EXPLICIT FIXED POINTS OF THE SMOOTHING TRANSFORMATION

JACQUES PEYRIÈRE

Abstract. We deal with the equation

\[ Y \overset{\text{d}}{=} \frac{1}{b} \sum_{1 \leq j \leq N} W_j Y_j, \]

where the unknown is the distribution of \( Y \), the variables in the right hand side are independent, the \( Y_j \) are equidistributed with \( Y \), \( N \) is an integer valued random variable, and the \( W_j \) are equidistributed, nonnegative and of expectation 1. Usually a solution is obtained as the limit of a martingale. In some cases we give an explicit formula for the law of \( Y \).

Let \( N \) be a nonnegative integer valued random variable with finite second moment. Its expectation \( b \) is assumed to be larger than 1. Set \( \varphi(x) = \mathbb{E} x^N = \sum_{n \geq 0} \mathbb{P}(N = n) x^n \). Then \( b = \varphi'(1) \). Let \( \mu \) be a probability measure on \([0, +\infty)\) such that

\[ \int_0^\infty x \mu(dx) = 1 \quad \text{and} \quad \int_0^\infty x^2 \mu(dx) < b. \]

We consider the equation

\[ Y \overset{\text{d}}{=} \frac{1}{b} \sum_{1 \leq j \leq N} W_j Y_j \]

whose unknown is the probability law \( \nu \) on \([0, +\infty)\) common to \( Y \) and all the \( Y_j \), the variables on the right hand side are independent and all the variables \( W_j \) are distributed according to \( \mu \).

The Mandelbrot martingales. To get a solution to Equation (2) one way is to use the Mandelbrot construction \([7]\). We set

\[ Y_n = b^{-n} \sum_{1 \leq j_1 \leq N} W_{j_1} \sum_{1 \leq j_2 \leq N_{j_1}, j_2 \leq j_3 - 1} \ldots \sum_{1 \leq j_n \leq N_{j_1, j_2, \ldots, j_{n-1}}} W_{j_1, j_2, j_3, \ldots, j_n}, \]

where all the variables in the right hand side are independent, the \( W_{j_1, j_2, \ldots, j_n} \) are distributed according to \( \mu \), and \( N \) and the variables \( N_{j_1, j_2, \ldots, j_n} \) are equidistributed.

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One has

\[ Y_{n+1} = b^{-n} \sum_{1 \leq j_1 \leq N} W_{j_1} \times \]
\[ \sum_{1 \leq j_i \leq N_{j_1, j_2, \ldots, j_{i-1}}} W_{j_1, j_2, j_3 \ldots, j_n} b^{-1} \sum_{1 \leq j_{i+1} \leq N_{n+1}} W_{j_1, j_2, \ldots, j_{i+1}}. \]

Therefore \((Y_n)_{n \geq 1}\) is a martingale.

We also have

\[(3) \quad Y_{n+1} = b^{-1} \sum_{1 \leq j_1 \leq N} W_{j_1} Y_n(j_1) \]

where all the variables are independent and the \(Y_n(j)\) are equidistributed with \(Y_n\).

So, \(\mathbb{E} Y_{n+1}^2 = b^{-2} \mathbb{E} W^2 \mathbb{E} Y_n^2 \mathbb{E} N + b^{-2} \mathbb{E} N(N - 1)\), i.e.,
\[ \mathbb{E} Y_{n+1}^2 = b^{-1} \mathbb{E} W^2 \mathbb{E} Y_n^2 + b^{-2} \mathbb{E} N(N - 1). \]

We therefore see that if \(\mathbb{E} W^2 < b\) the martingale \((Y_n)\) is bounded in \(L^2\). Then it has a limit \(Y\), almost surely and in \(L^2\), \(\mathbb{E} Y = 1\), and
\[ \mathbb{E} Y^2 = \frac{\mathbb{E} N(N - 1)}{b(b - \mathbb{E} W^2)}. \]

Due to (3) we see that \(Y\) is a solution to Equation (2).

B. Mandelbrot [7, 8] introduced this construction (with constant \(N\)) to give a simplified statistical description of the dissipation of energy in a turbulent flow. Since then this model and Equation (2) have been extensively studied and generalized (for instance [5, 3, 4, 6]). See [1, 2] for a survey.

As a matter of fact, the necessary and sufficient condition for the uniform integrability of the martingale \(Y_n\) is \(\mathbb{E} W \log W < \log b\) (see [5] when \(N\) is constant and [6] when \(N\) is not constant).

Let \(\alpha = \mathbb{P}(W \neq 0)\) and \(\beta = \mathbb{P}(Y = 0)\). It results from Equation (2) that \(\beta = \varphi(\alpha \beta + 1 - \alpha)\). Due to convexity, the function \(t \mapsto \varphi(\alpha t + 1 - \alpha)\) has a fixed point less than 1 if and only if its derivative at 1 is larger than 1, which means \(\alpha b > 1\). Due to hypotheses \(\mathbb{E} W = 1\) and \(\mathbb{E} W^2 < b\) this condition is fulfilled.

**Laplace transform.** For \(t \geq 0\) set \(g(t) = \mathbb{E} e^{-tY}\). Then

\[(4) \quad g(0) = -g'(0) = 1. \]
From (2) we get

\[ E(e^{-\lambda Y} \mid N, (W_j)_{j \geq 1}) = \prod_{1 \leq j \leq N} g(tW_j/b) \]

\[ E(e^{-\lambda Y} \mid N) = \left( \int g(tx/b) \, d\mu(x) \right)^N \]

(5)

\[ g(t) = \varphi \left( \int g(tx/b) \, d\mu(x) \right). \]

Let \( u(t) = \int g(tx/b) \, d\mu(x) \). So, \( g(t) = \varphi(u(t)) \).

Relations (4) become

(6)

\[ u(0) = 1 \quad \text{and} \quad u'(0) = -\frac{1}{b}. \]

Also \( g \) and \( u \) are decreasing, \( \lim_{t \to +\infty} g(t) = P(Y = 0) = \beta, \lim_{t \to +\infty} u(t) = \varphi^{-1}(\beta) = \alpha \beta + 1 - \alpha, \) and \( g(0) = u(0) = 1. \)

**A particular case.** Now we consider the particular case when

\[ d\mu(x) = (1 - \alpha)\delta(dx) + \alpha(1 - \gamma)b^{\gamma - 1}x^{\gamma - 1}\mathbf{1}_{(0,b)}(x) \, dx, \]

with \( \gamma < 1, 0 < \alpha \leq 1, \) and where \( \delta \) stands for the unit Dirac mass at 0. If \( W \) is distributed according to \( \mu \), the condition \( \mathbb{E}W = 1 \), means \( \gamma = 1 - \frac{1}{\alpha b - 1} \) and \( \mathbb{E}W^2 < b \) means \( \gamma > 1 - \frac{2}{\alpha b - 1} \). So the only constraints on the parameters are

\[ \frac{1}{b} < \alpha \leq 1 \quad \text{and} \quad \gamma = 1 - \frac{1}{\alpha b - 1}. \]

We have \( u(t) = 1 - \alpha + \alpha(1 - \gamma)b^{\gamma - 1} \int_0^b g(tx/b)x^{\gamma - 1} \, dx. \)

Then

\[ u'(t) = \alpha(1 - \gamma)b^{\gamma - 1} \int_0^b g'(tx/b)b^{\gamma - 1}x^{\gamma - 1} \, dx \]

\[ = \alpha(1 - \gamma) \frac{g(t)}{t} - \left( 1 - \gamma \right) \frac{1}{t} \int_0^b g(tx/b)x^{\gamma - 1} \, dx \]

\[ = \alpha(1 - \gamma) \frac{g(t)}{t} - \frac{1 - \gamma}{t} (u(t) - (1 - \alpha)). \]

We see that \( u \) satisfies the following differential equation

(7)

\[ (\alpha b - 1)u'(t) = \frac{1}{t} \left( \alpha \varphi(u(t)) - u(t) + 1 - \alpha \right). \]

Let \( \omega(x) = \frac{\alpha b - 1}{\alpha \varphi(x) - x + 1 - \alpha} + \frac{1}{1 - x}. \) As \( \varphi(x) = 1 + b(x - 1) + O((x-1)^2), \) \( \omega \) is bounded in a neighborhood of 1. Indeed, by continuity,
we have \( \omega(1) = \frac{\mathbb{E} N(N - 1)}{2(\alpha b - 1)} \). Then (7) rewrites as

\[
(\omega(u) - \frac{1}{1 - u}) \, du = \frac{dt}{t}.
\]

Let \( \Omega(x) = \exp \int_1^x \omega(\tau) \, d\tau \). The function \( x \mapsto \alpha \varphi(x) - x + 1 - \alpha \) is convex and vanishes for \( x = 1 \) and \( x = \varphi^{-1}(\beta) \); so it is negative on the interval \( (\varphi^{-1}(\beta), 1) \). This means that, on this interval, the derivative of \((1 - x)\Omega(x)\) is negative.

It follows that there is a constant \( c \) such that, for \( t \geq 1 \), \( u(t) \) is the unique solution to equation

\[
(1 - u(t))\Omega(u(t)) = ct, \quad \text{with} \quad u(0) = 1,
\]

in the interval \( (\varphi^{-1}(\beta), 1) \). By taking in account the initial conditions (6) we see that \( c = 1/b \). Finally, \( u \) is implicitly defined by

\[
b(1 - u)\Omega(u) = t.
\]

**Examples.** We give six examples of computations. The fourth one is interesting because it shows that the mapping \((N, \mu) \mapsto \nu\) is not one-to-one.

**Example 1.** We take \( \varphi(x) = x^{n+1} \) (where \( n \geq 1 \) is an integer) and \( \alpha = 1 \). Then \( b = n + 1 \), \( \gamma = 1 - \frac{1}{n} \), \( \beta = 0 \), and

\[
(1 - x)\Omega(x) = \frac{1 - x^n}{nx^n},
\]

\[
u(ds) = \Gamma \left( \frac{n + 1}{n}, \frac{n + 1}{n} \right) \frac{n+1}{s^n} \exp \left( \frac{(n+1)s}{n} \right) ds.
\]

This situation has been independently studied by G. Letac and the author. It is mentioned in [11] (page 264), but up to now [9], p.387–388 seems to be the only written trace of this formula.
Example 2. This time \( \varphi(x) = 1 - \rho + \rho x^2 \) with \( 0 < \rho \leq 1 \).
Then \( b = 2\rho \) (therefore \( \alpha > \frac{1}{2\rho} \)), \( \beta = 1 - \frac{2\rho - 1}{\rho^2 \alpha^2} \), \( \gamma = 1 - \frac{1}{2\rho - 1} \),
\( \omega(x) = -\frac{\alpha}{\alpha x + \rho - 1} \).

\[
\Omega(x) = \frac{2\alpha \rho - 1}{\alpha \rho x + \alpha - 1},
\]

\[
u(ds) = \frac{(\alpha^2 \rho - 2\alpha \rho + 1)}{\rho \alpha^2} \delta(ds)
+ \frac{4(2\alpha \rho - 1)^2((2\alpha \rho - 1)^2s + \alpha(1 - \alpha \rho))}{\rho \alpha^4} e^{-4\rho s + \frac{2s}{\rho}} ds.
\]

Example 3. \( \varphi = (1 - \rho)x + \rho x^{n+1} \), \( 0 < \rho \leq 1 \) and \( \alpha = 1 \). Then \( b = \rho n + 1 \), \( \gamma = (\rho n - 1)/\rho n \), \( (1 - x)\Omega(x) = (x^n - 1)/n \),
\( u(t) = \left(1 + \frac{nt}{\rho n + 1}\right)^{-1/n} \), and
\( g(t) = (1 - \rho)\left(1 + \frac{nt}{\rho n + 1}\right)^{-1/n} + \rho\left(1 + \frac{nt}{\rho n + 1}\right)^{-\frac{n+1}{n}} \).

Finally \( \nu \) is a barycenter of Gamma distributions.

Example 4. This time \( \varphi(x) = \frac{1 - p}{1 - px} \), with \( p > 1/2 \).

Then \( b = \frac{p}{1-p} \) (so \( \alpha > \frac{1-p}{p} \)), \( \gamma = 1 - \frac{1-p}{\alpha p + p - 1} \), \( \omega = -\frac{\alpha p^2}{(\alpha p + px - 1)(1-p)} \), and
\( \Omega(x) = \left(\frac{\alpha p + px - 1}{\alpha p + p - 1}\right)^{-\frac{\alpha p}{1-p}}. \)

By taking \( \alpha = 2(1 - p)/p \) the calculation can be pushed forward. In this condition
\[
\Omega(x) = \left(\frac{1 - p}{1 + p(x - 2)}\right)^2,
\]

\[
u(ds) = \frac{1}{2} \delta(ds) + \frac{e^{-\frac{p}{4\sqrt{\pi}s}}}{\sqrt{\pi s}} ds.
\]
Example 5. This time \( \varphi = \frac{(1-p)x}{1-px} \), with \( 0 < p < 1 \) and \( \alpha > 1 - p \).

Then \( b = 1 + \frac{p}{1-p} \), \( \gamma = 1 - \frac{1-p}{\alpha + 1} \), and

\[
\omega(x) = -\frac{\alpha p}{(px + \alpha - 1)(1-p)},
\]

\[
\Omega(x) = \left(\frac{px + \alpha - 1}{p + \alpha - 1}\right)^{-\frac{\alpha p}{1-p}}.
\]

Let us end the computation in case when \( \alpha = 2(1-p) \). Then we have

\[
\Omega(x) = \left(\frac{1-p}{1+p(x-2)}\right)^2,
\]

\[
u(\sigma) = \frac{2p-1}{2p} \delta(\sigma) + \frac{e^{-\frac{\sigma}{4p}}}{4p^2 \sqrt{\pi}} \, d\sigma.
\]

Example 6. This time \( \varphi(x) = \frac{(1-p)x^2}{1-px} \). Then \( b = 2 + \frac{p}{1-p} \) and \( \alpha > 1 - \frac{1}{2-p} \).

Also \( \gamma = 1 + \frac{1-p}{\alpha p - 2\alpha - p + 1} \), \( \omega = -\frac{\alpha}{((\alpha + p - \alpha p)\alpha + \alpha - 1)(1-p)} \), and

\[
\Omega(x) = \left(\frac{(\alpha + p - \alpha p)x + \alpha - 1}{2\alpha + p - \alpha p - 1}\right)^{-(1-p)(\alpha + p - \alpha p)}.
\]

Now take \( \alpha = \frac{2p(1-p)}{2p^2 - 4p + 1} \). Due to \( 1 - \frac{1}{(2-p)} < \alpha \leq 1 \), we have to assume that \( p > 1/2 \). Then \( \gamma = 2(1-p)^2 \),

\[
\Omega(x) = \left(\frac{1-p}{1+p(x-2)}\right)^2,
\]

\[
\Omega(x) = \left(\frac{2p-1}{2p^2} \right)^2 + \frac{1}{2p^2 \sqrt{\frac{t}{2-p} + 1}} - \frac{2(1-p)^2}{2p^2 \sqrt{\frac{t}{2-p} + 1} + \frac{t}{2-p + 1}}.
\]
and
\[ \nu(ds) = \frac{(2p-1)^2}{2p^2} \delta(ds) + \frac{(2p-1)(3-2p)}{2p^2} \sqrt{\frac{2-p}{\pi s}} e^{(-2+p)s} ds \\
+ \frac{2(2-p)(1-p)^2}{p^2} \text{erfc}\left(\sqrt{(2-p)s}\right) ds. \]

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