Perturbative self-interacting scalar field theory: a differential equation approach.

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We investigate the partition function related to a $\phi^4$-scalar field theory on an $n$-dimensional Minkowski spacetime, which is shown to be a self-interacting scalar field theory at least in 4-dimensional Minkowski spacetime. After revisiting the analytical calculation of the perturbative expansion coefficients and also the approximate values for suitable limits using Stirling’s formulæ, we investigate a spherically symmetric scalar field in an $n$-dimensional Minkowski spacetime. For the first perturbative expansion coefficient it is shown how it can be derived a modified Bessel equation (MBE), which solutions are investigated in one, four, and eleven-dimensional Minkowski spacetime. The solutions of MBE are the first expansion coefficient of the series associated with the partition function of $\phi^4$-scalar field theory. All results are shown graphically.

I. INTRODUCTION

The results presented in this article are based on Witten’s proposed questions, solved in [2], by P. Deligne, D. Freed, L. Jeffrey, and S. Wu, concerning perturbative $\phi^4$-scalar field theory (PSFT). We extend their solutions and show that the first perturbative coefficient can be obtained from modified Bessel functions of first kind. This article is organized as follows: In Sec. (II) the partition functional is exhibited in the context of a self-interacting $\phi^4$-scalar field theory and a perturbative solution for the partition function is obtained, in the light of the solutions obtained by Deligne, Freed, Jeffrey, and Wu [2]. In Sec. (III) the method presented is shown to hold for a spherically symmetric scalar field in $n$-dimensional Minkowski spacetime. The perturbative expansion coefficients are derived, and the first coefficient is shown to constrain the scalar field in a MBE, which solutions are used to obtain the first perturbative expansion coefficient associated with the partition function of PSFT, for the particular cases of 1-, 4-, and 11-dimensional spacetime. Numerical integration permits us to accomplish the expansion coefficient for 2- and 4-dimensional spacetime. In Appendix we plot the main results.

II. PERTURBATIVE $\phi^4$-SCALAR FIELD THEORY

We begin with the general form of the scalar Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi),$$

(1)
where \( \phi = \phi(x) \) is a Hermitian scalar field and \( V = V(\phi(x)) \) denotes the scalar potential. For instance, in a self-interacting 4-dimensional scalar field theory it is well-known that

\[
V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4,
\]

(2)

where \( \lambda \) is the self-coupling constant and \( m^2 \) denotes the mass parameter. Euler-Lagrange equation of motion gives, from eq.(1),

\[
\partial_\mu \partial^\mu \phi = - \frac{\partial V(\phi)}{\partial \phi}.
\]

(3)

The partition function associated with eqs.(1,2), can be written as

\[
Z(\lambda) = \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \phi^2 - \frac{1}{4!} \lambda \phi^4 \right) d\phi.
\]

(4)

As we want to investigate the perturbative character of the scalar field theory given by the Lagrangian density in eq.(1) with scalar potential given by eq.(2), it is now possible to expand the partition function \( Z(\lambda) \) in terms of a series as

\[
Z(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^k, \quad c_k \in \mathbb{R}.
\]

(5)

Using Taylor’s formulæ it can be shown that

\[
c_k = \frac{(-1)^k}{(4!)^k k!} \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \phi^2 - \frac{1}{4!} \phi^{4k} \right) d\phi.
\]

(6)

Now define

\[
f(A) = \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \phi^2 + A \phi \right) d\phi
\]

\[
= \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} (\phi + A)^2 \right) \exp \left( \frac{1}{2} A^2 \right) d\phi
\]

\[
= \sqrt{2\pi} \exp \left( \frac{1}{2} A^2 \right).
\]

(7)

It is clear that the integral involving the coefficient \( c_k \) in eq.(6) can be written as

\[
\frac{\partial^{(4k)} f(A)}{\partial A^{(4k)}} \bigg|_{A=0} = \sqrt{2\pi} \frac{(4k)!}{(2k!)^k 4^k},
\]

(8)

from which it follows that

\[
c_k = \frac{(-1)^k (4k)! \sqrt{2\pi}}{k! (24)^k (2k!)^k 4^k} = (-1)^k \frac{\sqrt{2\pi}}{24^k} (4k - 1)!!
\]

(9)

Then, perturbatively for large \( k \) the expansion in eq.(8) gives for the coefficient \( c_k \) the expression

\[
c_k \approx (-1)^k \sqrt{2k}^{-1/2} \left( \frac{2k}{3} \right)^k e^{-k}
\]

(10)

where Stirling’s formulæ \( n! \approx \sqrt{2\pi n} n^n e^{-n} \) is used.

2
III. **n-DIMENSIONAL SPHERICALLY SYMMETRIC PATH INTEGRAL: PERTURBATIVE METHOD**

Considering a spherically symmetric scalar field $\phi = \phi(r)$ in a $n$-dimensional spacetime, where $r$ denotes the radial coordinate, the Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2} \|d\phi\|^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \alpha^{4-n} \lambda \phi^4,$$

(11)

where $m$ denotes the mass associated with the scalar field and $\alpha$ is an arbitrary mass *constant* parameter introduced in order to leave the self-coupling constant $\alpha$ dimensionless in $n$-dimensions. The term $d\phi$ denotes the differential operator acting on the scalar field $\phi$ as $d\phi = \frac{\partial \phi}{\partial r} dr$.

The standard path integral representation of a theory defined by an action $S = \int \mathcal{L}$ is obtained by defining the quantity $Z$, the partition function, as

$$Z = \int \exp(iS) \mathcal{D}\phi$$

(12)

where the $\int \mathcal{D}\phi$ expression is just a shorthand for the product

$$\int \mathcal{D}\phi = \prod_x \int d\phi(x).$$

(13)

This product is taken over all infinite spacetime points $x$ in the volume of the system being described. We immediately spot similarities between the path integral in eq.(12) and the partition function of statistical mechanics $Z = \text{tr} \left[ \exp(-S) \right]$ [4]. The above integral over all field configurations is precisely analogous to a trace over all degrees of freedom of a system. However, the imaginary exponential factor in the path integral does not lend itself to a probabilistic interpretation in the same way that with the partition function. A standard trick used for formulating theories on a lattice is employed, which is to move into imaginary or Euclidean time [4]. The Euclidean spacetime metric tensor is given by $g_{ij} = \delta_{ij}$, where $\delta_{ij}$ denotes the Kronecker tensor. The path integral is written as

$$Z = \int \exp(-S) \mathcal{D}\phi$$

(14)

where

$$S = \int \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \alpha^{4-n} \lambda \phi^4 \right) dt \wedge dx_1 \wedge \cdots \wedge dx_{n-1}$$

(15)

with $\partial_\mu \partial^\mu = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial t^2}$. The exponential function in the path integral is now free of imaginary factors and can be interpreted as a probability distribution, thereby making the connection with the standard partition function from statistical mechanics. The Euclidean formulation is crucial for the operation of lattice Monte Carlo investigations [4]. Hereon

$$d\eta = dt \wedge dx_1 \wedge \cdots \wedge dx_{n-1}$$

(16)

denotes the $n$-dimensional volume element in a given local coordinate chart.

Proceeding as in Sec. [II], by expanding the partition function as in eq. [6] it follows that

$$c_k = \frac{(-1)^k}{(4!)^k k!} \int \left[ \exp \left( -\frac{1}{2} \int (\|d\phi\|^2 + m^2 \phi^2) d\eta \right) \left( \int \alpha^{n-4} \phi^4 d\eta \right)^k \right] \mathcal{D}\phi$$

(17)

$$= \frac{(-1)^k}{(4!)^k k!} \int \exp \left( -\frac{1}{2} \int (\|d\phi\|^2 + m^2 \phi^2) d\eta \right) + k \ln \left( \int \alpha^{n-4} \phi^4 d\eta \right) \left( \int \alpha^{n-4} \phi^4 d\eta \right)^k \mathcal{D}\phi.$$

(18)
In order to find the exponent critical points in eq. (18), which is equivalent to have knowledge of the large $k$ behavior of the coefficients $c_k$, one takes the derivative with respect to the $r$ coordinate in both terms of the exponent $-\frac{1}{2} \int (\|d\phi^2\| + m^2 \phi^2) \, d\eta + k \ln \left( \int \alpha^{n-4} \phi^4 \, d\eta \right)$, yielding:

$$
\int (\Delta \phi - m^2 \phi) \frac{d\phi}{dr} \, d\eta + \frac{4k \int \phi^3 \, d\eta}{\int \phi^4 \, d\eta}
$$

(19)

where $\Delta$ denotes the Laplacian operator. At the critical points of the exponent, the integrand in eq. (19) equals zero, and then

$$(\Delta + m^2)\phi = \frac{4k \phi^5}{\int \phi^4 \, d\eta}.$$  

(20)

As $\phi = \phi(r)$ is a radial scalar field, the Laplacian is given by

$$\Delta \phi = -\frac{1}{r^{n-1}} \frac{d}{dr} \left( r^{n-1} \frac{d\phi}{dr} \right),$$

(21)

and eq. (20) can be written as

$$\frac{d^2 \phi}{dr^2} + \frac{n-1}{r} \frac{d\phi}{dr} - m^2 \phi + \frac{k \phi^3}{\pi^2 \int_0^\infty r^3 \phi^4(r) \, dr} = 0.$$  

(22)

which can be led to

$$\frac{d^2 \phi}{dr^2} + \frac{n-1}{r} \frac{d\phi}{dr} - m^2 \phi + k \phi^3 = 0.$$  

(23)

if the rescaling $\phi \mapsto \left( \frac{\pi^2}{\int_0^\infty r^3 \phi^4(r) \, dr} \right)^{-2} \phi$ is imposed. This rescaling must be finite, and so the condition $\lim_{r \to 0} \phi(r) = 0$ must holds. Besides, $\lim_{r \to 0} \frac{d\phi}{dr} = 0$ in order that the derivative $d\phi/dr$ to make sense in $r = 0$. Denoting hereon

$$V(\phi) = \frac{k}{4} \phi^4 - \frac{m^2}{2} \phi^2, \tag{24}$$

for $n = 1$ eq. (23) is Newton’s second law associated with the potential $V(\phi)$ given by eq. (24):

$$\frac{d^2 \phi}{dr^2} + \frac{dV(\phi(r))}{d\phi} = 0.$$  

(25)

For $n > 1$ we have

$$\frac{d^2 \phi}{dr^2} + \frac{dV(\phi(r))}{d\phi} + \frac{n-1}{r} \frac{d\phi}{dr} = 0.$$  

(26)

When $k = 0$, eq. (23) can be written as

$$\frac{d^2 \phi}{dr^2} + \frac{n-1}{r} \frac{d\phi}{dr} - m^2 \phi = 0.$$  

(27)

which can be exactly solved if we first multiply eq. (23) by $r^2$, yielding

$$r^2 \frac{d^2 \phi}{dr^2} + r(n-1) \frac{d\phi}{dr} - m^2 r^2 \phi = 0.$$  

(28)
Now, let us consider \( \phi(r) = r^\beta \xi(r) \), and on substituting this ansatz in the equation above, which seems to be the best adapted to our investigation, it follows that

\[
 r^2 \frac{d^2 \xi(r)}{dr^2} + (2\beta + n - 1)r \frac{d\xi(r)}{dr} + [\beta(\beta - 1) + (n - 1)\beta] \xi(r) - m^2r^2 \xi(r) = 0. \tag{29}
\]

We now are left with the task of determining a suitable choice of the parameter \( \beta \) in eq. (29) in such a way that the coefficient of the term \( r \frac{d\xi(r)}{dr} \) to be 1, i.e., we choose \( \beta \) in order that eq. (29) can be led to a MBE. With this choice, it can be immediately shown that \( \beta = 1 - \frac{n}{2} \). Eq. (29) is then expressed as

\[
 r^2 \frac{d^2 \xi(r)}{dr^2} + r \frac{d\xi(r)}{dr} - \left[ (1 - \frac{n}{2})^2 + m^2r^2 \right] \xi(r) = 0. \tag{30}
\]

This is the MBE, which has as solutions the modified Bessel functions of the first kind \( I_{1-\frac{n}{2}}(mr) \) and of the second kind \( K_{1-\frac{n}{2}}(mr) \). Finally the self-interacting radial scalar field is given, for \( k = 0 \), by

\[
 \phi(r) = r^{1-\frac{n}{2}} I_{1-\frac{n}{2}}(mr). \tag{31}
\]

It can be shown that the solution above is regular at the origin, at least for \( n \) even. The graphics in Appendix show the behavior of such function for each \( n \). We use unitary field mass, and assume \( m = 1 \) without loss of generality.

From eq. (31) it follows that the first partition function perturbative expansion coefficient \( c_0 \) in eq. (5) is given by

\[
 c_0 = \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \exp \left( -\frac{1}{2} r^{2-n} I_{1-\frac{n}{2}}(mr) \right) \frac{dI_{1-\frac{n}{2}}}{dr} dr. \tag{32}
\]

For \( n = 1 \) we have

\[
 c_0 = \sqrt{\frac{2}{\pi m}} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{1}{r} \exp \left[ -\sqrt{\frac{r}{2\pi m}} \sinh(mr) \right] \left( -\frac{\sinh(mr)}{2r} + m \cosh(mr) \right) dr. \tag{33}
\]

For \( n = 4 \), consisting of the usual Minkowski spacetime \( \mathbb{R}^{1,3} \), it follows that \( c_0 \) is given by

\[
 c_0 = \int_{0}^{\infty} \exp \left[ -\frac{m^3}{16} \left( \sum_{j=0}^{\infty} \frac{(mr)^{2j}}{4j!(2+j)!} \right)^2 \left( \sum_{p=0}^{\infty} \frac{(2p+1)(mr)^{2p}}{4p!(2+p)!} \right) \right] dr, \tag{34}
\]

and numerical integration gives us

\[
 c_0 = 3.85378. \tag{35}
\]

For \( n = 2 \), it follows from eq. (32) that

\[
 c_0 = 0.39769. \tag{36}
\]

Finally, for \( n = 11 \), where now the scalar field is a 11-dimensional Minkowski spacetime-valued function, it follows that

\[
 c_0 = \sqrt{\frac{1}{2\pi m^9}} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{dr}{r} \exp \left[ \sqrt{\frac{2}{\pi (mr)^9}} \left( m^4r^4 + 45m^3r^2 + 105 \sinh(mr) - mr(10m^2r^2 + 105) \cosh(mr) \right) \right] \times \{ \sinh(mr)(-14m^4r^5 - 9m^4r^4 - 30m^3r^3 - 405m^2r^2 - 945) + mr \cosh(mr)(2m^4r^5 + 30m^2r^3 + 90m^2r^2 + 945) \}.
\]

This formula can be useful in investigating the KK gravitational waves propagation modes in brane-world scenario.
IV. CONCLUDING REMARKS

The first perturbative expansion coefficient of the partition function associated with a self-interacting $n$-dimensional Minkowski spacetime-valued scalar field is obtained from the exact solution of MBE, and the particular cases of one, four, and eleven-dimensional spacetimes are obtained in terms of modified Bessel functions of first kind. The 11-dimensional Minkowski spacetime case is shown to be very useful when we compactify it to an $\text{AdS}_5 \times S^7$ spacetime, which will be discussed and investigated in the context of Kaluza-Klein modes of gravitational waves propagation \[9\].

Anomalous dimensions associated with the operators of type $\phi^d$, $d \in \mathbb{N}$, in a 1- and 2-loop approximations \[1, 2\] are to be investigated in the light of the other coefficients $c_k, k > 0$ in a forthcoming paper.

Appendix

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FIG. 3: $\phi(r) \times r$ evaluated for $n = 3$.

FIG. 4: $\phi(r) \times r$ evaluated for $n = 4$.

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FIG. 5: $\phi(r) \times r$ evaluated for $n = 5$.

FIG. 6: $\phi(r) \times r$ evaluated for $n = 6$.

FIG. 7: $\phi(r) \times r$ evaluated for $n = 7$.  

FIG. 8: $\phi(r) \times r$ evaluated for $n = 8$.

FIG. 9: $\phi(r) \times r$ evaluated for $n = 9$.

FIG. 10: $\phi(r) \times r$ evaluated for $n = 10$. 
FIG. 11: $\phi(r) \times r$ evaluated for $n = 11$.

FIG. 12: $\phi(r) \times r$ evaluated for $n = 26$.

FIG. 13: $\phi(r) \times r$ plotted for $1 \leq n \leq 11$ and $n = 26$ together.