DIMENSION OF PLURIHARMONIC MEASURE AND POLYNOMIAL ENDOMORPHISMS OF $\mathbb{C}^n$.

I. BINDER AND L. DEMARCO

ABSTRACT. Let $F$ be a polynomial endomorphism of $\mathbb{C}^n$ which extends holomorphically to $\mathbb{CP}^n$. We prove that the dimension of $\mu_F$, the pluriharmonic measure on the boundary of the filled Julia set of $F$, is bounded above by $2n - 1$.

1. INTRODUCTION

The dimension of a probability measure on a metric space is defined as the minimal Hausdorff dimension of a set of full measure. In this paper, we show that the dimension of pluriharmonic measure in $\mathbb{C}^n$ is bounded above by $2n - 1$ when it arises as the measure of maximal entropy for a regular polynomial endomorphism.

For a compact set $K$ in $\mathbb{C}^n$, pluriharmonic measure is defined as

$$\mu_K := dd^c G_K \wedge \cdots \wedge dd^c G_K,$$

where $G_K$ is the pluricomplex Green’s function of $K$ with pole at infinity, $d = \partial + \bar{\partial}$ and $d^c = \frac{1}{2\pi i}(\partial - \bar{\partial})$. The support of $\mu_K$ is contained in the Shilov boundary of $K$. See §2.

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a regular polynomial endomorphism; i.e., one which extends holomorphically to $\mathbb{CP}^n$. The filled Julia set of $F$ is the compact set

$$K_F = \{ z \in \mathbb{C}^n : F^m(z) \not\to \infty \text{ as } m \to \infty \}.$$

Pluriharmonic measure $\mu_F$ on $K_F$ is $F$-invariant and of maximal entropy [FS95]. It is not difficult to construct examples where the Hausdorff dimension of the support of $\mu_F$ is any value up to and including $2n$. In answer to a question posed in [FS01], we prove:

Theorem 1. The dimension of pluriharmonic measure on the filled Julia set of a regular polynomial endomorphism of $\mathbb{C}^n$ is at most $2n - 1$.

In [Oks81], Oksendal conjectured that the dimension of harmonic measure in $\mathbb{C}$ would never exceed 1, though the Hausdorff dimension of its support can take values up to and including 2. Makarov addressed this question for simply connected domains, showing that the dimension of harmonic measure is always equal to one [Mak85]. The theorem was extended by Jones and Wolff, establishing that the dimension is no greater than 1 for general planar domains [JW88]. Moreover, Wolff

Date: June 8, 2002.

1991 Mathematics Subject Classification. Primary 32H50; Secondary 32U99.

The first author is supported by N.S.F. Grant DMS-9970283.
proved that there is always a set of full harmonic measure with $\sigma$-finite Hausdorff 1-measure [Wol93].

The complex structure on the plane plays a crucial role in the proof of these theorems. Namely, one relies heavily on the subharmonicity of the function $\log|\nabla u|$ for harmonic $u$.

It is also possible to take a dynamical approach to the dimension estimates. It follows from results of Carleson, Jones, and Makarov ([CJ92] and [Mak89]) that any planar domain can be approximated in some sense by domains invariant under hyperbolic dynamical systems (the fractal approximation). In the special case of conformal Cantor sets, Carleson obtained dimension estimates using the dynamics [Car85]. Recently, it was shown that it suffices to consider polynomial Julia sets in the fractal approximation [BJ02].

Harmonic measure (evaluated at infinity) on the Julia set of a polynomial is the unique measure of maximal entropy [Bro65], [Lju83]. The estimate on dimension in this case follows from a relation to the Lyapunov exponent and the entropy. Indeed, for any polynomial map $F$ on $\mathbb{C}$, we have

$$\dim \mu_F = \frac{\log(\deg F)}{L(F)},$$

where $\mu_F$ denotes harmonic measure on the Julia set of $F$ and $L(F) = \int \log|F'| \, d\mu_F$ is the Lyapunov exponent [Mn88], [Man84]. The Lyapunov exponent of a polynomial is bounded below by $\log(\deg F)$ with equality if and only if the Julia set is connected [Prz85].

On the other hand, for harmonic measure in $\mathbb{R}^n$, the methods applied to the study of dimension in $\mathbb{C}$ fail dramatically. The logarithm of the gradient of a harmonic function in $\mathbb{R}^n$, $n > 2$, is not subharmonic, and there is no dynamical interpretation of harmonic measure. Furthermore, in [Wol95], Wolff showed that for each $n > 2$ there exists a domain in $\mathbb{R}^n$ with the dimension of harmonic measure strictly greater than $n - 1$. A result of Bourgain, however, gives an upper bound on the dimension of harmonic measure in the form $n - \varepsilon(n)$ [Bou87].

Because of the harmonicity of $|\nabla u|^{\frac{n}{n-2}}$ for a harmonic function $u$ in $\mathbb{R}^n$ (see [Ste70]), it is conjectured that the dimension of harmonic measure in $\mathbb{R}^n$ does not exceed $n - 1 + \frac{n^2}{n-1}$.

For pluriharmonic measure in $\mathbb{C}^n$, both of the observations which led to proofs of the Oksendal conjecture in $\mathbb{C}$ are valid: the measure depends on the complex structure of $\mathbb{C}^n$ and is the measure of maximal entropy for polynomial dynamics. Theorem 1 should be the first step in the proof of the following conjecture:

**Conjecture 1.** The dimension of pluriharmonic measure of domains in $\mathbb{C}^n$ is at most $2n - 1$.

The example of area measure on the unit sphere in $\mathbb{C}^n$ shows that this estimate is sharp.

We believe also that one can give a precise formula for the dimension of pluriharmonic measure in the dynamical case, just as in dimension one (see formula
For a diffeomorphism of a real compact manifold, such a formula was obtained by Ledrappier and Young in [LY85]:

\[
\dim \mu = \sum \frac{h_i(\mu)}{\lambda_i(\mu)},
\]

where \( \mu \) is an ergodic invariant measure without zero Lyapunov exponents, \( \lambda_i(\mu) \) are its positive Lyapunov exponents, and \( h_i(\mu) \) are the corresponding directional entropies (defined in [LY85]). A similar formula was established by Bedford, Lyubich, and Smillie for polynomial diffeomorphisms of \( \mathbb{C}^2 \) [BLS93]. We make the following conjecture, which would imply Theorem 1.

**Conjecture 2.** For any holomorphic \( F : \mathbb{CP}^n \to \mathbb{CP}^n \) of (algebraic) degree \( d \geq 1 \),

\[
\dim \mu_F = \log d \sum_{i=1}^{n} \frac{1}{\lambda_i},
\]

where \( \lambda_i, \ i = 1 \ldots n \) are the Lyapunov exponents of \( F \) with respect to \( \mu_F \) repeated with multiplicities.

**Sketch proof of Theorem 1.** We rely on estimates on the Lyapunov exponents of \( F \) with respect to \( \mu_F \). In particular, Briend and Duval showed that the minimal Lyapunov exponent \( \lambda_{\min} \) is bounded below by \( \frac{1}{2} \log d \) (where \( d \) is the degree of \( F \)) [BD99]. Bedford and Jonsson proved that the sum \( \Lambda \) of the Lyapunov exponents satisfies \( \Lambda \geq \frac{n+1}{2} \log d \) [BJ00]. Combining these, we have \( \Lambda \geq \lambda_0 + \frac{n-1}{2} \log d \), where \( \lambda_0 = \max \{ \lambda_{\max}, \log d \} \).

We define an invariant set \( Y \) of full measure so that preimages of small balls centered at points in \( Y \) scale in a way governed by the Lyapunov exponents. Namely, for each point \( y \in Y \) there exists an infinite set \( M_y \subset \mathbb{Z} \) such that if \( m \in M_y \) then the \( m \)-th preimage of a ball of radius \( r \) centered at \( F^m(y) \) should contain a ball of radius \( \approx re^{-m\lambda_{\max}} \) around \( y \). In addition, the component of the preimage containing \( y \) will have volume \( \approx r^{2n}e^{-2m\Lambda} \). The details of the construction are very similar to the methods of [BD99].

Let \( A_m = \{ y \in Y : m \in M_y \} \). Note that \( Y = \bigcap_k \bigcup_{m \geq k} A_m \). If we cover \( Y \) by \( N \) balls of radius \( r \), then the “good” (as described above) \( m \)-th preimages define a cover of \( A_m \) by at most \( Nd^{\min} \) regions of controlled shape. Their union contains an \( re^{-m\lambda_{\max}} \)-neighborhood of \( A_m \), and has volume \( \leq Nr^{2n}d^m e^{2m(n-1)\lambda_0} \), by the estimates above.

Finally, a standard connection between the rate of decay of volume of a neighborhood of \( Y \) and its dimension allows us to conclude that \( \dim Y \leq 2n - 1 \).

## 2. Pluriharmonic measure in \( \mathbb{C}^n \) and dynamics

In this section, we give some of the necessary background on pluripotential theory in \( \mathbb{C}^n \) and its relation to polynomial dynamics. More details can be found in [Kli91], [BT87] and [Bed93].
Let $\text{PSH}(\mathbb{C}^n)$ denote the class of plurisubharmonic functions in $\mathbb{C}^n$. For a compact set $K$ in $\mathbb{C}^n$, the pluricomplex Green’s function with pole at infinity is defined as

$$G_K(z) = \sup \{ v(z) : v \in \text{PSH}(\mathbb{C}^n), v \leq 0 \text{ on } K, \ v(z) \leq \log \|z\| + O(1) \text{ near } \infty \}.$$ 

See, for example, [Kli91] or [Bed93]. If $G_K$ is continuous, the set $K$ is said to be regular.

In contrast to the one dimensional setting, $G_K$ is not necessarily pluriharmonic (or even harmonic) outside of $K$. It is, however, maximal plurisubharmonic, meaning that if $v$ is any plurisubharmonic function on a domain $\Omega$ compactly contained in $\mathbb{C}^n - K$ with $v \leq G_K$ on $\partial \Omega$, then $v \leq G_K$ on $\Omega$. Equivalently, the Monge-Ampere mass of $G_K$,

$$\mu_K = (dd^c G_K)^n,$$

vanishes in $\mathbb{C}^n - K$. We call the measure $\mu_K$ the pluriharmonic measure on $K$, and note that it is supported in the Shilov boundary of $K$. In fact, if $K$ is regular, then its support is equal to the Shilov boundary [BT87].

Pluriharmonic measure arises in the study of dynamics just as in the one dimensional setting. A polynomial endomorphism $F : \mathbb{C}^n \to \mathbb{C}^n$ is called regular if it can be extended holomorphically to $\mathbb{C}P^n$. The degree of $F$ is the degree of its polynomial coordinate functions. We consider only those $F$ of degree $> 1$. The escape rate function of $F$ is defined by

$$G_F(z) = \lim_{m \to \infty} \frac{1}{d^m} \log^+ \|F^m(z)\|,$$

where $d$ is the degree of $F$ and $\log^+ = \max\{\log, 0\}$. The function $G_F$ is continuous and agrees with the pluricomplex Green’s function for the filled Julia set $K_F = \{ z \in \mathbb{C}^n : F^m(z) \not\to \infty \}$. Fornaess and Sibony showed that the pluriharmonic measure $\mu_F$ on $K_F$ is ergodic for $F$ and a measure of maximal entropy [FS95].

By the Oseledec Ergodic Theorem, $F$ has $n$ Lyapunov exponents $\lambda_{\text{min}} \leq \cdots \leq \lambda_{\text{max}}$ almost everywhere with respect to $\mu_F$ [Ose68]. We will only need the existence of the minimal, maximal, and the sum $\Lambda$ of the Lyapunov exponents, which we can define as follows:

$$\lambda_{\text{min}} = - \lim_{m \to \infty} \frac{1}{m} \int \log \| (DF^m)^{-1} \| \, d\mu_F,$$

$$\lambda_{\text{max}} = \lim_{m \to \infty} \frac{1}{m} \int \log \| DF^m \| \, d\mu_F,$$

$$\Lambda = \int \log |\det DF| \, d\mu_F.$$

Briend and Duval proved that the Lyapunov exponents are all positive [BD99]; they show

$$\lambda_{\text{min}} \geq \frac{1}{2} \log d,$$

(3)
where $d$ is the degree of $F$. Bedford and Jonsson studied the sum of the Lyapunov exponents, and demonstrate that ([BJ00])
\[
\Lambda \geq \frac{n + 1}{2} \log d.
\]

For the proof of Theorem 1, it is convenient to work in the natural extension $(\hat{X}, F)$ where $F$ is invertible (see [CFS82] and [BD99]). Let $P(F) = \bigcup_{m \geq 0} F^m(C(F))$ be the postcritical set of $F$ and set $X = C^n - \bigcup_{m \geq 0} F^{-m}(P(F))$. The space $(\hat{X}, F)$ is the set of all bi-infinite sequences
\[
\{\hat{x} = (\cdots x_{-1}x_0x_1 \cdots) \in \prod_{-\infty}^{\infty} X : F(x_i) = x_{i+1}\}.
\]

The map $F$ acts on $(\hat{X}, F)$ by the left shift. We define projections $\pi_i : (\hat{X}, F) \to X$ for all $i$ by $\pi_i(\hat{x}) = x_i$. Since $\mu_F$ does not charge the critical locus of $F$, we have $\mu_F(X) = 1$. The measure $\mu_F$ lifts to a unique $F$-invariant probability measure $\hat{\mu}$ on $(\hat{X}, F)$ so that $\pi_0^* \hat{\mu} = \mu_F$.

3. Proof of the main theorem.

In this section we give a proof of the following theorem which clearly implies Theorem 1.

**Theorem 2.** Pluriharmonic measure $\mu_F$ on the filled Julia set of a degree $d$ regular polynomial endomorphism $F : C^n \to C^n$ satisfies
\[
\dim \mu_F \leq 2n - 2 + \frac{\log d}{\max\{\log d, \lambda_{\max}\}},
\]
where $\lambda_{\max}$ is the largest Lyapunov exponent of $F$ with respect to $\mu_F$.

We begin with a classical lemma. Statements (a) and (c) are exactly as in [BD99, Lemma 2]. We first observe that there exists a constant $C(n)$ so that for any $n \times n$ matrix $A$ with $\|A - I\| < 1$, we have
\[
|\det A - 1| \leq \frac{C(n)}{2} \|A - I\|.
\]

**Lemma 1.** Let $g : \Omega \to C^n$ be a function with bounded $C^2$-norm on a domain $\Omega \subset C^n$ and set $M = C(n) (\|g\|_{C^2} + 1)$. Let $x \in \Omega$ be a noncritical point of $g$. Given $\varepsilon > 0$, let $r(x) = \frac{1}{2M\|D_xg\|^{-1}\|D_xg\|^{-1}}$, set $B_0 = B(g(x), r(x))$ and let $B_1$ be the preimage of $B_0$ under $g$ containing $x$. Then
(a) $g^{-1}$ is well defined in $B_0$,
(b) $\text{Lip}(g|B_1) \leq \|D_xg\|e^{\varepsilon/3}$,
(c) $\text{Lip}(g^{-1}|B_0) \leq \|D_xg\|^{-1}e^{\varepsilon/3}$, and
(d) $\inf_{y \in B_1} |\det(D_yg)| \geq |\det(D_xg)|e^{-\varepsilon/3}$.
Proof. Let us consider a ball $B_2 = B(x, \rho)$, where
$$
\rho = \frac{e^{\epsilon/3} - 1}{M\|(D_x g)^{-1}\|}.
$$
For each $y \in B_2$, we have
$$
\|I - (D_x g)^{-1}(D_y g)\| \leq (\|g\|_{C^2} + 1)\|(D_x g)^{-1}\|\rho \leq \frac{e^{\epsilon/3} - 1}{C(n)},
$$
and in particular, Lip $(I - (D_x g)^{-1} \circ g) < 1$. If $g(y_1) = g(y_2)$ for some $y_1 \neq y_2 \in B_2$, then
$$
\|y_1 - y_2\| = \|(y_1 - (D_x g)^{-1}g(y_1)) - (y_2 - (D_x g)^{-1}g(y_2))\| < \|y_1 - y_2\|,
$$
which is a contradiction, and therefore $g$ is injective on $B_2$.

To establish (a), we need to know that $B_0 \subset g(B_2)$. Map $g$ is open on $B_2$, so it is enough to check that if $|y_1 - x| = \rho$, then $|g(y_1) - g(x)| > r(x)$. But this is again a direct consequence of (6).

Now, since $B_1 \subset B_2$, we have for all $y \in B_1$,
$$
\|D_x g - D_y g\| \leq \|D_x g\| \|I - (D_x g)^{-1}(D_y g)\| \leq \|D_x g\| \frac{e^{\epsilon/3} - 1}{C(n)},
$$
and we conclude that
$$
\|D_y g\| \leq \|D_x g\| + \frac{e^{\epsilon/3} - 1}{C(n)}\|D_x g\| \leq e^{\epsilon/3}\|D_x g\|,
$$
establishing (b).

To prove (c), observe that by (6) for $y \in B_2$,
$$
\|(D_y g)^{-1}\| \leq \|(D_x g)^{-1}\|\|(I - (I - (D_x g)^{-1}D_y g)^{-1})\|
\leq \frac{\|(D_x g)^{-1}\|}{1 - \|(I - (D_x g)^{-1}D_y g)\|}
\leq \|(D_x g)^{-1}\|e^{\epsilon/3}.
$$
For (d), we compute for all $y \in B_1$ (using (5) above),
$$
|\det D_y g - \det D_x g| = |\det D_x g| \|1 - \det(D_x g)^{-1}D_y g\|
\leq |\det D_x g| \frac{C(n)}{2} \|I - (D_x g)^{-1}D_y g\|
\leq |\det D_x g| \frac{1}{2}(e^{\epsilon/3} - 1)
\leq |\det D_x g|(1 - e^{-\epsilon/3}),
$$
and therefore,
$$
\inf_{y \in B_1} |\det D_y g| \geq |\det D_x g| e^{-\epsilon/3}.
$$
\end{proof}
Let \( F \) be a regular polynomial endomorphism of \( \mathbb{C}^n \) and \( \mu_F \) the pluriharmonic measure on the boundary of the filled Julia set of \( F \). Denote by \( \lambda_{\text{min}}, \lambda_{\text{max}}, \) and \( \Lambda \), the minimal, maximal, and sum of the \( n \) Lyapunov exponents of \( F \) with respect to \( \mu_F \). The space \((\hat{X}, F)\) denotes the natural extension of \( F \). See §2.

**Lemma 2.** Given \( \varepsilon > 0 \), there exist measurable functions \( r \) and \( \kappa \) on \((\hat{X}, F)\) so that for almost every point \( \hat{x} \), we have \( r(\hat{x}) > 0 \) and \( \kappa(\hat{x}) < \infty \), and for each \( m \geq 0 \), a well-defined branch of \( F^{-m} \) sending \( x_0 \) to \( x_{-m} \) with

(a) \( F^{-m}(B(x_0, s)) \supset B(x_{-m}, \frac{s}{\kappa(\hat{x})} e^{-m(\lambda_{\text{max}} + \varepsilon)}) \) for all \( s \leq r(\hat{x}) \), and

(b) \( \text{Vol} F^{-m}B(x_0, r(\hat{x})) \leq \kappa(\hat{x}) e^{-m(2\Lambda - \varepsilon)} \).

**Proof.** Choose \( N \) so that

\[
0 < \lambda_{\text{min}} - \varepsilon \leq -\frac{1}{N} \int \log \| (DF)^N \|^{-1} \| d\mu_F \leq \lambda_{\text{min}},
\]

and

\[
\lambda_{\text{max}} \leq \frac{1}{N} \int \log \| DF^N \| \, d\mu_F \leq \lambda_{\text{max}} + \varepsilon.
\]

Observe that

\[
\Lambda = \frac{1}{N} \int \log |\det DF^N| \, d\mu_F
\]

for any \( N \geq 0 \).

For notational simplicity, set \( g = F^N \). Observe that it is enough to prove the statement of the lemma for \( g \) instead of \( F \).

Fix \( \hat{x} \in (\hat{X}, g) \). Let

\[
r(x_{-m}) = \frac{1 - e^{-\varepsilon/3}}{2M\| (DF_{x_0}^{-m})^{-1} \|^2},
\]
as in Lemma 1 where \( \Omega \) is a large ball containing the filled Julia set of \( F \). By the ergodic theorem applied to the function

\[
\hat{x} \mapsto \log \| (D_{x_0}g)^{-1} \|,
\]

we have

\[
\lim_{m \to \infty} \frac{1}{m} \log \| (D_{x_0}g)^{-1} \| = 0,
\]

and therefore there exists a measurable function \( \eta > 0 \) on \((\hat{X}, g)\) with

\[
r(x_{-m}) \geq \eta(\hat{x}) e^{-m\varepsilon/2}
\]

for all \( m \geq 0 \) and almost every \( \hat{x} \).

Let \( B_m = B(x_0, r(x_{-1})) \cap \cdots \cap g^m B(x_{-m}, r(x_{-m-1})) \). Let \( g^{-m} \) denote the inverse branch of \( g^m \) taking \( x_0 \) to \( x_{-m} \), well-defined on \( B_m \) by Lemma 1(a). Iterating results (b), (c), and (d) of Lemma 1, we have

\[
\text{Lip}(g^{-m}|B_m) \leq \| (D_{x_0}g)^{-1} \| \cdots \| (D_{x_{-1}}g)^{-1} \| e^{m\varepsilon/3},
\]

\[
\text{Lip}(g^m|g^{-1}B_m) \leq \| D_{x_0}g \| \cdots \| D_{x_{-1}}g \| e^{m\varepsilon/3},
\]
and
\[ \inf_{y \in g^{-m}B_m} |\det D_y g^m| \geq |\det D_{x-m} g^m e^{-m \varepsilon/3}|. \]

Applying the ergodic theorem to the functions \( \hat{x} \mapsto \log \|D_{x_0}g\|^{-1} \), \( \hat{x} \mapsto \log \|D_{x_0}g\| \), and \( \hat{x} \mapsto \log |\det D_{x_0}g| \), we see that there exists a measurable function \( 1 \leq C(\hat{x}) < \infty \) so that
\[ \text{Lip}(g^{-m}B_m) \leq C(\hat{x}) e^{-m(N\lambda_{\min} - \varepsilon/2)}, \]
(8)
\[ \text{Lip}(g^m B^{-m}B_m) \leq C(\hat{x}) e^{m(N\lambda_{\max} + \varepsilon/2)}, \]
(9)
and
\[ \inf_{y \in g^{-m}B_m} |\det D_y g^m| \geq \frac{1}{C(\hat{x})} e^{m(N\Lambda - \varepsilon/2)} \]
(10)
for almost every \( \hat{x} \).

Let \( r(\hat{x}) = \min\{\eta(\hat{x})/C(\hat{x}), 1\} \). By induction and the estimates (7) and (8), we establish that \( B(x_0, r(\hat{x})) \) is contained in \( B_m \) for all \( m \geq 0 \). By (9), we have
\[ B(x_0, r(\hat{x})) \supset g^m B(x_0, \frac{r(\hat{x})}{2C(\hat{x})} e^{-m(N\lambda_{\max} + \varepsilon/2)}). \]

By (10), the volume of \( g^{-m}B(x_0, r(\hat{x})) \) is bounded by
\[ \text{Vol}(g^{-m}B(x_0, r(\hat{x}))) \leq \text{Vol}(B(x_0, r(\hat{x}))) C(\hat{x}) ^2 e^{-m(2\Lambda - \varepsilon)}. \]

The Lemma is proved upon setting \( \kappa(\hat{x}) = 2C(\hat{x})^2 \text{Vol} B_1. \)

**Proof of Theorem 1.** For fixed \( \varepsilon > 0 \), let \( r \) and \( \kappa \) be as in Lemma 2. Let \( d \) be the degree of \( F \) and \( \lambda_{\min}, \lambda_{\max}, \) and \( \Lambda \) the minimal, maximal, and sum of the Lyapunov exponents of \( F \).

Choose a set \( \hat{A} \subset (\hat{X}, F) \) and \( r_0, \kappa_0 > 0 \) so that
\[ \hat{A} \subset \{ \hat{x} \in (\hat{X}, F) : r(\hat{x}) \geq r_0 \text{ and } \kappa(\hat{x}) \leq \kappa_0 \}, \]
\( \pi_0(\hat{A}) \) has compact closure in \( \mathbb{C}^n \) and \( \hat{\mu}(\hat{A}) > 0 \). Let \( \hat{Y} \subset (\hat{X}, F) \) be the set of all points whose forward orbit under \( F \) lands in \( \hat{A} \) infinitely often. By ergodicity, \( \hat{\mu}(\hat{Y}) = 1 \). Let \( Y = \pi_0(\hat{Y}) = \{ x_0 : \hat{x} \in \hat{Y} \} \), so \( \mu(Y) = 1 \), and let \( A_i = \pi_{-i}(\hat{A}) \).

Observe that
\[ Y = \bigcap_{l \geq 0} \bigcup_{m \geq l} A_m. \]

We will show that the Hausdorff dimension of \( Y \) is bounded above by \( 2n - 2 + \frac{\log d}{\lambda_0} + \frac{4\varepsilon}{\lambda_0} \), where \( \lambda_0 = \max\{\lambda_{\max}, \log d\} \). As \( Y \) has full measure and \( \varepsilon \) is arbitrary, this will prove the theorem.

For a ball \( B \) in \( \mathbb{C}^n \), let \( \frac{1}{2}B \) denote a concentric ball with half the radius. Let \( \Sigma \) denote a finite collection of balls \( B \) of radius \( r_0 \) so that the balls \( \frac{1}{2}B \) cover \( A_0 \). For each point \( y \in A_m \), select \( \hat{y} \in \hat{A} \) so that \( y = \pi_{-m}(\hat{y}) \). Choose element \( B \) of \( \Sigma \) so that \( \pi_0(\hat{y}) \) lies in \( \frac{1}{2}B \). Let \( B_y \) be the preimage of \( F^{-m}B \) containing \( y \). The collection of these \( B_y \) for all \( y \in A_m \) defines the finite cover \( \Sigma_m \) of \( A_m \).
If $\sigma$ is the number of elements in $\Sigma$, then the number of elements of $\Sigma_m$ is no greater than $\sigma d^{mn}$. Let $\lambda_0 = \max\{\log d, \lambda_{\text{max}}\}$. We will establish the following two properties of the cover $\Sigma_m$:

(I) The union $\bigcup_{B \in \Sigma_m} B$ contains an $\frac{r_0}{4\kappa_0} e^{-m(\lambda_0 + \varepsilon)}$-neighborhood of $A_m$, and

(II) $\text{Vol} \left( \bigcup_{B \in \Sigma_m} B \right) \leq \sigma d^{mn} \kappa_0 e^{-m(2\lambda_0 - \varepsilon)}$.

Observe first that for each $y \in A_m$, the set $B_y \in \Sigma_m$ contains a ball of radius $\frac{r_0}{4\kappa_0} e^{-m(\lambda_{\text{max}} + \varepsilon)}$ around $y$ by Lemma 2. Of course, $\lambda_{\text{max}} \leq \lambda_0$, giving (I).

To establish (II), we observe that as $F^m B_y \subset B(\pi_0(\hat{y}), r_0)$ for each $B_y \in \Sigma_m$, Lemma 2(b) implies that

$$\text{Vol} B_y \leq \kappa_0 e^{-m(2\lambda - \varepsilon)}.$$  

Summing over the volumes of all elements in $\Sigma_m$, we write

$$\text{Vol} \left( \bigcup_{B \in \Sigma_m} B \right) \leq \sigma d^{mn} \kappa_0 e^{-m(2\lambda - \varepsilon)}.$$  

By (4), $\Lambda$ is bounded below by $\frac{n+1}{2} \log d$, and by (3), each Lyapunov exponent is bounded below by $\frac{1}{2} \log d$. Combining these gives $\Lambda \geq \frac{n-1}{2} \log d + \lambda_0$ and we obtain statement (II).

We define a covering $\mathcal{M}_m$ of $A_m$ to be the collection of all mesh cubes of edge length $\frac{r_0}{2\sqrt{2}n(\kappa_0)} e^{-m(\lambda_0 + \varepsilon)}$ which intersect $A_m$. Let $c = \frac{1}{2\sqrt{2}} \frac{r_0}{\kappa_0}$. By property (I), each cube is contained in an element of $\Sigma_m$. The number of cubes in $\mathcal{M}_m$ is bounded above by the volume $\text{Vol} \left( \bigcup_{B \in \Sigma_m} B \right)$ divided by the volume of each cube. That is,

$$|\mathcal{M}_m| \leq \frac{\sigma d^{mn} \kappa_0 e^{-m(2\lambda_0 - \varepsilon)}}{c^{2n} e^{-2mn(\lambda_0 + \varepsilon)}} = \frac{\sigma \kappa_0}{c^{2n}} \frac{d^{m} e^{2(n-1)m\lambda_0} e^{(2n+1)m\varepsilon}}{e^{(2n-2)} d^{\log d/\lambda_0 + 4\varepsilon/\lambda_0}}.$$  

We now show that Hausdorff measure of $Y$ in dimension $2n - 2 + \frac{\log d}{\lambda_0} + \frac{4\varepsilon}{\lambda_0}$ is finite, completing the proof. Fix $\delta > 0$. Choose $l \geq 0$ so that the mesh cubes in $\mathcal{M}_m$ are of diameter $\delta_m \leq \delta$ for each $m \geq l$. The union of the elements of $\mathcal{M}_m$ for $m \geq l$ covers $Y$. Therefore,

$$m_{2n-2 + \frac{\log d}{\lambda_0} + \frac{4\varepsilon}{\lambda_0}}(Y) \leq \sum_{m \geq l} |\mathcal{M}_m|(\delta_m)^{2n-2 + \frac{\log d}{\lambda_0} + \frac{4\varepsilon}{\lambda_0}}$$

$$\leq C \sum_{m \geq l} d^m e^{2(n-1)m\lambda_0} e^{(2n+1)m\varepsilon} e^{-m(\lambda_0 + \varepsilon)} (2n-2 + \frac{\log d}{\lambda_0} + \frac{4\varepsilon}{\lambda_0})$$

$$= C \sum_{m \geq l} e^{-m\varepsilon} \left( \frac{1}{2n-2 + \frac{\log d}{\lambda_0} + \frac{4\varepsilon}{\lambda_0}} \right)$$

$$\leq C \sum_{0}^{\infty} e^{-m\varepsilon} < \infty$$

$$\square$$
References

[BD99] Jean-Yves Briend and Julien Duval, Exposants de Liapounoff et distribution des points périodiques d’un endomorphisme de $\mathbb{C}^k$, Acta Math. 182 (1999), no. 2, 143–157. MR 2000f:32023

[Bed93] Eric Bedford, Survey of pluri-potential theory. Several complex variables (Stockholm, 1987/1988), Princeton Univ. Press, Princeton, NJ, 1993, pp. 48–97. MR 94b:32014

[BJ00] Eric Bedford and Mattias Jonsson, Dynamics of regular polynomial endomorphisms of $\mathbb{C}^k$, Amer. J. Math. 122 (2000), no. 1, 153–212. MR 2001c:32012

[BJ02] I. Binder and P. Jones, In preparation, 2002.

[BLS93] Eric Bedford, Mikhail Lyubich, and John Smillie, Polynomial diffeomorphisms of $\mathbb{C}^2$. IV. The measure of maximal entropy and laminar currents, Invent. Math. 112 (1993), no. 1, 77–125. MR 94g:32035

[Bou87] J. Bourgain, On the Hausdorff dimension of harmonic measure in higher dimension, Invent. Math. 87 (1987), no. 3, 477–483. MR 88b:31004

[Bro65] Hans Brolin, Invariant sets under iteration of rational functions, Ark. Mat. 6 (1965), 103–144 (1965). MR 33 #2805

[BT87] Eric Bedford and B. A. Taylor, Fine topology, ˇSilov boundary, and $(dd^c)^n$, J. Funct. Anal. 72 (1987), no. 2, 225–251. MR 88g:32033

[Car85] L. Carleson, On the support of harmonic measure for the sets of Cantor type, Ann. Acad. Sci. Fenn. 10 (1985), 113–123.

[CFS82] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, Ergodic theory, Springer-Verlag, New York, 1982, Translated from the Russian by A. B. Sosinski. MR 87f:28019

[CJ92] Lennart Carleson and Peter W. Jones, On coefficient problems for univalent functions and conformal dimension, Duke Math. J. 66 (1992), no. 2, 169–206. MR 93c:30022

[FS95] John Erik Fornaess and Nessim Sibony, Complex dynamics in higher dimension. II, Modern methods in complex analysis (Princeton, NJ, 1992), Princeton Univ. Press, Princeton, NJ, 1995, pp. 135–182. MR 97g:32033

[FS01], Complex dynamics in higher dimension, Preprint, 2001.

[FW88] Peter W. Jones and Thomas H. Wolff, Hausdorff dimension of harmonic measures in the plane, Acta Math. 161 (1988), no. 1-2, 131–144.

[Kli91] Maciej Klimek, Pluripotential theory, The Clarendon Press Oxford University Press, New York, 1991, Oxford Science Publications. MR 93h:32021

[Lju83] M. Ju. Ljubich, Entropy properties of rational endomorphisms of the Riemann sphere, Ergodic Theory Dynam. Systems 3 (1983), no. 3, 351–385. MR 85k:58049

[LY85] F. Ledrappier and L.-S. Young, The metric entropy of diffeomorphisms. II. Relations between entropy, exponents and dimension, Ann. of Math. (2) 122 (1985), no. 3, 540–574. MR 87i:58101b

[Mak85] N. G. Makarov, On the distortion of boundary sets under conformal mappings, Proc. London Math. Soc. (3) 51 (1985), no. 2, 369–384.

[Man84] Anthony Manning, The dimension of the maximal measure for a polynomial map, Ann. of Math. (2) 119 (1984), no. 2, 425–430. MR 85i:58068

[Mn88] Ricardo Mañe, The Hausdorff dimension of invariant probabilities of rational maps, Dynamical systems, Valparaiso 1986, Springer, Berlin, 1988, pp. 86–117. MR 90j:58073

[Oks81] Bernt Oksendal, Brownian motion and sets of harmonic measure zero, Pacific J. Math. 95 (1981), no. 1, 179–192. MR 83c:60106

[Ose68] V. I. Oseledec, A multiplicative ergodic theorem. Characteristic Lyapunov, exponents of dynamical systems, Trudy Moskov. Mat. Obšč. 19 (1968), 179–210. MR 39 #1629

[Prz85] Feliks Przytycki, Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map, Invent. Math. 80 (1985), no. 1, 161–179. MR 86g:30035
[Ste70] Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, N.J., 1970. MR 44 #7280

[Wol93] Thomas H. Wolff, *Plane harmonic measures live on sets of $\sigma$-finite length*, Ark. Mat. 31 (1993), no. 1, 137–172.

[Wol95] _______, *Counterexamples with harmonic gradients in $\mathbb{R}^3$*, Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991), Princeton Univ. Press, Princeton, NJ, 1995, pp. 321–384. MR 95m:31010

I. Binder, Department of Mathematics, University of Illinois, Urbana-Champaign, 1409 W. Green Street (MC-382), Urbana, Illinois 61801, USA
*E-mail address*: ilia@math.uiuc.edu

L. DeMarco, Harvard University, Department of Mathematics, One Oxford St # 325, Cambridge, MA 02138, USA
*E-mail address*: demarco@math.harvard.edu