SO(3) massive gravity

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In this paper, we propose a massive gravity theory with 5 degrees of freedom (d.o.f). The mass term is constructed by 3 St"uckelberg scalar fields, which respects SO(3) symmetry in the fields' configuration. By the analysis on the linear cosmological perturbations, we found that such 5 d.o.f are free from ghost instability, gradient instability, and tachyonic instability.

Introduction. The search for a consistent theory of finite range gravity is a longstanding and well motivated problem. Whether there exists such a consistent extension of general relativity (GR) by a mass term is a basic question of classical field theory. After Fierz and Pauli’s pioneering work in 1939 [1], this question has been attracting a great deal of interest. However, its consistency has been a challenging problem for several decades.

In Fierz and Pauli’s model, the GR is extended by a linear mass term. However, such simplest massive gravity model gives rise to a discontinuity in the observables [2]. This problem can be alleviated by nonlinear terms [3]. However, since the lack of Hamiltonian constraint and momentum constraint, it ends up with six d.o.f in the gravity sector. The Poincare symmetry in the 3+1 space time implies that a massive spin-2 particle should contain 5 helicities modes. The rest sixth mode is the so-called Boulware-Deser (BD) ghost [4], spoiling the stability of the theory.

Only recently, a non-linear massive gravity theory (which is dubbed as dRGT gravity) has been found [5,6]. The BD ghost is removed by constructing a general action based on the time reparameterization invariance and residual SO(3) symmetry in the scalar fields’ configuration, to recover the general covariance.

In this theory, the GR is extended by a linear mass term [7]. However, the following up cosmological perturbation analysis revealed a new ghost instability among the rest five d.o.f [8] [9] [10]. On the other hand, this theory is only contains 5 d.o.f in the gravity sector, which is intrinsically free from BD ghost issue [11]. By adopting the St"uckelberg trick, we introduce 3 scalars, which respect residual SO(3) symmetry in the fields’ configuration, to recover the general covariance.

This paper is organized as follows: Firstly we write down a general action based on the time reparameterization invariance and residual SO(3) symmetry in the scalar fields’ configuration. Then we apply our theory to cosmology. The linear cosmological perturbation analysis reveals 5 healthy d.o.f on the perturbation spectrum.

Setup. Taking the time reparameterization invariance and residual SO(3) symmetry as our building principle, a general action with a mass term can be written as

\[ I_f = \int d^4x \sqrt{-g} \left\{ \frac{M_p^2}{2} R + m_1^2 G_{\mu\nu} f_{\mu\nu} - M_p^2 m_2^2 \left( c_0 + c_1 f + c_2 f^2 + d_2 f_\mu f_\mu + \ldots \right) \right\}, \]

where \( f_{\mu\nu} \equiv \partial_\mu \phi^a \partial_\nu \phi^b \delta_{ab} \), and \( a, b = 1, 2, 3 \). \( f_{\mu\nu} \equiv g^{\mu\rho} f_{\rho\nu} \), \( f \equiv f_{\mu}^\mu \). Notice that \( G_{\mu\nu} \) is the Einstein tensor and \( G_{\mu\nu} f_{\mu\nu} \) is the so called Horndeski term [12].

The equation of motion is still second derivative. On the other hand, as for the \( M_p^2 m_2^2 \) part, in principle we can add an infinitely polynomial series inside of round bracket. However, for the simplicity of calculations, we truncate the higher order term, just consider a constant term, a linear term and two quadratic terms in this paper. The cosmology in the presence of matter content will be discussed in our future work [13].

In the unitary gauge,

\[ \phi^a = \alpha x^a, \quad f_{\mu\nu} = (0, \alpha^2, \alpha^2, \alpha^2), \]

where \( \alpha \) is an un-normalized quantity, which can be absorbed into the redefinition of coefficients \( c_1, c_2, d_2 \). Without introducing any ambiguity, we set \( \alpha = 1 \) and \( f_{\mu\nu} = (0, 1, 1, 1) \).

Assume we start from a FRW background, with physical metric,

\[ ds^2 = -N^2 dt^2 + a(t)^2 dx^2, \]
FIG. 1: Infinitely strong coupling (ISC) issue in the SO(3) massive gravity. The horizontal axes denotes $f \equiv g^\mu\nu f_{\mu\nu}$, the blue curve denotes the mass term with $c_1 (3c_2 + d_2) < 0$, the green curve denotes the mass term with $c_1 (3c_2 + d_2) > 0$, and the red line, which overlaps with the horizontal axes, denotes an Einstein static universe which exhibit the infinitely strong coupling. Please notice $f$ is positive by definition.

and $f = 3a^{-2}$ in this case. It is important to notice that for the parameter region $c_1 (3c_2 + d_2) < 0$, when $f$ approaches to the bottom of the potential, one gets an Einstein static universe, provided a proper cosmological constant(see FIG[1]). The helicity 0 mode and helicity 1 mode exhibit the infinitely strong coupling in such Einstein static universe, thus it is not our interest (the detail will be present in [21]). The condition $c_1 (3c_2 + d_2) > 0$ is required to avoid such problem.

**Cosmological solution** We choose the FRW ansatz, and the space-time metric can be written as eq.4. By taking the variation of the action with respect to the metric $g^\mu\nu$, we get such two background Einstein equations,

$$3H^2 = \frac{r_2 m_2^2}{2} \left[ c_0 + 3c_1 a^{-2} + 3(3c_2 + d_2)a^{-4} \right], \quad (5)$$

$$\frac{\dot{H}}{N} = \frac{r_2^2 m_2^2}{6a^2} \left[ \left( r_1 c_0 - 3c_1 \right) - \frac{(2 + r_1 a^{-2})(3c_2 + d_2)}{a^4} \right], \quad (6)$$

where $r_1 \equiv m_1^2/M_p^2$ and $r_2 \equiv M_p^2/(M_p^2 + \omega_a^2)$. In addition to a bare cosmological constant, we can see our mass term contributes a curvature-like term, and a radiation-like term. Noting that since the the SO(3) symmetry in the fields’ configuration, the constraint equations of 3 St"uckelberg scalars $\phi^a$ are trivially satisfied.

**Scalar perturbations** We perturb the space-time metric and define the scalar perturbations by

$$g_{00} = -N^2(t)[1 + 2\phi], \quad (7)$$

$$g_{0i} = N(t)a(t)\partial_i \beta, \quad (8)$$

$$g_{ij} = a^2(t)[\delta_{ij} + 2\psi \delta_{ij} + (\partial_i \partial_j - \frac{1}{3} \partial^2)\phi], \quad (9)$$

Here we choose the unitary gauge, where $\phi^a = x^a$. After integrating out the non-dynamical degree, the quadratic action of scalar perturbation is

$$I_{\text{scalar}} = \frac{M_p^2}{2} \int dt d^3k N a^3 \left( K_s \frac{\dot{E}^2}{N^2} - \mathcal{M}_s E^2 \right), \quad (10)$$

where

$$K_s = \frac{k^4 \left[ a^2 (c_1 m_2^2 + 3H^2 r_1) + 2m_2^2 (3c_2 + d_2) \right]}{2r_2 \left[ a^2 (3c_1 m_2^2 + 9H^2 r_1 + k^2) + 6m_2^2 (3c_2 + d_2) + k^2 r_1 \right]} \quad (11)$$

The full expression of $\mathcal{M}_s$ is quite bulky, and we are not going to show it here.

To check if the scalar mode is ghosty, let’s substitute eq.(15) into the above formula, and then take the super horizon approximation, we get

$$K_s \simeq \frac{1}{6r_2} k^4. \quad (12)$$

The scalar mode is ghost free at super horizon scale as long as $r_2 \equiv M_p^2/(M_p^2 + \omega_a^2)$ is positive.

To see the situation in the small scale, let’s take the sub-horizon approximation, we get

$$K_s \simeq \frac{1}{2} r_1 r_2 k^2 m_2^2 c_0 + \frac{1}{2} c_1 r_2 m_2^2 k^2 (1 + 4r_1 a^{-2})$$

$$+ \frac{1}{2} (3c_2 + d_2) r_2 m_2^2 k^2 (2a^{-2} + 5r_1 a^{-4}). \quad (13)$$

At late time epoch where $a \rightarrow \infty$, $K_s > 0$ requires

$$(r_1 c_0 + c_1) m_2^2 > 0, \quad (14)$$

and for the early stage where $a \ll 1$, $K_s > 0$ requires

$$(3c_2 + d_2) m_2^2 > 0. \quad (15)$$

In order to check if our theory is free from gradient instability and tachyonic instability, we define a new canonical variable

$$\mathcal{E} \equiv \kappa E, \quad (16)$$

where $\kappa$ is defined in the following eq.(18) and eq.(20). The canonical normalized action can be rewritten in terms of this canonical variable as

$$I_g = \frac{1}{2} \int dt d^3k N a^3 \left( \frac{\dot{\mathcal{E}}^2}{N^2} - \omega_\mathcal{E}^2 \mathcal{E}^2 \right). \quad (17)$$

Under the super horizon approximation, at leading order we have,

$$\kappa \simeq \frac{k^2 \tilde{M}_p}{\sqrt{6}}, \quad (18)$$

$$\omega_\mathcal{E}^2 \simeq \frac{m_2^2 (4c_1 + 3r_1 c_0)}{a^2}, \quad (19)$$
where $\tilde{M}_p^2 = M_p^2 + \frac{n_2^2}{a^2}$. There is no tachyonic instability outside of horizon if $n_2^2(4c_1 + 3r_1c_0) > 0$.

During late time epoch, under the sub horizon approximation, at the leading order we have

$$\kappa \simeq km_2M_p\sqrt{\frac{(c_0r_1 + c_1)}{2}}, \quad (20)$$

$$\omega_v^2 \simeq \frac{k^2}{a^2} \cdot (21)$$

We can see that during the late time epoch, at leading order the sound speed of scalar mode at subhorizon scale is 1. Although we start from an action break the Lorentz invariance, Lorentz violation effect doesn’t show up at the leading order of our calculation.

**Vector perturbations** By performing the similar approach to vector perturbation, we can also check that the vector mode is also healthy under the same ghost free condition. Firstly, let’s define the vector perturbations of the metric as,

$$\delta g_{0i} = N(t)a(t)S_i, \quad (22)$$

$$\delta g_{ij} = \frac{1}{2}a^2(t)(\partial_i F_j + \partial_j F_i), \quad (23)$$

where the vector perturbations satisfy the transverse condition,

$$\partial_i S^i = \partial_i F^i = 0. \quad (24)$$

After integrating out the non-dynamical degree, we get the quadratic action of the vector mode as follows,

$$I_{vector} = \frac{M_p^2}{2} \int dtd^3k N a^3 \left[ \kappa_v \dot{F}_i \dot{F}_i \frac{1}{N^2} - \mathcal{M}_v F_i F^i \right], \quad (25)$$

where

$$\kappa_v = \frac{k^2}{8r_2} \cdot (26)$$

We are not going to show the full expression of $\mathcal{M}_v$ since it is too bulky. At super horizon scale, we have

$$\kappa_v \simeq \frac{k^2}{8r_2} \cdot (27)$$

Again, similar to the scalar case, $r_2 > 0$ ensures that the kinetic term is positive. At subhorizon scale, we have

$$\kappa_v \simeq \frac{1}{2}c_0r_1 r_2 m_2 \left[ 1 + \frac{4r_1}{a^2} \right]$$

$$+ \frac{1}{2a^2(3c_2 + d_2)} r_2 m_2 \left[ 2 + \frac{5r_1}{a^2} \right] \cdot (28)$$

By requiring that the vector mode is ghost free at two opposite limit $a \to \infty$ and $a \ll 1$, we get exactly the same ghost free condition as in eq. [14, 15].

In order to check the gradient instability and tachyonic instability, we write down the canonical normalized action for vector perturbations,

$$I_{vector} = \frac{1}{2} \int dtd^3k N a^3 \left[ \frac{\dot{F}_i \dot{F}_i}{N^2} - \omega_v^2 F_i F^i \right], \quad (29)$$

where at leading order,

$$\dot{F}_i \simeq \frac{\sqrt{2}k \tilde{M}_p F_i}{4},$$

$$\omega_v^2 \simeq \frac{m_2^2(4c_1 + 3c_0r_1)}{a^2}, \quad for \ k \ll aH, \quad (30)$$

and

$$\mathcal{F}_i \simeq \frac{\sqrt{2}(c_1 + c_0r_1)}{2} \tilde{M}_p m_2 F_i,$$

$$\omega_v^2 \simeq \frac{k^2}{a^2}, \quad for \ k \gg aH. \quad (31)$$

The result is quite similar to the scalar perturbations.

**Tensor perturbations** Tensor perturbations on the metric can be defined as

$$\delta g_{ij} = a(t)\gamma_{ij}, \quad (32)$$

where the transverse condition and traceless condition are satisfied,

$$\partial_i \gamma^{ij} = \gamma^i_i = 0. \quad (33)$$

The quadratic action of the tensor perturbations reads

$$I_{tensor} = \frac{M_p^2}{4} \int dtd^3k N a^3 \left( \frac{\dot{\gamma}_{ij}}{N^2} - \mathcal{M}_T \gamma_{ij}^2 \right), \quad (34)$$

where

$$\mathcal{K}_T = \frac{1}{r_2}, \quad (35)$$

$$\mathcal{M}_T^2 = \frac{k^2}{a^2} \left( 1 + \frac{3r_1}{a^2} \right) + M_{GW}^2 \cdot (36)$$

Different from GR, the dispersion relation of tensor mode is modified by an effective mass term $M_{GW}^2$. To see the
tensor mode propagating speed, we define a canonical variable to canonical normalize the action,

\[ \tilde{\gamma}_{ij} \equiv \sqrt{\frac{2}{r_2}} \cdot \gamma_{ij} . \]  

(37)

The quadratic action for tensor mode can be rewritten in terms of the canonical variables as,

\[ I_{\text{tensor}} = \frac{M^2}{8} \int dtd^3kN a^3 \left[ \tilde{\gamma}_{ij}^2 \frac{\tilde{\gamma}_{ij}}{N^2} - \left( \frac{c_s^2 k^2}{a^2} + \tilde{M}_G^2 \right) \tilde{\gamma}_{ij}^2 \right] , \]

where at late time epoch, at leading order

\[ c_s^2 \equiv \frac{M_p^2 + \frac{3m^2}{a^2}}{M_p^2 + \frac{m^2}{a^2}} \approx 1 , \]  

(39)

\[ \tilde{M}_G^2 \approx \frac{m^2 (4c_1 + 3c_0 r_1)}{a^2} . \]  

(40)

As pointed out in [20], the primary modification due to the mass term of tensor mode is a sharp peak in the gravitational spectrum.

**Decoupling limit** The physics in our SO(3) massive gravity becomes extremely simple by adopting the effective field theory approach in the decoupling limit,

\[ m_2 \to 0, \quad M_p \to \infty, \quad keeping \quad (M_p m_2) \quad fixed . \]  

(41)

In such decoupling limit, the corresponding action for helicity 0 mode reads schematically as follows,

\[ I_{\phi} = M_p^2 m_2^2 \int (k^2 \phi^2 - k^3 \phi^3 - k^4 \phi^4 - \ldots ) , \]  

(42)

where \( \phi \) is the helicity 0 mode of our Goldstone excitation. Notice that in order to go to canonical normalization for \( \phi \), we define

\[ \phi^c \equiv (M_p m_2 k) \phi , \]  

(43)

the canonical normalized action reads as

\[ I_{\phi} = \int \phi^c \frac{\dot{\phi}^2}{M_p m_2} - \frac{k^3 \phi^c}{M_p m_2} - \frac{k^4 \phi^c}{M_p^2 m_2^2} - \ldots . \]  

(44)

By comparing the quadratic term, cubic term and quartic term, we can see that at the energy scale higher than \( \Lambda_2 = \sqrt{M_p m_2} \), higher order terms become large and helicity 0 mode gets strongly coupled, thus effective field theory approach breaks down here.

**Conclusion and Discussion** In this short letter, we propose a massive gravity theory based on SO(3) symmetry and time reparameterization invariance. The time reparameterization invariance ensure that our SO(3) massive gravity is free from the BD ghost. The cost to this virtue is to introduce Lorentz violation. Fortunately, cosmological perturbations analysis tells us the Lorentz violation effect does not show up at the leading order of our calculation, and thus this exotic Lorentz violation effect can be totally negligible at sub horizon scale. On the other hand, by carefully checking the linear perturbations of scalar mode, vector mode and tensor mode, we found that our SO(3) massive gravity is free from ghost instability, gradient instability, and tachyonic instability.

By setting the cosmological constant \( \phi_0 \) to be zero, and taking the late time limit \( a \to \infty \), one gets an asymptotical Minkowskian background. Similar to the analysis of linear perturbations under the subhorizon approximation, one can easily check that such 5 d.o.f are also healthy in this asymptotical Minkowskian background.

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