A New Kind of Deformed Hermite Polynomials and Its Applications

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Abstract
A new kind of deformed calculus was introduced recently in studying of parabosonic coordinate representation. Based on this deformed calculus, a new deformation of Hermite polynomials is proposed, its some properties such as generating function, orthonormality, differential and integral representations, and recursion relations are also discussed in this paper. As its applications, we calculate explicit forms of parabose squeezed number states, derive a particularly simple subset of minimum uncertainty states for parabose amplitude-squared squeezing, and discuss their basic squeezing behaviours.

1 Introduction
Parastatistics was introduced by Green as an exotic possibility extending the Bose and Fermi statistics and for the long period of time the interest to it was rather academic. Nowadays it finds some applications in the physics of the quantum Hall effect and (probably) it is relevant to high temperature superconductivity. The paraquantization, carried out at the level of the algebra of creation and annihilation operators, involves trilinear(or double)

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commutation relations in place of the bilinear relations that characterize Bose and Fermi statistics. Recently, the trilinear commutation relations of single paraparticle systems was rewritten as bilinear commutation relations by virtue of the so-called R-deformed Heisenberg algebra. For instance, the trilinear commutation relations

\[
\begin{align*}
[a, \{a^\dagger, a\}] &= 2a, \\
[a, \{a^\dagger, a^\dagger\}] &= 4a^\dagger, \\
[a, \{a, a\}] &= 0,
\end{align*}
\]

where \(a^\dagger\) and \(a\) are parabose creation and annihilation operators respectively, can be replaced by

\[
\begin{align*}
[a, a^\dagger] &= 1 + (p - 1)R, \\
\{R, a\} &= \{R, a^\dagger\} = 0, \\
R^2 &= 1,
\end{align*}
\]

where \(R\) is a reflection operator and \(p\) is the paraquantization order (\(p = 1, 2, 3, \ldots\)). Obviously, the bilinear commutation relations (2) may be treated as some kind of deformation of the ordinary Bose commutator with deformation parameter \(p\).

From the experience of studying q-deformed oscillators, we know that it will be very useful if one introduces corresponding deformed calculus to analyse the parabose systems. This was done recently and based on the new deformed calculus, the parabosonic coordinate representation was developed. Since special functions play important roles in mathematical physics, it is reasonable to imagine that some deformation of the ordinary special functions based on the new deformed calculus will also play similar roles in studying the parabose systems. In this paper, we introduce a new kind of deformation for the ordinary Hermite polynomials and demonstrate its various useful properties.

It is well-known that squeezed state is one of the most important non-classical states for the usual bose system. In fact in phase space the simultaneous measurements of canonical variables \(x\) (position) and \(P\) (momentum) are restricted to a limit of accuracy due to the Heisenberg uncertainty principle, one may attempt to measure one component in a quadrature to much greater accuracy at the cost of high uncertainty in the other component. This has been realized in the case of squeezed states for bosons. Since the squeeze operator is identical for both the ordinary and parabose systems, it is natural to investigate squeezing behaviours of a parabose system. Applying the squeeze operator to parabose number states gives the so-called parabose squeezed number states. We derive explicit forms for these states and show they can be expressed by virtue of the deformed Hermite polynomial, whose argument is the parabose creation operator multiplied by a constant, acting.
on the vacuum. We discuss basic squeezing properties of these states. Especially, we show the parabose squeezed vacuum is a minimum uncertainty state for normal squeezing (i.e., either in $x$ and $P$ direction squeezing), as well as for amplitude-squared squeezing (i.e., squeezing in variables that are quadratic of the creation or annihilation operators). It is also interesting to ask, besides the para-squeezed vacuum, if there exist other states for a one mode parabose system, which are not only amplitude-squared squeezing but also minimum uncertainty states? We present a particularly simple subset of such states, that is, the deformed Hermite polynomial states. These states may or may not be squeezed in the normal sense.

The paper is organized as follows. In Section 2, for the sake of self-contained of the present paper, we briefly mention the basic idea of the new kind of deformed calculus. The deformed Hermite polynomials are introduced in section 3, their orthonormality, generating function, differential and integral representations, and recursion relations are also discussed in this section. In section 4, we show that the deformed Hermite polynomials can be used to give the explicit forms for parabose squeezed number states, which are constructed by applying the squeeze operator to the parabose number states, and analyse the basic squeezing properties of these states. In section 5 by solving an eigenvalue equation that allows one to find the minimum uncertainty states, we present the subset of its solutions which has a particularly simple form, and discuss the properties of these states.

2 Deformed calculus related to parabosonic coordinate representation

It is well-known that parabose algebra is characterized by the double commutation relations (1). If one demands that the usual relations

$$a = \frac{x + iP}{\sqrt{2}}, \quad a^\dagger = \frac{x - iP}{\sqrt{2}}$$

still work for the parabose case, where $x$ and $P$ stand for the coordinate and momentum operator respectively, it can be proved that the most general expression for the momentum operator $P$ in the coordinate $x$ diagonal representation is of

$$P = -i \frac{d}{dx} - i \frac{p - 1}{2x}(1 - R),$$

3
where \( p \) is the paraquantization order and \( R \) the reflection operator which has property \( R f(x) = f(-x) \) in the coordinate representation for any \( x \) dependent function \( f(x) \). From (4) a new derivative operator \( D \) can be defined which acts on function \( f(x) \) as

\[
Df(x) \equiv \frac{D}{Dx}f(x) = \frac{d}{dx}f(x) + \frac{p-1}{2x}(1-R)f(x)
\]

\[
= df(x) + \frac{p-1}{2x}(f(x) - f(-x)),
\]

(5)

where \( df = \frac{d}{dx}f \). Definition (5) implies that \( D \) acts on an even function \( f_e(-x) = f_e(x) \) as the ordinary derivative \( Df_e(x) = df_e(x) \), and \( D \) acting on an odd function \( f_o(-x) = -f_o(x) \) leads to \( Df_o(x) = df_o(x) + \frac{p-1}{2x}f_o(x) \). For \( p = 1 \) case, \( D \) reduces to the ordinary derivative operator \( d \). Since \( P = -iD \), Eq.(3) means that the pair \((x,D)\) in realization of parabose algebra for a single degree of freedom plays the same role as \((x,d)\) in realization of the ordinary Bose algebra. Like q-deformed calculus in which the q-analogue of the number system was defined by \([n]_q = \frac{q^n - 1}{q - 1}\) [7], such that when \( q \to 1 \), \([n]_q \to n\), in the present case, one can introduce a new kind of deformed number system which is defined by

\[
[n] = n + \frac{p-1}{2}(1 - (-)^n).
\]

(6)

Obviously, \([2k] = 2k\), \([2k + 1] = 2k + p\) for any integer \( k \) and when \( p \to 1 \), \([n] \to n\). So paraquantization order \( p \) may be referred to as a deformation parameter. In terms of the number system \([n]\), basis vectors of Fock space for single mode of parabose oscillators take the usual form

\[
|n\rangle = (a^\dagger)^n|0\rangle, \quad a^\dagger|n\rangle = \sqrt{[n+1]}|n+1\rangle, \quad a|n\rangle = \sqrt{[n]}|n-1\rangle,
\]

(7)

where \([n]! = [n][n-1]...[1]\), \([0]! \equiv 1\), and \(|0\rangle\) is the unique vacuum vector satisfying \( a|0\rangle = 0, aa^\dagger|0\rangle = p|0\rangle \). Generalization of the ordinary differential relation \( dx^n = nx^{n-1} \) reads

\[
Dx^n = [n]x^{n-1},
\]

(8)

which reveals the effect of the deformed derivative operator \( D \) on the polynomials of \( x \). If we introduce a notation \( E(x) \) defined by

\[
E(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!},
\]

(9)
we also have

\[ DE(x) = E(x) \]  \hspace{1cm} (10)

Therefore the \( E(x) \) is a deformation of the ordinary exponential function \( e^x \) in our case and it will reduce to \( e^x \) when \( p \to 1 \). It is worthy of mention that in some special situation the usual Leibnitz rule also works for the deformed operator \( D \)

\[ D(f g) = (Df) g + f (Dg), \]  \hspace{1cm} (11)

where either \( f(x) \) or \( g(x) \) is an even function of \( x \).

Of course, inversion of the deformed derivative operator \( D \) also leads to a new deformed integration which may be formally written as

\[
\int DxF(x) = \sum_{n=0}^{\infty} (-)^n \left( \int \frac{p - 1}{2x} (1 - R) \right)^n \int dx F(x) \\
= \int dx F(x) - \int dx \frac{p - 1}{2x} (1 - R) \int dx F(x) \\
+ \left( \int dx \frac{p - 1}{2x} (1 - R) \right)^2 \int dx F(x) - \cdots. \]  \hspace{1cm} (12)

From this expression, it is easily seen that if \( F(x) \) is an odd function of \( x \), its deformed integration will reduce to the ordinary integration, that is, \( \int Dx F(x) = \int dx F(x) \) for \( F(-x) = -F(x) \). Corresponding to Eq. (8), one has

\[ \int Dx x^n = \frac{x^{n+1}}{[n + 1]} + c, \]  \hspace{1cm} (13)

where \( c \) is an integration constant. Eq.(12) gives a formal definition for the deformed integration in the sense of indefinite integral. For definite integral, we have

\[
\int_a^b Dx F(x) = \int_a^b dx \sum_{n=0}^{\infty} (-)^n \left( \int_a^x \frac{p - 1}{2x} (1 - R) \int_a^x dx \right)^n F(x) \\
= \int_a^b dx F(x) - \int_a^b dx \frac{p - 1}{2x} (1 - R) \int_a^x dx F(x) \\
+ \int_a^b dx \frac{p - 1}{2x} (1 - R) \int_a^x dx \frac{p - 1}{2x} (1 - R) \int_a^x dx F(x) - \cdots. \]  \hspace{1cm} (14)

If either \( F(x) \) or \( G(x) \) is an even function of \( x \), one has a formula of integration by parts from Eq.(11)

\[
\int_a^b Dx \frac{DF}{Dx} G = FG|_a^b - \int_a^b Dx F \frac{DG}{Dx}. \]  \hspace{1cm} (15)

5
3 Deformed Hermite polynomials and their properties

Let us consider solutions of a second-order differential equation based on the deformed derivative operator \( D \) defined in the previous section

\[
D^2 f(x) - 2x D f(x) + \mu f(x) = 0. \tag{16}
\]

In terms of the ordinary derivative notation, Eq.(16) can be rewritten as

\[
\frac{d^2}{dx^2} f(x) - \left( 2x - \frac{p-1}{x} \right) \frac{df}{dx} f(x) - (p-1) \left( 1 + \frac{1}{2x^2} \right) f(x)
+ (p-1) \left( 1 + \frac{1}{2x^2} \right) f(-x) + \mu f(x) = 0. \tag{17}
\]

We find out that when the parameter \( \mu \) takes eigenvalues \( \mu = 2[n], n = 0, 1, 2, 3, ... \), for each given paraquantization order \( p \), the deformed second-order differential equation (16) has solutions (eigenfunctions) which form a set of orthogonal functions in the whole real \( x \) axis. In fact, it is not difficult to see that the following polynomials

\[
H_n^{(p)}(x) = [n!]^{\left[\frac{n}{2}\right]} \sum_{k=0}^{[n/2]} \frac{(-)^k (2x)^{n-2k}}{k!(n-2k)!} \tag{18}
\]

are the desired solutions of the deformed second-order differential equation (16) for \( \mu = 2[n] \) which will reduce to the usual Hermite polynomials when \( p \to 1 \), where \( [k] \) in the above of summation notation \( \sum \) stands for the largest integer smaller than or equal to \( k \). So the polynomials (18) may be considered as a deformation of the usual Hermite polynomials. The first few polynomials of \( H_n^{(p)}(x) \) have the following explicit forms

\[
H_0^{(p)}(x) = 1, \quad H_1^{(p)}(x) = 2x, \quad H_2^{(p)}(x) = 4x^2 - [2]!,
H_3^{(p)}(x) = 8x^3 - 4[3]x, \quad H_4^{(p)}(x) = 16x^4 - 16[3]x^2 + 2[3]!,
H_5^{(p)}(x) = 32x^5 - 32[5]x^3 + 8[5][3]x, \\
H_6^{(p)}(x) = 64x^6 - 96[5]x^4 + 48[5][3]x^2 - [5]!. \tag{19}
\]

In order to convince oneself that the polynomials (18) are indeed solutions of eq.(16), one may substitute (18) into (16) and check coefficients of all
powers of \( x \) being zero. To do this, more convenient forms for \( H_{n}^{(p)}(x) \) are

\[
H_{2l}^{(p)}(x) = (-)^l [2l]! \sum_{k=0}^{l} \frac{(-)^k (2x)^{2k}}{(l-k)! [2k]!} \tag{20}
\]

and

\[
H_{2l+1}^{(p)}(x) = (-)^l [2l+1]! \sum_{k=0}^{l} \frac{(-)^k (2x)^{2k+1}}{(l-k)! [2k+1]!}. \tag{21}
\]

where \( l \) are non-negative integers. Thus we have

\[
D^2 H_{n}^{(p)}(x) - 2x DH_{n}^{(p)}(x) + 2[n] H_{n}^{(p)}(x) = 0, \tag{22}
\]

or, by virtue of the Leibnitz rule (11), we also have

\[
D \left( e^{-x^2} D H_{n}^{(p)}(x) \right) + 2[n] e^{-x^2} H_{n}^{(p)}(x) = 0. \tag{23}
\]

Also from Eq.(18) we know that \( H_{n}^{(p)}(-x) = (-)^n H_{n}^{(p)}(x) \), which means that the deformed Hermite polynomial \( H_{n}^{(p)}(x) \) has its parity \((-)^n\).

As in the ordinary Hermite polynomials case, the deformed ones also have their generating function. Using the deformed exponential function \( E(x) \) defined by (9), we can write the generating function of \( H_{n}^{(p)}(x) \) as

\[
e^{-t^2} E(2tx) = \sum_{n=0}^{\infty} \frac{t^n}{[n]!} H_{n}^{(p)}(x). \tag{24}
\]

We also would like to point out a differential representation of the deformed Hermite polynomials \( H_{n}^{(p)}(x) \) being

\[
H_{n}^{(p)}(x) = (-)^n e^{x^2} D^n e^{-x^2}. \tag{25}
\]

Before writing out an integral representation of \( H_{n}^{(p)}(x) \), let us introduce a notation \([x+y]^n\) which is defined as

\[
[x+y]^n \equiv \sum_{k=0}^{n} \frac{[n]!}{[k]! [n-k]!} x^{n-k} y^k. \tag{26}
\]

Using this notation, the integral representation of \( H_{n}^{(p)}(x) \) can be written as

\[
H_{n}^{(p)}(x) = 2^n N_0^2 \int_{-\infty}^{\infty} Dt [x+it]^n e^{-t^2}, \tag{27}
\]
where \( N_0 \) is determined by

\[
N_0^{-2} = \int_{-\infty}^{\infty} Dt e^{-t^2}.
\]  

(28)

Using the following integral formula [8]

\[
N_0^2 \int_{-\infty}^{\infty} Dt t^{2n} e^{-t^2} = \frac{[1][3] \cdots [2n-1]}{2^n}, \quad (n > 0)
\]

(29)

it is easily to show that equation (27) works. In fact,

\[
2^n N_0^2 \int_{-\infty}^{\infty} D t [x + it]^n e^{-t^2} = 2^n N_0^2 \sum_{k=0}^{n} \frac{[n]! i^k x^{n-k}}{[k]! [n-k]!} \int_{-\infty}^{\infty} Dt^k e^{-t^2}.
\]  

(30)

Noticing that for odd \( k \), the deformed integration in the right-hand side of (30) will reduce to ordinary integration which has no any contribution to the summation, the above summation can be written as

\[
2^n N_0^2 \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-)^k x^{n-2k}}{[2k]! [n-2k]!} \int_{-\infty}^{\infty} Dt^k e^{-t^2} = H_n^{(p)}(x),
\]

(31)

where (29) has been used.

Now we turn to the question of demonstrating the orthonormality of \( H_n^{(p)}(x) \)

\[
\int_{-\infty}^{\infty} Dx e^{-x^2} H_n^{(p)}(x) H_m^{(p)}(x) = \frac{2^n [n]!}{N_0^2} \delta_{n,m}.
\]  

(32)

Firstly we show the orthogonality of \( H_n^{(p)}(x) \)

\[
\int_{-\infty}^{\infty} Dx e^{-x^2} H_n^{(p)}(x) H_m^{(p)}(x) = 0, \quad (n \neq m).
\]

(33)

For \( n + m \) odd case, Eq.(33) works obviously. In fact, from the parity of \( H_n^{(p)}(x) \) we have \( H_n^{(p)}(-x)H_m^{(p)}(-x) = -H_n^{(p)}(x)H_m^{(p)}(x) \), which means that the deformed integration in (33) will reduce to an ordinary integration with an odd integrand \( e^{-x^2} H_n^{(p)}(x) H_m^{(p)}(x) \) over a whole real \( x \) axis, therefore the integration should be zero. For \( n + m \) even case, because of (23), \( H_n^{(p)}(x) \) and \( H_m^{(p)}(x) \) satisfy the following equations

\[
D \left( e^{-x^2} DH_n^{(p)}(x) \right) + 2[n] e^{-x^2} H_n^{(p)}(x) = 0,
\]

(34)
\[ D \left( e^{-x^2} D H_{m}^{(p)}(x) \right) + 2[m]e^{-x^2} H_{m}^{(p)}(x) = 0 \]  

(35)

respectively. Multiplying Eq.(34) and Eq.(35) by \( H_{m}^{(p)}(x) \) and \( H_{n}^{(p)}(x) \) respectively, and substracting the resulting equations, then integrating it over the whole real \( x \) axis, we have

\[
\int_{-\infty}^{\infty} D x \left( H_{m}^{(p)} \frac{D}{D x} \left( e^{-x^2} D H_{n}^{(p)}(x) \right) - H_{n}^{(p)} \frac{D}{D x} \left( e^{-x^2} D H_{m}^{(p)}(x) \right) \right) \\
+ 2([n] - [m]) \int_{-\infty}^{\infty} D x e^{-x^2} H_{n}^{(p)}(x) H_{m}^{(p)}(x) = 0. \]  

(36)

Since the first integration in Eq.(36) satisfies condition of the deformed integration by parts either for \( n \) and \( m \) being all even or all odd, we can write it as

\[
\int_{-\infty}^{\infty} D x \frac{D}{D x} \left( e^{-x^2} (H_{m}^{(p)} \frac{D}{D x} H_{n}^{(p)} - H_{n}^{(p)} \frac{D}{D x} H_{m}^{(p)}) \right) \\
= e^{-x^2} \left( H_{m}^{(p)} \frac{D}{D x} H_{n}^{(p)} - H_{n}^{(p)} \frac{D}{D x} H_{m}^{(p)}) \right) \bigg|_{-\infty}^{\infty} = 0, \]  

(37)

here we have used a fact that \( H_{n}^{(p)}(x), H_{m}^{(p)}(x) \) and their deformed derivatives are all polynomials of \( x \), so the right-hand side of (37) equals to zero. Noticing that \( n \neq m \), we obtain Eq.(33).

Then we calculate integration

\[ I_{n} = \int_{-\infty}^{\infty} D x e^{-x^2} H_{n}^{(p)}(x) H_{m}^{(p)}(x). \]  

(38)

Substituting Eq.(25) into this integration, we have

\[ I_{n} = \int_{-\infty}^{\infty} D x H_{n}^{(p)}(x)(-)^{n} \frac{D^{n}}{D x^{n}} e^{-x^2}. \]  

(39)

Noticing that the integration (38) also satisfies the condition of the deformed integration by parts no matter what non-negative integer \( n \) is, we get

\[
I_{n} = (-)^{n} H_{n}^{(p)}(x) D^{n-1} e^{-x^2} \bigg|_{-\infty}^{\infty} - (-)^{n} \int_{-\infty}^{\infty} D x \frac{D}{D x} \left( H_{n}^{(p)}(x) \frac{D^{n-1}}{D x^{n-1}} e^{-x^2} \right) \\
= -e^{-x^2} H_{n}^{(p)}(x) H_{n-1}^{(p)}(x) \bigg|_{-\infty}^{\infty} - (-)^{n} \int_{-\infty}^{\infty} D x \frac{D}{D x} \left( H_{n}^{(p)}(x) \frac{D^{n-1}}{D x^{n-1}} e^{-x^2} \right). \]
Continuing the procedure of integration by parts, at last we arrive at

\[ I_n = \int_{-\infty}^{\infty} D_x \frac{D^n H^{(p)}_n(x)}{D x^n} e^{-x^2} = \frac{2^n [n]!}{N_0^2}. \]  

(40)

Thus we demonstrated the equation (32).

To conclude this section, let us mention that there are some definite relations between neighbouring deformed Hermite polynomials and their derivatives which are called recursion relations of \( H^{(p)}_n(x) \). The main recursion relations are the following two:

\[ D H^{(p)}_n(x) - 2[n] H^{(p)}_{n-1}(x) = 0, \]  

(41)

\[ H^{(p)}_{n+1}(x) - 2x H^{(p)}_n(x) + 2[n] H^{(p)}_{n-1}(x) = 0. \]  

(42)

It is straightforward to prove these relations by virtue of the definition (18) and the deformed differential relation (8).

4 Parabose squeezed number states

As an application of the deformed Hermite polynomials \( H^{(p)}_n(x) \), let us consider the resulting states from parabose squeezed operator \( S(r) = e^{\frac{r}{2} a^2 - \frac{r}{4} (a^\dagger)^2} \) acting on the parabose number states \( |n\rangle \)

\[ |r, n\rangle = S(r)|n\rangle, \quad (n = 0, 1, 2, 3, \ldots) \]  

(43)

here for the sake of simplicity, we take the squeezing parameter \( r \) as a real number. We call \( |r, n\rangle \) the parabose squeezed number states. Obviously, \( |r, n\rangle \) form a complete and orthonormal state-vector set for the single mode of parabose system:

\[ \langle r, n| r, m\rangle = \langle n|m\rangle = \delta_{n,m}, \quad \sum_{n=0}^{\infty} |r, n\rangle \langle r, n| = 1. \]  

(44)

Using the following transformations

\[ S(r) a S(r)^{-1} = \cosh r a + \sinh r a^\dagger, \]

\[ S(r) a^\dagger S(r)^{-1} = \cosh r a^\dagger + \sinh r a \]  

(45)
and the disentangling formula

\[ S(r) = \exp \left( -\frac{\tanh r}{2} (a^\dagger)^2 \right) \exp \left( -\frac{\ln \cosh r}{2} (a^\dagger a + aa^\dagger) \right) \exp \left( \frac{\tanh r}{2} a^2 \right), \]

we can write \(|r,n\rangle\) as

\[ |r,n\rangle = \frac{(sechr)^{p/2}}{\sqrt{n!}} \left( \cosh a^\dagger + \sinh r \right)^n e^{-\tanh r(a^\dagger)^2/2} |0\rangle. \]

In terms of the deformed Hermite polynomials \( H_n^{(p)}(x) \), the explicit form of \(|r,n\rangle\) is

\[ |r,n\rangle = \frac{(sechr)^{p/2}}{\sqrt{n!}} \left( -\frac{1}{2} \tanh r \right)^{n/2} H_n^{(p)} \left( \frac{a^\dagger}{i\sqrt{\sinh 2r}} \right) e^{-\tanh r(a^\dagger)^2/2} |0\rangle. \]

We prove (48) by induction. Firstly, from (47), and using

\[ e^{\tanh r(a^\dagger)^2/2} a^n e^{-\tanh r(a^\dagger)^2/2} = (a - \tanh r a^\dagger)^n, \]

for \( n = 1 \) case, we have

\[ |r,1\rangle = \frac{(sechr)^{p/2}}{\sqrt{1!}} \cosh r a^\dagger e^{-\tanh r(a^\dagger)^2/2} |0\rangle = \frac{(sechr)^{p/2}}{\sqrt{1!}} \left( -\frac{1}{2} \tanh r \right)^{1/2} H_1^{(p)}(\chi) e^{-\tanh r(a^\dagger)^2/2} |0\rangle, \]

where \( \chi \) stands for \( \frac{a^\dagger}{i\sqrt{\sinh 2r}} \). Then supposing

\[ |r,n-1\rangle = \frac{(sechr)^{p/2}}{\sqrt{|n-1|!}} \left( -\frac{1}{2} \tanh r \right)^{\frac{n-1}{2}} H_{n-1}^{(p)}(\chi) e^{-\tanh r(a^\dagger)^2/2} |0\rangle, \]

we have

\[ |r,n\rangle = \frac{1}{\sqrt{|n|}} \left( \cosh a^\dagger + \sinh r \right) |r,n-1\rangle = \frac{(sechr)^{p/2}}{\sqrt{|n|!}} \left( -\frac{1}{2} \tanh r \right)^{\frac{n-1}{2}} \cosh a^\dagger H_{n-1}^{(p)}(\chi) e^{-\tanh r(a^\dagger)^2/2} |0\rangle + \frac{(sechr)^{p/2}}{\sqrt{|n|!}} \left( -\frac{1}{2} \tanh r \right)^{\frac{n-1}{2}} \sinh r e^{-\tanh r(a^\dagger)^2/2} (a - \tanh r a^\dagger) H_{n-1}^{(p)}(\chi) |0\rangle. \]
Using the following relation
\[ [a, (a^\dagger)^n] = (a^\dagger)^{n-1} \left( n + \frac{p-1}{2} (1 - (-)^n)R \right) \] (53)

it is easily to find
\[ a H_{n-1}^{(p)}(\chi)|0\rangle = \frac{2[n-1]}{i\sqrt{\sinh 2r}} \frac{H_{n-2}^{(p)}(\chi)}{H_{n-1}^{(p)}(\chi)}|0\rangle. \] (54)

Substituting (54) into (52), we obtain
\[ |r, n\rangle = (\text{sech} r)^{p/2} \left( -\frac{1}{2} \tanh r \right)^{n/2} \left( 2\chi H_{n-1}^{(p)}(\chi) - 2[n-1]H_{n-2}^{(p)}(\chi) \right) \]
\[ e^{-\tanh r (a^\dagger)^2/2}|0\rangle. \] (55)

Thus (48) is proved by virtue of the recursion relation (42).

From (3) we have Hermitian operators \( x \) and \( P \) defined by
\[ x = \frac{a + a^\dagger}{\sqrt{2}}, \quad P = \frac{a - a^\dagger}{i\sqrt{2}}, \] (56)

which satisfy the commutator \([x, P] = i[a, a^\dagger] = i(1 + (p-1)R)\). The variances of the operators \( x \) and \( P \) in the parabose squeezed number states are of the form
\[ \langle r, n|\Delta x^2|r, n\rangle = \langle r, n|x^2|r, n\rangle - \langle r, n|x|r, n\rangle^2 = e^{-2r}(n + \frac{p}{2}), \]
\[ \langle r, n|\Delta P^2|r, n\rangle = \langle r, n|P^2|r, n\rangle - \langle r, n|P|r, n\rangle^2 = e^{2r}(n + \frac{p}{2}), \] (57)

which lead to
\[ \langle r, n|\Delta x^2|r, n\rangle \langle r, n|\Delta P^2|r, n\rangle = \left( n + \frac{p}{2} \right)^2. \] (58)

On the other hand, according to the uncertainty relation
\[ \langle r, n|\Delta x^2|r, n\rangle \langle r, n|\Delta P^2|r, n\rangle \geq \frac{1}{4} \left| \langle r, n|[a, a^\dagger]|r, n\rangle \right|^2, \] (59)

we also have
\[ \langle r, n|\Delta x^2|r, n\rangle \langle r, n|\Delta P^2|r, n\rangle \geq \left( \frac{1}{2} + \frac{p-1}{2}(-)^n \right)^2, \] (60)
which means that only for the parabose squeezed vacuum \( |r, 0\rangle \), the uncertainty relation reduces to an equality. Note that since \([a, a^\dagger]\) is in general not a c-number, the right-hand side of (59) itself depends on the given state.

In order to show the squeezing properties of the parabose squeezed number states \( |r, n\rangle \), let us consider the variances of the operators \( x \) and \( P \) in the parabose number states

\[
\langle n \mid (\Delta x)^2 \mid n \rangle = \langle n \mid (\Delta P)^2 \mid n \rangle = n + \frac{p}{2},
\]

which means that for \( r \geq 0 \), the states \( |r, n\rangle \) are squeezed in the \( x \) direction

\[
\langle r, n \mid (\Delta x)^2 |r, n\rangle \leq \langle n \mid (\Delta x)^2 \mid n \rangle,
\]

\[
\langle r, n \mid (\Delta P)^2 |r, n\rangle \geq \langle n \mid (\Delta P)^2 \mid n \rangle
\]

and for \( r < 0 \), the states \( |r, n\rangle \) are squeezed in the \( P \) direction

\[
\langle r, n \mid (\Delta x)^2 |r, n\rangle \geq \langle n \mid (\Delta x)^2 \mid n \rangle,
\]

\[
\langle r, n \mid (\Delta P)^2 |r, n\rangle \leq \langle n \mid (\Delta P)^2 \mid n \rangle
\]

respectively.

5 Parabose Hermite polynomial states

In the previous section we show that the parabose squeezed vacuum \( |r, 0\rangle \) is a minimum uncertainty state for the normal squeezing. In fact, \( |r, 0\rangle \) is also a minimum uncertainty state for parabose amplitude-squared squeezing. To see this, let us introduce two Hermitian operators

\[
Y_1 = \frac{1}{2}(a^2 + a^{12}), \quad Y_2 = \frac{1}{2i}(a^2 - a^{12}),
\]

which satisfy the commutator \([Y_1, Y_2] = 2iN\) and the uncertainty relation

\[
\langle \psi \mid (\Delta Y_1)^2 \mid \psi \rangle \langle \psi \mid (\Delta Y_2)^2 \mid \psi \rangle \geq (\langle \psi \mid N \mid \psi \rangle)^2,
\]

where \( N = \{a^\dagger, a\}/2 \) and \( \langle \psi \mid (\Delta Y)^2 \mid \psi \rangle \) stands for the variance of the operator \( Y \) in the state \( |\psi\rangle \). It is easily to calculate the variances of \( Y_1 \) and \( Y_2 \) in the parabose squeezed vacuum \( |r, 0\rangle \)

\[
\langle r, 0 \mid (\Delta Y_1)^2 \mid r, 0 \rangle = \frac{p}{2}(\cosh 2r)^2, \quad \langle r, 0 \mid (\Delta Y_2)^2 \mid r, 0 \rangle = \frac{p}{2}.
\]
Comparing with $\langle r, 0|N|r, 0 \rangle = \frac{p^2}{2} \cosh 2r$, one finds that

$$\langle r, 0|\Delta Y_1|^2|r, 0 \rangle \geq \langle r, 0|N|r, 0 \rangle, \quad \langle r, 0|\Delta Y_2|^2|r, 0 \rangle \leq \langle r, 0|N|r, 0 \rangle,$$  \hspace{1cm} (67)

which mean that the state $|r, 0 \rangle$ is squeezed in the $Y_2$ direction.

To search other minimum uncertainty states for parabose amplitude-squared squeezing, following [9], let us consider the eigenvalue equation

$$(Y_1 + i\lambda Y_2)|\psi \rangle = \beta |\psi \rangle,$$ \hspace{1cm} (68)

where $\lambda$ is real and $\beta$ is complex. Multiplying (68) by the operator $Y_1 - i\lambda Y_2$ and then taking the inner product with $|\psi \rangle$, one gets

$$\langle \psi |(\Delta Y_1)^2|\psi \rangle + \lambda^2 \langle \psi |(\Delta Y_2)^2|\psi \rangle = 2\lambda \langle \psi |N|\psi \rangle.$$ \hspace{1cm} (69)

Again multiplying (68) by $Y_1 + i\lambda Y_2$ and taking the inner product with $|\psi \rangle$ provides further information

$$\langle \psi |(\Delta Y_1)^2|\psi \rangle = \lambda^2 \langle \psi |(\Delta Y_2)^2|\psi \rangle.$$ \hspace{1cm} (70)

From these two equations we have

$$\langle \psi |(\Delta Y_1)^2|\psi \rangle = \lambda \langle \psi |N|\psi \rangle, \quad \langle \psi |(\Delta Y_2)^2|\psi \rangle = \frac{1}{\lambda} \langle \psi |N|\psi \rangle.$$ \hspace{1cm} (71)

These equations show that a solution of (68) is a minimum uncertainty state for parabose amplitude-squared squeezing, and the real $\lambda$ plays the role of squeezing parameter.

Now we begin to solve (68) for $|\psi \rangle$. First, in terms of creation and annihilation operators, we can rewrite (68) as

$$\left(\frac{1 - \lambda}{2} a^2 + \frac{1 + \lambda}{2} a^\dagger a\right)|\psi \rangle = \beta |\psi \rangle.$$ \hspace{1cm} (72)

Introducing a new state $|\psi' \rangle = S(z)|\psi \rangle$, where $S(z) = \exp\left(\frac{z^2}{2} a^2 - \frac{z^2}{2} a^\dagger a\right)$ is the squeezed operator and $z = r e^{i\theta}$, one finds that $|\psi' \rangle$ satisfies

$$\left(\frac{1 - \lambda}{2} \cosh^2 r + \frac{1 + \lambda}{2} e^{2i\theta} \sinh^2 r\right) a^\dagger a^2 |\psi' \rangle$$
$$+ \frac{1}{2} \left(\frac{1 - \lambda}{2} e^{-i\theta} + \frac{1 + \lambda}{2} e^{i\theta}\right) \sinh 2r \{a^\dagger, a\} |\psi' \rangle$$
$$+ \left(\frac{1 - \lambda}{2} e^{-2i\theta} \sinh^2 r + \frac{1 + \lambda}{2} \cosh^2 r\right) a^2 |\psi' \rangle = \beta |\psi' \rangle.$$ \hspace{1cm} (73)
We now expand $|\lambda\rangle$ and for a simple subset of solutions. Noting that for $0 < \lambda < 1$, we have 

$$
(a^2 + \frac{i}{2}\sqrt{1 - \lambda^2}\{a^\dagger, a\})|\psi'\rangle = \beta|\psi'\rangle,
$$

(74)

and for $\lambda \geq 1$, we have 

$$
(a^2 + \frac{1}{2}\sqrt{\lambda^2 - 1}\{a^\dagger, a\})|\psi'\rangle = \beta|\psi'\rangle.
$$

(75)

We now expand $|\psi'\rangle$ in terms of the parabase number states (7), $|\psi'\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$, and substitute this expression into (74) and (75). This leads to the recursion relations 

$$
c_{n+2} = \frac{\beta - i\sqrt{1 - \lambda^2}(n+p/2)}{\sqrt{[n+1][n+2]}} c_n
$$

for $0 < \lambda < 1$, and 

$$
c_{n+2} = \frac{\beta - \sqrt{\lambda^2 - 1}(n+p/2)}{\lambda\sqrt{[n+1][n+2]}} c_n
$$

for $\lambda \geq 1$. Here we want to consider a particularly simple subset of solutions. Noting that for $0 < \lambda < 1$, if one takes $\beta = i\sqrt{1 - \lambda^2}(m+\frac{p}{2})$, where $m$ is a non-negative integer, then one has a truncated series for $|\psi'\rangle$ with $c_m|m\rangle$ being the last term. Similarly, for $\lambda \geq 1$, and $\beta = \sqrt{\lambda^2 - 1}(m+\frac{p}{2})$, one also gets a polynomial expression for $|\psi'\rangle$ with all $c_n = 0$ when $n > m$. Thus if $0 < \lambda < 1$, for even $m$, taking $c_1 = 0$, we have 

$$
|\psi'(m, \lambda)\rangle = \sum_{k=0}^{m} \frac{(i\sqrt{1 - \lambda^2})^{2k}(m)!}{\sqrt{[2k]}!(m/2 - k)!} c_0|2k\rangle,
$$

(76)

and for odd $m$, taking $c_0 = 0$, we have 

$$
|\psi'(m, \lambda)\rangle = \sum_{k=0}^{m-1} \frac{\sqrt{p}(i\sqrt{1 - \lambda^2})^{2k}(m-1)!}{\sqrt{[2k+1]}!(m/2 - k)!} c_1|2k+1\rangle.
$$

(77)

If $\lambda \geq 1$, for even $m$, also taking $c_1 = 0$, we have 

$$
|\psi'(m, \lambda)\rangle = \sum_{k=0}^{m} \frac{(\sqrt{\lambda^2 - 1})^{2k}(m)!}{\lambda^k\sqrt{[2k]}!(m/2 - k)!} c_0|2k\rangle,
$$

(78)

and for odd $m$, taking $c_0 = 0$, we have 

$$
|\psi'(m, \lambda)\rangle = \sum_{k=0}^{m-1} \frac{\sqrt{p}(\sqrt{\lambda^2 - 1})^{2k}(m-1)!}{\lambda^k\sqrt{[2k+1]}!(m/2 - k)!} c_1|2k+1\rangle.
$$

(79)
Using the deformed Hermite polynomial $H_m^{(p)}$, we can express these states in a relatively compact form

$$|\psi'(m, \lambda)\rangle = c_m(\lambda)H_m^{(p)}(i\gamma(\lambda)a^\dagger)|0\rangle,$$

where $c_m(\lambda)$ is a normalization constant, and for $0 < \lambda < 1$, $\gamma(\lambda) = e^{i\pi/4}\sqrt{\sqrt{1 - \lambda^2}/2}$; for $\lambda \geq 1$, $\gamma(\lambda) = \sqrt{\sqrt{\lambda^2 - 1}/\lambda}$. Finally, combining this with the squeezing transformation we have the minimum uncertainty states

$$|\psi(m, \lambda)\rangle = S^{-1}(z)|\psi'(m, \lambda)\rangle = c_m(\lambda)S^{-1}(z)H_m^{(p)}(i\gamma(\lambda)a^\dagger)|0\rangle,$$

where $z$ is chosen as mentioned earlier.

We now want to examine some of the properties of the states $|\psi(m, \lambda)\rangle$. Eq.(71) shows that if we know the average value of the operator $N$ in the state $|\psi(m, \lambda)\rangle$, then we know the variances of the operators $Y_1$ and $Y_2$ in the same state. Using (54) and noticing that

$$\langle 0|H_m^{(p)}(-i\gamma^*(\lambda)a)RH_m^{(p)}(i\gamma(\lambda)a^\dagger)|0\rangle = (-)^m/c_m(\lambda)^2,$$

for $0 < \lambda < 1$ one can get

$$\langle \psi(m, \lambda)|N|\psi(m, \lambda)\rangle = \frac{1 - \lambda^2}{\lambda} (m + \frac{p}{2}) + \frac{1 + (p - 1)(-)^m}{2\lambda} + 2[m^2\lambda\sqrt{1 - \lambda^2}/c_m(\lambda)]^2/c_{m-1}(\lambda)^2, \quad (82)$$

and for $\lambda \geq 1$ one can get

$$\langle \psi(m, \lambda)|N|\psi(m, \lambda)\rangle = \frac{\lambda^2 - 1}{\lambda} (m + \frac{p}{2}) + \frac{1 + (p - 1)(-)^m}{2\lambda} + 2[m^2\lambda\sqrt{\lambda^2 - 1}/\lambda^2/c_m(\lambda)]^2/c_{m-1}(\lambda)^2. \quad (83)$$

A straightforward calculation gives

$$|c_0(\lambda)|^{-2} = 1, \quad |c_1(\lambda)|^{-2} = 4[1]|\gamma|^2,$$

$$|c_2(\lambda)|^{-2} = 16[2]!|\gamma|^4 + ([2]!)^2.$$
Let us consider the state $|\psi(1, \lambda)\rangle$. We find for $0 < \lambda < 1$ that
\[
\langle \psi(1, \lambda)|N|\psi(1, \lambda)\rangle = \frac{2 + p}{2\lambda},
\langle \psi(1, \lambda)|(\Delta Y_1)^2|\psi(1, \lambda)\rangle = 1 + \frac{p}{2},
\langle \psi(1, \lambda)|(\Delta Y_2)^2|\psi(1, \lambda)\rangle = \frac{2 + p}{2\lambda^2},
\] (84)
and for $\lambda \geq 1$ that
\[
\langle \psi(1, \lambda)|N|\psi(1, \lambda)\rangle = \frac{2 + p}{2},
\langle \psi(1, \lambda)|(\Delta Y_1)^2|\psi(1, \lambda)\rangle = \frac{2 + p}{2}\lambda^2,
\langle \psi(1, \lambda)|(\Delta Y_2)^2|\psi(1, \lambda)\rangle = 1 + \frac{p}{2},
\] (85)
It is obviously that as $\lambda \to 0$, $Y_1$ becomes increasingly squeezed, and
\[
\langle \psi(1, \lambda)|(\Delta Y_1)^2|\psi(1, \lambda)\rangle \text{ is equal to } 1 + \frac{p}{2}.
\] Similarly, as $\lambda \to \infty$, $Y_2$ becomes more and more squeezed and
\[
\langle \psi(1, \lambda)|(\Delta Y_2)^2|\psi(1, \lambda)\rangle \text{ is equal to } 1 + \frac{p}{2}.
\] Then we examine the state $|\psi(2, \lambda)\rangle$. For $0 < \lambda < 1$, we have
\[
\langle \psi(2, \lambda)|N|\psi(2, \lambda)\rangle = \frac{(2 + p)(4 + p) - (8 + 6p)\lambda^2}{2\lambda(2 + p - 2\lambda^2)},
\] (86)
and for $\lambda \geq 1$, we have
\[
\langle \psi(2, \lambda)|N|\psi(2, \lambda)\rangle = \frac{(2 + p)(4 + p)\lambda^3 - (8 + 6p)\lambda}{2(2 + p)\lambda - 4}.
\] (87)
In this case, however, as $\lambda$ goes to zero, $\langle \psi(2, \lambda)|(\Delta Y_1 - 1)^2|\psi(2, \lambda)\rangle$ goes to $2 + \frac{p}{2}$, and as $\lambda$ goes to infinity, $\langle \psi(2, \lambda)|(\Delta Y_2)^2|\psi(2, \lambda)\rangle$ goes to $2 + \frac{p}{2}$. Generally, for any fixed non-negative integer $m$, since when $\lambda \to 0$ or $\lambda \to \infty$, $|c_m(\lambda)|^2$ goes to a number which only depends on the given integer $m$ and the parastatistica order $p$, we have
\[
\lim_{\lambda \to 0} \langle \psi(m, \lambda)|(\Delta Y_1)^2|\psi(m, \lambda)\rangle = m + \frac{p}{2},
\] and
\[
\lim_{\lambda \to \infty} \langle \psi(m, \lambda)|(\Delta Y_2)^2|\psi(m, \lambda)\rangle = m + \frac{p}{2}.
\] A short calculation shows that the state $|\psi(1, \lambda)\rangle$ is not squeezed in the normal sense.
In summary, we have introduced a new kind of deformation of the usual Hermite polynomials and discussed their main properties in this paper. These deformed Hermite polynomials may have some applications in studying parabose systems. For instance, they can be used to give the explicit form for the parabose squeezed number states, as well as to present a particularly simple subset of the minimum uncertainty states for parabose amplitude-squared squeezing. It is reasonable to believe that the para-deformed Hermite polynomials have other applications in parastatistics, and this is the task of our next step.

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