HEIGHT ESTIMATES FOR H-SURFACES IN THE WARPED PRODUCT $\mathbb{M} \times f \mathbb{R}$

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Abstract. In this article, we consider compact surfaces $\Sigma$ having constant mean curvature $H$ ($H$-surfaces) whose boundary $\Gamma = \partial \Sigma \subset \mathbb{M}_0 = \mathbb{M} \times f \{0\}$ is transversal to the slice $\mathbb{M}_0$ of the warped product $\mathbb{M} \times f \mathbb{R}$, here $\mathbb{M}$ denotes a Hadamard surface. We obtain height estimate for a such surface $\Sigma$ having positive constant mean curvature involving the area of a part of $\Sigma$ above of $\mathbb{M}_0$ and the volume it bounds. Also we give general conditions for the existence of rotationally-invariant topological spheres having positive constant mean curvature $H$ in the warped product $\mathbb{H} \times f \mathbb{R}$, where $\mathbb{H}$ denotes the hyperbolic disc. Finally we present a non-trivial example of such spheres.

1. Introduction

It is a classical result that a compact graph with positive constant mean curvature $H$ in the Euclidean three-dimensional space $\mathbb{R}^3$ and boundary on a plane can reach at most a height $1/H$ from this plane. Actually, this estimate is optimal because it is attained by the hemisphere of radius $1/H$, this classical result was proved in [6]. More recently, in [1], the authors have obtained height estimates for compact embedded surfaces with positive constant mean curvature in a Riemannian product space $\mathbb{M} \times \mathbb{R}$ (here $\mathbb{M}$ denotes a Riemannian surface without boundary) and boundary on a slice. In particular, they have obtained optimal height estimate for the homogeneous space $\mathbb{H} \times \mathbb{R}$ (here $\mathbb{H}$ denotes the complete connected simply connected hyperbolic disc having constant curvature $\kappa_g = -1$). The existence of height estimates for surfaces in a 3-dimensional ambient space reveals, in general, important properties on the geometric behaviour of these surfaces as well as existence and uniqueness results.

On the other hand, in [3], the authors have obtained height estimates for positive constant mean curvature, compact embedded surfaces in the product space $\mathbb{M}^2 \times \mathbb{R}$, whose boundary lies in a slice $\mathbb{M}^2_0$, here $\mathbb{M}^2$ denotes a Hadamard surfaces. They have obtained a relation between the height, the area above this slice and the volume it bounds. They were inspired by [7]. In this article, we generalize this height estimate for the warped product $\mathbb{M} \times f \mathbb{R}$. We obtain the following estimate.

Theorem 1.1. Let $\mathbb{M}$ be a Hadamard surface whose sectional curvature $K(\mathbb{M})$ satisfies $K(\mathbb{M}) \leq -\kappa \leq 0$ and let $\Sigma$ be a compact $H$-surface embedded in the warped product $\mathbb{M} \times f \mathbb{R}$, with boundary belonging to the slice $\mathbb{M}_0 = \mathbb{M} \times f \{0\}$ and transverse to $\mathbb{M}_0$. If $h$ denotes the height of $\Sigma_1$ with respect to $\mathbb{M}_0$, we have that

$$h \leq \frac{HF A^+}{2\pi} - \kappa \frac{Vol(U_1)}{4\pi},$$

where $F$, $A^+$ and $U_1$ are defined in Section 3. The equality holds if, and only if, $K(\mathbb{M}) \equiv -\kappa$ inside $U_1$ and $\Sigma$ is foliated by circles. Moreover if equality holds, then $f$ is constant on $B$.

Also, we focus our attention in the study of existence of rotationally-invariant spheres, which are compact and embedded in the warped product $\mathbb{H} \times f \mathbb{R}$, having positive constant mean curvature $H$. We give conditions to the existence of such spheres (see Theorem 4.5) and we present a non-trivial warping function $f$, whose associated warped product $\mathbb{M} \times f \mathbb{R}$ admits constant mean curvature, embedded, compact spheres, see Corollary 4.6.

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This article is organized as follows. In Section 2, we collect some results which are used along this work. In Section 3 we establish our main result, the height estimate. In Section 4 we study the mean curvature equation for surfaces immersed in $\mathbb{M} \times_f \mathbb{R}$. On the other hand, we give conditions to the existence of rotationally invariant topological spheres in the warped product $\mathbb{H} \times_f \mathbb{R}$ having positive constant mean curvature. We conclude this section constructing an example of such a topological sphere.

2. Preliminaries

Let $\mathbb{M}$ be a Hadamard surface, that is, a complete, simply connected, two dimensional Riemannian manifold, whose sectional curvature $K(\mathbb{M})$ satisfies $K(\mathbb{M}) \leq -\kappa \leq 0$, for some constant $\kappa \geq 0$. We denote by $\mathbb{R}$ the set of real numbers. On a tri-dimensional Riemannian product $\mathbb{M} \times \mathbb{R}$ we consider the canonical projections $\pi_1 : \mathbb{M} \times \mathbb{R} \rightarrow \mathbb{M}$ and $\pi_2 : \mathbb{M} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $\pi_1(p,t) = p$ and $\pi_2(p,t) = t$ respectively.

Definition 2.1. Suppose $\mathbb{M}$ is a Hadamard surface endowed with Riemannian metric $g_\mathbb{M}$ and as usual, the real line $\mathbb{R}$ is endowed with the canonical metric $g_0 = dt^2$. Let $f : \mathbb{M} \rightarrow \mathbb{R}$ be a smooth real function. The warped product $\mathbb{M} \times_f \mathbb{R}$ is the product manifold $\mathbb{M} \times \mathbb{R}$ endowed with metric

$$g = \pi_1^*(g_\mathbb{M}) + (e^f \circ \pi_1)^2 \pi_2^*(g_0).$$

Explicitly, if $v$ is a tangent vector to $\mathbb{M} \times_f \mathbb{R}$ at $(p,t)$, then

$$g(v,v) = g_\mathbb{M}(d\pi_1(v),d\pi_1(v)) + e^{2f(p)}g_0(d\pi_2(v),d\pi_2(v)).$$

As usual $\mathbb{M}$ is called the base of the warped product $\mathbb{M} \times_f \mathbb{R}$, $\mathbb{R}$ the fiber and $f$ the warping function.

Note that

(i) For each $t$, the map $\pi_1|_{(\mathbb{M} \times_f \{t\})}$ is an isometry onto $\mathbb{M}$.

(ii) For each $p$, the map $\pi_2|_{\{p\} \times \mathbb{R}}$ is a positive homothety onto $\mathbb{R}$ with scale factor $e^{-f(p)}$.

(iii) For each $(p,t) \in \mathbb{M} \times_f \mathbb{R}$, the slice $\mathbb{M} \times_f \{t\}$ and the fiber $\{p\} \times_f \mathbb{R}$ are orthogonal at $(p,t)$.

We denote by $\mathcal{L}(\mathbb{M}) \subset \chi(\mathbb{M} \times_f \mathbb{R})$ and by $\mathcal{L}(\mathbb{R}) \subset \chi(\mathbb{M} \times_f \mathbb{R})$ the set of all lifts of vector fields of $\chi(\mathbb{M})$ and of $\chi(\mathbb{R})$ respectively. Let $\xi = \text{lift}(\partial t)$, where $\partial t$ denotes a unit tangent vector field to the real line $\mathbb{R}$. By abuse of notation, we will use the same notation for a vector field and for its lifting. Sometimes we will use the over-bar to emphasize the lift of a vector field. We say that a vector field $X \in \chi(\mathbb{M} \times_f \mathbb{R})$ is vertical if $X$ is a non-zero multiple of $\xi$. If $g(X,\xi) = 0$, $X$ is said to be horizontal. For a vector field $Z \in \chi(\mathbb{M} \times_f \mathbb{R})$, we denote $\|Z\| := g(Z,Z)^{1/2}$.

We denote by $\nabla, \nabla^\mathbb{R}, \nabla$ the Levi-Civita connection of $\mathbb{M}$, $\mathbb{R}$ and $\mathbb{M} \times_f \mathbb{R}$ respectively. A straightforward computation gives the following lemma. See [8].

Lemma 2.2. On $\mathbb{M} \times_f \mathbb{R}$, if $X,Y \in \mathcal{L}(\mathbb{M})$ and $V,W \in \mathcal{L}(\mathbb{R})$, then

1. $\nabla_X Y \in \mathcal{L}(\mathbb{M})$ is the lift of $\nabla_X Y$ on $\mathbb{M}$, that is, $\nabla_X Y = \nabla_X^\mathbb{M} Y$.

2. $\nabla_X V = \nabla_V X = (Xf) V$.

3. $\nabla_W V = -g(V,W)\text{grad}(f) + \nabla_W^\mathbb{R} V$.

As an immediate consequence of Lemma 2.2, the vertical field $\xi$ is a Killing vector field.

In this section we recall some important results which will be used in the proof of the main theorem.
2.1. **Coarea Formula.** Let $h$ be a proper smooth real function defined on a Riemannian manifold $(M, g)$. Then the set of critical values of $h$ is a null set of $\mathbb{R}$ and the set $O$ of regular values is an open subset of $\mathbb{R}$. For $t \in O$, $h^{-1}(t)$ is a compact hypersurface of $M$, and the gradient vector $\text{grad}(h)(q)$, $h(q) = t$, is perpendicular to $h^{-1}(t)$. Now, we set

$$\Omega_t = \{ p \in M ; h(p) > t \}, \quad V_t := \text{vol}(\Omega_t)$$

$$\Gamma_t = \{ p \in M ; h(p) = t \}, \quad A_t := \text{vol}_{n-1} (\Gamma_t)$$

See [?, Theorem 5.8]

**Theorem 2.3** (Coarea Formula). The map $t \mapsto V_t$ is of class $C^\infty$ at a regular value $t$ of $h$ such that $V_t < +\infty$, and

$$V_t' = - \int_{\Gamma_t} \| \text{grad}(h) \|^{-1} d\nu_{g_t}$$

where $g_t$ is the induced metric on $\Gamma_t$ from $g$ and $V'(t) = \frac{dV}{dt}(t)$.

2.2. **The Flux Formula.** Let $U$ be a bounded domain in a Riemannian three manifold $M$, whose boundary, $\partial U$, consists of a smooth connected surface $\Sigma$, and the union $Q$ of finitely many smooth, compact and connected surfaces. The closed surface $\partial U$ is piecewise-smooth and smooth except perhaps on $\partial \Sigma = \partial Q$. Let

- $n$ = the outward-pointing unit normal vector field on $\partial U$,
- $n_\Sigma = $ the restriction of $n$ to $\Sigma$,
- $n_Q = $ the restriction of $n$ to $Q$,
- $n_1 = $ the outward-pointing unit conormal to $\Sigma$ along $\partial \Sigma$.

Suppose $Y$ is a vector field defined on a region of $M$ that contains $U$. It was proved in [5, Proposition 3] the following Flux Theorem.

**Theorem 2.4** (Flux Theorem). If $Y$ is a Killing vector field on $M$ and $\Sigma$ is a surface having constant mean curvature $H = g(\vec{H}, n)$. Then

$$\int_{\partial \Sigma} Y \cdot n_1 + H \int_{Q} Y \cdot n_Q = 0$$

where $Q$, $n_1$ and $\Sigma$ are as defined above.

2.3. **Isoperimetric Inequality for Surfaces.** Let $M$ be a two-dimensional $C^2$-manifold endowed with a $C^2$-Riemannian metric. We say that $M$ is a generalized surface if the metric in $M$ is allowed to degenerate at isolated points; such points are called singularities of the metric.

**Theorem 2.5** (Theorem 1.2,[2]). [Isoperimetric Inequality] Let $M$ be a generalized surface. Let $D$ be a simply connected domain in $M$ with area $A$ and bounded by a closed piecewise $C^1$-curve $\Gamma$ with length $L$. Let $K$ be the Gaussian curvature and $K_0$ be an arbitrary real number. Assume that in a neighborhood of a singular point, $K$ is bounded above. Then

$$L^2 \geq 4\pi A \left( 1 - \frac{1}{2\pi} \int_D (K - K_0) dM - \frac{K_0 A}{4\pi} \right),$$

equality holds if, and only if, $K \equiv K_0$ in $D$ and $D$ is a geodesic disc.
3. The main result

Let $\Sigma$ be a compact $H$-surface embedded in $\mathbb{M} \times_f \mathbb{R}$ with boundary belonging to $\mathbb{M}_0 = \mathbb{M} \times_f \{0\}$. Let $\Gamma$ be the boundary of $\Sigma$, $\Gamma = \partial \Sigma$, and assume $\Sigma$ is transverse to $\mathbb{M}_0$ along $\Gamma$.

We denote by $\Sigma^+$ and $\Sigma^-$ the intersection of $\Sigma$ with half-space above and below of $\mathbb{M}_0$ respectively. There is a connected component of $\Sigma^+$ or $\Sigma^-$ that contains $\Gamma$. Without loss of generality, we can assume that $\Gamma \subset \Sigma^+$. We call $\Sigma_1$ the connected component of $\Sigma^+$ that contains $\Gamma$.

Let $\hat{\Sigma}_1$ be the symmetry of $\Sigma_1$ with respect to $\mathbb{M}_0$. So $\hat{\Sigma}_1 \cup \Sigma_1$ is a compact embedded surface with no boundary, with corners along $\partial \Sigma_1$; this surface bounds a domain $U$ in $\mathbb{M} \times_f \mathbb{R}$. Let $U_1$ the intersection of $U$ with the half-space above $\mathbb{M}_0$. Thus $U_1$ is a bounded domain in $\mathbb{M} \times_f \mathbb{R}$, whose boundary $\partial U_1$ consist of the smooth connected surface $\Sigma_1$ and the union $\Omega$ of finitely smooth, compact and connected surfaces in $\mathbb{M}_0$. Let $A^+$ be denote the area of $\Sigma_1$.

Recall, we have considered the projections $\pi_1 : \mathbb{M} \times_f \mathbb{R} \to \mathbb{M}^2$ and $\pi_2 : \mathbb{M} \times_f \mathbb{R} \to \mathbb{R}$ given by $\pi_1(p,t) = p$ and $\pi_2(p,t) = t$ respectively. Let $B = \pi_1(\Sigma_1)$ be the projection on $\mathbb{M}$ of $\Sigma$, since $f$ is smooth, we denote by

$$\tilde{F} = \sup_{p \in B} \left( e^{-2f} \right) \quad \text{and} \quad F = \tilde{F} \cdot \sup_{p \in B} \left( e^f \right).$$

Under these notations, we have the following theorem.

**Theorem 3.1.** Let $\mathbb{M}$ be a Hadamard surface whose sectional curvature $K(\mathbb{M})$ satisfies $K(\mathbb{M}) \leq -\kappa \leq 0$ and let $\Sigma$ be a compact $H$-surface embedded in the warped product $\mathbb{M} \times_f \mathbb{R}$, with boundary belonging to the slice $\mathbb{M}_0 = \mathbb{M} \times_f \{0\}$ and transverse to $\mathbb{M}_0$. If $h$ denotes the height of $\Sigma_1$ with respect to $\mathbb{M}_0$, we have that

$$h \leq \frac{HFA^+}{2\pi} - \kappa \frac{\text{Vol}(U_1)}{4\pi}.$$  \hfill \left(3.1\right)

where $F$, $A^+$ and $U_1$ are as defined above. The equality holds if, and only if, $K(\mathbb{M}) \equiv -\kappa$ inside $U_1$ and $\Sigma$ is foliated by circles. Moreover if equality holds, then $f$ is constant on $B$.

**Proof.** We consider the unit normal $N$ of $\Sigma_1$ pointing inside of $U_1$. Let $\hat{H}$ the mean curvature vector of $\Sigma_1$ and we are supposing that the mean curvature function $H = g(\hat{H}, N) > 0$ is a positive constant. We denote by $h : \Sigma^+ \to \mathbb{R}$ the height function of $\Sigma$, that is, $h(p) = \pi_2(p)$ and $h_1 = h|_{\Sigma_1}$.

In order to estimate the function $h_1$, let $A(t)$ be the area of $\Sigma_t = \{ p \in \Sigma_1; h_1(p) \geq t \}$ and $\Gamma_t = \{ p \in \Sigma_1; h_1(p) = t \}$. Then, by the Co-area Formula

$$A'(t) = -\int_{\Gamma_t} \frac{1}{||\text{grad}_f(h_1)||} dv_{\Sigma_t}, \quad t \in O,$$

where $O$ is the set of all regular values of $\Sigma_1$. And we denote by $L(t)$ the length of the planar curve $\Gamma(t)$, so by the Schwartz inequality

$$L^2(t) \leq \int_{\Gamma(t)} ||\text{grad}_f(h_1)|| dv_t \int_{\Gamma(t)} \frac{1}{||\text{grad}_f(h_1)||} dv_t = -A'(t) \int_{\Gamma(t)} ||\text{grad}_f(h_1)|| dv_t, \quad t \in O.$$  \hfill \left(3.2\right)

On the other hand, we can decompose the vertical Killing field $\xi$ in the tangent and normal projections over the surface $\Sigma$. That is, we can write

$$\xi = T + vN$$

here $T$ is the tangent projection of $\xi$ and $v = g(\xi, N)$ is the normal component of $\xi$ over $\Sigma$. Notice that $\xi = e^{2f} \text{grad}_f(h)$, it follows that

$$T = e^{2f} \text{grad}_f(h_1)$$

\hfill \left(3.3\right)

\hfill \left(3.4\right)
which implies \( \| \text{grad}(h_1) \| = e^{-2f} \| T \| \) and by definition of \( \tilde{F} \) the inequality (3.2) becomes

\[
L^2(t) \leq -\tilde{F}A'(t) \int_{\Gamma(t)} \| T \| ds, \quad t \in O.
\]

Furthermore

\[
\| T \|^2 = g(\eta^t, \xi)^2
\]

where \( \eta^t \) is the inner conormal of \( \Sigma_t \) along \( \partial \Sigma_t \). Since \( \Sigma_t \) is above the plane \( \mathbb{M}_0 \) we have \( g(\eta^t, \xi) \geq 0 \) and therefore \( \| T \| = g(\eta^t, \xi) \). Once here, from (3.5), we obtain

\[
L^2(t) \leq -\tilde{F}A'(t) \int_{\Gamma(t)} g(\eta^t, \xi) ds, \quad t \in O.
\]

Let \( N_{\Sigma_t}, N_{\Omega_t} \) be the unit normal fields to \( \Sigma_t \) and \( \Omega_t \), respectively, that point inside \( U(t) \). Denote by \( \eta^t \) the unit conormal to \( \Sigma_t \) along \( \partial \Sigma_t \), pointing inside \( \Sigma_t \). Finally assume that \( \Sigma_t \) is a compact surface with constant mean curvature \( H = g(\tilde{H}, N_{\Sigma_t}) > 0 \). Let \( Y \) be a Killing vector field in \( \mathbb{M} \times_f \mathbb{R} \). Then by the Flux Formula

\[
\int_{\partial \Sigma_t} g(Y, \eta^t) = 2H \int_{\Omega_{\Omega_t}} g(Y, N_{\Omega_t})
\]

taking \( Y = \xi \) in (3.8), we have

\[
\int_{\Gamma(t)} g(\xi, \eta^t) \leq 2H \cdot \sup_{p \in B} \left( e^f \right) \cdot \| \Omega(t) \|
\]

where \( \| \Omega(t) \| \) is the area of the planar region \( \Omega(t) \). Thus if we substitute in (3.7), we obtain

\[
L^2(t) \leq -2HF\tilde{F}A'(t)\|\Omega(t)\|, \quad \text{for almost every } t \geq 0, \quad t \in O.
\]

Using the Isoperimetric Inequality for Surfaces, it was proved in [3]

\[
L^2(t) \geq 4\pi\|\Omega(t)\| + \kappa \| \Omega \|^2
\]

From (3.9) and (3.10), we obtain

\[
4\pi\|\Omega(t)\| + \kappa \| \Omega \|^2 \leq -2HF\tilde{F}A'(t)\|\Omega(t)\|
\]

\[
\|\Omega(t)\| \left( 4\pi + \kappa \| \Omega \| \right) + 2HF\tilde{F}A'(t) \leq 0
\]

(3.11)

\[
4\pi + \kappa \| \Omega \| + 2HF\tilde{F}A'(t) \leq 0
\]

By integrating inequality (3.11) from 0 to \( h = \max_{p \in \Sigma} h_1(p) \geq 0 \), we obtain

\[
4\pi h + 2HF(A(h) - A(0)) + \kappa Vol(U_1) \leq 0
\]

therefore

\[
A^+ = A(0) \geq \frac{2\pi h}{HF} + \frac{\kappa Vol(U_1)}{2HF}
\]

which is equivalent o inequality (3.1).

If the equality holds, then all the above inequalities become equalities. In particular, by Isoperimetric Inequality for Surfaces Theorem, \( \Gamma(t) \) is the boundary of a geodesic disc in \( \mathbb{M} \times_f \{ t \} \), for every \( t \geq 0 \), and \( K(\mathbb{M})(p) \equiv -\kappa \) for all \( p \in U \).

On the other hand, if equality holds on (3.1), the inequality (3.5) is a equality which implies that \( e^{-2f} \equiv \tilde{F} \) and then the warping function \( f \) is constant on \( B \).

\[ \square \]

We have the following consequences.
Corollary 3.2. Let $\Sigma$ be a compact, without boundary, embedded surface in the warped product $M \times_f R$, having constant mean curvature $H > 0$ and area $A$. Let $U$ be the compact domain bounded by $\Sigma$, then $\Sigma$ lies in a horizontal slab having height less than $\frac{HF \cdot A}{\pi} - \frac{\kappa \cdot \text{Vol}(U)}{2\pi}$, where $F$ is defined in the previous theorem. Moreover, one has equality if, and only if, $K(M) \equiv \kappa$ inside $U$ and $\Sigma$ is foliated by circles.

Corollary 3.3. Let $\Sigma$ be a compact, embedded surface in the warped product $M \times_f R$, having constant mean curvature $H > 0$ with boundary in the slice $M_0 = M \times_f \{0\}$ and transverse to $M_0$. Then

$$\kappa \frac{\text{Vol}(U_1)}{4\pi} \leq \frac{HF \cdot A^+}{2\pi}$$

where $F$, $A^+$ and $U_1$ are defined in the previous theorem.

4. MEAN CURVATURE EQUATION

Let $\Omega \subset M$ be a domain and $u : \Omega \to R$ be a smooth function, The graph of $u$ in $M \times_f R$ is the set

$$(4.1) \quad \Sigma_u = \{(p, u(p)) \in M \times_f R; p \in \Omega\}$$

Let $\vec{H}$ denote the mean curvature vector field of $\Sigma_u$ and we choose a unit normal vector field $\vec{N}$ to $\Sigma_u$ satisfying $g(\vec{N}, \xi) \leq 0$. Throughout this article a surface having constant mean curvature $H$ will be called an $H$-surface. In order to obtain the mean curvature equation in the divergence form, we prove the next lemma.

Lemma 4.1. Let $X$ be a vector field in $M \times_f R$

$$(4.2) \quad e^f \text{div}_f(X) = \text{div}_M(e^f d\pi_1(X)) + \xi \left(e^{-f} g(X, \xi)\right),$$

where $\text{div}$ and $\text{div}_M$ are the divergence on $M \times_f R$ and $M$, respectively.

Proof. Let $\{x_1, x_2\}$ be local coordinates for $M$, and $\{x_3 = t\}$ a local coordinate for $R$ whose associated vectors fields are $\{\partial_{x_1}, \partial_{x_2}, \partial_{x_3}\}$. Their lifts to $M \times_f R$ are denoted by $\{\tilde{\partial}_{x_1}, \tilde{\partial}_{x_2}, \tilde{\partial}_{x_3} = \xi\}$, these are the associated vector field to the local coordinates $\{x_1, x_2, x_3\}$ of $M \times_f R$. Denoting by $g^{ij}$ the coefficients of the inverse matrix of $g$ and apply the definition of the divergence of a vector field $X$ on $M \times_f R$ we obtain

$$\text{div}_f(X) = \frac{1}{\sqrt{\det g}} \sum_{i,j=1,2,3} \left( \partial_{x_i} \left( \sqrt{\det g} g^{ij} g(X, \tilde{\partial}_{x_j}) \right) \right)$$

$$= \frac{1}{e^f \sqrt{\det g_M}} \left( \sum_{i,j=1,2} \partial_{x_i} \left( e^f \sqrt{\det g_M} g^{ij} g_M(d\pi_1(X), \partial_{x_j}) \right) + \partial_{x_3} \left( e^{-f} \sqrt{\det g_M} g(X, \tilde{\partial}_{x_3}) \right) \right)$$

$$= \frac{1}{\sqrt{\det g_M}} \left( \sum_{i,j=1,2} \partial_{x_i} \left( \sqrt{\det g_M} g^{ij} g_M(d\pi_1(X), \partial_{x_j}) \right) + \partial_{x_3} \left( e^{-2f} \sqrt{\det g_M} g(X, \tilde{\partial}_{x_3}) \right) \right) +$$

$$+ \frac{1}{e^f \sqrt{\det g_M}} \left( \sqrt{\det g_M} \sum_{i,j=1,2} \partial_{x_i}(e^f) g_M(d\pi_1(X), \partial_{x_j}) g^{ij} \right)$$

$$= \text{div}_M(d\pi_1(X)) + \frac{1}{e^f} g_M \left( d\pi_1(X), \text{grad}_M(e^f) \right) + \partial_{x_3} \left( e^{-2f} g(X, \tilde{\partial}_{x_3}) \right).$$

The last equality follows from de definition of the gradient on $M$ which is given by $\text{grad}_M(e^f) := \sum_{i=1,2} \partial_{x_i}(e^f) g^{ij}_M \partial_{x_j}$. Then,
\[ e^f \text{div}_f(X) = \text{div}_M (e^f d\pi_1(X)) + \xi \left( e^{-f} g(X, \xi) \right). \]

Taking the equation (4.2) into account we obtain the following mean curvature equation for vertical graphs in \( M \times_f \mathbb{R} \).

**Lemma 4.2.** Let \( \Sigma_u \subset M \times_f \mathbb{R} \) be the vertical graph of a smooth function \( u : \Omega \subset M \to \mathbb{R} \) having mean curvature function \( H \). Then, \( u \) satisfies

\[
-2He^f = \text{div}_M \left( e^f \frac{\text{grad}_M u}{W} \right),
\]

where \( W^2 = e^{-2f} + ||\text{grad}_M u||^2 \).

**Proof.** We consider a smooth function \( u^* : M \times_f \mathbb{R} \to \mathbb{R} \) defined by \( u^*(x, y, t) = u(x, y) \). Set \( F(x, y, t) = u^*(x, y, t) - t \), therefore zero is a regular value of \( F \) and \( F^{-1}(0) = \Sigma_u \). It is well-known that the function \( H \) satisfies

\[
2H = -\text{div}_f \left( \frac{\text{grad}_f(F)}{||\text{grad}_f(F)||} \right),
\]

where \( \text{div}_f \) and \( \text{grad}_f \) denote the divergence and gradient in \( M \times_f \mathbb{R} \), respectively. Let \( X \) be a vector field on \( M \), we denote its lift to \( M \times_f \mathbb{R} \) by \( \tilde{X} \). We have,

\[
\text{grad}_f(F) = \text{grad}_M(u) - e^{-2f} \xi.
\]

Setting \( W^2 = ||\text{grad}_f(F)||^2 = e^{-2f} + ||\text{grad}_M u||^2 \) and applying Lemma 4.1 in equation (4.4), we obtain

\[
2H e^f = -e^f \text{div}_f \left( e^f \frac{\text{grad}_M(u)}{W} \right)
\]

\[
= -\text{div}_M \left( e^f \frac{\text{grad}_M(u)}{W} \right) + \xi \left( e^{-f} g \left( \frac{e^{-2f} \xi}{W}, \xi \right) \right)
\]

\[
= -\text{div}_M \left( e^f \frac{\text{grad}_M(u)}{W} \right).
\]

\[ \Box \]

4.1. **Some \( H \)-surfaces in \( \mathbb{H} \times_f \mathbb{R} \).** Now let us focus on the case \( M = \mathbb{H} \), where \( \mathbb{H} \) is the connected, simply connected two-dimensional Hyperbolic disc \( \mathbb{H} = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\} \) having constant curvature \( \kappa = -1 \). We consider coordinates \( (\rho, \theta) \) in \( \mathbb{H} \), where \( \rho \) is the hyperbolic distance to the origin and \( \theta \) is the angle between a segment from the origin and the positive semi-axis \( x \). More precisely, we consider a parametrisation \( \varphi(\rho, \theta) = (\tanh(\rho/2) \cos \theta, \tanh(\rho/2) \sin \theta) \) from \( (0, +\infty) \times [0, 2\pi) \) in the hyperbolic disc \( \mathbb{H} \). For simplicity, we treat properties of the surfaces \( \Sigma_u \) using the disc model \( \mathbb{H} \) or the slab \( (0, +\infty) \times [0, 2\pi) \) (via the parametrisation \( \varphi \)). In these polar coordinates the metric on \( \mathbb{H} \) is given by

\[
g_\mathbb{H} = d\rho^2 + \sinh^2(\rho) \, d\theta^2
\]

Let \( u : \Omega \subset \mathbb{H} \to \mathbb{R} \) be a smooth function. A parametrization of the graph \( \Sigma_u \) given in (4.1) is

\[
\psi(\rho, \theta) = (\tanh(\rho/2) \cos \theta, \tanh(\rho/2) \sin \theta, u(\rho, \theta)),
\]

where \( (\rho, \theta) \in \varphi^{-1}(\Omega) \). Under this notation, we have the next lemma.
A straightforward computation give us

\[ -2H e^f = \frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \left[ g_H \left( e^f \frac{\text{grad}_H u}{W}, \partial_\rho \right) \sinh \rho \right] + \frac{1}{\sinh^2 \rho} \frac{\partial}{\partial \theta} \left[ g_H \left( e^f \frac{\text{grad}_H u}{W}, \partial_\theta \right) \right] \]

where \( f = f(\rho, \theta) \), \( W = e^{-2f} + \| \text{grad}_H u \|^2 \), and \( \frac{\partial}{\partial z} \) denotes the derivative with respect to \( z \).

**Proof.** From Lemma 4.5 and Divergence’s Theorem

\[ -\int_{\Omega} 2H e^f dA = \int_{\Omega} \text{div} \left( e^f \frac{\text{grad}_H u}{W} \right) dA = \int_{\partial \Omega} g_H \left( e^f \frac{\text{grad}_H u}{W}, \eta \right) d\gamma \]

where \( \Omega \) is a domain with boundary \( \partial \Omega \) and \( \eta \) is the unit outer-conormal to \( \Omega \).

Let us consider the domain \( \Omega = [\rho_0, \rho_1] \times [\theta_0, \theta_1] \) in the \( \rho\theta \)-plane. Notice \( \partial \Omega = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \), where

- \( \gamma_1(s) = (s, \theta_0), \quad \rho_0 \leq s \leq \rho_1 \),
- \( \gamma_2(s) = (\rho_1, s), \quad \theta_0 \leq s \leq \theta_1 \),
- \( \gamma_3(s) = (\rho_1 - s, \theta_1), \quad 0 \leq s \leq \rho_1 - \rho_0 \),
- \( \gamma_4(s) = (\rho_0, \theta_1 - s), \quad 0 \leq s \leq \theta_1 - \theta_0 \).

Therefore

\[ -2H e^f dA = \sum_{i=1}^{4} \int_{\gamma_i} g_H \left( e^f \frac{\text{grad}_H u}{W}, \eta_i \right) d\gamma \]

A straightforward computation give us

1. \( \int_{\gamma_1} g_H \left( e^f \frac{\text{grad}_H u}{W}, \eta_1 \right) d\gamma = -\int_{\rho_0}^{\rho_1} g_H \left( e^f \frac{\text{grad}_H u}{W}, \partial_\rho \right) (\rho, \theta_0) \frac{1}{\sinh \rho} d\rho \)
2. \( \int_{\gamma_2} g_H \left( e^f \frac{\text{grad}_H u}{W}, \eta_2 \right) d\gamma = \int_{\rho_0}^{\rho_1} g_H \left( e^f \frac{\text{grad}_H u}{W}, \partial_\rho \right) (\rho_1, \theta) \sinh \rho \frac{1}{\sinh \rho} d\rho \)
3. \( \int_{\gamma_3} g_H \left( e^f \frac{\text{grad}_H u}{W}, \eta_3 \right) d\gamma = \int_{\rho_0}^{\rho_1} g_H \left( e^f \frac{\text{grad}_H u}{W}, \partial_\rho \right) (\rho, \theta_1) \frac{1}{\sinh \rho} d\rho \)
4. \( \int_{\gamma_4} g_H \left( e^f \frac{\text{grad}_H u}{W}, \eta_4 \right) d\gamma = -\int_{\theta_0}^{\theta_1} g_H \left( e^f \frac{\text{grad}_H u}{W}, \partial_\theta \right) (\rho_0, \theta) \sinh \rho \frac{1}{\sinh \rho} d\theta \)

Using the last four expressions into equation (4.8), we obtain

\[ -\int_{\Omega} 2H e^f dA = \int_{\Omega} \frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \left[ g_H \left( e^f \frac{\text{grad}_H u}{W}, \sinh \rho \right) \right] + \frac{1}{\sinh^2 \rho} \frac{\partial}{\partial \theta} \left[ g_H \left( e^f \frac{\text{grad}_H u}{W}, \sinh \rho \right) \right] dA \]

Once equation (4.9) holds for any such \( \Omega \) we complete the proof.

\[ \square \]

4.2. Rotational spheres in \( \mathbb{H} \times f \mathbb{R} \). In this section we will construct rotationally-invariant spheres. In order to do that, the warping function \( f \) must depend only on \( \rho \) as well as the function \( u \), that is, \( f(\rho, \theta) \equiv f(\rho) \) and \( u(\rho, \theta) \equiv u(\rho) \).

Let \( I \) be an interval in the positive \( x \)-axis and \( u : I \rightarrow \mathbb{R} \) be a smooth function whose graph \( \Gamma_u \) lies in the \( xt \)-plane. Denote by \( \mathcal{G} \) the group of rotations around the origin of the hyperbolic disc \( \mathbb{H} \). The rotationally-invariant surface \( \Sigma_u \) obtained from \( \Gamma_u \) is the surface \( \Sigma_u = \mathcal{G} \Gamma_u \). We assume that \( \Sigma_u \)
has non-negative constant mean curvature $H$ with respect to the pointing downwards unit normal vector field $\overrightarrow{N}$ of $\Sigma_\rho$. In order to give the conditions to the existence of spheres in the warped product $\mathbb{H} \times_f \mathbb{R}$, we need the next definition.

**Definition 4.4.** Consider the warped product $\mathbb{H} \times_f \mathbb{R}$, with $f \equiv f(\rho)$. Let $H > 0$ be a positive constant and $d$ a constant depending on $H$ and $f$. We say the function $u = u(\rho)$ is an admissible solution if the following conditions hold

1. There exists an interval $[0, \rho_0]$, $\rho_0 < \infty$, such that the function
   \[
   G(\rho) = d - 2HF(\rho), \quad \text{with} \quad F_{\rho}(\rho) = e^{f(\rho)} \sinh \rho
   \]
   satisfies $G(0) = 0$, $G(\rho) > 0$ on $(0, \rho_0]$ and $G(\rho) < e^{f(\rho)} \sinh \rho$, on $(0, \rho_0]$. Where $F_{\rho}$ denotes the derivative of the function $F$ with respect to $\rho$.
2. The graph $\Gamma_\rho$ of the function $u$ is defined on $[0, \rho_0]$, $\rho_0 < \infty$ and is given by the ordinary differential equation (ODE)
   \[
   u_{\rho}(\rho) = \frac{G(\rho)}{e^{f(\rho)} \sqrt{e^{2f(\rho)} \sinh^2 \rho - G^2(\rho)}}
   \]
   which has zero derivative at $\rho = 0$, non-finite derivative at $q = (\rho_0, u(\rho_0))$ and finite geodesic curvature at the points $p$ and $q$ as long as the left side of equation (4.10) is well defined on $\rho = 0$.

Now we presented the main theorem of this section, the notation on Definition 4.4 will be used.

**Theorem 4.5 (Rotational spheres).** Let $f = f(\rho)$ be a warping function depending only on $\rho$. Suppose that for a positive constant $H > 0$ there exists a constant $d$ (depending on $H$ and $f$) and an admissible solution $u(\rho)$. Then there exists a rotational sphere $S^H$ having constant mean curvature $H$, which is invariant by the group $\mathcal{G}$ and up to vertical translations, is a bi-graph with respect to the slice $\mathbb{H}_0 = \mathbb{H} \times_f \{0\}$.

**Proof.** We have denoted by $\Sigma_u = \mathcal{G}_{\gamma_u}$ the rotationally-invariant surface which is a graph of a function $u(\rho, \theta) \equiv u(\rho)$. If $\Sigma_u$ has non-negative constant mean curvature $H$ with respect to the downwards pointing unit normal vector field $\overrightarrow{N}$ then, by Lemma 4.3 the function $u$ satisfies

\[
-2H e^{f} \sinh \rho = \frac{\partial}{\partial \rho} \left( e^{f} u_{\rho} \frac{u_{\rho}}{W} \sinh \rho \right)
\]
where $W^2 = e^{-2f} + u_{\rho}^2$. Recall that $F$, by Definition 4.4, satisfies the ODE $F_{\rho}(\rho) = e^{f(\rho)} \sinh \rho$. Intregating equation (4.11), we obtain

\[
d - 2H e^{f} F = \frac{e^{2f} u_{\rho}}{\sqrt{1 + (e^{f} u_{\rho})^2}} \sinh \rho
\]
where $d \in \mathbb{R}$ is a constant. Once we are assuming that there is an admissible solution, equation (4.12) is equivalent to

\[
u_{\rho}(\rho) = \pm \frac{d - 2HF}{e^{f} \sqrt{e^{2f} \sinh^2 \rho - (d - 2HF)^2}}
\]
The admissible solution $u$ is a solution for equation (4.13). We can glue together the graph of the $u$ with the graph of $-u$ in order to obtain the rotational sphere $S^H$. \qed
4.3. Examples of rotational spheres. Once rotationally-invariant spheres do not have to exists for every warping function \( f = f(\rho) \). To see that the set of admissible solutions is non-empty, we consider the non-trivial warping function \( f(\rho) \) given by

\[
 f(\rho) = \ln (2 \cosh(\rho))
\]

From Lemma 4.3, the function \( u \) which generates the rotationally-invariant surface \( \Sigma_u = \Phi \Gamma_u \) having constant mean curvature \( H \), satisfies

\[
 (4.14) - 2H \sinh(2\rho) = \partial_\rho \left( \frac{e^{2\rho} u_\rho}{\sqrt{1 + (e^\rho u_\rho)^2}} \sinh \rho \right)
\]

By integrating equation (4.14), we obtain

\[
 (4.15) u_\rho(\rho) = \pm \frac{d - H \cosh(2\rho)}{2 \cosh(\rho) \sqrt{\sinh^2(2\rho) - (d - H \cosh(2\rho))^2}}
\]

where \( d \in \mathbb{R} \). Once here, we have the next corollary.

**Corollary 4.6.** Consider the warping function \( f(\rho) = \ln (2 \cosh(\rho)), \) where \( \rho \geq 0 \) is the hyperbolic distance from the origin in the hyperbolic disk \( \mathbb{H} \). For each positive constant \( \rho > 1 \), there exists a rotational sphere \( S^H \), embedded in the warped product \( \mathbb{H} \times_f \mathbb{R} \) having constant mean curvature \( H \), which is invariant by the group \( \Phi \) and up to vertical translation, the sphere \( S^H \) is a bi-graph with respect to the slice \( \mathbb{H}_0 = \mathbb{H} \times_f \{0\} \).

**Proof.** For the product space \( \mathbb{H} \times \mathbb{R} \), it was proved in [10] (or [9]) the existence of an admissible solution which generates an embedded, compact, rotationally-invariant sphere \( S \) having constant mean curvature \( H \), for any constant satisfying \( 2H > 1 \). If we denote by \( u_0 \) this admissible solution, then \( u_0 \) satisfies the ODE

\[
 (4.16) (u_0)_\rho(\rho) = \frac{(2H - 2H \cosh(\rho))}{\sqrt{\sinh^2(\rho) - (2H - 2H \cosh(\rho))^2}},
\]

here \( \rho \in [0, \rho_0] \), for some fixed \( \rho_0 > 0 \).

Take \( d = H \), for \( H > 1 \) in equation (4.15), following the same ideas presented in [10], we see that the solution of equation (4.15) is an admissible solution over an interval \([0, \rho_1]\) for some fixed \( \rho_1 > 0 \). Notice that, there exists constants \( m \) and \( M \) such that

\[
 (4.17) \frac{H - H \cosh(2\rho)}{m \sqrt{\sinh^2(2\rho) - (H - H \cosh(2\rho))^2}} \leq u \leq \frac{H - H \cosh(2\rho)}{M \sqrt{\sinh^2(2\rho) - (d - H \cosh(2\rho))^2}}
\]
in \([0, \rho_1]\). By Theorem 4.5, the admissible solution \( u \) generates the sphere \( S^H \).

\[\square\]

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