Nature of Lieb’s “hole” excitations
and two-phonon states of a Bose gas

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It is generally accepted that the “hole” and “particle” excitations are two independent
types of excitations of a one-dimensional system of point bosons. We show that the Lieb’s
“hole” with the momentum \( p = j2\pi/L \) is \( j \) identical interacting phonons with the momentum
\( 2\pi/L \) (here, \( L \) is the size of the system, and \( \hbar = 1 \)). We strictly prove this assertion for
\( j = 1,2 \) by comparing solutions for a system of point bosons with solutions for a system
of nonpoint bosons (in the limit of the point interaction). The Lieb-Liniger equations in
Gaudin’s form imply that our conclusion is proper also for \( j > 2 \). Thus, the holes are
not a physically independent type of quasiparticles. Moreover, we find the solution for two
interacting phonons in a Bose system with an interatomic potential of the general form at a
weak coupling and any dimension (1, 2, or 3). It is also shown that the maximum possible
number of phonons in a Bose system is equal to the number of atoms \( N \). Finally, we discuss
the solitonic properties of holes.

Keywords: point bosons, interaction of phonons, hole-like excitations.

1 Introduction

This work is devoted to two main problems: the determination of the wave function and the
energy of two interacting phonons in a Bose gas with a potential of the general form and the
study of the nature of Lieb’s “holes”. The first problem was not solved, to our knowledge,
and can help one to solve the second problem.

The elementary excitations of a one-dimensional (1D) system of point bosons are usually
separated into two types: particle-like (“particles”) and hole-like (“holes”) [1, 2, 3, 4, 5, 6, 7].
At the weak coupling the dispersion law of “particles” coincides with the Bogolyubov law
[8, 9] and agrees with the Feynman’s solutions [10, 11, 12] and more later models [13, 14,
15, 16, 17, 18, 19, 20, 21] (other references can be found in reviews [22, 23]). Therefore, it
is natural to consider that the particles correspond to Bogolyubov–Feynman quasiparticles.
The dispersion law of holes was found only in the approach based on the Bethe ansatz \cite{1}. In this case, Lieb attacked the Bogolyubov’s and Feynman’s approaches and proposed some arguments in favor of that the holes are an independent type of elementary excitations \cite{1,2}. This point of view became traditional. Later on, it was found that the dispersion law of holes is close to that for the soliton solution of the 1D Gross–Pitaevskii equation \cite{24,25}. This became the main argument in favor of that the holes are a particular independent type of quasiparticles. However, such point of view does not agree with the models \cite{8,9,10,11,12,13,14,15,16,17,18,19,20,21}. It is important that the models \cite{9,10,11,16,17,18,19,20,21} work in 1D, since they do not use a condensate (we note that the Bogolyubov method also works in 1D at small \(\gamma\) and \(T\), if \(N\) is finite \cite{26}). If the holes would be a separate type of quasiparticles, this would mean the significant shortcoming in the models \cite{8,9,10,11,12,13,14,15,16,17,18,19,20,21} and in close ones. In addition, if the holes are an independent type of excitations, then they give a separate contribution to thermodynamic quantities (since holes interact with particles and, therefore, participate in the thermal equilibrium). Such analysis indicates that the question about the nature of holes is of fundamental importance.

The one-dimensional system differs qualitatively from a three-dimensional (3D) one by that the atom in a 1D system cannot get around another atom. The former can only pass through the latter. Despite this circumstance, Lieb believed that 1D and 3D systems are qualitatively similar \cite{1}. Therefore, he made conclusion \cite{1} that holes can exists also in 3D systems, at least at some values of parameters.

In what follows, we will study the structure of the wave functions of “particles” and holes and will strictly show that the hole is a collection of interacting “particles” (in this case, the hole can be a soliton). It was noted in the literature that the holes are not an independent type of excitations \cite{6,27,28}. This conclusion was based on the Lieb–Lininger equations. However, these equations are not enough to clarify the physical nature of holes.

Let us consider what the Lieb–Lininger equations can say about the nature of holes. These equations describe a periodic 1D system of point bosons \cite{29}. Gaudin wrote them in the form \cite{4,30}

\[
Lk_i = 2\pi n_i + 2 \sum_{j=1}^{N} \arctan \frac{c}{k_i - k_j} \Big|_{j \neq i}, \quad i = 1, \ldots, N, \tag{1}
\]

where \(N\) is the number of bosons, \(L\) is the size of the system, and \(n_i = 0, \pm 1, \pm 2, \ldots\). In the literature, the point bosons are usually described by the Lieb–Lininger equations in the Yang–Yang form \cite{3}:

\[
Lk_i = 2\pi I_i - 2 \sum_{j=1}^{N} \arctan \frac{k_i - k_j}{c}, \quad i = 1, \ldots, N. \tag{2}
\]
The equations (1) and (2) are equivalent [4, 30]: the formula

$$\arctan \alpha = (\pi/2)\text{sgn}(\alpha) - \arctan(1/\alpha)$$

allows one to rewrite Eqs. (2) in the form (1). In this case,

$$I_i = n_i + i - \frac{N + 1}{2}.$$  (4)

The ground state of the system corresponds to the quantum numbers \(\{I_i\} = (1 - \frac{N+1}{2}, 2 - \frac{N+1}{2}, \ldots, N - \frac{N+1}{2})\), the particle-like excitation with the momentum \(p = 2\pi j/L\) corresponds to \(\{I_i\} = (1 - \frac{N+1}{2}, \ldots, N - 1 - \frac{N+1}{2}, N - \frac{N+1}{2} + j)\), and a hole with the momentum \(p = 2\pi l/L\) \((l > 0)\) corresponds to the quantum numbers \(I_{i\leq N-l} = i - \frac{N+1}{2}, I_{i> N-l} = 1 + i - \frac{N+1}{2}\) (we set \(\hbar = 2m = 1\) in this section). In the language of Eqs. (1), those states correspond to the following collections of quantum numbers \(\{n_i\} = (n_1, \ldots, n_N)\): \((0, \ldots, 0), (0, \ldots, 0, j),\) and \((0, \ldots, 0, 1, \ldots, 1)\), where 1 is repeated \(l\) times. In this case, the state \((0, \ldots, 0, 1)\) is particular: it can be considered as a particle and as a hole. In the last case, any state \((n_1, \ldots, n_N)\) can be considered as a collection of interacting holes. If the state \((0, \ldots, 0, 1)\) is a particle, then any state \((n_1, \ldots, n_N)\) can be considered as a collection of interacting particles. Therefore, the physical nature of the state \((0, \ldots, 0, 1)\) is the key point. From physical reasonings, we may expect that the state \((0, \ldots, 0, 1)\) corresponds to a phonon with the wavelength \(\lambda = L\) (indeed, if the state \((0, \ldots, 0, 1)\) would correspond to a hole, then the phonon with \(\lambda = L\) would be absent in the system, which is strange). In this case, each state \((n_1, \ldots, n_N)\) can be considered as a collection of interacting phonons. In particular, the state \((0, \ldots, 0, j)\) should correspond to one phonon with the momentum \(p = 2\pi j/L\). As for the state with \(n_{j\leq N-l} = 0, n_{j\geq N-l+1} = 1\), it should correspond to \(l\) interacting phonons, each of them has the wavelength \(\lambda = L\) and momentum \(2\pi/L\). However, according to the Lieb’s classification [1], the state with the quantum numbers \(n_{j\leq N-l} = 0, n_{j\geq N-l+1} = 1\) corresponds to a hole with the momentum \(p = 2\pi l/L\). Therefore, the hole with the momentum \(p = 2\pi l/L\) \((l > 0)\) should coincide with \(l\) interacting phonons, each of them has the momentum \(2\pi/L\). This possibility is also seen from the analysis by Lieb [1].

To ascertain the nature of a hole, it is necessary to study the structure of \(N\)-boson wave functions of a hole and a particle. In what follows, we will prove that the state \((0, \ldots, 0, 1)\) corresponds to a phonon, and the hole with the momentum \(p = 4\pi/L\) coincide with two interacting phonons \((0, \ldots, 0, 1)\) and \((0, \ldots, 0, 1, 0)\). In addition, we will determine the largest number of quasiparticles in a Bose gas and discuss the interconnection between holes and solitons.

2 Phono with the quantum numbers \(\{n_i\} = (0, \ldots, 0, 1)\).

One can investigate the structure of wave functions of a “particle” and a hole in two ways: based on the wave functions of point bosons [4, 29] or on the wave functions of nonpoint
bosons (i.e., bosons with nonzero interaction radius) [9, 10, 11, 17, 21, 31, 32, 33, 34], by passing to a point potential in the last case. Let us consider the second way. The transition from the solutions for nonpoint bosons to solutions for point ones, based on the Bethe ansatz, was not studied in the literature in details.

Consider a periodic system of \( N \) bosons with interatomic potential of the general form \( U(r_j - r_l) \). The dimensionality can be equal to 1, 2, or 3. The ground state of a gas is described by the wave function [34]

\[
\Psi_0(r_1, \ldots, r_N) = Ae^{S(r_1, \ldots, r_N)},
\]

and the wave function of a one-phonon state reads [17]

\[
\Psi_p(r_1, \ldots, r_N) = \psi_p \Psi_0,
\]

Here, \( N \) is the total number of atoms, \( r_j \) are the coordinates of atoms, \( \rho_q \) are the collective variables

\[
\rho_q = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-iqr_j},
\]

and all wave vectors \( q, p, l \) are quantized in the 3D case by the rule

\[
q = 2\pi \left( \frac{j_x}{L_x}, \frac{j_y}{L_y}, \frac{j_z}{L_z} \right),
\]

where \( j_x, j_y, j_z \) are integers, and \( L_x, L_y, L_z \) are the sizes of the system.

The approximate solutions for the functions \( \Psi_0 \) and \( \Psi_p \) were first obtained by Feynman [10, 11], Bogolyubov and Zubarev [9], and Jastrow [31]. Then these methods were developed
For a 1D system the energy of a phonon with the momentum finite motion the solution is as follows [17]:

\[ a_2(p) \equiv a_2(p) = \frac{1 - \alpha_p}{2}, \quad \alpha_p = \sqrt{1 + \frac{2\rho \nu(p)}{\hbar^2 p^2/(2m)}}. \]  

\[ E(p) = \frac{\hbar^2 p^2}{2m}(1 - 2a_2(p)) = \sqrt{\left(\frac{\hbar^2 p^2}{2m}\right)^2 + 2\rho \nu(p)\left(\frac{\hbar^2 p^2}{2m}\right)} \equiv E_B(p). \]  

\[ \nu(p) = \int_{-L_x}^{L_x} dx \int_{-L_y}^{L_y} dy \int_{-L_z}^{L_z} dz U(r) e^{-ipr}. \]  

We have obtained the Bogolyubov dispersion law \( E_B(p) \). In this approximation, formula \( 49 \) from Appendix gives the known Bogolyubov solution for the ground-state energy \( E_0 \) [8].

In the zero approximation the sound velocity is \( v_s = \sqrt{\frac{\rho \nu(0)}{m}} = v_s^{(0)} \). In the next approximation the solution is as follows [17]:

\[ v_s = v_s^{(0)}(1 + \delta_s), \quad \delta_s = -\frac{\hbar^2}{32m^2(v_s^{(0)})^2} \frac{1}{N} \sum_{q \neq 0} q^2 \left( \frac{2\rho \nu(q)}{\hbar^2 q^2/(2m)} \right)^2. \]  

For a 1D system the energy of a phonon with the momentum \( \hbar p_1 = h2\pi/L \) is \( E(p_1) = h p_1 v_s \). In this case, for a finite system we should set \( v_s^{(0)} = \sqrt{\frac{\rho \nu(0)}{m}} + \frac{\hbar^2 v_s^2}{4m^2} \).

Consider a finite 1D system of point bosons \( U(r) = 2c\delta(r) \), \( \nu(p) = 2c \) and set \( \hbar = 2m = 1 \), \( \gamma = c/\rho \). The above-presented formulae give the energy of a phonon with the momentum \( p_1 = 2\pi/L \):

\[ E(p_1) = \sqrt{p_1^4 + 4\rho^2 \gamma p_1^2} \cdot (1 + \delta_s) = \frac{4\pi \rho \sqrt{\gamma}}{L} \sqrt{1 + \frac{\pi^2}{\gamma N^2}} \cdot (1 + \delta_s), \]  

\[ \delta_s = -\frac{1}{4N} \frac{1}{1 + \pi^2/(\gamma N^2)} \sum_{j=1,2,\ldots,\infty} \frac{1}{1 + \pi^2 j^2/(\gamma N^2)} \frac{1}{\sqrt{1 + \gamma N^2/(\pi^2 j^2)}}. \]  

These formulae are valid for \( N^{-2} \ll \gamma \ll 1 \).

Our task is to clarify the nature of the particle \((0, \ldots, 0, 1)\). It is known that the energy \( E_L(p) \) of Lieb’s particle for small \( p \) is close to the Bogolyubov energy \( E_B(p) \) [12]. The small deviation of the particle energy from \( E_B(p) \) contains the information about the nature of the particle. Let us represent the energy of the particle with the momentum \( p_1 = 2\pi/L \) in the
form (15):

\[ E_L(p_i) = \frac{4\pi \rho \sqrt{\gamma}}{L} \sqrt{1 + \frac{\pi^2}{\gamma N^2} \cdot (1 + \delta_{sL})} . \]  

(17)

The energy and momentum of the particle is given by the known formulae

\[ E_L(p) = \sum_{i=1}^{N} (\dot{k}^2_i - k_i^2) , \]  

(18)

\[ p = \sum_{i=1}^{N} (\dot{k}_i - k_i) = \frac{2\pi}{L} \sum_{i=1}^{N} (\dot{n}_i - n_i) . \]  

(19)

In our case, the collections \( \{\dot{k}_i\} \) and \( \{k_i\} \) are solutions of the Gaudin’s equations (1) for a state with one particle (\( \{\dot{n}_i\} = \{0, \ldots, 0, 1\} \)) and for the ground state (\( \{n_i\} = \{0, \ldots, 0, 0\} \)), respectively. The quasimomenta \( \{\dot{k}_i\} \) and \( \{k_i\} \) can be obtained numerically from Eqs. (1) by the Newton method (the Yang–Yang equations (2) give the same solution).

![Fig. 1: Functions \( \delta_s(\gamma) \) (circles) and \( \delta_{sL}(\gamma) \) (crosses) obtained from Eqs. (16) and (1), (17)–(19), respectively; \( \rho = 1, N = 1000. \)](image)

It is seen from Fig. 1 that the small quantity \( \delta_{sL} \) obtained from Eqs. (17)-(19), (1) coincides with high accuracy with \( \delta_s (16) \). The difference of \( \delta_{sL} \) and \( \delta_s \) is about 1% for \( \gamma = 0.0001–0.1 \). Since the function \( \psi_p = \rho_{-p} \) for small \( p \) describes a phonon in the interacting Bose gas \( [9, 11, 17] \), we conclude that Lieb’s particle \( \{n_i\} = \{0, \ldots, 0, 1\} \) (i.e., \( \{I_i\} = \{-\frac{N-1}{2}, -\frac{N-3}{2}, \ldots, N-5, \frac{N-3}{2}, 1 + \frac{N-1}{2}\} \)) is a phonon. In this case, the Gaudin’s equations (11) imply that the hole with the momentum \( p = 2\pi l/L \) \( (l > 1) \) should coincide with \( l \) interacting phonons with the momentum \( 2\pi l/L \). Let us verify this directly for \( l = 2 \).
3 Two interacting phonons vs a hole with the quantum numbers \( \{n_i\} = (0, \ldots, 0, 1, 1) \).

In the language of the Lieb–Lininger equations \([2]\), the hole with the momentum \( p = 4\pi/L \) is characterized by the quantum numbers \( \{I_i\} = (-\frac{N-1}{2}, -\frac{N-3}{2}, \ldots, \frac{N-5}{2}, 1 + \frac{N-3}{2}, 1 + \frac{N-1}{2}) \). In the language of the Lieb–Lininger equations in the Gaudin’s form \([1]\), such hole is described by the quantum numbers \( \{n_i\} = (0, \ldots, 0, 1, 1) \). In the previous section we proved that the state \( \{n_i\} = (0, \ldots, 0, 1) \) describes a phonon with the momentum \( p = 2\pi/L \). The state \( \{n_i\} = (0, \ldots, 0, 1, 0) \) is equivalent to \( \{n_i\} = (0, \ldots, 0, 1) \). Therefore, it is obvious that the state \( \{n_i\} = (0, \ldots, 0, 1, 1) \) is two interacting phonons with the momentum \( p = 2\pi/L \). We now verify this assumption independently, by using the method of collective variables.

Consider a Bose gas with weak coupling and dimensionality of 1, 2, or 3. Let us find the wave function and the energy of two interacting phonons with wave vectors \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \). Feynman noticed that the energy of interaction (\( \delta E \)) of two phonons should be by \( \sim N \) times less than the energy of one phonon \([10]\). However, the solutions for a wave function and \( \delta E \) were not found.

The ground state is described by the wave function \( \psi_0 \) satisfying the Schrödinger equation

\[
-\frac{\hbar^2}{2m} \sum_j \Delta_j \psi_0 + \frac{1}{2} \sum_{ij} U(|\mathbf{r}_i - \mathbf{r}_j|) \psi_0 = E \psi_0
\]

with energy \( E = E_0 \). The equations for \( E_0 \) and the functions \( a_j \) from \([6]\) are given in Appendix. If the system contains one phonon, then the wave function is \( \psi_\mathbf{p} \psi_0 \), where \( \psi_\mathbf{p} \) is given by formula \([8]\), and the solutions for the functions \( b_j \) and the energy of a quasiparticle are given in the previous section. If two phonons with wave vectors \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) are present, then the system is described by the wave function \( \psi_\mathbf{p}_1 \psi_\mathbf{p}_2 \psi_0 \). We substitute this function in the Schrödinger equation and take into account that \( \Psi_0 = A e^S \) satisfies this equation with energy \( E_0 \). As a result, we obtain the equation for the function \( \psi_\mathbf{p}_1 \psi_\mathbf{p}_2 \):

\[
-\frac{\hbar^2}{2m} \sum_j \left[ \Delta_j \psi_\mathbf{p}_1 \psi_\mathbf{p}_2 + 2(\nabla_j S)(\nabla_j \psi_\mathbf{p}_1 \psi_\mathbf{p}_2) \right] = E_\mathbf{p}_1 \mathbf{p}_2 \psi_\mathbf{p}_1 \psi_\mathbf{p}_2,
\]

where \( E_\mathbf{p}_1 \mathbf{p}_2 = E - E_0 \) is the energy of two interacting phonons. Since the interaction of two phonons should be weak, we seek \( \psi_\mathbf{p}_1 \psi_\mathbf{p}_2 \) in the form

\[
\psi_\mathbf{p}_1 \psi_\mathbf{p}_2 = \psi_\mathbf{p}_1 \psi_\mathbf{p}_2 + \frac{\delta \psi_\mathbf{p}_1 \mathbf{p}_2}{\sqrt{N}},
\]

where \( \psi_\mathbf{p}_1 \) and \( \psi_\mathbf{p}_2 \) are one-phonon solutions. We substitute \( \psi_\mathbf{p}_1 \psi_\mathbf{p}_2 \) \([22]\) in Eq. \([21]\) and take into account that the one-phonon functions \( \psi_\mathbf{p}_1 \) and \( \psi_\mathbf{p}_2 \) satisfy Eq. \([21]\) with the energies
\[ E(p_1) \text{ and } E(p_2), \text{ respectively. In this way we get the following equation for } \delta \psi_{p_1p_2}: \]

\[ -\frac{\hbar^2}{2m} \sum_j [2(\nabla_j \psi_{p_1})(\nabla_j \psi_{p_2}) + \Delta_j \delta \psi_{p_1p_2} + 2(\nabla_j S)(\nabla_j \delta \psi_{p_1p_2})] = \]

\[ = [E(p_1) + E(p_2) + \delta E] \delta \psi_{p_1p_2} + \delta E \psi_{p_1} \psi_{p_2}, \]

\[ E_{p_1p_2} = E(p_1) + E(p_2) + \delta E. \]  

(23)

Here, the energy \( E_{p_1p_2} \) of two interacting phonons is represented as a sum of the energies \( E(p_1) \) and \( E(p_2) \) of free phonons and the correction \( \delta E \).

The solution for the function \( \delta \psi_{p_1p_2} \) should have the form \( \psi_p \) with \( p = p_1 + p_2 \), since formula \( (8) \) describes the state with any number of quasiparticles possessing the total momentum \( \hbar p \):

\[ \delta \psi_{p_1p_2} = B_1(p_1, p_2) \rho_{-p} + \sum_{q \neq 0} \frac{B_2(q; p_1, p_2)}{2! N^{1/2}} \rho_q \rho_{-q-p} \]

\[ + \sum_{q_1, q_2 \neq 0} \frac{B_3(q_1, q_2; p_1, p_2)}{3! N^{1/2}} \rho_{q_1} \rho_{q_2} \rho_{-q_1-q_2-p} + \ldots \]

\[ + \sum_{q_1, \ldots, q_{N-1} \neq 0} \frac{B_N(q_1, \ldots, q_{N-1}; p_1, p_2)}{N! N^{(N-1)/2}} \rho_{q_1} \cdots \rho_{q_{N-1}} \rho_{-q_1-\ldots-q_{N-1}-p}, \]  

(25)

where \( p = p_1 + p_2 \). We substitute \( \delta \psi_{p_1p_2} \) in (23). The result is reduced to the form

\[ 0 = C_1(p_1, p_2) \rho_{-p} + \sum_{q \neq 0} \frac{C_2(q; p_1, p_2)}{N^{1/2}} \rho_q \rho_{-q-p} + \ldots + \]

\[ + \sum_{q_1, \ldots, q_{N-1} \neq 0} \frac{C_N(q_1, \ldots, q_{N-1}; p_1, p_2)}{N^{(N-1)/2}} \rho_{q_1} \cdots \rho_{q_{N-1}} \rho_{-q_1-\ldots-q_{N-1}-p} \]  

(26)

(\( p = p_1 + p_2 \)). Since \( \rho_{-p}, \rho_q \rho_{-q-p}, \rho_{q_1} \rho_{q_2} \rho_{-q_1-q_2-p}, \ldots \) are independent functions of the variables \( r_1, \ldots, r_N \), Eq. (26) is equivalent to the system of \( N \) equations

\[ C_j(q_1, \ldots, q_{j-1}; p_1, p_2) = 0, \quad j = 1, \ldots, N. \]

(27)

For the weak coupling, it is sufficient to consider the equations \( C_1 = 0 \) and \( C_2 = 0 \). They have the form

\[ B_1(p_1, p_2) 2m \hbar^2 [E(p_1) + E(p_2) + \delta E - E_1(p_1 + p_2)] = \]

\[ = 2 \left[ b_1(p_1)b_1(p_2)p_1p_2 - p_1^2b_1(p_1)b_2(p_1; p_2) - p_2^2b_1(p_2)b_2(p_2; p_1) \right] - \]

\[ - \frac{1}{N} \sum_{q \neq 0} B_2(q; p_1, p_2)q(q + p) - \frac{1}{N} \sum_{q \neq 0} B_3(q, -q; p_1, p_2)q^2, \]

(28)
\[
B_2(q; p_1, p_2) \frac{2m}{\hbar^2}[E(p_1) + E(p_2) + \delta E - E_1(q) - E_1(q + p)] = \\
= -b_2(q; p_1)b_2(q + p_1; p_2)(q + p_1)^2 - b_2(-q - p; p_1)b_2(-q - p_2; p_2)(q + p_2)^2 - \\
b_2(q; p_2)b_2(q + p_2; p_1)(q + p_2)^2 - b_2(-q - p; p_2)b_2(-q - p_1; p_1)(q + p_1)^2 + \\
\{b_2(q; p_1) + b_2(-q - p_1; p_1)]b_1(p_2)p_2(q + p_1) + \\
\{b_2(q; p_2) + b_2(-q - p_2; p_2)]b_1(p_1)p_1(q + p_2) - \\
\{b_2(-q - p; p_1) + b_2(q + p_2; p_1)]b_1(p_2)p_2(q + p_2) - \\
\{b_2(-q - p; p_2) + b_2(q + p_1; p_2)]b_1(p_1)p_1(q + p_1) - \\
2p_1^2b_1(q; p_1)p_1(q; p_2) - 2p_2^2b_1(p_2)b_3(q; p_2; p_2) - N\delta E b_1(p_1)b_2(p_2) \frac{2m}{\hbar^2}(\delta_{a, -p_1} + \delta_{a, -p_2}) + \\
2B_1(p_1, p_2)N_1(q; a_2(q) - (p + q)a_2(p + q) - p a_3(p, q)] - \\
\frac{1}{N} \sum_{q_1 \neq 0} B_3(q_1, q - q_1 - p; p_1, p_2)q_1(q + q_1 + p) + \\
\frac{1}{N} \sum_{q_1 \neq 0} B_3(q_1, q - q_1 - p; p_1, p_2)q_1(q - q_1) - \frac{1}{N} \sum_{q_1 \neq 0} B_4(q_1, q - q_1; q_1; p_1, p_2)q_1^2,
\]

where \( p = p_1 + p_2, E_1(q) = \frac{\hbar^2 q^2}{2m} (1 - 2a_2(q)), \) and \( \delta_{a, -p} \) is the Kronecker delta. In this case, \( B_2(q; p_1, p_2) = B_2(-q - p; p_1, p_2). \)

Let us present the functions \( \tilde{\psi}_{p_1}, \tilde{\psi}_{p_2}, \) and \( \delta \tilde{\psi}_{p_1, p_2} \) in (22) in the form of expansions (8) and (25). Then the “leading” term in the expansion of \( \tilde{\psi}_{p_1, p_2} \) is \( A\rho_{-p_1, \rho_{-p_2}}. \) It is convenient to consider the constant \( A \) normalizing. Let us write the functions \( \psi_{p_1}, \psi_{p_2} \) in the form \( b_1(p_1)\tilde{\psi}_{p_1}, b_1(p_2)\tilde{\psi}_{p_2}. \) Then we present \( \psi_{p_1}, \psi_{p_2} \) as a series, where the first term is \( b_1(p_1)\tilde{\psi}_{p_1}, b_1(p_2)\tilde{\psi}_{p_2}. \) The corresponding terms in the expansion of \( \delta \tilde{\psi}_{p_1, p_2} \) have the form \( \frac{B_2(-p_1, p_1, p_2)}{2N}\delta_{p_1, \rho_{-p_2}}. \) Eventually, the coefficient of \( \rho_{-p_1, \rho_{-p_2}} \) in the expansion of the function \( \psi_{p_1, p_2} \) (22) is \( A = b_1(p_1)b_1(p_2) - \frac{B_2(-p_1, p_1, p_2) + B_2(-p_2, p_2, p_2)}{2N}. \)

Let us represent the function \( \tilde{\psi}_{p_1, p_2} \) (22) in the form \( \tilde{\psi}_{p_1, p_2} = A\tilde{\psi}_{p_1, p_2}, \) where \( \tilde{\psi}_{p_1, p_2} = \frac{b_1(p_1)N_1[p_2]}{A} \tilde{\psi}_{p_1, p_2} + \frac{\delta \tilde{\psi}_{p_1, p_2}}{2N}. \) Since the interaction of phonons is very weak, the term \( \frac{B_2(-p_1, p_1, p_2) + B_2(-p_2, p_2, p_2)}{2N} \) in \( A \) should be less than \( b_1(p_1)b_1(p_2) \) by \( \sqrt{N} \) or even \( N \) times. Therefore, \( \frac{2N}{b_1(p_1)b_1(p_2)} \approx 1. \) As a result, \( \tilde{\psi}_{p_1, p_2} = \tilde{\psi}_{p_1} \tilde{\psi}_{p_2} + \frac{\delta \tilde{\psi}_{1, p_2}}{2N}. \) Here, \( \tilde{\psi}_{p} \) is a one-phonon function (8) with \( b_1 = 1. \)

In this case, \( b_{j \geq 2} \) satisfy the equations from Appendix, in which \( b_1 = 1. \) Represent the term \( \delta \tilde{\psi}_{p_1, p_2}/A \) in the form (25). Then we consider the factor \( A \) to be normalizing and set \( A = 1. \) Such transformations lead to the necessity to set \( b_1(p_1) = b_1(p_2) = 1 \) and \( B_2(-p_1; p_1, p_2) = B_2(-p_2; p_1, p_2) = 0 \) in Eqs. (28), (29) and the equations of Appendix.

We consider the coupling to be weak: \( \gamma \ll 1, \) but \( \gamma \gg N^{-2} \) (the latter is necessary for the linearity of the dispersion law at small \( p \)). In this case, we can seek \( \delta E \) and \( \delta \tilde{\psi}_{p_1, p_2} \) in the zero approximation. This means [17, 34] that all sums in the chain of equations for \( B_j \) and \( \delta E \) should be neglected (if we find \( B_2(q; p_1, p_2) \) from (29), by neglecting sums of the form \( \sum_{q_1 \neq 0} B_3 \), we convince ourselves that the sum \( \frac{1}{N} \sum_{q_1 \neq 0} B_2(q; p_1, p_2)q(q + p) \) is negligible.
relative to $2\mathbf{p}_1\mathbf{p}_2$ in Eq. (28). As a result, Eq. (28) takes the form

$$B_1(\mathbf{p}_1, \mathbf{p}_2) = \frac{\hbar^2}{m} \left[ \mathbf{p}_1 \mathbf{p}_2 - \mathbf{p}_1^2 \mathbf{b}_2(\mathbf{p}_1; \mathbf{p}_2) - \mathbf{p}_2^2 \mathbf{b}_2(\mathbf{p}_2; \mathbf{p}_1) \right].$$

Let us set in (29) $\mathbf{q} = -\mathbf{p}_1$. Then Eq. (29) reads

$$0 = -b_2(-\mathbf{p}_2; \mathbf{p}_1) - b_2(\mathbf{p}_1 - \mathbf{p}_2; \mathbf{p}_2)(\mathbf{p}_2 - \mathbf{p}_1)^2 - b_2(-\mathbf{p}_2; \mathbf{p}_1) b_2(\mathbf{p}_2 - \mathbf{p}_1; \mathbf{p}_1)(\mathbf{p}_2 - \mathbf{p}_1)^2$$

$$- [b_2(-\mathbf{p}_1; \mathbf{p}_2) + b_2(\mathbf{p}_1 - \mathbf{p}_2; \mathbf{p}_2)]b_1(\mathbf{p}_1 - \mathbf{p}_2) = - [b_2(-\mathbf{p}_1; \mathbf{p}_2) + b_2(\mathbf{p}_2 - \mathbf{p}_1; \mathbf{p}_1)]p_2(\mathbf{p}_2 - \mathbf{p}_1)$$

$$- 2p_1^2 b_3(-\mathbf{p}_1; \mathbf{p}_2) - 2p_2^2 b_3(-\mathbf{p}_1; \mathbf{p}_2) - N\delta E \frac{2m}{\hbar^2} (1 + \delta p_2, p_1)$$

$$+ 2B_1(\mathbf{p}_1, \mathbf{p}_2) p = -p_1 a_2(-\mathbf{p}_1) - p_2 a_2(\mathbf{p}_2) - p_3 a_3(\mathbf{p}_1, -\mathbf{p}_1)].$$

Equation (29) for $\mathbf{q} = -\mathbf{p}_2$ is also reduced to (31) (to sight this, one needs to consider the relations $a_2(-\mathbf{p}) = a_2(\mathbf{p})$, $a_3(\mathbf{p}, -\mathbf{p}_1) = a_3(\mathbf{p}, \mathbf{p}_1 - \mathbf{p})$ and $b_3(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3) = b_3(-\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3; \mathbf{p}_2; \mathbf{p}_3)$). Equations (30), (31) allow us to find $B_1(\mathbf{p}_1, \mathbf{p}_2)$ and $\delta E$. From Eq. (29) at $\mathbf{q} \neq -\mathbf{p}_1, -\mathbf{p}_2$ we can determine $B_2(\mathbf{q}; \mathbf{p}_1, \mathbf{p}_2)$.

Consider the case $\mathbf{p}_2 = \mathbf{p}_1$. According to (31), $\mathbf{q}$ in $b_2(\mathbf{q}; \mathbf{p})$ must be nonzero. Therefore, if (31) includes the term $b_2(0; \mathbf{p})$, this term should be dropped. Then relations (30), (31) yield

$$B_1(\mathbf{p}_1, \mathbf{p}_1) = \frac{p_1^2[2 - 4b_2(\mathbf{p}_1; \mathbf{p}_1)]}{(2m/\hbar^2)[2E(\mathbf{p}_1) + \delta E - E_1(\mathbf{p}_1)]},$$

$$B_1(\mathbf{p}_1, \mathbf{p}_1) = \frac{-2p_1^2b_3(-\mathbf{p}_1, \mathbf{p}_1; \mathbf{p}_1) + (2m/\hbar^2)N\delta E}{4p_1^2[a_2(\mathbf{p}_1) + a_3(\mathbf{p}_1, -\mathbf{p}_1)].}$$

Equations (32) and (33) give a square equation for $\delta E$ with the roots

$$\delta E_\pm = -\tilde{E}_+ \pm \frac{\sqrt{\tilde{E}_- - 8N^{-1}[\hbar^2 p_1^2/(2m)]^2[1 - 2b_2(\mathbf{p}_1; \mathbf{p}_1)][a_2(\mathbf{p}_1) + a_3(\mathbf{p}_1, -\mathbf{p}_1)]}}{2},$$

where

$$\tilde{E}_\pm = E(p_1) - \frac{E_1(2p_1)}{2} \pm b_3(-\mathbf{p}_1; \mathbf{p}_1; \mathbf{p}_1) \frac{\hbar^2 p_1^2}{2mN}.$$ 

At $p_1 \to 0$ and $\gamma \ll 1$, the formulae in Appendix yield

$$a_3(2\mathbf{p}_1; -\mathbf{p}_1) = a_3(\mathbf{p}_1, \mathbf{p}_1) \approx -a_2(\mathbf{p}_1)/4, \quad b_3(-\mathbf{p}_1, \mathbf{p}_1; \mathbf{p}_1) \approx 1/8.$$ 

Using the relation $a_4(-\mathbf{p}_1, \mathbf{p}_1; \mathbf{p}_1) \approx 7a_2(\mathbf{p}_1)/16$, we get $b_3(-\mathbf{p}_1, \mathbf{p}_1; \mathbf{p}_1) \approx -5/32$. Therefore, relations (34), (35) are reduced to

$$\delta E_\pm = -\tilde{E}_+ \pm \frac{\sqrt{\tilde{E}_- - 9/2N \left( \frac{\hbar^2 p_1^2}{2m} \right)^2 a_2(\mathbf{p}_1)}}{a_2(\mathbf{p}_1)},$$

$$\tilde{E}_\pm = E(p_1) - \frac{E_1(2p_1)}{2} \pm \frac{5}{32N} \frac{\hbar^2 p_1^2}{2m},$$

where $E(p_1) = \hbar p_1 v_s$, $a_2(\mathbf{p}_1) \approx -a_2(\mathbf{p}_1)/2 \approx -\frac{\sqrt{m}\nu_s(p_1)}{\hbar p_1}$, and $E_1(2p_1)$, $v_s$ are determined by formulae (12), (14). At $N \gg 1, \gamma \lesssim N^{-1}$, the corrections $\frac{9}{2N} \left( \frac{\hbar^2 p_1^2}{2m} \right)^2 a_2(\mathbf{p}_1)$ and $\frac{5}{32N} \frac{\hbar^2 p_1^2}{2m}$ in (37), (38) are negligible, and solutions (37), (38) take the simple form

$$\delta E_+ \approx 2|\tilde{E}|, \quad \delta E_- \approx -\frac{9E(p_1)}{8N} \frac{\hbar^2 p_1^2}{2m|\tilde{E}|}.$$
Since $\delta E_+ > \delta E_-$, namely the solution $\delta E_-$ should be realized in Nature. Thus, we have found the energy of interaction, $\delta E$, of two phonons with the same momentum $\hbar p_1$ at $p_1 \to 0$ and weak coupling ($N^{-2} \ll \gamma \ll 1$). This result is new.

At the considered parameters of the system we have $|\tilde{E}| \sim \frac{\hbar^2 p_1^2}{2m}$. Therefore, $\delta E_- \sim -E(p_1)/N$. In this case, relations (33), (36) yield $B_1(p_1, p_1) \sim -1$. It is natural to expect that $|B_1(p_1, p_2)| \sim 1$ also at $p_2 \neq p_1$. In this case, Eq. (29) yields $|B_2(q, p_1, p_2)| \sim 1$. That is, the term $\delta \psi_{p_1, p_2}/\sqrt{N}$ in formula (22) is less than the main term $\psi_{p_1} \psi_{p_2}$ by $\sim N$ times. These estimates show that the interaction of two phonons is indeed very weak.

Let us return to the question about the nature of a hole. In the above equations, we pass to a 1D point potential. Compare $\delta E_-$ with the quantity

$$\delta E_h = E_h(p = 4\pi/L) - 2E_p(p = 2\pi/L)$$

(41)
equal to the difference of the energy of a hole with the quantum numbers $\{I_i\} = (-\frac{N-1}{2}, -\frac{N-3}{2}, \ldots, \frac{N-5}{2}, 1 + \frac{N-3}{2}, 1 + \frac{N-1}{2})$ and two energies of a free “particle” (phonon) with the quantum numbers $\{I_i\} = (-\frac{N-1}{2}, -\frac{N-3}{2}, \ldots, \frac{N-5}{2}, \frac{N-3}{2}, 1 + \frac{N-1}{2})$. The quantities $p = 4\pi/L$ and $p = 2\pi/L$ in (41) are momenta. The values of $E_h(p = 4\pi/L)$ and $E_p(p = 2\pi/L)$ can be found numerically from the Yang–Yang equations (2) and formulae (18), (19). The value of $\delta E_-$ follows from Eqs. (37) and (38), where we set $\nu(p) = 2c$, $\hbar = 2m = 1$, $c/\rho = \gamma$, and $p_1 = 2\pi/L$.

![Fig. 2: Color online] Functions $\delta E_-(\gamma)$ (37), (38) (circles), $\delta E_-(\gamma)$ (39), (40) (crosses), and $\delta E_h(\gamma)$ (31) (stars); $n = 1$, $N = 1000$. All values of $\delta E$ are multiplied by $10^6$.

It is seen from Fig. 2 that the energy of interaction of two phonons ($\delta E_-$) is close to $\delta E_h$, if $N^{-2} \ll \gamma \lesssim N^{-1}$. The very small value of $\delta E_h$ is an indicator of the nature of a hole. The closeness of the values of $\delta E_-$ and $\delta E_h$ proves that the hole $\{I_i\} = (-\frac{N-1}{2}, -\frac{N-3}{2}, \ldots, \frac{N-5}{2}, 1 + \frac{N-3}{2}, 1 + \frac{N-1}{2})$ coincides with two interacting phonons, each characterized by the collection $\{I_i\} = (-\frac{N-1}{2}, -\frac{N-3}{2}, \ldots, \frac{N-5}{2}, \frac{N-3}{2}, 1 + \frac{N-1}{2})$. This is the main result of the present work.
In the region $N^{-1} \ll \gamma \ll 1$ the quantities $\delta E_-$ and $\delta E_h$ are considerably different, since we found a solution for $\delta E_-$ only in zero approximation. The error of the numerical calculation of $\delta E_h$ should also be significant in this case.

We note that, to obtain namely a two-phonon solution, it is necessary firstly to set the orders of the quantities $B_j$ and $\delta E$. Otherwise, we can arrive at another solution, since the function $\psi_{\{p_1 p_2\}}$ (22), (25) can describe any excited state with the total momentum $\hbar (p_1 + p_2)$ (see Section 5). We took the two-phonon nature of the state into account with the help of the condition $|B_2(-p_1; p_1, p_2) + B_2(-p_2; p_1, p_2)|/(2N) \ll |b_1(p_1)b_1(p_2)|$.

The above two-phonon solution should be contained in Eqs. (49)–(54) of Appendix, since any (not only one-phonon) excited state of the system with the total momentum $p$ is described by the function $\psi_{\{p\}}\Psi_0$ (7).

4 Additional arguments.

Consider a 1D Bose gas with point interaction. Let us find the limit $c \to 0$ for the Lieb–Lininger solutions [4, 29]

$$\psi_{\{k\}}(x_1, \ldots, x_N) = \text{const} \sum_P a(P) e^{i \sum_{l=1}^N k_P x_l},$$

$$a(P) = \prod_{j=1}^L \left( 1 + \frac{ic}{k_P_j - k_P_j} \right).$$

For the state $\{n_i\} = (0, \ldots, 0, 1)$, at $c \to 0$ we get $\{k_i\} = (0, \ldots, 0, 2\pi/L)$. Relations (42) and (43) yield $a(P) = 1$ and

$$\psi_{\{k\}} \equiv \psi_1 = c_1 \rho_{-k_N},$$

where $k_N = 2\pi/L$. For the state $\{n_i\} = (0, \ldots, 0, 1, 1)$, we get $\{k_i\} \approx (0, \ldots, 0, 2\pi/L, 2\pi/L)$. Then relations (42), (43) yield $a(P) = 1$ and

$$\psi_{\{k\}} \equiv \psi_{11} = c_{11} \left( \rho_{-k_N} \rho_{-k_N} - \rho_{-2k_N} \sqrt{N} \right).$$

Here, while calculating $a(P)$, we take into account that $(k_N - k_{N-1})|_{c\to0} \sim c^{1/2}$. Functions (44) and (45) coincide with the wave functions of a system of free bosons, in which one or two (respectively) atoms have the momentum $2\pi/L$. The normalizing coefficients are $c_1 = L^{-N/2}, c_{11} = \sqrt{\frac{N}{N-1}} c_{11}^{33}$. Since $\rho_{-k_N} \sim 1$ for the overwhelming majority of configurations $(x_1, \ldots, x_N)$, the comparison of $\psi_{11}$ (45) and $\psi_1$ (44) shows that in the limit $c \to 0$ the hole $(0, \ldots, 0, 1, 1)$ is two interacting particles $(0, \ldots, 0, 1)$, which agrees with the result of the previous section. At $c = 0$ we have, of course, free atoms instead of quasiparticles.

The one-phonon and two-phonon solutions (7) and (22) pass at $c = 0$ to solutions (44) and (45). To demonstrate this with the formulae in Sections 2 and 3, we take the relations

$$a_j = 0, b_{j \geq 2} = 0, B_{j \geq 2} = 0,$$

and $\delta E = 0$ into account. Relation (32) yields $B_1(p_1, p_1) = -1.$
Thus, Eqs. (7), (8), and (22) describe free bosons at the zero interaction and phonons at a nonzero one (if the interaction is switched-on, the functions $\psi_{p_1}, \psi_{p_1p_2}$ vary negligibly, but the dispersion law $E(p) \sim p^2$ transits into $E(p) \approx v_s p$ due to a change of $\Psi_0$).

It is clear that any Lieb–Lininger solution (42) can be presented in the form (7), (8). It would be of interest to get solutions (7), (8), and (22) directly from (42) at $c \neq 0$. This is a task for the future.

Both in the Gaudin’s numbering and in the method of collective variables, each excited state of a 1D system is described by the collection of quantum numbers $\{n_i\}$ ($i = 1, \ldots, N$) corresponding to the collection of quasiparticles with the momenta $p_1, \ldots, p_N$, where $p_j = 2\pi n_j/L$. That is, there is one-to-one correspondence between solutions in the method of collective variables at $\nu(p) = 2c$ and solutions in the Lieb–Lininger approach. In this case, the uniqueness of a solution for each collection $\{n_i\}$ was proved only for the Lieb–Lininger approach [5].

The calculation of the statistical sum of a 1D system of point bosons at $N = \infty$ gives [35]

$$F \big|_{T \to 0} = E_0 + k_B T \sum_{l=\pm 1, \pm 2, \ldots} \ln \left(1 - e^{-\frac{E_p(p)}{k_B T}}\right),$$

where $E_p(p_l)$ is the dispersion law of particles. The calculation [35] involves all states of the system (including the ground state, particles, and holes). Formula (46) is exact at $N = \infty$ and $T \to 0$. Equation (46) is the known formula for the free energy of an ensemble of noninteracting Bose quasiparticles. The verification [28] indicates that formula (46) and the Yang–Yang approach [3] lead to identical thermodynamic solutions $F, S$. If we consider formally the state $\{n_i\} = (0, \ldots, 0, 1)$ as a hole, then any excited state $(n_1, \ldots, n_N)$ can be approximately considered as a collection of noninteracting holes. This leads again to formula (46) with the replacement of $E_p(p)$ by the dispersion law of holes $E_h(p)$. Such dualism of holes and particles is interesting but illusory, since the state $\{n_i\} = (0, \ldots, 0, 1)$ is physically a phonon, not a hole.

The analysis of Sections 1–4 clearly shows that the hole is simply a collection of identical interacting phonons with the momentum $2\pi/L$. This corresponds directly to the Gaudin’s numbering (see Eq. (1)). Therefore, the introduction of quasiparticles with the help of the Gaudin’s numbering [28, 35] is more physical. In this case, the curve of holes $E_h(p)$ describes the excited states with minimum energy for given $p$. The Yang–Yang numbering (see Eq. (2)) is also useful: using it, it is easy to find the energy of quasiparticles at strong coupling. We note that though at $\gamma \to \infty$ the energy of a particle is close to the energy of a free fermion (it is seen from Eq. (2)), the particle is described by the Bose statistics, due to the Bose symmetry of the wave function and Bose formula (46).

We recall also the arguments by Feynman [10, 11, 12]. According to them, only the single dispersion law, corresponding to phonons, should be in the region of low $E, p$. Such conclusion is in agreement with our analysis.
5 States with the largest number of quasiparticles.

Consider $N = 10^6$ Bose atoms placed in a vessel. How many quasiparticles can exist in such a system? At first sight, the number of quasiparticles should not be bounded from above, since a quasiparticle is similar to a wave in the probability field. However, it turns out that the number of quasiparticles cannot exceed $N$. This can be proved by two methods.

The most simple way is to use the Lieb–Lininger equations (1). In the Gaudin’s numbering, the creation of a quasiparticle is equivalent to a change in some $n_j$ from $n_j = 0$ to any $n_j = l \neq 0$. In this case, a Bogolyubov–Feynman quasiparticle with the momentum $p = 2\pi l/L$ is created. The largest number of quasiparticles is equal to the number of $n$’s with different $j$: it is the number of equations in system (1), which is equal to the number of atoms $N$. In this case, a hole is several Bogolyubov–Feynman quasiparticles. These properties were noted in [28, 35].

For nonpoint bosons it is necessary to note that a wave function of the form (7), (8) describes not only a state with one quasiparticle, but also the states with any number of quasiparticles. Indeed, the wave function of any stationary excited state can be written in the form $f(r_1, \ldots, r_N)\psi_0$. The periodic system has a definite momentum. The general form of the wave function of a state with the total momentum $\hbar p$ is set by formulae (7), (8). Therefore, the function $f(r_1, \ldots, r_N)$ should coincide with $\psi_p$ (8). In this case, $b_j$ are different for different states. For the state with one phonon, $b_j \sim 1$ for all $j$. For a state with two phonons with the momenta $\hbar p_1$ and $\hbar p_2$ we should set $p = p_1 + p_2$ in (7), (8). In this case, $b_{j \geq 3} \sim 1$, $b_1(p) \sim N^{-1/2}$, $b_2(q_1; p) \sim N^{-1/2}$ for $q_1 \neq -p_1, -p_2$, and $b_2(q_1; p) \sim N^{1/2}$ for $q_1 = -p_1, -p_2$. For a state with three phonons we have $p = p_1 + p_2 + p_3$. The lowest not small coefficients $b_j$ should be the coefficients $b_3(q_1, q_2; p)$ with such $q_1$ and $q_2$, for which $\rho_{q_1} \rho_{q_2} \rho_{-q_1-q_2-p} = \rho_{-p} \rho_{-p_2} \rho_{-p_3}$. For a state with $N$ quasiparticles the relation $p = p_1 + \ldots + p_N$ holds, and the coefficients $b_{j \leq N-1}$ are negligible: $b_{j \leq N-1} \sim N^{-a_j}$, $(a_j > 0)$. The coefficients $b_N(q_1, \ldots, q_{N-1}; p)$ are not small at such $q_1, \ldots, q_{N-1}$, for which $\rho_{q_1} \ldots \rho_{q_{N-1}} \rho_{-q_1-\ldots-q_{N-1}-p} = \rho_{-p} \ldots \rho_{-p_N}$.

Formulae (7), (8) imply that the largest number of quasiparticles equals $N$, since series (8) contains the terms $\rho_{-q_1} \ldots \rho_{-q_j}$ with at most $N$ factors $\rho_{-q}$. The last property is caused by that the functions $1, \rho_{-q_1}, \rho_{-q_1} \rho_{-q_2}, \ldots, \rho_{-q_1} \ldots \rho_{-q_N}$ form the complete (nonorthogonal) collection of functions, in which any Bose-symmetric function of the variables $r_1, \ldots, r_N$, which can be presented as the Fourier series, can be expanded [34]. Therefore, the product $\rho_{-q_1} \ldots \rho_{-q_N} \rho_{-q_{N+1}} \ldots \rho_{-q_{N+M}}$ containing more than $N$ factors $\rho_{-q}$ is reduced to an expansion of the form $\psi_p$ (8) with $p = q_1 + \ldots + q_{N+M}$. For example, for $N = 2$ we obtain

$$\rho_{q_1} \rho_{q_2} \rho_{q_3} = \frac{1}{\sqrt{N}}(\rho_{q_1+q_2} \rho_{q_3} + \rho_{q_1+q_3} \rho_{q_2} + \rho_{q_2+q_3} \rho_{q_1}) - \frac{2}{N}\rho_{q_1+q_2+q_3}.$$  \hspace{1cm} (47)

Thus, the largest number of quasiparticles in a Bose gas, being in some pure state $\Psi_p$, is equal to $N$. According to quantum statistics, the equilibrium number of quasiparticles for
the given temperature $T > 0$ is

$$
\bar{N}_Q(T) = \frac{1}{Z} \int d\mathbf{r}_1 \ldots d\mathbf{r}_N \sum_p e^{-E_p/k_BT} \Psi_p^* N_{Qp} \Psi_p,
$$

(48)

where $Z = \sum_l e^{-E_l/k_BT}$, $\{\Psi_p(x_1, \ldots, x_N)\}$ is the complete orthonormalized set of wave functions of a system with a fixed number of particles $N$, and $N_{Qp}$ is the number of quasiparticles in the state $\Psi_p$. According to the above analysis, the value of $N_p$ is determined by the structure of $\Psi_p(x_1, \ldots, x_N)$, and $N_p \leq N$ for any state. Therefore, $\bar{N}_Q(T) < N$. At low temperatures, the states with small $N_{Qp}$ make the main contribution to (48). Therefore, the average number of quasiparticles is small. In this case, $\bar{N}_Q(T)$ increases with $T$. It is clear that, as $T \to \infty$, we have $\bar{N}_Q(T) \to N$. Thus, in the gas at a high temperature, the number of quasiparticles is close to the number of atoms. This shows how a quantum Bose system transforms into a classical one.

6 Experiment

In the experiment [36], one point of the dispersion law $E(p)$ of a 1D Bose system was obtained for different $\gamma$ by means of measuring the dynamical structural factor. The results were compared with the theory [27, 37, 38, 39, 40], in which the bosons were considered point-like (bosons with zero radius of interaction). At small $p$ the experimental value of $E(p)$ is close to $E_p(p)$ of particle-like excitations. At larger $p$ the experimental value of $E(p)$ is significantly lower than the theoretical one $E_p(p)$, and the deviation increases with $p$. The authors have concluded that this deviation is related to the contribution of holes, since the dispersion curve for holes $E_h(p)$ lies lower than $E_p(p)$.

The experiment [36] is important, but the analysis [36] is insufficient to make conclusion about the contribution of holes. Many states with the quantum numbers $(n_1, \ldots, n_N)$ contribute to the dynamical structural factor. One particle corresponds to states of the form $(0, \ldots, 0, 1)$, and one hole corresponds to states of the form $(0, \ldots, 0, 1, \ldots, 1)$. But the majority of states (e.g., $(0, \ldots, 0, 1, 2, 3)$ or $(0, \ldots, 0, 2, 2, 2)$) can be divided into holes and particles by several (or many) different ways, irrespective of the nature of a hole. To determine the contribution of holes, we need indicate, for each state $(n_1, \ldots, n_N)$, a rule of separation of the state into definite numbers of holes and particles. Such rule was not given in [36]. Therefore, in our opinion, the results of this work do not allow one to ascertain whether the contribution of holes is large.

We have shown above that the hole is a collection of phonons. Therefore, there is no meaning to consider the holes as independent excitations. The difference between the experimental value of $E(p)$ and the theoretical one, $E_p(p)$, can be caused by that the real Bose atoms have a nonzero size. At $\gamma \lesssim 10$ the curve $E_p(p)$ is close to the Bogolyubov one $E_B(p)$ [12] [11, 11]. At the passage to a nonpoint interaction, $E_B(p)$ decreases, since $\nu(p) < \nu(0)$ for the potential-
tials of a reasonable shape. For example, for a 1D semitransparent ball $\nu(p) = \nu(0) \frac{\sin(pd_0)}{pd_0}$, where $d_0$ is the ball diameter. At $\pi/L \ll p \lesssim \pi/d_0$ the value of $\nu(0) - \nu(p)$ is not small and increases with $p$. Therefore, $\sqrt{\left(\frac{\hbar^2p^2}{2m}\right)^2 + 2n\nu(0)\frac{\hbar^2p^2}{2m}} - \sqrt{\left(\frac{\hbar^2p^2}{2m}\right)^2 + 2n\nu(p)\frac{\hbar^2p^2}{2m}}$ increases also with $p$, which agrees qualitatively with the experiment [36].

7 A hole and a soliton.

The Lieb’s hole is a stationary solution of the $N$-body Schrödinger equation for a cyclic system: $\tilde{\Psi}(x_1, \ldots, x_N, t) = e^{-iE_h(p)t/\hbar}\Psi(x_1, \ldots, x_N)$. This solution is characterized by a constant density: $\rho(x, t) = \text{const}$ [12]. However, the quasiclassical dark soliton, as a solution of the 1D Gross–Pitaevskii equation, is a solitary running density wave of the form $\Psi(x, t) = \Psi(x - vt)$, $\rho(x, t) = \rho(x - vt)$ [24, 25]. In this case, the wave package of one-hole states shows the properties of an immovable soliton [12, 43, 44] (though the density profile $\rho(x, t)$ of such package spreads, as $t$ increases, in contrast to a quasiclassical soliton [24, 25]). Moreover, the conditional probability density $\rho_N(x)$ in the hole state coincides with the stationary dark soliton profile [45]. Note also that the analysis in [25] refers to an infinite noncyclic system. In this case, classical and quantum momentums of the soliton are different. The dispersion curves of solitons and holes are close only in the classical definition of the soliton momentum [25]. If such properties hold for a cyclic system too, then a single hole is not a soliton (despite results in [45]), since the quantum definition of the momentum is primary. On the whole, the connection between a hole and a soliton is not quite clear [42, 43, 44, 45].

We have shown above that the hole is a collection of identical interacting phonons with the momentum $p = 2\pi/L$. Possibly, the collection of identical phonons with $p = 4\pi/L$ (or $p = 6\pi/L$, etc.) reveals also solitonic properties. Most probably, a hole has solitonic properties only for high momenta: in this case, the hole consists of a large number of identical phonons, and the collective effect is possible. The solitonic properties of holes are interesting, it is worth studying them in more details. In our opinion, it is better to use zero boundary conditions, because $\rho(x, t) \neq \text{const}$ in this case, and the density wave is possible.

8 Conclusion

We have shown that the hole with the momentum $p = jp_0$, where $p_0 = \pm\hbar2\pi/L$, is a collection of $j$ identical interacting phonons with the momentum $p_0$. Therefore, a hole is a composite excitation. If $j \sim N$, the hole corresponds to the condensate of phonons. Thus, Lieb’s excitations quite agree with the Bogolyubov’s and Feynman’s solutions. The traditional point of view, according to which a holes are an independent type of excitations, has survived for so long since the Lieb–Lininger wave functions was not compared with the wave functions of a system of nonpoint bosons. We think that fermionicity “penetrates” into the Bethe equations
since at $c = \infty$ the bosons are impenetrable and, therefore, are similar to the fermions.

We have also proved that the largest number of quasiparticles in a Bose gas is equal to the number of atoms $N$.

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9 Appendix.

The functions $a_j$ and $b_j$ from Eqs. (6) and (8) satisfy the Vakarchuk–Yukhnovskii equations \cite{17, 33}:

$$E_0 = \frac{N - 1}{2} m\nu(0) - \sum_{q \neq 0} \frac{m\nu(q)}{2} - \sum_{q \neq 0} \frac{\hbar^2 q^2}{2m} a_2(q),$$

$$\frac{m\nu(q)}{\hbar^2} + q^2 a_2(q) - q^2 a_2^2(q) - \frac{1}{N} \sum_{q_1 \neq 0} a_3(q, q_1)q_1(q + q_1) - \frac{1}{2N} \sum_{q_1 \neq 0} a_4(q, -q_1, q_1)q_1^2 = 0,$$

$$a_3(q_1, q_2)[E_1(q_1) + E_1(q_2) + E_1(q_1 + q_2)] + 2q_1q_2a_2(q_1)a_2(q_2) -$$

$$-2q_1(q_1 + q_2)a_2(q_1)a_2(q_1 + q_2) - 2q_2(q_1 + q_2)a_2(q_2)a_2(q_1 + q_2) -$$

$$-\frac{1}{N} \sum_{q \neq 0} a_5(q_1, q_2, q, -q)q^2 + \frac{1}{N} \sum_{q \neq 0} [a_4(q_1 - q, q_2, q)(q_1 - q)q +$$

$$+ a_4(q_1, q_2 - q, q)(q_2 - q)q + a_4(q_1, q_2, -q_1 - q_2 - q)(-q_1 - q_2 - q)q] = 0,$$

$$b_1(p)E(p) = b_1(p)E_1(p) - \frac{1}{N} \sum_{q \neq 0} b_2(q; p)\frac{\hbar^2}{2m}(p + q)q - \frac{1}{N} \sum_{q \neq 0} b_3(q, -q; p)\frac{\hbar^2 q^2}{2m},$$

$$b_2(q; p)\frac{2m}{\hbar^2}[E_1(q) + E_1(p + q) - E(p)] + 2b_1(p)q_1a_2(q) - 2b_1(p)p^2a_3(p, q) -$$

$$-2b_1(p)p(p + q)a_2(p + q) - \frac{1}{N} \sum_{q_1 \neq 0} q_1^2b_4(q_1, -q_1, q; p) +$$

$$+ \frac{1}{N} \sum_{q_1 \neq 0} [b_3(q_1, q - q_1; p)q_1(q - q_1) + b_3(q_1, -q - q_1 - p; p)q_1(-q_1 - q - p)] = 0.$$
\[ b_3(q_1, q_2; p) \frac{2m}{\hbar^2} [E_1(q_1) + E_1(q_2) + E_1(p + q_1 + q_2) - E(p)] - 2b_1(p)p^2a_4(q_1, q_2, p) - \\
- 2b_1(p)[a_3(q_1 + p, q_2)p(q_1 + p) + a_3(q_2 + p, q_1)p(q_2 + p) - a_3(q_1, q_2)p(q_1 + q_2)] - \\
- 2b_2(q_1; p)a_3(q_1 + p, q_2)(p + q_1)^2 - 2b_2(q_2; p)a_3(q_2 + p, q_1)(p + q_2)^2 - \\
- 2b_2(-q_1 - q_2 - p; p)a_3(q_1, q_2)(q_1 + q_2)^2 - \frac{1}{N}\sum_{q_{l}\neq 0}q_{l}^2b_5(q_{1}, -q_{4}, q_{1}, q_{2}; p) - \\
- 2a_2(q_1)q_1[b_2(q_2; p)(-q_2 - p) + b_2(-q_1 - q_2 - p; p)(q_1 + q_2)] - \\
- 2a_2(q_2)q_2[b_2(q_1; p)(-q_1 - p) + b_2(-q_1 - q_2 - p; p)(q_1 + q_2)] - \\
- 2a_2(q_1 + q_2 + p)(q_1 + q_2 + p)[b_2(q_1; p)(q_1 + p) + b_2(q_2; p)(q_2 + p)] - \\
+ \frac{1}{N}\sum_{q_{l}\neq 0}q_{l}(-q_1 - q_2 - q - p)b_4(q_1, q_2, -q_1 - q_2 - q - p; p) + \\
+ \frac{1}{N}\sum_{q_{l}\neq 0}q_{l}(q_1 - q)b_4(q_1 - q, q_2, -q_1 - q_2 - p; p) + \\
+ \frac{1}{N}\sum_{q_{l}\neq 0}q_{l}(q_2 - q)b_4(q_1, q_2, -q_1 - q_2 - p; p) = 0. \tag{54} \]

Here, \( E_1(q) = \frac{\hbar^2q^2}{2m}(1 - 2a_2(q)) \). The equation for the function \( a_4 \) is given in \([17, 34]\). If one of the arguments of the functions \( a_j \) or \( b_j \) in \((49)-(54)\) is zero, then the corresponding \( a_j \) or \( b_j \) should be set zero.

The functions \( a_{j+1}(q_1, \ldots, q_j) \) and \( b_{j+1}(q_1, \ldots, q_j; p) \) are invariant relative to the permutations of two any arguments \( q_l, q_n \). The functions \( a_{j+1}(q_1, \ldots, q_j) \) are also invariant relative to the change \( q_l \rightarrow -q_1 - q_2 - \ldots - q_j \) for any \( j \) and \( l = 1, \ldots, j \). As for the functions \( b_{j+1}(q_1, \ldots, q_j; p) \), they are invariant relative to the change \( q_l \rightarrow -q_1 - q_2 - \ldots - q_j - p \) for any \( j \geq 1, l = 1, \ldots, j \).

In works \([17, 34]\) a one-phonon state was considered and Eqs. \((49)-(54)\) were deduced for \( b_1(p) = 1 \). We write these equations for any \( b_1(p) \), so that the equations can be used to describe the states with the number of phonons \( \geq 1 \).

Equations \((49)-(54)\) are exact for an infinite system: \( N, V = \infty \). For a finite system, the product \( \rho_{-q_1} \cdots \rho_{-q_N} \rho_{-q_{N+1}} \cdots \rho_{-q_{N+M}} (M = 1, 2, \ldots) \) is reduced to a sum of terms, each of which contains at most \( N \) factors of the form \( \rho_{-q} \) (see Section 5). One needs to take this property into account while deriving the equations for \( a_j \) and \( b_j \), which will cause the appearance of many additional terms in Eqs. \((49)-(54)\). However, for the weak coupling, these terms should be negligible. Apparently, they are negligible also for a nonweak coupling. Otherwise, the transition from the solutions for a very large finite system to solutions for the infinite one would occur by jump. However, we do not expect such a jump. One can verify that the solutions of the Lieb–Lininger equations \((11) \) or \((2)\) have no such jump. Those additional terms were not considered in the literature, and we omitted them in Sections 2, 3.
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