Quantized spaces are four-dimensional compact manifolds with de-Sitter ($O(1,4)$ or $O(2,3)$) group of motion

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Abstract

It is shown uniquely that quantized spaces are realised on four-dimensional compact manifolds. In the case of $O(1,5)$ quantized space this are four independent parameters of $O(5)$ unite vector; in the case of $O(2,4)$ these are parameters of one two-dimensional unite vector (1 parameter) and components of unite four-dimensional vector (3-parameters) and at last in the case $O(3,3)$ these are parameters of 2 independent 3-dimensional unite vectors (each have 2 parameters). This result follows directly only from the condition to have a correct limit to usual theory (correspondence principle).
1 Introduction

In the previous paper of the author [1] it was lost and not used very important condition which allow to reconstruct uniquely representation of six-dimensional rotation algebra of quantized space. It was not taking in account the fact that that generators of Lorenz rotations in the limit to the usual theory ($L^2 \to \infty, M^2 \to \infty, H \to \infty$) are algebraically connected with the generators of coordinates and impulses by the well known quadratical relation:

$$S_{i,j} \equiv F_{i,j} + p_i x_j - p_j x_i = 0$$ (1)

As a consequence two Kazimir operators of Lorenze group (generated by $S_{ij}$) $K_2 = \sum S_{i,j} S_{i,j}, K_3 = \sum \epsilon_{i,j,k,l} S_{i,j} S_{k,l}$ are also equal to zero. But inverse is not true. From the fact that two Kazimir generators of Lorenze group (it is not compact (!)) are equal to zero doesn’t follow that representation is the trivial one. Directly from (1) obvious additional relations follows $\tilde{x}_i \equiv \epsilon_{i,j,k,l} x_j F_{k,l} = 0, \tilde{p}_i \equiv \epsilon_{i,j,k,l} p_j F_{k,l} = 0, \epsilon_{i,j,k,l} F_{i,j} F_{k,l} = 0$, which will be important for consideration below.

In the the previous paper [1] it was assumed that representation of the algebra of quantized space must be choosen in such way that in the “classical limit” ($L^2 \to \infty, M^2 \to \infty, H \to \infty$) second order Kazimir operator passes to unity and both other ones pass to zero. But consideration above shows that really in classical limit must be satisfied more strong condition (1). Compare the similar consideration in [3] and [4]. Representation of this kind was called their as scalar one.

Fortunelly as it will be shown below such representations are existed. More other these representations are realised on four-dimensional space of compact parameters. Thus quantized spases are also four-dimensional but compact manifolds in comparision with noncompact space-time manifold of the usual theory.

A little additional explanation. Six-dimensional rotation groups of the theory of quantized space are 15-th parametrical. They posses 3 Kazimir operators. Thus in the general case their irreducible representation (of the general position with three independent Kazimir operators) may be realized on the space of $\frac{15-3}{2} = 6$ parameters. Exactly on this number of parameters was realized the algebras of the real forms $O(3,3), O(2,4), O(1,5)$ in [1]. But the boundary condition in [1] was choosen not correctly. The correct choice (1) leads to four-dimensional compact quantized spaces.
The goal of the present paper is to explain and clarify this situation. In what follows we preserve all notations of [1].

2 Kazimir operators and general strategy

For convinience of the reader we rewrite commutation relations of quantized space from [1],[2] We will work below with the algebra of quantum space proposed in [2] and containing 3 dimensional parameters of the square of the length $L^2$, square of impulse $M^2$ and action $H$. Commutation relations of such algebra have the following form

\[
[p_i, x_j] = i\hbar (g_{ij} I + \frac{F_{ij}}{H}), \quad [p_i, p_j] = i\hbar \frac{F_{ij}}{L^2}, \quad [x_i, x_j] = i\hbar \frac{F_{ij}}{M^2},
\]

\[
[I, p_i] = i\hbar (\frac{p_i}{H} - \frac{x_i}{L^2}), \quad [I, x_i] = i\hbar (\frac{p_i}{M^2} - \frac{x_i}{H}), \quad [I, F_{ij}] = 0 \quad (2)
\]

\[
[F_{ij}, x_s] = i\hbar (g_{js} x_i - g_{is} x_j), \quad [F_{ij}, p_s] = i\hbar (g_{js} p_i - g_{is} p_j)
\]

\[
[F_{ij}, F_{sk}] = i\hbar (g_{js} F_{ik} - g_{is} F_{jk} - g_{jk} F_{is} + g_{ik} F_{js})
\]

We present the explicit expressions for Kazimir operators from [1]

\[
K_2 = -L^2 + \frac{1}{L^2} p^2 - \frac{1}{M^2} p^2 - \frac{(px + xp)}{H} + \left(\frac{1}{H^2} - \frac{1}{L^2 M^2}\right)(l^2 - f^2) = \nu^2 \left(\frac{L^2}{\nu^2} + \frac{(x - \frac{L^2}{M^2} p^2)}{\nu^2 L^2} + \frac{L^2}{H^2} p^2 + \frac{(f^2 - l^2)}{H^2}\right)
\]

where $a^2 = a_1^2 + a_2^2 + a_3^2 - a_4^2$, $l^2 = (\bar{l})^2$, $f^2 = (\bar{f})^2$ squares of three dimensional generators of rotations and Lorenz boosts. In what follows $\bar{x} \equiv (x - \frac{L^2}{M^2} p)$. Generators of three-dimensional space rotation are a compact ones and thus all generators (their squares) with the same sign in $K_2$ are also compact, with the opposit are noncompact. In the case $0 \leq L^2, M^2, \nu^2 = \frac{H^2}{L^2 M^2} - 1$ this is $O(3,3)$ algebra and so on.

Two other operators of Kazimir in three and four dimensional notations are as follows

\[
K_3 = I (\bar{f} \bar{l}) + (\bar{f} \bar{p}) (\bar{x} - \frac{L^2}{H} \bar{p}) - (\bar{l}, (p_4 (\bar{x} - \frac{L^2}{H} \bar{p}) - \bar{p} (x_4 - \frac{L^2}{H} p_4))
\]
where generators of the "spin" variables is defined as
\[ S_{i,j} = \sum_{i \leq j} S_{i,j} \]
and four-dimensional vectors \( \tilde{x}_i, \tilde{p}_i \) ( of the pseudo coordinates and pseudo impulses) are defined as
\[ \tilde{x}_i = \sum \varepsilon_{ijkl} x_j F_{kl}, \tilde{p}_i = \sum \varepsilon_{ijkl} p_j F_{kl}. \]

Pseudovectors \( \tilde{x}_i, \tilde{p}_i \) are introduced in analogy of Pauli-Lubansky vector in representation theory of the Poincare algebra. In the classical limit the operators of Kazimir \( K_3, K_4 \) fixed quantum numbers of representation of the Lorentz \( O(1,3) \) algebra.

[Without any connection with what follows we pay attention of the reader on symmetry of Kazimir operators of the second and fourth order with respect to substitution \( \vec{l}, \vec{f} \rightarrow S, I \rightarrow (\vec{f}\vec{l}), x \rightarrow \tilde{x}, p \rightarrow \tilde{p}. \)]

As was mentioned in introduction the classical limit would be satisfied under the assumption that in representation of the quantum space algebra \( S_{i,j} = IF_{i,j} + p_i \tilde{x}_j - p_j \tilde{x}_i = 0. \) In the classical limit \( I \rightarrow 1 \) and condition (1) would be a direct consequent of the last choice of the representation of six-dimensional group of rotation. To have the zero value for the Kazimir operators of 3 and 4 order it is sufficient to assume additionally
\[ \tilde{x}_i = 0, \quad \tilde{p}_i = 0, \quad (lf) = 0 \]
Thus if it will be possible to satisfy 15 conditions above then we will have representation of six-dimensional algebra having correct limit to four dimensional coordinate space of the usual theory. Below we rewrite these equation in three-dimensional notations and after this will try to resolve them for all real forms of ortogonal group (find such representations in which they are satisfied):
\[ If_\alpha + p_\alpha \bar{x}_4 - p_4 \bar{x}_\alpha = 0, \quad IL_{\alpha,\beta} + p_\alpha \bar{x}_\beta - p_\beta \bar{x}_\alpha = 0 \]
\[ \bar{x}_4 L_{\alpha,\beta} + \bar{x}_\alpha f_\beta - \bar{x}_\beta f_\alpha = 0, \quad p_4 L_{\alpha,\beta} + p_\alpha f_\beta - p_\beta f_\alpha = 0 \]
\[ (f \bar{l}) = (p \bar{l}) = (\bar{x} \bar{l}) = 0 \]
Not all of this equations are independent but this is not essential for further consideration.

Now it is possible to say that that quantized space is described by commutation relations (2) and additional algebraic equations (3), which responsible for the correct limit to the usual theory (correspondence principle)
and which must be satisfied by the choice of corresponding representation of six-dimensional algebra of quantized space.

3 \( O(3, 3) \) case with \( SO(2, 3) \) group of motion

\[
K_2 = \nu^2 \left( \frac{p^2 L^2}{H^2} + \frac{x^2}{L^2 \nu^2} + \frac{f^2 - l^2}{H^2} - \frac{l^2}{\nu^2} \right), \quad 0 \leq L^2, \nu^2 = \frac{H^2}{L^2 M^2} - 1
\]

In this case 6 generators

\[
I = F_{65}, \quad p_4 = F_{45}, \quad \bar{x}_4 = F_{46}, \quad l_\alpha = \epsilon_{\alpha, \beta, \gamma} Q_{\beta, \gamma}
\]

are compact one and 9

\[
p_\alpha = F_{\alpha, 6}, \quad \bar{x}_\alpha = F_{\alpha, 5}, \quad f_\alpha = F_{\alpha, 4}
\]

are non compact.

In connection with [1]

\[
F_{\alpha, i} = \sum \rho_\sigma q_{\alpha}^\sigma p_i^\sigma + \sum q_{\alpha}^\delta p_i^j Q^{j, \delta} + \sum q_{\alpha}^\delta p_i^j P^{j, \delta} \quad \tag{4}
\]

Left and right shifts of compact \( O(3) \) groups are connected by the relations

\[
Q_{\alpha, \beta} = \sum_{\mu, \nu} q_{\alpha}^\nu q_{\beta}^\mu Q^{\nu, \mu}, \quad P_{i, j} = \sum_{k, l} p_i^k p_j^l P^{k, l}
\]

3.1 Reducibility of representation. Invariant subspaces

Let us first consider 3 three scalar equations (3) \((fl) = (pl) = (\bar{x}l) = 0\). Substituting all above expressions and taking into account the obvious equality

\[
\sum_{\alpha, \beta, \gamma} q_{\alpha}^\nu q_{\beta}^\mu q_{\gamma}^\delta = \epsilon_{\nu, \mu, \delta}
\]

we come to equation

\[
\sum \rho_\sigma \epsilon_{\sigma, \mu, \nu} p_i^\sigma Q^{\mu, \nu} + \sum \epsilon_{\delta, \mu, \nu} p_i^j Q^{\mu, \nu} P^{j, \delta} + \sum \epsilon_{\delta, \mu, \nu} p_i^j Q^{\mu, \nu} Q^{\delta, j} = 0
\]

3 components of this system transforms to 3 conditions

\[
(\rho_1 + 1)Q^{23} = 0, \quad \rho_2 Q^{31} + (P^{21} + Q^{21})Q^{23} = 0, \quad \rho_3 Q^{12} + P^{31} Q^{23} + Q^{31} P^{32} = 0
\]
By the same technique 3 equations

\[(\rho_1 + 1)P^{23} = 0, \quad \rho_2 P^{31} + (P^{21} + Q^{21})P^{23} = 0, \quad \rho_3 P^{12} + P^{31} Q^{32} + Q^{31} P^{23} = 0\]

lead to

\[(\rho_1^2 + 1)P^{23} = 0, \quad \rho_2 P^{31} + (P^{21} + Q^{21})P^{23} = 0, \quad \rho_3 P^{12} + P^{31} Q^{32} + Q^{31} P^{23} = 0\]

From the above results it follows that the representation with \(\rho_2 = \rho_3 = 0\) is space reducible. One of its irreducible components is realized on subspace defined by the conditions \(P^{23} = Q^{23} = 0\).

This representation is realised exactly on two three dimensional unites vectors \(p^1, q^1\). Substituting this condition into (4) we obtain

\[F_{\alpha,i} = \rho q^1_{\alpha} p^1_i + q^1_{\alpha} \sum_j p^j_i P^{j,1} + p^1_i \sum_{\delta} q^1_{\alpha} Q^{\delta,1}\]  

(5)

9 remaining equations follows from (3) in terms of (5) may be rewritten as follows

\[P_{i,j} Q_{\alpha,\beta} + F_{i,\alpha} F_{j,\beta} - F_{i,\beta} F_{j,\alpha} = 0\]

Direct check show that they are satisfied for representation generating by (5).

This is finally expression for representation of \(O(3, 3)\) which is realised on two unite 3-dimensional vectors \(q^1_{\alpha}, p^1_i\) or on 4 compact parameters. Explicit expression for these generators in terms of 4 parameters of two unite 3-dimensional vectors see in subsection below.

### 3.2 Unitary representations

As it follows directly from (5) noncompact generators are antihermmits under the choice \(\rho = -2 + i\sigma\). To check this it is necessary to consider equation \((F_{\alpha,i})^H = -F_{\alpha,i}\), keeping in mind that all compact generators are antihermmits \((P^{j,i})^H = -P^{j,i}, (Q^{\alpha,\delta})^H = -Q^{\alpha,\delta}\) and taking into account commutation relation \([Q^{\delta,1}, q^1_{\alpha}] = -q^1_{\alpha}\). But as was mentioned in [1] such unitary representation is in contradiction with classical limit \(I \to 1\). Fortunately representation (5) posses discrete serie which gives oposite sign for second Kazimir operator and leads to correct limit to usual theory.

To find this representation it is neessary to consider corresponding Hermitian form defined as

\[K = \int dp^1 dq^1 F^*(p^1, q^1)(K(p^1, q^1; \bar{p}^1, \bar{q}^1)F(p^1, q^1)dp^1 dq^1\]
were \( dp^1 = \sin \theta d\theta d\phi \), \( dq^1 = \sin \tau d\tau d\psi \) is the invariant measure on \( O(3) \) and \((K(p^1, q^1; \bar{p}^1, \bar{q}^1))\) is the kernel of hermitian form which have to defined from the condition of its invariance with respect to transformation of the considered representation.

Condition of invariance \( K \) with respect to compact transformations restricted the form the kernel function up to dependence of two parameters \( K = K((p^1 \bar{p}^1)[= x], (q^1 \bar{q}^1)[= y]) (\sum_i^3 p_1^i \bar{p}_1^i \text{ and so on}) \).

After not combersom calculations we obtain the following system of equation with respect to kernel function:

\[
\begin{align*}
[(\rho^* + 4) + (\rho + 4)xy - y(1 - x^2)\partial_x - x(1 - y^2)\partial_y]K &= 0, \\
[(\rho^* + 4)x + (\rho + 4)y - (1 - x^2)\partial_x - (1 - y^2)\partial_y]K &= 0, \\
[(\rho^* + 4)y + (\rho + 4)x - (1 - x^2)\partial_x - (1 - y^2)\partial_y]K &= 0.
\end{align*}
\]

Reducing first-second and third-fourth equations leads to equalities

\((\rho^* - \rho)(1 - xy) = 0, \ (\rho^* - \rho)(x - y) = 0\)

from which it follows \(\rho^* = \rho\) or \(x = 1, y = 1\) or in other words \( K = \delta(1 - x)\delta(1 - y) \) in the second case. The last possibility is equivalent to considered above unitary representation with \(\rho = -2 + i\sigma\). For the first one simple resolving of the sytem of equations leads to:

\[ K = c|x - y|^{-(\rho + 4)} \]

In the case of positive natural \(\rho\) the last expression have to be considered as a generalised function or distribution [5] and as it follows from the general theory it arised some number of invariant spaces on some of which unitary representation is realized (it is necessary additional check that hermitian form is positive defined).

3.3 Explicit expressions. P-representation

In this subsection we would like to rewrite all expessions for the generators of quantized space under considration in terms of only four compact angles
\[ p_1^1 = \cos \theta, p_2^1 = \sin \theta \cos \phi, p_3^1 = \sin \theta \sin \phi, q_1^1 = \cos \tau, q_2^1 = \sin \tau \cos \psi, q_3^1 = \sin \tau \sin \psi. \] In this notations

\[ P_{12} = -\cos \phi \partial_\theta + \sin \phi \cot \theta \partial_\phi, \quad P_{13} = -\sin \phi \partial_\theta - \cos \phi \cot \theta \partial_\phi, \quad P_{23} = -\partial_\phi \]

\[ Q_{12} = -\cos \psi \partial_\tau + \sin \psi \cot \tau \partial_\phi, \quad Q_{13} = -\sin \psi \partial_\tau - \cos \psi \cot \tau \partial_\phi, \quad Q_{23} = -\partial_\phi \]

For arising in (5) operators \( \hat{P}_i \equiv \sum_j p_j^i P^j,1 = \sum_j p_j^1 P_{j,i} \) with the help of the formulae above we have

\[ \hat{P}_1 = \sin \theta \partial_\theta, \quad \hat{P}_2 = -\cos \phi \cos \theta \partial_\theta + \frac{\sin \phi}{\sin \theta} \partial_\phi, \quad \hat{P}_3 = -\sin \phi \cos \theta \partial_\theta - \frac{\cos \phi}{\sin \theta} \partial_\phi \]

\[ \hat{Q}_1 = \sin \tau \partial_\tau, \quad \hat{Q}_2 = -\cos \psi \cos \tau \partial_\tau + \frac{\sin \psi}{\sin \tau} \partial_\phi, \quad \hat{Q}_3 = -\sin \psi \cos \tau \partial_\tau - \frac{\cos \psi}{\sin \tau} \partial_\phi \]

We present also explicit expression for d Alambek operator- Kazimir operator of the second order of de-Sitter \( O(2,3) \) algebra. In calculations below \( p_6^1 = p_1^1, p_5^1 = p_2^1, p_4^1 = p_3^1 \).

\[ m^2 = p^2 + q^2 - p_4^2 - l^2 = \sum [q_\alpha^1 (\rho p_6^1 + P_4) + p_4^1 Q_\alpha]^2 + \sum [q_\alpha^1 (\rho p_5^1 + P_5) + p_5^1 Q_\alpha]^2 - p_4^2 - l^2 = \]

(in formulae above \( P_i = \sum_j p_j^i P^j,1, Q_\alpha = \sum_\delta q_\alpha^\delta Q^{\delta,1} \))

\[ ((\rho + 1) p_4^1 + P_4)^2 + ((\rho + 1) p_5^1 + P_5)^2 - p_4^2 - (p_6^1)^2 l^2 - (1 + (p_6^1)^2) = \]

\[ (\rho + 1)(\rho + 3) + P_3^2 - [(\rho + 1) \cos \theta - \sin \theta \partial_\theta]^2 - p_4^2 - (p_6^1)^2 l^2 - (1 + (p_6^1)^2) \]

where now \( P_3 = \partial_\phi^2 + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 \) Kazimir operator of \( O(3) \) algebra constructed on unite vector \( p_i^1 \). Not combersome manipulations lead to following explicit expression for d Alambek operator

\[ (\cos \theta)^{-(\rho+1)} m^2 (\cos \theta)^{(\rho+1)} = (\cos \theta) \partial_\theta)^2 + \cot \theta \partial_\theta + (\cot \theta)^2 \partial_\phi^2 - (1 + (\cos \theta)^2 - (\cos \theta)^2 l^2 \]

### 3.4 X-representation

Terms \( P_i X \)-representations we use in usual sense for quantum mechanic. Propiar values of any 4 mutual commutative operators constructed from the elements of the algebra of quantized space may be used for construction of its basis. But basises such construction are not connected by point like transformations but only by canonical ones. To have the results in more
nearest to usual theory form it is suitable to consider basis connected with
the proper values of the following 4 operators
\[ x_4, \quad r^2 \equiv (\vec{x})^2 - \frac{(\vec{l})^2}{M^2}, \quad (\vec{l})^2, \quad l_3 \]

which can be interpreted physically as the time coordinate, direction in the
space and distance up to point of observation from the initial point. We would
not like to present

4 The $O(2, 4)$ case $0 \leq L^2$ connected with the
$O(2, 3)$ group of the motion

\[ K_2 = -\mu^2 \left( \frac{I^2}{\mu^2} - \frac{(x - \frac{L^2}{H^2} p)^2}{\nu^2 L^2} + \frac{L^2}{H^2} p^2 + \frac{(f^2 - l^2)}{H^2} \right), \quad 0 \leq L^2, \mu^2 = 1 - \frac{H^2}{L^2 M^2} \]

In this case 7 generators

\[ l_\alpha = \epsilon_{\alpha,\beta,\gamma} Q_{\beta,\gamma}, \quad p_4 = T_{4,2} = P_{2,1} \quad \bar{x}_\alpha = T_{\alpha,4} = Q_{\alpha,4} \]

are compact one and 8

\[ I = T_{1,4} = \rho^1 \cos \phi q_4^1 + \rho^2 \sin \phi q_4^2 + \cos \phi \sum_{2 \leq i} q_4^i Q^{i,1} + \sin \phi \sum_{2 \leq i} q_4^i Q^{i,2} + \sin \phi q_4^1 P^{2,1} \]

\[ \bar{x}_4 = T_{2,4} = -\rho^1 \sin \phi q_4^1 + \rho^2 \cos \phi q_4^2 - \sin \phi \sum_{2 \leq i} q_4^i Q^{i,1} + \cos \phi \sum_{2 \leq i} q_4^i Q^{i,2} + \cos \phi q_4^1 P^{2,1} \]

\[ p_\alpha = T_{1,\alpha} = \rho^1 \cos \phi q_\alpha^1 + \rho^2 \sin \phi q_\alpha^2 + \cos \phi \sum_{2 \leq i} q_\alpha^i Q^{i,1} + \sin \phi \sum_{2 \leq i} q_\alpha^i Q^{i,2} + \sin \phi q_\alpha^1 P^{2,1} \]

\[ f_\alpha = T_{2,\alpha} = -\rho^1 \sin \phi q_\alpha^1 + \rho^2 \cos \phi q_\alpha^2 - \sin \phi \sum_{2 \leq i} q_\alpha^i Q^{i,1} + \cos \phi \sum_{2 \leq i} q_\alpha^i Q^{i,2} + \cos \phi q_\alpha^1 P^{2,1} \]

are noncompact. Now all notations with $Q, q$ are connected with four-
dimensional compact group of rotation, $P, p$ with $O(2)$.  

8
4.1 Invariant subspaces

15 equations (3) are responsible for correct limit to the usual theory. It is suitable to begin resolving of this system from 3 scalar equations

\[(\vec{l}, \vec{f}) = 0, \quad (\vec{i}, \vec{p}) = 0, \quad (\vec{l}, \vec{x}) = 0\] (6)

Third equation means that the second Kazimir operator of four dimensional rotation group equal to zero. But Kazimir operators constructed from the generators of left or right regular representation are the same. Thus we have

\[(\vec{l}, \vec{f}) = Q^{12}Q^{34} + Q^{13}Q^{24} + Q^{14}Q^{23} = 0\]

Two second scalar equation after simple manipulations equivalent to

\[
\rho_1(q^{2\vec{l}}) + \sum_{i=1}^{4} (q^{2\vec{l}})Q^{i,1} = 0, \quad \rho_2(q^{2\vec{l}}) + \sum_{i=2}^{4} (q^{2\vec{l}})Q^{i,2} + (q^{1\vec{l}})P^{21} = 0
\]

Further

\[(q^{1\vec{l}}) = \sum \epsilon_{\alpha,\beta,\gamma} q^{1}_4 q^{\mu}_\gamma Q^{\nu,\mu} = \sum \epsilon_{i,\nu,\mu,t} q^{1}_t Q^{\nu,\mu}\]

Thus

\[(q^{1\vec{l}}) = \sum \epsilon_{1,\nu,\mu,t} q^{1}_t Q^{\nu,\mu}\]

which means that neither \(\nu\) no \(\mu\) not equal to 1 and \((q^{1\vec{l}})\) is the linear combination of the generators \(Q^{2,3}, Q^{2,4}, Q^{3,4}\) with commutation relation of \(O(3)\) algebra. By the same reasons

\[(q^{2\vec{l}}) = \sum \epsilon_{2,\nu,\mu,t} q^{4}_t Q^{\nu,\mu} = aQ^{3,4} + bQ^{1,4} + cQ^{1,3}\]

From this consideration it follows uniquely that under the additional condition

\[\rho_2 = 0, \quad Q^{2,3} = 0, \quad Q^{2,4} = 0, \quad Q^{3,4} = 0\]

3 scalar equations (6) are satisfied.

The last condition determines invariant subspace on which representation of \(O(2,4)\) algebra (not of the general position \(\rho_2 = 0\)) is realised. These conditions means that representation of \(O(4)\) algebra in its turn is realised only on one four-dimensional unite vector \(q^{1}_0\) parametrised by 3 compact parameters. And the representation of \(O(2,4)\) algebra of the considered type
is realised on four compact parameters (three parameters of unite four-dimensional \( q^1 \) vector and \( \phi \).

Now we substitute these results into the general formulae for non compact generators

\[
I = \cos \phi (\rho q^1_4 + \sum q^i_4 Q^{i,1}) + \sin \phi q^1_4 P^{2,1} = \cos \phi A_4 + \sin \phi B_4
\]

\[
\bar{x}_4 = -\sin \phi (\rho q^1_4 + \sum q^i_4 Q^{i,1}) + \cos \phi q^1_4 P^{2,1} = -\sin \phi A_4 + \cos \phi B_4
\]

\[
p_\alpha = -\cos \phi (\rho q^1_\alpha + \sum q^i_\alpha Q^{i,1}) - \sin \phi q^1_\alpha P^{2,1} = -\cos \phi A_\alpha - \sin \phi B_\alpha
\]

\[
f_\alpha = -\sin \phi (\rho q^1_\alpha + \sum q^i_\alpha Q^{i,1}) + \cos \phi q^1_\alpha P^{2,1} = -\sin \phi A_\alpha + \cos \phi B_\alpha
\]

and would like to show that 12 remaining equations (4) are also satisfied. As an example let us consider the first system of 3 equations. We have in a consequence

\[
I f_\alpha + p_\alpha \bar{x}_4 = (\cos \phi A_4 + \sin \phi B_4)(-\sin \phi A_\alpha + \cos \phi B_\alpha) - (-\sin \phi A_4 + \cos \phi B_4)(\cos \phi A_\alpha + \sin \phi B_\alpha) =
\]

\[
A_4 B_\alpha - B_4 A_\alpha - q^1_4 q^1_\alpha P^{12} = \sum (q^\mu_4 q^1_\alpha - q^1_4 q^\mu_\alpha) Q^{\mu,1} = \sum q^\mu_4 q^\nu_\alpha Q^{\mu,\nu} P^{12} = Q_{\alpha,4} P^{12} = p_4 \bar{x}_\alpha
\]

All other equations (3) may be checked by the same way.

4.2 Unitary representations

In this case as it follows from explicit expression for \( K_2 \) the unitary representation of the main continues serie with \( \rho = -2 + i\sigma \) gives correct sign in Kazimir operator to have a correct limit \( I \to 1 \) to classical limit. By this reason we would not like to consider other unitary representations which are exist in the case under consideration.

4.3 de Alamber equation

We will not present all generators of \( O(4) \) in explicit form as functions of three angles of unite \( O(4) \) vector but only present below explicit form of the d Alamber operator. In case under consideration the group of motion is \( O(2, 3) \) and consequently we have

\[
m^2 = p^2 + f^2 - p^2_4 - l^2 = (\cos \phi A_\alpha + \sin \phi B_\alpha)^2 + (\sin \phi A_\alpha - \cos \phi B_\alpha)^2 - p^2_4 - l^2 =
\]
\[
\rho (\rho + 3) - \rho q_1^1 + Q)^2 + \partial_{\varphi}^2 + 2 \cot \tau \partial_{\tau} + \cot^2 \tau l^2 - \cos^2 \tau \partial_{\phi}^2 \]

where \( Q = \sum q_i^i Q_{i^1} = -\sin \tau \partial_{\tau}, q_1^1 = \cos \tau \). After a little further manipulations we come to a finally expression

\[
(\cos \tau)^{-\rho} m^2 (\cos \tau)^\rho = (\cos \tau \partial_{\tau})^2 + 2 \cot \tau \partial_{\tau} + \cot^2 \tau l^2 - \cos^2 \tau \partial_{\phi}^2
\]

As in the previous case \((O(3, 3))\) variables in this equation are separated and without any difficulties it is possible obtain the specter mass and corresponding wave functions.

## 5 The \(O(2, 4)\) case \(L^2 \leq 0\) connected with the \(O(1, 4)\) group of motion

As it follows from the explicit expression for second Kazimir operator

\[
K_2 = -\mu^2 \left( \frac{I^2}{\mu^2} - \frac{(x - \frac{L^2}{2} p)^2}{\nu^2 L^2} + \frac{L^2}{H^2 \rho^2} + \frac{(f^2 - l^2)}{H^2} \right), \quad L^2 \leq 0 \leq \mu^2 = 1 - \frac{H^2}{L^2 M^2}
\]

is different from the previous one only by exchange \(p \to \bar{x}, \bar{x} \to p, f \to -f, l \to -l\). Thus corresponding formulae may be obtained by simple substitutions in corresponding formulae of the previous case.

7 generators

\[
l_\alpha = \epsilon_{\alpha, \beta, \gamma} Q_{\beta, \gamma}, \quad \bar{x}_4 = F_{4, 2} = P_{2, 1} \quad p_\alpha = F_{\alpha, 4} = Q_{\alpha, 4}
\]

are compact one and 8

\[
I = F_{1, 4} = \rho^1 \cos \phi q_1^1 + \rho^2 \sin \phi q_2^1 + \cos \phi \sum q_i^i Q_{i^1}^1 + \sin \phi \sum q_i^i Q_{i^2}^1 + \sin \phi q_4^1 P_{2, 1}^1
\]

\[
p_4 = F_{2, 4} = -\rho^1 \sin \phi q_1^1 + \rho^2 \cos \phi q_2^1 - \sin \phi \sum q_i^i Q_{i^1}^1 + \cos \phi \sum q_i^i Q_{i^2}^1 + \cos \phi q_4^1 P_{2, 1}^1
\]

\[
\bar{x}_\alpha = T_{1, \alpha} = \rho^1 \cos \phi q_1^1 + \rho^2 \sin \phi q_2^1 + \cos \phi \sum q_i^i Q_{i^1}^1 + \sin \phi \sum q_i^i Q_{i^2}^1 + \sin \phi q_4^1 P_{2, 1}^1
\]

\[
f_\alpha = F_{2, \alpha} = -\rho^1 \sin \phi q_1^1 + \rho^2 \cos \phi q_2^1 - \sin \phi \sum q_i^i Q_{i^1}^1 + \cos \phi \sum q_i^i Q_{i^2}^1 + \cos \phi q_4^1 P_{2, 1}^1
\]

are noncompact. Now all notations with \(Q, q\) are connected with four-dimensional compact group of rotation, \(P, p\) with \(O(2)\).
5.1 Invariant subspaces and unitary representation

It is not difficult to understand that the system of equations (3) is invariant with respect to substitution $p \rightarrow \bar{x}, \bar{x} \rightarrow p, f \rightarrow -f, l \rightarrow -l$ and thus all remains the same as in the previous section.

5.2 de Alamber operator

The explicit expressions for quantized space generators are the following ones

$$p_4 = -\sin \phi (\rho^1 q^1_4 + \sum q^i_1 Q^{i,1}) + \cos \phi q^1_4 P^{2,1}$$

$$f_\alpha = -\sin \phi (\rho^1 q^1_\alpha + \sum q^i_1 Q^{i,1}) + \cos \phi q^1_\alpha P^{2,1}$$

$$l_\alpha = \epsilon_{\alpha,\beta,\gamma} Q_{\beta,\gamma}, \quad p_\alpha = Q_{\alpha,4}$$

The second Kazimir operator of $O(1,4)$ has the form:

$$-m^2 = p_4^2 + f_\alpha^2 - p_\alpha^2 - l_\alpha^2 = \sum_{i=1}^{4} (-\sin \phi (\rho q_i^1 + \sum q^i_s Q^{s,1}) + \cos \phi q^1_\alpha P^{2,1})^2 - K_2(O(4)) =$$

$$(\sin \phi \rho + \cos \phi \partial_\phi)^2 + 3 \cos \phi (\sin \phi \rho + \cos \phi \partial_\phi) - \cos \phi^2 K_2(O(4))$$

Further regrouping of the terms lead to a final result

$$-(\cos \tau)^{-\frac{\rho+\frac{3}{2}}{2}} m^2 (\cos \tau)^{\rho+\frac{3}{2}} = (\cos \phi \partial_\phi)^2 + (l + \frac{1}{2}) (l + \frac{3}{2}) \cos \phi^2 - \frac{9}{4}$$

6 The $O(1,5)$ case

$$K_2 = \nu^2 \left( \frac{p^2 L^2}{H^2} + \frac{x^2}{L^2 \nu^2} + \frac{f^2 - l^2}{H^2} - \frac{I^2}{\nu^2} \right), \quad L^2 \leq 0 \leq \nu^2$$

In this case 10 generators are compact

$$I = Q_{65}, \quad p_\alpha = Q_{\alpha,5}, \quad x_\alpha = Q_{\alpha,6}, \quad l_\alpha = \epsilon_{\alpha,\beta,\gamma} Q_{\beta,\gamma}$$

(we use indexes (1, 2, 3, 5, 6) for numeration of all ingredientes of $O(5)$ algebra). And 5 generators are non compact

$$f_\alpha = \rho q^5_\alpha + \sum_{s=1}^{5} q^s_\alpha Q^{(s,5)}, \quad x_4 = \rho q^5_4 + \sum_{s=1}^{5} q^s_6 Q^{(s,5)}, \quad p_4 = \rho q^5_4 + \sum_{s=1}^{5} q^s_5 Q^{(s,5)}$$
6.1 Invariant subspaces

Solution of the main system (3) is equivalent to equality to zero all generators of compact $O(4)$ algebra with generators $Q^{a,b}$ with $a, b = 1, 2, 3, 4$. This is equivalent to realization of $O(5)$ algebra on unit vector $q^5_i$. This is not a big problem to check this fact directly.

6.2 Unitary representation

It is obvious that generators of the quantised space posses the unitary irreducible representation with $\rho = -2 + i\sigma$. Compact generators are always antihermitian ones, the condition of antihermitianes of noncompact generators leads directly to values for $\rho$ presented above. But this representation in contradicition with the classical limit $I \to 1$. This easily to see from the explicit expressions for second Kazimir operator, which for this value of $\rho$ takes negative values. Thus as in the case of $O(3,3)$ algebra it is necessary to consider the case of unitary representation possess the Hermitian form.

The condition that invariance of hermitian form with respect to compact transformation restricted the funtucional dependence of the kernal of hermitian up to one variable $K = K((q^5_i, \bar{q}^5_i)) \equiv \bar{K}(x)$. Equations responsible for invariance with respect of noncompact generators are the following ones

\[ [(\rho + 4)q^{5}_i + (\rho + 4)\bar{q}^{5}_i - (q^{5}_i - q^{5}_i x)\partial_x - (q^{5}_i - \bar{q}^{5}_i x)\partial_x]K = 0 \]

which are equivalent to the pair of equations

\[ [(\rho + 4) + (\rho + 4)x - (1 - x^2)\partial_x]K = 0, \quad [(\rho + 4)x + (\rho + 4) - (1 - x^2)\partial_x]K = 0 \]

The condition of selfconsistency leads to relation

\[ |\rho - \rho|(x - 1) = 0 \]

Condition $x = 1$ leads to singular solution which responsible for the existence of representation with $\rho = -2 + i\sigma$.

In the case of the real $\rho$ the kernal function has a form

\[ K = c|1 - x|^{-(\rho + 4)} \]

with all comments with respect to $O(3,3)$ section.
6.3 d Alamber operator

The group of motion in this case is $O(1, 4)$ and corresponding second order operator of Kazimir is the following one

$$-m^2 = \tilde{f}^2 + p_4^2 - \tilde{p}^2 - \ell^2 = \sum_{a=1,2,3,4}[(\rho+4)q_a^5 + \sum_{s=1}^{5}Q^{(s,5)}q_a^s][\rho q_a^5 + \sum_{s=1}^{5}q_a^sQ^{(s,5)}] - \sum_{a\leq b}Q_{a,b}Q_{a,b} =$$

$$\rho(\rho+4) - [(\rho+4) \cos \tau - \sin \tau \partial_{\tau}][\rho \cos \tau - \sin \tau \partial_{\tau}] + \sum_{i\leq j}Q_{i,j}Q_{i,j} - \sum_{a\leq b} Q_{a,b}Q_{a,b} =$$

Further regrouping of the terms leads to the finally expression

$$(\cos \tau)^{-\rho}m^2(\cos \tau)^{\rho} = (\cos \tau \partial_{\tau})^2 + 3 \cot \tau \partial_{\tau} - (\cot \tau)^2 l(l+2)$$

7 Outlook

After this paper and [1] are finished, the author hopes that his almost forty years long struggle with quantized spaces [6] is now over and our relations now will be more friendly.

We have now a selfconsistent mathematical scheme, which permits explicit calculations without making changes in the existing physical theory. The only small question that remains is whether the Nature will accept the rules of the game we offer.

The new constants that arise in our machinery are of very large scale; hence, if a confirmation of our constructions is to be found, it will come from cosmological observations. Unfortunately, the author is very far from this field.

An essentially new feature of our formalism are the new terms in the energy balance (Lorenz moments). This gives a hope that the dark matter problem can be tackled from a new angle.

On the microscopic scale, the appearance of the de Sitter group essentially changes the problem of the unification of the internal and spatial symmetries. The non-invariance with respect to the time reversal may have its consequences on the microscopic level as well.
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