Analyzing the Download Time of Availability Codes

MEHMET FATIH AKTAŞ, Rutgers University
SWANAND KADHE, University of California, Berkeley
EMINA SOLJANIN, Rutgers University
ALEX SPRINTSON, Texas A&M University

Availability codes have recently been proposed to facilitate efficient storage, management, and retrieval of frequently accessed data in distributed storage systems. Such codes provide multiple disjoint recovery groups for each data object, which makes it possible for multiple users to access the same object in a non-overlapping way. However in the presence of server-side performance variability, downloading an object using a recovery group takes longer than using a single server hosting the object. Therefore it is not immediately clear whether availability codes reduce latency to access hot data. Accordingly, the goal of this paper is to analyze, using a queuing theoretical approach, the download time in storage systems that employ availability codes. For data access, we consider the widely adopted Fork-Join model with redundancy. In this model, each request arrival splits into multiple copies and completes as soon as any one of the copies finishes service. We first carry out the analysis under the low-traffic regime in which case the system consists of at most one download request at any time. In this setting, we compare the download time in systems with availability, maximum distance separable (MDS), and replication codes. Our results indicate that availability codes can reduce download time in some settings, but are not always optimal. When the low-traffic assumption does not hold, system consists of multiple inter-dependent Fork-Join queues operating in parallel, which makes the exact analysis intractable. For this case we present upper and lower bounds on the download time. These bounds yield insight on system performance with respect to varying popularities over the stored objects. We also derive an M/G/1 queue approximation for the system, and show with simulations that it performs well in estimating the actual system performance.

1 INTRODUCTION

Distributed storage systems provide reliability against node failures by storing content redundantly. Because of its simplicity, replication has traditionally been a preferred way to implement storage redundancy in legacy systems such as Google File System [19], Amazon’s Dynamo [13] or in modern systems such as Apache Cassandra [33], and Redis [47]. However replication incurs large storage overhead which makes it prohibitively costly as big data applications become more prevalent. This has forced practitioners to start replacing replication with erasure codes in order to build storage-efficient reliable systems.

In storage systems that utilize erasure codes, the data is first split into \( k \) objects of equal size and then encoded into \( n \) objects. In this case we say that the data is encoded by an \([n, k]\) erasure code. Each of the encoded objects is stored on a separate storage node, which we refer to as server. Well-known Maximum Distance Separable (MDS) codes are an important family of erasure codes. They provide the highest storage efficiency for a given level of reliability. Storage efficiency of MDS codes has made them prevalent in practice. For instance HDFS [51], Azure Storage [10], Facebook’s F4 [38] use MDS codes to store data reliably.

Although MDS codes are optimal in terms of storage efficiency, they incur large recovery overhead. Recovering a single object in an \([n, k]\) MDS coded storage requires downloading the whole set of \( k \) objects. That is why recovering from a node failure requires moving large amounts of data within the MDS coded system. Excessive network traffic generated during recovery aggravates the contention at system resources and degrades the overall system performance [45]. This problem has motivated
coding theorists to look for novel erasure codes that are recovery efficient \cite{14, 20, 40, 44, 53, 54}. An important class of such codes is Locally Recoverable Codes (LRCs) \cite{20, 40}. LRCs enable an object to be recovered by accessing only a small group of servers, called a recovery group. The number of servers in a recovery group is referred to as the code locality. LRCs have made a significant impact not only on coding theory but also on storage in practice, and have become a part of large scale production systems. For instance, Microsoft’s Azure Storage \cite{23} and Facebook’s HDFS \cite{48} deploy a type of LRC to implement reliable storage.

A special class of LRCs, which are known as availability codes, have an additional property that each object has multiple, disjoint recovery groups \cite{41, 44, 53, 56}. Availability in storage allows serving the same object to multiple users in a non-overlapping manner using separate recovery groups. For instance, consider the \( [n = 7, k = 3] \) binary Simplex code \cite{36} that encodes three objects \( \{f_1, f_2, f_3\} \) into seven by binary addition as \( \{f_1, f_2, f_3, f_1 + f_2, f_1 + f_3, f_2 + f_3, f_1 + f_2 + f_3\} \). The resulting set of encoded objects is stored across seven storage nodes. This code is said to have availability three as each of the initial objects \( f_1, f_2 \) and \( f_3 \) has three disjoint recovery groups. For example, \( f_1 \) can be recovered by reading both \( f_2 \) and \( f_1 + f_2 \) from nodes 2 and 4, or by reading both \( f_3 \) and \( f_1 + f_3 \) from nodes 3 and 5, or by reading \( f_2 + f_3 \) and \( f_1 + f_2 + f_3 \) from nodes 6 and 7. This code furthermore has locality two as each recovery group consists of at most two servers.

The notion of availability was proposed primarily to make the stored data more available, as the name suggests (see, e.g., \cite{44}). Multiple, disjoint recovery groups enable high data availability since each recovery group provides another way to download the object (with appropriate decoding). Thus availability codes make it possible to assign multiple requests for the same object to different servers without blocking any request. With high data availability, it is argued that these codes can provide low-latency access for hot data, i.e., objects that are frequently and simultaneously accessed by multiple users \cite{44}.

However, it is not immediately clear how useful the recovery groups are in reducing the download latency. This is because accessing an object through one of its recovery groups requires downloading one object from each of the recovery servers. Therefore, the download from a recovery group is complete once the encoded objects from all the servers in the recovery group are fetched. This means that the download will be slow even if the service is slow at only one of the servers. In fact, service times in modern large scale systems are known to exhibit significant variability \cite{12, 37}, and thus, the download time at a recovery group can be significantly slower than that on a single server. As an example, if service times at the servers are independent and exponentially distributed, mean time to download from a recovery group of size \( r \) will scale (approximately) by \( \ln r \) for large \( r \). This motivates the main question that we ask in this paper: can availability codes indeed reduce the latency of downloading hot data when servers experience runtime performance variability?

Parallel to the development of codes for storage- and recovery-efficient redundancy, it has been recognized that storage redundancy can be exploited for faster content access \cite{25, 26}. The key idea is to request content from both the original and redundant storage simultaneously, and to only wait for the fastest subset of the initiated downloads that are sufficient to reconstruct the desired content. This has been shown to mitigate the impact of server-side performance variability \cite{18, 25, 26}. This is connected with the question we posed above. As discussed above, downloading from a recovery group suffers from the performance variability at the servers. However recovery groups can also be used to serve download requests redundantly. Can we then compensate for the slow service at the recovery groups by exploiting the code availability and redundant requests?

Our main focus is to analyze the latency for downloading individual data objects that are encoded using an availability code. We consider the fork-join access strategy, wherein the request is forked (replicated) to the systematic server containing the requested object and all its recovery groups. While other access schemes are possible, we focus our attention to the fork-join strategy for the
following reasons. First, a request for an object that is assigned to a single server (containing the object) will typically experience smaller service time than those that are assigned to its recovery groups. Thus a user whose request is assigned to a recovery group would experience high latency. On the other hand, the fork-join access strategy treats all the requests uniformly, resulting in a form of fairness. Second, the fork-join strategy is a well known access model, and it has been shown to achieve low latency for downloading complete data set in MDS coded storage [25, 26].

**Our contributions and organization of the paper:** We analyze the download time of individual data objects that are jointly encoded with an availability code. Our focus is on the \textit{Fork-Join (FJ)} access model in which every request is copied into its systematic node and all its recovery groups upon arrival, and a request is completed once any of its copies finishes service. We analyse the download time for two arrival regimes: (i) low-traffic regime in which download requests do not overlap in the system, (ii) high-traffic regime in which requests can overlap and resource sharing is controlled by the local queues present at the storage nodes.

First we study systems with availability, replication and MDS codes under the low-traffic regime. We derive closed-form expressions for the distribution and expected value of download time in each system, which allows us to compare availability codes with several state-of-the-art erasure codes used in commercial systems. We compare the reliability, storage efficiency and average download time achieved by these codes. Furthermore we analyze the impact of important code parameters, such as the number and size of recovery groups, on the download time.

Next we move on to the high-traffic regime, in which the system consists of multiple inter-dependent Fork-Join queues. System in this case suffers from the infamous state explosion problem, and this makes it formidable to perform an exact analysis. We then proceed by establishing upper and lower bounds on the download time. Using these bounds, we derive inner and outer bounds on the system’s stability condition. This amounts to finding necessary and sufficient conditions on the request arrival rate such that the download time stays bounded.

We pay special attention on understanding the system performance in the case with extremely skewed object popularities. The reason for this is twofold. First, it represents the skewed popularity scenario observed in practice. Second, download time in this case serves as an upper bound for the case with arbitrary popularities. We start our analysis by outlining a set of important characterizations on the system behavior. These findings show that system can be approximated as an M/G/1 queue. Note that the proposed approximation becomes exact when the code locality is one, that is when the Fork-Join system implemented on the replicated storage. We proceed by improving our approximation for systems with locality two, and further refine it for systems with availability one. In our analysis, we use techniques from the literature on Markov processes and Renewal theory. Combination of the techniques we present is useful to study other Fork-Join types of queues. For instance some of the ideas presented in a conference version of this paper [3] have been used to analyze the Fork-Join access model that is defined by an \((n, 2)\) MDS code [4].

In order to better understand the tradeoff between storage efficiency and download performance, we simulated the Fork-Join content access model on various practical coding schemes. We found that storage efficiency can be traded off for lower download time. Replication codes achieve the fastest download while incurring large storage overhead, while MDS codes are optimal in storage efficiency but achieve slower download. Performance of LRCs, specifically availability codes, lie between these two extremes. This suggests that code availability and locality are useful knobs, not only to regulate the reliability and recovery efficiency but also to achieve an interplay between storage efficiency and download performance. Although it is well-studied how code availability and locality affect the storage efficiency, so far we do not understand how exactly they impact the download time. In this paper we present a step towards understanding this.
This paper is organized as follows: The rest of this section reviews the literature on Fork-Join queues that arise in coded storage. Sec. 2 explains the storage and content access model. Sec. 3 and 4 present our results on the download time respectively under low and high traffic regimes. Sec. 4 presents bounds on the download time and an M/G/1 queue approximation for the system. Sec. 5 analyzes the case with extremely skewed object popularities. Finally, Sec. 8 obtains expressions for the average download time by iterating and improving on the M/G/1 queue approximation.

Related and prior work: Literature on content download in coded storage has focused on downloading the complete set of data objects that are jointly encoded with an MDS code. This is an important research question and has been studied extensively, see e.g. [30, 34, 42] and references therein. This paper differs from this literature in two important aspects. First, we are here concerned with hot data download. In practice users are typically interested in accessing only a subset of the stored data, which is referred to as hot data. This leads to skews in object popularities as shown by numerous cluster traces collected from production systems. For instance, trace collected from a large Microsoft Bing cluster shows that 90% of the stored data is not accessed by more than one task simultaneously, while the remaining 10% is observed to be frequently and simultaneously accessed [6]. Our model for accessing content incorporates the skewed object popularities and the notion of hot data. Second, we consider storage systems with availability codes. We strive to understand the role that various parameters of availability codes play in the download time. LRCs with availability have recently replaced MDS codes in production systems, e.g. [23, 48]. It is important to understand the dynamics and performance of content download in storage systems that use these new erasure coding schemes. However content download performance of these codes has not yet been studied.

This work is a continuation of our effort to understand the impact of erasure coding on download time [3, 26, 28, 29]. Other recent studies have analyzed mainly replication and MDS codes through the lens of queuing theory, and have shown that storage redundancy has the potential to reduce the download time. Several access schemes, such as the Fork-Join (FJ) model, have been considered for content download with redundancy [18, 26, 28, 29]. Several variants of FJ model, such as the one with a centralized queue, have been considered [49]. Various distributions have been used to model service times at the servers, namely, exponential [18, 26, 49], shifted-exponential [35], new-worse-than-used and new-better-than-used distributions [50], and more general service time distributions [29]. Download time has been studied under different traffic patterns and system loads, such as low-traffic [26] and high-traffic [34] regimes, and arbitrary arrival processes [52]. Besides latency, cost of service with redundancy has also been studied [5, 28].

2 SYSTEM MODEL

2.1 Data storage model

We denote the data to be stored by $F$. Prior to storage, we first split the content $F$ into $k$ equal sized objects: $F = \{f_1, \ldots, f_k\}$ where each object $f_i$ is a symbol in some finite field $\mathbb{F}$. Then, the $k$ data objects are encoded into $n$ coded objects of equal size by an $[n, k]$ linear erasure code. We refer the reader to [36] as a reference on erasure codes. We refer the reader to [36] as a reference on erasure codes. In a linear $[n, k]$ erasure code, each coded object is a linear combination of the $k$ data objects over $\mathbb{F}$. The code is said to be systematic, if the resulting $n$ objects contain the original $k$ data objects. Throughout the paper, we assume that objects in $F$ are encoded using a systematic $[n, k]$-code. The resulting set of $n$ objects are stored across $n$ servers. Storage overhead of the code is defined as $n/k$.

We are interested in a special class of erasure codes for which each systematic server has $(r, t)$-availability [44, 53, 57]. A systematic server is one that stores one of the initial $k$ objects. A code is said to have $(r, t)$-availability if it ensures that failure of any systematic server can be recovered using one of the $t$ disjoint recovery groups of other storage servers, each of size at most $r$, where
We consider the Fork-Join (FJ) model for content access. FJ model on a system with an where 7 objects are stored on 7 servers. Note that each systematic server can be recovered from

The second level of FJ queues is formed at each recovery group. A request copy that is assigned

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to a recovery group is forked into

copies that are in service or waiting in queue, or reconstructing an object from the coded copies

as a request is completed, all its outstanding copies will be immediately removed from the system.

Consider a

Example: Consider a \((7, 3, 2, 3)\)-LRC as follows: \(\{f_1, f_2, f_3, f_1 + f_2, f_1 + f_3, f_2 + f_3, f_1 + f_2 + f_3\}\), where 7 objects are stored on 7 servers. Note that each systematic server can be recovered from

The \((r, t)\)-availability property allows accessing an object \(f_i\) in \((t+1)\) ways: either by downloading from the systematic server or by downloading all symbols in one of its \(t\) recovery groups. For instance, in the system with \((7, 3, 2, 3)\)-LRC given in the example above, \(f_1\) can be accessed either by downloading it from the systematic server or by recovering it after downloading any of the following pairs of coded symbols \(\{f_2, f_1 + f_2\}, \{f_3, f_1 + f_3\}, \{f_2 + f_3, f_1 + f_2 + f_3\}\).

\[
\text{Request } f_i \text{ for } f_i \\rightarrow \text{Si} \quad R_i^1 \quad R_i^2 \quad R_i^t
\]

Fig. 1. Fork-Join content access model for a storage that employs \((n, k, r, t)\)-LRC. Each request upon arrival is split into its systematic node and each of its \(t\) recovery groups.

2.2 Content Access Model

We consider the Fork-Join (FJ) model for content access. FJ model on a system with an \((r, t)\)-availability code consists of two levels of FJ queues. In the first level, each request upon arrival is forked (replicated) into \(t + 1\) copies. These copies are then split across the systematic server that stores the requested object and its \(t\) recovery groups. The request is said to complete once either its copy at the systematic server or a copy at any one of its recovery groups finishes service. As soon as a request is completed, all its outstanding copies will be immediately removed from the system.

The second level of FJ queues is formed at each recovery group. A request copy that is assigned to a recovery group is forked into \(r\) sub-copies and each enters the queue at its respective server. Request copy at a recovery group finishes when all its \(r\) forked sub-copies finish service and join. See Fig. 1 for an illustration of the two-level FJ access model. We assume cancellation of request copies that are in service or waiting in queue, or reconstructing an object from the coded copies downloaded from a recovery group incurs negligible delay in comparison with other delays.\(^1\)

2.3 Request Arrival and Service Model

For tractability, download requests are assumed to arrive as a Poisson process of fixed rate \(\lambda\). We model object popularities as follows: each request arrival asks for object \(f_i\) independently with probability \(p_i\) for \(i = 1, \ldots, k\), where \(0 \leq p_i \leq 1\) and \(\sum_{i=1}^{k} p_i = 1\).

We consider two arrival regimes. First one is the low-traffic regime, in which system can complete serving the current request before the next one arrives. System can therefore contain at most one request at any time. This case serves as a starting point for the download time analysis. It is also useful to understand download performance under low offered load. Second one is the high-traffic regime, in which subsequent requests might arrive before the current request(s) are

\(^1\)Indeed, it is observed from the measurements collected from Windows Azure storage system [24] that latency in reconstructing an object is several orders of magnitude smaller than the overall latency.
finished. Therefore, there needs to be a way to share system resources across the requests. We model resource sharing at the nodes with First-come First-serve (FCFS) queues.

Request copies are served at the nodes by reading and streaming the requested object. We model this by a random service time. For tractability, service times are assumed to be Exponential random variables (r.v.’s) of rate $\mu$, which we denote by $\text{Exp}(\mu)$. They are also assumed to be independently and identically distributed (i.i.d.) across different requests and servers. It is worth noting that shifted-exponential distribution has been shown to well model the empirical service times in data centers [35]. Nevertheless, exponential distribution is analytically tractable due to its memoryless property. For that reason it is commonly used to model service times in queueing theoretic analysis, as we also use it here. Insights derived from this study can serve as a starting step towards understanding the system performance under general service time distributions.

2.4 Notation and Tools
The list below presents some of the notation that we frequently use throughout the paper.

- $S$ denotes an exponential r.v. with rate $\mu$.
- Beta function is denoted as $\beta(x, y)$ and is given by $\int_0^1 v^{x-1}(1 - v)^{y-1} dv$.
- For r.v.’s $X$ and $Y$, $X \leq Y$ if and only if $\Pr\{X > s\} \leq \Pr\{Y > s\}$ for all $s$.
- $X_{n,i}$ denotes the $i$th smallest (order statistic) of $n$ i.i.d. samples drawn from a r.v. $X$.

3 ANALYSIS FOR LOW-TRAFFIC REGIME
In this section we focus our attention on the Fork-Join access model under the low-traffic regime. We begin with analyzing the contribution of recovery groups to achieve faster download. Our main interest in that is to understand the impact of code locality $r$ and availability $t$ on the performance of recovery groups and on the download speed overall. We then analyze the download time for system with availability codes and compare it with that for systems with replication and MDS codes. We finally compare availability codes with several state-of-the-art erasure codes that are deployed in practice, in terms of the download speed, storage efficiency and reliability they achieve.

3.1 Download Performance of Recovery Groups
It is natural to ask how the availability in terms of multiple, disjoint recovery groups helps to reduce the download latency. We do so by studying the download performance of recovery groups with respect to the systematic server. Towards this end, we first compute the expected download time when requests are split only into the $t$ recovery groups (without using the systematic server). Next, we find the probability that systematic node is faster in download than its recovery groups.

3.1.1 Expected Download Time of Recovery Groups. It is important to understand the download performance when only the recovery groups participate in the download. This is because the download time at recovery groups determines the time for a degraded read request. More specifically, if a user requests the object stored on a server that is undergoing a transient failure, then the request is called degraded as it needs to be served using one or more recovery groups. Since more than 90% of failures in a data center are transient [16, 48], it is crucial for an erasure code to have a small download time at its recovery groups so that degraded read requests are served faster.

We start by noting that completion time $S_{r,t}$ for a request that is split into its $t$ recovery groups (excluding its systematic server) is given by $S_{r,t} = [S_{r,r}]_{1:1}$. We next find the mean of $S_{r,t}$.

**Proposition 1.** (Proved in Appendix 10.1) The average time to download an object from $t$ recovery groups of size $r$ is given as

$$
\mathbb{E}[S_{r,t}] = \frac{1}{\mu} \sum_{i=1}^{t} \binom{t}{i} (-1)^{i-1} H_{r,i},
$$

(1)
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Fig. 2. The expected download time $E[S_{r,t}]$ versus the number of recovery groups $t$ for various values of the recovery group sizes $r$ and $\mu = 1$. Lower and upper bounds are given by (2). Note that the scales of the two figures on the $y$-axis are different. Observe that the bounds get loose as $r$ increases and both bounds become exact for $r = 1$ (not shown in the figure). Further, observe that as $r$ goes from 2 to 4, the number of recovery groups should go from 2 to 4 to ensure the expected download time of 1 unit.

where $H_n$ is the $n$th Harmonic number. Furthermore, $E[S_{r,t}]$ can be bounded as

$$
\frac{1}{\mu \cdot t} H_r \leq E[S_{r,t}] \leq \frac{\beta(t, 1/r)}{\mu \cdot r} H_r.
$$

(2)

It is worth noting that, when $r = 1$ in (2), the equalities hold for all $t \geq 1$. In Fig. 2, we numerically evaluate the tightness of the bound in (2). Bounds become loose as the code locality $r$ increases.

Remark 1. Consider an $[n, k]$-MDS coded system in which every request is copied and split across all the $n$ servers. If the systematic server for a request is unavailable, the request will be completed once any $k$ out of the remaining $n-1$ servers finish serving. Therefore in this case expected download time of the recovery groups is the $k$th order statistic among $n-1$ i.i.d. exponential r.v.’s.

Using the well-known result on mean of $k$th order-statistics of exponential r.v.’s (see e.g. [46]), expected download time at the recovery groups is given for an MDS code as

$$
E[T_{\text{MDS-}(n-1,k)}] = \frac{1}{\mu} (H_{n-1} - H_{n-1-k})
$$

(3)

3.1.2 Probability that Systematic Server is Faster than $t$ Recovery Groups. We here analyze the probability that all the recovery groups together take more time to finish serving a download request than the systematic server alone.

Proposition 2. (Proof is given in Appendix 10.2) In a system with an $(r, t)$-availability code, probability that the systematic server is faster than its repair groups in serving a download request is given under low-traffic regime as

$$
\Pr\{S_{r,t} > S\} = \frac{1}{r} \beta\left(t + 1, \frac{1}{r}\right).
$$

(4)

One can observe that this probability expression increases with $r$ and decreases with $t$, which corroborates the intuition. Fig. 3 plots the value of this probability with varying $t$ and by fixing $r$ at various values. Observe that increasing $r$ greatly increases the probability that recovery groups perform slower than systematic server. For instance, if we increase $r$ from 1 to 3, we need to increase $t$ from 1 to 5 to keep the probability of success the same (i.e., 0.5).
Similar to the above discussion, we next state expression the probability that the systematic server is faster than the recovery groups in an \([n, k]\)-MDS coded system. The proof is similar to that of Lemma 2 and is omitted.

**Proposition 3.** In a Fork-Join system for \([n, k]\) MDS coded storage, probability that the parity servers are slower than the systematic server in serving a download request is given as

\[
\Pr \{S_{n-1, k} > S\} = \frac{k}{n}. \tag{5}
\]

### 3.2 Download Time for Availability Codes

Next, we analyze the time for downloading an object, wherein we completely leverage the redundancy by replicating the request to both the systematic server as well as all its recovery groups. In the following theorem, we characterize the distribution and the first moment of the download time in a system with an \((r, t)\)-availability code.

**Theorem 1.** In a system with an \((r, t)\)-availability code and service times distributed as \(\text{Exp}(\mu)\), the download time \(T_{\text{FJ}(r, t)}\) under low-traffic regime is distributed as

\[
\Pr \{T_{\text{FJ}(r, t)} > s\} = \exp(-\mu \cdot s) (1 - (1 - \exp(-\mu \cdot s))^r)^t. \tag{6}
\]

Moreover the average download time is given as

\[
\mathbb{E} \left[ T_{\text{FJ}(r, t)} \right] = \frac{1}{\mu \cdot r} \beta \left( t + 1, \frac{1}{r} \right). \tag{7}
\]

**Proof.** First, note that a request copy assigned to a recovery group finishes when all its \(r\) forked sub-copies finish service. Hence the service time at a recovery group is distributed as \(S_{r,t}\). Next, note that a request finishes service as soon as either its copy at the systematic server or any of its copies at one of the \(t\) recovery groups finishes. Thus we have

\[
T_{\text{FJ}(r, t)} = \min \{S, [S_{r,t}]_{1:t}\}. \tag{8}
\]

We can then write

\[
\Pr \{T_{\text{FJ}(r, t)} > s\} = \Pr \{\min \{S, [S_{r,t}]_{1:t}\} > s\} \overset{(a)}{=} \Pr \{S > s\} \Pr \{S_{r,t} > s\}^t \quad \overset{(b)}{=} \Pr \{S > s\} (1 - \Pr \{S \leq s\})^t \overset{(c)}{=} \exp(-\mu \cdot s) (1 - (1 - \exp(-\mu \cdot s))^r)^t, \tag{9}
\]
where (a) and (b) follow from the independence of service times across the servers, and (c) comes from substituting \( \Pr \{ S > s \} = \exp(-\mu \cdot s) \).

Observing that \( T_{\text{FJ}(r,t)} \) is a non-negative r.v., we obtain

\[
\mathbb{E} \left[ T_{\text{FJ}(r,t)} \right] = \int_0^\infty \Pr \left\{ T_{\text{FJ}(r,t)} > s \right\} ds \overset{(a)}{=} \frac{1}{\mu \cdot r} \int_0^1 v^t \left( 1 - v \right)^{(\frac{1}{r} - 1)} dv \overset{(b)}{=} \frac{1}{\mu \cdot r} \left( t + 1, \frac{1}{r} \right),
\]

where (a) comes from substituting (9) and \( 1 - (1 - \exp(-\mu \cdot s))^r = v \), and (b) follows from the definition of the beta function given in Sec. 2.4.

\[ \square \]

**Remark 2.** Expression (7) for \( \mathbb{E} \left[ T_{\text{FJ}(r,t)} \right] \) allows us to examine how increasing the code availability \( t \) affects the average download time while the code locality is fixed to \( r \). The property \( \beta(x, y + 1) = \frac{y}{x+1} \beta(x, y) \), makes it straightforward to verify that the relative reduction in average download time per increment in \( t \) is given by \( (r(t + 1) + 1)^{-1} \), which is \( \sim 1/(r \cdot t) \) for large \( t \) and \( r \). In other words, incrementing \( t \) yields diminishing returns in reducing the average download time.

### 3.3 Comparison with Replication Codes

We here compare the download performance of availability codes and replication. Replication schemes are an important contestant of availability codes as they are used in several practical systems to implement reliable storage, see e.g., [13, 19, 33, 47]). In particular, we consider a \( t_r \)-replication code which stores \( t_r \) copies for each of the \( k \) objects.

Replication has poor storage efficiency, hence requiring larger number of servers than an erasure code to store the same number of objects. For a fair comparison, we suppose that the cumulative service rate in the system is fixed and jointly provided by homogeneous nodes. In this case, in a system of \( n \) nodes with cumulative service rate of \( n \cdot \mu \), each server runs with a service rate of \( \mu \). Let us denote the download time in a system using \( t_r \)-replication code with \( T_{\text{FJ}-t_r} \).

**Lemma 1.** (Proved in Appendix 10.3) In a system with \( t_r \)-replication code and service times distributed as \( \text{Exp}(\mu) \), average download time is given under the low-traffic regime as \( \mathbb{E} \left[ T_{\text{FJ}-t_r} \right] = 1/(t_r \cdot \mu) \). If the service rates at the nodes are adjusted to implement the same cumulative service rate as that of a system with an \((n, k, r, t)\)-LRC, then

\[
\mathbb{E} \left[ T_{\text{FJ}-t_r} \right] = \frac{k}{n \cdot \mu}.
\]

**Remark 3.** The replication factor \( t_r \) determines the fault-tolerance of the storage system. This is measured with the number of server failures that the system can tolerate before any data loss. Larger \( t_r \) results in higher fault-tolerance, but incurs heavier storage overhead. It is well-known that erasure codes (including availability codes) achieve much higher fault-tolerance for the same storage overhead [20, 44]. As an example, the \((7, 3, 2, 3)\) availability code discussed in Sec. 2.1 has the
fault-tolerance of 3 for the storage overhead of 2.3×. Whereas, 3-replication has the fault-tolerance of 2 for the storage overhead of 3×. Somewhat interestingly, in the low-traffic regime, as long as the cumulative average service rate of a \(r\)-replication system is kept the same as for an \([n, k]\) erasure coded system, average download time in the \(r\)-replication system only depends on \(k/n\), which is the inverse of the storage overhead for the \([n, k]\)-coded system.

### 3.4 Comparison with Maximum Distance Separable (MDS) Codes

An \([n, k]\) erasure code is said to be Maximum Distance Separable (MDS) if any \(k\) out of the \(n\) coded objects are sufficient to reconstruct the original \(k\) objects. Reed-Solomon codes are a common example of MDS codes \([36]\). Recent works have shown that, when downloading the entire set of \(k\) objects (instead of individual objects), MDS codes can achieve a smaller download time as compared to replication (see, e.g., \([26, 27, 49]\)). It is therefore natural to ask how MDS codes would perform in download of the individual objects.

**Example:** A \((4, 2)\) MDS code over the finite field of size three encodes two objects \(f_1\) and \(f_2\) into four as \(\{f_1, f_2, f_1 + f_2, f_1 + 2f_2\}\). Note that any data object can be recovered from any two out of the four object codes. Also, each object \(f_1\) or \(f_2\) can be recovered from either its systematic copy or any two of the remaining three. For example, \(f_1\) can be recovered either directly from its systematic copy or from any of the following pairs \(\{f_2, f_1 + f_2\}\), \(\{f_2, f_1 + 2f_2\}\), \(\{f_1 + f_2, f_1 + 2f_2\}\).

In the FJ access model for MDS coded storage, each download request is replicated into \(n\) copies upon arrival. These copies are then split across all \(n\) servers as illustrated in Fig. 4. A request is completed once either its copy at the systematic server finishes service or any \(k\) out of the remaining \(n - 1\) recovery servers jointly finish serving the request. It is easy to see that download time \(T_{\text{FJ-}(n,k)}\) in the MDS coded system is given by

\[
T_{\text{FJ-}(n,k)} = \min \{S, S_{(n-1):k}\}.
\]

This allows us to derive the download time under low-traffic regime as follows. Here as well we assume service times at the servers are i.i.d. exponential r.v.s with rate \(\mu\).

**Lemma 2.** (Proved in Appendix 10.4) In a system with an \([n, k]\)-MDS code and service times distributed as \(\text{Exp}(\mu)\), average download time is given under the low-traffic regime by

\[
\mathbb{E}\left[T_{\text{FJ-}(n,k)}\right] = \frac{k}{n \cdot \mu}.
\]  

### 3.5 Performance Comparison of Erasure Codes

In this section we compare the average download time of availability codes with several state-of-the-art erasure codes that are used in practice. We consider the following erasure codes: (i) 3-replication, which is commonly used in many distributed storage systems, e.g., Amazon’s Dynamo \([13]\), Facebook storage \([48]\); (ii) \((9, 6)\)-MDS code, which is used in Google file system \([16]\); (iii) \((10, 6, 2, 1)\)-LRC, which is used in Windows Azure Storage \([24]\); and (iv) \((7, 3, 2, 3)\)-availability code, which is constructed using the binary \((7, 3)\) simplex code\(^2\).

Taking only the download time into account while comparing different coding schemes can give misleading conclusions. For instance, storing content with a large replication factor can easily reduce the download time but will incur large storage overhead, which is another important performance metric. In order to get a holistic performance comparison, we also consider the

---

\(^2\)Binary simplex code, which is a dual of the Hamming code, forms a well-known class of availability codes with locality 2 \([9, 31]\); see Remark 5 for details.
following performance metrics in addition to average download time: (i) storage overhead and (ii) reliability in terms of mean time to data loss (MTTDL).

Recall that the storage overhead of an \([n, k]\) erasure code is given by \(n/k\). This performance metric is naturally important, as it directly impacts the storage cost in a data center. Reliability of a distributed storage system measured in terms of its MTTDL is a well-accepted performance metric [11, 21, 43]. We emphasize that, even though the storage overhead and MTTDL have been known to be important metrics in evaluating real-world distributed storage systems (see, e.g., [16, 24, 48]), the critical metric of (mean) download time is missing in the literature.

Markov models are commonly used to compute the reliability of distributed storage systems in terms of MTTDL (see, e.g., [11, 16, 21, 24, 43, 48]). Consistent with the literature, we consider a standard Markov model to evaluate the reliability in our comparison. We use a typical set of parameters as observed in Windows Azure Storage [24] to construct the Markov chain. The details are given in Appendix 10.5. It is worth noting that MTTDL of a 3-replication is typically considered as the performance objective while choosing an erasure code for a storage scheme [24, 48].

We present a detailed comparison in Table 1. Observe that the (7, 3, 2, 3)-availability code achieves smaller mean download time as compared to the state-of-the-art erasure codes, at the expense of comparatively worse storage overhead. On the other hand, it has smaller storage overhead over 3-replication, at the cost of achieving slightly worse mean download time. Indeed, it achieves a favorable trade-off between the storage overhead and the mean download time, with highest MTTDL due to large fault-tolerance. Therefore, availability codes form attractive candidates for storing hot data that requires small download latency. However, when the cumulative service rate in the system is kept fixed, (9, 6)-MDS code achieves the smallest download time as well as the smallest storage overhead, with high MTTDL. Hence, for storage systems with limited cumulative service rate, MDS codes will be favorable candidates.

### Table 1. Performance evaluation for several codes in commercial systems, compared with a (7, 3, 2, 3)-availability code. (9, 6)-MDS code is used in GFS [16] and (10, 6, 3, 1)-LRC is used in Windows Azure [24].

| No. | Erasure Code | \(E[T] \cdot \mu\) | Storage Overhead | MTTDL (in years) |
|-----|-------------|-----------------|-----------------|-----------------|
| 1.  | 3-replication | 0.3333 (0.6667) | 3x | 3.4E + 09 |
| 2.  | (9, 6)-MDS code | 0.6667 (0.6667) | 1.5x | 2.4E + 10 |
| 3.  | (10, 6, 3, 1)-LRC | 0.6 (0.8333) | 1.67x | 1.7E + 11 |
| 4.  | (7, 3, 2, 3)-availability code | 0.4571 (0.711) | 2.33x | 1.3E + 12 |

4 RESULT STATEMENTS FOR THE HIGH ARRIVAL RATE REGIME

In this section we remove the assumption that at most one request is present in the system at any time. Queues at the storage servers therefore might build up in this case and we need to account for this in the download time analysis.

In order to highlight the structure of the queues in our system, let us first review the notion of an \([n, k]\)-Fork-Join (FJ) queue [27]. In an \([n, k]\)-FJ queue, each request is forked into \(n\) copies upon arrival, and a request completes once any \(k\) out of its \(n\) copies finish service. The remaining \(n - k\) tasks are immediately removed from the system. See [27] for details on \([n, k]\)-FJ queues. Under the Fork-Join content access model that we described in Sec. 2, there are two levels of FJ queues present in a system using availability codes. In the outer level, arriving requests are split into \(t + 1\)

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As we discuss in Appendix 10.5, an (7, 3, 2, 3)-availability code can tolerate any 3-server failures and 80% of 4-server failures.
copies and are sent across their systematic servers and $t$ recovery groups. A request completes as soon as any one of its copies finishes service, hence forming a $(t + 1, 1)$-FJ queue. In the inner level, request copies that are sent to a recovery group fork into $r$ sub-copies and complete once all the forked copies finish service. Therefore an $(r, r)$-FJ queue is implemented at each recovery group.

**Example:** Consider a simple $(3, 2, 2, 2, 1)$-LRC code $\{f_1, f_2, f_1 + f_2\}$. We call the FJ queue for this particular code as the butterfly queue, since the code resembles the one used over the butterfly network for network coding [1]. (See Fig. 5.) Requests arriving for object $f_1$ form a Poisson process with rate $\lambda \cdot p_1$, while those arriving for $f_2$ form a Poisson process with rate $\lambda \cdot p_2$, where $p_1, p_2 \geq 0$, $p_1 + p_2 = 1$. Each request is split into three copies. A request for object $f_1$ (resp. $f_2$) is completed once either $f_1$ (resp. $f_2$) is downloaded directly or both $f_2$ and $f_1 + f_2$ (resp. both $f_1$ and $f_1 + f_2$) are downloaded. If a systematic copy finishes first, the remaining recovery copies corresponding to the request are removed from the system. A systematic copy is removed from the system if the other two recovery copies finish first.

Analysis of the Fork-Join queues is a notoriously hard problem. Its average response time is known only for the simplest case $(2, 2)$-FJ queue [15, 39]. Moreover, the inner FJ queues present at the recovery groups in our system are inter-dependent. This is because the completion of a request at its systematic server or one of its recovery groups triggers the cancellation of its remaining outstanding copies, which might be either waiting in queue or in service at the other servers. This inter-dependence makes it formidable to exactly analyse our system with two layers of FJ queues. Our approach to understand the system performance is therefore deriving bounds and approximations on the download time. We start by presenting various bounds on the download time in the following subsection.

### 4.1 Bounds on Download Time

We here present bounds on the system performance by considering more restricted counterparts of our system. These restrictions are carefully imposed on the system to make it possible to find exact or approximate expressions for the average download time. First we consider the Split-Merge model. In this model, requests are buffered upon arrival in a centralized FCFS queue, and then they are let go to the system one by one only when the system is idle. Next we consider the Fixed-Arrival model. In this model, every request arrival within a busy period asks for the same file. A busy period refers to the time interval during which there is at least one request in the system. This will hold for instance if the object popularities exhibit extreme skew, that is when the probability that a request asks for a particular object is one while it is zero for the rest of the objects. Leveraging these two models, we find lower and upper bounds on the performance of our Fork-Join system under arrivals with general popularity distribution.

**FJ-GA, FJ-FA, SM:** We refer to our availability coded system under the Fork-Join content access model and object popularity distribution $[p_1, \ldots, p_k]$ as the Fork-Join General-Arrival system, and in short as *FJ-GA*. We refer to the system under Fixed-Arrival and Split-Merge models as *FJ-FA* and *FJ-SM* respectively. We give a detailed study of the FJ-FA and FJ-SM systems in sections 5 and 6. In
the following, we denote the download time of a system \( x \) under an aggregate arrival rate of \( \lambda \) as \( T_{x, \lambda} \). We avoid writing \( \lambda \) explicitly when request arrival rate is clear from the context.

First we find an approximation for FJ-FA system, which we elaborate on in Sec. 5. This approximation is helpful in interpreting the bounds we derive on \( T_{FJ-GA} \). These bounds will be stated soon below in Theorems 2 and 3. Note that the approximation we state in Prop. 4 becomes exact for the system with locality 1, that is, when the Fork-Join content access model is implemented on the system with replicated storage [27].

**Proposition 4.** FJ-FA system can be approximated as an M/G/1 queue, hence the Pollaczek-Khinichin (PK) formula [55] gives the following approximation for the average download time in the system as

\[
\mathbb{E} [T_{FJ-FA}] \approx \mathbb{E}[S] + \frac{\lambda \cdot \mathbb{E} [S^2]}{2(1 - \lambda \cdot \mathbb{E}[S])},
\]

where \( \mathbb{E}[S] = \sum_{v \in \mathcal{N}} f_v \cdot \mathbb{E}[S_v], \quad \mathbb{E}[S^2] = \sum_{v \in \mathcal{N}} f_v \cdot \mathbb{E}[S_v^2], \)

where the set \( \mathcal{N} \) is defined in (21), and \( f_v \) is defined in Lemma 4 as the probability that the service time of an arbitrary request is sampled from r.v. \( S_v \). Distribution of \( S_v \) is given in Lemma 3.

We next establish a relation between the download times for FJ-GA, FJ-FA, and FJ-SM systems.

**Theorem 2.**

\[
\Pr \{ T_{FJ-GA, \lambda} > x \} \geq (a) \sum_{i=1}^{k} p_i \cdot \Pr \{ T_{FJ-FA, \lambda p_i} > x \} \geq (b) \sum_{i=1}^{k} p_i \cdot \exp (-((t + 1)\mu - p_i \cdot \lambda)x),
\]

\[
\Pr \{ T_{FJ-GA, \lambda} > x \} \leq (c) \Pr \{ T_{FJ-FA, \lambda} > x \} \leq (d) \Pr \{ T_{FJ-SM, \lambda} > x \},
\]

Using this result, we derive the following bounds on the average download time in FJ-GA system.

**Theorem 3.**

\[
\sum_{i=1}^{k} \frac{p_i}{(t + 1)\mu - p_i \cdot \lambda} \leq \mathbb{E} [T_{FJ-GA}] \leq \mathbb{E} [T_{FJ-FA}] \leq \eta + \frac{\lambda \cdot \sum_{j=0}^{t} (\mu_j)^{-1} \cdot \sum_{i=0}^{\min \{ j, t \}} (\lambda_i)^{-1} \cdot (1 + 1)^{-2}}{\mu^2 \cdot (1 - \lambda \cdot \eta)}.
\]

where \( \eta = \frac{1}{\mu} + \beta (t + 1, \frac{1}{\lambda}) \). The lower bound is valid for \( \lambda < (t + 1)\mu / \max \{ p_1, \ldots, p_k \} \), and the upper bound is valid for \( \lambda < 1/\eta \).

We defer proving Theorems 2 and 3, and the results presented in the following to Sec. 7 after we elaborate on the characteristics of FJ-FA and FJ-SM systems in Sec. 5 and 6.

**Remark 4.** We here reflect on the bounds given on \( T_{FJ-GA} \) in (16) and (17). First notice that the upper bound (c), which is given in terms of \( T_{FJ-FA} \), is tighter than (d), which is given by the Split-Merge model. In addition, Prop. 4 helps interpreting the lower bound (a), which is given in terms of \( T_{FJ-FA} \), as follows. Prop. 4 tells us that FJ-FA system can be interpreted as an M/G/1 queue. This allows us to interpret the lower bound (a) as the response time in a system of \( k \) separate FJ-FA’s, which are fed by splitting the Poisson arrival process across with the probabilities \( p_1, \ldots, p_k \). Using this interpretation, one can see that the value of the lower bound (a) increases as the object popularity distribution \( \{p_1, \ldots, p_k\} \) becomes more skewed, that is when the object popularity distribution is skewed towards a set of objects. This can be seen more easily for lower bound (b) since it is a closed-form expression. In fact the lower bound (a) becomes equal to the upper bound (c) when the
object popularities are polarized, i.e., when every arrival asks for the same object. Upper bound (c) suggests that the performance of FJ-GA becomes worst when the object popularities are polarized.

4.2 Inner and Outer Bounds on System Stability

Bounds presented in Theorem 3 on the average download time in FJ-GA system allows us to characterize an inner and an outer bound on the system’s stability condition.

**Theorem 4.** For the stability of FJ-GA system, a sufficient condition is given by

\[ \lambda < \mu \cdot \frac{r}{\beta} (t + 1, 1/r), \]  

and a necessary condition is given by

\[ \lambda < \mu \cdot (t + 1)/\max\{p_1, \ldots, p_k\}. \]  

4.3 Simulation Results

We here simulate the Fork-Join systems that are defined by the coding schemes given in Table 1. Fig. 6 shows the average download time vs. the request arrival rate for each of these four systems. These systems are ordered in terms of their download time as follows

3-Replication < (7, 3, 2, 3)-Availability < (10, 6, 3, 1)-LRC < (9, 6)-MDS.

Notice that this is the same ordering given by their storage efficiency. This means availability codes in the Fork-Join access model serve as an intermediate point in the storage vs. download time tradeoff between the two extremes: MDS and replication codes. This observation holds regardless of whether the cumulative service rate of the system is kept fixed or not, and also regardless of the skews in object popularities.

5 FORK-JOIN SYSTEM IN THE FIXED-ARRIVAL MODEL (FJ-FA)

We here focus on the Fork-Join access model under the fixed arrival regime, i.e., FJ-FA system. Fixed arrival regime is useful to understand the performance of Fork-Join model under highly skewed object popularities, which is known to be the typical case in practice [6]. Moreover as discussed in Sec. 4.1, FJ-FA system allows us to find bounds on the download time in the general arrival model, i.e., for the FJ-GA system. Overall the dynamics of FJ-FA are more tractable than those of FJ-GA, which allows us to derive several important insights into the system behavior. It should also be noted that the analysis we present here for FJ-FA not only allows us to derive bounds on the performance of FJ-GA but also is interesting in its own right. The analysis presented in this section sheds light on the important characteristics of multi-layer Fork-Join queues.

To begin with, in the Fixed-Arrival model, all request arrivals within a busy period ask for the same file. Therefore each server in the system has a fixed role for every request within the same busy period. Each server acts either as systematic or as a recovery server.

First important property of FJ-FA system is that requests depart the system in the order they arrive. This follows from the fact that request copies depart in the order of arrival both at the systematic server and at any recovery group. Request copies depart in the order they arrive at the systematic servers because servers are shared by the request copies via a FCFS queue. Concluding that departure is in the order of arrival at the recovery groups requires a bit more explanation. A request copy that is assigned to a recovery group is forked into the \( r \) sibling servers that compose the recovery group. At any given time \( t \), some of the recovery servers can be ahead of its siblings in service. We refer to such servers as leading. We refer to the other servers that are behind with service as slow servers. For instance in the example shown in Fig. 9 (in Sec. 8.1), server that hosts
Analyzing the Download Time of Availability Codes

Fig. 6. Average download time vs. request arrival rate $\lambda$ for systems with the coding schemes given in Table 1. Top row is for when all stored data objects have equal popularity, while the bottom row is for when $1/3$ of the objects have $90\%$ of the popularity and the rest share $10\%$ of popularity equally, e.g., $p_1 = 0.9$ and $p_2 = p_3 = 0.05$ for (7, 3, 2, 3)-availability coded system. In each row, the plot on the left is for when the service rate at each server is kept at 1 and the one on the right is for when the cumulative service rate is fixed at 10 and evenly allocated across the servers.

Recall that only one server takes the role of systematic server in the fixed arrival model. Another important property of FJ-FA system is that the slow servers in each recovery group have the same queue length as the systematic server. At any time in the system, slow servers serve a copy of the same request that is currently in service at the systematic server. Within each recovery group, sum of the slow and leading servers is always $r$, and there is at least one slow server unless the system is idle. Therefore there can be at most $r - 1$ leading servers in each recovery group, and $t(r - 1)$ leading servers in the whole system at any time.

There are two other system properties due to the leading servers, and these properties significantly complicate the system analysis. First, up to $1 + t(r - 1)$ different requests can be in service simultaneously. In order to see this, suppose there are at least $1 + t(r - 1)$ different requests in the system. Given that there can be at most $t(r - 1)$ leading servers in the system, and each leading server can potentially be serving a copy of a distinct request. This together with the request copy being served at the systematic (and slow servers) makes it possible for $1 + t(r - 1)$ requests to be in
service simultaneously. Second, different copies of the same request can start service at different times. This is because the leading servers can go ahead and serve the request copies waiting in their queues and so different copies of the same request can go into service at different times. In order to eliminate these two complications, we redefine the request service start times as follows.

**Definition 1.** We say 1) that a request is at the head of the line (HoL) once all its copies still in the system are in service, and 2) the request service start time is the time at which it moves to HoL.

Under this definition, there can be at most one request in service at any time. This is because for two requests to be simultaneously in service, all remaining copies of each must be in service simultaneously, which is impossible given that requests depart the system in the FIFO order. This observation is crucial for the rest of this section and is formally stated below.

**Observation 1.** In FJ-FA system, requests depart the system in the order they arrive and there can be at most one request in service at any time.

Once a request starts service, the layout of its copies at the servers determines its service time distribution. For instance if all copies and sub-copies of a request remain in the system at the time of its service start, then its service time will be distributed as

$$\min \{ S, (S_{r:r})_{t:1} \}.$$  

where $S$ is the service time at the systematic server, $S_{r:r}$ is the service time at a recovery group and the minimum across all $t$ recovery groups is shown with the subscript $t : 1$. Notice that we here employ the memoryless property of Exponential distribution. That is we don’t worry about the time copies of a request spend in service before the whole request moves to HoL.

A request can have up to $r - 1$ of its copies depart at every recovery group before the request moves to HoL. From this point on, we will refer to the copies that finish service before the request moves to HoL as *early departing copies*. If $d$ copies of a request depart early at a recovery group, then when the request moves to HoL service time of the request copy at that recovery group will be distributed as $S_{r-d:r-d}$. Given that $S_{r-d':r-d'} < S_{r-d:r-d}$ for $d' > d$, the more copies of a request depart early the faster (stochastically) the request will be served at a recovery group.

Let $d_i$ denote for a request the number of its early departing copies at the $i$th recovery group. Then service time distribution for the request is given as

$$S_{d_1, \ldots, d_r} = \min \{ S, S_{r-d_1:r-d_1}, \ldots, S_{r-d_r:r-d_r} \}.$$  

For a multiset $\{d_1, \ldots, d_r\} \in \{0, 1, \ldots, r - 1\}^r$ and a number $d \in \{0, 1, \ldots, r - 1\}$, let $v_d$ denote the number of occurrences of $d$ in the multiset. Given that the recovery groups are indistinguishable, service time distribution $S_{d_1, \ldots, d_r}$ is solely determined by the vector $\nu = (v_0, v_1, \ldots, v_{r-1})$, and we say in this case that the request has *type-$\nu$* service time, and denote it as $S_\nu$. Notice that the set of all possible $\nu$’s is given by

$$\mathcal{N} = \left\{ (v_0, v_1, \ldots, v_{r-1}) \mid \sum_{i=0}^{r-1} v_i = t, \ v_i \text{ is a non-negative integer} \right\}.$$  

(21)

We formalize these observations in the following lemma.

**Lemma 3.** In FJ-FA system, the set of all possible request service time distributions is given as

$$S = \{ S_\nu \mid \nu \in \mathcal{N} \},$$

where $\mathcal{N}$ is given in (21) and $|S| = |\mathcal{N}| = \binom{t + r - 1}{r - 1}$. Type-$\nu$ service time distribution is given as

$$\Pr \{ S_\nu > s \} = \exp(-\mu \cdot s) \cdot \prod_{d=0}^{r-1} (1 - (1 - \exp(-\mu \cdot s))^{r-d})^{v_d}.$$  

(22)
As discussed above, the more copies of a request depart early (before the request moves to HoL), the faster its service will be. This is clearly seen in (22). First recall that $\sum _{i} v_i = t$ for any $\nu \in N$. Tail distribution of $S_\nu$ gets smaller as the mass $t$ of $\nu$ is shifted on $v_i$’s with larger $i$. We have the following partial ordering between request service time distributions

$$S_\nu > S_{\nu'},$$

if there exists a $j$ such that $v_i \leq v'_i$ for all $i \geq j$. This ordering implies that request service times when $\nu$ is equal to $(0, \ldots, 0, t)$ and $(t, 0, \ldots, 0)$ are respectively the fastest and slowest request service time distributions. They play an important role in deriving the upper and lower bounds on the performance of FJ-FA system. Consequently by Theorem 2, they are also useful to derive bounds on the performance of FJ-GA system. For easy reference, we refer to the fastest service time $S_{(0, \ldots, 0, t)}$ as $S_{\text{fastest}}$, and refer to the slowest service time $S_{(t, 0, \ldots, 0)}$ as $S_{\text{slowest}}$. Their distributions are

$$\Pr \{S_{\text{fastest}} > s\} = \exp(-\mu(t + 1)s), \quad \Pr \{S_{\text{slowest}} > s\} = \exp(-\mu \cdot s)(1 - (1 - \exp(-\mu \cdot s))^t)' .$$

It is then natural to ask: what is the probability that an arbitrary request arrival is served with type-$\nu$ service time for a given value of $\nu$? The limiting value of this probability is equal to its “time average” as stated in the following.

**Lemma 4.** (Proof is given in Appendix 10.6) In FJ-FA system, let $J_i$ be the type of service time for the $i$th request arrival. When the system is stable, we have

$$\lim_{i \to \infty} \Pr \{J_i = \nu\} = f_\nu,$$

where $f_\nu$ is the limiting fraction of the requests that is served with $S_\nu$.

The service time distribution of a request is dictated by the system state at its service start time epoch. Queue lengths carry memory between the service starts of the subsequent requests. For instance, at the service start time of the $i$th request, if the difference between the queue lengths at all the leading and slow servers is at least 2, then this difference will be at least 1 at the service start time for the $(i + 1)$th request. Therefore in general, service times are not independent across the subsequent requests. However, very importantly, request service times can only be loosely coupled. Every request departure from the system triggers cancellation of its outstanding copies that are still in service. This helps the slow servers to catch up with the leading servers, and it is “hard” for the leading servers to keep leading, since they also compete with every other server in the system. Queues across all the servers are thus expected to frequently level up. Time epochs at which the queues level up while they are non-empty corresponds to a request starting service (i.e., moving to HoL) with type-0 service time distribution. Requests that move to HoL before or after a time epoch at which the queues level up have independent service time distributions. Therefore these time epochs break the dependence between the request service time distributions. Given that these time epochs occur frequently, request service times constitute a series of independent small-size batches.

**Observation 2.** FJ-FA system experiences frequent time epochs across which the request service time distributions are independent.

**Argument for Prop. 4.** Observations we have made so far allow us to develop an approximate method for analyzing FJ-FA system. Requests depart the system in the order they arrive (Obv. 1), hence the system as a whole acts as a FCFS queue. There are $t \choose r-1$ possible distributions for the request service times as given in Lemma 3. Although request service times are not independent, they are loosely coupled (Obv. 2). Putting all these together implies that FJ-FA system approximately behaves as an M/G/1 queue.
6 FORK-JOIN SYSTEM IN SPLIT-MERGE MODEL (FJ-SM)

In FJ-GA and FJ-FA systems, once a server is done serving a request copy, it can start serving the subsequent request copy waiting in its local queue. This creates dependency between the waiting times experienced by the request copies across different servers. This hence makes a request’s service time distribution dependent on the time at which the previous request departs system. Unlike FJ-GA and FJ-FA systems, in FJ-SM system requests wait in a centralized FCFS queue and admitted to service one at a time. FJ-SM therefore does not introduce any dependence between the service time distributions of different requests.

Lemma 5. FJ-SM system implements an M/G/1 queue with an arrival rate of $\lambda$ and service time distribution given in (6).

Proof. Requests move to service in the order they arrive and their service times are i.i.d. with $T_{\text{FJ-}}(r,t)$, given in (6). Thus FJ-SM system can be described as a FCFS queue with service times i.i.d. as $T_{\text{FJ-}}(r,t)$. This together with the Poisson arrival process defines the described M/G/1 queue. □

In FJ-SM system, all servers are blocked until the request at the HoL finishes service, whereas in FJ-GA or FJ-FA system, each server can independently start serving the others waiting in its queue. FJ-SM therefore performs slower than both FJ-GA and FJ-FA. This serves us to find an upper bound on the performance of FJ-GA and FJ-FA as stated below. It should be note that Split-Merge model is also used in [26, 27] for deriving upper bounds on the response time of $[n,k]$-FJ queue.

Lemma 6. Pr $\{T_{\text{FJ-GA}} > t\} \leq \Pr \{T_{\text{FJ-SM}} > t\}$ and $\Pr \{T_{\text{FJ-FA}} > t\} \leq \Pr \{T_{\text{FJ-SM}} > t\}$.

Fast-Split-Merge Model: Request service time distribution in FJ-SM, which is given in (6), is the same as $S_{\text{slowest}}$ that we previously derived in (24). Recall that $S_{\text{slowest}}$ denotes the slowest possible request service time distribution in FJ-GA system. In other words, FJ-SM is a modification of FJ-FA such that it forces every request to be served with the slowest possible distribution.

We next consider the Fast-Split-Merge model and its corresponding system FJ-FSM. It operates at the other extreme of FJ-SM and serves every request with the fastest possible service time distribution $S_{\text{fastest}}$ that we previously stated in (24).

Lemma 7. FJ-FA system in the Fast-Split-Merge model, namely FJ-FSM system, implements an M/M/1 queue with an arrival rate of $\lambda$ and service time distribution $\text{Exp}((t + 1)\mu)$. The download time in FJ-FSM system is then distributed as $\text{Exp}((t + 1)\mu - \lambda)$.

Proof. Recall that FJ-FA system implements a FCFS queue (Obv. 1). FJ-FSM system therefore implements a FCFS queue with service times i.i.d. as $S_{\text{fastest}}$. We know from (24) that $S_{\text{fastest}} \sim \text{Exp}((t + 1)\mu)$. This shows that FJ-FSM model implements the described M/M/1 queue. □

Just as we used FJ-SM to find an upper bound on the performance of FJ-FA (Lemma 6), we next use FJ-FSM system to find a lower bound.

Lemma 8. For $\lambda < (t + 1)\mu$, we have $\Pr \{T_{\text{FJ-FA}} > x\} \geq \exp(-(t + 1)\mu - \lambda)x$.

Proof. FJ-FA implements a FCFS queue (Obv. 1), and each request is served with one of the $\binom{t+r-1}{r-1}$ distributions given in Lemma 3. Under Fast-Split-Merge model, all requests are served with the fastest possible service time. Response time of FJ-FSM therefore serves as a lower bound for $T_{\text{FJ-FA}}$. This observation together with Lemma 7 gives us the lower bound on $\Pr \{T_{\text{FJ-FA}} > x\}$. □

The following corollary of Lemma 8 gives a bound on the average download time in FJ-FA system.

Corollary 5. For $\lambda < (t + 1)\mu$, we have $\mathbb{E} [T_{\text{FJ-FA}}] \geq \frac{1}{(t+1)\mu - \lambda}$. 
7 PROOFS AND DISCUSSION OF THE RESULTS FOR HIGH ARRIVAL REGIME

7.1 Proofs

In Sec. 5 and 6, we built an understanding of the Fork-Join access model in the Fixed-Arrival (FJ-FA) and Split-Merge (FJ-SM) models. We also presented various bounds on the download time in these models. Observations made in these sections will enable us here to derive various bounds on the download time in the General-Arrival model (FJ-GA). It should be noted that these bounds were previously presented in Theorem 2 and 4.

Proof of Theorem 2.

\[ \Pr \{ T_{\text{FJ-GA}} > x \} \geq \sum_{i=1}^{k} p_i \cdot \Pr \{ T_{\text{FJ-FA,p_i}} > x \} \geq \sum_{i=1}^{k} p_i \cdot \exp \left( -(t+1)\mu - p_i \cdot \lambda \right) x, \]  

(26)

\[ \Pr \{ T_{\text{FJ-GA}} > x \} \leq \Pr \{ T_{\text{FJ-FA}} > x \} \leq \Pr \{ T_{\text{FJ-SM}} > x \}, \]  

(27)

Upper bound (c): In both FJ-FA and FJ-GA, requests start service in the order they arrive (in the sense of Def. 1), and service time distribution of a request is determined by its early departing copies at the leading servers. Greater the number of copies that depart early for a request, faster its service time will be when it moves to HoL. Only way for either system to exploit storage redundancy and achieve faster download is through early completions at the leading servers.

Only difference between FJ-FA and FJ-GA is the download role of the servers for the requests. In FJ-FA, one server acts as the systematic server for all requests, and all the remaining servers act as recovery servers. Any leading server is a recovery server, and the early departures from them can only reduce (stochastically) the service time of a request. However in FJ-GA, a leading server might serve the systematic copy of a request waiting in line. Early departing copies at the leading servers can then completely finish a request, even before the request moves to HoL. Completion of a request with an early departing copy allows the request to depart the system with 0 service time, i.e., without even moving to HoL. This also removes the remaining copies of the completed request from other queues, hence also reducing the waiting time of the subsequent request copies.

Overall in order to reduce download time, it is better for a leading server to complete a systematic request copy rather than a recovery copy. FJ-GA allows completion of the whole request at the leading servers while FJ-FA does not allow it. FJ-GA therefore has smaller response time than FJ-FA.

To make the arguments used above concrete, let us consider the following scenario in FJ-FA system. Suppose the system is currently serving request \( i \), and there is a leading server that is serving a copy of request \( j > i \). We refer to this leading server as the tagged server. Under fixed arrival model, copy of request \( j \) that is in service at the tagged server is a recovery copy and its early completion would accelerate the service time of request \( j \). Let us now consider the case in which the tagged server happens to be a systematic server for request \( j \), as it might happen in FJ-GA system. The question is: would this surely reduce the system’s response time?

As discussed above, early completion of request \( j \)‘s systematic copy is better than completion of one of its recovery copies. We now consider the worst case, in which copy of request \( j \) at the tagged server does not finish service until request \( j \) moves to HoL. Notice that this case would be the same as the case in which request \( j \) has a recovery copy at the tagged server (as it would be the case in FJ-FA system) and the recovery copy did not finish service until request \( j \) moves to HoL. That is why having requests asking for different objects only improves the response time compared to the case with all requests asking for the same object. In other words, FJ-GA performs faster than FJ-FA.

Upper bound (d): Follows from Lemma 6.
Lower bounds (a) and (b): Let us introduce the following enlarged system. Suppose we create $k$ copies of the Fork-Join system, which we index over as system-$i$ for $i = 1, \ldots, k$. Let us forward the arrivals asking for object $i$ only to system-$i$. Then, system-$i$ will be equivalent to a FJ-FA system operating under a request arrival rate of $p_i \cdot \lambda$. Obviously, an arbitrary request in this enlarged system will experience a smaller response time than the original system, which gives us lower bound (a). System in Fast-Simple-Merge model presents a lower bound on $T_{FJ-FA}$ (Lemma 8), which together with (a) gives us (b).

In the following, we first derive the bounds presented in Theorem 3 on average download time $E[T_{FJ-GA}]$ in FJ-GA system. We then derive the stability conditions presented in Theorem 4 again for FJ-GA. We do so using the performance bounds we previously derived via Split-Merge (FJ-SM) and Fast-Split-Merge (FJ-FSM) models, as respectively stated in (d) of (27) and (a) of (26).

Proof of Theorem 3. Lower bound (a) given on $T_{FJ-GA}$ in (26) is a mixture distribution with components distributed as $\text{Exp}((t + 1)\mu - p_i \cdot \lambda)$ for $i = 1, \ldots, k$. Expected value of this mixture distribution yields the lower bound for $E[T_{FJ-GA}]$.

By (d) given in (27), average response time $E[T_{FJ-SM}]$ in FJ-SM system yields an upper bound on $E[T_{FJ-GA}]$. FJ-SM implements an M/G/1 queue with the service time distribution $S_{\text{slowest}}$ (Lemma 5). Then PK formula gives us

$$E[T_{FJ-SM}] = E[S_{\text{slowest}}] + \frac{\lambda \cdot E[S_{\text{slowest}}^2]}{2(1 - \lambda \cdot E[S_{\text{slowest}}])}. \tag{28}$$

First moment of $S_{\text{slowest}}$ is given in (7). We find its second moment as

$$E[S_{\text{slowest}}^2] = \int_0^\infty 2s \cdot \Pr\{S_{\text{slowest}} > s\} ds = \int_0^\infty 2s \cdot \exp(-\mu \cdot s) (1 - (1 - \exp(-\mu \cdot s))^r) ds \tag{29}$$

$$= \int_0^\infty \sum_{i=0}^r \binom{r}{i} (-1)^i \cdot \sum_{l=0}^i \binom{i}{l} (-1)^l \cdot \mu^2(l + 1)^2,$$

where (i) follows from the fact that $S_{\text{slowest}}^2$ is a non-negative r.v., (ii) comes from substituting (24) in, and (iii) follows from the binomial expansion of $(1 - (1 - \exp(-\mu \cdot s))^r)$ and interchanging the order of integration and summation. Finally, substituting (28) in the first and second moments of $S_{\text{slowest}}$ gives us the upper bound in (18).

Proof of Theorem 4. Since $T_{FJ-GA} \leq T_{FJ-SM}$, stability condition for the SM system gives a sufficient condition for the stability of FJ-GA. Given that FJ-SM implements an M/G/1 queue with service time distribution $S_{\text{slowest}}$ (Lemma 5). FJ-SM is stable iff $\lambda < E[S_{\text{slowest}}]$, which gives us (20).

Necessary condition: The lower bound (b) on $T_{FJ-GA}$ in Theorem 3 is found via forming an enlarged system by i) creating $k$ copies of the whole system, ii) dedicating a separate system copy for each different object and forwarding the download requests arriving for each object to its designated system copy, which makes each system copy work as an FJ-FA system, iii) lower bounding the download time in each copy of FJ-FA system using Fast-Split-Merge (FJ-FSM) model. We refer the reader to the Proof of Theorem 3 for details. Each of the FJ-FA copies in FJ-FSM model implements an M/M/1 queue with service times i.i.d. as $S_{\text{fastest}}$ (Lemma 7). Hence the ith copy of FJ-FA is stable if and only if $p_i \cdot \lambda < E[S_{\text{fastest}}]$. All the $k$ FJ-FA copies will be stable if the FJ-FA copy that operates under the largest arrival rate is stable. This gives us the stability condition for the enlarged system
as max\{p_1, \ldots, p_k\} \cdot \lambda < \mathbb{E}[S_{\text{fastest}}]. \) If FJ-GA is stable then the enlarged system has to be stable. This therefore gives us the necessary condition given on the stability of FJ-GA in (19).

\[\Box\]

### 7.2 Some Implications of FJ-FA and Service Time Probabilities

In Sec. 5, we analyzed the performance FJ-FA system. The understanding we obtained on FJ-FA allowed us to relate the performance of FJ-GA system to that of FJ-FA (Theorem 2). This allowed us then to argue about the performance of FJ-GA with respect to object popularities (Remark 4).

As discussed in the first half of Sec. 5, FJ-FA is more tractable than FJ-GA for download time analysis. This is mainly because the role of each server in FJ-FA is fixed across all requests, i.e., each server acts either as systematic or a recovery node for every request, while servers in FJ-GA change roles based on the request being served. This property of FJ-FA turned out to be quite useful and allowed us to make important observations about the order in which requests depart the system and their service time distribution (Obv. 1 and 2).

Despite the significant progress we have made in understanding the dynamics of FJ-FA, we still could not exactly analyze it due to the state explosion problem. Fortunately our observations led us to argue that FJ-FA is well approximated as an M/G/1 queue (Prop. 4). M/G/1 queue is a very well understood object, which makes our proposed approximation a very convenient tool. However, in order to employ the approximation we still need to find the probability \( f_\nu \) that an arbitrary request is served with type-\( \nu \) service time distribution \( S_\nu \) (Lemma 3).

An exact expression for \( f_\nu \) is found as follows. Recall from Lemma 4 that \( f_\nu \) is given by the limiting fraction of the requests that is served with service time distribution \( S_\nu \). Sub-sequence of request arrivals that find the system empty upon arrival forms a renewal process [17, Theorem 5.5.8]. Let \( \mathbb{I} \) be the indicator function and let \( \{J(t) = \nu\} \) denote the event that the request at HoL at time \( t \) is served with service time distribution \( S_\nu \). Let us also define a renewal-reward function as \( R_\nu(t) = \mathbb{I}\{J(t) = \nu\} \). We can then express \( f_\nu \) in terms of this function as

\[
f_\nu = \lim_{t \to \infty} \Pr\{R_\nu(t) = 1\} = \lim_{t \to \infty} \mathbb{E}[R_\nu(t)] = \lim_{t \to \infty} \frac{1}{t} \int_{-\infty}^{t} R_\nu(\tau) \, d\tau = \mathbb{E}\left[ \frac{\int_{t_{n-1}}^{T_n} R_\nu(\tau) \, d\tau}{\mathbb{E}[X]} \right].
\]

In the derivation above: i) \( (a) \) and \( (b) \) are due to the equality of the limiting time and ensemble averages of the renewal-reward function \( R_f(t) \) [17, Theorem 5.4.5]; ii) \( t_{n-1} \) and \( t_n^\nu \) denote the \((n-1)\)th and \( n \)th renewal epochs, i.e., consecutive arrival epochs that find the system empty; and iii) \( X \) denotes the i.i.d. inter-renewal intervals. The expression given above clearly indicates that finding the values of \( f_\nu \)'s requires an exact analysis of the system. In the next section, we present a methodology to estimate \( f_\nu \)'s when the system employs an important sub-class of availability codes.

### 8 Approximations for FJ-FA System with Locality Two

We next focus on the FJ-FA model for systems using availability codes with locality \( r = 2 \). This class of codes are of interest mainly because they are minimally different from replication, which has locality \( r = 1 \). Recall that FJ-FA systems are difficult to analyse because each recovery group implements a separate FJ queue and there is dependence between these FJ queues. Code locality \( r \) implies that FJ queues at the recovery groups consist of \( r \) servers. When locality is one, each recovery group is simply a single replica server and implements a FCFS queue. This makes the whole system behave as an M/G/1 queue, as we mentioned in Sec. 4.1. As soon as locality becomes two, system state becomes intractable due to the complex dynamics implemented by the inter-dependent FJ queues. Further as the locality grows, system state becomes increasingly complex. A system
with locality two therefore is the least complex of all intractable FJ-FA systems for a given code availability \( t \).

**Remark 5.** Note that a binary simplex code is an \( (n = 2^k - 1, k, r = 2, t = 2^{k-1} - 1)\)-LRC and an important class of availability codes with locality two [9, 31]. Simplex codes are known to be optimal in several ways: i) they meet the upper bound on the distance of LRCs with a given locality [9]; ii) they are shown to achieve the maximum rate (i.e., storage efficiency) among the binary linear codes with a given availability and locality two [31]; iii) they meet the Griesmer bound and are therefore linear codes with the lowest possible length given the code distance [32]. In addition, binary simplex codes have been shown to implement storage-efficient robustness against the skews in content popularity [2]. Robustness here refers to system’s ability to perform under stability.

Given that recovery groups consist of two servers, there can be at most one leading server in each recovery group. Hence a request can have at most one early departure at each of the recovery groups. (See Sec. 5 for the definition of early departing request copies.) Request service type vector \( \nu \) in this case is given by \((\nu_0, \nu_1)\) where \( \nu_0 \) (resp. \( \nu_1 \)) denotes the number of recovery groups at which the request has no (resp. one) early departing copy. Given that \( \nu_0 + \nu_1 \) is fixed and equal to \( t \), it is sufficient to only keep track of \( \nu_1 \) to determine the service time distribution of a request. In other words, service time distribution for a request is defined by its number of early departing copies, i.e., number of recovery groups at which the request had one early departure. Given that \( \nu_1 \) can take values in \([0, t]\), the set \( S \) of all possible service time distributions is of size \( t + 1 \) in this case. If a request has an early departing copy at \( i \) recovery groups before it moves to HoL, i.e., \( \nu_1 = i \), then the request will be served with type-\( i \) service time distribution. (We here follow the convention introduced on service types in Sec. 5.) Type-\( i \) service time distribution is given by

\[
Pr \{ S_i > s \} = \exp(-\mu \cdot s)^{i+1} \cdot (1 - (1 - \exp(-\mu \cdot s))^{t-i} \quad (30)
\]

for \( i = 0, 1, \ldots, t \). We have for \( s > 0 \),

\[
Pr \{ S_1 > s \} = \exp(-\mu \cdot s) \cdot (1 - (1 - \exp(-\mu \cdot s))^{t-1}) < 1.
\]

This implies that service times \( S_i \)'s stochastically are ordered as \( S_0 > S_1 > \cdots > S_t \). Recall (23) where we had a partial ordering between the service time distributions for FJ-FA system with general locality \( r \). This partial ordering turns into a complete ordering in this case with \( r = 2 \).

As we elaborated on in Sec. 7.2, finding the request service time probabilities \( f_i \)'s requires an exact analysis of the system, which we know to be intractable due to state explosion problem. For the case with locality of two, in the following we conjecture a relation between \( f_i \)'s. This relation will be used later to derive various estimates for \( f_i \)'s.

**Conjecture 1.** In FJ-FA system with \( r = 2 \), we have \( f_{i-1} > f_i \) for \( i = 1, \ldots, t \).

We next briefly discuss the reasoning behind the conjecture. Obv. 2 states that servers frequently level up. This is because the leading servers in the recovery groups compete with every other server in the system to keep leading or to possibly advance further ahead in serving the request copies in their queues. For a request to have type-\( i \) distribution for service (once it moves to HoL), it needs to have one early departure (before the request moves to HoL) at exactly \( i \) recovery groups. This requires one server in each of the \( i \) recovery groups to be leading, and this becomes less likely for larger \( i \). We have validated the conjecture with extensive simulations. Fig. 7 shows the simulated values for \( f_i \)'s in a system that employs a binary simplex code with availability \( t = 1, 3 \) and locality \( r = 2 \). Moreover, we prove a strong pointer for the conjecture in Appendix 10.7. This pointer says that given a request is served with type-\( i \) distribution \( S_i \), the subsequent request is more likely to be served with type-\( j \) distribution \( S_j \) for \( j < i \). Indeed, we prove the conjecture for the system with availability one and locality two. This is implied by the bounds given on \( f_i \)'s in Theorem 8 in Sec. 8.
Analyzing the Download Time of Availability Codes

From the simulated values of $f_i$'s given in Fig. 7, one can see that the frequency of type-$i$ service times increases for smaller values of $i$ as the arrival rate (i.e., the offered load on the system) increases. This is aligned with the discussion given in the previous paragraph. For a request to have type-$i$ distribution for service, it needs to have one early departure at $i$ recovery groups, and this is possible only if there are $i$ leading servers in the system. Recovery servers can go ahead of their peers in serving requests only if there are request copies waiting in their queues. As the offered load on the system increases, queues at the servers become more likely to accumulate larger number of request copies, and this allows the leading servers to progress even further than they could while operating under smaller offered load.

In the following we make some observations on $f_i$'s that will be useful to estimate their values. First we rewrite the ordering given in Conj. 1 as

$$f_i = \rho_i \cdot f_{i-1} \quad \text{for } \rho_i < 1, \quad i = 1, \ldots, t.$$  

Using the normalization requirement, we obtain $\sum_{i=0}^{t} f_i = f_0 \left( 1 + \sum_{i=1}^{t} \prod_{j=1}^{i} \rho_j \right) = 1$. Then $f_i$'s are given in terms of $\rho_i$'s as

$$f_0 = \left( 1 + \sum_{i=1}^{t} \prod_{j=1}^{i} \rho_j \right)^{-1}, \quad f_i = f_0 \prod_{j=1}^{i} \rho_j. \quad (31)$$

Let us define an upper bound on the ratio of consecutive $f_i$'s as $\rho \geq \max \{ f_i / f_{i-1} \}_{i=1}^{t}$. Substituting each $\rho_i$ in (31) with the same $\rho$ and then solving for $f_i$'s gives the following approximations for $f_i$'s

$$\hat{f}_0 = \frac{1-\rho}{1-\rho^{t+1}}, \quad \hat{f}_i = \rho^i \cdot f_0, \quad \text{for } 1 \leq i \leq t. \quad (32)$$

Notice that the above approximations preserve the relation given in Conj. 1. It is easy to see $f_0 \geq \hat{f}_0$ and $f_i \leq \hat{f}_i$. This, together with the fact that both $f_i$'s and $\hat{f}_i$'s are decreasing sequences with a cumulative value of 1, tells us that for some index $j$ between 0 and $t$, we have

$$\hat{f}_i \leq f_i \text{ for } i \leq j \quad \text{and} \quad \hat{f}_i \geq f_i \text{ for } i \geq j.$$  

We next derive several estimates for $f_i$'s. We start from the simplest but worst estimate, and then move to more complex and better-performing ones. Substituting these estimates in the M/G/1 approximation in Prop. 4 will directly give us an approximation for FJ-FA system with locality two.

**Straightforward approximation:** In (32), we find approximations $\hat{f}_i$'s for $f_i$'s. We derived them by simply replacing $\rho_i = f_i / f_{i-1}$'s with their upper bound $\rho$, and then expressed $f_i$'s in terms of $\rho$. Naturally, the closer $\rho$ is to $\max \{ \rho_i \}_{i=1}^{t}$, the better the approximations $\hat{f}_i$'s will perform.
simplest approach to set $\rho$ is to assume $f_1 = \ldots = f_t$. In this case we have $\max \{ \rho_i \}_{i=1}^t = 1$, and so $\rho$ will be set to its naive maximum value of 1. Substituting $\rho = 1$ in (32), we obtain the estimates as $\hat{f}_i = 1/(t+1)$ for $i = 0, \ldots, t$. Finally, substituting these estimates into our M/G/1 approximation gives us the straightforward approximation for FJ-FA system with locality two.

**Proposition 5.** Straightforward approximation for FJ-FA system with locality $r = 2$ is given by the M/G/1 queue with the service time distribution (where $S_i$’s are defined in (30)):

$$\Pr \{ \hat{S}_{\text{simple}} > s \} = \sum_{i=0}^{t} \frac{1}{t+1} \Pr \{ S_i > s \}.$$ 

**Better approximation:** We start by deriving an inequality for $\rho$ as follows.

**Lemma 9.** Under stability, $\rho$ given in (32) holds the inequality

$$\left(1 - \lambda \cdot \mathbb{E} \left[ \hat{S}_{\text{simple}} \right] \right) \cdot \rho^{t+1} - \rho + \lambda \cdot \mathbb{E} \left[ \hat{S}_{\text{simple}} \right] \geq 0,$$

where $\hat{S}_{\text{simple}}$ is defined in Prop. 5. Hence $\mathbb{E} \left[ \hat{S}_{\text{simple}} \right] = \frac{1}{t+1} \sum_{i=0}^{t} \mathbb{E} \left[ S_i \right]$ where $S_i$’s are defined in (30).

**Proof.** Under stability, sub-sequence of request arrivals that find FJ-FA system empty forms a renewal process [17, Theorem 5.5.8]. Expected number of request arrivals $\mathbb{E} \left[ J \right]$ between successive renewal epochs (busy periods) is given as $\mathbb{E} \left[ J \right] = 1/(1 - \lambda \cdot \mathbb{E} \left[ S \right])$, which can be easily shown to hold for FJ-FA using the same arguments given for the M/G/1 queue in [17, Theorem 5.5.10].

Requests that find the system empty upon arrival are served with type-0 (the slowest) service time distribution $S_0$. Requests that arrive within a busy period can be served with any type-$i$ service time distribution for $i = 0, \ldots, t$. This observation reveals that $1/\mathbb{E} \left[ J \right]$ is a lower bound for $f_0$.

Computing the value of $\mathbb{E} \left[ J \right]$ requires the value of $\mathbb{E} \left[ S \right]$, i.e., expected value of the request service time. An upper bound on $\mathbb{E} \left[ J \right]$ is given by $\mathbb{E} \left[ J_{\text{ub}} \right] = 1/(1 - \lambda \cdot \mathbb{E} \left[ S_{\text{ub}} \right])$, where $\mathbb{E} \left[ S_{\text{ub}} \right]$ is a lower bound for $\mathbb{E} \left[ S \right]$. One possible value for $\mathbb{E} \left[ S_{\text{ub}} \right]$ is given by $\mathbb{E} \left[ S_{\text{ub}} \right] = \mathbb{E} \left[ \hat{S}_{\text{simple}} \right] = \frac{1}{t+1} \cdot \sum_{i=0}^{t} \mathbb{E} \left[ S_i \right]$, which we previously stated and used in Prop. 5. Thus we have

$$f_0 \geq 1/\mathbb{E} \left[ J \right] \geq 1/\mathbb{E} \left[ J_{\text{ub}} \right] = 1 - \lambda \cdot \mathbb{E} \left[ S_{\text{ub}} \right].$$

In the system for which the estimate $\hat{f}_0 = 1/(t+1)$ is exact, the lower bound obtained from renewal theory (i.e., $1 - \lambda \cdot \mathbb{E} \left[ S \right]$) holds as well under stability. For this system, $\mathbb{E} \left[ S_{\text{ub}} \right]$ is exact, hence $\hat{f}_0 = 1/(t+1) \geq 1 - \lambda \cdot \mathbb{E} \left[ S_{\text{ub}} \right]$.

One can see that $(1 - \rho)/(1 - \rho^{t+1}) \geq 1/(t+1)$ for $0 \leq \rho < 1$, so we have $\frac{1-\rho}{1-\rho^{t+1}} \geq \frac{1}{t+1} \geq 1 - \lambda \cdot \mathbb{E} \left[ S_{\text{ub}} \right]$ from which (33) follows.

Next we use (33) to get a tighter value for $\rho$ as follows. Solving for $\rho$ in (33) does not yield a closed form solution, so to get one we take the limit as

$$\lim_{t \to \infty} \left(1 - \lambda \cdot \mathbb{E} \left[ \hat{S}_{\text{simple}} \right] \right) \cdot \rho^{t+1} - \rho + \lambda \cdot \mathbb{E} \left[ \hat{S}_{\text{simple}} \right] \geq 0 \iff \rho \leq \lambda \cdot \mathbb{E} \left[ \hat{S}_{\text{simple}} \right].$$

**Proposition 6.** Better approximation for FJ-FA system with locality $r = 2$ is given by the M/G/1 queue with service time distribution

$$\Pr \{ \hat{S}_{\text{better}} > s \} = \frac{1-\rho}{1-\rho^{t+1}} \cdot \sum_{i=0}^{t} \rho^i \cdot \Pr \{ S_i > s \},$$

where $\rho = \lambda \cdot \mathbb{E} \left[ \hat{S}_{\text{simple}} \right]$ and $\mathbb{E} \left[ \hat{S}_{\text{simple}} \right]$ is given in Lemma 9.
Fine-grained approximation: While deriving the naive and better approximations presented above, we replaced all $\rho_i$’s with a single upper bound $\hat{\rho}$. We here take a more fine grained approach and derive estimates for each $\rho_i$ separately.

We set $\hat{f}_i = \hat{\rho}_0 \cdot \hat{f}_0$ for $i = 1, \ldots, t$. Then, using the normalization $\sum_{i=0}^t \hat{f}_i = 1$, we obtain the estimate $\hat{f}_0 = 1/(1 + t \cdot \hat{\rho}_0)$. The inequality we derived in the Proof of Lemma 9 $1/(1 + t \cdot \hat{\rho}_0) \geq 1/(t + 1) \geq 1 - \lambda \cdot \mathbb{E}[\hat{S}_{\text{simple}}]$ together with the expression above for $\hat{f}_0$, gives us

$$\hat{\rho}_0 \leq \lambda \cdot \mathbb{E}\left[\hat{S}_{\text{simple}}\right] \cdot \frac{1}{1 - \lambda \cdot \mathbb{E}\left[\hat{S}_{\text{simple}}\right]}.$$  \hspace{1cm} (34)

We use this upper bound on $\hat{\rho}_0$ as its estimator.

Our goal next is to estimate $\hat{\rho}_1$. Fixing $\hat{\rho}_0$ to its upper bound given in (34), we obtain

$$\hat{f}_1 = \hat{\rho}_0 \cdot \hat{f}_0, \quad \hat{f}_i = \hat{\rho}_1 \cdot \hat{\rho}_0 \cdot \hat{f}_0, \quad \text{for } i = 2, \ldots, t.$$  

Then an upper bound can be found on $\hat{\rho}_1$ taking the same steps that we took above for driving (34). Normalization requirement gives us $\hat{f}_0 = (1 + \hat{\rho}_0(1 + \hat{\rho}_1(t - 1)))^{-1}$ and we have $\hat{f}_0 \geq 1/(1 + t) \geq 1 - \lambda \cdot \mathbb{E}[\hat{S}_{\text{simple}}]$. Substituting these into the expression given above for $\hat{f}_1$, we find

$$\hat{\rho}_1 \leq \frac{1 - (1 - \lambda \cdot \mathbb{E}[\hat{S}_{\text{simple}}])(1 - \hat{\rho}_0)}{(t - 1)(1 - \lambda \cdot \mathbb{E}[\hat{S}_{\text{simple}}]) \cdot \hat{\rho}_0}.$$  \hspace{1cm} (35)

We use this upper bound on $\hat{\rho}_1$ as its estimator.

The process given above to estimate $\hat{\rho}_0$ and $\hat{\rho}_1$ can be repeated to find an estimate for $\hat{\rho}_2$, and generalized to estimate all $\hat{\rho}_i$’s. Given the estimates for $\{\hat{\rho}_i\}_{i=0}^{l-1}$, an upper bound on $\hat{\rho}_j$ follows:

$$\hat{\rho}_j \leq \frac{1 - \left(1 - \lambda \cdot \mathbb{E}[\hat{S}_{\text{simple}}]\right)\left(1 + \sum_{k=0}^{l-1} \hat{\rho}_k\right)}{\left(1 - \lambda \cdot \mathbb{E}[\hat{S}_{\text{simple}}]\right) \left(t - 1\right) \prod_{k=0}^{l-1} \hat{\rho}_k}.$$  \hspace{1cm} (36)

Again, we use this upper bound on $\hat{\rho}_j$ as its estimator.

Using these estimates, we can compute $\hat{f}_i$’s. Note that these $\hat{f}_i$’s comply with the ordering in Conj. 1. For systems with availability one and two, we plot $\hat{f}_i$’s together with the simulated values of $f_i$’s in Fig. 7. As can be seen from this plot, $\hat{f}_i$’s serve as reasonable estimates for $f_i$’s.

Proposition 7. Fine-grained approximation for FJ-FA system with locality $r = 2$ is given by the M/G/1 queue with service time distribution.

$$\Pr\{\hat{S}_{\text{fine-grained}} > s\} = \left(1 + \sum_{i=1}^{t-1} \prod_{j=0}^{i-1} \hat{\rho}_j\right)^{-1} \cdot \sum_{i=0}^{t-1} \Pr\{S_i > s\} \cdot \prod_{j=0}^{i-1} \hat{\rho}_j,$$

where $\hat{\rho}_j$’s are recursively computed as ($i = 1, \ldots, t$)

$$\hat{\rho}_0 = \frac{\lambda \cdot \mathbb{E}[\hat{S}_{\text{simple}}]}{t \left(1 - \lambda \cdot \mathbb{E}[\hat{S}_{\text{simple}}]\right)}, \quad \hat{\rho}_i = \frac{1 - \left(1 - \lambda \cdot \mathbb{E}[\hat{S}_{\text{simple}}]\right)\left(1 + \sum_{k=0}^{l-1} \hat{\rho}_k\right)}{\left(1 - \lambda \cdot \mathbb{E}[\hat{S}_{\text{simple}}]\right) \left(t - 1\right) \prod_{k=0}^{l-1} \hat{\rho}_k},$$

$\mathbb{E}[\hat{S}_{\text{simple}}]$ is defined in Lemma 9, and $S_i$’s are defined in (30).

Comparison of the M/G/1 queue approximations: Fig. 8 gives a comparison between the three approximations presented above and the bounds presented in Sec. 5. Notice that the straightforward, better, and fine-grained approximations are ordered as their names suggest in terms of their accuracy.
8.1 FJ-FA with locality two and availability one

So far in this section, we focused on availability codes with locality two an availability to one. This case represents the simplest possible availability code, that is storing two objects together with an XOR’ed copy over three servers as \([a, b, a + b]\). The corresponding FJ system is thus the simplest of all FJ systems with availability codes. Although, the exact analysis of the system is still intractable, the system state is simple enough to find tighter bounds on the request service time probabilities. Notice that requests in this case are served with either type-0 or type-1 service time distributions. Hence we only have to find two request service time probabilities \(f_0\) and \(f_1\).

So far we have modeled the service times as i.i.d. \(\text{Exp}(\mu)\) RVs. The simplicity of the setting here allows us to address a more general service time model as follows. We again model the service times to be independent across different servers and request copies. However this time, we let the service time at the systematic server to be distributed as \(\text{Exp}(\gamma)\), and let the service times at the recovery servers to be distributed as \(\text{Exp}(\alpha)\) and \(\text{Exp}(\beta)\). Parameters \(\gamma, \alpha\) and \(\beta\) denote some non-negative constants. In the following we will refer to server with service rate \(x\) as server-\(x\).

System state at time \(\tau\) can be described with a triple \(s(\tau) = (N(\tau), (n_\alpha(\tau), n_\beta(\tau)))\). \(N(\tau)\) denotes the total number of requests in the system at time \(\tau\). This is given by the number of request copies present in the systematic server. \(n_\alpha(\tau)\) (resp. \(n_\beta(\tau)\)) denotes by how many request copies server-\(\alpha\) (resp. server-\(\beta\)) is leading the other recovery server. In other words, denoting the queue length at server-\(x\) at time \(\tau\) with length-\(x(\tau)\), we can express \(n_\alpha(\tau)\) and \(n_\beta(\tau)\) as
\[
\begin{align*}
n_\alpha(\tau) &= \max\{\text{length-}\alpha(\tau) - \text{length-}\beta(\tau), 0\}, \\
n_\beta(\tau) &= \max\{\text{length-}\beta(\tau) - \text{length-}\alpha(\tau), 0\}.
\end{align*}
\]

It follows that \(n_\alpha(\tau) \cdot n_\beta(\tau) = 0\) for all \(\tau\). This is natural because there can only be one leading recovery server at any time since the system has one recovery group of two servers (availability one, locality two). (Recall the definition of leading server as discussed in the 2nd paragraph of Sec. 5.) System state is illustrated in Fig. 9 with two different snapshots of the system.

System state \(s(\tau)\) is a Markov process as illustrated in Fig. 10 (Top). Let us define \(Pr\{s(\tau) = (k, i, j)\} = p_{k,i,j}(\tau)\). Suppose that the system stability is imposed and \(\lim_{\tau \to \infty} p_{k,i,j}(\tau) = p_{k,i,j}\). Then balance equations for the system are summarized for \(k, i, j \geq 0\) as
\[
\begin{align*}
(\gamma \cdot 1(k \geq 1) + \alpha \cdot 1(i \geq 1) + \beta \cdot 1(j \geq 1)) \cdot p_{k,i,j} \\
= \lambda \cdot 1(k \geq 1, i \geq 1, j \geq 1) \cdot p_{k-1,i-1,j-1} + \gamma \cdot p_{k+1,i+1,j+1} + (\gamma + \alpha) \cdot p_{k+1,i+1,j} + (\gamma + \beta) \cdot p_{k+1,i,j+1}.
\end{align*}
\]

Balance equations given above does not allow computing the generating function \(P_{w,x,y} = \sum_{k,i,j \geq 0} p_{k,i,j} \cdot w^k x^i y^j\). It is therefore intractable to exactly analyse the system’s steady state behavior. Intractability arises from the fact that the state space is infinite in two dimensions.
Fig. 9. Fork-Join system with availability one and locality two under Fixed-Arrival regime. Snapshot on the left illustrates the time epoch at which request 1 starts service and its service time distribution $S_0$. Snapshot on the left illustrates the same for request 2 and its service time distribution $S_1$.

Fig. 10. Markov state process for FJ-FA system with availability one and locality two (left), and its high traffic approximation (right).

We use two approaches to analyze the pyramid Markov process given above, that are commonly used to for intractable Markov processes. One is the local balance method with a guess-based analysis, which we use in Appendix 10.8. Another is matrix analytic method [22, Chapter 21], which involves truncating the process appropriately and then numerically solving for the steady state probabilities with an iterative procedure, as we use in Appendix 10.9. This analysis gives us the following upper bound on the average download time. Note that this bound is provably tighter than the Split-Merge upper bound previously given in Thm. 3. This is because the Split-Merge model is equivalent to truncating the pyramid process and keeping only the central column, while in our application of the matrix analytic method, we keep the five central columns of the state process.

**Theorem 6.** (Proved in Appendix 10.9) For FJ-FA system with availability one and locality two, a strict upper bound on the average data download time is given as

$$
\mathbb{E} \left[ T_{FJ-FA} \right] < \frac{1}{\lambda} \left( \pi_0 \cdot 1_0^T - \pi_0,0,0 + \pi_1 \cdot (I - R)^{-2} + (I - R)^{-1} \right) \cdot 1_1^T, \tag{36}
$$

where the unknowns in the expression are described in Appendix 10.9.

We next present a method to analyze the system using our M/G/1 approximation (Prop. 4). As in Sec. 5, we first find estimates for the request service time probabilities $f_0$ and $f_1$, and then substitute these estimates in the M/G/1 approximation. The main difference here is that using the pyramid Markov process, we are able to find relatively tighter estimates for $f_0$ and $f_1$. 
High-traffic approximation: Suppose that the system operates very close to its stability limit, st. the queues at the servers are always nonempty and servers are always busy serving a request copy. This reduces the complexity of the system dynamics, as we can now describe the system state keeping track of \( n(\tau) = (n_\alpha(\tau), n_\beta(\tau)) \) defined in (35). System state in this case implements a birth-death Markov process as shown in Fig. 10 (right). Steady state balance equations are given as

\[
\alpha \cdot p_{i,0} = (\gamma + \beta) p_{i+1,0}, \quad \beta \cdot p_{0,i} = (\gamma + \alpha) p_{0,i+1}, \quad i \geq 0.
\]  

(37)

It is now straightforward to find the limiting state probabilities \( \lim_{\tau \to \infty} \Pr \{ n(\tau) = (i, j) \} = p_{i,j} \):

\[
p_{0,0} = \frac{\gamma^2 - (\alpha - \beta)^2}{\gamma(\alpha + \beta + \gamma)}, \quad p_{i,0} = \left( \frac{\alpha}{\beta + \gamma} \right)^i p_{0,0}, \quad p_{0,i} = \left( \frac{\beta}{\alpha + \gamma} \right)^i p_{0,0}, \quad \text{for } i \geq 1.
\]

We next use these results to obtain bounds on the request service time probabilities \( f_0 \) and \( f_1 \). Along the way, we find bounds on some other quantities that give insight into the system behavior.

For the sake of simplicity, we set both service rates \( \alpha \) and \( \beta \) at the recovery servers \( \mu \). We keep the service rate at the systematic server fixed at \( \gamma \). A request completes as soon as either its copy at the systematic server or both copies at the recovery group finish service. We next present bounds on the fraction of the requests completed by the systematic server or the recovery group.

**Theorem 7.** In FJ-FA system with availability one and locality two, let \( \hat{w}_s \) and \( \hat{w}_r \) be the fraction of requests completed by respectively the systematic server and the recovery group. Then we have

\[
\hat{w}_s \geq \frac{\gamma \cdot \nu}{\gamma \cdot \nu + 2\mu^2}, \quad \hat{w}_r \leq \frac{\mu^2}{\gamma \cdot \nu + 2\mu^2}; \quad \text{where } \nu = \gamma + 2\mu.
\]

(38)

**Proof.** Suppose that the system operates under high traffic. Then we will find expressions for \( \hat{w}_s \) and \( \hat{w}_r \) from the steady state probabilities of the Markov chain embedded in the state process \( n(\tau) \). Notice that we use \( \cdot \) on top of these variables to indicate that these expressions will only serve us to find bound for the actual values of \( w_s \) and \( w_r \). Recall that \( n(\tau) \) is illustrated in Fig. 10 (right).

System stays at each state for an exponential duration of rate \( \nu = \gamma + 2\mu \). Therefore, steady state probabilities \( p_i \)'s (i.e., limiting fraction of the time spent in state \( i \)) of \( n(\tau) \) and the steady state probabilities \( \pi_i \)'s (i.e., limiting fraction of the state transitions into state \( i \)) of the embedded Markov chain are equal. This is easily seen by the equality \( \pi_i = p_i \cdot \nu / \sum_{j \geq 0} p_j \cdot \nu = p_j \).

Let \( f_s \) be the limiting fraction of state transitions that represent request completions by the systematic server. Let \( f_r \) denote the same quantity for the recovery group. We have

\[
f_s = \pi_{0,0} \cdot \gamma / \nu + \sum_{i=1}^{\infty} \pi_{i,0} \cdot \gamma / \nu + \sum_{i=1}^{\infty} \pi_{0,i} \cdot \gamma / \nu = \gamma / \nu, \quad f_r = \sum_{i=1}^{\infty} (\pi_{0,i} + \pi_{i,0}) \cdot \mu / \nu = 2(\mu / \nu)^2.
\]

Limiting fraction of all state transitions that represent a request departure is then found as

\[
f_d = f_s + f_r = (\gamma \cdot \nu + 2\mu^2) / \nu^2.
\]

Thus the fractions of request completions by the systematic server and by the recovery group are

\[
\hat{w}_s = \frac{f_s}{f_d} = \frac{\gamma \cdot \nu}{\gamma \cdot \nu + 2\mu^2}, \quad \hat{w}_r = \frac{f_r}{f_d} = \frac{2\mu^2}{\gamma \cdot \nu + 2\mu^2}.
\]

Recall that these values are calculated using the high traffic approximation, where all queues have unlimited number of requests and servers are never idle. However when the system operates under stability, there will always be a finite number of requests in the system, and thus the recovery servers would regularly go idle. That is why the value we found using the high traffic approximation for fraction of the request completions at the recovery group is smaller than its actual value under stability. Hence we conclude \( w_r \leq \hat{w}_r \). Given that \( w_s = 1 - w_r \), we directly have \( w_s \geq \hat{w}_s \). \( \square \)
In FJ-FA system with availability 1 and locality 2, request service time probabilities are bounded as

\[ f_0 \geq \frac{\gamma \cdot v}{\gamma \cdot v + 2\mu^2}, \quad f_1 \leq \frac{2\mu^2}{\gamma \cdot v + 2\mu^2}; \quad \text{where } v = \gamma + 2\mu. \] (39)

Proof. Suppose that the system operates under high traffic approximation. Consider the state process \( n(\tau) \) under high traffic assumption. Again here we will use the steady state probabilities \( \pi_{i,j} \)'s for the Markov chain embedded in \( n(\tau) \). State transitions that are towards the center state \((0, 0)\) represent request completions. Let us define \( f_d \) as the fraction of such state transitions.

Under high traffic regime, a request always starts being serviced as soon as its predecessor at service start. On the other hand, every time system transitions into any other state, a new request makes a type-0 service start. Let \( f_{\rightarrow 0} \) and \( f_{\rightarrow 1} \) be the fractions of state transitions that represent type-0 and type-1 request service starts. We then find

\[ f_d = \pi_{0,0} \cdot \frac{\gamma}{v} + \sum_{i=1}^{\infty} (\pi_{0,i} + \pi_{i,0}) \cdot \frac{(\mu + \gamma)}{v} = \frac{(2\mu^2 + 2\mu \cdot \gamma + \gamma^2)}{v^2}; \]

\[ f_{\rightarrow 0} = \pi_{0,0} \cdot \frac{\gamma}{v} + \pi_{1,0} \cdot \frac{(\mu + \gamma)}{v} + \pi_{0,1} \cdot \frac{(\mu + \gamma)}{v} = \pi_{0,0} \left( \frac{\gamma}{v} + \frac{2\mu}{\mu + \gamma} \cdot \frac{\mu + \gamma}{v} \right) = \pi_{0,0} = \frac{\gamma}{\gamma + 2\mu}. \]

Thus, the limiting fraction of the requests that make type-0 (\( \hat{f}_0 \)) or type-1 (\( \hat{f}_1 \)) service start are

\[ \hat{f}_0 = \frac{f_{\rightarrow 0}}{f_d} = \frac{\gamma}{\gamma \cdot v + 2\mu^2}, \quad \hat{f}_1 = 1 - \hat{f}_0 = \frac{2\mu^2}{\gamma \cdot v + 2\mu^2}, \quad \text{where } v = \gamma + 2\mu. \]

Recall that we derived \( \hat{f}_0 \) and \( \hat{f}_1 \) using the high traffic approximation, where all queues are always busy. We essentially treat the system’s life cycle as a single, very long busy period. However under stability, system has to empty out regularly. Each request that finds the system empty makes type-0 service start. Idle periods seldom happen, which makes it less likely for a request to have type-0 service time distribution. That is why value we find for \( f_0 \) using the high traffic approximation is smaller than its actual value under stability. Thus we conclude \( \hat{f}_0 \leq f_0 \) and \( \hat{f}_1 \geq f_1 \).

Comparison of approximations: Prop. 4 approximates FJ-FA system as an M/G/1 queue, which together with the PK formula gives us an approximate expression for the average download time.
This approximation requires the first and second moments of the service time distribution for an arbitrary request arrival (15). Substituting the bounds in Thm. 8 in place of the actual probabilities $f_0$ and $f_1$ yields the following lower bounds on the service time moments, because $S_1 < S_0$:

\[
\mathbb{E}[S] \geq \frac{1}{3} \left( \frac{2}{y + \mu} - \frac{1}{y + 2\mu} \right) + \frac{2}{3} \cdot \frac{1}{y + \mu}, \quad \mathbb{E}[S^2] \geq \frac{1}{3} \left( \frac{4}{(y + \mu)^2} - \frac{2}{(y + 2\mu)^2} \right) + \frac{2}{3} \cdot \frac{2}{(y + \mu)^2}.
\]

Substituting these bounds for $\mathbb{E}[S]$ and $\mathbb{E}[S^2]$ in (14) gives a lower bound on the download time in FJ-FA with availability one and locality two. Since (14) is not exact but an approximation, lower bound derived on the download time as described above can only be claimed to be an approximate one. Simulated average data download times are compared with the approximation in Fig. 12. Approximation performs very well in predicting the actual average download time, especially compared to the Split-Merge and Fast-Split-Merge bounds given in Lemma 6 and 8.

9 CONCLUSIONS
We presented a queueing theoretic analysis for download time in storage systems with availability codes. We considered the well-known Fork-Join queueing model for data download with redundant requests. For the low-traffic regime where no queuing is involved, we gave a comparison between the download times in replicated, MDS and availability coded storage systems. We demonstrated that availability codes can reduce download time in some settings, but are not always optimal. In the absence of low-traffic assumption, availability codes give rise to multi-layer inter-dependent Fork-Join queues, which are intractable for exact analysis due to the infamous state explosion problem. For this case we derived bounds and approximations on the download time by devising systems that are tractable variants of the Fork-Join system. Further, by focusing on an important class of availability codes with locality two, we presented finer approximations by using ideas from Markov process analysis and Renewal theory. In the process, we conjectured an interesting relationship between the service time probabilities, each of which represents the probability with which a request experiences the service time distribution of a specific type. Our bounds and approximations allowed us to give insights on the system performance for skewed popularities, which is a practically appealing scenario.

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10 APPENDIX
10.1 Proof of Proposition 1

We first prove (1). We begin with computing the complementary CDF of \( S_{r,t} \) as follows.

\[
\Pr \{ S_{r,t} > s \} = \Pr \{ (S_{r})_{t+1} > s \}^{(a)} \Pr \{ S_{r} > s \}^{(b)} \equiv (1 - \Pr \{ S \leq s \})^{(c)} \equiv (1 - (1 - \exp(-\mu \cdot s)))^{(d)} \cdot (1 - (1 - \exp(-\mu \cdot s)))^{(e)} \cdot (1 - (1 - \exp(-\mu \cdot s)))^{(f)}.
\]

where (a) and (b) follow from the independence of service times, and (c) comes by substituting \( \Pr \{ S > s \} = \exp(-\mu \cdot s) \). Since \( S_{r,t} \) is a non-negative r.v., we have

\[
\mathbb{E} \left[ S_{r,t} \right] = \int_{0}^{\infty} \Pr \{ S_{r,t} > s \} \, ds = \int_{0}^{\infty} (1 - (1 - \exp(-\mu \cdot s)))^{(d)} \cdot (1 - (1 - \exp(-\mu \cdot s)))^{(e)} \cdot (1 - (1 - \exp(-\mu \cdot s)))^{(f)} \, ds.
\]

where (d) follows from the binomial expansion with respect to powers \( t \) and \( r \), (e) is obtained by gathering together the terms in inner summation for \( j = 0 \) and using \( \sum_{j=0}^{t} \binom{t}{j} (-1)^{j} \cdot \sum_{j=1}^{r} \binom{r}{j} (-1)^{j} \cdot \exp(-\mu \cdot s) = 0 \), (f) is obtained by interchanging the order of summation and integration, and (g) is obtained using the identity \( H_{n} = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i-1} / i \).

Upper bound: Here we derive (2). From (41) we have

\[
\mathbb{E} \left[ S_{r,t} \right] = \int_{0}^{\infty} (1 - (1 - \exp(-\mu \cdot s)))^{(d)} \cdot (1 - (1 - \exp(-\mu \cdot s)))^{(e)} \cdot (1 - (1 - \exp(-\mu \cdot s)))^{(f)} \, ds \leq \frac{1}{\mu} \int_{0}^{1} \frac{1 - u^{r}}{1 - u} \cdot (1 - u^{r})^{t-1} \, du = \frac{1}{\mu} \int_{0}^{1} \frac{1 - u^{r}}{1 - u} \cdot (1 - u^{r})^{t-1} \, du = \frac{\beta(t, 1/r)}{\mu \cdot r} \cdot H_{r},
\]

where (h) follows from substituting \( 1 - \exp(-\mu \cdot s) = u \). To obtain (i), we use Harris inequality [8, Theorem 2.15], which says that, for a r.v. \( X \), if \( g(X) \) is a non-decreasing function and \( h(X) \) is a non-increasing function, then \( \mathbb{E} [g(X) \cdot h(X)] \leq \mathbb{E} [g(X)] \cdot \mathbb{E} [h(X)] \). We consider the r.v. \( u \) distributed uniformly over \([0, 1]\), and functions \( g(u) = (1 - u^{r})/(1 - u) \) and \( h(u) = (1 - u^{r})^{t-1} \). To obtain (j), we use the integral representation of a harmonic number given as \( H_{r} = \int_{0}^{1} (1 - u^{r})/(1 - u) \, du \), and we obtain \( \int_{0}^{1} (1 - u^{r})^{t-1} \, du = \frac{1}{r} \beta(t, 1/r) \) by substituting \( 1 - u^{r} = v \).
Lower bound: First, observe that, being an exponential r.v., $S$ is a log-concave r.v. [7]. It is straightforward to show that $S_{r,r}$ is also log-concave. Next, we use a result derived in [29] that for a log-concave r.v. $X$, we have $E[X] \leq t \ E[X_{t,1}]$. To obtain the lower bound, recall that $S_{r,t} = (S_{r,r})_{t,1}$, and that $E{S_{r,r}} = H_r/\mu$ (see, e.g., [46]).

10.2 Proof of Proposition 2

The probability that repair groups are slower than the systematic server is given as

$$
Pr\{S_{r,t} > S\} = \int_0^\infty Pr\{S_{r,t} > s\} f_S(s)ds \overset{(a)}{=} \int_0^\infty (1 - (Pr\{S \leq s\})^t) f_S(s)ds \overset{(b)}{=} \int_0^\infty (1 - (1 - e^{-\mu s}))^t \mu e^{-\mu s}ds \overset{(c)}{=} \frac{1}{r} \beta(t + 1, \frac{1}{r}),
$$

where (a) comes from (40), (b) follows from $S \sim \text{Exp}(\mu)$, and (c) follows by substituting $1 - (1 - e^{-\mu s})^t = \nu$.

10.3 Proof of Lemma 1

Notice that, for fairness of comparison, we consider that the replication scheme has the same cumulative mean service time as that of the system using an $(n, k, r, t)$-LRC. Therefore, if $\mu_r$ is the mean service time of a node in the replication scheme, then we have $n_r \cdot \mu_r = n \cdot \mu$, where $n_r = t_r \cdot k$ is the total number of nodes in the replication scheme. Therefore, we have $\mu_r = (n \cdot \mu)/(t_r \cdot k)$.

Now, in the replication scheme, a request is forked into all the $t_r$ replicas of the systematic node containing the file, and the request is complete when any one of the replicas fetches the file. Thus, the mean download time behaves as the minimum of $t_r$ independent, exponential random variables each with rate $\mu_r$. Therefore, we have $E{T_{FJ,t_r}} = (t_r \cdot \mu_r)^{-1}$. Substituting the value of $\mu_r$ gives the result (11).

10.4 Proof of Lemma 2

We first find the complementary CDF of $T_{FJ,(n,k)}$ as follows.

$$
Pr\{T_{FJ,(n,k)} > s\} \overset{(a)}{=} Pr\{\min\{S, S_{(n-1),k}\} > s\} \overset{(b)}{=} Pr\{S > s\} \cdot Pr\{S_{(n-1),k} > s\} \overset{(c)}{=} Pr\{S > s\} \cdot \sum_{j=0}^{k-1} \binom{n-1}{j} Pr\{S \leq s\}^j \cdot Pr\{S > s\}^{n-j-1} \overset{(d)}{=} \exp(-\mu s) \cdot \sum_{j=0}^{k-1} \binom{n-1}{j} (1 - \exp(-\mu s))^j \cdot \exp(-\mu(n-j-1)s) = \sum_{j=0}^{k-1} \binom{n-1}{j} \exp(-\mu(n-j)s) \cdot (1 - \exp(-\mu s))^j,
$$

(44)

where (a) follows from (12) and (b) follows from the independence of service times at the servers. To obtain (c), we use the standard expression for the CDF of the $k$th order statistics of $n - 1$ i.i.d. random variables, which is given as

$$
Pr\{S_{(n-1),k} \leq s\} = \sum_{j=k}^{n-1} \binom{n-1}{j} Pr\{S \leq s\}^j \cdot Pr\{S > s\}^{n-j-1}.
$$

Finally, (d) follows from the fact that $S \sim \text{Exp}(\mu)$.
Now, observing that $T_{\text{FJ}-(n,k)}$ is a non-negative random variable, we have

$$\mathbb{E} \left[ T_{\text{FJ}-(n,k)} \right] = \int_{s=0}^{\infty} \Pr \{ T_{\text{FJ}-(n,k)} > s \} \, ds$$

\[ = \int_{s=0}^{\infty} \sum_{j=0}^{k-1} \binom{n-1}{j} \exp(-\mu(n-j)s) \cdot (1 - \exp(-\mu \cdot s))^j \, ds \]

\[ = \sum_{j=0}^{k-1} \binom{n-1}{j} \int_{s=0}^{\infty} \exp(-\mu(n-j)s) \cdot (1 - \exp(-\mu \cdot s))^j \, ds \]

\[ \leq \sum_{j=0}^{k-1} \binom{n-1}{j} \frac{1}{\mu} \int_{v=0}^{1} v^j (1-v)^{n-j-1} dv \]

where (e) follows by substituting (44), (f) follows from changing the order of summation and integration, (g) follows from change of variables $v = 1 - \exp(-\mu \cdot s)$, and (h) follows from the definition of $\beta(x, y)$ function. Finally, (i) is obtained by using $\beta(x, y) = \frac{(x-1)!}{(x+y-1)!}$, when $x$ and $y$ are positive integers.

### 10.5 Details of Reliability Analysis in Sec. 3.5

We consider the standard Markov model to analyze the reliability. In particular, the model focuses on the reliability of a representative stripe consisting of $n$ blocks obtained by encoding the $k$ data blocks. Each state in the Markov chain represents the number of available blocks out of the $n$ blocks. The Markov chain transitions are specified by the rates at which block failures and recoveries happen. We assume independent failures, and the time between failures is exponentially distributed. It is worth noting that these assumptions are considered to be reasonable approximations for analyzing real-world distributed storage systems (see [16, 24, 48] for details).

As an example, Figure 13 shows the Markov chain corresponding to the $(7, 3, 2, 3)$-simplex code. Here $\lambda_f$ denotes the failure rate of a single block. The transition from the state of 4 available blocks warrants more explanation. Unlike MDS codes, wherein the possibility of system recovery depends only on the number of failures, the possibility of recovery in availability codes (and in general in LRCs) depend on which blocks are failed. Let $3F$ represent a state with four non-recoverable failures, while 3 denote a state with four recoverable failures. Let $p_d$ denote the fraction of recoverable four failures cases. Then, the transition rate from state 4 to 3 is $4\lambda_f p_d$, and the transition rate from 4 to $3F$ is $4\lambda_f (1 - p_d)$.

![Fig. 13. The Markov model for (7, 3, 2, 3)-simplex code.](image)

The rate at which a block is repaired depends on how many blocks need to be downloaded for the repair, the block size, and the download rate. To compute the recovery rates, we use the same assumptions as in [24]. Specifically, we assume that there are $M$ storage nodes in the system, each having $S$ storage space, and $B$ network bandwidth. When a storage node fails, we assume that the repair load is evenly distributed among the remaining $(M - 1)$ nodes. In addition, we assume that
the repair traffic is throttled so that it only uses $\epsilon$ fraction of the network bandwidth on each node. The average number of blocks required to repair a single block is denoted as the repair cost $C$, which is a function of the locality of the code. Then, the average repair rate of single-block failures for is $\rho_1 = \epsilon(M - 1)B/(SC)$.

For multiple block failures, we consider a fixed repair rate determined by the time $T$ needed to detect and trigger the repair. This is because, in practice, multiple block repairs are prioritized over single-failure ones, and as a result, multi-failure repairs are fast and take considerably less time (see [24] for details). Therefore, we set $\rho_6 = \rho_5 = \rho_4 = 1/T$.

For $(7,3,2,3)$-code, the locality for any block is 2, and thus the repair cost is $C = 2$. Further, we enumerate all four block failures to obtain that the fraction of recoverable four block failures is $p_d = 80\%$.

We consider the same approach for other erasure codes. For MDS codes, we use $C = k$, since all the $k$ blocks are required to repair any block; whereas for 3-replication, we use $C = 1$. For $(10,6,2,1)$-LRC the Markov chain can be found in [24, Figure 3]. The Markov chain used for MDS codes is similar to [48, Figure 3]. The 3-replication scheme can be considered as a (3,1)-MDS code.

To evaluate the MTTDL, we use the following typical set of parameters used in [24]: $M = 400$, $S = 16TB$, $B = 1Gbps$, $\epsilon = 0.1$, $T = 30$ minutes, and $\lambda_F = 450$ days.

10.6 Proof of Lemma 4

State of the system is defined by the queue lengths across the servers. System state defines a Markov process because the service times at the servers and the request inter arrival times are Exponentially distributed. Let $Q_i$ be the system state seen by the $i$th request arrival. Life cycle of the $i$th request from its arrival to departure is determined by $Q_i$. That means if $Q_i$ and $Q_j$ are equal, then the life cycle of both $i$th and $j$th request arrivals will be governed by the same stochastic process. Therefore $\Pr\{J_i = \nu | Q_i = q\}$ does not depend on $i$. Probability that an arbitrary arrival in the steady state will have type-$\nu$ service time is given as

$$\lim_{i \to \infty} \Pr\{J_i = \nu\} = \sum_q \Pr\{J = \nu | Q = q\} \cdot \lim_{i \to \infty} \Pr\{Q_i = q\}.$$ 

Fraction of the requests that have type-$\nu$ service time is given as

$$f_\nu = \sum_q \Pr\{J = \nu | Q = q\} \cdot \Pr\{Q = q\}$$

where $\Pr\{Q = q\}$ denotes the fraction of arrivals that find system in state $q$. Defining $p_q$ as the fraction of the time system spends in state $q$, we know by ergodicity and PASTA [58] that

$$\lim_{i \to \infty} \Pr\{Q_i = q\} = \Pr\{Q = q\} = p_q.$$ 

This gives us (25). It is worthy to note that ergodicity holds for the system state under stability.

10.7 On the Conjecture 1

We here present a Theorem that helps to build an intuition for Conj. 1. Recall that we consider FJ-FA system with locality $r = 2$. Let us redefine the system state as the service type (see Lemma 3 for details) of service start for the request at the head of the system, so $S \in \{0, \ldots , t\}$. System state transitions correspond to time epochs at which a request moves into service (see Def. 1 for the definition of request service start times).

Given that Conj. 1 holds i.e., $f_j > f_{j+1}$ for $j = 0, \ldots , t - 1$, one would expect the average drift at state-$j$ to be towards the lower rank state-$i$'s with $i < j$. In the Theorem below we prove this indeed holds. However, it poses neither a necessary nor a sufficient condition for Conj. 1. The biggest in
Theorem 9. In FJ-FA system, let $J_i$ be the type of service time distribution for request $i$. Then
\[ \Pr \{ J_{i+1} > j \mid J_i = j \} < 0.5. \]
In other words, given that a request with type-$j$ service time distribution, the subsequent request is more likely to be served with type-$i$ distribution such that $i < j$.

Proof. Let us define $L_k(t)$ as the absolute difference of queue lengths at repair group $k$ and time $t$. Suppose that the $i$th request moves in service at time $\tau$ and its service time is type-$j$; namely $J_i = j$. Let also $A$ denote the event that $L_k(\tau) > 1$ for every repair group $k$ that has a leading server at time $\tau$, we refer to other recovery groups as non-leading. Then, the following inequality holds
\[ \Pr \{ J_{i+1} > j \mid J_i = j, A \} > \Pr \{ J_{i+1} > j \mid J_i = j \}. \]
(45)

Event $A$ guarantees that $J_{i+1} \geq j$ i.e., $\Pr \{ J_{i+1} \geq j \mid J_i = j, A \} = 1$, because even when none of the leading servers advances before the $i$th request departs, the next request will make at least type-$j$ start.

We next try to find
\[ \Pr \{ J_{i+1} > j \mid J_i = j, A \} = 1 - \Pr \{ J_{i+1} = j \mid J_i = j, A \}. \]

Suppose that the $i$th request departs at time $\tau'$ and without loss of generality, let the repair group $k$ has a leading server only if $k \leq j$. The event $\{ J_{i+1} = j \mid J_i = j, A \}$ is equivalent to the event
\[ B_j = \{ L_k(\zeta) < 2; \, \zeta \in [\tau, \tau'] , \, j < k \leq t \} \quad \text{for } 0 < j < t - 1. \]
(46)

This is because for the $(i + 1)$th request to have type-$(j + 1)$ service time, one of the servers in at least one of the non-leading recovery groups should advance by at least two replicas before the $i$th request terminates.

Event $B_j$ can be re-expressed as
\[ B_j = \bigcup_{l=0}^{t-j} C_l \quad \text{where } C_l = \{ L_k(\tau) = 1; \, 1 \leq i \leq l , j < k_i \leq t \}. \]

Event $C_l$ describes that $l$ non-leading recovery groups start leading by one replica before the departure of $i$th request. Given that there are currently $j$ leading recovery groups, let $p_{j+1}^l$ be the probability that a non-leading repair group starts to lead by one before the departure of $i$th request, and $p_j^l$ be the probability that $i$th request departs first. Then, we can write
\[ \Pr \{ C_l \} = p_{j+1}^l \cdot \prod_{k=j}^{j+l-1} p_k^1. \]

Since the events $C_l$ for $l = 0, \ldots, t-j$ are disjoint, $\Pr \{ B_j \} = \sum_{l=0}^{t-j} \Pr \{ C_l \}$ and we get the recurrence relation
\[ \Pr \{ B_j \} = p_j^l + p_{j+1}^l \cdot \Pr \{ B_{j+1} \}. \]

Since the service times at the servers are exponentially distributed, the probabilities are easy to find as
\[ p_j^l = \frac{1 + j}{1 + 2t}, \quad p_{j+1}^l = \frac{2(t-j)}{1 + 2t}. \]
Then we find
\[
\Pr\{B_{t-1}\} = p_{t-1}^i + p_{t-1}^j \cdot p_t^i = \frac{1 + (t-1)}{1 + 2t} + \frac{2}{1 + 2t} \cdot \frac{1 + t}{1 + 2t} = \frac{1 + t}{1 + 2t} \cdot \frac{1}{(1 + 2t)^2} > \frac{1}{2}
\]

Suppose \(\Pr\{B_{j+1}\} > 0.5\), then we have
\[
\Pr\{B_j\} = \frac{1 + j}{1 + 2t} + \frac{2(t - i)}{1 + 2t} \cdot \Pr\{B_{j+1}\} > \frac{1 + i}{1 + 2t} + \frac{(t - i)}{1 + 2t} = \frac{1 + t}{1 + 2t} \geq \frac{1}{2}
\]

Knowing that \(\Pr\{B_{t-1}\} > 0.5\) together with \(\Pr\{B_k\} > 0.5\), and given that \(\Pr\{B_{k+1}\} > 0.5\) we have \(\Pr\{B_j\} > 0.5\) for each \(j\).

By (45) and (46), we find
\[
\Pr\{J_{i+1} > j \mid J_i = j\} < \Pr\{J_{i+1} > j \mid J_i = j, A\} = 1 - \Pr\{B_j\} < 0.5
\]
which tells us that for any request with type-\(j\) service time, the subsequent request is more likely to have service time with type less than \(j\).

\section{10.8 Approximate analysis of the Markov process for FJ-FA with availability one and locality two}

We here give an approximate analysis of the Markov process illustrated in Fig. 10 with a guessing based local balance equations approach. Consider the case in which both recovery servers operate at the same rate, i.e., \(\alpha = \beta = \mu\), which makes the pyramid process symmetric around the center column, i.e., \(p_{k,(i,0)} = p_{k,(0,i)}\) for \(1 \leq i \leq k\). As we do in the following, it is sufficient to present the analysis only for the states on the right side of the pyramid.

Observe that under low offered load, system spends almost the entire time in states \((0, (0, 0))\), \((1, (0, 0))\), \((1, (0, 1))\) and \((1, (1, 0))\). Given this observation, notice that the rate entering into \((1, (0, 0))\) due to request arrivals is equal to the rate leaving the state due to request departures. To help with guessing the steady-state probabilities, we start with the assumption that rate entering into a state due to request arrivals is equal to the rate leaving the state due to request departures. This gives us the following relation between the steady-state probabilities of the column-wise subsequent states:
\[
p_{k,(i,0)} = \frac{\lambda}{\gamma + 2\mu} \cdot p_{k-1,(i,0)}, \quad \text{for } 0 \leq i \leq k. \tag{47}
\]

Let us define \(\tau = \lambda/(\gamma + 2\mu)\), which allows us to write \(p_{k,(i,0)} = \tau^k \cdot p_{k,(i,0)}\). However this obviously won’t hold for higher arrival rates since at higher arrival rates some requests wait in the queue, which requires the rate entering into a state due to request arrivals to be higher than the rate leaving the state due to the completion of request copies. To be used in the following discussion, first we write \(p_{1,(i,0)}\) in terms of \(p_{0,(0,0)}\) from the global balance equations as
\[
\lambda \cdot p_{0,(0,0)} = \gamma \cdot p_{1,(0,0)} + 2(\gamma + \mu) \cdot p_{1,(1,0)} \quad \text{and} \quad p_{1,(1,0)} = \frac{\lambda - \gamma \cdot \tau}{2(\gamma + \mu)} \cdot p_{0,(0,0)}.
\]

For the nodes at the far right side of the pyramid, we can write the global balance equations as
\[
p_{l,(i,0)} \cdot (\lambda + \mu + \gamma) = p_{l,(i-1,0)} \cdot \mu + p_{l+1,(i+1,0)} \cdot (\mu + \gamma), \quad \text{for } i \geq 1,
\]
\[
p_{l+2,(i+2,0)} = b \cdot p_{l+1,(i+1,0)} + a \cdot p_{l,(i,0)}, \quad \text{for } i \geq 0,
\]

and
where \( b = 1 + \lambda/(\mu + \gamma) \) and \( a = -\tau \cdot \mu/(\gamma + \mu) \). We can then solve the recurrence relation given by the above balance equations as

\[
\begin{align*}
\psi_{l,j}(i,0) &= \frac{A}{r_0^i} + \frac{B}{r_1^i}, \\
\psi_{k,j}(i,0) &= \psi_{k,j}(0,i) = \tau^{k-i} \left( \frac{A}{r_0^i} + \frac{B}{r_1^i} \right), \quad \text{for } 0 \leq i \leq k.
\end{align*}
\]

where \( A \) and \( B \) are given by

\[
A = \psi_{0,0}(0,0) - B, \quad B = \frac{r_0 \cdot \psi_{0,0}(0,0) + (\psi_{1,0}(0,0) - b \cdot \psi_{0,0}(0,0)) \cdot r_0 \cdot r_1}{r_0 - r_1}
\]

for \( r_0 \) and \( r_1 \) that are given as

\[
r_0 = \frac{-b - \sqrt{\Delta}}{2a}, \quad r_1 = \frac{-b + \sqrt{\Delta}}{2a}, \quad \text{where } \Delta = b^2 + 4a.
\]

Even though the algebra does not permit much cancellation, one can find the unknowns \( A \) and \( B \) above by computing \( \psi_{0,0}(0,0) \) as follows.

\[
\sum_{k=0}^{\infty} \psi_{k,0}(0,0) + \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \left( \psi_{k,0}(l,0) + \psi_{0,0}(k,0) \right) = \frac{\psi_{0,0}(0,0)}{1 - \tau} + \frac{2}{1 - \tau} \sum_{l=1}^{\infty} \psi_{l,0}(0,0) \quad (\tau < 1)
\]

\[
= \frac{\psi_{0,0}(0,0)}{1 - \tau} + \frac{2}{1 - \tau} \sum_{l=1}^{\infty} \left( \frac{A}{r_0^l} + \frac{B}{r_1^l} \right) = \frac{\psi_{0,0}(0,0)}{1 - \tau} + \frac{2}{1 - \tau} \left( \frac{A}{r_0 - 1} + \frac{B}{r_1 - 1} \right) \quad \text{(from above; } r_0, r_1 > 1 \text{)}
\]

\[
= \frac{\psi_{0,0}(0,0)}{1 - \tau} + \frac{2}{1 - \tau} \left( \frac{(\psi_{0,0}(0,0) - B)(r_1 - 1) + B(r_0 - 1)}{(r_1 - 1)(r_0 - 1)} \right)
\]

\[
= \frac{\psi_{0,0}(0,0)}{1 - \tau} + \frac{2}{1 - \tau} \left( \frac{B(r_0 - r_1) + \psi_{0,0}(0,0)(r_1 - 1)}{(r_1 - 1)(r_0 - 1)} \right)
\]

\[
= \frac{\psi_{0,0}(0,0)}{1 - \tau} + \frac{2}{1 - \tau} \left( \frac{r_0 \psi_{0,0} + r_0 r_1 (\psi_{1,0}(0,0) - b \psi_{0,0}(0,0)) + \psi_{0,0}(0,0)(r_1 - 1)}{(r_1 - 1)(r_0 - 1)} \right)
\]

\[
= \psi_{0,0}(0,0) \left[ 1 + \frac{2}{1 - \tau} \frac{r_0 + r_0 r_1 ((\lambda - \gamma) r_1)/(2(\gamma + 2\mu) - b) + r_1 - 1}{(r_1 - 1)(r_0 - 1)} \right] = 1.
\]

Simulation results show that the model for \( \psi_{k,0}(l,0) \) discussed above is proper in structure i.e., \( \psi_{k,0}(l,0) \) decreases exponentially in \( k \) and \( l \). However, simulations also show that the correct form of \( \tau \) is given as \( \tau(\lambda) = k(\gamma, \mu) \cdot \lambda/(\gamma + 2\mu) \). For instance, when the service rates are equal across all servers, i.e., \( \gamma = \alpha = \beta = \mu \) we found that \( k(\gamma, \mu) \approx 0.3 \). Nevertheless this does not permit to find a general expression for \( k(\gamma, \mu) \).

### 10.9 Matrix analytic solution

We here find an upper bound on the average download time in FJ-FA with \( r = 2 \) and \( t = 1 \), which is provably tighter than the Split-Merge upper bound given for general values of \( r \) and \( t \) in Theorem 3. Let us truncate the system state process, which is shown in Fig. 10, such that the pyramid is limited to only the five central columns and infinite only in the vertical dimension. This choice is intuitive given Conj. 1; most requests have type-0 request service time distribution because the system spends most of its time at the central states, the frequency of requests with type-\( j \) service time distribution gets smaller for increasing values of \( j \) since the system spends increasingly less time at the states which are further away in the wings of the pyramid Markov process. In addition the approximate analysis of the pyramid Markov process given in Appendix 10.8 suggests that the most frequently visited system states are located at the central columns.
10.9.1 Computing the steady state probabilities. Finding a closed form solution for the steady state probabilities of the states in the truncated process is as challenging as the original problem. However one can solve the truncated process numerically with an arbitrarily small error using the Matrix Analytic method [22, Chapter 21]. In the following, we denote the vectors and matrices in bold font. Let us first define the limiting steady state probability vector as

\[
\pi = \begin{bmatrix}
\pi_{0,0}, & \pi_{1,0}, & \pi_{1,1}, & \pi_{1,2}, & \pi_{2,0}, & \pi_{2,1}, & \pi_{2,2}, & \pi_{3,0}, & \pi_{3,1}, & \pi_{3,2}, & \cdots
\end{bmatrix},
\]

where \(\pi_{k,i,j}\) is the steady state probability for state \((k, (i, j))\) and

\[
\pi_0 = [\pi_{0,0}, \pi_{1,0}, \pi_{1,1}, \pi_{1,2}],
\]

\[
\pi_i = [\pi_{i+1,0}, \pi_{i+1,1}, \pi_{i+1,2}, \pi_{i+1,3}],
\]

for \(i \geq 1\). One can write the balance equations governing the limiting probabilities in the form below

\[
\pi \cdot Q = 0
\]

For the truncated process, \(Q\) has the following form

\[
Q = \begin{bmatrix}
F_0 & H_0 \\
L_0 & F \\
L & F \\
0 & L & F & H \\
\cdot & \cdot & \cdot & \cdot
\end{bmatrix}
\]

where the sub-matrices \(F_0, H_0, L_0, F, L\) and \(H\) are given for \(\delta = \alpha + \beta + \gamma + \lambda\) as

\[
F_0 = \begin{bmatrix}
-\lambda & 0 & \lambda & 0 \\
\alpha + \gamma & -\delta & 0 & 0 \\
0 & \alpha + \gamma & 0 \\
0 & 0 & \beta + \gamma
\end{bmatrix}, \quad H_0 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & \lambda \\
\beta - \delta & 0 & 0 & 0
\end{bmatrix},
\]

\[
L_0 = \begin{bmatrix}
0 & 0 & \alpha + \gamma & 0 \\
0 & \alpha + \gamma & 0 \\
0 & 0 & \beta + \gamma \\
0 & 0 & 0 & \beta + \gamma
\end{bmatrix}, \quad F = \begin{bmatrix}
\beta - \delta & 0 & 0 & 0 \\
0 & \beta & -\delta & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

Using (49) and (50), we get the following system of equations in matrix form,

\[
\pi_0 \cdot F_0 + \pi_1 \cdot L_0 = 0,
\]

\[
\pi_0 \cdot H_0 + \pi_1 \cdot F + \pi_2 \cdot L = 0,
\]

\[
\pi_i \cdot H + \pi_{i+1} \cdot F + \pi_{i+2} \cdot L = 0, \quad i \geq 1.
\]
In order to solve the system above, we first assume the steady state probability vectors to be of the form
\[ \pi_i = \pi_1 \cdot R^{i-1}, \quad \text{for } i \geq 1 \text{ and } R \in \mathbb{R}^{5 \times 5}. \] (52)
Combining (51) and (52), we get
\[
\begin{align*}
\pi_0 \cdot F_0 + \pi_1 \cdot L_0 &= 0, \\
\pi_0 \cdot H_0 + \pi_1 \cdot (F + R \cdot L) &= 0, \\
\pi_i \cdot (H + R \cdot F + R^2 \cdot L) &= 0, \quad i \geq 1.
\end{align*}
\] (53)
From (53) we have common conditions for the system to hold
\[ H + R \cdot F + R^2 \cdot L = 0, \quad R = -(R^2 \cdot L + H) \cdot F^{-1}. \] (54)

The inverse of \( F \) in (54) exists since \( \det(F) = -\delta^3(\delta - \alpha)(\delta - \beta) \neq 0 \) assuming \( \delta = \alpha + \beta + \gamma + \lambda \) and \( \lambda > 0 \). Using (54), an iterative algorithm to compute \( R \) is given in Algorithm 1. The norm \( \|R_i - R_{i-1}\| \) corresponds to the absolute value of the largest element of the difference matrix \( R_i - R_{i-1} \). Therefore, the algorithm terminates when the largest difference between the elements of the last two computed matrices is smaller than the threshold \( \epsilon \). The initial matrix \( R_0 \) could take any value, not necessarily 0. The error threshold \( \epsilon \) could be fixed to any arbitrary value, but the lower this value the slower the convergence. Computing \( R \), vectors \( \pi_0 \) and \( \pi_1 \) are remaining to be found in order to deduce

**Algorithm 1 Computing matrix \( R \)**

1: procedure Computing \( R \)
2: \( \epsilon \leftarrow 10^{-6}, \quad R_0 \leftarrow 0, \quad i \leftarrow 1 \)
3: while true do
4: \( R_i \leftarrow -(R_i^{2} \cdot L + H) \cdot F^{-1} \)
5: if \( \|R_i - R_{i-1}\| > \epsilon \) then
6: \( i \leftarrow i + 1 \)
7: else return \( R_i \)

the values of all limiting probabilities. Recall that in (53), the first two equations are yet to be used. Writing these two equations in matrix form
\[
\begin{bmatrix}
\pi_0 & \pi_1
\end{bmatrix}
\begin{bmatrix}
F_0 & H_0 \\
L_0 & R \cdot L + F
\end{bmatrix} = 0,
\] (55)
where 0 is a \( 1 \times 9 \) zeros vector and
\[
\Phi = \begin{bmatrix}
F_0 & H_0 \\
L_0 & R \cdot L + F
\end{bmatrix} \in \mathbb{R}^{9 \times 9}.
\]
In addition, we have the normalization equation to take into account. Let \( \mathbf{v}^T \) be the transpose of vector \( \mathbf{v} \). Denoting \( 1_0 = [1, 1, 1, 1] \), \( 1_1 = [1, 1, 1, 1, 1] \) and using (52), we obtain
\[
\begin{align*}
\pi_0 \cdot 1_0^T + \sum_{i=1}^{\infty} \pi_i \cdot 1_1^T &= 1, \\
\pi_0 \cdot 1_0^T + \sum_{i=1}^{\infty} \pi_1 \cdot R^{i-1} \cdot 1_1^T &= 1, \\
\pi_0 \cdot 1_0^T + \pi_1 \cdot (I - R)^{-1} \cdot 1_1^T &= 1,
\end{align*}
\]
\[
\begin{bmatrix}
\pi_0 & \pi_1
\end{bmatrix}
\begin{bmatrix}
1_0^T \\
(I - R)^{-1} \cdot 1_1^T
\end{bmatrix} = 1,
\]
where \( I \) is the \( 5 \times 5 \) identity matrix. In order to find \( \pi_0 \) and \( \pi_1 \), we solve the following system
\[
\begin{bmatrix}
\pi_0 \\
\pi_1
\end{bmatrix} \cdot \Psi = [1, 0, 0, 0, 0, 0, 0, 0, 0].
\] (56)
where \( \Psi \) is obtained by replacing the first column of \( \Phi \) with \([1_0, 1_1(I - R^T)^{-1}]^T\). Hence, (56) is a linear system of 9 equations with 9 unknowns. After solving (56), we obtain the remaining limiting probabilities vector using (52).

10.9.2 Bounding the average download time. Let \( N_{ma} \) be the number of requests in the truncated system. First notice that
\[
\Pr \{N_{ma} = 0\} = \pi_{0,0},
\]
\[
\Pr \{N_{ma} = 1\} = \pi_{1,0} + \pi_{1,1} + \pi_{1,2} = \pi_0 \cdot 1^T_0 - \pi_{0,0},
\]
\[
\Pr \{N_{ma} = i\} = \pi_{i,0} + \pi_{i,1} + \pi_{i,2} = \pi_{i-1} \cdot 1^T_1, \quad i \geq 2.
\]
Then, the average number of requests in the truncated system is computed as
\[
\mathbb{E} [N_{ma}] = \sum_{i=0}^{\infty} i \cdot \Pr \{N_{ma} = i\} = \pi_0 \cdot 1^T_0 - \pi_{0,0} + \sum_{i=2}^{\infty} i \cdot \pi_{i-1} \cdot 1^T_1
\]  
\[= \pi_0 \cdot 1^T_0 - \pi_{0,0} + \sum_{i=2}^{\infty} i \cdot \pi_1 \cdot R^{i-2} \cdot 1^T_1
\]  
\[= \pi_0 \cdot 1^T_0 - \pi_{0,0} + \pi_1 \cdot \left( \sum_{i=2}^{\infty} (i-1)R^{i-2} + R^{i-2} \right) \cdot 1^T_1
\]  
\[= \pi_0 \cdot 1^T_0 - \pi_{0,0} + \pi_1 \cdot \left( \sum_{j=1}^{\infty} jR^{i-1} + \sum_{i=0}^{\infty} R^i \right) \cdot 1^T_1
\]  
\[= \pi_0 \cdot 1^T_0 - \pi_{0,0} + \pi_1 \cdot ((I - R)^{-2} + (I - R)^{-1}) \cdot 1^T_1.
\] (57)
Equation (57) shows that we only need \( \pi_0, \pi_1 \) and \( R \), thus no need to calculate the infinite number of limiting probabilities.

Now we are ready to show the upper bound presented on the download time in FJ-FA system with locality \( r = 2 \) and availability \( t = 1 \).

**Proof of Theorem 6.** Truncation of the state process is equivalent to imposing a blocking on the recovery group whenever one of the recovery server is ahead of its sibling by two replicas, which works slower than the actual system. Therefore, the average download time found for the truncated system is an upper bound on the actual download time and by Little’s law it is expressed as \( \mathbb{E} [N_{ma}] / \lambda \). \hfill \( \square \)