Quantitative Isoperimetry à la Levy-Gromov

F. CAVALLETTI
SISSA

F. MAGGI
University of Texas at Austin

AND

A. MONDINO
University of Warwick

Abstract

On a Riemannian manifold with a positive lower bound on the Ricci tensor, the distance of isoperimetric sets from geodesic balls is quantitatively controlled in terms of the gap between the isoperimetric profile of the manifold and that of a round sphere of suitable radius. The deficit between the diameters of the manifold and of the corresponding sphere is likewise bounded. These results are actually obtained in the more general context of (possibly nonsmooth) metric measure spaces with curvature-dimension conditions through a quantitative analysis of the transport rays decompositions obtained by the localization method. © 2018 Wiley Periodicals, Inc.

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1 Introduction

Comparison theorems are an important part of Riemannian geometry \([29, 30, 55]\). The typical result asserts that a complete Riemannian manifold with a pointwise curvature bound retains some geometric properties of the corresponding simply connected model space. We are interested here in the \textit{Levy-Gromov comparison theorem}, which states that, under a positive lower bound on the Ricci tensor, the isoperimetric profile of the manifold is bounded from below by the isoperimetric profile of the sphere. More precisely, define the isoperimetric profile of a smooth Riemannian manifold \((M, g)\) by

\[
I_{(M, g)}(v) = \inf \left\{ \frac{P(E)}{\text{vol}_g(M)} : \frac{\text{vol}_g(E)}{\text{vol}_g(M)} = v \right\}, \quad 0 < v < 1,
\]

where \(P(E)\) denotes the perimeter of a region \(E \subset M\). The Levy-Gromov comparison theorem states that, if \(\text{Ric}_g \geq (N - 1)g\), where \(N\) is the dimension of \((M, g)\), then

\[
I_{(M, g)}(v) \geq I_{(S^N, g_{S^N})}(v) \quad \forall v \in (0, 1),
\]

where \(g_{S^N}\) is the round metric on \(S^N\) with unit sectional curvature; moreover, if equality holds in (1.1) for some \(v \in (0, 1)\), then \((M, g)\) is isometric to \((S^N, g_{S^N})\).

Our main result is a quantitative estimate, in terms of the gap in the Levy-Gromov inequality, on the shape of isoperimetric sets in \((M, g)\). We show that isoperimetric sets are close to geodesic balls. Since the classes of isoperimetric sets and geodesic balls coincide in the model space \((S^N, g_{S^N})\), one can see our main result as a \textit{quantitative comparison theorem}. In detail, we show that if \(\text{Ric}_g \geq (N - 1)g\) and \(E \subset M\) is an isoperimetric set in \(M\) with \(\text{vol}_g(E) = v \text{vol}_g(M)\), then there exists \(x \in M\) such that

\[
\frac{\text{vol}_g\left( E \Delta B_{r_N}(v)(x) \right)}{\text{vol}_g(M)} \leq C(N, v) \left( I_{(M, g)}(v) - I_{(S^N, g_{S^N})}(v) \right)^{O(1/N)},
\]

where \(B_r(x)\) denotes the geodesic ball in \((M, g)\) with radius \(r\) and center \(x\), \(r_N(v)\) is the radius of a geodesic ball in \(S^N\) with volume \(v \text{vol}_{g_{S^N}}(S^N)\), and \(\Delta\) denotes the symmetric difference of sets. More generally, the same conclusion holds for every \(E \subset M\) with \(\text{vol}_g(E) = v \text{vol}_g(M)\), provided \(I_{(M, g)}(v)\) on the right-hand side of (1.2) is replaced by \(P(E)/\text{vol}_g(M)\).

We approach the proof of (1.2) from the synthetic point of view of metric measure geometry. For the sake of this introduction, a metric measure space is a triple \((X, d, m)\) where \((X, d)\) is a compact metric space and \(m\) is a Borel probability measure, playing the role of reference volume measure. Using optimal-transport techniques, Sturm [59, 60] and Lott-Villani [45] introduced the curvature-dimension condition \(\text{CD}(K, N)\); the rough geometric picture is that a \(\text{CD}(K, N)\) space should be thought of as a possibly nonsmooth metric measure space with Ricci curvature bounded below by \(K \in \mathbb{R}\) and dimension bounded above by \(N \in (1, \infty)\) in a synthetic sense. The basic idea of such a synthetic point of view is to analyze weighted
convexity properties of certain entropy functionals along geodesics in the space of probability measures endowed with the quadratic transportation distance.

A key technical assumption throughout the paper is the so-called essentially nonbranching property \[57\], which roughly amounts to requiring that the $L^2$-optimal transport between two absolutely continuous (with respect to the reference measure $\mu$) probability measures moves along a family of geodesics with no intersections, i.e., a nonbranching set of geodesics (for the precise definitions, see Section 2.1). Examples of essentially nonbranching spaces are Riemannian manifolds, Alexandrov spaces, Ricci limits, and more generally $\text{RCD}(K,N)$-spaces, Finsler manifolds endowed with a strongly convex norm; a standard example of a space failing to satisfy the essential nonbranching property is $\mathbb{R}^2$ endowed with the $L^1$-norm. At the end of the introduction, when discussing the main steps of the proof, we will mention where the essentially nonbranching property is used.

Our approach to establish (1.2) is to regard an $N$-dimensional Riemannian manifold $(M,g)$ with $\text{Ric}_g \geq (N-1)g$ as an essentially nonbranching metric measure space $(X,d,m)$ satisfying the $\text{CD}(N-1,N)$ curvature-dimension condition, "\text{e.n.b. CD}(N-1,N)$-space" for short. Considering this extension is natural, because the class of e.n.b. $\text{CD}(N-1,N)$-spaces contains measured Gromov-Hausdorff limits of $N$-dimensional Riemannian manifolds with Ricci tensor bounded from below by the constant $N-1$. And, in turn, a sequence of such Riemannian manifolds $(M_h,g_h)$ such that the right-hand side of (1.2) tends to 0 as $h \to \infty$ may develop singularities and admits a limit only in the measured Gromov-Hausdorff sense to an e.n.b. $\text{CD}(N-1,N)$-space.

In the enlarged class of e.n.b. $\text{CD}(N-1,N)$-spaces, round spheres are no longer the only equality cases in the Levy-Gromov comparison theorem, which instead coincide with the whole family of the so-called spherical suspensions. In addition, as proved in [25] and recalled in Theorem 2.5 below, the Levy-Gromov comparison theorem holds on essentially nonbranching metric measure spaces verifying the $\text{CD}(N-1,N)$ condition with any real number $N > 1$. In this general setting, the comparison isoperimetric profile is the one defined by the model space

$$
(1.3) \quad \left(0, \pi], |\cdot|, \frac{\sin^{N-1}(t)}{\omega_N} \mathcal{L}^1 \right) \quad \text{where } \omega_N = \int_0^\pi \sin^{N-1}(t) dt,
$$

and $|\cdot|$ denotes the euclidean distance on $\mathbb{R}$. Denoting by $\mathcal{I}_{N-1,N,\pi}$ the isoperimetric profile of this comparison model space (see (2.7)), we notice that

$$
\mathcal{I}_{N-1,N,\pi}(v) = \mathcal{I}_{(S^N, g_{S^N})}(v) \quad \forall v \in (0,1), \ \forall N \in \mathbb{N}, N \geq 2.
$$

With this notation in force, we state our main theorem.

**Theorem 1.1.** For every real number $N > 1$ and $v \in (0,1)$ there exists a real constant $C(N,v) > 0$ with the following property: If $(X,d,m)$ is an essentially nonbranching metric measure space satisfying the $\text{CD}(N-1,N)$ condition and
$\mathfrak{m}(X) = 1$ with $\text{supp}(\mathfrak{m}) = X$, then

\begin{equation}
(1.4) \quad \pi - \text{diam}(X) \leq C(N, v)(\mathcal{I}_{(X, d, \mathfrak{m})}(v) - \mathcal{I}_{N-1, N, \pi}(v))^{1/N}.
\end{equation}

Moreover, for every Borel set $E \subset X$ with $\mathfrak{m}(E) = v$ there exists $\bar{x} \in X$ such that

\begin{equation}
\mathfrak{m}(E \Delta B_{r_N}(\bar{x})) \leq C(N, v)\left(\mathcal{P}(E) - \mathcal{I}_{N-1, N, \pi}(\mathfrak{m}(E))\right)^\eta,
\end{equation}

where $r_N(v)$ is defined by

\begin{equation*}
\int_0^{r_N(v)} \sin^{N-1}(t) dt = v\omega_N.
\end{equation*}

Finally, if $(X, d)$ is isometric to a smooth Riemannian manifold $(M, g)$ (endowed with any measure $\mathfrak{m}$ such that the assumptions of the theorem hold), one can take $\eta = N/(N^2 + N - 1)$ in (1.5).

Let us first discuss the first claim (1.4) by recalling the celebrated Myers theorem [51]: if $(M, g)$ is an $N$-dimensional smooth Riemannian manifold with $\text{Ric}_g \geq (N - 1)g$, then $\text{diam}(M) \leq \pi$. A refined version of the Myers theorem involving the Levy-Gromov isoperimetric deficit very similar to (1.4) was established by Berard-Besson-Gallot [14], still in the framework of a smooth Riemannian manifold with $\text{Ric}_g \geq (N - 1)g$.

Theorem 1.1 shall be seen as a further step in two directions: first of all the space is not assumed to be smooth; secondly, not only is an estimate on the diameter of the space expressed in terms of the Levy-Gromov isoperimetric deficit but also a quantitative estimate on the closeness of the competitor subset to the metric ball is established in (1.5). Indeed, as the reader will realize, the claim (1.4) will be set along the way of proving the much harder (1.5).

Inequality (1.5) naturally fits in the context of quantitative isoperimetric inequalities. The basic result in this area is the improved euclidean isoperimetric theorem proved in [37], and stating that if $E \subset \mathbb{R}^n$ is a Borel set of positive and finite volume, then there exists $\bar{x} \in \mathbb{R}^n$ such that

\begin{equation}
|E \Delta B_{r_E}(\bar{x})| \leq C(N)\left(\frac{\mathcal{P}(E)}{\mathcal{P}(B_{r_E})} - 1\right)^{1/2},
\end{equation}

where $r_E$ is such that $|B_{r_E}| = |E|$; see also [32, 36]. A closer estimate to (1.5) is the improved spherical isoperimetric theorem from [18]: this result actually is (1.5) in the special case that $(X, d, \mathfrak{m}) = (S^n, g_{S^n})$ but with the sharp exponent $\eta = \frac{1}{2}$.

Taking variations in the broad context of metric measure spaces makes the prediction on the sharp exponent $\eta$ of (1.5) a hard task. Even formulating a conjecture is challenging, and at the present stage it could actually be that $\eta = O(1/N)$ as $N \to \infty$ is already sharp. In the direction of this guess, we notice that the exponent
$1/N$ in (1.4) is indeed optimal in the class of metric measure spaces, as a direct computation on the model one-dimensional space (1.3) shows.

Another question we do not address here is the explicit dependence of the constant $C(N, v)$ in Theorem 1.1 with respect to the parameters $N \in (1, \infty)$ and $v \in (0, 1)$; we just observe that, for a fixed $N \in (1, \infty)$ and a fixed $v_0 \in (0, \frac{1}{2}]$ one has $\sup_{v \in [v_0, 1]} C(N, v) < \infty$.

A challenging feature of Theorem 1.1 is that none of the three general methods to approach quantitative isoperimetry seems applicable in this context. This is evident for the approach in [37], based on symmetrization inequalities. The approach developed in [32] to address (1.6) and used in [18] to prove (1.5) with $D = 1$ in the case $(X, d, m) = (S^N, g_{S^N})$ has a vast domain of applicability. Essentially, the approach of [32] has a reasonable chance to work on every variational problem with a sufficiently smooth regularity theory and with strictly stable minimizers. (Depending on the problem, it may be quite nontrivial to implement one or both of these points.) In our context, of course, there are no regularity theories and no second variation formulae to be exploited.

Finally, the approach to (1.6), and more generally to the quantitative Wulff inequality, developed in [36] is based on the Gromov-Knothe proof of the (Wulff) isoperimetric inequality [44, 48]. But, at present, proving isoperimetry with the Gromov-Knothe argument beyond the case of euclidean spaces seems to be an open problem: for example, to the best of our knowledge, it is not known how to adapt the Gromov-Knothe argument for proving the isoperimetric theorem on, say, the sphere.

Before discussing the main steps of the proof of Theorem 1.1 it is worth including notable examples of spaces fitting in the assumptions of the result. Let us stress that our main theorem seems new in all of them. The class of essentially nonbranching $CD(N - 1, N)$-spaces includes many remarkable families of spaces, among them:

- **Measured Gromov-Hausdorff limits of Riemannian $N$-dimensional manifolds satisfying $\text{Ric}_g \geq (N - 1)g$ and more generally the class of $\text{RCD}(N - 1, N)$-spaces.** Indeed, measured Gromov-Hausdorff limits of Riemannian $N$-manifolds satisfying $\text{Ric}_g \geq (N - 1)g$ are examples of $\text{RCD}(N - 1, N)$-spaces (see, for instance, [39]) and $\text{RCD}(N - 1, N)$-spaces are essentially nonbranching $\text{CD}(N - 1, N)$ (see [57]).

- **Alexandrov spaces with curvature $\geq 1$.** Petrunin [56] proved that the lower curvature bound in the sense of comparison triangles is compatible with the optimal transport type lower bound on the Ricci curvature given by Lott-Sturm-Villani (see also [63]). Moreover, geodesics in Alexandrov spaces with curvature bounded below do not branch. It follows that Alexandrov spaces with curvature bounded from below by 1 are nonbranching $\text{CD}(N - 1, N)$-spaces.

- **Finsler manifolds where the norm on the tangent spaces is strongly convex, and which satisfy lower Ricci curvature bounds.** More precisely, we consider a
$C^\infty$-manifold $M$, endowed with a function $F: TM \to [0, \infty]$ such that $F|_{TM \setminus \{0\}}$ is $C^\infty$ and for each $p \in M$ it holds that $F_p := T_p M \to [0, \infty]$ is a strongly convex norm, i.e.,

$$g^{ij}_p(v) := \frac{\partial^2 (F^2_p)}{\partial v^i \partial v^j}(v)$$

is a positive definite matrix at every $v \in T_p M \setminus \{0\}$.

Under these conditions one can characterize geodesics as solutions to a system ODE and prove that no branching occurs: in other words, strongly convex Finsler manifolds are nonbranching. We also assume $(M, F)$ to be geodesically complete and endowed with a $C^\infty$ probability measure $m$ in such a way that the associated metric measure space $(m, m.s.) (X, F, m)$ satisfies the CD$(N - 1, N)$ condition. This class of spaces has been investigated by Ohta [53], who established the equivalence between the curvature dimension condition and a Finsler-version of Bakry-Emery $N$-Ricci tensor bounded from below.

We conclude this introduction by briefly illustrating the main steps in the proof of Theorem 1.1. The starting point of our approach is the metric-measured version of the classical localization technique. First introduced in the study of sharp Poincaré inequalities on convex domains by Payne and Weinberger [54], the localization technique has been developed into a general dimension reduction tool for geometric inequalities in symmetric spaces in the works of Gromov-Milman [41], Lovász-Simonovits [46], and Kannan-Lovász-Simonovits [42].

More recently, Klartag [43] bridged the localization technique with the Monge-Kantorovich optimal transportation problem, extending the range of applicability of the method to general Riemannian manifolds. The extension to the metric setting was finally obtained in [25]; see Section 2.4.

Given $E \subset X$, the localization theorem (Theorem 2.6) gives a decomposition of $X$ into a family of one-dimensional sets $\{X_q\}_{q \in Q}$ formed by the transport rays of a Kantorovich potential associated to the optimal transport of (the normalized restriction of $m$ to) $E$ into its complement in $X$; each $X_q$ is in particular isometric to a real interval. A first crucial property of such a decomposition is that each ray $X_q$ carries a natural measure $m_q$ (given by the disintegration theorem) in such a way that $(X_q, d, m_q)$ is a CD$(N - 1, N)$-space and $m_q(E \cap X_q) = m(E)$ so that both the geometry of the space and the constraint of the problem are localized into a family of one-dimensional spaces. A key ingredient used in the proof of such a decomposition is the essentially nonbranching property, which, coupled with CD$(K, N)$ (actually the weaker measure contraction would suffice here), guarantees that the rays form a partition of $X$ (up to an $m$-negligible set).

As a first step, we observe that most such rays are sufficiently long (Proposition 4.2). This shows the first part of Theorem 1.1, that is, estimate (1.4) (see Theorem 4.3).

A second crucial property of the decomposition $\{X_q\}_{q \in Q}$, inherited by the variational nature of the construction, is the so-called cyclical monotonicity. This is key to show that most of the transport rays $X_q$ have their starting point close to a “south
pole” \( \bar{x} \), and end up near a “north pole” \( \bar{y} \) (in particular, the distance between \( \bar{x} \) and \( \bar{y} \) is close to \( \pi \)) (Corollary 5.4). Then we observe that a one-dimensional version of Theorem 1.1 (see Section 6) forces most of the fibers \( E_q := E \cap X_q \) (that is, the intersection of \( E \) with the corresponding one-dimensional element of the partition) to be \( L^1 \) close to intervals centered either at the north pole or at the south pole of \( X_q \) (Lemma 6.2). To conclude the argument, a delicate step is to show that either most of the fibers \( E_q \) start from the south pole or most of them start from the north pole. In the smooth setting the proof can be obtained using a relative isoperimetric inequality. In our general framework we have to give a self-contained argument (to overcome the lack of convex neighborhoods) using an additional localization.

We conclude with a few additional remarks.

First, although Theorem 1.1 is formulated for \( CD(N - 1, N) \)-spaces, a statement for \( CD(K, N) \)-spaces with \( K > 0 \) is easily obtained by scaling. Indeed, \( (X, d, m) \) satisfies \( CD(K, N) \) if and only if, for any \( \alpha, \beta \in (0, \infty) \), the scaled metric measure space \( (X, \alpha d, \beta m) \) satisfies \( CD(\alpha^{-2}K, N) \); see [60, prop. 1.4].

Second, it would be interesting to understand quantitative isoperimetry in metric measure spaces in the regime \( N \to \infty \). The question is motivated by the validity of dimension-independent quantitative isoperimetric estimates on Gaussian spaces (see [11, 12, 31, 50] for a full account of this problem), and, of course, it is beyond the reach of Theorem 1.1 as the exponent \( \eta \) in (1.5) vanishes as \( N \to \infty \).

Third, we recall that in [25, cor. 1.6] the first and third author have proved the convergence to a spherical suspension (in the metric-measured Gromov-Hausdorff sense) of any sequence of spaces \( (X_i, d_i, m_i) \) satisfying the \( CD(N - 1, N) \) condition and such that \( I_{(X_i, d_i, m_i)}(\nu) \to I_{N-1,N,\pi}(\nu) \) for a fixed \( \nu \in (0, 1) \). It does not seem obvious, from this information alone, to deduce the convergence of isoperimetric regions \( E_i \) with \( m(E_i) = \nu m(X_i) \) to geodesic balls in \( X_i \) with radius \( r_N(\nu) \). Thus, (1.5) in Theorem 1.1 besides being a quantitative estimate, provides new information even without taking rates of convergence into account.

Fourth, in the smooth Riemannian case it is tempting to guess that using regularity theory one can bootstrap the \( L^1 \)-estimate (1.5) into a smooth \( C^k \)-estimate. We wish to stress that this seems not so trivial: indeed, for such an argument one would need a fixed Riemannian metric (which in this context would be the round metric on the sphere) or at least uniform \( C^k \)-estimates on the Riemannian metrics; this seems too much to hope for, as already proving a quantitative measured Gromov-Hausdorff estimate (which in turn is weaker than \( C^0 \)-closeness of the metrics) seems challenging.

Finally, we notice that the wide range of functional inequalities that can be proved via the localization technique (see, e.g., [24]) suggests a broad range of applicability for the constructions described in this paper.
2 Background Material

In this section we recall the main constructions needed in this paper. The reader familiar with curvature-dimension conditions and metric measure spaces will just need to check Sections 2.4 and 2.5 for the decomposition of $X$ into transport rays (localization), which is going to be used throughout the paper. In Section 2.1 we review geodesics in the Wasserstein distance, in Section 2.2 curvature-dimension conditions, and in Section 2.3 isoperimetric inequalities in the metric setting.

2.1 Geodesics in the $L^2$-Wasserstein Distance

A triple $(X, d, m)$ is a metric measure space (m.m.s.), if $(X, d)$ is a complete and separable metric space and $m$ a Borel nonnegative measure over $X$. We shall always assume that $m(X) = 1$. The space of all Borel probability measures over $X$ will be denoted by $\mathcal{P}(X)$, while $\mathcal{P}_2(X)$ stands for the space of probability measures with finite second moment. On the space $\mathcal{P}_2(X)$ we define the $L^2$-Wasserstein distance $W^2$ by setting, for $\mu_0, \mu_1 \in \mathcal{P}_2(X)$,

$$W^2(\mu_0, \mu_1)^2 = \inf_{\pi} \int_{X \times X} d^2(x, y)\pi(dx \, dy).$$

Here the infimum is taken over all $\pi \in \mathcal{P}(X \times X)$ with $\mu_0$ and $\mu_1$ as the first and the second marginal, i.e., $(P_1)_\# \pi = \mu_0, (P_2)_\# \pi = \mu_1$. Of course, $P_i, i = 1, 2$, is the projection on the first (resp., second) factor and $(P_i)_\#$ denotes the corresponding push-forward map on measures. As $(X, d)$ is complete, also $(\mathcal{P}_2(X), W^2)$ is complete.

Denote the space of geodesics of $(X, d)$ by $\text{Geo}(X) := \{\gamma \in C([0, 1], X) : d(\gamma_s, \gamma_t) = |s - t|d(\gamma_0, \gamma_1) \text{ for every } s, t \in [0, 1]\}.

Recall that a metric space is a geodesic space if and only if for each $x, y \in X$ there exists $\gamma \in \text{Geo}(X)$ so that $\gamma_0 = x, \gamma_1 = y$. A basic fact on the $L^2$-Wasserstein distance is that if $(X, d)$ is geodesic, then $(\mathcal{P}_2(X), W^2)$ is geodesic. Any geodesic $(\mu_t)_{t \in [0, 1]}$ in $(\mathcal{P}_2(X), W^2)$ can be lifted to a measure $\nu \in \mathcal{P}(\text{Geo}(X))$, so that $(e_t)_\# \nu = \mu_t$ for all $t \in [0, 1]$. Here for any $t \in [0, 1], e_t$ denotes the evaluation map:

$$e_t : \text{Geo}(X) \to X, \quad e_t(\gamma) := \gamma_t.$$ 

Given $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, we denote by OptGeo$(\mu_0, \mu_1)$ the space of all $\nu \in \mathcal{P}(\text{Geo}(X))$ for which $(e_0, e_1)_\# \nu$ realizes the minimum in (2.1). If $(X, d)$ is geodesic, then the set OptGeo$(\mu_0, \mu_1)$ is nonempty for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$.

A set $F \subset \text{Geo}(X)$ is a set of nonbranching geodesics if and only if for any $\gamma^1, \gamma^2 \in F$, the following holds:

$$\exists \overline{t} \in (0, 1) \text{ such that } \forall t \in [0, \overline{t}] \gamma^1_t = \gamma^2_t \quad \iff \quad \gamma^1_s = \gamma^2_s \quad \forall s \in [0, 1].$$

(Recall that a measure $\nu$ on a measurable space $(\Omega, \mathcal{F})$ is said to be concentrated on $A \subset \Omega$ if $\exists B \subset A$ with $B \in \mathcal{F}$ so that $\nu(\Omega \setminus B) = 0.$) With this terminology, we recall from [57] the following definition.
DEFINITION 2.1. A metric measure space \((X, d, m)\) is essentially nonbranching if and only if for any \(\mu_0, \mu_1 \in P_2(X)\), with \(\mu_0, \mu_1\) absolutely continuous with respect to \(m\), any element of \(\text{OptGeo}(\mu_0, \mu_1)\) is concentrated on a set of nonbranching geodesics.

2.2 Curvature-Dimension Conditions for Metric-Measure Spaces

The \(L^2\)-transport structure just described allows us to formulate a generalized notion of Ricci curvature lower bound coupled with a dimension upper bound in the context of metric measure spaces. This is the \(\text{CD}(K, N)\) condition introduced in the seminal works of Sturm [59,60] and Lott-Villani [45], which here is reviewed only for an m.m.s. \((X, d, m)\) with \(m \in P(X)\) and for \(K > 0\) and \(1 < N < \infty\) (the basic setting of the present paper).

For \(N \in (1, \infty)\), the \(N\)-Rényi relative-entropy functional \(E_N : P(X) \to [0, 1]\) is defined as

\[
E_N(\mu) := \int \rho^{1-\frac{1}{N}} \, dm,
\]

where \(\mu = \rho m + \mu^\text{sing}\) is the Lebesgue decomposition of \(\mu\) with \(\mu^\text{sing} \perp m\).

DEFINITION 2.2 (\(\tau_{K,N}\)-coefficients). Given \(K \in (0, \infty)\), \(N \in (1, \infty)\), and \(t \in [0, 1]\), define \(\sigma_{K,N}(t) : [0, \infty) \to [0, \infty]\) by setting

\[
\sigma_{K,N}(0) = t, \quad \sigma_{K,N}(\theta) := \frac{\sin(t \theta \sqrt{K/N})}{\sin(\theta \sqrt{K/N})}, \quad 0 < \theta < \frac{\pi}{\sqrt{K/N}},
\]

and \(\sigma_{K,N}(\theta) = +\infty\) otherwise; and define

\[
\tau_{K,N}(\theta) := t^{1/N} \sigma_{K,N-1}(\theta)^{1-1/N}.
\]

DEFINITION 2.3 (\(\text{CD}(K, N)\)). An m.m.s. \((X, d, m)\) is said to satisfy \(\text{CD}(K, N)\) if for all \(\mu_0, \mu_1 \in P_2(X, d, m)\), there exists \(v \in \text{OptGeo}(\mu_0, \mu_1)\) so that for all \(t \in [0, 1]\), \(\mu_t := (e_t)_\# v < m\), and for all \(N' \geq N\):

\[
E_{N'}(\mu_t) \geq \int_{X \times X} \left( \tau_{K,N'}(d(x_0, x_1)) \rho_0^{1/N'}(x_0) + \tau_{K,N'}(d(x_0, x_1)) \rho_1^{1/N'}(x_1) \right) \pi(dx_0, dx_1),
\]

where \(\pi = (e_0, e_1)_\# (v)\) and \(\mu_i = \rho_i m, i = 0, 1\).

If \((X, d, m)\) verifies the \(\text{CD}(K, N)\) condition, then the same is valid for \((\text{supp}[m], d, m)\); hence we directly assume \(X = \text{supp}[m]\).

The following pointwise density inequality is a known equivalent definition of \(\text{CD}(K, N)\) on essentially nonbranching spaces (the equivalence follows from [23]; see also [60, prop. 4.2]).

DEFINITION 2.4 (\(\text{CD}(K, N)\) for essentially nonbranching spaces). An essentially nonbranching m.m.s. \((X, d, m)\) satisfies \(\text{CD}(K, N)\) if and only if for all \(\mu_0, \mu_1 \in P_2(X, d, m)\),
\( \mathcal{P}_2(\mathbb{X}, \mathbb{d}, \mathbb{m}) \) there exists a unique \( \nu \in \text{OptGeo}(\mu_0, \mu_1) \), \( \nu \) is induced by a map (i.e., \( \nu = S_\#(\mu_0) \)) for some map \( S : \mathbb{X} \to \text{Geo}(\mathbb{X}) \)), \( \mu_t := (e_t)_\# \nu \ll \mathbb{m} \) for all \( t \in [0, 1] \), and writing \( \mu_t = \rho_t \mathbb{m} \), we have for all \( t \in [0, 1] \):

\[
\rho_t^{-1/N}(\gamma_t) \geq \tau^{(1-t)}_{K,N}(d(\gamma_0, \gamma_1))\rho_0^{-1/N}(\gamma_0) + \tau^{(t)}_{K,N}(d(\gamma_0, \gamma_1))\rho_1^{-1/N}(\gamma_1) \quad \text{for } \text{v-a.e. } \gamma \in \text{Geo}(\mathbb{X}).
\]

For the general definition of \( \text{CD}(K, N) \), see [45, 59, 60]. It is worth recalling that if \( (\mathbb{M}, g) \) is a Riemannian manifold of dimension \( n \) and \( h > 0 \), then the m.m.s. \( (\mathbb{M}, d_g, h \text{ vol}) \) verifies \( \text{CD}(K, N) \) with \( N \geq n \) if and only if (see theorem 1.7 of [60]):

\[
\text{Ric}_{g,h,N} \geq Kg, \quad \text{Ric}_{g,h,N} := \text{Ric}_g - (N - n)\frac{\nabla^2_g h \frac{1}{n-1}}{h \frac{1}{n-1}};
\]

in other words, if and only if the weighted Riemannian manifold \((\mathbb{M}, g, h \text{ vol})\) has a Bakry-Émery Ricci tensor bounded below by \( K \). Note that if \( N = n \) the Bakry-Émery Ricci tensor \( \text{Ric}_{g,h,N} = \text{Ric}_g \) makes sense only if \( h \) is constant.

We will also use several times the following terminology: a nonnegative function \( h \) defined on an interval \( I \subset \mathbb{R} \) is called a \( \text{CD}(K, N) \) density on \( I \), for \( K \in \mathbb{R} \) and \( N \in (1, \infty) \), if for all \( x_0, x_1 \in I \) and \( t \in [0, 1] \):

\[
h(tx_1 + (1-t)x_0)^{\frac{1}{n-1}} \geq \sigma_{K,N-1}^{(1-t)}(|x_1 - x_0|)h(x_1)^{\frac{1}{n-1}} + \sigma_{K,N-1}^{(t)}(|x_1 - x_0|)h(x_0)^{\frac{1}{n-1}},
\]

(recalling the coefficients \( \sigma \) from Definition 2.2).

The link with the definition of \( \text{CD}(K, N) \) for m.m.s. can be summarized as follows [22, theorem A.2]: if \( h \) is a \( \text{CD}(K, N) \) density on an interval \( I \subset \mathbb{R} \), then the m.m.s. \((I, | \cdot |, h(t)dt)\) verifies \( \text{CD}(K, N) \); conversely, if the m.m.s. \((\mathbb{R}, | \cdot |, \mu)\) verifies \( \text{CD}(K, N) \) and \( I = \text{supp}(\mu) \) is not a point, then \( \mu \ll L^1 \) and there exists a representant of the density \( h = d\mu/dL^1 \) that is a \( \text{CD}(K, N) \) density on \( I \).

In particular, if \( I \subset \mathbb{R} \) is any interval, \( h \in C^2(I) \), the m.m.s. \((I, | \cdot |, h(t)dt)\) verifies \( \text{CD}(K, N) \) if and only if

\[
(h^{1/N-1})'' + \frac{K}{N-1}h^{1/N-1} \leq 0;
\]

see also Appendix [A] for further properties of \( \text{CD}(K, N) \) densities.

The lack of the local-to-global property of the \( \text{CD}(K, N) \) condition (for \( K/N \neq 0 \)) led Bacher and Sturm in 2010 to introduce in [10] the reduced curvature-dimension condition, denoted by \( \text{CD}^*(K, N) \). The \( \text{CD}^*(K, N) \) condition asks for the same inequality (2.2) of \( \text{CD}(K, N) \) to hold but the coefficients \( \tau_{K,N}^{(s)}(d(\gamma_0, \gamma_1)) \) are replaced by the slightly smaller \( \sigma_{K,N}^{(s)}(d(\gamma_0, \gamma_1)) \).
A subsequent breakthrough in the theory was obtained with the introduction of the Riemannian curvature dimension condition $\text{RCD}^*(K,N)$: in the infinite-dimensional case, $N = \infty$ was introduced in [7] for finite measures $\mu$ and in [6] for $\sigma$-finite ones. The class $\text{RCD}^*(K,N)$ with $N < \infty$ (technically more involved) has been proposed in [8] and extensively investigated in [8, 33]. We refer to these papers and references therein for a general account on the synthetic formulation of the latter Riemannian-type Ricci curvature lower bounds. Here we only briefly recall that it is a stable strengthening of the reduced curvature-dimension condition: an m.m.s. verifies $\text{RCD}^*(K,N)$ if and only if it satisfies $\text{CD}^*(K,N)$ and is infinitesimally Hilbertian [38, def. 4.19 and prop. 4.22], meaning that the Sobolev space $W^{1,2}(X,\mu)$ is a Hilbert space (with the Hilbert structure induced by the Cheeger energy).

To conclude we recall also that the first author together with E. Milman in [22] recently proved the equivalence of $\text{CD}(K,N)$ and $\text{CD}^*(K,N)$ (and also of the $\text{CD}^e(K,N)$ and $\text{CD}^1(K,N)$), together with the local-to-global property for $\text{CD}(K,N)$, in the framework of essentially nonbranching m.m.s. having $\text{m}(X) < \infty$. As we will always assume the aforementioned properties to be satisfied by our ambient m.m.s. $(X,d,\mu)$, we will use both formulations with no distinction. It is also worth mentioning that an m.m.s. verifying $\text{RCD}^*(K,N)$ is essentially nonbranching (see [57, cor. 1.2]), implying also the equivalence of $\text{RCD}^*(K,N)$ and $\text{RCD}(K,N)$ (see [22] for details).

We shall always assume that the m.m.s. $(X,d,\mu)$ is essentially nonbranching and satisfies $\text{CD}(K,N)$ from some $K > 0$ with $\text{supp}(\mu) = X$. It follows that $(X,d)$ is a geodesic and compact metric space.

2.3 Isoperimetric Inequality for Metric Measure Spaces

In [25] the Lévy-Gromov-Milman isoperimetric inequality has been obtained for an essentially nonbranching m.m.s. $(X,d,\mu)$ verifying $\text{CD}(K,N)$ with $\text{m}(X) = 1$; also the rigidity statement has been obtained in the smaller class of $\text{RCD}(K,N)$-spaces. What follows is a short overview of the statements as obtained in the subsequent publication [27], where the results of [25] are obtained replacing the outer Minkowski content with the perimeter functional; see also [5] for the general relation between the outer Minkowski content with the perimeter functional.

Denote by $\text{Lip}(X)$ the space of real-valued Lipschitz functions over $X$. Given $u \in \text{Lip}(X)$, its slope $|\nabla u|(x)$ at $x \in X$ is defined by

\[
|\nabla u|(x) := \lim_{y \to x} \frac{|u(x) - u(y)|}{d(x,y)}.
\]

Following [1,2,49] and the more recent [4], given a Borel subset $E \subset X$ and $A$ open, the perimeter of $E$ relative to $A$ is denoted by $P(E,A)$ and is defined as
follows

\begin{equation}
\mathcal{P}(E, A) := \inf \left\{ \liminf_{n \to \infty} \int_A |\nabla u_n| \, m : u_n \in \text{Lip}(A), u_n \to \chi_E \text{ in } L^1(A, m) \right\}.
\end{equation}

We say that \( E \subset X \) has finite perimeter in \( X \) if \( \mathcal{P}(E, X) < \infty \). We also recall few properties of the perimeter functions:

(a) (locality) \( \mathcal{P}(E, A) = \mathcal{P}(F, A) \) whenever \( m((E \Delta F) \cap A) = 0 \);

(b) (l.s.c.) the map \( E \mapsto \mathcal{P}(E, A) \) is lower-semicontinuous with respect to the \( L^1_{\text{loc}}(A) \) convergence;

(c) (complementation) \( \mathcal{P}(E, A) = \mathcal{P}(E^c, A) \).

Moreover, if \( E \) is a set of finite perimeter, then the set function \( A \mapsto \mathcal{P}(E, A) \) is the restriction to open sets of a finite Borel measure \( \mathcal{P}(E, \cdot) \) in \( X \) (see lemma 5.2 of \[4\]), defined by

\[ \mathcal{P}(E, B) := \inf \{\mathcal{P}(E, A) : A \supseteq B, A \text{ open}\} . \]

Sometimes, for ease of notation, we will write \( \mathcal{P}(E) \) instead of \( \mathcal{P}(E, X) \).

The isoperimetric profile function of \((X, d, m)\), denoted by \( \mathcal{I}_{(X, d, m)} \), is defined as the pointwise maximal function so that \( \mathcal{P}(A) \geq \mathcal{I}_{(X, d, m)}(m(A)) \) for every Borel set \( A \subset X \), that is,

\begin{equation}
\mathcal{I}_{(X, d, m)}(v) := \inf \{\mathcal{P}(A) : A \subset X \text{ Borel}, m(A) = v\}.
\end{equation}

**Theorem 2.5 (Lévy-Gromov-Milman in \( \text{CD}(K,N) \)-spaces, \[25,27\]).** Let \((X, d, m)\) be an essentially nonbranching metric measure space with \( m(X) = 1 \) and having diameter \( D \in (0, +\infty) \). Assume it satisfies the \( \text{CD}(K,N) \) condition for some \( K \in \mathbb{R}, N \in (1, \infty) \). Then for every Borel set \( E \subset X \) it holds that

\[ \mathcal{P}(E) \geq \mathcal{I}_{K,N,D}(m(E)), \]

where \( \mathcal{I}_{K,N,D} \) are the model isoperimetric profile functions obtained in \[47\], i.e., \( \mathcal{I}_{(X, d, m)}(v) \geq \mathcal{I}_{K,N,D}(v) \) for every \( v \in [0, 1] \).

If \((X, d, m)\) satisfies \( \text{RCD}(N-1,N) \) for some \( N \in [2, \infty) \) and there exists \( \bar{v} \in (0, 1) \) such that \( \mathcal{I}_{(X, d, m)}(\bar{v}) = \mathcal{I}_{N-1,N,\infty}(\bar{v}) \), then \((X, d, m)\) is a spherical suspension: there exists an \( \text{RCD}(N-2,N-1) \) space \((Y, d_Y, m_Y)\) with \( m_Y(Y) = 1 \) such that \( X \) is isomorphic as a metric measure space to \([0, \pi] \times_{\sin^{-1}}^N Y\).

As reported above, the model spaces for general \( K, N \) were discovered by E. Milman \[47\], who extended the Lévy-Gromov isoperimetric inequality to smooth manifolds with densities, i.e., a smooth Riemannian manifold whose volume measure has been multiplied by a smooth nonnegative integrable density function. Milman detected a model isoperimetric profile \( \mathcal{I}_{K,N,D} \) such that if a Riemannian manifold with density has diameter at most \( D > 0 \), generalized Ricci curvature at least \( K \in \mathbb{R} \), and generalized dimension at most \( N \geq 1 \), then the isoperimetric profile function of the weighted manifold is bounded below by \( \mathcal{I}_{K,N,D} \).
In this paper, we will make extensive use of $\mathcal{I}_{K,N,D}$, at least in the case $K > 0$; we now therefore review their definitions (and refer to [47] for all the details; see in particular theorem 1.2 and corollary A.3):

**Case 1.** $K > 0$ and $D < \sqrt{\frac{N-1}{K}} \pi$.

$$\mathcal{I}_{K,N,D}(v) = \min_{\xi \in \left[0, \sqrt{\frac{N-1}{K}} \pi - D\right]} \mathcal{I}_{\left[\xi, \xi + D\right], \sin\left(\frac{K}{N-1}t\right)^{N-1}}(v) \quad \forall v \in [0, 1].$$

**Case 2.** $K > 0$ and $D \geq \sqrt{\frac{N-1}{K}} \pi$.

$$\mathcal{I}_{K,N,D}(v) = \mathcal{I}_{\left[0, \sqrt{\frac{N-1}{K}} \pi \right], \sin\left(\frac{K}{N-1}t\right)^{N-1}}(v) \quad \forall v \in [0, 1].$$

In both cases we have used the following notation: given $f$ on a closed interval $L \subset \mathbb{R}$, we denote by $\mu_{f,L}$ the probability measure supported in $L$ with density (with respect to the Lebesgue measure) proportional to $f$ there and $\mathcal{I}_{L,f}$ stands for $\mathcal{I}_{(L,|f|,\mu_{f,L})}$. Note that when $N$ is an integer,

$$\mathcal{I}_{\left[0, \sqrt{\frac{N-1}{K}} \pi \right], \sin\left(\frac{K}{N-1}t\right)^{N-1}}(v) = \mathcal{I}_{\left[0, \sqrt{\frac{N-1}{K}} \pi \right], \sin\left(\frac{K}{N-1}t\right)^{N-1}}(v)$$

by the isoperimetric inequality on the sphere, and so Case 2 with $N$ integer corresponds to the Lévy-Gromov isoperimetric inequality. In order to keep the notation concise, we will often write $\mathcal{I}_{D}$ in place of $\mathcal{I}_{N-1,1,D}$.

### 2.4 Localization

Theorem 2.5 has been proved obtaining a dimensional reduction of the isoperimetric inequality via the so-called localization theorem, proved for essentially non-branching metric measure spaces verifying the $\text{CD}(K, N)$ condition.

The localization theorem has its roots in a work by Payne-Weinberger [54] and has been developed by Gromov-Milman [41], Lovász-Simonovits [46], and Kannan-Lovász-Simonovits [42]; it consists of reducing an $n$-dimensional problem via tools of convex geometry to lower-dimensional problems that one can handle. In the previous papers the symmetric properties of the spaces were necessary to obtain such a dimensional reduction. In the recent paper [43], Klartag found a bridge between $L^1$-optimal transportation problems and the localization technique yielding the localization theorem in the framework of smooth Riemannian manifolds. Inspired by this approach, the first and the third author in [25] proved the following localization theorem for essentially nonbranching metric measure spaces verifying the $\text{CD}(K, N)$ condition.

**Theorem 2.6 ([25]).** Let $(X, d, m)$ be an essentially nonbranching metric measure space verifying the $\text{CD}(K, N)$ condition for some $K \in \mathbb{R}$ and $N \in [1, \infty)$. Let $f : X \rightarrow \mathbb{R}$ be $m$-integrable such that $\int_X f \, m = 0$ and assume the existence of $x_0 \in X$ such that $\int_X |f(x)| d(x, x_0) m(dx) < \infty.$
Then the space $X$ can be written as the disjoint union of two sets $Z$ and $T$ with $T$ admitting a partition \{$X_q\}_{q \in Q}$ and a corresponding disintegration of $m_{\mathcal{T}}$, $\{m_q\}_{q \in Q}$ such that we have the following:

- For any $m$-measurable set $B \subset \mathcal{T}$ it holds that
  \[ m(B) = \int_Q m_q(B)q(dq), \]
  where $q$ is a probability measure over $Q$ defined on the quotient $\sigma$-algebra $\mathcal{Q}$.
- For $q$-almost every $q \in Q$, the set $X_q$ is a geodesic and $m_q$ is supported on it. Moreover, $q \mapsto m_q$ is a CD($K$, $N$) disintegration.
- For $q$-almost every $q \in Q$, it holds $\int_{X_q} f m_q = 0$ and $f = 0$ $m$-a.e. in $Z$.

We refer to Appendix B for the disintegration theorem and its link with partitions of the space. Here we only mention that saying that $q \mapsto m_q$ is a CD($K$, $N$) disintegration means that for $q$-a.e. $q \in Q$, $m_q = h_q \mathcal{H}^1_{X_q}$, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure and $h_q \circ X_q$ is a CD($K$, $N$) density in the sense of (2.3): here, with a slight abuse of notation, $X_q$ also denotes the map with image $X_q$.

In the next section we recall all the needed terminology and objects from the theory of $L^1$-optimal transportation used to obtain Theorem 2.6. This will also serve as basis for establishing the main result of the present paper, i.e., a quantitative isoperimetric inequality.

### 2.5 $L^1$ Optimal Transportation

In this section we recall only some facts from the theory of $L^1$ optimal transportation that are of some interest for this paper; we refer to \cite{3, 9, 16, 21, 22, 34, 35, 43, 61} and references therein for more details on the theory of $L^1$ optimal transportation.

Following the approach of \cite{43}, Theorem 2.6 has been proven in \cite{25} by studying the following optimal transportation problem: define $\mu_0 := f^+ m$ and $\mu_1 := f^- m$, where $f^\pm$ denote the positive and the negative part of $f$, respectively, and study the $L^1$-optimal transport problem associated with it,

\[
\inf \left\{ \int_{X \times X} d(x, y)\pi (dx \, dy) : \pi \in \mathcal{P}(X \times X), (P_1)_\# \pi = \mu_0, (P_2)_\# \pi = \mu_1 \right\},
\]

where $P_i$ denotes the projection onto the $i^{th}$ component. Then the relevant object to study is given by the dual formulation of the previous minimization problem. By the summability properties of $f$ (see the hypothesis of Theorem 2.6), there exists a 1-Lipschitz function $\varphi: X \to \mathbb{R}$ such that $\pi$ is a minimizer in (2.8) if and only if $\pi(\Gamma) = 1$, where

\[
\Gamma := \{(x, y) \in X \times X : \varphi(x) - \varphi(y) = d(x, y)\}.
\]
is the naturally associated $d$-cyclically monotone set, i.e., for any $(x_1, y_1), \ldots, (x_n, y_n) \in \Gamma$ it holds that
\[
\sum_{i=1}^{n} d(x_i, y_i) \leq \sum_{i=1}^{n} d(x_i, y_{i+1}), \quad y_{n+1} = y_1,
\]
for any $n \in \mathbb{N}$. The set $\Gamma$ induces a partial order relation whose maximal chains produce a partition made of one-dimensional sets of a certain subset of the space, provided the ambient space $X$ verifies some mild regularity properties.

We now review how to obtain the partition from $\Gamma$; this procedure has already been presented and used in several contributions [9, 16, 35, 43, 61] when the ambient space is the euclidean space, a manifold, or a nonbranching metric space (see [16, 19] for extended metric spaces); the analysis in our framework started with [21] and has been refined and extended in [22]; we will follow the notation of [22] to which we refer for more details.

The transport relation $R$ and the transport set $T$ are defined as
\[
(2.9) \quad R := \Gamma \cup \Gamma^{-1} = \{ |\varphi(x) - \varphi(y)| = d(x, y) \}, \quad T := P_1(R \setminus \{ x = y \}),
\]
where $\{ x = y \}$ denotes the diagonal $\{ (x, y) \in X^2 : x = y \}$ and $\Gamma^{-1} = \{ (x, y) \in X \times X : (y, x) \in \Gamma \}$. Since $\varphi$ is 1-Lipschitz, the sets $\Gamma$, $\Gamma^{-1}$, and $R$ are closed, and therefore compact, thanks to the compactness of $(X, d)$ (recall $C(D(K, N)$ with $K > 0$). In particular, $T$ is $\sigma$-compact.

It is immediate to verify (see [3, prop. 4.2]) that if $(\gamma_0, \gamma_1) \in \Gamma$ for some $\gamma \in \text{Geo}(X)$, then $(\gamma_s, \gamma_t) \in \Gamma$ for all $0 \leq s \leq t \leq 1$. To exclude possible branching we need to consider the following sets, introduced in [21]:
\[
A_+ := \{ x \in T : \exists z \in \Gamma(x), (z, w) \notin R \},
\]
\[
A_- := \{ x \in T : \exists z, w \in \Gamma^{-1}(x), (z, w) \notin R \};
\]
where $\Gamma(x) = \{ y \in X : (x, y) \in \Gamma \}$ denotes the section of $\Gamma$ through $x$ in the first coordinate, and similarly for $R(x)$ (through either coordinate by symmetry). $A_\pm$ are called the sets of forward and backward branching points, respectively. Note that both $A_\pm$ are $\sigma$-compact sets. Then the nonbranched transport set is defined as
\[
T^b := T \setminus (A_+ \cup A_-)
\]
and is a Borel set; accordingly, the nonbranched transport relation is given by
\[
R^b := R \cap (T^b \times T^b).
\]
It was shown in [21] (cf. [16]) that $R^b$ is an equivalence relation over $T^b$ and that for any $x \in T^b$, $R(x) \subset (X, d)$ is isometric to a closed interval in $(\mathbb{R}, | \cdot |)$. Now, from the first part of the disintegration theorem (see Theorem B.2) applied to $(T^b, B(T^b), m_{\leq T^b})$, we obtain an essentially unique disintegration of $m_{\leq T^b}$ consistent with the partition of $T^b$ given by the equivalence classes $\{ R^b(q) \}_{q \in Q}$ of $R^b$:
\[
m_{\leq T^b} = \int_Q m_q(dq).
\]
with corresponding quotient space \((Q, \varnothing, q)\) \((Q \subset T^b\) may be chosen to be any section of the above partition). In what follows, we will also use the notation \(X_q\) to denote the transport ray \(R^b(q)\).

The next step is to show that the disintegration is strongly consistent. By the disintegration theorem, this is equivalent to the existence of a \(m_{L-T^b}\)-section \(\tilde{Q} \in \mathcal{B}(T^b)\) (which by a mild abuse of notation we will call a \(m\)-section) such that the quotient map associated to the partition is \(m\)-measurable, where we endow \(\tilde{Q}\) with the trace \(\sigma\)-algebra. This has already been shown in [16, prop. 4.4] in the framework of nonbranching metric spaces; since its proof does not use any nonbranching assumption, we can conclude that

\[
m_{L-T^b} = \int_Q m_q dq, \quad \text{and for } q - \text{a.e. } q \in Q, \quad m_q(R^b(q)) = 1,
\]

where now \(Q \supset \tilde{Q} \in \mathcal{B}(T^b)\) with \(\tilde{Q}\) an \(m\)-section for the above partition (and hence \(q\) is concentrated on \(\tilde{Q}\)). Moreover, the existence of an \(m\)-measurable quotient map permits us to conclude that the quotient \(\sigma\)-algebra on \(\tilde{Q}\), which we denote by \(\varnothing \cap \tilde{Q}\), is contained in \(\mathcal{B}(\tilde{Q})\), the completion with respect to \(q\) of the Borel \(\sigma\)-algebra over \(\tilde{Q}\).

The existence of an \(m\)-section also permits us to construct a measurable parametrization of the transport rays. We can define

\[
g : \text{Dom}(g) \subset \tilde{Q} \times \mathbb{R} \rightarrow T^b
\]

that associates to \((q, t)\) the unique \(x \in \Gamma(q)\) with \(d(q, x) = t\) provided \(t > 0\) or the unique \(x \in \Gamma^{-1}(q)\) with \(d(q, x) = -t\) otherwise. Then

\[
\text{graph}(g) = \{(q, t, x) \in \tilde{Q} \times [0, \infty) \times T^b : (q, x) \in \Gamma, \ d(q, x) = t\}
\]

\[
\cup \{(q, t, x) \in \tilde{Q} \times (-\infty, 0) \times T^b : (q, x) \in \Gamma^{-1}, \ d(q, x) = -t\},
\]

showing that graph\((g)\) is Borel; in particular, \(g : \text{Dom}(g) \rightarrow T^b\) is a Borel map with \(\text{Dom}(g)\) an analytic set and image \(\bigcup_{\alpha \in \tilde{Q}} R^b(\alpha)\), which is analytic as well. To conclude we also notice that \(g\) is injective and

\[
m\left(T^b \setminus \bigcup_{q \in \tilde{Q}} R^b(q)\right) = 0.
\]

A priori the nonbranched transport set \(T^b\) can be much smaller than \(T\). However, under fairly general assumptions one can prove that the sets \(A_{\pm}\) of forward and backward branching are both \(m\)-negligible. In [21], this was shown for an m.m.s. \((X, \varnothing, m)\) verifying \(\text{RCD}(K, N)\) and \(\text{supp}(m) = X\). The proof only relies on the following two properties, which hold for the latter spaces:

- \(\text{supp}(m) = X\).
- Given \(\mu_0, \mu_1 \in \mathcal{P}_2(X)\) with \(\mu_0 \ll m\), there exists a unique optimal transference plan for the \(W_2\)-distance, and it is induced by an optimal transport map.
These properties are also verified for an essentially nonbranching m.m.s. \((X, d, m)\) satisfying \(\text{CD}(K, N)\) and \(\text{supp}(m) = X\) (see [23]).

We summarize the above discussion as follows:

**Corollary 2.7.** Let \((X, d, m)\) be an essentially nonbranching m.m.s. satisfying \(\text{CD}(K, N)\) and \(\text{supp}(X) = m\). Then for any 1-Lipschitz function \(\psi : X \to \mathbb{R}\), we have \(m(T \setminus T^b) = 0\). In particular, we obtain the following essentially unique disintegration \((Q, \mathcal{Q}, \bar{q})\) of \(m_{\mathcal{T}} = m_{\mathcal{T}^b}\) strongly consistent with the partition of \(T^b\) given by the equivalence classes \(\{R^b(q)\}_{q \in Q}\) of \(R^b\):

\[
m_{\mathcal{T}} = \int_Q m_q(dq) \quad \text{and for } q - \text{a.e. } q \in Q, \quad m_q(R^b(q)) = 1.
\]

Here \(Q\) may be chosen to be a section of the above partition so that \(Q \supseteq \bar{Q} \in \mathcal{B}(T^b)\) with \(\bar{Q}\) an \(m\)-section with an \(m\)-measurable quotient map. In particular, \(\mathcal{Q} \supseteq \mathcal{B}(\bar{Q})\) and \(q\) is concentrated on \(\bar{Q}\).

To obtain Theorem 2.6 one still needs to show that the constraint \(\int_X f \, m = 0\) is localized, i.e., \(\int_Q f \, m_q = 0\) for \(q\)-a.e. \(q \in Q\), together with the curvature bound: \(q \mapsto m_q\) is a \(\text{CD}(K, N)\) disintegration. The first property it verified almost “by construction”; the second one is the more subtle and to prove it one should study the interplay between \(L^2\)-Wasserstein geodesics and the transport set \(\mathcal{T}\); we refer to [25, theorem 4.2] for the details.

Finally, we recall that \(q\)-a.e. \(R^b(q)\) is actually maximal, meaning that it coincides with \(R(q)\). This can be restated as follows: \(\Gamma\) induces a partial order relation on \(X\):

\[
y \leq x \iff (x, y) \in \Gamma,
\]
and for \(x \in T^b\), \((R(x), d)\) is isometric to a closed interval in \((\mathbb{R}, |\cdot|)\). This isometry induces a total ordering on \(R(x)\) that must coincide with either \(\leq\) or \(\geq\), implying that \((R(x), \leq)\) is totally ordered; in particular, it is a chain; the previous maximality property means that \(R^b(q) = R(q) \cap T^b\) is a maximal chain in the partially ordered set \((X, \leq)\).

To rigorously state this property, we use the classical definition of initial and final points, \(A\) and \(B\), respectively:

\[
A := \{x \in T : \exists y \in T, (y, x) \in \Gamma, y \neq x\},
B := \{x \in T : \exists y \in T, (x, y) \in \Gamma, y \neq x\}.
\]

Note that

\[
A = T \setminus P_1(\{\Gamma \setminus \{x \neq y\}\}),
\]
so \(A\) is Borel (since \((X, d)\) is compact, the set \(\Gamma \setminus \{x = y\}\) is \(\sigma\)-compact), and similarly for \(B\).

**Theorem 2.8 ([22]).** Let \((X, d, m)\) be an essentially nonbranching m.m.s. verifying \(\text{CD}(K, N)\) and \(\text{supp}(m) = X\). Let \(\varphi : (X, d) \to \mathbb{R}\) be any 1-Lipschitz function, with \(\text{supp}(\varphi)\) the associated disintegration of \(m_{\mathcal{T}}\).
Then there exists $\hat{Q} \subset Q$ such that $q(Q \setminus \hat{Q}) = 0$ and for any $q \in \hat{Q}$ it holds that
\[ R(q) \setminus T^b \subset A \cup B. \]
In particular, for every $q \in \hat{Q}$,
\[ R(q) = \overline{R^b(q)} \subset R^b(q) \supset \hat{R}(q) \]
(with the latter interpreted as the relative interior).

Possibly taking a full q-measure subset of $\hat{Q}$, we can assume $\hat{Q}$ to be Borel. In this paper we will make use of the map associating to each point $q \in \hat{Q}$ the starting point of the ray and the endpoint of the ray. As we are assuming $\mathbb{C}D(K, N)$ with $K > 0$, we will think of the starting point as a “south pole” and the ending point as a “north pole”; this justifies the following notation:
\[ P_S: \hat{Q} \to \mathcal{T}, \quad P_N: \hat{Q} \to \mathcal{T}, \]
with graphs
\[ \text{graph}(P_S) := (\hat{Q} \times A) \cap \Gamma^{-1}, \quad \text{graph}(P_N) := (\hat{Q} \times B) \cap \Gamma, \]
also showing that both $P_S$ and $P_N$ are Borel maps; this implies that the map $\hat{Q} \ni q \mapsto |X_q| = \delta(P_S(q), P_N(q))$, is also Borel.

From the measurability of the disintegration, one also obtains that
\[ \hat{Q} \times \mathbb{R} \ni (q, t) \mapsto h_q(t) \in [0, \infty) \]
is $q \otimes \mathcal{L}^1$-measurable (see, for instance, [22, prop. 10.4]); here with an abuse of notation we have denoted by $h_q$ the density function $h_q \circ g(q, \cdot)$ where $m_q = h_q \mathcal{H}^1_{\mathcal{L}X_q}$. It is fairly standard (using, for example, [58, theorem 3.1.30]) to restrict ourselves to a Borel subset of $\hat{Q}$ of the same q-measure, which for ease of notation we again denote by $\hat{Q}$, such that
\[ \hat{Q} \times \mathbb{R} \ni (q, t) \mapsto h_q(t) \in [0, \infty) \]
is Borel. Then we can compose it with a translation, Borel in $q$, to obtain that for each $q \in \hat{Q}$, $h_q: [0, |X_q|] \to [0, \infty)$, and still obtain a jointly Borel function.

One can also restrict the Borel map $g$ to the following Borel subset of its domain:
\[ \{(q, s) \in \hat{Q} \times \mathbb{R}: s \in (0, |X_q|)\}, \]
and by construction, the restriction of $g$ is injective.

We conclude this part mentioning that we will directly write $Q$ instead of $\hat{Q}$, and we summarize the measurability properties obtained:

- The disintegration formula holds: for a suitably chosen $Q \subset T^b$ Borel, it holds that
\[ m_{\mathcal{L}T} = \int_Q m_q(dq) \quad \text{and} \quad m_q(R(q)) = 1, \quad q\text{-a.e. } q \in Q; \]
• For q.a.e. \( q \in Q \), \( m_q = (g(q, \cdot))_\# h_q \mathcal{L}^1_{\cdot \in [0,|X_0|]} \) with \( h_q \) a \( \text{CD}(K, N) \) density, and the maps \( g : \text{Dom}(g) \to X \), \( h : \text{Dom}(h) \to [0, \infty) \) with \( \text{Dom}(g), \text{Dom}(h) \subset Q \times \mathbb{R} \) are Borel measurable.

### 3 Quantitative One-Dimensional Estimates

In this section we obtain all the one-dimensional results concerning the quantitative isoperimetric inequality that will then be used in the general framework of metric measure spaces.

We start considering the one-dimensional metric measure space \( (\mathbb{R}, \mathcal{L}^1, |\cdot|, h \cdot \mathcal{L}^1) \) verifying \( \text{CD}(N - 1, N) \) over \( \mathbb{R} \), i.e.,

\[
(h^{1/(N-1)})'' + h^{1/(N-1)} \leq 0
\]

in the sense of distributions and such that \( \int_{[0,D]} h(t)dt = 1 \); notice that, as by construction \( h \geq 0 \), then necessarily \( h > 0 \) over \( (0, D) \). Since \( \text{CD}(N - 1, N) \) densities are log-concave (meaning that \(-\log h\) is convex), by a result of Bobkov [17, prop. 2.1] for each \( v \in (0, 1) \) there exists an isoperimetric minimizer either of the form \([0, r^-_h(v)]\) or \([r^+_h(v), D]\), where

\[
v = \int_0^{r^-_h(v)} h(t)dt = \int_{r^-_h(v)}^{D} h(t)dt.
\]

The perimeter functional associated to \( h \) will be denoted by \( P_h \). Then it is natural to define the deficit associated to \( h \) as follows:

\[
\delta_h(E) := P_h(E) - \mathcal{I}_h(v), \quad \mathcal{I}_h(v) := \min\{h(r^-_h(v)), h(r^+_h(v))\}.
\]

Then, defining \( m := h \cdot \mathcal{L}^1 \), we obtain the next quantitative statement.

**Proposition 3.1.** For each \( v \in (0, 1) \) there exists \( \varepsilon(N, v), C(N, v) > 0 \) such that for each \( \varepsilon := \pi - D \in (0, \varepsilon(N,v)) \)

\[
P_h(E) - \mathcal{I}_h(v) \geq C(N, v) \min\{m(E \Delta [0, r^-_h(v)]), m(E \Delta [r^+_h(v), D])\},
\]

for any \( m = h \mathcal{L}^1 \) with \( h \) a \( \text{CD}(N - 1, N) \) density, supported over \( [0, D] \) that integrates to \( 1 \) and \( E \in \mathcal{B}(\mathbb{R}) \) with \( m(E) = v \).

**Proof.**

**Step 1.** Suppose by contradiction that the claim is false. Then we can find sequences \( \varepsilon_j \to 0 \), \( h_j \) of densities verifying \( \text{CD}(N - 1, N) \) over \([0, \pi - \varepsilon_j]\), and \( E_j \subset [0, D_j] \) with \( D_j := \pi - \varepsilon_j \) such that

\[
\lim_{j \to \infty} \frac{P_{h_j}(E_j) - \mathcal{I}_{h_j}(v)}{\min\{h_j \mathcal{L}^1(E_j \Delta [0, r^-_{h_j}(v)]), h_j \mathcal{L}^1(E_j \Delta [r^+_{h_j}(v), \pi])\}} = 0.
\]

Possibly passing to a subsequence, we assume that

\[
\lim_{j \to \infty} \frac{P_{h_j}(E_j) - \mathcal{I}_{h_j}(v)}{h_j \mathcal{L}^1(E_j \Delta [0, r^-_{h_j}(v)])} = 0;
\]

it will be clear from the proof that the other case follows similarly.
As $P_{h_j}(E_j)$ is uniformly bounded, we can find a representative of $E_j$ (i.e., having the same $P_{h_j}$), which we denote with the same symbol, such that

$$E_j = \bigcup_{i \in \mathbb{N}} (a^i_j, b^i_j).$$

By Proposition A.3 we deduce that $h_j \to h_N$ uniformly over any compact subset of $(0, \pi)$; moreover, since $h_j \geq 0, h_j^{1/N}$ is concave and $\int_0^{D_j} h_j = 1$, then $\sup_{t \in [0, D_j]} h_j < \infty$. In particular, $\mathcal{I}_{h_j}(v) \to \mathcal{I}_\pi(v)$ also yields that $P_{h_j}(E_j)$ converges to $\mathcal{I}_\pi(v)$; hence we infer by compactness that $h_j \chi_{E_j} \to h_N \chi_{[0, r_N^{-1}(v)]}$ pointwise over $[0, 1]$ and in $L^1_{\text{loc}}(0, \pi)$. Possibly passing to a subsequence, it follows that $\chi_{E_j} \to \chi_{[0, r_N^{-1}(v)]}$ pointwise (recall that also $r_N^{-1}(v) \to r_N^{-1}(v)$ as $j \to \infty$). For ease of notation $r_N^{-1}(v) = r_N^{-1}(v)$ and the same for $r_N^{-1}(v)$.

From Proposition A.3 we deduce that $E_j$ can be decomposed as

$$E_j = E_j^0 \cup (\beta_j, r_N^{-1}(v) + \gamma_j) \cup E_j^{D_j},$$

and where, without restriction, $\beta_j$ and $\gamma_j > 0$ are positive and such that

$$\eta_j < \beta_j \to 0, \quad r_N^{-1}(v) + \gamma_j < D_j - \eta_j, \gamma_j \to 0.$$ 

Step 2. Using again Proposition A.3, the unique maximum $x_j \in [0, D_j]$ of $h_j$ given by Lemma A.4 necessarily converges to $\pi/2$; hence if we replace $E_j^0$ with $[0, \alpha_j]$ and $E_j^{D_j}$ with $[\xi_j, D_j]$ such that

$$h_j \mathcal{L}^1(E_j^0) = h_j \mathcal{L}^1([0, \alpha_j]), \quad h_j \mathcal{L}^1(E_j^{D_j}) = h_j \mathcal{L}^1([\xi_j, D_j]),$$

and we call again the new sequence $E_j$, then the perimeter will be decreased and the symmetric difference with $[0, r_N^{-1}(v)]$ will remain the same.

Hence pick as a (possibly) new sequence of sets

$$E_j := [0, \alpha_j] \cup (\beta_j, r_N^{-1}(v) + \gamma_j) \cup [\xi_j, D_j]$$

and from the volume constraint

$$\int_{(0, \alpha_j)} h_j \, dt + \int_{(\beta_j, r_N^{-1}(v) + \gamma_j)} h_j \, dt + \int_{(\xi_j, D_j)} h_j \, dt = \int_{(0, r_N^{-1}(v))} h_j \, dt$$

giving

$$\int_{(r_N^{-1}(v), r_N^{-1}(v) + \gamma_j)} h_j \, dt + \int_{(\xi_j, D_j)} h_j \, dt = \int_{(\alpha_j, \beta_j)} h_j \, dt.$$
This permits us to obtain the next identity:

\[
\frac{h_j}{L^1(E_j \Delta [0, r_{h_j}(v)])} = \int_{(\alpha_j, \beta_j)} h_j(t) \, dt + \int_{(r_{h_j}(v), r_{h_j}(v) + \gamma_j)} h_j(t) \, dt + \int_{(\xi_j, D_j)} h_j(t) \, dt
\]

(3.2)

\[
= 2 \int_{(\alpha_j, \beta_j)} h_j(t) \, dt;
\]

note that by monotonicity of \(h_j\) (recall that \(\beta_j \to 0\))

\[
\int_{(\alpha_j, \beta_j)} h_j(t) \, dt \leq \beta_j h_j(\beta_j).
\]

**Step 3.** Define \(F_j := [\alpha_j, r_{h_j}(v) + \gamma_j] \cup [\xi_j, D_j]\) and notice that

\[
v_j := \int_{F_j} h_j(t) \, dt = v + \int_{(\alpha_j, \beta_j)} h_j(t) \, dt - \int_{(0, \alpha_j)} h_j(t) \, dt,
\]

giving by monotonicity \(|v_j - v| \leq \beta_j h_j(\beta_j)|. Hence expanding \(P_{h_j}(E_j)\) as

\[
P_{h_j}(E_j) = P_{h_j}(F_j) + h_j(\beta_j) \geq \mathcal{I}_{h_j}(v_j) + h_j(\beta_j)
\]

one obtains

\[
P_{h_j}(E_j) - \mathcal{I}_{h_j}(v) \geq h_j(\beta_j) + \mathcal{I}_{h_j}(v_j) - \mathcal{I}_{h_j}(v).
\]

To conclude we observe that the map \((0, 1) \ni v \to r_{h_j}(v)\) is differentiable with derivative equal to \(1/h_j(r_{h_j}(v))\); this together with the Lipschitz regularity of \(h_j\) (with Lipschitz constant uniform on \(j\) and depending just on \(N\) and \(v\); see Corollary A.2) implies that

\[
|\mathcal{I}_{h_j}(v_j) - \mathcal{I}_{h_j}(v)| \leq C_{N, v} |v_j - v| \leq C_{N, v} \beta_j h_j(\beta_j).
\]

(3.3)

Then we obtain a contradiction, noticing that the combination of (3.2) and (3.3) gives

\[
\frac{P_{h_j}(E_j) - \mathcal{I}_{h_j}(v)}{h_j L^1(E_j \Delta [0, r_{h_j}(v)])} \geq \frac{h_j(\beta_j)(1 - \beta_j C_{N, v})}{2 \int_{(\alpha_j, \beta_j)} h_j(t) \, dt};
\]

in particular, for \(j\) large, the right-hand side is arbitrarily large (recall that \(\beta_j \to 0\) and \(\int_{(\alpha_j, \beta_j)} h_j(t) \, dt \leq \beta_j h_j(\beta_j))\). The claim follows. \(\Box\)

### 3.1 Isoperimetric Profile

We now study the behavior of the model isoperimetric profile functions in terms of the diameter upper bounds. For any \(D \in [0, \pi]\) there exists \(\xi \in [0, \pi - D]\) such that the one-dimensional CD(N − 1, N) model space with diameter \(D\) is given by \( ([\xi, \xi + D], | \cdot |, \sin(t)^{N-1}/(\omega_N \lambda_D)) \) where

\[
\lambda_D := \frac{1}{\omega_N} \int_{\xi}^{\xi + D} \sin(t)^{N-1} \, dt;
\]

(3.4)
\( \omega_N \) being the renormalization constant (i.e., the volume of the \( N \)-dimensional sphere of unit radius, in case \( 2 \leq N \in \mathbb{N} \)); in particular, \( \lambda_\pi = 1 \). For ease of notation we set \( h_D := \sin(t)^{N/2}/(\omega_N \lambda_D) \) and we omit the dependence on \( D \) of \( \xi \); moreover, for coherence \( h_\pi \) will be denoted by \( h_N \).

For each volume \( v \in (0,1) \), consider the two intervals \([\xi, r_D^-(v)]\) and \([r_D^+(v), D + \xi] \subset [\xi, \xi + D] \) with

\[
\int_{\xi}^{r_D^-(v)} h_D(t) dt = \int_{r_D^-(v)}^{D + \xi} h_D(t) dt = v.
\]

Then [17 prop. 2.1] implies that

\[
\min \{ h_D(r_D^-(v)), h_D(r_D^+(v)) \} = I_D(v) := I_{N-1,N,D}(v).
\]

LEMMA 3.2. The following holds:

\[
I_D(v) = \frac{1}{\lambda_D} \min \left\{ \varpi \left( \lambda_D v + \int_0^\xi h_N d\tau \right) \varpi \left( \lambda_D (1 - v) + \int_0^\xi h_N d\tau \right) \right\}.
\]

PROOF. Observe that

\[
\lambda_D v = \int_{\xi}^{r_D^-(v)} h_N(t) dt = \int_0^{r_D^-(v)} h_N(t) dt - \int_0^\xi h_N(t) dt,
\]

and that \( \lambda_D h_D(r_D^-(v)) = h_N(r_D^-(v)) = \varpi(\lambda_D v + \int_0^\xi h_N) \). Analogously

\[
\lambda_D v = \int_{r_D^-(v)}^{D + \xi} h_N(t) dt = \int_{r_D^-(v)}^{r_D^+(v)} h_N(t) dt - \int_{D + \xi}^{r_D^+(v)} h_N(t) dt
\]

\[
= \int_{r_D^-(v)}^{\xi} h_N(t) dt - \left( 1 - \lambda_D - \int_0^\xi h_N(t) dt \right),
\]

showing that

\[
\lambda_D h_D(r_D^+(v)) = h_N(r_D^+(v)) = \varpi \left( \lambda_D (1 - v) + \int_0^\xi h_N(t) dt \right).
\]

The claim follows. \( \square \)

As we are going to consider the deficit, the following inequalities will be the relevant ones.

LEMMA 3.3 (Concavity of \( I_\pi \)). The following estimate holds:

\[
\min \left\{ \varpi \left( \lambda_D v + \int_0^\xi h_N d\tau \right) \varpi \left( \lambda_D (1 - v) + \int_0^\xi h_N d\tau \right) \right\} - \lambda_D I_\pi(v) \geq C_{N,v} \min \left\{ \lambda_D^{-1}, 1 - \lambda_D \right\}.
\]
where $C_{N,v}$ is an explicit constant depending just on $N$ and $v$:

$$C_{N,v} := \min \left\{ I_\pi(v) - v I'_\pi(v), I_\pi(v) + (1-v) I'_\pi(v) \right\}.$$

$$\lim_{t \to 0} \frac{I_\pi(t)}{t^{(N-1)/N}} \min \left\{ v \frac{N^2}{N+1}, (1-v) \right\}.$$

**Proof.**

**Step 1.** We first study the asymptotics for $\lambda_D \to 1$. Note that

$$I_\pi \left( \lambda_D v + \int_0^\xi h_N \right) - \lambda_D I_\pi(v)$$

$$= I_\pi \left( \lambda_D v + \int_0^\xi h_N \right) - I_\pi(v) + (1-\lambda_D) I_\pi(v)$$

$$= \left( v(\lambda_D - 1) + \int_0^\xi h_N \right) I'_\pi(P_{\lambda_D}) + (1-\lambda_D) I_\pi(v),$$

for some $P_{\lambda_D}$ between $\lambda_D v + \int_0^\xi h_N$ and $v$. In particular, for $\lambda_D \to 1$, necessarily $D \to \pi$ and $\xi \to 0$; hence

$$\lim_{\lambda_D \to 1} \frac{\int_0^\xi h_N}{1-\lambda_D} = \lim_{\lambda_D \to 1} \frac{\int_0^\xi h_N}{\int_0^\xi h_N + \int_0^\pi h_N dx} = c$$

with $0 \leq c \leq 1$. In particular,

$$\lim_{\lambda_D \to 1} \frac{I_\pi(\lambda_D v + \int_0^\xi h_N) - \lambda_D I_\pi(v)}{1-\lambda_D} = I_\pi(v) + (c-v) I'_\pi(v).$$

Being a linear function of $c \in [0,1]$, the last quantity is larger than the minimum between $I_\pi(v) - v I'_\pi(v)$ and $I_\pi(v) + (1-v) I'_\pi(v)$. Since $I_\pi(v) = I_\pi(1-v)$, the strict concavity of $I_\pi$ yields that both are strictly positive.

Using again the symmetry $I_\pi(v) = I_\pi(1-v)$, with the arguments above one can prove that the same lower bounds hold for

$$\lim_{\lambda_D \to 1} \frac{I_\pi(\lambda_D (1-v) + \int_0^\xi h_N) - \lambda_D I_\pi(v)}{1-\lambda_D}.$$

**Step 2.** Again from the strict convexity of $I_\pi$, for $\lambda_D$ sufficiently close to 0, we have that $I_\pi(\lambda_D v + \int_0^\xi h_N) \geq I_\pi(\lambda_D v)$, which yields

$$\lim_{\lambda_D \to 0} \frac{I_\pi(\lambda_D v + \int_0^\xi h_N)}{(\lambda_D)(N-1)/N} \geq \left( \lim_{t \to 0} \frac{I_\pi(t)}{t^{(N-1)/N}} \right) v \frac{N^2}{N+1}.$$ 

Analogously we have

$$\lim_{\lambda_D \to 0} \frac{I_\pi(\lambda_D (1-v) + \int_0^\xi h_N)}{(\lambda_D)(N-1)/N} \geq \left( \lim_{t \to 0} \frac{I_\pi(t)}{t^{(N-1)/N}} \right) (1-v) \frac{N^2}{N+1},$$

completing the proof. □
4 Reduction to the One-Dimensional Case

From now on we consider fixed a metric measure space \( (X, d, m) \) that is essentially nonbranching and it verifies \( CD(K, N) \) with \( K > 0 \). Recall that with no loss in generality, we can assume \( \text{supp}[m] = X \) and \( K = N - 1 \), giving \( \text{diam}(X) \leq \pi \); moreover, \( (X, d) \) is compact.

We also fix once and for all \( E \subset X \) together with the associated localization given by the \( L^1 \)-optimal transport problem between

\[
0 \leq \nu E \leq 1
\]

with \( \nu = m(E) \). Following Section 2.4 and Section 2.5 we also fix a \( 1 \)-Lipschitz Kantorovich potential \( \varphi : X \to \mathbb{R} \) (it is actually unique up to a constant) such that if

\[
(4.1) \quad \Gamma := \{(x, y) \in X \times X : \varphi(x) - \varphi(y) = d(x, y)\},
\]

then a transport plan \( \pi \) is optimal if and only if \( \nu (\Gamma) = 1 \). Then from \( \varphi \) one obtains the family of transport rays \( \{X_q \}_{q \in Q} \) with \( Q \) a Borel subset of the transport set \( T \); it is also immediate (see Theorem 2.6) to observe that \( m(X \setminus T) = 0 \), so we have the following disintegration formula:

\[
m = \int_Q m_q dq, \quad q \in \mathcal{P}(Q).
\]

Moreover, for \( q \)-a.e. \( q \in Q \):

- \( m_q(X_q) = 1 \);
- \( (X_q, d, m_q) \) is a \( CD(N - 1, N) \)-space (see Section 2);
- \( m_q(E) = m(E) = \nu \in (0, 1) \).

Define the deficit \( \delta(E) := \mathcal{P}(E) - \mathcal{I}_\pi(v) \).

**Lemma 4.1.** The following inequalities hold true:

\[
(4.2) \quad \delta(E) \geq \int_Q \mathcal{P}_q(E_q) - \mathcal{I}_\pi(v) q(dq) \geq \int_Q \mathcal{I}_{D_q}(v) - \mathcal{I}_\pi(v) q(dq),
\]

where \( \mathcal{I}_D = \mathcal{I}_{N-1,N,D} \), for any \( D \in [0, \pi] \) and \( D_q = |X_q| = d(P_S(q), P_N(q)) \).

**Proof.** As observed in Section 2.5 the map \( Q \ni q \mapsto |X_q| \) is Borel and therefore the same holds for \( Q \ni q \mapsto \lambda_q := \lambda_{|X_q|} \), where \( \lambda_D \) has been defined in (3.4). In particular, the last integral makes sense.

To obtain the measurability of \( q \mapsto \mathcal{P}_q(E_q) \) one can argue as follows. Consider any sequence \( u_n \to \chi_E \) such that

\[
\mathcal{P}(E) + \varepsilon \geq \lim_{n \to \infty} \int_Q |\nabla u_n| m = \lim_{n \to \infty} \int_Q |\nabla u_n| m_q(q(dq)) \geq \int_Q \liminf_{n \to \infty} |\nabla u_n| m_q(q(dq)).
\]
where the last inequality follows from Fatou’s lemma. This implies that for \( q \)-a.e. \( q \in Q \)

\[
\liminf_{n \to \infty} \int |\nabla u_n| m_q < \infty.
\]

Hence \( P_q(E_q) < \infty \) \( q \)-a.e. and therefore we can deduce that \( q \)-a.e. \( E_q \) has representative given by a countable union of intervals; in particular,

\[
P_q(E_q) = \sum_i h_q(x_{q,i}),
\]

where \( x_{q,i} \) belongs to the boundary of \( E_q \) in \( X_q \). To obtain \( q \)-measurability of \( q \mapsto P_q(E_q) \), it is then enough to prove \( q \)-measurability of \( \sum_i h_q(x_{q,i}) \).

For this purpose we consider \( \tilde{Q} \subset Q \) Borel such that for each \( q \in \tilde{Q} \) the set \( \{x_{q,i}\}_i \) is countable. Moreover, for each \( q \in \tilde{Q} \), points of the boundary of \( E_q \) in \( X_q \) can be obtained as follows: for each \( n \) consider

\[
\Lambda_n := \{(q, x, y, z) \in \tilde{Q} \times T^b \times T^b \times T^b: (q, x), (q, y) \in R^b, y \in E, z \in E^c, d(x, y), d(x, z) \leq 1/n\},
\]

and define \( \Lambda := \bigcap_n P_{1,2}(\Lambda_n) \), where \( P_{1,2} \) stands for the projection on the first and second component; notice that for each \( n \in \mathbb{N} \) \( P_{1,2}(\Lambda_n) \) is analytic and therefore \( \Lambda \) is analytic as well. Now we notice that each section \( \Lambda(q) := \{x \in T^b: (q, x) \in \Lambda\} \) is countable; then we can invoke a classical result of Lusin (see [58, theorem 5.10.3]) yielding the existence of countably many \( x_i: \tilde{Q} \to T^b \) such that

\[
\Lambda = \bigcup_{i \in \mathbb{N}} \{(q, x_i(q)): q \in \tilde{Q}\},
\]

with \( \{(q, x_i(q)): q \in \tilde{Q}\} \) analytic; in particular, for each \( i \in \mathbb{N} \) the map \( \tilde{Q} \ni q \mapsto x_i(q) \) is Borel; it follows that for each \( q \in \tilde{Q} \), the map \( \sum_i h_q(x_{i}(q)) = P_q(E_q) \) is \( q \)-measurable. Hence the second integral also makes sense. To conclude, it is enough to use Fatou’s lemma as before and observe that \( \int_{\nabla u_n \circ X_q} \geq \int |(u_n \circ X_q)'| \) \( q \)-a.e. (here, with a slight abuse of notation, we denote by \( X_q: [0, D_q] \to X \) the ray map).

4.1 Long Rays

For any \( v \in (0, 1) \) clearly \( \mathcal{I}_\pi(v) = \mathcal{I}_\pi(1 - v) \), then Lemma 3.3 applied together with Lemma 3.2 yields from (4.2) the following inequality:

\[
\delta(E) \geq C_{N,v} \int_{Q} \frac{1}{\lambda_q} \min\{\lambda_q^{(N-1)/N}, 1 - \lambda_q\} q(dq)
\]

\[
= C_{N,v} \left( \int_{Q_s} \lambda_q^{-1/N} q(dq) + \int_{Q_t} \frac{1 - \lambda_q}{\lambda_q} q(dq) \right),
\]

(4.3)
where $s$ and $\ell$ stand for “short” and “long,” and
\[
Q_s := \{ q \in Q : \lambda_q^{(N-1)/N} \leq 1 - \lambda_q \} = \{ q \in Q : \lambda_q \leq \eta_N \},
\]
\[
Q_\ell := \{ q \in Q : \lambda_q^{(N-1)/N} > 1 - \lambda_q \} = \{ q \in Q : \lambda_q > \eta_N \},
\]
where $0 < \eta_N < 1$ is the unique solution of $x^{(N-1)/N} = 1 - x$. Note that $\eta_N \to \frac{1}{2}$ as $N \to \infty$ and $Q_s, Q_\ell$ are Borel subsets of $Q$.

We continue by noticing
\[
\delta(E) \geq C_{N,v} \int_{Q_s} \eta_N^{-1/N} q(dq) = C_{N,v} \eta_N^{-1/N} q(Q_s),
\]
and in particular
\[
(4.4) \quad m \left( E \cap \left( \bigcup_{q \in Q_s} X_q \right) \right) \leq q(Q_s) \leq \eta_N^{1/N} C_{N,v}^{-1} \delta(E);
\]
this means that most of the $m$-measure of $E$ must be contained in the set spanned by the family of rays denoted by $Q_\ell$.

Since $Q_\ell \subset \{ q \in Q : D_q \geq D_N \}$ for some $D_N \in (0, \pi)$, we obtain the next proposition.

**Proposition 4.2.** The following estimate holds
\[
(4.5) \quad \delta(E) \geq C_{N,v} C_{N,v}' \int_{Q_\ell} (\pi - D_q)^N q(dq).
\]

**Proof.** From (4.3)
\[
\delta(E) \geq C_{N,v} \int_{Q_\ell} \frac{1 - \lambda_q}{\lambda_q} q(dq) \geq C_{N,v} \int_{Q_\ell} (1 - \lambda_q) q(dq).
\]
We now study the behavior of $1 - \lambda_q$ when $D_q$ approaches $\pi$ and $D_N$. Since
\[
1 - \lambda_q = 1 - \int_{\xi_q}^{D_q + \xi_q} h_N(t)dt = \int_0^{\xi_q} h_N(t)dt + \int_{\xi_q}^{D_q + \xi_q} h_N(t)dt,
\]
it follows that
\[
(4.6) \quad \lim_{D_q \to \pi} \frac{1 - \lambda_q}{(\pi - D_q)^N} = C_N > 0;
\]
notice indeed that $\xi_N^N + (\pi - D_q - \xi_q)^N \geq 2^{-N}(\pi - D_q)^N$.

It follows therefore the existence of an explicit, strictly positive constant $C_{N,v}'$ such that
\[
1 - \lambda_q \geq C_{N,v}' (\pi - D_q)^N.
\]
The claim follows.

We can therefore obtain the first main result of the paper (first claim in Theorem 1.1).
**Theorem 4.3.** There exists at least one $\bar{q} \in Q$ such that

\[(\pi - D\bar{q})^N \leq \delta(E) \frac{1}{C_{N,v}'(C_{N,v} - \delta(E)\eta^1/(N-1))} \leq C_{N,v}''\delta(E).\]

In particular, there exists a constant $C(N, v)$ depending only on $N > 1$ and $v \in (0, 1)$ such that

\[\pi - \text{diam}(X) \leq C(N, v)\delta(E)^{1/N}.\]

**Proof.** Just observe that from Proposition 4.2 there exists at least one $q \in Q$ such that

\[(\pi - Dq)^N \leq \delta(E) \frac{1}{q(Q\ell)C_{N,v}'C_{N,v}''}.\]

Then, from (4.4), we get that

\[q(Q\ell) = 1 - q(Q_\ell) \geq 1 - \delta(E)\frac{\eta^1/(N-1)}{C_{N,v}}.\]

The claim follows. \square

Since $D\bar{q}$ of Theorem 4.3 will play a key role, from now on we consider $\bar{q}$ fixed and given by Theorem 4.3.

5 **Structure of the Transport Set**

So far we have observed that the distance of $|X_q|$ from $\pi$ is controlled by the deficit $\delta(E)$ (see Proposition 4.2). In this section we use this information to prove that most of the rays start close to $P_S(\bar{q})$ and finish close to $P_N(\bar{q})$. The optimality of $\varphi$ will be crucial. We will use the following result (which is of interest in itself) giving a bound on the diameter of the complement of a metric ball.

**Proposition 5.1.** Given $N > 1$ there exists $C_N > 0$ such that the following holds:

Let $(X, d, m)$ be a $\text{CD}(N-1, N)$ space (actually $\text{MCP}(N-1, N)$ would be enough). Let $x, y, z \in X$ be such that $d(x, y) = d(x, z) \geq D$. Then $d(y, z) \leq C_N(\pi - D)$. In particular, for every $x_1 \in X$, there exists $x_2 = x_2(x_1)$ such that

\[X \setminus B_D(x_1) \subset B_{C_N(\pi - D)}(x_2).\]

**Proof.**

*Step 1.* Without loss of generality we can assume $m(X) = 1$. Define $r := \frac{1}{2}d(y, z)$ and let $\mu$ be the $(N-1, N)$-model measure on $[0, \pi]$, i.e., $\mu = h_N\mathcal{L}^1$.

First of all, by the Bishop-Gromov inequality we have

\[m(B_r(y)) \geq \mu([0, r]), \quad m(B_r(z)) \geq \mu([0, r]), \quad m(B_{D-r}(x)) \geq \mu([0, D - r]).\]
Moreover, by construction the sets \( B_{D-r}(x) \), \( B_r(y) \), and \( B_r(z) \) are pairwise disjoint. Thus

\[
1 = m(X) \geq m(B_{D-r}(x)) + m(B_r(y)) + m(B_r(z))
\]

\[
\geq \mu([0, D-r]) + 2\mu([0,r])
\]

\[
= (\mu([0, \pi-r]) - \mu([D-r, \pi-r])) + \mu([\pi-r, \pi]) + \mu([0, r])
\]

\[
= (\mu([0, \pi-r]) + \mu([\pi-r, \pi])) + \mu([0, r]) - \mu([D-r, \pi-r])
\]

\[
= \mu([0, \pi]) + \mu([0, r]) - \mu([D-r, \pi-r])
\]

\[
= 1 + \mu([0, r]) - \mu([D-r, \pi-r]).
\]

It follows that \( \mu([0, r]) \leq \mu([D-r, \pi-r]) \).

**Step 2.** We consider two different cases: if \( r \leq \pi - D \), then we are done. So we can restrict to the case \( r > \pi - D \). Then we have

\[
\int_0^r \sin^{N-1}(t) dt \leq \int_{D-r}^{\pi-r} \sin^{N-1}(t) dt \leq \int_{D-r}^{\pi-r} (\pi-t)^{N-1} dt
\]

\[
= \frac{r^N}{N} \left( \frac{\pi-D}{r} + 1 \right)^N - 1.
\]

By convexity

\[
(1 + s)^N - 1 \leq s(2^N - 1)
\]

provided \( s \leq 1 \), yielding

\[
\frac{N}{r^N} \int_0^r \sin^{N-1}(t) dt \leq (2^N - 1) \frac{\pi-D}{r}.
\]

Since \( \inf_{r \in (0,\pi)} \int_0^r \sin^{N-1}(t) dt / r^N \) is strictly positive, we prove the claim. \( \square \)

The \( d \)-monotonicity of \( \Gamma \), i.e., for any \((x_1, y_1), \ldots, (x_n, y_n) \in \Gamma\),

\[
\sum_{i=1}^n d(x_i, y_i) \leq \sum_{i=1}^n d(x_i, y_{i+1}), \quad y_{n+1} = y_1,
\]

for any \( n \in \mathbb{N} \), is crucial to obtain the next step.

**Lemma 5.2.** The following estimate holds true:

\[
2^{N-1} \left( \frac{1}{C_{N,\nu} C_{N,\nu}'} + C_{N,\nu}'' \right) \delta(E) \geq 
\int_{Q_{\ell}} \left( (\pi - d(P_S(q), P_N(q))) + (\pi - d(P_S(q), P_N(q))) \right)^N q(dq).
\]
PROOF. The $d$-monotonicity of the transport set implies that, for any ray $X_q$ and any $x, y \in X_q$ with $(x, y) \in \Gamma$,

$$2\pi - d(x, y) - d(P_S(q), P_N(q)) \geq 2\pi - d(x, P_N(q)) - d(y, P_S(q)).$$

We can rewrite this as

$$\pi - d(x, y) + \pi - D_q \geq \pi - d(x, P_N(q)) + \pi - d(y, P_S(q)).$$

In particular, if we take $x = P_S(q)$ and $y = P_N(q)$, we deduce that

$$\pi - D_q + \pi - D_q \geq \pi - d(P_S(q), P_N(q)) + \pi - d(P_S(q), P_N(q)).$$

Then from Theorem 4.3 it follows that

$$2^{-N}(\pi - d(P_S(q), P_N(q)) + \pi - d(P_S(q), P_N(q)))^N.$$

Recalling (4.5), we deduce that

$$\int_{Q_\ell} \left(\pi - d(P_S(q), P_N(q)) + \pi - d(P_S(q), P_N(q))\right)^N q(dq),$$

proving the claim. \qed

For $\beta \in (0, 1)$ to be fixed later, it is then natural to consider the following sets of rays:

$$Q_\ell^1 := \{q \in Q_\ell: d(P_S(q), P_N(q)) \leq \pi - \delta(E)^{\beta/N}\},$$

$$Q_\ell^2 := \{q \in Q_\ell: d(P_S(q), P_N(q)) \leq \pi - \delta(E)^{\beta/N}\},$$

notice that both $Q_\ell^1$ and $Q_\ell^2$ are Borel sets.

**Lemma 5.3.** The following estimates hold true:

$$q(Q_\ell^1) \leq C(N, \nu)\delta(E)^{1-\beta}, \quad q(Q_\ell^2) \leq C(N, \nu)\delta(E)^{1-\beta}.$$

**Proof.** From Lemma 5.2 we deduce that

$$\int_{Q_\ell} \left(\pi - d(P_S(q), P_N(q)) + \pi - d(P_S(q), P_N(q))\right)^N q(dq)$$

for $i = 1, 2$, proving the claim. \qed
We can therefore restrict our analysis to the following family of rays:
\begin{equation}
Q^g_\ell := Q_\ell \setminus (Q^1_\ell \cup Q^2_\ell),
\end{equation}
where “g” stands for “good.” Clearly $Q^g_\ell$ is Borel and
\[
q(Q^g_\ell) \geq 1 - C(N,v)\delta(E)^{1-\beta}.
\]

We can now prove that distances between initial and final points are also controlled by the deficit, provided $q$ belongs to the set of “good” rays. Proposition 5.1 will be now used in a crucial way.

**Corollary 5.4.** There exists a strictly positive constant $C(N,v)$ depending only on $N$ and $v \in (0,1)$ such that
\[
d(P_S(q), P_S(\bar{q})), d(P_N(q), P_N(\bar{q})) \leq C(N,v)\delta(E)^{\beta/N}
\]
for each $q \in Q^g_\ell$.

**Proof.** By definition for each $q \in Q^g_\ell$
\[
\delta(E)^{\frac{\beta}{N}} > \pi - d(P_S(q), P_N(\bar{q})), \quad \delta(E)^{\frac{\beta}{N}} > \pi - d(P_S(\bar{q}), P_N(q)).
\]
Moreover, from Theorem 4.3 we have that $C''_{N,v} \delta(E)^{1/N} \geq \pi - d(P_S(\bar{q}), P_N(\bar{q}))$. Hence Proposition 5.1 implies that
\[
C_N C''_{N,v} \delta(E)^{\frac{\beta}{N}} \geq d(P_S(q), P_S(\bar{q})), \quad C_N C''_{N,v} \delta(E)^{\frac{\beta}{N}} \geq d(P_N(q), P_N(\bar{q})),
\]
proving the claim. \qed

We summarize all the properties obtained so far for the set of good rays $Q^g_\ell$:
\[
q(Q^g_\ell) \geq 1 - C(N,v)\delta(E)^{1-\beta};
\]
\begin{equation}
\text{for each } q \in Q^g_\ell, d(P_S(q), P_S(\bar{q})), d(P_N(q), P_N(\bar{q})) \leq C(N,v)\delta(E)^{\frac{\beta}{N}};
\end{equation}
\[
\text{for each } q \in Q^g_\ell, D_q = d(P_S(q), P_N(q)) \geq \pi - C(N,v)\delta(E)^{\frac{\beta}{N}};
\]
where $C(N,v) > 0$ is a positive constant depending only on $N > 1$ and $v \in (0,1)$, and $\bar{q} \in Q$ is the distinguished ray from Theorem 4.3.

### 6 Quantitative Isoperimetric Inequality

We then want to separate the rays such that $E_q := E \cap X_q$ has an optimal competitor staying at the south pole from the rays having it at the north pole. For $\gamma \in (0,1)$ to be chosen later, we therefore continue considering the following subsets of rays:
\[
Q^S_\ell := \{q \in Q^g_\ell: m_q(E_q \Delta [0,r_{\bar{q}}]) \leq \delta(E)^{\gamma}\},
\]
\[
Q^N_\ell := \{q \in Q^g_\ell: m_q(E_q \Delta [r^+_q,D_q]) \leq \delta(E)^{\gamma}\}.
\]
where $D_q = |X_q|$ and $r_q^± := r_{h_q}^±(v) \in (0, D_q)$, with $m_q = h_q L^1$, are the unique points such that
\[
v = \int_0^{r_{h_q}^- (v)} h_q(t)dt = v = \int_{r_{h_q}^+ (v)}^{D_q} h_q(t)dt.\]

We will show that at least one of the previous sets of rays must have small measure.

First we need to prove that the $Q^S, Q^N$ are measurable. We start with the following measurability result:

**Lemma 6.1.** For any $v \in [0, 1]$, the maps
\[
Q \ni q \mapsto r_{h_q}^±(v)
\]
are Borel.

**Proof.** First we recall from Section 2.5 that the density $(q,t) \mapsto h_q(t)$ is Borel. Then from Fubini’s theorem, for each $r \in \mathbb{R}$,
\[
Q \ni q \mapsto \int_0^r h_q(t)dt
\]
is Borel; since for each $q \in Q$, $r \mapsto \int_0^r h_q(t)dt$ is continuous, it follows that $Q \times \mathbb{R} \ni (q,r) \mapsto \int_0^r h_q(t)dt$ is Borel. Then
\[
\text{graph}(r^-) = \{(q,r) \in Q \times \mathbb{R} : v = \int_0^r h_q(t)dt\}
\]
yields that graph$(r^-)$ is Borel and the claim follows. The same holds true for $r^+$. \qed

Then we can conclude as follows that $Q^S, Q^N$ are measurable: from the proof of the previous lemma, we get that for each $v \in (0, 1)$
\[
\Lambda := g((q,t) \in Q \times [0,\infty) : t \leq r_{h_q}^-(v))
\]
is an analytic set (graph$(r^-)$ is Borel); since
\[
m_q(E_q \Delta [0, r_{h_q}^- (v)]) = m_q(E \Delta \Lambda),
\]
from the measurability of the disintegration, it follows that $Q^S$ is $q$-measurable. The same holds for $Q^N$; possibly passing to subsets with the same $q$-measure, we can assume both of them to be Borel.

We now analyze $Q^S$ and $Q^N$. We first show the next lemma.

**Lemma 6.2.** Using the notation above it holds that
\[
q(Q^S \setminus (Q^N \cup Q^S)) \leq \frac{1}{C(N, v)} \delta^{1-\gamma}(E).
\]
PROOF. From Proposition 3.1 we know that for $\delta(E)$ sufficiently small
\[ P_{h,q}(E) - \mathcal{I}_{h,q}(v) \geq C(N,v) \min\{m_q(E_\Delta[0, r^-_q(v)]), m_q(E_\Delta[r^+_q(v), D_q])\} \].
We infer
\[ \delta(E) = P(E) - \mathcal{I}_{\pi}(v) \geq \int_Q (P_{h,q}(E) - \mathcal{I}_{h,q}(v))q(dq) \]
\[ \geq C(N,v) \int_{Q^c \setminus (Q^N \cup Q^S)} \min\{m_q(E_\Delta[0, r^-_q(v)]), m_q(E_\Delta[r^+_q(v), D_q])\}q(dq) \]
\[ \geq C(N,v)\delta(E)^\gamma q(Q^c \setminus (Q^N \cup Q^S)), \]
proving the claim. \(\square\)

For the reader’s convenience we include here an easy one-dimensional result.

**Lemma 6.3.** Let $f : X \to [0, 1]$ be a Borel function such that $\int f(x)\xi(dx) = c > 0$, with $\xi$ positive finite Borel measure. Then
\[ \xi(\{x \in X : f(x) \geq a\}) \geq \frac{c - a K}{1 - a}, \]
where $K = \xi(X)$.

**Proof.** Just note that
\[ c = \int_{\{f \geq a\}} f \xi + \int_{\{f < a\}} f \xi \leq \xi(\{f \geq a\}) + a \xi(\{f < a\}) \]
\[ = \xi(\{f \geq a\}) + a(K - \xi(\{f \geq a\})) \]
\[ = \xi(\{f \geq a\})(1 - a) + aK, \]
and the claim follows. \(\square\)

**Proposition 6.4.** For any $C > 0$ and any $\alpha > 0$ such that
\[ \alpha < \frac{N}{2N - 1} \min\{\gamma, 1 - \gamma, 1 - \beta\}, \]
there exists $\delta > 0$ such that, whenever $\delta(E) < \delta$, then the following inequality holds:
\[ \min\{\eta(Q^S), \eta(Q^N)\} \leq C \delta(E)^\alpha. \]

**Proof.** Suppose by contradiction the claim is false:
\[ \eta(Q^S), \eta(Q^N) > C \delta(E)^\alpha, \]
with $\alpha$ verifying the inequality of the statement; then we argue as follows.
Consider the set 
\[ E_S := \bigcup_{q \in Q_\ell^S} E_q, \quad E_N := \bigcup_{q \in Q_\ell^N} E_q, \quad E_b := \bigcup_{q \in Q_\ell^B} E_q, \]
where \( Q_\ell^B := Q_\ell^1 \cup Q_\ell^2 = Q_\ell \setminus Q_\ell^S \); notice that \( E_b \) coincides up to a set of m-measure zero with \( E \setminus E_S \cup E_N \). We will accordingly decompose the perimeter of \( E \) and eventually find a contradiction to a small deficit.

Step 1. Consider a ball \( B_r(P_S(\tilde{q})) \), which for ease of notation we simply denote by \( B_r^S \), with \( r > 0 \) such that

- for each \( q \in Q_\ell^N \) the interval \( [r_q^+, D_q] \cap B_{3r}^S = \emptyset \);
- for each \( q \in Q_\ell^S \), \( d(P_S(q), \partial B_{3r}^S) < r_q^- - \varepsilon \), for some \( \varepsilon > 0 \),

where \( B_{3r}^S \) denotes the ball centered in \( P_S(\tilde{q}) \) as well with radius \( 3r \).

For \( q \in Q_\ell^S \) it holds that \( D_q \geq \pi - C(N, v)\delta(E)^{\beta/N} \), which implies (see Proposition A.3) that

\[ |r_q^- - r_N^-(v)|, |r_q^+ - r_N^+(v)| \leq C(N, v)\delta(E)^{\frac{\beta}{N}}, \]

showing that we can choose \( r \) sufficiently small so that the previous properties are verified, at least for \( \delta(E) \) sufficiently small. Notice that as \( \delta(E) \) approaches 0, \( r \) can be considered fixed.

We now estimate the amount of mass of \( E \) contained in \( B_r^S \):

\[ m(E \cap B_r^S) \geq m(E_S \cap B_r^S) = \int_{Q_\ell^S} m_q(E_q \cap B_r^S) q(dq) \]
\[ \geq \int_{0}^{Q_\ell^S} m_q([0, r_q^-] \cap B_r^S) - m_q(E_q \Delta [0, r_q^-]) q(dq). \]

By the triangle inequality,

\[ [0, r_q^-] \cap B_r^S \supset [0, r - d(P_S(\tilde{q}), P_S(q))] \supset [0, r - C(N, v)\delta(E)^{\frac{\beta}{N}}]; \]

then we can continue as follows:

\[ \geq C(r)q(Q_\ell^S) - \delta(E)^{\gamma} \geq C(r)\delta(E)^{\alpha} - \delta(E)^{\gamma}, \]

with \( C(r) \) only depending on the radius of \( B_r^S \) remaining positive when \( \delta(E) \) approaches 0 (see again Proposition A.3).

Moreover, since \( B_r^S \cap [r_q^-, D_q] = \emptyset \), it follows that

\[ B_r^S \setminus E = (B_r^S \setminus (E \setminus [r_q^+, D_q])) \]

and therefore

\[ m(B_r^S \setminus E) \geq \int_{Q_\ell^N} m_q(B_r^S \setminus (E \setminus [r_q^+, D_q])) q(dq) \]
\[ \geq \int_{Q_\ell^N} m_q(B_r^S) q(dq) - \delta(E)^{\gamma} \geq C(r)\delta(E)^{\alpha} - \delta(E)^{\gamma}. \]
We will find two contributions to the perimeter of $E$: one coming from the relative perimeter of $E$ inside $B^S_{3r}$ and one coming from the relative perimeter of $E$ inside $X \setminus B^S_{3r+\varepsilon}$. In other words, we decompose (see Section 2.3)

$$\mathcal{P}(E) \geq \mathcal{P}(E, B^S_{3r}) + \mathcal{P}(E, X \setminus B^S_{3r+\varepsilon}).$$

The second contribution will be obtained in Step 3 using the localization of $E$ discussed above; for the first one instead, since we may not have an isoperimetric inequality inside $B^S_r$ (which may not be a convex subset of $X$), we will consider a new localization whose associated transport set is contained in $B^S_{3r}$. This will be discussed in Step 2.

**Step 2.** Consider the localization of the function

$$f = \frac{\chi_{E \cap B^S_r}}{m(E \cap B^S_r)} - \frac{\chi_{B^S_r \setminus E}}{m(B^S_r \setminus E)}.$$

Denote by $\mathcal{T}^1$ the corresponding transport set and consider the associated disintegration

$$m_{\mathcal{T}^1} = \int_{Q^1} m_q^1 d(q),$$

verifying for $q^1$-a.e. $q \in Q^1$ the following properties:

- $m_q^1(X_q^1) = 1$,
- $(X_q^1, d, m_q^1)$ verifies CD$(N - 1, N)$,
- $\int f m_q^1 = 0$.

Since the ray $X_q^1$ starts inside $E \cap B^S_r$ and arrives inside $B^S_r \setminus E$, one can modify the definition of the transport set. This new transport set will still be denoted by $X_q^1 \subset B^S_{3r}$, at the price of obtaining a decomposition of a strict subset of the original transport set, which, however, will be still denoted by $\mathcal{T}^1$ and still contain $B^S_r$, up to an $m$-negligible subset; see for more details [26, sec. 3].

By definition of $\mathcal{P}(E, B^S_{3r})$, for some $\{u_n\}_{n \in \mathbb{N}} \subset \text{Lip}(B^S_{3r})$ with $u_n \to \chi_E$ in $L^1(B^S_{3r}, m)$,

$$\mathcal{P}(E, B^S_{3r}) = \lim_{n \to \infty} \int_{B^S_{3r}} |\nabla u_n|(x)m(dx) \geq \lim_{n \to \infty} \int_{B^S_{3r} \cap \mathcal{T}^1} |\nabla u_n|(x)m(dx) \geq \lim_{n \to \infty} \int_{Q^1} \int_{B^S_{3r} \cap \mathcal{T}^1} |\nabla u_n|(x)m_q^1 dx m_q^1(q)(dq).$$

Notice now that for $q^1$-a.e. $q \in Q^1$, the map $u_n$ restricted to the ray $X_q^1$ is still Lipschitz and converges to $\chi_{E \cap X_q^1}$ in $L^1(B^S_{3r} \cap X_q^1, m_q^1)$; as observed before, $X_q^1 \subset B^S_{3r}$, hence the previous chain of inequalities can be continued using Fatou's
lemma as follows:
\[ P(E, B^S_{\delta r}) \geq \int_{Q^1} P_q(E, X^1_{d}) dq^1(dq) \geq \int_{Q^1} I_{N-1, N, \pi}(m^1_q(E)) dq^1(dq). \]

where \( P_q(E, X^1_{d}) \) is the one-dimensional perimeter of \( E \) in the one-dimensional open set \( X^1_{d} \) with respect to the one-dimensional measure \( m^1_q \). The last inequality holds thanks to the fact that \( (X^1_{d}, d, m^1_q) \) is a CD\((N - 1, N)\)-space.

Now from the localization, it follows that for \( q^1 \)-a.e. \( q \in Q^1 \)
\[ m^1_q(E \cap B^S_r) = \frac{m(E \cap B^S_r)}{m(B^S_r \setminus E)} m^1_q(B^S_r \setminus E). \]

As \( m^1_q \) is a probability measure \( q^1 \)-a.e. and
\[ m(E \cap B^S_r) = \int_{Q^1} m^1_q(E \cap B^S_r) dq^1(dq), \]
from Lemma 6.3 we deduce the next inequality, for any \( a \in [0, 1) \):
\[ q^1 \left( \{q \in Q^1 : m^1_q(E \cap B^S_r) \geq a \} \right) \geq \frac{m(E \cap B^S_r) - am(T^1)}{1 - a} \geq \frac{m(E \cap B^S_r) - a}{1 - a}. \]

Choosing \( a = \frac{m(E \cap B^S_r)}{2} \) and denoting
\[ \tilde{Q}^1 := \{q \in Q^1 : m^1_q(E \cap B^S_r) \geq \frac{m(E \cap B^S_r)}{2} \}, \]
we obtain the inequality
\[ q^1(\tilde{Q}^1) \geq \frac{m(E \cap B^S_r)}{2}. \]

From the definition of \( \tilde{Q}^1 \) and (6.5) it follows that
\[ \frac{1}{2} \frac{m(E \cap B^S_r)}{m(B^S_r \setminus E)} \leq m^1_q(E \cap B^S_r) \leq \frac{1}{2} m(B^S_r \setminus E) \leq m^1_q(B^S_r \setminus E), \]
\( q^1 \)-a.e. \( q \in \tilde{Q}^1 \).

Hence we obtain immediately that
\[ m^1_q(E) \geq m^1_q(E \cap B^S_r) \geq \frac{1}{2} m(E \cap B^S_r), \]
\( q^1 \)-a.e. \( q \in \tilde{Q}^1 \).

and
\[ m^1_q(E) \leq 1 - m^1_q(B^S_r \setminus E) \leq 1 - \frac{1}{2} m(B^S_r \setminus E), \]
\( q^1 \)-a.e. \( q \in \tilde{Q}^1 \).

Combining the last estimate with (6.2) and (6.3) we obtain that
\[ \frac{1}{2}(C(r)\delta(E)^a - \delta(E)^a) \leq m^1_q(E) \leq 1 - \frac{1}{2}(C(r)\delta(E)^a - \delta(E)^a), \]
\( q^1 \)-a.e. \( q \in \tilde{Q}^1 \).
Assuming \( \gamma \in (\alpha, 1) \), this implies that for \( q \)-a.e. \( q \in \mathcal{Q}^1 \),

\[
\mathcal{T}_\pi (m_1^q (E)) \geq \tilde{C} (N, v, r) \delta (E)^{\gamma \frac{N-1}{N}} ;
\]

hence, recalling (6.4), we obtain

\[
P(E, B_{3r}^S) \geq \tilde{C} (N, v, r) \delta (E)^{\gamma} \frac{N-1}{N} \cdot q(\mathcal{Q}^1)
\geq \tilde{C} (N, v, r) \delta (E)^{\gamma} \frac{2N-1}{N} \geq \tilde{C} (N, v) \delta (E)^{\gamma} \frac{2N-1}{N} ,
\]

where the second inequality follows from (6.2) and (6.6).

**Step 3.** Now we take into account the contribution to the perimeter of \( E \) inside \( X \setminus B_{3r+\epsilon}^S \). Reasoning as we did at the beginning of Step 2, we use the one-dimensional rays of the localization of \( E \) to obtain the next inequality

\[
P(E, X \setminus B_{3r+\epsilon}^S)
\geq \int_{Q_\ell} P_q (E_q, X_q \setminus B_{3r+\epsilon}^S) q( dq)
= \int_{Q_\ell} P_q (E_q, X_q \setminus B_{3r+\epsilon}^S) q( dq) + \int_{Q_\ell} P_q (E_q, X_q \setminus B_{3r+\epsilon}^S) q( dq),
\]

where \( P_q (E_q, X_q \setminus B_{3r+\epsilon}^S) \) is the one-dimensional relative perimeter of \( E \) in the one-dimensional open set \( X_q \setminus B_{3r+\epsilon}^S \subset X_q \) with respect to the one-dimensional measure \( m_q \).

For \( q \in Q_\ell^S \), by definition we know that \( m_q (E_q \Delta [0, r_q^-]) \leq \delta (E)^{\gamma} \); in particular,

\[
m_q (E \setminus [0, r_q^-]) \leq \delta (E)^{\gamma}.
\]

Notice that since \( P_q (E_q) < \infty \), up to a set of \( m_q \)-measure zero, we can assume it to be the countable union of closed sets. This will not affect any of the quantities involved in this proof. We now claim that

\[
[r_q^- - 2\delta (E)^{\gamma} / C_{N,q,v}, r_q^- + 2\delta (E)^{\gamma} / C_{N,q,v}] \cap \partial E_q \neq \emptyset
\]

where

\[
C_{N,q,v} := \min \{ h_q (t) : t \in [r_q^- / 2, r_q^- + (D_q - r_q^-) / 2] \}
\]

is uniformly positive for \( q \in Q_\ell \) and \( \delta (E) \in (0, \tilde{\delta} (N)) \), by Proposition [A.3]

We start the proof of (6.9) by showing that

\[
[r_q^- - 2\delta (E)^{\gamma} / C_{N,q,v}, r_q^-] \cap E_q \neq \emptyset.
\]

So suppose by contradiction that (6.10) is false. Since

\[
m_q (\{ r_q^- - 2\delta (E)^{\gamma} / C_{N,q,v}, r_q^- \}) \geq 2 \delta (E)^{\gamma},
\]
we deduce that

\[
 m_q(E_q \cap [0, r_q^-]) = m_q(E_q \cap [0, r_q^- - 2\delta(E)^\gamma]) \leq m([0, r_q^- - 2\delta(E)^\gamma]) \\
 \leq v - m([r_q^- - 2\delta(E)^\gamma, r_q]) \\
 \leq v - 2\delta(E)^\gamma.
\]

It follows that

\[
 v = m_q(E_q \setminus [0, r_q^-]) + m_q(E_q \cap [0, r_q^-]) \leq m_q(E_q \setminus [0, r_q^-]) + v - 2\delta(E)^\gamma,
\]

thus contradicting (6.8).

Hence (6.10) is proved. To obtain (6.9), observe analogously that

\[
 m_q([r_q^-, r_q^- + 2\delta(E)^\gamma / C_{N,q,v}] \cap \partial E_q.
\]

From (6.9), we deduce that for \( q \in \mathcal{Q}_i^S \) it holds that

\[
 \mathcal{I}(v) \geq \mathcal{P}(q) = \mathcal{I}(v) - \mathcal{I}(v) - (\mathcal{I}(v) - \mathcal{I}(v))
\]

Step 4. We now collect all the steps to reach a contradiction as follows:

\[
 \delta(E) \geq \mathcal{P}(E) - \mathcal{I}(v)
\]

\[
 \geq \mathcal{P}(E; B_{3r}^S) + \mathcal{P}(E; X \setminus B_{3r+e}^S) - \mathcal{I}(v)
\]

\[
 \geq \hat{C}(N, v)\delta(E)^{2N-1} + \int_{\mathcal{Q}_i^S} \mathcal{P}(E_q; X \setminus B_{3r+e}^S) q(dq)
\]

\[
 + \int_{\mathcal{Q}_i^S} \mathcal{P}(E_q; X \setminus B_{3r+e}^S) q(dq) - \mathcal{I}(v)
\]

\[
 \geq \hat{C}(N, v)\delta(E)^{2N-1} + q(Q_i^N \cup Q_i^S)(\mathcal{I}(v) - C\delta(E)^\gamma) - \mathcal{I}(v)
\]

\[
 \geq \hat{C}(N, v)\delta(E)^{2N-1} + \left(1 - C(N, v)\delta(E)^{1-\beta} - \frac{1}{C(N, v)}\delta(E)^{1-\gamma}\right)(\mathcal{I}(v) - C\delta(E)^\gamma)
\]

\[- \mathcal{I}(v),
\]
where in the last estimate we made use of (5.5) and (6.1). Since all the constants are stable for \( \delta(E) \) approaching 0, the last inequality shows a contradiction provided

\[
\alpha \frac{2N - 1}{N} < \min\{\gamma, 1 - \gamma, 1 - \beta\}
\]

for \( \delta(E) \) below a threshold depending only on \( N \) and \( v = m(E) \). \( \square \)

**Remark 6.5.** In case \((X, d)\) is the metric space associated to a smooth Riemannian manifold, then \( r \) can be chosen small enough so that \( B^S_r \) is a convex. In this case it follows that \((B^S_r, d|_{B^S_r}, m|_{B^S_r})\) is a nonbranching \( \text{CD}(N - 1, N) \)-space. Therefore Step 2 above can be simplified as we can directly apply the Levy-Gromov inequality stated in Theorem 2.5 and get the better estimate

\[
P(E, B) \geq \hat{C}(N, v)\delta(E)^{\alpha \frac{N-1}{N}}.
\]

Repeating verbatim the other steps of the proof, we reach a contradiction provided \( \alpha \) verifies the less restrictive inequality

\[
\alpha < \frac{N}{N - 1} \min\{\gamma, 1 - \gamma, 1 - \beta\}.
\]

In particular, we have the next result.

**Proposition 6.6.** Suppose \((X, d, m)\) is a \( \text{CD}(N - 1, N) \)-space, \( N \geq 2 \), with a \((X, d)\) metric space associated to a smooth Riemannian manifold. Then there exists \( \bar{\delta} > 0 \) such that, whenever \( \delta(E) < \bar{\delta} \), then the following inequality holds:

\[
\min\{q(Q^S_\ell), q(Q^N_\ell)\} \leq C\delta(E)^\alpha \quad \text{for any} \quad \alpha < \frac{N}{N - 1} \min\{\gamma, 1 - \gamma, 1 - \beta\}.
\]

We are now in position to prove the other main result of the paper.

**Proof of Theorem 1.1.** We first observe that the first claim has already been obtained in Theorem 4.3. We are therefore left with showing the second claim.

From Proposition 6.4 we assume with no loss in generality that \( q(Q^N_\ell) \leq C \delta(E)^\alpha \), and consequently we pick \( \bar{x} = P_S(\bar{q}) \).

We will now use the following notation:

\[
Q_{\text{bad}} = Q \setminus Q^S_\ell.
\]

From (5.5) and (6.1) we have

\[
q(Q^S_\ell) \geq q(Q^g_\ell) - q((Q^N_\ell \cup Q^S_\ell)) - q(Q^N_\ell) \\
\geq 1 - C(N, v)\delta(E)^{1-\beta} - \frac{1}{C(N, v)}\delta(E)^{1-\gamma} - C\delta(E)^\alpha.
\]

Recalling that \( \alpha < \min\{1 - \gamma, 1 - \beta\} \), we get \( q(Q_{\text{bad}}) \leq C_{\text{fin}}\delta(E)^\alpha \) with \( C_{\text{fin}} \) depending on \( N \) and \( v \), and therefore

\[
m(X \setminus \Omega^{-1}(Q_{\text{bad}})) \leq C_{\text{fin}}\delta(E)^\alpha.
\]
Using the definition of $Q^S_\ell$, we obtain
\[
m((E \Delta B_{r_N}(\nu))(\Omega^{-1}(Q^S_\ell)))
= \int_{Q^S_\ell} m_q(E \Delta B_{r_N}(\nu)(\Omega^{-1}(Q^S_\ell))) d\nu
\leq \int_{Q^S_\ell} m_q(E_q \Delta[0, r^-_q]) d\nu + \int_{Q^S_\ell} m_q([0, r^-_q] \Delta B_{r_N}(\nu)) d\nu
\leq \delta(E)^v + \int_{Q^S_\ell} m_q([0, r^-_q] \Delta B_{r_N}(\nu)) d\nu.
\]
From Corollary 5.4 we have that $d(P_S(q), P_S(\bar{q})) \leq C(N, v) \delta(E)^{\beta/N}$; it follows that
\[
[0, r^-_q] \Delta B_{r_N}(\nu) \subset [r^-_q - C(N, v)\delta(E)^{\beta \over N}, r^-_q + C(N, v)\delta(E)^{\beta \over N}],
\]
which implies that
\[
m_q([0, r^-_q] \Delta B_{r_N}(\nu)) \leq 2 \|h_q\|_{\infty} C(N, v)\delta(E)^{\beta \over N}.
\]
Since the rays of $Q_\ell$ are uniformly long once the deficit is assumed to be smaller than 1/10, we have $\|h_q\|_{\infty} \leq C(N)$, and therefore
\[
\int_{Q^S_\ell} m_q([0, r^-_q] \Delta B_{r_N}(\nu)) d\nu \leq 2C(N)C(N, v)\delta(E)^{\beta \over N}.
\]
We conclude that
\[
m(E \Delta B_{r_N}(\nu)) \leq m((E \Delta B_{r_N}(\nu)) \cap \Omega^{-1}(Q^S_\ell)) + m(X \setminus \Omega^{-1}(Q_{bad}))
\leq 2C(N)C(N, v)\delta(E)^{\beta \over N} + C_{\text{fin}}\delta(E)^{\alpha},
\]
giving the claim for $\mathcal{C}D(N - 1, N)$-spaces for
\[
\eta := \min \left\{ \frac{N}{2N - 1}, \min\{\gamma, 1 - \gamma, 1 - \beta\}, \frac{\beta}{N} \right\}.
\]
It is easy to check that for $\beta \in (0, 1)$, $\gamma \in (0, 1)$, the right-hand side is maximized for $\gamma = \frac{1}{2}$ and $\beta = \frac{N^2}{N^2 + 2N - 1}$ giving $\eta = \frac{N}{N^2 + 2N - 1}$.

For smooth Riemannian manifolds, we can follow the same arguments by using Proposition 6.6 to improving the claim with
\[
\eta := \min \left\{ \frac{N}{N - 1}, \min\{\gamma, 1 - \gamma, 1 - \beta\}, \frac{\beta}{N} \right\}.
\]
It is easy to check that for $\beta \in (0, 1)$, $\gamma \in (0, 1)$, the right-hand side is maximized for $\gamma = \frac{1}{2}$, and $\beta = \frac{N^2}{N^2 + N - 1}$, giving $\eta = \frac{N}{N^2 + N - 1}$. □
Appendix A Technical Lemmas

Let $h: [0, D] \to [0, \infty)$ be a CD$(N - 1, N)$ density; i.e., we have, for any $0 \leq t_0 < t_1 \leq D$,

$$\sin((t_1 - t_0))h^{\frac{1}{N-1}}((1 - s)t_0 + st_1) \geq \sin((1 - s)(t_1 - t_0))h^{\frac{1}{N-1}}(t_0)$$
\[+ \sin(s(t_1 - t_0))h^{\frac{1}{N-1}}(t_1).\]
(A.1)

Note that the above inequality is nothing but the synthetic version of the differential inequality

$$h^{\frac{1}{N-1}} \geq h^{\frac{1}{N-1}}(0);$$

the previous condition together with $\int_0^D h(t)dt = 1$ implies that $h > 0$ over $[0, D]$.

Here we are interested in studying the behavior of a general CD$(N - 1, N)$ density $h$ with almost maximal domain in the sense $\pi - D = \varepsilon \ll 1$. In particular, we look for uniform estimates, i.e., only depending on $\varepsilon$.

We recall the definition of the model CD$(N - 1, N)$ density:

$$h_N(t) := \frac{1}{\omega_N} \sin^{N-1}(t), \quad t \in [0, \pi],$$
$$\omega_N := \int_0^\pi \sin^{N-1}(t)dt.$$

Then the following estimates hold.

Lemma A.1. For any $t \in (0, D)$ the following holds: if $s > 0$ is such that $t + s \leq D$, then

$$\frac{h_N(t + s + \varepsilon)}{h_N(t + \varepsilon)} \leq \frac{h(t + s)}{h(t)} \leq \frac{h_N(t + s)}{h_N(t)}.$$  \hspace{1cm} (A.2)

Proof.

Step 1. Assume $t_1 = D = \pi - \varepsilon$ and $t_0$ is any element of $(0, D)$:

$$\sin(\pi - \varepsilon - t_0)h^{\frac{1}{N-1}}(t_0 + s(\pi - \varepsilon - t_0)) \geq \sin((1 - s)(\pi - \varepsilon - t_0))h^{\frac{1}{N-1}}(t_0).$$

Defining $\tau_0 := s(\pi - \varepsilon - t_0)$, we have

$$\frac{h^{\frac{1}{N-1}}(t_0 + \tau_0)}{h^{\frac{1}{N-1}}(t_0)} \geq \frac{\sin((1 - s)(\pi - \varepsilon - t_0))}{\sin(\pi - \varepsilon - t_0)} = \frac{\sin(\pi - \varepsilon - t_0 - \tau_0)}{\sin(\pi - \varepsilon - t_0)} = \frac{\sin(t_0 + \tau_0 + \varepsilon)}{\sin(t_0 + \varepsilon)},$$

for every $\tau_0 > 0$ such that $t_0 + \tau_0 \leq D$, thus yielding the first inequality in (A.2).

Step 2. Now fix $t_0 = 0$ and any $t_1$ to any element of $(0, D)$:

$$\sin(t_1)h^{\frac{1}{N-1}}(st_1) \geq \sin(st_1)h^{\frac{1}{N-1}}(t_1).$$
Now define \( t = st_1 \) and \( \tau := (1 - s)t_1 \) and obtain

\[
\frac{h_{N^{\frac{1}{n-1}}}(t + \tau)}{h_{N^{\frac{1}{n-1}}}(t)} \leq \frac{\sin(t + \tau)}{\sin(t)}
\]

for any \( t, \tau > 0 \) such that \( t + \tau \leq D \). Hence the second inequality in (A.2) is also proved. \( \square \)

Applying the logarithm to (A.2) and taking the limit as \( s \to 0 \), one obtains the next bound on the derivative of a general \( h \). Before stating it, recall that we set \( \lambda_D := \int_0^D h_N(t)dt \).

**Corollary A.2.** Any \( CD(N - 1, N) \) density \( h: [0, D] \to [0, \infty) \) is locally Lipschitz and for any \( t \in (0, D) \) point of differentiability of \( h \),

\[
\frac{h_N'(t + \varepsilon)}{h_N(t + \varepsilon)} \leq \frac{h'(t)}{h(t)} \leq \frac{h_N'(t)}{h_N(t)},
\]

where \( \varepsilon = \pi - D \). In particular, for any \( t \in (0, D) \) there exists \( \varepsilon_0 = \varepsilon_0(t) \) such that

\[
\text{(Lip } h)(t) \leq C(N, t)
\]

for any \( \varepsilon \leq \varepsilon_0(t) \).

From Lemma [A.1] it is fairly easy to obtain the following result:

**Proposition A.3.** For any \( t \in (0, D) \)

\[
\left( \frac{\omega_N}{\omega_N \lambda_D + \varepsilon} \right) \min\{h_N(t), h_N(t + \varepsilon)\} \leq h(t) \leq \left( \frac{\omega_N}{\omega_N - \varepsilon} \right) \max\{h_N(t), h_N(t + \varepsilon)\},
\]

with \( C = C(N, D) \).

**Proof.** Fix \( t \in (0, D) \); then from Lemma [A.1]

\[
h(t)h_N(t + s) \geq h(t + s)h_N(t)
\]

for any \( s > 0 \) such that \( t + s < D \). In particular,

\[
(A.3) \quad h(t) \int_t^D h_N(r)dt \geq h_N(t) \int_t^D h(r)dt.
\]

Similarly, for any \( s \leq 0 \) with \( t + s > 0 \),

\[
h(t)h_N(t + s + \varepsilon) \geq h(t + s)h_N(t + \varepsilon),
\]

which implies

\[
(A.4) \quad h(t) \int_t^{t+\varepsilon} h_N(r)dr \geq h_N(t + \varepsilon) \int_0^t h(r)dr.
\]
Adding (A.3) and (A.4) one obtains
\[
\left( \int_{t}^{t+\varepsilon} h_{N}(r)dr + \int_{t}^{D} h_{N}(r)dr \right) h(t) \geq \min\{h_{N}(t), h_{N}(t+\varepsilon)\}.
\]
Since \(\int_{t}^{t+\varepsilon} h_{N}(r)dr + \int_{t}^{D} h_{N}(r)dr \leq \lambda_{D} + \varepsilon/\omega_{N}\), it follows that
\[
\left( 1 + \frac{\varepsilon}{\omega_{N}\lambda_{D}} \right) h(t) \geq \frac{1}{\lambda_{D}} \min\{h_{N}(t), h_{N}(t+\varepsilon)\},
\]
which implies the first part of the claim.

The second part follows analogously: for \(s > 0\)
\[
h(t)h_{N}(t+s+\varepsilon) \leq h(t+s)h_{N}(t+\varepsilon),
\]
which implies
\[
h(t) \int_{t+\varepsilon}^{\pi} h_{N}(r)dr \leq h_{N}(t+\varepsilon) \int_{t}^{D} h(r)dr.
\]
For any \(s \leq 0\) with \(t + s > 0\),
\[
h(t)h_{N}(t+s) \leq h(t+s)h_{N}(t),
\]
yielding \(h(t) \int_{0}^{\pi} h_{N}(r)dr \leq h_{N}(t) \int_{t}^{\pi} h(r)dr\); summing the two contributions one obtains
\[
h(t) \left( 1 - \int_{t}^{t+\varepsilon} h_{N}(r)dr \right) \leq \max\{h_{N}(t), h_{N}(t+\varepsilon)\};
\]
since \(\int_{t}^{t+\varepsilon} h_{N}(r)dr \leq \varepsilon/\omega_{N}\), we have proved the claim. \(\square\)

We will also use the following easy monotonicity property.

**Lemma A.4.** Any density \(h: [0, D] \to (0, \infty)\) verifying \(\mathbb{C}D(N - 1, N)\) that integrates to 1 has a unique maximum \(x_{0} \in [0, D]\). Moreover, \(h\) is strictly increasing on \([0, x_{0}]\) and strictly decreasing over \([x_{0}, D]\).

**Proof.** By definition, for any \(t_{0}, t_{1}\) it holds that
\[
h^{\frac{1}{N-1}} ((1-s)t_{0} + st_{1}) \geq \sigma_{N-1,N}^{(1-s)}(t_{1} - t_{0})h^{\frac{1}{N-1}}(t_{0})
\]
\[
+ \sigma_{N-1,N}^{(s)}(t_{1} - t_{0})h^{\frac{1}{N-1}}(t_{1});
\]
since
\[
\sigma_{N-1,N}^{(s)}(t_{1} - t_{0}) = \frac{\sin(s(t_{1} - t_{0}))}{\sin(t_{1} - t_{0})} > s.
\]
In particular, \(h^{\frac{1}{N-1}}\) is strictly concave, implying the claim. \(\square\)
Appendix B  Disintegration Theorem

Given a measure space \((X, \mathcal{X}, m)\), suppose a partition of \(X\) is given into disjoint sets \(\{X_\alpha\}_{\alpha \in Q}\) so that \(X = \bigcup_{\alpha \in Q} X_\alpha\). Here \(Q\) is the set of indices and \(\Omega: X \to Q\) is the quotient map, i.e.,
\[
\alpha = \Omega(x) \iff x \in X_\alpha.
\]
We endow \(Q\) with the push-forward \(\sigma\)-algebra \(\mathcal{Q}\) of \(\mathcal{X}\):
\[
C \in \mathcal{Q} \iff \Omega^{-1}(C) \in \mathcal{X},
\]
i.e., the biggest \(\sigma\)-algebra on \(Q\) such that \(\Omega\) is measurable. Moreover, a measure \(q\) on \((Q, \mathcal{Q})\) can be obtained by pushing forward \(m\) via \(\Omega\), i.e., \(q := \Omega_* m\), obtaining the quotient measure space \((Q, \mathcal{Q}, q)\).

**Definition B.1 (Consistent and Strongly Consistent Disintegration).** A disintegration of \(m\) consistent with the partition is a map
\[
Q \ni \alpha \mapsto m_\alpha \in \mathcal{P}(X, \mathcal{X})
\]
such that the following requirements hold:

1. For all \(B \in \mathcal{X}\), the map \(\alpha \mapsto m_\alpha(B)\) is \(q\)-measurable.
2. For all \(B \in \mathcal{X}\) and \(C \in \mathcal{Q}\), the following consistency condition holds:
\[
m(B \cap \Omega^{-1}(C)) = \int_C m_\alpha(B) q(d\alpha).
\]
A disintegration of \(m\) is called strongly consistent if in addition

3. for \(q\)-a.e. \(\alpha \in Q\), \(m_\alpha\) is concentrated on \(X_\alpha = \Omega^{-1}(\alpha)\).

We now formulate the disintegration theorem (it is formulated for probability measures but clearly holds for any finite nonzero measure):

**Theorem B.2** (theorem A.7, prop. A.9 of [15]). Assume that \((X, \mathcal{X}, m)\) is a countably generated probability space and that \(\{X_\alpha\}_{\alpha \in Q}\) is a partition of \(X\).

Then the quotient probability space \((Q, \mathcal{Q}, q)\) is essentially countably generated, and there exists an essentially unique disintegration \(\alpha \mapsto m_\alpha\) consistent with the partition.

If, in addition, \(\mathcal{X}\) contains all singletons, then the disintegration is strongly consistent if and only if there exists an \(m\)-section \(S_m \in \mathcal{X}\) of the partition such that the \(\sigma\)-algebra on \(S_m\) induced by the quotient map contains \(\mathcal{B}(S_m)\).

Let us expand on the statement of Theorem B.2. Recall that a \(\sigma\)-algebra \(\mathcal{A}\) is countably generated if there exists a countable family of sets so that \(\mathcal{A}\) coincides with the smallest \(\sigma\)-algebra containing them. In the measure space \((Q, \mathcal{Q}, q)\), the \(\sigma\)-algebra \(\mathcal{Q}\) is called essentially countably generated if there exists a countable family of sets \(Q_n \subset Q\) such that for any \(C \in \mathcal{Q}\) there exists \(\hat{C} \in \hat{\mathcal{Q}}\), where \(\hat{\mathcal{Q}}\) is the \(\sigma\)-algebra generated by \(\{Q_n\}_{n \in \mathbb{N}}\) such that \(q(C \Delta \hat{C}) = 0\). Moreover, from [58, prop. 3.3.2] every countably generated measurable space having singletons as
atoms is isomorphic to a subset of \((0, 1], B([0, 1])\); in particular, there exists a topology over \(X\) such that \(\mathcal{X}\) coincide with the Borel \(\sigma\)-algebra of the topology; so the notation \(B(S_m)\) is justified.

Essential uniqueness is understood above in the following sense: if \(\alpha \mapsto m_1^\alpha\) and \(\alpha \mapsto m_2^\alpha\) are two consistent disintegrations with the partition, then \(m_1^\alpha = m_2^\alpha\) for \(\mathfrak{q}\)-a.e. \(\alpha \in Q\).

Finally, a set \(S \subset X\) is a section for the partition \(X = \bigcup_{\alpha \in Q} X_\alpha\) if for any \(\alpha \in Q\), \(S \cap X_\alpha\) is a singleton \(\{x_\alpha\}\). By the axiom of choice, a section \(S\) always exists, and we may identify \(Q\) with \(S\) via the map \(Q \ni \alpha \mapsto x_\alpha \in S\). A set \(S_m\) is an \(m\)-section if there exists \(Y \subset X\) with \(m(X \setminus Y) = 0\) such that the partition \(Y = \bigcup_{\alpha \in Q_m} (X_\alpha \cap Y)\) has section \(S_m\), where \(Q_m = \{\alpha \in Q : X_\alpha \cap Y \neq \emptyset\}\). As \(q = Q \setminus Q_m\), clearly \(q(Q \setminus Q_m) = 0\). As usual, we identify between \(Q_m\) and \(S_m\), so that now \(Q_m\) carries two measurable structures: \(\mathcal{Q} \cap Q_m\) (the push-forward of \(\mathcal{X} \cap Y\) via \(Q\)), and also \(\mathcal{X} \cap S_m\) via our identification. The last condition of Theorem \[B.2\] is that \(\mathcal{Q} \cap Q_m \supset \mathcal{X} \cap S_m\), i.e., that the restricted quotient map \(\Omega|_Y : (Y, \mathcal{X} \cap Y) \to (S_m, \mathcal{X} \cap S_m)\) is measurable, so that the full quotient map \(\Omega : (X, \mathcal{X}) \to (S, \mathcal{X} \cap S)\) is \(m\)-measurable.

We will typically apply the disintegration theorem to \((E, B(E), m_{\leq X})\), where \(E \subset X\) is an \(m\)-measurable subset (with \(m(E) > 0\)) of the m.m.s. \((X, d, m)\). As our metric space is separable, \(B(E)\) is countably generated, and so Theorem \[B.2\] applies.

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