CONFINEMENT - DECONFINEMENT ORDER PARAMETERS

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Abstract. We study numerically the monopole creation operator proposed recently by Fröhlich and Marchetti. The operator is defined with the help of a three dimensional model which generates random Mandelstam strings. These strings imitate the Coulombic magnetic field around the monopole. We show that if the Mandelstam strings are condensed the creation operator discriminates between the phases with condensed and non-condensed monopoles in the Abelian Higgs model with the compact gauge field.

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1. Introduction

The order parameters are very important for the investigation of the phase transitions. In case of the temperature phase transition in lattice QCD with dynamical quarks there is no good definition of the order parameter up to now. The traditional order parameters like Polyakov line and string tension which work well in the quenched case are not valid in case of the full QCD. Even at zero temperature the string between quark and anti-quark can be broken by sea quarks.

We discuss below a quantity which may serve as the order parameter for full QCD if the monopole (or, "dual superconductor") confinement mechanism [1] is valid. In this picture the monopoles – defined with the help of an Abelian projection [2] – are supposed to be condensed in the confinement phase. The monopole condensate causes a dual analogue of the Abrikosov vortex to be formed between quarks and anti-quarks. As a result the quarks and anti–quarks are confined into the colorless states. In the deconfinement phase the monopoles are not condensed and quarks are not confined. Thus
the natural confinement-deconfinement order parameter is the value of the monopole condensate, which should be nonzero in the confinement phase and zero in the deconfinement phase.

There are two (formal) difficulties in the definition of the monopole condensate. At first, the monopole condensate is defined as an expectation value of the monopole field. However, the monopoles are the topological defects in the compact Abelian gauge field and, as a result, the immediate output of the lattice simulations is an information about the monopole trajectories. Then one can apply a known procedure which allows us to rewrite the path integral over monopole trajectories as an integral over the monopole fields. The expectation value of the latter defines the monopole condensate.

The next difficulty is the following. The expectation value of the (e.g., scalar) field $\phi$ should always be zero regardless whether this field is condensed or not. The reason is very simple: the path integral includes the integration over all possible gauges while the charged field is gauge dependent. These two problems were solved in paper [3], where the gauge invariant monopole creation operator for compact QED (cQED) was explicitly constructed. The numerical calculations in lattice cQED [4] and in the Maximal Abelian projection of lattice $SU(2)$ gluodynamics, Ref. [5], show that this operator provides the order parameter for confinement – deconfinement phase transition. Later the other forms of the monopole creation operators were constructed and investigated numerically [6] in lattice gluodynamics.

But recently it appears the claim [7] that the monopole creation operator suggested in [3] depends on the shape of the Dirac string if dynamical electrically charged fields are present in the considered theory. Just this situation appears in the Abelian projection of QCD: the off-diagonal gluons become electrically charged dynamical fields while the diagonal gluons become compact Abelian gauge fields which contain monopoles. The authors of [7] suggested a “new” monopole creation operator, which does not depend on the shape of the Dirac string even in the presence of the dynamical electric charges.

Below we study this new creation operator in the compact Abelian Higgs model, having in mind the future application of this operator for the full QCD.

In the next Section we give the explicit construction of the “new” and the “old” monopole creation operator. In Section 3 we present the results of the numerical calculations in the compact Abelian Higgs model.
2. “Old” and “new” monopole creation operators.

The gauge invariant creation operator $\Phi$ was suggested by Dirac [8]:

$$\Phi = \phi(x) \exp \left\{ i \int E_k(x \to \vec{y}) A_k(\vec{y}) d^3 y \right\},$$  \hspace{1cm} (1)

here $\phi(x)$ and $A_k(x)$ are the electrically charged field and gauge potential, which transform under the gauge transformations as

$$\phi(x) \to \phi(x) e^{i \alpha(x)} \; , \; \; A_k(x) \to A_k(x) + \partial_k \alpha(x).$$  \hspace{1cm} (2)

The Coulomb field, $E_k(x)$, satisfies the equation:

$$\partial_k E_k = \delta^{(3)}(x).$$  \hspace{1cm} (3)

It is easy to see that the operator $\Phi$, eq. (1), is invariant under the gauge transformations (2).

Now we describe the Fröhlich–Marchetti construction [3] of the monopole creation operator in cQED. At first step the partition function of cQED is transformed to a dual representation. For the general form of the cQED action it can be shown [5] that the dual theory is an Abelian Higgs model (AHM) in the limit when the Higgs boson mass and the gauge boson mass are infinite. In this theory the Higgs field, $\phi_x$, corresponds to the monopole in the original cQED. The gauge field $^*B$ is dual to the original gauge field $\theta$. Thus the gauge invariant creation operator (1) for the AHM model, corresponds to the monopole creation operator in the original cQED. The explicit expression for this operator on the lattice is (cf. eq.(2)):

$$\Phi_{x}^{\text{mon}} = \phi_x e^{i(^*B,^*H_x)},$$  \hspace{1cm} (4)

where $^*H_x$ is the Coulomb field of the monopole, $^*H_x = ^*\delta_x$, and $^*\delta_x$ is the discrete $\delta$–function defined on the dual lattice. Here and below we will use the differential form notations on the lattice: $(a, b) = \sum_c a_c b_c$ is the scalar product of the forms $a$ and $b$ defined on the $c$–sells; $(a, a) \equiv ||a||^2$ is the norm of the form $a$; $d$ is the forward derivative (an analog of the gradient); $\delta$ is the backward derivative (an analog of the divergence) and $\delta$–operation transfers a form to the dual lattice. For a description of the language of the differential forms on the lattice see, e.g., review [9].

Performing the inverse duality transformation for the expectation value of the creation operator (4) we get the expectation value, $\langle \Phi^{\text{mon}} \rangle$, of this operator in cQED. The explicit expression in the lattice notations is:

$$\langle \Phi^{\text{mon}} \rangle = \frac{1}{Z} \int_{-\pi}^{\pi} D\theta \exp\{-S(d\theta + W)\},$$

$$Z = \int_{-\pi}^{\pi} D\theta \exp\{-S(d\theta)\},$$  \hspace{1cm} (5)
here $d\theta$ is the plaquette angle, the lattice action is a periodic function: $S(d\theta + 2\pi n) = S(d\theta), \ n \in \mathbb{Z}$; $W = 2\pi\delta\Delta^{-1}(H_x - \omega_x)$ and $*\omega_x$ is the Dirac string which starts at the monopole: $\delta^*\omega_x = \delta_x$. The Dirac string $*\omega_x$ is defined on the dual lattice. The numerical investigation of this creation operator in cQED shows [4] that it can be used as the confinement–deconfinement order parameter.

The operator (4) is well defined for the theories without dynamical matter fields. However, if an electrically charged matter is added, then the creation operator (4) depends on the position of the Dirac string. To see this fact let us consider the compact Abelian Higgs model with the Villain form of the action:

$$Z_{AHM} = \int_{-\pi}^{\pi} D\theta \int_{-\pi}^{\pi} D\varphi \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-\beta||d\theta + 2\pi n||^2 - \gamma||d\varphi + q\theta + 2\pi l||^2}.$$  \hspace{1cm} (6)

Here $\theta$ is the compact Abelian gauge field and $\varphi$ is the phase of the dynamical Higgs field. The integer $q$ is the charge of the Higgs field. For the sake of simplicity we consider the London limit (the Higgs mass is infinitely large while the Higgs condensate is constant).

Let us perform the Berezinsky-Kosterlitz-Thouless (BKT) transformation [10] with respect to the compact gauge field $\theta$:

$$d\theta + 2\pi n = dA + 2\pi\delta\Delta^{-1}m[j] + 2\pi k.$$  \hspace{1cm} (7)

Here $A$ is the non–compact gauge field, $*m[j]$ is a surface on the dual lattice spanned on the monopole current $*j$ ($\delta^*m[j] = *j$), $\Delta$ is the lattice Laplacian and $k$ is the integer–valued vector form$^1$. We substitute eqs. (7) in eq. (6) and make the shift of the integer variable, $l \rightarrow l + qk$.

Next we perform the BKT transformation with respect to the compact scalar field $\varphi$:

$$d\varphi + 2\pi l = d\vartheta + 2\pi\delta\Delta^{-1}s[\sigma], \ \text{with} \ \vartheta = \varphi + 2\pi\delta\Delta^{-1}s[\sigma] + 2\pi p.$$  \hspace{1cm} (8)

Here $\vartheta$ is the non–compact scalar field, $*s[\sigma]$ is a 3D hyper–surface on the dual lattice spanned on the closed surface $*\sigma$ ($\delta^*s[\sigma] = *\sigma$) and $p$ is the integer–valued scalar form.

Substituting eqs. (7,8) into the partition function (6) and integrating the fields $A$ and $\varphi$ we get the representation of the compact AHM in terms of the monopoles and strings ("the BKT–representation"):

$$Z_{AHM} \propto Z_{BKT} = \sum_{*j \in \mathbb{Z}(c_3)} \sum_{*\sigma_j \in \mathbb{Z}(c_2)} \exp\left\{ -4\pi^2\beta \left(j, (\Delta + m^2)^{-1}j\right) \right\}$$  \hspace{1cm} (9)

$^1$A detailed description of the duality and BKT transformations in terms of the differential forms on the lattice can be found, e.g., in Ref. [9].
\[-4\pi^2 \gamma \left( \sigma_j, (\Delta + m^2)^{-1} \sigma_j \right) \],

where we have introduced the new dual surface variable \(*\sigma_j = *\sigma + q * m[j]*\) which is spanned \(q\)-times on the monopole current \(j\): \(\delta^* \sigma_j = q^* j\). The flux of the unit charged magnetic monopole can be taken out by \(q\) strings carrying the unit flux. The mass of the gauge boson \(\theta\) is \(m = q \sqrt{\gamma / \beta}\).

The BKT–representation (9) of the AHM partition function (6 ) can be also transformed to the dual representation using simple Gaussian integrations. We use two dual compact fields \(*B\) (vector field) and \(*\xi\) (scalar field) in order to represent the closeness properties of the currents \(*\sigma_j\) and \(*j\), respectively. We also introduce two dual non–compact fields, \(*F\) (vector field) and \(*G\) (rank-2 tensor field) in order to get a linear dependence, correspondingly, on the currents \(*\sigma_j\) and \(*j\) under the exponential function:

\[
Z_{BKT} = \text{const.} \int_{-\pi}^{\pi} D^*F \int_{-\pi}^{\pi} D^*G \int_{-\pi}^{\pi} D^*B \int_{-\pi}^{\pi} D^*\xi 
\sum_{*j \in \mathbb{Z}(\ast c_3)} \sum_{*\sigma_j \in \mathbb{Z}(\ast c_2)} \exp \left\{ - *\beta (G, (\Delta + m^2) G) - *\gamma (F, (\Delta + m^2) F) 
+ i(F, *\sigma_j) + i(G, *j) + i(B, \delta^* \sigma_j - q^* j) - i(\xi, \delta^* j) \right\},
\]

where

\[
*\beta = \frac{1}{16 \pi^2 \gamma}, \quad *\gamma = \frac{1}{16 \pi^2 \beta}.
\]

Note that in this representation the integer variables \(*\sigma_j\) and \(*j\) are no more restricted by the closeness relations. Therefore we can use the Poisson summation formula with respect to these variables and integrate out the fields \(*F\) and \(*G\). Finally, we obtain the dual field representation of the partition function (6):

\[
Z_{BKT} \propto Z_{\text{dual field}} = \int_{-\pi}^{\pi} D^*B \int_{-\pi}^{\pi} D^*\xi \sum_{*u \in \mathbb{Z}(\ast c_3)} \sum_{*v \in \mathbb{Z}(\ast c_2)} \exp \left\{ - *\beta \left( d^*B + 2\pi^* u, (\Delta + m^2) \left( d^*B + 2\pi^* u \right) \right) \right\}
\]

\[
- *\gamma \left( d^*\xi + q^* B + 2\pi^* v, (\Delta + m^2) \left( d^*\xi + q^* B + 2\pi^* v \right) \right) \right\},
\]

where \(*u\) and \(*v\) are the integer valued forms defined on the plaquettes and links of the dual lattice, respectively. Clearly, this is the dual Abelian Higgs model with the modified action. The gauge field \(*B\) is compact and the radial variable of the Higgs field is frozen. The model is in the London limit and the dynamical scalar variable is the phase of the Higgs field \(*\xi\).
Thus in the presence of the dynamical matter the dual gauge field $^*B$ becomes compact. The compactness of the dual gauge field implies that it is transforming under the gauge transformations of the following form:

$$^*B \to ^*B + d^*\alpha + 2\pi^*k,$$  \hspace{1cm} (13)

where the integer valued field $k$ is chosen in such a way that $^*B \in (-\pi, \pi]$.

One can easily check that the operator (4) is not invariant under these gauge transformations:

$$\Phi_{\text{mon}}^x \to \Phi_{\text{mon}}^x e^{2\pi i(^*k, ^*H_x)}.$$ \hspace{1cm} (14)

The invariance of the operator (4) under the gauge transformations (13) can be achieved if and only if the function $^*H_x$ is an integer–valued form.

Thus, if we take into account the Maxwell equation $\delta^*H_x = ^*\delta_x$, we find that $^*H_x$ should be a string attached to the monopole ("Mandelstam string"): $^*H_x \to ^*j_x, ^*j_x \in \mathbb{Z}$, $^*\delta_j = ^*\delta_x$. The string must belong to the three–dimensional time–slice. However, one can show [7] that for a fixed string position the operator $\Phi$ creates a state with an infinite energy. This difficulty may be bypassed [7] by summation over all possible positions of the Mandelstam strings with a measure $\mu(^*j)$:

$$\Phi_{\text{mon, new}}^x = \phi_x \sum_{^*j_x \in \mathbb{Z}} \mu(^*j_x) e^{i(^*B, ^*j_x)}.$$ \hspace{1cm} (15)

If Higgs field $\phi$ is $q$–charged ($q \in \mathbb{Z}$), the summation in eq.(15) should be taken over $q$ different strings with the unit flux.

An example of a “reasonable” measure $\mu(j_x)$ is [7]:

$$\mu(^*j_x) = \exp\left\{-\frac{1}{2\kappa}||^*j_x||^2\right\}.$$ \hspace{1cm} (16)

This measure corresponds to the dual formulation of the 3D XY–model with the Villain action:

$$S(\chi, r) = \frac{\kappa}{2}||d\chi - 2\pi B + 2\pi r||^2.$$ \hspace{1cm} (17)

Due to the compactness of the spin variables $\chi$ the model (17) possesses vortex defects which enter the XY–partition function with measure (16).

We thus defined “old” (4) and “new” (15) monopole creation operators.

\[\text{Another way to establish this fact is to realize that the pure compact gauge model is dual to the non–compact } U(1) \text{ with matter fields (referred above as the (dual) Abelian Higgs model). Reading this relation backwards one can conclude that the presence of the matter field leads to the compactification of the dual gauge field } ^*B.\]
3. Numerical results

Below we present results of the numerical simulation of the new monopole creation operator. We investigate it in the simplest model which contains both the monopoles and the electrically charged fields: the Abelian Higgs model with compact gauge field and with the potential on the Higgs field corresponding to the London limit. The partition function for this model is given in eq.(6). The model has a nontrivial phase structure and we study both the phase where monopoles are condensed and the phase where monopoles are not condensed.

First we substitute the monopole creation operator (15,16) into the dual representation of the compact AHM (12). Then we perform the transformations back to the original representation:

\[
\langle \Phi_{\text{mon},\text{new}} \rangle = \frac{1}{Z} \sum_{j \in \mathbb{Z} \times \mathbb{Z}} \int_{-\pi}^{\pi} D\theta \exp\left\{ -\frac{1}{2\kappa} ||d^* j||^2 - \beta \cos(d\theta + \frac{2\pi q}{q} j) - \gamma \cos(q \theta) \right\},
\]

where we used the Wilson form of the action which is more suitable for the numerical simulations. The current \( j \equiv *^{(3)} *^{(4)} j \) means that duality operation was first applied in the 3D time slice and then in the full 4D space. We have fixed the unitary gauge therefore the Higgs field was eaten up by the corresponding gauge transformation.

The value of the monopole order parameter, \( \langle \phi \rangle \), corresponds to the minimum of the (effective constraint) potential on the monopole field. This potential can be estimated as follows:

\[
V_{\text{eff}}(\Phi) = -\ln \left( \langle \delta(\Phi - \Phi_{\text{mon},\text{new}}) \rangle \right).
\]

We simulated the 4D Abelian Higgs model on the \( 4^4, 6^4, 8^4 \) lattices, for \( \gamma = 0.3 \). The larger the charge of the Higgs field, \( q \), the easier the numerical calculation of \( V_{\text{eff}}(\Phi) \) is. We performed our calculations for \( q = 7 \). For each configuration of 4D fields we simulated 3D model to get the ensemble of the Mandelstam strings with the weight \( \mu(j_x) \). We generated 60 statistically independent 4D field configurations, and for each of these configurations we generated 40 configurations of 3D Mandelstam strings. We imposed the anti-periodic boundary conditions in the 3D space (the single monopole charge can not exist in the finite volume with periodic boundary condition).

As we have stated above the weight function (16) corresponds to the 3D XY–model with the Villain action. This model has the phase transition at \( \kappa_c(B = 0) \approx 0.32 \) [11]. Our numerical observation has shown that in
presence of the external field $B$, the critical coupling constant gets shifted: \( \kappa_c^B \approx 0.42 \).

Two configurations of the Mandelstam strings which correspond to condensed (large $\kappa$) and non-condensed (small $\kappa$) phases of these strings are shown in Figures 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1}
\caption{Charged loops appearing in auxiliary theory for (a) non-condensed strings ($\kappa = 0.3$) and (b) condensed strings ($\kappa = 0.5$).}
\end{figure}

We can expect that the operator (15) plays the role of the order parameter in the $\kappa > \kappa_c$ phase, where the Mandelstam strings are condensed ($\kappa > \kappa_c$). In Figures 2 we present the effective potential (19) in the con-

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig2}
\caption{The effective monopole potential (19) in (a) confinement and (b) deconfinement phases.}
\end{figure}
finement ($\beta = 0.85$) and deconfinement ($\beta = 1.05$) phases. The potential is shown for two values of the $3D$ coupling constants $\kappa > \kappa_c$ corresponding to high densities of the Mandelstam strings. In the confinement phase, Figure 2(a), the potential $V(\Phi)$ has a Higgs form signaling the monopole condensation. According to our numerical observations this statement does not depend on the lattice volume. In the deconfinement phase, Figure 2(b), the potential has minimum at $\Phi = 0$ which indicates the absence of the monopole condensate.

![Figure 3. The effective monopole potential (19) in the low-\(\kappa\) region of the 3D model.](image)

For small values of the $3D$ coupling constant $\kappa$ (in the phase where Mandelstam strings $j_x$ are not condensed), we found (Figure 3) that the potential $V(\Phi)$ has the same behaviour for the both phases of $4D$ model. Thus the operator (15) serves as the order parameter for the deconfinement phase transition, if Mandelstam strings are condensed, i.e. $\kappa$ should be larger than $\kappa_c(B)$.

Summarizing, the new operator suggested in [7] can be used as a test of the monopole condensation in the theories with electrically charged matter fields. Our calculations indicate that the operator should be defined in the phase where the Mandelstam strings are condensed. The value of the monopole field which corresponds to the minimum of the effective potential, is zero in deconfinement phase and non zero in the confinement phase.
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