Emergent quantization and possible measurements of Planck’s constant from the saturation of extrapolated high temperature liquid viscosity and other transport properties

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Quantum effects in material systems are often pronounced at low energies and become insignificant at high temperatures. As we elaborate here, this common occurrence might not, however, hold universally. We find that, perhaps counterintuitively, certain quantum effects may follow the opposite route and become progressively sharper so as to emerge in the “classical” high temperature limit. In the current work, we invoke and extend simple elements of the WKB approximation as applied to general Hamiltonians, extend the usual kinetic theory by taking into account a possible fundamental quantum time scale, transition state theory, and the equipartition theorem. On average, the extrapolated high temperature viscosity of general liquids may tend to a value set by the product of the particle number density \( n \) and Planck’s constant \( h \). We compare this theoretical result with experimental measurements of an ensemble of 23 metallic fluids where this seems to indeed be the case. The extrapolated high temperature viscosity of each of these liquids \( \eta \) divided by its value of \( n h \) veers towards a Gaussian with an ensemble average value that is close to unity up to an error of size 0.6%.

In the course of our analysis, we further trivially apply the maximum entropy principle to suggest a way to infer which rules might govern general measured data sets. We invoke similar ideas to discuss other transport properties to suggest how simple behaviors may appear including resistivity saturation and linear \( T \) resistivity may appear very naturally. Our approach suggests that minimal time lags may be present in fluid dynamics (which in the continuum follows the Navier-Stokes equation).

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I. INTRODUCTION

As long known, at atomic and smaller length scales, quantum mechanical effects are typically extremely important. Collectively in crystals and elsewhere, these effects may trigger striking (and often potentially useful) novel low temperature behaviors on far larger spatial scales as in, e.g., semiconductors, superconductors, superfluids, the quantum Hall effects, and countless other systems. Common lore asserts that at high temperatures, quantization is largely inconsequential. In the current work, we suggest that the opposite may occur in rather general high temperature fluids. Such an effect may have consequences for the understanding of the behavior of high temperature metals and insulators as well as basic minimal time scales in “classical” fluid dynamics.

It is clear that such a high temperature \((T)\) occurrence might not be entirely unexpected. At extrapolated asymptotically high temperatures such that \( k_B T \) is larger than any interaction energy scale (or energy associated with a particle having a wavelength determined by the inverse volume density, etc.), the dominant energy scale is set by \( k_B T \). Consequently, the only frequency scale is set by \( (k_B T)/\hbar \) with \( \hbar \) being Planck’s constant. Thus, even though it may seem counter-intuitive, in the very “classical” extreme high temperature limit, Planck’s constant may come in and clear signatures of quantum mechanics may emerge.

It is generally asserted that “classical transport”, such as that manifest in the viscosity of high temperature liquids, has little to do with quantum mechanics (apart from the latter setting the scale of the specific atomic interaction strength in the liquid that may vary dramatically from one liquid to another). Nearly ten years ago, string theory, which is a fundamentally quantum theory, made its debut in the field of physical kinetics by introducing the concept of perfect fluid. Perfect fluidity would be obtained by saturating the following lower bound for the ratio between the viscosity and the entropy density,

\[
\frac{\eta}{s} \geq \frac{\hbar}{4 \pi k_B},
\]

as conjectured by Kovtun and co-workers [1][2]. A perfect fluid as established by the above inequality is inherently a quantum fluid. We note that the light velocity does not arise in the bound above, so it is in principle possible to conclude that it applies to non-relativistic systems as well, although it has been derived for a relativistic theory. Experimental evidence for perfect fluidity appears in ultracold gases at unitarity and in the quark-gluon plasma [3]. It has also been argued that graphene might be an almost perfect fluid [4]. On the other hand, in classical physics perfect fluidity is associated to an ideal gas having \( \eta = 0 \). However, even classically, a weakly interacting system typically has a large viscosity, with a small viscosity being associated with a strongly interacting system. This supports the view that non-interacting particles have an infinite mean free path, leading in this way to infinite viscosity. Nevertheless, if the viscosity of an ideal classical or quantum gas is calculated via correlation functions, a vanishing result is obtained (this point will be reviewed in Section [VI]), in agreement with the expectations of classical fluid dynamics.

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In general, non-relativistic quantum fluids exhibit a viscosity that diverges as $T \to 0$, a behavior contrasting with dilute gases, where the viscosity vanishes as $T \to 0$. In dilute gases at high temperatures the viscosity is uniquely determined by two fundamental lengths governing the collisional dynamics, namely, for such fluids composed of particles of mass $m$, the non-relativistic thermal de Broglie wavelength, \( \lambda_{T}^{nr} = \sqrt{2\pi \hbar^2/(mk_BT)} \) and the scattering length, \( a_{sc} \), with the diluteness condition \( \lambda_{T}^{nr} a \ll 1 \) holding for a three-dimensional system, where \( n \) is the particle density per unit volume. Since the viscosity has dimensions of momentum divided by area, we can use the de Broglie relation with the thermal wavelength to write,

\[
\eta \sim \frac{\hbar}{\lambda_{T}^{nr} a_{sc}^2} \sim \frac{\sqrt{2\pi mk_BT}}{a_{sc}^2},
\]

and there is no dependence on the Planck constant or the density in this case. At low temperatures, quantum effects in a gas become relevant and a third length, the coherence length, plays also a role. For instance, for a dilute Bose gas in three dimensions and low temperatures such that \( na_{sc} (\lambda_{T}^{nr})^2 \ll 1 \), an explicit dependence on the superfluid density arises \( \eta \propto a_{sc}^4 \)[7] and contains a factor \( 1/a_{sc}^2 \), similar to Eq. (2). By contrast, at sufficiently low temperatures such that \( na_{sc} (\lambda_{T}^{nr})^2 \gg 1 \), the viscosity depends on the total density and exhibits a behaviour \( \eta \sim T^{-5} \). Interestingly, in this low temperature regime the viscosity is proportional to \( a_{sc}^4 \)[7], indicating a vanishing viscosity in the non-interacting limit. Calculations of the viscosity for \(^4\)He \( \sim \text{He} \) yield at low temperature, \( \eta \sim (a_{sc}/T)^5 \), in agreement with the previous result, while \( \eta \sim \hbar/(ka_BT) \) for high temperatures \( \eta \sim \Delta T \), where \( \Delta \) is the energy height of the roton minimum of the superfluid spectrum. A result that decreases with the temperature is also obtained for a Fermi liquid, where we have \( \eta \sim 1/T^2 \) \( \sim \text{He} \).

The high temperature regime of a relativistic system can be also analyzed similarly to our analysis of the high-temperature dilute gas. The main difference in the argument is the relativistic thermal de Broglie wavelength \( \lambda_{T}^{r} \), which in this case is dependent on the light velocity in vacuum \( c \) and the spatial dimensionality \( D \),

\[
\lambda_{T}^{r} = 2 \sqrt{\frac{\Gamma(D/2)}{2\Gamma(D)}} \frac{1^{1/D}}{D} \frac{hc}{k_BT}.
\]

The above thermal de Broglie wavelength holds for a massless relativistic particle, but it serves our purpose even for massive particles, since the high-temperature regime corresponds to \( k_BT \gg mc^2 \). Taking a \( \phi^4 \) scalar theory of dimensionless interaction strength \( u \) in \( D = 3 \) (in units where \( \hbar = c = 1 \) as an example, it is seen that the thermal de Broglie wavelength is the only length scale at high temperatures. Therefore, it is easy to obtain the high-temperature behavior,

\[
\eta \sim \frac{2(k_BT)^3}{\pi \hbar^2 c^2 u^2},
\]

This form attests to the subtle (and unexpected seemingly intertwined) “classical” non-quantum (i.e., \( \hbar \to 0 \)) and non-relativistic (\( c \to \infty \)) limits. An expression having precisely the above behavior has been indeed derived long time ago using Feynman diagrams within a non-equilibrium correlation function formalism \( \eta \). Although the high-temperature relativistic result \( \eta \) seems to diverge in the non-interacting limit, it vanishes in the non-relativistic limit, independently from the interaction strength. Of course, this result cannot be entirely trusted, as we have already seen in the example of the dilute non-relativistic Bose gas at low temperatures. Indeed, there we would have concluded from the moderately low temperature regime, corresponding to \( n a_{sc} \lambda_{T}^{nr} \ll 1 \), that the viscosity diverges in the non-interacting limit. However, in the even lower temperature regime, the behavior with the scattering length points out to a vanishing result instead.

As a weaker and more generic suggestive consideration regarding the specter of quantum effects sharply manifesting at high temperatures, we next invoke the energy-time uncertainty relations

\[
(\Delta E)(\Delta t) \geq \frac{\hbar}{2},
\]

If the broadening \( \Delta E \) is set by the only scale in the system, the thermal energy \( (k_BT) \), then this will indeed similarly motivate the appearance of a minimal uncertainty in time that is of the order of \( \hbar/(k_BT) \). More specific and detailed than simple uncertainty relations alone, a plethora of related well-known arguments may be applied to the high temperature limit in which the classical thermal de Broglie wavelength \( \lambda_{T}^{r} \) formally becomes asymptotically small at high temperatures. In the context of transport measurements, that form the principal motivation of this work, the Ioffe-Regel criterion \( \eta \) and other considerations like it applied to this small wavelength limit may lead to conclusions akin to those adduced from the above dimensional analysis arguments. Physically, for transport to occur the mean-free path \( \ell \) must be greater than or equal to the de-Broglie wavelength. For smaller \( \ell \), the notion of a particle (or quasi-particle) would be ill-defined. In such cases, constructive interference may lead to localization (similar to that found in electronic systems \( \eta \)). Thus, the formal vanishing of the de Broglie wavelength is impossible. Rather, in the liquid phase when quasi-particles are still defined, the de-Broglie wavelength can only be as small as the mean-free path (which in turn can be no smaller than the inter-particle separation). Thus, the uncertainty relations or related de Broglie wavelength scale mandate that at high temperature saturation occurs so long as no phase transitions appear as would be further anticipated for analytic high temperature extrapolations of transport functions within a liquid phase.

II. OUTLINE

In the sections that follow, we will explicitly suggest how this may occur and may have already been measured in material systems. The outline of the remainder of this article is as follows. In section III we demonstrate how, by virtue of the WKB relations, in the extreme high temperature “classical limit”, Planck’s constant must make an appearance when computing sums over all quantum states. Armed with this
In section VII, we demonstrate that the prefactors \(Z\) in the classical canonical partition function for a system of \(N\) particles in \(D\) spatial dimensions at an inverse temperature \(\beta\) and with a Hamiltonian \(H\) is
\[
Z = h^{\text{DN}} \int d^{\text{DN}}x \, d^{\text{DN}}p \, \exp(-\beta H) \tag{6}
\]
with \(h\) the Planck constant, \(n\) an integer labeling the state, and \(C\) a constant that vanishes for steep “hard boundaries”, assumes a value of \(C = 1/2\) for soft potentials, and \(C = 3/4\) for one dimensional potentials with one hard and one soft boundary. Historically, the hard boundaries setting captures the single valuedness of the wave function (or coherent phase contributions from periodic motion) when the momentum is set by harmonics of the de Broglie wavelength. The leading order WKB result of Eq. (6) becomes progressively more precise as the classical limit is approached. With an eye towards things to come, we remark that in systems such as fluids (that form the focus of our interest in the current work), atomic motion is indeed bounded. In Eq. (6), the momentum \(p_n\) is that associated with the \(n\)-th state and the integral is performed over a closed orbit (i.e., a complete periodic one dimensional semi-classical trajectory). Eq. (6) is the micro canonical classical phase space volume associated with all states of energy less than \(E_n\) in a system with a single remnant degree of freedom. The same holds for theories
in which the \( x \) coordinate is decoupled from all others. For ensembles with multiple decoupled cyclic coordinates, higher powers of \( h \) appear. From Eq. 6\(^{16}\), the volume of the phase space annulus between two consecutive values of \( n \) is none other than Planck’s constant \( h \), thus establishing the appearance of this exact fundamental constant in the micro-canonical one-dimensional one particle “ensemble” and all ensembles derived from it.

Furthermore, it is useful to recall the relation between \( J_n \) and the density of states of the quantum system \(^{21}\). Since,

\[
J_n = J(E_n),
\]

where,

\[
J(E) = \oint |dx| \sqrt{2m[E - V(x)]},
\]

is the action-angle variable describing the classical trajectory, the expression,

\[
\rho(E) = \frac{\tau(E)}{h} \sum_n \delta \left( \frac{J(E)}{h} - 2\pi n + C \right),
\]

yields precisely the density of states provided Eq. 9\(^{16}\) holds and Eq. 10\(^{16}\),

\[
\tau(E) = \frac{\partial J(E)}{\partial E} = \sqrt{\frac{m}{2}} \oint \frac{|dx|}{\sqrt{E - V(x)}}.
\]

We briefly remind some readers where the very standard, yet in some disciplines not often used, expression of Eq. 10\(^{16}\) for the period \( \tau(E) \) comes from. The speed of a classical particle of energy \( E \) in a potential energy field \( V(x) \) is \( v = \sqrt{2(E - V(x))}/m \) and the time increment required to traverse a distance \( |dx| \) is \( |dx|/v \). From this it is seen that the period \( \oint |dx|/v \) is given by Eq. 10\(^{16}\). From Eqs. 9\(^{16}\) and 10\(^{16}\) we obtain the well-known result that the classical frequency

\[
\nu(E) = \frac{h}{\tau(E)} = \frac{1}{\rho_{cl}(E)},
\]

where \( \rho_{cl}(E) \) is the classical density of states. Therefore, an immediate corollary of the WKB type relations of Eq. 6\(^{16}\) and the fact that \( J_n \) is, geometrically, the phase space volume of states bounded by an energy \( E_n \), is that sums over energy eigenstates \( n \) can be replaced by

\[
\sum_n \frac{1}{h} \oint dJ = \frac{1}{h} \oint dx \int dp,
\]

since \( J(E) \) is related to the number of states, \( N(E) \), bounded by \( E \) through \( J(E) = hN(E) \). Similar expressions follow relating quantum systems to their classical counterparts. For instance, replacing the partial derivative in Eq. 10\(^{16}\) by finite differences, \( \nu_n \sim (E_{n+1} - E_{n-1})/(J_{n+1} - J_{n-1}) \) and invoking the quantization condition of Eq. 6\(^{16}\) for both the \( n \)-th and the \((n + 1)\)-th levels, we arrive at an expression for the semi-classical frequency of the \( n \)-th state,

\[
y_n \sim \frac{E_{n+1} - E_{n-1}}{2h}.
\]

The constant \( C \) in Eq. 6\(^{16}\) drops out in the subtraction between \( J_n+1 \) and \( J_n-1 \). Eq. 13\(^{16}\) applies to general potentials. To establish trivial intuition, we briefly regress to a harmonic oscillator of resonant frequency \( \bar{\omega} \), with energies \( E_n = (n + 1/2)\hbar \bar{\omega} \) and for which (as indeed apparent in Eq. 13\(^{16}\)) as it must be) the angular frequency of oscillations of all semi-classical levels is \( \bar{\omega} \).

Albeit exceedingly simple, we are not aware of an earlier derivation of Eq. 12\(^{16}\) with \( h \) being Planck’s constant that instantly follows from the WKB type relation of Eq. 6\(^{16}\). To summarize, Eq. 12\(^{16}\) follows by noting that consecutive semi-classical trajectories \( n \) and \( (n + 1) \) have, by Eq. 6\(^{16}\), a variation in their action of size \( (J_n+1 - J_n) = \hbar \). This difference in the action associated with consecutive levels immediately gives rise to the fraction of \( 1/h \) multiplying the integrals on the right-hand side of Eq. 12\(^{16}\). It is in the semi-classical high temperature (or high energy) limit with a divergent number of states \( n \) that the sum over the many viable quantum states \( n \) can be replaced by the continuous integral of Eq. 12\(^{16}\). For a sum containing \( M = (n_2 - n_1) + 1 \) discrete \( n \) values in the sum of Eq. 12\(^{16}\), there are corrections of typical relative strength \( O(1/M) \) augmenting the integral on the righthand side as is seen from the well known Euler-MacLaurin formula,

\[
\sum_{n=n_1}^{n_2} f(n) \sim \int_{n_1}^{n_2} f(x) \, dx + \frac{f(n_1) + f(n_2)}{2} + \sum_{n_1}^{n_2} B_{2n'}/(2n')! \left( f^{(2n' - 1)}(n_1) - f^{(2n' - 1)}(n_2) \right),
\]

with \( B_{2n'} \) the Bernoulli numbers. Thus, from the Euler-MacLaurin formula, in the semi-classical limit, not only is Eq. 6\(^{16}\) asymptotically exact but also the ensuing conversion from a discrete sum to a continuous integral in Eq. 12\(^{16}\) may emerge as an exact relation with vanishing corrections.

IV. TRANSITION RATES IN SEMI-CLASSICAL SYSTEMS

Eqs. 10\(^{16}\) and 12\(^{16}\), provide means to compute the oscillation frequency and quantities relying on it within the semi-classical limit. We now combine this relation with an expression for the probability of transition from a given non-equilibrium initial state (or states) to a final lower energy state via intermediate transition states \(^{15\ldots20}\). In the current derivation, we assume, in a semi-classical spirit, that once a state via intermediate transition states \(^{15\ldots20}\) escapes, then full transition occurs and the system may overcome the energy barrier and go to a final lower energy state, see Fig. 1\(^{1}\). In the present case, excited states of energy \( E > E_{\text{escape}} \) (corresponding to “transition states” in the parlance of physical chemistry) are short-lived resonant states that may veer and “decay” to the low energy final state. We mark the spatial boundaries of the transition state region by \( x_{\text{final}} > x_{\text{initial}} \) (i.e., the left potential wall
within our framework, the semi-classical transition rate will veer towards the lower energy final state if the momentum $p > 0$ (assuming $x_{\text{final}} > x_{\text{initial}}$). The particle will move from its initial position due to energy absorbed from thermal fluctuations (these fluctuations are captured by a Boltzmann factor that we will shortly introduce). In those states with $p < 0$, the particle may retrace back to the initial state (or other state with lying to the left of $x_{\text{initial}}$). In the semi-classical limit, we assume that once the particle in the initial state is excited into a state $n$, it still starts off at an original physical $x = x_{\text{initial}}$ coordinate location while acquiring a higher energy (and momentum) and then starts moving from that initial spatial location. Within any resonant transition state $n$, the particle spends, on average, a time given by $\nu_n^{-1}/2$ before transitioning into the lower energy final state. This is so as the motion between $x_{\text{initial}}$ and $x_{\text{final}}$ constitutes half of a full period if infinite potential barriers existed at $x = x_{\text{initial}}$ and $x_{\text{final}}$. For the sake of computing this time, we may invoke Eq. (10) for a system in which the original potential $V$ was modified only at $x = x_{\text{initial}}, x_{\text{final}}$ by the insertion of an infinite potential barrier. Such an insertion will not modify the “time of flight” between $x_{\text{initial}}$ and $x_{\text{final}}$ with the true potential (i.e., that does not diverge at $x = x_{\text{final}}$). That this is true is evident by, e.g., examining half of Eq. (10), i.e., the integral of eq. (10) will clearly not be affected by change of potential $V(x)$ only at point $x = x_{\text{final}}$. Putting a hard wall boundary conditions corresponds to a fixed value of $C$ in Eq. (6) for all levels and just as for the undeformed system, the measure $dJ$ of Eq. (12) is unchanged. Similarly, when invoking Eq. (12), to rewrite quantities in terms of semi-classical integrals, the classical energy $H(x, p)$ will be unchanged for all $x_{\text{initial}} < x < x_{\text{final}}$. Let us denote the Hamiltonian resulting from this hard wall deformation by $H_{\text{transition}}$ (i.e., that describing the transition region and that cannot allow for classical paths beyond $x_{\text{final}}$ (the point at which, any way, the system will “collapse” into the lower energy final state). The probability of occupying any resonant transition state $n$ is given by a Boltzmann type factor $\text{Prob}_n(T) = \exp[-\beta(E_{n,\text{transition}} - F_{\text{initial}})]$ where $F_{\text{initial}}$ is the effective free energy of the system in the initial state as it makes transitions into the transition states. Following the logic laid out by Eyring [15] and many subsequent works in transition state theory [18] and applying it within our framework, the semi-classical transition rate $r$ from the initial to final state

$$r(T) = \sum_{E_n > E_{\text{escape}}} (2\nu_n) \text{Prob}_n(T) \Theta(p_n),$$

where the Heaviside function $\Theta(p)$ ensures that the momentum $p$ is positive. The factor of two multiplies $\nu_n$ as the time of flight from $x_{\text{initial}}$ to $x_{\text{final}}$ is, as noted above, half of an entire period, and thus the frequency of this motion to the right is double the frequency associated with a full oscillation. Invoking the simple correspondence of Eq. (12) for half of the states $n$ (i.e., those with positive momentum $p$ in Eq. (12)), and replacing the frequency $\nu_n$ by $\partial H_{\text{transition}}/\partial J$ evaluated at an energy $E = E_n$ (Eq. (10)) applied to the deformed classical potential with hard wall boundaries, the sum in Eq. (15) becomes

$$r(T) = \frac{1}{\hbar} \int_{E_{\text{escape}}}^{\infty} dJ e^{-\beta(E_{n,\text{transition}} - F_{\text{initial}})} = \frac{1}{\hbar} \int_{E_{\text{escape}}}^{\infty} dJ e^{-\beta(F_{\text{final}} - E_{\text{escape}})} = k_B T \frac{e^{\beta(F_{\text{final}} - E_{\text{escape}})}}{\hbar}.$$ (16)

We pause to briefly comment on the relation between Eq. (16) to known results. Similar expressions are found in transition state theory, e.g., [15][18]. The prefactor of $(k_B T/\hbar)$ drops out in the most common application of transition state theory to harmonic systems as we will briefly expand on shortly; this then gives rise to an Arrhenius form with a “classical” (i.e., Planck constant independent) constant that is fixed at all temperatures. In Kramers’ original work [17], Planck’s constant never appeared. Within transition state theory, the energy is often taken to be a conserved quantity during time evolution only in the “weak friction limit” of a classical system where noise is provided by thermal fluctuations. Within our derivation, the inverse temperature $\beta$ serves (as it often does) as a Lagrange multiplier for the energy with increasing fluctuations in the low $\beta$ (or high temperature) limit. The energy of the system with its surrounding thermal bath is that of a closed system for which the quantization of the energy holds. The energy was an “integral of motion” in our derivation for a
closed system starting from the Bohr-Sommerfeld relation of Eq. (6) as it is for any time translationally invariant quantum (or classical) system. Lastly, although seemingly innocuous, the physical meaning and precise definition of the effective free energy $F_{\text{initial}}$ is subtle as naively the exponentiated thermal dependence of typical free energy (i.e., a partition function as written in numerous works, e.g., [15]) might be thought to overwhelm any linear temperature dependence stemming from the $(k_B T / \hbar)$ prefactor. We will return to this point and argue that simple partition function forms might be misleading. That is, the rate need not be directly related to the ratio of partition functions as it typically is in transition state theory. Instead, the initial dynamics is of a non-equilibrium nature and may be considered of the Monte Carlo type or other forms that equilibrate at long times but are of a non-equilibrium nature at finite times.

Putting all of the pieces together, in attaining Eq. (16) for quantum systems via a very simple derivation, we invoked the WKB approximation, of Eq. (6), the semi-classical rate $\nu$ with all transition states. Eq. (16) is completely general for arbitrary Hamiltonians in both non-relativistic and relativistic systems and is not confined to that of free particles having translational degrees of freedom or of harmonic oscillators, etc., on which nearly all of the work has been done to date. Historically, e.g., [15] motion of the one dimensional coordinate from the initial state to “the left” (i.e., away from $x_{\text{final}}$) was ignored as such motion was regarded as an obstruction to the transition the final state. In what follows, we account for such displacements. If we account also for those particles that initially go the left from the initial state, are then “bounced off” the left potential wall and ricochet back to the right then, of course, the rate will be larger for that accounted for by Eq. (16). The additional contribution is

$$
\Delta r_{\text{ricochet}} = \sum_n (2\nu_n') \text{Prob}_n(T) \Theta(-p_n). \quad (17)
$$

In Eq. (17), the quantity $\nu'$ is the frequency of the motion from the initial state to the left boundary of the potential wall and the bounce back to right to go to the final state. A moment’s reflection reveals that $\nu' < 1/('time of flight’ between $x_{\text{left}}$ and $x_{\text{final}}$ for the semi-classical $n$-th state) $\equiv \nu$ where $x_{\text{left}}$ is the coordinate of the semi-classical bounce point. In order to have a bounce at $x_{\text{left}}$ that enables a transition to $x_{\text{final}}$ on the right, the energy $E_{\text{escape}} < E_n < E_{\text{escape left}}$ where $E_{\text{escape left}}$ is the energy threshold required for the particle to escape indefinitely to the left (i.e., $E_n < E_{\text{escape left}}$ ensures that a bounce will occur) and, as before, $E_{\text{escape}}$ is the energy required to escape over any barriers on the right-hand side branch $(x > x_{\text{initial}})$ of the potential barrier in order to reach the final state. Similar to the derivation of Eq. (16), we can deform the potential and invoke the WKB result of Eq. (6) in order to bound Eq. (17). The result is

$$
\Delta r_{\text{ricochet}} \leq \Theta(E_{\text{escape left}} - E_{\text{escape}}) \times \sum_n (2\nu_n) \text{Prob}_n(T) \Theta(-p_n) \\
= \frac{1}{\hbar} \int_{E_{\text{escape}}} e^{-\beta(\tilde{H} - F_{\text{initial}})} \Theta(E_{\text{escape left}} - E_{\text{escape}}) \, d\tilde{H} \\
= \frac{1}{\hbar} \int_{E_{\text{escape}}} e^{-\beta(\tilde{H} - F_{\text{initial}})} \times \Theta(E_{\text{escape left}} - E_{\text{escape}}) \\
= \frac{k_B T}{\hbar} (e^{\beta(F_{\text{initial}} - E_{\text{escape}})} - e^{\beta(F_{\text{initial}} - E_{\text{escape left}})}) \\
\times \Theta(E_{\text{escape left}} - E_{\text{escape}}). \quad (18)
$$

The full transition rate $r + \Delta r_{\text{ricochet}}$ will be trivially bounded from below by the extreme case of assuming that no particles going to the left will ever ricochet back to the right and, from above, by setting the frequency of the reflected particles from the left barrier to be equal to $\tilde{\nu}$. That is,

$$
\frac{k_B T}{\hbar} e^{\beta(F_{\text{initial}} - E_{\text{escape}})} \leq r + \Delta r_{\text{ricochet}} \leq \frac{k_B T}{\hbar} \left[ \Theta(E_{\text{escape left}} - E_{\text{escape}}) \left( 2e^{\beta(F_{\text{initial}} - E_{\text{escape}})} - e^{\beta(F_{\text{initial}} - E_{\text{escape left}})} \right) \right] \\
+ \Theta(E_{\text{escape}} - E_{\text{escape left}}) e^{\beta(F_{\text{initial}} - E_{\text{escape left}})} \right]. \quad (19)
$$

If $E_{\text{escape}}$ is not much larger than $E_{\text{escape left}}$, as will occur in systems symmetric along $x$, then the inequalities in Eq. (19) will become equalities and the rate will be given by Eq. (16).

We briefly note that our one-dimensional WKB borne analysis can be trivially extended to, e.g., the sequence of $s$-wave states in spherically symmetric Hamiltonians in three spatial dimensions to obtain results identical to those in Eq. (16) for one-dimensional systems with the same radial Hamiltonian. The radial coordinate plays the role of the one-dimensional coordinate $x$ in our analysis thus far. For higher (i.e., non $s$-wave) angular momentum eigenstates, the centrifugal barrier at the origin needs to be augmented in the effective Hamil-
tonian, $H_{ff} = H + \hat{L}^2/(2mR^2)$ with $\hat{L}$ the orbital angular momentum operator for the function $\phi = R\psi(R)$ with $\psi$ the wavefunction and $R = |\vec{R}|$ the radial coordinate.

Now, if we take $F_{\text{initial}}$ to be the equilibrated free energy of the system for all eigenstates of the potential whose minimum is the initial state or whether we consider $F_{\text{initial}}$ the free energy of the entire system, the exponential factor in Eq. (16) will have a temperature dependence that may overwhelm or partially cancel the linear in $T$ prefactor. For instance, if $F_{\text{initial}}$ corresponds to the free energy of, e.g., a simple harmonic oscillator potential of frequency $\omega_{\text{initial}}$ about a minimum of $E_{\text{initial}}$ at $x_{\text{initial}}$, then Eq. (16) will become $r(T) = \omega_{\text{initial}} e^{-\beta E_{\text{initial}}}/(2\pi)$, with $\Delta E = (E_{\text{escape}} - E_{\text{initial}})$, i.e., a pure Arrhenius type behavior with a prefactor set by the classical oscillator frequency $\omega_{\text{initial}}$.

If, however, the initial system is not in equilibrium, then $F_{\text{initial}}$ may be replaced by the energy of the initial state $E_{\text{initial}}$ to yield the relevant transition probability as in Monte-Carlo type dynamics. In such a case, $r(T) = k_B T^2 e^{-\beta E_{\text{initial}}}$, in which the natural frequency prefactor is set by the ratio of the thermal energy $(k_B T)$ to Planck’s constant $\hbar$. The exponential factor tends to unity in the extrapolated high temperature limit. Thus, from Eqs. (16)(19) the extrapolated rate

$$r(T) \sim \frac{k_B T}{\hbar}$$

asymptotically, at high $T$. (20)

It is important to emphasize that in the emergent extrapolated classical high temperature limit, Eq. (20) is an exact asymptotic equality with no undetermined prefactors of order unity. This equation is significant and will play a central role in our derivations of Section VII. We now briefly further comment on the physical situation in which more than a single perturbative effect may be possible about several saddles each with each own escape energy and total weight (a normalized probability distribution). However, irrespective of whether or not a saddle point approximation is valid, Eq. (20) must hold. So far we considered only uncorrelated single particle motion in a fixed background (leading to a particular time independent Hamiltonian $H$). In reality, correlated dynamics occur although at asymptotically high temperatures, single particle motion is not as constrained at it is at low temperatures where collective effects may be necessary for motion (as in, e.g., supercooled liquids). The drop of the correlations with temperature may be evinced via, e.g., a $(1/T)$ type expansions (23) in disparate arenas. In low temperature liquids, such correlated dynamics is made manifest by four point correlation functions that quantitatively elucidate heterogeneous liquid dynamics. Thus, we anticipate in general systems the average of the correlations to vanish. For each such trajectory (if independent of all others) there would be an effective local barrier $E_i$ which may be trajectory dependent in non-uniform liquids (especially non monoatomic ones). There is also a dependence on the initial state as that sets the energy gap.

We note that the energy difference $\Delta E_i$ is typically bounded as for each channel there is a particular minimal energy $E_{\text{escape}}$ to overcome in order to reach a viable final state: $E_{\text{escape}}$ does not reflect an unbounded possible energy of a single particle but rather the minimal energy required to transition. Thus, albeit appearances, the integral in the expression for $r_{\text{independent}}(T)$ is not a typical partition function type sum of a system with states with arbitrarily high energies may appear.

We now very simply review known facts to cogently highlight the use of non-equilibrium Monte-Carlo type dynamics. Canonically, in equilibrated systems with a ground state energy that is set equal to zero, the scaling of the partition function $Z = \exp(-\beta F)$ with temperature is that of the number of micro-states with energy $E \leq k_B T$. In order to have no temperature dependence multiplying the factor of $k_B T/\hbar$ in equilibrated systems, the range of temperature values $T$ is such that there are no energy levels of energies of order $k_B T$ so that either all states $i$ are easily thermalized and has a unity Boltzmann weight $\exp(-\beta E_i)$ when $E_i \ll k_B T$ or have a vanishing weight (with $E_i \gg k_B T$). To provide simple textbook type examples, for the single one-dimensional harmonic oscillator where for $n \gg 1$, the energies $E_n \propto n$. As the number of harmonic oscillator states whose energy is lower than $k_B T$ is proportional to $n$, it is seen, sans any calculations that the partition function trivially scales linearly in $T$ in the high temperature limit. Similarly, for a single free quantum particle in a one-dimensional box with infinite potential barriers, the energies $E_n \propto n^2$ and counting the number of states of energy less than $O(k_B T)$ it is seen that the corresponding partition function scales as $\sqrt{T}$ (as is easily calculated exactly). The exponentiated free energy $F_{\text{initial}}$, i.e., $\exp(-\beta E_{\text{initial}})$, does not correspond to such standard partitions with a non-trivial temperature ($T$) dependence. In the formal $\beta \to 0$ limit, $\exp(-\beta F_{\text{initial}})$
is a partition function and similarly counts the number of viable states starting from the single particle state at \( x_{\text{initial}} \). If we were to insist on assuming thermal equilibrium and on assigning an unconstrained partition function meaning to partition function \( \exp(-\beta F_{\text{initial}}) \) and not assume Monte Carlo dynamics as we have above, then at sufficiently high enough temperatures \( T \), the free energy \( F_{\text{initial}} \) would be arbitrarily low and \( F_{\text{initial}} \) would be ill-defined as all states could be occupied, and states would no longer be bound to a basin of states around a minimum at \( x_{\text{initial}} \). In such a case the “sea of states” centered about \( x_{\text{initial}} \) would “leak out” in all directions and any standard partition function would be that of the entire equilibrated system. Towards that end, we have invoked Monte Carlo type dynamics in our analysis above. That is, the system at the initial state is out of the equilibrium and would veer towards thermal equilibrium given sufficient time.

When invoked, the use of partition functions assumes that the systems are ergodic in the long time dynamics. In our current problem, we interested if the system transitions from an initial state to a final lower energy state at any particular time (and to find the local (in time) rate for such a process). For any such process to occur, the system has to be in the basic of the “initial” state. The energy of any such state at a given time will \( E_{\text{initial}}(t) \). This (stochastic) energy may vary yet, by the very definition of the escape problem, will be lower than the escape energy \( E_{\text{escape}} \) (for otherwise it would not be in the initial state energy basin). In the low \( \beta \) (or high temperature) limit, the escape rate will universally approach \( k_B T/\hbar \) regardless of the distribution of initial energies \( E_{\text{initial}} \) (as the probability distribution integrates to unity and for each choice of an initial energy the rate of transition to the final state is, at asymptotically high extrapolated temperatures, \( k_B T/\hbar \) and how these initial states vary with time to other state (so long as these states lie at energies lower than \( E_{\text{escape}} \) and still lie in the same energy basin of the initial state). With this in mind, we may formally define \( F_{\text{initial}} \) by

\[
Z_{\text{initial}}(\beta) = \exp(-\beta F_{\text{initial}}) = \int D\phi_{\text{initial}} \exp(-\beta E_{\text{initial}}) \]

where \( D\phi_{\text{initial}} \) is the measure including the distribution for states in the initial state basin. The average of \( \exp(\beta E_{\text{initial}}) \) for all initial states that still remain and lie within the energy basin that includes the initial state is simply \( \langle \exp(\beta E_{\text{initial}}) \rangle = Z_{\text{initial}}(0)/Z_{\text{initial}}(\beta) \). Similarly, \( \langle \exp(-\beta E_{\text{final}}) \rangle \) averaged over all possible paths over the escape energy barrier along different directions is \( Z_{\text{escape}}(\beta)/Z_{\text{escape}}(0) \) where “partition function” \( Z_{\text{escape}} \equiv \int D\phi_{\text{escape}} \exp(-\beta E_{\text{escape}}) \) where \( D\phi_{\text{escape}} \) is the measure of escape states lying at the brim of the basin surrounding the initial state. However, of course, the overall average in Eq. (21) is not given by the ratio of partition functions such as that in the product

\[
\frac{k_B T}{\hbar} \frac{Z_{\text{initial}}(0)}{Z_{\text{initial}}(\beta)} \frac{Z_{\text{escape}}(\beta)}{Z_{\text{escape}}(0)}. \quad (22)
\]

This is so as for each escape path \( \lambda \) from an initial state there is a unique corresponding escape energy. That is, given the initial energy and the direction of motion, there is a unique trajectory that the system follows and one may not average over all possible escape energy states. Stated alternately, there are correlations between the initial and escape energies and consequently \( \langle \exp(-\beta(E_{\text{initial}} - E_{\text{escape}})) \rangle \neq \langle \exp(-\beta E_{\text{initial}}) \rangle \times \langle \exp(\beta E_{\text{escape}}) \rangle \).

V. QUANTUM KINETIC THEORY OF THE VISCOSITY

We now return to the evaluation of the viscosity of “classical” fluids in their extrapolated high temperature limit. Armed with Eqs. (16–20), we next turn to the calculation of the viscosity by employing the standard Boltzmann equation. We briefly review elements essential to our analysis, e.g., [26]. In what follows, \( f^{(0)}(\vec{r}, \vec{v}, t) \) denotes a single particle equilibrium distribution (i.e., the equilibrium probability of finding a particle at location in a region of size \( d^3 r \) about \( \vec{r} \) and velocity that is within \( d^3 v \) at time \( t \) is \( f^{(0)}(\vec{r}, \vec{v}, t) \ d^3 r \ d^3 v \); this equilibrium distribution is set by the Boltzmann weights associated with the Hamiltonian \( H = \sum p_i^2/(2m) + V \). The single particle distribution \( f \) results from an integral over all but one of the particle coordinates. We will assume that the probability that a molecule will not experience a collision in a time increment \( dt \) is \((dt/\tau)e^{-t/\tau}\). The real (i.e., non equilibrium) distribution function

\[
f(\vec{r}, \vec{v}, t) = \int_0^\infty f^{(0)}(\vec{r}_0, \vec{v}_0, t - t')e^{-t'/\tau} \frac{dt'}{\tau}, \quad (23)
\]

where \( \vec{r}_0 = \vec{r}(t_0) \) and \( \vec{v}_0 = \vec{v}(t_0) \). The linear order relation of Eq. (23) expresses that particles reaching \((\vec{r}, \vec{v})\) at time \( t \) could have been in equilibrium for any amount of time \( 0 < t' \leq t \) beforehand and that the probability for staying in equilibrium for such an amount of time is set by \( e^{-t'/\tau} \) with \( \tau = 1/(r T) \) of Eq. (16). Multi-particle corrections to single particle motion are, e.g., often afforded by a BBGKY type hierarchy. To compute the viscosity within the Boltzmann equation formalism, one examines the system response of the system to a \( z \) dependent shear force along the \( x \) direction that leads to a constant average drift velocity. The shear produces an associated local equilibrium distribution function \( f^{(0)}(v_x - u_s(z), u_y, u_z) \equiv g(U_x, U_y, U_z) \). We will forgo repeating standard steps as those may be found in excellent standard texts. The final result, e.g., [26], is that the viscosity

\[
\eta = -m \int d^3 U \frac{\partial g}{\partial U_x} U_x^2 \tau(U) = m \tau \int d^3 U \ U_x^2 g \\
= m \tau \eta_0(U_x^2) = n k_B T \tau. \quad (24)
\]

The relaxation time \( \tau \) of Eqs. (16–20) is, in the high temperature limit, velocity independent and may be pulled out in front of the integral. The particle density per unit volume, \( = \int d^3 U \ g(U) \). Inserting Eqs. (16–20) we see that the extrapolated high temperature viscosity

\[
\lim_{T \to \infty} \eta = n \eta_0. \quad (25)
\]

Although we have specialized to \( D = 3 \) dimensions in the spatial measure of the above integrals to conform to the case of empirical interest, the result of Eq. (25) is \( D \) independent. As far as we are aware, the above Boltzmann equation derivation of Eq. (25) is new. Due to the limit of Eq. (20) that is
valid for each individual path $\lambda$ (and any weighted such distribution of escape paths), the relaxation time $\tau$ is universally $h/(k_BT)$ is the extrapolated high temperature limit. This rather exact value coincides with simpler more naive considerations regarding quantum saturation. It is interesting to note that Eq. (25) also follows from Maxwell’s equation $\eta \propto \rho \ell$ for the viscosity of a dilute gas (of mass density $\rho$, the velocity $v$, and mean free path $\ell$) [27] if it is “quantum limited” meaning that the mean free path saturates at high temperature and is set equal to the de Broglie wavelength $h/(mv)$. Physically, for transport to occur the mean-free path $\ell$ must be greater than or equal to the de Broglie wavelength; for smaller values of $\ell$, constructive interference may lead to localization, e.g., [14] occurring when the mean free path is of the order of the inter particle distance.

The linear Boltzmann equation ignores multi-particle correlations that lead to anisotropies for single particle motion. Higher order correlations and ensuing corrections to the lowest order application of the Boltzmann equation are reviewed in standard texts (see, e.g., chapter 14 of Ref. [26] for a pedagogical rudimentary account in simple monatomic systems). In the absence of known potentials and universal system details, not much can be said of corrections that will indeed differ from one system to another. In the below, we conceptually highlight the origins of the corrections that we found in Eq. (25).

For a fixed background of ions, the relaxation time of a single ion $i$ may be velocity direction dependent (as the classical trajectory $\lambda$ and associated $E_{\text{escape}}$ may vary); consequently an effective temperature dependent correction may multiply $\eta$ as if we look for a single specific particle in an assumed constant background, $\int d^3g(U_i)\tau(U_i) \neq \tau \int d^3U_i g(U_i)$ with $g(U_i)$ the distribution for the particular particle $i$ assuming a fixed background of external particles (which, of course, is not the same as the average single particle distribution $g(U)$ for the entire system). The last equality in Eq. (24) is a trivial consequence of the equipartition theorem for general non-relativistic Hamiltonians $H$ (i.e., those with with a $p^2$ momentum dependence with arbitrary potentials $V$) and their associated Boltzmann distributions averages which can be captured by weighted expectation values computed via $g$ (not solely those describing free non-interacting gas particles as in most elementary treatments). It would hold unchanged in the presence of different ion species with varying masses in complex liquids as for each the mass cancels by the use of the equipartition theorem. In general, we may expand $\tau(U_i)$ for a single particle $i$, in a fixed background of other particles, by a Taylor series in $\bar{U}_i$ about an average value $\bar{U}$ for all particles and note that the spatial or angular dependence on the direction of $\bar{U}$ may cancel when summing over all liquids when different geometries are concerned. In evaluating the viscosity, the terms in the Taylor series will lead to integrals of the derivative of $g$ relative to $U_i$ multiplied by powers of $U_\perp$ (and $U_z$); beyond the first constant term (giving rise to $nh$) an integration by parts will not be possible to get the simple single particle density $\eta$ and additional corrections appear. If the prefactors in the Taylor series for different liquids are random relative to each other then these will cancel when performing an ensemble average.

If, for any particular liquid, say a liquid of type “$a$”, only a single path $\lambda$ with an energy barrier $\bar{E}_a$ is relevant in Eq. (21) then the corresponding finite temperature $\eta_a$ will be the form

$$\eta_{\text{single-exponent } a} = A_a n \hbar \exp(\beta \bar{E}_a).$$ (26)

This form, first suggested by Eyring [28] is weaker than the exact result of Eq. (25). In Eyring’s original theory, the prefactor $A_a$ is set by the ratio of products of lengths associated with particle displacement, and the spread of applied shear forces along the different Cartesian directions. If, for simplicity, all of these lengths are symmetrically set equal to one another and further assumed to all be given by the average interatomic separation then according to Eyring’s theory [28] $A = 1$ in Eq. (26). Eyring posited the existence of “holes” in a liquid to which particles may move to (leaving a hole in their former location). He then modeled the rate of particle motion to such holes as a reaction process and subtracted the rate of backward motion (i.e., that in a direction opposite to an applied shear) from that of forward motion. We remark that although holes [28] in a liquid might be ill-defined, the notion of examining, in a broader and more rigorous context, pairs of time reversed states (any motion with or without imaginary holes) as a way to sum over the evolution of all states is viable. Repeating calculations mutatis mutandis, such a sum will yield an $h$ pre factor for each particle after linearizing the Boltzmann weight in Eq. (16) for all processes $\lambda_x$ for the rate of forward and backwards motion relative to the applied external shear directions (these time reversed states have relative energy shifts $\Delta E_{\lambda_x}$, proportional the applied size force multiplied by the particle displacements). Such a linearization is valid when the energy scale associated with applied shear is far smaller than the thermal energy so that the argument in the Boltzmann weight is small. We then have, at high temperatures,

$$(r_{\text{forward}} - r_{\text{backward}}) \sim \frac{k_B T}{h} (\Delta E_{\lambda_x} - \Delta E_{\lambda_\perp}).$$ (27)

If we keep the approach of [28] general without assuming “holes” nor special ratios of lengths, we find Eq. (26) with $A_a$ a dimensionless geometrical ratio associated with typical deformations of the liquid under shear. While different fluids $a$ may have individual favored shapes associated with deformations, one may anticipate that the angular “spherical average” related with the ensemble of all liquids may correspond to effectively symmetric length scales. In attempting to make contact between the Boltzmann equation derivation that we outlined in this section with Eyring’s older approximate treatment, we remark that employing a constant $\tau$ independent of local geometry (velocity direction) in our calculations might emulate equal isotropic geometrical scales for “hole” motion as in Eyring’s treatment [28].

As we underscored earlier, in any single liquid, the rate and equilibration times depend on velocity directions which may lead to the appearance of several saddle point energies $E^n_a$ (or a smooth distribution of such escape energies along different velocity directions). In fitting a process having many saddles by a single uniform constant saddle point energy, random errors
are made. All calculations are for monatomic uniform fluid systems for which all directions are isotropic. Locally, barriers along some velocity directions are different than other due to local structure (thus, for finite temperatures, the rate and time \( \tau \) are not velocity independent and \( \tau \) cannot be pulled out in front of the integral in Eq. (24). Different ion types in a liquid may also experience different effective barriers. Averaging the escape barrier \( E_{\text{escape}} \) over different ion types and velocity directions emulates an isotropic system. For a large ensemble we might expect the distribution of \( E_{\text{escape}} \) to approach a Gaussian distribution about a uniform average. We focused on a single particle description within a time independent potential \( V(\mathbf{x}) \). In the system, all of the particles move and consequently the potential as well as the final state location (and the energy \( E_{\text{escape}} \) required to reach it) may change. As the other particles may move in a way that can either increase or decrease the path length and time to reach the final state, we may assume that, on average, the single particle result of Eq. (25) will still hold.

VI. THE CURRENT CORRELATION FUNCTION AND THE VISCOSITY

Thus far, we highlighted the role of discrete quantum processes and their associated rates (or time scales) in governing dynamics. In this section, we will depart from this microscopic point of view and first analyze the viscosity carefully via continuous current correlations functions in which discrete dynamics are not enforced from the outset. We will then turn to examine what occurs for finite relaxation times. The form of the resulting expressions for the viscosity are similar to those of Eq. (24, 25, 26) yet due to the underlying premise are not identical to these.

While response theory calculations of the viscosity in relativistic system must necessarily be calculated via a stress tensor correlation function, the current correlation function can be used in the case of non-relativistic systems, and we have the Kubo formulas [5],

\[
\eta = m^2 \lim_{\omega \to 0} \lim_{q \to 0} \frac{\omega}{q^2} \text{Im} C_{ij}(\omega, \mathbf{q}),
\]

(28)

where \( C_{ij}(\omega, \mathbf{q}) \) is the transverse component of the retarded current correlation function,

\[
C_{ij}(t, \mathbf{r}, t'; \mathbf{r}') = -i \theta(t - t') \langle j_i(t, \mathbf{r}) j_j(t', \mathbf{r}') \rangle.
\]

(29)

A related formula allows to also calculate the bulk viscosity, \( \zeta \),

\[
\frac{4}{D} \eta + \zeta = m^2 \lim_{\omega \to 0} \lim_{q \to 0} \frac{\omega}{q^2} \text{Im} C_{ij}(\omega, \mathbf{q}),
\]

(30)

where \( D \) is the dimension and \( C_{ij}(\omega, \mathbf{q}) \) is the longitudinal component of the current correlation function.

A. The ideal semiclassical case

Let us verify explicitly that \( \eta = 0 \) in the case of ideal semiclassical gases. In this case the current correlation function is given in imaginary time simply by,

\[
C_{ij}(i\omega, \mathbf{q}) = \frac{\hbar g}{4m^2 k_B T} \sum_n \int \frac{d^D k}{(2\pi)^D} (2k_i + q_i)(2k_j + q_j) G_0(i\Omega_n, \mathbf{k}) G_0(i\omega - i\Omega_n, \mathbf{k} + \mathbf{q}),
\]

(31)

where \( g \) is the degeneracy, and

\[
G_0(i\omega, \mathbf{k}) = \frac{\hbar}{i\hbar \omega + \mu - \frac{\hbar^2 \mathbf{k}^2}{2m}},
\]

(32)

and the Matsubara sum is either fermionic or bosonic. After evaluating the Matsubara sum and assuming a classical regime such that the equilibrium distribution function is given by the Maxwell-Boltzmann one, we obtain,

\[
C_{ij}(i\omega, \mathbf{q}) = \frac{\hbar n(\mathcal{L}_T^w)^D}{4m^2} \int \frac{d^D k}{(2\pi)^D} \exp \left[ -\beta \left( \frac{\hbar^2 k^2}{2m} - \mu \right) \right] \frac{1}{i\omega - \frac{\hbar^2}{2m}(q^2 + 2\mathbf{k} \cdot \mathbf{q})} \delta((\hbar/m)\mathbf{k} \cdot \mathbf{q} - F_s(\omega, q)),
\]

(33)

where

\[
F_s(\omega, q) = \omega \pm \frac{\hbar q^2}{2m}.
\]

(35)

The integral can now be evaluated exactly and we obtain in the “classical” regime (i.e., \( \hbar \to 0 \)), for instance,

\[
\text{Im} C_{ij}(\omega, \mathbf{q}) = \frac{n \rho^w}{2m} \sqrt{\frac{\pi \beta}{2q^2}} \frac{(\omega)}{q} \exp \left[ -\frac{m \beta}{2q^2} \omega^2 + \frac{\hbar q^2}{2m} \right],
\]

(36)
where \( n \) is the density, and we see that \( \eta \) vanish. As we have already mentioned, this differs from the relativistic result where \( \eta \to \infty \) in the non-interacting case. Note that \( \eta = 0 \) immediately violates the inequality \( [1] \). At this point we recall that Eq. \([1]\) was derived assuming a strongly interacting system. Moreover, it is frequently argued that viscosity in non-interacting systems is an ill-defined concept. However, this argument has been frequently given in the framework of relativistic theories.

\[ \Delta_\eta(\omega, q) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i t h/m |q|^2} \left| B_j(\omega |q|) - \frac{\epsilon}{\omega} \right|^2, \quad (37) \]

B. The case with a finite relaxation time

The easiest way to account for a finite relaxation time, \( \tau \), arising due to collisions between the particles is to replace the

\[ C_i(\omega, q) = \frac{\sqrt{2\pi n} e^{\omega q/\hbar}}{m^3 h} \sinh \left[ \frac{\hbar \omega q^2}{2m (1 + q^2/m)} \right] \exp \left\{ -\frac{\tau^2}{2 (1 + q^2/m)} \right\} \left\{ \omega^2 + \frac{q^2}{2m} \right\}. \quad (38) \]

Note that when \( \tau \to \infty \) and \( \hbar \to 0 \) we recover Eq. \([36]\), as it should.

It is easy to see that the \( q \to 0 \) limit in Eq. \([28]\) does not vanish any longer for the case of Eq. \([38]\). Thus, let us consider the frequency-dependent viscosity,

\[ \eta(\omega) = m^2 \lim_{q \to 0} \frac{\omega}{q^2} \text{Im} C_i(\omega, q) = \sqrt{\frac{\pi}{2}} k_B T \eta \tau^3 \exp \left( \frac{\mu}{k_B T} - \frac{\omega^2 \tau^2}{2} \right). \quad (39) \]

If \( \tau \) is \( \omega \)-independent, we still have a vanishing viscosity, just like in the free case. From Eq. \([39]\) we see that if the relaxation time scales like \( \tau \sim \omega^{-2/3} \), a finite viscosity is obtained. We will see below, that \( \tau \sim \omega^{-2/3} \) is also possible beyond the simple toy model above.

Next, we discuss a technically more correct way to derive the viscosity in non-relativistic systems, akin to the quantum theory of screening featuring a relaxation time \([31]\). In this case, we obtain,

\[ C_i(\omega, q) = -\frac{\hbar g}{4m^2} \int \frac{d^d k}{(2\pi)^d} f(k) \left[ \frac{(2k_i + q_i)(2k_j + q_j)}{\omega - \frac{\hbar}{2m}(q^2 + 2k \cdot q) + i \tau^{-1}} - \frac{(2k_i - q_i)(2k_j - q_j)}{\omega + \frac{\hbar}{2m}(q^2 - 2k \cdot q) + i \tau^{-1}} \right]. \quad (40) \]

where \( m^* \) is the effective mass and \( f(k) \) is the Fermi-Dirac distribution function. Note that indeed current conservation can be used to relate the longitudinal component of \( C_i \) to the electric susceptibility, yielding in this way the standard dielectric function \([30]\).

\[ \varepsilon(\omega, q) = 1 + \frac{4\pi e^2}{\hbar q^2} \int \frac{d^d k}{(2\pi)^d} f(k) \left[ \frac{1}{\omega - \frac{\hbar}{2m}(q^2 + 2k \cdot q) + i \tau^{-1}} - \frac{1}{\omega + \frac{\hbar}{2m}(q^2 - 2k \cdot q) + i \tau^{-1}} \right]. \quad (41) \]

The transverse component of Eq. \([40]\) is obtained by contracting it with \((D - 1)^{-1}(\delta_{ij} - k_i k_j/k^2)\), which yields,

\[ C_\perp(\omega, q) = -\frac{\hbar g}{(D - 1)m^2} \int \frac{d^d k}{(2\pi)^d} f(k) \left[ k_i^2 - \frac{(k \cdot q)^2}{q^2} \right] \left( \frac{\hbar/m]{q^2}}{(\omega - \frac{\hbar}{m}(q \cdot k + i \tau^{-1}))^2 - \left( \frac{\hbar}{2m} \right)^2} \right). \quad (42) \]

Since \( \eta(\omega) = m^2 \lim_{q \to 0} \omega q^2 \text{Im} C_i(\omega, q) \), it is enough to expand the equation above up to order \( q^2 \) and take the imaginary part of the result. Thus, we obtain in this way the “optical” viscosity,

\[ \eta(\omega) = \frac{4\varepsilon K}{D} \frac{\omega^2 \tau^3}{(1 + \omega^2 \tau^2)^2}, \quad (43) \]

where \( \varepsilon K \) is the average effective kinetic energy density. If we use the Maxwell-Boltzmann distribution as the high-
temperature limit of the Fermi-Dirac distribution, we have
\[ \epsilon_K = Dn e^{\beta \mu} k_B T / 2, \]
Eq. (43) becomes,
\[ \eta(\omega) = 4k_B T n e^{\beta \mu} \frac{\omega^2 T^3}{(1 + \omega^2 T^2)^2}, \] (44)
where we have assumed \( g = 2 \), such that \( n = N/V \) is the total particle density without the spin degeneracy. Thus, the maximum, as a function of \( \omega \), of Eq. (44), occurs when \( \omega = 1/\tau \). When this is inserted, we have
\[ \eta_{\text{max}} = k_B T n \pi e^{\beta \mu}, \] (45)
obtaining in this way once more a viscosity with a single exponent reminiscent of Eyring’s formula of Eq. (26), with the chemical potential playing the role of the activation energy in this case. If we insert Eq. (20), at high temperatures, if \( \mu \) is bounded, \( \eta_{\text{max}} \rightarrow n h \).

The formula (43) can also be applied to a Fermi liquid in three dimensions at low temperatures, where typically \( \tau \approx \tilde{A} (\hbar / e_F) [k_B T / (\hbar \omega)]^2 \), with \( \tilde{A} \) being a numeric coefficient, in which case we obtain for the viscosity,
\[ \eta = \frac{4 h}{5 \tilde{A}} \left( \frac{e_F}{k_B T} \right)^2, \] (46)
where we have specialized to a spin degeneracy \( g = 2 \). The above formula corresponds to the well-known \( 1/T^2 \) behavior for the viscosity of a Fermi liquid [10].

We note once more that if the system is such that \( \tau \approx \omega^{-2/3} \), a finite \( \omega \rightarrow 0 \) limit of \( \eta(\omega) \) is also obtained. Such a behaviour is characteristic of an Ising-Nematic system [31]. However, the propagator we used to derive \( \eta(\omega) \) is not typical of Ising-Nematic systems.

It is interesting to note that the optical conductivity behaves in this case in a way very similar to the optical viscosity. This is more easily seen by recalling the formula relating the dielectric function to the complex conductivity, \( e(\omega, 0) = 1 + 4\pi i \sigma(\omega) / \omega \). Thus, we have,
\[ \frac{\eta(\omega)}{\Re \sigma(\omega)} = \frac{4 \epsilon_K \tau}{D \tau_0}, \] (47)
where \( \tau_0 \) is the conductivity as given by the Drude formula. Using the high-temperature arguments before maximizing the viscosity and accounting for Eq. (20), we obtain a resistance linear in the temperature. We will come back to this linear in \( T \) behavior for the resistivity at high-temperature later. Note that in the Fermi liquid regime nothing surprising happens within this approach and a resistivity \( \sim T^2 \) is obtained.

VII. NORMAL DISTRIBUTION OF THE EXTRAPOLATED INFINITE TEMPERATURE VISCOSITY

For the reasons that we outlined when computing the viscosity via the Boltzmann equation considerations in quantum kinetic theory in section [4] the exact viscosity in each liquid will include higher order correlation effects that might be partially emulated by spatially non-uniform barriers \( \Delta E \). These augment the existence of multiple dominant saddle point paths anticipated for a multi-ion liquid in which disparate interatomic potentials (and ensuing dominant barriers) may exist. Clearly, in replacing a sum of exponentials (associated with multiple ions, non symmetric configurations) by a single exponential in the form of Eq. (26), information is lost. Over an ensemble of liquids, entropy in the fitting forms is expected to be maximized (i.e., the liquids have no particular additional correlations that they adhere to) subject to the global condition that in all of these liquids “on average” the high temperature limiting viscosity must scale as Eq. (25) in the exact form will approximately do so in the single exponent form of Eq. (26). In the single exponent estimate of Eq. (26), the limiting high temperature viscosity is given by \( \lim_{T \rightarrow \infty} \eta_{\text{single-exponent}} = A n h \). Following the spirit of Jaynes [29], given a large ensemble liquids \( \{a\} \), we can examine the probability distribution \( \text{Prob}(A) \) for having a particular prefactor in Eq. (26). If we write \( \text{Prob}(A) \propto \exp[-w Q(A)] \) and require that the average value \( \int dA \text{Prob}(A) A = 1 \) (that may be mandated by a choice of the Lagrange multiplier \( w \) as it is in canonical ensembles for the average energy instead of an average value of \( A \)) and that \( \text{Prob}(A) \) has its maximum at \( A = 1 \) (requiring that \( Q \) have a quadratic minimum at \( A = 1 \), we find a Gaussian form for the probability distribution of \( A \) values. Stated alternatively, the exact relation \( A_g = 1 \) in the physical system for all liquids will become inexact when we use the approximate form of Eq. (26). This unbiased approximate form will satisfy this relation only on average as information is lost when we use the less detailed approximate form of Eq. (26). In general data sets, one may work backwards from the distribution of measured values such as those of \( A_a \) to find a corresponding “effective energy” \( Q \).

In Fig. 2 we compare these general expectations with the data reported in [13]. Figure 2 gives the probability of the deviation from the theoretical value of \( n \). The \( s = (A - 1) \) values are contrasted with those anticipated from a Gaussian distribution. The comparison suggests that the measured values of \( A \) are indeed normally distributed around the theoretical value of unity. Over the ensemble of measured 23 liquids, the mean value, when averaged over all liquids \( (1 \leq a \leq 23) \), of \( A_g - 1 = 0.00647 \), and the width of the distribution \( \sigma = 0.453 \). For the average value \( A = 1 \sum_{a=1}^{23} A_a \) the corresponding standard deviation about the mean value is \( \sigma_{A} = \sigma / \sqrt{22} \sim 0.09 \).

VIII. THE NAVIER STOKES EQUATION AND POSSIBLE MINIMAL TIME SCALE FOR DISCRETE DYNAMICS

Liquid dynamics adheres to the Navier Stokes equation,
\[ \rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla P + \nabla \cdot T + \mathbf{f}. \] (48)
Here, \( \rho \) the mass density, \( \mathbf{v} \) the velocity, \( P \) the pressure, \( \mathbf{f} \) external body forces (per unit volume) and \( T \) stress tensor. In the canonical setting, \( \nabla \cdot T = \eta \nabla \cdot \mathbf{v} \). Albeit its simplicity, the regularity of general solutions in \( D = 3 \) dimensions is
not known and currently constitutes an open problem. Given the results that we presented in the current work, \( \eta \) might be bounded form below by its limiting high temperature value in Eq. (25). This implies that the Reynolds number,

\[
Re = \frac{\rho v \ell}{\eta},
\]

with \( \ell \) the typical mean free path and \( v \) the average speed, may saturate to \( \rho/(nh) \times (vL) = p\ell/h \) (with \( p = mv \) the linear momentum scale) at high temperatures. In this limit, \( Re \sim L/\lambda \) with \( \lambda = h/p \) the de-Broglie wavelength. For particles to be well defined, the scale \( \ell \) cannot be significantly smaller than the de-Broglie wavelength.

While the Navier Stokes equations are, of course, those of a continuous velocity field, the results that we discussed highlight that transitions and motion on a microscopic scale are not continuous but rather set by minimal time increments of order \( \tau \sim h/(k_B T) \) for basic processes to occur. Taken literally, this smallest time step set by the reciprocals of Eqs. (16,20) suggests that in fluids motion cannot be considered as continuous and an effective time lag may be microscopically present between molecular transitions/motions. Such a requisite time separation is reminiscent to that introduced in recent analysis of the Navier-Stokes equation [32] in studying (ir)regularity properties of viable solutions.

**IX. ELECTRICAL CONDUCTIVITY**

The approach that we invoked in discussing viscosity may, in principle, be replicated for other transport measurements.

To conceptually illustrate the basic premise (with no pretenses of rigor), we consider a dirty insulator with charges (e) of volume density \( n \) in the presence of an applied electric field \( E \).

If the mean free path of between the ions is \( \ell = n^{-1/3} \) in \( D \) dimensions then the difference between the forward and backwards flow along and opposite the applied field will be

\[
\Delta r(T) = r_{\text{forward}}(T) - r_{\text{backwards}}(T) = r(T)\beta e|E|/\ell \quad (50)
\]

where \( r(T) \) is the rate of Eq. (16) in the absence of an applied field, and we Taylor expand the exponential of Eq. (16) for both forward and backward motion \( (f_{\text{forward}}, f_{\text{backwards}}) \) to obtain the factor of \( \beta e|E|/\ell \). The lowering/increase in the energy barrier for backwards and forward motion is linear in the distance \( \ell \) between charges on adjacent ions and the applied field. Pictorially, the applied field will tilt the potential barrier in Fig. 1. In the high temperature limit, we may invoke Eq. (20), and Eq. (50) becomes \( \Delta r(T) \sim e|E|/h \). The charge velocity between ions is \( v = \beta\sigma \) and the total current \( j = nev \) becomes, at high temperatures, \( j \sim ne^2\ell^2|E|/h \) and the conductivity is thus

\[
\sigma \sim n^{-1/3}(e^2/h) \equiv \sigma_{\text{sr}}. \quad (51)
\]

Thus, at high temperatures, within this simple approach, in two dimensions, the resistance will decrease with increasing temperature and tend, in the high temperature limit, to \( h/e^2 \) a value akin to the quantum of resistance.

We next consider a classical metal with a mean free path \( \ell \) that is a function of temperature and is generally larger than the inter particle separation \( n^{-1/3} \). We now examine what occurs as temperature is increased within our framework. Towards this end, we invoke the Drude formula, \( \sigma = ne^2\tau/m \) with \( m \) the particle mass and \( \tau = 1/r(T) \) of Eq. (16). In this case, at asymptotically high temperatures, the resistivity is linear in temperature and at finite temperatures, this is multiplied by a factor of \( \exp(\beta E) \); this exponential factor tends to unity in the high temperature limit,

\[
\rho = \left(\frac{mk_B T}{ne^2h}\right)e^{-\beta E}. \quad (52)
\]

At temperatures \( T \gg B/k_B \), the resistivity may increase linearly with temperature until the mean free path saturates to the inter particle distance \( n^{-1/3} \). When this saturation occurs, \( \sigma \sim \sigma_{\text{sr}} \) as in the insulating case above.

Thus, our simple considerations above suggest that the conductivities of both insulators and metals may veer to similar limiting high temperature behavior, \( \sigma \sim \sigma_{\text{sr}} \). For insulators, the conductivity tends to this value from below (with the conductivity scaling as \( r(T) \) of Eq. (16)) whereas for metals, the conductivity approaches this value from above (scaling as \( 1/r(T) \)). Interestingly, in \( D = 2 \) dimensional systems similar behavior is found as in Fig. 3. In “bad metals” [34], a linear \( T \) increase of the resistivity persists to high temperatures. An alternate and more rigorous derivation of a similar result is afforded by Eq. (47).
C 8 FET for different of a two dimensional electronic system in a disordered MOS-FET. Reproduced from [33]. The resistivity i.e., the Lorentz number $L$ between the thermal conductivity and the Drude conductivity, $V$ generally, classical $D$ current generated by a thermal gradient, $\vec{\rho}$ being the particle density and $\hbar$ Planck’s constant for single component fermions. This value turns out to be close to the minimal value of the extrapolated sample dependent viscosity measured in the experiments.

(2) Illustrated that the transition state theory type result of Eq. (16) may be derived for general Hamiltonians by invoking the WKB results and deformation of the potentials. In particular, we found that in the high temperature limit, irrespective of the complexity of the possible system trajectories, the relaxation time $\tau$ universally scales as Eq. (20). This latter equation is precise and involves no unknown prefactors of order unity.

(3) Showed that a current correlation function calculation for the viscosity as a function of frequency for non-interacting non-relativistic systems in the high temperature limit arrives at the maximum of viscosity at the frequency scale $\omega = 1/\tau$ being $\eta = \frac{1}{2}\eta_0$ with $n$ being the particle density and $\hbar$ Planck’s constant for single component fermions. This value turns out to be close to the minimal value of the extrapolated sample dependent viscosity measured in the experiments.

(4) By the use of the Boltzmann equation for the non-equilibrium problem, we found that the viscosity is a sum of exponentials with the limiting high temperature value set by $\eta = 7.12\times 10^{-6}$ cm$^2$. At finite temperatures, this is multiplied by an Arrhenius type factor $\exp(\beta E)$.

(5) Higher order multi-particle correlations, as well as the existence of several non-unique pathways for particle motion may modify the Boltzmann equation result is in an effective scatter of the prefactor in the fit of Eq. (26). We found that when fitting the experimentally measured viscosity of 23 metallic glass formers, the ensemble average of the extrapolated high temperature viscosity is exceedingly close to that suggested by the Boltzmann equation (with an error of 0.6% and standard deviation of order 9%).

(6) We suggested how to extend how own analysis as well as the more rudimentary ideas of transition state theory to the calculation of other transport properties including conductivities. Using very simple considerations we almost immediately found that possibility of a resistivity saturation values set by the quantum of resistance $e^2/h$ with $e$ the electronic charge. We furthermore found that linear in temperature resistance (such as that found in “bad metals”) is quite natural within this framework.

(7) We noted that our results suggest bounds on the extrapolated values of the Reynolds numbers at high temperatures and that there is a minimal time scale for discrete molecular dynamics that may amend the continuous fluid dynamics de-

X. THERMAL CONDUCTIVITY

The thermal conductivity $\kappa$ is set by the size of the heat current generated by a thermal gradient, $\vec{f}_Q = -\kappa \vec{\nabla} T$. In a classical $D = 3$ dimensional metal with free particles, the ratio between the thermal conductivity and the Drude conductivity, i.e., the Lorentz number $L = \kappa/(\sigma T) = (3/2) \times (k_B/e)^2$. More generally, $V\kappa = \langle \vec{p}^2 \rangle \tau C_v/D$ with $V$ the system volume and $C_v$ the specific heat. Replicating earlier steps, i.e., inserting Eqs. (16,20) for $\tau$ and employing the equipartition theorem, we trivially have that prior to the saturation of the mean free path (i.e., when the mean free path is larger than inter particle distance),

$$\kappa \sim \frac{h C_v v e^2}{m V \beta E}. \quad (53)$$

XI. CONCLUSIONS AND OUTLOOK.

While certain prevailing wisdom suggests a dichotomy of low temperature quantum and “classical” high temperature effects, our analysis illustrates that the two may be more “entangled” than is widely appreciated and that quantum phenomena may emerge rather sharply in the high temperature limit. We showed how this may occur in computing the viscosity and sketched how similar results may appear for other transport response. Our calculations and suggesting principles lead to results that are summarized below. Principally, we

(1) Established that at high temperatures, WKB type quantization leads to the exact well known substitution of the form of Eq. (12). This celebrated correspondence has earlier been motivated by heuristic considerations for which we now suggest an underlying rigorous basis.

(2) Illustrated that the transition state theory type result of Eq. (16) may be derived for general Hamiltonians by invoking the WKB results and deformation of the potentials. In particular, we found that in the high temperature limit, irrespective of the complexity of the possible system trajectories, the relaxation time $\tau$ universally scales as Eq. (20). This latter equation is precise and involves no unknown prefactors of order unity.

(3) Showed that a current correlation function calculation for the viscosity as a function of frequency for non-interacting non-relativistic systems in the high temperature limit arrives at the maximum of viscosity at the frequency scale $\omega = 1/\tau$ being $\eta = \frac{1}{2}\eta_0$ with $n$ being the particle density and $\hbar$ Planck’s constant for single component fermions. This value turns out to be close to the minimal value of the extrapolated sample dependent viscosity measured in the experiments.

(4) By the use of the Boltzmann equation for the non-equilibrium problem, we found that the viscosity is a sum of exponentials with the limiting high temperature value set by $\eta = 7.12\times 10^{-6}$ cm$^2$. At finite temperatures, this is multiplied by an Arrhenius type factor $\exp(\beta E)$.

(5) Higher order multi-particle correlations, as well as the existence of several non-unique pathways for particle motion may modify the Boltzmann equation result is in an effective scatter of the prefactor in the fit of Eq. (26). We found that when fitting the experimentally measured viscosity of 23 metallic glass formers, the ensemble average of the extrapolated high temperature viscosity is exceedingly close to that suggested by the Boltzmann equation (with an error of 0.6% and standard deviation of order 9%).

(6) We suggested how to extend how own analysis as well as the more rudimentary ideas of transition state theory to the calculation of other transport properties including conductivities. Using very simple considerations we almost immediately found that possibility of a resistivity saturation values set by the quantum of resistance $e^2/h$ with $e$ the electronic charge. We furthermore found that linear in temperature resistance (such as that found in “bad metals”) is quite natural within this framework.

(7) We noted that our results suggest bounds on the extrapolated values of the Reynolds numbers at high temperatures and that there is a minimal time scale for discrete molecular dynamics that may amend the continuous fluid dynamics de-
scribed by the Navier-Stokes equation.

We caution that while our calculations are suggestive and lie in close proximity to the measured experimental values of the viscosity, the “average value” arguments regarding corrections to the universal leading order Boltzmann equation result for the viscosity (Eq. (25)) that we have repeatedly alluded to in this work have not been established with rigor. We have not shown that corrections to the leading order result of Eq. (25) are not only experimentally but also theoretically distributed in a Gaussian type fashion. It is, of course, possible that additional effects in the analysis of the experimental data lead to a Gaussian type distribution about the computed theoretical values. The same may hold true for other transport functions such as the conductivities that we briefly discussed. On the experimental and experimental data analysis side, it would be interesting to further very carefully measure and analyze the viscosity of additional fluids, other than the 23 metallic fluids examined here and in [13], to see the distribution of their high temperature values to see if our results to do not change.

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Appendix A: Experimental details

In section VII (and in Fig. 2, in particular), we compared theoretical results to those recently reported for numerous metallic glass forming liquids [13]. In what follows, for self-completeness, we review important aspects of these experiments as they pertain to our analysis of the viscosity of high temperature fluids. In [13], samples were levitated and melted in the high-vacuum containerless environment of the Washington University Beamline ElectroStatic Levitation Facility (WU-BESL), see Fig. 4 [35].

The viscosity was subsequently measured via the oscillating drop method [36]. Specifically, for the $l$ th multipole mode, the viscosity

$$\eta = \frac{\rho R_0^2}{(l-1)(2l+1)P} \quad (A1)$$

where $P$ attenuation time, $R_0$ unperturbed radius of the sample, and $\rho$ the mass density. The droplet size oscillated as

$$R = R_0 \left(1 + \delta (\cos \Omega t) e^{-t/P} \right).$$
where $\delta$ is a constant, $\sigma$ is the surface tension, $Q$ is the charge, and $\epsilon$ the dielectric constant. The voltage on the vertical electrode was modulated at a frequency that was close to the $l = 2$ spherical harmonic mode resonant frequency (typically 120-140 Hz) of the liquid to induce surface vibrations. A high-speed camera (1560 frames per second) recorded the shadow of the oscillating sample, as depicted in Figs. 5[6]. After a sufficiently long time during which spurious transient oscillations faded, the perturbative voltage was removed and the time-dependent amplitude of the decaying surface harmonic oscillations was probed. The viscosity was determined from the decay time for the oscillation, $P$ via Eqs. (A1, A2).

The measured viscosity data (including that reported by works other than Ref. 13) of numerous metallic fluids collapses onto a single master curve, see Fig. 7 with $\eta_0$, the extrapolated high temperature viscosity. This value of $\eta_0$ was compared with theory in Fig. 2. The value of $\eta_0$ as well as the temperature $T_{coop}$ above which Arrhenius type dynamics is manifest is severely constrained by the data, see Fig. 8. Thus, the displayed points in Fig. 2 have small error bars if good fit of the data is to be achieved.

FIG. 6. (Color online.) A later snapshot oscillation of the shadow of the $l = 2$ mode oscillations.
FIG. 7. (Color online.) The universal collapse of the viscosity of metallic glass forming fluids. The collapse is obtained by scaling the viscosity by a system dependent constant $\eta_0$ (which close to $n\hbar$) and by scaling the temperature by $T_{coop}$ a temperature above which, as seen from the straight line at high temperatures, the viscosity is well described by a single exponential form of Eq. (26).

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FIG. 8. (Color online.) Fit parameters and associated standard deviations in fitting the experimental data of a metallic fluid (Vit1) [13]. A standard deviation of 1 sigma separates consecutive shown ovals. The data severely constrain the extrapolated high temperature viscosity $\eta_0$ and the scaling temperature $T_{\text{coop}}$ above which the dynamics are well described by Eq. (26).