Parameter Estimates of General Failure Rate Model:
A Bayesian Approach

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Abstract

The failure rate function plays an important role in studying the lifetime distributions in reliability theory and life testing models. A study of the general failure rate model \( r(t) = a + bt^{\theta-1} \), under squared error loss function taking \( a \) and \( b \) independent exponential random variables has been analyzed in the literature. In this article, we consider \( a \) and \( b \) not necessarily independent. The estimates of the parameters \( a \) and \( b \) under squared error loss, linex loss and entropy loss functions are obtained here.

Key Words and Phrases: Farlie-Gumbell-Morgenstern family of distributions, IFR and DFR classes, Markov Chain Monte Carlo simulation, Type-II censoring.

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1 Introduction

Failure is an unavoidable phenomenon with technological products and systems. Reliability is a measure of failure uncertainty. The failure rate function plays an important role in reliability theory. The failure rate function and the distribution function are equivalent in the sense that knowing one, other can be uniquely determined by the relationship

$$\bar{F}_X(t) = 1 - F_X(t) = e^{-\int_0^t r_X(u)du},$$

or equivalently,

$$f_X(t) = r_X(t)e^{-\int_0^t r_X(u)du},$$

where $f_X$, $F_X$, $\bar{F}_X$ and $r_X$ are the probability density, the distribution, the survival and the failure rate functions, respectively.

In reliability, lifetime distributions are often specified by choosing a particular failure rate function. The constant failure rate characterizes exponential distribution. The linear failure rate (LFR) distribution arises often in reliability literature and is motivated by its application to human survival data, see, for instance, Kodlin [5] and Carbone, Kellerhouse and Gehan [3]. Various distributional properties and applications of the LFR distribution to life testing and reliability studies have been described by Sen [12] and the references therein. Estimation of the unknown parameters in the lifetime distributions of the individual components belonging to a multi-component system is an interesting problem in reliability analysis. These estimators may be extremely useful in some ways, since they reflect the component reliability after being assembled into an operational system (cf. Usher and Hodgson, [15]). Such estimators can be used under appropriate conditions to predict the reliability of new configurations of the components of the system. The estimation of the LFR model $r_X(t) = a + bt$ with $a$ and $b$ nonnegative constants, using classical method, has been studied in the literature by Bain [2], Shaked [14], Sen and Bhattacharyya [13] among others.

Ashour and Youssef [1] have investigated the Bayesian estimators for the parameters of the LFR model based on Type-II censored samples. However, their derivation of the marginal posterior distributions seems to be erroneous (cf. Lin, Wu and Balakrishnan [7]). Pandey, Singh and Zimmer [9] have discussed the same problem with a simpler assumption on the joint prior distribution. The Bayesian estimation and prediction problems for the LFR model under general progressively Type-II censored samples are considered by Lin, Wu and Balakrishnan [7]. The estimation of LFR distribution based on records and inter-record times has been discussed in Lin, Wu and Balakrishnan [8]. In life testing and reliability studies, LFR distributions are useful in modeling the life length of a system or component when failures occur at random, and also from ageing or wear-out. But this failure rate model cannot describe other failure rate models except the linearly increasing one. To overcome this difficulty and to accommodate more varieties of failure rate models, Sarhan [10] has analyzed a more general failure rate model.
of the form
\[ r_X(t) = a + bt^{\theta-1}, \] (1.1)
for some nonnegative constants \(a, b\) and \(\theta\). (1.1) generalizes exponential distribution \((b = 0\) or \(\theta = 1\)), Rayleigh distribution \((a = 0, \ \theta = 2)\), Weibull distribution \((a = 0)\) and LFR model \((\theta = 2)\). For \(0 \leq \theta \leq 1\), (1.1) gives DFR (decreasing in failure rate) distribution, whereas for \(\theta \geq 1\), it gives IFR (increasing in failure rate) distributions. The general failure rate model given in (1.1) is a useful model to specify the lifetime distributions in reliability theory and life testing.

Sarhan [10] has obtained the Bayes’ estimators of \((a, b)\) under the squared error loss function taking \(a\) and \(b\) independent exponential random variables with known parameters, and \(\theta\) a known constant. But, \(a\) and \(b\) may not always be independent. Also, several common situations may arise when over estimation is more serious than under estimation and vice versa. In that case the loss function cannot be symmetric (e.g., squared error loss). Here we consider the case when \(a\) and \(b\) are not necessarily independent and the data are Type-II censored. We assume that \(\theta\) is known and \((a, b)\) have joint probability density function
\[ h(a, b) = f(a)g(b) + \rho f(a)g(b)[1 - 2F(a)][1 - 2G(b)], \quad -1 \leq \rho \leq 1 \] (1.2)
where \(f\) and \(g\) are the marginal probability densities of \(a\) and \(b\), and \(F\) and \(G\) are the distribution functions corresponding to \(f\) and \(g\), respectively. The probability density function given in (1.2) is the well known Farlie-Gumbell-Morgenstern bivariate density (cf. Farlie, [4]). We assume that the value of \(\rho\) in (1.2) is known. Clearly, when \(\rho = 0\), we get the result of Sarhan [10]. It is to be mentioned here that parameter estimation of the general failure rate model (1.1) using masked data is considered in Sarhan [11].

The paper is arranged as follows. In Section 2, we formulate the problem. The Bayes’ estimators of the parameters \(a\) and \(b\) under different loss functions viz. squared error loss function, linex loss function and entropy loss function are obtained in Section 3. In Section 4 some simulation results are presented.

2 Formulation of the problem

In the present context, we use the following assumptions:

**Assumption 2.1** A fixed number of units, say \(n\), are put on test and the data collected on the life of the units are Type-II censored. That is, the test is terminated once \(r\) (prespecified number) units fail. The failure times of the first \(r\) units are known. Let they be \(t_1, t_2, \ldots, t_r\).

**Assumption 2.2** No two units fail together. This means, \(t_1 < t_2 < \ldots < t_r\).

**Assumption 2.3** The marginal distributions of \(a\) and \(b\) are exponential with known means \(1/\lambda_1\) and \(1/\lambda_2\), respectively.
Assumption 2.4 Failure times of the units are statistically independent.

Under Assumption 2.3, (1.2) becomes
\[ h(a, b) = \lambda_1 \lambda_2 e^{-\lambda_1 a - \lambda_2 b} + \lambda_1 \lambda_2 \rho \left( 2e^{-2\lambda_1 a} - e^{-\lambda_1 a} \right) \left( 2e^{-2\lambda_2 b} - e^{-\lambda_2 b} \right). \]

Given \( a \) and \( b \), the survival function corresponding to (1.1) is given by
\[ \bar{F}_X(t|a, b) = \exp \left[ - \left( at + bt^\theta / \theta \right) \right], \quad t \geq 0. \tag{2.3} \]

If \( T \) denotes the random variable having failure rate function given by (1.1), then the joint probability density function of \((a, b)\) and \( T \) is given by
\[ g_1(t, a, b) = \left( a + bt^\theta - 1 \right) \exp \left[ - \left( at + bt^\theta / \theta \right) \right] \left[ \lambda_1 \lambda_2 e^{-\lambda_1 a - \lambda_2 b} + \lambda_1 \lambda_2 \rho \left( 2e^{-2\lambda_1 a} - e^{-\lambda_1 a} \right) \left( 2e^{-2\lambda_2 b} - e^{-\lambda_2 b} \right) \right], \]

which gives the marginal probability density function of \( T \), after some simplifications, as
\[ f_T(t) = \int_0^\infty \int_0^\infty g_1(t, a, b) da \, db = \lambda_1 \lambda_2 \int_0^\infty \int_0^\infty \left( a + bt^\theta - 1 \right) \exp \left[ -(at + bt^\theta / \theta) \right] e^{-\lambda_1 a - \lambda_2 b} da \, db + \lambda_1 \lambda_2 \rho \left( 4 \int_0^\infty \int_0^\infty \left( a + bt^\theta - 1 \right) \exp \left[ -(at + bt^\theta / \theta) \right] e^{-2\lambda_1 a - 2\lambda_2 b} da \, db \right. \]
\[ - 2 \left[ \int_0^\infty \int_0^\infty \left( a + bt^\theta - 1 \right) \exp \left[ -(at + bt^\theta / \theta) \right] e^{-\lambda_1 a - 2\lambda_2 b} da \, db \right] \]
\[ - 2 \left[ \int_0^\infty \int_0^\infty \left( a + bt^\theta - 1 \right) \exp \left[ -(at + bt^\theta / \theta) \right] e^{-2\lambda_1 a - \lambda_2 b} da \, db \right] \]
\[ + \left[ \int_0^\infty \int_0^\infty \left( a + bt^\theta - 1 \right) \exp \left[ -(at + bt^\theta / \theta) \right] e^{-\lambda_1 a - \lambda_2 b} da \, db \right], \]

where
\[ I_1 = \int_0^\infty \int_0^\infty \left( a + bt^\theta - 1 \right) \exp \left[ -(at + bt^\theta / \theta) \right] e^{-\lambda_1 a - \lambda_2 b} da \, db \]
\[ = \frac{\theta}{(t + \lambda_1)^2(t^\theta + \theta \lambda_2)} + \frac{\theta^2 t^\theta - 1}{(t + \lambda_1)^2(t^\theta + \theta \lambda_2)^2}, \]
\[ I_2 = \int_0^\infty \int_0^\infty \left( a + bt^\theta - 1 \right) \exp \left[ -(at + bt^\theta / \theta) \right] e^{-2\lambda_1 a - 2\lambda_2 b} da \, db \]
\[ = \frac{\theta}{(t + 2\lambda_1)^2(t^\theta + 2\theta \lambda_2)} + \frac{\theta^2 t^\theta - 1}{(t + 2\lambda_1)^2(t^\theta + 2\theta \lambda_2)^2}, \]
\[ I_3 = \int_0^\infty \int_0^\infty \left( a + bt^\theta - 1 \right) \exp \left[ -(at + bt^\theta / \theta) \right] e^{-\lambda_1 a - 2\lambda_2 b} da \, db \]
\[ = \frac{\theta}{(t + \lambda_1)^2(t^\theta + 2\theta \lambda_2)} + \frac{\theta^2 t^\theta - 1}{(t + \lambda_1)^2(t^\theta + 2\theta \lambda_2)^2}. \]
and
\[
I_4 = \int_0^\infty \int_0^\infty (a + bt^{\theta - 1}) e^{-(at + bt^{\theta}/\theta)} e^{-2\lambda_1 a - \lambda_2 b} da \, db \\
= \frac{\theta}{(t + 2\lambda_1)^2(t^\theta + \theta\lambda_2)} + \frac{\theta^2 t^{\theta - 1}}{(t + 2\lambda_1)(t^\theta + \theta\lambda_2)^2}.
\]

Thus, we have
\[
f_T(t) = \frac{\lambda_1 \lambda_2 \theta \{\theta + 1\} t^\theta + \theta \lambda_1 t^{\theta - 1} + \theta \lambda_2}{(\lambda_1 + t)^2(t^\theta + \theta\lambda_2)^2} \\
+ \frac{\rho \lambda_1 \lambda_2 \theta t^\theta}{(2\lambda_2 \theta + t^\theta)(\lambda_2 \theta + t^\theta)(2\lambda_1 + t)(\lambda_1 + t)} \\
\left[ \frac{t^2 - 2\lambda_1^2}{(2\lambda_1 + t)(\lambda_1 + t)} + \frac{\theta(2\lambda_2 t^\theta - 2\lambda_2^2 t^{2\theta})}{(2\lambda_2 \theta + t^\theta)(\lambda_2 \theta + t^\theta)} \right].
\] (2.4)

**Remark 2.1** If it is assumed that \( \rho \) is unknown with a uniform prior distribution in \((-1, 1)\), then the probability density function of \( T \) becomes \( f_T(t) = \lambda_1 \lambda_2 I_1 \), which is same as (2.4) with \( \rho = 0 \).

### 3 Bayes’ estimators of \( a \) and \( b \)

Once the values of \( t_1, t_2, \ldots, t_r \) and that of \( n \) are known, the likelihood function can be written as (cf. Lawless [6])
\[
L(t|a, b) = \prod_{i=1}^r f_X(t_i|a, b) \left[ \bar{F}_X(t_r|a, b) \right]^{n-r},
\] (3.5)
where \( t = (t_1, t_2, \ldots, t_r) \). On using (2.3), (3.5) reduces to
\[
L(t|a, b) = \left[ \prod_{i=1}^r \left( a + bt_i^{\theta - 1} \right) \right] e^{-aS_1 - \frac{b}{\theta}S_2} \\
= \sum_{j=0}^r a^{-j} b^j M_j(t) e^{-aS_1 - \frac{b}{\theta}S_2},
\]
where
\[
S_1 = \sum_{i=1}^r t_i + (n-r)t_r, \\
S_2 = \sum_{i=1}^r t_i^\theta + (n-r)t_r^\theta,
\]
and
\[
M_j(t) = \sum_{1 \leq i_1 \leq \ldots \leq i_j \leq r} \sum_{1 \leq i_{j+1} \leq \ldots \leq i_k \leq r} t_{i_1}^{\theta - 1} t_{i_2}^{\theta - 1} \ldots t_{i_j}^{\theta - 1},
\]
for \( j = 1, 2, \ldots, r \) with \( M_0(t) = 1 \). A similar kind of expression may be obtained in Sarhan [10].

Now, the joint distribution of \( t, a \) and \( b \) is

\[
f^\star(a, b, t) = L(t|a, b)h(a, b)
\]

\[
= \sum_{j=0}^{r} a^{r-j} b^j M_j(t) e^{-aS_1-bS_2} \left[ \lambda_1 \lambda_2 e^{-\lambda_1 a - \lambda_2 b} + \lambda_1 \lambda_2 \rho \left\{ 4e^{-2\lambda_1 a - 2\lambda_2 b} - 2e^{-\lambda_1 a - 2\lambda_2 b} - 2e^{-2\lambda_1 a - \lambda_2 b} + e^{-\lambda_1 a - \lambda_2 b} \right\} \right]
\]

\[
= \lambda_1 \lambda_2 \sum_{j=0}^{r} a^{r-j} b^j M_j(t) \left[ e^{-a(S_1+\lambda_1)-b(S_\theta + \lambda_2)} + \rho \left\{ 4e^{-a(S_1+2\lambda_1)-b(S_\theta + 2\lambda_2)} - 2e^{-a(S_1+\lambda_1)-b(S_\theta + 2\lambda_2)} - 2e^{-a(S_1+2\lambda_1)-b(S_\theta + \lambda_2)} + e^{-a(S_1+\lambda_1)-b(S_\theta + \lambda_2)} \right\} \right]
\]

which gives the marginal density function of \( t \) as

\[
f_1^\star(t) = \int_0^\infty \int_0^\infty f^\star(a, b, t) da \ db
\]

\[
= \int_0^\infty \int_0^\infty \lambda_1 \lambda_2 \sum_{j=0}^{r} a^{r-j} b^j M_j(t) \left[ e^{-a(S_1+\lambda_1)-b(S_\theta + \lambda_2)} + \rho \left\{ 4e^{-a(S_1+2\lambda_1)-b(S_\theta + 2\lambda_2)} - 2e^{-a(S_1+\lambda_1)-b(S_\theta + 2\lambda_2)} - 2e^{-a(S_1+2\lambda_1)-b(S_\theta + \lambda_2)} + e^{-a(S_1+\lambda_1)-b(S_\theta + \lambda_2)} \right\} \right] da \ db
\]

\[
= \lambda_1 \lambda_2 [I_5 + \rho \{ 4I_6 - 2I_7 - 2I_8 + I_5 \}], \text{ say,}
\]

where

\[
I_5 = \int_0^\infty \int_0^\infty \sum_{j=0}^{r} a^{r-j} b^j M_j(t) e^{-a(S_1+\lambda_1)-b(S_\theta + \lambda_2)} da \ db
\]

\[
= \sum_{j=0}^{r} M_j(t) \int_0^\infty a^{r-j} e^{-a(S_1+\lambda_1)} da \int_0^\infty b^j e^{-b(S_\theta + \lambda_2)} db
\]

\[
= \sum_{j=0}^{r} M_j(t) \frac{\Gamma(r-j+1)}{(S_1 + \lambda_1)^{r-j+1}} \frac{\Gamma(j+1)}{(S_\theta + \lambda_2)^{j+1}}
\]

\[
= \Phi(1, 1, 1, 1),
\]

\[
I_6 = \int_0^\infty \int_0^\infty \sum_{j=0}^{r} a^{r-j} b^j M_j(t) e^{-a(S_1+2\lambda_1)-b(S_\theta + 2\lambda_2)} da \ db
\]

\[
= \sum_{j=0}^{r} M_j(t) \frac{\Gamma(r-j+1)}{(S_1 + 2\lambda_1)^{r-j+1}} \frac{\Gamma(j+1)}{(S_\theta + 2\lambda_2)^{j+1}}
\]

\[
= \Phi(1, 1, 2, 2),
\]
\[ I_7 = \int_0^\infty \int_0^\infty \sum_{j=0}^r a^{r-j}b^j M_j(t) e^{-a(S_1+\lambda_1)-b\left(\frac{S_2}{\theta}+2\lambda_2\right)} da \, db \]

\[
= \sum_{j=0}^r M_j(t) \frac{\Gamma(r-j+1)}{(S_1+\lambda_1)^{r-j+1}} \frac{\Gamma(j+1)}{(\frac{S_2}{\theta}+\lambda_2)^{j+1}}
= \Phi(1, 1, 1, 2)
\]

and

\[ I_8 = \int_0^\infty \int_0^\infty \sum_{j=0}^r a^{r-j}b^j M_j(t) e^{-a(S_1+2\lambda_1)-b\left(\frac{S_2}{\theta}+\lambda_2\right)} da \, db \]

\[
= \sum_{j=0}^r M_j(t) \frac{\Gamma(r-j+1)}{(S_1+2\lambda_1)^{r-j+1}} \frac{\Gamma(j+1)}{(\frac{S_2}{\theta}+\lambda_2)^{j+1}}
= \Phi(1, 1, 2, 1)
\]

so that

\[ f_1^*(t) = \lambda_1 \lambda_2 \left[ \Phi(1, 1, 1, 1) + \rho \left\{ 4\Phi(1, 1, 2, 2) - 2\Phi(1, 1, 1, 2) - 2\Phi(1, 1, 2, 1) + \Phi(1, 1, 1, 1) \right\} \right] = K^{-1}, \text{ say,} \]

where

\[
\Phi(l, m, p, q) = \sum_{j=0}^r M_j(t) \frac{\Gamma(r-j+l)}{a_{1p}^{r-j+l}} \frac{\Gamma(j+m)}{a_{2q}^{j+m}},
\]

\[ a_{1p} = S_1 + p\lambda_1 \text{ and } a_{2q} = \frac{S_2}{\theta} + q\lambda_2. \]

Hence the posterior joint probability density function of \((a, b)\) is given by

\[
f_2(a, b|t) = \frac{f_1^*(a, b, t)}{f_1^*(t)}
= K\lambda_1 \lambda_2 \sum_{j=0}^r a^{r-j}b^j M_j(t) \left[ e^{-a(S_1+\lambda_1)-b\left(\frac{S_2}{\theta}+\lambda_2\right)} + \rho \left\{ 4e^{-a(S_1+2\lambda_1)-b\left(\frac{S_2}{\theta}+2\lambda_2\right)} - 2e^{-a(S_1+\lambda_1)-b\left(\frac{S_2}{\theta}+2\lambda_2\right)} - 2e^{-a(S_1+2\lambda_1)-b\left(\frac{S_2}{\theta}+\lambda_2\right)} + e^{-a(S_1+\lambda_1)-b\left(\frac{S_2}{\theta}+\lambda_2\right)} \right\} \right].
\]

### 3.1 Estimation under squared error loss function

Let us consider the loss function of the form

\[ L_1^*(\psi, \hat{\psi}) = k_1(a - \hat{a})^2 + k_2(b - \hat{b})^2, \tag{3.6} \]

where \(k_1, k_2 > 0\), \(\psi = (a, b)\), and \(\hat{\psi} = (\hat{a}, \hat{b})\) is the estimator of \(\psi\). It is well known that under the loss function of the form \((3.6)\), Bayes’ estimator of a parameter is its posterior mean. Thus, Bayes’ estimators of \(a\) and \(b\) are given by

\[ \hat{a}_{BS} = E_{f_2}(a) = \int_0^\infty \int_0^\infty a f_2(a, b|t) da \, db \]
These, after simplification, reduce respectively to
\[ \hat{a}_{BS} = K\lambda_1\lambda_2[\Phi(2, 1, 1, 1) + \rho\{4\Phi(2, 1, 2, 2) - 2\Phi(2, 1, 1, 2) - 2\Phi(2, 1, 2, 1) + \Phi(2, 1, 1, 1)\}] \]
and
\[ \hat{b}_{BS} = K\lambda_1\lambda_2[\Phi(1, 2, 1, 1) + \rho\{4\Phi(1, 2, 2, 2) - 2\Phi(1, 2, 1, 2) - 2\Phi(1, 2, 2, 1) + \Phi(1, 2, 1, 1)\}] \].

### 3.2 Estimation under linex loss function

Here we consider the loss function of the form
\[ L_2(\psi, \hat{\psi}) = l_1 \left[ e^{c_1(\hat{a} - a)} - c_1(\hat{a} - a) - 1 \right] + l_2 \left[ e^{c_2(\hat{b} - b)} - c_2(\hat{b} - b) - 1 \right], \]
where \( c_1, c_2, l_1, l_2 \) are constants, and \( \psi \) and \( \hat{\psi} \) are as defined earlier.

The Bayes’ estimators of \( a \) and \( b \) are then given by
\[
\hat{a}_{BL} = -\frac{1}{c_1} \ln E_{f_2} \left( e^{-c_1a} | t \right) = -\frac{1}{c_1} \ln \left[ \int_0^\infty \int_0^\infty e^{-c_1a} f_2(a, b | t) da db \right]
\]
and
\[
\hat{b}_{BL} = -\frac{1}{c_2} \ln E_{f_2} \left( e^{-c_2b} | t \right) = -\frac{1}{c_2} \ln \left[ \int_0^\infty \int_0^\infty e^{-c_2b} f_2(a, b | t) da db \right],
\]
respectively. These, after simplification, reduce respectively to
\[
\hat{a}_{BL} = -\frac{1}{c_1} \ln \{ K\lambda_1\lambda_2[\Phi^*(1, 1, 1, 1) + \rho\{4\Phi^*(1, 1, 2, 2) - 2\Phi^*(1, 1, 2, 1) - 2\Phi^*(1, 1, 1, 2) + \Phi^*(1, 1, 1, 1)\}] \}
\]
and
\[
\hat{b}_{BL} = -\frac{1}{c_2} \ln \{ K\lambda_1\lambda_2[\Phi^{**}(1, 1, 1, 1) + \rho\{4\Phi^{**}(1, 1, 2, 2) - 2\Phi^{**}(1, 1, 2, 1) - 2\Phi^{**}(1, 1, 1, 2) + \Phi^{**}(1, 1, 1, 1)\}] \},
\]
where \( \Phi^* = \Phi \) with \( a_{1p} \) replaced by \( a^*_{1p} = c_1 + S_1 + p\lambda_1 \) and \( \Phi^{**} = \Phi \) with \( a_{2q} \) replaced by \( a^*_{2q} = c_2 + \frac{S^*}{\rho} + q\lambda_2 \).

### 3.3 Estimation under entropy loss function

Here we consider the loss function of the form
\[ L_3(\psi, \hat{\psi}) = m_1 \left[ \frac{\hat{a}}{\hat{a}} - \ln \frac{\hat{a}}{\hat{a}} - 1 \right] + m_2 \left[ \frac{\hat{b}}{\hat{b}} - \ln \frac{\hat{b}}{\hat{b}} - 1 \right], \]
where \( m_1 \) and \( m_2 \) are constants, and \( \psi \) and \( \hat{\psi} \) are as defined earlier.

The Bayes’ estimators of \( a \) and \( b \) are then given by

\[
\hat{a}_{BE} = \frac{1}{E_{f_2} (\frac{1}{a} | t)}
\]

and

\[
\hat{b}_{BE} = \frac{1}{E_{f_2} (\frac{1}{b} | t)}
\]

respectively, which, after simplification, reduce respectively to

\[
\hat{a}_{BE} = \left[ K \lambda_1 \lambda_2 \{ \Phi(0, 1, 1, 1) + \rho(4\Phi(0, 1, 2, 2) - 2\Phi(0, 1, 2, 1) - 2\Phi(0, 1, 1, 2) + \Phi(0, 1, 1, 1)) \} \right]^{-1}
\]

and

\[
\hat{b}_{BE} = \left[ K \lambda_1 \lambda_2 \{ \Phi(0, 1, 1, 1) + \rho(4\Phi(0, 1, 2, 2) - 2\Phi(0, 1, 2, 1) - 2\Phi(0, 1, 1, 2) + \Phi(0, 1, 1, 1)) \} \right]^{-1}
\]

4 Simulation and Conclusion

Once the Bayes’ estimators of the parameters \( a \) and \( b \) are obtained it might be of interest to know how the estimators behave for different values of the parameters of the underlying model. This is done in this section through a simulation study as is detailed below.

In the tables given in the Appendix, the values of \( \hat{a}_{BS} \), \( \hat{b}_{BS} \), \( \hat{a}_{BL} \), \( \hat{b}_{BL} \), \( \hat{a}_{BE} \) and \( \hat{b}_{BE} \) are given for different values of \( n \), \( r \), \( \theta \), \( \lambda_1 \), \( \lambda_2 \), \( \rho \), \( c_1 \) and \( c_2 \). Keeping the other parameters fixed, the values of \( \hat{a}_{BS} \), \( \hat{b}_{BS} \), \( \hat{a}_{BL} \), \( \hat{b}_{BL} \), \( \hat{a}_{BE} \) and \( \hat{b}_{BE} \), for different values of \( n \), \( r \), \( \theta \), \( \rho \), \( \lambda_1 \) and \( \lambda_2 \), are given in Tables 2-7 respectively. Note that here we have simulated more number of values to get a clear picture of \( \hat{a}_{BE} \). Those values are reported at the bottom portion of Table 3. It is to be mentioned here that \( \hat{a}_{BS} \), \( \hat{b}_{BS} \), \( \hat{a}_{BE} \) and \( \hat{b}_{BE} \) are independent of \( c_1 \) and \( c_2 \), whereas \( \hat{a}_{BL} \) and \( \hat{b}_{BL} \) vary with \( c_1 \) and \( c_2 \) respectively.

Table 8 shows the values of \( \hat{a}_{BL} \) and \( \hat{b}_{BL} \) for different values of \( c_1 \) and \( c_2 \) respectively, taking the other parameters fixed. In Table 1 given below the conclusions about the monotonicity of \( \hat{a}_{BS} \), \( \hat{b}_{BS} \), \( \hat{a}_{BL} \), \( \hat{b}_{BL} \), \( \hat{a}_{BE} \) and \( \hat{b}_{BE} \) have been made on the basis of the tabulated values. Once the values show non-monotone behavior, we have simulated for more number of values, and the specific non-monotonic behavior has been noted in Table 1.

To compute these values following methodology has been adopted and the simulation works have been done using R-Software.

(1) Markov Chain Monte Carlo (MCMC) method (using Metropolis-Hastings algorithm) has been adopted to generate samples from \( f_T(t) \), the target distribution, given in (2.4). We take proposal distribution as exponential with rate \((\lambda_1 + \lambda_2)/2\). In the simulation process, we take 5000 burn-in observations. Then, a sample of size \( n \) is chosen and out of these observations, Type-II censored sample \((t_1, t_2, \ldots, t_r)\) is collected censoring at the point \( r \).
1000 such repeated samples are collected and required calculations made. Between two consecutive samples, 100 observations are discarded to minimize the dependency.

(2) For \( j = 1, 2, \ldots, r \), the values of \( M_j(t) \) are calculated from the values of \( t_1, t_2, \ldots, t_r \).

(3) Taking some fixed values of \( n, r, \theta, \lambda_1, \lambda_2, \rho, c_1 \) and \( c_2 \), the values of \( \Phi(l, m, p, q) \), \( \Phi^*(l, m, p, q) \) and \( \Phi^{**}(l, m, p, q) \) are calculated for different values of \( l, m, p \) and \( q \), which give the values of \( \hat{a}_{BS}, \hat{b}_{BS}, \hat{a}_{BL}, \hat{b}_{BL}, \hat{a}_{BE} \) and \( \hat{b}_{BE} \).

(4) Averages are calculated from these 1000 values of \( \hat{a}_{BS}, \hat{b}_{BS}, \hat{a}_{BL}, \hat{b}_{BL}, \hat{a}_{BE} \) and \( \hat{b}_{BE} \), which give their final values. Variances of these \( \hat{a}_{BS}, \hat{b}_{BS}, \hat{a}_{BL}, \hat{b}_{BL}, \hat{a}_{BE} \) and \( \hat{b}_{BE} \) are also calculated and it is clear from Table 9 that all are decreasing functions of \( n \), as expected.

(5) Empirical Bayes risk for estimating \( a \) and \( b \) under different loss functions are also calculated using the definition

\[
R^E_i = \frac{1}{1000} \sum_{j=1}^{1000} L^*_i(\bar{\psi}, \hat{\psi}_j), \quad \text{where} \quad \bar{\psi} = \frac{1}{1000} \sum_{j=1}^{1000} \hat{\psi}_j,
\]

\( i = 1, 2, 3 \). Here \( i = 1 \) gives \( R^E_{BS} \), \( i = 2 \) gives \( R^E_{BL} \), and \( i = 3 \) gives \( R^E_{BE} \). Accordingly we define \( R^E_{BS}, R^E_{BL} \) and \( R^E_{BE} \) as empirical Bayes’ risk under squared error loss function, linex loss function and entropy loss function respectively. Without any loss of generality, we choose each of \( k_1, k_2, l_1, l_2, m_1, m_2 \) as unity. Taking fixed values of \( n, r, \theta, \lambda_1, \lambda_2, c_1 \) and \( c_2 \), as in Table 4 the values of \( R^E_{BS}, R^E_{BL} \) and \( R^E_{BE} \) are obtained under 1000 repetitions for different values of \( \rho \). It is observed from Table 10 that \( R^E_{BS} \) is decreasing in \( \rho \). It is observed from Table 10 that the values of \( R^E_{BS} \) is less for any \( \rho > 0 \) than that when \( a \) and \( b \) are taken as independent, i.e., \( \rho = 0 \). Similarly, we can find some \( \rho \) for which \( R^E_{BL} \) and \( R^E_{BE} \) are less than the risk for \( \rho = 0 \), which exhibits the necessity of taking \( a \) and \( b \) not independent.

We have chosen basically two types of loss functions - symmetric (squared error loss) and asymmetric (linex loss and entropy loss). Two asymmetric loss functions have been chosen for their distinctive features - linex loss function has different shapes depending on \( c_1 \) and \( c_2 \), whereas entropy loss function has no change in shape. It may be possible to choose a number of loss functions but these three are chosen as representatives.

Once the estimates of \( a \) and \( b \) are obtained, the estimates of the reliability function and the failure rate function can be computed using the relationships

\[
\hat{R}_X(t) = \exp \left[ -\left( \hat{a}t + \hat{b}t^\theta / \theta \right) \right]
\]

and

\[
\hat{r}(t) = \hat{a} + \hat{b}t^{\theta - 1}.
\]

For different values of \( t \), the estimates of these two reliability measures can be calculated.
Table 1: Conclusions

| Parameters | $a_{BS}$ | $b_{BS}$ | $a_{BL}$ | $b_{BL}$ | $a_{BE}$ | $b_{BE}$ |
|------------|----------|----------|----------|----------|----------|----------|
| $n$        | RC       | increasing | increasing | increasing | RC       | increasing |
| $r$        | decreasing | decreasing | BT       | decreasing | RC       | decreasing |
| $\theta$  | RC       | increasing | UBT      | BT       | RC       | increasing |
| $\rho$    | decreasing | decreasing | RC       | decreasing | RC       | decreasing |
| $\lambda_1$ | decreasing | increasing | decreasing | increasing | decreasing | BT       |
| $\lambda_2$ | BT       | decreasing | increasing | decreasing | BT       | decreasing |
| $c_1$      | unchanged | unchanged | decreasing | unchanged | unchanged | unchanged |
| $c_2$      | unchanged | unchanged | unchanged | decreasing | unchanged | unchanged |

The following abbreviations have been used in the above table.

BT: Bathtub shaped;
UBT: Upside down bathtub shaped;
RC: Roller coaster.

References

[1] Ashour, S.K. and Youssef, A. (1991). Bayesian estimation of a linear failure rate. *IAPQR Transactions*, 16, 9-16.

[2] Bain, L.J. (1974). Analysis of linear failure rate life-testing distributions. *Technometrics*, 16, 551-560.

[3] Carbone, P. P., Kellerhouse, L.E. and Gehan, E.A. (1967). Plasmacytic myeloma: A study of the relationship of survival to various clinical manifestations and anomalous protein type in 112 patients. *American Journal of Medicine*, 42(6), 937-948.

[4] Farlie, D.J.G. (1960). The performance of some correlation coefficients for a general bivariate distribution. *Biometrika*, 47, 307-323.

[5] Kodlin, D. (1967). A new response time distribution. *Biometrics*, 23(2), 227-239.

[6] Lawless, J.F. (1982). *Statistical Models and Methods for Lifetime Data*. John Wiley and Sons, New York.

[7] Lin, C.T., Wu, S.J.S. and Balakrishnan, N. (2006a). Monte Carlo methods for Bayesian inference on the linear hazard rate distribution. *Communications in Statistics – Simulation and Computation*, 35(3), 575-590.
[8] Lin, C.T., Wu, S.J.S. and Balakrishnan, N. (2006b). Parameter estimation for the linear hazard rate distribution based on records and inter-record times. *Communications in Statistics-Theory and Methods*, **32**, 729-748.

[9] Pandey, A., Singh, A. and Zimmer, W.J. (1993). Bayes estimation of the linear hazard rate model. *IEEE Transactions on Reliability*, **42**, 636-640.

[10] Sarhan, A.M. (1999). Bayes estimation of the general hazard rate model. *Reliability Engineering and System Safety*, **66**, 85-91.

[11] Sarhan, A.M. (2004). Parameter estimations in a general hazard rate model using masked data. *Applied Mathematics and Computation*, **153**, 513-536.

[12] Sen, A. (2006). Linear hazard rate distribution. In: *Encyclopedia of Statistical Sciences*, Kotz, S., Balakrishnan, N., Read, C. B. and Vidakovic, B. (Eds), 2nd ed., **6**, 4212-4217, Hoboken, New Jersey: John Wiley and Sons.

[13] Sen, A. and Bhattacharyya, G.K. (1995). Inference procedures for linear failure rate model. *Journal of Statistical Planning and Inference*, **44**, 59-76.

[14] Shaked, M. (1978). Accelerated life testing for a class of linear hazard rate type distributions. *Technometrics*, **20(4)**, 457-466.

[15] Usher, J.S. and Hodgson, T.J. (1988). Maximum likelihood analysis of component reliability using masked system life-test data. *IEEE Transactions on Reliability*, **37**, 550-555.
Appendix 1

Table 2: Table for different values of $n$ ($r = 15$, $\theta = 1.5$, $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\rho = 0.5$, $c_1 = 5$, $c_2 = 10$).

| $n$ | $\hat{a}_{BS}$ | $\hat{b}_{BS}$ | $\hat{a}_{BL}$ | $\hat{b}_{BL}$ | $\hat{a}_{BE}$ | $\hat{b}_{BE}$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 20  | 2.428505       | 2.470713       | 0.9571143      | 0.3315961      | 1.841096       | 1.108504       |
| 30  | 2.47891        | 2.809811       | 0.9602245      | 0.3504222      | 1.869938       | 1.24303        |
| 50  | 2.570068       | 3.526479       | 1.031021       | 0.3728826      | 1.981042       | 1.544343       |
| 80  | 2.702363       | 3.833807       | 1.106235       | 0.3769399      | 2.144507       | 1.685688       |
| 90  | 2.651358       | 3.922087       | 1.117268       | 0.3771751      | 2.123916       | 1.721286       |
| 100 | 2.668780       | 3.99165        | 1.149213       | 0.3780229      | 2.161693       | 1.764691       |

\(^1\)Some additional simulation results in support of Table 1 for estimates which show non-monotone behavior are placed on the website: ftp://210.212.53.189:4777 (user id: ftpadmin, password: rgipt@7890)
Table 3: Table for different values of $r$ ($n = 50$, $\theta = 1.5$, $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\rho = 0.5$, $c_1 = 5$, $c_2 = 10$).

| $r$ | $\hat{a}_{BS}$ | $b_{BS}$ | $\hat{a}_{BL}$ | $b_{BL}$ | $\hat{b}_{BE}$ |
|-----|----------------|----------|----------------|----------|---------------|
| 10  | 2.889158       | 3.872732 | 1.046990       | 0.3765202| 1.92015       |
| 15  | 2.588836       | 3.444316 | 1.045380       | 0.3690954| 1.511899      |
| 20  | 2.476374       | 3.041744 | 1.073117       | 0.3585487| 1.254549      |
| 25  | 2.303992       | 2.494415 | 1.144468       | 0.3398418| 1.003384      |
| 35  | 2.153625       | 1.575009 | 1.260905       | 0.2871699| 0.620085      |
| 45  | 2.056847       | 0.8700868| 1.416823       | 0.2131185| 0.3430589     |

| $r$ | $\hat{a}_{BE}$ | $\hat{b}_{BE}$ | $\hat{a}_{BE}$ | $\hat{b}_{BE}$ | $\hat{a}_{BE}$ | $\hat{b}_{BE}$ | $\hat{a}_{BE}$ | $\hat{b}_{BE}$ | $\hat{a}_{BE}$ | $\hat{b}_{BE}$ | $\hat{a}_{BE}$ | $\hat{b}_{BE}$ | $\hat{a}_{BE}$ | $\hat{b}_{BE}$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 5   | 2.767692       | 2.413989       | 2.11847        | 2.017851       | 2.010617       | 1.991297       | 1.970913       | 2.041148       | 2.019716       | 2.00211        | 2.005149       | 2.060117       | 2.058064       | 2.024053       |
| 10  | 2.196753       | 1.991297       | 1.970913       | 2.041148       | 2.019716       | 2.00211        | 2.005149       | 2.060117       | 2.058064       | 2.024053       | 1.953117       | 1.900956       | 1.787256       |

Table 4: Table for different values of $\theta$ ($n = 50$, $r = 15$, $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\rho = 0.5$, $c_1 = 5$, $c_2 = 10$).

| $\theta$ | $\hat{a}_{BS}$ | $\hat{b}_{BS}$ | $\hat{a}_{BL}$ | $\hat{b}_{BL}$ | $\hat{a}_{BE}$ | $\hat{b}_{BE}$ |
|-----------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1.2       | 2.53878        | 2.698973       | 0.7468843      | 0.3592404      | 1.484473       | 1.118551       |
| 1.3       | 2.498865       | 2.978176       | 0.8089464      | 0.3638382      | 1.619551       | 1.247953       |
| 1.5       | 2.612741       | 3.455287       | 1.043905       | 0.3682661      | 2.023602       | 1.511676       |
| 1.7       | 2.655687       | 3.853542       | 1.295282       | 0.3755059      | 2.250713       | 1.815087       |
| 2.0       | 2.617994       | 4.128254       | 1.586975       | 0.3744949      | 2.357008       | 2.203706       |
| 2.5       | 2.649079       | 4.335498       | 1.803262       | 0.3745662      | 2.548854       | 2.838067       |
| 3.0       | 2.569748       | 4.389099       | 1.803354       | 0.3730589      | 2.400356       | 3.306292       |
| 3.5       | 2.531871       | 4.358319       | 1.802254       | 0.3713863      | 2.370501       | 3.552629       |
| 4.0       | 2.546643       | 4.336425       | 1.814499       | 0.3706115      | 2.386205       | 3.736094       |
| 4.5       | 2.504744       | 4.322051       | 1.792483       | 0.370061      | 2.347539       | 3.79589       |
| 5.0       | 2.474104       | 4.302419       | 1.776677       | 0.3694796      | 2.319187       | 3.846737       |
| 7.0       | 2.445197       | 4.281886       | 1.761036       | 0.3689414      | 2.292343       | 3.894736       |
| 10.0      | 2.466696       | 4.282187       | 1.775499       | 0.3689829      | 2.312524       | 3.904676       |
Table 5: Table for different values of $\rho$ ($n = 50, r = 15, \theta = 1.5, \lambda_1 = 0.1, \lambda_2 = 0.2, c_1 = 5, c_2 = 10$).

| $\rho$ | $\hat{a}_{BS}$ | $\hat{b}_{BS}$ | $\hat{a}_{BL}$ | $\hat{b}_{BL}$ | $\hat{a}_{BE}$ | $\hat{b}_{BE}$ |
|--------|----------------|----------------|----------------|----------------|----------------|----------------|
| -1.0   | 3.082916       | 5.472054       | 1.067603       | 0.4478139      | 3.009227       | 2.636043       |
| -0.8   | 2.993178       | 5.280714       | 1.038365       | 0.4372742      | 2.833177       | 2.469603       |
| -0.5   | 2.877378       | 4.904747       | 1.039639       | 0.4222705      | 2.763539       | 2.234952       |
| -0.1   | 2.747384       | 4.355944       | 1.021720       | 0.4007587      | 2.598687       | 1.927057       |
| 0.0    | 2.707445       | 4.114485       | 1.057667       | 0.3925779      | 2.736950       | 1.81324        |
| 0.1    | 2.687283       | 3.867259       | 1.045336       | 0.3841437      | 2.6258         | 1.699069       |
| 0.2    | 2.680645       | 3.844324       | 1.056197       | 0.3826008      | 2.664627       | 1.689498       |
| 0.5    | 2.675599       | 3.460527       | 1.073620       | 0.3685548      | 2.652695       | 1.527531       |
| 0.8    | 2.556038       | 3.089788       | 1.117351       | 0.3578624      | 2.623406       | 1.379521       |
| 1.0    | 2.53029        | 2.902948       | 1.166945       | 0.3519428      | 2.651116       | 1.306628       |

Table 6: Table for different values of $\lambda_1$ ($n = 50, r = 15, \theta = 1.5, \lambda_2 = 0.2, \rho = 0.5, c_1 = 5, c_2 = 10$).

| $\lambda_1$ | $\hat{a}_{BS}$ | $\hat{b}_{BS}$ | $\hat{a}_{BL}$ | $\hat{b}_{BL}$ | $\hat{a}_{BE}$ | $\hat{b}_{BE}$ |
|-------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0.05        | 4.05366        | 3.646147       | 1.435573       | 0.3633854      | 3.447897       | 1.70214        |
| 0.1         | 2.443404       | 3.356118       | 0.9997084      | 0.368385       | 1.866178       | 1.464352       |
| 0.2         | 1.656511       | 3.069585       | 0.761078       | 0.3770261      | 1.136215       | 1.360706       |
| 0.4         | 1.156696       | 2.816409       | 0.5889041      | 0.3915007      | 0.71128        | 1.323718       |
| 0.7         | 0.893846       | 2.762379       | 0.486466       | 0.4145108      | 0.5056924      | 1.411568       |
| 1.0         | 0.7624006      | 2.843336       | 0.4256194      | 0.4462848      | 0.4065848      | 1.580224       |
| 1.5         | 0.6312949      | 3.169195       | 0.3620832      | 0.4954608      | 0.3158953      | 1.916876       |
| 2.0         | 0.535375       | 3.588683       | 0.3153288      | 0.5511547      | 0.2584508      | 2.311455       |
Table 7: Table for different values of $\lambda_2$ ($n = 50$, $r = 15$, $\theta = 1.5$, $\lambda_1 = 0.1$, $\rho = 0.5$, $c_1 = 5$, $c_2 = 10$).

| $\lambda_2$ | $\hat{a}_{BS}$ | $\hat{b}_{BS}$ | $\hat{a}_{BL}$ | $\hat{b}_{BL}$ | $\hat{a}_{BE}$ | $\hat{b}_{BE}$ |
|-------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0.05        | 3.292485       | 10.68008       | 0.9125832      | 0.5066623      | 2.151261       | 4.835694       |
| 0.1         | 2.926236       | 5.977912       | 0.9462215      | 0.4347056      | 2.061211       | 2.609884       |
| 0.2         | 2.596785       | 3.441812       | 1.032488       | 0.3696412      | 2.017182       | 1.512799       |
| 0.4         | 2.485133       | 1.934206       | 1.192755       | 0.3050227      | 2.088596       | 0.8874038      |
| 0.7         | 2.507843       | 1.179365       | 1.427955       | 0.2527033      | 2.221971       | 0.5899946      |
| 1.0         | 2.621771       | 0.850314       | 1.557053       | 0.2211348      | 2.378278       | 0.4535783      |
| 1.5         | 2.847125       | 0.5858818      | 1.835657       | 0.1876845      | 2.630853       | 0.3536515      |
| 2.0         | 3.064069       | 0.444568       | 2.00267        | 0.164858       | 2.849913       | 0.2923578      |

Table 8: Table for different values of $c_1$ and $c_2$ ($n = 50$, $r = 15$, $\theta = 1.5$, $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\rho = 0.5$).

| $c_1$ | 5      | 10     | 20     | 40     | 70     | 100    |
|-------|--------|--------|--------|--------|--------|--------|
| $\hat{a}_{BL}$ | 1.043131 | 0.6219532 | 0.3624302 | 0.1955869 | 0.1200798 | 0.198313 |
| $\hat{c}_{BL}$ | 1      | 2.5    | 40     | 70     | 100    |
| $\hat{b}_{BL}$ | 0.2280337 | 0.1650221 | 0.1116809 | 0.07163291 | 0.0484698 | 0.03745599 |

Table 9: Variance table for different values of $n$ ($r = 15$, $\theta = 1.5$, $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\rho = 0.5$, $c_1 = 5$, $c_2 = 10$).

| $n$  | $\sigma_{BS}$ | $\sigma_{BL}$ | $\sigma_{BE}$ | $\sigma_{BE}$ |
|------|----------------|----------------|----------------|----------------|
| 20   | 2.786046       | 2.801144       | 0.1687665      | 0.167134       |
| 50   | 1.88847        | 1.857697       | 0.1675985      | 0.167134       |
| 80   | 1.476776       | 1.527406       | 0.1652423      | 0.1411586      |
| 90   | 1.398129       | 1.306582       | 0.1686950      | 0.1620072      |
| 100  | 1.398129       | 1.306582       | 0.1686950      | 0.1620072      |
Table 10: Risk table for different values of $\rho \ (n = 50, \ r = 15, \ \theta = 1.5, \ \lambda_1 = 0.1, \ \lambda_2 = 0.2, \ c_1 = 5, \ c_2 = 10)$.

| $\rho$ | $R^E_{BS}$ | $R^E_{BL}$ | $R^E_{BE}$ |
|--------|------------|------------|------------|
| -1.0   | 5.378928   | 37.31374   | 0.2712732  |
| -0.8   | 5.012633   | 68.0811    | 0.2837661  |
| -0.5   | 4.726565   | 75.09191   | 0.2942592  |
| -0.2   | 4.696967   | 27.32544   | 0.3198501  |
| -0.1   | 4.579706   | 32.56944   | 0.3070936  |
| 0.0    | 4.320078   | 123.9033   | 0.3094569  |
| 0.1    | 4.137500   | 21.84179   | 0.2806128  |
| 0.2    | 4.097972   | 48.70084   | 0.2945915  |
| 0.5    | 3.628694   | 36.62481   | 0.3146239  |
| 0.8    | 3.261719   | 22.28436   | 0.3165357  |
| 1.0    | 3.206366   | 78.3708    | 0.3034506  |