Quantumness Beyond Entanglement: The Case of Symmetric States

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It is nowadays accepted that truly quantum correlations can exist even in the absence of entanglement. For the case of symmetric states, a physically trivial unitary transformation can alter a quantum state from entangled to separable and vice versa. We propose to certify the presence of quantumness via an average over all physically relevant modal decompositions. We investigate extremal states for such a measure: SU(2)-coherent states that nevertheless exhibit traits without counterpart in the classical world [22]. Third, significantly entangled states are not unique. Actually, a mode transformation can alter a quantum state from being entangled to being separable and vice versa [25][26]. One might rightly argue that the physics of entanglement in this case should not change just by altering the basis [4], as changing the basis is here akin to producing entanglement by tilting one’s head: a waveplate can enact this transformation. In other words, there is more to quantumness than entanglement: entanglement relies on a preferred decomposition of Hilbert space, whereas quantumness persists in all sensible decompositions. In consequence, a bona fide criterion of quantumness should assign no preference to any of these modal decompositions.

In this paper, we analyze this question for the relevant case of pure two-mode symmetric states, which are permutationally invariant. They can be written as a superposition

$$|\psi\rangle = \sum_{n=0}^{2S} \psi_n \ |n\rangle_a \ |2S-n\rangle_b ,$$

which shows that they contain exactly 2S excitations. In this decomposition over modes a and b, which may for example correspond to horizontally and vertically polarized states of light, the state is entangled if and only if more than a single coefficient $\psi_n$ is nonzero. The existence of a symmetry simplifies the mathematical description and makes the states experimentally interesting, largely because symmetrically manipulating the system generally requires fewer resources than addressing individual constituents. In particular, symmetric states are relevant to many experimental situations, such as spin squeezing [27]. These states are also numerically tractable, in that the size of their Hilbert spaces grows only linearly with S, as opposed to exponentially. For these reasons, there have been numerous attempts to characterize the entanglement properties of symmetric (i.e., bosonlike) states [28–37].

A considerable amount of work has since been done using a multipartite description of symmetric states. In this scenario, the Hilbert space of the systems is considered as a tensor product of 2S single-qubit Hilbert spaces. In such a partitioning, changing the modal decomposition as above amounts to a series of local operations and thus does not affect the overall entanglement properties [38–40]. This has consequently led to entanglement measures defined from the perspective of a multipartite entanglement [41–43].

Still, it seems natural to address the entanglement properties of states such as (1) from a bipartite perspective to properly describe the quantumness found in, e.g., arbitrary spin-S systems. In addition, we should have a mode-independent quantification of the total quantumness present in such a system. In this Letter, we tackle this problem by averaging over all modal decompositions to provide a covariant notion of quantumness.
Our measure is given by a sum of multipole moments of a state, which allows us to connect quantumness to its geometrical properties. Exploiting the Majorana representation [44, 45], the problem appears to be closely related to distributing points over the surface of the Bloch (or Poincaré) sphere. We recall that the question of distributing points uniformly over a sphere has not only inspired mathematical research [46, 47], but it has been attracting the attention of physicists working in a variety of fields [48–60]. We find that the most quantum states have these points maximally spread, whereas the most classical states are the SU(2)-coherent states, thus giving no privilege to a particular basis, such as horizontal and vertical or diagonal and antidiagonal, for analyzing the entanglement.

SU(2)-covariant measure of bipartite entanglement. — To assess the amount of entanglement present in a pure state (1) we shall use the linear entropy of the reduced density matrices \( \varrho_{i} : E(\varrho_{i}) = 1 - \text{Tr}(\varrho_{i}^{2}) \) (\( i \in \{a, b\} \)), where \( \varrho_{a} = \text{Tr}_{b}(\varrho) \) (analogously for \( \varrho_{b} \)) and \( \varrho = |\psi\rangle \langle \psi| \) is the density matrix of the total system. In terms of the Schmidt coefficients \( \psi_{n} \), the linear entropy is given by [45]

\[
E(|\psi\rangle) = 1 - \sum_{n=0}^{2S} |\psi_{n}|^{2},
\]

where \( E = 0 \) implies a separable state and \( E = \frac{2S}{2S+1} \) a fully entangled one, where the former have a single non-zero Schmidt coefficient and the latter, like Bell states, have \( 2S + 1 \) Schmidt coefficients with equal magnitude [61]. Since the linear entropy is an entanglement monotone for bipartite pure states, it fully characterizes the entanglement present in this partition.

Changing the partition changes the Schmidt coefficients of a state, thereby changing the linear entropy \( E \).

Transforming the modes is represented by a unitary transformation \( R \in \text{SU}(2) \). To make an SU(2)-covariant measure, we average \( E \) over all of the relevant partitions of Hilbert space.

Using the normalized Haar measure [62] \( dR \) for SU(2), our averaged entanglement measure reads

\[
\bar{E}(|\psi\rangle) = \int dR E(R |\psi\rangle).
\]

In the language of polarization, this is equivalent to averaging the entanglement found after passing through a random wave plate, thus giving no privilege to a particular basis, such as horizontal and vertical or diagonal and antidiagonal, for analyzing the entanglement.

The action of \( R \) on the coefficients \( \psi_{n} \) is not straightforward, so we instead evaluate this quantity resorting to a parametrization of symmetric quantum states that is better suited to describing SU(2) transformations. To this end, we start by expressing the density matrix \( \varrho \) as

\[
\varrho = \sum_{K=0}^{2S} \sum_{q=-K}^{K} \varrho_{K,q} T_{K,q},
\]

where the irreducible tensors (also called polarization operators) associated with spin-\( S \) are given by [63, 64]

\[
T_{K,q} = \sqrt{\frac{2K+1}{2S+1}} \sum_{m,m'=-S}^{S} C_{S_{m},K,q}^{S_{m}',K,q'} |S_{m'}\rangle \langle S_{m}|,
\]

with \( C_{S_{m_{1}},S_{m_{2}}}^{S_{m}} \), denoting the Clebsch-Gordan coefficients [65] that couple a spin \( S_{1} \) and a spin \( S_{2} \) to a total spin \( S \) and vanish unless the usual angular momentum coupling rules are satisfied: \( 0 \leq K \leq 2S \) and \(-K \leq q \leq K\). These are \((2S+1)^{2}\) operators that constitute a basis of the space of linear operators acting on the Hilbert space. The expansion coefficients \( \varrho_{K,q} = \text{Tr}(\varrho T_{K,q}^\dagger) \) are called the state multipoles and contain the complete information about the state sorted in the appropriate way: they are the \( K \)th-order moments of the generators. Normalization dictates that \( \varrho_{00} = 1/\sqrt{2S+1} \), and Hermiticity implies \( \varrho_{K,q} = (-1)^{q} \varrho_{-K,-q} \).

Due to their very same definition, the multipoles inherit the proper transformation under SU(2); that is, if the state experiences the unitary transformation \( \varrho = R \varrho R^{\dagger} \), the multipoles transform as

\[
\bar{\varrho}_{K,q} = \sum_{q'=-K}^{K} D_{K,q}^{K',q'}(R) \varrho_{K,q'},
\]

where \( D_{K,q}^{K',q'}(R) \) are the Wigner D-matrices [65].

The linear entropy of the transformed state can be computed via the reduced density matrix

\[
\bar{E}(R |\psi\rangle) = 1 - \sum_{K,K'} \bar{\varrho}_{K,0} \bar{\varrho}_{K',0} \delta_{KK'}. \tag{8}
\]

We can then average over the rotations using properties of the D-matrices. To this end, we note that

\[
\int dR \bar{\varrho}_{K,0} \bar{\varrho}_{K',0} = \sum_{q,q'} \bar{\varrho}_{K,q} \bar{\varrho}_{K',q'} \int dR D_{q}^{K,0} D_{q'}^{K',0} = \frac{\delta_{KK'}}{2K+1} \sum_{q=-K}^{K} |\varrho_{K,q}|^{2}. \tag{9}
\]

The averaged entanglement thus becomes

\[
\bar{E}(|\psi\rangle) = 1 - \sum_{K=0}^{2S} \frac{1}{2K+1} \sum_{q=-K}^{K} |\varrho_{K,q}|^{2}, \tag{10}
\]

providing a simple metric for analyzing the quantumness. As \( \bar{E}(|\psi\rangle) \) involves all the multipoles, it provides a complete characterization of the state. For the case of pure states we are
dealing with, we expand in the angular-momentum basis as
\[ |\psi\rangle = \sum_m \psi_m |S, m\rangle, \]
so (10) takes the form
\[ \bar{E}(|\psi\rangle) = 1 - \frac{1}{2S+1} \sum_{K=0}^{2S} \sum_{q=-K}^{K} \sum_{m,m'=-S}^{S} \mathcal{C}^{S}_{Kq} \psi_{m} \psi_{m'}^{*}, \tag{11} \]
which is the quantumness measure we advocate.

Extremal states.— The averaged linear entropy (11) can be regarded as a nonlinear functional of the density matrix. The higher the value of \( \bar{E} \), the greater the value of the average entanglement. Some pure states give the maximal value of \( \bar{E} \) for a given partition but no pure state achieves \( \overline{E} = \frac{2S}{2S+1} \) for all partitions. Maximally mixed states, in contrast, give the maximum value of \( \bar{E} \), but linear entropy is only an entanglement measure and not a quantumness measure. To this end, we will use the concept of Majorana representation [44, 45], which maps every \((2S+1)\)-dimensional pure state \( |\psi\rangle \) into the polynomial
\[ \psi(\theta, \phi) = \langle \theta, \phi|\psi \rangle \propto \sum_{m=-S}^{S} \sqrt{\frac{(2S)!}{(S-m)!(S+m)!}} \psi_{m} \alpha^{S+m}. \tag{13} \]
Up to a global unphysical factor, \( |\psi\rangle \) is determined by the set \{\( \alpha_i \)\} of the \( 2S \) complex zeros of \( \psi(\theta, \phi) \), suitably completed by points at infinity if the degree of \( \psi(\theta, \phi) \) is less than \( 2S \). A nice geometrical representation of \( |\psi\rangle \) by \( 2S \) points on the unit sphere (often called the constellation) is obtained by an inverse stereographic map of \( \{\alpha_i\} \). Two states with the same constellation are the same, up to a global phase. For example, the SU(2)-coherent states have all \( 2S \) of the “stars” in their constellation co-located at angular coordinates \( (\theta, \phi) \). Several decades after its conception, this stellar representation has recently attracted a great deal of attention in several fields [48–60].

Intimately related to the Majorana polynomial \( \psi(\theta, \phi) \) is the SU(2) \( Q \)-function, defined as \( Q(\theta, \phi) = |\psi(\theta, \phi)|^2 \). Obviously, the stars \( \{\alpha_i\} \) are also the zeros of \( Q(\theta, \phi) \), so the \( Q \)-function is an attractive way of depicting the state to help appreciate the symmetries of \( |\psi\rangle \). It is not surprising that it has gained popularity in modern quantum information [20].

The \( Q \)-functions and the corresponding Majorana constellations for a few examples of extremal states are shown in Fig. 1, with many more given in Ref. [69]. The resulting constellations have the points symmetrically placed on the unit sphere, which agrees with other previous notions of quantumness, such as states of maximal Wehrl-Lieb entropy [70].

In particular dimensions, the constellations show a remarkable additional degree of symmetry, some of which are sum-
TABLE I. Symmetries of the constellations associated to the maximal states for the values of $S$.

| $S$ | Group | Order | Constellation          |
|-----|-------|-------|------------------------|
| 1   | $C_2$ | 2     | —                      |
| 3   | $S_3$ | 6     | triangle               |
| 5   | $S_4$ | 24    | Platonic               |
| 7   | $D_{12}$ | 12   | triangle + poles       |
| 5   | $C_2 \times S_4$ | 48   | Platonic               |
| 7   | $D_{20}$ | 20   | pentagon + poles       |
| 4   | $D_{16}$ | 16   | twisted cube           |
| 5   | $D_{16}$ | 16   | twisted cube + poles   |
| 6   | $C_2 \times A_3$ | 120  | Platonic               |
| 7   | $D_{24}$ | 24   | twisted hexagon + poles|
| 8   | $A_4$  | 12    | —                      |
| 12  | $S_4$  | 24    | —                      |

In Fig. 2 we plot the value of the averaged entropy $\bar{E}$ for the maximal states found numerically as a function of the dimension $S$. For comparison, we have also included the corresponding values for the minimal states, which correspond to coherent states. As we can appreciate, $\bar{E}$ approaches the limit value of unity as $S$ grows. One can easily guess that $\bar{E} \sim 1 - 1/(2S)$, which shows that the higher the value of $S$, the more quantum the extremal state is.

Concluding remarks.— We have comprehensively examined the notion of average entanglement for symmetric states, which is the physically relevant quantity for these states. We have proven that SU(2) coherent states are minimal. Their opposite counterparts, maximizing the average entanglement, have interesting properties. Apart from their indisputable geometrical beauty, there surely is plenty of room for the application of these states.

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FIG. 2. Average entanglement $\bar{E}$ for the states of maximal (blue bars) and of minimal average entanglement (yellow bars) as a function of $S$. The continuous red line represents the upper limit $\bar{E} = \frac{2S}{2S+1}$ attainable in (2).
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