CANONICAL STABILITY OF 3-FOLDS OF GENERAL TYPE WITH $p_g \geq 3$

MENG CHEN

Abstract. We study the canonical stability of a smooth projective 3-fold $V$ of general type. We prove that $|5K_V|$ gives a birational map onto its image provided $p_g \geq 4$. For those $V$ with $p_g = 3$, $|6K_V|$ gives a birational map. Known examples show that both the results are optimal.

1. Introduction

Throughout the base field is $\mathbb{C}$. Let $V$ be a smooth projective variety with $\kappa(V) > 0$. For all integer $m > 0$, one may define the so-called $m$-canonical map $\Phi_m$, which is nothing but the rational map corresponding to the complete linear system $|mK_V|$. Within birational geometry, to study the behavior of $\Phi_m$ has been one of the classical and important aspects. When $\dim V \leq 2$, the behavior of $\Phi_m$ is known quite well according to works by a long list of authors. When $\dim V \geq 3$, known results on $\Phi_m$ are partial and there remains a lot of open problems on this topic. In this paper, we study a 3-fold of general type.

Assume $\dim V = \kappa(V) = 3$. According to Mori’s MMP, $V$ has a minimal model $X$ with only $\mathbb{Q}$-factorial terminal singularities. Denote by $K_X$ the canonical Weil divisor on $X$. One may also define the $m$-canonical map $\varphi_m$ corresponding to the complete linear system $|mK_X|$. Modulo birational equivalence, both $\Phi_m$ and $\varphi_m$ share lots of common properties, e.g. birationality. In general, it doesn’t make sense to study the basepoint freeness of $|mK_X|$ unless $m$ is divisible by the canonical index $r(X)$. However, one may consider the birationality of $\Phi_m$ or $\varphi_m$. It’s well-known that, for a given $V$, $\Phi_{m(V)}$ is birational onto its image whenever $m(V) \gg 0$. Therefore quite an interesting thing to do is to answer the following question which is still open

Question 1.1. Let $V$ be a smooth projective 3-fold of general type.

1. Do there exist a universal lower bound $N$ ($N$ doesn’t depend on $V$) such that $\Phi_m$ is birational onto its image for all $m \geq N$?

2. For a given $V$, what is the optimal lower bound $m_0(V)$ such that $\Phi_{m_0(V)}$ is birational?

The paper is partially supported by the National Natural Science Foundation of China (Key Project No. 10131010), Shanghai Scientific & Technical Commission (Grant 01QA14042) and SRF for ROCS, SEM.
Apart from the generally accepted importance of 1.1(1) above, we would like to emphasize here that 1.1(2) is equally important. For example, it is strongly related to 3-fold geography (see [12]) and, also, it can be applied to determine the automorphism group of $V$ (see the Remark in [23]). There has been partial results on Question 1.1. According to [4, 8, 17], $\varphi_m$ is a birational morphism onto its image if $X$ is a minimal Gorenstein 3-fold of general type. For a general 3-fold $V$ with $q(V) > 0$ or $p_k(V) \geq 2$, J. Kollár ([15]) gave a positive answer to 1.1(1). For regular 3-folds of general type, T. Luo ([18]) partially answered 1.1(1). There is, however, hardly no result on 1.1(2). The aim of this paper is to study 1.1(2) under an extra assumption. My main result is the following

**Theorem 1.2.** Let $V$ be a smooth projective 3-fold of general type.

(1) Assume $p_g(V) = 3$. Then $\Phi_6$ is birational onto its image.

(2) Assume $p_g(V) \geq 4$. Then $\Phi_5$ is birational onto its image.

The following examples show that Theorem 1.2 is optimal.

**Example 1.3.** In [6], Chiaruttini and Gattazzo has found a smooth projective 3-fold $V$ of general type with $q(V) = h^2(\mathcal{O}_V) = 0, p_g(V) = 3$. They verified, on $V$, that $\Phi_m$ is birational if and only if $m \geq 6$ and that $\Phi_m$ is generically finite for $2 \leq m \leq 5$. It is, in fact, not difficult to see that $V$ is canonically fibred by curves of genus two.

**Example 1.4.** Denote by $S$ a smooth minimal surface of general type with $(K_S^2, p_g(S)) = (1, 2)$. Pick up a smooth curve $C_k$ of genus $k \geq 2$. Set $X_k := S \times C_k$. Then $p_g(X_k) = 2k \geq 4$. It is obvious that the $\Phi_4$ of $X_k$ is not birational. This is of course a trivial example.

**Example 1.5.** On $\mathbb{P}^3_C$, take a smooth hypersurface $S$ of degree 10. $S \sim 10H$ where $H$ is a hyperplane. Let $X$ be a double cover over $\mathbb{P}^3$ with branch locus along $S$. Then $X$ is a nonsingular canonical model. $K_X^3 = 2$ and $p_g(X) = 4$ and $\Phi_1$ is a finite morphism onto $\mathbb{P}^3$ of degree 2. One may easily check that $\Phi_4$ is also a finite morphism of degree 2.

# 2. The general method

## 2.1. Brief review on curves.

We recall several facts which will be applied in the paper.

(2.1.1) Let $C$ be a smooth curve of genus $\geq 2$. Then $\varphi_m$ is an embedding for all $m \geq 3$.

(2.1.2) Let $C$ be a smooth curve of genus $\geq 2$. Assume $D$ is a divisor on $C$. Then $K_C + D$ is very ample whenever $\deg(D) \geq 3$ and $|K_C + D|$ is basepoint free whenever $\deg(D) \geq 2$.

(2.1.3) Let $C$ be a smooth non-hyperelliptic curve. Assume $D$ is a divisor on $C$ with $\deg(D) \geq 2$. Then the rational map corresponding to $|K_C + D|$ gives a birational morphism onto its image.
2.2. Brief review of relevant results on surfaces. We only recall those cited in the paper. Let $S$ be a smooth minimal surface of general type. By [1 2 3 7], one has

- (2.2.1) $\varphi_m$ is a birational morphism onto its image for all $m \geq 5$.
- (2.2.2) $\varphi_4$ is a birational morphism if and only if $(K_S^2, p_g(S)) \neq (1, 2)$.
- (2.2.3) $\varphi_3$ is birational if and only if $(K_S^2, p_g(S)) \neq (1, 2)$ and (2.3).
- (2.2.4) $|2K_S|$ is basepoint free if $p_g(S) > 0$.

**Proposition 2.3.** Let $S$ be a smooth projective surface of general type with $(K_{S_0}^2, p_g(S)) = (1, 2)$, where $\sigma : S \to S_0$ is the contraction onto the minimal model. Assume $L$ is a nef and big $\mathbb{Q}$-divisor on $S$. Then the rational map corresponding to $|K_S + 3\sigma^*(K_{S_0}) + \gamma L|$ is birational onto its image.

**Proof.** First, it is easy to reduce to the case that the movable part of $|\sigma^*(K_{S_0})|$ is basepoint free. So we may assume, from now on, that the movable part of $|\sigma^*(K_{S_0})|$ is basepoint free. Denote by $|G|$ the movable part of $|\sigma^*(K_{S_0})|$. Then $|G|$ is composed of a rational pencil of curves of genus 2 and $h^0(S, G) = 2$ (see [1]). Let $C \in |G|$ be a general member.

The Kawamata-Viehweg vanishing theorem (see 2.4 below) gives the surjective map

$$H^0(S, K_S + \sigma^*(K_{S_0}) + \gamma L + G) \longrightarrow H^0(C, K_C + D_0),$$

where $\deg(D_0) > 0$. This means that $K_S + \sigma^*(K_{S_0}) + \gamma L + G \geq 0$ and that $K_S + 3\sigma^*(K_{S_0}) + \gamma L \geq G$. So $|K_S + 3\sigma^*(K_{S_0}) + \gamma L|$ separates different general members of $|G|$.

Applying the vanishing theorem again, we get the surjective map

$$H^0(S, K_S + 2\sigma^*(K_{S_0}) + \gamma L + G) \longrightarrow H^0(C, K_C + D),$$

where $\deg(D) \geq 3$. So $|K_S + 3\sigma^*(K_{S_0}) + \gamma L|^C$ gives a birational map.

We are done. $\square$

2.4. Vanishing theorem. We always apply the Kawamata-Viehweg vanishing theorem (see [13] or [22]), which plays very effective roles throughout the whole context. It is also well-known that, on surfaces, one may apply the vanishing theorem without the assumption for ”normal crossings” (see [20]).

2.5. Set up for $\varphi_1$. Because of the successful 3-dimensional MMP, one may always study a minimal 3-fold. Let $X$ be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. Suppose $p_g(X) \geq 2$. We study the canonical map $\varphi_1$ which is usually a rational map. Take the birational modification $\pi : X' \to X$, according to Hironaka, such that

(i) $X'$ is smooth;

(ii) the movable part of $|K_{X'}|$ is basepoint free. (Sometimes we even call for such a modification that those movable parts of a finite number of linear systems are all basepoint free.)
(iii) $\pi^*(K_X)$ is linearly equivalent to a divisor supported by a divisor of normal crossings.

Denote by $g$ the composition $\varphi_1 \circ \pi$. So $g : X' \to W' \subseteq \mathbb{P}^d(X)^{-1}$ is a morphism. Let $g : X' \xrightarrow{f} B \xrightarrow{s} W'$ be the Stein factorization of $g$. We may write

$$K_{X'} =_\mathbb{Q} \pi^*(K_X) + E_1 =_\mathbb{Q} M_1 + Z_1,$$

where $M_1$ is the movable part of $|K_{X'}|$, $Z_1$ the fixed part and $E_1$ an effective $\mathbb{Q}$-divisor which is a $\mathbb{Q}$-sum of distinct exceptional divisors. Throughout we always mean $\pi^*(K_X)$ by $K_{X'} - E_1$. We fix a divisor $K_{X'}$ from the beginning. So, whenever we take the round up of $m\pi^*(K_X)$, we always have $\lceil m\pi^*(K_X) \rceil \leq mK_{X'}$ for all positive number $m$. We may also write

$$\pi^*(K_X) =_\mathbb{Q} M_1 + E'_1,$$

where $E'_1 = Z_1 - E_1$ is actually an effective $\mathbb{Q}$-divisor.

If $\dim \varphi_1(X) = 2$, we see that a general fiber of $f$ is a smooth projective curve of genus $g \geq 2$. We say that $X$ is canonically fibred by curves of genus $g$.

If $\dim \varphi_1(X) = 1$, we see that a general fiber $S$ of $f$ is a smooth projective surface of general type. We say that $X$ is canonically fibred by surfaces with invariants $(c_2(S_0), p_g(S))$, where $S_0$ is the minimal model of $S$. We may write $M_1 \equiv a_1 S$ where $a_1 \geq p_g(X) - 1$.

A generic irreducible element $S$ of $|M_1|$ means either a general member of $|M_1|$ whenever $\dim \varphi_1(X) \geq 2$ or, otherwise, a general fiber of $f$.

**Theorem 2.6.** Let $X$ be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities and assume $p_g(X) \geq 2$. Keep the same notations as in 2.5. Pick up a generic irreducible element $S$ of $|M_1|$. Suppose, on the smooth surface $S$, there is a movable linear system $|G|$ and denote by $C$ a generic irreducible element of $|G|$. Set $\xi := (\pi^*(K_X) \cdot C)_X$, and

$$p := \begin{cases} 
1 & \text{if } \dim \varphi_1(X) \geq 2 \\
a_1 \geq p_g(X) - 1 & \text{otherwise.}
\end{cases}$$

Assume

(i) there is a positive integer $m$ such that the linear system

$$|K_S + \lceil (m - 2)\pi^*(K_X) \rceil|$$

separates different generic irreducible elements of $|G|$;

(ii) there is a rational number $\beta > 0$ such that $\pi^*(K_X)|_S - \beta C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor;

(iii) either the inequality $\alpha := (m - 1 - \frac{1}{p} - \frac{1}{\beta})\xi > 1$ holds (set $\alpha_0 := \lceil \alpha \rceil$) or $C$ is non-hyperelliptic, $m - 1 - \frac{1}{p} - \frac{1}{\beta} > 0$ and $C$ is an even divisor on $S$. 
Then we have the inequality \( m\xi \geq 2g(C) - 2 + \alpha_0 \). Furthermore, \( \varphi_m \) of \( X \) is birational onto its image provided either \( \alpha > 2 \) or \( \alpha_0 = 2 \) and \( C \) is non-hyperelliptic or \( C \) is non-hyperelliptic, \( m - 1 - \frac{1}{p} - \frac{1}{\beta} > 0 \) and \( C \) is an even divisor on \( S \).

**Proof.** We consider the sub-system

\[ |K_{X'} + \gamma(m - 1)\pi^*(K_X) - \frac{1}{p}E_1'| \subset |mK_{X'}|. \]

This system obviously separates different generic irreducible elements of \( |M_1| \). By the birationality principle (P1) and (P2) of \( [5] \), it is sufficient to prove that \( |mK_{X'}|_S \) gives a birational map. Noting that \((m - 1)\pi^*(K_X) - \frac{1}{p}E_1' - S\) is nef and big, the vanishing theorem gives the surjective map

\[ H^0(X', K_{X'} + \gamma(m - 1)\pi^*(K_X) - \frac{1}{p}E_1') \longrightarrow H^0(S, K_S + \gamma(m - 1)\pi^*(K_X) - \frac{1}{p}E_1'|_S). \]

We are reduced to prove that \( |K_S + \gamma(m - 1)\pi^*(K_X) - \frac{1}{p}E_1'|_S \) gives a birational map. We still apply the principle (P1) and (P2) of \( [5] \). Because

\[ K_S + \gamma(m - 1)\pi^*(K_X) - \frac{1}{p}E_1'|_S \geq K_S + \gamma(m - 2)\pi^*(K_X)|_S, \]

the linear system \( |K_S + \gamma(m - 1)\pi^*(K_X) - \frac{1}{p}E_1'|_S \) separates different irreducible elements of \( |G| \) by assumption (i). Now pick up a generic irreducible element \( C \in |G| \). By assumption (ii), there is an effective \( \mathbb{Q} \)-divisor \( H \) on \( S \) such that

\[ \frac{1}{\beta}\pi^*(K_X)|_S \equiv C + H. \]

By the vanishing theorem, we have the surjective map

\[ H^0(S, K_S + \gamma(m - 1)\pi^*(K_X) - \frac{1}{p}E_1'|_S - H) \longrightarrow H^0(C, D), \]

where \( D := \gamma((m - 1)\pi^*(K_X) - \frac{1}{p}E_1')|_S - C - H|_C \) is a divisor on \( C \). Noting that

\[ ((m - 1)\pi^*(K_X) - \frac{1}{p}E_1')|_S - C - H \equiv (m - 1 - \frac{1}{p} - \frac{1}{\beta})\pi^*(K_X)|_S \]

and that \( C \) is nef on \( S \), we have \( \deg(D) \geq \alpha \) and thus \( \deg(D) \geq \alpha_0 \). Whenever \( C \) is non-hyperelliptic, \( m - 1 - \frac{1}{p} - \frac{1}{\beta} > 0 \) and \( C \) is an even divisor on \( S \), \( \deg(D) \geq 2 \) automatically follows. If \( \deg(D) \geq 3 \), then \( |K_S + \gamma((m - 1)\pi^*(K_X) - \frac{1}{p}E_1')|_S - H|_C \) gives a birational map by (2.1.2). Because

\[ |K_S + \gamma((m - 1)\pi^*(K_X) - \frac{1}{p}E_1')|_S - H|_C \]

\[ \subset |K_S + \gamma(m - 1)\pi^*(K_X) - \frac{1}{p}E_1'|_S|, \]
we see that the right linear system in above gives a birational map. So $\varphi_m$ of $X$ is birational.

Whenever $\deg(D) \geq 2$, $|K_C + D|$ is basepoint free. Denote by $|M_m|$ the movable part of $[mK_X]$ and by $|N_m|$ the movable part of $|K_S + \Gamma((m - 1)\pi^*(K_X) - S - \frac{1}{p}E_1)|_S - H'|$. By Lemma 2.7 of [5], we have

$$m\pi^*(K_X)|_S \geq N_m \text{ and } (N_m \cdot C)_S \geq 2g(C) - 2 + \alpha_0.$$ 

The theorem is proved. $\square$

3. Proof of the main theorem

We study $\varphi_m$ according to the value $d := \dim \varphi_1(X)$. Obviously $1 \leq d \leq 3$.

**Theorem 3.1.** Let $X$ be a minimal 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $p_g(X) \geq 3$ and $d \geq 2$. Then

(i) $\varphi_5$ is birational onto its image for $p_g(X) \geq 4$.

(ii) $\varphi_6$ is birational onto its image for $p_g(X) = 3$.

**Proof.** We take $m = 5$ and will apply Theorem 2.6. In this case, $S$ is a general member of $|M_1|$. So $S$ is a smooth surface of general type. Set $G := M_1|_S$. Then $|G|$ is a basepoint free system on $S$, since $g$ is a morphism. Take a generic irreducible element $C$ of $|G|$. $C$ is a smooth curve of genus $\geq 2$. We have $p = 1$.

Case 1. If $d = 3$, then we may take $\beta = 1$. Because $\pi^*(K_X)|_S \geq G$ and $|G|$ is not composed of a pencil of curves, Theorem 2.6(i) is satisfied. Noting that $C^2 \geq 2$ in this case, we have

$$\xi = (\pi^*(K_X)|_S \cdot C)_S \geq C^2 \geq 2.$$ 

Thus $\alpha = 2\xi \geq 4$. Theorem 2.6 implies that $\varphi_5$ is birational.

Case 2. If $d = 2$, then we may take $\beta = 1$ whenever $p_g(X) = 3$ and $\beta = 2$ whenever $p_g(X) \geq 4$. Noting that $|G|$ is composed of a pencil, we have $G \equiv qC$ where $q \geq p_g(X) - 2$.

When $|G|$ is a rational pencil, since $\pi^*(K_X)|_S \geq C$, Theorem 2.6(i) is satisfied.

When $|G|$ is an irrational pencil, then $q \geq 2$. Pick up two different generic irreducible elements $C_1$ and $C_2$ in $|G|$. Then $G - C_1 - C_2$ is nef. Note that

$$K_S + \Gamma 3\pi^*(K_X)|_S \geq K_S + \Gamma 2\pi^*(K_X)|_S + G.$$ 

Applying the vanishing theorem, we have the surjective map $H^0(S, K_S + \Gamma 2\pi^*(K_X)|_S + G) \to H^0(C_1, K_{C_1} + D_1) \oplus H^0(C_2, K_{C_2} + D_2)$, where $D_i$ are of positive degree for all $i$. Thus $H^0(C_i, K_{C_i} + D_i) \neq \emptyset$. So the linear system $[K_S + \Gamma 3\pi^*(K_X)|_S]$ separates different generic irreducible elements of $|G|$. Theorem 2.6(i) is also satisfied.
By Proposition 5.1 of [5], \( \varphi_4 \) is generically finite. This means that \( |4\pi^*(K_X)|_S \) maps a general \( C \) onto a curve. Thus \( 4\pi^*(K_X)|_S \cdot C \geq 2 \) and so \( \xi \geq \frac{2}{3} \).

Now take \( m_1 = 6 \). Then \((m_1 - 3)\xi > 1\). Applying Theorem 2.6 one has \( \xi \geq \frac{2}{3} \). Take \( m_2 = 5 \). Then \((m_2 - 3)\xi > 1\). Theorem 2.6 gives \( \xi \geq \frac{5}{6} \). Take \( m_3 = 6 \). Then \((m_3 - 3)\xi > 2\). Theorem 2.6 gives \( \xi \geq \frac{5}{6} \).

Similarly, if we take \( m_4 = 7 \), then we get \( \xi \geq \frac{6}{7} \).

When \( p_g(X) \geq 4 \), \( \alpha = (5 - 1 - 1 - \frac{1}{2})\xi > 2 \). Therefore, by Theorem 2.6 \( \varphi_5 \) is birational.

When \( p_g(X) = 3 \), \( \alpha = (6 - 3)\xi > 2 \). Thus \( \varphi_6 \) is birational. \( \square \)

Remark 3.2. Examples 1.3 and 1.5 show that Theorem 3.1 is sharp.

Theorem 3.3. Let \( X \) be minimal 3-fold of general type with only \( \mathbb{Q} \)-factorial terminal singularities. Assume \( p_g(X) \geq 2 \), \( d = 1 \) and \( g(B) > 0 \). Then \( \varphi_5 \) is birational onto its image.

Proof. This is the simple case. Because \( g(B) > 0 \), the movable part of \( |K_X| \) is already basepoint free on \( X \) and \( M_1 \equiv aS \) with \( a \geq 2 \). So one always has \( \pi^*(K_X)|_S = \sigma^*(K_{S_0}) \), where \( S \) is the general fiber of the derived fibration \( f : X' \to B \) and \( \sigma : S \to S_0 \) is the contraction onto minimal model. Note that

\[
\pi^*(K_X) - S - \frac{1}{a}E'_1 \equiv (1 - \frac{1}{a})\pi^*(K_X).
\]

Applying the vanishing theorem, we have the surjective map

\[
H^0(X', K_{X'} + (4 - \frac{1}{a})\pi^*(K_X)|_S) \twoheadrightarrow H^0(S, K_S + (4 - \frac{1}{a})\pi^*(K_X)|_S).
\]

Also note that

\[
K_S + (4 - \frac{1}{a})\pi^*(K_X)|_S \geq K_S + 3\sigma^*(K_{S_0}) + (1 - \frac{1}{a})E'_1|_S \geq 0.
\]

If \( (K_{S_0}, p_g(S)) \neq (1, 2) \), then \( |K_S + 3\sigma^*(K_{S_0}) + (1 - \frac{1}{a})E'_1|_S|_S \) defines a birational map by 2.2 and so does \( \varphi_5|_S \). Otherwise, because \( E'_1|_S \equiv \pi^*(K_X)|_S \) is nef and big and effective, we have the same conclusion according to Proposition 2.3.

Because \( |5K_X| \) separates different fibers of \( f \), we may conclude that \( \varphi_5 \) is birational. \( \square \)

Proposition 3.4. Let \( X \) be a minimal 3-fold of general type with only \( \mathbb{Q} \)-factorial terminal singularities. Assume \( p_g(X) \geq 3 \), \( d = 1 \) and \( B = \mathbb{P}^1 \). Pick up a general fiber \( S \) of \( f : X' \to \mathbb{P}^1 \) and let \( \sigma : S \to S_0 \) be the contraction onto minimal model. Suppose that there is a movable linear system \( |G| \) on \( S \) with \( G \leq 2\sigma^*(K_{S_0}) \) and that \( |G| \) is not composed of an irrational pencil of curves. Then

(i) \( |4\pi^*(K_X)|_S \) separates different generic irreducible elements of \( |G| \) provided \( p_g(X) \geq 4 \).

(ii) \( |5\pi^*(K_X)|_S \) separates different generic irreducible elements of \( |G| \) provided \( p_g(X) \geq 3 \).
Proof. If \( p_g(X) \geq 4 \), then we have \( \mathcal{O}(3) \hookrightarrow f_*\omega_{X'} \). So we get
\[
\mathcal{O}(1) \otimes f_*\omega_{X'/\mathbb{P}^1}^2 \hookrightarrow f_*\omega_{X'}^4.
\]
This means that \( \varphi_4|_S \) dominates the bicanonical map of \( S \). So
\[
\lceil 4\pi^*(K_X)|S^3 \rceil \geq 2\sigma^*(K_{S_0}), \text{ because } |2\sigma^*(K_{S_0})| \text{ is basepoint free by (2.2.4)}. \]
Thus \( \lceil 4\pi^*(K_X)|S^3 \rceil \geq G \). We are done.

If \( p_g(X) = 3 \), then we have \( \mathcal{O}(2) \hookrightarrow f_*\omega_{X'} \). So we get
\[
\mathcal{O}(1) \otimes f_*\omega_{X'/\mathbb{P}^1}^2 \hookrightarrow f_*\omega_{X'}^5.
\]
This means that \( \varphi_5|_S \) dominates the bicanonical map of \( S \). So
\[
\lceil 5\pi^*(K_X)|S^3 \rceil \geq 2\sigma^*(K_{S_0}), \text{ because } |2\sigma^*(K_{S_0})| \text{ is basepoint free}. \text{ Thus}
\[
\lceil 5\pi^*(K_X)|S^3 \rceil \geq G \). We are done. \( \square \)

**Theorem 3.5.** Let \( X \) be a minimal 3-fold of general type with only \( \mathbb{Q} \)-factorial terminal singularities. Assume \( p_g(X) \geq 3 \), \( d = 1 \), \( B = \mathbb{P}^1 \) and \( S \) is a surface with invariants \( (K_S^2, p_g(S)) \neq (1,2) \) and \( (2,3) \) where \( S_0 \) is the minimal model of \( S \). Then

(i) \( \varphi_5 \) is birational whenever \( p_g(X) \geq 4 \).

(ii) \( \varphi_6 \) is birational whenever \( p_g(X) = 3 \).

Proof. In order to apply Theorem 2.6 we set \( G := 2\sigma^*(K_{S_0}) \) where \( \sigma : S \to S_0 \) is the contraction onto minimal model. By (2.2.4), \( |G| \) is basepoint free. So \( |G| \) is not composed of a pencil of curves. A general member \( C \in |G| \) is actually non-hyperelliptic by (2.2.3). We set \( m = 6 \), \( p = 2 \) whenever \( p_g(X) = 3 \) and \( m = 5 \), \( p = 3 \) whenever \( p_g(X) \geq 4 \).

By the previous proposition, Theorem 2.6(i) is satisfied. What we need is a suitable value \( \beta \) in order to perform the calculation. Denote by \( M_k \) the movable part of \( |kK_X| \) for all \( k > 0 \). Note that the total number of linear systems we come across in our argument is finite. So we may suppose, by modifying \( \pi \), that \( |M_k| \) is basepoint free for upper bounded \( k \). Pick up a number \( n > 0 \). Because \( \mathcal{O}(p) \hookrightarrow f_*\omega_{X'} \), we have
\[
\mathcal{O}(1) \otimes f_*\omega_{X'/\mathbb{P}^1}^m \hookrightarrow f_*\omega_{X'}^{(p+2)n+1}.
\]
This means that \( M_{(p+2)n+1}|_S \geq pm\sigma^*(K_{S_0}) \), because \( |pm\sigma^*(K_{S_0})| \) is basepoint free in general. Thus, for all \( n > 0 \), there is an effective \( \mathbb{Q} \)-divisor \( E(n) \) on \( S \) such that
\[
(k(p+2)n+1)|_S = \frac{pm}{(p+2)n+2} \sigma^*(K_{S_0}) + E(n).
\]
So we may choose a \( \beta \) in the number sequence \( \{ \frac{pm}{2(p+2)n+2} \mid n > 0 \} \).

Suppose \( n \) is sufficient large, one may take a number \( \beta \mapsto \beta^+_{2p+4} \), but \( \beta < \frac{p}{2p+4} \).

Now if \( p_g(X) \geq 4 \), then one may choose \( \beta \) such that \((5-\frac{1}{p} - \frac{1}{\beta}) > 0 \). By Theorem 2.6 and because \( C \) is an even divisor, \( \varphi_5 \) is birational.

If \( p_g(X) = 3 \), we may also choose a \( \beta \) such that \( 6 - 1 - \frac{1}{2} - \frac{1}{\beta} > 0 \). By Theorem 2.6 \( \varphi_6 \) is birational. \( \square \)
Theorem 3.6. Let $X$ be a minimal 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $p_g(X) \geq 3$, $d = 1$, $B = \mathbb{P}^1$ and $S$ is a surface with invariants $(K_{S_0}^2, p_g(S)) = (2, 3)$ where $S_0$ is the minimal model of $S$. Then

(i) $\varphi_5$ is birational whenever $p_g(X) \geq 4$.

(ii) $\varphi_6$ is birational whenever $p_g(X) = 3$.

Proof. We will still apply Theorem 2.6. We may take $p = 2$ whenever $p_g(X) = 3$ and $p = 3$ whenever $p_g(X) \geq 4$. Pick up a general fiber $S$. Take $|G|$ to be the movable part of $|\sigma^*(K_{S_0})|$. According to [11], we know that $|G|$ is not composed of a pencil and a general member $C \in |G|$ is a smooth curve of genus 3. By the vanishing theorem, we have the surjective map

$$H^0(X', K_{X'} + \Gamma \pi^*(K_X) - \frac{1}{p}E_1') \longrightarrow H^0(S, K_S + \Gamma \frac{p - 1}{p}E_1'|_S).$$

So we have $2\pi^*(K_X)|_S \geq C$ by Lemma 2.7 of [5]. One may take $\beta = \frac{1}{2}$. Note also that $\xi \geq \frac{1}{2}C^2 = 1$, because $C^2 = 2$.

If $p_g(X) \geq 4$, take $m_1 = 5$ and then $(m_1 - 1 - \frac{1}{p} - \frac{1}{\beta})\xi \geq \frac{5}{3} > 1$. Theorem 2.6 gives $\xi \geq \frac{6}{5}$. Take $m_2 = 6$, we will get $\xi \geq \frac{4}{3}$. So $\alpha = (5 - 1 - \frac{1}{p} - \beta)\xi > 2$, which means $\varphi_5$ is birational.

If $p_g(X) = 3$, then $\alpha = (6 - 1 - \frac{1}{p} - \beta)\xi > 2$. So, by Theorem 2.6 $\varphi_6$ is birational. □

3.7. The last case. Assume $p_g(X) \geq 3$, $d = 1$, $B = \mathbb{P}^1$ and $S$ is a surface with invariants $(K_{S_0}^2, p_g(S)) = (1, 2)$, where $S_0$ is the minimal model of $S$. Denote by $\sigma : S \rightarrow S_0$ the contraction map. Let $f : X' \rightarrow \mathbb{P}^1$ be the derived fibration. Let $\mathcal{L}_0$ be the saturated sub-bundle of $f_*\omega_{X'}$ which is generated by $H^0(W, f_*\omega_{X'})$. Because $|K_{X'}|$ is composed of a pencil of surfaces and $\varphi_1$ factors through $f$, we see that $\mathcal{L}_0$ is a line bundle on $\mathbb{P}^1$. Denote $\mathcal{L}_1 := f_*\omega_{X'}/\mathcal{L}_0$. Then we have the exact sequence

$$0 \longrightarrow \mathcal{L}_0 \longrightarrow f_*\omega_{X'} \longrightarrow \mathcal{L}_1 \longrightarrow 0.$$ 

Noting that $\text{rk}(f_*\omega_{X'}) = 2$, we see that $\mathcal{L}_1$ is also a line bundle. Noting that $H^0(\mathbb{P}^1, \mathcal{L}_0) \cong H^0(\mathbb{P}^1, f_*\omega_{X'})$, we have $h^1(\mathbb{P}^1, \mathcal{L}_0) \geq h^0(\mathbb{P}^1, \mathcal{L}_1)$. Note that $\text{deg}(\mathcal{L}_0) = p_g(X) - 1 \geq 2$. We have $h^1(\mathbb{P}^1, \mathcal{L}_0) = 0$. So $h^0(\mathbb{P}^1, \mathcal{L}_1) = 0$. On the other hand, it’s well-known that $f_*\omega_{X'}/\mathbb{P}^1$ is semi-positive (see [11]). Thus $\text{deg}(\mathcal{L}_1 \otimes \omega_{\mathbb{P}^1}^{-1}) \geq 0$. This means $\text{deg}(\mathcal{L}_1) \geq -2$. Using the R-R, we may easily deduce that $h^1(\mathcal{L}_1) \leq 1$. So

$$h^1(\mathbb{P}^1, f_*\omega_{X'}) \leq 1.$$

So $h^2(\mathcal{O}_X) = h^1(\mathbb{P}^1, f_*\omega_{X'}) \leq 1$.

Claim 3.8. Keep the same assumption as in 3.7. Fix two smooth fibers $S_1$ and $S_2$ of $f$, the restriction map

$$H^0(X', K_{X'} + S_1 + S_2) \longrightarrow H^0(S, K_S)$$
is surjective.

Proof. Considering the exact sequence:

\[ 0 \to \mathcal{O}_{X'}(K_{X'} + S_1 - S) \to \mathcal{O}_{X'}(K_{X'} + S_1) \to \mathcal{O}_S(K_S) \to 0, \]
we have the long exact sequence

\[ \cdots \to H^0(X', K_{X'} + S_1) \to H^0(S, K_S) \xrightarrow{\delta_1} H^1(X', K_{X'}) \to \]
\[ \to H^1(X', K_{X'} + S_1) \to H^1(S, K_S) = 0, \]

If, for a general fiber \( S \), \( \alpha_1 \) is surjective, then we see that

\[ \dim \Phi_{K_{X'}+S_1}(S) = \dim \Phi_{K_S}(S) = 1 \text{ and } \dim \Phi_{K_{X'}+S_1}(X) = 2. \]

So \( \dim \Phi_{K_{X'}+S_1+S_2}(X) = 2 \). We are done. Otherwise, \( \alpha_1 \) is not surjective. Because \( h^2(O_{X'}) = h^1(X', K_{X'}) > 1 \).

Lemma 3.9. Keep the same assumption as in 3.7. Denote by \( |G| \) the movable part of \( |K_S| \). Pick up a general member \( C \in |G| \). Then

(i) \( \pi^*(K_X)|_S - \frac{8}{13}C \) is numerically equivalent to an effective \( \mathbb{Q} \)-divisor whenever \( p_g(X) \geq 4 \).

(ii) \( \pi^*(K_X)|_S - \frac{4}{13}C \) is numerically equivalent to an effective \( \mathbb{Q} \)-divisor whenever \( p_g(X) = 3 \).

Proof. We may assume that, on \( X' \), all movable parts of a finite number of linear systems are basepoint free. Denote by \( M_k \) the movable part of \( |kK_{X'}| \) for all \( k > 0 \). Denote by \( M_0 \) the movable part of \( |K_{X'} + 2S_1| \), where \( S_1 \) is a fixed smooth fiber of \( f \). Because \( \pi^*(K_X) \geq 2S_1 \), by Claim 3.8 we always have

\[ 2\pi^*(K_X)|_S \geq M_2|_S \geq M_0|_S \geq G \sim C, \]

where \( G \) is the movable part of \( |K_S| \) and \( C \in |G| \) is a general member. So \( C \) is a smooth curve of genus two according to \([4]\).

Case 1. \( p_g(X) \geq 4 \). We consider the sub-system

\[ |K_{X'} + 5(K_{X'} + 2S_1) + 2(K_{X'} + 2S_1) + S_1| \subset |13K_{X'}|. \]

Denote by \( M_{00} \) the movable part of \( |5(K_{X'} + 2S_1)| \). By \([5]\), we know already that \( \varphi_5 \) is generically finite. So \( M_{00} \) is nef and big. By the vanishing theorem, we have the surjective map

\[ H^0(X', K_{X'} + M_{00} + 2M_0 + S_1) \longrightarrow H^0(S, K_S + M_{00}|_S + 2M_0|_S). \]

Noting that \( M_{00}|_S \geq 5M_0|_S \), we see that \( M_{13}|_S \geq 8C \) by Lemma 2.7 of \([5]\), because \( |8C| \) is movable on \( S \). Thus \( 13\pi^*(K_X)|_S \geq 8C \).

Case 2. \( p_g(X) = 3 \). First, we have the surjective map by the vanishing theorem

\[ H^0(X', K_{X'} + \lceil 2\pi^*(K_X) \rceil + S_1) \longrightarrow H^0(S, K_S + \lceil 2\pi^*(K_X) \rceil |_S). \]
Denote by $M_{01}$ the movable part of $|3K_{X'}+S_1|$. Then we have $M_{01}|_S \geq 2C$ by Lemma 2.7 of [5]. We consider the sub-system

$$|K_{X'}+3(3K_{X'}+S_1)+S_1| \subset |12K_{X'}|.$$ 

Because $\varphi_9$ is generically finite, the movable part $M_{000}$ of $|3(3K_{X'}+S_1)|$ is nef and big. Thus we have the surjective map

$$H^0(K_{X'}+M_{000}+S_1) \longrightarrow H^0(S, K_S + M_{000}|_S).$$

Noting that $M_{000} \geq 3M_{01}$, we see that $12\pi^*(K_X)|_S \geq M_{12}|_S \geq 7C$. We are done. \hfill \square

**Theorem 3.10.** Let $X$ be a minimal 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $p_g(X) \geq 3$, $d = 1$, $B = \mathbb{P}^1$ and $S$ is a surface with invariants $(K_{S_0}^2, p_S(S)) = (1, 2)$, where $S_0$ is the minimal model of $S$. Then

(i) $\varphi_5$ is birational whenever $p_g(X) \geq 4$.

(ii) $\varphi_6$ is birational whenever $p_g(X) = 3$.

**Proof.** By Proposition 3.4, Theorem 2.6(i) is satisfied. From the claim of Proposition 5.3 of [5], we know that $\xi \geq \frac{2}{5}$.

If $p_g(X) \geq 4$, we may take $p = 3$ and $\beta = \frac{8}{13}$ by Lemma 3.9. Take $m_1 = 5$. Then $(5 - 1 - \frac{1}{p} - \frac{13}{8})\xi \geq \frac{49}{59} > 1$. Theorem 2.6 gives $\xi \geq \frac{4}{5}$. Take $m_2 = 6$. Then Theorem 2.6 gives $\xi \geq \frac{5}{6}$. \ldots Take $m_{46} = 50$. Theorem 2.6 gives $\xi \geq \frac{49}{59}$. Now $\alpha = (5 - 1 - \frac{1}{3} - \frac{13}{8})\xi \geq \frac{2431}{1200} > 2$. Theorem 2.6 implies that $\varphi_5$ is birational.

If $p_g(X) = 3$, we may take $p = 2$ and $\beta = \frac{7}{12}$ by Lemma 3.9. Take $m_1 = 6$. Then $(m_1 - 1 - \frac{1}{2} - \frac{13}{7})\xi \geq \frac{87}{70} \geq 1$. Theorem 2.6 gives $\xi \geq \frac{2}{3}$. Take $m_2 = 5$. Then $(5 - 1 - \frac{1}{2} - \frac{13}{7})\xi \geq \frac{59}{21} > 1$. Theorem 2.6 gives $\xi \geq \frac{4}{5}$. Now $\alpha = (6 - 1 - \frac{1}{2} - \frac{13}{7})\xi \geq \frac{78}{35} > 2$. Thus $\varphi_6$ is birational. \hfill \square

Theorems 3.1, 3.3, 3.5, 3.6 and 3.10 imply Theorem 1.2

4. Applications to the case $p_g = 2$

We present an application of our method to the case $p_g = 2$. Throughout this section, we always assume that $X$ is a minimal 3-fold of general type with $p_g = 2$. Naturally, $|K_X|$ is composed of a pencil of surfaces. We keep the same notations as in 2.5. So we have a derived fibration $f : X' \to B$.

4.1. Suppose $g(B) > 0$. By Theorem 3.3, $\varphi_5$ is birational. So we only need to study the situation with $g(B) = 0$.

4.2. Generalized birationality principle. We formulate a generalized birationality principle here in order to state the birationality of $\varphi_7$.

(GP). Let $W$ be a smooth projective variety. Suppose that we have a linear system $\Lambda$ which is not necessarily a complete one and that
we have a movable linear system \(|G|\) on \(W\). Denote by \(H\) a generic irreducible element of \(|G|\). Assume that

(i) either \(H\) is smooth or \(H\) is a curve (not necessarily smooth);

(ii) \(\Lambda\) separates different generic irreducible elements of \(|G|\);

(iii) \(\Lambda|_H\) gives a birational map onto its image.

Then \(\Lambda\) gives a birational map.

Now we consider the system \(|7K_{X'}|\). Let \(S\) be a general fiber of \(f\). Suppose \(g(B) = 0\). In order to prove the birationality of \(\varphi_7\), we have to study the system \(|7K_{X'}|\) which is not necessarily a complete linear system. On the surface \(S\), we always take a movable system \(|G|\) where either \(G := 2\sigma^*(K_{S_0})\) or \(G \leq 2\sigma^*(K_{S_0})\) and \(|G|\) is not composed of an irrational pencil of curves.

Because

\[
\mathcal{O}(1) \otimes f_*\omega_{X'/B}^2 \rightarrow f_*\omega_{X'}^7,
\]

so \(|G| \subset |7K_{X'}|\). This means that (ii) is satisfied. Note also that (i) is automatically satisfied. We may replace Theorem 2.6(i) by (GP) and verify Theorem 2.6(ii) and Theorem 2.6(iii) case by case as follows.

4.3. Finding \(\beta\). We have

\[
\mathcal{O}(1) \otimes f_*\omega_{X'/B}^4 \rightarrow f_*\omega_{X'}^{13}.
\]

Because \(|4\sigma^*(K_{S_0})|\) is base point free and by Lemma 2.7 of [5], we have \(M_{13}|_S \geq 4\sigma^*(K_{S_0})\). We still call for a good \(\pi\) such that a finite number of movable linear systems on \(X'\) are all base point free. Noting that \(M_{13}\) is nef and big, by the vanishing theorem, we have the surjective map

\[
H^0(X', K_{X'} + M_{13} + S) \rightarrow H^0(K_S + M_{13}|_S).
\]

Using Lemma 2.7 of [5] again, we have \(M_{15}|_S \geq 5\sigma^*(K_{S_0})\). Repeating this procedure, we may get, for all \(t > 0\),

\[
(2t + 13)\sigma^*(K_X)|_S \geq M_{2t+13}|_S \geq (t + 4)\sigma^*(K_{S_0}).
\]

This means, in fact, that we may choose a \(\beta\) such that \(\beta \mapsto \frac{1}{\beta}\) and that \(\pi^*(K_X)|_S - \beta\sigma^*(K_{S_0})\) is numerically equivalent to an effective \(\mathbb{Q}\)-divisor.

4.4. Suppose \((K_{S_0}^2, p_g(S)) \neq (1, 2)\) and \((2, 3)\). Set \(m = 7, p = 1\) and \(G := 2\sigma^*(K_{S_0})\). Then a general \(C \in |G|\) is a smooth non-hyperelliptic curve. By (4.3), we may take \(\beta \mapsto \frac{1}{\beta}\). So \(m - 1 - 1 - \frac{1}{\beta} > 0\). Thus Theorem 2.6 gives the birationality of \(\varphi_7\).

4.5. Suppose \((K_{S_0}^2, p_g(S)) = (2, 3)\). We set \(m = 7, p = 1\) and \(G\) to be the movable part of \(|K_S|\). Then \(|G|\) is not composed of a pencil and a general member \(C \in |G|\) is a curve of genus 3 and \(C^2 = 2\). By (4.3), we may take \(\beta \mapsto \frac{1}{\beta}\). Also we have \(\xi \geq \frac{1}{2}C^2 = 1\). Now \(\alpha = (m - 1 - 1 - \frac{1}{\beta})\xi > 2\). Thus \(\varphi_7\) is birational.
4.6. Suppose \((K^2_{S_0}, p_g(S)) = (1, 2)\). We set \(m = 8\), \(p = 1\) and \(G\) to be the movable part of \(|K_S|\). Then a general member \(C \in |G|\) is a smooth curve of genus 2. By (4.3), we may take \(\beta \mapsto \frac{1}{\beta}\). By the Claim of Proposition 5.3 of [5], we know \(\xi \geq 3\). Now \(\alpha = (m - 1 - 1 - \frac{1}{\beta})\xi \mapsto \frac{12}{5} > 2\). Thus \(\varphi_8\) is birational.

4.7. Summary and remarks. What we get is the birationality of \(\varphi_8\) for 3-folds of general type with \(p_g = 2\). Unfortunately, we don’t know whether this is optimal. We don’t have any supporting examples.

Acknowledgement

This note was written while I was visiting the Universität Essen in the winter of 2002, which is subject to the joint German-Chinese project "Komplexe Geometrie" supported by the DFG and the NSFC. I am very grateful for the hospitality as well as encouragements from both Professor Hélène Esnault and Professor Eckart Viehweg. This paper benefited a lot from my frequent discussions with Professor Viehweg whom I would like to thank for his generous helps. Especially, Professor Viehweg pointed out the fact (2.1.3) to me, which greatly simplifies the proof of Theorem 3.5. Finally I appreciate valuable comments from the referee who points out several unclear arguments in the proof of Theorem 2.6 in the original draft.

References

[1] W. Barth, C. Peter, A. Van de Ven, Compact complex surface, Springer-Verlag, 1984.
[2] E. Bombieri, Canonical models of surfaces of general type, Publications I.H.E.S. 42(1973), 171-219.
[3] F. Catanese, Surfaces with \(K^2 = p_g = 1\) and their period mapping, Springer Lecture Notes in Math. 732 (1979), 1-29.
[4] M. Chen, On pluricanonical maps for threefolds of general type, J. Math. Soc. Japan 50 (1998), 615-621.
[5] —–, Canonical stability in terms of singularity index for algebraic threefolds, Math. Proc. Camb. Phil. Soc. 131 (2001), 241-264.
[6] S. Chiaruttini, R. Gattazzo, Examples of birationality of pluricanonical maps, Rend. Sem. Mat. Univ. Padova 107 (2002), 81-94.
[7] C. Ciliberto, The bicanonical map for surfaces of general type, Proc. Symposia in Pure Math. 62 (1997), 57-83.
[8] L. Ein, R. Lazarsfeld, Global generation of pluricanonical and adjoint linear systems on smooth projective threefolds, J. Amer. Math. Soc. 6 (1993), 875-903.
[9] H. Esnault, E. Viehweg, Lectures on Vanishing Theorems. DMV-Seminar 20 (1992), Birkhäuser, Basel-Boston-Berlin.
[10] T. Fujita, On Kahler fiber spaces over curves, J. Math. Soc. Japan 30 (1978), 779-794.
[11] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, I, Ann. of Math. 79 (1964), 109-203, II, ibid., 205-326.
[12] B. Hunt, Complex manifold geography in dimension 2 and 3, J. Differential Geom. 30 (1989), 51-153.
[13] Y. Kawamata, *A generalization of Kodaira-Ramanujam’s vanishing theorem*, Math. Ann. 261(1982), 43-46.
[14] Y. Kawamata, K. Matsuda, K. Matsuki, *Introduction to the minimal model problem*, Adv. Stud. Pure Math. 10(1987), 283-360.
[15] J. Kollár, *Higher direct images of dualizing sheaves*, I, Ann. of Math. 123(1986), 11-42; II, ibid. 124(1986), 171-202.
[16] J. Kollár, S. Mori, Birational geometry of algebraic varieties, 1998, Cambridge Univ. Press.
[17] S. Lee, *Remarks on the pluricanonical and adjoint linear series on projective threefolds*, Commun. Algebra 27(1999), 4459-4476.
[18] T. Luo, *Plurigenera of regular threefolds*, Math. Z. 217(1994), 37-46.
[19] Y. Miyaoka, *The pseudo-effectivity of $3c_2 - c_1^2$ for varieties with numerically effective canonical classes*, Algebraic Geometry, Sendai, 1985. Adv. Stud. Pure Math. 10(1987), 449-476.
[20] F. Sakai, *Weil divisors on normal surfaces*, Duke Math. J. 51(1984), 877-887.
[21] V. Shokurov, *3-fold log flips*, Izv. Russ. A. N. Ser. Mat. 56(1992), 105-203.
[22] E. Viehweg, *Vanishing theorems*, J. reine angew. Math. 335(1982), 1-8.
[23] G. Xiao, *Linear bound for abelian automorphisms of varieties of general type*, J. reine angew. Math. 476(1996), 201-207.

DEPARTMENT OF APPLIED MATHEMATICS, TONGJI UNIVERSITY, SHANGHAI, 200092, PR CHINA
E-mail address: mchen@tongji.edu.cn