Weak order in averaging principle for two-time-scale stochastic partial differential equations

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Abstract

This work is devoted to averaging principle of a two-time-scale stochastic partial differential equation on a bounded interval $[0, l]$, where both the fast and slow components are directly perturbed by additive noises. Under some regular conditions on drift coefficients, it is proved that the rate of weak convergence for the slow variable to the averaged dynamics is of order $1 - \varepsilon$ for arbitrarily small $\varepsilon > 0$. The proof is based on an asymptotic expansion of solutions to Kolmogorov equations associated with the multiple-time-scale system.

Keywords: Stochastic partial differential equation; Averaging principle; Invariant measure; Weak convergence; Asymptotic expansion.

MSC: primary 60H15, secondary 70K70

1. Introduction

In a previous paper [4], Bréhier exhibited the strong and weak order of an averaging principle for the following class of slow-fast stochastic reaction-diffusion equation on a bounded interval $D = [0, l]$ of $\mathbb{R}$:

$$
\begin{cases}
\frac{\partial}{\partial t} x_t^\varepsilon(\xi) = \Delta x_t^\varepsilon(\xi) + F(x_t^\varepsilon(\xi), y_t^\varepsilon(\xi)), \xi \in D, t > 0, \\
\frac{\partial}{\partial t} y_t^\varepsilon(\xi) = \frac{1}{\varepsilon} \Delta y_t^\varepsilon(\xi) + \frac{1}{\varepsilon} G(x_t^\varepsilon(\xi), y_t^\varepsilon(\xi)) + \frac{1}{\sqrt{\varepsilon}} W_t(\xi), \xi \in D, t > 0, \\
x_0^\varepsilon(\xi) = x(\xi), y_0^\varepsilon(\xi) = y(\xi), \xi \in D, \\
x_t^\varepsilon(0) = x_t^\varepsilon(l) = 0, t \geq 0, \\
y_t^\varepsilon(0) = y_t^\varepsilon(l) = 0, t \geq 0,
\end{cases}
$$

(1.1)

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where the leading linear operator \( \Delta = \frac{\partial^2}{\partial x^2} \) is the Laplacian operator. The ratio of time-scale separation is described by the positive and small parameter \( \varepsilon \).

With this time scale, the process \( x_\varepsilon^t(\xi) \) is always called as the slow component and \( y_\varepsilon^t(\xi) \) as the fast component. The drift coefficients \( F \) and \( G \) are suitable mappings from \( L^2(D) \) to itself. The stochastic perturbation \( W_\varepsilon \) is an \( L^2(D) \)-valued Wiener process with respect to a filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \).

In many applications, it is of interest to describe dynamics of the slow variable. Since exact solution is hard to be known, a simple equation without fast component, which can capture the essential dynamics of slow variable, is highly desirable. The fundamental method to approximate slow solution (1.1) is the so-called averaging procedure. Under some conditions, it has been proven that the slow solution (1.1) is the so-called averaging procedure. Under some conditions, it has been proved that the slow solution \( x_\varepsilon^t(\xi) \) to equation (1.1) is the so-called averaging procedure. Under some conditions, it has been proven that the slow solution \( x_\varepsilon^t(\xi) \) to problem (1.1) converges (as \( \varepsilon \) tends to 0), in a suitable sense, to solution of the so-called averaged equation, obtained by eliminating fast component via taking the average of the coefficient \( F \) over the slow equation.

Once one has proved the validity of averaging principle, a critical question arises as to how do we determine the rate of convergence for this procedure. In Brêhier [4], it has been proved that the strong convergence (approximation in pathwise) order is \( 1 - \varepsilon \) while the weak convergence (approximation in law) order is \( 1 - \varepsilon \), for arbitrarily small \( \varepsilon > 0 \), on condition that the slow motion equation is a deterministic parabolic equation. Due to the arbitrariness of \( \varepsilon \) we may say that strong (resp. weak) convergence order is \( \frac{1}{\varepsilon} \) (resp. \( 1^{-} \)). If an additive noise is included in the slow motion equation, the strong convergence order will decrease to \( \frac{1}{2} \). In this case, unfortunately, the methods used to prove the weak order will be invalid. The main difficulty is due to the fact that tactics depend on the time derivative of solution to the averaged equation, which does not exist in any general way with the case where the slow motion equation is perturbed with a noise. In a more recent work following the procedure inspired by Brêhier [4], Dong et al. [8] establish weak order \( 1^{-} \) in stochastic averaging for one dimensional Burgers equation only in the particular case of an additive noise on the fast component.

Unlike in the above-mentioned papers, where the noise acts only in the fast motion, in the present paper we study a class of stochastic partial differential equations on the bounded interval \( D = [0, l] \) of \( \mathbb{R} \), involving two separated time scales, which can be written as:

\[
\begin{align*}
\frac{\partial}{\partial t} x_\varepsilon^t(\xi) &= \Delta x_\varepsilon^t(\xi) + F(x_\varepsilon^t(\xi), y_\varepsilon^t(\xi)) + \sigma_1 \dot{W}_1^t(\xi), \quad \xi \in D, \quad t > 0, \\
\frac{\partial}{\partial t} y_\varepsilon^t(\xi) &= \frac{1}{\varepsilon} \Delta y_\varepsilon^t(\xi) + \frac{1}{\varepsilon} G(x_\varepsilon^t(\xi), y_\varepsilon^t(\xi)) + \sigma_2 \dot{W}_2^t(\xi), \quad \xi \in D, \quad t > 0, \\
x_\varepsilon^0(\xi) &= x(\xi), \quad y_\varepsilon^0(\xi) = y(\xi), \quad \xi \in D, \\
x_\varepsilon^t(0) &= x_\varepsilon^t(l) = 0, \quad t \geq 0, \\
y_\varepsilon^t(0) &= y_\varepsilon^t(l) = 0, \quad t \geq 0.
\end{align*}
\]

Assumptions on regularity of the drift coefficients \( F \) and \( G \) will be given below. The noises \( \dot{W}_1^t(\xi) \) and \( \dot{W}_2^t(\xi) \) are independent Wiener processes which will be detailed in next section. The coefficients of noise strength \( \sigma_1 \) and \( \sigma_2 \) are positive.
constants. The coupled stochastic partial differential equation in form of (1.2) arises from many physical systems when random spatio-temporal forcing is taken into account, such as diffusive phenomena in media, epidemic propagation and transport process of chemical species.

So far, the explicit order for weak convergence in averaging has not be extended to the general situation when both the fast and slow components are directly perturbed with some noises. In the current article, we will show that the weak order $1^{-}$ can be achieved even when there is a noise in the slow motion equation. More precisely, we prove that for any $T > 0$ and a class of test functions $\phi : L^2(D) \to \mathbb{R}$, with continuous and bounded derivatives up to the third order,

$$
|\mathbb{E}\phi(x^T) - \mathbb{E}\phi(\bar{X}_T)| \leq C\epsilon^{1-r}
$$

for any $r \in (0, 1)$, where $C$ is a constant independent of $\epsilon$ (see Theorem 3.1). In the estimate above, the averaged motion $\bar{X}_t$ solves the equation

$$
\begin{align*}
\frac{d}{dt} \bar{X}_t(\xi) &= \Delta \bar{X}_t(\xi) + \bar{F}(\bar{X}_t(\xi)) + \sigma_1 W_1^t(\xi), \quad \xi \in D, \quad t > 0, \\
\bar{X}_0(\xi) &= x(\xi), \quad \xi \in D, \\
\bar{X}_t(0) &= \bar{X}_t(l) = 0, \quad t \geq 0,
\end{align*}
$$

with an averaged drift $\bar{F}(x) := \int_{L^2(D)} F(x, y) \mu^x(dy)$, where $\mu^x$ is the unique mixing invariant measure for fast variable with frozen slow component (see equation (3.1)).

In order to prove (1.3), we adopt an asymptotic expansion scheme as in [4] to decompose $\mathbb{E}\phi(x^T)$ with respect to the scale parameter $\epsilon$ in form

$$
\mathbb{E}\phi(x^T) = u_0 + \epsilon u_1 + r^\epsilon
$$

where the functions $u_0$, $u_1$ and $r^\epsilon$ are determined recursively and obey certain linear evolutionary equations. First of all, we identify leading term $u_0$ as $\mathbb{E}\phi(\bar{X}_t)$ by a uniqueness argument. To this purpose, we introduce the Kolmogorov operators with parameter to construct an evolutionary equation that describes both $u_0$ and $\mathbb{E}\phi(\bar{X}_t)$. Moreover, this also allows us to characterize the expansion coefficient $u_1$ by a Poisson equation associated with the generator of fast process so that we obtain an explicit expression of $u_1$. As a result, some a priori estimates guarantee the boundedness of function $u_1$.

The next key step consists in estimate for the remainder term described by a linear equation depending on $\mathcal{L}_2 u_1$ and $\frac{\partial u_1}{\partial t}$, where $\mathcal{L}_2$ is the Kolmogorov operator for slow motion equation with frozen fast component. Due to the presence of unbounded operator $\mathcal{L}_2$, we have to reduce the problem to its Galerkin finite dimensional version. Since the noise is included in the slow motion equation, the Itô formula is employed to derive an explicit expression for $\frac{\partial u_1}{\partial t}$, which is related to the third derivative of $\phi$. This is the reason why we have to require the test function to be 3-times differentiable. After bounding the terms $\mathcal{L}_2 u_1$ and $\frac{\partial u_1}{\partial t}$, the remainder $r^\epsilon$ in the expansions can be estimated by standard evolution
equation method and the weak error with an explicit order is achieved, where Itô’s formula is used again to overcome the non-integrability of $r^e$ at zero point. We would like to stress that, due to the regular conditions imposed on noise in slow component (see (2.4) and (2.5)), the solution process to slow equation enjoys values in the domain of dominating linear operator. This allows estimates using techniques similar to those in Bréhier [4].

Up to now, to our knowledge, this is the first to obtain the weak convergence order for averaging of stochastic partial differential equations in the case of a noise acting directly on the slow motion equation. It is certainly believable that our method can be applied to stochastic Burgers equation with regular noise such that weak order $1^{-}$ in averaging is obtained. This will extend work of Dong et al. [8], as we do not require the slow motion equation is deterministic.

Averaging method plays a prominent role in the study of qualitative behavior of dynamical systems with two time scales and has a long and rich history. Their rigorous mathematical justification was due to Bogoliubov [5] for the deterministic dynamical system. Further developments to ordinary differential equations of the averaging theory can be found in Volosov [23], Besjes [2] and Gikhman [13]. The averaging result for stochastic differential equations of Itô type was firstly introduced in Khasminskii [16], being an extension of the theory to stochastic case. Since then, much progress has been made for multiple-time-scale stochastic dynamical systems in finite dimensions, see for instance [3, 10, 14, 17, 18, 20, 21, 22, 26, 27, 29, 31] and the references therein. In particular, averaging for finite dimensional stochastic systems with non-Gaussian noise may be found in [32, 33, 34, 35]. In a series of recent papers, Cerrai and Freidlin [4] and Cerrai [6] studied an infinite dimensional version of averaging principle for partial differential equations of reaction-diffusion type with additive and multiplicative Wiener noise, respectively, where global Lipschitz conditions were imposed. In contrast to Lipschitz setting, averaging principle for parabolic equations with polynomial growth coefficients was explored in Cerrai [7]. For the extensions to stochastic parabolic equations with non-Gaussian stable noise, we are referred to Bao et al. [1]. For related works on averaging for infinite dimensional stochastic dynamical systems we refer the reader to [30, 11, 12, 23, 25, 15].

The rest of the paper is arranged as follows. Section 2 is devoted to the general notation and framework. The ergodicity of fast process and the main result are introduced in Section 3. Then some a priori estimates is presented in Section 4. In Section 5, we present an asymptotic expansion scheme. In the final section, we state and prove technical lemmas applied in the preceding section.

Throughout the paper, the letter $C$ below with or without subscripts will denote positive constants whose value may change in different occasions. We will write the dependence of constant on parameters explicitly if it is essential.

2. Notations and preliminary results

To rewrite the system (1.2) as an abstract evolutionary equation, we present some notations and recall some well-known facts for later use.
For a fixed domain $D = [0, l]$, let $H$ be the real, separable Hilbert space $L^2(D)$, endowed with the usual scalar product $\langle \cdot, \cdot \rangle_H$. The corresponding norm is denoted by $\| \cdot \|$. Denote by $\mathcal{L}(H)$ the Banach space of linear and bounded operators from $H$ to itself, equipped with usual operator norm.

Let $\{e_k(\xi)\}_{k \geq 1}$ denote the complete orthonormal system of eigenfunctions in $H$ such that, for $k = 1, 2, \ldots,$

$$-\Delta e_k(\xi) = \alpha_k e_k(\xi), \quad e_k(0) = e_k(l) = 0,$$

with $0 < \alpha_1 \leq \alpha_2 \leq \cdots \alpha_k \leq \cdots$. We would like to recall the fact that $e_k(\xi) = \sqrt{\frac{2}{l}} \sin \frac{k\pi x}{l}$ and $\alpha_k = -\frac{k^2 \pi^2}{l^2}$ for $k = 1, 2, \ldots$.

Let $A$ be the Laplacian operator $\Delta$ satisfying zero Dirichlet boundary condition, with domain $\mathcal{D}(A) = H^2_0(D) \cap H^2(D)$, which generates a strongly continuous semigroup $\{S_t\}_{t \geq 0}$ on $H$, defined by, for any $h \in H$,

$$S_t h = \sum_{k \in \mathbb{N}} e^{-\alpha_k t} e_k(h)$$

Here, for the sake of brevity, we omit to write the dependence of the spatial variable $\xi$. It is straightforward to check that $\{S_t\}_{t \geq 0}$ is a contractive semigroup on $H$. For $\gamma \in [0, 1]$ we defined the operator $(-A)^\gamma$ by

$$(-A)^\gamma x = \sum_{k \in \mathbb{N}} \alpha_k^\gamma x_k e_k \in H$$

with domain

$$\mathcal{D}((-A)^\gamma) = \left\{ x = \sum_{k \in \mathbb{N}} x_k e_k \in H; \|x\|_{(-A)^\gamma} := \sum_{k \in \mathbb{N}} \alpha_k^{2\gamma} x_k^2 < \infty \right\}.$$

Using the spectral decomposition of $A$, the semigroup $\{S_t\}_{t \geq 0}$ enjoys the following smooth property.

**Proposition 2.1.** For any $\gamma \in [0, 1]$ there exists a constant $C_\gamma > 0$ such that

$$\|S_t x\|_{(-A)^\gamma} \leq C_\gamma t^{-\gamma} e^{-\frac{\alpha_1}{2} t} \|x\|, \quad t > 0, x \in H, \quad (2.1)$$

$$\|S_t x - S_\tau x\| \leq C_\gamma \frac{|t - \tau|^{\gamma}}{\tau^{\gamma}} e^{-\frac{\alpha_1}{2} \tau} \|x\|, \quad t, \tau > 0, x \in H, \quad (2.2)$$

$$\|S_t x - S_\tau x\| \leq C_\gamma |t - \tau|^{\gamma} e^{-\frac{\alpha_1}{2} \tau} \|x\|_{(-A)^\gamma}, \quad t, \tau > 0, x \in \mathcal{D}((-A)^\gamma). \quad (2.3)$$

For the perturbation noises we suppose the following setting. For $i = 1, 2$, let $W^i_t$ be the Wiener process on a stochastic base $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with a bounded covariance operator $Q_i : H \to H$ defined by $Q_i e_k = \lambda_{i,k} e_k$, where $\{\lambda_{i,k}\}_{k \in \mathbb{N}}$ are nonnegative and $\{e_k\}_{k \in \mathbb{N}}$ is the complete orthonormal basis in $H$. Formally, for $i = 1, 2$, Wiener processes $W^i_t$ can be written as the infinite sums (cf. Da Prato and Zabczyk [24])

$$W^i_t = \sum_{k \in \mathbb{N}} \sqrt{\lambda_{i,k}} B^i_{t,k} e_k,$$
where \( \{B^{(i)}_{t,k}\}_{k \in \mathbb{N}} \) are mutually independent real-valued Brownian motions on stochastic base \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\). For the sake of simplicity we prefer to assume that both \(Q_1\) and \(Q_2\) have finite trace, that is there exists a positive constant \(C\) such that

\[
Tr(Q_i) = \sum_{k \in \mathbb{N}} \lambda_{i,k} \leq C, \quad i = 1, 2.
\]  

(2.4)

Moreover, we also assume

\[
Tr((-A)Q_1) = \sum_{k \in \mathbb{N}} \lambda_{1,k} \alpha_k \leq C.
\]  

(2.5)

Concerning the drift coefficients \(F\) and \(G\) we shall impose the following conditions.

(H.1) For each fixed \(u \in H\), the mapping \(F(u, \cdot) : H \to H\) is of a class \(C^3\), with bounded derivatives, uniformly with respect \(u \in H\). Also suppose that for any \(v \in H\), the mapping \(F(\cdot, v) : H \to H\) is of class \(C^3\), with bounded derivatives, uniformly for \(v \in H\).

(H.2) For each fixed \(u \in H\), the mapping \(G(u, \cdot) : H \to H\) is of a class \(C^2\), with bounded derivatives, uniformly with respect \(u \in H\). Also suppose that for any \(v \in H\), the mapping \(G(\cdot, v) : H \to H\) is of class \(C^2\), with bounded derivatives, uniformly with respect \(v \in H\). Moreover, we assume that

\[
\sup_{u \in H} \|G'_v(u, v)\|_{\mathcal{L}(H)} := L_g < \alpha_1,
\]

where \(G'_v\) denotes the derivative with respect to \(v\) and \(\| \cdot \|_{\mathcal{L}(H)}\) denotes the operator norm on \(\mathcal{L}(H)\).

**Remark 2.1.** Under (H.1) and (H.1), it is not difficult to verify that there exist positive constants \(K_F\) and \(K_G\) such that

\[
\|F(u_1, v_1) - F(u_2, v_2)\| \leq K_F(\|u_1 - u_2\| + \|v_1 - v_2\|), \quad u_1, u_2, v_1, v_2 \in H, \quad \text{(2.6)}
\]

and

\[
\|G(u_1, v_1) - G(u_2, v_2)\| \leq K_G(\|u_1 - u_2\| + \|v_1 - v_2\|), \quad u_1, u_2, v_1, v_2 \in H, \quad \text{(2.7)}
\]

which means \(F, G : H \times H \to H\) are Lipschitz continuous.

Once introduced the main notations, system (1.2) can be written as

\[
\begin{align*}
    dX^\epsilon_t &= AX^\epsilon_t dt + F(X^\epsilon_t, Y^\epsilon_t)dt + \sigma_1 dW^1_t, \quad X^\epsilon_0 = x, \\
    dY^\epsilon_t &= \frac{1}{\epsilon}AY^\epsilon_t dt + \frac{1}{\epsilon}G(X^\epsilon_t, Y^\epsilon_t)dt + \frac{\sigma_2}{\sqrt{\epsilon}}dW^2_t, \quad Y^\epsilon_0 = y.
\end{align*}
\]

(2.8)

By virtue of conditions (2.6) and (2.7), it is easy to check that system (2.8) admits a unique mild solution, which, in order to emphasize the dependence
on the initial data, is denoted by \((X_t^ε(x,y), Y_t^ε(x,y))\). This means that for any \(t > 0\), it holds \(P - a.s.\) that

\[
\begin{aligned}
X_t^ε(x,y) &= S_t x + \int_0^t S_{t-s} F(X_s^ε(x,y), Y_s^ε(x,y)) \, ds + \sigma_t \int_0^t S_{t-s} dW_s^1, \\
Y_t^ε(x,y) &= S_2 y + \frac{1}{T} \int_0^t S_{t-s} G(X_s^ε(x,y), Y_s^ε(x,y)) \, ds + \frac{\sigma_2}{T} \int_0^t S_{t-s} dW_s^2.
\end{aligned}
\tag{2.9}
\]

Moreover, by using standard arguments, we have the following lemma.

**Lemma 2.1.** Under (H.1) and (H.2), for any \(T > 0\) and \(x,y \in H\), there exists a positive constant \(C_T\) such that for any \(y \in H\) and \(\epsilon \in (0,1]\),

\[
\begin{aligned}
\sup_{t \in [0,T]} E\|X_t^ε(x,y)\|^2 &\le C_T (1 + \|x\|^2 + \|y\|^2), \\
\sup_{t \in [0,T]} E\|Y_t^ε(x,y)\|^2 &\le C_T (1 + \|x\|^2 + \|y\|^2).
\end{aligned}
\]

To study weak convergence, we need to introduce some notations in connection with the test function. If \(\mathcal{X}\) is a Hilbert space equipped with inner product \((\cdot, \cdot)_{\mathcal{X}}\), we denote by \(\mathcal{C}^k(\mathcal{X}, \mathbb{R})\) the space of all \(k\)-times continuously Fréchet differentiable functions \(\phi : \mathcal{X} \to \mathbb{R}, x \mapsto \phi(x)\). By \(\mathcal{C}_b^k(\mathcal{X}, \mathbb{R})\) we denote the subspace of functions from \(\mathcal{C}^k(\mathcal{X}, \mathbb{R})\) which are bounded together with their derivatives. For \(\phi \in \mathcal{C}^m(\mathcal{X}, \mathbb{R})\), we use the notation \(D_{\text{m-times}}^m, \ldots, x^\phi(x)\) for its \(m\)-th derivative at the point \(x\).

Thanks to Riesz representation isomorphism, we may get the identity for \(x,h \in \mathcal{X}\):

\[
D_x \phi(x) \cdot h = (D_x \phi(x), h)_{\mathcal{X}}.
\]

For \(\phi \in C^2(\mathcal{X}, \mathbb{R})\), we will identify the second derivative \(D_{xx}^2 \psi(x)\) with a bilinear operator from \(\mathcal{X} \times \mathcal{X}\) to \(\mathbb{R}\) such that

\[
D_{xx}^2 \psi(x) \cdot (h,k) = (D_{xx}^2 \psi(x)h, k)_{\mathcal{X}}, \quad x, h, k \in \mathcal{X}.
\]

On some occasions, we also use the notations \(\phi', \phi''\) and \(\phi'''\) instead of \(D_x \phi, D_{xx}^2 \psi\) and \(D_{xxx}^3 \psi\), respectively.

### 3. Ergodicity of \(Y_t^ε\) and averaging dynamics

For fixed \(x \in H\) consider the problem associate to fast motion with frozen slow component

\[
\begin{aligned}
dY_t^ε &= AY_t^ε \, dt + G(x,Y_t^ε) \, dt + \sigma_2 dW_t^2, \\
Y_0^ε &= y.
\end{aligned}
\tag{3.1}
\]

Notice that the drift \(G : H \times H \to H\) is Lipschitz continuous. By arguing as before, for any fixed slow component \(x \in H\) and any initial data \(y \in H\), problem \((3.1)\) has a unique mild solution denoted by \(Y_t^ε(y)\). Now, we consider
the transition semigroup \( P_t^x \) associated with process \( Y_t^x(y) \), by setting for any \( \psi \in \mathcal{B}_b(H) \) the space of bounded functions on \( H \\
\),

\[
P_t^x \psi(y) = \mathbb{E}[Y_t^x(y)].
\]

By adopting a similar approach used in [11], we can show that

\[
\mathbb{E}[\|Y_t^x(y)\|^2] \leq C \left( e^{-(\alpha_1 - L_g) t} \|y\|^2 + \|x\| + 1 \right), \quad t > 0
\]

(3.2)

for some constant \( C > 0 \). This implies the existence of an invariant measure \( \mu^x \) for the Markov semigroup \( P_t^x \) associated with system (3.1) on \( H \) such that

\[
\int_H P_t^x \psi d\mu = \int_H \psi d\mu, \quad t \geq 0
\]

for any \( \psi \in \mathcal{B}_b(H) \) (for a proof, see, e.g., [6], Section 2.1). We recall that in [7] it is proved the invariant measure has finite 2–moments:

\[
\int_H \|y\|^2 \mu^x(dy) \leq C(1 + \|x\|^2).
\]

(3.3)

Let \( Y_t^x(y') \) be the solution of problem (3.1) with initial value \( Y_0 = y' \), it is not difficult to show that for any \( t \geq 0 \),

\[
\mathbb{E}[\|Y_t^x(y) - Y_t^x(y')\|^2] \leq C\|y - y'\|^2 e^{-\beta t}
\]

(3.4)

with \( \beta = (\alpha_1 - L_g) > 0 \), which implies that \( \mu^x \) is the unique invariant measure for \( P_t^x \). This allows us to define an \( H \)-valued mapping \( \bar{F} \) by averaging the coefficient \( F \) with respect to the invariant measure \( \mu^x \), that is,

\[
\bar{F}(x) := \int_H F(x,y) \mu^x(dy), \quad x \in H,
\]

and then, by using the condition (2.6), it is immediate to check that

\[
\|\bar{F}(x_1) - \bar{F}(x_2)\| \leq K_F \|x_1 - x_2\|, \quad x_1, x_2 \in H.
\]

(3.5)

According to the invariant property of \( \mu^x \), (3.3) and (2.6), we have

\[
\|\mathbb{E}F(x, Y_t^x(y)) - \bar{F}(x)\|^2 = \|\int_H \mathbb{E}(F(x, Y_t^x(y)) - F(x, Y_t^x(z))) \mu^x(dz)\|^2 \\
\leq \int_H \mathbb{E}\|Y_t^x(y) - Y_t^x(z)\|^2 \mu^x(dz) \\
\leq e^{-\beta t} \int_H \|y - z\|^2 \mu^x(dz) \\
\leq C e^{-\beta t} (1 + \|x\|^2 + \|y\|^2),
\]

(3.6)

which means that

\[
\bar{F}(x) = \lim_{t \to +\infty} \mathbb{E}F(x, Y_t^x(y)), \quad x \in H.
\]

(3.7)
Using this limit and Assumptions (H.1), it is possible to show that
\[ \| \bar{F}'(x) \cdot h \| \leq C \| h \|, \quad x, h \in H. \]  \hspace{1cm} (3.8)

Now we introduce the effective dynamics:
\[
\begin{cases}
\partial_t \bar{X}_t(\xi) = \Delta \bar{X}_t(\xi) + \bar{F}(\bar{X}_t(\xi)) + \sigma_1 \dot{W}_1^2(\xi), \quad \xi \in D, t > 0, \\
\bar{X}_t(0) = \bar{X}_t(l) = 0, \quad t \geq 0, \\
\bar{X}_0(\xi) = x(\xi), \quad \xi \in D.
\end{cases}
\]

By using the notations introduced in Section 2 it can be written as an abstract evolutionary equation in form
\[
\begin{cases}
d\bar{X}_t = A\bar{X}_t dt + \bar{F}(\bar{X}_t) dt + \sigma_1 dW_1^1, \quad t > 0, \\
\bar{X}_0 = x.
\end{cases}
\]  \hspace{1cm} (3.9)

For any initial datum \( x \in H \), the equation (3.9) admits a unique mild solution, which means that there exists a unique adapt process \( \bar{X}_t(x) \) such that
\[
\bar{X}_t(x) = S_t x + \int_0^t S_{t-s} \bar{F}(\bar{X}_s(x)) ds + \sigma_1 \int_0^t S_{t-s} dW_1^1, \quad P - a.s., \quad t \geq 0.
\]

Moreover, for any \( T > 0 \) it can be easily proved that
\[
E \| \bar{X}_t(x) \|^2 \leq C_T (1 + \| x \|^2), \quad t \in [0, T]. \]  \hspace{1cm} (3.10)

Thanks to averaging principle (see Cerrai [6] for details), we have that the limit
\[
\lim_{\epsilon \to 0^+} \sup_{0 \leq t \leq T} E \| \bar{X}_t(\epsilon) - X^\epsilon_t(x, y) \|^2 = 0 \]  \hspace{1cm} (3.11)

for any fixed \( T > 0 \). This means that the slow process \( X^\epsilon_t(x, y) \) enjoys strong convergence to the averaging process \( \bar{X}_t(x) \). Moreover, the strong order in averaging is \( \frac{1}{5} \) (Bréhier [4]). The weak convergence using test functions is obvious. Our aim is to establish rigorously weak error bounds for the limit of slow process with respect to scale parameter \( \epsilon \). The main result of this paper is the following, whose proof is postponed in the end of Section 5.

**Theorem 3.1.** Assume that \( x \in \mathcal{D}((-A)^{\theta}) \) for some \( \theta \in (0, 1) \) and \( y \in H \). Then, under (H.1) and (H.2), for any \( r \in (0, 1) \), \( T > 0 \) and \( \phi \in C^3_b(H, \mathbb{R}) \), there exists a constant \( C_{\theta, r, T, \phi, x, y} \) such that
\[
|E\phi(X^\epsilon_T(x, y)) - E\phi(\bar{X}_T(x))| \leq C_{\theta, r, T, \phi, x, y} \epsilon^{1-r}.
\]

**4. Some a priori estimates**

Before proving the main results, we need to state some technical lemmas used in subsequent section.
Lemma 4.1. Let the conditions (H.1) and (H.2) be satisfied and fix $x, y \in H$ and $T > 0$. Then for any $r \in (0, \frac{1}{2})$ there exists a constant $C_{r,T} > 0$ such that for any $0 < s \leq t \leq T$, we have
\[
(\mathbb{E}|X_t^r(x,y) - X_s^r(x,y)|^2)^{\frac{1}{2}} \leq C_{r,T} \frac{|t-s|^{1-r}}{s^{1-r}} \|x\| \\
+ C_{r,T} (|t-s|^\frac{1}{2} + |t-s|^{1-r} + |t-s|^r)(1 + \|x\| + \|y\|).
\]
Proof. We can write
\[
X_t^r(x,y) - X_s^r(x,y) = (S_t - S_s)x + \int_s^t S_{t-s} F(X_{\tau}^r(x,y), Y_{\tau}^r(x,y)) d\tau \\
+ \int_0^s (S_{t-s} - S_{s-\tau}) F(X_{\tau}^r(x,y), Y_{\tau}^r(x,y)) d\tau \\
+ \int_s^t S_{t-s} dW_t^1 + \int_0^s (S_{t-s} - S_{s-\tau}) dW_1^1. \tag{4.1}
\]
In the next, we estimate separately the different terms in (4.1). By using (2.2), for the first term we have
\[
\|(S_t - S_s)x\| \leq C_{r,T} \frac{|t-s|^{1-r}}{s^{1-r}} \|x\|. \tag{4.2}
\]
For the second term, by Lemma 2.1 and Hölder’s inequality, we obtain
\[
\mathbb{E}\| \int_s^t S_{t-s} F(X_{\tau}^r(x,y), Y_{\tau}^r(x,y)) d\tau \|^2 \\
\leq |t-s| \int_s^t \mathbb{E}\|S_{t-s} F(X_{\tau}^r(x,y), Y_{\tau}^r(x,y))\|^2 d\tau \\
\leq C |t-s| \int_s^t (1 + \|X_{\tau}^r(x,y)\|^2 + \|Y_{\tau}^r(x,y)\|^2) d\tau \\
\leq C_T |t-s|(1 + \|x\|^2 + \|y\|^2). \tag{4.3}
\]
Concerning the third term, by using (2.2), we can deduce that
\[
\mathbb{E}\| \int_0^s (S_{t-s} - S_{s-\tau}) F(X_{\tau}^r(x,y), Y_{\tau}^r(x,y)) d\tau \|^2 \\
\leq \mathbb{E}\left[ \int_0^s \|(S_{t-s} - S_{s-\tau}) F(X_{\tau}^r(x,y), Y_{\tau}^r(x,y))\|^2 d\tau \right]^2 \\
\leq C_r \mathbb{E}\left[ \int_0^s \frac{(t-s)^{1-r}}{(s-\tau)^{1-r}} e^{-\frac{r}{2}(s-\tau)} \|F(X_{\tau}^r(x,y), Y_{\tau}^r(x,y))\|^2 d\tau \right]^2.
\]
In view of Lemma 2.1 and Minkowski inequality, we get
\[
\mathbb{E}\| \int_0^s (S_{t-s} - S_{s-\tau}) F(X_{\tau}^r(x,y), Y_{\tau}^r(x,y)) d\tau \|^2
\]
That for any $r$ here the last inequality following from fact $\| \sum_{k=1}^{\infty} e_{\lambda} \| \leq C(1 + \| x \|^2 + \| y \|^2)$, we directly have

$$E \| \int_{s}^{t} S_{t-\tau} dW \|^2 = \sum_{k=1}^{\infty} \lambda_{1,k} \int_{s}^{t} \| S_{t-\tau} e_{k} \|^2 d\tau \leq Tr(Q_1) |t-s|.$$  

The final term on the right-hand side of the (4.1) can be treated as follows:

$$E \| \int_{0}^{s} (S_{t-\tau} - S_{s-\tau}) dW \|^2 = \sum_{k=1}^{\infty} \lambda_{1,k} \int_{0}^{s} \| (S_{t-\tau} - S_{s-\tau}) e_{k} \|^2 d\tau = \sum_{k=1}^{\infty} \lambda_{1,k} \int_{0}^{s} \| \int_{s-\tau}^{t-\tau} AS_{\rho} e_{k} d\rho \|^2 d\tau \leq C \sum_{k=1}^{\infty} \lambda_{1,k} \int_{0}^{s} \int_{s-\tau}^{t-\tau} \frac{1}{\rho} d\rho d\tau,$$

here the last inequality following from fact $\| AS_{\lambda} \|_{\mathcal{L}(H)} \leq Ct^{-1}$ for $t > 0$. Note that for any $r \in (0, \frac{1}{2})$, it holds

$$\int_{0}^{s} \int_{s-\tau}^{t-\tau} \frac{1}{\rho} d\rho d\tau \leq \int_{0}^{s} (s-\tau)^{-2r} \int_{s-\tau}^{t-\tau} \frac{1}{\rho^{1-r}} d\rho d\tau = r^{-2} \int_{0}^{s} (s-\tau)^{-2r} [(t-\tau)^{1-r} - (s-\tau)^{1-r}]^2 d\tau \leq r^{-2} \int_{0}^{s} (s-\tau)^{-2r} (t-s)^{2r} d\tau \leq r^{-2} |t-s|^{2r} \left( \frac{1}{1-2r} \right)^{1-2r} \leq C_r |t-s|^{2r} T^{1-2r},$$

which implies that

$$E \| \int_{0}^{s} (S_{t-\tau} - S_{s-\tau}) dW \|^2 \leq C_{r,T} |t-s|^{2r}.$$  

(4.6)

By taking (4.2)-(4.6) into account, we can deduce that

$$\left( \frac{\lambda_{1,k}}{\lambda_{1,k}} \right)^{2}.$$
Concerning $J$ we get

$$\leq C_{r,T}^r \frac{|t-s|^{1-r}}{s^{1-r}} \|x\|$$

$$+ C_{r,T}^r \frac{|t-s|^r + |t-s|^{1-r} + |t-s|^r}{(1 + \|x\| + \|y\|)}.$$ 

**Lemma 4.2.** Let the conditions (H.1) and (H.2) be satisfied and fix $x, y \in H$ and $T > 0$. Then for any $r \in (0, \frac{1}{2})$ there exists a constant $C_{r,T} > 0$ such that for any $0 < s \leq t \leq T$, we have

$$(\mathbb{E} \|Y_t^r(x, y) - Y_s^r(x, y)\|^2)^{\frac{1}{2}} \leq C_{r,T}(1 + \|x\| + \|y\|) \left[ \frac{|t-s|^r}{s^r} + \frac{|t-s|^r}{e^r} \right].$$

**Proof.** We have the decomposition

$$Y_t^r(x, y) - Y_s^r(x, y) = [S_y^r y - S_{2y}^r y] + \frac{1}{\epsilon} \int_s^t S_{1-\epsilon}^r G(X_{t-\epsilon}^r(x, y), Y_{t-\epsilon}^r(x, y)) d\tau$$

$$+ \frac{1}{\epsilon} \int_0^s (S_{1-\epsilon}^r - S_{1-2\epsilon}^r) G(X_{t-\epsilon}^r(x, y), Y_{t-\epsilon}^r(x, y)) d\tau$$

$$+ \frac{1}{\epsilon^2} \int_s^t S_{1-\epsilon}^r dW_2^2 + \frac{1}{\epsilon^2} \int_0^s (S_{1-\epsilon}^r - S_{1-2\epsilon}^r) dW_2^2$$

$$= \sum_{k=1}^5 J_k^r(t, s).$$ (4.7)

By (2.2), it is immediate to check that

$$\|J_1^r(t, s)\| \leq C_{r}^r \frac{|t-s|^r}{s^r} \|y\|. \tag{4.8}$$

By Minkowski inequality and Lemma 2.1, one can estimate $J_2^r(t, s)$ as follows:

$$\mathbb{E} \|J_2^r(t, s)\|^2 \leq \mathbb{E} \left( \frac{C}{\epsilon} \int_s^t e^{-\alpha_1\frac{t-s}{\epsilon}} (1 + \|X_{t-\epsilon}^r(x, y)\| + \|Y_{t-\epsilon}^r(x, y)\|) d\tau \right)^2$$

$$\leq C \epsilon^2 \left( \int_0^s e^{-\alpha_1\frac{t-s}{\epsilon}} (1 + \|X_{t-\epsilon}^r\| + \|Y_{t-\epsilon}^r\|) d\tau \right)^2$$

$$\leq C \left( \int_0^s e^{-\alpha_1\frac{t-s}{\epsilon}} \left( \mathbb{E} (1 + \|X_{t-\epsilon}^r(x, y)\| + \|Y_{t-\epsilon}^r(x, y)\|)^2 \right)^{\frac{1}{2}} d\tau \right)^2$$

$$= C_{T}(1 + \|x\|^2 + \|y\|^2) \left( 1 - e^{-\alpha_1\frac{t-s}{\epsilon}} \right)^2$$

$$\leq C_{r,T}(1 + \|x\|^2 + \|y\|^2) \frac{|t-s|^{2r}}{e^{2r}}, \tag{4.9}$$

where, the last step is due to the inequality $1 - e^{-a} \leq C_a a^r$ for $a > 0$, $r \in (0, 1)$.

Concerning $J_3^r(t, s)$, according to (2.2), Lemma 2.1 and Minkowski inequality, we get

$$\mathbb{E} \|J_3^r(t, s)\|^2 \leq$$

$$\leq C_{r,T}(1 + \|x\|^2 + \|y\|^2) \frac{|t-s|^{2r}}{e^{2r}}.$$
For \( J_4(t, s) \) we have

\[
E\|J_4(t, s)\|^2 = \frac{1}{\epsilon} \int_0^t \sum_{k \in \mathbb{N}} e^{-2(t-\tau)\alpha_k/\epsilon} d\tau
\]

\[
= \sum_{k \in \mathbb{N}} \int_0^{(t-s)/\epsilon} e^{-2\alpha_k \tau} d\tau
\]

\[
= \sum_{k \in \mathbb{N}} \frac{1}{2\alpha_k} (1 - e^{-2\alpha_k (t-s)/\epsilon})
\]

\[
\leq C_r \frac{|t-s|^{2r}}{\epsilon^{2r}} \sum_{k \in \mathbb{N}} \frac{1}{\alpha_k^{2r}}.
\]

Recalling that we have assumed \( r \in (0, \frac{1}{2}) \), it follows that \( \frac{1}{\alpha_k^{2r}} < +\infty \).

Therefore, we obtain

\[
E\|J_4(t, s)\|^2 \leq C_r \frac{|t-s|^{2r}}{\epsilon^{2r}}.
\]

(4.11)

For \( J_5(t, s) \) we have

\[
E\|J_5(t, s)\| = \frac{1}{\epsilon} \int_0^t \sum_{k \in \mathbb{N}} e^{-2(s-\tau)\alpha_k/\epsilon} (1 - e^{-(t-s)\alpha_k/\epsilon})^2 d\tau
\]

\[
\leq \sum_{k \in \mathbb{N}} (1 - e^{-(t-s)\alpha_k/\epsilon})^2 \frac{1}{2\alpha_k} (1 - e^{-2\alpha_k/\epsilon})
\]

\[
\leq \sum_{k \in \mathbb{N}} (1 - e^{-(t-s)\alpha_k/\epsilon})^2 \frac{1}{2\alpha_k}
\]

\[
\leq C_r \sum_{k \in \mathbb{N}} \frac{|t-s|^{2r}}{\epsilon^{2r}} \frac{1}{\alpha_k^{1-2r}}
\]
\[
\begin{align*}
&\leq C_r \frac{|t-s|^{2r}}{e^{2r}}. \\
&\text{(4.12)}
\end{align*}
\]

Collecting together (4.8)-(4.12), we obtain

\[
(E\|Y_t^r(x,y) - Y_t^s(x,y)\|^2)^{\frac{1}{2}} \leq C_{r,T}(1 + \|x\| + \|y\|) \left[ \frac{|t-s|}{s^r} + \frac{|t-s|^{r}}{e^{r}} \right].
\]

\[\square\]

**Lemma 4.3.** Assume that \( x \in \mathcal{D}((-A)^{\theta}) \) for some \( \theta \in (0,1] \). Then, under conditions \((H.1)\) and \((H.2)\), we have that \( X_t^r(x, y) \in \mathcal{D}(-A) \), \( P \) - a.s., for any \( t > 0 \) and \( \epsilon > 0 \). Moreover, for any \( r \in (0, \frac{1}{\theta}) \) it holds that

\[
(E \| AX_t^r(x,y) \|^2)^{\frac{1}{2}} \leq C_{r,T} t^{\theta-1} \|x\|_{(-A)^{\theta}} + C_{r,T}(1 + \|x\| + \|y\|)(1 + \frac{1}{e^r}), \ t \in [0,T].
\]

**Proof.** For any \( t \in [0,T] \) we write \( X_t^r(x,y) \) as

\[
X_t^r(x,y) = [S_t x + \int_0^t S_{t-s} F(X_t^r(x,y), Y_t^r(x,y)) ds]
+ \int_0^t S_{t-s} [F(X_t^r(x,y), Y_t^r(x,y)) - F(X_t^s(x,y), Y_t^s(x,y))] ds
+ \int_0^t S_{t-s} dW_s^r
:= X_t^{r,1}(x,y) + X_t^{r,2}(x,y) + X_t^{r,3}(x,y).
\]

For \( X_t^{r,1}(x,y) \) we have

\[
\| AX_t^{r,1}(x,y) \| = \| AS_t x \| + \| (S_t - I) F(X_t^r(x,y), Y_t^r(x,y)) \|
\leq C t^{\theta-1} \|x\|_{(-A)^{\theta}} + C(1 + \|x\| + \|y\|),
\]

so that, thanks to Lemma \([2,2]\), we obtain

\[
(E \| AX_t^{r,1}(x,y) \|^2)^{\frac{1}{2}} = C_{r,T} t^{\theta-1} \|x\|_{(-A)^{\theta}} + C_{r,T}(1 + \|x\| + \|y\|). \quad (4.13)
\]

From \([2,1]\), we have

\[
\| AX_t^{r,2}(x,y) \|
\leq C \int_0^t \frac{e^{-\frac{(t-s)}{t-s}}}{t-s} \| F(X_t^r(x,y), Y_t^r(x,y)) - F(X_t^s(x,y), Y_t^s(x,y)) \| ds
\leq C \int_0^t \frac{e^{-\frac{(t-s)}{t-s}}}{t-s} \| X_t^r(x,y) - X_t^s(x,y) \| + \| Y_t^r(x,y), Y_t^s(x,y) \| ds,
\]

which implies

\[
E \| AX_t^{r,2}(x,y) \|^2 \leq C \int_0^t \frac{e^{-\frac{(t-s)}{t-s}}}{t-s} (E \| X_t^r(x,y) - X_t^s(x,y) \|^2)^{\frac{1}{2}} ds.
\]

\[14\]
+ C \left[ \int_0^t \frac{e^{-\frac{\alpha}{2}(t-s)}}{t-s} \left( \mathbb{E} \| Y_t^x(x,y) - Y_s^x(x,y) \|^2 \right)^{\frac{r}{2}} ds \right]^2,

by making use of Minkowski inequality. If we take \( r \in (0, \frac{1}{4}) \) as in Lemma 4.1, we get

\[
\left[ \int_0^t \frac{e^{-\frac{\alpha}{2}(t-s)}}{t-s} \left( \mathbb{E} \| X_t^x(x,y) - X_s^x(x,y) \|^2 \right)^{\frac{r}{2}} ds \right]^2
\leq C_{r,T}(1 + \|x\| + \|y\|)^2 \left[ \int_0^t \frac{e^{-\frac{\alpha}{2}(t-s)}}{(t-s)^r s^{1-r}} ds + \int_0^t \frac{e^{-\frac{\alpha}{2}(t-s)}}{(t-s)^{1-r}} ds \right]^2
\leq C_{r,T}(1 + \|x\| + \|y\|)^2 \left[ \int_0^{t/2} \frac{1}{(t-s)^r s^{1-r}} ds + \int_{t/2}^t \frac{1}{(t-s)^{1-r}} ds \right]^2
\leq C_{r,T}(1 + \|x\| + \|y\|)^2 \left[ 1 + \int_0^{t/2} \frac{1}{(t-s)^r s^{1-r}} ds + \int_{t/2}^t \frac{1}{(t-s)^{1-r}} ds \right]^2
\leq C_{r,T}(1 + \|x\| + \|y\|)^2 \left[ 1 + \int_0^{t/2} \frac{1}{s^{1-r}} ds + \int_{t/2}^t \frac{1}{s^{1-r}} ds \right]^2
\leq C_{r,T}(1 + \|x\| + \|y\|)^2 \left[ 1 + \frac{1}{r} + \frac{1}{1-r} \right]
\leq C_{r,T}(1 + \|x\| + \|y\|)^2(1 + \frac{1}{r}).
\]

By using a completely analogous way, due to Lemma 4.2, it is possible to show that

\[
\left[ \int_0^t \frac{e^{-\frac{\alpha}{2}(t-s)}}{t-s} \left( \mathbb{E} \| Y_t^x(x,y) - Y_s^x(x,y) \|^2 \right)^{\frac{r}{2}} ds \right]^2
\leq C_{r,T}(1 + \|x\| + \|y\|)^2 \left[ \int_0^t \frac{e^{-\frac{\alpha}{2}(t-s)}}{(t-s)^r s^{1-r}} ds + \frac{1}{e^r} \int_0^t \frac{e^{-\frac{\alpha}{2}(t-s)}}{(t-s)^{1-r}} ds \right]^2
\leq C_{r,T}(1 + \|x\| + \|y\|)^2 \left[ 1 + \frac{1}{e^r} \right],
\]

which, together with (4.14), allows us to get the estimate

\[
\mathbb{E} \| AX_t^{r,2}(x,y) \|^2 \leq C_{r,T}(1 + \|x\| + \|y\|)^2 \left( 1 + \frac{1}{e^r} \right), \quad (4.15)
\]

Thus, it remains to estimate \( AX_t^{r,3}(x,y) \). By straightforward computations and condition 2.5, we get

\[
\mathbb{E} \| AX_t^{r,3}(x,y) \|^2 = \mathbb{E} \left\| \sum_{k \in \mathbb{N}} \sqrt{\lambda_{1,k}} \alpha_k e_k \int_0^t e^{-\alpha k(t-s)} dE_{s,k}^{(1)} \right\|^2.
\]
\[
\sum_{k \in \mathbb{N}} \lambda_{1,k} \alpha_k \int_0^t e^{-2\alpha_1,k(t-s)} \, ds \\
\leq C \sum_{k \in \mathbb{N}} \lambda_{1,k} \alpha_k \\
\leq C.
\]

This, together with (4.13) and (4.15), yields

\[
(\mathbb{E}\|AX_t^\delta(x,y)\|_2^2)^{\frac{1}{2}} \leq C_r,T \frac{|t-s|^{1-r}}{s^{1-r}} \|x\| + C_r,T (1 + \|x\| + \|y\|)(1 + \frac{1}{e^r}).
\]

**Lemma 4.4.** Let the conditions (H.1) and (H.2) be satisfied and fix \(x \in H\) and \(T > 0\). Then for any \(r \in (0, \frac{1}{4})\) there exists a constant \(C_{r,T} > 0\) such that for any \(0 < s \leq t \leq T\), we have

\[
\begin{align*}
(\mathbb{E}\|\hat{X}_t(x) - \hat{X}_s(x)\|_2^2)^{\frac{1}{2}} & \leq C_{r,T} \frac{|t-s|^{1-r}}{s^{1-r}} \|x\| \\
& + C_{r,T} (|t-s|^\frac{1}{2} |t-s|^{1-r} + |t-s|^r)(1 + \|x\|).
\end{align*}
\]

**Proof.** It holds that

\[
\hat{X}_t(x) - \hat{X}_s(x) = S_t x - S_s x + \int_s^t S_{t-\tau} \hat{F}(\hat{X}_\tau(x)) \, d\tau \\
+ \int_0^s [S_{t-\tau} \hat{F}(\hat{X}_\tau(x)) - S_{s-\tau} \hat{F}(\hat{X}_\tau(x))] \, d\tau \\
+ \int_s^t S_{t-\tau} dW^1_\tau + \int_0^s (S_{t-\tau} - S_{s-\tau}) dW^1_\tau. \quad (4.16)
\]

According to (2.24), we obtain

\[
\|(S_t - S_s)x\| \leq C_r \frac{|t-s|^{1-r}}{s^{1-r}} \|x\|.
\]

For the second term on the right-hand side of (4.16), by using (3.10) we have

\[
\begin{align*}
\mathbb{E}\|\int_s^t S_{t-\tau} \hat{F}(\hat{X}_\tau(x)) \, d\tau\|^2 & \leq |t-s| \int_s^t \mathbb{E}\|S_{t-\tau} \hat{F}(\hat{X}_\tau(x))\|^2 \, d\tau \\
& \leq C |t-s| \int_s^t \mathbb{E}(1 + \|\hat{X}_\tau(x)\|^2) \, d\tau \\
& \leq C_T |t-s|(1 + \|x\|^2).
\end{align*}
\]

Concerning the third term on the right-hand side of (4.16), we deduce

\[
\mathbb{E}\|\int_0^s (S_{t-\tau} - S_{s-\tau}) \hat{F}(\hat{X}_\tau(x)) \, d\tau\|^2
\]
The proof is analogous to that of the previous Lemma 4.3. We write

\[
\text{Proof.}
\]

Assume that \( \bar{X} \) and then, by using once more (3.10), this yields

\[
\mathbb{E}\left[ \int_0^s \| (S_{t-r} - S_{s-r}) \bar{F}(\bar{X}_r(x)) \| d\tau \right]^2
\]

and then, by using arguments analogous to those used in Lemma 4.1, we have

\[
\mathbb{E}\left[ \int_0^s \left( t - s \right)^{1-r} e^{-\frac{\alpha}{(s-r)}} \| \bar{F}(\bar{X}_r(x)) \| d\tau \right]^2
\]

By using arguments analogous to those used in Lemma 4.1, we have

\[
\mathbb{E}\left[ \int_s^t \left( S_{t-r} - S_{s-r} \right) \bar{F}(\bar{X}_r(x)) d\tau \right]^2
\]

and

\[
\mathbb{E}\left[ \int_0^s \left( S_{t-r} - S_{s-r} \right) dW_r^1 \right]^2 \leq C_{r,T} |t - s|^{2(r)}
\]

Therefore, collecting all estimate of terms appearing on the right-hand side of (1.10), we obtain

\[
(\mathbb{E}\| \bar{X}_t(x) - \bar{X}_s(x) \|^2)^{\frac{1}{2}} \leq C_{r,T} \frac{|t - s|^{1-r}}{s^{1-r}} \| x \|
\]

\[
+ \ C_{r,T} (|t - s|^{\frac{1}{2}} + |t - s|^{1-r} + |t - s|^r) (1 + \| x \|).
\]

**Lemma 4.5.** Assume that \( x \in \mathcal{D}((-A)^{\theta}) \) for some \( \theta \in (0, 1) \). Then, under conditions (H.1) and (H.2), we have that \( \bar{X}_t \in \mathcal{D}((-A)) \), \( \mathbb{P} \) a.s., for any \( t \in [0, T] \) and \( \epsilon > 0 \). Moreover, it holds that

\[
(\mathbb{E}\| A\bar{X}_t(x) \|^2)^{\frac{1}{2}} \leq C T t^{\theta - 1}\| x \|_{(-A)^{\theta}} + C_T (1 + \| x \|), \quad t \in [0, T].
\]

**Proof.** The proof is analogous to that of the previous Lemma 4.3. We write

\[
\bar{X}_t(x) = [S_t x + \int_0^t S_{t-s} \bar{F}(\bar{X}_s(x)) ds] + \int_0^t S_{t-s} \bar{F}(\bar{X}_s(x)) - \bar{F}(\bar{X}_t'(x)) ds + \int_0^t S_{t-s} dW^1_s
\]
\[ := \bar{X}_t^{(1)}(x) + \bar{X}_t^{(2)}(x) + \bar{X}_t^{(3)}(x). \] (4.17)

For \( \bar{X}_t^{(1)}(x) \), we have
\[
\| A\bar{X}_t^{(1)}(x) \| = \| AS_t x \| + \| (S_t - I)\bar{F}(\bar{X}_t(x)) \| \\
\leq C t^{\theta - 1} \| x \| (-A) \theta + C (1 + \| \bar{X}_t(x) \|),
\]
and then, thanks to (3.11), we get
\[
(\mathbb{E} \| A\bar{X}_t^{(1)}(x) \|^2)^{\frac{1}{2}} = Ct^{\theta - 1} \| x \| (-A) \theta + C T (1 + \| x \|).
\]

Concerning \( \bar{X}_t^{(2)}(x) \), we have
\[
\| A\bar{X}_t^{(2)}(x) \| \leq C \int_0^t e^{-\frac{\alpha}{t-s}} \| \bar{F}(\bar{X}_s(x)) - \bar{F}(\bar{X}_t(x)) \| ds \\
\leq C \int_0^t e^{-\frac{\alpha}{t-s}} \| \bar{X}_t(x) - \bar{X}_s(x) \| ds,
\]
and then, according to Minkowski inequality and Lemma 4.4, for a fixed \( r_0 \in (0, \frac{1}{4}) \) we obtain
\[
\mathbb{E} \| A\bar{X}_t^{(2)}(x) \|^2 \leq C r_0, T (1 + \| x \|)^2.
\]

On the other hand, as shown in the Lemma 4.3 we have
\[
\mathbb{E} \| A\bar{X}_t^{(3)}(x) \|^2 \leq C.
\]

Therefore, collecting all estimates of terms appearing on the right-hand side of (4.17), we can conclude the proof.

5. Asymptotic expansions

One of the main tools that we are using in order to prove the main result is Itô’s formula. On the other hand, here the operator \( A \) is unbounded, and then we can not apply directly Itô’s formula. Therefore we have to proceed by Galerkin approximation procedure, to this purpose we need to introduce some notations. For arbitrary \( n \in \mathbb{N} \), let \( H^{(n)} \) denote the finite dimensional subspace
of $H$, generated by the set of eigenvectors \(\{e_1, e_2, \ldots, e_n\}\). Let \(P_n : H \to H^{(n)}\) denote the orthogonal projection defined by

\[
P_n h = \sum_{k=1}^{n} (h, e_k)_H e_k, \ h \in H.
\]

We define \(A_n : H^{(n)} \to H^{(n)}\) by

\[
A_n h = AP_n h = P_n Ah = \sum_{k=1}^{n} (-\alpha_k) (h, e_k)_H e_k, \ h \in H^{(n)},
\]

which is the generator of a strongly semigroup \(\{S_{t,n}\}_{t \geq 0}\) on \(H^{(n)}\) taking the form

\[
S_{t,n} h = \sum_{k=1}^{n} e^{-\alpha_k t} (e_k, h)_H e_k.
\]

Similarly, for arbitrary \(n \in \mathbb{N}\) and \(\gamma \in \mathbb{R}\), one can define the \((-A_n)^{\gamma} : H^{(n)} \to H^{(n)}\) as

\[
(-A_n)^{\gamma} h = \sum_{k=1}^{n} \alpha_k^{\gamma} (e_k, h)_H e_k, \ h \in H^{(n)}.
\]

For each \(n\) we consider the approximating problem of (2.8):

\[
\begin{align*}
  dX^{\varepsilon,n}_t &= A_n X^{\varepsilon,n}_t \ dt + F_n (X^{\varepsilon,n}_t, Y^{\varepsilon,n}_t) dt + \sigma_1 P_n dW^1_t, \\
  dY^{\varepsilon,n}_t &= \frac{1}{\varepsilon} A_n Y^{\varepsilon,n}_t \ dt + \frac{1}{\varepsilon} G_n (X^{\varepsilon,n}_t, Y^{\varepsilon,n}_t) dt + \frac{\sigma_2}{\sqrt{\varepsilon}} P_n dW^2_t,
\end{align*}
\]

with initial conditions \(X^{\varepsilon,n}_0 := x^{(n)} = P_n x, \ Y^{\varepsilon,n}_0 := y^{(n)} = P_n y\), where \(F_n\) and \(G_n\) are respectively defined by

\[
F_n (u, v) = P_n F(u, v), \ u, v \in H^{(n)}, \quad G_n (u, v) = P_n G(u, v), \ u, v \in H^{(n)}.
\]

Such a problem is the finite dimensional problem with Lipschitz coefficients. Under the assumption (H.1) and (H.2), it is easy to show that the problem (5.1)-(5.2) admits a unique strong solution taking values in \(H^{(n)} \times H^{(n)}\), which is denoted by \((X^{\varepsilon,n}_t(x^{(n)},y^{(n)})), Y^{\varepsilon,n}_t(x^{(n)},y^{(n)})\). Moreover, for any fixed \(\varepsilon > 0\) and \(x, y \in H\) it holds that

\[
\lim_{n \to +\infty} \mathbb{E}(\|X^{\varepsilon}_t(x, y) - X^{\varepsilon,n}_t(x^{(n)}, y^{(n)})\|^2) = 0 \quad (5.3)
\]

and

\[
\lim_{n \to +\infty} \mathbb{E}(\|Y^{\varepsilon}_t(x, y) - Y^{\varepsilon,n}_t(x^{(n)}, y^{(n)})\|^2) = 0.
\]
For any fixed $x \in H$, we consider frozen problem associate with equation (5.2) in form
\[
dY_t^{x,n} = A_n Y_t^{x,n} dt + G_n(y^{(n)}_t) dt + \sigma_2 P_n dW_t^2, \quad Y_0^{x,n} = x^{(n)}. \tag{5.4}
\]
Under (H.1) and (H.2), it is easy to check that such a problem admits a unique strong solution denoted by $Y_t^{x,n}(y^{(n)})$, which has a unique invariant measure $\mu^{x,n}$ on finite dimensional space $H^{(n)}$.

The averaged equation for finite dimensional approximation problem (5.1) can be defined as follows:
\[
d\bar{X}^n_t(x^{(n)}) = A_n \bar{X}^n_t(x^{(n)}) dt + \bar{F}_n(\bar{X}^n_t(x^{(n)})) dt + 1P_n dW^1_t, \quad \bar{X}^n_0 = x^{(n)}, \tag{5.5}
\]
with
\[
\bar{F}_n(u) = \int_{\mathbb{H}^{(n)}} F_n(u,v) d\mu^{x,n}(dv), \quad u \in H^{(n)}.
\]

The averaging principle guarantees
\[
\lim_{\epsilon \to 0^+} \left\{ \mathbb{E}\|X_t^{x,n}(x^{(n)}, y^{(n)}) - \bar{X}^n_t(x^{(n)})\|^2 \right\}^{1/2} = 0, \tag{5.6}
\]
and the above limit is uniform with respect to $n \in \mathbb{N}$. By triangle inequality we obtain
\[
\mathbb{E}\|\bar{X}_t(x) - \bar{X}^n_t(x^{(n)})\| \leq \mathbb{E}\|\bar{X}_t(x) - X_t^{x}(x,y)\| + \mathbb{E}\|X_t^{x}(x,y) - X_t^{x,n}(x^{(n)}, y^{(n)})\| + \mathbb{E}\|X_t^{x,n}(x^{(n)}, y^{(n)}) - \bar{X}^n_t(x^{(n)})\|,
\]
which, together with (5.1) and (5.6), yields
\[
\lim_{n \to \infty} \mathbb{E}\|\bar{X}_t(x) - \bar{X}^n_t(x^{(n)})\| = 0. \tag{5.7}
\]

**Remark 5.1.** Note that for any $T > 0$ and $\phi \in C^3_b(H, \mathbb{R})$ we have
\[
|\mathbb{E}\phi(X_T^{x}(x,y)) - \mathbb{E}\phi(\bar{X}_T(x))| \leq |\mathbb{E}\phi(X_T^{x}(x,y)) - \mathbb{E}\phi(X_T^{x,n}(x^{(n)}, y^{(n)}))| + |\mathbb{E}\phi(X_T^{x,n}(x^{(n)}, y^{(n)})) - \mathbb{E}\phi(\bar{X}_T^{x}(x^{(n)}))| + |\mathbb{E}\phi(\bar{X}_T^{x}(x^{(n)})) - \mathbb{E}\phi(\bar{X}_T(x))|.
\]

According to the approximation results (5.6) and (5.7) the first and last terms above converge to zero as $n$ goes to infinity. In order to prove Theorem 3.7 we have only to show that for any $r \in (0,1)$, it holds
\[
|\mathbb{E}\phi(X_T^{x,n}(x^{(n)}, y^{(n)})) - \mathbb{E}\phi(\bar{X}_T^{x}(x^{(n)}))| \leq C_{\theta,r,T,\phi,x,y} \epsilon^{1-r} \tag{5.8}
\]
for some constant $C_{\theta,r,T,\phi,x,y}$ independent of the dimension index $n$. 

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Remark 5.2. For all $n \in \mathbb{N}$, the regular conditions on drift coefficients $F$ and $G$ presented in (H.1) and (H.2) are still valid for $F_n$ and $G_n$, respectively, but replacing $H$ by $H^{(n)}$. In particular, the boundedness on derivatives associated with $F_n$ and $G_n$ are uniform with respect to dimension $n$. As a result, all properties satisfied by $(X_1^n, Y_1^n)$, $Y_1^n$ and $P_1^n$ are still valid for $(X_t^{x,n}, Y_t^{x,n})$, $Y_t^{x,n}$ and for the transition semigroup $P_t^{x,n}$ corresponding to (5.4), respectively. Moreover, all estimates for $(X_1^n, Y_1^n)$, $Y_1^n$ and $P_1^n$ are uniform with respect to $n \in \mathbb{N}$. Similarly, $F_n$ and $X_t^n$ inherit all properties described for $F$ and $X_t$, respectively, with all estimates uniform with respect to $n \in \mathbb{N}$.

Remark 5.3. In what follows, the letter $C$ below with or without subscripts will denote generic positive constants independent of $\epsilon$ and dimension $n$, whose value may change from one line to another.

Let $\phi$ be the test function as in Theorem 3.1. As usual, we use the notation $(X_1^{x,n}(x,y), Y_1^{x,n}(x,y))$ to denote the solution to equation (5.5), and initial value $(X_0^{x,n}(x,y), Y_0^{x,n}(x,y)) = (x,y) \in H^{(n)} \times H^{(n)}$. For any $n \in \mathbb{N}$, we define a function $u_n^\epsilon : [0,T] \times H^{(n)} \times H^{(n)} \to \mathbb{R}$ by

$$u_n^\epsilon(t,x,y) = \mathbb{E}\phi(X_1^{x,n}(x,y)).$$

We now introduce two differential operators associated with the fast variable system (5.2) and slow variable system (5.1) in finite dimensional space, respectively:

$$\mathcal{L}_1^n \varphi(y) = \left( A_n y + G_n(x,y), D_y \varphi(y) \right)_H$$

$$+ \frac{1}{2} \sigma_2^2 \text{Tr}(D^2_{yy} \varphi(y) Q_{2,n}^1 (Q_{2,n}^2)^*), \varphi \in C_b^2(H^{(n)}, \mathbb{R})$$

and

$$\mathcal{L}_2^n \varphi(x) = \left( A_n x + F_n(x,y), D_x \varphi(x) \right)_H$$

$$+ \frac{1}{2} \sigma_1^2 \text{Tr}(D^2_{xx} \varphi(x) Q_{1,n}^1 (Q_{1,n}^2)^*), \varphi \in C_b^2(H^{(n)}, \mathbb{R}),$$

where $Q_{1,n} := Q_1 P_n$ and $Q_{2,n} := Q_2 P_n$ for any $n \in \mathbb{N}$. It is known that $u_n^\epsilon$ is a solution to the forward Kolmogorov equation:

$$\begin{cases}
\frac{\partial}{\partial t} u_n^\epsilon(t,x,y) = \mathcal{L}_1^n u_n^\epsilon(t,x,y), \\
u_n^\epsilon(0,x,y) = \phi(x),
\end{cases}$$

(5.9)

where $\mathcal{L}_1^{x,n} := \frac{1}{\tau} \mathcal{L}_1^n + \mathcal{L}_2^n$.

Also recall the Kolmogorov operator for the averaged system (5.3) is defined as

$$\mathcal{L}_n \varphi(x) = \left( A_n x + F_n(x), D_x \varphi(x) \right)_H$$

$$+ \frac{1}{2} \sigma_1^2 \text{Tr}(D^2_{xx} \varphi(x) Q_{1,n}^1 (Q_{1,n}^2)^*), \varphi \in C_b^2(H^{(n)}, \mathbb{R}).$$
If we set
\[ \bar{u}_n(t, x) = \mathbb{E} \phi(\bar{X}_n^T(x)), \]
we have
\[ \begin{cases} \frac{\partial}{\partial t} \bar{u}_n(t, x) = \mathcal{L}_n \bar{u}_n(t, x), \\ \bar{u}_n(0, x) = \phi(x). \end{cases} \tag{5.10} \]

Then the weak difference at time $T$ can be rewritten as
\[ \mathbb{E} \phi(X_{T,n}^\epsilon) - \mathbb{E} \phi(\bar{X}_n^T) = u_\epsilon^n(T, x, y) - \bar{u}_n(T, x). \]

Henceforth, for the sake of brevity, we will omit to write the dependence of the temporal variable $t$ and spatial variables $x$ and $y$ in some occasion. For example, we often write $u_\epsilon^n$ instead of $u_\epsilon^n(t, x, y)$. Our aim is to seek an expansion for $u_\epsilon^n(T, x, y)$ with the form
\[ u_\epsilon^n = u_{0,n} + \epsilon u_{1,n} + r_\epsilon^n, \tag{5.11} \]
where $u_{0,n}$ and $u_{1,n}$ are smooth functions which will be constructed below, and $r_\epsilon^n$ is the remainder term.

The rest of this section is devoted to the proof of Theorem 3.1. We will proceed in several steps, which have been structured as subsections.

5.1. The leading term

Let us first determine the leading term. Now, substituting expansions (5.11) into (5.9) yields
\[ \begin{align*} \frac{\partial u_{0,n}}{\partial t} + \epsilon \frac{\partial u_{1,n}}{\partial t} + \frac{\partial r_\epsilon^n}{\partial t} &= \frac{1}{\epsilon} \mathcal{L}_1^n u_{0,n} + \mathcal{L}_1^n u_{1,n} + \frac{1}{\epsilon} \mathcal{L}_1^n r_\epsilon^n \\ &\quad + \mathcal{L}_2^n u_{0,n} + \epsilon \mathcal{L}_2^n u_{1,n} + \mathcal{L}_2^n r_\epsilon^n. \end{align*} \]

By comparing orders of $\epsilon$, we obtain
\[ \mathcal{L}_1^n u_{0,n} = 0 \tag{5.12} \]
and
\[ \frac{\partial u_{0,n}}{\partial t} = \mathcal{L}_1^n u_{1,n} + \mathcal{L}_2^n u_{0,n}. \tag{5.13} \]

It follows from (5.12) that $u_{0,n}$ is independent of $y$, which means
\[ u_{0,n}(t, x, y) = u_{0,n}(t, x). \]

We also impose the initial condition $u_{0,n}(0, x) = \phi(x)$. Since $\mu^{x,n}$ is the invariant measure of a Markov process with generator $\mathcal{L}_1^n$, we have
\[ \int_{H^{(n)}} \mathcal{L}_1^n u_{1,n}(t, x, y) \mu^{x,n}(dy) = 0, \]
which, by invoking (5.13), implies
\[
\frac{\partial u_{0,n}(t,x)}{\partial t} = \int_{H^{(n)}} \frac{\partial u_{0,n}(t,x)}{\partial t} \mu_{x,n}(dy)
\]
\[
= \int_{H^{(n)}} \mathcal{L}_{2}^{n} u_{0,n}(t,x) \mu_{x,n}(dy)
\]
\[
= \left( A_{n} u_{0,n}(t,x) + \int_{H^{(n)}} F_{n}(x,y) \mu_{x,n}(dy), D_{x} u_{0,n}(t,x) \right)_{H}
\]
\[
+ \frac{1}{2} \sigma_{2}^{2} Tr(D_{xx}^{2} u_{0,n}(t,x) Q_{1,n}^{\frac{1}{2}}(Q_{1,n}^{\frac{1}{2}})^{*})
\]
\[
= \mathcal{L}^{n} u_{0,n}(t,x),
\]
so that \(u_{0,n}\) and \(\bar{u}_{n}\) satisfy the same evolution equation. By using a uniqueness argument, such \(u_{0,n}\) has to coincide with the solution \(\bar{u}_{n}\) and we have the following lemma:

**Lemma 5.1.** Assume (H.1) and (H.2). Then for any \(x,y \in H^{(n)}\) and \(T > 0\), we have \(u_{0,n}(T,x,y) = \bar{u}_{n}(T,x,y)\).

5.2. **Construction of \(u_{1,n}\)**

Let us proceed to carry out the construction of \(u_{1,n}\). Thanks to Lemma 5.1 and (5.10), the equation (5.13) can be rewritten as
\[
\bar{L}^{n} \bar{u}_{n} = L_{1}^{n} u_{1,n} + L_{2}^{n} \bar{u}_{n},
\]
and hence we get an elliptic equation for \(u_{1,n}\) with form
\[
L_{1}^{n} u_{1,n}(t,x,y) = \left( \bar{F}_{n}(x) - F_{n}(x,y), D_{x} \bar{u}_{n}(t,x) \right)_{H} := -\rho_{n}(t,x,y),
\]
where \(\rho_{n}\) is of class \(C^{2}\) with respect to \(y\), with uniformly bounded derivatives. Moreover, it satisfies for any \(t \geq 0\) and \(x \in H^{(n)}\),
\[
\int_{H^{(n)}} \rho_{n}(t,x,y) \mu_{x,n}(dy) = 0.
\]
For any \(y \in H^{(n)}\) and \(s > 0\) we have
\[
\frac{\partial}{\partial s} P_{s,n}^{x} \rho_{n}(t,x,y) = \left( A_{n} y + G_{n}(x,y), D_{y} P_{s,n}^{x} \rho_{n}(t,x,y) \right)_{H}
\]
\[
+ \frac{1}{2} \sigma_{2}^{2} Tr[D_{yy}^{2} P_{s,n}^{x} \rho_{n}(t,x,y)]Q_{2,n}^{\frac{1}{2}}(Q_{2,n}^{\frac{1}{2}})^{*},
\]
here
\[
P_{s,n}^{x} \rho_{n}(t,x,y) = \mathbb{E} \rho_{n}(t,x,Y^{x,n}_{s}(y))
\]
satisfying
\[
\lim_{s \to +\infty} \mathbb{E} \rho_{n}(t,x,Y^{x,n}_{s}(y)) = \int_{H^{(n)}} \rho_{n}(t,x,z) \mu_{x,n}(dz) = 0. \quad (5.14)
\]
Indeed, by the invariant property of $\mu^{x,n}$ and Lemma 6.5 in the next section,
\[
\begin{align*}
E\rho_n(t, x, Y_s^{x,n}(y)) - \int_{H^{(n)}} \rho_n(t, x, z) \mu^{x,n}(dz) & = \left| \int_{H^{(n)}} E[\rho_n(t, x, Y_s^{x,n}(y)) - \rho_n(t, x, Y_s^{x,n}(z)) \mu^{x,n}(dz)] \right| \\
& \leq \int_{H^{(n)}} \left| E\left[ F_n(x, Y_s^{x,n}(z)) - F_n(x, Y_s^{x,n}(y)), D_x u_n(t, x) \right] \right| H \mu^{x,n}(dz) \\
& \leq C \int_{H^{(n)}} \left| Y_s^{x,n}(z) - Y_s^{x,n}(y) \right| \mu^{x,n}(dz).
\end{align*}
\]
This, in view of (3.4) and (3.3), yields
\[
\begin{align*}
\left\| E\rho_n(t, x, Y_s^{x,n}(y)) - \int_{H^{(n)}} \rho_n(t, x, z) \mu^{x,n}(dz) \right\| & \leq C e^{-\frac{\beta_s}{2}} (1 + \|x\| + \|y\|),
\end{align*}
\]
which implies the equality (5.14). Therefore, we get
\[
\begin{align*}
& \left( A_n y + G_n(x, y), D_y \int_0^{+\infty} P_s^{x,n} \rho_n(t, x, y) ds \right)_H \\
& + \frac{1}{2} \sigma_2^2 Tr[D_yy] \int_0^{+\infty} (P_s^{x,n} \rho_n(t, x, y)) Q^{x,n}_2 (Q^{x,n}_2)^* ds \\
& = \int_0^{+\infty} \frac{\partial}{\partial s} P_s^{x,n} \rho_n(t, x, y) ds \\
& = \lim_{s \to +\infty} E\rho_n(t, x, Y_s^{x,n}(y)) - \rho_n(t, x, y) \\
& = \int_{H^{(n)}} \rho_n(t, x, y) \mu^{x,n}(dy) - \rho_n(t, x, y) \\
& = -\rho_n(t, x, y),
\end{align*}
\]
which means $L_1^n(\int_0^{+\infty} P_s^{x,n} \rho_n(t, x, y) ds) = -\rho_n(t, x, y)$. Therefore, we can set
\[
\begin{align*}
u_{1,n}(t, x, y) = \int_0^{+\infty} E\rho_n(t, x, Y_s^{x,n}(y)) ds.
\end{align*}
\]
Lemma 5.2. Assume (H.1) and (H.2). Then for any $x, y \in H^{(n)}$ and $T > 0$, we have
\[
\begin{align*}
|\nu_{1,n}(t, x, y)| \leq C_T (1 + \|x\| + \|y\|), \quad t \in [0, T].
\end{align*}
\]
Proof. As known from (5.15), we have
\[
\begin{align*}
\nu_{1,n}(t, x, y) = \int_0^{+\infty} E\left( F_n(x) - F_n(x, Y_s^{x,n}(y)), D_x u_n(t, x) \right)_H ds.
\end{align*}
\]
This implies that
\[ |u_{1,n}(t,x,y)| \leq \int_{0}^{+\infty} \|\bar{F}_n(x) - \mathbb{E}[F_n(x,Y_s^n(y))]\| \cdot \|D_x u_n(t,x)\| ds. \]

Then, in view of Lemma 3.5 and (3.6), this implies:
\[ |u_{1,n}(t,x,y)| \leq C_T (1 + \|x\| + \|y\|) \int_{0}^{+\infty} e^{-\frac{s}{2}} ds \leq C_T (1 + \|x\| + \|y\|). \]

5.3. Determination of remainder \( r^n \)

Once \( u_{0,n} \) and \( u_{1,n} \) have been determined, we can carry out the construction of the remainder \( r^\varepsilon_n \). It is known that
\[ (\partial_t - L^{\varepsilon,n}) u^\varepsilon_n = 0, \]
which, together with (5.12) and (5.13), implies
\[ (\partial_t - L^{\varepsilon,n}) r^\varepsilon_n = -(\partial_t - L^{\varepsilon,n}) u_{0,n} - \epsilon (\partial_t - L^{\varepsilon,n}) u_{1,n} \]
\[ = -(\partial_t - \frac{1}{\epsilon} L^1_n - \frac{1}{\epsilon} L^2_n) u_{0,n} - \epsilon (\partial_t - \frac{1}{\epsilon} L^1_n - \frac{1}{\epsilon} L^2_n) u_{1,n} \]
\[ = \epsilon (L^2_n u_{0,n} - \partial_t u_{1,n}). \]

In order to estimate the remainder term \( r^\varepsilon_n \) we need the following crucial lemmas.

**Lemma 5.3.** Assume that \( x,y \in H^n \). Then, under conditions (H.1) and (H.2), for any \( T > 0 \) and \( \theta \in (0,1] \) we have
\[ \left| \frac{\partial u_{1,n}}{\partial t}(t,x,y) \right| \leq C_T (1 + \frac{1}{t} + t^{\theta-1})(1 + \|x\| + \|y\| + \|x\|(-A_n)e)^2, \quad t \in [0,T]. \]

**Proof.** According to (5.15), we have
\[ \frac{\partial u_{1,n}}{\partial t}(t,x,y) = \int_{0}^{+\infty} \mathbb{E} \left( \bar{F}_n(x) - F_n(x,Y_s^n(y)), \frac{\partial}{\partial t} D_x u_n(t,x) \right)_H ds. \quad (5.17) \]

For any \( h \in H^n \),
\[ D_x u_n(t,x) \cdot h = \mathbb{E}[\phi'(X^n_t(x)) \cdot D_x X^n_t(x) \cdot h] \]
\[ = \mathbb{E} \left( \phi'(X^n_t(x)), \eta_{t}^{h,x,n} \right)_H, \quad (5.18) \]

here \( \eta_{t}^{h,x,n} \) is the mild solution (also strong solution) of variation equation corresponding to the problem \( (5.5) \) in form
\[ \left\{ \begin{array}{l}
\frac{d\eta_{t}^{h,x,n}}{dt} = \left( A_n \eta_{t}^{h,x,n} + \bar{F}_n'(X^n_t(x)) \cdot \eta_{t}^{h,x,n} \right) dt, \\
\eta_{0}^{h,x,n} = h.
\end{array} \right. \]
Keep in mind that $\bar{X}_t^n(x)$ is the strong solution of equation \ref{eq:5.5} with initial value $X_0^n(x) = x$. By Itô's formula in finite dimensional spaces, we get

$$\phi'(\bar{X}_t^n(x)) = \phi'(x) + \int_0^t \phi''(\bar{X}_s^n(x)) \cdot [A_n \bar{X}_s^n(x) + \bar{F}_n(\bar{X}_s^n(x))] ds$$

$$+ \int_0^t \phi''(\bar{X}_s^n(x)) dW_s^{1,n}$$

$$+ \frac{1}{2} \sum_{k=1}^n \int_0^t \phi'''(\bar{X}_s^n(x)) \cdot (\sqrt{\lambda_{1,k}} e_k, \sqrt{\lambda_{1,k}} e_k) ds,$$

where $W_t^{1,n} := \sum_{k=1}^n \sqrt{\lambda_{1,k}} B_t^{i(k)} e_k$ denotes the $Q^{1,n}$–Wiener process in $H^{(n)}$.

Then, by using again Itô’s formula, after taking the expectation we have

$$\mathbb{E}\left(\phi'(\bar{X}_t^n(x)), \eta_{t,x}^{h,n}\right)_H$$

$$= \left(\phi'(x), h\right)_H$$

$$+ \mathbb{E} \int_0^t \left(\eta_{s,x}^{h,n}, \phi''(\bar{X}_s^n(x)) \cdot [A_n \bar{X}_s^n(x) + \bar{F}_n(\bar{X}_s^n(x))] \right)_H ds$$

$$+ \mathbb{E} \int_0^t \left(\phi'(\bar{X}_s^n(x)), A_n \eta_{s,x}^{h,n} + \bar{F}_n'(\bar{X}_s^n(x)) \cdot \eta_{s,x}^{h,n}\right)_H ds$$

$$+ \frac{1}{2} \mathbb{E} \sum_{k=1}^n \int_0^t \left(\phi'''(\bar{X}_s^n(x)) \cdot (\sqrt{\lambda_{1,k}} e_k, \sqrt{\lambda_{1,k}} e_k), \eta_{s,x}^{h,n}\right)_H ds.$$ 

Now, returning to \ref{eq:5.18} and differentiating with respect to $t$, we obtain

$$\frac{\partial}{\partial t} (D_x \bar{u}_n(t,x) \cdot h) = \mathbb{E}\left(\eta_{t,x}^{h,n}, \phi''(\bar{X}_t^n(x)) \cdot [A_n \bar{X}_t^n(x) + \bar{F}_n(\bar{X}_t^n(x))] \right)_H$$

$$+ \mathbb{E}\left(\phi'(\bar{X}_t^n(x)), A_n \eta_{t,x}^{h,n} + \bar{F}_n'(\bar{X}_t^n(x)) \cdot \eta_{t,x}^{h,n}\right)_H$$

$$+ \frac{1}{2} \mathbb{E} \sum_{k=1}^n \left(\phi'''(\bar{X}_t^n(x)) \cdot (\sqrt{\lambda_{1,k}} e_k, \sqrt{\lambda_{1,k}} e_k), \eta_{t,x}^{h,n}\right)_H,$$

so that

$$\left|\frac{\partial}{\partial t} (D_x \bar{u}_n(t,x) \cdot h)\right| \leq C \mathbb{E} \left[||\eta_{t,x}^{h,n}|| (||A_n \bar{X}_t^n(x)|| + ||\bar{F}_n(\bar{X}_t^n(x))||)\right]$$

$$+ C \mathbb{E} ||A_n \eta_{t,x}^{h,n}|| + C \mathbb{E} ||\bar{F}_n'(\bar{X}_t^n(x)) \cdot \eta_{t,x}^{h,n}||$$

$$+ \mathbb{E} \sum_{k=1}^\infty \lambda_k ||\eta_{t,x}^{h,n}||.$$

Then, as \ref{eq:3.8} holds, by using Lemma \ref{lem:4.5}, Lemma \ref{lem:6.1} and Lemma \ref{lem:6.3}, it follows

$$\left|\frac{\partial}{\partial t} (D_x \bar{u}_n(t,x) \cdot h)\right| \leq C_T ||h|| (t^{\beta-1}||x||(-A_n)^\theta + 1 + ||x||)$$

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Let directly, we have Lemma 5.4. Assume \( H \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \). Then, under conditions (H.1) and (H.2), for any \( T > 0 \) we have

\[
|L_2^u u_{1,n}(t, x, y)| \leq C_T \left( 1 + \|A_n x\| + \|x\| + \|y\| \right) \left( 1 + \|x\| + \|y\| \right), \quad t \in [0, T].
\]

Proof. As known, for any \( x \in H^{(n)} \) it holds

\[
L_2^u u_{1,n}(t, x, y) = \left( A_n x + F_n(x, y), D_x u_{1,n}(t, x, y) \right)_H + \frac{1}{2} \sigma_2^2 \text{Tr} \left( D_{xx} u_{1,n}(t, x, y) Q_{1,n}^2 (Q_{1,n}^2)^* \right).
\]

We will carry out the estimate of \( |L_2^u u_{1,n}(t, x, y)| \) in two steps. (Step 1) Estimate of \( \left( A_n x + F_n(x, y), D_x u_{1,n}(t, x, y) \right)_H \).

For any \( k \in H^{(n)} \), we have

\[
D_x u_{1,n}(t, x, y) \cdot k = \left( D_x u_{1,n}(t, x, y) \right)_H \cdot k = \int_0^{t} \left( D_x(F_n(x) - EF_n(x, Y^{x,n}_s(y))) \cdot k, D_x u_{1,n}(t, x) \right)_H ds
\]

\[
+ \int_0^{t} \left( F_n(x) - EF_n(x, Y^{x,n}_s(y)), D_{xx} u_{1,n}(t, x) \cdot k \right)_H ds
\]

\[
:= I_{1,n}(t, x, y, k) + I_{2,n}(t, x, y, k).
\]

Directly, we have

\[
|I_{1,n}(t, x, y, k)|
\]

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By making use of (6.8), the above yields

\[
H \leq \int_0^{+\infty} \left|(D_x(\bar{F}_n(x) - EF_n(x, Y_s^{x,n}(y))) \cdot k, D_x\bar{u}_n(t, x))\right|_H ds.
\]

By differentiating twice with respect to \( x \), which means

\[
|I_{1,n}(t, x, y, k)| \leq C\|k\| \cdot \|D_x\bar{u}_n(t, x)\| \cdot \int_0^{+\infty} e^{-cs}(1 + \|x\| + \|y\|)ds
\]

\[
\leq C\|k\| \cdot \|D_x\bar{u}_n(t, x)\|(1 + \|x\| + \|y\|)
\]

\[
\leq C\|k\|(1 + \|x\| + \|y\|), \quad (5.19)
\]

where we used Lemma 6.5 in the last step. By Lemma 6.6 and (3.6), we have

\[
I_{1,n}(t, x, y, k) \leq C\|k\| \cdot \|D_x\bar{u}_n(t, x)\| \cdot \int_0^{+\infty} e^{-\frac{s}{2}}ds
\]

\[
\leq C\|k\|(1 + \|x\| + \|y\|).
\]

Together with (5.19), this allows us to get

\[
|D_xu_{1,n}(t, x, y) \cdot k| \leq C\|k\|(1 + \|x\| + \|y\|)
\]

which means

\[
\left|(A_n x + F_n(x, y), D_xu_{1,n}(t, x, y))\right|_H \leq C(1 + \|A_n x\| + \|x\| + \|y\|)(1 + \|x\| + \|y\|).
\]

(Step 2) Estimate of \( Tr \left(D_{xx}^2u_{1,n}(t, x, y)Q_{1,n}^T(Q_{1,n}^T)^*\right)\).

By differentiating twice with respect to \( x \in H^{(n)}\) in \( u_{1,n}(t, x, y)\), for any \( x, h, k \in H^{(n)}\) we have

\[
D_{xx}u_{1,n}(t, x, y) \cdot (h, k)
\]

\[
= \int_0^{+\infty} (D_x^2(\bar{F}_n(x) - EF_n(x, Y_s^{x,n}(y))) \cdot (h, k), D_x\bar{u}_n(t, x))_H ds
\]

\[
+ \int_0^{+\infty} \left(D_x(\bar{F}_n(x) - EF_n(x, Y_s^{x,n}(y))) \cdot h, D_x^2\bar{u}_n(t, x) \cdot k\right)_H ds
\]

\[
+ \int_0^{+\infty} \left(D_x(\bar{F}_n(x) - EF_n(x, Y_s^{x,n}(y))) \cdot k, D_x^2\bar{u}_n(t, x) \cdot h\right)_H ds
\]

\[
+ \int_0^{+\infty} \left(\bar{F}_n(x) - EF_n(x, Y_s^{x,n}(y)), D_x^3\bar{u}_n(t, x) \cdot (h, k)\right)_H ds
\]

\[
:= \sum_{i=1}^{4} J_{i,n}(t, x, y, h, k).
\]
By taking Lemma 6.9 and Lemma 6.5 into account, we can deduce
\[ |J_{1,n}(t, x, y, h, k)| \leq C \|h\| \cdot \|k\| \cdot \int_0^{\infty} e^{-cs}(1 + \|x\| + \|y\|) ds \]
\[ \leq C \|h\| \cdot \|k\| (1 + \|x\| + \|y\|). \] (5.21)

Next, thanks to Lemma 6.8 and Lemma 6.6 it holds
\[ |J_{2,n}(t, x, y, h, k)| \leq C \|h\| \cdot \|k\| \cdot \int_0^{\infty} e^{-cs}(1 + \|x\| + \|y\|) ds \]
\[ \leq C \|h\| \cdot \|k\| (1 + \|x\| + \|y\|). \] (5.22)

Parallel to (5.22), we can obtain the same estimate for \( J_{3,n}(t, x, y, h, k) \), that is,
\[ |J_{3,n}(t, x, y, h, k)| \leq C \|h\| \cdot \|k\| (1 + \|x\| + \|y\|). \] (5.23)

Thanks to Lemma 6.7 and (3.6), we get
\[ |J_{4,n}(t, x, y, h, k)| \leq C \|h\| \cdot \|k\| \cdot \int_0^{\infty} e^{-\beta s}(1 + \|x\| + \|y\|) ds \]
\[ \leq C \|h\| \cdot \|k\| (1 + \|x\| + \|y\|). \] (5.24)

Collecting together (5.21), (5.22), (5.23) and (5.24), we obtain
\[ |D_{xx}^2 u_{1,n}(t, x, y) \cdot (h, k)| \leq C \|h\| \cdot \|k\| (1 + \|x\| + \|y\|), \]
so that, as the operator \( Q_1 \) has finite trace (see (2.4)), we get
\[ \left| \text{Tr} \left( D_{xx}^2 u_{1,n}(t, x, y) Q_1^{\frac{1}{r}}(Q_1^{\frac{1}{r}})^* \right) \right| \]
\[ = \sum_{k=1}^n \left| D_{xx}^2 u_{1,n}(t, x, y) \cdot (\sqrt{\lambda_{1,k} e_k}, \sqrt{\lambda_{1,k} e_k}) \right| \]
\[ \leq C (1 + \|x\| + \|y\|). \] (5.25)

Finally, by taking inequalities (5.21) and (5.25) into account, we can conclude the proof of the lemma.

As a consequence of Lemma 5.3 and 5.4, we have the following fact for the remainder term \( r_n^e \).

Lemma 5.5. Under the conditions of Lemma 5.3, for any \( r \in (0, \frac{1}{4}) \) we have
\[ r_n^e(T, x, y) \leq C_{r,T,\theta} \epsilon^{1-2r}(1 + \|x\|^2 + \|y\|^2 + \|x\|^2_{\epsilon^{1-\gamma} A_n}). \]
Proof. By a variation of constant formula, we have

\[ r_n^ε(T,x,y) = E[r_n^ε(δ_n,X_{T-δ_n}^ε(x,y),Y_{T-δ_n}^ε(x,y))] \]

\[ + \epsilon E \left[ \int_{δ_n}^{T} (L_2 u_{1,n} - \frac{∂u_{1,n}}{∂s})(s,X_{T-s}^ε(x,y),Y_{T-s}^ε(x,y)) ds \right], \quad (5.26) \]

where \( δ_n \in (0, \frac{T}{2}) \) is a constant, only depending on \( ε > 0 \), to be chosen later. Now, we estimate the two terms in the right hand side of (5.26). Firstly, note that \( u_{n}^ε(0,x,y) = \bar{u}_n(0,x) \), it holds

\[ r_n^ε(δ_n,x,y) = u_n^ε(δ_n,x,y) - \bar{u}_n(δ_n,x,y) - ε u_{1,n}(δ_n,x,y) \]

\[ = -ε u_{1,n}(δ_n,x,y) + [u_n^ε(δ_n,x,y) - u_n^ε(0,x,y)] \]

\[ -[\bar{u}_n(δ_n,x) - \bar{u}_n(0,x)]. \]

By lemma 4.2 we have

\[ |ε u_{1,n}(δ_n,x,y)| \leq C_T ε(1 + \|x\| + \|y\|). \quad (5.27) \]

By using Itô’s formula and taking the expectation we obtain

\[ u_n^ε(δ_n,x,y) - u_n^ε(0,x,y) \]

\[ = E \int_{0}^{δ_n} φ'(X_{s}^ε(x,y)) \cdot [A_n X_{s}^ε(x,y) + F_n(X_{s}^ε(x,y),Y_{s}^ε(x,y))] ds \]

\[ + \frac{1}{2} E \sum_{k=1}^{n} \int_{0}^{δ_n} φ''(X_{s}^ε(x,y)) \cdot \left( \sqrt{λ_{1,k} e_k}, \sqrt{λ_{1,k} e_k} \right). \]

Then, due to Lemma 2.1 and 4.3 for any \( r \in (0, \frac{1}{4}) \) we have

\[ |u_n^ε(δ_n,x,y) - u_n^ε(0,x,y)| \]

\[ \leq C \int_{0}^{δ_n} \left[ E\|A_n X_{s}^ε(x,y)\| + 1 + E\|X_{s}^ε(x,y)\| + E\|Y_{s}^ε(x,y)\| \right] ds \]

\[ + C Tr(Q_1) δ_n \]

\[ \leq C_{r,T}(δ_n + \frac{δ_n}{θ} + \frac{δ_n}{ε})(1 + \|x\|(-A_n)^r + \|x\| + \|y\|). \quad (5.28) \]

By using again Itô’s formula, we get

\[ \bar{u}_n(δ_n,x) - \bar{u}_n(0,x) \]

\[ = E \int_{0}^{δ_n} φ'(X_{s}^n(x)) \cdot [A_n X_{s}^n(x) + F_n(X_{s}^n(x))] ds \]

\[ + \frac{1}{2} E \sum_{k=1}^{n} \int_{0}^{δ_n} φ''(X_{s}^n(x)) \cdot \left( \sqrt{λ_{1,k} e_k}, \sqrt{λ_{1,k} e_k} \right). \]

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Then, thanks to Lemma 4.3 and (5.10) it holds
\[
|\bar{u}_n(\delta, x) - \bar{u}_n(0, x)| \\
\leq C \int_0^{\delta} \left[ \mathbb{E}\|A_n \mathcal{X}_n^\epsilon(x)\| + 1 + \mathbb{E}\|\mathcal{X}_n^\epsilon(x)\| \right] ds \\
+ CTR(Q_1)\delta \\
\leq C_T(\delta + \delta^0 + \delta^0 \epsilon)(1 + \|x\| (-A_n)^\epsilon + \|x\|),
\]
which, in view of (5.27) and (5.28), means that
\[
|r_n^\epsilon(\delta, x, y)| \leq C_{r,T}(\epsilon + \delta + \delta^0 \epsilon + \delta^0 \epsilon^2)(1 + \|x\| (-A_n)^\epsilon + \|x\| + \|y\|),
\]
so that, due to Lemma 2.1 and 4.3 this easily implies that
\[
\mathbb{E}[r_n^\epsilon(\delta, X_{T-\delta}^{\epsilon,n}(x, y), Y_{T-\delta}^{\epsilon,n}(x, y))] \\
\leq C_{r,T}(\epsilon + \delta + \delta^0 \epsilon + \delta^0 \epsilon^2)(1 + \mathbb{E}\|X_{T-\delta}^{\epsilon,n}(x, y)\| (-A_n)^\epsilon + \mathbb{E}\|Y_{T-\delta}^{\epsilon,n}(x, y)\|) \\
\leq C_{r,T}(\epsilon + \delta + \delta^0 \epsilon + \delta^0 \epsilon^2)(|T - \delta|^\epsilon + |T - \delta|^2)(1 + \|x\| + \|y\|)(1 + \frac{1}{\epsilon^2}) \\
\leq C_{r,T,\theta}(\epsilon + \delta + \delta^0 \epsilon + \delta^0 \epsilon^2)(1 + \frac{1}{\epsilon^2})(|x| (-A)^\epsilon + 1 + \|x\| + \|y\|).
\]
If we pick \(\delta = \epsilon^2 \leq \epsilon\), we get
\[
\mathbb{E}[r_n^\epsilon(\delta, X_{T-\delta}^{\epsilon,n}(x, y), Y_{T-\delta}^{\epsilon,n}(x, y))] \\
\leq C_{r,T,\theta}(1 + |x| (-A_n)^\epsilon + \|x\| + \|y\|).
\]
The proof of Theorem 3.1 is completed.

\[
\begin{align*}
+C_{r,T} & \int_{\delta_s}^T \left[ \mathbb{E} \left\| A_n X_{T-s}^{\epsilon,n}(x,y) \right\|^2 \right]^{\frac{1}{2}}
\cdot \left[ \mathbb{E} (1 + \| X_{T-s}^{\epsilon,n}(x,y) \| + \| X_{T-s}^{\epsilon,n}(x,y) \| ) \right]^{\frac{1}{2}} ds
\leq C_{r,T} \epsilon (1 + \| x \|^2 + \| y \|^2 + \| x \|^2 (-A_\epsilon)^\theta)
\cdot \int_{\delta_s}^T \left( 1 + \frac{1}{c^r} + \frac{s}{\theta} + |T - s|^{\theta - 1} \right) ds
\leq C_{r,T} \epsilon (T + \frac{T^\theta}{\theta} + |\log T| + |\log(\delta_s)| + \frac{T}{c^r})
\cdot (1 + \| x \|^2 + \| y \|^2 + \| x \|^2 (-A_\epsilon)^\theta)
\leq C_{r,T} \epsilon^{1-r} (1 + \| x \|^2 + \| y \|^2 + \| x \|^2 (-A_\epsilon)^\theta),
\end{align*}
\]

which, together with \(6.20\), completes the proof.

\[\square\]

5.4. Proof of Theorem 3.1

Now we finish proof of main result introduced in Section 3

Proof. We stress that we need only to prove \(5.3\). With the notations introduced above, by Lemma 5.1, Lemma 5.2 and Lemma 5.5 for any \( r \in (0,1) \), \( x \in \mathcal{D}(\mathbb{R}) \) and \( y \in H \) we have

\[
\begin{align*}
\left| \mathbb{E} \phi(X_{T}^{\epsilon,n}(x^{(n)},y^{(n)})) - \mathbb{E} \phi(X_{T}^{\epsilon}(x^{(n)})) \right|
&= \left| u_n(T,x^{(n)},y^{(n)}) - \bar{u}_n(T,x^{(n)}) \right|
= \left| u_n(T,x^{(n)},y^{(n)}) - r_n(T,x^{(n)},y^{(n)}) \right|
\leq C_{r,T} \epsilon^{1-r} (1 + \| x^{(n)} \|^2 + \| y^{(n)} \|^2 + \| x^{(n)} \|^2 (-A_\epsilon)^\theta)
\leq C_{r,T} \epsilon^{1-r} (1 + \| x \|^2 + \| y \|^2 + \| x \|^2 (-A_\epsilon)^\theta),
\end{align*}
\]

where \( C_{r,T} \epsilon \) is a constant independent of the dimension \( n \).

The proof of Theorem 3.1 is completed.

\[\square\]

6. Appendix

In this section, we state and prove some technical lemmas used in the previous sections. We first study the differential dependence on initial datum for the solution \( X_t^{\epsilon,n}(x) \) of the averaged system \( \{5.5\} \). In what follows we denote by \( \eta_t^{h,x,n} \) the derivative of \( X_t^{\epsilon,n}(x) \) with respect to \( x \) along direction \( h \in H^{(n)} \).

**Lemma 6.1.** Under (H.1) and (H.2), for any \( x, h \in H^{(n)} \) and \( T > 0 \) there exists a constant \( C_T > 0 \) such that for any \( x, h \in H^{(n)} \),

\[
\| \eta_t^{h,x,n} \| \leq C_T \| h \|, \quad t \in [0,T].
\]

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Proof. Note that $\eta_{t}^{h,x,n}$ is the mild solution of the first variation equation associated with the problem (5.5):

$$
\begin{cases}
    d\eta_{t}^{h,x,n} = \left( A_{n}\eta_{t}^{h,x,n} + \bar{F}'(\bar{X}_{t}^{n}(x)) \cdot \eta_{t}^{h,x,n} \right) dt, \\
    \eta_{0}^{h,x,n} = h.
\end{cases}
$$

This means that $\eta_{t}^{h,x,n}$ is the solution of the integral equation

$$
\eta_{t}^{h,x,n} = S_{t,n}h + \int_{0}^{t} S_{t-s,n}\bar{F}'(\bar{X}_{s}(x)) \cdot \eta_{s}^{h,x,n} ds,
$$

and then, due to (3.8) and contractive property of $S_{t,n}$, we get

$$
\|\eta_{t}^{h,x,n}\| \leq \|h\| + C \int_{0}^{t} \|\eta_{s}^{h,x,n}\| ds.
$$

Then by Gronwall lemma it follows that

$$
\|\eta_{t}^{h,x,n}\| \leq C_{T}\|h\|, \quad t \in [0,T].
$$

Lemma 6.2. Under the conditions of Lemma 6.1, for any $T > 0$ and $r \in (0,1)$ there exists a constant $C_{r,T} > 0$ such that for any $x,h \in H^{(n)}$ and $0 < s \leq t \leq T$,

$$
\|\eta_{t}^{h,x,n} - \eta_{s}^{h,x,n}\| \leq C_{r,T}|t-s|^{1-r}(1 + \frac{1}{s^{1-r}})\|h\|.
$$

Proof. See Proposition B.5 in [4].

Lemma 6.3. Under the conditions of Lemma 6.1, for any $T > 0$ there exists a constant $C_{T} > 0$ such that for any $x,h,k \in H^{(n)}$,

$$
\|A_{n}\eta_{t}^{h,x,n}\| \leq C_{T}(1 + \frac{1}{t})(1 + \|x\|)\|h\|, \quad t \in [0,T].
$$

Proof. See Proposition B.6 in [4].

After we have study the first order derivative of $\bar{X}_{t}^{n}(x)$, we introduce the second order derivative of $\bar{X}_{t}^{n}(x)$ with respect to $x$ in directions $h,k \in H^{(n)}$ denoted by $\zeta^{h,k,x,n}$, which is the solution of the second variation equation

$$
\begin{cases}
    d\zeta_{t}^{h,k,x,n} = \left[ A_{n}\zeta_{t}^{h,k,x,n} + \bar{F}''(\bar{X}_{t}^{n}(x)) \cdot (\eta_{t}^{h,x,n},\eta_{t}^{k,x,n}) \\
    + \bar{F}'(\bar{X}_{t}^{n}(x)) \cdot \zeta_{t}^{h,k,x,n} \right] dt, \\
    \zeta_{0}^{h,k,x} = 0.
\end{cases}
$$

Lemma 6.4. Under the conditions of Lemma 6.1, for any $T > 0$ there exists a constant $C_{T} > 0$ such that for any $x,h,k \in H^{(n)}$,

$$
\|\zeta_{t}^{h,k,x,n}\| \leq C_{T}\|h\| \cdot \|k\|, \quad t \in [0,T].
$$
Proof. See Proposition B.7 in [4].

We now introduce the regular results for $\bar{u}_n(t,x)$ defined in Section 5.

**Lemma 6.5.** For any $T > 0$, there exists a constant $C_T > 0$ such that for any $x \in H^{(n)}$ and $t \in [0,T]$, we have

$$\|D_x \bar{u}_n(t,x)\| \leq C_{T,\phi}.$$ 

**Proof.** Note that for any $t \in [0,T]$ and $h \in H^{(n)}$,

$$D_x \bar{u}_n(t,x) \cdot h = E \left( \phi'(\bar{X}^n_t(x)), \eta^{h,x,n}_t \right)_H.$$ 

By Lemma 6.1 we have

$$|D_x \bar{u}_n(t,x) \cdot h| \leq C_T \sup_{z \in H} \|\phi'(z)\| \cdot \|h\|,$$ 

so that

$$\|D_x \bar{u}_n(t,x)\| \leq C_{T,\phi}.$$ 

**Lemma 6.6.** For any $T > 0$, there exists a constant $C_{T,\phi} > 0$ such that for any $x,h,k \in H^{(n)}$ and $t \in [0,T]$, we have

$$|D^2_{xx} \bar{u}_n(t,x) \cdot (h,k)| \leq C_{T,\phi} \|h\| \cdot \|k\|.$$ 

**Proof.** For any $h,k \in H^{(n)}$, we have

$$D^2_{xx} \bar{u}_n(t,x) \cdot (h,k) = E \left[ \phi''(\bar{X}^n_t(x)) \cdot (\eta^{h,x,n}_t, \eta^{k,x,n}_t) + \phi'(\bar{X}^n_t(x)) \cdot \zeta^{h,k,x,n}_t \right],$$ 

where $\zeta^{h,k,x,n}$ is governed by variation equation (6.1). By invoking Lemma 6.1 and Lemma 6.4, we can get

$$|D^2_{xx} \bar{u}_n(t,x) \cdot (h,k)| \leq C_{T,\phi} \|h\| \cdot \|k\|.$$ 

**Lemma 6.7.** For any $T > 0$, there exists $C_T > 0$ such that for any $x,h,k,l \in H^{(n)}$ and $t \in [0,T]$, we have

$$D^3_{xxx} \bar{u}_n(t,x) \cdot (h,k,l) \leq C_T \|h\| \cdot \|k\| \cdot \|l\|.$$ 

Finally, we introduce some regular results which is crucial in order to prove some important estimates in Section 5.
Lemma 6.8. There exist constants $C, c > 0$ such that for any $x, y, h \in H^{(n)}$ and $t > 0$ it holds
\[
\|D_x(\hat{F}_n(x) - \mathbb{E}F_n(x, Y^x_t(y))) \cdot h\| \leq Ce^{-ct}\|h\|(1 + \|x\| + \|y\|).
\]

Proof. We shall follow the approach of [4, Proposition C.2]. For any $t_0 > 0$, we set
\[
\hat{F}_{t_0,n}(x, y, t) = \hat{F}_n(x, y, t) - \hat{F}_n(x, y, t + t_0),
\]
where
\[
\hat{F}_n(x, y, t) := \mathbb{E}F_n(x, Y^x_t(y)).
\]

Thanks to Markov property we may write that
\[
\hat{F}_{t_0,n}(x, y, t) = \hat{F}_n(x, y, t) - \mathbb{E}F_n(x, Y^{x,n}_{t+t_0}(y)) = \hat{F}_n(x, y, t) - \mathbb{E}\hat{F}_n(x, Y^{x,n}_{t_0}(y), t).
\]

In view of the assumption (H.1), $\hat{F}_n$ is Gâteaux-differentiable with respect to $x$ at $(x, y, t)$. Therefore, we have for any $h \in H^{(n)}$ that
\[
D_x\hat{F}_{t_0,n}(x, y, t) \cdot h = D_x\hat{F}_n(x, y, t) \cdot h - \mathbb{E}D_x\left(\hat{F}_n(x, Y^{x,n}_{t+t_0}(y), t) \cdot h\right) = \hat{F}'_{n,x}(x, y, t) \cdot h - \mathbb{E}\hat{F}'_{n,x}(x, Y^{x,n}_{t_0}(y), t) \cdot h
\]
\[
- \mathbb{E}\hat{F}'_{n,y}(x, Y^{x,n}_{t_0}(y), t) \cdot (D_xY^{x,n}_{t_0}(y) \cdot h),
\]  
(6.3)

where we use the symbol $\hat{F}'_{n,x}$ and $\hat{F}'_{n,y}$ to denote the derivative with respect to $x$ and $y$, respectively. Note that the first derivative $\hat{F}'_{x,y,h,n} = D_xY^{x,n}_t(y) \cdot h$, at the point $x$ and along the direction $h \in H^{(n)}$, is the solution of variation equation
\[
d\hat{F}'_{x,y,h,n} = \left(A_n\hat{F}'_{x,y,h,n} + G'_{n,x}(x, Y^{x,n}_t(y)) \cdot h + G'_{n,y}(x, Y^{x,n}_t(y)) \cdot \hat{F}'_{x,y,h,n} \right) dt
\]

with initial data $\hat{F}'_{x,y,h,n}|_{t=0} = 0$. Hence, thanks to (H.2), it is immediate to check that for any $t \geq 0$,
\[
\mathbb{E}\|\hat{F}'_{x,y,h,n}\| \leq C\|h\|.  \tag{6.4}
\]

Note that there exists a constant $c > 0$, such that, for any $y_1, y_2 \in H^{(n)}$, it holds
\[
\|\hat{F}_n(x, y_1, t) - \hat{F}_n(x, y_2, t)\| = \|\mathbb{E}F_n(x, Y^{x,n}_t(y_1)) - \mathbb{E}F_n(x, Y^{x,n}_t(y_2))\|
\]
\[
\leq C\mathbb{E}\|Y^{x,n}_t(y_1) - Y^{x,n}_t(y_2)\|
\]
\[
\leq Ce^{-ct}\|y_1 - y_2\|,
\]
which implies
\[
\|\hat{F}'_{n,y}(x, y, t) \cdot k\| \leq Ce^{-ct}\|k\|, \quad k \in H.  \tag{6.5}
\]
Therefore, thanks to (6.4) and (6.5), we can conclude that
\[
\|E\left[F'_{n,y}(x, Y_{t_0}^{x,n}(y), t) \cdot (D_x Y_{t_0}^{x,n}(y) \cdot h)\right]\| \leq Ce^{-ct}\|h\|. \tag{6.6}
\]

Then, we directly have
\[
\hat{F}'_{n,x}(x, y_1, t) \cdot h - \hat{F}'_{n,x}(x, y_2, t) \cdot h
= E\left(F'_{n,x}(x, Y_{t}^{x,n}(y_1)) \cdot h - E(F'_{n,x}(x, Y_{t}^{x,n}(y_2))) \cdot h\right)
\]
\[
+ E\left(F'_{n,y}(x, Y_{t}^{x,n}(y_1)) \cdot \varsigma^{x,y_1,h,n} - F'_{n,y}(x, Y_{t}^{x,n}(y_2)) \cdot \varsigma^{x,y_2,h,n}\right)
\]
\[
= E\left(F'_{n,x}(x, Y_{t}^{x,n}(y_1)) \cdot h - E(F'_{n,x}(x, Y_{t}^{x,n}(y_2))) \cdot h\right)
\]
\[
+ E\left([F'_{n,y}(x, Y_{t}^{x,n}(y_1)) - F'_{n,y}(x, Y_{t}^{x,n}(y_2))] \cdot \varsigma^{x,y_1,h,n}\right)
\]
\[
+ E\left(F'_{n,y}(x, Y_{t}^{x,n}(y_2)) \cdot (\varsigma^{x,y_1,h,n} - \varsigma^{x,y_2,h,n})\right). \tag{6.7}
\]

First it is easy to show
\[
\|E\left(F'_{n,x}(x, Y_{t}^{x,n}(y_1)) \cdot h - E(F'_{n,x}(x, Y_{t}^{x,n}(y_2))) \cdot h\right)\|
\leq E\|\left(F'_{n,x}(x, Y_{t}^{x,n}(y_1)) \cdot h - (F'_{n,x}(x, Y_{t}^{x,n}(y_2))) \cdot h\|\|
\leq CE\|Y_{t}^{x,n}(y_1) - Y_{t}^{x,n}(y_2)\| \cdot \|h\|
\leq Ce^{-ct}\|y_1 - y_2\| \cdot \|h\|. \tag{6.8}
\]

Next, by Assumption (H.2) we have
\[
\|E\left([F'_{n,y}(x, Y_{t}^{x,n}(y_1)) - F'_{n,y}(x, Y_{t}^{x,n}(y_2))] \cdot \varsigma^{x,y_1,h,n}\right)\|
\leq C\|E\|\varsigma^{x,y_1,h,n}\|2^{\frac{1}{2}} \cdot \{E\|Y_{t}^{x,n}(y_1) - Y_{t}^{x,n}(y_2)\|2\}^{\frac{1}{2}}
\leq Ce^{-ct}\|h\| \cdot \|y_1 - y_2\|. \tag{6.9}
\]

By making use of Assumption (H.1) again, we can show that there exists a constant $c' > 0$ such that one has
\[
\|E\left(F'_{n,y}(x, Y_{t}^{x,n}(y_2)) \cdot (\varsigma^{x,y_1,h,n} - \varsigma^{x,y_2,h,n})\right)\|
\leq E\|\left(F'_{n,y}(x, Y_{t}^{x,n}(y_2)) \cdot (\varsigma^{x,y_1,h,n} - \varsigma^{x,y_2,h,n})\right)\|
\leq CE\|\varsigma^{x,y_1,h,n} - \varsigma^{x,y_2,h,n}\|
\leq Ce^{-c't}\|y_1 - y_2\| \cdot \|h\|. \tag{6.10}
\]

Collecting together (6.7), (6.8), (6.9) and (6.10), we get
\[
\|\hat{F}'_{n,x}(x, y_1, t) \cdot h - \hat{F}'_{n,x}(x, y_2, t) \cdot h\|
\leq Ce^{-ct}\|y_1 - y_2\| \cdot \|h\|.
\]
which means
\[
\| \hat{F}'_{n,x}(x,y,t) \cdot h - \mathbb{E}\hat{F}'_{n,x}(x,Y_{t_0}^{x,n}(y),t) \cdot h \|
\leq Ce^{-ct}(1 + \|y\|) \cdot \|h\|
\] (6.11)
since
\[
\mathbb{E}\|Y_{t_0}^{x,n}(y)\| \leq C(1 + \|x\| + \|y\|).
\]

Returning to (6.3), by (6.6) and (6.11) we conclude that
\[
\|D_x \hat{F}_{t_0,n}(x,y,t) \cdot h\| \leq Ce^{-ct}(1 + \|x\| + \|y\|)\|h\|.
\]

By taking the limit as \(t_0\) goes to infinity we obtain
\[
\|D_x (\hat{F}_n(x) - \mathbb{E}F_n(x,Y_t^n(y))) \cdot h\| \leq Ce^{-ct}\|h\| (1 + \|x\| + \|y\|).
\]

Proceeding with similar arguments above we can obtain similar result concerning the second order differentiability.

**Lemma 6.9.** There exist constants \(C, c > 0\) such that for any \(x, y, h, k \in H^{(n)}\) and \(t > 0\) it holds
\[
\|D_{xx} (\hat{F}_n(x) - \mathbb{E}F_n(x,Y_t^n(y))) \cdot (h,k)\| \leq Ce^{-ct}\|h\| \cdot \|k\| (1 + \|x\| + \|y\|).
\]

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