INTRINSIC RANK IN CAT(0) SPACES

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Abstract. Let $X$ be a proper, geodesically complete CAT(0) space which satisfies Chen and Eberlein’s duality condition. We show the existence of a strong notion of rank for $X$ by proving that the parallel sets $P_v$ of geodesics $v$ in $X$ are generically flat. More precisely, let $GX$ be the space of parametrized unit-speed geodesics in $X$. There is a unique $k$ and a dense $G_δ$ set $A$ in $GX$ such that $P_v$ is isometric to flat Euclidean space $\mathbb{R}^k$, for all $v \in A$. It follows that $\mathbb{R}^k$ isometrically embeds in $P_v$ for every $v \in GX$.

1. Introduction

Let $M$ be a Hadamard manifold, that is, a complete simply connected Riemannian manifold of nonpositive curvature. We will denote the set of (unit-speed parametrized) geodesic lines in $M$ by $GM$. For a geodesic line $v$ in $M$ the parallel set $P_v$ of $v$ is the subset of $M$ formed by the union of the images of all geodesic lines parallel to $v$. The set $P_v$ is a closed convex subset of $M$. Recall that the rank of a geodesic line $v$ is the number of linearly independent parallel Jacobi fields along $v$, and the rank of $M$ is the minimum rank of all geodesic lines in $M$.

If $P_v$ is isometric to some Euclidean space $\mathbb{R}^k$, then clearly the rank of $v$ is at least $k$. The converse is true generically, under a condition called the duality condition (see remark after theorem below). We state the result explicitly.

Theorem A (Theorem 2.2 of [13]). Let $M$ be a Hadamard manifold. Assume that $M$ satisfies the duality condition. If the rank of $M$ is $k$, then there is an open dense set $A \subseteq GM$ such that $P_v$ is isometric to $\mathbb{R}^k$, for all $v \in A$. Moreover, Euclidean space $\mathbb{R}^k$ isometrically embeds in $P_v$ for every $v \in GX$.

Remark. The duality condition is equivalent to the following property: the set of $\Gamma$-recurrent geodesics is dense, for some (not necessarily discrete) subgroup $\Gamma$ of the isometry group of $M$. If $M$ admits a geometric action—that is, properly discontinuous, cocompact group action by isometries—then it satisfies the duality condition [2, p.39] due to the presence of the Liouville measure on $GM$. However, if $M$ is homogeneous then $M$ satisfies the duality condition only if $M$ is a symmetric space [12, Proposition 4.9].

In this paper, we are concerned with generalizing the result above to proper, geodesically complete CAT(0) spaces. Our first comment in this regard is that Theorem A is not true in the CAT(0) setting, even if the underlying space is a manifold. In fact, one can construct a proper, geodesically complete CAT(0) metric on $\mathbb{R}^2$ under a rank one geometric action (which satisfies the duality condition by [21, Proposition 3.6]), but the set of geodesics $v$ with $P_v$ isometric to $\mathbb{R}$ does not contain an open dense set. Of course this metric is not Riemannian. We give the idea of this construction in the following paragraph.
Take a surface $S$ of genus $\geq 2$, with a hyperbolic metric. Enumerate the closed geodesics $c_1, c_2, \ldots$ and add thinner and thinner cylinders around each $c_i$. If we do this with some care (see the Appendix for a few more details), we obtain a well defined nonpositively curved geodesic metric; we denote the surface with this new metric by $S'$. Each $c_i$ will be homotopic to a closed geodesic $c'_i$ in $S'$. Moreover each $c'_i$ is contained, by construction, in a thin cylinder. Now just take the universal cover $X$ of $S'$, which is homeomorphic to $\mathbb{R}^2$. Let $B$ be the set of all possible liftings of the $c'_i$ (with varying base points). Then $B$ is dense in $GX$ and for every $v \in B$, $P_v$ is an infinite strip, hence not isometric to any Euclidean space. Therefore the set $A$ of all $v \in GX$ with $P_v$ isometric to $\mathbb{R}^k$ does not contain an open set.

The metric in the example above is not smoothly Riemannian, but it is $C^0$-Riemannian, and it is the limit of smooth Riemannian metrics.

However, in the example above one can prove that the set $A$ is a dense $G_\delta$ set. Recall that a dense $G_\delta$ set is a countable intersection of dense open sets. These sets behave like open dense sets in the sense that the intersection of two dense $G_\delta$ sets is also a dense $G_\delta$ set. Moreover, any countable intersection of dense $G_\delta$ sets is also a dense $G_\delta$ set. This is why a property is called generic if the set of objects satisfying the property is a dense $G_\delta$ set. The main result of this paper is a generalization of Theorem A.

**Main Theorem.** Let $X$ be a proper, geodesically complete CAT(0) space. Assume that $X$ satisfies the duality condition. Then there is a unique $k$ and a dense $G_\delta$ set $A \subseteq GX$ such that $P_v$ is isometric to $\mathbb{R}^k$, for all $v \in A$. Moreover, Euclidean space $\mathbb{R}^k$ isometrically embeds in $P_v$ for every $v \in GX$.

Note that now, assuming the duality condition, we can define the rank of $X$ as the number $k$ given by the theorem. We call this number the intrinsic rank of $X$.

We remark that a result of Ballmann (Theorem III.2.4 in [2]) states the following for a proper, geodesically complete CAT(0) space that satisfies the duality condition: If the geodesic flow on $GX$ does not have a dense orbit mod $\text{Isom}(X)$—equivalently, the $\text{Isom}(X)$-action on $\partial X$ is not minimal—then the intrinsic rank of $X$ is at least 2. (In other words, every geodesic in $X$ is contained in a full 2-flat.) Our Main Theorem not only applies even if the boundary action is not known to be non-minimal, but it also provides a dense set of geodesics for which the parallel set is exactly a $k$-flat instead of some larger space.

It is interesting to note that one can propose several natural ways of generalizing the concept of Riemannian rank to geodesic spaces. For instance, the rank of a geodesic $v$ is two if it is contained in a flat plane, or it is contained in the interior of an infinite strip, or if it bounds a half plane, or half bounds an infinite strip. One can also fix the width of the strip, or take strips with varying widths. In higher dimensions there are even more choices: $v$ is contained in an Euclidean space $\mathbb{R}^k$, or $v$ is contained in the boundary of a half Euclidean space, or in the boundary of a quarter Euclidean space (which could be “at the vertex” or not), and so on. Our result shows that, under the duality condition, all these possible generalizations are equivalent; they yield the same concept of the rank of a space.

In particular, if one can find a single geodesic line $v$ in $X$ such that $P_v$ does not contain a flat plane, then $X$ has rank one and there is a dense $G_\delta$ set of geodesics $w$ with $w$ being parallel only to itself. It follows (still assuming the duality condition, of course) that $X$ contains a dense set of so-called rank one axes—geodesics on which nontrivial elements of $\text{Isom}(X)$ act by translation and which do not bound
a flat half-plane \cite{2} Theorem III.3.4]. Under such conditions \(X\) is known to exhibit a fair degree of hyperbolic behavior.

This generalization of the concept of rank is also relevant in the formulation of a CAT(0) version of the Rank Rigidity Theorem of Ballmann, Brin, Burns, Eberlein, Heber, and Spatzier for Hadamard manifolds \cite{2,7,13}. This celebrated result states that if a Hadamard manifold satisfies the duality condition and has rank at least two, then it is either a symmetric space or a Riemannian product. A version of this result holds for CAT(0) cube complexes, by Caprace and Sageev \cite{9}. It is conjectured that an appropriate generalization also holds for CAT(0) spaces. Using the intrinsic rank we can state this conjecture in the following way.

**CAT(0) Rank Rigidity Conjecture.** Let \(X\) be a proper, geodesically complete CAT(0) space that satisfies the duality condition. If the intrinsic rank of \(X\) is at least two, then \(X\) is either a product, or a symmetric space or Euclidean building.

In Section 8, we prove (Corollary 29) a weak version of the CAT(0) Rank Rigidity Conjecture, subject to a dimension restriction.

**Main Corollary.** Let \(X\) be a proper, geodesically complete CAT(0) space that satisfies the duality condition. Let \(X = X_1 \times \cdots \times X_n\) be the maximal de Rham decomposition of \(X\), so that each \(X_i\) is neither compact nor a product. Assume \(\text{rank}(X) = 1 + \dim(\partial_T X)\). Then for each de Rham factor \(X_i\) of \(X\), either

(i) \(\text{Isom}(X_i)\) acts minimally on \(\partial X_i\) and the geodesic flow on \(GX_i\) has a dense orbit mod \(\text{Isom}(X_i)\), or

(ii) \(X_i\) is a symmetric space or Euclidean building of rank at least two.

We remark that Ballmann \cite{2} p.7 identified three problems to solve in extending the Rank Rigidity Theorem to proper, geodesically complete CAT(0) spaces. The first of these problems was to define the rank of a CAT(0) space (that satisfies the duality condition) in such a way that rank \(k \geq 2\) if and only if every geodesic is contained in a Euclidean \(k\)-flat (a subspace of \(X\) isometric to flat Euclidean \(\mathbb{R}^k\)), and rank one implies hyperbolic behavior. The second was to show that if every geodesic of \(X\) was contained in a \(k\)-flat, \(k \geq 2\) (and \(X\) satisfies the duality condition), then \(X\) is a product, a symmetric space, or a Euclidean building. The third was to show that if the space admits a properly discontinuous, cocompact group action by isometries, then it satisfies the duality condition. Our Main Theorem solves the first problem (and even provides a dense \(G_\beta\) set of geodesics whose parallel set is precisely a \(k\)-flat); Corollary 29 gives a partial solution to the second; the third is still open.

We make a final observation about the duality condition. This is equivalent to requiring the geodesic flow to be nonwandering. It is thus a very natural hypothesis from a dynamical perspective, and forms an essential ingredient in the proof of all the Rank Rigidity results for Riemannian manifolds mentioned previously. In fact, the earliest results assumed more—compactness or finite volume, both of which imply the geodesic flow is nonwandering (the proof going back to Poincaré). The most general version of Rank Rigidity for manifolds (due to Eberlein and Heber) assumes only the duality condition. The exception here is telling: Only by relying on the very strong combinatorial structure of the hyperplanes in CAT(0) cube complexes were Caprace and Monod able to elide the duality condition in their proof of Rank Rigidity. Thus, since the duality condition is the essential hypothesis
for proving Rank Rigidity for manifolds, this same dynamical information is the appropriate choice to prove Rank Rigidity for general CAT(0) spaces.

2. Preliminaries

Let $X$ be a metric space. A geodesic in $X$ is an isometric embedding $v: \mathbb{R} \to X$, a geodesic ray is an isometric embedding $\alpha: [0, \infty) \to X$, and a geodesic segment is an isometric embedding $\sigma: [0, r] \to X$ for some $r > 0$. The space $X$ is called geodesic if every pair of distinct points is connected by a geodesic segment; $X$ is uniquely geodesic if the segment is always unique (up to reversing parametrization).

Also, $X$ is called geodesically complete if every geodesic segment in $X$ extends to a full (not necessarily unique) geodesic in $X$.

We write $GX$ for the set of all geodesics in $X$, endowed with the compact-open topology (i.e. the topology of uniform convergence on compact subsets). This space is completely metrizable when $X$ is complete. There is also a canonical geodesic flow $g^t$ on $GX$ given by $(g^t v)(s) = v(s+t)$.

A uniquely geodesic metric space $X$ is called CAT(0) if, for every triple of distinct points $x, y, z \in X$, the geodesic triangle $\triangle(x, y, z) \subset X$ is no fatter than the corresponding comparison triangle $\triangle(x, y, z)$ in Euclidean $\mathbb{R}^2$ (the triangle with the same edge lengths). For more on CAT(0) spaces, see [2] or [5].

Let $X$ be a complete CAT(0) space. The visual boundary (written $\partial X$) of $X$ is the set of equivalence classes of asymptotic geodesic rays. Equivalently, one can fix a basepoint $x_0 \in X$ and take all geodesic rays emanating from $x_0$. The standard topology on $\partial X$ is the compact-open topology, often called the cone or visual topology. (This topology does not depend on choice of basepoint.)

Viewing each point $x \in X$ as a geodesic segment from a fixed basepoint to $x$, the topology of uniform convergence on compact subsets naturally gives a topology on $X = X \cup \partial X$. If $X$ is proper (meaning all closed balls are compact), then both $\partial X$ and $\overline{X} = X \cup \partial X$ are compact metrizable spaces. For each $v \in GX$, we will write $v(\infty) = \lim_{t \to +\infty} v(t) \in \partial X$ and $v(-\infty) = \lim_{t \to -\infty} v(t) \in \partial X$.

We now define parallel sets and cross sections; a version of each exists both on $X$ and $GX$. Let $v \in GX$. A geodesic $w \in GX$ is parallel to $v$ if the map $t \mapsto d(v(t), w(t))$ is constant. Let $\mathcal{P}_v \subset GX$ be the set of geodesics parallel to $v$, and let $P_v$ be the set of points in $X$ that lie on some $w \in \mathcal{P}_v$. We call $\mathcal{P}_v$ the parallel set of $v$ in $GX$, and $P_v$ the parallel set of $v$ in $X$. It is a standard fact of CAT(0) geometry that $P_v$ is a convex subset of $X$, isometric to $C_v \times \mathbb{R}$, where $C_v$ is a closed convex subset of $P_v$ containing $v(0)$. We call $C_v$ the cross section of $P_v$ in $X$, or just the cross section of $v$ in $X$. We call the set $\mathcal{CS}_v = \{w \in \mathcal{P}_v \mid w(0) \in C_v\}$ the cross section of $v$ in $GX$. Notice that footpoint projection $GX \to X$ (given by $w \mapsto w(0)$) bijectively carries $\mathcal{P}_v$ to $P_v$ and $\mathcal{CS}_v$ to $C_v$.

There is a simple and useful metric on $GX$, defined by

$$d(v, w) = \sup_{t \in \mathbb{R}} e^{-|t|} d(v(t), w(t)) \quad \text{for all } v, w \in GX.$$ 

This metric is complete if $X$ is complete, and proper if $X$ is proper (by the Arzelà-Ascoli theorem). It is also isometry-invariant and induces the topology of uniform convergence on compact subsets. Moreover, within each parallel set $\mathcal{P}_v$, it is flow-invariant and therefore restricts to the metric on $P_v$. Thus footpoint projection restricts to an isometry $\mathcal{P}_v \to P_v$ and $\mathcal{CS}_v \to C_v$ for each $v \in GX$. 

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3. Duality and Recurrence

We first describe the duality condition, introduced by Chen and Eberlein [10] for Hadamard manifolds. The results in this section should be familiar to the experts.

Let $X$ be a complete CAT(0) space. Write $\text{Isom}(X)$ for the isometry group of $X$. For a subgroup $\Gamma \leq \text{Isom}(X)$, two points $\xi, \eta \in \partial X$ are called $\Gamma$-dual if there exists a sequence $(\gamma_n)$ in $\Gamma$ such that $\gamma_n x \to \xi$ and $\gamma_n^{-1} x \to \eta$ for some (hence any) $x \in X$. The subgroup $\Gamma$ is said to satisfy the duality condition if $v(\infty)$ and $v(-\infty)$ are $\Gamma$-dual for every $v \in GX$. We will say that $X$ satisfies the duality condition if $\text{Isom}(X)$ does.

Notice $X$ satisfies the duality condition whenever any subgroup $\Gamma$ of $\text{Isom}(X)$ does. In fact, if the duality condition holds for a group $\Gamma$, then it holds not only for arbitrary supergroups but also for finite-index subgroups. Moreover, when $X$ is proper and geodesically complete, if $X$ splits as a CAT(0) product $X = X_1 \times X_2$ and satisfies the duality condition, then $X_1$ and $X_2$ also satisfy the duality condition. These facts are proved in [2, Remark III.1.10].

The reason the duality condition is of interest dynamically is its relationship to recurrence and nonwandering, which we describe below. A geodesic $v \in GX$ is called (forward) $\Gamma$-recurrent if there exist sequences $t_n \to +\infty$ and $\gamma_n \in \Gamma$ such that $\gamma_n g^{t_n}(v) \to v$ as $n \to \infty$; it is called $\Gamma$-nonwandering if there exist sequences $v_n \in GX$, $t_n \to +\infty$, and $\gamma_n \in \Gamma$ such that $v_n \to v$ and $\gamma_n g^{t_n}(v_n) \to v$ as $n \to \infty$. (Notice $\Gamma$-recurrent implies $\Gamma$-nonwandering.)

These notions are related to the usual notions of recurrence and nonwandering as follows. If $\Gamma \leq \text{Isom}(X)$ is discrete, then $v \in GX$ is $\Gamma$-recurrent (respectively, $\Gamma$-nonwandering) if and only if its projection onto $\Gamma \backslash GX$ is recurrent (respectively, nonwandering) under the geodesic flow $g^t_\Gamma$ on $\Gamma \backslash GX$. The situation with nonwandering is completely similar.

**Definition 1.** We will say $v \in GX$ is recurrent if $v$ is $\text{Isom}(X)$-recurrent, and nonwandering if $v$ is $\text{Isom}(X)$-nonwandering. Note this departure from standard usage creates minimal potential for confusion because no $v \in GX$ can ever be $\langle \text{id}\rangle$-recurrent or $\langle \text{id}\rangle$-nonwandering in a CAT(0) space.

The relationship with duality is derived from the following result, originally due to Eberlein [11] in the case of nonpositively curved smooth Riemannian manifolds.

**Lemma 2** (Lemma III.1.1 of [2]). Let $X$ be a geodesically complete CAT(0) space, and let $\Gamma$ be a subgroup of $\text{Isom}(X)$. If $v, w \in GX$ are such that $v(\infty)$ and $w(-\infty)$ are $\Gamma$-dual, then there exist $\gamma_n \in \Gamma$, $t_n \to +\infty$, and $v_n \in GX$ such that $v_n \to v$ and $\gamma_n g^{t_n}(v_n) \to v$ as $n \to \infty$.

In particular (see the discussion preceding Corollary III.1.4 in [2]):

**Corollary 3.** Let $X$ be a CAT(0) space. If every $v \in GX$ is nonwandering, then $X$ satisfies the duality condition. The converse holds if $X$ is geodesically complete.

We will want two results later, which we record here. The first is the following standard result (see [2, Corollary III.1.5], for instance, for a proof).

**Lemma 4.** Let $X$ be a complete CAT(0) space. If every $v \in GX$ is nonwandering, then the recurrent geodesics form a dense $G_\delta$ set in $GX$.

**Proof.** For each $k, n \in \mathbb{N}$, define $U_{k,n} := \bigcup_{t \geq n} \bigcup_{\gamma \in \Gamma} \{v \in GX \mid d(v, \gamma g^t v) < \frac{1}{k}\}$. Each $U_{k,n}$ is open and dense by the nonwandering hypothesis, hence $\bigcap_{k,n \in \mathbb{N}} U_{k,n}$ is
dense $G_{\delta}$ in $GX$ because $GX$ is complete. But $\bigcap_{k,n\in\mathbb{N}} U_{k,n}$ is precisely the set of recurrent geodesics. Thus the conclusion of the lemma holds.

The next lemma follows easily from Lemma 6.7 and Lemma 6.10 in [21], but we provide a proof here for the convenience of the reader.

**Lemma 5.** Let $X$ be a proper CAT(0) space. If $v \in GX$ is recurrent, then for every $w \in GX$ such that $w(\infty) = v(\infty)$, there is an isometric embedding $(\mathcal{CS}_w, w) \hookrightarrow (\mathcal{CS}_v, v')$ for some $v' \parallel v$ such that $d(v, v') \leq d(v, w)$.

(Here we have adopted the notation $(\mathcal{CS}_w, w) \hookrightarrow (\mathcal{CS}_v, v')$ to mean $\mathcal{CS}_w \rightarrow \mathcal{CS}_v$ is an isometric embedding that sends $w \mapsto v'$.)

**Proof.** Let $w \in GX$ with $w(\infty) = v(\infty)$. Since $v$ is recurrent, there exist $\gamma_n \in \text{Isom}(X)$ and increasing $t_n \to +\infty$ such that $\gamma_n g^{t_n}(v) \to v$. Since $w$ and $v$ are forward asymptotic, $d(\gamma_n g^{t_n}(v), \gamma_n g^{t_n}(w))$ is nonincreasing. By Arzela-Ascoli, we may pass to a subsequence for which the isometric embeddings $\gamma_n g^{t_n}|_{\mathcal{CS}_w}: \mathcal{CS}_w \hookrightarrow GX$ converge to an isometric embedding $\psi: (\mathcal{CS}_w, w) \hookrightarrow (\mathcal{CS}_u, u)$ for some $u \parallel v$.

The desired isometric embedding is now $g^r \circ \psi: (\mathcal{CS}_w, w) \hookrightarrow (\mathcal{CS}_v, g^ru)$, where $r \in \mathbb{R}$ is chosen so that $g^ru \in \mathcal{CS}_v$. \qed

A variation on Lemma 5 is the following property of recurrent geodesics, which is due to Guralnik and Swenson [10, Corollary 3.24] in the case that the sequence $(\gamma_n)$ lies completely in some discrete subgroup of Isom(X).

**Lemma 6.** Let $X$ be a proper CAT(0) space. Let $w \in GX$ be recurrent, so there exist $\gamma_n \in \text{Isom}(X)$ and $t_n \to +\infty$ such that $\gamma_n g^{t_n}(w) \to w$. Then for every $p \in \partial X$, every accumulation point of $(\gamma_n p)$ lies in $\partial P_w$.

**Proof.** Let $q \in \partial X$ be an accumulation point of $(\gamma_n p)$. Passing to a subsequence, we may assume $\gamma_n p \to q$. Note that $\gamma_n(w(\infty)) \to w(\infty)$. Now put $x = w(0)$, and for each $n$, put $x_n = w(t_n) = g^{t_n}w(0)$. Then

$\angle_x(q, w(\infty)) \geq \limsup \angle_{x_n}(\gamma_n p, \gamma_n w(\infty)) = \limsup \angle_{x_n}(p, w(\infty)) = \angle(p, w(\infty))$ by standard CAT(0) geometry. But by lower semicontinuity of $\angle$, we know

$\angle(p, w(\infty)) = \angle(\gamma_n p, \gamma_n w(\infty)) \geq \angle(q, w(\infty)) \geq \angle_x(q, w(\infty)).$

Therefore, $\angle(q, w(\infty)) = \angle(p, w(\infty))$. Papasoglu and Swenson’s $\pi$-convergence theorem (stated for a discrete group of isometries in [21, Lemma 19], but the proof does not use this assumption) then shows

$\angle(q, w(-\infty)) \leq \angle(q, w(-\infty)) \leq \pi - \angle(p, w(\infty)) = \pi - \angle_x(q, w(\infty)).$

Since $\angle(q, w(\infty)) = \pi$, we see that either $q = w(\pm \infty)$ or $q$ lies in the ideal boundary of a flat half-plane bounded by $w$, i.e. in either case, $q \in \partial P_w$. \qed

4. Complete Approachability

Call $v \in GX$ completely approachable if, for every $x \in C_v$ and sequence $v_n \to v$ in $GX$, there exists $x_n \to x$ in $X$ such that each $x_n \in C_{v_n}$. This terminology reflects the idea that every $x \in C_v$ is approachable by $x_n \in C_{v_n}$ for every $v_n \to v$.

For a metric space $Z$, write $\mathcal{C}(Z)$ for the space of closed subsets of $Z$, with the Hausdorff topology.
Lemma 7. Let $X$ be a proper CAT(0) space, and let $v \in GX$. The following are equivalent.

(i) $v$ is completely approachable.
(ii) For every $x \in C_v$ and sequence $v_n \to v$ in $GX$, there exist $x_n \to x$ in $X$ such that each $x_n \in C_{v_n}$.
(iii) For every $w \in CS_v$ and sequence $v_n \to v$ in $GX$, there exist $w_n \to w$ in $GX$ such that each $w_n \in CS_{v_n}$.
(iv) The extended cross-section map $\overline{CS} : GX \to \mathcal{C}(GX)$ is continuous at $v$, where $GX$ is one-point compactification of $GX$ and $\overline{CS}(w) := CS_w \cup \{\infty\}$.

Proof. (ii) is the definition of (i). The equivalence of (iii) $\iff$ (iii) is trivial, and (iii) $\iff$ (iv) because $CS$ is upper semicontinuous. \hfill $\Box$

It is a standard fact (see, for example, [1] Theorem A.1) that every upper semicontinuous map $Y \to C(Z)$, where $Y$ is a complete metric space and $Z$ a compact metric space, has a dense $G_\delta$ set of continuity points. Thus we have the following.

Lemma 8. The completely approachable geodesics form a dense $G_\delta$ set in $GX$.

Corollary 9. Assume every geodesic $v \in GX$ is nonwandering. The set $U \subseteq GX$ of geodesics that are both completely approachable and recurrent is dense $G_\delta$ in $GX$.

We will denote the set of completely approachable geodesics by $A$.

Lemma 10. Let $v \in A$ and $w \in GX$. If $v(\infty)$ and $w(-\infty)$ are $\text{Isom}(X)$-dual, then there is an isometric embedding $(CS_v, v) \hookrightarrow (CS_w, w)$.

Proof. By Lemma 2 there exist $\gamma_n \in \Gamma$, $t_n \to +\infty$, and $v_n \in GX$ such that $v_n \to v$ and $\gamma_n g^{t_n} v_n \to w$. Because $v$ is completely approachable, $CS_{v_n} \to CS_v$. By upper semicontinuity of $CS$ we find $CS_w \supseteq \lim \gamma_n g^{t_n} CS_{v_n}$. Since each $\gamma_n g^{t_n}$ is an isometry on $GX$ which preserves cross sections, the limit of isometries $\varphi_n = \gamma_n g^{t_n}|_{CS_{v_n}}$ is the desired isometric embedding $(CS_v, v) \hookrightarrow (CS_w, w)$.

\hfill $\Box$

Corollary 11. If $v \in GX$ is completely approachable and nonwandering then for every $w \in GX$ with $w(\infty) = v(\infty)$, there is an isometric embedding $(P_v, v) \hookrightarrow (P_w, w)$.

Lemma 12. Let $Y$ and $Z$ be proper metric spaces, and let $y_0 \in Y$ and $z_0 \in Z$. If both $f : (Y, y_0) \to (Z, z_0)$ and $g : (Z, z_0) \to (Y, y_0)$ are isometric embeddings, then both $f$ and $g$ are isometries.

Remark. It may be that $g \neq f^{-1}$.

Proof. The composition $g \circ f$ is an isometry on each closed metric ball $\overline{B}_Y(y_0, R)$ in $Y$ by compactness [2] Theorem 1.6.14, hence $g$ is surjective. Similarly for $f$. \hfill $\Box$

Corollary 13. Assume every geodesic $v \in GX$ is nonwandering. If $v, w \in A$ and $v(\infty) = w(\infty)$, then there is an isometry $(P_v, v) \to (P_w, w)$.

5. Lifting convergent sequences

A map $f : X \to Y$ is called open at $x \in X$ if, for every open neighborhood $U$ of $x$ in $X$, the image $f(U)$ contains an open neighborhood of $f(x)$ in $Y$. 
Lemma 14. Let \( f: X \to Y \) be a continuous map between metric spaces, and let \( x \) be a point of \( X \). The following are equivalent.

(i) \( f \) is open at \( x \).

(ii) For every open neighborhood \( U \) of \( x \) in \( X \), the image \( f(U) \) contains an open neighborhood of \( f(x) \) in \( Y \).

(iii) For every sequence \( y_n \to f(x) \) in \( Y \), there are subsequences \( n_k \), and \( x_k \in X \) with \( f(x_k) = y_{n_k} \), and \( x_k \to x \).

(iv) For every sequence \( y_n \to f(x) \) in \( Y \), there exists a sequence \( x_n \to x \) in \( X \) such that \( f(x_n) = y_n \) for all \( n \).

Proof. The proof is straightforward and left as an exercise to the reader. \( \square \)

Our interest in open maps comes from Theorem 16, which allows us to take dense \( G_\delta \) slices of dense \( G_\delta \) sets. The following lemma is the dense open set version.

Lemma 15. Let \( X \) and \( Y \) be complete metric spaces with \( X \) separable. Let \( C \subseteq X \times Y \) and let \( U \subseteq C \) be relatively open and dense in \( C \). Let \( \pi_Y: X \times Y \to Y \) be coordinate projection onto \( Y \). Assume the restriction \( \pi_Y|_C : C \to \pi_Y(C) \) is open at every point of \( U \). Then

\[
Y_U^C := \{ y \in Y \mid (X \times \{ y \}) \cap U \text{ is (open and) dense in } (X \times \{ y \}) \cap C \}
\]

contains a dense \( G_\delta \) subset of \( Y \).

Proof. Let \( (V_n) \) be a countable basis for \( X \). Let

\[
E_n = \{ y \in Y \mid (V_n \times \{ y \}) \cap U = \emptyset \text{ but } (V_n \times \{ y \}) \cap C \neq \emptyset \}.
\]

Then \( Y \setminus Y_U^C = \bigcup_n E_n \).

Now let \( C_n = (V_n \times Y) \cap C \) and \( U_n = (V_n \times Y) \cap U \). Notice that \( E_n = f_n(C_n) \setminus f_n(U_n) \), where \( f_n = \pi_Y|_{C_n} \) is the restriction of \( \pi_Y \) to \( C_n \to \pi_Y(C) \). Since \( V_n \times Y \) is open, \( U_n \) is relatively open and dense in \( C_n \); moreover, \( f_n \) is continuous and open at every point of \( U \). Thus \( f_n(U_n) \) is relatively open and dense in \( f_n(C_n) \). It follows that \( E_n \) is relatively nowhere dense in \( f_n(C_n) \), hence nowhere dense in \( Y \). Thus \( Y \setminus Y_U^C \) is the countable union of nowhere dense sets. \( \square \)

Taking countable intersections we obtain the following.

Theorem 16. Let \( X \) and \( Y \) be complete metric spaces with \( X \) separable. Let \( C \subseteq X \times Y \) and let \( A \subseteq C \) contain a subset which is dense \( G_\delta \) in \( C \). Let \( \pi_Y: X \times Y \to Y \) be coordinate projection onto \( Y \). Assume the restriction \( \pi_Y|_C : C \to \pi_Y(C) \) is open at every point of \( C \). Then

\[
Y_A^C := \{ y \in Y \mid (X \times \{ y \}) \cap A \text{ contains a dense } G_\delta \text{ subset of } (X \times \{ y \}) \cap C \}
\]

contains a dense \( G_\delta \) subset of \( Y \).

Remark. The slice \( (X \times \{ y \}) \cap A \) may be empty for \( y \in Y_A^C \), but only if \( (X \times \{ y \}) \cap C \) is empty. Thus for many applications one must show that \( (X \times \{ y \}) \cap C \) is not empty for some dense \( G_\delta \) set of \( y \in Y \), and then one finds that the set

\[
\hat{Y}_A^C := \{ y \in Y \mid (X \times \{ y \}) \cap A \text{ is nonempty and contains a dense } G_\delta \text{ subset of } (X \times \{ y \}) \cap C \}
\]

contains a dense \( G_\delta \) subset of \( Y \).
Our application for Theorem [16] is to actually to the forward-endpoint map $\partial X \to \partial X$ taking $v \mapsto v(\infty)$. The topological embedding $e: GX \to \mathbb{R}^* \times \partial X \times \partial X$ given by $e(v) = (v(0), v(-\infty), v(\infty))$ provides the ambient product structure, and the following lemma shows that the map is open at every point of $GX$.

**Lemma 17.** Let $X$ be a proper, geodesically complete CAT(0) space. Let $v \in GX$ and $p = v(\infty)$. Let $p_n \in \partial X$ with $p_n \to p$. Then there are subsequences $n_k$, and $v_k \in GX$ with $v_k(\infty) = p_{n_k}$, and $v_k \to v$.

**Proof.** Write $x_k = v(-k)$. For each $k, n$, choose $w_{k,n}$ such that $w_{k,n}(-k) = x_k$ and $w_{k,n}(\infty) = p_n$. So for each fixed $k$, the geodesics $w_{k,n}$ keep $w_{k,n}(-k) = x_k$ while $w_{k,n}(\infty) \to p$. Thus we may find for each $k$ some $n_k \geq k$ such that $w_{k,n_k}(0) < 1/k$. It follows that $v_k := w_{k,n_k} \to v$. \hfill \square

Recall from Lemma [9] that the set $\mathcal{U} := \{v \in \mathcal{A} \mid v \text{ is recurrent}\}$ is a dense $G_\delta$ subset of $GX$. For $p \in \partial X$, let $GX_p$ be the set of $v \in GX$ such that $v(\infty) = p$.

**Corollary 18.** Assume $X$ is geodesically complete and every geodesic $v \in GX$ is nonwandering. There is a set $b\mathcal{U}$ in $\partial X$ that contains a dense $G_\delta$ subset of $\partial X$, such that for every $p \in b\mathcal{U}$ the set $\mathcal{U}_p := \mathcal{U} \cap GX_p$ contains a subset that is dense $G_\delta$ in $GX_p$.

**Proof.** Combine Corollary [9], Lemma [17], and Theorem [16]. \hfill \square

### 6. Isometric Transitivity

**Lemma 19.** Let $X$ be a proper, geodesically complete CAT(0) space that satisfies the duality condition. Then the isometry group $\text{Isom}(\mathcal{CS}_v)$ is transitive for all $v \in GX$ such that $v(\infty) \in b\mathcal{U}$.

**Proof.** Let $p \in b\mathcal{U}$. By Lemma [12] it suffices to construct an isometric embedding $(\mathcal{CS}_v, v) \to (\mathcal{CS}_w, w)$ for all $v \in GX_p$ and $w \parallel v$. So let $v \in GX_p$ and $w \parallel v$.

By density of $\mathcal{U}_p$ in $GX_p$, there is a sequence $(v_n)$ in $\mathcal{U}_p$ such that $v_n \to v$. By Lemma [5] and Corollary [11] for each $n$ we can find isometric embeddings $\varphi_n: (\mathcal{CS}_v, v) \to (\mathcal{CS}_{v'_n}, v'_n)$, for some $v'_n \parallel v_n$ such that $d(v_n, v) \leq d(v_n, v)$, and $\psi_n: (\mathcal{CS}_{v'_n}, v_n) \to (\mathcal{CS}_w, w)$. Thus $\psi_n \circ \varphi_n: (\mathcal{CS}_v, v) \to (\mathcal{CS}_w, w)$ is a sequence of isometric embeddings with $w_n = \psi_n(v'_n) \to w$. A subsequence of $\psi_n \circ \varphi_n$ converges to an isometric embedding $(\mathcal{CS}_v, v) \to (\mathcal{CS}_w, w)$, as desired. \hfill \square

A variation on the preceding proof gives us the following corollary.

**Corollary 20.** Let $X$ be a proper, geodesically complete CAT(0) space that satisfies the duality condition. Then $\mathcal{CS}_v$ is isometric to $\mathcal{CS}_w$ for all $v, w \in GX$ such that $v(\infty) = w(\infty) \in b\mathcal{U}$.

**Proof.** Fix $p \in b\mathcal{U}$ and $w \in \mathcal{U}_p$. Let $v \in GX_p$. Lemma [5] gives us an isometric embedding $\varphi: (\mathcal{P}_v, v) \to (\mathcal{P}_{w'}, w')$ for some $w' \parallel w$. By Lemma [19] we may assume $w' = w$. And Corollary [11] gives us an isometric embedding $\psi: (\mathcal{P}_w, w) \to (\mathcal{P}_v, v)$. By Lemma [12] $\varphi$ and $\psi$ are isometries. The corollary follows. \hfill \square

### 7. Intrinsic Rank

Throughout this section, $X$ is a proper, geodesically complete CAT(0) space.
7.1. Parallel sets. We use the same notation for a geodesic \( v \in GX \) and its image \( \nu([R]) \). Recall that for \( v \in GX \), the parallel set \( P_v \subset GX \) is the set of geodesics parallel to \( v \), and \( P_v = \bigcup_{v \in P_v} v \subset X \) is isometric to \( P_v \) under footpoint projection.

If we want to specify the space \( X \), we will write \( P_v^X \) instead of \( P_v \).

Recall that \( P_v \) splits isometrically as \( P_v = C_v \times v \), where \( C_v \) is the cross section of \( v \). Specifically, the isometry \( P_v \rightarrow C_v \times v \) is given by \( x \mapsto (\pi_{C_v}(x), \pi_v(x)) \), where each coordinate is convex projection. Call these the \( v \)-coordinates of \( x \in P_v \).

Sometimes we will write \( x = (y, v(a)) \) and identify \( P_v \) with \( C_v \times v \).

We collect some facts about parallel sets.

(i) Let \( w \) be a geodesic in the CAT(0) space \( P_v \), with \( w \) not parallel to \( v \). There exist a geodesic \( u \) in \( C_v \) and an angle \( \theta \in (0, \pi) \), such that \( u(t) = (u(t \sin \theta), v(a + t \cos \theta)) \) for all \( t \in \mathbb{R} \). Here we have \( v(a) = \pi_v(w(0)) \).

Note that \( u(t) = \pi_{C_v}(w(t \csc \theta)) \). Call \( u \) the normalized projection of \( w \) in \( C_v \). Of course \( w \) is contained in the 2-flat \( u \times v \subseteq P_v \). (Note: \( u \times v \) may not contain \( v \), despite the notation. But it does if \( w(0) = v(0) \).)

(ii) Let \( w \) be a geodesic in \( P_v \) which is not parallel to \( v \). Let \( u \) be the normalized projection of \( w \). Assume \( v(0) = w(0) \). Since \( w \) is contained in the 2-flat \( u \times v \subseteq P_v \), we see that \( v \) is contained in \( P_w \). By the same argument, every \( v' \parallel v \) such that \( v'(0) = w'(0) \) for some \( v' \parallel w \) is contained in \( P_w = P_w^v \).

(iii) Let \( w_1 \) and \( w_2 \) be two geodesics in \( P_v \) which are not parallel to \( v \). Let \( u \) be the normalized projection of \( w_1 \). If \( w_1 \parallel w_2 \) then \( v_1 \parallel v_2 \). This is because the projection \( \pi_{C_v} \) does not increase distances.

**Lemma 21.** Let \( w \) be a geodesic in \( P_v \) that is not parallel to \( v \), such that \( v(0) = w(0) \). Let \( u \) be the normalized projection of \( w \). Then, using the identification \( P_v \rightarrow C_v \times v \), we can write

\[
P_v \cap P_w = P_u^{C_v} \times v.
\]

**Remark.** Notice that the right-hand side of the equation above does not depend directly on \( w \), only on the normalized projection \( u \) of \( w \).

**Proof.** The set \( P_v \cap P_w \) is a convex subset of \( P_v \) that contains \( v \), along with every \( v' \parallel v \) such that \( v'(0) \in P_w \) (see (ii) above). Therefore we can write \( P_v \cap P_w = E \times v \) for some convex subset \( E \subset C_v \). Since \( w \subset E \times v \) and \( u \) is the normalized projection of \( w \), we see that \( u \) is a geodesic in \( E \).

We now prove \( E \times v \subset P_u^{C_v} \times v \). Let \( x \in E \times v = P_v \cap P_w \). Then \( x \in w' \), for some \( w' \parallel w \). Since \( x \in P_v \) and the distance from \( w' \) to \( P_v \) is bounded (because \( w' \parallel w \) and \( w \subset P_v \)), we see that \( w' \subset P_v \). By (iii) above, \( w' \parallel v \). Then \( v \) is parallel to \( u \), where \( v' \) is the normalized projection of \( w' \). Therefore \( x \in v' \times v \), with \( v' \parallel v \). This proves \( E \times v \subset P_u^{C_v} \times v \). The other inclusion follows from the definitions and the fact that if \( w' \parallel w' \) then the 2-flats \( v' \times v \) and \( u \times v \) are parallel.

\[
7.2. The Decomposition Lemma. \ We need a lemma about convex sets.
\]

**Lemma 22.** Let \( F \) be a closed convex set in the CAT(0) space \( X \), and let \( v, w \) be parallel geodesics in \( X \) such that \( w \) is contained in \( F \). Then \( t \mapsto d_X(v(t), F) \) is constant, and there is a geodesic \( w' \in F \) such that \( w' \parallel v \) and \( d_X(v, F) = d_X(v, w') \).

**Proof.** The distance to \( F \) is constant because it is a convex and bounded function on \( \mathbb{R} \). Define \( w' \) by \( w'(t) = \pi_F(v(t)) \), where \( \pi_F \) is the convex projection \( X \rightarrow F \). This geodesic satisfies the desired conditions.

\[
\square
\]
Lemma 23 (Decomposition Lemma). Let \( v \in GX. \) Assume that \( C_v \) contains a geodesic \( u \) with \( u(0) = v(0) \). Further assume there is a sequence \( v_n \rightarrow v \) in \( u \times v \subseteq P_v \) with \( v_n \neq v \) such that for every \( x \in C_v \), there is a sequence \( x_n \rightarrow x \) in \( X \) where each \( x_n \in C_{v_n} \). Then \( C_v = P_u^{C_v}. \) Hence we can write \( C_v = E \times u \) for some proper \( CAT(0) \) space \( E. \)

Proof. By reversing the orientation of \( u \) if necessary, and passing to a subsequence of \( v_n \), we may assume each \( v_n \) is the geodesic \( t \rightarrow (u(t \sin \theta_n), v(t \cos \theta_n)) \) in \( u \times v \) for some \( \theta_n \in (0, \pi/2), \) and \( \theta_n \rightarrow 0. \) Note that the normalized projection of \( v_n \) is always \( u. \) Thus by Lemma 24 we have that

\[
P_v \cap P_{v_n} = P_u^{C_v} \times v.
\]

Observe that the right-hand side does not depend on \( n. \)

Let \( x \in C_v. \) We will prove that \( x \in P_u^{C_v}. \) By hypothesis, there exist \( x_n \in C_{v_n} \) such that \( x_n \rightarrow x. \) Let \( w_n \) be the geodesic parallel to \( v_n \) with \( w_n(0) = x_n. \) Since \( v_n \in P_v \) and \( P_v \) is convex, by Lemma 24 we can project \( w_n \) onto \( P_v \) to obtain a geodesic \( w'_n \) in \( P_v \) which is parallel to \( w_n \) and such that

\[
d_X(w_n, w'_n) = d_X(w_n, P_v) \leq d_X(x_n, x).
\]

Thus \( d_X(x, P_v) \leq d_X(x, w'_n) + d_X(w'_n, w_n') \leq 2d_X(x, x_n) \rightarrow 0. \) This, together with \( 1 \) and the fact that \( x \in C_v \) implies that \( d_X(x, P_u^{C_v}) = 0. \) Since \( P_u^{C_v} \) is closed, we see that \( C_v \subseteq P_u^{C_v}. \) The reverse inclusion is obvious.

Corollary 24. Let \( v \in A. \) Assume \( C_v \) contains a geodesic \( u \) with \( u(0) = v(0). \) Then \( C_v = P_u^{C_v}. \) Hence we can write \( C_v = E \times u \) for some proper \( CAT(0) \) space \( E. \)

7.3. Approachable cross sections are Euclidean.

Theorem 25. Let \( X \) be a proper, geodesically complete \( CAT(0) \) space that satisfies the duality condition. There is a nonnegative integer \( k \) such that the cross section \( C_v \) of every \( v \in A \) is a \( k \)-flat.

Proof. We prove the theorem in three steps. We start by proving the theorem for \( v \in A \) such that \( v(\infty) \in blU, \) but allow \( k \) to depend on \( v. \) We then remove dependence of \( k \) on \( v, \) for \( v \in U \) with \( v(\infty) \in blU. \) We finally extend to all \( v \in A. \)

Step I. We first prove \( C_v \) is flat for all \( v \in A \) such that \( v(\infty) \in blU \) for \( v \in U \) and \( v \in U_g. \) By [5] Theorem 6.15(6)], \( C_v \) admits a canonical product splitting \( C_v = Y \times H, \) where \( H \) is a Hilbert space and \( Y \) does not admit nontrivial Clifford translations; furthermore, every isometry of \( C_v \) preserves the product splitting. By Theorem 19 we see that \( Isom(Y) \) acts transitively on \( Y. \) Thus \( Y \) is either a single point or is unbounded; we claim the former case holds.

For suppose \( Y \) is unbounded. Since \( Isom(Y) \) is transitive, \( Y \) is cocompact. It follows from [13] that every point \( q \in \partial Y \) can be joined to some \( q' \in \partial Y \) by a geodesic in \( Y. \) In particular, \( Y \) contains a geodesic. By transitivity, there is a geodesic \( u \) in \( Y \) such that \( u(0) = v(0) \) (we may, of course, assume \( v(0) \in Y \).) By Lemma 24 we therefore have nontrivial Clifford translations on \( Y, \) a contradiction.

Thus \( C_v = H. \) Notice that \( H \) must be finite dimensional because \( X \) is proper. Thus every \( v \in A \) with \( v(\infty) \in blU \) is isometric to some Euclidean space \( \mathbb{R}^k. \)

Step II. The dimension of \( C_v \) does not depend on \( v \in U \): Let \( p, q \in blU, v \in U, \) and \( w \in U_g. \) Let \( k = \dim(C_v) \) and \( m = \dim(C_w). \) Since \( w \) is recurrent, there exist \( \gamma_n \in Isom(X) \) and \( t_n \rightarrow +\infty \) such that \( \gamma_n g^{t_n}(w) \rightarrow w. \) By Lemma 4 we see that
approachable, CS is a sequence of the v sequence (CS previous two steps, each (12 PEDRO ONTANEDA AND RUSSELL RICKS parallel set of every complete the proof. Let v parallel set P parallel set GX Corollary 27. completely approachable then so is every geodesic forward asymptotic to v. □

Write rank(X) for dim(P_v) = 1 + dim(CS_v) of some (any) v ∈ A. Thus the parallel set P_v of every v ∈ A is a flat of dimension rank(X). In particular, the parallel set of every w ∈ GX contains a flat of dimension rank(X) by density of the completely approachable geodesics. Thus we have proved the Main Theorem.

We close this section with two observations about A which are only now clear.

Corollary 26. v ∈ A if and only if P_v is a k-flat, where k = rank(X).

Corollary 27. GX_v(∞) ⊂ A for all v ∈ U. I.e. for recurrent v ∈ GX, if v is completely approachable then so is every geodesic forward asymptotic to v.

Proof. Lemma [5]

8. APPLICATION: A LITTLE BIT OF RANK RIGIDITY

Write rank(X) for the intrinsic rank of X and dim(∂T X) for the geometric dimension of its Tits boundary. We now show that if rank(X) = 1 + dim(∂T X), then we have some rigidity.

Theorem 28. Let X be a proper, geodesically complete CAT(0) space. Assume some subgroup Γ ≤ Isom(X) satisfies the duality condition, and that rank(X) = 1 + dim(∂T X). Then one of the following holds.

(i) Γ acts minimally on ∂X.
(ii) X is a symmetric space or Euclidean building of rank ≥ 2.
(iii) X splits as a nontrivial product.

Proof. Assume case (i) does not hold. Our plan is to use Lytchak’s rigidity theorem [19 Main Theorem] on the Tits boundary ∂T X of X. This theorem says that if ∂T X is geodesically complete and contains proper closed involutive set (a set A being involutive meaning for every p ∈ A and q ∈ ∂T X with ∠(p, q) = π, we have q ∈ A), then ∂T X is a spherical building or join.

So we first show ∂T X is geodesically complete. Since rank(X) = 1 + dim(∂T X), the Tits boundary ∂T X of X is covered by Euclidean unit spheres of dimension dim(∂T X). Thus ∂T X is geodesically complete (by applying [3] Lemma 3.1] to the link of each point).

Next we find a proper closed involutive subset of ∂T X. Since Γ satisfies the duality condition, the orbit-closure Γp in ∂X of every point p ∈ X is a minimal nonempty closed invariant subset of ∂X [2 Proposition III.1.9]; these minimal sets are all pairwise disjoint. (Note that by closed we mean here closed in the cone.
topology. But by lower semicontinuity of the Tits metric, they are then also closed under the Tits metric.) So fix an arbitrary \( v \in GX \), and consider the minimal sets \( M = \overline{\Gamma v(-\infty)} \) and \( N = \overline{\Gamma v(\infty)} \) in \( \partial X \). Now the set \( M \cup N \) is clearly closed in \( \partial_T X \).

In [22] (Lemma 27 and first remark following), it is shown that \( M \cup N \) is proper and involutive, assuming \( \Gamma \) is discrete. However, the same arguments apply without that assumption, by simply passing to subsequences instead of using ultrafilters, so we conclude that \( M \cup N \) is a proper closed and involutive subset of \( \partial_T X \).

Thus Lytchak’s rigidity theorem [19, Main Theorem] applies, and we conclude that \( \partial_T X \) is a spherical join or building of dimension at least 1. By Leeb’s theorem [18, Main Theorem], either case (ii) or (iii) holds.

**Remarks.**

(1) If Isom(\( X \)) acts cocompactly on \( X \), then \( 1 + \dim(\partial_T X) \) coincides with the dimension of a maximal flat in \( X \) by Kleiner [17, Theorem C]. Thus in this case, the condition \( \text{rank}(X) = 1 + \dim(\partial_T X) \) is equivalent to the condition \( \text{rank}(X) = \max \{ \dim F \mid F \text{ is a flat in } X \} \).

(2) Let \( X \) be a proper, geodesically complete CAT(0) space satisfying the duality condition. By Ballmann [2, Theorem III.2.3], case (i) is equivalent to the geodesic flow on \( GX_i \) having a dense orbit mod \( \text{Isom}(X_i) \).

Using the deRham decomposition of \( X \) (which exists and is unique by Foertsch and Lytchak [14, Theorem 1.1]), we can state the following corollary.

**Corollary 29.** Let \( X \) be a proper, geodesically complete CAT(0) space that satisfies the duality condition. Let \( X = X_1 \times \cdots \times X_n \) be the maximal de Rham decomposition of \( X \), so that each \( X_i \) is neither compact nor a product. Assume \( \text{rank}(X) = 1 + \dim(\partial_T X) \). Then for each de Rham factor \( X_i \) of \( X \), either

(i) \( \text{Isom}(X_i) \) acts minimally on \( \partial X_i \) and the geodesic flow on \( GX_i \) has a dense orbit mod \( \text{Isom}(X_i) \), or

(ii) \( X_i \) is a symmetric space or Euclidean building of rank at least two.

**Proof.** Each \( X_i \) satisfies the hypotheses of Theorem 28 with \( \Gamma_i = \text{Isom}(X_i) \), but none splits as a nontrivial product by hypothesis. □

**Remark.** Let \( X = X_1 \times \cdots \times X_n \) be as in Corollary 29 except that \( \text{rank}(X) \neq 1 + \dim(\partial_T X) \). If the CAT(0) Rank Rigidity Conjecture holds, then at least one of the de Rham factors \( X_i \) must admit a rank one axis.

**Appendix A. A surface example**

**Theorem 30.** Let \( S \) be an orientable closed surface of genus \( > 1 \), and let \( g = g_0 \) be a \( C^\infty \) nonpositively curved Riemannian metric on \( S \). Then there is a sequence \( g_n \) of \( C^\infty \) nonpositively curved Riemannian metrics on \( S \) such that \( g_n \) \( C^0 \)-converges to a \( C^0 \) nonpositively curved Riemannian metric \( g_\infty \) on \( S \) under which every closed \( g_\infty \)-geodesic is contained in an isometrically immersed flat cylinder.

**Remarks.**

(1) The \( C^0 \) metric \( g_\infty \) induces a geodesic metric on \( S \). In this case “nonpositively curved” means locally CAT(0). (See Theorem 4.11 in [8]. Here Burstschier proves \( C^0 \) Riemannian manifolds are length spaces. But compact length spaces are geodesic spaces; see [5, p.35] for this.)

(2) With more care one can possibly arrange to have a \( C^\infty \) path \( g_t \) of such metrics \( C^0 \)-converging to \( g_\infty \).
Proof. Let $S$ and $g$ as in the Theorem, and give $S$ an orientation. Let $\gamma : X \to S$ be a nontrivial geodesic in $(S, g)$, where $X$ is either the circle $S^1$, or the interval $[0, 1]$. Also, we will denote by $X(\ell)$ the circle of length $\ell$ or the interval $[0, \ell]$. For each $u \in X$ let $V(u) \subset T_{\gamma(u)}S$ be the unit vector perpendicular to $\gamma'(u)$ and such that $(\gamma'(u), V(u))$ is positively oriented. We get a map $E = E_{\gamma,g} : X \times \mathbb{R} \to S$ given by $E(u, s) = \exp_{\gamma(u)}(sV(u))$. Then $E$ is an immersion near $\gamma$, that is, there is $\varepsilon > 0$ such that $E = E_{\gamma,g}$ restricted to $X \times [-2\varepsilon, 2\varepsilon]$ is an immersion. The supremum of all such $\varepsilon$ will be denoted by $\varepsilon_{\gamma,g}$. Therefore, for all $\varepsilon < \varepsilon_{\gamma,g}$, the map $E$ is an immersion on $X \times [-2\varepsilon, 2\varepsilon]$. In this case the pullback of $g$ to $X \times [-2\varepsilon, 2\varepsilon]$ is a Riemannian metric, which we denote by $g_\varepsilon$. If $X$ is an interval we will assume $\gamma$ extends to a larger interval $X'$; if $X = S^1$ then $X' = X$.

For two Riemannian metrics $g_1, g_2$ we write $g_1 \leq g_2$ if $g_1(x, x) \leq g_2(x, x)$ for all tangent vectors $x$. An arc in $X$ is a subspace $A \subset X$ homeomorphic to a closed interval. The set $\gamma(A)$ will also be called an arc. We will call the set $E(A \times [-\varepsilon, \varepsilon])$ the $\varepsilon$-rectangle of $A$. We will need the following lemma.

Lemma 31. Let $S$, $\gamma$, $g$ and $E$ as above. Let $\varepsilon < \varepsilon_{\gamma,g}/2$, and $\delta > 0$. Then there is a $C^\infty$ nonpositively curved Riemannian metric $g_1$ on $S$ such that

1. $(1 - \delta)g_0 \leq g_1 \leq (1 + \delta)g_0$.
2. $g_1 = g$ outside $E(X' \times [-\varepsilon/2, \varepsilon/2])$.
3. The curve $\gamma$ is a $g_1$-geodesic.
4. There is $\varepsilon' \in (0, \varepsilon/4)$ such that $X \times [-\varepsilon', \varepsilon']$ with metric $(g_1)_\gamma$ is isometric to the (flat) Euclidean product of $X(\ell)$ with $[-\varepsilon', \varepsilon']$, where $\ell$ is the $g_1$-length of $X$.
5. Let $A$ be an arc in the interior of $X$ and assume that the curvature $K_\gamma$ is zero on an open set containing $E(A \times [-\varepsilon, \varepsilon])$. Then we can arrange that $g_\gamma = (g_1)_\gamma$ on $A \times [-\varepsilon, \varepsilon]$.

Postponing the proof of Lemma 31 for the moment, we proceed with our proof of Theorem 30. Enumerate the free homotopy classes of loops in $S$: $C_1, C_2, C_3, \ldots$. Write $\delta_n = \frac{1}{2^{n+2}}$. As our first step just choose a closed $g$-geodesic $\gamma_1$ in $C_1$, and apply Lemma 31 with $X = S^1$, $\gamma = \gamma_1$, $\delta = \delta_1$, and any $\varepsilon < \varepsilon_{\gamma_1,g}/2$ to obtain a metric $g_1$. Write $\varepsilon' = 2\eta_1 > \eta_1$, where $\varepsilon'$ is as in Lemma 31. That is, $\gamma_1$ is contained in an isometrically immersed flat cylinder $C_1(2\eta_1)$ of width $2\eta_1 > \eta_1$, with respect to the metric $g_1$. (We will denote the image of $C_1$ also by $C_1$.) The next step is to choose a closed $g_1$-geodesic $\gamma_2$ in $C_2$, and apply Lemma 31 with $X = S^1$, $\gamma = \gamma_2$, $\delta = \delta_2$, and $\varepsilon = \varepsilon_2$ small (how small will be determined below) to obtain a metric $g_2$. Write $E_2 = E_{\gamma_2,g_2}$ and $E_2(\varepsilon_2) = E_2(S^1 \times [-\varepsilon_2, \varepsilon_2])$. The geodesic $\gamma_2$ is contained in a flat immersed cylinder $C_2(2\eta_2)$ of some width $2\eta_2 > \eta_2$. But this step may change the cylinder of step 1. This is an unavoidable problem, but we can minimize the problem: because of (5) of Lemma 31 and the fact that the width of the cylinder in step 1 is strictly larger than $\eta_1$, we can choose $\varepsilon_2$ so small that the new $g_2$-width of the cylinder of $\gamma_1$ is still $> \eta_1$ (even though the width decreases a bit). Here is a more detailed description of how to do this. Let $C_1(\frac{3}{2}\eta_1) \subset C(2\eta_1)$ be the flat isometrically immersed cylinder of width $\frac{3}{2}\eta_1$. The intersection of the image of $\gamma_2$ with the cylinder $C_1(\frac{3}{2}\eta_1)$ of step 1 is a finite set of arcs $\gamma_2(A_i)$. Choose $\varepsilon_2$ small enough so that the $\varepsilon_2$-rectangles of the $A_i$ are contained in the interior of $C_1(2\eta_1)$.
and the set
\[
\left(C_1(2\eta_1) \cap E_2(\varepsilon_2)\right) \setminus \bigcup_i \varepsilon_2\text{-rectangle of } A_i
\]
is outside the \(\eta_i\) neighborhood of \(\gamma_i\). Since the curvature is zero on \(C_1(2\eta_1)\), (5) of Lemma 31 implies that we can arrange for the metrics \(g_1\) and \(g_2\) to coincide on the \(\varepsilon_1\)-rectangles of the \(A_i\). In this way, after step 2, \(\gamma_i\) is still contained in a flat isometrically immersed cylinder of width \(> \eta_i\).

Now, proceed inductively to obtain \(g_n\) and \(\gamma_n\) contained in an isometrically immersed flat cylinder of width \(> \eta_n\). For the \(n+1\) step we proceed similarly, choosing \(\varepsilon_{n+1}\) so small that all \(\gamma_i, i \leq n\), are still contained in isometrically immersed flat cylinders of width \(> \eta_i\). In this way we define \(g_n\) for \(n = 1, 2, 3, \ldots\)

Next we prove convergence.

Recall \(\delta_n = \frac{1}{2n+1+2}\), so \(1 - \delta_n = \frac{2^{n+1}+1}{2^{n+1}+2}\) and \(1 + \delta_n = \frac{2^{n+1}+3}{2^{n+1}+2} \leq \frac{2^{n+1}+2}{2^{n+1}+1} = \frac{1}{\varepsilon_i}\).

Now, from (1) of Lemma 31 we have (1)
\[
(1 - \delta_n)g_n \leq g_{n+1} \leq (1 + \delta_n)g_n.
\]
Hence \(a_n g \leq g_{n+1} \leq \frac{1}{a_n}g\), where \(a_n = \prod_{i=1}^n (1 - \delta_i)\). One can show, by induction, that \(a_n = \frac{2^{n+1}+1}{2^{n+1}+2} \geq \frac{1}{2}\). Hence \(\frac{1}{2}g \leq g_{n+1} \leq 2g\).

Let \(x\) be a tangent vector. Then (1) of Lemma 31 implies
\[
-\delta_ng_n(x, x) \leq g_{n+1}(x, x) - g_n(x, x) \leq \delta_ng_n(x, x).
\]
Therefore
\[
|g_{n+1}(x, x) - g_n(x, x)| \leq \delta_n g_n(x, x) \leq 2\delta_n g(x, x) = \frac{2}{2+2^{n+1}}g(x, x) \leq \frac{1}{2g}(x, x).
\]
Replacing \(x\) by \(x + y\) and using the triangular inequality we obtain
\[
|g_{n+1}(x, y) - g_n(x, y)| \leq \frac{1}{2g}(g(x, x) + g(y, y)).
\]
Therefore, for each pair \(x, y\) the sequence \(g_n(x, y)\) is a Cauchy sequence, hence converges. This gives a \(C^\infty\)-symmetric bilinear form \(g_\infty\) on \(TS\). We certainly have \(g_\infty(x, x) \geq 0\). But we have showed that \(\frac{1}{2}g \leq g_{n+1}\).

This shows \(\frac{1}{2}g(x, x) \leq g_\infty(x, x)\). Therefore \(g_\infty\) is nondegenerate. This proves the theorem.

**Proof of Lemma 31.** We assume \(\gamma\) has speed 1. Let \(\ell\) be the length of \(\gamma\) (recall \(X\) is an interval or a circle). For simplicity we change a bit the domains of \(\gamma\) and \(E\): We replace \(X\) by \(X(\ell)\). We use coordinates \((u, v) \in X(\ell) \times [-\varepsilon, \varepsilon]\). The velocity vectors of the \(u\)-lines \(u \mapsto (u, v_0)\) and \(v\)-lines \(v \mapsto (u_0, v)\) will be denoted by \(\partial_u\) and \(\partial_v\), respectively. Recall that \(g_\gamma\) is the pullback of \(g\) by the immersion \(E\); we consider \(X(\ell) \times [-\varepsilon, \varepsilon]\) with this metric. Note that the \(v\)-lines are speed one geodesics, and the \(g\)-geodesic \(\gamma\) corresponds to the \(u\)-line \(u \mapsto (u, 0)\). We have \(g_\gamma(\partial_u, \partial_v) = 1\) and \(g_\gamma(\partial_u, \partial_u) = 0\). Write \(g_\gamma(\partial_u, \partial_u) = f^2 > 0\). Note that \(f(u, 0) = 1\), for all \(u\). Hence the metric \(g_\gamma\) on \(X(\ell) \times [-\varepsilon, \varepsilon]\) can be written as \(f^2(u, v)du^2 + dv^2\). The curvature of this metric is \(-\frac{f''}{f}\). Hence \(f_{vv} \geq 0\). Also, since \(u \mapsto (u, 0)\) is a geodesic, one can deduce from the equations of a geodesic that \(f_{vv}(u, 0) = 0\), for all \(u\). Hence \(v \mapsto f(u, v)\) has a minimum at \(v = 0\), and \(f(u, v) \geq 1\), for all \((u, v)\).

To construct the metric \(g_1\) we will need the following functions. For \(t \in [0, 1]\), let \(\rho_t : \mathbb{R} \to \mathbb{R}\) be \(C^\infty\) and such that (1) \(\rho_t(-z) = -\rho_t(z)\), (2) \(\rho_t(z) = z - 2t\) for all \(z \geq 3\), (3) \(\rho'_t(z) \geq 0\) for all \(z \geq 0\), (4) \(|z - \rho_t(z)| \leq 2t\). Note that property (4) for \(t = 0\) implies \(\rho_0 = 1\). We also demand (5) \(\rho_1(z) = 0\) whenever \(|z| \leq 1\). For \(\eta > 0\) define \(\rho_{\eta, t}(z) = \eta \rho_t(z)\). We write \(\rho_\eta = \rho_{\eta, 1}\).
We will also need the following functions. For \( \eta > 0 \) small and \( t \in [0, 1] \), let \( \sigma_{n,t} : \mathbb{R} \to \mathbb{R} \) be \( C^\infty \) such that 
1. \( \sigma_{n,t}(z) = z \), for \( |z| \leq \eta \), 
2. \( \sigma_{n,t}(-z) = -\sigma_{n,t}(z) \), 
3. \( \sigma_{n,t}(z) = z + 2\eta t \), for \( z \geq \sqrt{\eta} \), 
4. \( 1 \leq \frac{d}{dz} \sigma_{n,t}(z) \leq 1 + 3t\sqrt{\eta} \), 
5. \( |\sigma_{n,t}(z) - z| \leq 2\eta t \), 
6. \( |\frac{d^2}{dz^2} \sigma_{n,t}(z)| \leq 3\eta \), for all \( z \) and \( t \in [0,1] \). Note that it follows that \( \sigma_{n,0} = 1 \). We will write \( \sigma_n = \sigma_{n,1} \).

We assume \( 0 < \varepsilon < \frac{\eta}{4} \), and \( \eta > 0 \) with \( \sqrt{\eta} \leq \varepsilon/4 \). Define the diffeomorphism \( S_\eta : X \times \mathbb{R} \to X \times \mathbb{R} \) by \( S_\eta(u,v) = (u, \sigma_n(v)) \). On \( X \times [-\eta, \eta] \) define the metric \( h_\eta = f^2(u, \rho_\eta(v)) du^2 + dv^2 \). Finally, on \( X \times [-\varepsilon, \varepsilon] \) define the metric \( g_\eta = S_\eta^* h_\eta \), that is

\[
g_\eta(u,v) = f^2\left(u, \rho_\eta(\sigma_n(v))\right) du^2 + \left(\sigma'_n(v)\right)^2 dv^2
\]

We have the following properties.

(a) For \( |v| \geq \sqrt{\eta} \) and all \( u \) we have \( g_\eta(u,v) = g_\eta(u,v) \).
(b) \( \frac{\partial g_\eta}{\partial v} f(u, \rho_\eta(v)) \geq 0 \), hence \( h_\eta \) is nonpositively curved. Therefore \( g_\eta \) is nonpositively curved.
(c) \( (1 - C\sqrt{\eta}) g_\eta \leq g_\eta \leq (1 + C\sqrt{\eta}) g_\eta \), for some constant \( C \).
(d) On \( X \times [-\eta, \eta] \) we have \( h_\eta = du^2 + dv^2 \), hence \( X \times [-\eta, \eta] \) with metric \( h_\eta \) is isometric to a flat cylinder. Since \( S_\eta \) sends \( X \times [-\eta, \eta] \) to itself, the same is true for \( X \times [-\eta, \eta] \) with metric \( g_\eta \).

Properties (a), (b) and (d) follow directly from the definitions. We prove (c). From the definitions we have

\[
|\rho_\eta(\sigma_n(v)) - v| \leq |\rho_\eta(\sigma_n(v)) - \sigma_n(v)| + |\sigma_n(v) - v| \leq 2\eta + 2\eta = 4\eta.
\]

Let \( C_1 \) be the \( C^1 \)-norm of \( f \). Then \( |f(u, \rho_\eta(\sigma_n(v))) - f(u, v)| \leq 4\eta C_1 \). Therefore \( |f^2(u, \rho_\eta(\sigma_n(v))) - f^2(u, v)| \leq 4\eta C_1^2 \). We write \( C = C_1 + 1 = 8C_1^2 \). Let \( x = a\partial_u + b\partial_v \) be a tangent vector. Then

\[
g_\eta(u,v)(x,x) - g_\eta(u,v)(x,x) = \left|f^2(u, \rho_\eta(\sigma_n(v))) - (\sigma'_n(v))^2 b^2 - f^2(u, v) a^2 - b^2\right|
\]

\[
\leq f^2(u, \rho_\eta(\sigma_n(v))) - f(u, v) a^2 + \left(|\sigma'_n(v)| - 1\right) b^2
\]

\[
\leq C_\eta a^2 + 9\sqrt{\eta} b^2
\]

\[
\leq C_\eta \sqrt{(f^2(u, v) a^2 + b^2)} = C_\eta g_\eta(u,v)(x,x).
\]

In the last inequality we are using \( \eta < 1 \), \( f \geq 1 \), \( C \geq 9 \). Also, in the second inequality we are using (4) of the definition of \( \sigma \). This proves (c).

We prove one more property of the metric \( g_\eta \) on \( X \times [-\varepsilon, \varepsilon] \) given by (4).

**Lemma 32.** Let \( A \) be an arc in the interior of \( X \), and assume the curvature is zero on an open set containing the \( \varepsilon \)-rectangle of \( A \). Then we can modify \( g_\eta \) so that \( g_\eta = g_\gamma \) on \( A \times [-\varepsilon, \varepsilon] \).

**Proof.** Let \( A = [a,b] \) be an arc. Then there is \( \chi > 0 \) such that on \( U = (a - \chi, b + \chi) \times [-\varepsilon, \varepsilon] \), the curvature of \( g_\gamma \) is zero. We have to prove that we can modify \( g_\eta \) so that \( g_\eta = g_\gamma \) on \( A \times [-\varepsilon, \varepsilon] \). Since the curvature is zero we have \( f_{uv} = 0 \). But we also have \( f_{u}(u,0) = 0 \) and \( f(u,0) = 1 \). Therefore \( f \equiv 1 \) on \( U \), hence \( g = du^2 + dv^2 \) on \( U \). On the other hand, from the definitions, one can see that on \( U \) we have \( h_\eta = du^2 + dv^2 \) and \( g_\eta = du^2 + (\sigma'_n)^2 dv^2 \), which is isometric to \( h_\eta = du^2 + dv^2 \) via \( S_\eta \). We now change \( S_\eta \). Let \( \theta : X \to [0,1] \) such
has exactly one self-intersection at the point \( p \) with more self-intersections is similar. Let \( \chi \) be an arc in \( \gamma \) with the image of \( \gamma \) outside the image of \( E \). Define \( \tilde{g}_\eta = S^*_{\gamma} h_\eta \). It can now be shown from the definitions that properties (a), (b) and (d) still hold for \( \tilde{g}_\eta \); moreover we also have \( \tilde{g}_\eta = du^2 + dv^2 = g_\gamma \) on \( A \times [-\varepsilon, \varepsilon] \), as required. One may have now a new problem with property (c) since there is a new term \( \frac{d}{du} \sigma_{\gamma \theta(u)}(v) \) in the derivative of \( \tilde{g}_\eta \) that could be large. To solve this note that property (6) in the definition of \( \sigma \) implies \( |\frac{d}{du} \sigma_{\gamma \theta(u)}(v)| = |\theta'(u)||\frac{d}{du} \sigma_{\gamma \theta(u)}(v)|_{\theta(u)}| \leq 3\eta |\theta'(u)| \). Hence, we can just fix \( \chi \) and \( \theta \) and take \( \eta \) very small. In this way it is straightforward to show that (c) still holds, maybe with a larger \( C \) which depends on the fixed number \( \chi \) and fixed function \( \chi \).

We divide the remainder of the proof in three cases.

1. \( X = S^1 \) and \( E \) is an Embedding.

   If \( E \) is an embedding we can define the metric \( g_1 \) by demanding \( g_1 = g \) outside the image of \( E \) and equal to \( E \cdot g_\eta \) inside the image of \( E \), where \( g_\eta \) is as in equation (†). By property (a) this metric is well defined. By choosing \( \eta \) small we get that this \( g_1 \) satisfies properties (1)-(4) of Lemma 31. Property (5) follows from Lemma 32.

2. \( X = [0, \ell] \) is an Interval and \( E \) an Embedding.

   First we have to extend the domain of \( E \). Since \( \varepsilon < \varepsilon_{\gamma, g}/2 \) there is \( \chi > 0 \) such that \( E \) extends to an embedding \( E : I \times [-\varepsilon, \varepsilon] \to S \), where \( I = [-\chi, \ell + \chi] \). Let \( \theta : I \to [0, 1] \) be smooth and such that \( \theta \equiv 1 \) on a neighborhood of \( [0, \ell] \) and \( \theta \equiv 0 \) near the end points of \( I \). Now consider the following extensions of \( h_\eta \) and \( S_\eta \). Define \( h_\eta(u, v) = \int f^2(u, \rho_{\eta \theta(u)}(v))du^2 + dv^2 \), and \( S_\eta(u, v) = (u, \sigma_{\gamma \theta(u)}(v)) \). Finally define \( g_\eta = S^*_\eta h_\eta \). It can be directly checked from the definitions that properties (a), (b) and (d) above still hold. Also, from the definition of the newly extended \( g_\eta \) we have that \( g_\eta = g_\gamma \) near \( \{-\chi\} \times [-\varepsilon, \varepsilon] \) and \( \{\ell + \chi\} \times [-\varepsilon, \varepsilon] \). Property (c) can be proven as in the proof of Lemma 32. Fix \( \chi \) and take \( \eta \) sufficiently small. Define \( g_1 = g \) outside the image of \( E \) and \( g_1 = E \cdot g_\eta \) on the image of \( E \). By (a)-(d) \( g_1 \) is well defined and satisfies (1)-(4) of Lemma 31. In fact a bit more than (3) holds:

   \[ \gamma : I \to S \] is still a geodesic. Also, (5) follows from Lemma 32.

3. General Case.

   We assume \( X = S^1(\ell) \). The case of \( X \) being an interval is similar. We now allow the closed geodesic \( \gamma \) to have self-intersections. To simplify our argument we assume \( \gamma \) has exactly one self-intersection at the point \( p = \gamma(u_1) = \gamma(u_2) \), \( u_1 \neq u_2 \); the case with more self-intersections is similar. Let \( A \) be an arc in \( S^1(\ell) \) containing \( u_1 \) as middle point, and \( \varepsilon \) small such that \( E \) restricted to \( A \times [-\varepsilon, \varepsilon] \) is an embedding. By case 2, and taking \( \varepsilon \) even smaller if necessary, we can assume (1) the curvature on the \( \varepsilon' \)-rectangle of \( A \) is zero, where \( \varepsilon' < \varepsilon \), (2) \( \gamma \) still has exactly one self-intersection at some \( q = \gamma(u_1') = \gamma(u_2') \) near \( p \), with \( u_1' \in A \). (3) the intersection of the \( \varepsilon' \)-rectangle of \( A \) with the image of \( \gamma \) is exactly two arcs \( \gamma(A) \) and \( \gamma(A') \), where \( A \cap A' = \emptyset \), \( u_2' \in A' \), and \( \gamma(A') \cap E(\partial A \times [-\varepsilon, \varepsilon]) = \emptyset \). After applying case 2 the length of \( \gamma \) may change a bit, but we will still denote it by \( \ell \).

Remark. Note that after applying the (already-proved) Case 2 of Lemma 31 \( \gamma \) may change a bit, but the new \( \gamma \) can be chosen as close as the old \( \gamma \) by taking \( \delta \) in Lemma 31 (Case 2) as small as needed.
Let $A'' \subset A'$ such that $\gamma(A'') = \gamma(A') \cap \gamma(A \times [-\varepsilon', \varepsilon'])$. Note that $u'' \in \overline{A''}$. Now, let $\varepsilon'' > 0$ be small so that (1) the $\varepsilon''$-rectangle of $A''$ is contained in the interior of the $\varepsilon'$-rectangle of $A$, (2) the intersection of the $\varepsilon''$-rectangles of $A$ and $A''$ is disjoint from $E(\partial A \times [-\varepsilon'', \varepsilon'']) \cup E(\partial A' \times [-\varepsilon'', \varepsilon''])$. Let $g_\varepsilon$ be the metric on $X \times [-\varepsilon, \varepsilon]$ given in equation (\infty), with the new $\varepsilon = \varepsilon''$ and $X = S^1(\ell)$. By Lemma 32 we can assume that $g_{\varepsilon} = g_{\varepsilon}$ on $A \times [-\varepsilon'', \varepsilon'']$ and on $A'' \times [-\varepsilon'', \varepsilon'']$. As before we define $g_1 = g$ outside the image of $E$ and $g_1 = E, g_{\varepsilon}$ on the image of $E$. Note that this metric is well defined because $E, g_{\varepsilon} = g$ on the $\varepsilon''$-rectangles of $A$ and $A''$. As in the previous cases, $g_{1}$ satisfies (1)-(5) in the statement of Lemma 31. □

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