HYPERGEOMETRIC EVALUATION IDENTITIES AND SUPERCONGRUENCES

LING LONG

Abstract. In this article, we provide an application of hypergeometric evaluation identities, including a strange valuation of Gosper, to prove several supercongruences related to special evaluations of truncated hypergeometric series. In particular, we prove a conjecture of van Hamme.

1. Introduction

In this article, we use \( p \) to denote an odd prime. In [Zud09], Zudilin proved several Ramanujan-type supercongruences using the Wilf-Zeilberger (WZ) method. One of them, conjectured by van Hamme, is of the form

\[
\sum_{k=0}^{p-1} (4k + 1) \left( \frac{1}{2} \right)_k \left( 1 \right)_k!^3 (-1)^k \equiv (-1)^{p-1} \frac{p}{2} \pmod{p^3},
\]

where \((a)_k = a(a+1)\cdots(a+k-1)\) is the rising factorial when \(a \in \mathbb{C}\) and \(k \in \mathbb{N}\).

The first proof of (1) was given by Mortenson in [Mor08]. It is said to be of Ramanujan-type because it is a \( p \)-adic version of the following formula of Ramanujan.

\[
\sum_{k=0}^{\infty} (4k + 1) \left( \frac{1}{2} \right)_k \left( 1 \right)_k!^3 (-1)^k = \frac{2}{\pi}.
\]

See [Zud09] for more Ramanujan-type supercongruences.

In this short note, we will present a new proof of (1), which summarizes our strategy in proving similar type of supercongruences.

In [MO08], McCarthy and Osburn proved the following conjecture of van Hamme [vH97].

\[
\sum_{k=0}^{\infty} \left( \frac{1}{2} \right)_k \left( 1 \right)_k!^5 (-1)^k \equiv \begin{cases} \frac{-p}{\Gamma_p(\frac{5}{2})^5} \pmod{p^3} & \text{if } p \equiv 1 \pmod{4}; \\ 0 \pmod{p^3} & \text{if } p \equiv 3 \pmod{4}, \end{cases}
\]

where \(\Gamma_p(\cdot)\) denotes the \(p\)-adic Gamma function.

Another comparable conjecture of van Hamme is as follows: for any prime \( p > 3 \)

\[
\sum_{k=0}^{\frac{p-1}{2}} (6k + 1) \left( \frac{1}{2} \right)_k \left( 1 \right)_k!^3 4^{-k} \equiv (-1)^{\frac{p-1}{2}} \frac{p}{2} \pmod{p^4}. \tag{2}
\]

van Hamme said “we have no real explanation for our observations”. In our exploration, it becomes clear that such particular kind of supercongruences reflects extra symmetries, which we are able to interpret using hypergeometric evaluation identities. Of course, they can also be seen from other perspectives, such as the WZ method.

Meanwhile, it is known that some of the truncated hypergeometric series are related to the number of rational points on certain algebraic varieties over finite fields and further to coefficients.

Date: Sep. 17, 2010.

1991 Mathematics Subject Classification. \text{33C20.}

The research was supported by an NSA grant H98230-08-1-0076.
of modular forms. For instance, based on the result of Ahlgren and Ono in [AO00], Kilbourn [Kil06] proved that
\[
\sum_{k=0}^{p-1} \left( \frac{1}{k!} \right)^4 (4k+1) \equiv a_p \mod p^3,
\]
where \(a_p\) is the \(p\)th coefficient of a weight 4 modular form
\[
\eta(2z)^4 \eta(4z)^4 := q \prod_{n \geq 1} (1 - q^{2n})^4 (1 - q^{4n})^4, \quad q = e^{2\pi iz}.
\]

This is one instance of supercongruences conjectured by Rodriguez-Villegas [RV03] which relate special truncated hypergeometric series values and coefficients of Heck eigenforms. In [McC09], the author proved another supercongruence of this type and his approach provides a general combinatorial framework for all these congruences.

In this note, we will establish a few supercongruences via mainly hypergeometric evaluation identities, and combinatorics. Since there exist many amazing hypergeometric evaluation identities in the literature, we expect that our approach can be used to prove other interesting congruences.

Here is a summary of our results.

**Theorem 1.** Let \(p > 3\) be a prime and \(r\) be a positive integer. Then
\[
\sum_{k=0}^{p^r-1} (4k+1) \left( \frac{1}{k!} \right)^4 \equiv p^r \mod p^{3+r}.
\]

**Theorem 2.** Let \(p > 3\) be a prime. Then
\[
\sum_{k=0}^{p^r-1} (4k+1) \left( \frac{1}{k!} \right)^6 \equiv p \cdot a_p \mod p^4.
\]

**Conjecture 1.** Let \(p > 3\) be a prime and \(r\) be a positive integer. Then
\[
\sum_{k=0}^{p^r-1} (4k+1) \left( \frac{1}{k!} \right)^6 \equiv p^r \cdot a_{p^r} \mod p^{3+r},
\]
where \(a_{p^r}\) is the \(p^r\)th coefficient of \(\eta(2z)^4 \eta(4z)^4\).

**Theorem 3.** van Hamme’s conjecture [2] is true.

**Theorem 4.** Let \(p > 3\) be a prime, then
\[
\sum_{k=0}^{p^r-1} (6k+1) \left( \frac{1}{k!} \right)^3 \left( \frac{-1}{8} \right)^k \equiv (-1)^{\frac{r^2-1}{2} + \frac{p^2-1}{2} + p} \mod p^2.
\]

The author would like to thank Heng Huat Chan and Wadim Zudilin for their encouragements, enlightening discussions and valuable comments. In particular, Wadim Zudilin pointed out to the author a few useful ideas and the reference [GS82]. The author further thanks the anonymous referees for their detailed comments, including pointing out a reference [Cai02], on an earlier version of this article.
2. Preliminaries

2.1. Hypergeometric series. For any positive integer \( r \),
\[
\begin{align*}
\binom{r+1}{r}[a_1, a_2, \ldots, a_r+1; b_1, \ldots, b_r] z & = \sum_{k \geq 0} \frac{(a_1)_k \cdots (a_r+1)_k z^k}{k!(b_1)_k \cdots (b_r)_k},
\end{align*}
\]
where \((a)_k\) is the rising factorial and \( z \in \mathbb{C} \). A hypergeometric series terminates if it is well-defined and at least one of the \( a_i \)'s is a negative integer. We will make use of this fact to produce various truncated hypergeometric series.

By the definition of rising factorial,
\[
\binom{1}{2} k! = 2^{-2k} \binom{2k}{k}, \tag{9}
\]

2.2. Gamma function. Let \( \Gamma(x) \) denote the usual Gamma function which is defined for all \( x \in \mathbb{C} \) except non-positive integers. It satisfies some well-known properties such as
\[
\Gamma(x+1) = x\Gamma(x).
\]

Thus, \((a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}\) when \( \Gamma(a) \neq 0 \) and \( \Gamma(a+k) \) are defined.

Another formula we need is the Euler’s reflection formula
\[
\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.
\]

2.3. Some combinatorics. We gather here some results in combinatorics to be used later. It is the author’s pleasure to acknowledge that the approaches used in \([10]\) to \([11]\) are due to Zudilin.

Here is a key idea of Zudilin for rising factorials, see also \([CLZ10, Lemma 1]\)
\[
\left(\frac{1}{2} + \varepsilon\right)_k^2 = \left(\frac{1}{2} + \varepsilon\right)\left(\frac{1}{2} + \varepsilon + 1\right) \cdots \left(\frac{1}{2} + \varepsilon + k - 1\right) \tag{10}
\]
\[
= \left(\frac{1}{2}\right)_k^2 \left(1 + 2\varepsilon \sum_{j=1}^{k} \frac{1}{2j - 1} + 4\varepsilon^2 \sum_{1 \leq i < j \leq k} \frac{1}{(2i - 1)(2j - 1)} + O(\varepsilon^3)\right).
\]

Hence, \(\left(\frac{1}{2} + \varepsilon\right)_k^2\), as a power series of \(\varepsilon^2\) can be expanded as follows:
\[
\left(\frac{1}{2} + \varepsilon\right)_k^2 \left(\frac{1}{2} - \varepsilon\right)_k^2 = \left(\frac{1}{2}\right)_k^2 \left(1 - 4\varepsilon^2 \sum_{j=1}^{k} \frac{1}{(2j - 1)^2} + O(\varepsilon^4)\right). \tag{11}
\]

Similarly,
\[
(1 + \varepsilon)_k(1 - \varepsilon)_k = (1)_k^2 \left(1 - \varepsilon^2 \sum_{j=1}^{k} \frac{1}{j^2} + O(\varepsilon^4)\right) \tag{12}
\]

Letting \( \varepsilon = -\frac{p}{2} \) and \( \varepsilon = \frac{p}{2} \) respectively in \([10]\) and taking \( k \) to be an integer between 1 and \( \frac{p^2 - 1}{2} \), we obtain that
\[
(-1)^k \left(\frac{p^2 - 1}{2}\right)_k \equiv \left(\frac{1}{2}\right)_k \mod p, \quad \left(\frac{p^2 - 1}{2}\right)_k + k \equiv \left(\frac{1}{2}\right)_k \mod p.
\]

Similarly, letting \( \varepsilon = \frac{p^r}{2} \) in \([11]\) and \( k \) be an integer between 1 and \( \frac{p^r - 1}{2} \), we have
\[
(-1)^k \left(\frac{p^r - 1}{2}\right)_k \left(\frac{p^r - 1}{2}\right)_k + k \equiv \left(\frac{1}{2}\right)_k^2 \mod p^2 \tag{13}
\]
Lemma 1. For any positive integer \( n > 1 \),
\[
(2n+1) \sum_{k=0}^{n} \frac{1}{2k+1} \binom{n}{k} \binom{n+k}{k} (-1)^k = 1.
\]  
\[
(14)
\]

Proof. We use the following partial fraction decomposition
\[
\frac{(t-1)(t-2) \cdots (t-n)}{t(t+1) \cdots (t+n)} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} \frac{1}{t+k}.
\]

Letting \( t = \frac{1}{2} \), it becomes
\[
(-1)^n \frac{2}{2n+1} = 2 \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} \frac{1}{1+2k},
\]
which is equivalent to the claim of the Lemma. \( \square \)

Lemma 2. Let \( n \) be an odd positive integer. Then
\[
\frac{\left( \frac{3}{4} - \frac{r}{2} \right)^{n-1} \left( 1 - \frac{r}{2} \right)^{n-1}}{\left( 2 - \frac{r}{2} \right)^{n-1} \left( 1 - \frac{r}{2} \right)^{n-1}} = (-1)^{\frac{n-1}{2}} n.
\]

Proof. Using \( (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \), we have
\[
\frac{\left( \frac{3}{4} - \frac{r}{2} \right)^{n-1} \left( 1 - \frac{r}{2} \right)^{n-1}}{\left( 2 - \frac{r}{2} \right)^{n-1} \left( 1 - \frac{r}{2} \right)^{n-1}} = \frac{\Gamma\left( \frac{3}{4} - \frac{n}{2} + \frac{n-1}{2} \right) \Gamma\left( \frac{1}{2} \right) \Gamma\left( 2 - \frac{n}{2} \right) \Gamma\left( 1 - \frac{n}{2} \right)}{\Gamma\left( \frac{3}{4} - \frac{n}{2} \right) \Gamma\left( 1 - \frac{n}{2} \right) \Gamma\left( \frac{1}{2} \right) \Gamma\left( 1 - \frac{n}{2} + \frac{n-1}{2} \right)}
\]
\[
= \frac{\left( 1 - \frac{r}{2} \right) \Gamma\left( \frac{n}{4} \right) \Gamma\left( 1 - \frac{n}{4} \right)}{\frac{1}{2} \cdot \frac{3}{4} \cdot \Gamma\left( \frac{n}{4} \right) \Gamma\left( \frac{3}{4} \right) \Gamma\left( 1 - \frac{n}{4} \right) \Gamma\left( \frac{1}{4} \right)}
\]
\[
= n \cdot \frac{\sin(\pi/2 - \pi n/4)}{\sin(\pi n/4)} = (-1)^{\frac{n-1}{2}} n.
\]
\( \square \)

Lemma 3. Let \( n \) be an odd integer. Then
\[
\frac{\left( \frac{3}{4} - \frac{r}{2} \right)^{n-1} \left( 1 - \frac{r}{2} \right)^{n-1}}{\left( 2 - \frac{r}{2} \right)^{n-1} \left( 1 - \frac{r}{2} \right)^{n-1}} 2^{\frac{n-1}{2}} = (-1)^{\frac{n^2-1}{8} + \frac{n-1}{2}} n.
\]

Proof.
\[
\frac{\left( \frac{3}{4} - \frac{r}{2} \right)^{n-1} \left( 1 - \frac{r}{2} \right)^{n-1}}{\left( 2 - \frac{r}{2} \right)^{n-1} \left( 1 - \frac{r}{2} \right)^{n-1}} 2^{\frac{n-1}{2}} = \frac{(3 - \frac{r}{2})(5 - \frac{r}{2}) \cdots \frac{n}{2}}{(2 - \frac{r}{2})(3 - \frac{r}{2}) \cdots \frac{n}{2}} = \text{sgn} \cdot n,
\]
where \( \text{sgn} = (-1)^\# \) and \( \# \) is the number of negative terms appearing in the above fraction. It is easy to see that \( \# = \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor - 2 = \frac{n^2-1}{8} + \frac{n-1}{2} \mod 2. \)
\( \square \)

Lemma 4 (Cai, [Cai02]). For any prime \( p > 3 \) and positive integer \( r \),
\[
(-1)^{\frac{p^r-1}{2}} \left( \frac{p^r-1}{2} \right) \equiv \left( \frac{\left( \frac{1}{2} \right)^{p^r-1}}{2^{p^r-1}} \right)^2 \mod p^3.
\]  
\[
(15)
\]
Using (9), the congruence (15) is equivalent to
\[
\left( \frac{p^r - 1}{p^r - 1} \right) \equiv (-1)^{\frac{p^r - 1}{4}} 2^{2(p^r - 1)} \mod p^3.
\]
When \( r = 1 \), it is proved by Morley in [Mor].

2.4. A generalized harmonic sum. Let \( H_k^{(2)} := \sum_{j=1}^{k} \frac{1}{j^2} \).

**Lemma 5.** Let \( p > 3 \) be a prime. We have
\[
H_k^{(2)} \equiv 0 \mod p,
\]
and
\[
\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{(2j - 1)^2} \equiv 0 \mod p. \tag{16}
\]

**Proof.** See [Mor]. \( \square \)

Using arguments in [Mor] or elementary congruence, it is easy to see the following Lemma holds.

**Lemma 6.** Let \( p > 3 \) be a prime, then for every integer \( k \) between 1 and \( p - 2 \)
\[
H_k^{(2)} + H_{p-1-k}^{(2)} \equiv 0 \mod p.
\]

**Lemma 7.** Let \( p > 3 \) be a prime and \( s \) be a positive integer. Then
\[
\sum_{k=0}^{\frac{p-1}{2}} \left( \frac{1}{k!} \right)^2 \cdot H_{2k}^{(2)} \equiv 0 \mod p.
\]

**Proof.** Using the fact that \((-1)^k \left( \frac{p-1}{2} \right) \equiv \left( \frac{1}{k!} \right) \mod p\), we have
\[
\sum_{k=0}^{\frac{p-1}{2}} \left( \frac{1}{k!} \right)^2 H_{2k}^{(2)} \equiv \sum_{k=0}^{\frac{p-1}{2}} \left( \frac{p-1}{2} \right)^{2s} H_{2k}^{(2)} \mod p
\]
\[
= \frac{1}{2} \left( \sum_{k=0}^{\frac{p-1}{2}} \left( \frac{p-1}{2} \right)^{2s} H_{2k}^{(2)} \right) + \sum_{k=0}^{\frac{p-1}{2}} \left( \frac{p-1}{2} \right)^{2s} H_{p-1-2k}^{(2)}
\]
\[
= \frac{1}{2} \left( \sum_{k=0}^{\frac{p-1}{2}} \left( \frac{p-1}{2} \right)^{2s} \left( H_{2k}^{(2)} + H_{p-1-2k}^{(2)} \right) \right)
\]
\[
\equiv 0 \mod p. \square
\]
2.5. **An elementary p-adic analysis.** Let $F(x_1, \ldots, x_t; z)$ be a $(t+1)$-variable formal power series. For instance, it could be a scalar multiple of a terminating hypergeometric series as follows:

$$C \cdot r+1F_r \left[ \begin{array}{c} a_1, \ a_2, \ \cdots, \ a_r, \ -n; \ z \\ b_1, \ \cdots, \ b_{r-1}, \ b_r \end{array} \right].$$

Assume that by specifying values $x_i = a_i, i = 1, \cdots, t$ and $z = z_0$,

$$F(a_1, \cdots, a_t; z_0) \in \mathbb{Z}_p.$$

Now we fix $z_0$ and deform the parameters $a_i$ into polynomials $a_i(x) \in \mathbb{Z}_p[x]$ such that $a_i(0) = a_i$ for all $1 \leq i \leq t$, and assume that the resulting function $F(a_1(x), \cdots, a_t(x); z_0)$ is a formal power series in $x^2$ with coefficients in $\mathbb{Z}_p$, i.e.

$$\begin{align*}
F(a_1(x), \cdots, a_t(x); z_0) &= A_0 + A_2x^2 + A_4x^4 + \cdots, \\
A_i &\in \mathbb{Z}_p,
\end{align*}$$

where $A_0 = F(a_1, \cdots, a_t; z_0)$.

**Lemma 8.** Under the above setting, if $p^s \mid A_2$ for $s = 1, 2$, then

$$F(a_1(p), \cdots, a_t(p); z_0) \equiv A_0 \mod p^{2+s}.$$

3. A NEW PROOF OF (1)

Now we briefly outline our method for proving the next few supercongruences, which is motivated by the papers [MO08] and [Mor08]. To each congruence, we first identify a corresponding hypergeometric evaluation identity, which with specified parameters is congruent to a target truncated hypergeometric series evaluation up to some power of $p$. Usually the power of $p$ hence obtained is weaker than the conjectural exponent. In our cases, we reduce the optimal congruences to some combinatorial identities, which are established using additional hypergeometric evaluation identities or combinatorics.

Our strategy can be best implemented in the following new proof of (1). An identity of Whipple [Whi26 (5.1)] says

$$\begin{align*}
4F3 \left[ \begin{array}{c} a, \ 1+a/2, \ c, \ d; \\ a/2, \ 1+a-c, \ 1+a-d \end{array} \right] &= \frac{\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a)\Gamma(1+a-c-d)}.
\end{align*}$$

Letting $a = \frac{1}{2}$, $c = \frac{1}{2} + \frac{p}{2}$, $d = \frac{1}{2} - \frac{p}{2}$, we conclude immediately that

$$\sum_{k=0}^{p-2} (4k + 1) \left( \frac{\frac{1}{2} k}{k!} \right)^3 (-1)^k \equiv \frac{\Gamma(1 - \frac{p}{2})\Gamma(1 + \frac{p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} = (-1)^{p+1} p \mod p^2.$$

To achieve the congruence modulo $p^3$, we consider the expansion of the terminating hypergeometric series (it terminates as $-\frac{1}{2}$ is a negative integer):

$$\begin{align*}
4F3 \left[ \begin{array}{c} \frac{1-p}{2}, \ \frac{5}{4}, \ \frac{1-x}{4}, \ \frac{1-x}{2}; \\ \frac{1}{2}, \ 1 + \frac{x}{2}, \ 1 - \frac{x}{2} \end{array} \right] &= \sum_{k=0}^{p-1} (4k + 1) \left( \frac{\frac{1}{2} k}{k!} \right)^3 (-1)^k + A_2x^2 + \cdots, \quad (17)
\end{align*}$$

for some $A_2 \in \mathbb{Z}_p$.

By Lemma 8, if $p \mid A_2$, we are done. Now we follow Mortenson [Mor08] to use another hypergeometric evaluation identity, which is a specialization of Whipple’s $7F_6$ formula (see [Bai35, pp. 28]).

$$\begin{align*}
6F5 \left[ \begin{array}{c} a, \ 1 + \frac{a}{2}, \ b, \ c, \ d, \ e; \\ \frac{a}{2}, \ 1 + a - b, \ 1 + a - c, \ 1 + a - d, \ 1 + a - e \end{array} \right] &= \frac{\Gamma(1 + a - d)\Gamma(1 + a - e)}{\Gamma(1 + a)\Gamma(1 + a - d - e)} \cdot
\end{align*}$$

$$\begin{align*}
3F2 \left[ \begin{array}{c} 1 + a - b - c, \ d, \ e; \\ 1 + a - b, \ 1 + a - c \end{array} \right].
\end{align*}$$
Letting $a = \frac{1}{2}$, $b = \frac{1-x}{2}$, $c = \frac{1+x}{2}$, $e = \frac{1-p}{2}$, $d = 1$, we have

$$6F_5\left[\frac{1}{2}, \frac{5}{2}, \frac{1-x}{2}, \frac{1+x}{2}, \frac{1-p}{2}, \frac{1}{2}; 1, -\frac{p}{2}, -1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{p}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{p}{2}\right)} \cdot \left[\frac{1}{2}, 1, 1, -1\right].$$  \hspace{1cm} (18)

Since $\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{p}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{p}{2}\right)} = p$, every $x$-coefficient of the above is in $p\mathbb{Z}_p$. Moreover, modulo $p$ the left hand side of (17) is congruent to that of (18). So when we expand the left hand side of (17) in terms of $x$, the coefficients are all in $p\mathbb{Z}_p$. In particular, $p \mid A_2$ and this concludes the proof of (1).

4. Proofs of Theorems 1, 2, 3, and 4

We first recall the following identity of Whipple [Whi26 (7.7)]:

$$7F_6\left[a, 1+\frac{1}{2}a, \frac{c}{2a}, 1+a-c, 1+a-d, 1+a-e, 1+a-f, 1+a-g; 1\right]$$

$$= \frac{\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-e-f-g)}{\Gamma(1+a)\Gamma(1+a-f-g)\Gamma(1+a-e-f)\Gamma(1+a-e)\times}$$

$$4F_3\left[1+a-c-d, 1+a-e, 1+a-f, 1+a-g; 1\right],$$

subject to the condition that the $4F_3$ is a terminating series.

4.1. Proof of Theorem 1. Let $r$ be a positive integer and $p > 3$ a prime. In the identity (19), we let

$$a = \frac{1}{2}, c = \frac{1}{2} + \frac{ip^r}{2}, d = \frac{1}{2} - \frac{ip^r}{2}, e = \frac{1}{2} + \frac{ip^r}{2}, f = \frac{1}{2} - \frac{ip^r}{2}, g = 1,$$

where $i = \sqrt{-1}$ and thereafter, then following McCarthy and Osburn’s argument we know the left hand side of (19) is congruence to

$$\sum_{k=0}^{p^r-1} (4k+1) \left(\frac{1}{2}k\right)^4 \mod p^{4r}$$

and the right hand side of (19) equals

$$\frac{\Gamma(1-\frac{p^r}{2})\Gamma(1+\frac{p^r}{2})\Gamma(-\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(-\frac{p^r}{2})\Gamma(\frac{p^r}{2})} 4F_3\left[\frac{1}{2}, \frac{1}{2} + \frac{ip^r}{2}, \frac{1}{2} - \frac{ip^r}{2}, 1; 1\right].$$

Since

$$\frac{\Gamma(1-\frac{p^r}{2})\Gamma(1+\frac{p^r}{2})\Gamma(-\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(-\frac{p^r}{2})\Gamma(\frac{p^r}{2})} = p^{2r},$$

it suffices to prove

$$p^r \cdot \sum_{k=0}^{p^r-1} \frac{1}{2k+1} \left(\frac{1}{2}k\right)^2 \equiv 1 \mod p^3, \text{ for } p > 3.$$

Recall that Lemma 1 says for any odd integer $n > 1$,

$$(2n+1) \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \left(\begin{array}{c}n \\ k\end{array}\right) \left(\begin{array}{c}n+k \\ k\end{array}\right) = 1.$$
Therefore, combining this identity, congruence (11), and Lemma 4, we have

\[
p^r \cdot \sum_{k=0}^{\nu^r-1} \frac{1}{2k+1} \left( \frac{1}{k!} \right)^2 = p^r \cdot \sum_{k=0}^{\nu^r-1} \frac{1}{2k+1} \left( \frac{1}{k!} \right)^2 + \left( \frac{1}{2} \right)^{\nu^r-1} \left( \frac{1}{k!} \right)^2
\]

\[
\equiv p^r \cdot \sum_{k=0}^{\nu^r-1} \frac{1}{2k+1} \left( \frac{1}{k!} \right)^2 + \left( \frac{1}{2} \right)^{\nu^r-1} \left( \frac{1}{k!} \right)^2 + (-1)^{\nu^r-1} \left( \frac{p^r - 1}{2} \right) \mod p^3
\]

\[
\equiv 1 \mod p^3.
\]

This concludes our proof of Theorem 1.

4.2. Proof of Theorem 2. In the formula (19), take

\[
a = \frac{1}{2}, c = \frac{1}{2} + ip^2, d = \frac{1}{2} - ip^2, e = \frac{1}{2} - p, f = \frac{1}{2} + p, g = \frac{1}{2} - p^4,
\]

then the left hand side of (19) is congruent to

\[
\sum_{k=0}^{\nu^r-1} (4k + 1) \left( \frac{1}{k!} \right)^6 \mod p^4.
\]

Meanwhile, the right hand side of (19) is congruent to

\[
\frac{\Gamma(1 - \frac{p}{2})\Gamma(1 + \frac{p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} \frac{\Gamma(1 + p^4)\Gamma(p^4)}{\Gamma(\frac{1}{2} + \frac{p}{2} + p^4)\Gamma(\frac{1}{2} - \frac{p}{2} + p^4)} \sum_{k=0}^{\nu^r-1} \frac{1}{k!^2(1 - \frac{p}{2})^k(1 + \frac{p}{2})^k}
\]

\[
\equiv \frac{\Gamma(1 - \frac{p}{2})\Gamma(1 + \frac{p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} \equiv (-1)^{\nu^r-1} \frac{1}{p},
\]

where

\[
\Gamma(1 - \frac{p}{2})\Gamma(1 + \frac{p}{2}) = (-1)^{\nu^r-1} \frac{1}{p},
\]

and

\[
\frac{\Gamma(1 + p^4)\Gamma(p^4)}{\Gamma(\frac{1}{2} + \frac{p}{2} + p^4)\Gamma(\frac{1}{2} - \frac{p}{2} + p^4)} = \frac{(p^4 - \frac{p}{2})^{\nu^r-1} \frac{1}{p}}{(1 + p^4)^{\nu^r-1}}
\]

\[
\equiv \frac{(-\frac{p}{2})^{\nu^r-1}(-\frac{p}{2})^{\nu^r-1} + 1 \cdots (-1)}{1 \cdot 2 \cdots (\frac{p}{2})} \mod p = (-1)^{\nu^r-1}.
\]

Therefore, Theorem 2 follows from the result of Kilbourn (see (3)) and the next Lemma.

Lemma 9. Let \( p > 3 \) be a prime, then

\[
\sum_{k=0}^{\nu^r-1} \frac{1}{k!^2(1 - \frac{p}{2})^k(1 + \frac{p}{2})^k} \equiv \sum_{k=0}^{\nu^r-1} \left( \frac{1}{k!} \right)^4 \mod p^3.
\]

Proof. Expand

\[
\sum_{k=0}^{\nu^r-1} \frac{1}{k!^2(1 - \frac{p}{2})^k(1 + \frac{p}{2})^k} = \sum_{k=0}^{\nu^r-1} \left( \frac{1}{k!} \right)^4 (1 + b_{2,k}x^2 + b_{4,k}x^4 + \cdots).
\]

Using (11) and (12), we have

\[
b_{2,k} = -\sum_{j=1}^{k} \frac{1}{(2j - 1)^2} - \frac{1}{4} \sum_{j=1}^{k} \frac{1}{j^2} = -\sum_{j=1}^{2k} \frac{1}{j^2}.
\]

The claim of the Lemma is valid by using Lemma 8 and taking \( s = 2 \) in Lemma 7. □
4.3. The proof of Theorem \[ \text{3} \] We start with the following combinatorial identity.

**Lemma 10.**
\[
\sum_{k=0}^{\frac{p-1}{2}} (6k+1) \frac{\left( \frac{1}{2} \right)_k \left( \frac{1}{2} - \frac{x}{4} \right)_k \left( \frac{1}{2} + \frac{x}{4} \right)_k}{\left( 1 + \frac{1}{4} \right)_k \left( 1 - \frac{2}{4} \right)_k} 4^k \frac{1}{4} = \left( -1 \right)^{\frac{p-1}{2}} p.
\]

**Proof.** Recall that (31.1) of Gessel [Ges95] says
\[
_5F_4 \left[ \begin{array}{cc} \frac{1}{2} + a - c, & -n, \\ 2 - c + n, & \frac{3}{2} - \frac{2c}{3} + \frac{n}{3} \\ \frac{2}{3} - \frac{2c}{3} + \frac{n}{3}, & n - 2a + 2, & \frac{2}{3} - c \end{array} \right] = \frac{(2-c)_n(2-2a)_n}{(3-2c)_n(\frac{2}{3} - a)_n}. 
\]

Letting \( a = \frac{1}{2} + \frac{q}{4} \), \( c = \frac{1}{2} + \frac{2q}{4} \), \( n = \frac{p-1}{2} \) and using Lemma \[ \text{2} \] we have
\[
_5F_4 \left[ \begin{array}{cc} \frac{1}{2}, & \frac{1}{2} - \frac{2q}{4}, \\ \frac{1}{2} - \frac{2q}{4}, & \frac{1}{2} + \frac{2q}{4} \\ 1 - \frac{1}{4}, & 1 + \frac{1}{4} \end{array} \right] = \frac{(\frac{3}{4} - \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}} (1 - \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}}}{(2 - \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}} (1 - \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}}} = \left( -1 \right)^{\frac{p-1}{2}} p. \tag{20}
\]

**Lemma 11.** The function
\[
\left( \sum_{k=0}^{\frac{p-1}{2}} \frac{6k+1}{4^k} \frac{\left( \frac{1}{2} \right)_k \left( \frac{1}{2} - \frac{x}{4} \right)_k \left( \frac{1}{2} + \frac{x}{4} \right)_k}{\left( 1 + \frac{1}{4} \right)_k \left( 1 - \frac{2}{4} \right)_k} 4^k \right)^3
\]
is a formal power series in \( x^2 \) with coefficients in \( \mathbb{Z}_p \). Its \( x^2 \) coefficient is zero modulo \( p \).

**Proof.** We use the following strange valuation of Gosper (cf. [GSS82] (1.2))
\[
_5F_4 \left[ \begin{array}{cc} 2a, & 2b, \\ a + b - 1, & a + b + \frac{1}{2}, \\ 1 - 2b, & 1 + \frac{2a}{3}, \\ 1 + 2a + 2n & -n; \frac{1}{4} \end{array} \right] = \frac{(a + \frac{1}{2})_n(a + 1)_n}{(a + b + \frac{1}{2})_n(a + b - 1)_n}. 
\]

Letting \( a = \frac{1}{4}, b = \frac{1}{4} - \frac{2}{3}, n = \frac{p-1}{2} \), the left hand side of the above equals
\[
_5F_4 \left[ \begin{array}{cc} \frac{1}{2}, & \frac{1}{2} - \frac{q}{2}, \\ \frac{1}{2} - \frac{q}{2}, & \frac{1}{2} + \frac{q}{2} \\ 1 - \frac{1}{4}, & 1 + \frac{1}{4} \end{array} \right] = \frac{(\frac{3}{4} - \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}} (\frac{5}{4} - \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}}}{(1 - \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}} (1 + \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}}}. \tag{21}
\]

We remark that
\[
_5F_4 \left[ \begin{array}{cc} \frac{1}{2}, & \frac{1}{2} - \frac{q}{2}, \\ \frac{1}{2} - \frac{q}{2}, & \frac{1}{2} + \frac{q}{2} \\ 1 - \frac{1}{4}, & 1 + \frac{1}{4} \end{array} \right] = \sum_{k=0}^{\frac{p-1}{2}} \frac{6k+1}{4^k} \frac{\left( \frac{1}{2} \right)_k \left( \frac{1}{2} - \frac{q}{2} \right)_k \left( \frac{1}{2} + \frac{q}{2} \right)_k}{\left( 1 + \frac{1}{4} \right)_k \left( 1 - \frac{2}{4} \right)_k} \mod p. \tag{22}
\]

When \( x = 0 \), the right hand side of \[ \text{21} \] equals \( \frac{(\frac{3}{4} - \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}} (\frac{5}{4} - \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}}}{(1 - \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}} (1 + \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}}} \), which is in \( p\mathbb{Z}_p \). As a matter of fact, if \( p \equiv 1 \mod 4 \) then \( \frac{3}{4} - \frac{q}{2} - 1 = \frac{q}{2} \); and if \( p \equiv 3 \mod 4 \) then \( \frac{3}{4} + \frac{q-3}{2} = \frac{q}{2} \), while \( \frac{p-1}{2} \frac{1}{4} \) is a \( p \)-adic unit. It is not difficult to see that \( p \) divides \( \frac{(\frac{3}{4} - \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}} (\frac{5}{4} - \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}}}{(1 - \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}} (1 + \frac{q}{2})^{\frac{p-1}{2} \frac{1}{4}}} \) exactly. Consequently, if we expand \( _5F_4 \left[ \begin{array}{cc} \frac{1}{2}, & \frac{1}{2} - \frac{q}{2}, \\ \frac{1}{2} - \frac{q}{2}, & \frac{1}{2} + \frac{q}{2} \\ 1 - \frac{1}{4}, & 1 + \frac{1}{4} \end{array} \right] \) in terms of formal power series of \( x \) (in fact, \( x^2 \)), each coefficient is in \( p\mathbb{Z}_p \). Thus the coefficients of the right hand side of \[ \text{22} \], including the coefficient of \( x^2 \), are all divisible by \( p \). By Lemmas \[ \text{3} \] and \[ \text{10} \]
\[
\sum_{k=0}^{\frac{p-1}{2}} \frac{6k+1}{4^k} \frac{\left( \frac{1}{2} \right)_k}{k!} \equiv \left( -1 \right)^{\frac{p-1}{2}} p \mod p^3. \tag{23}
\]
Namely,
\[\sum_{k=0}^{p-1} \frac{6k+1}{4^k} \left( \frac{1}{2} \right)_k^3 = (-1)^{p-1} p + ap^3,\]
for some \(a \in \mathbb{Z}_p\). The statement of Theorem 4 is equivalent to \(a \in p\mathbb{Z}_p\).

The quotient
\[\left( \sum_{k=0}^{p-1} \frac{6k+1}{4^k} \left( \frac{1}{2} \right)_k \left( \frac{1}{2} - \frac{x}{2} \right)_k \left( \frac{1}{2} + \frac{y}{2} \right)_k \right) / \left( \sum_{k=0}^{p-1} \frac{6k+1}{4^k} (1)_k \left( \frac{1}{2} \right)_k \left( 1 - \frac{z}{2} \right)_k \right)\]
is a formal power series in \(x^2\) with \(p\)-integral coefficients, as the denominators are divisible by \(p\) exactly. The same conclusion applies to the following
\[\left( 5F_4 \left[ \frac{1}{2}, \frac{1}{2} + \frac{x}{2}, \frac{1}{2} + \frac{y}{2}, \frac{1}{2} - \frac{z}{2}; 1 - \frac{a}{2}, 1 + \frac{a}{2} \right] \right) / \left( 5F_4 \left[ \frac{1}{2}, \frac{1}{2} + \frac{x}{2}, \frac{1}{2} + \frac{y}{2}, \frac{1}{2} - \frac{z}{2}; 1 - \frac{a}{2}, 1 + \frac{a}{2} \right] \right)\]
is a scalar multiple of \(H^{(2)}_{\frac{a-1}{2}}\), which is in \(p\mathbb{Z}_p\) by Lemma 5 so is the \(x^2\) coefficient of (24).

By Lemma 5 and the above analysis,
\[\frac{(-1)^{p-1}}{(-1)^{p-1}} p = (1)^{p-1} \equiv 1 \mod p^3,\]
hence \(a \in p\mathbb{Z}_p\), which concludes the proof of Theorem 3.

4.4. The proof of Theorem 4. Recall we want to prove
\[\sum_{k=0}^{p-1} (6k+1) \left( \frac{1}{2} \right)_k^3 \left( \frac{1}{2} \right)^k \equiv -1 \frac{\frac{\frac{1}{2} - \frac{a}{2} - \frac{1}{8} + \frac{1}{2}}{2}}{8^k} \equiv (-1)^{\frac{1}{2} + \frac{1}{2}} p \mod p^3.\]

It is a consequence of the following combinatorial identity.

Lemma 12.
\[\sum_{k=0}^{p-1} (6k+1) \left( \frac{1}{2} \right)_k \left( \frac{1}{2} - \frac{a}{2} \right)_k \left( \frac{1}{2} + \frac{y}{2} \right)_k \left( -1 \right)^k \frac{1}{8^k} = (-1)^{\frac{a^2}{8} + \frac{1}{2}} p.\]

Proof. This time, we use the following identity in Gessel [Ges95, pp. 544, last identity]
\[\left( \begin{array}{c} a+n+1 \\ a+\frac{1}{2}+n \end{array} \right) = \frac{\left( \begin{array}{c} a+n+1 \\ a+\frac{1}{2}+n \end{array} \right) \left( \begin{array}{c} a+n+\frac{3}{2} \\ a+\frac{3}{2}+n \end{array} \right)}{\left( \begin{array}{c} a+n+1 \\ a+\frac{1}{2}+n \end{array} \right) \left( \begin{array}{c} a+n+\frac{3}{2} \\ a+\frac{3}{2}+n \end{array} \right)}.\]

Letting \(a = -\frac{p}{2}\) and \(n = \frac{p-1}{2}\) and using Lemma 3 we have
\[\left( \begin{array}{c} a+n+1 \\ a+\frac{1}{2}+n \end{array} \right) = \frac{\left( \begin{array}{c} a+n+\frac{3}{2} \\ a+\frac{3}{2}+n \end{array} \right)}{\left( \begin{array}{c} a+n+1 \\ a+\frac{1}{2}+n \end{array} \right) \left( \begin{array}{c} a+n+\frac{3}{2} \\ a+\frac{3}{2}+n \end{array} \right)} = \frac{\left( \begin{array}{c} a+n+\frac{3}{2} \\ a+\frac{3}{2}+n \end{array} \right)}{\left( \begin{array}{c} a+n+1 \\ a+\frac{1}{2}+n \end{array} \right) \left( \begin{array}{c} a+n+\frac{3}{2} \\ a+\frac{3}{2}+n \end{array} \right)} = \frac{(-1)^{\frac{a^2}{8} + \frac{1}{2}} p}{\left( \begin{array}{c} a+n+1 \\ a+\frac{1}{2}+n \end{array} \right) \left( \begin{array}{c} a+n+\frac{3}{2} \\ a+\frac{3}{2}+n \end{array} \right)}.\]

□
Remark 1. The following conjecture of van Hamme
\begin{equation}
\sum_{k=0}^{p-1} (6k+1) \left( \frac{1}{2} \right)_k \frac{1}{k!} \equiv (-1)^{\frac{p^2-1}{8}} + \frac{p-1}{8} \mod p^3 \tag{26}
\end{equation}
holds subject to the following:
\begin{equation}
\sum_{k=0}^{p-1} (6k+1) \left( \frac{1}{2} \right)_k \frac{1}{k!} \equiv \left( \sum_{j=1}^{k} \frac{1}{(2j-1)^2} - \frac{1}{16} \sum_{j=1}^{k} \frac{1}{j^2} \right) \frac{(-1)^k}{8^k} \equiv 0 \mod p. \tag{27}
\end{equation}

The proof of (27) is left to the interested reader.

Remark 2. Note that using the method in [Zud09], Zudilin proved the congruence (27) modulo $p^2$ and the congruence (26) modulo $p$.

REFERENCES

[AO00] S. Ahlgren and K. Ono, A Gaussian hypergeometric series evaluation and Apéry number congruences. J. Reine Angew. Math. 518 (2000), 187–212.

[Bai35] W. N. Bailey, Generalized Hypergeometric Series, Cambridge Tracts in Mathematics and Mathematical Physics, no. 32, Cambridge University Press (1935).

[Cai02] T. Cai, A congruence involving the equations of Euler and its applications (I), Acta Arith. 103 (2002), 313–320.

[CLZ10] H.H. Chan, L. Long, and W. Zudilin, A supercongruence motivated by the Legendre family of elliptic curves, to appear in Mat. Zametki 88:4 (2010).

[Ges95] I. M. Gessel, Finding identities with the WZ method, J. Symbolic Comput. 20 (1995), no. 5-6, 537–566, Symbolic computation in combinatorics $\Delta_1$ (Ithaca, NY, 1993).

[GS82] I. M. Gessel and D. Stanton, Strange evaluations of hypergeometric series, SIAM J. Math. Anal. 13 (1982), no. 2, 295–308.

[Kil06] T. Kilbourn, An extension of the Apéry number supercongruence, Acta Arith. 123 (2006), no. 4, 335–348.

[McC09] D. McCarthy, Supercongruence Conjectures of Rodriguez-Villegas, arXiv:0907.5089, preprint, (2009)

[MO08] D. McCarthy and R. Osburn, A $p$-adic analogue of a formula of Ramanujan, Arch. Math. (Basel) 91 (2008), no. 6, 492–504.

[Mor] F. Morley, Note on the congruence $2^{4n} \equiv (-1)^n(2n)!/(n!)^2$, where $2n+1$ is prime, Annals of Math., 9 (1895), pages 168-170.

[Mor08] E. Mortenson, A $p$-adic supercongruence conjecture of van Hamme, Proc. Amer. Math. Soc. 136 (2008), no. 12, 4321–4328.

[RV03] F. Rodriguez-Villegas, Hypergeometric families of Calabi-Yau manifolds, Calabi-Yau varieties and mirror symmetry (Toronto, ON, 2001), Fields Inst. Commun., vol. 38, Amer. Math. Soc., Providence, RI, 2003, 223–231.

[vH97] L. van Hamme, Some conjectures concerning partial sums of generalized hypergeometric series, $p$-adic functional analysis (Nijmegen, 1996), Lecture Notes in Pure and Appl. Math., vol. 192, Dekker, New York, 1997, 223–236.

[Whi26] F.I.W. Whipple, On well-posed series, generalised hypergeometric series having parameters in pairs, each pair with the same sum, Proc. London Math. Soc. 24 (1926), no. 2, 247–263.

[Zud09] W. Zudilin, Ramanujan-type supercongruences, J. Number Theory 129 (2009), no. 8, 1848–1857.

Mathematics Department, Iowa State University, Ames, Iowa, 50011, USA
E-mail address: linglong@iastate.edu