EQUATION OF STATE IN NUMERICAL RELATIVISTIC HYDRODYNAMICS

DONGSU RYU,¹ INDRANIL CHATTOPADHYAY,¹ AND EUNWOO CHOI²

Received 2006 January 20; accepted 2006 May 15

ABSTRACT

Relativistic temperature of gas raises the issue of the equation of state (EOS) in relativistic hydrodynamics. We study
the EOS for numerical relativistic hydrodynamics, and propose a new EOS that is simple and yet approximates very
closely the EOS of the single-component perfect gas in relativistic regime. We also discuss the calculation of primitive
variables from conservative ones for the EOSs considered in the paper, and present the eigenstructure of relativistic
hydrodynamics for a general EOS, in a way that they can be used to build numerical codes. Tests with a code based
on the total variation diminishing (TVD) scheme are presented to highlight the differences induced by different EOSs.
Subject headings: hydrodynamics — methods: numerical — relativity

1. INTRODUCTION

Relativistic flows are involved in many high-energy astrophysical
phenomena. Examples include relativistic jets from Galactic
sources (see Mirabel & Rodríguez [1999] for review), extragalactic
jets from active galactic nuclei (see Zensus [1997] for review), and
gamma-ray bursts (see Meszaros [2002] for review). In relativistic
jets from some Galactic microquasars, intrinsic beam velocities
larger than 0.9c are typically required to explain the observed
superluminal motions. In some powerful extragalactic radio sources,
ejections from galactic nuclei produce true beam velocities of
more than 0.98c. In the general fireball model of gamma-ray
bursts, the internal energy of gas is converted into the bulk kinetic
energy during expansion and this expansion leads to relativistic
outflows with high bulk Lorentz factors ≳100. The flow motions
in these objects are usually highly nonlinear and intrinsically com-
plex. Understanding such relativistic flows is important for cor-
rectly interpreting the observed phenomena, but often studying
them is possible only through numerical simulations.

Numerical codes for special relativistic hydrodynamics (RHD)
have been successfully built, based on explicit finite difference up-
wind schemes that were originally developed for codes of non-
relativistic hydrodynamics. These schemes utilize approximate or
exact Riemann solvers and the characteristic decomposition of the
hyperslabic system of conservation equations. RHD codes based
on upwind schemes are able to capture sharp discontinuities robustly in complex flows, and to describe the physical solution reli-
ably. A partial list of such codes includes Falle & Komissarov (1996)
based on the van Leer scheme; Martí & Müller (1996), Aloy et al. (1999),
and Mignone et al. (2005) based on the piece-
wise parabolic method (PPM) scheme; Sokolov et al. (2001) based
on the Godunov scheme; Choi & Ryu (2005) based on the TVD
scheme; Dolezal & Wong (1995), Donat et al. (1998), DelZanna
& Bucciantini (2002); and Rahman & Moore (2005) based on the
essentially nonscillatory (ENO) scheme; and Mignone & Bodo
(2005) based on the Harten, Lax, and van Leer (HLL) scheme.
Reviews of some numerical approaches and test problems can be
found in Martí & Müller (2003) and Wilson & Mathews (2003).

Gas in RHD is characterized by relativistic fluid speed (v ~ c)
and/or relativistic temperature (internal energy much greater than
rest energy), and the latter brings us to the issue of the equation
of state (EOS) of the gas. The EOS most commonly used in
numerical RHD, which is designed for the gas with constant
ratio of specific heats, however, is essentially valid only for the
gas of either subrelativistic or ultrarelativistic temperature. It is
because that is not derived from relativistic kinetic theory. On
the other hand, the EOS of the single-component perfect gas in
relativistic regime can be derived from thermodynamics. But its
form involves the modified Bessel functions (see Synge 1957)
and is too complicated to be implemented in numerical schemes.

In this paper, we study EOSs for numerical RHD. We first
revisit two EOSs previously used in numerical codes, one with
constant ratio of specific heats, and the other first used by Mathews
(1971) and later proposed for numerical RHD by Mignone et al.
(2005). We then propose a new EOS, which is simple to implement
in numerical codes with minimum effort and minimum computa-
tional cost, but which at the same time approximates very closely
the EOS of the single-component perfect gas in relativistic regime.
We also discuss the calculation of primitive variables from cons-
ervative ones for the three EOSs. Then we present the entire
eigenstructure of RHD for a general EOS, in a way to be used to
build numerical codes. In order to see the consequence of dif-
f erent EOSs, shock tube tests performed with a code based on
the TVD scheme are presented. The tests demonstrate the dif-
f erences in flow structure due to different EOSs. Employing a
correct EOS should be important for getting quantitatively cor-
rect results in problems involving a transition from nonrelativ-
istic temperature to relativistic temperature or vice versa.

This paper is organized as follows. In §§ 2 and 3 we discuss
three EOSs and the calculation of primitive variables from con-
servative ones for those three. In § 4 we present the eigenstruc-
ture of RHD with a general EOS. In §§ 5 and 6 we present a code
based on the TVD scheme and shock tube tests with the code.
Concluding remarks are drawn in § 7.

2. RELATIVISTIC HYDRODYNAMICS

2.1. Basic Equations

The special RHD equations for an ideal fluid can be written in
the laboratory frame of reference as a hyperbolic system of con-
servation equations

\[ \partial D \over \partial t + \frac{\partial}{\partial x_j} (Dv_j) = 0, \]  

(1a)
\[
\frac{\partial M_i}{\partial t} + \frac{\partial}{\partial x_j} (M_{ij} + p \delta_{ij}) = 0, \\
\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j} [(E + p)v_j] = 0,
\]

where \( D, M_i, \) and \( E \) are the mass density, momentum density, and total energy density, respectively (see, e.g., Landau & Lifshitz 1959; Wilson & Mathews 2003). The conserved quantities in the laboratory frame are expressed as

\[
D = \Gamma \rho,
\]

\[
M_i = \Gamma^2 \rho v_i,
\]

\[
E = \Gamma^2 \rho h - p,
\]

where \( \rho, v_i, p, \) and \( h \) are the proper mass density, fluid three-velocity, isotropic gas pressure, and specific enthalpy, respectively, and the Lorentz factor is given by

\[
\Gamma = \frac{1}{\sqrt{1 - v^2}}, \quad \text{with} \quad v^2 = v_x^2 + v_y^2 + v_z^2. \tag{3}
\]

In the above, the Latin indices (e.g., \( i \)) represent spatial coordinates and conventional Einstein summation is used. The speed of light is set to unity (\( c = 1 \)) throughout this paper.

2.2. Equation of State

The above system of equations is closed with an EOS. Without loss of generality it is given as

\[
h \equiv h(p, \rho). \tag{4}
\]

Then the general form of polytropic index, \( n \), and the general form of sound speed, \( c_s \), can be written as

\[
n = \rho \frac{\partial h}{\partial p} - 1, \quad c_s^2 = -\frac{\rho}{nh} \frac{\partial h}{\partial \rho}. \tag{5}
\]

In addition we use a variable \( \gamma_h \) to present the EOS property conveniently,

\[
\gamma_h = \frac{h - 1}{\Theta}, \tag{6}
\]

where \( \Theta = p/\rho \) is a temperature-like variable.

The most commonly used EOS, which is called the ideal EOS (hereafter ID), is given as

\[
p = (\gamma - 1)(e - \rho) \quad \text{or} \quad h = 1 + \frac{\gamma \Theta}{\gamma - 1}, \tag{7}
\]

with a constant \( \gamma \). Here \( \gamma = c_p/c_v \) is the ratio of specific heats and \( e \) is the sum of the internal and rest-mass energy densities in the local frame and is related to the specific enthalpy as

\[
h = \frac{e + p}{\rho}. \tag{8}
\]

For ID, \( \gamma_h = \gamma/(\gamma - 1) \) does not depend on \( \Theta \). ID may be correctly applied to the gas of either subrelativistic temperature with \( \gamma = 5/3 \) or ultrarelativistic temperature with \( \gamma = 4/3 \). But ID is borrowed from nonrelativistic thermodynamics, and hence it is not consistent with relativistic kinetic theory. For example, we have

\[
n = \frac{1}{\gamma - 1}, \quad c_s^2 = \frac{\gamma \Theta (\gamma - 1)}{\gamma \Theta + \gamma - 1}. \tag{9}
\]

In the high-temperature limit, i.e., \( \Theta \to \infty \), and for \( \gamma > 2, c_v > 1; i.e., sound speed is superluminal. More importantly, using relativistic kinetic theory, Taub (1948) showed that the choice of EOS is not arbitrary and has to satisfy the inequality

\[
(h - \Theta)(h - 4\Theta) \geq 1. \tag{10}
\]

This rules out ID for \( \gamma > 4/3 \), if applied for \( 0 < \Theta < \infty \).

The correct EOS for the single-component perfect gas in relativistic regime (hereafter RP) can be derived (see Synge 1957); it is given as

\[
h = \frac{K_2(1/\Theta)}{K_3(1/\Theta)}, \tag{11}
\]

where \( K_2 \) and \( K_3 \) are the modified Bessel functions of the second kind of order two and three, respectively. In the nonrelativistic temperature limit (\( \Theta \to 0 \), \( \gamma_h \to 5/2 \), and in the ultrarelativistic temperature limit (\( \Theta \to \infty \), \( \gamma_h \to 4 \)). However, using the above EOS comes with a price of extra computational costs (Falle & Komissarov 1996), since the thermodynamics of the fluid is expressed in terms of the modified Bessel functions.

There have been efforts to find approximate EOSs that are simpler than RP but more accurate than ID. For example, Sokolov et al. (2001) proposed

\[
\Theta = \frac{1}{4} (h - 1/h) \quad \text{or} \quad h = 2\Theta + 4\Theta^2 + 1. \tag{12}
\]

But this EOS neither satisfies Taub’s inequality nor is consistent with the value of \( \gamma_h \) in the nonrelativistic temperature limit.

In a recent paper, Mignone et al. (2005) proposed for numerical RHD an EOS that fits RP well. The EOS, which was first used by Mathews (1971), is given as

\[
p/\rho = \frac{1}{3} \left( e/\rho - \rho \right) \quad \text{or} \quad h = \frac{5}{2} \Theta + \frac{3}{2} \sqrt{\Theta^2 + 4}. \tag{13}
\]

and is abbreviated as TM following Mignone et al. (2005). With TM the expressions for \( n \) and \( c_s \) become

\[
n = \frac{3}{2} + \frac{3}{2} \sqrt{\Theta^2 + 4/9}, \quad c_s^2 = \frac{5\Theta \sqrt{\Theta^2 + 4/9 + 3\Theta^2}}{12\Theta \sqrt{\Theta^2 + 4/9 + 12\Theta^2 + 2}}. \tag{14}
\]

TM corresponds to the lower bound of the Taub inequality, i.e., \( (h - \Theta)(h - 4\Theta) = 1 \). It produces the correct asymptotic values for \( \gamma_h \).

In this paper we propose a new EOS, which is a simpler algebraic function of \( \Theta \) and is also a better fit of RP compared to TM. We abbreviate our proposed EOS as RC and give it as

\[
p/\rho = \frac{3p + 2\rho}{9p + 3\rho} \quad \text{or} \quad h = \frac{2}{3} \Theta^2 + 4\Theta + 1. \tag{15}
\]
Schneider et al. (1993) showed that equations (2a)–(2c) with the EOS in equation (7) reduce to a single quartic equation for $v$,

$$v^4 + b_1 v^3 + b_2 v^2 + b_3 v + b_4 = 0,$$

where

$$b_1 = -\frac{2\gamma(\gamma - 1)ME}{(\gamma - 1)^2(M^2 + D^2)},$$

$$b_2 = \frac{\gamma^2 E^2 + 2(\gamma - 1)M^2 - (\gamma - 1)^2 D^2}{(\gamma - 1)^2(M^2 + D^2)},$$

$$b_3 = -\frac{2\gamma ME}{(\gamma - 1)^2(M^2 + D^2)},$$

$$b_4 = \frac{M^2}{(\gamma - 1)^2(M^2 + D^2)},$$

and $M = (M_x^2 + M_y^2 + M_z^2)^{1/2}$. The quartic equation (18) can be solved numerically or analytically. In Choi & Ryu (2005) the analytical solution was used for the very first time, although the exact nature of the solution was not presented.

The general form of analytical roots for quartic equations can be found in Abramowitz & Stegun (1972). One may even use software such as Mathematica or Maxima to find the roots. We found that out of the four roots of the quartic equation (18), two are complex and two are real. The two real roots are

$$z_1 = \frac{-B + \sqrt{B^2 - 4C}}{2}, \quad z_2 = \frac{-B - \sqrt{B^2 - 4C}}{2},$$

where

$$B = \frac{1}{2} \left( b_1 + \sqrt{b_1^2 - 4b_2 + 4x_1} \right), \quad C = \frac{1}{2} \left( x_1 - \sqrt{x_1^2 - 4b_4} \right),$$

$$x_1 = \left( R + T^{1/2} \right)^{1/3} + \left( R - T^{1/2} \right)^{1/3} - \frac{a_1}{3},$$

$$R = \frac{9a_1 a_2 - 27a_3 - 2a_1^3}{54}, \quad S = \frac{3a_2 - a_1^2}{9}, \quad T = R^2 + S^3,$$

$$a_1 = -b_2, \quad a_2 = b_1 b_3 - 4b_4, \quad a_3 = 4b_2 b_4 - b_3^2 - b_1^2 b_4.$$
3.2. TM

Combining equations (2a)–(2c) with the EOS in equation (13), we get a cubic equation for \( W = \Gamma^2 - 1 \):

\[
W^3 + c_1 W^2 + c_2 W + c_3 = 0, \quad (23)
\]

where

\[
c_1 = \frac{(E^2 + M^2)[4(E^2 + M^2) - (M^2 + D^2)] - 14M^2E^2}{2(E^2 - M^2)^2},
\]

\[
c_2 = \frac{[4(E^2 + M^2) - (M^2 + D^2)]^2 - 57M^2E^2}{16(E^2 - M^2)^2},
\]

\[
c_3 = -\frac{9M^2E^2}{16(E^2 - M^2)^2}.
\]

Cubic equations admit analytical solutions simpler than quartic equations (see also Abramowitz & Stegun 1972). We found that out of the three roots of the cubic equation (23), two are unphysical, giving \( \Gamma < 1 \), and only one gives the physical solution, which is

\[
W = 2\sqrt{-J} \cos \left(\frac{\nu}{3}\right) - \frac{c_1}{3}, \quad (25)
\]

where

\[
J = \frac{3c_2 - c_1^2}{9}, \quad \cos \nu = \frac{H}{\sqrt{-J^3}}, \quad H = \frac{9c_1c_2 - 27c_3 - 2c_1^3}{54}.
\]

Then the fluid speed is calculated by

\[
v = \frac{W}{\sqrt{W^2 + 1}}, \quad (27)
\]

and the quantities \( \rho, v_i, \) and \( p \) are calculated by

\[
\rho = \frac{D}{\Gamma}, \quad (28a)
\]

\[
v_x = \frac{M_x}{M} v_i, \quad v_y = \frac{M_y}{M} v_i, \quad v_z = \frac{M_z}{M} v_i. \quad (28b)
\]

\[
p = \frac{(E - M_xv_x - M_yv_y - M_zv_z)^2 - \rho^2}{3(E - M_xv_x - M_yv_y - M_zv_z)}. \quad (28c)
\]

3.3. RC

Combining equations (2a)–(2c) with the EOS in equation (15), we get

\[
M\sqrt{(\Gamma^2 - 1)[3E\Gamma(8\Gamma^2 - 1) + 2D(1 - 4\Gamma^2)]}
\]

\[
= 3\Gamma^2[4(M^2 + E^2)\Gamma^2 - (M^2 + 4E^2)]
\]

\[
- 2D(4E\Gamma - D)(\Gamma^2 - 1). \quad (29)
\]

Further simplification reduces it into an equation of eighth power in \( \Gamma \).

Although equation (29) has to be solved numerically, it behaves very well. We first analyzed the nature of the roots with a root-finding routine in the IMSL library. As noted by Schneider et al. (1993), the physically meaningful solution should be between the upper limit, \( \Gamma_u \),

\[
\Gamma_u = \frac{1}{\sqrt{1 - v_w^2}} \quad \text{with} \quad v_w = \frac{M}{E}, \quad (30)
\]

and the lower limit, \( \Gamma_l \), that is derived by inserting \( D = 0 \) into equation (29):

\[
16(M^2 - E^2)^2\Gamma_u^8 - 8(M^2 - E^2)(M^2 - 4E^2)\Gamma_u^4
\]

\[
+ (M^4 - 9M^2E^2 + 16E^4)\Gamma_u^2 + M^2E^2 = 0 \quad (31)
\]

(a cubic equation of \( \Gamma_l^2 \)). Out of the eight roots of equation (29), four are complex and four are real. Out of the four real roots, two are negative and two are positive. Out and out of the two real and positive roots, one is always larger than \( \Gamma_u \), and the other is between \( \Gamma_l \) and \( \Gamma_u \), and so is the physical solution.

Inside RHD codes the physical solution of equation (29) can be easily calculated by the Newton-Raphson method. With an initial guess \( \Gamma = \Gamma_l \) or any value smaller than it including 1, iteration can proceed upward. Since the equation is extremely well behaved, the iteration converges within a few steps. Once \( \Gamma \) is known, the fluid speed is calculated by

\[
v = \frac{\sqrt{\Gamma^2 - 1}}{\Gamma}, \quad (32)
\]

and the quantities \( \rho, v_i, \) and \( p \) are calculated by

\[
\rho = \frac{D}{\Gamma}, \quad (33a)
\]

\[
v_x = \frac{M_x}{M} v_i, \quad v_y = \frac{M_y}{M} v_i, \quad v_z = \frac{M_z}{M} v_i \quad (33b)
\]

\[
p = \frac{(E - M_xv_x - M_yv_y - M_zv_z)^2 - \rho^2}{3(E - M_xv_x - M_yv_y - M_zv_z)}. \quad (33c)
\]

where

\[
M_xv_x = M_xv_x + M_xv_y + M_xv_z. \quad (34)
\]

4. EIGENVALUES AND EIGENVECTORS

In building a code based on Roe-type schemes, such as the TVD and ENO schemes, that solves a hyperbolic system of conservation equations, the eigenstructure (eigenvalues and eigenvectors of the Jacobian matrix) is required. The eigenstructure for RHD was previously described, for instance, in Donat et al. (1998). However, with the parameter vector different from that of Donat et al. (1998), the eigenvectors become different. Here we present our complete set of eigenvalues and eigenvectors without assuming any particular form of EOS.

Equations (1a)–(1c) can be written as

\[
\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x_j} = 0, \quad (35)
\]
with the state and flux vectors
\[
\mathbf{q} = \begin{pmatrix} D \\ M_i \\ E \end{pmatrix}, \quad \mathbf{F}_j = \begin{bmatrix} D_{ij} \\ M_{ij} + p \delta_{ij} \\ (E + p)v_j \end{bmatrix},
\]
(36)
or as
\[
\frac{\partial \mathbf{q}}{\partial t} + A_j \frac{\partial \mathbf{q}}{\partial x_j} = 0, \quad A_j = \frac{\partial \mathbf{F}_j}{\partial \mathbf{q}}.
\]
(37)
Here \( A_j \) is the \( 5 \times 5 \) Jacobian matrix composed with the state and flux vectors. The construction of the matrix \( A_j \) can be simplified by introducing a parameter vector, \( \mathbf{u} \), as
\[
A_j = \frac{\partial \mathbf{F}_j}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{q}}.
\]
(38)

4.1. One Velocity Component

The eigenstructure is simplified if only a single component of velocity is chosen, i.e., \( v = v_s \). In principle it can be reduced from that with three components of velocity in \( \S \ 4.2 \). Nevertheless we present it, for the case in which the simpler eigenstructure with one velocity component can be used.

The explicit form of the Jacobian matrix, \( A \), is presented in Appendix A. The eigenvalues of \( A \) are
\[
a_- = \frac{v - c_s}{1 - c_s v}, \quad a_0 = v, \quad a_+ = \frac{v + c_s}{1 + c_s v}.
\]
(40)
The right eigenvectors are
\[
\mathbf{R}_- = \begin{bmatrix} 1 \\ \Gamma h(v - c_s) \\ \Gamma h(1 - c_s v) \end{bmatrix}, \quad \mathbf{R}_0 = \begin{bmatrix} 1 \\ \Gamma h(1 - nc_s^2) \\ \Gamma h(1 - nc_s^2) \end{bmatrix}, \quad \mathbf{R}_+ = \begin{bmatrix} 1 \\ \Gamma h(v + c_s) \\ \Gamma h(1 + c_s v) \end{bmatrix},
\]
(41)
and the left eigenvectors are
\[
\mathbf{L}_- = -\frac{1}{2hc_s^2} \left[ h(1 - nc_s^2), \ (\Gamma v + nc_s), \ -\Gamma(1 + nc_s v) \right], \quad \mathbf{L}_0 = \frac{1}{hnc_s^2} \left[ h, \ \Gamma v, \ -\Gamma \right], \quad \mathbf{L}_+ = -\frac{1}{2hc_s^2} \left[ h(1 - nc_s^2), \ (\Gamma v - nc_s), \ -\Gamma(1 - nc_s v) \right].
\]
(42)

4.2. Three Velocity Components

The \( x \)-component of the Jacobian matrix, \( A_x \), when all the three components of velocity are considered, is presented in Appendix B. The eigenvalues of \( A_x \) are
\[
a_1 = \frac{(1 - c_s^2)v_s - c_s/\sqrt{Q}}{1 - c_s^2v^2},
\]
(43a)
\[
a_2 = v_s,
\]
(43b)
\[
a_3 = v_s,
\]
(43c)
\[
a_4 = v_s,
\]
(43d)
\[
a_5 = \frac{(1 - c_s^2)v_s + c_s/\sqrt{Q}}{1 - c_s^2v^2},
\]
(43e)
where \( Q = 1 - v_s^2 - c_s^2(v_s^2 + v_y^2) \). The eigenvalues represent the five characteristic speeds associated with two sound wave modes (\( a_1 \) and \( a_5 \)) and three entropy modes (\( a_2, a_3, \) and \( a_4 \)). A remarkable feature is that the eigenvalues do not explicitly depend on \( h \) and \( n \), but only on \( v_s \) and \( c_s \). Hence the eigenvalues are the same regardless of the choice of EOS once the sound speed is defined properly.

The corresponding right eigenvectors (\( \mathbf{A}_x \mathbf{R} = \mathbf{a}_x \mathbf{R} \)), however, depend explicitly on \( h \) and \( n \), and the complete set is given by
\[
\mathbf{R}_1 = \begin{bmatrix} 1 - a_1v_s \\ a_1h(1 - v_s^2), \ h(1 - a_1v_s)v_y, \\ h(1 - a_1v_s)v_z, \ h(1 - v_s^2) \end{bmatrix}^T,
\]
(44a)
\[
\mathbf{R}_2 = \hat{\mathbf{X}}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5)^T,
\]
(44b)
\[
R_3 = \frac{1}{1 - v_s^2} \left( \frac{v_y}{\Gamma h}, \ 2v_xv_y, \ 1 - v_s^2 + v_y^2, \ v_yv_z, \ 2v_y \right)^T,
\]
(44c)
\[
R_4 = \frac{1}{1 - v_s^2} \left( \frac{v_x}{\Gamma h}, \ 2v_xv_y, \ v_yv_z, \ 1 - v_s^2 + v_x^2, \ 2v_x \right)^T,
\]
(44d)
\[
\mathbf{R}_5 = \begin{bmatrix} 1 - a_5v_s \\ a_5h(1 - v_s^2), \ h(1 - a_5v_s)v_y, \\ h(1 - a_5v_s)v_z, \ h(1 - v_s^2) \end{bmatrix}^T,
\]
(44e)
where
\[
\hat{\mathbf{X}} = \frac{\sqrt{D}}{nc_s^2(1 - v_s^2)}.
\]
(45f)
The complete set of the left eigenvectors \((\mathbf{L} \mathbf{A}_i = \mathbf{aL})\), which are orthonormal to the right eigenvectors, is

\[
\mathbf{L}_1 = \frac{1}{Y_1}(Y_{11}, Y_{12}, Y_{13}, Y_{14}, Y_{15}),
\]

\[
\mathbf{L}_2 = \left( \frac{h}{\Gamma}, v_x, v_y, v_z, -1 \right),
\]

\[
\mathbf{L}_3 = (-\Gamma h v_y, 0, 1, 0, 0),
\]

\[
\mathbf{L}_4 = (-\Gamma h v_z, 0, 0, 1, 0),
\]

\[
\mathbf{L}_5 = \frac{1}{Y_5}(Y_{51}, Y_{52}, Y_{53}, Y_{54}, Y_{55}),
\]

where

\[
Y_{1i} = -\frac{h}{\Gamma} (1 - a_i v_i) (1 - \nu_i^2),
\]

\[
Y_{2i} = n a_i (1 - \nu_i^2) + a_i (1 + \nu_i^2) v_i^2 - (1 + n) v_i,
\]

\[
Y_{3i} = -(1 + \nu_i^2) (1 - a_i v_i) v_y,
\]

\[
Y_{4i} = -(1 + \nu_i^2) (1 - a_i v_i) v_z,
\]

\[
Y_{5i} = (1 + \nu_i^2) v_i^2 + (1 - \nu_i^2) m_i^2 - a_i (1 + n) v_i,
\]

\[
\bar{Y}_i = h n \left( (a_i - v_i^2) Q + \nu_i^2 \right),
\]

and index \(i = 1\) and 5.

We note that with three degenerate modes that have same eigenvalues, \(a_2 = a_3 = a_4\), we have a freedom to write down the right and left eigenvectors in a variety of different forms. We chose to present the ones that produce the best results with the TVD code described next.

5. ONE-DIMENSIONAL FUNCTIONING CODE

To demonstrate the differences in flow structure due to different EOSs, a one-dimensional functioning code based on the total variation diminishing (TVD) scheme was built. The code utilizes the eigenvalues and eigenvectors given in the previous section and can employ arbitrary EOSs, including those in § 2.2.

5.1. The TVD Scheme

The TVD scheme, originally developed by Harten (1983), is an Eulerian, finite-difference scheme with second-order accuracy in space and time. The second-order accuracy in time is achieved by modifying numerical flux using the quantities in five grid cells (see below and Harten [1983] for details). The scheme is basically identical to that previously used in Ryu et al. (1993) and Choi & Ryu (2005). But for completeness, the procedure is concisely shown here.

The state vector \(\mathbf{q}_i^n\) at the cell center \(i\) at the time step \(n\) is updated by calculating the modified flux vector \(\tilde{f}_{x,i+1/2}\) along the \(x\)-direction at the cell interface \(i \pm 1/2\) as follows:

\[
\mathbf{L}_x \mathbf{q}_i^n = \mathbf{q}_i^n + \frac{\Delta t^n}{\Delta x} (\tilde{f}_{x,i+1/2} - \tilde{f}_{x,i-1/2}),
\]

\[
\tilde{f}_{x,i+1/2} = \frac{1}{2} \left[ \mathbf{F}_x(\mathbf{q}_i^n) + \mathbf{F}_x(\mathbf{q}_{i+1}^n) \right] - \frac{\Delta x}{2 \Delta t^n} \sum_{k=1}^{5} \beta_{k,i+1/2} \mathbf{R}_k^n \mathbf{q}_{k,i+1/2},
\]

\[
\beta_{k,i+1/2} = Q_k \left( \frac{\Delta t^n}{\Delta x} a_k^n \mathbf{q}_{k,i+1/2} + \gamma_k,i+1/2 \right) \alpha_{k,i+1/2} - (g_{k,i} + g_{k,i+1}),
\]

\[
\gamma_{k,i+1/2} = \begin{cases} \frac{(g_{k,i+1} - g_{k,i})}{\alpha_{k,i+1/2}} & \text{for } \alpha_{k,i+1/2} \neq 0, \\ 0 & \text{for } \alpha_{k,i+1/2} = 0, \end{cases}
\]

\[
g_{k,i} = \text{sign}(\tilde{g}_{k,i+1/2}) \max \left\{ 0, \min \left[ \tilde{g}_{k,i+1/2}, \text{sign}(\tilde{g}_{k,i+1/2}) \tilde{g}_{k,i-1/2} \right] \right\},
\]

\[
\tilde{g}_{k,i+1/2} = \frac{1}{2} \left[ Q_k \left( \frac{\Delta t^n}{\Delta x} a_k^n \mathbf{q}_{k,i+1/2} \right) - \left( \frac{\Delta t^n}{\Delta x} a_k^n \right)^2 \right] \alpha_{k,i+1/2},
\]

\[
\alpha_{k,i+1/2} = \mathbf{L}_x^k \mathbf{q}_{k,i+1/2} - (\mathbf{q}_{k,i} - \mathbf{q}_{k,i+1}),
\]

\[
Q_k(x) = \begin{cases} x^2/4 & \text{for } |x| < 2 \varepsilon_k, \\ |x| & \text{for } |x| \geq 2 \varepsilon_k. \end{cases}
\]

Here \(k = 1\)–5 stand for the five characteristic modes. The internal parameters \(\varepsilon_k\) implicitly control numerical viscosity, and they are defined for \(0 \leq \varepsilon_k < 0.5\). The flux limiters in equations (52a)–(52c) are the minmod, monotinized central difference, and superbee limiters, respectively, a partial list of the limiters that are consistent with the TVD scheme, and one of them has to be employed.

5.2. Quantities at Cell Interfaces

To calculate the fluxes, we need to define the local quantities at the cell interfaces, \(i \pm 1/2\). The TVD scheme originally used the Roe linearization technique (Roe 1981) for it. Although it is possible to implement this linearizion technique in the relativistic domain in a computationally feasible way (see Eulderink & Mellema 1995), there is unlikely to be a significant advantage from the computational point of view. Instead, we simply use the algebraic averages of quantities at two adjacent cell centers to define the fluid three-velocity and specific enthalpy at the cell interfaces:

\[
v_{x,i+1/2} = \frac{v_{xi} + v_{xi+1}}{2}, \quad v_{y,i+1/2} = \frac{v_{yi} + v_{yi+1}}{2},
\]

\[
v_{z,i+1/2} = \frac{v_{zi} + v_{zi+1}}{2},
\]

\[
v_{z,i+1/2} = \frac{h_i + h_{i+1}}{2}.
\]
Defining $n$ and $c_s$ for the calculation of eigenvalues and eigenvectors at the cell interfaces depends on the EOS. For ID, $n$ is constant and

$$c_{s,i+1/2} = \left( \frac{h_{i+1/2} - 1}{nh_{i+1/2}} \right)^{1/2}.$$  \hfill (58)

For TM, we first compute from equation (13)

$$\Theta_{i+1/2} = \frac{5h_{i+1/2} - \sqrt{9h_{i+1/2}^2 + 16}}{8},$$  \hfill (59)

then define $n_{i+1/2}$ and $c_{s,i+1/2}$ according to equation (14). For RC, we first compute from equation (15)

$$\Theta_{i+1/2} = \frac{3h_{i+1/2} - 8 + \sqrt{9h_{i+1/2}^2 + 48h_{i+1/2} - 32}}{24},$$  \hfill (60)

then define $n_{i+1/2}$ and $c_{s,i+1/2}$ according to equation (16).

6. NUMERICAL TESTS

The differences induced by different EOSs are illustrated through a series of shock tube tests performed with the code described in the previous section. We use the tests used in previous works (e.g., Martí & Müller 2003; Mignone et al. 2005), instead of inventing our own. Two sets are considered, one being purely one-dimensional with only the velocity component parallel to structure propagation, and the other with transverse velocity component.

For the first set with parallel velocity component only, two tests are presented, P1: $\rho_L = 10^4, \rho_R = 10^{-2}$, and $v_{p,L} = v_{p,R} = 0$ initially, and $t_{end} = 0.4$. The box covers the region of $0 \leq x \leq 1$. Here the subscripts $L$ and $R$ denote the quantities in the left and right states of the initial discontinuity at $x = 0.5$, and $t_{end}$ is the time when the solutions are presented. These two tests have been extensively used

![Fig. 1 — Relativistic shock tube with parallel component of velocity only (P1) with RC (red), TM (blue), and ID (green and cyan).](image)

![Fig. 2 — Relativistic shock tube with parallel component of velocity only (P2) with RC (red), TM (blue), and ID (green and cyan).](image)

![Fig. 3 — Relativistic shock tube with transverse component of velocity (T1) with RC (red), TM (blue), and ID (green and cyan).](image)

![Fig. 4 — Relativistic shock tube with transverse component of velocity (T1) with RC (red), TM (blue), and ID (green and cyan).](image)
for tests of RHD codes with the ID EOS (see Martí & Müller 2003), and the analytic solutions were described in Martí & Müller (1994).

Figures 2 and 3 show the numerical solutions with RC and TM and the analytic solutions with ID and $\gamma = 5/3$ and 4/3. The numerical solutions with RC and TM were obtained using the version of the TVD code having one velocity component (see § 4.1), and the analytic solutions with ID come from the routine described in Martí & Müller (1994). The numerical solutions with ID are almost indistinguishable from the analytic solutions, once they are calculated.

The ID solutions with $\gamma = 4/3$ and 5/3 show noticeable differences. The density shell between the contact discontinuity (CD) and the shock becomes thinner and taller with smaller $\gamma$, because the post shock pressure is lower, and so is the shock propagation speed. The rarefaction wave is less elongated with $\gamma = 4/3$, because the sound speed is lower. Those solutions with ID are also clearly different from the solutions obtained with RC and TM. The ID solution with $\gamma = 4/3$ better approximates the solutions with RC and TM in the left region of the CD, because the flow has relativistic temperature of $\Theta \approx 1$ there. The difference is, however, obvious in the shell between the CD and the shock. On the other hand, the solutions obtained with RC and TM look very much alike. It reflects the similarity in the distributions of specific enthalpy in equations (13) and (15). Yet there is a noticeable difference, especially in the shell between the CD and the shock, and the difference in density reaches up to $\sim 5\%$.

For the second set with transverse velocity component, four tests, where different transverse velocities were added to the test P2, are presented. T1: initially $v_{r,R} = 0.99$ to the right state, $t_{\text{end}} = 0.45$; T2: initially $v_{r,L} = 0.9$ to the left state, $t_{\text{end}} = 0.55$; T3: initially $v_{r,L} = v_{r,R} = 0.99$ to the left and right states, $t_{\text{end}} = 0.18$; and T4: initially $v_{r,L} = 0.9$ and $v_{r,R} = 0.99$ to the left and right states, $t_{\text{end}} = 0.75$. The notations are the same ones used in P1 and P2. These are subsets of the tests originally suggested by Pons et al. (2000) with the ID EOS and later used by Mignone et al. (2005).

Figures 4, 5, 6, and 7 show the numerical solutions with RC and TM and the analytic solutions with ID and $\gamma = 5/3$ and 4/3. The numerical solutions with RC and TM were obtained using the version of the TVD code having three velocity components (see § 4.2), and the analytic solutions with ID come from the routine described in Pons et al. (2000).

Again the ID solutions with $\gamma = 4/3$ and 5/3 show noticeable differences. Especially with transverse velocity initially on the left side of the initial discontinuity (Figs. 5, 6, and 7), the parallel velocity reaches lower values, while the transverse velocity achieves higher values, with higher $\gamma = 5/3$ in the region to the left of the CD. As a result, the density shell between the CD and the shock has propagated less. As in the P tests, the solutions with ID are clearly different from the solutions obtained with RC and TM, most noticeably in the shell between the CD and the shock. The solutions with RC and TM look very much alike with differences in the density in the shell between the CD and the shock of about $\sim 5\%$.

We note that this paper is intended to focus on the EOS in numerical RHD, not intended to present the performance of the code. Hence, one-dimensional tests of high resolution (with $2^{16}$ grid cells for the P tests and $2^{17}$ grid cells for the T tests) are presented to manifest the difference induced by different EOSs.
The performance of the code such as capturing of shocks and CDs will be discussed elsewhere.

7. SUMMARY AND DISCUSSION

The conservation equations for both Newtonian hydrodynamics and RHD are strictly hyperbolic, rendering the apt use of upwind schemes for numerical codes. The actual implementation to RHD is, however, complicated, partly due to the EOS. In this paper we study three EOSs for numerical RHD, two being previously used and the other being newly proposed. The new EOS is simple and yet approximates the enthalpy of single-component perfect gas in relativistic regime with accuracy better than 0.8%. Then we discuss the calculation of primitive variables from conservative ones for the EOSs considered. We also present the eigenvalues and eigenvectors of RHD for a general EOS, in a way that they are ready to be used to build numerical codes based on the Roe-type schemes such as the TVD and ENO schemes. Finally we present numerical tests to show the differences in flow structure due to different EOSs.

The most commonly used, the ideal EOS, can be used for the gas of entirely nonrelativistic temperature ($\Theta \ll 1$) with $\gamma = 5/3$ or for the gas of entirely ultrarelativistic temperature ($\Theta \gg 1$) with $\gamma = 4/3$. However, if the transition from nonrelativistic to relativistic or vice versa with $\Theta \sim 0.1$ is involved, the ideal EOS produces incorrect results and its use should be avoided. The EOS proposed by Mignone et al. (2005), TM, produces reasonably correct results with error of a few percent at most. The most preferable advantage of using TM is that the calculation of primitive variables admits analytic solutions, thereby making its implementation easy. The newly suggested EOS, RC, which approximates the EOS of the relativistic perfect gas, RP, most accurately, produces thermodynamically the most accurate results. At the same time it is simple enough to be implemented in numerical codes with minimum efforts and minimum computational costs. With RC the primitive variables should be calculated numerically by an iteration method such as the Newton-Raphson method. However, the equation for the calculation of primitive variables behaves extremely well, so the iteration converges in a few steps without any trouble.

In Galactic and extragalactic jets and gamma-ray bursts, as the flows travel relativistic fluid speeds ($v \sim 1$ but $\Theta \ll 1$), they would hit the surrounding media. Then shocks are produced and the gas can be heated up to $\Theta \gg 1$. These kind of transitions, continuous or discontinuous, between relativistic bulk speeds and relativistic temperatures are intrinsic in astrophysical relativistic flows, and so a correct EOS is required to simulate the flows correctly. The correctness as well as the simplicity make RC suitable for astrophysical applications like these.

The work of D. R. and I. C. was supported by the KOSEF grant R01-2004-000-10005-0. The work of E. C. was supported by RPE funds to PEGA at GSU.

APPENDIX A

JACOBIAN MATRIX WITH ONE VELOCITY COMPONENT

The Jacobian matrix, when only one component of velocity is included, is given as follows:

$$A = \frac{1}{N} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & 0 \end{pmatrix} ,$$

$$A_{11} = v^2 n h (1 - c_r^2) + \frac{v h}{\Gamma} ,$$

$$A_{12} = -\frac{1}{\Gamma^3} + \frac{1 + n}{\Gamma} ,$$

$$A_{13} = -\frac{v (1 + n)}{\Gamma} .$$
\[
A_{21} = -\frac{h^2}{\Gamma_1} (1 - nc_s^2),
A_{22} = -\frac{vh}{\Gamma_1} (1 - nc_s^2) + 2vhn(1 - c_s^2),
A_{23} = -v^2hn(1 - c_s^2) + \frac{h}{\Gamma_1},
A_{32} = hn(1 - c_s^2v^2),
N = hn(1 - c_s^2v^2).
\]

**APPENDIX B**

**JACOBIAN MATRIX WITH THREE VELOCITY COMPONENTS**

The Jacobian matrix, when all the three components of velocity are included, is given as follows:

\[
A_x = \frac{1}{N} \begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\
A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\
A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\
A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\
0 & A_{52} & 0 & 0 & 0
\end{pmatrix};
\]  
(B1)

\[
A_{11} = v_yhn(1 - c_s^2) + \frac{hv_x}{\Gamma_2},
A_{12} = \frac{1}{\Gamma} \left[ n + v_x^2 - nc_s^2 (v_y^2 + v_z^2) \right],
A_{13} = \frac{1}{\Gamma} v_xv_y (1 + nc_s^2),
A_{14} = \frac{1}{\Gamma} v_xv_y (1 + nc_s^2),
A_{15} = -\frac{1}{\Gamma} v_x(1 + n),
A_{21} = -\frac{1}{\Gamma} (1 - v_y^2)h^2 (1 - nc_s^2),
A_{22} = v_xh \left[ 2n(1 - c_s^2v^2) - (1 - v_y^2)(1 + nc_s^2) \right],
A_{23} = -v_xh(1 - v_y^2)(1 + nc_s^2),
A_{24} = -v_xh(1 - v_y^2)(1 + nc_s^2),
A_{25} = -v_y^2h(1 + n) + h(1 + nc_s^2v^2),
A_{31} = \frac{1}{\Gamma} v_xv_yh^2 (1 - nc_s^2),
A_{32} = v_yh \left[ n(1 - c_s^2v^2) + v_y^2 (1 + nc_s^2) \right],
A_{33} = v_yh \left[ n(1 - c_s^2v^2) + v_y^2 (1 + nc_s^2) \right],
A_{34} = v_xv_yv_zh(1 + nc_s^2),
A_{35} = -v_xv_yh(1 + n),
A_{41} = \frac{1}{\Gamma} v_xv_yh^2 (1 - nc_s^2),
\]
\[ A_{42} = v_z h \left[ n \left( 1 - c_s^2 v^2 \right) + v_x^2 \left( 1 + nc_s^2 \right) \right], \]

\[ A_{43} = v_x v_y v_z h \left( 1 + nc_s^2 \right), \]

\[ A_{44} = v_x h \left[ n \left( 1 - c_s^2 v^2 \right) + v_z^2 \left( 1 + nc_s^2 \right) \right], \]

\[ A_{45} = -v_x v_y h (1 + n), \]

\[ A_{52} = h n (1 - c_s^2 v^2), \]

\[ N = h n \left( 1 - c_s^2 v^2 \right). \]

REFERENCES

Abramowitz, M. A., & Stegun, I. A. 1972, Handbook of Mathematical Functions (Dover: Dover Publishing Company)
Aloy, M. A., Ibáñez, J. M., Martí, J. M., & Müller, E. 1999, ApJS, 122, 151
Choi, E., & Ryu, D. 2005, NewA, 11, 116
DelZanna, L., & Bucciantini, N. 2002, A&A, 390, 1177
Dolezal, A., & Wong, S. S. M. 1995, J. Comput. Phys., 120, 266
Donat, R., Font, J. A., Ibáñez, J. M., & Marquina, A. 1998, J. Comput. Phys., 146, 58
Eulderink, F., & Mellema, G. 1995, A&AS, 110, 587
Falle, S. A. E. G., & Komissarov, S. S. 1996, MNRAS, 278, 586
Harten, A. 1983, J. Comput. Phys., 49, 357
Landau, L. D., & Lifshitz, E. M. 1959, Fluid Mechanics (New York: Pergamon)
Martí, J. M., & Müller, E. 1994, J. Fluid Mech., 258, 317
———. 1996, J. Comput. Phys., 123, 1
———. 2003, Living Rev. Relativity, 6, 7
Mathews, W. G. 1971, ApJ, 165, 147
Mészáros, P. 2002, ARA&A, 40, 137
Mignone, A., & Bodo, G. 2005, MNRAS, 364, 126
Mignone, A., Plewa, T., & Bodo, G. 2005, ApJS, 160, 199
Mirabel, I. F., & Rodríguez, L. F. 1999, ARA&A, 37, 409
Pons, J. A., Martí, J. M., & Müller, E. 2000, J. Fluid Mech., 422, 125
Rahman, T., & Moore, R. 2005, preprint (astro-ph/0512246)
Roe, P. L. 1981, J. Comput. Phys., 43, 357
Ryu, D., Ostriker, J. P., Kang, H., & Cen, R. 1993, ApJ, 414, 1
Schneider, V., Katscher, U., Rischke, D. H., Waldhauser, B., Maruhn, J. A., & Munz, C.-D. 1993, J. Comput. Phys., 105, 92
Sokolov, I., Zhang, H. M., & Sakai, J. I. 2001, J. Comput. Phys., 172, 209
Synge, J. L. 1957, The Relativistic Gas (Amsterdam: North-Holland)
Taub, A. H. 1948, Phys. Rev., 74, 328
Wilson, J. R., & Mathews, G. J. 2003, Relativistic Numerical Hydrodynamics (Cambridge: Cambridge Univ. Press)
Zensus, J. A. 1997, ARA&A, 35, 607