ANALYTICITY AND CROSSING SYMMETRY OF THE EIKONAL AMPLITUDES IN GAUGE THEORIES

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Abstract

After a brief review and a more refined analysis of some relevant analyticity properties (when going from Minkowskian to Euclidean theory) of the high–energy parton–parton and hadron–hadron scattering amplitudes in gauge theories, described nonperturbatively, in the eikonal approximation, by certain correlation functions of two Wilson lines or two Wilson loops near the light cone, we shall see how these same properties lead to a nice geometrical interpretation of the crossing symmetry between quark–quark and quark–antiquark eikonal amplitudes and also between loop–loop eikonal amplitudes. This relation between Minkowskian–to–Euclidean analyticity properties and crossing symmetry is discussed in detail and explicitly tested in the first orders of perturbation theory. Some nonperturbative examples existing in the literature are also discussed.

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1. Introduction

A big effort has been made in the last fifteen years (since the seminal paper by Nachtmann in 1991 [1]) in the nonperturbative study, from the first principles of QCD, of the high-energy parton–parton and hadron–hadron elastic scattering amplitudes (for a review, see Refs. [2, 3]): these can be described, in the so-called eikonal approximation (and, therefore, they will be sometimes called “eikonal scattering amplitudes”), by certain correlation functions of two Wilson lines or two Wilson loops near the light cone.

The section 2 of this paper contains a brief review (for the benefit of the reader) and a more refined analysis of some relevant analyticity properties of the line–line and loop–loop correlation functions in gauge theories, when going from Minkowskian to Euclidean theory: these properties make it possible to reconstruct the eikonal scattering amplitudes starting from the Euclidean correlation functions, which can be computed with nonperturbative techniques (some examples existing in the literature will be discussed in section 5).

In section 3 we will show (always in a nonperturbative way, using the functional integral approach) how these same properties also lead to a nice geometrical interpretation of the crossing symmetry between quark–quark and quark–antiquark correlators and also between loop–loop correlation functions. This relation between Minkowskian–to–Euclidean analyticity properties and crossing symmetry is the main novel result of this paper and is discussed in detail also in the two last sections of it.

In particular, in section 4 (and appendix A) it is explicitly tested in the first orders of perturbation theory, which is the only available technique for computing (from first principles) both the Minkowskian and the Euclidean line–line and loop–loop correlation functions. As already stressed in Ref. [4] (but see also Refs. [5, 6] and references therein), such perturbative expansions of the line–line and loop–loop correlation functions, when considered in the Minkowskian theory in the limit of very large rapidity gap, must be eventually compared (as a non–trivial check!) to the well–known results obtained when computing the high–energy scattering amplitudes with usual perturbative techniques [7, 8, 9].

Finally, some nonperturbative examples existing in the literature and also the necessity of a real nonperturbative foundation of the above–mentioned analyticity properties are discussed as concluding remarks in section 5 (and appendix B), together with some prospects for the future.
2. Eikonal scattering amplitudes

The parton–parton elastic scattering amplitude, at high squared energies $s$ in the center of mass and small squared transferred momentum $t$ (that is to say: $|t| \leq 1 \text{ GeV}^2 \ll s$), can be described by the expectation value of two infinite lightlike Wilson lines, running along the classical trajectories of the colliding particles \[1, 10, 11, 12, 13\]. However, this description is affected by infrared (IR) divergences \[10, 11\], which are typical of 3 + 1 dimensional gauge theories. One can regularize this IR problem by letting the Wilson lines coincide with the classical trajectories for partons with a non–zero mass $m$ (so forming a certain finite rapidity gap, i.e., a certain finite hyperbolic angle $\chi$ in Minkowskian space–time: of course [see Eq. (2.4) below], $\chi \simeq \log(s/m^2) \to \infty$ when $s \to \infty$) and, in addition, by considering finite Wilson lines, extending in proper time from $-T$ to $T$ (and eventually letting $T \to +\infty$) \[4, 10, 11, 14, 15\]. For example, the high–energy quark–quark elastic scattering amplitude $M_{\text{qq}}(s,t)$ is (explicitly indicating the colour indices $i,j$ [initial] and $i',j'$ [final] and the spin indices $\alpha,\beta$ [initial] and $\alpha',\beta'$ [final] of the colliding quarks):

\[
M_{\text{qq}}^{\text{qq}}(s;t) \sim -2s \delta_{\alpha'\alpha} \delta_{\beta'\beta} \ g_{M}^{\text{qq}}(p_1, p_2; T \to \infty; t), \quad (2.1)
\]

with $g_{M}^{\text{qq}}$ defined as:

\[
g_{M}^{\text{qq}}(p_1, p_2; T; t)_{i'j'i''j''} \equiv \frac{1}{[Z_M(T)]^2} \int d^2 z_\perp e^{i \vec{q}_\perp \cdot \vec{z}_\perp} \langle [W_{p_1}^{(T)}(\vec{z}_\perp) - \mathbb{I}]_{i'i'} [W_{p_2}^{(T)}(\vec{0}_\perp) - \mathbb{I}]_{j'j'} \rangle, \quad (2.2)
\]

where $t = -|\vec{q}_\perp|^2$, $\vec{q}_\perp$ being the transferred momentum, and $\vec{z}_\perp = (z^2, z^3)$ is the distance between the two trajectories in the transverse plane (impact parameter). We are taking the two colliding quarks (with mass $m$) moving (in the center–of–mass system) with speed $V$ and $-V$ along, for example, the $x^1$–direction and so having four–momenta $p_1$ and $p_2$ given by:

\[
p_1 = m(\cosh \frac{\chi}{2}, \sinh \frac{\chi}{2}, 0), \quad p_2 = m(\cosh \frac{\chi}{2}, -\sinh \frac{\chi}{2}, 0), \quad (2.3)
\]

where $\chi = 2 \arctanh V$ is the hyperbolic angle between the two trajectories (i.e., $p_1 \cdot p_2 = m^2 \cosh \chi$). Therefore:

\[
s \equiv (p_1 + p_2)^2 = 2m^2 (\cosh \chi + 1). \quad (2.4)
\]
The two IR–regularized Wilson lines are defined as \( [z = (0, 0, z_\perp)] \):

\[
W_{p_1}^{(T)}(\vec{z}_\perp) \equiv T \exp \left[ -ig \int_{-T}^{+T} A_\mu(z + \frac{p_1}{m} \tau) \frac{p_\mu}{m} d\tau \right],
\]

\[
W_{p_2}^{(T)}(\vec{0}_\perp) \equiv T \exp \left[ -ig \int_{-T}^{+T} A_\mu(\frac{p_2}{m} \tau) \frac{p_\mu}{m} d\tau \right],
\]

(2.5)

where \( T \) stands for “time ordering” and, for a non–Abelian gauge theory with \( N_c \) colours, \( A_\mu = A^a_\mu T_a \), \( T^a \ (a = 1, \ldots, N_c^2 - 1) \) being the generators of the \( SU(N_c) \) algebra in the fundamental representation. The two Wilson lines are schematically shown in Fig. 1.

Finally, \( Z_{M}(T) \) is a sort of Wilson–line renormalization constant:

\[
Z_{M}(T) \equiv \frac{1}{N_c} \langle \text{Tr}[W_{p_1}^{(T)}(\vec{0}_\perp)] \rangle = \frac{1}{N_c} \langle \text{Tr}[W_{p_2}^{(T)}(\vec{0}_\perp)] \rangle.
\]

(2.6)

The expectation values \( \langle W_{p_1} W_{p_2} \rangle \), \( \langle W_{p_1} \rangle \) and \( \langle W_{p_2} \rangle \) are averages in the sense of the QCD functional integrals:

\[
\langle O[A] \rangle = \frac{1}{Z} \int [dA] \det(Q[A]) e^{iS_A} O[A],
\]

\[
Z = \int [dA] \det(Q[A]) e^{iS_A},
\]

(2.7)

where \( S_A \) is the pure–gauge (Yang–Mills) action and \( Q[A] \) is the quark matrix, coming from the functional integration over the fermion degrees of freedom.

The correlation function (2.2), with the four–vectors \( p_1 \) and \( p_2 \) defined by Eq. (2.3), will be also denoted (with a slight abuse of notation) as:* \( g^{qq}_{M}(p_1, p_2; T; t)_{\nu_i \nu_i' j j'} \equiv g^{qq}_{M}(\chi; T; t)_{\nu_i \nu_i' j j'} \).

(2.8)

By virtue of the invariance under parity transformations and \( O(3) \) spatial rotations, the domain of the function \( g_M \) in the variable \( \chi \) can be restricted to the real positive axis, \( \chi \in \mathbb{R}^+ \). In fact, a parity transformation together with a 180° rotation around the \( x^1 \)

*We remark that only the asymptotic behaviour for \( \chi \simeq \log(s/m^2) \to \infty \) of the correlator \( g^{qq}_{M} \) describes the high–energy quark–quark elastic scattering amplitude by virtue of Eq. (2.1). The correlator \( g^{qq}_{M} \) as a function of the generic hyperbolic angle \( \chi \) between the two Wilson lines, defined by Eqs. (2.2) and (2.3), must not be identified with the scattering amplitude at every \( \chi \), i.e., at every \( s = 2m^2(\cosh \chi + 1) \).
axis, i.e., a transformation

\[ x \to x' = \Lambda x, \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.9} \]

brings \( \chi \) into \( -\chi \) without modifying the functional integral:

\[ g_M^{qq}(\chi; T; t)_{i'i'; j'j} = g_M^{qq}(-\chi; T; t)_{i'i'; j'j}, \quad \forall \chi \in \mathbb{R}. \tag{2.10} \]

Turning now to the Euclidean theory, one can consider the corresponding quantity \( g_E^{qq} \), defined as a (properly normalized) correlation function of two (IR–regularized) Euclidean Wilson lines \( \widetilde{W}_{\ell 1 E} \) and \( \widetilde{W}_{\ell 2 E} \), i.e.,

\[ g_E^{qq}(p_{1 E}, p_{2 E}; T; t)_{i'i'; j'j} = \frac{1}{Z_E(T)^2} \int d^2 z_\perp e^{i \vec{q} \cdot \vec{z}_\perp} \langle [\widetilde{W}_{\ell 1 E}^{(T)}(z_\perp)] - \mathbb{I} \mid [\widetilde{W}_{\ell 2 E}^{(T)}(\vec{0}_\perp)] - \mathbb{I} \rangle_E, \]

\[ Z_E(T) = \frac{1}{N_c \langle \text{Tr}[\widetilde{W}_{\ell 1 E}^{(T)}(\vec{0}_\perp)] \rangle_E} = \frac{1}{N_c \langle \text{Tr}[\widetilde{W}_{\ell 2 E}^{(T)}(\vec{0}_\perp)] \rangle_E}, \tag{2.11} \]

where \( [z_E = (0, \vec{z}_\perp, 0)] \):

\[ \widetilde{W}_{\ell 1 E}^{(T)}(\vec{z}_\perp) \equiv \mathcal{T} \exp \left[ -ig \int_{-T}^{T} A_{\mu}^{(E)}(z_E + \frac{p_{1 E} \mu}{m} \frac{p_{1 E}}{m} d\tau \right], \]

\[ \widetilde{W}_{\ell 2 E}^{(T)}(\vec{0}_\perp) \equiv \mathcal{T} \exp \left[ -ig \int_{-T}^{T} A_{\mu}^{(E)}(\frac{p_{2 E} \mu}{m} \frac{p_{2 E}}{m} d\tau \right], \tag{2.12} \]

and:

\[ \langle \mathcal{O}[A^{(E)}] \rangle_E = \frac{1}{Z^{(E)}} \int [dA^{(E)}] \det(Q^{(E)}[A^{(E)}]) e^{-S_A^{(E)}} \mathcal{O}[A^{(E)}], \]

\[ Z^{(E)} = \int [dA^{(E)}] \det(Q^{(E)}[A^{(E)}]) e^{-S_A^{(E)}}, \tag{2.13} \]

\( S_A^{(E)} \) being the Euclidean pure–gauge (Yang–Mills) action and \( Q^{(E)}[A] \) being the Euclidean quark matrix, coming from the functional integration over the fermion degrees of freedom. The two Euclidean four–vectors \( p_{1 E} \) and \( p_{2 E} \) are chosen to be:

\[ p_{1 E} = m(\sin \frac{\theta}{2}, \vec{0}_\perp, \cos \frac{\theta}{2}), \]

\[ p_{2 E} = m(-\sin \frac{\theta}{2}, \vec{0}_\perp, \cos \frac{\theta}{2}), \tag{2.14} \]
θ being the angle formed by the two trajectories in the Euclidean four-space (i.e., \( p_1 \cdot p_2 = m^2 \cos \theta \)).

The correlation function \( g_E \) in (2.11), with the four-vectors \( p_1 \) and \( p_2 \) defined by Eq. (2.14), will be also denoted (with a slight abuse of notation) as:

\[ g_{qq}^E(p_1E, p_2E; T; t) \equiv g_{qq}^E(\theta; T; t) \equiv g_{qq}^E(\theta; T; t) \equiv g_{qq}^E(\theta; T; t); \quad i' \equiv g_{qq}^E(\theta; T; t) \equiv g_{qq}^E(\theta; T; t) \equiv g_{qq}^E(\theta; T; t). \tag{2.15} \]

By virtue of the \( O(4) \) symmetry of the Euclidean theory, the domain of the function \( g_E \) in the variable \( \theta \) can be restricted to the interval \((0, \pi)\). In fact, the invariance of the functional integral under the following \( O(4) \) transformation:

\[ x_E \rightarrow x_E' = \mathcal{R}_1 x_E, \quad \mathcal{R}_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.16} \]

leads to the following relation:

\[ g_{qq}^E(\theta; T; t) \equiv g_{qq}^E(\theta; T; t) \equiv g_{qq}^E(\theta; T; t) \equiv g_{qq}^E(\theta; T; t); \quad \forall \theta \in \mathbb{R}. \tag{2.17} \]

Similarly, the invariance of the functional integral under the following \( O(4) \) transformation:

\[ x_E \rightarrow x_E' = \mathcal{R}_2 x_E, \quad \mathcal{R}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.18} \]

leads to the following relation:

\[ g_{qq}^E(\theta; T; t) \equiv g_{qq}^E(\theta; T; t) \equiv g_{qq}^E(\theta; T; t) \equiv g_{qq}^E(\theta; T; t); \quad \forall \theta \in \mathbb{R}. \tag{2.19} \]

These two relations imply the possibility of restricting the domain in the angular variable \( \theta \) to the interval \((0, \pi)\), as said above.\(^\dagger\)

\(^\dagger\)After substituting \( \theta \rightarrow \theta + 2\pi \) into Eq. (2.19) and using also Eq. (2.17), one finds that \( g_E(\theta + 2\pi; T; t) = g_E(\theta; T; t) = g_E(\theta; T; t) \). Moreover, after substituting \( \theta \rightarrow 2\pi - \theta \) into Eq. (2.17) and using also Eq. (2.19), one finds that \( g_E(\theta - 2\pi; T; t) = g_E(2\pi - \theta; T; t) = g_E(\theta; T; t) \). Therefore we conclude that: \( g_E(\theta + 2\pi k; T; t) = g_E(\theta; T; t), \forall k \in \mathbb{Z} \).
The quantity $g_{qq}^{\chi}(\chi; T; t)$ with $\chi \in \mathbb{R}^+$ can be reconstructed from the corresponding Euclidean quantity $g_{qq}^{\theta}(\theta; T; t)$, with $\theta \in (0, \pi)$, by an analytic continuation in the angular variables and in the IR cutoff [4, 14, 15]:

$$g_{qq}^{\chi}(\chi; T; t) = g_{qq}^{\theta}(\chi \to i\theta; T \to -iT; t),$$

$$g_{qq}^{\theta}(\theta; T; t) = g_{qq}^{\chi}(\theta \to -i\chi; T \to iT; t).$$

Eq. (2.20) is then intended to be valid for every $\chi \in D_M$ (i.e., for every $\theta \in D_E$):

$$\overline{g}_{qq}^{\chi}(\theta; T; t) = \overline{g}_{qq}^{\theta}(\theta \to -i\chi; T \to iT; t), \quad \forall \chi \in D_M.$$  

This result is valid both for Abelian and non–Abelian gauge theories. We stress the fact that the regularized quantities $g_M(\chi; T; t)$ and $g_E(\theta; T; t)$, while being finite at any given value of $T$, are divergent in the limit $T \to \infty$ (even if in some cases this IR divergence can be factorized out and one thus ends up with an IR–finite physical quantity).

Differently from the parton–parton scattering amplitudes, which are known to be affected by IR divergences, the elastic scattering amplitude of two colourless states in gauge theories, e.g., two $q\bar{q}$ meson states, is expected to be an IR–finite physical quantity [16]. It was shown in Refs. [17, 18, 19] (for a review, see Refs. [2, 3]) that the high–energy meson–meson elastic scattering amplitude can be approximately reconstructed by
first evaluating, in the eikonal approximation, the elastic scattering amplitude of two \( q\bar{q} \) pairs (usually called “dipoles”), of given transverse sizes \( \vec{R}_{1\perp} \) and \( \vec{R}_{2\perp} \) respectively, and then averaging this amplitude over all possible values of \( \vec{R}_{1\perp} \) and \( \vec{R}_{2\perp} \) with two proper squared wave functions \(|\psi_1(\vec{R}_{1\perp})|^2\) and \(|\psi_2(\vec{R}_{2\perp})|^2\), describing the two interacting mesons.\(^5\) (For the treatment of baryons, a similar, but, of course, more involved, picture can be adopted, using a genuine three–body configuration or, alternatively and even more simply, a quark–diquark configuration: we refer the interested reader to the above–mentioned original references \[2, 3, 17, 18, 19\].)

The high–energy elastic scattering amplitude of two dipoles is governed by the (properly normalized) correlation function of two Wilson loops \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \), which follow the classical straight lines for quark (antiquark) trajectories:

\[
\mathcal{M}^{(\text{el})}(s, t; \vec{R}_{1\perp}, \vec{R}_{2\perp}) = -i 2s \int d^2 \vec{z} e^{iq_{\perp} \cdot \vec{z}} \left[ \frac{\langle \mathcal{W}_1 \mathcal{W}_2 \rangle}{\langle \mathcal{W}_1 \rangle \langle \mathcal{W}_2 \rangle} - 1 \right], \tag{2.22}
\]

where \( s \) and \( t = -|q_{\perp}|^2 \) (\( q_{\perp} \) being the transferred momentum) are the usual Mandelstam variables. More explicitly the Wilson loops \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) are so defined:

\[
\mathcal{W}_1^{(T)} = \frac{1}{N_c} \text{Tr} \left\{ \mathcal{P} \exp \left[ -ig \oint_{C_1} A_\mu(x) dx^\mu \right] \right\},
\]

\[
\mathcal{W}_2^{(T)} = \frac{1}{N_c} \text{Tr} \left\{ \mathcal{P} \exp \left[ -ig \oint_{C_2} A_\mu(x) dx^\mu \right] \right\}, \tag{2.23}
\]

where \( \mathcal{P} \) denotes the “path ordering” along the given path \( C \); \( C_1 \) and \( C_2 \) are two rectangular paths which follow the classical straight lines for quark \( [X_+](\tau) \), forward in proper time \( \tau \) and antiquark \( [X_-](\tau) \), backward in \( \tau \) trajectories, i.e.,

\[
C_1 \rightarrow X_\mu^{(\pm 1)}(\tau) = z^\mu + \frac{p_1^\mu}{m} \tau \pm \frac{R_1^\mu}{2},
\]

\[
C_2 \rightarrow X_\mu^{(\pm 2)}(\tau) = \frac{p_2^\mu}{m} \tau \pm \frac{R_2^\mu}{2}, \tag{2.24}
\]

\(^5\)Here and in what follows we take, for simplicity, the longitudinal–momentum fractions \( f_1 \) and \( f_2 \) of the two quarks in the two dipoles (and, therefore, also the longitudinal–momentum fractions \( 1 - f_1 \) and \( 1 - f_2 \) of the two antiquarks in the two dipoles) to be fixed to 1/2: this is known to be a good approximation for hadron–hadron interactions (see Refs. \[2, 3 \] and references therein). However, the dependence on the longitudinal–momentum fractions \( f_1 \) and \( f_2 \) could be easily implemented in the hadron wave functions \( \psi_1(\vec{R}_{1\perp}, f_1) \) and \( \psi_2(\vec{R}_{2\perp}, f_2) \) and in the loop–loop correlator itself (see again Refs. \[2, 3 \] and references therein for more details), without altering any relevant formula or conclusion in our paper.
and are closed by straight-line paths at proper times $\tau = \pm T$, where $T$ plays the role of an IR cutoff, which must be removed in the end ($T \to \infty$). Here $p_1$ and $p_2$ are the four-momenta of the two dipoles with mass $m$, moving with speed $V$ and $-V$ along, for example, the $x^1$–direction. Their expression is given by Eq. (2.3), where $\chi = 2 \arctanh V$ is the hyperbolic angle between the two trajectories (+1) and (+2). Moreover, $R_1 = (0, 0, \vec{R}_{1\perp})$, $R_2 = (0, 0, \vec{R}_{2\perp})$ and $z = (0, 0, \vec{z}_{\perp})$, where $\vec{z}_{\perp} = (z^2, z^3)$ is the impact-parameter distance between the two loops in the transverse plane.

In the Euclidean theory, one considers the correlation function of two Euclidean Wilson loops $\tilde{W}_1$ and $\tilde{W}_2$ running along two rectangular paths $\tilde{C}_1$ and $\tilde{C}_2$ which follow the following straight-line trajectories:

\begin{align*}
\tilde{C}_1 & \to X^{(\pm 1)}_{E \mu}(\tau) = z_{E \mu} + \frac{p_{1E \mu} \tau \pm R_{1E \mu}}{2}, \\
\tilde{C}_2 & \to X^{(\pm 2)}_{E \mu}(\tau) = \frac{p_{2E \mu} \tau \pm R_{2E \mu}}{2},
\end{align*}

(2.25)

and are closed by straight-line paths at proper times $\tau = \pm T$. Here $R_{1E} = (0, \vec{R}_{1\perp}, 0)$, $R_{2E} = (0, \vec{R}_{2\perp}, 0)$, $z_E = (0, \vec{z}_{\perp}, 0)$, and the Euclidean four-vectors $p_{1E}$ and $p_{2E}$ are defined by Eq. (2.14), where $\theta$ is the angle formed by the two trajectories (+1) and (+2) in Euclidean four-space.

Let us introduce the following notations for the normalized correlators $\langle W_1 W_2 \rangle / \langle W_1 \rangle \langle W_2 \rangle$ in the Minkowskian and in the Euclidean theory, in the presence of a finite IR cutoff $T$:

\begin{align*}
G_M(\chi; T; \vec{z}_{\perp}, \vec{R}_{1\perp}, \vec{R}_{2\perp}) & \equiv \frac{\langle W_1^{(T)} W_2^{(T)} \rangle}{\langle W_1^{(T)} \rangle \langle W_2^{(T)} \rangle}, \\
G_E(\theta; T; \vec{z}_{\perp}, \vec{R}_{1\perp}, \vec{R}_{2\perp}) & \equiv \frac{\langle \tilde{W}_1^{(T)} \tilde{W}_2^{(T)} \rangle_E}{\langle \tilde{W}_1^{(T)} \rangle_E \langle \tilde{W}_2^{(T)} \rangle_E}.
\end{align*}

(2.26)

As already stated in Ref. [15], and formally proved in Ref. [20], the two quantities in Eq. (2.26) are expected to be connected by the same analytic continuation in the angular variables and in the IR cutoff which was already derived in the case of Wilson lines; i.e., with analogous hypotheses of analyticity in the angular variables and in the IR cutoff $T$ and using the same notation already introduced for the line-line case:

\begin{align*}
\overline{G}_E(\theta; T; \vec{z}_{\perp}, \vec{R}_{1\perp}, \vec{R}_{2\perp}) & = \overline{G}_M(i\theta; iT; \vec{z}_{\perp}, \vec{R}_{1\perp}, \vec{R}_{2\perp}), \quad \forall \theta \in \mathcal{D}_E; \\
\overline{G}_M(\chi; T; \vec{z}_{\perp}, \vec{R}_{1\perp}, \vec{R}_{2\perp}) & = \overline{G}_E(-i\chi; iT; \vec{z}_{\perp}, \vec{R}_{1\perp}, \vec{R}_{2\perp}), \quad \forall \chi \in \mathcal{D}_M.
\end{align*}

(2.27)

The analytic continuation (2.27) (as the corresponding result for the line-line case) is an exact, i.e., nonperturbative result, valid both for the Abelian and the non-Abelian case.
As we have said above, the loop–loop correlation functions (2.26), both in the Minkowskian and in the Euclidean theory, are expected to be IR–finite quantities, i.e., to have finite limits when \( T \to \infty \), differently from what happens in the case of Wilson lines. One can then define the following loop–loop correlation functions with the IR cutoff removed:

\[
\begin{align*}
C_M(\chi; \bar z_\perp, \bar R_{1\perp}, \bar R_{2\perp}) & \equiv \lim_{T \to \infty} \left[ \mathcal{G}_M(\chi; T; \bar z_\perp, \bar R_{1\perp}, \bar R_{2\perp}) - 1 \right], \\
C_E(\theta; \bar z_\perp, \bar R_{1\perp}, \bar R_{2\perp}) & \equiv \lim_{T \to \infty} \left[ \mathcal{G}_E(\theta; T; \bar z_\perp, \bar R_{1\perp}, \bar R_{2\perp}) - 1 \right].
\end{align*}
\] (2.28)

It has been proved in Ref. [20] that, under certain analyticity conditions in the complex variable \( T \) [conditions which are also sufficient to make the relations (2.27) meaningful], the two quantities (2.28), obtained after the removal of the IR cutoff (\( T \to \infty \)), are still connected by the usual analytic continuation in the angular variables only:

\[
\begin{align*}
\mathcal{C}_E(\theta; \bar z_\perp, \bar R_{1\perp}, \bar R_{2\perp}) & = \mathcal{C}_M(i\theta; \bar z_\perp, \bar R_{1\perp}, \bar R_{2\perp}), \quad \forall \theta \in \mathcal{D}_E; \\
\mathcal{C}_M(\chi; \bar z_\perp, \bar R_{1\perp}, \bar R_{2\perp}) & = \mathcal{C}_E(-i\chi; \bar z_\perp, \bar R_{1\perp}, \bar R_{2\perp}), \quad \forall \chi \in \mathcal{D}_M.
\end{align*}
\] (2.29)

This is a highly non–trivial result, whose general validity is discussed in Ref. [20].

As said in Ref. [20], if \( \mathcal{G}_M \) and \( \mathcal{G}_E \), considered as functions of the complex variable \( T \), have in \( T = \infty \) an “eliminable isolated singular point” [i.e., they are analytic functions of \( T \) in the complex region \( |T| > R \), for some \( R \in \mathbb{R}^+ \), and the finite limits (2.28) exist when letting the complex variable \( T \to \infty \)], then, of course, the analytic continuation (2.29) immediately derives from Eq. (2.27) (with \( |T| > R \)), when letting \( T \to +\infty \). But the same result (2.29) can also be derived under different conditions. For example, let us assume that \( \mathcal{G}_E \) is a bounded analytic function of \( T \) in the sector \( 0 \leq \text{arg} \ T \leq \frac{\pi}{2} \), with finite limits along the two straight lines on the border of the sector: \( \mathcal{G}_E \to \mathcal{G}_{E1} \), for \( (\Re T \to +\infty, \Im T = 0) \), and \( \mathcal{G}_E \to \mathcal{G}_{E2} \), for \( (\Re T = 0, \Im T \to +\infty) \). And, similarly, let us assume that \( \mathcal{G}_M \) is a bounded analytic function of \( T \) in the sector \( -\frac{\pi}{2} \leq \text{arg} \ T \leq 0 \), with finite limits along the two straight lines on the border of the sector: \( \mathcal{G}_M \to \mathcal{G}_{M1} \), for \( (\Re T \to +\infty, \Im T = 0) \), and \( \mathcal{G}_M \to \mathcal{G}_{M2} \), for \( (\Re T = 0, \Im T \to -\infty) \). We can then apply the “Phragmén–Lindelöf theorem” (see, e.g., Theorem 5.64 in Ref. [21]) to state that \( \mathcal{G}_{E2} = \mathcal{G}_{E1} \) and \( \mathcal{G}_{M2} = \mathcal{G}_{M1} \). Therefore, also in this case, the analytic continuation (2.29) immediately derives from Eq. (2.27) when \( T \to \infty \).

\footnote{For example, if \( \mathcal{G}_M \) and \( \mathcal{G}_E \) are analytic functions of \( T \) in the complex region \( |T| > R \), for some \( R \in \mathbb{R}^+ \), and they are bounded at large \( T \), i.e., \( \exists B_{M,E} \in \mathbb{R}^+ \) such that \( |\mathcal{G}_{M,E}(T)| < B_{M,E} \) for \( |T| > R \), then \( T = \infty \) is an “eliminable singular point” for both of them.}
3. Analyticity and crossing symmetry

In this section we will show how the analytic–continuation relations from the Minkowskian to the Euclidean theory lead to a nice geometrical interpretation of the so–called crossing symmetry between the quark–quark and quark–antiquark scattering amplitudes (and also between dipole–dipole scattering amplitudes) in the eikonal approximation.

In such an approximation the scattering amplitudes factorize in a product of Kronecker’s deltas in the spin variables, expressing spin conservation at high energies, and in a term that is essentially the (normalized) correlator of two Wilson lines in the appropriate representation. According to the results found in [18] and [13], changing from a quark to an antiquark just corresponds, in our formalism, to substitute the corresponding Wilson line (in the fundamental representation) with its complex conjugate (i.e., the Wilson line in the complex conjugate representation, $T_a = -\bar{T}_a$). Therefore, the eikonal amplitude for the soft elastic scattering of a quark $q$ and an antiquark $\bar{q}$ with given spin and colour quantum numbers,

$$q(p_1, \alpha, i) + \bar{q}(p_2, \beta, j) \rightarrow q(p_1' \simeq p_1, \alpha', i') + \bar{q}(p_2' \simeq p_2, \beta', j'),$$  \hspace{1cm} (3.1)

where the particles four–momenta $p_1' \simeq p_1$ and $p_2' \simeq p_2$ are defined in Eq. (2.3), is given by the formula:

$$\mathcal{M}^{q\bar{q}}(p_1, p_2; t)_{ij'i''j''} \sim -i 2 s \delta_{\alpha'\alpha} \delta_{\beta'\beta} g_M^{q\bar{q}}(p_1, p_2; T \rightarrow \infty; t)_{ij'i''j''},$$  \hspace{1cm} (3.2)

where the correlator $g_M^{q\bar{q}}(p_1, p_2; T; t)_{ij'i''j''}$ is defined as:

$$g_M^{q\bar{q}}(p_1, p_2; T; t)_{ij'i''j''} = \frac{1}{[Z_M(T)]^2} \int d^2 \vec{z}_\perp e^{i \vec{q}_\perp \cdot \vec{z}_\perp} \langle [W_{p_1}^{(T)}(\vec{z}_\perp)] - \mathbb{I}_{i'i'} [W_{p_2}^{(T)}(\vec{0}_\perp)] - \mathbb{I}_{j'j'} \rangle. \hspace{1cm} (3.3)$$

Crossing symmetry relates the amplitude of this process to the amplitude of the “crossed” process, defined as:

$$q(p_1, \alpha, i) + q(-p_2' \simeq -p_2, \beta', j') \rightarrow q(p_1' \simeq p_1, \alpha', i') + q(-p_2, \beta, j).$$  \hspace{1cm} (3.4)

*The Wilson–line renormalization constants in the complex conjugate and in the fundamental representations are equal because of the invariance of the functional integral under charge conjugation of the gluon fields, i.e., $A_\mu \rightarrow A_\mu' = -A_\mu^T = -A_\mu^*; \frac{1}{N_c} \langle \text{Tr}[W_{p_1}^{(T)}(\vec{0}_\perp)] \rangle = \frac{1}{N_c} \langle \text{Tr}[W_{p_2}^{(T)}(\vec{0}_\perp)] \rangle = Z_M(T).$
Using the fact that the generators $T_a$ are hermitian and the variables $A^a_\mu$ are real, the complex conjugate Wilson line $W_{p_2}^{(T)*}(\vec{0}_\perp)$ corresponding to the antiquark can also be re-written as follows:

\[
\left[ W_{p_2}^{(T)*}(\vec{0}_\perp) \right]_{j'j} = \left[ \mathcal{T} \exp \left( ig \int_{-T}^{+T} A^*_\mu \left( \frac{p_2}{m} \right) \frac{p_\mu}{m} d\tau \right) \right]_{j'j} \\
= \sum_{n=0}^{\infty} \int_{-T}^{+T} d\tau_1 \ldots \int_{-T}^{+T} d\tau_n \theta(\tau_1 - \tau_2) \ldots \theta(\tau_n - 1 - \tau_1) \\
\times \left\{ \left[ ig A^*_\mu_1 \left( \frac{p_2}{m} \tau_1 \right) \frac{p^\mu_1}{m} \right] \ldots \left[ ig A^*_\mu_n \left( \frac{p_2}{m} \tau_n \right) \frac{p^\mu_n}{m} \right] \right\}_{j'j} \\
= \sum_{n=0}^{\infty} \int_{-T}^{+T} d\tau_1 \ldots \int_{-T}^{+T} d\tau_n \theta(\tau_1 - \tau_2) \ldots \theta(\tau_n - 1 - \tau_1) \\
\times \left\{ \left[ ig A^*_\mu_1 \left( \frac{p_2}{m} \tau_1 \right) \frac{p^\mu_1}{m} \right] \ldots \left[ ig A^*_\mu_1 \left( \frac{p_2}{m} \tau_1 \right) \frac{p^\mu_1}{m} \right] \right\}_{jj'} \\
= [\mathcal{T} \exp \left( ig \int_{-T}^{+T} A^*_\mu \left( \frac{p_2}{m} \frac{p^\mu}{m} d\tau \right) \right)]_{jj'} = \left[ W_{p_2}^{(T)}(\vec{0}_\perp) \right]_{jj'}, \tag{3.5}
\]

where $\mathcal{T}$ exp$(\ldots)$ is the “anti $\mathcal{T}$–ordered” exponential. Replacing the integration variables $\tau_i = -\tau'_i$ in the last expression we immediately get:

\[
\left[ W_{p_2}^{(T)*}(\vec{0}_\perp) \right]_{jj} = \left[ W_{p_2}^{(T)}(\vec{0}_\perp) \right]_{jj'} \\
= \sum_{n=0}^{\infty} \int_{-T}^{+T} d\tau'_1 \ldots \int_{-T}^{+T} d\tau'_n \theta(\tau'_n - \tau'_n - 1) \ldots \theta(\tau'_1 - \tau'_1) \\
\times \left\{ \left[ -ig A^*_\mu \left( -\frac{p_2}{m} \tau'_n \right) \left( -\frac{p^\mu}{m} \right) \right] \ldots \left[ -ig A^*_\mu_1 \left( -\frac{p_2}{m} \tau'_1 \right) \left( -\frac{p^\mu_1}{m} \right) \right] \right\}_{jj'} \\
= \left[ W_{-p_2}^{(T)}(\vec{0}_\perp) \right]_{jj'}. \tag{3.6}
\]

Summarizing:

\[
\left[ W_{p_2}^{(T)*}(\vec{0}_\perp) \right]_{jj} = \left[ W_{p_2}^{(T)}(\vec{0}_\perp) \right]_{jj'} = \left[ W_{-p_2}^{(T)}(\vec{0}_\perp) \right]_{jj'}. \tag{3.7}
\]

We can now write the correlator $g_{M}^{qq}$ in the form:

\[
g_{M}^{qq}(p_1, p_2; T; t) \equiv \frac{1}{|Z_M(T)|^2} \int d^2 z_\perp e^{i\vec{z}_\perp \cdot \vec{z}_\perp} \langle [W_{p_1}^{(T)}(z_\perp) - \mathbb{1}]_{j'j} [W_{-p_2}^{(T)}(\vec{0}_\perp) - \mathbb{1}]_{jj'} \rangle; \tag{3.8}
\]
that is, reminding the definition of the quark–quark correlator:

$$g_{q\bar{q}}^M(p_1, p_2; T; t)_{i', i}^{i, j'} = g_{q\bar{q}}^M(-p_1, -p_2; T; t)_{i', i}^{i, j'}.$$  \hspace{1cm} (3.9)

This relation is the direct expression of crossing symmetry, once we have formulated the appropriate analyticity conditions on \( g_{q\bar{q}}^M \) as a function of the four–momenta making the right–hand side meaningful, and is valid for every value of the IR cutoff \( T \).

We want to give now a “geometrical” interpretation of this relation, expressing it in terms of the hyperbolic angle \( \chi \) between the four–momenta \( p_1 \) and \( p_2 \); using this interpretation we will be able to discuss in details the analyticity hypotheses on \( g_{q\bar{q}}^M \) and \( g_{q\bar{q}}^M \) that make the relation (3.9) meaningful.

We shall denote the left–hand side of (3.9) (with a slight abuse of notation) also as \( g_{q\bar{q}}^M(\chi; T; t)_{i', i}^{i, j'} \); in the right–hand side we have instead the function \( g_{q\bar{q}}^M(p_1, \tilde{p}_2; T; t)_{i', i}^{i, j'} \) calculated at four–momenta \( p_1 \) and \( \tilde{p}_2 = -p_2 \); the substitution of \( p_2 \) with the (unphysical) four–momentum \( \tilde{p}_2 \) corresponds to the substitution \( \cosh \chi \to -\cosh \chi \). To determine unambiguously which complex values of \( \chi \) this substitution corresponds to, we will make use of the analytic–continuation relation between the Minkowskian and the Euclidean theory and of the \( O(4) \) symmetry of the latter.

The relation (3.7) is evidently valid also for Euclidean Wilson lines, i.e.,

$$\left[ \tilde{W}_{p_{2E}}^{(T)}(\vec{0}_\perp) \right]_{j' j} = \left[ \tilde{W}_{\tilde{p}_{2E}}^{(T)}(\vec{0}_\perp) \right]_{j' j}, \hspace{1cm} (3.10)$$

and so relation (3.9) is extended to the Euclidean case:

$$g_{q\bar{q}}^E(p_{1E}, p_{2E}; T; t)_{i', i}^{i, j'} = g_{q\bar{q}}^E(-p_{1E}, -p_{2E}; T; t)_{i', i}^{i, j'}, \hspace{1cm} (3.11)$$

where the Euclidean four–momenta \( p_{1E} \) and \( p_{2E} \) are given by Eq. (2.14). In our notation the left–hand side of (3.11) is denoted as \( g_{q\bar{q}}^E(\theta; T; t)_{i', i}^{i, j'} \), where \( \theta \) is the angle between the Euclidean four–momenta \( p_{1E} \) and \( p_{2E} \). The right–hand side can be written as \( g_{q\bar{q}}^E(\pi + \theta; T; t)_{i', i}^{i, j'} \), using the invariance under the \( O(4) \) 90° clockwise “rotation” in the \((x_{E1}, x_{E4})\) plane:

$$x_E \to x'_E = \mathcal{R}_3 x_E, \hspace{1cm} \mathcal{R}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \hspace{1cm} (3.12)$$

13
and also, using the relation (2.19), as $g_{E}^{qg}(\pi - \theta; T; t)_{i'j'j'}$. In this way relation (3.11) takes the form (see Fig. 2):

$$g_{E}^{qg}(\theta; T; t)_{i'j'j'} = g_{E}^{qg}(\pi - \theta; T; t)_{i'j'j'}, \quad \forall \theta \in \mathbb{R}. \quad (3.13)$$

For $\theta \in (0, \pi)$ the two functions $g_{E}^{qg}$ and $g_{E}^{qg}$ are calculated in points belonging to the respective analyticity domains. Suppose now that the relation (3.13) can be analytically extended to complex values of $\theta$ in a common analyticity domain $D_{E}$ for $\overline{g}_{E}^{qg}$ and $\overline{g}_{E}^{qg}$ (and for every value of the complex variable $T$ in an appropriate analyticity domain). This domain must then have the property that, if $\theta \in D_{E}$, then also $\pi - \theta \in D_{E}$, i.e., it has to be symmetric with respect to the point $\theta_{0} = (\text{Re}\theta_{0} = \pi/2, \text{Im}\theta_{0} = 0)$, and it has to include the segment $(0 < \text{Re}\theta < \pi, \text{Im}\theta = 0)$, the negative imaginary axis $(\text{Re}\theta = 0, \text{Im}\theta < 0)$ and the semiaxis $(\text{Re}\theta = \pi, \text{Im}\theta > 0)$: it is schematically shown in Fig. 3.

Using the notation previously introduced, we write:

$$\overline{g}_{E}^{qg}(\theta; T; t)_{i'j'j'} = \overline{g}_{E}^{qg}(\pi - \theta; T; t)_{i'j'j'}, \quad \forall \theta \in D_{E}. \quad (3.14)$$

Now, using repeatedly the analytic–continuation relations (2.21), we get the following relation between the Minkowskian correlators:

$$\overline{g}_{M}^{qg}(\chi; T; t)_{i'j'j'} = \overline{g}_{E}^{qg}(-i\chi; iT; t)_{i'j'j'} = \overline{g}_{E}^{qg}(\pi + i\chi; iT; t)_{i'j'j'} = \overline{g}_{E}^{qg}(i\pi - \chi; T; t)_{i'j'j'}, \quad \forall \chi \in D_{M}, \quad (3.15)$$

where $D_{M} = \{\chi \in \mathbb{C} | -i\chi \in D_{E}\}$ is the common analyticity domain of $\overline{g}_{M}^{qg}$ and $\overline{g}_{M}^{qg}$, with the property that, if $\chi \in D_{M}$, then also $i\pi - \chi \in D_{M}$, i.e., it is symmetric with respect to the point $\chi_{0} = (\text{Re}\chi_{0} = 0, \text{Im}\chi_{0} = \pi/2)$, and it includes the real positive axis, $(\text{Re}\chi > 0, \text{Im}\chi = 0)$, the imaginary segment $(\text{Re}\chi = 0, 0 < \text{Im}\chi < \pi)$ and the semiaxis $(\text{Re}\chi < 0, \text{Im}\chi = \pi)$: it is schematically shown in Fig. 4.

In particular, for $\chi \in \mathbb{R}^{+}$ we have:

$$g_{M}^{qg}(\chi; T; t)_{i'j'j'} = \overline{g}_{M}^{qg}(i\pi - \chi; T; t)_{i'j'j'}, \quad \forall \chi \in \mathbb{R}^{+}. \quad (3.16)$$

This is the “geometrical transcription” in terms of the angular variable $\chi$ of relation (3.9), and states that the quark–antiquark correlator can be obtained from the quark–quark one.
by the analytic–continuation $\chi \rightarrow i\pi - \chi$ in the hyperbolic angle and by the colour index exchange $j \leftrightarrow j'$.

Reminding the relation (2.4) between the Mandelstam variable $s$ and the hyperbolic angle $\chi$, we see that the substitution $\chi \rightarrow i\pi - \chi$ corresponds, taking the limit $\chi \rightarrow \infty$, to the substitution

$$ s \rightarrow e^{-i\pi} s, \quad (3.17) $$

while the Mandelstam variable $t$ doesn’t change going over to the crossed channel. This is in agreement with what we expect from crossing symmetry: the exchange $p_2 \leftrightarrow -p'_2$ implies the exchange $s = (p_1 + p_2)^2 \leftrightarrow u = (p_1 - p'_2)^2$, while $t = (p_1 - p'_1)^2$ remains unchanged; moreover, in our limit, because of the relation $s + t + u = 4m^2$, we have $u \simeq -s$.

In a perfectly analogous way we can obtain a crossing–symmetry relation for loop–loop correlators. Let us consider a certain Wilson loop

$$ \mathcal{W}_p^{(T)}(\vec{b}_\perp, \vec{R}_\perp) = \frac{1}{N_c} \text{Tr} \left\{ \mathcal{P} \exp \left[ -ig \oint_{C(p,b,R)} A_\mu(x)dx^\mu \right] \right\}, \quad (3.18) $$

defined on the rectangular path $C(p,b,R)$, consisting of the straight–line trajectories $[b = (0,0,\vec{b}_\perp), R = (0,0,\vec{R}_\perp)]$

$$ X_\mu^{(\pm)}(\tau) = b_\mu + \frac{p_\mu}{m} \tau \pm \frac{R_\mu}{2} \quad (3.19) $$

doing the quark $[X_\mu^{(+))(\tau)}$, with $\tau$ going from $-T$ to $+T]$ and of the antiquark $[X_\mu^{(-)}(\tau)$, with $\tau$ going from $+T$ to $-T]$, joined by straight–line paths at $\tau = \pm T$ ("links"), so making the loop a gauge invariant operator. Let us define the corresponding *antiloop* $\overline{\mathcal{W}}$ by exchanging the quark and the antiquark trajectories (and reversing the links direction in order to preserve gauge invariance). Clearly this corresponds to reverse the direction of the path of the initial loop $\mathcal{W}$, i.e., to make the substitution $p \rightarrow -p$:

$$ \overline{\mathcal{W}}_p^{(T)}(\vec{b}_\perp, \vec{R}_\perp) = \frac{1}{N_c} \text{Tr} \left\{ \mathcal{P} \exp \left[ -ig \oint_{C(p,b,R)} A_\mu(x)dx^\mu \right] \right\}, \quad (3.20) $$

†In Ref. [28] the crossing–symmetry relation for line–line correlators was instead identified with $\chi \rightarrow \chi - i\pi$. The correct relation $\chi \rightarrow i\pi - \chi$ has been guessed, but not properly justified, in Ref. [11], apparently only on the basis of the correspondence rule $\cosh \chi \rightarrow -\cosh \chi$, which however, as we have already said, cannot unambiguously fix the correspondence rule for the hyperbolic angle $\chi$ alone.
where:
\[ \mathcal{C}(p, b, R) = \mathcal{C}(-p, b, R). \] (3.21)

Evidently, the transition from a loop to the corresponding antiloop can also be made by keeping \( p \) fixed and substituting \( R \to -R \). Consequently:
\[ \mathcal{C}(p, b, R) = \mathcal{C}(-p, b, R) = \mathcal{C}(p, b, -R), \] (3.22)

and:
\[ \mathcal{W}^{(T)}_p(\vec{b}_\perp, \vec{R}_\perp) = \mathcal{W}^{(T)}_{-p}(\vec{b}_\perp, \vec{R}_\perp) = \mathcal{W}^{(T)}_p(\vec{b}_\perp, -\vec{R}_\perp). \] (3.23)

Let us define the loop–antiloop correlator \( G^{(l)}_M \) substituting in the loop–loop correlator \( G_M \) the loop \( \mathcal{W}_2 \) with the corresponding antiloop:
\[ G^{(l)}_M(\chi; T; \vec{z}_\perp, \vec{R}_1\perp, \vec{R}_2\perp) = \frac{\langle \mathcal{W}^{(T)}_1 \mathcal{W}^{(T)}_2 \rangle}{\langle \mathcal{W}^{(T)}_1 \rangle \langle \mathcal{W}^{(T)}_2 \rangle}. \] (3.24)

Going on as we have done in the line–line case, we immediately verify that the first equality in (3.23) leads to the crossing–symmetry relation:
\[ G^{(l)}_M(\chi; T; \vec{z}_\perp, \vec{R}_1\perp, \vec{R}_2\perp) = G_M(i\pi - \chi; T; \vec{z}_\perp, \vec{R}_1\perp, \vec{R}_2\perp), \quad \forall \chi \in \mathbb{R}^+. \] (3.25)

As before, it is derived from the Euclidean space relation obtained from the Euclidean version of (3.23), i.e.,
\[ G^{(l)}_E(\theta; T; \vec{z}_\perp, \vec{R}_1\perp, \vec{R}_2\perp) = G_E(\pi - \theta; T; \vec{z}_\perp, \vec{R}_1\perp, \vec{R}_2\perp), \quad \forall \theta \in \mathbb{R}, \] (3.26)

with appropriate analyticity hypotheses on \( G_E \) as a function of the angular variable \( \theta \) (or on \( G_M \) as a function of the angular variable \( \chi \)), completely analogous to the hypotheses made in the line–line case. Moreover, the second equality in (3.23) implies that:
\[ G^{(l)}_M(\chi; T; \vec{z}_\perp, \vec{R}_1\perp, \vec{R}_2\perp) = G_M(\chi; T; \vec{z}_\perp, \vec{R}_1\perp, -\vec{R}_2\perp); \] (3.27)

\(^1\)Also in this case the charge–conjugation invariance (or, more simply, the rotation invariance) imposes that the vacuum expectation values of the loop and the antiloop are equal: \( \langle \mathcal{W} \rangle = \langle \mathcal{W} \rangle \).
and, in the Euclidean case:

$$G_E^{(l)}(\theta; T; \vec{z}_\perp, \vec{R}_1\perp, -\vec{R}_2\perp) = G_E(\theta; T; \vec{z}_\perp, \vec{R}_1\perp, -\vec{R}_2\perp). \quad (3.28)$$

These two relations, together with the relations (3.25) and (3.26) found above, allow us to deduce non trivial properties of the Minkowskian correlator $G_M$ under the exchange $\chi \to i\pi - \chi$ and of the Euclidean correlator $G_E$ under the exchange $\theta \to \pi - \theta$. In the Minkowskian case:

$$G_M(i\pi - \chi; T; \vec{z}_\perp, \vec{R}_1\perp, \vec{R}_2\perp) = G_M(\chi; T; \vec{z}_\perp, -\vec{R}_1\perp, \vec{R}_2\perp), \quad \forall \chi \in \mathbb{R}^+; \quad (3.29)$$

while, in the Euclidean case:

$$G_E(\pi - \theta; T; \vec{z}_\perp, \vec{R}_1\perp, \vec{R}_2\perp) = G_E(\theta; T; \vec{z}_\perp, -\vec{R}_1\perp, \vec{R}_2\perp), \quad \forall \theta \in \mathbb{R}. \quad (3.30)$$

[The last two equalities in (3.29) and (3.30) are obtained considering the exchange $\mathbf{W}_1 \to -\mathbf{W}_1$ instead of $\mathbf{W}_2 \to -\mathbf{W}_2$. These two relations are valid for every value of the IR cutoff $T$ and so completely analogous relations also holds for the loop–loop correlation functions $C_M$ and $C_E$ with the IR cutoff removed ($T \to \infty$), defined in Eq. (2.28).

4. Perturbative expansion of the eikonal amplitudes

As the exact (i.e., nonperturbative) calculation from first principles of the line–line and loop–loop correlators is beyond our possibilities (but see also the discussion in section 5), we cannot verify directly if they satisfy the desired analyticity conditions. A way to study the analytic structure of such correlators is to use perturbation theory. Perturbation theory is the only calculation technique from first principles available both in the Minkowskian and in the Euclidean theory, and although the properties of the perturbative series to any given order do not allow us to get conclusive results, they can however give us some useful insights about the analytic structure of the real (nonperturbative) correlation functions. Let us start considering the loop–loop correlation functions.
As a pedagogic example to illustrate these considerations, we shall first consider the simple case of QED, in the so-called quenched approximation, where vacuum polarization effects, arising from the presence of loops of dynamical fermions, are neglected: this amounts to putting the fermion–matrix determinant equal to 1, i.e., \( \det(Q[A]) = 1 \) in Eq. (2.7) and \( \det(Q^{(E)}[A^{(E)}]) = 1 \) in Eq. (2.13). In such an approximation the functional integrals become simple Gaussian integrals and therefore the calculation of the normalized loop–loop correlators (2.20) can be performed exactly (i.e., nonpertubatively) both in Minkowskian and in Euclidean theory. One finds [20] that i) the two quantities \( G_M \) and \( G_E \) are indeed connected by the analytic continuation (2.27), and ii) the two quantities are finite in the limit when the IR cutoff \( T \) goes to infinity, the two limits (2.28) being:

\[
C_M(\chi; \vec{z}_1, \vec{R}_{1\perp}, \vec{R}_{2\perp}) = \exp \left[ -i4e^2 \coth \chi \ t(\vec{z}_1, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \right] - 1, \\
C_E(\theta; \vec{z}_1, \vec{R}_{1\perp}, \vec{R}_{2\perp}) = \exp \left[ -4e^2 \frac{\cos \theta}{\sin \theta} \ t(\vec{z}_1, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \right] - 1,
\]

(4.1)

where the coupling constant is now the electric charge \( e \) and

\[
t(\vec{z}_1, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \equiv \int \frac{d^2k_\perp e^{-i\vec{k}_\perp \cdot \vec{z}_\perp}}{(2\pi)^2} \sin \left( \frac{k_\perp \cdot \vec{R}_{1\perp}}{2} \right) \sin \left( \frac{k_\perp \cdot \vec{R}_{2\perp}}{2} \right) \\
= \frac{1}{8\pi} \log \left( \frac{|\vec{z}_1 + \vec{R}_{1\perp} + \vec{R}_{2\perp}|}{|\vec{z}_1 + \vec{R}_{1\perp} - \vec{R}_{2\perp}|} \frac{|\vec{z}_1 - \vec{R}_{1\perp} - \vec{R}_{2\perp}|}{|\vec{z}_1 - \vec{R}_{1\perp} + \vec{R}_{2\perp}|} \right).
\]

(4.2)

One immediately sees that the analytic extension \( \overline{C}_M \) of the Minkowskian correlator from the positive real axis \( \chi \in \mathbb{R}^+ \) and the analytic extension \( \overline{C}_E \) of the Euclidean correlator from the real segment \( \theta \in (0, \pi) \) are given by:

\[
\overline{C}_M(\chi; \vec{z}_1, \vec{R}_{1\perp}, \vec{R}_{2\perp}) = \exp \left[ -i4e^2 \coth \chi \ t(\vec{z}_1, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \right] - 1, \quad \forall \chi \in \mathcal{D}_M; \\
\overline{C}_E(\theta; \vec{z}_1, \vec{R}_{1\perp}, \vec{R}_{2\perp}) = \exp \left[ -4e^2 \cot \theta \ t(\vec{z}_1, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \right] - 1, \quad \forall \theta \in \mathcal{D}_E.
\]

(4.3)

The analyticity domain \( \mathcal{D}_M \) of \( \overline{C}_M \) in the complex variable \( \chi \) is the entire complex plane with the exception of the points \( ik\pi, \ k \in \mathbb{Z} \), and, equivalently, the analyticity domain \( \mathcal{D}_E \) of \( \overline{C}_E \) in the complex variable \( \theta \) is the entire complex plane with the exception of the points \( k\pi, \ k \in \mathbb{Z} \), i.e.,

\[
\mathcal{D}_M = \{ \chi \in \mathbb{C} | \chi \neq ik\pi, \ k \in \mathbb{Z} \}; \\
\mathcal{D}_E = \{ \theta \in \mathbb{C} | i\theta \in \mathcal{D}_M \} = \{ \theta \in \mathbb{C} | \theta \neq k\pi, \ k \in \mathbb{Z} \}.
\]

(4.4)
These domains have precisely the characteristics, described in the previous sections, which are sufficient to guarantee both the analytic–continuation relations (2.29) and the crossing–symmetry relations (3.29) and (3.30), with $T \to \infty$. And these relations are indeed realized by the explicit expressions (4.3). (Also the presence of the singularities for $\chi = ik\pi$, $k \in \mathbb{Z}$, or, equivalently, for $\theta = k\pi$, $k \in \mathbb{Z}$, are not unexpected and they are discussed in appendix B.)

As shown in Ref. [20], the results (4.1) can be used to derive the corresponding results in the case of a non–Abelian gauge theory with $N_c$ colours, up to the order $O(g^4)$ in perturbation theory (see also Refs. [6, 22, 23]):

$$C_M(\chi; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \mid_{g^4} = -2g^4 \left( \frac{N_c^2 - 1}{N_c^2} \right) \coth^2 \chi \left[ t(\vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \right]^2,$$

$$C_E(\theta; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \mid_{g^4} = 2g^4 \left( \frac{N_c^2 - 1}{N_c^2} \right) \cot^2 \theta \left[ t(\vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \right]^2. \quad (4.5)$$

In this case, obviously, these are also the expressions for the analytic extension $\mathcal{C}_M$ from the positive real axis $\chi \in \mathbb{R}^+$ and the analytic extension $\mathcal{C}_E$ from the real segment $\theta \in (0, \pi)$, with analyticity domains $\mathcal{D}_M$ and $\mathcal{D}_E$ still given by Eq. (4.4). Both the analytic–continuation relations (2.29) and the crossing–symmetry relations (3.29) and (3.30), with $T \to \infty$, are of course trivially satisfied. (Indeed, the validity of the relation (2.29) for the loop–loop correlators has been also recently verified in Ref. [6] by an explicit calculation up to the order $O(g^6)$ in perturbation theory.)

In the case of the line–line correlators we cannot simply remove the IR cutoff $T$, as we have done in Eq. (2.28) for the loop–loop case, since the limits $T \to \infty$ are divergent. Nevertheless, we can remove the IR cutoff $T$, by letting $T \to \infty$, provided that another IR cutoff $\lambda$ has been introduced to regularize the line–line correlators. This is exactly what one usually does when one computes the correlators in perturbation theory by giving a small mass $\lambda$ to the gluons (or photons) exchanged in each graph. In this way one can define the new IR–regularized line–line correlators $g_M^{(\lambda)}$ and $g_E^{(\lambda)}$ by removing the nonperturbative IR cutoff $T$ ($T \to \infty$), while keeping the perturbative IR cutoff $\lambda$ fixed, i.e.:

$$g_M^{(\lambda)}(\chi; t) \equiv \lim_{T \to \infty} g_M^{(\lambda)}(\chi; T; t),$$

$$g_E^{(\lambda)}(\theta; t) \equiv \lim_{T \to \infty} g_E^{(\lambda)}(\theta; T; t). \quad (4.6)$$

Then we can repeat what we have said and done above, at the end of section 2, for the
loop–loop correlators and thus conclude that, under certain analogous analyticity conditions in the complex variable $T$ for the two IR–regularized line–line correlators $g_M^{(\lambda)}(\chi; T; t)$ and $g_E^{(\lambda)}(\theta; T; t)$, the two quantities (4.6), obtained after the removal of the nonperturbative IR cutoff $T$, are still connected by the usual analytic continuation in the angular variables only:

$$g_E^{(\lambda)}(\theta; t) = g_M^{(\lambda)}(i\theta; t), \quad \forall \theta \in \mathcal{D}_E;$$
$$g_M^{(\lambda)}(\chi; t) = g_E^{(\lambda)}(-i\chi; t), \quad \forall \chi \in \mathcal{D}_M. \quad (4.7)$$

For example, in quenched QED the calculation gives, in the Minkowskian and in the Euclidean case respectively, the following results for the fermion–fermion correlation functions in the Feynman gauge (where the gauge–fixing parameter $\alpha$ is put equal to 1∗), whenever $\vec{q}_\perp \neq \vec{0}_\perp$ (i.e., $t = -|\vec{q}_\perp|^2 < 0$) [4]:

$$g_{ff}^{(\lambda)}(\chi; t) = \int d^2\vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} \exp \left[ -ie^2 |\coth \chi| \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\vec{k}_\perp \cdot \vec{z}_\perp} \frac{1}{k_\perp^2 + \lambda^2} \right],$$
$$g_{ff}^{(\lambda)}(\theta; t) = \int d^2\vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} \exp \left[ -e^2 \cos \theta \ |\sin \theta| \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\vec{k}_\perp \cdot \vec{z}_\perp} \frac{1}{k_\perp^2 + \lambda^2} \right]. \quad (4.8)$$

For obtaining the fermion–antifermion correlation function it is clearly sufficient to exchange $e^2$ with $-e^2$ in the fermion–fermion correlator, getting:

$$g_{\bar{f}f}^{(\lambda)}(\chi; t) = \int d^2\vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} \exp \left[ ie^2 |\coth \chi| \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\vec{k}_\perp \cdot \vec{z}_\perp} \frac{1}{k_\perp^2 + \lambda^2} \right],$$
$$g_{\bar{f}f}^{(\lambda)}(\theta; t) = \int d^2\vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} \exp \left[ e^2 \cos \theta \ |\sin \theta| \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\vec{k}_\perp \cdot \vec{z}_\perp} \frac{1}{k_\perp^2 + \lambda^2} \right]. \quad (4.9)$$

These correlators, seen as functions of the complex angular variables $\chi$ (Minkowskian) and $\theta$ (Euclidean), have the same analytic structure of the loop–loop correlators discussed above. In fact, both $g_{\bar{f}f}^{(\lambda)}$ and $g_{ff}^{(\lambda)}$ can be analytically extended from the positive real

---

*The free photon propagator in the generic $\alpha$–gauge, also including the IR cutoff $\lambda$ in the form of a photon mass, is given by:

$$\tilde{P}_{\mu\nu}(k) = -i \left( g_{\mu\nu} - (1 - \alpha) \frac{k_{\mu}k_{\nu}}{k^2 - \alpha \lambda^2 + i\varepsilon} \right) \frac{1}{k^2 - \lambda^2 + i\varepsilon}.$$
axis $\chi \in \mathbb{R}^+$ to the same domain $\mathcal{D}_M$ defined in Eq. (4.4) and, similarly, both $g_{f\bar{f}}^f$ and $g_{f\bar{f}}^E$ can be analytically extended from the real segment $\theta \in (0, \pi)$ to the same domain $\mathcal{D}_E$ defined in Eq. (4.4). The analytic extensions $\mathcal{G}_M^f$, $\mathcal{G}_M^{f\bar{f}}$, $\mathcal{G}_E^f$ and $\mathcal{G}_E^{f\bar{f}}$ are obtained from the expressions (4.8) and (4.9) by the simple substitutions $|\coth \chi| \to \coth \chi$ and $\cos \theta/|\sin \theta| \to \cot \theta$. The analytic–continuation relations (4.7) are trivially satisfied and so is the crossing–symmetry relation (3.15) or (3.16):

$$g_{f\bar{f}}^f(\chi; t)^{(\lambda)} = \mathcal{G}_M^f(i\pi - \chi; t)^{(\lambda)}, \quad \forall \chi \in \mathbb{R}^+.$$ (4.10)

Let us now address the more interesting and surely more complicated question of the computation of the line–line correlators in QCD perturbation theory.

The perturbative calculation of the quark–antiquark Minkowskian correlator is completely analogous to the quark–quark one: as we shall see below, it comes out that the contribution of every Feynman graph is the same as in the quark–quark calculation, with the only difference that the colour factor coming from the second Wilson line has to be changed according to a simple crossing rule. In fact, when expanding each Wilson line in the numerator of the correlation function $g_{f\bar{f}}^q$ in powers of $g$ (according to the definition of the time–ordered exponential), one finds:

$$g_{f\bar{f}}^q(\chi; T; t)_{i'j'} - g_{f\bar{f}}^q(\chi; T; t)_{i'j'} = \frac{1}{[Z_M(T)]^2} \sum_{r,s=1}^{\infty} (-ig)^{r+s} \frac{p_1^{\mu_1}}{m} \cdots \frac{p_1^{\mu_r}}{m} \frac{p_2^{\nu_1}}{m} \cdots \frac{p_2^{\nu_s}}{m} \times (T_{a_1} \cdots T_{a_r})_{i'j'} \times (T_{b_1} \cdots T_{b_s})_{i'j'} \times \int d^2 \vec{z}_1 e^{i\vec{q}_1 \cdot \vec{z}_1} \int_{-T}^{+T} d\tau_1 \cdots \int_{-T}^{+T} d\tau_r \int_{-T}^{+T} d\sigma_1 \cdots \int_{-T}^{+T} d\sigma_s \times \theta(\tau_1 - \tau_2) \cdots \theta(\tau_{r-1} - \tau_r) \theta(\sigma_1 - \sigma_2) \cdots \theta(\sigma_{s-1} - \sigma_s) \times G_{\mu_1 \cdots \mu_r \nu_1 \cdots \nu_s}^{a_1 \cdots a_r b_1 \cdots b_s} (z + \frac{p_1}{m} \tau_1, \ldots, z + \frac{p_1}{m} \tau_r, \frac{p_2}{m} \sigma_1, \ldots, \frac{p_2}{m} \sigma_s),$$ (4.11)

having denoted with

$$G_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_p}^{a_1 \cdots a_p} (x_1, \ldots, x_p) \equiv \langle A_{\mu_1}^{a_1} (x_1) \cdots A_{\mu_p}^{a_p} (x_p) \rangle$$ (4.12)

the (complete) $p$–point gluonic Green function.

The analogous expansion for the quark–antiquark correlator is, of course [see also Eq.
\[ g_M^{qq}(\chi; T; t)_{i,i'; j,j'} = \frac{1}{[Z_M(T)]^2} \sum_{r,s=1}^{\infty} (-ig)^{r+s} \frac{p_1^r}{m} \frac{p_2^s}{m} \ldots \frac{p_{2}^{s}}{m} \times (T_{a_1} \ldots T_{a_r})_{i'i}(-1)^s(T_{b_1}^* \ldots T_{b_s}^*)_{j'j} \times \left(\int d^2z_1 e^{iq \cdot z_1} \int_{-T}^{+T} d\tau_1 \ldots \int_{-T}^{+T} d\tau_r \int_{-T}^{+T} d\sigma_1 \ldots \int_{-T}^{+T} d\sigma_s \times (T_{r_1}^* \ldots T_{r_s}^*)_{jj'} \right) \theta(\tau_1 - \tau_2) \ldots \theta(\tau_{r-1} - \tau_r) \theta(\sigma_1 - \sigma_2) \ldots \theta(\sigma_{s-1} - \sigma_s) \times C_{\mu_1 \ldots \mu_r \nu_1 \ldots \nu_s}^{a_1 \ldots a_r b_1 \ldots b_s} \left( z + \frac{p_1}{m} \tau_1, \ldots, z + \frac{p_1}{m} \tau_r, \frac{p_2}{m} \sigma_1, \ldots, \frac{p_2}{m} \sigma_s \right). \] (4.13)

From a comparison with the previous relation (4.11), we immediately see that the same gluonic Green function \( G_{a_1 \ldots a_r b_1 \ldots b_s}^{\mu_1 \ldots \mu_r \nu_1 \ldots \nu_s} \) (and, therefore, also every given Feynman graph originated from its perturbative expansion, taking into account also the squared renormalization constant \( [Z_M(T)]^2 \) at the denominator) comes out to be contracted with the same colour factor \((T_{a_1} \ldots T_{a_r})_{i'i}\) for the first Wilson line and a different colour factor \((-1)^s(T_{b_1} \ldots T_{b_s})_{j'j}\) for the second Wilson line [see also Eq. (3.5): we have used the fact that the generators \( T_a \) are hermitian]. The change

\[(T_{b_1} \ldots T_{b_s})_{j'j} \rightarrow (-1)^s(T_{b_1} \ldots T_{b_s})_{j'j}\] (4.14)

for each given Feynman graph from the quark–quark to the quark–antiquark case represents what we can call the "crossing relation" for Feynman graphs.

Let us see, in particular, how this works in the case of correlators evaluated in QCD perturbation theory up to the fourth order in the (renormalized) coupling constant. The calculation of the quark–quark Minkowskian and Euclidean correlators up to order \( g^4_{R} \) in the renormalized coupling constant has been already carried out in Ref. [4], with the results (always in the Feynman gauge \( \alpha = 1 \)):

\[ g_M^{qq}(\chi; t)_{i'i;j'j} |g^2_R| = i g^2_R \frac{1}{T} \coth \chi \left[ 1 - g^2_R \left( F^{(2)}(t) + \frac{2N_c B}{(4\pi)^2} + \frac{N_c t I(t)|\chi||\coth \chi|}{4\pi} \right) \right] \cdot (G_1)_{i'i;j'j} \]

\[ - \frac{1}{2} g^2_R I(t) \coth^2 \chi \cdot (G_2)_{i'i;j'j}; \] (4.15)

22
correlator are satisfied too. Moreover, since the result obtained when computing the high-energy quark-quark scattering amplitude with usual perturbative techniques \(7, 8, 9\).)

The property (2.10) for the Euclidean correlator is trivially satisfied and, since \(\{\theta\} = \{\theta\} \) and \(\{(2\pi - \theta)\} = \{\theta\}\), the properties (2.17) and (2.19) for the Euclidean correlator are satisfied too. Moreover, since \(\{\theta\} = \theta \) for \(\theta \in (0, \pi)\), one immediately finds that the analytic extension \(\tilde{G}_M^{\theta\theta}\) of the Minkowskian correlator from the positive real axis \(\chi \in \mathbb{R}^+\) and the analytic extension \(\tilde{G}_E^{\theta\theta}\) of the Euclidean correlator from the real segment \(\theta \in (0, \pi)\) are given by:

\[
\tilde{G}_M^{\theta\theta}(\chi; t; \nu^{(\lambda)}_{\nu'; j'})|_{\theta_R} = ig_R^2 \frac{1}{t} \coth \chi \left[ 1 - g_R^2 \left( F^{(2)}(t) + \frac{2N_c B}{(4\pi)^2} + \frac{N_c}{4\pi} t I(t) \coth \chi \right) \right] \cdot (G_1)_{\nu'i;j'}
\]

\[
-\frac{1}{2} g_R^4 I(t) \cot^2 \theta \cdot (G_2)_{\nu'i;j'}, \quad \forall \chi \in \mathcal{D}_M, (4.20)
\]

\[
\tilde{G}_E^{\theta\theta}(\theta; t; \nu^{(\lambda)}_{\nu'; j'})|_{\theta_R} = g_R^2 \frac{1}{t} \cot \theta \left[ 1 - g_R^2 \left( F^{(2)}(t) + \frac{2N_c B}{(4\pi)^2} + \frac{N_c}{4\pi} t I(t) \cot \theta \right) \right] \cdot (G_1)_{\nu'i;j'}
\]

\[
+ \frac{1}{2} g_R^4 I(t) \cot^2 \theta \cdot (G_2)_{\nu'i;j'}, \quad \forall \theta \in \mathcal{D}_E, (4.21)
\]
with the usual analyticity domains $\mathcal{D}_M$ and $\mathcal{D}_E$ defined in Eq. (4.4). From the explicit expressions (4.20) and (4.21) one immediately verifies that the analytic–continuation relations (4.7) are verified.

Let us now turn our attention to the quark–antiquark correlator. As we have already said above, it is not necessary to repeat the calculation from the beginning, but we can simply use the crossing relation (4.14) derived above to convert the contribution of each given Feynman graph from the quark–quark to the quark–antiquark case. In practice, it comes out that those which do not vanish, with the exception of the two–gluon–exchange graphs $b$ and $c$ in Fig. 5, have (apart from a multiplicative constant) the same colour factor (4.17) of the one–gluon–exchange graph $a$ in Fig. 5 and, apart from the colour–index exchange $j \leftrightarrow j'$, they simply take an extra minus sign, exactly as graph $a$:

\[
(G_1)_{i'i'jj'} = (T_a)_{i'i'}(T_a)_{jj'} \rightarrow -(T_a)_{i'i'}(T_a)_{jj'} = -(G_1)_{i'i'jj'}. \tag{4.22}
\]

(These graphs contribute to the pieces containing $B$ and $F^{(2)}(t)$ in Eqs. (4.15) and (4.16). The whole set of graphs contributing to the order $\mathcal{O}(g_4^2)$ are reported in Figs. 2 and 3 of Ref. [4].)

Instead, always by virtue of the crossing relation (4.14), graphs $b$ and $c$ simply enter in the quark–quark and quark–antiquark correlators with exchanged colour factor (and exchanged colour index exchange $j \leftrightarrow j'$):

\[
M(b) \cdot (T_aT_b)_{i'i'}(T_aT_b)_{jj'} \rightarrow M(b) \cdot (T_aT_b)_{i'i'}(T_bT_a)_{jj'},
\]

\[
M(c) \cdot (T_aT_b)_{i'i'}(T_bT_a)_{jj'} \rightarrow M(c) \cdot (T_aT_b)_{i'i'}(T_aT_b)_{jj'}. \tag{4.23}
\]

In Ref. [4] it was found that:

\[
M(b) = \frac{ig_4^2}{2\pi} I(t)(i\pi - |\chi|) \coth^2 \chi,
\]

\[
M(c) = \frac{ig_4^2}{2\pi} I(t)|\chi| \coth^2 \chi, \tag{4.24}
\]

so that: $M(b) + M(c) = -\frac{1}{2}g_4^2 I(t) \coth^2 \chi$.

Making use of the following relation for the colours factors [with the definitions (4.17) and (4.18)]:

\[
(T_aT_b)_{ij}(T_bT_a)_{kl} = (T_aT_b)_{ij}(T_aT_b)_{kl} + \frac{N_c}{2}(T_c)_{ij}(T_c)_{kl}
\]

\[
\equiv (G_2)_{ij;kl} + \frac{N_c}{2}(G_1)_{ij;kl}, \tag{4.25}
\]

24
we obtain that the contribution of graphs $b$ and $c$ to the quark–antiquark correlator is given by the following expression:

$$M(b) \cdot (T_a T_b)_{\nu_i \nu_j} + M(c) \cdot (T_a T_b)_{\nu_i \nu_j} = \frac{N_c}{2} M(b) \cdot (G_1)_{\nu_i \nu_j} + [M(b) + M(c)] \cdot (G_2)_{\nu_i \nu_j}$$

Summing all contributions, the following result is found for the quark–antiquark correlator at order $\mathcal{O}(g_4^4 R)$, for positive hyperbolic angle $\chi > 0$:

$$g_{q \bar{q}}(\chi; t)_{\nu_i \nu_j} |_{g_R^4} = -i g_R^2 \frac{1}{t} \coth \chi \left[ 1 - g_R^2 \left( F^{(2)}(t) + \frac{2 N_c B}{(4 \pi)^2} - \frac{N_c}{4 \pi} t I(t) (i \pi - \chi) \coth \chi \right) \right] \cdot (G_1)_{\nu_i \nu_j}$$

This is also the expression of the analytic extension $g_{q \bar{q}}^M$ from the real positive $\chi$–axis to the same analyticity domain $\mathcal{D}_M = \{ \chi \in \mathbb{C} | \chi \neq i k \pi, k \in \mathbb{Z} \}$ introduced above for the quark–quark correlator $g_{qq}^M$ written in Eq. (4.20). A comparison of (4.27) with (4.20) shows that the crossing–symmetry relation is verified:

$$g_{q \bar{q}}^M(\chi; t)_{\nu_i \nu_j} |_{g_R^4} = g_{q \bar{q}}^M(i \pi - \chi; t)_{\nu_i \nu_j} |_{g_R^4}, \quad \forall \chi \in \mathbb{R}^+.$$  (4.28)

Going to larger perturbative orders, many more and much more complicated Feynman diagrams are involved: however, there is apparently no reason why the results that we have found and discussed above concerning analyticity and crossing symmetry should not be true also in these cases. In the appendix A, for example, we discuss correlators at orders $\mathcal{O}(g_R^6)$, limiting ourselves (for simplicity) to the (physically interesting) diffractive part (defined according to Ref. [1]) of the three–gluon–exchange graphs (diagrams $d \div i$ in Fig. 5).
5. Concluding remarks and prospects

The main result of this paper has been to clarify the relation between Minkowskian–to–Euclidean analyticity properties and crossing symmetry both for line–line and loop–loop eikonal amplitudes. In sections 2 and 3 we have shown, in a nonperturbative way, using the functional integral approach, how certain apparently reasonable analyticity hypotheses of the line–line and loop–loop correlation functions in the angular variables and in the IR cutoff $T$, which are known to imply the Minkowskian–to–Euclidean analytic–continuation relation, also imply (directly from this) the crossing–symmetry relation, of which a nice geometrical interpretation has been provided. The reasonableness of the above–mentioned analyticity hypotheses comes essentially from the explicit tests that have been done in perturbation theory and which are presented in section 4 and appendix A. Of course, as we have already said at the beginning of section 4, perturbation theory is the only available technique for computing (from first principles) both the Minkowskian and the Euclidean correlation functions and so explicitly testing all the above–mentioned analyticity and crossing–symmetry properties.

A real nonperturbative foundation of these properties is at the moment out of our reach, but this is really the kind of effort that one should make in order to fully understand (and so fully trust!) the nonperturbative results which derive from them. (In appendix B, for example, we have discussed the connection between some singularities in the correlators and the emergence of certain parton–parton or dipole–dipole bound states. It would be nice if one could generalize this kind of arguments and find a nonperturbative way of identifying all type of singularities in the correlators and so have a complete description of their analyticity structure.)

There exist in the literature some nonperturbative computations of the Euclidean correlation functions, obtained using some specific models in the Euclidean theory. (They can then be continued to the corresponding Minkowskian correlation functions using the analytic–continuation relation in the angular variables and in the IR cutoff $T$ and so one can in principle address, from a fully nonperturbative point of view, the formidable [and unfortunately still unsolved!] problem of the asymptotic $s$–dependence of hadron–hadron scattering amplitudes and total cross sections.)

For example, in Ref. [23] the loop–loop Euclidean correlation functions have been evaluated in the context of the so–called “loop–loop correlation model” [24], in which the QCD vacuum is described by perturbative gluon exchange and the nonperturbative
“Stochastic Vacuum Model”: the result is a loop–loop Euclidean correlation function 

\[ C_E(\theta; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \]

which, for \( \theta \in (0, \pi) \), is an analytic function of \( \cot \theta \) and can then be 
analytically extended to the entire complex plane with the exception of the singularities of 
\( \cot \theta \), i.e., to the same domain given in Eq. (4.4), 

\[ D_E = \{ \theta \in \mathbb{C} | \theta \neq k\pi, k \in \mathbb{Z} \} \]

[including the real segment \((0, \pi)\), the negative imaginary axis \( \theta = -i\chi, \chi > 0 \), and the semiaxis 
\((\text{Re} \theta = \pi, \text{Im} \theta > 0)\)]. The Euclidean–to–Minkowskian analytic continuation can then be 
safely applied and the crossing–symmetry relation comes out to be trivially satisfied.

The same also happens adopting a different Euclidean approach [25], consisting in 
evaluating the one–instanton contribution to both the line–line (see also Ref. [26]) and 
the loop–loop Euclidean correlation functions: one finds that the colour–elastic line–line 
and loop–loop Euclidean correlation functions for \( \theta \in (0, \pi) \) scale as \( 1/\sin \theta \), while the 
colour–changing inelastic line–line Euclidean correlation function scales as \( \cot \theta \).

Finally, we want to comment on a third Euclidean approach existing in the literature, 
in which one computes the line–line/loop–loop Euclidean correlation functions in strongly 
coupled gauge theories using the AdS/CFT correspondence [27, 28, 29]. In a first paper 
[27] this approach was used to study the loop–loop Euclidean correlation function in the 
\( \mathcal{N} = 4 \) SYM theory in the limit of large number of colours \((N_c \to \infty)\) and strong coupling: 
one finds that the Euclidean correlation function for \( \theta \in (0, \pi) \) is a combination of various 
pieces scaling as \( 1/\sin \theta, \cot \theta \) and \( \cos^2 \theta/\sin \theta \), which can be analytically extended to 
the same complex domain \( D_E \) which was considered in the two previous cases and in Eq. (4.4).

A different situation appears, instead, when one tries to extend the approach based 
on the AdS/CFT correspondence in order to study the line–line/loop–loop Euclidean 
correlation functions in strongly coupled confining (i.e., nonconformal) gauge theories, as 
was done in Refs. [28, 29]. In this case the analytic structure of the Euclidean correlation 
functions involves branch cuts in the complex \( \theta \) and \( T \) planes, coming from logarithms 
and square roots, which lead to an ambiguity in the Euclidean–to–Minkowskian analytic 
continuation. It is not clear to us, at the moment, if this analytic structure is something 
peculiar to the specific model considered there or, vice versa, if it is a more general 
characteristic, maybe related to the presence of confinement (as it seems to be indicated 
by the authors in Refs. [28, 29]). In the case of this second possibility, one is immediately 
faced with the following series of questions:

1) Are the analyticity hypotheses considered in section 2 maybe too strong and to what
degree can they be relaxed?

ii) What about the Euclidean–to–Minkowskian analytic continuation if weaker analyticity hypotheses are kept in place of those discussed in section 2?

iii) What about, finally, the crossing symmetry relation in the presence of a more complicated analytic structure, when the Euclidean–to–Minkowskian analytic continuation cannot be trusted, at least in the form presented in section 2?

We have not, at the moment, the answers for all these questions but we hope that future work along the lines indicated in this paper will shed some light on these and other related problems.
Appendix A: Three–gluon–exchange diffractive contributions to $qq$ and $q\bar{q}$ correlators

Order $\mathcal{O}(g_R^6)$ calculations involve much more, and more complicated, Feynman graphs than order $\mathcal{O}(g_R^4)$. In this appendix we shall limit ourselves to the study of the diffractive part of the three–gluon–exchange graphs only. The diffractive parts of the quark–quark and quark–antiquark correlators are defined according to Ref. [1] as:

$$g_{M}^{qq}(\chi; t)^{(D)} \equiv N_c \sum_{i,j=1}^{N_c} g_{M}^{qq}(\chi; t)_{ii; jj},$$

$$g_{M}^{q\bar{q}}(\chi; t)^{(D)} \equiv N_c \sum_{i,j=1}^{N_c} g_{M}^{q\bar{q}}(\chi; t)_{ii; jj},$$

(A.1)

i.e., as the traces over the colour factors of each line, and thus correspond to a process without exchange of colour. (By virtue of Eq. (2.1) and of the optical theorem, their real parts, in the limit of very large rapidity gap $\chi \simeq \log(s/m^2) \to \infty$ and vanishing squared transferred momentum $t \to 0$, are related to the colour–averaged quark–quark and quark–antiquark total cross sections at high energies.)

Because of the cyclicity property of the trace, diagrams differing only by an even permutation of vertices have the same colour factor. Therefore, the diffractive contributions of the six three–gluon–exchange diagrams $d \div i$ in Fig. 5 have only two different colour factors, one ($S_3$) for the “ladder” diagrams $d, e, f$, and another ($S'_3$) for the “crossed” diagrams $g, h, i$, given by:

$$S_3 \equiv \text{Tr} [T_a T_b T_c] \text{Tr} [T_a T_b T_c] = -\frac{N_c^2 - 1}{4N_c},$$

(A.2)

$$S'_3 \equiv \text{Tr} [T_a T_b T_c] \text{Tr} [T_b T_a T_c] = S_3 + \frac{N_c}{2} S_2,$$

(A.3)

where:

$$S_2 \equiv \text{Tr} [T_a T_b] \text{Tr} [T_a T_b] = \frac{N_c^2 - 1}{4}.$$  

(A.4)

The diffractive contributions of the six diagrams to the quark–quark correlator $g_M^{qq}$ can then be written in the form:

$$\Delta g_{M}^{qq}(\chi, t)^{(D)}_{3\text{gluon}} = S_3 L(\chi, t) + S'_3 X(\chi, t)$$

$$= S_3 [L(\chi, t) + X(\chi, t)] + \frac{N_c}{2} S_2 X(\chi, t).$$

(A.5)
The diffractive contribution of the same six diagrams to the quark–antiquark correlator \( g_{M}^{q\bar{q}} \) is, according to the crossing relation (4.14):

\[
\Delta g_{M}^{q\bar{q}}(\chi, t)_{3\text{gluon}}^{(D)} = (-1)^{3} \left[ S_{3}^2 L(\chi, t) + S_{3} X(\chi, t) \right] \\
= -S_{3} [L(\chi, t) + X(\chi, t)] - \frac{N_{c}}{2} S_{2} L(\chi, t).
\]

(A.6)

The three–gluon–exchange contributions are given by the expressions (for \( \chi \in \mathbb{R}^{+} \)):

\[
\Delta g_{M}^{qq}(\chi; t)_{3\text{gluon}}^{(D)} = \left. ig_{R}^{6} \coth^{3} \chi \left\{ S_{3} \left[ \frac{1}{6} I_{1}(t) \right] + \frac{N_{c}}{2} S_{2} \left[ \frac{i}{2\pi} \left( \chi - \frac{2\pi}{3} \right) I_{1}(t) + \frac{1}{2\pi^{2}} H(\chi) \right] \right\} \right|_{\text{at } t = 0},
\]

(A.7)

\[
\Delta g_{M}^{q\bar{q}}(\chi; t)_{3\text{gluon}}^{(D)} = \left. -ig_{R}^{6} \coth^{3} \chi \left\{ S_{3} \left[ \frac{1}{6} I_{1}(t) \right] - \frac{N_{c}}{2} S_{2} \left[ \frac{i}{2\pi} \left( \chi - \frac{\pi}{3} \right) I_{1}(t) + \frac{1}{2\pi^{2}} H(\chi) \right] \right\} \right|_{\text{at } t = 0},
\]

(A.8)

where:

\[
I_{1}(t) \equiv \int \frac{d^{2}\vec{k}_{1\perp}}{(2\pi)^{2}} \frac{d^{2}\vec{k}_{2\perp}}{(2\pi)^{2}} \frac{d^{2}\vec{k}_{3\perp}}{(2\pi)^{2}} \frac{1}{\lambda^{2} (\vec{k}_{1\perp})^{2} + \lambda^{2} (\vec{k}_{2\perp})^{2} + \lambda^{2} (\vec{k}_{3\perp})^{2}} \left[ \frac{1}{\xi^{2} + \eta^{2} - 2\eta \cosh \chi + \lambda^{2} \kappa_{3\perp}^{2} + \lambda^{2} - i\epsilon} \right]
\]

(A.9)

and the function \( H(\chi) \) is defined by the integral:

\[
H(\chi) = \int \frac{d^{2}\vec{k}_{1\perp}}{(2\pi)^{2}} \frac{d^{2}\vec{k}_{2\perp}}{(2\pi)^{2}} \frac{d^{2}\vec{k}_{3\perp}}{(2\pi)^{2}} (2\pi)^{2} \delta^{(2)} (\vec{q}_{\perp} - \vec{k}_{1\perp} - \vec{k}_{2\perp} - \vec{k}_{3\perp})
\times h\left( \chi; \vec{k}_{1\perp}, \vec{k}_{2\perp}, \vec{k}_{3\perp} \right),
\]

\[
h\left( \chi; \vec{k}_{1\perp}, \vec{k}_{2\perp}, \vec{k}_{3\perp} \right) = \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\eta P_{\frac{1}{\xi}} P_{\frac{1}{\eta}}
\times \left( \frac{1}{\xi^{2} + \kappa_{1\perp}^{2} + \lambda^{2} \eta^{2} + \kappa_{2\perp}^{2} + \lambda^{2} \kappa_{3\perp}^{2} + \lambda^{2}} - \frac{1}{\kappa_{1\perp}^{2} + \lambda^{2} \kappa_{2\perp}^{2} + \lambda^{2}} \right)
\times \frac{1}{\xi^{2} + \eta^{2} - 2\xi\eta \cosh \chi + \kappa_{3\perp}^{2} + \lambda^{2} - i\epsilon},
\]

(A.10)

having denoted with

\[
P_{\frac{1}{\xi}} \equiv \frac{1}{2} \left( \frac{1}{\xi - i\epsilon} + \frac{1}{\xi + i\epsilon} \right)
\]

(A.11)

the “Cauchy principal part” of \( 1/\xi \). The explicit form of \( H(\chi) \) is not too enlightening; anyway, it can be continued analytically from the positive real axis \( \chi \in \mathbb{R}^{+} \) into a domain
including also the imaginary segment \((\text{Re}\chi = 0, 0 < \text{Im}\chi < \pi)\) and the semiaxis \((\text{Re}\chi < 0, \text{Im}\chi = \pi)\). Using the notation previously introduced, we denote such an extension as \(\mathcal{H}(\chi)\); as one immediately sees from (A.10), it has the property:

\[
\mathcal{H}(i\pi - \chi) = -\mathcal{H}(\chi).
\]

(A.12)

Repeating the calculation in the Euclidean case, we obtain the result, for \(\theta \in (0, \pi)\):

\[
\Delta g_E^{q\bar{q}}(\theta; t)^{(D)}_{3\text{gluon}} = -g_R^6 \cot^3 \theta \left\{ S_3 \left[ \frac{1}{6} I_1(t) \right] + \frac{N_c}{2} S_2 \left[ \frac{i}{2\pi} \left( i\theta - \frac{2\pi}{3} \right) I_1(t) + \frac{1}{2\pi^2} \mathcal{H}(i\theta) \right] \right\},\quad (A.13)
\]

immediately verifying the analytic–continuation relation:

\[
\Delta g_E^{q\bar{q}}(\theta; t)^{(D)}_{3\text{gluon}} = \Delta g_M^{q\bar{q}}(i\pi - \chi; t)^{(D)}_{3\text{gluon}}.
\]

(A.14)

Moreover, by virtue of the property (A.12), the contributions (A.4) and (A.8) of the three–gluon–exchange diagrams to the \(qq\) and \(q\bar{q}\) correlators satisfy the crossing–symmetry relation:

\[
\Delta g_M^{q\bar{q}}(\chi; t)^{(D)}_{3\text{gluon}} = \Delta g_M^{q\bar{q}}(i\pi - \chi; t)^{(D)}_{3\text{gluon}}.
\]

(A.15)

The three–gluon–exchange diffractive contributions to the \(qq\) and \(q\bar{q}\) eikonal scattering amplitudes are readily obtained once we know the asymptotic behaviour of the function \(H(\chi)\) in the limit of very large rapidity gap \(\chi \simeq \log(s/m^2) \to \infty\). From the explicit expressions (A.10) reported above, it is easy to see that the derivative \(dH(\chi)/d\chi\) tends to zero when \(\chi\) goes to infinity and, therefore, \(H(\chi)\) tends to a constant in the same limit:

\[
\lim_{\chi \to +\infty} H(\chi) = H_0.
\]

(A.16)

The high–\(\chi\) diffractive contribution of the three–gluon–exchange diagrams to the quark–quark correlator is then:

\[
\Delta g_M^{qq}(\chi \simeq \log(s/m^2) \to +\infty; t)^{(D)}_{3\text{gluon}} \simeq i g_R^6 \left\{ S_3 \left[ \frac{1}{6} I_1(t) \right] + \frac{N_c}{2} S_2 \left[ \frac{i}{2\pi} \left( \log \left( \frac{s}{m^2} \right) - \frac{2\pi}{3} \right) I_1(t) + \frac{1}{2\pi^2} H_0 \right] \right\} \\
= -g_R^6 \frac{N_c S_2}{4\pi} I_1(t) \log \left( \frac{s}{m^2} \right) + \text{constant imaginary part},
\]

(A.17)
while the high–χ diffractive contribution of the same diagrams to the quark–antiquark correlator is:

\[ \Delta g_M^{q\bar{q}}(\chi) \simeq \log(s/m^2) \to +\infty; t)^{(D)}_{\text{gluon}} \]

\[ \simeq -ig^6_R \left\{ S_3 \left[ \frac{1}{6} I_1(t) \right] - \frac{N_c}{2} S_2 \left[ \frac{i}{2\pi} \left( \log \left( \frac{s}{m^2} \right) - i\frac{\pi}{3} \right) I_1(t) + \frac{1}{2\pi^2} H_0 \right] \right\} \]

\[ = -g^6_R \frac{N_c S_2}{4\pi} I_1(t) \log \left( \frac{s}{m^2} \right) + \text{constant imaginary part}. \quad (A.18) \]

As in the case of the (full) \( \mathcal{O}(g^4_R) \) results discussed in section 4, also the (partial) \( \mathcal{O}(g^6_R) \) results (A.17) and (A.18) agree with the corresponding (i.e., three–gluon–exchange and diffractive) results obtained when computing the high–energy quark–quark and quark–antiquark scattering amplitudes with usual perturbative techniques \([7, 8, 9]\). This agreement as well as the analytic–continuation relation (A.14) and the crossing–symmetry relation (A.15) are, of course, expected to hold also for all other \( \mathcal{O}(g^6_R) \) contributions, i.e., for the full \( \mathcal{O}(g^6_R) \) results. As we have already said in section 4, in the case of the loop–loop correlation function the full \( \mathcal{O}(g^6_R) \) perturbative expansion has been computed in Ref. \([6]\) and found to be in agreement with both the BFKL results \([7, 8, 9]\) (in the limit of very large rapidity gap) and with the analytic–continuation relation (2.29).
Appendix B: Angular singularities vs. bound states

In this appendix we want to show that the singularities of the Euclidean correlation functions (when $T \to \infty$) in the points $\theta = k\pi, k \in \mathbb{Z}$, that we have found in all examples described in section 4, are indeed expected on general (i.e., nonperturbative) grounds as the consequence of the relation of these quantities with the potential of certain static dipole–dipole or parton–parton bound states.

For example, considering in particular the loop–loop Euclidean correlation function $G_E(\theta; T; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp})$, it is well known that in the case $\theta = 0$ and $T \to \infty$ this quantity is related to the van der Waals potential $V_{12}(\vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp})$ between two static fermion–antifermion dipoles (one positioned in $\vec{z}_\perp \pm \vec{R}_{1\perp}/2$ and the other positioned in $\pm \vec{R}_{2\perp}/2$) by the following expression [30, 31]:

$$G_E(\theta = 0; T; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \approx \exp \left[ -2T V_{12}(\vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \right].$$

(B.1)

This van der Waals potential can be studied both in QCD perturbation theory [30, 31, 32] and with nonperturbative techniques (see, e.g., Ref. [23] and references therein).

As a pedagogic example, in quenched QED this quantity can be easily calculated from the expressions for $G_E = \exp[-(\Phi'_1(T) + \Phi'_2(T))]$ reported in Ref. [20], Eqs. (2.10), (2.11) and (2.13), where we have to put $\theta = 0$. It is immediate to see that the integral $\Phi'_2(T)$ (with $\theta = 0$) continues to vanish in the large–$T$ limit, while the integral $\Phi'_1(T)$ behaves, for $\theta = 0$ and in the large–$T$ limit, exactly as $2T V_{12}$ where:

$$V_{12}(\vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) = \frac{e^2}{4\pi} \left( \frac{1}{|\vec{z}_\perp + \vec{R}_{1\perp} - \vec{R}_{2\perp}|} + \frac{1}{|\vec{z}_\perp - \vec{R}_{1\perp} + \vec{R}_{2\perp}|} \right) - \frac{1}{|\vec{z}_\perp + \vec{R}_{1\perp} + \vec{R}_{2\perp}|} - \frac{1}{|\vec{z}_\perp - \vec{R}_{1\perp} - \vec{R}_{2\perp}|}. $$

(B.2)

This is indeed the electromagnetic van der Waals potential between two static fermion–antifermion dipoles, in the quenched approximation.

Coming back to the more general case, Eq. (B.1) tells us that the correlator $G_E$ when $T \to \infty$ has a singularity in $\theta = 0$. The use of the crossing–symmetry relation (3.30) then immediately tells us that $G_E$ when $T \to \infty$ has also a singularity in $\theta = \pi$ and therefore, by virtue of the periodicity in $\theta$, a singularity is expected in each point $\theta = k\pi, k \in \mathbb{Z}$. 

33
A similar result (obtained using a similar approach) is expected to hold also for the quark–quark and quark–antiquark Euclidean correlation functions. (The Euclidean colour–singlet quark–antiquark correlator at $\theta = 0$ with impact parameter $\vec{z}_\perp$ is essentially the expectation value of a single Euclidean Wilson loop with transverse separation $\vec{z}_\perp$. This quantity is known to be related, in the large-$T$ limit, to the potential $V_{qq}(\vec{z}_\perp)$ between a static quark and a static antiquark separated by $\vec{z}_\perp$.)
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**FIGURE CAPTIONS**

**Fig. 1.** The space–time configuration of the two Wilson lines $W_{p_1}$ and $W_{p_2}$ entering in the expression for the quark–quark elastic scattering amplitude in the high–energy limit.

**Fig. 2.** The sequence of Euclidean transformations (leaving the functional integral unchanged) which shows how Eq. (3.11) can be written in the form (3.13).

**Fig. 3.** The common analyticity domain of $\tilde{g}_{E}^{qq}$ and $\tilde{g}_{E}^{q\bar{q}}$ in the complex variable $\theta$.

**Fig. 4.** The common analyticity domain of $\tilde{g}_{M}^{qq}$ and $\tilde{g}_{M}^{q\bar{q}}$ in the complex variable $\chi$.

**Fig. 5.** The Feynman diagrams with exchange of one, two and three gluons which contribute to the quark–quark and quark–antiquark correlators.
Fig. 1. The space–time configuration of the two Wilson lines $W_{p_1}$ and $W_{p_2}$ entering in the expression for the quark–quark elastic scattering amplitude in the high–energy limit.
Fig. 2. The sequence of Euclidean transformations (leaving the functional integral un-
changed) which shows how Eq. (3.11) can be written in the form (3.13).
Fig. 3. The common analyticity domain of $\mathcal{G}_E^{\eta}$ and $\mathcal{G}_E^{\bar{\eta}}$ in the complex variable $\theta$. 
Fig. 4. The common analyticity domain of $\mathcal{J}_M^\nu$ and $\overline{\mathcal{J}}_M^\nu$ in the complex variable $\chi$. 
Fig. 5. The Feynman diagrams with exchange of one, two and three gluons which contribute to the quark–quark and quark–antiquark correlators.