A FAMILY OF RIESZ DISTRIBUTIONS FOR DIFFERENTIAL FORMS ON EUCLIDIAN SPACE

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Abstract. In this paper we introduce a new family of operator-valued distributions on Euclidian space acting by convolution on differential forms. It provides a natural generalization of the important Riesz distributions acting on functions, where the corresponding operators are \((-\Delta)^{-\alpha/2}\), and we develop basic analogous properties with respect to meromorphic continuation, residues, Fourier transforms, and relations to conformal geometry and representations of the conformal group.

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1. Introduction

Important aspects of analysis in Euclidian space are studied via the Riesz distributions \(|x|^\lambda\), see [Rie49], i.e. the complex powers of the Euclidian norm. As convolution operators these allow a rigorous treatment of the Laplace operator \(\Delta\) and its complex powers \(I_\alpha = (-\Delta)^{-\alpha/2}\), acting on functions as a natural semigroup, and there are many classical and recent results related to this family of operators, sometimes referred to as fractional Laplacians.

Generalizations of those ideas to norms induced by indefinite metrics or spinor valued distributions where studied in [KV91] and [CØ14].
In this paper we introduce a natural family of similar operators acting on differential forms in Euclidian space; however, the semigroup property (as above) has to be relaxed. The corresponding family of distributions seems to deserve the name of Riesz distributions for differential forms, and it allows a certain extra flexibility in the complex parameters involved. We shall start by giving the basic definitions and calculations in the most general cases, and later specialize to some specific cases of particular interest. One of the important special cases is not new by any means, since it corresponds to some of the cases of the intertwining operators treated by Knapp and Stein in detail, when they introduced their celebrated kernel operators \cite{KS71}. Here our calculations make these operators more explicit; in particular we give the Euclidian Fourier transform of these Knapp-Stein operators and simplifying proofs for the unitarity of the complementary series of representations of the conformal group. For this see Remark \ref{rem:4.10}.

Another interesting case which arises by specializing parameters in our family is the Beurling-Ahlfors operator $S$ (in even dimension, acting on forms of middle degree), see \cite{IM93}; this operator is in several ways an analogue of the Hilbert transform.

In summary, we shall for the Riesz distributions for differential forms \((3.1)\) be interested in their

- Fourier transforms (Theorem 3.2)
- Bernstein-Sato identities (Theorem 3.3)
- residues (Theorem 3.4)
- convolution formulas (Theorem 3.6)

and for example obtain the Branson-Gover operators \cite{BG05}

$$L_{2N}^{(p)} = \left(\frac{n}{2} - p + N\right)(\delta d)^N + \left(\frac{n}{2} - p - N\right)(d\delta)^N$$

on $p$-forms in Euclidian space (well-known from conformal geometry) as residues, see Corollary \ref{cor:4.4}. Here $d$ is the usual derivative on forms and $\delta$ its $L^2$-adjoint, and these (and their symbols) are the basic building blocks in our study. This corresponds well with the conjectural fact that in general (on general Riemannian manifolds) the Branson-Gover operators may be obtained as residues of the scattering operator \cite{AG11, GZ03}. While we shall carry out most of the analysis in Euclidian space, it is also possible to work on the conformal compactification, i.e. the standard sphere of the same dimension; for this we shall apply some of the formulas in \cite{BO0}, where the so-called compact picture of the bundles in question are studied.

We see the present paper as laying the groundwork for further studies of Euclidian analysis on differential forms, such as for example Sobolev inequalities, giving the basic facts towards finding fundamental solutions of natural differential operators on differential forms, and elliptic boundary value problems. Also we hope our study might contribute to the branching program of T. Kobayashi, in particular to his theory of symmetry-breaking operators and branching problems for complementary series representations.

### 2. Preliminaries about differential forms

Let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be the Euclidian vector space of dimension $n \in \mathbb{N}$ and $\{e_j\}_{j=1}^n$ its standard basis. The space of differential $p$-forms on $\mathbb{R}^n$, for $0 \leq p \leq n$, is denoted by $\Omega^p(\mathbb{R}^n)$. The Euclidian scalar product extends to $\Omega^p(\mathbb{R}^n)$ and will be denoted with
the same symbol. We introduce two algebraic actions on \( \omega \in \Omega^p(\mathbb{R}^n) \) which will be important: For \( x \in \mathbb{R}^n \) one defines

\[
i_x \omega \overset{\text{def}}{=} \sum_{k=1}^n x_k i_{e_k} \omega, \quad \varepsilon_x \omega \overset{\text{def}}{=} \sum_{k=1}^n x_k \varepsilon_{e_k} \wedge \omega.
\]

Mainly these are the interior and exterior product by \( x \in \mathbb{R}^n \) and are related as symbols to the exterior differential \( d : \Omega^p(\mathbb{R}^n) \to \Omega^{p+1}(\mathbb{R}^n) \) and the negative of the co-differential \( \delta : \Omega^p(\mathbb{R}^n) \to \Omega^{p-1}(\mathbb{R}^n) \), the \( L^2 \)-adjoint of the differential. They satisfy the well known identities:

**Lemma 2.1** The algebraic actions \( i_x \) and \( \varepsilon_x \) are nilpotent of degree 2 and formally adjoint to each other with respect to the scalar product \( \langle \cdot , \cdot \rangle \) on \( \Omega^p(\mathbb{R}^n) \), i.e. \( \langle i_x \omega, \eta \rangle = \langle \omega, \varepsilon_x \eta \rangle \) for all \( x \in \mathbb{R}^n \) and \( \omega \in \Omega^p(\mathbb{R}^n) \), \( \eta \in \Omega^{p-1}(\mathbb{R}^n) \). Furthermore, it holds

\[
i_x \varepsilon_x = \sum_{k,l=1}^n x_k x_l i_{e_k} (e_l \wedge \cdot), \quad \varepsilon_x i_x = \sum_{k,l=1}^n x_k x_l e_l \wedge i_{e_k} (\cdot), \quad i_x \varepsilon_x + \varepsilon_x i_x = \sum_{k=1}^n x_k^2.
\]

In the next section it will be important to know some differential actions on the distribution

\[
r^\lambda(x) \overset{\text{def}}{=} (x_1^2 + \cdots + x_n^2)^{\frac{\lambda}{2}} \tag{2.1}
\]

defined for \( x \in \mathbb{R}^n \) and \( \lambda \in \mathbb{C} \) with \( \Re(\lambda) > -n \). This distribution is termed Riesz distribution and was for example studied in [Rie49, GS64].

The following is the key lemma in order to obtain the Fourier transform of the Riesz distribution for differential forms.

**Lemma 2.2** Let \( \beta \) be a constant \( p \)-form. Then, for fixed \( x \in \mathbb{R}^n \),

\[
\delta d(r^{\lambda+2}(x-y)\beta) = - (\lambda + 2)(n - p)r^{\lambda}(x-y)\beta - (\lambda + 2)\lambda r^{\lambda-2}(x-y)i_{x-y}\varepsilon_{x-y}\beta, \\
d\delta(r^{\lambda+2}(x-y)\beta) = - (\lambda + 2)p r^{\lambda}(x-y)\beta - (\lambda + 2)\lambda r^{\lambda-2}(x-y)\varepsilon_{x-y} i_{x-y}\beta.
\]

Derivatives are taken with respect to the \( y \)-variable.

**Proof.** In order to prove the lemma we first observe

\[
\partial_k r^\nu(x-y) = -\nu(x_k - y_k) r^{\nu-2}(x-y), \\
\sum_{k=1}^n e_k \wedge i_{e_k} \beta = p \beta, \quad \sum_{k=1}^n i_{e_k} (e_k \wedge \beta) = (n - p) \beta. \tag{2.2}
\]

Then a straightforward computation shows

\[
\delta d(r^{\lambda+2}(x-y)\beta) = - \sum_{k,l=1}^n \partial_l \partial_k (r^{\lambda+2}(x-y)) i_{e_k} (e_l \wedge \beta) \\
= - (\lambda + 2)r^{\lambda}(x-y) \sum_{k=1}^n i_{e_k} (e_k \wedge \beta)
\]
Using the above observations (2.2) and Lemma 2.1 we may conclude

$$\delta d(r^{\lambda+2}(x - y)\beta) = -(\lambda + 2)(n - p)r^\lambda(x - y)\beta - (\lambda + 2)\lambda r^{\lambda-2}(x - y)i_{x-y}e_x \varepsilon_{x-y}\beta.$$  

The second claim runs along the same line, which completes the proof. □

Let us denote by $S^p(\mathbb{R}^n) \overset{\text{def}}{=} S(\mathbb{R}^n) \otimes \Lambda^p(\mathbb{R}^n)$ the space of Schwartz functions with value in $\Lambda^p(\mathbb{R}^n)^*$. We follow the convention for the Fourier transform [GS64] on Schwartz functions $f \in S(\mathbb{R}^n)$:

$$\mathcal{F}(f)(\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^n} f(x)e^{i\langle x, \xi \rangle} dx,$$

and extend it to a Fourier transform on $\omega \in S^p(\mathbb{R}^n)$ by acting on the coefficients of $\omega$ in some arbitrarily chosen basis. Our normalization of the Fourier transform is chosen in such a way that $\mathcal{F}(\delta_0) = 1$, where $\delta_0$ is the Dirac-distribution centered at the origin.

Recall that for a polynomial $P$ in $n$ variables we have the identities

$$P(\partial_{\xi_1}, \ldots, \partial_{\xi_n})\mathcal{F}(\omega)(\xi) = \mathcal{F}(P(ix_1, \ldots, ix_n)\omega)(\xi),$$

$$\mathcal{F}(P(\partial_{x_1}, \ldots, \partial_{x_n})\omega)(\xi) = P(-i\xi_1, \ldots, -i\xi_n)\mathcal{F}(\omega)(\xi) \quad (2.3)$$

for $\omega \in S^p(\mathbb{R}^n)$.

3. A class of Riesz distributions for differential forms

We introduce a class of tempered distributions $R_{A_{\lambda}, B_{\lambda}}^{\lambda}(x)$ valued in endomorphisms of differential forms, which generalizes the Riesz distribution (2.1). Based on an explicit formula for its Fourier transform, we give a corresponding Bernstein-Sato identity, compute its residues and some identities concerning their convolutions. Finally, this distribution when acting by convolution on differential forms induces an integral operator.

Let $A_{\lambda}, B_{\lambda}$ be holomorphic functions in $\lambda \in \mathbb{C}$. For $\lambda \in \mathbb{C}$ with $\Re(\lambda) > -n$ a distribution valued in the endomorphisms of differential forms, termed Riesz distribution for differential forms, is defined by

$$R_{A_{\lambda}, B_{\lambda}}^{\lambda}(x) \overset{\text{def}}{=} r^{\lambda-2}(x)(A_{\lambda}i_x e_x + B_{\lambda} e_x i_x). \quad (3.1)$$

The parameters $A_{\lambda}, B_{\lambda}$ can also depend on the form-degree $p$ and the dimension $n$, but we will suppress such dependencies.

**Remark 3.1** One could also assume that $A_{\lambda}$ and $B_{\lambda}$ are meromorphic in $\lambda$ without disturbing the results obtained here. Of course, possible poles of $A_{\lambda}$ and $B_{\lambda}$ will influence the assumptions and statements in this paper.

3.1. Fourier transform. It is well known that the Fourier transform, a special version of an integral transform, plays an important rule in the analysis of functions. Into that branch falls for example the study of solutions of differential equations, detecting possible poles and computing their residues.
In terms of $A_\lambda, B_\lambda$ we define holomorphic functions
\[
C_\lambda \overset{\text{def}}{=} (\lambda + p)A_\lambda - pB_\lambda, \quad D_\lambda \overset{\text{def}}{=} -(n - p)A_\lambda + (\lambda + n - p)B_\lambda.
\] (3.2)

The next theorem states that the Fourier transform $\mathcal{F}$ preserves the class of distribution $R_{A_\lambda, B_\lambda}^\lambda(x)$ by changing (in general) the parameters $A_\lambda, B_\lambda.$

**Theorem 3.2** The Fourier transform of $R_{A_\lambda, B_\lambda}^\lambda(x)$ is given by
\[
\mathcal{F}(R_{A_\lambda, B_\lambda}^\lambda)(\xi) = -2^{\lambda+n-1}\pi^{\frac{n}{2}} \frac{\Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma\left(-\frac{\lambda+n}{2}\right)} R_{C_\lambda, D_\lambda}^{-\lambda-n}(\xi),
\] (3.3)
where $C_\lambda, D_\lambda$ are given by (3.2).

**Proof.** For $\omega \in \mathcal{S}^p(\mathbb{R}^n)$ it is enough to verify that both expressions
\[
\mathcal{F}\left(\int_{\mathbb{R}^n} r^{\lambda-2}(x - y)(A_\lambda i_{x - y} \varepsilon_{x - y} + B_\lambda \varepsilon_{x - y} i_{x - y})\omega(y)dy\right)(\xi),
\]
which is actually $\mathcal{F}(R_{A_\lambda, B_\lambda}^\lambda \ast \omega)(\xi),$ and
\[
r^{-\lambda-2-n}(\xi)(\tilde{C}_\lambda i_\xi \varepsilon_\xi + \tilde{D}_\lambda \varepsilon_\xi i_\xi)\mathcal{F}(\omega)(\xi),
\]
for $\tilde{C}_\lambda, \tilde{D}_\lambda \in \mathbb{C}$ to be determined, agree up to a constant. We start to compute the latter one. First note the identities:
\[
r^{-\lambda-2-n}(\xi) = c_\lambda^{-1}(r^{\lambda+2})(\xi),
\]
\[
(\tilde{C}_\lambda i_\xi \varepsilon_\xi + \tilde{D}_\lambda \varepsilon_\xi i_\xi)\mathcal{F}(\omega)(\xi) = \mathcal{F}((\tilde{C}_\lambda \delta d + \tilde{D}_\lambda d\delta)\omega)(\xi),
\]
see [GS64, Chapter II, Section 3.3] and (2.3), and
\[
c_\lambda \overset{\text{def}}{=} 2^{\lambda+2+n}\pi^{\frac{n}{2}} \frac{\Gamma\left(\frac{\lambda+2+n}{2}\right)}{\Gamma\left(-\frac{\lambda+2+n}{2}\right)}.
\] (3.4)

Hence we obtain
\[
r^{-\lambda-2-n}(\xi)(\tilde{C}_\lambda i_\xi \varepsilon_\xi + \tilde{D}_\lambda \varepsilon_\xi i_\xi)\mathcal{F}(\omega)(\xi) = c_\lambda^{-1}\mathcal{F}\left(\int_{\mathbb{R}^n} r^{\lambda+2}(x - y)(\tilde{C}_\lambda \delta d + \tilde{D}_\lambda d\delta)\omega dy\right)(\xi).
\]

Taking a constant $p$-form $\beta$ we have by partial integration
\[
\int_{\mathbb{R}^n} \langle \beta, r^{\lambda+2}(x - y)(\tilde{C}_\lambda \delta d + \tilde{D}_\lambda d\delta)\omega \rangle dy
\]
\[
= \int_{\mathbb{R}^n} \langle (\tilde{C}_\lambda \delta d + \tilde{D}_\lambda d\delta)(r^{\lambda+2}(x - y)\beta), \omega \rangle dy. \quad (3.5)
\]

By Lemma 2.2 we find
\[
\int_{\mathbb{R}^n} \langle \delta d(r^{\lambda+2}(x - y)\beta), \omega \rangle dy = - (\lambda + 2) \int_{\mathbb{R}^n} \langle \beta, (n - p)r^\lambda(x - y)\omega 
\]
\[
+ \lambda r^{\lambda-2}(x - y)i_{x - y} \varepsilon_{x - y}\omega \rangle dy,
\]
\[
\int_{\mathbb{R}^n} \langle d\delta(r^{\lambda+2}(x - y)\beta), \omega \rangle dy = - (\lambda + 2) \int_{\mathbb{R}^n} \langle \beta, pr^\lambda(x - y)\omega 
\]
\[
+ \lambda r^{\lambda-2}(x - y)\varepsilon_{x - y}i_{x - y}\omega \rangle dy.
\]
In turn, Equation (3.3) is equivalent by collecting the coefficients of \( r^{\lambda-2}(x-y)i_{x-y}\varepsilon_{x-y} \) and \( r^{\lambda-2}(x-y)\varepsilon_{x-y}i_{x-y} \) in (3.5) to the following linear system for \( \tilde{C}_\lambda, \tilde{D}_\lambda \):

\[
(\lambda + n - p)\tilde{C}_\lambda + p\tilde{D}_\lambda = A_\lambda, \\
(n - p)\tilde{C}_\lambda + (\lambda + p)\tilde{D}_\lambda = B_\lambda.
\]  

(3.6)

The unique solution is given by

\[
\tilde{C}_\lambda = \frac{(\lambda + p)A_\lambda - PB_\lambda}{\lambda(\lambda + n)}, \quad \tilde{D}_\lambda = \frac{-(n - p)A_\lambda + (\lambda + n - p)B_\lambda}{\lambda(\lambda + n)}
\]  

(3.7)

This implies, since \( \beta \) was an arbitrary \( p \)-form,

\[
c_\lambda^{-1}F(\int_{\mathbb{R}^n} r^{\lambda+2}(x-y)(\tilde{C}_\lambda\delta d + \tilde{D}_\lambda d\delta)\omega dy)(\xi)
\]

\[
= -(\lambda + 2)c_\lambda^{-1}\int_{\mathbb{R}^n} r^{\lambda-2}(x-y)(A_\lambda i_{x-y}\varepsilon_{x-y} + B_\lambda\varepsilon_{x-y}i_{x-y})\omega(y)dy.
\]

Note that the factor \( \lambda + n \) in \( \tilde{C}, \tilde{D} \) will be absorbed into corresponding Gamma functions arising from \( c_\lambda \). The proof is complete.

\[ \square \]

3.2. Bernstein-Sato identity. For a given polynomial \( f \in \mathbb{R}[x_1, \ldots, x_n] \) and complex number \( s \in \mathbb{C} \) one can consider \( f_+^s(x) := f(x)^s \) for \( f(x) > 0 \) and zero otherwise. If \( \Re(s) > 0 \) this is locally integrable on \( \mathbb{R}^n \). Bernstein [Ber71] proves that \( f_+^s \) admits a meromorphic continuation to \( \mathbb{C} \) with poles given by the zero locus of a certain polynomial, the Bernstein polynomial. Roughly speaking one can construct a differential operator \( P(s) \) with polynomial coefficients such that \( P(s)f_+^{s+1}(x) = b(s)f_+^s(x) \), where \( b(s) \) is the Bernstein-Sato polynomial. This differential equation is termed Bernstein-Sato identity and was independently discovered in [SS72].

Now we present a vector-valued Bernstein-Sato identity which in turn implies the mereomorphic continuation of \( R_{A_\lambda, B_\lambda}^\lambda(x) \) to \( \lambda \in \mathbb{C} \).

**Theorem 3.3**  **The distribution** \( R_{A_\lambda, B_\lambda}^\lambda(x) \) **satisfies the differential equation:**

\[
D_2(\lambda; n, p)R_{A_{\lambda+2}, B_{\lambda+2}}^{\lambda+2}(x) = -\lambda(\lambda + n)C_{\lambda+2}D_{\lambda+2}R_{A_\lambda, B_\lambda}^\lambda(x)
\]

where \( D_2(\lambda; n, p) \) is a second-order differential operator given by

\[
D_2(\lambda; n, p) \overset{\text{def}}{=} C_\lambda D_{\lambda+2}\delta d + D_\lambda C_{\lambda+2}d\delta.
\]

**Proof.** We apply the Fourier transform to Equation (3.8) and use Theorem 3.2. Hence, it remains to verify that

\[
-2^{\lambda+n+1}\pi^{\frac{\lambda+n+2}{2}}\frac{\Gamma(\frac{\lambda+n+2}{2})}{\Gamma(-\frac{1}{2})}r^{-\lambda-n-4}(\xi)(C_\lambda D_{\lambda+2}i_\xi\varepsilon_\xi + D_\lambda C_{\lambda+2}\varepsilon_\xi i_\xi)(C_\lambda C_{\lambda+2}i_\xi\varepsilon_\xi + D_{\lambda+2}\varepsilon_\xi i_\xi)
\]

\[
= -2^{\lambda+n+1}\pi^{\frac{\lambda+2+n}{2}}\frac{\Gamma(\frac{\lambda+2+n}{2})}{\Gamma(-\frac{1}{2})}r^{-\lambda-n-4}(\xi)(C_\lambda C_{\lambda+2}D_{\lambda+2}(i_\xi\varepsilon_\xi)^2 + C_{\lambda+2}D_{\lambda}D_{\lambda+2}(\varepsilon_\xi i_\xi)^2)
\]

and
The poles at \( \lambda \)

First note that the residues of \( \text{Res}_\lambda \) and \( \text{Res}_\lambda \) do agree.

\[ \lambda(\lambda + n)C_{\lambda+2}D_{\lambda+2}2^{\lambda+n-1}\pi^2 \frac{\Gamma(\frac{\lambda+n}{2})}{\Gamma(-\frac{\lambda+2}{2})}r^{-\lambda-n-2}(\xi)(C_\lambda i\xi_\xi + D_\lambda \xi_\xi) \]

\[ = -2^{\lambda+n+1}\pi^2 \frac{\Gamma(\frac{\lambda+n+2}{2})}{\Gamma(-\frac{\lambda}{2})}r^{-\lambda-n-4}(\xi)(C_\lambda C_{\lambda+2}D_{\lambda+2}(i\xi_\xi)^2 + C_\lambda+2D_\lambda D_{\lambda+2}(\xi_\xi)^2) \]

An iteratively application of Equation (3.8) gives

\[ R_{\lambda,B_\lambda}^\lambda(x) = \frac{(-1)^k}{4^k(k!)^2(k+n+2)k}D_{2k}(\lambda; n, p)R_{\lambda+2k,B_{\lambda+2k}}^{\lambda+2k}(x), \tag{3.9} \]

where \( D_{2k}(\lambda; n, p) \) is a differential operator of order \( 2k \), for \( k \in \mathbb{N} \), given by

\[ D_{2k}(\lambda; n, p) \overset{\text{def}}{=} C_\lambda D_{\lambda+2k}(\delta d)^k + C_{\lambda+2k}D_\lambda(d\delta)^k. \tag{3.10} \]

By convention we set \( D_0(\lambda; n, p) \overset{\text{def}}{=} C_\lambda D_{\lambda} \text{Id}. \)

The tempered distribution \( R_{\lambda,B_\lambda}^\lambda(x) \) originally defined for \( \Re(\lambda) > -n \) can now, by Equation (3.9), be meromorphically extended to \( \lambda \in \mathbb{C} \) with simple poles at \( \lambda = -n - 2\mathbb{N}_0 \).

### 3.3. Residues

In the last subsection we have identified the location of the poles of \( R_{\lambda,B_\lambda}^\lambda \) and their type. In general their residues, when acting as convolution operators, will not be differential operators. However, under some assumptions they reduce to differential operators given as a weighted sum of powers of \( \delta d \) and \( d\delta \).

**Theorem 3.4** The residue of \( \mathcal{F}(R_{\lambda,B_\lambda}^\lambda)(\xi) \) at \( \lambda = -n - 2k \), for \( k \in \mathbb{N}_0 \), is given by

\[ \text{Res}_{\lambda=-n-2k}(\mathcal{F}(R_{\lambda,B_\lambda}^\lambda)(\xi)) = \frac{(-1)^k}{4^k(k!)^2(k+n+2)k}C_{n-2k}(i\xi_\xi)^k(\frac{D_{n-2k}(\xi_\xi)^k}{D_{n-2k}(\xi_\xi)^k})R_{\lambda=-n-2k}^\lambda(\xi). \tag{3.11} \]

Especially, in case \( R_{\lambda,B_\lambda}^\lambda(x) \) acts on 0-forms we have

\[ \text{Res}_{\lambda=-n-2k}(\mathcal{F}(R_{\lambda,B_\lambda}^\lambda)(\xi)) = \frac{(-1)^k}{4^k(k!)^2(k+n+2)k}C_{n-2k}(\xi_\xi)^k. \tag{3.12} \]

**Proof.** First note that the residues of \( \mathcal{F}(R_{\lambda,B_\lambda}^\lambda)(\xi) \) coincide with the residues of \( R_{\lambda,B_\lambda}^\lambda(x) \). The poles at \( \lambda = -n - 2k \) are simple. Hence its residue is given by

\[ \text{Res}_{\lambda=-n-2k}(\mathcal{F}(R_{\lambda,B_\lambda}^\lambda)(\xi)) = \lim_{\lambda \to -n-2k} (\lambda + n + 2k)\mathcal{F}(R_{\lambda,B_\lambda}^\lambda)(\xi). \]

From Equation (3.9) and Theorem 3.2 we get

\[ \mathcal{F}(R_{\lambda,B_\lambda}^\lambda)(\xi) = \frac{(-1)^k\mathcal{F}(D_{2k}(\lambda; n, p))R_{\lambda+2k,B_{\lambda+2k}}^{\lambda+2k}(\xi)}{4^k(k!)^2(k+n+2)kC_{\lambda+2k}D_{\lambda+2k}(-\frac{\lambda+2k}{2})} \]

\[ = \frac{(-1)^k\mathcal{F}(D_{2k}(\lambda; n, p))R_{\lambda+2k,B_{\lambda+2k}}^{\lambda+2k}(\xi)}{4^k(k!)^2(k+n+2)kC_{\lambda+2k}D_{\lambda+2k}(-\frac{\lambda+2k}{2})} \]

\[ \times \left[ C_\lambda D_{\lambda+2k}(i\xi_\xi)^k + C_{\lambda+2k}D_\lambda(\xi_\xi)^k \right] \mathcal{F}(R_{\lambda+2k,B_{\lambda+2k}}^{\lambda+2k}(\xi)). \]
Combining the factor \((\lambda + n + 2k)\) with the Gamma function \(\Gamma\left(\frac{\lambda+n+2k}{2}\right)\) and taking the limit \(\lambda \to -n - 2k\) gives

\[
\text{Res}_{\lambda=-n-2k} \left( \mathcal{F}(R_{A,\lambda}^\lambda B_\lambda)(\xi) \right) = \frac{(-1)^{k+1} \pi^{\frac{n}{2}}}{4^k k! \Gamma(n+2k+2)} \left[ C_{n-2k} D_{-n}(i\xi \varepsilon \xi)^k + C_{n} D_{-n-2k}(\varepsilon \xi i\xi)^k \right] \delta_{C_{n-2k},D_{-n}}(\xi),
\]

which implies (3.11). The case when \(\mathcal{F}(R_{A,\lambda}^\lambda B_\lambda)(\xi)\) acts on 0-forms is an easy consequence. The proof is complete.

\[\square\]

**Corollary 3.5** Let \(A\) and \(B\) holomorphic in \(\lambda\) such that \(C_{-n} = D_{-n}\). Then the residue of \(R_{A,\lambda}^\lambda B_\lambda(x)\) at \(\lambda = -n - 2k\), for \(k \in \mathbb{N}_0\), is given by

\[
\text{Res}_{\lambda=-n-2k} \left( R_{A,\lambda}^\lambda B_\lambda(x) \right) = \frac{(-1)^{k+1} \pi^{\frac{n}{2}}}{4^k k! \Gamma(n+2k+2)} \left[ C_{n-2k}(\delta d)^k + D_{-n-2k}(d\delta)^k \right] \delta_{0}(x). \tag{3.13}
\]

**Proof.** This is a direct consequence of Theorem 3.4 and

\[\text{Res}_{\lambda=-n-2k} \left( R_{C_{-n},D_{-n}}^0 (\xi) \right) = C_{-n} r^{-2}(\xi)(i\xi \varepsilon \xi + \varepsilon \xi i\xi) = C_{-n} \text{Id} \]

and \(\mathcal{F}(\delta_0)(\xi) = \text{Id}(\xi)\).

\[\square\]

The assumption in Corollary 3.5 is not empty. For example, the pairs \((A, B)\) defined by \((1, 1), (1, -1)\) or \((\alpha, \beta)\), where \(\alpha, \beta\) are given by (4.5), satisfy the condition \(C_{-n} = D_{-n}\), cf. (3.2).

### 3.4. Convolutions

Already mentioned before, the operation between function known as convolution is important to define certain integral operators. It also is important in the theory of Fourier transform, since the Fourier transform transforms convolutions into products.

We proceed with an identity concerning convolutions of \(R_{A,\lambda}^\lambda B_\lambda(x)\).

**Theorem 3.6** Let \(A\) and \(B\) such that \(C_{2(\lambda-n)}C_{-2\lambda} = D_{2(\lambda-n)}D_{-2\lambda}\). Then the distribution \(R_{A,\lambda}^\lambda B_\lambda(x)\) satisfies

\[
R_{A_{2(\lambda-n)},B_{2(\lambda-n)}}^{2(\lambda-n)} * R_{A_{-2\lambda},B_{-2\lambda}}^{-2\lambda}(x) = \pi^n \frac{\Gamma(\frac{2\lambda-n}{2})\Gamma(\frac{-2\lambda+n}{2})}{2^{2\lambda} \Gamma(-\lambda + n + 1) \Gamma(\lambda + 1)} C_{2(\lambda-n)}C_{-2\lambda} \delta_0(x). \tag{3.14}
\]

**Proof.** By application of the Fourier transform, see Theorem 3.2 to (3.14) we obtain

\[
\mathcal{F}\left( R_{A_{2(\lambda-n)},B_{2(\lambda-n)}}^{2(\lambda-n)} * R_{A_{-2\lambda},B_{-2\lambda}}^{-2\lambda} \right)(\xi) = 2^{-\frac{n}{2}} \pi^n \frac{\Gamma(\frac{2\lambda-n}{2})\Gamma(\frac{-2\lambda+n}{2})}{\Gamma(-\lambda + n + 1) \Gamma(\lambda + 1)} r^{-4}(\xi)(C_{2(\lambda-n)} i\xi \varepsilon \xi + D_{2(\lambda-n)} \varepsilon i\xi i\xi)(C_{-2\lambda} \xi i\xi + D_{-2\lambda} \xi \xi i\xi).
\]

Now the assumption \(C_{2(\lambda-n)}C_{-2\lambda} = D_{2(\lambda-n)}D_{-2\lambda}\) and \(i\xi \varepsilon \xi + \varepsilon i\xi i\xi = r^2(\xi)\), cf. Lemma 2.7 complete the proof.

\[\square\]
The assumption $C_{2(\lambda-n)}C_{-2\lambda} = D_{2(\lambda-n)}D_{-2\lambda}$ is not empty. For example take the values $(A_\lambda, B_\lambda) \overset{\text{def}}{=} (1, 1), (1, -1)$ or $(\alpha_\lambda, \beta_\lambda)$, cf. (4.5). In each case the assumption of Theorem 3.6 is satisfied.

A semi-group structure of $R_{A_\lambda, B_\lambda}(x)$ in full generality is not possible, see Remark 5.1 for an example and a general statement.

3.5. Integral operators. Integral transforms are widely used in mathematics. Most constructions of integral operators based on integrating or taking convolutions by kernel functions. The Fourier transform is such an example, it arises by integration against $e^{i\langle x, \xi \rangle}$. Also pseudo-differential operators arise as integral operators. Special cases are differential operators, namely those having polynomial kernels. Furthermore, some pseudo-differential operators have a certain symmetry, for example the Knapp-Stein integral operators [KS71] intertwine the corresponding principal series.

Recall that $R_{A_\lambda, B_\lambda}(x)$ is meromorphic in $\lambda \in \mathbb{C}$ with only simple poles at $\lambda = -n - 2\mathbb{N}_0$. The integral operator

$$T_{\nu, (A_\nu, B_\nu)} : \Omega^p(\mathbb{R}^n) \to \Omega^p(\mathbb{R}^n)$$

is defined as convolution operator with $R_{A_\lambda, B_\lambda}(x)$, i.e.

$$(T_{\nu, (A_\nu, B_\nu)}\omega)(x) \overset{\text{def}}{=} \int_{\mathbb{R}^n} R_{A_{2(\nu-n)}, B_{2(\nu-n)}}(x - y)\omega(y)dy.$$ 

These family $T_{\nu, (A_\nu, B_\nu)}$ of pseudo-differential operators is meromorphic in $\nu \in \mathbb{C}$ with only simple poles at $\nu = \frac{n}{2} - \mathbb{N}_0$. Now we describe their residues for a class of integral operators.

**Theorem 3.7** Let $A_\lambda, B_\lambda$ holomorphic in $\lambda \in \mathbb{C}$ such that $C_{-n} = D_{-n}$. Then the residue of $T_{\nu, (A_\nu, B_\nu)}$ at $\nu = \frac{n}{2} - k$, for $k \in \mathbb{N}_0$, is

$$\text{Res}_{\nu = \frac{n}{2} - k}(T_{\nu, (A_\nu, B_\nu)}\omega) = \frac{(-1)^{k+1}n^{\frac{2n}{2}}}{4^{k}k!\Gamma(\frac{n}{2} + k + 1)}(C_{-n-2k}(\delta d)^k + D_{-n-2k}(d\delta)^k)\omega,$$ 

for $\omega \in \Omega^p(\mathbb{R}^n)$.

**Proof.** This is a direct consequence of Corollary 3.5.

4. Further results and applications

We focus on Riesz distributions $R_{A_\lambda, B_\lambda}(x)$ for certain $A_\lambda, B_\lambda$, and we show how to recover the Knapp-Stein intertwining operators [KS71] on functions and differential forms. Furthermore, we discuss their relation to the well-known GJMS- and Branson-Gover operators [GJMS92, BG05], respectively.

4.1. Knapp-Stein intertwining operator on functions. We show that the distribution

$$R_{1,0}^\lambda(x) = r^{\lambda-2}(x)i_x \varepsilon_x$$

when acting on 0-forms reduces to the Riesz distribution (2.1). The Knapp-Stein intertwining operator on functions is defined as convolution operator with the Riesz distribution.
Let us briefly recall the Riesz distribution [Ric49, GS64], which is given for $\lambda \in \mathbb{C}$ with $\Re(\lambda) > -n$ by

$$r^\lambda(x) = (x_1^2 + \ldots + x_n^2)^{\lambda/2}.$$ 

It extends to a tempered distribution meromorphic in $\lambda \in \mathbb{C}$ with simple poles at $\lambda = -n - 2k$ for $k \in \mathbb{N}_0$, and obeys the following properties:

1. Fourier transform: $\mathcal{F}(r^\lambda)(\xi) = 2^{\lambda+n}n^{n/2}\frac{\Gamma(\frac{n+\lambda}{2})}{\Gamma(-\frac{\lambda}{2})} r^{-\lambda-n}(\xi)$.
2. Bernstein-Sato identity: $\Delta(r^{\lambda+2}(x)) = (\lambda + 2)(\lambda + n)r^\lambda(x)$, where $\Delta = \sum_i \partial^2_i$.
3. Convolutionary inverse: $(r^{2(\lambda-n)}r^{-2\lambda})(x) = \pi^n \frac{\Gamma(\frac{n+\lambda}{2})\Gamma(\frac{n+2}{2})}{\Gamma(\lambda)} \delta_0(x)$, where $\delta_0(x)$ is the Dirac-distribution centered at the origin.
4. Residues at $\lambda = -n - 2k$: $\text{Res}_{\lambda=-n-2k}(r^\lambda(x)) = \frac{2\pi^{\frac{n}{2}}}{2^nk!(\frac{n}{2})^k}\Delta^k\delta_0(x)$, where $k \in \mathbb{N}_0$.

By Lemma 2.1 we obtain

$$R^\lambda_{1,0}(x) = r^{\lambda-2}(x)i_x\varepsilon_x = r^\lambda(x).$$

Hence, $R^\lambda_{A_\lambda,B_\lambda}(x)$ is a generalization of $r^\lambda(x)$. Note that one could keep the parameters $(A_\lambda, B_\lambda)$ different from $(1, 0)$ which leads, as long as acting on functions, to a different normalization of $r^\lambda(x)$ by the factor $A_\lambda$, since $B_\lambda$ has no contribution on functions due to $i_xf = 0$ for any function $f$.

It is an easy exercise, using the parameters $C_\lambda = \lambda$ and $D_\lambda = -n$ (again the latter one has no contribution on functions), cf. (3.2), that Theorems 3.2, 3.3, 3.6 and 3.4 specialize for $R^\lambda_{1,0}(x)$ to (1)-(4) given above.

**Remark 4.1** Note that we have only used the knowledge of the Fourier transform to achieve Theorem 3.1. In [GS64], a different computation for $\text{Res}_{\lambda=-n-2k}(r^\lambda(x))$ is presented, which doesn’t based on the Fourier transform.

The integral operator (3.15) reduces to

$$(\mathbb{T}_{\nu,(1,0)}f)(x) = \int_{\mathbb{R}^n} |x-y|^{2(\nu-n)}f(y)dy,$$

(4.2)

which is exactly the Knapp-Stein intertwining operator. Its major property is that it intertwines the corresponding principal series [KS71]. Furthermore, from Theorem 3.7 it follows that the residue of $\mathbb{T}_{\nu,(1,0)}$ at $\nu = \frac{n}{2} - k$, $k \in \mathbb{N}_0$, is given by the differential operator

$$\text{Res}_{\nu=\frac{n}{2}-k}(\mathbb{T}_{\nu,(1,0)}) = \frac{2\pi^{\frac{n}{2}}}{4^{k}k!(\frac{n}{2}+k)}\Delta^k\delta_0.$$ 

Note that our convention implies $\delta d = -\Delta$ when action on functions. Hence, residues of $\mathbb{T}_{\nu,(1,0)}$ recover the well-known conformal powers of the Laplacian on function, [GJMS92].

**4.2. Knapp-Stein intertwining operator for differential forms.** We next investigate the distribution

$$R^\lambda_{1,-1}(x) = r^{\lambda-2}(x)(i_x\varepsilon_x - \varepsilon_xi_x).$$

(4.3)
We show that the residues of integral operators $\mathbb{T}_{\nu(1,-1)}$ are the Branson-Gover operators of order $2N \in \mathbb{N}$, cf. [BG05], which are given by

$$L_{2N}^{(p)} = \alpha_N(\delta d)^N + \beta_N(d\delta)^N : \Omega^p(\mathbb{R}^n) \to \Omega^p(\mathbb{R}^n),$$

with coefficients

$$\alpha_\lambda \overset{\text{def}}{=} \frac{n}{2} - p + \lambda, \quad \beta_\lambda \overset{\text{def}}{=} \frac{n}{2} - p - \lambda. \quad (4.5)$$

Furthermore, we extend the intertwining property of $L_{2N}^{(p)}$, see for example [FJS16], to $\mathbb{T}_{\nu(1,-1)}$. We also could get the intertwining property by a direct comparison of $\mathbb{T}_{\nu(1,-1)}$ with the Knapp-Stein intertwining operator studied by [SV11], see Remark 4.9. Finally, we present an elementary proof of positive-definiteness of scalar products induced by $\mathbb{T}_{\nu(1,-1)}$, cf. [SV11].

Now, for $(A_\lambda, B_\lambda) = (1, -1)$ the coefficients (3.2) are

$$C_\lambda = \lambda + 2p = -2\alpha_{-\frac{\lambda+n}{2}}, \quad D_\lambda = -(\lambda + 2n - 2p) = -2\beta_{-\frac{\lambda+n}{2}}.$$  

From Theorems 3.2, 3.3, 3.6 and Corollary 3.5 we obtain the following corollaries.

**Corollary 4.2**  The Fourier transform of $R_{1,-1}^\lambda(x)$ is given by

$$\mathcal{F}(R_{1,-1}^\lambda)(\xi) = 2^{\lambda+n} \pi^{\frac{n}{2}} \frac{\Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma\left(\frac{\lambda+2n+2k}{2}\right)} R_{\lambda-n,\frac{\lambda+n}{2},\frac{\lambda+n}{2}}(\xi). \quad (4.6)$$

**Corollary 4.3**  The distribution $R_{1,-1}^\lambda(x)$ satisfies the following Bernstein-Sato identity:

$$\left[(\lambda + 2n - 2p)(\lambda + 2p - 2)\delta d + (\lambda + 2p)(\lambda + 2n - 2p - 2)d\delta\right] R_{1,-1}^\lambda(x) = -(\lambda - 2)(\lambda + n - 2)(\lambda + 2p)(\lambda + 2n - 2p) R_{1,-1}^{\lambda-2}(x).$$

Especially, the last corollary implies

$$R_{1,-1}^\lambda(x) = \frac{(-1)^k}{4^k(\frac{\lambda}{2})_k(\frac{\lambda+n}{2})_k} \left[\frac{(\lambda + 2p)}{\lambda + 2p + 2k}(\delta d)^k + \frac{(\lambda + 2n - 2p)}{(\lambda + 2n - 2p + 2k)}(d\delta)^k\right] R_{1,-1}^{\lambda+2k}(x). \quad (4.7)$$

**Corollary 4.4**  The residue of $R_{1,-1}^\lambda(x)$ at $\lambda = -n - 2k$ for $k \in \mathbb{N}_0$ is

$$\text{Res}_{\lambda=-n-2k}(R_{1,-1}^\lambda(x)) = \frac{(-1)^k 2\pi^{\frac{n}{2}}}{4^k k! \Gamma\left(\frac{n}{2} + k + 1\right)} [\alpha_k(\delta d)^k + \beta_k(d\delta)^k] \delta_0(x), \quad (4.8)$$

which is up to a constant the Branson-Gover operator $L_{2k}^{(p)}$ acting on the Dirac-distribution.
Corollary 4.5  The distribution $R^{2\lambda-n}_{1,-1}(x)$ satisfies
\begin{equation}
R^{2\lambda-n}_{1,-1} * R^{-2\lambda}_{1,-1}(x) = \pi^n \frac{\Gamma(\frac{2\lambda-n}{2})\Gamma(-\frac{2\lambda+n}{2})}{\Gamma(-\lambda + n + 1)\Gamma(\lambda + 1)} x^{-2\lambda+n} \delta_0(x) \tag{4.9}
\end{equation}

The integral operator (3.15) specializes to
\begin{equation}
(T_{\nu,(1,-1)} \omega)(x) = \int_{\mathbb{R}^n} |x-y|^{2(\nu-n-1)}(i_{x-y}\varepsilon_{x-y} - \varepsilon_{x-y}i_{x-y})\omega(y)dy \tag{4.10}
\end{equation}
for $\omega \in \Omega^p(\mathbb{R}^n)$. From Theorem 3.7 we obtain the following.

Corollary 4.6  The residue of $T_{\nu,(1,-1)}$ at $\nu = \frac{n}{2} - k$, $k \in \mathbb{N}_0$, is given by
\begin{equation}
\text{Res}_{\nu=\frac{n}{2}-k} \left( (T_{\nu,(1,-1)}) \right) = \frac{(-1)^k 2\pi^2}{4^k k! \Gamma(\frac{n}{2} + k + 1)} L^{(p)}_{2k} \omega, \tag{4.11}
\end{equation}
where $\omega \in \Omega^p(\mathbb{R}^n)$.

Now we show the intertwining property of $T_{\nu,(1,-1)}$ with respect to the principal series, acting on differential forms. For that let us introduce some notation. Let $G = SO_0(n+1,1,\mathbb{R})$ the connected component of the group preserving the inner product
\[ 2x_0x_{n+1} + x_1^2 + \cdots + x_n^2 \]
on $\mathbb{R}^{n+2}$. Let $P_+ \overset{\text{def}}{=} \text{Stab}(\mathbb{R}e_0) \subset G$ and $P_- \overset{\text{def}}{=} \text{Stab}(\mathbb{R}e_{n+1}) \subset G$ be the stabilizer of the line $\mathbb{R}e_0$, respectively $\mathbb{R}e_{n+1}$, where $\{e_0, \ldots, e_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+2}$. The groups $P_\pm$ are parabolic in $G$ and have Langlands decompositions $P_\pm = MAN_\pm$ with $M \simeq SO(n,\mathbb{R})$, $A \simeq \mathbb{R}^+$ and $N_\pm \simeq \mathbb{R}^n$. Corresponding Lie algebras are denoted by $\mathfrak{g}(\mathbb{R})$ and $\mathfrak{p}_\pm(\mathbb{R}) = \mathfrak{m}(\mathbb{R}) \oplus \mathfrak{a}(\mathbb{R}) \oplus \mathfrak{n}_\pm(\mathbb{R})$ with $\mathfrak{m}(\mathbb{R}) \simeq \mathfrak{so}(n,\mathbb{R})$, $\mathfrak{a}(\mathbb{R}) \simeq \mathbb{R}$ and $\mathfrak{n}_\pm(\mathbb{R}) \simeq \mathbb{R}^n$. We define a representation of $P_+$ as an trivial extention to $N_+$ of the tensorproduct of the standard representation $(\Lambda^p(\mathbb{R}^n)^*, \sigma_p)$ of $M$ and a 1-dimensional represenation $(\mathbb{C}_\lambda, \xi_\lambda)$ of $A$, i.e. $P_+$ acts on $V_{\lambda,p} \overset{\text{def}}{=} \Lambda^p(\mathfrak{g}(\mathbb{R}))^* \otimes \mathbb{C}_\lambda \simeq \Lambda^p(\mathbb{R}^n)^* \otimes \mathbb{C}_\lambda$ by
\[ \rho_{\lambda,p} : P_+ = MAN_+ \rightarrow GL(V_{\lambda,p}) \]
\[ \text{man} \mapsto \{(v \otimes 1) \mapsto \sigma_p(m)v \otimes a^{-\lambda}\} \]Now, the space of sections $\Gamma(G \times_{P_+,\rho_{\lambda,p}} V_{\lambda,p})$ is equivalent to the space of $P_+$-equivariant functions
\[ C^\infty(G, V_{\lambda,p})^{P_+} = \{ f : G \rightarrow V_{\lambda,p} \mid f(gp') = \rho_{\lambda,p}((g')^{-1})f(g) \quad \forall p' \in P_+ \}. \]
Let us denote by $\pi_{\lambda,p}$ the regular left representation (strictly speaking anti-representation) of $G$ on $C^\infty(G, V_{\lambda,p})^{P_+}$, i.e. $\pi_{\lambda,p}(g)f(g') \overset{\text{def}}{=} f(g \cdot g')$.

Lemma 4.7  [FJS16, Lemma 2.3.3]  The infinitesimal action $d\pi_{\lambda,p}(E_j^+)$ on $u \otimes \omega \in C^\infty(\mathfrak{n}_-(\mathbb{R})) \otimes \Lambda^p(\mathfrak{n}_-(\mathbb{R}))^* \simeq C^\infty(\mathbb{R}^n) \otimes \Lambda^p(\mathbb{R}^n)^*$ is given by
\[ d\pi_{\lambda,p}(E_j^+)(u \otimes \omega)(x) = \left( -\frac{1}{2} \sum_{k=1}^{n} x_k^2 \partial_{x_j} + x_j(\lambda + \sum_{k=1}^{n} x_k \partial_{x_k}) \right)(u) \otimes \omega \]
We show that the relation (4.12) extends to $T$ contribution of $r$ and difference become

We prove the claim in the Fourier image. First note the identity and definition

\[ \text{Proposition 4.8} \quad \text{The integral operators } T_{\nu,(1,-1)} \text{ satisfy} \]

\[ d\pi_{-\nu,n,p}(X)T_{\nu,(1,-1)} = T_{\nu,(1,-1)}d\pi_{-\nu,p}(X), \quad \forall X \in \mathfrak{so}(n + 1, 1, \mathbb{R}). \]  \hfill (4.13)

\[ \text{Proof.} \quad \text{We prove the claim in the Fourier image. First note the identity and definition} \]

\[ \mathcal{F}(d\pi_{\lambda,p}(X)\omega) = -i\left( \frac{1}{2}\xi_j \Delta \xi - ((\lambda + n) + \sum_{k=1}^n \xi_k \partial_{\xi_k}) \partial_{\xi_j} \]

\[ + \sum_{k=1}^n \partial_{\xi_k}((E_j^+)\omega - (E_k^+)\omega) \right) \mathcal{F}(\omega) \]

\[ \overset{\text{def}}{=} D_2(\lambda, j) \mathcal{F}(\omega). \]

Hence $D_2(\lambda, j)$ is a second order differential operator on differential forms. In order to show (4.13) it remains to verify that the difference of

\[ \mathcal{F}(d\pi_{-\nu-n,p}(X)T_{\nu,(1,-1)})(\xi) = D_2(\nu - n, j)\mathcal{F}(R_{1,-1}^{2(\nu-n)})(\xi) \mathcal{F}(\omega)(\xi) \]

\[ = c_1 D_2(\nu - n, j)((r^{-2\nu+n-2}(\alpha_{-2\nu+n}i_{\xi} \epsilon_{\xi} + \beta_{-2\nu+n} \epsilon_{\xi})) \mathcal{F}(\omega)(\xi) \]

and

\[ \mathcal{F}(T_{\nu,(1,-1)}d\pi_{-\nu,p}(X))(\xi) = \mathcal{F}(R_{1,-1}^{2(\nu-n)})(\xi) D_2(-\nu, j) \mathcal{F}(\omega)(\xi) \]

\[ = c_1 r^{-2\nu+n-2}(\alpha_{-2\nu+n}i_{\xi} \epsilon_{\xi} + \beta_{-2\nu+n} \epsilon_{\xi}) D_2(-\nu, j) \mathcal{F}(\omega)(\xi). \]

vanishes. Here $c_1$ is the constant arising from the Fourier transform of $R_{1,-1}^{2(\nu-n)}(x)$. Performing the differentiation with $D_2(\nu - n, j)$ and $D_2(-\nu, j)$, canceling the common contribution of $r^\mu$ for appropriate $\mu$ and expanding everything in powers of $\nu$, the difference become

\[ P_3(\omega) r^3 + P_2(\omega) r^2 + P_1(\omega) r^1 + P_0(\omega) = 0, \]

for some coefficients $P_i(\omega)$, $i = 0, \ldots, 3$, which are differential operators acting on $\omega$. That it is of third degree in $\nu$ comes from the fact that $D_2(\mu, j)$ is of second order containing a first order contribution with coefficient $\mu$ and the coefficients $\alpha_\mu$ and $\beta_\mu$ are linear in $\mu$. Since the left-hand side is a polynomial of order 3, it remains to check its vanishing at 4 different points. Since Equation (4.13) is satisfied at all residues of $T_{\nu,(1,-1)}$, that means at infinitely many, due to Equation (4.12), the proof is complete. \hfill \Box
Remark 4.9. The intertwining property of the integral operator $T_{\nu,(1,-1)}$ was already studied in [SV11, Section 4]. Actually they have studied the Knapp-Stein intertwining operator for differential forms. It just remains to show that our formula for $T_{\nu,(1,-1)}$ matches with that in [SV11]. This can be seen by noting that, when acting on a vector $Y$ in $\mathbb{R}^n$ (a dual 1-form), our algebraic action

$$i_x \varepsilon_x - \varepsilon_x i_x \frac{Y}{|x|^2} = Y - 2\frac{\langle Y, x \rangle}{|x|^2} x$$

becomes a reflection of $Y$ in the hyperplane (through the origin) orthogonal to $x$, compare with [SV11, Lemma 2.1]. Note that this formula extends naturally to $p$-forms. Note also that our notation for the algebraic action enables us to compute explicitly the residues of $T_{\nu,(1,-1)}$, or even in a more general setting for $T_{\nu,(A\lambda,B\lambda)}$.

Remark 4.10. The intertwining operator $T_{\nu,(1,-1)}$ defines an invariant pairing between two induced representations in natural duality (via the natural $L^2$ pairing), namely $\pi_{\nu-n,p}$ and $\pi_{-\nu,p}$ - here we consider real values of $\nu$, and from Corollary 4.2 we see that the interval where this pairing is positive-definite (i.e. the interval for the unitary complementary series) is exactly $|\lambda| < (n/2) - p$, with $\lambda = \nu - (n/2)$. Indeed, the invariant Hermitian form is on the Fourier transform side given as the natural $L^2$ expression (the Fourier transform of $(T_{\nu,(1,-1)}\omega, \omega)$)

$$2^{2\lambda} \pi^{n/2} \frac{\Gamma(\lambda)}{\Gamma((n/2) + 1 - \lambda)} \int_{\mathbb{R}^n} |\xi|^{-2\lambda - 2}(((n/2) - p - \lambda)i_\xi \varepsilon_\xi + ((n/2) - p + \lambda)\varepsilon_\xi i_\xi)\hat{\omega}, \hat{\omega}) d\xi$$

which we see as positive definite in the interval indicated; furthermore, for

$$0 < \lambda < (n/2) - p$$

the density here is locally integrable. Note that in this case $T_{\nu,(1,-1)}$ gives an equivalence between two unitary representations in the complementary series.

This is consistent with the formulas in [BOØ], and gives a new and elementary proof of the size of the unitary complementary series corresponding to $p$-forms. Indeed, in [BOØ] the eigenvalues of the intertwining operator in its compact picture on the sphere are shown to be labeled by integers $j \geq 1$ and $q = 0, 1$ (corresponding to exact and co-exact differential forms respectively) as

$$Z(j, q, \lambda) = \frac{\Gamma((n/2) + j + \lambda) \Gamma((n/2) - p + q + \lambda)}{\Gamma((n/2) + j - \lambda) \Gamma((n/2) - p + q - \lambda)}$$

suitably normalized. The pair $(j, q)$ corresponds to the highest weight

$$(j, 1, 1, \ldots, 1, q, 0, \ldots)$$

of an irreducible representation of $K = SO(n+1)$, and $\lambda$ is the same parameter as before. Note that the ratio between this eigenvalue for $q = 0$ and $q = 1$ is exactly $((n/2) - p - \lambda)/((n/2) - p + \lambda)$.

Let us close this subsection with a comment about the impact of the Bernstein-Sato identity, see Corollary 4.3, for $R^1_{1-1}(x)$ to a recurrence relation among Branson-Gover operators. This recurrence relation can also be directly obtained from (4.4). However, we want to demonstrate its appearance from the Bernstein-Sato identity.
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**Proposition 4.11** The Branson-Gover operators \(L_{2N}^{(\nu)}\), for \(N \in \mathbb{N}\), on \(\mathbb{R}^n\) satisfy

\[
L_{2N}^{(\nu)} = \left( \frac{\alpha_N}{\alpha_{N-1}} d + \frac{\beta_N}{\beta_{N-1}} d\delta \right) L_{2N-2}^{(\nu)}. \tag{4.14}
\]

By convention we set \(L_0^{(\nu)} \overset{\text{def}}{=} \alpha_0 \text{Id}\).

**Proof.** First observe that Corollary 4.3 implies

\[
\mathbb{T}_{\nu,(1-1)}(x) = \int_{\mathbb{R}^n} R_{1,-1}^{2(\nu-n)}(x-y) \omega(y) dy = \frac{1}{(2\nu-2n)(2\nu-n)} \left( \frac{2\nu - 2n + 2p}{2\nu - 2n + 2p + 2} d + \frac{2\nu - 2p}{2\nu - 2p + 2} d\delta \right)
\times \int_{\mathbb{R}^n} R_{1,-1}^{2(\nu-n+1)}(x-y) \omega(y) dy
\]

Now we take the residue at \(\nu = \frac{n}{2} - N\) for \(N \in \mathbb{N}_0\) using Equation (4.11). The residue of the left-hand side is

\[
\frac{(-1)^N 2\pi^{\frac{n}{2}}}{4^N N! \Gamma(\frac{n}{2} + N + 1)} L_{2N}^{(\nu)},
\]

while the right-hand side has the residue

\[
\frac{(-1)^N 2\pi^{\frac{n}{2}}}{4^N N! \Gamma(\frac{n}{2} + N + 1)} \left( \frac{n - 2p + 2N}{n - 2p + 2N - 2} d + \frac{n - 2p - 2N}{n - 2p - 2N + 2} d\delta \right) L_{2N-2}^{(\nu)}.
\]

Now a cancelation of common factors completes the proof. \qed

5. SOME REMARKS AND FURTHER APPLICATIONS

This sections collect some observations concerning \(R_{A,\lambda}^\lambda(x)\).

**Remark 5.1** The Riesz distribution \(R_{1,1}^\lambda(x)\), when appropriately normalized, satisfies a semi-group property, i.e. for

\[
\mathbb{R}_{1,1}^\lambda(x) = \frac{\Gamma(-\frac{\lambda-n}{2})}{2^{\lambda - \frac{n}{2}} \pi^{\frac{n}{2}} \Gamma(\frac{\lambda}{2})} R_{1,1}^{\lambda - n}(x)
\]

we have, whenever it makes sense,

\[
\mathbb{R}_{1,1}^\lambda * \mathbb{R}_{1,1}^{\nu} = \mathbb{R}_{1,1}^{\lambda + \nu}(x).
\]

In order to prove a semi-group structure for the family of Riesz distributions \(R_{A,\lambda}^\lambda(x)\) one has to allow \(A, B\) to be meromorphic in \(\lambda \in \mathbb{C}\). More precisely, define

\[
\mathbb{R}_{A,\lambda}^\lambda(x) \overset{\text{def}}{=} \frac{\Gamma(-\frac{\lambda-n}{2})}{2^{\lambda - 1} \pi^{\frac{n}{2}} \Gamma(\frac{\lambda}{2})} R_{A,\lambda-n}^{\lambda - n}(x),
\]

and show in the Fourier image that

\[
\mathcal{F}(\mathbb{R}_{A,\lambda}^\lambda * \mathbb{R}_{A',\lambda}^{\nu})(\xi) = \mathcal{F}(\mathbb{R}_{A,\lambda}^\lambda)(\xi) \mathcal{F}(\mathbb{R}_{A',\lambda}^{\nu})(\xi) = r^{-(\lambda + \nu) - 2}(\xi)(C_{\lambda-n} C_{\nu-n} i\xi \xi + D_{\lambda-n} D_{\nu-n} \xi \xi)
\]
and
\[
\mathcal{F}(\tilde{R}_{A_{\lambda+\nu},B_{\lambda+\nu}}^{\lambda+\nu})(\xi) = r^{-(\lambda+\nu)-2}(\xi)(C_{\lambda+\nu-n}'\xi + D_{\lambda+\nu-n}'\xi + D_{\lambda+\nu-n}')^{\xi}
\]
do agree iff \(C_{\lambda+\nu-n}' = C_{\lambda-n}'\) and \(D_{\lambda+\nu-n}' = D_{\lambda-n}'\) hold. Note that a tuple \((A_\lambda, B_\lambda)\) will give a tuple \((C_\lambda, D_\lambda)\), see (3.2). Equivalently, using (5.2) we have to solve
\[
C_{\lambda-n} C_{\nu-n} = (\lambda + \nu - n + p)A_{\lambda+\nu-n} - pB_{\lambda+\nu-n},
\]
\[
D_{\lambda-n} D_{\nu-n}' = -(n-p)A_{\lambda+\nu-n} + (\lambda + \nu - p)B_{\lambda+\nu-n}
\]
for \(A_{\lambda+\nu-n}', B_{\lambda+\nu-n}'\). The unique solution is given by
\[
A_{\lambda+\nu-n} \overset{\text{def}}{=} \frac{(\lambda + \nu - p)C_{\lambda-n}C_{\nu-n} + pD_{\lambda-n}D_{\nu-n}'}{\lambda + \nu - n}
\]
\[
B_{\lambda+\nu-n} \overset{\text{def}}{=} \frac{(n-p)C_{\lambda-n}C_{\nu-n} + (\lambda + \nu - n + p)D_{\lambda-n}D_{\nu-n}'}{\lambda + \nu - n}
\]
Hence we obtain the semi-group property, whenever it makes sense,
\[
\tilde{R}_{A_{\lambda},B_{\lambda}}^{\lambda} * \tilde{R}_{A'_{\nu},B'_{\nu}}^{\nu} = \tilde{R}_{A_{\lambda+\nu},B_{\lambda+\nu}}^{\lambda+\nu}.
\]

**Remark 5.2** We have seen in Theorem 3.2 that the Fourier transform preserves the family of Riesz distribution \(R_\nu(x)\). In general it holds that the pairs \((A_\lambda, B_\lambda)\) and \((C_\lambda, D_\lambda)\), see (3.2), have nothing in common. However, for \(A_\lambda \overset{\text{def}}{=} \alpha_\lambda\) and \(B_\lambda \overset{\text{def}}{=} \beta_\lambda\) we have \(C_\lambda = -\lambda \alpha_{-\lambda-n}\) and \(D_\lambda = -\lambda \beta_{-\lambda-n}\), hence by Theorem 3.2 we have
\[
\mathcal{F}(R_{\alpha,\beta}^{\lambda-n})(\xi) = c_{\lambda} R_{-\alpha_{-\lambda-n},-\beta_{-\lambda-n}}^{\lambda-n}(\xi)
\]
for some constant \(c_\lambda\).

**Remark 5.3** Similar operators to the ones we study here have been considered in connection with Ahlfors operators and generalized Beurling-Ahlfors operators, see [IM93]; these arise in the study of quasi-regular mappings generalizing quasi-conformal mappings in two dimensions. Of interest here are new \(L^p\) - estimates for the Beurling-Ahlfors operator on differential forms.

**Remark 5.4** (Structure of GJMS operators) Let \((M, g)\) be a manifold. The GJMS operators, [GJMS92],
\[
P_{2N} : C^\infty(M) \to C^\infty(M),
\]
arises as residues of the scattering operator \(S(s) : C^\infty(M) \to C^\infty(M)\), [GZ03]. The latter one is a generalization of the intertwining integral operators \(T_{\nu,(1,0)} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)\) considered on 0-forms, see [4.2]. Since each \(P_{2N}\) is polynomial in second order differential operators \(M_{2k}\) [J10], \(k = 0, \ldots, 2N\), does this structure extend to \(S(s)\)? Explicit formulae for those second order differential operators exist on flat and on Einstein manifolds.
Remark 5.5 (Structure of Branson-Gover operators and conformal powers of the Dirac operator) The Branson-Gover operators [BG05], again conjectural, and conformal powers of the Dirac operator [GMP12] are also residues of certain scattering operators. Less is known about their structure. One might expect that some aspects of our results here in the model case carry over to the case of Riemannian manifolds and conformal geometry here; in particular the nature of a scattering operator should resemble our convolutions with Riesz kernels.

Remark 5.6 It is clear that given the large amount of analysis based on the Riesz distributions on functions, one may expect similar applications (e.g. fundamental solutions, wave equations, heat equations, $L^p$-mapping properties) of our formulas here; we plan to consider some of these applications in another paper. Along similar lines: can one study heat equations for the integral kernels $T_{\nu, (A_{\nu}, B_{\nu})}$?

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