A dual 2D model for the Quantum Hall Fluid *

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Abstract

We present a dual 2D statistical model of a Quantum Hall Fluid which depends on a coupling constant $g$ and an angular variable $\theta$, parametrizing a surface term. We show that such a model has topologically non trivial vacua (corresponding to rational values of the filling), which are infrared stable fixed points of the renormalization group. Moreover its partition function has a dual infinite discrete symmetry, $SL(2, \mathbb{Z})$, which reproduces the phenomenological laws of corresponding states. Such a symmetry reproduces the universality of the phase diagram of the Hall Fluid and allows for an unified description of its fixed points in terms of a 2D Conformal Field Theory with central charge $c = 1$.  

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Our understanding of the conduction properties of a Quantum Hall Fluid at the plateaux has greatly improved recently by means of the two-dimensional Conformal Field Theory (2D CFT) [1]. In particular a theoretical framework has been proposed, which basically substantiates Laughlin physical idea of associating a magnetic flux to a charge identifying the anyons as basic excitations [2]. Furthermore both the Integer Quantum Hall Effect and the Fractional one now appear strictly related to each other, sharing similar many-body properties. Then we can ask the question of the form of the phase diagram and its critical points [3,4]. In this context there has been a theoretical as well as a phenomenological attempt [3,5] to uncover the physical properties of such a diagram: its dependence on two parameters (and their physical meaning), its nested character and the universality shared by the fixed points (attractive or repulsive) describing the Hall fluid at different fillings.

We are able to build a 2D statistical model, which, besides the usual g-dependance of the kynetic term and a surface term, parametrized by the angle $\theta$, [6,7], contains newly generated electric and magnetic background terms.

We extend the Renormalization Group (RG) analysis to such a model sorting out its new interesting properties, as, for example, the existence of Infrared (IR) stable fixed points which are well described by the presence of both electric and magnetic backgrounds. These are interpreted as topologically non-trivial vacua of the field theory in the continuum which takes the form of a generalized dual Coulomb Gas with background, reproducing, for rational values of $\theta$, the phenomenologically observed properties of a Hall fluid at the plateaux at corresponding fillings.

Also we observe the existence of an infinite discrete symmetry $SL(2, Z)$
generalizing the well known Kramers-Wannier duality) which acts as in ref. [7], mapping those non trivial fixed points one into another. This suggests a unified picture of the IR fixed points in terms of a 2D Conformal Field Theory with central charge $c = 1$.

We stress that there is a simple connection between the duality transformations of the model and the phenomenological ”laws of corresponding states” advocated by the authors of ref. [5].

Let us first remind how Cardy and Rabinovici build their 2D model starting from a $U(1)$ gauge theory with both electric and magnetic matter coupled by a surface $\theta$ term [6]. Then we will search for a non trivial extension of such a model in which a fixed background is generated.

The explicit action is given by:

$$S[A, S, n] \equiv \frac{1}{2g} \int d^2r (\partial_\beta A - S_\beta)^2 - i \int d^2rnA - i\frac{\theta}{2\pi} \int d^2r \epsilon_{\beta\gamma} S_\beta \partial_\gamma A ,$$

where $A$ is the gauge potential, $S_\beta(\vec{r})$ is the ”magnetic frustration field”, defined by the constraint:

$$\epsilon_{\beta\gamma} \partial_\beta S_\gamma(\vec{r}) - m(\vec{r}) = 0, \quad (2)$$

while $n(\vec{r})$ and $m(\vec{r})$ are defined in terms of the electric and magnetic charge density; $\epsilon_{\beta\gamma}$ is the antisymmetric tensor in 2D. In particular, the magnetic charge density is $Q_m = m/\sqrt{g}$ while the electric one is obtained by integrating out the field $S_\beta$ with the constraint (2) and is given by:

$$Q_e = \sqrt{g} \left(n + \frac{\theta}{2\pi}m\right).$$

The densities $(n, m)$ are constrained by the neutrality condition (required to make the system infrared stable):
\[ \int d^2r \ n(\vec{r}) = \int d^2r \ m(\vec{r}) = 0 \ . \] (3)

In the following we need two representations of our model, one as a Coulomb gas where the role of the electric and magnetic charges is emphasized, and the gauge representation, where both charges have gauge interaction.

The Coulomb gas representation is defined by:

\[ e^{-S_{CG}(n,m)} = \int \mathcal{D}A \int \prod_{\alpha=1,2} \mathcal{D}S_{\alpha} \delta(\epsilon_{\beta\gamma} \partial_{\beta} S_{\gamma} - m) e^{-S[A,S,n]} \ . \] (4)

It is straightforward to evaluate the path integrals in eq.(4) and the result is given by:

\[ S_{CG}[n, m] = \frac{g}{2} \int d^2r d^2r' (n(\vec{r}) + \frac{\theta}{2\pi} m(\vec{r}))(n(\vec{r'}) + \frac{\theta}{2\pi} m(\vec{r'})) G(\vec{r} - \vec{r'}) + \frac{1}{2g} \int d^2r d^2r' m(\vec{r}) m(\vec{r'}) G(\vec{r} - \vec{r'}) + \int d^2r d^2r' n(\vec{r}) m(\vec{r'}) \varphi(\vec{r} - \vec{r'}) \ , \] (5)

where \( G(\vec{r}) \) and \( \varphi(\vec{r}) \) are the “longitudinal” and “transverse” Green-Feynman functions in 2D given by:

\[ G(\vec{r}) = \ln \left( \frac{|\vec{r}|}{a} \right) , \quad \varphi(\vec{r}) = \arctan \left( \frac{y}{x} \right) \ , \] (6)

where \( a \) is a cutoff. The last term in eq.(5) is the (imaginary) Bohm-Aharonov term [8]. Also notice that eq.(5) defines for \( \theta = 0 \) the standard Coulomb gas for both electric and magnetic charges( see ref.[9]).

To obtain the gauge representation we solve the constraint (2) as:

\[ \delta(\epsilon_{\beta\gamma} \partial_{\beta} S_{\gamma} - m) = \int \mathcal{D}A_D \exp \left( -i \int d^2r A_D(\vec{r})(\epsilon_{\beta\gamma} \partial_{\beta} S_{\gamma} - m) \right) \ . \] (7)

Then by evaluating explicitly the functional integral in eq.(4) one gets the action:

\[ S[A, A_D] = \frac{g}{2} \int d^2r (\partial_{\beta} A_D)^2 - i \int d^2r \epsilon_{\beta\gamma} \partial_{\beta} A_D \partial_{\gamma} A_D \ . \] (8)
It turns out that the relevant 2-points functions are:

\[ \langle A(\vec{r})A(\vec{r}') \rangle = g G(\vec{r} - \vec{r}') , \quad (9.a) \]

\[ \langle A_D(\vec{r})A_D(\vec{r}') \rangle = \frac{1}{g} G(\vec{r} - \vec{r}') , \quad (9.b) \]

\[ \langle A(\vec{r})A_D(\vec{r}') \rangle = i \varphi(\vec{r} - \vec{r}') . \quad (9.c) \]

The dual form of the Green functions in eqs.(9.a), (9.b) should be noticed. Naturally one can recover the Coulomb gas representation eq.(5) by choosing for the charge densities:

\[ n(\vec{r}) = \sum_i n_i \delta(\vec{r} - \vec{r}_i) ; \quad m(\vec{r}) = \sum_i m_i \delta(\vec{r} - \vec{r}_i) . \quad (10) \]

We now search for solutions where the gauge fields can be splitted in two parts: a ‘background” one and a “fluctuating” one, as:

\[ A(\vec{r}) \equiv \bar{A}(\vec{r}) + a(\vec{r}) , \quad \partial^2 \bar{A}(\vec{r}) = -ig(\bar{n} + \frac{\theta}{2\pi} \bar{m}) \quad (11) \]

\[ A_D(\vec{r}) \equiv \bar{A}_D(\vec{r}) + a_D(\vec{r}) , \quad \partial^2 \bar{A}_D(\vec{r}) = \frac{i}{g} \bar{m} \quad (12) \]

\[ S_\beta(\vec{r}) \equiv \bar{S}_\beta(\vec{r}) + s_\beta(\vec{r}) , \quad \epsilon_{\beta\gamma} \partial_\beta \bar{S}_\gamma(\vec{r}) = \bar{m} . \]

The equations above can be easily solved obtaining for the background gauge fields:

\[ \bar{A}(\vec{r}) = -\frac{ig}{4}(\bar{n} + \frac{\theta}{2\pi} \bar{m}) \vec{r}^2 \quad (13) \]

\[ \bar{A}_D(\vec{r}) = \frac{i}{4g} \bar{m} r^2 , \]

which reproduces the usual harmonic form for the neutralizing background.
We can now integrate over the fluctuating field \( s_\beta \), eq.(12), by taking into account the constraint given by eq.(7), and obtain:

\[
Z[\bar{n}, \bar{m}; \mu, \nu] = \int \mathcal{D}a \int \mathcal{D}a_D e^{-S_f[a,a_D]} \exp(i \int d^2r (\nu(\vec{r}) + \mu(\vec{r})) - i \int d^2r \mu(\vec{r})(\bar{A}_D(\vec{r}) + a_D(\vec{r}))) ,
\]

where \( \mu(\vec{r}), \nu(\vec{r}) \) are the fluctuating charges,

\[
S_f[a, a_D] = \frac{g}{2} \int d^2r [\partial_\beta(a_D)]^2 - i \epsilon_\beta, \int d^2r [\partial_\beta(a) \partial_\gamma(a_D)]
\]

and we define the background term \( S_B \) as:

\[
S_B = -i \int d^2r [\nu(\vec{r}) + \mu(\vec{r})] \bar{A}(\vec{r}) + i \int d^2r \mu(\vec{r}) \bar{A}_D(\vec{r}) =
\]

\[
\frac{g}{4}(\bar{n} + \frac{\theta}{2\pi} \bar{m}) \int d^2r r^2 [\nu(\vec{r}) + \frac{\theta}{2\pi} \mu(\vec{r})] - \frac{1}{4g} \bar{m} \int d^2r r^2 \mu(\vec{r}) .
\]

Eq.(16) gives a “background term” whose meaning may be derived by rewriting the partition function at fixed backgrounds \( \bar{N} = \int d^2r \bar{n}(\vec{r}) \) and \( \bar{M} = \int d^2r \bar{m}(\vec{r}) \) where the fluctuating charges are given by:

\[
\mu(\vec{r}) = \sum_i \mu_i \delta(\vec{r} - \vec{r}_i) , \nu(\vec{r}) = \sum_i \nu_i \delta(\vec{r} - \vec{r}_i) .
\]

We then get:

\[
Z[\bar{N}, \bar{M}, \{\mu\}, \{\nu\}] = Z_f e^{-S_B} \langle \prod_i \exp \left( i \sqrt{g}(\nu_i + \frac{\theta}{2\pi} \mu_i) a(\vec{r}_i) - i \sqrt{g} \mu_i a_D(\vec{r}_i) \right) \rangle ,
\]

where

\[
Z_f \equiv \int \mathcal{D}a \mathcal{D}a_D e^{-S_f[a,a_D]}
\]

and \( \langle \rangle \) denotes the averaged value with respect to the “weight” \( \exp(-S_f) \).
In eq.(17) a non neutral correlator between vertices appears. In fact, the neutrality condition looks like:

\[
\bar{M} = - \sum_{i=1}^{N_p} \mu_i, \quad \bar{N} = - \sum_{i=1}^{N_p} \nu_i. \tag{18}
\]

In other words the background charges neutralize the Coulomb charges; therefore the “splitting” of the charge densities in an uniform and a “fluctuating” part has allowed us to describe a non neutral Coulomb gas which generalizes the usual Coulomb gas largely analyzed in the literature [6,9].

As a result of explicit RG analysis for this generalized Coulomb gas (at first order in the relevant parameters) we find that the IR stable fixed points correspond to the maximum value of the critical exponents as it should be.

Being \(\bar{M}\) the magnetic background and \(\bar{N}\) the electric one we assume that the most probable condensate of \(N_p\) particles is the one which corresponds to the maximum value of the critical exponent which, in the presence of magnetic and electric background, is given by:

\[
x(\nu, \mu) = 2 + g \left[ \nu(\bar{N} - \nu) + \frac{\theta}{2\pi} \mu(\bar{M} - \mu) \right] + \frac{1}{g} \mu(\bar{M} - \mu). \tag{19}
\]

Then we maximize the above exponent with respect to \(\mu\) and \(\nu\) by imposing the double constraint, given by eq.(18), obtaining:

\[
\mu_k = \bar{\mu} = -\frac{\bar{M}}{N_p} \quad \forall k = 1, \ldots, N_p \tag{20}
\]

\[
\nu_k = \bar{\nu} = -\frac{\bar{N}}{N_p} \quad \forall k = 1, \ldots, N_p
\]

We see that the charges of the condensate are the same for each particle and their values given above are fixed by the background only.

Furthermore it is easy to show that the model is invariant under the following discrete \(SL(2, Z)\) transformations defined in terms of the complex variable
\[ \zeta = 1/g + i\theta/2\pi \]

as:

\[ S : \zeta \to \zeta - i; \quad \bar{N} \to \bar{N} + \bar{M}, \quad \bar{M} \to \bar{M} \quad (21.a) \]

\[ T : \zeta \to -\frac{1}{\zeta}; \quad \bar{N} \to \bar{M}, \quad \bar{M} \to -\bar{N} \quad (21.b) \]

\[ C : \zeta \to \zeta^*; \quad \bar{N} \to -\bar{N}, \quad \bar{M} \to \bar{M} \quad . \quad (21.c) \]

The above infinite discrete symmetry \( SL(2,Z) \) allows us to generate all non trivial vacua (IR fixed points) starting from a given one. In fact the system defined by the background charges \( \bar{M}, \bar{N} \) transforms in a new one as given by eqs.(21.a,b,c). At this stage we can compare the previous discrete symmetry \( SL(2,Z) \) of our model with the phenomenological “laws of corresponding states” introduced in ref.[5]:

1) \( f \to f + 1 \)

2) \( 1/f \to 1/f + 2 \)

3) \( f \to 1 - f \) ( for \( f < 1 \) ),

where the filling factor \( f = N_e / N_s \) rewritten in our units is given by \( f = \bar{N} / \bar{M} \). We then easily prove that the above laws can be expressed in terms of the duality transformation, eqs.(21.a,b,c) as 1)\( S \), 2)\( TS^2T \), 3)\( SC \).

To be precise the above transformation laws generate only a subgroup of the duality transformations, which preserves both the oddness of the denominator in the filling \( f \) and the sign of the condensed electric and magnetic charges( [5]). All the previous properties of the model here proposed seem to generate the complete phase diagram of the Quantum Hall system and its hypothized universality properties( [3,5,10]).
Here we only briefly discuss some properties of the critical points postponing a detailed analysis to a subsequent paper.

Let us start by considering the IR fixed points previously found. We have proved that they are stable against CFT perturbations and define non trivial field theories, which have the following properties: for a subset of them the charges $Q_e, Q_m$, previously defined satisfy the “chirality” condition: $Q_e = Q_m, \nu = 0$, i.e. the field theory is chiral and describes the plateaux at filling $f = 1/\mu$, $\mu$ odd. Then the relevant vertex operators of the left (chiral) sector associated to these charges take the simple form

$$\hat{V}_\mu(z) = \exp(i \frac{\mu}{\sqrt{g}} \phi_L(z))$$  \hspace{1cm} (22)

when expressed in terms of a scalar field $\phi_L$ depending only on $z$ (1).

Furthermore for these plateaux it is known that $\sigma_H$ is a topological invariant which takes the values $\sigma_H = \frac{1}{\mu} = \frac{Q_e}{Q_m}$ if one imposes periodic boundary conditions(11).

Finally these points are described by a CFT, whose primary fields are the vertices of eq.(22) realized in terms of a chiral field $\phi_L(z)$ compactified on a radius $R$ such that $R^2 = \mu$, with central charge of Virasoro algebra $c$ equal to one.

The simplest IR non trivial fixed point, which corresponds to $f = 1$ is given by $(1/\bar{g} = 0, \bar{\theta}/2\pi = -1)$. It is not difficult to show that the renormalization group flow for the two parameters($g, \frac{\theta}{2\pi}$) defines a circle given by:

$$\left(\frac{1}{g} - \frac{1}{2}\right)^2 + \left(\frac{\theta}{2\pi} + 1\right)^2 = \frac{1}{4}$$  \hspace{1cm} (23)

which has radius equal to 1/2 and is tangent to the $1/g$ axis at $\bar{\theta}/2\pi = -1$. The interior of the circle defines a phase whose attractive point is the above one and where the condensate is a dyon with electric and magnetic charge equal
to unity and $\sigma_H = 1$. Then by using the duality transformations given by eqs. (21.a,b,c) we can get all the other non trivial IR fixed points associated to the fillings $\nu = p/\mu$ (with $p$ and $\mu$ prime factors). The $SL(2, Z)$ matrix which realizes that is determined by the finite fraction representation of the rational $p/\mu$ ([7]).

The same matrix maps circles into circles in such a way that the tangent points of two of them are unstable repulsive fixed points describing the physics of the transition region between plateaux.

Then all these fixed points are in the same universality class of the $c = 1$ CFT just mentioned; that is a crucial point concerning the universal character of the transition between two plateaux. In particular it would be very interesting to find the appropriate operator which, in Conformal Field Theory, drives such a transition. Then by the use of the well known techniques of 2D CFT we should be able to evaluate explicitly the critical exponents.

Finally, a clever use of the duality transformation laws (21a,b,c) should give us precise informations about the other phases (as for ex. the Hall insulator one) of our physical system.

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