Curves on Oeljeklaus-Toma Manifolds

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Abstract

Oeljeklaus-Toma manifolds are complex non-Kähler manifolds constructed by Oeljeklaus and Toma from certain number fields, and generalizing the Inoue surfaces $S_m$. We prove that Oeljeklaus-Toma manifolds contain no compact complex curves.

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1 Introduction.

Oeljeklaus-Toma manifolds (defined in [O–T]) are compact complex manifolds that are generalizing Inoue surfaces (defined in [I]). Let us describe them in detail.

1.1 Oeljeklaus-Toma manifolds

Let $K$ be a number field (a finite extension of $\mathbb{Q}$), $s$ be a number of its real embeddings and $2t$ be number of its complex embeddings. One can easily prove that for each $s$ and $t$ there exists a field $K$ which has these numbers of real and complex embeddings (see e.g. [O–T]).

**Definition 1.1:** Ring of algebraic integers $O_K$ is a subring of $K$ that consists of all roots of polynomials with integer coefficients which lie in $K$. Unit group $O_K^*$ is a multiplicative subgroup of invertible elements of $O_K$.

Let $m$ be $s + t$. Let $\sigma_1, \ldots, \sigma_s$ be real embeddings of the field $K$, $\sigma_{s+1}, \ldots, \sigma_{s+2t}$ be complex embeddings such that $\sigma_{s+i}$ and $\sigma_{s+t+i}$ are complex conjugate for each $i$ from $1$ to $t$. Now we can define a map $l : O_K^* \to \mathbb{R}^m$ where $l(u) = (\ln |\sigma_1(u)|, \ldots, \ln |\sigma_s(u)|, 2\ln |\sigma_{s+1}(u)|, \ldots, 2\ln |\sigma_{m}(u)|)$. Denote $O_K^{s+*} = \{a \in O_K^* : \sigma_i(a) > 0, i = 1, \ldots, s\}$. Let’s consider following definitions:

**Definition 1.2:** A lattice $\Lambda$ in $\mathbb{R}^n$ is a discrete additive subgroup such that $\Lambda \otimes \mathbb{R} = \mathbb{R}^n$.

**Definition 1.3:** Group $U \subset O_K^{s+*}$ of rank $s$ is called admissible for the field $K$ if the projection of $l(U)$ to the first $s$ components is a lattice in $\mathbb{R}^s$.

Consider a linear space $L = \{x \in \mathbb{R}^m \mid \sum_{i=1}^m x_i = 0\}$. The projection of $L \subset \mathbb{R}^m$ to the first $s$ coordinates is surjective, because $s < m$. Using the Dirichlet unit theorem (see e.g. [Mii09]) one can prove that $l(O_K^{s+*})$ is a full lattice in $L$. Therefore there exists a group $U$ that is admissible.

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \text{im} z > 0\}$. Let $U \subset O_K^{s+*}$ be a group which is admissible for $K$. Let $O_K$ be an additive group of algebraic integers. The group $U$ acts on $O_K$ multiplicatively. This defines a structure of the semidirect product $U' := U \times O_K$. Define the action of $U'$ on $\mathbb{H}^s \times \mathbb{C}^t$ as follows. The element $u \in U$ acts on $\mathbb{H}^s \times \mathbb{C}^t$ mapping $(z_1, \ldots, z_m)$ to $(\sigma_1(u)z_1, \ldots, \sigma_m(u)z_m)$. Since $U$ lies in $O_K^{s+*}$, the action $U$ on the first $s$ coordinates preserves $\mathbb{H}$. The additive group $O_K$ acts on $\mathbb{H}^s \times \mathbb{C}^t$ by parallel translations: $a \in O_k$ is mapping $(z_1, \ldots, z_m)$ to $(\sigma_1(a) + z_1, \ldots, \sigma_m(a) + z_m)$. Since the first $s$ embeddings are real, this action preserves $\mathbb{H}$ in the first $s$ coordinates.
One can see that \((u, a) \in U \ltimes O_K\) maps \((z_1, \ldots, z_m)\) to \((\sigma_1(u)z_1 + \sigma_1(a), \ldots, \sigma_m(u)z_m + \sigma_m(a))\).

Let us show that this action is compatible with the group operation in the semidirect product.

By definition, one has \((u, a) \ast (u_1, a_1) = (uu_1, ua_1 + a)\), and

\[
(u, a) \circ (u_1, a_1)(z_1, \ldots, z_m) = (u, a)((\sigma_1(u_1)z_1 + \sigma_1(a_1), \ldots, \sigma_m(u_1)z_m + \sigma_m(a_1)) =
= (\sigma_1(u)\sigma_1(u_1)z_1 + \sigma_1(u)\sigma_1(a_1) + \sigma_1(a), \ldots, \sigma_m(u)\sigma_m(u_1)z_m + \sigma_m(u)\sigma_m(a_1) + \sigma_m(a)) =
= (\sigma_1(uu_1)z_1 + \sigma_1(uu_1 + a), \ldots, \sigma_m(uu_1)z_m + \sigma_m(uu_1 + a)) = (uu_1, uu_1 + a)(z_1, \ldots, z_m) =
= ((u, a) \ast (u_1, a_1))(z_1, \ldots, z_m),
\]

This proves the compatibility.

**Definition 1.4:** *Oeljeklaus-Toma manifold* is a quotient of \(\mathbb{H}^s \times \mathbb{C}^t\) by the action of the group \(U \ltimes O_K\) which was defined above.

This quotient exists because \(U \ltimes O_K\) acts properly discontinuously on \(\mathbb{H}^s \times \mathbb{C}^t\). Therefore \(\mathbb{H}^s \times \mathbb{C}^t/\ltimes O_K\) is a compact complex manifold. To prove it, let \(U\) be admissible for \(K\). The quotient \(\mathbb{H}^s \times \mathbb{C}^t/\ltimes (O_K)\) is obviously diffeomorphic to the trivial toric bundle \((\mathbb{R}_{>0})^s \times (S^1)^n\). The group \(U\) acts properly discontinuously on the base \((\mathbb{R}_{>0})^s\). Therefore it acts properly discontinuously on \(\mathbb{H}^s \times \mathbb{C}^t/\ltimes O_K\). Also, groups \(U\) and \(O_K\) act holomorphically on \(\mathbb{H}^s \times \mathbb{C}^t\). Therefore the quotient has a holomorphic structure.

## 2 Curves on the Oeljeklaus-Toma manifolds

In this section we shall prove that there are no complex curves on the Oeljeklaus-Toma manifolds, just as on the Inoue surfaces (see [1]).

### 2.1 The exact positive (1,1)-form on the Oeljeklaus-Toma manifold

Let \(M\) be a smooth complex manifold, \(z_1, \ldots, z_n\) — local complex coordinates in the open neighborhood of the point \(x \in M\).

**Definition 2.1:** The *Hodge decomposition* of the Grassmanian algebra \(\Lambda^p M\) is a decomposition into the direct sum \(\Lambda^p M = \oplus_p \Lambda^p M\), where \(\Lambda^p M = \Lambda^p M \wedge \Lambda^0 M\); space \(\Lambda^p M\) of the real differential forms is generated by \(dz_1, \ldots, dz_{p}\), and \(\Lambda^p M\) of the real differential forms is generated by \(\omega_1, \ldots, \omega_{n}\).

Therefore \(\Lambda^2 M = \Lambda^2 M \wedge \Lambda^0 M\).

**Definition 2.2:** (1,1)-form on a complex manifold \(M\) is a section of \(\Lambda^1 M\).

**Definition 2.3:** (1,1)-form \(\omega\) on a complex manifold \(M\) is positive (or semipositive) if \(\sqrt{-1} \omega(x, \bar{x}) \geq 0\) for each tangent vector \(x \in T^1 M\).

As in [O-V], we consider a certain positive (1,1)-form on Oeljeklaus-Toma manifold \(M = \mathbb{H}^s \times \mathbb{C}^t/\ltimes O_K\). Firstly, we introduce a (1,1)-form \(\bar{\omega}\) on \(\tilde{M} = \mathbb{H}^s \times \mathbb{C}^t\) which is preserved by the action of the group \(\Gamma = (U \ltimes O_K)\) and since then it would be a (1,1)-form on \(M\).

Let \((z_1, \ldots, z_m)\) be complex coordinates on \(\tilde{M}\). Define \(\varphi(z) = \Pi_{i=1}^s \text{im}(z_i)\). Since the first \(s\) components of \(\tilde{M}\) correspond to upper half-planes \(\mathbb{H} \subset \mathbb{C}\), this function is positive on \(\tilde{M}\).

Let us now consider a form \(\bar{\omega} = \sqrt{-1} \bar{\omega} \log \varphi\). Using standard coordinates on \(\tilde{M}\) one can write this form as \(\bar{\omega} = \sqrt{-1} \bar{\omega} \sum_{i=1}^s \frac{dz_i \wedge d\bar{z}_i}{(\text{im} z_i)^2}\). Therefore \(\bar{\omega}\) is a positive (1,1)-form on \(\tilde{M}\).

Let us show that this form is \(\Gamma\)-invariant.

We denote by \(\Gamma = (U \ltimes O_K)\). \(\Gamma\) is a semidirect product of the additive group \(O_K\) and the multiplicative group \(U\). Additive group acts on the first \(s\) components of \(\tilde{M}\) (which correspond to upper half-planes \(\mathbb{H} \subset \mathbb{C}\)) by translations along the real line. Therefore it does not change \(\text{im} z_i\) for \(i = 1 \ldots s\). Hence the function \(\log \varphi\) is preserved by the action of the additive component.
The multiplicative component acts on the first \( s \) coordinates \( \tilde{M} \) by multiplying them by a real number (since the first \( s \) embeddings of the number field \( K \) are real). Then every \( \text{im} z_i \) is multiplied by a real number and so there is a real number added to \( \log(\text{im} z_i) \). Since \( \log \varphi(z) = \sum_{i=1}^s \log(\text{im} z_i) \), there is a real number added to \( \log \varphi \). Operator \( \bar{\partial} \) is zero on the constants, so \( \bar{\omega} = \sqrt{-1} \bar{\partial} \log \varphi \) is preserved by action of group \( \Gamma \).

Since (1,1)-form \( \bar{\omega} \) is \( \Gamma \)-invariant it is a pullback of (1,1)-form \( \omega \) on the Oeljeklaus-Toma manifold \( M = \tilde{M}/\Gamma \).

Let’s now show that the form \( \bar{\omega} \) is exact on \( \tilde{M} \). For that we define operator \( d^c \).

**Definition 2.4:** Define the twisted differential \( d^c = I^{-1} dI \) where \( d \) is a De Rham differential and \( I \) is the operator of the almost complex structure.

Since \( d d^c = 2\sqrt{-1} \bar{\partial} (\text{see } [G–H]) \), one can see that \( \bar{\omega} = \sqrt{-1} \bar{\partial} \log \varphi = \frac{1}{2} d d^c \log \varphi \) and so \( \bar{\omega} \) is exact as a form on \( \tilde{M} \). Also since the operator \( d^c \) vanishes on the constants the form \( d^c \log \varphi \) is \( \Gamma \)-invariant, so \( \omega \) is exact on \( M \).

### 2.2 (1,1)-form \( \omega \) and curves on the Oeljeklaus-Toma manifold

Since the form \( \omega \) on manifold \( M \) is positive, its integral on any complex curve \( C \subset M \) is nonnegative. The form \( \omega \) is exact. Hence the Stokes’ theorem implies that its integral on any complex curve vanishes. So, if \( C \subset M \) is a closed complex curve, \( \omega \) vanishes on it.

To find out on which curves \( \omega \) vanishes, let us define the zero foliation of the form \( \omega \).

**Definition 2.5:** Involutive distribution (or foliation) on \( M \) is a subbundle \( B \subset TM \) in the tangent bundle that is closed under commutator: \[ \{B, B\} \subset B. \]

**Definition 2.6:** A leaf of a foliation is a submanifold in \( M \) such that its dimension equal \( \dim B \) and that is tangent to \( B \) at every point (not necessarily closed).

**Theorem 2.7:** (Frobenius) Let \( B \subset TM \) be an involutive distribution. Then for each point of a manifold \( M \), there is at most one leaf of this distribution that contains this point (see e.g. [Boch] Section IV. 8. Frobenius’s Theorem).

**Definition 2.8:** The zero foliation of (1,1)-form \( \omega \) on \( M \) is a subbundle of \( TM \) that consists of tangent vectors \( x \in TM \) such that \( \omega(x, Ix) = 0 \) where \( I \) is an operator of complex structure.

Consider the zero foliation of \( \bar{\omega} \) on \( \tilde{M} \).

The form \( \bar{\omega} \) is strictly positive on each vector \( v = (z_1, \ldots, z_n) \) such that at least one of \( z_i \) is nonzero. Such a vector cannot be in the leaf of the zero foliation. Therefore on each leaf of the zero foliation of the form \( \bar{\omega} \) the first \( s \) coordinates are constant.

Hence a leaf of the zero foliation of \( \bar{\omega} \) on \( M \) is isomorphic to \( \mathbb{C}^s \).

Let us now consider the zero foliation of \( \omega \) on \( M \).

We show that the image of the action of \( \Gamma \) on any leaf \( L \) of the zero foliation of the form \( \bar{\omega} \) does not intersect with \( L \).

One can see that \( L \) is \( (z_1, \ldots, z_s) \times \mathbb{C}^t \) for some fixed \( (z_1, \ldots, z_m) \). Therefore, for any \( \gamma \in \Gamma \) such that \( L \cap \gamma(L) \neq \emptyset \), the first \( s \) coordinates of the points in \( L \) coincide with the first \( s \) coordinates of the points in \( \gamma(L) \). Therefore we have a following system of equations:

\[
\sigma_i(u) z_i + \sigma_i(a) = z_i, \quad i = 1 \ldots s,
\]

where \( \gamma = (u, a) \).

These equations imply that \( z_i = \frac{\sigma_i(a)}{1 - \sigma_i(u)} \). Therefore \( z_i \) is real but \( \mathbb{R} \) doesn’t have real elements. Therefore \( L \cap \Gamma(L) = \emptyset \).

Since \( \omega \) vanishes on each compact curve \( C \subset M \), each curve is contained in some leaf of the zero foliation of \( \omega \). Since \( \omega \) is \( \Gamma \)-invariant, each leaf of the zero foliation of \( \omega \) on \( M \) is a factor of the leaf of the zero foliation of \( \bar{\omega} \) on \( \tilde{M} \). Therefore, it is isomorphic to \( \mathbb{C}^t \). All the coordinate functions
\[ z_i, i = 1, \ldots, t \] are holomorphic and therefore constant on all compact connected subvarieties. Therefore, \( \mathbb{C}^t \) does not contain complex curves, and \( M \) does not contain complex curves either because it’s compact and every closed curve on a compact manifold is compact.

We proved the following theorem:

**Theorem 2.9:** There are no closed complex curves on the Oeljeklaus-Toma manifolds.

**References**

[Aus] Auslander L. *The structure of compact locally affine manifolds*. Topology 3 (1964), 131-139.

[B1] Bieberbach L. *Über die Bewegungsgruppen der Euklidischen Räume I*. Mathematische Annalen 70 (3), 297-336.

[B2] Bieberbach L. *Über die Bewegungsgruppen der Euklidischen Räume II*. Mathematische Annalen 72 (3), 400-412.

[Boo] Boothby W.M. *An Introduction to Differentiable Manifolds and Riemannian Geometry*. Academic Press, San Diego, California, 2003.

[G–H] Griffiths Ph., Yarris J. *Principles of Algebraic Geometry*. Wiley-Interscience, 1994.

[I] Inoue M. *On surfaces of Class VII0*, Invent. Math. 24 (1974), 269-310.

[Mat08] Milne J.S. *Fields and Galois Theory, September 2008*. This paper can be found on [http://www.jmilne.org/math/CourseNotes/ft.html](http://www.jmilne.org/math/CourseNotes/ft.html) version 4.21

[Mat09] Milne J.S. *Algebraic Number Theory, April 2009*. This paper can be found on [http://www.jmilne.org/math/CourseNotes/ant.html](http://www.jmilne.org/math/CourseNotes/ant.html) version 3.02

[O–T] Oeljeklaus K., Toma M. *Non-Kähler compact complex manifolds associated to number fields*. Ann. Inst. Fourier 55 (2005), 1291-1300.

[O–V] Ornea L., Verbitsky M. *Subvarieties in Oeljeklaus-Toma manifolds*.

[P–V] Parton M., Vuletescu V. *Examples of non-trivial rank in locally conformal Kähler geometry*. Math. Z. (2010), DOI 10.1007/s00209-010-0791-5, [arXiv:1001.4891](http://arxiv.org/abs/1001.4891)

[R] Raghunathan M.S. *Discrete subgroups of Lie groups*. Springer 1972.

[V] Voisin C. *Hodge Theory and Complex Algebraic Geometry Volume 1*. Cambridge University Press, 2002.

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