Novel rotating hairy black hole in (2+1)-dimensions

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Abstract

We present some novel rotating hairy black hole metric in (2 + 1) dimensions, which is an exact solution to the field equations of the Einstein-scalar-AdS theory with a non-minimal coupling. The scalar potential is determined by the metric ansatz and consistency of the field equations and cannot be prescribed arbitrarily. In the simplified, critical case, the scalar potential contains two independent constant parameters, which are respectively related to the mass and angular momentum of the black hole in a particular way. As long as the angular momentum does not vanish, the metric can have zero, one or two horizons. The case with no horizon is physically uninteresting because of the curvature singularity lying at the origin. We identified the necessary conditions for at least one horizon to be present in the solution, which imposes some bound on the mass-angular momentum ratio. For some particular choice of parameters our solution degenerates into some previously known black hole solutions.

Keywords: rotating hairy black hole, (2+1)-dimensional gravity, non-minimal coupling, scalar potential

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1 Introduction

In spite of the fact that our spacetime is (3 + 1)-dimensional, gravity in (2 + 1) spacetime dimensions is still an active field of study because of its relative simplicity and its applications in realizing and testing some fundamental ideas which may apply to all spacetime dimensions. For instance, (2+1)-dimensional gravity is often invoked in the studies of AdS/CFT duality [1, 2] and/or of basic properties black

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holes. The first known and the most famous black hole solution in (2+1) dimensions is the Bañados-Teitelboim-Zanelli (BTZ) black hole [3], which is an exact solution of the (2+1)-dimensional Einstein equation in the absence of matter source. Charged rotating variants of BTZ black hole was found in [4, 5], see also [6]. It is often tempting and challenging to include extra matter sources other than simply the Maxwell field and see whether novel black hole solutions will emerge and whether the nice properties of the BTZ black hole and its charged rotating variants (such as the duality relationships) still persist. In this respect, an extra scalar source is the simplest choice and has been studied extensively.

The study of hairy extensions of the BTZ black hole has a history which is almost as long as the age of BTZ black hole itself. To this date, many explicit hairy extensions of BTZ black hole has been found, and in most cases the extra hair is a scalar field which couples to gravity either minimally [7, 8, 9, 10, 11, 12] or non-minimally [13, 14], the scalar may or may not couple to itself via a scalar potential, and the hairy black holes can be static, charged or rotating, see [15] for more references on related studies. Charged static solutions are relatively easy to find, however it is very difficult to find charged rotating solutions due to the complexities involved in solving the Maxwell field equation in rotating background.

In this paper, we present some novel exact rotating black hole solutions in (2+1)-dimensional gravity with a non-minimally coupled scalar field. Rotating solution in a similar theory was previously studied in [14] in the absence of scalar potential. The metric ansatz used in [14] is different from ours, and hence the solution presented there is not the limit of our solution in the limit of vanishing scalar potential. In our case, the metric ansatz together with consistency of Einstein’s equation determines the scalar potential up to five adjustable integration constants, one of which is a negative cosmological constant as is required by the existence of a smooth event horizon [16]. In a simplified branch of solutions, the scalar potential behaves like $\phi^6 + \mathcal{O}(\phi^{10})$ when $\phi$ is small, and if the coefficient of the $\phi^6$ term is negative, the true vacuum energy gets an extra negative contribution and will differ from the bare cosmological constant.

The paper is organized as follows. In Section 2, we describe the action of the model and the corresponding field equations. That the metric ansatz together with consistency of Einstein’s equation determine the scalar field and its potential some integration constants is shown clearly. Fixing one of the constants to a special value, we are led to a much simplified, critical branch of solution which is described in Section 3. This branch of solution contains an essential singularity at the origin which has to be enclosed by some event horizons and the whole spacetime is not conformally flat. The behavior of the scalar potential is also briefly discussed in this section. In Section 4 we discuss the horizon structure and the conditions required to ensure the existence of horizons. If some particular values for the coupling strength for the scalar field or the integration constants in the metric are chosen, our solution will degenerate into some known solutions found in the literature. Meanwhile, some other specific choices of parameters may greatly simplify the solution and will allow us to discuss the properties of the solution in greater depth. These are discussed in
Section 5. Finally, in Section 6, some concluding remarks will be given.

### 2 Action, field equations and generic solution

The model we consider is the (2 + 1)-dimensional gravity with a non-minimally coupled scalar field, which is described by the action (omitting some appropriate boundary terms which are necessary to make the variation of the action consistent)

$$
I = \frac{1}{2} \int d^3x \sqrt{-g} \left[ R - g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{8} R \phi^2 - 2V(\phi) \right], \quad (1)
$$

where we have set the gravitational constant $\kappa = 8\pi G = 1$, and the coupling constant between gravity and the scalar field is fixed to be $\xi = \frac{1}{8}$. The reason for this choice is that when the scalar potential $V(\phi)$ is vanishing, the scalar action (i.e. the second and the third terms together) is conformally invariant. On the other hand, when $V(\phi)$ is nonconstant, this particular choice of gravity-scalar coupling allows us to obtain exact solutions in a neat form.

We did not prescribe a concrete form for the scalar potential $V(\phi)$. The reason for this lies in that, unlike the cases in Minkowski spacetime, the scalar potential in curved spacetime cannot be prescribed arbitrarily, just like we cannot simply prescribe an arbitrary metric and announce that it is a solution to Einstein’s equation by identifying the value of the corresponding Einstein tensor as the stress-energy tensor of some unknown matter source. In real physical interesting situations, the stress-energy tensor of the matter source must have a good field theoretic or fluid mechanical description. Likewise, a physically good matter source must allow for the existence of a solution for the metric field with reasonable symmetries.

In this paper, we are interested in studying solutions with a rotation symmetry. To reach our goal, we first need to write down the field equations associated with the action. These are given by

$$
E_{\mu\nu} \equiv G_{\mu\nu} - T_{\mu\nu}^{[\phi]} + V(\phi)g_{\mu\nu} = 0, \quad (2)
$$

$$
\Box \phi - \frac{1}{8} R \phi - \partial_\phi V(\phi) = 0, \quad (3)
$$

where

$$
T_{\mu\nu}^{[\phi]} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\rho \phi \nabla_\rho \phi + \frac{1}{8} (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu + G_{\mu\nu}) \phi^2
$$

is the stress-energy tensor associated with the scalar field $\phi$.

To obtain the rotating solution, we begin with the following metric ansatz

$$
ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 \left( d\psi + \omega(r) dt \right)^2, \quad (4)
$$
where the coordinate ranges are given by \(-\infty < t < \infty, \ r \geq 0, \ -\pi \leq \psi \leq \pi\). Using this metric ansatz, it is straightforward to get

\[
16r \left( E_t^t - E_r^r \right) = r \left( -2r^2 \phi \phi_{,rr} + 8r^2 \phi \omega_{,rr} + 24r \omega_{,rr} + 12f (\phi, r)^2 - r^2 \phi \phi_{,rr} - 3r \phi^2 \omega_{,rr} - 4f \phi_{,rr} \right) = 0.
\]

Meanwhile, we also have

\[
16E^t_\psi = -r \left( -24\omega_{,r} - 8r \omega_{,rr} + 2r \phi \phi_{,r} \omega_{,r} + 3\phi^2 \omega_{,r} + r \phi^2 \omega_{,rr} \right) = 0. \quad (5)
\]

These two equations together give

\[
\phi_{,rr} - 3(\phi_{,r})^2 = 0,
\]

which yields the solution

\[
\phi(r) = \pm \frac{1}{\sqrt{kr + b}}, \quad (6)
\]

with \(k\) and \(b\) being integration constants. It is remarkable that the value of the scalar field is solely determined by the metric ansatz and the consistency of Einstein’s equation, without referring to the scalar potential and the scalar field equation. Moreover, the dependence on the metric ansatz is quite weak, because the very same solution (6) has also appeared in our previous work [15] on a charged static circularly symmetric solution.

Now inserting (6) into (5) and solve the resulting equation, we get

\[
\omega(r) = c_1 \left( \frac{8k^2}{(8b - 1)^3} \log \frac{r}{8b - 1 + 8kr} - \frac{8b^2 - b - 2kr}{2(8b - 1)^2 r^2} \right) + c_2, \quad (7)
\]

where \(c_1\) and \(c_2\) are integration constants. In order for the metric to contain non-trivial rotation, we need \(c_1 \neq 0\). The asymptotic behavior of \(\omega(r)\) as \(r \to +\infty\) reads

\[
\omega(r) = c_2 - \frac{8k^2 \log(8k)}{(8b - 1)^3} c_1 - \frac{c_1}{16} r^{-2} - \frac{c_1}{192k} r^{-3} + \mathcal{O}(r^{-4}).
\]

Thus, removal of global rotations of the coordinate system, i.e. \(\omega(r) \big|_{r \to +\infty} = 0\), lead us to the choice

\[
c_2 = \frac{8k^2 \log(8k)}{(8b - 1)^3} c_1 \quad (8)
\]

provided \(k \neq 0\). If \(k = 0\), then the scalar field \(\phi\) is constant, and hence the action becomes that of the standard Einstein-Hilbert action with a cosmological constant. In this paper, we are not interested in this degenerated case.

Now consider the case \(k \neq 0\). Inserting (8) into (7) and setting \(c_1 = 8a\), we have

\[
\omega(r) = 8a \left( \frac{8k^2}{(8b - 1)^3} \log \frac{8kr}{8b - 1 + 8kr} - \frac{8b^2 - b - 2kr}{2(8b - 1)^2 r^2} \right). \quad (9)
\]
Therefore, we are left with only three independent constant parameters \( a, b, k \) in \( \omega(r) \). Here \( a \) is a rotation parameter and is related to the angular momentum of the solution. Inserting (6), (9) into the rest of the field equations, we can get a very complicated, yet exact solution for the metric function \( f(r) \), together with a more complicated, exact scalar potential \( V(\phi) \) which takes pages to be written down. These results are so complicated that we do not think it worth to reproduce the concrete expressions here. However, there are two important points to be noted:

- In the process of solving \( f(r) \), two new independent integration constants will arise, one of which also appear in the final scalar potential as a constant term. This constant can be easily identified to be related to the cosmological constant, and the existence of smooth black hole horizons will force the cosmological constant to be negative and best denoted as \( \Lambda = -\frac{1}{\ell^2} \), where \( \ell \) has the interpretation as AdS radius. The other integration constant arising in the solution for \( f(r) \) may be denoted \( \beta \) and is related to the mass parameter;

- The form of the scalar potential \( V(\phi) \) is completely determined by the metric ansatz and consistency of the field equations. The only adjustable part of the potential is the set of integration constants that are inherent from the scalar field and the metric functions \( f(r) \) and \( \omega(r) \). There are five such parameters, i.e. \( a, b, k, \beta \) and \( \ell \).

Due to the very complicated form of the solution, we are unable to proceed to make quantitative analysis on the solution for generic choices of the constants \( a, b, k, \beta \) and \( \ell \).

Notice, however, that the solution (9) is apparently divergent at \( b = \frac{1}{8} \). From either the action or the field equations, there is no sign of divergence for this particular choice of integration constant. Since the parameter \( b \) is related to the value of \( \phi(r) \) at the origin, the apparent divergence at \( b = \frac{1}{8} \) might signify some critical behavior of the solution. In order to understand more on this point, let us look at the asymptotic behavior of \( \omega(r) \) as \( b \to \frac{1}{8} \). We have

\[
\omega(r) = \frac{ak^2 \log \left( \frac{1}{8} \right) + ak^2 \log(8k)}{8 \left( b - \frac{1}{8} \right)^3} - \frac{a(12kr + 1)}{24 (kr^3)} + \frac{a \left( b - \frac{1}{8} \right)}{32k^2r^4} + O \left( \left( b - \frac{1}{8} \right)^2 \right). \tag{10}
\]

The first term on the right hand side actually vanishes because it is proportional to \( \log(1) \) (provided \( k \neq 0 \)). The third and rest terms are polynomial in \( b - \frac{1}{8} \) and they all vanish at \( b = \frac{1}{8} \). Therefore, we are left with only the second term which is completely regular at \( b = \frac{1}{8} \). One can check that the second term alone indeed solves eq.(5) if the parameter \( b \) is taken to be equal to \( \frac{1}{8} \). Proceeding with the particular choice \( b = \frac{1}{8} \) and solving the rest of the field equations, it turns out that the corresponding solution is surprisingly simple, so that \( f(r) \) and \( V(\phi) \) can both be written in a neat and concise form. This simplified, critical solution will be the subject of concern in the rest part of this paper.
3 Simplified solution and some of its properties

In this and the subsequent sections, we shall consider only the case \( b = \frac{1}{8} \). For convenience we shall also rewrite \( k = \frac{1}{8B} \) and take \( B \) as the only free parameter characterizing the scalar field. Clearly, we have

\[
\phi(r) = \pm \sqrt{\frac{8B}{r+B}}.
\]

(11)

Substituting this into the field equations, we get the following metric functions,

\[
f(r) = 3\beta + \frac{2B\beta}{r} + \frac{(3r + 2B)^2a^2}{r^4} + \frac{r^2}{\ell^2},
\]

(12)

\[\omega(r) = \frac{(3r + 2B)a}{r^3},\]

(13)

where (13) is exactly the second term on the right hand side of (10). In order that \( \phi(r) \) is nonsingular at finite \( r \), it is necessary to require \( B \geq 0 \). The parameter \( \beta \) is related to the black hole mass \( M \) via

\[
\beta = -\frac{M}{3},
\]

(14)

as can be seen in the \( B = 0 \) limit, in which case the solution degenerates into the BTZ black hole. Without loss of generality, we shall assume \( a \geq 0 \) in the rest of the paper. The opposite choice \( a < 0 \) can be recovered by a reflection of the angular coordinate, i.e. \( \psi \rightarrow -\psi \).

Before carrying out explicit analysis on the structure of the spacetime given by the solution (11)-(13), let us first describe some of the geometric characteristics of the solution. First let us calculate some of the curvature invariants. The Ricci scalar is singular at \( r = 0 \) if \( a \neq 0 \),

\[
R = \left( \frac{30B^2}{r^6} + \frac{36B}{r^5} \right) a^2 - \frac{6}{\ell^2}.
\]

Higher order curvature invariants, e.g. \( R_{\mu\nu}R^{\mu\nu} \) and \( R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \) are also singular at \( r = 0 \) even if \( a = 0 \):

\[
R_{\mu\nu}R^{\mu\nu} = \frac{12}{\ell^4} + \frac{6B^2\beta^2}{r^6} + \left( \frac{324B^4}{r^{12}} + \frac{792B^3}{r^{11}} + \frac{486B^2}{r^{10}} \right) a^4
\]

\[
+ \left( \frac{12B^3\beta}{r^9} + \frac{18B^2\beta}{r^8} + \frac{102B^2}{r^6\ell^2} + \frac{144B}{r^5\ell^2} \right) a^2,
\]

\[
R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{12}{\ell^4} + \frac{24B^2\beta^2}{r^6} + \left( \frac{396B^4}{r^{12}} + \frac{1008B^3}{r^{11}} + \frac{648B^2}{r^{10}} \right) a^4
\]

\[
+ \left( \frac{48B^3\beta}{r^9} + \frac{72B^2\beta}{r^8} + \frac{48B^2}{r^6\ell^2} + \frac{144B}{r^5\ell^2} \right) a^2.
\]

Therefore, the metric contains a curvature singularity at \( r = 0 \), which has to be enclosed by some event horizon in order for the solution to be physically acceptable.
On the other hand, by direct evaluation, we find that some of the components of the Cotton tensor
\[
C_{abc} = \nabla_c R_{ab} - \nabla_b R_{ac} + \frac{1}{4} (\nabla_b R g_{ac} - \nabla_c R g_{ab})
\]
are nonvanishing if \( B > 0 \). We give only one of the nonvanishing components below:
\[
C_{\psi r \psi} = \left( \frac{12 B^2}{r^5} + \frac{18 B}{r^4} \right) a^2 + 3 \beta B r^2.
\]
In \((2 + 1)\) dimensions, the nonvanishing Cotton tensor signifies that the metric is non conformally flat \[17\].

The scalar potential associated with the above solution is determined to be
\[
V(\phi) = \frac{2}{\ell^2} + U(\phi),
\]
\[
U(\phi) = X \phi^6 + Y \left( \frac{\phi^6 - 40 \phi^4 + 640 \phi^2 - 4608}{(\phi^2 - 8)^5} \right), \tag{15}
\]
where \( X, Y \) are given by
\[
X = \frac{1}{512} \left( \frac{1}{\ell^2} + \frac{\beta}{B^2} \right), \quad Y = \frac{1}{512} \left( \frac{a^2}{B^4} \right). \tag{16}
\]
This potential has a nice behavior in power series expansion,
\[
U(\phi) \simeq X \phi^6 + Y O(\phi^{10}), \tag{17}
\]
in which only even powers of \( \phi \) are present and all the coefficients in the \( O(\phi^{10}) \) part are positive as long as \( Y > 0 \). The \( \phi^6 \) potential in \((2 + 1)\)-dimensional gravity was previously known \[18\] to have good behavior in yielding exact black hole solutions. Our work shows that the addition of the infinite higher even power series will keep this feature.

In order that the scalar potential to be bounded from below, we may have the following choices for the constants \( X \) and \( Y \):

- \( Y = 0, \ X \geq 0 \);
- \( Y > 0 \), in which case \( X \) can be an arbitrary real number. In particular,
  - If \( X \geq 0 \), then \( U(\phi) \) has only a single extremum, i.e. the minimum at \( \phi = 0 \);

\[1\]According to \((16)\), it seems that the choice \( B = 0 \) is not allowed, because in this case the scalar self coupling constants \( X, Y \) diverges. However, we may substitute \((11)\) and \((16)\) into \((15)\) and then take the \( B \to 0 \) limit. Then we find that \( U(\phi) \) actually vanishes as \( B \to 0 \), which is consistent with the fact that the scalar field \( \phi \) is vanishing at \( B = 0 \).
If $X < 0$, then $U(\phi)$ has three extrema, i.e. one local maximum at $\phi = 0$ with $U(0) = 0$, and two minima at $\phi = \pm \phi_0$ for some $\phi_0$, with $U(\pm \phi_0) < 0$.

The qualitative behavior of the scalar potential is very similar to the scalar potential found in our previous work [15], though the concrete form of the potential is different. The similar qualitative behavior of the scalar potential may be responsible for some universal properties of $(2 + 1)$-dimensional gravitational theories with a non-minimally coupled scalar field, such as the same CFT dual for theories with different potential functions.

4 Horizon structure

Now let us go back to the exact solution (11)-(13). If the curvature singularity at $r = 0$ is not naked, $f(r)$ must contain some zeros which correspond to black hole horizons. It can be seen from eq.(12) that if $\beta \geq 0$, then $f(r)$ will be positive for all $r \in [0, +\infty)$. So, the existence of black hole horizons requires $\beta < 0$. Moreover, eq.(12) indicates that $f(r) \to +\infty$ as both $r \to 0$ and $r \to +\infty$. So, the function $f(r)$ must have at least one minimum, and the total number of extrema of $f(r)$ must be odd. Since the $a = 0$ case of the solution is just the static hairy black hole solution which is studied before [15], we will consider only the $a \neq 0$ solution below.

In order to identify the number of zeros of $f(r)$, we first need to know the exact number and values of its extrema. The condition for the extrema of $f(r)$ is just the condition for the zeros of $f'(r)$. From the expression

$$f'(r) = \frac{2r}{\ell^2} - \frac{2\beta B}{r^2} - \left( \frac{18}{r^3} + \frac{36B}{r^4} + \frac{16B^2}{r^5} \right) a^2,$$

we see that

$$f'(0) = -\infty, \quad f'(+\infty) = +\infty.$$

So, $f'(r)$ has to have some zero in the range $r \in [0, +\infty)$. However, it is difficult to find the zeros of $f'(r)$ analytically. In order to gain some information about the number of zeros of $f'(r)$, let us first try to find the number of extrema of $f'(r)$ and the value of $f'(r)$ at its extrema. We denote by $r_i$ the location of possible extrema of $f'(r)$ (which need not exist at all). At the hypothetic extrema, we should have

$$f'(r_i) = \frac{2r_i}{\ell^2} - \frac{2\beta B}{r_i^2} - \left( \frac{18}{r_i^3} + \frac{36B}{r_i^4} + \frac{16B^2}{r_i^5} \right) a^2,$$

$$f''(r_i) = \frac{2}{\ell^2} + \frac{4\beta B}{r_i^3} + \left( \frac{54}{r_i^4} + \frac{144B}{r_i^5} + \frac{80B^2}{r_i^6} \right) a^2 = 0.$$

It is easy to see that

$$f'(r_i) = \left( f'(r_i) + \frac{r_i}{2} f''(r_i) \right) = \frac{3r_i}{\ell^2} + \left( \frac{9}{r_i^3} + \frac{36B}{r_i^4} + \frac{24B^2}{r_i^5} \right) a^2 > 0.$$
This means that if $f'(r)$ has some extrema, it must be positive at all the extrema. An alternative possibility is that $f'(r)$ has no extrema at all. In either cases the curve for $f'(r)$ will cross zero only once. Therefore $f(r)$ has only one minimum. Let us denote the location of the minimum of $f(r)$ as $r_{\text{min}}$. Depending on the values of the parameters, we may encounter one of the following three possibilities: an extremal rotating hairy black hole, a non-extremal rotating hairy black hole or a naked singularity.

### 4.1 Extremal rotating hairy black hole

The extremal rotating hairy black hole corresponds to the case in which $f(r)$ has only one zero at its minimum, i.e. $f(r_{\text{min}}) = 0$ and $f'(r_{\text{min}}) = 0$. The joint solution of these two equations is more restrictive than just the solution of $f'(r_{\text{min}}) = 0$, so we introduce a novel notation $r_{\text{ex}}$ for the joint solution. If $r_{\text{ex}}$ exists, then it will correspond to the horizon radius of the extremal rotating hairy black hole.

To actually get the horizon radius, we introduce

$$P(r_{\text{ex}}) = \frac{B}{r_{\text{ex}} + B} \left( f(r_{\text{ex}}) + \frac{r_{\text{ex}}(9r_{\text{ex}} + 6B)}{6B} f'(r_{\text{ex}}) \right)$$

$$= -\left( \frac{27}{r_{\text{ex}}^2} + \frac{36B}{r_{\text{ex}}^3} + \frac{12B^2}{r_{\text{ex}}^4} \right) a^2 + \frac{3r_{\text{ex}}^2}{\ell^2} = 0,$$  

(18)

$$K(r_{\text{ex}}) = \frac{r_{\text{ex}}^3}{3B} P(r_{\text{ex}}) - \frac{r_{\text{ex}}^5}{2B} f'(r_{\text{ex}}) = \beta r_{\text{ex}}^3 + (6r_{\text{ex}} + 4B)a^2 = 0.$$  

(19)

Then $r_{\text{ex}}$ will be the joint solution of (18) and (19).

A necessary condition for (18) and (19) to have joint solution $r_{\text{ex}}$ is

$$\beta = -\frac{2a}{\ell}.$$  

(20)

Under this condition, $r_{\text{ex}}$ is the real positive solution of the equation

$$r_{\text{ex}}^3 - 3a\ell r_{\text{ex}} - 2\ell B = 0,$$

whose explicit value is given by

$$r_{\text{ex}} = z + \frac{a\ell}{z}, \quad z = \left[ a\ell \left( B + (B^2 - a\ell)^{1/2} \right) \right]^{1/3}.$$  

(21)

We can also think of eq.(18) as an equation for $B$. The solution reads

$$B = -\frac{r_{\text{ex}}}{2a\ell} \left( 3a\ell \pm r_{\text{ex}}^2 \right).$$

Since $B \geq 0$, only the minus sign choice in the bracket is allowed, and we have

$$r_{\text{ex}} \geq (3a\ell)^{1/2},$$

where the bound is saturated when $B = 0$. 

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4.2 Non-extremal rotating hairy black hole

The non-extremal rotating hairy black hole corresponds to the conditions \( f(r_{\text{min}}) < 0 \) and \( f'(r_{\text{min}}) = 0 \). Therefore, \( r_{\text{min}} \) must obey the inequalities

\[
P(r_{\text{min}}) < 0, \quad K(r_{\text{min}}) < 0,
\]

where \( P(r) \) and \( K(r) \) are given by the same expression in (18) and (19) but with \( r_{\text{ex}} \) changed into \( r \). Since \( P'(r) > 0 \) for all \( r \in [0, +\infty) \), we see that \( P(r) \) increases monotonically. So, the solution to the inequality \( P(r) < 0 \) must be smaller than the solution of the equation \( P(r) = 0 \), i.e \( r_{\text{min}} < r_{\text{ex}} \). Solving \( \beta \) from the inequality \( K(r_{\text{min}}) < 0 \), we find that the necessary condition for the existence of two disjoint horizons is

\[
\beta < -\frac{2a}{\ell}.
\]

However, due to the complicated form (12) of the function \( f(r) \), we are unable to find its zeros \( r = r_{\pm} \) analytically for generic choices of parameters. The only qualitative property we know about \( r_{\pm} \) is that \( r_- < r_{\text{min}}, r_+ > r_{\text{min}} \), and that \( r_+ \) is the event horizon for the non-extremal black hole. However, for the specific choice \( Y = -X > 0 \), we can indeed work out the explicit values of the two horizon radii. This will be shown in the next section.

4.3 Naked singularity

If \( f(r) > 0 \) at its minimum \( r_{\text{min}} \), then there will be no horizons at all in the solution. In this case the curvature singularity at \( r = 0 \) will be naked. This can happen if

\[
\beta > -\frac{2a}{\ell},
\]

which can be obtained by analyzing the allowed values of \( \beta \) as the solution to the inequality \( K(r) > 0 \). We will not expand our discussions in this case, because naked singularity is physically unappealing.

Summarizing the above three subsections, we find that the existence of black hole horizons imposes an upper bound for the parameter \( \beta \), which reads

\[
\beta \leq -\frac{2a}{\ell}.
\]

Since \( \beta \) is related to the mass of the black hole, we can also view this as a bound of the mass-angular momentum ratio,

\[
-\frac{\beta}{a} \geq \frac{2}{\ell}.
\]

Actually we can take this bound as an analogue of the famous Kerr bound in (2+1) dimensions.
5 Special cases

The scalar potential $U(\phi)$ contains two independent coupling constants $X$ and $Y$. If either (or both) of these constants vanish(es), tremendous simplification of the solution would occur. Moreover, taking special values for these constants or the parameters appearing the solution would also make the solution simplified, and there are cases in which the simplified versions of the solution have already been found in the literatures. In this section, we shall list some of the special cases of the solution and the corresponding conditions imposed on the parameters.

5.1 Degenerate cases which are previously known

In this subsection we list three degenerate cases which are known in the literatures. These are:

- Rotating BTZ black hole [3], which corresponds to the choice $B = 0$. As mentioned in the footnote right beneath eq.(14), taking the $B = 0$ limit effectively set both $X$ and $Y$ equal to zero, so, both the scalar field and its potential vanishes in this case. The solution reads

$$f(r) = 3\beta + \frac{9a^2}{r^2} + \frac{r^2}{\ell^2},$$
$$\omega(r) = -\frac{3a}{r^2}.$$

- Static hairy black hole with a $\phi^6$ potential [1, 18], which corresponds to the choice $a = 0$ or $Y = 0$. The solution reads

$$f(r) = 3\beta + \frac{2B\beta}{r} + \frac{r^2}{\ell^2},$$
$$\phi(r) = \pm \left( \frac{8B}{r+B} \right)^{1/2},$$
$$U(\phi) = X\phi^6,$$
$$\omega(r) = 0.$$

- Conformally dressed black hole[13], which corresponds to $X = 0$, $Y = 0$ but $B \neq 0$. Notice that $Y = 0$ implies $a = 0$, so the solution becomes static. Explicitly,

$$f(r) = -\frac{3B^2}{\ell^2} - \frac{2B^3}{\ell^2} \frac{1}{r} + \frac{r^2}{\ell^2},$$
$$\phi(r) = \pm \left( \frac{8B}{r+B} \right)^{1/2},$$
and both $U(\phi)$ and $\omega(r)$ vanishes.

Since these degenerate cases are already known in the literatures, we will not make any discussion about their properties.
### 5.2 Special cases which makes the solution simplify

In this subsection we shall look at a special case of the solution with $Y = -X > 0$, i.e. $\beta = -\frac{a^2 \ell^2 + B^4}{B^4 \ell^2}$. We are interested in this particular choice of parameter because the metric function $f(r)$ possesses a particular factorized form, which enables us to calculate the horizon radii and other properties of the black hole solution in analytic form.

The solution now reads

$$f(r) = \frac{(r + B)^2(r - 2B)(B^2 r^3 - 3a^2 \ell^2 r - 2Ba^2 \ell^2)}{B^2 \ell^2 r^4}, \tag{25}$$

$$\omega(r) = -\frac{(3r + 2B)a}{r^3}, \tag{26}$$

$$\phi(r) = \pm \left( \frac{8B}{r + B} \right)^{1/2}, \tag{27}$$

$$U(\phi) = \frac{1}{512} \left( \frac{a^2}{B^4} \right) \left( -\phi^6 + \frac{(\phi^6 - 40 \phi^4 + 640 \phi^2 - 4608) \phi^{10}}{(\phi^2 - 8)^5} \right). \tag{28}$$

The condition (20) for extremal black hole becomes $B^2 = a \ell$, with the horizon radius $r_{\text{ex}} = 2B$. More generally, for any $B > 0$, we have

$$-\beta - \frac{2a}{\ell} = \frac{(B^2 - a \ell)^2}{B^2 \ell^2} \geq 0,$$

i.e. $\beta \leq -\frac{2a}{\ell}$, in which the equality holds only in the extremal case. Therefore, under the choice $\beta = -\frac{a^2 \ell^2 + B^4}{B^4 \ell^2}$, the solution will always behave as a black hole solution, and the singularity at the origin can never become naked. The horizon radii can be worked out explicitly, which read

$$r_1 = 2 (\sigma a \ell)^{1/2}, \quad r_2 = \left( \frac{a \ell}{\sigma} \right)^{1/2} \left( \zeta + \frac{1}{\zeta} \right),$$

where we have set $B^2 = \sigma a \ell$, and

$$\zeta = (\sigma + (\sigma^2 - 1)^{1/2})^{1/3}.$$

If $0 < \sigma < 1$, $r_2$ is the outer horizon, if $\sigma > 1$, $r_1$ is the outer horizon. The critical choice $a = 1$ corresponds to the extremal case, with $r_1 = r_2 = r_{\text{ex}}$.

Incidentally, the extrema of $U(\phi)$ can also be obtained analytically in this case. Solving the condition $\frac{dU}{d\phi} = 0$ as an algebraic equation, we get either $\phi = 0$, which corresponds to a five-fold degenerate local maxima, or $\phi = \pm \phi_0 = \pm \frac{2\sqrt{3}}{3}$, which correspond to a pair of reflection symmetric minima of $U(\phi)$. The value of $U(\phi)$ at the minima reads

$$U(\pm \phi_0) = -\frac{7}{3125} \left( \frac{a^2}{B^4} \right),$$

which adds an extra negative contribution to the bare cosmological constant $-\frac{1}{\ell^2}$.
If we fix $\phi$ to take the constant value $\phi_0$ in the action, then the system becomes a pure Einstein-AdS gravity theory and the true cosmological constant will become

$$\Lambda_{\text{eff}} = -\frac{1}{\ell^2} + U(\pm \phi_0) = \frac{1}{\ell^2} - \frac{7}{3125} \left( \frac{a^2}{B^4} \right) \equiv \frac{1}{\ell_{\text{eff}}^2}.$$  

Notice that the effective cosmological constant $\Lambda_{\text{eff}}$ is negative even if the bare value $-\frac{1}{\ell^2}$ vanishes, i.e. $\ell \to \infty$. The rotating BTZ black hole mentioned in the previous subsection is an exact solution to this theory, but with $\ell \to \ell_{\text{eff}}$. However, this solution cannot be obtained from our solution (25)-(28), because the scalar field $\phi$ as given in (27) can never become a constant.

What will happen if we take the $\ell \to \infty$ limit in the solution (25)-(28)? It is clear that $\phi(r), \omega(r)$ and $U(\phi)$ are not affected by this limit, but the outer horizon in the metric will be pushed to $r = +\infty$, because the $r^3$ term in the last factor in $f(r)$ drops off while taking the limit. Consequently, there will be no “outside region” of the black hole. Any observer must be located either inside the inner horizon with $r < r_1 = 2B$, or in between the two horizons with $2B < r < +\infty$. In the former case, the spacetime patch in which the observer is inhabited is stationary, with a “cosmological horizon” at $r = 2B$, but also contains a naked singularity at the origin, which is physically uninteresting. In the latter case, the function $f(r)$ becomes negative in the spacetime patch the observer is located in, so, the coordinate $r$ becomes timelike and $t$ spacelike. The metric is time dependent in this patch, with two temporal horizons at $r = 2B$ and $r = +\infty$ respectively. This means that time has both a beginning and an end. Moreover, the metric is no longer rotating, but with some spacial twisting, because the non-diagonal elements of the metric live purely in the spacial directions. The spacetime in this situation cannot be understand as a usual black hole solution.

6 Concluding remarks

The study of $(2 + 1)$-dimensional gravity with matter source turns out to be very fruitful and many exact solutions with black hole solutions have been found in the recent years. Naturally one expects that such solutions will play some role in the forthcoming constructions for the dual theories in the spirit of AdS/CFT duality. The present work add some more input in this picture.

It is worth mentioning that although the solution we presented in this paper is a rotating hairy black hole without electromagnetic charge, the scalar field takes exactly the same value as in the solution to the static charged hairy black hole case. This coincidence of scalar solution actually lies in the heart of the whole construction. As long as we impose the condition that $\phi$ depends only on $r$, the sourced Einstein’s equation will imply that the value of the scalar field $\phi$ is forced to take the form (6), of which (11) is a particular choice. Moreover, this procedure actually determines the allowed type of the scalar potential $U(\phi)$ up to several adjustable constants. This phenomenon has also been observed in other studies of hairy black holes in four-
The lesson to be learnt here is that, interaction between gravity and the scalar field imposes strong constraints on the allowed form of scalar potential. Such constraints, when applied in more realistic field theoretic models, will greatly help us in understanding the dynamics of fundamental scalar matter which is otherwise difficult to determine in flat spacetime.

After the first version of this paper has appeared in arXiv, we were notified of the reference [23], in which an algorithm for generating stationary axi-symmetric solutions of 3D dilaton gravity in the Einstein frame was proposed. In there, the scalar potential in the Einstein frame with the ADM metric ansatz can contain a general function besides several integration functions. It will be interesting to see the relationship between the result of [23] with ours.

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