Generalized Browder’s and Weyl’s Theorems for Generalized Derivations

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Abstract

Given Banach spaces $X$ and $Y$ and Banach space operators $A \in L(X)$ and $B \in L(Y)$, let $\rho: L(Y, X) \to L(Y, X)$ denote the generalized derivation defined by $A$ and $B$, i.e., $\rho(U) = AU - UB$ $(U \in L(Y, X))$. The main objective of this article is to study Weyl and Browder type theorems for $\rho \in L(L(Y, X))$. To this end, however, first the isolated points of the spectrum and the Drazin spectrum of $\rho \in L(L(Y, X))$ need to be characterized. In addition, it will be also proved that if $A$ and $B$ are polaroid (respectively isoloid), then $\rho$ is polaroid (respectively isoloid).

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1. Introduction

In the recent past several authors have studied Weyl and Browder type theorems and the condition of being polaroid or isoloid for tensor product and elementary operators; see for example [2, 13, 14, 23, 26, 27, 29, 31, 36]. Concerning the above mentioned research area, generalized derivations have been studied mainly for particular classes of operators defined on Hilbert spaces; see [16, 17, 20, 21, 24, 25, 32, 33].

The main objective of this article is to study Weyl and Browder type theorems for generalized derivations in the context of Banach spaces. In fact, given two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, two Banach space operators $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$ and $\rho \in L(L(\mathcal{Y}, \mathcal{X}))$ the generalized derivation defined by $A$ and $B$, using an approach similar to the one in [14, 23, 26], in section 4 the problem of transferring (generalized) Browder’s theorem from $A$ and $B$ and (generalized) $a$-Browder’s theorem from $A$ and $B^*$ to $\rho$ will be studied. Furthermore, in section 5, when $A$ and $B^*$ are isoloid (respectively $a$-isoloid) operators satisfying generalized Weyl’s (respectively generalized $a$-Weyl’s) theorem, necessary and sufficient conditions for $\rho$ to satisfy generalized Weyl’s (respectively generalized $a$-Weyl’s) theorem will be given; what is more, Weyl’s and $a$-Weyl’s theorems will be also studied. In addition, it will be proved that the condition of being polaroid (respectively isoloid) transfers from $A$ and $B$ (respectively from $A$ and $B^*$) to $\rho$.

However, to this end, in section 3, after having recalled some preliminary definitions and facts in section 2, the isolated points of the spectrum and the Drazin spectrum of $\rho$ will be fully characterized.

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2. Preliminary definitions and facts

From now on $X$ and $Y$ will denote infinite dimensional complex Banach spaces and $L(Y, X)$ the algebra of all bounded linear maps defined on $Y$ and with values in $X$; as usual, $L(X) = L(Y, X)$. Given $A \in L(Y, X)$, $N(A) \subseteq Y$ and $R(A) \subseteq X$ will stand for the null space and the range of $A$, respectively. In addition, $X^*$ will denote the dual space of $X$ while $A^* \in L(X^*)$ will stand for the adjoint operator of $A \in L(X)$. Recall that $A \in L(X)$ is said to be bounded below, if $N(A) = 0$ and $R(A)$ is closed. Denote the approximate point spectrum of $A$ by $\sigma_a(A) = \{\lambda \in \mathbb{C}: A - \lambda$ is not bounded below\}, where $A - \lambda$ stands for $A - \lambda I$, $I$ the identity map of $L(X)$. Let $\sigma(A)$ denote the spectrum of $A$.

Recall that $A \in L(X)$ is said to be a Fredholm operator if $\alpha(A) = \dim N(A)$ and $\beta(A) = \dim X/R(A)$ are finite dimensional, in which case its index is given by

$$\text{ind}(A) = \alpha(A) - \beta(A).$$

If $R(A)$ is closed and $\alpha(A)$ is finite, then $A \in L(X)$ is said to be upper semi-Fredholm, while if $\alpha(A)$ and $\beta(A)$ are finite and equal, so that the index is zero, $A$ is said to be a Weyl operator. These classes of operators generate the Fredholm or essential spectrum, the upper semi-Fredholm spectrum and the Weyl spectra of $A \in L(X)$, which will be denoted by $\sigma_e(A)$, $\sigma_{SF+}(A)$ and $\sigma_w(A)$, respectively. On the other hand, $\Phi(A)$ and $\Phi+(A)$ will denote the complement in $\mathbb{C}$ of the Fredholm spectrum and of the upper semi-Fredholm spectrum of $A$, respectively.

In addition, the Weyl essential approximate point spectrum of $A \in L(X)$ is the set $\sigma_{aw}(A) = \{\lambda \in \sigma_a(A): A - \lambda$ is not upper semi-Fredholm or $0 < \text{ind}(A - \lambda)\}$, see [34].

Recall that the concept of Fredholm operator has been generalized. An operator $A \in L(X)$ will be said to be $B$-Fredholm, if there exists $n \in \mathbb{N}$ for which $R(A^n)$ is closed and the induced operator $A_n \in L(R(A^n))$ is Fredholm. In a similar way it is possible to define upper $B$-Fredholm operators. Note that if for some $n \in \mathbb{N}$, $A_n \in L(R(A^n))$ is Fredholm, then $A_m \in L(R(A^m))$ is Fredholm for all $m \geq n$; moreover $\text{ind}(A_n) = \text{ind}(A_m)$, for all $m \geq n$. Therefore, it makes sense to define the index of $A$ by $\text{ind}(A) = \text{ind}(A_n)$. Recall that $A$ is said to be $B$-Weyl, if $A$ is $B$-Fredholm and $\text{ind}(A) = 0$. Naturally, from this class of operators the $B$-Weyl spectrum of $A \in L(X)$ can be derived, which will be denoted by $\sigma_{BW}(A)$. In addition, set $\sigma_{SBF+}(A) = \{\lambda \in \mathbb{C}: A - \lambda$ is not upper semi $B$-Fredholm or $0 < \text{ind}(A - \lambda)\}$, see [8].

On the other hand, the ascent (respectively descent) of $A \in L(X)$ is the smallest non-negative integer $a$ (respectively $d$) such that $N(A^a) = N(A^{a+1})$ (respectively $R(A^d) = R(A^{d+1})$); if such an integer does not exist, then $\text{asc}(A) = \infty$ (respectively $\text{dsc}(A) = \infty$). The operator $A$ will be said to be Browder, if it is Fredholm and its ascent and descent are finite. Then, the Browder spectrum of $A \in L(X)$ is the set $\sigma_b(A) = \{\lambda \in \mathbb{C}: A - \lambda$ is not Browder\}. It is well known that

$$\sigma_e(A) \subseteq \sigma_w(A) \subseteq \sigma_b(A) = \sigma_e(A) \cup \text{acc } \sigma(A),$$

where if $K \subseteq \mathbb{C}$, then acc $K$ denotes the limit points of $K$ while iso $K$ stands for the isolated points of $K$, i.e., iso $K = K \setminus \text{acc } K$.

In addition, the Browder essential approximate point spectrum of $A \in L(X)$ is the set $\sigma_{ab}(A) = \{\lambda \in \sigma_a(A): \lambda \in \sigma_{aw}(A)$ or $\text{asc}(A - \lambda) = \infty\}$, see [34]. It is clear that $\sigma_{aw}(A) \subseteq \sigma_{ab}(A) \subseteq \sigma_a(A)$.

Recall that a Banach space operator $A \in L(X)$ is said to be Drazin invertible, if there exists a necessarily unique $B \in L(X)$ and some $m \in \mathbb{N}$ such that

$$A^m = A^m BA, \quad BAB = B, \quad AB = BA,$$

see for example [19, 30]. If $DR(L(X)) = \{A \in L(X): A$ is Drazin invertible\}$, then the Drazin spectrum of $A \in L(X)$ is the set $\sigma_{DR}(A) = \{\lambda \in \mathbb{C}: A - \lambda \notin DR(L(X))\}$, see [9, 11]. It is well
known that if \( A \in L(\mathcal{X}) \) is Drazin invertible, then there is \( k \in \mathbb{N} \) such that \( \mathcal{X} = R(A^k) \oplus N(A^k) \) and \( A_k \in L(R(A^k)) \) is invertible, see for example [28]. In particular, \( \sigma_{BW}(A) \subseteq \sigma_{DR}(A) \subseteq \sigma(A) \).

To introduce the definitions of the main notions studied in this work, some notation is needed. Let \( A \in L(\mathcal{X}) \) and denote by \( \Pi(A) = \{ \lambda \in \mathbb{C} : \text{asc}(A - \lambda) = \text{dsc}(A - \lambda) < \infty \} \) (respectively \( \Pi_0(A) = \{ \lambda \in \Pi(A) : \alpha(A - \lambda) < \infty \} \) the set of poles of \( A \) (respectively the poles of finite rank of \( A \)). Similarly, denote by \( \Pi^a(A) = \{ \lambda \in \text{iso} \sigma_A(a = \text{asc}(A - \lambda) < \infty \) and \( R(A - \lambda)^{a+1} \) is closed} \) (respectively \( \Pi^0_\text{iso}(A) = \{ \lambda \in \Pi^0(A) : \alpha(A - \lambda) < \infty \} \)) the set of left poles of \( A \) (respectively, the left poles of finite rank of \( A \)).

In addition, given \( A \in L(\mathcal{X}) \), let \( E(A) = \{ \lambda \in \text{iso} \sigma(A) : 0 < \alpha(A - \lambda) \} \) (respectively \( E_0(A) = \{ \lambda \in E(A) : \alpha(A - \lambda) < \infty \} \)) the set of eigenvalues of \( A \) which are isolated in the spectrum of \( A \) (respectively, the eigenvalues of finite multiplicity isolated in \( \sigma(A) \)). Similarly, let \( E^a(A) = \{ \lambda \in \sigma_a(A) : 0 < \alpha(A - \lambda) \} \) (respectively \( E^0_\text{iso}(A) = \{ \lambda \in E^a(A) : \alpha(A - \lambda) < \infty \} \)) the set of eigenvalues of \( A \) which are isolated in \( \sigma_a(A) \) (respectively the eigenvalues of finite multiplicity isolated in \( \sigma_a(A) \)).

Next the definitions of the main notions studied in this article will be given.

**Definition 2.1.** Consider a Banach space \( \mathcal{X} \) and \( A \in L(\mathcal{X}) \). Then it will be said that
(i) Browder’s theorem holds for \( A \), if \( \sigma_w(A) = \sigma(A) \setminus \Pi_0(A) \),
(ii) generalized Browder’s theorem holds for \( A \), if \( \sigma_{BW}(A) = \sigma(A) \setminus \Pi(A) \),
(iii) \( a \)-Browder’s theorem holds for \( A \), if \( \sigma_{aw}(A) = \sigma_a(A) \setminus \Pi^0_\text{iso}(A) \),
(iv) generalized \( a \)-Browder’s theorem holds for \( A \), if \( \sigma_{SBF^+}(A) = \sigma_a(A) \setminus \Pi^a(A) \).

**Definition 2.2.** Consider a Banach space \( \mathcal{X} \) and \( A \in L(\mathcal{X}) \). Then it will be said that
(i) Weyl’s theorem holds for \( A \), if \( \sigma_w(A) = \sigma(A) \setminus E_0(A) \),
(ii) generalized Weyl’s theorem holds for \( A \), if \( \sigma_{BW}(A) = \sigma(A) \setminus E(A) \),
(iii) \( a \)-Weyl’s theorem holds for \( A \), if \( \sigma_{aw}(A) = \sigma_a(A) \setminus E^0_\text{iso}(A) \),
(iv) generalized \( a \)-Weyl’s theorem holds for \( A \), if \( \sigma_{SBF^+}(A) = \sigma_a(A) \setminus E^a(A) \).

Finally, recall that an operator \( T \in L(\mathcal{X}) \) is said to have SVEP, the single–valued extension property, at a (complex) point \( \lambda_0 \), if for every open disc \( D \) centered at \( \lambda_0 \) the only analytic function \( f : D \rightarrow \mathcal{X} \) satisfying \( (T - \lambda)f(\lambda) = 0 \) is the function \( f \equiv 0 \). We say that \( T \) has SVEP on a subset \( K \) of the complex plane if it has SVEP at every point of \( K \). Trivially, every operator \( T \) has SVEP at points of the resolvent \( \rho(A) = \mathbb{C} \setminus \sigma(T) \). Also \( T \) has SVEP at points \( \lambda \in \text{iso} \sigma(T) \) and \( \lambda \in \text{iso} \sigma_a(T) \). See [11] Chapters 2-3 for more information on operators with SVEP.

3. Spectra of generalized derivations

In this section the Browder spectrum, the Browder essential approximate point spectrum and the Drazin spectrum of a generalized derivation will be characterized. To this end, let \( \mathcal{X} \) and \( \mathcal{Y} \) be two Banach spaces and consider \( A \in L(\mathcal{X}) \) and \( B \in L(\mathcal{Y}) \). Let \( \rho : L(\mathcal{Y}, \mathcal{X}) \rightarrow L(\mathcal{Y}, \mathcal{X}) \) be the generalized derivation defined by \( A \) and \( B \), i.e., \( \rho(U) = AU - UB, U \in L(\mathcal{Y}, \mathcal{X}) \). In other words, \( \rho = L_A - R_B \), where if \( S \in L(\mathcal{X}) \) (respectively \( S \in L(\mathcal{Y}) \)), then \( L_S \in L(L(\mathcal{Y}, \mathcal{X})) \) (respectively \( R_S \in L(L(\mathcal{Y}, \mathcal{X})) \)) is the operator defined by left (respectively right) multiplication by \( S \), i.e., for \( U \in L(\mathcal{Y}, \mathcal{X}) \), \( L_S(U) = SU \) (respectively \( R_S(U) = US \)). In first place, the Browder spectrum of \( \rho \in L(L(\mathcal{Y}, \mathcal{X})) \) will be studied.

**Remark 3.1.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two Banach spaces and consider \( A \in L(\mathcal{X}) \) and \( B \in L(\mathcal{Y}) \). Let \( \rho \in L(L(\mathcal{Y}, \mathcal{X})) \) be the generalized derivation defined by \( A \) and \( B \).
(i) According to [28] Corollary 3.4, \( \sigma(\rho) = \sigma(A) - \sigma(B) \).
(ii) It is not difficult to prove that
\[
\text{acc } \sigma(\rho) = (\text{acc } \sigma(A) - \sigma(B)) \cup (\sigma(A) - \text{acc } \sigma(B))
\]
\[
= (\text{acc } \sigma(A) - \text{acc } \sigma(B)) \cup (\text{acc } \sigma(A) - \text{iso } \sigma(B)) \cup (\text{iso } \sigma(A) - \text{acc } \sigma(B)).
\]

What is more,
\[
\text{iso } \sigma(\rho) = (\text{iso } \sigma(A) - \text{iso } \sigma(B)) \setminus \text{acc } \sigma(\rho).
\]

(iii) Recall that according to [28, Corollary 3.4], \(\sigma_e(\rho) = (\sigma(A) - \sigma_e(B)) \cup (\sigma_e(A) - \sigma(B))\). Now well, since \(\sigma_b(\rho) = \sigma_e(\rho) \cup \text{acc } \sigma(\rho)\), according to (ii),
\[
\sigma_b(\rho) = (\sigma(A) - \sigma_b(B)) \cup (\sigma_b(A) - \sigma(B)).
\]

Next the Browder essential approximate point spectrum of \(\rho \in L(L(\mathcal{Y}, \mathcal{X}))\) will be characterized. However, first some preparation is needed.

**Remark 3.2.** Let \(\mathcal{X}\) and \(\mathcal{Y}\) be two Banach spaces and consider \(A \in L(\mathcal{X})\) and \(B \in L(\mathcal{Y})\). Let \(M\) be the two-tuple of commuting operators \(M = (L_A, R_B), L_A\) and \(R_B \in L(L(\mathcal{Y}, \mathcal{X})).\) Recall that the *approximate point joint spectrum* and the *upper semi-Fredholm joint spectrum* of \(M\) are the sets
\[
\sigma_{a}(M) = \{ (\mu, \nu) \in \mathbb{C}^2 : \sigma_{a}(A - \mu B - \nu) \text{ is not bounded below}\}
\]
and
\[
\sigma_{\Phi_{a}}(M) = \{ (\mu, \nu) \in \mathbb{C}^2 : \sigma_{a}(A - \mu B - \nu) \text{ is upper semi-Fredholm}\},
\]
respectively, where \(\sigma_{a}(A - \mu B - \nu) : L(\mathcal{Y}, \mathcal{X}) \to L(\mathcal{Y} \mathcal{X}) \times L(\mathcal{Y}, \mathcal{X})\), \(\sigma_{a}(A - \mu B - \nu)(S) = (L_{A-\mu}(S), R_{B-\nu}(S)) = ((A-\mu)S, S(B-\nu))\). Note that the conditions of being bounded below and upper semi-Fredholm for operators defined between two different Banach spaces is similar to the one given in section 2. Concerning the properties of these joint spectra, see for example [10, 15, 35].

**Proposition 3.3.** Let \(\mathcal{X}\) and \(\mathcal{Y}\) be two Banach spaces and consider \(A \in L(\mathcal{X})\) and \(B \in L(\mathcal{Y})\). Let \(\rho \in L(L(\mathcal{Y}, \mathcal{X}))\) be the generalized derivation defined by \(A\) and \(B\). Then the following statement hold.
(i) \(\sigma_{a}(\rho) = \sigma_{a}(A) - \sigma_{a}(B^{*})\).
(ii) \(\sigma_{SF_{a}}(\rho) = (\sigma_{SF_{a}}(A) - \sigma_{a}(B^{*})) \cup (\sigma_{a}(A) - \sigma_{SF_{a}}(B^{*}))\).
(iii) \(\text{acc } \sigma_{a}(\rho) = (\text{acc } \sigma_{a}(A) - \sigma_{a}(B^{*})) \cup (\sigma_{a}(A) - \text{acc } \sigma_{a}(B^{*}))\).
(iv) \(\text{iso } \sigma_{a}(\rho) = (\text{iso } \sigma_{a}(A) - \text{iso } \sigma_{a}(B^{*})) \setminus \text{acc } \sigma_{a}(\rho)\).
(v) \(\sigma_{ab}(\rho) = (\sigma_{ab}(A) - \sigma_{a}(B^{*})) \cup (\sigma_{a}(A) - \sigma_{ab}(B^{*}))\).

**Proof.** (i). Recall that according to the proof of [14 Proposition 4.3(i)],
\[
\sigma_{\pi}(L_{A}, R_{B}) = \sigma_{a}(A) \times \sigma_{a}(B^{*}).
\]
To prove the statement under consideration, apply the spectral mapping theorem to the polynomial mapping \(P : \mathbb{C}^2 \to \mathbb{C}, P(X, Y) = X - Y\) ([35 Theorem 2.9]).
(ii). According to the proof of [14 Proposition 4.3(ii)],
\[
\sigma_{\Phi_{a}}(L_{A}, R_{B}) = \sigma_{SF_{a}}(A) \times \sigma_{a}(B^{*}) \cup \sigma_{a}(A) \times \sigma_{SF_{a}}(B^{*}).
\]
However, the statement under consideration can be derived applying the spectral mapping theorem to the polynomial mapping \(P : \mathbb{C}^2 \to \mathbb{C}, P(X, Y) = X - Y\) ([15 Theorem 7]).
(iii).-(iv). These statements can be easily deduced from statement (i).

(v). Denote by $S$ the set $S = (\sigma_{ab}(A) - \sigma_{ab}(B^{*})) \cup (\sigma_{a}(A) - \sigma_{ab}(B^{*}))$. Let $\lambda \in \sigma_{a}(\rho) \setminus S$. Then, according to what has been proved and to [34, Corollary 2.2], $\lambda \notin \sigma_{SF_{a}}(\rho)$ and $\lambda \in \sigma_{a}(\rho)$.

In particular, according to [1, Theorem 3.16], $asc(\rho - \lambda)$ is finite. Therefore, according to [34 Corollary 2.2], $\lambda \in \sigma_{a}(\rho) \setminus \sigma_{ab}(\rho)$. Hence, $\sigma_{ab}(\rho) \subseteq S$.

Now let $\lambda \in \sigma_{a}(\rho) \setminus \sigma_{ab}(\rho)$. Then, according to [34 Corollary 2.2] and statement (iv), there exist $\mu \in \sigma_{a}(A)$ and $\nu \in \sigma_{a}(B^{*})$ such that $\lambda = \mu - \nu$. In addition, since $\sigma_{SF_{a}}(\rho) \subseteq \sigma_{ab}(\rho)$, according to statement (ii), $\mu \notin \sigma_{SF_{a}}(\sigma_{a}(B^{*}))$. In particular, according to [1 Theorem 3.16] and [34 Corollary 2.2], $\mu \notin \sigma_{ab}(A)$ and $\nu \notin \sigma_{ab}(B^{*})$. Consequently, $\lambda \in \sigma_{a}(\rho) \setminus S$. Hence, $S \subseteq \sigma_{ab}(\rho)$.

To fully characterize the Drazin spectrum of a generalized derivation $\rho : L(\mathcal{Y}, \mathcal{X}) \rightarrow L(\mathcal{Y}, \mathcal{X})$, it is first necessary to characterize the set of poles $\Pi(\rho)$ and its complement in the isolated points of the spectrum of $\rho$, i.e., $I(\rho) = \sigma(\rho) \setminus \Pi(\rho)$. To this end, however, the following result need to be considered.

**Theorem 3.4.** Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and suppose that $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$ are such that $\sigma(A) = \{\mu\}$ and $\sigma(B) = \{\nu\}$. Consider $\rho \in L(L(\mathcal{Y}, \mathcal{X}))$, the generalized derivation defined by $A$ and $B$. Then, the following statements hold.

(i) If $\Pi(A) = \{\mu\}$ and $\Pi(B) = \{\nu\}$, then $\sigma(\rho) = \Pi(\rho) = \{\mu - \nu\}$.

(ii) If $I(A) = \{\mu\}$ or $I(B) = \{\nu\}$, then $\sigma(\rho) = I(\rho) = \{\mu - \nu\}$.

**Proof.** Clearly, $\sigma(\rho) = \{\mu - \nu\}$. In addition, Note that $\rho - (\mu - \nu) = L(A - \mu) - R(B - \nu)$.

(i) Suppose that $\Pi(A) = \{\mu\}$ and $\Pi(B) = \{\nu\}$. Since the operators $L(A - \mu)$ and $R(B - \nu)$ are nilpotent ([12, Remark 3.1(i)]) and commute, $\rho - (\mu - \nu)$ is nilpotent, equivalently, $\sigma(\rho) = \Pi(\rho) = \{\mu - \nu\}$ ([12, Remark 3.1(i)])).

(ii) Let $x \in \mathcal{X}$ and $f \in \mathcal{Y}^{*}$. Define the operator $U_{x,f} \in L(\mathcal{Y}, \mathcal{X})$ as follows: $U_{x,f}(z) = xf(z)$, where $z \in \mathcal{Y}$. Note that since $\sigma(A) = \{\mu\}$, there exist a sequence $(x_{n})_{n \in \mathbb{N}} \subseteq \mathcal{X}$ such that $\|x_{n}\| = 1$, for all $n \in \mathbb{N}$, and $(\{A - \mu\}(x_{n}))_{n \in \mathbb{N}}$ converges to 0. Suppose that $I(B) = \{\nu\}$. Since $\sigma(B^{*}) = I(B^{*}) = \{\nu\}$, for each $k \in \mathbb{N}$, there is $f_{k} \in \mathcal{Y}^{*}$ such that $\|((B - \nu)^{*})k(f_{k})\| = 2$ ([12, Remark 3.1(ii)])

Now, well, note that given $k \in \mathbb{N}$,

$$(\rho - (\mu - \nu))^{k} = \sum_{j=1}^{k} c_{k,j} L(A - \mu)^{j} R(B - \nu)^{k-j} + (-1)^{k} R(B - \nu)^{k},$$

where $c_{k,j} = (-1)^{k-j} \frac{k!}{j!(k-j)!}$. However, using $(U_{x_{n},f_{k}})_{n \in \mathbb{N}} \subseteq L(\mathcal{Y}, \mathcal{X})$, it is not difficult to prove that there is $n_{0} \in \mathbb{N}$ such that

$$\| \left( \sum_{j=1}^{k} c_{k,j} L(A - \mu)^{j} R(B - \nu)^{k-j} (U_{x_{n},f_{k}}) \right) \| < 1,$$

for all $n \geq n_{0}$, $n \in \mathbb{N}$. Since $\| R(B - \nu)^{k} (U_{x_{n},f_{k}}) \| = 2$,

$$\| (\rho - (\mu - \nu))^{k} (U_{x_{n},f_{k}}) \| \geq 1,$$

$n \geq n_{0}$. Therefore, since $k \in \mathbb{N}$ is arbitrary, $\sigma(\rho) = I(\rho) = \{\mu - \nu\}$ ([12, Remark 3.1(ii)])

Interchanging $A$ with $B$, it is possible to prove the theorem under the assumption $I(A) = \{\mu\}$. □
Next, the isolated points of a spectrum of a generalized derivation will be characterized.

**Theorem 3.5.** Let $X$ and $Y$ be two Banach spaces and consider $A \in L(X)$ and $B \in L(Y)$. Let $\rho \in L(L(Y, X))$ be the generalized derivation defined by $A$ and $B$. Then, the following statements hold.

(i) $I(\rho) = ((I(A) - \text{iso } \sigma(B)) \cup (\text{iso } \sigma(A) - I(B))) \setminus \text{acc } \sigma(\rho)$.

(ii) $\Pi(\rho) = (\Pi(A) - \Pi(B)) \setminus \sigma_{\text{DR}}(\rho)$.

**Proof.** Let $\lambda \in \text{iso } \sigma(\rho)$. Note that there exist $n \in \mathbb{N}$ and finite spectral sets $\{\mu_i\} = \{\mu_1, \ldots, \mu_n\} \subseteq \text{iso } \sigma(A)$ and $\{\nu_i\} = \{\nu_1, \ldots, \nu_n\} \subseteq \text{iso } \sigma(B)$ such that $\lambda = \mu_i - \nu_i$ for all $1 \leq i \leq n$ and that if there are $\mu' \in \text{iso } \sigma(A)$ and $\nu' \in \text{iso } \sigma(B)$ such that $\lambda = \mu' - \nu'$, then there is $i$, $1 \leq i \leq n$, such that $\mu' = \mu_i$ and $\nu' = \nu_i$. Corresponding to these spectral sets there are closed subspaces $M_1$, $M_2$ and $(M_i)_{i=1}^n$ of $X$ invariant for $A$ and closed subspaces $N_1$, $N_2$ and $(N_i)_{i=1}^n$ of $Y$ invariant for $B$ such that $X = M_1 \oplus M_2$, $M_1 = \oplus_{i=1}^n M_i$, $Y = N_1 \oplus N_2$, $N_1 = \oplus_{i=1}^n N_i$, $\sigma(A_1) = \{\mu_i\}$, $\sigma(A_2) = \sigma(A) \setminus \{\mu_i\}$, $\sigma(A_1) = \{\nu_i\}$, $\sigma(A_2) = \sigma(B) \setminus \{\nu_i\}$ and $\sigma(B_1) = \{\nu_i\}$, where $A_1 = A|_{M_1}$, $A_2 = A|_{M_2}$, $A_1 = A|_{M_1}$, $B_1 = B|_{N_1}$, $B_2 = B|_{N_2}$ and $B_1 = B|_{N_1}$. In addition, $\rho - \lambda$ is invertible on the closed invariant subspaces $L(N_2, M_1)$, $L(N_1, M_2)$, $L(N_2, M_2)$ and $L(N_i, M_{j\neq i})$, $1 \leq j \neq k \leq n$. Moreover, $L(Y, X)$ is the direct sum of these subspaces and $L(N_i, M_i)$, $1 \leq i \leq n$. To prove statement (i), define $Z = (((I(A) - \text{iso } \sigma(B)) \cup (\text{iso } \sigma(A) - I(B)))) \setminus \text{acc } \sigma(\rho)$.

Let $\lambda \in I(\rho)$ and consider $n = n(\lambda) \in \mathbb{N}$ and $\mu_i \in \text{iso } \sigma(A)$ and $\nu_i \in \text{iso } \sigma(B)$ such that $\lambda = \mu_i - \nu_i$, $i = 1, \ldots, n$. Now, if $\lambda \notin Z$, then for each $i = 1, \ldots, n$, $\mu_i \in \Pi(A)$ and $\nu_i \in \Pi(B)$. However, according to Theorem 3.4(i) and [12, Remark 3.1], $\lambda \notin \Pi(\rho)$, which is impossible.

Suppose that $\lambda \in Z \subseteq \text{iso } \sigma(\rho)$. Then, there exist $\mu \in \text{iso } \sigma(A)$ and $\nu \in \text{iso } \sigma(B)$ such that $\lambda = \mu - \nu$ and either $\mu \in I(A)$ or $\nu \in I(B)$. Applying to $\lambda \in Z$ what has been done in the first paragraph of this proof, there exist an $n = n(\lambda) \in \mathbb{N}$ and an $i$, $1 \leq i \leq n$, such that $\mu = \mu_i$ and $\nu = \nu_i$. Therefore, according to Theorem 3.4(ii) and [12, Remark 3.1], $\lambda \in I(\rho)$.

To prove statement (ii), apply what has been proved, the fact that $\text{iso } \sigma(\rho) = (\text{iso } \sigma(A) - \text{iso } \sigma(B)) \cup \text{acc } \sigma(\rho)$ and [30, Theorem 4].

Recall that given a Banach space $X$ and $A \in L(X)$, $A$ is said to be polaroid, if $\text{iso } \sigma(A) = \Pi(A)$, equivalently $I(A) = \emptyset$. As a first application of Theorem 3.5 it will be proved that this condition transfers to generalized derivations.

**Theorem 3.6.** Let $X$ and $Y$ be two Banach spaces and consider $A \in L(X)$ and $B \in L(Y)$ such that $A$ and $B$ are polaroid operators. Then, the generalized derivation defined by $A$ and $B\rho \in L(L(Y, X))$ is polaroid.

**Proof.** According to the proof of Theorem 3.5 if $I(A) = \emptyset = I(B)$, then $I(\rho) = \emptyset$ and $\text{iso } \sigma(\rho) = \Pi(\rho) = (\Pi(A) - \Pi(B)) \setminus \text{acc } \sigma(\rho)$.

In the following theorem, the Drazin spectrum of a generalized derivation will be characterized.

**Theorem 3.7.** Let $X$ and $Y$ be two Banach spaces and consider $A \in L(X)$ and $B \in L(Y)$. Then, if $\rho: L(Y, X) \to L(Y, X)$ is the generalized derivation defined by $A$ and $B$,

\[ \sigma_{\text{DR}}(\rho) = (\sigma_{\text{DR}}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{\text{DR}}(B)). \]

**Proof.** Recall that according to [30, Theorem 4, $\sigma_{\text{DR}}(\rho) = I(\rho) \cup \text{acc } \sigma(\rho)$. To conclude the proof, apply Theorem 3.5(i) and the fact that $\text{acc } \sigma(\rho) = (\text{acc } \sigma(A) - \sigma(B)) \cup (\sigma(A) - \text{acc } \sigma(B)$.

\[\Box\]
4. Browder’s theorems

In this section Browder type theorems will be studied for generalized derivations. Recall that Browder’s theorem (respectively $a$-Browder’s theorem) is equivalent to generalized Browder’s theorem (respectively to generalized $a$-Browder’s theorem), see [1] Theorems 2.1-2.2. In first place, (generalized) Browder’s theorem will be considered.

**Theorem 4.1.** Let $X$ and $Y$ be two Banach spaces and consider $A \in L(X)$ and $B \in L(Y)$ such that (generalized) Browder’s theorem holds for $A$ and $B$. If $\rho : L(Y,X) \to L(Y,X)$ is the generalized derivation defined by $A$ and $B$, then the following statements are equivalent.

(i) (generalized) Browder’s theorem holds for $\rho$.
(ii) $(\text{acc } \sigma(A) - \sigma(B)) \cup (\sigma(A) - \text{acc } \sigma(B)) \subseteq \sigma_w(\rho)$.
(iii) $\sigma_w(\rho) = (\sigma_w(A) - \sigma(B)) \cup (\sigma(A) - \sigma_w(B))$.
(iv) $(\text{acc } \sigma(A) - \sigma(B)) \cup (\sigma(A) - \text{acc } \sigma(B)) \subseteq \sigma_{BW}(\rho)$.
(v) $\sigma_{BW}(\rho) = (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B))$.
(vi) Given $\lambda \in \sigma(\rho) \setminus \sigma_w(\rho)$, for all $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$ such that $\lambda = \mu - \nu$, $\mu \in \Phi(A)$, $\nu \in \Phi(B)$, $A$ has the SVEP at $\mu$ and $B^*$ has the SVEP at $\nu$.

**Proof.** Recall that according to [5] Proposition 2], $\rho \in L(Y,X)$ satisfies Browder’s theorem if and only if $\text{acc } \sigma(\rho) \subseteq \sigma_w(\rho)$. In particular, according to Remark 3.1(ii), statements (i) and (ii) are equivalent.

Note that according again to [5] Proposition 2], Browder’s theorem is equivalent to the identity $\sigma_w(\rho) = \sigma_b(\rho)$. Therefore, according to Remark 3.1(iii),

$$\sigma_b(\rho) = (\sigma_w(A) - \sigma(B)) \cup (\sigma(A) - \sigma_w(B)).$$

As a result, Browder’s theorem holds for $\rho$ if and only if statement (iii) holds.

According to [7] Theorem 2.3], necessary and sufficient for $\rho \in L(Y,X)$ to satisfy generalized Browder’s theorem is the fact that $\text{acc } \sigma(\rho) \subseteq \sigma_{BW}(\rho)$. Consequently, according to Remark 3.1(ii), statements (i) and (iv) are equivalent.

Applying again [7] Theorem 2.3], it is not difficult to prove that generalized Browder’s theorem is equivalent to the fact that $\sigma_{BW}(\rho) = \sigma_{DR}(\rho)$. In particular, according to Theorem 4.7

$$\sigma_{DR}(\rho) = (\sigma_{BW}(A) - \sigma(B)) \cup (\sigma(A) - \sigma_{BW}(B)).$$

Consequently, (generalized) Browder’s theorem is equivalent to statement (v).

Next suppose that Browder’s theorem holds for $\rho \in L(Y,X)$ and consider $\lambda \in \sigma(\rho) \setminus \sigma_w(\rho)$. Let $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$ such that $\lambda = \mu - \nu$. Since $\sigma_w(\rho) = \sigma_b(\rho)$, $\lambda \in \Pi_0(\rho) \subseteq \text{iso } \sigma(\rho)$. In particular, $\mu \in \text{iso } \sigma(A)$ and $\nu \in \text{iso } \sigma(B) = \text{iso } \sigma(B^*)$, which implies that $A$ has the SVEP at $\mu$ and $B^*$ has the SVEP at $\nu$. In addition, since $\sigma_c(\rho) \subseteq \sigma_b(\rho)$, according to [28] Corollary 3.4], $\mu \in \Phi(A)$ and $\nu \in \Phi(B)$.

On the other hand, suppose that statement (vi) holds. Let $\lambda \in \sigma(\rho) \setminus \sigma_w(\rho)$ and consider $M_\lambda = \{(\mu, \nu) \in \sigma(A) \times \sigma(B): \lambda = \mu - \nu\}$. According to [28] Lemma 4.1] applied to the two tuple of operators $M = (L_A, R_B) \in L(L(Y,X))^2$ and the polynomial mapping $P : \mathbb{C}^2 \to \mathbb{C}$, $P(X,Y) = X - Y$, $M_\lambda$ is a finite set. Let $m \in \mathbb{N}$ such that $M_\lambda = \{(\mu_i, \nu_i): 1 \leq i \leq m\}$. Then, according to the paragraphs before [28] Theorem 4.2], there is $n \in \{0, \dotsc, m\}$ such that:

(i) $\mu_i \in \text{iso } \sigma(A)(1 \leq i \leq n)$; (ii) if $n < m$, then $\nu_i \in \text{iso } \sigma(B)(n + 1 \leq i \leq m)$.

Moreover, since $\text{ind } (\rho - \lambda) = 0$, according to [28] Theorem 4.2] applied to $M$ and $P$,

$$\sum_{k=n+1}^{m} (\dim X_B(\nu_k)) \text{ind}(A - \mu_k) = \sum_{k=1}^{n} (\dim X_A(\mu_k)) \text{ind}(B - \nu_k),$$

where $X_A(\mu_k)$ and $X_B(\nu_k)$ are the eigenspaces of $A$ and $B$ corresponding to $\mu_k$ and $\nu_k$, respectively.
where $X_A(\mu_i)$ (respectively $X_B(\nu_i)$) is the spectral subspace of $A$ (respectively $B$) associated to the isolated point $\mu_i$ ($1 \leq i \leq n$) (respectively $\nu_i$ ($n + 1 \leq i \leq m$)).

Note that since $\mu_i \in \Phi(A) \cap \text{iso } \sigma(A)$ (respectively $\nu_i \in \Phi(B)$ (respectively $\nu_i \in \Pi_0(B)$) ($1 \leq i \leq n$) (respectively $n + 1 \leq i \leq m$)). In addition, according to [18] Theorem 1.52, $0 < \dim X_A(\mu_i) < \infty$ ($1 \leq i \leq n$) and $0 < \dim X_B(\nu_i) < \infty$ ($n + 1 \leq i \leq m$).

Now well, since $\mu_i \in \Phi(A)$ and $A$ has the SVEP at $\mu_i$, according to [11] Theorem 3.19(i), $\text{ind}(A - \mu_i) \leq 0$ ($n + 1 \leq i \leq m$). Similarly, since $\nu_i \in \Phi(B)$ and $B^*$ has the SVEP at $\nu_i$, according to [11] Theorem 3.19(ii), $\text{ind}(B - \nu_i) \geq 0$ ($1 \leq i \leq n$). Then, using the identity derived from [28] Theorem 4.2, it is not difficult to prove that $\text{ind}(A - \mu_i) = 0$ ($n + 1 \leq i \leq m$) and $\text{ind}(B - \nu_i) = 0$ ($1 \leq i \leq n$). Therefore, $\mu_i \in \Pi_0(A)$ and $\nu_i \in \Pi_0(B)$ ($1 \leq i \leq m$), which implies that $\lambda \in \Pi_0(\rho)$ (Remark 3.1(iii)). As a result, $\sigma(\rho) \subseteq \sigma_w(\rho)$, equivalently $\sigma_b(\rho) = \sigma_w(\rho)$.

Next (generalized) $a$-Browder theorem will be studied.

**Theorem 4.2.** Let $X$ and $Y$ be two Banach spaces and consider $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ such that (generalized) $a$-Browder’s theorem holds for $A$ and $B^*$. If $\rho : L(Y, X) \to L(Y, X)$ is the generalized derivation defined by $A$ and $B$, then the following statements are equivalent.

(i) (generalized) $a$-Browder’s theorem holds for $\rho$.

(ii) $(\text{acc } \sigma_a(A) - \sigma_a(B^*)) \cup (\sigma_a(A) - \text{acc } \sigma_a(B^*)) \subseteq \sigma_{aw}(\rho)$.

(iii) $\sigma_{aw}(\rho) = (\sigma_{aw}(A) - \sigma_a(B^*)) \cup (\sigma_a(A) - \sigma_{aw}(B^*))$.

(iv) $(\text{acc } \sigma_a(A) - \sigma_a(B^*)) \cup (\sigma_a(A) - \text{acc } \sigma_a(B^*)) \subseteq \sigma_{SBF^*}(\rho)$.

(v) Given $\lambda \in \sigma_a(\rho) \ \cap \ \sigma_{aw}(\rho)$, for all $\mu \in \sigma_a(A)$ and $\nu \in \sigma_a(B^*)$ such that $\lambda = \mu - \nu$, $\mu \in \Phi_+(A)$, $\nu \in \Phi_+(B^*)$, $A$ has the SVEP at $\mu$ and $B^*$ has the SVEP at $\nu$.

**Proof.** According to [34] Corollary 2.2 and [34] Corollary 2.4, necessary and sufficient for the $a$-Browder’s theorem to hold for $\rho : L(Y, X) \to L(Y, X)$ is that $\text{acc } \sigma_a(\rho) \subseteq \sigma_{aw}(\rho)$. In particular, to prove the equivalence between statements (i) and (ii), apply Proposition 3.3(iii).

Note that according to [34] Corollary 2.2, $a$-Browder’s theorem holds for $\rho$ if and only if $\sigma_{aw}(\rho) = \sigma_{ab}(\rho)$. However, according to Proposition 3.3(v), $\sigma_{ab}(\rho) = (\sigma_{aw}(A) - \sigma_a(B^*)) \cup (\sigma_a(A) - \sigma_{aw}(B^*))$. Consequently, statements (i) and (iii) are equivalent.

Recall that, according to [8], Theorem 2.8, the generalized $a$-Browder’s theorem holds for $\rho \in L(L(Y, X))$ if and only if $\text{acc } \sigma_a(\rho) \subseteq \sigma_{SBF^*}(\rho)$. Therefore, to prove the equivalence between statements (i) and (iv), apply Proposition 3.3(iii).

To prove the equivalence between statements (i) and (v), suppose that $a$-Browder’s theorem holds for $\rho$ and consider $\lambda \in \sigma_a(\rho) \ \cap \ \sigma_{aw}(\rho)$. Let $\mu \in \sigma_a(A)$ and $\nu \in \sigma_a(B^*)$ such that $\lambda = \mu - \nu$. Then, according to statement (iii), $\mu \in \sigma_a(A) \ \cap \ \sigma_{ab}(A)$ and $\nu \in \sigma_a(B) \ \backslash \ \sigma_{ab}(B^*)$. In particular, according to [34] Theorem 2.1, $\mu \in \Phi_+(A)$, $\nu \in \Phi_+(B^*)$ and $\text{asc } (A - \mu)$ and $\text{asc } (B^* - \nu)$ are finite, which implies that $A$ has the SVEP at $\mu$ and $B^*$ has the SVEP at $\nu$ (Theorem 3.8).

Next suppose that statement (v) holds and consider $\lambda \in \sigma_a(\rho) \ \cap \ \sigma_{aw}(\rho)$. Then, for every $\mu \in \sigma_a(A)$ and $\nu \in \sigma_a(B^*)$ such that $\lambda = \mu - \nu$, $\mu \in \Phi_+(A)$, $\nu \in \Phi_+(B^*)$ and $A$ has the SVEP at $\mu$ and $B^*$ has the SVEP at $\nu$. According to [11] Theorems 3.16 and 3.23, $\mu \in \text{iso } \sigma_a(A)$, $\nu \in \text{iso } \sigma_a(B^*)$ and $\text{asc } (A - \mu)$ and $\text{asc } (B^* - \nu)$ are finite. However, according to [34] Corollary 2.2, $\mu \in \sigma_a(A) \ \backslash \ \sigma_{ab}(A)$ and $\nu \in \sigma_a(B^*) \ \backslash \ \sigma_{ab}(B^*)$. Thus, according to Proposition 3.3(v), $\lambda \notin \sigma_{ab}(\rho)$. Hence, $\lambda \in \sigma_a(\rho) \ \backslash \ \sigma_{ab}(\rho)$, which implies that $\sigma_{ab}(\rho) \subseteq \sigma_{aw}(\rho)$, equivalently, $\sigma_{ab}(\rho) = \sigma_{aw}(\rho)$.

5. **Weyl’s theorems**

Before considering Weyl type theorems for generalized derivations, some preparation is needed. Recall first that given $A \in \mathcal{L}(X)$, $X$ a Banach space, $A$ is said to be *isoloid* (respectively *a-isoloid*), if $\text{iso } \sigma(A) = E(A)$ (respectively $\text{iso } \sigma_a(A) = E^a(A)$). These conditions transfer to generalized derivations.
Proposition 5.1. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two Banach spaces and consider \( A \in L(\mathcal{X}) \) and \( B \in L(\mathcal{Y}) \). Let \( \rho : L(\mathcal{Y}, \mathcal{X}) \to L(\mathcal{Y}, \mathcal{X}) \) be the generalized derivation defined by \( A \) and \( B \). Then, if \( A \) and \( B^* \) are isoloid (respectively a-isoloid), \( \rho \in L(L(\mathcal{Y}, \mathcal{X})) \) is isoloid (respectively a-isoloid).

Proof. Given \( \lambda \in \sigma(\rho) \), according to Remark 3.1(ii), there exist \( \mu \in \sigma(A) \) and \( \nu \in \sigma(B) \) such that \( \lambda = \mu - \nu \). Since \( A \) and \( B^* \) are isoloid, there are \( x \in \mathcal{X} \), \( x \neq 0 \), and \( f \in \mathcal{Y}^* \), \( f \neq 0 \), such that \( A(x) = \mu x \) and \( B^*(f) = \nu f \). Consider the operator \( U_{x,f} \in L(\mathcal{Y}, \mathcal{X}) \) defined in the proof of Theorem 3.4(ii). Then, an easy calculation proves that \( \rho(U_{x,f}) = \lambda U_{x,f} \).

Therefore, iso \( \sigma(\rho) = E(\rho) \).

A similar argument, using in particular Proposition 3.3(iv), proves the a-isoloid case. \( \square \)

Next (generalized) Weyl’s theorem for generalized derivations will be studied. However, first some fact need to be recalled.

Remark 5.2. Let \( \mathcal{X} \) be a Banach space and consider \( A \in L(\mathcal{X}) \) and \( A^* \in L(\mathcal{X}^*) \). Then, the following statements are equivalent.

(i) \( A \) is isoloid and generalized Weyl’s theorem holds for \( A \).

(ii) \( A \) is polaroid and generalized Browder’s theorem holds for \( A \).

(iii) \( A^* \) is isoloid and generalized Browder’s theorem holds for \( A^* \).

(iv) \( A^* \) is isoloid and generalized Weyl’s theorem holds for \( A^* \).

(v) \( A \) is polaroid and Weyl’s theorem holds for \( A \).

(vi) \( A \) is polaroid and Browder’s theorem holds for \( A \).

(vii) \( A^* \) is polaroid and Browder’s theorem holds for \( A^* \).

(viii) \( A^* \) is polaroid and Weyl’s theorem holds for \( A^* \).

The proof can be easily derived from the notions involved in the statements and from well known results. Although the details are left to the reader, some indications will be given.

To prove the equivalence between statements (i) and (ii), use [7] Corollary 2.6]. The equivalence between statements (ii) and (iii) can be proved using [3] Theorem 2.8(iii) and [6] Remark B. To prove the equivalence between statements (iii) and (iv), apply what has been proved. According to [4] Theorem 2.1, statements (ii) and (vi) (respectively statements (iii) and (vii)) are equivalents. Finally, according to [22] Theorem 2.2, statements (v) and (vi) (respectively statements (vii) and (viii)) are equivalent.

Theorem 5.3. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two Banach spaces and consider \( A \in L(\mathcal{X}) \) and \( B \in L(\mathcal{Y}) \) such that \( A \) and \( B^* \) are isoloid and generalized Weyl’s theorem holds for \( A \) and \( B^* \). If \( \rho : L(\mathcal{Y}, \mathcal{X}) \to L(\mathcal{Y}, \mathcal{X}) \) is the generalized derivation defined by \( A \) and \( B \), then the following statements are equivalent.

(i) generalized Weyl’s theorem holds for \( \rho \in L(L(\mathcal{Y}, \mathcal{X})) \).

(ii) (generalized) Browder’s theorem holds for \( \rho \in L(L(\mathcal{Y}, \mathcal{X})) \).

(iii) (generalized) Browder’s theorem holds for \( \rho^* \in L(L(\mathcal{Y}, \mathcal{X}^*)) \).

(iv) generalized Weyl’s theorem holds for \( \rho^* \in L(L(\mathcal{Y}, \mathcal{X}^*)) \).

(v) Weyl’s theorem holds for \( \rho \in L(L(\mathcal{Y}, \mathcal{X})) \).

(vi) Weyl’s theorem holds for \( \rho^* \in L(L(\mathcal{Y}, \mathcal{X}^*)) \).

Proof. According to Remark 5.2, \( A \) and \( B \) are polaroid. Hence, according to Theorem 3.6 \( \rho \in L(L(\mathcal{Y}, \mathcal{X})) \) is polaroid. Moreover, according to [3] Theorem 2.8(iii), \( \rho^* \in L(L(\mathcal{Y}, \mathcal{X}^*)) \) is polaroid. To conclude the proof, apply Remark 5.2. \( \square \)

Note that according to Remark 5.2, the hypothesis in Theorem 5.3 is equivalent to the fact that \( A \) and \( B^* \) are polaroid operators satisfying Weyl’s theorem. The following lemma will be useful to study a-Weyl’s theorem.
Lemma 5.4. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and consider $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$ two operators such that $A$ and $B^*$ are a-isoloid. Consider $\rho: L(\mathcal{Y}, \mathcal{X}) \to L(\mathcal{Y}, \mathcal{X})$ the generalized derivation defined by $A$ and $B$ and let $\lambda \in E_0^a(\rho) \subseteq \text{iso } \sigma_a(\rho)$. If $\mu \in \text{iso } \sigma_a(A)$ and $\nu \in \text{iso } \sigma_a(B^*)$ are such that $\lambda = \mu - \nu$, then $\mu \in E_0^a(A)$ and $\nu \in E_0^a(B^*)$.

Proof. Note that according to Proposition 5.1, $\rho \in L(\mathcal{Y}, \mathcal{X})$ is a-isoloid. In particular, according to Proposition 3.3(iv), if $\lambda \in \text{iso } \sigma_a(\rho) = E_a(\rho)$, then there are $\mu \in \text{iso } \sigma_a(A) = E_a(A)$ and $\nu \in \text{iso } \sigma_a(B^*) = E_a(B^*)$ such that $\lambda = \mu - \nu$. Suppose that $\lambda \in E_0^a(\rho)$. To prove the Lemma, it is enough to show that $\dim N(A - \mu)$ and $\dim N(B^* - \nu)$ are finite dimensional.

If $\dim N(A - \mu)$ is not finite dimensional, then there is a sequence of linearly independent vectors $(x_n)_{n \in \mathbb{N}} \subseteq N(A - \mu)$ such that $\|x_n\| = 1$. Let $f \in N(B^* - \nu)$, $f \neq 0$ and consider the sequences of operators $(U_{x_n,f})_{n \in \mathbb{N}} \subseteq L(\mathcal{Y}, \mathcal{X})$ (see the proof of Theorem 3.4(ii)). Then, it is not difficult to prove that $(U_{x_n,f})_{n \in \mathbb{N}} \subseteq L(\mathcal{Y}, \mathcal{X})$ is a sequence of linearly independent operators such that $(U_{x_n,f})_{n \in \mathbb{N}} \subseteq N(\rho - \lambda)$, which is impossible for $\lambda \in E_0^a(\rho)$.

A similar argument proves that if $\dim N(B^* - \nu)$ is not finite dimensional, then $\lambda \notin E_0^a(\rho)$. \hfill $\blacksquare$

Next a-Weyl’s theorem will be considered.

Theorem 5.5. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and consider two operators $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$ such that $A$ and $B^*$ are a-isoloid and a-Weyl’s theorem holds for $A$ and $B^*$. If $\rho: L(\mathcal{Y}, \mathcal{X}) \to L(\mathcal{Y}, \mathcal{X})$ is the generalized derivation defined by $A$ and $B$, then the following statements are equivalent.

(i) a-Weyl’s theorem holds for $\rho \in L(\mathcal{Y}, \mathcal{X})$.

(ii) a-Browder’s theorem holds for $\rho \in L(\mathcal{Y}, \mathcal{X})$.

Proof. According to [34] Corollary 2.2] and [34] Corollary 2.4], statement (i) implies statement (ii).

On the other hand, since $\Pi_0^0(\rho) \subseteq E_0^a(\rho)$, if statement (ii) holds, then to prove statement (i), it is enough to show that $E_0^a(\rho) \subseteq \Pi_0^0(\rho)$. Let $\lambda \in E_0^a(\rho) \subseteq \text{iso } \sigma_a(\rho)$. Then, according to Proposition 3.3(iv), there exist $\mu \in \text{iso } \sigma_a(A)$ and $\nu \in \text{iso } \sigma_a(B^*)$ such that $\lambda = \mu - \nu$. In particular, according to Lemma 5.4 $\mu \in E_0^a(A) = \sigma_a(A) \setminus \text{aw}(A)$ and $\nu \in E_0^a(B^*) = \sigma_a(B^*) \setminus \text{aw}(B^*)$. Consequently, according to Theorem 4.2(iii), $\lambda \notin \sigma_{aw}(\rho) = \sigma_{ab}(\rho)$. Hence, $\lambda \in \Pi_0^0(\rho)$ (34 Corollary 2.2]). \hfill $\blacksquare$

In the following theorem generalized a-Weyl’s theorem for generalized derivations will be studied.

Theorem 5.6. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and consider two operators $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$ such that $A$ and $B^*$ are a-isoloid and generalized a-Weyl’s theorem holds for $A$ and $B^*$. If $\rho: L(\mathcal{Y}, \mathcal{X}) \to L(\mathcal{Y}, \mathcal{X})$ is the generalized derivation defined by $A$ and $B$, then the following statements are equivalent.

(i) generalized a-Weyl’s theorem holds for $\rho \in L(\mathcal{Y}, \mathcal{X})$.

(ii) generalized a-Browder’s theorem holds for $\rho \in L(\mathcal{Y}, \mathcal{X})$ and $\sigma_{SBF^+}(\rho) = (\sigma_a(A) - \sigma_{SBF^+}(B^*)) \cup (\sigma_{SBF^-}(A) - \sigma_a(B^*))$.

Proof. Suppose that statement (i) holds. Then, generalized a-Browder’s theorem holds for $\rho$ (3.3 Corollary 3.3).

Let $\lambda \in \sigma_a(\rho) \setminus \sigma_{SBF^-}(\rho) = E_a(\rho) = \text{iso } \sigma_a(\rho)$. According to Proposition 3.3(iv), $\lambda = \mu - \nu$, where $\mu \in \text{iso } \sigma_a(A)$ and $\nu \in \text{iso } \sigma_a(B^*)$. However, since for every $\mu$ and $\nu$ such that $\lambda = \mu - \nu$, $\mu$ is $\text{iso } \sigma_a(A) = E_a(A) = \sigma_a(A) \setminus \sigma_{SBF^+}(A)$ and $\nu$ is $\text{iso } \sigma_a(B^*) = E_a(B^*) = \sigma_a(B^*) \setminus \sigma_{SBF^+}(B^*)$, $\lambda \in \sigma_a(\rho) \setminus ((\sigma_a(A) - \sigma_{SBF^+}(B^*)) \cup (\sigma_{SBF^-}(A) - \sigma_a(B^*)))$. Consequently, $(\sigma_a(A) - \sigma_{SBF^+}(B^*)) \cup (\sigma_{SBF^-}(A) - \sigma_a(B^*)) \subseteq \sigma_{SBF^-}(\rho)$. \hfill $\blacksquare$
Next suppose that $\lambda \in \sigma_a(\rho) \setminus ((\sigma_a(A) - \sigma_{SBF}^-(B^*)) \cup (\sigma_{SBF}^+(A) - \sigma_a(B^*) ))$. Note that according to [8, Lemma 2.12], acc $\sigma_a(A) \subseteq \sigma_{SBF}^-(A)$ and acc $\sigma_a(B^*) \subseteq \sigma_{SBF}^+(B^*)$. Thus, according to Proposition 3.3(iii), $\lambda \in \text{iso} \sigma_a(A) = \sigma_{SBF}^-(A)$ and acc $\sigma_a(\rho) = \sigma_a(\rho) \setminus \sigma_{SBF}^+(\rho)$.

Hence, $\sigma_{SBF}^-(\rho) \subseteq (\sigma_a(A) - \sigma_{SBF}^-(B^*)) \cup (\sigma_{SBF}^+(A) - \sigma_a(B^*))$.

Suppose that statement (ii) holds. Since $\Pi^a(\rho) \subseteq E^a(\rho)$, according to [8, Corollary 3.2], it is enough to prove that $E^a(\rho) \subseteq \Pi^a(\rho)$. Let $\lambda \in E^a(\rho)$. Then, according to Proposition 3.3(iv), there exist $\mu \in \text{iso} \sigma_a(A)$ and $\nu \in \text{iso} \sigma_a(B^*)$ such that $\lambda = \mu - \nu$. However, as in the first paragraph of the proof, $\mu \in \sigma_a(A) \setminus \sigma_{SBF}^-(A)$ and $\nu \in \sigma_a(B^*) \setminus \sigma_{SBF}^+(B^*)$. Therefore, $\lambda \in \sigma_a(\rho) \setminus \sigma_{SBF}^+(\rho) = \Pi^a(\rho)$.

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