Kramers degeneracy theorem in nonrelativistic QED

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Abstract

Degeneracy of the eigenvalues of the Pauli-Fierz Hamiltonian with spin 1/2 is proven by the Kramers degeneracy theorem. The Pauli-Fierz Hamiltonian at fixed total momentum is also investigated.

1 Introduction

Among the fundamental observations in physics the anomalous Zeeman effect must be high on anybody’s list. Due to the spin of the electron the degeneracy of energy levels is lifted by the interaction with an external magnetic field. On the other hand in zero external field, there is no splitting of the energy levels observed, despite the fact that electrons carry their magnetic radiation field with them. In purely physical terms the total system, i.e., the radiation included, has a time reversal symmetry which can be used to explain that energy levels must be degenerate. This is important for understanding the spectrum of atoms but also justifies the use of effective spin Hamiltonians. It is the aim of this little note to explain all this on the basis of non-relativistic QED using the Kramers degeneracy theorem.

We consider an electron coupled to the quantized radiation field described what is sometimes called the Pauli-Fierz Hamiltonian. The existence of a ground state for non-relativistic QED is by now rather well understood. The reader may consult [1, 2, 6, 8, 12] for the use of various techniques. In the absence of spin the ground state can be shown to be unique, either by estimating the overlap between ground states and the Fock vacuum [2] or relying on the Perron-Frobenius theorem via the functional integral formula for the heat kernel [8].
If spin is included, the ground state is no longer unique. Exact double degeneracy of the ground state was first proven by Hiroshima and Spohn [11]. They prove first that the eigenvalue is at least doubly degenerate by a perturbative argument and then show that the degeneracy cannot be more than two by calculating the overlap with the Fock vacuum as in [2]. Both calculations require that the fine structure constant (for a given ultra violet cutoff) is sufficiently small.

In this note we give a simple proof of the at least double degeneracy based on the Kramers degeneracy theorem which first appeared in [14]. In view of its importance of the result and the simplicity of the argument we decided to present this argument in a separate paper. The reader will see that our proof clarifies an essential structure behind Hiroshima and Spohn's proof. It should be remarked that not only ground states but also all eigenvectors, should they exist, are at least doubly degenerate. As far as we know, no one has applied the Kramers theorem for this purpose to nonrelativistic QED before. Our arguments also apply to $N$-electron system coupled to the Maxwell field provided $N$ is odd.

The Hamiltonian $H$ of an electron interacting with the radiation field is translation invariant and hence the total momentum is conserved. Thus $H$ can be written as a direct integral $H = \int_{\mathbb{R}^3} H(P) \, dP$, where $H(P)$ is the Hamiltonian with a fixed total momentum $P$. The existence of the ground state of $H(P)$ is established by [3, 5, 17] under suitable conditions. Exact double degeneracy of it was shown by Hiroshima and Spohn [11], too. Their proof is a modification of the proof of the corresponding statement for $H$. Kramers degeneracy theorem applies to this case as well and yields the degeneracy of every eigenvalue.

Recently Hiroshima gave another proof of the ground state degeneracy by using the fact that $H(P)$ commutes with rotations [10]. (We remark that, his method is motivated by Sasaki [15].) This argument, however, depends on the choice of the polarization vectors of the quantized radiation field. This makes the arguments somewhat complicated. Our method is free from the choice of the polarization vector. Hiroshima’s proof, however, clarifies the symmetry property of the ground states.

This paper is organized as follows. In section 2 we establish an abstract framework of the Kramers degeneracy theorem. We apply this abstract theory to the Pauli-Fierz Hamiltonian with spin $1/2$ in section 3 and to the Pauli-Fierz Hamiltonian at fixed total momentum in section 4. In section 5 we remark about the $N$-electron system coupled to the radiation field. Section 6 is devoted to discussion of Hiroshima-Spohn’s lemma.

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2 Abstract theory

2.1 Reality preserving operators

Let $\mathfrak{h}$ be a complex Hilbert space and $j$ be an involution on $\mathfrak{h}$. Namely (i) $j$ is antilinear, (ii) $j^2 = 1$, the identity on $\mathfrak{h}$, (iii) $\|jx\| = \|x\|$ for all $x \in \mathfrak{h}$. Let $\mathfrak{h}^j = \{x \in \mathfrak{h} \mid jx = x\}$. Then $\mathfrak{h}^j$ is a real Hilbert space. A vector $x$ in $\mathfrak{h}$ is said to be $j$-real if $x \in \mathfrak{h}^j$ holds.

A linear operator $a$ on $\mathfrak{h}$ is called to be reality preserving with respect to $j$ if $j \text{dom}(a) \subseteq \text{dom}(a)$ (equivalently $j \text{dom}(a) = \text{dom}(a)$) and $ajx = jax$ for all $x \in \text{dom}(a)$. We remark that $a$ preserves reality w.r.t. $j$ if and only if $a \mathfrak{h}^j \cap \text{dom}(a) \subseteq \mathfrak{h}^j$ holds. We denote the set of all reality preserving operators w.r.t. $j$ by $\mathfrak{A}_j(\mathfrak{h})$. Basic property of $\mathfrak{A}_j(\mathfrak{h})$ is stated as below.

**Proposition 2.1** $\mathfrak{A}_j(\mathfrak{h})$ is a real algebra, namely, we have the following.

(i) $\forall a, b \in \mathfrak{A}_j(\mathfrak{h}) \forall \alpha, \beta \in \mathbb{R}, \alpha a + \beta b \in \mathfrak{A}_j(\mathfrak{h})$.

(ii) $\forall a, b \in \mathfrak{A}_j(\mathfrak{h}), ab \in \mathfrak{A}_j(\mathfrak{h})$ provided the product is defined.

**Proof.** This is an easy exercise. □

2.2 Abstract Kramers degeneracy theorem

The following proposition is an abstract version of the Kramer’s degeneracy, found in the physical literature.

**Proposition 2.2** Let $\vartheta$ be an antiunitary operator with $\vartheta^2 = -1$. Let $H$ be a self-adjoint operator. Assume that $H$ commutes with $\vartheta$. Then each eigenvalue of $H$ is at least doubly degenerate.

**Proof.** Let $\varphi$ be an eigenvector for the eigenvalue $E$. Since $H$ commutes with $\vartheta$, one sees

$$H\vartheta\varphi = \vartheta H\varphi = E\vartheta\varphi.$$ 

Hence $\vartheta\varphi$ is also eigenvector for $E$. By the antiunitarity, $\langle \vartheta\psi_1, \vartheta\psi_2 \rangle = \langle \psi_2, \psi_1 \rangle$ for all $\psi_1, \psi_2$. Thus, using $\vartheta^2 = -1$,

$$-\langle \varphi, \vartheta\varphi \rangle = \langle \vartheta(\vartheta\varphi), \vartheta\varphi \rangle = \langle \varphi, \vartheta\varphi \rangle.$$

Hence $\langle \varphi, \vartheta\varphi \rangle = 0$. □

The following lemma is a direct consequence of the functional calculus.

**Proposition 2.3** Assume that $H$ and $\vartheta$ satisfy the conditions in Proposition 2.2. Let $f$ be a real-valued measurable function on $\mathbb{R}$. Then $f(H)$ defined by the functional calculus commutes with $\vartheta$ too.
2.3 Kramers degeneracy in the system with spin 1/2

Let us consider a direct sum Hilbert space \( \mathcal{H} = \mathfrak{h} \oplus \mathfrak{h} \). Let \( j \) be an involution on \( \mathfrak{h} \). Then \( J = j \oplus j = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \) is an involution on \( \mathcal{H} \) too. Let \( \sigma_1, \sigma_2, \sigma_3 \) be the 2 \( \times \) 2 Pauli matrices on \( \mathcal{H} \):

\[
\sigma_1 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.
\]

We restrict our attention to the following case. Let \( H_0 \) be a semibounded self-adjoint operator on \( \mathcal{H} \) having a form

\[
H_0 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.
\]

Hence \( A \) is self-adjoint and bounded from below. We assume the following.

(\textbf{H.1}) \( A \in \mathfrak{A}_j(\mathfrak{h}) \).

Clearly each eigenvalue of \( H_0 \) is doubly degenerate. We consider the perturbation by a symmetric operator \( H_1 \) of the form:

\[
H_1 = \sigma \cdot B = \sum_{j=1}^{3} \sigma_j B_j = \begin{pmatrix} B_3 & B_1 - iB_2 \\ B_1 + iB_2 & -B_3 \end{pmatrix},
\]

where each \( B_i, \ i = 1, 2, 3 \) is a symmetric operator on \( \mathfrak{h} \) possessing the following properties:

(\textbf{H.2}) Each \( B_i \) is infinitesimally small with respect to \( A \).

(\textbf{H.3}) \( iB_i \in \mathfrak{A}_j(\mathfrak{h}), \ i = 1, 2, 3 \).

Remark that (H.3) is equivalent to \( j B_i x = -B_i j x \) for all \( x \in \text{dom}(B_i), \ i = 1, 2, 3 \). The condition (H.2) guarantees the self-adjointness of the following operator

\[
H = H_0 + H_1.
\]

Define an antiunitary operator by

\[
\vartheta = \sigma_2 J.
\]

Then \( \vartheta \) is an antiunitary operator satisfying \( \vartheta^2 = -\mathbb{1} \).

**Theorem 2.4** Under the assumptions (H.1), (H.2) and (H.3) each eigenvalue of \( H \) is at least doubly degenerate.

**Proof.** Noting the facts \( \vartheta \sigma_i = -\sigma_i \vartheta, \ jB_i = -B_i j, \ i = 1, 2, 3 \) and the assumption (H.1), one has

\[
\vartheta H_0 = H_0 \vartheta,
\]

\[
\vartheta \sigma \cdot B = \sigma \cdot B \vartheta
\]

which implies \( \vartheta H = H \vartheta \). Hence, by Proposition 2.2, each eigenvalue of \( H \) is at least doubly degenerate. \( \square \)
3 Pauli-Fierz Hamiltonian with spin 1/2

The Pauli-Fierz Hamiltonian is given by

$$H_{PF} = \frac{1}{2} \left( \frac{-i\nabla_x + eA(x)}{\sqrt{2}} \right)^2 + \frac{e}{2} \sigma \cdot B(x) + V(x) + H_f$$

acting in $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathfrak{F}$, where $\mathfrak{F}$ is the photon Fock space

$$\mathfrak{F} = \sum_{n \geq 0} \mathbb{L}^2(\mathbb{R}^3 \times \{1, 2\})^\otimes n,$$

$\mathbb{L}^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathfrak{F}$ means the $n$-fold symmetric tensor product of $\mathbb{L}^2(\mathbb{R}^3; \mathbb{C}^2)$ with the convention $\mathbb{L}^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathfrak{F}$.

The quantized vector potential $A(x) = (A_1(x), A_2(x), A_3(x))$ is given by

$$A(x) = \sum_{\lambda=1,2} \int_{|k| \leq \Lambda} \frac{dk}{\sqrt{2(2\pi)^3|k|}} \varepsilon(k, \lambda) \left( e^{ik \cdot x} a(k, \lambda) + e^{-ik \cdot x} a(k, \lambda)^* \right),$$

where $\varepsilon(k, \lambda)$ is a polarization vector which is real valued and measurable, $\Lambda$ is the ultraviolet cutoff. Here $a(k, \lambda), a(k, \lambda)^*$ are the annihilation and creation operators which satisfy the standard commutation relations

$$[a(k, \lambda), a(q, \mu)^*] = \delta_{\lambda \mu} \delta(k - q),$$

$$[a(k, \lambda), a(q, \mu)] = 0 = [a(k, \lambda)^*, a(q, \mu)^*].$$

$B(x)$ is the quantized magnetic field defined by

$$B(x) = \text{rot} A(x) = i \sum_{\lambda=1,2} \int_{|k| \leq \Lambda} \frac{dk}{\sqrt{2(2\pi)^3|k|}} (k \times \varepsilon(k, \lambda)) \left( e^{ik \cdot x} a(k, \lambda) - e^{-ik \cdot x} a(k, \lambda)^* \right).$$

$H_f$ is the field energy given by

$$H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk |k| a(k, \lambda)^* a(k, \lambda).$$

Throughout this section, we assume the following:

(V) $V$ is infinitesimally small with respect to $-\Delta_x$.

Then, by [7, 9], $H_{PF}$ is self-adjoint on $\text{dom}(-\Delta_x) \cap \text{dom}(H_f)$, bounded from below.

Our Hilbert space $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathfrak{F}$ is naturally identified with

$$L^2(\mathbb{R}^3, \mathfrak{F}) \oplus L^2(\mathbb{R}^3, \mathfrak{F}).$$
Under this identification, $H_{PF}$ is understood as follows:

$$H_{PF} = H_0 + \frac{e}{2} \sigma \cdot B(x),$$

$$H_0 = \begin{pmatrix} H_{\text{Spinless}} & 0 \\ 0 & H_{\text{Spinless}} \end{pmatrix},$$

$$H_{\text{Spinless}} = \frac{1}{2} \left( -i \nabla_x + eA(x) \right)^2 + V(x) + H_f.$$ 

Note the following facts:

(i) $\sigma \cdot B(x)$ is infinitesimally small w.r.t. $H_0$.

(ii) $H_{\text{Spinless}}$ is self-adjoint on $\text{dom}(-\Delta_x) \cap \text{dom}(H_f)$ and bounded from below by [7, 9].

On $L^2(\mathbb{R}^3; \mathfrak{F})$, we take the following involution:

$$j\varphi = \sum_{n \geq 0}^{\oplus} \overline{\varphi^{(n)}}(-x; k_1, \lambda_1, \ldots, k_n, \lambda_n),$$

$$x \in \mathbb{R}^3, \ (k_i, \lambda_i) \in \mathbb{R}^3 \times \{1, 2\}$$

for $\varphi = \sum_{n \geq 0}^{\oplus} \varphi^{(n)}(x; k_1, \lambda_1, \ldots, k_n, \lambda_n) \in L^2(\mathbb{R}^3; \mathfrak{F})$. Since the annihilation operator $a(k, \lambda)$ acts by

$$a(k, \lambda)\varphi = \sum_{n \geq 0}^{\oplus} \sqrt{n + 1} \varphi^{(n+1)}(x; k, \lambda, k_1, \lambda_1, \ldots, k_n, \lambda_n)$$

for $\varphi \in L^2(\mathbb{R}^3; \mathfrak{F})$, one has

$$ja(k, \lambda) = a(k, \lambda)j, \quad ja(k, \lambda)^* = a(k, \lambda)^*j.$$ 

Namely the annihilation and creation operators are reality preserving w.r.t. $j$.

As a consequence, we obtain

$$j(-i\nabla_x) = (-i\nabla_x)j,$$

$$jA(x) = A(x)j,$$

$$jB(x) = -B(x)j,$$

$$jH_f = H_fj,$$

$$jV(x) = V(-x)j.$$ 

By the above relations, one arrives at the following:

Lemma 3.1 Assume that $V(-x) = V(x)$.

(i) The spinless Hamiltonian $H_{\text{Spinless}}$ preserves the reality w.r.t. $j$, equivalently $H_{\text{Spinless}} \in \mathfrak{A}_j(L^2(\mathbb{R}^3; \mathfrak{F}))$. This is corresponding to (H.1).
Kramers degeneracy

(ii) \( jB(x) = -B(x)j \), that is, \( jB(x) \in \mathfrak{A}_j(L^2(\mathbb{R}^3; \mathfrak{F})) \). This corresponds to (H.3).

Thus we can apply Theorem 2.4 to obtain the following.

**Theorem 3.2** Let \( \vartheta = \sigma_2 J \) with \( J = j \oplus j \). Then \( \vartheta \) is an antiunitary operator satisfying \( \vartheta^2 = -1 \). Assume that (V) holds. Moreover suppose that \( V(x) = V(-x) \). Then we obtain

\[
\vartheta H_{PF} = H_{PF} \vartheta.
\]

In particular, each eigenvalue of \( H_{PF} \) is degenerate.

### 4 Pauli-Fierz Hamiltonian at fixed total momentum

Let us consider the Hamiltonian at fixed total momentum

\[
H_{PF}(P) = \frac{1}{2} (P - P_t + eA(0))^2 + \frac{e}{2} \sigma \cdot B(0) + H_f, \quad P \in \mathbb{R}^3,
\]

where \( P_t \) is the field momentum defined by

\[
P_t = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk \ k a(k, \lambda)^* a(k, \lambda).
\]

Our Hilbert space is \( \mathbb{C}^2 \otimes \mathfrak{F} \).

Under the natural identification \( \mathbb{C}^2 \otimes \mathfrak{F} = \mathfrak{F} \oplus \mathfrak{F} \), our Hamiltonian is represented as

\[
H_{PF}(P) = H_0(P) + \sigma \cdot B(0),
\]

\[
H_0(P) = \begin{pmatrix}
H_{Spinless}(P) & 0 \\
0 & H_{Spinless}(P)
\end{pmatrix},
\]

\[
H_{Spinless}(P) = \frac{1}{2} (P - P_t + eA(0))^2 + H_f.
\]

By [10, 13], \( H_{Spinless}(P) \) is positive and self-adjoint on \( \text{dom}(P_t^2) \cap \text{dom}(H_f) \). Moreover \( \sigma \cdot B(0) \) is infinitesimally small with respect to \( H_0(P) \). We choose an involution \( j \) by

\[
j \varphi = \sum_{n \geq 0} \varphi(n)(k_1, \lambda_1, \ldots, k_n, \lambda_n), \quad (k_i, \lambda_i) \in \mathbb{R}^3 \times \{1, 2\}
\]

for each \( \varphi = \sum_{n \geq 0} \varphi(n)(k_1, \lambda_1, \ldots, k_n, \lambda_n) \in \mathfrak{F} \). Then the annihilation and creation operators \( a(k, \lambda), a(k, \lambda)^* \) are reality preserving w.r.t. \( j \) again, because the action of \( a(k, \lambda) \) on \( \varphi = \sum_{n \geq 0} \varphi(n)(k_1, \lambda_1, \ldots, k_n, \lambda_n) \in \mathfrak{F} \) is given by

\[
a(k, \lambda) \varphi = \sum_{n \geq 0} \sqrt{n+1} \varphi^{(n+1)}(k, \lambda, k_1, \lambda_1, \ldots, k_n, \lambda_n).
\]
Accordingly one can easily see that
\[ jA(0) = A(0)j, \]
\[ jB(0) = -B(0)j, \]
\[ jH_t = H_tj, \]
\[ jP_t = P_tj, \]
which imply that \( H_{\text{Spinless}}(P) \) and \( iB_i(0), \ i = 1, 2, 3 \) are in \( \mathfrak{H}_j(\mathfrak{F}) \). Thus we can apply Theorem 2.4 and obtain the following:

**Theorem 4.1** Let \( \vartheta = \sigma_2 J \) with \( J = j \oplus j \). Then \( \vartheta \) is an antiunitary operator with \( \vartheta^2 = -\mathbb{1} \). Moreover we obtain
\[ \vartheta H_{PF}(P) = H_{PF}(P) \vartheta. \]

In particular, each eigenvalue of \( H_{PF}(P) \) is at least doubly degenerate.

## 5 \( N \)-electron system with one fixed nucleus

In this section, we remark on an \( N \)-electron system governed by the following Hamiltonian
\[
H_N = \sum_{j=1}^{N} \left\{ \frac{1}{2} \left( \sigma^{(j)} \cdot \left( -i \nabla_{x_j} + eA(x_j) \right) \right)^2 - \frac{Ze^2}{|x_j|} \right\} + \sum_{1 \leq i < j \leq N} \frac{e^2}{|x_i - x_j|} + H_t.
\]

\( H_N \) is acting in \( \mathcal{H}_N = \left( L^2(\mathbb{R}^3; \mathbb{C}^2)^{\otimes N} \right) \otimes \mathfrak{F} \) and self-adjoint on \( \cap_{j=1}^{N} \text{dom}(\Delta_{x_j}) \cap \text{dom}(H_t) \) by [4]. \( L^2(\mathbb{R}^3; \mathbb{C}^2)^{\otimes N} \) means the \( N \)-fold antisymmetric tensor product of \( L^2(\mathbb{R}^3; \mathbb{C}^2) \). \( \sigma^{(l)} = (\sigma_1^{(l)}, \sigma_2^{(l)}, \sigma_3^{(l)}) \), \( l = 1, \ldots, N \) is given by
\[
\sigma_i^{(l)} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \sigma_i \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}.
\]

Each \( \varphi \in \mathcal{H}_N \) is expressed as
\[
\varphi = \sum_{n \geq 0} \varphi^{(n)}(x_1, \ldots, x_N; \tau_1, \ldots, \tau_N; k_1, \lambda_1, \ldots, k_n, \lambda_n),
\]
where \( x_i \in \mathbb{R}^3, \tau_i = 1, 2 \) and \( (k_i, \lambda_i) \in \mathbb{R}^3 \times \{1, 2\} \). Let us define
\[
J \varphi = \sum_{n \geq 0} \overline{\varphi^{(n)}}(-x_1, \ldots, -x_N; \tau_1, \ldots, \tau_N; k_1, \lambda_1, \ldots, k_n, \lambda_n)
\]
and
\[
\vartheta = \Pi_{i=1}^{N} \sigma_2^{(i)} J.
\]
Clearly \( \vartheta \) is antiunitary. If \( N \) is odd, then \( \vartheta \) satisfies \( \vartheta^2 = -\mathbb{1} \). Passing through similar arguments as in Section 4 one arrives at the following.
Theorem 5.1 Let \( \vartheta \) be defined as above. Then \( \vartheta \) is antiunitary and one has
\[
\vartheta H_N = H_N \vartheta.
\]
Moreover if \( N \) is odd, \( \vartheta^2 = -1 \) holds. Hence each eigenvalue of \( H_N \) is at least doubly degenerate.

Remark 5.2 We can treat the \( N \)-electron Hamiltonian with a fixed total momentum discussed in \([13]\) by the similar way.

6 Discussion of Hiroshima-Spohn’s lemma

Let us consider the Hamiltonian \( H(P) \) in this section. In \([11]\) the following lemma is a key ingredient of their proof. Here we derive the lemma from our viewpoint, because this lemma is itself interesting.

Lemma 6.1 (Hiroshima-Spohn \([11]\)) Let \( x \in \mathbb{C}^2 \). Then there exists \( a(t) \in \mathbb{R} \) independent of \( x \) such that for all \( t \geq 0 \)
\[
\langle x \otimes \Omega, e^{-tH(P)} x \otimes \Omega \rangle = a(t) \| x \|_{\mathbb{C}^2}^2,
\]
where \( \Omega = 1 \oplus 0 \oplus 0 \oplus \cdots \in \mathfrak{F} \), the Fock vacuum.

Proof. Since \( H(P) \) commutes with \( \vartheta \) defined in Theorem 4.1, \( e^{-tH(P)} \) also commutes with \( \vartheta \) for all \( t \geq 0 \) by Proposition 2.3. Hence for any \( x, y \in \mathbb{C}^2 \), one sees
\[
\langle x \otimes \Omega, e^{-tH(P)} y \otimes \Omega \rangle = \langle \vartheta y \otimes \Omega, e^{-tH(P)} \vartheta x \otimes \Omega \rangle.
\]
If we choose \( x = (1_0) \) and \( y = (0_1) \), one obtains
\[
\langle (1_0) \otimes \Omega, e^{-tH(P)} (0_1) \otimes \Omega \rangle = -\langle (1_0) \otimes \Omega, e^{-tH(P)} (0_1) \otimes \Omega \rangle
\]
which implies
\[
\langle (1_0) \otimes \Omega, e^{-tH(P)} (0_1) \otimes \Omega \rangle = 0.
\]
Similarly if we choose \( x = (1_0) \) and \( y = (1_0) \), one gets
\[
\langle (1_0) \otimes \Omega, e^{-tH(P)} (1_0) \otimes \Omega \rangle = \langle (0_1) \otimes \Omega, e^{-tH(P)} (0_1) \otimes \Omega \rangle.
\]
Hence we have the desired result. \( \Box \)

Similarly we can also show a slightly generalized version.

Proposition 6.2 Let \( x \in \mathbb{C}^2 \) and \( f \in L^\infty(\mathbb{R}) \). If \( f \) is real valued, then we have
\[
\langle x \otimes \varphi, f(H(P)) x \otimes \varphi \rangle = \| x \|_{\mathbb{C}^2}^2 \langle (1_0) \otimes \varphi, f(H(P)) (1_0) \otimes \varphi \rangle
\]
\[
= \| x \|_{\mathbb{C}^2}^2 \langle (0_1) \otimes \varphi, f(H(P)) (0_1) \otimes \varphi \rangle
\]
for each \( \varphi \in \mathfrak{F} \) with \( j\varphi = \varphi \).
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