Coherent Quantum Dynamics: What Fluctuations Can Tell

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Coherent states provide a natural connection of quantum systems to their classical limit and are employed in various fields of physics. Here we derive general systematic expansions, with respect to quantum parameters, of expectation values of products of arbitrary operators within both oscillator coherent states and SU(2) coherent states. In particular, we generally prove that the energy fluctuations of an arbitrary Hamiltonian are in leading order entirely due to the time dependence of the classical variables. These results add to the list of wellknown properties of coherent states and are applied here to the Lipkin-Meshkov-Glick model, the Dicke model, and to coherent intertwiners in spin networks as considered in Loop Quantum Gravity.

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I. INTRODUCTION

Coherent states are at the heart of semiclassical descriptions of generic quantum systems and have proven to be a versatile tool in a multitude of physical problems. In the general literature, mainly two types of coherent states are typically distinguished: The first type, the coherent states of the harmonic oscillator, was already investigated by Schrödinger shortly after the birth of quantum mechanics, while SU(2) coherent states in the Hilbert space of a spin of general length $S$ were added in the early 1970s.

Both types of coherent states share a list of wellknown properties which constitute the basis for their prominent role in semiclassics: (i) The coherent states can be generated by a unitary transformation from an appropriate reference state. In the oscillatory case this state is the ground state of an harmonic, while for spins one uses the heighest-weight state in some arbitrary basis. As a result, the coherent states are (ii) (over-)complete, (iii) eigenstates of simple operators generic to the system, and (iv) they have minimum uncertainty products with respect to an obvious choice of variables. Moreover, (v) coherent states show a coherent time evolution perfectly mimicking the classical limit under appropriate Hamiltonians. For oscillator coherent states such a Hamiltonian is the one of the harmonic oscillator itself, and for the spin case the Zeeman Hamiltonian (coupling the spin to an external magnetic field) plays an analogous role.

In the present work we argue that one can extend the above list of properties of coherent states. Reflecting the widespread use of the latter objects, we apply our findings to the Lipkin-Meshkov-Glick model originating from nuclear physics, to the Dicke model describing superradiance in quantum optics, and to coherent intertwiners of spin networks occurring in the loop approach to quantum gravity.

This paper is organized as follows: In section II we review and summarize important properties of oscillator coherent states and SU(2) coherent states. The announced results on the coherent expectation values of arbitrary operator products are derived in section III and discussed there on a general footing. Some technical details of the calculations are deferred to appendix A. Section IV contains the application of our general findings to the Lipkin-Meshkov-Glick model, and the Dicke model is treated in section V. In section VI we turn to the study of coherent intertwiners of spin networks investigated in the covariant approach advocated by Loop Quantum Gravity. Here we derive semiclassical corrections to expectation values in terms of universal expansion coefficients depending only on the network geometry. We close with a summary and an outlook in section VII.

II. COHERENT STATES

We now briefly review, using standard notation, distinctive properties of oscillator coherent states and SU(2) coherent states.

A. Coherent Oscillator States

The harmonic oscillator is described by

$$\mathcal{H}_h = \frac{1}{2} \left( p^2 + \omega^2 q^2 \right) = \hbar \omega \left( a^+ a + \frac{1}{2} \right)$$

(1)

with

$$a = \frac{1}{\sqrt{\hbar}} \left( \frac{\omega}{\hbar} q + \frac{i}{\sqrt{\hbar \omega}} p \right), \quad a^+ = (a)^+$$

(2)
fulfilling
\[ [p, q] = \hbar \frac{i}{\hbar} \quad \Leftrightarrow \quad [a, a^\dagger] = 1. \quad (3) \]
The system has an equidistant spectrum labelled by \( n \in \{0, 1, 2, \ldots\} \),
\[ \mathcal{H}_n |n\rangle = \hbar \omega \left( n + \frac{1}{2} \right) |n\rangle. \quad (4) \]
Coherent states of the harmonic oscillator are eigenstates of the lowering operator \( a \) with complex eigenvalues \( \alpha \),
\[ a|\alpha\rangle = \alpha |\alpha\rangle. \quad (5) \]
They are generated from the ground state via
\[ |\alpha\rangle = \exp \left( \alpha a^\dagger - \alpha^* a \right) |0\rangle \quad (6) \]
\[ = \exp \left( -\frac{1}{2} |\alpha|^2 \right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (7) \]
The parameter \( \alpha \) is naturally decomposed into its real and imaginary part as
\[ \alpha = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\omega}{\hbar}} \xi + \frac{i}{\sqrt{\hbar \omega}} \pi \right). \quad (8) \]
Denoting an expectation value within a coherent state \( \mathbb{C} \) by \( \langle \cdot \rangle \) it holds
\[ \langle q \rangle = \xi, \quad \langle p \rangle = \pi. \quad (9) \]
Coherent states maintain their shape in the time evolution of the harmonic oscillator,
\[ e^{-i\frac{\mathcal{H}_t}{\hbar}} |\alpha\rangle = e^{-i\frac{\omega t}{\hbar}} |\alpha \rangle e^{-i\omega t}, \quad (10) \]
and the time dependence of the expectation values \( \mathbb{C} \) follows exactly the classical motion of the harmonic oscillator. This fact justifies the term 'coherent states' and relies on the equidistance of the spectrum. The latter property is shared by a quantum spin of arbitrary length and the time dependence of the expectation values \( \mathbb{C} \) follows exactly the classical motion of the harmonic oscillator, as we will discuss in section II B.
Moreover, coherent states minimize uncertainty products,
\[ \Delta p \Delta q = \hbar / 2 \quad (11) \]
and fulfill an (over-)completeness relation,
\[ \frac{1}{\pi} \int d^2 \alpha |\alpha\rangle \langle \alpha | = 1. \quad (12) \]

B. SU(2) coherent states

In the Hilbert space of a spin of length \( S \) an SU(2) (or spin) coherent state \( |\vartheta, \varphi\rangle \) is defined by the equation
\[ \hat{S} \cdot \hat{S} |\vartheta, \varphi\rangle = \hbar S |\vartheta, \varphi\rangle \quad (13) \]
for the direction \( \vec{s} = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta) \). For generic systems expectation values within these states provide a natural approach to the classical limit given by \( \hbar \to 0, S \to \infty \) while \( \hbar S \) is kept constant.
Introducing the usual basis of eigenstates of \( S^z \) \( (S^z |m\rangle = \hbar m |m\rangle) \) coherent states can be generated from \( |S\rangle \) by a unitary rotation,
\[ |\vartheta, \varphi\rangle = U(\vartheta, \varphi) |S\rangle = \frac{1}{\sqrt{1 + |z|^2}} e^{z S^+ - i \pi \varphi} |S\rangle \quad (14) \]
with \( z, \varphi \)
\[ U(\vartheta, \varphi) = \exp \left( \frac{i}{\hbar} \vartheta \left( \sin \varphi S^x - \cos \varphi S^y \right) \right) \quad (16) \]
\[ = e^{z S^+/\hbar} e^{i \varphi S^y / \hbar} e^{-z S^+ / \hbar} \quad (17) \]
\[ = e^{-z S^+/\hbar} e^{-i \varphi S^y / \hbar} e^{z S^+ / \hbar} \quad (18) \]
and
\[ z(\vartheta, \varphi) = \tan \frac{\vartheta}{2} e^{i \varphi}, \quad \eta(\vartheta) = 2 \ln \cos \frac{\vartheta}{2}. \quad (19) \]
Expanded in the above basis SU(2) coherent states read
\[ |\vartheta, \varphi\rangle = \frac{1}{\sqrt{1 + |z|^2}} \sum_{m=-S}^{S} \left( \begin{array}{c} 2S \\ S + m \end{array} \right)^{1/2} z^m |m\rangle \quad (20) \]
\[ \sum_{m=-S}^{S} \left[ \left( \begin{array}{c} 2S \\ S + m \end{array} \right)^{1/2} \left( \cos \frac{\vartheta}{2} \right)^{S+m} \\ \left( \sin \frac{\vartheta}{2} \right)^{S-m} e^{i \varphi (s+m)} \right] |m\rangle. \quad (21) \]
The analog of the harmonic oscillator for SU(2) coherent states is the Zeeman Hamiltonian
\[ \mathcal{H}_z = -\vec{S} \cdot \vec{h} \quad (22) \]
coupling the spin to a magnetic field \( \vec{h} \). The spectrum consists of \( 2S + 1 \) equidistant energy levels, and the corresponding time evolution of SU(2) coherent states is a coherent Larmor precession, which is most easily seen when putting, without loss of generality, the field direction along the \( z \)-axis,
\[ e^{-i \frac{\mathcal{H}_z t}{\hbar}} |\vartheta, \varphi\rangle = e^{-i \varphi S^y t} |\vartheta, \varphi + h t\rangle. \quad (23) \]
The latter finding is completely analogous to the harmonic oscillator having a semi-infinite equidistant spectrum.
As further standard properties shared with coherent oscillator states, SU(2) coherent states have a minimum uncertainty product
\[ \Delta (\hat{e}_1 \cdot \vec{S}) \Delta (\hat{e}_2 \cdot \vec{S}) = \frac{\hbar^2 S}{2} \quad (24) \]
with \( \vec{e}_1, \vec{e}_2, \vec{s} \) being an orthonormal system, and their (over-) completeness can be expressed as

\[
1 = \frac{2S + 1}{4\pi} \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta |\vartheta, \varphi \rangle \langle \vartheta, \varphi| \tag{25}
\]

\[
= \frac{2S + 1}{\pi} \int \frac{d^2 z}{(1 + |z|^2)^2} |z\rangle \langle z|
\tag{26}
\]

\[
= \frac{2S + 1}{\pi} \int d^2 z e^{iS^*/\hbar}|S\rangle \langle S|e^{iS^*/\hbar} \frac{1}{(1 + |z|^2)(S+1)} , \tag{27}
\]

where \( |z\rangle = |\vartheta, \varphi \rangle \). Further below it will be useful to change reference state \( |S\rangle \) in Eq. (27) to an arbitrary SU(2) coherent state by applying the unitary transformation given in Eqs. (15): \( 1 \rightarrow SU \)

\[
1 = \frac{2S + 1}{\pi} \int d^2 w e^{iS^*/\hbar}|S\rangle \langle S|e^{iS^*/\hbar} \frac{1}{(1 + |w|^2)(S+1)} U^+ \tag{28}
\]

with \( \vec{S} = U \vec{S} U^+ \).

\section*{III. CORRELATIONS}

We now derive general theorems for the expectation values of operator products within coherent states.

\subsection*{A. Oscillatory Systems}

\subsubsection*{1. General Correlation Functions}

Let \( A, B \) be two operators being functions of the two canonical operators \( p, q \) (or, equivalently, \( a, a^\dagger \)). Using the completeness relation (12) the expectation value of \( AB \) within a coherent state can be formulated as

\[
\langle \alpha | AB | \alpha \rangle = \frac{1}{\pi} \int d^2 \beta e^{-|\beta|^2} \langle 0 | U^+_\alpha A U_\alpha U^+_\alpha e^{\beta a^\dagger} | 0 \rangle
\]

\[
\cdot \langle 0 | e^{\bar{\beta} a} U_\alpha U^+_\alpha B U_\alpha | 0 \rangle , \tag{29}
\]

where \( U_\alpha \) is the unitary operator on the r.h.s. of Eq. (3), and

\[
U^+_\alpha e^{\beta a^\dagger} | 0 \rangle = e^{\frac{i}{\hbar} |a|^2 + \beta \bar{a}} e^{i(\beta - a^\dagger) a^\dagger} | 0 \rangle \tag{30}
\]

such that

\[
\langle \alpha | AB | \alpha \rangle = \frac{1}{\pi} \int d^2 \beta e^{-|\beta|^2} \langle 0 | U^+_\alpha A U_\alpha e^{i(\beta - a^\dagger) a^\dagger} | 0 \rangle
\]

\[
\cdot \langle 0 | e^{\bar{\beta} a} U_\alpha U^+_\alpha B U_\alpha | 0 \rangle
\]

\[
= \frac{1}{\pi} \int d^2 \beta e^{-|\beta|^2} \langle 0 | e^{i(\beta - a^\dagger) a^\dagger} U^+_\alpha A U_\alpha e^{\beta a^\dagger} | 0 \rangle
\]

\[
\cdot \langle 0 | e^{\bar{\beta} a} U_\alpha U^+_\alpha B U_\alpha e^{-\beta a} | 0 \rangle , \tag{31}
\]

where we have shifted the integration variable and used \( e^{-\beta a^\dagger} | 0 \rangle = | 0 \rangle \). The remaining operator products can be expanded into series of iterated commutators according to

\[
e^{X} Y e^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!} [X, Y]_n \tag{32}
\]

with \( [X, Y]_0 = Y \) and \( [X, Y]_n = [X, [X, Y]_{n-1}] \). Upon performing the integration the two infinite series shrink to a single one yielding

\[
\langle \alpha | AB | \alpha \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle 0 | [-a^+, U^+_\alpha A U_\alpha]_n | 0 \rangle
\]

\[
\cdot \langle 0 | [a, U^+_\alpha B U_\alpha]_n | 0 \rangle \tag{33}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \langle \alpha | [i U_\alpha a^+ U^+_\alpha , A]_n | \alpha \rangle
\]

\[
\cdot \langle \alpha | [i U_\alpha a U_\alpha , B]_n | \alpha \rangle
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \langle \alpha | [i a^+, A]_n | \alpha \rangle
\]

\[
\cdot \langle \alpha | [i a, B]_n | \alpha \rangle . \tag{34}
\]

In the last step we took into account that \( U_\alpha a^+ U^+_\alpha \) and \( a^+ \) differ just by a constant which commutes with any operator. Thus, we have arrived at an expression for the expectation value of product of two operators within coherent states in terms of a sum over products of such coherent-state expectation values which involve only one of the operators. An alternative form of the above expansions can be given via Eq. (33) as

\[
\langle \alpha | AB | \alpha \rangle = \sum_{n=0}^{\infty} \langle 0 | U^+_\alpha A U_\alpha | n \rangle \langle n | U^+_\alpha B U_\alpha | 0 \rangle \tag{35}
\]

which of course just expresses the completeness of the states \( |n\rangle \) and provides an alternative way to derive Eq. (34).

Moreover, using the definition (2) Eq. (34) can be rewritten as

\[
\langle \alpha | AB | \alpha \rangle = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left[ \left\langle \frac{i}{\hbar} \left( \sqrt{\omega q - i \frac{p}{\sqrt{\omega}} \right) , A \right| \right| n \right] \left. \left\langle \frac{i}{\hbar} \left( \sqrt{\omega q + i \frac{p}{\sqrt{\omega}} \right) , B \right| \right| \alpha \right) , \tag{36}
\]

Since each commutation of \( p, q \) with \( A \) or \( B \) yields a factor of \( \hbar \) all expectation value on the r.h.s. are of the same order in \( \hbar \). Thus, Eq. (33) is indeed systematic expansion in \( \hbar \) of the coherent-state expectation value of an arbitrary product of two operators. The zeroth order equals the classical result, and for a general correlation...
function one has the semiclassical expansion
\[ C_{AB} := \langle \alpha | AB | \alpha \rangle - \langle \alpha | A | \alpha \rangle \langle \alpha | B | \alpha \rangle = \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \langle \alpha \left| \frac{i}{\hbar} \left( \sqrt{\omega q - \frac{p}{\sqrt{\omega}}} \right) A \right| n \rangle \langle n \left| \alpha \right. \rangle \cdot \left. \langle \alpha \left| \frac{i}{\hbar} \left( \sqrt{\omega q + \frac{p}{\sqrt{\omega}}} \right) B \right| n \rangle \langle n \left| \alpha \right. \rangle \right. \]
\[ \cdot \left. \langle \alpha | A | \alpha \rangle \langle \alpha | B | \alpha \rangle \right) . \] (37)

Choosing \( A = B \) we obtain a general expression for the variance of an hermitian operator \( A \),
\[ (\Delta A)^2 = \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left( \langle \alpha \left| \frac{i}{\hbar} \left( \sqrt{\omega q - \frac{p}{\sqrt{\omega}}} \right) A \right| n \rangle \langle n \left| \alpha \right. \rangle \langle \alpha | A | \alpha \rangle \right) . \] (38)
where each term in the semiclassical expansion is non-negative.

2. Energy Fluctuations

Considering \( A = H \) as an Hamiltonian, the corresponding energy fluctuation reads in leading order in \( \hbar \)
\[ (\Delta H)^2 = \frac{\hbar}{2} \left( \left( \frac{i}{\hbar} \left[ \sqrt{\omega q}, H \right] \right)^2 + \left( \frac{i}{\hbar} \left[ \frac{p}{\sqrt{\omega}}, H \right] \right)^2 \right) \]
+ \( O(\hbar^2) \) \[ + \frac{\hbar}{2} \left( \omega (\partial_t q)^2 + \frac{(\partial_t p)^2}{\omega} \right) , \] (39)
where we have replaced, according to the Heisenberg equations of motion, the commutators with time derivatives. Indeed, if the system is prepared at some initial time \( t = t_i \) in a coherent state we have (cf. Eq. [3])
\[ \langle \partial_t q \rangle = \partial_t \xi \] , \[ \langle \partial_t p \rangle = \partial_t \pi \] (41)
and
\[ (\Delta H)^2 = \frac{\hbar}{2} \left( \omega (\partial_t \xi)^2 + \frac{(\partial_t \pi)^2}{\omega} \right) + O(\hbar^2) \] (42)
at \( t = t_i \). In the subsequent time evolution governed by the Hamiltonian \( H \) the state of the system will, for not too large times, approximately be coherent with time-dependent parameters \( \xi(t) \), \( \pi(t) \) playing the approximate role of classical Hamiltonian variables. Thus, in this semiclassical regime the fact that a coherent state has a finite energy variance, i.e. it is not an eigenstate of the Hamiltonian, is in leading order in \( \hbar \) just expressed by the fact that the classical Hamiltonian variables have a nontrivial time dependence, i.e. the system is moving. This result complements the historical Ehrenfest theorem stating that expectation values of observables follow the classical equations.

Relations of the type (40), (42) were already found in Ref. 13 on the example of specific Hamiltonians. The results here are derived for arbitrary systems and are based on the very general expansions (37), (38) for correlation functions and fluctuations.

The fact that the system will in its time evolution in general not strictly remain in a coherent state, i.e. decoherence occurs, is reflected by the higher contributions to the energy variance. Indeed, for a harmonic oscillator [14] the time evolution is strictly coherent and we have as an identity
\[ (\Delta H_h)^2 = \frac{\hbar}{2} \left( \omega (\partial_t \xi)^2 + \frac{(\partial_t \pi)^2}{\omega} \right) \] (43)
for all times \( t \geq t_i \) and without any higher correction.

Finally, it is straightforward to extend the above results for general operator products to the case of \( N > 1 \) degrees of freedom; details are sketched in appendix A.

For the energy variance one finds in leading order in \( \hbar \)
\[ (\Delta H)^2 = \frac{\hbar}{2} \sum_{a=1}^{N} \left( \left( \frac{i}{\hbar} \left[ \sqrt{\omega_a q_a}, H \right] \right)^2 + \left( \frac{i}{\hbar} \left[ \frac{p_a}{\sqrt{\omega_a}}, H \right] \right)^2 \right) \]
+ \( O(\hbar^2) \) \[ = \frac{\hbar}{2} \sum_{a=1}^{N} \left( \omega_a (\partial_t q_a)^2 + \frac{(\partial_t p_a)^2}{\omega_a} \right) \] (44)
where \( H \) the state of the system will, for not too large times, approximately be coherent with time-dependent parameters \( \xi(t) \), \( \pi(t) \) playing the approximate role of classical Hamiltonian variables. Thus, in this semiclassical regime the fact that a coherent state has a finite energy variance, i.e. it is not an eigenstate of the Hamiltonian, is in leading order in \( \hbar \) just expressed by the fact that the classical Hamiltonian variables have a nontrivial time dependence, i.e. the system is moving. This result complements the historical Ehrenfest theorem stating that expectation values of observables follow the classical equations.

1. General Correlation Functions

We consider again two arbitrary operators \( A, B \) which are now functions of a spin operator \( \vec{S} \). The expectation value of the product \( AB \) within an SU(2) coherent state \( |z\rangle \) for spin length \( S \) can be formulated as
\[ \langle z \left| AB \right| z \rangle = \frac{2S + 1}{\pi} \int \frac{d^2w}{(1 + |w|^2)^{2(S+1)}} \]
\[ \langle z | e^{-w S^- / \hbar} A e^{w S^- / \hbar} | z \rangle \]
\[ \cdot \langle z | e^{w S^+ / \hbar} B e^{-w S^+ / \hbar} | z \rangle , \] (47)
where we have used the completeness relation in the form [25] and the observation
\[ e^{-w S^+ / \hbar} | z \rangle = U e^{-w S^+ / \hbar} | 0 \rangle = | z \rangle \] (48)
with \( U \) given in Eqs. (13), (15). Employing now again the expansion (32) and performing the integration leads to

\[
\langle z | AB | z \rangle = \sum_{n=0}^{2S} \frac{(2S-n)!}{n!(2S)!} \int \left[ \left| \frac{i}{\hbar} S^- , A \right| _n \right] \left| z \right> \left< z \right| \left[ \frac{i}{\hbar} S^+ , B \right] _n \left| S \right> .
\]  

(49)

The above equation is the spin analog of the result (36). Again all iterated commutators are of the same order in \( \hbar \) and \( S \) whereas the prefactor of the \( n \)-th term carries a product \( 2S(2S-1) \cdots (2S-n+1) \) in its denominator. Thus, Eq. (49) is essentially an expansion in the quantum parameter \( 1/S \). Note that the spin components \( \hat{S}^x, \hat{S}^y \) represent the direction perpendicular to the spin polarization of the coherent state \( |z\rangle \). Alternatively, the result (49) can be written as

\[
\langle z | AB | z \rangle = \sum_{n=0}^{2S} \frac{(2S-n)!}{n!(2S)!} \left< S \left| \left[ \frac{i}{\hbar} S^- , U^+ AU \right] _n \right| S \right> \left< S \left| \left[ \frac{i}{\hbar} S^+ , U^+ BU \right] _n \right| S \right> \left< S \right| \left| S-n \right| U^+ BU \right> \left| S \right> .
\]  

(50)

Analogously to Eq. (35a) for oscillatory systems, the last formulation is just the completeness relation for the states \( |n\rangle \) and allows for an alternative derivation of the central result (49). Using the latter, arbitrary correlation functions within SU(2) coherent states can be expressed in full analogy to Eq. (37).

2. Fluctuations

For the variance of an hermitian operator \( A \) we have

\[
(\Delta A)^2 = \sum_{n=1}^{2S} \frac{(2S-n)!}{n!(2S)!} \left< z \left| \left[ \frac{i}{\hbar} S^- , A \right] _n \right| z \right> ^2 .
\]  

(52)

The expectation values occurring in leading order can be rewritten as

\[
\left< z \left| \left[ \frac{i}{\hbar} S^- , A \right] _n \right| z \right> ^2 = \sum_{i=1}^{3} \left< z \left| \left[ i S^i , A \right] _n \right| z \right> ^2 = \sum_{i=1}^{3} \left< z \left| \left[ i S^i , A \right] \right| z \right> ^2 ,
\]  

(53)

(54)

where we have observed that \( |z\rangle \) is an eigenstate of \( \hat{S}^z \), and that \( \hat{S} \) and \( \hat{S} \) are related by an orthogonal matrix,

\[
\hat{S}^i = \sum_{j=1}^{3} O_{ij} S^j .
\]  

(55)

Thus, we have

\[
(\Delta A)^2 = \frac{1}{2S} \sum_{i=1}^{3} \left< z \left| \left[ \frac{i}{\hbar} S^i , A \right] \right| z \right> ^2 + O \left( \frac{1}{S^2} \right) ,
\]  

(56)

and by a slight generalization of the above arguments one finds for the expectation value of a product of commuting operators \( A, B \)

\[
\langle z | AB | z \rangle = \frac{1}{2} \langle z | AB + BA | z \rangle = \langle z | A | z \rangle \langle z | B | z \rangle + \frac{1}{2S} \sum_{i=1}^{3} \left< z \left| \left[ \frac{i}{\hbar} S^i , A \right] \right| z \right> \left< z \left| \left[ \frac{i}{\hbar} S^i , B \right] \right| z \right> + O \left( \frac{1}{S^2} \right) .
\]  

(57)

The requirement here for a symmetrized operator product stems from the fact that for an identity analogous to Eq. (54) to hold products of expectation values involving both \( \hat{S}^x \) and \( \hat{S}^y \) should drop out.

Choosing now in Eq. (54) \( A \) to be the Hamiltonian \( \hat{H} \) of the underlying system we can write by the same arguments as for Eq. (40)

\[
(\Delta \hat{H})^2 = \frac{1}{2S} \langle \partial_t \hat{S}, \partial_t \hat{S} \rangle ^2 + O \left( \frac{1}{S^2} \right)
\]  

(58)

with \( \langle \cdot \rangle = \langle z | \cdot | z \rangle \). To this result the same comments apply as to its oscillatory counterpart Eq. (40): If the system is initially in an SU(2) coherent state it holds (cf. Eq. (13))

\[
\langle \partial_t \hat{S} \rangle = hS \partial_t \hat{s}
\]  

(59)

and

\[
(\Delta \hat{H})^2 = (hS)^2 \left( \frac{1}{2S} (\partial_t \hat{s})^2 + O \left( \frac{1}{S^2} \right) \right)
\]  

(60)

at initial time \( t = t_i \), and for not too large times \( t > t_i \) the system will approximately remain coherent in its time evolution under \( \hat{H} \) with \( \hat{s}(t) \) being a classical vector. Our finding (60) is again a manifestation of our previous result (12): In leading order in the quantum parameter \( \hbar \) or \( 1/S \) the variance of the energy is due to the classical motion of the system. Findings of the type (60) were also obtained previously in Ref. [13] on the example of specific Hamiltonians. Here we provide a generalization to arbitrary systems based on the very general expansions (13), (23) for correlation functions and fluctuations.

Decoherence effects, i.e. deviations from the coherent state with time-dependent parameters \( \hat{s}(t) \) are again indicated by the higher-order terms in the energy variance, as we shall investigate on a specific example in section 1VA. Conversely, the Zeeman Hamiltonian (22) generates a strictly coherent time evolution with

\[
(\Delta \hat{H}_z)^2 = \frac{(hS)^2}{2S} (\partial_t \hat{s})^2
\]  

(61)
as an identity for arbitrary times $t \geq t_1$.

Similarly as for oscillator systems, the above results for general operator products are easily generalized to the situation of $N > 1$ spins of various lengths; details can be found in appendix A. The leading order of the energy variance is given by

$$\langle \Delta \mathcal{H} \rangle^2 = \sum_{a=1}^{N} \left[ \frac{1}{2S_a} \left( \frac{1}{\hbar} \langle \partial_t S_a \rangle \right)^2 + \mathcal{O} \left( \frac{1}{S_a^2} \right) \right] ,$$

and Eq. (60) is generalized to

$$\langle \Delta \mathcal{H} \rangle^2 = \sum_{a=1}^{N} \left[ \left( \frac{1}{2S_a} \left( \frac{1}{\hbar} \langle \partial_t S_a \rangle \right)^2 + \mathcal{O} \left( \frac{1}{S_a^2} \right) \right) \right] .$$

(63)

IV. THE LIPKIN-MESHKOV-GLICK MODEL

The Lipkin-Meshkov-Glick (LMG) model is an approximate description of $N$ interacting spin-$1/2$ systems and was originally inspired by nuclear physics15,16. More recently this model has been argued to describe two-mode Bose-Einstein condensates17,18, phase transitions in optical cavity QED19,20, and molecular magnet21,22. Moreover it has been employed to model a spin bath23,24 and in studies of quenched dynamics25. In the last decade a flurry of publications investigating various aspects of the LMG model has appeared; as an entry point to the recent literature we refer to Refs. 26,27.

Concentrating on the sector of maximal spin $S = N/2$, the LMG Hamiltonian reads

$$\mathcal{H} = -\hbar S^z - \frac{1}{2\hbar S} \left( \gamma_x S^x S^z + \gamma_y S^y S^y \right) ,$$

(64)

where $h$ can be interpreted as a magnetic field coupling to the $z$-component of the spin while $\gamma_x, \gamma_y$ parametrize an anisotropic interaction among the perpendicular components. The factor $\hbar S$ in the denominator is a convention common to the literature and leads to a linear scaling of energies as a function of $S \gg 1$. The expectation value with in an SU(2) coherent state is given by (neglecting a constant contribution)

$$\langle \mathcal{H} \rangle = \hbar S \left( -\hbar \cos \vartheta \right)$$

$$+ \frac{\gamma_x}{2} \sin^2 \vartheta \cos^2 \varphi + \frac{\gamma_y}{2} \sin^2 \vartheta \sin^2 \varphi \right)$$

and equals the classical energy expression up the renormalized parameters $\bar{\gamma}_i = \gamma_i (1 - 1/(2S))$. Taking coherent expectation values of both sides of the Heisenberg equations of motion one obtains the (semi-)classical equations

$$\frac{dS^z}{dt} = \hbar \sin \vartheta \sin \varphi - \bar{\gamma}_y \cos \vartheta \sin \vartheta \sin \varphi ,$$

(66)

$$\frac{dS^y}{dt} = -\hbar \sin \vartheta \cos \varphi + \bar{\gamma}_x \cos \vartheta \sin \vartheta \cos \varphi ,$$

(67)

$$\frac{dS^z}{dt} = -\left( \bar{\gamma}_x - \bar{\gamma}_y \right) \sin^2 \vartheta \cos \vartheta \sin \varphi .$$

(68)

For the energy variance one finds by a direct (and somewhat tedious) calculation

$$\langle \Delta \mathcal{H} \rangle^2 = \Omega_1 + \Omega_2$$

(69)

with

$$\Omega_1 = (\hbar S)^2 \frac{1}{2S} \left[ \hbar^2 \sin^2 \vartheta - 2\hbar \bar{\gamma}_x \cos \vartheta \sin^2 \vartheta \cos^2 \varphi - 2\hbar \bar{\gamma}_y \cos \vartheta \sin^2 \vartheta \sin^2 \varphi \right]$$

$$+ \left( \bar{\gamma}_x \left( 1 - \sin^2 \vartheta \cos^2 \varphi \right) + \bar{\gamma}_y \left( 1 - \sin^2 \vartheta \sin^2 \varphi \right) \right)$$

(64)

and

$$\Omega_2 = (\hbar S)^2 \frac{1}{8S^2} \left( 1 - \frac{1}{2S} \right) \left[ -4\bar{\gamma}_x \bar{\gamma}_y \cos^2 \vartheta \right.$$}

$$+ \left( \gamma_x \left( 1 - \sin^2 \vartheta \cos^2 \varphi \right) + \gamma_y \left( 1 - \sin^2 \vartheta \sin^2 \varphi \right) \right) \right]$$

(70)

being of leading order $1/S$ while the contributions summarized in

$$\Omega_1 = (\hbar S)^2 \frac{1}{2S} \left( \frac{dS^z}{dt} \right)^2 ,$$

(72)

in accordance with the general result [60]. The subleading contributions $\Omega_2$ indicate decoherence effects, i.e. departures from the submanifold of the coherent states in the Hamiltonian time evolution, as we now discuss explicitly on the example of the isotropic LMG model.

A. The isotropic case

Putting $\gamma_x = \gamma_y =: \gamma$ the Hamiltonian becomes diagonal in the states $|m\rangle$ with eigenvalues

$$\varepsilon_m = -\hbar m + \frac{\gamma}{2S} m^2 - \frac{\gamma}{2} (m + 1) .$$

(73)

This eigensystem is simple enough to analytically compute the exact time evolution of coherent expectation values $\langle S(t) \rangle$: Due to symmetry, the $z$-component is constant,

$$\langle S^z(t) \rangle \equiv \hbar S \cos \vartheta$$

(74)

while for the perpendicular components one finds

$$\langle S^+(t) \rangle = \hbar S \sin \vartheta e^{i \left( \frac{\gamma}{2S} - 1 \right) (m + 1)}$$

$$\cdot \left( e^{-i \frac{\gamma \cos \vartheta}{2S} t} \left[ \cos \left( \frac{\gamma t}{2S} \right) + \cos \vartheta \sin \left( \frac{\gamma t}{2S} \right) \right] \right)^{2S-1}$$

(75)

The above closed result relies on the fact that $S^+$ couples only eigenstates with neighboring indices such that all
occurring energy differences are, apart from a constant term, linear in \( m \). The first line in Eq. (75) describes a classical rotation of the spin according to Eqs. (66)-(68) whereas the second line contains quantum effects: The "spin length"

\[
|\langle S^+(t) \rangle| = \hbar S \sin \vartheta \left( 1 - \sin^2 \vartheta \sin^2 \left( \frac{\varphi}{2S} \right) \right)^{S-1/2}
\]

(76)

composed from the perpendicular components breathes sinusoidally in time. Quantum (quasi-)revivals occur at times at \( t = 2\pi k S/\gamma \) for any integer \( k \) where the state returns precisely to the submanifold of the coherent states. These times are large in the semiclassical regime as they are proportional to \( S \).

Regarding small times, we define \( t =: \sqrt{S} \tau \) and consider the regime \( \gamma \tau \ll \sqrt{S} \), such that for large \( S \gg 1 \) it follows

\[
\frac{|\langle S^+(t) \rangle|}{\hbar S \sin \vartheta} \approx \left( 1 - \frac{\langle \gamma \tau \sin \vartheta \rangle^2/4}{S} \right)^{S-1/2}
\]

\[
\approx e^{-\langle (\gamma \tau \sin \vartheta)^2/4 \rangle},
\]

(77)
i.e. the spin expectation value \( \langle S(t) \rangle \) shows a gaussian decay with time scale \( \Delta t = \sqrt{2S}/(\gamma \sin \vartheta) \). On this time scale, sometimes known as Ehrenfest time\(^49\), departures between classical and quantum dynamics become sizable. The above finding for \( \Delta t \) is consistent with a heuristic uncertainty argument in the following sense: Replacing in \( \Delta H \Delta t \geq \hbar \) the energy uncertainty with \( \sqrt{\langle H^2 \rangle} \) one obtains a lower bound for \( \Delta t \) being proportional to \( \hbar S \) which is a constant independent of \( S \) in the semiclassical regime. Thus, this lower bound is consistent with the above result which grows with the square root of \( S \).

V. THE DICKE MODEL

The Dicke model describes the superradiant interaction of a single cavity mode of a radiation field with \( N \) two level systems (atoms\(^40\)). Although introduced already in the 1950s, this model continues to be investigated under various aspects; as a guide to the recent literature see e.g. Refs.\(^40-43\).

Focusing again on the sector of maximal spin \( S = N/2 \), the Dicke Hamiltonian can be formulated as

\[
H = \hbar \omega a^+ a + \Omega S^z + \frac{\lambda}{\sqrt{2S}} S^z (a^+ + a)
\]

(78)

\[
= \frac{1}{2} \left( p^2 + \omega^2 q^2 \right) + \Omega S^z + \lambda \sqrt{\frac{\omega}{\hbar S}} S^z q,
\]

(79)

where the parameters \( \omega, \Omega \), and \( \lambda \) have all dimension of inverse time. In the classical limit, the superradiant phase, characterized by a finite bosonic occupation in the ground state, occurs for \( \lambda^2 > \Omega \omega \). The expectation value of the Hamiltonian within a tensor product of an oscillator and an SU(2) coherent state reads

\[
\langle H \rangle = \hbar \omega |\alpha|^2 + \Omega \hbar S \cos \vartheta + \frac{\lambda}{\sqrt{2S}} \hbar S \sin \vartheta \cos \varphi \left( \bar{\alpha} + \alpha \right),
\]

(80)

which perfectly matches the classical expression. The (semi-)classical equations of motion can be obtained analogously as Eqs. \(^40-43\),

\[
\begin{align*}
\frac{d\alpha}{dt} &= i \omega \alpha + \frac{i}{\hbar} \lambda \sqrt{2S} \hbar S \sin \vartheta \cos \varphi, \\
\frac{ds}{dt} &= -\Omega \sin \vartheta \sin \varphi, \\
\frac{ds^v}{dt} &= \Omega \sin \vartheta \cos \varphi - \frac{\lambda}{\sqrt{2S}} \cos \varphi \left( \bar{\alpha} + \alpha \right), \\
\frac{ds^z}{dt} &= \frac{\lambda}{\sqrt{2S}} \sin \vartheta \sin \varphi \left( \bar{\alpha} + \alpha \right),
\end{align*}
\]

(81)-(84)

and a direct (but again quite lengthy) calculation of the energy variance yields

\[
(\Delta H)^2 = \Omega_1 + \Omega_2
\]

(85)

with the leading-order term

\[
\Omega_1 = (\hbar \omega)^2 |\alpha|^2 + (\hbar S)^2 \Omega^2 \sin^2 \vartheta
\]

\[
+ (\hbar S)^2 \frac{\lambda^2}{2S} \left[ \frac{1}{2S} (\sin^2 \vartheta \sin^2 \varphi + \cos^2 \varphi) \right] (\bar{\alpha} + \alpha)^2
\]

\[
+ \sin^2 \vartheta \cos^2 \varphi
\]

\[
+ \hbar \omega \frac{\lambda}{\sqrt{2S}} \hbar S \sin \vartheta \sin \varphi \left( \bar{\alpha} + \alpha \right)
\]

\[
- \Omega \frac{\lambda}{\sqrt{2S}} \frac{(\hbar S)^2}{S} \cos \vartheta \sin \vartheta \cos \varphi \left( \bar{\alpha} + \alpha \right)
\]

(86)

and the subleading contributions

\[
\Omega_2 = (\hbar S)^2 \frac{\lambda^2}{(2S)^2} (\sin^2 \vartheta \sin^2 \varphi + \cos^2 \varphi)
\]

\[
= \frac{\lambda^2}{(2S)^2} \left( \langle S^\parallel \rangle^2 + \langle S^\perp \rangle^2 \right)
\]

(87)

Finally, comparison with Eqs. \(^40-43\) shows

\[
\Omega = \frac{\hbar}{2} \left( \omega \langle \partial_t \xi \rangle^2 + \langle \partial_t \pi \rangle^2 \right) + \frac{(\hbar S)^2}{2S} \langle \partial_t \delta \xi \rangle^2
\]

(89)

in accordance with the general results \(^40-43\), and \(^47\).

Note that the higher-order contributions\(^48\) describing decoherence effects depend only on the coupling parameter \( \lambda \) but not on the frequencies \( \omega, \Omega \), in accordance with the fact that spin and oscillator show perfectly coherent time evolutions in the absence of coupling. Another distinctive feature of the result\(^48\) (compared to
e.g. Eq. (71) is its simplicity which calls for further applications. In fact, an extensive numerical study of the dynamics of the Dicke model in the semiclassical regime was performed recently in Ref. 46. Here the initial condition was, as in the present work, a tensor product of an oscillator and a spin coherent state, and it is straightforward to evaluate the above $Ω_2$ in terms of such dynamical data. In particular, it is an interesting speculation whether or not $Ω_2$ behaves differently in the regular versus (quantum) chaotic regime as studied in Ref. 46. Another aspect is to compare the Ehrenfest times $∆_t$ found numerically with estimates according to $\sqrt{I_2}∆t ≥ ℏ$.

VI. COHERENT INTERTWINERS IN SPIN NETWORKS

We now apply our general findings on coherent expectation values of operator products to spin network states as studied in Loop Quantum Gravity (LQG) 12, 46, 47. In brief, a spin network is a collection of points (called vertices or nodes) in (typically) three-dimensional space connected by one-dimensional curves (edges). Each edge is assigned a spin of individual length, and a spin network state in the tensor product of all SU(2) representations is defined by the additional requirement that all spins joining in a given node are coupled to a total singlet. The latter property implements the Gauss constraint.

A convenient parametrization of spin network states are coherent intertwiners as introduced by Livine and Speziale 48. Fixing an $N$-valent node (connecting $N$ edges), one considers a tensor product

$$|Φ⟩ := \bigotimes_{a=1}^N |θ_a, φ_a⟩$$

of SU(2) coherent states describing the spin on each edge. A coherent intertwiner is then defined by the projection of this object onto the singlet subspace 49

$$|Φ⟩_s = \frac{P|Φ⟩}{\langle Φ|P|Φ⟩}$$

where the denominator takes care of the normalization. The projection operator can be formalized by a Haar integration over all uniform rotations of the $N$ spins (group averaging),

$$P = \int_{SU(2)} dμ \exp \left( iψ\bar{ν} \sum_a \vec{S}_a \right)$$

$$= \frac{1}{4π^2} \int_0^π dθ \sin θ \int_0^{2π} dφ \int_0^{2π} dψ \sin^2 ψ \left( \exp \left( iψ\bar{ν} \sum_a \vec{S}_a \right) \right)$$

with $\bar{ν} = (\sin θ \cos φ, \sin θ \sin φ, \cos θ)$. Here and in what follows we take all spin operators to be dimensionless (as a factor of $ℏ$ will occur below in the Planck length squared). In particular, a coherent intertwiner is by construction invariant under arbitrary rotations of all spins meaning

$$\left( \sum_{a=1}^N \vec{S}_a \right) |Φ⟩_s = 0 .$$

Moreover, nodes in a spin network allow for a geometric interpretation in terms of convex polyhedra 48. From a classical point of view, this relies on a theorem due to Minkowski 46. It states that given $N$ unit vectors $\vec{s}_a$ and $N$ positive numbers $A_a$ fulfilling $\sum_a A_a \vec{s}_a = 0$, there is a unique convex polyhedron with $N$ faces such that $\vec{s}_a$ is the normal to the $a$-th face and $A_a$ is its area. Thus choosing as areas the quantum numbers $S_a$, the classical closure relation

$$\sum_{a=1}^N S_a \vec{s}_a = 0$$

ensures that the geometric information contained in the state (90) encodes a convex polyhedron. The quantum counterpart of the relation (95) is equation (96) giving rise to the notion of a quantum polyhedron 48. In the framework of LQG, the spin operators representing the faces of the polyhedron are, up to a prefactor, considered to be flux operators 46.

$$\vec{E}_a = 8πγ\ell_P^2 \vec{s}_a$$

with $γ$ being the Immirzi parameter and the squared Planck length $\ell_P^2 = ℏG/c^3$.

Let us now explore expectation values within coherent intertwiners. Here one can concentrate without loss of generality on operators unchanged by uniform rotations since for any operator being the sum of a rotationally invariant part and terms without this property, only the former will contribute. Any rotationally invariant operator $Q$ commutes with the projector onto the singlet space, $[Q,P] = 0$, such that $PQP = QP = PQ$. Therefore we can use the result 47 to obtain a semiclassical approximation to the expectation value within a coherent intertwiner,

$$\langle Φ|Q|Φ⟩_s = \frac{\langle Φ|PQ + QP|Φ⟩}{2 \langle Φ|P|Φ⟩} \langle Φ|Q|Φ⟩$$

$$+ \sum_{a=1}^N \frac{1}{2S_a} \sum_{i=1}^3 \langle Φ| [iS^i_a, Q] |Φ⟩ C^i_a(Φ) + \cdots$$

with

$$C^i_a(Φ) = \frac{\langle Φ| [iS^i_a, P] |Φ⟩}{\langle Φ|P|Φ⟩} .$$

Thus, the expectation value of $Q$ is in leading order just given by the expectation value of the unprojected state
where the normalization factor in the definition \( \delta \) drops out. For the subleading corrections one needs to determine the coefficients \( \delta \). Here both numerator and denominator are conveniently formulated in terms of Haar integrations as shown explicitly in Eq. \((104)\). In the semiclassical regime studied here where all spins are long, \( \forall a S_a \gg 1 \), these integrals become amenable to a saddle-point approximation as worked out in Refs. \cite{11}. For the denominator one finds for a general \( N \)-valent node

\[
\langle \Phi | P | \Phi \rangle = \frac{1}{\sqrt{\pi \det H}} + \frac{\text{tr} (H^{-1})}{4 \sqrt{\pi \det H}} + \cdots \tag{100}
\]

where

\[
H^{ij} = \sum_{a=1}^{N} S_a (\delta^{ij} - s_a^1 s_a^2) \tag{101}
\]

is twice the negative Hessian of the saddle point expression, and the details of the calculation can be found in appendix \ref{B}. Since \( H \) is a linear combination of geometric projection operators \((\delta^{ij} - s_a^1 s_a^2)\) with positive coefficients, its eigenvalues are nonnegative, and zero eigenvalues only occur in the degenerate case where all vectors \( s_a \) are collinear, which we shall not consider here. Thus, the eigenvalues of \( H \) can be taken to be positive, and the determinant can be formulated more explicitly as

\[
\det H = \frac{T}{2} \sum_{ab} S_a S_b (s_a^1 \times s_b^1)^2 - \frac{1}{6} \sum_{abc} S_a S_b S_c [(s_a^1 \times s_b^1) \cdot s_c^1]^2 \tag{102}
\]

with \( T = \sum_a S_a \). The semiclasical limit of a quantum polyhedron is obtained by rescaling all quantum numbers as \( S_a \mapsto \lambda S_a \) with some integer \( \lambda \gg 1 \). Thus the leading term in Eq. \((100)\) (already obtained in Ref. \cite{22}) is of order \( \lambda^{-3/2} \) while the subleading correction scales like \( \lambda^{-5/2} \). The numerator in Eq. \((99)\) can be evaluated via saddle point approximation in a similar fashion (see appendix \ref{B} giving, again for a general \( N \)-valent node fulfilling the classical closure relation \((95)\),

\[
\langle \Phi | [ \tilde{s}^a, P ] | \Phi \rangle = S_a (s_a^1 \times (H^{-1} s_a^1)/\sqrt{\pi \det H} \tag{103}
\]

such that for the coefficients themselves we have the amazingly simple result

\[
C_a^{\ell} (\Phi) = S_a s_a^1 \times (H^{-1} s_a^1) \tag{104}
\]

The expression \((103)\) is of order \( \lambda^{-3/2} \) while the coefficients \((104)\) are independent of \( \lambda \) and vanish if the matrix \( H \) is proportional to the unit matrix. Thus, polyhedra where all eigenvalues of \( H \) are degenerate enjoy an enhanced classical character in the sense that the leading order of semiclassical corrections to general expectation values \((98)\) vanishes. In the general case, Eq. \((98)\) tells us that the coherent-intertwiner expectation value of any (rotationally invariant) operator is in leading order given by the expectation value of the unprojected tensor product of SU(2) coherent states, and the leading correction scales with the inverse of the spin lengths.

The coefficients \((104)\) are universal in the sense that they are the same for any operator \( Q \). Making use of the symmetry of \( H \) they can also be formulated as

\[
C_a^{\ell} (\Phi) = S_a \sum_{jk l} \epsilon^{ijk} (H^{-1})^{kl} (s_a^1 s_b^1 - \delta^{ij}) \tag{105}
\]

implying the sum rule

\[
\sum_{a=1}^{N} C_a^{\ell} (\Phi) = - \sum_{jk l} \epsilon^{ijk} (H^{-1})^{kl} H^{lj} = 0 , \tag{106}
\]

which also follows from the definition \((99)\) and the quantum closure relation \((54)\). Thus, the sum rule \((106)\) holds independently of the fulfillment of the classical closure relation \((95)\) which underlies the explicit result \((104)\).

Moreover, it is interesting to note that the matrix \( H \) can be interpreted as the inertia tensor of a distribution of masses \( S_a \), whose positions are given by the unit vectors \( s_a^1 \). By the same token, the classical closure relation \((95)\) states that the center of mass of this distribution lies in the origin of the chosen coordinate system. In particular, \( H \) is proportional to the unit matrix (such that the expansion coefficients \((99)\) vanish) if the node has the shape of an archimedian body such as a regular tetrahedron. We leave it to further studies to explore further possible consequences of the above analogy.

Very typical examples of rotationally invariant operators are volume operators of polyhedra \cite{25,26,27}. The simplest nontrival case of a quantum polyhedron is given by a tetrahedron, i.e. a 4-valent node \cite{22}. The volume operator can be formulated as

\[
V = \frac{\sqrt{2}}{3} \sqrt{\sum_{j=1}^{4} (\vec{E}_{j} \cdot (\vec{E}_{j} \times \vec{E}_{j}))} \tag{107}
\]

using any three of the four flux operators. Squaring this expressions and stripping all prefactors one is led to consider the expression

\[
Q = \tilde{S}_1 \cdot (\tilde{S}_2 \times \tilde{S}_3) \tag{108}
\]

acting on the Hilbert space defined by the constraint \((14)\). The study of this operator in the semiclasical limit has attracted quite a deal of interest recently \cite{95,96,97}. For the expectation value within coherent intertwiners one finds from Eqs. \((98)\), \((104)\)

\[
\langle S | \hat{Q} | S \rangle = S_1 S_2 S_3 \left[ \tilde{s}_1 \cdot (\tilde{s}_2 \times \tilde{s}_3) \right. \nonumber \\
- \frac{1}{2} \left( \tilde{s}_1 \times (\tilde{s}_2 \times \tilde{s}_3) \cdot (\tilde{s}_1 \times (H^{-1} \tilde{s}_1)) + \text{c.p.} \right) \nonumber \\
+ \cdots \tag{109}
\]

As before, the form of the subleading corrections here holds if the classical closure relation \((95)\) is fulfilled.
VII. SUMMARY AND OUTLOOK

We have derived general systematic expansions with respect to quantum parameters of expectation values of products of arbitrary operators within both oscillator coherent states and SU(2) coherent states. These results are a versatile tools for the study of the semiclassical regime of generic quantum systems. In particular, we prove that the energy fluctuations of an arbitrary Hamiltonian are in leading order entirely due to the time dependence of the classical variables, a result very general and very intuitive at the same time.

Our findings offer many possibilities for application in various fields of physics. Here we have specifically studied the Dicke model stemming from quantum optics, and the LMG model originating from nuclear physics. For the latter system we have investigated decoherence effects (i.e. deviations from the submanifold of coherent states) via an exact solution of the dynamics, which also appears novel to the literature. Finally we have applied our general results to coherent intertwiners in spin networks as investigated in LQG. For expectation values of rotationally invariant operators (and these are the only ones contributing) one finds here a subleading correction to the classical limit given in terms of universal (i.e. operator-independent) expansion coefficients which contain only geometric information about the network node.

Appendix A: $N > 1$ degrees of freedom

Let us now extend our results on the coherent-state expectation values of operator products to systems with $N > 1$ degrees of freedom. We start by two oscillatory degrees of freedom $q_a, p_a$ with frequencies $\omega_a, a \in \{1, 2\}$. Iterating the arguments leading to Eq. (36) one finds

$$\langle \alpha | AB | \alpha \rangle = \sum_{m,n=0}^{\infty} \frac{\hbar^{m+n}}{m! n!} \langle \alpha | [Q_2, [Q_1, A]]_m | \alpha \rangle \cdot \langle \alpha | [Q_2^+, [Q_1^+, B]]_n | \alpha \rangle,$$

(A1)

with $|\alpha\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle$ and

$$Q_a = \frac{i}{\sqrt{2\hbar}} \left( \sqrt{\omega_a} q_a - i \frac{p_a}{\sqrt{\omega_a}} \right).$$

(A2)

Since $Q_1, Q_2$ commute, the corresponding left arguments in the above nested commutators can be freely interchanged such that

$$\langle \alpha | Q^+_{P_1^{(1)}} , Q^+_{P_2^{(1)}} , \cdots , Q^+_{P_n^{(1)}} , B ] [ Q^-_{P_1^{(2)}} , Q^-_{P_2^{(2)}} , \cdots , Q^-_{P_n^{(2)}} , A ] [ Q^-_{P_1^{(3)}} , Q^-_{P_2^{(3)}} , \cdots , Q^-_{P_n^{(3)}} , A ] \cdots ] | \alpha \rangle,$$

(A3)

where the sum goes over all functions $P_{mn} : \{1, \ldots, m+n\} \rightarrow \{1, 2\}$ taking $m$ times the value 2 and $n$ times the value 1. Thus we arrive at

$$\langle \alpha | AB | \alpha \rangle = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{P_n} \langle \alpha | [Q_{P_1^{(1)}}, [Q_{P_2^{(2)}}, \cdots , [Q_{P_n^{(n)}}, A] \cdots ] | \alpha \rangle \cdot \langle \alpha | [Q^+_{P_1^{(1)}}, Q^+_{P_2^{(2)}}, \cdots , Q^+_{P_n^{(n)}}, B ] [ Q^-_{P_1^{(2)}}, Q^-_{P_2^{(3)}}, \cdots , Q^-_{P_n^{(3)}}, A ] \cdots ] | \alpha \rangle,$$

(A4)

where the second sum extends now over all functions $P_n : \{1, \ldots, n\} \rightarrow \{1, 2\}$. Moreover, it is straightforward to see that the above expression also holds for an arbitrary number $N$ of oscillatory degrees of freedom with $|\alpha\rangle = |\alpha_1\rangle \otimes \cdots \otimes |\alpha_N\rangle$ and functions $P_n : \{1, \ldots, n\} \rightarrow \{1, \ldots, N\}$. In particular, the variance of an hermitian operator $A$ can be expressed as
\[ (\Delta A)^2 = \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \sum_{P_n} |\langle \alpha || [Q_{P_n(1)}, [Q_{P_n(2)}, \cdots [Q_{P_n(n)}, A] \cdots] || \alpha \rangle|^2, \]  

and the leading-order results for energy fluctuations are given in Eqs. (44)-(46).

\[ \langle z|AB|z \rangle = \sum_{m=0}^{2S_1} \sum_{n=0}^{2S_2} \frac{(2S_2 - m)! (2S_1 - n)!}{m!(2S_2)! n!(2S_1)!} \left\langle z \left| \left[ \frac{i}{\hbar} \hat{S}_2^-, A \right] \right| z \right\rangle \left\langle z \left| \left[ \frac{i}{\hbar} \hat{S}_1^-, B \right] \right| z \right\rangle. \]  

The counterpart of Eq. (A1) for two spins \( \hat{S}_1, \hat{S}_2 \) reads

with \( |z\rangle = |z_1\rangle \otimes |z_2\rangle \). Due to the more complicated prefactors, a similarly compact form as in Eq. (A1) for the full expansion seems to be unachievable for spin systems. The leading terms of energy fluctuations given in Eqs. (62),(63), however, are again rather simple and allow for an intuitive interpretation.

Finally, combining both types of systems, the leading-order contribution to the fluctuation of an Hamiltonian depending on \( N \) oscillatory degrees of freedom and \( M \) spins reads

(\Delta H)^2 \approx \frac{\hbar}{2} \sum_{a=1}^{N} \left[ \omega_a \langle \partial_t q_a \rangle^2 + \langle \partial_t p_a \rangle^2 \right] + O(\hbar^2) + \frac{M}{2} \left[ \langle \hbar S_b^+ \rangle^2 \left( \frac{1}{2S_b^+} \langle \partial_t \hat{S}_b^- \rangle^2 + O \left( \frac{1}{S_b^+} \right) \right) \right]. \]  

(A7)

Appendix B: The Normalization of Coherent Intertwiners and Related Integrals

In order to evaluate the normalization integral of coherent intertwiners in the semiclassical regime, we shall use a slightly different version of SU(2) coherent states generated by

\[ V(\theta, \varphi) = e^{-\varphi \hat{S}_z} e^{-\theta \hat{S}_y} \]  

such that compared to Eq. (14) one has (dropping again factors of \( \hbar \))

\[ V(\theta, \varphi)|S\rangle = e^{i\varphi \hat{S}_z} U(\theta, \varphi)|S\rangle, \]  

i.e. the coherent states generated by the operators \( |16\rangle \) and \( |17\rangle \) just differ by a phase factor which drops out from all expectation values. The operator \( |17\rangle \) fulfills

\[ V^+ \hat{S} V = \hat{u} S^x + \hat{v} S^y + \hat{s} S^z \]  

with

\[ \vec{u} = \frac{\vec{e} \times \hat{s}}{|\vec{e} \times \hat{s}|}, \quad \vec{v} = \frac{\vec{e} \times \hat{s}}{|\vec{e} \times \hat{s}|} \]  

such that \( \vec{u}, \vec{v}, \hat{s} \) form an orthonormal system. If one had used the original operator \( |16\rangle \) for generating coherent states the form of the vectors \( \vec{u}, \vec{v} \) would be less transparent. Now the normalization integral can be written as

\[ \langle \Phi | P | \Phi \rangle = \int d\mu \prod_{a=1}^{N} \left\langle S_a | e^{i\psi \hat{n}_a S_z} | S_a \right\rangle \]  

with

\[ n^x_a = \bar{n} \hat{u}_a, \quad n^y_a = \bar{n} \hat{v}_a, \quad n^z_a = \bar{n} \hat{s}_a \]  

(B6)

where \( \bar{n} \) is the rotation axis occurring in Eqs. (62),(63). Taking into account the explicit form of the rotation matrix element \( 27 \)

\[ \langle S | e^{i\psi \hat{n} S_z} | S \rangle = \left( \cos \frac{\psi}{2} + i n^z \sin \frac{\psi}{2} \right)^{2S} \]  

(B7)

elementary manipulations lead to

\[ \langle \Phi | P | \Phi \rangle = \frac{1}{2\pi^2} \int_0^\pi d\vartheta \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\psi \sin^2 \psi \cdot \sum_{\eta = \pm 1} \prod_{a=1}^{N} \left( \eta \cos \psi + i \bar{n} \hat{s}_a \sin \psi \right)^{2S_a}, \]  

(B8)

where the cosine of \( \psi \) is nonnegative in the entire integration interval. Following Ref. we introduce \( \bar{n} := \bar{n} \sin \psi \) fulfilling

\[ d^3 p = \sin \vartheta \sin^2 \psi \cos \psi d\vartheta d\varphi d\psi \]  

such that

\[ \langle \Phi | P | \Phi \rangle = \frac{1}{2\pi^2} \sum_{\eta = \pm 1} \int_{p_1} d^3 p \widehat{e^{S_a(\bar{n})}} \]  

(B10)
where
\[ S_0(\vec{p}) = \sum_{a=1}^{N} 2S_a \ln \left( \eta \sqrt{1 - p^2} + i\bar{\mu}a \right). \quad (B11) \]

In this form the integral can be evaluated via saddle point approximation to \( S_0(\vec{p}) \). As discussed in detail in Ref. \[15\], provided that the classical closure relation \[15\] holds, the maximum of \( S_0(\vec{p}) \) occurs at \( \bar{p} = 0 \) with
\[ S_+(0) = 0, \quad S_-(0) = 2\pi i \sum_{a=1}^{N} S_a \] \quad (B12)

and since the latter sum must be integer for a nontrivial singlet space we have \( \exp(S_+(0))=1 \). The Hessian is given by (cf. Eq. \[11\])
\[ \left( \frac{\partial^2 S_\pm(\vec{p})}{\partial p^i \partial p^j} \right)_{\bar{p}=0} = -2H^{ij} = -2 \sum_{a=1}^{N} S_a \left( \delta^{ij} - s_a^i s_a^j \right). \quad (B13) \]

Extending now the integration domain in Eq. \[11\] to the infinite space (as the integrand falls off rapidly), we are left with simple gaussian integrals leading to the result \[11\] where the leading first term was already obtained in Ref. \[15\] while the subleading correction stems from expanding the square root in Eq. \[11\].

To compute the numerator of the coefficients \[19\] we consider
\[ \langle \Phi \left| iV_a S_a^x V_a^+ + P \right| \Phi \rangle = \int d\mu \left\langle S_a \left| i\bar{\mu}a e^{i\bar{\varphi}a} S_a \right| S_a \right\rangle \times \prod_{b \neq a} \left\langle S_b \left| e^{i\bar{\varphi}b S_b} \right| S_b \right\rangle \quad (B14) \]

with \( V_a = V(\theta_a, \varphi_a) \). With the help of the rotation matrix element \[27\] one derives
\[ \langle S - 1 \left| e^{i\bar{\varphi}S} \right| S \rangle = \sqrt{2S} \left( \left. \cos \frac{\psi}{2} + in^2 \sin \frac{\psi}{2} \right)^{2S-1} \right. \]
\[ \times \left( \left. (n^x + in^y) \sin \frac{\psi}{2} \right)^{2S-1} \right. \]
\[ \left. \cdot (n^x + in^y) \sin \frac{\psi}{2} \right) \quad (B15) \]

Now proceeding as before the two nontrivial expectation values can be formulated as
\[ \langle \Phi \left| iV_a S_a^x V_a^+ \right| \Phi \rangle \quad (B19) \]
\[ = \frac{-iS_a}{\eta^2} \sum_{\eta=\pm} \int \frac{d^3p}{\sqrt{1 - p^2} \eta \sqrt{1 - p^2} + i\bar{\mu}a} \bar{\mu}a, e^{S_a(\vec{p})}, \]
\[ \langle \Phi \left| iV_a S_a^y V_a^+ \right| \Phi \rangle \quad (B20) \]
\[ = \frac{iS_a}{\eta^2} \sum_{\eta=\pm} \int \frac{d^3p}{\sqrt{1 - p^2} \eta \sqrt{1 - p^2} + i\bar{\mu}a} \bar{\mu}a, e^{S_a(\vec{p})}. \]

Performing again a saddle point approximation to the exponential and expanding the remaining integrand in quadratic order around \( \bar{p} = 0 \) leads to
\[ \langle \Phi \left| iV_a S_a^x V_a^+ \right| \Phi \rangle = -S_a \left( \bar{v}_a H^{-1} \bar{\varphi}a \right) \quad (B21) \]
\[ \langle \Phi \left| iV_a S_a^y V_a^+ \right| \Phi \rangle = S_a \left( \bar{u}_a H^{-1} \bar{\varphi}a \right) \quad (B22) \]

and using Eq. \[13\] along with elementary geometric relations it follows for the coefficients \[19\]
\[ \bar{C}_a(\Phi) = -S_a \left( \bar{u}_a \left( \bar{v}_a^T H^{-1} \bar{\varphi}a \right) - \bar{v}_a \left( \bar{u}_a^T H^{-1} \bar{\varphi}a \right) \right) \]
\[ = S_a \bar{\varphi}a \times \left( \bar{v}_a \left( \bar{v}_a^T H^{-1} \bar{\varphi}a \right) \right. \]
\[ + \bar{u}_a \left( \bar{u}_a^T H^{-1} \bar{\varphi}a \right) \right. \] \quad (B23)

Finally, observing that \( \bar{u}_a, \bar{v}_a \) span the plane perpendicular to \( \bar{\varphi}a \) we obtain the result \[10\], and the numerator of Eq. \[19\] is given by Eq. \[11\].
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