Managing $\gamma_5$ in Dimensional Regularization and ABJ Anomaly

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Abstract

An integral representation is proposed for the trace involving $\gamma_5$ in dimensional regularization. Lorentz covariance is preserved. ABJ anomaly naturally follows. The Local Functional Equation associated to $x$-dependent chiral transformations is verified.

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1 Introduction

In dimensional regularization (Ref. [1], [2] and [3]) $\gamma_5$ has always been a very difficult object to deal with. Many important contributions to the topics are present in the literature. We provide an uncommented list of works [4], [1], [5]-[27], which is far from being complete.

In this paper an integral representation of the trace involving $\gamma_5$ is suggested

$$
\text{Tr}(\not p_1 \ldots \not p_N \gamma_5) = i^D \frac{1}{D-1} K \int d^D \chi d^N \bar{c} \exp \left( \sum_{\mu=1}^D \sum_{i=1}^N \bar{c}_i p_{i\mu} \chi_\mu + \sum_{i<j}^N \bar{c}_i (p_i p_j) \bar{c}_j \right),
$$

(1)

where $\bar{c}_i$ and $\chi_\mu$ are Grassmannian variables. $K$ is a real parameter related to the normalization of the trace.

The strategy is the following: a) we show that the integral representation obeys the algebra of the gamma’s; b) for integer $D$ we evaluate the algebra of $\gamma_5$ and verify the cyclicity of the trace; c) we require cyclicity of the trace operation (for any dimension $D$); d) we provide a closed formula for the evaluation of the trace.

Once these results are established, we can dimensionally regularize any Feynman amplitude and use those manipulations that are consistent with the regularization in eq. (1).

We test our proposal with the evaluation of the axial anomaly. ABJ result [28] and [29] is obtained in a very natural and elegant way.

Moreover a Local Functional Equation (LFE) [30]-[33] is derived and checked at one loop level. This fact is noteworthy: the LFE doesn’t suffer of anomalies.

In order to make easy the notations we use Euclidean metric for indices and we change the name ($\chi$ for chiral)

$$
\gamma_5 \rightarrow \gamma_\chi.
$$

(2)

2 Integral Representation of Trace of Gamma’s Product

We consider the trace of a generic product of gamma’s [34]-[38], where the indices are saturated by vectors $p_{j\mu}$ ($\not p_j$ is a standard notation for $\gamma_{\mu} p_{j\mu}$).
The discrete index $j$ runs over the set of integers $\{1 \ldots N\}$, while the “component” $\mu$ is an element of the set $\{1, D\}$ (of $\mathbb{R}$). Our aim is to find an integral representation for the trace with and without $\gamma_X$.

We prove the integral representation by showing the validity of the gamma’s algebra (Clifford) (with no mention to the dimension $D$). For integer $D$ we prove the property of cyclicity and evaluate the algebra of $\gamma_X$.

We use the standard properties of integration on Grassmannian real coordinates

$$
\int d\bar{c} = 0
$$

$$
\int d\bar{c} \bar{c} = 1
$$

$$
\bar{c}' \equiv -\bar{c}, \quad \Rightarrow \quad \int d\bar{c}' \bar{c}' = 1.
$$

(3)

2.1 If $N$ is even and no $\gamma_X$ is present

The trace can be written in terms of an integral over a set of Grassmannian variables $\bar{c}_j$

$$
\text{Tr}(\mathbf{p}_1 \ldots \mathbf{p}_N) = K \int d^N \bar{c} \exp \left( \sum_{i<j} \bar{c}_i (p_i p_j) \bar{c}_j \right)
$$

$$
= K \int d^N \bar{c} \exp \left( \frac{1}{2} \sum \bar{c}_i h_{ij} \bar{c}_j \right),
$$

(4)

where

$$
h_{ij} = -h_{ji} \equiv (p_i p_j) \quad \text{for } i < j.
$$

(5)

In the expansion of the exponential only the monomials containing all $\bar{c}$ (and only once) yield non-zero result under integration. The monomials have the form

$$
[\bar{c}_{i_1} (p_{i_1} p_{j_1}) \bar{c}_{j_1}] [\bar{c}_{i_2} (p_{i_2} p_{j_2}) \bar{c}_{j_2}] \ldots
$$

(6)

where $\{i_1, j_1, i_2, j_2 \ldots\}$ is any permutation $\mathcal{P}$ of $\{1, 2, \ldots, N\}$ conditioned by

$$
i_1 < i_2 < i_3 \ldots
$$

$$
i_1 < j_1, \quad i_2 < j_2, \quad i_3 < j_3, \quad \ldots.
$$

(7)

The integration over $d\bar{c}_N \ldots d\bar{c}_1$ yields

$$
\delta_{\mathcal{P}}[(p_{i_1} p_{j_1})] [(p_{i_2} p_{j_2})] [(p_{i_3} p_{j_3})] \ldots,
$$

(8)
where $\delta_T$ is the signature of the permutation.

We prove the validity of eq. (4) by showing that the algebra of the gamma’s and cyclicity are obeyed.

2.2 The Algebra for even $N$ (and no $\gamma^\chi$)

We evaluate the algebra of the gamma’s.

We start with

\[
Tr\left(\{\hat{p}_1, \hat{p}_2 \} \hat{p}_3 \ldots \hat{p}_N\right) = K \int d^N \vec{c} \exp \left( \sum_{i<j}^N \bar{c}_i (p_i p_j) \bar{c}_j \right)
\]

\[
+ K \int d^N \vec{c} \exp \left( \bar{c}_1 (p_2 p_1) \bar{c}_2 + \sum_{j=3}^N \bar{c}_1 (p_2 p_j) \bar{c}_j + \sum_{j=3}^N \bar{c}_2 (p_1 p_j) \bar{c}_j \right.
\]

\[
+ \left. \sum_{i=3, i<j}^N \bar{c}_i (p_i p_j) \bar{c}_j \right) .
\]

(9)

In the second integral we rename $\bar{c}_1 \leftrightarrow \bar{c}_2$. The measure changes sign and therefore

\[
Tr\left(\{\hat{p}_1, \hat{p}_2 \} \hat{p}_3 \ldots \hat{p}_N\right) = K \int d^N \vec{c} \exp \left( \sum_{i<j}^N \bar{c}_i (p_i p_j) \bar{c}_j \right)
\]

\[
- K \int d^N \vec{c} \exp \left( - \bar{c}_1 (p_2 p_1) \bar{c}_2 + \sum_{j=3}^N \bar{c}_1 (p_2 p_j) \bar{c}_j + \sum_{j=3}^N \bar{c}_2 (p_1 p_j) \bar{c}_j \right.
\]

\[
+ \left. \sum_{i=3, i<j}^N \bar{c}_i (p_i p_j) \bar{c}_j \right)
\]

\[
= K \int d^N \vec{c} \exp \left( \sum_{i<j}^N \bar{c}_i (p_i p_j) \bar{c}_j \right) \left(1 - \exp \left( - 2\bar{c}_1 (p_1 p_2) \bar{c}_2 \right) \right)
\]

\[
= K \int d^N \vec{c} \left[2\bar{c}_1 (p_1 p_2) \bar{c}_2 \right] \exp \left( \sum_{2<i<j}^N \bar{c}_i (p_i p_j) \bar{c}_j \right) .
\]

(10)

By integrating over $d\vec{c}_2 d\bar{c}_1$ we finally get

\[
Tr\left(\{\hat{p}_1, \hat{p}_2 \} \hat{p}_3 \ldots \hat{p}_N\right) = 2(p_1 p_2) Tr(\hat{p}_3 \ldots \hat{p}_N) .
\]

(11)

Thus the algebra (Clifford) of the gamma’s is in agreement with the representation in eq. (4)

\[
\{\hat{p}, \hat{q}\} = 2(p q)\mathbb{I}
\]

(12)
with

\[ Tr(1) = K. \]  \hfill (13)

### 2.3 Cyclicity for even \( N \) (and no \( \gamma \_\chi \))

We check now cyclicity

\[
Tr(\bar{p}_2 \cdots \bar{p}_N \bar{p}_1) = K \int d\bar{c}_N \cdots d\bar{c}_1 \exp \left( \sum_{i=1, i<j}^{N-1} \bar{c}_i (p_{i+1}p_{j+1})\bar{c}_j \right.
\]

\[ + \sum_{i=1}^{N-1} \bar{c}_i (p_{i+1}p_1)\bar{c}_N \right) \]  \hfill (14)

We rename

\[
\bar{c}_j \to \bar{c}_{j+1} \quad \bar{c}_N \to \bar{c}_1
\]  \hfill (15)

and get

\[
Tr(\bar{p}_2 \cdots \bar{p}_N \bar{p}_1) = K \int d\bar{c}_N d\bar{c}_N \cdots d\bar{c}_2 \exp \left( \sum_{i=2, i<j}^{N} \bar{c}_i (p_{i+1}p_{j+1})\bar{c}_j \right.
\]

\[ - \sum_{j=2}^{N} \bar{c}_1 (p_1p_j)\bar{c}_j \right) \]  \hfill (16)

Now the minus sign emerging by the ordering of the measure is compensated by the change of sign of the \( p_1 \)-dependent terms in the exponential. Finally we obtain the identity

\[ Tr(\bar{p}_2 \cdots \bar{p}_N \bar{p}_1) = Tr(\bar{p}_1 \bar{p}_2 \cdots \bar{p}_N). \]  \hfill (17)

Both eqs. (11) and (17) have been derived by using only the symmetry properties of the measure of the integral. This implies that these properties are valid for generic values of \( D \).

### 3 If \( N \) is odd or \( N \) is even but \( \gamma \_\chi \) is present: the Clifford Algebra

We represent the trace involving a single \( \gamma \_\chi \) or an odd number of gamma’s by an integral over Grassmann variables \{\( \bar{c}_j \)\} and \{\( \chi_\mu \)\}.

\[
Tr(\bar{p}_1 \cdots \bar{p}_N \gamma_\chi)
\]

\[ = i^{D(D-1)/2} K \int d^D \bar{c} d^N \bar{c} \exp \left( \sum_{i=1}^{D} \sum_{\mu=1}^{N} \bar{c}_i p_{i\mu} \chi_\mu + \sum_{i<j}^{N} \bar{c}_i (p_i p_j)\bar{c}_j \right) \]  \hfill (18)
As before the algebra of the gamma’s is correctly implemented; in fact

\[
Tr(\{p_2, p_1, p_3 \ldots p_N, \gamma\}) = \frac{iD(D-1)}{2} K \int d^D \chi \, d^N \bar{c} \, \exp \left[ \sum_{\mu=1}^D (\bar{c}_1 p_{2\mu} \chi_\mu + c_2 p_1 \mu \chi_\mu + \sum_{i=3}^N \bar{c}_i p_{i\mu} \chi_\mu) + \bar{c}_1 (p_2 p_1) \bar{c}_2 + \sum_{j=3}^N \bar{c}_1 (p_2 p_j) \bar{c}_j \right] + \sum_{j=3}^N \bar{c}_2 (p_1 p_j) \bar{c}_j + \sum_{i=3, j>i}^N \bar{c}_i (p_i p_j) \bar{c}_j.
\]

Again we rename \( \bar{c}_1 \leftrightarrow \bar{c}_2 \) and change the order in the measure

\[
Tr(\{p_2, p_1, p_3 \ldots p_N, \gamma\}) = -i \frac{D(D-1)}{2} K \int d^D \chi \, d^N \bar{c} \, \exp \left[ \sum_{\mu=1}^D \left( \sum_{i=3}^N \bar{c}_i p_{i\mu} \chi_\mu \right) - 2 \bar{c}_1 (p_2 p_1) \bar{c}_2 \right] + \sum_{j=3}^N \bar{c}_2 (p_2 p_j) \bar{c}_j + \sum_{j=2}^N \bar{c}_1 (p_1 p_j) \bar{c}_j + \sum_{i=3, j>i}^N \bar{c}_i (p_i p_j) \bar{c}_j.
\]

Thus one gets

\[
Tr(\{p_1, p_2\} \{p_3 \ldots p_N, \gamma\}) = \frac{iD(D-1)}{2} K \int d^D \chi \, d^N \bar{c} \, \exp \left[ \sum_{\mu=1}^D \left( \sum_{i=1}^N \bar{c}_i p_{i\mu} \chi_\mu \right) \right] \left( 1 - \exp(-2 \bar{c}_1 (p_2 p_1) \bar{c}_2) \right) \exp \left[ \sum_{i=1, j>i}^N \bar{c}_i (p_i p_j) \bar{c}_j \right]
\]

\[
= \frac{D(D-1)}{2} K \int d^D \chi \, d^N \bar{c} \bar{c}_1 (p_2 p_1) \bar{c}_2 \exp \left[ \sum_{\mu=1}^D \left( \sum_{i=3}^N \bar{c}_i p_{i\mu} \chi_\mu \right) \right] \exp \left[ \sum_{i=3, j>i}^N \bar{c}_i (p_i p_j) \bar{c}_j \right] = 2(p_2 p_1)Tr(\{p_3 \ldots p_N, \gamma\}),
\]

after the integration over \( d\bar{c}_2 d\bar{c}_1 \).

The result of eq. (21) has been obtained for generic values of \( D \), in fact the integration over \( \chi \)'s is not involved in the proof.
4 Cyclicity of Trace for generic $D$ with $\gamma_\chi$ or odd $N$?

In case of non-integer $D$ the algebra of $\gamma_\chi$ and cyclicity are more complex issues.

First of all one should notice that the proof, leading to eq. (17), fails now, due to the fact that the new term in the exponential (i.e. $c_N \, p_{1\mu} \chi_\mu$) under the exchange (15) becomes $\bar{c}_1 p_{1\mu} \chi_\mu$. In eq. (16) all $\bar{c}_1$ necessary to give a non-zero integration on $d\bar{c}_1$ come from the bilinear form where all $\bar{c}_1$ have a common minus sign, that compensate for the reordering of the measure. In presence of $\gamma_\chi$ there are other possible $\bar{c}_1$ factors coming from $\bar{c}_1 p_{1\mu} \chi_\mu$ where the minus sign is not present. That proof does not work here. Strategy must change.

The new strategy is: let us assume that cyclicity works. We see now what are the consequences of this. From repeated use of eq. (21) it follows that ( means that the factor has to be omitted )

$$
Tr(\hat{p}_1 \hat{p}_2 \hat{p}_3 \ldots \hat{p}_N \gamma_\chi) = 2 \sum_{k=2}^{N} (-)^k (p_1 p_k) Tr(\hat{p}_1 \ldots \hat{p}_k \ldots \hat{p}_N \gamma_\chi) \\
+(-)^{(N-1)} Tr(\hat{p}_2 \ldots \hat{p}_N \hat{p}_1 \gamma_\chi). 
$$

Then cyclicity implies:

- for even $N$

  $$
  Tr(\hat{p}_2 \hat{p}_3 \ldots \hat{p}_N \{\gamma_\chi, \hat{p}_1\}) = Tr(\{\hat{p}_1, \hat{p}_2 \hat{p}_3 \ldots \hat{p}_N\} \gamma_\chi) \\
  = 2 \sum_{k=2}^{N} (-)^k (p_1 p_k) Tr(\hat{p}_1 \ldots \hat{p}_k \ldots \hat{p}_N \gamma_\chi) 
  $$

- for odd $N$

  $$
  Tr(\hat{p}_2 \hat{p}_3 \ldots \hat{p}_N [\gamma_\chi, \hat{p}_1]) = Tr(\{\hat{p}_1, \hat{p}_2 \hat{p}_3 \ldots \hat{p}_N\} \gamma_\chi) \\
  = 2 \sum_{k=2}^{N} (-)^k (p_1 p_k) Tr(\hat{p}_1 \ldots \hat{p}_k \ldots \hat{p}_N \gamma_\chi). 
  $$

It should be stressed that eqs. (23) and (24) have been obtained with no conditions on $D$. Moreover it is worth noticing that both eqs. (23) and (24) are consistent with Lorentz covariance.

The algebra of $\gamma_\chi$ will be tested in the evaluation of the ABJ anomaly and in the validity of the LFE.
5 Moving $\gamma_{\chi}$ around

Let us elaborate on the conclusions of Section 4 and in particular on the implications of cyclicity. We now demonstrate that cyclicity allows us to represent a situation where $\gamma_{\chi}$ is in arbitrary position.

We have made the assumption: represent a trace with one $\gamma_{\chi}$ to the right (eq. (18)) by

$$\text{Tr}(\not p_1 \not p_2 \ldots \not p_N \gamma_{\chi}) = i^{\frac{D(D-1)}{2}} K \int d^D \chi \, d^N \bar{\epsilon} \exp \left( \sum_{\mu=1}^{D} \sum_{i=1}^{N} \bar{c}_i p_{i\mu} \chi_{\mu} \right)$$

$$+ \sum_{i=1, i<j}^{N} \bar{c}_i (p_i p_j) \bar{c}_j.$$  \hfill (25)

With the same tools we want to represent

$$\text{Tr}(\not p_1 \not p_2 \ldots \not p_{N-1} \gamma_{\chi} \not p_N).$$  \hfill (26)

To achieve this, we consider the expression as in eq. (25) but with $\not p_N$ in the first position

$$\text{Tr}(\not p_N \not p_1 \not p_2 \ldots \not p_{N-1} \gamma_{\chi}) = i^{\frac{D(D-1)}{2}} K \int d^D \chi \, d^N \bar{\epsilon} \exp \left( \sum_{\mu=1}^{D} \sum_{i=2}^{N} \bar{c}_i p_{(i-1)\mu} \chi_{\mu} \right)$$

$$+ \sum_{\mu=1}^{D} \bar{c}_1 p_{N\mu} \chi_{\mu} + \sum_{i=2, i<j}^{N} \bar{c}_i (p_{i-1} p_j-1) \bar{c}_j + \sum_{j=2}^{N} \bar{c}_1 (p_{N} p_j-1) \bar{c}_j.$$  \hfill (27)

Finally we use cyclicity to obtain

$$\text{Tr}(\not p_1 \not p_2 \ldots \not p_{N-1} \gamma_{\chi} \not p_N) = i^{\frac{D(D-1)}{2}} K \int d^D \chi \, d^N \bar{\epsilon} \exp \left( \sum_{\mu=1}^{D} \sum_{i=2}^{N} \bar{c}_i p_{(i-1)\mu} \chi_{\mu} \right)$$

$$+ \sum_{\mu=1}^{D} \bar{c}_1 p_{N\mu} \chi_{\mu} + \sum_{i=2, i<j}^{N} \bar{c}_i (p_{i-1} p_{j-1}) \bar{c}_j + \sum_{j=2}^{N} \bar{c}_1 (p_{N} p_{j-1}) \bar{c}_j.$$  \hfill (28)

We can write a different expression if we rename the dummy integration variables

$$\bar{c}_j \rightarrow \bar{c}_{j-1} \quad \text{for} \quad j > 1$$

$$\bar{c}_1 \rightarrow \bar{c}_N.$$  \hfill (29)
We get
\[ \text{Tr}(\hat{\varphi}_1 \hat{\varphi}_2 \ldots \hat{\varphi}_{N-1}\gamma \hat{\varphi}_N) \]
\[ = \left(-\right)^{N-1} C^D \left( \sum_{i=2}^{N} \sum_{\mu} \bar{c}_{i-1}p_{(i-1)}\mu \gamma \mu \right) \]
\[ + \sum_{\mu=1}^{D} \bar{c}_{\mu}p_{\mu} \gamma \mu \]
\[ + \sum_{i=2}^{N} \bar{c}_{i-1}(p_{i-1}p_{j-1})\bar{c}_{j-1} + \sum_{j=2}^{N} \bar{c}N(p_{i}p_{j})\bar{c}_{j-1} \right) \]
\[ = \left(-\right)^{N-1} C^D \left( \sum_{i=2}^{N} \sum_{\mu} \bar{c}_{i-1}p_{(i-1)}\mu \gamma \mu \right) \]
\[ + \sum_{i=1}^{N-1} \bar{c}_i(p_{i}p_{j})\bar{c}_j + \sum_{j=1}^{N-1} \bar{c}_j(p_{j}p_{N})\bar{c}_{j} \right). \quad (30) \]

Comment: eq. (30) can be written with the usual trick as in eq. (21)

\[ \text{Tr}(\hat{\varphi}_1 \hat{\varphi}_2 \ldots \hat{\varphi}_{N-1}\gamma \hat{\varphi}_N) \]
\[ = \left(-\right)^{N-1} C^D \left( \sum_{i=2}^{N} \sum_{\mu} \bar{c}_{i-1}p_{(i-1)}\mu \gamma \mu \right) \]
\[ + \sum_{\mu=1}^{D} \bar{c}_{\mu}p_{\mu} \gamma \mu \]
\[ + \sum_{i=1}^{N-1} \bar{c}_{i}(p_{i}p_{j})\bar{c}_{j-1} + \sum_{j=1}^{N-1} \bar{c}_{j}(p_{j}p_{N})\bar{c}_{j-1} \right) \]
\[ = \left(-\right)^{N-1} C^D \left( \sum_{i=2}^{N} \sum_{\mu} \bar{c}_{i-1}p_{(i-1)}\mu \gamma \mu \right) \]
\[ + \sum_{i=1}^{N-1} \bar{c}_i(p_{i}p_{j})\bar{c}_j + \sum_{j=1}^{N-1} \bar{c}_j(p_{j}p_{N})\bar{c}_{j} \right) \quad (31) \]

The last expression is an alternative definition of the shifted \( \gamma \):

\[ \text{Tr}(\hat{\varphi}_1 \hat{\varphi}_2 \ldots \hat{\varphi}_{N-1}\gamma \hat{\varphi}_N) \]
\[ = \left(-\right)^{N-1} C^D \left( \sum_{i=2}^{N} \sum_{\mu} \bar{c}_{i-1}p_{(i-1)}\mu \gamma \mu \right) \]
\[ + \sum_{\mu=1}^{D} \bar{c}_{\mu}p_{\mu} \gamma \mu \]
\[ + \sum_{i=1}^{N-1} \bar{c}_{i}(p_{i}p_{j})\bar{c}_{j} + \sum_{j=1}^{N-1} \bar{c}_{j}(p_{j}p_{N})\bar{c}_{j} \right) \quad (32) \]

For even \( N \)

\[ \text{Tr}(\hat{\varphi}_1 \hat{\varphi}_2 \ldots \hat{\varphi}_{N-1}\gamma \hat{\varphi}_N) \]
\[ = -\text{Tr}(\hat{\varphi}_1 \hat{\varphi}_2 \ldots \hat{\varphi}_{N-1}\gamma \hat{\varphi}_N) + \text{Tr}(\{ \hat{\varphi}_N, \hat{\varphi}_1 \hat{\varphi}_2 \ldots \hat{\varphi}_{N-1} \} \gamma \) \quad (33) \]

For odd \( N \)

\[ \text{Tr}(\hat{\varphi}_1 \hat{\varphi}_2 \ldots \hat{\varphi}_{N-1}\gamma \hat{\varphi}_N) \]
\[ = \text{Tr}(\hat{\varphi}_1 \hat{\varphi}_2 \ldots \hat{\varphi}_{N-1}\gamma \hat{\varphi}_N) + \text{Tr}(\{ \hat{\varphi}_N, \hat{\varphi}_1 \hat{\varphi}_2 \ldots \hat{\varphi}_{N-1} \} \gamma \) \quad (34) \]

Eqs. (33) and (34) are in agreement with eqs. (23) and (24).
6 When $D$ is an Integer

For integer values of $D$, eq. (18) tells that the trace is zero, unless $N \geq D$ and $N - D$ is even. In particular for $N = D$

$$\text{Tr}(\hat{p}_1 \hat{p}_2 \ldots \hat{p}_D \gamma_\chi) = i^{\frac{D(D-1)}{2}} K \det[p] = i^{\frac{D(D-1)}{2}} K \epsilon_{\mu_1 \ldots \mu_D} p_{\mu_1} \ldots p_{\mu_D}. \quad (35)$$

For the general case $N > D$ it is convenient to write eq. (18) in a different fashion. We first change notation: for $i = 1, \ldots, D$

$$\tilde{c}_{N+i} \equiv \chi_i,$$

$$p_{(N+i)\mu} = \delta_{i\mu}. \quad (36)$$

Then we have

$$(p_ip_j) = \delta_{ij}, \quad \text{for } i, j > N$$

$$p_{i\mu} = (p_ip_{(N+\mu)}), \quad \text{for } i < N. \quad (37)$$

With the new variables, eq. (18) becomes

$$\text{Tr}(\hat{p}_1 \ldots \hat{p}_N \gamma_\chi) = i^{\frac{D(D-1)}{2}} K \int d\tilde{c}_{(N+D)} \ldots d\tilde{c}_1 \exp \left( \sum_{i=1,i<j}^{(N+D)} \tilde{c}_i (p_ip_j) \tilde{c}_j \right) \quad (38)$$

or

$$\text{Tr}(\hat{p}_1 \ldots \hat{p}_N \hat{p}_{N+1} \ldots \hat{p}_{N+D}) = K \int d\tilde{c}_{(N+D)} \ldots d\tilde{c}_1 \exp \left( \sum_{i=1,i<j}^{(N+D)} \tilde{c}_i (p_ip_j) \tilde{c}_j \right). \quad (39)$$

Once the trace is written in the canonical form eq. (38) or eq. (39) one can apply the results of Section 2 about the Clifford algebra in eq. (12) and of Section 2.3 on cyclicity as in eq. (17).

This argument show that the properties of the integral representation are identical for integer $D$ to those of the standard irreducible representation in terms of matrices.

6.1 When $D$ is an Integer: $\gamma_\chi$

The above argument is the final answer to the check of the integral representation for integer $D$. However we shall explicitly verify some of the
formulae in order to get used to the relevant algebraic manipulations. In particular we consider again the issue of cyclicity for integer $D$ and $N > D$.

By using cyclicity as in eq. (17) and the gamma’s algebra one gets

$$Tr(p_1 p_2 \ldots p_N \gamma_x) = (-)^{(D-1)}Tr(p_2 \ldots p_N p_1 \gamma_x).$$

(40)

Let us give a formal proof of the above equation by using the representation (39)

$$Tr(p_2 \ldots p_N \bar{p}_1 \bar{p}_{N+1} \ldots \bar{p}_{N+D})$$

$$= K \int d\bar{c}_{(N+D)} \ldots d\bar{c}_N \ldots d\bar{c}_1 \exp \left( \sum_{i=1, i<j \leq N}^{N-1} \left[ \bar{c}_i(p_{i+1}p_{j+1})\bar{c}_j \right] \right)_{p_{N+1}=p_1}$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{D} \bar{c}_i(p_{i+1})\bar{c}_{N+D} + \sum_{i=1}^{N-1} \sum_{j=1}^{D} \bar{c}_i(p_{i+1}p_{N+j})\bar{c}_{N+j-1}.$$  

(41)

and a similar one for

$$Tr(p_2 \ldots p_N \bar{p}_{N+1} \ldots \bar{p}_{N+D} \bar{p}_1)$$

$$= K \int d\bar{c}_{(N+D)} \ldots d\bar{c}_N \ldots d\bar{c}_1 \exp \left( \sum_{i=1, i<j < N}^{N} \left[ \bar{c}_i(p_{i+1}p_{j+1})\bar{c}_j \right] \right)_{p_{N+1}=p_1}$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{D} \bar{c}_i(p_{i+1}p_1)\bar{c}_{N+D} + \sum_{i=1}^{N-1} \sum_{j=1}^{D} \bar{c}_i(p_{i+1}p_{N+j})\bar{c}_{N+j-1}. 

(42)

We need

$$Tr(p_2 \ldots p_N \bar{p}_1 \bar{p}_{N+1} \ldots \bar{p}_{N+D}) + (-)^D Tr(p_2 \ldots p_N \bar{p}_{N+1} \ldots \bar{p}_{N+D} \bar{p}_1)$$

$$= K \int d\bar{c}_{(N+D)} \ldots d\bar{c}_N \ldots d\bar{c}_1 \exp \left( \sum_{i=1, i<j \leq N}^{N-1} \left[ \bar{c}_i(p_{i+1}p_{j+1})\bar{c}_j \right] \right)_{p_{N+1}=p_1}$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{D} \bar{c}_i(p_{i+1})\bar{c}_{N+D} + \sum_{i=1}^{N-1} \sum_{j=1}^{D} \bar{c}_i(p_{i+1}p_{N+j})\bar{c}_{N+j-1}$$

$$+ (-)^D K \int d\bar{c}_{(N+D)} \ldots d\bar{c}_N \ldots d\bar{c}_1 \exp \left( \sum_{i=1, i<j < N}^{N} \left[ \bar{c}_i(p_{i+1}p_{j+1})\bar{c}_j \right] \right)$$

$$+ \sum_{i=1}^{N+D-1} \bar{c}_i(p_{i+1}p_1)\bar{c}_{N+D} + \sum_{i=1}^{N-1} \sum_{j=1}^{D} \bar{c}_i(p_{i+1}p_{N+j})\bar{c}_{N+j-1}. 

(43)

In the second term in eq. (43) we rename $\bar{c}$ according to the following table

$$\bar{c}_N \rightarrow \bar{c}_{N+1}$$

11
\[ \tilde{c}_{N+j} \rightarrow \tilde{c}_{N+j+1}, \quad j < D \]
\[ \tilde{c}_{N+D} \rightarrow \tilde{c}_N. \quad (44) \]

Next, we recover the order of the product in the measure. Thus we get a factor \((-)^D\)

\[ Tr(p_2 \ldots p_N \ p_1 \ p_{N+1} \ldots p_{N+D}) + (-)^D Tr(p_2 \ldots p_N \ p_{N+1} \ldots p_{N+D} \ p_1) \]
\[ = K \int d\tilde{c}_{(N+D)} \ldots d\tilde{c}_N \ldots d\tilde{c}_1 \left\{ \exp \left( \sum_{i=1, i<j \leq N} \tilde{c}_i(p_{i+1}p_{j+1})\tilde{c}_j \right) \right\}_{p_{N+1}=p_1} \\
+ \sum_{i=1}^{N} \sum_{j=1}^{D} \tilde{c}_i(p_{i+1}) \left( \tilde{c}_{N+j} \right)_{p_{N+1}=p_1} \}
\[ + \exp \left( \sum_{i=1, i<j \leq N} \tilde{c}_i(p_{i+1}p_{j+1})\tilde{c}_j \right) + \sum_{i=1}^{N-1} \tilde{c}_i(p_{i+1}p_1)\tilde{c}_N \\
- \sum_{i=1}^{D} \tilde{c}_N(p_1\ p_{N+i})\tilde{c}_{N+i} + \sum_{i=1}^{N-1} \sum_{j=1}^{D} \tilde{c}_i(p_{i+1}p_{N+j})\tilde{c}_{N+j} \right\} \quad (45) \]
\[ = K \int d\tilde{c}_{(N+D)} \ldots d\tilde{c}_N \ldots d\tilde{c}_1 \left\{ 1 + \prod_{i=1}^{D} \left[ 1 - 2\tilde{c}_N(p_1p_{N+i})\tilde{c}_{N+i} \right] \right\} \\
\[ \exp \left\{ \sum_{i=1, i<j \leq N} \tilde{c}_i(p_{i+1}p_{j+1})\tilde{c}_j \right\}_{p_{N+1}=p_1} + \sum_{i=1}^{N-1} \sum_{j=1}^{D} \tilde{c}_i(p_{i+1}) \left( \tilde{c}_{N+j} \right)_{p_{N+1}=p_1} \}
\[ p_{N+j} \tilde{c}_{N+j} \right\}. \quad (46) \]

The result can be easily interpreted as

\[ eq. (43) = 2Tr \left( p_2 \ldots p_N \ p_1 \ p_{N+1} \ldots p_{N+D} \right) \]
\[-2 \sum_{j=1}^{D} (p_1, p_{N+j})(-)^{j-1} Tr \left( p_2 \ldots p_N \ p_1 \ p_{N+1} \ldots p_{N+j} \ldots p_{N+D} \right) \quad (47) \]

For even \(D\) eq. (47) becomes

\[ Tr \left( p_2 \ldots p_N \left[ p_1 \ p_{N+1} \ldots p_{N+D} \right] \right) \]
\[ = 2 \sum_{j=1}^{D} (p_1, p_{N+j})(-)^{j-1} Tr \left( p_2 \ldots p_N \ p_1 \ p_{N+1} \ldots p_{N+j} \ldots p_{N+D} \right) \quad (48) \]
and for odd $D$

$$Tr\left( \hat{p}_2 \cdots \hat{p}_N \left\{ \hat{p}_1, \hat{p}_{N+1} \cdots \hat{p}_{N+D} \right\} \right)$$

$$= 2 \sum_{j=1}^{D} (p_1, p_{N+j}) (-)^{j-1} Tr\left( \hat{p}_2 \cdots \hat{p}_N \hat{p}_1 \hat{p}_{N+1} \cdots \hat{p}_{N+j} \cdots \hat{p}_{N+D} \right)$$ (49)

Finally eq. (410) is obtained from eqs. (48) and (49) by using the identity

$$2 \sum_{j=1}^{D} (p_1, p_{N+j}) (-)^{j-1} \hat{p}_{N+1} \cdots \hat{p}_{N+j} \cdots \hat{p}_{N+D}$$

$$= 2 \sum_{j=1}^{D} (p_1)_{j} (-)^{j-1} \hat{p}_{N+1} \cdots \hat{p}_{N+j} \cdots \hat{p}_{N+D}$$

$$= 2 \sum_{j=1}^{D} (p_1)_{j} (-)^{j-1} \hat{p}_{N+1} \cdots \hat{p}_{N+j} \hat{\gamma}_j^2 \cdots \hat{p}_{N+D}$$

$$= 2 \hat{p}_1 \hat{p}_{N+1} \cdots \hat{p}_{N+D}.$$

(50)

The relation (410) implies

$$\left\{ \left\{ \gamma_\chi, \gamma_\nu \right\} = 0, \quad \forall \nu = 1 \ldots D \right\}$$

for even $D$

$$\left\{ \left\{ \gamma_\chi, \gamma_\nu \right\} = 0, \quad \forall \nu = 1 \ldots D \right\}$$

for odd $D$. (51)

We match the representation of the trace in eq. (18) with the standard matrix expressions. The matrix representation of the gamma’s is assumed to be irreducible thus we choose the phase

$$\gamma_\chi = \left( i \frac{D(D-1)}{2} \right) \gamma_1 \ldots \gamma_D \quad \text{for even } D$$

$$\gamma_\chi = \left( i \frac{D(D-1)}{2} - \frac{1}{2} \right) \quad \text{for odd } D.$$ (52)

7 When $D$ is an Integer: Trace in Closed Form

Now we study the integral representation in eq. (18) for integer $D$. By using a theorem in general matrix theory [39], the rectangular matrix $p_{j\mu}, j=1, \ldots, N$ can be brought to a diagonal form $\Sigma$ by suitable unitary transformations $U$ and $V^\dagger$

$$p = U \Sigma V^\dagger.$$ (53)

$U$ is a $N \times N$ and $V$ is $D \times D$ matrix (singular value decomposition). Both $U$ and $V$ can be orthogonal matrices if $p$ is a real matrix. $\Sigma$ is unique if the
eigenvalues are positive or zero and ordered
\[
\Sigma_{i\mu} = \sigma_i \delta_{i\mu}, \quad \sigma_i \geq \sigma_{i+1} \geq 0, \quad 1 \leq i < \min(D, N). \tag{54}
\]
We can change variable of integration by unitary transformation
\[
\bar{c}_i \rightarrow \bar{c}_j U_{ji}^\dagger
\]
\[
\chi_{\mu} \rightarrow V_{\mu\nu} \chi_{\nu}'. \tag{55}
\]
The ensuing integrations on the \(\chi\)’s can be non-zero only if the rank of the matrix \(\{p\}\) is equal to \(D\). Thus it is necessary that \(N \geq D\). Consequently if \(N > D\) the integration over \(\bar{c}_j\) is non zero only if \(N - D\) is even. Thus
\[
n = \frac{1}{2}(N - D) \tag{56}
\]
is a positive integer.

The bilinear term in \(\bar{c}\) can be written (as in eq. (4))
\[
\sum_{i,j=1, i<j}^N \bar{c}_i (p_i p_j) \bar{c}_j = \frac{1}{2} \sum_{i,j} \bar{c}_i h_{ij} \bar{c}_j,
\]
\[
h_{ij} |_{i<j} = (p_i p_j) = U_{ii'} \sigma_{i'}^2 U_{i'j}^\dagger. \tag{57}
\]
The integral representation (18) becomes
\[
\text{Tr}(\hat{\rho}_1 \ldots \hat{\rho}_N \gamma \chi) = i \frac{D(D-1)}{2} K \int d^D \chi d^N \bar{c} \exp \left( \sum_{\mu=1}^D \sum_{i=1}^N \bar{c}_i \Sigma_{i\mu} \chi_{\mu} + \frac{1}{2} \sum_{ab=D+1}^N \sum_{i,j=1}^N \bar{c}_a U_{ai}^\dagger h_{ij} U_{jb} \bar{c}_b \right), \tag{58}
\]
where the limits on the sums are dictated by the form of \(\Sigma\) as in eq. (54).

In the present problem both \(U\) and \(V\) can be chosen to be orthogonal matrices, then the matrix
\[
H_{ab} \equiv \sum_{i,j=1}^N U_{ai}^\dagger h_{ij} U_{jb}, \quad a, b = D + 1, \ldots, N \tag{59}
\]
is skew-symmetric. The integration over \(\bar{c}\) is a standard result in matrix theory (Pfaffian) [54]
\[
Pf(H) = \int d^{(N-D)} \bar{c} \exp \frac{1}{2} \left( \sum_{a,b=D+1}^N \bar{c}_a H_{ab} \bar{c}_b \right)
\]
\[
= \sum_{\mathcal{P}} \delta_\mathcal{P} H_{i_1j_1} H_{i_2j_2} H_{i_3j_3} \ldots \tag{60}
\]
where $\mathcal{P}$ is any permutation of $D + 1, \ldots, N$ and $\delta_{\mathcal{P}}$ is its signature. The sum $\sum'$ is restricted to the permutations satisfying the conditions

$$i_k < i_{k+1} \text{ and } i_k < j_k, \forall k.$$  \hfill (61)

The above expression can be evaluated by using the relation (Thomas Muir)

$$[Pf(H)]^2 = \det[H].$$ \hfill (62)

Thus the singular value decomposition of the matrix $p_{i\mu}$ allows a straightforward evaluation of the trace in even dimension ($\gamma_\chi$ present) or in odd dimensions.

### 8 ABJ Anomaly

We use the algebra for $\gamma_\chi$ developed in Section 4 in order to evaluate the ABJ anomaly \cite{28} \cite{29}. We consider a massless fermion triangle, where one vertex is given by an axial current. Thus we consider the integral ($p$ is the incoming momentum on the vertex $\sigma$ and $k$ on $\rho$; crossed graph will be considered at the end)

$$T_{\mu\rho\sigma}(k,p) = \int \frac{d^Dq}{(2\pi)^D} \frac{Tr \left\{ \gamma_\mu \gamma_5 (q-k)_{\alpha} \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma (q+p)_{\chi} \right\}}{(q-k)^2 q^2 (q+p)^2}$$ \hfill (63)

Now we use Feynman parametrization and get

$$\begin{align*}
Tr \left( \gamma_\mu \gamma_5 \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\chi \right) 2 \int_0^1 dx \int_0^x dy \int \frac{d^Dq}{(2\pi)^D} \\
(q + r - k)_{\alpha} (q + r)_{\beta} (q + r + p),
\end{align*}$$

$$\begin{align*}
\left[ q^2 + k^2 y + p^2 x - p^2 y - (k y - p x + p y)^2 \right]^{-3},
\end{align*}$$ \hfill (64)

$$r_{\nu} \equiv (y k - x p + y p)_{\nu}. \hfill (65)$$

We use the simplified case

$$k^2 = p^2 = 0.$$ \hfill (66)

After symmetric integration over $q$ we can split the integral into a divergent

$$\frac{2}{D} \int_0^1 dx \int_0^x dy \left( \delta_{\alpha\beta} (r + p)_{\chi} + \delta_{\alpha\chi} r_{\beta} + \delta_{\beta\chi} (r - k)_{\alpha} \right)$$

$$\left( \frac{i}{(4\pi)^2} \right) \left[ \frac{2}{D - 4} + \gamma + 2 - \ln 4\pi + \ln 2pk(y - x) \right]$$ \hfill (67)
and finite part
\[
2 \int_0^1 dx \int_0^x dy (r - k) \delta (r + p) \int \frac{d^D q}{(2\pi)^D (q^2 - 2kpy(y - x))^\delta} \frac{1}{2y(y - x)}.
\]

In front of the two amplitudes (67) and (68) the gamma’s trace must be expanded in powers of \((D - 4)\) as required by eq. (63). For the finite part in eq. (68) we can use the \(D = 4\) expression, but for the divergent part one needs also the linear part. Let us use the representation of \(\gamma_\chi\) provided by eq. (18) in order to tackle the problem. According to the discussion of the Section 4 we know that the algebra of \(\gamma_\chi\) for non-integer \(D\) is rather complicated as shown in eqs. (23) and (24). Then we use instead the algebra of the gamma’s in eq. (11) which has been proved valid for generic \(D\); i.e. we do not change the relative position of \(\gamma_\chi\) with respect to the remaining factors in the trace. Thus in eq. (63), according to eq. (67), we evaluate

\[
Tr \left( \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\gamma \gamma_\chi \gamma_\rho \gamma_\gamma \gamma_\mu \right) \delta_{\alpha\beta} = [-2 - (D - 4)] Tr \left( \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\gamma \gamma_\mu \right) \\
Tr \left( \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\gamma \gamma_\chi \gamma_\rho \gamma_\gamma \gamma_\mu \right) \delta_{\beta\chi} = [-2 - (D - 4)] Tr \left( \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\gamma \gamma_\rho \gamma_\sigma \right) \\
Tr \left( \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\gamma \gamma_\chi \gamma_\rho \gamma_\gamma \gamma_\mu \right) \delta_{\alpha\chi} = Tr \left( \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\gamma \gamma_\rho \gamma_\sigma \right) \\
\left[ (2 - (D - 4)) \gamma_\rho \gamma_\gamma \gamma_\sigma - 4(\delta_{\rho\beta} \gamma_\sigma - \delta_{\rho\sigma} \gamma_\beta + \delta_{\sigma\beta} \gamma_\rho) \right].
\]

We collect the non-zero part associated to the amplitude (67).

8.1 Contribution of the Divergent Integral

Start with eq. (67)

\[
\frac{2}{D} \int_0^1 dx \int_0^x dy Tr \left( \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\gamma \gamma_\mu \gamma_\rho \gamma_\gamma \gamma_\chi \right) \left( \delta_{\alpha\beta}(r + p) + \delta_{\alpha\beta}(r + p - k) \right) \\
\left( - \frac{i}{(4\pi)^2} \right) \left[ \frac{2}{D - 4} + \gamma + 2 - \ln 4\pi + \ln 2pky(y - x) \right]
\]

\[
= \frac{2}{(4\pi)^2} \frac{2}{D} \int_0^1 dx \int_0^x dy \\
\left( 2(3r + p - k) + (D - 4)(r + p - k) \right) \\
\left[ \frac{2}{D - 4} + \gamma + 2 - \ln 4\pi + \ln 2pky(y - x) \right].
\]
By using
\[
\int_0^1 dx \int_0^x dy \left[ 3(yk - xp + yp) + p - k \right] = 0
\]
\[
\int_0^1 dx \int_0^x dy (yk - xp + yp + p - k) = \frac{1}{3} (p - k)
\]  
(71)
one gets
\[
\frac{i}{(4\pi)^2} \left( \frac{2}{D} \right) \left( \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_i \right) \int_0^1 dx \int_0^x dy \left( 2(3r + p - k) + (D - 4)(r + p - k) \right) \left[ \frac{2}{D - 4} + \gamma + 2 - \ln 4\pi + \ln 2pk(x - y) \right]
\]
\[
= \frac{i}{(4\pi)^2} \left( \frac{1}{2} \right) \left( \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_i \right)
\]
\[
\left( \int_0^1 dx \int_0^x dy (3r + p - k)_i \ln y(x - y) + \frac{2}{3} (p - k)_i \right).
\]  
(72)
Finally we use
\[
\int_0^1 dx \int_0^x dy \ln |y(x - y)| = \frac{3}{2}
\]
\[
\int_0^1 dx \int_0^x dy y \ln |y(x - y)| = -\frac{4}{9}
\]
\[
\int_0^1 dx x \int_0^x dy \ln |y(x - y)| = -\frac{8}{9}
\]  
(73)
and get
\[
\frac{i}{(4\pi)^2} \left( \frac{1}{2} \right) \left( \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_i \right)
\]
\[
\left( \int_0^1 dx \int_0^x dy (3r + p - k)_i \ln y(x - y) + \frac{2}{3} (p - k)_i \right)
\]
\[
= \frac{i}{(4\pi)^2} \left( \frac{1}{2} \right) \left( \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_i \right) \left( -\frac{1}{3} (p - k)_i + \frac{2}{3} (p - k)_i \right)
\]
\[
= \frac{i}{(4\pi)^2} \left( \frac{1}{2} \right) \left( \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_i \right) \frac{1}{3} (p - k)_i.
\]  
(74)
The divergence of the axial current is obtained by multiplying with \((p + k)^\mu\)
\[
(k + p)^\mu \frac{i}{(4\pi)^2} \left( \frac{1}{2} \right) \left( \gamma_\mu \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_i \right) \frac{1}{3} (p - k)_i
\]
\[
= -\frac{i}{(4\pi)^2} \left( \gamma_\chi \gamma_\rho \gamma_\sigma \gamma_\theta \right) \frac{1}{3}.
\]  
(75)
By adding the crossed graph
\[ (k + p) \mu \left( T_{\mu\rho\sigma}^{\text{DIV}}(k, p) + T_{\mu\sigma\rho}^{\text{DIV}}(p, k) \right) = \frac{-2}{3 (4\pi)^2} Tr \left( \gamma_5 \gamma_\rho \not{p} \gamma_\sigma \not{k} \right). \] (76)

### 8.2 Contribution of the Convergent Integral

Further finite terms can be evaluated directly at \( D = 4 \). From eq. (64) the finite integral contribution to the triangular graph is (always in the case \( k^2 = p^2 = 0 \))

\[ T_{\mu\rho\sigma}^{\text{FIN}}(k, p) = Tr \left\{ \gamma_\mu \gamma_5 \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right\} 2 \int_0^1 dx \int_0^x dy (k(y - 1) + p(y - x))_\alpha \]

\[ (ky + p(y - x))_\beta (ky + p(y - x + 1))_\iota \left( -\frac{i}{(4\pi)^2} \frac{1}{4pk(y - x)} \right) \] (77)

By repeated use of the identity

\[ \not{k} \gamma_\rho \not{k} = -k^2 \gamma_\rho + 2k_\rho \not{k}. \] (78)

one shows that only two forms

\[ Tr \left\{ \gamma_\mu \gamma_5 \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right\} p^\alpha k^\beta p^\iota = -2p_\mu Tr \left\{ \gamma_5 \gamma_\rho \not{k} \gamma_\sigma \not{p} \right\} \]

\[ Tr \left\{ \gamma_\mu \gamma_5 \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right\} k^\alpha p^\beta k^\iota = -2k_\mu Tr \left\{ \gamma_5 \gamma_\rho \not{p} \gamma_\sigma \not{k} \right\} \] (79)

give non-zero contribution to the divergence of the current.

\[ (k + p) \mu T_{\mu\rho\sigma}^{\text{FIN}}(k, p) \]

\[ = Tr \left\{ (k + p) \mu \gamma_\mu \gamma_5 \gamma_\alpha \gamma_\rho \gamma_\beta \gamma_\sigma \gamma_\iota \right\} 2 \int_0^1 dx \int_0^x dy (k(y - 1) + p(y - x))_\alpha \]

\[ (ky + p(y - x))_\beta (ky + p(y - x + 1))_\iota \left( -\frac{i}{(4\pi)^2} \frac{1}{4pk(y - x)} \right) \]

\[ = \frac{i}{(4\pi)^2} Tr \left\{ \gamma_5 \gamma_\rho \not{p} \gamma_\sigma \not{k} \right\} \int_0^1 dx \int_0^x dy ((y - 1) - (y - x + 1)) \]

\[ = -\frac{i}{(4\pi)^2} Tr \left\{ \gamma_5 \gamma_\rho \not{p} \gamma_\sigma \not{k} \right\} \frac{2}{3} \] (80)

The contribution of the finite integral to the divergence is then

\[ (k + p) \mu (T_{\mu\rho\sigma}^{\text{FIN}}(k, p) + T_{\mu\sigma\rho}^{\text{FIN}}(p, k)) = -\frac{i}{(4\pi)^2} Tr \left\{ \gamma_5 \gamma_\rho \not{p} \gamma_\sigma \not{k} \right\} \frac{4}{3} \] (81)
Finally the sum of the contributions in eq. (76) and (81) is
\[(k + p)^{\mu}(T_{\mu\rho}(k, p) + T_{\mu\rho}(p, k)) = -\frac{2i}{(4\pi)^2} Tr \left\{ \gamma_5 \gamma_{\rho} \not{p} \gamma_{\sigma} \not{k} \right\}, \tag{82} \]
which agrees with the ABJ anomaly.

9 Local Functional Equation

Once we have discovered that \(\gamma_\chi\) has a complicated behavior in \(D\) dimension, we must test our formalism in the path integral. The LFE has been discussed at length in Ref. [30]. Here we give the essential steps. The functional is
\[Z[A] = \int \prod_x \prod_\mu d\bar{\psi}_\mu(x) \prod_{\mu'} d\psi_{\mu'}(x) e^{iS[A]} \tag{83} \]
where the action \((e = 1)\) is function of the external vector field \(A_\mu(x)\)
\[S = \int d^D x \bar{\psi}(i \not{\partial} - \not{A})\psi. \tag{84} \]
The path integral measure is Lorentz invariant. Moreover it is invariant under the \(U(1)\) local chiral transformations
\[\psi \rightarrow e^{i\alpha(x)\gamma_\chi}\psi \]
\[\psi^\dagger \rightarrow \psi^\dagger e^{-i\alpha(x)\gamma_\chi} \tag{85} \]
since the Jacobian of the transformation is equal one. In fact
\[\prod_\mu d\bar{\psi}_\mu \rightarrow \det(e^{i\alpha(x)\gamma_\chi}) \prod_\mu d\psi_{\mu} = e^{i\alpha(x)Tr(\gamma_\chi)} \prod_\mu d\psi_{\mu}. \tag{86} \]
Thus if we perform a substitution in the path integral variables according to eq. (85) the functional \(Z\) does not change. For infinitesimal parameter \(\alpha\) one gets
\[\left\langle \left( - \bar{\psi}\gamma_0\gamma_\chi\gamma_0(i \not{\partial} - \not{A})\psi + \bar{\psi}(i \not{\partial} - \not{A})\gamma_\chi\psi - i\partial^\mu(\bar{\psi}\gamma_\mu\gamma_\chi\psi) \right) \right\rangle = 0, \tag{87} \]
where the brackets \(\langle \cdot \cdot \cdot \rangle\) denote the mean value with the path integral measure of eq. (83).

If one uses the naive commutation relations of \(\gamma_\chi\) (i.e. \(\{\gamma_\chi, \gamma_\mu\} = 0\)) the first two first terms from the left in eq. (87) cancel out.
In Section 2 we have found that $\gamma_{\chi}$ has complicated behavior. Then one must evaluate at one loop the expressions in eq. (87) according the rules of eqs. (23) and (24). The results will be compared with eq. (82).

A single interaction insertion gives zero since it depends only on $k$ or $p$; never on both. Consequently no completely antisymmetric tensor can emerge. We need two insertions: the triangular graph. We consider only the one that can provide some non-zero contributions

$$T_{\rho\sigma}(k, p) = \int \frac{d^D q}{(2\pi)^D} \frac{Tr \left[ (q - k)^\rho \gamma_{\chi} (q - k) \gamma_{\rho} \gamma_\sigma (q + p) \right]}{(q - k)^2 q^2 (q + p)^2}.$$  \hspace{1cm} (88)

The gamma’s algebra gives

$$Tr \left[ (q - k)^\rho \gamma_{\chi} (q - k) \gamma_{\rho} \gamma_\sigma (q + p) \right] = -Tr \left[ \gamma_\chi (q - k)^\rho \gamma_\rho (q + p) \right] + Tr \left[ \gamma_\chi \left\{ (q - k)^\rho \gamma_{\rho} \gamma_\sigma (q + p) \right\} \right]$$  \hspace{1cm} (89)

In the first term of the RHS the dependence on $k$ disappears, thus it can be neglected. We consider the remaining terms

$$Tr \left[ \gamma_\chi \left\{ (q - k)^\rho \gamma_{\rho} \gamma_\sigma (q + p) \right\} \right] = 2(q - k)^2 Tr \left[ \gamma_\chi \gamma_\rho \gamma_\sigma (q + p) \right] - 2(q - k)^2 Tr \left[ \gamma_\chi (q - k)^\rho \gamma_\rho \gamma_\sigma (q + p) \right] + 2(q - k)^2 q Tr \left[ \gamma_\chi (q - k)^\rho \gamma_\rho (q + p) \right] - 2(q - k)^2 q Tr \left[ \gamma_\chi (q - k)^\rho \gamma_\rho (q + p) \right] + 2(q - k)(q + p) Tr \left[ \gamma_\chi (q - k)^\rho \gamma_\rho \gamma_\sigma \right]$$  \hspace{1cm} (90)

All terms containing $(q^2)^2$ or $( (q - k)^2 )^2$ or $( (q + p)^2 )^2$ should be neglected since no $\epsilon$ term can emerge. Thus

$$Tr \left[ \gamma_\chi \left\{ (q - k)^\rho \gamma_{\rho} \gamma_\sigma (q + p) \right\} \right] = 2(q - k)^2 Tr \left[ \gamma_\chi \gamma_\rho \gamma_\sigma p \right] + 2(q - k)^2 Tr \left[ \gamma_\chi k^\rho \gamma_\sigma p \right] - 2(q - k)^2 q Tr \left[ \gamma_\chi (p + k)^\rho \gamma_\rho \gamma_\sigma (q + p) \right] + 2(q - k)^2 q Tr \left[ \gamma_\chi k^\rho \gamma_\rho \gamma_\sigma \right] - 2(q - k)(q + p) Tr \left[ \gamma_\chi k^\rho \gamma_\rho \gamma_\sigma \right]$$  \hspace{1cm} (91)
Now we shift 

\[ q \rightarrow q + r \quad r \equiv yk - xp + yp \]  

(92)

and we drop all terms that are zero as a result of the symmetric integration

\[
\text{Tr} \left[ \gamma \chi \left\{ (q-k), (q-k) \gamma_{\rho} \gamma_{\sigma} (q+p) \right\} \right] = \frac{q^2}{D} \left( 2y - 1 \right) \text{Tr} \left[ \gamma \chi \gamma_{\rho} \gamma_{\rho} \gamma_{\sigma} \gamma_{p} \right] + \text{Tr} \left[ \gamma \chi \gamma_{p} \gamma_{\rho} \gamma_{\sigma} \gamma_{p} \right] \\
+ D \text{Tr} \left[ \gamma \chi \gamma_{p} \gamma_{\rho} \gamma_{\sigma} \gamma_{p} \right] - D(1-x) \text{Tr} \left[ \gamma \chi \gamma_{p} \gamma_{\rho} \gamma_{\sigma} \gamma_{p} \right] + D(1-x) \text{Tr} \left[ \gamma \chi \gamma_{\rho} \gamma_{\sigma} \gamma_{p} \right] \\
- 2(2y - 1) \text{Tr} \left[ \gamma \chi \gamma_{p} \gamma_{\rho} \gamma_{\sigma} \gamma_{p} \right] + D(x-y) \text{Tr} \left[ \gamma \chi \gamma_{p} \gamma_{\rho} \gamma_{\sigma} \gamma_{p} \right] \\
\approx \frac{q^2}{D} \text{Tr} \left[ \gamma \chi \gamma_{p} \gamma_{\rho} \gamma_{\sigma} \gamma_{p} \right] \left\{ 2(2y - 1) + yD - 1 + 2(2y - 1) + (1-x)D - 1 + (x-y)D \right\} \\
= \frac{q^2}{D} \text{Tr} \left[ \gamma \chi \gamma_{p} \gamma_{\rho} \gamma_{\sigma} \gamma_{p} \right] \left\{ -4 + D \right\}.  
\]  

(93)

The factor \( D - 4 \) is expected, since in four dimensions \( \gamma_{\chi} \) anti-commutes with all \( \gamma_{\mu} \) and therefore \( T_{\mu}(k,p) \) is zero from start in eq. (88).

The integration over \( x, y \) gives

\[
= \frac{q^2}{D} \text{Tr} \left[ \gamma \chi \gamma_{p} \gamma_{\rho} \gamma_{\sigma} \gamma_{p} \right] \frac{D - 4}{2} \\
= \frac{q^2 D - 4}{D} \text{Tr} \left[ \gamma \chi \gamma_{p} \gamma_{\rho} \gamma_{\sigma} \gamma_{p} \right]  
\]  

(94)

Now we multiply by 2 (the Feynman parameter), 2 (the crossed graph), \(-\frac{i}{(4\pi)^2} \frac{2}{D-4}\) from \( q \) integration. We get

\[
2 \frac{i}{(4\pi)^2} \text{Tr} \left[ \gamma \chi \gamma_{p} \gamma_{\rho} \gamma_{\sigma} \gamma_{p} \right].  
\]  

(95)
Thus the results in eqs. (82) and (95) do satisfy the LFE identity in eq. (87).

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