GENERALISED TEMPERLEY-LIEB ALGEBRAS OF TYPE $G(r,p,n)$

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Abstract. In an earlier work, we defined a “generalised Temperley-Lieb algebra” $T_{L_{r,1,n}}$ corresponding to the imprimitive reflection group $G(r,1,n)$ as a quotient of the cyclotomic Hecke algebra. In this work we introduce the generalised Temperley-Lieb algebra $T_{L_{r,p,n}}$ which corresponds to the complex reflection group $G(r,p,n)$. Our definition identifies $T_{L_{r,p,n}}$ as the fixed-point subalgebra of $T_{L_{r,1,n}}$ under a certain automorphism $\sigma$. We prove the cellularity of $T_{L_{r,p,n}}$ by proving that $\sigma$ induces a special shift automorphism with respect to the cellular structure of $T_{L_{r,1,n}}$. Finally, we give a description of the cell modules of $T_{L_{r,p,n}}$ and their decomposition numbers.

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1. INTRODUCTION

The Temperley-Lieb algebra was initially introduced in [13] in connection with transition matrices in statistical mechanics. Since then, it has been linked with many diverse areas of mathematics, such as operator algebras, quantum groups, categorification and representation theory. It can be defined in several ways, including as a quotient of the Hecke algebra and an associative diagram algebra.

In [8], we defined a generalised Temperley-Lieb algebra $T_{L_{r,1,n}}$ corresponding to the complex reflection group $G(r,1,n)$ as a generalisation of the Temperley-Lieb algebras of types $A_{n-1}$ and $B_n$. In that work, we give a cellular structure for

2020 Mathematics Subject Classification. Primary 16G20, 20C08; Secondary 20G42.
Key words and phrases. Temperley-Lieb algebra, Hecke algebras, KLR algebras, cellular basis, decomposition numbers.
We generalise that work in this paper. Specifically, we introduce a generalised Temperley-Lieb algebra $TL_{r,p,n}$ which corresponds to an arbitrary imprimitive complex reflection group $G(r,p,n)$. It is defined as the fixed-point subalgebra of a suitably specialised generalised Temperley-Lieb algebra $TL_{r,1,n}$ under an automorphism $\sigma_{TL}$ defined in (4.3).

Moreover, we introduce a modified multipartition which is called a 3-dimensional multipartition in Section 5, and construct a new cellular structure of our specialised $TL_{r,1,n}$ in Theorem 7.8. Inspired by the skew cellularity introduced by Hu, Mathas and Rostam in [7], we show that $\sigma_{TL}$ induces a shift automorphism with respect to this cellular basis in Theorem 7.18 and deduce the cellularity of $TL_{r,p,n}$ in Theorem 7.24. Using this cellular structure of $TL_{r,p,n}$ inherited from $TL_{r,1,n}$, we determine the decomposition matrix of the cell modules of $TL_{r,p,n}$ in Theorem 8.22.

2. The cyclotomic Hecke algebra $H(r,p,n)$

To define the generalised Temperley-Lieb algebra $TL_{r,p,n}$, we first recall the definition of the cyclotomic Hecke algebra $H(r,p,n)$ corresponding to the imprimitive reflection group $G(r,p,n)$. In this section, we also recall Ariki’s construction in [1] showing that $H(r,p,n)$ is the fixed point subalgebra under an automorphism $\sigma$ of $H_n(q,\zeta)$, a specialization of $H(r,1,n)$. We will use this automorphism to define $TL_{r,p,n}$.

The following lemma gives a presentation of the imprimitive reflection group $G(r,p,n)$.

**Lemma 2.1.** (cf. [2], Appendix 2) Let $r, p$ and $n$ be positive integers such that $p | r$ and $d = \frac{r}{p}$. The complex reflection group $G(r,p,n)$ is the group generated by $s_0, s_1, s_1', s_2, s_3, \ldots, s_{n-1}$ subject to the following relations:

\[
\begin{align*}
    s_0^d &= s_1^d = s_1'^{d} = s_2^d = \cdots = s_{n-1}^d = 1; \\
    s_0s_1s_0s_1 &= s_1s_0s_1s_0; \\
    s_0s_1's_0s_1' &= s_1's_0s_1's_0; \\
    s_0s_1s_1' &= s_1s_1's_0; \\
    s_0s_i &= s_is_0 \text{ for } i \geq 2; \\
    s_1's_2s_1' &= s_2s_1's_2; \\
    s_1's_i &= s_is_1' \text{ for } i \geq 3; \\
    s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} \text{ for } i \geq 1; \\
    s_is_j &= s_js_i \text{ for } |i - j| \geq 2 \text{ and } i, j \geq 1; \\
    s_0s_1's_1's_1' &= s_1's_0s_1's_1's_1' \cdots = s_1's_0s_1's_1's_1' \cdots.
\end{align*}
\]

$G(r,p,n)$ may be viewed as the group consisting of all $n \times n$ monomial matrices such that each non-zero entry is an $r^{th}$ root of unity and the product of the non-zero entries is a $d^{th}$ root of unity, $w$. 
$d = \frac{\zeta}{p}$. More precisely, let $\zeta \in \mathbb{C}^*$ be a primitive $r$th root of unity. The complex reflection group $G(r, p, n)$ is generated by the following elements:

$$s_0 = \zeta^p E_{1,1} + \sum_{k=2}^{n} E_{k,k},$$

$$s_1' = \zeta E_{1,2} + \zeta^{-1} E_{2,1} + \sum_{k=3}^{n} E_{k,k},$$

$$s_i = \sum_{1 \leq k \leq i-1} E_{k,k} + E_{i+1,i} + E_{i,i+1} + \sum_{i+2 \leq k \leq n} E_{k,k} \text{ for all } 1 \leq i \leq n - 1$$

where $E_{i,j}$ is the elementary $n \times n$ matrix with $(i,j)$-entry equal to 1 and all other entries equal to zero. Since the complex reflection group $G(r, 1, n)$ is generated by $s_i(1 \leq i \leq n - 1)$ as well as the following matrix:

$$t_0 = \zeta E_{1,1} + \sum_{k=2}^{n} E_{k,k},$$

the complex reflection group $G(r, p, n)$ is a subgroup of $G(r, 1, n)$.

The Hecke algebra of type $G(r, p, n)$ is a deformation of its complex group ring. Following [2], the Hecke algebra $H(r, p, n)$ may be defined as follows:

**Definition 2.2.** (2.2) Let $R$ be a commutative ring with 1 and $q, u_1, u_2, \ldots, u_d \in R^*$. The Hecke algebra $H(r, p, n)$ is the unitary associative $R$-algebra generated by $S, T_1', T_2', \ldots, T_{n-1}'$ subject to the following relations:

$$(S - u_1)(S - u_2) \cdots (S - u_d) = 0;$$

$$(T_i' - q)(T_i' + 1) = (T_i - q)(T_i + 1) = 0 \text{ for } 1 \leq i \leq n - 1;$$

$$ST_i T_i = T_i S T_i S;$$

$$ST_i T_i' = T_i' S T_i' S;$$

$$ST_i T_i' = T_i T_i' S;$$

$$ST_i T_i = T_i S T_i \text{ for } i \geq 2;$$

$$T_i' T_2 T_i' = T_2 T_i' T_2;$$

$$T_i' T_i = T_i T_i' \text{ for } i \geq 3;$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \text{ for } i \geq 1;$$

$$T_i T_j = T_j T_i \text{ for } |i - j| \geq 2 \text{ and } i, j \geq 1;$$

$$\underbrace{ST_i T_i T_i T_i'}_{p+1} \cdots = \underbrace{ST_i T_i T_i T_i'}_{p+1}.\ldots$$

In analogy with the restrictions on the parameters in [3], we require that $R$ is a field of characteristic 0 with a $p^{th}$ primitive root of unity and $q, u_1, u_2, \ldots, u_d \in R^*$ such that

(2.3) \[ \frac{u_i}{u_j} \neq 1, q \text{ or } q^2 \]

for all $i \neq j$ and

(2.4) \[ (1 + q)(1 + q + q^2) \neq 0. \]

Moreover, we assume that there exists $w_i \in R$ such that $w_i^p = u_i$ for $i = 1, 2, \ldots, d$. 

We see from their matrix realisations that $G(r, p, n)$ is a subgroup of $G(r, 1, n)$. It is a natural question to ask whether there is a connection between the two Hecke algebras. To review the construction of the Hecke algebra of $G(r, 1, n)$ by Ariki in [1] which indicates the connection between $H(r, 1, n)$ and $H(r, p, n)$, we recall the definition of $H(r, 1, n)$.

**Definition 2.5.** ([2], Definition 4.21) Let $R$ be as defined above and $q, v_1, v_2, \ldots, v_r \in R^*$. The cyclotomic Hecke algebra $H(r, 1, n)$ corresponding to $G(r, 1, n)$ over $R$ is the associative algebra generated by $T_0, T_1, \ldots, T_{n-1}$ subject to the following relations:

$$(T_0 - v_1)(T_0 - v_2) \ldots (T_0 - v_r) = 0; \quad (2.6)$$

$$(T_i - q)(T_i + 1) = 0 \quad \text{for} \quad 1 \leq i \leq n - 1; \quad (2.7)$$

$$T_0T_1T_0 = T_1T_0T_1; \quad (2.8)$$

$$T_{i+1}T_i = T_{i+1}T_i \quad \text{for} \quad 1 \leq i \leq n - 2; \quad (2.9)$$

$$T_iT_j = T_jT_i \quad \text{for} \quad |i - j| \geq 2. \quad (2.10)$$

We remind readers that in the original definition of $H(r, 1, n)$, the base field $R$ is only required as a commutative ring with unity. The restrictions on $R$ are for the following specialisation:

Let $\zeta \in R$ be a primitive root of unity of order $p$. Choose the parameters $v_1, v_2, \ldots, v_r$ in (2.6) such that the equation is of the following form:

$$(2.11) \quad \prod_{k=1}^{d} \prod_{i=1}^{p} (T_0 - \zeta^i w_k) = 0,$$

where $w_k$ is an element in $R$ such that $w_k^p = u_k$. Then this equation can be written as

$$(2.12) \quad \prod_{k=1}^{d} (T_0^p - u_k) = 0.$$

Denote by $H_n(q, u_k)$ the algebra $H(r, 1, n)$ with the specialization above. Then we have an automorphism $\sigma : H_n(q, u_k) \mapsto H_n(q, u_k)$ such that:

$$(2.13) \quad \sigma(T_0) = \zeta T_0, \sigma(T_i) = T_i \quad \text{for all} \quad 1 \leq i \leq n - 1.$$

Ariki shows in [1]:

**Proposition 2.14.** ([1], Proposition 1.6) The algebra homomorphism $\phi : H(r, p, n) \to H_n(q, u_i)$ given by:

$$\phi(S) = T_0^p,$$

$$\phi(T_1^i) = T_0^{-1}T_1T_0,$$

$$\phi(T_i) = T_i \quad \text{for} \quad 1 \leq i \leq n - 1$$

is well-defined and one-to-one. And $H(r, p, n) \cong \phi(H(r, p, n))$.

Further, we have

**Proposition 2.15.** ([1], Corollary 1.18) The algebra $H(r, p, n)$ is isomorphic to $H_n(q, u_k)^{\sigma}$ via $\phi$, where the latter algebra is the fixed point subalgebra of $H_n(q, u_k)$ under $\sigma$. 
This proposition implies that the cyclotomic Hecke algebra $H(r, p, n)$ can be realised as the subalgebra of a specialisation of the cyclotomic Hecke algebra $H(r, 1, n)$, consisting of the points fixed by the automorphism $\sigma$ defined in (2.19). In analogy with this situation, we will define the generalised Temperley-Lieb algebra $\mathcal{TL}_{r,p,n}$ as a fixed point subalgebra of $\mathcal{TL}_{r,1,n}$ under $\sigma$ in Section [1].

In the remaining part of this section, we interpret the automorphism $\sigma$ into the language of quivers which are used to define a KLR algebra, from which we obtain the cellularity of $\mathcal{TL}_{r,p,n}$. This interpretation was first discussed by Rostam in Section 3.1 in [11].

Let $q \in \mathbb{R}^*$ be the parameter in definition 2.2 and $e$ be the smallest positive integer such that $1 + q + \cdots + q^{e-1} = 0$, setting $e := 0$ if no such integer exists. Let $\Gamma_{e,p}$ be the quiver with vertex set $K := I \times J$ where $I = \mathbb{Z}/p\mathbb{Z}$ and $J = \mathbb{Z}/e\mathbb{Z}$, and there is a directed edge from $(i_1, j_1)$ to $(i_1, j_1 + 1)$. As there is no edge between $(i, j)$ and $(i', j')$ if $i \neq i'$ in the quiver $\Gamma_{e,p}$, it decomposes as $p$ layers which are isomorphic to each other. For example, $\Gamma_{0,3}$ consists of three layers, each of them is a quiver of type $A_{\infty}$:

\[
\Gamma_{0,3} : \quad \cdots \rightarrow (0, -1) \rightarrow (0, 0) \rightarrow (0, 1) \rightarrow (0, 2) \rightarrow \cdots
\]

\[
\cdots \rightarrow (1, -1) \rightarrow (1, 0) \rightarrow (1, 1) \rightarrow (1, 2) \rightarrow \cdots
\]

\[
\cdots \rightarrow (2, -1) \rightarrow (2, 0) \rightarrow (2, 1) \rightarrow (2, 2) \rightarrow \cdots
\]

Let $(c_{(i,j)(i',j')})$ be the Cartan matrix associated with $\Gamma_{e,p}$, so that

\[
(2.16) \quad c_{(i,j)(i',j')} = \begin{cases} 
2 & \text{if } (i, j) = (i', j'); \\
-1 & \text{if } e \neq 2, i = i' \text{ and } j = j' \pm 1; \\
-2 & \text{if } e = 2, i = i' \text{ and } j = j' \pm 1; \\
0 & \text{otherwise.}
\end{cases}
\]

Let $\{\alpha_{(i,j)}| (i, j) \in K\}$ be the associated set of simple roots and $\{\Lambda_{(i,j)}| (i, j) \in K\}$ be the set of fundamental weights. Let $(,)$ be the bilinear form determined by

\[
(\alpha_{(i,j)}, \alpha_{(i',j')}) = c_{(i,j)(i',j')}, \quad \text{and } (\Lambda_{(i,j)}, \alpha_{(i',j')}) = \delta_{(i,j)(i',j')},
\]

Let $P_+ = \oplus_{(i,j) \in K} \mathbb{N} \alpha_{(i,j)}$ be the dominant weight lattice and $Q_+ = \oplus_{(i,j) \in K} \mathbb{N} \alpha_{(i,j)}$ be the positive root lattice. For $\Lambda \in P_+$, define the length of $\Lambda$, $l(\Lambda) := \sum_{(i,j) \in K} \langle \Lambda, \alpha_{(i,j)} \rangle$ and for $\alpha \in Q_+$, the height of $\alpha$, $ht(\alpha) := \sum_{(i,j) \in K} \langle \alpha, \Lambda_{(i,j)} \rangle$.

Fix a dominant weight $\Lambda$ such that $l(\Lambda) = r$ and $(\Lambda, \alpha_{(i,j)}) = (\Lambda, \alpha_{(i',j')})$ for all $i, i' \in I$ and $j \in J$. Let $\zeta$ be a $p^{th}$ primitive root of unity and choose the parameters $v_1, v_2, \ldots, v_r$ so that the equation (2.6) becomes:

\[
(2.17) \quad \prod_{i \in I} \prod_{j \in J} (T_0 - \zeta^i q^j)^{c_{(i,j)(i',j')}}(\Lambda, \alpha_{(i,j)}) = 0,
\]

which can be written as

\[
(2.18) \quad \prod_{j \in J} (T_0^p - q^{pj})^{c_{(i,j)(i',j')}}(\Lambda, \alpha_{(i,j)}) = 0.
\]

Denote by $H_n^\Lambda(q, \zeta)$ the algebra $H(r, 1, n)$ specialised as above. Then $H_n^\Lambda(q, \zeta)$ can be also regarded as a specialisation of $H_n(q, u_k)$ which is given by (2.11) with
$w_k = q^{j_k}$ for some $j_k \in J$. Then we have an automorphism $\sigma : H_n^\Lambda(q, \zeta) \rightarrow H_n^\Lambda(q, \zeta)$ such that:

$$\sigma(T_0) = \zeta T_0, \sigma(T_i) = T_i \quad \text{for all} \quad 1 \leq i \leq n - 1.$$  

Similarly, choose the parameters $u_1, \ldots, u_d$ such that the first equation in Definition 2.2 becomes

$$\prod_{j \in J} (S - q^{p_j})^{(\Lambda, \alpha_{(i,j)})} = 0$$

and denote by $H_{p,n}^\Lambda(q)$ the new algebra with this specialization. As a direct consequence of Proposition 2.15, the algebra $H_{p,n}^\Lambda(q)$ is isomorphic to $H_n^\Lambda(q, \zeta)^\sigma$ via the automorphism $\phi$ given in Proposition 2.14.

We next recall the interpretation of the automorphism $\sigma$ as that of the cyclotomic KLR algebra by Rostam in [11]. Let $\sigma_0 : \Gamma_{e,p} \rightarrow \Gamma_{e,p}$ be the quiver automorphism given by

$$\sigma_0((i,j)) = (i-1, j)$$

and $\sigma_1$ be the map on $K^n$ induced by $\sigma_0$, that is the map such that

$$\sigma_1(k_l) = \sigma_0(k_l)$$

for all $k \in K^n$ and $1 \leq l \leq n$. Let $\sigma'$ be the algebra automorphism of the cyclotomic KLR algebra $R_n^\Lambda(\Gamma_{e,p})$ given by

$$\sigma'(e(k)) = e(\sigma_1(k));$$  

$$\sigma'(y_i) = y_i \quad \text{for} \quad 1 \leq i \leq n$$  

$$\sigma'(\psi_i) = \psi_i \quad \text{for} \quad 1 \leq i \leq n - 1.$$  

Then we have

**Theorem 2.22.** (\cite[Theorem 4.14]{[11]}) Let $f : R_n^\Lambda(\Gamma_{e,p}) \rightarrow H_n^\Lambda(q, \zeta)$ be the isomorphism between the cyclotomic KLR algebra and Hecke algebra given in Theorem 1.1 in \cite{8} and $\sigma$ be the isomorphism of $H_n^\Lambda(q, \zeta)$ defined in (2.14). Let $\sigma'$ be the isomorphism of $R_n^\Lambda(\Gamma_{e,p})$ defined above. Then we have $f \circ \sigma' = \sigma \circ f$.

The following corollary is a direct consequence of this theorem:

**Corollary 2.23.** The Hecke algebra $H_{p,n}^\Lambda(q)$ is isomorphic to $R_n^\Lambda(\Gamma_{e,p})^\sigma$, the fixed point subalgebra of $R_n^\Lambda(\Gamma_{e,p})$ under $\sigma$.

We shall not distinguish below between the two automorphisms $\sigma$ and $\sigma'$.

### 3. The generalised Temperley-Lieb algebra $TL_{r,1,n}$

In this section, we recall the definition of the generalised Temperley-Lieb algebra $TL_{r,1,n}$ constructed in \cite{8}. Our new generalised Temperley-Lieb algebra $TL_{r,p,n}$ will later be defined as the subalgebra of $TL_{r,1,n}$ fixed by automorphism induced by $\sigma$.

Let $H(r, 1, n)$ be the cyclotomic Hecke algebra defined in Definition 2.5 and $H_{i,i+1}$ be the subalgebra of $H(r, 1, n)$ which is generated by two non-commuting generators, $T_i$ and $T_{i+1}$ where $0 \leq i \leq n - 2$. We call $H_{i,i+1}$ a parabolic subalgebra of $H(r, 1, n)$. The parabolic subalgebra $H_{0,1}$ has $2r$ 1-dimensional representations corresponding to the multipartitions $(0, 0, \ldots, (2), \ldots, 0)$ and $(0, 0, \ldots, (1, 1), \ldots, 0)$ where $(2)$ and $(1,1)$ are in the $j^{th}$ component of the $r$-partition with $1 \leq j \leq r$. 

Denote by \( E_0^{(j)} \) and \( F_0^{(j)} \) the corresponding primitive central idempotents in \( H_{0,1} \). These exist because of our choice of parameters, which implies semisimplicity.

For \( i \) such that \( 1 \leq i \leq n-2 \), the parabolic subalgebra \( H_{i,i+1} \) is a Hecke algebra of type \( A_2 \). It has 2 1-dimensional representations and the corresponding primitive central idempotents are

\[
E_i = a(T_iT_{i+1}T_i + T_iT_{i+1} + T_i + T_{i+1}) + 1);
F_i = b(T_iT_{i+1}T_i - qT_iT_{i+1} - qT_{i+1}T_i + q^2T_i + q^2T_{i+1} - q^4),
\]

where \( a = (q^3 + 2q^2 + 2q + 1)^{-1} \) and \( b = -a \). The generalised Temperley-Lieb algebra \( TL_{r,1,n} \) is defined as the quotient of \( H(r,1,n) \) by the two-sided ideal generated by half of the central idempotents listed above; specifically, the definition is as follows.

**Definition 3.1.** ([8], Definition 3.3) Let \( H(r,1,n) \) be the cyclotomic Hecke algebra defined in [2,5] and \( E_i(E_i^{(j)} \), if \( i = 0 \) be the primitive central idempotents of \( H_{i,i+1} \) listed above. The generalised Temperley-Lieb algebra \( TL_{r,1,n} \) is

\[
TL_{r,1,n} := H(r,1,n)/\langle E_0^{(1)}, \ldots, E_0^{(r)}, E_1, \ldots, E_{n-2} \rangle.
\]

According to Theorem 1.1 in [8], the Hecke algebra \( H(r,1,n) \) is isomorphic to a cyclotomic KLR algebra \( R_n^\Lambda \) with \( \Lambda \) a dominant weight decided by the parameters \( q, v_1, v_2, \ldots, v_r \). Therefore, we have the following alternative definition of \( TL_{r,1,n} \) as a quotient of the cyclotomic KLR algebra \( R_n^\Lambda \):

**Theorem 3.2.** ([8], Theorem 3.24) Let \( TL_{r,1,n} \) be the generalised Temperley-Lieb algebra in Definition 3.1, and \( R_n^\Lambda \) be the cyclotomic KLR algebra isomorphic to the corresponding Hecke algebra \( H(r,1,n) \). Then

\[
(3.3) \quad TL_{r,1,n} \cong R_n^\Lambda / \mathcal{J}_n,
\]

where the ideal \( \mathcal{J}_n \) is defined by

\[
\begin{align*}
\mathcal{J}_n^\Lambda = \sum_{\alpha \in I_n^{-2}, (\alpha, \Lambda) > 0} & \langle e(i, i + 1, i) \rangle_{R_n^\Lambda} \\
+ \sum_{\alpha \in I_n^{-3}, (\alpha, \Lambda) > 0} & \langle e(i_1, i_2, i_3, i) \rangle_{R_n^\Lambda},
\end{align*}
\]

with \( e(i) \) the KLR generators of \( R_n^\Lambda \) (cf. Definition 2.27 in [8]).

4. THE DEFINITION OF \( TL_{r,p,n} \)

In this section, we define the generalised Temperley-Lieb algebra \( TL_{r,p,n} \) corresponding to \( G(r,p,n) \) using the automorphism \( \sigma \). In analogy with the case \( p = 1 \), we will show that \( TL_{r,p,n} \) is a quotient of the corresponding Hecke algebra \( H(r,p,n) \).

Denote by \( TL_{r,1,n}(q, \zeta) \) the specialisation of the generalised Temperley-Lieb algebra \( TL_{r,1,n} \) corresponding to \( H_n^\Lambda(q, \zeta) \).

By Theorem 3.24 in [8], \( TL_{r,1,n}(q, \zeta) = R_n^\Lambda(\Gamma_{r,p}) / \mathcal{J} \) where \( \mathcal{J} \) is the two-sided ideal generated by all the \( e(k) \), \( k \in K^n \) such that:

\[
i_1 = i_2, j_1 = j_2 - 1 \text{ if } k_1 = (i_1, j_1) \text{ and } k_2 = (i_2, j_2)
\]

or

\[
(4.2) \quad (\alpha_{k_l}, \Lambda) > 0 \text{ for } l = 1, 2, 3.
\]
As $\sigma_0(i, j) = (i - 1, j)$, $e(k)$ satisfies (4.1) if and only if $\sigma(e(k))$ does. The restriction on $\Lambda$ implied by $(\Lambda, \alpha(i, j)) = (\Lambda, \alpha(i', j))$ guarantees that $e(k)$ satisfies (4.2) if and only if $\sigma(e(k))$ does. Therefore, we have

$$\sigma(\mathcal{J}) = \mathcal{J}$$

which implies that $\sigma$ induces an automorphism of $TL_{r,1,n}(q, \zeta)$, which we denote by $\sigma_{TL}$. In other words, $\sigma = \sigma_{TL}$ is the automorphism of $TL_{r,1,n}(q, \zeta)$ such that

(4.3) \quad $\sigma_{TL}(e((i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n))) = e((i_1 - 1, j_1), (i_2 - 1, j_2), \ldots, (i_n - 1, j_n))$;

$$\sigma_{TL}(y_i) = y_i \text{ for } 1 \leq i \leq n$$

$$\sigma_{TL}(\psi_i) = \psi_i \text{ for } 1 \leq i \leq n - 1.$$ 

Now we can state the main theorem in this section.

**Theorem 4.4.** Let $\zeta \in R$ be a primitive root of unity of order $p$ and $TL_{r,1,n}(q, \zeta)$ be the specialization of the generalized Temperley-Lieb algebra $TL_{r,1,n}(q, \zeta)$ such that the relation (2.6) is transformed into (2.7). Let $\sigma_{TL}$ be the automorphism of $TL_{r,1,n}(q, \zeta)$ defined above, $TL_{r,1,n}(q, \zeta)^{\sigma_{TL}}$ be the fixed point subalgebra under $\sigma_{TL}$ and let $\mathcal{J}^\sigma = \mathcal{J} \cap R^\Lambda_n(\Gamma_{e,p})^\sigma$. Then we have

(4.5) \quad $TL_{r,1,n}(q, \zeta)^{\sigma_{TL}} \cong R^\Lambda_n(\Gamma_{e,p})^\sigma / \mathcal{J}^\sigma$.

Thus the fixed subalgebra $TL_{r,1,n}(q, \zeta)^{\sigma_{TL}}$ is a quotient of $R^\Lambda_n(\Gamma_{e,p})^\sigma$, which is isomorphic to the cyclotomic Hecke algebra of type $G(r, p, n)$.

**Proof.** Let $f : R^\Lambda_n(\Gamma_{e,p})^\sigma / \mathcal{J}^\sigma \to TL_{r,1,n}(q, \zeta)^{\sigma_{TL}}$ be the map such that $f(a + \mathcal{J}^\sigma) = a + \mathcal{J}$ for all $a \in R^\Lambda_n(\Gamma_{e,p})^\sigma$. We only need to prove the surjectivity.

For any $a + \mathcal{J} \in TL_{r,1,n}(q, \zeta)^{\sigma_{TL}}$, we have $a + \mathcal{J} = \sigma(a + \mathcal{J}) = \sigma(a) + \mathcal{J}$. Thus, $a + \mathcal{J} = \sigma(a) + \mathcal{J} = \cdots = \sigma^{p-1}(a) + \mathcal{J}$. As the characteristic of the field is 0, $a' = a + \mathcal{J} = a + \mathcal{J} + \cdots + \mathcal{J} = a + \mathcal{J}$ for all $a \in R^\Lambda_n(\Gamma_{e,p})^\sigma$ and $a' + \mathcal{J} = a + \mathcal{J}$. Further, $a' + \mathcal{J} \in R^\Lambda_n(\Gamma_{e,p})^\sigma / \mathcal{J}^\sigma$ and $f(a' + \mathcal{J}^\sigma) = a + \mathcal{J}$. So $f$ is a surjection. Therefore, we have $TL_{r,1,n}(q, \zeta)^{\sigma_{TL}} \cong R^\Lambda_n(\Gamma_{e,p})^\sigma / \mathcal{J}^\sigma$.

We now define our generalized Temperley-Lieb algebra of type $G(r, p, n)$ as follows:

**Definition 4.6.** Let $R$ be a field of characteristic 0 and let $TL_{r,1,n}(q, \zeta)$ be the generalized Temperley-Lieb algebra defined over $R$ as in subsection 4. Let $\sigma_{TL}$ be the automorphism of $TL(r, 1, n)(q, \zeta)$ as in Theorem 4.4. We define the fixed subalgebra $TL_{r,1,n}(q, \zeta)^{\sigma_{TL}}$ as $TL_{r,p,n}$ and refer to it as the Temperley-Lieb algebra corresponding to the complex reflection group $G(r, p, n)$.

Since the map $\sigma_{TL}$ is trivial when $p = 1$, $TL_{r,1,n}(q, \zeta)$ can be regarded as a special case of the generalized Temperley-Lieb algebra $TL_{r,p,n}$. In particular, the Temperley-Lieb algebras of types $A_{n-1}$ and $B_n$ are both special cases of our $TL_{r,p,n}$, where $r = p = 1$ and $r = 2$, $p = 1$, respectively. As another special case of $TL_{r,p,n}$ with $r = p = 2$, the algebra $TL_{2,2,n}$ is a quotient of the Temperley-Lieb algebra of type $D_n$ in the sense of [12] and [9]. This quotient is called a forked Temperley-Lieb algebra. We refer readers to [6] for details.
5. 3-DIMENSIONAL MULTIPARTITIONS AND THE CELLULAR STRUCTURE OF $TL_{r,1,n}$

As a specialization of the Temperley-Lieb algebra $TL_{r,1,n}$, $TL_{r,1,n}(q,\zeta)$ is a cellular algebra according to Theorem 4.19 in [8]. In this section we focus on this specialization. As we see from the interpretation above, this specialisation transforms the quiver with a single layer in section 2.4 in [8] into one with $p$ layers. Accordingly, we need to rearrange the partitions in a multipartition. More precisely, we introduce 3-dimensional multipartitions and tableaux. They are the primary tool we will later use to prove the cellularity of the generalised Temperley-Lieb algebra $TL_{r,p,n}$.

Let $K = I' \times L$ be an index set where $I' = \{0, 1, \ldots, p - 1\}$ and $L = \{0, 1, \ldots, d - 1\}$. Note that this index set $K$ is different from the vertex set $K$ of the quiver $\Gamma_{e,p}$.

We first construct a table with $d$ rows and $p$ columns and label the boxes in the table with the elements of $K$. We call this table the floor of a 3D multipartition. Figure 1 shows the floor with $p = 5$ and $d = 3$:

\begin{align*}
(0,2) & | (1,2) & | (2,2) & | (3,2) & | (4,2) \\
(0,1) & | (1,1) & | (2,1) & | (3,1) & | (4,1) \\
(0,0) & | (1,0) & | (2,0) & | (3,0) & | (4,0)
\end{align*}

**Figure 1.** The floor of $5 \times 3$

A 3D multipartition of $n$ consists of a floor labelled with $K$ and $n(i,l)$ nodes in a column in the box $(i,l)$ on the floor such that

$$\sum_{i=0}^{p-1} \sum_{l=0}^{d-1} n(i,l) = n.$$ 

Denote by $\mathcal{P}_n$ the set of 3D multipartitions of $n$ in which at most two boxes on the floor are non-empty. Figures 2 and 3 give two examples of such 3D multipartitions. We next introduce some notation relating to the 3D multipartitions in $\mathcal{P}_n$.

For $\lambda \in \mathcal{P}_n$, if $\lambda$ has only one non-empty component, which is in the box $(i,l)$, call $\lambda$ a single multipartition, and denote it by $\lambda_{(i,l)}$. Otherwise, let $(i_1,l_1)$ and $(i_2,l_2)$ be the two non-empty boxes with $l_1 < l_2$ or $l_1 = l_2$ and $i_1 < i_2$. If there are $a$ nodes in the partition in box $(i_1,l_1)$ and $b$ nodes in $(i_2,l_2)$, denote $\lambda$ by $\lambda_{(i_1,l_1),(i_2,l_2)}^{[a-b]}$. As $\lambda$ is a multipartition of $n$, we have $a + b = n$. Therefore, the multipartition $\lambda$ is determined by the indices $(i_1,l_1), (i_2,l_2)$ and the number $a - b$.

We are now in a position to define the reducibility of a 3D multipartition.

**Definition 5.1.** Let $\lambda \in \mathcal{P}_n$ be a 3D multipartition. If $\lambda$ is of the form $\lambda_{(i_1,l_1),(i_2,l_2)}^{[a-b]}$ with $i_1 \neq i_2$, we call it an irreducible multipartition. Otherwise, we call it reducible.

In other words, if the two non-empty boxes are in the same column of the floor, we call the multipartition a reducible one; otherwise it is irreducible. In the last section of this chapter, we will show that the cell modules of $TL_{r,p,n}$ corresponding to
irreducible multipartitions are simple. Figure 2 gives an irreducible multipartition of 5 and Figure 3 gives a reducible one.

Figure 2. The irreducible multipartition $\lambda_{(2,0),(4,2)}^{[-1]}$

Figure 3. The reducible multipartition $\lambda_{(2,0),(2,2)}^{[-3]}$

We next define the residue of each node in a multipartition. This residue provides a connection between the multipartitions and the quivers we introduced above. Let $j_l \in J = \mathbb{Z}/e\mathbb{Z}$ be such that the dominant weight $\Lambda$ associated with the parameters in (2.17) is of the following form

$$\Lambda = \sum_{i=0}^{p-1} \sum_{l=0}^{d-1} \Lambda_{(i,j_l)}.$$  

(5.2)

For $\lambda \in \mathfrak{P}_n$, let $\gamma = (a, 1, (i, l))$ be the node in the $a^{th}$ row of the $(i, l)^{th}$ component of $\lambda$. We denote

$$Res^\lambda(\gamma) = (i, 1 - a + j_l) \in K$$
where $K = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/e\mathbb{Z}$ is the vertex set of the quiver $\Gamma_{e,p}$.

A tableau corresponding to a 3D multipartition of $n$ is defined as a filling of the multipartition with numbers $1, 2, \ldots, n$; we call it standard if these numbers increase from the floor table along each column. For a 3D multipartition $\lambda$, denote by $\text{Tab}(\lambda)$ the set of tableaux of shape $\lambda$ and by $\text{Std}(\lambda)$ the standard ones. For $t \in \text{Tab}(\lambda)$ and $1 \leq m \leq n$, set $\text{Res}^\lambda_t(m) = \text{Res}^\lambda_t(\gamma)$, where $\gamma$ is the unique node such that $t(m) = \gamma$. Define the residue sequence of $t$ as follows:

$$\text{res}^\lambda(t) := (\text{Res}^\lambda_t(1), \text{Res}^\lambda_t(2), \ldots, \text{Res}^\lambda_t(n)) \in K^n.$$  

For two nodes $\gamma = (a, 1, (i, l))$ and $\gamma' = (a', 1, (i', l'))$ in a 3D multipartition $\lambda$ we say $\gamma < \gamma'$ if one of the following conditions holds:

(i) $a < a'$;
(ii) $a = a'$ and $l < l'$;
(iii) $a = a'$, $l = l'$ and $i < i'$.

We write $\gamma \leq \gamma'$ if $\gamma < \gamma'$ or $\gamma = \gamma'$. The order $\leq$ is a total order on the set of nodes. Denote by $t^\lambda$ the unique tableau of shape $\lambda$ such that $t^\lambda(i) < t^\lambda(j)$ if $1 \leq i < j \leq n$. Let $e_\lambda = e(\text{res}^\lambda(t^\lambda))$ be the idempotent in $R^\lambda_n(\Gamma_{e,p})$.

Now the total order $\leq$ on nodes leads to a partial order $\preceq$ on $P_n$, the set of 3D-multipartitions in the following way: for $\lambda$ and $\mu$, we say $\lambda \preceq \mu$ if for each $\gamma_0 \in \mathbb{N} \times \{1\} \times K$ we have

$$\{|\gamma \in [\lambda]: \gamma \leq \gamma_0\| \geq \{|\gamma \in [\mu]: \gamma \leq \gamma_0\|,$$

where $\leq$ is the total order on nodes.

Let $s, t \in \text{Std}(\lambda)$ and fix reduced expressions $d(s) = s_{i_1}s_{i_2}\ldots s_{i_k}$ and $d(t) = s_{j_1}s_{j_2}\ldots s_{j_m}$ such that $s = t^\lambda \circ d(s)$ and $t = t^\lambda \circ d(t)$ where $\circ$ is the natural $S_n$-action on the tableaux. Let $*$ be the unique $R$-linear anti-automorphism of the KLR algebra $R^\lambda_n$ introduced by Brundan and Kleshchev in section 4.5 of [4] which fixes all the generators of KLR type.

For any 3D-multipartition $\lambda \in P_n$ and $s, t \in \text{Std}(\lambda)$, let

$$C^\lambda_{s,t} = \psi^*_{d(s)}e_\lambda\psi_{d(t)}.$$

Then we have

**Theorem 5.6.** ([5], Theorem 4.19) The set $\{C^\lambda_{s,t}| \lambda \in P_n, s, t \in \text{Std}(\lambda)\}$ forms a (graded) cellular basis of the generalised Temperley-Lieb algebra $\text{TL}_{r,1,n}(q, \zeta)$ with respect to the partial order $\preceq$.

### 6. Skew cellular algebras

In this section, we review some general results on graded skew cellular algebras, which were introduced by Hu, Mathas and Rostam in [7] as a slight generalisation of the cellular algebras by Graham and Lehrer in [5]. We will demonstrate the cellularity of our generalised Temperley-Lieb algebra $\text{TL}_{r,p,n}$ using the theory of skew cellularity.

In a similar way to cellular algebras, skew cellular algebras are defined in terms of a skew cell datum, which consists of the terms of a cell datum as well as a poset involution.

**Definition 6.1.** Let $\mathcal{P}$ be a finite poset with $\leq$ as the partial order. A poset automorphism of $(\mathcal{P}, \leq)$ is a permutation $\sigma$ of $\mathcal{P}$ such that $\lambda \preceq \mu$ if and only if $\sigma(\lambda) \preceq \sigma(\mu)$.
for all \( \lambda, \mu \in \mathfrak{P} \). If \( \sigma = \iota \) is an involution, we say that \( \iota \) is a poset involution of \( \mathfrak{P} \).

Graded skew cellular algebras are defined as follows.

**Definition 6.2.** ([7], Definition 2.2) Let \( R \) be an integral domain and \( A \) be a \( \mathbb{Z} \)-graded algebra over \( R \) which is a free module of finite rank over \( R \).

\( A \) is a graded skew cellular algebra if it has a graded skew cellular datum \((\mathfrak{P}, \iota, T, C, \deg)\) where \((\mathfrak{P}, \leq)\) is a poset, \( \iota \) is a poset involution and for each \( \lambda \in \mathfrak{P} \), there is a finite set \( T(\lambda) \) together with a bijection \( \iota_\lambda : T(\lambda) \to T(\iota(\lambda)) \) such that

\[
\iota_\lambda \circ \iota_\lambda = id_{T(\lambda)}.
\]

The injection

\[
\begin{align*}
C : \cup_{\lambda \in \mathfrak{P}} T(\lambda) \times T(\lambda) & \to A \\
(S, T) & \mapsto C_{S,T}^\lambda
\end{align*}
\]

and the degree map

\[
\deg : \cup_{\lambda \in \mathfrak{P}} T(\lambda) \to \mathbb{Z}
\]

satisfy the following conditions:

(C1) \( C_{S,T}^\lambda \) is homogeneous of degree \( \deg(S) + \deg(T) \);

(C2) The image of \( C \) forms an \( R \)-basis of \( A \);

(C3) Let \( A_{\iota\lambda} \) be the \( R \)-span of all the elements of form \( C_{X,Y}^\mu \) with \( \mu \triangleleft \lambda \) in the poset. Then for all \( a \in A \),

\[
a C_{S,T}^\lambda = \sum_{S' \in T(\lambda)} r_a(S', S) C_{S',T}^\lambda \mod A_{\iota\lambda}
\]

with the coefficients \( r_a(S', S) \) independent of \( T \).

(C4) There is a unique anti-isomorphism \( * : A \to A \) such that

\[
(C_{S,T}^\lambda)^* = C_{\iota_\lambda(T),T}^{\iota_\lambda(S)}.
\]

The basis \( \{C_{S,T}^\lambda | \lambda \in \mathfrak{P}, S, T \in T(\lambda)\} \) is called a \( \mathbb{Z} \)-graded skew cellular basis of \( A \).

The following is a baby example of the concept of a skew cellular structure.

**Example 6.5.** ([7] Example 2.5) Let \( R \) be a ring and \( x, y \) be two indeterminates. For any integer \( m \geq 1 \), let \( A = R[x]/(x^m) \oplus R[y]/(y^m) \). Let \( \mathfrak{P} = \mathbb{Z}_2 \times \{0, 1, \ldots, m-1\} \) and \( \leq \) be the partial order on \( \mathfrak{P} \) such that \((i_1, k_1) \leq (i_2, k_2)\) only if \( i_1 = i_2 \) and \( k_1 \leq k_2 \). Let \( \iota \) be the poset involution on \( \mathfrak{P} \) such that \( \iota(i, k) = (i + 1, k) \). Define \( T(i, k) = \{k\} \) and \( \deg(k) = k \), that is, there exists a unique ‘tableau’ of each element in \( \mathfrak{P} \). Define

\[
C_{i,k}^{i,k} = \begin{cases} 
x^i & \text{if } i = 0; \\
y^k & \text{if } i = 1.
\end{cases}
\]

Then \((\mathfrak{P}, \iota, T, C, \deg)\) is a graded cellular datum of \( A \).

By applying the anti-isomorphism \( * \) to (6.3) and relabelling, we obtain

\[
C_{S,T}^\lambda a = \sum_{T' \in T(\lambda)} r_{a^*}(\iota_\lambda(T'), \iota_\lambda(T)) C_{S,T'}^\lambda \mod A_{\iota\lambda}
\]

where the coefficients \( r_{a^*}(\iota_\lambda(T'), \iota_\lambda(T)) \) are the same ones as in (6.3) and do not depend on \( S \).
If it should be remarked that if the poset involution \( \iota = id_P \) and \( \iota_\lambda = id_{T(\lambda)} \), the graded skew cellular algebra is graded cellular. We will show in this way that the apparent skew cellularity of our \( TL_{r,p,n} \) is actually cellularity.

Similarly to cellular structures, skew cellular ones are also a useful tool to study the representations of algebras which are not semisimple. Since our generalised Temperley-Lieb algebra \( TL_{r,p,n} \) will later turn out to be a cellular algebra, we omit details of skew cell modules and decomposition matrices. We refer the reader to \([7]\) for further information.

Next recall the definition of shift automorphisms which provide a general method for the construction of skew cellular algebras from cellular ones.

**Definition 6.7.** ([7, Definition 2.22]) Let \( A \) be a \( \mathbb{Z} \)-graded cellular algebra with graded cell datum \( (\mathcal{P}, T, C, \text{deg}) \). A shift automorphism of \( A \) is a triple of automorphisms \( \sigma = (\sigma_A, \sigma_P, \sigma_T) \) where \( \sigma_A \) is an algebra automorphism of \( A \), \( \sigma_P \) is a poset automorphism of \( \mathcal{P} \) and \( \sigma_T \) is a bijection on \( T = \bigsqcup_{\lambda \in \mathbb{P}} T(\lambda) \) such that:

(a) For \( S \in T(\lambda) \), \( \sigma_T(S) \in T(\sigma_P(\lambda)) \) and \( \text{deg}(\sigma_T(S)) = \text{deg}(S) \);

(b) For \( S, T \in T(\lambda) \), \( \sigma_A(C_{S,T}) = C_{\sigma_T(S),\sigma_T(T)} \);

(c) For \( S, T \in T(\lambda) \), \( \sigma_T^k(T) = T \) if and only if \( \sigma_T^k(S) = S \).

Before stating the main theorem of this section, we introduce some notation.

Denote by \( A^\sigma \) the subalgebra of \( A \) consisting of \( \sigma_A \)-fixed points. Let \( p \) be the order of \( \sigma_A \) and \( p' \) be the order of \( \sigma_P \). Let \( \mathcal{P}_\sigma \) be a set of representatives for the \( \langle \sigma_P \rangle \)-orbits in \( \mathcal{P} \). Define a partial order \( \preceq_\sigma \) on \( \mathcal{P}_n \) by

\[ \lambda \preceq_\sigma \mu \quad \text{if and only if} \quad \sigma_P^k(\lambda) \preceq \mu \quad \text{for some} \quad k \in \mathbb{Z}. \]

Denote by \( o_\lambda \) the size of the \( \langle \sigma_P \rangle \)-orbit through \( \lambda \). Let \( \sigma^p = \sigma_{p,o_\lambda}^\pi \). Similarly, let \( T_\sigma(\lambda) \) be a set of representatives for the \( \langle \sigma_P \rangle \)-orbit on \( T(\lambda) \). The condition (c) above implies that for a \( \lambda \in \mathcal{P} \), all the orbits in \( T(\lambda) \) are of the same size. Denote this common value by \( o_{T(\lambda)} \). Then \( o_\lambda = o_\mu \) and \( o_{T(\lambda)} = o_{T(\mu)} \) for \( \lambda \) and \( \mu \) in the same \( \langle \sigma_P \rangle \)-orbit on \( \mathcal{P} \).

Let \( \mathcal{P}_{\sigma,p} := \{(\lambda, k)|\lambda \in \mathcal{P}_\sigma, k \in \mathbb{Z}/o_{T(\lambda)}\mathbb{Z}\} \) be the poset with partial order \( \preceq_\sigma \) given by

\[ (\lambda, k) \preceq_\sigma (\mu, l) \quad \text{if and only if} \quad (\lambda, k) = (\mu, l) \quad \text{or} \quad \lambda \preceq_\sigma \mu, \]

for all \( (\lambda, k), (\mu, l) \in \mathcal{P}_{\sigma,p} \).

Define \( T_\sigma(\lambda, k) = T_\sigma(\lambda) \) for \( (\lambda, k) \in \mathcal{P}_{\sigma,p} \). Assuming there exists a primitive \( p'^{th} \) root of unity \( \zeta \), set

\[ C_\sigma(S, T) = \sum_{j=0}^{\sigma_T(\lambda) - 1} \sum_{l=0}^{p-1} \zeta^{k^j} \sigma_A^l(C_{S,\sigma_T(T)}) \]

where \( \zeta = \zeta^p/o_{T(\lambda)} \) and \( S, T \in T_\sigma(\lambda, k) \). Let \( \deg_\sigma(S) = \deg(S) \). Finally, let \( \iota_\sigma \) be the poset involution such that \( \iota_\sigma(\lambda, k) = (\lambda, -k) \) and \( (\iota_\sigma)(\lambda, k) : T_\sigma(\lambda, k) \mapsto T_\sigma(\lambda, -k) \) be the map given by the identity map on \( T_\sigma(\lambda) \).

Then we have a shift automorphism that leads to a skew cellular algebra. More precisely,

**Theorem 6.8.** ([7, Theorem 2.28]) Suppose \( A \) is a \( \mathbb{Z} \)-graded cellular algebra with graded cell datum \( (\mathcal{P}, T, C, \text{deg}) \) over the integral domain \( R \). Let \( \sigma = (\sigma_A, \sigma_P, \sigma_T) \) be a shift automorphism of \( A \). Denote by \( p \) the order of \( \sigma_A \). If \( R \) contains a
primitive $p^{th}$ root of unity and $p \in R^*$, then $A^*$ is a graded skew cellular algebra with skew cellular datum $(\mathcal{P}_{\sigma,p}, \iota_\sigma, T_\sigma, C_\sigma, \deg_\sigma)$.

7. The skew cellularity of $TL_{r,p,n}$

In [7], Hu, Mathas and Rostam introduced skew cellular algebras as a generalisation of the original cellular algebras. They define the notion of a “shift automorphism” of a cellular algebra and show that the fixed subalgebra has a skew cellular structure (cf. Theorem 6.8 above). In this section, we prove that $TL_{r,p,n}$ is a skew cellular by constructing a shift automorphism on $TL_{r,1,n}(q, \zeta)$ and identifying $TL_{r,p,n}$ as the subalgebra consisting of the fixed points of the shift automorphism.

We first define the shift automorphism on the specialised Temperley-Lieb algebra $TL_{r,1,n}(q, \zeta)$. As shown in Theorem 6.2, $TL_{r,p,n}$ is the fixed-point subalgebra of $TL_{r,1,n}(q, \zeta)$ under $\sigma_{TL}$, which is induced by the automorphism $\sigma'$ of the cyclotomic KLR algebra $R^A_n(T_{r,p})$ defined above Theorem 2.2. So we choose the algebra automorphism $\sigma_{TL}$ as the first term in the shift automorphism $\sigma$.

Condition (b) in Definition 2.7 indicates that the poset automorphism $\sigma_\mathcal{P}$ and the bijection $\sigma_\gamma$ are determined by the cellular structure of $A$ and the algebra automorphism $\sigma_A$. However, the poset map $\sigma_\mathcal{P}$ induced by the algebra automorphism $\sigma_{TL}$ with respect to the cellular structure in Theorem 5.6 is not an automorphism of the poset $(\mathcal{P}_n, \leq)$ with the partial order $\leq$ given by (5.4). As a counterexample, let $\lambda_{(0,0)}$ and $\lambda_{(1,0)} \in \mathcal{P}_n$ be the 3D multipartitions of $n$ with the $(0,0)^{th}$ and $(1,0)^{th}$ components are the only non-empty ones respectively. We have $\lambda_{(0,0)} \leq \lambda_{(1,0)}$ but $\sigma_\mathcal{P}(\lambda_{(1,0)}) = \lambda_{(0,0)} \leq \lambda_{(p-1,0)} = \sigma_\mathcal{P}(\lambda_{(0,0)})$.

We therefore introduce a new cellular structure on the specialised algebra $TL_{r,1,n}(q, \zeta)$.

It should be pointed out that the following construction does not work for the original $TL_{r,1,n}$ without the specialisation to the case we treat. Let $\mathcal{P}_n$ be the set of 3D multipartitions with at most two non-empty components. We first introduce a new partial order on the indices on the floor table.

**Definition 7.1.** Let $\mathcal{K} = \{0, 1, \ldots, p-1\} \times \{0, 1, \ldots, d-1\}$ be the index set introduced in Section 5. Define $\leq$ as the total order on $\mathcal{K}$, such that for $(i_1, l_1), (i_2, l_2) \in \mathcal{K}$:

$$ (i_1, l_1) \leq (i_2, l_2) \text{ if and only if } l_1 < l_2 \text{ or } l_1 = l_2 \text{ and } i_1 \leq i_2. $$

And define $\leq'$ as the partial order on $\mathcal{K}$, such that for $(i_1, l_1), (i_2, l_2) \in \mathcal{K}$:

$$ (i_1, l_1) \leq' (i_2, l_2) \text{ if and only if } l_1 < l_2 \text{ or } l_1 = l_2 \text{ and } i_1 = i_2. $$

It is obvious that $(i_1, l_1) \leq' (i_2, l_2)$ implies $(i_1, l_1) \leq (i_2, l_2)$. Similarly to the notation in section 5 denote a multipartition $\lambda \in \mathcal{P}_n$ by:

(1) $\lambda_{(i,j)}$, if the $(i,l)^{th}$ component is the only non-empty one in $\lambda$. Equivalently, $\lambda$ consists of a column of $n$ nodes in the $(i,l)^{th}$ position.

(2) $\lambda_{[i_1,l_1],(i_2,l_2)}$, if the $(i_1,l_1)^{th}$ and $(i_2,l_2)^{th}$ components in $\lambda$ are non-empty with $(i_1,l_1) < (i_2,l_2)$ in $(\mathcal{K}, \leq)$ and $a = a_{i_1,l_1} - a_{i_2,l_2}$ where $a_{i,l}$ is the number of nodes in the $(i,l)^{th}$ component. It is obvious that $2 - n \leq a \leq n - 2$ and $a \equiv n (\text{mod } 2)$.

We next define two partial orders $\leq$ and $\leq'$ on $\mathcal{P}_n$, which respectively correspond to $\leq$ and $\leq'$ on $\mathcal{K}$, as follows:
Definition 7.4. Let $\mathcal{P}_n$ be the set of 3D multipartitions of $n$, each containing at most two non-empty components. Define the partial order $\leq$ (resp. $\leq'$) to be the unique partial order satisfying:

1. If $\lambda$ and $\mu$ of the form $\lambda_{(i_1,l_1)}$ and $\mu_{(i_2,l_2)},$ then
   \[ \lambda_{(i_1,l_1)} \leq \lambda_{(i_2,l_2)} \text{ (resp. } \lambda_{(i_1,l_1)} \leq \lambda_{(i_2,l_2)}') \text{ if and only if } (i_1,l_1) \leq (i_2,l_2) \text{ in } \mathcal{K}. \]

2. If $\lambda$ and $\mu$ are of the form $\lambda_{(i_1,l_1),(i_2,l_2)}$ and $\mu_{(i_3,l_3)},$ then
   \[ \lambda_{(i_1,l_1),(i_2,l_2)} \leq \lambda_{(i_3,l_3)} \text{ (resp. } \lambda_{(i_1,l_1),(i_2,l_2)} \leq \lambda_{(i_3,l_3)}') \text{ if and only if } (i_1,l_1) \leq (i_3,l_3) \text{ in } \mathcal{K}. \]

3. If $\lambda$ and $\mu$ are of the form $\lambda_{(i_1,l_1),(i_2,l_2)}$ and $\mu_{(i_3,l_3),(i_4,l_4)},$ then
   \[ \lambda_{(i_1,l_1),(i_2,l_2)} \leq \lambda_{(i_3,l_3),(i_4,l_4)} \text{ (resp. } \lambda_{(i_1,l_1),(i_2,l_2)} \leq \lambda_{(i_3,l_3),(i_4,l_4)}') \text{ if and only if } (i_1,l_1) \leq (i_3,l_3), \]
   \[ (i_2,l_2) \leq (i_4,l_4) \text{ in } \mathcal{K} \text{ and one of the following cases holds:} \]
   \[ (i) \ |a| < |b|; \]
   \[ (ii) \ |a| = |b| \text{ and } a \geq b; \]
   \[ (iii) \ |a| = |b|, \ a < b \text{ and } (i_2,l_2) \leq (i_3,l_3). \]

Theorem 5.6 implies $\{C^\lambda_{a,i} \}$ is a cellular basis with respect to the partial order $\leq$. We next use the other partial order $\leq'$ to define a new cellular structure on $TL_{e,1,n}(q, \zeta).$ As a direct consequence of the fact that $\leq'$ is covered by $\leq$, $\lambda \leq' \mu$ implies $\lambda \leq \mu$ for any $\lambda, \mu \in \mathcal{P}_n$. The converse is not always true. Nevertheless, we have the following two lemmas:

Lemma 7.5. Let $\lambda \triangleleft \mu \in \mathcal{P}_n$ be two 3D multipartitions. If there exists a standard tableau $t$ of shape $\lambda$ such that
\[ e(t) = e_{\mu}, \]
where $e_{\mu} = e(\text{res}^A(\mu)),$ then we have $\lambda \triangleleft' \mu \in \mathcal{P}_n$ where $\triangleleft'$ is the finer partial order defined above.

Proof. As $\mu \in \mathcal{P}_n,$ it has one of the following three shapes:

(a) $\mu$ has only one non-empty component, that is $\mu = \mu_{(i,l)}.$ In this case, we have
\[ (7.6) \ \text{res}^A(t) = \left( (i,j_1), (i,j_1 - 1), \ldots, (i,j_1 - n) \right). \]
So $\text{res}^A(t) = \text{res}^A(\mu) = \left( (i,j_1), (i,j_1 - 1), \ldots, (i,j_1 - n) \right),$ which implies that the shape of $t$ is either of the form $\lambda_{(i_1,l_1)}$ or $\lambda_{(i_1,l_1),(i_2,l_2)}.$ By Definition 7.4 $\lambda \triangleleft \mu$ implies $(i_1,l_1) \leq (i,l)$ in the first case and $(i_1,l_1) \leq (i_2,l_2) \leq (i,l)$ in the second. Comparing the two orders in Definition 7.4, we have $(i,l_1) \leq (i,l)$ in the first case and $(i_3,l_3) \leq (i,l)$ in the second. As a immediate consequence, $\lambda \triangleleft' \mu$ in both cases.

(b) $\mu$ has two non-empty components and is of the form $\mu = \mu_{(i_1,l_1),(i_2,l_2)}.$ With the same argument as in the first case, we have $\lambda = \lambda_{(i_3,l_3),(i_4,l_4)}.$ As the first terms in all the indices are the same, the two partial orders on the indices, $\leq$ and $\leq',$ are equivalent. Therefore, $\lambda_{(i_1,l_1),(i_2,l_2)} \triangleleft \mu_{(i_1,l_1),(i_2,l_2)}$ implies $\lambda_{(i_3,l_3),(i_4,l_4)} \triangleleft' \mu_{(i_1,l_1),(i_2,l_2)}.$

(c) $\mu$ is of the form $\mu = \mu_{(i_1,l_1),(i_2,l_2)}.$ There are $\frac{1}{2}(n + a)$ terms of the form $(i_1,x)$ and $\frac{1}{2}(n - a)$ terms of the form $(i_2,y)$ in the residue of $\mu.$ As $\text{res}^A(t) = \text{res}^A(\mu),$ the number of nodes in $\lambda$ with residue of the form $(i_1,x)$ is $\frac{1}{2}(n + a)$ and that of the form
\[(i_2, y) = \frac{1}{2}(n - a)\]. The restriction on the parameters imposed in (2.3) guarantees that the nodes in the first two rows of a 3D multipartition have different residues. In other words, the nodes in the first two rows are uniquely determined by their residue. As \(t\) is a standard tableau of shape \(\lambda\), the number 1 in \(t\) is in the first row. Further, \(\text{res}^\lambda(t) = \text{res}^\lambda(t^\mu)\) implies that \(\text{Res}_s(1) = \text{Res}_s(1^\mu)\). Therefore, \(t(1) = t^\mu(1) = (1, 1, (i_1, l_1))\) where \(t(1)\) is the node labelled with 1 in \(t\). Similarly, \(t(2) = t^\mu(2) = (1, 1, (i_2, l_2))\). As \(\lambda\) consists of at most two non-empty components, we have \(\lambda = \lambda^{(a)}_{(i_1, l_1), (i_2, l_2)} = \mu\) which contradicts \(\lambda \triangleleft \mu\). So there is no standard tableau \(t\) of shape \(\lambda\) such that \(e(t) = e_\mu\) in this case. Therefore, if \(\lambda \triangleleft \mu \in \mathcal{P}_n\) and there exists a standard tableau \(t\) of shape \(\lambda\) such that
\[e(t) = e_\mu,\]
the 3D multipartition \(\mu\) is of the form either \(\mu_{(i, j)}^{[a]}\) or \(\mu_{(i_1, l_1), (i_2, l_2)}^{[a]}\). In both of these cases, \(\lambda \prec' \mu \in \mathcal{P}_n\) where \(\prec'\) is the finer partial order in Definition 7.4.

**Lemma 7.7.** Let \(\lambda \prec \mu \in \mathcal{P}_n\) be two 3D multipartitions. We have \(\lambda \prec' \mu\) if there exists a standard tableau \(t\) of shape \(\lambda\) and a Garnir tableau \(g\) of shape \(\mu\) such that
\[e(t) = e(g).\]

**Proof.** The proof of cases (a) and (b) is the same as in the last lemma. We claim that the third case does not occur. If \(\mu\) is of form \(\mu_{(i_1, l_1), (i_2, l_2)}^{[a]}\). There are \(\frac{1}{2}(n + a)\) terms of the form \((i_1, x)\) and \(\frac{1}{2}(n - a)\) terms of the form \((i_2, y)\) in the residue of \(t^\theta\). Let \(i\) be such that \(\text{Res}_s(1) = (i_j, i)\) and \(c\) be the smallest number such that \(\text{Res}_s(c) = (i', j')\) where \(i' \neq i\). As \(g\) is a Garnir tableau, \(s_k \circ g\) is standard for some \(1 \leq k \leq n - 1\). If \(k = 1\) (or \(c\), 1 (or \(c\)) is in the second row and 2 (or \(c + 1\)) is in the first row of the same component, otherwise it is in the first row of \(g\). As \(\text{res}^\lambda(t) = \text{res}^\lambda(g), \text{Res}_s(1) = \text{Res}_s(1)\) and \(\text{Res}_s(c) = \text{Res}_s(c)\). Both 1 and \(c\) should be in the first row of \(t\) as \(t\) is standard. As the nodes in the first two rows are uniquely determined by their residue, the nodes containing 1 and \(c\) are in the same position in \(t\) and \(g\). So we have \(\lambda = \lambda^{[a]}_{(i_1, l_1), (i_2, l_2)} = \mu\), which contradicts \(\lambda \prec \mu\). So there is no standard tableau \(t\) of shape \(\lambda \prec \mu\) such that \(e(t) = e_g\) for some Garnir tableau \(g\) of shape \(\mu\). □

By Theorem 5.4 for \(\lambda \in \mathcal{P}_n\) and \(s, t \in \text{Std}(\lambda)\), let \(C^\lambda_{s,t} \in TL_{r,1,n}(q, \zeta)\) be the element defined in (5.5). Then \(\{C^\lambda_{s,t} | \lambda \in \mathcal{P}_n, s, t \in \text{Std}(\lambda)\}\) is a graded cellular basis of \(TL_{r,1,n}(q, \zeta)\) with respect to the partial order \(\triangleleft\). We next show that this cellularity still holds with respect to the finer partial order \(\triangleleft'\), that is,

**Theorem 7.8.** Let \(TL_{r,1,n}(q, \zeta)\) be the specialization of \(TL_{r,1,n}\) in section 2. Let \((\mathcal{P}_n, \triangleleft')\) be the poset defined in Definition 7.4 and \(C^\lambda_{s,t}\) be the element defined in (5.5). Then \(\{C^\lambda_{s,t} | \lambda \in \mathcal{P}_n, s, t \in \text{Std}(\lambda)\}\) is a graded cellular basis of \(TL_{r,1,n}(q, \zeta)\) over \(R\) with respect to with respect to the cell datum \((\mathcal{P}_n, \text{Std}, C, \text{deg})\).

**Proof.** Comparing with the cellular structure in Theorem 5.4 it is enough to show that for any \(a \in TL_{r,1,n}(q, \zeta), \lambda \in \mathcal{P}_n\) and \(s, t \in \text{Std}(\lambda)\), we have
\[(7.9) \quad aC^\lambda_{s,t} = \sum_{s' \in \text{Std}(\lambda)} r_a(s', s)C^\lambda_{s', t} + \sum_{\mu' \triangleleft' \lambda, u, v \in \text{Std}(\mu)} c_a(s, t, u, v)C^\mu_{u,v}\]
where \( r_a(s', s) \in R \) does not depend on \( t \) and \( c_a(s, t, u, v) \in R \). We only need to check the cases where \( a \) is in the generating set of \( TL_{r,1,n}(q,\zeta) \) as a KLR algebra.

If \( a = e(i) \), \( aC_{s,t}^\lambda \) is either 0 or \( C_{s,t}^\lambda \), because all \( e(i) \)'s are orthogonal idempotents.

If \( a = y_k \), the following equation can be obtained by direct calculation using the generating relations of KLR algebras.

\[
(7.10) \quad y_kC_{s,t}^\lambda = y_i\psi_d(s)\psi_d(t) = \psi_d(t)y_j\psi_d(t) + \sum_{l(V) < l(d(t))} c(V)\psi_d^* e\lambda \psi_d(t)
\]

where \( c(V) \) is 1 or -1. The first term is in the two-sided ideal generated by \( y_j e\lambda \).

The other terms can be transformed by Lemma 6.7 in [8] as follows:

\[
(7.11) \quad \psi_d^* e\lambda \psi_d(t) = \sum_{s' \in \text{Std}(\lambda)} r_a(s', s)C_{s',t}^\lambda + \sum_{i=1}^n Y_i + \sum_{g \in \text{Gar}(\lambda)} D_g
\]

where \( Y_i \) is in the two-sided ideal generated by \( y_i e\lambda \) and \( D_g \) is in the one generated by \( e(g) \) with \( g \) running over the Garnir tableaux of shape \( \lambda \).

If \( a = \psi_k \), Lemma 6.7 in [8] indicates that

\[
(7.12) \quad aC_{s,t}^\lambda = \sum_{s' \in \text{Std}(\lambda)} r_a(s', s)C_{s',t}^\lambda + \sum_{i=1}^n Y_i + \sum_{g \in \text{Gar}(\lambda)} D_g
\]

where \( Y_i \) is in the two-sided ideal generated by \( y_i e\lambda \) and \( D_g \) is in the one generated by \( e(g) \) with \( g \) running over the Garnir tableaux of shape \( \lambda \).

Therefore, it is enough to show that

\[
(7.13) \quad y_i e\lambda = \sum_{\mu \in \lambda, u,v \in \text{Std}(\mu)} c_i(u,v)C_{u,v}^\mu
\]

for \( 1 \leq i \leq n \), and

\[
(7.14) \quad e(g) = \sum_{\mu \in \lambda, u,v \in \text{Std}(\mu)} c_g(u,v)C_{u,v}^\mu
\]

for \( g \in \text{Gar}(\lambda) \). We treat \( y_i e\lambda \) first. By Lemma 17 and Lemma 37 in [10], we have

\[
(7.15) \quad y_i e\lambda = \sum_{\mu \in \lambda, u,v \in \text{Std}(\mu)} c_i(u,v)C_{u,v}^\mu.
\]
As $y_i \epsilon_\lambda = \epsilon_\lambda y_i$ and $\epsilon_\lambda$ is an idempotent, we have

$$y_i \epsilon_\lambda = \epsilon_\lambda y_i \epsilon_\lambda$$

$$= \sum_{\mu \triangleleft \lambda, u, v \in \text{Std}(\mu)} c_i(u, v) \epsilon_\lambda C^\mu_{u,v}$$

$$= \sum_{\mu \triangleleft \lambda, u, v \in \text{Std}(\mu)} c_i(u, v) \epsilon_\lambda \psi^*_d(u) \epsilon_\mu \psi_d(v)$$

$$= \sum_{\mu \triangleleft \lambda, u, v \in \text{Std}(\mu)} c_i(u, v) \epsilon_\lambda \epsilon(u) \psi^*_d(u) \psi_d(v)$$

$$= \sum_{\mu \triangleleft \lambda, u, v \in \text{Std}(\mu), \epsilon_\lambda = \epsilon(u)} c_i(u, v) C^\mu_{u,v}$$

$$= \sum_{\mu \triangleleft \lambda, u, v \in \text{Std}(\mu)} c_i(u, v) C^\mu_{u,v}$$

where the last step is by Lemma 7 and the coefficients $c_i(u, v)$ are zero if $\epsilon_\lambda \neq \epsilon(u)$ in the last summand.

For $\epsilon(g)$, Lemma 35 in [10] implies that

$$\epsilon(g) = \sum_{\mu \triangleleft \lambda, u, v \in \text{Std}(\mu)} c_g(u, v) C^\mu_{u,v}.$$

As $\epsilon(g)$ is an idempotent, we have

$$\epsilon(g) = \epsilon(g)^2$$

$$= \epsilon(g) \sum_{\mu \triangleleft \lambda, u, v \in \text{Std}(\mu)} c_g(u, v) C^\mu_{u,v}$$

$$= \sum_{\mu \triangleleft \lambda, u, v \in \text{Std}(\mu)} c_g(u, v) \epsilon(g) \psi^*_d(u) \epsilon_\mu \psi_d(v)$$

$$= \sum_{\mu \triangleleft \lambda, u, v \in \text{Std}(\mu)} c_g(u, v) \epsilon(g) \psi^*_d(u) \psi_d(v)$$

$$= \sum_{\mu \triangleleft \lambda, u, v \in \text{Std}(\mu), \epsilon(u) = \epsilon(g)} c_g(u, v) \epsilon(u) \psi^*_d(u) \psi_d(v)$$

$$= \sum_{\mu \triangleleft \lambda, u, v \in \text{Std}(\mu), \epsilon(u) = \epsilon(g)} c_g(u, v) C^\mu_{u,v}$$

$$= \sum_{\mu \triangleleft \lambda, u, v \in \text{Std}(\mu)} c_g(u, v) C^\mu_{u,v}$$

where the last step is by Lemma 7 and the coefficients $c_g(u, v)$ are zero if $\epsilon_g \neq \epsilon(u)$ in the last sum.

So for any $a \in TL_{r,1,n}(q, \zeta)$, $\lambda \in \mathfrak{P}_n$, and $s, t \in \text{Std}(\lambda)$, we have

$$a C^\lambda_{s,t} = \sum_{s' \in \text{Std}(\lambda)} r_a(s', s) C^\lambda_{s', t} + \sum_{\mu \triangleleft \lambda, u, v \in \text{Std}(\mu)} c_a(s, t, u, v) C^\mu_{u,v},$$

(7.16)
Therefore, \( \{ C_{s,t}^\lambda | \lambda \in \mathcal{P}_n, s, t \in \text{Std}(\lambda) \} \) is a graded cellular basis of \( TL_{r,1,n}(q, \zeta) \) over \( R \) with respect to \( \leq' \).

Now consider the corresponding poset automorphism \( \sigma_{P_n} \) on the poset \( (\mathcal{P}_n, \leq') \). Let \( \sigma_{P_n} : \mathcal{P}_n \to \mathcal{P}_n \) be the bijective map given by:

\[
\sigma_{P_n}(\lambda_{(i,l)}) = \lambda_{(i-1,l)}, \\
\sigma_{P_n}(\lambda_{[a]}_{(i_1,l_1), (i_2,l_2)}) = \lambda_{[a]_{(i_1-1,l_1), (i_2-1,l_2)}},
\]

where \( i - 1 = p - 1 \) when \( i = 0 \). In other words, \( \sigma_{P_n} \) shifts the 3D-multipartitions forward by one column. We may now prove:

**Lemma 7.17.** \( \sigma_{P_n} \) is a poset automorphism of \( (\mathcal{P}_n, \leq') \).

**Proof.** The bijectivity is obvious. We prove that \( \sigma_{P_n} \) preserves the partial order \( \leq' \). For any \( (i_1, l_1) <' (i_2, l_2) \in K \), we have \( l_1 < l_2 \) which implies that \( (i_1 - 1, l_1) <' (i_2 - 1, l_2) \). Therefore, \( (i_1, l_1) \leq' (i_2, l_2) \) implies \( (i_1 - 1, l_1) \leq' (i_2 - 1, l_2) \). By Definition 7.3, \( \lambda <' \mu \in \mathcal{P}_n \) implies \( \sigma_{P_n}(\lambda) <' \sigma_{P_n}(\mu) \). Therefore, \( \sigma_{P_n} \) is a poset automorphism of \( (\mathcal{P}_n, \leq') \). \( \square \)

For \( t \in \text{Std}(\lambda) \) where \( \lambda \in \mathcal{P}_n \), define \( \sigma_T(t) \) as the standard tableau with shape \( \sigma_{P_n}(\lambda) \) and all the numbers in the same relative positions as those in \( t \).

**Theorem 7.18.** Let \( \sigma_{TL} \) be the automorphism of \( TL_{r,1,n}(q, \zeta) \) defined by (6.8), \( \sigma_{P_n} \) and \( \sigma_T \) be the corresponding automorphisms described above. Then \( \sigma = (\sigma_{TL}, \sigma_{P_n}, \sigma_T) \) is a shift automorphism (cf. Definition 6.7) with respect to the cell datum \( (\mathcal{P}_n, \text{Std}, C, \text{deg}) \) of \( TL_{r,1,n}(q, \zeta) \) where \( \mathcal{P}_n \) is the poset with \( \leq' \) (cf. Theorem 7.8).

**Proof.** By the argument above, \( \sigma_{TL}, \sigma_{P_n}, \) and \( \sigma_T \) are well defined. The conditions (a) and (b) in Definition 6.7 can be checked directly. For any \( t \in \text{Std}(\lambda), \sigma_T^k(t) = t \) if and only if \( p|k \). So (c) holds for all \( \lambda \in \mathcal{P}_n \). \( \square \)

We next identify the skew cell datum of the point-wise fixed subalgebra \( TL_{r,p,n} \) under the shift automorphism \( \sigma \). Let \( \equiv \) be the equivalence relation on \( \mathcal{P}_n \) such that

\[
\lambda_{(i,l)} \equiv \lambda_{(i+k,l)}, \\
\mu_{[a]_{(i_1,l_1), (i_2,l_2)}} \equiv \mu_{[a]_{(i_1+k,l_1), (i_2+k,l_2)}},
\]

for all \( k \in I \) and \( \lambda_{(i,l)}, \mu_{[a]_{(i_1,l_1), (i_2,l_2)}} \in \mathcal{P}_n \). Notice that \( \lambda \equiv \sigma_{P_n}(\lambda) \) for all \( \lambda \in \mathcal{P}_n \).

Denote by \([\lambda]\) the equivalence class in \( \mathcal{P}_n \) containing \( \lambda \). Then \([\lambda]\) is the orbit of \( \lambda \) under \( \sigma_{P_n} \). Define the set of orbits

\[
\mathcal{P}_{n,p} := \{ [\lambda]| \lambda \in \mathcal{P}_n \}.
\]

Define \([\lambda] \leq_p [\mu] \in \mathcal{P}_{n,p} \) if and only if there exist \( \lambda_1 \equiv \lambda \) and \( \mu_1 \equiv \mu \) such that \( \lambda_1 \leq' \mu_1 \in \mathcal{P}_n \). We should remind readers that \([\lambda] \leq_p [\mu]\) does not imply \( \lambda \leq' \mu \), they may not be comparable with respect to \( \leq' \). We next give a counterexample.

**Example 7.21.** Let \( r = 9, p = 3 \) and \( n = 5 \). Then \( \lambda_{(1,1),(2,3)}^1 \) and \( \mu_{(0,1)} \) are two 3D multipartitions in \( \mathcal{P}_5 \). We have

\[
[\lambda_{(1,1),(2,3)}^1] \leq_p [\mu_{(0,1)}]
\]
because $\lambda_{(1,1),(2,3)}^{[1]} \equiv \lambda_{(0,1),(1,3)}^{[1]} \leq'$ $\mu_{(0,1)}$ according to (2) in Definition 4.4. But $\lambda_{(1,1),(2,3)}^{[1]}$ and $\mu_{(0,1)}$ are incomparable with respect to $\leq'$.

By definition, for any $\lambda \in \mathfrak{P}_n$ and $t \in \text{Std}(\lambda)$, the size of the $\langle \sigma_{\Psi_n} \rangle$-orbit containing $\lambda$ and the size of the $\langle \sigma_T \rangle$-orbit containing $t$ are both $p$. Therefore, the $\sigma_\lambda$ in the general theory before Theorem 6.8 is identity which implies that the poset involution $t_p = id_{\Psi_n}$. This implies the skew cellular structure we construct is actually cellular.

In analogy to the equivalence relation among the multipartitions, we define an equivalence relation $\equiv$ on $\mathcal{T} = \bigsqcup_{\lambda \in \mathfrak{P}_n} T(\lambda)$ as follows:

For $s, t \in \mathcal{T}$, we say $s \equiv t$ if and only if $\text{shape}(s) \equiv \text{shape}(t)$ and the numbers in $s$ are in the same relative positions as those in $t$. Denote by $[t]$ the equivalence class containing $t$ in $\mathcal{T}$. For $s \equiv t \in \mathcal{T}$, as the numbers in $s$ are in the same relative positions as those in $t$, we have $d(s) = d(t)$ and $\deg(s) = \deg(t)$, where $d(s)$ is the element in $S_n$ transforming $s$ to the unique standard tableau $t^\lambda$. Therefore, the element $d([t])$ can be defined as $d(s)$ and $\deg_p([t])$ can be defined as $\deg(s)$ for any $s \equiv t$. For $[\lambda] \in \mathfrak{P}_n$, define:

$$T_p([\lambda]) := \{[t] | t \in \text{Std}(\lambda)\}.$$  

Finally, for $[s], [t] \in T_p([\lambda])$, define

$$C_p^{[\lambda]}([s], [t]) = \psi_{d([s])}(\sum_{\mu \equiv \lambda} e_\mu \psi_{d([t])}).$$

The following Theorem is a direct consequence of Theorem 6.8.

**Theorem 7.24.** Let $TL_{r,p,n}$ be the Temperley-Lieb algebra of type $G(r,p,n)$ defined in Definition 4.4. $\mathfrak{P}_{n,p}$ be the poset of equivalence classes of multipartitions defined in (7.20) and $T_p, C_p, \deg_p$ be as described above. Then $TL_{r,p,n}$ is a skew cellular algebra with skew cell datum $(\mathfrak{P}_{n,p}, id_T, T_p, C_p, \deg_p)$. Moreover, since the poset involution is trivial, $TL_{r,p,n}$ is a cellular algebra with cell datum $(\mathfrak{P}_{n,p}, T_p, C_p, \deg_p)$.

8. The representations of $TL_{r,p,n}$

In this section, we study the representations of $TL_{r,p,n}$ from the cellular point of view. We first calculate the dimensions of the cell modules and define reducible and irreducible elements in the poset $\mathfrak{P}_{n,p}$. These names come from the fact that the cell modules corresponding to the irreducible elements are simple. Then we show that the cell modules of $TL_{r,p,n}$ corresponding to reducible elements in $\mathfrak{P}_{n,p}$ can be regarded as cell modules of some $TL_{d,1,n}$ where $d = \frac{p}{2}$ with special parameters. Finally we calculate the decomposition numbers of the reducible cell modules.

8.1. The cell modules of $TL_{r,p,n}$ and irreducible cells. In this subsection, we give a description of the cell modules of $TL_{r,p,n}$. Let $[\lambda] \in \mathfrak{P}_{n,p}$ be an equivalence class of multipartitions in $\mathfrak{P}_n$ with respect to the equivalence relation $\equiv$ defined in (7.19). The (left) cell module of $TL_{r,p,n}$ corresponding to $[\lambda]$, denoted $W([\lambda])$, is the free module over $R$ with basis $\{C_s | [s] \in T([\lambda])\}$. The $TL_{r,p,n}$-action is defined by

$$aC_s = \sum_{[s'] \in T([\lambda])} r_a([s'], [s]) C_{s'}$$

for $a \in TL_{r,p,n}$.
for all \( a \in TL_{r,p,n} \) and \( [s] \in T([\lambda]) \), where \( r_{a}([s'], [s]) \) is the coefficient uniquely defined by

\[
aC_{[s],[t]}^{[\lambda]} = \sum_{[s'] \in T([\lambda])} r_{a}([s'], [s]) C_{[s],[t]}^{[\lambda]} + \sum_{[\mu] \in P \cap T([\mu])} dC_{[s],[v]}^{[\mu]}.
\]

The right cell modules of \( TL_{r,p,n} \) are defined similarly. We next calculate the dimensions of the cell modules. For \( [\lambda] \in \Psi_{n,p} \), the dimension of the cell module \( W([\lambda]) \) equals to the number of standard tableaux of shape \( \lambda \). Therefore, for \( \lambda \in \Psi_{n,p} \), we have

\[
dim(W([\lambda, i,j])) = 1
\]

\[
dim(W([\mu]_{i,j})) = \left( \frac{n-a}{2} \right).
\]

Further, denote by \( TL_{\Phi_{p}[\lambda]} \) and \( TL_{\subseteq_{p}[\lambda]} \) the subspaces spanned by \( \{ C_{[s],[t]}^{[\mu]} | [\mu] \subseteq_{p}[\lambda] \} \) and \( \{ C_{[s],[t]}^{[\mu]} | [\mu] \subseteq_{p}[\lambda] \} \) respectively. These are \( TL_{r,p,n} \)-bimodules by definition of a cell datum. Define the \( TL_{r,p,n} \)-bimodule corresponding to \( [\lambda] \) as

\[
TL([\lambda]) := TL_{\subseteq_{p}[\lambda]} / TL_{\Phi_{p}[\lambda]}.
\]

Then we have

\[
\dim(TL([\lambda])) = \begin{cases} 
1 & \text{if } \lambda = \lambda_{i,j}; \\
\left( \frac{n-a}{2} \right)^{2} & \text{if } \lambda = \lambda_{i,j}^{(a)}.
\end{cases}
\]

This permits calculation of the dimension of \( TL_{r,p,n} \):

\[
\dim(TL_{r,p,n}) = \sum_{[\lambda] \in \Psi_{n,p}} \dim(TL([\lambda]))
\]

\[
= \sum_{[\lambda] \in \Psi_{n,p}(1)} \dim(TL([\lambda])) + \sum_{[\lambda] \in \Psi_{n,p}(2)} \dim(TL([\lambda]))
\]

\[
= \frac{d+1}{p} \left( \frac{r}{2} \right) \left( \sum_{a_{1}=1}^{n-1} \left( \begin{array}{c} n \\ a_{1} \end{array} \right)^{2} \right)
\]

\[
= \frac{d+1}{p} \left( \frac{r}{2} \right) \left( \begin{array}{c} 2n \\ n \end{array} \right) - 2
\]

\[
= \frac{1}{p} \left( \left( \frac{r}{2} \right) \left( \begin{array}{c} 2n \\ n \end{array} \right) - r^{2} + 2r \right)
\]

where \( \Psi_{n,p}(i) \) consists of the equivalence classes of 3D multipartitions with exactly \( i \) non-empty components. By Definition 7.1, the reducibility of \( \lambda \) is equivalent to that of any \( \lambda' \in [\lambda] \). So we can say \( [\lambda] \in \Psi_{n,k} \) is reducible if \( \lambda \in \Psi_{n} \) is.

The poset \( \Psi_{n,p} \) into two parts, viz. those subsets \( \Psi_{0,n,p} \) consisting of the reducible orbits and the \( \Psi_{1,n,p} \) consisting of the irreducible orbits. We will show that the cell modules corresponding to these two parts have different properties. We begin by showing that the bimodules corresponding to the irreducible orbits are irreducible two-sided ideals of \( TL_{r,p,n} \).

**Lemma 8.5.** Let \( [\lambda] \in \Psi_{n,p} \) be irreducible. For any \( [\mu] \neq [\lambda], [s], [t] \in T([\mu]) \) and \( [u], [v] \in T([\lambda]) \), we have

\[
C_{p}^{[\mu]}([s],[t])C_{p}^{[\lambda]}([u],[v]) = 0.
\]
Proof. By Theorem 2.23, we have

\[ C_p^{[\mu]}([s], [t]) C_p^{[\lambda]}([u], [v]) = \psi_{d([s])}(\sum_{\mu' \equiv \mu} e_{\mu'} \psi_{d([u])}\psi_{d([s])}(\sum_{\lambda' \equiv \lambda} e_{\lambda'} \psi_{d([v])})) = \psi_{d([s])}\psi_{d([u])}(\sum_{t' \equiv t} e(t'))(\sum_{u' \equiv u} e(u')) \psi_{d([u])}\psi_{d([s])}(\psi_{d([v])}). \]

It suffices to show

\[ (\sum_{t' \equiv t} e(t'))(\sum_{u' \equiv u} e(u')) = 0. \]

As the distinct \( e(i) \) are mutually orthogonal, we need only show that \( e(t) \neq e(u) \) for all \( u \in \text{Std}(\lambda) \) and \( t \in \text{Std}(\mu) \) with \( \mu \neq \lambda \). Let \( \lambda \) be of the form \( \lambda_{(i_1, j_1), (i_2, j_2)}^{[a]} \) with \( i_1 \neq i_2 \) and \( e(u) = e(k_1, k_2, \ldots, k_n) \). As \( u \) is a standard tableau of shape \( \lambda \), the \( \lambda \)'s are of the form \( (i_1, x) \) and \( \frac{a}{2} \lambda \)'s are of the form \( (i_2, x) \). If \( e(t) = e(u) \), there should be \( \frac{a}{2} \lambda \) nodes in \( \mu \) with residues of the form \( (i_1, x) \) and \( \frac{a}{2} \lambda \) nodes in \( \mu \) with residues of the form \( (i_2, x) \). As \( \mu \) has at most 2 non-empty components, it is irreducible. For any standard tableau \( t' \equiv t \), we have \( \text{Shape}(t') \equiv \mu \neq \lambda \). Thus,

\[ \left( \sum_{t' \equiv t} e(t') \right) e(u') = 0. \]

But for any standard tableau \( u' \equiv u \), \( \text{Shape}(u') \equiv \lambda \neq \mu' \). We have

\[ \left( \sum_{t' \equiv t} e(t') \right) \left( \sum_{u' \equiv u} e(u') \right) = 0. \]

Therefore, \( C_p^{[\mu]}([s], [t]) C_p^{[\lambda]}([u], [v]) = 0. \)

By applying the anti-involution of TLr,p,n which fixes the KLR generators, we further obtain:

**Lemma 8.6.** Let \( [\lambda] \in \mathfrak{P}_{n,p} \) be irreducible. For any \( [\mu] \neq [\lambda] \), \( [s], [t] \in T([\mu]) \) and \( [u], [v] \in T([\lambda]) \), we have

\[ C_p^{[\lambda]}([u], [v]) C_p^{[\mu]}([s], [t]) = 0. \]

The following proposition is a direct consequence of the two lemmas above:

**Proposition 8.7.** Let \( [\lambda] \in \mathfrak{P}_{n,p} \) be irreducible and \( W([\lambda]) \) be the cell module of TLr,p,n corresponding to \( \lambda \). For any \( [\mu] \neq [\lambda] \) and \( [s], [t] \in T([\mu]) \), we have

\[ C_p^{[\mu]}([s], [t]) W([\lambda]) = 0. \]
The following proposition is where the name “irreducible multipartition” comes from. Let \( L([\lambda]) \) be the simple module corresponding to \([\lambda] \in \mathfrak{P}_{n,p}\). We have:

**Proposition 8.8.** Let \([\lambda] \in \mathfrak{P}_{n,p} \) be an irreducible multipartition, then \( W([\lambda]) = L([\lambda]) \). In other words, the cell module \( W([\lambda]) \) is simple.

**Proof.** Let \([\mu] \in \mathfrak{P}_{n,p} \) be a equivalence class different from \([\lambda] \). By Proposition 8.7, \( C_p^{[\mu]}([s],[t])W([\lambda]) = 0 \) for all \([s],[t] \in T([\mu])\). On the other hand, since \( L([\mu]) = W([\mu])/\text{rad}(\phi_{[\mu]}) \), for any \( x \in L([\mu]) \), there exist \([s],[t] \in T([\mu])\) such that \( C_p^{[\mu]}([s],[t])x \neq 0 \). Therefore, we have

\[
[W([\lambda]) : L([\mu])] = 0
\]

for all \([\mu] \neq [\lambda] \). So \( W([\lambda]) \) is simple and \( W([\lambda]) = L([\lambda]) \). \( \square \)

Note that the converse of the proposition is not true. As a counterexample, the multipartitions with exactly one non-empty component are reducible by definition, but the corresponding cell modules are always simple since they are of rank 1.

### 8.2. The reducible cells and their decomposition numbers.

As shown in the last subsection, the cell modules corresponding to irreducible orbits are simple. To determine the decomposition numbers for all the cell modules, we concentrate on those corresponding to reducible orbits in this subsection.

Let \( \Lambda \) be the dominant weight we choose for the specialisation \( H_n^\lambda(q,\zeta) \) (cf. Theorem 3.2) and \( TL_{r,1,n}^\lambda(q,\zeta) \) be the corresponding Temperley-Lieb algebra. For \( d = 2 \), let \( \Lambda^0 \) be the dominant weight of length \( d \) such that \((\Lambda^0,\alpha_{(0,j)}) = (\Lambda,\alpha_{(0,j)}) \) for all \( j \in J \). Let \( \mathcal{R}_{n}^{\Lambda^0} \) be the cyclotomic KLR algebra corresponding to \( \Lambda^0 \) and \( TL_{d,1,n}^{\Lambda^0} \) be the Temperley-Lieb quotient we define in Theorem 3.2. The following lemma indicates that \( TL_{d,1,n}^{\Lambda^0} \) is a quotient of \( TL_{r,1,n}^\lambda(q,\zeta) \):

**Lemma 8.9.** Let \( TL_{r,1,n}^\lambda(q,\zeta) \) be the Temperley-Lieb quotient of \( H_n^\lambda(q,\zeta) \) (cf. Theorem 3.2) and \( TL_{d,1,n}^{\Lambda^0} \) be the Temperley-Lieb algebra as defined above. Then \( TL_{d,1,n}^{\Lambda^0} \) is a quotient of \( TL_{r,1,n}^\lambda(q,\zeta) \) by the two-sided ideal generated by all \( e(1) = e(i_1,i_2,\ldots,i_n) \) where \( i_j \) are in the vertex set \( K = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/e\mathbb{Z} \) and \((\Lambda^0,\alpha_{i_1}) = 0 \).

**Proof.** Denote by \( \mathcal{R}_{n}^{\Lambda^0} \) the KLR algebra with respect to the dominant weight \( \Lambda^0 \). Let \( \mathcal{R}_{n}^{\Lambda} \) be the KLR algebra isomorphic to \( H_n^\lambda(q,\zeta) \) (cf. Theorem 1.1 in [3]). By comparing the generators and relations, we notice that \( \mathcal{R}_{n}^{\Lambda^0} \) is a quotient of \( \mathcal{R}_{n}^{\Lambda} \) by the two-sided ideal generated by all \( e(1) \) such that \((\Lambda^0,\alpha_{i_1}) = 0 \) where \( i \in K^n \) and \( \alpha_{i_1} \) is the simple root in section 2.

By Theorem 3.2, \( TL_{r,1,n}^\lambda(q,\zeta) \) is a quotient of \( \mathcal{R}_{n}^{\Lambda} \) by the two-sided ideal \( \mathcal{J}_n(\Lambda) \) in 3.3, and \( TL_{d,1,n}^{\Lambda^0} \) is a quotient of \( \mathcal{R}_{n}^{\Lambda^0} \) by the two-sided ideal \( \mathcal{J}_n(\Lambda^0) \) in 3.4. We only need to show that generators of \( \mathcal{J}_n(\Lambda) \) in 3.4 are in the two-sided ideal of \( \mathcal{R}_{n}^{\Lambda^0} \) generated by all \( e(1) \) such that \((\Lambda^0,\alpha_{i_1}) = 0 \) and \( \mathcal{J}_n(\Lambda^0) \). This is obvious except for the idempotent \( e(i_1,i_2,i_3,i) \) where \((\Lambda^0,\alpha_{i_1}) > 0 \) but \((\Lambda^0,\alpha_{i_2}) = 0 \) or \((\Lambda^0,\alpha_{i_3}) = 0 \). Without losing generality, assume \((\Lambda^0,\alpha_{i_2}) = 0 \). In this case, we use the fact, \( e(i_1,i_2,i_3,i) = \psi_1 e(i_2,i_1,i_3,i)\psi_1 \), to show that \( e(i_1,i_2,i_3,i) \) is in the two-sided ideal generated by all \( e(1) \) such that \((\Lambda^0,\alpha_{i_1}) = 0 \).
Therefore, the ideal of $R^\lambda_n$ corresponding to $TL^\lambda_{r,1,n}(q, \zeta)$ is in that corresponding to $TL^\lambda_{d,1,n}$. So $TL^\lambda_{d,1,n}$ is a quotient of $TL^\lambda_{r,1,n}(q, \zeta)$. Moreover, we have:

$$TL^\lambda_{d,1,n} = R^\lambda_n / \mathcal{J}_n(\lambda^0)$$

(8.10)

where $i$ runs over all $i \in K^n$ such that $(\lambda^0, \alpha_i) = 0$. 

Let $f : TL^\lambda_{r,1,n}(q, \zeta) \to TL^\lambda_{d,1,n}$ be the natural quotient map. We have

$$f(y_k) = y_k, \text{ for } 1 \leq k \leq n;$$

$$f(\psi_l) = \psi_l, \text{ for } 1 \leq l \leq n - 1;$$

(8.11)

$$f(e_\lambda) = \begin{cases} e_\lambda, & \text{if } \lambda \in \Psi^1_n; \\ 0, & \text{if } \lambda \in \Psi_n - \Psi^1_n, \end{cases}$$

where $\Psi_n$ is the set of 3D multipartitions of $n$ with at most two non-empty components, and $\Psi^1_n$ is the subset consisting of those 3D multipartitions with all the non-empty components in the first column of the floor table. Since $TL^\lambda_{r,p,n}$ is a subalgebra of $TL^\lambda_{r,1,n}(q, \zeta)$, $f(TL^\lambda_{r,p,n})$ is a subalgebra of $TL^\lambda_{d,1,n}$. We show the map $f$ is surjective. By Theorem 5.6, $TL^\lambda_{d,1,n}$ has a cellular basis $\{C^\lambda_{s,t}\}$ corresponding to the cell datum $(\mathfrak{B}^{(d)} - \mathfrak{D}^{(d)}, Std, C, deg)$. For any $\lambda_0 \in \mathfrak{B}^{(d)} - \mathfrak{D}^{(d)}$ and $s_0, t_0 \in Std(\lambda_0)$, let $j_1, j_2, \ldots, j_n \in \mathbb{Z}/c\mathbb{Z}$ be such that $e_{\lambda_0} = e((0, j_1), (0, j_2), \ldots, (0, j_n))$. We have $C^\lambda_{p,\lambda_0}([s_0], [t_0]) \in TL^\lambda_{r,p,n}$ and

$$f(C^\lambda_{p,\lambda_0}([s_0], [t_0])) = f(\psi_{d([s_0])}(\sum_{\lambda' \equiv \lambda_0} e_{\lambda'} \psi_{d([t_0])}))$$

$$= \psi_{d([s_0])}(\sum_{\lambda' \equiv \lambda_0} e_{\lambda'} \psi_{d([t_0])})$$

(8.12)

$$= \psi_{d([s_0])}(\sum_{i=0}^{p-1} e((i, j_1), (i, j_2), \ldots, (i, j_n)) \psi_{d([t_0])})$$

$$= \psi_{d([s_0])} e((0, j_1), (0, j_2), \ldots, (0, j_n)) \psi_{d([t_0])} = C^\lambda_{s_0, t_0}.$$

Therefore, the restriction of the natural map $f$ on $TL^\lambda_{r,p,n}$ is surjective, that is $f(TL^\lambda_{r,p,n}) = TL^\lambda_{d,1,n}$.

The next lemma shows that the kernel of this map is the subspace spanned by the elements corresponding to irreducible orbits in the cellular basis in $\{7, 23\}$.

**Lemma 8.13.** Let $TL^1_{r,p,n}$ be the subspace spanned by all elements $C^\lambda_{p,\lambda}([s], [t])$ in the cellular basis of $TL^\lambda_{r,p,n}$ in $\{7, 23\}$, where $[\lambda]$ is irreducible. Then $TL^1_{r,p,n}$ is the kernel of $f|_{TL^\lambda_{r,p,n}}$ where $f$ is the natural quotient map defined above. Moreover, $TL^1_{r,p,n}$ is an ideal of $TL^\lambda_{r,p,n}$ and

$$TL^\lambda_{d,1,n} \cong TL^\lambda_{r,p,n}/TL^1_{r,p,n}$$

(8.14)
Proof. We first show that \( f(C_p^\lambda([u],[v])) = 0 \) if \([\lambda]\) is irreducible. Using the notation introduced before Definition 7.4, let \( \lambda = \lambda^{(i_1,i_2)}(i_2) \). As \( \lambda \) is irreducible, we have \( i_1 \neq i_2 \) and \( e_\lambda = e((i_1,j_1),(i_2,j_2),k) \) with \( k \in K_n^{-2} \). We have

\[
\begin{align*}
e_\lambda &= \sum_{\lambda' \equiv \lambda} e_{\lambda'} \\
&= \sum_{i=0}^{p-1} e((i_1-i,j_1),(i_2-i,j_2),\sigma_1(k)),
\end{align*}
\]

where \( \sigma_1 \) is the map defined in (2.21). For \( i \neq i_1 \), \( e((i_1-i,j_1),(i_2-i,j_2),\sigma_1(k)) \) is obviously in the ideal described in Lemma 8.1. If \( i = i_1 \), we have \( i_2-i \neq 0 \) and \( e((i_1-i,j_1),(i_2-i,j_2),\sigma_1(k)) = \psi_1 e((i_2-i,j_2),(i_1-i,j_1),\sigma_1(k)\psi_1). \) So \( e((i_1-i,j_1),(i_2-i,j_2),\sigma_1(k)) \) is in that ideal. Then we have \( e_{\lambda'} \in \ker(f) \), thus \( C_p^\lambda([u],[v]) \in \ker(f) \) for any \( u, v \in \text{Std}(\lambda) \). Therefore, we have \( TL_{r,p,n}^1 \subseteq \ker(f) \).

On the other hand, for any reducible orbit \([\mu] \in \mathfrak{P}_{n,p} \) and \([s],[t] \in T_p^r([\mu])(cf.7.22)\), equation (8.12) shows that \( C_p^\mu([s],[t]) \) is not in the kernel of \( f \). So \( TL_{r,p,n}^1 = \ker(f|_{TL_{r,p,n}^1}) \). Since \( f|_{TL_{r,p,n}^1} \) is a surjective algebra homomorphism, \( TL_{r,p,n}^1 \) is an ideal of \( TL_{r,p,n} \) and

\[
TL_{d,1,n}^0 \cong TL_{r,p,n}^1/TL_{r,p,n}^1.
\]

\( \square \)

The next lemma is an immediate consequence of the definition of cellular algebras and shows that \( TL_{d,1,n}^0 \) inherits a cellular structure from \( TL_{r,p,n}^1 \) as a quotient.

**Lemma 8.16.** Let the algebras \( TL_{r,p,n}^1 \) and \( TL_{r,p,n}^1 \) be as defined above. Then \( TL_{r,p,n}^1/TL_{r,p,n}^1 \) is a graded cellular algebra with cellular datum \((\mathfrak{P}_{n,p}^0,T_p,\mathfrak{C}_p,\text{deg}_p)\),\]

where \( \mathfrak{P}_{n,p}^0 \) is the subset of \( \mathfrak{P}_{n,p} \) consisting of all the reducible orbits, \( T_p,\text{deg}_p \) are the same as the ones in Theorem 7.24 and \( \mathfrak{C}_p^\mu([s],[t]) = C_p^\mu([s],[t]) + TL_{r,p,n}^1 \) for all \([\mu] \in \mathfrak{P}_{n,p}^0 \) and \([s],[t] \in T_p([\mu])\).

For \([\mu] \in \mathfrak{P}_{n,p}^0 \) and \([\lambda] \in \mathfrak{P}_{n,p}^1 \), Lemma 8.6 implies

\[
C_p^\lambda([u],[v])W([\mu]) = 0.
\]

Thus the kernel of the map \( f \) above acts trivially on \( W([\mu]) \), whence the action of \( TL_{r,p,n} \) on the cell module \( W([\mu]) \) is exactly the same as that of \( TL_{d,1,n}^0 \). Hence we need only to determine the decomposition numbers for the cell modules \( W([\mu]) \), regarded as \( TL_{d,1,n}^0 \)-modules.

By Theorem 5.6, the generalised Temperley-Lieb algebra \( TL_{d,1,n}^0 \) has another cell datum \((\mathfrak{B}_{n,d}^0,\mathfrak{D}_n^{(d)},\text{Std},\text{C},\text{deg})\) where \( \mathfrak{B}_{n,d}^0,\mathfrak{D}_n^{(d)} \) consists of the 3D multipartitions with at most two non-empty components and both of them are in the first column of the floor table. The decomposition numbers corresponding to this cell datum are given in the last chapter. To show that these two cell data are the same, we first choose a special representative in each reducible orbit.

**Definition 8.18.** For any reducible orbit \([\mu] \in \mathfrak{P}_{n,p}^0 \), define the original representative \( \mu_0 \) in \([\mu] \) as the multipartition such that all the non-empty components are in the first column of the floor table.
Since the non-empty components in a reducible multipartition are in the same column, and the map \( \sigma_{\Psi_n} \) moves multipartitions one column ahead, there exists a unique original representative \( \mu_0 \) in each reducible orbit \( [\mu] \). For \( [t] \in T_p([\mu]) \), denote by \( t_0 \in [t] \) the standard tableau of shape \( \mu_0 \). Let \( f : TL^L_{r,p,n} \to TL^L_{d,1,n} \) be the natural quotient map. Then we have

\[
f(C_p^{[\mu]}([s],[t])) = f(\psi_d([s]) \left( \sum_{\mu' \equiv \mu_0} e_{\mu'} \psi_d([t]) \right)
= \psi_d([s]) \left( \sum_{i=0}^{p-1} e_{\mu_i} \psi_d([t]) \right)
= \psi_d([s_0]) f(e_{\mu_0}) \psi_d([t_0])
= C^{\mu_0}(s_0,t_0).
\]

Therefore, we have

\[
C_p^{[\mu]}([s],[t]) = C^{\mu_0}(s_0,t_0),
\]

for all \([\mu] \in \Psi^0_{n,p}\) and \([s],[t] \in T_p([\mu])\). As a direct consequence, we have

**Lemma 8.20.** For any reducible orbit \([\mu]\), let \( W([\mu]) \) (resp. \( L([\mu]) \)) be the cell (resp. simple) module of \( TL^L_{d,1,n} \) with respect to the cell datum \((\Psi^0_{n,p}, T_p, C_p, deg_p)\). Denote by \( \mu_0 \) the original representative in \([\mu]\). Let \( W(\mu_0) \) (resp. \( L(\mu_0) \)) be the cell (resp. simple) module with respect to the cell datum \((\Psi^{(d)}_n - \Xi^{(d)}_n, Std, C, deg)\). Then we have

\[
W([\mu]) \cong W(\mu_0); L([\mu]) \cong L(\mu_0).
\]

Moreover, let \([\lambda],[\mu]\) be two reducible orbits. As \( TL^L_{d,1,n} \)-modules, we have

\[
[W([\lambda]) : L([\mu])] = [W(\lambda_0) : L(\mu_0)].
\]

Since \( TL_{r,p,n} \) acts trivially on \( W([\lambda]) \) where \([\lambda]\) is reducible, the left hand side equals to the decomposition numbers for the corresponding cell modules of \( TL_{r,p,n} \). The right-hand side can be obtained from Theorem 6.23 in [8]. We are now in a position to describe the decomposition numbers of \( TL(r,p,n) \).

**Theorem 8.22.** Let \( TL_{r,p,n} \) be the Temperley-Lieb algebra for the imprimitive complex reflection group \( G(r,p,n) \) as defined in Definition 4.6. According to Theorem 7.24, \( TL_{r,p,n} \) is a cellular algebra with respect to the poset \( \Psi_{n,p} \) in (7.20). For an orbit of multipartitions \([\lambda] \in \Psi_{n,p} \), let \( W([\lambda]) \) and \( L([\lambda]) \) be the cell and simple modules corresponding to \([\lambda]\), respectively.

If \( \lambda \) is irreducible (cf. Definition 3.7), then

\[
W([\lambda]) = L([\lambda]).
\]

If \( \lambda \) is reducible, \( W([\lambda]) \) has no decomposition factors of the form \( L([\mu]) \) where \( \mu \) is irreducible.

For any reducible orbit \([\mu] \in \Psi_{n,p} \), let \( \lambda_0 \) and \( \mu_0 \) be the original representatives as defined in Definition 8.13. Then

\[
[W([\lambda]), L([\mu])] = \begin{cases} 
1 & \text{if } \lambda_0 \leq \mu_0 \text{ and there exists } t_0 \in \text{Std}(\lambda_0) \text{ such that } e(t_0) = e_{\mu_0}; \\
0 & \text{otherwise,}
\end{cases}
\]

where \( e(t_0) \) and \( e_{\mu_0} \) are the KLR generators.
Proof. If $\lambda$ is irreducible, $W([\lambda])$ is simple by Proposition 8.8, so we have
$$W([\lambda]) = L([\lambda]).$$

If $\lambda$ is reducible but $\mu$ is irreducible, Lemma 8.6 implies that
$$C_p^{|\lambda|}([s], [t]) W([\mu]) = 0,$$
for all $[s], [t] \in T([\lambda])$. On the other hand, for any non-zero element $x \in L([\lambda])$, there exist $[s], [t] \in T([\lambda])$ such that $C_p^{|\lambda|}([s], [t]) x \neq 0$. Therefore, we have
$$[W([\mu]) : L([\lambda])] = 0.$$ 

If both $\lambda$ and $\mu$ are reducible, then (8.21) applies. Hence by Theorem 6.23 in [8],
$$[W([\lambda]), L([\mu])] = \begin{cases} 1 & \text{if } \lambda_0 \preceq \mu_0 \text{ and there exists } t_0 \in \text{Std}(\lambda_0) \text{ such that } e(t_0) = e(\mu_0); \\ 0 & \text{otherwise}. \end{cases}$$

□

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