Proof. Consider the following diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \ker(f) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & F_\omega R \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \omega_R \\
\downarrow & & \downarrow \\
0 & \longrightarrow & L \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \omega_R^{b_i} \\
\end{array}
\]

where \(L\) is the push out of the diagram. Since \(L\) is in \(\text{Ext}^1(\omega_R^{b_i}, \omega_R) = 0\), the exact sequence splits. \(L\) is isomorphic to \(\omega_R^{b_i+1}\).

Proposition 3.8. Let \((R, m, k)\) be a reduced \(F\)-finite Cohen-Macaulay local ring. Then \(s(\omega_R) \geq s(R)\).

Proof. Assume we have

\[
0 \longrightarrow \ker(f) \longrightarrow F_\omega R \longrightarrow R^{b_i} \longrightarrow 0
\]

Since \(R^{b_i}\) is projective, this exact sequence splits. Therefore \(\ker(f)\) is maximal Cohen-Macaulay module. We also have

\[
0 \longrightarrow R^{b_i} \longrightarrow F_\omega R \longrightarrow \ker(f) \longrightarrow 0
\]

and this implies \(s(\omega_R) \geq s(R)\) by Lemma 3.6.

\[
\square
\]
**Theorem 3.9.** Let \((R, \mathfrak{m}, k)\) be a reduced F-finite Cohen-Macaulay local ring. Then \(s(\omega_R) = 1\) if and only if \(R\) is regular.

**Proof.** If \(R\) is regular, then the result follows from Kunz’s theorem. Therefore it is enough to prove the converse. Let \(\mathfrak{x}\) be a system of parameters generating a minimal reduction of \(\mathfrak{m}\), then \(l(\mathfrak{m}/\mathfrak{x}) = e(\mathfrak{m}) - 1\). By the definition of \(b_q\) and Lemma 3.6, we have

\[
0 \longrightarrow R^{b_q} \longrightarrow F^q R \longrightarrow K \longrightarrow 0
\]

where \(K\) is a Cohen-Macaulay module. By tensoring \(R/\mathfrak{m}\) and \(R/(\mathfrak{x})\), we have

\[
0 \longrightarrow (R/(\mathfrak{x}))^{b_q} \longrightarrow F^q (R/(\mathfrak{x})^{[q]}) \longrightarrow K/(\mathfrak{x})K \longrightarrow 0
\]

The injectivity of the second map of the first line follows from

\[
\text{Tor}_1^R(K, R/\mathfrak{m}) \cong \text{Ext}_R^{d-1}(R/(\mathfrak{x}), K) = 0
\]

The vertical maps are surjective. We have

\[
l(F^q(R/(\mathfrak{x})^{[q]}) - l(F^q(R/\mathfrak{m}^{[q]}) \geq b_q(l(R/(\mathfrak{x}) - l(R/\mathfrak{m})) + l(K/(\mathfrak{x})K) - l(K/\mathfrak{m}K) \\
\geq b_q(e - 1)
\]

Dividing both side by \(q^{d+\alpha}\) and taking the limit, we obtain

\[
e_{\text{HK}}((\mathfrak{x}), R) - e_{\text{HK}}(\mathfrak{m}, R) \geq e - 1
\]

Since \((\mathfrak{x})\) is minimal reduction for \(\mathfrak{m}\) and \(R\) is Cohen-Macaulay, \(e_{\text{HK}}((\mathfrak{x}), R) = e(\mathfrak{m}, R) = e\). This gives \(e_{\text{HK}}(\mathfrak{m}, R) = 1\) and \(R\) is regular. \(\square\)

**Proposition 3.10.** Let \((R, \mathfrak{m}, k)\) be a reduced F-finite local ring such that \(s(\omega_R) > 0\). Then \(R\) is Gorenstein if and only if \(s(R) = s(\omega_R)\).

**Proof.** It is enough to show the converse. Since \(s(R) = s(\omega_R) > 0\), \(R\) is F-regular. Assume \(R\) is not Gorenstein. Let

\[
F^q \omega_R \cong R^{a_q} \oplus \omega_R^{b_q} \oplus M
\]

be a direct summand decomposition of \(F^q \omega_R\) such that \(M\) has no direct summand of \(R\) or \(\omega_R\). We obtain \(\lim_{q \to \infty} \frac{a_q}{q^{d+\alpha}} = s(R)\) by Theorem 2.7. Taking \(\omega_R\)-dual, we have

\[
F^q R \cong R^{b_q} \oplus \omega_R^{a_q} \oplus \text{Hom}(M, \omega_R)
\]
This implies \( \lim_{q \to \infty} \frac{b_q}{q^{d+a}} = s(R) \). Let us denote the number of minimal generators of \( \omega_R \) by \( \mu(\omega_R) \). There is a surjective homomorphism from \( R^{\mu(\omega_R)} \) to \( \omega_R \). It follows that \( s(\omega_R) \geq \frac{s(R)}{\mu(\omega_R)} + s(R) > s(R) \). This is contradiction. \( \square \)

**Remark 3.11.** We remark the assumption in Theorem 3.10 is essential. Let \( R \) be a non-Gorenstein, non-F-injective local ring, then \( s(R) = s(\omega_R) = 0 \). But \( R \) is not Gorenstein.

**Corollary 3.12.** Let \( (R, m, k) \) be a reduced F-finite Cohen-Macaulay local ring. Then \( s(R) = 1 \) if and only if \( R \) is regular.

**Proof.** Assume \( s(R) = 1 \), then \( s(\omega_R) = 1 \) by Proposition 3.8. This implies that \( R \) is regular by Theorem 3.9. \( \square \)

**Lemma 3.13.** Let \( (R, m, k) \) be an Artinian local ring of infinite residue field and \( M \) be an \( R \)-module. Let \( V \) be a sub \( k \)-vector space of \( \text{soc}(R) \). Assume \( l(\Delta M) \geq \dim V \) for any \( \Delta \in \text{soc}R \), then there is an \( R \)-homomorphism \( \phi : R \to M \) such that \( \phi \) is injective on \( V \). In particular, if \( V = \text{soc}(R) \), then there is an injective \( R \)-homomorphism \( \phi : R \to M \).

**Proof.** We prove the claim by induction on \( \dim V \). Assume \( \dim V = 1 \), then there is an socle element \( \Delta_1 \) which generates \( V \). By the assumption, there is an element \( m \) in \( M \) such that \( \Delta_1 m \neq 0 \). This \( m \) gives the map from \( R \) to \( M \). Assume \( \dim V = n \), then there are socle elements \( \Delta_1, ..., \Delta_n \) which generate \( V \). Let \( W \) be the \( k \)-vector space generated by \( \Delta_1, ..., \Delta_{n-1} \). By induction, there is a \( R \)-homomorphism \( \phi : R \to M \) such that \( \phi \) is injective on \( W \). We put \( \phi(1) = m \). If this map is injective on \( V \), there is nothing to prove. So we may assume \( \phi \) is not injective on \( V \). Then there is an element \( \Delta \in V \) such that \( \Delta m = 0 \). In this case, \( \Delta_1, ..., \Delta_{n-1}, \Delta \) is a basis of \( V \). Since \( l(\Delta M) > \dim W \), there is an element \( m' \) in \( M \) such that \( \Delta_1 m, ..., \Delta_{n-1} m, \Delta m' \) are linearly independent in \( \text{soc}(R)M \). This implies that a minor of the matrix \( (\Delta_1 m, ..., \Delta_{n-1} m, \Delta m') \) is not zero. Let us take \( c \in R \) and put \( n = m + cm' \). If \( c \) is general, \( c \) gives the map from \( R \) to \( M \) which satisfy the desired condition. To confirm this, we denote the corresponding minor of the matrix \( A \) by \( \text{minor} A \). With this notation,

\[
\text{minor}(\Delta_1 n, ..., \Delta_{n-1} n, \Delta n) = \text{minor}(\Delta_1 m + c\Delta_1 m', ..., \Delta_{n-1} m + c\Delta_{n-1} m', c\Delta m') = c \cdot \text{minor}(\Delta_1 m, ..., \Delta_{n-1} m, \Delta m') + c^2 f(c)
\]

where \( f \) is a polynomial with a variable.

Since \( \text{minor}(\Delta_1 m, ..., \Delta_{n-1} m, \Delta m') \) is not zero, \( \text{minor}(\Delta_1 n, ..., \Delta_{n-1} n, \Delta n) \)
is non zero polynomial with the variable $c$. Therefore for general $c$, the determinant is not zero. This implies $\Delta_1 n, ..., \Delta_{n-1} n, \Delta n$ is linearly independent and the map induced by $n$ is injective on $V$.

\[ \square \]

**Lemma 3.14.** Let $R$ be a Noetherian local ring and $M$ be an $R$-module of finite length. Denote the injective hull of the residue field by $E$. Then $l(xM) = l(\text{Hom}(xM, E)) = l(x(\text{Hom}(M, E)))$.

**Proof.** The first equality follows from the Matlis duality. It is enough to show $\text{Hom}(xM, E)$ is isomorphic to $x(\text{Hom}(M, E))$. From the $R$-module structure of $\text{Hom}(M, E)$, we can regard $x(\text{Hom}(M, E))$ as submodule of $\text{Hom}(xM, E)$. Conversely, if we take $\phi \in \text{Hom}(xM, E)$, we can extend $\phi$ to $\tilde{\phi} \in \text{Hom}(M, E)$ since $E$ is injective module. This implies that the inclusion is surjective.

\[ \square \]

**Lemma 3.15.** Let $R$ be a Noetherian local ring and $M$ be an $R$-module. Let $x$ be a system of parameters. There is a natural number $c$ such that $c \geq l(\text{Tor}_1^R(\omega_R, R/(x, \Delta)))$ for any $\Delta \in \text{Soc}R/(x)$.

**Proof.** Let $X$ be a free resolution of $\omega_R$. Then there is a natural surjection from $X_1 \otimes R/(x)$ to $X_1 \otimes R/(x, \Delta)$. Since $\text{Tor}_1^R(\omega_R, R/(x, \Delta))$ is sub-quotient of $X_1 \otimes R/(x, \Delta)$, $l(X_1 \otimes R/(x)) \geq l(\text{Tor}_1^R(\omega_R, R/(x, \Delta)))$ holds.

\[ \square \]

**Theorem 3.16.** Let $(R, m)$ be a reduced $F$-finite Cohen-Macaulay local ring. Then $s(\omega_R) > 0$ if and only if $R$ is $F$-rational.

**Proof.** Let $x$ be arbitrary system of parameters and $\Delta$ be a element in the socle of $R/(x)$. By the definition of $b_q$, we have

\[ 0 \longrightarrow K \longrightarrow F^q_R \omega_R \longrightarrow \omega_R^{b_q} \longrightarrow 0 \]

where $K$ is the kernel of the surjective map. Tensoring $R/(x)$ and $R/(x, \Delta)$, we have

\[ 0 \longrightarrow K/(x)K \longrightarrow F^q_R(\omega_R/(x)^{(l_q)} \omega_R) \longrightarrow (\omega_R/(x) \omega_R)^{(b_q)} \longrightarrow 0 \]

\[ \text{Tor}_1^R(\omega_R, R/(x, \Delta))^{b_q} \longrightarrow K/(x, \Delta)K \longrightarrow F^q_R(\omega_R/(x, \Delta)^{(l_q)} \omega_R) \longrightarrow (\omega_R/(x, \Delta) \omega_R)^{(b_q)} \longrightarrow 0 \]

The injectivity of the first map of the first line follows from

\[ \text{Tor}_1^R(\omega_R, R/(x)) \cong \text{Ext}_R^{d-1}(R/(x), \omega_R) = 0 \]
The vertical maps are surjective. We have
\[ l(F^*_i(\omega_R/(x)^{[q]}\omega_R)) - l(F^*_i(\omega_R/(x,\Delta)^{[q]}\omega_R)) \geq \]
\[ b_q(l(\omega_R/(x)^{[q]}\omega_R)) - l(\omega_R/(x,\Delta)\omega_R) + l(K/(x,\Delta)K) - l(K/(x,\Delta)K) \geq b_q \]
Dividing both side by \( q^{d+a} \) and taking the limit, we obtain
\[ e_{HK}(x,R) - e_{HK}(x,\Delta),R) \geq s(\omega_R) \]
Assume \( R \) is not F-rational, then there exists \( \Delta \) such that \( \Delta \) is in the tight closure of \( (x) \). This implies \( s(\omega_R) = 0 \). Conversely, assume \( R \) is F-rational. From the diagram, we have
\[ b_q(l(\omega_R/(x)^{[q]}\omega_R)) - l(\omega_R/(x,\Delta)\omega_R) + l(K/(x,\Delta)K) - l(K/(x,\Delta)K) \]
\[ \geq l(F^*_i(\omega_R/(x)^{[q]}\omega_R)) - l(F^*_i(\omega_R/(x,\Delta)^{[q]}\omega_R)) \]
If \( s(\omega_R) = 0 \), then this means the order of \( b_q \) is less than \( q^{d+a} \). Since \( R \) is F-rational, the order of the right hand side of the inequality is \( q^{d+a} \).

**Lemma 3.15** and **Theorem 2.5** implies that there is a \( q \) such that for all \( \Delta \in soc(R) \), \( l(\Delta(K/(x)K)) = l(K/(x)K) - l(K/(x,\Delta)K) \) is bigger than \( l(soc(R/x)) \). By **Lemma 3.14** and **Lemma 3.13**, there is an injective homomorphism \( \phi : R/(x) \rightarrow Hom(K/(x),E(k)) \). This implies that there is an surjective homomorphism \( \phi : K \rightarrow \omega_R \) by **Lemma 3.6**. This contradicts to the maximality of \( b_q \) by the Remark 3.7.

**Example 3.17.** Let \( R = k[[x^n,x^{n-1}y,\ldots,y^n]] \), the \( n \)-th Veronese subring of \( k[[x,y]] \), where \( k \) is a perfect field of positive characteristic \( p \). Assume that \( n \geq 2 \) and \( p/n \). Then \( R \) has finite CM type. The indecomposable nonfree MCM \( R \)-modules are the fractional ideals \( I_1 = (x,y) \), \( I_2 = (x^2,xy,y^2) \), \ldots, \( I_{n-1} = (x^{n-1},x^{n-2}y,\ldots,y^{n-1}) \). Denote \( R \) also by \( I_0 \). Then we can decompose \( F_\ast I_i \) by using these modules.
\[ F_\ast I_i \cong \bigoplus_{i=0}^{n-1} I_i \]
We can also show the order of \( a_i, j \)'s are same and equal to \( q^2/n \). Furthermore, we can easily show the following things.

1. If \( \ell < k \), then any \( R \)-module homomorphism \( f : I_k \rightarrow I_l \) factor through \( mI_i \).
2. If \( \ell > k \), then there is no surjective homomorphism \( g : I_k \rightarrow I_l \)
3. If \( \ell > k \), then there exists a surjective homomorphism \( h : I_k \oplus I_{\ell-k-1} \rightarrow I_l \)

We compute \( s(I_l) \). By (1), there is no contribution from \( I_k \) (\( \ell < k \)). Since there is a natural surjection \( id : I_l \rightarrow I_l \), the contribution from \( I_l \) is one
to one. By (2) and (3), the contribution from $I_k$ ($l > k$) is two to one. Therefore,

$$s(I_l) = \frac{1}{n} + \frac{l}{2} \cdot \frac{1}{n} = \frac{l + 2}{2n}$$

In particular, $s(R) = s(I_0) = 1/n$, $s(\omega_R) = s(I_{n-2}) = 1/2$. $R$ is Gorenstein if and only if $n = 2$ by Proposition 3.10.

4. Questions

In this section, we define the dual $F$-signature of pairs and the $F$-surjective ratio to give questions about these.

The $F$-signature of pairs was introduced by M. Blickle, K. Schwede and K.Tucker in [BST11] to solve the conjecture about the $F$-splitting ratio by M. Aberbach and F. Enescu. Firstly, We define the $F$-surjective ratio which is a generalization of the $F$-splitting ratio defined by M. Aberbach and F. Enescu [AE05].

**Definition 4.1.** Let $(R, m, k)$ be a reduced $F$-finite local ring of characteristic $p > 0$ and $M$ be an $R$-module. For each natural number $e$, Put $q = p^e, \alpha = \log_p[k : k^p], \text{and } b_q = \max\{ n \mid 3^e F^e M \to M^n \}$ and for each $i \in \mathbb{N}$ define

$$r_i(M) = \limsup_{e \to \infty} \frac{b_q}{q^{i+\alpha}}$$

Consider about the set

$$D = \{r_i(M)\mid i \in \mathbb{N}\} \in \mathbb{R} \cup \{\infty\}$$

We define the $F$-surjective ratio by

$$r(M) = \max\{D \setminus \{\infty\}\}$$

We call the minimal number $i$ such that $r_i(M) = r(M)$ holds $F$-surjective dimension of $M$.

**Example 4.2.** Let $S_n = k[[x_1, x_2, \ldots, x_n]]$ be a formal power series ring of $n$ variables. There are natural surjections $\phi_i : S_n \to S_i = k[[x_1, x_2, \ldots, x_i]]$ for $i \leq n$. We regard $S_i$ as $S_n$-module by using these maps. Then the $F$-surjective dimension of $S_i$ is $i$ and $r(S_i) = 1$.

We can easily see that $r(R)$ coincide with the $F$-splitting ratio defined by M. Aberbach and F. Enescu. M. Blickle, K. Schwede and K.Tucker proved the positivity of the $F$-splitting ratio characterize the $F$-purity of $R$ in [BST11]. The conjecture is the same thing holds for the $F$-surjective ratio.
Conjecture 4.3. Let \((R, m, k)\) be a reduced Cohen-Macaulay \(F\)-finite local ring of characteristic \(p > 0\) and \(\omega_R\) be a canonical module. Then \(R\) is \(F\)-injective if and only if \(r(\omega_R)\) is positive.

The case of \(F\)-splitting ratio was solved by considering about the pair of \(F\)-signature. Not only for this application, it is important to consider about the dual \(F\)-signature of pairs.

Definition 4.4. Let \((R, m, k)\) be a reduced \(F\)-finite local ring of characteristic \(p > 0\) and \(M\) an \(R\)-module. Let \(\mathcal{D}_M\) be a Cartier sub algebra of \(M\). For each natural number \(e\), Put \(q = p^e, \alpha = \log_p [k : k^p] \), and \(b_q = \max \{ n | n \phi : F^e M \to M^n, pr_i \cdot \phi \in \mathcal{D}_M \) for any \(i \} \) and define

\[
s(M, \mathcal{D}_M) = \limsup_{e \to \infty} \frac{b_q}{q^{\dim k + \alpha}}\]

We call \(b_q\) \(q\)-th \(F\)-surjective number of \(M\) along \(\mathcal{D}_M\) and call \(s(M, \mathcal{D}_M)\) the dual \(F\)-signature of the pair \(M\) and \(\mathcal{D}_M\).

Example 4.5. Let \((R, m, k)\) be a reduced \(F\)-finite local ring of characteristic \(p > 0\) and \(M\) be a finitely generated \(R\)-module. We denote \(\mathcal{D}^s_M\) as the Cartier multiplicative closed sub set such that all homogeneous elements consist of split-surjective maps. Then \(s(M, \mathcal{D}^s_M)\) is equivalent to the generalized \(F\)-signature of \(M\) defined in [HN]. More concretely, in the Example 3.17, \(s(I_t, \mathcal{D}^s_{I_t}) = 1/n\).

Remark 4.6. By using the canonical duality, we can easily check \(s(R) = s(\omega_R, \mathcal{D}^s_{\omega_R})\).

Question 4.7. Can we generalize the results in Theorem 1.2 by using the dual \(F\)-signature of pairs?

References

[AE05] I. M. Aberbach and F. Enescu: *The structure of F-pure rings*, Math. Z. 250 (2005), no. 4, 791–806. MR2180375

[AL03] I. M. Aberbach and G. J. Leuschke: *The F-signature and strong F-regularity*, Math. Res. Lett. 10 (2003), no. 1, 51–56. MR1960123 (2004b:13003)

[BST11] M. Blickle, K. Schwede, and K. Tucker: *F-signature of pairs and the asymptotic behavior of Frobenius splittings*, preprint, arxiv:1107.1082

[BST11] M. Blickle, K. Schwede, and K. Tucker: *F-signature of pairs: Continuity, p-fractals and minimal log discrepancies*, preprint, arxiv:1111.2762

[G04] O. Gabber: *Notes on some t-structures*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 711-734.

[HH89] M. Hochster and C. Huneke: *Tight closure and strong F-regularity*, Mém. Soc. Math. France (N.S.) (1989), no. 38, 119–133, Colloque en l’honneur de Pierre Samuel (Orsay, 1987). MR1044348 (91i:13025)
[HH90] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. 3 (1990), 31–116.

[HH94] , *F-regularity and smooth base change*, Trans. Amer. Math. Soc. 346 (1994), 1–60.

[HN] M. Hashimoto and Y. Nakajima: *Generalized F-signature of invariant subrings*, In preparation

[HR74] M. Hochster and J. L. Roberts: *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math. 13 (1974), 115–175. MR0347810 (50 #311)

[HR76] M. Hochster and J. L. Roberts: *The purity of the Frobenius and local cohomology*, Advances in Math. 21 (1976), no. 2, 117–172. MR0417172 (54 #5230)

[HY] M. Hochster and Y. Yao: *The F-rational signature and drops in the Hilbert-Kunz multiplicity*, preprint

[Hun96] C. Huneke: * Tight closure and its applications*, CBMS Regional Conference Series in Mathematics, vol. 88, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996, With an appendix by Melvin Hochster. MR1377268 (96m:13001)

[HL02] C. Huneke and G. J. Leuschke: *Two theorems about maximal Cohen-Macaulay modules*, Math. Ann. 324 (2002), no. 2, 391–404. MR1933863 (2003j:13011)

[Kun69] E. Kunz: *Characterizations of regular local rings for characteristic p*, Amer. J. Math. 91 (1969), 772–784. MR0252389 (40 #5609)

[Kun76] E. Kunz: *On Noetherian rings of characteristic p*, Amer. J. Math. 98 (1976), no. 4, 999–1013. MR0432625 (55 #5612)

[Mon83] P. Monsky: *The Hilbert-Kunz function*, Math. Ann. 263 (1983), no. 1, 43–49. MR697329 (84k:13012)

[Mon08] P. Monsky: *Rationality of Hilbert-Kunz multiplicities: a likely counterexample*, Michigan Math. J. 57 (2008), 605–613, Special volume in honor of Melvin Hochster. 2492471 (2010g:13026)

[Sch08] K. Schwebel: *Centers of F-purity*, arXiv:0807.1654, to appear in Mathematische Zeitschrift.

[SVdB97] K. E. Smith and M. Van den Bergh: *Simplicity of rings of differential operators in prime characteristic*, Proc. London Math. Soc. (3) 75 (1997), no. 1, 32–62. MR1444312 (98d:16039)

[Tuc] K. Tucker: *F-signature exists*, Inventiones Mathematicae, 1–23, 10.1007/s00222-012-0389-0.

[WWY04] K.-i. Watanabe and K.-i. Yoshida: *Minimal relative Hilbert-Kunz multiplicity*, Illinois J. Math. 48 (2004), no. 1, 273–294. 2048225 (2005b:13033)

[Yao06] Y. Yao: *Observations on the F-signature of local rings of characteristic p*, J. Algebra 299 (2006), no. 1, 198–218. MR2225772 (2007k:13007)

Graduate School of Mathematics, Nagoya University Chikusa-ku, Nagoya 4648602 JAPAN.

E-mail address: x12004h@math.nagoya-u.ac.jp