Discrete Gravitational Dimensions

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We study the physics of a single discrete gravitational extra dimension using the effective field theory for massive gravitons. We first consider a minimal discretization with 4D gravitons on the sites and nearest neighbor hopping terms. At the linear level, 5D continuum physics is recovered correctly, but at the non-linear level the theory becomes highly non-local in the discrete dimension. There is a peculiar UV/IR connection, where the scale of strong interactions at high energies is related to the radius of the dimension. These new effects formally vanish in the limit of zero lattice spacing, but do not do so quickly enough to reproduce the continuum physics consistently in an effective field theory up to the 5D Planck scale. Nevertheless, this model does make sense as an effective theory up to energies parametrically higher than the compactification scale. In order to have a discrete theory that appears local in the continuum limit, the lattice action must have interactions between distant sites. We speculate on the relevance of these observations to the construction of finite discrete theories of gravity in four dimensions.

In a recent paper [1], a technique was introduced for studying gravitational theories in discrete “theory spaces” [2, 3]. These spaces are defined by sites, with separate four-dimensional metrics, and their associated general coordinate invariances, and “link fields” that map between sites. In analogy with the Callan-Coleman-Wess-Zumino formalism for gauge theories, this technique allows us to understand effective field theories with multiple interacting spin two fields in a transparent way. In the simplest case [1], a single link is “eaten”, leading to a single massive graviton. The physics of massive gravitons is qualitatively different than that of massive gauge bosons, owing to the peculiar properties of the scalar longitudinal component of the graviton. Nevertheless, there is a sensible effective theory that makes sense to energies parametrically above the graviton mass. In this letter, we study what happens when we string together many sites and links to generate what looks like a gravitational extra dimension. Such models have been considered before [4, 5, 6, 7, 8, 9, 10], but not at the level of a consistent effective field theory. Given the peculiarities associated with massive gravity, we should expect surprises, and we indeed encounter a number of them.

We will begin by considering the minimal discretizations, with nearest neighbor interactions. The discretized dimension can be taken to be either a circle or an interval:

\[ S = S_{\text{site}} + S_{\text{link}} \] (1)

contains a part

\[ S_{\text{site}} = \sum_j \int d^4x M^2 \sqrt{g} R[g] \] (2)

which has \( N \) copies of general coordinate invariance (GC). We can see this by choosing a new dummy variable \( x_j \) for each integral in the above sum. The other part of \( S \) involves interactions between neighboring sites, and will produce mass terms of Fierz-Pauli form [4]:

\[ S_{\text{link}}^U = \sum_j \int d^4x \sqrt{g} M^2 m^2 (g_{\mu\nu}^j - g_{\mu\nu}^{j+1})(g_{\alpha\beta}^j - g_{\alpha\beta}^{j+1}) \times (g^{j\mu\nu} g^{j\alpha\beta} - g^{j\mu\alpha} g^{j\nu\beta}) \] (3)

The \( U \) in \( S_{\text{link}}^U \) stands for Unitary gauge. Indeed, we can see that a gauge is chosen because \( S_{\text{link}}^U \) breaks all but one copy of GC.

The action we have constructed is just a naive discretization of compactified 5D Einstein gravity. To see this, start with a 5D metric \( G_{MN}(x, z) \), with \( z \) the compact extra dimension. Ignoring the radion and graviphoton degrees of freedom (they do not affect the following discussion) it is possible to choose a gauge where the metric takes the form

[1] G_{MN} = \begin{pmatrix} g_{\mu\nu}(x, z) & 0 \\ 0 & 1 \end{pmatrix} \] (4)

In this gauge, the 5D action \( \int d^5x M_5^3 \sqrt{G} R[G] \) is

\[ S = \int d^4xdz \sqrt{M_5^3} \left( R_{4D}[g] + \frac{1}{4} \partial_z g_{\mu\nu}(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\rho\delta}) \partial_z g_{\alpha\beta} \right) \] (5)

Then, a naive discretization with lattice spacing \( a \) instructs us to replace

\[ \int dz \to a \sum_j \partial_z g_{\mu\nu} \to \frac{1}{a} (g_{\mu\nu}^j - g_{\mu\nu}^{j+1}) \] (6)
which directly produces our unitary gauge action \(3\). And we can read off that the radius \(R\), lattice spacing \(a\), and effective 5D Planck scale \(M_{5D}\) are given by:

\[
R = Na, \quad a = m^{-1}, \quad M_{5D} = M^2 m
\]  
(7)

Now, as explained in \(\textbf{1}\), we can restore the broken \(GC\) symmetries in \(\textbf{6}\) by adding link fields \(Y_j^µ(x)\) between the sites. It is useful to think of the link as a map from a point on site \(j\) with coordinate \(x_j^µ\) to a point on site \(j+1\) with coordinate \(Y_j^µ(x_j)\). Under general coordinate transformations generated by \(x_j\) to \(f_j(x_j)\), the link fields transform as:

\[
Y_j \rightarrow f_j^{-1}Y_jf_j
\]  
(8)

which allows us to compare fields on adjacent sites.

These link fields can be used to construct objects which transform as tensors under \(GC\) and are invariant under \(GC_{j+1}\) out of objects which are invariant under \(GC_j\) and tensors under \(GC_{j+1}\):

\[
G^{j+1}_\mu\nu(x_j) = \partial_\mu Y_j^\sigma \partial_\nu Y_j^\beta g^{\sigma\beta}(Y_j(x_j))
\]  
(9)

\(G^{j+1}_\mu\nu(x_j)\) can be thought of as a pull-back of the metric on site \(j+1\) to site \(j\) using the maps \(Y_j\). Thus, the link action becomes:

\[
S_{\text{link}} = \sum_j \int d^4x_j \sqrt{g} M^2 m^2 (g^{j}_\mu_\nu(x_j) - G^{j+1}_\mu\nu(x_j))
\]  
(10)

\[
\times (g^{j}_\alpha_\beta(x_j) - G^{j+1}_\alpha_\beta(x_j))(g^{j\mu_\nu}g^{j\alpha_\beta} - g^{j\mu\alpha}g^{j\nu_\beta})
\]

We can now replace the dummy variables \(x_j\) by a common set of coordinates \(x\). By construction this 4D action is explicitly invariant under \(N\) copies of \(GC\).

It is useful to expand the link fields about the identity as

\[
Y_j^\mu(x) = x^\mu + \pi^\mu_j(x)
\]  
(11)

The vector fields \(\pi^\mu\) are the Goldstone bosons that are eaten in producing a collection of massive spin two fields. Indeed, in unitary gauge we set \(\pi_j = 0\) \((Y_j^\mu(x) = x^\mu)\) and \(\textbf{10}\) reduces to \(\textbf{3}\).

The spectrum of this minimal theory space is that of standard latticizations: there is a massless 4D graviton, and tower of massive spin two fields, with a characteristic lattice spectrum \(m_n = m \sin(n/N)\). For \(n \ll N\) the spectrum approaches a KK tower of compactified 5D theory. At the linear level, the exchange of these modes generates the correct 5D graviton propagator up to small corrections, and so, for example, 5D Newtonian gravity is reproduced.

However, at the non-linear level peculiar new features are revealed, which can be traced to the interactions of the longitudinal modes of the massive gravitons, as in

\[\textbf{11}\]. Expanding the metrics about flat space as \(g^{j}_\mu_\nu = \eta_\mu_\nu + h^j_\mu_\nu\), the hopping terms involve

\[
g^{j}_\mu_\nu - G^{j+1}_\mu_\nu = h^{j}_\mu_\nu - h^{j+1}_\mu_\nu + \pi^{j}_\mu_\nu + \pi^{j}_\nu_\mu + \pi^{j}_\nu_\mu \pi^{j}_\nu_\mu + \cdots
\]  
(12)

where the \(\cdots\) refer to terms involving more powers of the \(h\). As in the study of a single massive graviton, it is useful to decompose the \(\pi^{j}_\mu_\nu\) as

\[
\pi^{j}_\mu_\nu = A^{j}_\mu_\nu + \partial_\mu \phi^j_\nu + \partial_\nu \phi^j_\mu + \cdots
\]  
(13)

The dynamics of the \(\phi^j\) is that of the scalar longitudinal components of the gravitons, and they produce the amplitudes that grow most dangerously with energy.

Inserting \(\textbf{12}\) into \(\textbf{10}\) and going to momentum space displays the kinetic terms and interactions for all the Goldstone modes. Schematically:

\[
S = \int d^4x N M^2 h_0 \partial_\mu h_0 + NM^2 m^2 \left(\frac{n^2}{N^2} h_n^2 + \frac{n}{N} h_0 \partial_\mu \phi_n + (\partial_\mu \phi_n)(\partial_\nu \phi_m)(\partial_\nu \phi_{n+m}) + \cdots\right)
\]  
(14)

where \(h_0\) is the massless graviton and \(h_n\) are the graviton and scalar Goldstone at the \(n^{th}\) mass level. Now, \(\phi_n\) picks up a kinetic term of the form \(M^2 m^4 n^2 N^{-1} \phi_n \partial_\mu \phi_n\) from mixing with \(h_n\) and the strongest interactions come from the \(\partial_\mu \phi^3\) vertex for the lowest modes, just as in \(\textbf{11}\). For instance the amplitude for \(\phi_1\)-\(\phi_1\) scattering goes as:

\[
A = \phi_1 \phi_1 \sim \frac{E^{10}}{\Lambda^{10}}
\]

(15)

Where, expressed in terms of the low energy 4D Planck scale \(M_{P1} = \sqrt{N} M\) and the mass of the first KK mode \(m_1 = m/N\):

\[
\Lambda = (N m_1 M_{P1})^{1/5}
\]

(16)

This is higher by a factor of \(N^{1/5}\) than the scale that the theory of a single graviton of mass \(m_g = m_1\) breaks down.

In terms of 5D variables:

\[
\Lambda = \left(\frac{M_{5D}}{R^3 a^2}\right)^{1/10}
\]

(17)

Note that bizarrely, the UV scale at which the theory becomes strongly coupled depends on an IR scale, the size of the extra dimension!

Naturally, as we decrease the lattice spacing, \(\Lambda\) increases. However, for a consistent effective theory, we should require that all the states in the theory be lighter than the UV cutoff \(\Lambda\). In particular, the heaviest KK mode, of mass \(\sim a^{-1}\), should be lighter than \(\Lambda\). This
means that the lattice spacing cannot be decreased beyond a certain point, and the highest UV cutoff the theory can possibly have is

$$\Lambda_{\text{max}} \sim a_\text{min}^{-1} \sim \left( \frac{M_{5D}^3}{R^2} \right)^{1/8}$$  \hspace{1cm} (18)$$

Again, this strikingly exhibits a UV/IR connection: the highest possible UV cutoff $\Lambda_{\text{max}}$ decreases as the size $R$ of the dimension is increased in such a way that $\Lambda_{\text{max}}^3 R^5 = M_{5D}^3$ stays fixed. Note that for any radius larger than the 5D Planck length, $\Lambda_{\text{max}}$ is always smaller than $M_{5D}$. In other words, the minimal lattice cannot look like 5D gravity at the non-linear level all the way up to the 5D Planck scale.

It is useful to understand what is going on directly in unitary gauge, where the amplitude $\langle a \rangle$ is that of scattering scalar longitudinal ($sL$) polarizations of the lightest massive gravitons. The amplitude can be written in the instructive form:

$$A \sim \frac{E_1^{10}}{\Lambda_{10}^{10}} \sim \frac{E_1^{10}}{M_{5D}^3 (1/R)^2} \times \frac{a^2}{R^2}$$  \hspace{1cm} (19)$$

In the case of a single massive graviton of mass $m_g$ and Planck scale $M_{Pl}$, there are two contributions to the amplitude for graviton scattering, one from graviton exchange and one from the direct 4-point graviton vertex.

There is no cancellation between these two contributions, so the amplitude grows as $E_1^{10}/(m_g^8 M_{Pl}^2)$. Our scattering amplitude has exactly the same form, with $m_g \to 1/R$, however there is a suppression factor of $a^2/R^2$. Evidently, in the continuum theory, there is an exact cancellation between the two contributions, ensured by the 5D gravitational Ward identities. However, in the discretized theory, the spectrum and interactions are modified by $\sim (a/R)$ effects, and so the cancellation is imperfect, reflecting the breaking of the 5D general coordinate invariance by our discretization.

Needless to say, this behavior is dramatically different than for gauge theories. The same theory space for a non-Abelian gauge theory would become strongly coupled at an energy scale which is always higher than than the mass of all the modes. So the discretized theory can be made to look identical to a higher dimensional gauge theory all the way up to scale where the 5D (non-renormalizable) gauge theory would naturally break down.

Nevertheless, the minimal gravitational model does define a sensible effective field theory of gravity valid to energies parametrically above the compactification radius $1/R$. The strong interactions formally vanish in the limit of zero lattice spacing, they just do not vanish quickly enough to reproduce 5D gravity all the way up to $M_{5D}$ in a consistent effective field theory.

Let us now study this model in position space, where the physics is more transparent. Working directly in continuum language, the action $S$ becomes:

$$S = \int d^4 x dz M_{5D}^3 \left( (\partial h)^2 + (\partial \pi)^2 + \frac{1}{a} h^2 \partial_z \phi + \frac{1}{a^2} (\partial^2 \phi)^3 + \cdots \right)$$  \hspace{1cm} (20)$$

We have done an integration by parts to bring the bilinear mixing between $h$ and $\phi$ to the above form. Note also that because the extra dimension is an interval, the boundary conditions at the ends are that $\partial_z h$ and $\pi$ vanish, so that there are no Goldstone zero modes (corresponding to the fact that they are all eaten).

We can remove the kinetic mixing between $h$ and $\phi$ by defining

$$h_{\mu \nu} = h_{\mu \nu} - \eta_{\mu \nu} \psi, \text{ where } \psi = \frac{1}{a} \partial_z \phi$$  \hspace{1cm} (21)$$

which generates a kinetic term for $\phi$. We add gauge-fixing terms directly for $h$, the precise form of which will not be important here. The $\phi$ action then becomes:

$$S = \int d^4 x dz M_{5D}^3 \left( (\partial \psi)^2 + (\partial \pi)^2 + \frac{1}{a^2} (\partial^2 \phi)^3 + \cdots \right)$$

Already we see that this action is strange. The combination $\psi = \frac{1}{a} \partial_z \phi$ has a normal kinetic term. $\psi$ also couples directly to the trace of the energy momentum tensor in the way required to produce the correct tensor structure for the propagator in 5D. So, at the linear level, $\psi$ is the physical propagating field. Observe, however, that the self-interactions involve $\phi$ and are therefore highly non-local with respect to $\psi$. To see this, we can formally write $\phi = \frac{1}{a} \psi$; with the boundary conditions that $\phi$ vanishes at the ends of the interval, $\partial_z$ is invertible and this can be done unambiguously. The interaction Lagrangian for $\psi$ is then

$$\int d^4 x dz M_{5D}^3 \left( \frac{\partial^2}{\partial_z^2} \psi \right)^3 + \cdots$$  \hspace{1cm} (22)$$

which is manifestly non-local in the $z$ direction.

Note that the interaction formally vanishes in the zero lattice spacing limit, just as $\Lambda \to \infty$ as $a \to 0$ in (17). However for any finite lattice spacing $a$, at large enough distances this term can become important. On a finite interval of size $R$, the largest wavelength modes of size $\sim R$ will suffer the strongest interactions. These modes $\sim \psi_R$ correspond to the longitudinal polarizations of the lowest KK modes, and are described by the effective 4D action:

$$\int d^4 x \left( M_{5D}^3 R (\partial \psi_R)^2 + M_{5D}^3 a R^4 (\partial^2 \psi_R)^3 + \cdots \right)$$  \hspace{1cm} (23)$$
We can see immediately that the amplitude for $\psi_R \psi_R \rightarrow \psi_R \psi_R$ is the same as $\psi \rightarrow \psi_R \psi_R$. It is precisely the non-local interactions of $\psi$ which lead to the strong amplitudes which force $\Lambda \ll M_{5D}$.

We have seen that, while our starting point appears extremely local, with only nearest neighbor hopping terms for the gravitons on the sites, it actually induces highly non-local interactions in the discretized dimension. Why did this happen? Presumably, these effects are related to the fact that we have broken the full 5D diffeomorphism invariance of the theory by our discretization. Usually, however, when gauge symmetries are explicitly broken, a theory gains new degrees of freedom (which correspond to pure gauge modes in the gauge invariant theory). Here (ignoring the irrelevant radion), the number of physical degrees of freedom beneath $a^{-1}$ match exactly between the discrete and continuum theories. Indeed, despite the absence of the full 5D diffeomorphism invariance and Lorentz invariance, the theory still has an exactly “massless” graviton even in the 5D sense, by which we mean gapless excitations with $\omega \rightarrow 0$ as $k \rightarrow 0$. However, the interactions are qualitatively different in the two theories.

So far we have discussed compactification on an interval. When we compactify on a circle, we have to contend with the Goldstone zero mode, which is no longer killed by the boundary conditions. The zero mode has $\psi_0 = a^{-1} \partial_z \phi_0 = 0$ (equivalently, the $n = 0$ mode in $\phi$) and therefore has no kinetic term at all. However, it does appear in interactions! This means that it is inconsistent not to include a kinetic term for $\phi_0$ in the theory, which becomes a mass term for the graviphoton $\pi_0^\mu$ (more specifically, the generally covariant plaquette operator $\Pi$ made from the “Wilson line” $Y_1 \circ \cdots \circ Y_N$ is generated, which contains the mass term for $\pi_0^\mu$ as well as various non-linear interactions). It is not surprising that such a term should be generated. In the continuum, the massless graviphoton is associated with a $U(1)$ gauge symmetry inherited from the 5D reparameterization invariance of the circle. This invariance is clearly broken by the discretization. We can see this concretely, because in the continuum $\phi_0$ is a pure gauge mode with no dynamics at all, while in our discretization it does appear in interactions. Therefore, the $U(1)$ gauge invariance is explicitly broken and there is no symmetry to prevent $\pi_0^\mu$ from picking up a mass.

A similar analysis can be done for the case where we start with 3D sites and go up to a 4D theory. Amusingly, in this case, the 3D gravity on the sites does not have any propagating degrees of freedom – all of the local degrees of freedom in the 4D theory come from the links. Once again, the $\phi$ fields only acquire a kinetic term by mixing with the site metrics, and the same sorts of non-local interactions arise. The maximum cutoff for a consistent effective theory in this case is

$$\Lambda_{\text{max}} \sim \left( \frac{M_{5D}^2 R}{R^5} \right)^{1/7}$$

Of course, the discretization of one of three spatial dimensions in our universe leads to the breaking of rotational invariance and there are already very severe limits on such effects. But even ignoring this the above cutoff is still incredibly low; taking $R \approx 10^{28}$ cm to be about the size of the universe today, $\Lambda_{\text{max}} \approx 10^{12}$ cm! Clearly such discretizations are not sensible to describe our 4D world.

Are there discretizations that avoid inducing non-local interactions and correctly reproduce local continuum physics? Simply adding next-to-nearest neighbor interactions, or anything similar which is essentially local on the lattice, cannot possibly work. We can get a hint of what is needed by looking at a discretization that is guaranteed to work, at least at tree-level. Take a continuum theory on an interval and simply truncate the KK tower, keeping only $N$ of the modes. Of course, we do not need an infinite number of states for an effective theory. By KK momentum conservation, the tree-level scattering amplitudes for the lowest KK modes cannot involve the truncated modes, and therefore their scattering amplitudes coincide with the (healthy) ones of the continuum theory. The strongest tree level amplitudes come from the scattering of the modes between $\sim N/2$ and $N$ near the top of the tower, of mass $\sim N/R$. As is discussed in detail in [11], the strong coupling scale in this case is determined by the $\Lambda_3$ scale associated with this mass:

$$\Lambda \sim \left( \frac{N^2}{R^2 M_{Pl}} \right)^{1/3} \sim \left( \frac{M_{5D}^3 R}{a^4} \right)^{1/6}$$

So, for any lattice spacing $a$, the model makes sense as an effective theory up to energies parametrically higher than the top of the KK tower. Therefore in contrast with our minimal discretization, we can take a limit where the new strong amplitudes induced by the discretization are no more important than those associated with the 5D Planck scale $M_{5D}$. At least at tree-level, this model can make sense all the way up to $M_{5D}$, reproducing local 5D physics.

Starting with these $N$ modes in momentum space, we can go back to an $N$-site theory in position space by a discrete Fourier transform. The sharp momentum truncation induces interactions between sites in position space that are not strictly local:
The Fourier transform of a step function has a rapid oscillatory behavior, but dies off as a power with distance in position space \[12\], rather than exponentially as in genuinely local theories.

The non-local interactions we found in the minimal discretization have the same origin as the strong-coupling effects for a single massive graviton discussed in \[1]. These strong coupling effects were softened in curved backgrounds (such as AdS), so it would interesting to look for similar improvements in the context of discrete dimensions, where the sites are taken to have AdS geometries. Another straightforward exercise is to discretize warped geometries (such as AdS\(_5\)). Since quantum gravity in AdS\(_5\) is dual to a 4D field theory, it would be interesting to see whether a naive discretization can do a better job in describing the continuum physics in this case, at least at scales larger than the AdS\(_5\) length.

We have seen that a naive local discretization of gravity induces non-local interactions at long distances, in sharp contrast to gauge theories, where the minimal discretization is perfectly well-behaved. This is apparent already at tree-level for a single discrete dimension, and makes it impossible to reproduce the correct continuum physics within a consistent effective field theory. We have also seen that a less local discretization, comprising very specific interactions between distant sites in position space, does better than the minimal discretization, and can successfully reproduce local physics at low energies at least at tree-level. This discretization follows from a simple truncation of the continuum Kaluza-Klein tower. In gauge theory, such a truncation is artificial and the corresponding non-local interactions in position space give rise to various pathologies; it is interesting that in the gravity case this “artificial” discretization is superior to the minimal one.

Ultimately, it is of interest to consider full space or space-time lattices that may provide a sensible definition of quantum gravity in four dimensions. However, despite the many hints of discrete structures underlying quantum gravity, lattice approaches have all suffered from the inability to reproduce the correct continuum physics at low energies. Our simple examples suggest that a successful discretization of gravity cannot look local, but should involve a special set of non-local interactions between distant sites. It is tempting to speculate that this is a reflection of a UV/IR connection in quantum gravity, and that the overly-local nature of most approaches to lattice gravity is responsible for their failures at low energy. Of course, further progress requires finding a principle that dictates the structure of the required non-local interactions.

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[1] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, arXiv:hep-th/0210184.
[2] N. Arkani-Hamed, A. G. Cohen and H. Georgi, Phys. Rev. Lett. 86, 4757 (2001) [arXiv:hep-th/0104005].
[3] C. T. Hill, S. Pokorski and J. Wang, Phys. Rev. D 64, 105005 (2001) [arXiv:hep-th/0104035].
[4] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A 173, 211 (1939).
[5] T. Damour and I. I. Kogan, arXiv:hep-th/0206042.
[6] A. Sugamoto, Prog. Theor. Phys. 107, 793 (2002) [arXiv:hep-th/0104241].
[7] M. Alishahiha, Phys. Lett. B 517, 406 (2001) [arXiv:hep-th/0105153].
[8] M. Bander, Phys. Rev. D 64, 105021 (2001) [arXiv:hep-th/0107130].
[9] V. Jejjala, R. G. Leigh and D. Minic, arXiv:hep-th/0212057.
[10] N. Kan and K. Shiraishi, arXiv:gr-qc/0212113.
[11] M. J. Duff, C. N. Pope and K. S. Stelle, Phys. Lett. B 223, 386 (1989).
[12] M. D. Schwartz, arXiv:hep-th/0303114.