Block-transitive automorphism groups on 3-designs with small block size

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Abstract

The paper is an investigation of the structure of block-transitive automorphism groups of a 3-design with small block size. Let $G$ be a block-transitive automorphism group of a nontrivial 3-$(v, k, \lambda)$ design $D$ with $k \leq 6$. We prove that if $G$ is point-primitive then $G$ is of affine or almost simple type. If $G$ is point-imprimitive then $D$ is a 3-$(16, 6, \lambda)$ design with $\lambda \in \{4, 12, 24, 28, 48, 56, 84, 96, 112, 140\}$, and $\text{rank}(G) = 3$.

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1 Introduction

A $t$-$(v, k, \lambda)$ design is a pair $(\mathcal{P}, \mathcal{B})$ in which $\mathcal{P}$ is a $v$-set of points and $\mathcal{B}$ is a collection of $k$-sets of $\mathcal{P}$ called blocks, such that every $t$-set of $\mathcal{P}$ is contained in precisely $\lambda$ blocks. If $t < k < v - 1$ holds, then we speak of a nontrivial $t$-design. It is simple if no two blocks are identical. All of the $t$-designs in this paper will be simple and nontrivial.

An automorphism of $D$ is a permutation of $\mathcal{P}$ which leaves $\mathcal{B}$ invariant. The full automorphism group of $D$ consists of all automorphisms of $D$ and is denoted by $\text{Aut}(D)$. A subgroup $G$ of the automorphism group of $D$ is block-transitive if it acts transitively on $\mathcal{B}$; $D$ is said to be block-transitive if $\text{Aut}(D)$ is. Point- and flag-transitivity are defined similarly.

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A set of blocks of $\mathcal{D}$ is called a set of base blocks with respect to an automorphism group $G$ of $\mathcal{D}$ if it contains exactly one block from each $G$-orbit on the block set. In particular, if $G$ is a block-transitive automorphism group of $\mathcal{D}$, then any block $B$ is a base block of $\mathcal{D}$.

Block-transitivity is just one of many conditions that can be imposed on the automorphism group $G$ of a $t$-design $\mathcal{D}$. It is well known that if $G$ is block-transitive, then $G$ is also point-transitive (Block’s Lemma [3]). It is elementary that the flag-transitivity of $G$ on a linear space $(2-(v,k,1)$ design) implies its point-primitivity. By a result of Davies [8], for 2-$(v,k,\lambda)$ designs, this implication remains true if $(r,\lambda) = 1$ (where $r$ denotes the number of blocks containing a given point). However, block-transitivity does not necessarily imply point-primitivity. For example, let $\mathcal{D}$ be a 2-design consisting of the points and hyperplanes of any Desarguesian projective space $PG(n,q)$ where $n \geq 2$ and $(q^{n+1} - 1)/(q - 1)$ is not a prime, and take $G$ as the group generated by a Singer cycle.

If the automorphism group $G$ of $\mathcal{D}$ is point-primitive, then $G$ is of one of the following five types by O’Nan-Scott theorem (see [14] for details).

(i) Affine.

(ii) Almost simple.

(iii) Product.

(iv) Simple diagonal.

(v) Twisted wreath product.

In 1984, Camina and Gagen [6] proved that if $G$ is block-transitive on a 2-$(v,k,1)$ design $\mathcal{D}$ with $k \mid v$, then $G$ is either point-primitive of affine or almost simple type. Inspired by the proof, several others [4, 9, 17] generalised the result in [6] to prove that groups acting flag-transitively on 2-$(v,k,1)$ designs are affine or almost simple. It is worth noting that both [9] and [17] generalised the result to the situation of 2-designs with $(r,\lambda) = 1$. For a $t$-$(v,k,\lambda)$ design, Cameron and Praeger [5] proved the following result in 1993.

Proposition 1.1 Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a $t$-$(v,k,\lambda)$ design with $t \geq 2$. Then the following holds:

(i) If $G \leq Aut(\mathcal{D})$ acts block-transitively on $\mathcal{D}$, then $G$ also acts $\lfloor t/2 \rfloor$-homogeneously on $\mathcal{P}$.
(ii) If $G \leq \text{Aut}(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$, then $G$ also acts $[(t+1)/2]$-homogeneously on $\mathcal{P}$.

According to this result, if $G$ acts block-transitively on a $t-(v, k, \lambda)$ design $\mathcal{D}$ with $t \geq 4$ then $G$ is either point-primitive of affine or almost simple type as $G$ is 2-homogeneous on the points of $\mathcal{D}$. Therefore, it is necessary to study the block-transitive $t-(v, k, \lambda)$ designs with $t \leq 3$.

The main aim of this paper is to study 3-$(v, k, \lambda)$ designs admitting a block-transitive automorphism group $G$. Firstly, we analyse the case in which the automorphism group $G$ is point-primitive, and we prove a reduction theorem for small values of $k$.

**Theorem 1** Let $G$ be a block-transitive automorphism group of a nontrivial 3-$(v, k, \lambda)$ design with $k \leq 6$. If $G$ is point-primitive, then $G$ is of affine type, or almost simple type.

In fact, there exist many 3-designs admitting a block-transitive, point-primitive automorphism group of affine or almost simple type. Here are some examples (cf. [13]):

**Example 1.1**

(i) Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ and $\mathcal{B}$ are the points and planes of the affine space $AG(d, 2)$ with $d \geq 3$. Then $\mathcal{D}$ is isomorphic to the 3-$(2^d, 4, 1)$ design admitting $G = AGL(d, 2)$ as its flag-transitive (block-transitive), point-primitive automorphism group of affine type.

(ii) Let $\mathcal{D}$ be the Mathieu-Witt 3-$(22, 6, 1)$ design, and $G \geq M_{22}$. Then $G$ is a flag-transitive (block-transitive), point-primitive automorphism group of $\mathcal{D}$ with almost simple action.

For the point-imprimitive case, Delandtsheer and Doyen have shown in [10] that if $\mathcal{D}$ is a $t-(v, k, \lambda)$ design admitting a block-transitive point-imprimitive automorphism group $G$ then $v \leq \left(\left(\begin{array}{c} k \\ 2 \end{array}\right) - 1\right)^2$. Assume that $G$ has a system of $d$ blocks of imprimitivity each of size $c$. In [5] Corollaries 3.2 and 3.4, it was shown that for a block-transitive, point-imprimitive 3-$(v, k, \lambda)$ design with $d = 2$ or $c = 2$ then $v \leq \left(\begin{array}{c} k \\ 2 \end{array}\right) + 1$. Thus, for a fixed block size $k$, there are only finitely many $t-(v, k, \lambda)$ designs with a block-transitive automorphism group which is point-imprimitive.

Secondly, the other purpose of this paper is to study 3-$(v, k, \lambda)$ designs admitting a block-transitive point-imprimitive automorphism group and prove the following theorem:
Theorem 2 Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial $3$-$(v, k, \lambda)$ design with $k \leq 6$ and admitting a block-transitive automorphism group $G$. If $G$ is point-imprimitive then $\text{rank}(G) = 3$, and $\mathcal{D}$ is a $3$-$(16, 6, \lambda)$ design with

$$\lambda \in \{4, 12, 16, 24, 28, 48, 56, 64, 84, 96, 112, 140\}.$$ 

The paper is organized as follows. In Section 2, we introduce some preliminary results that are important for the remainder of the paper. In Sections 3 and 4, we shall give the proofs of the Theorems 1 and 2 respectively.

2 Preliminaries

The notation and terminology used is standard and can be found in [7, 11] for design theory and in [12, 15] for group theory. In particular, if $G$ is a permutation group on point set $\mathcal{P}$, and $\alpha \in B \subseteq \mathcal{P}$, then $G_\alpha$ denotes the stabilizer of a point $\alpha$ in $G$, and $G_B$ denotes the setwise stabilizer of $B$ in $G$, and $G_{\alpha B}$ denotes the stabilizer of a flag $(\alpha, B)$ in $G$.

Lemma 2.1 [7, 1.2, 1.9] The parameters $v, b, r, k, \lambda$ of a 3-design satisfy the following conditions:

(i) $vr = bk$.

(ii) $\lambda v(v - 1)(v - 2) = bk(k - 1)(k - 2)$.

The following lemma is useful for the study of block-transitive $3$-$(v, k, \lambda)$ designs.

Lemma 2.2 Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial $3$-$(v, k, \lambda)$ design with $k \leq 6$ and admitting a block-transitive automorphism group $G$. Then $r$ divides $k|G_\alpha|$. Furthermore, $r$ divides $k \lambda d(d - 1)$, and $(v - 1)(v - 2)$ divides $k(k - 1)(k - 2)d(d - 1)$, for all nontrivial subdegrees $d$ of $G$.

Proof. Let $B$ be a block of $\mathcal{D}$ containing the point $\alpha$. The point-transitivity and block-transitivity imply

$$|G : G_{\alpha B}| = |G : G_\alpha||G_\alpha : G_{\alpha B}| = v|G_\alpha : G_{\alpha B}|,$$
and

\[ |G : G_{αB}| = |G : G_B||G_B : G_{αB}| = b|G_B : G_{αB}|.\]

Hence, \( |G_α : G_{αB}| = \frac{r|G_B : G_{αB}|}{k} \) by Lemma 2.1(i), and so \( r \) divides \( k|G_α| \). In order to prove the remaining result, here we prove only the case \( k = 4 \), and the result in lemma can be proved imitate to the proof of the case \( k = 4 \) for the other values of \( k \).

Clearly, \( |G_B : G_{αB}| = 1, 2, 3 \) or \( 4 \) as \( k = |B| = 4 \). We will analyze each of these cases separately.

(1) Let \( |G_B : G_{αB}| = 1 \), then \( |G_α : G_{αB}| = \frac{r}{4} \). Suppose that \( G_α \) has four orbits with same size on pencil \( P(α) \) (i.e. blocks containing a given point \( α \)), and denoted by \( O_1, O_2, O_3 \) and \( O_4 \), respectively. Let \( Γ \neq \{α\} \) be a nontrivial \( G_α \)-orbit with \( |Γ| = d \). Set \( μ_i = |Γ \cap B_i| \) where \( B_i \in O_i \) \((i = 1, 2, 3, 4)\). Clearly, \( 0 ≤ μ_i ≤ 3 \). Counting the number of set \( \{\{β, γ\}, B\} \mid \{β, γ\} ∈ B \cap Γ \} \) in two ways, and we get

\[ \frac{r}{4} \sum_{i=1}^{4} \left( \frac{μ_i}{2} \right) = \lambda \binom{d}{2}. \]

So \( r \) divides \( 4λd(d − 1) \).

Suppose that \( G_α \) has three orbits \( O_1, O_2, \) and \( O_3 \) with sizes \( \frac{r}{4}, \frac{r}{4} \) and \( \frac{r}{2} \) on pencil \( P(α) \) respectively. Then

\[ \frac{r}{4} \binom{μ_1}{2} + \frac{r}{4} \binom{μ_2}{2} + \frac{r}{2} \binom{μ_3}{2} = λ \binom{d}{2}. \quad (1) \]

Also, \( r \) divides \( 4λd(d − 1) \).

Assume that \( G_α \) has two orbits \( O_1 \) and \( O_2 \) with \( |O_1| = \frac{r}{4} \) and \( |O_2| = \frac{3r}{4} \) on pencil \( P(α) \), we obtain

\[ \frac{r}{4} \binom{μ_1}{2} + \frac{3r}{4} \binom{μ_2}{2} = λ \binom{d}{2}. \quad (2) \]

Hence, \( r \) divides \( 4λd(d − 1) \).

(2) Let \( |G_B : G_{αB}| = 2 \), then \( |G_α : G_{αB}| = \frac{r}{2} \). If \( G_α \) has three orbits with sizes \( \frac{r}{2}, \frac{r}{4}, \frac{r}{4} \) then the Equation (1) holds. If \( G_α \) has two orbits \( O_1 \) and \( O_2 \) with sizes \( \frac{r}{2} \) and \( \frac{r}{2} \) respectively, then

\[ \frac{r}{2} \binom{μ_1}{2} + \frac{r}{2} \binom{μ_2}{2} = λ \binom{d}{2}. \]

In both of cases we have \( r \) divides \( 4λd(d − 1) \).

(3) Let \( |G_B : G_{αB}| = 3 \), then \( |G_α : G_{αB}| = \frac{3r}{4} \), and \( G_α \) has two orbits with sizes \( \frac{3r}{4} \) and \( \frac{r}{4} \) respectively. Thus, \( r \) divides \( 4λd(d − 1) \) by Equation (2).
(4) Let $|G_B : G_{\alpha B}| = 4$, then $|G_{\alpha} : G_{\alpha B}| = r$ and so $G_{\alpha}$ acts transitively on $P(\alpha)$. Set $\mu = |\Gamma \cap B|$ where $B \in P(\alpha)$. We obtain

$$r\left(\frac{\mu}{2}\right) = \lambda\left(\frac{d}{2}\right).$$

Hence $r$ divides $\lambda d (d - 1)$, and so $4\lambda d (d - 1)$ is divisible by $r$.

By Lemma 2.1(i)(ii), $r = \frac{\lambda(v - 1)(v - 2)}{(k-1)(k-2)}$ and so $(v - 1)(v - 2)$ divides $24d(d - 1)$. \hfill \Box

From the proof of [1, Lemma 2.3] we get the following:

**Lemma 2.3** There does not exist a non-abelian finite simple group satisfying

$$((|T| - 1)(|T| - 2) < 480|\text{Out}(T)|.$$ 

In the study of point-imprimitive case, the basis of our method is the following elementary result.

**Lemma 2.4** [5, Proposition 1.1] Let $D = (\mathcal{P}, \mathcal{B})$ be a $t$-$(v, k, \lambda)$ design, admitting a block-transitive automorphism group $G$. Let $H$ be a permutation group with $G \leq H \leq S_v$, and $\mathcal{B}^* = \mathcal{B}^H$ the set of images of blocks in $\mathcal{B}$ under $H$. Then $(\mathcal{P}, \mathcal{B}^*)$ is a $t$-$(v, k, \lambda^*)$ design, for some $\lambda^*$, admitting the block-transitive automorphism group $H$.

## 3 Primitivity

The principal tool used in the proof is the O’Nan-Scott theorem for finite primitive groups proved by Liebeck, Praeger and Saxl in [14]. We will prove Theorem 1 by dealing with the cases of product action, simple diagonal action and twisted wreath product action separately. The proof of the Theorem 1 is inspired by the proof of [16, Theorem 1.1].

### 3.1 Product action

Here, we suppose that $G$ has a product action on $\mathcal{P}$. Then $G \leq K^m \rtimes S_m = K \rtimes S_m$ with $m \geq 2$, where $K$ is a primitive group (of almost simple or diagonal type) on $\Omega$ of size $v_0 \geq 5$, and $\mathcal{P} = \Omega^n$. 
Proposition 3.1 Let \( D = (\mathcal{P}, \mathcal{B}) \) be a nontrivial 3-(\( v, k, \lambda \)) design with \( k \leq 6 \) admitting a block-transitive point-primitive automorphism group \( G \). Then \( G \) is not of product action type.

Proof. Assume the contrary, suppose that \( H = K \wr S_m \) with \( S_m \) acting on the set \( M = \{1, 2, \ldots, m\} \). Let \( \alpha \) and \( \beta \) be two distinct points of \( \mathcal{P} \). Then \( d = |\beta G_\alpha| \) is a subdegree of \( G \). Since \( G \) is a subgroup of \( H \), it follows that

\[
d = |G_\alpha : G_{\alpha \beta}| \leq |H_\alpha : H_{\alpha \beta}|.
\]

Let \( \alpha = (\gamma, \gamma, \ldots, \gamma) \in \mathcal{P} \), \( \beta = (\delta, \gamma, \ldots, \gamma) \in \mathcal{P} \) with \( \delta \neq \gamma \) and let \( B \cong K^m \) be the base group of \( H \). Then \( B_\alpha = K_\gamma^m \), \( B_{\alpha \beta} = K_\gamma^m \times K_\gamma^{m-1} \). Now \( H_\alpha = K_\gamma \wr S_m \), and \( H_{\alpha \beta} \geq K_\gamma \times (K_\gamma \wr S_{m-1}) \). Suppose \( K \) has rank \( s \) on \( \Omega \) with \( s \geq 2 \). We can choose a \( \delta \in \Omega \) satisfying \( |K_\gamma : K_\gamma^\delta| \leq \frac{v_0 - 1}{s - 1} \), so that

\[
|H_\alpha : H_{\alpha \beta}| = \frac{|H_\alpha|}{|H_{\alpha \beta}|} \leq \frac{|K_\gamma|^m \cdot m!}{|K_\gamma^\delta||K_\gamma|^{m-1} \cdot (m - 1)!} \leq \frac{m v_0 - 1}{s - 1},
\]

and hence \( d \leq \frac{m v_0 - 1}{s - 1} \) by Equation (3). From Lemma 2.2 we have

\[(v - 1)(v - 2) \leq k(k - 1)(k - 2) \cdot \frac{m v_0 - 1}{s - 1} \cdot (m v_0 - 1) - 1).
\]

Combining this with \( v = v_0^m \) and \( k \leq 6 \) we get all possible \((v_0, m, s)\) as in Table 1.

Table 1: All possible values of \( v_0, m, s \) with \( k \leq 6 \)

| \( (m, s) \) | \( k = 4 \) | \( k = 5 \) | \( k = 6 \) |
|---|---|---|---|
| \( (2, 2) \) | \( \emptyset \) | \( \emptyset \) | \( \emptyset \) |
| \( (2, 3) \) | \( \{5, 6, 7, 8\} \) | \( \{5, 6, \ldots, 14\} \) | \( \{5, 6, \ldots, 20\} \) |
| \( (3, 2) \) | \( \emptyset \) | \( \emptyset \) | \( v_0 = 5 \) |

First, assume that \( (m, s) = (2, 2) \). Then \( K \) acts 2-transitively on \( \Omega \), and \( H = K \wr S_2 \) has rank 3 with subdegrees 1, 2\((v_0 - 1)\), \((v_0 - 1)^2\) on the point set \( \mathcal{P} = \Omega \times \Omega \). Note that \( G \leq H \), so each subdegree of \( H \) is the sum of some subdegrees of \( G \), by Lemma 2.2 we conclude that \( (v_0^2 - 1)(v_0^2 - 2) \) divides \( k(k - 1)(k - 2)(2v_0 - 2)(2v_0 - 3) \), it is impossible.

For the case \( (m, s) = (2, 3) \), \( K \) is a primitive group with rank 3 on \( \Omega \). From 9.62 Table], there is no such group \( K \) with a primitive action (of almost simple or diagonal type) and rank 3 on a set \( \Omega \) of size \( v_0 \in \{5, 6, 7, 8, 9\} \).
Now, assume that \((m, s) = (3, 2)\). Then \(H = K \wr S_3\) has rank 4 with subdegrees 1, 3\((v_0 - 1)\), 3\((v_0 - 1)^2\), \((v_0 - 1)^3\) on the point set \(\mathcal{P} = \Omega \times \Omega \times \Omega\). This contradicts the fact that \((v_0^3 - 1)(v_0^3 - 2)\) divides \(k(k - 1)(k - 2)(3v_0 - 3)(3v_0 - 4)\) as \(v_0 = 5\) and \(k = 6\). \(\square\)

### 3.2 Simple diagonal action

Suppose that \(G\) is a primitive group of simple diagonal type. Then \(M = \text{Soc}(G) = T_1 \times \cdots \times T_m \cong T^m\) and \(M_{\alpha} \cong T\) is a diagonal subgroup of \(M\), where \(T_i \cong T\) is a non-abelian finite simple group, for \(i = 1, \ldots, m\) and \(m \geq 2\). Here \(G_{\alpha}\) is isomorphic to a subgroup of \(\text{Aut}(T) \times S_m\) and has an orbit \(\Gamma\) in \(\mathcal{P} - \{\alpha\}\) with \(|\Gamma| \leq m|T|\).

**Proposition 3.2** Let \(\mathcal{D} = (\mathcal{P}, \mathcal{B})\) be a nontrivial 3-(\(v, k, \lambda\)) design with \(k \leq 6\) admitting a block-transitive point-primitive automorphism group \(G\). Then \(G\) is not of simple diagonal type.

**Proof.** If \(G\) is of simple diagonal type, then \(|\mathcal{P}| = |T|^{m-1}\) and \(G\) has a subdegree \(d\) less than \(m|T|\). From Lemma 2.2, we have

\[
(|T|^{m-1} - 1)(|T|^{m-1} - 2) \leq k(k - 1)(k - 2) \cdot m|T| \cdot (m|T| - 1).
\]

It is easy to get \(m = 2\) as \(k \leq 6\) and \(|T| \geq 60\).

Also by Lemma 2.2, we have that \(r\) divides \(k|G_{\alpha}|\), and so \(k|\text{Aut}(T)||S_2|\) is divisible by \(r\). Since \(\text{Out}(T) \cong \text{Aut}(T)/\text{Inn}(T)\) and \(\text{Inn}(T) \cong T/Z(T)\), it yields that \(r\) divides \(2k|T||\text{Out}(T)|\) as \(T\) is a non-abelian simple group. Combining this with Lemma 2.1(ii), we get that \((|T| - 1)(|T| - 2)\) divides \(2k(k - 1)(k - 2)|T||\text{Out}(T)|\). Then \((|T|, |T| - 1) = 1\) and \((|T|, |T| - 2) = 2\) imply

\[
(|T| - 1)(|T| - 2) < 4k(k - 1)(k - 2)|\text{Out}(T)| \leq 480|\text{Out}(T)|.
\]

This violates Lemma 2.3. \(\square\)

### 3.3 Twisted wreath product action

Next, we suppose that \(G\) is a primitive group of twisted wreath product type on \(\mathcal{P}\). Let \(\alpha \in \mathcal{P}\). Then \(G \cong QB \times P\), where \(P = G_{\alpha}\) is a transitive permutation group on \(\{1, \ldots, m\}\) with \(m \geq 6\), and \(QB = \text{Soc}(G) = T_1 \times \cdots \times T_m \cong T^m\) is regular for some nonabelian simple groups \(T\). Thus, \(v = |\mathcal{P}| = |T|^m\). Moreover, \(G_{\alpha}\) has an orbit \(\Gamma\) with \(|\Gamma| \leq m|T|\).
Proposition 3.3 Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial 3-(v, k, \lambda) design with $k \leq 6$ admitting a block-transitive point-primitive automorphism group $G$. Then $G$ is not of twisted wreath product type.

Proof. If $G$ is of twisted wreath product type, then the argument here is similar to the proof of Proposition 3.2. By Lemma 2.2 we easily observe that

$$ (|T|^{m-1} - 1)(|T|^{m-1} - 2) \leq k(k - 1)(k - 2) \cdot m|T|(m|T| - 1). $$

Then the inequalities $k \leq 6$ and $|T| \geq 60$ imply $m \leq 2$, this contradicts the fact that $m \geq 6$. \hfill \Box

Proof of Theorem 1 It follows from Propositions 3.1-3.3.

4 Imprimitivity

Suppose that $G$ is an imprimitive group on the point set $\mathcal{P}$. Then $\mathcal{P}$ can be partitioned into $d$ nontrivial blocks of imprimitivity $\Delta_j$, $j = 1, \ldots, d$, each of size $c$, and so $v = |\mathcal{P}| = cd$, with $c, d > 1$. Let $B$ be a $k$-set of $\mathcal{P}$, and let $\mathcal{B}^* = B^G$. Then the sizes of the intersections of each element of $\mathcal{B}^*$ with the imprimitivity classes determine a partition of $k$, say $x = (x_1, x_2, \ldots, x_d)$ with $x_1 \geq x_2 \geq \ldots \geq x_d$ and $\sum_{i=1}^{d} x_i = k$. Set $b_t = \sum_{i=1}^{d} x_i(x_i - 1) \cdots (x_i - t + 1)$. Note that $b_1 = k$. By [5, Proposition 2.2], the following lemma holds.

Lemma 4.1 Let $\mathcal{D}^* = (\mathcal{P}, \mathcal{B}^*)$. Then

(i) $\mathcal{D}^*$ is a 2-design if and only if

$$ b_2 = \sum_{i=1}^{d} x_i(x_i - 1) = \frac{k(k - 1)(c - 1)}{(v - 1)}. $$

(ii) $\mathcal{D}^*$ is a 3-design if and only if it is a 2-design and

$$ b_3 = \sum_{i=1}^{d} x_i(x_i - 1)(x_i - 2) = \frac{k(k - 1)(k - 2)(c - 1)(c - 2)}{(v - 1)(v - 2)}. $$

Proposition 4.1 Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial 3-(v, k, \lambda) design with $k \leq 6$ admitting a block-transitive point-imprimitive automorphism group $G$. Then $k = 6$ and $v = 16$. 

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Proof. Since \( S_c \wr S_d \) is a point-imprimitive maximal subgroup of symmetric group \( S_v \), we only need to consider the case that the group \( G = S_c \wr S_d \) acts point-imprimitively on \( D \) by Lemma \([2,4]\).

Suppose that the block size \( k = 4 \). Then the partition \( x \) of \( k \) is \((3,1,0,\ldots,0)\) as \( D \) is a 3-design. If \( x = (3,1,0,\ldots,0) \) then there is no such pair \((c,d)\) satisfying Lemma \([4.1]\)(i) as \( c > 1 \) and \( d > 1 \).

Now assume that \( k = 5 \). By Lemma \([4.1]\)(i), the partitions \( x \) of \( k \) and parameters \( c, d \) are listed in Table \([2]\). By using Lemma \([4.1]\)(ii), we get that \( x = (3,2) \) and \((c,d) = (3,2)\).

| \( x \)   | \((4,1,\ldots,0)\) | \((3,1,1)\) | \((3,2)\) | \((2,2,1,\ldots,0)\) |
|----------|----------------|----------|----------|----------------|
| \((c,d)\) | \(\emptyset\)  | \((7,3)\) | \((3,2)\) | \((2,3),(4,4)\) |

Then \( v = cd = 6 \), contradicts the nontriviality of \( D \).

Finally, we assume that \( k = 6 \). Similarly, the the case \((5,1,\ldots,0)\) does not happen by Lemma \([4.1]\)(i). Other partitions \( x \) of \( k \) and parameters \( c, d \) are listed in Table \([3]\). From Lemma \([4.1]\)(ii) and the nontriviality of \( D \), we get that \( x = (4,2) \) and \((c,d) = (8,2)\) and so \( v = 16 \).

| \( x \)   | \((4,1,1,\ldots,0)\) | \((4,2)\) | \((3,3)\) | \((3,2,1,\ldots,0)\) | \((3,1,1,1\ldots,0)\) |
|----------|----------------|----------|----------|----------------|
| \((c,d)\) | \((3,2)\)      | \((8,2)\) | \((3,2)\) | \(\emptyset\)   | \((2,3),(4,4)\) |

Corollary 4.1 Let \( D = (\mathcal{P}, \mathcal{B}) \) be a 3-(16,6,\(\lambda\)) design admitting \( G \) as its block-transitive, point-imprimitive automorphism group. Then \( \text{rank}(G) = 3 \) with subdegrees 1, 7, 8, and \( \lambda \in \{4, 12, 16, 24, 28, 48, 56, 64, 84, 96, 112, 140\} \).

Proof. By the proof of Proposition \([4.1]\) we have that \( G \leq S_8 \wr S_2 \). From Lemma \([2,4]\) there exists a 3-(16,6,\(\lambda^*\)) design \( D^* \) admitting \( S_8 \wr S_2 \) as a block-transitive, point-imprimitive automorphism group. Let \( \Delta_1 \) and \( \Delta_2 \) be the blocks of imprimitivity of \( \mathcal{P} \), and let \( \alpha \in \Delta_1 \). Clearly, \( \Delta_1^{G_\alpha} = \Delta_1 \), so \(|\Delta_1|\) is sum of some subdegrees \( d \) of \( G \). On the other hand, it follows
from Lemma 2.2 that 7 divides \(d(d-1)\), and then we easily observe that \(G_\alpha\) has subdegrees 1, 7, 8.

Let \(B\) be any base block of \(D^*\). Since the partition of block size \(k\) is \(x = (4,2)\), without loss of generality, we set \(|B \cap \Delta_1| = 4\) and \(|B \cap \Delta_2| = 2\). Let \(D^* = (\mathcal{P}, \mathcal{B}^*)\). Then each block of \(D^*\) is the 6-set in \(\mathcal{P}\) with partition \(x = (4,2)\) as \(S_8 \wr S_2\) acts transitively on \(\{\Delta_1, \Delta_2\}\) and 4-transitively on \(\Delta_i\) \((i = 1, 2)\). Thus, the number of blocks in \(D^*\) is

\[
|\mathcal{B}^*| = \binom{2}{1} \binom{8}{4} \binom{8}{2} = 3920.
\]

We further obtain \(\lambda^* = 140\) by Lemma 2.1(ii), and so \(\lambda \leq 140\) as \(B \subseteq \mathcal{B}^*\).

By using the software package MAGMA [2]-command TransitiveGroups(16), we know that there are 1954 transitive groups on \(\mathcal{P} = \{1,2,3,\ldots,16\}\), exactly 22 of which are primitive. Here we only consider that \(G\) is one of the remaining 1932 imprimitive groups. Note that, if \(B\) is a base block of \(\mathcal{B}\), then \(B \in \mathcal{B}^*\). A simple calculation by using command Design<3,16|BG>, we get that \(\lambda \in \{4, 12, 16, 24, 28, 48, 56, 64, 84, 96, 112, 140\}\). \(\square\)

**Proof of Theorem 2** It follows from Proposition 4.1 and Corollary 4.1.

**Remark** Up to isomorphism, there are 28 different block-transitive point-imprimitive nontrivial 3-(\(v, k, \lambda\)) designs with \(k \leq 6\) by using command IsIsomorphic(D1,D2) (see Table 4). The notation “\(n\)” in Table 4 means that there are \(n\) pairwise non-isomorphic block-transitive point-imprimitive 3-(16, 6, \(\lambda\)) designs.

| \(\lambda\) | 4   | 12  | 16  | 24  | 28  | 48  | 56  | 64  | 84  | 96  | 112 | 140 |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \(n\)    | 5   | 4   | 5   | 1   | 1   | 6   | 1   | 1   | 1   | 1   | 1   | 1   |

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