An Analog Analogue of a Digital Quantum Computation*

Edward Farhi†
Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge, MA 02139

Sam Gutmann‡
Department of Mathematics
Northeastern University
Boston, MA 02115

MIT-CTP-2593, quant-ph/9612026 December 1996

Abstract

We solve a problem, which while not fitting into the usual paradigm, can be viewed as a quantum computation. Suppose we are given a quantum system described by an $N$ dimensional Hilbert space with a Hamiltonian of the form $E |w\rangle \langle w|$ where $|w\rangle$ is an unknown (normalized) state. We show how to discover $|w\rangle$ by adding a Hamiltonian (independent of $|w\rangle$) and evolving for a time proportional to $N^{1/2}/E$. We show that this time is optimally short. This process is an analog analogue to Grover’s algorithm, a computation on a conventional (!) quantum computer which locates a marked item from an unsorted list of $N$ items in a number of steps proportional to $N^{1/2}$.

---

*This work was supported in part by The Department of Energy under cooperative agreement DE-FC02-94ER40818
†farhi@mitlns.mit.edu
‡sgutm@muhub.neu.edu
Although a quantum computer, beyond certain elementary gates, has not yet been constructed, a paradigm [1] for quantum computation is in place. A quantum computer is envisaged as acting on a collection of spin 1/2 particles sitting at specified sites. Each elementary operation is a unitary transformation which acts on the spins at one or two sites. A quantum computer program, or algorithm, is a definite sequence of such unitary transformations. For a given initial spin state, the output of the program is the spin state after the sequence of transformations has acted. The length of the algorithm is equal to the number of elementary unitary transformations which make up the algorithm.

This framework for quantum computation is general enough that any ordinary digital computer program can be turned into a quantum computer algorithm. (It is required that the ordinary program be reversible; however any ordinary computer program can be written in reversible code.) Quantum computers can go beyond ordinary computers when they act on superpositions of states and take advantage of interference effects. An example of a quantum algorithm which outperforms any classical algorithm designed to solve the same problem is the Grover algorithm [2]. Here we are given a function \( f(a) \) defined on the integers \( a \) from 1 to \( N \). The function has the property that it takes the value 1 on just a single element of its domain, \( w \), and it has the value 0 for all \( a \neq w \). With only the ability to call the function \( f \), the task is to find \( w \). On a classical computer this requires, on average, \( N/2 \) calls of the function \( f \). However Grover showed that with a quantum computer \( w \) can be found with of order \( N^{1/2} \) function calls. This remarkable speed-up illustrates the power of quantum computation. (In the appendix we explain how the Grover algorithm works.)

In this paper we consider quantum computation differently, as controlled Hamiltonian time evolution of a system, obeying the Schrödinger equation

\[
i \frac{d}{dt} \left| \psi \right> = H(t)\left| \psi \right>, \tag{1}
\]

which is designed to solve a specified problem. We illustrate this with an example. Suppose we are given a Hamiltonian in an \( N \) dimensional vector space and we are told that the Hamiltonian has one eigenvalue \( E \neq 0 \) and all the others are 0. The task is to find the eigenvector \( \left| w \right> \) which has eigenvalue \( E \). We now give a solution to this problem and then explain in what sense it is optimal.

We are given

\[
H_w = E\left| w \right><w \right|, \tag{2}
\]

with \( \left| w \right> \) unspecified and \( \left< w \right|w \right> = 1 \). Pick some normalized vector \( \left| s \right> \) which of course does not depend on \( \left| w \right> \) since we don’t yet know what \( \left| w \right> \) is. Now add to \( H_w \) the “driving” Hamiltonian

\[
H_D = E\left| s \right><s \right|, \tag{3}
\]

so that the full Hamiltonian is

\[
H = H_w + H_D. \tag{4}
\]

We now calculate the time evolution of the state \( \left| \psi_w, t \right> \) which at \( t = 0 \) is \( \left| s \right> \),

\[
\left| \psi_w, t \right> = e^{-iHt} \left| s \right>. \tag{5}
\]
It suffices to confine our attention to the two dimensional subspace spanned by $|s\rangle$ and $|w\rangle$. The vectors $|s\rangle$ and $|w\rangle$ are (generally) not orthogonal and we call their inner product $x$,

$$\langle s|w \rangle = x$$  \hspace{1cm} (6)

where $x$ can be taken to be real and positive since any phase in $\langle s|w \rangle$ can ultimately be absorbed in $|s\rangle$. We will discuss the expected size of $x$ shortly. Now the vectors

$$|r\rangle = \frac{1}{\sqrt{1-x^2}} (|s\rangle - x|w\rangle)$$  \hspace{1cm} (7)

and $|w\rangle$ are orthonormal. In the $|w\rangle$, $|r\rangle$ basis the Hamiltonian (4) is

$$H = E \left[ \frac{1+x^2}{x\sqrt{1-x^2}} \frac{x\sqrt{1-x^2}}{1-x^2} \right]$$  \hspace{1cm} (8)

and

$$|s\rangle = \left[ \frac{x}{\sqrt{1-x^2}} \right]$$  \hspace{1cm} (9)

Now a simple calculation gives

$$|\psi_{w,t}\rangle = e^{-iEt} \left[ \frac{x \cos(Ext) - i \sin(Ext)}{\sqrt{1-x^2}\cos(Ext)} \right].$$  \hspace{1cm} (10)

Thus we see that at time $t$ the probability of finding the state $|w\rangle$ is

$$P(t) = \sin^2(Ext) + x^2 \cos^2(Ext)$$  \hspace{1cm} (11)

and that at a time $t_m$ given by

$$t_m = \frac{\pi}{2Ex}$$  \hspace{1cm} (12)

the probability is one.

How big do we expect $x$ to be? In an $N$ dimensional complex vector space, if you pick two normalized vectors at random (uniformly on the 2N-1 dimensional unit sphere), then the expected value of the inner product squared is $1/N$ so we know that the expected value of $x$ is of order $N^{-1/2}$. Thus starting with $|s\rangle$, for the probability of finding $|w\rangle$ to be appreciable we must wait a time of order $N^{1/2}/E$. This is the analog analogue of the Grover algorithm result.

Note that the eigenvalues of the Hamiltonian (8) are $E(1 \pm x)$. Thus the difference in eigenvalues is $(2xE)$ which is of order $E/N^{1/2}$. By the time-energy uncertainty principle, the time required to evolve substantially, that is from $|s\rangle$ to $|w\rangle$, must be of order $N^{1/2}/E$ which is the time we found. You might think that by increasing the energy difference, that is for example, by using $H_D = E'|s\rangle\langle s|$ with $E' \gg E$ you could speed up the procedure for finding $|w\rangle$. However the next result shows that this is not the case.

We now show that our procedure for finding $|w\rangle$, in a time which grows like $N^{1/2}/E$, is optimally short. The proof we give here is the analog analogue of the oracle proof [3] which
can be used to show that the Grover algorithm is optimal for the problem it sets out to solve. Again we are given the Hamiltonian $H_w = E|w\rangle\langle w|$ and we wish to add some Hamiltonian $H_D(t)$ to it which drives the system to a state which allows us to determine $|w\rangle$. In an $N$ dimensional vector space, there are $N$ linearly independent choices for $|w\rangle$. We can pick these to be a basis for the vector space and we then have

$$\sum_w H_w = E \sum_w |w\rangle\langle w| = E.$$  \hspace{1cm} (13)

The idea of the proof is this: Start with some initial $|w\rangle$-independent state $|i\rangle$ and evolve it with the Hamiltonian

$$H = H_w + H_D(t).$$  \hspace{1cm} (14)

After a time $t$ the state we get must be substantially different from what we would have gotten using $H_w + H_D(t)$ or else we can not tell $|w\rangle$ from $|w'\rangle$. Let

$$i \frac{d}{dt} |\psi_w, t\rangle = (H_w + H_D(t))|\psi_w, t\rangle$$  \hspace{1cm} (15)

with

$$|\psi_w, 0\rangle = |i\rangle.$$

In order for $|\psi_w, t\rangle$ to differ sufficiently from $|\psi_w, t\rangle$ it is certainly necessary that, for all (but one) $w$, $|\psi_w, t\rangle$ differs sufficiently from any $|w\rangle$-independent vector. (If some of the $|\psi_w, t\rangle$ were very close to a particular $|w\rangle$-independent vector, we could not tell them apart.) Let $|\psi, t\rangle$ evolve with $H_D(t)$, that is,

$$i \frac{d}{dt} |\psi, t\rangle = H_D(t)|\psi, t\rangle$$  \hspace{1cm} (16)

with

$$|\psi, 0\rangle = |i\rangle.$$

We will use $|\psi, t\rangle$ as a $|w\rangle$-independent vector which the $|\psi_w, t\rangle$ must differ from. We require $t$ to be large enough that $\left\| |\psi_w, t\rangle - |\psi, t\rangle \right\|^2 \geq \varepsilon$ for some fixed $\varepsilon$ which implies

$$\sum_w \left\| |\psi_w, t\rangle - |\psi, t\rangle \right\|^2 \geq N\varepsilon.$$  \hspace{1cm} (17)

Now consider

$$\frac{d}{dt} \left\| |\psi_w, t\rangle - |\psi, t\rangle \right\|^2 = -2 \text{Re} \frac{d}{dt} \langle \psi_w, t | \psi, t \rangle$$  \hspace{1cm} (18)

which upon using (13) and (16) gives

$$\frac{d}{dt} \left\| |\psi_w, t\rangle - |\psi, t\rangle \right\|^2 = 2 \text{Im} \langle \psi_w, t | H_w | \psi, t \rangle \leq 2 |\langle \psi_w, t | H_w | \psi, t \rangle| \leq 2 \| H_w | \psi, t \|.$$  \hspace{1cm} (19)
We now sum on \( w \) and use the fact that if \( \sum_{i=1}^{N} |a_i|^2 = 1 \) then \( \sum_{i=1}^{N} |a_i| \leq N^{1/2} \) along with (14) to obtain
\[
\frac{d}{dt} \sum_{w} \left\| \psi_{w}, t \right\| - \left\| \psi, t \right\| \leq 2EN^{1/2}.
\] (20)

Since \( |\psi_{w}, 0\rangle = |\psi, 0\rangle \) we have
\[
\sum_{w} \left\| \psi_{w}, t \right\| - \left\| \psi, t \right\|^2 \leq 2EN^{1/2}t.
\] (21)

Therefore in order to satisfy (17) we must have
\[
t \geq \frac{\varepsilon N^{1/2}}{2E}.
\] (22)

This shows that the \( H_D \) we have chosen allows us to determine \( |w\rangle \) as quickly as possible in terms of \( N \).

**Appendix: The Grover Algorithm**

We are given a function \( f(a) \) with \( a = 1, \ldots N \) such that \( f(w) = 1 \) and \( f(a) = 0 \) for \( a \neq w \). We assume that the function \( f(a) \) can be calculated using ordinary (reversible) computer code. The goal is to find \( w \). Classically this requires, on average, \( N/2 \) evaluations of the function \( f \).

Now consider the vector
\[
|s\rangle = \frac{1}{N^{1/2}} \sum_{a} |a\rangle.
\] (A3)

It is also possible to write quantum computer code which implements the unitary operator
\[
U_s = 2|s\rangle\langle s| - 1.
\] (A4)

The number of two bit operations required to implement \( U_s \) grows more slowly than \( N \) to any positive power.
The Grover algorithm consists of letting the operator $U_s U_f$ act $k$ times on the vector $|s\rangle$. To see what happens we can restrict our attention to the two dimensional subspace spanned by $|s\rangle$ and $|w\rangle$. Let

$$|r\rangle = \frac{1}{\sqrt{N-1}} \sum_{a \neq w} |a\rangle$$  \hfill (A5)$$

so that $|w\rangle$ and $|r\rangle$ form an orthonormal basis for the relevant subspace. In the $|w\rangle$, $|r\rangle$ basis the operator $U_s U_f$ takes the form

$$U_s U_f = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$  \hfill (A6)$$

where $\cos \theta = 1 - 2/N$. This implies that

$$(U_s U_f)^k = \begin{bmatrix} \cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{bmatrix}.$$  \hfill (A7)$$

Now for $N$ large $\theta \sim 2N^{-1/2}$ so each application of $U_s U_f$ is a rotation by an angle $\sim 2N^{-1/2}$. In the $|w\rangle$, $|r\rangle$ basis, the initial state $|s\rangle$ is

$$|s\rangle = \begin{bmatrix} \sqrt{N-1} / 2 \\ (1 - 1/N)^{1/2} \end{bmatrix}$$  \hfill (A8)$$

which is very close to $|r\rangle$. However after $k$ steps where $k\theta = \pi/2$ the algorithm has rotated the initial state to lie (almost) along $|w\rangle$. This requires $k \sim \pi N^{1/2}/4$ steps. Each step actually requires two evaluations of $f$ so the number of evaluations of $f$ required to find $w$ grows like $N^{1/2}$. Accordingly the number of two bit operations required to implement the algorithm also grows like $N^{1/2}$.

Acknowledgement

E. F. would like to thank the theory group at Università di Roma 1 for their hospitality and discussions as well as the INFN for partial support.

References

[1] For a summary and references see A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, Phys. Rev. A 52, 3457 (1995).

[2] L. K. Grover, A fast quantum mechanical algorithm for database search, quant-ph/9605043.

[3] C. H. Bennett, E. Bernstein, G. Brassard, and U. V. Vazirani, Strengths and weaknesses of quantum computing, SIAM Journal on Computing, to appear.

[4] M. Boyer, G. Brassard, P. Hoeyer, and A. Tapp, Tight bounds on quantum searching, quant-ph/9605034.