Application of Tomita-Takesaki theory in algebraic euclidean field theories

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Abstract

The construction of the known interacting quantum field theory models is mostly based on euclidean techniques. The expectation values of interesting quantities are usually given in terms of euclidean correlation functions from which one should be able to extract information about the behavior of the correlation functions of the Minkowskian counterpart.

We think that the C*-algebraic approach to euclidean field theory gives an appropriate setup in order to study structural aspects model independently. A previous paper deals with a construction scheme which relates to each euclidean field theory a Poincaré covariant quantum field theory model in the sense of R. Haag and D. Kastler.

Within the framework of R. Haag and D. Kastler, the physical concept of PCT symmetry and spin and statistics is related to the Tomita-Takesaki theory of von Neumann algebras and this important aspects has been studied by several authors.

We express the PCT symmetry in terms of euclidean reflexions and we explicitly identify the corresponding modular operator and the modular conjugation of the related Tomita-Takesaki theory. Locality, wedge duality, and a geometric action of the modular group of the von Neumann algebra of observables, localized within a wedge region in Minkowski space, are direct consequences.
1 Introduction

Concerning the known examples for non-free quantum field theory models, the construction of them, by means of euclidean techniques, is the most successful method which is known. Not surprisingly, most of the interesting quantities of such a model are explicitly given only in terms of euclidean correlation functions. It is therefore natural to ask the following question:

Given a euclidean field theory from which a quantum field theory can be constructed. Which properties of the quantum field theory can directly read off from the euclidean data?

This motivates the development of tools which analyze structural aspects of euclidean field theories in a systematic manner and we think that the C*-algebraic approach of euclidean field theory gives an appropriate setup in order to follow this program. Analogously to the famous Osterwalder-Schrader Theorem [27], it can be shown [28] that to each euclidean field theory, formulated in the C*-algebraic framework, a quantum field theory model in the sense of R. Haag and D. Kastler [21, 22] can be associated. We give a brief description of the corresponding construction scheme later.

Based on this work, a tool for investigating the high energy behavior of a quantum field theory model, by only looking at its euclidean counterpart, has already been discussed in [30]. Compared to the scaling limit analysis of D. Buchholz and R. Verch [8, 9, 10, 11, 12] one finds the expected results, namely that the high energy behavior of the euclidean model reflects the high energy behavior of the corresponding quantum field theory.

In addition to that, the C*-algebraic point of view provides new strategies for constructing euclidean field theory models, as it is laid out in [29, 31].

Within the framework of algebraic quantum field theory, the physical concept of PCT symmetry and spin and statistics is deeply linked to the Tomita-Takesaki theory of von Neumann algebras and has been studied in several papers, see for example [2, 3, 4, 20, 25, 36, 35, 5, 7]. Moreover, Tomita-Takesaki theory might also be important for constructive purposes as it is, for example, proposed in [32].

From the point of view of euclidean field theory, it would be desirable to express the modular data of a wedge algebra of the quantum field theoretical counterpart directly in terms of euclidean correlation functions. This might be of importance, since according to the discussion in [32], there is hope, that an analysis of the modular data related to the euclidean field theory models, which are constructed in [29, 31] by an abstract procedure, can be used as a tool in order to decide whether a theory describes interaction.
Organization of the paper. In the second part of the present section, we make some preliminary remarks on the algebraic approach to euclidean field theory in order to introduce notations and conventions which are used. Starting from a given euclidean field, we present in Section 2 the main results which state in particular that the square-root of the modular operator of a wedge algebra of the quantum field theory, constructed from the euclidean data, can explicitly be identified with a particular euclidean rotation with rotation angle \( s = \pi \). Furthermore, the corresponding modular conjugation is a PCT symmetry which can be expressed in terms of euclidean reflexions. We close the paper by Section 3, mentioning some work in progress. We feel obliged to postpone most of the technical details and the proofs to the appendix in order to keep the paper more readable.

Preliminary remarks on the algebraic approach to euclidean field theory. The starting point within the C*-algebraic approach to euclidean field theory is a so called euclidean net of C*-algebras \((\mathfrak{B}, \beta)\). Such a net is an inclusion preserving prescription \( \mathcal{U} \mapsto \mathfrak{B}(\mathcal{U}) \subset \mathfrak{B} \), which assigns to each bounded convex region \( \mathcal{U} \) in \( \mathbb{R}^d \) a C*-algebra \( \mathfrak{B}(\mathcal{U}) \). This assignment has to fulfill several assumptions, according to physical principles.

Two operators commute if they are localized in disjoint regions, more specifically, if the intersection \( \mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset \) is empty, then the commutator \([b_1, b_2] = 0\) vanishes for all operators \( b_1 \in \mathfrak{B}(\mathcal{U}_1) \) and \( b_2 \in \mathfrak{B}(\mathcal{U}_2) \). The net \( \mathcal{U} \mapsto \mathfrak{B}(\mathcal{U}) \) is euclidean covariant, i.e. there exists a group homomorphism \( \beta \) form the euclidean group into the automorphism group of \( \mathfrak{B} \) such that for a euclidean transformation \( h \in E(d) \) the algebra of a local region \( \mathcal{U} \) is mapped, via \( \beta_h \), onto the algebra of the transformed region \( h\mathcal{U} : \beta_h \mathfrak{B}(\mathcal{U}) = \mathfrak{B}(h\mathcal{U}) \).

We have to mention that, although the analogy between the C*-algebraic approach to euclidean field theory and the Haag-Kastler framework for quantum field theory is obvious, the euclidean C*-algebras have to be interpreted in a different manner. Within the Haag-Kastler program, the dynamics of a given quantum field theory is usually contained in the relative inclusion of local algebras, whereas a euclidean net of C*-algebras only describes kinematical aspects. The dynamics of euclidean field theory model is encoded in the choice of a particular (euclidean invariant and reflexion positive regular) state on the corresponding euclidean C*-algebra:

Euclidean invariance: A state \( \eta \) on \( \mathfrak{B} \) is called euclidean invariant if for each \( h \in E(d) \) the identity \( \eta \circ \beta_h = \eta \) holds true.
**Reflexion positivity:** A state $\eta$ on $\mathcal{B}$ is called reflexion positive if exists a euclidean direction $e \in S^{d-1}$ such that the sesquilinear form

$$\mathcal{B}(H_e) \otimes \mathcal{B}(H_e) \ni b_0 \otimes b_1 \mapsto \langle \eta, j_e(b_0)b_1 \rangle$$

is positive semi definite. Here, $H_e$ is the half space $\mathbb{R}^d_+ + e^\perp$ with respect to a given euclidean direction $e \in S^{d-1}$, $j_e$ is the anti-linear involution which is given by $j_e(b) = \beta e^\theta_e(b^*)$ where $\theta_e$ is the euclidean reflexion $\theta_e x = x - 2\langle e, x \rangle e$.

**Regularity:** A state $\eta$ on $\mathcal{B}$ is called regular if for each $b_0, b_1, b_2 \in \mathcal{B}$ the map

$$h \mapsto < \eta, b_0 \beta_h(b_1)b_2 >$$

is continuous.

A triple $(\mathcal{B}, \beta, \eta)$ consisting of a euclidean net of C*-algebras $(\mathcal{B}, \beta)$ and a euclidean invariant reflexion positive regular state $\eta$ is called a euclidean field.

**From euclidean field theory to quantum field theory.** We briefly describe here, as it has carried out in [28], how to construct from a given euclidean field a quantum field theory.

**Step 1:** According to the reflexion positivity, for a given direction $e \in S^{d-1}$ there exists a Hilbert space $\mathcal{H}$ and a linear map

$$\Psi : \mathcal{B}(H_e) \mapsto \mathcal{H}$$

which is uniquely determined by

$$\langle \Psi[b_0], \Psi[b_1] \rangle = < \eta, j_e(b_0)b_1 >$$

and we have a distinguished vector $\Omega := \Psi[1]$, the vacuum vector. Following the analysis presented in [18], a unitary strongly continuous representation of the Poincaré group $U$ on $\mathcal{H}$ can be constructed (see also [33, 28]). The vacuum vector $\Omega$ is invariant under the action of $U$ and, in addition to that, the spectrum of the the generator of the translations $x \mapsto U(x)$ is contained in the closed forward light cone $V_+ = \{ x \in \mathbb{R}^{1,d-1} | x^2 \geq 0; x_0 \geq 0 \}$. A more detailed description of the construction of $U$ is given in Appendix A.
Step 2: For a subset $\mathcal{V}$ of the hyperplane $e^\perp$, perpendicular to $e$, we introduce the algebra $\mathfrak{B}(\mathcal{V})$ of time zero operators, localized in $\mathcal{V}$, by the intersection

$$\mathfrak{B}(\mathcal{V}) := \bigcap_{s \in \mathbb{R}_+} \mathfrak{B}([0, s)e \times \mathcal{V}).$$

The algebra $\mathfrak{B}(e^\perp)$ is then the $C^*$-algebra which is generated by all local time-zero algebras $\mathfrak{B}(\mathcal{V})$ with $\mathcal{V} \subset e^\perp$.

There exists a $*$-representation $\pi$ of the time-zero algebra $\mathfrak{B}(e^\perp)$ on $\mathcal{H}$, which is uniquely determined by the relation

$$\pi(b)\Psi[b_1] = \Psi[bb_1]$$

for each $b_1 \in \mathfrak{B}(\mathcal{H}_c)$.

The algebra $\pi(\mathfrak{B}(e^\perp))$ can be regarded as the Cauchy data of the quantum field theory model by identifying the hyperplane $e^\perp$ with the spacelike hyperplane $x^0 = 0$ in Minkowski space $\mathbb{R}^{1,d-1}$.

For a double cone $\mathcal{O}$ (a causally complete bounded set in Minkowski space), we define $\mathfrak{A}(\mathcal{O})$ to be the von Neumann algebra on $\mathcal{H}$ which is generated by all operators $\Pi[g,b] := U(g)\pi(b)U(g)^*$ with $b \in \mathfrak{B}(\mathcal{V})$, such that $g\mathcal{V} \subset \mathcal{O}$ for the Poincaré transformation $g$.

The prescription

$$\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$$

is an isotonous net of $C^*$-algebras. We define a group homomorphism $\alpha$ from the Poincaré group to the automorphism group of $\mathfrak{A}$ by

$$\alpha_g := \text{Ad}(U(g))$$

which is, by construction, covariant, i.e. $\alpha_g \mathfrak{A}(\mathcal{O}) = \mathfrak{A}(g\mathcal{O})$ for each double cone $\mathcal{O}$ and for each Poincaré transformation $g$.

The state $\omega$ on $\mathfrak{A}$, which is given by

$$\langle \omega, a \rangle = \langle \Omega, a\Omega \rangle$$

has the following properties:

Poincaré invariance: For each Poincaré transformation $g$ the identity $\omega \circ \alpha_g = \omega$ holds true which is a consequence of the invariance of $\Omega$ under $U(g)$.
**Positivity of the energy:** For each \( a_1, a_2 \in \mathfrak{A} \) and for each test function \( f \in S(\mathbb{R}^{1,d-1}) \) on Minkowski space whose Fourier transform \( \hat{f} \) has support in the complement of the closed forward light cone \( \bar{V}_+ \), the identity

\[
\int dx \, f(x) < \omega, a_1 a_x a_2 > = 0
\]

is valid. This is nothing else but expressing the fact that the generators of the translations \( x \mapsto U(x) \) have joint spectrum in the closed forward light cone.

**Locality:** If \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are spacelike separated regions, then the commutator \( \left[ a_1, a_2 \right] = 0 \) vanishes for \( a_1 \in \mathfrak{A}(\mathcal{O}_1) \) and \( a_2 \in \mathfrak{A}(\mathcal{O}_2) \).

**Remark.** Whereas Poincaré invariance as well as the positivity of the energy follow directly from the construction of the representation \( U \), the fact that locality is fulfilled is not directly visible. The proof, carried out in [28], is quite lengthy. We shall see later, that the use of Tomita-Takesaki theory for wedge algebras, leads to a more elegant proof of locality.

### 2 Modular data for wedge algebras as geometric operations in euclidean space

The present section is concerned with the discussion of the Tomita-Takesaki theory of wedge algebras of the net \( \mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) \) of von Neumann algebras emerging from a given euclidean field theory model. We present here the main results of the paper, which relate the modular operator and the modular conjugation of a particular wedge algebra to geometric operations in euclidean space. The technical details and the proofs are postponed to the appendix.

#### 2.1 KMS states associated with wedge algebras

For any wedge region \( \mathcal{W} \) in Minkowski space the wedge algebra with respect to \( \mathcal{W} \) is the von Neumann algebra

\[
\mathfrak{A}(\mathcal{W}) := \bigvee_{\mathcal{O} \subset \mathcal{W}} \mathfrak{A}(\mathcal{O})
\]

generated by all local von Neumann algebras \( \mathfrak{A}(\mathcal{O}) \) with \( \mathcal{O} \subset \mathcal{W} \).

Particular wedge algebras are related to euclidean directions \( e_0 \in S^{d-1} \) which are perpendicular \( e_0 \perp e \) to \( e \). An application of the boosts
in \((\varepsilon, \varepsilon_0)\) direction to the half hyperplane \(P_{(\varepsilon,\varepsilon_0)} := H_{\varepsilon} \cap H_{-\varepsilon} \cap H_{\varepsilon_0}\) yields the wedge region

\[
\mathcal{W}_{(\varepsilon,\varepsilon_0)} := \bigcup_{t \in \mathbb{R}} \exp(tB_{(\varepsilon,\varepsilon_0)})P_{(\varepsilon,\varepsilon_0)} .
\]

Obviously, the spacelike complement \(\mathcal{W}'_{(\varepsilon,\varepsilon_0)}\) of the wedge \(\mathcal{W}_{(\varepsilon,\varepsilon_0)}\) is just the wedge \(\mathcal{W}_{(-\varepsilon,\varepsilon_0)}\) with respect to the reflected direction \(-\varepsilon_0\). Writing \(\alpha_{(\varepsilon,\varepsilon_0)} : \mathbb{R} \to \text{Aut}(\mathfrak{A})\) for the one-parameter automorphism group of boosts

\[
\alpha_{(\varepsilon,\varepsilon_0),t} := \text{Ad} \left[ U_{(\varepsilon,\eta)}(\exp(tB_{(\varepsilon,\varepsilon_0)})) \right]
\]

we obtain a \(W^\ast\)-dynamical system \((\mathfrak{A}(\mathcal{W}_{(\varepsilon,\varepsilon_0)}), \alpha_{(\varepsilon,\varepsilon_0)})\) together with a \(\alpha_{(\varepsilon,\varepsilon_0)}\)-invariant state

\[
\omega_{(\varepsilon,\varepsilon_0)} := \omega|_{\mathfrak{A}(\mathcal{W}_{(\varepsilon,\varepsilon_0)})} ,
\]

the restriction of the vacuum state \(\omega\) to the corresponding wedge algebra.

**Theorem 2.1**: For each direction \(\varepsilon_0 \perp \varepsilon\), the state \(\omega_{(\varepsilon,\varepsilon_0)}\) is a KMS state at inverse temperature \(\beta = 2\pi\) with respect to the \(W^\ast\)-dynamical system

\[
(\mathfrak{A}(\mathcal{W}_{(\varepsilon,\varepsilon_0)}), \alpha_{(\varepsilon,\varepsilon_0)}) .
\]

The proof of Theorem 2.1 can be obtained in complete analogy to the analysis of [23] and we give a version of it within the Appendix B.

### 2.2 The PCT symmetry and complex Lorentz boosts as geometric operations in euclidean space

**The \(\varepsilon_0\)-PCT operator.** For a euclidean direction \(\varepsilon_0 \in S^{d-1}\) which is perpendicular to \(\varepsilon\), the euclidean reflexion

\[
\theta_{\varepsilon_0} : x \mapsto x - 2\langle \varepsilon_0, x \rangle \varepsilon
\]

is contained in the stabilizer group of \(\varepsilon\) and hence it gives rise to an anti-unitary operator \(\mathcal{J}_{(\varepsilon,\varepsilon_0)}\), called the \(\varepsilon_0\)-PCT operator. It is defined according to the prescription

\[
\mathcal{J}_{(\varepsilon,\varepsilon_0)} \Psi[b] := \Psi[j_{\varepsilon_0}(b)]
\]

and it has the geometric property (see Appendix B for the proof):

**Proposition 2.2**: For each \(\varepsilon_0 \in S^{d-1} \cap P_{\varepsilon}\) the identity

\[
\mathcal{J}_{(\varepsilon,\varepsilon_0)} \mathfrak{A}(\mathcal{W}_{(\varepsilon,\varepsilon_0)}) \mathcal{J}_{(\varepsilon,\varepsilon_0)} = \mathfrak{A}(\mathcal{W}_{(-\varepsilon,\varepsilon_0)})
\]

is valid.
Complex Lorentz boosts as euclidean rotations. For a direction \( e_0 \in S^{d-1}, e \perp e_0 \), the one-parameter group \( t \mapsto U(\exp(tB(e,e_0))) \) can be extended analytically to complex parameters on a appropriate dense subspace of \( \mathcal{H} \). This is based on remarkable facts which have been established by J. Fröhlich [17], on one hand, and by A. Klein and L. J. Landau [24], on the other hand.

The generator \( L(e,e_0) \in \mathfrak{so}(d) \) of the euclidean rotations within the \( e-e_0 \) plane yields a one-parameter group of automorphisms on \( \beta_{(e,e_0,s)} := \beta_{\exp(sL(e,e_0))} \).

For each \( s \in (-\pi/2, \pi/2) \) an operator \( V(e,e_0)(s) \) is uniquely determined by

\[
V(e,e_0)(s)\Psi[b] = \Psi[\beta_{(e_1,e,s)}b]
\]

for each \( b \in \mathfrak{B}(\mathfrak{H}_e) \) with \( \beta_{(e_1,e,s)}b \in \mathfrak{B}(\mathfrak{H}_e) \). On an appropriate dense subspace \( D \subset \mathfrak{H} \), the operator \( V(e,e_0)(s) \) is related to the one-parameter group \( t \mapsto U(\exp(tB(e,e_0))) \) by

\[
V(e,e_0)(s) = U(\exp(isB(e,e_0)))
\]

2.3 The modular operator and the modular conjugation for wedge algebras.

From the fact that \( \omega_{(e,e_0)} \) is a KMS state for the \( \mathcal{W}^* \)-dynamical system \( (\mathfrak{A}(\mathcal{W}(e,e_0)), \alpha_{(e,e_0)}) \), one concludes (e.g. [1]) that the vector \( \Omega \) is separating for the algebra \( \mathfrak{A}(\mathcal{W}(e,e_0)) \).

Therefore there exists a modular operator \( \Delta_{(e,e_0)} \) and a modular conjugation \( J_{(e,e_0)} \) with respect to the pair \( (\mathfrak{A}(\mathcal{W}(e,e_0)), \Omega) \). The subsequent theorem, which is proven in the Appendix, states that the modular data can be expressed in terms of geometric actions in euclidean space:

**Theorem 2.3**: For the modular operator \( \Delta_{(e,e_0)} \) and the modular conjugation \( J_{(e,e_0)} \) the identities

\[
\Delta^{1/2}_{(e,e_0)} = V(e,e_0)(\pi)
\]

\[
J_{(e,e_0)} = J_{(e,e_0)}
\]

hold true for each \( e_0 \in S^{d-1} \cap P_e \).

Locality and wedge duality. The statement of Theorem 2.3 has some direct implications. As already mentioned, a proof for locality of the net \( \mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) \) has already been established in [28]. The proof of
Theorem 2.3, which we postpone to the appendix, do not make use of this fact and by means of Theorem 2.3 we derive an independent proof for locality which is, compared to [28], much more elegant and straightforward.

**Corollary 2.4 :** The net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ fulfills wedge duality, i.e. for each wedge region $\mathcal{W}$ the identity

$$\mathfrak{A}(\mathcal{W})' = \mathfrak{A}(\mathcal{W}')$$

is valid. In particular, locality holds true.

**Proof.** As a consequence of Theorem 2.3 we get

$$\mathfrak{A}(\mathcal{W}(e,e_0))' = \mathfrak{A}(\mathcal{W}(e,-e_0))$$

and the Poincaré covariance of the net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ implies wedge duality for each wedge $\mathcal{W}$. Therefore, the net fulfills locality since for two spacelike separated double cones $\mathcal{O}_1 \subset \mathcal{O}_2'$ there is a wedge $\mathcal{W}$ with $\mathcal{O}_1 \in \mathcal{W}$ and $\mathcal{O}_2 \in \mathcal{W}'$. Now wedge duality implies for operators $a_1 \in \mathfrak{A}(\mathcal{O}_1)$ and $a_2 \in \mathfrak{A}(\mathcal{O}_2)$ that $[a_1, a_2] = 0$. $\square$

### 3 Concluding remarks

We have shown, that the modular operator and the modular conjugation of a particular wedge algebra $\mathfrak{A}(\mathcal{W})$, associated to a given euclidean field $(\mathfrak{F}, \beta, \eta)$, do not only have the meaning as geometric action on Minkowski space in terms of Lorentz boosts and reflexions, they also can be identified with geometric operations in euclidean space, namely particular euclidean rotations and euclidean reflexions. This fact can be used to conclude wedge duality for the net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$. As a consequence we get, compared to the analysis carried out in [28], an improved method in order to prove locality.

Keeping in mind that the minkowskian analogue of the euclidean $d$-sphere $S^d \subset \mathbb{R}^{d+1}$ is the de Sitter space, it should be possible, by exploring the analytic structure of de Sitter space, to construct from a given euclidean field theory $(\mathfrak{F}, \beta, \eta)$ on the sphere $S^d$ a quantum field theory $(\mathfrak{A}, \alpha, \omega)$ in de Sitter space (A forthcoming preprint is in preparation). For an example, we refer the reader to the work of R. Figari, R. Høegh-Krohn, and C. R. Nappi [14]. According to Theorem 2.1, we conjecture that the reconstructed state $\omega$ fulfills the so called geodesic KMS condition, in the sense of H. J. Borchers and D. Buchholz [10], i.e. for any geodesic observer the state $\omega$ looks like an equilibrium state. In order to prove locality for the constructed quantum field theory in de Sitter space, the method of [28], can not directly be applied since
here the euclidean translations are used. Establishing the analogous results of Theorem 2.3 for the theory in de Sitter space, locality would also follow here.

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A Construction of a representation of the Poincaré group

In order to keep the paper self contained, we review here the construction procedure of a representation of the Poincaré group from a given eucliden field \((\mathfrak{B}, \beta, \eta)\) in more detail.

We first give a list of facts which are consequences of the axioms for a euclidean field.

**Fact 1:** For each \(e \in S^d\) there is a strongly continuous one-parameter semi-group of contractions \(T_e\) with positive generator \(H_e \geq 0\) which is given according to

\[ T_e(s)\Psi[b] = \Psi[\beta_e b] \]

for \(s > 0\) and for \(b \in \mathfrak{B}(\mathfrak{H}_e)\).

**Fact 2:** For \(e \in S^{d-1}, w \in (0, \pi/2)\), the conic region \(\Gamma(e, w)\) is defined to be the \(O_e(d-1)\) invariant cone in \(e\) direction with opening angle \(w\). Moreover, for a pair euclidean time directions \(e, e_1 \in S^{d-1}, e \perp e_1\), the generator of the rotations within the \((e, e_1)\) plane is denoted by \(L(e, e_1) \in \mathfrak{s}o(d)\). Hence the corresponding one-parameter group yields a one-parameter group of automorphisms on \(\mathfrak{B}\)

\[ \beta_{(e, e_1, s)} := \exp(sL(e, e_1)) \]

and a local one-parameter group

\[ V_{(e, e_1)} = \{ V_{(e, e_1)}(s) | s \in (-\pi/2, \pi/2) \} \]

of selfadjoint operators. For each \(s \in (-\pi/2, \pi/2)\) the domain of \(V_{(e, e_1)}(s)\) is

\[ \mathcal{D}(e, |s|) := \Psi[\mathfrak{B}(\Gamma(e, \pi/2 - |s|))] \]

for each \(s \in (-\pi/2, \pi/2)\). The operator \(V_{(e, e_1)}(s)\) is uniquely determined by

\[ V_{(e, e_1)}(s)\Psi[b] = \Psi[\beta_{(e_1, e, s)} b] \]

and there exists an anti-selfadjoint operator \(B_{(e, e_1)}\) on \(\mathcal{H}\) such that

\[ V_{(e, e_1)}(s) = \exp(isB_{(e, e_1)}) . \]

For each angle \(w \in (0, \pi/2)\), the vectors in \(\mathcal{D}(e, w)\) are analytic for \(B_{(e_1, e)}\). This remarkable facts have been established by J. Fröhlich [17], on one hand, and by A. Klein and L. J. Landau [24], on the other hand.
**Fact 3:** There is a unitary strongly continuous representation $W$ on $\mathcal{H}$ of the stabilizer subgroup $E_e(d-1)$ fulfilling

$$W(g)\Psi[b] = \Psi[\beta_gb]$$

for $g \in E_e(d-1)$ and for $b \in \mathfrak{B}(H_e)$.

**A representation of the Poincaré group.** By making use of the analysis of virtual representations [18] a strongly continuous unitary representation

$$U \in \text{Hom}[\mathbb{P}_+^1, U(\mathcal{H})]$$

of the Poincaré group can be constructed. The paper [18] exploits the facts, listed above, and proceeds in several steps:

**Step 1:** According to Fact 3 the stabilizer subgroup $E_e(d-1) \subset E(d)$ is represented by $W$ and we put

$$U(g) := W(g)$$

for each $g \in E_e(d-1)$.

**Step 2:** By using Fact 2 for each $e_1 \perp e$ there exists a anti-selfadjoint operator $B_{(e,e_1)}$ on $\mathcal{H}_{(e,e)}$ such that

$$V_{(e,e_1)}(s) = \exp(-isB_{(e,e_1)}) .$$

Let $B_{(e,e_1)} \in \mathfrak{o}(d-1,1)$ be the boost generator in $e_1 - e$ direction we define a unitary operator by

$$U(\exp(tB_{(e_1,e)})) := \exp(itB_{(e_1,e)}) .$$

**Step 3:** Finally, the translations with respect to the time-like direction which corresponds to the $e$-direction in euclidean space are represented by

$$U(te) := \exp(itH_e)$$

according to Fact 1.

**B Proof of Theorem 2.1**

The main steps of the proof can be performed in complete analogy to the the analysis of [23]. We consider a family of operators $b_1, \ldots, b_n$ which are contained in the time slice algebra, where $b_j \in \mathfrak{B}(V_j)$ is localized in a convex subset $V_j \subset H_{(e,e_0)}$. This implies that $\beta_{(e,e_0,s)}b_j \in$
for each $s \in (0, \pi)$ where we have chosen the condition $\exp(2\pi L_{(e,e_0)}) = 1$. We introduce the open subset in $\mathbb{R}^2$

$$I(\mathcal{Y}_j) := \{ (\tau, s) \in \mathbb{R}^2 | \forall e_2 : \exp(s L_{(e,e_0)}) \exp(\tau L_{(e,e_2)}) \mathcal{Y}_j \subset H_e \}$$

which contains in particular the set $\{0\} \times (0, \pi) \subset I(\mathcal{Y}_j)$. We put $V := V_{(e,e_0)}$ and $\Omega = \Psi[1]$. By introducing the operators

$$b_j(\tau_j) := V_{(e,e_j)}(\tau_j) \pi(b_j)V_{(e,e_j)}(-\tau_j)$$

and

$$b_j(\tau_j, s_j) := \beta(e,e_0, s_j) \beta(e,e_j, \tau_j)b_j$$

we obtain

$$V(s_k)b_k(\tau_k) \cdots V(s_1)b_1(\tau_1)\Omega$$

$$= V(s_k)b_k(\tau_k)V(-s_k)V(s_k + s_{k-1})b_{k-1}(\tau_{k-1})V(-s_k - s_{k-1})\cdots$$

$$\cdots V(s_1)b_1(\tau_1)\Omega$$

$$= V(s_k)b_k(\tau_k)V(-s_k)\cdots V(s_1 + \cdots + s_k)b_1(\tau_1)\Omega$$

$$= \Psi[b_k(\tau_k, s_k) \cdots b_1(\tau_1, s_1 + \cdots + s_k)] \ .$$

We compute for $s_1, \ldots, s_n \in \mathbb{R}_+$, $(\tau_j, s_n + \cdots + s_j) \in I(\mathcal{Y}_j)$ for $n \leq j \leq k + 1$ and $(\tau_i, s_k + \cdots + s_i) \in I(\mathcal{Y}_i)$ for $k \leq i \leq 1$:

$$(V(s_n)b_n(\tau_n) \cdots V(s_{k+1})b_{k+1}(\tau_{k+1})\Omega, V(s_k)b_k(\tau_k) \cdots$$

$$\cdots V(s_1)b_1(\tau_1)\Omega)$$

$$= \langle \eta, j_c[b_n(\tau_n, s_n) \cdots b_{k+1}(\tau_{k+1}, s_n + \cdots + s_{k+1})]$$

$$\times b_k(\tau_k, s_k) \cdots b_1(\tau_1, s_k + \cdots + s_1) > \rangle$$

$$= \langle \eta, b_{k+1}(-\tau_{k+1}, -s_n - \cdots - s_{k+1}) \cdots b_1(-\tau_0, -s_n)$$

$$\times b_k(\tau_k, s_k) \cdots b_1(\tau_1, s_k + \cdots + s_1) > \rangle .$$

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Furthermore, we find due to our periodicity condition $\exp(2\pi L) = 1$:

$$< \eta, b^*_k (-\tau_{k+1}, -s_n - \cdots - s_{k+1}) \cdots b^*_n (-\tau_n, -s_n) \times b_k (\tau_k, s_k) \cdots b_1 (\tau_1, s_1 + \cdots + s_1) >$$

$$= < \eta, b^*_k (-\tau_{k+1}, -\pi - s_n - \cdots - s_{k+1}) \cdots b^*_n (-\tau_n, -\pi - s_n) \times b_k (\tau_k, -\pi + s_k) \cdots b_1 (\tau_1, -\pi + s_1 + \cdots + s_1) >$$

$$= < \eta, b_k (\tau_k, -\pi + s_k) \cdots b_1 (\tau_1, -\pi + s_1 + \cdots + s_1)$$

$$\times b^*_k+1 (-\tau_{k+1}, -(\pi + s_n + \cdots + s_{k+1})) \cdots b^*_n (-\tau_n, -(\pi + s_n)) > .$$

In the last step we have used the locality of the euclidean net $\mathcal{B}$, i.e. operators which are localized in disjoint regions commute. According to the definition of $I(\mathcal{Y}_j)$ we have for $(\tau, s) \in I(\mathcal{Y}_j)$

$$\exp(sL_{(e,e_0)}) \exp(\tau L_{(e,e_j)}) \mathcal{Y}_j \subset H_e$$

and hence

$$\exp((-\pi + s)L_{(e,e_0)}) \exp(\tau L_{(e,e_j)}) \mathcal{Y}_j \subset H_{-e}$$

and therefore $(-\tau, \pi - s)) \in I(\mathcal{Y}_j)$ which implies

$$b_k (\tau_k, -\pi + s_k) \cdots b_1 (\tau_1, -\pi + s_1 + \cdots + s_1) \in \mathcal{B}(H_{-e})$$

$$b^*_k+1 (-\tau_{k+1}, \pi - (s_n + \cdots + s_{k+1})) \cdots b^*_n (-\tau_n, \pi - s_n) \in \mathcal{B}(H_e) .$$

Keeping in mind that we have

$$b_j (\tau_j)^* = V_{(e,e_j)} (-\tau_j) \pi (b^*_j V_{(e,e_j)})(\tau_j)$$

the identity

$$< V(s_n) b_n (\tau_n) \cdots V(s_{k+1}) b_{k+1} (\tau_{k+1}) \Omega, V(s_k) b_k (\tau_k) \cdots b_1 (\tau_1) \Omega >$$

$$= < V(\pi - (s_1 + \cdots + s_k)) b_1 (\tau_1)^* V(s_1) b_2 (\tau_2)^* \cdots V(s_{k-1}) b_k (\tau_k)^* \Omega, \times V(\pi - (s_{k+1} + \cdots + s_n)) b_{k+1} (\tau_{k+1}) \cdots V(s_{k+1}) b_{k+2} (\tau_{k+2})^* \cdots \cdots V(s_{n-1}) b_n (\tau_n)^* \Omega >$$

14
follows which expresses the KMS condition in the euclidean points. Finally, a straight forward application of the analysis of [23] proves the theorem. □

C Proof of Proposition 2.2

The \( e_0 \)-PCT operator \( J(e,e_0) \) commutes with the local one-parameter group \( V(e,e_1) \) for \( e_0 \perp e_1 \) and it fulfills the relation

\[
J(e,e_0) V(e,e_0)(s) J(e,e_0) = V(e,e_0)(-s)
\]

as easily can be verified. For a time-zero operator \( b \in \mathfrak{B}(e, P(e,e_0)) \), for a family of directions \( e = (e_0, \cdots, e_n) \) with \( e_i = e_j \) or \( e_i \perp e_j \), \( i,j = 0, \cdots, n \) and for \( t = (t_1, \cdots, t_n) \in \mathbb{R}^n \), we introduce an operator \( \Phi(e,e)[t,b] \in \mathfrak{A}(\mathcal{W}(e,e_0)) \), localized in the wedge \( \mathcal{W}(e,e_0) \):

\[
\Phi(e,e)[t,b] := \alpha(e,e_1,t_1) \cdots \alpha(e,e_n,t_n) \pi(b).
\]

The wedge algebra \( \mathfrak{A}(\mathcal{W}(e,e_0)) \) is generated by these operators and since

\[
J(e,e_0) \Phi(e,e)[t,b] J(e,e_0) = \Phi(e,e)[\sigma(t), j e_0 b]
\]

is contained in \( \mathfrak{A}(\mathcal{W}(e,-e_0)) \), the result follows, where \((\sigma t)_j = t_j\), if \( e_j \perp e_0 \), and \((\sigma t)_j = -t_j\), if \( e_j = e_0 \). □

D Proof of Theorem 2.3

By following the analysis of [23], we choose a family of operators \( b_1, \cdots, b_n \) which are contained in the time slice algebra, where \( b_j \in \mathfrak{B}(\mathcal{Y}_j) \) is localized in a convex subset \( \mathcal{Y}_j \subset H(e,e_0) \). By using the same notations as for the proof of Theorem 2.1, we obtain by putting \( V := V(\eta,e,e_0) \) and \( J := J(e,e_0) \):

\[
V(s_k) b_k(\tau_k) \cdots V(s_1) b_1(\tau_1) \Omega
= V(s_k) b_k(\tau_k)V(-s_k)V(s_k + s_{k-1}) b_{k-1}(\tau_{k-1})V(-s_k - s_{k-1}) \cdots
\]

\[
\cdots V(s_1) b_1(\tau_1) \Omega
= V(s_k) b_k(\tau_k)V(-s_k) \cdots V(s_1 + \cdots + s_k) b_1(\tau_1) \Omega
= \Psi[b_k(\tau_k, s_k) \cdots b_1(\tau_1, s_1 + \cdots + s_k)].
\]
We compute for \( s_1, \ldots, s_k \in \mathbb{R}_+ \) and \( (\tau_1, s_k + \cdots + s_i) \in I(\mathcal{V}_i) \) for \( k \leq i \leq 1 \):

\[
\mathcal{J} V(s_k) b_k(\tau_k) \cdots V(s_1) b_1(\tau_1) \Omega
\]

\[
= \Psi[j_{\mathcal{V}_0}(b_k(\tau_k, s_k) \cdots b_1(\tau_1, s_1 + \cdots + s_k))]
\]

\[
= \Psi[b_1(\sigma_1 \tau_1, \pi - (s_1 + \cdots + s_k))^* \cdots b_k(\sigma_k \tau_k, \pi - s_k)^*]
\]

\[
= V(\pi - s_1 + \cdots + s_k) b_1(\sigma_1 \tau_1) V(s_1) \cdots b_k(\sigma_k \tau_k)^* V(s_k) \Omega
\]

with \( \sigma_j = 1 \) if \( e_j \perp e_0 \) and \( \sigma_j = -1 \) if \( e_j = e \). Performing an analytic continuation within the parameter \( s_1, \ldots, s_k \) and \( \tau_1, \ldots, \tau_k \) and taking boundary values at \( s_j = \tau_j = 0 \) yields the relation (compare [23] as well as [18] and [28])

\[
\mathcal{J}(e, e_0) \prod_{j=1}^{k} \exp(t_j B(e, e_{j})) b_j \exp(-t_j B(e, e_{j})) \Omega
\]

\[
= V(\pi, e, e_0)(\pi) \prod_{j=1}^{k} \exp(t_j B(e, e_{j})) b_j \exp(-t_j B(e, e_{j})) \Omega
\]

which implies that the Tomita operator is

\[
J_{(e, e_0)} \Delta_{(e, e_0)}^{1/2} = \mathcal{J}(e, e_0) V(e, e_0)(\pi).
\]

Moreover, according to Theorem 2.1, the automorphism group

\[
\alpha_{(e, e_0)} : t \mapsto \text{Ad}[U(\exp(tB(e, e_0)))]
\]

maps \( \mathfrak{A}(\mathcal{W}(e, e_0)) \) into itself and the state

\[
\omega_{(e, e_0)} = \omega |_{\mathfrak{A}(\mathcal{W}(e, e_0))}
\]

is a KMS state at inverse temperature \( \beta = 2\pi \) and the theorem follows. □
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