Constructive Renormalization Theory

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Abstract

These notes are the second part of a common course on Renormalization Theory given with Professor P. da Veiga. I emphasize here the rigorous non-perturbative or constructive aspects of the theory. The usual formalism for the renormalization group in field theory or statistical mechanics is reviewed, together with its limits. The constructive formalism is introduced step by step. Taylor forest formulas allow to perform easily the cluster and Mayer expansions which are needed for a single step of the renormalization group in the case of Bosonic theories. The iteration of this single step leads to further difficulties whose solution is briefly sketched. The second part of the course is devoted to Fermionic models. These models are easier to treat on the constructive level, so they are very well suited to beginners in constructive theory. It is shown how the Taylor forest formulas allow to reorganize perturbation theory nicely in order to construct the Gross-Neveu model without any need for cluster or Mayer expansions. Finally applications of this technique to condensed matter and renormalization group around Fermi surface are briefly reviewed.

1 The Renormalization Group: an overview

1.1 Functional Integration and its problems

In this section we restrict ourselves to the bosonic $\phi^4$ field theory in $d$ Euclidean space time dimensions. The model, introduced in P. Da Veiga’s lectures, is defined by the (formal) measure

$$d\mu_C(\phi)e^{-S(\phi)}, \quad S(\phi) = \lambda \int \phi^4(x)d^dx$$

(1.1)

where $d\mu_C(\phi)$ represents the Gaussian measure for the free field. Gaussian measures are characterized by their covariance, or propagator, which for a massive theory is, in Fourier space:

$$C(p) = (p^2 + m^2)^{-1},$$

(1.2)

and $S$ is the (bare) action. In dimension $d = 2,3$ the model is superrenormalizable, and its rigorous construction was the first major achievement of constructive theory [GJ].

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dimension $d = 4$ the model is just renormalizable, hence a more general action, including mass and wave function counterterms is needed:

$$S(\phi) = \lambda \int \phi^4(x)d^dx + \mu \int \phi^2(x)d^dx + a \int (\nabla \phi)^2(x)d^dx \quad (1.3)$$

We recall the three main problems associated with giving a mathematical precise sense to the formal measure (1.1).

A) **The ultra-violet problem** The propagator (1.2) has not a well defined kernel in direct space for $d \geq 2$, since $\int d^dp(p^2 + m^2)^{-1}$ is not absolutely convergent. This does not prevent mathematically to define the Gaussian measure associated to this propagator, for instance through Minlos theorem [GJ], since the propagator is positive in Fourier space. However it entails that this measure is supported by distributions. The immediate consequence is that a local interaction like $\phi^4(x)$ is ill defined, since multiplication of distributions in general is ill-defined. Therefore we must truncate the propagator at high momenta by means of a cutoff, introducing for instance

$$C_\kappa(p) = \int_\kappa^\infty e^{-\alpha(p^2 + m^2)}d\alpha$$

which has a well defined kernel

$$C_\kappa(x, y) = \int d^dp e^{ip\cdot(x-y)} \int_\kappa^\infty e^{-\alpha(p^2 + m^2)}d\alpha = \int_\kappa^\infty e^{-\alpha m^2 - |x-y|^2/\alpha} \frac{d\alpha}{\alpha^{d/2}}$$

and to let $\kappa \to 0$ later.

As a consequence the support of $d\mu_{C_\kappa}(\phi)$ is now made of perfectly smooth functions, and $\phi^4(x)$ becomes well-defined. An other popular ultraviolet cutoff, especially for gauge theories, is the lattice cutoff, but I will not use it here.

B) **The infra-red problem** Even with an ultraviolet cutoff, if $\phi^4(x)$ is well defined, with probability one it is not decreasing at infinity, hence it is certainly not integrable on $\mathbb{R}^d$. Hence the quantity $S(\phi) = \int_{\mathbb{R}^d} \phi^4(x)d^dx$ with probability 1 is ill-defined. The solution is to restrict the interaction to a finite volume (usually a box $\Lambda$ of size $L$), and to let $L \to \infty$ later.

The measure with these two cutoffs becomes

$$d\mu_{C_\kappa}(\phi)e^{-\lambda \int_\Lambda \phi^4(x)d^dx}$$

C) **The large field problem** In (1.6) the integrand is now well defined on a set of measure one. But it does not mean that it can always be integrated! Even in one-dimensional integration, $F(\lambda) = \int_{-\infty}^{+\infty} e^{-x^2/2-\lambda x^4} dx$ converges only for $\lambda \geq 0$. We cannot hope the infinite dimensional integral (1.6) to be better behaved than this one dimensional integral. Therefore Bosonic functional integrals require some stability for the potential (here e.g. $\lambda \geq 0$). As dicussed by P. da Veiga, the perturbation series for $F$ do not converge. $F$ is not analytic, but Borel summable. This is also the best we can hope for the $\phi^4$ theory, and what has been proved in dimension 2 or 3.

Convergence of the functional integral itself, and the divergence of perturbation theory can be considered as generic “large field” problems, because they are related to the fact that a
bosonic field is an unbounded variable. Physically a large field corresponds to a large number of excitations or particles being produced, and large field problems are generic in Bosonic theories because Bosons, instead of Fermions, can pile up in large numbers at the same place.

### 1.2 Thermodynamic limit

Having reviewed the main mathematical problems of field theory in the functional framework, we remark that cutoffs by themselves alone cannot solve any problem. For instance the theory with cutoffs does not satisfy any reasonable axioms. Some manipulations have to be performed, *cancellations* have to occur, so that the limits $\kappa \to 0$ and $L \to \infty$ which looked hopeless at first sight become finite and well defined.

The conceptually simplest of these manipulations is the thermodynamic limit $\Lambda \to \mathbb{R}^d$, which is particularly easy when the theory has a mass. We know from statistical mechanics that only the “intensive” or thermodynamic quantities should have a limit as $\Lambda \to \infty$, the extensive ones, proportional to the volume, should diverge. Hence the only “manipulation” in this case is to restrict our attention to quantities such as the pressure $p$ or the normalized Schwinger functions $S_n$ of the theory:

\[
p = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \log Z(\Lambda); \quad Z(\Lambda) = \int e^{-S_\Lambda(\phi)} d\mu_{\mathcal{C}}(\phi) \quad (1.7)
\]

\[
S_N = \lim_{\Lambda \to \infty} \frac{1}{Z(\Lambda)} \int \phi(x_1)\cdots\phi(x_N) e^{-S_\Lambda(\phi)} d\mu_{\mathcal{C}}(\phi) \quad (1.8)
\]

In these quotients both quantities should diverge as $\Lambda \to \infty$, but their ratio should have a finite limit. This is clear in perturbation theory: the series for extensive or unnormalized quantities correspond to general graphs, which may contain in particular vacuum graphs which by translation invariance, give rise to infinite integrals when the spatial integration is over all $\mathbb{R}^d$. The intensive quantities such as $p$ or $S_N$ have power series made of connected graphs, with at least one vertex fixed, so that if the propagator itself is integrable, as is the case in a massive theory, the corresponding amplitudes are finite in the thermodynamic limit. In the next sections we will see how the thermodynamic limit can be performed non-perturbatively. For Bosons it requires a cluster and a Mayer expansion, whereas for Fermions, it requires almost nothing: simply reorganizing the finite dimensional integrals of perturbation series in terms of trees rather than Feynman graphs.

### 1.3 Renormalization

The ultraviolet problem increases in difficulty when $d$ increases. In general we cannot expect a finite $\kappa \to 0$ limit for quantities such as the Schwinger functions if we keep the bare parameters fixed as $\kappa \to 0$. But physically these bare parameters are not observable in low energy realistic accelerators! P. da Veiga has already explained, at least at the perturbative level, how to get rid of the ultraviolet divergences that occur in the Feynman graphs of the theory. For $d = 4$ where the theory is just renormalizable, perturbation theory suggests that if we allow the bare parameters of the theory $\lambda$, $\mu$ and $a$ in (1.3) to depend on the cutoff $\kappa$ in such a way as to diverge as $\kappa \to 0$, cancellations can take place so that the Schwinger
functions $S_N$ might have a well defined limit as $\kappa \to 0$. I say that it only suggests this, since bare quantities in standard perturbative renormalization are only defined as formal power series in the renormalized coupling. These series are not necessarily summable, as explained in da Veiga’s lectures, both because of the large number of graphs, and because perturbative renormalization creates additional sources for divergences, namely renormalons. It is very risky to speculate whether or not an infinite non summable series of diverging terms really produces a quantity diverging with the cutoffs (in fact in asymptotically free cases it does not!).

I will not add further remarks to the lectures of da Veiga on perturbative renormalization, except to recall that historically many mathematicians felt renormalized perturbation theory “pulls the infinities of quantum field theory under the rug” by hiding them into the bare constants. Physicists also felt in the early days of renormalization theory that it was not “natural”; in particular it did not explain why nature seems to prefer renormalizable field theories. Professor Dyson, one of the founding fathers of renormalization, once told me that for him the main surprise of the theory was that it lived so long! In short perturbative renormalization, although beautiful and encouraging, and certainly useful for many phenomenological computations (QED, QCD in some regimes...) is a partial and unsatisfactory answer to the problems of quantum field theory both for mathematicians and for physicists.

1.4 The Renormalization Group

A key progress in the theory of renormalization was the introduction of the philosophy of the Renormalization Group (hereafter called RG) by Wilson and followers. The basic idea is the following: since the limit $\kappa \to 0$ is so hard to grasp, let us not perform it at once, but in steps. A single step can be well understood, and the problem is reduced to the hopefully simpler problem of iterating many times a relatively simple transformation. Hence RG does not solve renormalization, but it replaces it by a problem in dynamical systems. This seems an almost trivial idea, but it is really also a tremendous change in perspective which immediately led to progress. For instance it replaced the old “Landau ghost” or its modern version, the renormalon problem, by a different question: is the theory asymptotically free (or safe), so that the flow for the coupling constant remains bounded at all scales? Also the renormalization group philosophy can be applied directly to infrared problems and statistical mechanics with a lot of success.

As you surely know, the $\phi_4^4$ theory for the right sign of the coupling constant is not asymptotically free. At least until now this means that we do not know how to perform its limit $\kappa \to 0$ and end up with a non trivial interacting theory (although $\phi_4^4$ remains interesting for pedagogical discussions). But the good news are that other theories such as gauge theories or the two dimensional Gross-Neveu and $\sigma$ models are asymptotically free. Also the renormalization group explained quickly why nature seemed to prefer renormalizable theories: indeed in a generic interaction at a very high momentum scale, non renormalizable terms are irrelevant; washed out by the RG flow, their presence cannot be detected in the effective theory at low energy.

Let us briefly sketch what would be the RG strategy for the $\phi_4^4$ theory:

The key idea is to split the space of all frequencies into discrete slices, following a geometric
progression. This can be done on lattices with the popular tool called block-spinning, but we can also simply cut the propagator into “momentum slices”. Taking some fixed ratio \( M \) for the geometric progression, and \( \kappa = M^{-2\rho} \), \( \rho \) being an integer, we have:

\[
C_\rho(p) = \sum_{j=0}^{\rho} C^j(p) \tag{1.9}
\]

\[
C^0(p) = \int_1^\infty e^{-\alpha(p^2+m^2)}d\alpha \quad ; \quad C^j(p) = \int_{M^{-2j}}^{M^{-2(j-1)}} e^{-\alpha(p^2+m^2)}d\alpha \quad \text{for} \; j \geq 1 \tag{1.10}
\]

There is a corresponding separation of the field into a sum of independent random variables:\n\[
\phi = \sum_j \phi_j, \; \phi_j \text{ distributed according to } C^j. \quad \phi_j \text{ is called the (j-th) fluctuation field and } \sum_{k=0}^{j-1} \phi_k \text{ the (j-th) background field.}
\]

**Exercise 1.1** Prove that for some constant \( K \):

\[
C^j(x,y) \leq KM^{2j}e^{-M|x-y|/K} \tag{1.11}
\]

Now introduce the operation \( * \) by

\[
\mu_j * Z(\phi) = \int d\mu_{C_j}(\zeta)Z(\phi + \zeta) \tag{1.12}
\]

where \( \zeta \) plays the role of a fluctuation and \( \phi \) of a background field.

If we define \( S_\rho(\phi) \) as the bare action and \( Z_\rho(\phi) = e^{-S_\rho(\phi)} \), we have:

\[
Z = \int d\mu_{C_\rho}(\phi)Z_\rho(\phi) = (\mu_0 * ... * (\mu_{\rho-1} * (\mu_{\rho} * Z_\rho)) \tag{1.13}
\]

Let us define

\[
Z_j = (\mu_j * ... * (\mu_{\rho-1} * (\mu_{\rho} * Z_\rho)) \quad ; \quad Z_{j-1} = \mu_{j-1} * Z_j; \tag{1.14}
\]

and

\[
S_j(\phi) = -\log Z_j(\phi).
\]

We see that constructing the ultraviolet limit is the same as finding a bare action \( S_\rho(\phi) \) such that the effective, or renormalized action \( S_0(\phi) \) converges as \( \rho \to \infty \).

Remark that the \( C^j \) satisfy approximate scaling laws: \( C^{j+1}(x) \simeq M^2C^j(Mx) \). They would even satisfy exact scaling if we consider a massless theory with \( m = 0 \) which is often done. In this case we can use the scaling \( C^{j+1}(x) = C^j(Mx) \), to perform the famous trivial but slightly confusing rescalings of the renormalization group. Defining \( \phi_M(x) = M^{(d-2)/2}\phi(Mx) \), and \( \tilde{Z}_j(\phi) = \tilde{Z}_j(\phi_M) \) we obtain indeed the equation

\[
\tilde{Z}_j(\phi) = (\mu_1 * \tilde{Z}_{j+1})(\phi_{M-1}).
\]

This defines the \((j \text{ independent}) \mathcal{R} \) operation on the action \( S \) as the composition of 4 steps:

- spatial rescaling by \( M^{-1} \)
-field rescaling by $M^{-(d-2)/2}$
-integration over a fixed (scale 1) fluctuation slice
-taking the logarithm of the result to reexpress it as an action for the next step.

In this way the problem of the ultra-violet limit reduces to convergence of the $\rho$ times iterated transformation $R\circ R\ldots\circ RS_\rho$ as $\rho \to \infty$. The problem of iteration of a fixed map can be analyzed as a discrete dynamical system: if a fixed point appears, for instance the free field or a theory close to it, one can hopefully trust a perturbative analysis of the vicinity of this fixed point.

In particular we see that for instance polynomial terms such as $\int \phi^n(x)d^4x$ scale with a factor (corresponding to power counting) which is $M^{(4-n)j}$ after $j$ steps. For $n > 4$ they are “irrelevant” and die out in the RG flow; the $\phi^4$ term is marginal, and its flow is governed by the sign of the “bubble graph”, hence by the sign of the beta function at small coupling. Finally the mass term, quadratically growing, is “relevant”. Adding derivative couplings, we find that only the wave function term $\int (\nabla \phi)^2(x)d^4x$ is not irrelevant, but marginal. From this very simple analysis follows the classification of renormalizable theories into asymptotically free (Gaussian fixed points) and not asymptotically free, and the discovery of non-Gaussian fixed points of the RG close to Gaussians, if one modifies for instance slightly the canonical scaling of the fields (mimicking non-integer dimensions) for an asymptotically free model.

Study of infrared critical points by RG is very similar to that of ultraviolet limits, except that in an infrared problem, the ultraviolet cutoff is fixed, hence it is natural to give it an index 0 or 1; and the slices run towards momentum $p = 0$, so that as $j$ grows, it indexes smaller and smaller momenta.

Sometimes here a confusion arises: the renormalization group does not simply “exchange” infrared and ultraviolet limits. The basic fact to keep in mind is that it flows always in the same direction: from small spatial scales to larger ones. Indeed it deduces macroscopic actions from microscopic ones, as is traditional in the physical analysis of a phenomenon, where large scale effects are explained by smaller ones. Hence the direction of flow of RG never changes, only the point of view of the observer. In an ultraviolet problem in some sense it is the source of the RG river which goes to infinity, whereas in an infrared problem it is the mouth of the river which goes to infinity relatively to the observer.

1.5 Constructive RG is necessary!

After this blitz overview of the standard renormalization group of textbooks remark that although it clarifies enormously the ultraviolet problem, it is not yet formulated in a correct mathematical way. In particular starting form any polynomial action it creates an effective action which is obviously no longer polynomial, and this even after a single step! Therefore the large field problem (integration on $\phi$, or $\zeta$), appears! More precisely the behavior of $S_{\text{eff}}(\phi)$ at large $\phi$ is unclear, so that starting from a stable interaction, even the second step of the RG may be already ill-defined. This point has to be stressed to physicists!

The solution is to reconsider the single step of the RG, hence the theory in a finite slice with a background field and to analyze it in a way which can be iterated correctly.

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2We are aware that this traditional view is put in question by duality in string theory, but in this lecture we nevertheless stick to it!
mathematically. This is the “constructive” version of the RG, as introduced in the 80’s by Gallavotti and coworkers BCGNOPS, and developed in [GK1-2]. It has been now well developed and systematized into a coherent mathematical formalism, by Pordt, Brydges and collaborators [P][B]. A related formalism is the “multiscale phase space expansion” of [FMRS1-2], now redesigned into a more transparent and explicit formalism in [A][AR2]. The main difference is that in the phase space expansion formalism the rescalings are not performed, so that the iteration of the RG steps is fully developed. Phase space expansion therefore leads to an explicit description of the solution of the RG induction, somewhat like Zimmermann’s forest formulas “solves” Bogoliubov’s induction.

In these formalisms the main tool consists in performing the key step of integration of fluctuation (the mathematically well defined version being called a “cluster expansion”) and taking the logarithm (the mathematically well defined version being called a “Mayer expansion”) only in regions where the background field is small. In the other so-called large field regions one simply “waits and sees”. Although a full presentation of the constructive RG formalism for bosonic theories is beyond the scope of this course, we give in the next section in detail the treatment of a single scale model with a cluster + Mayer expansion.

The constructive RG will be an iteration of this treatment, with the added complication (alas very nontrivial) that one cannot expand the “large field regions” and must exploit the fact that their weights are small in the probabilistic sense.

Using the constructive RG approach or the related “multiscale” expansions some models have been built, giving the first concrete examples of renormalizable quantum field theories fulfilling all Wightman axioms, such as the Gross-Neveu$_2$ model [GK1] [FMRS1]. New models not perturbatively renormalizable but asymptotically safe become also accessible, such as the Gross-Neveu$_3$ model built by P. da Veiga and collaborators [dV]. In the infrared regime bosonic models of renormalizable power counting such as the critical (massless) $\phi^4$ with an infrared cutoff [GK2] [FMRS2], or 4 dimensional weakly self avoiding polymers have been controlled [IM1], and their asymptotics at large distance established. Nonperturbative mass generation has been established in the Gross-Neveu$_2$ model and in the nonlinear $\sigma$ model at large number of components [KMR][K]. Finally the RG when applied to condensed matter give rise to many rigorous results and programs, and this is the subject of the last section.

2 Single Scale Bosonic Model: the cluster and Mayer expansions

2.1 Tree and Forest formulas

In statistical mechanics, the key step of the thermodynamic limit is to take a logarithm to pass to intensive quantities which correspond to connected quantities. The minimal discrete connecting structures between points are trees. It is therefore worth devoting some time to the precise combinatorics of trees, and to the way to generate them systematically between points.

Let $n \geq 2$ be an integer, $I_n = \{1, \ldots, n\}$, $\mathcal{P}_n = \{ \{i,j\}/i,j \in I_n, i \neq j\}$ (the set of unordered pairs in $I_n$). An element $l$ of $\mathcal{P}_n$ will be called a link, a subset of $\mathcal{P}_n$, a graph. A
graph \( \mathcal{F} = \{ l_1, \ldots, l_\tau \} \) containing no loops, i.e. no subset \( \{ \{ i_1, i_2 \}, \{ i_2, i_3 \}, \ldots, \{ i_k, i_1 \} \} \) with \( k \geq 2 \) elements, is called a forest. A connected forest is called a tree. A forest is a union of disconnected trees \( \mathcal{T} \), the supports of which are disjoint subsets of \( I_n \) called the connected components or clusters of \( \mathcal{F} \).

The first basic result on trees goes back to the XIXth century:

**Theorem 1 (Cayley’s Theorem)** There are exactly \( n^{n-2} \) labeled trees on a set of \( n \) points.

**Exercise 2.1:** Prove Cayley’s theorem by induction, using the multinomial expansion, and showing that there are \( n! / \prod (d_i - 1)! \) trees with coordination number \( d_i \) at vertex \( i \), using the fact that trees always have “leaves”.

Let \( \mathcal{S} \) be the space of smooth functions of \( n(n-1)/2 \) variables \( x = (x_l)_{l \in \mathcal{P}_n} \) associated to the links of \( \mathcal{P}_n \). A Forest formula is a Taylor formula with integral remainder, which expands a function \( H \) of \( \mathcal{S} \) to search for the explicit dependence on given link variables \( x_l \). Taylor formulas with remainders in general are provided with a “stopping rule” and forest formulas stop at the level of connected sets. This means that two points which are joined by a link are treated as a single block. (More sophisticated formulas with higher stopping rules are very useful for higher particle irreducibility analysis, or renormalization group computations but are no longer forest formulas in the strict sense [AR2]).

Any forest is a union of connected trees. Therefore any forest formula has an associated tree formula for its connected components. And therefore, at least formally, any forest formula solves the problem of computing normalized correlation functions. Indeed applying the forest formula to the functional integral for the unnormalized functions, the connected functions are simply given by the connected pieces of the forest formula, hence by the corresponding tree formula. It is in this sense that forest formulas exactly solve the well known snag that makes connected functions difficult to compute. This snag is that since typically there are many trees in a graph, one does not know “which one to choose” when one tries to compute connected functions in the (desirable) form of tree sums. Any forest formula gives a particular answer to that problem, associating a weakening factor \( w \) to the links which remain underived (the potential “loops”). This weakening factor \( w \) tells us exactly by how much our pondered “tree choice” has “weakened” the remaining loop lines.

Several forest formulas exist, with different rules for \( w \); they correspond to different ways of letting forests grow. In one logic, the most “symmetric” one, the forest grows in a random way: it leads to a formula first established by Brydges and Kennedy [BK]. In another logic, which leads to the “rooted formula” each tree grows layer by layer from a preferred root [AR1]. Our presentation here follows [AR1].

Applied to an element \( H \) of \( \mathcal{S} \), the Taylor Rooted Forest formula takes the form (the vector with all entries equal to 1 being denoted by \( \mathbb{1} \)):

**Symmetric Taylor Forest Formula**

\[
H(\mathbb{1}) = \sum_{\mathcal{F} \in \mathcal{F}} \left( \prod_{l \in \mathcal{F}} \int_0^1 dw_l \right) \left( \prod_{l \in \mathcal{F}} \frac{\partial}{\partial x_l} \right) H \left( X_{\mathcal{F}}(w) \right) .
\]
where the summation extends over all possible forests \( \mathfrak{F} \), including \( \tau = 0 \) hence the empty forest. To each link of \( \mathfrak{F} \) is attached a variable of integration \( w_l \). The vector \( X_\mathfrak{F}(w) \) is the value at which we evaluate the derivative of \( H \); it is the vector \( (x_l)_{l \in \mathcal{P}_n} \) defined by \( x_l = w_l^\mathfrak{F}(w) \), where the weakening factor \( w^\mathfrak{F}_{ij}(w) \) is \( \inf \{ w_l, l \in L_\mathfrak{F}(ij) \} \) if \( L_\mathfrak{F}(ij) \) is the unique path in the forest \( \mathfrak{F} \) connecting \( i \) to \( j \), and is 0 if no such path exists (hence if \( i \) and \( j \) belong to different clusters of \( \mathfrak{F} \)).

The notation \( u \)-forest stands for regular “unordered” forests. Decomposing the integration domain into \( \tau! \) subdomains according to a complete ordering of the parameters \( w_l \) we obtain a related formula where the sum runs over so called “ordered” forests in which the links are ordered, and the integration parameters \( w_l \) follow the ordering.

**Exercise 2.2** Write down the general ordered forest formula. Remark that for \( n = 2 \) it reduces to the ordinary Taylor formula \( H(1) = H(0) + \int_0^1 xH'(wx)dw \). Performing a series of ordinary Taylor interpolations at each step restricted to the links joining different existing clusters, prove by induction the ordered formula, from which the unordered one follows easily by regluing the integration domains for the \( w \) parameters.

**Exercise 2.3** Write down the ordered and unordered formulas for \( n = 3 \) and \( 4 \) (very instructive).

Although we won’t use it let us stress that these formulas are not unique. For instance there exists an absolutely identical formula, the rooted formula, with a different rule for the weakening factor \( w \), now called \( w^\mathfrak{F}_{ij}(w) \) (the superscript \( R \) standing for “rooted”). It is a less symmetric formula since we have to give a rule for choosing a root in each cluster. For each non empty subset or cluster \( C \) of \( I_n \), choose \( r_C \), the least element in the natural ordering of \( I_n = \{1, \ldots, n\} \), to be the root of all the trees with support \( C \) that appear in the following expansion. Now if \( i \) is in some tree \( \mathfrak{T} \) with support \( C \) we call the height of \( i \) the number of links in the unique path of the tree \( \mathfrak{T} \) that goes from \( i \) to the root \( r_C \). We denote it by \( l^\mathfrak{T}(i) \).

The set of points \( i \) with a fixed height \( k \) is called the \( k \)-th layer of the tree. The Rooted Taylor Forest Formula is then absolutely identical to the Symmetric one, except changing the superscript \( S \) to \( R \) (as “rooted”) and defining the weakening parameter \( w^R \) differently, by the following rule:

**Rooted weakening factors**

\[
\begin{align*}
w^\mathfrak{F}_{ij}(w) &= 0 \text{ if } i \text{ and } j \text{ are not connected by the } \mathfrak{F}. \text{ If } i \text{ and } j \text{ fall in the support } C \text{ of the same tree } \mathfrak{T} \text{ of } \mathfrak{F} \text{ then} \\
w^\mathfrak{F}_{ij}(w) &= 0 \text{ if } |l^\mathfrak{T}(i) - l^\mathfrak{T}(j)| \geq 2 \text{ (} i \text{ and } j \text{ in distant layers)} \\
w^\mathfrak{F}_{ij}(w) &= 1 \text{ if } l^\mathfrak{T}(i) = l^\mathfrak{T}(j) \text{ (} i \text{ and } j \text{ in the same layer)} \\
w^\mathfrak{F}_{ij}(w) &= w_{i(i')} \text{ if } l^\mathfrak{T}(i) - 1 = l^\mathfrak{T}(j) = l^\mathfrak{T}(i'), \text{ and } \{ii'\} \in \mathfrak{T}. \text{ (} i \text{ and } j \text{ in neighboring layers, } i' \text{ is then unique and is called the ancestor of } i \text{ in } \mathfrak{T}. \text{ In particular, if } \{ij\} \in \mathfrak{F}, \text{ then } w^\mathfrak{F}_{ij}(w) = w_{\{ij\}}. \\
\end{align*}
\]

**Exercise 2.4** Prove the rooted formula.
**Theorem 2 (Positivity of the Symmetric Formula)**

Extended to a symmetric matrix by the convention \( w_{ii}^{\mathcal{F}}(w) = 1 \forall i \), the matrix \( w_{ij}^{\mathcal{F}}(w) \) is positive.

**Exercise 2.5** Prove this theorem (hint: show that \( w_{ij}^{\mathcal{F}}(w) \) is a convex combination of block matrices with 1 everywhere).

### 2.2 Cluster Expansion

Let us apply the previous formula to study the thermodynamic limit of the single slice \( \phi^4 \) theory. The idea of the *cluster expansion* is that since perturbation theory diverges we must keep most of it in the form of functional integrals. However we can test whether distant regions of space are joined or not by propagators, and this will allow to rewrite the theory as a polymer gas (with hardcore interactions). When the coupling constant is small, the activities for the polymers are small, and the technique of the Mayer expansion which compares the hard core gas to a perfect gas, allows to perform the thermodynamic limit.

For instance consider the free bosonic (massless) Gaussian measure \( d\mu_C \) in \( \mathbb{R}^d \) defined by the single slice covariance

\[
C(x,y) = \int_{M^{-1}} \frac{d\alpha}{\alpha^{d/2}} e^{-|x-y|^2/4\alpha}.
\]

This propagator is obviously integrable in \( y \) at fixed \( x \).

Put now the regular interaction \( e^{-\lambda \int_{\Lambda} \phi^4(x) dx} \) in a finite volume \( \Lambda \) and for simplicity let us study the pressure

\[
p = \lim_{\Lambda \to \mathbb{R}^d} \frac{1}{|\Lambda|} \log Z(\Lambda),
\]

where we recall that the partition function \( Z(\Lambda) \) in a finite volume \( \Lambda \) is

\[
Z(\Lambda) = \int d\mu_C(\phi) e^{-\lambda \int_{\Lambda} \phi^4(x) dx}.
\]

Let us explain how the Taylor formula (2.3) performs the task of rewriting the partition function as a dilute gas of clusters with hard core interaction.

The set \( I_n \) is defined as a partition of \( \Lambda \) into (unit size) cubes, and clusters are subsets of such cubes. We write \( \Lambda = \bigcup_{i \in I_n} b_i \), where each \( b_i \) is a unit cube, and define \( \chi_b \) as the characteristic function of \( b \), and \( \chi_\Lambda = \sum_{i \in I_n} \chi_{b_i} \). Since the interaction lies entirely within \( \Lambda \), the covariance \( C \) can be replaced by \( C_\Lambda = \chi_\Lambda(x)C(x,y)\chi_\Lambda(y) \) without changing the value of \( Z(\Lambda) \). Moreover \( C_\Lambda \) can be interpolated, defining for \( l = \{i,j\} \in P_n \)

\[
C_\Lambda((x_l)_{l \in P_n})(x,y) = \sum_{i=1}^n \chi_{b_i}(x)C(x,y)\chi_{b_i}(y)
\]

\[
+ \sum_{\{i,j\} \in P_n} x_{(ij)}(\chi_{b_i}(x)C(x,y)\chi_{b_j}(y) + \chi_{b_j}(x)C(x,y)\chi_{b_i}(y))
\]

Remark that \( C_\Lambda(\mathbb{1}) = C_\Lambda \). Now we apply the Taylor formula (2.3) with the function \( H \) being the partition function obtained by replacing in (2.6) the covariance \( C \) by \( C_\Lambda((x_l)_{l \in P_n}) \). Here
it is crucial to use the positivity theorem, in order for the interpolated covariance to remain positive, hence for the corresponding normalized Gaussian measure to remain well-defined.

From the rules of Gaussian integration of polynomials [GJ], we can compute the effect of deriving with respect to a given \( x_l \) parameter, and we obtain that (2.3) in this case takes the form

\[
Z(\Lambda) = H(\mathbb{1}) = \sum \int d\mu_{C_\Lambda}(x_\mathbb{1}(w)) \left( \prod_{l \in \mathcal{S}} \int_0^1 dw_l \right) \left\{ \prod_{l=\{ij\} \in \mathcal{T}} \int dx dy \chi_{b_i}(x) \chi_{b_j}(y) C(x, y) \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(y)} e^{-\lambda \int \phi^4(x) dx} \right\}
\]

where \( b_i \) and \( b_j \) are the two ends of the line \( l \). Since both the local interaction and the covariance as a matrix factorize over the clusters of the forest \( \mathcal{S} \), the corresponding contributions in (2.7) themselves factorize, which means that (2.7) can also be rewritten as a gas of non-overlapping clusters, each of which has an amplitude given by a tree formula:

\[
Z(\Lambda) = \int d\mu_{C_\Lambda}(\phi) e^{T_\Lambda(\phi)} = \sum_{\text{sets } \{Y_1, \ldots, Y_n\} \text{ such that } \cup_{Y_i} \neq \emptyset} \prod_{i=1}^n A(Y_i)
\]

\[
A(Y) = \sum_{\mathcal{T} \text{ on } Y} (\prod_{l \in \mathcal{T}} \int_0^1 dw_l) \int d\mu_{C_Y}(x_\mathcal{T}(w))(\phi) \left\{ \prod_{l=\{ij\} \in \mathcal{T}} \int dx dy \chi_{b_i}(x) \chi_{b_j}(y) C(x, y) \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(y)} e^{-\lambda \int \phi^4(x) dx} \right\}
\]

where the sum is over trees \( \mathcal{T} \) which connect together the set \( Y \), hence have exactly \(|Y| - 1\) elements (if \(|Y| = 1, \mathcal{T} = \emptyset \) connects \( Y \)). The measure \( d\mu_{C_Y}(\{x_\mathcal{T}(h)\})(\phi) \) is the normalized Gaussian measure with (positive) covariance

\[
C_Y(X_{\mathcal{T}}(w))(x, y) = \chi_Y(x) w_{\mathcal{T}}(w)(x, y) C(x, y) \chi_Y(y)
\]

where \( w_{\mathcal{T}}(w)(x, y) \) is 1 if \( x \) and \( y \) belong to the same cube, and otherwise it is the infimum of the parameters \( w_l \) for \( l \) in the unique path \( L_{\mathcal{T}}(b(x), b(y)) \) which in the tree \( \mathcal{T} \) joins the cube \( b(x) \) containing \( x \) to the cube \( b(y) \) containing \( y \).

**Exercise 2.6** Prove that if \( a_{ij} \) and \( b_{ij} \) are two positive matrices, their Hadamard product \( c_{ij} = a_{ij} b_{ij} \) is again a positive matrix (hint: use square roots) Complete the proof that \( C_Y(X_{\mathcal{T}}(w))(x, y) \) is, as announced, a positive covariance, hence that it has a well-defined associated Gaussian measure.

The following bound now summarizes that polymer activities are small enough so that one can sum over all polymers containing a fixed point (and absorb a fixed constant per cube of the polymer). This will be used in the next section.

**Theorem 3 (Bound on Polymer Activities)**

*Given any constant \( K \), for small enough \( \lambda \) with \( \text{Re} \ \lambda > 0 \) we have

\[
\sum_{Y \text{ such that } 0 \in Y} |A(Y)| K^{|Y|} \leq 1
\]

(2.11)
Exercise 2.7 Prove this theorem (hint: use Formula (2.9). One needs notations to compute the action of the functional derivatives in (2.9) by Leibniz rule, the result being cumbersome. Then it is useful to remark that the propagators exponential decay (see (2.4)) can absorb any “local factorial” of the coordination numbers $d_i$ of the tree $\Sigma$. This allows to achieve the proof. ♦

2.3 The Mayer expansion

The Mayer expansion allows to deduce from (2.11) the existence and e.g. the Borel summability in $\lambda$ of thermodynamic functions such as the pressure $p$.

In the cluster expansion (2.8), the condition that the disjoint union of all clusters is $\Lambda$ is a global annoying constraint. Remark that the polymer amplitudes are translation invariant. In particular the trivial amplitude of a singleton cluster $Y = \{b\}$ is a number $A_0$ independent of $b$. Redefining $A_r(Y) = A(Y)/A_0^{\left|Y\right|}$ and $Z_r(\Lambda) = Z(\Lambda)/A_0^{\left|\Lambda\right|}$ we quotient out all the trivial clusters so that

$$Z_r(\Lambda) = 1 + \sum_{n \geq 1} \sum_{\substack{\text{sets } (Y_1, \ldots, Y_n) \quad \left|Y_i\right| \geq 2 \quad Y_i \cap Y_j = \emptyset \quad 1 \leq i < j \leq n}} \prod_{i=1}^{n} A_r(Y_i)$$

This is the partition function of a polymer gas: the sums over individual polymers would be independent were it not for the hard core constraints $Y_i \cap Y_j = \emptyset$. Adding in an infinite number of vanishing terms, we can replace the sum in (2.12) by a sum over ordered sequences $(Y_1, \ldots, Y_n)$ of polymers with hard core interaction and a symmetrizing factor $1/n!$ coming from the replacement of sets by sequences.

$$Z_r(\Lambda) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{\text{sequences } (Y_1, \ldots, Y_n) \quad \left|Y_i\right| \geq 2 \quad Y_i \cap Y_j = \emptyset \quad 1 \leq i < j \leq n} \prod_{i=1}^{n} A_r(Y_i) \prod_{1 \leq i < j \leq n} \eta(Y_i, Y_j)$$

where the two-body hard core interaction is expressed by the factors $\eta(X, Y) = 1$ if $X \cap Y = \emptyset$, and $\eta(X, Y) = 0$ if $X \cap Y \neq \emptyset$. To factorize these hardcore interactions we apply again the symmetric Taylor forest formula (2.3)! More precisely for a fixed sequence $(Y_1, \ldots, Y_n)$ of polymers, we define $I_n$ as the set of these polymers, and define $\epsilon_l = \epsilon_{\{ij\}} = \eta(Y_i, Y_j) - 1$, for $i \neq j$. We consider the function

$$H((x_l)_{l \in I_n}) = \prod_{l \in I_n} (1 + x_l \epsilon_l)$$

so that $H(\mathbb{1}) = \prod_{1 \leq i < j \leq n} \eta(Y_i, Y_j)$. Rewrite (2.13) as

$$Z_r(\Lambda) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{\text{sequences } (Y_1, \ldots, Y_n) \quad \left|Y_i\right| \geq 2 \quad Y_i \cap Y_j = \emptyset \quad 1 \leq i < j \leq n} \eta(Y_i, Y_j)$$

This method is identical to the one in [R], part III. But remark that although the full amplitudes $A(Y)$ defined in (2.9) must be identical to those in [R], the subcontributions associated to particular trees are different, since the formula used in [R] was not the symmetric one.

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\[
1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{|Y_i| \geq 2} \prod_{i=1}^n A_r(Y_i) \sum_{l \in \mathcal{E}} \left( \prod_{l \in \mathcal{E}} \int_0^1 dw_l \right) \left( \prod_{l' \in \mathcal{E}} \epsilon_l \right) \left( \prod_{l' \notin \mathcal{E}} (1 + w_l^T(w)\epsilon_l) \right)
\]

\[
= \sum_{n \geq 0} \frac{1}{n!} \left( \sum_{k \geq 1} \frac{1}{k!} \sum_{|Y_i| \geq 2} \left( \prod_{i=1}^k A_r(Y_i) \right) C^T(Y_1, ..., Y_k) \right)^n
\]

where

\[
C^T(Y_1, ..., Y_k) = \sum_{\mathfrak{T} \text{ tree on } \{1, ..., k\}} \left( \prod_{l' \in \mathcal{E}} \int_0^1 dw_{l'} \right) \left( \prod_{l' \in \mathcal{E}} \epsilon_{l'} \right) \prod_{l' \in \mathcal{E}} (1 + w_l^T(w)\epsilon_l)
\]

where, as before, \( w_l^T(w) \) is, if \( l = \{ij\} \), the infimum of the parameters \( w_{l'} \) for \( l' \) in the unique path \( L_\mathfrak{T}(ij) \) which in the tree \( \mathfrak{T} \) joins \( i \) to \( j \).

Exercise 2.8 Check that the connecting factor \( C^T(Y_1, ..., Y_k) \) does not depend of the particular tree formula, since we have

\[
C^T(Y_1, ..., Y_k) = \sum_{\text{connected graph } G} \prod_{l \in G} \epsilon_l.
\]

We obtain immediately that

\[
\log Z_r(\Lambda) = \sum_{k \geq 1} \frac{1}{k!} \sum_{|Y_i| \geq 2} \left( \prod_{i=1}^k A_r(Y_i) \right) C^T(Y_1, ..., Y_k)
\]

These formulas can be used together with (2.11) to control the thermodynamic limit \( p = A_0 + \lim_{\Lambda \to \mathbb{H}^d} \frac{1}{|\Lambda|} \log Z_r(\Lambda) \) (hint [R]): Every tree coefficient forces the necessary overlaps along the links of the tree, and is bounded by 1, since \(|(1 + w_l^T(w)\epsilon_l)| \leq 1\). Start from the \( d_i - 1 \) leaves of the tree hooked to a given polymer of index \( i \), fix the \( d_i - 1 \) squares of overlap of these leaves with the polymer (at a cost \(|Y_i|^{d_i-1}\)), then sum over the leaves using the decay of (2.11) in the size of polymers. Use the factor \( \frac{1}{(d_i-1)!} \) in Cayley’s theorem (Exercise 2.1) and the fact that \( \sum_{d_i} |Y_i|^{d_i-1}/(d_i - 1)! = e^{|Y_i|} \). Then absorb the factor \( e^{|Y_i|} \) in the constant \( K \) of (2.11). Finally iterate to conclude, fixing the last sum to contain the origin, since this cancels out the volume factor \(|\Lambda|\) in (2.5), up to boundary terms which vanish as \(|\Lambda| \to \mathbb{H}^d\).

Exercise 2.9 Prove Borel summability of the theory in a single slice, using the Nevanlinna theorem of P.da Veiga’s lectures (hint: the analyticity for Re\(\lambda > 0\) and \(\lambda\) small is easy. For the \( K^n n! \) bounds on the Taylor remainder at order \( n \), simply rewrite the same analysis than above but with \( n \) explicit vertices expanded first.

A full renormalizable model such as the infrared critical point of \( \phi^4_4 \) can be built by extending this single slice analysis to the full multislice model. Let us sketch very briefly how this model would be treated in the multislice expansion of [AR2]. The analysis decomposes into three main steps.
The multislice model contains propagators $C^j$ in each momentum slice, and space has to be therefore decomposed at each slice into a scaled lattice $(M^j\mathbb{Z})^4$. The union $\cup_j (M^j\mathbb{Z})^4$ is the full phase space for the theory, each cube being some kind of degree of freedom for the appropriate field $\phi^j$. The first expansion step is a “small versus large field decomposition” which tests, for each cube $b \in \cup_j (M^j\mathbb{Z})^4$ of scale $j(b)$, whether a quantity such as $\int_b (\phi^j(b))^4$ is small or not. The maximal connected regions of large field cubes will be treated as single blocks, i.e., no cluster expansion will be performed inside them. This step, although perhaps not strictly necessary, simplifies a lot the bounds later.

The second step is the multiscale “cluster expansion”. It has to decouple now all the small field cubes and all the large field blocks of all slices $j = 0, \ldots, \rho$ in (1.10) at the same time. This means that it derived both propagators $C^j$ in each momentum slice, this time joining cubes of scaled lattices $(M^j\mathbb{Z})^4$. But it has also to derive new links which join the cubes of different slices at a same point, this time through vertices. These new “vertex links” can join up to 4 cubes at the same time, since a vertex can hook to at most 4 propagators in different slices. Therefore the symmetric Taylor formula has to be properly extended to accommodate such new links [AR2]. Also the formulas in the “momentum slice” direction have to be pushed further than simple connectedness, since we want to distinguish convergent activities (those with more than 5 external lower momentum legs) from divergent ones (with 2 and 4 external lower momentum legs), which require renormalization. As a result the formula in the slice (or “vertical”) direction is not a tree formula, but rather some kind of a “5 particle irreducible” formula, with an expansion rule which stops the vertical or vertex interpolations only when 5 or more links are derived. In spite of these complications, at the end of the multiscale cluster expansion the theory is factorized as a gas of polymers swimming in the full phase space, hence in $\cup_j (M^j\mathbb{Z})^4$, with hardcore interactions.

The third step is the generalized Mayer expansion, whose role is no longer only to factorize and quotient out the normalization (vacuum graphs), but also to factorize and renormalize the 2 and 4 point functions. Indeed only the convergent polymers, those which do not contain subpolymers with 2 and 4 external lower momentum legs, of the previous expansion have small summable activities. Divergent polymers require renormalization, the appropriate counterterms being absorbed into effective constants. This generates more complicated formulas, but the logic remains the same (interpolate the hardcore links). The result is some complicated but totally explicit formula for expanding the model. It can be viewed as the correct constructive analog of Zimmermann’s forest formula for perturbative renormalization [A].

This method, although still very heavy and very technical, is nevertheless clearly more explicit and conceptually more transparent than previous methods in which cluster and Mayer expansion steps were mixed together in an inductive way (such as in [FMRS2]).

3 Fermionic theories

3.1 Introduction

The initial constructions of renormalizable Fermionic models such as the Gross-Neveu$_2$ model [GK1] or [FMRS1] used the same heavy apparatus as Bosonic models, namely slicings in mo-
mentum space and direct space, plus cluster and Mayer expansions. As a consequence the renormalization group equations obtained were discrete difference equations instead of differential equations, a detail which was considered annoying by some physicists. On the other hand perturbation theory for Fermion systems is often said to converge, whereas for boson systems it is said to diverge. But what does this mean exactly? Unnormalized Fermionic perturbation series with cutoffs are not only convergent but entire, whereas Bosonic perturbation series, even unnormalized and with cutoffs, have zero radius of convergence. Using this property there ought to be constructive versions of renormalization for Fermions very close to the perturbative concepts. In particular it was advocated recently in [S] that there ought to exist continuous flows and differential equations of the renormalization group for Fermions.

Progressively we realized that one can avoid the use of the cluster and Mayer expansions of the previous section for Fermionic models (for an early example see [FMRT1]). In particular the symmetric forest formula of the previous section, when the interpolated parameters are directly applied to the Feynman lines of a graph, gives a particularly simple “three lines” construction of interacting Fermions in a single momentum slice [AR3], which is given below. In [DR] we applied similar ideas to give a new construction of the Gross-Neveu 2 model. This construction gives a solution of the theory in the form of explicit sums of finite dimensional integrals, containing, however, effective constants which are defined as the (non explicit) solutions of the differential renormalization group equations.

The positivity property of the symmetric Taylor forest formula (Theorem 3 of the previous section) was crucial for Bosonic models. But here again it is useful, since it allows to apply Gram’s estimates:

**Lemma 1 (Interpolated Gram inequality)** Let $A = a_{ij} = < f_i \cdot g_j >$ be a Gram matrix. Gram’s inequality remains true for the matrix $B = b_{ij} = w^3_{ij}(w) < f_i \cdot g_j >$, namely:

$$|\det B| \leq \prod_i ||f_i|| \prod_j ||g_j|| \quad \forall w \quad (3.4)$$

**Proof:** Indeed for fixed $\delta$ and $(w)$ we can take the symmetric square root $v$ of the positive matrix $w^3(w)$, so that $w_{ij} = \sum_k v_{ik} v_{kj}$. Defining the components of the vectors $f$ and $g$ in an orthonormal basis for the scalar product $< \cdot >$ to be $f_i^m$ or $g_j^m$, we define the tensorized vectors $F_i$ and $G_j$ with components $F_i^{km} = v_{ik} f_i^m$ and $G_j^{km} = v_{jk} g_j^m$ and we have for the tensor scalar product $< \cdot >: b_{ij} = < F_i \cdot G_j >$, so that $\det B \leq \prod_i ||F_i||_T \prod_j ||G_j||_T$. But obviously $||F_i||^2_T = \sum_{km} (F_i^{km})^2 = \sum_{km} v_{ik}^2 (f_i^m)^2 = w_{ii} \sum_m (f_i^m)^2 = ||f_i||^2$ (we recall that by definition $w^3_{ii}(w) = 1$).

Let us now apply the Symmetric Taylor Forest formula to the computation of the connected functions of a Fermionic theory. The corresponding Grassmann integral is:

$$\frac{1}{Z} \int d\mu C(\psi, \bar{\psi}) P(\bar{\psi}_a, \psi_a) e^{S(\bar{\psi}_a, \psi_a)} \quad (3.5)$$
where $C$ is the covariance or propagator, $P$ is a particular monomial (set of external fields) and $S$ is some general action. We take as simplest example the massive Gross-Neveu model with cutoff, for which the action is local and quartic in a certain number of Fermionic fields. In a finite box $\Lambda$ the action is

$$S_{\Lambda} = \frac{\lambda}{N} \int_{\Lambda} dx (\sum_a \bar{\psi}_a(x) \psi_a(x))^2$$

(3.6)

where $a$ runs over some finite set of $N$ “color” indices. $\lambda$ is the coupling constant. There may be also a spinor index which we forget in our notations since it plays no role in what follows. The covariance $C$ is massive and has an ultraviolet cutoff, hence in Fourier space it is for instance $\eta(p)/(\not p + m)$ where $\eta$ is a cutoff function on large momenta. We only need to know that $C$ is diagonal in color space

$$C(x,a; y,a') = 0 \text{ for } a \neq a'$$

and that it can be decomposed as

$$C(x,y) = \int_{\mathbb{R}^d} d\mu C_{\sigma}(\bar{w}) \int dx_1...dx_n \delta(x_1 = 0) \left\{ \prod_{l=1}^{n+1} C_{\sigma(l)}(x_i, y_j) \right\} \int d\mu_{C_{X_{\bar{w}}}(\bar{w})} \psi(\bar{w}) \prod_{r=1}^{n+1} \bar{\psi}(x_{i(r)}, a(r)) \psi(x_{j(r)} a'(r))$$

(3.8)

where the sum is over trees $\Sigma$ which connect together the $n$ points $x_1,...,x_n$. These trees also contain an arrow information $\sigma$ which for each line determines which end was a field and which was an antifield. The coloring $C$ contains the color indices of each line of the tree and of
each remaining field or antifield. This information completely determines the remaining set of uncontracted fields \(\prod_{r=1}^{n+1} \bar{\psi}(x_{i(r)},a(r))\psi(x_{j(r)},a'(r))\), that is it determines the functions \(i(r)\) and \(j(r)\) which tell to which vertex the remaining fields are hooked, and the coloring index \(C\) completely determines the functions \(a(r), a'(r)\) (hence their color). Finally \(\epsilon(\Xi, \sigma)\) is some sign \(\pm 1\) which we do not need to compute.

**Exercise 3.1** Find first a similar but simpler formula for the number of connected graphs of the \(\phi^4\) theory at order \(n\) as a sum over trees with weakening factors. (Counting graphs with their correct combinatoric factor is equivalent to field theory in zero dimension). Check explicitly in the cases \(n = 3\) and \(n = 4\) how the weakening factors when integrated restore the right combinatorics for the ordinary graphs (very instructive).

**Exercise 3.2** Check that (3.8) is indeed the result of applying the Symmetric Taylor Forest formula to the Grassmann functional integral (3.5), interpolating the Fermionic covariance \(C\) and using the Grassmannian rules of “integration by parts”. For the courageous reader: find an explicit formula for the sign \(\epsilon(\Xi, \sigma)\).

**Lemma 2** The index \(C\) in the sum (3.8) runs over a set of exactly \(2^n N^{n+1}\) elements.

**Proof:** At each vertex the circulation of color indices is fixed by a factor 2 (which tells which of the two fields \(\psi\) is paired with a \(\bar{\psi}\) i.e. forced to have the same color by the form (3.6) of the action). Once these circulation rules are fixed, the attribution of color indices costs \(N^2\) for the first vertex, and by induction a factor \(N\) for each vertex of the tree. Indeed climbing inductively into the tree layer by layer, at every vertex there is one color already fixed by the line joining the vertex to the root, hence one remaining color to fix, except for the root, for which two colors have to be fixed. This proves the lemma.

Then the expression

\[
\int d\mu_{C_{X}(w)}(\psi, \bar{\psi}) \prod_{r=1}^{n+1} \bar{\psi}(x_{i(r)},a(r))\psi(x_{j(r)},a'(r))
\]

is nothing but an \(n + 1\) by \(n + 1\) determinant, with matrix element the covariance

\[
w_l^T(w)\delta(a(r), a'(r'))C(x_{i(r)}, x_{j(r')})
\]

between the line \(r\) and the column \(r'\), where \(l = (i(r), j(r'))\).

Let us now use these Fermionic tree formulas for proofs of convergence.

### 3.3 Convergence of the tree formulas

A typical constructive result for this Gross-Neveu model with cutoff is to prove:

**Theorem 4** The pressure and the connected functions of the cut-off Gross-Neveu model are analytic in \(\lambda\) in a disk of radius \(R\) independent of \(N\).
**Proof:** Apply Gram’s inequality to the loop lines in (3.8). By Lemma I and (3.7b), this determinant is bounded by $K^{n+1}$. The spatial integrals are bounded, using (3.7a) by $K^{n-1}$. The number of trees on $n$ vertices is $n^{n-2}$ by Cayley’s theorem. In the sum over colors, the coloring factor $N^{n+1}$ of Lemma 2 almost cancels with the factor $1/N^n$ in (3.8). Hence the power series in $\lambda$ for the pressure is bounded by $N^{\frac{n^2-2}{n}}K'$ for some constant $K'$.

This is perhaps the shortest and most transparent proof of constructive theory yet. The theorem above is interesting not only for the analysis of the Gross-Neveu model but also for that of the two-dimensional interacting Fermions considered in the next section. In this latter case, the “colors” correspond to angular sectors on the Fermi sphere and the factor $1/N$ in the coupling is provided by power counting.

### 3.4 Renormalization, an overview

In this subsection we give a brief summary of how to remove the ultraviolet cutoff $\eta$ and perform renormalization of the Gross-Neveu model, following [DR].

The covariance $C$ in Fourier space with infrared cutoff $\Lambda$ (this is no longer the volume!) and ultraviolet cutoff $\Lambda_0$ can be written as

$$C_{\Lambda_0}^{\Lambda} = \frac{(-\hat{\mathbf{p}} + m)}{p^2 + m^2} (e^{-\Lambda_0^{-2}(p^2+m^2)} - e^{-\Lambda^{-2}(p^2+m^2)}) = \int_{\Lambda_0^{-2}}^{\Lambda^{-2}} (-\hat{\mathbf{p}} + m)e^{-\alpha(p^2+m^2)} \quad (3.11)$$

Our goal is to prove that the vertex functions of the theory have a non-perturbative limit as $\Lambda_0 \to \infty$, and that they are the Borel sum of their renormalized power series in the renormalized coupling constant (because of renormalons they cannot be analytic).

The main idea is to apply the interpolation forest formula not directly to the lines of the Feynman graphs, but to the continuous scale parameter $\alpha$ introduced in (3.11). In this way an *ordered* forest formula is built, in which the ordering of the tree lines plays the role of the necessary ordering of momenta in any phase space analysis, namely to distinguish higher momenta from lower momenta. This expansion alone however, does not put explicitly into display the divergent subgraphs with two and four external legs. An additional construction (classes of “chains” in [DR]) has to be performed, to expand further the remaining loop determinant in (3.8) so that this structure becomes visible. This remaining expansion has to be performed with some care. Indeed we cannot simply sum over the attributions of the $\alpha$ parameters of the loop lines to the bands defined by the intervals left between the $\alpha$ parameters of the tree lines. This would make visible not only the 2 and 4 point high energy subgraphs but also all the other high energy subgraphs, with arbitrary number of external legs. And it is a general rule of this kind of mathematical problems that information has a price. This particular information would lead to uncontrolled divergences at large order. The solution found in [DR] is to reglue together many band attributions into so-called

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4One can also prove the same result using the rooted formula rather than the symmetric one [AR3]. Although the proof is somewhat longer, it is interesting since the weakening factor in the rooted formula completely factorizes out of the determinant. This second formula may therefore be useful in problems for which Gram’s inequality is not applicable and the method of “comparison of rows and columns” of [IM2] or [FMRT1] has to be used. This method roughly corresponds to Taylor expanding around a middle point further and further when fields or antifields accumulate in any given cube of unit size of a lattice covering $\mathbb{R}^d$.  

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“chain classes” so as to expand much less the loop determinant. After this regluing, the only information on a high energy subgraph displayed by the expansion is whether this subgraph has 2, 4, or more than 4 low energy legs, and this information alone does not introduce large order divergences.

After this subtle point is settled, the rest of the analysis is completely similar to perturbative renormalization theory. Counterterms for 0 momentum values are introduced which subtract the two and four point high energy subgraphs and create flows for the three relevant or marginal operators of this theory, namely the coupling constant, the wave function constant and the mass. Technically the subtractions are performed in direct \( x \) space, since again we do not want to know the exact structure of loop lines, which would be necessary for Zimmermann’s subtraction scheme in momentum space.

Remark that the initial ordering of the tree lines in this formalism reminds of the ordering of Hepp’s sectors in perturbative renormalization theory (see [R], and references therein); but it orders only about “half” the lines of a graph (namely the tree lines) whereas Hepp’s sectors ordered all the lines of a graph.

The formalism has also some similarities to the Gallavotti-Nicolò tree expansion [GN], but beware that the basic trees in [DR] are not the Gallavotti-Nicolò trees, which correspond rather to the auxiliary trees considered for the construction of the “chain classes”.

4 Renormalization Group in Condensed Matter

4.1 Many Interacting Fermions and the BCS Problem

Conducting electrons in a metal at low temperature are well described by Fermi liquid theory. However we know that the Fermi liquid theory is not valid until temperature 0. Indeed below the BCS critical temperature the dressed electrons or holes which are the excitations of the Fermi liquid are bound into Cooper pairs and the metal becomes superconducting.

During the last ten years a program has been designed to investigate rigorously this phenomenon by means of field theoretic methods [BG][FT1-2][FMRT1][S]. In particular the renormalization group of Wilson and followers has been extended to models with surface singularities such as the Fermi surface. The ultimate goal is to create a mathematically rigorous theory of the BCS transition and of similar phenomena of solid state physics. From the start we know that nonperturbative effects must be incorporated in the analysis of superconductivity since the BCS gap is nonperturbative. But this model is the only one we know of in which nonperturbative effects should be accessible in the near future to rigorous mathematical control, without any ad hoc modifications (such as introducing an artificially large number of colors). Indeed an amazing property of this model is that angular variables around the Fermi surface play a role analogous to that of colors, so that an expansion of the \( 1/N \) type should control the BCS regime [FMRT2], in which ordinary perturbation is no longer valid. We can call this situation a “dynamical \( 1/N \)” effect.

Nevertheless the full construction of the BCS model is a long and difficult program which requires to glue together several ingredients. The main new idea was to extend the renormalization group of Wilson (which analyzes the point singularity \( p = 0 \) in momentum space) to more general extended singularities. This very natural and general idea is susceptible
of many applications in various domains, including field theory (in Minkowski space). But this idea has also to be combined with rigorous control of spontaneous continuous symmetry breaking, and this generates a lot of technical complications. In these lectures we therefore restrict us to the study of the concept of RG around the Fermi surface in a simpler model which is at a temperature higher than the temperature of the BCS phase transition.

The key feature which differentiates electrons in condensed matter from Euclidean field theory, and makes the subject in a way mathematically richer, is that Lorentz invariance is broken, and density is finite. The field theory formalism remains the best tool to isolate the fundamental issues such as the existence of non-perturbative effects. Imaginary (Euclidean) time (in the form of a circle, with antiperiodic b.c. for Fermions) corresponds to finite temperature. The fundamental state of the theory corresponds to the limit of the temperature going to 0. In this limit the imaginary time circle grows to $\mathbb{R}$. Finite density creates the Fermi sea and surface. In the simplest case where rotation invariance is preserved (isotropic jellium) this surface is simply a sphere.

The free Fermi liquid theory is therefore defined by Fermion fields with two spin indices, and propagator

$$C_{ab}(k) = \delta_{ab} \frac{1}{ik_0 - e(k)} \quad ; \quad e(k) = \frac{k^2}{2m} - \mu$$

(4.1)

where $a, b \in \{1, 2\}$ are the spin indices. The vector $k$ is in $d$ spatial dimensions. Adding one time dimension there are really $d + 1$ dimensions. The parameters $m$ and $\mu$ correspond to the effective mass and the chemical potential (which fixes the Fermi energy). To simplify notation we put $2m = 1, \mu = 1$ so $e(k) = k^2 - 1$.

This propagator is rotation invariant, a feature which simplifies considerably the study of the renormalization group flows after branching the interaction. In particular it has a spherical Fermi surface. This jellium isotropic model is realistic for instance in solid state physics in the limit of weak electron densities (where the Fermi surface becomes approximately spherical), but in general a propagator with a more complicated energy function $e(k)$ (such as a lattice Laplacian) has to be considered. This is not necessary for our purpose here which is to test the constructive validity of perturbation theory.

Since we work at finite temperature and since Fermionic fields have to satisfy antiperiodic boundary conditions, the component $k_0$ can take only discrete values (called the Matsubara frequencies):

$$k_0 = \pm \frac{2n + 1}{\beta \hbar}$$

(4.2)

so the integral over $k_0$ is really a discrete sum over $n$. As $k_0 \neq 0 \forall n$ the denominator in $C(k)$ can never be 0. This is why the temperature provides a natural infrared cut-off. When $T \to 0$ $k_0$ becomes a continuous variable and the propagator diverges on the Fermi surface, defined by $k_0 = 0$ and $|k| = 1$.

The interaction term in the action is defined by:

$$S_{\Lambda} = \frac{\lambda}{2} \int_{\Lambda} d^3 x \left( \sum_a \bar{\psi}_a \psi_a \right)^2$$

(4.3)
Physically this interaction may represent the effective interaction due to phonons. A more realistic interaction would not be completely local to include the short range nature of the phonon propagator, but we can consider the local action (4.3) as an idealization which captures all essential mathematical difficulties.

4.2 Renormalization around the Fermi Surface

It is convenient to add a continuous ultraviolet cut-off (at a fixed scale) to the propagator (4.1) because it makes its Fourier transformed kernel in position space well defined, and because a non relativistic theory does not make sense anyway at high energies.

The basic difference between this theory and an ordinary critical point in statistical mechanics or field theory, is that the singularity of the propagator is of codimension 2 in the \(d + 1\) dimensional space-time. This changes dramatically the power counting. Instead of changing with dimension, power counting in this kind of models is essentially independent of the dimension, and is the one of a just renormalizable theory. This can be understood basically in the following way. In a graph with 4 external legs, there are \(n\) vertices, \(2n - 2\) internal lines and \(L = n - 1\) independent loops. Each independent loop momentum gives rise to two transverse variables \(k_0\) and \(|\vec{k}|\) and \(d - 1\) inessential bounded angles. Hence the \(2L = 2(n - 1)\) dimensions of integration for the loop momenta exactly balance the \(2n - 2\) singularities of the internal propagators, as is the case in a just renormalizable theory.

To justify this very crude picture, it turns out that it is very convenient to further decompose the propagator into discrete slices and each slice into discrete angular sectors:\footnote{A continuous slicing in the style of [DR] is also possible; but the discrete slicing is more in Wilson's spirit.}

\[
C = \sum_{j=1}^{\infty} C^j \quad \text{; \quad } C^j(k) = \frac{f^j(k)}{ik_0 - e(\vec{k})}\tag{4.4}
\]

where

\[
f^j(k) = f \left( M^{2j} \left( k_0^2 + e(\vec{k})^2 \right) \right)\tag{4.4}
\]

effectively forces \(|ik_0 - e(\vec{k})| \sim M^{-j}\). The function \(f\) is in \(C_0^\infty([1, M^4])\). The parameter \(M\) is strictly bigger than one so that the slices pinch more and more the Fermi surface as \(j \to \infty\).

The slice propagator is further decomposed into sectors:

\[
C^{(j)}(k) = \sum_{\sigma \in \Sigma^j} C^{j,\sigma}(k) \quad \text{; \quad } C^{j,\sigma}(k) = \frac{f^{j,\sigma}(k)}{ik_0 - e(\vec{k})}\tag{4.5}
\]

where \(\Sigma^j\) is a set of roughly \(M^{(d-1)j}\) angular patches, called sectors, which cover the Fermi sphere, with linear dimensions of order \(M^{-j}\). For instance if \(d = 2\) we simply cut the circle into intervals of length \(2\pi M^{-j}\).

The RG philosophy applied to this problem is now clear. As before, higher slices give rise to local effects relatively to lower slices. After more and more slice integrations one obtains effective actions which govern longer and longer distance physics. These effective actions are however more complicated than in the field theory context. In rotation invariant models such
as the one above, renormalization of the two point function can be absorbed in a change of normalization of the Fermi radius. It removes all infinities from perturbation theory at generic momenta [FT1]. But the flow for the four point function is a flow for an infinite set of coupling constants describing the momentum zero channel of the Cooper pairs [FT2]. In the case of attractive interaction for \( \lambda \), if the temperature cutoff is lowered to zero, this flow diverges at the BCS scale, where the symmetry linked to particle number conservation is spontaneously broken, giving rise to the effective BCS theory for the Cooper pairs.

Like in the previous section the key problem from the constructive point of view is to understand the resummation of perturbation theory in a single slice, in order to iterate the renormalization group step.

Curiously, although power counting does not depend on the dimension, momentum conservation in terms of sectors in a fixed slice depends on it and this has crucial constructive consequences. In \( d = 2 \) we have the “rhombus rule”: four sectors meeting at a vertex must be roughly two by two equal. This means that two dimensional condensed matter in a slice is directly analogous to an \( N \)-vector model in which angles on the Fermi surface play the role of colors [FMRT3]. This allows to complete the proof that the radius of convergence of perturbation theory is independent of the slice index \( j \). Roughly speaking, the model in slice \( j \) is a vector model with \( N = M^j \) colors. Power counting at a vertex costs \( M^{3j} \) for the space-time integration, and the propagators in fixed sectors earn a scaling factor \( M^{-2j} \) each. Since there are in average two propagators per vertex, we remain with a factor \( M^{-j} = 1/N \) left per vertex, which is exactly what is necessary to pay for the color sums and obtain a uniform radius of convergence by Theorem 4 of section 3. This completes the sketch of the proof of:

**Theorem 5 [FMRT1]** In two dimensions, there exists a finite \( \kappa > 0 \) independent of \( j \) such that the power series in \( \lambda \) of the Schwinger function for the interacting Fermionic measure with propagator (4.5) and interaction (4.3) has a convergence radius of at least \( \kappa \).

Furthermore it ought to be possible in two dimensions to resum perturbation theory to build an effective action until the BCS temperature. As an example, the convergent part of the expansion (which does not include 2 or 4 point subgraphs) has been resummed in [FMRT1].

In three dimensions, two momenta at a vertex in a slice do not determine the third and fourth: there is an additional torsion angle, since four momenta of same length adding to 0 are not necessarily coplanar. The radius of convergence of perturbation theory is still independent of the slice index \( j \), so that Theorem 5 also holds in three dimension [MR], but it is much harder to prove than in dimension 2, and until now it is not clear that this partial result alone allows a full constructive analysis of the model up to the scale where the BCS symmetry breaking takes place.

In dimension 1 the Fermi surface reduces to two points, and there is also no proper BCS theory since there is no continuous symmetry breaking in two dimensions (by the “Mermin-Wagner theorem”). Nevertheless the many Fermion system in 1 dimension gives rise to an

\(^6\)A precise version is given in [FMRT1]. Remark that there is a logarithmic correction to this rhombus rule, due to the case of a nearly collapsed rhombus. A nice improvement to avoid further problems with this logarithm is to define angular sectors longer in the tangential than in the radial direction.
interesting non-trivial behavior, that of the Luttinger liquid [BG].

4.3 The Weakly Coupled Anderson Model

This model describes a single electron in a random potential. This is not strictly speaking a field theory. But renormalization group and field theoretic methods can also be applied to models such as self repelling walks [IM1]. This model lies in this category. The two-point function is given by

\[ \mathcal{S}(x,y; E; \lambda V; \epsilon) = \left( \frac{1}{\Delta - E + \eta\lambda V + i\epsilon} \right)(x,y) \]

\[ = \left( \frac{1}{p^2 - E + i\epsilon} \right) \frac{1}{1 + \lambda \frac{1}{p^2 - E + i\epsilon} \eta V} \]

\[ (4.5) \]

where \( \eta \) is an ultraviolet cutoff and \( V \) is a random potential (multiplication operator in \( x \)-space), for instance distributed with a Gaussian (regularized) white noise for which the covariance is a (regularized) delta function. Indeed in this model the singularity in Fourier space lies on the surface \( |p| = \sqrt{E} \), just as for the interacting Fermions of the previous subsection. The perturbative expansion for the averaged Green’s function is a resolvent expansion in which the integration over the potential creates structures similar to the \( \phi^4 \) graphs, as for self avoiding polymers; the main difference is in the combinatorics for the graphs, which is the one of a “\( N = 0 \) component” theory (in particular vacuum parts are forbidden).

The main mathematical question of physical interest is to prove the existence or non-existence of localized states. This in turn amounts to study the behavior of the average of the modulus square of the two-point function

\[ \lim_{\epsilon \to 0} \int |\mathcal{S}(x,y; E; \lambda V; \epsilon)|^2 \, d\mu(V) \]

\[ (4.6) \]

The rigorous results in more than one dimension are restricted up to now to the strong coupling regime, where it has been proved that this average of the modulus square of the two-point function decays exponentially [FS],[AM]. This in turn implies localization of all states.

At small coupling it is expected, that

- i) in two dimensions the average of the modulus square of the two-point function decays exponentially with a rate dependent on the coupling

- ii) in three dimensions it decays only like a power which means that some states are delocalized.

One expects also that for all (non zero) couplings the mean value of the two-point function

\[ \lim_{\epsilon \to 0} \int \mathcal{S}(x,y; E; \lambda V; \epsilon) \, d\mu(V) \]

\[ (4.7) \]

decreases exponentially, and that it is real analytic in \( E \).

As a first step in this direction, we have proved [MPR]:

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Theorem 6  In two dimensions, there exists some \( \kappa > 0 \) such that for \( \lambda \) small enough

\[
\int S(x,y;E;\lambda V;\lambda^{2+\kappa}) \, d\mu(V) \sim e^{-\text{cst}(\kappa)} \lambda^2 |x-y|
\]

which is the expected rate of decrease.

To suppress the regularization \( \epsilon = \lambda^{2+\kappa} \) in our theorem, hence to prove the exponential decay of (4.7) is well under way.

We can only give a flavor of the arguments used in Theorem 6. A key new ingredient is to use the rhombus rule in two dimensions for the part of the theory in which the momenta are close to the singularity \( p^2 = E \), to create loops with approximate momentum conservation. These loops are then smaller than expected because of a Ward identity which is somewhat involved technically [MPR].

But a more immediately accessible and instructive point is to apply the decomposition of the critical surface into discrete angular sectors also to this problem. One finds that the random potential sandwiched between sector cutoffs of a given slice

\[
\eta_\sigma(p) \tilde{V}(p-q) \eta_{\sigma'}(q) \simeq \mathcal{V}_{\sigma,\sigma'} \quad \text{for} \quad p \in \sigma, \, q \in \sigma'
\]

is a random matrix whose elements are labeled by the shell cells. In two dimensions, since all momenta in a white noise distribution are equiprobable, and since the momentum \( p-q \) of \( \tilde{V} \) approximately determines \( p \) and \( q \) on the circle \( |p| = \sqrt{E} \) (because of the rhombus rule), this random matrix is approximately a self adjoint matrix with random \textit{independently distributed} entries

\[
\mathcal{V}(\sigma,\sigma') = \tilde{\mathcal{V}}(\sigma',\sigma) = \tilde{\mathcal{V}}(-\sigma,-\sigma').
\]

Therefore we can use the mathematical theory of such matrices for instance to bound the deviation or probability that eigenvalues become anomalously large [M]. In three dimensions this \( N \times N \) matrix is more complicated because

\[
V_{\sigma,\sigma'} = V_{\tau,\tau'} \quad \text{iff} \quad \sigma + \sigma' = \tau + \tau'
\]

so that approximately \( \sqrt{N} \) matrix elements corresponding to the same momentum transfer are equal. The study of such matrices with nonindependent entries, and more generally the study of the weakly coupled 3 dimensional Anderson model remains a mathematical challenge.

5  Further topics

In this last section we give a brief list of open problems for constructive methods and rigorous renormalization group studies.

- **Mass generation in the Gross-Neveu model without cutoff.** Prove that at large number \( N \) of colors, the model without ultraviolet cutoff has a spontaneously generated mass, for instance by “gluing together” the analysis of the ultraviolet limit in [DR] and the mass generation with ultraviolet cutoff in [KMR]. This could be called “constructive dimensional transmutation”, after Coleman.
- **Ultraviolet limit of the nonlinear $\sigma$ model in two dimension.** This is a long standing problem. Although there are partial results, for instance for the hierarchical model, there is no clear rigorous construction using asymptotic freedom of this model. Once it is built, there is also the problem of gluing the ultraviolet analysis to mass generation (at large $N$) to obtain “dimensional transmutation” also in this model. This would be interesting in view of the controversial issues raised by Patrascioiu and Seiler on this kind of model.

- **Fermi liquid and BCS theory in dimension 2:** Since [FMRT1], the full construction of the BCS vacuum at zero temperature can be considered a sound mathematical constructive program, although its technical realization is very hard, and implies a tricky constructive analysis of the infrared problems associated with the Goldstone boson of the BCS continuous symmetry breaking.

  A more accessible task is to precise the mathematical status of Fermi liquid theory itself. This is important also in view of the debate around the nature of high-temperature superconductivity. Fermi liquid theory is not valid at zero temperature because of the BCS instability. Even when the dominant electron interaction is repulsive, the Kohn-Luttinger instabilities prevent the Fermi liquid theory to be generically valid until zero temperature. There are nevertheless two proposals for a mathematically rigorous Fermi liquid theory:

  - one can block the BCS and Kohn-Luttinger instabilities by considering models in which the Fermi surface is not invariant under $p \rightarrow -p$ (we suggest to call this, for obvious reasons, the “egg model”). In two dimensions it should be possible to prove (nonperturbatively) that in this case the Fermi liquid theory remains valid at zero temperature, and the corresponding program is well under way [FKLT]. This program requires to control rigorously the stability of a non-spherical Fermi surface under the renormalization group flow, a difficult technical issue [FST].

  - a simpler proposal, advocated in [S], is to study the Fermi liquid theory at finite temperature above the BCS transition temperature. As seen above, the temperature acts as an infrared cutoff on the propagator in the field theory description of the model. Hence in this point of view, the nontrivial theorem consists in showing that stability (i.e. summability of perturbation theory) holds for all temperatures higher than a certain critical temperature whose dependence in terms of the initial interaction is proved to be of the correct BCS form [S].

- **Fermi liquid and BCS theory in dimension 3:** Here the constructive analog of [FMRT1] must be found first. [MR] is only a first step in this direction. Having worked for several years with J. Magnen on this problem, it seems to me the most beautiful I met in constructive theory until now. It is well posed and simple. For instance one could ask: is the sum of all convergent contributions to the theory analytic in a finite disk? Perturbative power counting suggests that the answer should be “yes”, but because of the torsion angle, the constructive problem is surprisingly hard. Again an understanding of this constructive issue should clear up the way for many subprograms, such as the Fermi liquid theory in 3 dimensions for “egg models” or above the BCS transition, and the BCS theory itself.

  **Anderson Model, Anderson-Mott phase transition in dimension 3** See the previous subsection.

- **Continuation to Minkowski space, constructive scattering theory** Here again we feel that the key for instance to a constructive analysis of asymptotic completeness in
Field theory might be related to renormalization group analysis around a surface, this time the mass shell. The “constructive return” to Minkowski space is of course full of interest and of important issues for the physics of the models (scattering, bound states, time dependent problems). Constructive study of time dependent problems in statistical mechanics or condensed matter theory is also clearly a potential field for many developments.

- In the future one could focus more systematically on clarifying the mathematical status of the fast developed theories of the recent decades (Conformal theories, Non Abelian Gauge Theories, in particular Supersymmetric ones, Topological Theories). Some constructive tools, of course presumably with unexpected additions, might be of help. In this line one can also dream of Constructive Duality and Constructive String or Membrane Theory.

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References

[A] A. Abdesselam, PhD thesis (1997).

[AM] M. Aizenman and A. Molchanov, Localization at large disorder and at extreme energies: an elementary derivation, Commun. Math. Phys. 157, 245 (1993).

[AR1] A. Abdesselam and V. Rivasseau, Trees, forests and jungles: a botanical garden for cluster expansions, in Constructive Physics, Lecture Notes in Physics 446, Springer Verlag, 1995.

[AR2] A. Abdesselam and V. Rivasseau, An Explicit Large Versus Small Field Multiscale Cluster Expansion, Rev. Math. Phys. Vol. 9 No 2, 123 (1997) (and references therein).

[AR3] A. Abdesselam and V. Rivasseau, Explicit Fermionic Tree Expansion, Letters in Math. Phys. 44, 77 (1998)

[BCGNOPS] G. Benfatto, M. Cassandro, G. Gallavotti, F. Nicolò, E. Olivieri, E. Presutti and E. Scacciatelli, Comm. Math. Phys. 59, 1433 (1978) and 71, 95 (1980).

[B] David Brydges, Weak Perturbations of the Massless Gaussian Measure, in Constructive Physics, Lecture Notes in Physics 446, Springer Verlag, 1995 (and references therein).

[BG] G. Benfatto and G. Gallavotti Perturbation theory of the Fermi surface in a quantum liquid. A general quasi-particle formalism and one dimensional systems, Journ. Stst. Physics 59, 541, 1990.

[BK] D. Brydges and T. Kennedy, Mayer Expansions and the Hamilton-Jacobi Equation, Journ. Stat. Phys. 48, 19 2064 (1978).
[dV] P. Faria da Veiga, PhD thesis.

[DR] M. Disertori and V. Rivasseau, Continuous Constructive Fermionic Renormalization, hep-th/9802145

[FKLT] J. Feldman, D. Lehmann, H. Knörer and E. Trubowitz, Fermi Liquids in Two Space Dimensions, in Constructive Physics, Lecture Notes in Physics 446, Springer Verlag, 1995.

[FMRS1] J. Feldman, J. Magnen, V. Rivasseau and R. Sénéor, A renormalizable field theory: the massive Gross-Neveu model in two dimensions, Comm. Math. Phys. 103, 67 (1986).

[FMRS2] J. Feldman, J. Magnen, V. Rivasseau and R. Sénéor, Construction of infrared $\phi^4$ by a phase space expansion, Comm. Math. Phys. 109, 437 (1987).

[FMRT1] J. Feldman, J. Magnen, V. Rivasseau and E. Trubowitz, An Infinite Volume Expansion for Many Fermion Green’s Functions Helv. Phys. Acta Vol. 65, 679 (1992).

[FMRT2] J. Feldman, J. Magnen, V. Rivasseau and E. Trubowitz, An Intrinsic $1/N$ Expansion for Many Fermion Systems, Europhys. Letters 24, 437 (1993).

[FMRT3] J. Feldman, J. Magnen, V. Rivasseau and E. Trubowitz, Two dimensional Many Fermion Systems as Vector Models, Europhys. Letters 24, 521 (1993).

[FS] J. Fröhlich and T. Spencer, Absence of diffusion in the Anderson tight binding model for large disorder or low energy, Commun. Math. Phys. 88, 151 (1983).

[FST] J. Feldman, M. Salmhofer and E. Trubowitz, Regularity of interacting nonspherical Fermi surfaces: The full self-energy. Commun. Pure Appl. Math. to appear (1998).

[FT1] J. Feldman and E. Trubowitz, Perturbation Theory for Many Fermions Systems, Helv. Phys. Acta 63, 156 (1990).

[FT2] J. Feldman and E. Trubowitz, The Flow of an Electron-Phonon System to the Superconducting State, Helv. Phys. Acta 64, 213 (1991).

[GJ] Glimm, J., Jaffe, A.M.: Quantum Physics: A Functional Integral Point of View, Springer, New York, 1988; Positivity of the $\phi^4_3$ Hamiltonian, Fortschr. Phys. 21, 327 (1973).

[GK1] K. Gawedzki and A. Kupiainen, Gross-Neveu model through convergent perturbation expansions, Comm. Math. Phys. 102, 1 (1985).

[GK2] K. Gawedzki and A. Kupiainen, Massless $\phi^4_3$ theory: Rigorous control of a renormalizable asymptotically free model, Comm. Math. Phys. 99, 197 (1985).

[GN] G. Gallavotti, Renormalization Theory and ultraviolet stability via renormalization group methods, Rev. Mod. Phys. 57, 471 (1985)
[K] C. Kopper, *Mass Generation in the Large N Non-linear $\sigma$ Model*, to appear in Comm. Math. Phys. (1999)

[KMR] C. Kopper, J. Magnen and V. Rivasseau, *Mass Generation in the Large N Gross-Neveu Model*, Comm. Math. Phys. **169** 121 (1995)

[IM1] D. Iagolnitzer and J. Magnen, *Polymers in a weak random potential in dimension 4: Rigorous Renormalization Group Analysis*, Commun. Math. Phys. **162**, 85 (1994).

[IM2] D. Iagolnitzer and J. Magnen, *Asymptotic Completeness and Multiparticle Structure in Field Theories, II, Theories with Renormalization*, Commun. Math. Phys. **111**, 81 (1987).

[M] M. L. Mehta, Random Matrices, Academic Press, Boston 1991.

[MPR] J. Magnen, G. Poirrot and V. Rivasseau, *Ward type Identities for the 2d Anderson Model at weak Disorder*, to appear in Journ. Stat. Phys.

[MR] J. Magnen and V. Rivasseau, *A single scale Infinite Volume Expansion for Three-Dimensional Many Fermion Green’s Functions*, Mathematical Physics Electronic Journal Vol. 1, n3 (1995).

[P] A. Pordt, *On Renormalization Group Flows and Polymer Algebras*, in Constructive Physics, Lecture Notes in Physics **446**, Springer Verlag, 1995

[R] V. Rivasseau, *From Perturbative to Constructive Renormalization*, Princeton University Press, 1991

[S] M. Salmhofer, *Continuous renormalization for fermions and Fermi liquid theory*, Comm. Math. Phys. **194**, 249 (1998).