QBF Solving by Counterexample-guided Expansion

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Abstract. We introduce a novel generalization of Counterexample-Guided Inductive Synthesis (CEGIS) and instantiate it to yield a novel, competitive algorithm for solving Quantified Boolean Formulas (QBF). Current QBF solvers based on counterexample-guided expansion use a recursive approach which scales poorly with the number of quantifier alternations. Our generalization of CEGIS removes the need for this recursive approach, and we instantiate it to yield a simple and efficient algorithm for QBF solving.
Lastly, this research is supported by a competitive, though straightforward, implementation of the algorithm, making it possible to study the practical impact of our algorithm design decisions.

1 Introduction

Quantified Boolean Formulas (QBFs) are propositional logic formulas with existential and universal quantification. We consider QBF written in the prenex normal form \( \exists x_0. \forall y_0. \ldots. \forall y_n. \exists x_{n+1}. Q(x_0, \ldots, x_{n+1}, y_0, \ldots, y_n) \), with \( x_i \) and \( y_i \) representing disjoint sets of Boolean variables, and \( Q \) a propositional formula; any QBF formula can be easily and efficiently converted to prenex normal form.

Just as deciding satisfiability of a propositional-logic formula is the prototypical NP-complete problem, QBF satisfiability is the prototypical PSPACE-complete problem. Improvements in QBF solving will thus have a strong impact in many application domains with problems that can be translated to QBF, including planning [30], model checking [21], circuit design [25] and synthesis [7,32], in particular synthesis of distributed systems [15,16]. Research into engineering QBF solvers has seen considerable progress since initial work in the mid-nineties [10,22], with most solvers either implementing a quantified version of CDCL (e.g., [11,24,33]) or an expansion-based approach (e.g., [3]).

Since the first complete solver in 1998, based on DPLL [10], an incredible amount of work has gone into QBF: an analogue of propositional CDCL was discovered in 2002 [17,23], the very same year as expansion-based solving [2] was discovered. Expansion-based solving has known tremendous success, and is the basis for counterexample-guided methods such as RAReQS [20], a state-of-the-art expansion-based solver. In 2016, a completely new approach to QBF solving, based on Skolem

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function computation, was introduced \cite{28}. Still, expansion-based methods remain prevalent.

In this paper, we follow an expansion-based approach to solving QBFs that is inspired by counterexample-guided inductive synthesis (CEGIS) \cite{1, 32}. CEGIS attempts to learn an input–output relation using a learning algorithm interacting with a verification oracle. It keeps a database of examples of input-output pairs. The learning algorithm proposes a candidate relation that is consistent with the examples in the database and sends it to the oracle (or gives up if no such relation can be found). The oracle adds counterexample to the candidate to the database (or declares that the proposed relation is correct).

We describe a novel relaxation of CEGIS that relies on an approximate verifier, i.e. a verifier which may produce spurious counterexamples: this makes the verifier simpler and faster. In exchange, the learner must provide values which prove that the candidate solution fulfils all the counterexamples produced so far.

The advantage of our approach is two-fold. First, it is extremely simple: the description of the algorithm is extremely small, and so is its proof of correctness. Second, it is efficient: our prototype implementation does not rely on any formula optimisation or implementation technique not described in this paper, yet it is competitive with state-of-the-art QBF solvers.

Our approach is similar to the one implemented in the very successful solver RAReQS \cite{20}, which instantiates the outer existential quantifiers using a counterexample-guided approach. When solving a formula of the form \( \exists x_0 \forall y_0 \ldots \forall y_n \exists x_{n+1}. Q \), RAReQS obtains the counterexample using a recursive call for \( \exists y_0 \forall x_1 \ldots \forall y_n \exists x_{n+1}. \neg Q \).

To show that our approach is not simply a reformulation of RAReQS, we introduce in Section 4.3 a class of simple formulas on which RAReQS needs an exponential number of SAT queries while our approach needs only a small, constant number.

We have implemented our approach in a tool called alejtehad. We show that alejtehad is competitive with RAReQS, one of the best current QBF solvers, and that it is significantly outperforms RAReQS in some classes of benchmarks, especially those with high quantifier alternation depth.

## 2 Counterexample-Guided Inductive Synthesis (CEGIS)

### 2.1 Concepts

Counterexample-guided inductive synthesis (CEGIS) is a generic framework, initially devised in the context of syntax-guided synthesis \cite{1}, involving iterations between two components:

- The verifier provides, given a candidate solution, a counterexample that disproves it, or (correctly) proves the solution to be valid.
- The learner generates candidate solutions that are consistent with all counterexamples provided so far, or provides a (correct) unrealisability answer.

Although the worst-case complexity guarantees usually devolve to the size of the set of counterexamples or of candidate solution, this approach has been wildly successful in practice \cite{20, 20, 32}. 

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In a QBF setting, the standard approach is to consider that the solutions to be learned are values for the first quantifier; they must make the formula true, if the quantifier is \(\exists\), or falsify it in the case of the \(\forall\) quantifier.

However, one issue surfaces immediately when dealing with problems involving multiple quantifier alternations: the problem solved by the verifier is only marginally easier than the original one in practice.

For example, \(\exists x. \forall y. \exists z. Q(x, y, z)\) is a type of QBF problem that occurs in synthesis of resilient systems, program diagnosis and repair, finite-state games, ... There, the verifier’s problem is almost as hard as the original problem:

\[ \exists y. \forall z. \neg Q(x, y, z) \]

One of the earliest applications of CEGIS\(^1\) to QBF solving, RAReQS \([20]\), introduced a recursive approach where the whole solver is used as a verifier, each nested solver dealing with one block of quantifiers.

There has been many refinements of this technique since RAReQS, from RAReQS’ recent reimplementation from scratch for 2-QBF \([3]\) to the clause abstraction performed by CAQE \([29]\). However, all of those share RAReQS’ recursive approach to deal with quantifier alternation. This is an issue for efficiency, for multiple reasons:

- Performance degrades quickly as the quantifier depth increases, even on trivial instances, as shown in Section 4.3 and Section 5.
- Incremental solving, where SAT problems are constructed lazily by adding new conjuncts and solved without starting from scratch, becomes exceedingly difficult when sub-formulas change based on decisions made at arbitrary depths in the nested solvers.
- The solver does not share information deduced at different quantifier depths, resulting in duplicate work.
- The solver is forced, at any given point, to make decisions on variables belonging to a specific quantifier depth; we exploit this in Theorem\(^3\) to highlight algorithmic differences between our approach and RAReQS.

Alternatively, the candidate solutions that are being discovered could be Skolem functions describing the values taken by the existentially-quantified variables. In that case, the natural form for counterexamples are traces: assignment of values to the universally-quantified variables that falsify the formula (under a given choice of Skolem functions).

However, this introduces a major difficulty: synthesis of Boolean (Skolem) functions is a hard problem on its own, that in general cannot be efficiently reduced to SAT (or a similarly-efficient decision procedure). Moreover, this would replace a situation where the learning is easy and the verification hard, with a problem where the learning is hard and the verification easy.

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\(^1\) Many of those are described in terms of Counterexample-Guided Abstraction Refinement (CEGAR), despite CEGIS being a closer fit to the algorithms.
2.2 Contribution

We take a slightly different, but related, approach: a candidate is a set $E$ of witnesses, i.e. assignments to the existentially quantified variables; the learner’s claim is that for any counterexample (i.e. any assignment of the universally quantified variables), the formula can be made true by instantiating the existentially-quantified variables to some element of $E$. Conversely, the database of counterexamples is a set $A$ of assignments to all universally-quantified variables.

This means that the algorithm is symmetric between the universal and the existential variables: in order to find a new set of counterexamples (resp. candidate solution), we expand the formula using the candidate (resp. set of counterexamples) and call a SAT solver to find an assignment to the universal (resp. existential) variables. If the formula can be decided using a small set of candidates or a small set of counterexamples, this approach is fast.

This can be thought of as a relaxation of CEGIS in the sense that we allow an approximate verifier, i.e. a verifier which might produce spurious counterexamples. To appropriately deal with this, we must require that the learner not only produces a solution (an assignment to the outermost quantifier), but also auxiliary values (the witnesses) that prove that the proposed solution is consistent with the known counterexamples.

More explicitly, in the previous example ($\exists x. \forall y. \exists z. Q(x, y, z)$) the learner’s problem becomes, at the $n$th iteration

$$\exists x, z_1 \ldots z_n. \land_{i=1}^{n} Q(x, y_i, z_i)$$

while the verifier’s problem becomes

$$\exists y_{n+1}. \land_{i=1}^{n} \neg Q(x, y_{n+1}, z_i)$$

While this enables us to iterate more rapidly between two simpler components, this comes at a cost: the solver may iterate over a correct value for $x$ several times before providing enough witnesses for the verifier to determine that the solution is correct.

We show in Section 5 that, in practice, we outperform “traditional” CEGIS approaches on certain classes of problems, while being slower on others. This shows that this is a very promising approach: our algorithm is extremely simple and so are the associated proofs; yet the prototype implementation, written in less than 2 months, can compete with state-of-the-art solvers and outperforms them in some classes of problems, without using any optimisation not described in this paper.

3 A novel algorithm for QBF solving

We give a particular instantiation of this approach for QBF solving. We indifferently call our algorithm and its prototype implementation alejtehad, after the eponymous concept in Islamic philosophy: it refers both to the argumentation
process through which case law is produced, and to the state of knowing that one
knows nothing.

This hints at the minimalistic nature of the algorithm: *alejtehad* is simple, in
the sense that it is concise to describe and has few prerequisites, both in the the-
ory and in its implementation. It relies on an iterative process, in which argu-
ments and counterarguments are produced until a solution is found.

Beyond the novelty and simplicity of its solving algorithm, *alejtehad* is un-
usual in that it efficient as described, without relying on complex formula optimi-
isations or implementation strategies not described in Section 3.3.

3.1 Notation

In the entirety of this paper, $\Phi$ will denote a QBF formula and $Q$ the matrix *(i.e.
the propositional body)* of a QBF formula. We consider, without loss of gener-
ality, prenex formulas where each quantifier ranges over a single bitvector:

$$
\Phi := \exists x_0. \forall y_0. \ldots \forall y_n. \exists x_{n+1}. Q(x_0, \ldots, x_{n+1}, y_0, \ldots, y_n)
$$

We denote a single bitvector by $z_i$, a tuple of bitvectors by $Z$, and their as-
signed values (in a given model) by $z_i$ and $Z$. Under this notation, we can now
precisely define witnesses and counterexamples:

- Witnesses are tuples $X := (x_0, \ldots, x_{n+1})$ of concrete values for the existentially-
quantified variables. $X$ is said to be a witness for a counterexample $Y$ if
$Q(X, Y)$ holds.
- Counterexamples are tuples $Y := (y_0, \ldots, y_n)$ of concrete values for the
universally-quantified variables. $Y$ is a counterexample of a witness $X$ if
$\neg Q(X, Y)$ holds.
- Witness variables are tuples $X^Y := (x_0, x_1^y_0, x_2^y_0, \ldots, x_{n+1}^{y_n} \ldots, y_n)$ associated
to a given counterexample $Y$. Given a model, $X^Y$ denotes its concrete value
(in a given model), *i.e.* a *witness*.
- Counterexample variables are likewise defined, denoted $Y^X$, and their con-
crete values, denoted $Y^X$, are counterexamples.

For example, for the QBF $\Phi := \forall x. \exists y. Q(x, y)$ with $Q(x, y) := x = y + 1$ (in-
terpreting $x$ and $y$ as $n$-bit integers), the counterexample $0$ has a corresponding
witness variable $x^0$. By solving $Q(x^0, 0)$ (*i.e.* $x^0 = 1$), a matching witness $x^0 = 1$
is discovered, and has a corresponding counterexample variable $y^1$. Solving the
SAT instance $\neg Q(x^0, y^x)$ then yields a new counterexample $y^1$.

If a prenex QBF formula as defined above is true, it has an associated *Skolem
model* [31]: functions $f_{x_{i+1}}(y_0, \ldots, y_i)$ such that, for any counterexample $Y$, $(f_{x_0}(), f_{x_1}(y_0), \ldots)$
is a matching witness. Conversely, false formulas have *Herbrand models* [18]: func-
tions $f_y(x_0, \ldots x_i)$ such that for any witness $X$, $(f_{y_0}(x_0), f_{y_1}(x_0, x_1), \ldots)$ is a match-
ing counterexample.

Thinking of the QBF formula as a 2-players game, a Skolem (resp. Herbrand)
model is a winning strategy for the existential (resp. universal) player. We use the

\[\text{This is a propositional SAT problem.}\]
notions of Skolem and Herbrand models later, in Section 4, while proving that 
alejtehad is sound.

3.2 The alejtehad algorithm

As described previously, our algorithm is composed of a learner and an approxi-
mate verifier. It maintains a database of counterexamples (denoted $A$) and a database 
of witnesses (denoted $E$).

At each iteration, the learner produces witnesses for all counterexamples ob-
served so far, at which time the verifier responds with counterexamples for all 
examples; both consist of a single query to Boolean SAT. The solver keeps iter-
ating until the learner or the verifier returns UNSAT, at which point the solver 
returns either a value for $x_0$, if there are no counterexamples, or false if there are 
no satisfying witnesses.

A more precise description is provided in Algorithm 1, while Section 3.3 ex-
poses the differences between this description and an efficient implementation.

\begin{algorithm}
 Result: false or a satisfying value $x_0$
 Initially, $E := \emptyset$ and $A := \{Y_0\}$ contains a single, arbitrary counterexample;
\begin{algorithmic}
\While {true}
  \Comment{Learner}
  \If {$\varphi := \bigwedge_{Y \in A} Q(X^Y, Y)$ is satisfiable}
    \State Add each witness $X^Y$ to $E$;
  \Else
    \State return false;
  \EndIf
  \Comment{Verifier}
  \If {$\psi := \bigwedge_{X \in E} \neg Q(X, Y^X)$ is satisfiable}
    \State Add each counterexample $Y^X$ to $A$;
  \Else
    \State return $x_0$;
  \EndIf
\EndWhile
\end{algorithmic}
\end{algorithm}

\begin{figure}[h]
\centering
\caption{ijtihad algorithm}
\end{figure}

3.3 Implementation concerns

Obviously, Algorithm 1 only gives a high-level description of the solving algorithm, 
which isn’t quite the description of a fast implementation. Here are notable as-
pcts of our implementation that makes it perform well:

- At each iteration, new propositional conjuncts are introduced in $\varphi$ and $\psi$: 
  we systematically use incremental SAT solving.
Existing clauses aren’t introduced in duplicate in \( \varphi \) or \( \psi \).
Likewise, when converting \( \neg Q \) to CNF, naively using Tseitin’s results in a large (though linear) blowup: instead, we maintain a cache that associates to each clause a Tseitin variable; when a clause is encountered was already processed, no new clauses are added to \( \psi \) and the existing Tseitin variable is reused.

While RAReQS, against which we compare our implementation in Section 5 uses similar techniques in its implementation, it also includes optimisations performed on the QBF formula, both before and during solving, that we do not implement: unit propagation, pure literals elimination, universal reduction and blocking clauses. Indeed, alejtehad does not consider explicitly QBF subproblems, so it is unclear whether existing QBF simplification techniques can be applied thorough the execution of the algorithm.

4 Formal arguments

As mentioned earlier, Algorithm 1 is very simple and so are its termination and soundness arguments. We provide no complexity guarantees beyond the immediate upper-bound that follows from the termination argument (Theorem 1): while counterexample-guided methods perform well in practice, they do so at the cost of worst-case bounds.

However, we exhibit certain classes of problems that alejtehad can deal within 3 SAT queries, whereas existing CEGIS-based methods, represented by RAReQS, require exponentially many.

4.1 Termination

Lemma 1. At every iteration of Algorithm 1, \( A \) increases strictly (with regards to set inclusion).

Proof. The only operation affecting \( A \) is adding to it, so \( A \) increases.

Hence, it is sufficient to prove that at least one new counterexample is produced at each iteration. For a given \( A \), if \( \Phi := \bigwedge_{Y \in A} Q(Y, Y) \) is satisfiable, \( Q(X^Y, Y) \) holds in the resulting model, for each counterexample \( Y \in A \).

Hence, \( \psi := \land_{X \in E} \neg Q(X, Y^X) \) cannot be satisfied using only existing counterexamples (concrete values that are already in \( A \)): either \( \psi \) is unsatisfiable (and the algorithm terminates) or the produced model contains new values that are added to \( A \), which thus grows (strictly) larger.

Q.e.d.

Theorem 1. Algorithm 1 terminates after at most \( 2^{1 + \sum_{i=0}^n |y_i|} \) SAT queries.

Proof. The set of all possible counter-examples, a superset of \( A \), is finite and of size \( 2^{\sum_{i=0}^n |y_i|} \).

From Lemma 1 at least one new counterexample is produced at each iteration: there are at most \( 2^{\sum_{i=0}^n |y_i|} \) iterations.
Lastly, two SAT queries are performed at each iteration, except for the last which may perform one or two queries.

Q.e.d.

4.2 Soundness

Lemma 2. If the QBF formula $\Phi$ is true, Algorithm 1 returns some $x_0$ such that $\Phi$ admits a Skolem model of the formula where the value of $x_0$ is $x_0$.

Proof. If the original formula is true, there is a Skolem model $f_{y_0}(\cdot), f_{y_1}(\cdot), \ldots, f_{y_n}(\cdot)$ such that $\forall y_0, \ldots, y_n, Q(f_{y_0}(\cdot), y_0, f_{y_1}(y_0), \ldots, y_{n-1}, f_{y_n}(y_0, \ldots, y_n))$

Then, $\varphi := \bigwedge_{Y \in A} Q(X^Y, Y)$ is always satisfiable: $x_i^0, \ldots, x_i^n := f_{y_i}(y_0, \ldots, y_i)$ (for each $(y_0, \ldots, y_n) \in A$) is a satisfying model. Hence, Algorithm 1 cannot return false, and it terminates (Theorem 1), so it must returns some $x_0$.

Let us consider the following formula:

$\forall y_0. \exists x_1 \forall y_1, \ldots, \exists x_{n+1}. Q(f_{y_0}, x_1, \ldots, x_{n+1}, Y)$

If this formula is true, then adding $x_0 := f_{y_0}$ to one of its Skolem models yields a Skolem model of the original QBF with the desired property.

If it is false, then a model for $\psi := \bigwedge_{X \in E} \neg Q(X, Y)$ can be constructed from its Herbrand model, using the same construction as in Lemma 3; thus, $\psi$ is satisfiable and the algorithm cannot have returned at that point.

Q.e.d.

Lemma 3. If the QBF formula is false, Algorithm 1 returns false.

Proof. If the original QBF is false, it has a Herbrand model $f_{y_0}(\cdot), f_{y_1}(\cdot), \ldots, f_{y_n}(\cdot)$ such that, for each $x_1, \ldots, x_{n+1}, \neg Q(x_0, f_{y_0}(x_0), \ldots, x_n, f_{y_n}(x_0, \ldots, x_n), x_{n+1})$

Hence, $\psi$ has a satisfying model, $y_i^0, \ldots, x_i := f_{y_i}(x_0, \ldots, x_i)$ (for each $(x_0, \ldots, x_{n+1}) \in A$), so Algorithm 1 cannot return some $x_0$.

Since it must return (Theorem 1), it must then return false.

Q.e.d.

The soundness result follows immediately:

Theorem 2. The result returned by Algorithm 1 is consistent with the value of the input.

4.3 Distinguishability from RAReQS

As mentioned earlier, counterexample-guided methods usually have exponential worst-case complexity, which is disconnected from their performance in practice. This makes it difficult to study those methods through the lens of complexity theory.
However, we can give here a family of examples which highlights the algorithmic differences between our approach and “traditional” CEGIS-based solvers, such as RAReQS.

Note that we compare against the algorithmic description of RAReQS, without considering additional optimizations, especially in-processing, which are not described in the RAReQS paper.

**Theorem 3.** There exists an infinite family of QBF terms such that ijtihad performs $O(1)$ SAT queries solving them, whereas RAReQS’ algorithm performs $\Omega(2^n)$ queries.

We define those formulas to have the following structure, with $Q^X_n$ satisfiable and $Q^Y_n$ falsifiable, $X$ being $(x_0, \ldots, x_{n+1})$ and $Y$ being $(y_0, \ldots, y_n)$:

$$\Phi_n := \exists x_0. \forall y_0. \ldots \forall y_n. \exists x_{n+1}. Q^X_n(X) \land Q^Y_n(Y)$$

**Lemma 4.** ijtihad performs either 1 or 3 SAT queries to solve $\Phi_n$, for any $n$.

**Proof.**
1. $Y_0$ is taken arbitrarily.
2. A first SAT call is performed to obtain a witness $X_0$.
   - If $Y_0$ falsifies $Q^Y_n$, the result is UNSAT, and ijtihad returns.
   - Otherwise, $X_0$ is such that $Q^X_n(X_1)$ holds.
3. Since $Q^X_n(X_0)$ holds, the counterexample $Y_1$ must falsify $Q^Y_n$.
4. The next SAT call is guaranteed to yield UNSAT, as no witness exists.

Q.e.d.

For convenience, we define auxiliary families of formulas:

$$\phi_{n,i}^{(x_0, y_0, \ldots, x_i, y_i)} := \exists x_{i+1}. \forall y_{i+1}. \exists x_{i+2} \ldots \exists x_{n+1}. Q^X_n(x_0, \ldots, x_i, x_{i+1}, \ldots, x_{n+1}) \land Q^Y_n(y_0, \ldots, y_i, y_{i+1}, \ldots, y_n)$$

$$\phi_{n,i}^{(x_0, y_0, \ldots, x_i, y_i)} := \exists y_i. \forall x_{i+1}. \exists y_{i+1} \ldots \exists x_{n+1}. \neg Q^X_n(x_0, \ldots, x_i, x_{i+1}, \ldots, x_{n+1}) \lor \neg Q^Y_n(y_0, \ldots, y_{i-1}, y_i, \ldots, y_n)$$

Note that $\Phi_n = \phi_{n,i}^{(x_0, y_0, \ldots, x_i, y_i)} = \forall x_{i+1}. \phi_{n,i}^{(x_0, y_0, \ldots, x_i, y_i, x_{i+1})}$.

**Lemma 5.** RAReQS’ algorithm performs $\Omega(2^{n-i})$ SAT queries when called on $\phi_{n,i}^{(x_0, y_0, \ldots, x_i, y_i)}$.

**Proof.** The execution of RAReQS on $\phi_{n,i}^{(X, Y)}$ results in a single SAT call, as there is only a single quantifier.

The execution of RAReQS on $\phi_{n,i}^{(x_0, y_0, \ldots, x_i, y_i)}$ goes as follows:

1. An arbitrary value $x_{i+1}$ is picked. Let us assume that is leaves $Q^X_n$ satisfiable.
2. RAReQS is recursively called on $\phi_{n,i}^{(x_0, y_0, \ldots, x_i, y_i, x_{i+1})}$

\[\text{This is the best case for RAReQS, but it doesn’t matter for the sake of our argument.}\]
(a) An arbitrary value $y_{i+1}$ is picked. Let us assume it leaves $Q_n^y$ not falsifiable.
(b) RAReQS is recursively called on $\Phi_n^{(x_0,...,x_i,y_{i+1})}$, yielding $x_{i+1}$.
(c) RAReQS is recursively called on $\exists y_{i+1} \neg \Phi_n^{(x_0,...,x_i,y_{i+1},y_{i+1})}$, yielding $y_{i+1}$ such that $Q_n^y$ is refutable.
(d) RAReQS is called on $\Phi_n^{(x_0,...,x_i,y_{i+1})}$, yielding false.
(e) $y_{i+1}$ is returned from the recursive RAReQS call.

3. The execution of RAReQS on $\Phi_n^{(x_0,...,x_i,y_{i+1})}$ resumes.

RAReQS performs, while solving $\Phi_n^{(x_0,...,x_i,y_{i+1})}$, at least two recursive calls on $\Phi_n^{(x_0,...,x_{i+1},y_{i+1})}$ with different values of $y_{i+1}$: it follows that $\Omega(2^{n-i})$ SAT calls are performed.

Q.e.d.

5 Experimental results

In this section we look at the performance of alejtehad against RAReQS, which won several recent QBF competitions. We also point out different families of problems where we do better, as well as those where we do very poorly. Furthermore, we point out the impact of various formula parameters on our solving time.

5.1 Experimental setting

The optimised alejtehad algorithm is implemented in C++ and uses MiniSat\textsuperscript{[12]}\textsuperscript{[14]} as its backing SAT solver. The code is based on RAReQS\textsuperscript{1}, so as to minimise the differences (data structures, cache optimisations, ...) that could explain performance differences, aside from the implemented algorithm. It is available, under a free software licence (GPLv3), at our laboratory’s source code repository\textsuperscript{4}.

The input QBF is given in the standard QDIMACS format. All experiments were run on a dedicated Nehalem machine with 192GB of RAM and 24 CPU threads, running at 2.8GHz each. Each execution was allocated a single virtual core and 10 minutes per problem, while the machine was otherwise unused.

RAReQS was run using its default configuration, whereas our solver does not have tweakable parameters.

5.2 Performance on Application Benchmarks

We ran both solvers on the 825 benchmarks from the main track of the QBF-Eval\textsuperscript{16} competition. We also evaluated the impact of the preprocessor Bloqker\textsuperscript{6}\textsuperscript{[19]}, which yields an overall performance increase for both solvers but is more helpful to RAReQS, as seen in Table\textsuperscript{1}.

We then analysed how the number of quantifiers affects the performance of the solvers, which can be seen in Figure\textsuperscript{2}. It becomes clear that alejtehad tends

\textsuperscript{4}https://extgit.iak.tugraz.at/scos/ijtihad
\textsuperscript{5}Available on http://www.qbflib.org/eval16.html at the time of writing.
| Solver        | Solved instances | SAT   | UNSAT | Solving time (s) Total | Average |
|--------------|------------------|-------|-------|------------------------|---------|
| RAReQS + b   | 627              | 306   | 321   | 16316.9                | 26.0    |
| alejtehad + b| 545              | 261   | 284   | 5153.9                 | 9.5     |
| RAReQS       | 331              | 125   | 206   | 8505.9                 | 25.7    |
| alejtehad    | 322              | 124   | 198   | 8739.3                 | 27.1    |

Table 1. Number of solved test cases, time needed for solving them, and the average solving time of solved test cases for each of the solvers (not including timeouts).

to have an advantage over RAReQS for QBFs with more quantifier alternations. Using Bloqker, however, generally decreases the number of quantifiers [6], which explains why preprocessing helps RAReQS more.

The biggest difference in performance can be seen on the benchmark families described by Pan and Vardi [27], and those described by Mneimneh and Sakallah [26]. As shown in Figure 3 and Table 2, alejtehad is faster on the Pan family, which has higher quantifier depth. On the other hand, RAReQS outperforms it on the Mneimneh-Sakallah family, which exhibits many variables and low quantification depth. In fact, one could argue that the Mneimneh-Sakallah QBF family is the main reason RAReQS’ results seem better in Table 1 as alejtehad almost always runs into a timeout in this specific class, which is overrepresented, accounting for approximately 12% of the entire set of benchmarks.

6 Only quantifier depths with more than one test case are shown in order to reduce noise.
Table 2. Number of solved test cases and time needed for solving them, for the QBF families Pan and Mneimneh-Sakallah

| Benchmarks          | # Test Cases | Solver          | # Solved | Solving time |
|---------------------|--------------|-----------------|----------|--------------|
| Pan                 | 181          | RAREQS + b      | 171      | 879.681      |
|                     |              | alejtehad + b   | 173      | 710.27       |
| Mneimneh-Sakallah   | 100          | RAREQS + b      | 73       | 9748         |
|                     |              | alejtehad + b   | 12       | 323.19       |

Fig. 3. Log-log time scatter plots of the Pan family on the left, and Mneimneh-Sakallah family on the right. alejtehad’s solving time is shown on the y axis, while RAREQS’s is on the x axis.
5.3 Performance on Random QBFs with a fixed structure

Experimental evidence shown in [9] suggests that there is a crossover point in the size of random QBF problems, where the formula is equally likely to be SAT and UNSAT, and where verifying or falsifying it becomes as hard as possible. We use this to argue that alejtehad and RAReQS scale differently as quantifier depth increases.

In order to further support the claim that alejtehad outperforms RAReQS on QBFs with many quantifier alternations, we ran both solvers without external preprocessing on hard random QBFs. All examined QBFs were generated by BlocksQBF [8] and had a fixed structure in which each clause of the quantified CNF had one variable from every quantified variable vector.

We produced benchmarks with increasing numbers of variables under each quantifier until both solvers ran into timeouts, and then compared the overall results. Figure 4 shows the performance of the solvers for the number of variables which produced the biggest difference in behaviour, for a given number of quantifiers.

As can be observed, on problems with more than 4 quantifiers, we consistently solve more problems than RAReQS, and faster. One important thing to note is that timeouts weight in the solving time, with no particular penalty beyond the timeout value; this explains that our speedup seems to taper off at quantification depths 12 to 14: alejtehad takes more time to solve instances, and so seems closer to RAReQS despite the fact that it doesn’t solve any instances at all.

![Fig. 4. Plot showing the difference in number of solved test cases by depth, as well as the relative time. In the graph, \(n_i\) and \(n_r\) denote the number of test cases solved by alejtehad and RAReQS respectively, whereas \(n\) denotes the total number of test cases (with the given quantifier depth).](image)

7 The number of quantifiers is important and Bloqger would change it
Conclusion

To summarise our contribution, we provide a new perspective on Counterexample-Guided Inductive Synthesis (CEGIS), which yields a novel QBF-solving algorithm, which is both extremely simple to describe and prove correct, and which our experiments show to be efficient.

Those findings are supported by an implementation of this algorithm, \textit{alejtehad}, using no optimisation or implementation techniques not described in the paper. This implementation is competitive with RAReQS, a state-of-the-art solver which won recent QBF competitions. In particular, Section 5 shows that our approach outperforms RAReQS in problems involving more than 4 nested quantifiers.

On the other hand, our approach shares a weakness with other expansion-based methods (including RAReQS): very simple formulas like $\forall x. \exists y. y = x$ take $2^n$ steps to enumerate the $(x, y)$ pairs \cite{15}, with $x$ and $y$ being vectors of $n$ Boolean variables. Alternative approaches may not suffer from this specific issue. Most notably, Rabe \textit{et al.} have been investigating synthesising directly Skolem functions \cite{15,16}, which can efficiently deal with this kind of QBF instances.

There are many venues of research this opens for future work:

- Are there commonplace QBF optimisations that are applicable here, and do they improve the performance of \textit{alejtehad}?
- Preprocessors such as Bloqer are designed for solvers for which deeper quantification is problematic. As exhibited in Section 5.3, current preprocessors do not benefit \textit{alejtehad} as much as other solver: can Bloqer be tuned or improved to better support our approach?
- Can our approach be combined with Skolem function synthesis \cite{15,16} so as to handle problems where the Skolem or Herbrand models contain many concrete values?
- Outputting small certificates is a known challenge for counterexample-guided methods. More generally, can we extract small Skolem and Herbrand models?
- Can we generalise our implementation to NNF (Negation Normal Form) without losing efficiency? NNF is interesting, especially for QBF, as encoding problems into NNF does not require introducing new variables, and so does negating NNF formulas.
- How does search-based solving compare to our approach? Can we combine both approaches?
- The \textit{alejtehad} algorithm is intrinsically sequential, due to its data dependencies. Can we find a variation on it that can be efficiently parallelised?

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