Alternating sums of the powers of Fibonacci and Lucas numbers

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ALTERNATING SUMS OF THE POWERS OF FIBONACCI AND LUCAS NUMBERS

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Abstract. We shall consider alternating Melham’s sums for Fibonacci and Lucas numbers of the form
\[\sum_{k=1}^{n} (-1)^k \frac{F_{2m+e}}{2k+\delta} \] and
\[\sum_{k=1}^{n} (-1)^k \frac{L_{2m+e}}{2k+\delta},\]
where \(e, \delta \in \{0, 1\} \).

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1. Introduction

The Fibonacci \(F_n\) and Lucas numbers \(L_n\) are defined by the following recursions: for \(n > 0\),
\[F_{n+1} = F_n + F_{n-1}\] and \(L_{n+1} = L_n + L_{n-1}\),
where \(F_0 = 0\), \(F_1 = 1\) and \(L_0 = 2\), \(L_1 = 1\), respectively.

If the roots of the characteristic equation \(x^2 - x - 1 = 0\) are \(\alpha\) and \(\beta\), then the Binet formulas are
\[F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}\] and \(L_n = \alpha^n + \beta^n\).

Wiemann and Cooper [6] raised certain conjectures for the Melham sum:
\[\sum_{k=1}^{n} \frac{F_{2m+1}}{2k+\delta}.\]

Ozeki [2] considered Melham’s sum and he gave an explicit expansion for it as a polynomial in \(F_{2n+1}\).

More generally, Prodinger [3] derived a formula for the sum:
\[\sum_{k=0}^{n} \frac{F_{2m+e}}{2k+\delta},\]
where \(e, \delta \in \{0, 1\} \). He also evaluated the corresponding sums for the Lucas numbers.
In this paper, we consider the alternating analogs of Melham’s sums. We derive explicit formulas for the sums:

\[
\sum_{k=1}^{n} (-1)^k \frac{2^m+\varepsilon}{2k+\delta} \quad \text{and} \quad \sum_{k=1}^{n} (-1)^k \frac{2^m+\varepsilon}{2k+\delta},
\]

where \(\varepsilon, \delta \in \{0, 1\}\).

2. **Alternating Melham’s Sums for Fibonacci Numbers**

In this section we will start with some lemmas and then we shall derive our results about the alternating Melham’s sum.

**Lemma 1.** For positive integers \(n, m\) and \(t\) such that \(m \geq t\),

i) \((-1)^{t+1} F_{(2m-2t+1)n} + F_{(2m-2t+1)(n+1)} = \sum_{j=0}^{2m-2t} (-1)^{(t-2)} F_{(2m-2t+1)n+j + (1-3(-1)^t)/2}\)

and

ii) \(F_{2(m-t)(n+1)} - (-1)^t F_{2(m-t)n} = \begin{cases} F_{2(m-t)n+m-t} F_{m-t} & \text{if } m \text{ is odd}, \\ L_{2(m-t)n+m-t} & \text{if } m \text{ is even}. \end{cases}\)

**Proof.** i) We can write

\[
(-1)^{t+1} F_{(2m-2t+1)n} + F_{(2m-2t+1)(n+1)} = (-1)^{t+1} \left( \frac{\alpha^{(2m-2t+1)n} - \beta^{(2m-2t+1)n}}{\alpha - \beta} \right) + \left( \frac{\alpha^{(2m-2t+1)(n+1)} - \beta^{(2m-2t+1)(n+1)}}{\alpha - \beta} \right)
\]

\[
= \frac{\alpha^{(2m-2t+1)n} (-1)^{t+1} + \alpha^{2m-2t+1}}{\alpha - \beta} - \frac{\beta^{(2m-2t+1)n} (-1)^{t+1} + \beta^{2m-2t+1}}{\alpha - \beta}
\]

\[
= \begin{cases} \sum_{j=0}^{2m-2t} (-1)^j \left( F_{(2m-2t+1)n+j} + F_{(2m-2t+1)n+j+1} \right) & \text{if } t \text{ is odd}, \\ - \sum_{j=0}^{2m-2t} F_{(2m-2t+1)n+j} - F_{(2m-2t+1)n+j+1} & \text{if } t \text{ is even}, \end{cases}
\]
\[ D_{2m-2t} \sum_{j=0}^{2m-2t} (-1)^j F_{(2m-2t+1)n+j+2} \quad \text{if } t \text{ is odd,} \]
\[ = \sum_{j=0}^{2m-2t} F_{(2m-2t+1)n+j-1} \quad \text{if } t \text{ is even,} \]
\[ = \sum_{j=0}^{2m-2t} (-1)^{j(t-2)} F_{(2m-2t+1)n+j+(1-3(-1)^t)/2}. \]

\[ \text{ii) The proof is similar to the ones for the Binet formulas of } \{F_n\} \text{ and } \{L_n\}. \square \]

From [4], we have the following result for the Gibonacci sequence \( \{G_n\} \), defined by for \( n>0 \)
\[ G_{n+1} = G_n + G_{n-1}. \]
with arbitrary initial values \( G_0 \) and \( G_1 \).

**Lemma 2.** Let \( a, p \neq 0, q \) be arbitrary integers. Then for \( n>0 \)
\[ n \sum_{i=a}^{n} G_{pi+q} = \frac{G_{p(n+1)+q} + (-1)^{p+1} G_{pn+q} + (-1)^p G_{p(a-1)+q} - G_{pa+q}}{L_p - 1 - (-1)^p}, \]
and
\[ \sum_{i=a}^{n} (-1)^i G_{pi+q} = \frac{(-1)^n G_{p(n+1)+q} + (-1)^{p+n} G_{pn+q} + (-1)^a G_{pa+q} + (-1)^{a+p} G_{p(a-1)+q}}{1 + (-1)^p + L_p}. \]

As a consequence of Lemma 2, for further use we state the following result:

**Corollary 1.** For any integer \( r \) and positive even integer \( t \),
\[ n \sum_{k=1}^{n} (-1)^{kr} L_{kt} = \frac{(-1)^{nr} F_{t(n+1)} + (-1)^{nr-r} F_{tn} - F_t}{F_t} \]
and
\[ n \sum_{k=1}^{n} (-1)^{kr} F_{k(t+1)} = \frac{(-1)^{(n+1)r} F_{(t+1)n} + (-1)^{nr} F_{(t+1)(n+1)} - F_{t+1}}{L_{t+1}}. \]

**Proof.** Clearly
\[ n \sum_{k=1}^{n} (-1)^{kr} L_{kt} = (-1)^r L_t + L_{2t} + \ldots + (-1)^{nr} L_{nt}. \]
For the first claim, we consider two cases: the first case is when \( n \) is an odd integer. Here

\[
\sum_{k=1}^{n} (-1)^k r L_{kt} = (-1)^{r} \sum_{j=1}^{(n+1)/2} L_{(2j-1)t} + \sum_{j=1}^{(n-1)/2} L_{2jt}.
\] (2.1)

If we take \( a = 1, p = 2t, q = -t \) and \( n \rightarrow \frac{n+1}{2} \) in Lemma 2, then we get

\[
\sum_{j=1}^{(n+1)/2} L_{(2j-1)t} = \frac{L_{t(n+2)} - L_{tn} + L_{-t} - L_{t}}{L_{2t} - 2}.
\]

The following identities are well known \([1, 5]\):

\[
L_{c+t} - L_{c-t} = 5F_c F_t
\] (2.2)

for even \( t \), and

\[
L_{2c} - (-1)^c 2 = 5F_c^2 \quad \text{and} \quad L_{-c} = (-1)^c L_c
\] (2.3)

for any integer \( c \). Thus we have

\[
\sum_{j=1}^{(n+1)/2} L_{(2j-1)t} = \frac{5F_t(n+1)F_t}{5F_t^2} = \frac{F_t(n+1)}{F_t}.
\] (2.4)

Meanwhile, if we take \( a = 1, p = 2t, q = 0 \) and \( n \rightarrow \frac{n-1}{2} \) in Lemma 2, then we get

\[
\sum_{j=1}^{(n-1)/2} L_{2jt} = \frac{L_{t(n+1)} - L_{t(n-1)} + L_0 - L_{2t}}{L_{2t} - 2}.
\]

Since \( t \) is even, by (2.2) and (2.3), we rewrite the last equation as

\[
\sum_{j=1}^{(n-1)/2} L_{2jt} = \frac{5FtnF_t}{5F_t^2} - 1 = \frac{Ftn}{F_t} - 1.
\] (2.5)

If we substitute (2.4) and (2.5) in (2.1), then we obtain

\[
\sum_{k=1}^{n} (-1)^k r L_{kt} = (-1)^r \left( \frac{F_t(n+1)}{F_t} \right) + \left( \frac{Ftn}{F_t} - 1 \right)
\]

\[
= \frac{(-1)^r F_t(n+1) + Ftn - F_t}{F_t}.
\] (2.6)

For the second case, let \( n \) be an even integer, thus

\[
\sum_{k=1}^{n} (-1)^k r L_{kt} = (-1)^r \sum_{j=1}^{n/2} L_{(2j-1)t} + \sum_{j=1}^{n/2} L_{2jt}.
\] (2.7)
By taking $a = 1$, $p = 2t$, $q = -t$ and $n \to \frac{n}{2}$ and $a = 1$, $p = 2t$, $q = 0$ and $n \to \frac{n}{2}$ in Lemma 2, respectively, we obtain the following result by (2.2) and (2.3), for even $t$,

$$\sum_{j=1}^{n/2} L_{(2j-1)t} = \frac{F_{tn}}{F_t},$$

(2.8)

$$\sum_{j=1}^{n/2} L_{2jt} = \frac{F_{t(n+1)}}{F_t} - 1.$$

(2.9)

If we put (2.8) and (2.9) in (2.7), we get

$$\sum_{k=1}^{n} (-1)^{kr} L_{kt} = (-1)^r \left( \frac{F_{tn}}{F_t} \right) + \left( \frac{F_{t(n+1)}}{F_t} - 1 \right)$$

$$= \frac{(-1)^r F_{tn} + F_{t(n+1)} - F_t}{F_t}.$$  \hspace{1cm} (2.10)

Combining (2.6) and (2.10), we get the final result:

$$\sum_{k=1}^{n} (-1)^{kr} L_{kt} = \frac{(-1)^{nr} F_{t(n+1)} + (-1)^{nr-r} F_{tn} - F_t}{F_t},$$

as claimed.

Finally by taking $a = 1$, $p = t + 1$, $q = 0$ in Lemma 2, the second claim is obtained similarly to the first claim. \hfill \Box

**Theorem 1.** i) For positive odd $m$,

$$\sum_{k=1}^{n} (-1)^k F_k^{2m} = \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{i(n+1)+n} \binom{2m}{i} \frac{F_{(m-i)(2n+1)}}{F_{m-i}}$$

$$= -\frac{1}{5^m} \binom{2m-1}{m} - \frac{1}{5^m} \binom{2m}{m} n,$$

and, for positive even $m$,

$$\sum_{k=1}^{n} (-1)^k F_k^{2m}$$

$$= \begin{cases} \frac{1}{5^m} \left( \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} + \binom{2m-1}{m} \right) \quad &\text{if } n \text{ is even,} \\
-\frac{1}{5^m} \left( \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} + \binom{2m-1}{m-1} \right) \quad &\text{if } n \text{ is odd,} 
\end{cases}$$
and

\[ i \ i \ \sum_{k=1}^{n} (-1)^k F_k^{2m+1} = \sum_{i=0}^{m} \binom{2m+1}{i} \frac{(-1)^{n+1} + n}{2m-2i+1} \times \sum_{j=0}^{2(m-i)} (-1)^{i(j-2)} F_{n(2m-2i+1) + j + \frac{1-3(-1)^i}{2}} \frac{1}{5m} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} \frac{F_{2m-2i+1}}{2m-2i+1}. \]

**Proof.**  For odd \( m \), consider

\[ \sum_{k=1}^{n} (-1)^k F_k^{2m} \]

\[ = \sum_{k=1}^{n} (-1)^k \left( \frac{\alpha^k - \beta^k}{\alpha - \beta} \right)^{2m} \]

\[ = \sum_{k=1}^{n} (-1)^k \left( \frac{1}{(\alpha - \beta)^{2m}} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \alpha^{ki} \beta^{k(2m-i)} \right) \]

\[ = \sum_{k=1}^{n} \frac{(-1)^k}{5m} \left( \sum_{i=0}^{m} (-1)^i \binom{2m}{i} \left( \alpha^{k(2m-i)} \beta^{ki} + \alpha^{ki} \beta^{k(2m-i)} \right) \right) \]

\[ - (-1)^m \binom{2m}{m} (\alpha \beta)^{km} \]

\[ = \frac{1}{5m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{k=1}^{n} (-1)^k (i+1) L_{2k(m-i)} \]

\[ + \binom{2m}{m} \sum_{k=1}^{n} (-1)^m (k+1) + k. \]

By taking \( i + 1 = r \) and \( 2(m-i) = t \) in Corollary 1, we write

\[ \sum_{k=1}^{n} (-1)^k F_k^{2m} \]

\[ = \frac{1}{5m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \]

\[ \times \left( (-1)^{n+1} F_{2(m-i)(n+1)} + (-1)^{(n+1)(i+1)} F_{2(m-i)n} - F_{2(m-i)} \right) \frac{F_{2(m-i)}}{F_{2(m-i)}} \]
Using \((ii)\) in Lemma 1, we have the claimed result. For even \(m\), the desired result is also obtained.

\(\text{\(ii\)}\) Consider

\[
\sum_{k=1}^{n} (-1)^k F_k^{2m+1}
\]

\[
= \sum_{k=1}^{n} (-1)^k \left( \frac{\alpha^k - \beta^k}{\alpha - \beta} \right)^{2m+1}
\]

\[
= \sum_{k=1}^{n} \frac{(-1)^k}{(\alpha - \beta)^{2m+1}} \sum_{i=0}^{2m+1} \binom{2m+1}{i} (-1)^{i+1} \alpha^k \beta^{k(2m+1-i)}
\]

\[
= \sum_{k=1}^{n} \frac{(-1)^k}{(\alpha - \beta)^{2m+1}} \sum_{i=0}^{m} \binom{2m+1}{i} (-1)^i \left( \alpha^{k(2m+1-i)} \beta^{ki} - \alpha^{ki} \beta^{k(2m+1-i)} \right)
\]

\[
= \frac{1}{2m} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} \sum_{k=1}^{n} (-1)^{k(i+1)} F_k(2m-2i+1).
\]

By taking \(r = i + 1\) and \(t = 2(m-i)\) in Corollary 1, we write

\[
\sum_{k=1}^{n} (-1)^k F_k^{2m+1}
\]

\[
= \frac{1}{2m} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} \times
\]

\[
\frac{(-1)^{(n+1)(i+1)} F_{(2m-2i+1)n} + (-1)^{n(i+1)} F_{(2m-2i+1)(n+1)} - F_{2m-2i+1}}{L_{2m-2i+1}}
\]

\[
= \frac{1}{2m} \sum_{i=0}^{m} (-1)^{i+n(i+1)} \binom{2m+1}{i} \times \frac{(-1)^{i+1} F_{(2m-2i+1)n} + F_{(2m-2i+1)(n+1)}}{L_{2m-2i+1}}
\]
By Lemma 1, we obtain the claimed result. □

For further use, we state a consequence of Lemma 2:

**Corollary 2.** For even positive integer \( t \) and \( n > 0 \),
\[
\sum_{k=1}^{n} (-1)^k \frac{L_{2tk}}{L_t} = \frac{(-1)^n L_t(2tn+1)}{L_t} - 1.
\]

**Proof.** Substituting \( a = 1 \), \( p = 2t \) and \( q = 0 \) in Lemma 2, we get
\[
\sum_{k=1}^{n} (-1)^k \frac{L_{2tk}}{L_t} = \frac{(-1)^n L_{2t(n+1)} - (-1)^n L_{2tn} - 2}{2 + L_{2t}}
= \frac{(-1)^n (L_{2t(n+1)} + L_{2tn}) - (L_{2t} + 2)}{2 + L_{2t}}.
\]

For even \( t \), from [1, 5], we have that
\[
L_{c+t} + L_{c-t} = L_c L_t
\]
and for any \( c \),
\[
L_{2c} + (-1)^c 2 = L_c^2.
\]
Thus we obtain
\[
\sum_{k=1}^{n} (-1)^k \frac{L_{2tk}}{L_t} = (-1)^n \frac{L_t(2tn+1)}{L_t} - 1,
\]
as claimed. □

**Theorem 2.** For \( m > 0 \),
\[
i) \quad \sum_{k=1}^{n} (-1)^k \frac{F_{2m}}{2_k} = \begin{cases} \frac{1}{5^m} \sum_{i=0}^{m} (-1)^i \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} + (-1)^{m} \frac{2m-1}{5^m} \binom{m}{m-1} & \text{if } n \text{ is even,} \\ \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{i+1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - (-1)^{m} \frac{2m-1}{5^m} \binom{m}{m-1} & \text{if } n \text{ is odd,} \end{cases}
\]
and
\[
ii) \quad \sum_{k=1}^{n} (-1)^k \frac{F_{2m+1}}{2_k}
\]
\[= \frac{1}{2^{m+1}} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} (-1)^n \frac{L_{(2m-2i+1)(2n+1)} - L_{2m-2i+1}}{F_{2m-2i+1}}.\]

Proof. We write

\[\begin{align*}
\sum_{k=1}^{n} (-1)^k F_{2k}^{2m} &= \sum_{k=1}^{n} (-1)^k \left( \frac{\alpha^{2k} - \beta^{2k}}{\alpha - \beta} \right)^{2m} \\
&= \sum_{k=1}^{n} (-1)^k \frac{1}{(\alpha - \beta)^{2m}} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \alpha^{2ki} \beta^{2k(2m-i)} \\
&= \sum_{k=1}^{n} \frac{(-1)^k}{2^{m}} \left( \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \alpha^{2k(2m-i)} \beta^{2ki} + \alpha^{2ki} \beta^{2k(2m-i)} \right) \\
&\quad + (-1)^m \binom{2m}{m} (\alpha \beta)^{2km} \\
&= \sum_{k=1}^{n} \frac{(-1)^k}{2^{m}} \left( \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} L_{4k(m-i)} + (-1)^m \binom{2m}{m} \sum_{k=1}^{n} (-1)^k \right).
\end{align*}\]

If we take \(t = 2(m - i)\) in Corollary 2, we write

\[\begin{align*}
\sum_{k=1}^{n} (-1)^k F_{2k}^{2m} &= \sum_{k=1}^{n} \frac{(-1)^{m-i}}{2^{m}} \binom{2m}{i} (-1)^i \frac{(-1)^n L_{2(m-i)(2n+1)} - 1}{L_{2(m-i)}} \\
&\quad + \frac{1}{2^{m}} \binom{2m}{m} (-1)^m \sum_{k=1}^{n} (-1)^k \\
&= \frac{1}{2^{m}} \sum_{i=0}^{m-1} (-1)^{n+i} \binom{2m}{i} L_{2(m-i)(2n+1)} - \frac{1}{2^{m}} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \\
&\quad + \frac{1}{2^{m}} \binom{2m}{m} (-1)^m \sum_{k=1}^{n} (-1)^k \\
&= \frac{1}{2^{m}} \sum_{i=0}^{m-1} (-1)^{n+i} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)} - L_{2(m-i)}}{L_{2(m-i)}}.
\end{align*}\]
\[
+ \frac{1}{2^m} (-1)^m \left( \frac{2m-1}{m} \right) + \frac{2m}{m} \left( (-1)^n - \frac{1}{2} \right).
\]

According to the choice of \( n \) as an odd or an even number, we have the conclusion.

\( i i ) \) The proof is obtained similarly to the proof of (i).

□

For further use, we state the following result:

**Corollary 3.** For positive even integer \( t \) and \( n > 0 \),

\[
\sum_{k=1}^{n} (-1)^k L_{(2k+1)t} = \frac{(-1)^n L_{2t(n+1)} - L_{2t}}{L_t}.
\]

**Proof.** Substituting \( a = 1 \), \( p = 2t \) and \( q = t \) in Lemma 2, we get

\[
\sum_{k=1}^{n} (-1)^k L_{(2k+1)t} = \frac{(-1)^n L_{(2n+3)t} + (-1)^n L_{(2n+1)t} - L_{3t} - L_t}{2 + L_{2t}}
\]

\[
= \frac{(-1)^n (L_{(2n+3)t} + L_{(2n+1)t}) - (L_{3t} + L_t)}{2 + L_{2t}}.
\]

By (2.11) and (2.12), we rewrite the last equation as for even \( t \),

\[
\sum_{k=1}^{n} (-1)^k L_{(2k+1)t} = (-1)^n \frac{L_t L_{2t(n+1)}}{L_t^2} - \frac{L_{3t} + L_t}{L_t^2}
\]

\[
= (-1)^n \frac{L_{2t(n+1)}}{L_t} - \frac{L_{2t}}{L_t}.
\]

Thus we have the conclusion. □

**Theorem 3.** For \( m > 0 \),

\( i i ) \) \( \sum_{k=1}^{n} (-1)^k F_{2m+1} \frac{2m}{2k+1} = \)

\[
\left\{ \begin{array}{ll}
\frac{-1}{2^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)(n+1)} + L_{4(m-i)}}{L_{2m}(m-i)} - \frac{1}{2^m} \binom{2m}{m} & \text{if } n \text{ is odd} \\
\frac{1}{2^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2m}(m-i)} & \text{if } n \text{ is even},
\end{array} \right.
\]

and

\( i i ) \) \( \sum_{k=1}^{n} (-1)^k F_{2m+1} \frac{2m+1}{2k+1} = \)

\[
\frac{1}{2^{m+1}} \sum_{i=0}^{m} \binom{2m+1}{i} \frac{(-1)^n L_{(4m-4i+2)(n+1)} - L_{4m-4i+2}}{F_{2m-2i+1}}.
\]


Proof. i) Consider
\[
\sum_{k=1}^{n} (-1)^k F_{2k+1}^{2m} = \sum_{k=1}^{n} \frac{(-1)^k}{(2^k + 1)(2^{m-k})} \left( \frac{\alpha^{2k+1} - \beta^{2k+1}}{\alpha - \beta} \right)^{2m}.
\]
\[
= \sum_{k=1}^{n} \frac{(-1)^k}{\alpha - \beta} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \alpha^{(2k+1)i} \beta^{(2k+1)(2m-i)}
\]
\[
= \frac{(-1)^k}{\alpha - \beta} \sum_{i=0}^{m-1} \binom{2m}{i} \alpha^{(2k+1)(2m-i)} \beta^{(2k+1)i} + \alpha^{(2k+1)i} \beta^{(2k+1)(2m-i)}
\]
\[
+ \binom{2m}{m} (\alpha \beta)^{2km}
\]
\[
= \sum_{k=1}^{n} \frac{(-1)^k}{5^m} \left( \sum_{i=0}^{m-1} \binom{2m}{i} L_{2(2k+1)(m-i)} + \binom{2m}{m} \right)
\]
\[
= \frac{1}{5^m} \left( \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=1}^{n} (-1)^k L_{2(2k+1)(m-i)} + \binom{2m}{m} \right) \sum_{k=1}^{n} (-1)^k
\]

If we take \( t = 2(m - i) \) in Corollary 3, we write
\[
\sum_{k=1}^{n} (-1)^k F_{2k+1}^{2m} = \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \left( (-1)^n \frac{L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} \right) + \frac{1}{5^m} \binom{2m}{m} \sum_{k=1}^{n} (-1)^k
\]
\[
= \frac{(-1)^n}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)(n+1)}}{L_{2(m-i)}} - \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)}}{L_{2(m-i)}} + \frac{1}{5^m} \binom{2m}{m} \left( \frac{(-1)^n - 1}{2} \right).
\]

According to the choice of \( n \) as an odd or as an even number, we have the conclusion.

ii) The proof of (ii) is obtained similarily to the proof of (i). \( \square \)
3. Alternating Melham’s Sum for Lucas Numbers

Theorem 4. i) For positive odd $m$,

$$
\sum_{k=1}^{n} (-1)^k L_k^{2m} = \sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{F_{(m-i)(2n+1)}}{F_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} + \binom{2m}{m}^n.
$$

and, for even $m$,

$$
\sum_{k=1}^{n} (-1)^k L_k^{2m} = \begin{cases}
\sum_{i=0}^{m-1} \binom{2m}{i} L_{(m-i)(2n+1)} \frac{1}{L_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} & \text{if } n \text{ is even,}

\sum_{i=0}^{m-1} (-1)^{i+1} \binom{2m}{i} L_{(m-i)(2n+1)} \frac{1}{L_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} - \binom{2m}{m} & \text{if } n \text{ is odd,}
\end{cases}
$$

and

$$
\sum_{k=1}^{n} (-1)^k L_k^{2m+1} = \sum_{i=0}^{m} \binom{2m+i}{i} (-1)^{(i+1)n+1} \frac{1}{L_{2m-2i+1}}
$$

$$
\times \sum_{j=0}^{2(m-i)} (-1)^{j+1} L_{n(2m-2i+1)+j} \frac{1-3(-1)^j}{2} + 2 \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} \frac{1}{L_{2m-2i+1}} - 2^{2m}.
$$

Proof. i) By the Binet formula of $\{L_n\}$, we have

$$
\sum_{k=1}^{n} (-1)^k L_k^{2m} = \sum_{k=1}^{n} (-1)^k (\alpha^k + \beta^k)^{2m}
$$

$$
= \sum_{k=1}^{n} (-1)^k \left( \sum_{i=0}^{m-1} \binom{2m}{i} (\alpha^{(2m-i)} \beta^i + \alpha^i \beta^{(2m-i)}) \right) + \binom{2m}{m} (\alpha \beta)^{km}.
$$
Since $\alpha \beta = -1$, we get
\[
\sum_{k=1}^{n} (-1)^k L_k^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=1}^{n} (-1)^{k(i+1)} L_{2k(m-i)} + \binom{2m}{m} \sum_{k=1}^{n} (-1)^{k(m+1)}.
\]
From Corollary 1, we write
\[
\sum_{k=1}^{n} (-1)^k L_k^{2m}
= \sum_{i=0}^{m-1} \binom{2m}{i} \left( (-1)^{(i+1)n} F_{2(m-i)(n+1)} + (-1)^{(n+1)(i+1)} F_{2(m-i)n} - F_{2(m-i)} \right)
\]
\[\frac{F_{2(m-i)}}{F_{2(m-i)}}\]
\[+ \binom{2m}{m} \sum_{k=1}^{n} (-1)^{k(m+1)}
= \sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{F_{2(m-i)(n+1)} + (-1)^{i+1} F_{2(m-i)n}}{F_{2(m-i)}}
- \sum_{i=0}^{m-1} \binom{2m}{i} + \binom{2m}{m} \sum_{k=1}^{n} (-1)^{k(m+1)}
\]
\[
\begin{cases}
\sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{F_{(m-i)(2n+1)}}{F_{m-i}} - 2^{2m-1} \\
+ \binom{2m-1}{m} + \binom{2m}{m} n \\
\sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} - 2^{2m-1} \\
+ \binom{2m-1}{m} + \binom{2m}{m} \left( \frac{(-1)^n - 1}{2} \right)
\end{cases}
\]
if $m$ is even.
According to the choice of $n$, the claim is obtained.

$ii)$ The proof is similar to the proof of $i$.

**Theorem 5.** $i)$ For $m > 0$ and even $n > 0$,
\[
\sum_{k=1}^{n} (-1)^k L_k^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} + \binom{2m-1}{m}.
\]
and, for odd $n > 0$,
\[
\sum_{k=1}^{n} (-1)^k L_{2k}^{2m} = -\sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} \binom{2m-1}{m}.
\]

i) For positive integers $m$ and $n$,
\[
\sum_{k=1}^{n} (-1)^k L_{2k}^{2m+1} = (-1)^{n} \sum_{i=0}^{m} \binom{2m+1}{i} \frac{F_{2(m-2i+1)(2n+1)}}{F_{2m-2i+1}} - 2^{2m}.
\]

Proof. i) We write
\[
\sum_{k=1}^{n} (-1)^k L_{2k}^{2m+1} = (-1)^{n} \sum_{k=1}^{m} \binom{2m}{k} \frac{\alpha^{2k} \beta^{2k}}{\alpha^{2k} \beta^{2k} + \alpha^{2k} \beta^{2k} + \alpha^{2k} \beta^{2k}} + \binom{2m}{m} \frac{\alpha^{2k} \beta^{2k}}{\alpha^{2k} \beta^{2k} + \alpha^{2k} \beta^{2k}}
\]

If we take $t = 2(m-i)$ in Corollary 2, we can write
\[
\sum_{k=1}^{n} (-1)^k L_{2k}^{2m+1} = \sum_{i=0}^{m-1} \binom{2m}{i} \frac{(-1)^{n} L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 1 + \binom{2m}{m} \sum_{k=1}^{n} (-1)^k
\]
\[
= (-1)^{n} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=1}^{n} (-1)^k
\]
\[
= (-1)^{n} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} + \binom{2m}{m-1} + \binom{2m}{m} \frac{(-1)^{n-1}}{2}.
\]
According to the choice of \( n \), the proof is complete by the fact that

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.
\]

\( \Box \)

**Theorem 6.** For \( m > 0 \),

\( i \) \( \sum_{k=1}^{n} (-1)^k L_{2k+1}^{2m} = \)

\[
\begin{cases}
\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} & \text{if } n \text{ is even}, \\
(-1)^m \binom{2m}{m} - \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)(n+1)} + L_{4(m-i)}}{L_{2(m-i)}} & \text{if } n \text{ is odd},
\end{cases}
\]

and

\( ii \) \( \sum_{k=1}^{n} (-1)^k L_{2k+1}^{2m+1} = \)

\[
\sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} \frac{(-1)^n F_{2(2m-2i+1)(n+1)} - F_{2(2m-2i+1)}}{F_{2m-2i+1}}.
\]

**Proof.** \( i \) Using the Binet formula of \( \{L_n\} \), we have

\[
\sum_{k=1}^{n} (-1)^k L_{2k+1}^{2m} =
\sum_{k=1}^{n} (-1)^k \left( \alpha^{2k+1} + \beta^{2k+1} \right)^{2m}
\]

\[
= \sum_{k=1}^{n} (-1)^k \left( \sum_{i=0}^{m-1} \binom{2m}{i} \left( \alpha^{(2k+1)(2m-i)} + \beta^{(2k+1)(2m-i)} \right) \right)
\]

\[
+ \binom{2m}{m} (\alpha \beta)^{(2k+1)m}
\]

\[
= \sum_{k=1}^{n} (-1)^k \left( \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \left( \alpha^{(2k+1)(2m-2i)} + \beta^{(2k+1)(2m-2i)} \right) \right)
\]
If we take \( t = 2(m-i) \) in Corollary 3, we get
\[
\sum_{k=1}^{n} (-1)^{k} L_{2k+1}^{2m} \left( \frac{\left(-1\right)^{n} L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} \right) + \binom{2m}{m} (-1)^{m} \sum_{k=1}^{n} (-1)^{k}
\]
\[
= \sum_{i=0}^{m-1} (-1)^{i} \binom{2m}{i} \left( \frac{\left(-1\right)^{n} L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} \right) + \binom{2m}{m} (-1)^{m} \sum_{k=1}^{n} (-1)^{k}
\]
\[
= - \sum_{i=0}^{m-1} (-1)^{i} \binom{2m}{i} \left( \frac{\left(-1\right)^{n} L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} \right) - \sum_{i=0}^{m-1} (-1)^{i} \binom{2m}{i} \frac{L_{4(m-i)}}{L_{2(m-i)}}
\]
\[
+ \binom{2m}{m} (-1)^{m} \left( \frac{\left(-1\right)^{n} - 1}{2} \right)
\].

According to the choice of \( n \), the claimed result is clear.

(ii) The proof is similar to the proof of (i). \( \square \)

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