**The Large $N$ Reduction in Matrix Quantum Mechanics**  
— a Bridge between BFSS and IKKT —

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The large $N$ reduction is an equivalence between large $N$ gauge theories and matrix models discovered by Eguchi and Kawai in the early 80s. In particular the continuum version of the quenched Eguchi-Kawai model may be useful in studying supersymmetric and/or chiral gauge theories non-perturbatively. We apply this idea to matrix quantum mechanics, which is relevant, for instance, to nonperturbative studies of the BFSS Matrix Theory, a conjectured nonperturbative definition of M-theory. In the bosonic case we present Monte Carlo results confirming the equivalence directly, and discuss a possible explanation based on the Schwinger-Dyson equations. In the supersymmetric case we argue that the equivalence holds as well although some care should be taken if the rotational symmetry is spontaneously broken. This equivalence provides an explicit relation between the BFSS model and the IKKT model, which may be used to translate results in one model to the other.

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**Introduction.** — The large $N$ reduction $^1$ states that large $N$ gauge theories are equivalent to matrix models obtained by dimensionally reducing those theories to a point. Recent developments in the exact calculation of the low-energy superpotential in $\mathcal{N} = 1$ supersymmetric gauge theories $^2$ may also be regarded as a particular application of the large $N$ reduction $^2$. We apply the original idea to matrix quantum mechanics, which is relevant, for instance, to nonperturbative studies of the BFSS Matrix Theory $^4$ proposed as a nonperturbative definition of M-theory. The same model may also be interpreted as a low-energy effective theory of D0-branes in type IIA superstring theory. (In the cases where fermions are absent, interesting results are already obtained by a direct lattice approach $^5$. See Ref. $^4$ for a lattice formulation in the supersymmetric case.)

In Ref. $^1$ it was conjectured that the U($N$) gauge theory on an infinite lattice is equivalent in the large $N$ limit to a one-site model called the Eguchi-Kawai (EK) model $^1$. This conjecture was based on the observation that the Schwinger-Dyson equations of the two theories coincide under the assumption that the U(1)$^d$ symmetry $U_\mu \to e^{i\alpha_\mu}U_\mu$ of the EK model remains unbroken. The U(1)$^d$ symmetry, however, turned out to be spontaneously broken in the weak coupling region $^2$$^3$$^4$, and the quenched EK (QE) model was proposed as a remedy $^5$. The “quenched” eigenvalues of $U_\mu$ were interpreted later as lattice momenta $^6$, and this interpretation enabled an extension to non-gauge theories. The equivalence for the QE model was confirmed both perturbatively $^10$ and nonperturbatively $^2$$^3$$^4$.

The continuum version of the QE model (cQE model) has been proposed in Ref. $^12$. In spite of its potential usefulness in studying supersymmetric and/or chiral gauge theories, the EK equivalence for the cQE model has not been checked directly so far.

The twisted EK model $^{13}$, which was proposed as an alternative way to cure the problem of the original EK model, has been interpreted recently as field theories on a noncommutative geometry $^{14}$. However, this model is not useful for our purpose since the twisting procedure is applicable only to even space-time dimensions, and moreover the continuum version can be defined only in the strict $N = \infty$ limit $^{15}$. As another recent development, it has been found that the breaking of the U(1)$^d$ symmetry can be avoided by keeping the size of the lattice sufficiently large, and that the critical size in physical units becomes finite in the continuum limit $^{10}$.

In this letter we consider the cQE model for matrix quantum mechanics, and discuss its EK equivalence. In fact the corresponding cQE model is analogous to the IKKT model $^{17}$ proposed as a nonperturbative formulation of type IIB superstring theory. Based on this observation, we discuss an explicit relation between the BFSS model and the IKKT model.

The models. — We study a one-dimensional U($N$) gauge theory, or matrix quantum mechanics (QM), with adjoint scalars $X_i(t)$ ($i = 1, \cdots, d-1$), which are $N \times N$ Hermitian matrices. The action is given by

$$S_{\text{mQM}} = \frac{1}{2g^2} \int dt \text{tr} \left\{ \frac{1}{2} (DX_i(t))^2 - \frac{1}{4} [X_i(t), X_j(t)]^2 \right\},$$

(1)

where $DX_i = \frac{\partial}{\partial t} X_i - i[A, X_i]$ represents the covariant derivative with $A$ being the 1d gauge field. The model can be obtained formally by the dimensional reduction of $d$-dimensional Yang-Mills theory to one dimension. In what follows we fix the gauge to $A(t) = 0$. This model is UV finite and hence it does not require any UV regularization. As a result, the parameter $g$ in (1) can always...
be scaled out by an appropriate rescaling of the matrices and the time coordinate $t$. We take $g = \frac{1}{N}$, which turns out to be convenient when we discuss the large $N$ limit.

Applying the quenching prescription [9, 12] to the matrix QM (1), we obtain the corresponding cQEK model variables to be integrated over the region $\Omega$ and $\mathbf{X}$.

\[ \mathbf{X} \equiv \text{matrices and } \Omega \]

As $U(N)$ invariant operators in the matrix QM (1), let us consider the corresponding operator $\mathcal{O} (3)$ directly by Monte Carlo simulation. In order to simulate the 1D model (1), we discretize the time "$\tau$", and obtain the action

\[ S_{\text{lat}} = Na \left\{ \frac{1}{2} \sum_{n=1}^{T-1} \left( \frac{X_i(n+1) - X_i(n)}{a} \right)^2 - \sum_{n=1}^{T} \frac{1}{4} \text{tr} [X_i(n), X_j(n)]^2 \right\} , \]

where $a$ represents the lattice spacing. The $N \times N$ Hermitian matrices $X_i(n) (n = 1, \cdots, T)$ sit on each site, and we do not impose periodic boundary conditions. In order to retrieve the original model (1), we have to take the $a \to 0$ and $\tau \equiv aT \to \infty$ limits, where the order of the limits should not matter. The system is invariant under $X_i(n) \to X_i(n) + \beta \mathbf{1}$, and we fix the corresponding zero mode by imposing the condition $\sum_{n=1}^{T} \text{tr} X_i(n) = 0$.

Monte Carlo simulation of the models (1) and (2) can be performed by the heat bath algorithm [13]. For the cQEK model, we first take an average over the ensemble $\{ \mathbf{X} \}$ with the action (2) for each $\Omega$, and then perform the averaging over $\Omega$ by using random numbers distributed uniformly within the region $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Correspondingly there are two types of statistical errors. We estimate the total error by taking an average over the error of the first type and adding the one of the second type.

We focus on the $d = 4$ case in what follows. Let us first consider the observable $\mathcal{O}_1 = \frac{1}{N} \sum_{n=1}^{T} \text{tr} (X_i(n)^2)$ for the lattice model (1). In Fig. 1 we plot the results for $\langle \mathcal{O}_1 \rangle_{\text{lat}}$ against $a$ for $\tau = 5.0, 7.0, 10.0, 15.0$ at $N = 16$. We fit the results to $\langle \mathcal{O}_1 \rangle_{\text{lat}} = C_1 + C_2 a + C_3 a^2$, from which we obtain the continuum limit.

Monte Carlo simulation. — We test the equivalence directly by Monte Carlo simulation. In order to simulate the 1D model (1), we can be fitted to $\lim_{\tau \to \infty} \mathcal{O}_1 \to \mathcal{O}_1 (\tau \to \infty)$.

\[ \lim_{\tau \to \infty} \langle \mathcal{O} \rangle_{\text{mQM}} = \lim_{\Lambda \to \infty} \lim_{N \to \infty} \int d\Omega \langle \mathcal{O} \rangle_{\text{cQEK}}, \]

where $\langle \cdot \rangle_{\text{mQM}}$ and $\langle \cdot \rangle_{\text{cQEK}}$ represent the expectation values with respect to the actions (1) and (2), respectively, and $\int d\Omega \equiv \prod_{i=1}^{N} \int_{-\Lambda}^{\Lambda} \frac{d\omega}{2\pi}$. The order of the two limits on the r.h.s. of (3) cannot be inverted.

The results obtained in the continuum limit for various $\tau$ can be fitted to $\lim_{\tau \to \infty} \langle \mathcal{O}_1 \rangle_{\text{lat}} = C_4 + C_5 \tau^{-1}$, from which we obtain the $\tau \to \infty$ limit. We redo this analysis for $N = 20, 32$. The results can be fitted to $C_6 + C_7 N^{-2}$ as one can see from Fig. 2. This large $N$ behavior is analogous to the one observed in the bosonic IKKT model [13]. Thus we can obtain the final result for the matrix QM (1).

Let us move on to the cQEK model (2). The corresponding observable is $\mathcal{O}_1 = \frac{1}{N} \text{tr} (X_i^2)$. In Fig. 3 we plot the results $\langle \mathcal{O}_1 \rangle_{\text{cQEK}}$ obtained for $N = 32, 48, 64$ with $\epsilon = 0.2, 0.25, 0.3, 0.5$. For fixed $\epsilon$ the large $N$ behavior is given by $C_8 + C_9 N^{-2}$. The coefficient $C_9$, however, diverges as $\frac{1}{\epsilon}$. This can be understood if we recall that the
density of $\omega_i$ in the region $[-\frac{1}{2}, \frac{1}{2}]$ is $\rho = \frac{N}{\pi^2}$. Since finite $N$ effects at small $\epsilon$ come mainly from the finiteness of $\rho$, any power of $N$ should be associated with the same power of $\epsilon$. This motivated us to plot the results against $\frac{1}{(N\epsilon)^2}$ in Fig. 3, where the results indeed lie on a single straight line for $\epsilon \leq 0.3$. Fitting all the data for $\epsilon \leq 0.3$, we obtain the final result for the cQEκ model except for the terms $(s = 1, \cdots, n)$

\[
\frac{1}{\epsilon^{s-1}} \int d\Omega \langle \Omega \rangle_{\text{cQEκ}} \equiv \frac{1}{\epsilon^{s-1}} \int d\Omega \langle \Omega \rangle_{\text{cQEκ}} = N^s \times A_{\mu \nu}^2, \quad (s = 1, \cdots, n)
\]

where $A_{\mu \nu}^2$ is the eigenvalue distribution of $\Omega$. The cross on the vertical axis (the error bar is almost invisible) represents the final result for the matrix QM.

We repeat this analysis for other observables and summarize the results in Table I, where we find good agreement within error bars. We consider this as a compelling evidence for the EK equivalence.

\[\text{Schwinger-Dyson equations. — Let us discuss a possible explanation for the equivalence based on the Schwinger-Dyson (SD) equations} \]

In the matrix QM, we consider

\[\langle \text{tr} (\lambda^s X_{i_1}(t_1) \cdots X_{i_n}(t_n)) \rangle_{\text{QM}} = 0.2, 0.25, 0.3, 0.5. \]

the straight line for $\epsilon \leq 0.3$. Fitting all the data for $\epsilon \leq 0.3$, we obtain the final result for the cQEκ model.

We summarize the results in Table I, where we find good agreement within error bars. We consider this as a compelling evidence for the EK equivalence.

\[\text{Relation to the IKKT model. — The cQEκ model}\]

\[\text{for the matrix QM}\]

\[\text{has the action of the same form as the bosonic IKKT model}\]

\[S_{\text{IKKT}} = -\frac{N}{\pi} \text{tr} [A_{\mu}, A_{\nu}]^2 \]

\[\text{if we identify}\]

\[A_i \equiv e^{i/4} X_i \quad (i \leq d - 1), \quad A_d \equiv e^{i/4} \Omega.\]

\[\text{The only difference is that}\]

\[A_d \text{ has to be quenched. In the bosonic IKKT model it is known that}\]

\[\langle \frac{1}{2} \text{tr} (A_{\mu}^2) \rangle_{\text{IKKT}} \]

\[\text{is of O(1)} \quad (\text{all directions}), \quad \text{and the extent of the eigenvalue distribution}\]

\[\text{is of O(1)} \quad (\text{all directions}). \]

\[\text{The quenching stretches the eigenvalue distribution of}\]

\[A_d \text{ to have an extent of O} (\epsilon^{-3/4}), \quad \text{and as a result the eigenvalue distribution}\]

\[\text{of}\]

\[A_i (i \leq d - 1) \text{ shrinks to O(1)} \quad (\text{all directions}).\]

\[\text{This implies that, in spite of their formal similarity, the bosonic BFSS model and the bosonic IKKT model probe}\]

\[\text{totally different regions of the configuration space.}\]

\[\text{How about the supersymmetric case? In d = 4, since}\]

\[\text{the situation is qualitatively the same (no spontaneous breaking of the rotational symmetry and}\]

\[\langle \frac{1}{2} \text{tr} (A_{\mu}^2) \rangle \sim O(1) \quad (\text{all directions})], \quad \text{all the statements for the bosonic case should}\]

\[\text{equally apply.}\]

\[\text{In d = 10 (i.e., the IKKT model), there are certain evidences that the eigenvalue distribution of}\]

\[A_{\mu} \text{ collapses dynamically to a four-dimensional hypersurface}\]

\[\text{Let us assume further that the eigenvalue distribution extends to infinity in four directions at large N, and that}\]

\[\text{the distribution is uniform in these directions, as expected if the IKKT model really describes our 4d} \]
time. When we quench the eigenvalues of \( A_{10} \), the 4d hypersurface will arrange itself to have the required extent in the 10th direction. In this situation a vector in the 10th direction may have non-zero components in directions orthogonal to the 4d hypersurface, which implies that the system is not invariant under a shift in the 10th direction. We therefore suspect that the EK equivalence between the BFSS model and the corresponding cQEK model does not hold as it stands.

We may, however, add a “mass term” \( \frac{1}{2} N m^2 \int dt \, tr \, X_t^2 \) to the BFSS model, and consider the corresponding cQEK model, in which the mass term forces the 10th direction to be included in the 4d hypersurface. This guarantees the translational invariance in the 10th direction except at the boundary, which should be fine again due to the UV finiteness. Therefore the EK equivalence in this case is expected to hold at any finite \( m \), and it should hold, too, even if we send \( m \) to zero eventually.

We speculate that actually quenching is not needed since the 10th direction, in which we do not have the mass term, will extend to infinity as \( N \to \infty \) anyway. This implies that the IKKT model with a small mass term in 9 directions, which should be removed after taking the large \( N \) limit, is equivalent (in the sense of Eguchi-Kawai) to the BFSS model. In particular if the IKKT model has 4 extended directions, the BFSS model should have 3 out of 9 transverse directions extended, thus breaking the SO(9) symmetry down to SO(3) spontaneously. This equivalence is somewhat reminiscent of the T-duality, which relates the BFSS model on a circle to the IKKT model with Taylor’s compactifying condition \( N \to \infty \). Note, however, that our conjecture is different in that the two models can be both considered in the decompactified limit.

Concluding remarks.— In this letter we have provided the first direct confirmation of the EK equivalence for the cQEK model in the case of matrix QM or the 1d gauge theory. Whether it holds also in higher dimensions is an interesting open question. This is relevant to non-perturbative studies of Matrix String Theory and it is also important in view of the difficulties in formulating supersymmetric and/or chiral gauge theories on the lattice. (See Refs. \cite{13} and \cite{14} for recent developments.) As is clear from the discussion on the SD equations, the question is highly nontrivial in the presence of UV divergences. In Ref. \cite{12} it is speculated that the equivalence holds if the UV divergence is at most logarithmic. A natural starting point in this direction would be to study the cQEK model for the 2d pure Yang-Mills theory, and see whether the known exact results can be reproduced.

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