An $\mathcal{N}=8$ superconformal particle in the half-plane

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Abstract

By imposing global supersymmetry and scale invariance we construct an $\mathcal{N}=8$ superconformal mechanical system based on the inhomogeneous $(2,8,6)$ linear multiplet. The unique action describes a special Kähler sigma model with a Calogero-type potential and Fayet-Iliopoulos terms. The classical dynamics of the two propagating bosons is restricted to a (warped) half-plane and bounded. We numerically inspect typical trajectories of this special particle.
1 Introduction and summary

For classical mechanics (field theory in 0+1 dimensions) there exists a rich landscape of $\mathcal{N}=8$ supersymmetric models, distinguished by the number $b$ of propagating bosonic degrees of freedom and by the nature of the supersymmetry transformations (linear or non-linear) \([1, 2, 3]\). Restricting to the linear type, the notation $(b, \mathcal{N}, \mathcal{N}-b)$ counts their propagating bosonic, fermionic and auxiliary components. As was already observed in \([4, 5]\), an important role is played by a potential inhomogeneity in the supersymmetry transformation of the fermions. The parameters appearing there may be viewed as a constant shift of the auxiliary components and are introduced through the superfield constraints. Together with Fayet-Iliopoulos terms, they create a bosonic potential, lead to central charges and partial supersymmetry breaking.

To accommodate these inhomogeneous terms, we apply the techniques discussed in \([6]\) and \([7]\) and produce the most general inhomogeneous linear supermultiplets compatible with the ordinary supersymmetry algebra $\{Q_i, Q_j\} = \delta_{ij} H$ (without central extensions).

Here, we concentrate on the classical mechanics of a $(2,8,6)$ particle. The Lagrangian and Hamiltonian of this model has been formulated for a general prepotential $F$ in \([8]\) (without inhomogeneity) and in \([9]\) (with inhomogeneity). Here, we specialize to the conformal case and investigate the classical dynamics of the conformal $(2,8,6)$ particle.

The inhomogeneous $(2,8,6)$ $\mathcal{N}=8$ supermultiplet, under the requirement of scale-invariance for the action, defines a unique superconformal mechanical system. The only free parameters are the the scale-setting Fayet-Iliopoulos coupling and the dimensionless shift entering the inhomogeneous supersymmetry transformations.

We review the inhomogeneous supersymmetry transformations for $\mathcal{N}\leq8$ and rederive the invariant conformal action for the inhomogeneous $(2,8,6)$ multiplet including Fayet-Iliopoulos terms, without using superspace technology. After eliminating the auxiliary components we arrive at a very specific (non-isotropic and indefinite) Weyl factor and bosonic potential in the two-dimensional target space. It proves to be legitimate (at least classically) to restrict to a (positive-definite) half-space, where we present some typical particle trajectories.

The inhomogeneous supersymmetry transformations that we investigate here close the ordinary supersymmetry algebra without central extensions. This is the case because we work within the Lagrangian framework. Central extensions of the supersymmetry algebra can arise, both in the classical and quantum cases, as a consequence of the Hamiltonian formulation and the closure of the Noether-(super)charge algebra under the Poisson bracket structure \([4]\).

It is tempting to push the idea of this paper to even higher-extended supersymmetry. For example, by coupling two inhomogeneous $(2,8,6)$ multiplets linked by an extra, 9th, supersymmetry, one should be able to construct an $\mathcal{N}=9$ superconformal mechanics model with a four-dimensional target. This might be related with the standard reduction of $\mathcal{N}=4$ super Yang-Mills to an off-shell multiplet of type $(9,16,7)$ in one dimension.
2 Inhomogeneous minimal linear supermultiplets

Minimal linear supermultiplets of extended supersymmetry in one dimension are usually formulated with homogeneous transformations for their component fields. However, in some cases it is possible to extend the supersymmetry transformations by the addition of an inhomogeneous term. This is admissible at

- $\mathcal{N} = 2$ for the supermultiplet $(0, 2, 2)$
- $\mathcal{N} = 4$ for the supermultiplets $(0, 4, 4)$ and $(1, 4, 3)$
- $\mathcal{N} = 8$ for the supermultiplets $(0, 8, 8)$ and $(1, 8, 7)$ and $(2, 8, 6)$

The remaining $\mathcal{N} = 2, 4, 8$ supermultiplets do not admit an inhomogeneous extension, as can be easily verified by investigating the closure of the ordinary $\mathcal{N}$-extended supersymmetry algebra.

Let $x$ and $y$ be physical bosons, $\psi, \psi_i, \lambda$ and $\lambda_i$ denote fermions, and $g, g_i, f$ and $f_i$ describe auxiliary fields. Here, the isospin index $i$ runs over a range depending on the number of supersymmetries. The presence of an inhomogeneous term requires the following mass dimension for the fields:

$$[t] = -1 \quad \rightarrow \quad [x] = -1, \quad [\psi] = -\frac{1}{2}, \quad [g] = 0.$$  \hspace{1cm} (1)

In all the above cases, by a suitable $R$ transformation, the inhomogeneous terms can be rotated to point only in a specific iso-direction. We choose the one with the highest iso-index, i.e. $i = 2, 3$ or $7$, depending on the case. With this choice, let us list the various supersymmetry transformations $Q_i$ for the six cases listed above.

$(0,2,2)$. For the inhomogenous $\mathcal{N} = 2$ $(0, 2, 2)$ supermultiplet, the two supersymmetry transformations, without loss of generality, can be expressed as $(j, k = 1, 2, \epsilon_{12} = 1)$

$$Q_1 \psi_j = g_j, \quad Q_1 g_j = \dot{\psi}_j,$$
$$Q_2 \dot{\psi}_j = \epsilon_{jk} \tilde{g}_k, \quad Q_2 g_j = \epsilon_{jk} \dot{\psi}_k,$$  \hspace{1cm} (2)

where the inhomogeneous extension hides in

$$\tilde{g}_k := g_k + c_k \quad \text{with} \quad c_k \in \mathbb{R},$$  \hspace{1cm} (3)

and we rotate to $c_1 = 0, c_2 \equiv c > 0$.

$(0,4,4)$. For the $\mathcal{N} = 4$ $(0, 4, 4)$ multiplet, we have $(i, j, k = 1, 2, 3, \epsilon_{123} = 1)$

$$Q_0 \psi = g, \quad Q_0 \dot{\psi}_j = g_j, \quad Q_0 g = \dot{\psi}, \quad Q_0 \dot{g}_j = \dot{\psi}_j,$$
$$Q_i \psi = g_i, \quad Q_i \dot{\psi}_j = -\delta_{ij} g + \epsilon_{ijk} \tilde{g}_k, \quad Q_i g = -\dot{\psi}_i, \quad Q_i \dot{g}_j = \delta_{ij} \dot{\psi} - \epsilon_{ijk} \dot{\psi}_k,$$  \hspace{1cm} (4)

and we may choose

$$\tilde{g}_1 = g_1, \quad \tilde{g}_2 = g_2 \quad \text{but} \quad \tilde{g}_3 = g_3 + c.$$  \hspace{1cm} (5)
(1,4,3). The \( \mathcal{N}=4 \) (1, 4, 3) multiplet looks slightly different,

\[
\begin{align*}
Q_0 x &= \psi , & Q_0 \psi &= \dot{x} , & Q_0 \psi_j &= g_j , & Q_4 g_j &= \dot{\psi}_j , \\
Q_i x &= \psi_i , & Q_i \psi &= -g_i , & Q_i \psi_j &= \delta_{ij} \dot{x} + \epsilon_{ijk} \dot{g}_k , & Q_i g_j &= -\delta_{ij} \dot{\psi} - \epsilon_{ijk} \dot{\psi}_k ,
\end{align*}
\]

(6)

with the same \( \dot{g}_k \) as in (0,4,4).

(0,8,8). Without loss of generality, we can generate the \( \mathcal{N}=8 \) multiplets from the \( \mathcal{N}=4 \) ones by replacing the quaternionic structure constants \( \epsilon_{ijk} \) by the (totally antisymmetric) octonionic structure constants \( c_{ijk} \), with \( i, j, k = 1, \ldots , 7 \) and

\[
c_{123} = c_{147} = c_{165} = c_{246} = c_{257} = c_{354} = c_{367} = 1 ,
\]

(7)
together with \( c_{ijk} = 0 \) for all other index combinations. Therefore, the case of (0,0,8) yields

\[
\begin{align*}
Q_0 \psi &= g , & Q_0 \psi_j &= g_j , & Q_0 g &= \dot{\psi} , & Q_0 g_j &= \dot{\psi}_j , \\
Q_i \psi &= g_i , & Q_i \psi_j &= -\delta_{ij} g + c_{ijk} \dot{g}_k , & Q_i g &= -\dot{\psi}_i , & Q_i g_j &= \delta_{ij} \dot{\psi} - c_{ijk} \dot{\psi}_k ,
\end{align*}
\]

(8)

and we take

\[
\dot{g}_k = g_k + \delta_{k7} c .
\]

(9)

(1,8,7). In analogy with (1,4,3), we get

\[
\begin{align*}
Q_0 x &= \psi , & Q_0 \psi &= \dot{x} , & Q_0 \psi_j &= g_j , & Q_0 g &= \dot{\psi} , & Q_0 g_j &= \dot{\psi}_j , \\
Q_i x &= \psi_i , & Q_i \psi &= -g_i , & Q_i \psi_j &= \delta_{ij} \dot{x} + \epsilon_{ijk} \dot{g}_k , & Q_i g_j &= -\delta_{ij} \dot{\psi} - \epsilon_{ijk} \dot{\psi}_k ,
\end{align*}
\]

(10)

and again \( \dot{g}_k = g_k \) except for \( \dot{g}_7 = g_7 + c \) with \( c > 0 \).

(2,8,6). This is the most interesting multiplet. It is convenient to present it in quaternionic form, by fusing \((1,4,3) \oplus (1,4,3) = (2,8,6)\), with components labeled by \((x, \psi_i, g_{ij})\) and \((y, \lambda_i, f_{ij})\), respectively, where \( i = 1, 2, 3 \). It is convenient to present the supersymmetry transformations in the following table,

\[
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3 Invariant action for a (2,8,6) particle

To investigate the dynamics of superconformal particles on a line, based on the various inhomogeneous supermultiplets, we shall need to construct invariant actions for them. For $\mathcal{N} \geq 4$ and the presence of at least one physical boson, there exists a canonical method [7] to generate such actions, by setting

$$S = \int dt \mathcal{L} = \int dt \, Q_1 Q_2 Q_3 Q_4 F(x, y, \ldots), \quad (12)$$

where $F(x, y, \ldots)$ is an unconstrained prepotential. In order to obtain conformally invariant mechanics, the action should not contain any dimensionful coupling parameter, and therefore, due to $[Q_i] = \frac{1}{2}$, we demand that $[F] = -1$. One can prove that the ensuing scale invariance extends to full conformal invariance.

Without the inhomogeneous extension, (12) yields only a kinetic term with some metric. It is the inhomogeneous term which will give rise to a Calogero-type potential. The action may be complemented by the addition of a Fayet-Iliopoulos term

$$S_{FI} = \int dt \sum_i (q_i g_i + r_i f_i) \quad \text{with} \quad [g_i] = [r_i] = 1, \quad (13)$$

introducing dimensionful couplings compatible with conformal invariance. These Fayet-Iliopoulos terms produce an oscillatorial damping, via the DFF trick of conformal mechanics [10].

For the (1,4,3) multiplet (only $x$ and $g_i$, no $y$ or $f_i$), the proper choice for the prepotential is

$$F(x) = \frac{1}{4} x \ln x \quad \rightarrow \quad \mathcal{L} + \mathcal{L}_{FI} = F''(x) \left( \dot{x}^2 + g_i^2 + c g_3 \right) + q_i g_i \text{ fermions}. \quad (14)$$

After eliminating the auxiliary components $g_i$ via their equations of motion and putting the fermions to zero, one gets

$$\mathcal{L}_{bos}' = F''(x) \left( \dot{x}^2 - \frac{1}{4} c^2 \right) - \frac{1}{4} q_i^2 / F''(x) - \frac{1}{2} c g_3$$

$$= \frac{1}{4} \left( \dot{x}^2 - \frac{1}{4} c^2 \right) / x - g_i^2 x - \frac{1}{2} c g_3$$

$$= \frac{1}{2} w^2 - \frac{1}{8} c^2 w^{-2} - \frac{1}{2} g_i^2 w^2 - \frac{1}{2} c g_3, \quad (15)$$

and we have recovered the standard conformal action after the coordinate change $x = \frac{1}{2} w^2$.

Stepping up to $\mathcal{N}=8$, we change the iso-labelling to make $Q_0, Q_1, Q_2, Q_3$ manifest,

$$S = \int dt \mathcal{L} = \int dt \, Q_0 Q_1 Q_2 Q_3 F(x, y, \ldots). \quad (16)$$

Demanding invariance under the additional four supersymmetries by requiring

$$Q_l \mathcal{L} = \partial_t W_l \quad \text{for} \quad l = 4, 5, 6, 7 \quad (17)$$
imposes severe constraints on $F$. In fact, for the \((1,8,7)\) multiplet no action can be invariant under the inhomogeneous supersymmetry transformations.\footnote{In the homogeneous case the constraint reads $F''''(x) = 0$, which produces $L = (ax + b)x^2 + \ldots$.}

However, the situation is much more interesting for the \((2,8,6)\) multiplet. Here, the constraint (17) says that, like in the homogeneous case [11], the prepotential $F(x, y)$ must be harmonic,

$$\Box F \equiv F_{xx} + F_{yy} = 0 .$$

The general solution is encoded in a meromorphic function $H(z)$ via

$$F(x, y) = H(z) + \overline{H(z)} = 2 \mathrm{Re}H(z) ,$$

where it is convenient to pass to complex coordinates,

$$z = x + iy , \quad \partial_z = \frac{1}{2}(\partial_x - i\partial_y) , \quad h_i = g_i + if_i , \quad \chi(i) = \psi(i) + i\lambda(i)$$

$$\bar{z} = x - iy , \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) , \quad \bar{h}_i = g_i - if_i , \quad \bar{\chi}(i) = \psi(i) - i\lambda(i) .$$

Inserting (19) into (16), we obtain

$$\mathcal{L} = 2 \mathrm{Re}\{Hzz(\bar{z}\bar{z} + \bar{h}_ih_i + c\chi^3 + \frac{i}{2}\bar{\chi}\chi - \frac{1}{2}\bar{\chi}\chi - \frac{1}{2}\bar{\chi}\chi - \frac{1}{2}\bar{\chi}\chi) + H_{zzz}(\chi\bar{\chi}h_i + \frac{i}{2}e_{ijk}\chi_i\chi_jh_k + c\chi\chi^3) + \frac{i}{8}H_{zzzz}e_{ijk}\chi_i\chi_j\chi_k\} ,$$

where the inhomogeneous extension is clearly visible in the terms containing the parameter $c$. The bosonic metric $g_{zz} = H_{zz} + H_{\bar{z}\bar{z}}$ is special Kähler of rigid type [12]. Reverting to real notation and introducing the Weyl factors

$$\Phi = 2 \mathrm{Re}H_{zz} = \frac{1}{2}(F_{xx} - F_{yy}) \quad \text{ and } \quad \tilde{\Phi} = -2 \mathrm{Im}H_{zz} = F_{xy} ,$$

the Lagrangian reads

$$\mathcal{L} = \Phi(x^2 + y^2 + g_i^2 + f_i^2 - \psi\dot{\psi} - \lambda\dot{\lambda} - \psi_i\dot{\psi}_i - \lambda_i\dot{\lambda}_i) + \Phi_x(\psi\dot{\psi}g_i - \lambda\dot{\lambda}f_i - \lambda\dot{\lambda}g_i - \psi\dot{\psi}g_i - \epsilon_{ijk}(\frac{i}{2}g_i\psi_j\psi_k - \frac{1}{2}g_i\lambda_j\lambda_k - f_i\lambda_j\psi_k)) + \Phi_y(\lambda\dot{\lambda}g_i - \lambda\dot{\lambda}f_i + \psi\dot{\psi}f_i + \psi\dot{\psi}g_i + \epsilon_{ijk}(\frac{i}{2}f_i\psi_j\psi_k - \frac{1}{2}f_i\lambda_j\lambda_k + g_i\lambda_j\psi_k)) + \frac{1}{2}(\Phi_{xx} - \Phi_{yy})\epsilon_{ijk}(\frac{i}{2}\psi\dot{\psi}\psi_j\psi_k + \frac{1}{2}\lambda\dot{\lambda}\lambda_j\lambda_k - \frac{1}{2}\psi\dot{\psi}\psi_j\lambda_k - \frac{1}{2}\lambda\dot{\lambda}\psi_j\psi_k) + \Phi_{xy}\epsilon_{ijk}(\frac{1}{2}\lambda\dot{\lambda}\psi_j\psi_k - \frac{1}{2}\lambda\dot{\lambda}\psi_j\lambda_k + \frac{1}{2}\psi\dot{\psi}\psi_j\psi_k - \frac{1}{2}\psi\dot{\psi}\psi_j\lambda_k) + c(\Phi g_3 + \tilde{\Phi} f_3 + \Phi_x(\psi\dot{\psi} - \lambda\dot{\lambda}) + \Phi_y(\lambda\dot{\lambda} + \psi\dot{\psi})) ,$$

to which we add the Fayet-Iliopoulos terms

$$\mathcal{L}_{FI} = g_i g_i + r_i f_i .$$

The harmonic prepotential with the correct scaling dimension $[H] = -1$ is \footnote{Multiplying $H$ with a phase corresponds to an irrelevant rotation in the complex plane.}

$$H(z) = \frac{1}{8} z \ln z \quad \leftrightarrow \quad F(x, y) = \frac{1}{8} x \ln(x^2 + y^2) - \frac{1}{4} y \arctan \frac{y}{x} ,$$

$$\Box H \equiv H_{xx} + H_{yy} = 0 .$$
and the corresponding Weyl factors read

$$\Phi = \frac{1}{4} \text{Re} \frac{1}{z} = \frac{1}{4} \frac{x}{x^2 + y^2} \quad \text{and} \quad \Phi = -\frac{1}{4} \text{Im} \frac{1}{z} = \frac{1}{4} \frac{y}{x^2 + y^2}. \quad (26)$$

Note that the corresponding metric is an indefinite one, as it must be for any harmonic Weyl factor.

In the bosonic limit, obtained by setting all fermions equal to zero, we obtain

$$L_{\text{bos}} + L_{\text{FI}} = \Phi (\dot{x}^2 + \dot{y}^2 + g_i^2 + f_i^2) + c (\Phi g_3 + \Phi f_3) + q_i g_i + r_i f_i. \quad (27)$$

We eliminate the auxiliary fields via their algebraic equations of motion,

$$g_1 = -\frac{q_1}{\Phi}, \quad g_2 = -\frac{q_2}{\Phi}, \quad g_3 = -\frac{q_3 + c\Phi}{2\Phi}, \quad f_1 = -\frac{r_1}{\Phi}, \quad f_2 = -\frac{r_2}{\Phi}, \quad f_3 = -\frac{r_3 + c\Phi}{2\Phi}, \quad (28)$$

and arrive at

$$L'_{\text{bos}} = \Phi (\dot{x}^2 + \dot{y}^2) - \frac{1}{4\Phi} \left( q_1^2 + q_2^2 + (q_3 + c\Phi)^2 + r_1^2 + r_2^2 + (r_3 + c\Phi)^2 \right)$$

$$= \frac{x}{x^2 + y^2} \frac{\dot{x}^2 + \dot{y}^2}{4} - \frac{(q_1^2 + r_1^2)(x^2 + y^2)}{x} - c \frac{g_3 x + r_3 y}{2x} - \frac{c^2}{16x} \quad (29)$$

making explicit the effect of both the inhomogeneous supersymmetry transformation ($c$) and the Fayet-Iliopoulos terms ($q_i, r_i$) on the potential $V$.

It is tempting to perform the same coordinate change as for the $(1,4,3)$ multiplet, $x = \frac{1}{2} w^2$, which yields

$$L'_{\text{bos}} = \frac{1}{2} (1 + \gamma^2)^{-1} \left( \dot{w}^2 + \frac{\dot{y}^2}{w^2} \right) - \frac{1}{2} (1 + \gamma^2) (q_i^2 + r_i^2) w^2 - \frac{1}{2} c (q_3 + r_3 \gamma) - \frac{c^2}{8w^2}, \quad (30)$$

where $\gamma = 2y/w^2$. This form reveals both the oscillator and Calogero terms, but also shows the added complexity in two dimensions (mostly hidden in $\gamma$). Putting $y \equiv 0$ (also $\gamma=0$) brings back the $(1,4,3)$ result (15).

## 4 Trajectories of a $(2,8,6)$ particle

Without loss of generality, let us drop inessential Fayet-Iliopoulos terms and put

$$q_1 = q_2 = r_1 = r_2 = 0 \quad \text{and} \quad q_3 =: q, \quad r_3 =: r, \quad q + ir =: s. \quad (31)$$

In complex coordinates, the kinetic and potential energies then read

$$K = \Phi \dot{z} \dot{\bar{z}} = \frac{1}{8} \frac{z + \bar{z}}{z \bar{z}} \dot{z} \dot{\bar{z}}, \quad (32)$$

$$V = \left( (q + c\Phi)^2 + (r + c\Phi)^2 \right)/4\Phi = \frac{1}{8} \frac{1}{z + \bar{z}} \left( 4s\bar{z} + c \right) \left( 4\bar{s}z + c \right). \quad (33)$$
The level curves of this potential are circles of center and radius
\[ z_0(V) = \frac{2V - cs}{4(q^2 + r^2)} \quad \text{and} \quad r(V) = \frac{\sqrt{V(V - cq)}}{2(q^2 + r^2)}, \tag{34} \]
respectively, and its only minimum \( V_{\min} = cq \) is located at
\[ z_{\min} = z_0(cq) = \frac{c\bar{s}}{4(q^2 + r^2)}. \tag{35} \]
The parameter \( r \) governs the asymmetry under \( y \rightarrow -y \). The reflection \( x \rightarrow -x \) flips the sign of \( V - \frac{1}{2}cq \). Due to the factor of \( z + \bar{z} = 2x \), both the Weyl factor and the potential are strictly positive on the right half-space \( x > 0 \) and strictly negative for \( x < 0 \). Therefore, the \( (2,8,6) \) particle is a reasonable dynamical system only if its trajectories do not cross the \( x = 0 \) dividing line. Seen from the right half-space, the potential barrier for \( x \rightarrow 0 \) has a hole at \( y = 0 \) if \( c = 0 \), but the Weyl factor explodes precisely there. For large coordinate values, the potential grows linearly with \( x \) and quadratically with \( y \), so the \( x > 0 \) trajectories remain bounded.

The equation of motion takes the form
\[
0 = \Phi^3 \ddot{z} + \Phi^2 \Phi_z \dot{z}^2 - \frac{1}{4} \Phi_x (q^2 + (r + 2icH_{zz})^2) \\
\quad \propto (z + \bar{z})^3 z \ddot{z} - (z + \bar{z})^2 \dddot{z}^2 + z^2 \dddot{z}^2 ((4qz)^2 + (4rz + ic)^2), \tag{36}
\]
which in real coordinates reads
\[
0 = \ddot{x} - \frac{1}{2x} \frac{x^2 - y^2}{x^2 + y^2} (x^2 - y^2) - \frac{2y}{x^2 + y^2} \dot{x} \dot{y} + \frac{x^2 + y^2}{x^3} (2(q^2 + r^2)(x^2 - y^2) - cry - \frac{1}{2}c^2), \\
0 = \ddot{y} + \frac{y}{x^2 + y^2} (x^2 - y^2) - \frac{1}{x} \frac{x^2 - y^2}{x^2 + y^2} \dot{x} \dot{y} + \frac{x^2 + y^2}{x^3} (4(q^2 + r^2) x y + cr x). \tag{37}
\]
The only constant of motion of this system is the energy $E = T + V$, so the generic particle motion is not integrable. Figure 2 shows the trajectory for the $(c, q, r)$-value chosen in Figure 1 and a couple of initial conditions. One sees that the curve does not fill out the

region $V(x, y) \leq E$, on effect of the position-dependent effective mass $M = 2\Phi(x, y)$. It is also clear that the $x=0$ barrier is impenetrable. Therefore, it makes sense to substitute $w = \sqrt{2x}$ and introduce the dynamics in the $wy$-plane according to (30). The trajectories of Figure 2 get somewhat distorted in these variables, but their qualitative behavior is unchanged.

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