ERROR ESTIMATE IN SECOND-ORDER CONTINUOUS-TIME SIGMA-DELTA MODULATORS

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ABSTRACT
Continuous-time Sigma-Delta (CT-ΣΔ) modulators are oversampling Analog-to-Digital converters that may provide a higher sampling rate and lower power consumption than their discrete counterpart. While approximation errors are established for high order discrete time ΣΔ modulators, theoretical analysis of the error between the filtered output and the input remain scarce. This paper presents a general framework to study this error: under regularity assumptions on the input and the filtering kernel, we prove for a second-order CT-ΣΔ that the error estimate may be in $o(1/N^2)$, where $N$ is the oversampling ratio. The whole theory is ultimately validated through numerical experiments.

Index Terms— Sigma-Delta modulator, Continuous, Analog-to-Digital conversion (ADC), Approximation

1. INTRODUCTION
Introduced by Inose and Yasude [1], Sigma-Delta (ΣΔ) modulators are nowadays a widely used Analog-to-Digital converter. Such an 1-bit ADC operates at many times the Nyquist rate while achieving the same resolution of Nyquist modulators [2]. For k-th order discrete-time ΣΔ modulators, works from Daubechies, Güntürk and al. [3–5] provide estimates for the error between the filtered output and input. Typically, they obtain a mean-squared error estimate in $O(1/N^k)$ for a time-varying input, where $N$ is the oversampling ratio, using for example a sinc$^{k+1}$ filter [6]. But for continuous-time ΣΔ (CT-ΣΔ) modulators, such general results remain partial. These CT-ΣΔ modulators deliver more power-efficient operations than their discrete-time equivalent, as well as higher sampling rates [2][8].

Privileged in high performance motor control [9], the ΣΔ ADC is used to retrieve the phase currents which carry information on the rotor position if correctly filtered [10]. Indeed, the Pulse-Width Modulation (PWM) of the input voltage creates ripples in the current measurements [11] that we can extract through a demodulation procedure using linear combination of iterated moving averages [12]. Therefore, knowing the error estimate of the ΣΔ modulator is of utmost importance for this type of application.

We present a general technique to study higher-order CT-ΣΔ modulators. Under regularity assumptions on the input and the filtering kernel, we prove for a second-order CT-ΣΔ that the error estimate may be in $o(1/N^2)$.

This paper is organized as follows: we first detail the required definitions and technical lemmas; then we prove the error estimate on a specific second-order ΣΔ modulator. The theory is finally validated on numerical examples.

2. ERROR ESTIMATE FOR A CT-ΣΔ MODULATOR

2.1. Notations, definitions, preliminary results
We consider the second-order CT-ΣΔ modulator depicted in figure [1] $u$ denotes the input of the modulator which varies in a timescale $1/T_{pwm}$, $v \in \{0, 1\}$ its output, $T_s$ its the sampling time, $x_{1,2}$ the states of the modulator, $N := T_{pwm}/T_s$ the oversampling ratio, $\beta(Nt) := u(t) - v(t)$. We assume the stability of the modulator, which means both $x_1$ and $x_2$ are bounded.

The notation $O$ denotes the "big O" of analysis, i.e. $f(t,\varepsilon) = O(\varepsilon)$ if there exists $K > 0$ independent of $t$ and $\varepsilon$ such that $\|f(t,\varepsilon)\| \leq K \varepsilon$. Likewise, the notation $o$ is the "small o" of analysis, i.e. $f(t,\varepsilon) = o(\varepsilon)$ if $\|f(t,\varepsilon)\| \leq o(\varepsilon)$ where $\lim_{\varepsilon \to 0} g(\varepsilon) = 0$.

The proof in subsection 2.2 relies on the application of a generalization of the classical Riemann-Lebesgue lemma:

Lemma 1 (Generalized Riemann-Lebesgue lemma [13]). Let consider $\beta \in L^\infty[0, +\infty)$ such that $\beta$ has a mean value $\overline{\beta}$, with

$$\overline{\beta} := \lim_{T \to +\infty} \frac{1}{T} \int_0^T \beta(t) \; dt.$$ 

Then for every $f \in L^1[0, +\infty)$,

$$\lim_{N \to +\infty} \int_0^\infty \beta(Nt) f(t) \; dt = \overline{\beta} \int_0^\infty f(t) \; dt.$$

The input $u$ is assumed to be either $AC^1$ or piecewise $AC^1$, as defined below, as well as a rewriting of the integration by parts for piecewise $AC^1$ functions.

Definition 1 ($AC^p$ functions). A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is $AC^p$ on an interval $I$ if it is $p$-times differentiable and its
\( p \)-th order derivative \( f^{(p)} \) is absolutely continuous. It is piecewise \( AC \) if \( f \) is \( p \) times differentiable and \( f^{(p)} \) is piecewise absolutely continuous.

**Lemma 2** (Integration by parts for piecewise \( AC^0 \) functions). Let consider \( f \in L^1[a, b] \) with \(-\infty \leq a < b \leq +\infty\), \( F \) a primitive of \( f \), and \( g \) a piecewise \( AC^0 \) function. Write \( I = \bigcup_{0 \leq i \leq m} [x_i, x_{i+1}] \), with \( a = x_0 < x_1 < \ldots < x_m = b \), such that \( g \) is \( AC^0 \) on each \([x_i, x_{i+1}]\). \( g \) being piecewise \( AC^0 \), it is differentiable almost everywhere, and we note \( g' \) this derivative. Then

\[
\int_a^b f(\sigma)g'(\sigma)\,d\sigma = \sum_{i=0}^{m-1} \left[ F(x_{i+1})g(x_{i+1}) - F(x_i)g(x_i) \right] - \int_a^b F(\sigma)g'(\sigma)\,d\sigma.
\]

**2.2. Second-order CT-ΣΔ ADC**

We consider the modulator depicted in figure 1. The state of the modulator then reads

\[
\begin{align*}
T_s \dot{x}_1(t) &= u(t/T_{\text{pwm}}) - v(t/T_s) \\
T_s \dot{x}_2(t) &= x_1(t) \\
y(t) &= x_2(t) + \frac{3}{2} x_1(t)
\end{align*}
\]

which reads, in the normalized time \( \tau := t/T_{\text{pwm}} \),

\[
\begin{align*}
\frac{1}{N} \dot{x}_1(\tau) &= u(\tau) - v(N\tau) \quad (1a) \\
\frac{1}{N} \dot{x}_2(\tau) &= x_1(\tau) \\
y(\tau) &= x_2(\tau) + \frac{3}{2} x_1(\tau) \quad (1c)
\end{align*}
\]

In this subsection we prove that \( \beta(N\tau) := u(\tau) - v(N\tau) \) admits a zero-mean primitive \( \beta^{(-1)} \), which also has a zero-mean primitive \( \beta^{(-2)} \). Integrating (1a) from 0 to \( t \) yields

\[
\frac{1}{Nt}(x_1(t) - x_1(0)) = \frac{1}{t} \int_0^t u(\sigma)\,d\sigma - \frac{1}{t} \int_0^t v(N\sigma)\,d\sigma.
\]

The modulator is assumed to be stable, so \( x_1 \) is bounded; the left-hand side of the previous equation vanishes when \( t \) tends to infinity, and

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t [u(\sigma) - v(N\sigma)]\,d\sigma = 0
\]

i.e., by definition, \( \overline{\beta} = 0 \). Integrating (1b) from 0 to \( t \) yields

\[
\frac{1}{Nt}(x_2(t) - x_2(0)) = \frac{1}{t} \int_0^t x_1(\sigma)\,d\sigma
\]

Since \( x_2 \) is bounded as we consider the modulator is stable,

\[
\overline{x}_1 = \lim_{t \to +\infty} \frac{1}{t} \int_0^t x_1(\sigma)\,d\sigma = 0.
\]

So \( \frac{1}{N} x_1(\tau) \) has zero mean, and by (1a), it is the primitive of \( \beta(N\tau) \). Thus \( \beta^{(-1)}(N\tau) := \frac{1}{N} x_1(\tau) \) is the zero-mean primitive of \( \beta(N\tau) \). Now integrating equation (1b) from 0 to \( t \) gives

\[
\frac{1}{Nt^2}(x_2(t) - x_2(0)) = \int_0^t \frac{1}{N} x_1(\sigma)\,d\sigma = \int_0^t \beta^{(-1)}(N\sigma)\,d\sigma
\]

The left-hand side is bounded, so every primitive of \( \beta^{(-1)} \) is bounded as well. Consequently, \( \beta^{(-2)} \), the zero-mean primitive of \( \beta^{(-1)} \) is well-defined.

**2.3. Filtering process**

Theorem 3 provides an estimate for functions \( \beta \) such that \( \beta^{(-2)} \) and \( \beta^{(-1)} \) with zero mean exist.

**Theorem 3.** Let consider \( \beta \in L^2[0, +\infty) \) such that the zero-mean primitive \( \beta^{(-1)} \) of \( \beta \) exists, as well as the zero-mean primitive \( \beta^{(-2)} \) of \( \beta^{(-1)} \). Consider as well \( K_k \) a two-times differentiable kernel whose support is \( [0, k] \), and such that \( K_k(0) = K_k(k) = (K_k)'(0) = (K_k)'(k) = 0 \).

If \( s \) is \( AC^1 \), then for \( t \geq 0 \),

\[
I(t) := \int_0^t \beta(N\sigma)s(\sigma)K_k^p(\sigma)\,d\sigma = o(1/N^2),
\]

with \( K_k^p(\sigma) = K_k(t - \sigma) \). If \( s \) is piecewise \( AC^1 \), then for \( t \geq 0 \), \( I(t) = O(1/N^2) \).

In other words, the instantaneous difference between the filtered input and the filtered output is in \( o(1/N^2) \) under some regularity assumptions on the kernel \( K_k \) and the input \( u \).

**Proof.** If \( s \) is \( AC^1 \) (resp. piecewise \( AC^1 \)), then \( f_t : \sigma \mapsto s(\sigma)K_k^p(\sigma) \) is also \( AC^1 \) (resp. piecewise \( AC^1 \)). In any case,
\[ f_i \] is differentiable with support \([t-k, t]\) and a basic integration by parts gives
\[
I(t) = \frac{1}{N} \left[ \beta^{(1)}(N(t)) f_i(t) - \beta^{(1)}(N(t-k)) f_i(t-k) \right] - \frac{1}{N} \int_{t-k}^t \beta^{(1)}(N\sigma) f'_i(\sigma) d\sigma,
\]
where the first term is zero since \( f_i(t) = f_i(t-k) = 0 \).

We write \( t - k = \sigma_0 < \ldots < \sigma_m = t \) the locations of the loss of regularity of \( s \). The integration by parts, given by lemma \([2]\) yields
\[
I(t) = -\frac{1}{N^2} \sum_{i=0}^{m-1} \left[ \beta^{(2)}(N\sigma_{i+1}) f''_i(\sigma_{i+1}) - \beta^{(2)}(N\sigma_i) f''_i(\sigma_i) \right] + \frac{1}{N^2} \int_{t-k}^t \beta^{(1)}(N\sigma) f'_i(\sigma) d\sigma.
\]
The limit of the integral term in \([2]\), by lemma \([1]\) is
\[
\lim_{t \to +\infty} \int_0^{+\infty} \beta^{(2)}(N\sigma) f''_i(\sigma) d\sigma = \beta^{(2)} \int_0^{+\infty} f''_i(\sigma) d\sigma = 0,
\]
i.e.
\[
\frac{1}{N^2} \int_{t-k}^t \beta^{(2)}(N\sigma) f''_i(\sigma) d\sigma = o(1/N^2).
\]
If \( f \) is \( AC^1 \), the sum in \([2]\) is zero since \( f''_i(t) = f''_i(t-k) \); therefore \( I(t) = O(1/N^2) \). Otherwise the sum in \([2]\) is non-necessarily zero, and \( I(t) = O(1/N^2) \), which concludes the proof. \( \square \)

3. NUMERICAL RESULTS

The estimate obtained in section \([2]\) is now validated on a numerical example. We consider the modulator as in figure \([1]\) with \( T_s = 5 \times 10^{-3} \) s. The tests are conducted with three different inputs \( u_i(t) := z(t)s_i(t) \), with \( z(t) := 0.04 \cos \left( \frac{t}{10} \right) - 0.06 \sin \left( \frac{t}{3\pi} \right) \), and
\[
s_1(t) := \frac{1}{\sqrt{0.03}} \left( \tau 1_{[0,0.6]}(t) + 1.5(1 - \tau)1_{[0.6,1]}(t) - 0.3 \right),
\]
\[
s_2(t) := \sqrt{2} \cos(2\pi t), \quad s_3(t) := 1_{[0,0.5]}(t) - 1_{[0.5,1]}(t),
\]
where \( \tau = \text{mod}(t, T_{\text{pwm}})/T_{\text{pwm}}, T_{\text{pwm}} = 1 \) s and \( t \in [0, 250] \) s. Illustrated in figure \([2]\) the \( s_i \)'s are respectively piecewise \( AC^1 \) \( (s_1) \), \( AC^1 \) \( (s_2) \) and discontinuous \( (s_3) \), and such that \( ||s_1||_2 = 1 \). A kernel satisfying the hypotheses of theorem \([3]\) is the convolution power of the characteristic function \( 1_{[0,1]} \), \( K^3 := 1_{[0,1]} \ast 1_{[0,1]} \ast 1_{[0,1]} \), as \( \text{supp } K = [0, 3] \) and \( K^3(0) = (K^3)'(0) = (K^3)'(3) = 0 \) (see for example \([14]\)).

Define \( \hat{z} \) (resp. \( \hat{z}_{\Sigma \Delta} \)) the filtered input (resp. output) as
\[
\hat{z}(t) := \int_0^{+\infty} u(\sigma) s(\sigma) K^3(t - \sigma) d\sigma,
\]
\[
\hat{z}_{\Sigma \Delta}(t) := \int_0^{+\infty} v(\sigma) s(\sigma) K^3(t - \sigma) d\sigma,
\]
so that \( I(t) = \hat{z}(t) - \hat{z}_{\Sigma \Delta}(t) \). The estimates \( \hat{z} \) and \( \hat{z}_{\Sigma \Delta} \) are illustrated in figure \([3]\) as well as \( I(t) \); their accordance shows the commutation of the filtering process with the \( \Sigma \Delta \) modulator.

To confirm the asymptotic behavior described by theorem \([3]\) the same simulation is carried out for each input \( u_i \) and different values of \( N \), and study the L2-error \( ||I||_2 := \left( \int_0^{+\infty} I(\sigma)^2 d\sigma \right)^{1/2} \) of \( I \). Figure \([4]\) shows these behaviors for the three inputs \( u_i \) and validates the approximation orders. Indeed, when \( s = s_1 \) is piecewise \( AC^1 \), the approximation order is in \( O(1/N^2) \); it is slightly better when \( s = s_1 \) is \( AC^1 \), with \( ||I||_2 = O(1/N^{2.3}) = o(1/N^2) \); when \( s = s_3 \) is discontinuous, we only have an estimate in \( O(1/N) \).

4. CONCLUSION

Depending on the regularity of the input, and assuming the modulator is stable, we proved the error between the filtered output and filtered input decreases at a rate which is \( o(1/N^2) \) if the input is differentiable with a derivative that is absolutely continuous. Such an approximation error is crucial for precise operations, in particular, for ripple extraction in sensorless control of electric motors.
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