Existence of weak solutions to the two-dimensional incompressible Euler equations in the presence of sources and sinks

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Abstract

A classical model for sources and sinks in a two-dimensional perfect incompressible fluid occupying a bounded domain dates back to Yudovich’s paper [11] in 1966. In this model, on the one hand, the normal component of the fluid velocity is prescribed on the boundary and is nonzero on an open subset of the boundary, corresponding either to sources (where the flow is incoming) or to sinks (where the flow is outgoing). On the other hand the vorticity of the fluid which is entering into the domain from the sources is prescribed.

In this paper we investigate the existence of weak solutions to this system by relying on a priori bounds of the vorticity, which satisfies a transport equation associated with the fluid velocity vector field. Our results cover the case where the vorticity has a $L^p$ integrability in space, with $p$ in $[1, +\infty]$, and prove the existence of solutions obtained by compactness methods from viscous approximations. More precisely we prove the existence of solutions which satisfy the vorticity equation in the distributional sense in the case where $p > \frac{4}{3}$, in the renormalized sense in the case where $p > 1$, and in a symmetrized sense in the case where $p = 1$. 
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1 Introduction

This paper focuses on the mathematical analysis of a 2D perfect incompressible fluid occupying a bounded domain in the presence of sources and sinks. A classical model dates back to Yudovich’s paper [44] in 1966 where, on the one hand, the normal component of the fluid velocity is prescribed on the whole boundary of the fluid domain, and on the other hand the vorticity is prescribed on the part of the boundary where the fluid is entering into the domain. The connected components of this part are called the sources whereas the connected components of the part of the boundary where the flow is exiting of the domain are called the sinks.

More precisely let $Ω$ an open bounded connected simply-connected non-empty subset of $\mathbb{R}^2$ with smooth boundary. Let $N \geq 2$, $1 \leq n \leq N - 1,$

$$I^+ = \{1, \ldots, n\}, \quad I^- = \{n + 1, \ldots, N\} \quad \text{and} \quad I = I^+ \cup I^-.$$ 

For $i \in I$, let $S^i$ an open connected simply-connected non-empty subset of $\mathbb{R}^2$ compactly contained in $Ω$ with smooth boundary. We assume that the closures of the sets $S^i$ are pairwise disjoint. The domain occupied by the fluid is

$$F = Ω \setminus \bigcup_{i \in I} S^i,$$

and we split the boundary of the fluid domain into two parts:

$$\partial F^+ = \bigcup_{i \in I^+} \partial S^i \quad \text{and} \quad \partial F^- = \bigcup_{i \in I^-} \partial S^i,$$

respectively called outlet and inlet. An example of fluid domain is shown in Figure 1. Finally note that it is possible to deal with the case where the fluid is allowed to enter or exit through the exterior domain $\partial Ω$ but in this work we assume, for sake of simplicity, that the boundary $\partial Ω$ of $Ω$ is impermeable.

The equations in the unknown $(v, p)$ that model the dynamics read as

$$\begin{align*}
\partial_t v + v \cdot \nabla v + \nabla p &= 0 \quad \text{in } \mathbb{R}^+ \times F, \quad (1a) \\
\text{div } v &= 0 \quad \text{in } \mathbb{R}^+ \times F, \quad (1b) \\
v \cdot n &= g \quad \text{on } \mathbb{R}^+ \times \partial F, \quad (1c) \\
\text{curl } v &= \omega^+ \quad \text{on } \mathbb{R}^+ \times \partial F^+, \quad (1d) \\
v(0, \cdot) &= v^{in} \quad \text{in } F, \quad (1e)
\end{align*}$$

where $v : \mathbb{R}^+ \times F \rightarrow \mathbb{R}^2$ is the fluid velocity field and $p : \mathbb{R}^+ \times F \rightarrow \mathbb{R}$ is the fluid pressure. In (1c), $n$ is the unit normal vector field exiting from the domain $F$. The data $g$ for the normal component of the velocity on the boundary is assumed to satisfy $g < 0$ on $\mathbb{R}^+ \times \partial S^i$ with $i \in I^+$, $g > 0$ on $\mathbb{R}^+ \times \partial S^i$ with $i \in I^-$, and $g = 0$ on $\mathbb{R}^+ \times \partial Ω$. Because of these sign conditions we say that $S^i$ is a source if $i \in I^+$ and a sink if $i \in I^-$. The condition that $g = 0$ on $\mathbb{R}^+ \times \partial Ω$ encodes that
the external boundary $v$ is impermeable. We also assume that, at any time $t$, the function $g(t, \cdot)$ has zero average on $\partial F$, which is the compatibility condition associated with the incompressibility, obtained from the integration of the divergence free condition (1b) over the whole fluid domain $F$. In (1d),

$$\omega^+ : \mathbb{R}^+ \times \partial F^+ \to \mathbb{R},$$

is the entering vorticity. Finally the initial data $v^{in}$ for the fluid velocity is assumed to satisfy

$$\text{div} v^{in} = 0 \text{ in } F.$$

**Definition 1** (Source-sink compatible). We say that a vector field $v : \mathbb{R}^+ \times F \to \mathbb{R}^2$, respectively a function $g : \mathbb{R}^+ \times \partial F \to \mathbb{R}$, is source-sink compatible (SSC) if

$$\text{div}(v) = 0 \text{ in } F, \quad \int_{\partial F} v \cdot n = 0, \quad v \cdot n = 0 \text{ on } \partial \Omega, \quad v \cdot n < 0 \text{ on } \partial F_+, \quad v \cdot n > 0 \text{ on } \partial F_-,$$

and respectively if

$$\int_{\partial F} g = 0, \quad g = 0 \text{ on } \partial \Omega, \quad g < 0 \text{ on } \partial F_+, \quad g > 0 \text{ on } \partial F_-.$$

Moreover, for a SSC vector field $v$, we usually denote the normal trace on the boundary by $g = v \cdot n$ on $\partial F$.

**Remark 1.** The choice of completing the system (1) by prescribing the entering vorticity is not the only one. Another possibility is to prescribe a condition on the pressure, see [33].

**Remark 2.** Systems such as (1) were extensively used in control theory, see for instance [12, Section 6.2], [23], [25], [24]. Let us highlight that the sign conventions may differ here from some of these papers.

### 1.1 Transport of vorticity

The velocity formulation (1) is not well-adapted to a weak formulation and to energy estimates because of the pressure term, in particular of its trace on the permeable part of the boundary. On the other hand Yudovich’s boundary conditions [13-14] are very well adapted to a formulation in terms of the vorticity. In this paragraph we formally derive such a formulation by performing some computations from the velocity formulation above, assuming that we handle a smooth solution.

First we apply the curl operator to the first equation of (1) to obtain the following transport equation for the vorticity $\omega$:

$$\partial_t \omega + v \cdot \nabla \omega = 0 \quad \text{in } \mathbb{R}^+ \times F,$$

$$\omega = \omega^+ \quad \text{on } \mathbb{R}^+ \times \partial F^+, \quad \omega(0, \cdot) = \omega^{in} \quad \text{in } F.$$  

(2a)

(2b)

(2c)

Formally one may solve this transport equation by the methods of characteristics. Because some fluid is entering through a part of the boundary one needs to distinguish two kinds of characteristics. First for any $x \in F$, we consider the position $X^i(t, x)$ at time $t > 0$ of the fluid particle which is at the position $x$ at time $t = 0$ and which moves following the velocity field $v$, that is we consider the ODE

$$\begin{cases}
\partial_t X^i(t, x) = v(t, X^i(t, x)), & t > 0 \\
X^i(0, x) = x.
\end{cases} \quad (3)$$

On the other hand, for any $z \in \partial F_+$ and $s \geq 0$, we consider the position $X^b(t, z, s)$ at time $t > 0$ of the fluid particle which is at the position $z$ at time $t = s$ and which moves following the velocity field $v$, that is we consider the ODE

$$\begin{cases}
\frac{d}{dt} X^b(t, z, s) = v(t, X^b(t, z, s)), & t > s \\
X^b(s, z, s) = z.
\end{cases} \quad (4)$$

4
Then the vorticity at time $t$ can be formally recovered from the initial vorticity and from the entering vorticity as follows. On $[0,T] \times \mathcal{F}$, we define:

- $\omega^i$ by setting, for $(t,x) \in [0,T] \times \mathcal{F}$, $\omega^i(t,x) = \omega_0(y)$ when there is $y \in \mathcal{F}$ such that $x = X^i(t,y)$ and $\omega^i(t,x) = 0$ otherwise,

- $\omega^b$ by setting, for $(t,x) \in [0,T] \times \mathcal{F}$, $\omega^b(t,x) = \omega_+(t,z)$ when there is $(s,z) \in [0,T] \times \partial \mathcal{F}_+$ such that $x = X^b(t,z,s)$ and $\omega^b(t,x) = 0$ otherwise.

Then for any $(t,x)$ in $[0,T] \times \mathcal{F}$, the vorticity can be obtained as

$$\omega(t,x) = \omega^i(t,x) + \omega^b(t,x).$$

Of course, this discussion is very formal since we did not not care about the Cauchy problem for $\omega^i$ and $\omega^b$; indeed even in a smooth setting the flow has to be stopped or extended when the characteristics cross the outlet $\partial \mathcal{F}_+$. However this formal approach gives some insights on the a priori bounds which may be true for the vorticity. In particular it sounds reasonable to expect the following $L^p$ a priori bounds on the vorticity: for $1 \leq q < \infty$,

$$\|\omega(t)\|_{L^q(\mathcal{F})}^q + \int_0^t \int_{\partial \mathcal{F}_-} g|\omega|^q ds dt \leq \|\omega^{in}\|_{L^q(\mathcal{F})}^q + \int_0^t \int_{\partial \mathcal{F}_+} (-g)|\omega^+|^q ds dt, \quad (5)$$

and

$$\max \left\{ \|\omega\|_{L^\infty((0,t) \times \partial \mathcal{F}_-)}, \|\omega^-\|_{L^\infty((0,t) \times \partial \mathcal{F}_-)} \right\} \leq \max \left\{ \|\omega^{in}\|_{L^\infty(\mathcal{F})}, \|\omega^i\|_{L^\infty((0,t) \times \partial \mathcal{F}_+)} \right\}. \quad (6)$$

In view of these a priori bounds, we introduce the following definition regarding the integrability of the data of the problem concerning the vorticity, that is the initial vorticity $\omega^{in}$ and the vorticity $\omega^+$ entering though the inlet $\partial \mathcal{F}_+$.

**Definition 2** (Couple of input vorticities). For $p$ in $[1,+\infty]$, we say that $(\omega^{in}, \omega^+)$ is a couple of input vorticities in $L^p$ or shortly (CIV) in $L^p$ if

$$\omega^{in} \in L^p(\mathcal{F}, dx) \quad \text{and} \quad \omega^+ \in L^p_{loc}(\mathbb{R}^+; L^p(\partial \mathcal{F}^+ \times gds)). \quad (CIV)$$

Similarly, we introduce the following definition regarding the integrability of the vorticity $\omega$ in the domain $\mathcal{F}$ and the one of the the exiting vorticity that is the trace $\omega^-$ of $\omega$ on the outlet $\mathcal{F}_-$.

**Definition 3** (Continuous and weakly continuous couple). For $p$ in $[1, +\infty]$, we say that $(\omega, \omega^-)$ is a continuous couple with values in $L^p$ if

$$\omega \in C(\mathbb{R}^+; L^p(\mathcal{F})) \quad \text{and} \quad \omega^- \in L^p_{loc}(\mathbb{R}^+; L^p(\partial \mathcal{F}^- \times gds)), \quad \text{if } p < +\infty,$$

$$\omega \in C(\mathbb{R}^+; L^\infty(\mathcal{F}) - w^*) \quad \text{and} \quad \omega^- \in L^p_{loc}(\mathbb{R}^+; L^\infty(\partial \mathcal{F}^- \times gds)), \quad \text{if } p = +\infty. \quad (CC)$$

Moreover for $p \in [1, +\infty)$ we say that $(\omega, \omega^-)$ is a weakly continuous couple with values in $L^p$ if

$$\omega \in C(\mathbb{R}^+; L^p(\mathcal{F}) - w) \quad \text{and} \quad \omega^- \in L^p_{loc}(\mathbb{R}^+; L^p(\partial \mathcal{F}^- \times gds)). \quad (WCC)$$

Above the notation $L^\infty(\mathcal{F}) - w^*$ refers to the weak star topology of $L^\infty(\mathcal{F})$ viewed as the topological dual space of $L^1(\mathcal{F})$.

### 1.2 Velocity field as solution of an elliptic problem

The velocity field can be recovered by solving the following div-curl system:

\[
\begin{align*}
\text{div}\ v &= 0 \quad \text{in } \mathcal{F}, \\
\text{curl}\ v &= \omega \quad \text{in } \mathcal{F}, \\
v \cdot n &= g \quad \text{on } \partial \mathcal{F}, \\
\int_{\partial S_i}(v(t,\cdot) \cdot \tau) &= C_i(t) \quad \text{for } i \in \mathcal{I},
\end{align*}
\]
where we denote by $\tau$ the counterclockwise tangent vector to the boundary. The quantities in the last equation are the circulations of the velocity vector field $v$ around the connected components $\partial S^i$. The reason why these circulations are important in the discussion is linked to the multiply-connectedness of the fluid domain $\mathcal{F}$, this will be detailed below, in Section 1.4, after the analysis of the dynamics of these circulations.

1.3 Dynamics of the circulations around the sources and sinks

For each $i \in I$, the circulation $C_i(t)$ evolves in time according to the following Cauchy problem:

$$C_i'(t) = -\int_{\partial S^i} \omega(t,.) g(t,.) ds, \quad C_i(0) = \int_{\partial S^i} v^{in} \cdot nds. \quad (8)$$

This follows from (1a) recast as

$$\partial_t v + \omega v^\perp + \nabla (p + \frac{1}{2} |v|^2) = 0, \quad (9)$$

Indeed, by (7d) and (9),

$$C_i' = \int_{\partial S^i} (\partial_t v) \cdot \tau = -\int_{\partial S^i} \omega \tau \cdot v^\perp - \int_{\partial S^i} \tau \cdot \nabla (p + \frac{1}{2} |v|^2). \quad (10)$$

Since $\partial S^i$ is a closed curve, the second integral is zero whereas the first one can be converted in the right hand side of the first equation in (8) by observing that $\tau \cdot v^\perp = -n \cdot v = -g$. On the other hand the second equality in (8) is another compatibility condition for the initial data $v^{in}$ with the boundary conditions. The identities in (8) are known at least since the paper of Yudovich mentioned above, see [44, Lemma 1.2]. By integration in time of (8), we arrive at the following formula for the circulations at time $t$:

$$C_i(t) = C^{in}_i - \int_0^t \int_{\partial S^i} \omega^+ g \quad \text{for } i \in I^+, \quad \text{and} \quad C_i(t) = C^{in}_i - \int_0^t \int_{\partial S^i} \omega^- g \quad \text{for } i \in I^- \quad (11)$$

Above we have separated the circulations around the sources and the ones around the sinks because the first ones are deduced from the boundary data $\omega^+$ (the entering vorticity) and $g$ (the entering normal velocity); they are therefore themselves to be considered as prescribed data for this problem. On the other hand the second ones are unknowns of the problem since they involve the exiting vorticity that is the trace $\omega^-$ of $\omega$ on $\mathcal{F}^-$. Moreover a computation similar to (10) for the circulation

$$C_{\partial \Omega} = \int_{\partial \Omega} v(t,.) \cdot \tau, \quad (12)$$

around the external boundary $\partial \Omega$ proves that $C_{\partial \Omega}$ is constant in time, because of the impermeability condition on $\partial \Omega$. This is the standard case of Kelvin’s theorem. Finally, by integration of the equation (7b) over the whole fluid domain $\mathcal{F}$, we obtain the following identity, which holds at any time $t$,

$$\int_\mathcal{F} \omega(t,.) = C_{\partial \Omega} + \sum_{i \in I} C_i(t). \quad (13)$$

Therefore the circulation $C_{\partial \Omega}$ does not contain any new information and will not intervene in the sequel.

1.4 Decomposition of the velocity

Let us first consider the following potential lift of the boundary data $g$ for the normal velocity: with any smooth enough $g$ we associate $v_g = \nabla \varphi$, where $\varphi_g$ is the unique solution of

$$\begin{cases}
-\Delta \varphi_g = 0 & \text{in } \mathcal{F}, \\
\nabla \varphi_g \cdot n = g & \text{in } \partial \mathcal{F}.
\end{cases}$$


The regularity of the vector field \( v_g \) depends on the boundary data \( g \). For more in this direction we refer for example to [21] and [28].

Let us also recall that for any smooth function \( \omega \) there is a unique vector field \( K_H[\omega] \) satisfying

\[
\begin{align*}
\text{div} \, K_H[\omega] &= 0 \quad \text{in } F, \quad (14a) \\
\text{curl} \, K_H[\omega] &= \omega \quad \text{in } F, \quad (14b) \\
K_H[\omega] \cdot n &= 0 \quad \text{on } \partial F, \quad (14c) \\
\int_{\partial S_i} K_H[\omega] \cdot \tau &= 0 \quad \text{for } i \in I. \quad (14d)
\end{align*}
\]

The mapping \( \omega \mapsto K_H[\omega] \) is called the hydrodynamical Biot-Savart law. It can be written as an integral operator of the form

\[
K_H[\omega](x) = \int_F K(x, y) \omega(y) \, dy. \quad (15)
\]

Moreover it follows from the Hodge-De Rham theory that the vector space of the vector fields \( v \) satisfying \( \text{div} \, v = 0 \) and \( \text{curl} \, v = 0 \) in \( F \), and \( v \cdot n = 0 \) on \( \partial F \) is of dimension \( N \) (i.e. the number of holes in the fluid domain), and a basis of this vector space is given by the unique vector fields \( (X_i)_{i \in I} \) satisfying

\[
\begin{align*}
\text{div} \, X_i &= 0 \quad \text{in } F, \quad (16a) \\
\text{curl} \, X_i &= 0 \quad \text{in } F, \quad (16b) \\
X_i \cdot n &= 0 \quad \text{on } \partial F, \quad (16c) \\
\int_{\partial S_i} X_i \cdot \tau &= \delta_{ij} \quad \text{for } j \in I, \quad (16d)
\end{align*}
\]

where the notation \( \delta_{ij} \) stands for the Kronecker symbols. In view of the \textit{a priori} bounds \( 5 \) and \( 6 \) some important estimates regarding the operator \( \omega \mapsto K_H[\omega] \) are the following: for \( 1 < q < \infty \), there exists a constant \( C > 0 \) such that

\[
\|K_H[\omega]\|_{W^{1,q}(F)} \leq C \|\omega\|_{L^q(F)}, \quad (17)
\]

and a constant \( C > 0 \) such that

\[
\|K_H[\omega]\|_{L^L(F)} \leq C \|\omega\|_{L^\infty(F)}, \quad (18)
\]

Moreover for any smooth functions \( g \) and \( \omega \), there is a unique solution \( v \) to \( 7 \) and \( v \) can be decomposed into

\[
v = v_g + \sum_{i \in I} C_i(t) X_i + K_H[\omega]. \quad (19)
\]

We refer here to [26, 32, 20, 35] for more.

### 1.5 Formal vorticity formulation

Gathering \( 2 \), \( 11 \) and \( 12 \), we deduce that the system \( 1 \) is formally equivalent to following vorticity-based reformulation:

\[
\begin{align*}
\partial_t \omega &+ v \cdot \nabla \omega = 0 \quad \text{in } \mathbb{R}^+ \times F, \quad (20a) \\
\omega &= \omega^+ \quad \text{on } \mathbb{R}^+ \times \partial F^+, \quad (20b) \\
\omega(0, \cdot) &= \omega^{in} \quad \text{in } F, \quad (20c) \\
v &= v_g + \sum_{i \in I} C_i(t) X_i + K_H[\omega] \quad \text{in } \mathbb{R}^+ \times F, \quad (20d) \\
C_i(t) &= C_i^{in} - \int_0^t \int_{\partial S_i} \omega^+ g \quad \text{for } i \in I^\pm. \quad (20e)
\end{align*}
\]

A few comments are in order.
Let us insist of the fact that there are two unknowns to the system (20) which are $\omega$ and $\omega^-$. In particular this is an additional feature of the present setting where the fluid exits through some holes in the domain that the exiting vorticity $\omega^-$ is necessary to determine the velocity vector field $v$, see (20). Let us mention that in the case, which is not considered in this paper, where the fluid exits from the domain only through a part of the external boundary $\partial \Omega$ then the determination of the velocity $v$ inside the fluid domain $\mathcal{F}$ is decoupled from the exiting vorticity $\omega^-$, and so is the the determination of the vorticity $\omega$ inside the fluid domain $\mathcal{F}$.

Above the discussion has been quite formal; we did not care about the regularity or kind of solution for which the equivalence of the system (1) and of the system (20) holds true. Indeed the formulation (20) seems much more appropriate to formulate rigorous mathematical results on the problem at stake. In this direction, it is worth to highlight that the velocity vector field $v$ being divergence free in $\mathcal{F}$, the first equation of (20) can be rewritten in the conservative form:

$$\partial_t \omega + \text{div}(\omega v) = 0 \quad \text{in } \mathbb{R}^+ \times \mathcal{F}. \quad (21)$$

2 A first glance on the main results

To avoid the reader to wait too long for an exposition of the main results of this paper, we first state the following informal statement which offers in a single glance some assertions regarding the existence of solutions of some different appropriate weak formulations of the system (20) with input vorticities in $L^p$.

**Theorem 1.** Let $p$ in $[1, +\infty]$ and $(\omega^{in}, \omega^+)$ a couple of input vorticities (CIV) in $L^p$. Then there is $(\omega, \omega^-)$ a continuous couple (CC) with values in $L^p$ solution of the system (20). This solution have to be understood in different ways depending on the range of $p$ according to the following cases:

(i) for $p$ in $(4/3, +\infty]$, there exists a solution in a distributional sense,

(ii) for $p$ in $(1, +\infty]$, there exists a solution in a renormalized sense,

(iii) for $p = 1$, there exists a solution of a symmetrized formulation.

In each of these cases, these solutions can be obtained as vanishing viscosity limits.

These results on the existence of weak solutions complement the existence (and uniqueness) of smooth (typically with vorticity in $W^{2,p}$) solutions obtained in in the work [44] by Yudovich. We refer to Theorem 2, Theorem 3, and Theorem 4 for precise statements corresponding respectively to the three cases above. Yet, let us already state a few remarks.

- In the part (i) of Theorem 1 we refer to the existence of solutions of a weak formulation with test functions supported up to the boundary, making a slight abuse of language by using the terminology of distributional solutions. Indeed such an existence result has already been proved in [34] by Mamontov and Uvarovskaya. Their proof makes use of smooth solutions of (20), given in the work [44] by Yudovich, corresponding to regularized input data. Here we will take a slightly different path by considering some parabolic approximations, that is some Navier-Stokes type equations with vanishing viscosity. This suggest that these solutions are perhaps more physical. However we consider an artificial boundary condition for these parabolic approximations. A further step in the direction of the construction of more physical solutions could be to consider more physical boundary conditions for such viscous approximations. In this direction, let us mention the papers [9] and [10], where some Navier slip-with-friction boundary conditions are considered.

- In the part (ii) of Theorem 1 we refer to the renormalization theory as initiated by Di Perna and Lions in [17] for the transport equations. Moreover the part (ii) of Theorem 1 can be seen
as an extension of the result [14] by Crippa and Spirito in the case of the two-dimensional incompressible Euler equations without any source nor sink. It extends the part (i) in the sense that for $p$ in $(4/3, +\infty]$, the two types of solutions: distributional and renormalized, are equivalent.

1. In the part (iii) of Theorem 1, we refer to a weak reformulation of the problem inspired by the works [15] and [38] respectively by Delort and Schochet where the case of a diffuse positive Radon measure as initial vorticity is addressed, in the case without any source nor sink. A crucial point in these works is that an energy estimate allows to prevent from a vorticity concentration in Dirac masses at some positive times. Such an argument seems difficult to reproduce in our setting, what leads us to deal with the easier case where the vorticity is $L^1$, for which an argument of propagation of compactness can be used to prevent from any concentration. However, even with this restriction, some difficulties appear with the boundary conditions. Indeed the symmetrized formulation hinted here suffers from a loss of information regarding the prescription of the entering vorticity: only the vorticity fluxes integrating on the whole boundary of each sink are encoded in the formulation, not their pointwise values, see Remark 4 below. Let us therefore highlight that the part (iii) of Theorem 1 has to be seen only as a partial result in the $L^1$ case.

2. In the three cases, the regularity of the velocity field is enough to give a sense to the circulations [16] and [12]. Moreover, for the solutions which are constructed in the proof of Theorem 1 the time evolution of the circulations around the sources and sinks is given by [20c] whereas the circulation around the external boundary is constant. Finally the conservation law (13) holds at any time.

**Remark 3.** The issue of the uniqueness of weak solutions to the system above is a delicate topic, which requires some different types of argument. It is the object of current investigations.

### 3 Precise existence results of distributional and renormalized solutions

In this section we precisely state the existence results of distributional and renormalized solutions: the parts (i) and (ii) of Theorem 1.

#### 3.1 Existence of distributional solutions

Let us start with Part (i), that is the existence of distributional solutions with $L^p$ vorticity when $p > 4/3$. The terminology “distributional solutions” refers to the transport equation for the vorticity: we will require it to be satisfied in the sense that for any $\varphi$ in $C^\infty_0([0, +\infty) \times \mathcal{F}; \mathbb{R})$,

\[
\int_\mathcal{F} \omega^i \varphi(0, .)dx + \int_{\mathbb{R}^+} \int_\mathcal{F} \omega(\partial_t \varphi + v \cdot \nabla \varphi) dx dt = \int_{\partial \mathcal{F}^+} \int_{\mathbb{R}^+} g_\omega^+ \varphi ds dt + \int_{\partial \mathcal{F}^-} \int_{\mathbb{R}^+} g_\omega^- \varphi ds dt. \tag{22}
\]

For any test function $\varphi$ in $C^\infty_0([0, +\infty) \times \mathcal{F}; \mathbb{R})$, the equation (22) is obtained from the equation (21) by multiplying it by $\varphi$ and integrating by parts taking into account the boundary conditions: $v \cdot n = g$ on $\partial \mathcal{F}$ and $\omega = \omega^+$ on $\partial \mathcal{F}^+$. As already mentioned the function $\omega^-$ is an unknown, in particular because the existence of a trace on $\partial \mathcal{F}^-$ of a function $\omega$ which is only in $L^\infty (\mathbb{R}^+; L^p(\mathcal{F}))$ does not follow from standard trace theorems. On the other hand let us highlight that if $\omega$ is a smooth solution of the transport equation (21) then an integration by parts provides the equation (22) with the trace of $\omega$ on $\partial \mathcal{F}^-$ instead of $\omega^-$. The result that is alluded to in Part (i) of Theorem 1 is the following.
Theorem 2. Let \( p \in (4/3, \infty] \). Let \( C_{\text{in}}^{n} \) in \( \mathbb{R} \) for each \( i \) in \( I \) the initial circulations around \( \partial S^{1} \). Let \( g \) a source-sink compatible function \( \text{SSC}_{\text{loc}} \) in \( L_{1, \text{loc}}^{1}(\mathbb{R}^{+}; W^{-1/p,p}(\partial F)) \) in the case where \( p \in (1, \infty) \) and in \( L_{1, \text{loc}}^{1}(\mathbb{R}^{+}; W^{-1/\infty,\infty}(\partial F)) \) in the case where \( p = \infty \). Let \( (\omega^{n}, \omega^{+}) \) a couple of input vorticities \( \text{CI}^{+} \) in \( L^{p} \). Then there exists \( (\omega, \omega^{-}) \) a continuous couple \( \text{CC} \) with values in \( L^{p} \) such that for any \( \varphi \in C_{c}^{\infty}([0, +\infty) \times \overline{F}; \mathbb{R}) \), the identity (22) is satisfied with \( v \) given by (20a) and (20e), satisfying \( \textit{5} \) with equal sign and with \( q = p \) in the case where \( p < +\infty \) and \( \textit{6} \) in the case where \( p = +\infty \).

As already mentioned above this result has been obtained by Mamontov and Uvarovskaya, see [34]. However we will provide in Section 7 a slightly different proof based on some viscous approximations which are introduced in Section 6.

3.2 Existence of renormalized solutions

Now let us turn our attention to the part (ii) of Theorem 1 that is to the existence of renormalized solutions with \( L^{p} \) vorticity when \( p > 1 \). It is based on the observation that for smooth vorticities satisfying the transport equation (21a) and for any function \( \beta \in C_{c}^{1}(\mathbb{R}) \), by the chain rule,

\[
\partial_{t}\beta(\omega) + v \cdot \nabla \beta(\omega) = 0.
\]

Then, for any test function \( \varphi \) in \( C_{c}^{\infty}([0, +\infty) \times \overline{F}; \mathbb{R}) \), by multiplying the previous equation by \( \varphi \) and integrating by parts taking into account the boundary condition: \( v \cdot n = g \) on \( \partial F \) and \( \omega = \omega^{+} \) on \( \partial F^{+} \), we arrive at

\[
\int_{F} \beta(\omega^{n})\varphi(0, .)dx + \int_{\mathbb{R}^{+}} \int_{F} \beta(\omega)(\partial_{t}\varphi + v \cdot \nabla \varphi)dxdt = \int_{\mathbb{R}^{+}} \int_{\partial F^{+}} g\beta(\omega^{+})\varphi ds dt (23) + \int_{\mathbb{R}^{+}} \int_{\partial F^{-}} g\beta(\omega^{-})\varphi ds dt.
\]

The terminology “renormalized solution” precisely refers to a vorticity satisfying such identities.

Theorem 3. Let \( p \in (1, \infty] \). Let \( C_{\text{in}}^{n} \in \mathbb{R} \) for \( i \in I \) the initial circulations around \( \partial S^{1} \). Let \( g \in L_{1, \text{loc}}^{1}(\mathbb{R}^{+}; W^{-1/p,p}(\partial F)) \) in the case where \( p \in (1, \infty) \), and \( g \in L_{1, \text{loc}}^{1}(\mathbb{R}^{+}; W^{-1/\infty,\infty}(\partial F)) \) in the case where \( p = \infty \). Assume that \( g \) is source-sink compatible, see \( \text{SSC}_{\text{loc}} \). Let \( (\omega^{n}, \omega^{+}) \) a couple of input vorticities \( \text{CI}^{+} \) in \( L^{p} \). Then there exists a continuous couple \( (\omega, \omega^{-}) \) with values in \( L^{p} \) (see \( \text{CC} \)) is a renormalized solution, that is, it satisfies (23) for any test function \( \varphi \) in \( C_{c}^{\infty}([0, +\infty) \times \overline{F}; \mathbb{R}) \), with \( v \) given by (20a) and (20e), Moreover it satisfies \( \textit{5} \) with equal sign and with \( q = p \) for \( p < \infty \) and \( \textit{6} \) for \( p = \infty \).

The proof of Theorem 3 is given in Section 8.

4 Precise existence results of symmetrized solutions

In this section we give a precise statement of the point (iii) of Theorem 1. We first recall the symmetrization argument, which has appeared in the works [40], [15] and [38], and which leads to the formulation hinted in the point (iii) of Theorem 1. The starting point of this argument is to recast the nonlinear term \( \omega v \) of (21) in a weak sense by using the integral expression of \( v \) in terms of \( \omega \).

4.1 Case of a full plane

To recall this idea in a simple way let us consider the case where \( v \) is given in terms of \( \omega \) by the usual Biot-Savart law in \( \mathbb{R}^{2} \):

\[
v(x) = \int_{\mathbb{R}^{2}} K_{\mathbb{R}^{2}}(x, y)\omega(y) dy \quad \text{where } K_{\mathbb{R}^{2}}(x, y) := \frac{(x - y)^{\perp}}{2\pi|x - y|^{2}},
\]

(24)
Observe that we have dropped here the time variable to simplify the exposition of the core of the argument for which it only plays the role of a parameter. Then for any test function \( \varphi \in C_c^\infty(\mathbb{R}^2) \),

\[
\begin{align*}
\int_{\mathbb{R}^2} \omega(x)v(x) \cdot \nabla \varphi(x) \, dx &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_{\mathbb{R}^2}(x,y) \cdot \nabla \varphi(x) \omega(x) \omega(y) \, dx \, dy \\
&= \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_{\mathbb{R}^2}(x,y) \cdot (\nabla \varphi(x) - \nabla \varphi(y)) \omega(x) \omega(y) \, dx \, dy,
\end{align*}
\]

(25)

by symmetrization. The interest of the symmetrization is that, when \( x \) and \( y \) are close, one has, by Taylor’s expansion,

\[
\nabla \varphi(x) - \nabla \varphi(y) \sim D^2 \varphi(x)(x-y),
\]

(26)

and therefore the term

\[
K_{\mathbb{R}^2}(x,y) \cdot (\nabla \varphi(x) - \nabla \varphi(y)) \sim \frac{1}{2\pi} D^2 \varphi(x) \left( \frac{x-y}{|x-y|}, \frac{(x-y)^{\perp}}{|x-y|} \right),
\]

(27)

remains bounded for \((x,y)\) in \(\mathbb{R}^2 \times \mathbb{R}^2\). As a consequence the right-hand-side of (25) makes sense for a vorticity \(\omega\) in \(L^1(\mathcal{F})\).

4.2 Case of a general domain

For a general domain \(\mathcal{F}\), by (14), and using a symmetrisation with respect to \(x\) and \(y\) as above, we obtain that, for any test function \(\varphi \in C^\infty(\mathcal{F})\),

\[
\int_{\mathcal{F}} \omega K_\mathcal{F}[\omega] \cdot \nabla \varphi = \int_{\mathcal{F}} \int_{\mathcal{F}} H_\varphi(x,y)\omega(t,x)\omega(t,y) \, dx \, dy,
\]

(28)

where \(H_\varphi\) is the following auxiliary function

\[
H_\varphi(x,y) = \frac{1}{2} \left( \nabla_x \varphi(x) \cdot K(x,y) + \nabla_y \varphi(y) \cdot K(y,x) \right).
\]

(29)

It is classical, see [8 18 22], that there exists a constant \(C\) such that for every \(x,y \in \mathcal{F}\),

\[
|K(x,y)| \leq \frac{C}{|x-y|}.
\]

(30)

Moreover, since in the case of the full plane the counterpart of the function \(H_\varphi\) is half the function given in the left hand side of (27), which is bounded, one could wonder whether in the general case \(H_\varphi\) is bounded or not. Since \(H_\varphi\) can be decomposed as

\[
H_\varphi(x,y) = \frac{1}{2} \nabla_x \varphi(x) \cdot \left( K(x,y) + K(y,x) \right) - \frac{1}{2} \left( \nabla_x \varphi(x) - \nabla_y \varphi(y) \right) \cdot K(y,x),
\]

(31)

where the second term is bounded thanks to \(26\) and \(30\), the question reduces to determine whether \(K(x,y) + K(y,x)\) is bounded or not. Indeed in the interior of the domain, the desingularization still occurs because the Biot-Savart law associated with any domain is, away from the boundary, a regular perturbation of the Biot-Savart law associated with the full plane and given in (24). More precisely \(K(x,y)\) can be decomposed as

\[
K(x,y) = K_{\mathbb{R}^2}(x,y) + R(x,y),
\]

(32)

with \(R\) smooth in the interior set \(\mathcal{F} \times \mathcal{F}\). Thus

\[
K(x,y) + K(y,x) = R(x,y) + R(y,x),
\]

is bounded on any compact subset of \(\mathcal{F} \times \mathcal{F}\). As a consequence the right-hand-side of (28) makes sense for a vorticity \(\omega\) in \(L^1(\mathcal{F})\) and for any test function \(\varphi \in C_c^\infty(\mathcal{F})\).
On the other hand, close to the boundary, some caution is needed. To illustrate the difficulty at stake, let us first consider the case where the fluid occupies a half-space. The Green function associated with the Laplace operator in the right half-plane with the Dirichlet condition on \( \mathbb{R} \times \{0\} \) is given for any \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) in \( F \) by

\[
\frac{1}{4\pi} \ln \left( 1 + \frac{4x_2y_2}{|x - y|^2} \right).
\]

The gradient of this function with respect to its first variable, is then given by

\[
\frac{1}{\pi(|x - y|^2 + 4x_2y_2)} \left( (0, y_2) - 2x_2y_2 \frac{x - y}{|x - y|^2} \right),
\]

and its symmetrization is therefore

\[
\frac{(0, x_2 + y_2)}{\pi(|x - y|^2 + 4x_2y_2)}.
\]

On the one hand the tangential component vanishes, on the other hand the normal component is singular when \( x \) and \( y \) are close to each other and to the boundary. Thus the cancellation observed in the case of the full plane is maintained in the case of a half-plane up to the boundary for the tangential component but not for the normal one. Looking back to \( \text{(31)} \) we observe that for any test function \( \varphi \) in \( C^\infty(\overline{F}) \) with \( \nabla \varphi \) normal to the boundary, the cancellation observed above in the case of the full plane is maintained and the auxiliary function \( H_\varphi \) is again bounded, whereas it is not the case for a general test function \( \varphi \) in \( C^\infty(\overline{F}) \).

This conclusion can be extended to any bounded domain with smooth boundary by using the mirror method, which is the local approximation of the Green function of a general bounded domain associated with the Dirichlet on the boundary condition on \( \partial F \) by

\[
G(x, y) = \frac{1}{2\pi} \ln \frac{|x - y|}{|x - y|} + O(1),
\]

as \( x, y \to x_0 \in \partial F \), in \( C^1 \), where \( \overline{x} \) is the mirror image of \( x \) through \( \partial F \). This mirror image of \( x \) is well-defined for \( x \) sufficiently close to the boundary \( \partial F \) by the formula \( \overline{x} := 2p(x) - x \), where \( p(x) \) is the orthogonal projection of \( x \) on \( \partial F \) that is the element of \( \partial F \) which satisfies \( |p(x) - x| = d(x) \), where \( d(x) \) denotes the distance to the boundary. Moreover there is a neighborhood \( \mathcal{V} \) of \( \partial F \) where \( p \) is \( C^1 \) and its derivative can be explicitly computed in terms of \( d(x) \), of the unit tangent vector to \( \partial F \) at \( p(x) \), and of the curvature of \( \partial F \) at \( p(x) \). This allows to approximate the function \( H_\varphi(x, y) \) in \( C^0 \) as \( x, y \to x_0 \in \partial F \) and to conclude that the following property holds for the auxiliary function \( H_\varphi(x, y) \) when the test function \( \varphi \) is in the space \( \mathcal{C}_0(\mathcal{F}) \) of the functions in \( C^\infty(\overline{F}) \) which are constant on any connected component of the boundary (with a constant depending on each connected component) and equal to 0 on \( \partial \Omega \).

**Lemma 1.** For any test function \( \varphi \) in \( \mathcal{C}_0(\mathcal{F}) \), the function \( H_\varphi \) is bounded on \( \mathcal{F} \times \mathcal{F} \setminus \{(x, x) : x \in \mathcal{F}\} \).

Thanks to \( \text{(19)} \), we can recast the nonlinear term \( \omega v \) of \( \text{(21)} \) in a weak sense for any time-dependent test function \( \varphi \) in \( C^\infty_c([0, +\infty); \mathcal{C}_0(\mathcal{F})) \),

\[
\int_F \omega v \cdot \nabla \varphi = \int_F \omega v_y \cdot \nabla \varphi + \sum_i C_i(t) \int_F \omega X_i \cdot \nabla \varphi + \int_F \int_F H_\varphi(x, y)\omega(t, x)\omega(t, y) \, dx \, dy.
\]

### 4.3 Symmetrized formulation with \( L^1 \) vorticity

With the previous considerations in hand we are now ready to precise the sense in which the solutions are considered in the point \( \text{(18)} \) of Theorem \( \text{1} \). Combining \( \text{(19)}, \text{(22)}, \text{(28)} \) and \( \text{(34)} \) we are led to the following definition.
Definition 4. We say that a weakly continuous couple \((\omega, \omega^-)\) with values in \(L^1\), see [WCC], is a symmetrized solution to (38) if for any \(\varphi\) in \(C_c^\infty([0, +\infty); \mathcal{C}_0(\mathcal{F}))\),
\[
\int_F \omega^i \varphi(0,.)dx + \int_{\mathbb{R}^+} \int_F \omega(\partial_t \varphi + v_g \cdot \nabla \varphi) dx dt + \sum_i \int_{\mathbb{R}^+} C_i(t) \int_F \omega X_i \cdot \nabla \varphi dx dt \tag{35}
\]
\[
+ \int_{\mathbb{R}^+} \int_F \int_F H_\varphi(x,y)\omega(t,x)\omega(t,y) dx dy dt = \int_{\mathbb{R}^+} \int_{\partial F^+} g^+ \varphi ds dt \\
+ \int_{\mathbb{R}^+} \int_{\partial F^-} g^- \varphi ds dt,
\]
where
\[
C_i(t) = C_i^{in} - \int_0^t \int_{\partial S^i} \omega^g \text{ for } i \in I^+ \text{ and } C_i(t) = C_i^{in} - \int_0^t \int_{\partial S^i} \omega^- g \text{ for } i \in I^- \tag{36}
\]
Remark 4. Observe that the weak formulation (38), when restricted to test functions \(\varphi(t,.)\) in \(\mathcal{C}_0\) suffers from a loss of information on the boundary data since it only depends on \(\omega^+\) through the integrals
\[
\int_{\partial F^+} g \omega^+ dz, \tag{37}
\]
where the time variable \(t\) plays the role of implicit parameter and takes its values in \(\mathbb{R}_+\). In particular if one considers two sets of smooth data, for the initial and boundary conditions, for which the entering vorticities are distinct but with the same value for the total entering vorticity flux given by (37), then by the existence and uniqueness result of Yudovich in [44] there are corresponding smooth solutions to the system (20). These two solutions satisfy the weak formulation (38) for any test functions \(\varphi\) in \(C_c^\infty([0, +\infty); \mathcal{C}_0(\mathcal{F}))\), but are distinct since their respective traces on \(\partial F^+\) are supposed to be different. On the other hand, in this approach, this unfortunate restriction of the definition of the data seems mandatory because of the discussion in Section 4.2.

The precise statement hinted in the point (31) of Theorem 1 is the following.

Theorem 4. Let \(C_i^{in} \in \mathbb{R}\) for \(i \in I\) the initial circulations around \(\partial S^i\). Let \(g \in L^1_{loc}(\mathbb{R}^+; L^1(\partial F))\) source-sink compatible, see [SSC]. Let \((\omega^{in}, \omega^+)\) a couple of input vorticities (CIV) in \(L^1\). Then there is a symmetrized solution \((\omega, \omega^-)\) to (38) in the sense of Definition 4 and satisfying the inequality (33) with \(q = 1\).

The proof of Theorem 4 is given in Section 9.

5 Remainder on the transport equation with given non-tangential velocity

In this section we recall a few instrumental facts regarding the transport equation
\[
\partial_t \omega + v \cdot \nabla \omega = 0 \quad \text{in } \mathbb{R}^+ \times \mathcal{F}, \tag{38a}
\]
\[
\omega = \omega^+ \quad \text{on } \mathbb{R}^+ \times \partial F^+, \tag{38b}
\]
\[
\omega(0,.) = \omega^{in} \quad \text{in } \mathcal{F}, \tag{38c}
\]
where the [SSC] vector field \(v\) is assumed to be given and to satisfy the assumptions:
\[
v \in L^1_{loc}(\mathbb{R}^+; W^{1,q}(\mathcal{F})) \quad \text{and } \text{div } v = 0 \text{ in } \mathcal{F}, \tag{39}
\]
for some \(q \in [1, +\infty]\).

This is a quite classical topic in the case where the boundary \(\partial \mathcal{F}\) is impermeable. In the case where the fluid can enter into and exit of the boundary \(\partial \mathcal{F}\) of the domain, a nice reference is [1] where trace issues, as well as existence and uniqueness of solutions in various senses for the system
is considered. Note that in [6], the author assumes extra regularity of the normal component of the velocity on the boundary, namely \( v \cdot n \in L^p_{\text{loc}}(\mathbb{R}^+ \times \partial F) \) for some \( \alpha > 1 \). In our work this condition is removed thanks to the peculiar geometry at stake, as explained in the following remark.

**Remark 5.** In this paper we will assume the velocity fields to be \([SSC]\), which implies that the regions where the normal component of the velocity field has distinguished signs are well-separated, in the sense that there exists a smooth cut-off \( \psi \) that is identically equal to 1 in a neighborhood of \( \partial F^+ \) and identically equal to 0 a neighborhood of the remaining boundary. With the help of this cut-off it is then possible to deduce information on \( \omega \) on \( \partial F^+ \), and on \( \partial F^- \), from the value of \( \omega \) in \((0, T) \times F\). To do that is enough to multiplying \([SSA]\) with the smooth cut-off \( \psi \) (or \(1 - \psi\)), to integrate in \((0, T) \times F\) and to do some integrations by parts. In the general case when the normal component of the velocity does not have a sign, it is not clear how to define a cut-off that separates the regions where \( v \cdot n \) has different signs. In [6], this difficulty is tackled by assuming that \( v \cdot n \in L^p_{\text{loc}}(\mathbb{R}^+ \times \partial F) \) for some \( \alpha > 1 \). This hypothesis is used in an essential manner in [6] (3.14]). Within our geometrical setting, with the separation of the regions of the boundary where the normal component of the velocity field has different signs, the results showed in [6] are valid without the extra integrability assumption \( v \cdot n \in L^p_{\text{loc}}(\mathbb{R}^+ \times \partial F) \) for some \( \alpha > 1 \).

### 5.1 Distributional and renormalized solutions

In this subsection we recall the definitions of distributional and renormalized solutions to the transport equation \([SSA]\). We recall first the following terminology.

**Definition 5.** We say that \( p \) and \( q \) in \([1, +\infty]\) are conjugated if \( \frac{1}{p} + \frac{1}{q} = 1 \).

Distributional solutions of the transport equations are then defined as follows.

**Definition 6.** Let \( p \) in \([1, +\infty]\), let \( v \) satisfying the hypothesis \([SSA]\), with \( q \) such that \( p \) and \( q \) in \([1, +\infty]\) are conjugated, and let \((\omega^m, \omega^+)^{(CIV)}\) in \(L^p\). We say that a continuous couple \((CC)\) with values in \(L^p(\omega, \omega^-)\) is a distributional solution to \([SSA]\) if for any \( \varphi \) in \(C^\infty_0((0, +\infty) \times F; \mathbb{R})\), the identity \([22]\) is satisfied.

On the other hand renormalized solutions of the transport equations are defined as follows whatever \( p \) and \( q \) are conjugated or not.

**Definition 7.** Let \( p \) in \([1, +\infty]\), let \( v \) satisfying hypothesis \([SSA]\) and \((\omega^m, \omega^+)\) \((CIV)\) in \(L^p\). We say that a continuous couple \((CC)\) with values in \(L^p(\omega, \omega^-)\) is a renormalized solution to \([SSA]\) if for any \( \beta \) in \(C^\infty_0(\mathbb{R})\) and for any \( \varphi \) in \(C^\infty_0((0, +\infty) \times F; \mathbb{R})\), the identity \([22]\) is satisfied.

### 5.2 Existence and uniqueness results

The following result gathers some results regarding the existence and uniqueness of distributional and renormalized solutions to the transport equation \([SSA]\) and a duality formula.

**Proposition 1.** Let \( p \) and \( q \) in \([1, +\infty]\), let \( v \) satisfying the hypothesis \([SSA]\), let \((\omega^m, \omega^+)\) \((CIV)\) in \(L^p\). Then we have the following results.

1) If \( p \) and \( q \) in \([1, +\infty]\) are conjugated, there exists a unique distributional solution \((\omega, \omega^-)\) in \(L^p\) to \([SSA]\) (in the sense of Definition \(6\)).

2) If \( p \) and \( q \) in \([1, +\infty]\) are conjugated, any distributional solution \((\omega, \omega^-)\) in \(L^p\) to \([SSA]\) (in the sense of Definition \(6\)) is a renormalized solution to \([SSA]\) (in the sense of Definition \(7\)).

3) There exists a unique renormalized solution \((\omega, \omega^-)\) in \(L^p\) to \([SSA]\) (whatever \( p \) and \( q \) are conjugated or not).
4) If \( p \) and \( q \) in \([1, +\infty]\) are conjugated, if \((\omega, \omega^-)\) is a \(L^p\) renormalized solution to \(\Psi\), \(\Psi\) is in \(L^q([0, T] \times \partial F^-; g d\omega d\tau)\), \(\phi_T\) is in \(L^q(F)\), \(\chi\) is in \(L^1([0, T]; L^q(F))\), and \((\phi, \phi^+)\) is a \(L^q\) renormalized solution of the backward transport equation

\[
-\partial_t \phi - v \cdot \nabla \phi = \chi \quad \text{in } [0, T] \times F, \\
\phi = \Psi \quad \text{on } [0, T] \times \partial F^-,
\]

\(\phi(T, x) = \phi_T(x)\),

then

\[
\int_0^T \int_F \omega \chi + \int_0^T \int_{\partial F^-} g \omega^- \Psi = \int_F \omega^\alpha \phi(0, .) - \int_F \omega(T, .) \phi_T - \int_0^T \int_{\partial F^+} g \omega^+ \phi^+. 
\]

**Proof of 1.** The case where \( p \) is in \((1, +\infty)\) is already proved in [6]. In particular [6] Th. 4.1] and [6, Th. 4.1] deal with respectively \( p = +\infty \) and \( p \) in \((1, +\infty)\). Moreover the case where \( p = 1 \) can be tackled along the same way. We briefly recall the sketch of the proof in [6] for sake of completeness. Regarding existence it is enough to consider a viscous approximation, quite similar to the one we will introduce in Section 4 and to pass to the limit in the weak formulation from a priori bounds. On the other hand, at a formal level, uniqueness follows from the linearity of the transport equation and of the a priori estimates. To justify rigorously these steps it is enough to consider the regularization of the solutions, see [6, Sec. 2.2], which satisfy pointwise almost everywhere the transport equation associated with the vector field \( v \) and with a source term that converge to zero in \(L^1\). The \(L^1\) a priori bound for the difference of two regularized solutions implies uniqueness.

**Proof of 2.** It is a consequence of point 2 of [6, Th. 3.1]. Indeed, in this reference, a stronger result is proved, since \( \omega \) is there only assumed to satisfy \(\Psi\) for test functions supported away from the boundary. Then the existence of traces on the boundary such that the weak formulation holds for test functions supported up to the boundary is proved as a consequence of the fact that \( \omega \) satisfies the transport equation in the distributional sense inside. As presented in Remark 5 in [6] the velocity field satisfies the extra hypothesis \( v \cdot n \in L^p_{loc}(\mathbb{R}^+ \times \partial F) \) for some \( \alpha > 1 \), which is replaced in our setting by the fact that the regions where \( v \cdot n \) has different sign are well-separated.

**Proof of 3.** The existence of a renormalized solution \((\omega, \omega^-)\) in \(L^p\) in the case where \(1/p + 1/q = 1\) follows from points 1) and 2). In the case where \(1/p + 1/q > 1\), consider an injective function \( b \in C^1_b(\mathbb{R}) \). Since \((b(\omega^m), b(\omega^+)) \in L^\infty \cap L^1\) and \(1/\infty + 1/1 < 1\), there exists \((b, b^-)\) a renormalized solution to the transport associated with the data \((b(\omega^m), b(\omega^+))\). Using the injectivity of \( b \) we define \((\omega, \omega^-) := (b^{-1}(b), b^{-1}(b^-))\) and we observe that it is a renormalized solution to the transport equation. Let us now prove the uniqueness part of the statement. Suppose that there exists another renormalized solution \((\tilde{\omega}, \tilde{\omega}^-)\) associated with the same initial data. By definition, for the same function \( b \), the couple \((b(\tilde{\omega}), b(\tilde{\omega}^-))\) is a distributional solution to the transport equation. Uniqueness of point 1) we deduce that \((b(\omega), b(\omega^-)) = (b(\tilde{\omega}), b(\tilde{\omega}^-))\). Since \( b \) is injective, this implies that \((\omega, \omega^-) = (\tilde{\omega}, \tilde{\omega}^-)\).

**Proof of 4.** At a formal level it is enough to test the weak formulation satisfied by \((\omega, \omega^-)\) with \((\phi, \phi^+)\). To show this point rigorously, we proceed as previously by considering an appropriate regularization of \( \phi \) with a smoothing kernel, see [6, Sec. 2.2], which in particular does not change boundary data. This regularization of \( \phi \) satisfies the transport equation associated with the velocity field \( v \) and with right hand side a function that converge to \( \chi \) in \(L^1\). Then we consider the weak formulation of the renormalized transport equation satisfied by \((\omega, \omega^-)\) with as test function the previous regularization of \( \phi \) and we pass to the limit to obtain (40) for \((b(\omega), b(\omega^-))\). Finally we conclude by approximating \( b(x) = x \) via an appropriate approximation-truncation process.
6 Smooth viscous approximations

This section is devoted to an auxiliary system, a transport-diffusion equation with a small parameter in front of the diffusion term, which is useful to construct solutions to the transport equation [21].

6.1 A family of viscous approximated models

We consider, for $\nu$ in $(0,1)$,
\begin{align*}
\partial_t \omega_{\nu} + v_{\nu} \cdot \nabla \omega_{\nu} &= \nu \Delta \omega_{\nu} \quad \text{in } \mathbb{R}^+ \times \mathcal{F}, \\
\omega_{\nu}(0, \cdot) &= \omega_{\nu}^{in} \quad \text{in } \mathcal{F}, \\
\nu \partial_n \omega_{\nu} &= (\omega_{\nu} - \omega_{\nu}^+) g_{\nu} \mathbb{1}_{\partial \mathcal{F}_+} \quad \text{in } \mathbb{R}^+ \times \partial \mathcal{F},
\end{align*}
\hfill (41a)
\begin{align*}
\omega_{\nu}^{in} \quad \text{and } g_{\nu} = v_{\nu} \cdot n, \text{ are given. The notation } \mathbb{1}_{\partial \mathcal{F}_+} \text{ stands for the set function associated with the inlet } \partial \mathcal{F}_+, \text{ which is equal to 1 when } x \text{ is in } \partial \mathcal{F}_+, \text{ and 0 otherwise.}
\end{align*}

Regarding the family of vector fields $(v_{\nu})_{\nu \in (0,1)}$, we will consider several cases:

(i) the family is constant equal to a given vector field: there is a given vector field $v$ such that $v_{\nu} = v$ for each $\nu$ in $(0,1)$,

(ii) the family $(v_{\nu})_{\nu \in (0,1)}$ is not constant but is given,

(iii) the family $(v_{\nu})_{\nu \in (0,1)}$ is related to the family of vorticity $(\omega_{\nu})_{\nu \in (0,1)}$ by [19].

In the two first cases we assume that the sequence of vector fields $(v_{\nu})_{\nu \in (0,1)}$ satisfy the assumptions:
\begin{align*}
\text{div } v_{\nu} = 0 \text{ in } \mathcal{F} \quad \text{and} \quad v_{\nu} \cdot n = g_{\nu} \text{ on } \partial \mathcal{F}.
\end{align*}
\hfill (42)

In the case (iii), the system [11] is a Navier-Stokes type system with a non-standard boundary condition which corresponds to a penalisation of the Yudovich’s boundary conditions [13]. A similar approximation system was used in [6]. The interest to choose such boundary conditions appears when considering the weak formulation of the system [11]: assuming that $\omega_{\nu}$ is a smooth solution of (11) and multiplying by a test function $\varphi$ in $C^\infty_c(\mathbb{R}^+ \times \mathcal{F}; \mathbb{R})$, we obtain after some integrations by parts, using (42), and some simplifications:
\begin{align*}
\int_{\mathcal{F}} \omega_{\nu}^{in} \varphi(0,\cdot) dx + \int_{\mathbb{R}^+} \int_{\mathcal{F}} \omega_{\nu}(\partial_t + v_{\nu} \cdot \nabla) \varphi - \nu \int_{\mathbb{R}^+} \int_{\mathcal{F}} \nabla \omega_{\nu} \cdot \nabla \varphi \\
= \int_{\mathbb{R}^+} \int_{\partial \mathcal{F}^+} g_{\nu} \omega_{\nu}^+ \varphi + \int_{\mathbb{R}^+} \int_{\partial \mathcal{F}^-} g_{\nu} \omega_{\nu^-} \varphi.
\end{align*}
\hfill (43)

Observe in particular that the integrand in the first integral in the right hand side contains the prescribed value $\omega_{\nu}^+$ rather than the trace of the unknown $\omega_{\nu}$.

6.2 Reminder on the weak compactness in $L^1$

One of the ingredients to deal with the case $p = 1$ is the so called De la Vallée Poussin’s lemma which establishes the equivalence of different definitions of uniform integrability.

**Lemma 2** (De la Vallée Poussin’s lemma). Let $O \subset \mathbb{R}^n$ bounded and measurable and let $f_j : O \longrightarrow \mathbb{R}$, with $j \in \mathbb{N}$, a sequence of measurable functions. The following definitions of uniform integrability are equivalent.

1. It holds
\begin{align*}
\lim_{n \to \infty} \sup_{j \in \mathbb{N}} \int_{\{|f_j| > n\}} |f_j| = 0.
\end{align*}
2. For any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any measurable \( A \subset O \) with Lebesgue measure \( \mu(A) < \delta \),

\[
\sup_{j \in \mathbb{N}} \int_A |f_j| < \varepsilon.
\]

3. There exists a convex even smooth function \( G : \mathbb{R} \rightarrow \mathbb{R}^+ \) such that

\[
\lim_{|s| \rightarrow +\infty} \frac{G(s)}{|s|} = +\infty \quad \text{and} \quad \sup_{j \in \mathbb{N}} \int O |f_j| < +\infty.
\]

Proof. We recall how to prove the implication \( 1. \Rightarrow 3. \), which is the most difficult one, to prepare the sequel. Assumption 1 implies the existence of an increasing sequence \( N_i \) with \( i \in \mathbb{N} \) such that \( N_0 = 0 \) and such that for any \( i \geq 1 \),

\[
\sup_{j \in \mathbb{N}} \int_{|f_j| > N_i} |f_j| < \frac{1}{2^i}.
\]

We consider the unique even function \( G \) such that for any \( i \geq 1 \),

\[
G(s) = iN_i + \frac{s - N_i}{N_{i+1} - N_i}((i + 1)N_{i+1} - iN_i) \quad \text{for} \quad s \in [N_i, N_{i+1}).
\]

Note that the above function is increasing, convex and superlinear. Moreover

\[
\sup_{j \in \mathbb{N}} \int O |f_j| = \sup_{j} \sum_i \int_{N_i \leq |f_j| < N_{i+1}} G(|f_j|) \leq \sup_{j} \sum_i \int_{N_i \leq |f_j| < N_{i+1}} (i + 1)|f_j|,
\]

since \( G(s) \leq (i + 1)s \) for \( s \) in \( [N_i, N_{i+1}) \). Thus, by \( 44 \), we arrive at

\[
\sup_{j \in \mathbb{N}} \int O G(|f_j|) \leq \sup_{j} \left( N_1|O| + \sum_i \frac{i + 1}{2^i} \right) < +\infty.
\]

To conclude it is sufficient to regularize \( G \) by a suitable convolution process.

The following corollary will be useful to deal with the incoming vorticity.

**Corollary 1.** Let \( O \subset \mathbb{R}^n \) bounded and measurable and let \( f_j, h_j : O \rightarrow \mathbb{R} \), with \( j \in \mathbb{N} \), two sequences of measurable functions such that \( h_j > 0 \), the sequence \( h_j \) is uniformly integrable and

\[
\lim_{\delta \rightarrow 0} \sup_{j \in \mathbb{N}} \mu\{h_j \leq \delta\} = 0.
\]

Then the following assertions are equivalent:

i. The sequence \( f_jh_j \) is uniformly integrable.

ii. There exists an even convex function \( G : \mathbb{R} \rightarrow \mathbb{R}^+ \) such that

\[
\lim_{|s| \rightarrow +\infty} \frac{G(s)}{|s|} = +\infty \quad \text{and} \quad \sup_{i} \int O G(f_j)h_j < +\infty.
\]

Proof.
i. ⇒ ii. By (45) and assumption i., we have
\[
\sup_{j \in \mathbb{N}} \int_{\{h_j \leq \delta\}} |f_j h_j| < \varepsilon,
\]
where we use the definition 2. of uniform integrability from Lemma 2.
Again from assumption i., we have
\[
\lim_{n \to +\infty} \sup_{j \in \mathbb{N}} \int_{\{|f_j h_j| > \delta n\}} |f_j h_j| = 0.
\]
Since
\[
\{f_j > n\} \subset \{f_j h_j > \delta n\} \cup \{h_j \leq \delta\},
\]
it follows from (46) and (47) that
\[
\lim_{n \to +\infty} \sup_{j \in \mathbb{N}} \int_{\{|f_j| > n\}} |f_j h_j| = 0.
\]
Now to conclude the proof of Corollary 1 it is sufficient to mimick the implication 1. ⇒ 3. of the previous proof of De la Vallée Poussin’s lemma.

ii. ⇒ i. Suppose by contradiction that i. is false. From definition 2. of uniform integrability from Lemma 2, there exists \(\varepsilon > 0\) such that for any \(\delta > 0\) there exists \(A_\delta \subset \mathcal{O}\) with \(\mu(A_\delta) < \delta\) and \(j_\delta = j \in \mathbb{N}\) for which
\[
\int_{A_\delta} |f_j h_j| > \varepsilon.
\]
Now from assumption ii. there exists an increasing and convex function \(G\) for which we have, by the Jensen inequality,
\[
G\left(\frac{\varepsilon}{\int_{A_\delta} |h_j|}\right) \leq G\left(\frac{\int_{A_\delta} |f_j h_j|}{\int_{A_\delta} |h_j|}\right) \leq \int_{A_\delta} G(|f_j|) h_j
\]
and therefore
\[
G\left(\frac{\varepsilon}{\int_{A_\delta} |h_j|}\right) \leq \frac{C}{\varepsilon}.
\]
with \(C = \sup_j \int_{A_\delta} G(f_j) h_j < +\infty\). Note that \(\varepsilon\) is fixed and from uniform integrability of the sequence \(h_j\), we have \(\int_{A_\delta} |h_j| \to 0\) as \(\delta \to 0\). Thus from assumption ii. the left hand side of (48) tends to \(+\infty\) which is in contradiction to the fact that the right hand side is bounded. \(\square\)

Let us also recall the Dunford-Pettis theorem which links the uniform integrability to the weak compactness in \(L^1\), that is, the Dunford-Pettis theorem asserts that a subset of \(L^1\) is weakly relatively compact if and only if it is uniformly integrable.

**Remark 6.** Let us stress that it follows from Egoroff’s theorem and the Dunford-Pettis theorem that a sequence \((h_j)_j\) such that \(h_j \to h\) in \(L^1(O)\) with \(h > 0\) satisfies the assumptions required in the above corollary.

Moreover to address the issue of compactness in \(C([0, T]; L^1 - w)\) we recall the following variant of the Arzelà-Ascoli theorem, see [42, Theorem 1.3.2].

**Theorem 5.** Let \(X\) be a real Banach space. Let \(C([0, T]; X - w)\) the locally convex topological vector space of the functions continuous on \([0, T]\) with values in \(X\) endowed with its weak topology. Let \(K\) a subset of \(C([0, T]; X - w)\). Then \(K\) is relatively sequentially compact in \(C([0, T]; X - w)\) if and only if the two following assertions hold true:
• $K$ is weakly equicontinuous on $[0,T]$,
• there exists a dense subset $D$ in $[0,T]$ such that for any $t$ in $D$, $K(t) := \{f(t)/ f \in K\}$ is weakly relatively compact in $X$.

Observe that Theorem 5 can be applied to some cases where the closed unit ball of $X - w$ is not metrizable, in particular to the case where $X$ is the space $L^1$.

**Corollary 2.** A sequence $(f_n)_n$ in $C([0,T]; L^1)$ is relatively sequentially compact in $C([0,T]; L^1 - w)$ if and only if the two following assertions hold true:

• the sequence $(f_n)_n$ is weakly equicontinuous on $[0,T]$,
• there exists a dense subset $D$ in $[0,T]$ such that for any $t$ in $D$, $(f_n(t))_n$ is weakly relatively compact in $L^1$.

### 6.3 Existence of compatible data for the viscous model

The following result states that the input of the problem (20) can be approximated by a family of inputs, for $
u$ in $(0,1)$, which satisfy the compatibility conditions for the problems (11).

**Lemma 3.** Let $p$ in $[1, +\infty]$ and $q \in (1, +\infty)$. Let $(\omega^\nu, \omega^+) \in (\mathcal{H})$ in $L^p$. Let also

- either $\nu \in L^1_{loc}(\mathbb{R}^+; W^{1,q}(F))$ and $(\text{SSC})_1$, and $\nu \equiv (\text{SSC})_2$, or
- or $\nu \in L^1_{loc}(\mathbb{R}^+; W^{1-1/q,q}(\partial F))$ and $(\text{SSC})_2$, and such that for $p < \infty$

$$
\omega^\nu \rightarrow \omega^+ \quad \text{in} \quad L^p(F) \quad \text{and} \quad (g_\nu)^{1/p} \omega^\nu \rightharpoonup g^{1/p} \omega^+ \quad \text{in} \quad L^p((0,T) \times \partial F^+),
$$

- either $\nu \rightarrow \nu$ in $L^1_{loc}(\mathbb{R}^+; W^{1,q}(F))$ if we assume (49),
- or $g_\nu \rightarrow g$ in $L^1_{loc}(\mathbb{R}^+; W^{1-1/q,q}(\partial F))$ if we assume (50),

as $\nu$ tends to $0$. Moreover

$$
\|\omega^\nu\|_{L^2}^2 + \|g_\nu^{1/2} \omega^\nu\|_{L^2((0,T) \times \partial F_+)}^2 \leq \frac{C_T}{\nu^2} \left( \|\omega^\nu\|_{L^p} + \|g_\nu^{1/p} \omega^\nu\|_{L^p((0,T) \times \partial F^+)} \right)^2.
$$

Finally when $p = \infty$ the above convergence holds for any $l < p = \infty$.

**Proof.** Let first consider the case where we assume (51). We use the density of $C_c^\infty$ in $L^p$ for $p < \infty$ to show the existence of

$$
\omega^\nu \rightarrow \omega \quad \text{in} \quad L^p(F) \quad \text{and} \quad f_\nu \rightarrow g^{1/p} \omega^+ \quad \text{in} \quad L^p((0,T) \times \partial F),
$$
in particular we choose the approximations such that (52) holds. Moreover we define $g_\nu$ by convoluting $g$ with an appropriate positive smoothing kernel of integral 1 such that $g_\nu$ is $(\text{SSC})_2$. We define $\omega^\nu := f_\nu g_\nu^{-1/p}$.

In the case we assume (49), we use the same construction as in the previous step with $g = v \cdot n$. Finally we define $v_\nu$ as the unique smooth solution of

$$
\begin{align*}
\text{div} v_\nu &= 0 \quad \text{in} \quad F,
\text{curl} v_\nu &= \eta_\nu \ast \text{curl} v \quad \text{in} \quad F,
 v_\nu \cdot n &= g_\nu \quad \text{in} \quad \partial F,
 \int_{\partial S_i} v_\nu \cdot \tau &= \tilde{\eta}_\nu \ast \int_{\partial S_i} v \cdot \tau \quad \text{for} \ i \in I,
\end{align*}
$$

where $\eta_\nu$ and $\tilde{\eta}_\nu$ are respectively a spacial and time smoothing convolution kernel.
6.4 Existence of smooth solutions to the viscous model

This subsection is devoted to the existence of smooth solutions to the problems (20).

**Lemma 4.** Let \( \omega_{i}^{n} \in C_{c}^{\infty}(\mathcal{F}) \), let \( \omega_{v}^{\tau} \in C_{c}^{\infty}((0, +\infty) \times \partial\mathcal{F}) \) and let

\[
\text{either } v_{v} \in C_{c}^{\infty}(\mathbb{R}^{+} \times \mathcal{F}) \text{ and } (SSC_{1}), \quad (53) \\
or \quad g_{v} \in C_{c}^{\infty}(\mathbb{R}^{+} \times \partial\mathcal{F})) \text{ and } (SSC_{2}). \quad (54)
\]

Then there exists a global unique smooth solution \( \omega_{v} \) of the system (11) associated with the data \( \omega_{i}^{n}, \omega_{j}^{\tau} \) and \( v_{v} \) if we assume (53) and of the system (11) together with (19) associated with the data \( \omega_{v}^{n}, \omega_{i}^{\tau} \) and \( g_{v} \) if we assume (54).

This result is part of the mathematical folklore on boundary values problem for parabolic equation, see for instance [27, Chapters 8 and 10] and [29, Chapter 4 and 7]. However, since the boundary conditions are quite unusual and for sake of clarity, we give a sketch of proof.

**Proof.** Let us start by dealing with the case where the hypothesis (53) holds true. Existence of weak solutions was shown in [6] via a Galerkin method. It is then enough to show a priori estimates for higher derivatives. Without loss of generality we assume that \( \nu = 1 \) and we do not write the index of the approximation. We prove by induction on \( n \in \mathbb{N} \) that

\[
\omega \in \bigcap_{i=0}^{n} H^{i}(0, T; H^{2n-2i}(\mathcal{F})). \quad (55)
\]

This implies that \( \omega \) is smooth. The case \( n = 0 \) was proved in [6], it relies on an energy estimate, which is obtained by testing (13) with the solution \( \omega \) and by some integration by parts, so that we arrive at

\[
\frac{1}{2} \int_{\mathcal{F}} |\omega(t,.)|^{2} + \frac{1}{t} \int_{0}^{t} \int_{\mathcal{F}} g \frac{|\omega|^{2}}{2} + \frac{1}{t} \int_{\mathcal{F}} |\nabla \omega|^{2} \leq \int_{\mathcal{F}} |\omega(t,.)|^{2} + \int_{0}^{t} \int_{\partial\mathcal{F}^{+}} (-g) \frac{|\omega^{+}|^{2}}{2}. \quad (56)
\]

Before to move on the iteration step, we first tackle the case where \( n = 1 \) to display the method.

**Proof of** (55) **in the case where** \( n = 1 \). We test the weak formulation (13) with \( \partial_{t} \omega \) and after some integrations by parts, we have

\[
\frac{1}{2} \int_{\mathcal{F}} |\nabla \omega|^{2}(t,.) + \frac{1}{4} \int_{0}^{t} \int_{\mathcal{F}} |\partial_{t} \omega|^{2} + \frac{1}{8} \int_{0}^{t} \int_{\mathcal{F}} |\Delta \omega|^{2} \leq \frac{1}{4} \int_{0}^{t} \int_{\mathcal{F}} |v \cdot \nabla \omega|^{2} + \int_{0}^{t} \int_{\partial\mathcal{F}^{+}} (\omega - \omega^{+}) \partial_{t} \omega. \quad (57)
\]

This leads to

\[
\frac{1}{2} \int_{\mathcal{F}} |\nabla \omega|^{2}(t,.) + \frac{1}{4} \int_{0}^{t} \int_{\mathcal{F}} |\partial_{t} \omega|^{2} + \frac{1}{8} \int_{0}^{t} \int_{\mathcal{F}} |\Delta \omega|^{2} + \frac{1}{2} \int_{\partial\mathcal{F}^{+}} (-g) |\omega - \omega^{+}|^{2}(t,.)
\]

\[
\leq \frac{3}{4} \int_{0}^{t} \left( \|v\|_{L^{\infty}(\mathcal{F})} \int_{\mathcal{F}} |\nabla \omega|^{2} \right) - \int_{0}^{t} \int_{\partial\mathcal{F}^{+}} g \omega^{n} - \omega^{+} \omega^{+}^{2} + \int_{0}^{t} \int_{\partial\mathcal{F}^{+}} |\partial_{t} g|^{2} + \frac{1}{2} \int_{0}^{t} \int_{\partial\mathcal{F}^{+}} |g| |\partial_{t} \omega^{+}|^{2} + \frac{1}{2} \int_{\mathcal{F}} |\nabla \omega^{n}|^{2}.
\]

All the terms in the right hand side above only depend on the data, except the first one, which can be handled by a Gronwall lemma. Using the classical elliptic estimate:

\[
\int_{\mathcal{F}} |\nabla^{2} \omega|^{2} \leq C \int_{\mathcal{F}} (|\Delta \omega|^{2} + |\nabla \omega|^{2} + |\omega|^{2}),
\]

we deduce that (55) holds true for \( n = 1 \).
Inductive step. We suppose that (55) holds true up to some order \( n \) and we are going to prove that it also holds true at the order \( n + 1 \). To do that we apply \( \partial_t^n \) to (43). Then \( \tilde{\omega} := \partial_t \omega^n \) satisfies

\[
\partial_t \tilde{\omega} + v \cdot \nabla \tilde{\omega} = \nu \Delta \tilde{\omega} + \sum_{i=1}^n \partial_t^i v \cdot \nabla \partial_t^{n-i} \omega, \quad \text{in } \mathbb{R}^+ \times \mathcal{F}, \tag{58}
\]

\[
\partial_n \tilde{\omega} = (\tilde{\omega} - \tilde{\omega}^+)g + \sum_{i=1}^n \partial_t^{n-i}(\omega - \omega^+)\partial_t^i g, \quad \text{in } \mathbb{R}^+ \times \partial \mathcal{F}. \tag{59}
\]

For this system we have an energy estimate similar to (56), obtained by multiplying (58) by \( \tilde{\omega} \), except that two new terms appear due to the last terms in the right hand side above. Proceeding as before, we have only to estimates the two new terms

\[
I = \int_0^t \int_{\mathcal{F}} \sum_{i=1}^n \partial_t^i v \cdot \nabla (\partial_t^{n-i} \omega) \tilde{\omega} \quad \text{and} \quad II = \int_0^t \int_{\partial \mathcal{F}^+} \sum_{i=1}^n \partial_t^{n-i}(\omega - \omega^+)\partial_t^i g \tilde{\omega}.
\]

For the first term, it holds

\[
|I| \leq \int_0^t \int_{\mathcal{F}} |\tilde{\omega}|^2 + \sum_{i=1}^n \int_0^t \int_{\mathcal{F}} |\partial_t^i v \cdot \nabla \partial_t^{n-i} \omega|^2 \\
\leq \int_0^t \int_{\mathcal{F}} |\tilde{\omega}|^2 + \sum_{i=1}^n \int_0^t \|\partial_t^i v\|_{L^\infty(\mathcal{F})} \int_{\mathcal{F}} |\nabla \partial_t^{n-i} \omega|^2 \\
\leq \int_0^t \int_{\mathcal{F}} |\tilde{\omega}|^2 + C,
\]

where \( C \) depends on the data, thanks to the previous steps. The term \( II \) can be handled similarly.

Let now multiply the equation (58) by \( \partial_t \tilde{\omega} \) and integrate over \((0, t) \times \mathcal{F}\). This provides an estimate similar to (57) with two extra terms due to the last terms in the right hand side of (58). These two terms are:

\[
I^{im} = \int_0^t \int_{\mathcal{F}} \sum_{i=1}^n \partial_t^i v \cdot \nabla \partial_t^{n-i} \omega \partial_t \tilde{\omega} \quad \text{and} \quad II^{im} = \int_0^t \int_{\partial \mathcal{F}^+} \sum_{i=1}^n \partial_t^{n-i}(\omega - \omega^+)\partial_t^i g \partial_t \tilde{\omega}.
\]

As before we have

\[
|I^{im}| \leq \frac{1}{8} \int_0^t \int_{\mathcal{F}} |\partial_t \tilde{\omega}|^2 + C.
\]
The second one is more technical. We rewrite

\[ II^{im} = \sum_{i=1}^{n} \int_{\partial^{+}} \partial_{i}^{n-i} (\omega(t, \cdot) - \omega^{+}(t, \cdot)) \partial_{i}^{n} g(t, \cdot) \partial_{t}^{n} \tilde{\omega}(t, \cdot) - \sum_{i=1}^{n} \int_{\partial^{+}} \partial_{i}^{n-i} (\omega^{im} - \omega^{+}(0, \cdot)) \partial_{i}^{n} g(0, \cdot) \tilde{\omega}(0, \cdot) \]

\[- \sum_{i=1}^{n} \int_{0}^{t} \int_{\partial^{+}} \partial_{i}^{n-i+1} (\omega^{+} - \omega^{+}) \partial_{i}^{n} g \partial_{t}^{i+1} \tilde{\omega} - \sum_{i=1}^{n} \int_{0}^{t} \int_{\partial^{+}} \partial_{i}^{n-i} (\omega^{+} - \omega^{+}) \partial_{i}^{n} g \partial_{t}^{i+1} \tilde{\omega} \]

\[= \sum_{i=1}^{n} \int_{\partial^{+}} \partial_{i}^{n-i} (\omega(t, \cdot) - \omega^{+}(t, \cdot)) \partial_{i}^{n} g(t, \cdot) \partial_{t}^{n} \tilde{\omega}(t, \cdot) - \sum_{i=1}^{n} \int_{\partial^{+}} \partial_{i}^{n-i} (\omega^{im} - \omega^{+}(0, \cdot)) \partial_{i}^{n} g(0, \cdot) \tilde{\omega}(0, \cdot) \]

\[- \int_{0}^{t} \int_{\partial^{+}} \partial_{i}^{n-i} (\omega^{+} - \omega^{+}) \partial_{i}^{n} g \partial_{t}^{i+1} \tilde{\omega} - \sum_{i=1}^{n} \int_{0}^{t} \int_{\partial^{+}} \partial_{i}^{n-i+1} (\omega^{+} - \omega^{+}) \partial_{i}^{n} g \partial_{t}^{i+1} \tilde{\omega} \]

Using the above equality we deduce that

\[|II^{im}| \leq \frac{1}{4} \|\tilde{\omega}\|_{H^{1}(F)}^{2} + \frac{1}{4} \|\tilde{\omega}^{im}\|_{H^{1}(F)}^{2} + \sum_{i=1}^{n} \|\partial_{i}^{n} g\|_{C^{0}} \|\partial_{i}^{n-i} (\omega^{+} - \omega^{+})\|_{C^{0}(0,T;H^{1}(F))}^{2} \]

\[+ \|\partial_{i}^{n} g\|_{C^{0}(0,T)} \int_{0}^{t} \|\tilde{\omega}\|_{H^{1}(F)}^{2} \int_{0}^{t} \|\tilde{\omega}\|_{H^{1}(F)}^{2} + \int_{0}^{t} \int_{\partial^{+}} |\tilde{\omega}\|_{H^{1}(F)}^{2} \int_{0}^{t} \int_{\partial^{+}} |\tilde{\omega}|_{H^{1}(F)}^{2} \]

\[+ \sum_{i=2}^{n} \int_{0}^{t} \int_{\partial^{+}} |\partial_{i}^{n-i-1} (\omega^{+} - \omega^{+})\|_{H^{1}(F)}^{2} \sum_{i=2}^{n} \int_{0}^{t} \int_{\partial^{+}} |\partial_{i}^{n-i} (\omega^{+} - \omega^{+})\|_{H^{1}(F)}^{2} \]

The first term of the right hand side can be absorbed by the left hand side, the first two terms of the second line are tackled by the Grönwall argument, and all the remaining ones are bounded.

We deduce that \(\tilde{\omega} = \partial_{i}^{n} \omega\) is a priori bounded in \(H^{1}(0,T;L^{2}(F)) \cap L^{2}(0,T;H^{2}(F))\). The desired regularity follows by using the equation (13). This allows to conclude that (13) is true for every \(n\) in \(\mathbb{N}\). This achieves the proof of Lemma 4 in the case where the hypothesis (53) holds true.

To prove existence in the case where hypothesis (54) holds, we are going to use the Schauder fixed point theorem which asserts that if \(B\) is a nonempty convex closed subset of a normed space \(X\) and \(F : B \rightarrow B\) is a continuous mapping such that \(F(B)\) is contained in a compact subset of \(B\), then \(F\) has a fixed point.

**Definition of an appropriate operator.** Let \(n \in \mathbb{N}, R > 0\) and

\[Z = \left\{ \omega \in \bigcap_{i=0}^{n} H^{i}(0,T;H^{2n-2i}(F)) \text{ such that } \|\omega\|_{Z} := \|\omega\|_{\bigcap_{i=0}^{n} H^{i}(0,T;H^{2n-2i}(F))} \leq R \right\}.\]
Define the map \( F : Z \rightarrow \bigcap_{i=0}^{n-1} H^i(0, T; H^{2n-2i}(\mathcal{F})) \) where \( F(\omega) = \bar{\omega} \) is the solution of
\[
\begin{align*}
\partial_t \bar{\omega} + v_\omega \cdot \nabla \bar{\omega} - \Delta \omega &= 0 & \text{in } \mathbb{R}^+ \times \mathcal{F}, \\
\partial_n \bar{\omega} &= (\bar{\omega} - \omega^+)g & \text{in } \mathbb{R}^+ \times \partial \mathcal{F}, \\
\text{div } v_\omega &= 0 & \text{in } \mathbb{R}^+ \times \mathcal{F}, \\
\text{curl } v_\omega &= \omega & \text{in } \mathbb{R}^+ \times \mathcal{F}, \\
v_\omega \cdot n &= g & \text{in } \mathbb{R}^+ \times \partial \mathcal{F}, \\
\int_{\partial \mathcal{S}^i} v_\omega \cdot \tau &= C_i^{in} + \int_0^t \int_{\partial \mathcal{S}^i} \omega g & \text{for any } i \in I_-, \\
\int_{\partial \mathcal{S}^i} v_\omega \cdot \tau &= C_i^{in} + \int_0^t \int_{\partial \mathcal{S}^i} \omega^+ g & \text{for any } i \in I_+.
\end{align*}
\]

For any \( \omega \in Z \), we have that \( v_\omega \) is in \( \bigcap_{i=0}^{n-1} H^i(0, T; H^{2n-2i}(\mathcal{F})) \). Then, by using the \textit{a priori} estimates above, we observe that the \( Z \)-norm of \( \bar{\omega} \) depends on the \( \bigcap_{i=0}^{n-1} H^i(0, T; H^{2n-2i}(\mathcal{F})) \) of \( v_\omega \), which converges, as \( t \) tend to zero, to zero uniformly with respect to \( \omega \in Z \). Therefore for \( T \) small enough, we conclude that \( F(Z) \subset Z \).

Let us now prove that \( F \) is relatively compact. Let \( (\omega_j)_j \) a bounded sequence in \( Z \). Then, up to a subsequence, \( \omega_j \rightharpoonup \omega \) in \( Z \). By Rellich’s theorem the convergence is strong in
\[
\bigcap_{i=0}^{n-1} H^i(0, T; H^{2n-2i-1}(\mathcal{F})).
\]
We deduce that the corresponding velocity \( v_j = v_{\omega_j} \) converges to \( v \) in \( \bigcap_{i=0}^{n-1} H^i(0, T; H^{2n-2i}(\mathcal{F})) \). Moreover for any \( j \), the function \( w_j = \bar{\omega} - \omega_j \) satisfies the system
\[
\begin{align*}
\partial_t w_j + v \cdot \nabla w_j - \Delta w_j &= -(v - v_j) \cdot \nabla \bar{\omega} & \text{in } \mathbb{R}^+ \times \mathcal{F}, \\
\partial_n w_j &= w_j g & \text{in } \mathbb{R}^+ \times \partial \mathcal{F},
\end{align*}
\]
with zero initial data. We observe that
\[
\| (v - v_j) \cdot \nabla \bar{\omega} \|_{H^{n-1}(0,T;L^2(\mathcal{F}))} \leq \sum_{i=0}^{n-2} \| \partial^i_t (v - v_j) \|_{L^\infty(0,T;L^\infty(\mathcal{F}))} \| \partial^{n-1-i}_t \nabla \bar{\omega} \|_{L^2(0,T;L^2(\mathcal{F}))} \\
+ \| \partial^{n-1}_t (v - v_j) \|_{L^2(0,T;L^\infty(\mathcal{F}))} \| \partial^{n-1-i}_t \nabla \bar{\omega} \|_{L^\infty(0,T;L^2(\mathcal{F}))} \rightarrow 0.
\]

Proceeding in the same way for higher derivatives, we obtain that
\[
\sum_{i=0}^{n-1} \| (v - v_j) \cdot \nabla \bar{\omega} \|_{H^{i}(0,T;H^{2n-2-2i}(\mathcal{F}))} \rightarrow 0.
\]
Then by using the \textit{a priori} estimates above, we deduce that \( w_j \) converges to \( 0 \) in \( Z \). Thus \( F \) is relatively compact. The continuity of \( F \) can be proved along the same lines. Thus Schauder’s fixed point theorem can be applied. It implies that \( F \) has a fixed point in \( Z \). This has proved the local in time existence of smooth solutions. Moreover the existence to all \([0,T]\) can be deduced from the \textit{a priori} estimates. In fact if we suppose by contradiction that there exists a maximal time of existence \( t_\ast < T \), the \textit{a priori} estimates ensure that \( \omega(t_\ast) \) is enough regular to apply again the local existence result and we obtain a contradiction. Uniqueness follows from the energy estimate and Grönnwall’s lemma.

\section{Convergence of the approximations}
In this subsection we establish the convergence of the solutions to the problems \cite{20} in the vanishing viscosity limit.
Proposition 2. Let $p$ in $[1, +\infty]$. Let $q = p$ in the case where $p > 1$ and $q > 2$ in the case where $p = 1$. Let $g$ in $L^1_{loc}(\mathbb{R}^+; W^{1-1/q, q}(\mathcal{F}))$ and $(\text{SSC}_1)$. Let $(\omega_t^n, \omega^+)$ a $(\mathcal{CIV})$ in $L^p$. Let some families $\omega_t^{n_\nu}$, $\omega^+_\nu$ and $g_\nu$ for $\nu$ in $(0, 1)$, as in Lemma 3. Let $\omega_t$, the corresponding global unique smooth solution of the system $(\ref{eq:system})$ associated with the data $\omega_t^{n_\nu}$, $\omega^+_\nu$ and $g_\nu$, from Lemma 4. Then there are a subsequence of $\omega_t$ and a subsequence $v_\nu$ which we still denote $\omega_t$ and $v_\nu$ and which satisfy

$$
\omega_t \rightarrow \omega \quad \text{in } C_w([0, T]; L^p(\mathcal{F})) \quad g_\nu^{1/p} \omega_t \rightarrow g^{1/p} \omega^- \quad \text{in } L^p((0, T) \times \partial \mathcal{F}^-), \tag{60}
$$

and

$$\sqrt{\nu} \nabla \omega_t \quad \text{bounded in } L^2((0, T); L^2(\mathcal{F})), \tag{61}
$$

$$v_\nu - v_g \rightarrow v - v_g \quad \text{in } C_w([0, T]; W^{1, p}(\mathcal{F})), \tag{62}
$$

in the case where $p < \infty$ and which satisfy $(\ref{eq:bound})$ with any real number greater than 1 instead of $p$ in the case where $p = +\infty$. Moreover $v$ and $\omega$ satisfy $(\ref{eq:prop2})$ and $(\ref{eq:prop2a})$.

Proof of Proposition 2. We will proceed in six steps.

Step 1. A priori bounds. Let $G$ a positive even convex function. Formally, multiplying $(\ref{eq:system})$ by $G(|\omega_t|) \frac{\omega_t}{|\omega_t|}$ and using the rule

$$\partial_t |\omega_t| = \frac{\omega_t}{|\omega_t|} \partial_t \omega_t,$$

for $i = t, x_1$ or $x_2$, we arrive at

$$\partial_t G(\omega_t) + v_\nu \cdot \nabla G(\omega_t) = \nu \Delta G(\omega_t) - \nu G''(\omega_t)|\nabla \omega_t|^2. \tag{63}
$$

Integrating on $\mathcal{F}$ and using the boundary conditions $(\ref{eq:boundary})$ we have that

$$\int_{\mathcal{F}} v_\nu \cdot \nabla G(\omega_t) \, dx = \int_{\mathcal{F}} \text{div}(G(\omega_t) v_\nu) \, dx = \int_{\partial \mathcal{F}} G(\omega_t) g_\nu \, ds,
$$

and that

$$\nu \int_{\mathcal{F}} \Delta G(\omega_t) \, dx = \nu \int_{\partial \mathcal{F}} \partial_n G(\omega_t) \, dx = \int_{\partial \mathcal{F}} \frac{\omega_t}{|\omega_t|} (\omega_t - \omega_t^+) g_\nu \, ds.
$$

Thus, by integrating $(\ref{eq:63})$ on $[0, T] \times \mathcal{F}$, we obtain:

$$\int_{\mathcal{F}} G(|\omega_t(t, \cdot)|) \, dx + \int_0^t \int_{\partial \mathcal{F}} g_\nu G(|\omega_t|) \, dx + \nu \int_0^t \int_{\mathcal{F}} G''(|\omega_t|)|\nabla \omega_t|^2 \, dx \leq \int_{\mathcal{F}} G(|\omega_t^{n_\nu}|)
$$

$$+ \int_0^t \int_{\partial \mathcal{F}} (-g_\nu) G(|\omega_t|) - \frac{\omega_t}{|\omega_t|} (\omega_t^+ - \omega_t^-) \, ds.
$$

Using the convexity of the function $G$, we have that

$$G(|x|) - G'(|x|) \frac{x}{|x|} (y - x) - G(|y|) \leq 0 \quad \text{and} \quad G''(|x|) \geq 0.
$$

We use these inequalities with $x = \omega_t$ and $y = \omega_t^+$ to bound the two last terms of $(\ref{eq:63})$ and the last term of the right hand side and we arrive at:

$$\int_{\mathcal{F}} G(|\omega_t(t, \cdot)|) + \int_0^t \int_{\partial \mathcal{F}} g_\nu G(|\omega_t|) \leq \int_{\mathcal{F}} G(|\omega_t^{n_\nu}|) + \int_0^t \int_{\partial \mathcal{F}} (-g_\nu) G(|\omega_t^+|). \tag{65}
$$

This bound can be rigorously justified by considering $\sqrt{x^2 + \varepsilon}$ as an appropriate sequence of regularizations of the absolute value and by passing to the limit as $\varepsilon$ goes to 0. Using the peculiar cases of the power functions we have that $\omega_t$ satisfies the a priori bounds $(\ref{eq:bound})$ and $(\ref{eq:bound2})$. 

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Step 2. Convergence of the vorticity in the case where $p > 1$. In the case where $p > 1$, we use the a priori bound \( (65) \) in the case where $G(x) = x^p$. This implies that $\omega_\nu$ converges to $\omega$ in $L^\infty([0,T]; L^p(F) - w)$ and $g_\nu^{1/p} \omega_\nu$ converges to $f$ in $L^p((0,T) \times \partial F^-)$. The second convergence in \( (60) \) follows from the fact that $g > 0$ on $\partial F^-$ and defining $\omega^- = f/g^{1/p}$. To reinforce the convergence of $\omega_\nu$ as a strong convergence in time such as stated in the first convergence in \( (60) \), we are going to establish some bounds on the time derivative.

In the case where $p \geq 4/3$ it follows from \( (11) \) from \( (17) \) and from the Sobolev embedding theorem that the sequence $(v_\nu \omega_\nu)_\nu$ is uniformly bounded in $L^1((0,T); L^q(F))$ with $q = 2p/(2-p)$, so that by Hölder’s inequality, the term $v_\nu \omega_\nu$ is uniformly bounded in $L^1((0,T) \times F)$. Then it follows from \( (11a) \) and from the Sobolev embedding theorem that the sequence $(\partial_t \omega_\nu)_\nu$ is bounded in a Sobolev space of negative order.

In the case where $p \in (1,4/3]$ is more tricky. For each $\nu$ in $(0,1)$, the smooth solution $\omega_\nu$ of the system \( (11) \) also satisfies the weak formulation \( (13) \), and by using \( (28) \) and \( (19) \), we obtain that it also satisfies the following viscous weak symmetrized formulation: for any $\varphi \in C^\infty_c(\mathbb{R}^+ \times F)$,

\[
\int_F \omega_\nu^n \varphi(0,.dx) + \int_{\mathbb{R}^+} \int_F \omega_\nu(\partial_t \varphi + v_\nu \cdot \nabla \varphi) \, dx \, dt + \sum_i \int_{\mathbb{R}^+} \int_F C_{i,\nu}(t) \, \int_F \omega_\nu X_i \cdot \nabla \varphi \, dx \, dt \tag{66}
\]

\[
+ \int_{\mathbb{R}^+} \int_F \int_F H_\varphi(x,y) \omega_\nu(t,x) \omega_\nu(t,y) \, dx \, dy \, dt - \nu \int_{\mathbb{R}^+} \int_F \nabla \omega_\nu \cdot \nabla \varphi = 0,
\]

where

\[
C_{i,\nu}(t) = C_{i,\nu}^n - \int_0^t \int_{\partial S_i} \omega_\nu^\pm g_\nu \quad \text{for } i \in I^\pm.
\tag{67}
\]

Then we deduce the uniform bound for $\partial_t \omega_\nu$, thanks to Lemma \( (1) \) and

\[
\int_F v_{g_\nu} \cdot \nabla \varphi \omega_\nu = \int_F v_{g_\nu} \cdot \nabla \varphi \Delta \eta_\nu = \int_F \Delta v_{g_\nu} \cdot \nabla \varphi \eta_\nu + \int_F \nabla v_{g_\nu} : \nabla^2 \varphi \eta_\nu - \int_F v_{g_\nu} \cdot \nabla \eta_\nu : \nabla^2 \varphi,
\]

where $\Delta \eta_\nu = \omega_\nu$ in $F$ and $\eta_\nu = 0$ in $\partial F$. Moreover recall that $v_{g_\nu} = \nabla \phi_{g_\nu}$ solution of $\Delta \phi_{g_\nu} = 0$ in $F$ and $\nabla \phi_{g_\nu} \cdot n = g_\nu$, in particular $\Delta v_{g_\nu} = 0$. It follows

\[
\left| \int_F v_{g_\nu} \cdot \nabla \varphi \omega_\nu \right| \leq \left| \int_F \nabla v_{g_\nu} : \nabla^2 \varphi \eta_\nu \right| + \left| \int_F v_{g_\nu} \cdot \nabla \eta_\nu : \nabla^2 \varphi \right|
\]

\[
\leq \| \nabla v_{g_\nu} \|_{L^p(F)} \| \nabla^2 \varphi \|_{L^\infty(F)} \| \eta_\nu \|_{L^{p-2}(F)}
\]

\[
+ \| \nabla v_{g_\nu} \|_{L^2(F)} \| \eta_\nu \|_{L^2(F)} \| \nabla^2 \varphi \|_{L^\infty(F)}
\]

\[
\leq \| v_{g_\nu} \|_{L^1(F)} \| \omega_\nu \|_{L^p(F)} \| \varphi \|_{W^{2,\infty}(F)},
\]

where from the Sobolev embedding in dimension two we have $W^{2,k} \subset L^\infty$ and $W^{1,p} \subset L^2$ for $p > 1$.

We now apply the following version of the Aubin-Lions lemma, see for example \( (39) \) Lemma 11, to $f_\nu = \omega^{\nu}$ with $X = L^{p-2}(F)$ (respectively $L^1(F)$ if $p = +\infty$) and $Y = H^{M+1}_0(F)$, with $M$ large enough.

**Lemma 5.** Let $X$ and $Y$ be two Banach spaces such that $Y$ is dense in $X$ and $X$ is separable. Assume that $(f_n)_n$ is a bounded sequence in $L^\infty(0,T; X')$ such that $(\partial_t f_n)_n$ is bounded in $L^1(0,T; Y')$. Then $(f_n)_n$ is relatively compact in $C([0,T]; X' - w^*)$.

In particular this proves the first convergence in \( (60) \) in the case where $p > 1$.

Step 3. Convergence of the vorticity in the case where $p = 1$. In the case where $p = 1$, by \( (51) \) and Dunford-Pettis’ theorem we have that $(\omega_\nu^n)_\nu$ and $(\omega_\nu^\pm g_\nu)_\nu$ are uniformly integrable respectively in $L^p(F)$ and in $L^p((0,T) \times \partial F^+)$, Therefore, by Lemma \( (2) \) and Corollary \( (11) \) there exists an even convex function $G : \mathbb{R} \to \mathbb{R}^+$ such that

\[
\lim_{|s| \to +\infty} \frac{G(s)}{|s|} = +\infty, \quad \sup_{\nu} \int G(|\omega_\nu^n|) < +\infty, \quad \text{and} \quad \sup_{\partial F^+} G(\omega_\nu^\pm) (-g_\nu) < +\infty.
\]

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Then by \((64)\), we deduce that
\[
\sup_t \sup_{\nu} \int G(\omega_{\nu}(t, \cdot)) < +\infty, \quad \text{and} \quad \sup_{\nu} \int_{\partial\mathcal{F}} G(\omega_{\nu}) g_{\nu} < +\infty.
\]
Therefore, using again Lemma 2 and Dunford-Pettis’ theorem (more precisely the parts regarding the reverse statements) for any \(t \in [0, T]\) the sequence \((\omega_{\nu}(t, \cdot))_{\nu}\) is weakly relatively compact in \(L^1\). On the other hand we can obtain a uniform bound of \(\partial_t \omega_{\nu}\) in a Sobolev space of negative order by proceeding as in the case \(p \in (1, 4/3)\) by using that \(v_{g_{\nu}} \in L^\infty(\mathcal{F})\), a consequence of the regularity of \(g_{\nu}\). Then it suffices to apply Corollary 2 to conclude that the first convergence in \((61)\) holds true in the case where \(p = 1\).

**Step 4. Endgame.** Since \(\omega_{\nu}\) and \(v_{\nu}\) are related by \((19)\) the convergence of the velocity in \((62)\) straightforwardly follows from the convergence of the vorticity, see in particular the property \((17)\). Moreover we can pass to the limit in the relation \((19)\) and we obtain that \(\omega\) and \(v\) are related by \((19)\). For more details on the convergence of the part of the velocity associated with the circulations see \((20)\).

The bound of the gradient of the vorticity in \((61)\) follows from the \(L^p\) a priori bounds, corresponding to the case where \(G(x) = x^2\) and from the hypothesis \((52)\) on the data.

\[\square\]

7 **Proof of Theorem 2** on distributional solutions

Let, for \(\nu \in (0, 1)\), \(\omega_{\nu}^m\), \(\omega_{\nu}^+\) and \(g_{\nu}\), as in Lemma 3 and \(\omega_{\nu}\) the corresponding global unique smooth solution of the system \((11)\) as in Lemma 1 These solutions satisfy \((43)\) for any \(\varphi \in C^\infty_c(\mathbb{R}^+ \times \mathcal{F})\). By applying Proposition 2 in the case where the family \((v_{\nu})_{\nu \in (0, 1)}\) is related to the family of vorticity \((\omega_{\nu})_{\nu \in (0, 1)}\) by \((19)\), we obtain that, up to a subsequence, \((\omega_{\nu})_{\nu}\) and \((g_{\nu}^{1/p} \omega_{\nu})_{\nu}\) satisfy \((43)\) and \((v_{\nu})_{\nu \in (0, 1)}\) satisfies \((42)\), and the limit vorticity \(\omega\) and the limit velocity \(v\) satisfy \((19)\) and \((20a)\).

By the Rellich-Kondrachov theorem, using that \(2p/(2-p) > p/(p-1)\) when \(p > 4/3\), we obtain that, up to a subsequence, \((v_{\nu})_{\nu \in (0, 1)}\) converges, strongly, in
\[L^1_{\text{loc}}(\mathbb{R}^+; L^{p/(p-1)}(\mathcal{F})).\]

These convergences allow to pass to the limit in \((43)\) and to arrive at \((22)\). Moreover the continuity in time with values in \(L^p\) of \(\omega\) and the equality \((3)\) follow from the fact that \((\omega, \omega^-)\) is also a renormalized solution which will be proved in the next section. The bound \((6)\) follows from the lower semi-continuity of the weak limits.

8 **Proof of Theorem 3** on renormalized solutions

This section is devoted to the proof of Theorem 3 on the existence of renormalized solutions to the Euler equations in presence of sources and sinks in the case where the input vorticities are in \(L^p\), with \(p \in (1, \infty]\). The proof also relies on the viscous approximations built in the previous section. The next subsection is devoted to the convergence of a subsequence of these approximations and to the proof that the limit is a renormalized solution. Then in Subsection 8.3 we prove the strong convergence of the approximated vorticities \(\omega_{\nu}\) in \(C_{\text{loc}}(\mathbb{R}^+; L^p(\mathcal{F}))\), not only in \(C_{\text{loc}}(\mathbb{R}^+; L^p(\mathcal{F})-w)\), which concludes the proof of Theorem 3. The proof makes uses of the transport equation satisfied by the vorticity, where the velocity vector field is associated with the vorticity by \((20a)\) and \((20c)\). With a few adaptations, it is also possible to deal with the case of a transport equation when the velocity field \(v\) is given rather than associated with the vorticity, this is explained in Subsection 8.3.
8.1 Convergence of the approximations to a renormalized solution

We start as in the proof of Theorem 2 for \( \nu \) in \((0, 1)\), we consider \( \omega^m, \omega^+ \) and \( g_\nu \), as in Lemma 3 and \( \omega^+ \) the corresponding global unique smooth solution of the system (41) as in Lemma 4. By applying Proposition 2 in the case where the family \( (v_\nu)_{\nu \in (0, 1)} \) is related to the family of vorticity \( (\omega_\nu)_{\nu \in (0, 1)} \) by \( \nu \), we obtain that, up to a subsequence, \( (\omega_\nu) \) and \( (g_\nu^p, \omega_\nu) \), satisfy (40), and \( (v_\nu)_{\nu \in (0, 1)} \) satisfies (42). Moreover the limit vorticity \( \omega \) and the limit velocity \( v \) satisfy (19) and (20). However we will not try to pass to the limit in the renormalized formulation (23) of the evolution equation but rather proceed by duality, following the strategy used in [14] to prove the corresponding result in the case without source nor sink. We will proceed in 2 steps.

**Step 1. Reduction to a duality formula.** By Proposition 1 there is a renormalized solution \((\bar{\omega}, \bar{\omega}^-)\) to the transport equation associated with the vector field \( v \) and the data \( \omega^m \) and \( \omega^+ \) and to prove Theorem 3 it is sufficient to establish that
\[
(\omega, \omega^-) = (\bar{\omega}, \bar{\omega}^-).
\]
To prove (68) it is sufficient to prove that for any \( T > 0 \), for any smooth functions \( \phi_- \) and \( \phi_T \),
\[
\int_\mathcal{F} (\omega(T, \cdot) - \bar{\omega}(T, \cdot))\phi_T + \int_0^T \int_{\partial \mathcal{F}^-} g(\omega^- - \bar{\omega}^-)\phi_- = 0,
\]
which we are going to prove thanks to the duality formula. Indeed by Proposition 1 the renormalized solution \((\bar{\omega}, \bar{\omega}^-)\) of the transport equation associated with the vector field \( v \) satisfies
\[
\int_\mathcal{F} \bar{\omega}(T, \cdot)\phi_T + \int_0^T \int_{\partial \mathcal{F}^-} g\bar{\omega}^- \phi_- = \int_\mathcal{F} \omega^m \phi(0, \cdot) - \int_0^T \int_{\partial \mathcal{F}^+} g\omega^+ \phi^+,
\]
where \((\phi, \phi^+)\) is the unique renormalized solution of
\[
\begin{align}
-\partial_t \phi - v \cdot \nabla \phi &= 0 & \text{in } (0, T] \times \mathcal{F}, \\
\phi &= \phi_- & \text{on } (0, T] \times \partial \mathcal{F}^-, \\
\phi(T, \cdot) &= \phi_T.
\end{align}
\]
It is therefore sufficient to prove that \((\omega, \omega^-)\) satisfies the same equation, that is
\[
\int_\mathcal{F} \omega(T, \cdot)\phi_T + \int_0^T \int_{\partial \mathcal{F}^-} g\omega^- \phi_- = \int_\mathcal{F} \omega^m \phi(0, \cdot) - \int_0^T \int_{\partial \mathcal{F}^+} g\omega^+ \phi^+,
\]
since the difference of (72) and of (70) leads to (69).

**Step 2. Proof of the duality formula.** To prove (72) we are going to establish first a similar duality formula for the viscous approximations, and then we will pass to the limit as \( \nu \) converges to 0. By using Lemma 1 in the second case with \( v_\nu \) as above, there exist a smooth solution \( \phi_\nu \) to the backward viscous transport equation
\[
\begin{align}
-\partial_t \phi_\nu - v_\nu \cdot \nabla \phi_\nu - \nu \Delta \phi_\nu &= 0 & \text{in } (0, T] \times \mathcal{F}, \\
\partial_t \phi_\nu &= -(\phi_\nu - \phi_-)g_\nu 1_{\partial \mathcal{F}^-} & \text{on } (0, T] \times \partial \mathcal{F}, \\
\phi_\nu(T, \cdot) &= \phi_T.
\end{align}
\]
Moreover, by Proposition 2 up a subsequence, the functions \( \phi_\nu \) satisfy
\[
\phi_\nu \rightharpoonup \bar{\phi} \quad \text{in } C_w([0, T]; L^p(\mathcal{F})) \quad \text{and} \quad g_\nu^{1/p} \phi_\nu \rightharpoonup g^{1/p} \bar{\phi}^+ \quad \text{in } L^p((0, T) \times \partial \mathcal{F}^+) \quad \text{as } \nu \to 0,
\]
where \( p^* \) is the dual exponent of \( p \). Thanks to (42) and (73), by passing to the limit in (74), we obtain that \((\bar{\phi}, \bar{\phi}^+)\) satisfies (71) in the distributional sense. By Proposition 1 it is also a
renormalized solution of (71), and by uniqueness, \((\tilde{\phi}, \tilde{\phi}^+) = (\phi, \phi^+)\). Now, for \(\nu\) in \((0, 1)\), since \(\omega\) is a smooth solution of the system (111), it also satisfies the weak formulation (113). Considering in particular the test function \(\varphi = \phi\nu\) and by an integration by parts of the term containing \(\nabla\) we deduce that for any \(\nu > 0\),

\[
-\int_F \omega_\nu(T, \cdot)\phi_\nu(T, \cdot)dx + \int_F \omega^\text{in}_\nu\phi_\nu(0, \cdot)dx + \int_0^T \int_F \omega_\nu(\partial_t\phi_\nu + v_\nu \cdot \nabla\phi_\nu + \nu \Delta\phi_\nu) \quad (75)
\]

\[
= \int_0^T \int_{\partial F^+} g_\nu \omega^+_\nu \phi_\nu + \int_0^T \int_{\partial F^-} g_\nu \omega^-_\nu \phi_\nu + \nu \int F \omega_\nu \partial_n \phi_\nu.
\]

Let us mention that the time integrals in (143) can be converted into the integrals on \([0, T]\) as above by a standard approximation process by smooth functions of the truncation \(1_{(0,T)}\). Using (73) we deduce from (75) that for any \(\nu > 0\),

\[
\int_F \omega_\nu(T, \cdot)\phi_\nu(T, \cdot)dx + \int_0^T \int_{\partial F^-} g_\nu \omega^-_\nu \phi_\nu = \int_F \omega^\text{in}_\nu\phi_\nu(0, \cdot)dx - \int_0^T \int_{\partial F^+} g_\nu \omega^+_\nu \phi_\nu. \quad (76)
\]

Using (51), (60) and (74) we deduce (72) and therefore conclude the proof of Theorem 8.

Remark 7. In the previous proof we use that \(\omega_\nu\) converges to \(\omega\) in \(C^0_w(0, T; L^p(F))\) to identify \(\omega\) with \(\tilde{\omega}\). Let us note that following the same strategy, it is possible to show the same result in the case where the convergence holds weakly-star in \(L^\infty(0, T; L^p(F))\). To do that is enough to consider an inverse flow \(\phi_\nu\) which is zero at final time and with a source term. More precisely \(\phi_\nu\) satisfies

\[
-\partial_t \phi_\nu - v_\nu \cdot \nabla \phi_\nu - \nu \Delta \phi_\nu = \chi \quad \text{in} \ [0, T] \times F,
\]

\[
\partial_n \phi_\nu = -(\phi_\nu - \phi^-) g_\nu 1_{\partial F^-} \quad \text{on} \ [0, T] \times \partial F,
\]

\[
\phi_\nu(T, \cdot) = 0.
\]

where \(\chi \in C^\infty_c((0,T) \times F)\). Using the properties of \(\phi_\nu\), we deduce

\[
\int_0^T \int_F (\omega - \tilde{\omega}) \chi + \int_0^T \int_{\partial F^-} g(\omega^- - \tilde{\omega}^-) \phi_\nu = 0,
\]

which is the analogous of (69). This approach will be used in the proof of Theorem 7. Finally under the hypothesis that

\[
\omega_\nu \xrightarrow{w^*} \omega \text{ in } L^\infty(0, T; L^p(F)) \quad \text{and} \quad \omega_\nu(T, \cdot) \xrightarrow{w} \omega_T \text{ in } L^p(F),
\]

then by taking an inverse flow \(\phi_\nu\) with source term and non-zero initial data, we deduce

\[
\int_0^T \int_F (\omega - \tilde{\omega}) \chi + \int_F (\omega(T, \cdot) - \tilde{\omega}(T, \cdot)) \phi_T + \int_0^T \int_{\partial F^-} g(\omega^- - \tilde{\omega}^-) \phi_\nu = 0, \quad (77)
\]

in particular also \(\omega_T = \omega(T, \cdot) = \tilde{\omega}(T, \cdot)\).

8.2 Strong convergence of the vorticity

Above we have proven that the sequence of approximate solutions \(\omega_\nu\) converges to \(\omega\) as the parameter \(\nu\) goes to 0, in \(C^0_{\text{loc}}(\mathbb{R}^+; L^p(F) - w)\) and \(\omega_\nu g_\nu^{1/p} \xrightarrow{w} \omega^- g^{1/p}\) in \(L^p_{\text{loc}}(\mathbb{R}^+ \times \partial F^-)\) where \((\omega, \omega^-)\) is the unique renormalized solution to the Euler system in vorticity form. Indeed we can prove that the convergences of \(\omega_\nu\) and of \(\omega_\nu g_\nu^{1/p}\) are strong respectively in \(C^0_{\text{loc}}(\mathbb{R}^+; L^p(F))\) and \(L^p_{\text{loc}}(\mathbb{R}^+ \times \partial F^-)\), in the case where \(p\) is in \((1, \infty)\).

Theorem 6. Let \(p\) in \((1, \infty)\). As \(\nu\) goes to 0, the sequence \(\omega_\nu\) converges to \(\omega\) in \(C^0_{\text{loc}}(\mathbb{R}^+; L^p(F))\) and the sequence \(\omega_\nu g_\nu^{1/p}\) converges to \(\omega^- g^{1/p}\) in \(L^p_{\text{loc}}(\mathbb{R}^+ \times \partial F^-)\).

Proof of Theorem 7. The proof is divided into seven steps.
Step 1. Let \( G(x) := x^2/\sqrt{x^2 + 1} \). The function \( G \) is strictly convex function. Moreover, since \( G(x) \leq |x| \), we deduce from the bounds on the sequences \( \omega_\nu \) and \( \omega_\nu g_\nu^{1/p} \) that the sequences \( G(\omega_\nu) \) and \( G(\omega_\nu) g_\nu^{1/p} \) are uniformly bounded respectively in \( L^\infty(\mathbb{R}^+; L^p(\mathcal{F})) \) and \( L^p(\mathbb{R}^+; \partial \mathcal{F}^-) \). As a consequence, up to subsequences, for any \( T > 0 \),

\[
G(\omega_\nu) \xrightarrow{w*} \mathcal{G} \quad \text{in} \quad L^\infty(0, T; L^p(\mathcal{F})) \quad \text{and} \quad G(\omega_\nu) \xrightarrow{w} \mathcal{g} \quad \text{in} \quad L^p(0, T; L^p(\partial \mathcal{F})).
\]

Step 2. Recall that \( G(\omega_\nu) \) satisfies the system:

\[
\partial_t G(\omega_\nu) + v_\nu \cdot \nabla G(\omega_\nu) = \nu \Delta G(\omega_\nu) - \nu G''(\omega_\nu)|\nabla \omega_\nu|^2 \quad \text{in} \quad \mathbb{R}^+ \times \mathcal{F}, \tag{79}
\]

\[
\nu \partial_n G(\omega_\nu) = (\omega_\nu - \omega_\nu^+) G'(\omega_\nu) g_\nu 1_{\mathcal{F}+} \quad \text{in} \quad \mathbb{R}^+ \times \partial \mathcal{F}+. \tag{80}
\]

By some integration by parts, we deduce that

\[
\int_\mathcal{F} G(\omega_\nu(t, \cdot)) + \int_0^t \int_{\partial \mathcal{F}^-} g_\nu G(\omega_\nu) + \nu \int_0^t \int_\mathcal{F} G''(\omega_\nu)|\nabla \omega_\nu|^2 \tag{81}
\]

\[
+ \int_0^t \int_{\partial \mathcal{F}^+} (G'(\omega_\nu^+) - G(\omega_\nu) + (\omega_\nu - \omega_\nu^+) G'(\omega_\nu))(-g_\nu) = \int_\mathcal{F} G(\omega_\nu^m) + \int_0^t \int_{\partial \mathcal{F}^+} g_\nu G(\omega_\nu^+).
\]

Observe that, since \( G \) is convex and \( g_\nu \) is negative on \( \mathcal{F}^+ \), the two last terms in the left hand side are nonnegative. Using on the one hand, the weak convergence of the sequences \( \omega_\nu \) and \( \omega_\nu g_\nu^{1/p} \) and the weakly lower semicontinuity of the functionals

\[
\mathcal{G}(f) := \int_\mathcal{F} G(f) \quad \text{and} \quad \mathcal{G}_b(f_1, f_2) := \int_0^t \int_{\partial \mathcal{F}^+} G(f_1) |f_2|,
\]

for the left hand side, and on the other hand the strong convergence of the data to handle the right hand side, we deduce that

\[
\int_\mathcal{F} G(\omega(t, \cdot)) + \int_0^t \int_{\partial \mathcal{F}^-} g G(\omega^-) + \nu \liminf \nu \int_0^t \int_\mathcal{F} G''(\omega_\nu)|\nabla \omega_\nu|^2 \tag{82}
\]

\[
+ \liminf \left( \int_0^t \int_{\partial \mathcal{F}^+} (G(\omega_\nu^+) - G(\omega_\nu) + (\omega_\nu - \omega_\nu^+) G'(\omega_\nu))(-g_\nu) \right)
\]

\[
\leq \int_\mathcal{F} G(\omega_\nu^m) + \int_0^t \int_{\partial \mathcal{F}^+} g G(\omega^+).
\]

Moreover \((\omega, \omega^-)\) being a renormalized solution to the transport equation associated with the velocity \( v \) and the data \( \omega_\nu^m \) and \( \omega^+ \), it holds in particular that

\[
\int_\mathcal{F} G(\omega(t, \cdot)) + \int_0^t \int_{\partial \mathcal{F}^-} g G(\omega^-) = \int_\mathcal{F} G(\omega_\nu^m) + \int_0^t \int_{\partial \mathcal{F}^+} g G(\omega^+). \tag{83}
\]

Combining (82) and (83) we arrive at

\[
\nu \liminf \nu \int_0^t \int_\mathcal{F} G''(\omega_\nu)|\nabla \omega_\nu|^2 + \nu \liminf \nu \int_0^t \int_{\partial \mathcal{F}^+} (G(\omega_\nu^+) - G(\omega_\nu) + (\omega_\nu - \omega_\nu^+) G'(\omega_\nu))(-g_\nu) \leq 0.
\]

Therefore

\[
\nu G''(\omega_\nu)|\nabla \omega_\nu|^2, (G(\omega_\nu^+) - G(\omega_\nu) + (\omega_\nu - \omega_\nu^+) G'(\omega_\nu)) g_\nu \to 0 \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^+; L^4(\mathcal{F})) \times L^1_{\text{loc}}(\mathbb{R}^+ \times \partial \mathcal{F}^+). \tag{84}
\]
Step 3. Let $q$ such that $p$ and $q$ are conjugated. Let $\Psi$ in $L^q([0,T] \times \partial F^-; g \, dz \, ds)$ and $\chi$ in $L^1([0,T]; L^q(F))$. Let $(\phi, \phi^+)$ the $L^q$ renormalized solution of the backward transport equation:

$$
-\partial_t \phi - v \cdot \nabla \phi = \chi \quad \text{in } [0,T] \times F,
$$

$$
\phi = \Psi \quad \text{on } [0,T] \times \partial F^-,
$$

$$
\phi(T,x) = 0.
$$

Then, proceeding as in the proof of Theorem 3, with the help of the convergences (78) and (78), we obtain:

$$
\int_0^T \int_F \Theta \chi + \int_0^T \int_{\partial F^-} g \phi \Psi = \int_F G(\omega^d) \phi(0,.) - \int_0^T \int_{\partial F^+} g G(\omega^d) \phi^+. \tag{85}
$$

On the other hand, since $(\omega, \omega^-)$ with the velocity $v$ converges to $\omega^d$, from Lemma 6 and the weak convergence of $G$, that passing to subsequences $G_{\omega}$ of Proposition 2 we deduce that the sequences $G$, $\omega^d$ converge almost everywhere to $\omega$ pointwise.

Step 4. We recall the following result which is proved in the first step of the proof of Lemma 3.34 of [36].

Lemma 6. Let $O \subset \mathbb{R}^2$ measurable, let $f_n, f$ a sequence of $L^1(O)$ functions and $G$ a strictly convex function. If $f_n \overset{w}{\rightharpoonup} f$ and $G(f_n) \overset{w}{\rightharpoonup} G(f)$ in $L^1(O)$. Then, up to a subsequence, $f_n$ converges almost everywhere to $f$ pointwise.

To handle the convergence on the boundary we will also use the following corollary, which is proved in an appendix.

Corollary 3. Let $p$ in $(1, +\infty)$, let $f_n, f$ a sequence of $L^p((0,T) \times \partial F^-; g \cdot dt \cdot ds)$ functions and $G$ a strictly convex function from $\mathbb{R}$ to $\mathbb{R}$ with bounded derivative. If $f_n g_n \overset{w}{\rightharpoonup} fg$ and $G(f_n) g_n \overset{w}{\rightharpoonup} G(f) g$ in $L^p((0,T) \times \partial F^-)$. Then, up to subsequence, $f_n$ converges almost everywhere to $f$.

From the previous steps, Lemma 6 and Corollary 3 we deduce the almost everywhere convergences of the sequence $\omega_n$ to $\omega$ in $(0,t) \times F$ and of the sequence $\omega_n$ to $\omega^d$ in $[0,T] \times \partial F^-$. 

Step 5. From De la Vallée Poussin’s lemma and Corollary 3 together with the estimates from Step 1 of Proposition 2 we deduce that the sequences $|\omega_n| p$ and $g_n \omega_n |p|_{\partial F^-}$ are uniformly integrable. Then, by the previous step and Vitali’s Lemma (see for example Theorem 1.18 of [36]), we have that $\omega_n \to \omega$ in $L^q(0,T; L^p(F))$ for any $q \in [1, +\infty)$ and $\omega_n g_n^{1/p} \to \omega g^{1/p}$ in $L^p((0,T) \times \partial F^-)$.

Step 6. The convergence $\omega_n$ to $\omega$ in $L^q(0,T; L^p(F))$, implies that $\omega_n(t,.)$ converges to $\omega(t,.)$ in $L^p(F)$ for almost any time. Let now prove by contradiction that the convergence holds for any time. Suppose that there exists $s \in (0,T]$ such that the convergence does not hold. In particular there is $\varepsilon > 0$ and there exists a subsequence of $(\omega_n)$, which we do not relabel, such that $\|\omega_n(s,.) - \omega(s,.)\|_{L^p(F)} \geq \varepsilon > 0$. From the a priori estimates [65], with $t = s$, we deduce that passing to subsequences $G(\omega_n(s,)) \overset{w}{\rightharpoonup} G(s,)$ in $L^p(F)$. Moreover proceeding as mentioned in Remark 7 we deduce the identification formula (77). In particular we identify $G(s,)$ with $G(\omega(s,))$. From Lemma 6 and the weak convergence of $\omega_n(s,.)$ to $\omega(s,.)$, we obtain that up to subsequence $\omega_n(s,.)$ converges strongly to $\omega(s,.)$ in $L^p(F)$ which is a contradiction. We have shown that $\omega_n(t,.)$ converges to $\omega(t,.)$ for any $t \in [0,T]$. 

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Step 7. We conclude by showing that the convergence holds uniformly in time. Suppose by contradiction that the convergence is not uniform. Then there exists $\delta > 0$, there exists a subsequence of $\{\omega_n\}$, which we do not relabel, and a sequence of times $t_n$ such that $\|\omega_n(t_n) - \omega(t_n)\|_{L^p(F)} \geq \delta$. The interval $[0, T]$ is compact. Passing to a subsequence we can assume that $t_n \to t$. Using the continuity of $\omega$ from $n$ large enough we have that $\|\omega_n(t_n) - \omega(t_n)\|_{L^p(F)} \geq \delta$. On the other hand it follows from the $C_w(L^p)$ convergence of $\omega$, to $\omega$ that $\omega_n(t_n)$ converges to $\omega(t)$ weakly in $L^p(F)$ and combining (31) and (33) we obtain that $\|\omega_n(t_n)\|_{L^p(F)} \geq \|\omega(t)\|_{L^p(F)}$. Since for $1 < p < \infty$, the space $L^p$ is uniformly convex, and we deduce that $\omega_n(t_n)$ converges to $\omega(t)$ in $L^p(F)$, which is the desired contradiction. This concludes the proof of Theorem 6.

8.3 A note on the vanishing viscosity solution to the transport equation

Since the proof above uses the transport equation satisfied by the vorticity, one may wonder if similar results are true for the transport equation when the velocity field $v$ is given rather than associated with the vorticity. Indeed in the case where the fluid occupies the whole space $R^d$, it is known, see [17] IV.1 and more recently in [37], that the vanishing viscosity approximations converge in $C([0,T]; L^p)$ to the renormalized solution to the transport equation. These results can be extended to the present setting, where sources and sinks are present. However, compared to the proof above, some adaptations are needed. In particular, in Subsection 8.2, we took advantage of the convergence of the vorticity in $C([0,T]; L^p - w)$, which is not clear in the case where the velocity field is not related to the vorticity through a $div$-curl system. For sake of completeness, we state and sketch the proof of the corresponding result.

Theorem 7. Let $p \in [1, +\infty]$ and $q \in (1, +\infty)$. Let $v$ in $L^1_{loc}(R^d; W^{1,q}(F))$ a SSC vector field. Let $(\omega^m, \omega^+)$ a (CM) in $L^p$. Let some families $\omega^m_n, \omega^+_n, v^+_n$ as in Lemma 3. Let $\omega_n$ the corresponding global unique smooth solution of the system (11) associated with the data $\omega^m_n, \omega^+_n$ and $v_n$, as in Lemma 4. Then there exists a subsequence of $\omega_n$ which we still denote $\omega_n$ and $v_n$ and which satisfy for $p < \infty$

$$\omega_n \overset{w}{\rightharpoonup} \omega \quad \text{in} \quad L^\infty([0,T]; L^p(F)), \quad g_1/p \omega_n \overset{w}{\rightharpoonup} g_1/p \omega^- \quad \text{in} \quad L^p((0,T) \times \partial F^-),$$

and

$$\sqrt{\nu} \nabla \omega_n \longrightarrow 0 \quad \text{in} \quad L^2([0,T]; L^2(F)).$$

Moreover $(\omega, \omega^-)$ is the unique renormalized solution to the transport associated with the velocity field $v$ and the data $(\omega^m, \omega^+)$. Finally, if $p, q \in (1, \infty)$, then the two first convergences in (87) can be improved into the following strong convergences: $\omega_n$ converges to $\omega$ in $C^0_{loc}(\mathbb{R}^d; L^p(F))$ and $\omega_n g_1/p$ converges to $\omega^{-g_1/p}$ in $L^p_{loc}(\mathbb{R}^d \times \partial F^-)$.

Proof. The proof of (87) can be performed as in the proof of Proposition 2. Moreover, by using the duality method as in the proof of Theorem 3 and taking account Remark 6 we prove that $(\omega, \omega^-)$ is the unique renormalized solution to the transport associated with the velocity field $v$ and the data $(\omega^m, \omega^+)$. Finally it is possible to show the strong convergence of the vorticity as follows. For $q \geq 2$, we deduce that $\omega_n \rightarrow \omega$ in $C^0_w([0,T]; L^p(F))$ by using the equations so that one may proceed as in the case of the Euler system, see Subsection 8.2. For $q < 2$, we can proceed as in [37], by considering a sequence $(\omega^m_n, \omega^-) \in L^q$ converging to $(\omega^m, \omega^-)$ in $L^p$. From Lemma 3 there exist sequences $(\omega^m_n, \omega^+_n)$ of compatible regular data. Since $q < 2$, the estimates (32) hold true and we can furthermore impose the extra condition

$$\sup \left\| \omega^m_n - \omega^m \right\|_{L^p(F)} + \left\| g_1/p \omega^+_n - g_1/p \omega^+ \right\|_{L^p((0,T) \times \partial F^-)} \longrightarrow 0 \quad \text{as } \nu \longrightarrow 0.$$
Note that for the first and the last line of the right hand side the norm can be bounded by a constant times the size of the initial data, and therefore converge to zero. Moreover the middle term converges to zero for any fixed \( l \), thanks to the previous step since \( q' > p \).

\[ \square \]

9 Proof of Theorem 4 on symmetrized solutions

We start as in the proof of Theorem 2 and of Theorem 3: for \( \nu \) in \((0,1)\), we consider \( \omega^n_\nu, \omega^+_\nu \) and \( g_\nu \), as in Lemma 3 with \( p = 1 \), and \( \omega_i \), the corresponding global unique smooth solution of the system (41) as in Lemma 4. Lemma 3, and \( \omega_i \) the corresponding global unique smooth solution of the system (41) as in Lemma 4. These solutions satisfy the weak formulation (66) for any \( \nu \) how to pass to the limit (41) and (67) as \( \nu \to 0 \).

- Thanks to (61),
  \[
  \int \omega^n_\nu \varphi(0,.dx \to \int \omega^n_\nu \varphi(0,.dx \text{ and } \int R^+ \int _{\partial R^+} g_\nu \omega^+_\nu \varphi \text{dsdt} \to \int R^+ \int _{\partial R^+} g_\nu \omega^+ \varphi \text{dsdt}. \tag{88}
  \]
  In particular, from the last convergence we deduce that
  \[
  C_{i,\nu} \to C_i \text{ in } C([0,T]), \text{ with } C_i(t) := C_i^n(t) - \int_0^t \int _{\partial S^i} \omega^+ g \text{ for } i \in \mathcal{I}^+. \tag{89}
  \]

- Thanks to (60), up to a subsequence,
  \[
  \int R^+ \int _{\partial R^+} g^{-1}_\nu \omega^{-1}_\nu \varphi \text{dsdt} \to \int R^+ \int _{\partial R^+} g^{-1}_\nu \omega^{-1} \varphi \text{dsdt}, \tag{92}
  \]
  In particular, from the last convergence we deduce that
  \[
  C_{i,\nu} \to C_i \text{ in } C([0,T]), \text{ with } C_i(t) := C_i^n(t) - \int_0^t \int _{\partial S^i} \omega^- g \text{ for } i \in \mathcal{I}^- . \tag{93}
  \]

Therefore (60) is already proved and moreover, from (60), (41) and (93), we deduce that

\[
\sum_i \int_{R^+} C_{i,\nu}(t) \int _{\mathcal{F}} \omega_\nu \varphi \text{dx dt} \to \sum_i \int_{R^+} C_i(t) \int _{\mathcal{F}} \omega X_i \cdot \nabla \varphi \text{dx dt}. \tag{94}
\]

- Thanks to (60), the viscous term converges to zero:
  \[
  \nu \int_{R^+} \int _{\mathcal{F}} \nabla \omega_\nu \cdot \nabla \varphi \to 0. \tag{95}
  \]

- We are now going to prove that for any test function \( \varphi \) in \( \mathcal{C}_0(\mathcal{F}) \) (recall the definition above Lemma 4),
  \[
  \int R^+ \int _{\mathcal{F}} H_\varphi(x,y)\omega_\nu(t,x) \omega_\nu(t,y) \text{dx dy dt} \to \int R^+ \int _{\mathcal{F}} H_\varphi(x,y) \omega(t,x) \omega(t,y) \text{dx dy dt}. \tag{96}
  \]
To prove that we consider a $C^\infty$ function $\zeta : [0, +\infty) \to [0, +\infty)$ such that $\zeta(x) = 1$ for $x \leq 1$ and $\zeta(x) = 0$ for $x \geq 2$. Let $\delta > 0$ and $\zeta : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ such that $\zeta(x, y) = \zeta\left(\frac{|x-y|}{\delta}\right)$. The family containing the measures $dt \otimes d\omega_\nu \otimes d\omega_\mu$, for $\nu$ in $(0, 1)$, and $dt \otimes d\omega \otimes d\omega$ are uniformly integrable in $L^1((0, T) \times \mathcal{F} \times \mathcal{F})$. Therefore for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \int_{\mathbb{R}^+} \int_{\mathcal{F}} \int_{\mathcal{F}} \zeta\left(\frac{|x-y|}{\delta}\right)H_\psi(x, y)\omega_\nu(t, x)\omega_\nu(t, y) \, dx \, dy \, dt \right| \leq \varepsilon,$$

using that the Lebesgue measure of the set of the couples $(x, y)$ in $\mathcal{F} \times \mathcal{F}$ such that $|x-y| \leq 2\delta$ goes to zero as $\delta$ goes to zero and Lemma [11] regarding the boundedness of $H_\psi$. Moreover, for this $\delta$, since the function $(x, y) \mapsto (1 - \zeta\left(\frac{|x-y|}{\delta}\right))H_\psi(x, y)$ is continuous on $(0, T) \times \mathcal{F} \times \mathcal{F}$ and the tensor product $\omega_\nu \otimes \omega_\nu$ converges to $\omega \otimes \omega$ in $C(0, T; \mathcal{M}(\mathcal{F} \times \mathcal{F}))$, there exists $\nu > 0$ small enough for

$$\left| \int_{\mathbb{R}^+} \int_{\mathcal{F}} \int_{\mathcal{F}} (1 - \zeta\left(\frac{|x-y|}{\delta}\right))H_\psi(x, y)\omega_\nu(t, x)\omega_\nu(t, y) \, dx \, dy \, dt \right| \leq \varepsilon,$$

Gathering (97) and (98) we obtain (99).

Combining (96), (97), (98), (99), (100), (101), (102), (103), (104), and (105) we arrive at (106). This concludes the proof of Theorem [1].

A Proof of Corollary [3]

In this appendix we prove Corollary [3] which is a variation Lemma 3.34 of [30]. For the reader’s convenience we recall the statement of Corollary [3] for $p$ in $(1, +\infty)$, for any sequence $f_\nu$ in $L^p((0, T) \times \partial F^+; g_\nu \, dt \, ds)$, for any $f$ in $L^p((0, T) \times \partial F^+; g \, dt \, ds)$ and for any strictly convex function $G$ from $\mathbb{R}$ to $\mathbb{R}$ with bounded derivative, if $f_\nu, g_\nu \xrightarrow{\text{w*}} f$ and $G(f_\nu)g_\nu \xrightarrow{\text{w*}} G(f)g$ in $L^p((0, T) \times \partial F^+; dt \, ds)$, then, up to a subsequence, $f_\nu$ converges to $f$ almost everywhere.

Let us also recall that $g_\nu$ and $g$ are positive functions on $\mathbb{R}^+ \times \partial F^+$ such that $g_\nu$ converges to $g$ in $L^1((0, T) \times \partial F^+; dt \, ds)$.

Proof of Corollary [3]. By the hypothesis that $g > 0$ and by Egoroff’s theorem, we obtain that, for any $\varepsilon > 0$, there exists a measurable set $E \subset (0, T) \times \partial F$ with Lebesgue measure $\mu(E) < \varepsilon$ and a constant $c_\varepsilon > 0$ such that $g \geq c_\varepsilon$ in $E^c := ((0, T) \times \partial F) \setminus E$, and such that $g_\nu$ converges to $g$ uniformly in $E^c$.

Since the function $G$ is strictly convex, for any $x$ in $\mathbb{R}$ there exists a strictly increasing function $\psi_x$ on $\mathbb{R}^+$ with $\psi_x(0) = 0$, such that for any $y$ in $\mathbb{R}$,

$$G(y) - G(x) - G'(x)(y - x) \geq \psi_x(|y - x|).$$

We refer here to the proof of Lemma 3.34 in [30] for more on the construction of such a function $\psi_x$. We deduce that

$$\int_{E^c} G(f_\nu)g - G(f)g - G'(f)(f_\nu - f)g \geq \int_{E^c} \psi_x(|f_\nu - f|)g,$$

Since $g$ is positive, it is therefore sufficient to prove that the left-hand side converges to zero to conclude the proof of Corollary [3]. We have

$$\int_{E^c} G(f_\nu)g - G(f)g - G'(f)(f_\nu - f)g = \int_{E^c} G(f_\nu)g_\nu - G(f)g - G'(f)(f_\nu g_\nu - fg)$$

$$+ \int_{E^c} G(f_\nu)(g - g_\nu) - G'(f)f_\nu(g - g_\nu).$$
The first line of the right hand side converges to zero from the weak convergences of \( f_\nu g_\nu \) to \( fg \) and of \( G(f_\nu)g_\nu \) to \( G(f)g \) in \( L^p((0,T) \times \partial F^-) \). For the second one we first use the lower bound of \( g \) and the uniform convergence of \( g_\nu \) to \( g \) to infer from the weak convergences above that \( f_\nu \overset{w}{\rightharpoonup} f \) and that \( G(f_\nu) \overset{w}{\rightharpoonup} G(f) \) in \( L^p((0,T) \times \partial F^-) \). This entails that the term on the second line converges to 0. This concludes the proof.

\[ \square \]

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