THE ∂-COMPLEX ON THE FOCK SPACE

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Abstract. We study certain densely defined unbounded operators on the Fock space. These are the annihilation and creation operators of quantum mechanics. In several complex variables we have the ∂-operator and its adjoint ∂∗ acting on (p, 0)-forms with coefficients in the Fock space. We consider the corresponding ∂-complex and study spectral properties of the corresponding complex Laplacian ˜□ = ∂∂∗ + ∂∗∂. Finally we study a more general complex Laplacian ˜□D = DD∗ + D∗D, where D is a differential operator of polynomial type, to find the canonical solutions to the inhomogeneous equations Du = α and D∗v = β.

1. Introduction

Purpose of this paper is to consider the ∂-complex and to use the powerful classical methods of the ∂-complex based on the theory of unbounded densely defined operators on Hilbert spaces, see [7], [15]. The main difference to the classical theory is that the underlying Hilbert space is now not an L2-space but a closed subspace of an L2-space - the Fock space of entire functions A2(Cn, e−|z|2). It is well known that the differentiation with respect to zj defines an unbounded operator on A2(Cn, e−|z|2). We consider the operator

∂f = ∑n j=1 (∂f/∂zj) dzj,

which is densely defined on A2(Cn, e−|z|2) and maps to A2,0(Cn, e−|z|2), the space of (1, 0)-forms with coefficients in A2(Cn, e−|z|2). In general, we get the ∂-complex

A2(p−1,0)(Cn, e−|z|2) → A2(p,0)(Cn, e−|z|2) → A2(p+1,0)(Cn, e−|z|2),

where 1 ≤ p ≤ n − 1 and ∂∗ denotes the adjoint operator of ∂.

We will choose the domain dom(∂) in such a way that ∂ becomes a closed operator on A2(Cn, e−|z|2). In addition we get that the corresponding complex Laplacian

□p = ∂∗∂ + ∂∂∗,

with dom(□p) = {f ∈ dom(∂) ∩ dom(∂∗) : ∂f ∈ dom(∂∗) and ∂f∗ ∈ dom(∂)} acts as unbounded self-adjoint operator on A2(p,0)(Cn, e−|z|2). We point out that in this case the
complex Laplacian is a differential operator of order one. Nevertheless we can use the general features of a Laplacian for these differential operators of order one.

Using an estimate which is analogous to the basic estimate for the \( \bar{\partial} \)-complex, we obtain that \( \Box_p \) has a bounded invers \( \tilde{N}_p : A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \to A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \) and we show that \( \tilde{N}_p \) is even compact. In addition we compute the spectrum of \( \tilde{\Box}_p \).

The inspiration for this comes from quantum mechanics, where the annihilation operator \( a_j \) can be represented by the differentiation with respect to \( z_j \) on \( A^2(\mathbb{C}^n, e^{-|z|^2}) \) and its adjoint, the creation operator \( a_j^* \), by the multiplication by \( z_j \), both operators being unbounded densely defined, see [5], [4]. One can show that \( A^2(\mathbb{C}^n, e^{-|z|^2}) \) with this action of the \( a_j \) and \( a_j^* \) is an irreducible representation \( M \) of the Heisenberg group, by the Stone-von Neumann theorem it is the only one up to unitary equivalence. Physically \( M \) can be thought of as the Hilbert space of a harmonic oscillator with \( n \) degrees of freedom and Hamiltonian operator

\[
H = \sum_{j=1}^n \frac{1}{2}(P_j^2 + Q_j^2) = \sum_{j=1}^n \frac{1}{2}(a_j^*a_j + a_ja_j^*).
\]

In the second part we consider a general plurisubharmonic weight function \( \varphi : \mathbb{C}^n \to \mathbb{R} \) and the corresponding weighted space of entire functions \( A^2(\mathbb{C}^n, e^{-\varphi}) \). The \( \partial \)-complex has now the form

\[
A^2_{(p-1,0)}(\mathbb{C}^n, e^{-\varphi}) \overset{\partial}{\longrightarrow} A^2_{(p,0)}(\mathbb{C}^n, e^{-\varphi}) \overset{\partial}{\longrightarrow} A^2_{(p+1,0)}(\mathbb{C}^n, e^{-\varphi}),
\]

where \( 1 \leq p \leq n-1 \). Finally we prove a formula which is analogous to the Kohn-Morrey formula for the classical \( \bar{\partial} \)-complex, see [7], [13] or [1]. We will show that

\[
\|\partial u\|_{\varphi}^2 + \|\partial^* u\|_{\varphi}^2 = \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left( \frac{\partial u_j}{\partial z_k} \right)^2 e^{-\varphi} \, d\lambda + \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left( \frac{\partial^2 \varphi}{\partial z_k \partial z_j} \right) u_j \bar{u}_k e^{-\varphi} \, d\lambda + T,
\]

for \( u \in \text{dom}(\partial) \cap \text{dom}(\partial^*) \), where the term \( T \) is non-positive.

Finally we investigate operators of the \( Du = \sum_{j=1}^n p_j(u) \, dz_j \), where \( u \in A^2(\mathbb{C}^n, e^{-|z|^2}) \) and \( p_j(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}) \) are polynomial differential operators with constant coefficients. Differential operators of polynomial type on the Fock space were also investigated by J.D. Newman and H. Shapiro in [11] and [12] and by H. Render [13] in the real analytic setting. Replacing \( \partial \) by \( D \) one gets a corresponding complex Laplacian \( \Box_D = DD^* + D^* D \), for which one can use duality and the machinery of the \( \bar{\partial} \)-Neumann operator \( \{\Box, \Box^*\} \) in order to prove existence and boundedness of the inverse to \( \Box_D \) and to find the canonical solutions to the inhomogeneous equations \( Du = \alpha \) and \( D^* v = \beta \).
We consider the Fock space $A^2(\mathbb{C}^n, e^{-|z|^2})$ consisting of all entire functions $f$ such that
\[ \|f\|^2 = \int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} \, d\lambda(z) < \infty. \]

It is clear, that the Fock space is a Hilbert space with the inner product
\[ (f, g) = \int_{\mathbb{C}^n} f(z) \overline{g(z)} e^{-|z|^2} \, d\lambda(z). \]

Setting $n = 1$, we obtain for $f \in A^2(\mathbb{C}, e^{-|z|^2})$ that
\[ |f(z)| \leq \frac{1}{\pi r^2} \int_{D_r(z)} e^{[w]^2/2} |f(w)| e^{-[w]^2/2} \, d\lambda(w) \]
\[ \leq \frac{1}{\pi r^2} \left( \int_{D_r(z)} e^{[w]^2} \, d\lambda(w) \right)^{1/2} \left( \int_{D_r(z)} |f(w)|^2 e^{-[w]^2} \, d\lambda(w) \right)^{1/2} \]
\[ \leq C \left( \int_{\mathbb{C}} |f(w)|^2 e^{-[w]^2} \, d\lambda(w) \right)^{1/2} \]
\[ \leq C \|f\|, \]
where $C$ is a constant only depending on $z$. In addition, for each compact subset $L$ of $\mathbb{C}$ there exists a constant $C_L > 0$ such that
\[ \sup_{z \in L} |f(z)| \leq C_L \|f\|, \]
for all $f \in A^2(\mathbb{C}, e^{-|z|^2})$.

For several variables one immediately gets an analogous estimate. This implies that the Fock space $A^2(\mathbb{C}^n, e^{-|z|^2})$ has the reproducing property. The monomials \( \{z^\alpha\} \) constitute an orthogonal basis, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is a multiindex, and the norms of the monomials are
\[ \|z^\alpha\|^2 = \int_{\mathbb{C}} |z_1|^{2\alpha_1} e^{-|z_1|^2} \, d\lambda(z_1) \cdots \int_{\mathbb{C}} |z_n|^{2\alpha_n} e^{-|z_n|^2} \, d\lambda(z_n) \]
\[ = (2\pi)^n \int_0^\infty r^{2\alpha_1+1} e^{-r^2} \, dr \cdots \int_0^\infty r^{2\alpha_n+1} e^{-r^2} \, dr \]
\[ = \pi^n \alpha_1! \cdots \alpha_n!. \]

It follows that each function $f \in A^2(\mathbb{C}^n, e^{-|z|^2})$ can be written in the form
\[ f = \sum_\alpha f_\alpha \varphi_\alpha, \]
where
\[ \varphi_\alpha(z) = \frac{z^\alpha}{\sqrt{\pi^n \alpha!}} \quad \text{and} \quad \sum_\alpha |f_\alpha|^2 < \infty \]
and $\alpha! = \alpha_1! \cdots \alpha_n!$. 

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Hence the Bergman kernel of $A^2(\mathbb{C}^n, e^{-|z|^2})$ is of the form
\[
K(z, w) = \sum_{\alpha} z^\alpha \overline{w}^\alpha \frac{1}{\|z\|^{2\alpha} \prod_{\alpha=1}^n \alpha!} = \frac{1}{\pi^n} \exp(z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n).
\]

See [17] for an extensive study of the Fock space.

We point out that the space $A^2(\mathbb{C}, e^{-|z|^2})$ serves for a representation of the states in quantum mechanics (see [4]), where $a(f) = df/dz$ is the annihilation operator and $a^*(f) = z f$ is the creation operator, both of them being densely defined unbounded operators on $A^2(\mathbb{C}, e^{-|z|^2})$. The span of the finite linear combinations of the basis functions $\varphi_\alpha$ is dense in $A^2(\mathbb{C}, e^{-|z|^2})$. Hence both operators $a$ and $a^*$ are densely defined.

The function
\[
F(z) = \sum_{k=2}^{\infty} \frac{\varphi_k(z)}{\sqrt{k(k-1)}} \in A^2(\mathbb{C}, e^{-|z|^2}),
\]
but
\[
F'(z) = \sum_{k=1}^{\infty} \frac{\varphi_k(z)}{\sqrt{k}} \notin A^2(\mathbb{C}, e^{-|z|^2}),
\]
and
\[
G(z) = \sum_{k=0}^{\infty} \frac{\varphi_k(z)}{k+1} \in A^2(\mathbb{C}, e^{-|z|^2}),
\]
but
\[
zG(z) = \sum_{k=1}^{\infty} \frac{\varphi_k(z)}{\sqrt{k}} \notin A^2(\mathbb{C}, e^{-|z|^2}),
\]
hence both operators $a$ and $a^*$ are unbounded operators on $A^2(\mathbb{C}, e^{-|z|^2})$.

But taking a primitive of a function $f \in A^2(\mathbb{C}, e^{-|z|^2})$ yields a bounded operator
\[
T : A^2(\mathbb{C}, e^{-|z|^2}) \longrightarrow A^2(\mathbb{C}, e^{-|z|^2});
\]
let
\[
f(z) = \sum_{k=0}^{\infty} f_k \frac{z^k}{\sqrt{\pi} \sqrt{k!}}, \quad \sum_{k=0}^{\infty} |f_k|^2 < \infty.
\]
Then the function
\[
h(z) = \sum_{k=0}^{\infty} f_k \frac{z^{k+1}}{\sqrt{\pi} \sqrt{k+1} \sqrt{(k+1)!}}.
\]
defines a primitive of \( f \) and we can write
\[
T(f) = h = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k!}} (f, \tilde{\varphi}_k) \varphi_k,
\]
where \( \tilde{\varphi}_k = \varphi_{k-1} \) and the constant term in the Taylor series expansion of the primitive is always 0. This implies immediately that \( T \) is even a compact operator.

This is also a special result from the theory of Volterra-type integration operators on the Fock space of the form
\[
T_g f(z) = \int_0^z f' g' \, d\zeta,
\]
see [2].

Next we define the domain of the operator \( a \) to be
\[
\text{dom}(a) = \{ f \in A^2(\mathbb{C}, e^{-|z|^2}) : f' \in A^2(\mathbb{C}, e^{-|z|^2}) \}.
\]
Then \( \text{dom}(a^*) \) consists of all functions \( g \in \text{dom}(a) \) such that the densely defined linear functional \( L(f) = (a(f), g) \) is continuous on \( \text{dom}(a) \). This implies that there exists a function \( h \in A^2(\mathbb{C}, e^{-|z|^2}) \), such that \( L(f) = (a(f), g) = (f, h) \).

Next we show that
\[
\text{dom}(a^*) = \{ g \in A^2(\mathbb{C}, e^{-|z|^2}) : zg \in A^2(\mathbb{C}, e^{-|z|^2}) \}.
\]
Let \( f \in \text{dom}(a) \) and \( g \in \text{dom}(a^*) \). Then
\[
(a(f), g) = \int_C \frac{df(z)}{dz} g(z) e^{-|z|^2} \, d\lambda(z)
\]
\[
= -\int_C f(z) \frac{dg(z)}{dz} e^{-|z|^2} \, d\lambda(z)
\]
\[
= \int_C f(z) zg(z) e^{-|z|^2} \, d\lambda(z)
\]
\[
= (f, a^*(g)),
\]
where we used integration by parts in the first step (see [3] for a detailed proof) and that
\[
\frac{d}{dz} (g(z) e^{-|z|^2}) = g(z) (-\pi e^{-|z|^2}).
\]
An alternative proof uses Taylor series expansion: let
\[
f(z) = \sum_{k=0}^{\infty} f_k \varphi_k(z) \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} g_k \varphi_k(z),
\]
where \( (f_k), (g_k) \in l^2 \). Then we have
\[
(a(f), g) = \sum_{k=0}^{\infty} \sqrt{k+1} f_{k+1} \overline{g_k} = (f, a^*(g)).
\]

**Remark 2.1.** (a) We point out that the commutator satisfies \([a, a^*] = I\), which is of importance in quantum mechanics, see [4].
(b) For \( f \in \text{dom}(a) \cap \text{dom}(a^*) \) we get from the last results that
\[
\|a(f)\|^2 + \|a^*(f)\|^2 = 2\|a(f)\|^2 + \|f\|^2.
\]
Lemma 2.2. The operators \( a \) and \( a^* \) are densely defined operators on \( A^2(\mathbb{C}, e^{-|z|^2}) \) with closed graph.

**Proof.** By general properties of unbounded operators, it suffices to prove the assertion for \( a \) (see [7] or [16]). Let \((f_j) \) be a sequence in \( \text{dom}(a) \) such that \( \lim_{j \to \infty} f_j = f \) and \( \lim_{j \to \infty} a(f_j) = g \). We have to show that \( f \in \text{dom}(a) \) and \( a(f) = g \). By (2.1) it follows that \((f_j) \) converges uniformly on each compact subset of \( \mathbb{C} \) to \( f \) and the same is true for the derivatives \( \lim_{j \to \infty} f'_j = f' \). This implies that \( f' = a(f) = g \). We supposed that \( g \in A^2(\mathbb{C}, e^{-|z|^2}) \) and we know that taking primitives does not leave \( A^2(\mathbb{C}, e^{-|z|^2}) \). Therefore we get that \( f \in \text{dom}(a), \) which proves the assertion. \( \square \)

For the rest of this section we consider the Fock space in several variables with the weight \( \varphi(z) = |z_1|^2 + \cdots + |z_n|^2 \). We will denote the derivative with respect to \( z \) by \( \partial \) and in the following we will consider the \( \partial \)-complex for the Fock space in several variables

\[
A^2_{(p-1,0)}(\mathbb{C}^n, e^{-|z|^2}) \xrightarrow{\partial} A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \xrightarrow{\partial^*} A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2}),
\]

where \( A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \) denotes the Hilbert space of \((p,0)\)-forms with coefficients in \( A^2(\mathbb{C}^n, e^{-|z|^2}) \), and

\[
\partial f = \sum_{|J|=p} \sum_{j=1}^n \frac{\partial f_j}{\partial z_j} dz_j \wedge dz_J
\]

for a \((p,0)\)-form

\[
f = \sum_{|J|=p} f_J dz_J
\]

with summation over increasing multiindices \( J = (j_1, \ldots, j_p), \; 1 \leq p \leq n-1; \) and we take

\[
\text{dom}(\partial) = \{ f \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) : \partial f \in A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2}) \}.
\]

Now let

\[
\tilde{\Box}_p = \partial^* \partial + \partial \partial^*,
\]

with \( \text{dom}(\tilde{\Box}_p) = \{ f \in \text{dom}(\partial) \cap \text{dom}(\partial^*) : \partial f \in \text{dom}(\partial^*) \) and \( \partial^* f \in \text{dom}(\partial) \}. \)

Then \( \tilde{\Box}_p \) acts as unbounded self-adjoint operator on \( A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \), see [7].

**Remark 2.3.** (a) It is pointed out that a \((1,0)\)-form \( g = \sum_{j=1}^n g_j \; dz_j \) with holomorphic coefficients is invariant under the pull back by a holomorphic map \( F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n \). We have

\[
F^* g = \sum_{l=1}^n g_l \; dF_l = \sum_{j=1}^n \left( \sum_{l=1}^n g_l \frac{\partial F_l}{\partial z_j} \right) \; dz_j,
\]

where we used the fact that

\[
dF_l = \partial F_l + \overline{\partial} F_l = \sum_{j=1}^n \frac{\partial F_l}{\partial z_j} \; dz_j + \sum_{j=1}^n \frac{\partial F_l}{\partial \overline{z_j}} \; d\overline{z_j} = \sum_{j=1}^n \frac{\partial F_l}{\partial z_j} \; dz_j.
\]

The expressions \( \frac{\partial F_l}{\partial z_j} \) are holomorphic.
(b) For \( p = 0 \) and a function \( f \in \text{dom}(\tilde{\Box}_0) \) we have
\[
\tilde{\Box}_0 f = \partial^* \partial f = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}.
\]
If \( 1 \leq p \leq n - 1 \) and \( f = \sum_{|J|=p} f_J dz_J \in \text{dom}(\tilde{\Box}_p) \) is a \((p,0)\)-form, we have
\[
(2.5) \quad \tilde{\Box}_p f = \sum_{|J|=p} \left( \sum_{k=1}^{n} z_k \frac{\partial f_J}{\partial z_k} + pf_J \right) dz_J.
\]
For \( p = n \) and a \((n,0)\)-form \( F \in \text{dom}(\tilde{\Box}_n) \) (here we identify the \((n,0)\)-form with a function), we have
\[
\tilde{\Box}_n F = \partial \partial^* F = \sum_{j=1}^{n} z_j \frac{\partial F}{\partial z_j} + nF.
\]
Before we continue the study of the box operator \( \tilde{\Box}_p \), we collect some facts about Fock spaces with more general weights.

3. Generalized Fock spaces

Let \( \varphi : \mathbb{C}^n \rightarrow \mathbb{R} \) be a plurisubharmonic \( C^\infty \) function. Let
\[ A^2(\mathbb{C}^n, e^{-\varphi}) = \{ f : \mathbb{C}^n \rightarrow \mathbb{C} \text{ entire} : \| f \|_\varphi^2 = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} d\lambda < \infty \}, \]
with inner product
\[ (f, g)_\varphi = \int_{\mathbb{C}^n} f \overline{g} e^{-\varphi} d\lambda. \]
It is easily seen that \( A^2(\mathbb{C}^n, e^{-\varphi}) \) is a Hilbert space with the reproducing property. Hence it has a reproducing kernel \( K_\varphi(z, w) \) (Bergman kernel) which has the following properties:
\[ K_\varphi(w, z) = \overline{K_\varphi(z, w)}, \]
the function \( z \mapsto K_\varphi(z, w) \) belongs to \( A^2(\mathbb{C}^n, e^{-\varphi}) \) and
\[ f(z) = \int_{\mathbb{C}^n} K_\varphi(z, w) f(w) e^{-\varphi(w)} d\lambda(w), \]
for each \( f \in A^2(\mathbb{C}^n, e^{-\varphi}) \).
The Bergman projection \( P_\varphi : L^2(\mathbb{C}^n, e^{-\varphi}) \rightarrow A^2(\mathbb{C}^n, e^{-\varphi}) \) can be written in the form
\[ P_\varphi F(z) = \int_{\mathbb{C}^n} K_\varphi(z, w) F(w) e^{-\varphi(w)} d\lambda(w), \]
for \( F \in L^2(\mathbb{C}^n, e^{-\varphi}) \).

**Remark 3.1.** We indicate that \( A^2(\mathbb{C}^n, e^{-\varphi}) \) is infinite dimensional, if the lowest eigenvalue \( \mu_\varphi \) of the Levi matrix
\[ \left( \frac{\partial^2 \varphi}{\partial z_k \partial \bar{z}_j} \right)_{j,k=1}^{n} \]
satisfies $\lim_{|z| \to \infty} |z|^2 \mu_\varphi(z) = +\infty$, see [14] or [7].

We study the $\partial$-complex

$$A^2_{(p-1,0)}(\mathbb{C}^n, e^{-\varphi}) \xrightarrow{\partial} A^2_{(p,0)}(\mathbb{C}^n, e^{-\varphi}) \xrightarrow{\partial} A^2_{(p+1,0)}(\mathbb{C}^n, e^{-\varphi}),$$

where $A^2_{(p,0)}(\mathbb{C}^n, e^{-\varphi})$ denotes the Hilbert space of $(p,0)$-forms with coefficients in $A^2(\mathbb{C}^n, e^{-\varphi})$, and

$$\partial f = \sum_{|J|=p} \sum_{j=1}^n \frac{\partial f_j}{\partial z_j} dz_j \wedge dz.$$

for a $(p,0)$-form

$$f = \sum_{|J|=p} f_J dz.$$

with summation over increasing multiindices $J = (j_1, \ldots, j_p)$, $1 \leq p \leq n-1$; and we take

$$\dom(\partial) = \{ f \in A^2_{(p,0)}(\mathbb{C}^n, e^{-\varphi}) : \partial f \in A^2_{(p+1,0)}(\mathbb{C}^n, e^{-\varphi}) \}.$$

The adjoint operator to $\partial$ depends on the weight:

$$\dom(\partial^*_\varphi) = \{ g \in A^2_{(p+1,0)}(\mathbb{C}^n, e^{-\varphi}) : f \mapsto (\partial f, g)_\varphi \text{ is continuous on } \dom(\partial) \}.$$

We use the Gauß–Green Theorem in order to compute the adjoint $\partial^*_\varphi$.

Let $\Omega = \{ z \in \mathbb{C}^n : r(z) < 0 \}$, where $r$ is a real valued $C^1$-function with

$$\nabla_z r := (\frac{\partial r}{\partial z_1}, \ldots, \frac{\partial r}{\partial z_n}) \neq 0$$

on $b\Omega = \{ z : r(z) = 0 \}$. Without loss of generality we can suppose that $|\nabla_z r| = |\nabla r| = 1$ on $b\Omega$. For $u, v \in C^\infty(\Omega)$ and

$$(u, v) = \int_\Omega u(z) \overline{v(z)} \, d\lambda(z).$$

The Gauß–Green Theorem implies that

$$\left( \frac{\partial u}{\partial z_k}, v \right) = -\left( u, \frac{\partial v}{\partial z_k} \right) + \int_{b\Omega} u(z) \overline{v(z)} \frac{\partial r}{\partial z_k}(z) \, d\sigma(z),$$

where $d\sigma$ is the surface measure on $b\Omega$.

In our case we have holomorphic components $f_J$ and $g_{J_J}$ and the inner product

$$\left( \frac{\partial f_J}{\partial z_j}, g_{J_J} \right)_\varphi = \int_{\mathbb{C}^n} \frac{\partial f_J}{\partial z_j} g_{J_J} e^{-\varphi} \, d\lambda.$$

Now let $\Omega = \{ z : |z| < R \}$ and take $r(z) = \frac{|z|^2 - R^2}{R}$ and apply (3.1) to get

$$\left( \int_{|z| \leq R} \frac{\partial f_J}{\partial z_j} g_{J_J} e^{-\varphi} \, d\lambda \right) - \int_{|z| \leq R} \frac{\partial f_J}{\partial z_j} g_{J_J} e^{-\varphi} \, d\lambda = \int_{|z|=R} f_J \overline{g_{J_J}} \frac{e^{-\varphi}}{R} \, d\sigma.$$

By Cauchy-Schwarz we get

$$\left| \int_{|z|=R} f_J \overline{g_{J_J}} \frac{e^{-\varphi}}{R} \, d\sigma \right|^2 \leq \int_{|z|=R} |f_J|^2 e^{-\varphi} \, d\sigma \int_{|z|=R} |g_{J_J}|^2 e^{-\varphi} \, d\sigma,$$
and as
\[ \|f_j\|_\varphi^2 = \int_{\mathbb{C}^n} |f_j|^2 e^{-\varphi} \, d\lambda = \int_0^\infty R^{2n-1} \int_{|z|=R} |f_j|^2 e^{-\varphi} \, d\sigma \, dR < \infty \]
the right hand side of (3.2) tends to zero as \( R \) tends to \( \infty \). So we obtain for the components of the \((p, 0)\)-form \( f \) and the \((p + 1, 0)\)-form \( g \)
\[
\left( \frac{\partial f_j}{\partial z_j}, g_{jJ} \right)_\varphi = \left( f_j, \frac{\partial \varphi}{\partial z_j} g_{jJ} \right)_\varphi = \left( P_\varphi(f_j), \frac{\partial \varphi}{\partial z_j} g_{jJ} \right)_\varphi = \left( f_j, P_\varphi \left( \frac{\partial \varphi}{\partial z_j} g_{jJ} \right) \right)_\varphi,
\]
where we used the fact that the components \( f_j \) are holomorphic. Hence we obtain
\[
(3.3) \quad \varphi^* u = \sum_{|K|=p-1} \sum_{j=1}^n P_\varphi \left( \frac{\partial \varphi}{\partial z_j} u_{jK} \right) dz_K,
\]
for a \((p, 0)\)-form \( u \in \text{dom}(\varphi^*) \).

Similar to Lemma 2.2 one shows that \( \varphi : \text{dom}(\varphi) \to A_{p+1,0}^2(\mathbb{C}^n, e^{-\varphi}) \) has closed graph. Now let
\[
\tilde{\Box} = \varphi^* \varphi + \partial \varphi^*,
\]
with \( \text{dom}(\tilde{\Box}) = \{ f \in \text{dom}(\varphi) \cap \text{dom}(\varphi^*) : \varphi f \in \text{dom}(\varphi^*) \text{ and } \varphi^* f \in \text{dom}(\varphi) \} \).

Then \( \tilde{\Box} \) acts as unbounded self-adjoint operator on \( A_{p,0}^2(\mathbb{C}^n, e^{-\varphi}) \), see [1], [7], [15].

In the following we prove an identity which is analogous to the Kohn-Morrey formula for the \( \overline{\partial} \)-complex (see [15], [7]). Now an additional non-positive term appears, which vanishes for the weighted \( \overline{\partial} \)-complex.

**Theorem 3.2.** Let \( u = \sum_{j=1}^n u_j dz_j \in A_{(1,0)}^2(\mathbb{C}^n, e^{-\varphi}) \) and suppose that \( u \in \text{dom}(\varphi) \cap \text{dom}(\varphi^*) \). Then
\[
\|\varphi^* u\|_\varphi^2 = \sum_{j<k} \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial z_k} - \frac{\partial u_k}{\partial z_j} \right|^2 e^{-\varphi} \, d\lambda + \sum_{j,k=1}^n \int_{\mathbb{C}^n} \frac{\partial^2 \varphi}{\partial z_k \partial z_j} u_j \overline{u_k} e^{-\varphi} \, d\lambda
\]
\[
- \sum_{j,k=1}^n \left( \frac{\partial \varphi}{\partial z_j} u_j - P_\varphi \left( \frac{\partial \varphi}{\partial z_j} u_j \right), \frac{\partial \varphi}{\partial z_k} u_k \right)_\varphi.
\]

**Proof.** We get since
\[
\partial u = \sum_{j<k} \left( \frac{\partial u_j}{\partial z_k} - \frac{\partial u_k}{\partial z_j} \right) dz_j \wedge dz_k \text{ and } \varphi^* u = \sum_{j=1}^n P_\varphi \left( \frac{\partial \varphi}{\partial z_j} u_j \right)
\]
that
\[
\|\varphi^* u\|_\varphi^2 = \int_{\mathbb{C}^n} \sum_{j<k} \left| \frac{\partial u_j}{\partial z_k} - \frac{\partial u_k}{\partial z_j} \right|^2 e^{-\varphi} \, d\lambda + \int_{\mathbb{C}^n} \sum_{j,k=1}^n P_\varphi \left( \frac{\partial \varphi}{\partial z_j} u_j \right) \overline{P_\varphi \left( \frac{\partial \varphi}{\partial z_k} u_k \right)} e^{-\varphi} \, d\lambda
\]

\[ \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \left( \frac{\partial u_j}{\partial z_k} \right)^2 e^{-\varphi} \, d\lambda + \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \left( P_\varphi \frac{\partial \varphi}{\partial z_j} u_j P_\varphi \frac{\partial \varphi}{\partial z_k} u_k - \frac{\partial u_j}{\partial z_k} \frac{\partial u_k}{\partial z_j} \right) e^{-\varphi} \, d\lambda \]

\[ = \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \left( \frac{\partial u_j}{\partial z_k} \right)^2 e^{-\varphi} \, d\lambda + \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \left[ \frac{\partial}{\partial z_k} P_\varphi \circ \frac{\partial \varphi}{\partial z_j} \right] u_j \overline{u_k} e^{-\varphi} \, d\lambda, \]

where we used the fact that for \( f, g \in A^2(\mathbb{C}^n, e^{-\varphi}) \) we have

\[ \left( \frac{\partial f}{\partial z_k}, g \right)_\varphi = \left( f, P_\varphi \frac{\partial \varphi}{\partial z_k} g \right)_\varphi \]

and hence

\[ \left( \left[ \frac{\partial}{\partial z_k}, P_\varphi \circ \frac{\partial \varphi}{\partial z_j} \right] u_j, u_k \right)_{\varphi} = \left( \left( \frac{\partial}{\partial z_k} P_\varphi \right) \left( \frac{\partial \varphi}{\partial z_j} u_j \right), u_k \right)_{\varphi} + \left( \frac{\partial^2 \varphi}{\partial z_k \partial z_j} u_j, u_k \right)_{\varphi}. \]

Since we have

\[ \left( \left[ \frac{\partial}{\partial z_k}, P_\varphi \circ \frac{\partial \varphi}{\partial z_j} \right] u_j, u_k \right)_{\varphi} = \left( \left( \frac{\partial}{\partial z_k} P_\varphi \right) \left( \frac{\partial \varphi}{\partial z_j} u_j \right), u_k \right)_{\varphi} + \left( \frac{\partial^2 \varphi}{\partial z_k \partial z_j} u_j, u_k \right)_{\varphi}, \]

so we get the desired result.

**Remark 3.3.** The last term

\[ \sum_{j,k=1}^{n} \left( \frac{\partial \varphi}{\partial z_j} u_j - P_\varphi \frac{\partial \varphi}{\partial z_j} u_j, \frac{\partial \varphi}{\partial z_k} u_k \right)_{\varphi} \]

vanishes for \( \varphi(z) = |z_1|^2 + \cdots + |z_n|^2 \), and we obtain

\[ \sum_{j,k=1}^{n} \left( \frac{\partial \varphi}{\partial z_j} u_j - P_\varphi \frac{\partial \varphi}{\partial z_j} u_j, \frac{\partial \varphi}{\partial z_k} u_k \right)_{\varphi} = 0 \]

so we obtain

\[ ||\partial u||^2_\varphi + ||\partial^*_\varphi u||^2_\varphi = \sum_{j=1}^{n} \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial z_j} \right|^2 e^{-|z|^2} \, d\lambda + \sum_{j=1}^{n} \int_{\mathbb{C}^n} |u_j|^2 e^{-|z|^2} \, d\lambda \]

If \( n = 1 \) and \( u \) is a \((1,0)\)-form, we have \( \partial u = 0 \) and

\[ ||\partial u||^2_\varphi + ||\partial^*_\varphi u||^2_\varphi = ||\partial^*_\varphi u||^2_\varphi = ||u||^2_\varphi + ||u'||^2_\varphi. \]

**Theorem 3.4.** The last term in Theorem

\[ \sum_{j,k=1}^{n} \left( \frac{\partial \varphi}{\partial z_j} u_j - P_\varphi \frac{\partial \varphi}{\partial z_j} u_j, \frac{\partial \varphi}{\partial z_k} u_k \right)_{\varphi} \]

is always non-negative; we have

\[ \sum_{j,k=1}^{n} \left( \frac{\partial \varphi}{\partial z_j} u_j - P_\varphi \frac{\partial \varphi}{\partial z_j} u_j, \frac{\partial \varphi}{\partial z_k} u_k \right)_{\varphi} \]
\[
\sum_{j,k=1}^{n} \left( \left[ \frac{\partial}{\partial z_k}, \frac{\partial \varphi}{\partial z_j} \right] u_j, u_k \right) \varphi - \sum_{j,k=1}^{n} \left( \left[ \frac{\partial}{\partial z_k}, P_{\varphi} \circ \frac{\partial \varphi}{\partial z_j} \right] u_j, u_k \right) \varphi
\]

\[
= \| R_{\varphi} v_1 + \cdots + R_{\varphi} v_n \|_{\varphi}^2 = \| V \|_{\varphi}^2 - \| P_{\varphi} V \|_{\varphi}^2,
\]

where \( R_{\varphi} \) denotes the orthogonal projection \( R_{\varphi} = I - P_{\varphi} \) and

\[
V = \sum_{j=1}^{n} v_j = \sum_{j=1}^{n} \frac{\partial \varphi}{\partial z_j} u_j.
\]

**Proof.** Since

\[
\sum_{j,k=1}^{n} \left( \left[ \frac{\partial}{\partial z_k}, \frac{\partial \varphi}{\partial z_j} \right] u_j, u_k \right) \varphi - \sum_{j,k=1}^{n} \left( \left[ \frac{\partial}{\partial z_k}, P_{\varphi} \circ \frac{\partial \varphi}{\partial z_j} \right] u_j, u_k \right) \varphi
\]

\[
= - \sum_{j,k=1}^{n} \left( \left[ \frac{\partial}{\partial z_k}, P_{\varphi} \right] \left( \frac{\partial \varphi}{\partial z_j} u_j, u_k \right) \right) \varphi,
\]

which equals

\[
\sum_{j=1}^{n} \left( \frac{\partial \varphi}{\partial z_j} u_j - P_{\varphi} \left( \frac{\partial \varphi}{\partial z_j} u_j \right), \frac{\partial \varphi}{\partial z_k} u_k \right) \varphi,
\]

by the last computation in the proof of Theorem 3.2. This term can be written in the form

\[
\sum_{j,k=1}^{n} (R_{\varphi} v_j, v_k)_{\varphi} = \sum_{j,k=1}^{n} (R_{\varphi} v_j, R_{\varphi} v_k)_{\varphi}
\]

\[
= (R_{\varphi} v_1 + \cdots + R_{\varphi} v_n, R_{\varphi} v_1 + \cdots + R_{\varphi} v_n)_{\varphi}
\]

\[
= \| R_{\varphi} v_1 + \cdots + R_{\varphi} v_n \|_{\varphi}^2
\]

\[
= \| V \|_{\varphi}^2 - \| P_{\varphi} V \|_{\varphi}^2,
\]

and we are done. \( \square \)

**Remark 3.5.** Notice that for \( u = \sum_{j=1}^{n} u_j d z_j \in dom(\partial) \cap dom(\partial^*) \) we have

\[
\left\| \frac{\partial u_j}{\partial z_k} \right\|_{\varphi}^2 = \left\| \frac{\partial \varphi}{\partial z_k} u_j \right\|_{\varphi}^2 - \int_{C^n} \frac{\partial^2 \varphi}{\partial z_k \partial z_k} |u_j|^2 e^{-\varphi} \, d\lambda.
\]
This follows from
\[ \left\| \frac{\partial u_j}{\partial z_k} \right\|_\varphi^2 = (P_\varphi \left( \frac{\partial \varphi}{\partial z_k} \frac{\partial u_j}{\partial z_k} \right), u_j)_\varphi = (\frac{\partial \varphi}{\partial z_k}, u_j) \varphi = (\frac{\partial u_j}{\partial z_k}, \frac{\partial \varphi}{\partial z_k} u_j) \varphi = -(u_j, \frac{\partial \varphi}{\partial z_k} (\frac{\partial \varphi}{\partial z_k} u_j e^{-\varphi})) = -(u_j, \frac{\partial^2 \varphi}{\partial z_k^2} u_j) \varphi + (u_j, \frac{\partial \varphi}{\partial z_k} \frac{\partial \varphi}{\partial z_k} u_j) \varphi \]
= \left\| \frac{\partial \varphi}{\partial z_k} u_j \right\|_\varphi^2 - \int_{\mathbb{C}^n} \frac{\partial^2 \varphi}{\partial z_k^2} |u_j|^2 e^{-\varphi} d\lambda, \]
where we used again that the components of $u_j$ are holomorphic.

Now we generalize Theorem 3.2 for $(p,0)$-forms $u = \sum_{|J|=p} u_J dz_J$ with coefficients in $A^2(\mathbb{C}^n, e^{-\varphi})$ where $1 \leq p \leq n - 1$. We notice that
\[ \partial u = \sum'_{|J|=p} \sum_{j=1}^n \frac{\partial u_j}{\partial z_j} dz_j \wedge dz_J, \]
and
\[ \partial^* u = \sum'_{|K|=p-1} \sum_{j=1}^n P_\varphi \left( \frac{\partial \varphi}{\partial z_j} u_{jK} \right) dz_K. \]
We obtain
\[ \|\partial u\|_\varphi^2 + \|\partial^* u\|_\varphi^2 = \sum'_{|J|=p} \sum_{j=1}^n \int_{\mathbb{C}^n} \frac{\partial u_j}{\partial z_j} \frac{\partial u_j}{\partial z_k} e^{-\varphi} d\lambda \]
+ \sum'_{|K|=p-1} \sum_{j,k=1}^n \int_{\mathbb{C}^n} P_\varphi \left( \frac{\partial \varphi}{\partial z_j} u_{jK} \right) P_\varphi \left( \frac{\partial \varphi}{\partial z_k} u_{kK} \right) e^{-\varphi} d\lambda, \]
where $\epsilon_{j,k}^{KM} = 0$ if $j \notin J$ or $k \notin M$ or if $k \cup M \neq j \cup J$, and equals the sign of the permutation $(j,k)$ otherwise. The right-hand side of the last formula can be rewritten as (3.7)
\[ \sum'_{|J|=p} \sum_{j=1}^n \left\| \frac{\partial u_j}{\partial z_j} \right\|_\varphi^2 + \sum'_{|K|=p-1} \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left( P_\varphi \left( \frac{\partial \varphi}{\partial z_j} u_{jK} \right) P_\varphi \left( \frac{\partial \varphi}{\partial z_k} u_{kK} \right) - \frac{\partial u_j}{\partial z_k} \frac{\partial u_k}{\partial z_j} \right) e^{-\varphi} d\lambda, \]
In order to prove this we first consider the (nonzero) terms where $j = k$ (and hence $M = J$). These terms result in the portion of the first sum in (3.7) where $j \notin J$. On the other hand, when $j \neq k$, then $j \in M$ and $k \in J$, and deletion of $j$ from $M$ and $k$ from $J$ results in the strictly increasing multi-index $K$ of length $p - 1$. Consequently, these terms can be collected into the second sum in (3.7) (in the part with the minus sign, we have interchanged the summation indices $j$ and $k$). In this sum, the terms where $j = k$ compensate for the terms in the first sum where $j \in J$.
Now one can use the same reasoning as in the last proof to get
\[ \|\partial u\|_\varphi^2 + \|\partial^* u\|_\varphi^2 = \sum'_{|J|=p} \sum_{j=1}^n \left\| \frac{\partial u_j}{\partial z_j} \right\|_\varphi^2 + \sum'_{|K|=p-1} \sum_{j,k=1}^n \int_{\mathbb{C}^n} \frac{\partial^2 \varphi}{\partial z_k \partial z_j} u_{jK} u_{kK} e^{-\varphi} d\lambda \]
Remark 3.6. For \( \varphi(z) = |z_1|^2 + \cdots + |z_n|^2 \) we obtain

\[
(3.9) \quad \| \partial u \| ^2 + \| \partial^* u \| ^2 = \sum_{|J|=p} \sum_{j=1}^{n} \| \partial u_J / \partial z_j \| ^2 + p \sum_{|J|=p} \int_{C^n} |u_J|^2 e^{-|z|^2} d\lambda.
\]

4. The \( \partial \)-Neumann operator on the Fock space

As an immediate consequence of (3.5) and (3.9) we get what is called the basic estimates.

Lemma 4.1. Let \( 1 \leq p \leq n - 1 \) and let \( u = \sum'_{|J|=p} u_J dz_J \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \) and suppose that \( u \in \text{dom}(\partial) \cap \text{dom}(\partial^*) \). Then

\[
(4.1) \quad \| u \| ^2 \leq \frac{1}{p} (\| \partial u \| ^2 + \| \partial^* u \| ^2).
\]

The proof of the last results follows easily from the corresponding results for general Fock spaces, see Theorem 3.2 and (3.9).

Now we can use the machinery of the classical \( \partial \)-Neumann operator to show the following results.

Lemma 4.2. Both operators \( \partial \) and \( \partial^* \) have closed range.

If we endow \( \text{dom}(\partial) \cap \text{dom}(\partial^*) \) with the graph-norm \( (\| \partial f \| ^2 + \| \partial^* f \| ^2)^{1/2} \), the dense subspace \( \text{dom}(\partial) \cap \text{dom}(\partial^*) \) of \( A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \) becomes a Hilbert space.

Proof. We notice that \( \ker \partial = (\text{im} \partial^*)^\perp \), which implies that

\[
(ker \partial)^\perp = \text{im} \partial^* \subseteq \ker \partial^*.
\]

If \( u \in \ker \partial \cap \ker \partial^* \), we have by (4.1) that \( u = 0 \). Hence

\[
(4.2) \quad (ker \partial)^\perp = \ker \partial^*.
\]

If \( u \in \text{dom}(\partial) \cap (ker \partial)^\perp \), then \( u \in \ker \partial^* \), and (4.1) implies

\[
\| u \| \leq \frac{1}{p} \| \partial u \|.
\]

Now we can use general results of unbounded operators on Hilbert spaces (see for instance [7] Chapter 4) to show that \( \text{im} \partial \) and \( \text{im} \partial^* \) are closed. The last assertion follows again by (4.1), see [7] Chapter 4. \( \square \)

The next result describes the implication of the basic estimates (4.1) for the \( \bar{\partial} \)-operator.
Theorem 4.3. The operator $\tilde{\square} : \text{dom}(\tilde{\square}) \to A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ is bijective and has a bounded inverse

$$\tilde{N} : A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \to \text{dom}(\tilde{\square}).$$

In addition

$$(4.3) \quad \|\tilde{N}u\| \leq \frac{1}{p} \|u\|,$$

for each $u \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$.

Proof. Since $(\tilde{\square}u, u) = \|\partial u\|^2 + \|\partial^* u\|^2$, it follows that for a convergent sequence $(\tilde{\square}u_n)_n$ we get

$$\|\tilde{\square}u_n - \tilde{\square}u_m\| \|u_n - u_m\| \geq \|\tilde{\square}(u_n - u_m), u_n - u_m\| \geq \|u_n - u_m\|^2,$$

which implies that $(u_n)_n$ is convergent and since $\tilde{\square}$ is a closed operator we obtain that $\tilde{\square}$ has closed range. If $\tilde{\square}u = 0$, we get $\partial u = 0$ and $\partial^* u = 0$ and by (4.1) that $u = 0$, hence $\tilde{\square}$ is injective. Using again general results on unbounded operators on Hilbert spaces we get that the range of $\tilde{\square}$ is dense, therefore $\tilde{\square}$ is surjective.

We showed that

$$\tilde{\square} : \text{dom}(\tilde{\square}) \to A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$$

is bijective and therefore has a bounded inverse

$$\tilde{N} : A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \to \text{dom}(\tilde{\square}).$$

For $u \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ we use (4.1) for $\tilde{N}u$ to obtain

$$\|\tilde{N}u\|^2 \leq \frac{1}{p} (\|\partial \tilde{N}u\|^2 + \|\partial^* \tilde{N}u\|^2)$$

$$= \frac{1}{p} ((\partial^* \partial \tilde{N}u, \tilde{N}u) + (\partial \partial^* \tilde{N}u, \tilde{N}u))$$

$$= \frac{1}{p} (u, \tilde{N}u)$$

$$\leq \frac{1}{p} \|u\| \|\tilde{N}u\|,$$

which implies (4.3). \hfill \Box

Following the classical $\bar{\partial}$-Neumann calculus we obtain

Theorem 4.4. Let $\tilde{N}_p$ denote the inverse of $\tilde{\square}$ on $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$. Then

$$(4.4) \quad \tilde{N}_{p+1} \partial = \partial \tilde{N}_p,$$

on $\text{dom}(\partial)$ and

$$(4.5) \quad \tilde{N}_{p-1} \partial^* = \partial^* \tilde{N}_p,$$

on $\text{dom}(\partial^*)$.

In addition we have that $\partial^* \tilde{N}_p$ is zero on $(\ker \partial)^\perp$. 

Proof. For \(u \in \text{dom}(\partial)\) we have \(\partial u = \partial \partial^* \tilde{\partial} N_p u\) and
\[
\tilde{\partial} N_{p+1} u = \tilde{\partial} N_{p+1} \partial \partial^* \partial \tilde{\partial} N_p u = \tilde{\partial} N_{p+1} (\partial \partial^* + \partial^* \partial) \partial \tilde{\partial} N_p u = \partial \tilde{\partial} N_p u,
\]
which proves (5.4). In a similar way we get (5.5).

Now let \(k \in (\ker \partial)^\perp\) and \(u \in \text{dom}(\partial)\), then
\[
(\partial^* \tilde{\partial} N_p k, u) = (\tilde{\partial} N_p k, \partial u) = (k, \tilde{\partial} N_p \partial u) = (k, \partial \tilde{\partial} N_{p-1} u) = 0,
\]
since \(\partial \tilde{\partial} N_{p-1} u \in \ker(\partial)\), which gives \(\partial^* \tilde{\partial} N_p k = 0\). □

Now we can also prove a solution formula for the equation \(\partial u = \alpha\), where \(\alpha\) is a given \((p,0)\)-form in \(A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})\) with \(\partial \alpha = 0\).

**Theorem 4.5.** Let \(\alpha \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})\) with \(\partial \alpha = 0\). Then \(u_0 = \partial^* \tilde{\partial} N_p \alpha\) is the canonical solution of \(\partial u = \alpha\), this means \(\partial u_0 = \alpha\) and \(u_0 \in (\ker \partial)^\perp = \text{im} \partial^*\), and
\[
\|\partial^* \tilde{\partial} N_p \alpha\| \leq p^{-1/2} \|\alpha\|.
\]

**Proof.** For \(\alpha \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})\) with \(\partial \alpha = 0\) we get
\[
\alpha = \partial \partial^* \tilde{\partial} N_p \alpha + \partial^* \partial \tilde{\partial} N_p \alpha.
\]
If we apply \(\partial\) to the last equality we obtain:
\[
0 = \partial \alpha = \partial \partial^* \tilde{\partial} N_p \alpha,
\]
and since \(\partial \tilde{\partial} N_p \alpha \in \text{dom}(\partial^*)\) we have
\[
0 = (\partial \partial^* \partial \tilde{\partial} N_p \alpha, \partial \tilde{\partial} N_p \alpha) = (\partial \partial^* \partial \tilde{\partial} N_p \alpha, \partial \partial^* \partial \tilde{\partial} N_p \alpha) = \|\partial \partial^* \tilde{\partial} N_p \alpha\|^2.
\]
Finally we set \(u_0 = \partial^* \tilde{\partial} N_p \alpha\) and derive from (4.7) and (4.8) that for \(\partial \alpha = 0\)
\[
\alpha = \partial u_0,
\]
and we see that \(u_0 \perp \ker \partial\), since for \(h \in \ker \partial\) we get
\[
(u_0, h) = (\partial^* \tilde{\partial} N_p \alpha, h) = (\tilde{\partial} N_p \alpha, \partial h) = 0.
\]

It follows that
\[
\|\partial^* \tilde{\partial} N_p \alpha\|^2 = (\partial \partial^* \tilde{\partial} N_p \alpha, \tilde{\partial} N_p \alpha) = (\partial \partial^* \partial \tilde{\partial} N_p \alpha, \partial \tilde{\partial} N_p \alpha) + (\partial \partial^* \tilde{\partial} N_p \alpha, \tilde{\partial} N_p \alpha) = (\alpha, \tilde{\partial} N_p \alpha) \leq \|\alpha\| \|\tilde{\partial} N_p \alpha\|
\]
and using (4.3) we obtain
\[
\|\partial^* \tilde{\partial} N_p \alpha\| \leq p^{-1/2} \|\alpha\|.
\]

Now we discuss a different approach to the operator \(\tilde{\partial} N\) which is related to the quadratic form
\[
Q(u, v) = (\partial u, \partial v) + (\partial^* u, \partial^* v).
\]
For this purpose we consider the embedding

\[ \iota : \text{dom}(\partial) \cap \text{dom}(\partial^*) \to A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}), \]

where \( \text{dom}(\partial) \cap \text{dom}(\partial^*) \) is endowed with the graph-norm

\[ u \mapsto (\|\partial u\|^2 + \|\partial^* u\|^2)^{1/2}. \]

The graph-norm stems from the inner product

\[ Q(u, v) = (u, v)_Q = (\partial u, \partial v) + (\partial^* u, \partial^* v). \]

The basic estimates (1.1) imply that \( \iota \) is a bounded operator with operator norm

\[ \|\iota\| \leq \frac{1}{\sqrt{p}}. \]

By (1.1) it follows in addition that \( \text{dom}(\partial) \cap \text{dom}(\partial^*) \) endowed with the graph-norm \( u \mapsto (\|\partial u\|^2 + \|\partial^* u\|^2)^{1/2} \) is a Hilbert space, see Lemma 1.2

Since \((u, v) = (u, v)_Q\), we have that \((u, v) = (\iota u, v)_Q\).

For \( u \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \) and \( v \in \text{dom}(\partial) \cap \text{dom}(\partial^*) \) we get

\[ (u, v) = (\tilde{\nabla}N u, v) = ((\partial \partial^* + \partial^* \partial)\tilde{N} u, v) = (\partial^* \tilde{N} u, \partial^* v) + (\partial \tilde{N} u, \partial v). \]

Equation (1.9) suggests that as an operator to \( \text{dom}(\partial) \cap \text{dom}(\partial^*) \), \( \tilde{N} \) coincides with \( \iota^* \) and as an operator to \( A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \), \( \tilde{N} \) is equal to \( \iota \circ \iota^* \), see [7] or [15] for the details.

Hence \( \tilde{N} \) is compact if and only if the embedding

\[ \iota : \text{dom}(\partial) \cap \text{dom}(\partial^*) \to A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}), \]

is compact. This will be used to prove the following theorem.

**Theorem 4.6.** The operator \( \tilde{N} : A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \to A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}), \ 1 \leq p \leq n, \) is compact.

**Proof.** First we consider the case when \( p = 1 \). We use (3.5) for the graph norm on \( \text{dom}(\partial) \cap \text{dom}(\partial^*) \) and indicate that it suffices to consider one component \( u_j \) of the 

(1, 0)-form \( u \). For this purpose we will denote \( u_j \) by \( f \). We have to handle

\[ \sum_{k=1}^{n} \int_{\mathbb{C}^n} \left| \frac{\partial f}{\partial z_k} \right|^2 e^{-|z|^2} d\lambda + \int_{\mathbb{C}^n} |f|^2 e^{-|z|^2} d\lambda \]

for the graph-norm. We use the complete orthonormal system \( (\varphi_\alpha)_\alpha \) of \( A^2(\mathbb{C}^n, e^{-|z|^2}) \).

Let \( f = \sum_\alpha f_\alpha \varphi_\alpha \) be an element of \( \text{dom}(\partial) \cap \text{dom}(\partial^*) \). We have \( \iota(f) = f \) and hence

\[ \iota(f) = \sum_\alpha (f, \varphi_\alpha) \varphi_\alpha \]

in \( A^2(\mathbb{C}^n, e^{-|z|^2}) \). The basis elements \( \varphi_\alpha \) are normalized in \( A^2(\mathbb{C}^n, e^{-|z|^2}) \). First we have to compute the graph-norm of the basis elements \( \varphi_\alpha \). Notice that

\[ \frac{\partial \varphi_\alpha}{\partial z_k} = \frac{1}{\sqrt{\pi^n} \sqrt{\alpha_1!} \cdots \sqrt{\alpha_k!} \sqrt{\alpha_{k+1}!} \cdots \sqrt{\alpha_n!}} z_{\alpha_1} \cdots z_{\alpha_k-1} \zeta_{\alpha_{k+1}} \cdots \zeta_{\alpha_n} = \sqrt{\alpha_k} \varphi(ak-1), \]

where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \alpha_k \) is the component in the \( k \)-th position.
where \((\alpha k - 1) = (\alpha_1, \ldots, \alpha_{k-1}, \alpha_k - 1, \alpha_{k+1}, \ldots, \alpha_n)\). Hence the graph-norm of the basis elements \(\varphi_{\alpha}\) equals
\[
\|\varphi_{\alpha}\|_Q = (\|\varphi_{\alpha}\|^2 + \sum_{k=1}^{n} \left| \frac{\partial \varphi_{\alpha}}{\partial z_k} \right|^2)^{1/2} = \sqrt{1 + |\alpha|},
\]
where \(|\alpha| = \alpha_1 + \cdots + \alpha_n\).

Now let
\[
\psi_{\alpha} = \frac{\varphi_{\alpha}}{\sqrt{1 + |\alpha|}}.
\]
Then \((\psi_{\alpha})_\alpha\) constitutes a complete orthonormal system in the Hilbert space \(\text{dom}(\partial) \cap \text{dom}(\partial^*)\) endowed with the graph-norm, notice that
\[
(f, \psi_{\alpha})_Q = \sum_{k=1}^{n} \frac{\alpha_k}{\sqrt{1 + |\alpha|}} f_{\alpha} + \frac{1}{\sqrt{1 + |\alpha|}} f_{\alpha} = \sqrt{1 + |\alpha|} f_{\alpha},
\]
and we have
\[
\iota(f) = f = \sum_{\alpha} (f, \psi_{\alpha})_Q \psi_{\alpha}.
\]

For the norm of \(A^2(\mathbb{C}^n, e^{-|z|^2})\) we have
\[
\|\iota(f) - \sum_{|\alpha| \leq N} (f, \psi_{\alpha})_Q \psi_{\alpha}\|^2 = \|\sum_{|\alpha| \geq N+1} (f, \psi_{\alpha})_Q \psi_{\alpha}\|^2
\]
\[
= \|\sum_{|\alpha| \geq N+1} \frac{1}{\sqrt{1 + |\alpha|}} (f, \psi_{\alpha})_Q \psi_{\alpha}\|^2
\]
\[
= \sum_{|\alpha| \geq N+1} \left| \frac{1}{\sqrt{1 + |\alpha|}} (f, \psi_{\alpha})_Q \right|^2
\]
\[
\leq \frac{\|f\|_Q^2}{N + 2},
\]
where we finally used Bessel’s inequality for the Hilbert space \(\text{dom}(\partial) \cap \text{dom}(\partial^*)\) endowed with the graph-norm. This proves that
\[
\iota : \text{dom}(\partial) \cap \text{dom}(\partial^*) \rightarrow A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2})
\]
is a compact operator and the same is true for
\[
\tilde{N} : A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}) \rightarrow A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}).
\]
For arbitrary \(p\) between 1 and \(n\) we can use (3.9) and the same reasoning as before to get the desired conclusion. 

Compare with the \(\overline{\partial}\)–Neumann operator \(N\) on \(L^2(\mathbb{C}^n, e^{-|z|^2})\): in this case \(N\) fails to be compact, see [7]. This is related to the fact that the kernel of \(\overline{\partial}\) is large (it is the Fock space) in case of the \(\overline{\partial}\)-complex, but the kernel of \(\partial\) consists just of the constant functions in case of the \(\partial\)-complex on the Fock space.

In order to compute the spectrum of the operator \(\square_p\) we will use the following
Lemma 4.7. Let \( A \) be a symmetric operator on a Hilbert space \( H \) with domain \( \text{dom}(A) \), and suppose that \( (x_k)_k \) is a complete orthonormal system in \( H \). If each \( x_k \) lies in \( \text{dom}(A) \), and there exist \( \lambda_k \in \mathbb{R} \) such that
\[
Ax_k = \lambda_k x_k
\]
for every \( k \in \mathbb{N} \), then \( A \) is essentially self-adjoint and the spectrum of \( \overline{A} \) is the closure in \( \mathbb{R} \) of the set of all \( \lambda_k \).

See [3] or [7].

Theorem 4.8. The spectrum of \( \tilde{\Box}_p \), where \( 0 \leq p \leq n \), consists of all numbers \( m + p \) for \( m = 0, 1, 2, \ldots, \) where \( m + p \) has multiplicity \((n+|\alpha|-1)\binom{n}{p}\).

Proof. Recall that the monomials \((\varphi_\alpha)_\alpha\), where \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \) is a multiindex, constitute a complete orthonormal system in \( A^2(\mathbb{C}^n, e^{-|z|^2}) \). We use (2.5) and compute
\[
\sum_{k=1}^n z_k \frac{\partial \varphi_\alpha}{\partial z_k} + p \varphi_\alpha = (|\alpha| + p) \varphi_\alpha.
\]
We use Lemma 4.7 and indicate that there are \((n+|\alpha|-1)\) monomials of degree \( |\alpha| \). Hence we get the assertion about the multiplicity from the fact that \( A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \) is the direct sum of \( \binom{n}{p} \) copies of \( A^2(\mathbb{C}^n, e^{-|z|^2}) \).

The last result implies also that the inverse \( \tilde{N}_p \) of \( \tilde{\Box}_p \) is a compact operator with the eigenvalues \( 1/(m + p) \).

Note that the complex Laplacian \( \Box_q \) of the \( \overline{\partial} \)-complex on \( L^2(\mathbb{C}^n, e^{-|z|^2}) \) has also the eigenvalues \( q + m \) for \( m = 0, 1, 2, \ldots, \) but each of them have infinite multiplicity, see [10], [6], [7].

5. The general \( \partial \)-complex

Now we return to the classical Fock space but replace a single derivative with respect to \( z_j \) by a differential operator of the form \( p_j(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}) \), where \( p_j \) is a complex polynomial on \( \mathbb{C}^n \), see [11], [12]. We consider the densely defined operators
\[
(5.1) \quad Du = \sum_{j=1}^n p_j(u) \, dz_j,
\]
where \( u \in A^2(\mathbb{C}^n, e^{-|z|^2}) \) and \( p_j(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}) \) are polynomial differential operators with constant coefficients.

More general we define
\[
(5.2) \quad Du = \sum_{|J|=p} ' \sum_{k=1}^n p_k(u_J) \, dz_k \wedge dz_J,
\]

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where \( u = \sum_{|J|=p} u_J d z_j \) is a \((p,0)\)-form with coefficients in \( A^2(\mathbb{C}^n, e^{-|z|^2}) \).

It is clear that \( D^2 = 0 \) and that we have

\[
(Du, v) = (u, D^*v),
\]

where \( u \in \text{dom}(D) = \{ u \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) : Du \in A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2}) \} \) and

\[
D^*v = \sum_{|K|=p-1} \left( \sum_{j=1}^n p_j^* v_{jK} dz_K \right)
\]

for \( v = \sum_{|J|=p} v_J d z_J \) and where \( p_j^* (z_1, \ldots, z_n) \) is the polynomial \( p_j \) with complex conjugate coefficients, taken as multiplication operator.

Now the corresponding \( D \)-complex has the form

\[
A^2_{(p-1,0)}(\mathbb{C}^n, e^{-|z|^2}) \xrightarrow{D} A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \xrightarrow{D^*} A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2}).
\]

In the sequel we consider the generalized box operator

\[
\tilde{\Box}_{D,p} := D^* D + D D^*
\]

as a densely defined self-adjoint operator on \( A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \) with \( \text{dom}(\tilde{\Box}_{D,p}) = \{ f \in \text{dom}(D) \cap \text{dom}(D^*) : Df \in \text{dom}(D^*) \text{ and } D^* f \in \text{dom}(D) \} \).

We want to find conditions under which \( \tilde{\Box}_{D,1} \) has a bounded inverse. For this purpose we have to consider the graph norm \( (\|Du\|^2 + \|D^*u\|^2)^{1/2} \) on \( \text{dom}(D) \cap \text{dom}(D^*) \).

**Theorem 5.1.** Let \( u = \sum_{j=1}^n u_j d z_j \in \text{dom}(D) \cap \text{dom}(D^*) \) and suppose that there exists a constant \( C > 0 \) such that

\[
\|u\|^2 \leq C \sum_{j,k=1}^n (|p_k, p_j^*| u_j, u_k).
\]

Then

\[
\|u\|^2 \leq C (\|Du\|^2 + \|D^*u\|^2).
\]

**Proof.** First we have

\[
Du = \sum_{j<k} (p_k(u_j) - p_j(u_k)) dz_j \wedge dz_k \quad \text{and} \quad D^*u = \sum_{j=1}^n p_j^* u_j,
\]

hence

\[
\|Du\|^2 + \|D^*u\|^2 = \int_{\mathbb{C}^n} \sum_{j<k} |p_k(u_j) - p_j(u_k)|^2 e^{-|z|^2} d\lambda + \int_{\mathbb{C}^n} \sum_{j,k=1}^n p_j^* u_j \overline{p_k u_k} e^{-|z|^2} d\lambda
\]

\[
= \sum_{j,k=1}^n \int_{\mathbb{C}^n} |p_k(u_j)|^2 e^{-|z|^2} d\lambda + \sum_{j,k=1}^n \int_{\mathbb{C}^n} (p_j^* u_j \overline{p_k u_k} - p_k(u_j) \overline{p_j(u_k)}) e^{-|z|^2} d\lambda
\]

\[
= \sum_{j,k=1}^n \int_{\mathbb{C}^n} |p_k(u_j)|^2 e^{-|z|^2} d\lambda + \sum_{j,k=1}^n \int_{\mathbb{C}^n} |p_k p_j^* u_j \overline{u_k} e^{-|z|^2} d\lambda,
\]

where we used \((5.3)\). Now the assumption \((5.4)\) implies the desired result. \(\square\)
Let $1 \leq p \leq n - 1$ and let $u = \sum_{|J|=p} u_J dz_J \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ and suppose that $u \in \text{dom}(D) \cap \text{dom}(D^*)$. In a similar way as in (\ref{3.8}) we get

\begin{equation}
\|Du\|^2 + \|D^*u\|^2 = \sum_{|J|=p} \left( \sum_{k=1}^n \|p_k(u_J)\|^2 + \sum_{|K|=p-1} \sum_{j,k=1}^n \int_{\mathbb{C}^n} |p_k, p_j^* u_{jK} u_{kK}| e^{-|z|^2} d\lambda, \right.
\end{equation}

and if we suppose that

\begin{equation}
\|u\|^2 \leq C \sum_{|K|=p-1} \left( \sum_{j,k=1}^n (|p_k, p_j^*| u_{jK}, u_{kK}) \right),
\end{equation}

we get the basic estimate \((5.5)\), which also implies that both $\text{im}D$ and $\text{im}D^*$ are closed, see for instance \cite{7}, Chapter 4. With the basic estimate \((5.5)\) we are now able to use the machinery of the corresponding Neumann operator - the bounded inverse of $\tilde{D}_{D,p}$ - (see Theorem 4.3 and Theorem 4.5) and get the following results

**Theorem 5.2.** Let $D$ be as in (\ref{5.2}) and suppose that

\[
\|u\|^2 \leq C \sum_{|K|=p-1} \left( \sum_{j,k=1}^n (|p_k, p_j^*| u_{jK}, u_{kK}) \right),
\]

for all $u \in \text{dom}(D) \cap \text{dom}(D^*)$. Then $\tilde{D}_{D,p}$ has a bounded inverse

\[\tilde{N}_{D,p} : A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \rightarrow \text{dom}(\tilde{D}_{D,p}).\]

If $\alpha \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ satisfies $D\alpha = 0$, then $u_0 = D^*\tilde{N}_{D,p}\alpha$ is the canonical solution of $Du = \alpha$, this means $Du_0 = \alpha$ and $u_0 \in (\ker D)^\perp = \text{im}D^*$, and $\|D^*\tilde{N}_{D,p}\alpha\| \leq C\|\alpha\|$, for some constant $C > 0$ independent of $\alpha$.

**Example 5.3.** a) Let $p_k = \frac{\partial^2}{\partial z_j^2}$. Then $p_j^*(z) = z_j^2$ and we have

\[
\sum_{j,k=1}^n (|p_k, p_j^*| u_j, u_k) = \sum_{j,k=1}^n (2\delta_{j,k} u_j, u_k) + \sum_{j,k=1}^n (4\delta_{j,k} z_j \frac{\partial u_j}{\partial z_k}, u_k) = 2\|u\|^2 + 4 \sum_{j=1}^n \left( \frac{\partial u_j}{\partial z_j} \right)^2,
\]

for $u = \sum_{j=1}^n u_j dz_j \in \text{dom}(D) \cap \text{dom}(D^*)$. Hence (\ref{5.3}) is satisfied.

b) Let $n = 2$ and take $p_1 = \frac{\partial^2}{\partial z_1 \partial z_2}$ and $p_2 = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2}$. Then $p_1^*(z) = z_1 z_2$ and $p_2^*(z) = z_1^2 + z_2^2$ and we have

\[
\begin{align*}
([p_1, p_1^*]u_1, u_1) &= (u_1, u_1) + \left( \frac{\partial u_1}{\partial z_1}, \frac{\partial u_1}{\partial z_1} \right) + \left( \frac{\partial u_1}{\partial z_2}, \frac{\partial u_1}{\partial z_2} \right), \\
([p_1, p_2^*]u_2, u_1) &= 2 \left( \frac{\partial u_2}{\partial z_1} \frac{\partial u_1}{\partial z_1} + 2 \left( \frac{\partial u_2}{\partial z_2}, \frac{\partial u_1}{\partial z_1} \right), \\
([p_2, p_1^*]u_1, u_2) &= 2 \left( \frac{\partial u_1}{\partial z_1}, \frac{\partial u_2}{\partial z_1} \right) + 2 \left( \frac{\partial u_1}{\partial z_2}, \frac{\partial u_2}{\partial z_1} \right), \\
([p_2, p_2^*]u_2, u_2) &= 4(u_2, u_2) + 4 \left( \frac{\partial u_2}{\partial z_1}, \frac{\partial u_2}{\partial z_1} \right) + 4 \left( \frac{\partial u_2}{\partial z_2}, \frac{\partial u_2}{\partial z_2} \right).
\end{align*}
\]

So we obtain

So we obtain
\[
\sum_{j,k=1}^{2} ([p_k, p_j^*] u_j, u_k) = \int_{\mathbb{C}^2} \left( |u_1|^2 + 4|u_2|^2 + \left| \frac{\partial u_1}{\partial z_1} + 2 \frac{\partial u_2}{\partial z_2} \right|^2 + \left| \frac{\partial u_1}{\partial z_2} + 2 \frac{\partial u_2}{\partial z_1} \right|^2 \right) e^{-|z|^2} \, d\lambda,
\]

for \( u = \sum_{j=1}^{2} u_j d z_j \in \text{dom}(D) \cap \text{dom}(D^*) \). Again, (5.4) is satisfied.

We remark that we can interchange the roles of \( D \) and \( D^* \) and obtain

**Theorem 5.4.** Suppose that \( n > 1 \) and \( 1 \leq p \leq n - 1 \). Let \( D \) be as in (5.2) and suppose that

\[
\|u\|^2 \leq C \sum_{|K| = p-1} \sum_{j,k=1}^{n} ([p_k, p_j^*] u_{jK}, u_{kK}),
\]

for all \( u \in \text{dom}(D) \cap \text{dom}(D^*) \). If \( \beta \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \) satisfies \( D^* \beta = 0 \), then \( v_0 = D\tilde{N}_{D,p} \beta \in A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2}) \) is the canonical solution of \( D^* v = \beta \), this means \( D^* v_0 = \beta \) and \( v_0 \in (\ker D^*)^\perp = \text{im} D, \) and \( \|D\tilde{N}_{D,p} \beta\| \leq C\|\beta\| \), for some constant \( C > 0 \) independent of \( \beta \).

**Proof.** As in Theorem 5.2 we get that \( \square_{D,p} \) has a bounded inverse

\[
\tilde{N}_{D,p} : A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \to \text{dom}(\square_{D,p}).
\]

Now, using the \( \partial \)-Neumann calculus, we obtain that

\[
0 = D^* \beta = D^*(D^* D + DD^*) \tilde{N}_{D,p} \beta = D^* DD^* \tilde{N}_{D,p} \beta,
\]

hence

\[
0 = (D^* DD^* \tilde{N}_{D,p} \beta, D^* \tilde{N}_{D,p} \beta) = (DD^* \tilde{N}_{D,p} \beta, DD^* \tilde{N}_{D,p} \beta).
\]

This implies that \( DD^* \tilde{N}_{D,p} \beta = 0 \) and we get

\[
D^* v_0 = D^* D \tilde{N}_{D,p} \beta = (DD^* + D^* D) \tilde{N}_{D,p} \beta = \beta,
\]

and \((v_0, f) = (\tilde{N}_{D,p} \beta, D^* f) = 0\), for all \( f \in \ker D^* \).

\[\square\]

**Example 5.5.** We take Example 5.3 b) and consider \( f = f_1 \, dz_1 + f_2 \, dz_2 \in A^2_{(1,0)}(\mathbb{C}^2, e^{-|z|^2}) \) such that \( D^* f = p_1^* f_1 + p_2^* f_2 = 0 \). By Theorem 5.4 we get

\[
g = g \, dz_1 \wedge dz_2 = D\tilde{N}_{D,1} f \in A^2_{(2,0)}(\mathbb{C}^2, e^{-|z|^2})
\]

such that \( D^* g = -p_2^* g \, dz_1 + p_1^* g \, dz_2 = f \) and \( \|D\tilde{N}_{D,1} f\| \leq C\|f\| \), for some constant \( C > 0 \).

**Remark 5.6.** Finally we point out that the \( \partial \)-Neumann operator \( \tilde{N}_{D,p} \) exists and is bounded on \( A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \) if and only if the basic estimate (5.5) holds, see for instance [7], Remark 9.12., for the details.
References

1. So-Chin Chen and Mei-Chi Shaw, *Partial differential equations in several complex variables*, Studies in Advanced Mathematics, vol. 19, Amer. Math. Soc., 2001.
2. Olivia Constantin, *A Volterra-type integration operator on Fock spaces*, Proc. AMS 140 (2012), 4247–4257.
3. E.B. Davies, *Spectral theory and differential operators*, Cambridge studies in advanced mathematics, vol. 42, Cambridge University Press, Cambridge, 1995.
4. L.D. Faddeev and O.A. Yakubovskii, *Lectures on quantum mechanics for mathematics students*, Student Mathematical Library, vol. 47, American Mathematical Society, 2009.
5. G.B. Folland, *Harmonic analysis in phase space*, Annals of Mathematics Studies 122, Princeton University Press, Princeton, 1989.
6. F. Haslinger, *Spectrum of the $\overline{\partial}$-Neumann Laplacian on the Fock space*, J. of Math. Anal. and Appl. 402 (2013), 739–744.
7. ______, *The $\overline{\partial}$-Neumann problem and Schrödinger Operators*, de Gruyter Expositions in Mathematics, vol. 59, Walter de Gruyter, 2014.
8. J. Kohn, *Harmonic integrals on strongly pseudoconvex manifolds, I*, Ann. of Math. 78 (1963), 112–148.
9. ______, *Harmonic integrals on strongly pseudoconvex manifolds, II*, Ann. of Math. 79 (1964), 450–472.
10. X. Ma and G. Marinescu, *Holomorphic Morse Inequalities and Bergman Kernels*, Progress in Mathematics, vol. 254, Birkhäuser, 2007.
11. D.J. Newman and H.S. Shapiro, *Certain Hilbert spaces of entire functions*, Bull. Amer. Math. Soc. 72 (1966), 971–977.
12. ______, *Fischer spaces of entire functions*, Entire functions and related parts of analysis, Proc. Pure Math., La Jolla, Calif. 1966, Amer. Math. Soc., Providence, R.I., 1968, pp. 360–369.
13. H. Render, *Real Bargmann spaces, Fischer decompositions and sets of uniqueness for polyharmonic functions*, Duke Math. J. 142 (2008), 313–352.
14. I. Shigekawa, *Spectral properties of Schrödinger operators with magnetic fields for a spin 1/2 particle*, J. of Functional Analysis 101 (1991), 255–285.
15. E. Straube, *The $L^2$-Sobolev theory of the $\overline{\partial}$-Neumann problem*, ESI Lectures in Mathematics and Physics, EMS, 2010.
16. J. Weidmann, *Linear Operators in Hilbert Spaces*, Graduate Texts in Mathematics, Springer, 1980.
17. K. Zhu, *Analysis on Fock spaces*, Graduate Texts in Mathematics, Springer, 2012.

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