Smooth projective horospherical varieties with Picard number 1

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Abstract

We describe smooth projective horospherical varieties with Picard number 1. Moreover we prove that the automorphism group of any such variety acts with at most two orbits and we give a geometric characterisation of non-homogeneous ones.

Introduction

A horospherical variety is a normal algebraic variety where a reductive algebraic group acts with an open orbit which is a torus bundle over a flag variety. For example, toric varieties and flag varieties are horospherical.

In this paper we describe all smooth projective horospherical varieties with Picard number 1. In particular, they are Fano varieties, i.e. their anticanonical bundle is ample. Fano horospherical varieties are classified in terms of some rational polytopes in [10]. The study of those with Picard number 1 is partly motivated by Theorem 0.1 of [10], that gives an upper bound of the degree of smooth Fano horospherical varieties. Indeed, the expression of this upper bound is different if the Picard number is 1.

One can remark that the smooth projective toric varieties with Picard number 1 are exactly the projective spaces. This is not the case for horospherical varieties: the first examples of smooth projective horospherical varieties with Picard number 1 are flag varieties $G/P$ with $P$ a maximal parabolic subgroup of $G$.

Moreover, smooth projective horospherical varieties with Picard number 1 are not necessarily homogeneous. For example, let $\omega$ be a skew-form of maximal rank on $\mathbb{C}^{2m+1}$. For $i \in \{1, \ldots, m\}$, define the odd symplectic grassmannian $Gr_{i,2m+1}$ as the variety of $i$-dimensional $\omega$-isotropic subspaces of $\mathbb{C}^{2m+1}$. Odd symplectic grassmannians are horospherical varieties (see Proposition 4.2) and have two orbits under the action of their automorphism group which is a connected non-reductive linear group [9]. In fact, the latter property is satisfied by all non-homogeneous smooth projective horospherical varieties with Picard number 1 (see Theorem 4.1).
Two-orbits varieties (i.e. normal varieties with two orbits under the action of a connected linear group) have already been studied by D. Akhiezer and S. Cupit-Foutou. Indeed, D. Akhiezer has classified the ones whose closed orbit is of codimension 1 and he has proved in particular that they are horospherical when the group is not semi-simple. S. Cupit-Foutou has classified two-orbits varieties when the acting group is semi-simple, and she has also proved that they are spherical (i.e. they admit a dense orbit of a Borel subgroup) [6]. It could be interesting to look at smooth two-orbits varieties with Picard number 1. Note also that a classification of smooth projective symmetric varieties with Picard number 1 has been recently given by A Ruzzi [11].

The paper is organized as follows. In Section 1 we recall the known results on horospherical varieties which we will use in all the paper. In particular we briefly summarize the Luna-Vust theory [8] in the case of horospherical varieties.

In Section 2, we prove that a given horospherical homogeneous space admits at most one smooth compactification with Picard number 1. Then we give the list of horospherical homogeneous spaces that admit a smooth compactification not isomorphic to a projective space and with Picard number 1. We obtain a list of 8 cases (Theorem 2.5).

In Section 3, we prove that in 3 of these cases, the smooth compactification is homogeneous (under the action of a larger group).

In Section 4, we study the 5 remaining cases. We compute the automorphism group of the corresponding smooth compactification with Picard number 1. We prove that this variety has two orbits under the action of its automorphism group and the latter is connected and not reductive.

In Section 5, we obtain a purely geometric characterization of the varieties obtained in Section 4.

1 Notation

Let $G$ be a reductive and connected algebraic group over $\mathbb{C}$, $B$ a Borel subgroup of $G$, $T$ a maximal torus of $B$ and $U$ the unipotent radical of $B$. Denote by $C$ the center of $G$ and by $G'$ the semi-simple part of $G$ (so that $G = C.G'$). Denote by $S$ the set of simple roots of $(G, B, T)$, and by $\Lambda$ (respectively $\Lambda^+$) the group of characters of $B$ (respectively the set of dominant characters). Denote by $W$ the Weyl group of $(G, T)$ and, when $I \subset S$, denote by $W_I$ the subgroup of $W$ generated by the reflections associated to the simple roots of $I$. If $\alpha$ is a simple root, we denote by $\check{\alpha}$ its coroot and $\omega_\alpha$ the fundamental weight corresponding to $\alpha$. Denote by $P(\omega_\alpha)$ the maximal parabolic subgroup containing $B$ such that $\omega_\alpha$ extends to $P(\omega_\alpha)$. Let $\Gamma$ be the Dynkin diagram of $G$. When $I \subset S$, we denote by $\Gamma_I$ the subgraph of $\Gamma$ with vertices the elements of $I$ and with edges those joining two elements of $I$. Let
\( \lambda \in \Lambda^+ \), then we denote by \( V(\lambda) \) the irreducible representation of \( G \) of highest weight \( \lambda \) and by \( v_\lambda \) a highest weight vector of \( V(\lambda) \).

Let \( H \) be a closed subgroup of \( G \). Then \( H \) is said to be horospherical if it contains the unipotent radical of a Borel subgroup of \( G \). In that case we also say that the homogeneous space \( G/H \) is horospherical. Up to conjugation, one can assume \( H \supseteq U \). Denote by \( P \) the normalizer \( N_G(H) \) of \( H \) in \( G \), it is a parabolic subgroup of \( G \) such that \( P/H \) is a torus. Then \( G/H \) is a torus bundle over the flag variety \( G/P \). The dimension of the torus is called the rank of \( G/H \) and denoted by \( n \).

All varieties are irreducible algebraic varieties over \( \mathbb{C} \). Let \( X \) be a normal variety where \( G \) acts. Then \( X \) is said to be a horospherical variety if \( G \) has an open orbit isomorphic to an horospherical homogeneous space \( G/H \). In that case, \( X \) is also said to be a \( G/H \)-embedding. The classification of \( G/H \)-embeddings (due in a more general situation to D. Luna et Th. Vust [8]) is detailed in [10, Chap.1].

Let me summarize here the principal points of this theory. Let \( G/H \) be a fixed horospherical homogeneous space of rank \( n \). Then it defines a set of simple roots

\[ I := \{ \alpha \in S \mid \omega_\alpha \text{ is not a character of } P \} \]

where \( P \) is the unique parabolic subgroup defined above. We also introduce a lattice \( M \) of rank \( n \) as the sublattice of \( \Lambda \) consisting of all characters \( \chi \) of \( P \) such that the restriction of \( \chi \) to \( H \) is trivial. Denote by \( N \) the dual lattice to \( M \).

For all \( \alpha \in S \setminus I \), we denote by \( \check{\alpha}_M \) the element of \( N \) defined as the restriction to \( M \) of \( \check{\alpha} : \Lambda \longrightarrow \mathbb{Z} \). The point \( \check{\alpha}_M \) is called the image of the color corresponding to \( \alpha \). See [10, Chap.1] for the definition of a color and note that the set of colors is in bijection with \( S \setminus I \).

**Definition 1.1.** A colored cone of \( N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \) is an ordered pair \((C, F)\) where \( C \) is a convex cone of \( N_{\mathbb{R}} \) and \( F \) is a set of colors called the set of colors of the colored cone, such that

1. \( C \) is generated by finitely many elements of \( N \) and contains the image of the colors of \( F \),
2. \( C \) does not contain any line and the image of any color of \( F \) is zero.

One can define a colored fan as a set of colored cones such that any two of them intersect in a common colored face (see [10, def.1.14] for the precise definition).

Then \( G/H \)-embeddings are classified in terms of colored fans. Define a simple embedding as an open \( G \)-stable subvariety of \( X \) containing exactly one closed \( G \)-orbit. Let \( X \) be a \( G/H \)-embedding and \( F \) its colored fan. Then \( X \) is covered by its simple subembeddings, and each of them corresponds to a maximal colored cone of \( F \). Denote by \( D_X \) the set of simple roots in \( S \setminus I \) which correspond to colors of \( F \).

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1 When \( G/H \) is spherical. See for example [7] or [5].
2 Classification of smooth projective embeddings with Picard number 1

The Picard number $\rho$ of a smooth $G/H$-embedding $X$ satisfies

$$\rho = r + \#(S \setminus I) - \#(D_X)$$

where $r$ is the number of rays of the colored fan of $X$ minus the rank $n$ [10, (4.5.1)]. Since $X$ is projective, its colored fan is complete (i.e. it covers $N_\mathbb{R}$) and then $r \geq 1$. Moreover $D_X \subset S \setminus I$, so $\rho = 1$ if and only if $r = 1$ and $D_X = S \setminus I$. In particular the colored fan of $X$ has exactly $n + 1$ rays.

Lemma 2.1. Let $G/H$ be a horospherical homogeneous space. Up to isomorphism of varieties, there exists at most one smooth projective $G/H$-embedding with Picard number 1.

Proof. Let $X$ and $X'$ be two smooth projective $G/H$-embeddings with respective colored fans $\mathbb{F}$ and $\mathbb{F}'$ and with Picard number 1. Denote by $e_1, \ldots, e_{n+1}$ the primitive elements of the $n + 1$ rays of $\mathbb{F}$. By the smoothness criterion of [10, Chap.2], $(e_1, \ldots, e_n)$ is a basis of $N$, $e_{n+1} = -e_1 - \cdots - e_n$, the images in $N$ of the colors are disjoint and contained in $\{e_1, \ldots, e_{n+1}\}$. The same happens for $\mathbb{F}'$. Then there exists an automorphism $\phi$ of the lattice $N$ which stabilizes each image $\tilde{\alpha}_M$ of a color and satisfies $\mathbb{F} = \phi(\mathbb{F}')$. Thus the varieties $X$ and $X'$ are isomorphic [10, Prop. 3.10].

If it exists, we call $X^1$ the unique smooth projective $G/H$-embedding with Picard number 1 and we say that $G/H$ is "special".

2.1 Projective space

The following result asserts in particular that $X^1$ is a projective space when $n \geq 2$.

Theorem 2.2. Let $G/H$ be a "special" homogeneous space. Then $X^1$ is isomorphic to a projective space in the following cases:

(i) $\#(D_{X^1}) \leq n$,

(ii) $n \geq 2$

(iii) $n = 1$, $\#(D_{X^1}) = 2$ and the two simple roots of $D_{X^1}$ are not in the same connected component of $\Gamma$.

Proof. (i) In that case, a maximal colored cone of the colored fan of $X$ contains all colors. Then the corresponding simple subembedding of $X^1$, whose closed orbit is a point [10, Lem.2.8], is affine [7, th.3.1] and smooth. So it is necessarily a horospherical $G$-module.
Thus $\mathbb{P}(\mathbb{C} \oplus V)$ is a smooth projective $G/H$-embedding with Picard number 1. Then by Lemma 2.1, $X^1$ is isomorphic to $\mathbb{P}(\mathbb{C} \oplus V)$.

(ii) We may assume that $\sharp(D_{X^1}) = n + 1$. Denote by by $\alpha_1, \ldots, \alpha_{n+1}$ the elements of $S \setminus I$ and by $\Gamma_i$ the Dynkin diagram $\Gamma_{S \setminus \{\alpha_i\}}$. The smoothness criterion of horospherical varieties [10 Chap.2] applied to $X^1$ tells us two things.

First, for all $i \in \{1, \ldots, n+1\}$ and for all $j \neq i$, $\alpha_j$ is a simple end ("simple" means not adjacent to a double edge) of a connected component $\Gamma^j_i$ of $\Gamma_i$ of type $A_m$ or $C_m$. Moreover the $\Gamma^j_i$ are distinct, in other words, each connected component of $\Gamma_i$ has at most one vertex among the $(\alpha_i)_{i \in \{1, \ldots, n+1\}}$.

Secondly, $(\tilde{\alpha}_i M)_{i \in \{1, \ldots, n\}}$ is a basis of $N$ and $\tilde{\alpha}_{(n+1)M} = -\tilde{\alpha}_1 M - \cdots - \tilde{\alpha}_n M$. Thus a basis of $M$ (dual of $N$) is of the form

$$(\omega_{\alpha_1} - \omega_{\alpha_{n+1}} + \chi_i)_{i \in \{1, \ldots, n\}}$$

where $\chi_i$ is a character of the center $C$ of $G$, for all $i \in \{1, \ldots, n\}$.

Let us prove that a connected component of $\Gamma$ contains at most one vertex among the $(\alpha_i)_{i \in \{1, \ldots, n+1\}}$. Suppose the contrary: there exist $i, j \in \{1, \ldots, n+1\}, i \neq j$ such that $\alpha_i$ and $\alpha_j$ are vertices of a connected component of $\Gamma$. One can choose $i$ and $j$ such that there is no vertex among the $(\alpha_k)_{k \in \{1, \ldots, n+1\}}$ between $\alpha_i$ and $\alpha_j$. Since $n \geq 2$, there exists an integer $k \in \{1, \ldots, n+1\}$ different from $i$ and $j$. Then we observe that $\Gamma^j_k$ does not satisfy the condition that each of its connected component has at most one vertex among the $(\alpha_i)_{i \in \{1, \ldots, n+1\}}$ (because $\Gamma^i_k = \Gamma^j_k$).

Thus we have proved that

$$\Gamma = \bigsqcup_{j=0}^{n+1} \Gamma^j$$

such that for all $j \in \{1, \ldots, n+1\}$, $\Gamma^j$ is a connected component of $\Gamma$ of type $A_m$ or $C_m$ in which $\alpha_j$ is a simple end.

For all $\lambda \in \Lambda^+$, denote by by $V(\lambda)$ the simple $G$-module of weight $\lambda$. Then Equation 2.2.1 tells us that the projective space

$$\mathbb{P}(V(\omega_{\alpha_{n+1}}) \oplus V(\omega_{\alpha_1} + \chi_1) \oplus \cdots \oplus V(\omega_{\alpha_n} + \chi_n))$$

is a smooth projective $G/H$-embedding with Picard number 1. Thus $X^1$ is isomorphic to this projective space.

(iii) As in case (ii), one checks that $X^1$ is isomorphic to $\mathbb{P}(V(\omega_{\alpha_2} \oplus V(\omega_{\alpha_1} + \chi_1))$ (where $\chi_1$ is a character of $C$).
2.2 When $X^1$ is not isomorphic to a projective space

According to Theorem 2.2 we have to consider the case where the rank of $G/H$ is 1 and where there are two colors corresponding to simple roots $\alpha$ and $\beta$ in the same connected component of $\Gamma$. As we have seen in the proof of Theorem 2.5, the lattice $M$ (here of rank 1) is generated by the character $\omega_\alpha - \omega_\beta + \chi$ where $\chi$ is a character of the center $C$ of $G$. Moreover, $H$ is the kernel of $\omega_\alpha - \omega_\beta + \chi : P(\omega_\alpha) \cap P(\omega_\beta) \rightarrow \mathbb{C}^*$.

Let us reduce to the case where $G$ is semi-simple.

Proposition 2.3. Let $H' = G' \cap H$. Then $G/H$ is isomorphic to $G'/H'$.

Proof. We are going to prove that $G/H$ and $G'/H'$ are both isomorphic to a horospherical homogeneous space $(G' \times \mathbb{C}^*)/H''$. In fact $G/H$ is isomorphic to $(G' \times P/H)/\tilde{H}$ [10, Proof of Prop.3.10], where

$$\tilde{H} = \{(g, pH) \in G' \times P/H \mid gp \in H\}.$$  

Similarly $G'/H'$ is isomorphic to $(G' \times P'/H')/\tilde{H}'$ where $P' = P \cap G'$ and $\tilde{H}'$ defined as the same way as $\tilde{H}$. Moreover the morphisms

$$P/H \rightarrow \mathbb{C}^* \quad \text{and} \quad P'/H' \rightarrow \mathbb{C}^*$$

are isomorphisms. Then

$$\tilde{H} = \{(p', c) \in P' \times \mathbb{C}^* \mid (\omega_\alpha - \omega_\beta + \chi)(p') = c^{-1}\}$$

$$= \{(p', c) \in P' \times \mathbb{C}^* \mid (\omega_\alpha - \omega_\beta)(p') = c^{-1}\} = \tilde{H}'. $$

This completes the proof. $\square$

Remark 2.4. In fact $P/H \simeq \mathbb{C}^*$ acts on $G/H$ by right multiplication, so it acts on the $\mathbb{C}^*$-bundle $G/H \rightarrow G/P$ by homotheties on fibers. Moreover, this action extends to $X^1$.

We may assume that $G$ is semi-simple. Let $G_1, \ldots, G_k$ the normal simple subgroups of $G$, so that $G$ is the quotient of the product $G_1 \times \cdots \times G_k$ by a central finite group $C_0$. We can suppose that $C_0$ is trivial because $G/H \simeq G.C_0/H.C_0$. If $\alpha$ and $\beta$ are simple roots of the connected component corresponding to $G_i$, denote by $H_i$ the kernel of $\omega_\alpha - \omega_\beta$ in the parabolic subgroup $P(\omega_\alpha) \cap P(\omega_\beta)$ of $G_i$. Then

$$H = G_1 \times \cdots \times G_{i-1} \times H_i \times G_{i+1} \times \cdots \times G_k$$

and $G/H = G_i/H_i$.

So from now, without loss of generality, we suppose that $G$ is simple.
**Theorem 2.5.** With the assumptions above, $G/H$ is "special" if and only if $(\Gamma, \alpha, \beta)$ appears in the following list (up to exchanging $\alpha$ and $\beta$)\footnote{The numerotation of the simple roots is that of \cite{3}.}

1. $(A_m, \alpha_1, \alpha_m)$
2. $(A_m, \alpha_i, \alpha_{i+1})$ with $i \in \{1, \ldots, m-1\}$
3. $(B_m, \alpha_{m-1}, \alpha_m)$
4. $(B_3, \alpha_1, \alpha_3)$
5. $(C_m, \alpha_i, \alpha_{i+1})$ with $i \in \{1, \ldots, m-1\}$
6. $(D_m, \alpha_{m-1}, \alpha_m)$
7. $(F_4, \alpha_2, \alpha_3)$
8. $(G_2, \alpha_2, \alpha_1)$

**Proof.** The Dynkin diagrams $\Gamma_{S\backslash\{\alpha\}}$ and $\Gamma_{S\backslash\{\beta\}}$ respectively. They are of type $A_m$ or $C_m$ by the smoothness criterion \cite[Chap.2]{10}. Moreover $\alpha$ and $\beta$ are simple ends of $\Gamma_{S\backslash\{\beta\}}$ and $\Gamma_{S\backslash\{\alpha\}}$ respectively.

Suppose $\Gamma$ is of type $A_m$. If $\alpha$ equals $\alpha_1$ then, looking at $\Gamma_{S\backslash\{\alpha\}}$, we remark that $\beta$ must be $\alpha_2$ or $\alpha_m$. So we are in Case 2 or 1. If $\alpha$ equals $\alpha_m$ the argument is similar. Now if $\alpha$ is not an end of $\Gamma$, in other words if $\alpha = \alpha_i$ for some $i \in \{2, \ldots, m-1\}$ then, looking at $\Gamma_{S\backslash\{\alpha\}}$, we see that $\beta$ can be $\alpha_1$, $\alpha_{i-1}$, $\alpha_{i+1}$ or $\alpha_m$. The cases where $\beta$ equals $\alpha_1$ or $\alpha_m$ are already done and the case where $\beta$ equals $\alpha_{i-1}$ or $\alpha_{i+1}$ is Case 2.

The study of the other cases is analogous and left to the reader. \hfill \Box

In the next two sections we are going to study the variety $X^1$ for each case of this theorem. In particular we will see that $X^1$ is not isomorphic to a projective space in all cases.

### 3 Homogeneous varieties

In this section, with the notation of Section 2.2 we are going to prove that $X^1$ is homogeneous in Cases 1, 2 and 6.

In all cases (1 to 8), there are exactly 4 projective $G/H$-embeddings and there are all smooth; they correspond to the 4 colored fans consisting of the two half-lines of $\mathbb{R}$, without
color, with one of the two colors and with the two colors, respectively (see [10, Ex.1.19] for a similar example).

Let us realize $X^1$ in a projective space as follows. The homogeneous space $G/H$ is isomorphic to the orbit of the point $[v_{\omega_\beta} + v_{\omega_\alpha}]$ in $\mathbb{P}(V(\omega_\beta) \oplus V(\omega_\alpha))$, where $v_{\omega_\alpha}$ and $v_{\omega_\beta}$ are highest weight vectors of $V(\omega_\alpha)$ and $V(\omega_\beta)$ respectively. Then $X^1$ is the closure of this orbit in $\mathbb{P}(V(\omega_\beta) \oplus V(\omega_\alpha))$, because both have the same colored cone (i.e. that with two colors)\(^3\).

We will describe the other $G/H$-embeddings later in the proof of Lemma 4.7.

**Proposition 3.1.** In Case 1, $X^1$ is isomorphic to the quadric $Q^{2m} = SO_{2m+2}/P(\omega_\alpha)$.

**Proof.** Here, the fundamental representations $V(\omega_\alpha)$ and $V(\omega_\beta)$ are the simple $SL_{m+1}$-modules $C^{m+1}$ and its dual $(C^{m+1})^*$, respectively. Let denote by $Q$ the quadratic form on $C^{m+1} \oplus (C^{m+1})^*$ defined by $Q(u, u^*) = \langle u^*, u \rangle$. Then $Q$ is invariant under the action of $SL_{m+1}$. Moreover $Q(v_{\omega_\alpha} + v_{\omega_\beta}) = 0$, so that $X^1$ is a subvariety of the quadric ($Q = 0$) in $\mathbb{P}(C^{m+1} \oplus (C^{m+1})^*) = \mathbb{P}(C^{2m+2})$.

We complete the proof by computing the dimension of $X^1$:

$$\dim X^1 = \dim G/H = 1 + \dim G/P = 1 + \#(R^+ \setminus R^+_I) \quad (3.1.1)$$

where $R^+$ is the set of positive roots of $(G, B)$ and $R^+_I$ is the set of positive roots generated by simple roots of $I$. So $\dim X^1 = \dim Q^{2m} = 2m$ and $X^1 = Q^{2m}$.

**Proposition 3.2.** In Case 2, $X^1$ is isomorphic to the grassmannian $Gr_{i+1,m+2}$.

**Proof.** The fundamental representations of $SL_{m+1}$ are the simple $SL_{m+1}$-modules

$$V(\omega_\alpha_i) = \bigwedge^i C^{m+1}$$

and a highest weight vector of $V(\omega_\alpha_i)$ is $e_1 \wedge \cdots \wedge e_i$ where $e_1, \ldots, e_{m+1}$ is a basis of $C^{m+1}$.

We have

$$\xymatrix{ X^c \ar[r] & \mathbb{P}(\bigwedge^i C^{m+1} \oplus \bigwedge^{i+1} C^{m+1}) \ar[dl] \ar[l] \ar[u] \\
G/H \ar[r] & G.[e_1 \wedge \cdots \wedge e_i + e_1 \wedge \cdots \wedge e_{i+1}] \ar[u] \ar[l]}
$$

Complete $(e_1, \ldots, e_{m+1})$ to obtain a basis $(e_0, \ldots, e_{m+1})$ of $C^{m+2}$, then the morphism

$$\bigwedge^i C^{m+1} \oplus \bigwedge^{i+1} C^{m+1} \quad \xrightarrow{x + y} \quad \bigwedge^{i+1} C^{m+2}$$

\(^3\)See [10, Chap.1] for the construction of the colored fan of a $G/H$-embedding.
is an isomorphism. Then $X^1$ is a subvariety of the grassmannian

$$G_{i+1,m+2} \cong SL_{m+2} \cdot [e_1 \wedge \cdots \wedge e_i \wedge (e_0 + e_{i+1})] \subset \mathbb{P}(\wedge^{i+1} \mathbb{C}^{m+1}).$$

Moreover the dimension of $X^1$ is the same as the dimension of $G_{i+1,m+2}$. Indeed, one checks that $\dim X^1 = (i+1)(m+1-i)$ using Formula 3.1.1. \qed

**Proposition 3.3.** In Case 6, $X^1$ is isomorphic to the spinor variety $SO(2m+1)/P(\omega_{\alpha_m})$.

**Proof.** The direct sum $V(\omega_\alpha) \oplus V(\omega_\beta)$ of the two half-spin representations of $SO(2m)$ is isomorphic to the spin representation of $SO(2m+1)$. Moreover $v_{\omega_\alpha} + v_{\omega_\beta}$ is in the orbit of a highest weight vector of the spin representation of $SO(2m+1)$. Thus we deduce that $X^1$ is a subvariety of $SO(2m+1)/P(\omega_{\alpha_m})$. Then we complete the proof by comparing dimensions. In fact $\dim X^1 = \frac{m(m+1)}{2}$ (by Formula 3.1.1). \qed

## 4 Non-homogeneous varieties

With the notation of Section 2.2 we prove in this section the following result.

**Theorem 4.1.** In Cases 3, 4, 5, 7 and 8 (and only in these cases), $X^1$ is not homogeneous.

Moreover the automorphism group of $X^1$ is $(SO(2m+1) \times \mathbb{C}^*) \ltimes V(\omega_{\alpha_m})$, $(SO(7) \times \mathbb{C}^*) \ltimes V(\omega_{\alpha_3})$, $((Sp(2m) \times \mathbb{C}^*)/\{\pm 1\}) \ltimes V(\omega_{\alpha_1})$, $(F_4 \times \mathbb{C}^*) \ltimes V(\omega_{\alpha_4})$ and $(G_2 \times \mathbb{C}^*) \ltimes V(\omega_{\alpha_1})$ respectively.

Finally, $X^1$ has two orbits under its automorphism group.

One can remark that in Case 5, Theorem 4.1 follows from [9, Chap.3 and Prop.5.1] and the following result.

**Proposition 4.2.** In Case 5, $X^1$ is isomorphic to the odd symplectic grassmannian $Gr_{i+1,2m+1}$.

**Proof.** As in the proof of Proposition 3.2, $X^1$ is a subvariety of the odd symplectic grassmannian

$$G_{i+1,2m+1} \cong Sp_{2m+1} \cdot [e_1 \wedge \cdots \wedge e_i \wedge (e_0 + e_{i+1})] \subset \mathbb{P}(\Lambda^{i+1} \mathbb{C}^{m+1}).$$

Again we complete the proof by comparing the dimensions. Indeed, by Formula 3.1.1

$$\dim X^1 = (i+1)(2m-i) - \frac{i(i+1)}{2} = \dim Gr_{i+1,2m+1} \quad [9 \text{ Prop 4.1}].$$

\qed
Now, let $X$ be one of the varieties $X^1$ in Cases 3, 4, 7 and 8.

Then $X$ has three orbits under the action of $G$ (the open orbit $X_0$ isomorphic to $G/H$ and two closed orbits). Recall that $X$ can be seen as a subvariety of $\mathbb{P}(V(\omega_\alpha) \oplus V(\omega_\beta))$. Let $P_Y := P(\omega_\alpha)$, $P_Z := P(\omega_\beta)$ and denote by $Y$ and $Z$ the closed orbits, isomorphic to $G/P_Y$ and $G/P_Z$ respectively. (In Case 8 where $G$ is of type $G_2$, we have $\alpha = \alpha_2$ and $\beta = \alpha_1$.)

Let $X_Y$ be the simple $G/H$-embedding of $X$, we have $X_Y = X_0 \cup Y$. Then $X_Y$ is a homogeneous vector bundle over $G/P_Y$ [10, Chap.2] in the sense of the following definition.

**Definition 4.3.** Let $P$ be a parabolic subgroup of $G$ and $V$ a $P$-module. Then the homogeneous vector bundle $G \times P V$ over $G/P$ is the quotient of the direct product $G \times V$ by the equivalence relation $\sim$ defined by

$$\forall g \in G, \forall p \in P, \forall v \in V, \quad (g, v) \sim (gp^{-1}, p.v).$$

Specifically, $X_Y = G \times P_Y V_Y$ where $V_Y$ is a simple $P_Y$-module of highest weight $\omega_\beta - \omega_\alpha$, and similarly, $X_Z = G \times P_Z V_Z$ where $V_Z$ is a simple $P_Z$-module of highest weight $\omega_\alpha - \omega_\beta$.

Denote by $\text{Aut}(X)$ the automorphism group of $X$ and $\text{Aut}^0(X)$ the connected component of $\text{Aut}(X)$ containing the identity.

**Remark 4.4.** Observe that $\text{Aut}(X)$ is a linear algebraic group. Indeed $\text{Aut}(X)$ acts on the Picard group of $X$ which equals $\mathbb{Z}$ (the Picard group of a horospherical variety is free [4]). This action is necessarily trivial. Then $\text{Aut}(X)$ acts on the projectivization of the space of global sections of a very ample bundle. This gives a faithful representation of $\text{Aut}(X)$.

We now complete the proof of Theorem 4.1 by proving several lemmas.

**Lemma 4.5.** The closed orbit $Z$ of $X$ is stable under $\text{Aut}^0(X)$.

**Proof.** We are going to prove that the normal bundle $N_Z$ of $Z$ in $X$ has no nonzero global section. This will imply that $\text{Aut}^0(X)$ stabilises $Z$, because the Lie algebra $\text{Lie}(\text{Aut}^0(X))$ is the space of global sections $H^0(X, T_X)$ of the tangent bundle $T_X$ of $X$ [2, Chap.2.3] and we have the following exact sequence

$$0 \rightarrow T_{X,Z} \rightarrow T_X \rightarrow N_Z \rightarrow 0$$

where $T_{X,Z}$ is the subsheaf of $T_X$ consisting of vector fields that vanish along $Z$. Moreover $H^0(X, T_{X,Z})$ is the Lie algebra of the subgroup of $\text{Aut}^0(X)$ that stabilizes $Z$.

The total space of $N_Z$ is the vector bundle $X_Z$. So using the Borel-Weil theorem [2, 4.3], $H^0(G/P_Z, N_Z) = 0$ if and only if the smallest weight of $V_Z$ is not antidominant. The smallest weight of $V_Z$ is $w_0^\beta(\omega_\alpha - \omega_\beta)$ where $w_0^\beta$ is the longest element of $W_{S\setminus\{\beta\}}$. Let $\gamma \in S$, then

$$\langle w_0^\beta(\omega_\alpha - \omega_\beta), \gamma \rangle = \langle \omega_\alpha - \omega_\beta, w_0^\beta(\gamma) \rangle.$$
If $\gamma$ is different from $\beta$ then $w_0^\beta(\gamma) = -\delta$ for some $\delta \in S\setminus\{\beta\}$. So we only have to compute $w_0^\beta(\tilde{\beta})$.

In Case 3, $\beta = \alpha_m$, so $w_0^\beta$ maps $\alpha_i$ to $-\alpha_{m-i}$ for all $i \in \{1, \ldots, m-1\}$\footnote{The numerotation of simple roots is still that of \cite{[3]}.}. Here, $\omega_\beta = \alpha_1+2\alpha_2+\cdots+m\alpha_m$. Then, using the fact that $w_0^\beta(\omega_\beta) = \omega_\beta$, we have $w_0^\beta(\beta) = \alpha_1+\cdots+\alpha_m$ so that $w_0^\beta(\tilde{\beta}) = 2(\alpha_1+\cdots+\alpha_{m-1}) + \alpha_m$ and $\langle \omega_\alpha - \omega_\beta, w_0^\beta(\gamma) \rangle = 1$ (because $\alpha = \alpha_{m-1}$).

The computation of $w_0^\beta(\tilde{\beta})$ in the other cases is similar and left to the reader. In all four cases, $\langle \omega_\alpha - \omega_\beta, w_0^\beta(\tilde{\beta}) \rangle > 0$ (this equals 1 in Cases 3, 4, 7 and 2 in Case 8). This proves that $w_0^\beta(\omega_\alpha - \omega_\beta)$ is not antidominant.

\begin{remark}
Using the Borel-Weil theorem we can also compute $H^0(G/P_Y, N_Y)$ by the same method. We find that in Cases 3, 4, 7 and 8, this $G$-module is isomorphic to the simple $G$-module $V(\omega_\beta)$, $V(\omega_\beta)$, $V(\omega_\alpha)$ and $V(\omega_\beta)$ respectively.
\end{remark}

We now prove the following lemma.

\begin{lemma}
In Cases 3, 4, 7 and 8, Aut$^0(X)$ is $(\SO(2m+1) \times \C^*) \ltimes V(\omega_{\alpha_m})$, $(\SO(7) \times \C^*) \ltimes V(\omega_{\alpha_4})$, $(F_4 \times \C^*) \ltimes V(\omega_{\alpha_4})$ and $(G_2 \times \C^*) \ltimes V(\omega_{\alpha_1})$ respectively.
\end{lemma}

\begin{remark}
By Remark \cite{[2.4]} we already know that the action of $G$ on $X$ extends to an action of $G \times \C^*(\simeq G \times P/H)$. Moreover $C := \{(c, c^{-1}H) \in C \times P/H\} \subset G \times \C^*$ acts trivially on $X$.
\end{remark}

\begin{proof}
Let $\pi : \tilde{X} \rightarrow X$ be the blowing-up of $Z$ in $X$. Since $Z$ and $X$ are smooth, $\tilde{X}$ is smooth; it is also a projective $G/H$-embedding. In fact $\tilde{X}$ is the projective bundle

$$\phi : G \times P_Y \mathbb{P}(V_Y \oplus \C) \rightarrow G/P_Y$$

where $P_Y$ acts trivially on $\C$ (because both are projective $G/H$-embeddings with exactly the same color). Moreover the exceptional divisor $\tilde{Z}$ of $\tilde{X}$ is $G/P$.

Let us remark that Aut$^0(\tilde{X})$ is isomorphic to Aut$^0(X)$. Indeed, it contains Aut$^0(X)$ because $Z$ is stable under the action of Aut$^0(X)$ and we know, by a result of A. Blanchard, that Aut$^0(\tilde{X})$ acts on $X$ such that $\pi$ is equivariant \cite[Chap.2.4]{[2]}.

Now we are going to compute Aut$^0(\tilde{X})$. Observe that $H^0(G/P_Y, N_Y)$ acts on $\tilde{X}$ by translations on the fibers of $\phi$:

$$\forall s \in H^0(G/P_Y, N_Y), \ \forall (g_0, [v_0, \xi]) \in G \times P_Y \mathbb{P}(V_Y \oplus \C), \ s(g_0, [v_0, \xi]) = (g_0, [v_0 + \xi v(g_0), \xi])$$

where $v(g_0)$ is the element of $V_Y$ such that $s(g_0) = (g_0, v(g_0))$.

Then the group $((G \times \C^*)/\tilde{C}) \ltimes H^0(G/P_Y, N_Y)$ acts effectively on $\tilde{X}$ (the semi-product is defined by $((g', c'), s'),((g, c), s) = ((g'g, c'c), c'g's + s'))$. In fact we are going to prove that

$$\text{Aut}^0(\tilde{X}) = ((G \times \C^*)/\tilde{C}) \ltimes H^0(G/P_Y, N_Y).$$
By [2 Chap.2.4] we know that $\text{Aut}^0(\tilde{X})$ exchanges the fibers of $\phi$ and induces an automorphism of $G/P_Y$. Moreover we have $\text{Aut}^0(G/P_Y) = G/C$ in our four cases [2 Chap.3.3]. So we have an exact sequence

$$0 \longrightarrow A \longrightarrow \text{Aut}^0(\tilde{X}) \longrightarrow G/C \longrightarrow 0$$

where $A$ is the set of automorphisms which stabilise each fiber of the projective bundle $\tilde{X}$. In fact $A$ consists of affine transformations in fibers.

Then $H^0(G/P_Y, N_Y)$ is the subset of $A$ consisting of translations. Let $A_0$ be the subset of $A$ consisting of linear transformations in fibers. Then $A_0$ fixes $Y$ so that $A_0$ acts on the blowing-up $\tilde{\tilde{X}}$ of $Y$ in $\tilde{X}$. Moreover $\tilde{\tilde{X}}$ is a $\mathbb{P}^1$-bundle over $G/P$, it is in fact the toroidal $G/H$-embedding (i.e. without colors) [10 Ex.1.13 (2)]. As before we know that $\text{Aut}^0(\tilde{\tilde{X}})$ exchanges the fibers of that $\mathbb{P}^1$-bundle and induces an automorphism of $G/P$. Moreover we have $\text{Aut}^0(G/P) = G/C$ and then $\text{Aut}^0(\tilde{\tilde{X}}) = (G \times \mathbb{C}^*)/\tilde{C}$. We deduce that $A_0 = \mathbb{C}^*$.

We complete the proof by Remark 4.6. □

**Remark 4.9.** We can use the same arguments for the odd symplectic grassmannian (Case 5). Then $Y$ is stabilised by $\text{Aut}^0(X)$ and $H^0(G/P_Z, N_Z) = V(\omega_{a_1}) \simeq \mathbb{C}^{2m}$. We deduce that $\text{Aut}^0(X) = ((\text{Sp}(2m) \times \mathbb{C}^*)/\{\pm 1\}) \ltimes \mathbb{C}^{2m}$.

To complete the proof of Theorem 4.1 we prove the following lemma.

**Lemma 4.10.** The automorphism group of $X$ is connected.

**Proof.** Let $\phi$ be an automorphism of $X$. We want to prove that $\phi$ is in $\text{Aut}^0(X)$. It acts by conjugation on $\text{Aut}^0(X)$. Let $L$ be a Levi subgroup of $\text{Aut}^0(X)$. Then $\phi^{-1}L\phi$ is again a Levi subgroup of $\text{Aut}^0(X)$. But all Levi subgroups are conjugated in $\text{Aut}^0(X)$. So we can suppose, without loss of generality, that $\phi$ stabilises $L$.

Then $\phi$ induces an automorphism of the direct product of $\mathbb{C}^*$ with a simple group $G$ of type $B_m, C_m, F_4$ or $G_2$. It also induces an automorphism of $G$ which is necessarily an inner automorphism of $G$ (because there is no non-trivial automorphism of the Dynkin diagram of $G$). So we can now that $\phi$ commutes with all elements of $G$.

Then $\phi$ stabilises the open orbit $G/H$ of $X$. Let $x_0 := H \in G/H \subset X$ and $x_1 := \phi(x_0) \in G/H$. Since $\phi$ commutes with the elements of $G$, the stabilizer of $x_1$ is also $H$. So $\phi$ acts on $G/H$ as an element of $N_G(H)/H = P/H \simeq \mathbb{C}^*$ (where $N_G(H)$ is the normalizer of $H$ in $G$). Then $\phi$ is an element of $\mathbb{C}^* \subset \text{Aut}^0(X)$. □

5 On two-orbit varieties

Here we give a characterization of the two-orbit varieties obtained in Section 4.
Theorem 5.1. Let $X$ be a smooth projective variety with Picard number 1 and put $G := \text{Aut}^0(X)$. Suppose that $G$ is not semi-simple and that $X$ has two orbits under the action of $G$. Denote by $Z$ the closed orbit.

Then the codimension of $Z$ is at least 2.

Suppose in addition that the blowing-up of $X$ along $Z$ has also two orbits under the action of $G$. Then $X$ is one of the varieties $X^1$ obtained in the cases 3, 4, 5, 7 and 8.

Remark 5.2. The converse implication holds by Section 4.

To prove this result we need a result of D. Akhiezer on two-orbits varieties with codimension one closed orbit.

Lemma 5.3 (Th.1 of [1]). Let $X$ a smooth complete variety with an effective action of the (connected linear non semi-simple) group $G$. Suppose that $X$ has two orbits under the action of $G$ and that the closed orbit is of codimension 1. Let $G$ be a maximal semi-simple subgroup of $G$.

Then there exist a parabolic subgroup $P$ of $G$ and a $P$-module $V$ such that:

(i) the action of $P$ on $\mathbb{P}(V)$ is transitive;

(ii) there exists an irreducible $G$-module $W$ and a surjective $P$-equivariant morphism $W \rightarrow V$;

(iii) $X = G \times^P \mathbb{P}(V \oplus \mathbb{C})$.

In particular, $X$ is a horospherical $G$-variety of rank 1.

Remarks 5.4. It follows from (i) that $V$ is an (irreducible) horospherical $L$-module of rank 1, where $L$ is a Levi subgroup of $G$. This is why $X$ is horospherical of rank 1.

The $G$-module $W$ is the set of global sections of the vector bundle $G \times^P V$.

Proof of Theorem 5.1. If $Z$ is of codimension 1, Lemma 5.3 tells us that $X$ is a horospherical $G$-variety. Moreover the existence of the irreducible $G$-stable divisor $Z$ tells us that one of the two rays of the colored fan of $X$ has no color [10, Chap 1]. Then $X$ satisfies the condition (i) of Theorem 2.2, and $X = \mathbb{P}(V \oplus \mathbb{C})$. Since $X$ is not homogeneous we conclude that the codimension of $Z$ is at least 2.

Denote by $\tilde{X}$ the blowing-up of $X$ along $Z$. Then $\tilde{X}$ is a horospherical $G$-variety by Lemma 5.3. Moreover $X$ and $\tilde{X}$ have the same open $G$-orbit, so that $X$ is also a horospherical $G$-variety. We conclude by Theorem 4.1.

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