Gravity on Conformal Superspace

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To my family
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Abstract

The configuration space of general relativity is *superspace* - the space of all Riemannian 3-metrics modulo diffeomorphisms. However, it has been argued that the configuration space for gravity should be *conformal superspace* - the space of all Riemannian 3-metrics modulo diffeomorphisms *and* conformal transformations. Taking this conformal nature seriously leads to a new theory of gravity which although very similar to general relativity has some very different features particularly in cosmology and quantisation. It should reproduce the standard tests of general relativity. The cosmology is studied in some detail. The theory is incredibly restrictive and as a result admits an extremely limited number of possible solutions. The problems of the standard cosmology are addressed and most remarkably the cosmological constant problem is resolved in a natural way. The theory also has several attractive features with regard to quantisation particularly regarding the problem of time.
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Chapter 1

Introduction

1.1 Introduction

As formulated by Einstein, the natural arena for gravity as represented by general relativity (GR) is spacetime. We have a purely 4-dimensional structure and the 4-geometry reigns. (The invention of GR was a truly \textit{monumental} achievement and no offence is intended by any attempt here to suggest an alternative theory.) Dirac \cite{Dirac} and Arnowitt, Deser and Misner (ADM) \cite{ADM} reformulated the theory in canonical form which is more in-keeping with other areas of modern physics. This formulation led to Wheeler’s identification of the configuration space as superspace and GR as the theory of the evolution of the 3-geometry which led to the coining (again by Wheeler) of geometrodynamics. To get superspace one first considers $\text{Riem}$ the space of all Riemannian 3-geometries. Superspace is then $\text{Riem}$ modulo diffeomorphisms, that is, we identify all 3-geometries related by diffeomorphisms.

York \cite{York} went further and identified the conformal 3-geometry with the dynamical degrees of freedom of the gravitational field. The correct configuration space for gravity should not be superspace but rather \textit{conformal superspace} - superspace modulo conformal transformations. Barbour and Ó Murchadha (BOM) \cite{Barbour} went further again and formulated a theory with conformal superspace at the very core.

We’ll begin with a brief review of GR as found from the Einstein-Hilbert action and the ADM formulation. We’ll then discuss the York approach and the original BOM theory. All of this will serve as a warm up (albeit, a necessary warm up) to the real focus of
1.2 General Relativity

Although Einstein developed GR using beautiful physical reasoning and principles it is
the Hilbert derivation from an action principle which is more instructive to us. (We will
however refer to the action as the Einstein-Hilbert action as it was Einstein’s work which
inspired Hilbert to find the action to begin with.)

The Einstein-Hilbert action of general relativity is well known. It has the form

\[ S = \frac{1}{16\pi} \int \sqrt{-g} R \, d^4x \quad (1.1) \]

where \( g_{\alpha\beta} \) is the 4-metric and \( R \) is the four dimensional Ricci scalar. The action is varied
with respect to \( g_{\alpha\beta} \) and the resulting equations are the (vacuum) Einstein equations

\[ G^{\alpha\beta} = \left( R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) = 0 \quad (1.2) \]

matter sources may be included in the action and the resulting equations of motion are
the full Einstein equations

\[ G^{\alpha\beta} = 8\pi T^{\alpha\beta} \quad (1.3) \]

where \( T^{\alpha\beta} \) is the energy-momentum tensor of the matter field given by

\[ T^{\alpha\beta} = g^{\alpha\beta} L_{\text{matter}} - 2 \frac{\delta L_{\text{matter}}}{\delta g_{\alpha\beta}} \quad (1.4) \]

1.3 (3+1)-Decomposition

Before we consider the new theory it will be instructive to recall the ADM treatment of
general relativity as much of this will carry straight over to the new theory.

The idea in the ADM treatment is that a thin-sandwich 4-geometry is constructed from
two 3-geometries separated by the proper time \( d\tau \). The 4-metric found from the ADM
construction is

\[
\begin{pmatrix}
(4) g_{00} & (4) g_{0k} \\
(4) g_{i0} & (4) g_{ik}
\end{pmatrix}
= \begin{pmatrix}
(N^s N_s - N^2) & N_k \\
N_i & g_{ik}
\end{pmatrix} \quad (1.5)
\]
$N = N(t, x, y, z)$ is the lapse function given by

$$d\tau = N(t, x, y, z) dt$$  \hspace{1cm} (1.6)

and $N^i = N^i(t, x, y, z)$ are the shift functions given by

$$x^i_2(x^m) = x^i_1 - N^i(t, x, y, z) dt$$  \hspace{1cm} (1.7)

where $x^i_2$ is the position on the “later” hypersurface corresponding to the position $x^i_1$ on the “earlier” hypersurface. The indices in the shift are raised and lowered by the 3-metric $g_{ij}$.

The reciprocal 4-metric is

$$\begin{vmatrix}
(4)g^{00} & (4)g^{0k} \\
(4)g^{i0} & (4)g^{ik}
\end{vmatrix}
= \begin{vmatrix}
-1/N^2 & N^k/N^2 \\
N^i/N^2 & g^{ik} - N^iN^k/N^2
\end{vmatrix}$$  \hspace{1cm} (1.8)

The volume element has the form

$$\sqrt{(4)g} \, d^4x = N\sqrt{g} \, dt \, d^3x$$  \hspace{1cm} (1.9)

This construction of the four metric also automatically determines the components of the unit timelike normal vector $n$. We get

$$n_\beta = (-N, 0, 0, 0)$$  \hspace{1cm} (1.10)

and raising the indices using $(4)g^{\alpha\beta}$ gives us

$$n^\alpha = (1/N, -N^m/N)$$  \hspace{1cm} (1.11)

Consider now the Einstein-Hilbert action

$$S = \int \sqrt{-^{(4)}g} \, ^{(4)}R \, d^4x$$  \hspace{1cm} (1.12)

Using the Gauss-Codazzi relations we get

$$(^{(4)}R = R - (trK)^2 + K^{ab}K_{ab} - 2A^{\alpha}_{,\alpha}$$  \hspace{1cm} (1.13)

where $A^\alpha$ is given by (as earlier)

$$A^\alpha = \left( n^\alpha trK + a^\alpha \right)$$  \hspace{1cm} (1.14)
\( n^\alpha \) is the unit timelike normal and

\[
a^\alpha = n^\alpha,^\beta n^\beta \tag{1.15}
\]
is the four-acceleration of an observer travelling along \( n \). It is easily verified that \( a^0 = 0 \) and that \( a^i = \nabla_i N \). Substituting into the action gives

\[
S = \int N \sqrt{g} (R - (trK)^2 + K^{ab} K_{ab}) dt d^3x \tag{1.16}
\]

where the total divergence \( A^\alpha_{\;\alpha} \) has been discarded. \( K \) is the extrinsic curvature given by

\[
K = -\frac{1}{2} \mathcal{L}_n g \tag{1.17}
\]

the Lie derivative of the 3-metric \( g \) along \( n \). In the coordinates we are using here the extrinsic curvature takes the form

\[
K_{ab} = -\frac{1}{2N} \left( \frac{\partial g_{ab}}{\partial t} - N_{a,b} - N_{b,a} \right) \tag{1.18}
\]

The action is varied with respect to \( \frac{\partial g_{ab}}{\partial t} \) to get the canonical momentum

\[
\pi^{ab} = \sqrt{g} (g^{ab} trK - K^{ab}) \tag{1.19}
\]

and varied with respect to \( N \) and \( N_a \) to give the initial value equations

\[
\mathcal{H} = 0 \quad \text{and} \quad \mathcal{H}^a = 0 \tag{1.20}
\]

respectively, where

\[
\mathcal{H} = \sqrt{g} \left( \pi^{ab} \pi_{ab} - \frac{1}{2} (tr \pi)^2 \right) - \sqrt{g} R \tag{1.21}
\]

and

\[
\mathcal{H}^a = -2 \pi^{ab} ;_b \tag{1.22}
\]

these are known as the Hamiltonian constraint and the momentum constraint respectively. The Hamiltonian of the theory is then given by

\[
H = \int \left( N \mathcal{H} + N_i \mathcal{H}^i \right) d^3x \tag{1.23}
\]

We get evolution equations for \( g_{ab} \) and \( \pi^{ab} \) by varying the Hamiltonian with respect to \( \pi^{ab} \) and \( g_{ab} \) respectively, using Hamilton’s equations

\[
\frac{\partial g_{ab}}{\partial t} = \frac{\delta H}{\delta \pi^{ab}} \tag{1.24}
\]

\[
\frac{\partial \pi^{ab}}{\partial t} = -\frac{\delta H}{\delta g_{ab}} \tag{1.25}
\]

These equations propagate the constraints. Solutions are a pair \( \{ g_{ab}, \pi^{ab} \} \) which satisfy the constraints and are then evolved using the evolution equations. The lapse and shift \( (N \text{ and } N^i) \) are specified initially but after that are freely specifiable. This is the 4-dimensional covariance.
1.4 York’s Approach

The Hamiltonian and momentum constraints correspond to the 00 and 0a components of Einstein’s equations (1.3). They are equivalently initial-value constraints. We need to be able to find initial data which satisfy these. One method was proposed by Baerlein, Sharp and Wheeler (BSW) [5]. This is known as the thin-sandwich conjecture. First the pair \( \{ g_{ab}, \frac{\partial g_{ab}}{\partial t} \} \) are freely specified and then the momentum constraint is solved for the shift \( N^i \). Although progress has been made, a regular method to solve this has not been found. A second method is York’s conformal approach.

There are actually two different York methods (although they are very intimately related). In the first we begin with a maximal hypersurface. That is, the trace of the momentum is zero

\[
tr \pi = 0 \tag{1.26}
\]
everywhere on the hypersurface. Under these conditions the momentum constraint (1.22) is invariant under a conformal transformation of the form

\[
g_{ab} \rightarrow \phi^4 g_{ab} \tag{1.27}
\]

\[
\pi^{ab} \rightarrow \phi^{-4} \pi^{ab} \tag{1.28}
\]

Transforming the Hamiltonian constraint under the same transformation gives the Lichnerowicz equation

\[
\pi^{ab} \pi_{ab} \phi^{-7} - R \phi + 8 \nabla^2 \phi = 0 \tag{1.29}
\]

York’s approach is to solve the momentum constraint in a conformally invariant way (and such a way is well known) and then to solve the Lichnerowicz equation for \( \phi \). The physical data is then \( \{ \phi^4 g_{ab}, \phi^{-4} \pi^{ab} \} \).

It turns out that the decoupling of the two constraints is still simple when the initial hypersurface has constant mean curvature (CMC) rather than being maximal. The CMC condition is that

\[
tr p = \frac{tr \pi}{\sqrt{g}} = \text{spatial constant} \tag{1.30}
\]

We should introduce some new terminology here. The tracefree part of the momentum is

\[
s^{ab} = \pi^{ab} - \frac{1}{3} g^{ab} tr \pi \tag{1.31}
\]

Now, if the CMC condition holds then the momentum constraint reduces to

\[
\nabla_b s^{ab} = 0 \tag{1.32}
\]
Now the tracefree part is transverse-traceless (TT). This property is invariant under the conformal transformation

$$g_{ab} \rightarrow \omega^4 g_{ab} \quad (1.33)$$

$$\sigma^{ab} \rightarrow \omega^{-4} \sigma^{ab} \quad (1.34)$$

It is important here that \( tr\pi \) now transforms in a different way to the tracefree part \( \sigma^{ab} \). For the momentum constraint to be conformally invariant we need to define

$$trp = \frac{tr\pi}{\sqrt{g}} \rightarrow trp \quad (1.35)$$

That is, \( trp \) transforms as a conformal scalar. Since there is a well known method to find a TT tensor we can find the pair \( \{g_{ab}, \sigma^{ab}\} \) easily. The Hamiltonian constraint transforms to become

$$\sigma^{ab}\sigma_{ab}\phi^7 - \frac{1}{6}(tr\pi)^2\phi^5 - R\phi + 8 \nabla^2\phi = 0 \quad (1.36)$$

the extended Lichnerowicz equation. Specifying \( g_{ab}, \sigma^{ab} \) and \( trp \) we can solve for \( \phi \) and then our physical data is \( \{\phi^4 g_{ab}, \omega^{-4} \sigma^{ab}, trp\} \).

In GR the conditions (1.26) and (1.30) are gauge conditions. If we are dealing with a manifold which is compact without boundary then we cannot have the maximal condition more than once. However, we may have the CMC condition holding always. It yields a foliation that is extremely convenient in the case of globally hyperbolic spacetimes. It is unique and the value of \( trp \) increases monotonically either from \(-\infty\) to \(+\infty\) in the case of a big bang to big crunch cosmological solution or from \(-\infty\) to 0 in the case of eternally expanding universes. In the first case the volume of the universe increases monotonically from 0 to a point of maximum expansion at which the hypersurface is maximal. From this point on it decreases monotonically back to 0. The volume cannot stay constant except (momentarily) at the maximum expansion when \( trp = 0 \). Thus, in GR the volume is dynamic which is of course the standard explanation of the cosmological redshift. One further point is that the quantity

$$\tau = \frac{2}{3} trp \quad (1.37)$$

is often interpreted as a notion of time, the York time, due to the properties of \( trp \) noted above.

### 1.4.1 Gauge Fixing in GR

It is important to notice the difference between a single use of the CMC condition to find initial data and subsequent use of the condition when the data is propagated. This is by
no means guaranteed. As noted earlier, once the initial data has been specified the lapse and shift are freely specifiable. To maintain the CMC slicing during the evolution it is necessary to choose the lapse in a particular way. Using the evolution equations we get

$$\frac{\partial \text{tr}_p}{\partial t} = 2NR - 2 \nabla^2 N + \frac{N(\text{tr}_p)^2}{4} + \nabla_c \text{tr}_\pi \sqrt{g} N^c$$

(1.38)

To ensure CMC slicing we need to set $\nabla_c \text{tr}_\pi = 0$ and $\frac{\partial \text{tr}_p}{\partial t} = C$. That is

$$\frac{\partial \text{tr}_p}{\partial t} = 2NR - 2 \nabla^2 N + \frac{N(\text{tr}_p)^2}{4} = C$$

(1.39)

where $C$ is a spatial constant but not necessarily a temporal constant. If we wish to maintain a maximal slicing we must have $\text{tr}_\pi = 0$ and $\frac{\partial \text{tr}_p}{\partial t} = 0$. Thus

$$\frac{\partial \text{tr}_p}{\partial t} = 2NR - 2 \nabla^2 N = 0$$

(1.40)

As mentioned earlier, this particular condition cannot be maintained in a spacetime which is compact without boundary but can be maintained in an asymptotically flat spacetime. This will be dealt with in more detail later and will be of huge significance in the conformal theories developed. It should be noted that the conditions (1.39) and (1.40) do not fix the lapse uniquely since they are homogeneous in the lapse. They fix $N$ up to a global reparameterisation

$$N \rightarrow f(t)N$$

(1.41)

where $f(t)$ is an arbitrary monotonic function of $t$. These lapse-fixing equations arise naturally in the conformal theory as we shall see.

We should be ready now to move onto the original BOM conformal theory. We shall present it in a slightly different form however. The BOM action was of the Jacobi form. We shall derive a Lagrangian in the spirit of the traditional ADM ($3 + 1$)-dimensional action of GR.

### 1.5 Lagrangian and Hamiltonian Formulations

#### 1.5.1 The Lagrangian

To begin with let’s recall the ADM action of GR (1.16)

$$S = \int N \sqrt{g}(R - (\text{tr} K)^2 + K^{ab}K_{ab}) dt d^3x$$

(1.42)
To find the conformal action we simply transform the Lagrangian under the transformation
\[ g_{ab} \longrightarrow \psi^4 g_{ab} \]  
\( (1.43) \)

We need to define how the lapse and shift are transformed under such a transformation. In a later chapter we will see that this theory can be found using a 4-dimensional action where
\[ g_{\alpha\beta} \longrightarrow \psi^4 g_{\alpha\beta} \]  
\( (1.44) \)

and under this we would have
\[ N \longrightarrow \psi^2 N \]  
\( (1.45) \)

and
\[ N_i \longrightarrow \psi^4 N_i \]  
\( (1.46) \)

Let’s adopt these as our transformation rules. Under such a transformation
\[ R \longrightarrow \psi^{-4} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) \]  
\( (1.47) \)

The extrinsic curvature is a little more tricky. We have
\[ K_{ab} = -\frac{1}{2N} \left( \frac{\partial g_{ab}}{\partial t} - (KN)_{ab} \right) \]  
\( (1.48) \)

where
\[ (KN)_{ab} = \nabla_a N_b + \nabla_b N_a \]  
\( (1.49) \)

Under a conformal transformation the various quantities behave as
\[ N \longrightarrow \psi^2 N \]
\[ \frac{\partial g_{ab}}{\partial t} \longrightarrow \psi^4 \frac{\partial g_{ab}}{\partial t} + 4 \psi^3 g_{ab} \frac{\partial \psi}{\partial t} \]
\( (1.50) \)

\[ (KN)_{ab} \longrightarrow \psi^4 (KN)_{ab} + 4 \psi^3 g_{ab} N^c \nabla_c \psi \]

Thus
\[ K_{ab} \longrightarrow \psi^2 B_{ab} = -\frac{\psi^2}{2N} \left( \frac{\partial g_{ab}}{\partial t} - (KN)_{ab} - \theta g_{ab} \right) \]  
\( (1.51) \)

where
\[ \theta = -\frac{4}{\psi} \left( \frac{\partial \psi}{\partial t} - N^c \nabla_c \psi \right) \]  
\( (1.52) \)

The Lagrangian is thus
\[ \mathcal{L} = N \sqrt{g} \psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} + B^{ab} B_{ab} - (tr B)^2 \right) \]  
\( (1.53) \)
Note: We had $\theta$ in terms of $\psi$: $\theta = -\frac{4}{\psi}(\dot{\psi} - \psi, N^i)$. We can also find a coordinate independent form for $B$. This is

$$B = -\frac{1}{2}\psi^{-1}\mathcal{L}_2(\psi^4g) \quad (1.54)$$

This is analogous to the expression

$$K = -\frac{1}{2}\mathcal{L}_2(g) \quad (1.55)$$

for the extrinsic curvature $K$ in general relativity.

### 1.5.2 Constraints and Evolution Equations

We can perform the usual variations to find the constraints of the theory. Let’s vary with respect to $N$ first. This gives us,

$$R - 8\nabla^2\psi + (trB)^2 - B^{ab}B_{ab} = 0 \quad (1.56)$$

Varying with respect to $N^a$ gives us,

$$\nabla^b\left(\psi^4\left(g_{ab}trB - B_{ab}\right)\right) - 4\psi^3\psi^a trB = 0 \quad (1.57)$$

As noted earlier we may vary with respect to $\psi$ and $\dot{\psi}$ independently. The $\dot{\psi}$ variation gives us,

$$trB = 0 \quad (1.58)$$

This greatly simplifies equation (1.57) which now becomes

$$\nabla^b\left(\psi^4B_{ab}\right) = 0 \quad (1.59)$$

The $\psi$ variation gives us,

$$N\psi^3\left(R - 7\nabla^2\psi\right) - \nabla^2\left(N\psi^3\right) = 0 \quad (1.60)$$

where we have used the other constraints to simplify. The constraints may appear more familiar if we write them in terms of the canonical momentum rather than $B_{ab}$. We find the canonical momentum, $\pi^{ab}$ by varying the action with respect to $\frac{g_{ab}}{\partial t}$. We get

$$\pi^{ab} = \sqrt{g}\psi^4\left(g^{ab}trB - B^{ab}\right) \quad (1.61)$$

Then using equation (1.58) we get

$$\pi^{ab} = -\sqrt{g}\psi^4B^{ab} \quad (1.62)$$
The constraints are then,

\[ \pi^{ab} \pi_{ab} - g \psi^8 \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) = 0 \]  
\[ (1.63) \]

\[ \nabla_b \pi^{ab} = 0 \]  
\[ (1.64) \]

\[ tr \pi = 0 \]  
\[ (1.65) \]

\[ N \psi^3 \left( R - 7 \frac{\nabla^2 \psi}{\psi} \right) - \nabla^2 \left( N \psi^3 \right) = 0 \]  
\[ (1.66) \]

Equation (1.63) corresponds to the Hamiltonian constraint of General Relativity. Equation (1.64) is the usual momentum constraint of general relativity which represents diffeomorphism invariance. Equation (1.65) is new and represents conformal invariance. Our initial data consists of a pair \((g_{ab}, \pi^{ab})\) which must satisfy equations (1.64) and (1.65). These are the initial value equations. Equation (1.63) is used to find the “conformal field” \(\psi\) once we have specified the initial data. Equation (1.66) is a lapse-fixing equation which is used to determine \(N\) throughout. We must check if these constraints are propagated under evolution.

The evolution equations are found in the usual way. They are

\[ \frac{\partial g_{ab}}{\partial t} = Ng^{-\frac{1}{2}} \psi^{-4} \pi_{ab} + (KN)_{ab} - \theta g_{ab} \]  
\[ (1.67) \]

and

\[ \frac{\partial \pi^{ab}}{\partial t} = -\sqrt{g} N \psi^4 \left( R^{ab} - g^{ab} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) \right) - 2Ng^{-\frac{1}{2}} \psi^{-4} \pi^{ac} \pi^b_c \]
\[ + \sqrt{g} \psi \left( \nabla^a \nabla^b (N \psi^3) - g^{ab} \nabla^2 (N \psi^3) \right) \]
\[ + N \sqrt{g} \psi^3 \left( \nabla^a \nabla^b \psi + 3g^{ab} \nabla^2 \psi \right) \]
\[ + 4 \sqrt{g} g^{ab} \nabla_c (N \psi^3) \nabla^c \psi - 6 \sqrt{g} \nabla^{(a} (N \psi^3) \nabla^{b)} \psi \]
\[ + \nabla_c \left( \nabla^c \pi^{ab} \right) - \pi^{bc} \nabla_c N^a - \pi^{ac} \nabla_c N^b - \theta \pi^{ab} \]  
\[ (1.68) \]

It can be verified that these equations do indeed preserve the constraints.

We can see how similar the results are to those in York’s approach. The Hamiltonian constraint has become the Lichnerowicz equation. The momentum is TT. Also, the lapse
fixing equation is the gauge requirement of GR to preserve the $tr\pi = 0$ constraint. Of course, those equations are all secondary in GR whereas here they have arisen directly through a variational procedure!

### 1.5.3 The Hamiltonian

Now that we have found the momentum it is straightforward to find the Hamiltonian. As usual we have

$$H = \int \left( \pi^{ab} \frac{\partial g_{ab}}{\partial t} - \mathcal{L} \right) d^3x \quad (1.69)$$

We must write $\mathcal{L}$ in terms of the momentum $\pi^{ab}$. This is

$$\mathcal{L} = N \sqrt{g} \psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} + \frac{\pi^{ab} \pi_{ab} - \frac{1}{2} (tr\pi)^2}{g^{\psi^8}} \right) d^3x \quad (1.70)$$

This leads to

$$H = \int \left[ N \left( \frac{1}{\sqrt{g}\psi^4} \left( \pi^{ab} \pi_{ab} - \frac{1}{2} (tr\pi)^2 \right) - \sqrt{g} \psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) \right) - 2N_a \nabla \pi^{ab} + \theta tr\pi \right] d^3x \quad (1.71)$$

Recalling the constraints we see that yet again, as found by Dirac and ADM, the Hamiltonian is a sum of the constraints with Lagrange multipliers.

### 1.6 Jacobi Action

Baerlein, Sharp and Wheeler [5] constructed a Jacobi Action for general relativity. Their action was,

$$S = \pm \int d\lambda \int \sqrt{g} \sqrt{R} \sqrt{T_{GR}} d^3x \quad (1.72)$$

where

$$T_{GR} = \left( g^{ac} g^{bd} - g^{ab} g^{cd} \right) \left( \frac{\partial g_{ab}}{\partial t} - (KN)_{ab} \right) \left( \frac{\partial g_{cd}}{\partial t} - (KN)_{cd} \right) \quad (1.73)$$

Variation with respect to $\frac{\partial g_{ab}}{\partial t}$ gives

$$\pi^{ab} = \sqrt{\frac{g R}{T_{GR}}} \left( g^{ac} g^{bd} - g^{ab} g^{cd} \right) \left( \frac{\partial g_{cd}}{\partial t} - (KN)_{cd} \right) \quad (1.74)$$

This expression is squared to give the Hamiltonian constraint. The variation with respect to $N_a$ gives the momentum constraint. The evolution equations are found in the usual way. The equations found with the Jacobi action are those of general relativity if we identify $2N$ and $\sqrt{\frac{R}{T}}$. We want to construct the analogous case in conformal gravity. Let us return to our $(3+1)$ Lagrangian,

$$\mathcal{L} = N \sqrt{g} \psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} - (trB)^2 + B^{ab} B_{ab} \right) \quad (1.75)$$
We can write this as
\[ \mathcal{L} = \sqrt{g} \psi^4 \left[ N \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) + \frac{1}{4N} \left( \beta^{ab} \beta_{ab} - (tr \beta)^2 \right) \right] \] (1.76)
where \( \beta_{ab} = -2NB_{ab} = \left( \frac{\partial g_{ab}}{\partial t} - (KN)_{ab} - \theta g_{ab} \right) \). We now extremise with respect to \( N \). This gives us,
\[ N = \pm \frac{1}{2} \left( \beta^{ab} \beta_{ab} - (tr \beta)^2 \right)^{\frac{1}{2}} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right)^{-\frac{1}{2}} \] (1.77)
Substituting this back into the action gives us
\[ S = \pm \int d\lambda \int \sqrt{g} \psi^4 \sqrt{R - 8 \frac{\nabla^2 \psi}{\psi} \sqrt{T}} d^3x \] (1.78)
where \( T = \left( \beta^{ab} \beta_{ab} - (tr \beta)^2 \right) \). This is the conformal gravity version of the BSW action (1.72).

We can do all the usual variations here: \( N^a, \dot{\psi}, \text{and } \psi \). These give the momentum constraint, the conformal constraint and the lapse-fixing equation respectively. Because of the independent variations of \( \dot{\psi} \) and \( \psi \), it turns out that we may vary with respect to \( \theta \) and \( \psi \) to get the conformal constraint and the lapse-fixing equation respectively. When we find the canonical momentum \( \pi^{ab} \) we can “square” it to give the “Hamiltonian constraint.”

Actually, this is precisely the BOM action found by starting with the BSW action and conformalising it under conformal transformations of the 3-metric
\[ g_{ab} \longrightarrow \psi^4 g_{ab} \] (1.79)
The Jacobi action is manifestly 3-dimensional and its configuration space is naturally conformal superspace - the space of all 3-D Riemannian metrics modulo diffeomorphisms and conformal rescalings.

1.7 Conformally Related Solutions

In conformal superspace conformally related metrics are equivalent. Thus conformally related solutions of the theory must be physically equivalent and so it is crucial that we have a natural way to relate such solutions. Suppose we have one set of initial data \( (g_{ab}, \pi^{ab}) \). These must satisfy the constraints (1.64) and (1.65). We solve the Hamiltonian constraint (1.63) for our “conformal field” \( \psi \). Suppose now we start with a different pair \( (h_{ab}, \rho^{ab}) \) where \( h_{ab} = \alpha^4 g_{ab} \) and \( \rho^{ab} = \alpha^{-4} \pi^{ab} \). Our new initial data is conformally
related to the original set of initial data. This is allowed as “transverse-traceless”-ness is conformally invariant and so our initial data constraints are satisfied. All we must do is solve the new Hamiltonian constraint for our new conformal field $\chi$ say. This constraint is now

$$\rho^{ab} \rho_{ab} = h \chi^8 \left( R_h - 8 \nabla^2_h \chi \right)$$

(1.80)

The subscript $h$ on $R$ and $\nabla$ is because we are now dealing with the new metric $h_{ab}$. We now solve this for $\chi$. It can be shown that we must have $\chi = \frac{\psi}{\alpha}$. That is, $\psi$ is automatically transformed when our initial data is transformed.

Now,

$$\chi^4 h_{ab} = \frac{\psi^4}{\alpha^4} \alpha^4 g_{ab} = \psi^4 g_{ab}$$

(1.81)

and

$$\chi^{-4} \rho^{ab} = \psi^{-4} \alpha^{ab}$$

(1.82)

If we label these as $\tilde{g}_{ab} = \psi^4 g_{ab}$ and $\tilde{\pi}^{ab} = \psi^{-4} \pi^{ab}$ than we can write our constraints as

$$\tilde{\pi}^{ab} \tilde{\pi}_{ab} = \tilde{g} \tilde{R}$$

(1.83)

$$\tilde{\nabla}^b \tilde{\pi}^{ab} = 0$$

(1.84)

$$\tilde{tr} \tilde{\pi} = 0$$

(1.85)

$$\tilde{N} \tilde{R} - \tilde{\nabla}^2 \tilde{N} = 0$$

(1.86)

All conformally related solutions are identical in this form and thus we shall refer to this as the physical representation. The momentum constraint is identical in the two theories. The Hamiltonian constraint of general relativity on a maximal slice is identical to that here. The lapse-fixing equation in this representation looks just like the maximal slicing equation of general relativity. In this representation the evolution equations are

$$\frac{\partial g_{ab}}{\partial t} = Ng^{-\frac{1}{2}} \pi_{ab} + (KN)_{ab}$$

(1.87)

and

$$\frac{\partial \pi^{ab}}{\partial t} = -\sqrt{g} N \left( R^{ab} - g^{ab} R \right) + \nabla^a \nabla^b \left( \sqrt{g} N \right)$$

$$-\sqrt{g} g^{ab} \nabla^2 N - 2Ng^{-\frac{1}{2}} \pi^{ac} \pi^b_c$$

$$- \nabla_c N^a \pi^{bc} - \nabla_c N^b \pi^{ac} + \nabla_c \left( N^c \pi^{ab} \right)$$

(1.88)
These are exactly those of general relativity on a maximal slice. Thus, solutions of general relativity in maximal slicing gauge are also solutions here. There are of course solutions of general relativity which do not have a maximal slicing and these are not solutions of the conformal theory.

1.8 Topological Considerations

So far we have not considered any implications which the topology of the manifold may have. In an asymptotically flat case we have no problems with the theory as it stands. This is not the case however in a topology which is compact without boundary.

1.8.1 Integral Inconsistencies

Recall the lapse-fixing equation of the theory in the physical representation (removing the “hats” for simplicity),

\[ NR - \nabla^2 N = 0 \]  \hspace{1cm} (1.89)

Let’s integrate this equation:

\[ \int \sqrt{g} NR \ d^3x - \int \sqrt{g} \nabla^2 N \ d^3x = 0 \]  \hspace{1cm} (1.90)

The second term integrates to zero and so we just have

\[ \int \sqrt{g} NR \ d^3x = 0 \]  \hspace{1cm} (1.91)

and so we must have that \( N \) is sometimes positive and sometimes negative or else is identically zero. Suppose the first of these possibilities is true. Let’s now restrict our integration of the lapse-fixing equation to the positive values of \( N \) only. This has a real boundary, namely, \( N = 0 \). We thus have

\[ \int \sqrt{g} NR \ d^3x - \int \sqrt{g} \nabla^2 N \ d^3x = 0 \]  \hspace{1cm} (1.92)

again. Now, the first integral

\[ \int \sqrt{g} NR \ d^3x \]  \hspace{1cm} (1.93)

is positive definite. The second integral is

\[ - \int \sqrt{g} \nabla^2 N \ d^3x \]  \hspace{1cm} (1.94)

which becomes a surface integral after integrating by parts

\[ - \int \sqrt{g} N \nabla^c N \ d\Sigma_c \]  \hspace{1cm} (1.95)
where $\Sigma_c$ is the boundary on which $N = 0$. Since $N$ is decreasing on the boundary we have that this term is positive definite. This means however that we have a vanishing sum of two positive definite quantities. This is a contradiction. Thus we must have $N \equiv 0$. We get frozen dynamics. (This is not the case with a manifold which is asymptotically flat so the earlier analysis works in that case.) Frozen dynamics also arises in general relativity if one imposes a fixed $tr\pi = 0$ gauge condition. However, this is a problem of the gauge rather than a problem of the theory as with conformal gravity. (See [6] for a treatment of this problem.)

The easiest way to resolve this problem involves a slight change to the action. We introduce a volume term. The inspiration for this term comes from the Yamabe theorem. The action is

$$S = \int \frac{N\sqrt{g}\psi^4}{V^{2/3}} \left( R - 8\frac{\nabla^2\psi}{\psi} + B^{ab}B_{ab} - (trB)^2 \right) \ d^3x \ dt \quad (1.96)$$

The $V$ in the denominator is the volume

$$V = \int \sqrt{g}\psi^6 \ d^3x \quad (1.97)$$

The power of $\frac{2}{3}$ on the volume leaves the action homogeneous in both $\psi$ and $g$. The constraints arising from this action are not very different from the original constraints. We get firstly that

$$\pi^{ab} = \frac{\sqrt{g}\psi^4}{V^{2/3}} \left( g^{ab}trB - B^{ab} \right) \quad (1.98)$$

The constraints are then

$$\pi^{ab}\pi_{ab} = \frac{g\psi^8}{V^{4/3}} \left( R - 8\frac{\nabla^2\psi}{\psi} \right) \quad (1.99)$$

$$\nabla_b\pi^{ab} = 0 \quad (1.100)$$

$$tr\pi = 0 \quad (1.101)$$

$$N\psi^3 \left( R - 7\frac{\nabla^2\psi}{\psi} \right) - \nabla^2 \left( N\psi^3 \right) = C\psi^5 \quad (1.102)$$

The term $C$ is given by

$$C = \int \frac{N\sqrt{g}\psi^4}{V} \left( R - 8\frac{\nabla^2\psi}{\psi} \right) \ d^3x \quad (1.103)$$

which arises due to the variation of the volume. In the physical representation these constraints become

$$\pi^{ab}\pi_{ab} = \frac{gR}{V(\psi)^{4/3}} \quad (1.104)$$
\[ \nabla_b \pi^{ab} = 0 \]  \hspace{1cm} (1.105)

\[ tr \pi = 0 \]  \hspace{1cm} (1.106)

\[ NR - \nabla^2 N = C \]  \hspace{1cm} (1.107)

where \( C \) is now

\[ C = \frac{1}{2} \int \frac{\sqrt{g} \sqrt{R} \sqrt{T}}{V} d^3x = \langle NR \rangle \]  \hspace{1cm} (1.108)

**Note:** \( \langle A \rangle \) is the average of \( A \) given by the usual notion of average

\[ \langle A \rangle = \frac{\int \sqrt{g} A d^3x}{\int \sqrt{g} d^3x} \]  \hspace{1cm} (1.109)

In this form, the lapse-fixing equation looks just like the constant mean curvature slicing equation of general relativity on a maximal slice. To check if we have any inconsistency this time, we integrate our lapse-fixing equation again. We need,

\[ \int \sqrt{g} NR \, d^3x - \int \nabla^2 N \, d^3x - \int \sqrt{g} C \, d^3x = 0 \]  \hspace{1cm} (1.110)

This becomes

\[ \int \sqrt{g} NR \, d^3x - \int \sqrt{g} \langle NR \rangle \, d^3x = 0 \]  \hspace{1cm} (1.111)

removing the second term which integrates to zero. The left hand side is then just

\[ V \langle NR \rangle - V \langle NR \rangle = 0 \]  \hspace{1cm} (1.112)

as required. Thus, with the introduction of the volume term, we have removed the problem.

**Note:** Although we have only used the physical representation in our integral tests it can be verified easily that everything also works out in the general representation. Of course, in EVERY situation, this *must* be true. We are losing *nothing* by working in the physical representation.
We should consider the evolution equations again now that we have changed the action. The evolution equations become
\[
\frac{\partial g_{ab}}{\partial t} = Ng^{-\frac{1}{2}}V(\psi)^{\frac{4}{3}}\psi^{-4}\pi_{ab} + (KN)_{ab} + \theta g_{ab} \tag{1.113}
\]
and
\[
\frac{\partial \pi_{ab}}{\partial t} = -N \sqrt{g} \psi^3 \left( R_{ab} - g_{ab} \left( R - 8 \nabla^2 \psi \right) \right) - 2 NV^{4/3} \sqrt{g} \psi \pi^{ac} \pi^{b}_{c} \\
+ \frac{\sqrt{g} \psi}{V^{2/3}} \left( \nabla^a \nabla^b (N \psi^3) - g_{ab} \nabla^2 (N \psi^3) \right) \\
+ \frac{N \sqrt{g} \psi^3}{V^{2/3}} \left( \nabla^a \nabla^b \psi + 3g_{ab} \nabla^2 \psi \right) \\
+ \frac{4 \sqrt{g} g_{ab}}{V^{2/3}} \nabla_c (N \psi^3) \nabla^c \psi - \frac{6 \sqrt{g}}{V^{2/3}} \nabla^a (N \psi^3) \nabla^b \psi \\
+ \nabla_c \left( N^c \pi^{ab} \right) - \pi^{bc} \nabla_c N^a - \pi^{ac} \nabla_c N^b - \theta \pi^{ab} - \frac{2}{3} \frac{\sqrt{g} g_{ab} C \psi^5}{V^{2/3}} \tag{1.114}
\]
where \( C \) is as in (1.103). As usual we can write these in the physical representation. In this form the evolution equations are
\[
\frac{\partial g_{ab}}{\partial t} = Ng^{-\frac{1}{2}}V^{2/3} \pi_{ab} + (KN)_{ab} \tag{1.115}
\]
and
\[
\frac{\partial \pi_{ab}}{\partial t} = -N \sqrt{g} \psi^3 \left( R_{ab} - g_{ab} \left( R - 8 \nabla^2 \psi \right) \right) - 2 NV^{4/3} \sqrt{g} \psi \pi^{ac} \pi^{b}_{c} \\
+ \frac{\sqrt{g} \psi}{V^{2/3}} \left( \nabla^a \nabla^b (N \psi^3) - g_{ab} \nabla^2 (N \psi^3) \right) \\
+ \frac{N \sqrt{g} \psi^3}{V^{2/3}} \left( \nabla^a \nabla^b \psi + 3g_{ab} \nabla^2 \psi \right) \\
- \pi^{bc} \nabla_c N^a - \pi^{ac} \nabla_c N^b - \theta \pi^{ab} - \frac{2}{3} \frac{\sqrt{g} g_{ab} C \psi^5}{V^{2/3}} \tag{1.116}
\]
where now \( C = \langle NR \rangle \).

Let’s define
\[
\hat{\pi}^{ab} = V^{2/3} \pi^{ab} \tag{1.117}
\]
Then the constraints are
\[
\hat{\pi}^{ab} \hat{\pi}_{ab} - gR = 0 \tag{1.118}
\]
\[
\nabla_{\mu} \hat{\pi}^{ab} = 0 \tag{1.119}
\]
\[
\hat{\imath} \nabla \hat{\pi} = 0 \tag{1.120}
\]

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The lapse-fixing equation is
\[ NR - \nabla^2 N = C \] (1.121)

These are precisely the constraints and gauge fixing conditions for propagated maximal slicing in GR. The evolution equations are
\[ \frac{\partial g_{ab}}{\partial t} = N g^{\frac{1}{2}} \pi^{ab} + (KN)_{ab} \] (1.122)
and
\[ \frac{\partial \pi^{ab}}{\partial t} = -N \sqrt{g} \left( R^{ab} - g^{ab} R \right) - \frac{2N}{\sqrt{g}} \pi^{ac} \pi^{b}_c 
+ \sqrt{g} \left( \nabla^a \nabla^b N - g^{ab} \nabla^2 N \right) + \nabla_c \left( N^c \pi^{ab} \right) \] (1.123)

which are identical to those in GR apart from the global \( C \) term in the equation for \( \pi^{ab} \).

We can easily find the Hamiltonian and the Jacobi action for the new form. They are
\[ H = \int \left[ N \frac{V^{2/3}}{\sqrt{g} \psi^4} \left( \pi^{ab} \pi_{ab} - \frac{1}{2} (tr \pi)^2 - \frac{1}{\sqrt{g} \psi^4} \left( R - \frac{8}{\psi} \nabla^2 \psi \right) \right) \right] d^3x \] (1.124)
and
\[ S = \int d\lambda \int \frac{\sqrt{g} \psi^4 \sqrt{R - \frac{8\nabla^2 \psi}{\psi}}}{V(\psi)^{\frac{2}{3}}} d^3x \] (1.125)

Note again the homogeneity throughout in \( \psi \).

### 1.9 Other Results

There has been work on other aspects of this theory not described here. It is unnecessary from the point of view of this work while, of course, being valuable in itself with a number of worthwhile results most notably on the constraint algebra and the Hamilton-Jacobi theory. The interested reader can find this in [7].

### 1.10 Problem

Although the theory has emerged beautifully and easily form very natural principles we can find at least one major problem immediately. Consider the volume of a hypersurface \( V \)
\[ V = \int \sqrt{g} \, d^3x \] (1.126)
Taking the time derivative of this we get

$$\frac{\partial V}{\partial t} = \int \frac{1}{2} \sqrt{g} g^{ab} \frac{\partial g_{ab}}{\partial t} \, d^3 x$$  \hspace{1cm} (1.127)

This becomes

$$\frac{\partial V}{\partial t} = \int \left( NV^{2/3} tr\pi + 2 \nabla_c N^c \right) \, d^3 X$$  \hspace{1cm} (1.128)

Now since $tr\pi = 0$ the entire expression is zero. That is

$$\frac{\partial V}{\partial t} = 0$$  \hspace{1cm} (1.129)

and the volume of the universe is static. This rules out expansion and thus the standard cosmological solution is lost. In particular, the red-shift, an experimental fact, is unexplained. This is a serious shortcoming. All is not lost however...
Chapter 2

A New Hope

2.1 The Need For A Change

Despite all the promising features of the theory there is at least one major drawback. We can find the time derivative of the volume quite easily and get that it is proportional to $tr\pi$ and thus is zero. That is, the volume does not change and so the theory predicts a static universe and we cannot have expansion. This is quite a serious problem as the prediction of expansion in GR is considered to be one of the theory’s greatest achievements. We are left with the following options:

(a) Abandon the theory;
(b) Find a new explanation of the red-shift (among other things);
(c) Amend the theory to recover expansion.

The first option seems quite drastic and the second, while certainly the most dramatic, also seems to be the most difficult. Thus, let’s check what we can find behind door (c).

2.1.1 Resolving The problem(s)

Any change to the theory needs to be made at the level of the Lagrangian and so we’ll return to our earlier expression for $\mathcal{L}$

$$\mathcal{L} = N \sqrt{g} \psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} + B_{ab}B^{ab} - (trB)^2 \right)$$  \hspace{1cm} (2.1)
but naively change the form of $B_{ab}$ to
\[ B_{ab} = -\frac{1}{2N} \left( \frac{\partial g_{ab}}{\partial t} - (KN)_{ab} - \nabla_c \xi^c g_{ab} \right) \] (2.2)

Let’s vary the action with respect to $\xi^c$. We get
\[
\delta \mathcal{L} = N \sqrt{g} \psi^4 \left( 2B^{ab} - 2trB g^{ab} \right) \delta B_{ab} \\
= 2N \sqrt{g} \psi^4 \left( B^{ab} - 2trB g^{ab} \right) \left( -\frac{1}{2N} \right) \left( -\nabla_c \delta \xi^c g_{ab} \right) \\
= -2 \sqrt{g} \psi^4 trB \nabla_c \delta \xi^c 
\] (2.3)

Integrating by parts gives
\[
\delta \mathcal{L} = 2 \sqrt{g} \nabla_c (trB \psi^4) \delta \xi^c 
\] (2.4)
and so
\[
\nabla_c (trB \psi^4) = 0 
\] (2.5)

Recall that we had
\[
tr\pi = \psi^4 trB 
\] (2.6)
and so the constraint is
\[
\nabla_c tr\pi = 0 
\] (2.7)
the CMC condition.

However, since we have the same form for $\mathcal{L}$ as before our lapse-fixing equation is unchanged and as a result, the constraint is not propagated unless $tr\pi = 0$. Thus we haven’t gained anything. We need a further change.

It will prove instructive to split $B_{ab}$ into its trace and tracefree parts. (The reason for this will become clear quite soon.) We label the tracefree part as $S_{ab}$. Thus we have
\[
B_{ab} = S_{ab} + \frac{1}{3} g_{ab} trB 
\] (2.8)

We shall retain the new form of $B_{ab}$ as defined above in (2.2) all the same. The Lagrangian now reads
\[
\mathcal{L} = N \sqrt{g} \psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} + S_{ab} S^{ab} - \frac{2}{3} (trB)^2 \right) 
\] (2.9)

We still need to make one further change. We’ll simply stick in an additional $\psi$ term to the $trB$ part. (This is equivalent to redefining our conformal transformation so that $S_{ab}$ and $trB$ transform in different ways.) The Lagrangian takes the form
\[
\mathcal{L} = N \sqrt{g} \psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} + S_{ab} S^{ab} - \frac{2}{3} \psi^n (trB)^2 \right) 
\] (2.10)
Before we continue, one interesting point about $S_{ab}$ is the following. We have

$$S_{ab} = B_{ab} - \frac{1}{3}g_{ab}trB \quad (2.11)$$

Let’s write this out explicitly. We have

$$S_{ab} = -\frac{1}{2N}\left(\frac{\partial g_{ab}}{\partial t} - (KN)_{ab} - \nabla_c \xi^c g_{ab}\right) - \frac{1}{3}g_{ab}\left(g^{cd}\frac{\partial g_{cd}}{\partial t} - g^{cd}(KN)_{cd} - 3\nabla_c \xi^c\right) \quad (2.12)$$

Splitting this up further gives

$$S_{ab} = -\frac{1}{2N}\left(\frac{\partial g_{ab}}{\partial t} - (KN)_{ab}\right) + \frac{3}{2N}g_{ab}\nabla_c \xi^c - \frac{1}{3}g_{ab}\left(g^{cd}\frac{\partial g_{cd}}{\partial t} - g^{cd}(KN)_{cd}\right) - \frac{3}{2N}g_{ab}\nabla_c \xi^c \quad (2.13)$$

and with a simple cancellation

$$S_{ab} = -\frac{1}{2N}\left(\frac{\partial g_{ab}}{\partial t} - (KN)_{ab}\right) - \frac{1}{3}g_{ab}\left(g^{cd}\frac{\partial g_{cd}}{\partial t} - g^{cd}(KN)_{cd}\right) \quad (2.14)$$

Of course, this is

$$S_{ab} = K_{ab} - \frac{1}{3}g_{ab}trK \quad (2.15)$$

That is, $S_{ab}$ is the tracefree part of the extrinsic curvature and is independent of any conformal fields.

Let us find $\pi^{ab}$. This is done as usual by varying with respect to $\frac{\partial g_{ab}}{\partial t}$. We get

$$\delta \mathcal{L} = 2N\sqrt{g}\psi^4\left(2S^{ab}\delta S_{ab} - \frac{4}{3}\psi^n trBg^{ab}\delta B_{ab}\right)$$

$$= 2N\sqrt{g}\psi^4\left(S^{ab}\left(\delta B_{ab} - \frac{1}{3}g_{ab}g^{cd}\delta B_{cd}\right) - \frac{2}{3}\psi^n trBg^{ab}\delta B_{ab}\right)$$

$$= 2N\sqrt{g}\psi^4\left(S^{ab} - \frac{2}{3}\psi^n trBg^{ab}\right)\delta B_{ab}$$

$$= -\sqrt{g}\psi^4\left(S^{ab} - \frac{2}{3}\psi^n S^{ab} trB\right)\delta \frac{\partial g_{ab}}{\partial t}$$

Thus,

$$\pi^{ab} = -\sqrt{g}\psi^4 S^{ab} + \frac{2}{3}\sqrt{g}\psi^{n+4}g^{ab}\text{tr}B \quad (2.16)$$

Splitting $\pi^{ab}$ into its trace and tracefree parts will further clear things up. We’ll label the split as

$$\pi^{ab} = \sigma^{ab} + \frac{1}{3}g^{ab} tr\pi \quad (2.18)$$

Thus the tracefree part of $\pi^{ab}$ is

$$\sigma^{ab} = -\sqrt{g}\psi^4 S^{ab} \quad (2.19)$$
and the trace is given by

$$tr\pi = 2\psi^{n+4}trB$$  \hspace{1cm} (2.20)

Note that our value of $n$ is undefined as yet.

The constraints are found by varying with respect to $\xi^c$, $\psi$, $N$ and $N^a$. The conformal constraint and the lapse-fixing equation are given by varying with respect to $\xi^c$ and $\psi$ respectively. These give

$$\nabla_c tr\pi = 0$$  \hspace{1cm} (2.21)

and

$$N\psi^3 \left( R - 7\frac{\nabla^2 \psi}{\psi} \right) - \nabla^2 (N\psi^3) + \left( \frac{(trp)^2\psi^7}{4} \right) = 0$$  \hspace{1cm} (2.22)

respectively. From the variation with respect to $N$ we get

$$S_{ab}S^{ab} - \frac{2}{3} \psi^n (trB)^2 - g\psi^8 \left( R - 8\frac{\nabla^2 \psi}{\psi} \right) = 0$$  \hspace{1cm} (2.23)

which in terms of the momentum is

$$\sigma_{ab}\sigma^{ab} - \frac{1}{6}\psi^{-n}(tr\pi)^2 - g\psi^8 \left( R - 8\frac{\nabla^2 \psi}{\psi} \right) = 0$$  \hspace{1cm} (2.24)

and finally, from the variation with respect to $N^a$ we get

$$\nabla_b\pi^{ab} = 0$$  \hspace{1cm} (2.25)

We require conformal invariance in our constraints. Under what conditions is the momentum constraint (2.25) invariant? The tracefree part of the momentum, $\sigma^{ab}$, has a natural weight of $-4$ (from the original theory). That is

$$\sigma^{ab} \rightarrow \omega^{-4}\sigma^{ab}$$  \hspace{1cm} (2.26)

If $tr\pi = 0$ then we have conformal invariance. If not however, we require various further conditions. We need

$$\nabla_b\sigma^{ab} = 0$$  \hspace{1cm} (2.27)

$$\nabla_c tr\pi = 0$$  \hspace{1cm} (2.28)

and that

$$trp = \frac{tr\pi}{\sqrt{g}} \rightarrow trp$$  \hspace{1cm} (2.29)

under a conformal transformation. In our theory we have the first two conditions emerging directly and naturally from the variation. Thus we simply define $trp$ to transform as a conformal scalar as required. With this done our momentum constraint is conformally invariant.
Transforming the constraint (2.24) gives

\[
\sigma_{ab}\sigma^{ab} - \frac{1}{6}\psi^{-n}g(tr\pi)^2\omega^{12+n} - g\psi^8\left(R - 8\frac{\nabla^2\psi}{\psi}\right) = 0
\] (2.30)

and so we must have \( n = -12 \) for conformal invariance. The constraint then becomes

\[
\sigma_{ab}\sigma^{ab} - \frac{1}{6}\psi^{12}g(tr\pi)^2 - g\psi^8\left(R - 8\frac{\nabla^2\psi}{\psi}\right) = 0
\] (2.31)

(Note: This is exactly the Lichnerowicz equation from GR. However, we have found it directly from a variational procedure.)

Thus we have determined the unique value of \( n \) and our constraints are

\[
\sigma_{ab}\sigma^{ab} - \frac{1}{6}\psi^{12}(tr\pi)^2 - g\psi^8\left(R - 8\frac{\nabla^2\psi}{\psi}\right) = 0
\] (2.32)

\[
\nabla_b\pi^{ab} = 0
\] (2.33)

\[
\nabla_c tr\pi = 0
\] (2.34)

\[
N\psi^3\left(R - 7\frac{\nabla^2\psi}{\psi}\right) - \nabla^2(N\psi^3) + \frac{(tr\pi)^2\psi^7}{4} = 0
\] (2.35)

Let’s proceed to the Hamiltonian formulation.

### 2.2 The Hamiltonian Formulation

The earlier expression for \( \pi^{ab} \) can be inverted to get \( \frac{\partial g_{ab}}{\partial t} \). We get

\[
\frac{\partial g_{ab}}{\partial t} = \frac{2N}{\sqrt{g}\psi^4}\left(\sigma_{ab} - \frac{1}{6}g_{ab}tr\pi\psi^{12}\right) + (KN)_{ab} + g_{ab}\nabla_c \xi^c
\] (2.36)

The Hamiltonian may then be found in the usual way. We get

\[
\mathcal{H} = \frac{N}{\sqrt{g}\psi^4}\left[\sigma^{ab}\sigma_{ab} - \frac{1}{6}(tr\pi)^2\psi^{12} - g\psi^8\left(R - 8\frac{\nabla^2\psi}{\psi}\right)\right] - 2N_a \nabla_b \pi^{ab} - \xi^c \nabla_c tr\pi
\] (2.37)
As a consistency check let’s find $\frac{\partial g_{ab}}{\partial t}$ from this by varying with respect to $\pi^{ab}$. We get

$$\frac{\partial g_{ab}}{\partial t} = \frac{2N}{\sqrt{g}} \left( \sigma_{ab} - \frac{1}{6} g_{ab} tr \pi \psi^{12} \right) + (KN)_{ab} + g_{ab} \nabla_{\epsilon} \xi^{c} \tag{2.38}$$

as before. Thus, all is well. We may do all the usual variations here to get the constraints. Varying the Hamiltonian with respect to $g_{ab}$ gives us our evolution equation for $\pi^{ab}$. We get

$$\frac{\partial \pi^{ab}}{\partial t} = -N \sqrt{g} \psi^{4} \left( R^{ab} - g^{ab} \left( R - 8 \frac{\nabla^{2} \psi}{\psi} \right) \right)$$

$$- \frac{2N}{\sqrt{g}} \psi^{4} \left( \pi^{ac} \pi^{b}_{c} - \frac{1}{3} \pi^{ab} tr \pi - \frac{1}{6} \pi^{ab} tr \pi \psi^{12} \right)$$

$$+ \sqrt{g} \psi \left( \nabla^{a} \nabla^{b} (N \psi^{3}) - g^{ab} \nabla^{2} (N \psi^{3}) \right)$$

$$+ N \sqrt{g} \psi^{3} \left( \nabla^{a} \nabla^{b} \psi + 3g^{ab} \nabla^{2} \psi \right) \tag{2.39}$$

$$+ 4g^{ab} \sqrt{g} \nabla_{c} (N \psi^{3}) \nabla^{c} \psi - 6\sqrt{g} \nabla^{(a} (N \psi^{3}) \nabla^{b)} \psi$$

$$+ \nabla_{c} (\pi^{ab} N^{c}) - \pi^{bc} \nabla_{c} N^{a} - \pi^{ac} \nabla_{c} N^{b}$$

$$- \left( \pi^{ab} - \frac{1}{2} g^{ab} tr \pi \right) \nabla_{\epsilon} \xi^{c}$$

We may use the evolution equations to find $\frac{\partial tr\pi}{\partial t}$ quite easily. (Of course, we need the evolution equations to propagate all of the constraints. We will deal with the others later.) We find that

$$\frac{\partial tr\pi}{\partial t} = 0 \tag{2.40}$$

using the lapse-fixing equation. Thus we have that $tr\pi = \text{constant}$ both spatially and temporally!! We could proceed to check propagation of the constraints here but it will be easier and more instructive to do a little more work first.

Since $tr\pi$ is identically a constant our dynamical data consists of $g_{ab}$ and $\sigma^{ab}$. Thus, it may prove useful to have an evolution equation for $\sigma^{ab}$ rather than the full $\pi^{ab}$. It is reasonably straightforward to do this. Firstly we note that

$$\frac{\partial \sigma_{ab}}{\partial t} = \frac{\partial \pi_{ab}}{\partial t} - \frac{1}{3} \frac{\partial g_{ab} tr \pi}{\partial t} \tag{2.41}$$
Working through the details gives us
\[ \frac{\partial \sigma_{ab}}{\partial t} = -N\sqrt{g}\psi^4 \left( R^{ab} - \frac{1}{3} g^{ab} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) \right) - \frac{2N}{\sqrt{g}\psi^4} \sigma^{ac} \sigma^b_c \\
+ \sqrt{g}\psi \left( \nabla^a \nabla^b (N\psi^3) - \frac{1}{3} g^{ab} \nabla^2 (N\psi^3) \right) \\
+ N\sqrt{g}\psi^3 \left( \nabla^a \nabla^b \psi + \frac{7}{3} g^{ab} \nabla^2 \psi \right) \\
+ 4g^{ac} \sqrt{g} \nabla_c (N\psi^3) \nabla^c \psi - 6 \sqrt{g} \nabla^{(a} (N\psi^3) \nabla^{b)} \psi \\
+ \nabla_c (\sigma^{ab} N^c) - \sigma^{bc} \nabla_c N^a - \sigma^{ac} \nabla_c N^b \\
- \sigma^{ab} \nabla_c \xi^c + \frac{N\psi^8}{3\sqrt{g}} \sigma^{ab} tr\pi \]

(2.42)

2.3 Jacobi Action

We can also find the Jacobi action of this theory. Recall the (3+1) Lagrangian,
\[ \mathcal{L} = N\sqrt{g}\psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} + S^{ab} S_{ab} - \frac{2}{3} \psi^{-12} (trB)^2 \right) \]  

(2.43)

We can write this as
\[ \mathcal{L} = \sqrt{g}\psi^4 \left[ N \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) + \frac{1}{4N} \left( \Sigma^{ab} \Sigma_{ab} - \frac{2}{3} \psi^{-12} (tr\beta)^2 \right) \right] \]

(2.44)

where \( \Sigma_{ab} = -2NS_{ab} \) and \( \beta_{ab} = -2NB_{ab} \). We now extremise with respect to \( N \). This gives us,
\[ N = \pm \frac{1}{2} \left( \Sigma^{ab} \Sigma_{ab} - \frac{2}{3} \psi^{-12} (tr\beta)^2 \right)^{1/2} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right)^{-1/2} \]

(2.45)

Substituting this back into the action gives us
\[ S = \pm \int d\lambda \int \sqrt{g}\psi^4 \sqrt{R - 8 \frac{\nabla^2 \psi}{\psi} \sqrt{T} d^3 x} \]

(2.46)

where \( T = \left( \Sigma^{ab} \Sigma_{ab} - \frac{2}{3} \psi^{-12} (tr\beta)^2 \right) \).

We can do all the usual variations here: \( N^a, \xi^c \) and \( \psi \). These give the momentum constraint, the conformal constraint and the lapse-fixing condition respectively. When we find the canonical momentum \( \pi^{ab} \) we can “square” it to give the “Hamiltonian constraint.”

So far, so good. We shall rarely use the Jacobi form of the action here but from a thin-sandwich point of view it is important and may well be of use in future work. Let’s move on.
We can do almost exactly the same thing here as we did in the section with the same name in Chapter 1. Suppose we start with initial data \( \{g_{ab}, \sigma^{ab}, trp\} \) obeying the initial data conditions (2.33) and (2.34). We then solve (2.32) for \( \psi \)

Suppose instead that we start with the conformally related initial data \( \{h_{ab}, \rho^{ab}, trp\} = \{\alpha^{4}g_{ab}, \alpha^{-4}\sigma^{ab}, trp\} \). These automatically satisfy the initial data conditions by the conformal invariance. We now solve the Hamiltonian constraint for the conformal “field” \( \chi \), say. Just like before it can be shown that \( \chi = \frac{\psi}{\alpha} \). Thus, yet again,

\[
\psi^{4}g_{ab} = \chi^{4}h_{ab} \tag{2.47}
\]

and

\[
\psi^{-4}\sigma^{ab} = \chi^{-4}\rho^{ab} \tag{2.48}
\]

Again we label these as \( \tilde{g}_{ab} \) and \( \tilde{\rho}^{ab} \) and put a hat over \( trp \) also (for clarity). The constraints become

\[
\tilde{\sigma}_{ab}\tilde{\sigma}^{ab} - \frac{1}{6} (\tilde{tr}\tilde{\pi})^{2} - \tilde{g}\tilde{R} = 0 \tag{2.49}
\]

\[
\tilde{\nabla}^{b}\tilde{\pi}^{ab} = 0 \tag{2.50}
\]

\[
\tilde{\nabla}^{c}tr\tilde{\pi} = 0 \tag{2.51}
\]

\[
\tilde{N}\tilde{R} - \tilde{\nabla}^{2}\tilde{N} + \frac{(trp)^{2}}{4} = 0 \tag{2.52}
\]

Consider GR in the CMC gauge. The constraints are

\[
\sigma_{ab}\sigma^{ab} - \frac{1}{6} (tr\pi)^{2} - gR = 0 \tag{2.53}
\]

\[
\nabla^{b}\pi^{ab} = 0 \tag{2.54}
\]

\[
\nabla^{c}tr\pi = 0 \tag{2.55}
\]

Evolution of the CMC condition gives

\[
NR - \nabla^{2}N + \frac{(trp)^{2}}{4} = C \tag{2.56}
\]

The similarities are quite striking.
2.4.1 What of $\xi^c$?

Precious little has been revealed about what $\xi^c$ may be or even how it transforms. This needs to be addressed. First let’s recall that we demanded that

$$\text{tr}B \rightarrow \omega^{-8}\text{tr}B$$  \hspace{1cm} (2.57)

under a conformal transformation. This will be enough to reveal the transformation properties of $\xi^c$. Taking the trace gives us

$$\text{tr}B = -\frac{1}{2N}\left(g^{ab}\frac{\partial g_{ab}}{\partial t} - g^{ab}(KN)_{ab} - 3\nabla_c\xi^c\right)$$  \hspace{1cm} (2.58)

Under a conformal transformation we get

$$\omega^{-8}\text{tr}B = -\frac{1}{2\omega^2N}\left(g^{ab}\frac{\partial g_{ab}}{\partial t} + 12\frac{\dot{\omega}}{\omega} - \omega^{-4}g^{ab}\left(\omega^A(KN)_{ab} + 4\omega^3\omega_cN^c g_{ab}\right) - 3\nabla_c\xi^c\right)$$

$$= -\frac{1}{2\omega^2N}\left(g^{ab}\frac{\partial g_{ab}}{\partial t} - g^{ab}(KN)_{ab} - 3\nabla_c\xi^c\right)$$

$$- \frac{3}{2\omega^2N}\nabla_c\xi^c + \frac{3}{2\omega^2N}\nabla_c\xi^c - \frac{6}{\omega^3N}\left(\dot{\omega} - \omega_cN^c\right)$$

$$= \omega^{-2}\text{tr}B + \frac{3}{2\omega^2N}\left(\nabla_c\xi^c - 3\nabla_c\xi^c - \frac{4}{\omega}\left(\dot{\omega} - \omega_cN^c\right)\right)$$  \hspace{1cm} (2.59)

Thus,

$$\frac{3}{2\omega^2N}\left(\nabla_c\xi^c - 3\nabla_c\xi^c - \frac{4}{\omega}\left(\dot{\omega} - \omega_cN^c\right)\right) = -\frac{1}{\omega^2N}\text{tr}B\left(1 - \omega^{-6}\right)$$  \hspace{1cm} (2.60)

and so

$$\nabla_c\xi^c = \nabla_c\xi^c + \frac{4}{\omega}\left(\dot{\omega} - \omega_cN^c\right) - \frac{2N}{3}\text{tr}B\left(1 - \omega^{-6}\right)$$  \hspace{1cm} (2.61)

This tells us how things transform but not what $\xi^c$ itself actually is. We can find this though.

Let’s write the evolution equations in the physical representation. It can be verified that they are

$$\frac{\partial \tilde{g}_{ab}}{\partial t} = \frac{2\tilde{N}}{\sqrt{g}}\left(\tilde{\sigma}_{ab} - \frac{1}{6}\tilde{g}_{ab}\text{tr}\tilde{\pi}\right) + (\widetilde{KN})_{ab} + \tilde{g}_{ab}\nabla_c\tilde{\xi}^c$$  \hspace{1cm} (2.62)
and

\[
\frac{\partial \tilde{\sigma}^{ab}}{\partial t} = -\tilde{N} \sqrt{\tilde{\gamma}} \left( \tilde{R}^{ab} - \frac{1}{3} \tilde{g}^{ab} \tilde{R} \right) - \frac{2\tilde{N}}{\sqrt{\tilde{g}}} \sigma^{ac} \sigma^{b}_{c} \\
+ \sqrt{\tilde{g}} \left( \tilde{\nabla}^{a} \tilde{\nabla}^{b} \tilde{N} - \frac{1}{3} \tilde{g}^{ab} \tilde{\nabla}^{2} \tilde{N} \right) \\
+ \tilde{\nabla}_{c} \left( \tilde{\sigma}^{ab} \tilde{N}_{c} \right) - \tilde{\sigma}^{bc} \tilde{\nabla}_{c} \tilde{N}_{a} - \tilde{\sigma}^{ac} \tilde{\nabla}_{c} \tilde{N}_{b} \\
+ \frac{\tilde{N}}{3 \sqrt{\tilde{g}}} \tilde{\sigma}^{ab} tr \pi - \tilde{\sigma}^{ab} \tilde{\nabla} \tilde{\xi}^{c} \] (2.63)

We require the evolution equations to propagate the constraints. However, when we check this it turns out that we are forced to set \( \tilde{\nabla}_{c} \tilde{\xi}^{c} \) to zero. However, this means that we have

\[
\nabla_{c} \xi^{c} + \frac{4}{\omega} (\dot{\psi} - \psi_{,c} N^{c}) - \frac{2N}{3} tr B (1 - \psi^{-6}) = 0 \] (2.64)

by (2.61). Thus we have

\[
\nabla_{c} \xi^{c} = -\frac{4}{\dot{\psi}} (\dot{\psi} - \psi_{,c} N^{c}) + \frac{2N}{3} tr B (1 - \psi^{-6}) \] (2.65)

That is,

\[
\nabla_{c} \xi^{c} = \theta + \frac{2N}{3} tr B (1 - \psi^{-6}) \] (2.66)

where \( \theta \) is as in the original theory. Thus, the exact form of \( \xi^{c} \) is determined. We needed \( \nabla_{c} \xi^{c} \) to be zero in the physical representation for constraint propagation and so we should check that this is the case with our newly found expression for \( \nabla_{c} \xi^{c} \). We can check this easily. In the physical representation \( \theta = 0 \) and \( \psi = 1 \). Thus, we do have that \( \tilde{\nabla}_{c} \tilde{\xi}^{c} \) is zero.

It is vital to note that this is strictly a POST-VARIATION identification. If we use this form for \( \xi^{c} \) in the action we will run into problems, not least an infinite sequence in the variation of \( tr B \) with respect to \( \xi^{c} \). (This is because we would have \( tr B \) defined in terms of \( tr B \) itself.) We see that \( \xi^{c} \) is intimately related with how \( \psi \) changes from slice to slice.

Our constraints in the physical representation are

\[
\sigma_{ab} \sigma^{ab} - \frac{1}{6} (tr \pi)^{2} - gR = 0 \] (2.67)

\[
\nabla_{b} \pi^{ab} = 0 \] (2.68)

\[
\nabla_{c} tr \pi = 0 \] (2.69)
\[ NR - \nabla^2 N + \frac{N(trp)^2}{4} = 0 \]  \hspace{1cm} (2.70)

and our evolution equations are

\[
\frac{\partial g_{ab}}{\partial t} = \frac{2N}{\sqrt{g}} \left( \sigma_{ab} - \frac{1}{6} g_{ab} tr \pi \right) + \left( KN \right)_{ab} \hspace{1cm} (2.71)
\]

\[
\frac{\partial \sigma^{ab}}{\partial t} = -N \sqrt{g} \left( R^{ab} - \frac{1}{3} g^{ab} R \right) - \frac{2N}{\sqrt{g}} \sigma^{ac} \sigma^{b}c
\]
\[
+ \sqrt{g} \left( \nabla^a \nabla^b N - \frac{1}{3} g^{ab} \nabla^2 N \right)
\]
\[
+ \nabla_c (\sigma^{ab} N^c) - \sigma^{bc} \nabla_c N^a - \sigma^{ac} \nabla_c N^b
\]
\[
+ \frac{N}{3 \sqrt{g}} \sigma^{ab} tr \pi \hspace{1cm} (2.72)
\]

(The hats are removed for simplicity.) These are identical to those in GR in the CMC gauge (with \( trp \) a temporal constant).

### 2.5 Topological Considerations

Mimicking the section in Chapter one we find in the same way that if the manifold is compact without boundary we get frozen dynamics. In the asymptotically flat case we have no such problem and this will prove to be important in solar system tests of the theory. In the problematic case we can resolve the issue in the same manner as before although, it is a little more complicated this time.

#### 2.5.1 Integral Inconsistencies (Slight Return)

The root of the integral inconsistency is in the lapse-fixing equation. If we integrate this equation we find that the only solution is \( N \equiv 0 \). That is, we have frozen dynamics. The resolution to this in regular CG was to introduce a volume term in the denominator of the Lagrangian. Actually, the key is to keep the Lagrangian homogeneous in \( \psi \) using different powers of the volume. The volume of a hypersurface here is given by

\[ V = \int \sqrt{g} \psi^6 \, d^3x \]  \hspace{1cm} (2.73)

In the original theory the Lagrangian has an overall factor of \( \psi^4 \) and so we need to divide by \( V^{2/3} \) to keep homogeneity in \( \psi \). There is no such overall factor in the new theory.
and so it is not as straightforward. The key is to treat the two parts of the Lagrangian separately. We try
\[ L_1 = \frac{N \sqrt{g}}{V^n} \left( R - 8 \frac{\nabla^2 \psi}{\psi} + S^{ab} S_{ab} \right) \tag{2.74} \]
and
\[ L_2 = -\frac{2}{3} N \frac{\sqrt{g}}{V} \psi^{-8} (tr B)^2 \tag{2.75} \]
and we determine \( n \) and \( m \) from the homogeneity requirement. Thus we have that \( n = \frac{2}{3} \) and \( m = -\frac{4}{3} \). Using this result our Lagrangian is now
\[ L = \frac{N \sqrt{g}}{V^{2/3}} \left( R - 8 \frac{\nabla^2 \psi}{\psi} + S^{ab} S_{ab} - \frac{2}{3} \psi^{-12} (tr B)^2 V^2 \right) \tag{2.76} \]

### 2.5.2 New Constraints

We go about things in exactly the same manner as before. The momentum is found to be
\[ \pi^{ab} = -\frac{\sqrt{g}}{V^{2/3}} S^{ab} + \frac{2}{3} \sqrt{g} \psi^{-8} V^{4/3} g^{ab} tr B \tag{2.77} \]
The constraints are (almost) unchanged. They are
\[ \sigma_{ab} \sigma^{ab} - \frac{\psi^{12} (tr \pi)^2}{6V^2} - \frac{g \psi^{8}}{V^{4/3}} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) = 0 \tag{2.78} \]
\[ \nabla_b \pi^{ab} = 0 \tag{2.79} \]
\[ \nabla_c tr \pi = 0 \tag{2.80} \]
The lapse-fixing equation is
\[ \frac{N \sqrt{g}}{V^{2/3}} \left( R - 7 \frac{\nabla^2 \psi}{\psi} \right) - \sqrt{g} \frac{\nabla^2 (N \psi^3)}{V^{2/3}} + \sqrt{g} C \psi^5 - \frac{2}{3} \sqrt{g} D \psi^5 V^{4/3} = 0 \tag{2.81} \]
where
\[ C = \int \frac{N \sqrt{g}}{V} \left( R - 8 \frac{\nabla^2 \psi}{\psi} + S^{ab} S_{ab} \right) \, d^3 x \tag{2.82} \]
and
\[ D = \int \frac{N \sqrt{g} \psi^{-8}}{V} (tr B)^2 \, d^3 x \tag{2.83} \]
The \( C \) and \( D \) terms result from the variations of the volume. Rearranging the lapse-fixing equation we get
\[ \frac{N \sqrt{g}}{V^{2/3}} \left( R - 7 \frac{\nabla^2 \psi}{\psi} \right) - \sqrt{g} \frac{\nabla^2 (N \psi^3)}{V^{2/3}} \sqrt{g} N \psi^{-9} (tr B)^2 V^{4/3} = \frac{\sqrt{g}}{2V^{2/3}} \left( C + \frac{4}{3} DV^2 \right) \tag{2.84} \]
Integrating across this expression gives no problem. The inconsistency has been removed.
2.6 The Hamiltonian Formulation

We should consider the evolution equations again now that we have changed the action. First of all the momentum is now given by

\[ \pi_{ab} = -\sqrt{g}\psi^4 S_{ab} + \frac{2}{3} \sqrt{g}\psi^{-8} g^{ab} trBV^{4/3} \] (2.85)

The new Hamiltonian is

\[ H = \frac{NV^{2/3}}{\sqrt{g}\psi^4} \left[ \sigma_{ab} \sigma_{ab} - \frac{(tr\pi)^2 \psi^{12}}{6V^2} - \frac{g\psi^8}{V^{4/3}} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) \right] - 2N_a \nabla_b \pi_{ab} - \xi_c \nabla_c tr\pi \] (2.86)

The evolution equations are then

\[ \frac{\partial g_{ab}}{\partial t} = \frac{2NV^{2/3}}{\sqrt{g}\psi^4} \left( \sigma_{ab} - \frac{g_{ab} tr\pi \psi^{12}}{6V^2} \right) + (KN)_{ab} + g_{ab} \nabla_c \xi^c \] (2.87)

and

\[ \frac{\partial \pi_{ab}}{\partial t} = \frac{-N \sqrt{g}\psi^4}{V^{2/3}} \left( R^{ab} - g^{ab} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) \right) \]

\[ - \frac{2NV^{2/3}}{\sqrt{g}\psi^4} \left( \pi_{ac} \pi_{cb} \frac{1}{3} \pi_{ab} tr\pi - \frac{\pi_{ab} tr\pi \psi^{12}}{6V^2} \right) \]

\[ + \frac{N\psi^3 \sqrt{g}}{V^{2/3}} \left( \nabla^a \nabla^b \psi + 3g^{ab} \nabla^2 \psi \right) \]

\[ + 4 \frac{\sqrt{g}}{V^{2/3}} g^{ab} \nabla_c (N\psi^3) \nabla^c \psi - \frac{6}{V^{2/3}} \nabla^a (N\psi^3) \nabla^b \psi \]

\[ + \nabla_c \left( \pi_{ab} N^c \right) - \pi_{bc} \nabla_c N^a - \pi_{ac} \nabla_c N^b \]

\[ - \left( \pi_{ab} - \frac{1}{2} g_{ab} tr\pi \right) \nabla_c \xi^c - \frac{2}{3} \frac{\sqrt{g}\psi^6 g^{ab}}{V^{2/3}} C \]

where

\[ C = \left\langle N \sqrt{g}\psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} + \frac{\psi^4 (trp)^2}{4V^{2/3}} \right) \right\rangle \] (2.89)

We can again take the time derivative of trp and find yet again that

\[ \frac{\partial trp}{\partial t} = 0 \] (2.90)
Thus, our dynamic data will once again be \( \{g_{ab}, \sigma^{ab}\} \) and so we want to find the evolution equation for \( \sigma^{ab} \) again. Slogging through we get

\[
\frac{\partial \sigma^{ab}}{\partial t} = -\frac{N\sqrt{g}\psi^4}{V^{2/3}} \left( R^{ab} - \frac{1}{3}g^{ab}\left( R - 8\nabla^2\psi \right) \right) - \frac{2NV^{2/3}}{\sqrt{g}\psi^4}\sigma^{ac}\sigma^{bc} \\
+ \frac{\sqrt{g}\psi}{V^{2/3}} \left( \nabla^a \nabla^b (N\psi^3) - \frac{1}{3}g^{ab} \nabla^2 (N\psi^3) \right) \\
+ \frac{N\sqrt{g}\psi^3}{V^{2/3}} \left( \nabla^a \nabla^b \psi + \frac{7}{3}g^{ab} \nabla^2 \psi \right) \\
+ \frac{4\sqrt{g}}{V^{2/3}} g^{ab} \nabla_c (N\psi^3) \nabla_c \psi - \frac{6\sqrt{g}}{V^{2/3}} \nabla^a (N\psi^3) \nabla^b \psi \\
+ \nabla_c \left( \sigma^{ab} N^c \right) - \sigma^{bc} \nabla_c N^a - \sigma^{ac} \nabla_c N^b \\
- \sigma^{ab} \nabla_c \xi^c + \frac{N\psi^8}{3\sqrt{g}V^{4/3}} \sigma^{ab} tr\pi 
\]

(2.91)

Note that the term with \( C \) has dropped out.

The physical representation is achieved either by the naive substitution of \( \psi = 1 \) and \( \nabla_c \xi^c = 0 \) or by doing it the longer more correct way. The result is the same in either case. The momentum is

\[
\pi^{ab} = -\frac{\sqrt{g}}{V^{2/3}} S^{ab} + \frac{2}{3}\sqrt{g}g^{ab}trK V^{4/3} 
\]

(2.92)

Thus

\[
\sigma^{ab} = -\frac{\sqrt{g}}{V^{2/3}} S^{ab} \quad \text{and} \quad tr\pi = 2\sqrt{g}(trK)V^{4/3} 
\]

(2.93)

The constraints are

\[
\sigma^{ab}\sigma_{ab} - \frac{1}{6} \left( tr\pi \right)^2 = \frac{gR}{V^{4/3}} 
\]

(2.94)

\[
\nabla_b \pi^{ab} = 0 
\]

(2.95)

\[
\nabla_c trp = 0 
\]

(2.96)

\[
NR - \nabla^2 N + \frac{N(trp)^2}{4V^{2/3}} = C 
\]

(2.97)

where we now have \( C = \left\langle N \left( R + \frac{(trp)^2}{4V^{2/3}} \right) \right\rangle \). The evolution equations are

\[
\frac{\partial g_{ab}}{\partial t} = \frac{2NV^{2/3}}{\sqrt{g}} \left( \sigma_{ab} - \frac{g_{ab}tr\pi}{6V^2} \right) + (KN)_{ab} 
\]

(2.98)
\[
\frac{\partial \sigma_{ab}}{\partial t} = -\frac{N\sqrt{g}}{V^{2/3}} \left( R_{ab} - \frac{1}{3}g^{ab}R \right) - \frac{2NV^{2/3}}{\sqrt{g}} \sigma^{ac} \sigma_{cb} \\
+ \frac{\sqrt{g}\psi}{V^{2/3}} \left( \nabla^a \nabla^b N - \frac{1}{3}g^{ab} \nabla^2 N \right) \\
+ \nabla_c \left( \sigma_{ab} N^c \right) - \sigma_{bc} \nabla_c N^a - \sigma_{ac} \nabla_c N^b \\
+ \frac{N}{3\sqrt{g}V^{4/3}} \sigma_{ab} \text{tr} \pi
\]

(2.99)

2.7 The Volume

This theory was inspired by the need to recover expansion. After all this work, have we succeeded? The time derivative of the volume is

\[
\frac{\partial V}{\partial t} = \int \frac{1}{2} \sqrt{g} g^{ab} \frac{\partial g_{ab}}{\partial t} \, d^3x \\
= - \int \frac{1}{2} \frac{N\sqrt{g} \text{tr} \pi}{V^{4/3}} \, d^3x \\
= - \frac{\text{tr} \langle N \rangle}{2V^{1/3}}
\]

(2.100)

Thus, we have recovered expansion. The big test of the compact without boundary theory will be to study the cosmological solutions and this will be the focus of a later chapter.

2.8 Jacobi Action

For completeness let's find the Jacobi action for the compact theory. Without going through each step let's simply require homogeneity in \( \psi \). The Jacobi action for the non-compact theory was \( (2.46) \)

\[
S = \pm \int d\lambda \int \sqrt{g} \psi^4 \sqrt{R - 8\frac{\nabla^2 \psi}{\psi}} \sqrt{T} d^3x
\]

(2.101)

where \( T = \left( \Sigma^{ab} \Sigma_{ab} - \frac{2}{3} \psi^{-12} (\text{tr} \beta)^2 \right) \) where \( \Sigma^{ab} = -2NS^{ab} \) and \( \beta^{ab} = -2NB^{ab} \). Applying the homogeneity requirement gives

\[
S = \pm \int d\lambda \int \frac{\sqrt{g} \psi^4}{V^{2/3}} \sqrt{R - 8\frac{\nabla^2 \psi}{\psi}} \sqrt{T} d^3x
\]

(2.102)
where \( T = \left( \sum_{ab} \Sigma_{ab} - \frac{2}{3} \psi^{-12} (tr \beta)^2 V^2 \right) \).

Everything else emerges as before.

### 2.9 Comparison with GR

In the earlier “static” conformal theory we saw that the labelling

\[
\widehat{\pi}^{ab} = V^{2/3} \pi^{ab}
\]

made the theory appear incredibly similar to GR. A similar labelling is possible here. Define

\[
\widehat{\sigma}^{ab} = V^{2/3} \sigma^{ab}
\]

and

\[
\widehat{tr} \pi = \frac{tr \pi}{V^{1/3}}
\]

With this rebelling the constraints are

\[
\widehat{\sigma}^{ab} \widehat{\sigma}_{ab} - \frac{1}{6} (\widehat{tr} \pi)^2 = gR
\]

\[
\nabla_b \widehat{\pi}^{ab} = 0
\]

\[
\nabla_c \widehat{tr} p = 0
\]

and the lapse-fixing equation is

\[
NR - \nabla^2 N + \frac{N(\widehat{tr} p)^2}{4} = C
\]

where \( C = \left\langle N \left( R + \frac{(\widehat{tr} p)^2}{4} \right) \right\rangle \). These are identical to GR in the CMC gauge. The evolution equations are

\[
\frac{\partial g_{ab}}{\partial t} = \frac{2N}{\sqrt{g}} \left( \widehat{\sigma}_{ab} - \frac{g_{ab} \widehat{tr} \pi}{6V} \right) + (KN)_{ab}
\]
\[
\frac{\partial \sigma^{ab}}{\partial t} = V^{2/3} \frac{\partial \sigma^{ab}}{\partial t} + \frac{2}{3V^{1/3}} \frac{\partial V}{\partial t}
\]
\[
= -N \sqrt{g} \left( R^{ab} - \frac{1}{3} g^{ab} R \right) - \frac{2N}{\sqrt{g}} \sigma^{ac} \sigma^{b}_c
\]
\[
+ \sqrt{g} \psi \left( \nabla^a \nabla^b N - \frac{1}{3} g^{ab} \nabla^2 N \right)
\]
\[
+ \nabla_c (\sigma^{ab} N^c) - \sigma^{bc} \nabla_c N^a - \sigma^{ac} \nabla_c N^b
\]
\[
+ \frac{\left( N - \langle N \rangle \right)}{3 \sqrt{g}V} \sigma^{ab} \text{tr} \pi
\]

(2.111)

In GR these are
\[
\frac{\partial g_{ab}}{\partial t} = 2N \sqrt{g} \left( \sigma_{ab} - \frac{g_{ab} \text{tr} \pi}{6} \right) + (KN)_{ab}
\]

(2.112)

and
\[
\frac{\partial \sigma^{ab}}{\partial t} = -N \sqrt{g} \left( R^{ab} - \frac{1}{3} g^{ab} R \right) - \frac{2N}{\sqrt{g}} \sigma^{ac} \sigma^{b}_c
\]
\[
+ \sqrt{g} \psi \left( \nabla^a \nabla^b N - \frac{1}{3} g^{ab} \nabla^2 N \right)
\]
\[
+ \nabla_c (\sigma^{ab} N^c) - \sigma^{bc} \nabla_c N^a - \sigma^{ac} \nabla_c N^b
\]
\[
+ \frac{N}{3 \sqrt{g}} \sigma^{ab} \text{tr} \pi
\]

(2.113)

There are very few differences between the theories. We shall compare the two theories later in purely geometric terms, that is, in terms of the metric and curvature rather than the momentum. Let us leave this for now.

**2.10 Time**

Recall the York time (1.37)
\[
\tau = \frac{2}{3} \text{tr} \psi
\]

(2.114)

In GR this was a good notion of time. However in this theory we have that \( \text{tr} \psi \) is identically constant. Thus it cannot be used as a notion of time. We note now though that unlike in GR, for us the volume is monotonically increasing. (In fact, it goes from 0 to \( \infty \) as we shall see when discussing cosmology later.) Of course, the volume must be constant on any hypersurface and so the volume is a good notion of time in this theory. This may be extremely beneficial in a quantisation program.
### 2.11 Light Cones

So far the theory is quite promising. There are a number of things that must carry over from GR though if it is to be taken seriously. One of these is that the speed of propagation of the wave front must be unity (the speed of light). The easiest way to check this is to consider the evolution equations. Let’s consider the case in GR briefly. The corresponding case in the conformal theory will work in almost exactly the same way.

The evolution equation for $g_{ab}$ in GR is

$$
\frac{\partial g_{ab}}{\partial t} = \frac{2N}{\sqrt{g}} \left( \pi_{ab} - \frac{1}{2} g_{ab} \text{tr} \pi \right) + (KN)_{ab}
$$

(2.115)

Inverting this we find that

$$
\pi_{ab} = \frac{\sqrt{g}}{2N} \frac{\partial g_{ab}}{\partial t}
$$

(2.116)

We will be working here to leading order in the derivatives which is the reason for only omitting the other terms. Differentiating both sides gives

$$
\frac{\partial \pi_{ab}}{\partial t} = \frac{\sqrt{g}}{2N} \frac{\partial^2 g_{ab}}{\partial t^2}
$$

(2.117)

Now substituting this into the evolution equation for $\pi_{ab}$ gives us

$$
\frac{\sqrt{g}}{2N} \frac{\partial^2 g_{ab}}{\partial t^2} = -N \sqrt{g} \left( R_{ab} - \frac{1}{2} g_{ab} R \right)
$$

(2.118)

(Note: The alternate form of the evolution equation is used here with the factor of $\frac{1}{2}$ on $R$.)

Now,

$$
\left( R_{ab} - \frac{1}{2} g_{ab} R \right) = \frac{1}{2} g^{cd} \left( g_{bd,ac} + g_{ac,bd} - g_{ab,cd} - g_{cd,ab} - g_{ab} g^{ef} \left( g_{ec,fd} - g_{ef,cd} \right) \right)
$$

(2.119)

again only using leading order in the derivatives. We are concerned with the transverse traceless part of $g_{ab}$ which we’ll label as $g^{TT}_{ab}$. The only relevant part is then

$$
- \frac{1}{2} g^{cd} g^{TT}_{ab,cd}
$$

(2.120)

which we’ll write as

$$
- \frac{1}{2} \frac{\partial^2 g^{TT}_{ab}}{\partial x^2}
$$

(2.121)

All the other terms are cancelled either through the transverse or traceless properties. Using only the $TT$ part in the time derivatives also gives us

$$
\frac{1}{2N^2} \frac{\partial^2 g^{TT}_{ab}}{\partial t^2} = \frac{1}{2} \frac{\partial^2 g^{TT}_{ab}}{\partial x^2}
$$

(2.122)
This is a wave equation with wave speed 1. Thus we get gravitational radiation! Various
details are omitted here but the essence of the idea is quite clear. Let’s consider the
conformal theory. We’ll use the compact without boundary theory (that is, the one with
the volume terms).

The evolution equation for $g_{ab}$ can be inverted to get

$$\sigma_{ab} \frac{\sqrt{g}}{2NV^{2/3}} \frac{\partial g_{ab}}{\partial t} + ...$$  \hspace{1cm} (2.123)

Differentiating both sides gives

$$\frac{\partial \sigma_{ab}}{\partial t} = \frac{\sqrt{g}}{2NV^{2/3}} \frac{\partial^2 g_{ab}}{\partial t^2}$$  \hspace{1cm} (2.124)

again, working only to leading order in the derivatives. Substituting this into the evolution
equation for $\sigma_{ab}$ gives us

$$\frac{\sqrt{g}}{2NV^{2/3}} \frac{\partial^2 g_{ab}}{\partial t^2} = \frac{N}{V^{2/3}} \left( R_{ab} - \frac{1}{2} g_{ab} R \right)$$  \hspace{1cm} (2.125)

The volume terms cancel and we are left with the same equation as (2.118) above. In
exactly the same way this becomes

$$\frac{1}{N^2} \frac{\partial^2 g_{ab}^{TT}}{\partial t^2} = \frac{\partial^2 g_{ab}^{TT}}{\partial x^2}$$  \hspace{1cm} (2.126)

Yet again, we have found a wave equation with speed 1. Thus we have recovered gravita-
tional radiation with wavefronts propagating at the speed of light. All is still well.

### 2.12 Matter in General Relativity

While the aim is to couple matter to gravity in the new theory it will be instructive
to warm up by reviewing the corresponding cases in GR. We’ll treat various different
sources, namely, a cosmological constant, electromagnetism and dust dealing with each
one in turn.

#### 2.12.1 Cosmological Constant

This is the easiest of all matter sources. Let’s take the Lagrangian for vacuum GR to be

$$\mathcal{L} = N \sqrt{g} \left( R + K_{ab} \mathring{K}^{ab} - \left( tr K \right)^2 \right)$$  \hspace{1cm} (2.127)

The Lagrangian for a cosmological constant is simply

$$\mathcal{L}_{cc} = -N \sqrt{g} \Lambda$$  \hspace{1cm} (2.128)
and our full Lagrangian is
\[ L = N \sqrt{g} \left( R - \Lambda + K^{ab} K_{ab} - (tr K)^2 \right) \] (2.129)

There is no change to the momentum constraint. The Hamiltonian constraint changes in an easy way becoming
\[ K^{ab} K_{ab} - (tr K)^2 - \left( R - \Lambda \right) = 0 \] (2.130)

In terms of the momentum this is
\[ \pi^{ab} \pi_{ab} - \frac{1}{2} \left( tr \pi \right)^2 - g \left( R - \Lambda \right) = 0 \] (2.131)

The momentum constraint is unchanged (\( \Lambda \) has no conjugate momentum). In fact, to see the changes here one simply substitutes \( R - \Lambda \) wherever there was \( R \). Thus there is no change to the evolution equation for \( g_{ab} \) and there is only one simple change to the evolution equation for \( \pi^{ab} \). Let’s move on to electromagnetism.

### 2.12.2 Electromagnetism

The Lagrangian for electromagnetism is
\[ L_{em} = N \sqrt{g} \left( U + T \right) \] (2.132)

where
\[ U = -\frac{1}{4} \left( \nabla_b A_a - \nabla_a A_b \right) \nabla^b A^a \] (2.133)

and
\[ T = \frac{1}{4N^2} g^{ab} \left( \frac{\partial A_a}{\partial t} - \mathcal{L}_N A_a - \nabla_a \Phi \right) \left( \frac{\partial A_b}{\partial t} - \mathcal{L}_N A_b - \nabla_b \Phi \right) \] (2.134)

The full Lagrangian is then
\[ L = N \sqrt{g} \left( R + U + K^{ab} K_{ab} - (tr K)^2 + T \right) \] (2.135)

The Hamiltonian constraint is
\[ \pi^{ab} \pi_{ab} - \frac{1}{2} \left( tr \pi \right)^2 + \pi^c \pi_c - g \left( R + U \right) = 0 \] (2.136)

where \( \pi^c \) is the momentum conjugate to \( A_c \) given by
\[ \pi^c = \frac{\sqrt{g}}{2N} g^{ac} \left( \frac{\partial A_a}{\partial t} - \mathcal{L}_N - \nabla^a \Phi \right) \] (2.137)
In more familiar language this becomes
\[ \pi^{ab}\pi_{ab} - \frac{1}{2}(tr\pi)^2 - g\left(R - 16\pi\rho_r\right) = 0 \] (2.138)
where \(\rho_r\) is the energy density of the radiation.

The variation with respect to \(\Phi\) gives us the electromagnetic Gauss constraint
\[ \nabla c\pi^c = 0 \] (2.139)
and using this the momentum constraint is unchanged
\[ \nabla_b\pi^{ab} = 0 \] (2.140)

### 2.12.3 Dust

The Lagrangian for dust is given by
\[ \mathcal{L}_d = \sqrt{-g}\rho_d\left(g^{\alpha\beta}U_\alpha U_\beta + 1\right) \] (2.141)
where \(\rho_d\) is the rest mass density and \(U_\alpha\) is the four velocity of the dust. This is written here in 4-D form to show some of the properties. In particular the constraint which arises on variation with respect to \(M\) is
\[ g^{\alpha\beta}U_\alpha U_\beta + 1 = 0 \] (2.142)

Keeping this constraint will be important when we attempt to couple dust to the conformal theory. In any case the full Lagrangian is
\[ \mathcal{L} = N\sqrt{g}\left(R + K^{ab}K_{ab} - (tr K)^2 + \rho_d\left(g^{\alpha\beta}U_\alpha U_\beta + 1\right)\right) \] (2.143)

The Hamiltonian constraint is
\[ \pi^{ab}\pi_{ab} - \frac{1}{2}(tr\pi)^2 - g\left(R - \rho_d\right) = 0 \] (2.144)
The momentum constraint is unchanged.

### 2.13 Matter and Conformal Gravity

To approach this problem we’ll proceed as in the GR cases above by simply considering each type of matter in turn. The changes are very straightforward in any case. We’ll consider the case without the volume terms for simplicity. Inserting the volume terms will be easy by just requiring homogeneity in \(\psi\).
2.13.1 Cosmological Constant

The easiest thing to do is to take the GR Lagrangian and conformalise it. In GR we had

$$\mathcal{L} = N \sqrt{g} \left( R - \Lambda + K^{ab} K_{ab} - (tr K)^2 \right)$$  \hspace{1cm} (2.145)

This becomes

$$\mathcal{L} = N \psi^2 \sqrt{g} \psi^6 \left( \psi^{-4} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) - \Lambda + \psi^{-4} S^{ab} S_{ab} - \psi^{-16} (tr B)^2 \right)$$  \hspace{1cm} (2.146)

Simplifying this gives

$$\mathcal{L} = N \sqrt{g} \psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} - \psi^4 \Lambda + S^{ab} S_{ab} - \psi^{-12} (tr B)^2 \right)$$  \hspace{1cm} (2.147)

The Hamiltonian constraint becomes

$$\sigma^{ab} \sigma_{ab} - \frac{1}{6} \psi^{12} (tr p)^2 - g \psi^8 \left( R - 8 \frac{\nabla^2 \psi}{\psi} - \psi^{-4} \Lambda \right) = 0$$  \hspace{1cm} (2.148)

If we want the compact without boundary case we simply substitute in volumes so as to achieve homogeneity in $\psi$. We get

$$\sigma^{ab} \sigma_{ab} - \frac{1}{6} \psi^{12} (tr p)^2 - g \psi^8 \left( R - 8 \frac{\nabla^2 \psi}{\psi} - \psi^{-4} \Lambda \right) = 0$$  \hspace{1cm} (2.149)

In the physical representation this is

$$\sigma^{ab} \sigma_{ab} - \frac{1}{6} \left( tr p \right)^2 - g \left( R - \frac{\Lambda}{V^{2/3}} \right) = 0$$  \hspace{1cm} (2.150)

So far so good.

2.13.2 Electromagnetism

The Lagrangian in GR was

$$\mathcal{L} = N \sqrt{g} \left( R + U + K^{ab} K_{ab} - (tr K)^2 + T \right)$$  \hspace{1cm} (2.151)

Conformalising this gives

$$\mathcal{L} = N \psi^2 \sqrt{g} \psi^6 \left( \psi^{-4} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) + \psi^{-8} U + \psi^{-4} S^{ab} S_{ab} - \psi^{-16} (tr B)^2 + \psi^{-8} T \right)$$  \hspace{1cm} (2.152)

where $U$ and $T$ are unchanged from the GR case. Simplifying this gives us

$$\mathcal{L} = N \sqrt{g} \psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} + \psi^{-4} U + S^{ab} S_{ab} - \psi^{-12} (tr B)^2 + \psi^{-4} T \right)$$  \hspace{1cm} (2.153)
The Hamiltonian constraint is
\[ \sigma^{ab} \sigma_{ab} - \frac{1}{6} \psi^{12} (trp)^2 + \psi^4 \pi_c \pi_c - g \psi^8 \left( R - 8 \frac{\nabla^2 \psi}{\psi} + \psi^{-4} U \right) = 0 \] (2.154)
where \( \pi_c \) is the momentum conjugate to \( A_c \) as before. In the compact without boundary case we get
\[ \sigma^{ab} \sigma_{ab} - \frac{1}{6} \psi^{12} (trp)^2 + \psi^4 \pi_c \pi_c - g \psi^8 \left( R - 8 \frac{\nabla^2 \psi}{\psi} + \psi^{-4} U V^{2/3} \right) = 0 \] (2.155)
In the physical representation this is
\[ \sigma^{ab} \sigma_{ab} - \frac{1}{6} \psi^{12} (trp)^2 + \frac{\psi^4 \pi_c \pi_c}{V^{2/3}} - g \psi^8 \left( R + UV^{2/3} \right) = 0 \] (2.156)
In slightly more usual language the Hamiltonian constraint is
\[ \sigma^{ab} \sigma_{ab} - \frac{1}{6} \psi^{12} (trp)^2 - \frac{g \psi^8}{V^{4/3}} \left( R - 16 \pi \rho_r V^{2/3} \right) = 0 \] (2.157)
where \( \rho_r \) is the energy density of the radiation.

2.13.3 Dust

The Lagrangian in GR was
\[ \mathcal{L} = N \sqrt{g} \left( R + K^{ab} K_{ab} - (trK)^2 + \rho_d \left( g^{\alpha\beta} U_\alpha U_\beta + 1 \right) \right) \] (2.158)
Conformalising this is a little less straightforward than the earlier cases. Firstly, what weight should we give to \( \rho_d \)? Well, Dicke prescribes that under a conformal transformation of the form
\[ g_{ab} \rightarrow \omega^4 g_{ab} \] (2.159)
we have
\[ \rho_d \rightarrow \omega^{-2} \rho_d \] (2.160)
Another point is that we wish the constraint
\[ g^{\alpha\beta} U_\alpha U_\beta + 1 = 0 \] (2.161)
to hold in the new theory also and so we demand that
\[ \left( g^{\alpha\beta} U_\alpha U_\beta + 1 \right) \rightarrow \omega^{-4} \left( g^{\alpha\beta} U_\alpha U_\beta + 1 \right) \] (2.162)
Then the conformal Lagrangian is
\[ \mathcal{L} = N \psi^2 \sqrt{g} \psi^6 \left( \psi^{-4} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) + \psi^{-4} S^{ab} S_{ab} - \psi^{-16} (trB)^2 + \rho_d \psi^{-2} \left( g^{\alpha\beta} U_\alpha U_\beta + 1 \right) \psi^{-4} \right) \] (2.163)
Simplifying we get
\[ \mathcal{L} = N \sqrt{g} \psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} + S^{ab} S_{ab} - \psi^{-12} (tr B)^2 + \psi^{-2} \rho_d \left( g^{\alpha\beta} U_{\alpha} U_{\beta} + 1 \right) \right) \] (2.164)

The Hamiltonian constraint becomes
\[ \sigma^{ab} \sigma_{ab} - \frac{1}{6} \psi^{12} (trp)^2 - g \psi^8 \left( R - 8 \frac{\nabla^2 \psi}{\psi} - \psi^{-2} \rho_d \right) = 0 \] (2.165)

where \( \rho_d \) is the mass density of the dust. In the compact without boundary case this becomes
\[ \sigma^{ab} \sigma_{ab} - \frac{1}{6} \psi^{12} (trp)^2 - g \psi^8 \frac{R - 8 \frac{\nabla^2 \psi}{\psi}}{V^{4/3}} \rho_d V^{1/3} = 0 \] (2.166)

In the physical representation this is
\[ \sigma^{ab} \sigma_{ab} - \frac{1}{6} \frac{(trp)^2}{V^2} - g \frac{R - \rho_d V^{1/3}}{V^{4/3}} = 0 \] (2.167)

Phew! At last. These results will ALL be used when we consider the cosmological implications of the conformal theory and so that will all have been worth it soon.
Chapter 3

Four Dimensions!

3.1 Introduction

Einstein’s formulation of relativity, both special and general, was beautifully embodied in the 4-dimensional spacetime. Although we have been dealing with a \((3 + 1)\)-dimensional picture throughout it is possible for us to consider things in 4-dimensions. The 4-dimensional picture emerges very naturally in the static theory. As for the new theory, we do not attempt to find an action but deduce the field equations nonetheless.

Of course, at the end we will always have a breaking of full 4-dimensional covariance due to the lapse fixing equations but it may be instructive to consider the 4-dimensional picture even as a tool for making comparisons with GR.

3.2 BOM Conformal Gravity

In this section we construct a 4-dimensional action based on conformal transformations of the 4-metric. We then decompose this to a \((3 + 1)\)-dimensional form and from this we find the Jacobi action of the theory. Incredibly, it turns out to be the same as that of Barbour and Ó Murchadha.
3.2.1 The Action

As given earlier (1.1) the Einstein-Hilbert action of general relativity is

\[ S = \int \sqrt{-g} \, (4) R \, d^4x \] (3.1)

where \( g_{\alpha\beta} \) is the 4-metric and \( (4) R \) is the four dimensional Ricci scalar. The action is varied with respect to \( g_{\alpha\beta} \) and the resulting equations are the (vacuum) Einstein equations

\[ G^{\alpha\beta} = \left( R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) = 0 \] (3.2)

We would like to construct an action which is invariant under conformal transformations of the metric

\[ g_{\alpha\beta} \rightarrow \Omega^2 g_{\alpha\beta} \] (3.3)

where \( \Omega \) is a strictly positive function using the Einstein-Hilbert action as a guide. First we need to develop some machinery for dealing with conformal transformations.

3.2.2 Dimensional Properties of Conformal Transformations

A supposed problem with conformal transformations and different numbers of dimensions is that various coefficients change when the number of dimensions changes. This turns out not to be a problem in this analysis as will be shown.

Let us consider conformal transformations and the scalar curvature. If we make a conformal transformation of the metric

\[ g_{\alpha\beta} \rightarrow \Omega^2 g_{\alpha\beta} \] (3.4)

then the Ricci tensor transforms as

\[
(n) R_{\alpha\beta} \rightarrow (n) R_{\alpha\beta} + 2(n - 2) \frac{(\nabla_\alpha \Omega) \nabla_\beta \Omega}{\Omega^2} - (n - 2) \frac{\nabla_\alpha \nabla_\beta \Omega}{\Omega} \\
+ (3 - n) g_{\alpha\beta} \frac{(\nabla_\gamma \Omega) \nabla_\gamma \Omega}{\Omega^2} - g_{\alpha\beta} \frac{\nabla_\gamma \nabla_\gamma \Omega}{\Omega} \] (3.5)

and the scalar curvature transforms as

\[
(n) R \rightarrow \Omega^{-2} \left( (n) R - 2(n - 1) g^{\alpha\beta} \frac{\nabla_\alpha \nabla_\beta \Omega}{\Omega} + (n - 1)(4 - n) \frac{\nabla_\gamma \Omega \nabla_\gamma \Omega}{\Omega^2} \right) \] (3.6)

where \( n \) is the number of dimensions. A consequence is that the combination

\[
\phi^{2/s} \left( (n) R - \frac{4(n - 1)}{(n - 2)} g^{\alpha\beta} \frac{\nabla_\alpha \nabla_\beta \phi}{\phi} \right) \] (3.7)
is conformally invariant for any scalar function $\phi$ under the combined transformation
\[ g_{\alpha\beta} \rightarrow \Omega^2 g_{\alpha\beta} \quad \phi \rightarrow \Omega^s \phi \quad (3.8) \]
where $s = 1 - \frac{n}{2}$. While this is true in any number of dimensions we are of course most concerned with the 3-dimensional and 4-dimensional cases. In 3 dimensions we have $s = -\frac{1}{2}$. Thus we get that
\[ \phi^{-4} \left( (3) R - \frac{8 \nabla^2 \phi}{\phi} \right) \quad (3.9) \]
is conformally invariant under the transformation
\[ g_{ab} \rightarrow \Omega^2 g_{ab} \quad \phi \rightarrow \frac{\phi}{\sqrt{\Omega}} \quad (3.10) \]
In four dimensions $s = -1$ and the combination
\[ \phi^{-2} \left( (4) R - \frac{6 \Box \phi}{\phi} \right) \quad (3.11) \]
is conformally invariant under the transformation
\[ g_{\alpha\beta} \rightarrow \Omega^2 g_{\alpha\beta} \quad \phi \rightarrow \frac{\phi}{\Omega} \quad (3.12) \]
Then the combination
\[ \sqrt{-g} \phi^2 \left( (4) R - \frac{6 \Box \phi}{\phi} \right) \quad (3.13) \]
is also conformally invariant. This will be our Lagrangian density $\mathcal{L}$. Thus our action is
\[ S = \int \mathcal{L} d^4x \quad (3.14) \]
Before we decompose this to a $(3 + 1)$ form let us consider the 4-dimensional structure and see what emerges.

### 3.2.3 Varying with respect to $g_{\alpha\beta}$

The variation with respect to $g_{\alpha\beta}$ is quite straightforward. The resulting equations are
\[ -\phi^2 \left( R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) + 4 \nabla^\alpha \phi \nabla^\beta \phi - g^{\alpha\beta} \nabla \gamma \phi \nabla^\gamma \phi - 2\phi \nabla^\alpha \nabla^\beta \phi + 2g^{\alpha\beta} \phi \Box \phi = 0 \quad (3.15) \]
This looks quite complicated but it is actually just
\[ \bar{G}^{\alpha\beta} = 0 \quad (3.16) \]
where $\bar{G}^{\alpha\beta}$ is just the Einstein tensor conformally transformed with conformal factor $\phi$. Equivalently, this is the Einstein tensor for the metric $\phi^2 g_{\alpha\beta}$. This interpretation will prove useful later.
3.2.4 Varying with respect to $\phi$

Again, this variation is fairly straightforward. We get

$$(4) R - \frac{6 \Box \phi}{\phi} = 0 \quad (3.17)$$

This is actually the trace of (3.16) and so, as such, is redundant. This can be viewed as a result of $\phi$ being pure gauge. The notion of free end-point variation of gauge variables shows that for a pure gauge variable $\psi$ say, we may vary the action with respect to both $\psi$ and its time derivative $\dot{\psi}$ independently. Because $\phi$ is pure gauge here we may vary the action with respect to $\phi$ and $\dot{\phi}$ independently. This will be crucial in the theory. We shall return to this.

3.2.5 A note on the action

The form of the action as it stands is not conventional as it contains second time derivatives of the metric. However, the combination

$$(4) R + 2A^\alpha_{;\alpha} \quad (3.18)$$

where $A^\alpha = (n^\alpha tr K + a^\alpha)$, $n^\alpha$ is the unit timelike normal and $a^\alpha$ is the four-acceleration of an observer travelling along $\mathbf{n}$ contains no second time derivatives. (The coordinates $\alpha$ are general.) We write our Lagrangian as

$$\mathcal{L} = \sqrt{-g} \phi^2 \left( (4) R + 2A^\alpha_{;\alpha} - 2A^\alpha_{;\alpha} - \frac{6 \Box \phi}{\phi} \right) \quad (3.19)$$

which then becomes

$$\mathcal{L} = \sqrt{-g} \phi^2 \left( (4) R + 2A^\alpha_{;\alpha} + 4\phi\phi_{;\alpha}A^\alpha + 6g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right) \quad (3.20)$$

after some integration by parts.

This Lagrangian contains no second time derivatives of the metric. Varying this with respect to $\phi$ and $\dot{\phi}$ gives two conditions which combine to give equation (3.17). Although we may do these variations here in a general coordinate form it will be more instructive to do a (3 + 1)-dimensional decomposition and get the corresponding equations there.

3.2.6 (3+1)-Decomposition

We are now ready to consider the new action. This is

$$S = \int \sqrt{-g} \phi^2 \left( (4) R - \frac{6 \Box \phi}{\phi} \right) d^4x \quad (3.21)$$
The 4-dimensional scalar curvature decomposes as earlier. The action becomes

$$ S = \int \sqrt{\mathcal{g}} \phi^2 \left( R - (\text{tr} K)^2 + K^{ab} K_{ab} - 2A^\alpha_{;\alpha} - 6 \frac{\Box \phi}{\phi} \right) d^4 x \quad (3.22) $$

Let’s separate this into two terms $S_1$ and $S_2$ where,

$$ S_1 = \int \sqrt{\mathcal{g}} \phi^2 \left( R - (\text{tr} K)^2 + K^{ab} K_{ab} - 2A^\alpha_{;\alpha} \right) d^4 x \quad (3.23) $$

and

$$ S_2 = -\int 6 \sqrt{\mathcal{g}} \phi \Box \phi d^4 x \quad (3.24) $$

Consider the first term. In the ADM theory $A^\alpha_{;\alpha}$ leads to a total divergence which is discarded. However, the presence of the $\phi^2$ here changes this. Integrating by parts we get

$$ -2\phi^2 A^\alpha_{;\alpha} \rightarrow 2(\phi^2)_{;\alpha} A^\alpha \quad (3.25) $$

discarding the total divergence again. Decomposing this gives

$$ 2 \left( \dot{\phi}^2 \left( n^0 \text{tr} K + a^0 \right) + \left( \phi^2 \right)_{,i} \left( n^i \text{tr} K + a^i \right) \right) = 4\phi \left( \dot{\phi} n^0 \text{tr} K + \phi_{,i} n^i \text{tr} K \right) + 4\phi \phi_{,i} a^i $$

$$ = \frac{4\phi}{N} \left( \dot{\phi} - \phi_{,i} N^i \right) \text{tr} K + 4\phi \phi_{,i} a^i $$

using the fact that

$$ n^\alpha = \left( 1/N, -N^m/N \right) \quad (3.27) $$

Then,

$$ S_1 = \int N \sqrt{\mathcal{g}} \phi^2 \left( R - (\text{tr} K)^2 + K^{ab} K_{ab} \right) d t d^3 x $n $$

$$ + \int 4\sqrt{\mathcal{g}} \phi \left[ \left( \dot{\phi} - \phi_{,i} N^i \right) \text{tr} K + N \phi_{,i} a^i \right] d t d^3 x \quad (3.28) $$

We must now deal with $S_2$. This is,

$$ S_2 = -\int 6 \sqrt{\mathcal{g}} \phi \Box \phi d^4 x \quad (3.29) $$

After a little integration by parts this is

$$ S_2 = \int 6 \sqrt{\mathcal{g}} \ g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi d^4 x \quad (3.30) $$

Decomposing this gives

$$ S_2 = \int 6N \sqrt{\mathcal{g}} \left( -\frac{1}{N^2} \left( \phi \right)^2 + \frac{2N^i}{N^2} \phi_{,i} + \left( g^{ij} - \frac{N^i N^j}{N^2} \right) \phi_{,i} \phi_{,j} \right) d t d^3 x \quad (3.31) $$
The full action is now

\[
S = \int N \sqrt{g} \phi^2 \left( R - (trK)^2 + K^{ab} K_{ab} \right) dt \, d^3x \\
+ \int 4 \sqrt{g} \phi \left[ \left( \dot{\phi} - \phi, N^i \right) trK + N \phi_i a^i \right] dt \, d^3x \\
+ \int 6N \sqrt{g} \left( -\frac{1}{N^2} \left( \dot{\phi} \right)^2 + \frac{2N^i \dot{\phi} N_{i}}{N^2} + \left( g^{ij} - \frac{N^i N^j}{N^2} \right) \phi_i \phi_j \right) dt \, d^3x
\]

(3.32)

This looks like a much more complicated object than we began with. There will, however, be much simplification. First, let's write it as

\[
S = \int N \sqrt{g} \phi^2 \left( R - (trK)^2 + K^{ab} K_{ab} \right) dt \, d^3x \\
+ \int 4 \sqrt{g} \phi \left[ \left( \dot{\phi} - \phi, N^i \right) trK + \nabla_i \phi \nabla^i N \right] dt \, d^3x \\
- \int \frac{6}{N} \sqrt{g} \left( \dot{\phi} - \phi, N^i \right)^2 dt \, d^3x + \int 6N \sqrt{g} \nabla_i \phi \nabla^i \phi dt \, d^3x
\]

(3.33)

where we have used \( a^i = \nabla_i N \). If we set \( \theta = -\frac{2}{\phi} \left( \dot{\phi} - \phi, N^i \right) \) then we get

\[
S = \int N \sqrt{g} \phi^2 \left( R - (trK)^2 + K^{ab} K_{ab} \right) dt \, d^3x \\
- \int 2\sqrt{g} \theta \phi^2 trK dt \, d^3x - \int \frac{3}{2} \frac{\sqrt{g} \theta^2 \phi^2}{N} dt \, d^3x \\
+ \int 6N \sqrt{g} \nabla_i \phi \nabla^i \phi dt \, d^3x + \int 4\sqrt{g} \phi \nabla_i \phi \nabla^i N dt \, d^3x
\]

(3.34)

This becomes

\[
S = \int N \sqrt{g} \phi^2 \left( R - (trK)^2 + K^{ab} K_{ab} \right) dt \, d^3x \\
- \int 2\sqrt{g} \theta^2 trK dt \, d^3x - \int \frac{3}{2} \frac{\sqrt{g} \theta^2 \phi^2}{N} dt \, d^3x \\
+ \int 2N \sqrt{g} \nabla_i \phi \nabla^i \phi dt \, d^3x - \int 4N \sqrt{g} \phi \nabla^2 \phi dt \, d^3x
\]

(3.35)

after some integration by parts. We notice that there might be a possibility of “completing some squares” with terms involving \( K \) and those involving \( \theta \). We have,

\[
-N(trK)^2 + NK^{ab} K_{ab} - 2\theta trK - \frac{3}{2} \frac{\theta^2}{N}
\]

(3.36)

Let’s try the combination,

\[
-N \left( trK + A \frac{\theta}{N} \right)^2 + N \left( K^{ab} + B \frac{\theta g^{ab}}{N} \right) \left( K_{ab} + B \frac{\theta g_{ab}}{N} \right)
\]

(3.37)
This gives us,

$$-N(trK)^2 - 2A\theta trK - A^2 \frac{\theta^2}{N} + NK^{ab}K_{ab} + 2B\theta trK + 3B^2\frac{\theta^2}{N}$$  \hspace{1cm} (3.38)

Comparing coefficients with equation (3.36) gives us,

$$-2A + 2B = -2 \quad \text{and} \quad -A^2 + 3B^2 = -\frac{3}{2}$$  \hspace{1cm} (3.39)

Solving here gives $A = \frac{3}{2}$ and $B = \frac{1}{2}$ and so we have,

$$-N\left(trK + \frac{3}{2N}\theta\right)^2 + N\left(K^{ab} + \frac{1}{2N}g_{ab}\right)\left(K^{ab} + \frac{1}{2N}g^{ab}\right)$$  \hspace{1cm} (3.40)

Finally, let us set

$$B_{ab} = \left(K_{ab} + \frac{\theta}{2N}g_{ab}\right)$$  \hspace{1cm} (3.41)

Thus we get,

$$-N(trB)^2 + NB^{ab}B_{ab}$$  \hspace{1cm} (3.42)

overall. Our full action is now,

$$S = \int N\sqrt{g}\psi^4\left(R - (trB)^2 + B^{ab}B_{ab}\right) dt d^3x$$  

$$+ \int 2N\sqrt{g} \nabla_i \phi \nabla^i \phi dt d^3x$$  \hspace{1cm} (3.43)

$$- \int 4N\sqrt{g}\phi \nabla^2 \phi dt d^3x$$

We are now in a $(3 + 1)$-dimensional form and so we would like to use the power of $\phi$ which is appropriate in 3 dimensions. From the earlier discussion of conformal invariance in different numbers of dimensions we find that we should use $\psi = \phi^{1/2}$ (or $\psi^2 = \phi$). This is no more than a relabelling to make things look neater. There is no real change to the theory in this mere relabelling. We get,

$$S = \int N\sqrt{g}\psi^4\left(R - (trB)^2 + B^{ab}B_{ab}\right) dt d^3x$$  

$$+ \int 8N\sqrt{g}\psi^2 \nabla_i \psi \nabla^i \psi dt d^3x - \int 8N\sqrt{g}\psi^2 \nabla_i \psi \nabla^i \psi dt d^3x$$  \hspace{1cm} (3.44)

$$- \int 8N\sqrt{g}\psi^3 \nabla^2 \psi dt d^3x$$

Thus the action is

$$S = \int N\sqrt{g}\psi^4\left(R - 8\frac{\nabla^2 \psi}{\psi} - (trB)^2 + B^{ab}B_{ab}\right) dt d^3x$$  \hspace{1cm} (3.45)

This looks much better! In fact, this is precisely the action we found in Chapter 1 (1.53)! By demanding 4-dimensional conformal invariance we have constructed the exact theory BOM found by demanding only 3-dimensional conformal invariance.
3.3 Conformally Related Solutions

We should consider again the issue of conformally related solutions. Suppose we have a solution of the equations \( g_{\alpha\beta} \) and \( \phi \). If we perform a conformal transformation on this metric with conformal factor \( \alpha \), say, the new metric \( h_{\alpha\beta} = \alpha^2 g_{\alpha\beta} \) must still be a solution. We find that the conformal factor this time is \( \eta = \frac{\phi}{\alpha} \) and so we have \( \phi^2 g_{\alpha\beta} = \eta^2 h_{\alpha\beta} \) which is yet another demonstration of the identification of conformally related solutions.

In the physical representation the 4-dimensional equations take the form of the Einstein equations in vacuum

\[
G^{\alpha\beta} = 0
\]  

(3.46)

However, these are supplemented with the conformal conditions breaking the 4-dimensional covariance and thus setting it apart from general relativity.

3.3.1 Topological Considerations

We found earlier that if the manifold was compact without boundary that we had to make a change to the action by adding in a volume term. Of course, with the introduction of the volume term we have a change in the original 4-dimensional action also. This becomes,

\[
\int \sqrt{\left(-g^{(4)}\right)} g^{\phi^2 \left(\frac{(4)R}{6} - \frac{\Box \phi}{\phi}\right)} \frac{1}{V(\phi)^{\frac{1}{2}}} d^4x
\]

(3.47)

We have an implicit \((3+1)\) split here because \( V \) is a purely three-dimensional quantity. We vary with respect to \((4)g_{\alpha\alpha}\) and \((4)g_{ij}\) separately. (We vary with respect to the lower index case as \((4)g_{ij} = g_{ij}\) and so both the numerator and the denominator may be varied with respect to the spatial part of the metric.) The variations give

\[
\bar{G}^{0\alpha} = 0
\]

(3.48)

and

\[
N \sqrt{g} \phi^2 \bar{G}^{ij} + \frac{2}{3} g^{ij} \sqrt{g} C \phi^3 = 0
\]

(3.49)

where

\[
C = \int \sqrt{g} \psi^4 \left( R - 8 \frac{\Box \psi}{\psi} \right) V(\psi)^{-\frac{1}{2}} d^3x
\]

(3.50)

arises, as usual, due to variation of the volume. As earlier, \( \bar{G}^{\alpha\beta} \) is the Einstein tensor of the metric \( \phi^2 g_{\alpha\beta} \) and \( \psi^2 = \phi \). We have used the Hamiltonian constraint to simplify \( C \).

We can combine the equations to get

\[
\bar{G}^{\alpha\beta} + \frac{2}{3N} h^{\alpha\beta} C \phi = 0
\]

(3.51)
where $h^{\alpha\beta}$ is the induced 3-metric. This has the form

$$
\begin{bmatrix}
h^{00} & h^{0k} \\
h^{i0} & h^{ik}
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 \\
0 & g^{ik}
\end{bmatrix}
$$

(3.52)

We may lower the indices using $g_{\alpha\beta}$ to get

$$
\begin{bmatrix}
h_{00} & h_{0k} \\
h_{i0} & h_{ik}
\end{bmatrix}
= 
\begin{bmatrix}
N^s N_s & N_k \\
N_i & g_{ik}
\end{bmatrix}
$$

(3.53)

In the physical representation equation (3.51) becomes

$$
G^{\alpha\beta} + \frac{2}{3N}h^{\alpha\beta}C = 0
$$

(3.54)

where now $C = \left\langle NR \right\rangle$.

Of these ten equations, the four $0\alpha$ equations are identical to those in general relativity while the remaining six differ by the new term which arose due to the variation of the volume. This new term is both time dependent and position dependent and so behaves like a “non-constant cosmological constant.” It will undoubtedly lead to new features, particularly in cosmology. However we shall not delve into this here.

We must also do the variations with respect to $\phi$ and $\dot{\phi}$. The volume is independent of $\dot{\phi}$ and so this variation gives us exactly the same result as earlier, namely

$$
trB = 0
$$

(3.55)

However the volume is not independent of $\phi$ and so we will have a slight change. Varying with respect to $\phi$ gives us exactly what we found when we did the variation on the original form of the action (of course)

$$
N\psi^3 \left( R - 7\frac{\nabla^2 \psi}{\psi} \right) - \nabla^2 \left( N\psi^3 \right) = C\psi^5
$$

(3.56)

where

$$
C = \int \frac{N\sqrt{g}\psi^4 \left( R - 8\frac{\nabla^2 \psi}{\psi} \right) d^3x}{V(\psi)}
$$

(3.57)
This becomes
\[ NR - \nabla^2 N = \left\langle NR \right\rangle \] (3.58)
in the physical representation.

Let us consider equation (3.54) again. Taking the trace gives us
\[ -^{(4)}R + 2 \frac{\left\langle NR \right\rangle}{N} = 0 \] (3.59)
or
\[ N^{(4)}R = 2 \left\langle NR \right\rangle \] (3.60)
If we average both sides of this equation we get
\[ \frac{\int N \sqrt{g}^{(4)} R d^3x}{\int \sqrt{g} d^3x} = 2 \left\langle NR \right\rangle \] (3.61)
Now, decomposing \(^{(4)}R\) as earlier we get
\[ \int \frac{N \sqrt{g} \left( R - (trK)^2 + K^{ab}K_{ab} - 2A_{\alpha}^0 \right) d^3x}{V} = 2 \left\langle NR \right\rangle \] (3.62)
This gives us
\[ \int \frac{N \sqrt{g} \left( R - (trK)^2 + K^{ab}K_{ab} \right) d^3x}{V} - 2 \int \frac{N \sqrt{g} A_{\alpha}^0 d^3x}{V} = 2 \left\langle NR \right\rangle \] (3.63)
We notice that \( N \sqrt{g} = \sqrt{-(4)g} \) and so we may write
\[ \int \frac{N \sqrt{g} \left( R - (trK)^2 + K^{ab}K_{ab} \right) d^3x}{V} - 2 \int \frac{\sqrt{-(4)g} A_{\alpha}^0 d^3x}{V} = 2 \left\langle NR \right\rangle \] (3.64)
Thus we have
\[ \int \frac{N \sqrt{g} \left( R - (trK)^2 + K^{ab}K_{ab} \right) d^3x}{V} - 2 \int \frac{\left( \sqrt{-(4)g} A_{\alpha}^0 \right) d^3x}{V} = 2 \left\langle NR \right\rangle \] (3.65)
The second term on the left hand side is a total 4-divergence. We can discard the spatial part to leave us with
\[ \int \frac{N \sqrt{g}(R - (trK)^2 + K^{ab}K_{ab}) d^3x}{V} - 2 \int \frac{\left( \sqrt{-(4)g} A_{\alpha}^0 \right) d^3x}{V} = 2 \left\langle NR \right\rangle \] (3.66)
Using the Hamiltonian constraint we get
\[
\int \sqrt{g}(2NR)\frac{d^3x}{V} - 2 \int \frac{\left(\sqrt{-g}A^0\right)_{,0} d^3x}{V} = 2\langle NR\rangle
\]  
(3.67)

This is
\[
2\langle NR\rangle - 2 \int \frac{\left(\sqrt{-g}A^0\right)_{,0} d^3x}{V} = 2\langle NR\rangle
\]  
(3.68)

and so
\[
\int \left(\sqrt{-g}A^0\right)_{,0} d^3x = 0
\]  
(3.69)

Thus
\[
\left(\sqrt{-g}A^0\right)_{,0} = 0
\]  
(3.70)

But using the form of \( A^\alpha \) which we gave earlier
\[
A^\alpha = \left(n^\alpha trK + a^\alpha\right)
\]  
(3.71)

we have
\[
\left(\sqrt{-g}trK\right)_{,0} = 0
\]  
(3.72)

Recall once more that \( \sqrt{g} = N\sqrt{\bar{g}} \) to get
\[
\left(\sqrt{\bar{g}}trK\right)_{,0} = 0
\]  
(3.73)

which from the definition of \( \pi^{ab} \) is
\[
\frac{\partial tr\pi}{\partial t} = 0
\]  
(3.74)

Of course, this is already known from the propagation of the \( tr\pi \) constraint. Thus we have demonstrated that there is no inconsistency in the equations.

**Note:** Although we have demonstrated this only in the physical representation it is equally valid in the general representation.

Of course, despite all these nice outcomes we know that the theory is flawed. Can we find a suitable 4-dimensional picture for the new conformal theory?

### 3.4 New Conformal Gravity

Despite the natural way in which the 4-dimensional picture emerged in the non-expanding theory we could have simply realised what the 4-dimensional equations would have been
by comparing with the GR cases. This is exactly how we will proceed here. (The change from the simple \( \theta \) term in the original theory to the more complicated \( \nabla_c \xi^c \) makes finding an action much more difficult. Indeed whether anything is actually to be gained in finding a 4-dimensional action remains to be seen.)

### 3.4.1 Non-Compact

Let’s begin with the non-compact theory. We want to work with geometric quantities here rather than the Hamiltonian quantities. That is, in terms of the extrinsic curvature rather than the momentum. The three constraints are

\[
A^{ab}A_{ab} - \frac{2}{3}(trK)^2 - R = 0 \tag{3.75}
\]

\[
\nabla_b (A^{ab} + \frac{2}{3}g^{ab}trK) = 0 \tag{3.76}
\]

\[
\nabla_c trK = 0 \tag{3.77}
\]

We also have the lapse-fixing equation

\[
NR - \nabla^2 N + \frac{N(trp)^2}{4} = 0 \tag{3.78}
\]

It is well known that the 0\( \alpha \) components of the Einstein tensor are the Hamiltonian and momentum constraints of GR. (Actually, this is true for the \( G^0_\alpha \) components strictly speaking.) Let’s consider the conformal constraints. The first two (3.75) and (3.76) are exactly the same as the GR constraints. We simply have

\[
G^{0\alpha} = 0 \tag{3.79}
\]

as in GR. As for the \( ij \) components, we have exactly the same as in GR also since the evolution equations are identical in both theories. Thus overall

\[
G^{\alpha\beta} = 0 \tag{3.80}
\]

These are supplemented by the conformal constraints (3.77) and (3.78). These break the 4-covariance. Nonetheless, the two theories are incredibly similar. Every solution of the conformal theory is a solution of GR although there are solutions of GR which are not solutions of the conformal theory (namely those which do not have a constant CMC slicing as prescribed by the conformal theory). When the volume terms are introduced it is not quite so simple.
### 3.4.2 Compact Manifold

As noted above, the constraints are the one up one down 0α components of the Einstein tensor \( G^0_α \). When both indices are raised we have

\[
G^{00} = -\frac{1}{2N^2} \left( A^{ab} A_{ab} - \frac{2}{3} (trK)^2 - R \right) + \frac{N^a}{N^2} \nabla_b \left( A^b_a - \frac{2}{3} g^b_a trK \right)
\]

and

\[
G^{0c} = \frac{N^c}{2N^2} \left( A^{ab} A_{ab} - \frac{2}{3} (trK)^2 - R \right) + \nabla_b \left( A^{ab} - \frac{2}{3} g^{ab} trK \right) - \frac{N^c N^a}{N^2} \nabla_b \left( A^b_a - \frac{2}{3} g^b_a trK \right)
\]

Although we have raised the indices both equations are still combinations of the constraints which are

\[
A^{ab} A_{ab} - \frac{2}{3} (trK)^2 V^2 - gR = 0 \tag{3.83}
\]

\[
\nabla_b \left( -\frac{A^{ab}}{V^{2/3}} + \frac{2}{3} g^{ab} trK V^{4/3} \right) = 0 \tag{3.84}
\]

\[
\nabla_c \left( trK V^{4/3} \right) = 0 \tag{3.85}
\]

We also have the lapse-fixing equation

\[
NR - \nabla^2 N + N (trK)^2 V^2 = C \tag{3.86}
\]

where \( C = \left\langle N \left( R + (trK)^2 V^2 \right) \right\rangle \).

The first two constraints (3.83) and (3.84) are very similar to the components of \( G^{0a} \). Indeed, just by adding and subtracting things we can get the constraints to appear explicitly as components of \( G^{0a} \) with extra terms. Consider the first constraint (3.83). This can be written as

\[
A^{ab} A_{ab} - \frac{2}{3} (trK)^2 - gR + \frac{2}{3} (trK)^2 - \frac{2}{3} (trK)^2 V^2 = 0 \tag{3.87}
\]

That is

\[
A^{ab} A_{ab} - \frac{2}{3} (trK)^2 - gR = \frac{2}{3} (trK)^2 (V^2 - 1) \tag{3.88}
\]

The second constraint (3.84) can be written as

\[
\frac{1}{V^{2/3}} \nabla_b \left( -A^{ab} + \frac{2}{3} g^{ab} trK V^2 \right) = 0 \tag{3.89}
\]

Then

\[
\frac{1}{V^{2/3}} \nabla_b \left( -A^{ab} + \frac{2}{3} g^{ab} trK \right) = \frac{1}{V^{2/3}} \frac{2}{3} \nabla_b \left( g^{ab} trK (1 - V^2) \right) \tag{3.90}
\]
Thus, referring to the components of $G^{0\alpha}$ (3.81) and (3.82) above we get
\[ G^{00} = \frac{1}{3N^2}(trK)^2(1 - V^2) + \frac{2}{3} \frac{N^c}{N^2}(V^2 - 1) \nabla_c trK \] (3.91)
and
\[ G^{0c} = \frac{N^c}{3N^2}(trK)^2(V^2 - 1) + 2 \left( \frac{N^c}{N^2} - \frac{N^c N^d}{N^2} \right) \nabla_d trK \] (3.92)

Of course, by the first conformal constraint (3.85) we have $\nabla_c trK = 0$. Thus we get
\[ G^{00} = \frac{1}{3N^2}(trK)^2(1 - V^2) \] (3.93)
and
\[ G^{0c} = \frac{N^c}{3N^2}(trK)^2(V^2 - 1) \] (3.94)

We can further simplify (well, a little at least). Let’s define
\[ 8\pi \rho_{\text{ex}} = \frac{1}{3N^2}(trK)^2(1 - V^2) \] (3.95)
and
\[ 8\pi s^c_{\text{ex}} = \frac{N^c}{3N^2}(trK)^2(V^2 - 1) \] (3.96)
which gives
\[ s^c_{\text{ex}} = -N^c \rho_{\text{ex}} \] (3.97)

We can now write
\[ G^{0\alpha} = 8\pi T^{0\alpha}_{\text{ex}} \] (3.98)
where
\[ T^{00}_{\text{ex}} = \rho_{\text{ex}} \] (3.99)
and
\[ T^{0c}_{\text{ex}} = s^c_{\text{ex}} \] (3.100)

That was the easy part, so to speak. Finding the $ab$ components is more tricky. The most straightforward way to get this part is to look at the evolution equation for $\pi^{ab}$. For us, the most convenient way to express this is as
\[ \frac{\partial \pi^{ab}}{\partial t} = \frac{\partial \sigma^{ab}}{\partial t} + 1 \frac{\partial g^{ab} tr\pi}{\partial t} \] (3.101)

This is
\[ \frac{\partial \pi^{ab}}{\partial t} = \frac{\partial \sigma^{ab}}{\partial t} + \frac{1}{3} tr\pi \frac{\partial g^{ab} tr\pi}{\partial t} + \frac{1}{3} g^{ab trp} \frac{\partial \sqrt{g}}{\partial t} \frac{\partial trp}{\partial t} + \frac{1}{3} g^{ab trp} \frac{\partial trp}{\partial t} \] (3.102)

Consider now each term on its own. There are quite a lot of calculations here which are all quite straightforward although cumbersome. We want to get each term expressed as
the equivalent term in GR plus whatever extra terms there are. It is easiest explained by actually performing the calculations. The first term proceeds as follows. In Chapter 2 we had

\[
\frac{\partial \sigma_{ab}}{\partial t} = -N \sqrt{g} \left( R^{ab} - \frac{1}{3} g^{ab} R \right) - \frac{2N V^{2/3}}{\sqrt{g}} \sigma^{ac} \sigma^{b}_c \\
+ \sqrt{g} \left( \nabla^a \nabla^b N - \frac{1}{3} g^{ab} \nabla^2 N \right) + \nabla_c \left( \sigma^{ab} N^c \right)
\]

(3.103)

Let’s factor out \( \frac{1}{V^{2/3}} \). We get

\[
\frac{\partial \sigma_{ab}}{\partial t} = \frac{1}{V^{2/3}} \left( -N \sqrt{g} \left( R^{ab} - \frac{1}{3} g^{ab} R \right) - \frac{2N V^{4/3}}{\sqrt{g}} \sigma^{ac} \sigma^{b}_c \\
+ \sqrt{g} \left( \nabla^a \nabla^b N - \frac{1}{3} g^{ab} \nabla^2 N \right) + V^{2/3} \nabla_c \left( \sigma^{ab} N^c \right) \\
- V^{2/3} \sigma^{bc} \nabla_c N^a - V^{2/3} \sigma^{ac} \nabla_c N^b \\
+ \frac{N}{3 \sqrt{g} V^{4/3}} \sigma^{ab} tr \pi \right)
\]

(3.104)

Now we need to change momenta to curvatures. Recall that we had (2.93)

\[
\sigma_{ab} = -\sqrt{g} \frac{S_{ab}}{V^{2/3}}
\]

(3.105)

and

\[
tr \pi = 2\sqrt{g}(tr K) V^{4/3}
\]

(3.106)

We get

\[
\frac{\partial \sigma_{ab}}{\partial t} = \frac{1}{V^{2/3}} \left( -N \sqrt{g} \left( R^{ab} - \frac{1}{3} g^{ab} R \right) - \frac{2N g V^{4/3}}{\sqrt{g}} S^{ac} S^{b}_c \\
+ \sqrt{g} \left( \nabla^a \nabla^b N - \frac{1}{3} g^{ab} \nabla^2 N \right) - V^{2/3} \nabla_c \left( \sqrt{g} S^{ab} \frac{S_{bc}}{V^{2/3}} N^c \right) \\
+ V^{2/3} \sqrt{g} S^{bc} \frac{S_{ab}}{V^{2/3}} \nabla_c N^a + V^{2/3} \sqrt{g} S^{ac} \frac{S_{bc}}{V^{2/3}} \nabla_c N^b \\
- \frac{2N g S^{ab}}{3 \sqrt{g} V^{2/3}} tr KV^{4/3} \right)
\]

(3.107)
Simplifying we get
\[
\frac{\partial \sigma^{ab}}{\partial t} = \frac{1}{V^{2/3}} \left( -N \sqrt{g} \left( R^{ab} - \frac{1}{3} g^{ab} R \right) - 2N \sqrt{g} S^{ac} S^{b}_{c} \right.
\]
\[
+ \sqrt{g} \left( \nabla^{a} \nabla^{b} N - \frac{1}{3} g^{ab} \nabla^{2} N \right) - \sqrt{g} \nabla_{c} \left( \sigma^{ab} N^{c} \right)
\]
\[
+ \sqrt{g} S^{c} \nabla_{c} N^{a} + \sqrt{g} S^{ac} \nabla_{c} N^{b}
\]
\[
- \frac{2N \sqrt{g}}{3} S^{ab} \text{tr} K \right)
\]
(3.108)

The quantity inside the outer brackets is exactly what we have in GR.
\[
\left( \frac{\partial \sigma^{ab}}{\partial t} \right)_{CG} = \frac{1}{V^{2/3}} \left( \frac{\partial \sigma^{ab}}{\partial t} \right)_{GR}
\]
(3.109)

This shows the correspondence between the two theories as closely as possible and there are no extra terms. Considering (3.102) again and taking the second term on the right hand side we get
\[
\frac{1}{3} \text{tr} \pi \frac{\partial g^{ab}}{\partial t} = -\frac{1}{3} \text{tr} \pi g^{ac} g^{bd} \frac{\partial g^{cd}}{\partial t}
\]
\[
= -\frac{1}{3} \text{tr} \pi \left( \frac{2N V^{2/3}}{\sqrt{g}} \sigma^{ab} - \frac{1}{3} \frac{N \text{tr} \pi g^{ab} V^{2/3}}{V^{2}} + (KN)^{ab} \right)
\]
\[
= -\frac{2N V^{2/3}}{3 \sqrt{g}} \sigma^{ab} \text{tr} \pi + \frac{N (tr \pi)^{2}}{9 \sqrt{g} V^{4/3}} - \frac{\text{tr} \pi (KN)^{ab}}{3}
\]
\[
= \frac{4N}{3} \sqrt{g} S^{ab} \text{tr} K V^{4/3} + \frac{4N}{9} V^{2/3} - \frac{2}{3} \sqrt{g} \text{tr} K V^{2} (KN)^{ab}
\]
(3.110)
\[
= \frac{1}{V^{2/3}} \left( \frac{4N}{3} \sqrt{g} S^{ab} \text{tr} K V^{2} + \frac{4N}{9} V^{2/3} - \frac{2}{3} \sqrt{g} \text{tr} K V^{2} (KN)^{ab} \right)
\]
\[
= \frac{1}{V^{2/3}} \left( -\frac{1}{3} \left( \text{tr} \pi g^{ac} g^{bd} \frac{\partial g^{cd}}{\partial t} \right)_{GR} + \frac{4N}{3} \sqrt{g} S^{ab} \text{tr} K (V^{2} - 1)
\]
\[
+ \frac{4N}{9} V^{2/3} - \frac{2}{3} \sqrt{g} \text{tr} K (V^{2} - 1) (KN)^{ab} \right)
\]
This shows the relationship between the two theories as closely as possible for this term.

The third term in (3.102) is
\[
\frac{1}{6} g^{ab} tr \pi g^{cd} \frac{\partial g_{cd}}{\partial t} = \frac{1}{6} g^{ab} tr \pi \left( -\frac{N tr \pi}{\sqrt{g} \sqrt{V}} + 2 \nabla_c N^c \right)
\]
\[
= - \frac{N}{6 \sqrt{g} \sqrt{V}} g^{ab} (tr \pi)^2 + \frac{1}{3} g^{ab} tr \pi \nabla_c N^c
\]
\[
= \frac{1}{V^{2/3}} \left( - \frac{2}{3} N \sqrt{g} g^{ab} (tr K)^2 V^2 + \frac{2}{3} \sqrt{g} g^{ab} tr K V^2 \nabla_c N^c \right)
\]  
\[
= \frac{1}{V^{2/3}} \left( \left( \frac{1}{6} g^{ab} tr \pi g^{cd} \frac{\partial g_{cd}}{\partial t} \right)_{GR} - \frac{2}{3} N \sqrt{g} g^{ab} (tr K)^2 (V^2 - 1) + \frac{2}{3} \sqrt{g} g^{ab} C \right) \tag{3.111}
\]

The final term in (3.102) is the most straightforward. We have
\[
\frac{1}{3} g^{ab} \sqrt{g} \frac{\partial tr p}{\partial t} = \frac{1}{V^{2/3}} \left( \frac{1}{3} g^{ab} \sqrt{g} \frac{\partial tr p}{\partial t} \right)_{GR} + \frac{2}{3} N \sqrt{g} (tr K)^2 (V^2 - 1) - \frac{2}{3} \sqrt{g} g^{ab} C \tag{3.112}
\]

where \( C = \left< N \left( R + (tr K)^2 \right)^2 \right> \). The overall result is
\[
\left( \frac{\partial \pi^{ab}}{\partial t} \right)_{CG} = \frac{1}{V^{2/3}} \left[ \left( \frac{\partial \pi^{ab}}{\partial t} \right)_{GR} + \frac{4}{3} N \sqrt{g} S^{ab} tr K (V^2 - 1) + \frac{4}{9} N \sqrt{g} g^{ab} (tr K)^2 (V^2 - 1) + \frac{2}{3N} \sqrt{g} tr K (V^2 - 1) \left( g^{ab} \nabla_c N^c - (KN)^{ab} \right) - \frac{2}{3} \sqrt{g} g^{ab} C \right] \tag{3.113}
\]

Of course,
\[
\left( \frac{\partial \pi^{ab}}{\partial t} \right)_{GR} = \frac{\delta L_{GR}}{\delta g^{ab}} = -N \sqrt{g} G^{ab} \tag{3.114}
\]

Thus we get
\[
G^{ab} = \frac{4}{3} S^{ab} tr K (V^2 - 1) + \frac{4}{9} g^{ab} (tr K)^2 (V^2 - 1) \tag{3.115}
\]
\[
+ \frac{2}{3N} tr K (V^2 - 1) \left( g^{ab} \nabla_c N^c - (KN)^{ab} \right) - \frac{2}{3N} g^{ab} C
\]

Let’s label the right hand side as \( 8 \pi s_{ex}^{ab} \). Thus we have
\[
G^{ab} = 8 \pi T_{ex}^{ab} \tag{3.116}
\]

where \( T_{ex}^{00} = \rho_{ex} \), \( T_{ex}^{0i} = s_{ex}^i \) and \( T_{ex}^{ab} = s_{ex}^{ab} \). Here we are assigning an energy-momentum tensor to the expansion.
We could change notation here and instead of using an energy-momentum tensor relabel again. We’ll define

\[-C^{\alpha\beta} = 8\pi T_{ex}^{\alpha\beta}\] (3.117)

The field equations are

\[G^{\alpha\beta} - C^{\alpha\beta} = 0\] (3.118)

In both cases, of course, the equations are supplemented by the conformal constraint

\[\nabla_c tr K = 0\] (3.119)

and the lapse-fixing equation

\[NR - \nabla^2 N + N(tr K)^2 V^2 = C\] (3.120)

### 3.5 Special Case

A special case of the theory is when \(tr K = 0\). We would expect the theory to reduce to the original static theory. If we look at the 4-dimensional equations we see that \(tr K = 0\) gives exactly [3.54]

\[G^{\alpha\beta} + \frac{2}{3N} h^{\alpha\beta} C = 0\] (3.121)

where now \(C = \left\langle NR \right\rangle\). The conformal constraint and lapse-fixing equations also reduce to those of the static theory. We could check this in any of the various formulations we have considered and it would work out in each and every one.

### 3.6 The Solar System

A necessary result for any theory of gravity is that it reproduce the well-tested (talk about an understatement!) solar system results. The field equations as presented here offer a good chance to do just this. If we take the solar system to be isolated and asymptotically flat then clearly we must use the asymptotically flat version of this theory. In that case we have complete agreement. The maximally sliced solar system result of GR is well known and thus we reproduce the results.

However, we could treat the solar system as part of a larger solution. Clearly then we cannot treat it as an isolated, asymptotically flat system. However, we still assume it to be static. Then we have \(tr K = 0\). We have just seen that the only difference between our field equations and those of GR for a static region is the term \(\frac{2}{3N} h^{\alpha\beta} C\) where \(C = \left\langle NR \right\rangle\).
The 0α equations are exactly the same here. Indeed, if \( C \) is small. In a closed FRW universe this term will certainly be very small as the curvature is very small. Thus we recover the solar system results quite easily.

### 3.7 Comment

It is very interesting that the 4-dimensional picture is so similar to the standard GR picture. Of course, the (3 + 1) form is also very similar to that of GR but when we move to strictly geometrical quantities the similarities show up all the more so. We could do this in the (3 + 1) formalism also by replacing the momentum terms with their corresponding curvature terms. However, doing this a second time (since we have done it here already) seems excessive.
Chapter 4

Cosmology

4.1 Introduction

Despite the successes enjoyed by the original theory, it suffers from the fact that it predicts a static universe and so expansion is automatically prohibited. As a result, all of the successes of the big bang picture are lost and in particular the cosmological redshift - an experimental fact - is unexplained. In the new theory we have succeeded in recovering expansion and with this renewed confidence we should examine further the cosmological implications of the new theory. To begin with it will be instructive to briefly review cosmology à la GR.

4.2 Cosmology In General Relativity

In GR the dynamics are all in the Hamiltonian constraint. This is written in terms of the geometry and sources in question and from this the cosmological dynamics of the universe are determined. We’ll assume the standard Friedmann-Robertson-Walker (FRW) metrics and consider each in turn beginning with the open universe.

4.2.1 Open Universe

The FRW metric for the open universe is

\[ d\sigma^2 = a(t)^2 \left[ d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \right] \]  \hspace{1cm} (4.1)
The tracefree part of the extrinsic curvature, $A_{ab}$, is zero and the trace part is $-\frac{3\dot{a}}{a}$. Then

$$trp = 2trK = -\frac{6\dot{a}}{a}$$  \hspace{1cm} (4.2)$$

The Hamiltonian constraint is

$$R + \frac{1}{6}(trp)^2 = 0$$  \hspace{1cm} (4.3)$$

In cosmological terms this becomes

$$-\frac{6}{a^2} + \frac{6\dot{a}^2}{a^2} = 0$$  \hspace{1cm} (4.4)$$

This can be solved easily to give

$$a = t + a_i$$  \hspace{1cm} (4.5)$$

where $a_i$ is the value of $a$ at $t = 0$. Let’s move to the flat case.

### 4.2.2 Flat Universe

The FRW metric for a flat universe is given by

$$d\sigma^2 = a(t)^2 \left[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$  \hspace{1cm} (4.6)$$

The extrinsic curvature is unchanged from the above case. The Hamiltonian constraint is now

$$0 + \frac{6\dot{a}^2}{a^2} = 0$$  \hspace{1cm} (4.7)$$

and so we get

$$\dot{a} = 0$$  \hspace{1cm} (4.8)$$

a static universe.

Of course, we haven’t tried adding matter to the system in either of these two cases. We could do this easily and get different results for $a(t)$. The whole point here is to compare (and contrast) the predictions of the two theories and for reasons that will become apparent soon, we need only concern ourselves with the closed universe in any detail.

### 4.2.3 Closed Universe

The FRW metric for the closed universe is

$$d\sigma^2 = a(t)^2 \left[d\chi^2 + \sin^2\chi^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$  \hspace{1cm} (4.9)$$
The extrinsic curvature is the same yet again. This time the Hamiltonian constraint is

$$R + \frac{1}{6}(trp)^2 = 0 \quad (4.10)$$

$$\frac{6}{a^2} + \frac{6\dot{a}^2}{a^2} = 0 \quad (4.11)$$

Thus

$$\dot{a}^2 = -1 \quad (4.12)$$

which is a contradiction. Thus we cannot have a closed and matter-free FRW universe.

Let’s try adding matter. We will try three different types, namely a cosmological constant, radiation and dust.

**Cosmological Constant**

With the introduction of a cosmological constant the Hamiltonian constraint becomes

$$R + \frac{1}{6}(trp)^2 = \Lambda \quad (4.13)$$

and so

$$\frac{6}{a^2} + \frac{6\dot{a}^2}{a^2} = \Lambda \quad (4.14)$$

and then

$$\dot{a}^2 = \frac{\Lambda a^2}{6} \quad (4.15)$$

We can now solve this for \(a(t)\) (if we wish).

**Radiation**

Radiation couples to gravity in GR in the Hamiltonian constraint as

$$R - \sigma^{ab}\sigma_{ab} + \frac{1}{6}(trp)^2 = 16\pi \rho_r \quad (4.16)$$

where \(\rho_r\) is the radiation energy density. For radiation

$$\rho_r = \rho_{r0} \frac{a_0^4}{a^4} \quad (4.17)$$

where a subscript 0 denotes the value today. Thus the Hamiltonian constraint becomes

$$\frac{6}{a^2} + \frac{6\dot{a}^2}{a^2} = 16\pi \rho_{r0} \frac{a_0^4}{a^4} \quad (4.18)$$

Again, we can solve this for \(a(t)\).
Dust

Dust couples to gravity in the Hamiltonian constraint as

\[ R - \sigma^{ab}\sigma_{ab} + \frac{1}{6}(tr_p)^2 = 16\pi\rho_d \] (4.19)

where \( \rho_d \) is the dust energy density. For dust

\[ \rho_d = \rho_{d0} \frac{a_0^3}{a^3} \] (4.20)

where a subscript 0 again denotes the value today. Thus the Hamiltonian constraint becomes

\[ \frac{6}{a^2} + \frac{6\dot{a}^2}{a^2} = 16\pi\rho_{d0} \frac{a_0^3}{a^3} \] (4.21)

Again, we can solve this for \( a(t) \).

Although we have glossed over some details and not solved explicitly for \( a(t) \) each time, the important features should be clear. Let’s move to the conformal theory and see what happens there.

4.3 Cosmology in the Conformal Theory

In the conformal theory we will need to consider two constraints. The Hamiltonian constraint and the conformal constraint \( tr_p \equiv \text{constant} \) will both have significance. Again, we will assume the standard FRW metrics and consider each in turn. We will use the results for the Hamiltonian constraints from the last chapter in every case.

4.3.1 Open Universe

The metric in this case is

\[ ds^2 = a(t)^2 \left[ d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \right] \] (4.22)

The first thing we should consider is the constant \( tr_p \) constraint. The extrinsic curvature is of course the same as in GR: the tracefree part, \( A_{ab} \), is zero and we get \( trK = -\frac{3\dot{a}}{a} \). Thus, the equation to be solved is

\[ \int \frac{1}{a} \, da = \int C \, dt \] (4.23)

This gives us

\[ a = Ae^{Bt} \] (4.24)
Let’s examine the Hamiltonian constraint now.

The Hamiltonian constraint here is

\[ R + \frac{(trp)^2}{6} = 0 \]  

(4.25)

Thus we get that

\[ R = -\frac{(trp)^2}{6} \]  

(4.26)

that is, \( R \) is constant. Thus, \( a \) must be constant. This is not a great result as we have again lost expansion. Let’s try to couple matter to the system.

**Cosmological Constant**

Adding in a cosmological constant here does not change the essence of the above result. We still have no expansion. The next obvious matter to try is dust.

**Dust**

Dust couples just as in GR to give a Hamiltonian constraint

\[ R + \frac{(trp)^2}{6} = \rho \]  

(4.27)

In cosmological terms we get

\[ -\frac{6}{a^2} + \frac{(trp)^2}{6} = \frac{\rho_0 a^3}{a^3} \]  

(4.28)

Clearly, the solution found above \([4.24]\) does not work here unless, again, \( a \) is constant in time. Thus, we have lost expansion again. We could attempt to couple in radiation but we would find the same problem arising. The open FRW universe seems a lost cause. Let’s move on.

**4.3.2 Flat Universe**

The metric in this case is

\[ ds^2 = a(t)^2 \left[ d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \]  

(4.29)

Applying the constant \( trp \) constraint gives us the same as in the open universe case.

\[ a = Ae^{Bt} \]  

(4.30)

Without wasting any more time, it isn’t too difficult to see that we will come up against the very same problems as in the open universe. Thus, both the standard open and flat FRW universes seem to be lost causes. This is worrying. Will this trend continue? Will the theory fail yet again? Let’s find out.
4.3.3 The Closed Universe

The metric here is

\[ d\sigma^2 = a(t)^2 \left[ d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (4.31) \]

We find \( K_{ab} \) and \( trK \) as usual. Yet again we get

\[ A_{ab} = 0 \quad (4.32) \]

and

\[ trK = -\frac{3\dot{a}}{a} \quad (4.33) \]

However, in this case we have a totally new expression for finding the momentum

\[ trp = 2(trK)V^{4/3} \quad (4.34) \]

The volume of a hypersurface in this universe is

\[ V = 2\pi^2 a^3 \quad (4.35) \]

Thus we get

\[ trp = 2 \left( -\frac{3\dot{a}}{a} \right) (2\pi^2)^{4/3} a^4 \]

\[ = -6(2\pi^2)^{4/3} a^3 \dot{a} \quad (4.36) \]

The constraint becomes

\[ -6(2\pi)^{4/3} a^3 \dot{a} = D \quad (4.37) \]

where \( D \) is a constant. Integrating across we get

\[ -6(2\pi)^{4/3} \int a^3 \ da = \int D \ dt \quad (4.38) \]

Thus we get an expression for \( a(t) \).

\[ a^4 = Ct + a_i^4 \quad (4.39) \]

where \( C \) is a constant both spatially and temporally and \( a_i \) is the radius of the universe at \( t = 0 \).

This must hold regardless of what matter we include which is precisely the point which caused the other cases to fail. What will happen here?
Matter free

Let’s first see what happens if there is no matter. The Hamiltonian constraint is

\[ R + \frac{1}{6} \frac{(trp)^2}{V^{2/3}} = 0 \]  

(4.40)

We know that \( R = \frac{6}{a^2} \) and so we get

\[ \frac{6}{a^2} + \frac{1}{6} \frac{(trp)^2}{(2\pi^2)^{2/3}a^2} = 0 \]  

(4.41)

The \( a^2 \) cancels in each term and we get

\[ 6 + \frac{1}{6} \frac{(trp)^2}{(2\pi^2)^{2/3}} = 0 \]  

(4.42)

We get an identity for \( trp \) consistent with the constant \( trp \) constraint! Of course, this particular identity cannot hold for real \( trp \) but it is a step in the right direction at least.

Let’s try some matter. We will consider three types of matter here: a cosmological constant, dust and radiation. These have been treated with regard to the conformal theory in an earlier chapter but the beauty of what happens (from a cosmological point of view) was not noticed. The various Hamiltonian constraints which we have found earlier will be examined. Of course, we will work purely in the physical representation.

**Cosmological Constant**

The Hamiltonian constraint here is

\[ R + \frac{1}{6} \frac{(trp)^2}{V^{2/3}} + \frac{\Lambda}{V^{2/3}} = 0 \]  

(4.43)

Substituting in the quantities in terms of \( a \) we get

\[ \frac{6}{a^2} + \frac{1}{6} \frac{(trp)^2}{(2\pi^2)^{2/3}a^2} + \frac{\Lambda}{(2\pi^2)^{2/3}a^2} = 0 \]  

(4.44)

and yet again the \( a^2 \) cancels across the entire expression to give

\[ (trp)^2 = -6\Lambda - 36(2\pi^2)^{2/3} \]  

(4.45)

This is entirely consistent with the earlier expression for \( a(t) \). Also, we notice that since both sides must be positive we get a condition on \( \Lambda \)

\[ \Lambda \leq -6(2\pi^2)^{2/3} \]  

(4.46)

This is very encouraging. Let’s try electromagnetism.
The Hamiltonian constraint here was found to be

$$\sigma_{ab} \sigma_{ab} - \frac{1}{6} \frac{(trp)^2}{V^2} + \frac{\mu^i \mu_i}{V^{2/3}} - \frac{U}{V^{2/3}} - \frac{gR}{V^{4/3}} = 0$$ \hspace{1cm} (4.47)

Moving things about a little gives us

$$R - \sigma_{ab} \sigma_{ab} V^{4/3} + \frac{1}{6} \frac{(trp)^2}{V^{2/3}} = \left( \mu^i \mu_i - U \right) \frac{V^{2/3}}{g}$$ \hspace{1cm} (4.48)

This becomes

$$R - \sigma_{ab} \sigma_{ab} V^{4/3} + \frac{1}{6} \frac{(trp)^2}{V^{2/3}} = 16\pi \rho_r V^{2/3}$$ \hspace{1cm} (4.49)

where $\rho_r$ is the energy density of radiation in the universe. For radiation we have that

$$\rho_r = \frac{\rho_r a_0^4}{a^4}$$ \hspace{1cm} (4.50)

where $\rho_r$ is the energy density of radiation today and $a_0$ is the radius of the universe today. The constraint becomes

$$\frac{6}{a^2} + \frac{(trp)^2}{6(2\pi)^{2/3} a^2} = 16\pi \frac{\rho_r a_0^4}{a^4} (2\pi)^{2/3} a^2$$ \hspace{1cm} (4.51)

Yet again, the $a^2$ cancels right across the board and we get an identity involving the collective “energies” which is completely devoid of dynamical content, namely

$$6 + \frac{(trp)^2}{6(2\pi)^{2/3} a^2} = 28^{2/3} \pi^{5/3} \rho_r a_0^4$$ \hspace{1cm} (4.52)

This is completely consistent with the solution of $a(t)$ from earlier. In fact, just like with the cosmological constant we can get the total “radiation energy” in the universe in terms of trp (or vice versa). Let’s now consider dust.

**Dust**

Our Hamiltonian constraint here is

$$R - \sigma_{ab} \sigma_{ab} V^{4/3} + \frac{1}{6} \frac{(trp)^2 V^{-2/3}}{V^{1/3}} - 16\pi \rho_d V^{1/3} = 0$$ \hspace{1cm} (4.53)

where $\rho_d$ is the energy density of dust. For dust we have that

$$\rho_d = \frac{\rho_d a_0^3}{a^3}$$ \hspace{1cm} (4.54)

where $\rho_{do}$ is the energy density of dust today and $a_0$ is the radius of the universe today. In cosmological terms this becomes

$$\frac{6}{a^2} + \frac{1}{6(2\pi)^{2/3} a^2} = 16\pi \rho_d a_0^3 (2\pi)^{1/3} a$$ \hspace{1cm} (4.55)
Yet again, all the $a$ terms cancel and we get
\[ 6 + \frac{1}{6} \frac{(trp)^2}{(2\pi)^{2/3}} = 8192^{1/3} \pi^{4/3} \rho_d a_0^3 \]
(4.56)

Thus, yet again, it is an identity involving strictly non-time evolving terms. The dynamical content is still in $trp = \text{constant}$.

**General Case**

We can put all the results together here to get
\[ 6 + \frac{1}{6} \frac{(trp)^2}{(2\pi)^{2/3}} = 8192^{1/3} \pi^{4/3} \rho_m a_0^3 + 128^{2/3} \pi^{5/3} \rho_r a_0^4 + \frac{\Lambda}{(2\pi)^{2/3}} \]
(4.57)

(Here, $\rho_m$ is the matter mass density in the universe. It behaves just like dust from the point of view of the Hamiltonian constraint.) However, the evolution of the universe is still governed by the equation found earlier
\[ a^4 = Dt + a_i^4 \]
(4.58)

We get an ever-expanding decelerating universe. The Hamiltonian constraint seems to have been promoted to an identity for the various energies. This needs further examination.

(Note: We should note that although we have chosen to treat the constant $trp$ condition separately from the Hamiltonian constraint that in the closed universe they amount to the same thing and the form of the Hamiltonian constraint actually determines that constant!)

### 4.4 Cosmological Parameters

Very often cosmological scenarios are described using various parameters. The most important two being the Hubble parameter and the deceleration parameter. These are defined as
\[ H = \frac{1}{3} \frac{d}{dt} (\ln \sqrt{g}) \]
(4.59)

for the Hubble parameter and
\[ q = -\frac{\dot{a}}{a} \frac{1}{H^2} \]
(4.60)

for the deceleration parameter. For the FRW metrics the Hubble parameter is just
\[ H = \frac{\dot{a}}{a} \]
(4.61)

What does the conformal theory say about these?
4.4.1 Hubble Parameter

We’ll take the same definition for the Hubble parameter

\[ H = \frac{1}{3} \frac{d \ln \sqrt{g}}{dt} \quad (4.62) \]

At first glance, with all the volume terms which have been introduced to the theory it seems unlikely that the result will be the same. It is not too difficult to go through the calculation to find that, indeed, the result does work out just as in GR to give

\[ H = \frac{\dot{a}}{a} \quad (4.63) \]

4.4.2 Deceleration Parameter

We can find an identity for the deceleration parameter explicitly. We have that

\[ trp \equiv \text{constant} \quad (4.64) \]

that is

\[ a^3 \ddot{a} \equiv \text{constant} \quad (4.65) \]

Differentiating across and rearranging slightly we find that

\[ \ddot{a} \equiv -\frac{3\dot{a}^2}{a} \quad (4.66) \]

Substituting this into the formula for the deceleration parameter \( q \) gives

\[ q = -\frac{\ddot{a}}{\frac{a}{H^2}} = -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{-3\dot{a}^2 a}{a\ddot{a}^2} = 3 \quad (4.67) \]

4.5 Problems of the Standard Cosmology

There are a number of well known problems with the standard cosmology of GR. How does the new theory stand up to these? Let’s consider them each in turn.
This is probably the best known problem. In GR we have the following. There is a
discrepancy of at least 120 orders of magnitude between the possible value of $\Lambda$ today
and what is expected at the Planck epoch taking the interpretation of $\Lambda$ as a vacuum
energy. In GR the cosmological constant appears with the scalar curvature in the form
$R + \Lambda$. However, in the new theory here it appears with a volume coefficient in the form
$R + \frac{\Lambda}{V^{2/3}}$. Now, let us consider the change in volume since the Planck epoch. The radius
of the universe was approximately $10^{-35}$ cm at the Planck epoch. Today, the radius of
the universe is about $10^{28}$ cm. Thus the ratio of the volume at the Planck epoch to the
volume today is about
$$\frac{V_0}{V_{Pl}} = \left(\frac{10^{28}}{10^{-35}}\right)^3 = 10^{189} \quad (4.68)$$

Thus
$$\frac{V_0^{2/3}}{V_{Pl}^{2/3}} = 10^{126} \quad (4.69)$$

If we were to consider the quantity $\frac{\Lambda}{V^{2/3}}$ as the cosmological “constant” we would have the
problem encountered in GR. However, the recognition of $\Lambda$ as the constant and recognising
the presence of the volume term removes any problem whatsoever from this theory.

We should point out that this is fundamentally different from postulating a “time-varying
cosmological constant”. The constant enters at the same level in his theory as in GR and
it is the behaviour of the scalar curvature which changes things.

### 4.5.2 The Flatness Problem

The Flatness Problem In GR is entirely a product of the Hamiltonian constraint. In GR
the Hamiltonian constraint determines the dynamics and different energy values change
the dynamics. However, in new CG, the Hamiltonian constraint is purely an identity
and the flatness problem simply doesn’t exist. The universe expands eternally according
to the conformal constraint. This also has implications for the notion of dark matter.
Although dark matter is sometimes employed to explain the otherwise strange behaviour
of particular systems, one of the major reasons is to explain the apparent lack of matter
needed to provide the observed flatness of the universe. The conformal theory needs no
such strange explanations.
4.5.3 The Horizon Problem

To address this problem let’s do a more in-depth analysis. To begin with let’s return to GR and discuss the problem.

Arc Parameter Time

The (4–dimensional) FRW metric for the closed universe is

$$ds^2 = -dt^2 + a(t)^2 \left[ d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right]$$ \hspace{1cm} (4.70)

Of course, we can reparameterise this as

$$ds^2 = a(t)^2 \left( d\eta^2 + d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right)$$ \hspace{1cm} (4.71)

The parameter \( \eta \) is the *arc parameter time*. It is a measure of the distance travelled by a photon along the surface of the three-sphere. In GR we can get values for two different times in particular. The Hubble time \( H^{-1} \) and the proper time \( t \) since the beginning of expansion. The Hubble time in terms of \( \eta \) is

$$H^{-1} = \frac{a(\eta)^2}{da/d\eta}$$ \hspace{1cm} (4.72)

This is given by [9]

$$H^{-1} = \frac{a_{\text{max}}^2 (1 - \cos \eta)^2}{2 \sin \eta}$$ \hspace{1cm} (4.73)

where

$$a_{\text{max}} = \frac{8\pi}{3} \rho_m a_o^3$$ \hspace{1cm} (4.74)

The actual time since the beginning of expansion is given by [9]

$$t = \frac{a_{\text{max}}^2 (1 - \cos \eta)}{2}$$ \hspace{1cm} (4.75)

If we approximate these numbers as

$$H^{-1} = 20 \times 10^9 \text{lyr}$$ \hspace{1cm} (4.76)

and

$$t = 10 \times 10^9 \text{lyr}$$ \hspace{1cm} (4.77)

then we can find the total distance travelled by a photon travelling on the three sphere since the very start of expansion

$$\frac{20 \times 10^9 \text{lyr}}{10 \times 10^9 \text{lyr}} = \frac{H^{-1}}{t}$$

$$= \frac{a_{\text{max}}^2 (1 - \cos \eta)^2}{2 \sin \eta}$$

$$= \frac{a_{\text{max}}^2}{2} (1 - \cos \eta)$$ \hspace{1cm} (4.78)
From this we find that

\[ \eta = 1.975^\circ \]  

(4.79)

Thus the horizon size at decoupling is only about 2\(^o\). However, there are about 10\(^5\) different regions of this size in the cosmic microwave background (CMB) sky. That is, about 10\(^6\) causally disconnected regions and yet we observe isotropy to about one part in 10\(^5\). This is the horizon problem in GR.

What can we discover about this from the conformal theory? We need to find \( a(\eta) \) and \( t(\eta) \). We found earlier the equation for \( a(t) \) (4.39). This was

\[ a(t)^4 = Ct + a_i^4 \]  

(4.80)

where \( C \) was a constant (spatially and temporally) and \( a_i \) is the radius at time \( t = 0 \). Let’s take \( a_i \) to be zero. Thus

\[ a(t)^4 = Ct \]  

(4.81)

now

\[ \eta = \int \frac{dt}{a(t)} \]  

(4.82)

Solving this we get

\[ \eta = \frac{4}{3C^{1/4}} \eta^{3/4} \]  

(4.83)

Inverting this to get \( t(\eta) \) gives

\[ t = \left( \frac{4}{3} \right)^{4/3} C^{1/3} \eta^{4/3} \]  

(4.84)

Then we find \( a(\eta) \) easily

\[ a(\eta) = C^{1/4} \left( \frac{3}{4} \right)^{1/3} C^{1/12} \eta^{1/3} \]  

(4.85)

Differentiating we get

\[ \frac{da}{d\eta} = \frac{1}{3} C^{1/3} \left( \frac{3}{4} \right)^{1/3} \eta^{-2/3} \]  

(4.86)

Then

\[ H^{-1} = \frac{a^2}{da/d\eta} = 3C^{1/3} \left( \frac{3}{4} \right)^{1/3} \eta^{4/3} \]  

(4.87)

Finding the same ratio of \( H^{-1} \) and \( t \) as in GR we get

\[ \frac{H^{-1}}{t} = \left( \frac{3}{4} \right)^{4/3} C^{1/3} \eta^{4/3} \]  

(4.88)
This simplifies very easily to give just

\[ \frac{H^{-1}}{t} = 4 \]  \hspace{1cm} (4.89)

There is no dependence on \( \eta \) whatsoever!

Indeed, if we look at either of the expressions for \( H^{-1} \) and \( t \) in terms of \( \eta \) and demand that the distance travelled by a photon since the beginning of expansion satisfy the seeming experimental value then what we get is a bound on the constant \( C \) which in turn gives a bound on \( trp \) which is already bounded by the values of \( \Lambda, \rho_d \) and \( \rho_m \). There is no horizon problem if the figures match up!! Of course, this removes the need to look for things like inflation which may in turn throw up a problem of its own in looking for large-scale structure formation. At first thought however, it does not seem like there will be a significant problem.

Another speculative idea which would require further work is one which could give rise to inflation. If at some stage in the past the conformal symmetry were to be broken we could envisage a situation where the then physical field \( \psi \) might take on the role of the inflaton before decaying to the purely gauge field we treat in the theory. Again, I must stress that this is pure speculation rather than the “solid” prediction of a crank...

### 4.6 Some Numbers

We can actually make some concrete calculations very easily. The ratio of the cosmological constant at the Planck epoch to that of today is

\[ \frac{\Lambda_{\text{Pl}}}{\Lambda_0} = 10^{121} \]  \hspace{1cm} (4.90)

at the very least. This can give us the ratio of the volume of the universe today to that of the Planck epoch

\[ \left( \frac{V_0}{V_{\text{Pl}}} \right)^{2/3} = 10^{121} \]  \hspace{1cm} (4.91)

and from this the ratio of the radius today to the Planck length

\[ \frac{R_0}{R_{\text{Pl}}} = \sqrt{10^{121}} = 3.2 \times 10^{60} \]  \hspace{1cm} (4.92)

where \( R_{\text{Pl}} \) is the Planck length \( 1.7 \times 10^{-35} \text{m} \). Thus we get

\[ R_0 = 5.4 \times 10^{25} \text{m} \]  \hspace{1cm} (4.93)
for the minimum radius of the universe today.

The relationship found in (4.89)

$$t = \frac{1}{4H}$$

(4.94)
can be used to get an estimate for the age of the universe. Taking the Hubble constant today to be

$$H_0 = 60\text{km/s/Mpc} = 1.9 \times 10^{-18}\text{s}^{-1}$$

(4.95)
gives

$$t_0 = 1.32 \times 10^{17}\text{s}$$

(4.96)
A quick check of the value of \(ct_0\) gives

$$ct_0 = 4 \times 10^{25}\text{m}$$

(4.97)
which is very close to our value of \(R_0 = 5.4 \times 10^{25}\text{m}\). The age of the universe is predicted from other means to be approximately \(4.4 \times 10^{17}\text{s}\). This is about 3 times our value. All the same, we have found it from very elementary reasoning and to exactly the same order. One point here is that we found the deceleration parameter \(q_0\) to be exactly 3. This is higher than expected. However, \(q_0\) is notoriously difficult to measure and perhaps in the light of the predictions here it should be re-examined. If indeed the commonly used value of \(q_0\) (about 1.5) were too low than the value of \(H_0\) would be too high. Then our value of \(t_0\) would go up by a factor similar to the correcting factor for \(q_0\). Thus we see how a factor of 3 might arise.

Suppose for a while that \(H_0\) is indeed lower than accepted. Consider the following expression

$$H_0 = \frac{c}{d_L}f(q_0, z)$$

(4.98)
where \(f(q_0, z)\) is a function of \(q_0\) and \(z\) (the redshift) only. From this we see that a large \(H_0\) acts like a small \(d_L\). Thus, if our value of \(H_0\) is too large we might interpret it as saying that \(d_L\) is too small. A smaller than expected \(d_L\) is exactly what is found in the recent supernovae experiments resulting in an apparent acceleration of the universe. The higher value of \(q_0\) might possibly reconcile this with the theory here as the function \(f\) behaves very roughly like \(q_0^{-1}\).

Of course, a more obvious explanation may be that the simple FRW metric is just not a perfect model for the universe. Perhaps we need a non-standard cosmology.
4.7 A Non-Standard Cosmology: Anisotropy

The standard FRW universes are all homogeneous and isotropic. What, if anything, does the new theory say about the subject of anisotropy. One obvious anisotropic model we can examine quite easily is the Kasner model.

4.7.1 The Kasner Universe

The Kasner metric is

\[ d\sigma^2 = t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2 \]  

(4.99)

where

\[ p_1 + p_2 + p_3 = (p_1)^2 + (p_2)^2 + (p_3)^2 = 1 \]  

(4.100)

Each \( t = \) constant hypersurface of this cosmological model is a flat three-dimensional space. It represents an expanding universe since

\[ \sqrt{g} = t \]  

(4.101)

is constantly increasing. However, its expansion is anisotropic. Consider two standard observers. If only their \( x \)–coordinates differ than their separation is given by \( t^{2p_1} \Delta x \). Thus distances parallel to the \( x \)–axis expand at one rate \( l_1 \propto t^{p_1} \) while those along the \( y \)–axis expand at a different rate \( l_2 \propto t^{p_2} \). A truly remarkable feature is that along one of the axes distances contract rather than expand. This is because one of the \( p_i \) must be non-positive. Let’s calculate the extrinsic curvature of this model. This is very straightforward. We find that

\[ K_{ii} = -2p_i t^{2p_i - 1} \]  

(4.102)

Thus we get

\[ trK = \sum_i t^{-2p_i} (-2p_i t^{2p_i - 1}) \]

\[ = -2t^{-1} \sum p_i \]

\[ = -2t^{-1} \]  

(4.103)

If the universe we are considering is not closed then we know that

\[ trp = 2trK \]  

(4.104)

and here that means that \( trp \) is time-dependent which of course is not allowed by the theory. Thus we cannot have a non-closed Kasner model. Let’s try now the same metric
but where the \( x, y \) and \( z \) coordinates are interpreted as angles with period \( 4\pi \). This model is closed. The volume of a hypersurface is given by

\[
V = \int \sqrt{g} \, d^3x \\
= \int t \, d^3x \\
= t(xyz)|^{4\pi}_0 \\
= 64t\pi^3
\]

The form of \( trK \) is unchanged but now

\[
trp = 2trKV^{4/3}
\]

and so we have

\[
trp = 1024\pi^4t^{1/3}
\]

However, this is still a time-varying quantity! Thus the Kasner model also seems a lost cause.

### 4.7.2 Effective Anisotropic Energy Density

The physics of the anisotropic scenario can be discussed in other language. In GR the idea is to write the Hamiltonian constraint as

\[
\pi^{ab}\pi_{ab} - \frac{1}{2}(tr\pi)^2 - g(R - \rho_m - \rho_{an}) = 0
\]

where \( \rho_{an} \) is the anisotropy energy density and \( \rho_m \) is the energy density of matter (whatever that matter happens to be). The anisotropy energy density is found to have an equation of state

\[
\rho_{an} \propto g^{-1}
\]

That is

\[
\rho_{an} \propto V^{-2}
\]

Thus we can write

\[
\rho_{an} = \rho_{an0} \frac{V_0^2}{V^2}
\]

where the subscript 0 means the value today. In the conformal Hamiltonian constraint this appears with \( R \) as

\[
\frac{R}{V^{4/3}} - \rho_{an0} \frac{V_0^2}{V^2} = \frac{1}{V^{4/3}} \left( R - \rho_{an0} \frac{V_0^2}{V^{2/3}} \right)
\]

Thus we get the characteristic \( V^{-2/3} \) factor even for the anisotropy which is encouraging.
4.8 Discussion

The cosmological scenario presented here is very interesting. Not alone have we managed to recover expansion (our initial inspiration for the new theory) but we have found many other desirable and exciting features. Among these are the following: the theory seems incredibly restrictive: we are forbidden to have an open or flat universe (at least of the FRW type); we have specific bounds on the various matter sources in terms of each other; some of the major problems of the standard GR cosmology have been resolved (and some that haven’t may yet succumb to this theory); we have a definite prediction for the deceleration parameter; we are restricted in the types of anisotropy we may have. Such successes at such an early stage are promising and further work must surely be warranted.
Chapter 5

Discussion

The path of general relativity has led from Einstein’s spacetime to the (space + time) of Dirac and ADM, to Wheeler’s superspace and to finally to York’s conformal superspace. Rather than taking this route we place conformal superspace in the central position to begin with and find a theory which gives a lot while taking very little.

Of course, one might argue that just because this theory is self consistent is not reason enough to demand further attention. However, a recent result of Ó Murchadha \[10\] shows that if we demand a constrained Hamiltonian with a closed constraint algebra then we are severely limited in our options. In fact, there are essentially only 4 options. These are

(i) Regular GR
(ii) A maximally sliced theory
(iii) A constant mean curvature sliced theory
(iv) Strong gravity

The first and fourth are known and exist in their own right. However, the second and third are seen to arise naturally in the conformal approach adopted here. (In fact, we could consider option (ii) to be a special case of option (iii) and see the options decrease even further.) We have found a consistent theory and since these are so rare, it is surely a worthy result in its own right.

Of course, experiment will always have the last word and any theory is only as good as its predictions. How does our theory stand up to experiment? The solar system results
seem to hold and this is also expected to be true for the binary pulsar. However, it is in cosmology that the beauty of this theory is most apparent.

The theory seems to accept very few cosmological solutions. It is incredibly restrictive. We are denied the FRW open and flat universes. We are also denied the Kasner universe. The fact that $trK$ is (spatially) constant places a very severe restriction on what anisotropy (if any) is possible. The constant $trK$ is a property of the theory itself and not simply a property of a particular cosmological solution. The Hamiltonian constraint is elevated to an identity and the dynamics are found in the conformal constraint $trp = \text{constant}$.

Most of the problems of the standard cosmology of GR do not occur and in fact there is a possibility that all of the problems may be removed. Of course, there is also the chance that the predictions of the deceleration parameter and the age of the universe may prove to be the downfall of the theory. The fact that these predictions are so easy to find however is a positive thing and as they say, “hope springs eternal.”

From a quantisation viewpoint the theory has several attractive features. Firstly, with regard to the static theory, the absence of $tr\pi$ removes various problems since the quantity $\pi^{ab}\pi_{ab}$ is positive definite unlike the quantity $\pi^{ab}\pi_{ab} - \frac{1}{2}(tr\pi)^2$ in GR. The fact that the configuration space is smaller is also attractive. While this theory seems unlikely to be a good model of reality it may nonetheless teach some valuable lessons with regard to the quantisation of gravity.

The non-static theory is also attractive from a quantisation point of view. We have a similar advantage with the reduced configuration space. The most attractive feature may be the emergence of a physically preferred slicing. The problem of time is the major stumbling block in the path to a quantisation of gravity and this preferred slicing may prove to be invaluable. We also have an added bonus in that the volume of the universe is monotonically increasing from 0 to $\infty$ and is trivially constant on any hypersurface. Thus the volume may be of use as a notion of time in the theory.

As regards cosmology again, the elevation to identity of the Hamiltonian constraint may be crucial. The constant $trp$ constraint is a far more elementary quantity and this may be of no-small help with regard to quantising the cosmological solution.

All this indicates that a quantisation program for the theory would be beneficial regardless of the eventual fate of the theory as a classical competitor to GR. Of course, if
that fate were to be a positive one then I for one won’t be complaining...
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