RIEMANN–HILBERT PROBLEM FOR THE MODIFIED LANDAU–LIFSHITZ EQUATION WITH NONZERO BOUNDARY CONDITIONS

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We study a matrix Riemann–Hilbert (RH) problem for the modified Landau–Lifshitz (mLL) equation with nonzero boundary conditions at infinity. In contrast to the case of zero boundary conditions, multivalued functions arise during direct scattering. To formulate the RH problem, we introduce an affine transformation converting the Riemann surface into the complex plane. In the direct scattering problem, we study the analyticity, symmetries, and asymptotic behavior of Jost functions and the scattering matrix in detail. In addition, we find the discrete spectrum, residue conditions, trace formulas, and theta conditions in two cases: with simple poles and with second-order poles present in the spectrum. We solve the inverse problems using the RH problem formulated in terms of Jost functions and scattering coefficients. For further studying the structure of the soliton waves, we consider the dynamical behavior of soliton solutions for the mLL equation with reflectionless potentials. We graphically analyze some remarkable characteristics of these soliton solutions. Based on the analytic solutions, we discuss the influence of each parameter on the dynamics of the soliton waves and breather waves and propose a method for controlling such nonlinear phenomena.

Keywords: modified Landau–Lifshitz equation, matrix Riemann–Hilbert problem, nonzero boundary condition, soliton solution

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1. Introduction

Nonlinear Schrödinger equations are an important model from the mathematical and physical standpoints. They are applied in many fields such as magnetism and optics. Nevertheless, for describing the complexity of phenomena, just the nonlinear Schrödinger equation itself is insufficient, which leads to adding some specific terms, for example, as in the Sasa–Satsuma equation, the Hirota equation, and others. To

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describe the dynamical behavior of local magnetization in electromagnetism, the modified Landau–Lifshitz (mLL) equation [1], [2] is used,

$$\frac{\partial M}{\partial t} = -\gamma M \times M_{\text{eff}} + \frac{\rho}{M_s} \times \frac{\partial M}{\partial t} + \tau_t,$$

(1.1)

where $M \equiv M(x,t)$ is the localized magnetization, $\gamma$ is the gyromagnetic ratio, $\rho$ is the Gilbert damping parameter, $\tau_t$ is the spin torque vector, $M_s$ is the saturation magnetization, and $M_{\text{eff}}$ is the effective magnetic field including the external field, the anisotropy field, the demagnetization field, and the exchange field. Equation (1.1) is equivalent to

$$M_{\text{eff}} = \frac{2A}{M_s^2} \frac{\partial^2 M}{\partial x^2} + \left[ \left( \frac{H_k}{M_s} - 4\pi \right) M_z + H_{\text{ext}} \right] e_z,$$

(1.2)

where $A$ is the exchange constant, $H_k$ is the anisotropy field, $H_{\text{ext}}$ is the external field, $M_z$ is the demagnetization field, and $e_z$ is the unit vector in the $z$ direction. Setting $m = M/M_s$, we obtain another expression for (1.1):

$$\frac{\partial m}{\partial t} = -m \times \frac{\partial^2 m}{\partial x^2} + m \times \frac{\partial m}{\partial x} + \frac{b_J}{t_0} \frac{\partial m}{\partial x} - \left( m + \frac{H_{\text{ext}}}{H_k - 4\pi M_s} \right) m \times e_z,$$

(1.3)

where $t = t'/t_0$, $x = x'/l_0$, $t_0 = 1/\gamma(H_k - 4\pi M_s)$ is the characteristic time, $l_0 = \sqrt{2A/[M_s(H_k - 4\pi M_s)]}$ is the characteristic length, and $b_J = P_j e \mu_B / e M_s$ is the spin-torque, where the spin polarization parameters are the current $P$ and the current density $j_e$. Here, $\mu_B$ is the Bohr magneton, and $e$ is the magnitude of the electron charge. Obviously, $m \equiv (m_x, m_y, m_z) = (0, 0, 1)$ corresponds to the system ground state, and we can obtain two kinds of nonlinear excited states, a spin wave and a magnetic soliton. We next consider the case where the magnetic field is sufficiently large and the deviation of the magnetization of the two excited states from the ground state is small. It is reasonable to make the transformation

$$\hat{q} = m_x + im_y, \quad m_z = \sqrt{1 - |\hat{q}|^2},$$

(1.4)

Substituting these quantities in (1.3) yields

$$i \frac{\partial \hat{q}}{\partial t} - m \frac{\partial^2 \hat{q}}{\partial x^2} + \hat{q} \frac{\partial^2 m_z}{\partial x^2} + \delta \left( m \frac{\partial \hat{q}}{\partial t} - q \frac{\partial m_z}{\partial t} \right) - \frac{b_J}{t_0} \frac{\partial m_z}{\partial x} + \left( m_z + \frac{H_{\text{ext}}}{H_k - 4\pi M_s} \right) m \times e_z = 0.$$

(1.5)

Here, we seek different kinds of soliton solutions under nonzero boundary conditions using the Riemann–Hilbert (RH) problem in a uniaxial ferromagnetic nanowire with spin torque. But (1.5) is not integrable. To overcome this problem, we consider the case of the absence of damping and the long-wave approximation [3]. Keeping only nonlinear terms of the order of $|\hat{q}|^2 \hat{q}$, we find that we can write (1.5) as the integrable equation

$$i \hat{q}_t - \hat{q}_{xx} - \frac{1}{2} |\hat{q}|^2 \hat{q} + \left( 1 + \frac{H_{\text{ext}}}{H_k - 4\pi M_s} \right) \hat{q} - \frac{b_J}{t_0} \hat{q}_x = 0,$$

(1.6)

where the subscripts $x$ and $t$ indicate derivatives with respect to $x$ and $t$. Equation (1.6) has been studied in many publications. Soliton solutions were found using the Hirota method [4]–[6]: conservation laws, modulation instability, and rogue waves were obtained in [7]. A direct Darboux transformation [8] was used in [9] when discussing the properties of soliton solutions of the mLL equation on a spin-wave background, and a general soliton solution was constructed using that transformation. Rogue waves obtained in solving Eq. (1.6) were studied in [10], and it was noted that the accumulation of energy plays a vital role in generating magnetic rogue waves.
Nonlinear integrable equations and boundary value problems with nonzero boundary conditions, including conservation laws and Hamiltonian structures, were studied in [11] using the inverse scattering method (ISM). But we note that symmetries, the asymptotic behavior, trace formula, etc., were not presented in that paper.

Boundary value problems with nonzero boundary conditions have attracted the attention of scholars in recent years. For example, special attention was given to applying the ISM for nonlinear integrable equations with nonzero boundary conditions [12]–[22] based on constructing the RH problem [23]–[32]. In fact, the ISM, which is a special variant of the Fourier method, allows solving the Cauchy problem for integrable models. After Gardner, Greene, Kruskal, and Miura proposed using the ISM to solve the Cauchy problem for the integrable KdV model [33], the ISM has been widely applied to other equations, for example, the Camassa–Holm equation [34], the Kundu–Eckhaus equation [35], the Gerdjikov–Ivanov type of the derivative nonlinear Schrödinger equation [36], the coupled mKdV system [37], a generalized Sasa–Satsuma equation [38], the mKdV equation [39], the Maxwell–Bloch equation [40], etc.

But as far as we know, the ISM and soliton solutions of Eq. (1.6) with nonzero boundary conditions have not yet been studied. There are many problems to be solved: how to introduce an affine transformation to overcome the multiplicity of eigenfunctions; how to appropriately transform the boundary function $q e^{(\delta_1 - 2iq_0^2)t}$ into a constant function identically equal to $q$; how to introduce an invertible matrix to transform the spectral problem of the asymptotic Lax pair into a problem with a diagonal matrix; how to study the analyticity of Jost functions and the scattering matrix based on this. Our main goal is to study these problems and construct a generalized RH problem for Eq. (1.6) in terms of the Jost functions and scattering data. We then analyze the residue conditions at discrete spectral points and also obtain solutions of Eq. (1.6) in the case of simple and double poles. We note that the trace formulas and theta conditions differ for simple and double poles.

Structure of the paper. In Sec. 2, we introduce an affine transformation converting the boundary functions into constants and then obtain asymptotic Lax pairs. In addition, using spectral analysis, we study the analytic and symmetry properties of the Jost functions and scattering matrix. To analyze the inverse transformation, we introduce residue conditions. In Sec. 3, we study the RH problem with simple poles and reconstruct its potential function by solving the inverse problem. In Sec. 4, appropriately choosing parameters, we discuss the behavior of different kinds of solutions and briefly analyze them. In addition, in Sec. 4, we construct solutions and derive the trace formula and theta condition for the RH problem with second-order poles.

2. Direct scattering problem with nonzero boundary conditions

In this section, we consider the analyticity and asymptotic behavior of the Jost functions, the asymptotic behavior of the scattering matrix, symmetries, the discrete spectrum, and the residue conditions in direct scattering. In addition, in contrast to the problem with zero boundary conditions, a multivalued function appears in the solution process, and we must determine an appropriate transformation converting it into a single-valued function, which allows simplifying the study.

2.1. Lax pair of the mLL equation. Changing $q = \dot{q}/2$ in mLL equation (1.6), we obtain

$$i q_t - q_{xx} - 2|q|^2 q + \delta_1 q - \delta_2 q_x = 0, \quad \delta_1 = 1 + \frac{H_{\text{ext}}}{H_k - 4\pi M_s}, \quad \delta_2 = \frac{b_j t_0}{l_0}.$$ 

We consider the nonzero boundary conditions as $x \to \pm \infty$

$$\lim_{x \to \pm \infty} q(x, t) = q_{\pm} e^{(\delta_1 - 2iq_0^2)t}, \quad (2.1)$$

1613
where \(|q_\pm| = q_0 \neq 0\). We have the equivalent Lax pair

\[
\begin{align*}
\phi_x &= X \phi, \quad X = ik \sigma_3 + Q, \\
\phi_t &= T \phi, \quad T = \left[2ik^2 + i \delta_2 k + \frac{1}{2}(\delta_1 - 2|q|^2) + iQ_x\right] \sigma_3 + 2kQ + \delta_2 Q,
\end{align*}
\] (2.2)

where the function \(\phi\) is a \(2 \times 2\) matrix, \(k\) is a spectral parameter,

\[
Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and the superscript \(^*\) denotes the complex conjugate.

**Theorem 1.** We introduce the transformation

\[
q = u e^{(\delta_1 - 2iq_0^2)t}, \quad \phi = e^{-(i/2)(2q_0^2 + i\delta_1)t} \sigma_3 \psi.
\] (2.3)

The boundary conditions then change as \(u(x, t) \to u_\pm\) as \(x \to \pm \infty\) with \(|u_\pm| = q_0 \neq 0\).

For convenience, we again let \(q\) denote \(u\) in what follows.

As a result, taking Theorem 1 into account, we can obtain the asymptotic scattering problem for the Lax pair of the mLL equation at infinity:

\[
\begin{align*}
\psi_{\pm, x} &= X_{\pm} \psi_{\pm}, \quad X_{\pm} = \lim_{x \to \pm \infty} X = ik \sigma_3 + Q_\pm, \\
\psi_{\pm, t} &= T_{\pm} \psi_{\pm}, \quad T_{\pm} = \lim_{x \to \pm \infty} T = (2k + \delta_2)X_{\pm},
\end{align*}
\] (2.4)

where \(Q_\pm = \lim_{x \to \pm \infty} Q = \begin{pmatrix} 0 & q_{\pm} \\ -q_{\pm}^* & 0 \end{pmatrix}\).

### 2.2. Riemann surface and uniformization coordinates.

Obviously, the eigenvalues \(\pm i\sqrt{k^2 + q_0^2}\) of the asymptotic matrices \(X_{\pm}\) are multivalued functions, and in contrast to the case of zero boundary conditions, we must introduce a two-sheeted Riemann surface defined by

\[
\lambda^2 = k^2 + q_0^2,
\] (2.5)

which is obtained by sewing two copies of \(S_1\) and \(S_2\) of the extended complex-\(k\) plane along the cut \(iq_0[-1, 1]\) between the branch points \(k = \pm iq_0\), which are solutions of the equation \(\sqrt{k^2 + q_0^2} = 0\). Introducing the local polar coordinates

\[
k + iq_0 = r_1 e^{i \theta_1}, \quad k - iq_0 = r_2 e^{i \theta_2}, \quad -\frac{\pi}{2} < \theta_1, \theta_2 < \frac{3\pi}{2},
\] (2.6)

we obtain a single-valued analytic function on the Riemann surface

\[
\lambda(k) = \begin{cases} (r_1 r_2)^{1/2} e^{(\theta_1 + \theta_2)/2} & \text{on } S_1, \\ -(r_1 r_2)^{1/2} e^{(\theta_1 + \theta_2)/2} & \text{on } S_2. \end{cases}
\] (2.7)

We define the uniformization variable \(z\) by the conformal map [41]

\[
z = k + \lambda
\] (2.8)

and obtain two single-valued functions

\[
k(z) = \frac{1}{2} \left(z - \frac{q_0^2}{z}\right), \quad \lambda(z) = \frac{1}{2} \left(z + \frac{q_0^2}{z}\right)
\] (2.9)

from (2.5).
Theorem 2. Under conformal map (2.8),

- $\Im k > 0$ of sheet $S_1$ and $\Im k < 0$ of sheet $S_2$ are mapped into $\Im \lambda > 0$,
- $\Im k < 0$ of sheet $S_1$ and $\Im k > 0$ of sheet $S_2$ are mapped into $\Im \lambda < 0$,
- the interval $[-iq_0, iq_0]$ on the $k$ plane becomes the interval $[-q_0, q_0]$ on the $\lambda$ plane,
- as $k \to \infty$, we have $z \to \infty$ on the sheet $S_1$ and $z \to 0$ on the sheet $S_2$, and
- under the Joukowsky map $\lambda(z)$,
  - the half-plane $\Im \lambda > 0$ becomes the domain
    \[ D^+ = \{ z \in \mathbb{C} : (|z|^2 - q_0^2) \Im z > 0 \}, \]
    which means that the upper half-plane of $\lambda$ is mapped into the exterior half of the half-circle of radius $q_0$ in the upper half-plane of $z$ and the interior of the half-circle of the lower half-plane (see Fig. 1), and
  - the half-plane $\Im \lambda < 0$ becomes the domain
    \[ D^- = \{ z \in \mathbb{C} : (|z|^2 - q_0^2) \Im z < 0 \}, \]
    which means that the lower half-plane of $\lambda$ is mapped into the interior of the half-circle of radius $q_0$ in the upper half-plane of $z$ and the exterior of the half-circle in the lower half-plane (see Fig. 1).

2.3. Jost functions and their analyticity. Using asymptotic Lax pair (2.4), we can obtain an invertible matrix $E_\pm$ composed of asymptotic eigenvectors to diagonalize the matrices $X_\pm$ and $T_\pm$:

\[
X_\pm(x, t; z) = E_\pm(z)(i\lambda\sigma_3)E_\pm^{-1}(z),
\]
\[
T_\pm(x, t; z) = E_\pm(z)[i\lambda(2k^2 + \delta_2)\sigma_3]E_\pm^{-1}(z).
\]
As usual, the continuous spectrum $\Sigma_k$ comprises all values $k$ on both planes $S_1$ and $S_2$ for which $\lambda(k) \in \mathbb{R}$, namely, $\Sigma_k = \mathbb{R} \cup iq_0[-1, 1]$. Under map (2.8), the continuous spectrum becomes $\Sigma_\pm = \mathbb{R} \cup C_0$, where $C_0$ is the circle of radius $q_0$. For brevity, we write $\Sigma_\pm$ as $\Sigma$. For all $z \in \Sigma$, we can now introduce solutions $\phi_\pm$ of Lax pair (2.2) satisfying the asymptotic condition
\[
\phi_\pm(x, t; z) \sim \psi_\pm(x, t; z) = E_\pm(z)e^{i\theta(x,t;\pm\lambda)} , \quad x \to \pm \infty ,
\]  
where $\theta(x, t; z) = \lambda(z)[x + (2k(z) + \delta_2)t]$ and
\[
E_\pm(z) = \left( \begin{array}{c} 1 \\ \frac{iq_\pm}{k + \lambda} \frac{iq_\pm}{k + \lambda} \\ k + \lambda \end{array} \right) = \mathbb{I} + \frac{i}{z} \sigma_3 Q_\pm . \tag{2.12}
\]
The solutions $\phi_\pm$ are called Lax functions.

Let
\[
u_\pm(x, t; z) = \phi_\pm(x, t; z)e^{-i\theta(x,t;\pm\lambda)} . \tag{2.13}
\]
Then
\[
\lim_{x \to \pm \infty} \nu_\pm(x, t; z) = E_\pm(z) . \tag{2.14}
\]
The Lax pair of $\nu_\pm$ has the form
\[
[(E_\pm^{-1}\nu_\pm)(z)]_x + i\lambda[E_\pm^{-1}(z)\nu_\pm(z), \sigma_3] = (E_\pm^{-1}\Delta Q_\pm\nu_\pm)(z) ,
\]
\[
[(E_\pm^{-1}\nu_\pm)(z)]_y + i\lambda(2k + \delta_2)[E_\pm^{-1}(z)\nu_\pm(z), \sigma_3] = (E_\pm^{-1}\Delta T_\pm\nu_\pm)(z) , \tag{2.15}
\]
where $\Delta Q_\pm(z) = Q - Q_\pm$ and $\Delta T_\pm(z) = T - T_\pm$. Further, we write Lax pair (2.15) as a full derivative. We choose the contours $(-\infty, x)$ and $(x, \infty)$ and rewrite the ordinary differential equations for $\nu_\pm$ as equations for the eigenfunctions
\[
u_-(x, t; z) = E_- + \int_{-\infty}^{x} E_- e^{i\lambda(x-y)\sigma_3} E_\pm^{-1}\Delta Q_-(y, t)u_-(y, t; z)e^{-i\lambda(x-y)\sigma_3} dy ,
\]
\[
u_+(x, t; z) = E_+ - \int_{x}^{\infty} E_+ e^{i\lambda(x-y)\sigma_3} E_\pm^{-1}\Delta Q_+(y, t)u_+(y, t; z)e^{-i\lambda(x-y)\sigma_3} dy . \tag{2.16}
\]
We note that if $q(x) - q \in L^1(-\infty, a)$ for some $a \in \mathbb{R}$, then the Neumann series converges absolutely and uniformly in $x \in (-\infty, a)$ and $z \in D_\varepsilon^-$ for all $\varepsilon > 0$. Here, $D_\varepsilon^-$ is the domain $D^-$ with the two $\varepsilon$-neighborhoods of the points $\pm iq_0$ excluded. Because a uniformly convergent series of analytic functions converges to an analytic function [42], [43], the corresponding column of the matrix eigenfunctions is analytic in this domain (see [44] for the details). We thus obtain the analyticity of the functions $\nu_\pm$.

**Proposition 1.** In the matrices $u_\pm(x, t; z)$, the columns $u_{+,1}(x, t; z)$ and $u_{-2}(x, t; z)$ are analytic in the domain $D^+$ on the $z$ plane, and the columns $u_{-,1}(x, t; z)$ and $u_{+,2}(x, t; z)$ are analytic in the domain $D^-$. 

**Corollary 1.** Let the unique solution $\phi_\pm(x, t; z)$ of Eqs. (2.4) satisfy condition (2.11). Then the analyticity of the Jost functions $\phi_\pm(x, t; z)$ and the functions $u_\pm(x, t; z)$ is consistent, i.e., the columns $\phi_{+,1}(x, t; z)$ and $\phi_{-,2}(x, t; z)$ are analytic in the domain $D^+$, and the columns $\phi_{+,2}(x, t; z)$ and $\phi_{-,1}(x, t; z)$ are analytic in the domain $D^-$. 

1616
Theorem 3 (Liouville formula). Let the \( n \)-dimensional vector \( Y(x) \) be a solution of the homogeneous linear differential equation \( Y'(x) = M(x)Y(x) \), where \( M(x) \) is an \( n \times n \) matrix. Then \( (\det Y)_x = \text{tr} M \det Y \), and consequently

\[
\det Y(x) = \det Y(x_0) \exp \left\{ \int_{x_0}^{x} \text{tr} M(\zeta) \, d\zeta \right\}.
\]

Corollary 2. Let the matrix \( M(x) \) in the equation in Theorem 3 have a zero trace. Then \( (\det Y)_x = 0 \).

2.4. Scattering matrix, scattering coefficients, and reflection coefficients. In this subsection, we discuss the relation between the Jost functions \( \phi_\pm \) and obtain the reflection coefficients.

With the matrices \( X(z) \) and \( T(z) \) in Lax pair (1.2) taken into account, we have \( (\det \phi)_x = (\det \phi)_t = 0 \) by Theorem 3, and hence

\[
(\det \phi_{\pm}(x; t; z)) = E_{\pm}(z) = \gamma(z), \quad z \in \Sigma.
\]

Setting \( \Sigma_0 = \Sigma - \{ \pm iq_0 \} \), from the Liouville formula, we find that \( \phi_{\pm} \) are fundamental solutions of Lax pair (2.2). This means there exists a constant matrix \( S(z) \) (i.e., independent of \( x \) and \( t \)) such that

\[
\phi_{+}(x; t; z) = \phi_{-}(x; t; z)S(z), \quad z \in \Sigma_0,
\]

and consequently

\[
\phi_{+,1} = s_{11}\phi_{-,1} + s_{21}\phi_{-,2}, \quad \phi_{+,2} = s_{12}\phi_{-,1} + s_{22}\phi_{-,2},
\]

where \( s_{ij} \ (i, j = 1, 2) \) are elements of the matrix \( S(z) \). From (2.11), we obtain \( \det S(z) = 1 \). Further, we discuss the analyticity of the matrix \( S(z) \).

Proposition 2. Let \( q - q_\pm \in L^1(\mathbb{R}^+) \). Then the elements \( s_{11} \) and \( s_{22} \) are respectively analytic in \( D^+ \) and \( D^- \) and continuous in \( D^+ \cup \Sigma_0 \) and \( D^- \cup \Sigma_0 \). The off-diagonal elements of \( S(z) \), although not analytic, are continuous in \( \Sigma_0 \).

Proof. In accordance with (2.18), we have

\[
s_{11}(z) = \frac{\text{Wr}(\phi_{+,1}; \phi_{-,2})}{\gamma}, \quad s_{22}(z) = \frac{\text{Wr}(\phi_{-,1}; \phi_{+,2})}{\gamma},
\]

\[
s_{12}(z) = \frac{\text{Wr}(\phi_{+,2}; \phi_{-,2})}{\gamma}, \quad s_{21}(z) = \frac{\text{Wr}(\phi_{-,1}; \phi_{+,1})}{\gamma},
\]

where \( \gamma(z) = \det E_{\pm}(z) = 1 + q_0^2/z^2 \). Applying Corollary 1 completes the proof of the proposition.

Finally, we introduce the reflection coefficients, which play an important role in the inverse scattering problem:

\[
\rho(z) = \frac{s_{21}(z)}{s_{11}(z)}, \quad \bar{\rho}(z) = \frac{s_{12}(z)}{s_{22}(z)}, \quad z \in \Sigma.
\]

2.5. Symmetry of the scattering matrix and the Jost eigenfunction. We note that the scattering problem contains two symmetries, which are related to the values of the Jost eigenfunctions on each of the sheets of the Riemann surface, and the values of these Jost eigenfunctions eventually affect the discrete spectral and residual conditions. The two symmetries are \((k, \lambda) \rightarrow (k^*, \lambda^*)\) and \((k, \lambda) \rightarrow (k, -\lambda)\) in the \( k \) plane, and they map to \( z \rightarrow z^* \) and \( z \rightarrow -q_0^2/z \) in the \( z \) plane by (2.9).
Proposition 3. The symmetries of the Jost functions $\phi_\pm$ for $z \in \Sigma$ are given by the formulas

$$
\phi_\pm(z) = -\sigma_0 \phi_\pm^*(z^*) \sigma_0, \quad \phi_\pm(z) = \frac{i}{z} \phi_\pm \left(-\frac{q_0^2}{z}\right) \sigma_3 Q_\pm,
$$

(2.22)

where $\sigma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In terms of the columns, these formulas are written as

$$
\phi_{\pm,1}(z) = \sigma_0 \phi_{\pm,2}^*(z^*), \quad \phi_{\pm,1}(z) = \frac{i q_0^2}{z} \phi_{\pm,2} \left(-\frac{q_0^2}{z}\right),
$$

(2.23)

$$
\phi_{\pm,2}(z) = -\sigma_0 \phi_{\pm,1}^*(z^*), \quad \phi_{\pm,2}(z) = \frac{i q_0^2}{z} \phi_{\pm,1} \left(-\frac{q_0^2}{z}\right).
$$

Proposition 4. The symmetries of the scattering matrix $S(z)$ for $z \in \Sigma$ are

$$
S^*(z^*) = -\sigma_0 S(z) \sigma_0, \quad S(z) = (\sigma_3 Q_-)^{-1} S \left(-\frac{q_0^2}{z}\right) \sigma_3 Q_+.
$$

(2.24)

Corollary 3. The relations of the scattering coefficients and reflection coefficients for $z \in \Sigma$ can be written as

$$
s_{22}(z) = s_{11}^*(z^*), \quad s_{12}(z) = -s_{21}^*(z^*),
$$

(2.25)

$$
s_{11}(z) = q_{11}(z) s_{22} \left(-\frac{q_0^2}{z}\right), \quad s_{21}(z) = q_{21}(z) s_{11} \left(-\frac{q_0^2}{z}\right),
$$

(2.26)

and

$$
\rho(z) = -\bar{\rho}^*(z^*) = \frac{q_0^2}{q_+} \rho \left(-\frac{q_0^2}{z}\right).
$$

(2.27)

2.6. Discrete spectrum and residue conditions. The discrete spectrum of the scattering problem comprises all values $k \in \mathbb{C} \setminus \Sigma$ for which there exist eigenfunctions in $L^2(\mathbb{R})$.

Further, we discuss the fact that the discrete spectrum is formed by zeros of $s_{11}(z)$ and $s_{22}(z)$ respectively with $z \in D^+$ and $z \in D^-$. We assume that $s_{11}(z)$ has $N$ simple zeros in $D^+ \cap \{z \in \mathbb{C} : \text{Im } z > 0\}$, denoted by $z_n$. In other words, $s_{11}(z_n) = 0$, but $s_{11}(z_n) \neq 0$, $n = 1, \ldots, N$ (here and hereafter, a prime denotes the derivative with respect to $z$). Taking symmetry properties (2.25) and (2.26) into account, we obtain $s_{22}(z_n^*) = s_{22}(-q_0^2/z_n) = s_{11}(-q_0^2/z_n^*) = 0$ if $s_{11}(z) = 0$, which yields the set of points of the discrete spectrum

$$
Z^d = \left\{ z_n, -\frac{q_0^2}{z_n}, \frac{q_0^2}{z_n}, -\frac{q_0^2}{z_n^*}, -\frac{q_0^2}{z_n^*} \right\}_{n=1}^N, \quad s_{11}(z_n) = 0.
$$

(2.28)

In what follows, we consider the residue conditions required for solving the inverse problem. If $z_n$ is a simple zero of $s_{11}(z)$, then we can derive the sought relation from the first equality in (2.20),

$$
\phi_{+,1}(z_n) = b_+(z_n) \phi_{-,2}(z_n)
$$

(2.29)

or, equivalently,

$$
u_{+,1}(z_n) = e^{-2i \theta(z_n)} b_+(z_n) u_{-,2}(z_n),
$$

(2.30)

where $b_+(z_n)$ is a normalization constant. Consequently,

$$
\text{Res} \lim_{z \to z_n} \frac{u_{+,1}(z)}{s_{11}(z)} = \frac{u_{+,1}(z_n)}{s_{11}^*(z_n)} = \frac{b_+(z_n)}{s_{11}^*(z_n)} e^{-2i \theta(z_n)} u_{-,2}(z_n).
$$

(2.31)
Similarly, if $s_{22}(z_n^*) = 0$ and $s'_{22}(z_n^*) \neq 0$ for $D^- \cap \{z \in \mathbb{C} : \text{Im} z < 0\}$, then

$$u_{+,2}(z_n^*) = b_-(z_n^*)e^{2i\theta(z_n^*)}u_{-,1}(z_n^*),$$

and we obtain the residue condition from the second equality in (2.20),

$$\text{Res}_{z=z_n^*} \left[ \frac{u_{+,2}(z)}{s_{22}(z)} \right] = \frac{u_{+,2}(z_n^*)}{s_{22}(z_n^*)} = \frac{b_-(z_n^*)}{s'_{22}(z_n^*)} e^{2i\theta(z_n^*)}u_{-,1}(z_n^*).$$

(2.33)

For brevity, we introduce the notation

$$C_+[z_n] = \frac{b_+(z_n)}{s'_{11}(z_n)}, \quad C_-[z_n] = \frac{b_-(z_n^*)}{s'_{22}(z_n^*)}.$$ 

(2.34)

As a result, we obtain the following statement.

**Proposition 5.** The coefficients of the residue conditions are related by

$$C_+[z_n] = -C_-[z_n], \quad C_+[z_n] = \frac{z_n^2}{q_n} C_- \left[ -\frac{q_n^2}{z_n} \right].$$

(2.35)

Using the symmetry of the scattering matrix and the Jost function, we hence obtain the relations

$$C_+[z_n] = -C_-[z_n] = \frac{z_n^2}{q_n} C_- \left[ -\frac{q_n^2}{z_n} \right] = -\frac{z_n^2}{q_n} C_+ \left[ -\frac{q_n^2}{z_n} \right].$$

(2.36)

2.7. Asymptotic analysis. We can use the asymptotic property of the Jost function and the scattering data to formulate the corresponding RH problem, whose solution allows reconstructing the sought potential. In connection with this, we consider the asymptotic behavior as $z \to \infty$ and $z \to 0$. We consider the Neumann series

$$u_{\pm}(x, t; z) = \sum_{n=0}^{\infty} u_{\pm}^{[n]}(x, t; z),$$

(2.37)

where $u_{\pm}^{[0]}(x, t; z) = E_{\pm}(z)$ and

$$u_{\pm}^{[n+1]}(x, t; z) = \int_{x}^{\infty} E_{\pm}(z)e^{i\lambda(x-y)\sigma_3} \left[ E_{\pm}^{-1}(z) \Delta Q_{\pm}(y, t) u_{\pm}^{[n]}(y, t; z) \right] e^{-i\lambda(x-y)\sigma_3} dy.$$ 

Similarly to what was done in [44], using the results in [45], we obtain

$$u_{\pm}^{[n+1],d}(x; z) = \left\{ \begin{array}{ll}
0(1 + (q_0/z)^2) & z \to \infty, \\
\left( \frac{u_{\pm}^{[n],d}(z)}{z} \right) + O \left( \frac{u_{\pm}^{[n],d}(z)}{z^2} \right), & z \to 0,
\end{array} \right.$$ 

$$O(z u_{\pm}^{[n],d}(z)) + O(z^2 u_{\pm}^{[n],d}(z)) = 0(z u_{\pm}^{[n],d}(z)) + O(z u_{\pm}^{[n],d}(z)), \quad z \to 0,$$
Moreover, for all \( n \)\):

\[
\begin{align*}
& u_{[n+1],0}^\prime(x, z) = \\
& = \frac{i\sigma_3 Q_\pm(y)}{z(1 + (q_0/y^2)^2)} \int_{x \to \pm \infty} \Delta Q_\pm(y) u_{[n],0}^\prime(y, z) dy + \\
& + \frac{1}{1 + (q_0/y^2)^2} \int_{x \to \pm \infty} e^{i\lambda(x-y)\sigma_3} \left( \Delta Q_\pm(y, t) u_{[n],d}^\prime(y, z) - \frac{i\sigma_3 Q_\pm(y)}{z} \Delta Q_\pm(y) u_{[n],0}^\prime(y, z) \right) dy.
\end{align*}
\]

where the superscripts \( d \) and \( o \) on \( u_\pm \) denote the diagonal and off-diagonal elements of \( u_\pm \) and we introduce the notation \( e^{i\lambda(x-y)\sigma_3} A = e^{i\lambda(x-y)\sigma_3} A e^{i\lambda(x-y)\sigma_3} \) for the \( 2 \times 2 \) matrix \( A \). At this stage, we also temporarily regard \( t \) as a dummy variable. From the formulas presented above, we can conclude that

\[
\begin{align*}
u_{[n],d}^0(x, t, z) &= O(1), & u_{[n],0}^0(x, t, z) &= O(1/z), & z \to \infty, \\
u_{[n],d}^0(x, t, z) &= O(1), & u_{[n],0}^0(x, t, z) &= O(1/z), & z \to 0.
\end{align*}
\tag{2.38}
\]

Moreover, for all \( n \in \mathbb{N} \), we have

\[
\begin{align*}
u_{[n+1],d}^{2n}(z) &= O\left(\frac{1}{z^n}\right), & u_{[n+1],0}^{2n}(z) &= O\left(\frac{1}{z^n+1}\right), \\
u_{[n+1],d}^{2n+1}(z) &= O\left(\frac{1}{z^n+1}\right), & u_{[n+1],0}^{2n+1}(z) &= O\left(\frac{1}{z^n+1}\right)
\end{align*}
\tag{2.39}
\]

as \( z \to \infty \) and

\[
\begin{align*}
u_{[n+1],d}^{2n}(z) &= O(z^n), & u_{[n+1],0}^{2n}(z) &= O(z^{n-1}), \\
u_{[n+1],d}^{2n+1}(z) &= O(z^n), & u_{[n+1],0}^{2n+1}(z) &= O(z^n).
\tag{2.40}
\end{align*}
\]

as \( z \to 0 \).

**Proposition 6.** The asymptotic behaviors of the Jost functions in the corresponding limits with respect to \( z \) have the forms

\[
u_\pm(x, t, z) = \begin{cases} \\ I + O\left(\frac{1}{z}\right), & z \to \infty, \\ i\frac{\sigma_3 Q_\pm}{z} + O(1), & z \to 0. \\
\end{cases}
\tag{2.41}
\]

**Proof.** We present the proof for \( z \to \infty \); the proof in the case \( z \to 0 \) is similar. We consider the series

\[
u_\pm = \nu_\pm^{(0)} + \frac{\nu_\pm^{(1)}}{z} + O\left(\frac{1}{z}\right), & z \to \infty.
\tag{2.42}
\]

Substituting it in (2.15) and comparing functions of like powers of \( z \), we find that \( \nu_\pm^{(0)} \) is a diagonal matrix independent of \( x \) and \( t \). Therefore, we can exchange the integration order for the function \( \nu_\pm(z) \) and let \( z \) and \( x \) tend to infinity. As a result of this analysis, we obtain

\[
\lim_{z \to \infty} \lim_{x \to \pm \infty} \nu_\pm = \lim_{x \to \pm \infty} \lim_{z \to \infty} \left( \nu_\pm^{(0)} + O\left(\frac{1}{z}\right) \right) = \nu_\pm^{(0)},
\tag{2.43}
\]

and (2.14) implies that \( \nu_\pm^{(0)} = I \). We note that the function \( E_\pm \) is also a function of \( z \). The proposition is thus proved.
Proposition 7. The asymptotic behavior of the scattering matrix in different limits on the $z$ plane has the form

$$S(z) = \begin{cases} I + O\left(\frac{1}{z}\right), & z \to \infty, \\ \text{diag}\left(\frac{q_-}{q_+}, \frac{q_+}{q_-}\right) + O(z), & z \to 0. \end{cases} \tag{2.44}$$

Proof. Taking Proposition 5 and the first formula in (2.20) into account, we obtain

$$s_{11}(z) = \frac{\text{Wr}(\phi_{+1}, \phi_{-2})}{\gamma} = \frac{\text{Wr}(u_{+1}, u_{-2})}{1 + q_0^2/z^2} = \begin{cases} \det\left(1 + O(1/z)\right) = 1 + O\left(\frac{1}{z}\right), & z \to \infty, \\ \det\left(\frac{O(1)}{1 + O(1/z^2)}\right) = 1 + O\left(\frac{1}{z}\right), & z \to \infty, \\ \det\left(\frac{O(1)}{1 + O(1/z^2)}\right) = 1 + O\left(\frac{1}{z}\right), & z \to \infty, \\ \frac{q_-}{q_+} + O(z), & z \to 0. \end{cases} \tag{2.45}$$

The proof for the other elements is similar.

3. Inverse scattering problem in the case of simple poles

3.1. Generalized RH problem. To formulate a generalized RH problem, we must redefine the piecewise-analytic functions in accordance with the analyticity of the scattering matrix and the eigenfunctions to be able to study the behavior of the scattering matrix and the eigenfunctions in the regions $D^+$ and $D^-$. We define the piecewise-meromorphic matrices

$$M^+(z) = \begin{cases} M^+(z), & z \in D^+, \\ M^-(z), & z \in D^-. \end{cases} \tag{3.1}$$

$$M^+(z) = \left(\frac{u_{+1}(z)}{s_{11}(z)}, u_{-2}(z)\right), \quad M^-(z) = \left(\frac{u_{-1}(z)}{s_{11}(z)}, \frac{u_{+2}(z)}{s_{22}(z)}\right).$$

The RH Problem. The multiplicative matrix RH problem is posed as follows. Find a matrix $M(z)$ of form (3.1) that satisfies the equation

$$M^-(x, t; z) = M^+(x, t; z)(I - G(x, t; z)) \tag{3.2}$$

with the jump matrix

$$G(x, t; z) = e^{i\theta(z)\sigma_3} \begin{pmatrix} 0 & -\bar{\rho}(z) \\ \rho(z) & \rho(z)\bar{\rho}(z) \end{pmatrix} \tag{3.3}$$

where $e^{i\theta(z)\sigma_3} A = e^{i\theta(z)\sigma_3} e^{i\theta(z)\sigma_3}$, and has the asymptotic behavior

$$M^\pm(x, t; z) = \begin{cases} \begin{pmatrix} 1 + O\left(\frac{1}{z}\right) \\ \frac{i}{\sigma_3}Q_+ + O(1) \end{pmatrix}, & z \to \infty, \\ \begin{pmatrix} I + O\left(\frac{1}{z}\right) \\ \frac{i}{\sigma_3}Q_+ + O(1) \end{pmatrix}, & z \to 0. \end{cases} \tag{3.4}$$
For convenience in subsequently solving the RH problem, we introduce the notation

\[ \xi_n = \begin{cases} 
  z_n, & n = 1, \ldots, N, \\
  -q_0^2/z_n^* & n = N+1, \ldots, 2N, 
\end{cases} \quad \dot{\xi}_n = -\frac{q_0^2}{\xi_n^n} \] (3.5)

where the \( z_n \) are zeros of \( s_{11}(z) \).

**Theorem 4.** The solution of matrix RH problem (3.2)–(3.4) with \( z \in \mathbb{C} \setminus \Sigma \) can be written as

\[ M(x, t; z) = I + \frac{i}{z} \sigma_3 Q_- + \sum_{n=1}^{2N} \left( \text{Res}_{z=\xi_n} M(z) \right) \frac{1}{z - \xi_n} + \frac{1}{2\pi i} \int_{\Sigma} \frac{(MG)(x, t; \zeta)}{\zeta - z} \, d\zeta, \] (3.6)

where \( \Sigma \) is the contour shown in Fig. 1.

**Proof.** Matrix RH problem (3.2)–(3.4) can be regularized by subtracting the asymptotic behaviors and pole contributions. We then have

\[ M^-(x, t; z) = I - \frac{i}{z} \sigma_3 Q_- - \sum_{n=1}^{2N} \frac{\text{Res}_{z=\xi_n} M^-(z)}{z - \xi_n} = M^+(x, t; z) - \frac{i}{z} \sigma_3 Q_- - \sum_{n=1}^{2N} \left[ \frac{\text{Res}_{z=\xi_n} M^-(z)}{z - \xi_n} + \frac{\text{Res}_{z=\xi_n} M^+(z)}{z - \xi_n} \right] - M^+(z) G(z). \] (3.7)

We know that the left-hand side of this equation is analytic in \( D^- \) and the first four terms in the right-hand side are analytic in \( D^+ \). We can then solve (3.7) using the Plemelj formula. We note that the Cauchy projectors \( P_\pm \) over \( \Sigma \) are defined by

\[ P_\pm[f](z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\zeta)}{\zeta - (z \pm i0)} \, d\zeta, \]

where \( z \pm i0 \) denotes the left/right limits for \( z \in \Sigma \). According to the properties of the Cauchy operator, if \( f^\pm \) are respectively analytic in \( D^\pm \), then we have \( P_+ f^\pm = f^\pm \) and \( P_- f^+ = P_- f^- = 0 \). Applying operator (3.7), we obtain (3.6). The theorem is proved.

### 3.2. Reconstructing the potential.

We can express the solution of the RH problem for a closed system in terms of the residue condition in (3.6). It can be seen from definition (3.1) that the first column of \( M \) has poles at the points \( z = z_n \) and \( z = -q_0^2/z_n^* \) and the second column has poles at the points \( z = z_n^* \) and \( z = -q_0^2/z_n \):

\[ \text{Res}_{z=\xi_n} M^+ = (C_+[\xi_n] e^{-2\theta(\xi_n)} u_{-2}(\xi_n), 0), \]

\[ \text{Res}_{z=\xi_n} M^- = (0, C_-[\xi_n] e^{2\theta(\xi_n)} u_{-1}(\xi_n)), \] (3.8)

Further, for \( z = \xi_s \) (\( s = 1, \ldots, 2N \)), we have

\[ u_{-2}(\xi_s) = \left( \begin{array}{c} iq_s/\xi_s \\ 1 \end{array} \right) + \sum_{n=1}^{2N} C_-[\xi_s] e^{2\theta(\xi_s)} u_{-1}(\xi_n) + \frac{1}{2\pi i} \int_{\Sigma} \frac{(M^+ G)_2(\xi)}{\xi - \xi_s} \, d\xi. \] (3.9)

The last relation in (2.23) then leads to

\[ u_{-2}(x, t; \xi_s) = \frac{iq_s}{\xi_s} u_{-1}(x, t; \xi_s), \quad s = 1, \ldots, 2N. \] (3.10)
Substituting this in (3.9), we obtain

\[
\sum_{n=1}^{2N} \left( \frac{C_-[\xi_n]e^{2i\theta(\xi_n)}}{\xi_s - \xi_n} - \frac{iq_-/\xi_s}{\xi_s} \delta_{sn} \right) u_{-1}(x,t;\hat{\xi}_n) + \left( -\frac{iq_-/\xi_s}{1} \right) + \frac{1}{2\pi i} \int_{\Sigma} \left( \frac{M^+G(z;\xi)}{\xi - \xi_s} \right) d\xi = 0, \tag{3.11}
\]

where \(\delta_{sn}\) is the Kronecker symbol. Formula (3.11) gives \(2N\) equations for \(2N\) unknowns. As a result, we obtain \(2N\) solutions \(u_{-1}(x,t;\hat{\xi}_n) (s = 1, \ldots, 2N)\), and the solutions \(u_{-2}(x,t;\xi_s)\) are obtained using formula (3.10). The function \(M(x,t;z)\) is thus derived from the scattering data by substituting \(u_{-1}(x,t;\hat{\xi}_n)\) and \(u_{-2}(x,t;\xi_s)\) in (3.8) and then substituting (3.8) in (3.6).

We complete the task of reconstructing the potential.

**Theorem 5.** The potential function with simple poles for the mLL equation with nonzero boundary conditions is obtained by the formula

\[
q(x,t) = q_0 \sum_{n=1}^{2N} C_-[\xi_n]e^{2i\theta(\xi_n)} + \frac{1}{2\pi i} \int_{\Sigma} (M^+(z;\xi)G(z;\xi)) d\xi, \tag{3.12}
\]

where \(\xi_n\) is given by (3.5) and the \(u_{-1,1}(\hat{\xi}_n)\) are determined by the equation

\[
\sum_{n=1}^{2N} \left( \frac{C_-[\xi_n]e^{2i\theta(\xi_n)}}{\xi_s - \xi_n} - \frac{iq_-/\xi_s}{\xi_s} \delta_{sn} \right) u_{-1}(\hat{\xi}_n) + \frac{1}{2\pi i} \int_{\Sigma} \left( \frac{M^+(z;\xi)G(z;\xi)}{\xi - \xi_s} \right) d\xi = 0.
\]

**Proof.** Let \(M(x,t;z)e^{\sigma_3(x,t,z)z}\) be a solution of (3.2). It hence follows that

\[
M_x(x,t;z) + M(x,t;z) \left( \frac{1}{2} i\sigma_3 z + \frac{1}{2} i\sigma_0^2 \sigma_3 \right) = \left( \frac{1}{2} i\sigma_3 z - \frac{1}{2} i\sigma_0^2 \sigma_3 + Q \right) M(x,t;z). \tag{3.13}
\]

Expanding \(M(x,t;z)\) in a Taylor series,

\[
M(x,t;z) = I + \frac{1}{z} M^{(1)}(x,t;z) + O \left( \frac{1}{z^2} \right), \quad z \to \infty, \tag{3.14}
\]

we obtain an expression for \(M^{(1)}(x,t;z)\) using (3.6) and (3.8):

\[
M^{(1)}(x,t;z) = i\sigma_3 q_0 - \frac{1}{2\pi i} \int_{\Sigma} M^+(x,t;\zeta)G(x,t;\zeta) d\zeta + 
\sum_{n=1}^{2N} \left( C_+[\xi_n]e^{-2i\theta(\xi_n)}u_{-2}(\xi_s), C_-[\xi_n]e^{2i\theta(\xi_n)}u_{-2}(\hat{\xi}_n) \right). \tag{3.15}
\]

Substituting (3.14) in (3.13) and comparing the coefficients of \(z^0\), we obtain the result \((i/2) [M^{(1)}, \sigma_3] = Q\), which means that \(q(x,t) = -i(M^{(1)})_{12}\). Using this equation and (3.15), we complete the proof of the theorem.

**3.3. Trace formula and theta condition.** In this section, we express the scattering coefficients \(s_{11}(z)\) and \(s_{22}(z)\) in terms of the reflection coefficients \(r(z)\) and \(\hat{r}(z)\) and the points of the discrete spectrum \(Z^d\) and derive the so-called trace formula. We simultaneously calculate the phase difference of the boundary conditions \(q_+\) and \(q_-\), i.e., obtain the theta conditions [41].
It follows from Proposition 1 that \( s_{11}(z) \) and \( s_{22}(z) \) are respectively analytic in \( D^+ \) and \( D^- \). We consider the functions

\[
\vartheta^+(z) = s_{11}(z) \prod_{n=1}^{2N} \frac{(z - z_n^*)(z + q_0^2/z_n)}{(z - z_n)(z + q_0^2/z_n^*)},
\]

\[
\vartheta^-(z) = s_{22}(z) \prod_{n=1}^{2N} \frac{(z - z_n)(z + q_0^2/z_n^*)}{(z - z_n^*)(z + q_0^2/z_n)}.
\]

We can show that \( \vartheta^\pm(z) \) are analytic functions and have no zeros in the corresponding regions \( D^\pm \). Asymptotic formula (2.44) as \( z \to \infty \) implies that \( \vartheta^\pm(z) \to 1 \) as \( z \to \infty \). Calculating the determinants of (2.18), we obtain

\[
\det S(z) = s_{11}(z)s_{22}(z) - s_{12}(z)s_{21}(z) = 1, \quad z \in \Sigma,
\]

which leads to

\[
\vartheta^+(z)\vartheta^-(z) = \frac{1}{1 - \rho(z)\bar{\rho}(z)}, \quad z \in \Sigma.
\]

We take the logarithms,

\[
\log \vartheta^+(z) + \log \vartheta^-(z) = -\log[1 - \rho(z)\bar{\rho}(z)], \quad z \in \Sigma.
\]

Further, using the Plemelj formula and Cauchy projectors, we obtain

\[
\log \vartheta^\pm(z) = \mp \frac{1}{2\pi i} \int_{\Sigma} \frac{\log[1 - \rho(\zeta)\bar{\rho}(\zeta)]}{\zeta - z} d\zeta, \quad z \in D^\pm.
\]

Substituting this relation in (3.16) yields the trace formula

\[
s_{11} = \exp\left(\frac{1}{2\pi} \int_{\Sigma} \frac{\log[\rho(\zeta)]}{\zeta - z} d\zeta \right) \prod_{n=1}^{2N} \frac{(z - z_n)(z + q_0^2/z_n)}{(z - z_n^*)(z + q_0^2/z_n^*)},
\]

\[
s_{22} = \exp\left(\frac{1}{2\pi} \int_{\Sigma} \frac{\log[1 - \rho(\zeta)\bar{\rho}(\zeta)]}{\zeta - z} d\zeta \right) \prod_{n=1}^{2N} \frac{(z - z_n^*)(z + q_0^2/z_n)}{(z - z_n)(z + q_0^2/z_n^*)}.
\]

We now discuss the asymptotic phase conditions for \( q_+ \) and \( q_- \). Condition (2.44) as \( z \to 0 \) shows that \( s_{11}(z) \to q_-/q_+ \). We note that

\[
\prod_{n=1}^{2N} \frac{(z - z_n)(z + q_0^2/z_n)}{(z - z_n^*)(z + q_0^2/z_n^*)} \to 1, \quad z \to 0.
\]

Hence, from (3.21), we obtain the relation as \( z \to 0 \)

\[
\frac{q_-}{q_+} = \exp\left(\frac{i}{2\pi} \int_{\Sigma} \frac{\log[1 - \rho(\zeta)\bar{\rho}(\zeta)]}{\zeta} d\zeta \right),
\]

which we can bring to the form of the theta condition

\[
\arg \frac{q_-}{q_+} = \frac{1}{2\pi} \int_{\Sigma} \frac{\log[1 - \rho(\zeta)\bar{\rho}(\zeta)]}{\zeta} d\zeta + 4 \sum_{n=1}^{N} \arg z_n.
\]

If we consider the case of a reflectionless potential, i.e., \( \rho(z) = 0 \), then the phase difference is defined by

\[
\arg \frac{q_-}{q_+} = \arg q_- - \arg q_+ = 4 \sum_{n=1}^{N} \arg z_n.
\]
3.4. Reflectionless potentials. In this section, we consider a solution of Eq. (3.2) in the particular case where the reflection coefficients are absent, $\rho(z) = \tilde{\rho}(z) = 0$. This means that there is no jump ($J = 0$) in passing from $M^+(x, t; z)$ to $M^-(x, t; z)$ along the continuous spectrum. Then (3.11) reduces to

$$\sum_{n=1}^{2N} \left( C_n \frac{\hat{\xi}_n e^{2i\theta(\xi_n)}}{\xi_s - \hat{\xi}_n} - \frac{iq}{\xi_s} \delta_{s n} \right) u_{-1}(x, t; \hat{\xi}_n) + \frac{q}{\xi_s} = 0. \quad (3.27)$$

In this case, we can solve the closed system algebraically and obtain soliton solutions.

**Proposition 8.** The solution of Eq. (3.2) with nonzero boundary conditions and without reflection coefficients is expressed as

$$q(x, t) = q_- + i \frac{\text{det} \tilde{M}}{\text{det} M}, \quad \tilde{M} = \begin{pmatrix} M & \nu \\ \omega^T & 0 \end{pmatrix}, \quad (3.28)$$

where

$$\omega = (\omega_j)_{(2N) \times 1}, \quad \omega_j = C_- [\hat{\xi}_j e^{2i\theta(x, t; \hat{\xi}_j)}, \quad \nu = (\nu_j)_{(2N) \times 1}, \quad \nu_j = \frac{iq}{\xi_j},$$

$$M = (m_{sj})_{(2N) \times (2N)}, \quad m_{sj} = \frac{\omega_j}{\xi_s - \xi_j}, \quad Y = (y_j)_{(2N) \times 1}, \quad y_j = u_{-1,1}(x, t; \hat{\xi}_j).$$

3.5. Soliton solutions. We note that for some fixed parameters $\delta_1, \delta_2, z_n$, and $N$ and coefficients $C_+ [z_n]$ in the residue conditions, we can obtain solution (3.28) in explicit form using the program Maple. Generally speaking, the form of this solution is very complicated. For brevity, we do not write the explicit form of formulas (3.28) with the chosen parameter values and only show graphically how the solutions behave and briefly analyze these images.

**Case 1.** If $N = 1$ and $z_1 = 3i/2$, then the asymptotic phase difference is equal to $2\pi$ (see (3.26)), which means that there is no phase transition. We obtain a breather solution of the mLL equation with a simple pole (see Fig. 2).

It can be seen from Fig. 2 that as the parameter $\delta_2$ decreases, the direction of wave propagation gradually turns to become parallel to the time axis. When the parameter is equal to zero, the solution is exactly parallel to the time axis, and we obtain a so-called stationary breather solution (a time-periodic breather). Because this solution has good behavior independently of the parameter values, we consider it in detail.

How does the boundary condition affect the solution behavior? From Fig. 3, we see that the periodic part of the solution gradually moves up as the boundary condition decreases, i.e., the periodic behavior of the breather solution with a simple pole is manifested in its upper part, and as the boundary condition tends to zero, the breather is finally transformed into a bright soliton.

In this case, we chose one purely imaginary eigenvalue as the discrete spectrum. We next consider a case where the spectral point has a real part.

**Case 2.** For $N = 1$, $q_- = 1$, and $C_+ [z_1] = 1$, we choose two complex eigenvalues $z_1 = 2e^{i\pi/3}$ and $z_1 = 2e^{i\pi/3}$, and the corresponding phase differences are then equal to $4\pi/3$ and $8\pi/3$ (see Fig. 4).

As can be seen from Fig. 4, the propagation of the solution is not parallel to the time axis nor the space axis ($x$ axis), and we obtain a so-called nonstationary soliton solution. With a special choice of the parameters, we can also obtain a space-periodic breather-type solution.
Fig. 2. Breather-type solutions with $N = 1$, $z_1 = 3i/2$, $C_+[Z_1] = 1$, and $q_- = 1$: (a) Tajiri–Watanabe breather with $\delta_2 = 1$, (b) breather with $\delta_2 = 0.5$, (c) breather with $\delta_2 = 0.2$, and (d) Kuznetsov–Ma breather with $\delta_2 = 0$.

Fig. 3. Breather-type solutions with $N = 1$, $z_1 = 3i/2$, and $C_+[z_1] = 1$: breathers with (a) $q_- = 0.8$, (b) $q_- = 0.5$, and (c) $q_- = 0.2$ and (d) a bright soliton with $q_- = 0.01$.

Case 3. For $N = 1$, $q_- = 1$, and $z = e^{i\pi/4}$, the asymptotic phase difference is equal to $\pi$, and the behavior of the solution can be seen in Fig. 5.
Fig. 4. Breather-type solutions with $N = 1$, $C_+ [z_1] = 1$, and $\delta_2 = 0$: (a) breather with $q_- = 1$ and $z_1 = 2e^{i\pi/3}$ and (b) its two-dimensional density plot; (c) breather with $q_- = 1$ and $z_1 = 2e^{2i\pi/3}$ and (d) its two-dimensional density plot.

Fig. 5. Breather-type solutions with $N = 1$ and $C_+ [z_1] = 1$: (a) space-breather (Akhmediev breather) with $q_- = 1$, $z_1 = e^{i\pi/4}$, and $\delta_2 = 0$ and (b) its two-dimensional density plot; (c) breather with $q_- = 1$, $z_1 = e^{i\pi/4}$, and $\delta_2 = 1$ and (d) its two-dimensional density plot.

Figure 5 shows that when there is no parameter influence ($\delta_2 = 0$), the solution of the equation changes periodically along the spatial axis, and we have a so-called space-periodic breather. When the modulus of the parameter $\delta_2$ increases, the solution is no longer parallel to the $x$ axis, and a change of sign of the parameter only changes the propagation direction of the solution.
**Case 4.** For $N = 2$, we see a weak interaction of three different solutions with a certain choice of the parameters (see Figs. 6–8).

Similarly to Case 1, the weak interaction between two breathers (see Fig. 6) gradually changes into the interaction between bright solitons (see Fig. 8) as the value of $q_-$ decreases, and the periodic behavior gradually moves up and appears in the upper part of the plots.
Fig. 8. Solution of the type of two bright solitons with the same parameters as in Fig. 6 in the case $q_- = 0.01$.

**Case 5.** Based on Case 4, we study the influence of the parameter $\delta_2$ on propagation of the solution with $N = 2$

Comparing Figs. 9a and 9b shows that the sign of the parameter $\delta_2 = \pm 1$ affects the direction of solution propagation without changing the shape and size of the solution. Comparing Figs. 10a and 10b shows that as $\delta_2$ gradually increases, the shape of solution changes and becomes irregular. In Fig. 6 with $\delta_2 = 0$, we saw that the energy of two breathers is distributed equally between them when they interact. But in Fig. 9, we understand that with $\delta_2 \neq 0$, energy is transferred between the two breathers during the interaction process.

4. Nonzero boundary conditions with second-order poles

4.1. Residue conditions. Based on the preceding analysis, we discuss the case of the equation under nonzero boundary conditions with second-order poles. This discussion is similar to the discussion in the case of simple poles, but there are many differences, including the form of the solution, the theta condition, etc. In what follows, we assume that the discrete spectrum $Z^d$ comprises second-order zeros of the scattering coefficient $s_{11}(z)$, which means that $s_{11}(z_0) = s'_{11}(z_0) = 0$ with $s''_{11}(z_0) \neq 0$. In the subsequent calculation, we use the following statement on the residue condition [46].

**Proposition 9.** If functions $f$ and $g$ are analytic in a complex region $\Omega \subset \mathbb{C}$ and $g$ has a second-order zero $z_0 \in \Omega$ and $f(z_0) \neq 0$, then the residue condition for $f/g$ can be derived from the Laurent expansion at $z = z_0$:

$$
\text{Res}_{z = z_0} \left[ \frac{f}{g} \right] = \frac{2f'(z_0)}{g''(z_0)} - \frac{2f(z_0)g'''(z_0)}{3(g''(z_0))^2}, \quad P_{-2} \left[ \frac{f}{g} \right] = \frac{2f(z_0)}{g''(z_0)}.
$$

(4.1)

The discrete spectrum points discussed below are all second-order poles. For all $z_n \in Z^d \cup D^+$, we
Fig. 9. Breather-type solutions with $C_1[z_1] = C_1[z_2] = 1$, $z_1 = 1/2 + 3i/2$, and $z_2 = -1/2 + 3i/2$ in the cases (a) $\delta_2 = 1$ and (b) $\delta_2 = -1$.

Fig. 10. Breather-type solutions with the same parameters as in Fig. 9 in the cases (a) $\delta_2 = 2$ and (b) $\delta_2 = -2$.

have $s_{11}(z_n) = s'_{11}(z_n) = 0$ with $s''_{11}(z_n) \neq 0$. With (2.20) taken into account, the normalization constants $b_+[z_n]$ (we use the same notation as above, but the expression itself differs) satisfy the equalities

$$\phi_{+,1}(x, t; z_n) = b_+[z_n] \phi_{-,2}(x, t; z_n),$$

$$\phi'_{+,1}(x, t; z_n) = d_+[z_n] \phi_{-,2}(x, t; z_n) + b_+[z_n] \phi'_{-,2}(x, t; z_n).$$

(4.2)
We have similar relations for $z_n^* \in Z^d \cup D^-$, where $s_{11}(z_n^*) = s'_{11}(z_n^*) = 0$ with $s''_{11}(z_n^*) \neq 0$:

$$
\phi_{+,2}(x, t; z_n^*) = b_+[z_n^*]\phi_{-1}(x, t; z_n^*),
$$
$$
\phi'_{+,2}(x, t; z_n^*) = d_+[z_n^*]\phi_{-1}(x, t; z_n^*) + b_+[z_n^*]\phi'_{-1}(x, t; z_n^*).
$$

(4.3)

Using (4.1) and (4.2), we consider the residue condition for $u_{+,1}(x, t; z)/s_{11}(z)$:

$$
\text{Res}_{z=z_n} \left[ \frac{u_{+,1}(z)}{s_{11}(z)} \right] = A_+[z_n] (\phi'_{-1}(z_n) + B_+[z_n]\phi_{-1}(z_n)),
$$
$$
\text{P}_{-2} \left[ \frac{u_{+,1}(z)}{s_{11}(z)} \right] = 2u_{+,1}(z_n) = A_+[z_n]\phi_{-1}(z_n),
$$

(4.4)

where for brevity, we introduce the notation

$$
A_+[z_n] = \frac{2b_+[z_n]}{s''_{11}(z_n)}, \quad B_+[z_n] = \frac{d_+[z_n]}{b_+[z_n]} - \frac{s''_{11}(z_n)}{3s''_{11}(z_n)}.
$$

Similarly, the residual condition for $u_{+,2}(x, t; z)/s_{22}(z)$ with $z_n^* \in Z^d \cup D^-$ has the form

$$
\text{Res}_{z=z_n^*} \left[ \frac{u_{+,2}(z)}{s_{22}(z)} \right] = A_-[z_n^*] (\phi'_{-1}(z_n^*) + B_-[z_n^*]\phi_{-1}(z_n^*)),
$$
$$
\text{P}_{-2} \left[ \frac{u_{+,2}(z)}{s_{22}(z)} \right] = 2u_{+,2}(z_n^*) = A_-[z_n^*]\phi_{-1}(z_n^*),
$$

(4.5)

where

$$
A_-[z_n^*] = \frac{2b_-[z_n^*]}{s''_{22}(z_n^*)}, \quad B_-[z_n^*] = \frac{d_-[z_n^*]}{b_-[z_n^*]} - \frac{s''_{22}(z_n^*)}{3s''_{22}(z_n^*)}.
$$

Using relations (2.23), we derive the symmetry condition (see [46], [47] for the details) for $z_n \in Z_d \cap D^+$

$$
A_+[z_n] = -A^*_+[z_n^*] = \frac{\Delta_n q}{\Delta_n q'} A_--\hat{\Delta}_n A_+ \left[ \frac{q_n}{z_n} \right] = \frac{\Delta_n q}{\Delta_n q'} A_+ \left[ \frac{q_n^*}{z_n^*} \right],
$$
$$
B_+[z_n] = B^*_+[z_n^*] = \frac{\Delta_n q}{\Delta_n q'} B_--\hat{\Delta}_n B_+ \left[ \frac{q_n^*}{z_n^*} \right] + \frac{2}{z_n} = \frac{\Delta_n q}{\Delta_n q'} B_+ \left[ \frac{q_n}{z_n} \right] + \frac{2}{z_n}.
$$

From (2.13), (3.5), (4.4), and (4.5), we obtain the residue conditions

$$
\text{Res} M^+_1(z) = \text{Res}_{z=\zeta_n} \left[ \frac{u_{+,1}(z)}{s_{11}(z)} \right] = A_+ [\zeta_n] e^{-2i\theta(\zeta_n)} (u'_{+,2}(\zeta_n) + \xi u_{-,2}(\zeta_n)),
$$
$$
\text{P}_{-2} M^+_1(z) = \text{P}_{-2} \left[ \frac{u_{+,1}(z)}{s_{11}(z)} \right] = A_+ [\zeta_n] e^{-2i\theta(\zeta_n)} u_{-,2}(\zeta_n),
$$
$$
\text{Res} M^-_2(z) = \text{Res}_{z=\zeta_n} \left[ \frac{u_{+,2}(z)}{s_{22}(z)} \right] = A_- [\zeta_n] e^{-2i\theta(\zeta_n)} (u'_{-,1}(\zeta_n) + \hat{\xi} u_{-,1}(\zeta_n)),
$$
$$
\text{P}_{-2} M^-_2(z) = \text{P}_{-2} \left[ \frac{u_{+,2}(z)}{s_{22}(z)} \right] = A_- [\zeta_n] e^{2i\theta(\zeta_n)} u_{-,1}(\zeta_n),
$$

where $\xi = B_+[\zeta_n] - 2i\theta'(\zeta_n)$ and $\hat{\xi} = B_- [\zeta_n] + 2i\theta'(\zeta_n)$.
4.2. The RH problem with second-order poles. In the case of second-order poles, RH problem (3.2)–(3.4) is unchanged,

\[ M^-(x, t; z) = M^+(x, t; z)(\mathbb{I} - G(x, t; z)), \]

and the asymptotic behavior and the jump matrix \( G(x, t; z) \) are the same as above. To solve regularized RH problem (4.6), we also subtract asymptotic values (3.4) and the singular contributions from the poles \( z \) in \( \mathbb{Z} \) and the asymptotic behavior and the jump matrix \( \Sigma \) is the contour shown in Fig. 1b.

Substituting symmetry (2.23) in the left-hand side of this equality, we can express its left-hand side in terms of only \( u \) of only \( M \).

Applying the Cauchy projectors and the Plemelj formulas, we rewrite RH problem (4.7) as

\[ M^+ - \mathbb{I} - \frac{i}{z} \sigma_3 Q_1 - \frac{1}{z} \sigma_3 Q_2 = \left( \sum \right) \frac{\text{Res}_{z = \xi_n} M^+}{z - \xi_n} + \frac{(P_{-2})_{z = \xi_n} M^+}{(z - \xi_n)^2} + \frac{\text{Res}_{z = \xi_n} M^-}{z - \xi_n} + \frac{(P_{-2})_{z = \xi_n} M^-}{(z - \xi_n)^2} = \]

where \( \Sigma \) is the contour shown in Fig. 1b.

4.3. Formula for reconstructing the potential with second-order poles.

Proposition 10. The potential with second-order poles for the mLL equation is defined by

\[ q(x, t) = q_+ - i \sum_{n=1}^{2N} A_n, [\xi_n] e^{2i\theta(\xi_n)} (u_{-1,1}(\xi_n) + \hat{D}_n u_{-1,1}(\xi_n)) + \frac{1}{2\pi i} \int_{\Sigma} (M^+ G)_{12}(\xi) d\xi. \]

Proof. Taking \( M = m^+ \) and considering the second column in matrix (4.8) at the discrete spectrum points \( z = \xi_s (s = 1, \ldots, 2N) \), we obtain

\[ u_{-2}(z) = \left( \frac{iq_+/z}{1} \right) + \sum_{n=1}^{2N} \hat{C}_n(z) \left[ u_{-1,1}(\xi_n) + \left( \hat{D}_n + \frac{1}{z - \xi_n} \right) u_{-1,1}(\xi_n) \right] + \frac{1}{2\pi i} \int_{\Sigma} \frac{(M^+ G)_{12}(\xi)}{\xi - z} d\xi. \]

Substituting symmetry (2.23) in the left-hand side of this equality, we can express its left-hand side in terms of only \( u_{-1} \) with \( z = \xi_s (s = 1, \ldots, 2N) \):

\[ \sum_{n=1}^{2N} \left\{ \hat{C}_n(\xi_s) u_{-1,1}(\xi_n) + \left[ \hat{C}_n(\xi_k) \left( \hat{D}_n + \frac{1}{\xi_s - \xi_n} \right) - \frac{iq_+}{i\xi_s} \delta_{s,n} \right] \right\} u_{-1,1}(\xi_n) = \]

\[ = - \left( \frac{iq_+/z}{1} \right) - \frac{1}{2\pi i} \int_{\Sigma} \frac{(M^+ G)_{12}(\xi)}{\xi - \xi_k} d\xi. \]
Here,\[ C_n(z) = \frac{A_+{\hat{\xi}_n}}{z - \xi_n} e^{-2i\theta(\xi_n)}, \quad D_n = B_+{\hat{\xi}_n} - 2i\theta'(\xi_n), \]
\[ \hat{C}_n(z) = \frac{A_-{\hat{\xi}_n}}{z - \xi_n} e^{2i\theta(\xi_n)}, \quad \hat{D}_n = B_-{\hat{\xi}_n} + 2i\theta'(\hat{\xi}_n). \]

Further, we find the derivative of the function \( u_{-2} \) with respect to \( z \) at the discrete spectrum point \( z = \xi_s \),\[ u_{-2}'(z) = \left( -\frac{i\sigma_- \hat{Q}_-}{1} \right) - \sum_{n=1}^{2N} \frac{\hat{C}_n(z)}{z - \xi_n} \left[ u_{-1}'(\xi_n) + \left( \hat{D}_n + \frac{2}{z - \xi_n} \right) u_{-1}(\xi_n) \right] + \frac{1}{2\pi i} \int_{\Sigma} \frac{(M^+G)^2(\xi)}{(\xi - z)^2} d\xi. \quad (4.12) \]

Differentiating the right-hand side of (2.23) with respect to \( z \), we then obtain the relation
\[ u_{-2}'(z) = -\frac{i\sigma_- \hat{Q}_-}{z^2} u_{-1}(\xi_s) + \frac{i\sigma_- \hat{Q}_-}{z^2} u_{-1}'(\xi_s) + \frac{1}{2\pi i} \int_{\Sigma} \frac{(M^+G)^2(\xi)}{(\xi - \xi_s)^2} d\xi. \quad (4.14) \]

Similarly to the proof of Theorem 5, because of the asymptotic behavior of \( M(x, t; z) \) for the term of the Taylor expansion, we obtain
\[ M^{(1)}(x, t; z) = i\sigma_3 Q_- - \frac{1}{2\pi i} \int_{\Sigma} M^+(x, t; \zeta) G(x, t; \zeta) d\zeta + \sum_{n=1}^{2N} \left[ A_+{\hat{\xi}_n} e^{-2i\theta(\xi_n)} \left( u_{-2}'(\xi_n) + \frac{1}{2\pi i} \int_{\Sigma} \frac{(M^+G)^2(\xi)}{(\xi - \xi_s)^2} d\xi \right) \right]. \]

Comparing the coefficients of \( z^0 \), we can write the final expression for \( q(x, t) \) in form (4.9).

4.4. Trace formula and theta condition. The trace formula and the theta condition in the case of second-order poles differ from those in the case of simple poles. We derive the trace formula. The scattering coefficients \( s_{11}(z) \) and \( s_{22}(z) \) are expressed in terms of discrete eigenvalues and the reflection coefficient. When the discrete spectrum points \( z_n \) and \(-q_0^2/z_n^* \) are second-order zeros of \( s_{11}(z) \), the function
\[ \Xi^+(z) = s_{11}(z) \prod_{n=1}^{2N} \frac{(z - z_n^*)^2(z + q_0^2/z_n^*)}{(z - z_n)^2(z + q_0^2/z_n^*)} \quad (4.16) \]
is analytic in \( D^+ \) and has no zeros in that region. Similarly,
\[ \Xi^-(z) = s_{22}(z) \prod_{n=1}^{2N} \frac{(z - z_n^*)^2(z + q_0^2/z_n^*)}{(z - z_n)^2(z + q_0^2/z_n^*)} \quad (4.17) \]
is analytic in $D^-$ and has no zeros in that region. Obviously, \( \Xi^+(z)\Xi^-(z) = s_{11}(z)s_{22}(z) \) for \( z \in \Sigma \). Moreover, it follows from (3.18) that

\[
\Xi^+(z)\Xi^-(z) = \frac{1}{1 - \rho(z)\bar{\rho}(z)}, \quad z \in \Sigma. \tag{4.18}
\]

Similarly to the preceding discussion, taking logarithms and applying the Cauchy operator, we obtain

\[
s_{11}(z) = \exp \left( -\frac{1}{2\pi i} \log[1 - (\rho\bar{\rho})(\zeta)] \right) \prod_{n=1}^{2N} \frac{(z - z_n)^2(z + q_n^2/z_n^*)^2}{(z - z_n^*)^2(z + q_n^2/z_n)^2},
\]

\[
s_{22}(z) = \exp \left( -\frac{1}{2\pi i} \log[1 - (\rho\bar{\rho})(\zeta)] \right) \prod_{n=1}^{2N} \frac{(z - z_n^*)^2(z + q_n^2/z_n^*)^2}{(z - z_n)^2(z + q_n^2/z_n)^2}.
\]

Finally, we can write the so-called theta condition, i.e., the asymptotic phase difference that determines the boundary value, in terms of the scattering coefficients,

\[
\arg \frac{q_-}{q_+} = \arg q_- - \arg q_+ = \frac{1}{2\pi} \int_{\Sigma} \frac{\log[1 - \rho(\zeta)\bar{\rho}(\zeta)]}{\zeta} \, d\zeta + 8 \sum_{n=1}^{N} \arg z_n.
\]

4.5. Soliton solutions in the case of second-order poles.

**Proposition 11.** The solution of the mLL equation with second-order poles in the reflectionless case can be written as

\[
q(x, t) = q_- + i \frac{\det(M_{\nu})}{\det M}, \tag{4.19}
\]

where

\[
M = \begin{pmatrix} M^{(11)} & M^{(12)} \\ M^{(21)} & M^{(22)} \end{pmatrix}, \\
M^{(ij)} = (m_{kn}^{(ij)})_{(2N) \times (2N)},
\]

\[
m_{kn}^{(11)} = \hat{C}_n(\xi_k) \left( \hat{D}_n + \frac{1}{\xi_k - \xi_n} \right) - i q_- \frac{\xi_k}{\xi_k^2} \delta_{k,n}, \quad m_{kn}^{(12)} = \hat{C}_n(\xi_k),
\]

\[
m_{kn}^{(21)} = \frac{\hat{C}_n(\xi_k)}{\xi_k - \xi_n} \left( \hat{D}_n + \frac{2}{\xi_k - \xi_n} \right) - i q_- \frac{\xi_k}{\xi_k^2} \delta_{k,n}, \quad m_{kn}^{(22)} = \frac{\hat{C}_n(\xi_k)}{\xi_k - \xi_n} + \frac{i q_- q_n^2}{\xi_k^2} \delta_{k,n},
\]

\[
\mu_n^{(1)} = A_- [\xi_n] e^{2i \theta(\xi_n)} \hat{D}_n, \quad \mu_n^{(2)} = A_- [\xi_n] e^{2i \theta(\xi_n)}, \quad \nu_n^{(1)} = -\frac{i q_-}{\xi_n}, \quad \nu_n^{(2)} = -\frac{i q_-}{\xi_n^2}.
\]

We chose appropriate parameters and used the program Maple to construct plots of solution (4.19), based on which we analyzed the influence of the parameters on the solution behavior.

It can be seen from Fig. 11 that with \( \delta_2 = 0 \), the solution behaves as two interacting breathers. As \( \delta_2 \) gradually increases, the solution behavior becomes irregular. If \( \delta_2 < 0 \), then the wave propagation direction changes, but its shape and size remain unchanged.

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Fig. 11. Wave solutions of the breather type with $N = 1$, $z_1 = 3i/2$, $A_+|z_1| = B_+|z_1| = 1$, and $q_- = 1$: breathers with (a) $\delta_2 = 0$, (b) $\delta_2 = 1$, (c) $\delta_2 = -1$, and (d) $\delta_2 = 1.5$.

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