ASYMPTOTICS OF \( H \)-IDENTITIES FOR ASSOCIATIVE ALGEBRAS WITH AN \( H \)-INARIANT RADICAL

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Abstract. We prove the existence of the Hopf PI-exponent for finite dimensional associative algebras \( A \) with a generalized Hopf action of an associative algebra \( H \) with 1 over an algebraically closed field of characteristic 0 assuming only the invariance of the Jacobson radical \( J(A) \) under the \( H \)-action and the existence of the decomposition of \( A/J(A) \) into the sum of \( H \)-simple algebras. As a consequence, we show that the analog of Amitsur’s conjecture holds for \( G \)-codimensions of finite dimensional associative algebras over a field of characteristic 0 with an action of an arbitrary group \( G \) by automorphisms and anti-automorphisms and for differential codimensions of finite dimensional associative algebras with an action of an arbitrary Lie algebra by derivations.

1. Introduction

Amitsur’s conjecture on asymptotic behaviour of codimensions of ordinary polynomial identities was proved by A. Giambruno and M. V. Zaicev [10, Theorem 6.5.2] in 1999.

When an algebra is endowed with a grading, an action of a group \( G \) by automorphisms and anti-automorphisms, an action of a Lie algebra by derivations or a structure of an \( H \)-module algebra for some Hopf algebra \( H \), it is natural to consider, respectively, graded, \( G \)-, differential or \( H \)-identities [4, 5, 6, 14]. The analog of Amitsur’s conjecture for finite dimensional associative algebras with a \( \mathbb{Z}_2 \)-action was proved by A. Giambruno and M.V. Zaicev [10, Theorem 10.8.4] in 1999. In 2010–2011, E. Aljadeff, A. Giambruno, and D. La Mattina [1, 2, 9] obtained the validity of the analog of Amitsur’s conjecture for associative PI-algebras with an action of a finite Abelian group by automorphisms as a particular case of their result for graded algebras.

In 2012, the analog of the conjecture was proved [12, 13] for finite dimensional associative algebras with a rational action of a reductive affine algebraic group by automorphisms and anti-automorphisms, with an action of a finite dimensional semisimple Lie algebra by derivations or an action of a semisimple Hopf algebra. These results were obtained as a consequence of [12, Theorem 5] and [13, Theorem 6], where the authors considered finite dimensional associative algebras with a generalized Hopf action of an associative algebra \( H \) with 1. In the proof, they required the existence of an \( H \)-invariant Wedderburn — Mal’cev and Wedderburn — Artin decompositions. Here we remove the first restriction. This enables us to prove the analog of Amitsur’s conjecture for \( G \)-codimensions of finite dimensional associative algebras with an action of an arbitrary group \( G \) by automorphisms and anti-automorphisms and for differential codimensions of finite dimensional associative algebras with an action of an arbitrary Lie algebra by derivations.

2010 Mathematics Subject Classification. Primary 16R10; Secondary 16R50, 16W20, 16W22, 16W25, 16T05, 20C30.

Key words and phrases. Associative algebra, polynomial identity, derivation, group action, Hopf algebra, \( H \)-module algebra, codimension, cocharacter, Young diagram.

Supported by Fonds Wetenschappelijk Onderzoek — Vlaanderen Pegasus Marie Curie post doctoral fellowship (Belgium) and RFBR grant 13-01-00234a (Russia).
2. POLYNOMIAL $H$-IDENTITIES AND THEIR CODIMENSIONS

Let $H$ be a Hopf algebra over a field $F$. An algebra $A$ over $F$ is an $H$-module algebra or an algebra with an $H$-action, if $A$ is endowed with a homomorphism $H \to \text{End}_F(A)$ such that $h(ab) = (h(a_1)b_1)(h(a_2)b_2)$ for all $h \in H$, $a, b \in A$. Here we use Sweedler’s notation $\Delta h = h(a_1) \otimes h(a_2)$ where $\Delta$ is the comultiplication in $H$.

In order to embrace an action of a group by anti-automorphisms, we consider a generalized Hopf action [6, Section 3].

Let $H$ be an associative algebra with 1 over $F$. We say that an associative algebra $A$ is an algebra with a generalized $H$-action if $A$ is endowed with a homomorphism $H \to \text{End}_F(A)$ and for every $h \in H$ there exist $h'_i, h''_i, h'''_i, h''''_i \in H$ such that

$$h(ab) = \sum_i ((h'_i(a)(h''_ib) + (h'''_ib)(h''''_ia)) \text{ for all } a, b \in A. \quad (1)$$

Choose a basis $(\gamma_\beta)_{\beta \in \Lambda}$ in $H$ and denote by $F\langle X|H \rangle$ the free associative algebra over $F$ with free formal generators $x_i^{\gamma_\beta}$, $\beta \in \Lambda, i \in \mathbb{N}$. Let $x_i^b := \sum_{\beta \in \Lambda} \alpha_\beta x_i^{\gamma_\beta}$ for $h = \sum_{\beta \in \Lambda} \alpha_\beta \gamma_\beta$, $\alpha_\beta \in F$, where only finite number of $\alpha_\beta$ are nonzero. Here $X := \{x_1, x_2, x_3, \ldots\}$, $x_i := x_i^1$, $1 \in H$. We refer to the elements of $F\langle X|H \rangle$ as $H$-polynomials. Note that here we do not consider any $H$-action on $F\langle X|H \rangle$.

Let $A$ be an associative algebra with a generalized $H$-action. Any map $\psi : X \to A$ has a unique homomorphic extension $\hat{\psi} : F\langle X|H \rangle \to A$ such that $\hat{\psi}(x_i^h) = h\psi(x_i)$ for all $i \in \mathbb{N}$ and $h \in H$. An $H$-polynomial $f \in F\langle X|H \rangle$ is an $H$-identity of $A$ if $\hat{\psi}(f) = 0$ for all maps $\psi : X \to A$. In other words, $f(x_1, x_2, \ldots, x_n)$ is an $H$-identity of $A$ if and only if $f(a_1, a_2, \ldots, a_n) = 0$ for any $a_i \in A$. In this case we write $f \equiv 0$. The set $\text{Id}_H^n(A)$ of all $H$-identities of $A$ is an ideal of $F\langle X|H \rangle$. Note that our definition of $F\langle X|H \rangle$ depends on the choice of the basis $(\gamma_\beta)_{\beta \in \Lambda}$ in $H$. However such algebras can be identified in the natural way, and $\text{Id}_H^n(A)$ is the same.

Denote by $P_n^H$ the space of all multilinear $H$-polynomials in $x_1, \ldots, x_n, n \in \mathbb{N}$, i.e.

$$P_n^H = \langle x_1^{h_1}x_2^{h_2}\cdots x_n^{h_n} \mid h_i \in H, \sigma \in S_n \rangle_F \subset F\langle X|H \rangle.$$

Then the number $c_n^H(A) := \dim \left( \frac{P_n^H}{\text{Id}_H^H(A)} \right)$ is called the $n$th codimension of polynomial $H$-identities or the $n$th $H$-codimension of $A$.

The analog of Amitsur’s conjecture for $H$-codimensions of $A$ can be formulated as follows.

**Conjecture.** There exists $\text{Pl}exp_H^H(A) := \lim_{n \to \infty} \sqrt[n]{c_n^H(A)} \in \mathbb{Z}_+$.

We call $\text{Pl}exp_H^H(A)$ the Hopf $PI$-exponent of $A$.

**Example 1.** Every algebra $A$ is an $H$-module algebra for $H = F$. In this case the $H$-action is trivial and we get ordinary polynomial identities and their codimensions.

**Example 2.** Let $A$ be an associative algebra with an action of a group $G$ by automorphisms and anti-automorphisms. Then $A$ is an algebra with a generalized $H$-action where $H = FG$. We introduce the free $G$-algebra $F\langle X|G \rangle := F\langle X|H \rangle$, the ideal of polynomial $G$-identities $\text{Id}_G^H(A) := \text{Id}_H^H(A)$, and $G$-codimensions $c_n^G(A) := c_n^H(A)$.

**Example 3.** If $H = U(g)$ is the universal enveloping algebra of a Lie algebra $g$, then an $H$-module algebra is an algebra with a $g$-action by derivations. The corresponding $H$-identities are called differential identities or polynomial identities with derivations.

**Theorem 1.** Let $A$ be a finite dimensional non-nilpotent associative algebra with a generalized $H$-action where $H$ is an associative algebra with 1 over an algebraically closed field $F$. 

\[ \text{Pl}exp_H^H(A) \]
of characteristic 0. Suppose that the Jacobson radical \( J := J(A) \) is an \( H \)-submodule. Let \( A/J = B_1 \oplus \ldots \oplus B_q \) (direct sum of \( H \)-invariant ideals)

where \( B_i \) are \( H \)-simple algebras and let \( \pi \colon A/J \rightarrow A \) be any homomorphism of algebras (not necessarily \( H \)-linear) such that \( \pi \sigma = \id_{A/J} \) where \( \pi \colon A \rightarrow A/J \) is the natural projection. Then there exist constants \( C_1, C_2 > 0, r_1, r_2 \in \mathbb{R} \) such that

\[
C_1 n^{r_1} d^n \leq c_n^H(A) \leq C_2 n^{r_2} d^n \quad \text{for all } n \in \mathbb{N}
\]

where

\[
d = \max \dim \left( B_{i_1} \oplus B_{i_2} \oplus \ldots \oplus B_{i_r} \right) \quad r \geq 1,
\]

\[
(H \pi(B_{i_1}))A^+ (H \pi(B_{i_2}))A^+ \cdots (H \pi(B_{i_{r-1}}))A^+ (H \pi(B_{i_r})) \neq 0
\]

and \( A^+ := A + F \cdot 1 \).

**Remark.** If \( A \) is nilpotent, i.e. \( x_1 x_2 \ldots x_p \equiv 0 \) for some \( p \in \mathbb{N} \), then \( P_n^H \subseteq \id^H(A) \) and \( c_n^H(A) = 0 \) for all \( n \geq p \).

**Corollary.** The analog of Amitsur’s conjecture holds for such codimensions.

**Remark.** The existence of the map \( \pi \) follows from the ordinary Wedderburn—Mal’cev theorem.

Theorem 1 will be proved in Sections 4 and 5.

3. Applications

Here we list some important corollaries from Theorem 1.

**Theorem 2.** Let \( A \) be a finite dimensional non-nilpotent associative \( H \)-module algebra for a Hopf algebra \( H \) over a field \( F \) of characteristic 0. Suppose that the antipode of \( H \) is bijective and the Jacobson radical \( J(A) \) is an \( H \)-submodule. Then there exist constants \( d \in \mathbb{N}, C_1, C_2 > 0, r_1, r_2 \in \mathbb{R} \) such that

\[
C_1 n^{r_1} d^n \leq c_n^H(A) \leq C_2 n^{r_2} d^n \quad \text{for all } n \in \mathbb{N}.
\]

**Proof.** Let \( K \ni F \) be an extension of the field \( F \). Then

\[
(A \otimes_F K)/(J \otimes_F K) \cong (A/J) \otimes_F K
\]

is again a semisimple algebra and \( J \otimes_F K \) is nilpotent and \( H \otimes_F K \)-invariant.

Now we notice that \( H \)-codimensions do not change upon an extension of the base field. The proof is analogous to the case of ordinary codimensions [10, Theorem 4.1.9]. Hence we may assume \( F \) to be algebraically closed. By [13, Lemma 1], \( A/J = B_1 \oplus \ldots \oplus B_q \) (direct sum of \( H \)-invariant ideals) for some \( H \)-simple algebras \( B_i \). Now we apply Theorem 1.

**Theorem 3.** Let \( A \) be a finite dimensional non-nilpotent associative algebra over a field \( F \) of characteristic 0 with an action of a Lie algebra \( g \) by derivations. Then there exist constants \( d \in \mathbb{N}, C_1, C_2 > 0, r_1, r_2 \in \mathbb{R} \) such that

\[
C_1 n^{r_1} d^n \leq c_n^{U(g)}(A) \leq C_2 n^{r_2} d^n \quad \text{for all } n \in \mathbb{N}.
\]

**Proof.** By [7, Lemma 3.2.2], the Jacobson radical (which coincides with the prime radical) of a finite dimensional associative algebra is invariant under all derivations. Hence we may apply Theorem 2.

**Theorem 4.** Let \( A \) be a finite dimensional non-nilpotent associative algebra over a field \( F \) of characteristic 0 with an action of a group \( G \) by automorphisms and anti-automorphisms. Then there exist constants \( d \in \mathbb{N}, C_1, C_2 > 0, r_1, r_2 \in \mathbb{R} \) such that

\[
C_1 n^{r_1} d^n \leq c_n^G(A) \leq C_2 n^{r_2} d^n \quad \text{for all } n \in \mathbb{N}.
\]
Proof. Again, $G$-codimensions do not change upon an extension of the base field. Hence we may assume $F$ to be algebraically closed. The radical is invariant under all automorphisms and anti-automorphisms. Now we apply [13, Lemma 2] and Theorem [3] □

The algebra in the example below has no $G$-invariant Wedderburn — Mal’cev decomposition, however it satisfies the analog of Amitsur’s conjecture.

Example 4 (Yuri Bahturin). Let $F$ be a field of characteristic $0$ and let

$$A = \left\{ \left( \begin{array}{cc} C & D \\ 0 & 0 \end{array} \right) \right\} \mid C, D \in M_n(F) \subseteq M_{2n}(F),$$

$m \geq 2$. Consider $\varphi \in \text{Aut}(A)$ where

$$\varphi \left( \begin{array}{cc} C & D \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} C & C + D \\ 0 & 0 \end{array} \right).$$

Then $A$ is an algebra with an action of the group $G = \langle \varphi \rangle \cong \mathbb{Z}$ by automorphisms. There is no $G$-invariant Wedderburn — Mal’cev decomposition for $A$, however there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} m^{2n} \leq c_n^G(A) \leq C_2 n^{r_2} m^{2n} \text{ for all } n \in \mathbb{N}.$$ 

Proof. Note that

$$J(A) = \left\{ \left( \begin{array}{cc} 0 & D \\ 0 & 0 \end{array} \right) \right\} \mid D \in M_{m}(F) \right\}.$$ 

Suppose $A = B \oplus J(A)$ (direct sum of $G$-invariant subspaces) for some maximal semisimple subalgebra $B$. Since $\varphi(a) - a \in J(A)$ for all $a \in A$, we have $\varphi(a) = a$ for all $a \in B$. Thus $B \subseteq J(A)$ and we get a contradiction. Therefore, there is no $G$-invariant Wedderburn — Mal’cev decomposition for $A$.

Again, $G$-codimensions do not change upon an extension of the base field. Moreover, upon an extension of $F$, $A$ remains an algebra of the same type. Thus without loss of generality we may assume $F$ to be algebraically closed.

Note that $A/J \cong M_m(F)$ is a simple algebra. Hence $\text{Pexp}^G(A) = \dim M_m(F) = m^2$ by Theorems [1] and [4] □

Example 5. Let $A$ be the associative algebra from Example 4. Denote by $\mathfrak{g}$ the corresponding Lie algebra with the commutator $[x, y] = xy - yx$ and consider the adjoint action of $\mathfrak{g}$ on $A$ by derivations. Then there is no $\mathfrak{g}$-invariant Wedderburn — Mal’cev decomposition for $A$, however there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} m^{2n} \leq c_n^{U(\mathfrak{g})}(A) \leq C_2 n^{r_2} m^{2n} \text{ for all } n \in \mathbb{N}.$$ 

Proof. Suppose $A = B \oplus J(A)$ (direct sum of $\mathfrak{g}$-submodules) for some maximal semisimple associative subalgebra $B$. Then $B$ is a Lie ideal of $\mathfrak{g}$. By [3], $J(A)$ is Abelian as a Lie algebra. Thus the center of $\mathfrak{g}$ contains $J(A)$, which is not true. We get a contradiction. Hence there is no $\mathfrak{g}$-invariant Wedderburn — Mal’cev decomposition for $A$.

Again, without loss of generality we may assume $F$ to be algebraically closed. Since $A/J \cong M_m(F)$ is a simple algebra, $\text{Pexp}^{U(\mathfrak{g})}(A) = \dim M_m(F) = m^2$ by Theorems [1] and [3] □

Remark. The radical of Sweedler’s algebra with an action of its dual is not $H$-invariant, however the analog of Amitsur’s conjecture holds for its $H$-identities [12, Section 7.4].
4. $S_n$-COCHARACTERS AND UPPER BOUND

One of the main tools in the investigation of polynomial identities is provided by the representation theory of symmetric groups.

Let $A$ be an associative algebra with a generalized $H$-action where $H$ is an associative algebra over a field $F$ of characteristic 0. The symmetric group $S_n$ acts on the spaces $\frac{P_n^H}{\text{Id}^H(A)}$, by permuting the variables. Irreducible $FS_n$-modules are described by partitions $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$ and their Young diagrams $D_\lambda$. The character $\chi^H_n(A)$ of the $FS_n$-module $\frac{P_n^H}{\text{Id}^H(A)}$ is called the $n$th cocharacter of polynomial $H$-identities of $A$. We can rewrite $\chi^H_n(A)$ as a sum

$$\chi^H_n(A) = \sum_{\lambda \vdash n} m(A, H, \lambda) \chi(\lambda)$$

of irreducible characters $\chi(\lambda)$. Let $e_{T_\lambda} = a_{T_\lambda} b_{T_\lambda}$ and $e_{T_\lambda}^* = b_{T_\lambda} a_{T_\lambda}$ where $a_{T_\lambda} = \sum_{\pi \in R_T^\lambda} \pi$ and $b_{T_\lambda} = \sum_{\sigma \in C_{T_\lambda}} \pi_{\sigma} \sigma$ be Young symmetric correspond to a Young tableau $T_\lambda$. Then $M(\lambda) = FS_n e_{T_\lambda} \cong FS_n e_{T_\lambda}^*$ is an irreducible $FS_n$-module corresponding to a partition $\lambda \vdash n$.

Let $\sigma : H \rightarrow \text{End}_F(A)$ be the homomorphism corresponding to the $H$-action, and let $(\sigma(\gamma_j))_{j=1}^m$ be a basis in $\sigma(H)$. Let $\varphi : S_n \rightarrow S_m$, $\varphi(\sigma) = \begin{pmatrix} 1 & 2 & \ldots & n & n+1 & n+2 & \ldots & 2n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) & n+\sigma(1) & n+\sigma(2) & \ldots & n+\sigma(n) \end{pmatrix}$ and the $S_n$-homomorphism $\pi : (P_m \downarrow \varphi(S_n)) \rightarrow P_n^H$ defined by $\pi(x_{n(i-1)+t}) = x_i^{\gamma_j}$, $1 \leq i \leq m$, $1 \leq t \leq n$. Note that $\pi(P_m \cap \text{Id}(A)) \subseteq P_n^H \cap \text{Id}^H(A)$ and $x^h - \sum_{j=1}^m \alpha_j x^{\gamma_j} \in \text{Id}^H(A)$ for all $h \in H$ and $\alpha_j \in F$ such that $\zeta(h) = \sum_{j=1}^m \alpha_j \zeta(\gamma_j)$. Hence the $FS_n$-module $\frac{P_n^H}{\text{Id}^H(A)}$ is
a homomorphic image of the $FS_n$-module $\left(\frac{P_m}{P_m \cap \text{Id}(A)}\right) \downarrow \varphi(S_n)$. Denote by $\text{length}(M)$ the number of irreducible components of a module $M$. Then

$$\sum_{\lambda \vdash n} m(A, H, \lambda) = \text{length} \left(\frac{P_m}{P_m \cap \text{Id}(A)}\right) \leq \text{length} \left(\frac{P_m}{P_m \cap \text{Id}(A)}\right) \downarrow \varphi(S_n).$$

Therefore, it is sufficient to prove that $\text{length} \left(\frac{P_m}{P_m \cap \text{Id}(A)}\right) \downarrow \varphi(S_n)$ is polynomially bounded. Replacing $|G|$ with $m$ in [11] Lemma 10 and 12 (or, alternatively, using the proof of [6, Theorem 13 (b)]), we derive this from (4) and [10, Theorem 4.6.2].

In the next two lemmas we consider a finite dimensional associative algebra with a generalized $H$-action having an $H$-invariant nilpotent ideal $J$ where $H$ is an associative algebra with 1 over a field $F$ of characteristic 0 and $J^p = 0$ for some $p \in \mathbb{N}$. Fix a decomposition $A/J = B_1 \oplus \ldots \oplus B_q$ where $B_i$ are some subspaces. Let $\pi: A/J \to A$ be an $F$-linear map such that $\pi \chi = \text{id}_{A/J}$ where $\pi: A \to A/J$ is the natural projection. Define the number $d$ by (2).

**Lemma 1.** Let $n \in \mathbb{N}$ and $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$. Then if $\sum_{k=d+1}^s \lambda_k \geq p$, we have $m(A, H, \lambda) = 0$.

**Proof.** It is sufficient to prove that $e_{T_\lambda}^* f \in \text{Id}^H(A)$ for all $f \in P_n$ and for all Young tableaux $T_\lambda$ corresponding to $\lambda$.

Fix a basis in $A$ that is a union of bases of $\chi(B_1), \ldots, \chi(B_q)$ and $J$. Since $e_{T_\lambda}^* f$ is multilinear, it is sufficient to prove that $e_{T_\lambda}^* f$ vanishes under all evaluations on basis elements. Fix some substitution of basis elements and choose $1 \leq i_1, \ldots, i_r \leq q$ such that all the elements substituted belong to $\chi(B_{i_1}) \oplus \ldots \oplus \chi(B_{i_r}) \oplus J$, and for each $k$ we have an element being substituted from $\chi(B_{i_k})$. Then we may assume that $\dim(B_{i_1} \oplus \ldots \oplus B_{i_r}) \leq d$, since otherwise $e_{T_\lambda}^* f$ is zero by the definition of $d$. Note that $e_{T_\lambda}^* f = b_{T_\lambda} a_{T_\lambda}$ and $b_{T_\lambda}$ alternates the variables of each column of $T_\lambda$. Hence if $e_{T_\lambda}^* f$ does not vanish, this implies that different basis elements are substituted for the variables of each column. Therefore, at least $\sum_{k=d+1}^s \lambda_k \geq p$ elements must be taken from $J$. Since $J^p = 0$, we have $e_{T_\lambda}^* f \in \text{Id}^H(A)$. \qed

**Lemma 2.** If $d > 0$, then there exist constants $C_2 > 0$, $r_2 \in \mathbb{R}$ such that $c_n^H(A) \leq C_2r_2^d n^m$ for all $n \in \mathbb{N}$. In the case $d = 0$, the algebra $A$ is nilpotent.

**Proof.** Lemma [1] and [10, Lemmas 6.2.4, 6.2.5] imply

$$\sum_{m(A,H,\lambda) \neq 0} \dim M(\lambda) \leq C_3r_3^d n^m$$

for some constants $C_3, r_3 > 0$. Together with Theorem [5] this inequality yields the upper bound. \qed

5. LOWER BOUND

As usual, in order to prove the lower bound, it is sufficient to provide a polynomial alternating on sufficiently many sufficiently large sets of variables. Lemma 3 below is a generalization of [12, Lemma 10].

**Lemma 3.** Let $A$, $J$, $\chi$, $B_i$, and $d$ be the same as in Theorem 7. If $d > 0$, then there exists a number $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exist disjoint subsets $X_1, \ldots, X_{2k} \subseteq \{x_1, \ldots, x_n\}$, $k := \lceil \frac{n-n_0}{2d} \rceil$, $|X_1| = \ldots = |X_{2k}| = d$ and a polynomial $f \in P_n^H \setminus \text{Id}^H(A)$ alternating in the variables of each set $X_j$. 
Proof. Without lost of generality, we may assume that $d = \dim(B_1 \oplus B_2 \oplus \ldots \oplus B_r)$ where $(H \kappa(B_1))^{A^+}(H \kappa(B_2))^{A^+} \ldots (H \kappa(B_{r-1}))^{A^+}(H \kappa(B_r)) \neq 0$.

Since $J$ is nilpotent, we can find maximal $\sum q_i$, $q_i \in \mathbb{Z}_+$, such that

$$\left( a_1 \prod_{i=1}^{q_1} j_{i_1} \right) \gamma_1 \kappa(b_1) \left( a_2 \prod_{i=1}^{q_2} j_{i_2} \right) \gamma_2 \kappa(b_2) \ldots \left( a_r \prod_{i=1}^{q_r} j_{i_r} \right) \gamma_r \kappa(b_r) \left( a_{r+1} \prod_{i=1}^{q_{r+1}} j_{r+1,i} \right) \neq 0$$

for some $j_{i_k} \in J$, $a_i \in A^+$, $b_i \in B_i$, $\gamma_i \in H$. Let $j_i := a_i \prod_{k=1}^{q_i} j_{i_k}$.

Then

$$j_1 \gamma_1 \kappa(b_1) j_2 \gamma_2 \kappa(b_2) \ldots j_r \gamma_r \kappa(b_r) j_{r+1} \neq 0 \quad (5)$$

for some $b_i \in B_i$, $\gamma_i \in H$, however

$$j_1 \tilde{b}_1 j_2 \tilde{b}_2 \ldots j_r \tilde{b}_r j_{r+1} = 0 \quad (6)$$

for all $\tilde{b}_i \in A^+ (H \kappa(B_i))^{A^+}$ such that $\tilde{b}_k \in J (H \kappa(B_i))^{A^+} (H \kappa(B_i)) J$ for at least one $k$.

Recall that $\kappa$ is a homomorphism of algebras. Moreover $\pi (h \kappa (a) - \kappa (ha)) = 0$ implies $h \kappa (a) - \kappa (ha) \in J$ for all $a \in A$ and $h \in H$. Hence, by (5), if we replace $\kappa (b_i)$ in the left-hand side of (5) with a product of $\kappa (b_i)$ and an expression involving $\kappa$, the map $\kappa$ will behave like a homomorphism of $H$-modules. We will exploit this property further.

In virtue of [12] Theorem 7, there exist constants $m_t \in \mathbb{Z}_+$ such that for any $k$ there exist multilinear polynomials

$$f_t = f_t (x_1^{(1, 1)}, \ldots, x_1^{(t, 1)}, \ldots, x_1^{(t, 2k)}, \ldots, x_{d_t}^{(t, 1)}, \ldots, x_{d_t}^{(t, 2k)}, \ldots, z_1, \ldots, z_{m_t}, z_t) \in \mathbb{P}^{H}_{2kd_t + m_t + 1}$$

alternating in the variables from disjoint sets $X_{t}^{(1)} = \{x_1^{(t, 1)}, x_2^{(t, 1)}, \ldots, x_{d_t}^{(t, 1)}\}$, $1 \leq t \leq 2k$. There exist $z_t^{(1)} \in B_t$, $1 \leq \alpha \leq m_t$, such that

$$f_t (a_1^{(1)}, \ldots, a_1^{(t, 1)}, \ldots, a_{d_t}^{(1)}, \ldots, a_{d_t}^{(t, 1)}, \ldots; z_1, \ldots, z_{m_t}; z_t) = z_t$$

for any $z_t \in B_t$.

Let $n_0 = 2r + 1 + \sum_{i=1}^{r} m_i$, $k = \left\lceil \frac{n_0 - m_0}{2d} \right\rceil$, $\tilde{k} = \left\lceil \frac{(n - 2kd - n_0) - m_0}{2d} \right\rceil + 1$. We choose $f_t$ for $B_t$ and $k$, $1 \leq t \leq r$. In addition, again by [12] Theorem 7, we take $\tilde{f}_1$ for $B_1$ and $\tilde{k}$. Let

$$\tilde{f}_1 := v_1 \gamma_1 \kappa \left( f_1 \left( x_1^{(1, 1)}, \ldots, x_1^{(1, 1)}, \ldots, x_1^{(1, 2k)}, \ldots, x_{d_1}^{(1, 1)}, \ldots, x_{d_1}^{(1, 2k)}, \ldots; z_1, \ldots, z_{m_1}^{(1)} \right) \right) \Bigg|_{x_1^{(t, 1)} = \hat{v}_t, x_1^{(1, 1)} = \hat{v}_1, \ldots, x_1^{(t, 2k)} = \hat{v}_t, z_1 = \hat{v}_1, \ldots, z_{m_1}^{(1)} = \hat{v}_t, z_1 = \hat{v}_1, \ldots, z_{m_1}^{(1)} = \hat{v}_t, z_t = \tilde{z}_t}$$

The value of the multilinear function $\tilde{f}_1$ under the substitution $x^{(t, \alpha)} = a^{(t)}_\beta$, $z^{(t)} = z^{(t)}_\beta$, $v_t = j_t$, $z_t = b_t$, $y^{(\alpha)}_1 = a^{(1)}_\alpha$, $u_\beta = z^{(1)}_\beta$ equals the left-hand side of (5), which is nonzero.

Note that $f_t$ are multilinear in $z_i$ and $\tilde{f}_1$ is multilinear in $z_1$. Therefore we may assume that for a fixed $i$ in all entries of $z_i$ in $f_i$ the element $h \in H$ is the same. Analogously, we may assume that in all entries of $z_i^k$ in $\tilde{f}_1$ the element $h \in H$ is the same. Furthermore, we can hide these $h$ inside the elements substituted for $z_i$ in the case of $\tilde{f}_1$ and $f_i$, $i \geq 2$, and inside $\tilde{f}_1$ in the case of $f_1$, and the value $b$ of $\tilde{f}_0$ is still nonzero under the substitution $x^{(t, \alpha)} = a^{(t)}_\beta$, $z^{(t)} = z^{(t)}_\beta$, $v_t = j_t$, $z_t = h_t b_t$, $y^{(\alpha)}_1 = a^{(1)}_\alpha$, $u_\beta = z^{(1)}_\beta$ for some $h_t \in H$.

As we have mentioned, $\kappa$ is a homomorphism of algebras and, by (5), behaves like a homomorphism of $H$-modules. Note that do not need the last property for $z^k_i$ since we may assume that in all such entries we have $h = 1$. Hence the value of

$$f_0 := v_1 \left( f_1 \left( x_1^{(1, 1)}, \ldots, x_1^{(1, 1)}, \ldots, x_1^{(1, 2k)}, \ldots, x_{d_1}^{(1, 1)}, \ldots, x_{d_1}^{(1, 2k)}, z_1, \ldots, z_{m_1}^{(1)} \right) \right) \Bigg|_{x_1^{(t, 1)} = \hat{v}_t, x_1^{(1, 1)} = \hat{v}_1, \ldots, x_1^{(t, 2k)} = \hat{v}_t, z_1 = \hat{v}_1, \ldots, z_{m_1}^{(1)} = \hat{v}_t, z_1 = \hat{v}_1, \ldots, z_{m_1}^{(1)} = \hat{v}_t}$$
under the substitution \( u \)

\[
H \rightarrow H
\]

proof.

\[
\begin{align*}
&= \text{Alt} \\
&\text{on each} \\
&\kappa
\end{align*}
\]

\[
X
\]

Now we repeat verbatim the proofs of [12, Lemma 11 and Theorem 5] using Lemmas 1 and 3 instead of [12, Lemma 10 and Theorem 6].

Note that without additional manipulations a composition of \( H \)-polynomials is only a multilinear function but not an \( H \)-polynomial. However, using (1), we can always represent such function by an \( H \)-polynomial. Here we make such manipulations at the very end of the proof.

Let \( X_\ell = \bigcup_{t=1}^{r} X^{(t)}_\ell \) and let \( \text{Alt}_t \) be the operator of alternation on the set \( X_\ell \). Denote \( \tilde{f} := \text{Alt}_1 \text{Alt}_2 \ldots \text{Alt}_{2k} f_0 \). Note that the alternations do not change \( z_t \), and \( f_t \) is alternating on each \( X^{(t)}_\ell \). Hence the value of \( \tilde{f} \) under the substitution \( \Xi \) equals \((d_1)!/(d_2)! \ldots (d_r)!)^{2k} b \neq 0 \) since \( \mathcal{X}(B_1) \oplus \ldots \oplus \mathcal{X}(B_r) \) is a direct sum of (not necessarily \( H \)-invariant) ideals and if the alternation puts a variable from \( X^{(t)}_\ell \) on the place of a variable from \( X^{(t')}_\ell \) for \( t \neq t' \), the corresponding \( h \mathcal{X}(a^{(t)}_t), h \in H \), annihilates \( h \mathcal{X}(b^{(t)}_t) \).

Note that \( \tilde{f}_1 \) is a linear combination of multilinear monomials \( W \), and one of the terms

\[
\text{Alt}_1 \text{Alt}_2 \ldots \text{Alt}_{2k} v_1 \left( f_1(x^{(1),1}_1, \ldots, x^{(1),2}_d, \ldots; x^{(1,2k)}_{d_1}, x^{(1,2k)}_{d_2}; z^{(1)}_1, \ldots, z^{(1)}_{m_1}; W) \right) \gamma_1 v_2.
\]

in \( \tilde{f} \) does not vanish under the substitution \( \Xi \). Moreover,

\[
\deg \tilde{f} = 2kd + (2kd_1 + m_1) + n_0 > n
\]

and \( \deg W = \deg \tilde{f}_1 = 2kd_1 + m_1 + 1 \). Let \( W = w_1 w_2 \ldots w_{2kd_1+m_1+1} \) where \( w_i \) are variables from the set \{ \( y^{(1)}_1, \ldots, y^{(1)}_{d_1}; \ldots; y^{(2k)}_1, \ldots, y^{(2k)}_{d_1}; u_1, \ldots, u_{m_1}; z_1 \} \) replaced under the substitution \( \Xi \) with \( \bar{w}_i \in \mathcal{X}(B_1) \). Let

\[
f := \text{Alt}_1 \text{Alt}_2 \ldots \text{Alt}_{2k} v_1 \left( f_1(x^{(1),1}_1, \ldots, x^{(1),2}_d, \ldots; x^{(1,2k)}_{d_1}, x^{(1,2k)}_{d_2}; z^{(1)}_1, \ldots, z^{(1)}_{m_1}; w_1 w_2 \ldots w_{n-2kd-n_0} z) \right)^{\gamma_1} v_2.
\]

where \( z \) is an additional variable. Then using (1), we may assume \( f \in P^H_n \). Note that \( f \) is alternating in \( X_\ell \), \( 1 \leq \ell \leq r \), and does not vanish under the substitution \( \Xi \) with \( z = \bar{w}_n-2kd-n_0+1 \ldots w_{2kd_1+m_1+1} \). Thus \( f \) satisfies all the conditions of the lemma.

\[\square\]

\textbf{Proof of Theorem 11} Now we repeat verbatim the proofs of [12] Lemma 11 and Theorem 5] using Lemmas 1 and 3 instead of [12] Lemma 10 and Theorem 6].

\[\square\]

\textbf{ACKNOWLEDGEMENTS}

This work started while I was an AARMS postdoctoral fellow at Memorial University of Newfoundland, whose faculty and staff I would like to thank for hospitality. I am grateful to Yuri Bahturin, who suggested that I study polynomial \( H \)-identities, and to Mikhail Zaicev, who suggested that I consider algebras without an \( H \)-invariant Wedderburn — Mal’cev decomposition.
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