One-loop Type II Seesaw Neutrino Model with Stable Dark Matter Candidates

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Abstract

Opening up the Weinberg operator at 1-loop level using a scalar triplet, two scalar doublets and one fermion gives rise to T4-2-i one-loop topology. Neutrino masses generated from this topology is always accompanied by the tree level Type II seesaw contribution. In this work, we propose a radiative Majorana neutrino mass model based on this topology where to avoid the tree level Type II seesaw mechanism, we extend the model by a $G_f = \mathbb{D}_4 \times Z_3 \times Z_2$ flavor symmetry and we promote the fermion inside the loop to three right-handed neutrinos. In this scenario, the tree level Dirac neutrino masses resulted from these right-handed neutrinos is also prevented by the $G_f$ group. Moreover, in order for T4-2-i topology to fully function, the scalar sector is extended by two flavon fields where after $G_f$ symmetry breaking, the model accounts successfully for the observed neutrino masses and mixing as well as allows for the existence of stable dark matter (DM) candidates. Indeed, all the particles running in the loop are potential dark matter candidates as their stability is guaranteed by the unbroken discrete group $Z_2 \subset G_f$.

Key words: Neutrino masses and mixing, Flavor symmetries, Dark matter stability.
1 Introduction

The developments in the field of neutrino physics in the past two decades have been undoubtedly impressive. Neutrinos which rarely interact with ordinary matter have been identified in the Standard Model (SM) as massless particles. However, many neutrino oscillation experiments performed in the past twenty years confirmed that neutrinos have nonzero masses, thus making these particles as the current best probe for new physics beyond the SM (BSM) [1,2]. Another matter that requires going BSM and which has been explored at length in the literature is the existence of dark matter where amongst its known properties, an appropriate candidate must has zero electric charge, produce the correct relic abundance and must be stable over cosmological time scales [3]. This stability asserts the existence of a new kind of charge carried by the DM particle, and in model building, the stability is usually guaranteed by imposing new symmetries like $Z_2$ which is the most commonly used symmetry in literature.

In recent years, there have been a growing interest in radiative neutrino mass models that provide an interconnection between the neutrino and the DM sectors. Indeed, these models predict neutrino masses at the loop level as well as the existence of DM candidates in the form of one of the intermediate particles running in the loop. One class of these models is the n-loop realizations of the well-known $d = 5$ Weinberg operator $O_5 = LLHH$ where $L$ stands for the $SU(2)_L$ lepton doublets while $H$ denotes the $SU(2)_L$ Higgs doublet of the SM. A popular one-loop realization of $O_5$ is the scotogenic model which extends the SM particle content by three right-handed neutrinos and an extra inert scalar doublet [4], while an exact $Z_2$ symmetry prevents the tree level Dirac masses for neutrinos as well as allowing for stable DM candidates. This model has been studied in detail using the same and in many times different set of particles inside the loop; see, for instance, Refs. [5–45]. The full possible one-loop diagrams induced from this operator can be found in [46] while a systematic study of two and three-loop realizations of $O_5$ is done in [47] and [48], respectively. For a detailed review on radiative neutrino mass models and their classification see [49] and the references therein. To explain neutrino data along with providing a good DM candidate in the context of radiative models, the particle content and the gauge symmetry of the SM, $G_{SM} = SU(3)_C \times SU(2)_L \times U(1)_Y$, need to be extended. Actually, there are no restrictions concerning whether the extra symmetries should be Abelian or non-Abelian, discrete or continuous, simple or multiple. On the other hand, it is well-known that non-Abelian discrete groups are well justified by the large leptonic mixing angles measured by oscillation experiments, and when radiative models are extended by a non-Abelian flavor symmetry, an interesting implication is that the stability of DM candidate may be ensured by one of the subgroups obtained after breaking the flavor symmetry, see for instance Refs. [50–60]. Therefore, non-Abelian flavor symmetries are an effective tool to address both neutrino and dark matter issues.

While most of the finite one-loop diagrams are studied extensively in model-building BSM, there is in particular one topology that have never been realized in a field theory; it is denoted by T4-2-i as illustrated in figure [1,2,46]. This topology involves a scalar triplet $T$—with hypercharge
\[ Y = 2 \] — two extra inert scalars \( \phi \) and \( \rho \) and one fermion \( \psi \) running in the loop. The Higgs triplet \( T \) coupled to the SM Higgs doublet \( H \) through the interaction \( HT^\dagger H \), and thus, it will always involves the usual tree level Type II seesaw contribution to neutrino masses \( LTL \) that cannot be prevented by any additional \( U(1) \) or \( Z_N \) symmetries \cite{22, 46}. The authors in reference \cite{46} stated that to prevent the tree level contributions, two things are required: (i) Promoting the fermion \( \psi \) inside the loop to be Majorana fermion; and (ii) assuming that all couplings conserve lepton number.

In this paper, our purpose is to cure the difficulties encountered when building a field theory with topology \( T4-2-i \). To achieve this, we propose a radiative Majorana neutrino mass model within an extension of the SM based on the \( G_f = \mathbb{D}_4 \times Z_3 \times Z_2 \) flavor symmetry. Furthermore, as previously mentioned, in order to obtain neutrino masses and mixing consistent with the current neutrino data along with providing a stable DM candidate, the obvious implication is that we must extend the particle content of the SM as well. Therefore, we proceed with the first requirement in \cite{46} and we promote the fermion inside the loop to three right-handed neutrino singlets \( N_k \), while we discard the second one; which means that we do not assume that all couplings must conserve lepton number. The alternative for the second requirement—which ensures the suppression of the tree level Type II seesaw contribution to neutrino masses \( LTL \)—is fulfilled by the choice of the particle assignment under \( Z_3 \in G_f \). Actually, our \( Z_3 \) charge assignments given in Tables \( \text{2 and 3} \) prevent the tree level Type II seesaw contribution as well as the tree level Dirac Yukawa coupling \( y_{ij} \tilde{L}_i \tilde{H} N_j \), and eventually, the Type I seesaw contribution to neutrino masses. Thus, the only possibility for neutrino mass generation in our model is at the loop level in the scotogenic fashion. However, the price to pay with the \( Z_3 \) charge assignments is that the two Yukawa couplings connecting \( N_k \), \( L \) and the two inert scalars in the loop of topology \( T4-2-i \) carry non trivial \( Z_3 \) charge. Moreover, the usual vertex connecting two Higgs doublets with the scalar triplet \( T \) (upper vertex in figure \( \text{11} \)) is also prevented by the \( Z_3 \) symmetry. To restore \( Z_3 \) invariance, we have enlarged the scalar sector by adding two flavon fields \( F \) and \( \chi \) carrying quantum numbers under \( G_f \); thus, fixing the issues of topology \( T4-2-i \). When the flavon \( F \) acquires its vacuum expectation value (VEV), the \( \mathbb{D}_4 \) group gets broken down to a subgroup \( Z_2' \) leading to a neutrino mass matrix compatible with the well-known trimaximal mixing matrix \cite{61–67}. We have studied numerically the phenomenology associated with neutrino sector in the normal mass hierarchy (NH) case. Finally, for the DM candidates, all the particles running in the loop—right-handed neutrino \( N_k \) and the scalars \( \rho \) and \( \phi \)—are odd under the discrete group \( Z_2 \in G_f \) whilst all SM particles are even. Therefore, the lightest odd particle will be stable and can play the role of the DM candidate. We have provided information on the possible processes for two cases; (a) Fermionic DM candidate with \( N_3 \) being the lightest odd particle, and (b) Bosonic DM candidate with \( \rho \) being the lightest odd particle. On the other hand, although the \( Z_2' \) subgroup of \( \mathbb{D}_4 \) is unbroken, it is not responsible for DM stability; however, there might be processes allowed by \( Z_2 \) but forbidden by \( Z_2' \) since the residual symmetry that survives the \( G_f \) symmetry breaking is given by the group \( Z_2' \times Z_2 \). Thus, we have checked the invariance of the various DM processes under \( Z_2 \) as well as \( Z_2' \).
The paper is organized as follows. In Sec. II we start by a general discussion on topology T4-2-i, then we present our field content and the solution to the problems of topology T4-2-i. In Sec. III we start by studying in details the neutrino sector and then describe the phenomenology associated with neutrino masses and mixing. In Sec. IV we discuss all possible DM candidates of the model and provide their decay, annihilation and co-annihilation processes. In Sec. V, we give our conclusion. Finally, we add an Appendix which contains some useful tools on the dihedral \( \mathbb{D}_4 \) group.

2 Genuine one-loop Type II seesaw using \( G_f \) flavour symmetry

In this section, we first describe the particles involved in topology T4-2-i and all their possible charge assignments under the electroweak group and we provide the necessary requirements to fix the issues associated with topology T4-2-i. Then, we present our scenario to account for this topology by implementing the \( G_f \) flavor symmetry accompanied with extra flavon fields.

2.1 One loop Type II seesaw topology

There are several approaches to generate neutrino masses beyond the SM, among which are the radiative models where neutrino masses arise at the loop level. These models are rather interesting because they not only account for the tiny neutrino masses naturally, but also provide a DM candidate given by one of the new fields running in the loop. One of the most effective ways to classify these models is through the topology of the loop diagrams which generate neutrino masses \([46, 47, 49, 68, 69]\). The majority of these models are the one-loop realizations of the well-known dimension-5 Weinberg operator \( LLHH \). While most of the finite one-loop diagrams are studied extensively in building BSM physics models, there is in particular one topology that have never been realized in a field theory; it is denoted by T4-2-i as illustrated in Fig. 1. In this topology,

![Figure 1: One-loop neutrino mass generation from an SU(2)_L scalar triplet like in the Type II seesaw mechanism. This diagram is denoted as T4-2-i in reference 46.](image_url)
there are four new particles compared to the SM; an $SU(2)_L$ scalar triplet with hypercharge $Y = 2$ which couples to the SM Higgs doublets $H$ (bottom vertex), two scalars $\phi$ and $\rho$ and one fermion $\psi$ running in the loop. From this, we deduce five different field assignments leading to five different models generating neutrino masses at one-loop. These five possibilities are reported in Table I using $SU(2)_L$ representations to differentiate between different models.

| Fields | Model I | Model II | Model III | Model IV | Model V |
|--------|---------|----------|-----------|----------|---------|
| $\phi$  | 3       | 2        | 2         | 1        | 3       |
| $\rho$  | 1       | 2        | 2         | 3        | 3       |
| $\psi$  | 2       | 1        | 3         | 2        | 2       |

Table 1: Different $SU(2)_L$ assignments for the fields $\rho$, $\phi$ and $\psi$ leading to five possible one-loop neutrino mass models from topology T4-2-i.

On the other hand, it was mentioned in Refs. [22,46] that topology T4-2-i will always involves the usual tree level Type II seesaw contribution to neutrino masses $LTL$ that cannot be prevented by any additional $U(1)$ or $Z_n$ symmetries. This can be easily shown by considering the hypercharge quantum numbers of the different particles involved in the tree level contribution as well as topology T4-2-i. Therefore, for the Type II seesaw mass term $LTL$ we have the condition

$$2Y_L + Y_T = 0 \quad \text{with} \quad Y_L = -1 \text{ and } Y_T = 2,$$

where $Y_X$ is the hypercharge of field $X$ under the $U(1)_Y$ group. For topology T4-2-i, the loop in the diagram of Fig. I consists of three vertices with the following conditions on $Y_X$

$$Y_L - Y_{\rho} + Y_\psi = 0 \rightarrow \text{vertex connecting } L, \phi \text{ and } \psi$$
$$Y_L + Y_{\phi} - Y_\psi = 0 \rightarrow \text{vertex connecting } L, \rho \text{ and } \psi$$
$$Y_T + Y_{\rho} - Y_\phi = 0 \rightarrow \text{vertex connecting } T, \phi \text{ and } \rho.$$

The sum of these three equations leads to the condition (2.1) which implies that a neutrino mass generated by topology T4-2-i is always accompanied by the tree-level Type II seesaw mechanism. This is true for any $U(1)$ or $Z_n$ quantum charges $q_X$. On the other hand, the authors in Refs. [22,46] stated that to prevent the tree level contributions, two things are required: (i) Promoting the fermion $\psi$ inside the loop to be Majorana fermion; and (ii) assuming that all couplings conserve lepton number. In this regard, once we impose these two conditions, the tree level Type II seesaw contribution $LTL$ will be eliminated as it violates lepton number conservation while the Majorana mass term for the fermion running in the loop $M_{\psi^c_{\psi}}$ will be the only term allowed to break lepton number. Moreover, these two conditions narrow down the number of $SU(2)_L \times U(1)_Y$ assignments for the fermion $\psi$ to only two options: a fermion singlet or a fermion triplet both with hypercharge $Y = 0$. As a result, only the assignments in the models II and III from Table I are allowed in this scenario. However, building models and taking into account these prerequisites—especially the
condition of imposing lepton number conservation—is not an easy task; thus, a call for additional symmetries and particles seems necessary. In this regard, we propose in the next subsection a solution to the issues of topology T4-2-i by extending the SM by a $\mathbb{D}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ flavor symmetry.

2.2 Implementing $G_f$ flavour symmetry in T4-2-i model

As mentioned above, the first step to forbid the tree level Type II seesaw coupling $\lambda LTL$ is by promoting the fermion $\psi$ inside the loop to a Majorana fermion. In this work, we consider three right-handed neutrino singlets $N_k$ which correspond to model II in Table (1). In a second step, to ensure lepton number conservation we extend the SM gauge group with an additional $G_f = \mathbb{D}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ flavor symmetry along with extra flavon fields allowing us to control the couplings in the 1-loop diagram. Actually, the choice of this additional symmetry in our model is introduced not only to forbid the tree level Type II seesaw contribution, but also to satisfy the following requirements: (i) forbid the tree level Type I seesaw contribution coming from the Dirac operator $y_{ij} L_i \tilde{H} N_j$; (ii) obtain neutrino masses and mixing angles consistent with the current neutrino data; and (iii) stabilize the dark matter candidate against decay. Now we turn to present

![Figure 2: One-loop feynman diagram responsible for the neutrino mass matrix in our $\mathbb{D}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ model.](image)

the particle content of the model and describe the $G_f$ quantum numbers of the leptons as well as for new extra fields. Recall first that the discrete $\mathbb{D}_4$ group has five irreducible representations: four singlets $1_{p,q}$ with indices $p,q = \pm, \pm$; and one doublet $2_{0,0}$ indexed by the characters $\chi(S), \chi(T)$ of the two non-commuting generators $S$ and $T$ of the dihedral $\mathbb{D}_4$; see appendix for more details. For the lepton sector, as stated in the beginning of this subsection we have added three right-handed neutrinos to the usual $SU(2)$ lepton doublets $L_i$ and lepton singlets $l_i^R$ of the SM, here $i$ run over the three lepton families. Their quantum numbers under the SM gauge group and the $G_f$ flavor group are as given in Table 2. For the scalar sector, besides the usual SM Higgs doublet $H = (h^+, h^0)$, the model involves five additional scalar fields; two inert Higgs doublets
Yukawa couplings, which is considered very unnatural. On the other hand, in flavor symmetries among charged leptons at the leading order where we need to impose a hierarchical values on the model, the extra right-handed neutrinos \(N_k\) and the inert Higgs doublets \(\rho\) and \(\phi\) are running in the loop as in the original topology of Fig. 1. However, based on the chosen \(\mathbb{D}_4\) representations and \(Z_3\) charge assignments shown in Table 3 the two Yukawa couplings \(y_{ik}\bar{L}_i\rho N_k\) and \(y'_{jk}\bar{L}_i\tilde{\phi}N_k\) behave as doublets under \(\mathbb{D}_4\) group and they carry non zero \(Z_3\) charge \(\omega^2\). To restore the invariance under the \(\mathbb{D}_4 \times Z_3\) symmetry, we have added the flavon field \(\chi\) which transforms as a \(\mathbb{D}_4\) doublet and carries a \(Z_3\) charge \(\omega\). On the other hand, the one-loop vertex \(\mu_T H T^\dagger H\) connecting two Higgs doublets with the scalar triplet in Fig. 1 is prevented in our model by the \(Z_3\) symmetry since its charge is \(\omega^2\), the invariance is restored by the flavon field \(\chi\) which carries the charge \(\omega\), see Table 3. The above couplings are invariant under the \(Z_2\) symmetry which will be only used to stabilize the dark matter candidate. Therefore, the \(\mathbb{D}_4 \times Z_3\) group and the new flavon fields are sufficient to address the challenge of Topology T4-2-i, leading subsequently to the modified one-loop radiative diagram shown in Fig. 2. In the following section, we will study in details the neutrino masses and mixing and their corresponding phenomenological consequences.

Before we describe the neutrino sector, let us comment briefly on the charged lepton masses. With respect to the chosen \(\mathbb{D}_4\) particle assignments—see Tables 2 and 3—the charged lepton mass matrix is diagonal. This can easily be seen by considering the leading order terms responsible for the charged lepton masses. These terms invariant under \(G_f\) are \(y_e (\bar{L}_e)^{++} (e_R)^{++} (H)^{++}\), \(y_{\mu} (\bar{L}_\mu)^{+-} (\mu_R)^{++} (H)^{++}\) and \(y_{\tau} (\bar{L}_\tau)^{--} (\tau_R)^{++} (H)^{++}\). Therefore, after the Higgs field takes its VEV as \(\langle H \rangle = \left( 0 \ 1 \ \sqrt{2} (v_H + h + i A_1) \right)^T\), we obtain a diagonal charged lepton mass matrix as \(m_l = v_H / \sqrt{2} \text{diag}(y_e, y_{\mu}, y_{\tau})\). However, it is clear that it is not trivial to produce the mass hierarchy among charged leptons at the leading order where we need to impose a hierarchical values on the Yukawa couplings, which is considered very unnatural. On the other hand, in flavor symmetries

| Fermions | \(L_e\) | \(L_\mu\) | \(L_\tau\) | \(l_e^R\) | \(l_\mu^R\) | \(l_\tau^R\) | \(N_1\) | \(N_2\) | \(N_3\) |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| \(G_{SM}\) | \((1,2)_{-1}\) | \((1,1)_{-2}\) | \((1,1)_{0}\) | | | | | | |
| \(\mathbb{D}_4\) | \(1_{+,+}\) | \(1_{+-}\) | \(1_{-+}\) | \(1_{++,}\) | \(1_{+-}\) | \(1_{--}\) | \(1_{+,+}\) | \(1_{+-}\) | \(1_{--}\) |
| \((Z_3, Z_2)\) | \((\omega, 1)\) | \((\omega, 1)\) | \((\omega, 1)\) | | | | | | |

Table 2: Gauge and flavor quantum numbers for leptons and right-handed neutrino fields, where \(\omega = e^{\frac{2\pi i}{3}}\).

| Scalars | \(H\) | \(T\) | \(\rho\) | \(\phi\) | \(\Gamma\) | \(\chi\) |
|---------|----------|----------|----------|----------|----------|----------|
| \(G_{SM}\) | \((1,2)_1\) | \((1,3)_2\) | \((1,2)_{-1}\) | \((1,2)_1\) | \((1,1)_0\) | \((1,1)_0\) |
| \(\mathbb{D}_4\) | \(1_{+,+}\) | \(1_{+,+}\) | \(2_{0,0}\) | \(2_{0,0}\) | \(2_{0,0}\) | \(1_{+,+}\) |
| \((Z_3, Z_2)\) | \((1,1)\) | \((\omega, 1)\) | \((\omega, -1)\) | \((\omega^2, -1)\) | \((\omega, 1)\) | \((\omega, 1)\) |

Table 3: Gauge and flavor quantum numbers for all scalar fields of the model.
based models, the mass hierarchy can be achieved by taking into account corrections in the the charged lepton mass matrix from higher-dimensional operators involving flavon fields. An example of such operators can be written as $\bar{L}^i_R H (\frac{\Omega}{\Lambda})^n (\frac{\zeta}{\Lambda})^m$ with $n + m \geq 1$ and $\Lambda$ is a cutoff scale while $\Omega$ and $\zeta$ denote the flavon fields needed also to ensure the invariance under $G_f$. Another attractive method used to explain this hierarchy is the Froggatt-Nielsen mechanism which relies on the spontaneous breaking of a $U(1)_F$ flavor symmetry, for details on this method see Ref. [70].

3 Neutrino model building based on topology T4-2-i

In this section, we study the neutrino masses and mixing in the framework described in the previous subsection. Neutrino masses are generated radiatively while we considered the trimaximal mixing matrix scheme. Then, by using the $3\sigma$ experimental values of the oscillation parameters, we show by means of scatter plots the physical observables $m_{ee}$ and $m_{\nu_e}$ related respectively to neutrinoless double beta decay and tritium beta decay experiments, and we also provide scatter plot predictions on the sum of neutrino masses as well as on the Dirac $CP$ violating phase.

3.1 Neutrino masses and mixing

In our model, the $G_f$ flavor symmetry forbids the usual SM tree level Dirac term $y \bar{L} \tilde{H} N_k$, and since the neutral component of the scalar fields $\rho$ and $\phi$ do not acquire VEVs, the usual seesaw mechanism is no longer responsible for neutrino masses. Nonetheless, the light neutrino masses are generated radiatively through the one-loop diagram which involves $\rho$, $\phi$ and $N_k$ in the internal lines; see Fig. 2. According to the field assignments in Tables 2 and 3, the relevant couplings in the neutrino sector, invariant under gauge and $D_4 \times Z_3 \times Z_2$ symmetries are given by the following lagrangian

$$\mathcal{L} = \frac{y_{\rho}}{\Lambda} \bar{L}_i N_k \rho F + \frac{y_{\phi}}{\Lambda} \bar{L}_j N_k \phi F + \frac{M_k}{2} N_k \bar{N}_k + h.c., \quad (3.1)$$

Here $y_{\rho}$ and $y_{\phi}$ are Yukawa couplings and $\tilde{\phi} = i \sigma_2 \phi^*$. The first two terms in this lagrangian are the leading order contributions to Dirac neutrino masses while the third one is the Majorana mass term for $N_k$. For example, the first coupling transforms under the $D_4$ discrete symmetry as

$$\bar{L}_i N_k \rho F \sim 1_{a,b} \otimes 1_{c,d} \otimes 2_{0,0} \otimes 2_{0,0}, \quad (3.2)$$

with indices $a, b, c, d = \pm$. Thus, to obtain the desired $D_4$ trivial singlet, the tensor product between the $D_4$ doublets—which decomposes into the direct sum of the four $D_4$ singlets; see the Appendix—should transform in the same manner as the product between the two singlet $1_{a,b} \otimes 1_{c,d}$. This can easily be seen in the following examples

$$\bar{L}_e N_1 \rho F \sim (1_{+,+} \otimes 1_{+,+})|_{1_{+,+}} \otimes (2_{0,0} \otimes 2_{0,0})|_{1_{+,+}},$$

$$\bar{L}_e N_2 \rho F \sim (1_{+,+} \otimes 1_{+,-})|_{1_{+,-}} \otimes (2_{0,0} \otimes 2_{0,0})|_{1_{+,-}}. \quad (3.3)$$
The same discussion holds for the second term in (3.2). To break the flavor symmetry, the flavon doublet \( F \) acquires its VEV along the direction \( \langle F \rangle = v_f (1, 1) \) while the scalar fields \( \rho \) and \( \phi \) do not acquire VEVs and may be expressed as

\[
\rho = \left( \frac{1}{\sqrt{2}} (\rho_1 + i A_2) \right), \quad \phi = \left( \frac{1}{\sqrt{2}} (\phi_1 + i A_3) \right), \tag{3.4}
\]

with \( \rho_1 (\phi_1) \) and \( A_2(A_3) \) present respectively the scalar and the pseudoscalar parts of the neutral component of \( \rho (\phi) \). At the first sight, it seems that \( \rho \) and \( \phi \) are adjoint of each other as they transform as \( \mathbb{D}_4 \) doublets and carry \( Z_3 \) charges \( \omega \) and \( \bar{\omega} \) respectively. They are placed, however, in different \( \mathbb{D}_4 \) doublet components

\[
\rho \equiv \begin{pmatrix} \rho \\ 0 \end{pmatrix} \quad \text{and} \quad \phi \equiv \begin{pmatrix} 0 \\ \phi \end{pmatrix}. \tag{3.5}
\]

This difference is due to the vertex connecting \( T, \phi \) and \( \rho \) in the diagram of Fig. 2 where by asking for a non-vanishing coupling \( \lambda_T T \rho \phi \) the bilinear term \( (\rho \phi) \) must transform as a trivial singlet (since \( T \sim 1_{++} \)). Using the \( \mathbb{D}_4 \) tensor product, the product between two mass matrices deduced from the two first terms in (3.2) is given by

\[
\frac{v^2_F}{\Lambda^2} y^i_k y^j_k = \frac{v^2}{\Lambda^2} \left( \begin{array}{ccc} \rho^{e1} & \rho^{e2} & \rho^{e3} \\ \rho^{\mu1} & \rho^{\mu2} & \rho^{\mu3} \\ \rho^{\tau1} & \rho^{\tau2} & \rho^{\tau3} \end{array} \right) \left( \begin{array}{ccc} y^{e1}_{\phi} & y^{e2}_{\phi} & -y^{e3}_{\phi} \\ y^{\mu1}_{\phi} & y^{\mu2}_{\phi} & -y^{\mu3}_{\phi} \\ -y^{\tau1}_{\phi} & -y^{\tau2}_{\phi} & y^{\tau3}_{\phi} \end{array} \right). \tag{3.6}
\]

As for the Majorana mass term \( M_k \sum y^i_k N_k \), since the three right-handed neutrinos transform trivially under \( \mathbb{D}_4 \times Z_3 \), we obtain a diagonal Majorana neutrino mass matrix \( M_k = \text{diag}(M_1, M_2, M_3) \). Consequently, neutrino masses induced via the one-loop diagram in Fig. 2 are given by

\[
(M_{\nu})_{ij} = -\frac{\mu_T \lambda_T v^2_F}{m^2_\Delta} \Lambda^2 y^i_k y^j_k J(m^2_{\rho}, m^2_{\phi}, M_k) J(m^2_{\rho}, m^2_{\phi}, M_k) \tag{3.7}
\]

where \( v_\chi \) is the VEV of the flavon \( \chi \) while \( \Gamma_k \) is defined as follows

\[
\Gamma_k = -\frac{\mu_T v^2_H \lambda_T M_k}{m^2_\Delta} \Lambda J(m^2_{\rho}, m^2_{\phi}, M_k) \tag{3.8}
\]

while the loop function \( J \) is defined as

\[
J(m^2_{\rho}, m^2_{\phi}, M_k) = -\frac{1}{(4\pi)^2} \left[ \frac{m^2_{\rho}}{(m^2_{\rho} - m^2_{\phi})(m^2_{\rho} - M_k^2)} \ln \frac{M_k^2}{m^2_{\rho}} + \frac{m^2_{\phi}}{(m^2_{\phi} - m^2_{\phi})(m^2_{\phi} - M_k^2)} \ln \frac{M_k^2}{m^2_{\phi}} \right]. \tag{3.9}
\]
Assuming for simplicity that we have a quasi-degenerate right-handed neutrino masses with $M_3 \simeq M_2 \simeq M_1$ implying $\Gamma_3 \simeq \Gamma_2 \simeq \Gamma_1$. In this case, the total neutrino mass matrix can be expressed as $M_\nu = \Gamma_1 \left[ \frac{\nu_s}{X} y^k \bar{y}^l \right]$, and by imposing the following conditions on the Yukawa couplings

$$
y_{\phi}^{\mu 2} = y_{\phi}^{\tau 2} = y_{\phi}^{\mu 3} = y_{\phi}^{\tau 3} = 0 \ , \ y_{\rho}^{\mu 1} = - y_{\rho}^{\tau 1} = y_{\rho}^{\mu 2} = - y_{\rho}^{\tau 2} ,\ y_{\rho}^{\mu 3} = y_{\rho}^{\tau 3} = - y_{\rho}^{\nu 3} \ , \ y_{\rho}^{\nu 2} y_{\rho}^{\nu 2} = y_{\rho}^{\nu 3} y_{\rho}^{\nu 3} ,
$$

we obtain the total neutrino mass matrix expressed as

$$M_\nu = \Gamma_1 \begin{pmatrix} 2a + b & -a & -b \\ -a & a & a \\ -b & a & b \end{pmatrix} , \tag{3.11}$$

where we have introduced the following parametrization $a = \frac{\nu_s}{X} y_{\rho}^{\nu 2} y_{\rho}^{\nu 2}$ and $b = \frac{\nu_s}{X} y_{\rho}^{\nu 3} y_{\rho}^{\nu 3}$ to avoid heavy notations. This matrix exhibits the magic symmetry referring to the equality of the sum of each row and the sum of each column in $M_\nu$. It is well known that the mass matrix acquiring this property is diagonalized by the trimaximal mixing matrix $U_{TM_2}$ which accounts naturally for the the non-zero $\theta_{13}$ as well as a possible determination of the $\theta_{23}$ octant. Therefore, $M_\nu$ is diagonalized as $U_{TM_2}^\dagger M_\nu U_{TM_2} = \text{diag}(m_1, m_2, m_3)$ with $U_{TM_2}$ is expressed following the PDG parametrization for the lepton mixing matrix as

$$U_{TM_2} = \begin{pmatrix} \sqrt{\frac{2}{3}} \cos \theta & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \sin \theta e^{-i\sigma} \\ -\frac{\cos \theta}{\sqrt{2}} - \frac{\sin \theta}{\sqrt{2}} e^{i\sigma} & \frac{1}{\sqrt{2}} & \frac{\cos \theta}{\sqrt{2}} - \frac{\sin \theta}{\sqrt{2}} e^{-i\sigma} \\ -\frac{\cos \theta}{\sqrt{2}} + \frac{\sin \theta}{\sqrt{2}} e^{i\sigma} & \frac{1}{\sqrt{2}} & -\frac{\cos \theta}{\sqrt{2}} - \frac{\sin \theta}{\sqrt{2}} e^{-i\sigma} \end{pmatrix} U_P \tag{3.12}$$

Here $\theta$ is an arbitrary angle that will be related to the observed neutrino mixing angles $\theta_{ij}$, $\sigma$ is an arbitrary phase that will be related later on to the Dirac CP phase $\delta_{CP}$ and $U_P = \text{diag}(1, e^{i\frac{\alpha_2}{2}}, e^{i\frac{\alpha_3}{2}})$ is a diagonal matrix that encodes the Majorana phases $\alpha_{21}$ and $\alpha_{31}$. The Yukawa couplings in the parameters $a$ and $b$ are complex number of order one, hence $M_\nu$ is a complex mass matrix. To diagonalize $M_\nu$ we take $b$ to be real without loss of generality while the parameter $a$ remains complex; $a \to |a| e^{i\phi_a}$ where $\phi_a$ is CP violating phase. As a result, we obtain the three active light neutrino masses $m_1, m_2$ and $m_3$ expressed explicitly as

$$|m_1| = \Gamma \sqrt{a^2 + 4b^2 + 9a^4 \frac{4b^2}{4b^2} + |a| \left( \frac{3a^2}{b} + 4b \right) \cos \phi_a + 6a^2 \cos 2\phi_a} ,$$

$$|m_2| = \Gamma |a| ,$$

$$|m_3| = \Gamma \sqrt{a^2 + 9a^4 \frac{4b^2}{4b^2} - \frac{3 |a|^3}{b} \cos \phi_a} , \tag{3.13}$$

provided that $a < b$ and the following conditions on $\theta$ and $\sigma$ hold

$$\tan 2\theta = -\frac{\sqrt{3} (a^2 - b^2)^2 + 12a^2 b^2 \sin^2 \phi_a}{4 |a| b \cos \phi_a + 3a^2 + b^2} , \quad \tan \sigma = \frac{2 |a| b \sin \phi_a}{a^2 - b^2} . \tag{3.14}$$
Regarding the mixing angles, we use the PDG standard parametrization of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [72], then we calculate the three observed neutrino mixing angles in terms of the trimaximal mixing parameters, we find

\[
\sin^2 \theta_{13} = \frac{2}{3} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1}{3 - 2 \sin^2 \theta}
\]

\[
\sin^2 \theta_{23} = \frac{1}{2} - \frac{3 \sin 2 \theta}{2 \sqrt{3} (3 - 2 \sin^2 \theta)} \cos \sigma.
\]

(3.15)

Since the recent experimental data signals that a normal mass ordering is more preferred than an inverted ordering [73, 74], we will perform our numerical study in the normal hierarchy case. Therefore, we use as input data the results of the global analysis by NuFIT 4.0 of the neutrino oscillation parameters at 3\(\sigma\) interval [74] in the NH case; we have

\[
\sin^2 \theta_{13} \in [0.02044 \rightarrow 0.02437], \quad \sin^2 \theta_{23} \in [0.428 \rightarrow 0.624], \quad \sin^2 \theta_{12} \in [0.275 \rightarrow 0.350], \quad \Delta m_{21}^2 \ [\text{eV}^2] \in [6.79 \rightarrow 8.01]
\]

(3.16)

The trimaximal matrix is described by two unknown parameters \(\theta\) and \(\sigma\) which are in turn linked to the free parameters \(\Gamma, a, b\) and \(\phi_a\) appearing in the neutrino mass matrix (3.11). First, by using the 3\(\sigma\) experimental range of \(\sin^2 \theta_{13}\) and the first equation in (3.15) we find the permitted values of \(\theta\) as \(0.176 \lesssim \theta \ [\text{rad}] \lesssim 0.193\). Inserting this constraint on \(\theta\) in the formula of the solar mixing angle in (3.15) allows to restrict the interval of \(\theta_{12}\) compared to its 3\(\sigma\) allowed range (see Eq. (3.16)) where we obtain \(\sin^2 \theta_{12} \in [0.334 \rightarrow 0.341]\). Then, by using the experimental values of three mixing angles \(\sin^2 \theta_{ij}\) at 3\(\sigma\) range, we show in the left panel of Fig. 3 the correlation between \(\theta\) and the arbitrary phase \(\sigma\) which is randomly varied in the range \([-\pi \rightarrow \pi]\). Accordingly, we find a more constrained range for \(\sigma\) given by

\[
\sigma \ [\text{rad}] \in [-3.139313 \rightarrow -0.827825] \cup [0.846743 \rightarrow 3.141149].
\]

(3.17)

In the right panel of Fig. 3, we show the correlation among the parameters \(a, b\) and \(\phi_a\) where we used as input parameters the 3\(\sigma\) allowed ranges of the mass squared differences \(\Delta m_{21}^2\) and \(\Delta m_{31}^2\) given in (3.16). The scan ranges of the free parameters are \(0 \lesssim \Gamma \lesssim 1, -1 \lesssim a, b \lesssim 1\) and \(0 \lesssim \phi_a \lesssim 2\pi\). As a result, we find

\[
a \in [-0.99784 \rightarrow -0.02587], \quad b \in [-0.99955 \rightarrow 0.98998], \quad \phi_a \ [\text{rad}] \in [0.00151 \rightarrow 6.28196].
\]

(3.18)

### 3.2 Neutrino phenomenology

Given that neutrino oscillation experiments depend only on the squared-mass splittings \(\Delta m_{21}^2\) and \(\Delta m_{31}^2\), there are three different approaches employed to determine the absolute scale of neutrino masses: (1) the sum of the three active neutrino masses from cosmological observations \(\sum m_i \equiv \)
The effective neutrino mass $m_{\nu e} = \sqrt{\sum_i |U_{ei}|^2 m_i^2}$ using kinematic effects in beta decay experiments, and (3) the effective Majorana neutrino mass $|m_{\nu e}| = \sum_i U_{ei}^2 m_i$ in neutrinoless double beta decay ($0\nu\beta\beta$) where $m_i$ are the three neutrino masses and $U_{ei}$ are the elements of the first row of the mixing matrix. In our numerical study, we use the latest result from the Planck data which when combined with measurements of the baryon acoustic oscillations (BAO) provides an upper limit on $m_{\text{sum}}$ given by $m_{\text{sum}} < 0.12\text{eV}$ at 95\% C.L \cite{75}. We start by substituting the elements of the mixing matrix and the masses defined in the above observables by
their expressions given in Eqs. (3.12) and (3.13) respectively. Hence, this shows the dependence of these observables on our model parameters $a$, $b$, $\Gamma$ and $\phi_a$ as well as the parameters involved in the trimaximal mixing matrix (5.12). Then, we present our predictions using scatter plots. At first, we show in the left panel of Fig. 4 the correlations of the three neutrino masses $m_{i=1,2,3}$ and their sum $m_{\text{sum}}$ versus the lightest neutrino mass $m_1$ where we find

$$
0.01300 \lesssim m_1 \text{[eV]} \lesssim 0.03009 , \quad 0.01561 \lesssim m_2 \text{[eV]} \lesssim 0.03124 \\
0.05103 \lesssim m_3 \text{[eV]} \lesssim 0.05867 , \quad 0.07990 \lesssim m_{\text{sum}} \text{[eV]} \lesssim 0.11997.
$$

(3.19)

From the interval of $m_1$, we take the lightest (largest) value and we replace $m_2$ and $m_3$ by $\sqrt{m_1^2 + \Delta m_{21}^2}$ and $\sqrt{m_1^2 + \Delta m_{31}^2}$ respectively, we find that the sum of neutrino masses in the normal mass hierarchy—using the best fit values of $\Delta m_{21}^2$ and $\Delta m_{31}^2$ given in (74)—requires $m_{\text{sum}} \gtrsim 0.039 \text{eV} \ (m_{\text{sum}} \gtrsim 0.09 \text{eV})$. While the constraint on $m_{\text{sum}}$ corresponding to the lightest $m_1$ is far from any current experiment, the upper bound on $m_{\text{sum}}$ corresponding to the largest $m_1$ may be achieved in the upcoming experiments such as CORE+BAO targeting a bound on $\sum m_i$ around $0.062 \text{eV}$ [76]. In the right panel of Fig. 4 we show the correlation between the effective mass of the electron neutrino $m_{\nu_e}$ and $m_1$ where the horizontal gray band indicates the expected sensitivity of $m_{\nu_e}$ from the KATRIN collaboration [77,78]. We find that $m_{\nu_e}$ varies in the following range

$$
0.01574 \lesssim m_{\nu_e} \text{[eV]} \lesssim 0.03133.
$$

(3.20)

Clearly, the values in this interval are very small when compared with the forthcoming $\beta$-decay experiment sensitivities such as KATRIN [77,78], HOLMES [79], and Project 8 [80] which will investigate $m_{\nu_e}$ at 0.2eV, 0.1eV and 0.04eV respectively. If none of these experiments would measure $m_{\nu_e}$, our predicted values could be probed by future experiments aiming to reach improved sensitivities around 0.01eV. Now, let us explore the effective Majorana neutrino mass parameter of neutrinoless double beta decay $|m_{ee}|$. A positive signal of $0\nu\beta\beta$ would assert the Majorana nature of neutrinos as well as provide a measure of the absolute neutrino mass scale. There are many ongoing and upcoming experiments around the world setting as their purpose the detection of this process, where the present bounds on $|m_{ee}|$ come from the KamLAND-Zen [81], CUORE [82] and GERDA [83] experiments corresponding to $|m_{ee}| < (0.061 - 0.165) \text{eV}$, $|m_{ee}| < (0.11 - 0.5) \text{eV}$ and $|m_{ee}| < (0.15 - 0.33) \text{eV}$ respectively. In Fig. 5 we show the correlation between $|m_{ee}|$ and the lightest neutrino mass $m_1$ where we use the known $3\sigma$ ranges of the oscillation parameters while we allow all phases to vary between 0 and $2\pi$. Thus, the interval of $|m_{ee}|$ is given by

$$
0.00316 \lesssim |m_{ee}| \text{[eV]} \lesssim 0.02949.
$$

(3.21)

As a result, our predicted region is far from the current sensitivities as can be seen from the horizontal dashed lines in Fig. 5 displaying the bounds on $|m_{ee}|$ from some of the ongoing $0\nu\beta\beta$ decay experiments. On the other hand, the anticipated sensitivities of the next-generation experiments
Figure 5: The effective Majorana mass $|m_{ee}|$ as a function of the lightest neutrino mass $m_1$. The vertical gray region indicates the upper limit on the sum of the three light neutrino masses from Planck+BAO data.

such as GERDA Phase II ($|m_{ee}| \sim (0.01-0.02)\,\text{eV}$) [84] and nEXO ($|m_{ee}| \sim 0.005\,\text{eV}$) [85] will cover our model predictions on $|m_{ee}|$.

On a different note, the expression for Jarlskog rephasing quantity $J_{CP} = \text{Im}(U_{e1}U_{\mu1}^*U_{\mu2}U_{e2}^*)$ which is a measure of $CP$ violation, is given in terms of the trimaximal parameters $\sigma$ and $\theta$ as follows

$$J_{CP} = \frac{\sin 2\theta \sin \sigma}{6\sqrt{3}}. \quad (3.22)$$

On the other hand, by matching the expressions of the rephasing invariant $J_{CP}$ in the standard parametrisation of the PMNS matrix and in the trimaximal matrix defined in Eq. (3.12), we derive the relation between the Dirac $CP$ phase $\delta_{CP}$ and the arbitrary phase $\sigma$ given as $\sin \sigma = \sin 2\theta_{23} \sin \delta_{CP}$. Moreover, recall that the trimaximal mixing approach used in this model restricts the atmospheric angle $\theta_{23}$ around its maximal value (but not exactly maximal $\theta_{23} \neq 45$) while the $\delta_{CP}$ phase falls in the close vicinity of $\delta_{CP} \simeq 0.5\pi$ and $\delta_{CP} \simeq -0.5\pi$. In this regards, we show in left panel of Fig. the correlation between $J_{CP}$ and the $\delta_{CP}$ phase where we used the equation relating $\delta_{CP}$ with $\sigma$ as an input, and since we have the same parameter scan as before, the obtained range for the arbitrary phase $\sigma$ is as in Eq. (3.17) while for $\delta_{CP}$ we find

$$-0.499\pi \lesssim \delta_{CP} \,[\text{rad}] \lesssim 0.498\pi. \quad (3.23)$$

Based on Eq. (3.22) and the obtained ranges of $\theta$ and $\sigma$; it is straightforward to verify that $\sin 2\theta \neq 0$ and $\sin \sigma \neq 0$, subsequently leading to a non vanishing $J_{CP} \neq 0$. On the other hand, it is clear from Eq. (3.15) that the maximal value of the atmospheric angle ($\sin^2 \theta_{23} = 1/2$) is
Figure 6: Left: Correlation between the Jarlskog invariant $J_{CP}$ and the CP-violating phase $\delta_{CP}$. Right: Correlation between $\delta_{CP}$ and the atmospheric angle $\sin^2 \theta_{23}$.

excluded in our model. Moreover, since the $CP$ phase is correlated significantly with $\sin^2 \theta_{23}$ than the other mixing angles, we show in the right panel of Fig. 6 the predicted regions of $\delta_{CP}$ versus $\sin^2 \theta_{23}$ at $3\sigma$. As can be seen, our model allows $\sin^2 \theta_{23}$ to vary randomly in the interval $[0.428 \to 0.612]$ where the corresponding region of $\delta_{CP}$ is as in Eq. (3.23).

4 Dark Matter candidates

Before we provide the possible DM candidates in the present model, let us discuss the breaking pattern of the $G_f$ flavor group. First, recall that our model involves three scalar fields that acquire VEVs; the usual $SU(2)_L$ Higgs doublet $H$ and two flavon fields $\phi$ and $\chi$. The Higgs doublet $H$ transforms trivially under $G_f$ and thus, it only contributes to the electroweak symmetry breaking. On the other hand, the flavons $\phi$ and $\chi$ transform respectively as a doublet and a trivial singlet under $D_4$. Therefore, only the non-trivial VEV of $\phi$ is responsible for breaking the $D_4$ symmetry down to one of its subgroups $G_r \subset D_4$. Moreover, since these three scalar fields are chosen to be even under the additional $Z_2$ symmetry, this latter remains unbroken. To determine the remnant $G_r$ symmetry that survives the $D_4$ breaking, we recall that $D_4$ is isomorphic to the semidirect product $Z_4 \rtimes Z'_2$ and has two generators $S$ and $T$ where $S$ generates $Z_4$ and $T$ generates $Z'_2$ symmetries satisfying the relations $S^4 = T^2 = 1d$ and $STS = T$. Recall also that the VEV alignment of the flavon $\phi$—$\langle \phi \rangle = v_\phi (1, 1)$—is chosen to reproduce the observed neutrino masses and mixing. This specific VEV direction breaks $D_4$ down to $Z'_2$ with broken part given by the $Z_4$ group. This means that the VEV structure of the flavon $\phi$ preserves the generator $T$ while changes the generator $S$;

$^1$Notice that since both flavon fields $\phi$ and $\chi$ carries the charge $\omega$ under the additional $Z_3$ symmetry; thus, this group is spontaneously broken to the identity.
we have

\[ T \langle F \rangle = \langle F \rangle, \quad S \langle F \rangle \neq \langle F \rangle. \]  

(4.1)

Therefore, the spontaneous breaking of the full discrete flavor symmetry \( G_f \) is given by

\[ \mathbb{D}_4 \times Z_3 \times Z_2 \xrightarrow{(f)} \mathbb{Z}_2' \times Z_2. \]  

(4.2)

Now, we are in position to discuss the stabilization of the DM candidate by the remnant \( \mathbb{Z}_2' \times Z_2 \) symmetry. For this purpose, let us first briefly comment these reflection symmetries individually. On the one hand, for the residual \( \mathbb{Z}_2' \) symmetry, it is useful to study the decomposition of \( \mathbb{D}_4 \) irreducible representations into those of its subgroup \( \mathbb{Z}_2' \). The latter has two singlet representations \( 1_+ \) (trivial) and \( 1_- \), and from the characters of the \( \mathbb{D}_4 \) group (see Table 4 in the Appendix), it is easy to check that the singlet representations \( 1_{+,+} \) and \( 1_{+,-} \) of \( \mathbb{D}_4 \) correspond to \( 1_+ \) of \( \mathbb{Z}_2' \), while \( 1_{-,+} \) and \( 1_- \) of \( \mathbb{D}_4 \) correspond to \( 1_- \) of \( \mathbb{Z}_2' \). For the \( \mathbb{D}_4 \) doublet \( 2_{0,0} \), it decomposes into \( \mathbb{Z}_2' \) representations as \( 2_{0,0} = 1_+ + 1_- \) where the first component of \( 2_{0,0} \) is associated to \( 1_+ \) while the second one is associated to \( 1_- \). Therefore, the particles running in the loop transform under the \( \mathbb{Z}_2' \) symmetry as

\[ N_1 \to N_1, \quad N_2 \to N_2, \quad N_3 \to -N_3 \]
\[ \rho \to \rho, \quad \phi \to -\phi \]  

(4.3)

On the other hand, these particles are odd under the extra symmetry \( \mathbb{Z}_2 \) whilst all SM particles are even under it. This clearly shows that this extra symmetry stabilizes these potential DM particles against decay into SM ones. Therefore, the DM candidate in our model is the lightest among the fermionic right-handed neutrinos \( N_k \) and the scalars \( \rho \) and \( \phi \). On the surface, it looks like \( \mathbb{Z}_2 \) symmetry is sufficient to stabilize the DM candidate—which is true—however, since the full residual flavor symmetry in the neutrino sector is \( \mathbb{Z}_2' \times Z_2 \) group, there might be processes allowed by \( \mathbb{Z}_2 \) but forbidden by \( \mathbb{Z}_2' \). Thus, it is important to verify the invariance of the various DM processes under \( Z_2 \) as well as \( Z_2' \). Here, we will only provide information on the possible processes of these candidates while a thorough calculation of their properties such as annihilation cross section, lifetime and relic abundance is beyond the purpose of the present work.

These processes are mainly extracted from the lagrangian (4.1) and the scalar potential \( V \) of the model. Here, we provide only the relevant cubic and quartic couplings of \( V \) contributing to the

---

2See the matrix representation of the \( \mathbb{D}_4 \) generators in the Appendix.
annihilation, co-annihilation and decay processes. They can be written as

\[
\mathcal{V} \supset [\mu_1 \phi \dagger T \rho + \mu_2 (2F_1 F_1 \chi + 2F_1 F_2 \chi) + \mu_3 \chi^4 + \text{h.c.}]
\]

\[
+ \lambda_1 (H^\dagger H)^2 + \lambda_2 (\rho \dagger \rho)^2 + \lambda_3 (\phi \dagger \phi)^2 + \lambda_4 (Tr T^\dagger T)^2 + \lambda_5 Tr (T^\dagger T)^2
\]

\[
+ 2\lambda_6 \left[ (F_1^\dagger F_1)^2 + (F_2^\dagger F_2)^2 + (F_1^\dagger F_2)^2 \right] + \lambda_7 (\chi \dagger \chi)^2
\]

\[
+ \lambda_8 (H^\dagger H) Tr (T^\dagger T) + \lambda_9 H^\dagger TT^\dagger H + \lambda_10 (H^\dagger F_1)(H F_2^\dagger) + \lambda_11 (H^\dagger \chi)(H \chi^\dagger)
\]

\[
+ \lambda_12 (2\phi \dagger F_1 \phi F_1^\dagger) + \lambda_13 (F_1^\dagger \chi)(F_2 \chi^\dagger) + \lambda_14 (2\rho \dagger F_1 \rho F_1^\dagger) + 2\rho \dagger F_2 \rho F_2^\dagger
\]

\[
+ \lambda_15 Tr (T^\dagger T)(F_1^\dagger F_2) + \lambda_16 Tr (T^\dagger T)(\chi \dagger \chi) + \lambda_17 [(H^\dagger H)(\rho \phi) + \text{h.c.}]
\]

\[
+ \lambda_18 [(\rho \phi)(F_1^\dagger F_2) + \text{h.c.}] + \lambda_{19} [(\rho \phi)(\chi \dagger \chi) + \text{h.c.}] + \lambda_T [HT^\dagger H \chi + \text{h.c.}],
\]

where the \( \mathbb{D}_4 \) decomposition is carried out for all the products between \( \mathbb{D}_4 \) doublets. Here we find that terms like \( \rho^i \sigma^2 T \rho, \phi^i \sigma^2 T^\dagger \phi, (\rho \dagger \rho)(\phi \dagger \phi), (H^\dagger \rho)(\rho \dagger H), (H^\dagger \phi)(\phi \dagger H), (\phi \dagger \chi)(\rho \dagger \chi), (\phi \dagger \chi)(\phi \dagger \chi), Tr (T^\dagger T)(\rho \dagger \rho), \) and \( Tr (T^\dagger T)(\phi \dagger \phi) \) are \( G_f \)-invariant; however, they vanish due to the specific \( \mathbb{D}_4 \) assignments of \( \rho \) and \( \phi \) given in Eq. (3.5). As mentioned above, all the particles running in the loop diagram of Fig. 2 are potential DM candidates. In the following, we present two possibilities:

**Case I: Fermionic dark matter candidate**

In order to facilitate the engineering of neutrino masses and mixing, we have considered the case where \( M_3 \simeq M_2 \simeq M_1 \). We assume here for simplicity that \( N_3 \) is the only fermionic DM candidate. In this scenario, from the first Yukawa interaction in (3.1), we have the decay channel \( \rho \to N_3 l_i \), and by taking into consideration the charges of the residual \( Z_2^\prime \times Z_2 \) symmetry, this decay channel can be expressed using the components of \( l_i = (\nu_i, l_i) \) and the components of \( \rho \) given in (3.4), where we find \( \rho_1(A_2) \to N_3 \tilde{\nu}_3 \) and \( \rho^\pm \to N_3 l_{3}^\pm \). The second Yukawa interaction in (3.1) leads to the same decays but with \( \phi \) instead of \( \rho \), and in terms of the components we find \( \phi_1(A_3) \to N_3 \tilde{\nu}_{1,2} \) and \( \phi^\pm \to N_3 l_{1,2}^\pm \). Moreover, from the same couplings, there are reactions by the t-channel of annihilation modes \( N_3 N_3 \to L_i L_i \) where the \( i \) index depends on whether \( \rho \) or \( \phi \) is exchanged.

In general, the dark matter annihilations into SM particles arises through a Higgs or Z-boson portal which are the only SM particles able to serve as mediator between the visible and the dark sector. Since \( N_3 \) is \( SU(2)_L \) singlet, it has no interactions with the electroweak gauge bosons, while in the case of the Higgs boson portal, there is no direct coupling between \( N_3 \) and \( H \) in the lagrangian (3.1); however, non-renormalizable terms suppressed by powers of the cut-off \( \Lambda \), can be added to the Majorana mass term \( M_3 \overline{N_3} N_3 \) and lead to additional DM processes. In this way, terms such as \( (\lambda_X/\Lambda) \overline{N_3} N_3 (X^\dagger X) \) where \( X \) stands for \( H, \chi \) and \( F \) give rise to annihilation processes through Higgs and flavon portals; these are deduced from Eqs. (3.11) and (4.4)

\[
\begin{align*}
\tilde{N}_3 N_3 & \to SM \text{ through } H \text{ exchange with } SM \equiv \tilde{f} f, W^+ W^-, ZZ, HH \\
\tilde{N}_3 N_3 & \to YY \text{ through } \chi \text{ exchange with } YY \equiv \chi \chi, \chi H, \chi T, HH, T \tilde{T}, TH \\
\tilde{N}_3 N_3 & \to PP \text{ through } F_1 \text{ exchange with } PP \equiv F^\dagger T, F^\dagger_{1,2} F_{1,2}, \rho \rho, \phi \phi, \chi \chi, T \tilde{T}.
\end{align*}
\]
Besides these processes, when the masses of $N_1$ and/or $N_2$ are close to that of $N_3$ the co-annihilation of the DM $N_3$ becomes significant and thus, the allowed co-annihilation process $\bar{N}_3 N_2 \rightarrow KK$ through the exchange of $F_2$ with $KK \equiv F_2 \chi, F_1 F_2, H F_2, \rho F_2, \phi F_2, F_2 T$ must be taken into account.

**Case II: Scalar dark matter candidate**

Assuming a mass hierarchy $m_\rho < m_\phi, m_{N_k}$, the lightest chargeless component of the inert Higgs doublet $\rho$ ($\rho_1$ or $A_2$) is stable and plays the role of a DM candidate. In this framework, from the first Yukawa interaction in (3.1), there is a production of the scalar DM $\rho$ via the decay channel $N_k \rightarrow \rho L_k$, and by taking account of $Z'_2 \times Z_2$ charges, we find two different channels; $N_{1,2} \rightarrow \rho L_{1,2}$ and $N_3 \rightarrow \rho L_3$. In addition, there is another production process of $\rho$ derived from the $\lambda_{14}$ coupling in Eq. (4.4); this decay channel is given by $F_1 \rightarrow \rho \rho$. On the other hand, from the first term in (4.4), a pair of $\rho$ can annihilate into a pair of $\phi$ by the t-channel process $\rho \rho \rightarrow \phi \phi$ mediated by members of the $SU(2)_L$ triplet $T$. There are also many annihilation modes by the s-channel process $\rho \rho \rightarrow PP$ through $F_1$ exchange ($PP$ is the same final state as in Eq. (4.5) excluding $\rho \rho$). All these different pairs in the final states may easily deduced from the quartic coupling in the scalar potential (4.4). Moreover, as in the previous case where in the presence of non-renormalizable terms such as $(\lambda_1 / \Lambda) \bar{N}_3 N_3(FF)$, there will be another annihilation process of $\rho$ into pair of right-handed neutrinos $\rho \rho \rightarrow N_k \bar{N}_k$ through $F_1$ exchange. Finally, there are several co-annihilation processes of the DM $\rho$ with $\phi$ into different pairs in final states. Specifically, there are processes mediated by the Higgs boson and others mediated by the flavon $F_2$; we have

$$\begin{align*}
\rho \phi & \rightarrow SM, TT, TH, F_1 H, \chi \chi, \chi H, T \chi \text{ through } H \text{ exchange} \\
\rho \phi & \rightarrow F_2 \chi, F_1 F_2, H F_2, \rho F_2, F_1 \phi, T F_2 \text{ through } F_2 \text{ exchange},
\end{align*}$$

(4.6)

where $SM$ is given in Eq. (4.5). Notice by the way that all the processes provided in the present work are presented based upon their invariance under the remnant $Z'_2 \times Z_2$ symmetry. However, a detailed study of different processes taking into account constraints from kinematics may lead to exclude or suppress several processes. Notice also that all the processes in case II assuming $\rho$ as DM can also be applied to the case of the scalar $\phi$ as a dark matter candidate with few differences in the decay processes.

## 5 Conclusion

In this work, we have proposed a radiative neutrino model based on topology T4-2-i providing an explanation for the observed neutrino masses and mixing as well as allowing for stable dark matter candidates. In order to avoid the tree level Type I and Type II seesaw contributions always accompanying topology T4-2-i, fitting the neutrino data as well as to ensure the stability of DM candidates, we have extended the SM gauge symmetry with the $G_f = \mathbb{D}_4 \times Z_3 \times Z_2$ flavor group. For this purpose, besides promoting the SM fermion in topology T4-2-i to singlet right-handed
neutrinos $N_k$, we have added two flavon fields $F$ and $\chi$ to guarantee the invariance of neutrino Yukawa couplings and the preexisting vertex $\mu_T H T^\dagger H$ connecting two Higgs doublets with scalar triplet $T$. Therefore, the neutrino masses are radiatively generated at one-loop level while their mixing is described by the well known TM$_2$ pattern due to $G_f$ symmetry breaking.

Using the recent experimental neutrino data, we have performed our numerical study in the normal mass hierarchy case where we have shown through several scatter plots the allowed ranges of our model parameters $\{\sigma, \theta, a, b, \phi_a\}$, the $CP$ violating phase $\delta_{CP}$ as well as the non-oscillatory observables $m_{\nu_e}$, $m_{ee}$ and $m_{sum}$ that fit the experimental values of the three mixing angles $\theta_{ij}$ and the mass square differences $\Delta m^2_{ij}$ at $3\sigma$ range.

Another matter considered in the present work is the dark matter candidates given by one of the fields running in the loop. On the basis of our considerations, the DM candidates can be manifested by the neutral components of the scalar doublets $\rho$ and $\phi$, and the right-handed neutrino $N_3$, the lightest of which can play the role of DM as they carry odd charge under $Z_2$. We have briefly listed the several DM processes (decays, annihilations and co-annihilations) for the case of the fermionic DM candidate $N_3$ and the bosonic DM candidates $\rho$ and $\phi$. We showed that the stability of DM is guaranteed by the unbroken $Z_2$ symmetry while the interactions relevant for decays, annihilations and co-annihilations processes are controlled by the residual group $Z_2' \times Z_2$ after $G_f$ symmetry breaking. We should mention however that a thorough study of the dark sector requires performing further studies by analysing two particular experimental constraints; the observed DM relic density and the cross section for direct detection of DM scattering off nucleon. This analysis, however, goes beyond the scope of this paper.

6 Appendix: Dihedral $D_4$ group

In this appendix, we briefly review the basic features of the dihedral group $D_4$ as well as the decomposition of its representations into those of the $Z_2$ subgroup. Recall first that the discrete group $D_4$ is generated by the two elements $S$ and $T$ which fulfill the relations $S^4 = T^2 = Id$ and $STS = T$. It has five irreducible representations; four singlets $1_{+,+}, 1_{+-}, 1_{-,+}$ and $1_{--,}$, and one doublets $2_{0,0}$ where the indices in the representations refer to their characters under $S$ and $T$ as in the following table. The generators $S$ and $T$ of the two-dimensional representations can be

| $\chi_{i,j}$ | $\chi_{1,+}$ | $\chi_{1,-}$ | $\chi_{1,-}$ | $\chi_{1,-}$ | $\chi_{20,0}$ |
|--------------|---------------|---------------|---------------|---------------|---------------|
| $Id$        | +1            | +1            | +1            | +1            | 2             |
| $S$         | +1            | -1            | +1            | -1            | 0             |
| $T$         | +1            | +1            | -1            | -1            | 0             |

Table 4: Character table of the dihedral group $D_4$. 

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expressed by the following $2 \times 2$ matrices

$$S = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  \hfill (6.1)

Now, we consider tensor products of $\mathbb{D}_4$ irreducible representations. The tensor product of two doublets $2_x = (x_1, x_2)^T$ and $2_y = (y_1, y_2)^T$ is decomposed into a sum of $\mathbb{D}_4$ singlet representations as $2_x \times 2_y = 1_{+,+} + 1_{+,−} + 1_{−,+} + 1_{−,−}$, where

$$1_{+,+} = x_1 y_2 + x_2 y_1, \quad 1_{+,−} = x_1 y_1 + x_2 y_2$$

$$1_{−,+} = x_1 y_2 - x_2 y_1, \quad 1_{−,−} = x_1 y_1 - x_2 y_2$$  \hfill (6.2)

whereas the product between two singlets is as follows

$$1_{i,j} \times 1_{k,l} = 1_{ik, jl} \text{ with } i, j, k, l = \pm.$$  \hfill (6.3)

Finally, since the experimental data on neutrino masses and mixing angles require the breaking of $\mathbb{D}_4$ down to its remnant $\mathbb{Z}_2'$ subgroup, so we restrict our discussion only to the breaking pattern $\mathbb{D}_4 \xrightarrow{U} \mathbb{Z}_2'$. After this stage of breaking, it is obvious that the matter and scalar fields in our model will be charged under the unbroken discrete symmetry $\mathbb{Z}_2'$. Accordingly, we summarize in the following table the decompositions of $\mathbb{D}_4$ irreducible representations into those of the residual group $\mathbb{Z}_2'$ subgroup.

| Fields under $\mathbb{D}_4$ Irreps. | Decomposition into $\mathbb{Z}_2'$ Irreps. |
|-------------------------------------|---------------------------------------------|
| $(\alpha_1, \alpha_2)$ $\sim 2_{0,0}$ | $\alpha_1 \sim +1$ $\alpha_2 \sim -1$ |
| $\beta_1 \sim 1_{+,+}, \beta_2 \sim 1_{+,−}$ | $\beta_1, \beta_2 \sim +1$ |
| $\gamma_1 \sim 1_{−,+}, \gamma_2 \sim 1_{−,−}$ | $\gamma_1, \gamma_2 \sim -1$ |

(6.4)

where $\alpha_i$, $\beta_i$ and $\gamma_i$ can be any fermionic or bosonic field. For more details on the $\mathbb{D}_4$ Dihedral group see for instance Ref. [86].

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