SHELLING TOTALLY NONNEGATIVE FLAG VARIETIES

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Abstract. In this paper we study the partially ordered set $Q^J$ of cells in Rietsch’s [20] cell decomposition of the totally nonnegative part of an arbitrary flag variety $P^J_{\geq 0}$. Our goal is to understand the geometry of $P^J_{\geq 0}$. Lusztig [13] has proved that this space is contractible, but it is unknown whether the closure of each cell is contractible, and whether $P^J_{\geq 0}$ is homeomorphic to a ball. The order complex $\|Q^J\|$ is a simplicial complex which can be thought of as a combinatorial approximation of $P^J_{\geq 0}$. Using combinatorial tools such as Bjorner’s EL-labellings [1] and Dyer’s reflection orders [7], we prove that $Q^J$ is graded, thin and EL-shellable. As a corollary, we deduce that $Q^J$ is Eulerian and that the Euler characteristic of the closure of each cell is 1. Additionally, our results imply that $\|Q^J\|$ is homeomorphic to a ball, and moreover, that $Q^J$ is the face poset of some regular CW complex homeomorphic to a ball.

1. Introduction

The classical theory of total positivity concerns matrices in which all minors are nonnegative. In the past decade Lusztig has extended this subject [12, 13, 14] by introducing the totally nonnegative variety $G_{\geq 0}$ in an arbitrary reductive group $G$ as well as the totally nonnegative part $B_{\geq 0}$ of a real flag variety $B$, which he called a “remarkable polyhedral subspace” [14]. Rietsch has constructed a cell decomposition [19] for the totally nonnegative part of an arbitrary flag variety $P^J_{\geq 0}$, and has described the order relation for closures of cells. The partially ordered set (poset) of cells, which we denote by $Q^J$, is intimately connected to the Bruhat order of the corresponding Weyl group $W$: for example, when $P^J$ is the complete flag variety, $Q^J$ is the interval poset of the Bruhat order, and when $P^J$ is the type A Grassmannian, $Q^J$ is Postnikov’s cyclic Bruhat order.

Lusztig [12] has proved that the totally nonnegative part of the (full) flag variety is contractible, which implies the same result for any partial flag variety [21]. However, it is unknown whether these spaces are homeomorphic to balls. In addition, the topology of the individual cells is not well understood: for example, it is unknown whether the closure of a cell is contractible.

The goal of this paper is to apply combinatorial methods to $Q^J$ in order to better understand the geometry of $P^J_{\geq 0}$. Indeed, the past thirty years have seen a wealth of literature designed to facilitate the interplay between combinatorics and geometry (for references related to the very important property of shellability, see [6], [1], [4], [5], [2]). In particular, in a 1984 paper [2], Bjorner recognized that regular CW complexes are combinatorial objects in the following sense: if $Q$ is the poset of closed cells in a regular CW decomposition of a space $P$, then the order
complex (or nerve) $\| \mathcal{Q} \|$ is homeomorphic to $P$. Furthermore, he gave criteria [2] for recognizing when a poset is the face poset of a regular CW complex: for example, if a poset is thin and shellable then it is the face poset of some regular CW complex homeomorphic to a sphere.

This theory has been applied to intervals in the Bruhat order of a Coxeter group. Bjorner and Wachs [4] proved that such intervals are thin and CL-shellable. Dyer later used reflection orders to prove Bjorner’s conjecture that such intervals are actually EL-shellable. Bjorner [2] pointed out that such intervals are therefore face posets of regular CW complexes homeomorphic to spheres and asked for a natural geometric construction of such a CW complex. Sixteen years later, Fomin and Shapiro [8] had the insight that such a construction might come from total positivity, and conjectured that links in the Bruhat decomposition of the totally nonnegative part of the unipotent radical of a semisimple group $G$ have the desired properties. Although they could not prove regularity of these stratified spaces, they proved contractibility of both the closed and open strata in type A.

In this paper we begin a line of research that in many ways parallels the story of the Bruhat order. Our main results are the following.

**Theorem 1.1.** $\mathcal{Q}^J$ is thin.

**Theorem 1.2.** $\mathcal{Q}^J$ is EL-shellable.

As a corollary, we deduce that $\mathcal{Q}^J$ is Eulerian and that the Euler characteristic of the closure of each cell is 1. Additionally, it follows from our main results that $\| \mathcal{Q}^J \|$ is homeomorphic to a ball, and that moreover, $\mathcal{Q}^J$ is the face poset of some regular CW complex homeomorphic to a ball.

The structure of this paper is as follows. We begin by reviewing tools from the topology of partially ordered sets, especially Bjorner’s notion of EL-shellability [1]. In Section 3 we review properties of the Bruhat order and define the reflection orders of Dyer [7]. In Section 4 we introduce the relevant notions from total positivity, and Rietsch’s [19, 20] description of the poset $\mathcal{Q}^J$ of cells in her cell decomposition of the totally nonnegative part of a flag variety $\mathcal{P}_{\geq 0}^J$. In Sections 5 and 6 we prove our main results: that the poset $\mathcal{Q}^J$ is thin and EL-shellable. We then use Bjorner’s theorem [2] to conclude that $\mathcal{Q}^J$ is the face poset of a regular CW complex homeomorphic to a ball. We conjecture that in fact the totally nonnegative part of an arbitrary flag variety is a regular CW complex homeomorphic to a ball.

Additionally, we include an appendix which gives a detailed explanation of some of the combinatorics of $\mathcal{Q}^J$ in the type A Grassmannian case, relating our present work to Postnikov’s [16].

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Given an edge labeling $\lambda$, we say that $c$ is, the set $\lambda$ can be associated with a complex whose vertices are the elements of $P$, its order complex.

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2. Background on poset topology

Poset topology is the study of combinatorial properties of a partially ordered set, or poset, which reflect the topology of an associated simplicial complex. In this section we will review some of the basic definitions and results of poset topology.

Let $P$ be a poset with order relation $. We will use the symbol $\preceq$ to denote the covering relation in the poset: $x \preceq y$ means that $x < y$ and there is no $z$ such that $x < z < y$. Additionally, if $x < y$ then $[x, y]$ denotes the interval from $x$ to $y$; that is, the set $\{z \in P \mid x \leq z \leq y\}$.

The natural geometric object that one associates to a poset $P$ is the realization of its order complex (or nerve). The order complex $\Delta(P)$ is defined to be the simplicial complex whose vertices are the elements of $P$ and whose simplices are the chains $x_0 < x_1 < \cdots < x_k$ in $P$. We will denote the realization of $\Delta(P)$ by $|\Delta(P)|$, or just $|P|$, and if we say that $P$ possesses some topological property, we will mean that $|P|$ possesses that property. As we will see later, it is particularly useful to study $|P|$ when $P$ is the face poset of a CW complex.

Recall that a poset is said to be bounded if it has a least element and a greatest element. These elements are denoted $\hat{0}$ and $\hat{1}$, respectively. A finite poset is said to be pure if all maximal chains have the same length, and graded, if in addition, it is finite and bounded. Any element $x$ of a graded poset $P$ has a well-defined rank $\rho(x)$ equal to the common length of all unrefinable chains from $\hat{0}$ to $x$ in $P$.

We now introduce the property of being thin.

**Definition 2.1.** A poset $P$ is called thin if every interval of length 2 is a diamond, i.e. if for any $p < q$ such that $\rho(q) - \rho(p) = 2$, there are exactly two elements in the open interval $(p, q)$.

Another useful property is that of shellability. This property is desirable to have because it implies, for example, that $\Delta$ is Cohen-Macaulay.

**Definition 2.2.** A pure finite simplicial complex $\Delta$ is said to be shellable if its maximal faces can be ordered $F_1, F_2, \ldots, F_n$ in such a way that $F_k \cap (\bigcup_{i=1}^{k-1} F_i)$ is a nonempty union of maximal proper faces of $F_k$ for $k = 2, 3, \ldots, n$.

One technique that can be used to prove that an order complex $\Delta(P)$ is shellable is the notion of lexicographic shellability, or EL-shellability, which was first introduced by Bjorner [1]. Let $P$ be a graded poset, and let $\mathcal{E}(P)$ be the set of edges of the Hasse diagram of $P$, i.e. $\mathcal{E}(P) = \{(x, y) \in P \times P \mid x \gg y\}$. An edge labeling of $P$ is a map $\lambda : \mathcal{E}(P) \to \Lambda$ where $\Lambda$ is some poset (usually the integers). Given an edge labeling $\lambda$, each maximal chain $c = (x_0 \gg x_1 \gg \cdots \gg x_k)$ of length $k$ can be associated with a $k$-tuple $\sigma(c) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{k-1}, x_k))$. We say that $c$ is an increasing chain if the $k$-tuple $\sigma(c)$ is increasing; that is, if $\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{k-1}, x_k)$. The edge labeling allows us to order
the maximal chains of any interval of $P$ by ordering the corresponding $k$-tuples lexicographically. If $\sigma(c_1)$ lexicographically precedes $\sigma(c_2)$ then we say that $c_1$ lexicographically precedes $c_2$ and we denote this by $c_1 <_L c_2$.

**Definition 2.3.** An edge labeling is called an EL-labeling (edge lexicographical labeling) if for every interval $[x, y]$ in $P$,

1. there is a unique increasing maximal chain $c$ in $[x, y]$, and
2. $c <_L c'$ for all other maximal chains $c'$ in $[x, y]$.

If a graded poset $P$ admits an EL-labeling then its order complex is shellable [1]. Therefore a graded poset that admits an EL-labeling is said to be **EL-shellable**.

We now turn our attention to posets that come from CW complexes. First we review the definition of a **regular** CW complex.

**Definition 2.4.** A CW complex is regular if the closure $\overline{C}$ of each cell $C$ is homeomorphic to a closed ball and if additionally $\overline{C} \setminus C$ is homeomorphic to a sphere.

Given a CW complex $K$, we define its **face poset** $\mathcal{F}(K)$ to be the set of closed cells ordered by containment and augmented by a least element $\hat{0}$. In general, the order complex $\|\mathcal{F}(K) - \{\hat{0}\}\|$ does not reveal the topology of $K$. However, the following result shows that regular CW complexes are combinatorial objects in the sense that the incidence relation of cells determines their topology.

**Proposition 2.5.** [3, Proposition 4.7.8] Let $K$ be a regular CW complex. Then $K$ is homeomorphic to $\|\mathcal{F}(K) - \{\hat{0}\}\|$.

It is natural to ask when a poset is the face poset of a regular CW complex. Bjorner [2] gave a complete answer to this question, which we now explain.

**Definition/Theorem 2.6.** [2] A poset $P$ is said to be a CW poset if

1. $P$ has a least element $\hat{0}$,
2. $P$ is nontrivial, i.e. has more than one element, and
3. for all $x \in P - \{\hat{0}\}$, the open interval $(\hat{0}, x)$ is homeomorphic to a sphere.

Furthermore, $P$ is a CW poset if and only if it is isomorphic to the face poset of a regular CW complex.

Bjorner gave some additional criteria for determining when a poset $P$ is a CW poset. We list below the criterion which will be most useful to us later.

**Proposition 2.7.** [2] Let $P$ be a nontrivial, finite, pure poset of length $d + 1$ with least element $\hat{0}$. Let $\hat{P}$ denote the poset $P \cup \{1\}$, where $\hat{1}$ is a new greatest element. If $\hat{P}$ is shellable and thin then $P$ is isomorphic to $\mathcal{F}(K)$, where $K$ is a regular $d$-dimensional CW complex homeomorphic to the $d$-sphere.

A finite graded poset with $\hat{0}$ and $\hat{1}$ is called **Eulerian** if every interval of length at least one has the same number of elements of odd rank as of even rank. It is an elementary topological result [22] that if a poset is a CW poset, then it is Eulerian.

**Lemma 2.8.** [22] A CW poset is Eulerian.
3. Background on the Bruhat order

In this section we will review properties of the Bruhat order $\leq$ of a Coxeter group, define reflection orders, and prove some lemmas that we will need later. We will assume knowledge of the basic definitions of Coxeter systems and Bruhat order; we refer the reader to [9] for details.

Fix a Coxeter group $W$ generated by a set of simple reflections $\{s_i \mid i \in I\}$ and let $T$ be the set of all reflections. Let $\ell : W \to \mathbb{N}$ denote the length function: $\ell(w)$ is the length of a minimal reduced expression for $w$. Recall that length is the rank function for the Bruhat order of a Coxeter group (which is a graded poset). One of the key facts about the nature of reduced expressions in $W$ is the Strong Exchange Property, which is due to Verma [24].

**Theorem 3.1.** (Strong Exchange Property) Let $w = s_1 \ldots s_r$ (where $i \in I$ for $1 \leq i \leq r$), not necessarily a reduced expression. Suppose a reflection $t \in T$ satisfies $\ell(wt) < \ell(w)$. Then there is an index $i$ for which $wt = s_1 \ldots \hat{s}_i \ldots s_r$ (omitting $s_i$).

If the expression for $w$ is reduced, then $i$ is unique.

Without loss of generality, assume that $W$ is realized geometrically as a group of isometries of a real vector space $V$. Let $I$ denote the set of simple roots, $\Phi^+$ the set of positive roots, and for non-isotropic $\alpha \in V$, let $r_\alpha : V \to V$ denote the reflection in $\alpha$. Note that the map $\alpha \mapsto r_\alpha$ is a bijection between $\Phi^+$ and $T$.

Dyer [7] proved that intervals in the Bruhat order of a Coxeter group are EL-shellable, strengthening the earlier CL-shellability result of Bjorner and Wachs [4]. Dyer’s primary tool was his notion of “reflection orders,” certain total orderings of $T$. To describe reflection orders, we use the fact that positive roots are naturally in bijection with $T$. The orders on the positive roots which correspond to reflection orders on $T$ may be characterized as follows: “the restriction of the order to the positive roots lying on the plane spanned by any two positive roots is one of the two possible orders in which a ray from the origin, undergoing a full rotation in the plane beginning at a negative root, would sweep through the positive roots on that plane” [7]. The algebraic definition of a reflection order is the following.

**Definition 3.2.** [7] A total order $\preceq$ on $T$ is called a reflection order if for any dihedral reflection subgroup $W'$ of $W$ either $r \prec rsr \prec \cdots \prec srsrs \prec rs \prec s$ or $s \prec rs \prec srsrs \prec \cdots \prec rsrsr \prec r$.

Dyer proved the existence of many different reflection orders.

**Proposition 3.3.** [7] Let $J, K$ be disjoint subsets of $I$ and let $W_J = \langle s_j \mid j \in J \rangle$, $W_K = \langle s_k \mid k \in K \rangle$ be the corresponding parabolic subgroups of $W$. Then there is a reflection order $\preceq$ on $T$ such that

1. $t \prec t'$ if $t \in W_J \cap T$ and $t' \in T \setminus W_J$, and
2. $t \prec t'$ if $t \in T \setminus W_K$ and $t' \in W_K \cap T$.

In other words, there is a reflection order such that reflections contained in $W_J$ come first, and reflections contained in $W_K$ come last.

Dyer used these reflection orders to prove the following lemma.
Lemma 3.4. [7] Let $u, w \in W$ with $u \leq w$ and $\ell(w) - \ell(u) = 2$. Let $\preceq$ be a reflection order. Then there exist unique $x, y \in W$ such that $u \prec x \prec w$, $u \prec y \prec w$, $x^{-1}u \prec w^{-1}x$ and $y^{-1}u \succ w^{-1}y$. Moreover, $w^{-1}y \prec w^{-1}x$ and $x^{-1}u \prec y^{-1}u$.

In particular, this lemma implies the following result (originally proved in [4]):

Corollary 3.5. [4, 7] The Bruhat order of $W$ is thin.

Dyer then extended this result to EL-shellability, as follows.

Proposition 3.6. [7] Fix a reflection order $\preceq$ and regard the set $T$ of reflections as a poset under $\preceq$. Label each edge $x \succ y$ of the Bruhat order by the reflection $x^{-1}y$. Then this edge labeling is an EL-labeling; therefore the Bruhat order is EL-shellable.

We will now prove some further properties of Bruhat order which will be useful to us later. Let us fix $(W, I)$ and a subset $J \subset I$. We will denote by $W^J$ the set of minimal-length coset representatives of $W/W_J$. Recall that each element $w \in W$ has a unique factorization of the form $u_1w'$ where $w' \in W^J$ and $u \in W_J$. Also recall that the projection from $W$ to $W^J$ is order-preserving. Choose a reflection order $\preceq$ such that reflections contained in $W_J$ come last. As above, we will label the edge between two elements $v \succ u$ with the reflection $v^{-1}u$.

Lemma 3.7. Suppose that $w, w' \in W^J$, and $w'v \prec w$ for some $v \in W_J$. Let $\lambda$ be the label of the edge between $w$ and $w'v$. Then if $m$ is any chain in $W$ from $w$ to $w'v$ which passes through $w'v$, the labels of $m$ following $\lambda$ are greater than $\lambda$.

Proof. Clearly any chain in $W$ from $w$ to $w'v$ through $w'v$ will have the form $w \succ w'v_1 \succ w'v_2 \succ \cdots \succ w'v_m = w'$ where $v_i \in W_J$ and $v_m = e$. This is because the projection from $W$ to $W^J$ is order-preserving.

By the definition of our labeling, $\lambda = w^{-1}w'v$, which is clearly not in $W_J$. Now note that the edge between any $w'v_i$ and $w'v_{i+1}$ is labeled by the element $(w'v_i)^{-1}w'v_{i+1} = v_i^{-1}v_{i+1}$, which is in $W_J$. Because we chose $\preceq$ to be a reflection order in which elements of $W_J$ come last, the lemma follows.

This lemma implies the following results.

Corollary 3.8. Suppose that $w, w' \in W^J$, and $w \succ w'v$ for some $v \in W_J$. Let $\lambda$ be the label of the edge between $w$ and $w'v$. Then the unique chain from $w$ to $w'v$ in $W$ with an increasing label begins with $\lambda$.

Proof. By Lemma 3.7, we know that for any chain from $w$ to $w'$ which goes through $w'v$, the edge labels after the initial $\lambda$ are greater than $\lambda$. Furthermore, by Proposition 3.6, there exists a unique increasing chain from $w'v$ to $w'$. Therefore this gives rise to an increasing chain from $w$ to $w'$ whose initial edge label is $\lambda$. This increasing chain must necessarily be the unique one.

Corollary 3.9. Fix $w, w' \in W^J$ with $w \succ w'$. Then there can be at most one element in $W$ covered by $w$ which has the form $w'v$ (for any $v \in W_J$).
Suppose that there were another element covered by \( w \) of the form \( w'u \) for some \( u \in W_J \). Let the label of the edge from \( w \) to \( w'v \) be \( \lambda := w^{-1}w'v \) and let the label of the edge from \( w \) to \( w'u \) be \( \mu := w^{-1}w'u \). Then Corollary 3.8 implies that the unique increasing chain in \( W \) from \( w \) to \( w'u \) begins with \( \lambda \) and also begins with \( \mu \), which implies that \( \lambda = \mu \). But this implies that \( w'v = w'u \).

**Lemma 3.10.** Fix \( w', w \in W^J \) with \( w' < w \). Suppose that \( w'u < wv \) where \( u, v \in W_J \). Then we can write \( u \) in the form \( rv \) where \( r, v \in W_J \) and \( \ell(u) = \ell(r) + \ell(v) \).

**Proof.** Let us choose a reduced expression for \( wv \), say \( s_1s_2\ldots s_{m+1}\ldots s_n \) where \( w = s_1s_2\ldots s_m \) and \( v = s_{m+1}\ldots s_n \). Here, the \( s_i \) are simple reflections. Then by the Strong Exchange property, \( w'u = s_1s_2\ldots \hat{s}_a\ldots s_{m+1}\ldots s_n \) for a uniquely determined \( a \leq m \). Clearly \( s_1s_2\ldots \hat{s}_a\ldots s_m \) is reduced, and hence \( s_1s_2\ldots \hat{s}_a\ldots s_m < w \). We can now write \( s_1s_2\ldots \hat{s}_a\ldots s_m \) uniquely in the form \( tr \) where \( t \in W^J \) and \( r \in W_J \) and \( \ell(t) + \ell(r) = m - 1 \). But now \( w'u = trv \) which implies that \( t = w' \) and \( u = rv \). Since \( \ell(t) + \ell(r) = m - 1 \), \( \ell(v) = n - m \), and \( \ell(w'u) = n - 1 \), we must have that \( \ell(rv) = \ell(r) + \ell(v) \).

**Corollary 3.11.** Fix \( w', w \in W^J \) with \( w' < w \) and fix \( v \in W_J \). Then \( wv \) covers at most one element of the form \( w'v \) (for any \( u \in W_J \)).

**Proof.** This follows from Corollary 3.9 and Lemma 3.10.

**Corollary 3.12.** Fix \( w', w \in W^J \) with \( w' < w \) and fix \( u \in W_J \). Then \( w'u \) is covered by at most one element of the form \( wv \) (for any \( v \in W_J \)).

**Proof.** Assume that \( w'u \) is covered by two elements \( wv \) and \( w\hat{v} \). Then by Lemma 3.10, we can write \( u = rv \) and also \( u = \hat{r}\hat{v} \), where \( \ell(u) = \ell(r) + \ell(v) \) and \( \ell(u) = \ell(\hat{r}) + \ell(\hat{v}) \). Since \( wv \) and \( w\hat{v} \) cover \( w'rvw = w'\hat{r}\hat{v}w \), it follows that \( w \) covers \( w'r \) and \( w'\hat{r} \). But now by Corollary 3.9, we must have that \( r = \hat{r} \). It follows that \( v = \hat{v} \).

4. The cell decomposition of \( \mathcal{P}^+_{\geq 0} \) and the poset of cells \( Q^J \)

In this section we introduce the totally nonnegative part of a flag variety \( \mathcal{P}^+_{\geq 0} \) and its cell decomposition proved by Rietsch. Note that the reader interested purely in the combinatorial results of this paper may skip most of the content in this section, focusing on Definition 4.6 for the definition of the poset \( Q^J \).

Let \( G \) be a semisimple linear algebraic group over \( \mathbb{C} \) split over \( \mathbb{R} \), with split torus \( T \). We identify \( G \) (and related spaces) with their real points and consider them with their real topology. Let \( X(T) = \text{Hom}(T, \mathbb{R}^+) \) and \( \Phi \subset X(T) \) the set of roots. Choose a system of positive roots \( \Phi^+ \). We denote by \( B^+ \) the Borel subgroup corresponding to \( \Phi^+ \) and by \( U^+ \) its unipotent radical. We also have the opposite Borel subgroup \( B^- \) such that \( B^+ \cap B^- = T \), and its unipotent radical \( U^- \).

Denote the set of simple roots by \( \Pi = \{ \alpha_i \mid i \in I \} \subset R^+ \). For each \( \alpha_i \in \Pi \) there is an associated homomorphism \( \phi_i : \text{SL}_2 \rightarrow G \). Consider the 1-parameter subgroups in \( G \) (landing in \( U^+, U^- \), and \( T \), respectively) defined by

\[
x_i(m) = \phi_i\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \quad y_i(m) = \phi_i\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}, \quad \alpha_i^+(t) = \phi_i\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},
\]
where $m \in \mathbb{R}, t \in \mathbb{R}^*, i \in I$. The datum $(T, B^+, B^-, x_i, y_i; i \in I)$ for $G$ is called a *pinning*. The standard pinning for $\text{SL}_d$ consists of the diagonal, upper-triangular, and lower-triangular matrices, along with the simple root subgroups $x_i(m) = I_d + mE_{i,i+1}$ and $y_i(m) = I_d + mE_{i+1,i}$ where $I_d$ is the identity matrix and $E_{i,j}$ has a 1 in position $(i,j)$ and zeroes elsewhere.

The Weyl group $W = N_G(T)/T$ acts on $X(T)$ permuting the roots $\Phi$. The simple reflections $s_i \in W$ are given explicitly by $s_i := \hat{s}_i T$ where $\hat{s}_i := \phi_i \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$ and any $w \in W$ can be expressed as a product $w = s_{i_1} s_{i_2} \ldots s_{i_m}$ with $\ell(w)$ factors. We set $\hat{w} = s_{i_1} s_{i_2} \ldots s_{i_m}$.

We can identify the flag variety with the variety $B$ of Borel subgroups, via $gB^+ \iff g \cdot B^+ := gB^+ g^{-1}$.

We have the Bruhat decompositions

$$B = \bigsqcup_{w \in W} B^+ \hat{w} \cdot B^+ = \bigsqcup_{w \in W} B^- \hat{w} \cdot B^-$$

of $B$ into $B^+$-orbits called *Bruhat cells*, and $B^-$-orbits called *opposite Bruhat cells*.

**Definition 4.1.** For $v, w \in W$ define

$$R_{v,w} := B^+ \hat{w} \cdot B^+ \cap B^- \hat{v} \cdot B^-.$$ 

The intersection $R_{v,w}$ is non-empty precisely if $v \leq w$, and in that case is irreducible of dimension $\ell(w) - \ell(v)$, see [10].

Let $J \subset I$. The parabolic subgroup $W_J \subset W$ corresponds to a parabolic subgroup $P_J$ in $G$ containing $B^+$. Namely, $P_J = \bigsqcup_{w \in W_J} B^+ \hat{w} B^+$. Let $\mathcal{P}^J$ denote the variety of all parabolic subgroups of $G$ conjugate to $P_J$. This can be identified with the partial flag variety $G/P_J$ via

$$gP_J \iff gP_J g^{-1}.$$ 

We have the usual projection from the full flag variety to a partial flag variety which takes the form $\pi = \pi^J : B \to \mathcal{P}^J$, where $\pi(B)$ is the unique parabolic subgroup of type $J$ containing $B$.

We now give the relevant definitions from total positivity.

**Definition 4.2.** [14] The totally nonnegative part $U^-_{\geq 0}$ of $U^-$ is defined to be the semigroup in $U^-$ generated by the $y_i(t)$ for $t \in \mathbb{R}_{\geq 0}$.

The totally nonnegative part of $B$ is defined by

$$B_{\geq 0} := \{ u \cdot B^+ \mid u \in U^-_{\geq 0} \},$$ 

where the closure is taken inside $B$ in its real topology.

The totally nonnegative part $\mathcal{P}_J^J_{\geq 0}$ of a partial flag variety $\mathcal{P}^J$ is defined to be $\pi^J(B_{\geq 0})$.

There are natural decompositions of $B_{\geq 0}$ and $\mathcal{P}_J^J_{\geq 0}$ which were introduced by Lusztig [14, 13].
Theorem 4.5. a concrete description of the order relation on cells.

Definition 4.3. [14] For \( v, w \in W \) with \( v \leq w \), let\[ R_{v,w;} := R_{v,w} \cap B_{\geq 0}. \]

We write \( W^J \) (respectively \( W^{J}_{\text{max}} \)) for the set of minimal (respectively maximal) length coset representatives of \( W/W_J \).

Definition 4.4. [13] Let \( I^J \subset W^{J}_{\text{max}} \times W_J \times W^J \) be the set of triples \((x, u, w)\) with the property that \( x \leq wu \). Given \((x, u, w) \in I^J\), we define \( P_{x,u,w;} > 0 := \pi^J(R_{x,w;u;} > 0) = \pi^J(R_{x,u-1,w;} > 0) \).

Rietsch [19] proved that \( R_{v,w;} > 0 \) and \( P_{x,u,w;} > 0 \) are semi-algebraic cells of dimension \( \ell(w) - \ell(v) \) and \( \ell(wu) - \ell(x) \), respectively. Furthermore, Rietsch [20] has given a concrete description of the order relation on cells.

Theorem 4.5. [20] We have that \( P_{x,u,w;} > 0 \subset P_{x',u',w';} > 0 \) precisely if there exist \( u_1, u_2 \in W_J \) with \( u_1 u_2 = u \) and \( \ell(u) = \ell(u_1) + \ell(u_2) \), such that the following holds:
\[
x' u'^{-1} \leq u_1^{-1} u_2^{-1} \leq wu_1 \leq w'.
\]

Furthermore, the closure of a cell is a union of cells.

Definition 4.6. Fix a Coxeter system \((W, I)\) and a subset \( J \subset I \) such that \( W_J \) is finite. Let \( Q^J \) denote the poset with elements \( \{\hat{0}\} \cup \{Q_{x,u,w} \mid (x, u, w) \in I^J\} \) and with the order relation above. That is, \( Q_{x,u,w} < Q_{x',u',w'} \) if and only if there exist \( u_1, u_2 \in W_J \) with \( u_1 u_2 = u \) and \( \ell(u) = \ell(u_1) + \ell(u_2) \), such that the following holds:
\[
x' u'^{-1} \leq u_1^{-1} u_2^{-1} \leq wu_1 \leq w'.
\]

Additionally, we stipulate that \( \hat{0} < Q_{x,u,w} \) for all \( Q_{x,u,w} \in Q^J \).

Clearly when \( W \) is a Weyl group of a semisimple linear algebraic group, \( Q^J \) is the poset of closed cells of \( P^J_{\geq 0} \) augmented by a least element. However, we define \( Q^J \) as above because our combinatorial results are true in this generality; we thank Anders Bjorner for pointing this out to us.

Besides having a unique least element \( \hat{0} \), \( Q^J \) also has a unique greatest element: this element is \( Q_{w_0, u_0, w_0'} \), where \( w_0 \) is the longest element in \( W_J \) and \( w_0' \) is the longest element in \( W^J \). In Section 6 we will prove that \( Q^J \) is a graded poset, where the rank of \( Q_{x,u,w} \) is simply \( \ell(wu) - \ell(x) \).

See Example A.7 and Figure 5 for the case of the Grassmannian \( Gr_{2,4}(\mathbb{R}) \).

Remark 4.7. When \( P^J \) is the (type A) Grassmannian, the poset \( Q^J \) is the poset of cells of the totally nonnegative Grassmannian, studied by Postnikov [16] and the author [25]. This poset is also called the cyclic Bruhat order, and Postnikov [16] has shown that it can be described in terms of many different combinatorial objects, such as decorated permutations and \( J \)-diagrams (certain \( 0 \to 1 \) tableaux). See Appendix A for more details.

Remark 4.8. When \( P^J \) is the complete flag variety, the poset \( Q^J \) has an especially simple description. Recall that if \( P \) is a poset, then the interval poset \( Int(P) \) is defined to be the poset of intervals \([x, y]\) of \( P \), ordered by containment. In this special case, \( Q^J \) is simply the interval poset of the Bruhat order.
5. $Q^J$ is thin

In this section we will study cover relations in $Q^J$ as well as intervals of rank 2. In particular, we will prove that $Q^J$ is thin.

Recall that $Q_{x',v,w} < Q_{x,u,w}$ if and only if there exist $v_1$ and $v_2$ such that $v = v_1v_2$ (lengths add) and $xu^{-1} \leq x'v_2^{-1} \leq w'v_1 \leq w$. Here $w, w' \in W_J, u, v, v_1, v_2 \in W_J$, and $x, x' \in W_{max}$. There are three types of cover relations:

- **Type 1**: $Q_{x',v,w} \lessdot Q_{x,u,w}$ such that $xu^{-1} = x'v_2^{-1}$ and $w'v_1 \lessdot w$. This implies that $x = x'$, $u = v_2$, $w'v < wu$, and hence $w' < w$.
- **Type 2**: $Q_{x',v,w} \lessdot Q_{x,u,w}$ such that $xu^{-1} \lessdot x'v_2^{-1}$ and $w'v_1 = w$. This implies that $w' = w$ and $xu^{-1} \lessdot x'v^{-1}$.
- **Type 3**: $\hat{0} < Q_{x,u,w}$ where $Q_{x,u,w}$ is a 0-cell. This implies that $x = wu$.

We will now prove that intervals of rank 2 in $Q^J$ are diamonds.

**Theorem 1.1.** $Q^J$ is thin.

**Proof.** Let us fix two elements $Q_{a,b,c} < Q_{x,u,w} \in Q^J$, such that rank$(Q_{x,u,w}) -$ rank$(Q_{a,b,c}) = 2$. We must show that there are exactly two elements between $Q_{a,b,c} < Q_{x,u,w} \in Q^J$.

Since $Q_{a,b,c} < Q_{a',b',c'}$, there exist $b_1, b_2 \in W_J$ with $b_1b_2 = b$ (and the lengths add) such that

$$xu^{-1} \leq ab_2^{-1} \leq cb_1 \leq w.$$ \hfill (1)

Since the difference in rank between $Q_{a,b,c}$ and $Q_{x,u,w}$ is 2, there are three possibilities for the inequalities above:

- **Case 1**: $xu^{-1} = ab_2^{-1}$, and $cb_1 < w$ where $\ell(w) - \ell(cb_1) = 2$.
- **Case 2**: $xu^{-1} < ab_2^{-1}$ and $cb_1 < w$.
- **Case 3**: $xu^{-1} < ab_2^{-1}$ where $\ell(ab_2^{-1}) - \ell(xu^{-1}) = 2$, and $cb_1 = w$.

Let us first consider Case 1. Since $xu^{-1} = ab_2^{-1}$, we have that $x = a$ and $u = b_2$. The inequality $cb_1 < w$ implies that $c < w$. Finally, $cb_1 < w$ and $b_2 = u$ implies that $cb < wu$ where $\ell(wu) - \ell(cb) = 2$.

The fact that the Bruhat order is thin implies that there exist two elements in $W$ between $cb$ and $wu$. Let us factor these two elements (uniquely) as $tr$ and $t'r'$ where $t, t' \in W_J$ and $r, r' \in W_J$.

We now claim that

$$Q_{a,b,c} \lessdot Q_{a,r,t},$$ \hfill (2)

and

$$Q_{a,r,t} \lessdot Q_{a,u,w}.$$ \hfill (3)

(The same relations will hold for $Q_{a,r',t'}$.)

First note that $Q_{a,r,t} \in Q^J$, since $a \leq cb < tr$ implies that $a \leq tr$.

Let us prove (2). Note that either $c = t$ or $c < t$. If $c = t$, then $cb \lessdot tr$ implies that $b < r$. Clearly $ar^{-1} \lessdot ab^{-1} \leq c = c$ which implies that $Q_{a,b,c} \lessdot Q_{a,r,t}$.
If \( c < t \), then since \( cb < tr \), Lemma 3.10 implies that we can write \( b = b_1 r \) where \( \ell(b) = \ell(b_1) + \ell(r) \). Let \( b_2 = r \). Since \( cb < tr = tb_2 \), it follows that \( cb_1 < t \). We now have \( ar^{-1} = ab_2^{-1} \leq cb_1 < t \), which implies that \( Q_{a,b,c} < Q_{a,r,t} \).

Now let us prove (3). Note that either \( t = w \) or \( t < w \). If \( t = w \) then \( tr \leq wu \) implies that \( r \leq u \). Therefore \( au^{-1} < ar^{-1} \leq w = w \) which implies that \( Q_{a,r,t} = Q_{a,r,w} < Q_{a,u,w} \).

If \( t < w \), then since \( tr < wu \), Lemma 3.10 implies that we can write \( r = r_1 u \) where \( \ell(r) = \ell(r_1) + \ell(u) \). Let \( r_2 = u \). Now \( tr < wu \) implies that \( tr_1 u < wu \) which implies that \( tr_1 < w \). We now have \( au^{-1} = ar_2^{-1} \leq tr_1 < w \), and so \( Q_{a,r,t} < Q_{a,u,w} \).

We have now shown that there exist two elements \( Q_{a,r,t} \) and \( Q_{a,r',w} \) which lie between \( Q_{a,b,c} \) and \( Q_{a,u,w} = Q_{a,u,w} \) when we are in the situation of Case 1. To complete the proof in this case, we need to show that these are the only two elements which lie in this open interval.

Take any element \( Q_{d,h,f} \) such that \( Q_{a,b,c} < Q_{d,h,f} < Q_{a,u,w} \). By using the fact that the projection from \( W \) to \( W_J \) is order preserving, it is easy to see that we must have \( d = a \). Now the fact that \( Q_{a,b,c} < Q_{a,h,f} < Q_{a,u,w} \) implies that there exist \( b_1, b_2 \in W_J \) with \( b_1 b_2 = b \) such that

\[
(4) \quad ah^{-1} \leq ab_2^{-1} \leq cb_1 \leq f,
\]

and there exist \( h_1, h_2 \in W_J \) with \( h_1 h_2 = h \) such that

\[
(5) \quad au^{-1} \leq ah_2^{-1} \leq fh_1 \leq w.
\]

The fact that (4) and (5) represent cover relations in the poset implies that in each of (4) and (5) exactly one of the outer \( \leq \)'s is a \( < \) and one is an equality. If \( ah^{-1} = ab_2^{-1} \) and \( cb_1 < f \) then \( h = b_2 \) and \( cb_1 < f \) implies that \( cb < fh \). On the other hand, if \( ah^{-1} < ab_2^{-1} \) and \( cb_1 = f \) then \( c = f \) and \( b_1 = e \). Thus \( ah^{-1} < ab^{-1} \) which implies that \( b < h \) and hence \( cb < fh \). Similarly, it is easy to show from (5) that \( fh \leq wu \). But now since \( cb < fh < wu \), the element \( Q_{a,h,f} \) must be one of the two elements \( Q_{a,r,t} \) and \( Q_{a,r',w} \) which we already found.

Now we consider Case 2. Recall that we have \( xu^{-1} < ab_2^{-1} \) and \( cb_1 < w \), where \( b = b_1 b_2 \) and the lengths add. If \( x = a \), then as in the previous case, we will have \( cb < wu \) with \( \ell(wu) - \ell(cb) = 2 \), and the two elements between \( Q_{a,b,c} \) and \( Q_{a,u,w} \) will be of the form \( Q_{a,r,t} \) where \( cb < tr < wu \).

We now assume that \( x \neq a \). We claim that the only elements between \( Q_{a,b,c} \) and \( Q_{x,u,w} \) are \( Q_{a,b_2,w} \) and \( Q_{x,b_1 u,c} \).

To show that \( Q_{a,b_2,w} \) is well-defined (i.e. that \( (a, b_2, w) \) is in the indexing set for Rieisch's cell decomposition), note that \( cb_1 < w \) implies that \( cb < wb_2 \). This together with \( a \leq cb \) implies that \( a \leq wb_2 \), and so \( Q_{a,b_2,w} \) is well-defined. Now it is easy to see that \( ab_2^{-1} = cb_2^{-1} \leq cb_1 < w \) which implies that \( Q_{a,b,c} < Q_{a,b_2,w} \), and also \( xu^{-1} < ab_2^{-1} \leq w = w \) which implies that \( Q_{a,b_2,w} < Q_{x,u,w} \).

In order to show that \( Q_{x,b_1 u,c} \) is well-defined, we must first observe that \( \ell(b_1 u) = \ell(b_1) + \ell(u) \). By applying Lemma 3.10 to the fact that \( xu^{-1} < ab_2^{-1} \) (and remembering that each element of \( W_{max} \) has a unique expression as a product of an element of \( W_J \) and the longest element \( u_0 \in W_J \)), we see that we can write \( b_2 \) in the form \( ub' \) where \( \ell(b_2) = \ell(u) + \ell(b') \). But now since lengths add in the product \( b = b_1 b_2 \),
this implies that lengths add in the product $b_1u$. Therefore $x \leq cu$ implies that 
$x \leq cb_1u$, and hence $Q_{a,b_1u,c}$ is well-defined.

Now $xu^{-1} \prec ab_2^{-1}$ together with the fact that lengths add in the product $b_1u$ implies that 
$xu^{-1}b_1^{-1} \prec ab_2^{-1}b_1^{-1}$. Therefore $x(b_1u)^{-1} \prec ab^{-1} \leq c = c$ and so 
$Q_{a,b,c} \prec Q_{x,b_1u,c}$. Finally, we have $xu^{-1} = xu^{-1} \leq cb_1 \prec w$ and so $Q_{x,b_1u,c} \prec Q_{x,u,w}$.

To complete the proof for Case 2, we need to show that the only two elements 
between $Q_{a,b,c}$ and $Q_{x,u,w}$ are $Q_{a,b_2,w}$ and $Q_{x,b_2u,c}$.

Consider any $Q_{r,s,t}$ such that $Q_{a,b,c} \prec Q_{r,s,t} \prec Q_{x,u,w}$. This implies that there 
exist $b_1, b_2 \in W_J$ with $b_1b_2 = b$ and $s_1, s_2 \in W_J$ with $s_1s_2 = s$ such that 
$rs^{-1} \leq ab_2^{-1} \leq cb_1 \leq t$ and $xu^{-1} \leq rs_2^{-1} \leq ts_1 \leq w$. Since these inequalities represent 
cover relations in the poset, we see that if $t = c < w$ then $ts_1 < w$ and hence 
$xu^{-1} = rs_2^{-1}$ which implies that $r = x$. On the other hand, if $t > c$, then we must 
have $rs^{-1} = ab_2^{-1}$ which implies that $r = a$. Note that if $t > c$ then we must in fact 
have $t = w$, because as we saw before, $t < w$ implies that $r = x$. Since $r = a$ and 
we have assumed that $x \neq a$, this is a contradiction.

Therefore we know that elements between $Q_{a,b,c}$ and $Q_{x,u,w}$ either have the form 
$Q_{x,s,t}$ or $Q_{a,s,w}$. If $Q_{a,b,c} \prec Q_{x,s,t} \prec Q_{x,u,w}$ then $xu^{-1} \leq s_2^{-1} \leq cs_1 \leq w$ together 
with the fact that $c \neq w$ implies that $cs_1 \prec w$ and $xu^{-1} = xs_2^{-1}$. Corollary 3.9 
applied to $cs_1 \leq w$ implies that $s_1$ is uniquely determined, and since $s_2 = u$, we 
have that $s$ is uniquely determined. Therefore there is at most one element of the 
form $Q_{x,s,t}$ between $Q_{a,b,c}$ and $Q_{x,u,w}$.

Similarly, if $Q_{a,b,c} \prec Q_{a,s,w} \prec Q_{x,u,w}$ then $xu^{-1} \leq as_2^{-1} \leq ws_1 \leq w$ implies that 
$xu^{-1} < as_2^{-1}$ and $s_1 = e$. Corollary 3.12 implies that $s_2$ is uniquely determined, and 
therefore there is at most one element of the form $Q_{a,s,w}$ between $Q_{a,b,c}$ and $Q_{x,u,w}$. This completes the proof for Case 2.

The proof for Case 3 is the simplest of all. The fact that $cb_1 = w$ implies that 
b_1 = e and c = w. Therefore $xu^{-1} < ab^{-1}$ and $\ell(ab^{-1}) - \ell(xu^{-1}) = 2$.

Since the Bruhat order is thin, there are exactly two elements in the open interval 
$(xu^{-1}, ab^{-1})$, which we can factor uniquely as $rs^{-1}$ and $r's'^{-1}$ for $r, r' \in W^J$ 
and $s, s' \in W_J$. It is clear that $Q_{a,b,w} < Q_{r,s,w} < Q_{x,u,w}$ because $rs^{-1} < ab^{-1} \leq w \leq w$, 
and $xu^{-1} < rs^{-1} \leq w$. Conversely, it is easy to see that if some $Q_{r,s,w}$ satisfies 
$Q_{a,b,w} < Q_{r,s,w} < Q_{x,u,w}$, then $xu^{-1} \leq r's'^{-1} \leq ab^{-1}$.

To complete the proof, we must address the rank 2 intervals whose least element 
is 0. Let the greatest element of such an interval be $Q_{x,u,w}$. It follows that $x \prec w$.
It is now an easy exercise to see that there are exactly two elements in the open 
interval $(0, Q_{x,u,w})$: $Q_{wu_0,u_0,w}$ and $Q_{x,u_0,xu_0}$, where $u_0$ is the longest element in 
$W_J$. Therefore $Q_J$ is thin.

We now summarize the analysis of the previous proof.

Remark 5.1. In the situation of Case 1, the interval $[Q_{x,b,c}, Q_{x,u,w}]$ in $Q^J$ naturally corresponds to the interval $[cb, wu]$ in $W$. Note that because $c < w$, each chain 
from $Q_{x,u,w}$ to $Q_{x,b,c}$ must contain at least one Type 1 cover relation; however, a 
Type 2 cover relation may also occur in this chain.
In the situation of Case 2, one of the chains from \(Q_{x,u,w}\) to \(Q_{a,b,c}\) consists of a Type 1 cover relation followed by a Type 2 cover relation; the other chain consists of a Type 2 cover relation followed by a Type 1 cover relation.

In the situation of Case 3, the interval \([Q_{a,b,w}, Q_{x,u,w}]\) in \(Q^J\) naturally corresponds to the interval \([xu^{-1}, ab^{-1}]\) in \(W\). Note that all four edges of the diamond interval \([Q_{a,b,w}, Q_{x,u,w}]\) must have Type 2.

For rank 2 intervals of the form \([0, Q_{x,u,w}]\), one of the chains from \(Q_{x,u,w}\) to \(0\) has the form Type 1 – Type 3, and the other chain has the form Type 2 – Type 3.

6. \(Q^J\) is EL-shellable

In this section we will prove that \(Q^J\) is a graded poset and that it has an EL-labeling, which implies that \(Q^J\) is EL-shellable [1]. In particular, the order complex of \(Q^J\) is shellable.

Let us begin by recalling the three types of cover relations in \(Q^J\):

**Type 1:** \(Q_{x',w,v} \leq Q_{x,u,w}\) such that \(xu^{-1} = x'v_2^{-1}\) and \(wv_1 \leq w\). This implies that \(x = x', u = v_2,\) and \(w'v < wu,\) and hence \(w' < w\).

**Type 2:** \(Q_{x',v,w} \leq Q_{x,u,w}\) such that \(xu^{-1} \leq x'v_2^{-1}\) and \(wv_1 = w\). This implies that \(w' = w\) and \(xu^{-1} \leq x'v^{-1}\).

**Type 3:** \(0 < Q_{x,u,w}\) where \(Q_{x,u,w}\) is a 0-cell. This implies that \(x = wu\).

We now prove a lemma which describes a condition that diamond intervals in \(Q^J\) may not possess.

**Lemma 6.1.** There are no diamond intervals in \(Q^J\) in which the top two edges have Type 2 and the bottom two edges have Type 1.

**Proof.** Let \(P_1, P_2, P'_2, P_3\) be the elements of a diamond interval in \(Q^J\), such that \(P_3 \leq P_2 \leq P'_2\) and also \(P_3 \leq P'_2 \leq P_1\). The element \(P_1\) has the form \(Q_{x,u,w}\), and if the top two edges have Type 2, then we can write \(P_2 = Q_{x',u',w}\) and \(P'_2 = Q_{x'',u'',w}\). If additionally the bottom two edges have Type 1 then \(P_3 = Q_{\tilde{x}, \tilde{u}, w}\) where \(\tilde{x} = x'\) and \(\tilde{x} = x''\). Therefore \(x' = x''\). Furthermore, \(\tilde{w} \leq wu\) and \(\tilde{w} \leq wu''\), which implies by Corollary 3.12 that \(w' = w''\). But this shows that \(P_2 = P'_2\), which contradicts the fact that we were considering a diamond interval. Therefore the kind of diamond interval described in Lemma 6.1 is impossible. \(\square\)

**Proposition 6.2.** Suppose that \(Q_{a,b,c} < Q_{x,u,w}\) where \(c < w\). Then there exists some \(Q_{r,s,t}\) with \(t < w\) such that \(Q_{a,b,c} < Q_{r,s,t} < Q_{x,u,w}\).

**Proof.** We prove this by induction on \(\ell(w) - \ell(c)\).

The base case is when \(c < w\). In this situation we claim that \(Q_{a,b,c} < Q_{x,u,w}\). The fact that \(Q_{a,b,c} < Q_{x,u,w}\) implies that there exists a decomposition \(b = b_1b_2\) (lengths add) such that \(xu^{-1} \leq ab_2^{-1} \leq cb_1 \leq w\). Here we cannot have \(cb_1 = w\) because that would imply that \(c = w\). Therefore \(cb_1 < w\) and the fact that \(c < w\) implies that \(b_1 = e\). This implies that \(xu^{-1} \leq ab^{-1} \leq c < w\). In particular, \(xu^{-1} \leq c\) implies that \(Q_{x,u,c}\) is well-defined. Now it is obvious that \(Q_{x,u,c} < Q_{x,u,w}\) since \(xu^{-1} \leq xu^{-1} \leq c < w\). It remains to show that \(Q_{a,b,c} < Q_{x,u,c}\), i.e. that there
exist $b_1, b_2$ such that $b = b_1 b_2$ (lengths add) and $xu^{-1} \leq ab_2^{-1} \leq cb_1 \leq c$. Clearly we can take $b_1 = c$ and $b_2 = b$.

We now prove the general case. Consider the interval $I$ between $Q_{a,b,c}$ and $Q_{x,u,w}$. Note that any $Q_{r,s,t}$ that lies in this interval necessarily satisfies $c \leq t \leq w$. Now if any $Q_{r,s,t}$ satisfies $c < t < w$ then since $\ell(w) - \ell(t) < \ell(w) - \ell(c)$, we are done by induction. Therefore we are left to consider the case that for each $Q_{r,s,t}$ in this interval, either $t = c$ or $t = w$. Let us choose a $Q_{r,s,c}$ in this interval with maximal rank. If $Q_{r,s,c}$ is covered by $Q_{x,u,w}$, then we are done. If not, then all elements $Q_{c,f,g}$ of $I$ which are greater than $Q_{r,s,c}$ satisfy $g = w$. In particular, there is a diamond interval with $Q_{r,s,c}$ at the bottom in which the other three elements have the form $Q_{s,s,w}$. But this is impossible by Lemma 6.1.

**Lemma 6.3.** Suppose that $Q_{a,b,w} < Q_{x,u,w}$. Then there exists some $Q_{r,s,w}$ such that $Q_{a,b,w} < Q_{r,s,w} < Q_{x,u,w}$.

**Proof.** We have $xu^{-1} \leq ab_2^{-1} \leq wb_1 \leq w$ which implies that $xu^{-1} \leq ab^{-1} \leq w$. Because the Bruhat order is graded, there exists an element $v$ such that $xu^{-1} \leq v \leq ab^{-1}$ which can be factored uniquely in the form $v = rs^{-1}$ where $r \in W^a_{\text{max}}, s \in W_J$. It is now easy to see that $Q_{r,s,w} \in Q^J$ and that $Q_{a,b,c} < Q_{r,s,w} < Q_{x,u,w}$. □

**Corollary 6.4.** $Q^J$ is a graded poset, where the rank of $Q_{x,u,w} = \ell(wu) - \ell(x)$.

**Proof.** This follows from Proposition 6.2 and Lemma 6.3. □

We now propose an edge labelling for $Q^J$. See Example A.7 together with Figure 5 for the example of the Grassmannian $Gr_{2,4}(R)$.

**Definition 6.5.** Label Type 1 edges with the element $(wu)^{-1}w'v \in T$; label Type 2 edges with the element $(x'v^{-1})^{-1}xu^{-1} \in T$; and label Type 3 edges with the symbol $\emptyset$. Choose any reflection order $\preceq$ on $T$ such that elements of $T \cap W_J$ come last. We then choose the total order $\triangleright$ on labels of edges in $Q^J$ determined by the following conditions:

1. If $\lambda$ is any Type 1 label and $\mu$ is any Type 2 label, then $\lambda \triangleright \emptyset \triangleright \mu$.
2. If $\lambda$ and $\mu$ are Type 1 labels then $\lambda \triangleright \mu$ if and only if $\lambda < \mu$.
3. If $\lambda$ and $\mu$ are Type 2 labels then $\lambda \triangleright \mu$ if and only if $\lambda < \mu$.

**Remark 6.6.** Observe that the labels of Type 1 edges are never in $W_J$. This will be important for our arguments later.

**Theorem 1.2.** The labeling of edges of $Q^J$ described above is an EL-labeling.

**Proof.** Fix two elements $Q_{x,u,w}$ and $Q_{a,b,c}$ in $Q^J$, such that $Q_{a,b,c} < Q_{x,u,w}$. First we will show that the lexicographically minimal chain (with respect to $\triangleright$) from $Q_{x,u,w}$ to $Q_{a,b,c}$ is increasing.

By Proposition 6.2, it is clear that the lexicographically minimal chain $m$ will consist of a series of Type 1 edges followed by Type 2 edges; let the chain label be $(\lambda_1, \lambda_2, \ldots, \lambda_i, \mu_1, \mu_2, \ldots, \mu_j)$, where the $\lambda_k \in T$ are Type 1 edges and the $\mu_k \in T$ are Type 2 edges.
For the sake of contradiction, suppose that for some \( h \) we have \( \lambda_h \triangleright \lambda_{h+1} \). In other words, \( \lambda_h \succ \lambda_{h+1} \), where \( \lambda_h \) and \( \lambda_{h+1} \) are the labels, respectively, for the edges of the chain \( Q_{x,u_1,w_1} \succ Q_{x,u_2,w_2} \succ Q_{x,u_3,w_3} \). As shown in Section 5, the interval from \( Q_{x,u_1,w_1} \) to \( Q_{x,u_3,w_3} \) is a diamond. If we let \( Q_{x,u'_2,w'_2} \) denote the other middle element of this interval, then we know that \( w_3 u_3 \preceq w_2 u_2 \preceq w_1 u_1 \) and also \( w_3 u_3 \preceq w'_2 u'_2 \preceq w_1 u_1 \). Since \( \lambda_h \) and \( \lambda_{h+1} \) are Type 1 labels, we know that \( w_3 < w_2 < w_1 \). However, for \( w'_2 \), we know only that \( w_3 \leq w_2 \leq w_1 \). Observe that the labels \( \lambda_h \) and \( \lambda_{h+1} \) that we’ve used for our elements in \( Q' \) are also the labels used by Dyer for the edges of the interval \([w_3 u_3, w_1 u_1]\). Therefore Lemma 3.4 implies that the labels \( \gamma_h \) and \( \gamma_{h+1} \) for the chain \( w_1 u_1 \succ w'_2 u'_2 \succ w_3 u_3 \) satisfy \( \gamma_h \prec \gamma_{h+1} \) and also \( \gamma_h \prec \lambda_h \). We now claim that the chain \((\lambda_1, \lambda_2, \ldots, \gamma_h, \gamma_{h+1}, \ldots, \lambda_i, \mu_1, \mu_2, \ldots, \mu_j)\) is lexicographically smaller than \((\lambda_1, \lambda_2, \ldots, \lambda_i, \mu_1, \mu_2, \ldots, \mu_j)\). To complete the proof of the claim, it suffices to show that \( \gamma_h \) is the label of a Type 1 cover relation. It is not hard to see that \( \gamma_h \) is the label of a Type 1 cover relation if and only if \( \gamma_h \notin W_J \). Since \( w_3 < w_1 \), it is clear that we cannot have both \( \gamma_h \) and \( \gamma_{h+1} \) in \( W_J \); at least one is \( \text{not} \) in \( W_J \). And now because we’ve chosen a reflection order in which elements of \( W_J \) come \( \text{last} \), it follows that \( \gamma_h \) is \( \text{not} \) in \( W_J \), and hence is the label of a Type 1 cover relation. We’ve now found a lexicographically smaller chain, which is a contradiction.

Therefore the lexicographically minimal chain label \((\lambda_1, \lambda_2, \ldots, \lambda_i, \mu_1, \mu_2, \ldots, \mu_j)\) satisfies \( \lambda_1 \triangleright \lambda_2 \triangleright \cdots \triangleright \lambda_i \). Let \( Q_{r,s,c} \) denote the element of \( Q' \) that we reach if we start at \( Q_{x,u,w} \) and traverse the edges labeled by \( \lambda_1, \lambda_2, \ldots, \lambda_i \). We need to show that the lexicographically minimal chain from \( Q_{r,s,c} \) to \( Q_{a,b,c} \) is increasing. First note that all edge labels of this interval are Type 2 labels. Furthermore, by considering the order relation in \( Q' \), it is easy to see that any element \( Q_{d,e,c} \) in the interval \([Q_{a,b,c}, Q_{r,s,c}]\) satisfies \( rs^{-1} \leq de^{-1} \leq ab^{-1} \). Conversely, for any \( d \in W_{\text{max}}' \) and \( e \in W_J \) such that \( rs^{-1} \leq de^{-1} \leq ab^{-1} \), we have that \( Q_{a,b,c} \preceq Q_{d,e,c} \preceq Q_{r,s,c} \). Therefore the interval \([Q_{a,b,c}, Q_{r,s,c}]\) is isomorphic to the dual of the interval \([rs^{-1}, ab^{-1}]\). Recall that our edge labeling of \([Q_{a,b,c}, Q_{r,s,c}]\) is inherited from (the dual of) Dyer’s edge labeling of \([rs^{-1}, ab^{-1}]\). It follows – using EL-shellability of intervals in Bruhat order – that the lexicographically minimal chain (with respect to \( \triangleright \)) from \( Q_{r,s,c} \) to \( Q_{a,b,c} \) is increasing. Since we’ve chosen an ordering in which Type 1 labels precede Type 2 labels, we have now shown that the lexicographically minimal chain from \( Q_{x,u,w} \) to \( Q_{a,b,c} \) is increasing.

It remains to show that this is the unique increasing chain from \( Q_{x,u,w} \) to \( Q_{a,b,c} \). Clearly the labels on any increasing chain will again consist of a series of Type 1 edge labels followed by a series of Type 2 edge labels. Suppose that there are two increasing chains \( m_1 \) and \( m_2 \) from \( Q_{x,u,w} \) to \( Q_{a,b,c} \); let \( Q_{r,v,c} \) and \( Q_{r',v',c} \) be the two intermediate elements of these chains which we obtain after starting at \( Q_{x,u,w} \) and traversing the Type 1 edges of \( m_1 \) and \( m_2 \), respectively. It follows that \( cv < wu \) and \( cv' < wu \). Furthermore, the increasing chain labels from \( Q_{x,u,w} \) to \( Q_{r,v,c} \) and from \( Q_{x,u,w} \) to \( Q_{r',v',c} \) correspond to increasing chain labels from \( wu \) to \( cv \) and from \( wu \) to \( cv' \). Note that by Remark 6.6, the labels on these increasing chains from \( wu \) to \( cv \) and from \( wu \) to \( cv' \) are \( \text{not} \) in \( W_J \).
Now consider the interval \([c, wu]\). Clearly both \(cv\) and \(cv'\) are in this interval. By EL-shellability of the Bruhat order, we can find increasing chains from \(cv\) to \(c\) and also from \(cv'\) to \(c\). Clearly the labels on these chains will be elements of \(T \cap W_J\). And therefore by our choice of reflection ordering, the increasing chain from \(wu\) to \(cv\) extends to an increasing chain from \(wu\) to \(c\); similarly, the increasing chain from \(wu\) to \(cv'\) extends to an increasing chain from \(wu\) to \(c\). We have now found two increasing chains from \(wu\) to \(c\), which contradicts the fact that reflection orders give EL-labelings of the Bruhat order. Therefore there is a unique increasing chain from \(Q_{x,u,w}\) to \(Q_{a,b,c}\).

This would complete the proof that \(Q^J\) is EL-shellable except that we have so far ignored the chains from \(Q_{x,u,w}\) to \(\hat{0}\). We will now address these chains.

Consider all maximal chains from \(Q_{x,u,w}\) to a 0-cell, i.e. an element \(Q_{a,b,c}\) with \(\ell(cb) = \ell(a)\). We claim that among these, the lexicographically minimal chain \(m\) consists entirely of Type 1 edges and is increasing. First note that if any element \(Q_{a,b,c}\) is not a 0-cell, then there is a Type 1 edge from \(Q_{a,b,c}\) to an element below it. For example, if we choose some \(d\) such that \(a < d < cb\) and factor \(d\) uniquely as \(c' b'\) where \(c', b' \in W_J\) and \(Q_{a,b',c'} < Q_{a,b,c}\). Therefore by induction, the lexicographically minimal chain from \(Q_{x,u,w}\) to a 0-cell consists entirely of Type 1 edges. Moreover, it is increasing, by the argument that we used in the third paragraph of this proof. Therefore by adding to \(m\) the final edge to \(\hat{0}\), we have found an increasing chain from \(Q_{x,u,w}\) to \(\hat{0}\) which is lexicographically minimal in this interval.

It remains to show that there is a unique increasing chain from \(Q_{x,u,w}\) to \(\hat{0}\). Suppose that there are two. Both of these chains end with the label \(\emptyset\), so by our choice of ordering, these chains must consist of Type 1 edges followed by the \(\emptyset\) edge. Therefore the two increasing chains must both pass through 0-cells of the form \(Q_{x,b,c}\) and \(Q_{x,b',c'}\), where \(c, c' \in W_J\) and \(b, b' \in W_J\). Since these are 0-cells, we must have \(x = cb\) and \(x = c'b'\). It follows that \(c = c'\) and \(b = b'\), so the increasing chain from \(Q_{x,u,w}\) to \(\hat{0}\) is indeed unique.

This completes the proof that \(Q^J\) is EL-shellable. \(\square\)

Recall that \(Q_{u_0,u_0,w_0}^J\) is the unique maximal element of \(Q^J\). We now apply Theorem 2.6 and Proposition 2.7 to the poset \(P := Q^J \setminus \{Q_{u_0,u_0,w_0}^J\}\).

**Corollary 6.7.** \(\|Q^J \setminus \{Q_{u_0,u_0,w_0}^J\}\|\) is homeomorphic to a sphere. Moreover, the poset \(Q^J \setminus \{Q_{u_0,u_0,w_0}^J\}\) is the face poset of a regular CW complex which is homeomorphic to a sphere.

Because \(Q^J\) has a unique greatest element, Corollary 6.7 implies that \(Q^J\) is the face poset of a regular CW complex which is homeomorphic to a ball.

Now we apply Lemma 2.8 to \(Q^J\).

**Corollary 6.8.** \(Q^J\) is Eulerian.

Because Rietsch has proved that the closure of every cell in \(\mathcal{P}_{J,0}^J\) is a union of cells (see Theorem 4.5), we can deduce the following.
Corollary 6.9. The Euler characteristic of the closure of every cell in the cell decomposition of $\mathcal{P}_J^{\geq 0}$ is 1.

These results lead us to make the following conjecture.

Conjecture 6.10. The totally nonnegative part of an arbitrary flag variety together with its cell decomposition is a regular CW complex homeomorphic to a ball.

Appendix A. $Q^J$ for the type A Grassmannian

In independent work, Postnikov [16] has studied the poset of cells of a natural cell decomposition of the totally nonnegative part of the (type A) Grassmannian $Gr^+_k$ and showed that this poset can be described in terms of certain tableaux (the so-called J-diagrams) and also in terms of certain “decorated” permutations. He defined $Gr^+_k$ to be the subset of the real Grassmannian where all Plucker coordinates are non-negative, and defined cells to be subsets of $Gr^+_k$ with a given vanishing pattern of Plucker coordinates. It is not too hard to see that in the case of the Grassmannian, Postnikov’s cell decomposition is a special case of Rietzch’s cell decomposition [19]; although we will not prove that here, we will give bijections between Rietzch’s cells $Q_{x,u,w}$, J-diagrams, and decorated permutations. Additionally, we will describe in detail the case of the Grassmannian $Gr_{2,4}(\mathbb{R})$.

Recall that a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a weakly decreasing sequence of non-negative numbers. For a partition $\lambda$, where $\sum \lambda_i = n$, the Young diagram $Y_\lambda$ of shape $\lambda$ is a left-justified diagram of $n$ boxes, with $\lambda_i$ boxes in the $i$th row. Figure 1 shows a Young diagram of shape $(4, 2, 1)$.

![Figure 1. A Young diagram of shape $(4, 2, 1)$](image)

Definition A.1. Fix $k$ and $n$. A J-diagram $^1 (\lambda, D)_{k,n}$ is a partition $\lambda$ contained in a $k \times (n-k)$ rectangle, together with a filling $D : Y_\lambda \to \{0, +\}$ which has the J-property: there is no 0 which has a + above it and a + to its left.

(Here, “above” means above and in the same column, and “to its left” means to the left and in the same row.) In Figure 2 we give an example of a J-diagram.

The rank of $(\lambda, D)_{k,n}$ is the number of +’s in the filling $D$.

Definition A.2. A decorated permutation $\tilde{\pi} = (\pi, d)$ is a permutation $\pi$ in the symmetric group $S_n$ together with a coloring (decoration) $d$ of its fixed points $\pi(i) = i$ by two colors, “clockwise” and “counterclockwise.”

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$^1$The symbol J is meant to remind the reader of the shape of the forbidden pattern, and should be pronounced as [le], because of its relationship to the letter L.
We represent a decorated permutation \( \tilde{\pi} = (\pi, d) \), where \( \pi \in S_n \), by its chord diagram, constructed as follows. Put \( n \) equally spaced points around a circle, and label these points from 1 to \( n \) in clockwise order. If \( \pi(i) = j \) then this is represented as a directed arrow, or chord, from \( i \) to \( j \). If \( \pi(i) = i \) then we draw a chord from \( i \) to \( i \) (i.e. a loop), and orient it either clockwise or counterclockwise, according to \( d \).

For example, the decorated permutation \((3, 1, 5, 4, 8, 6, 7, 2)\) (written in list notation) with the fixed points 4, 6, and 7 colored in counterclockwise, clockwise, and counterclockwise, respectively, is represented by the chord diagram in Figure 3.

**Figure 3. A chord diagram for a decorated permutation**

Recall that \( i \) is a weak excedence of a permutation \( \pi \) if \( \pi(i) \geq i \). This definition can be extended to decorated permutations as follows: \( i \) is a weak excedence of a decorated permutation \((\pi, d)\) if either \( \pi(i) > i \) or if \( \pi(i) \) is a counterclockwise loop.

We will not review here the rank function on decorated permutations nor the order relations for these objects; for details, see [25]. However, we should recall Postnikov’s result [16] relating the \( J \)-diagrams and decorated permutations to \( Gr_{k,n}^+ \).

**Theorem A.3.** [16] There is an order-preserving bijection between the poset of cells of \( Gr_{k,n}^+ \) and the poset of \( J \)-diagrams \((\lambda, D)_{k,n}\). Additionally, there is an order-preserving bijection between the poset of cells of \( Gr_{k,n}^+ \) and the poset of decorated permutations on \( n \) letters with \( k \) weak excedences.

We now let \( W \) be the symmetric group on \( n \) letters, \( S = \{s_1, s_2, \ldots, s_{n-1}\} \) be the set of adjacent transpositions, and \( J = \{s_1, s_2, \ldots, \hat{s}_{n-k}, \ldots, s_{n-1}\} \). We will now use \( Q^J \) to denote the poset of cells defined in terms of this data.
Lemma A.4. There is an order-preserving bijection $\Phi_1$ from $Q^J$ to the poset of decorated permutations in $S_n$ with $k$ weak excedences, which is defined as follows. Let $Q_{x,u,w} \in Q^J$. Then $\Phi_1(Q_{x,u,w}) = (\pi, d)$ where $\pi = xu^{-1}w^{-1}$. To define $d$, we make any fixed point that occurs in one of the positions $w(1), w(2), \ldots, w(n-k)$ a clockwise loop, and we make any fixed point that occurs in one of the positions $w(n-k+1), \ldots, w(n)$ a counterclockwise loop.

Additionally, there is a natural bijection between $\Gamma$-diagrams $(\lambda, D)_{k,n}$ and $Q^J$: we thank Postnikov for explaining this to us.

Lemma A.5. [17] There is an order-preserving bijection $\Phi_2$ from the set of $\Gamma$-diagrams $(\lambda, D)_{k,n}$ to $Q^J$, defined as follows.

1. Take $(\lambda, D)_{k,n}$ and replace each $+$ with an elbow joint $\begin{array}{c} \hline \end{array}$, and each 0 with a cross $\Box$.
2. Note that the west and north borders, and the south and east borders, respectively, of $\lambda$ give rise to two length-$n$ paths from the north-east corner to the south-east corner of the $k \times (n-k)$ rectangle. Label each of these paths with the numbers 1 through $n$.
3. View the resulting “pipe dream” as a permutation $w_{\lambda,D} \in S_n$, as in A.6.
4. Repeat this procedure for the $\Gamma$-diagram $(\lambda, D_0)_{k,n}$, where $D_0$ denotes the filling of $\lambda$ which consists entirely of 0’s. Denote the resulting permutation by $w_{\lambda,D_0}$; this permutation is in $W^J$, i.e. it is a Grassmannian permutation.
5. Let $w := w_{\lambda,D_0}$. Since every element of $W$ has a unique factorization as the product of an element in $W^J_{\text{max}}$ and an element in $W_J$, define $x$ and $u$ by the equation $xu^{-1} = w_{\lambda,D}$. We now set $\Phi_2((\lambda, D)_{k,n}) := Q_{x,u,w}$.

Example A.6. Figure 4 shows a $\Gamma$-diagram $(\lambda, D)_{3,7}$ together with the related pipe dreams. This gives rise to the permutation $w_{\lambda,D} := (2, 1, 5, 4, 6, 3, 7)$ (written in list notation) and the permutation $w_{\lambda,D_0} := (2, 4, 5, 7, 1, 3, 6)$.

![Diagram](image-url)

Figure 4. The bijection $\Phi_2$

For a simple bijection between $\Gamma$-diagrams and decorated permutations (which is equal to $\Phi_2 \circ \Phi_1$), see [23].

Example A.7. We now explain the case of the Grassmannian $Gr_{2,4}(\mathbb{R})$ in detail. In that case, the Weyl group $W$ is $S_4$, the symmetric group on 4 letters, and the set $S$ of simple reflections is $\{s_1, s_2, s_3\}$ where $s_i$ is the transposition $(i, i+1)$ which exchanges $i$ and $i+1$. The subset $J$ is $\{s_1, s_3\}$ and the parabolic subgroup $W_J$ is $\langle s_1, s_3 \rangle$. A reflection order which puts elements of $W_J$ at the end is the following:

$$(23) \prec (24) \prec (13) \prec (14) \prec (34) \prec (12)$$
In Figure 5, we have drawn the Hasse diagram of the poset $Q_J$ for the Grassmannian $Gr_{2,4}(\mathbb{R})$. Elements $Q_{x,u,w}$ (where $x \in W_{\text{max}}^J$, $u \in W_J$, $w \in W_J$) are represented by $\Gamma$-diagrams, and below each $\Gamma$-diagram, we have listed the triple $(x,u,w)$ corresponding to $Q_{x,u,w}$. Note that in each of these triples we have abbreviated $s_i$ by $i$. Also note that we have labelled the unique increasing chain from the greatest element to the least element of $Q_J$; every element in this chain is the totally positive part of a Schubert variety.

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Figure 5. $\ quo$ for the Grassmannian $Gr^2_4(\mathbb{R})$. 