Catalysis of entanglement and other quantum resources

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In chemistry, a catalyst is a substance which enables a chemical reaction or increases its rate, while remaining unchanged in the process. Instead of chemical reactions, quantum catalysis enhances our ability to convert quantum states into each other under physical constraints. The nature of the constraints depends on the problem under study and can arise, e.g., from energy preservation. This article reviews the most recent developments in quantum catalysis and gives a historical overview of this research direction. We focus on the catalysis of quantum entanglement and coherence, and also discuss this phenomenon in quantum thermodynamics and general quantum resource theories. We review applications of quantum catalysis and also discuss the recent efforts on universal catalysis, where the quantum state of the catalyst does not depend on the states to be transformed. Catalytic embezzling is also considered, a phenomenon that occurs if the catalyst’s state can change in the transition.

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I. INTRODUCTION

In 1835, Jacob Berzelius observed a common feature among several chemical reactions: they all required an additional substance, which remained unchanged in the process. He described this feature with the word “catalysis”, which goes back to Greek, and can be translated as “down” or “loosen” (Berzelius, 1835; van Santen et al., 1999). A systematic study of catalytic processes in chemistry has been performed later by Wilhelm Ostwald, who gave a precise definition which is also used today: “Catalysis is the acceleration of a slow chemical process by the presence of a foreign substance” (Ostwald, 1894). First known examples of chemical catalysis used by humans are fermentation processes, which have been used for thousands of years (van Santen et al., 1999). Today, chemical catalysis is broadly used in the chemical industry and is essential for many industrial processes. In addition, the importance of catalytic reactions in biochemistry cannot be overstated: to the point that the ability of living organisms to “catalyse chemical reactions efficiently and selectively” via metabolic pathways could be called one of the “fundamental conditions for life” (Nelson et al., 2008).

Quantum catalysis is conceptually similar to chemical catalysis but differs from it in several important details. A simple analogy between quantum and chemical catalysis can be established by replacing “chemical reaction” with “quantum state transition”. With this, a quantum catalyst is a quantum system which enables otherwise impossible transitions be-
tween quantum states. First investigations of quantum catalysis go back to the work of Jonathan and Plenio (1999), who studied transitions between entangled states shared by two remote parties (Alice and Bob). It is commonly assumed that the remote parties have the complete freedom to manipulate their quantum systems locally, including all kinds of transformations and measurements which are compatible with the laws of quantum mechanics. Additionally, Alice and Bob can exchange the outcomes of their measurements via a classical communication channel. This allows each party to adjust their local setups to the outcomes of the previous measurements obtained by the other party. This setup describes everything Alice and Bob can do to their shared quantum state if they cannot exchange other quantum particles (Bennett et al., 1996c; Horodecki et al., 2009).

In the first example of entanglement catalysis given by Jonathan and Plenio (1999), Alice and Bob aim to convert their quantum state $|\psi_{AB}\rangle$ into another state $|\phi_{AB}\rangle$ under the limitations described above. A few months earlier, Nielsen (1999) presented conditions for such state transformations, which allow to check whether $|\phi_{AB}\rangle$ can be converted into $|\psi_{AB}\rangle$ for any pair of states. Jonathan and Plenio (1999) found that for some states an otherwise impossible transition can be enabled by using an additional pair of particles $A'$ and $B'$ held by Alice and Bob, respectively. More precisely, by choosing an appropriate entangled state $|\eta_{AB}'\rangle$, they showed that $|\phi_{AB}\rangle \otimes |\eta_{AB}'\rangle$ can be converted into $|\psi_{AB}\rangle \otimes |\eta_{AB}'\rangle$, even though a direct conversion from $|\phi_{AB}\rangle$ into $|\psi_{AB}\rangle$ is not possible. As a result the state $|\eta_{AB}'\rangle$ remains unchanged and the two-particle system $A' B'$ can be seen as a quantum catalyst.

In recent years, it became clear that entanglement is not the only quantum feature to exhibit catalytic behaviour, mostly thanks to the development of quantum resource theories (Chitambar and Gour, 2019), which goes back to the systematic study of quantum entanglement (Horodecki et al., 2009) as a resource for the emerging quantum technology. Every quantum resource theory is based on a set of free states and free operations, corresponding to states and transformations which are easy to prepare and implement within well-defined physical constraints. Apart from quantum entanglement, the framework of quantum resource theories gave significant insight into the thermodynamical properties of quantum systems (Brandão et al., 2013; Goold et al., 2016; Gour et al., 2015), and the role of purity (Gour et al., 2015; Horodecki et al., 2003b) and quantum coherence (Baumgratz et al., 2014; Streltsov et al., 2017a) for technological applications. Catalytic effects have been found in most resource theories studied in recent literature, allowing to overcome certain constraints of the theory by adding a particle which remains unchanged in the procedure.

The original formulation of quantum catalysis proposed by Jonathan and Plenio (1999) did not allow for any correlations between the catalyst and the primary system as a result of a catalytic process. This constraint has been lifted in recent literature, allowing the procedure to build up and preserve correlations. This fruitful approach has significantly improved our understanding of quantum catalysis, with applications in the resource theories of quantum coherence (Åberg, 2014), quantum thermodynamics (Müller, 2018; Shiraishi and Sagawa, 2021, 2022; Wilming et al., 2017), purity (Boes et al., 2019; Wilming, 2021), and entanglement (Datta et al., 2022; Kondra et al., 2021a). In many cases, it has significantly simplified the analysis of catalytic state transformations within the corresponding resource theory (Boes et al., 2019; Datta et al., 2023, 2022; Kondra et al., 2021a; Müller, 2018; Shiraishi and Sagawa, 2021, 2022; Wilming, 2021).

In this article, we review recent developments in the catalysis of quantum entanglement and other quantum resource theories. The article is structured as follows: In Section II we recall the main features of quantum resource theories. Exact and approximate catalysis are discussed in detail in Sections III and IV, respectively. In Section V we review applications of catalysis for quantum information processing. Section VI presents recent results on universal catalysis. Catalytic embrazing phenomena are discussed in Section VII, and perspectives and open problems are presented in Section VIII.

II. QUANTUM RESOURCE THEORIES

The study of quantum resource theories (Chitambar and Gour, 2019) goes back to the early research in quantum entanglement (Horodecki et al., 2009). After the first observation of quantum entanglement in nature (Aspect et al., 1982, 1981), it was suggested that entanglement might eventually find interesting technological applications. Among others, Bennett et al. (1993) established that maximally entangled singlet states could be regarded as a vehicle for quantum teleportation. And even before that, a cryptographic scheme with singlets was described in (Ekert, 1991). Thus, from the point of view of various technological applications, an important problem emerged, how to find an optimal method of transforming arbitrary quantum states into maximally entangled singlets:

$$|\psi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle). \quad (1)$$

This question is highly relevant also in the context of mixed, or noisy, quantum states, which are the right model of a quantum system in any real experiment. One of the goals of quantum resource theories is to develop optimal strategies for transforming arbitrary quantum states, states perhaps not so useful for some applications into ones that are deemed more “resourceful”.

Any quantum resource theory specifies a context in which certain states and transformations are considered free. Free states are always available at no cost – an additional subsystem as a free state can be added or discarded at will. For example, the resource theory of entanglement regards all separable states (Werner, 1989) as free:

$$\rho_{sep} = \sum_i p_i \rho_i^A \otimes \sigma_i^B, \quad (2)$$
where $\rho_A^i$ and $\sigma_B^i$ are local quantum states. A composite quantum system can be brought to a separable state without the requirement of having shared quantum entanglement in advance, using only local operations and classical communication (LOCC). For a general resource theory, free operations are those transformations of quantum systems that can be implemented without the need of having access to resourceful, non-free states. In the resource theory of entanglement, free operations are precisely the class of LOCC.

It was the resource theory of quantum entanglement that was historically the first to undergo a systematic study (Horodecki et al., 2009). Intuitively, entanglement is a valuable resource in systems that allow for clear spatial separation into several discrete subsystems. Not all prominent applications of quantum technologies, however, emphasise the role of such spatial separation. For example, an instance of quantum computation is typically assumed to be a local process. The role of entanglement in quantum computation has been the subject of numerous publications, and it has been shown that under certain assumptions, a quantum computer requires entanglement in order to exhibit an exponential speed-up compared to classical computation (Jozsa and Linden, 2003). As of today, it remains unclear, however, whether it is quantum entanglement that is a crucial component of computation. It is quantum entanglement that is a crucial component of quantum computation is typically assumed to be a resource (Takagi and Regula, 2019; Takagi et al., 2017a,b). In particular, an exact correlated catalytic transformation between two states $\rho^S$ and $\sigma^S$ of a system $S$ is possible when there exists a catalyst $C$ in a state $\tau^C$ and a free operation $\Lambda_f$ acting on the system $S$ and the catalyst $C$ such that (Jonathan and Plenio, 1999)

$$\Lambda_f(\rho^S \otimes \tau^C) = \sigma^S \otimes \tau^C.$$  

(3)

Exact catalytic transformation is very stringent, and over time relaxations of it have been considered in the literature, allowing the catalyst to build up correlations with the primary system (Äberg, 2014; Boes et al., 2018; Lostaglio and Müller, 2019; Rethinasamy and Wilde, 2020; Yadin et al., 2022). In particular, an exact correlated catalytic transformation between two states $\rho^S$ and $\sigma^S$ is possible if and only if there exists a catalyst state $\tau^C$, a correlated state $\sigma^{SC}$ and a free operation $\Lambda_f$ such that

$$\Lambda_f(\rho^S \otimes \tau^C) = \sigma^{SC} \text{ with } \text{Tr}_C \sigma^{SC} = \rho^S \text{ and } \text{Tr}_S \sigma^{SC} = \tau^C.$$  

(4)

Having defined exact catalysis for general quantum resource theories, we will focus on the catalysis of entanglement in the following.

A. Entanglement

Historically, Jonathan and Plenio (1999) introduced the notion of exact catalysis of quantum states first in the context
of the resource theory of entanglement. The idea is based on the following fundamental result, which characterises fully the necessary and sufficient conditions to be met by all admissible transformations of pure bipartite entangled states. Consider two states \( |\psi\rangle^A \) and \( |\phi\rangle^A \) with their respective Schmidt coefficients \( \{\alpha_i\} \) and \( \{\beta_i\} \) sorted in non-increasing order.

**Theorem 1** (Nielsen’s theorem, see (Nielsen, 1999)). The transformation \( |\psi\rangle^A \rightarrow |\phi\rangle^A \) is possible via LOCC if and only if the reduced states \( \rho^A \) and \( \phi^A \) fulfill the majorization relation \( \rho^A < \phi^A \), i.e., their Schmidt coefficients satisfy

\[
\sum_{i=0}^n \alpha_i \leq \sum_{i=0}^n \beta_i \quad \text{for all} \ 0 \leq n \leq d - 1, \tag{5}
\]

with \( d = \min\{d_A, d_B\} \), where \( d_A \) and \( d_B \) are the dimensions of the Hilbert space of Alice and Bob, respectively.

The same result as in Theorem 1 has been derived independently by Hardy (1999).

It is not difficult to find such states which do not satisfy the above relations, and hence an LOCC transition between them is not possible. As an example, consider the following two states (Jonathan and Plenio, 1999):

\[
|\psi\rangle = \sqrt{0.4} |00\rangle + \sqrt{0.1} |11\rangle + \sqrt{0.1} |22\rangle + \sqrt{0.3} |33\rangle, \tag{6}
\]

\[
|\phi\rangle = \sqrt{0.5} |00\rangle + \sqrt{0.25} |11\rangle + \sqrt{0.25} |22\rangle. \tag{7}
\]

It can be easily verified that the above states do not satisfy Nielsen’s theorem, and as a result, a deterministic conversion is not possible. When a deterministic transformation is not possible, one may consider probabilistic transformation with maximum success probability given by (Vidal, 1999):

\[
P_{\text{max}}(|\psi\rangle \rightarrow |\phi\rangle) = \min_{m=0, d-1} \frac{\sum_{i=m}^{d-1} \alpha_i}{\sum_{i=m}^{d-1} \beta_i}. \tag{8}
\]

For the states given in Eqs. (6) and (7), the optimal probability of transforming \( |\psi\rangle \) into \( |\phi\rangle \) is 0.8. Surprisingly, if the two parties have access to an additional entangled state \( |\eta\rangle = \sqrt{0.6} |00\rangle + \sqrt{0.4} |11\rangle \), then they can perform the transformation \( |\psi\rangle \otimes |\eta\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle \) with certainty (Jonathan and Plenio, 1999; Nielsen, 1999). After the transformation the state \( |\eta\rangle \) remains unchanged and can be used again for the same purpose. We can say that the state \( |\eta\rangle \) acts as a catalyst and enables a transformation that otherwise would not be possible. Note that in the context of quantum entanglement, what is nowadays predominantly called “catalysis”, used to be described as “entanglement-assisted LOCC operation” (ELOCC) (Eisert and Wilkens, 2000; Jonathan and Plenio, 1999).

A maximally entangled state of any dimension cannot catalyse a transformation between two arbitrary incomparable bipartite pure states (Jonathan and Plenio, 1999). A surprising result that already suggests a rich structure of catalytic transformations of quantum resources. Moreover, if for a pair of states \( |\psi\rangle \) and \( |\phi\rangle \), there exist catalytic transformations \( |\phi\rangle \rightarrow |\psi\rangle \) and \( |\psi\rangle \rightarrow |\phi\rangle \), i.e., if two states are interconvertible under catalytic LOCC, then there is a local unitary \( U \) such that \( |\phi\rangle = U |\psi\rangle \) (Jonathan and Plenio, 1999). As a consequence, if a direct, non-catalytic transformation from \( |\psi\rangle \) to \( |\phi\rangle \) is not possible, whereas a catalytic transformation exists in that case, then the reverse transition \( |\phi\rangle \rightarrow |\psi\rangle \) cannot be realised even as a catalytic one. Hence, catalysis can have an advantage over the direct transformation of quantum states only in the case when the two states are incomparable: |\phi⟩ ↔ |ψ⟩, i.e., there is neither a direct transformation |ϕ⟩ → |ψ⟩ nor |ψ⟩ → |ϕ⟩. However, note that the above result is based on assuming pure state catalysts, and how the result behaves with mixed catalyst states is not known. The result suggests that, as far as transformations of two-qubit pure entangled states are concerned, there is no advantage with a catalyst in a pure state, since it is always true that either |ψ⟩ → |ϕ⟩ or |ϕ⟩ → |ψ⟩ in this case.

A catalytic LOCC transformation with catalysts in pure states has no advantage in the case of two-qutrit pure states as well. In order to see this, Jonathan and Plenio (1999) prove that a bipartite state |ψ⟩ of local dimension \( d \) can be converted catalytically into another bipartite state |ϕ⟩, only if the following inequalities hold

\[
\alpha_0 \leq \beta_0 \quad \text{and} \quad \alpha_{d-1} \geq \beta_{d-1}, \tag{9}
\]

where the Schmidt coefficients \( \{\alpha_i\} \) and \( \{\beta_i\} \), \( i = 0, 1, \ldots, d-1 \), of respectively |ϕ⟩ and |ψ⟩ are sorted in non-increasing order. In the case of \( d = 3 \), the above necessary condition for catalysis can be stated as

\[
\alpha_0 \leq \beta_0 \quad \text{and} \quad \alpha_0 + \alpha_1 \leq \beta_0 + \beta_1, \tag{10}
\]

since the numbers \( \{\alpha_i\} \) and \( \{\beta_i\} \) sum up to identity. But this is exactly the condition (5) of Nielsen’s majorization theorem!
Hence, as in the two-qubit case, there is no advantage in catalysis for bipartite systems with $d = 3$: the existence of a catalytic conversion of entangled states implies that these states can be converted directly via LOCC. A genuine catalytic conversion of incomparable states occurs first in systems of dimension $d = 4$, as we have already seen in Eqs. (6) and (7).

When a catalytic transition from $|\psi\rangle$ to $|\phi\rangle$ is not possible with certainty, it can happen that by adding a catalyst state we will increase the probability of the conversion. As an example, consider two incomparable states (Jonathan and Plenio, 1999):

$$|\psi\rangle = \sqrt{0.6}|00\rangle + \sqrt{0.2}|11\rangle + \sqrt{0.2}|22\rangle,$$
$$|\phi\rangle = \sqrt{0.5}|00\rangle + \sqrt{0.4}|11\rangle + \sqrt{0.1}|22\rangle.$$  \hspace{1cm} (11)

The optimal probability to convert $|\psi\rangle$ into $|\phi\rangle$ is $P_{\text{max}}(\psi \rightarrow |\phi\rangle) = 0.8$ (Vidal, 1999), whereas with the help of another state $|\eta\rangle = \sqrt{0.65}|00\rangle + \sqrt{0.35}|11\rangle$, we have $P_{\text{max}}(|\psi\rangle \otimes |\eta\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle) = 0.904$. Of course, there are situations when even adding a catalyst in an arbitrary pure state cannot increase the optimal probability of transformation. This is the case, e.g. when the optimal probability of converting a bipartite state $|\psi\rangle$ into $|\phi\rangle$ is $P_{\text{max}}(\psi \rightarrow |\phi\rangle) = \alpha_d^{-1}/\beta_d^{-1}$. Indeed, let $\{\gamma_j\}_{j=0}^{d^2-1}$ be the ordered, non-zero Schmidt coefficients of $|\eta\rangle$. Since the Schmidt coefficients, not necessarily ordered, of the state $|\psi\rangle \otimes |\eta\rangle$ are given by $\{\alpha_i\gamma_j\}_{i=0}^{d^2-1,j=0}$, and similarly: $\{\beta_i\gamma_j\}_{i=0}^{d^2-1,j=0}$ for the state $|\phi\rangle \otimes |\eta\rangle$, from Eq. (8) we obtain (Jonathan and Plenio, 1999)

$$P_{\text{max}}(|\psi\rangle \otimes |\eta\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle) = \min_{0 \leq i,k \leq d^2-1} \left\{ \sum_{i=0}^{d^2-1} \alpha_i \gamma_j \sum_{k=0}^{d^2-1} \beta_k \gamma_j \right\} \leq \frac{\alpha_d^{-1}}{\beta_d^{-1}},$$ \hspace{1cm} (13)

where the inequality is obtained by setting $l = d-1, k = d'-1$, instead of minimising over all possible pairs $l, k$.

Of course, a pure catalyst state can also increase the transformation probability. Feng et al. (2004, 2005) have studied the advantage of pure catalysts in probabilistic entangled state transformations. Here, the goal is to find a pure catalyst state $|\eta\rangle$ increasing the optimal conversion probability, i.e.,

$$P_{\text{max}}(|\psi\rangle \otimes |\eta\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle) > P_{\text{max}}(|\psi\rangle \rightarrow |\phi\rangle).$$ \hspace{1cm} (14)

The following necessary and sufficient condition has been derived for the existence of a catalyst $|\eta\rangle$ for such an enhancement (Feng et al., 2004):

$$P_{\text{max}}(|\psi\rangle \rightarrow |\phi\rangle) < \min \left\{ \alpha_d^{-1} \beta_d^{-1}, 1 \right\},$$ \hspace{1cm} (15)

where $\alpha_d^{-1}$ and $\beta_d^{-1}$ are the smallest Schmidt coefficient of $|\psi\rangle$ and $|\phi\rangle$ respectively. Furthermore, they also discussed the scenario where the probability can be increased with the help of a two-qubit pure catalyst. In addition, an efficient algorithm has been proposed to find the most economical catalyst in such cases (Feng et al., 2005).

As mentioned earlier, if the local dimension of the given state is smaller than or equal to 3, a catalyst does not help in accomplishing a forbidden transformation (Jonathan and Plenio, 1999). Now one may wonder that for a given final state $|\phi\rangle$ of dimension at least 4 there exists an initial state $|\psi\rangle$ with $\psi^A \neq \phi^A$, such that $|\psi\rangle$ can be converted catalytically into $|\phi\rangle$. It turns out that almost for all final states $|\phi\rangle$ there exists a catalytic conversion from $|\psi\rangle$ (Daftuar and Klimesh, 2001). To be precise, the following has been shown: For a given final state $|\phi\rangle$, there exists a pure state catalyst and an initial state $|\psi\rangle$ with $\psi^A \neq \phi^A$, such that the transformation $|\psi\rangle \rightarrow |\phi\rangle$ is possible catalytically if and only if $\beta_0 \neq \beta_1$ and $\beta_m \neq \beta_{d-1}$ for some $l, m$ with $0 < l < m < d - 1$ (Daftuar and Klimesh, 2001). Here, $\beta_i$ are Schmidt coefficients of the state $|\phi\rangle$. This result emphasises that for almost all final states of dimension 4 or more, the sets of initial states to achieve a final state $|\phi\rangle$ using a pure state catalyst and without a catalyst are not the same, i.e., $T(|\phi\rangle) \neq S(|\phi\rangle)$. Here $T(|\phi\rangle)$ and $S(|\phi\rangle)$ represent the sets of possible initial states $|\psi\rangle$ that can be converted into $|\phi\rangle$ with a catalyst in an arbitrary pure state and without a catalyst, respectively. Note that no restriction has been imposed on the dimension of the catalyst. For a restriction on the dimension of the catalyst, the set of initial states is represented by $T_k(|\phi\rangle)$, where $k$ is the maximum dimension of the catalyst. Then it has been shown that there exists no $k$ such that $T_k(|\phi\rangle) = T(|\phi\rangle)$ when $T(|\phi\rangle) \neq S(|\phi\rangle)$ (Daftuar and Klimesh, 2001). Although the set $T_k(|\phi\rangle)$ is closed, the set $T(|\phi\rangle)$ is not closed in general (Daftuar and Klimesh, 2001). Therefore, the dimension of the catalyst is unbounded in general to determine which states can be catalytically converted into a given final state. Furthermore, as we have previously discussed maximally entangled state cannot be a catalyst. Naturally, one may ask which states are useful to be catalysts. As shown by Daftuar and Klimesh (2001), it turns out that almost all bipartite pure states are potential catalysts. Precisely the following has been shown: For a bipartite state $|\eta\rangle = \sum_{i=0}^{d^2-1} \sqrt{\beta_i} |ii\rangle$ of local dimension $d$ with a nonuniform distribution of Schmidt coefficients there exist states $|\psi\rangle$ and $|\phi\rangle$ in dimension 4 with $\psi^A \neq \phi^A$, such that $\psi^A \otimes \eta^A < \phi^A \otimes \eta^A$ (Daftuar and Klimesh, 2001).

Since the discovery of entanglement catalysis by Jonathan and Plenio (1999), an important problem was to characterize all pairs of states $|\psi\rangle^A \otimes |\phi\rangle^B$ allowing for a catalytic LOCC transformation. Results in this direction have been presented in (Aubrun and Nichehia, 2007, 2008; Klimesh, 2007; Sun et al., 2005; Turgut, 2007), focusing on pure catalyst states. When a catalyst enables a forbidden transformation between two bipartite pure states, the respective Schmidt coefficients satisfy a trumping condition. More precisely, a state $|\psi\rangle$ is trumped by another state $|\phi\rangle$, i.e., $\psi^A <_T \phi^A$ if there exists a catalyst in the state $|\eta\rangle$ such that $\psi^A \otimes \eta^A < \phi^A \otimes \eta^A$ holds (Daftuar and Klimesh, 2001). The trumping condition is an extension of Nielsen’s majorization relation presented earlier in Theorem 1.

A detailed characterization of the trumping condition is mathematically very complex (Daftuar and Klimesh, 2001). Sun et al. (2005) provided a necessary and sufficient condi-
tion for the existence of a pure two-qubit catalyst state for $4 \otimes 4$ incomparable pure states. Furthermore, they have given an efficient algorithm to decide whether a $k \otimes k$ pure catalyst state exists for a pair of $n \otimes n$ incomparable pure states. Aubrun and Nechita (2007, 2008) provided some more partial results in this direction by determining the closure of the set $T(\phi)$ consisting of initial states $|\phi\rangle$ for a given final state $|\phi\rangle$. Precisely, they showed that the closure of $T(\phi)$ in $l_1$ norm can be completely characterized by the following inequalities: $||\phi^A||_p \leq ||\phi^B||_p$ for all $p \geq 1$, where $|| \cdot ||_p$ represents $l_p$ norm. Despite these efforts, the characterization was not complete and a full characterization has been provided by Turgut (2007) and Klimesh (2007). To be precise, it has been shown that the trumping condition is equivalent to (Klimesh, 2007; Turgut, 2007)

$$f_\alpha(s(\phi)) > f_\alpha(s(\psi)) \forall \alpha \in (-\infty, \infty), \quad (16)$$

where $s(\phi)$ and $s(\psi)$ represent the Schmidt coefficients of the states $|\phi\rangle$ and $|\psi\rangle$, respectively. The function $f_\alpha(x)$ for a $d$-dimensional probability vector $x$ is expressed as

$$f_\alpha(x) = \begin{cases} \ln \sum_{i=1}^d x_i^\alpha & (\alpha > 1); \\ \sum_{i=1}^d x_i \ln x_i & (\alpha = 1); \\ -\ln \sum_{i=1}^d x_i^\alpha & (0 < \alpha < 1); \\ -\ln \sum_{i=1}^d \ln x_i & (\alpha = 0); \\ \ln \sum_{i=1}^d x_i^\alpha & (\alpha < 0). \end{cases} \quad (17)$$

If one allows an infinitesimally small error in the initial state, i.e., $||\phi^A - \phi^B||_1 \leq \epsilon$ with the trace norm $||M||_1 = \text{Tr} \sqrt{M^T M}$, then the trumping condition $\psi^A \prec \phi^A$ is equivalent to the monotonicity of the Rényi entropies (Brandão et al., 2015).

**Theorem 2** (See Proposition 4 of the Supplementary Information in (Brandão et al., 2015)). Consider two states $|\phi\rangle$ and $|\psi\rangle$ with Schmidt coefficients $s(\phi)$ and $s(\psi)$ respectively. Then for arbitrary $\epsilon > 0$, there exists a state $\psi^A$ with $||\psi^A - \phi^A||_1 \leq \epsilon$ and $\psi^A \prec \phi^A$ if and only if

$$H_\alpha(s(\phi)) \geq H_\alpha(s(\psi)) \quad (18)$$

is true for all $\alpha \in (-\infty, \infty)$.

Here, $H_\alpha$ represent the Rényi entropies which can be expressed as

$$H_\alpha(x) = \frac{\text{sgn}(\alpha)}{1 - \alpha} \ln \left( \sum_{i=1}^d x_i^\alpha \right) \text{ for } \alpha \in \mathbb{R} \setminus \{0, 1\}. \quad (19)$$

$H_\alpha(x)$ for $\alpha \in (-\infty, 0, 1, \infty)$ are defined by taking limits and expressed as

$$H_0(x) = \ln \text{rank}(x), \quad H_1(x) = -\sum_{i=1}^d x_i \ln x_i \quad (20)$$

$$H_{-\infty}(x) = \ln x_{\text{min}}, \quad H_{\infty}(x) = -\ln x_{\text{max}}, \quad (21)$$

where $x_{\text{max}}$ and $x_{\text{min}}$ are the maximum and minimum elements of the vector $x$, respectively.

In real life, noise is inevitable, and as a result, we are forced to deal with mixed states rather than pure states. Mixed state transformations are much more complicated (Terhal and Horodecki, 2000) as majorization relations described in Theorem 1 are not applicable in general. Consequently, the conversion of mixed states with the help of a catalyst gets convoluted. The reason is that we need to first check whether a mixed state transformation $\rho \rightarrow \sigma$ is possible. In general, it is a formidable task. Nevertheless, the problem has been solved for some classes of mixed states. Eisert and Wilkens (2000) explored transformations between a special class of rank two mixed states of the form

$$\rho = \gamma |\psi\rangle \langle \psi| + (1 - \gamma) |\omega\rangle \langle \omega| \quad \text{and} \quad (22)$$

$$\sigma = \gamma |\phi\rangle \langle \phi| + (1 - \gamma) |\omega\rangle \langle \omega|, \quad (23)$$

where $\gamma = \sqrt{\text{Tr} \Omega}$, $\Omega = \Pi |\psi\rangle \langle \psi| \Pi$, $\Pi = \mathbf{1} - |\omega\rangle \langle \omega|$ and $|\omega\rangle$ is a product state. Moreover, we take $|\langle \phi| \omega\rangle|^2 = 0$. With this an LOCC transformation from $\rho$ to $\sigma$ implies that (Eisert and Wilkens, 2000)

$$\frac{\text{Tr}_A \Omega}{\text{Tr} \Omega} \prec \phi^A. \quad (24)$$

Now let us choose

$$|\phi\rangle = \sqrt{0.38} |00\rangle + \sqrt{0.38} |11\rangle + \sqrt{0.095} |22\rangle + \sqrt{0.095} |33\rangle + \sqrt{0.05} |44\rangle, \quad (25)$$

$$|\psi\rangle = \sqrt{0.5} |00\rangle + \sqrt{0.25} |11\rangle + \sqrt{0.25} |22\rangle, \quad (26)$$

$$|\omega\rangle = |44\rangle, \quad (27)$$

and $\gamma = 0.95\nu$, where $1 > \nu > 0$ (Eisert and Wilkens, 2000). With this one can easily verify that Eq. (24) is not satisfied, and hence the transformation $\rho \rightarrow \sigma$ is not possible.

One may ask whether the forbidden transformation can be accomplished with the help of a catalyst. The answer is yes and the protocol is as follows (Eisert and Wilkens, 2000). Alice performs a von Neumann measurement on her subsystem with Kraus operators $K_1 = \sum_{i=0}^3 |i\rangle \langle i|$ and $K_2 = |4\rangle \langle 4|$. When she gets an outcome associated with $K_2$, no further operations are needed as we have state $|\omega\rangle$ with probability $(1 - \gamma)$. In the case of other outcomes, the final state is

$$|\varphi\rangle = \sqrt{0.4} |00\rangle + \sqrt{0.4} |11\rangle + \sqrt{0.1} |22\rangle + \sqrt{0.1} |33\rangle. \quad (28)$$

Note that, the reduced state of $|\varphi\rangle$ is not majorized by the reduced state of $|\psi\rangle$. Therefore, a direct transformation is not possible and a catalyst could help in the transformation $|\varphi\rangle \rightarrow |\psi\rangle$. In fact, by taking a catalyst in state $|\zeta\rangle = \sqrt{0.4} |00\rangle + \sqrt{0.6} |11\rangle$, we obtain

$$\phi^A \otimes \zeta^A \prec \phi^A \otimes \zeta^A. \quad (29)$$

Finally, discarding the classical information, we achieve the desired state $\sigma$. 
In (Li and Shi, 2011a), the authors analyse the catalytic transformation of mixed states of two qubits. Building on the results from (Gour, 2005b; Li and Shi, 2011b), their work sheds some light on the question of whether a mixed catalyst state could provide an advantage for entanglement catalysis. For a given mixed state $\rho$, the authors define a family of entanglement monotones:

$$
\hat{E}_{k,l}(\rho) = \min_{(p_j,|\psi_j\rangle)} \sum_j p_j \hat{E}_{k,l}(|\psi_j\rangle),
$$

with $l = 2, 3, \ldots, d$ and $k \in (0, 1]$ where the minimum is taken over all ensembles $(p_j,|\psi_j\rangle)$ of probability distributions $p_j$ and bipartite pure states $|\psi_j\rangle$ such that $\rho = \sum_j p_j |\psi_j\rangle\langle \psi_j|$, and

$$
\hat{E}_{k,l}(\psi) = \min \left\{ 1, \sum_{i=1}^d \alpha_i \right\},
$$

for any bipartite pure state $|\psi\rangle = \sum_i \sqrt{\alpha_i} |i\rangle$ with ordered Schmidt coefficients. The ensemble that minimises Eq. (30) is called optimal.

In another paper, (Li and Shi, 2011b), the same authors give a non-standard definition of an LOCC transformation between mixed states. According to Definition 2.6 in (Li and Shi, 2011b), a mixed state $\rho$ can be converted into another mixed state $\sigma$ via LOCC if there is an LOCC operation between optimal ensembles of $\rho$ and $\sigma$. For a precise definition of a transformation between ensembles of pure states, see (Li and Shi, 2011b), p. 97. For our discussion here, it is enough to notice that if there is an LOCC map $\Lambda$, such that $\Lambda(\rho) = \sigma$ in the ordinary sense, then $\rho$ can be converted to $\sigma$ via LOCC according to Definition 2.6 in (Li and Shi, 2011b) but not necessarily the other way round.

The main result from (Li and Shi, 2011a), namely Proposition 2.2 therein, can be now stated as follows: Suppose $\rho$, $\sigma$, and $\omega$ are rank 2 mixed bipartite states of a two-qubit system. Then the state $\rho \otimes \omega$ can be converted into $\sigma \otimes \omega$ via LOCC according to Definition 2.6 in (Li and Shi, 2011b) if and only if

$$
\hat{E}_{k,l}(\rho \otimes \omega) \geq \hat{E}_{k,l}(\sigma \otimes \omega),
$$

for all $k \in (0, 1]$ and $l = 2, 3, 4$. In other words, as long as we consider rank 2 mixed states, a catalytic transformation between two-qubit systems is governed by the family of entanglement monotones defined in Eq. (30). The result is a generalisation of the trumping condition for pure states, see the discussion above, p. 5.

Another important result in (Li and Shi, 2011a), Proposition 2.5, follows quickly from the proof of the Proposition 2.2, and gives us a partial answer to the question of whether mixed catalyst states could provide an advantage over pure states: If $\rho$, $\sigma$ are rank 2 mixed bipartite states of a two-qubit system that cannot be converted into each other via LOCC, but for which a catalyst’s state $\omega$ exists as in Eq. (32), then $\omega$ cannot be pure, even if we allowed for $\omega$ to have an arbitrary Schmidt rank. Note that this statement assumes that all LOCC transformations between mixed states are according to Definition 2.6 in (Li and Shi, 2011b), but as we have already discussed above, that implies that there is no catalytic LOCC transformation in the ordinary sense as well.

Lastly, the authors give an example of a mixed catalyst state that facilitates a transformation between rank 2 mixed two-qubit states as above, see Sec. 3 in (Li and Shi, 2011a). However, again, since the authors’ definition of a LOCC transformation between mixed states describes a larger family of maps than standard LOCC on the convex set of mixed states, the question of whether an ordinary map and a mixed catalyst state exist remains open. Given that we know that no pure two-qubit state can be a catalyst in this case, a positive answer to that question would be a strong argument for the advantages of mixed-state catalysis.

Next, we will discuss the advantage of a catalyst for small transformations where we want to change the state slightly. In particular, the goal is to convert a state $\rho$ into another state $\sigma$ with $F(\rho, \sigma) > 1 - \varepsilon$, where $F(\rho, \sigma) = \left[ \text{Tr} \sqrt{\sqrt{\rho} \sqrt{\sigma}} \right]^{1/2}$ is the fidelity (Uhlmann, 1976). In such a scenario, there exist states $\rho$ and $\sigma$ such that the transformation $\rho \rightarrow \sigma$ is impossible without a catalyst and can be accomplished with a suitable catalyst. Indeed, such states can be designed using Eqs. (22) and (23) (Eisert and Wilkens, 2000). On the contrary, a surprising result exists when we want to transform a pure state slightly. For all pure states $|\psi\rangle$ there exists an $\varepsilon > 0$, such that the following statement holds for all states $|\psi\rangle$ with $(|\langle \psi|\phi\rangle|^2 > 1 - \varepsilon$: the catalytic transformation $|\psi\rangle \rightarrow |\phi\rangle$ is impossible with a pure catalyst if it is impossible without the catalyst (Eisert and Wilkens, 2000).

The structure of a catalyst for a successful transformation between two incomparable pure states was studied by Zhou and Guo (2000). To be precise, the authors showed that successful transformation of $|\psi\rangle = \sum_{i=0}^{d-1} \sqrt{\alpha_i} |i\rangle$ into $|\phi\rangle = \sum_{i=0}^{d-1} \sqrt{\beta_i} |i\rangle$ can be accomplished with the help of a pure catalyst state having $m$ non-zero Schmidt coefficients, i.e., $|\eta\rangle = \sum_{i=0}^{m-1} \sqrt{\gamma_i} |i\rangle$, only if (Zhou and Guo, 2000)

$$
m_{\gamma_{m-1}} \leq \frac{P_{\max}(|\psi\rangle \rightarrow |\phi\rangle)}{P'},
$$

where $P_{\max}(|\psi\rangle \rightarrow |\phi\rangle)$ is described in Eq. (8), represents the optimal probability of transforming $|\psi\rangle$ into $|\phi\rangle$ (Vidal, 1999). Furthermore, they extended the above result to stochastic scenarios. As a matter of fact, they showed that the probability of the transformation $P_{\max}(|\psi\rangle \rightarrow |\phi\rangle)$ can be increased with the help of a pure state catalyst $|\eta\rangle$, only if (Zhou and Guo, 2000)

$$
m_{\gamma_{m-1}} \leq \frac{P_{\max}(|\psi\rangle \rightarrow |\phi\rangle)}{P'},
$$

where $P' \geq P_{\max}(|\psi\rangle \rightarrow |\phi\rangle)$ represents the enhancement in the probability for catalytic transformation. Their results provide some insight into the structure of the catalyst.

In (Feng et al., 2002), the concept of mutual catalysis has been introduced. For this, suppose there are two pairs of states $(|\psi\rangle, |\phi\rangle)$ and $(|\eta\rangle, |\zeta\rangle)$. The transformations $|\psi\rangle \rightarrow |\phi\rangle$ and
\[ |\eta\rangle \rightarrow |\zeta\rangle \] are not possible deterministically. However, sometimes it is possible that joint LOCC can transform \(|\psi\rangle \otimes |\eta\rangle\) to \(|\phi\rangle \otimes |\zeta\rangle\). Therefore, \(|\psi\rangle\) and \(|\eta\rangle\) help each other to achieve the desired target states. As an example, consider the following states (Feng et al., 2002)

\[ |\psi\rangle = \sqrt{0.4}|00\rangle + \sqrt{0.36}|11\rangle + \sqrt{0.14}|22\rangle \]  
\[ + \sqrt{0.1}|33\rangle, \]

\[ |\phi\rangle = \sqrt{0.5}|00\rangle + \sqrt{0.25}|11\rangle + \sqrt{0.25}|22\rangle, \]

\[ |\eta\rangle = \sqrt{0.6}|00\rangle + \sqrt{0.4}|11\rangle, \]

\[ |\zeta\rangle = \sqrt{0.55}|00\rangle + \sqrt{0.45}|11\rangle. \]

It is easy to check that \(|\psi\rangle \leftrightarrow |\phi\rangle\) and \(|\eta\rangle \rightarrow |\zeta\rangle\) (Nielsen, 1999). However, they can be transformed jointly into the desired target states. Necessary conditions for such transformations have been derived in (Feng et al., 2002). Furthermore, the authors have discussed the scenario where a conversion from \(|\psi\rangle\) to \(|\phi\rangle\) is not possible even with the help of a catalyst but can be realized with mutual catalysis (Feng et al., 2002).

According to Theorem 1, there exist states \(|\psi\rangle\) and \(|\phi\rangle\), such that \(|\psi\rangle \rightarrow |\phi\rangle\) cannot be realized with certainty. However, in some cases, if we possess multiple copies of \(|\psi\rangle\), they can be transformed to the same number of \(|\phi\rangle\) (Bandyopadhyay et al., 2002). Along the same line, a concept of multicopy catalysis has been introduced by Duan et al. (2005b). A state \(|\eta\rangle\) is called a multiple-copy catalyst when it cannot catalyze the transformation \(|\psi\rangle \rightarrow |\phi\rangle\), however the same can be accomplished with the help of \(|\eta\rangle^{\otimes k}\) for some \(k > 1\). As an example Duan et al. (2005b) consider the following case. Consider the initial state \(|\psi\rangle\) as given in Eq. (6) and the target state is

\[ |\phi\rangle = \sqrt{0.5}|00\rangle + \sqrt{0.25}|11\rangle + \sqrt{0.22}|22\rangle + \sqrt{0.03}|33\rangle. \]

The transformation \(|\psi\rangle \rightarrow |\phi\rangle\) cannot be realized with a pure two-qubit catalyst state (Sun et al., 2005). However, note that the above transformation can be realized with a pure two-qutrit catalyst state. Now consider a situation where Alice and Bob only have access to some copies of a two-qubit pure state. One may wonder whether the above transformation can be realized with multiple copies of a two-qubit catalyst. Indeed there exists a two-qubit state \(|\eta\rangle = \sqrt{0.6}|00\rangle + \sqrt{0.4}|11\rangle\), such that \(|\psi\rangle^{\otimes 5} \otimes |\eta\rangle < |\phi\rangle \otimes (|\eta\rangle^{\otimes 5})\) (Duan et al., 2005b). Furthermore, they have derived some necessary conditions for an arbitrary pure state to be a catalyst for a given transformation \(|\psi\rangle \rightarrow |\phi\rangle\). In addition, an advantage in probabilistic transformation with multiple copy catalysis has also been shown in (Duan et al., 2005b). Instead of a single copy of the initial state, if we have multiple copies of it and want to transform it into the same number of copies of the final state with the help of a multicopy catalyst, then a study has been done on the tradeoff between the number of copies of the source state and that of the catalyst state. To be precise, they have shown that the number of copies required for a catalyst state decreases with the number of copies of the source state (Duan et al., 2005b). In (Duan et al., 2005a, 2004) it has been shown that any multiple copy entangled state transformation (Bandyopadhyay et al., 2002) can be realized in single copy level with the help of a suitable pure catalyst. It is natural to wonder whether the opposite is also true. But the answer is negative. As shown in (Feng et al., 2006), the catalytic assisted transformation is strictly more powerful than multiple copy transformations. Now, it is natural to ask whether they are equivalent in the asymptotic scenario. Indeed it has been shown that catalytically assisted transformation is asymptotically equivalent to the multiple-copy transformation when the dimension of the catalyst and number of copies tend to infinity (Duan et al., 2005c). More precisely, these two kinds of transformations are equivalent in the sense that they can simulate each other’s ability to transform a given initial state to a given target state with the same optimal success probability. Optimal probability for multiple-copy state transformation from \(|\psi\rangle\) to \(|\phi\rangle\) can be defined as (Duan et al., 2005c)

\[ P_M(|\psi\rangle \rightarrow |\phi\rangle) = \sup_m P_{\max}(|\psi\rangle^{\otimes m} \rightarrow |\phi\rangle^{\otimes m}). \]

On the other hand, the optimal conversion probability for catalytically assisted transformation is given by (Duan et al., 2005c)

\[ P_C(|\psi\rangle \rightarrow |\phi\rangle) = \sup_{|\eta\rangle} P_{\max}(|\psi\rangle \otimes |\eta\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle). \]

In (Duan et al., 2005c), it has been shown that \(P_M(|\psi\rangle \rightarrow |\phi\rangle) = P_C(|\psi\rangle \rightarrow |\phi\rangle)\). Furthermore, Duan et al. (2005a) have shown that multiple copy transformation, with the aid of a pure catalyst, is equivalent to single copy catalytic transformation with an arbitrary pure catalyst state. Let \(V(\phi)\) be the set of all initial states \(|\psi\rangle\) such that \(|\psi\rangle \otimes |\eta\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle\) for some \(|\eta\rangle\), and \(V_M(\phi)\) is the set of all initial states \(|\psi\rangle\) such that \(|\psi\rangle^{\otimes m} \otimes |\eta\rangle \rightarrow |\phi\rangle^{\otimes m} \otimes |\eta\rangle\) for some \(m \geq 1\) and \(|\eta\rangle\). Obviously \(V(\phi) \subseteq V_M(\phi)\). In fact, it has been shown that they are equal \(V(\phi) = V_M(\phi)\) (Duan et al., 2005a). Therefore, the power of catalytic assisted transformation (assuming pure catalyst states) cannot be elevated by increasing the number of copies of the original state.

While the previous discussion showed when a catalytic transformation is possible, it does not provide any information about the catalyst. To be precise, it does not give information about the entanglement content or the dimension of the catalyst required for a transformation. Sanders and Gour (2009) provided a lower bound on the dimension of a possible catalyst state \(|\eta\rangle\) for a transformation \(|\psi\rangle \otimes |\eta\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle\). Suppose \(|\psi\rangle, |\phi\rangle\) are two incomparable pure states with \(d\) non-zero Schmidt coefficients and the transformation \(|\psi\rangle \otimes |\eta\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle\) is possible with the help of a pure catalyst state \(|\eta\rangle\) that has \(r\) non-zero Schmidt coefficients. Then it has been shown that the minimum number of non-zero Schmidt coefficients for the catalyst should be (Sanders and Gour, 2009)

\[ r \geq 1 + \frac{d - 1}{d} \log_2 C_{d-1}(\phi) - \frac{1}{d} \log_2 C_d(\phi), \]

\[ \frac{1}{d} \log_2 C_{d-1}(\phi) - \frac{1}{d} \log_2 C_d(\phi). \]
where \( C_k \) represents generalized \( k \)-th concurrence and can be expressed as

\[
C_k(\psi) = \left( \frac{f_k(s(\psi^*^*)]}{f_k(s(\psi^*)\}} \right)^{1/k},
\]

where \( \psi^* \) is a maximally entangled state of dimension \( d \) and \( f_k(x) \) represents \( k \)-th elementary symmetric function of \( d \) variables: \( f_k(x) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k} \) (Gour, 2005a).

So far, we have not discussed the amount of entanglement required for a catalyst to enable a transformation between a pair of incomparable states. However, the answer is not fully known and only some partial results are available in the literature. Duan et al. (2005b) derived some necessary conditions for a bipartite pure entangled state to be a catalyst. In fact, the following has been proved: Suppose \( |\psi\rangle \) and \( |\phi\rangle \) are two incomparable states of Schmidt rank \( d \) with Schmidt coefficients \( \{p\} \) and \( \{q\} \) respectively, and a catalyst state \( |\eta\rangle \) that has \( k \) number of non-zero Schmidt coefficients represented by \( \{t\} \) enables the transformation from \( |\psi\rangle \) to \( |\phi\rangle \). Then for any \( l \in L_{\psi,\phi} \) and \( 0 \leq i \leq k - 2 \), the Schmidt coefficients of the catalyst state satisfy the following three conditions:

\[
\begin{align*}
\frac{t_0}{t_{k-1}} &> \frac{q_1}{q_{i+1}}, \\
\frac{t_0}{t_i} &> \frac{q_{i+1}}{q_i} \quad \text{or} \quad \frac{t_i}{t_{i+1}} < \frac{q_0}{q_i}, \\
\frac{t_{i+1}}{t_{d-1}} &> \frac{q_l}{q_{i+1}} \quad \text{or} \quad \frac{t_l}{t_{i+1}} < \frac{q_{i+1}}{q_{d-1}},
\end{align*}
\]

(44a), (44b), (44c)

where \( L_{\psi,\phi} \) represents the set of all \( l \) such that

\[
L_{\psi,\phi} = \left\{ l : 0 \leq l \leq d - 1 \quad \text{and} \quad \sum_{j=0}^{l} p_j > \sum_{j=0}^{l} q_j \right\}.
\]

(45)

These restrictions on the Schmidt coefficients represent the deviation of state \( |\eta\rangle \) from the product and maximally entangled states.

Very recently, Grabowecky and Gour (2019) presented some stronger conditions than Eqs. (44). They showed that for any two incomparable states \( |\psi\rangle, |\phi\rangle \) with \( d \) non-zero Schmidt coefficients and a catalyst state \( |\eta\rangle \) having \( k \) non-zero Schmidt coefficients, if \( \psi^A \otimes \eta^A < \phi^A \otimes \eta^A \), then (Grabowecky and Gour, 2019)

\[
\begin{align*}
\max_{r \in \{0, \ldots, k-2\}} \frac{t_r}{t_{r+1}} < \min \left\{ \frac{q_0}{q_a}, \frac{q_{a+1}}{q_{a+1}} \right\} &\equiv a, \\
\frac{t_0}{t_{k-1}} &> \max_{r \in L_{\psi,\phi}} \frac{q_r}{q_{r+1}} \equiv b,
\end{align*}
\]

(46a), (46b)

where \( m = \min L_{\psi,\phi} \) and \( n = \max L_{\psi,\phi} \). These two conditions show how much a catalyst state can differ from both a product state and a maximally entangled state. Previously in Eq. (42), we have seen a lower bound on the dimension of a catalyst in terms of the generalised \( k \)-th concurrence (Sanders and Gour, 2009). However, calculating \( k \)-th concurrence for a higher-dimensional state is not easy in general. In that context, Grabowecky and Gour (2019) provided the following lower bound on the dimension of a catalyst, which can be easily calculated:

\[
k > \frac{\ln b}{\ln a} + 1.
\]

(47)

The constraints on the Schmidt coefficients of the catalyst state presented in Eqs. (44) and (46) depend only on the Schmidt coefficients of the target state. Recently, Guo et al. (2021) studied a similar scenario and showed that for \( \psi^A \otimes \eta^A < \phi^A \otimes \eta^A \) the following result holds:

\[
\frac{t_0}{t_{k-1}} > \max_{l \in L_{\psi,\phi}} \left\{ \min \left\{ \frac{p_t}{p_{t+1}}, \frac{q_t}{q_{t+1}} \right\} \right\} \equiv c,
\]

(48)

where \( k \) is the number of non-zero Schmidt coefficients of state \( |\eta\rangle \) and also represents the dimension of the catalyst. Furthermore, they put a bound on the minimum dimension of the catalyst state that can be expressed as (Guo et al., 2021)

\[
k > \frac{\ln c}{\ln(a \sqrt{b})} + 1.
\]

(49)

Unlike Eq. (47), the above bound depends on both \( \{p\} \) and \( \{q\} \). One can compare and check which bound provides a good lower bound.

Duarte et al. (2016) explored an interesting question about self-catalysis. Given two incomparable states \( |\psi\rangle \) and \( |\phi\rangle \), they studied whether \( |\psi\rangle \) can catalyze the transformation \( |\phi\rangle \otimes |\phi\rangle \rightarrow |\phi\rangle \otimes |\phi\rangle \). As an example, consider the following incomparable states (Duarte et al., 2016):

\[
|\psi\rangle = \sqrt{0.9} |00\rangle + \sqrt{0.08} |11\rangle + \sqrt{0.01} |22\rangle + \sqrt{0.009} |33\rangle,
\]

(50)

\[
|\phi\rangle = \sqrt{0.95} |00\rangle + \sqrt{0.03} |11\rangle + \sqrt{0.02} |22\rangle.
\]

(51)

The transformation of \( |\psi\rangle \) into \( |\phi\rangle \) can be accomplished with the help of \( |\phi\rangle \) as catalyst, i.e., \( |\psi\rangle \otimes |\phi\rangle \rightarrow |\phi\rangle \otimes |\phi\rangle \). One can modify it further by considering multi-copy self-catalysis, where \( |\phi\rangle \otimes |\phi\rangle \rightarrow |\phi\rangle \otimes |\phi\rangle \) is not possible but \( |\phi\rangle \otimes |\phi\rangle \rightarrow |\phi\rangle \otimes |\phi\rangle \) is possible for some \( N > 1 \) (Duarte et al., 2016). For instance, consider the initial state to be

\[
|\psi_1\rangle = \sqrt{0.918} |00\rangle + \sqrt{0.07} |11\rangle + \sqrt{0.006} |22\rangle + \sqrt{0.006} |33\rangle
\]

(52)

and the target state to be \( |\phi\rangle \) in Eq. (51). Then the transformation of \( |\psi_1\rangle \) into \( |\phi\rangle \) can be achieved using \( |\psi_1\rangle \otimes^N \) as a catalyst. Duarte et al. (2016) tried to answer the question of how typical the self-catalysis phenomenon is using numerical methods. Although they found that self-catalysis occurs in systems of any dimension, the phenomenon is relatively rare (Duarte et al., 2016). They also explored the self-catalysis of stochastic transformations and concluded that self-catalysis in this case occurs more often than in the deterministic one (Duarte et al., 2016). In a recent paper, Acácio and Duarte (2022) discussed in detail whether neural network models can detect catalysis or self-catalysis for bipartite pure entangled state conversions.
The practical implementation of majorization relations was first studied by Gagatsos et al. (2013). In fact, they showed that beam splitter can be used to achieve several classes of majorization relations. Furthermore, they constructed incomparable states and showed that these states can be catalytically converted. Such catalyst states can be easily implemented by using single-photon states or two-mode squeezed vacuum states (Gagatsos et al., 2013).

Recently, Santra and Malinovsky (2021a) studied the catalytic entanglement concentration of two-qubit pure states. They derived a formula for the maximum probability of converting a finite number of copies of a general two-qubit pure state into a singlet with the help of a two-qubit pure catalyst state. Furthermore, they have shown that any pure two-qubit entangled state can be a potential catalyst. Based on the approach given in (Santra and Malinovsky, 2021a), the authors further discussed the advantage of catalysis in the long-range entanglement distribution over a quantum network in (Santra and Malinovsky, 2021b).

In a recent numerical study, the entanglement properties of pure catalyst states have been explored (Gupta et al., 2022). The average amount of entanglement required to catalyze a forbidden transformation decreases with the increment in entanglement difference between the pure initial and target states. Furthermore, for multiple copy state transformations, the amount of entanglement required to catalyze the transformation is lower compared to the single copy catalytic transformation.

An intriguing role of entangled catalysts in the context of implementing some nonlocal operations has been discussed in (Vidal and Cirac, 2002). In fact, Vidal and Cirac (2002) have shown that certain nonlocal tasks cannot be implemented without using additional entanglement. Interestingly, this additional entanglement does not need to be consumed and can be recovered exactly, thus serving as a catalyst in the process. More precisely, there exists a certain two-qubit unitary gate $U_{AB}$ that cannot be implemented from the available two-qubit unitary gate $U_{AB}$ by means of LOCC only (Vidal and Cirac, 2002). However, the same can be implemented by means of local unitary operations and using a maximally entangled state as a catalyst (Vidal and Cirac, 2002).

Kay and Ericsson (2006) explored the cloning of nonmaximally entangled states with LOCC and a copy of a maximally entangled state. The authors completely characterised the set of clonable states. Furthermore, it has been shown that catalysts do not provide any advantage in the cloning process (Kay and Ericsson, 2006).

In general, multipartite entangled state transformation is more complex in nature (Walter et al., 2016) and that makes catalytic transformation even more complex. Unlike catalytic transformation in bipartite scenarios, very few results are known for catalytic state transformations in multipartite scenarios. One of the first results in this direction was provided by Chen et al. (2010). More precisely, they studied the advantage of catalysis in stochastic LOCC (SLOCC) transformation of multipartite pure states. In fact, it has been shown that there exist state transformations that cannot be realised by SLOCC but can be realised by both catalytic SLOCC and multicroy SLOCC. Catalytic SLOCC represents the probabilistic transformation of $|\psi\rangle$ to $|\phi\rangle$ with the help of a catalyst state $|\eta\rangle$, such that the transformation $|\psi\rangle \otimes |\eta\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle$ has non-vanishing probability. On the other hand, a transformation is possible via multicroy SLOCC if there exists a $k$, such that $|\phi\rangle^\otimes k$ can be converted into $|\psi\rangle^\otimes k$ with non-vanishing probability. As specific examples, consider the following transformations (Chen et al., 2010):

1. $|\psi\rangle_{ABC} \rightarrow |\phi\rangle^\otimes 1_{AB} \otimes |\phi\rangle^\otimes 2_{BC} \otimes |\phi\rangle^\otimes 3_{CA}$ where $|\phi\rangle_{ABC}$ is any generalised GHZ-type state of tensor rank 6.
2. $|\text{GHZ}^{n-1}_n\rangle \rightarrow |W_n\rangle$, where $|\text{GHZ}^{d}_n\rangle = \sum_{i=0}^{d-1} |\gamma^{i\alpha}_n\rangle / \sqrt{d}$ and $|W_n\rangle$ is $n$-partite generalisation of $|\text{W}_3\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$.

Both of the above state transformations are not possible even stochastically but can be accomplished via both catalytic SLOCC and multicroy SLOCC. Furthermore, it has been shown that if $|\psi\rangle$ can be converted to $|\phi\rangle$ via multicroy SLOCC then the same can be realised with catalytic SLOCC (Chen et al., 2010).

Recently Neven et al. (2021) found a majorization condition for a class of GHZ-like states, which completely characterises LOCC transformations. For this, suppose $g$ and $h$ are two invertible, complex and diagonal matrices satisfy $\text{Tr}(g^{T}) = \text{Tr}(h^{T})$. Then the transformation

$$1 \otimes \cdots \otimes 1 \otimes g |\text{GHZ}^{d}_n\rangle \rightarrow 1 \otimes \cdots \otimes 1 \otimes h |\text{GHZ}^{d}_n\rangle$$

(53)

is possible by LOCC if and only if

$$\left(|g_1|^{2}, \ldots, |g_d|^{2}\right)^{T} \prec \left(|h_1|^{2}, \ldots, |h_d|^{2}\right)^{T}$$

(54)

holds. Here, $(g_1, \ldots, g_d)$ and $(h_1, \ldots, h_d)$ are diagonal elements of the matrices $g$ and $h$, respectively. Note that $g$ and $h$ correspond to the GHZ-like states of the form $\sum_{i=0}^{d-1} \sqrt{r_i} |\gamma^{i\alpha}_n\rangle$. As a direct consequence of this we see that $|\psi\rangle$ and $|\phi\rangle$ characterised by

$$g = \text{diag}(\sqrt{0.45}, \sqrt{0.35}, \sqrt{0.12}, \sqrt{0.08}),$$
$$h = \text{diag}(\sqrt{0.56}, \sqrt{0.21}, \sqrt{0.17}, \sqrt{0.06}),$$

(55)
(56)

respectively, are incomparable as they fail to satisfy Eq. (54).

However, the transformation can be accomplished with another GHZ-like state $|\varphi\rangle$ characterised by

$$f = \text{diag}(\sqrt{0.63}, \sqrt{0.27}, \sqrt{0.07}, \sqrt{0.03}),$$

(57)

that acts as a catalyst in the process. In fact, the transformation $|\psi\rangle \otimes |\varphi\rangle \rightarrow |\phi\rangle \otimes |\varphi\rangle$ is equivalent to the following transformation:

$$1 \otimes \cdots \otimes 1 \otimes (g \otimes f) |\text{GHZ}^{16}_n\rangle \rightarrow 1 \otimes \cdots \otimes 1 \otimes (h \otimes f) |\text{GHZ}^{16}_n\rangle$$

(58)

This completes our discussion of exact entanglement catalysis, and we will consider the exact catalysis of coherence in the following.
B. Quantum coherence

Recent studies suggest that quantum coherence might be another useful resource for many information processing tasks that are otherwise impossible without it (Streltsov et al., 2017a). There are two different notions of coherence in the literature: unspreadable and spreadable (Marvian and Spekkens, 2016). In the case of unspreadable coherence, the levels of eigenspaces are important for the quantification of coherence, for example, energy. Whereas for the latter one, it is irrelevant which eigenspaces are responsible for the coherence. While the former type of coherence is characterized using the resource theory of asymmetry (Marvian and Spekkens, 2013), the latter is characterized using the usual notion of resource theory of coherence introduced by Baumgratz et al. (2014). In the following, we will discuss catalysis in both cases.

Quantum catalysis in the resource theory of coherence was introduced in Åberg (2014). In this paper, the author studied the question of how to use a catalyst to create superposition between different energy levels under energy-preserving unitary operations. As energy must remain conserved, a system in a definite energy level cannot be changed to a superposition of different energy levels. However, an additional system or reservoir with a high degree of coherence can help to achieve this. Due to this, the amount of coherence in the reservoir can decrease. In contrast, Åberg (2014) showed that the coherence in the reservoir can be used to turn the state of the system into a superposition of energy levels without consuming it in the process. Hence, coherence in this reservoir acts like a catalyst in the process. The advantage has been shown by constructing two examples: the doubly-infinite ladder and the half-infinite ladder (Åberg, 2014). Furthermore, as a direct consequence, it has been shown that this catalytic procedure has application in work extraction (Åberg, 2014). Note that in Åberg’s protocol the state of the reservoir does not remain the same. Nonetheless, the argument for catalysis is that the coherence resource in the reservoir does not degrade in the process and can be reused on another system to apply the same transformation.

The change in the reservoir’s state is unusual to the definition of catalysis. Hence, an alternative description to (Åberg, 2014) has been provided by Korzekwa et al. (2016), who preferred the term “repeatability” over catalysis as the state of the reservoir changes. They studied work extraction from the coherence of the given system while the reservoir can be used repeatedly. The amount of work that can be extracted from a system in the state $\rho$ is equal to the difference in free energy $\Delta F(\rho)$ when the allowed operations conserve the energy only on average (Skrzypczyk et al., 2014). Here $\Delta F(\rho) = F(\rho) - F(\gamma)$ with the Helmholtz free energy

$$F(\rho) = \text{Tr}[\rho H] - kT S(\rho),$$

where $H$ and $\gamma$ correspond to the system Hamiltonian and thermal state respectively, and $S(\rho) = -\text{Tr}[\rho \log \rho]$ is the von Neumann entropy. However, in (Korzekwa et al., 2016), the authors were interested in the work extraction only from the coherence of the system, and in that case,

$$F(\rho) - F(D[\rho]) = kT S(\rho \| D[\rho])$$

is the amount of free energy available for the extraction from the coherence of state $\rho$, where $D$ is a dephasing operation in the energy eigenbasis and $S(\rho \| D[\rho])$ represents a measure of coherence (Gour et al., 2009) with the quantum relative entropy $S(\rho \| \sigma) = \text{Tr}[\rho \log \rho] - \text{Tr}[\rho \log \sigma]$. Korzekwa et al. (2016) showed that the amount of work extracted from the coherence of state $\rho$ is arbitrarily close to the free energy extraction limit, i.e., $kT S(\rho \| D[\rho])$. However, all the coherence cannot be converted to work, as some of the work will be needed to repump the reservoir so that it can be reused. Nevertheless, they showed that the amount of extracted work can be arbitrarily close to the coherence provided when the coherence of the reference system is sufficiently large.

Recently, Vaccaro et al. (2018) analyzed the proposal of Åberg (2014) to check whether coherence can be really used repeatedly without any degradation in its performance. In particular, Vaccaro et al. (2018) showed that if the reservoir is used repeatedly then the final systems can have correlations that have not been considered by Åberg (2014). The correlations between the final systems actually affect the efficiency of repeated operations. To be precise, the coherence drawn from the reservoir by uncorrelated systems is completely different compared to the coherence drawn by correlated systems. Furthermore, they showed that neglecting these correlations among the final systems could lead to unphysical phenomena, for example, two non-orthogonal reservoir states can be discriminated almost perfectly (Vaccaro et al., 2018).

In the context of whether coherence is catalytic, an important study was done by Lostaglio and Müller (2019). In fact, it has been shown that coherence can neither be cloned nor broadcasted for finite dimensional systems. The coherence of a finite-dimensional system $\rho^S$ can be broadcast if there exists a time-translation covariant channel $\Lambda$ and an incoherent state $\rho^S$, such that $[\rho^S, H^S] = 0$ with

$$\Lambda(\rho^E \otimes \rho^S) = \sigma^{ES},$$

where $\rho^E = \text{Tr}_S(\sigma^{ES})$, $\sigma^S = \text{Tr}_E(\sigma^{ES})$ and $[\sigma^S, H^S] \neq 0$. The action of a time-translation covariant channel $\Lambda$ on a state $\rho$ with Hamiltonian $H$ is expressed as $\Lambda(\rho) = \text{Tr}_A[U(\rho \otimes \sigma^A)U^\dagger]$, where $\sigma^A$ is some ancillary state with Hamiltonian $H_A$ and $U$ represents some global unitary with $[U, H + H_A] = 0$. On the other hand, the coherence of the system $\rho^S$ can be cloned when there is no correlation between the sub-systems, i.e.,

$$\Lambda(\rho^E \otimes \rho^S) = \rho^E \otimes \sigma^S.$$

\footnote{If not stated otherwise, we define the von Neumann entropy with the natural logarithm.}
Here, the system $E$ acts like a catalyst. In such a scenario, correlated catalysis under covariant operations cannot broadcast or clone coherence in an initially incoherent state $\rho^S$ (Lostaglio and Müller, 2019). However, the questions of coherence cloning and broadcasting remain open for infinite-dimensional systems. Note that the results of Åberg (2014) suggest that a weaker version of coherence broadcasting, where the state $\rho^F$ is allowed to change in the procedure, is possible for infinite dimensional systems. Hence, it seems like broadcasting as defined earlier in Eq. (61), where the state $\rho^F$ remains unchanged may not be possible.

In the above result, we see that broadcasting of coherence to an initially incoherent state is not possible from a finite-dimensional system (Lostaglio and Müller, 2019). On the contrary, Åberg’s result suggests that it could be possible with infinite-dimensional systems (Åberg, 2014). Recently, Ding et al. (2021) tried to create a connection between the two results by studying the role of a finite-dimensional catalyst in enlarging the set of accessible states under time-translational covariant operations. In their protocol they allow correlation between the system and the catalyst after the transformation, however, the reduced state of the catalyst remains unchanged. As correlation is allowed, they first showed that pure state catalysis do not help in any state transformations under covariant operations. Precisely, if $\rho$ cannot be transformed into $\rho'$ under covariant operations, then there exists no pure catalyst state $|\psi\rangle$ that can catalyse the transformation $\rho \otimes \phi \rightarrow \rho' \otimes \phi$ under covariant catalytic transformations. This result is a generalisation of the previously studied result by Marvian and Spekkens (2013), where $\rho$ and $\rho'$ are considered to be pure. Now if one can allow correlation between the system and the catalyst then there exist coherent states such that the achievable set of states from it is strictly larger under covariant operations. This has been shown explicitly for qubit states (Ding et al., 2021). Furthermore, with the dimension of the catalyst the accessible set of states increases (Ding et al., 2021).

In the same spirit as Nielsen’s theorem in entanglement theory (see Theorem 1), a counterpart has been derived in coherence theory to address the conversions among pure states (Du et al., 2015a; Winter and Yang, 2016; Zhu et al., 2017). Before presenting the result, we introduce incoherent states and incoherent operations. To introduce these, we first fix a basis $\{i\}$ (called as an incoherent basis) in a finite-dimensional Hilbert space. The incoherent states are diagonal in the incoherent basis and can be represented as $\sum_i p_i |i\rangle\langle i|$, where $p_i \geq 0$ and $\sum_i p_i = 1$. The set of all such incoherent states is represented by $I$. A quantum operation characterised by a set of Kraus operators $\{K_a\}$ satisfying $\sum_a K_a^\dagger K_a = \mathbb{1}$, is called incoherent operation if the Kraus operators satisfy $K_a \rho K_a^\dagger \mathcal{E} \{(K_a \rho K_a^\dagger)\} = I$ for all $n$ and $\rho \in I$ (Baumgratz et al., 2014). Let us consider a pair of states: $|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle$ and $|\phi\rangle = \sum_{i=0}^{d-1} \beta_i |i\rangle$, where $\alpha_i$ and $\beta_i$ are arranged in the following way $|\alpha_0|^2 \geq \cdots \geq |\alpha_{d-1}|^2$ and $|\beta_0|^2 \geq \cdots \geq |\beta_{d-1}|^2$. Then $|\psi\rangle$ can be converted to $|\phi\rangle$ via incoherent operations if $\Delta|\psi\rangle < \Delta|\phi\rangle$, or equivalently the following holds (Du et al., 2015a; Winter and Yang, 2016; Zhu et al., 2017)

$$\sum_{i=0}^{d-1} |\alpha_i|^2 \leq \sum_{i=0}^{d-1} |\beta_i|^2 \quad \forall k = \{0, \ldots, d-1\},$$

where $\psi = |\psi\rangle\langle\psi|$ and $\Delta[\rho] = \sum_i \langle i|\rho| i\rangle |i\rangle\langle i|$ represents complete dephasing in the incoherent basis. Note that the above result also holds true for other classes of free operations considered in coherence theory (Winter and Yang, 2016).

Following similar ideas as for entanglement, it is not difficult to find examples of incomparable states where the above conditions are not satisfied. However, in some cases, an additional coherent state can help in accomplishing a forbidden transformation without being changed in the process (Du et al., 2015a). The above result in Eq. (63) sheds some light on the problem of coherence catalysis, similar to the previously discussed cases for entanglement (Jonathan and Plenio, 1999). In fact, maximally coherent states cannot catalyze a forbidden transformation between two arbitrary incomparable pure states (Du et al., 2015a). Furthermore, if a forbidden transformation $|\psi\rangle \rightarrow |\phi\rangle$ can be accomplished with the help of a pure catalyst state, then the reverse transformation $|\phi\rangle \rightarrow |\psi\rangle$ cannot be catalyzed with a pure catalyst state (Du et al., 2015a). Hence, pure catalysts can only catalyse transformations between incomparable pure states. Another important consequence of Eq. (63) is that conversion of $|\psi\rangle$ into $|\phi\rangle$ is possible with the help of a pure catalyst if $|\alpha_0| \leq |\beta_0|$ and $|\alpha_{d-1}| \geq |\beta_{d-1}|$ (Du et al., 2015a).

Bu et al. (2016) studied the necessary and sufficient conditions for catalytic coherence transformations considering pure catalyst states. They derived necessary and sufficient conditions for the existence of a catalyst for a pair of incomparable pure states. Suppose $|\psi\rangle$ and $|\phi\rangle$ are two incomparable states of dimension $d$. Then there exists a catalyst state $|\eta\rangle$ accomplishing the transformation $|\psi\rangle \otimes |\eta\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle$, if and only if (Bu et al., 2016)

$$H_\alpha(\Delta|\psi\rangle) > H_\alpha(\Delta|\phi\rangle) \forall \alpha \in (-\infty, \infty)/0 \text{ and } \text{Tr}(\log \Delta|\psi\rangle) > \text{Tr}(\log \Delta|\phi\rangle).$$

Here

$$H_\alpha(\rho) = \frac{\text{sgn}(\alpha)}{1-\alpha} \log \text{Tr}[(\rho^\alpha) \mathcal{E} \rho].$$

represents Rényi entropy. The strict inequalities in the above equations can be made less strict by allowing an infinitesimally small error in the initial or in the final state. In fact, for arbitrary $\varepsilon > 0$, there exists a state $|\psi_\varepsilon\rangle$ with $||\psi - \psi_\varepsilon||_1 \leq \varepsilon$ such that $|\psi_\varepsilon\rangle$ can be converted into $|\phi\rangle$ via catalytic incoherent operations (assuming pure catalyst states), if and only if (Bu et al., 2016)

$$H_\alpha(\Delta|\psi\rangle) \geq H_\alpha(\Delta|\phi\rangle) \forall \alpha \in (-\infty, \infty)/0 \text{ and } \text{Tr}(\log \Delta|\psi\rangle) \geq \text{Tr}(\log \Delta|\phi\rangle).$$

Note the analogy between this result and the case of entanglement as discussed in Eq. (18). Furthermore, Bu et al.
(2016) explored the structure of qubit catalyst states for four-dimensional state transformations. The necessary and sufficient conditions for the existence of a qubit catalyst $|\psi\rangle = \sqrt{a}|0\rangle + \sqrt{1-a}|1\rangle$ for two four dimensional incomparable states $|\psi\rangle = \sum_{i=0}^{3} \sqrt{\alpha_i}|i\rangle$ and $|\phi\rangle = \sum_{i=0}^{3} \sqrt{\beta_i}|i\rangle$ are

$$a_0 \leq \beta_0,$$

$$a_0 + a_1 > \beta_0 + \beta_1,$$

$$a_0 + a_1 + a_2 \leq \beta_0 + \beta_1 + \beta_2,$$

and additionally

$$\max \left\{ \frac{a_0 + a_1 - \beta_0}{\beta_1 + \beta_2}, 1 - \frac{a_3 - \beta_1}{\beta_2 - a_2} \right\} \leq a \leq \min \left\{ \frac{\beta_0 - a_0}{a_0 + a_1 - \beta_1}, \frac{\beta_0 - a_0}{a_0 + a_1 - \beta_1}, 1 - \frac{\beta_3}{a_2 + a_3} \right\}. \quad (72)$$

Here, without loss of generality we assume that $\{a_i\}$ and $\{\beta_i\}$ are real and they are arranged in descending order. Hence, Eq. (72) provides a range of the catalyst for the above transformation. Similar to the result in entanglement theory (Jonathan and Plenio, 1999), catalytic conversions (assuming pure catalysts) of incomparable pure states are not possible for two and three-dimensional systems, and the first nontrivial result can be found in dimension four as shown in Eq. (72), see (Bu et al., 2016).

So far we have discussed deterministic state transformations. If a pair of states cannot be transformed deterministically, i.e. they violate Eq. (63), they still can be transformed stochastically with some optimal probability $P_{\text{max}}$ (Du et al., 2015b). Bu et al. (2016) studied the advantage of a catalyst in stochastic state transformations. In fact, they showed that for two incomparable pure states $|\phi\rangle = \sum_{i=0}^{d-1} \alpha_i|i\rangle$ and $|\phi\rangle = \sum_{i=0}^{d-1} \beta_i|i\rangle$, there exists a catalyst state $|\psi\rangle$ such that $P_{\text{max}}(|\phi\rangle \rightarrow |\psi\rangle) < P_{\text{max}}(|\psi\rangle \otimes |\psi\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle)$, if and only if

$$P_{\text{max}}(|\psi\rangle \rightarrow |\psi\rangle) < \min \left\{ \frac{|\alpha_{d-1}|^2}{|\beta_{d-1}|^2}, 1 \right\}. \quad (73)$$

Note that a similar result is also present in entanglement theory (Jonathan and Plenio, 1999). By analogy to self-catalysis in entanglement (Duarte et al., 2016), in (Bu et al., 2016), the authors also explored self-catalysis in the context of coherence theory. As an example, consider the states

$$|\phi\rangle = \sqrt{0.05}|0\rangle + \sqrt{0.081}|1\rangle + \sqrt{0.001}|2\rangle + \sqrt{0.009}|3\rangle, \quad (74)$$

$$|\phi\rangle = \sqrt{0.95}|0\rangle + \sqrt{0.053}|1\rangle + \sqrt{0.02}|2\rangle. \quad (75)$$

It is easy to check that $|\phi\rangle \rightarrow |\phi\rangle$, while at the same time $|\phi\rangle \otimes |\phi\rangle \rightarrow |\phi\rangle \otimes |\phi\rangle$ (Bu et al., 2016).

Earlier, we discussed the advantage of catalysis-assisted probabilistic transformation of pure states. Naturally, one may wonder whether a catalyst exists such that it can enhance the probability of transforming a mixed state into a pure state. Recently, Liu and Zhou (2020) explored this scenario for strictly incoherent operations and derived necessary and sufficient conditions for the existence of a pure catalyst such that the optimal transformation probability is enhanced. A quantum operation characterised by Kraus operators $\{K_i\}$ is called a strictly incoherent operation if all $K_i$ and $K_i^\dagger$ are incoherent in the given incoherent basis (Winter and Yang, 2016; Yadin et al., 2016). For such operations Liu and Zhou (2020) showed that $P_{\text{max}}(\rho \otimes \eta \rightarrow |\psi\rangle \otimes |\eta\rangle) > P_{\text{max}}(\rho \rightarrow |\psi\rangle)$ is possible, if and only if the following holds at least for one $\mu$:

$$P_{\text{max}}(\phi^{\mu} \rightarrow |\psi\rangle) < \min \left\{ \frac{\theta^{\mu}_n}{\psi_n}, 1 \right\}. \quad (76)$$

where $\{|\phi^{\mu}\rangle\}$ correspond to the maximally dimensional pure coherent state subspaces of $\rho$, $|\phi^{\mu}\rangle = \sum_{i=0}^{d-1} \phi_i^{\mu}|i\rangle$, $|\phi\rangle = \sum_{i=0}^{d-1} \psi_i|i\rangle$, $n = \max(d_1 - 1, d_2 - 1)$, and $d_1$ and $d_2$ denote the dimensions of $|\phi^{\mu}\rangle$ and $|\psi\rangle$ respectively. The coefficients of the pure states are arranged in descending order. A state $\rho$ has a pure coherent state subspace of dimension $n$ if there exists an incoherent projector $\Pi$ such that $\sum_{i=0}^{d-1} \Pi^{\phi^{\mu}_i} \rho \Pi^{\phi^{\mu}_i} = \phi^{\mu}$ and $\phi^{\mu}$ has coherence rank $n$ (Liu and Zhou, 2020). Coherence rank corresponds to the number of nonzero terms $\phi^{\mu}_i$ in $|\phi^{\mu}\rangle$. The subspace is maximally coherent when the rank of $\phi^{\mu}$ cannot be increased.

Catalytic coherence transformation of some classes of mixed states has been studied by Xing and Yang (2020a). Consider the following rank-2 mixed states of the forms

$$\rho = \alpha |\psi\rangle \langle \psi| + (1 - \alpha) |4\rangle \langle 4| \quad \text{and} \quad (77)$$

$$\sigma = \beta |\psi\rangle \langle \psi| + (1 - \beta) |4\rangle \langle 4|, \quad (78)$$

where

$$|\psi\rangle = \sqrt{0.38} |0\rangle + \sqrt{0.095} |1\rangle + \sqrt{0.095} |2\rangle + \sqrt{0.095} |3\rangle, \quad (79)$$

$$|\phi\rangle = \sqrt{0.05} |0\rangle + \sqrt{0.25} |1\rangle + \sqrt{0.25} |2\rangle. \quad (80)$$

The transformation of $\rho$ into $\sigma$ is not possible via incoherent transformation (Xing and Yang, 2020a). However, as shown in (Xing and Yang, 2020a), the transformation can be achieved with the help of a catalyst in state $|\phi\rangle = \sqrt{0.6} |0\rangle + \sqrt{0.4} |1\rangle$. Note that the above states and the catalytic conversion protocol are similar to the entanglement case discussed in (Eisert and Wilkens, 2000). The above transformation of rank-2 mixed states can be translated to some special class of higher-rank states as well. In addition, for general mixed states catalytic coherence transformation is also discussed in (Xing and Yang, 2020a).

Xing and Yang (2020b) also studied supercatalytic coherence transformation. Suppose we have two incomparable states $|\phi\rangle$ and $|\phi\rangle$, and we want to achieve the transformation $|\phi\rangle \rightarrow |\phi\rangle$. We already see that such transformations can be realized with the help of some auxiliary state if Eqs. (64) and (65) are satisfied. In contrast to this, the transformation can be realized in another way where the auxiliary state changes in the process, but the coherence of the final auxiliary system is increased. More precisely, $|\phi\rangle \otimes |\phi_1\rangle \rightarrow |\phi\rangle \otimes |\phi_2\rangle$, where
\(|\varphi_1\rangle\) and \(|\varphi_2\rangle\) represent auxiliary states with \(C_r(\varphi_2) \geq C_r(\varphi_1)\). Here \(C_r\) represents the relative entropy of coherence (Baumgratz et al., 2014). Examples and some interesting results of such supercatalytic coherence transformation are discussed in (Xing and Yang, 2020b).

C. Quantum thermodynamics

Catalysis has also been broadly explored in quantum thermodynamics, investigating transformations of quantum systems under thermal operations in the presence of catalysts. Thermal operations model transformations of a quantum system \(S\) with Hamiltonian \(H_S\) interacting with a heat reservoir \(B\) with Hamiltonian \(H_B\) and in a thermal state \(\gamma^B = e^{-\beta H_S}/Z_B\), where \(\beta = 1/kT\) is the inverse temperature and \(Z_B = \text{Tr} e^{-\beta H_B}\) is the partition function. The interaction is described by a unitary \(U_{SB}\) that conserves the total energy: \([U_{SB}, H_S + H_B] = 0\) in the absence of catalyst. Moreover, the authors find necessary and sufficient conditions for such transformation to be possible in the thermodynamic limit, transitions are governed by free energy only, and its monotonicity provides the standard second law. Furthermore, Brandão et al. (2015) discussed a generalisation of the above result by considering transformations of arbitrary states and provided some necessary conditions. A state \(\rho\) can be transformed into \(\rho'\) with arbitrary accuracy only if (Brandão et al., 2015)

\[
F_\alpha'(\rho, \gamma_\beta) \geq F_\alpha'(\rho, \gamma_\beta) \forall \alpha \geq \frac{1}{2}, \tag{86a}
\]

\[
F_\alpha'(\gamma_\beta, \rho) \geq F_\alpha'(\gamma_\beta, \rho') \text{ for } \frac{1}{2} \leq \alpha \leq 1, \text{ and,} \tag{86b}
\]

\[
F_\alpha''(\rho, \gamma_\beta) \geq F_\alpha''(\rho', \gamma_\beta) \text{ for } 0 \leq \alpha \leq 2, \tag{86c}
\]

where \(F'\) and \(F''\) are defined as

\[
F'(\rho, \gamma_\beta) = \frac{kT}{\alpha - 1} \log \text{Tr} \left( [\gamma_\beta^{1/\alpha} \rho (\gamma_\beta)^{1/\alpha}] - kT \log Z, \tag{87}\right.
\]

\[
F''(\rho, \gamma_\beta) = \frac{kT \text{sgn}(\alpha)}{\alpha - 1} \log \text{Tr} \left( \rho^{\alpha-1} (\gamma_\beta)^{1-\alpha} \right) - kT \log Z. \tag{88}\]

van der Meer et al. (2017) studied how the above mentioned free energies behave in the asymptotic limit for block-diagonal states. As a result of their study, they found out that if \(F(\rho) > F(\rho')\), for any \(\epsilon > 0\) there exists a sufficiently large \(n\) and a catalytic thermal operation \(\Lambda\), taking \(\rho^{\otimes n}\) arbitrarily close to \(\rho^{\otimes n}\), i.e., \(|\Lambda(\rho^{\otimes n}) - \rho^{\otimes n}|_1 \leq \epsilon\).

In contrast to the previous result by Brandão et al. (2015), Lostaglio et al. (2015b) showed that only the monotonicity of free energy is required for state transformations if one allows correlations among the particles of the catalyst and an arbitrarily small error in the final state \(\rho'\). Their definition of catalytic state transformation, a block-diagonal state \(\rho\) is transformed to another block-diagonal state \(\rho'\) when for all \(\epsilon > 0\), there exists a thermal operation \(\Lambda\), a state \(\rho_\epsilon'\) with \(|\rho' - \rho_\epsilon'| < \epsilon\) and a correlated state \(\sigma_{C_1 \cdots C_n}\) with marginals \(\sigma_{C_1}, \cdots, \sigma_{C_n}\) such that (Lostaglio et al., 2015b)

\[
\Lambda(\rho \otimes \sigma_{C_1} \otimes \cdots \otimes \sigma_{C_n}) = \rho_\epsilon' \otimes \sigma_{C_1 \cdots C_n}. \tag{89}\]

Such a transformation is possible, if and only if \(F(\rho) \geq F(\rho')\) (Lostaglio et al., 2015b). However, note that although the local states of the particles remain the same, the global state of the catalyst changes in this procedure. This may lead to embrazing, a phenomenon discussed in more detail in Section VII.

The above result emphasizes that if a transformation is possible in the thermodynamic limit then it is also possible in the single copy settings. Hence, it provides an operational meaning to free energy. Motivated by this result, Müller and
Pastena (2016) derived a necessary and sufficient condition for the transformations of probability distributions. For this, they studied transformations of probability distributions under bistochastic maps, which are represented by bistochastic matrices that are composed of non-negative real numbers with every individual row and column summing up to one. In particular, Müller and Pastena (2016) considered transformations of probability distributions of the form

$$p \otimes r_1 \otimes \cdots \otimes r_N \rightarrow q \otimes r_{1,\cdots,N},$$

(90)

where the probability distribution $$p = (p_1, \ldots, p_m)$$ is mapped to another probability distribution $$q = (q_1, \ldots, q_m)$$ with the help of auxiliary systems that get correlated in the procedure, but at the same time, the marginals remain unchanged. In such a scenario, if $$p$$ and $$q$$ are not identical up to permutation and both contain non-zero elements, then the transformation from $$p$$ to $$q$$ is possible if and only if $$H(p) < H(q)$$ (Müller and Pastena, 2016), where $$H(p) = -\sum_i p_i \log p_i$$ represents the Shannon entropy.

In most of the previous studies (Brandão et al., 2015; Lostaglio et al., 2015b; van der Meer et al., 2017), only block-diagonal states are considered, which contain no coherence in the eigenbasis of the Hamiltonian. Although the conditions in Eqs. (86) hold for arbitrary states, the role of coherence in state transformations is not apparent from them. On the contrary, in (Lostaglio et al., 2015a) the authors considered arbitrary states and derived the necessary conditions for state transformations under catalytic thermal operations. They considered the exact catalytic thermal operation presented in Eq. (82). If a state $$\rho$$ can be converted to another state $$\rho'$$ via a catalytic thermal operation where the catalyst state is in the block-diagonal form, then we have (Lostaglio et al., 2015a)

$$A_{\alpha}(\rho) \geq A_{\alpha}(\rho') \forall \alpha \geq 0,$$

(91)

where $$A_{\alpha}(\rho) = S_\alpha(p||D[p])$$ represents free coherence of the state $$\rho$$ (Lostaglio et al., 2015a) and $$D$$ corresponds to the dephasing operation in the given energy eigenbasis.

In (Wilming et al., 2017), the authors considered marginal-catalytic transformation and correlated-catalytic transformation under Gibbs preserving operations and showed that free energy is the only monotonic function under such free operations. In their definition, the marginal-catalytic transformation has a similar structure as presented in Eq. (89), with the only difference that the final state is returned exactly and the maps are Gibbs preserving (Wilming et al., 2017). On the other hand, in the case of correlated-catalytic transformation of $$\rho^S$$ into $$\mu^S$$, we have $$\rho^S \otimes \sigma^C \rightarrow \mu^S \otimes \sigma^C$$, where the final state $$\mu^S \otimes \sigma^C$$ is correlated with marginals $$\mu^S$$ and $$\sigma^C$$ (Wilming et al., 2017). Then it has been shown that any monotone under these two kinds of catalytic transformations, which is additive on the tensor product and continuous, is proportional to the free energy difference (Wilming et al., 2017). Hence, free energy is the only monotone for such kind of catalytic transformations.

Recently, the role of exact uncorrelated catalysis for the Markovian thermal process has been investigated in (Korzekwa and Lostaglio, 2022). In fact, catalysis has been demonstrated to be advantageous in the cooling of systems. More specifically, the authors demonstrated that a qubit catalyst can cool a system initially attached to a bath to temperatures lower than the bath temperature, whereas without a catalyst, the system can only be cooled to the bath temperature (Korzekwa and Lostaglio, 2022).

In a recent paper (Son and Ng, 2022), the role of exact uncorrelated catalysis in elementary thermal operations has been discussed. Elementary thermal operations are a subset of thermal operations, and their practical realizations are apparent in comparison to thermal operations (Lostaglio et al., 2018). In (Son and Ng, 2022), the authors provided a few examples to illustrate that for a given initial qutrit state, the set of final states under thermal operations can almost be achieved with elementary thermal operations aided by qubit catalysts. Therefore, as elementary thermal operations can be experimentally realized, this method could provide an experimentally feasible way of achieving the set of final states under thermal operations using only elementary thermal operations and catalysts.

### D. Purity

In the resource theory of purity, a key question is to find the minimum amount of noise required to perform a given physical process on a quantum system (Boes et al., 2018; Gour et al., 2015). In this theory, the free operations are noisy operations. In general, a noisy operation can be implemented in the following steps: i) first, a given system is attached to an ancillary system prepared in a completely mixed state; ii) then, a unitary operation is performed to couple them; and iii) finally, we discard the ancillary system by taking partial trace over the ancillary system. The overall transformation on a state $$\rho^S$$ in the Hilbert space $$\mathcal{H}_S$$ can be written as

$$\Lambda[\rho^S] = \text{Tr}_A \left[ U \left( \rho^S \otimes \mathbb{1}_A \right) U^\dagger \right],$$

(92)

where $$U$$ is in $$\mathcal{H}_S \otimes \mathcal{H}_A$$, $$\mathcal{H}_A$$ represents the Hilbert space of the ancillary system, and $$d_A$$ corresponds to the dimension of the ancillary system. Note that this kind of operation was first reported in (Horodecki et al., 2003a,b). All the states except the maximally mixed state are considered resources in this theory. Furthermore, note that without loss of generality, we can assume the states to be diagonal matrices as unitary operations are free in this resource theory. The necessary and sufficient conditions for exact state transformations are given by the majorization relation. More precisely, a transformation from $$\rho$$ to $$\sigma$$ is possible under a noisy operation if and only if $$\sigma \prec \rho$$ (Gour et al., 2015). In contrast to the entanglement theory, the majorization relation is in the opposite direction.

An immediate consequence is that there exist transformations that are not achievable only via noisy operations but
could be accomplished with the help of a catalyst (Gour et al., 2015). To analyse state transformations under catalytic noisy operations, one needs to consider a trumping relation \( \sigma \prec \rho \), which states that there exists a catalyst \( \eta \) that helps in converting \( \rho \) into \( \sigma \) via noisy operations, i.e., \( \sigma \otimes \eta \prec \rho \otimes \eta \) (Gour et al., 2015). The noisy trumping relation is equivalent to (Gour et al., 2015; Klimesh, 2007)

\[
f_\alpha(\rho) > f_\alpha(\sigma) \quad \forall \alpha \in (-\infty, \infty),
\]

(93)

where the functions \( f_\alpha \) are the extensions of the functions in Eq. (17) to density matrices, i.e.,

\[
\begin{align*}
\ln \Tr[\rho^{\alpha}] & \quad (\alpha > 1); \\
\ln[\Tr[\rho \ln \rho]] & \quad (\alpha = 1); \\
\ln[\Tr[\rho^{\alpha}]] & \quad (0 < \alpha < 1); \\
- \ln \Tr[\rho^{\alpha}] & \quad (\alpha = 0); \\
\ln[\Tr[\rho^{\alpha}]] & \quad (\alpha < 0).
\end{align*}
\]

(94)

Similar to the case of entanglement and coherence, an alternative definition of noisy trumping is introduced (Gour et al., 2015), where an infinitesimally small error is allowed in the initial or final state. More formally a state \( \rho \) trumps noisily \( \sigma \), if there exists a state \( \eta \) such that \( \rho \otimes \eta \rightarrow \sigma \otimes \eta \) for all \( \delta > 0 \). Here \( \nu_\delta = (1/m, \ldots, 1/m, 0, \ldots, 0) \) is a sharp state of dimension \( n \) with \( m \leq n \) and satisfies \( \|\nu_\delta - \nu_n\| \leq \delta \). With this, Gour et al. (2015) provided necessary and sufficient conditions for noisy trumping as follows: A state \( \rho \) noisy trumps \( \sigma \), if and only if

\[
H_\alpha(\rho) \leq H_\alpha(\sigma) \quad \forall \alpha \geq 0,
\]

(95)

where \( H_\alpha \) represent the Rényi entropies defined earlier in Eq. (66). Furthermore, they also explored approximate catalysis where a state \( \rho \) is transformed into \( \sigma \) approximately via noisy operations and a catalyst \( \eta \). However, no further restriction is imposed on the catalyst except that the final state is close to \( \sigma \otimes \eta \). Therefore, the catalyst might change infinitesimally in the procedure. The slight change in the catalyst leads to embezzling, which is discussed in more detail in Section VII. In such a scenario, for a given \( \varepsilon > 0 \) the noisy operations are trivial, i.e., every state can be converted to every other state (Brandão et al., 2015; Gour et al., 2015). We also refer to Section IV for a more detailed analysis of approximate catalysis.

Previously, we discussed the necessary and sufficient conditions for state transformations under catalytic noisy operations. Now, one may wonder what is the minimum amount of randomness required to implement state transitions via noisy operations, and if there is any difference between classical and quantum randomness. Before answering these questions, an understanding of classical and quantum randomness is important. The nature of the source of randomness defines whether it is quantum or classical. A transformation from \( \rho \) to \( \sigma \) using \( m \) units of classical randomness is possible if there exists a set of unitaries \( \{U_i\} \) such that (Boes et al., 2018)

\[
\sigma = \frac{1}{m} \sum_{i=1}^{m} U_i \rho U_i^\dagger
\]

(96)

and the transformation is denoted by \( \rho \xrightarrow{\mathcal{C}} m \sigma \). On the other hand, a transformation from \( \rho \) to \( \sigma \) using \( m \) units of quantum randomness is possible if there exists a unitary \( \mathcal{U} \) and an ancillary system \( A \) of dimension \( m \) in the maximally mixed state \( \mathbb{I}/m \) such that (Boes et al., 2018)

\[
\sigma = \Tr_A \left( \rho \otimes \frac{\mathbb{I}}{m} \right) \mathcal{U}^\dagger,
\]

(97)

and the transformation is denoted by \( \rho \xrightarrow{\mathcal{Q}} m \sigma \). When there is no bound on the amount of available randomness (quantified by \( m \)), the set of transitions that can be achieved with quantum randomness sources can also be achieved with classical randomness sources (Boes et al., 2018; Marshall et al., 2011). Equivalently we have,

\[
\rho \xrightarrow{\mathcal{Q}} \sigma \iff \rho \xrightarrow{\mathcal{C}} \sigma \iff \rho > \sigma.
\]

(98)

The transformations of the form (97) in the limit \( m \to \infty \) have been previously studied in (Gour et al., 2015), where noise is treated as free or in other words we can use a maximally mixed state of arbitrary dimension to implement a noisy operation. On the contrary, Boes et al. (2018) treated noise as a resource and investigated the minimum amount of noise required to accomplish a given state transformation. For a given Hilbert space \( \mathcal{H}_S \) of dimension \( d \), the minimum amount of noise or randomness can be quantified as

\[
m^*_X(d) = \arg \min_{\rho} \left\{ \frac{X}{m} \sigma \mid \rho, \sigma \in \mathcal{H}_S \text{ with } \rho > \sigma \right\},
\]

(99)

where \( X \) corresponds to either \( C \) or \( Q \). It is important to note that for a specific transition, the required randomness could be smaller than the above-specified value in Eq. (99). Boes et al. (2018) provided an analytical expression for these quantities as following

\[
m^*_C(d) = d,
\]

(100)

\[
m^*_Q(d) = \lceil \sqrt{d} \rceil.
\]

(101)

Here, we will focus mainly on quantum randomness. In proving the optimality of \( m^*_Q \), the authors of (Boes et al., 2018), used the Schur-Horn lemma (Horn, 1954) with the majorisation relation \( \rho > \sigma \), which states that for every state transition under noisy operation there exists a map of the form

\[
\sigma = \mathcal{L}(\rho) = \mathcal{U}' \circ \Delta_B \circ \mathcal{U}(\rho)
\]

(102)

where, \( \mathcal{U}, \mathcal{U}' \) are unitary channels that are specific to the states \( \rho \) and \( \sigma \). Here \( \Delta_B \) represents a dephasing map in a fixed basis \( B = \{|i\rangle\}_{i=1}^d \) of dimension \( d \) and defined as \( \langle i | \Delta_B(\rho) | j \rangle = \langle i | \rho | j \rangle \delta_{ij} \). As the implementations of unitary channels do not require any noise, all the noise needed for a state transition is required to implement the dephasing map \( \Delta_B \) in Eq. (102). Next, Boes et al. (2018) provided an explicit protocol implementing dephasing map \( \Delta_B \) in any basis \( B \), using a noise model and requiring \( m^*_Q(d) = \lceil \sqrt{d} \rceil \). Specifically,
for any dimension $d$ and basis $B$, Boes et al. (2018) showed that there exists a unitary $U$, such that
\[
\text{Tr}_A \left[ U \left( \rho \otimes \frac{I}{\sqrt{d}} \right) U^\dagger \right] = \Delta_B(\rho),
\]
\[
\text{Tr}_S \left[ U \left( \rho \otimes \frac{I}{\sqrt{d}} \right) U^\dagger \right] = \frac{1}{\sqrt{d}} \forall \rho.
\]

Although the final state may have correlations between the system and the ancilla, the second condition implies that the ancillary system $A$ is returned exactly in the same state as before and thus acts as a catalyst in the process. As mentioned previously, Boes et al. (2018) considered noise as a resource, and interestingly, here the noise is not consumed in the process and can be reused again. Furthermore, note that Eq. (102) emphasises that every possible state transition utilising randomness can be accomplished if a dephasing map can be constructed and Boes et al. (2018) provided an explicit protocol for that.

In (Boes et al., 2018), only maximally mixed states have been used as a source of randomness. In contrast, Lie et al. (2021) considered any arbitrary randomness source $\eta$ and derived a necessary and sufficient condition such that the randomness source $\eta$ can be used to dephase an arbitrary input state $\rho$. In fact, it has been shown that any randomness source $\eta$ can dephase a $d^2$ dimensional input states catalytically if the following holds (Lie et al., 2021)
\[
S_{\min}(\eta) \geq \log_2 d,
\]
where $S_{\min}(\eta) = -\log_2 ||\eta||$ represents min-entropy and the operator norm $||\eta||$ corresponds to the largest singular value of $\eta$. However, note that in their formulation, the state of the catalyst $\eta$ can change, but the same catalyst can be used repeatedly to dephase arbitrary input states of the same dimension. In validation of this, the authors of (Lie et al., 2021) showed that randomness described by min-entropy is not consumed in the process and can be reused.

Based on the result in (Boes et al., 2018), Boes et al. (2019) proved that von Neumann entropy completely characterises single-shot state transitions under unitary operations. To achieve this, the authors in (Boes et al., 2019) used an ancillary system as a catalyst and an environment which dephases a system in a preferred basis. In particular, the following has been shown (Boes et al., 2019).

**Theorem 4** (See Theorem 1 in (Boes et al., 2019)). For any two states $\rho$ and $\sigma$ of the same finite dimension and different spectra, the following statements are equivalent:
(i) $S(\sigma) > S(\rho)$ and $\text{rank}(\sigma) \geq \text{rank}(\rho)$
(ii) There exists a unitary $U$ and a finite dimensional catalyst in the state $\eta$, which can be chosen to be diagonal in any basis $B$ such that
\[
\text{Tr}_A \left[ U \left( \rho \otimes \eta \right) U^\dagger \right] = \sigma,
\]
\[
\Delta_B(\text{Tr}_S \left[ U \left( \rho \otimes \eta \right) U^\dagger \right]) = \eta.
\]
Note that the result corresponds to Eqs. (103) and (104) is a special case of the above theorem. Naturally, one may wonder whether the dephasing map given in Eq. (107) is at all necessary to implement all state transitions such that the von Neumann entropy does not decrease. In support of this, Boes et al. (2019) provided the following catalytic entropy conjecture.

**Conjecture 1** (See Conjecture 1 in (Boes et al., 2019)). For any two states $\rho$ and $\sigma$ of the same finite dimension and different spectra, the following statements are equivalent:
(i) $S(\sigma) > S(\rho)$ and $\text{rank}(\sigma) \geq \text{rank}(\rho)$
(ii) There exists a finite dimensional state $\eta$ and a unitary $U$ such that,
\[
\text{Tr}_A \left[ U \left( \rho \otimes \eta \right) U^\dagger \right] = \sigma,
\]
\[
\text{Tr}_S \left[ U \left( \rho \otimes \eta \right) U^\dagger \right] = \eta.
\]
This conjecture has been proven very recently by Wilming (2022). In fact, Wilming (2022) showed the following: Let $\rho$ and $\rho'$ be two states of dimension $d$ in $\mathcal{H}_A$ that are not unitarily equivalent. Then there exist a natural number $n$ and a state $\rho'_n$ on $\mathcal{H}_d^{\otimes n}$ such that
\[
\rho_{\text{out}} \propto \rho'_n \quad \text{and} \quad \text{Tr}_{[1,\ldots,n]} \rho'_n = \rho' \forall i \in \{1,\ldots,n\}
\]
holds true if and only if $S(\rho) < S(\rho')$ and $\text{rank}(\rho) \leq \text{rank}(\rho')$. The validity of the catalytic entropy conjecture is a consequence of this result (Wilming, 2022).

Recently, in (Lie and Jeong, 2021a,b), randomness to implement quantum channels has been explored. In particular, Lie and Jeong (2021b) used an ancillary system as a source of randomness such that when this ancillary system is used to implement a quantum channel no information is leaked into it. They named this a randomness-utilising process, as no information is leaked into the ancillary system while implementing a quantum channel. As a consequence, the ancillary system after the implementation of the process should not depend on the input state of the channel. A randomness-utilising channel on $\mathcal{B}(\mathcal{H}_S)$ can be expressed as (Lie and Jeong, 2021b)
\[
\Lambda(\rho) = \text{Tr}_A \left[ U \left( \rho \otimes \sigma \right) U^\dagger \right],
\]
with a randomness source $\sigma$ on $\mathcal{H}_A$, some unitary $U$ on $\mathcal{H}_S \otimes \mathcal{H}_A$, and the state $\text{Tr}_S \left[ U(\rho \otimes \sigma) U^\dagger \right]$ is independent of the input state. Lie and Jeong (2021b) called this state of the ancillary system after the implementation of the randomness-utilising process residue randomness. If the residue randomness has the same spectrum as $\sigma$, the process is called catalytic. Hence, the randomness source can be reused to implement the same process on another input state. Equipped with this Lie and Jeong (2021b) showed that any dimension-preserving randomness-utilising process is catalytic. By “dimension-preserving” the authors of (Lie and Jeong, 2021b) meant that the input and output systems are of the same dimension. Naturally one may wonder whether all dimension-preserving channels can be implemented by randomness-utilising process.
turns out that only unital quantum channels can be implemented using randomness-utilising process (Lie and Jeong, 2021b). A unital quantum channel $\Lambda_P$ keeps the identity operator invariant, i.e., $\Lambda_P(\mathds{1}) = \mathds{1}$. However, whether all unital quantum channels are catalytic, or in other words, all unital quantum channels can be implemented using randomness-utilising processes, remained open in (Lie and Jeong, 2021b). A negative answer to this problem is given in (Lie and Jeong, 2021a) by showing that not all unital quantum channels can be implemented using randomness-utilising processes. Furthermore, Lie and Jeong (2021a) defined catalytic entropy to quantify the maximum amount of randomness that can be extracted catalytically from a randomness source $\sigma$. The catalytic entropy of a randomness source $\sigma$ with eigenspace decomposition $\sigma = \sum_i \lambda_i P_i$ is given in (Lie and Jeong, 2021a) by

$$S_C(\sigma) := -\sum_i \lambda_i r_i \log_2 \frac{\lambda_i}{r_i}. \quad (112)$$

Here, $P_i$ are the orthogonal projectors associated with the eigenvalue $\lambda_i$ and satisfy $P_i P_j = 0$ if $\lambda_i \neq \lambda_j$, and $r_i = \text{Tr} P_i$ represents degeneracy of $\lambda_i$. $S_C(\sigma)$ is the maximum amount of entropy or randomness that can be produced from $\sigma$ (Lie and Jeong, 2021a). In another recent work, the authors extended the role of catalytic quantum randomness for dynamical maps (Lie and Jeong, 2022).

Henao and Uzdin (2022) explored the role of catalysts in reducing the impact of information erasure on the environment. In any information erasing process, an initial state $\rho_s$ of a system $s$ with von Neumann entropy $S(\rho_s)$ is taken to another state $\rho'_s$ that has less entropy than the initial state $\rho_s$, i.e., $\Delta S_s = S(\rho_s) - S(\rho'_s) > 0$, and the change in entropy $-\Delta S_s$ is dumped as heat in the environment. For a finite-dimensional environment, the authors in (Henao and Uzdin, 2022) showed that the correlation between the system and the environment can be manoeuvred catalytically to reduce the entropy or heat waste in the environment. To set the scene, we introduce an environment $e$ in state $\rho_e$ that is attached to the state $\rho_s$ to reduce the entropy of the system $s$. By applying a global unitary transformation $U_{se}$ on the system and the environment, the uncorrelated state $\rho_s \otimes \rho_e$ is transformed to a final state $\sigma_{se} = U_{se}(\rho_s \otimes \rho_e) U_{se}^\dagger$, such that the entropy of the system decreases, i.e., $S(\rho_s) - S(\rho'_s) < 0$, where $\sigma_s = \text{Tr}_e \sigma_{se}$. As a result, the heat dumped in the environment is $Q_e = \text{Tr}[H_e(\sigma_s - \rho_s)]$, where $H_e$ represents the Hamiltonian associated with the environment and $\sigma_s = \text{Tr}_e \sigma_{se}$. Now with the help of a catalyst $c$ in the state $\rho_c$, Henao and Uzdin (2022) aimed to reduce the heat dissipation in the environment to a value $Q'_e$ such that $Q'_e < Q_e$. This leads to a two-step process where $\rho_s \otimes \rho_e$ is transformed to $\sigma_{se}$ and then $\sigma_{se}$ is transformed to $\rho'_{se}$ with the help of a suitable catalyst state $\rho_c$ and a global unitary $U_{sec}$ such that $Q'_e = \text{Tr}[H_e(\rho'_s - \rho'_e)] < Q_e$. Here, $\rho'_se = U_{sec}(\sigma_{se} \otimes \rho_c) U_{sec}^\dagger$ and $\rho'_e = \text{Tr}_s \rho'_{sec}$. Apart from ensuring $\rho'_s < \text{Tr}_e \rho'_{sec} = \sigma_s$, we need to make sure that the transformation is catalytic, i.e., $\text{Tr}_s \rho'_{sec} = \rho_c$. With this, Henao and Uzdin (2022) proved the following: There exists a unitary transformation $U_{se}$ and a catalyst state $\rho_c$ such that $\rho'_s > \sigma_s$, if and only if $\sigma_{se}$ is a correlated state of the system $s$ and the environment $e$. Furthermore, they showed that if $\rho'_s$ majorises $\sigma_s$, then $Q_e \geq Q'_e$ (Henao and Uzdin, 2022). Note that as the unitary is free in this process, the catalyst is the only party responsible for the reduction in heat dissipation to the environment. Furthermore, they showed $S(\rho'_s) \leq S(\sigma_s)$ which reflects that the entropy change of the environment can also be reduced. Therefore, in any information erasure process involving a correlated state $\sigma_{se}$, it is possible to reduce the heat and entropy dissipation to the environment by using a suitable catalyst state.

E. Other quantum resource theories

Symmetry arguments play a crucial part in understanding many important results in physics. For example, in many cases when the evolution of a system under some dynamics gets complicated to analyse, one can look for symmetry arguments to derive some useful results. In quantum theory, Marvian and Spekkens (2013) explored the role that symmetry plays in shaping the dynamics of a quantum system. More precisely, they explored state transformations in the context of time evolution with a given symmetry. To analyse this feature systematically, they introduced the resource theory of asymmetry in (Marvian and Spekkens, 2013). Suppose a group $G$ with elements $g$ represents a set of symmetry transformations. The action of each group element $g$ is described by a unitary transformation $U_g$. In this resource theory, the free states are those states that remain invariant under the action of all $g$, i.e., $\mathcal{U}_g(\rho) = \rho \forall g \in G$, where $\mathcal{U}_g(\rho) = U_g(\rho) U_g^\dagger$. In addition, a quantum operation $\Lambda$ is free if it respects the symmetry, i.e.,

$$\Lambda[U_g(\rho(U_g^\dagger)] = U_g^\dagger \Lambda[\rho](U_g^\dagger) \quad \forall g \in G, \quad (113)$$

where $U_g$ and $U_g^\dagger$ represent the input and output representation of $G$, respectively. These types of operations are named as $G$-covariant operations in the literature (Marvian and Spekkens, 2013).

In such a resource theory, Marvian and Spekkens (2013) explored the role of catalysis for pure state transformations. If a state $|\psi\rangle$ cannot be transformed into $|\phi\rangle$ under any deterministic $G$-covariant operation but the same transformation can be achieved deterministically with a $G$-covariant operation and a catalyst in a state $|\eta\rangle$ such that $|\psi\rangle \otimes |\eta\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle$, then such a transformation from $|\phi\rangle$ to $|\psi\rangle$ has been called a nontrivial example of catalysis (Marvian and Spekkens, 2013). With this Marvian and Spekkens (2013) showed that no nontrivial example of catalysis is possible using a finite-dimensional catalyst when the symmetries are associated with compact connected Lie groups. However, this does not work for finite groups and a nontrivial example of catalysis has been shown for finite groups (Marvian and Spekkens, 2013).

Recently Rethinasamy and Wilde (2020) studied the role of catalysis in relative majorization for classical probability distributions. A pair $(p', q')$ of probability distributions $p'$ and $q'$ is relatively majorised by another pair $(p, q)$ of probability
distributions \( p \) and \( q \), i.e., \( (p, q) > (p', q') \) if and only if there exists a classical channel or a doubly stochastic matrix that transforms \( p \) and \( q \) to \( p' \) and \( q' \), respectively (Buscemi and Gour, 2017; Dahl, 1999; Renes, 2016). In (Rethinasamy and Wilde, 2020), the authors reformulated the above-mentioned transformation protocol in the presence of a catalyst pair (Rethinasamy and Wilde, 2020), allowing an infinitesimally small error on the target probability distribution \( r \). As the catalyst is returned exactly it can be reused to transform another independent set of \( p, q \). With this, the authors in (Rethinasamy and Wilde, 2020) showed that relative entropy completely determines exact catalytic pair transformations. Suppose \( p, q, p', q' \) are probability distributions on the same alphabet with \( (p, q) \) and \( (p', q') \) having different relative spectra, and \( q', q'' \) have only rational entries and full support. Then the following two statements are equivalent: i) \( D(p||q) > D(p'||q') \) and \( D_{0}(p||q) \geq D_{0}(p'||q') \). ii) For all \( \delta > 0 \), there exists a probability distribution \( r \) and a joint probability distribution \( t' \) with marginals \( p' \) and \( r \) such that \( (p \otimes r, q \otimes v) > (t', q' \otimes v) \), \( D(t'||p \otimes r) < \delta \) (Rethinasamy and Wilde, 2020). Here \( D(p||q) = \sum_{x} p(x) \ln(p(x)/q(x)) \), \( D_{0}(p||q) = -\ln \sum_{x} q(x) \), and \( v \) is a uniform distribution that is used in place of \( s \). The relative spectrum of a pair \( (p, q) \) is defined as \( \{ p(x)/q(x) \} \). (Rethinasamy and Wilde, 2020). Therefore, relative entropy completely characterises exact transformations under correlated catalysis.

Note that the correlation can be made arbitrarily small such that \( t' \) is almost the same as \( p' \otimes r \). The constraints on different relative spectra, \( q, q' \) are rational, and \( D_{0}(p||q) \geq D_{0}(p'||q') \) can be lifted by allowing an infinitesimally small error on the target probability distribution \( p' \) (Rethinasamy and Wilde, 2020). Suppose \( p, q, p', q' \) are probability distributions on the same alphabet with \( q, q' \) have full support. Then the following two statements are equivalent: i) \( D(p||q) \geq D(p'||q') \). ii) For all \( \delta > 0 \) and \( \epsilon > 0 \), there exists a probability distribution \( r \) and a joint probability distribution \( t'_r \) with marginals \( p'_r \) and \( r \) such that \( (p \otimes r, q \otimes v) > (t'_r, q' \otimes v) \), \( \| p'_r - p' \| \leq 2\epsilon \), and \( D(t'_r||p'_r \otimes r) < \delta \). Here, \( \| p - q \|_1 = \sum_{i,j} |p_{ij} - q_{ij}| \). Considering \( q \) and \( q' \) to be the uniform distributions, a special case arises. Suppose \( p, p' \) are probability distributions on the same alphabet with \( p \neq \text{Per}(p') \) for all permutation matrices \( \text{Per} \). Then the following two statements are equivalent: i) \( H(p) \geq H(p') \). ii) For all \( \delta > 0 \) and \( \epsilon > 0 \), there exists a probability distribution \( r \) and a joint probability distribution \( t'_r \) with marginals \( p'_r \) and \( r \) such that \( (p \otimes r, q \otimes v) > (t'_r, q' \otimes v) \), \( \| p'_r - p' \| \leq 2\epsilon \), and \( D(t'_r||p'_r \otimes r) < \delta \) (Rethinasamy and Wilde, 2020). Here \( H \) represents the Shannon entropy of a probability distribution. Therefore, a single entropic condition completely dictates the approximate catalytic relative majorisation of classical probabilities. Note that this result can be applied to other quantum resource theories, for example, purity, quasi-classical thermodynamics, and asymmetric distinguishability (Rethinasamy and Wilde, 2020).

The role of catalysis has also been explored in the framework of Gaussian thermal resource theory (Yadin et al., 2022). Let us consider an \( n \)-mode Bosonic continuous variable quantum system on the Hilbert space \( \mathcal{H} = \otimes_{k=1}^{n} \mathcal{H}_k \), with corresponding Bosonic creation and annihilation field operators \( \hat{a}_k \) and \( \hat{a}_k^\dagger \), respectively. From the field operators, the quadrature operators (or position and momentum-like operators) are constructed as \( \hat{x}_k = (\hat{a}_k + \hat{a}_k^\dagger) / \sqrt{2} \) and \( \hat{p}_k = i(\hat{a}_k^\dagger - \hat{a}_k) / \sqrt{2} \). For convenience, all the quadrature operators are grouped together to make a column vector \( \hat{r} = (\hat{x}_1, \hat{p}_1, \cdots, \hat{x}_n, \hat{p}_n)^T \). For a given quantum state \( \rho \), the corresponding first and second moments are defined as (Weedbrook et al., 2012)

\[
\begin{align*}
\mathbf{r} & = (\langle \hat{x}_1 \rangle_\rho, \langle \hat{p}_1 \rangle_\rho, \cdots, \langle \hat{x}_n \rangle_\rho, \langle \hat{p}_n \rangle_\rho)^T \\
\mathbf{\sigma}_{ij} & = \frac{\langle [\hat{r}_i, \hat{r}_j]^T \rangle_\rho}{2} - \langle \hat{r}_i \rangle_\rho \langle \hat{r}_j \rangle_\rho \tag{114}
\end{align*}
\]

respectively, where the second moments \( \mathbf{\sigma}_{ij} \) are the elements of the covariance matrix \( \mathbf{\sigma} \). Gaussian states are completely characterized by their first moments and covariance matrix (Weedbrook et al., 2012).

After defining Gaussian states, we introduce Gaussian operations that preserve Gaussianity, or, in other words, take Gaussian states to Gaussian states. Generally, a Gaussian operation can be realised by allowing an interaction between an environment in a Gaussian state and a given system through a global Gaussian unitary and then performing a projective measurement on the environment (Weedbrook et al., 2012). In the case of Gaussian thermal operations, the global unitary and the state of the environment should be energy-preserving and in a thermal state respectively. As Gaussian states are completely characterised by \( \mathbf{r} \) and \( \mathbf{\sigma} \), it is enough to study the transformations of these two under Gaussian thermal operations. In (Yadin et al., 2022), the authors considered Gaussian states with \( \mathbf{r} = 0 \) and studied the transformation of \( \mathbf{\sigma} \) under strong and weak catalytic Gaussian thermal operations. Note that the covariance matrix can be characterised by two more matrices \( M \) and \( A \) with elements \( M_{ij} = \langle a_j^\dagger a_i \rangle - \delta_{ij} \) and \( A_{ij} = \langle a_i a_j \rangle \), respectively (Simon et al., 1994) and this has been used to study state transformations under Gaussian thermal operations (Serafini et al., 2020; Yadin et al., 2022). Here \( \nu = (\omega^2 + 1)/(\omega^2 - 1) \), \( \omega \) is the eigenfrequency of the modes and \( \beta = 1/kt \) is the inverse temperature. A strong catalytic Gaussian transformation of \( \rho \) to \( \rho' \) is possible when there exists a catalyst state \( \sigma \) such that \( \rho \otimes \sigma \rightarrow \rho' \otimes \sigma \) is achieved under Gaussian thermal operation. On the other hand, a weak catalytic Gaussian transformation of \( \rho \) to \( \rho' \) is possible when there exists a catalyst state \( \sigma \) and a joint final state \( \rho^F \) such that \( \rho \otimes \sigma \rightarrow \rho^F \) with \( \text{Tr}_G \nu = \nu \) and \( \text{Tr}_C \nu = \nu' \) is accomplished under Gaussian thermal operation. Equipped with these tools, the authors in (Yadin et al., 2022) studied transformations of single-mode Bosonic Gaussian states. The transition from \( \sigma \) to \( \sigma' \) under strong catalytic Gaussian thermal operation is possible if and only if there exists a \( p \in [0, 1] \).
such that the conditions
\[ \mu' = p \mu \text{ and } \alpha' = q \alpha \] (116)
hold (Yadin et al., 2022). Here \( \sigma \) and \( \sigma' \) are the covariance matrices of a single-mode Bosonic Gaussian state. For an \( n \)-mode Bosonic state, \( \mu \) and \( \alpha \) correspond to the eigenvalues and singular values of the matrix \( M \) and \( A \), respectively. Note that the conditions in Eq. (116) are exactly the same as the non-catalytic transition presented in (Serafimi et al., 2020). Hence, strong catalysis does not provide any advantage in this scenario. Now, if we allow correlations between the system and the catalyst then in the weak catalytic scenario, the above transition is possible if and only if there exists \( p, q \in [0, 1] \) with \( p \geq q \) such that the conditions
\[ \mu' = p \mu \text{ and } \alpha' = q \alpha \] (117)
are satisfied (Yadin et al., 2022). Furthermore, in (Yadin et al., 2022) multi-mode state transformations have been discussed as well.

The role of quantum catalysts in the distillation of magic states has also been investigated recently (Campbell, 2011). Magic states have been considered to be one of the key ingredients for achieving fault-tolerant quantum computation (Bravyi and Kitaev, 2005; Campbell et al., 2017; Knill, 2004). Generally, prepared states are noisy, and one needs to distill pure magic states from those to achieve fault-tolerant quantum computation. In magic state distillation a smaller number of more useful pure magic states have been distilled from multiple noisy quantum states (Bravyi and Kitaev, 2005; Campbell et al., 2017; Knill, 2004; Veitch et al., 2014). Quantum catalysis has been shown to be useful in magic state distillation as well (Campbell, 2011). Consider a state \( |\psi\rangle \propto |H_{0,0,0}\rangle + |H_{1,1,1}\rangle \), where \( |H_{0,0,0}\rangle = |H_0\rangle \otimes |H_0\rangle \otimes |H_0\rangle, |H_{1,1,1}\rangle = |H_1\rangle \otimes |H_1\rangle \otimes |H_1\rangle, |H_0\rangle \) and \( |H_1\rangle \) are qubit magic states and also eigenvectors of the Hadamard matrix. Note that \( |\psi\rangle \rightarrow |H_0\rangle \) cannot be achieved deterministically under Clifford unitary operations (Campbell, 2011). However, if we have an additional state \( |H_0\rangle \) in possession, the following transformation \( |\psi\rangle \otimes |H_0\rangle \rightarrow |H_0\rangle \otimes |H_0\rangle \) can be achieved deterministically with Clifford unitary operations (Campbell, 2011).

Another important aspect in the field of quantum computation is quantum error correction. Recently, the role of catalysis in it has been discussed by Brun et al. (2014). Consider the following time evolution channel \( N : L^{0n} \rightarrow L^{0k} \) between Alice and Bob, which takes quantum states on Alice’s system as inputs and provides quantum states on Bob’s system as outputs. Here \( L \) represents the space of linear operators acting on the qubit Hilbert space \( H \). Note that here Alice and Bob are separated in time. Suppose Alice and Bob also have access to a noiseless channel \( id^{0c} \), and Alice has an initial quantum state of \( k \) qubits. Equipped with this a catalytic quantum error correcting code can be defined as the following: i) Alice performs an encoding operation \( E : L^{0n} \otimes id^{0c} \rightarrow L^{0k} \); ii) After the time evolution Bob performs a decoding operation \( D : L^{0m} \otimes id^{0k} \rightarrow L^{0k} \) such that
\[ D \circ (N \otimes id^{0c}) \circ E = id^{0k} = id^{0k-c} \otimes id^{0c}. \] (118)
The net rate of this error correcting code is \( (k - c)/n \) and the noiseless channel \( id^{0c} \) served as a catalyst in the process. If we additionally have access to \( m \) copies of the channel \( N \), then the following result has been proved (Brun et al., 2014):
If Eq. (118) holds, then for any arbitrary integer \( m \geq 0 \) there exists an encoding operation \( E_m \) and a decoding operation \( D_m \) for the channel \( N^{0m} \) such that
\[ D_m \circ (N^{0m} \otimes id^{0c}) \circ E_m = id^{0m(k-c)} \otimes id^{0c}. \] (119)

The above result shows that the size of the catalyst can be reduced from \( mc \) to \( c \).

In a recent study (Karvonen, 2021), the role of catalysis has been explored in the resource theory of contextuality (Budroni et al., 2021; Kochen and Specker, 1967) and non-locality (Bell, 1966; Brunner et al., 2014). Karvonen (2021) considered the following kind of catalytic transformation: \( p \otimes c \leftrightarrow q \otimes c \), where \( p, q, c \) represent some correlations, and \( q \) can be simulated (\( \leftrightarrow \)) from \( p \) with the help of a catalyst \( c \) that remains intact in the process. Based on this, Karvonen (2021) proved that catalysis does not help in the resource theory of contextuality and non-locality, or, in other words, there exists no correlation that can be used to catalyse an otherwise impossible transformation. More precisely, if \( p \otimes c \leftrightarrow q \otimes c \) is possible in the resource theory of contextuality and non-locality, then \( p \leftrightarrow q \) is also possible (Karvonen, 2021).

IV. APPROXIMATE CATALYSIS

The definition of exact catalysis, initially introduced for the resource theory of entanglement (Jonathan and Plenio, 1999), does not allow for any errors in the transformation. In recent years, this notion has been extended, allowing for an error in the initial or final state (Brandão et al., 2015; Lostaglio et al., 2015b) while at the same time requiring that the catalyst be completely decoupled from the primary system at the end of the process. In parallel, there has been a growing interest in correlated catalysis, allowing the catalyst to build up correlations with the system, which may also be preserved at the end of the procedure (Åberg, 2014; Boes et al., 2019; Kondra et al., 2021a; Müller, 2018; Shiraiishi and Sagawa, 2021, 2022; Wilming, 2021; Wilming et al., 2017).

Recently, the framework of approximate catalysis has been proposed in (Datta et al., 2022; Kondra et al., 2021a). While this framework allows for correlations between the main system and the catalyst, it additionally requires that these correlations can be made arbitrarily small. In particular, an approximate catalytic transformation between two states \( \rho^S \) and \( \sigma^S \) of a system \( S \) is possible if and only if for any \( \epsilon > 0 \) there exists a catalyst state \( \tau^C \) and a free operation \( \Lambda_f \) acting on the system and the catalyst such that (Datta et al., 2022; Kondra et al., 2021a)
\[ \left\| \Lambda_f (\rho^S \otimes \tau^C) - \sigma^S \otimes \tau^C \right\|_1 \leq \epsilon. \] (120)
\[
\text{Tr}_S \left[ \Lambda_f \left( \rho^S \otimes \tau^C \right) \right] = \tau^C. \tag{121}
\]

In contrast to the exact catalysis discussed in Section III, the above definition allows for correlations between the system and the catalyst in the final state \(\rho^C = \Lambda_f (\rho^S \otimes \tau^C)\). However, Eq. (120) ensures that these correlations can be made arbitrarily small by choosing an appropriate state of the catalyst \(\tau^C\) and a free operation \(\Lambda_f\). The condition in Eq. (121) guarantees that the state of the catalyst does not change in the procedure. This ensures that the catalyst can be reused for the same transformation in the future. Moreover, this requirement makes it impossible to “embezzle” quantum resources by using small changes of the catalyst state, see also Section VII.

An alternative framework for approximate catalysis has been proposed in (Müller, 2018; Rethinasamy and Wilde, 2020; Shiraishi and Sagawa, 2022). While Eq. (120) represents that the correlations between the system and the catalyst can be made arbitrarily small as quantified by the trace norm, it is possible to formulate a similar requirement in terms of mutual information \(I^{S:B}(\rho^{AB}) = S(\rho^A) + S(\rho^B) - S(\rho^{AB})\). Then, an approximate catalytic transformation from \(\rho^S\) into \(\rho^S\) is possible if and only if for any \(\epsilon > 0\) and any \(\delta > 0\) there exists a free operation \(\Lambda_f\) and a catalyst state \(\tau^C\) such that (Müller, 2018; Rethinasamy and Wilde, 2020; Shiraishi and Sagawa, 2022)

\[
\left\| \text{Tr}_C \left[ \Lambda_f \left( \rho^S \otimes \tau^C \right) \right] - \sigma^S \right\| _1 \leq \epsilon, \tag{122}
\]

\[
\text{Tr}_S \left[ \Lambda_f \left( \rho^S \otimes \tau^C \right) \right] = \tau^C, \tag{123}
\]

\[
I^{S:C} \left[ \Lambda_f \left( \rho^S \otimes \tau^C \right) \right] \leq \delta. \tag{124}
\]

Recently, Rubboli and Tomamichel (2022) demonstrated equivalence between this definition and the definition in Eqs. (120) and (121).

Note that the term “approximate catalysis” has been previously used by Gour et al. (2015) in the resource theory of purity, referring to transformations which fulfill Eq. (120) for any \(\epsilon > 0\). Since the condition in Eq. (121) is not imposed, this type of transformations leads to embezzling phenomena (Gour et al., 2015), see also Section VII. Throughout this article, whenever we refer to approximate catalysis we mean transformations which obey Eqs. (120) and (121).

In the following, we will review recent results on approximate catalysis for concrete resource theories, including quantum thermodynamics, entanglement, and quantum coherence.

### A. Quantum thermodynamics

Approximate catalysis has been first investigated in quantum thermodynamics by Müller (2018). For the case that the initial and the final state are both block diagonal in the energy eigenbasis, it has been shown that state transformations are completely characterized by the Helmholtz free energy. In particular, a block diagonal state \(\rho\) can be converted into another block diagonal state \(\sigma\) via thermal operations with approximate catalysis if and only if (Müller, 2018)

\[
F(\rho) \geq F(\sigma). \tag{125}
\]

Shiraishi and Sagawa (2021, 2022) have investigated a closely related setting, replacing thermal operations with the more general class of Gibbs-preserving operations. In this case, Eq. (125) completely characterizes transformations between all quantum states, leading to the following theorem.

**Theorem 5** (See Theorem 1 in (Shiraishi and Sagawa, 2021)). A state \(\rho\) can be converted into another state \(\sigma\) via Gibbs-preserving operations with approximate catalysis if and only if \(F(\rho) \geq F(\sigma)\).

It is possible to express the Helmholtz free energy in terms of the quantum relative entropy as follows:

\[
F(\rho) = kT[S(\rho||\gamma) - \log Z], \tag{126}
\]

where \(Z\) is the partition function and \(\gamma\) is the Gibbs state. Using Eq. (126), we see that Eq. (125) is equivalent to

\[
S(\rho||\gamma) \geq S(\sigma||\gamma). \tag{127}
\]

The techniques developed by Shiraishi and Sagawa (2021, 2022) to prove the above theorem turned out to be applicable to other resource theories, and we will provide a sketch of the proof in the following. The major insight behind the proof is a connection between asymptotic transformations and approximate catalysis, which has been investigated earlier in (Duan et al., 2005a). Let us assume that \(\rho\) can be converted into \(\sigma\) via Gibbs-preserving operations in the asymptotic limit, i.e., for any \(\delta > 0\) there exists an integer \(n\) and a Gibbs-preserving operation \(\Lambda\) acting on \(n\) copies of \(\rho\) such that

\[
\left\| \Lambda \left( \rho^{\otimes n} \right) - \sigma^{\otimes n} \right\| _1 \leq \delta. \tag{128}
\]

In the following \(S_i\) will denote copies of the system \(S\), and we further define \(\Gamma = \Lambda(\rho^{\otimes n})\), which is a quantum state on \(S_1 \otimes S_2 \otimes \cdots \otimes S_n\). Shiraishi and Sagawa (2021, 2022) showed that Eq. (128) implies an approximate catalytic transformation from \(\rho\) into \(\sigma\). The catalyst achieving this transformation is given as

\[
\tau^C = \frac{1}{n} \sum_{k=1}^{n} \rho^{\otimes k-1} \otimes \Gamma_{n-k} |k\rangle\langle k|. \tag{129}
\]

Here, \(\Gamma_0 = 1\), and for \(i > 0\) the state \(\Gamma_i\) is the reduced state of \(\Gamma\) on \(S_1 \otimes S_2 \otimes \cdots \otimes S_i\). The Hilbert space of the catalyst is in \(S_1^{\otimes n-1} \otimes K\), where \(K\) represents an ancillary system of dimension \(n\) with a fully degenerated Hamiltonian and basis states \(|k\rangle\rangle_{n=1}^{K}\).

The Gibbs-preserving operation achieving the desired conversion consists of the following three steps applied onto the total state \(\rho^{\otimes n} \otimes \tau^C\) (Shiraishi and Sagawa, 2021, 2022):
1. A controlled Gibbs-preserving operation is performed, with the ancillary system $K$ as the control, and all other particles as the target. Conditioned on the ancilla being in the state $|n\rangle$, the Gibbs-preserving operation $\Lambda$ is applied onto $S_1 \otimes S_2 \otimes \cdots \otimes S_n$ such that Eq. (128) is fulfilled.

2. A unitary is applied onto the ancillary system that converts $|n\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |i+1\rangle$.

3. The systems are shifted as $S_i \rightarrow S_{i+1}$ and $S_n \rightarrow S_1$.

The initial state of the system and the catalyst has the form

$$\rho^S \otimes \tau^C = \frac{1}{n} \sum_{k=1}^{n} \rho^{\otimes k} \otimes \Gamma_{n-k} \otimes |k\rangle\langle k|.$$  \hspace{1cm} (130)

After applying 1. step of the procedure, we obtain the following state:

$$\mu_1 = \frac{1}{n} \sum_{k=1}^{n} \rho^{\otimes k} \otimes \Gamma_{n-k} \otimes |k\rangle\langle k| + \frac{1}{n} \Gamma \otimes |n\rangle\langle n|.$$  \hspace{1cm} (131)

In 2. step, $\mu_1$ is transformed into the state

$$\mu_2 = \frac{1}{n} \sum_{k=1}^{n} \rho^{\otimes k-1} \otimes \Gamma_{n-k+1} \otimes |k\rangle\langle k|.$$  \hspace{1cm} (132)

Tracing out $S_n$ from $\mu_2$ gives the initial state of the catalyst $\tau$, see Eq. (129). Therefore, the shift in 3. step transforms $\mu_2$ to the final state $\mu^{SC}$ having the property $Tr_S[\mu^{SC}] = \tau^C$, which implies that the condition (121) for approximate catalysis is fulfilled.

Using the protocol presented above, Shiraishi and Sagawa (2021) further showed that the final system state $\mu^S$ can be made arbitrarily close to the desired state $\sigma^S$, and also the mutual information between the system and the catalyst in the total final state $\mu^{SC}$ can be made arbitrarily small whenever $\rho^S$ can be converted into $\sigma^S$ via Gibbs-preserving operations in the asymptotic limit. Thus, the procedure fulfills the criteria for approximate catalysis in Eqs. (122)-(124). Recalling that these criteria are equivalent to Eqs. (120) and (121), we will now prove that the transformation presented above fulfills Eq. (120). For this, we will show that for any $\epsilon > 0$ the protocol described above can be performed such that

$$\|\mu^{SC} - \sigma^S \otimes \tau^C\|_1 \leq \epsilon.$$  \hspace{1cm} (133)

In this part of the proof we will use the techniques presented for entanglement catalysis in (Kondra et al., 2021a; Lipka-Bartosik and Skrzypczyk, 2021b), adjusting them to the problem under study. Note that $\mu^{SC}$ is equivalent to the state $\mu_2$ in Eq. (132) up to a cyclic shift. Thus, our figure of merit $\|\mu^{SC} - \sigma^S \otimes \tau^C\|_1$ is equal to $\|\mu_2 - \nu\|_1$, where the state $\nu$ has the form

$$\nu = \frac{1}{n} \sum_{k=1}^{n} \rho^{\otimes k-1} \otimes \Gamma_{n-k} \otimes \sigma \otimes |k\rangle\langle k|. \hspace{1cm} (134)$$

Using the triangle inequality, Eq. (128), and the monotonicity of trace distance under partial trace we obtain

$$\|\mu^{SC} - \sigma^S \otimes \tau^C\|_1 = \|\mu_2 - \nu\|_1 \hspace{1cm} (135)$$

$$\leq \frac{1}{n} \sum_{k=1}^{n} \|\Gamma_{n+1-k} - \Gamma_{n-k} \otimes \sigma\|_1 \hspace{1cm} (136)$$

$$\leq \frac{1}{n} \sum_{k=1}^{n} \|\Gamma_{n+1-k} - \sigma^{\otimes(n+1-k)}\|_1 \hspace{1cm} (137)$$

$$+ \frac{1}{n} \sum_{k=1}^{n} \|\Gamma_{n-k} \otimes \sigma - \sigma^{\otimes(n+1-k)}\|_1 \leq 2\delta.$$

Setting $\delta = \epsilon/2$ we see that Eq. (133) is fulfilled. This proves that the Gibbs-preserving operation presented above together with the catalyst state in Eq. (129) achieves the desired transformation. This holds true whenever $\rho$ can be converted into $\sigma$ via Gibbs-preserving operations in the asymptotic limit.

Shiraishi and Sagawa (2021, 2022) further proved that a conversion from $\rho$ into $\sigma$ via asymptotic Gibbs-preserving operations is possible whenever $F(\rho) > F(\sigma)$ is fulfilled. For proving this statement, Shiraishi and Sagawa (2021, 2022) make use of the quantum Stein’s lemma (Hiai and Petz, 1991; Ogawa and Nagaoka, 2000) together with the properties of quantum hypothesis testing relative entropy and relative Rényi entropies (Buscemi et al., 2019; Dutta, 2009; Wang and Renner, 2012). Note that this result has previously been proved in a more general setting, specifically for the resource theory of asymmetric distinguishability (Wang and Wilde, 2019), and Gibbs-preserving operations can be considered as a special scenario of this resource theory. With the arguments presented above, it follows that for any states $\rho$ and $\sigma$ fulfilling $F(\rho) > F(\sigma)$, a conversion from $\rho$ into $\sigma$ is possible via Gibbs-preserving operations with approximate catalysis (Shiraishi and Sagawa, 2021, 2022). By a simple continuity argument this result can be extended to the case $F(\rho) = F(\sigma)$ (Shiraishi and Sagawa, 2021, 2022).

It is left to show that a conversion via Gibbs-preserving operations with approximate catalysis is not possible if Eq. (125) is violated. For this, consider the following inequalities:

$$S(\rho^S||\gamma^S) + S(\tau^C||\gamma^C) = S(\rho^S \otimes \tau^C||\gamma^S \otimes \gamma^C) \geq S(\rho^S \otimes \tau^C||\Lambda(\gamma^S \otimes \gamma^C))$$

$$\geq S(\Lambda(\rho^S \otimes \tau^C)||\Lambda(\gamma^S \otimes \gamma^C)) = S(\mu^C||\gamma^C) \geq S(\mu^C||\gamma^S) + S(\tau^C||\gamma^C). \hspace{1cm} (136)$$

The above result follows from the properties of the quantum relative entropy (Capel et al., 2018; Wehrl, 1978): the second line follows from the data processing inequality, the third line from the fact that $\Lambda$ is a Gibbs-preserving operation, and the last line follows from superadditivity. Moreover, in the last line of Eq. (136) we have used the state $\mu^{SC} = \Lambda(\rho^S \otimes \tau^C)$, noting that $\mu^C = \tau^C$. Noting that the Gibbs states $\gamma^S$ and $\gamma^C$ have full rank, the quantum relative entropies appearing in Eq. (136) are all finite and continuous (Audenaert and Eijsert, 2005). We arrive at $S(\rho^S||\gamma^S) \geq S(\rho^S||\gamma^S)$ for any state.
\(\mu^c\) which is achievable from \(\rho^c\) via Gibbs-preserving operations with approximate catalysis. By continuity, it follows that the same inequality also holds for any state \(\sigma^c\) which can be achieved with an arbitrary small error as in Eq. (120). Recalling that Eq. (125) is equivalent to Eq. (127), the proof of Theorem 5 is complete.

The results of Shiraishi and Sagawa (2021, 2022) have also been used by Wilming (2021) to provide a proof of the catalytic entropy conjecture (Boes et al., 2019) in an approximate form, see also Section III.D. In particular, for any two system states \(\rho\) and \(\sigma\) with \(S(\rho) \leq S(\sigma)\) and any \(\varepsilon > 0\) there exists a catalyst state \(\tau\) and a unitary \(U\) acting on the system and the catalyst such that (Wilming, 2021)

\[
\begin{align*}
\text{Tr}_S \left[ U\rho \otimes \tau U^\dagger \right] &= \tau, \quad (137) \\
\left\| \text{Tr}_C \left[ U\rho \otimes \tau U^\dagger \right] - \sigma \right\|_1 &\leq \varepsilon. \quad (138)
\end{align*}
\]

Applications of catalysis in quantum thermodynamics have also been reported by Lipka-Bartosik et al. (2022), who proposed a notion of temperature of a non-equilibrium state, inspired by the “zeroth law of thermodynamics”. They showed that for every quantum system there exist two effective temperatures – hot and cold temperatures – quantifying the ability of a quantum system to cool down or heat up via thermal operations. Precisely, Lipka-Bartosik et al. (2022) showed that, if the temperature of the bath is less than the cold temperature, then the system can never gain energy via any thermal operation (and vice versa). Interestingly, Lipka-Bartosik et al. (2022) showed that approximate catalysis leads to much colder or hotter effective temperatures.

B. Entanglement

Approximate catalysis has also been studied within the resource theory of entanglement (Datta et al., 2022; Kondra et al., 2021a; Lipka-Bartosik and Skrzypczyk, 2021b). In this case, Kondra et al. (2021a) provided a full characterization of state transformations for all bipartite pure states.

Theorem 6 (See Eq. (4) in (Kondra et al., 2021a)). Using LOCC and approximate catalysis, Alice and Bob can convert a state \(|\psi\rangle^{AB}\) into \(|\phi\rangle^{AB}\) if and only if

\[
S(\psi^A) \geq S(\phi^A). \quad (139)
\]

Similar to the method of Shiraishi and Sagawa (2021, 2022) for quantum thermodynamics, the proof of Theorem 6 presented by Kondra et al. (2021a) uses an analogy between approximate catalysis and asymptotic state transformations via LOCC. For this, recall that the optimal rate for transforming \(|\psi\rangle^{AB}\) into \(|\phi\rangle^{AB}\) via LOCC in the asymptotic limit is given by \(S(\psi^A)/S(\phi^A)\) (Bennett et al., 1996a). If Eq. (139) is fulfilled, then asymptotic conversion from \(|\psi\rangle^{AB}\) into \(|\phi\rangle^{AB}\) is possible with rate at least one. It is then possible to construct a catalyst state and an LOCC protocol achieving the transformation with approximate catalysis, in analogy to the construction proposed by Shiraishi and Sagawa (2021, 2022) for quantum thermodynamics, see the previous Section IV.A for more details. We also note that the result in Theorem 6 does not assume that the state of the catalyst is pure, and in fact a catalyst in a mixed state is sometimes required to achieve the desired transformation (Datta et al., 2022; Kondra et al., 2021a).

Kondra et al. (2021a) further used the properties of squashed entanglement (Christandl and Winter, 2004) to prove that an approximate catalytic transformation is not possible if Eq. (139) is violated. Squashed entanglement is defined as (Christandl and Winter, 2004)

\[
E_{sq}(\rho^{AB}) = \inf \left\{ \frac{1}{2} I(A; B; E) : \rho_{AB}^{EE} \text{ extension of } \rho^{AB} \right\}, \quad (140)
\]

with the quantum conditional mutual information \(I(A; B|E)\). As has been shown by Christandl and Winter (2004), squashed entanglement does not increase under LOCC, and moreover it fulfills

\[
E_{sq}(\rho^{AA'}) \geq E_{sq}(\rho^{AB}) + E_{sq}(\rho^{BB'}) \quad (141)
\]

with equality if \(\rho^{BB'} = \rho^{BB'} \otimes \sigma^{BB'}\). For pure states, squashed entanglement coincides with the von Neumann entropy (with logarithm to the base 2) of the reduced state (Christandl and Winter, 2004). Using these properties, Kondra et al. (2021a) proved the following:

Theorem 7 (See Theorem 2 in (Kondra et al., 2021a)). If a bipartite state \(\rho^{AB}\) can be transformed into another state \(\nu^{AB}\) via LOCC with approximate catalysis, then

\[
E_{sq}(\rho^{AB}) \geq E_{sq}(\nu^{AB}). \quad (142)
\]

We will present the proof of this theorem following the arguments of Kondra et al. (2021a). If \(\rho^{AB}\) can be transformed into \(\nu^{AB}\) with approximate catalysis, then for any \(\varepsilon > 0\) there exists a catalyst state \(\tau^{BB'}\) and an LOCC protocol \(\Lambda\) such that the final state \(\sigma^{AA'BB'} = \Lambda(\rho^{AB} \otimes \tau^{BB'})\) fulfills

\[
\begin{align*}
\left|\left| \text{Tr}_A[B] \left( \sigma^{AA'BB'} - \nu^{AA'} \right) \right|\right|_1 &< \varepsilon, \quad (143) \\
\text{Tr}_{AB} \left( \sigma^{AA'BB'} \right) &= \tau^{AA'}. \quad (144)
\end{align*}
\]

We further obtain

\[
E_{sq}(\sigma^{AA'BB'}) \leq E_{sq}(\rho^{AB} \otimes \tau^{BB'}) = E_{sq}(\rho^{AB}) + E_{sq}(\tau^{BB'}), \quad (145)
\]

where in the inequality we used the fact that squashed entanglement does not increase under LOCC, and the equality follows from the additivity of \(E_{sq}\) on tensor products. From Eq. (141) we further have

\[
E_{sq}(\sigma^{AA'BB'}) \geq E_{sq}(\text{Tr}_{A'[B]}(\sigma^{AA'BB'})) + E_{sq}(\tau^{BB'}). \quad (146)
\]

Combining Eqs. (145) and (146) we obtain

\[
E_{sq}(\rho^{AB}) \geq E_{sq}(\text{Tr}_{AB}(\sigma^{AA'BB'})). \quad (147)
\]
The continuity of squashed entanglement (Alicki and Fannes, 2004) together with Eq. (143) implies \( E_{sq}(\rho^{AB}) \geq E_{sq}(\sigma^{AB}) \), which completes the proof of Theorem 7.

Theorem 7 together with the faithfulness of squashed entanglement (Li and Winter, 2014) implies that it is not possible to create entanglement from separable states by using LOCC and approximate catalysis.

The proof of Theorem 6 presented in (Kondra et al., 2021a) uses the fact that for bipartite pure states \(|\psi^{AB}\rangle\) and \(|\phi^{AB}\rangle\) fulfilling \( S(\psi^A) > S(\phi^A) \) asymptotic conversion from \(|\phi^{AB}\rangle\) is possible with unit rate (Bennett et al., 1996a). States with such properties have also been studied by Bennett et al. (2000), where this feature has been called asymptotic reducibility. A general relation between asymptotic reducibility and approximate catalysis has been established by Kondra et al. (2021a) for multipartite systems and multipartite LOCC protocols.

**Theorem 8** (See Theorem 1 in (Kondra et al., 2021a)). If \( \rho \) is asymptotically reducible to \( \sigma \), then \( \rho \) can be transformed into \( \sigma \) via LOCC with approximate catalysis.

A stronger form of asymptotic reducibility can be constructed where the parties can additionally use a sublinear amount of entanglement with the standard LOCC protocol to accomplish the same transformation, which is denoted by asymptotic LOCC reducibility (Bennett et al., 2000). Similarly, catalytic asymptotic LOCC reducibility \( \text{CLOC}_{c} \) can be defined as follows: A state \(|\phi\rangle\) is asymptotically LOCC reducible to a state \(|\phi\rangle\) if there exists a state \(|\eta\rangle\) and a \( n \) such that \(|\phi\rangle \otimes |\eta\rangle\rangle^{\otimes n} \) can be converted arbitrarily close to \(|\phi\rangle \otimes |\eta\rangle\rangle^{\otimes n} \). Bennett et al. (2000) showed that LOCC \( \text{CLOC}_{c} \) is at least as powerful as LOCC \( \text{CLOC}_{q} \). However, the converse i.e., whether LOCC \( \text{CLOC}_{c} \) can be simulated by LOCC \( \text{CLOC}_{q} \), has been shown only for some special cases (Bennett et al., 2000). In the context of LOCC reducibility, a recent result on the catalysis of arbitrary entangled states compares single-copy catalysis with asymptotic catalysis (Datta et al., 2022). A transformation \( \rho \rightarrow \sigma \) on system \( S \) is possible via asymptotic catalysis if and only if for any \( \epsilon > 0 \) and any \( \delta > 0 \) there exist integers \( n \) and \( m \) with \( n > m \), a catalyst state \( \tau^C \) and an LOCC protocol \( \Lambda \) such that \( 2 \)

\[
\| \Lambda [\rho^{\otimes n} \otimes \tau^C] - \sigma^{\otimes m} \otimes \tau^C \|_1 \leq \epsilon.
\]

\[
\text{Tr}_{S^{\otimes m}} [\Lambda [\rho^{\otimes n} \otimes \tau^C]] = \tau^C, \tag{148b}
\]

\[
\frac{m}{n} + \delta \geq 1. \tag{148c}
\]

With this Datta et al. (2022) showed that state transformations under asymptotic catalysis do not provide any advantage over single-copy catalysis. To be precise, if a state transformation \( \rho \rightarrow \sigma \) can be achieved via asymptotic catalysis with unit rate, then the same transformation can be realised with single-copy catalysis too.

Datta et al. (2022) also considered asymptotic settings beyond the independent and identically distributed (“iid”) case, where the quantum state to be transformed asymptotically is no longer assumed to be just the tensor product \(|\phi\rangle^{\otimes m} \). In particular, Datta et al. (2022) considered a pure-state distillation scenario, in which Alice and Bob share a more general total state \( |\psi_i\rangle_A^B \), where each \(|\psi_i\rangle_A^B \) are bipartite states that need not be identical. As we will see, approximate catalysis provides, in this setting, a significant advantage over non-catalytic LOCC transformations.

By definition, a singlet state \(|\phi_s^2\rangle\) can be extracted from a sequence of states \(|\psi_1\rangle^AB\) with fidelity \( f \) and probability \( p \), if there exists an integer \( n \) and a probabilistic LOCC protocol \( \Lambda \) such that (Datta et al., 2022)

\[
\text{Tr} \left[ \Lambda \left[ \otimes_{i=1}^{\infty} |\psi_i\rangle \right] \right] = p, \quad \langle \phi_s^2 | \Lambda \left[ \otimes_{i=1}^{n} |\psi_i\rangle \right] |\phi_s^2 \rangle = pf \tag{149}
\]

where we set \( \psi_i = |\psi_i\rangle_A^B \) for brevity. Compare also Eqs. (34) and (35) in (Datta et al., 2022). Moreover, in the context of pure state entanglement as the relevant resource, we can say that the same sequence allows for catalytic extraction of \( m \) singlets if the conditions provided in Eqs. (120) and (121) are satisfied, with \( \sigma^2 = |\phi_s^2\rangle \langle \phi_s^2 |^{\otimes m} \) and \( \Lambda \) being an LOCC protocol. With this, Datta et al. (2022) proved the following:

**Theorem 9** (See Theorem 3 in (Datta et al., 2022)). Given fidelity \( f > \frac{1}{2} \) and any \( \epsilon > 0 \), there is a sequence of two-qubit states \(|\phi_i\rangle^{AB}_1\) such that for any \( n \geq 1 \), the probability of converting the state \( \otimes_{i=1}^{n} |\psi_i\rangle^{AB} \) into a singlet, via LOCC with fidelity \( f \), is smaller than \( \epsilon \). At the same time, an unbounded number of singlets can be extracted from \( \otimes_{i=1}^{n} |\psi_i\rangle^{AB} \) with the help of catalysis, as \( n \rightarrow \infty \).

The proof is by explicit construction (Datta et al., 2022). The sequence \(|\psi_i\rangle^{AB}_1\) given in the proof is designed in such a way that although the state \( \otimes_{i=1}^{n} |\psi_i\rangle^{AB} \) stays arbitrarily close to a product state with respect to systems \( A \) and \( B \), its total local entropy grows indefinitely as \( n \rightarrow \infty \), see (Datta et al., 2022) for more details. The above results demonstrate clearly the advantage of an approximate catalytic transformation over all non-catalytic LOCC protocols. Whereas for the two-qubit sequence specified in the proof of Theorem 9, the amount of entanglement available via catalytic transformation is unbounded, and not a single maximally entangled state can be extracted with reasonable probability and fidelity without catalysis. It is worth mentioning that entanglement distillation from non-iid sequences of states has been investigated earlier in (Bowen and Datta, 2008; Buscemi and Datta, 2010; Waelderchen et al., 2016).

Eisert and Wilkens (2000) have investigated the role of catalysis for approximate transformations between quantum states via LOCC, requiring however that the catalyst is not correlated with the primary system at the end of the procedure. In particular, Eisert and Wilkens (2000) demonstrated

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2 The LOCC protocol \( \Lambda \) acts on \( n \) copies of \( S \) and the catalyst \( C \), and produces a state on \( m \) copies of \( S \) and the catalyst.
that catalysis is useful for approximately converting a mixed entangled state into a pure state. For this, Eisert and Wilkens (2000) considered the approximate conversion fidelity

$$ F_{\text{LOCC}}(\rho \rightarrow \sigma) = \max_{\Lambda \in \text{LOCC}} F(\Lambda(\rho), \sigma), $$

$$ F_{\text{CLLOCC}}(\rho \rightarrow \sigma) = \max_{\Lambda \in \text{CLLOCC}} F(\Lambda(\rho), \sigma), $$

(150)
(151)

where CLLOCC represents catalytic LOCC. Eisert and Wilkens (2000) proved that the conversion fidelity for CLLOCC can exceed the same for LOCC when some initial mixed states \( \rho \) are being converted into a pure target state \( |\phi\rangle \):

$$ F_{\text{CLLOCC}}(\rho \rightarrow |\phi\rangle) > F_{\text{LOCC}}(\rho \rightarrow |\phi\rangle). $$

(152)

To prove the statement, Eisert and Wilkens (2000) used the state \( \rho \) given in Eq. (22) with \( |\omega\rangle \) given in Eq. (27) and

$$ |\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle + \sqrt{\frac{1}{2}} (|22\rangle + |33\rangle) + \sqrt{1 - \epsilon^2} |44\rangle. $$

(153)

Setting \( |\zeta\rangle = |\psi\rangle \) given in Eq. (26), Eisert and Wilkens (2000) proved that the inequality given in Eq. (152) holds. The above equation holds for all \( \epsilon \in (\epsilon', 1) \) with some appropriate \( \epsilon' \in (0, 1) \) and independent of the value of \( \nu \in (0, 1) \).

Early results on the role of entanglement catalysis in the presence of noise have also been presented by Vidal et al. (2000). Suppose a catalyst in the state \( |\eta\rangle \) improves the performance of the approximate transformation \( |\psi\rangle \rightarrow |\phi\rangle \), i.e.,

$$ D(\rho \otimes \eta \rightarrow \phi \otimes \eta < D(\psi \rightarrow \phi), $$

(154)

where \( D(\rho \rightarrow \sigma) = \min_{\Lambda} D(\Lambda(\rho), \sigma), \Lambda \) represents the set of LOCC operations, and \( D(\rho, \sigma) = ||\rho - \sigma||_2 \) is the trace distance. The quantity

$$ \delta = D(\rho \rightarrow \sigma) - D(\phi \rightarrow \eta < D(\psi \rightarrow \phi) $$

(155)

can thus be regarded as a quantifier of the catalytic advantage in this procedure. Vidal et al. (2000) have investigated how much noise is allowed in the initial state so that the catalytic advantage remains, i.e., \( \delta > 0 \). Suppose the initial state \( |\psi\rangle \otimes |\eta\rangle \) is distorted to \( \rho \) such that \( D(\rho, \psi \otimes \eta) = \epsilon \). If \( \epsilon < \delta \), it has been shown that the catalytic advantage is preserved as

$$ D(\rho \rightarrow \phi \otimes \eta < D(\psi \rightarrow \phi) \) (Vidal et al., 2000).

The role of catalysis in the approximate transformation of bipartite quantum channels has been discussed recently in (Kim et al., 2021). In fact, Kim et al. (2021) explored the dynamic resource theory of entanglement using the superchannel theory and also discussed the one-shot catalytic dynamic entanglement cost of bipartite quantum channels. More precisely, the authors provided a lower and an upper bound for the one-shot catalytic dynamic entanglement cost of an arbitrary bipartite channel. Before presenting the bounds, in the following, we introduce the one-shot catalytic dynamic entanglement cost of a bipartite channel \( N_{AB} \) for a given \( \delta > 0 \) and \( \epsilon \geq 0 \) (Kim et al., 2021):

$$ E_{C,\delta,\text{SEPPSC}}^{\epsilon}(N_{AB}) := \min \{ \log_2 m^2 \} : $$

(156)

$$ \Theta_{A'B'CD}^{n} \rightarrow \Theta_{ABCD}^{m - n} = \Theta_{A'B'CD}^{m} \otimes F_{CD}^{n} = N_{AB}' \otimes F_{CD}^{n}, $$

$$ \Theta_{A'B'CD}^{n} \rightarrow \Theta_{ABCD}^{m - n} \in \delta - \text{SEPPSC}, $$

$$ ||N_{AB} - N_{AB}'||_2 \leq 2\epsilon, $$

where \( m, n \) represent natural numbers and \( \Theta_{A'B'CD}^{n} \rightarrow \Theta_{ABCD}^{m - n} \) is a \( \delta \)-separability-preserving superchannel (\( \delta \)-SEPPSC) from \( A'B'CD \) to \( ABCD \). For a given \( \delta > 0 \), a superchannel \( \Theta_{AB} \rightarrow \Theta_{A'B}' \in \delta - \text{SEPPSC} \) when \( R(\Theta_{AB} \rightarrow \Theta_{A'B}') (S_{AB}) \leq \delta \) holds for all separable channels \( S_{AB} \), where \( R \) is the generalised robustness with respect to the separable channels (Kim et al., 2021). Here \( F_{AB}^{n} \) denotes \( m \)-SWAP channel which can be represented by the application of \( m \)-SWAP gate \( F_{AB}^{n} = \sum_{i,j=0}^{n-1} |i⟩⟨j| f_{i AB} b_{i AB} \rangle \). With this, we are ready to introduce the following bounds. For \( \delta > 0, \epsilon \geq 0 \), there exists a natural number \( n > 0 \) with \( n^2 \geq (\delta + 1)/\delta \) such that the one-shot catalytic dynamic entanglement cost of an arbitrary bipartite channel \( N_{AB} \) is bounded from below and above as (Kim et al., 2021)

$$ R_{\epsilon}^{n}(N_{AB} \otimes F_{CD}^{n}) - \log_2 n^2 - \log_2 (1 + \delta) \leq E_{C,\delta,\text{SEPPSC}}^{\epsilon}(N_{AB}) \leq $$

$$ R_{\epsilon}^{n}(N_{AB} \otimes F_{CD}^{n}) - \log_2 n^2 - \log_2 (1 + 2\epsilon) + 2, $$

(157)

C. Quantum coherence

As discussed previously in Section III.B, correlated catalysis does not help in coherence broadcasting under covariant operations (Lostaglio and Müller, 2019; Marvian and Spekkens, 2019). Due to these limitations on coherence transformation under covariant operations, Takagi and Shiraishi (2022) investigated a different kind of catalyst known as marginal catalysts (Lostaglio et al., 2015b; Wilming et al., 2017) that could help in enhancing covariant transformations. Marginally catalytic covariant transformations are defined as follows: Suppose \( \rho \) is transformed into \( \rho' \) via a marginal catalytic conversion. Then there exists a covariant operation \( \Lambda \), a constant \( N \) and a finite set of states \( \otimes_{j=1}^{N} \tau_{C_j} \) such that (Takagi and Shiraishi, 2022)

$$ \Lambda \left( \rho \otimes_{j=1}^{N} \tau_{C_j} \right) = \rho' \otimes \tau_{C_1} \cdots \tau_{C_N} $$

(158)

$$ \text{Tr}_{C_j} \tau_{C_1} \cdots \tau_{C_{N-1}} \tau_{C_N} = \tau_{C_j} \text{ for all } j, $$

(159)

where \( C_j \) represents all the systems except \( C_j \). Although in the final state, there are correlations among different catalyst systems, the marginal state of each catalyst system remains unchanged. Therefore, the change in the global state of the catalyst produces an embezzling-like phenomenon. For more details on embezzling see Section VII. With this Takagi and Shiraishi (2022) showed that marginal catalysts can provide enormous power in state transformations under covariant transformations. Indeed for any state \( \rho, \rho' \) and \( \epsilon > 0, \rho \) can be transformed into a state \( \rho''_\epsilon \) that is arbitrarily close to \( \rho' \) in trace
distance, i.e., \(||\rho' - \rho||_1 \leq 2\varepsilon\) by a marginal-catalytic covariant transformation. Hence, marginal catalysts exceptionally simplify coherence transformations in this theory. Similar results for qubit states have been presented earlier by Ding et al. (2021).

Fang et al. (2018) studied one-shot probabilistic coherence distillation with catalytic assistance. The idea is to transform a coherent state \(\rho\) close to a maximally coherent state \(|\Psi_N\rangle = \frac{1}{\sqrt{N}} \otimes |1\rangle^N\) of dimension \(N\) with the help of a catalyst \(\eta\). In such a scenario, Fang et al. (2018) showed that the maximum success probability of coherence distillation is given by the maximum value of \(p\), such that

\[
\Lambda(\rho \otimes \eta) = \left(\frac{p}{2} |0\rangle \langle 0| \otimes \sigma + (1 - p) |1\rangle \langle 1| \otimes \eta\right) \otimes \eta, \quad (160)
\]

\[
F(\sigma, \Psi_m) \geq 1 - \varepsilon. \quad (161)
\]

The result holds if \(\Lambda\) is an incoherent operation, but it can also be applied to other classes of free operations studied in the resource theory of coherence (Fang et al., 2018; Streltsov et al., 2017a). Note that the catalyst state remains unchanged regardless of the outcome. However, this constraint was not enforced in the previous probabilistic transformation protocol (Bu et al., 2016). With the following example, a significant improvement has been shown in probabilistic coherence distillation. Consider the initial state

\[
\rho = \frac{1}{2} (|u_1\rangle \langle u_1| + |u_2\rangle \langle u_2|) \quad (162)
\]

with

\[
|u_1\rangle = \frac{1}{2} (|00\rangle - |01\rangle - |10\rangle + |11\rangle), \quad (163)
\]

\[
|u_2\rangle = \frac{1}{5 \sqrt{2}} (2|00\rangle + 6|01\rangle - 3|10\rangle + |11\rangle). \quad (164)
\]

The probability of distilling the state \(|\Psi_2\rangle\) (with an error \(\varepsilon \leq 0.01\)) from \(\rho\) via dephasing-covariant incoherent operations (DIO) (Chitambar and Gour, 2016a,b; Marvian and Spekkens, 2016) can be increased significantly with the help of a catalyst in the state \(|\Psi_2\rangle\) (Fang et al., 2018). In contrast to the deterministic-exact catalytic transformation where maximally coherent states are useless (Du et al., 2015a), this result shows that a maximally coherent state can be used as a catalyst in stochastic-approximate transformations (Kondra et al., 2021b).

Recently, Chen et al. (2019) studied the role of catalysis in one-shot coherence distillation under incoherent operations. In a one-shot catalytic coherence distillation protocol, the distillable coherence of \(\rho\) can be defined as follows:

\[
C_{\rho,\varepsilon}(\rho) = \max_{\Lambda \in O, \dim \rho < \infty} \{\log_2 N : P(\Lambda(\rho \otimes \sigma), \Psi_N \otimes \sigma) \leq \varepsilon\} \quad (165)
\]

where \(\rho\) is the initial state, \(O\) corresponds to different classes of incoherent operations, and \(P\) is the purified distance given as \(P(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)}\). This definition has been adapted from the recently introduced one-shot coherence distillation protocol (Regula et al., 2018; Zhao et al., 2019). Note that there is no further restriction on the catalyst \(\sigma\) except that the dimension is finite. In this setup, one can extract as much coherence as one wishes from any initial state \(\rho\) with some arbitrary small error \(\varepsilon\). As there is no restriction on the catalyst, in principle the catalyst can change slightly in this procedure keeping the error within \(\varepsilon\). Since the dimension of the catalyst could be arbitrarily large, a small change in coherence in it can provide a large value of \(C_{\rho,\varepsilon}(\rho)\). This phenomenon is known as embezzling and is discussed in more detail in Section VII. As the previous case trivializes the transformation, in the next case Chen et al. (2019) studied coherence distillation when the dimension of the catalyst is not more than \(M\). In this case, they provide a lower bound on the amount of coherence that can be distilled. In fact, for an integer \(M\) and a positive number \(\varepsilon\) with the constraint \(\varepsilon^2 \log_2 (M - 1) / 4 \geq 1\), the distillable coherence of a state \(\rho\) can be lower bounded as

\[
C_{\rho}(\rho) \geq \frac{\varepsilon^2 [\log_2 (M - 1) + 1] - 2}{2}. \quad (166)
\]

This result holds for incoherent operations, and can also be extended to other operations investigated in the resource theory of coherence (Chen et al., 2019). However, this bound appears to be loose as it is independent of the coherence of the initial state \(\rho\), and a better bound may be possible. To avoid such embezzling they proposed a perfect catalysis protocol with pure catalyst states where the catalyst is returned exactly, i.e., \(\text{Tr}_S \Lambda(\rho \otimes \eta) = \eta\) and the error is allowed only in the final state, i.e., \(P(\text{Tr}_C \Lambda(\rho \otimes \eta), \Psi_N) \leq \varepsilon\). As shown in (Chen et al., 2019), a catalyst in a pure state does not provide any advantage over standard one-shot coherence distillation where no catalyst is present to help. It is an open question whether the result is also true for a catalyst in a mixed state.

Recently, Char et al. (2021) explored the approximate catalytic state transformations in coherence theory. Their results provide a connection between asymptotic transformation and single copy transformation with a catalyst. In fact, the following has been shown: If there exists an asymptotic incoherent operation transforming a state \(\rho\) into \(\sigma\) with unit rate, then \(\rho\) can be transformed into \(\sigma\) via incoherent operations with approximate catalysis (Char et al., 2021). An analogous result in entanglement theory has been reported earlier by Kondra et al. (2021a), see also Theorem 8. Furthermore, Char et al. (2021) showed that the relative entropy of coherence is a monotone under incoherent operations with approximate catalysis. A direct consequence of these results gives a necessary and sufficient condition for pure state transformations under incoherent operations with approximate catalysis. Particularly, the transformation from \(|\psi\rangle\) to \(|\phi\rangle\) is possible if and only if

\[
S(\Delta|\psi\rangle) \geq S(\Delta|\phi\rangle). \quad (167)
\]

The above result is analogous to the result in entanglement theory describing catalytic transformations for pure bipartite states (Kondra et al., 2021a), see also Theorem 6. Char et al. (2021) prove Eq. (167) making use of the methods proposed.
by Shiraishi and Sagawa (2021) in quantum thermodynamics, see Section IV.A.

D. Other quantum resource theories

One of the ways to quantify the amount of resource present in a state is the minimum amount of noise or randomness required to make it a free state (Groisman et al., 2005). Recently, this idea has been put forward for general resource theories with the additional freedom of allowing free states to help in the process as a catalyst (Anshu et al., 2018). In such a scenario, we want to quantify the resource content of a state \( \rho \), or in other words, the minimum amount of noise needed to make it a free state. To achieve this, we add an extra system in a free state \( \rho_f \) to it and apply free operations on the joint state \( \rho \otimes \rho_f \), such that the final state \( \Lambda_f (\rho \otimes \rho_f) \) becomes arbitrarily close to a free state of the form \( \sigma_f \otimes \rho_f \), where \( \sigma_f \) is a free state. The last point ensures that the additional state \( \rho_f \) remains almost the same in the procedure and, thus, serves as a catalyst. However, note that there is no additional constraint on the catalyst, like Eq. (121), which ensures that the catalyst remains exactly the same. In any case, as free states (which are not expensive) are being used as catalysts, slight changes in them do not prove to be costly. Equipped with this, Anshu et al. (2018) showed that the asymptotic randomness rate of catalytic transformation, or the resource content of a state \( \rho \), is given by the regularised relative entropy of the state

\[
E^\alpha(\rho) = \lim_{n \to \infty} \frac{E(\rho^{\otimes n})}{n},
\]

where \( E(\rho) = \inf_{\sigma \in F} S(\rho\|\sigma) \) and \( F \) represents the set of free states. Furthermore, the resource content of a state in a single copy regime has also been discussed in (Anshu et al., 2018). As the above result has been derived in a general resource theoretic framework, it is applicable to resource theories of entanglement, coherence, purity, and asymmetry (Anshu et al., 2018).

While the preceding discussion shows when an approximate catalytic transformation is possible, it does not provide any information about the catalyst’s resource content. Rubboli and Tomamichel (2022) recently provided some positive answers in this direction for any general resource theory where certain monotones are additive on the tensor product. Assume that a quantity \( D \) completely characterizes approximate catalytic state transformations in a resource theory, i.e., a transformation from \( \rho \) to \( \sigma \) is possible under approximate catalysis iff \( D(\rho) \geq D(\sigma) \), where \( D(\rho) = \min_{\rho' \in H} S(\rho\|\rho') \). Keeping this in mind, we can consider \( D \) as a relevant resource measure to quantify the catalyst’s resourcefulness. In fact, the order of the resource content of a catalyst has been provided for a given error in (Rubboli and Tomamichel, 2022).

**Theorem 10** (See Theorem 1 in (Rubboli and Tomamichel, 2022)). Consider two quantum states \( \rho \) and \( \sigma \), and \( \alpha \in [1/2, 1] \) such that \( D_\alpha(\rho) < D_\alpha(\sigma) \) and \( D_\alpha \) is additive for the state \( \sigma \). Then, for any given error \( \varepsilon \), in an approximate catalytic transformation of \( \rho \) into \( \sigma \) with catalyst \( \eta \), we have

\[
D(\eta) = O\left(\log \frac{1}{\varepsilon}\right).
\]

Here, \( D_\alpha(\rho) = \min_{\rho' \in H} S_\alpha(\rho\|\rho') \) and \( S_\alpha(\cdot\|\cdot) \) is defined earlier in Eq. (85). Therefore, the catalyst’s resourcefulness grows as the error reduces, and in the limit of zero error, the catalyst’s resource content is unbounded. In (Rubboli and Tomamichel, 2022), the authors also derived a quantitative upper bound on the resource content of a catalyst. Furthermore, in (Rubboli and Tomamichel, 2022), it has been discussed that the additivity assumption, combined with \( D_\alpha(\rho) < D_\alpha(\sigma) \) for some \( \alpha \in [1/2, 1] \), implies that the approximate catalytic transformation of \( \rho \) into \( \sigma \) is only possible if a correlation between the system and the catalyst is allowed. Note that this kind of result has been discussed independently in the context of the resource theory of entanglement (Datta et al., 2022).

V. APPLICATIONS OF QUANTUM CATALYSIS

A. Quantum state merging

Catalysis has proven useful for quantum state merging, an important protocol in quantum information theory. Before we review catalytic quantum state merging as discussed by Kondra et al. (2021a), we consider the original quantum state merging introduced in (Horodecki et al., 2005, 2006). In the standard quantum state merging protocol, Alice (A), Bob (B), and a reference system (R) have access to many copies of a pure state \( |\psi\rangle^{ABR} \). By performing LOCC operations between Alice and Bob, Alice aims to transfer her part of the state to Bob while preserving correlation with the reference system. For this, Alice and Bob can share additional singlets. In the asymptotic limit, where many copies of the overall state \( |\psi\rangle^{ABR} \) are available, the minimal singlet rate for this procedure is given by the quantum conditional entropy\(^3\) (Horodecki et al., 2005, 2006):

\[
S(A|B) = S(|\psi^{AB}\rangle) - S(|\psi^B\rangle).
\]

Depending on the value of \( S(A|B) \) two situations can arise. When \( S(A|B) > 0 \), Alice and Bob need to share additional singlets at a rate \( S(A|B) \) to accomplish the state merging protocol. However, for \( S(A|B) \leq 0 \), Alice and Bob can achieve state merging and gain extra singlets at a rate \( |S(A|B)| \) to their disposal.

Catalytic quantum state merging (Kondra et al., 2021a) allows Alice and Bob to achieve merging for a single copy of the state \( |\psi\rangle^{ABR} \) by using approximate catalysis, see Section IV. In this case, Alice, Bob, and the reference system share only one

\(^3\) In Section V, the von Neumann entropy is defined with a logarithm to the base 2, i.e., \( S(\rho) = -\text{Tr}[\rho \log_2 \rho] \).
copy of the state $|\psi\rangle^{AB}$. Additionally, they share a tripartite catalyst state $\tau^{RRR}$, see also Fig. 2. In contrast to standard quantum state merging, in the catalytic case local unitaries are allowed on the reference system. However, no communication between the reference system and the other parties is required (Kondra et al., 2021a). Similar to standard quantum state merging, the performance of this procedure can be quantified via the conditional entropy given in Eq. (170). For $S(A|B) > 0$, Alice and Bob can achieve catalytic state merging if they additionally have access to a pure entangled state with entanglement entropy $S(A|B)$. For $S(A|B) \leq 0$ Alice can merge her part of the state with Bob without any additional entanglement. Apart from merging the state, Alice and Bob will gain a pure entangled state having entanglement entropy $S(A|B)$. The optimality of both scenarios is discussed in (Kondra et al., 2021a).

The standard quantum state merging protocol has also been studied in the context of incoherent operations, known as incoherent quantum state merging (Streltsov et al., 2016). In incoherent quantum state merging the main idea remains the same with some subtle changes. In comparison to LOCC in the standard state merging protocol (Horodecki et al., 2005, 2006), here, the only change is that Bob’s operations are restricted to local incoherent operations. These kinds of operations between Alice and Bob are denoted by local quantum-incoherent operations and classical communications (LQICC) (Chitambar et al., 2016; Streltsov et al., 2017b). Furthermore, Alice and Bob can have access to additional singlets at a rate $E$ and only Bob has access to maximally coherent states at a rate $C$. Then the question is what are the achievable pairs of $(E, C)$ such that the merging is accomplished in the asymptotic limit? In fact, for a tripartite pure state $|\psi\rangle^{ABB}$, an achievable pair of $(E, C)$ should satisfy $E + C \leq S(A|B)_{\psi^D}$, where $\psi^D = \Delta(\psi^{AB})$ and $\psi^{AB} = \text{Tr}_R|\psi\rangle\langle\psi|^{ABB}$. As shown in (Streltsov et al., 2016), the bound can be saturated with $(E_0, 0)$, where $E_0 = S(\psi^D) - S(\psi^D)$ of approximate catalysis (Char et al., 2021) and named as catalytic incoherent state merging. As a matter of fact, it has been shown that under catalytic LQICC incoherent state merging of a single copy of $|\psi\rangle^{ABB}$ is possible when Alice and Bob have access to an additional pure state of entanglement entropy $E_0$ (Char et al., 2021). Note that no additional coherence is needed on Bob’s side. Furthermore, optimality has been shown by verifying that $E_0$ is the minimum amount of entanglement required to achieve the state merging protocol. Therefore, catalysis helps us translate the asymptotic state merging protocol in the single copy scenario.

### B. Quantum teleportation

The role of approximate entanglement catalysis for quantum teleportation has been studied by Lipka-Bartosik and Skrzypczyk (2021b). Before presenting the result, we first briefly recapitulate the standard teleportation protocol (Bennett et al., 1993; Popescu, 1994). In general, there are two distant parties Alice and Bob sharing a quantum state $\rho^{AB}$. Additionally, Alice holds a particle $\tilde{A}$ in an unknown state $|\psi\rangle^A$. The goal of teleportation is to transfer Alice’s state $|\psi\rangle^A$ to Bob using only LOCC. After performing an LOCC protocol $\Lambda$, the final state of Bob can be expressed as

$$
\rho^B = \text{Tr}_{\tilde{A}A} \Lambda [\rho^{AB} \otimes |\psi\rangle \langle \psi|^A].
$$

(171)

The merit of the above protocol can be quantified by the teleportation fidelity (Horodecki et al., 1999; Popescu, 1994) expressed as

$$
F_\Lambda(\rho^{AB}) = \max_\Lambda \int d\psi \langle \rho^{AB} | \text{Tr}_{\tilde{A}A} \Lambda [\rho^{AB} \otimes |\psi\rangle \langle \psi|^A] |\psi\rangle.
$$

(172)

where the maximization is performed over all possible LOCC operations and the integral is taken over uniform distribution $d\psi$ with respect to all pure input states $|\psi\rangle$. The fidelity of teleportation can also be expressed as

$$
F_\Lambda(\rho^{AB}) = \frac{f(\rho^{AB})d_\Lambda + 1}{d_\Lambda + 1},
$$

(173)

where $d_\Lambda$ is the dimension of the input state $|\psi\rangle^A$ and $f(\rho^{AB}) = \max_\Lambda \langle \Phi^+ | \Lambda(\rho^{AB}) |\Phi^+ \rangle$ is the singlet fraction maximised over all LOCC protocols (Horodecki et al., 1999).

Recently, the standard teleportation protocol has been studied in a catalytic scenario to see whether there is an advantage in teleportation fidelity (Lipka-Bartosik and Skrzypczyk,
2021b). In contrast to the standard teleportation protocol, here in the catalytic teleportation protocol Alice and Bob share an additional entangled state $\tau^{A'B'}$ such that it remains the same after the process. To be precise, the catalytic teleportation fidelity can be expressed as (Lipka-Bartosik and Skrzypczyk, 2021b)

$$F_c(\rho_{AB}) = \max_{\Lambda, \psi} \int d\psi \langle \psi | \text{Tr}_{A\tilde{A}B} \Lambda \left[ \rho_{AB} \otimes \tau^{A'B'} \otimes \psi^\dagger \right] | \psi \rangle$$

with the constraint $\text{Tr}_{A\tilde{A}B} \Lambda [\rho_{AB} \otimes \tau^{A'B'} \otimes \psi^\dagger] = \tau^{A'B'}$, which ensures that the catalyst remains unchanged in the process. Furthermore, in (Lipka-Bartosik and Skrzypczyk, 2021b) an achievable lower bound of $F_c(\rho_{AB})$ has been derived:

$$F_c(\rho_{AB}) \geq \frac{f_c(\rho_{AB}) + 1}{d_{\tilde{A}} + 1}, \quad (175)$$

where $f_c(\rho_{AB}) = \lim_{n \to \infty} f_n(\rho_{B^n})/n$ is the regularised version of the singlet fraction and $f_n(\rho) = \max_{\Lambda} \sum_{i=1}^{n} \langle \Phi_i^n | \text{Tr}_i \Lambda (\rho_{\Phi_i^n}) \rangle$, where $\text{Tr}_i$ represents partial trace over systems 1 to $n$ except for $i$. As $f_c(\rho_{AB}) \geq f(\rho_{AB})$ for all states $\rho_{AB}$, we have $F_c(\rho_{AB}) \geq F(\rho_{AB})$ (Lipka-Bartosik and Skrzypczyk, 2021b). In addition, when $\rho_{AB}$ is pure, Lipka-Bartosik and Skrzypczyk (2021b) showed that the catalytic teleportation outperforms the standard teleportation protocol for a large variety of generic quantum states. Furthermore, they studied the advantage when the dimension of the catalyst is small.

C. Assisted distillation of entanglement and coherence

Catalytic-assisted entanglement distillation has been introduced and studied by Kondra et al. (2021a). In the standard assisted entanglement distillation scenario, Alice, Bob, and Charlie share many copies of a tripartite state $|\psi\rangle^{ABC}$, and their aim is to extract singlets between Alice and Bob by performing LOCC (DiVincenzo et al., 1999; Smolin et al., 2005). In (Smolin et al., 2005), the singlet rate has been given in the standard asymptotic scenario as $\lim_{n \to \infty} \frac{1}{n} \max \{ S(\rho^{A}), S(\rho^{B}) \}$. In catalytic assisted entanglement distillation (Kondra et al., 2021a) the parties have access to one copy of a state $|\psi\rangle^{ABC}$, and can make use of tripartite entangled catalysts. By using approximate catalysis as discussed in Section IV, a single copy of $|\psi\rangle^{ABC}$ can be converted into a pure entangled state shared by Alice and Bob with entanglement entropy given by $\min \{ S(\rho^{A}), S(\rho^{B}) \}$. The optimality of this procedure and more details are discussed in (Kondra et al., 2021a).

The role of catalysis has also been studied for assisted coherence distillation (Chitambar et al., 2016) in (Char et al., 2021). In assisted coherence distillation Alice and Bob share many copies of a state $\rho_{AB}$ and their target is to maximize the coherence on Bob’s side by performing LQICC (Chitambar et al., 2016). The quantifier of the performance $C_d^{AB}(\rho_{AB})$ represents the optimal rate at which maximally coherent states per copy of $\rho_{AB}$ can be obtained on Bob’s side. If the initial state is pure then it has been shown that (Chitambar et al., 2016)

$$C_d^{AB}(|\psi\rangle^B) = S(\Lambda|\psi\rangle^B). \quad (176)$$

With the help of an additional system as a catalyst, the above protocol can be translated to the single copy scenario where a single copy of state $|\psi\rangle^{AB}$ is used to maximize coherence on Bob’s side by performing catalytic LQICC (Char et al., 2021).

D. Catalysis of noisy quantum channels

The role of entanglement catalysis for quantum communication via noisy quantum channels has been studied by Datta et al. (2022). For this, Datta et al. (2022) introduced and studied catalytic quantum capacity, capturing the ability of a quantum channel to transmit qubits in the presence of entangled catalysts. In the standard asymptotic setting, where the communicating parties have access to many copies of a quantum channel which can be used in parallel, the performance is quantified by quantum capacity. It represents the maximum rate at which qubits can be faithfully transmitted to a distant receiver via a channel $\Lambda$ (Horodecki et al., 2000; Lloyd, 1997; Schumacher and Nielsen, 1996), and can be expressed as (Devetak, 2005; Shor, 2002)

$$Q(\Lambda) = \lim_{n \to \infty} \frac{1}{n} \max_{\Lambda_0} I(\rho_0, \Lambda_0^{\otimes n}). \quad (177)$$

Here, $I$ represents coherent information and can be expressed as $I(\rho, \Lambda) = S(\Lambda(\rho)) - S(\Lambda(\rho_{AB}))$ (Lloyd, 1997; Schumacher and Nielsen, 1996) and $|\psi\rangle_{AB}$ represents some purification of $\rho$.

If the communicating parties can use the quantum channel only once, they will typically not be able to transmit qubits faithfully through the channel. However, the situation changes if they have access to entangled catalysts (Datta et al., 2022). The figure of merit in this setup is quantified via catalytic quantum capacity (Datta et al., 2022)

$$Q_c(\Lambda) = \max \left\{ m : \lim_{n \to \infty} \| \rho_{n+1} - \rho_{n+1}^{\Lambda_0} \|_{\infty} = 0 \right\}, \quad (178)$$

where $\rho_{n+1}$ is a sequence of states obtained by a single use of the noisy quantum channel $\Lambda$ and approximate catalysis (Datta et al., 2022). Note that the catalytic quantum capacity is an integer, and can be zero even for channels which are not entanglement-breaking. A lower bound on the catalytic quantum capacity can be obtained as (Datta et al., 2022)

$$Q_c(\Lambda) \geq \mathbf{E}_\Lambda \left[ \sup_{\psi} E_d (\Lambda \otimes [\psi]) \right], \quad (179)$$
where $E_d$ is the distillable entanglement (Bennett et al., 1996a, b) and the supremum is taken over all bipartite pure states $\psi = |\psi\psi⟩$. For achieving this bound, Alice can prepare the two-particle state $|\psi\rangle$ locally, and send one half of it to Bob via the channel $\Lambda$. In this way, Alice and Bob end up sharing the state $\mathbb{I} \otimes \Lambda[\psi]$, which can be converted into any pure state with entanglement entropy $E_d(\mathbb{I} \otimes \Lambda[\psi])$ (Datta et al., 2022).

As distillable entanglement is lower bounded by $E_d(\rho^{AB}) \geq S(\rho^{A}) - S(\rho^{AB})$ (Devetak and Winter, 2005), we can obtain an easily computable lower bound for the catalytic quantum capacity (Datta et al., 2022):

$$Q_c(\Lambda) \geq \left[ \log_2 d - S\left(\mathbb{I} \otimes \Lambda\left[\phi_+^i\right]\right) \right],$$

(180)

where $d$ is the dimension of the Hilbert space on which $\Lambda$ is acting and $|\phi_+^i⟩ = \sum_{i=0}^{d-1} |ii⟩ / \sqrt{d}$. For a general quantum channel with Kraus operators $K_i$, Datta et al. (2022) further obtained the lower bound

$$Q_c(\Lambda) \geq \left[ \log_2 d - H(p_i) \right],$$

(181)

where $H(p)$ is the Shannon entropy of the probability distribution

$$p_i = \text{Tr}\left[\mathbb{I} \otimes K_i\right]|\phi_+^i⟩\langle \phi_+^i|\mathbb{I} \otimes K_i^†\right].$$

(182)

As an immediate result, we observe that a channel $\Lambda$ of dimension $d \geq 4$ has a catalytic capacity of at least one if it can be decomposed into at most two Kraus operators (Datta et al., 2022). An upper bound on the catalytic quantum capacity has also been obtained by Datta et al. (2022):

$$Q_c(\Lambda) \leq \Delta E_{sq}(\Lambda),$$

(183)

with

$$\Delta E_{sq}(\Lambda) = \sup_{\rho^{AB}} \left\{ E^{ABC}_{sq}(\Lambda^C[\rho^{ABC}]) - E^{ABC}_{sq}(\rho^{ABC}) \right\},$$

(184)

where $E_{sq}$ is the squashed entanglement (Christandl and Winter, 2004). $\Delta E_{sq}(\Lambda)$ can be interpreted as the amount of entanglement that the channel $\Lambda$ can transmit. Generally, computation of $\Delta E_{sq}(\Lambda)$ is a formidable task. However, for certain types of channels an upper bound on it can be easily computed, e.g., for multiple copies of a single-qubit Pauli channel $\Lambda_\rho(p) = \sum_i p_i \sigma_i \rho \sigma_i^†$. In particular, the catalytic quantum capacity of $\Lambda_{sq}^{\mathbb{I}}$ is bounded as follows (Datta et al., 2022):

$$\left[ n E_f(\mathbb{I} \otimes \Lambda_\rho[\phi_+^i]) \right] \geq Q_c(\Lambda_{sq}^{\mathbb{I}}) \geq \left[ n - n H(p_i) \right],$$

(185)

where $E_f$ represents entanglement of formation (Bennett et al., 1996c). The upper bound on $Q_c$ can be easily evaluated noting that entanglement of formation can be computed for all two-qubit states (Bennett et al., 1996c; Wootters, 1998).

Datta et al. (2022) also discussed the advantage of catalysis for entanglement distribution between two distance parties. Consider a scenario where Alice wants to distribute entanglement with Bob via a single qubit depolarising channel of the form

$$\Lambda_t(\rho) = e^{-at} \rho + \left(1 - e^{-at}\right) \mathbb{I} / 2,$$

(186)

where $l$ is the length of the channel and $a \geq 0$ represents a damping parameter. The channel is entanglement breaking when $l \geq \ln 3/\alpha$. Suppose there is a node in between Alice and Bob to assist them in the entanglement distribution. More precisely, the node is allowed to perform any local operations and communicate classically with Alice and Bob, see also Fig. 3. Nevertheless, entanglement distribution is still not possible for $l \geq \ln 3/\alpha$ as shown in (Datta et al., 2022). To see this, let us assume that Alice prepares a maximally entangled state locally and sends one half of it to Bob through the channel. As a result, the state between the node and Alice becomes mixed, and to ensure that the resulting state between them is entangled the position ($s$) of the node should satisfy $s < \ln 3/\alpha$. As all two-qubit entangled states are distillable (Horodecki et al., 1997), it is possible to create a pure entangled state between Alice and the node using the results in (Kondra et al., 2021a). After that, the particle at the node is sent to Bob via the channel, leading to an entangled state between Alice and Bob. Therefore, the introduction of an intermediate node and a suitable catalyst makes a useless depolarising channel for entanglement distribution useful.

E. Authentication protocols based on catalysis

Catalysis has shown to be useful in authentication protocols as well (Barum, 1999). Suppose there are two parties Alice, Bob and Bob needs to authenticate himself to Alice. As we know there exist incomparable states $|\psi⟩$ and $|\phi⟩$ which can not be transformed into each other via LOCC. Let us say, Alice prepares locally a bipartite state $|\psi⟩$ and sends one half of it to Bob. Additionally, Alice and Bob share a catalyst whose presence accomplishes the forbidden transformation $|\psi⟩ → |\phi⟩$. Now they perform catalytic LOCC on the state $|\psi⟩$, such that they achieved the desired final state. In the next step, Bob will send his half of the final state to Alice and Alice confirms the presence of Bob by performing a measurement in the basis $|\phi⟩$. Suppose, there is some third-party intruder Charlie on the other side of Alice’s communication channel. However, they...
will not be able to transform the state $|\psi\rangle$ to the desired final state $|\phi\rangle$ as Charlie does not possess the other half of the catalyst state. Note that the requirement of incomparable states is essential for this protocol. Otherwise, Alice and Charlie would have accomplished the desired transformation without the catalyst. For the error probability and security analysis we refer to (Barnum, 1999).

**F. Catalytic decoupling**

Catalysis has also been useful in the decoupling of quantum information (Majenz et al., 2017). In the standard decoupling technique, we want to approximately decouple $S$ and $E$ from a bipartite state $\rho^{SE}$ by erasing a part of $S$ (Dupuis et al., 2014; Horodecki et al., 2005, 2006). Precisely, the bipartite state $\rho^{SE}$ is $\epsilon$-decoupled, if there exists a unitary operation $U_S$ such that the following holds (Majenz et al., 2017)

$$\min_{\omega_S,\omega_E} P(\text{Tr}_{S_S} U_S \rho^{SE} U_S^\dagger, \omega_S \otimes \omega_E) \leq \epsilon,$$

with the purified distance $P(\rho, \sigma) = (1 - F(\rho, \sigma))^{1/2}$ and $S = S \bar{S}$. Therefore, we successfully decouple $S_1$ from $E$ at the price of losing $S_2$ in the process. The figure of merit is the minimum dimension of $S_2$ or to be precise $\log_2 |S_2|$ to achieve the $\epsilon$-decoupling and it is denoted by $R^\epsilon(S; E)_\rho$. In fact, the minimum system size satisfies the following (Majenz et al., 2017)

$$R^\epsilon(S; E)_\rho \geq \frac{1}{2} I_{\text{max}}(E; S)_\rho,$$

where $I_{\text{max}}(E; S)_\rho$ represents smooth max-mutual information in the initial state $\rho^{SE}$. It is defined as

$$I_{\text{max}}(E; S)_\rho = \min_{\rho} I_{\text{max}}(E; S)_\rho,$$

where the minimization is taken over all bipartite states $\rho^{SE}$ such that $P(\rho^{SE}, \rho^{SE}) \leq \epsilon$ and

$$I_{\text{max}}(E; S)_\rho = \log_2 \min_{\sigma, \rho} [\text{Tr}_{\sigma_S} (|\sigma_S \otimes \rho^E \geq \rho^{SE})].$$

However, the bound is not achievable for all states. On the contrary, if we allow catalytic decoupling the above bound can be achieved for any bipartite system (Majenz et al., 2017). As a formal definition of catalytic decoupling, we say that a bipartite state $\rho^{SE}$ is $\epsilon$-decoupled with the help of a catalyst in state $\rho^S$, if there exists a unitary operation $U_{\bar{S}}$ such that

$$\min_{\omega^{S'} \otimes \omega^E} P(\text{Tr}_{S_S} U_{\bar{S}} \rho^{SE} U_{\bar{S}}^\dagger, \omega^{S'} \otimes \omega^E) \leq \epsilon,$$

with $\bar{S} = S S' = S_1 S_2 S'$, and $\rho^{S'} = \rho^{SE} \otimes \rho^{S'}$. Here $\omega^{S'}$ represents uncorrelated state of $S_1$ and $S'$. As mentioned in (Majenz et al., 2017), the protocol is catalytic in the sense that the share of the ancillary state $\rho^{S'}$, which becomes part of the decoupled system $\bar{S} S'$, remains uncorrelated with $S_1$ after the decoupling. However, note that other than the catalyst remaining decoupled from the system, no further restriction has been imposed on the catalyst state $\rho^{S'}$, such that it remains unaltered after the protocol. Therefore, it may not be catalytic in the sense that the state of the catalyst should remain unchanged. The minimum system size required in this scenario can be quantified as

$$R^\epsilon_c(S; E)_\rho = \min_{\rho^{S'}} R^\epsilon(S S'; E)_\rho,$$

where the minimum is taken over all possible Hilbert spaces $\mathcal{H}_S'$ and all states $\sigma^{S'}$. It is straightforward to check that $R^\epsilon_c(S; E)_\rho \geq R^\epsilon(S; E)_\rho$. With this Majenz et al. (2017) showed that the bound in Eq. (188) is achievable as we have

$$R^\epsilon_c(S; E)_\rho \leq \frac{1}{2} I_{\text{max}}^\epsilon(S; E)_\rho,$$

where $1 \geq \epsilon \geq \delta > 0$ and “$\leq$” is up to terms of the order $O(\log(1/\epsilon))$. Furthermore, they discuss some applications of catalytic decoupling.

**G. Catalytic cooling**

Henao and Uzdin (2021) explored the advantage of catalysts to cool a system. In (Henao and Uzdin, 2021), the authors considered a passive state $\rho_b$ as an initial state that is to be cooled. Note that a passive state cannot be cooled alone as its average energy $\langle H_b \rangle = \text{Tr}(H_b \rho_b)$ cannot be lowered using any local unitary $U_b$, where $H_b$ is the corresponding Hamiltonian (Allahverdyan et al., 2004). More precisely a state $\rho_b$ is called passive if and only if the change in average energy under any unitary operation $U$ is greater or equal to zero, i.e., $\Delta(H_b) = \text{Tr}(H_b(U_b \rho_b U_b^\dagger - \rho_b)) \geq 0$ (Allahverdyan et al., 2004). As $\rho_b$ cannot be cooled alone, we need to attach some ancillary systems to it and perform a global unitary to cool the system. If we attach a hot object which is not in a passive state then cooling is possible. However, in (Henao and Uzdin, 2021), the authors considered a hot object in a passive state $\rho_r$ and used a catalyst in a non-passive state $\rho_c$ to cool the system in state $\rho_b$. Therefore, under a global unitary $U_{\text{bc}}$ the initial state $\rho = \rho_b \otimes \rho_r \otimes \rho_c$ transforms to $\rho' = U_{\text{bc}} \rho U_{\text{bc}}^\dagger$ and the process is catalytic if we have $\text{Tr}_{\rho_c} \rho' = \rho_c$ additionally. To ensure that the system in state $\rho_b$ cools we should have $\Delta(H_b) < 0$, and in such a scenario, the transformation is called a catalytic cooling transformation (Henao and Uzdin, 2021). Equipped with this, Henao and Uzdin (2021) showed that catalytic cooling is always possible with a sufficiently large dimensional catalyst state. More precisely, they showed the following: For a passive state $\rho_b \otimes \rho_r$ with $\rho_c \neq \mathbb{1}/d_r$ ($d_r$ is the dimension of $\rho_c$), there exists a global unitary $U_{\text{bc}}$ and a suitable catalyst state $\rho_c$ such that catalytic cooling transformation is possible (Henao and Uzdin, 2021). Furthermore, they also showed that cooling of a qubit system can be maximised with a hot qubit and a three-dimensional catalyst (Henao and Uzdin, 2021).
VI. UNIVERSAL CATALYSIS

So far, for the results presented above, the construction of the catalytic state for a non-trivial quantum transformation depends strongly on the choice of input and output states that are to be transformed. A natural question arises then, whether some states are more suitable than others to serve as a catalyst for a wider range of desired transformations. Perhaps one could even find a single quantum state that would be a catalyst for all allowed transformations within a particular resource theory? As we will see, such a universal catalyst state can be readily constructed as long as the allowed catalytic transformations induce a continuous partial order on the compact state space.

In this section, we will review the nascent theory of universal catalysis, when the catalyst state does not depend on the states undergoing the transition. The question about the existence of a universal catalysis state in the context of quantum thermodynamics was raised, and found an intriguing answer, by Lipka-Bartosik and Skrzypczyk (2021a). Soon afterwards, a similar question was posed by Kondra et al. (2021a) for the theory of pure state quantum entanglement, and an answer was given in the subsequent work (Datta et al., 2022). Here, we are going to focus our attention first on the results regarding quantum thermodynamics; then we will proceed to the case of quantum entanglement, leaving the general discussion about the implications of the universal catalysis to the final part of the section.

A. Quantum thermodynamics

For a given pair $\rho_S, \sigma_S$, of states that are block-diagonal in the energy eigenbasis of a Hamiltonian $H_S$ of the system $S$, let vectors $p = \text{diag}[\rho_S]$ and $q = \text{diag}[\sigma_S]$ denote their diagonal parts. Let also $[E_i]$ be the eigenvalues of the Hamiltonian $H_S$. We set $\gamma_S = \text{diag}[g] = (g_1, g_2, \ldots, g_d)$ be the system’s thermal state, diagonal in the same basis, with $g_i = e^{-\beta E_i}/Z_S$, $Z_S = \text{Tr} e^{-\beta H_S}$. Recall that, for a block-diagonal state in the energy eigenbasis $\rho_S$, such that $p = \text{diag}[\rho_S]$, with Hamiltonian $H$ and in contact with the environment at inverse temperature $\beta$, the generalized free energies are defined as

$$F_\alpha(p, g) = S_\alpha(p||g) - Z,$$  

where $S_\alpha(p||g) = \frac{\text{sgn}(\alpha)}{1-\alpha} \log \left( \sum_i p_i^\alpha g_i^{1-\alpha} \right)$ is the Rényi relative entropy (Müller-Lennert et al., 2013), and $Z = \text{Tr} e^{-\beta H}$ is the total partition function, compare also Eqs. (84) and (85).

Now, the main result concerning the universal catalytic transformation in quantum thermodynamics can be stated as follows.

**Theorem 11** (See Lipka-Bartosik and Skrzypczyk, 2021a, Theorem 2). Let $\rho_S$ and $\sigma_S$ be two states with corresponding representations $p = \text{diag}[\rho_S]$ and $q = \text{diag}[\sigma_S]$ which satisfy:

$$F_\alpha(p, g) > F_\alpha(q, g) \quad \forall \alpha \geq 0,$$  

Then, for any catalyst state $\omega_C$ with $\epsilon = \text{diag}[\omega_C]$ and sufficiently large $n$ there exists a thermal operation $T_{\omega_C}$ such that:

$$T_{\omega_C} \left[ \rho_S \otimes \omega_C^{\otimes n} \right] = \sigma_C^{\alpha},$$  

and the errors on the system and the catalyst satisfy:

$$\epsilon_C := \| \text{Tr} \left[ \sigma_C^{\alpha} - \omega_C^{\otimes n} \right] \| \leq O(e^{-\epsilon}),$$  

$$\epsilon_S := \| \text{Tr} \left[ \sigma_C^{\alpha} - \sigma_S \right] \| = 0,$$  

where $\kappa \in (0, 1)$ can be chosen arbitrarily.

A remark regarding the expression “any catalyst state” is in order here. In the proof of the above result, Lipka-Bartosik and Skrzypczyk (2021a) assume that the catalyst system of finite dimension $d_C$ is prepared in the state $\omega_C$ that is diagonal in the eigenbasis of the system’s Hamiltonian $H_C$. It is known that if a catalytic transformation between diagonal states $\rho_S$ and $\sigma_S$ is admissible, it is also possible to transform $\rho_S$ into $\sigma_S$ as in Eq. (196) with the help of a catalyst state that has a fully degenerate spectrum, compare the proof of Theorem 18, Supplementary Information in (Brandão et al., 2015). Hence, without loss of generality, we can assume that the Hamiltonian $H_C$ is trivial, $H_C \propto I$. This, in turn, means that any quantum state $\omega_C$ can be considered a viable catalyst state, since we require only that it be diagonal in the eigenbasis of $H_C$. In this sense, any quantum state $\omega_C$, as long as sufficiently many copies are available, can act as a universal catalyst.

The family of equations given in Eq. (195) are referred to as “second laws of quantum thermodynamics” (Brandão et al., 2015), as they induce a partial order on the state space, which describes allowed catalytic transformations, see also Section III.C. It is important to notice here that if one relaxes the strict condition in Eq. (197) on the amount of error accumulated by the transformation, in a sense that the error is allowed to vanish at the rate slower than exponential in $n$, the order will “collapse” and transformations between essentially all states become possible, as in (Lipka-Bartosik and Skrzypczyk, 2021a), Theorem 1. We will discuss this phenomenon, known in general as catalytic embezzling of quantum resources, thoroughly in the subsequent Section VII.

An intuitive explanation of the above result is the following. Because of the law of large numbers, for an arbitrary state $\omega_C$ that is diagonal in the energy eigenbasis, as the number of copies of the state increases in the asymptotic regime ($n \to \infty$), a subset of eigenvalues of $\omega_C^{\otimes n}$ forms a typical set: almost all probability weight is concentrated in a set that is uniformly distributed. Using a result from (Chubb et al., 2019), see Theorem 2 therein, Lipka-Bartosik and Skrzypczyk (2021a) claim that the state $\omega_C^{\otimes n}$ can be converted reversibly into multiple copies of yet another state, useful for catalytic transformation, at a rate given approximately by relative entropy. The exact quantification of the typicality property is the source of the error as in Eq. (197). The proof of Theorem 11 utilises this kind of intuition and focuses on specifying exactly the thermal operation $T_{\omega_C}$. The actual catalytic transformation allows for a change of the catalyst. To avoid embezzling
of the quantum resource, care needs to be taken in analysing the magnitude of that change, as in Eq. (197).

B. Pure state quantum entanglement

As pointed out in Section IV.B, for two bipartite pure states $|\psi\rangle^{AB}$, $|\phi\rangle^{AB}$, an approximate catalytic conversion is possible if and only if $S(\psi^A) \geq S(\phi^A)$. This result was first proven in (Kondra et al., 2021a), where an explicit construction for the catalyst state was provided. The state of the catalyst given in (Kondra et al., 2021a) depends strongly on the states $|\psi\rangle^{AB}$, $|\phi\rangle^{AB}$, and a natural question was raised in (Kondra et al., 2021a) whether there exists a catalyst state that can catalyze all allowed transformations, even if only to some arbitrary precision. Datta et al. (2022) give a positive answer to that question.

**Theorem 12** (See (Datta et al., 2022), Theorem 1). Consider a bipartite Hilbert space of arbitrary but finite dimension. For every $\varepsilon > 0$ there exists a universal catalyst state $\tau_{\varepsilon}^{AB}$ such that for every pair of pure states $|\psi\rangle^{AB}$ and $|\phi\rangle^{AB}$ with $S(\psi^A) \geq S(\phi^A)$ there is an LOCC protocol $\Lambda$ for which

$$\text{Tr}_{AB}\left[\Lambda\left(\psi^{AB} \otimes \tau_{\varepsilon}^{AB}\right)\right] = \tau_{\varepsilon}^{AB}, \quad (199a)$$

$$\left\|\Lambda\left(\psi^{AB} \otimes \tau_{\varepsilon}^{AB}\right) - \phi^{AB} \otimes \tau_{\varepsilon}^{AB}\right\|_1 < \varepsilon. \quad (199b)$$

As we can see, contrary to Eq. (197) in the previous section VI.A, Eq. (199a) assures that the universal catalyst state $\tau_{\varepsilon}^{AB}$ remains unchanged as a result of the LOCC transformation $\Lambda$. It is generally understood that the requirement in Eq. (199a) is essential if we want to talk meaningfully about universal catalysis for pure state entanglement. This is perhaps the most natural assumption to make if one wants to avoid “quite implausible and unphysical” (Müller, 2018) effects like embezzling of entanglement (cf. Sec. VII). The requirement for approximate catalysis, as in Sec. IV above, is encoded in Eq. (199b). We should notice here that the amount of error allowed during the transformation does not depend on the choice of the input and output states. Hence, an arbitrary precision $\varepsilon > 0$ can be achieved for all possible catalytic transformations using only one state $\tau_{\varepsilon}^{AB}$. In this sense, one can speak of a universal catalytic state for pure-state entanglement.

The proof of Theorem 12, reported in (Datta et al., 2022), makes use of the fact that the set of quantum states is compact. By discretizing appropriately the set of states, one can find a catalytic state for every pair of initial and final states from the finite collection of discrete “points” that allow a catalytic transformation as in Sec. IV. Then the universal catalytic state is obtained by taking the tensor product of that finite number of catalysts. By controlling how fine the discretization should be, a bound for the error of the universal catalytic transformation is achieved as in Eq. (199b).

C. Comparison of universal catalysis of entanglement and quantum thermodynamics

The two constructions of a universal catalytic state given above are rather different. On the one hand, the construction in (Lipka-Bartosik and Skrzypczyk, 2021a) for quantum thermodynamics utilizes the typicality of a sufficiently large tensor power of any quantum state; on the other, the method employed in (Datta et al., 2022) for the resource theory of pure state entanglement relies on the discretization of the compact state space. Both approaches, however, share a common assumption about what catalytic transformations are in general possible with a particular resource theory. That is the existence of a continuous function, or a family of functions, inducing a partial order of allowed transformations on the compact state space. Additionally, both results are believed to be extendable to a broader context beyond their respective resource theories: compare Section III.F in (Lipka-Bartosik and Skrzypczyk, 2021a) and Discussion in (Datta et al., 2022). It is then of merit to ask at this point if both methods can be unified somehow and expanded to a general theory of universal catalytic states. It seems like in both cases the universality of the construction is achieved by taking a state that in some sense contains all other catalysts, at least within an assumed margin of error. Hence, the existence of a universal catalyst should be considered as another manifestation of the “second laws” governing the allowed catalytic transformation between resourceful states.

VII. CATALYTIC EMBEZZLING PHENOMENA

In most parts of the previous discussion, we focused on catalytic transformations with the requirement that the state of the catalyst is left unchanged by the overall procedure. Here we will discuss phenomena which occur if a change of the catalyst state is allowed.

One of the first results in this direction has been presented by van Dam and Hayden (2003). The authors investigated exact entanglement catalysis, allowing the catalysts to change by an arbitrarily small amount, as measured by fidelity between the initial and final state of the catalyst system. In this setting, van Dam and Hayden (2003) proved that there exists a family of pure state $|\mu_n\rangle$ such that for any bipartite state $|\phi\rangle^{AB}$ there is an LOCC protocol: $|\mu_n\rangle \rightarrow |\phi\rangle^{AB} \otimes |0\rangle^A$ with $\langle \mu_n | \langle 0 | \rangle > 1 - \varepsilon$ and vanishing $\varepsilon > 0$ as $n \rightarrow \infty$. Thus, if we consider approximate entanglement catalysis with Eq. (120) being the only constraint, the transformations obtained in this way allow to transform any bipartite pure state into any other bipartite pure state. In particular, for any $\varepsilon > 0$ and any two states $|\psi\rangle^{AB}$ and $|\phi\rangle^{AB}$ there exists a catalyst state $\tau$ and an LOCC protocol $\Lambda$ such that Eq. (120) is fulfilled. This applies also to the setting where the initial state $|\psi\rangle^{AB}$ is product, and the final state $|\phi\rangle^{AB}$ corresponds to $n$ singlets, where $n$ is an arbitrary integer. Thus, the procedure allows to extract an unbounded number of singlets from a shared catalyst, inducing only a
small change in the quantum state of the catalyst. Due to this behavior, this phenomenon has also been termed “embezzling entanglement” (van Dam and Hayden, 2003).

The result of van Dam and Hayden (2003) indicates that perfect embezzlement with zero error is not possible in a finite-dimensional state. This result has recently been extended to infinite-dimensional systems, demonstrating that perfect embezzlement is also impossible in this case (Cleve et al., 2017). Furthermore, in contrast to the tensor product framework, Cleve et al. (2017) have demonstrated that perfect embezzlement is possible in a commuting operator framework.

van Dam and Hayden (2003) also proved that universal entanglement embezzling is possible, which means that for any given ε > 0 and given dimensions $d_A$ and $d_B$ there exists a catalyst achieving the conversion $|\psi\rangle^{AB} \rightarrow |\phi\rangle^{AB}$ for any initial and final state. Moreover, the conversion can be achieved by using local unitaries, and no communication between the parties is required. Additional families of states for universal entanglement embezzling have been presented by Leung and Wang (2014), and entanglement embezzling has also been used for an alternative proof for the quantum reverse Shannon theorem (Bennett et al., 2014; Berta et al., 2011).

A detailed study of catalytic embezzling phenomena in quantum thermodynamics has been performed by Brandão et al. (2015). The authors introduced catalytic thermal operations (see also Sec. III.C), and investigated different regimes for the change of the catalyst. For the case that the catalyst is returned unchanged, the transitions are characterized by the second laws as described in Sec. III.C. The same holds true if the state of the catalyst changes in such a way that the change can be corrected by investing a small amount of work (Brandão et al., 2015). The third regime considered by Brandão et al. (2015) requires that the state of the catalyst changes at most by $\varepsilon/\log d$ in trace norm, where $d$ is the dimension of the catalyst. In this case, it is shown that the transitions are completely characterized by the standard second law, under the assumption that the initial and the final states are diagonal in the energy eigenbasis.

For the resource theory of purity, a careful investigation of embezzling phenomena has been performed by Ng et al. (2015). The following condition must hold for the catalyst state, if the catalyst is to enable all state transformations via unital operations (Ng et al., 2015):

$$\frac{1}{2} ||\omega - \omega'||_1 \geq \frac{d_S - 1}{1 + (d_S - 1) \log d_{\omega}} (200)$$

where $d_S$ and $d_C$ are the dimensions of the system and the catalyst, respectively. In particular, for any system states $\rho$ and $\sigma$ there exist states of the catalyst $\omega$ and $\omega'$ fulfilling Eq. (200) such that $\rho \otimes \omega$ can be transformed into $\sigma \otimes \omega'$ via a unital operation. Moreover, this bound is optimal: it is not possible to enable all state transformations with a catalyst violating Eq. (200).

For general resource theory, an interesting result regarding the resource content of a catalyst in the case of embezzlement has recently been discussed in (Rubboli and Tomamichel, 2022). In fact, Rubboli and Tomamichel (2022) demonstrated that the Theorem 10 holds true even when the catalyst changes in the procedure, resulting in resource embezzlement. As a consequence, the results indicate that a highly resourceful catalyst is required to achieve small errors in state transformations.

In (Leung et al., 2008), catalytic embezzling phenomena have also been studied in the context of coherent state exchange. Suppose there are $m$ parties, and they want to transform a $m$-partite state $|\phi\rangle$ into another $m$-partite state $|\phi\rangle$ by a coherent process and without any communication, i.e.,

$$\alpha |0\rangle^{m\omega} |\gamma\rangle + \beta |1\rangle^{m\omega} |\phi\rangle \rightarrow \alpha |0\rangle^{m\omega} |\gamma\rangle' + \beta |1\rangle^{m\omega} |\phi\rangle'. \quad (201)$$

Note that in the above process the coherence remains intact. In general, the above task cannot be always accomplished for $m \geq 2$ (Leung et al., 2008). However, with an additional catalyst state shared between $m$ parties and allowed to change slightly in the process, it has been shown that coherent state exchange is always possible for any number of parties (Leung et al., 2008). Furthermore, Leung et al. (2008) showed that universal embezzling is possible for any $m$. In another recent study, universal embezzling has been explored for entangled projection games (Dinur et al., 2013).

VIII. PERSPECTIVES AND OPEN PROBLEMS

While significant progress has been achieved in the investigation of quantum catalysis in recent years, some fundamental problems in this research area are still open. One such open problem concerns exact entanglement catalysis, where the catalyst is left uncorrelated with the primary system at the end. To the best of our knowledge it is not known whether an entangled catalyst in a mixed state can offer an advantage over pure-state catalysts. This question is open even for transformations between bipartite pure states. In fact, the trumping conditions (Klimesh, 2007; Turgut, 2007) as well as the condition based on Rényi entropies (Brandão et al., 2015) all assume that the catalyst is in a pure state. It is not known whether or not relaxing this condition will enrich the possible state transformations. Thus, a complete characterization of exact catalytic transformations between bipartite pure states remains a pressing open question. It is worth mentioning that the above question has been solved for approximate entanglement catalysis, as Theorem 6 does not require the catalyst state to be pure. Even more, it has been shown in (Datta et al., 2022) that a catalyst in a mixed state is required to achieve some transitions which are not achievable otherwise. In quantum thermodynamics, most results on catalytic state transformations are limited to specific families of states, e.g., states which are block diagonal in the energy eigenbasis. A complete characterization of catalytic transformations via thermal operations between general quantum states has so far remained open.

Another open question concerns the potential equivalence between asymptotic state transformations without catalysis
and single-copy transformations with approximate catalysis. Isolated results demonstrating this equivalence have been presented recently, but a systematic understanding of this phenomenon is still lacking. For Gibbs-preserving operations the equivalence has been proven by Shiraishi and Sagawa (2021), showing that for any pair of asymptotically reducible states the transformation can also be achieved via approximate catalysis and vice versa. The equivalence between asymptotic reducibility and approximate catalysis has also been proven for pure-state entanglement in the bipartite setting (Kondra et al., 2021a). For general transformations between multipartite entangled states, it has been shown that asymptotic reducibility implies that the transition is also achievable on the single-copy level via approximate catalysis (Kondra et al., 2021a). However, it remains unclear whether catalysis can offer an advantage over asymptotic setups (without catalysis) in any of these scenarios. This leads to an intriguing question of whether bound entangled states (Horodecki et al., 1998) can be distilled into singlets with the aid of catalysis.

In the existing literature, entanglement catalysis has primarily been explored within the confines of bipartite, or tripartite frameworks. However, an emerging and fascinating avenue for research involves the study of entanglement in many-body systems (Amico et al., 2008; Frérot et al., 2023; Streltsov et al., 2020). To effectively explore catalysis in these multipartite configurations, it becomes essential to move beyond the confines of the traditional LOCC framework and embrace more advanced paradigms (Streltsov, 2023). We hope that these problems will be addressed and solved in the near future, thus significantly improving our understanding of quantum catalysis and quantum systems in general.

Many of the recent results discussed in this article make use of an analogy between asymptotic and catalytic setups, as developed by Shiraishi and Sagawa (2021) for quantum thermodynamics. This technique allows to build catalytic protocols in quantum thermodynamics and other quantum resource theories, by resorting to the tools developed for the asymptotic setting. When it comes to asymptotic analysis of resource theories, an important line of research aims to build reversible resource theories from minimal assumptions, and provide quantifiers for the corresponding state transition rates. Indeed, it is believed that under few reasonable assumptions, the conversion rates between quantum states are characterized by the regularized relative entropy of the corresponding resource:

\[
R(\rho \rightarrow \sigma) = \frac{E^\infty(\rho)}{E^\infty(\sigma)}, \quad (202)
\]

Here, \(E^\infty(\rho) = \lim_{n \rightarrow \infty} E(\rho^\otimes n)/n\), and \(E(\rho) = \inf_{\sigma \in \mathcal{F}} S(\rho||\sigma)\) is the relative entropy of the resource, where \(\mathcal{F}\) is the set of free states of the resource theory under study.

While Brandão and Gour (2015) claimed that Eq. (202) holds for a general class of quantum resource theories, the proof of this result presented in (Brandão and Gour, 2015) relies on the generalized quantum Stein’s lemma (Brandão and Plenio, 2010a), which has been recently called into question (Berta et al., 2022). It thus remains an important open problem to characterize all quantum resource theories for which Eq. (202) holds true. Resolving this problem is also important for the understanding of catalysis in general quantum resource theories. We also note that none of the results reviewed in this article are affected by the problem reported in (Brandão and Plenio, 2010a), more details are given in the Appendix.

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APPENDIX

Recently, Berta et al. (2022) uncovered a gap in the proof of the main result of (Brandão and Plenio, 2010a), a generalisation of the quantum Stein’s lemma. This puts directly into question the validity of the framework for general quantum resource theories, and the resource theory of quantum entanglement specifically, presented in (Brandão and Gour, 2015; Brandão and Plenio, 2010a,b), which tries to establish those theories as reversible. Briefly speaking, a reversible resource theory is such that it admits a unique entropy-like quantifier of the resource, which governs all possible asymptotic transformations between resourceful states (Chitambar and Gour, 2019). As of today, it is not known whether the generalised quantum Stein’s lemma holds, and hence one has to resort to other particular results, proven independently, that assure reversibility of individual resource theories. For example, it is true that the resource theory of coherence is reversible under a relevant class of asymptotically coherence-nonincreasing operations (Berta et al., 2022). We scrutinised all publications mentioned in this review and concluded that none of the results presented here are affected by the issue described in (Berta et al., 2022).
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