Drawing Partially Embedded and Simultaneously Planar Graphs

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Abstract

We investigate the problem of constructing planar drawings with few bends for two related problems, the partially embedded graph problem—to extend a straight-line planar drawing of a subgraph to a planar drawing of the whole graph—and the simultaneous planarity problem—to find planar drawings of two graphs that coincide on shared vertices and edges. In both cases we show that if the required planar drawings exist, then there are planar drawings with a linear number of bends per edge and, in the case of simultaneous planarity, with a number of crossings between any pair of edges which is bounded by a constant. Our proofs provide efficient algorithms if the combinatorial embedding of the drawing is given. Our result on partially embedded graph drawing generalizes a classic result by Pach and Wenger which shows that any planar graph can be drawn with a linear number of bends per edge if the location of each vertex is fixed.
1 Introduction

In many practical applications we wish to draw a planar graph while satisfying some geometric or topological constraints. One natural situation is that we have a drawing of part of the graph and wish to extend it to a planar drawing of the whole graph. Pach and Wenger [26] considered a special case of this problem. They showed that any planar graph can be drawn with its vertices lying at pre-assigned points in the plane and with a linear number of bends per edge. In this case the pre-drawn subgraph has no edges.

If the pre-drawn subgraph $H$ has edges, a planar drawing of the whole graph $G$ extending the given drawing $\mathcal{H}$ of $H$ may not exist. Angelini et al. [1] gave a linear-time algorithm for the corresponding decision problem; the algorithm returns, for a positive answer, a planar embedding of $G$ that extends that of $\mathcal{H}$ (i.e., if we restrict the embedding of $G$ to the edges and vertices of $H$, we obtain the embedding corresponding to $\mathcal{H}$). If one does not care about maintaining the actual planar drawing of $H$ this is the end of the story, since standard methods can be used to find a straight-line planar drawing of $G$ in which the drawing of $H$ is topologically equivalent to the one of $\mathcal{H}$. In this paper we show how to draw $G$ while preserving the actual drawing $\mathcal{H}$ of $H$, so that each edge has a linear number of bends. This bound is worst-case optimal, as proved by Pach and Wenger [26] in the special case in which $H$ has no edges.

A result analogous to ours was claimed by Fowler et al. [14] for the special case in which $H$ has the same vertex set as $G$. Their algorithm draws the edges of $G$ one by one, in any order so that edges connecting distinct connected components of $H$ precede edges within the same connected component of $H$; each edge is drawn as a curve with the minimum number of bends. Fowler et al. claim that their algorithm constructs drawings with a linear number of bends per edge. However, we prove that there exists a tree, a set of prescribed positions for its vertices, and an order of the edges of the tree, such that drawing the edges in the given order as curves with the minimum number of bends results in some edges having an exponential number of bends.

The second graph drawing problem we consider is the simultaneous planarity problem [5], also known as “simultaneous embedding with fixed edges” (SEFE). The SEFE problem is strongly related to the partially embedded graph problem and—in a sense we will make precise later—generalizes it. We are given two planar graphs $G_1$ and $G_2$ that share a common subgraph $G$ (i.e., $G$ is composed of those vertices and edges that belong to both $G_1$ and $G_2$). We wish to find a simultaneous planar drawing, i.e., a planar drawing of $G_1$ and a planar drawing of $G_2$ that coincide on $G$. Graphs $G_1$ and $G_2$ are simultaneously planar if they admit such a drawing. Both $G_1$ and $G_2$ may have private edges that are not part of $G$. In a simultaneous planar drawing the private edges of $G_1$ may cross the private edges of $G_2$; in fact, a private edge of $G_1$ may cross a private edge of $G_2$ several times. The simultaneous planarity problem arises in information visualization when we wish to display two relationships on two overlapping element sets.

The decision version of the simultaneous planarity problem is not known to
be $\text{NP}$-complete, or to be solvable in polynomial time, though it is known to be $\text{NP}$-complete if more than two graphs are given \[16\]. However, there is a combinatorial characterization of simultaneous planarity, based on the concept of a “compatible embedding”, due to Jünger and Schulz \[21\] (see below for details). Erten and Kobourov \[12\], who first introduced the problem, gave an efficient drawing algorithm for the special case where the two graphs share vertices but no edges. In this case, a simultaneous planar drawing on a polynomial-size grid always exists in which each edge has at most two bends and therefore any two edges cross at most nine times, see \[11, 12, 22\]. In this paper we show that if two graphs have a simultaneous planar drawing, then there is a drawing on a polynomial-size grid in which every edge has a linear number of bends and in which any two edges cross at most 24 times. Our result is algorithmic, assuming a compatible embedding is given.

1.1 Realizability Results

Our paper addresses the following two drawing problems:

**Planarity of a partially embedded graph (PEG).** Given a planar graph $G$ and a straight-line planar drawing $H$ of a subgraph $H$ of $G$, find a planar drawing of $G$ that extends $H$ (see \[1, 20\]).

**Simultaneous planarity (SEFE).** Given two planar graphs $G_1$ and $G_2$ that share a subgraph $G$, find a simultaneous planar drawing of $G_1$ and $G_2$ (see \[5\]).

We prove the following results:

**Theorem 1 (Realizing a Partially Embedded Graph)** Let $G$ be an $n$-vertex planar graph, let $H$ be a subgraph of $G$, and let $H$ be a straight-line planar drawing of $H$. Suppose that $G$ has a planar embedding $E$ that extends the one of $H$. Then we can construct a planar drawing of $G$ in $O(n^2)$-time which realizes $E$, extends $H$, and has at most $72|V(H)|$ bends per edge.

Theorem 1 generalizes Pach and Wenger’s classic result, which corresponds to the special case in which the pre-drawn subgraph has no edges.

**Theorem 2 (Realizing a Simultaneous Planar Embedding)** Let $G_1$ and $G_2$ be simultaneously planar graphs on a total of $n$ vertices with a shared subgraph $G$. If we are given a compatible embedding of the two graphs, then we can construct in $O(n^2)$ time a drawing that realizes the compatible embedding, and in which any private edge of $G_1$ and any private edge of $G_2$ intersect at most 24 times. In addition, we can ensure either one of the following two properties:

(i) each edge of $G$ is straight, and each private edge of $G_1$ and of $G_2$ has at most $72n$ bends; also, vertices, bends, and crossings lie on an $O(n^2) \times O(n^2)$ grid; or
(ii) each edge of \(G_1\) is straight and each private edge of \(G_2\) has at most \(72|V(G_1)|\) bends per edge.

Theorem 1 provides a weak form of Theorem 2: If \(G_1\) and \(G_2\) are simultaneously planar, they admit a compatible embedding. Take any straight-line planar drawing of \(G_1\) realizing that embedding and extend the induced drawing of \(G\) to a drawing of \(G_2\). By Theorem 1 we obtain a simultaneous planar drawing where each edge of \(G_1\) is straight and each private edge of \(G_2\) has at most \(72|V(G_1)|\) bends per edge. Our stronger result of 24 crossings between any two edges is obtained by modifying the proof of Theorem 1 rather than applying that result directly.

Grilli et al. [17] independently proved a result in some respect stronger than Theorem 2. They showed that two simultaneously planar graphs have a simultaneous planar drawing with at most 9 bends per edge, vastly better than our \(72n\) bound. On the other hand, our bound of 24 crossings per pair of edges is better than the bound of 100 that can be derived from their result. Also, our algorithm allows us to construct simultaneous planar drawings in which each edge of one graph is straight or in which vertices, bends, and crossings lie on a polynomial-size grid. The former feature is not achievable by means of Grilli et al.’s algorithm; the latter one could be obtained from Grilli et al.’s result, at the expense of increasing the number of bends per edge to 300\(n\) (which corresponds to the number of crossings on a single private edge).

Frati et al. [15] very recently proved that two simultaneously planar graphs have a simultaneous planar drawing with at most 6 bends per edge and 16 crossings per pair of edges. This result improves on Grilli et al.’s result [17] and at the same time on part (i) of our Theorem 2 where the 72\(n\) bound would be replaced by a 48\(n\) bound. On the other hand, Frati et al. [15] cannot guarantee the private edges of one graph to be straight.

1.2 Related Work

The decision version of simultaneous planarity generalizes partially embedded planarity: given an instance \((G, H, \mathcal{H})\) of the latter problem, we can augment \(\mathcal{H}\) to a drawing of a 3-connected graph \(G_1\) and let \(G_2 = G\). Then \(G_1\) and \(G_2\) are simultaneously planar if and only if \(G\) has a planar embedding extending \(\mathcal{H}\). In the other direction, the algorithm [1] for testing planarity of partially embedded graphs solves the special case of the simultaneous planarity problem in which the embedding of the common graph \(G\) is fixed (which happens, e.g., if \(G\) or one of the two graphs is 3-connected).

Several optimization versions of partially embedded planarity and simultaneous planarity are NP-hard. Patrignani showed that testing whether there is a straight-line drawing of a planar graph \(G\) extending a given drawing of a subgraph of \(G\) is NP-complete [27], so bend minimization in partial embedding extensions is NP-complete; Patrignani’s result holds even if a combinatorial embedding of \(G\) is given.\(^1\) Bend minimization in simultaneous planar drawings

\(^1\)Patrignani does not explicitly claim NP-completeness in the case in which the embedding
is \textbf{NP}-hard, since it is \textbf{NP}-hard to decide whether there is a straight-line simultaneous drawing \cite{13}. Crossing minimization in simultaneous planar drawings is also \textbf{NP}-hard, as follows from an \textbf{NP}-hardness result on \textit{anchored planar drawings} by Cabello and Mohar \cite{9}; see Theorem 1 in Section 4 for a slightly stronger result.

Di Giacomo et al. \cite{10} studied the special case of PEG in which the \(n\)-vertex graph \(G\) to be drawn is a tree. They showed that, given a drawing \(H\) of a subtree \(H\) of \(G\), a drawing of \(G\) extending \(H\) can be computed in \(O(n^2 \log n)\) time so that each edge of \(G\) has at most \(1 + 2\lceil|V(H)|/2\rceil\) bends.

Further, as mentioned above, the special cases of PEG and SEFE in which there are no edges in the pre-drawn subgraph and in the common subgraph have already been studied.

Concerning PEG, Pach and Wenger \cite{26} proved the following result: given an \(n\)-vertex planar graph \(G\) with fixed vertex locations, a planar drawing of \(G\) in which each edge has at most 120\(n\) bends can be constructed in \(O(n^2)\) time. They also proved that such a bound is asymptotically tight in the worst case. Regarding the constant, Badent et al. \cite{2} improved the bound to \(3n + 2\) bends per edge. Biedl and Floderus \cite{4} considered the more general problem of drawing an \(n\)-vertex planar graph on fixed vertex locations where the drawing is constrained to lie inside a \(k\)-vertex polygon. They show that there is a drawing with \(O(n + k)\) bends per edge.

Concerning SEFE, Di Giacomo and Liotta \cite{11} and independently Kammer \cite{22} proved the following result: given two planar graphs \(G_1\) and \(G_2\) sharing some vertices and no edge with a total number of \(n\) vertices, there exists an \(O(n)\)-time algorithm to construct a simultaneous planar drawing of \(G_1\) and \(G_2\) on a grid of size \(O(n^2) \times O(n^2)\), where each edge has at most 2 bends, hence there are at most 9 crossings between any edge of \(G_1\) and any edge of \(G_2\). This improves upon a previous result of Erten and Kobourov \cite{12}. The algorithms in \cite{11, 12, 22} make use of a drawing technique introduced by Kaufmann and Wiese \cite{23}.

Haeupler et al. \cite{18} showed that if two simultaneously planar graphs \(G_1\) and \(G_2\) share a subgraph \(G\) that is connected, then there is a simultaneous planar drawing in which no two edges intersect more than once. Introducing vertices at crossing points yields a planar graph, and a straight-line drawing of that graph provides a simultaneous planar drawing with \(O(n)\) bends per edge, \(O(n)\) crossings per edge, and with vertices, bends, and crossings on an \(O(n^2) \times O(n^2)\) grid. Our result generalizes this to the case where the common graph \(G\) is not necessarily connected.

### 1.3 Graph Drawing Terminology

A \textit{drawing} of a graph is a mapping of each vertex to a distinct point of the plane and of each edge to a Jordan arc between the endpoints of the edge. A \textit{planar}
Figure 1: A face in a planar drawing of a disconnected graph. The face is colored gray and is delimited by three facial walks of sizes 13, 11, and 4. The numbers on each facial walk indicate how to count its vertices to determine its size. The red dots indicate where the traversal of each walk was initiated.

drawing is such that no two edges intersect except, possibly, at common endpoints. A planar drawing of a graph determines a clockwise order of the edges incident to each vertex, called rotation system. A planar drawing of a graph partitions the plane into topologically connected regions, called faces. The unbounded face is the outer face, while the other faces are internal. For connected graphs, the rotation system uniquely defines the walk delimiting each face; this is called facial walk—it is the closed walk composed of all the vertices and edges incident to the face. Two drawings of the same connected graph are equivalent if they determine the same rotation system and they have the same walk delimiting the outer face. A planar embedding (or combinatorial embedding) is an equivalence class of planar drawings. We note that a planar embedding can be specified combinatorially, namely by giving the rotation system and the outer facial walk. Furthermore, a given rotation system corresponds to some planar embedding if and only if Euler’s formula holds, i.e., \( n - m + f = 2 \) where \( n \) is the number of vertices, \( m \) the number of edges, and \( f \) the number of facial walks.

The size \( |W| \) of a facial walk \( W \) is the number of vertices of \( W \), where we count vertex repetitions. That is, if \( W \) consists of a single vertex, its size is 1. Otherwise, the size of \( W \) is the number of vertices, or equivalently the number of edges, encountered when traversing \( W \) as follows (refer to Figure 1). Start traversing any edge \((a, b)\) from \( a \) to \( b \) and assume w.l.o.g. that the face is to the right during the traversal; when traversing an edge from a vertex \( u \) to a vertex \( v \), choose \((v, w)\) as the next edge to be traversed from \( v \) to \( w \), where \((v, w)\) is the edge following \((u, v)\) in the counter-clockwise order of the edges incident to \( v \) in \( W \) (note that \( w = u \) if the degree of \( v \) is one); stop the traversal when the edge \((a, b)\) is again being traversed from \( a \) to \( b \). Note that the same vertex might be encountered more than once in the described traversal, and every time it is encountered it is counted for the size of \( W \).

The definition of planar embedding as stated above does not handle the combinatorics of a planar drawing of a disconnected graph—namely it does not tell us how connected components nest into each other.
Following Jünger and Schulz \[21\], we define a topological embedding of a (possibly non-connected) graph as follows: We specify a planar embedding for each connected component. This determines a set of inner faces. For each connected component we specify a “containing” face, which may be an inner face of some other component or the unique outer face. Furthermore, we forbid cycles of containment—in other words, if a connected component is contained in an inner face, which is contained in a component, etc., then this chain of containments must lead eventually to the unique outer face.

A face in a topological embedding of a graph has several facial walks along its boundary. Each facial walk along the boundary of a face is also called a boundary component. Each face (unless it is the outer face) has a distinguished facial walk we call the outer facial walk separating the remaining inner facial walks from the outer face of the embedding; in Figure 1 the outer facial walk is the one with size 13. The size of a face $F$, denoted by $|F|$, is the sum of the sizes of its boundary components.

A compatible embedding of two planar graphs $G_1$ and $G_2$ consists of topological embeddings of $G_1$ and $G_2$ such that the common subgraph $G$ inherits the same topological embedding from $G_1$ as from $G_2$ (where a subgraph inherits a topological embedding in a straightforward way; in particular, if we remove an edge that disconnects the graph, the face containment is determined by the edge that was removed). Jünger and Schulz \[21\] proved that $G_1$ and $G_2$ are simultaneously planar if and only if they have a compatible embedding. For that proof, they construct a simultaneous planar drawing of $G_1$ and $G_2$ by extending a drawing of $G$ (thus proving a form of our Theorem 1). However, their method does not yield any bounds on the number of bends or crossings.

## 2 Partially Embedded Graphs

In this section we prove Theorem 1 that is, we show how to construct a planar drawing of $G$ that extends the planar straight-line drawing $\mathcal{H}$ and has a linear number of bends per edge assuming that we are given a planar embedding of $G$ extending the one of $\mathcal{H}$. It is sufficient to prove the result for a single face $F$ of $\mathcal{H}$ (possibly $F$ is the outer face of $\mathcal{H}$), since the embedding of $G$ is given, and we know for each vertex and edge of $G$ which face of $\mathcal{H}$ it lies in, so the drawings in different faces of $\mathcal{H}$ do not interfere with each other.

Pach and Wenger \[26\] proved their upper bound on the number of bends needed to draw a graph with fixed vertex locations by drawing a tree with its leaves at the fixed vertex locations, and “routing” all the edges close to the tree, sometimes crossing the tree but never crossing each other. We want to use their approach, but we have to deal with a more general problem. Instead of fixed vertex locations we have fixed facial boundaries. The solution is natural: We contract each facial walk $W_i$ of $F$ to a single vertex $v_i$, fix a position for vertex $v_i$ inside $F$ near $W_i$, and then apply the Pach-Wenger method to draw the contracted multigraph on the fixed vertex locations $v_i$. We ensure that the contracted multigraph is drawn inside $F$, indeed we stay a small distance
away from the boundary of \( F \), inside a polygonal region \( F' \) that is an \textquotedblleft inner approximation\textquotedblright{} of \( F \). Inside \( F' \) we draw a tree \( T \) with its leaves \( v_i \) at the fixed vertex locations, while suitably bounding the number of vertices of \( T \) so as to get our bound on the number of bends. We then route the edges of the contracted multigraph close to \( T \) as Pach and Wenger do. Finally, to retrieve the original, uncontracted graph, we route the edges incident to \( v_i \) to their true endpoint on the facial walk \( W_i \)—these routes use the empty buffer zone \( F - F' \).

We fill in the details of this argument in Section 2.3 but before doing so we introduce \textquotedblleft inner approximations\textquotedblright{} in Section 2.1 and formalize the tree argument in Section 2.2.

To simplify notation, we use \( n_A \) and \( m_A \) for the number of vertices and edges in a graph (or subgraph) \( A \).

### 2.1 Approximating Faces

In the drawing \( H \), the face \( F \) is a region of the plane homeomorphic to a disc with holes. Each facial walk of \( F \) appears in the drawing as a \textit{closed polygonal arc}, i.e. a sequence of straight-line segments joined in a path that returns to its starting point (repeated segments/vertices may occur); see Figure 2(a). We will refer to a facial walk and its drawing interchangeably.

We will approximate \( F \) by offsetting each of its facial walks into the interior of \( F \). See Figure 2(b). Let \( W_1 \) be the outer facial walk of \( F \), and let \( W_2, \ldots, W_b \) be the inner facial walks. An \textit{inner \( \varepsilon \)-approximation} of \( W_i \) is a simple polygon \( P_i \) (a closed polygonal arc with no self-intersections) such that:

1. \( P_i \) is \( \varepsilon \)-\textit{close} to \( W_i \), meaning that every point of \( P_i \) is within distance \( \varepsilon \) of a point of \( W_i \),

2. the inner facial walk \( W_i \) lies in the interior of \( P_i \) if \( 2 \leq i \leq b \), and

3. the outer facial walk \( W_1 \) lies in the exterior of \( P_1 \).

If in addition the \( P_i \)'s form a \textit{polygonal region} (a simple polygon with holes) with \( P_1 \) as the outer polygon, then we say that the polygonal region is an \textit{inner \( \varepsilon \)-approximation} of \( F \). The next lemma shows that we can build inner \( \varepsilon \)-approximations of \( F \).

**Lemma 1** For any \( \varepsilon > 0 \) we can construct an inner \( \varepsilon \)-approximation \( F' \) of \( F \) in time \( O(|F|) \).

See Figure 2 for an illustration of Lemma 1. To prove the lemma, we construct—for every sufficiently small \( \varepsilon > 0 \) and for every facial walk of \( F \)—an inner \( \varepsilon \)-approximating polygon \( P_\varepsilon \) which does not have too many bends, and so that the \( P_\varepsilon \) are \textit{nested} in the following sense: if \( 0 < \varepsilon' < \varepsilon \), then \( P_{\varepsilon'} \) lies in the interior of \( P_\varepsilon \) if \( F \) is the walk that \( P_\varepsilon \) and \( P_{\varepsilon'} \) approximate is an inner facial walk, and vice versa otherwise. There are various ways to achieve this. Pach and Wenger [26] use the Minkowski sum of the facial walk (in their case the facial walk of a tree) and a square diamond centered at 0. We use a slightly
different construction, because it seems easier (both computationally and conceptually) and it gives a slightly better bound on the number of bends (which is what we are most interested in): for the facial walk of an \( n \)-vertex tree, Pach and Wenger construct a polygon with \( 4n - 2 \) vertices, while ours have \( 2n - 2 \) vertices. Our construction does have one disadvantage: the resulting drawings are tight, placing elements close together, for sharp (acute or obtuse) angles (the Minkowski-sum construction has the same problem for highly obtuse angles only).

Lemma 2  Let \( W \) be a facial walk in a face \( F \) of a drawing of a graph \( G \) in the plane. We can construct a nested family of inner \( \varepsilon \)-approximating polygons \( P_\varepsilon \) so that each \( P_\varepsilon \) has at most \( \max\{3, |W|\} \) vertices. Each \( P_\varepsilon \) can be computed in time \( O(n) \).

Proof: Let \( e, v, f \) be a corner of \( W \), that is, two consecutive edges \( e, f \) and their shared vertex \( v \). At \( v \) erect the angle bisector of \( e \) and \( f \) of length \( \varepsilon \) (inside \( F \)), and let \( v' \) be the endpoint of the bisector different from \( v \). In order to avoid square root computations, we will use the \( \ell_1 \)-norm at this point. If \( (v_i)_{i=1}^k \) is the sequence of vertices along \( W \), with \( k = |W| \), then \( (v'_i)_{i=1}^k \) defines a closed polygonal chain. If \( \varepsilon \) is sufficiently small, namely less than half the distance between any vertex of \( W \) and a non-adjacent edge on \( W \), the polygonal chain is free of self-crossings, and therefore bounds a simple polygon with \( |W| \) vertices. There are two special cases in which this argument does not work: if the facial walk is a facial walk on an isolated vertex or an isolated edge. In both of these cases, we can approximate \( W \) using a triangle. \( \square \)

To prove Lemma 1 we can use Lemma 2 to efficiently construct an inner \( \varepsilon \)-approximating polygon for each facial walk of \( F \). The resulting polygons are disjoint and form a polygonal region as long as \( \varepsilon \) is less than half the distance between any two non-adjacent vertices or edges of \( \mathcal{H} \).
2.2 Extending Partial Embeddings

Our main technical tool in the proof of Theorem 1 is the following lemma. Multigraphs, in this paper, may have multiple edges and loops.

**Lemma 3** Let $G$ be a multigraph with a given planar embedding and fixed locations for a subset $U$ of its vertices. Suppose we are given a straight-line drawing of a tree $T$ whose leaves include all the vertices in $U$ at their fixed locations. Then for every $\varepsilon > 0$ there is a planar poly-line drawing of $G$ so that

1. the drawing is $\varepsilon$-close to $T$,
2. the drawing realizes the given embedding,
3. the vertices in $U$ are at their fixed locations, and
4. each edge has at most $12n_T$ bends and comes close to each vertex $u$ in $U$ at most six times, where coming close to $u$ means intersecting an $\varepsilon$-neighborhood of $u$. Furthermore, any edge that comes close to $u$ will either terminate at $u$ or enter the $\varepsilon$-neighborhood of $u$, bend at a point in this $\varepsilon$-neighborhood, and then leave it.

Our proof of Lemma 3 will follow closely the structure of Pach and Wenger’s algorithm [26] to draw a planar graph with fixed vertex locations. That algorithm has three ingredients: (i) making $G$ Hamiltonian, (ii) drawing the Hamiltonian cycle of $G$, and (iii) drawing the remaining edges of $G$. We use their result (i) directly:

**Lemma 4 (Pach, Wenger [26])** Given a planar graph $G$ we can in linear time construct a Hamiltonian planar graph $G'$ with $|E(G')| \leq 5|E(G)| - 10$ by adding and subdividing edges of $G$ (each edge is subdivided by at most two new vertices).

We will use a slightly stronger version of Lemma 4 in which $G$ is allowed to be a multigraph. Pach and Wenger’s proof of Lemma 4 works in the presence of multiple edges and loops.

For part (ii) Pach and Wenger show that a Hamiltonian cycle can be drawn at fixed vertex locations $\varepsilon$-close to a star connecting all the vertices. For our application, we replace their star with a straight-line drawing of a tree $T$ whose leaves are the vertices $v_i$ (recall that $v_i$ is the vertex to which we contract the facial walk $W_i$ of $F$). Lemma 5 shows how to draw the Hamiltonian cycle. Later we will see how to draw the remaining edges.

Independently of our result, the generalization of part (ii) to trees has essentially been shown by Chan et al. [8]. Since their goal was to minimize edge lengths, they did not give an estimate on the number of bends.

**Lemma 5** Let $C$ be a cycle with fixed vertex locations, and suppose we are given a straight-line planar drawing of a tree $T$, in which the vertices of $C$ are leaves of $T$ at their fixed locations. Then for every $\varepsilon > 0$ there is a planar poly-line drawing of $C$ with at most $2|E(T)| - 1$ bends per edge and $\varepsilon$-close to $T$. 
Figure 3: (a) A straight-line planar drawing of a tree $T$ (edges are black, leaves are red), together with polygons $\Theta_i$ (orange). In order to improve the readability, $\Theta_1$ is farther from $T$ than it should be. (b) A look at the situation after the construction of a poly-line drawing of $p_1, p_2$, which is represented by green lines. Polygon $\Theta_2'$ is represented by blue lines. The edges of $T$ not in $T_2 := Q_1 \cup Q_2$ are dotted. (c) Complete planar poly-line drawing of cycle $C$.

**Proof:** Let $p_1, \ldots, p_n$ be the vertices of $C$ in their order along the cycle. We build a planar poly-line drawing of $C$ as follows. Let $\Theta_i$ be an $\varepsilon/(n + 1)$-approximation of the given drawing of $T$ for $1 \leq i \leq n$ (which we construct using Lemma 2). Figure 3(a) shows polygons $\Theta_i$ drawn around $T$. We start at $p_1$. Suppose we have already built the poly-line drawing of $p_1, \ldots, p_i$, and we want to add $p_i, p_{i+1}$. For $1 \leq j \leq n - 1$, let $Q_j$ be the unique path in $T$ connecting $p_j$ to $p_{j+1}$. Create $\Theta_i'$ from $\Theta_i$ by keeping only the vertices of $\Theta_i$ close to (approximating) vertices in $T_i := \bigcup_{j \leq i} Q_j$. This removes parts of the walk along $\Theta_i$ which we patch up as follows (refer to Figure 3(b)): suppose $v$ is an interior vertex of $T_i$, and $v$ is incident to $e$ which does not lie on $T_i$. Then $v$ is approximated by two vertices $v_1$ and $v_2$ which lie on bisectors formed by $e$ with neighboring edges. Now $v_1$ and $v_2$ belong to $\Theta_i'$, but the path along $\Theta_i$ between them got removed (since $e$ does not belong to $T_i$). We add $v_1, v_2$ to $\Theta_i'$ to connect them. Note that $v_1, v_2$ does not pass through $v$ since $v$ is incident to at least three edges ($e$ and two edges of $T_j$), and it does not cross any edges of any $\Theta_j'$ with $j < i$, since $T_i$ is monotone: if $e \notin E(\Theta_i)$, then $e \notin E(\Theta_j)$ for $j < i$.

Now both $p_i$ and $p_{i+1}$ correspond to unique vertices on $\Theta_i'$ (since they are leaves), so we can pick the facial walk $v_1, \ldots, v_k$ on $\Theta_i'$ which connects $p_i$ to $p_{i+1}$ and which avoids passing by $p_1$. We now add line segments $p_i, v_2, v_2, v_3, \ldots, v_{k-2}, v_{k-1}, v_{k-1}, p_{i+1}$ to the poly-line drawing of $C$. We treat the final edge $p_n, p_1$ similarly, except that we move along $\Theta_n' = \Theta_n$ back to $p_1$ in the last step, which we can do since none of the intermediate paths passed by $p_1$. Figure 3(c) shows an example of application of the described algorithm for the construction of a planar poly-line drawing of $C$ that is $\varepsilon$-close to $T$. 

Figure 3: (a) A straight-line planar drawing of a tree $T$ (edges are black, leaves are red), together with polygons $\Theta_i$ (orange). In order to improve the readability, $\Theta_1$ is farther from $T$ than it should be. (b) A look at the situation after the construction of a poly-line drawing of $p_1, p_2$, which is represented by green lines. Polygon $\Theta_2'$ is represented by blue lines. The edges of $T$ not in $T_2 := Q_1 \cup Q_2$ are dotted. (c) Complete planar poly-line drawing of cycle $C$. 

**Proof:** Let $p_1, \ldots, p_n$ be the vertices of $C$ in their order along the cycle. We build a planar poly-line drawing of $C$ as follows. Let $\Theta_i$ be an $\varepsilon/(n + 1)$-approximation of the given drawing of $T$ for $1 \leq i \leq n$ (which we construct using Lemma 2). Figure 3(a) shows polygons $\Theta_i$ drawn around $T$. We start at $p_1$. Suppose we have already built the poly-line drawing of $p_1, \ldots, p_i$, and we want to add $p_i, p_{i+1}$. For $1 \leq j \leq n - 1$, let $Q_j$ be the unique path in $T$ connecting $p_j$ to $p_{j+1}$. Create $\Theta_i'$ from $\Theta_i$ by keeping only the vertices of $\Theta_i$ close to (approximating) vertices in $T_i := \bigcup_{j \leq i} Q_j$. This removes parts of the walk along $\Theta_i$ which we patch up as follows (refer to Figure 3(b)): suppose $v$ is an interior vertex of $T_i$, and $v$ is incident to $e$ which does not lie on $T_i$. Then $v$ is approximated by two vertices $v_1$ and $v_2$ which lie on bisectors formed by $e$ with neighboring edges. Now $v_1$ and $v_2$ belong to $\Theta_i'$, but the path along $\Theta_i$ between them got removed (since $e$ does not belong to $T_i$). We add $v_1, v_2$ to $\Theta_i'$ to connect them. Note that $v_1, v_2$ does not pass through $v$ since $v$ is incident to at least three edges ($e$ and two edges of $T_j$), and it does not cross any edges of any $\Theta_j'$ with $j < i$, since $T_i$ is monotone: if $e \notin E(\Theta_i)$, then $e \notin E(\Theta_j)$ for $j < i$.

Now both $p_i$ and $p_{i+1}$ correspond to unique vertices on $\Theta_i'$ (since they are leaves), so we can pick the facial walk $v_1, \ldots, v_k$ on $\Theta_i'$ which connects $p_i$ to $p_{i+1}$ and which avoids passing by $p_1$. We now add line segments $p_i, v_2, v_2, v_3, \ldots, v_{k-2}, v_{k-1}, v_{k-1}, p_{i+1}$ to the poly-line drawing of $C$. We treat the final edge $p_n, p_1$ similarly, except that we move along $\Theta_n' = \Theta_n$ back to $p_1$ in the last step, which we can do since none of the intermediate paths passed by $p_1$. Figure 3(c) shows an example of application of the described algorithm for the construction of a planar poly-line drawing of $C$ that is $\varepsilon$-close to $T$. 

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Now both $p_i$ and $p_{i+1}$ correspond to unique vertices on $\Theta_i'$ (since they are leaves), so we can pick the facial walk $v_1, \ldots, v_k$ on $\Theta_i'$ which connects $p_i$ to $p_{i+1}$ and which avoids passing by $p_1$. We now add line segments $p_i, v_2, v_2, v_3, \ldots, v_{k-2}, v_{k-1}, v_{k-1}, p_{i+1}$ to the poly-line drawing of $C$. We treat the final edge $p_n, p_1$ similarly, except that we move along $\Theta_n' = \Theta_n$ back to $p_1$ in the last step, which we can do since none of the intermediate paths passed by $p_1$. Figure 3(c) shows an example of application of the described algorithm for the construction of a planar poly-line drawing of $C$ that is $\varepsilon$-close to $T$.
Note that $\Theta'_i$ has at most as many edges as $\Theta_i$, which has at most $2|E(T)|$ edges. Hence, the polygonal arc we build along $\Theta'_i$ has at most $2|E(T)| - 1$ edges (since it is not closed). We conclude that each edge of $C$ is replaced by a polygonal arc with at most $2|E(T)| - 1$ bends.

The following lemma shows how to draw the remaining edges of $G$, assuming that $G$ is Hamiltonian. As mentioned earlier, this lemma is close to a result by Chan et al. [8], except for the claim about the number of bends, and the rotation system (which we need for our main result).

**Lemma 6** Let $G$ be a Hamiltonian multigraph with a given planar embedding and fixed vertex locations. Suppose we are given a straight-line drawing of a tree $T$ whose leaves include all the vertices of $G$ at their fixed locations. Then for every $\varepsilon > 0$ there is a planar poly-line drawing of $G$ so that

1. the drawing is $\varepsilon$-close to $T$,
2. the drawing realizes the given embedding,
3. the vertices of $G$ are at their fixed locations,
4. every edge has at most $4|E(T)| - 1$ bends, and
5. every edge comes close to any leaf of $T$ at most twice, and only does so by terminating at or bending near the leaf.

The obvious idea—routing edges along the Hamiltonian cycle $C$—only gives a quadratic bound on the number of bends, since each edge would follow the path of a linear number of edges of $C$, and each edge of $C$ has a linear number of bends. Pach and Wenger came up with an ingenious way to construct auxiliary curves with few bends based on the level curves $\Theta'_i$ which carry the cycle $C$ in the proof of Lemma 5.

**Proof:** Let $C$ be the Hamiltonian cycle of $G$ and let $G_1$ and $G_2$ be the two outerplanar graphs composed of $C$ and, respectively, of the edges of $G$ inside and outside $C$. Using Lemma 5, we find a planar poly-line drawing of $C$ on $V(G)$. We need to show how to draw $G_1$ and $G_2$ respecting the planar embeddings induced by the given embedding of $G$. Let $n = |V(G)|$ and $m_i = |E(G_i)|$. We only describe how to draw $G_1$, since $G_2$ can be handled analogously. Let $\Delta_{i,k}$, $1 \leq k \leq m_1 + m_2$, be a $\kappa \varepsilon/(n(m_1 + m_2 + 1))$-approximation of $\Theta'_i$ constructed using Lemma 2 (see Figure 4(a)). For a fixed $i$, each $\Delta_{i,k}$ crosses $C$ twice: when $C$ moves from $p_i$ to $\Theta'_{i+1}$, and when it finally moves back from $\Theta'_n$ to $p_1$. As in Pach and Wenger, we can then split $\Delta_{i,k}$ at the crossings and connect their free ends to $p_1$ and $p_i$, resulting (for each $k$) in two curves $\Delta'_{i,k}$ and $\Delta''_{i,k}$ connecting $p_1$ to $p_i$, where $\Delta'_{i,k}$ lies inside $C$ (these are the curves we use for $G_1$) and $\Delta''_{i,k}$ lies outside $C$ (these are the curves we use for $G_2$). Each such curve has at most $2|E(T)| - 1$ bends. As in the proof of Pach and Wenger, we can create edges $p_ip_j \in E(G_1)$ by concatenating $\Delta'_{i,k}$ with $\Delta'_{j,k}$. Since we chose $m_1 + m_2$ such approximations, we can do this for each edge in $G_1$. There are two problems
Figure 4: Drawing an edge of $G_1$ between $p_3$ and $p_5$. (a) Parts of polygons $\Delta_{3,k}$ and $\Delta_{5,k}$ are shown by blue lines. Note that there should be $m_1 + m_2$ polygons $\Delta_{3,k}$ (same for $\Delta_{5,k}$), however only one of them is shown, for the sake of readability. (b) Drawing a polygonal path between $p_3$ and $p_5$ (represented by blue lines) by concatenating the parts $\Delta_{3,k}'$ and $\Delta_{5,k}'$ of $\Delta_{3,k}$ and $\Delta_{5,k}$ inside $C$ and suitably introducing a bend close to $p_1$. Remaining: edges $p_ip_j$ now all pass through $p_1$ and they could potentially cross (rather than just touch) there. Pach and Wenger show that any two edges touch, so the drawing can be modified close to $p_1$ so as to separate all edges $p_ip_j$ from each other; see Figure 4(b). This introduces at most one more bend per edge, so that the resulting edges have $2(2|E(T)| - 1) + 1 = 4|E(T)| - 1$ bends. Finally, note that each edge $p_ip_j$ comes close to each leaf of $T$ (including $p_1$) at most twice, once for $\Delta_{i,k}'$ and once for $\Delta_{j,k}'$. Each time an edge comes close to a leaf of $T$ it either terminates at the leaf, or bends near the leaf.

We are finally ready to complete the proof of Lemma 3. We show how to apply Lemma 6 in case $G$ is not Hamiltonian, and not all its vertices are assigned fixed locations.

Proof of Lemma 3. By Lemma 4, we can construct a graph $G'$ with a Hamiltonian cycle $C$ by subdividing each edge of $G$ at most twice, and by adding some edges, where $G'$ has a planar embedding extending the embedding of (a subdivision of) $G$.

Next we deal with the issue that not all vertices lie in $U$, the set of vertices with fixed locations. Traverse $C$: whenever we encounter an edge of $C$ with at least one endpoint not in $U$, contract that edge. This yields a new Hamiltonian multigraph $G''$ with $V(G'') = U$ and a planar embedding induced by the planar embedding of $G'$. Use Lemma 9 to construct a planar poly-line drawing of $G''$ at the fixed vertex locations, and $\varepsilon$-close to $T$, so that each edge of $G''$ has at most $4|E(T)| - 1$ bends. Each vertex $u \in U$ of $G''$ corresponds to a set of vertices $V_u \subseteq V(G')$ which was contracted to $u$, so the subgraph $G'_u$ of $G'$ induced by $V_u$ is connected. Since we embedded $G''$ with the induced planar embedding of
The same idea was used in [18, Theorem 2].

This will involve introducing new vertices where edges cross into the disc. The same idea was used in [18, Theorem 2].

To this end, we define a graph $G_u^+$, which consists of $G'_u$, a cycle $C_u$ containing $G'_u$ in its interior, and some further edges. Each vertex of $C_u$ corresponds to an edge of $G'$ incident to $G'_u$, i.e., with an end-vertex in $V_u$ and an end-vertex not in $V_u$. Vertices appear in $C_u$ in the same order as the corresponding edges incident to $G'_u$ leave $G'_u$ (this order also corresponds to the cyclic order of the edges incident to $u$ in $G''$); each vertex of $C_u$ corresponding to an edge $e$ of $G'$ is connected to the end-vertex of $e$ in $V_u$. Finally, $G_u^+$ contains further edges that triangulate its internal faces.

Consider a small disk $\delta$ around $u$. We erase the part of the drawing of $G''$ inside $\delta$. We construct a straight-line convex drawing of $G_u^+$ in which each vertex of $C_u$ is mapped to the point in which the corresponding edge crosses the boundary of $\delta$. This drawing always exists (and can be constructed efficiently), since $G_u^+$ is 2-connected and internally-triangulated. Removing the edges that triangulate the internal faces of $G_u^+$ completes the reintroduction of $G'_u$.

Overall, we added one bend to an edge with exactly one endpoint in $V_u$. Since an edge can have endpoints in at most two $V_u$, this process adds at most two bends per edge, so every edge has at most $4|E(T)| + 1$ bends. Since each edge of $G$ was subdivided at most twice to obtain $G'$, each edge of $G$ has at most $3(4|E(T)| + 1) = 12|E(T)| + 3 < 12|V(T)|$ bends. Each edge of $G'$ comes close to each leaf of $T$ at most twice, so each edge of $G$ comes close to each vertex of $U$ at most six times. Each time an edge comes close to a leaf of $T$ it either terminates at the leaf, or bends near the leaf. This concludes the proof of Lemma 4.

\[ \Box \]

### 2.3 Proof of Theorem 1

As we mentioned earlier, it is sufficient to prove the result for each face of $\mathcal{H}$, so fix such a face $F$. Let $W_i$, with $1 \leq i \leq b$, be the facial walks of $F$. We distinguish between facial walks consisting of isolated vertices, indexed by $I := \{i : |W_i| = 1\}$, and facial walks consisting of more than one vertex, with indices in $N := \{1, \ldots, b\} \setminus I$. Temporarily remove the isolated vertices $W_i$, with $i \in I$, from $F$ and construct an inner $\varepsilon$-approximation $F_N$ of the resulting face using Lemma 4. Reinsert the isolated vertices and let $F'$ be the face bounded by the boundary components of $F_N$ and by the isolated vertices $W_i$ with $i \in I$. For $i \in N$, let $W_i'$ be the polygon in $F'$ that approximates $W_i$. Then $|W_i'| \leq \max\{3, |W_i|\} \leq |W_i| + 1$ by Lemma 2 and the fact that $|W_i| \geq 2$. Thus we have that $|F'| \leq \sum_{i \in N} |W_i| + |N| + |I|$. We remark that all the boundary components of $F'$ are either isolated vertices or simple polygons (thus the size of $F'$ is equal to the number of vertices in its boundary components).

We can triangulate $F'$ using at most $|F'| + 2|N| + |I| - 4$ triangles, applying the following lemma with $n = |F'|$, $h_1 = |I|$, and $h_2 = |N| - 1$. 


Figure 5: A face $F$ with outer facial walk $W_1$ and inner facial walk $W_2$. (a) The 5 edges of $G - H$. (b) The polygons $W'_1$ and $W'_2$ (in heavy blue) that bound the inner $\varepsilon$-approximation $F'$ of $F$; a triangulation of $F'$ (fine lines); and the dual spanning tree (dashed red) with extra vertices $v_1$ and $v_2$ close to $W_1$ and $W_2$, respectively.

**Lemma 7 (Based on O’Rourke [25, Lemma 5.2])** Any $n$-vertex polygonal region with $h_1$ point-holes and $h_2$ non-point-holes can be triangulated by adding chords in time $O(n \log n)$. The resulting triangulation has $n + h_1 + 2h_2 - 2$ triangles.

**Proof:** The time bound can be derived from the algorithm of O’Rourke [25, Lemma 5.1]. Consider the total sum of all angles in triangles of the triangulation. Suppose there are $n_0$ vertices on the outer face, $n_1 = h_1$ isolated vertices, and $n_2$ vertices on non-point-holes (of which there are $h_2$). Then the total angle sum is $[(n_0 - 2) + 2n_1 + (n_2 + 2h_2)]\pi$ which equals $t\pi$, where $t$ is the number of triangles. We conclude that $t = n + h_1 + 2h_2 - 2$. □

We use a result of Bern and Gilbert [3] to construct a straight-line drawing of the dual of the triangulation; refer to Figure 5. Bern and Gilbert place a vertex at the *incenter* of each triangle (where the angle bisectors of the triangle meet) and prove that the straight-line edge joining two vertices in adjacent triangles lies within the union of the two triangles. Now take a spanning tree $T$ of the dual. By Lemma 7, $T$ has $|F'| + 2|N| + |I| - 4$ vertices. For each facial walk $W_i$, $i \in N$, we augment $T$ with a new leaf $v_i$ close to $W_i$ and inside $F'$; for each facial walk $W_i$, $i \in I$, we add the isolated vertex of $W_i$ to $T$ as a new leaf $v_i$. This adds $|N| + |I|$ vertices to $T$, so the number of vertices of $T$ is now $n_T = |F'| + 3|N| + 2|I| - 4$.

Let $G_F$ be the embedded multigraph obtained by restricting $G$ to vertices and edges lying inside or on the boundary of $F$ and by contracting each facial walk $W_i$ of $F$ to a single vertex $v_i$. We can now use Lemma 3 to embed $G_F$ along $T$ so that vertices $v_i$ are drawn at their fixed locations. Each edge of $G_F$ has at most $12n_T$ bends.

We now want to connect edges in $G_F$ to the suitable vertices in the boundary
components of $F$ they are incident to in $G$. For facial walks $W_i$, $i \in I$, there is nothing to do, since we chose $v_i$ to coincide with the isolated vertex $W_i$. So we may assume that we are dealing with boundary components consisting of more than one vertex. We will use the buffer zone $F - F'$ to do this; note that this buffer zone is composed of $|N|$ connected regions, namely for each $i \in N$ such that $W_i$ is an inner facial walk of $F$, we have a connected region that is exterior to $W_i$ and interior to $W'_i$, and for the outer facial walk $W'_i$ of $F$ (if it exists, i.e. if $F$ is not the outer face of $G$) we have one connected region that is exterior to $W'_i$ and interior to $W_i$.

In order to route the edges in the buffer zone, we split the buffer zone into two, so we apply Lemma 1 a second time to obtain an inner $\varepsilon/2$-approximation $F''$ of $F$, so that $F' \subseteq F'' \subseteq F$. See Figure 6. Let $W'_i$ be the polygon that approximates $W_i$ in $F''$. Note that $|W''_i| = |W'_i| \leq |W_i| + 1$. Now for each walk $W_i$ we extend the edges ending at $v_i$ to their endpoint on $W_i$. Since the cyclic order in which the edges of $G$ are incident to $W_i$ is the same as the one in which they are incident to $v_i$ in $G_F$, we can simply route these edges around $W_i$ using approximations to $W_i$ via Lemma 1 and we can do so in the open connected region that is exterior to $W_i$ and interior to $W''_i$, if $W_i$ is an inner facial walk of $F$, or exterior to $W''_i$ and interior to $W'_i$, if $W_i$ is the outer facial walk of $F$.

This adds two bends to the edge near $v_i$, plus at most one bend for each vertex of $W''_i$ except the one corresponding to the final destination vertex on $W_i$. In total we add at most $2 + |W''_i| - 1 \leq |W_i| + 2$ bends. There is one difficulty: there are edges of $G_F$ that pass by $v_i$, separating it from the segment of $W'_i$ close to $v_i$ (which is our gate to $W_i$). To remedy this difficulty, we first route all of these edges around the whole obstacle $W_i$ in the $F'' - F'$ part of the buffer (more precisely in the open connected region delimited by $W'_i$ and $W''_i$), which adds $|W'_i| + 3 \leq |W_i| + 4$ bends to an edge every time it passes $v_i$ (see Figure 6(b), note that the edge starts with one bend close to the vertex).

Now we are free to route the edges of $G - H$ that have to be embedded in $F$ and are incident to $W_i$ to their endpoints along $W_i$. Since an edge can pass by and/or terminate at a vertex at most six times, the number of additional bends in each edge caused by going around $W_i$ is at most $6(|W_i| + 4) = 6|W_i| + 24$; totaling this number over all boundary components of $F$ yields a bound of at most $6 \sum_{i \in N} |W_i| + 24|N|$ bends along the whole edge (we can ignore $W_i$ with $i \in I$, since we do not reroute around those components). Since each edge started with $12n_T$ bends in the drawing of $G_F$, each edge of $G - H$ embedded in $F$ now has at most $12n_T + 6 \sum_{i \in N} |W_i| + 24|N|$ bends.

In order to derive a bound in terms of $n_H = |V(H)|$, we use:

(1) $n_T = |F'| + 3|N| + 2|I| - 4$ (as discussed in the first part of this subsection),
(2) $|F'| \leq \sum_{i \in N} |W_i| + |N| + |I|$ (as discussed in the first part of this subsection),
(3) $\sum_{i \in N} |W_i| \leq 2n_H$ (which can be easily proved by induction on $|N|$),
(4) $2|N| + |I| \leq n_H$ (since each facial walk $W_i$ with $i \in N$ consists of more than one vertex).
Figure 6: A close-up of the situation near inner facial walk $W_2$. The tree $T$ has an edge (drawn as a heavy dashed line) incident to vertex $v_2$. (a) After drawing the edges of $G_F$ around the tree $T$ edges 1, ..., 5 are incident to $v_2$ in the correct cyclic order, but two other edges $e$ and $f$ come near $v_2$, passing between $v_2$ and $W'_2$. (b) We add an $\epsilon/2$-approximation $F''$ of $F$ which introduces polygon $W''_2$, and we route the edges $e$ and $f$ (in dashed red) around $W_2$ in the buffer zone between $W'_2$ and $W''_2$. (c) We route the edges incident to $W_2$ in the buffer zone between $W''_2$ and $W_2$.

From (1) and (2) we get that $n_T \leq \sum_{i \in N} |W_i| + 4|N| + 3|I|$. Thus the number of bends in each edge of $G - H$ that is embedded in $F$ is at most

$$12n_T + 6 \sum_{i \in N} |W_i| + 24|N| \leq 12\left( \sum_{i \in N} |W_i| + 4|N| + 3|I| \right) + 6 \sum_{i \in N} |W_i| + 24|N|$$

$$\leq 18\left( \sum_{i \in N} |W_i| \right) + 72|N| + 36|I|$$

$$\leq 18\left( \sum_{i \in N} |W_i| \right) + 36(2|N| + |I|).$$

From (3) and (4), we conclude that each edge of $G - H$ has at most $36n_H + 36n_H = 72n_H$ bends.

Most of the steps in the construction can be performed in linear time. Building the triangulation takes time $O(n_H \log n_H)$. The overall running time is thus bounded by the size of the resulting drawing which contains a linear number of edges each with a linear number of bends, yielding the quadratic running time.

**Remark 1.** The algorithm we presented in this section provides a bound better than $72n_H$ bends per edge if the subgraph $H$ of $G$ for which a straight-line drawing $\mathcal{H}$ is given as part of the input is *induced*. If that is the case, then the embedded multigraph $G_F$ defined in this section contains no self-loops; consequently, a Hamiltonian planar graph $G_F'$ can be constructed in linear time by adding vertices and edges and by subdividing edges of $G_F$ so that each edge is subdivided by at most one new vertex (while in the general case we use two subdivision vertices per edge, see Lemma 4). This can be done by exploiting an
algorithm by Kaufmann and Wiese [23] for making embedded (simple) graphs 4-connected, as described in the following.

**Lemma 8** Let $G_F$ be an embedded multigraph with no self-loops. An embedded simple Hamiltonian graph $G'_F$ can be constructed from $G_F$ by adding vertices and edges and by subdividing each edge of $G_F$ with at most one new vertex.

**Proof:** A *separating triangle* in an embedded (multi-)graph is a cycle $(u,v,z)$ such that removing $u$, $v$, and $z$ and their incident edges disconnects the graph. We state two facts that we use for our proof.

First, it is a well-known theorem of Tutte [30] that a 4-connected simple maximal planar graph is Hamiltonian. Second, it has been shown by Kaufmann and Wiese [23] how to turn a simple maximal planar graph into a 4-connected simple maximal planar graph by subdividing each of its edges with at most one new vertex and by adding some edges to the resulting graph; moreover, an edge is subdivided with a new vertex only if it is an edge of a separating triangle.

Now starting from $G_F$, we add edges to it so that every face is delimited by a cycle with three vertices or by two parallel edges. Next, for each pair of vertices $u$ and $v$ such that there is more than one edge connecting $u$ and $v$, we subdivide all the parallel edges $(u,v)$ with one subdivision vertex; denote by $S$ the set of newly inserted vertices. We add a new vertex $v_f$ inside each face $f$ and we connect $v_f$ to all the vertices on the boundary of $f$, obtaining a simple maximal planar graph $H_f$. It is easy to note that no edge incident to a vertex in $S$ belongs to a separating triangle in $H_f$. Then we can complete the proof by using the previously mentioned results. Namely, by Kaufmann and Wiese’s result, $H_f$ can be turned into a 4-connected simple maximal planar graph $G'_F$ by subdividing some of its edges and inserting some new edges; since no edge incident to a vertex in $S$ belongs to a separating triangle, each original edge of $G_F$ is subdivided at most once. By Tutte’s result $G'_F$ is Hamiltonian, which completes the proof of the lemma. □

Subdividing each edge with one new vertex rather than two immediately allows us to improve the bounds in Lemma 3 on the number of bends per edge to $8n_T$ and on the number of times each edge comes close to each vertex $u$ to at most four. The same analysis as above and the improved bounds of Lemma 3 allow us to upper bound the number of bends per edge in Theorem 1 by $48n_H$.

**Remark 2.** An improvement upon the $72n_H$ bound of Theorem 1 can be obtained by modifying the placement of $v_i$, for each $i \in N$, and the route of the edges that go around $W_i$. This modification makes the algorithm slightly more involved, so we preferred to omit it from the proof and to sketch it here. The main idea is that vertex $v_i$ can be inserted not just at any point inside $F'_i$, but rather at a convex corner of $F'_i$ that approximates an occurrence $\sigma$ of a vertex of $W_i$. Then each edge that goes around $v_i$ and has to be “wrapped around” $W_i$ can save three bends (each time it passes by $v_i$) with respect to the route described in Figure 6(b). To achieve this, we bend the edge at its intersection points with $F'_i$ and then connect it directly to the suitable approximations of
bends each time an edge passes by $v$. A similar argument can be used for the edges that terminate at some vertex of $W_i$. This results in each edge of $G - H$ having at most $12n_H^2 + 6 \sum_{i \in N} |W_i| + 6|N|$ bends. Then the same calculations described above lead to a bound of $63n_H$ bends per edge.

## 3 Extending Partial Drawings Greedily

Let $G$ be a planar graph with a spanning subgraph $H$ for which we have fixed a straight-line planar drawing $\mathcal{H}$. For a given ordering $\sigma = [e_1, \ldots, e_m]$ of the edges in $G \setminus H$ we say that a drawing $\Gamma$ of $G$ greedily extends $\mathcal{H}$ with respect to $\sigma$ if it is obtained by drawing edges $e_1, \ldots, e_m$ in this order, so that $e_i$ is drawn as a polygonal curve that respects the embedding of $G$ and with the minimum number of bends, for $i = 1, \ldots, m$. Note that the graph $H$ might have no edges; in this case we call it the empty spanning subgraph of $G$.

Suppose $\sigma$ orders the edges of $G \setminus H$ so that the edges between distinct connected components of $H$ precede edges between vertices in the same connected component of $H$. For such orderings Fowler et al. claimed in [14] that there is a drawing $\Gamma$ of $G$ greedily extending $\mathcal{H}$ with respect to $\sigma$ in which each edge has $O(|V(G)|)$ bends. However, in the following we confirm a claim of Schaefer [29] stating that greedy extensions do not, in general, lead to drawings with a polynomial number of bends.

**Theorem 3** For every $n \geq 9$ there exists an $n$-vertex planar graph $G$, a planar drawing $\mathcal{H}$ of $H = (V(G), \emptyset)$, the empty spanning subgraph of $G$, and an order $\sigma$ of the edges in $G$ so that any drawing of $G$ that greedily extends $\mathcal{H}$ with respect to $\sigma$ has edges with $2^{\Omega(n)}$ bends.

**Proof:** We adapt an example by Kratochvíl and Matoušek [24]. Refer to Figure 2. Let $N = \left\lceil \frac{n}{3} \right\rceil - 2$, for any integer $n \geq 9$. Graph $H$ consists of $n$ isolated vertices, name them $u_1, \ldots, u_N$, $v_1, \ldots, v_N$, $w_1, \ldots, w_N$, $a, b, c, d, e, r_1, \ldots, r_{n-3N-5}$. Note that $N \geq 1$, given that $n \geq 9$, and $n - 3N - 5 \geq 1$. The first $n - N - 1$ edges in $\sigma$ are $(u_i, v_i)$ for $i = 1, \ldots, N$, $(w_i, w_{i+1})$ for $i = 1, \ldots, N - 1$, $(r_i, r_{i+1})$ for $i = 1, \ldots, n - 3N - 6$, $(e, w_1)$, $(b, c)$, $(c, e)$, $(e, d)$, $(a, d)$, and $(a, r_{n-3N-5})$. All these edges are straight-line segments in any drawing $\Gamma$ of $G$ that greedily extends $\mathcal{H}$ with respect to $\sigma$. The last $N$ edges in $\sigma$ are $(u_1, v_1), \ldots, (u_N, v_N)$ in this order.

Consider any drawing $\Gamma$ of $G$ that greedily extends $\mathcal{H}$ with respect to $\sigma$. We claim that edge $(u_i, v_i)$ has at least $2^{i-1}$ bends in $\Gamma$. In fact, it suffices to prove that $(u_i, v_i)$ has $2^{i-1}$ intersections with the straight-line segment $ab$ in $\Gamma$. Indeed, $(u_1, v_1)$ has exactly one intersection with $ab$ in $\Gamma$. Inductively assume that $(u_i, v_i)$ has $2^{i-1}$ intersections with $ab$ in $\Gamma$; we prove that $(u_{i+1}, v_{i+1})$ has $2^i$ intersections with $ab$ in $\Gamma$. This proof is accomplished by following Kratochvíl and Matoušek [24] almost verbatim. Since $(u_{i+1}, v_{i+1})$ does not cross $(u_i, v_i)$, it has a bend $b_{i+1}$ around $v_i$, i.e., inside the square defined by $u_{i-2}$, $w_{i-2}$, $w_{i-1}$, and $u_{i-1}$. Thus the polygonal curve representing $(u_{i+1}, v_{i+1})$ in $\Gamma$ consists of
Figure 7: A drawing $\Gamma$ of $G$ that greedily extends $\mathcal{H}$ with respect to $\sigma$. Drawing $\mathcal{H}$ consists of the black circles. The first $n - N - 1$ edges in $\sigma$ are (black) straight-line segments. The last $N$ edges $(u_i, v_i)$ are (colored) polygonal lines whose bends have been made smooth to improve the readability. Only four of the latter edges are shown.

two parts—one from $u_{i+1}$ to $b_{i+1}$, the other from $b_{i+1}$ to $v_{i+1}$. Both of these parts may be used as an edge joining $u_i$ and $v_i$, after contracting $u_{i+1}$ and $v_{i+1}$ into $u_i$, and $b_{i+1}$ into $v_i$. Hence, by induction, each of these two parts has $2^{i-1}$ intersections with $ab$, and the whole edge $(u_{i+1}, v_{i+1})$ has $2^i$ intersections with $ab$.

Hence, in any drawing $\Gamma$ of $G$ that greedily extends $\mathcal{H}$ with respect to $\sigma$, one edge has $2^{N-1} = 2^\lceil \frac{n}{3} \rceil - 3 \in 2\Omega(n)$ bends, which concludes the proof.

We remark that the graph $G$ in the proof of Theorem 3 is a tree, so every edge of $G$ connects vertices in distinct connected components of $H$.

4 Simultaneous Planarity

Before turning to our algorithm to draw simultaneously planar graphs, we justify our claim that minimizing the number of crossings in a simultaneous planar drawing is NP-hard. This result follows from Cabello and Mohar’s proof of NP-hardness for the anchored planarity problem [6, Theorem 2.1], but a more direct proof of a slightly stronger result is possible by reduction from the NP-complete crossing number problem.

Theorem 4 Minimizing the number of crossings in a simultaneous planar drawing of two graphs is NP-complete, even if one graph is the disjoint union of paths of length at most two and the other graph is a matching.

The result is sharp in the sense that if both $G_1$ and $G_2$ are matchings, the problem is easy, since the union of two matchings is always planar.

Proof: We use the fact that the (standard) crossing number problem is NP-hard for cubic graphs [19]. Let $K$ be a cubic graph with $m$ edges. Subdivide each edge $2m$ or $2m + 1$ times (we will shortly see which). At each of the
original vertices of $K$ choose two of the incident edges, and make them part of $G_1$; the third edge at each vertex is added to $G_2$. Now add the remaining edges to $G_1$ and $G_2$ so that along each path between original vertices $G_1$ and $G_2$ edges alternate. If such a path ends with two $G_1$-edges or two $G_2$-edges, we need to subdivide it $2m$ times to make this possible; if it ends with one $G_1$-edge and one $G_2$-edge, we subdivide it $2m + 1$ times. By this construction, $G_1$ is a disjoint union of paths of length at most two, and $G_2$ is a matching; further, the common subgraph of $G_1$ and $G_2$ has the same vertex set as $G_1$ and $G_2$, and contains no edge. Finally, the number of crossings in a simultaneous planar drawing of $G_1$ and $G_2$ is an upper bound on the crossing number of $K$, and, since we subdivided each edge of $K$ sufficiently often, the two numbers are equal: starting with a crossing-minimal drawing of $K$, we can realize each crossing by aligning a $G_1$-edge with a $G_2$-edge. □

We now turn to the proof of Theorem 2.

**Proof of Theorem 2.** We first note that it is easy to go from (ii) to (i): Suppose we have constructed, in time $O(n^2)$ a simultaneous planar drawing $\Gamma$ so that a private edge of $G_1$ and a private edge of $G_2$ intersect at most 24 times. We add dummy vertices at the locations of the $O(n^2)$ crossings points in $\Gamma$, thus obtaining a planar drawing of a graph $L$. Observe that $L$ might have parallel edges, either between two dummy vertices or between a vertex and a dummy vertex. In either case, no more than two edges are parallel to each other, because one comes from part of an edge of $G_1$ and one comes from part of an edge of $G_2$. We consider two cases. If there are two parallel edges between two dummy vertices, then we can swap those two parts of the original edges to eliminate the two crossings altogether. Doing this involves splitting each dummy vertex into two degree-2 vertices, one in the $G_1$ edge and one in the $G_2$ edge. Note that we still have a planar graph, and we have not altered the rotation system. If there are two parallel edges between a vertex $v$ and a dummy vertex then we will not perform a swap since it might change the rotation system at vertex $v$. Instead, we will introduce one extra dummy vertex near $v$ in one of the parallel edges. With these modifications $L$ becomes a simple planar graph. We then construct a straight-line drawing of $L$ on a small grid. The number of bends in an edge is equal to the number of dummy vertices we added along the edge. Each edge in $\Gamma$ intersects at most $3n - 6$ edges, and intersects each one of them at most 24 times. The number of dummy vertices we added along the edge is therefore at most $24(3n - 6) + 2 \leq 72n$, where the $+2$ takes into account the extra dummy vertices we may have added near each endpoint of the edge.

We are left with the proof of (ii). That is, we have to construct in time $O(n^2)$ a simultaneous planar drawing of $G$ in which private edges of $G_1$ and $G_2$ intersect at most 24 times, all edges of $G_1$ are straight, and every private edge of $G_2$ has at most $72|V(G_1)|$ bends.

Start with an arbitrary straight-line planar drawing $\Gamma_1$ of $G_1$. We now construct a drawing $\Gamma_2$ of $G_2$ using an approach similar to the proof of Theorem 1. Drawing $\Gamma_1$ induces a straight-line planar drawing $\Gamma$ of $G$. Thus, in order to
determine \( \Gamma_2 \), it remains to describe how to draw the private edges of \( G_2 \). We will accomplish this independently for each face \( F \) of \( G \).

We construct a triangulation \( \Sigma \) of \( F \) by using all the vertices and edges of \( G_1 \) that lie inside \( F \). Next, we execute the same algorithm we used in the proof of Theorem 2. Namely, we construct a straight-line drawing of the dual \( D \) of \( \Sigma \) and we take a spanning tree \( T \) of \( D \). For each facial walk \( W_i \) of \( F \), we augment \( T \) with a leaf \( v_i \) close to \( W_i \) and inside \( F' \), if \( |W_i| > 1 \), and coinciding with \( W_i \), if \( |W_i| = 1 \); here, \( F' \) is an inner \( \epsilon \)-approximation of \( F \) constructed as earlier. Let \( G_2^F \) be the embedded multigraph obtained by restricting \( G_2 \) to the vertices and edges inside or on the boundary of \( F \), and by contracting each facial walk \( W_i \) of \( F \) to a single vertex \( v_i \). We use Lemma 3 to construct a planar poly-line drawing of \( G_2^F \) that realizes the given embedding, that is \( \epsilon \)-close to \( T \), and in which vertices \( v_i \) maintain their fixed locations. Finally, for boundary components with \( |W_i| > 1 \), we reconnect edges in \( G_2^F \) to the boundary components they belong to. In order to do this, we first “wrap” the edges of \( G_2^F \) passing by a vertex \( v_i \) around \( W_i \), and we then extend the edges of \( G_2^F \) incident to \( v_i \) to their endpoint on \( W_i \), by routing them around \( W_i \).

By construction every edge of \( G_1 \) is straight. By Theorem 1 every private edge of \( G_2 \) has at most \( 72|V(G_1)| \) bends. Also, the algorithmic steps are the same as for the proof of Theorem 1, hence the algorithm runs in \( O(n^2) \) time. It remains to prove that any private edge of \( G_1 \) and any private edge of \( G_2 \) intersect at most 24 times.

Consider any private edge \( e \) of \( G_2 \) and any private edge \( e' \) of \( G_1 \). Recall that \( e' \) is an edge of \( \Sigma \). Denote by \( W_i \) and \( W_j \) the facial walks that the end-vertices of \( e' \) belong to. Edge \( e \) can only intersect edge \( e' \) in the following two situations: when passing by \( v_i \) or \( v_j \) and when passing by the point \( p_T \) in which the edge of \( D \) is sub-

5 Open Questions

We conclude with three open questions. We proved that if a graph has a planar drawing extending a straight-line planar drawing of a subgraph then there is
such a drawing with at most 72\(n\) bends per edge. This is asymptotically tight, but can the constant 72 be reduced? As sketched at the end of Section 2 a variation of our algorithm decreases this constant to 63, however new ideas seem to be needed in order to push the bound further down.

Our second result was that any two simultaneously planar graphs have a simultaneous planar drawing with at most 24 crossings per pair of edges, a bound which was recently improved to 16 crossings per pair of edges [15]. The only lower bound on the number of crossings between two edges in a simultaneous planar drawing is 2 (see [9] or the figure in the margin for the entry “simultaneous crossing number” in [28]). There is a large gap between 2 and 16. Can two edges be forced to cross more than twice in a simultaneous planar drawing?

As a third open question, we note that Frati et al. [15] proved that two simultaneously planar graphs have a drawing with at most 6 bends per edge and 16 crossings per pair of edges, though not on a grid. Is it possible to achieve a constant number of bends per edge, a constant number of crossings per pair of edges, and a nice grid?

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