Thermodynamic properties of Rashba Spin-Orbit Coupled Fermi Gas

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We investigate the thermodynamic properties of a superfluid Fermi gas subject to Rashba spin-orbit coupling and effective Zeeman field. We adopt a T-matrix scheme that takes beyond-mean-field effects — which are important for strongly interacting systems — into account. We focus on the calculation of two important quantities: the superfluid transition temperature and the isothermal compressibility. Our calculation shows very distinct influences of the out-of-plane and the in-plane Zeeman field on the Fermi gas. We also confirm that the in-plane Zeeman field induces a Fulde-Ferrell superfluid below the critical temperature and an exotic finite-momentum pseudo-gap phase above the critical temperature.

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I. INTRODUCTION

Ultracold Fermi gas\textsuperscript{1}, with tunable atom-atom interaction through Feshbach resonance\textsuperscript{2}, has been an ideal platform for the study of the crossover physics from weak coupling Bardeen-Cooper-Schrieffer (BCS) pairing to a Bose-Einstein condensate (BEC) of bound pairs\textsuperscript{3–5}. Recently synthetic gauge fields\textsuperscript{6} and spin-orbit (SO) coupling\textsuperscript{7–11} were realized in experiments, opening up a completely new avenue of research in superfluid Fermi gas. The interplay of Zeeman fields and SO coupling leads to many novel phenomena at both zero\textsuperscript{12–17} and finite temperatures\textsuperscript{18,19}, from mixed singlet-triplet pairing to topological phase transition\textsuperscript{12,20,24}.

Many of the interesting physics of SO coupled Fermi gas can be captured by mean-field theory. However, it is also true that mean-field theory may fail under many circumstances. For example, mean-field theory, which does not include the effect of non-condensed pairs, fails to describe accurately the phase transition from a superfluid to a normal gas, particularly for systems with strong interaction. Beyond-mean-field theoretical methods have been proposed\textsuperscript{27,31}. Here we present a theoretical investigation under the framework of the T-matrix scheme to address the superfluid properties of a Rashba SO coupled Fermi gas over the entire BCS-BEC crossover regime\textsuperscript{32–40}. In the absence of the Zeeman field, it was shown that the SO coupling enhances superfluid pairing\textsuperscript{16,40–42}. In the mean-field level, we know that the presence of both the SO coupling and a perpendicular (out-of-plane) Zeeman field give rise to effective p-wave pairing\textsuperscript{43,44}. Meanwhile introducing an in-plane Zeeman component creates an anisotropic Fermi surface which favors finite-momentum pairing, giving rise to Fulde-Ferrell superfluid\textsuperscript{21,22}. These previous studies motivated our present work, in which we investigate the thermodynamic properties of an SO coupled Fermi gas subject to a Zeeman field. We will focus our calculation on two important quantities that are measurable in experiment: the superfluid to normal transition temperature and the isothermal compressibility.

For this goal we organize this paper as follows. In Sec. II we give an introduction to the Rashba SO coupled model and its modifications to the conventional mean-field theory at finite temperature. Then in Sec. III we briefly review the zero-temperature properties of the system. The superfluid transition temperature $T_c$ in the BEC-BCS crossover is investigated in Sec. IV and its dependence on the Zeeman field is emphasized. We present the numerical results of compressibility in Sec. V before we provide a summary in Sec. VI. We show that both the superfluid transition temperature and the compressibility have distinct dependence on the out-of-plane and the in-plane Zeeman field. In the Appendix we present technical details of the T-matrix formalism used in the calculation.

II. MODEL

We consider a three-dimensional two-component degenerate Fermi gas with Rashba SO coupling together with effective Zeeman fields. This system can be described by the following Hamiltonian:

$$\mathcal{H} = \int d\mathbf{r} \psi^\dagger (\mathcal{H}_0 + \mathcal{H}_s + h_z \sigma_z + h_x \sigma_x) \psi (\mathbf{r}) + U \int d\mathbf{r} \psi_\uparrow (\mathbf{r}) \psi_\downarrow (\mathbf{r}) \psi_\uparrow (\mathbf{r}) \psi_\downarrow (\mathbf{r}),$$

where $\psi (\mathbf{r}) = (\psi_\uparrow, \psi_\downarrow)^T$ represents the fermionic field operator, $\mathcal{H}_0 = -\nabla^2 / (2m) - \mu$ is the single-particle Hamiltonian with $\mu$ being the chemical potential. The Rashba
SO coupling term takes the form $H_{so} = \alpha (k_x \sigma_x - k_y \sigma_y)$ in the $xy$-plane, with the parameter $\alpha$ characterizing the strength of the SO coupling. We consider both an out-of-plane Zeeman field $h_z$ and an in-plane Zeeman field $h_x$. The quantity $U$ represents the bare two-body interaction constant and in the calculation will be replaced by the $s$-wave scattering length $a_s$ through the standard regularization scheme: $1/U = -m/(4\pi \hbar^2 a_s) + \sum_k m/k^2$.

In the mean-field BCS theory, we introduce an order parameter $\Delta = U \sum_k \langle q_{k \uparrow} \psi_{k,-} \rangle$ to characterize the property of the superfluid. Here $Q$ represents the center-of-mass momentum of the pairs. A finite $Q$ arises from the presence of the in-plane Zeeman field $\hat{h}_{\|}$. At finite temperature in a T-matrix scheme, the order parameter $\Delta(T)$ can be divided into two parts: $\Delta^2 = \Delta_{sc}^2 + \Delta_{pg}^2$. Here $\Delta_{sc}$ is the superfluid gap arising from the condensed pairs and vanishes above the superfluid transition temperature $T_c$. The pseudo-gap $\Delta_{pg}^2 \sim (\Delta^2(T)) - (\Delta(T))^2$ describes the thermodynamic fluctuation of non-condensed pairs. Below the superfluid transition temperature $T_c$, the thermodynamic quantities are determined by the Thouless criterion within the T-matrix scheme instead of the BCS formalism in the mean-field model. This is because at finite temperature thermodynamic fluctuation plays a critical role with a tendency towards destroying the pairing condensation. The Thouless criterion for finite pairing momentum $Q$ takes the form:

$$U^{-1} + \chi(0,Q) = 0,$$

where $\chi(0,Q)$ is the spin symmetrized pair susceptibility. More technical details are given in the two Appendices for the general formalism and Rashba SO coupled case.

III. ZERO-TEMPERATURE PROPERTIES

We first briefly review the main features of the system at zero temperature. Note that the pseudo-gap tends to zero in the low temperature limit as $\Delta_{pg} \sim T^{3/4}$ [see Eq. (20)] and vanish at $T = 0$. Hence, the T-matrix scheme we adopted here reduces to the mean-field theory at zero temperature. The Rashba SO couplings and the Zeeman field have opposite effects on the magnitude of the gap parameter: the former tends to enhance the gap, while the latter tends to reduce the gap. Fig. 2(a) displays the gap parameter as a function of interaction strength for several different Zeeman fields. The in-plane Zeeman field is known to break the symmetry of the band structure and results in Cooper pairs with finite momentum, a signature of the Fulde-Ferrell superfluid.

In Fig. 3(a), we show how the magnitude of the momentum of Cooper pairs $Q = |Q|$ (where $Q$ is along the direction of the in-plane Zeeman field, i.e., along the $x$-axis) changes in the BEC-BCS crossover. $Q$ decreases quickly on the BEC side of the resonance as the two-body $s$-wave interaction, which favors zero-momentum pairing, dominates in that regime.

Another important feature induced by the Zeeman field is that it closes the bulk quasi-particle excitation gap at a critical value. Fig. 3(a) represents a phase diagram where we plot the critical Zeeman field at which the system changes from gapped to gapless. The region below each line represents the gapped phase. As the interaction strength varies from the BCS side to the BEC side, the order parameter increases [see Fig. 1(a)], and correspondingly the critical field strength increases and the gapped region enlarges. In the absence of the

![FIG. 1: (Color online) Thermodynamic quantities at zero temperature, (a) order parameter, and (b) pairing momentum, as functions of interaction strength characterized by $1/a_s K_F$ for different $\theta_h$. Here the SO coupling strength is $\alpha K_F = 2.0 E_F$, and the effective Zeeman field strength is $h = 0.5 E_F$. $\theta_h$ is the angle between the effective Zeeman field and the $z$-axis, such that $h_z = h \cos \theta_h$ and $h_x = h \sin \theta_h$. Therefore $\theta_h = 0$ represents a pure out-of-plane Zeeman field, while $\theta_h = \pi/2$ represents a pure in-plane Zeeman field. $K_F$ and $E_F$ are Fermi wavenumber and Fermi energy of the ideal Fermi gas, respectively.](image)

![FIG. 2: (Color online) Superfluid gap at $T = 0$ as functions of the Zeeman field strength for $1/a_s K_F = -0.5$ (a), 0 (b) and 0.5 (c). The solid, dashed, and dot-dashed lines correspond to $\theta_h = 0$, $\pi/4$, and $\pi/2$. The SO coupling strength is $\alpha K_F = 2.0 E_F$. The vertical arrows indicate the critical Zeeman field strength at which the system becomes gapless.](image)
plane Zeeman field (i.e., when $h_z = 0$), the system becomes gapless due to the presence of the discrete Fermi points located along the $k_z$-axis in momentum space at $k_z = \pm \sqrt{2m(\mu \pm \sqrt{h^2 - \Delta^2})}$. These Fermi points are topological defects. Hence the transition from the gapped to the gapless region in this case also represents a transition from a topologically trivial to a topologically non-trivial quantum phase. For finite $h_z$, by contrast, the gapless region features a nodal surface in momentum space on which the single particle excitation gap vanishes [17, 18, 52]. An example of such a nodal surface is plotted in Fig. 3(b). The vertical arrows in Fig. 2 indicate the critical Zeeman field strength at which the system becomes gapless. One can see that the sharp drop for $h_z = 0.0$, $h_z = 1.0E_F$, $\alpha K_F = 2.0E_F$ and $1/\alpha s K_F = 0.0$. The range of the plot is from $-2K_F$ to $2K_F$ along each direction.

IV. SUPERFLUID TRANSITION TEMPERATURE

We now turn our attention to the finite-temperature properties of the system. The first quantity we want to address is the superfluid transition temperature $T_c$, which is identified as the lowest temperature at which the superfluid gap $\Delta_{sc}$ vanishes. Fig. 4(a) shows an example how the gap varies as the temperature increases. As temperature increases, the superfluid gap $\Delta_{sc}$ decreases monotonically and vanishes at $T_c$. In contrast, the pseudo-gap $\Delta_{pg}$ is a monotonically increasing function $\Delta_{pg} \sim T^{3/4}$ below $T_c$ and decreases above $T_c$. On the other hand, Fig. 4(b) shows that the center-of-mass momentum $Q$ of the Cooper pairs is quite insensitive to the temperature for $T < T_c$, and slowly decreases for $T > T_c$. This indicates that the in-plane Zeeman field not only induces a Fulde-Ferrell superfluid below $T_c$, but also induces an exotic normal state above $T_c$ featuring finite-momentum pseudo-gap [26].

![Figure 3](image1.png)

**FIG. 3:** (Color online) (a) The critical magnetic field that separates the gapped region and the gapless region. The region below the lines is gapped. Here the SO coupling strength is $\alpha K_F = 2.0E_F$. (b) The nodal surface in the gapless region for $h_z = 0.0$, $h_z = 1.0E_F$, $\alpha K_F = 2.0E_F$ and $1/\alpha s K_F = 0.0$. The range of the plot is from $-2K_F$ to $2K_F$ along each direction.

![Figure 4](image2.png)

**FIG. 4:** (Color online) (a) The total gap ($\Delta$), pseudo-gap ($\Delta_{pg}$) and superfluid gap ($\Delta_{sc}$) as functions of the temperature. (b) The corresponding pairing momentum $Q$ as a function of the temperature. The parameters used are $1/\alpha s K_F = 0.0$, $\alpha K_F = 2.0E_F$, $h_z = 0.0$, and $h_z = 0.5E_F$. In this example, the green vertical dashed line indicates the critical temperature $T_c = 0.210E_F$.

![Figure 5](image3.png)

**FIG. 5:** Critical temperature $T_c$ in the BEC-BCS crossover for (a) different spin-orbit couplings (the values of the SO coupling strength $\alpha$, in units of $E_F/K_F$, are shown in figure legends) with no Zeeman field; and for (b) different ($h$, $\theta_h$) with $\alpha K_F = 2.0E_F$. $h$ is in units of $E_F$.

In Fig. 5 we show the superfluid transition temperature $T_c$ in the BEC-BCS crossover. In Panel (a), we plot $T_c$ as a function of the interaction strength for several different values of the SO coupling strength without the Zeeman field. For comparison, the mean-field result is also included. For weak interaction, where $1/(\alpha s K_F)$ is large and negative, the mean-field result agrees with the T-matrix result. As the interaction increases towards the BEC limit, the mean-field theory predicts an unphysically large critical temperature, a clear indication of its invalidity. In the BEC limit where the two-body attractive interaction is dominant, the system behaves like a condensation of weakly interacting tightly bound bosonic dimers with effective mass $2m$, regardless of the SO coupling strength [49]. The BEC transition temperature for this system is 0.218$E_F$ [28, 30].

In Fig. 5(b), we examine how the Zeeman field affects $T_c$. In the BEC limit, we have again $T_c = 0.218E_F$, insensitive to either the SO coupling strength or the Zeeman field strength [48]. On the BCS side, however, the Zeeman field tends to decrease $T_c$. This effect is much more pronounced with the in-plane Zeeman field than...
with the out-of-plane Zeeman field, indicating that the finite-momentum Fulde-Ferrell pairing is less robust than the zero-momentum Cooper pairing.

FIG. 6: Critical temperature \( T_c \) as functions of the Zeeman field strength for \( 1/a_s K_F = -0.5 \) (a), 0 (b) and 0.5 (c). The solid, dashed, and dot-dashed lines correspond to \( \theta_h = 0, \pi/4, \) and \( \pi/2 \). The SO coupling strength is \( \alpha K_F = 2.0 E_F \). The vertical arrows indicate the value of the Zeeman field strength at which the system becomes gapless.

To show the effect of the Zeeman field on the transition temperature more clearly, we plot in Fig. 6 \( T_c \) as functions of the Zeeman field strength at different orientations. Across the BEC-BCS crossover, \( T_c \) is not very sensitive to the out-of-plane Zeeman field over a large range of Zeeman field strength. When the in-plane Zeeman field is present, \( T_c \) starts to drop rather sharply near the critical Zeeman field strength where the system becomes gapless. This steep down turn of \( T_c \) is particularly pronounced on the BEC side of the resonance. We attribute this steep drop of \( T_c \) and the corresponding steep drop of zero-temperature superfluid gap (see Fig. 2) to the enhanced fluctuation as a result of the emergence of the nodal surface induced by a large in-plane Zeeman field.

V. ISOThermal COMPRESSIBILITY

The isothermal compressibility of system, defined as \( \kappa = \frac{1}{n} \frac{\partial n}{\partial P} \), measures the relative changes of the density \( n \) in response to the change of the pressure \( P \). Recent experiment has measured the compressibility of a Fermi gas across the superfluid phase transition [54, 55], which is found to be in good agreement with theory [55]. In the superfluid regime, it is found that \( \kappa \) increases with temperature, i.e., the gas becomes more compressible as temperature increases. This can be intuitively understood as a lower temperature yields a larger superfluid gap, hence a gas less likely to be compressed. Here we want to examine how SO coupling affects the behavior of \( \kappa \).

For our system, \( \kappa \) can be expressed as [41, 55, 56]

\[
\kappa = \frac{1}{N^2} \left[ \left( \frac{\partial N}{\partial \mu} \right)_{T,\Delta,Q} + \left( \frac{\partial N}{\partial \Delta} \right)_{T,\mu,Q} \left( \frac{\partial \Delta}{\partial \mu} \right)_{T,Q} \right. \\
+ \left. \left( \frac{\partial N}{\partial Q} \right)_{T,\mu,\Delta} \left( \frac{\partial Q}{\partial \mu} \right)_{T,\Delta} \right],
\]

(3)

where

\[
\frac{\partial \Delta}{\partial \mu} = \left( \frac{\partial N}{\partial \Delta} \right)_{T,\mu,Q} \left( \frac{\partial^2 \Omega}{\partial \Delta^2} \right)_{T,\mu,Q},
\]

\[
\frac{\partial Q}{\partial \mu} = \left( \frac{\partial N}{\partial Q} \right)_{T,\mu,\Delta} \left( \frac{\partial^2 \Omega}{\partial Q^2} \right)_{T,\mu,\Delta}.
\]

In Fig. 7 we show the compressibility \( \kappa \) as a function of temperature \( T \) in the superfluid regime on both sides of the Feshbach resonance. On the left panels, we set the in-plane Zeeman field \( h_z = 0 \). Across the Feshbach resonance, we see that regardless of the strength of the out-of-plane Zeeman field \( h_z \), \( \kappa \) increases smoothly as the temperature increases from 0 to \( T_c \), just like in a superfluid Fermi gas without SO coupling [51, 53]. By contrast, the presence of the in-plane Zeeman field gives...
rise to some surprising effect. On the right panels of Fig. 7 we set \( h_z = 0 \). One can see that for small \( h_x \) such that the system is gapped, \( \kappa \) is a monotonically increasing function of \( T \). However, once \( h_x \) exceeds the critical value such that the system becomes gapless, \( \kappa \) becomes a non-monotonic function of temperature. In certain regimes, \( \kappa \) even decreases as \( T \) increases.

![FIG. 8: Compressibility at \( T = 0 \) as functions of the Zeeman field strength. Dashed line corresponds to the in-plane Zeeman field (\( \theta_z = \pi/2 \)), and solid line to the out-of-plane Zeeman field (\( \theta_z = 0 \)). The dashed and solid arrows indicate the critical field strength at which the system becomes gapless in the presence of an in-plane and an out-of-plane Zeeman field, respectively. The SO coupling strength is \( \alpha K_F = 2.0 E_F \), and the scatter length is set at \( 1/K_F a_s = 0 \).](Image 100x523 to 253x649)

The drastically different effects on compressibility by the in-plane and the out-of-plane Zeeman field can also be seen in Fig. 8, where we plot \( \kappa \) at unitarity limit at zero temperature as functions of the Zeeman field strength. For out-of-plane Zeeman field, \( \kappa \) decreases smoothly as the field strength increases. In particular, it does not exhibit any strange behavior when the system changes from gapped to gapless. This is consistent with the fact that the gapped to gapless transition in the presence of a pure out-of-plane Zeeman field represents a topological phase transition which does not leave its trace in thermodynamic quantities. On the other hand, when the Zeeman field is in-plane with its strength increasing from zero, \( \kappa \) first decreases and then arises sharply near the critical field when the system becomes gapless. Hence the in-plane Zeeman field induced gapless superfluid, featuring Fermi nodal surface, is highly compressible.

The nodal surface resulting from the in-plane Zeeman field is not unique to Rashba SO coupling. For example, in the experimentally realized equal-weight Rashba-Dresselhaus coupling, a large in-plane Zeeman field can also induce a nodal surface \([57, 58]\). We have checked that for that system, the Zeeman-field dependence of the compressibility exhibits very similar behavior. As it is known, using the standard form of the fluctuation-dissipation theorem for a balanced system, the isothermal compressibility can be rewritten as \( \kappa = \frac{\hbar}{2K_F} \left( \frac{N^2 - N_z^2}{N^2} \right) \) \([59]\), i.e., \( \kappa \) is directly proportional to number fluctuation of the system. The increase of \( \kappa \) can therefore be interpreted as a consequence of enhanced number fluctuation induced by the nodal surface.

VI. SUMMARY

In summary, we have investigated the effect of the pairing fluctuation on the thermodynamic properties of a Rashba SO coupled superfluid Fermi gas by adopting a T-matrix scheme. We focus on the effect of the Zeeman field. In particular, the in-plane Zeeman component leads to finite-momentum Cooper pairing and, when its magnitude becomes sufficiently large, induces a nodal Fermi surface on which the quasi-particle excitation gap vanishes. The presence of the nodal surface has dramatic effects on the superfluid properties: it greatly suppresses the superfluid transition temperature and increases the isothermal compressibility. Both phenomena can be attributed to the enhanced fluctuation due to the presence of the nodal surface. By stark contrast, when only the out-of-plane Zeeman field is present, both \( T_c \) and \( \kappa \) exhibit smooth behavior when the Zeeman field strength is increased, even when the system becomes gapless at large Zeeman field strength. The key difference here is that a large out-of-plane Zeeman field, unlike its in-plane counterpart, does not give rise to a nodal surface, but only discrete Fermi points.

In the BEC limit, the system behaves like a condensate of bosonic molecules. Not only the SO coupling but also the Zeeman field take little influence on the tightly bound molecules, hence the phase transition temperature is the constant Bose condensation temperature \( T_c = 0.218 E_F \). In the BCS region, the transition temperature from a superfluid state to a normal one depends upon both the strength and the orientation of the Zeeman field. With the presence of a large out-of-plane Zeeman field, the quasiparticle spectrum of the system become gapless at discrete points along the \( k_z \)-axille. Those Fermi points are topological defects and their appearance does not leave their trace on thermodynamic quantities. When introducing large in-plane Zeeman component, the quasiparticle excitation vanishes on a surface in momentum space. The appearance of such nodal surfaces induce large number fluctuations, and as a consequence, dramatically lower the superfluid transition temperature and increase the compressibility. We also find an unconventional pseudo-gap state above the superfluid transition temperature, in which the non-condensed pairs possess non-zero center-of-mass momentum. We attribute it to the anisotropic Fermi surface induced by the in-plane Zeeman field, which is independent of temperature.

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**Appendix A: General Formalism**

Introducing the Nambu-Gorkov spinor \( \Psi = (\psi_\uparrow, \psi_\downarrow, \psi_\uparrow^\dagger, \psi_\downarrow^\dagger)^T \), the Green’s function is given by

\[
\mathcal{G}(K) = \begin{pmatrix} G(K) & F(K) \\ F(K) & G(K) \end{pmatrix} \quad (A1)
\]

where \( K \) is the four-dimensional vector and \( K = (i\omega_n, k) \) \((i\omega_n = (2n + 1)\pi T)\). The relation of the four Green’s functions is given by

\[
\mathcal{G}(i\omega_n, k) = -\left(G(-i\omega_n, -k)\right)^T, \quad (A2)
\]

\[
\mathcal{F}(i\omega_n, k) = \left(F(-i\omega_n, k)\right)^\dagger. \quad (A3)
\]

\[
\begin{align*}
t_{pg} & = \frac{1}{U} + \frac{G_0}{U} t_{pg} \\
\sum & = \frac{G_0}{U} + t_{pg}
\end{align*}
\]

**FIG. 9:** Feynman diagrams in a T-matrix scheme.

In a T-matrix scheme, the pair propagator consists in two parts,

\[
t(Q') = t_{sc}(Q') + t_{pg}(Q'). \quad (A4)
\]

The superfluid condensate part is

\[
t_{sc}(Q') = -\Delta_{sc}^2 \delta(Q' - Q), \quad (A5)
\]

in which \( Q = (0, Q) \) and \( Q \) is the pairing momentum of superfluid. The pseudo-gap part is

\[
t_{pg}^{-1}(Q') = U^{-1} + \chi(Q'), \quad (A6)
\]

where \( \chi(Q') \) is the pair susceptibility in \( G_0G \) formalism \([40]\).

\[
\chi(Q') = \frac{1}{2\beta} \text{Tr} \sum_K G(K) i\sigma_y \tilde{G}_0(K - Q') i\sigma_y. \quad (A7)
\]

At \( T \lesssim T_c \), the vanished chemical potential implies \([51]\)

\[
t_{pg}^{-1}(Q) = U^{-1} + \chi(Q) = 0. \quad (A8)
\]

It means that \( t_{pg}(Q') \) behaves a Dirac \( \delta \) function \([33, 40]\),

\[
t_{pg}(Q') = -\Delta_{sc}^2 \delta(Q' - Q). \quad (A9)
\]

The self-energy is given by

\[
\Sigma(K, Q) = \frac{1}{\beta} \sum_{Q'} t(Q') i\sigma_y \tilde{G}_0(K - Q') i\sigma_y
\]

\[
= -\left(\Delta_{sc}^2 + \Delta_{pg}^2\right) i\sigma_y \tilde{G}_0(K - Q) i\sigma_y
\]

\[
= -\Delta_{pg}^2 i\sigma_y \tilde{G}_0(K - Q) i\sigma_y \quad (A10)
\]

According to the Dyson’s equation in the Nambu-Gorkov spinor \( \Psi = \left(\psi_\uparrow(Q/2 + K), \psi_\downarrow(Q/2 + K), \psi_\uparrow^\dagger(Q/2 - K), \psi_\downarrow^\dagger(Q/2 - K)\right)^T \), the full Green’s function is

\[
\mathcal{G}^{-1}(K, Q) = \mathcal{G}_0^{-1}(K, Q) - \Sigma(K, Q), \quad (A11)
\]

so that thermodynamic potential is given by

\[
\Omega = -\frac{\Delta_{pg}^2}{U} - \frac{1}{2\beta} \sum_K \text{Tr} \ln \left(-\beta \mathcal{G}^{-1}(K, Q)\right) + \sum_k \xi_{Q/2 - k}. \quad (A12)
\]

Thermodynamic quantities are obtained by self-consistent solving the Thouless criterion,

\[
U^{-1} + \chi(0, Q) = 0, \quad (A13)
\]

the number density conversation,

\[
n = \sum_{\sigma} \psi_\sigma^\dagger(k) \psi_\sigma(k), \quad (A14)
\]

and the saddle point of paring susceptibility \( \chi(Q) \) \([53]\),

\[
\frac{\partial}{\partial Q'} \chi(0, Q') \bigg|_{Q'=Q} = 0. \quad (A15)
\]

In \( G_0G \) formalism, the above equations share the same forms with the gap, number, pairing momentum equations in the conventional mean-field method,

\[
\frac{\partial \Omega}{\partial \Delta} = 0, \quad \frac{\partial \Omega}{\partial \mu} = n, \quad \frac{\partial \Omega}{\partial Q} = 0. \quad (A16)
\]

Hence we can solve Eq. \([A10]\) instead to determine \( \Delta, \mu \) and \( Q \).

From Eq. \([A6]\) and Eq. \([A8]\), we can expand \( \chi(Q') \) at \( Q = (0, Q) \)

\[
\chi(Q') = \chi(Q) + Z \left( Q_0' - \sum_{i=1}^3 \frac{(Q_i' - Q_i)^2}{2m_i} \right), \quad (A17)
\]

where \( Q_0' = i\omega_n = 2n\pi T \) is the bosonic Matsubara frequency,

\[
Z = \frac{\partial \chi(Q')}{\partial Q_0'} \bigg|_{Q'=Q} \quad \text{and} \quad \frac{Z}{2m_i} = -\frac{1}{2} \frac{\partial^2 \chi(Q')}{\partial Q_i'^2} \bigg|_{Q'=Q}. \quad (A18)
\]
Then $t_{pg}(Q')$ is given by

$$t_{pg}(Q')^{-1} = Z \left( Q'_0 - \frac{1}{2} \sum_{i=1}^{\beta} (Q'_i - Q_i)^2 \right). \quad (A19)$$

Substituting Eq. (A19) into Eq. (A9), we obtain

$$\Delta_{pg}^2 = \frac{1}{\beta} \sum_{Q'_q} \Delta_{pg}^2 (Q' - Q) = \frac{1}{\beta} \sum_{Q'_q} t_{pg}(Q')$$

$$= \frac{1}{Z} \prod_{i=1}^{\beta} \left[ \sqrt{\frac{T_m}{2\pi}} \zeta \left( \frac{3}{2} \right) \right], \quad (A20)$$

where $\zeta(3/2) = 2.6123753 \ldots$ and $n_B(E) = 1/\exp(\beta E - 1)$ is the Bose-Einstein distribution. Self-consistently solving Eq. (A16) and Eq. (A20), we can obtain $\Delta_{sc}, \Delta_{pg}, \mu$ and $Q$ at $T \leq T_c$.

### Appendix B: Rashba SO coupled Case

In the Nambu-Gorkov spinor

$$\Psi = \begin{pmatrix} \psi_+(Q/2 + K) \\
\psi_-(Q/2 + K) \\
\psi_+(Q/2 - K) \\
\psi_-(Q/2 - K) \end{pmatrix},$$

the Green’s function reads

$$G^{-1}(K, Q) = \begin{pmatrix} G_0^{-1}(Q/2 + K) & -i\sigma_y \Delta \\
-i\sigma_y \Delta & [G_0^{-1}(Q/2 - K)]^T \end{pmatrix}. \quad (B1)$$

$G_0(K)$ is the bare Green’s function

$$G_0^{-1}(K) = i\omega_n - \xi_k - h_{so}(k). \quad (B2)$$

$i\omega_n = (2n + 1)\pi T$ is the fermionic Matsubara frequency. $h_{so}(k)$ is given by the SO coupling and Zeeman field items that contain Pauli matrices (i.e. $h_{so}(k)$ is no trace),

$$h_{so}(k) = \mathcal{H}_{so} + h_x \sigma_z + h_x \sigma_x = (\alpha k_x + h_x) \sigma_x - \alpha k_y \sigma_y + h_x \sigma_z. \quad (B3)$$

From the Green’s function (B1), we define $G^{-1} = i\omega_n I_{4 \times 4} - H_{BdG}$. As $H_{BdG}$ is hermit, we can find a unitary matrix $U$ to diagonalize it,

$$U^\dagger H_{BdG} U = \text{diag}(E_1, E_2, E_3, E_4). \quad (B4)$$

Hence

$$G = U \text{diag}(i\omega_n - E_1, \ldots)^{-1} U^\dagger. \quad (B5)$$

Substituted into (A17) and then making Matsubara sum, we can get the numeric results of the pair susceptibility $\chi(Q)$.

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