The Fundamental Commutator For Massless Particles

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Abstract

It is discussed that the usual Heisenberg commutation relation (CR) is not a proper relation for massless particles and then an alternative is obtained. The canonical quantization of the free electromagnetic (EM) fields based on the field theoretical generalization of this alternative is carried out. Without imposing the normal ordering condition, the vacuum energy is automatically zero. This can be considered as a solution to the EM fields vacuum catastrophe and a step toward managing the cosmological constant problem at least for the EM fields contribution to the state of vacuum.

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1. Introduction

Quantum mechanically the relation \([\hat{Q}_i, \hat{P}_j] = i\hbar\delta_{ij}\hat{1}\) is generally accepted as the canonical CR between position and momentum operators of any particle regardless of its individual properties. For massive particles, this can be derived by means of symmetry group transformations and the related Lie algebra commutators [1-3]. We assume the position operator for the particle to be \(\hat{Q}_i\) where by definition

\[
\hat{Q}_i | \vec{q} > = q_i | \vec{q} > \quad i = 1, 2, 3
\]  

has an unbounded continuous spectrum. Two assumptions are involved here, the space is a continuum and all three components of the position operator are mutually commutative and so possess a common set of eigenvectors. The space displacement of localized position eigenvectors by applying the operator, \((e^{-i\frac{\epsilon_j}{\hbar}\hat{P}_j})\), requires the displaced observables bear the same relationship to the displaced vectors as the original observables do to the original vectors, in particular

\[
\hat{Q}_i' = e^{i\frac{\epsilon_j}{\hbar}\hat{P}_j} \hat{Q}_i e^{-i\frac{\epsilon_j}{\hbar}\hat{P}_j}
\]  

Up to first order approximation in \(\epsilon\), we may write

\[
\hat{Q}_i' = (\hat{1} + i\frac{\epsilon_j}{\hbar}\hat{P}_j)\hat{Q}_i(\hat{1} - i\frac{\epsilon_j}{\hbar}\hat{P}_j)
\]  

or

\[
\hat{Q}_i' = \hat{Q}_i - i\frac{\epsilon_j}{\hbar}[\hat{Q}_i, \hat{P}_j]
\]  

By the substitution of \([\hat{Q}_i, \hat{P}_j] = i\hbar\delta_{ij}\), in (4) one gets

\[
\hat{Q}_i' = \hat{Q}_i + \epsilon_i \hat{1}
\]  

which means translation of reference frame is equivalent to translation of the particle. This equivalence has route in this fact that for a massive particle one can always transform to its rest frame. However for massless particles e.g. photons, this transformation is impossible. It is a property of Poincaré transformations that there is no rest frame for massless particles, they move with constant velocity in any reference frame. Consequently, the canonical CR between coordinates and momenta is not the same as massive for massless particles and we should look for the proper relation.
2. Alternative Canonical CR For Massless Particles

According to the Hamilton’s equation of motion we can write
\[
\frac{dQ_i}{dt} = \frac{\partial H}{\partial P_i} = \frac{\partial \sqrt{P^2C^2 + m^2}}{\partial P_i} = H^{-1}P_i
\] (6)

On the other hand by the Poisson brackets formalism, we have
\[
\frac{dQ_i}{dt} = [Q_i, H]_p
\] (7)

Therefore, classically (6) and (7) give
\[
H^{-1}P_i = [Q_i, H]_p
\] (8)

where \([,]_p\) represents the classical Poisson bracket.

Now, by means of Dirac’s canonical quantization rule \([,] \rightarrow \frac{[\hat{,}]}{i\hbar}\) we have the following operatorial relation
\[
\hat{H}^{-1}\hat{P}_i = \frac{[\hat{Q}_i, \hat{H}]}{i\hbar}
\] (9)

where \(\hat{H}\) and \(\hat{P}\) commute with each other. By means of the Heisenberg equation of motion for position operator based on the fact that the Hamilton operator is the generator of time translation, we have
\[
[\hat{Q}_i, \hat{H}] = i\hbar \frac{d\hat{Q}_i}{dt}
\] (10)

Comparision of (9) and (10) gives
\[
\hat{P}_i = \left( \frac{\hat{H}}{c^2} \right) \left( \frac{d\hat{Q}_i}{dt} \right)
\] (11)

Summing over square form of (11) leads to \(\hat{H}^2 = \hat{P}^2\) which is an accepted relation for massless particles. Using (11), we can write the commutator of \(\hat{Q}_i\) and \(\hat{P}_j\) as
\[
[\hat{Q}_i, \hat{P}_j] = \left[ \hat{Q}_i, \frac{\hat{H} d\hat{Q}_j}{c^2 dt} \right] = \frac{1}{c^2}[\hat{Q}_i, \hat{H}]\frac{d\hat{Q}_j}{dt} + \frac{\hat{H}}{c^2} \left[ \hat{Q}_i, \frac{d\hat{Q}_j}{dt} \right]
\] (12)

Insertion of (10) in (12) gives,
\[
[\hat{Q}_i, \hat{P}_j] = \frac{i\hbar}{c^2} \left( \frac{d\hat{Q}_i}{dt} \right) \left( \frac{d\hat{Q}_j}{dt} \right) + \frac{\hat{H}}{c^2} \left[ \hat{Q}_i, \frac{d\hat{Q}_j}{dt} \right]
\] (13)
For massless particles, we have
\[ \sum_{j=1}^{3} \left( \frac{d\hat{Q}_j}{dt} \right)^2 = c^2 \hat{1} \] (14)
\[ \frac{d^2\hat{Q}_j}{dt^2} = 0 \]

Then from (14) we may infer that
\[ \sum_{j=1}^{3} \left[ \hat{Q}_i, \left( \frac{d\hat{Q}_j}{dt} \right)^2 \right] = 0 \Rightarrow \sum_{j=1}^{3} \left( \hat{Q}_i, \frac{d\hat{Q}_j}{dt} \right) \frac{d\hat{Q}_j}{dt} + \frac{d\hat{Q}_j}{dt} \left[ \hat{Q}_i, \frac{d\hat{Q}_j}{dt} \right] = 0 \] (15)

But we have
\[ \frac{d}{dt} \left[ \hat{Q}_i, \frac{d\hat{Q}_j}{dt} \right] = \left[ \frac{d\hat{Q}_i}{dt}, \frac{d\hat{Q}_j}{dt} \right] + \left[ \frac{d\hat{Q}_j}{dt}, \frac{d\hat{Q}_j}{dt} \right] = [\hat{H}^{-1}\hat{P}_i, \hat{H}^{-1}\hat{P}_j] = 0 \] (16)

Therefore \( \left[ \hat{Q}_i, \frac{d\hat{Q}_j}{dt} \right] \) is a constant operator or at most a function of the operators \( \hat{H} \) and \( \hat{P} \). Since \( \frac{d\hat{Q}_j}{dt} = \hat{H}^{-1}\hat{P}_j \), we will have
\[ \left[ \hat{Q}_i, \frac{d\hat{Q}_j}{dt} \right] = 0 \] (17)

Using (17) and (15) will give
\[ \sum_{j=1}^{3} \left( \hat{Q}_i, \frac{d\hat{Q}_j}{dt} \right) \frac{d\hat{Q}_j}{dt} = 0 \] (18)

(18) may be written in the form
\[ \hat{Q}_i \sum_{j} \left( \frac{d\hat{Q}_j}{dt} \right)^2 - \sum_{j} \frac{d\hat{Q}_j}{dt} \hat{Q}_i \frac{d\hat{Q}_j}{dt} = 0 \] (19)

By using (14), (19) gives
\[ \hat{Q}_i = \frac{1}{c^2} \sum_{j} \frac{d\hat{Q}_j}{dt} \hat{Q}_i \frac{d\hat{Q}_j}{dt} \] (20)

\( \{ | \tilde{k} > \} \) is the complete set of eigenvectors of \( \frac{d\hat{Q}_i}{dt} \),
\[ \frac{d\hat{Q}_j}{dt} | \tilde{k} > = \frac{k_j}{|k|} | \tilde{k} > \] (21)
The matrix element of (20) between $\langle \vec{k}' |$ and $| \vec{k} >$ gives

$$\langle \vec{k}' | \hat{Q}_i | \vec{k} > = \langle \vec{k}' | \hat{Q}_i | \vec{k} > \sum_j \frac{k_j k'_j}{| \vec{k} |}$$

(22) holds if and only if $\langle \vec{k}' | \hat{Q}_i | \vec{k} > = 0$ for $\vec{k}' \neq \vec{k}$.

Thus there is only diagonal elements which means the complete set $\{ | \vec{k} > \}$ is simultaneous eigenvectors of $\hat{Q}_i$. Since $\hat{Q}_i$ and $\frac{d\hat{Q}_j}{dt}$ are self-adjoint and have a common complete set of eigenvectors we have

$$[\hat{Q}_i, \frac{d\hat{Q}_j}{dt}] = 0$$

(23)

Finally the desired CR for massless particles may be obtained by inserting (23) in (13).

$$[\hat{Q}_i, \hat{P}_j] = \frac{i\hbar}{c^2} \left( \frac{d\hat{Q}_i}{dt} \right) \left( \frac{d\hat{Q}_j}{dt} \right)$$

(24)

It should be noticed that if there were a rest frame for massless particles, then $\frac{d\hat{Q}_i}{dt}$, $\frac{d\hat{Q}_j}{dt}$, and $c^2$ could be set to go to zero and the right-hand side of (24) became $i\hbar \delta_{ij}$ (The limit of $\frac{a_i a_j}{a^2}$ as $a_i, a_j, a^2 \to 0$ is the Kronecker delta).

By means of (11), (24) can be also written in the following form

$$[\hat{Q}_i, \hat{P}_j] = i\hbar c^2 \hat{H}^{-2} \hat{P}_i \hat{P}_j$$

(25)

Another possible form for this CR is as follows. Let’s consider the eigenvalue equation for momentum operator $\hat{P}$

$$\hat{P} | \vec{k} > = \hbar \vec{k} | \vec{k} >$$

(26)

Since $\hat{H}$ commutes with $\hat{P}$, for free particles, the eigenvectors $| \vec{k} >$’s are simultaneous eigenvectors of $\hat{H}$ with the eigenvalue equation

$$\hat{H} | \vec{k} > = \hbar \omega | \vec{k} >$$

(27)

where, $\omega = c| \vec{k} |$.

Now, by means of the identity operator $\sum_{\vec{k}} | \vec{k} > < \vec{k} | = \hat{1}$, we can write

$$[\hat{Q}_i, \hat{P}_j] = i\hbar c^2 \sum_{\vec{k}} \sum_{\vec{k}'} | \vec{k} > < \vec{k} | \hat{H}^{-2} \hat{P}_i \hat{P}_j | \vec{k}' > < \vec{k}' |$$

(28)
Using (26), (27), and the orthonormality of eigenvectors \( | \vec{k} > \), (28) becomes

\[
[\hat{Q}_i, \hat{P}_j] = \hbar \sum_k \frac{k_i k_j}{|k|^2} | \vec{k} > < \vec{k} |
\]

(29)

If one were able to find a frame in which \( \vec{k} = 0 \) (the rest frame of the particle), then the right-hand side of (29) would become \( i\hbar \delta_{ij} \hat{1} \) which is the common result. But since there is no such rest frame for massless particles we should work with the new proposed CR or its equivalence when dealing with such particles.

It is evident from (26) and (27) that for any physical state whose energy-eigenvalue is zero, automatically its momentum-eigenvalue is zero too; this prevents the singularity in (25) due to the appearance of the operator \( \hat{H}^{-2} \).

3. The Canonical Quantization of the Free EM Fields

For free EM fields we have;

\[
L = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu}
\]

(30)

\[
F_{\mu \nu} = \partial^\nu A^\mu - \partial^\mu A^\nu
\]

(31)

\[
\Box A^\mu - \partial^\mu (\partial_\nu A^\nu) = 0
\]

(32)

In Lorentz gauge, Fermi Lagrangian density is

\[
L_f = -\frac{1}{2} (\partial_\nu A_\mu)(\partial^\nu A^\mu)
\]

(33)

then (32) gives

\[
\Box A^\mu = 0
\]

(34)

and

\[
\pi_\mu = \frac{\partial L}{\partial A^\mu} = -\frac{1}{c^2} \dot{A}_\mu
\]

(35)

where \( \pi_\mu \) is the canonical momentum operator conjugated to the field operator four-potential \( A_\mu \). The solutions of the equation \( \Box A^\mu = 0 \) can be expanded in terms of a complete set of solutions of the wave equation. Fourier expansion by imposing the periodic boundary condition gives

\[
A^\mu(x) = A^\mu_+ (x) + A^\mu_- (x)
\]

(36)
where

\[ A^{\mu^+}(x) = \sum_r \sum_k \left( \frac{\hbar c^2}{2V\omega} \right)^{\frac{1}{2}} \varepsilon^\mu_r(\bar{k}) \hat{a}_r(\bar{k}) e^{-ikx} \]  

(37)

\[ A^{\mu^-}(x) = \sum_r \sum_k \left( \frac{\hbar c^2}{2V\omega} \right)^{\frac{1}{2}} \varepsilon^{\mu}_r(\bar{k}) \hat{a}_r^+(\bar{k}) e^{+ikx} \]  

(38)

\( \omega = ck^0 = c|\bar{k}| \) and \( A^\mu \) is appropriately normalized.

The summation over \( r \), from \( r = 0 \) to \( r = 3 \), corresponds to the fact that for the field \( A^\mu(x) \) there exists four linearly independent polarization states for each \( \bar{k} \). These are described by the polarization vectors \( \varepsilon^\mu_r(\bar{k}) \) which we choose to be real and satisfy the orthonormality and completeness relations

\[ \varepsilon_r(\bar{k})\varepsilon_s(\bar{k}) = \varepsilon_{r\mu}(\bar{k})\varepsilon^\mu_s(\bar{k}) = -\xi_r\delta_{rs} \]  

(39)

\[ \sum_r \xi_r\varepsilon^\mu_r(\bar{k})\varepsilon^\nu_r(\bar{k}) = -g^{\mu\nu} \]  

(40)

\[ \begin{cases} 
\xi_0 = -1, \quad \xi_1 = \xi_2 = \xi_3 = 1 \\
g_{00} = -g_{11} = -g_{22} = -g_{33} = 1 
\end{cases} \]  

(41)

Construction of the canonical quantum field theories are based on the equal-time commutation relations of the field operators and their canonically conjugated momenta. For the free EM fields, the following equal-time commutation relation has been generally accepted and used

\[ [A^\mu(\bar{x}, t), \pi^\nu(\bar{x}' , t)] = -i\hbar c^2 g^{\mu\nu}\delta(\bar{x} - \bar{x}') \]  

(42)

Indeed, this relation is the field theoretical, continuous, generalization of the usual canonical commutation relation \([\hat{Q}_i, \hat{P}_j] = i\hbar\delta_{ij}\hat{1}\) between the conjugate coordinates and momenta of the discrete lattice approximation. But since we have already obtained an alternative CR for massless particles, we should modify the relation (42) and find the appropriate corresponding relation. Following the same procedure generally used to pass from quantum mechanics of discrete systems to canonical quantum field theories and by means of (29), we may write

\[ \left[ \hat{A}^\mu(\bar{x}, t), \hat{\pi}^\nu(\bar{x}' , t) \right] = i\hbar \sum_{\bar{K}} \frac{K^\mu K^\nu}{K_0^2} <\bar{K}|\delta(\bar{x} - \bar{x}') \]  

(43)

in which \( K_0 = |\bar{K}| \). If there were a rest frame for the system, the expression \( \frac{K^\mu K^\nu}{K_0} \) could be set equal to \( -g^{\mu\nu} \) and the relation (42) could be derived. Of course, there is not such a rest frame. After finding the proper CRs between \( \hat{a}_r \) and \( \hat{a}_r^+ \)'s, one
can show that the relation (43) preserves the local property of the theory and the microcausality relation can be verified (see Appendix -A).

Let us verify what the relation (43) gives for the CR between \(\hat{a}_r\) and \(\hat{a}_r^+\). Using (35), (43) becomes

\[
-\frac{1}{c^2} \left[ \hat{A}^\mu(x, t), \hat{A}^\nu(x', t) \right] = i\hbar \left( \sum_K \frac{K^\mu K^\nu}{K_0^2} \right) \delta(x - x') \quad (44)
\]

or

\[
-\frac{1}{c^2} \sum_r \sum_{r'} \sum_{\vec{k}} \sum_{\vec{k}'} \left[ \frac{\hbar c^2}{2V\omega} \right]^\frac{1}{2} \left[ \frac{\hbar c^2}{2V\omega'} \right]^\frac{1}{2} \varepsilon_r^\mu(\vec{k}) \varepsilon_r'^\nu(\vec{k}') [\hat{a}_r(\vec{k}) e^{-ikx} + \hat{a}_r^+(\vec{k}) e^{ikx}, \quad (45)
\]

\((-ik'^0 c)\hat{a}_{r'}(\vec{k}') e^{-ik'x'} + (ik^0 c)\hat{a}_{r'}^+(\vec{k}') e^{ik'x'} = i\hbar \left( \sum_K \frac{K^\mu K^\nu}{K_0^2} \right) \delta(x - x') \quad (46)\]

It should be noticed that \(t = t'\) and \(k^0 c = \omega\).

The application of \(\sum_{\vec{k}} \frac{1}{\sqrt{V}} e^{ikx} \frac{1}{\sqrt{V}} e^{-ik'x'} = \delta(\vec{x} - \vec{x}')\) leads us to find:

\[
\sum_r \varepsilon_r^\mu(\vec{k}) \varepsilon_r'^\nu(\vec{k}') [\hat{a}_r(\vec{k}), \hat{a}_{r'}^+(\vec{k}')] = -\left( \sum_K \frac{K^\mu K^\nu}{K_0^2} \right) \delta(\vec{k}, \vec{k}') \quad (47)
\]

Of course, one may easily find that \([\hat{a}_r(\vec{k}), \hat{a}_r(\vec{k}')] = [\hat{a}_r^+(\vec{k}), \hat{a}_r^+(\vec{k}')] = 0\) and all other commutators vanish for \(r \neq r'\).

A specific choice of polarization vectors in one given frame of reference often facilitates the interpretation. In the frame in which the photon (anticipating the particle interpretation!) is moving along the third axis, we have \(K^\mu = (K, 0, 0, K)\), and

\[
\varepsilon_0 = (1, 0, 0, 0), \varepsilon_1 = (0, 1, 0, 0), \varepsilon_2 = (0, 0, 1, 0), \varepsilon_3 = (0, 0, 0, 1) \quad (48)
\]

Since our aim is to find a scalar, (the vacuum energy), which is a Lorentz invariant and also gauge independent quantity, this special choice of polarization vectors does not restrict the validity of discussion. By means of the above polarization vectors, the following commutation relations can be found from (47)

\[
[\hat{a}_0(\vec{k}), \hat{a}_0^+(\vec{k}')] = -\hat{1} \delta_{\vec{k}\vec{k}'} \quad (49)
\]

\[
[\hat{a}_3(\vec{k}), \hat{a}_3^+(\vec{k}')] = +\hat{1} \delta_{\vec{k}\vec{k}'} \quad (50)
\]

\[
[\hat{a}_1(\vec{k}), \hat{a}_1^+(\vec{k}')] = [\hat{a}_2(\vec{k}), \hat{a}_2^+(\vec{k}')] = 0 \quad (51)
\]
For "scalar" photons, we have the CR (49) just the same as in the usual quantum field theory of radiation. A remarkable point is the minus sign in the right hand side of (49) for which a comparison with the harmonic oscillator shows that, contrary to the usual interpretation, \( \hat{a}_0^+(\vec{k}) \) and \( \hat{a}_0^-(\vec{k}) \) have the roles of creation and annihilation operators respectively. The CR (50) is the same result of the usual quantum field theory of radiation for longitudinal photons. The difference appears in CRs (51). In fact, (51) shows that the spectrum of the eigenvalues of the operators \( \hat{a}_i^+\hat{a}_i \) \((i=1,2)\) is a continuous spectrum of all real values from zero to infinity because these operators are Hermitian and positive definite. The continuity of the spectrum of these operators is reasonable because we are working with free fields without any bound or restriction unless for the special choice of reference frame whose effect is clear in the CR (50). Now, we are ready to find the expectation value of the Hamiltonian for the ground state i.e. vacuum. The Hamiltonian is

\[
\hat{H} = \int d^3x (\hat{\pi}_\mu \hat{A}_\mu - \hat{L})
\]  

(52)

Using (35)-(38) and (52) leads to

\[
\hat{H} = \sum_r \sum_{\vec{k}} \left[ \frac{\hbar \omega}{2} \xi_r \left[ \hat{a}_r^+ (\vec{k}) \hat{a}_r (\vec{k}) + \hat{a}_r (\vec{k}) \hat{a}_r^+ (\vec{k}) \right] \right]
\]  

(53)

Substitution of (49), (50), and (51) into the relation (53) leads to the following expression for the Hamiltonian

\[
\hat{H} = \sum_{\vec{k}} (\hbar \omega) \left[ \hat{a}_1^+ (\vec{k}) \hat{a}_1 (\vec{k}) + \hat{a}_2^+ (\vec{k}) \hat{a}_2 (\vec{k}) + \hat{a}_3^+ (\vec{k}) \hat{a}_3 (\vec{k}) - \hat{a}_0 (\vec{k}) \hat{a}_0^+ (\vec{k}) \right]
\]  

(54)

Since for the operators \( \hat{a}_1^+ \hat{a}_1 \) and \( \hat{a}_2^+ \hat{a}_2 \) the minimum possible value of their (continuous) spectrum is zero and since for scalar photons \( \hat{a}_0^+ (\vec{k}) \) has the role of annihilation operator then we have

\[
\langle 0 | \hat{H} | 0 \rangle = 0
\]  

(55)

This achievement has been obtained without imposing the normal ordering condition which is usually used for removing the infinity of the vacuum energy. For any case whose absolute value of energy is not relevant, normal ordering will appear to be adequate. But when the absolute value of energy is concerned, particularly in calculating the cosmological constant, normal ordering does not seem to be reasonable. A serious doubt about the above expression for the Hamiltonian is that for scalar particles one may deal with a state of negative infinite energy and we should have a satisfactory explanation for this problem (see Appendix -B).
4. Conclusion

Without imposing any condition, it turns out that the free EM fields vacuum energy is zero and this can be considered as the solution of vacuum catastrophe [4]. It may be also considered as a step toward managing the cosmological constant problem [5]. Although there are many works on the solution of the cosmological constant problem e.g. supersymmetry [6], quantum cosmology [7-10], supergravity [11-13] and so on [14]; in all of them there are either physically unknown assumptions and principles or mathematical difficulties such as unmeasurability of the path integral used in quantum cosmology. But in this treatment the cosmological constant problem turns out to have probably a simple and physically reasonable solution at least for the contribution of EM fields to the vacuum state.

An important question which may raise to mind is that whether our result is in challenge with the Casimir effect [15]? The answer is no because in this study we have dealt with the free electromagnetic fields Lagrangian density, but in the case of the Casimir effect one should enter the effect of boundary conditions. This, however, requires a detailed and independent study.

Appendix -A

In order to verify the microcausality condition, we will find the covariant commutator of the fields at two arbitrary points at first and then show that the result vanishes when the points have space-like separation. For the points \( x : (ct, \vec{x}) \) and \( x' : (ct', \vec{x'}) \), the covariant commutation relation between \( \hat{A}^\mu(x) \) and \( \hat{A}^\nu(x') \) is

\[
[\hat{A}^\mu(x), \hat{A}^\nu(x')] = \sum_r \sum_{r'} \sum_{\vec{k}} \sum_{\vec{k}'} \left( \frac{\hbar c^2}{2V\omega} \right)^{\frac{3}{2}} \left( \frac{\hbar c^2}{2V\omega'} \right)^{\frac{3}{2}} \varepsilon^\mu_r(\vec{k})\varepsilon^\nu_{r'}(\vec{k}') \left[ \hat{a}_r(\vec{k})e^{-ikx} + \hat{a}^+_r(\vec{k})e^{ikx}, \hat{a}_{r'}(\vec{k}')e^{-ik'x'} + \hat{a}^+_{r'}(\vec{k}')e^{ik'x'} \right] \tag{56}
\]

Taking the limit \( V \to \infty \) we must substitute \( \frac{1}{V} \sum_k \) by \( \frac{1}{(2\pi)^3} \int d^3k \) and (56) becomes

\[
[\hat{A}^\mu(x), \hat{A}^\nu(x')] = \frac{\hbar c^2}{2(2\pi)^3} \sum_r \varepsilon^\mu_r \varepsilon^\nu_{r'}[\hat{a}_r, \hat{a}^+_r] \int \frac{e^{-ik(x-x')}}{\omega} - \frac{e^{ik(x-x')}}{\omega} d^3k \tag{57}
\]

But, the integral \( \int \frac{e^{-ik(x-x')}}{\omega} - \frac{e^{ik(x-x')}}{\omega} d^3k \) is the \( \Delta \)-function up to a multiplicative constant. \( \Delta(x-x') \) is a Lorentz invariant function and \( \Delta(\vec{x} - \vec{x}', 0) = 0 \). Therefore \( \Delta(x-x') \) vanishes for \( (x-x')^2 < 0 \) and the final result

\[
[\hat{A}^\mu(x), \hat{A}^\nu(x')] = 0, \text{ for } (x-x')^2 < 0 \tag{58}
\]

is the microcausality condition.
Appendix -B

In this work, as in the usual quantization of the free EM fields, we cannot simply take the Lorentz condition as an operator identity because it is incompatible with the covariant form of the commutation relation (43). This problem may be resolved by replacing the Lorentz condition with the following weaker condition

$$< \Psi | \partial_\mu A^\mu | \Psi > = 0$$  \hspace{1cm} (59)

where $| \Psi >$ is the physical state-vector of the system. This ensures that the Lorentz condition and hence Maxwell’s equations hold as the classical limit of the theory. In order to understand the meaning of the above subsidiary condition, we express it in momentum space. On substituting the explicit form of $A_\mu (x)$, we obtain

$$< \psi | (\hat{a}_0 - \hat{a}_3) e^{-ikx} + (\hat{a}_0^+ - \hat{a}_3^+) e^{ikx} | \Psi > = 0$$  \hspace{1cm} (60)

for which we have used the choice mentioned in the text where $\varepsilon^\mu_r$ is orthogonal to $k_\mu$ for $r = 1, 2$. Now, let define the following operators

$$\hat{A}_0 = \hat{a}_0 e^{-ikx} \hat{A}_3 = \hat{a}_3 e^{-ikx}$$  \hspace{1cm} (61)

Since here $\hat{A}_0^+$ is an annihilation operator, at point x, if we demand to have the following condition

$$(\hat{A}_0^+ - \hat{A}_3) | \Psi > = 0$$  \hspace{1cm} (62)

then, the above expectation value for Lorentz condition will be automatically satisfied. Thus, the action of scalar particles compensates exactly that of the longitudinal ones and we can always work in a gauge for which we deal with an appropriate admixture of scalar and longitudinal photons not to have an infinite negative energy.
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