A relation for a class of Racah polynomials

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Abstract

In this paper we derive a relation for a class of Racah polynomials that appear in a conjecture of Kresch and Tamvakis. The relation follows from an inversion formula for a transformation of a discrete sequence of complex numbers \( \{x_n\}_{n=0}^{\infty} \). As a result of our inversion formula, we also obtain other combinatorial identities.

1 Introduction

Let \( p \) and \( q \) be non-negative integers. Let \( a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q, z \in \mathbb{C} \). The hypergeometric series of type \( \text{pFq} \) with numerator parameters \( a_1, a_2, \ldots, a_p \) and denominator parameters \( b_1, b_2, \ldots, b_q \) is defined by

\[
\text{pFq}\left[ a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{n!(b_1)_n(b_2)_n\cdots(b_q)_n} z^n, \tag{1.1}
\]

where the rising factorial \((a)_n\) is given by

\[
(a)_n = \begin{cases} 
  a(a + 1) \cdots (a + n - 1), & n > 0, \\
  1, & n = 0.
\end{cases}
\]

If no numerator parameter is a non-positive integer, we need no denominator parameter to be a non-positive integer. In this case, the series in (1.1) converges absolutely for all \( z \) if \( p < q + 1 \). If \( p > q + 1 \), the series converges only when \( z = 0 \). In the case \( p = q + 1 \), the series converges absolutely if \( |z| < 1 \) or if \( |z| = 1 \) and \( \text{Re}(\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i) > 0 \) (see [1, p. 8]).

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If a numerator parameter is a non-positive integer, then, letting $-n$ be the largest non-positive integer numerator parameter, only the first $n+1$ terms of the series (1.1) are non-zero and the series is said to terminate. In this case, we require that no denominator parameter be in the set $\{−n+1, −n+2, \ldots\}$. We note that (1.1) reduces to a polynomial in $z$ of degree $n$.

When $z = 1$, we say that the series is of unit argument and of type $pF_q(1)$. If $\sum_{i=1}^q b_i - \sum_{i=1}^p a_i = 1$, the series is called Saalschützian.

We will make use of the Chu-Vandermonde formula (see [1, p. 3]) for the sum of a terminating $2F_1(1)$ series:

$$2F_1\left[-n, a; b; 1\right] = \frac{(b-a)_n}{(b)_n}.$$  \hspace{1cm} (1.2)

We will also use the binomial coefficient identities

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \quad 1 \leq k \leq n,$$  \hspace{1cm} (1.3)

$$\binom{n}{k} \binom{k}{m} = \binom{n}{m} \binom{n-m}{k-m}, \quad 0 \leq m \leq k \leq n,$$  \hspace{1cm} (1.4)

and

$$\frac{\binom{m+k}{n+k}}{\binom{n+k}{m+k}} = \frac{\binom{n}{m}}{\binom{n+k}{m+k}}, \quad 0 \leq m \leq n, \quad k \geq 0.$$  \hspace{1cm} (1.5)

The Racah polynomials, first given by Wilson [7], are defined by (see also [1])

$$R_n(\lambda(x); \alpha, \beta, \gamma, \delta) = 4F_3\left[-n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1; \alpha+1, \beta+\delta+1, \gamma+1; 1\right],$$  \hspace{1cm} (1.6)

where

$$\lambda(x) = x(x+\gamma+\delta+1)$$

and

$$\alpha+1 = -N \text{ or } \beta+\delta+1 = -N \text{ or } \gamma+1 = -N, \text{ with } N \text{ a non-negative integer.}$$

We note that the Racah polynomials are terminating Saalschützian $4F_3(1)$ hypergeometric series.
The special case $\alpha = \beta = \gamma + \delta = 0, \gamma = T$, where $T$ is a positive integer leads to the definition of

$$R_n(s, T) := R_n(\lambda(s); 0, 0, T, -T) = \binom{-n, n + 1, -s, s + 1}{1, 1 - T, 1 + T; 1},$$

where $0 \leq n, s \leq T - 1$.

It is conjectured by Kresch and Tamvakis [5] that

$$|R_n(s, T)| \leq 1 \quad (1.8)$$

for all $0 \leq n, s \leq T - 1, T \geq 1$. Special cases of the conjecture are proven by Kresch and Tamvakis in [5]. Special cases of the conjecture are also proven by Ismail and Simeonov [2]. Furthermore, Ismail and Simeonov demonstrate asymptotics for $R_n(s, T)$ in [2] that are in agreement with the conjecture.

2 Main result

Definition 2.1. Let $\{x_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$. For each $n \geq 0$, we define

$$\tilde{x}_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n + k}{k} x_k. \quad (2.1)$$

We note that the transformation (2.1) is linear.

In view of the formulas

$$(-1)^k \binom{n}{k} = \frac{(-n)_k}{k!} \quad (2.2)$$

and

$$\binom{n + k}{k} = \frac{(n + 1)_k}{k!}, \quad (2.3)$$

equation (2.1) can also be written as

$$\tilde{x}_n = \sum_{k=0}^{n} \frac{(-n)_k(n + 1)_k}{k!} x_k. \quad (2.4)$$

We remark that the transformation in Definition 2.1 is inspired by the binomial transform (introduced by Knuth in [3]) of a sequence $\{x_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ defined by

$$\hat{x}_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x_k, \quad n \geq 0. \quad (2.5)$$
The inversion formula for the binomial transform is well-known (see [6]) and is
\[ x_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \xi_k, \quad n \geq 0. \] (2.6)

Certain terminating hypergeometric series can be considered as binomial transforms. For example, the Chu-Vandermonde formula (1.2) can be written as
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a}{k} \binom{b}{k} = (b-a)_n, \] and therefore we can conclude that the binomial transform of the sequence \( \{ (a)_n \} \) is the sequence \( \{ (b-a)_n \} \).

**Theorem 2.2.** Let \( \{ x_n \}_{n=0}^{\infty} \subseteq \mathbb{C} \). Let \( \{ \xi_n \}_{n=0}^{\infty} \subseteq \mathbb{C} \) be defined by (2.1). Then for each \( n \geq 0 \), we have
\[ x_n = \sum_{k=0}^{n} (-1)^k \binom{2k+1}{k} \binom{n}{k+1} \xi_k. \] (2.7)

Equation (2.7) gives us the inverse transformation of the transformation defined in (2.1).

Using the formulas (2.2) and (2.3), we can also write equation (2.7) as
\[ x_n = \sum_{k=0}^{n} \frac{(2k+1)(-n)_k}{(n+1)_{k+1}} \xi_k, \quad n \geq 0. \] (2.8)

Before we prove Theorem 2.2, we need the following lemma:

**Lemma 2.3.** For each \( n \geq 1 \), we have
\[ \sum_{k=0}^{n-1} (-1)^k (2k+1) \binom{2n+1}{n-k} = (-1)^{n-1} (2n+1). \] (2.9)

**Proof.** The result is directly verified when \( n = 1, 2, 3, 4 \). Assume \( n \geq 5 \). We let
\[ A = \sum_{k=0}^{n-1} (-1)^k (2k+1) \binom{2n+1}{n-k}. \]
We split off the first two terms and the last two terms of $A$ to get

$$A = \binom{2n+1}{n} - 3 \binom{2n+1}{n-1} + \sum_{k=2}^{n-3} (-1)^k (2k+1) \binom{2n+1}{n-k}$$

$$+ (-1)^{n-2}(2n-3) \binom{2n+1}{2} + (-1)^{n-1}(2n-1) \binom{2n+1}{1}. $$

For $0 \leq k \leq n-2$, we apply (1.3) twice to $\binom{2n+1}{n-k}$ and obtain

$$\binom{2n+1}{n-k} = \binom{2n}{n-k} + \binom{2n}{n-k-1} = \binom{2n-1}{n-k} + 2 \binom{2n-1}{n-k-1} + \binom{2n-1}{n-k-2}. $$

Also, by (1.3),

$$\binom{2n+1}{1} = \binom{2n}{1} + \binom{2n}{0} = \binom{2n-1}{1} + \binom{2n-1}{0} + \binom{2n}{0}. $$

Therefore,

$$A = \binom{2n-1}{n} + 2 \binom{2n-1}{n-1} + \binom{2n-1}{n-2}$$

$$- 3\left( \binom{2n-1}{n-1} + 2 \binom{2n-1}{n-2} + \binom{2n-1}{n-3} \right)$$

$$+ \sum_{k=2}^{n-3} (-1)^k (2k+1) \left( \binom{2n-1}{n-k} + 2 \binom{2n-1}{n-k-1} + \binom{2n-1}{n-k-2} \right)$$

$$+ (-1)^{n-2}(2n-3) \left( \binom{2n-1}{2} + 2 \binom{2n-1}{1} + \binom{2n-1}{0} \right)$$

$$+ (-1)^{n-1}(2n-1) \left( \binom{2n-1}{1} + \binom{2n-1}{0} + \binom{2n}{0} \right). $$

From here, we can write

$$A = A_1 + A_2 + A_3,$$

where

$$A_1 = \binom{2n-1}{n} + 2 \binom{2n-1}{n-1} - 3 \binom{2n-1}{n-1},$$
\[ A_2 = \binom{2n-1}{n-2} - 3 \left( 2 \binom{2n-1}{n-2} + \binom{2n-1}{n-3} \right) \]
\[ + \sum_{k=2}^{n-3} (-1)^k (2k+1) \left( \binom{2n-1}{n-k} + 2 \binom{2n-1}{n-k-1} + \binom{2n-1}{n-k-2} \right) \]
\[ + (-1)^{n-2}(2n-3) \left( \binom{2n-1}{2} + 2 \binom{2n-1}{1} \right) \]
\[ + (-1)^{n-1}(2n-1) \binom{2n-1}{1}, \]
and
\[ A_3 = (-1)^{n-2}(2n-3) \binom{2n-1}{0} + (-1)^{n-1}(2n-1) \left( \binom{2n-1}{0} + \binom{2n}{0} \right). \]

Since \( \binom{2n-1}{n} = \binom{2n-1}{n-1} \), we have that \( A_1 = 0 \).

To evaluate \( A_2 \), we have
\[ A_2 = \sum_{k=0}^{n-3} (-1)^k (2k+1) \binom{2n-1}{n-k-2} \]
\[ + \sum_{k=1}^{n-2} (-1)^k (2k+1)(2) \binom{2n-1}{n-k-1} + \sum_{k=2}^{n-1} (-1)^k (2k+1) \binom{2n-1}{n-k} \]
\[ = \sum_{k=1}^{n-2} (-1)^{k-1}(2k-1) \binom{2n-1}{n-k-1} + \sum_{k=1}^{n-2} (-1)^k (4k+2) \binom{2n-1}{n-k-1} \]
\[ + \sum_{k=1}^{n-2} (-1)^{k+1}(2k+3) \binom{2n-1}{n-k-1} = 0, \]
where the last equality follows by combining the three sums into one and factoring out \( (-1)^{k-1}(2n-1) \).

Finally,
\[ A_3 = (-1)^{n-2}(2n-3) + (-1)^{n-1}(4n-2) = (-1)^{n-1}(2n+1). \]

Therefore,
\[ A = A_1 + A_2 + A_3 = (-1)^{n-1}(2n+1), \]
which completes the proof. \( \Box \)
Proof of Theorem 2.2. We will prove that equation (2.8) holds for each \( n \geq 0 \). In our proof, we will use the equivalent form (2.4) of (2.1).

For \( n, m \geq 0 \), we define
\[
a_{n,m} = \frac{(2m + 1)(-n)_m}{(n + 1)_{m+1}}.
\]

We note that \( a_{n,m} = 0 \) for \( m > n \), since \((-n)_m = 0\) for \( m > n \). In order to prove the theorem, we need to show that for each \( m \geq 0 \), the transformation (2.4) of the sequence \( \{a_{n,m}\}_{n=0}^{\infty} \) is given by
\[
\tilde{a}_{n,m} = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases} \tag{2.10}
\]

When \( m = 0 \), we have \( \{a_{n,0}\}_{n=0}^{\infty} = \{\frac{1}{(n+1)}\}_{n=0}^{\infty} \). Using the Chu-Vandermonde formula (1.2), we have
\[
\tilde{a}_{n,0} = \sum_{k=0}^{n} \frac{(-n)_k(n+1)_k}{k!k!} \frac{1}{k+1} = \frac{1}{(p+2)!} \sum_{k=0}^{n} \frac{(-n)_k(n+1)_k}{k!(2)_k} = 2F_1 \left[ \begin{array}{c} -n, n+1; \\ 2; 1 \end{array} \right] = 1 \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}
\]

Therefore (2.10) holds for \( m = 0 \). Assume now that (2.10) holds for all \( m = 0, 1, \ldots, p \), for some \( p \geq 0 \). We will show that (2.10) holds for \( m = p + 1 \) and the result will follow by induction.

For \( n \geq 0 \), we define
\[
b_{n,p+1} = \frac{1}{(n+1)_{p+2}}.
\]

Using the Chu-Vandermonde formula (1.2), we have
\[
\tilde{b}_{n,p+1} = \sum_{k=0}^{n} \frac{(-n)_k(n+1)_k}{k!k!} \frac{1}{(k+1)_{p+2}} = \frac{1}{(p+2)!} \sum_{k=0}^{n} \frac{(-n)_k(n+1)_k}{k!(p+3)_k} = \frac{1}{(p+2)!} 2F_1 \left[ \begin{array}{c} -n, n+1; \\ p+3; 1 \end{array} \right] = \frac{1}{(p+2)!} \frac{(p+2-n)_n}{(p+3)_n} = \frac{(p+2-n)_n}{(p+n+2)!}.
\]
When $n > p + 1$, we have that $(p + 2 - n)_n = 0$ and so $\tilde{b}_{n,p+1} = 0$. When $n = p + 1$, we have

$$\tilde{b}_{p+1,p+1} = \frac{(1)_{p+1}}{(2p + 3)!} = \frac{1}{(p + 2)_{p+2}}.$$  

Using the induction hypothesis and the fact that the transformation (2.4) is linear, it follows that if we define

$$c_{n,p+1} = (p + 2)_{p+2}b_{n,p+1} - \sum_{m=0}^{p} \frac{(p + 2)_{p+2}(p + 2 - m)_m}{(p + m + 2)!}a_{n,m}, \quad n \geq 0,$$

then we will have

$$\tilde{c}_{n,p+1} = \begin{cases} 1, & n = p + 1, \\ 0, & n \neq p + 1. \end{cases}$$

It remains to show that $c_{n,p+1} = a_{n,p+1}$ for all $n \geq 0$.

We compute

$$\frac{(p + 2)_{p+2}(p + 2 - m)_m}{(p + m + 2)!} = \frac{(2p + 3)!}{(p + 1 - m)! (p + m + 2)!} = \binom{2(p + 1) + 1}{p + 1 - m}.$$  

Hence for each $n \geq 0$,

$$c_{n,p+1} = \frac{(p + 2)_{p+2}}{(n + 1)_{p+2}} - \sum_{m=0}^{p} \binom{2(p + 1) + 1}{p + 1 - m} \frac{(2m + 1)(-n)_m}{(n + 1)_{m+1}}. \quad (2.11)$$

By combining all terms under a common denominator, it follows that we can write $c_{n,p+1} = \frac{f(n)}{(n + 1)_{p+2}}$, where $f(n)$ is a polynomial in $n$ of degree at most $p + 1$. Now since $\tilde{c}_{n,p+1} = 0$ for $n = 0, 1, \ldots, p$, we must have $c_{n,p+1} = 0$ for $n = 0, 1, \ldots, p$. But then $f(n) = 0$ for $n = 0, 1, \ldots, p$ and so we must have $f(n) = \alpha \prod_{i=0}^{p}(n - i)$ for some $\alpha \in \mathbb{R}$. In view of (2.11),

$$\alpha = -\sum_{m=0}^{p} (-1)^m (2m + 1) \binom{2(p + 1) + 1}{p + 1 - m}.$$  

By Lemma 2.3, $\alpha = (-1)^{p+1}(2(p + 1) + 1)$. Therefore,

$$c_{n,p+1} = \frac{(-1)^{p+1}(2(p + 1) + 1) \prod_{i=0}^{p}(n - i)}{(n + 1)_{p+2}} = \frac{(2(p + 1) + 1)(-n)_{p+1}}{(n + 1)_{p+2}} = a_{n,p+1} \text{ for all } n \geq 0,$$
which shows that (2.10) holds for $m = p + 1$ and completes the proof by induction.

Interesting consequences of Theorem 2.2 are given below:

**Corollary 2.4.** We have the following identities:

(a) For $0 \leq m \leq n$,

$$
\sum_{k=m}^{n} (-1)^k \frac{(2k+1)\binom{n-m}{k-m}}{(n+k+1)\binom{n+k}{m+k}} = \begin{cases} (-1)^n, & m = n, \\ 0, & 0 \leq m < n. \end{cases} \quad (2.12)
$$

(b) For $n \geq 0$,

$$
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{n+k}{k} \frac{1}{2k+1} = \frac{1}{2n+1}. \quad (2.13)
$$

**Proof.** (a) Using Theorem 2.2 and then switching the order of summation, we have that for every $n \geq 0$,

$$
x_n = \sum_{k=0}^{n} (-1)^k \frac{(2k+1)\binom{n}{k}}{(n+k+1)\binom{n+k}{k}} \tilde{x}_k
$$

$$
= \sum_{k=0}^{n} (-1)^k \frac{(2k+1)\binom{n}{k}}{(n+k+1)\binom{n+k}{k}} \left( \sum_{m=0}^{k} (-1)^m \binom{k}{m} \binom{k+m}{m} x_m \right)
$$

$$
= \sum_{m=0}^{n} \left( (-1)^m \frac{\sum_{k=m}^{n} (-1)^k \binom{2k+1}{k} \binom{n}{k} \binom{k+m}{m}}{(n+k+1)\binom{n+k}{k}} x_m \right)
$$

$$
= \sum_{m=0}^{n} \left( (-1)^m \frac{n}{m} \sum_{k=m}^{n} (-1)^k \binom{2k+1}{k} \binom{n-m}{k-m} \binom{n+k}{m+k} \binom{n+k}{m+k} \binom{n+k}{m+k} \binom{n+k}{m+k} x_m \right),
$$

where the last equality follows from (1.4) and (1.5). Hence we must have

$$
(-1)^m \frac{n}{m} \sum_{k=m}^{n} (-1)^k \binom{2k+1}{k} \binom{n-m}{k-m} \binom{n+k}{m+k} \binom{n+k}{m+k} \binom{n+k}{m+k} = \begin{cases} 1, & m = n, \\ 0, & 0 \leq m < n, \end{cases}
$$

and this implies (2.12).
(b) It is enough to show that the sequence \( \left\{ \frac{1}{2n+1} \right\}_{n=0}^{\infty} \) is fixed by the inverse transformation of (2.1). Indeed, we have

\[
\sum_{k=0}^{n} \frac{(2k+1)(-n)_k}{(n+1)_{k+1}} \frac{1}{2k+1} = \sum_{k=0}^{n} \frac{(-n)_k}{(n+1)_{k+1}} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{(-n)_k}{(n+2)_k}
\]

\[
\frac{1}{n+1} {}_2F_1 \left[ -n, 1; 1 \right] = \frac{1}{n+1} \frac{(n+1)_n}{(n+2)_n} = \frac{1}{2n+1},
\]

where in the next-to-last step we used the Chu-Vandermonde formula (1.2).

The special case \( m = 0 \) in (2.12) gives

\[
\sum_{k=0}^{n} (-1)^k \frac{(2k+1)(n)_k}{(n+k+1)(n+k)_k} = \begin{cases} 1, & n = 0, \\ 0, & n > 0. \end{cases}
\tag{2.14}
\]

In view of (2.8), we can also write (2.14) as

\[
\sum_{k=0}^{n} \frac{(2k+1)(-n)_k}{(n+1)_{k+1}} = \begin{cases} 1, & n = 0, \\ 0, & n > 0. \end{cases}
\tag{2.15}
\]

We note that (2.13) implies that the sequence \( \left\{ \frac{1}{2n+1} \right\}_{n=0}^{\infty} \) is fixed by the transformation (2.1). In fact, since in the last term of the sum

\[
\tilde{x}_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} x_k
\]

the coefficient in front of \( x_n \) is \((-1)^n \binom{2n}{n} \neq 1\) for \( n > 0 \), it follows that, up to constant multiples, the sequence \( \left\{ \frac{1}{2n+1} \right\}_{n=0}^{\infty} \) is the only one fixed by the transformation (2.1).

**Corollary 2.5.** Let \( 0 \leq s \leq T - 1 \). Then for every \( m \) such that \( 0 \leq m \leq T - 1 \), we have

\[
\sum_{n=0}^{m} (-1)^n \frac{(2n+1)(m)_n}{(m+n+1)(m+n)_n} R_n(s, T) = \frac{(-s)_m(s + 1)_m}{(1-T)_m(1+T)_m}.
\tag{2.16}
\]
Proof. Let
\[ x_n = \begin{cases} 
\frac{(-s)_n(s+1)_n}{(1-T)_n(1+T)_n}, & 0 \leq n \leq T - 1 \\
0, & n > T - 1.
\end{cases} \]

Then for \( 0 \leq n \leq T - 1 \),
\[
R_n(s, T) = \binom{n + 1}{1} \frac{(-s)_n(s+1)_n}{(1-T)_n(1+T)_n} = \tilde{x}_n
\]

Theorem 2.2 now yields the result. \( \square \)

In view of (2.8), we can also write (2.16) as
\[
\sum_{n=0}^{m} \frac{(2n+1)(-m)_n}{(m+1)_n} R_n(s, T) = \frac{(-s)_m(s+1)_m}{(1-T)_m(1+T)_m}, \quad (2.17)
\]
for all \( m \) such that \( 0 \leq m \leq T - 1 \).

We note that \( \frac{(-s)_m(s+1)_m}{(1-T)_m(1+T)_m} = 0 \) if \( s + 1 \leq m \leq T - 1 \), and so we have
\[
\sum_{n=0}^{m} (-1)^n \frac{(2n+1)(m)}{(m+n+1)(m+n)} R_n(s, T) = 0, \quad s + 1 \leq m \leq T - 1, \quad (2.18)
\]
or, equivalently,
\[
\sum_{n=0}^{m} \frac{(2n+1)(-m)_n}{(m+1)_n} R_n(s, T) = 0, \quad s + 1 \leq m \leq T - 1. \quad (2.19)
\]

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