A Tale of Two-Timescale Reinforcement Learning with the Tightest Finite-Time Bound

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Abstract
Policy evaluation in reinforcement learning is often conducted using two-timescale stochastic approximation, which results in various gradient temporal difference methods such as GTD(0), GTD2, and TDC. Here, we provide convergence rate bounds for this suite of algorithms. Algorithms such as these have two iterates, \( \theta_n \) and \( w_n \), which are updated using two distinct stepsize sequences, \( \alpha_n \) and \( \beta_n \), respectively. Assuming \( \alpha_n = n^{-\alpha} \) and \( \beta_n = n^{-\beta} \) with \( 1 > \alpha > \beta > 0 \), we show that, with high probability, the two iterates converge to their respective solutions \( \theta^* \) and \( w^* \) at rates given by \( \| \theta_n - \theta^* \| = O( n^{-\alpha/2} ) \) and \( \| w_n - w^* \| = O( n^{-\beta/2} ) \); here, \( O \) hides logarithmic terms. Via comparable lower bounds, we show that these bounds are, in fact, tight. To the best of our knowledge, ours is the first finite-time analysis which achieves these rates. While it was known that the two timescale components decouple asymptotically, our results depict this phenomenon more explicitly by showing that it in fact happens from some finite time onwards. Lastly, compared to existing works, our result applies to a broader family of stepsizes, including non-square summable ones.

1 Introduction
Stochastic Approximation (SA) (Kushner and Yin 1997) is the name given to algorithms useful for finding optimal points or zeros of a function for which only noisy access is available. This makes SA theory vital to machine learning and, specifically, to Reinforcement Learning (RL). Here, we obtain tight convergence rate estimates for the special class of linear two-timescale SA, which involves two interleaved update rules with distinct stepsize sequences. In the context of RL, the analysis here applies to policy evaluation schemes with function approximation.

A generic linear two-timescale SA has the form:

\[
\theta_{n+1} = \theta_n + \alpha_n [ h_1(\theta_n, w_n) + M_n^{(1)} ] ,
\]

\[
w_{n+1} = w_n + \beta_n [ h_2(\theta_n, w_n) + M_n^{(2)} ] ,
\]

where \( \alpha_n, \beta_n \in \mathbb{R} \) are stepsizes and \( M_n^{(i)} \in \mathbb{R}^d \) denotes noise. Further, \( h_i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) has the form

\[
h_i(\theta, w) = v_i - \Gamma_i \theta - W_i w
\]

for a vector \( v_i \in \mathbb{R}^d \) and matrices \( \Gamma_i, W_i \in \mathbb{R}^{d \times d} \).

Within RL, this class of algorithms mainly concerns the suite of gradient Temporal Difference (TD) methods, which was introduced in (Sutton, Maei, and Szepesvári 2009) and has gradually gained increasing attention since then. That work presented a gradient descent variant of TD(0), called GTD(0). As it supports off-policy learning, GTD(0) is advantageous over TD(0). More recently, additional variants were introduced such as GTD2 and TDC (Sutton et al. 2009); while being better than TD(0), these are also faster than GTD(0). The above gradient TD methods have been shown to converge asymptotically in the case of linear and non-linear function approximation (Sutton, Maei, and Szepesvári 2009; Sutton et al. 2009; Bhatnagar et al. 2009). Separately, there are also a few convergence rate results for altered versions of the GTD family (Liu et al. 2015) and sparsely-projected variants (Dalal et al. 2018b). Both works apply projections to keep the iterates in a confined region around the solutions. However, in (Liu et al. 2015), the learning rates are set to a fixed ratio which makes the altered algorithms single-timescale variants of the original ones.

To place our work in the landscape of the existing literature on generic two-timescale SA, we now briefly review a few seminal papers. The first well-known use of the two-timescale idea is the Polyak-Ruppert averaging scheme (Ruppert 1988; Polyak 1990). There, iterate averaging is used to improve the convergence rate of a one-timescale algorithm, which is especially beneficial when the driving matrices have poor conditioning. The general two-timescale SA scheme is formulated in (Borkar 1997); this work provided conditions for convergence. Since then, relatively little work has been published on the topic; the main results obtained so far include weak convergence and asymptotic convergence rates (Gerencsér 1997; Konda and Tsitsiklis 2004; Mokkadem and Pelletier 2006), and stability (Lakshminarayanan and Bhatnagar 2017).

We now discuss two specific papers from the above list that are the closest to our work. Denote by \( \theta^* \) and \( w^* \) the respective solutions of (1) and (2); i.e., \( h_1(\theta^*, w^*) = 0 \). In (Konda and Tsitsiklis 2004), it was shown that both, \( (\theta_n - \theta^*)/\sqrt{\alpha_n} \) and \( (w_n - w^*)/\sqrt{\beta_n} \), are asymptotically normal. This result surprisingly tells us that...
eventually the two components do not influence each other’s convergence rates. However, one of the assumptions there is that the noise sequence is independent of its past values, and their variance-covariance matrices are constant across the iterations. This makes their results inapplicable to the RL methods of our interest. In (Mokkadem and Pelletier 2006), a similar weak convergence result has been derived in the context of nonlinear SA under the assumptions that the step-sizes are square sum-able. This result also explicitly establishes asymptotic independence (see (5) there) between the two components. A separate result in this last work is that of almost-sure asymptotic convergence rate. The issue with this last result is that it cannot be used to obtain explicit form for the constants. In fact, by its very nature, the constants involved depend on the sample paths.

In this work, we revisit the convergence rate question for two-timescale RL methods with a focus on finite-time behaviour. In order to highlight the merits of this work over existing literature, we first classify common types of convergence results. The first class is of asymptotic convergence, which is beneficial for the rudimentary verification that an algorithm converges after an infinite amount of time. The second class is asymptotic convergence rates; these are stronger in the sense of telling us that an algorithm would asymptotically converge at a certain rate, but again they have little practical implications; even given exact knowledge of all parameters of the problem, with these results one cannot numerically compute a bound on the distance from the solution with a corresponding numerical probability value. The third class, to which the results in this work belong, are finite time bounds. These contain explicit constants — both controllable such as stepsize parameters and uncontrollable such as eigenvalues — as well as finite-time rates, thereby revealing intriguing dependencies among such parameters that crucially affect convergence rates (e.g., $1/q_i$; see Table 3). Moreover, the constants are trajectory-independent and thus can be of help in obtaining stopping time theorems. We consider this a significant step forward in obtaining practical results that would enable to assuredly adapt algorithm parameters so as to maximize their efficiency.

Our Contributions In (Dalal et al. 2018b), the first finite time bound for the GTD family was proved. Here, we significantly strengthen it and, in fact, obtain a tight rate. Specifically, our key result (Theorem 3) is that the iterates $\theta_n'$ and $w_n'$, obtained by sparsely projecting $\theta_n$ and $w_n$, respectively, satisfy $||\theta_n' - \theta^*|| = O(n^{-\alpha/2})$ and $||w_n' - w^*|| = O(n^{-\beta/2})$ with high probability. Here, $O$ hides logarithmic terms and $\alpha$ and $\beta$ originate in the stepsize choice $\alpha_n = n^{-\alpha}$ and $\beta_n = n^{-\beta}$ with $1 > \alpha > \beta > 0$. We establish the tightness of this upper bound by deriving a matching lower bound.

We emphasize that we have explicit formulas for the constants hidden in these order notations and also bounds on the iteration index from where these rates apply. In particular, our bound shows how the convergence rate of a given GTD method depends on the parameters of the MDP itself; e.g., the eigenvalues of the driving matrix.

As in (Dalal et al. 2018b) which dealt with single-timescale algorithms, the bounds in this work are applicable for both square-summable and non-square-summable step-sizes. This was indeed also the case in (Konda and Tsitsiklis 2004); however, as pointed earlier, the noise assumptions there are significantly stronger than ours.

The sparse projection scheme used here is novel but is similar in spirit to the one used in (Dalal et al. 2018b). There, the iterates were only projected when the iteration indices were powers of 2, whereas here we project whenever the iteration index is of the form $k^k = 2^{k\log_2 k} k \geq 0$. The motivation for using projections is to keep the iterates bounded. However, projections also modify the original algorithm by introducing non-linearity. This highly complicates the analysis. Evidently, the literature almost doesn’t contain analyses of projected algorithms at all. Moreover, projections are often empirically found to be unnecessary. The advantages of using a sparse projection scheme is that we effectively almost never project and, more importantly, it makes the analysis oblivious to its non-linearity.

An additional novelty of this paper is its proof technique. At its heart lie two induction tricks— one inspired from (Thoppe and Borkar 2019) and the other, being rather non-standard, from (Mokkadem and Pelletier 2006). The first induction is on the iteration index $n$; together with projections it enables us to show that both $\theta_n'$ and $w_n'$ iterates are $O(1)$, i.e., bounded, with high probability. On each sample path where the iterates are bounded, we then use the second induction to show that the convergence rate of the $w_n'$ iterates can be improved from $O(n^{-\beta/2}[\ell \neq 0] + n^{-(\alpha-\beta)\ell})$ to $O(n^{-\beta/2}[\ell \neq 0] + n^{-(\alpha-\beta)(\ell+1)})$ for all suitable $\ell$. In particular, we use this to show that the bound on the behaviour of $w_n'$ iterates can be incrementally improved from $O(1)$, established above, to the desired $O(n^{-\beta/2})$. Finally, we use this latter result to show that $||\theta_n' - \theta^*|| = O(n^{-\alpha/2})$.

We end this section by describing the key insights that our main result in Theorem 3 provides.

Decoupling after Finite Time: Even though both $\theta_n'$ and $w_n'$ influence each other, our result shows that, from some finite time onwards, their convergence rates do not depend on $\beta$ and $\alpha$, respectively. While from the results in (Konda and Tsitsiklis 2004) and (Mokkadem and Pelletier 2006), one would expect the two-timescale components to indeed decouple asymptotically, our result shows that this in fact happens from some finite time that can conceptually be numerically evaluated. All of this is in sharp contrast to the former state-of-the-art finite-time result given in (Dalal et al. 2018b) which showed that the convergence rate is $O(n^{-\min(\alpha-\beta/2)}).

One vs Two-Timescale: A natural question for an RL practitioner is whether to run the algorithm given in (1) and (2) in the one-timescale mode, i.e., with $\alpha_n/\beta_n$ being constant, or in the two-timescale mode, i.e., with $\alpha_n/\beta_n \to 0$. Judging solely on the convergence rate order – based on this work and on single-timescale results from, e.g., (Liu et al. 2015), the answer is to pick the single timescale mode with $\alpha_n/\beta_n = 1/n$ case above would bring the condition number of the driving matrices into picture (Dalal et al. 2018a). To overcome this, one could use Polyak-Ruppert iterate averaging for two-timescale SA (Mokkadem and Pelletier 2006).
\( \alpha_n = \beta_n \approx 1/n \). This then brings forth an imperative question for future work: “what indeed are the provable benefits of two-timescale RL methods?” A comparison to recent gradient descent literature suggests that this question can be better answered via iteration complexity, i.e., the number of iterations required to hit some \( \epsilon \)-ball around the solution. In particular, we believe the eigenvalues of the driving matrices — hiding in the constants — can have dramatic influence on the actual rate. A predominant recent example is how the heavy-ball method, which is similar in nature to a two-timescale algorithm, has an \( O(\sqrt{\ln(1/\epsilon)}) \) iteration complexity as compared to the usual stochastic gradient descent which has \( O(\kappa \ln(1/\epsilon)) \) (Loizou and Richtárik 2017); here, \( \kappa \) is the condition number. Thus, we believe that finite-time analyses of two-timescale methods are crucial for understanding their potential merits over one-single variants.

### 2 Main Result

We state our main convergence rate result here. It applies to the iterates \( \theta_n \) and \( w_n \) which are obtained by sparsely-projecting \( \theta_n \) and \( w_n \) from (1) and (2). We begin by stating our assumptions and defining the projection operator.

\( \mathcal{A}_1 \) (Matrix Assumptions). \( W_2 \) and \( X_1 = \Gamma_1 - W_1 W_2^{-1} \Gamma_2 \) are positive definite (not necessarily symmetric).

\( \mathcal{A}_2 \) (Stepsize Assumption). \( \alpha_n = (n+1)^{-\alpha} \) and \( \beta_n = (n+1)^{-\beta} \), where \( 1 > \alpha > \beta > 0 \).

**Definition 1** (Noise Condition). \( \{M_n^{(1)}\} \) and \( \{M_n^{(2)}\} \) are said to be \( (\theta_n, w_n) \)-dominated martingale differences with parameters \( \alpha \) and \( \beta \), if they are martingale difference sequences w.r.t. the family of \( \sigma \)-fields \( \{\mathcal{F}_n\} \), where \( \mathcal{F}_n = \sigma(\theta_0, w_0, M_1^{(1)}, M_2^{(2)}, \ldots, M_n^{(1)}, M_n^{(2)}) \), and

\[
\begin{align*}
\left\| M_{n+1}^{(1)} \right\| &\leq m_1(1 + \left\| \theta_n \right\| + \left\| w_n \right\|), \\
\left\| M_{n+1}^{(2)} \right\| &\leq m_2(1 + \left\| \theta_n \right\| + \left\| w_n \right\|)
\end{align*}
\]

for all \( n \geq 0 \).

**Definition 2** (Sparse Projection). For \( R > 0 \), let \( \Pi_R(x) = \min \{ 1, R/\|x\| \} \cdot x \) be the projection into the ball with radius \( R \) around the origin. The sparse projection operator

\[
\Pi_{n,R} = \begin{cases} 
\Pi_R, & \text{if } n = k^k - 1 \text{ for some } k \in \mathbb{Z}_{>0}, \\
\mathcal{I}, & \text{otherwise.}
\end{cases}
\]

We call it sparse as it projects only on specific indices that are exponentially far apart.

Pick an arbitrary \( p > 1 \). Fix some constants \( R_0^{\text{proj}} > 0 \) and \( R_0 \) for the radius of the projection balls. Further, let

\[
\theta^* = X_1^{-1} b_1, \quad w^* = W_2^{-1} (v_2 - \Gamma_2 \theta^*)
\]

with \( b_1 = v_1 - W_1 W_2^{-1} v_2 \). Using (Borkar 2009) and (Lakshminarayanan and Bhatnagar 2017), it can be shown that \( (\theta_n, w_n) \) → \( (\theta^*, w^*) \) a.s.

**Theorem 3** (Main Result). Assume \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \). Let \( \theta_0', w_0 \in \mathbb{R}^d \) be arbitrary. Consider the update rules

\[
\theta'_{n+1} = \Pi_{n+1,R_{n+1}^{\text{proj}}} \left( \theta'_n + \alpha_n [h_1 (\theta'_n, w'_n) + M_{n+1}^{(1)}] \right), \quad (5)
\]

\[
\text{and } w'_{n+1} = \Pi_{n+1,R_{n+1}^{\text{proj}}} \left( w'_n + \beta_n [k_2 (\theta'_n, w'_n) + M_{n+1}^{(2)}] \right), \quad (6)
\]

where \( \{M_n^{(1)}\} \) and \( \{M_n^{(2)}\} \) are \( (\theta'_n, w'_n) \)-dominated martingale differences with parameters \( m_1 \) and \( m_2 \) (see Def. 1).

Then, with probability larger than \( 1 - \delta \), for all \( n \geq N_3 \)

\[
\left\| \theta'_n - \theta^* \right\| \leq C_{3,30} \sqrt{\ln(4d^2(n+1)^p)/\delta} / (n+1)^{\alpha/2},
\]

\[
\left\| w'_n - w^* \right\| \leq C_{3,30} \sqrt{\ln(4d^2(n+1)^p)/\delta} / (n+1)^{\beta/2}.
\]

Refer to Tables 1 and 3 for the constants.

**Comments on Main Result**

1. Our analysis goes through even if \( \theta_n \in \mathbb{R}^{d_1} \), \( w_n \in \mathbb{R}^{d_2} \) with \( d_1 \neq d_2 \). For brevity, we work with \( d_1 = d_2 = d \).

2. The constants in the above result equal infinity when \( \alpha = \beta \). This is because the algorithm then ceases to be two-timescale, thereby making our analysis invalid.

### 2.1 Tightness

Here, we accompany our upper bound by a lower bound. This bound is asymptotic and holds for unprojected algorithms. Nonetheless, a coupling argument as in the proof of Theorem 3 can be used to obtain a similar bound for projected ones. We thus establish the tightness (up to logarithmic terms) of the result in the Theorem 3.

**Proposition 4** (Lower Bound). Assume \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \). Consider (1) and (2) with \( \{M_n^{(1)}\} \) and \( \{M_n^{(2)}\} \) being \( (\theta_n, w_n) \)-dominated martingale differences (see Def. 1). Then, there exists an algorithm for which

\[
\left\| \theta_n - \theta^* \right\| = \Omega_p(n^{-\alpha/2}) \quad \text{and} \quad \left\| w_n - w^* \right\| = \Omega_p(n^{-\beta/2}),
\]

where \( X_n = \Omega_p(\gamma_n) \) means that for any \( \epsilon > 0 \), there are constants \( c \) and \( K \) such that \( \mathbb{P}\{ |X_n| / \gamma_n < c \} \leq \epsilon, \forall n \geq K \).

**Proof.** See Appendix A.

### 3 Applications to Reinforcement Learning

Here, we apply our results on the general linear two-timescale setup to the specific RL use case. Namely, we apply Theorem 3 to derive the tightest existing finite sample bound for the GTD family. This section relies on a similar procedure as in Section 5, (Dalal et al. 2018b). Nonetheless, we reiterate it here for completeness.
3.1 Background

A Markov Decision Processes (MDP) is a tuple $(S, A, P, R, \gamma)$ (Sutton 1988), where $S$ is the state space, $A$ is the action space, $R$ is the transition kernel, and $\gamma$ the discount factor. A policy $\pi : S \rightarrow A$ is a stationary mapping from states to actions, and $V^*(s) = E^s(\sum_{n=0}^{\infty} \gamma^n r_n | s_0 = s)$ is the value function at state $s$ w.r.t. $\pi$.

As mentioned above, our results apply to GTD, which is a suite of policy evaluation algorithms. These algorithms are used to estimate the value function $V^*(s)$ with respect to a given $\pi$ using linear regression, i.e., $V^*(s) \approx \phi^T \theta(s)$, where $\phi(s) \in \mathbb{R}^d$ is a feature vector at state $s$, and $\theta \in \mathbb{R}^d$ is a parameter vector. For brevity, we omit the notation $\pi$ and denote $\phi(s_n), \phi_n \in \phi(s_n)$ by $\phi_n, \phi'_n$. Finally, let $\delta_n = r_n + \gamma \phi_n^T \phi_n - \phi_n^T \phi_n, \alpha = E[\phi \phi^T],$ and $\beta = E[\phi \phi^T]$, where the expectations are w.r.t. the stationary distribution of the induced chain $\pi$.

We assume all rewards $r(s)$ and feature vectors $\phi(s)$ are bounded: $|r(s)| \leq 1, ||\phi(s)|| \leq 1 \forall s \in S$. Also, it is assumed that the feature vector $\Phi$ is full, so $A$ and $C$ are full rank. This assumption is standard (Maei et al. 2010; Sutton, Maei, and Szepesvári 2009). Therefore, due to its structure, $A$ is also positive definite (Bertsekas 2012). Moreover, by construction, $C$ is positive semi-definite; thus, by the full-rank assumption, it is actually positive definite.

3.2 The GTD(0) Algorithm

First introduced in (Sutton, Maei, and Szepesvári 2009), GTD(0) is designed to minimize the objective function $J^\text{NEU}(\theta) = \frac{1}{2}(b - A\theta)\top(b - A\theta)$. Its update rule is

$$\theta_{n+1} = \theta_n + \alpha_n (\phi_{n+1} - \phi_n^T \phi_n) \phi_n^T w_n,$$

$$w_{n+1} = w_n + \beta_n \gamma \phi_{n+1}^T \phi_n - \phi_n^T \phi_n \phi_n \phi_{n+1}^T \theta_n.$$

It thus takes the form of (1) and (2) with $h_1(\theta, w) = A^\top w$, $h_2(\theta, w) = b - A\theta - w - M_{n-1}^{(1)} = (\phi_n - \phi_n^T \phi_n) \phi_n^T w_n - A^\top w_n$, $M_{n-1}^{(2)} = r_n \phi_n + \phi_n^T \phi_n - \phi_n^T \phi_n \phi_n \phi_{n+1}^T \theta_n - (b - A\theta_n)$. That is, in case of GTD(0), the relevant matrices in the update rules are $\Gamma_1 = 0, W_1 = -A^\top$, $\Gamma_2 = 0$, and $W_2 = I, \nu_2 = b$. Additionally, $X_1 = 1 - W_1 W_2^{-1} \Gamma_2 = -A^\top$. By our assumption above, both $W_2$ and $X_1$ are symmetric positive definite matrices, and thus the real parts of their eigenvalues are also positive. Also, $||M_{n+1}^{(1)}|| \leq (1 + \gamma + ||A||)||w_n||$, $||M_{n+1}^{(2)}|| \leq 1 + ||b|| + (1 + \gamma + ||A||)||\theta_n||$. Hence, the noise condition in Defn. 1 is satisfied with constants $m_1 = 1 + \gamma + ||A||$ and $m_2 = 1 + \max(||b||, \gamma + ||A||)$.

We can now prove Theorem 3 to get the following result.

**Corollary 5.** Consider the Sparsely Projected variant of GTD(0) as in (5) and (6). Then, for $\alpha_n = 1/((n+1)^3)$, $\beta_n = 1/((n+1)^5)$, with probability larger than $1 - \delta$, for all $n \geq N_3$, we have

$$||\theta_n - \theta^*|| \leq C_{3, \delta} \sqrt{(4d^2(n+1)^3/\beta)} (n+1)^5/\delta).$$

$$||w_n - \theta^*|| \leq C_{3, \delta} \sqrt{(4d^2(n+1)^3/\beta)} (n+1)^5/\delta).$$

For GTD2 and TDC (Sutton et al. 2009), the above result can be similarly reproduced. The detailed derivation and relevant constants are provided in Appendix K.

4 Outline of the Main Result

Here, we first state an intermediary result in Theorem 6 and using that we sketch a proof of Theorem 3. The full proof is in Appendix C.

**Assume $A_1$ and $A_2$.** Consider (1) and (2) with $M_{n}^{(1)}$ and $M_{n}^{(2)}$ being $\{\theta_n, w_n\}$-dominated martingale differences with parameters $m_1$ and $m_2$ (see Defn. 1). Let $G_{n}^{(t)}$ be the event given by

$$G_{n_0}^{(t)} = \{||\theta_n - \theta^*|| \leq R_0, ||w_n - w^*|| \leq R_0\}$$

and let $\nu(\gamma; \gamma, \gamma) = (1 + \gamma)^{-\gamma/2} \sqrt{\ln (4d^2(n+1)^3/\beta)}$.

**Theorem 6.** Let $\delta \in (0, 1)$. Suppose that $n_0 \geq N_5$ and that the event $G_{n_0}^{(t)}$ holds. Then, with probability larger than $1 - \delta$,

$$||\theta_n - \theta^*|| \leq A_5, \nu(\gamma, \alpha)$$

$$||w_n - w^*|| \leq A_4, \nu(\gamma, \beta)$$

for all $n \geq n_0$.

**Sketch of Proof for Theorem 3.** Our idea is to use a coupling argument to show that the projected iterates, given in (5) and (6), and the unprojected iterates, given in (1) and (2), are identically distributed from some time on. This then allows us to use Theorem 6 to conclude Theorem 3.

The key steps in our argument are as follows.

1. First we note that, for the projected algorithm, the event $G_{n_0}^{(t)}$ holds whenever $n_0$ is of the form $k^5 - 1$.
2. Further, recalling (4), we observe that, for any $k \geq 0$, between projection steps $k^5 - 1$ and $(k+1)^5 - 1$, the projected iterates $\{\theta_n, w_n\}$ behave exactly as the unprojected iterates $\{\theta_n, w_n\}$ that are initiated at $(\theta_{k-1}^t, w_{k-1}^t)$.
3. It then follows from Theorem 6 that if $k$ is large enough so that $n_0 = k^5 - 1 \geq N_6$, then (11) and (12) apply to $\{\theta_n, w_n\}$ and $\{\theta_n, w_n\}$ for $k^5 - 1 \leq n < (k+1)^5 - 1$.
4. In fact, if $k$ is enlarged a bit more so that $n_0 = k^5 - 1 \geq N_5 \geq N_6$, then not only does the above claim hold, it is also true that the RSHs in (11) and (12) are less than $R_0^{\theta}$ and $R_0^{w}$, respectively, for $n \geq (k+1)^5 - 1$.
5. In turn, the latter implies that the projected iterates and unprojected iterates, starting from $(\theta_n, w_n)$, behave exactly the same $\forall n \geq k^5 - 1$. Consequently, (11) and (12) hold for the projected iterates $\forall n \geq N_5$. Substituting $n_0 = N_5$ then establishes Theorem 3.

See Appendix C for the actual proof.
Next, we discuss the proof of Theorem 6; note that this result only concerns the unprojected iterates. First, we introduce some further notations.

Fix any $p > 1$ and let $\mathcal{U}(n_0)$ be the event given by

$$
\mathcal{U}(n_0) := \bigcap_{n \geq n_0} \left\{ \|\theta_n - \theta^*\| \leq C_R^\theta R_{\text{proj}}^\theta, \|L_n(\theta_n)\| \leq \epsilon_n'(\theta), \right. \\
\left. \|w_n - w^*\| \leq C_R^w R_{\text{proj}}^w, \|L_n(w_n)\| \leq \epsilon_n'(w) \} \right., \quad (13)
$$

where

$$
\epsilon_n' = \sqrt{d^3L_0^\theta C_{14,\theta} \nu(n, \alpha),} \quad (14)
$$

$$
\epsilon_n'' = \sqrt{d^3L_0^w C_{14,\theta} \nu(n, \beta).} \quad (15)
$$

Further, let $L_n(\theta)$ and $L_n(w)$ be appropriate aggregates of the martingale noise terms given by

$$
L_{n+1}(\theta) = \sum_{k=n_0}^n \left[ \prod_{j=k+1}^n [I - \beta_j W_{n+1}] \right] \beta_k M_{k+1}^{(2)}, \quad (16)
$$

$$
L_{n+1}(w) = \sum_{k=n_0}^n \left[ \prod_{j=k+1}^n [I - \alpha_j X_{n+1}] \right] \alpha_k \times \left[ -W_1 W_2^{-1} M_{k+1}^{(2)} + M_{k+1}^{(1)} \right]. \quad (17)
$$

For the definition of the constants above, see Table 3.

As a first step in proving Theorem 6, we show that the co-occurrence of the events $\mathcal{G}_{n_0}^\theta$ and $\mathcal{U}(n_0)$ has small probability if $n_0$ is large enough. The proof, inspired from (Thoppe and Borkar 2019), uses induction on the iteration index $n$. Specifically, we show that if, at time $n$, the iterates are bounded and the aggregate noise is well-behaved (respectively bounded by $\epsilon_n'(\theta)$ and $\epsilon_n''(w)$), then the iterates continue to remain bounded at time $n+1$ as well w.h.p.

**Theorem 7.** Let $\delta \in (0, 1)$ and $n_0 \geq N_7$. Then,

$$
P\{\mathcal{U}(n_0)|\mathcal{G}_{n_0}^\theta\} \leq \delta.
$$

Next, we show that, on the event $\mathcal{U}(n_0)$, the convergence rates of $\{\theta_n\}$ and $\{w_n\}$ are $O(n^{-\alpha/2})$ and $O(n^{-\beta/2})$, respectively. The proof proceeds as follows. By refining an induction trick from (Mokkadem and Pelletier 2006), we first show that the convergence rate estimate for the $\{w_n\}$ iterates can be improved from $O(1)$ to $O(n^{-\beta/2})$. Using this, we then show that $\|\theta_n - \theta^*\| = \tilde{O}(n^{-\alpha/2})$. We emphasize that these results are deterministic.

**Theorem 8.** Let $n_0 \geq N_8$. Then,

$$
\mathcal{U}(n_0) \subseteq \{\|w_n - w^*\| \leq A_{4,n_0} \nu(n; \beta), \forall n \geq n_0\} \quad (18)
$$

and

$$
\mathcal{U}(n_0) \subseteq \{\|\theta_n - \theta^*\| \leq A_{5,n_0} \nu(n; \alpha), \forall n \geq n_0\}. \quad (19)
$$

**Proof of Theorem 6.** Theorems 7 and 8 together establish Theorem 6.

The next two subsections highlight the key steps in the proofs of these last two results.

### 4.1 Proof of Theorem 7

Let $C_R^\theta = 3$ and $C_R^w = 3/2 + (\epsilon_n'/\epsilon_n)\|\Gamma_2\|_{C_{16,w}} C_R^{R_{\text{proj}}^\theta}$. Further, let $\mathcal{G}_n$, $\mathcal{L}_n$, and $\mathcal{A}_n$ be the events given by

$$
\mathcal{G}_n = \bigcap_{k=n_0}^n \{\|\theta_k - \theta^*\| \leq C_R^\theta R_{\text{proj}}^\theta, \|w_k - w^*\| \leq C_R^w R_{\text{proj}}^w \}, \quad (20)
$$

$$
\mathcal{L}_n = \bigcap_{k=n_0}^n \{\|L_k(\theta)\| \leq \epsilon_k'(\theta), \|L_k(w)\| \leq \epsilon_k''(w) \}, \quad (21)
$$

and $\mathcal{A}_n = \mathcal{G}_n \cap \mathcal{L}_n$. Using (13), note that $\mathcal{U}(n_0) = \lim_{n \to \infty} \mathcal{A}_n = \bigcap_{n \geq n_0} \mathcal{A}_n$. Finally, define

$$
\mathcal{Z}_n = \{\|\theta_n - \theta^*\| \leq C_R^\theta R_{\text{proj}}^\theta, \|w_n - w^*\| \leq C_R^w R_{\text{proj}}^w, \|L_n(\theta)\| \leq \epsilon_n'(\theta), \|L_n(w)\| \leq \epsilon_n''(w) \}. \quad (22)
$$

**Proof of Theorem 7.** By adopting ideas from (Thoppe and Borkar 2019), we first decompose the event $\mathcal{G}_{n_0}^\theta \cap \mathcal{U}_{n_0}^\theta$. From (149) - (161) in the appendix, we have

$$
\mathcal{G}_{n_0}^\theta \cap \mathcal{U}^\theta(n_0) = (\mathcal{G}_{n_0}^\theta \cap \mathcal{Z}_{n_0}^c) \cup (\mathcal{G}_{n_0}^\theta \cap \mathcal{A}_{n_0} \cap \mathcal{Z}_{n_0+1}^c) \cup (\mathcal{G}_{n_0}^\theta \cap \mathcal{A}_{n_0+1} \cap \mathcal{Z}_{n_0+2}^c) \cup \ldots, \quad (23)
$$

and

$$
\mathcal{G}_{n_0}^\theta \cap \mathcal{A}_{n_0} \cap \mathcal{Z}_{n_0+1}^c \quad (25)
$$

$$
\subseteq \mathcal{G}_{n_0}^\theta \cap \mathcal{A}_{n_0} \cap \bigg\{\|\theta_{n+1} - \theta^*\| > C_R^\theta R_{\text{proj}}^\theta \bigg\} \cup \bigg\{\|w_{n+1} - w^*\| > C_R^w R_{\text{proj}}^w \bigg\} \quad (26)
$$

$$
\cup \bigg\{\mathcal{G}_{n+1} \cap \bigg\{\|L_{n+2}(\theta)\| > \epsilon_{n+1}'(\theta) \bigg\} \cup \bigg\{\|L_{n+2}(w)\| > \epsilon_{n+1}''(w) \bigg\} \bigg\}. \quad (27)
$$

With regards to (27), we also have the following fact.

**Lemma 9.** Let $n \geq n_0 \geq \max\{K_{15,\alpha}, K_{15,\beta}, K_{20,\alpha}(0), K_{21,\beta}, K_9\}$. Then,

$$
\mathcal{G}_{n_0}^\theta \cap \mathcal{A}_{n_0} \cap \bigg\{\|\theta_{n+1} - \theta^*\| > C_R^\theta R_{\text{proj}}^\theta \bigg\} \cup \bigg\{\|w_{n+1} - w^*\| > C_R^w R_{\text{proj}}^w \bigg\} \cap \bigg\{\|L_{n+2}(\theta)\| > \epsilon_{n+1}'(\theta) \bigg\} \cup \bigg\{\|L_{n+2}(w)\| > \epsilon_{n+1}''(w) \bigg\} = \emptyset.
$$

**Proof.** See Appendix F.

Therefore, it follows that for $n \geq n_0$

$$
\mathcal{G}_{n_0}^\theta \cap \mathcal{A}_{n_0} \cap \mathcal{Z}_{n_0+1} \subseteq \mathcal{G}_{n+1} \cap \bigg\{\|L_{n+2}(\theta)\| > \epsilon_{n+1}'(\theta) \bigg\} \cup \bigg\{\|L_{n+2}(w)\| > \epsilon_{n+1}''(w) \bigg\}. \quad (28)
$$

Equations (23), (24) and (28) together imply
\[ G'_n \cap U^{(n)} \subseteq \bigcup_{n \geq n_0} \left( G_n \cap \{|L_{n+1}^{(w)}| > e_n^{(w)}\} \right. \]
\[ \left. \cup \{|L_{n+1}^{(u)}| > e_n^{(u)}\} \right) \quad (29) \]

The usefulness of this decomposition lies in the fact that each term in the union contains the event \( G_n \) which ensures that the iterates are bounded. This, along with our noise assumption in Definition 1, implies that the Martingale differences are in turn bounded and the Azuma-Hoeffding inequality can now be invoked (see Lemma 29). Applying this on (29) after using the union bound gives

\[ P\{U^{(n)} | G'_n \} = P\{U^{(n)} \cap G'_n \} \cdot (30) \]
\[ \leq \sum_{n \geq n_0} P\left( G_n \cap \{|L_{n+1}^{(u)}| > e_n^{(w)}\} | G'_n \right) \quad (31) \]
\[ + \sum_{n \geq n_0} 2d^2 \exp \left( -\frac{(u^{(w)})^2}{d^2 L_{n+1}^{(w)}} \right) \quad (32) \]
\[ \leq \sum_{n \geq n_0} 2d^2 \exp \left( -\frac{(u^{(w)})^2}{d^2 L_{n+1}^{(w)}} \right) \quad (33) \]

Additionally, due to Lemma 14 in the Appendix,

\[ a_{n+1} \leq C_{14,a}(n+1)^{-\alpha}, \]
\[ b_{n+1} \leq C_{14,b}(n+1)^{-\beta}. \quad (34) \]

Substituting (34) and (14) in (33) gives

\[ P\left\{ G'_n \cap U^{(n)} \right\} \leq \sum_{n \geq n_0} \frac{\delta}{(n+1)^p} \leq \delta \frac{n_0^{-(p-1)}}{p-1}. \quad (35) \]

Now, since

\[ n_0 \geq \frac{(p-1)}{\delta} \quad (36) \]

it eventually follows that (35) \( \leq \delta \), as desired.

\[ \square \]

### 4.2 Proof of Theorem 8

For a sequence \( u \in \mathbb{R}_+^\infty \), let

\[ \mathcal{W}_n(u) := \{|w_k - w^*| \leq u_k \forall n_0 \leq k \leq n\}. \quad (37) \]

**Definition 10.** We say that \( u \in \mathbb{R}_+^\infty \) is \( \alpha \)-moderate from \( k_0 \) onwards if

\[ \frac{u_{k+1}}{u_{k}} \leq \frac{\alpha_{k+1}}{\alpha_{k}} \frac{\beta_{k+1}}{\beta_{k}} e^{\alpha_{k+1} / 2} \quad (\forall k \geq k_0). \]

**Definition 11.** We say that \( u \in \mathbb{R}_+^\infty \) is \( \beta \)-moderate from \( k_0 \) onwards if

\[ \frac{u_{k+1}}{u_{k}} \leq \frac{\alpha_{k+1}}{\alpha_{k}} \frac{\beta_{k+1}}{\beta_{k}} e^{\alpha_{k+1} / 2} \quad (\forall k \geq k_0). \]

We consider these definitions to be part of the novelty of this work. They characterize a sequence via the ratio of its consecutive terms. Ratios in a decaying sequence (such as the ones used in this paper) satisfying Defs. 10 or 11 will converge to 1. Examples of sequences satisfying these definitions are constant sequences and those that decay at an inverse polynomial rate. On the other hand, sequences that decay exponentially fast do not satisfy these conditions. These definitions play a crucial role in enabling our induction; i.e., they help us show that the estimates on the rate of convergence of \( |w_n - w^*| \) can be incrementally improved. One quick way to see this is via (43) given later; it shows that if the bound on \( |w_n - w^*| \) was \( u_n \), then it can be improved via induction to \( O(u_n) + O(u_{n_0}^{\alpha}) \). These definitions are motivated by Definitions 1 and 2 in (Mokkadem and Pelletier 2006). However, there they are expressed as a certain asymptotic behavior, while ours provide the exact sequence, including constants, and thereby enable finite time analysis.

For \( \ell \geq 0 \), let \( \mathcal{E}(n_0, \ell) := \cap_{n \geq n_0} \{ |w_n - w^*| \leq u_n(\ell) \} \), where

\[ u_n(\ell) := \left[ A_{1,n_0} \sum_{i=0}^{\ell} A_{2,i} + A_{3,2,1} \frac{\alpha_n}{\beta_n} \right]^\ell \quad (38) \]

all the constants are given in Table 3.

**Proof of Theorem 8.** Our proof idea inspired by (Mokkadem and Pelletier 2006) is as follows. We use induction to show that whenever \( U^{(n)} \) holds, the rate of convergence of \( w_n \) is bounded by (38) for all \( \ell \leq \ell^* \), where the latter is as in (39). Notice that there are two terms in (38) that depend on \( n \), one is \( \alpha_n \) and the other is \( \alpha_n / \beta_n \). As \( \ell \) increases, \( (\alpha_n / \beta_n)^\ell \) decays faster. Thus, eventually, for \( \ell = \ell^* \), the convergence rate of \( w_n \) would be dictated by \( \epsilon_n \), thereby giving us our desired result.

Formally, we begin with proving the following claim.

**Claim:** Let

\[ \ell^* = \left[ \frac{\beta}{2(\alpha - \beta)} \right]; \quad (39) \]

i.e., let \( \ell^* \) be the smallest integer \( \ell \) such that \( (\alpha - \beta)\ell \geq \beta / 2 \). Then, for \( 0 \leq \ell \leq \ell^* \),

\[ U(n_0) \subseteq \mathcal{E}(n_0, \ell). \quad (40) \]

**Induction Base:** By definition, \( U(n_0) \subseteq \mathcal{E}(n_0, 0) \).

**Induction Hypothesis:** Suppose (40) holds for some \( \ell \) such that \( 0 \leq \ell < \ell^* \).

**Induction Step:** For the \( \ell \) defined in the hypothesis above, we have \( (\alpha - \beta)\ell \leq \beta / 2 \). Making use of this, we now show that \( U(n_0) \subseteq \mathcal{E}(n_0, \ell + 1) \).

From the induction hypothesis, on \( \mathcal{U}(n_0) \), for \( n \geq n_0 - 1 \),

\[ |w_n - w^*| \leq u_{n+1}(\ell). \quad (41) \]

A useful result for improving this bound is the following.

**Lemma 12.** Let \( n_0 \in \mathbb{N} \). Let \( u \in \mathbb{R}_+^\infty \) be a monotonically decreasing sequence that is both \( \alpha \)-moderate and \( \beta \)-moderate from \( n_0 - 1 \) onwards. Let \( n \geq n_0 - 1 \). Suppose that the event \( \mathcal{W}_n(u) \) holds, \( |L_n^{(w)}| \mathbb{I}[n \geq n_0 + 1] \leq e_n^{(w)} \), and \( |L_n^{(u)}| \mathbb{I}[n \geq n_0 + 1] \leq e_n^{(u)} \). If \( n \geq n_0 \), then

\[ |\theta_n - \theta^*| \leq C_{32,6} \frac{\alpha_n - 1}{\beta_n - 1} e_n^{(1)} \quad (42) \]
+ C_{32,a} \left[ \| \theta_{n_0} - \theta^* \| + \frac{\alpha_n}{\beta_n} \| w_{n_0} - w^* \| \right] e^{-q_1 \sum_{j=n_0+1}^{n-1} \alpha_j}.

Additionally, if \( n_0 \geq \max\{K_{30,a}, K_{30,b}, K_{35,a}, K_{35,b}, K_{20,a}(\beta/2)\} + 1 \) and \( n \geq n_0 - 1 \), then
\[
\| w_{n+1} - w^* \| \leq A_1, n_0 \varepsilon_{n+1}^{(w)} + A_2 \frac{\alpha_{n+1}}{\beta_{n+1}} u_{n+1}. \tag{43}
\]

All the constants are as in Table 3.

**Proof.** See Appendix H.

We now verify the conditions necessary to apply this result. After substituting the value of \( \varepsilon_{n}^{(w)} \) from (14), and those of \( \alpha_n, \beta_n \) into (38), and then pulling out \( p \) from (14) to the constants, observe that \( u_n(\ell) \) is of the form
\[
u_n(\ell) = B_1 (n + 1)^{-\beta/2} \left( \mathbb{E}[B_2(n+1)] \right) + B_3 (n + 1)^{-\alpha - \beta \ell} \tag{44}
\]
for some suitable constants \( B_1, B_2 \) and \( B_3 \). Clearly, \( B_1 \) and \( B_3 \) are strictly positive, while \( B_2 = \left( 4d^2/\delta \right)^{1/p} \geq 1 \). Lemma 34 then shows \( \{ u_n(\ell) \} \) is \( \alpha \)-moderate, \( \beta \)-moderate, and monotonically decreasing from \( n - 0 \) onwards.

Additionally, notice that due to (41) the event \( W_n(u) \) holds for \( u = \{ u_n(\ell) \} \), while on \( \mathcal{U}(n_0) \) the events \( \{ \| L_n^{(\ell)} \| I[i \geq n_0 + 1] \leq \varepsilon_{n-1}^{(\ell)} \} \) and \( \{ \| L_n^{(\ell)} \| I[i \geq n_0 + 1] \leq \varepsilon_{n}^{(w)} \} \) hold. Hence, \( n_0 \geq N_8 \geq \max\{ K_{30,a}, K_{30,b}, K_{35,a}, K_{35,b}, K_{20,a}(\beta/2) \} + 1 \), we can now employ Lemma 12 with \( \{ u_n \} = \{ u_n(\ell) \} \) and obtain that, on the event \( \mathcal{U}(n_0) \),
\[
\| w_{n+1} - w^* \| \leq A_1, n_0 \varepsilon_{n+1}^{(w)} + A_2 \frac{\alpha_{n+1}}{\beta_{n+1}} u_{n+1}(\ell). \tag{45}
\]

By substituting the value of \( \varepsilon_{n+1}(\ell) \) from (38) and making use of the fact that \( \alpha_n / \beta_n \leq 1 \), we get
\[
u_{n+1}(\ell) = A_1, n_0 \varepsilon_{n+1}^{(w)} + A_2 \frac{\alpha_{n+1}}{\beta_{n+1}} u_{n+1}(\ell) \leq u_{n+1}(\ell + 1). \tag{46}
\]

This completes the proof of the induction step.

When \( \ell = \ell^* \), it now follows that \( \mathcal{U}(n_0) \subseteq \mathcal{E}(n_0; \ell^*) \).

That is, when the event \( \mathcal{U}(n_0) \) holds,
\[
\| w_{n+1} - w^* \| \leq u_{n+1}(\ell^*), \quad \forall n \geq n_0 - 1.
\]

We now bound \( u_n(\ell^*) \). Since \( \left[ \frac{\beta}{2(\alpha - \beta)} \right] \geq \frac{\beta}{2(\alpha - \beta)} \), we have
\[
\left( \alpha_n / \beta_n \right) \left[ \frac{\beta}{2(\alpha - \beta)} \right] \leq (n + 1)^{-\beta/2}. \tag{47}
\]
Substituting the value of \( \varepsilon_{n}^{(w)} \) and using the above relation along with the fact that \( 4 \geq e \) which implies \( \sqrt{\ln (4d^2(n+1)^2/p)} / \delta \geq 1 \), we have
\[
u_{n}(\ell^*) \leq \left[ A_{1,n_0} \sum_{i=0}^{\left[ \frac{\beta}{2(\alpha - \beta)} \right]} \right]^{-1} A_1^i \sqrt{d^3 L_n C_{14,w}} + A_2 A_2 \left[ \frac{\beta}{2(\alpha - \beta)} \right] \nu(n; \beta). \tag{48}
\]

Consequently, for \( n \geq n_0 - 1 \),
\[
\| w_{n+1} - w^* \| \leq u_{n+1}(\ell^*) \leq A_{4,n_0} \nu(n + 1; \beta) \tag{49}
\]

which establishes (18).

We now prove (19). On the event \( \mathcal{U}(n_0) \), we can apply (42) from Lemma 12 with \( \{ u_n \} = \{ u_n(\ell^*) \} \) and use the fact that \( \alpha_{n_0} / \beta_{n_0} \leq 1 \), as well as bound \( \| \theta_{n_0} - \theta^* \| \) and \( \| w_{n_0} - w^* \| \) using \( \mathcal{U}(n_0) \), to get
\[
\| \theta_n - \theta^* \| \leq C_{32,b} \frac{\alpha_{n-1}}{\beta_{n-1}} u_{n-1}(\ell^*) + C_{32,a} \left[ C_{R} R_{\text{proj}}^\theta + C_{R} R_{\text{proj}}^w \right] e^{-q_1 \sum_{j=n_0+1}^{n-1} \alpha_j + \varepsilon_{n-1}^{(\ell)}}. \tag{50}
\]

Now, Lemma 35 (see Appendix J) and the fact that \( q_1 \geq q_{\min} \) imply (in Lemma 35 we require \( n \geq n_0 \) but here we use it from \( n_0 - 1 \), which is justified since \( n_0 \geq K_{35,b} + 1 \), on \( \mathcal{U}(n_0) \),
\[
\| \theta_n - \theta^* \| \leq C_{32,b} \frac{\alpha_{n-1}}{\beta_{n-1}} u_{n-1}(\ell^*) + C_{32,a} \left[ C_{R} R_{\text{proj}}^\theta + C_{R} R_{\text{proj}}^w \right] / \varepsilon_{n_0-1} + 1 \varepsilon_{n-1}^{(\ell)}.
\]

Consequently, using (14), (46) and the facts that \( \alpha_{n-1} / \beta_{n-1} = n^{-\alpha / \beta} \) and \( \alpha / 2 = \alpha - \beta / 2 \), we have that, on \( \mathcal{U}(n_0) \),
\[
\| \theta_n - \theta^* \| \leq C_{32,b} \left[ A_{4,n_0} \nu(n - 1, \alpha) \right] + C_{32,a} \left[ C_{R} R_{\text{proj}}^\theta + C_{R} R_{\text{proj}}^w \right] / \varepsilon_{n_0-1} + 1 \varepsilon_{n-1}^{(\ell)}.
\]

Since \( \nu(n - 1, \alpha) \leq 2 \nu(n, \alpha) \), the theorem follows.

## 5 Discussion

Two-timescale SA lies at the foundation of RL in the shape of several popular evaluation and control methods. This work introduces the tightest finite sample analysis for the GTD algorithm suite. We provide it as a general methodology that applies to all linear two-timescale SA algorithms.

Extending our methodology to the case of GTD algorithms with non-linear function-approximation, in similar fashion to (Bhatnagar et al. 2009), would be a natural future direction to consider. Such a result could be of high interest due to the attractiveness of neural networks. Finite time analysis of non-linear SA would also be of use in better understanding actor-critic RL algorithms. An additional direction for future research could be finite sample analysis of distributed SA algorithms of the kind discussed in (Mathkar and Borkar 2016).

Lastly, it would also be interesting to see how adaptive stepsizes can help improve sample complexity in all the above scenarios.
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A Proof of Proposition 4: Lower Bound from the CLT

We first introduce the following necessary assumption.

\( \mathcal{A}_3 \) ((Mokkadem and Pelletier 2006)[Assumption (A4)(ii)]) There exists a positive definite matrix \( \Gamma \) such that

\[
\lim_{n \to \infty} \mathbb{E} \left( \begin{bmatrix} M_{n+1}^{(1)} \\ M_{n+1}^{(2)} \end{bmatrix} \begin{bmatrix} M_{n+1}^{(1) \top} \\ M_{n+1}^{(2) \top} \end{bmatrix} \right) = \Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}.
\] (47)

**Theorem 13.** Assume \( \mathcal{A}_1, \mathcal{A}_2, \) and \( \mathcal{A}_3. \) Consider (1) and (2) with \( \{M_n^{(1)}\} \) and \( \{M_n^{(2)}\} \) being \( \mathbb{R}^d \)-valued \( (\theta_n, w_n) \)-dominated martingale differences with parameters \( m_1 \) and \( m_2 \) (see Def. 1). Then,

\[
\|\theta_n - \theta^*\| = \Omega_p(n^{-\alpha/2}) \quad \text{and} \quad \|w_n - w^*\| = \Omega_p(n^{-\beta/2}),
\] (48)

where \( X_n = \Omega_p(\gamma_n) \) means that for every \( \epsilon > 0 \), there are constants \( c \) and \( K \) such that \( \mathbb{P}\{|X_n|/\gamma_n < c\} \leq \epsilon, \forall n \geq K. \) As a consequence, for any \( C \in (0, \infty) \) and positive sequence \( \{g_n\} \) s.t. \( \lim_{n \to \infty} g_n = 0, \)

\[
\lim_{n \to \infty} \mathbb{P}\left\{ \|\theta_n - \theta^*\| \leq Cn^{-\alpha/2}g_n \right\} = 0.
\] (49)

A similar expression holds for \( \|w_n - w^*\|. \)

**Proof.** The CLT in (Mokkadem and Pelletier 2006)[Theorem 1] shows that

\[
n^{\alpha/2}(\theta_n - \theta^*) \Rightarrow N(0, \Sigma_{\theta}),
\] (50)

\[
n^{\beta/2}(w_n - w^*) \Rightarrow N(0, \Sigma_w)
\] (51)

for some covariance matrices \( \Sigma_{\theta} \) and \( \Sigma_w. \)

Let \( \epsilon > 0. \) For any \( c > 0, \) we have

\[
\mathbb{P}\{n^{\alpha/2}|\theta_n - \theta^*| < c\} \leq u(c) + v_n(c),
\] (52)

where

\[
u(c) = \mathbb{P}\{\|N(0, \Sigma_{\theta})\| < c\}
\] (53)

and

\[
v_n(c) = |\mathbb{P}\{\|N(0, \Sigma_{\theta})\| < c\} - \mathbb{P}\{n^{\alpha/2}|\theta_n - \theta^*| < c\}|.
\] (54)

Pick \( c \equiv c_\epsilon \) so that \( u(c) \leq \epsilon/2. \) For this choice of \( c, \) pick \( K \) so that \( v_n(c) \leq \epsilon/2 \) for all \( n \geq K; \) such a choice is possible because of (50) and the fact that \( \|\cdot\| \) is continuous. From this, we can conclude that that \( \|\theta_n - \theta^*\| = \Omega_p(n^{-\alpha/2}), \) as desired.

Let \( C \) and \( \{g_n\} \) be as in (49). Then, for any given \( c > 0, \)

\[
\{n^{\alpha/2}|\theta_n - \theta^*| \leq Cg_n\} \subseteq \{n^{\alpha/2}|\theta_n - \theta^*| \leq c\}
\] (55)

for all sufficiently large \( n. \) From this, it is easy to see that (49) holds.

The statements on \( \{w_n\} \) can be proved similarly.

It remains to show that assumptions (A1)-(A4) in (Mokkadem and Pelletier 2006)[Section 2.1] hold in our setting as well; we do this now.

1. To show (A1), we first establish the stability of the iterates, i.e., \( \sup_n (\|\theta_n\| + \|w_n\|) \) is finite. For that, we employ (Lakshminarayan and Bhatacharia 2017)[Theorem 10] (whose conditions A1-A5 in that work can be easily verified). By invoking (Borkar 2009)[Theorem 6.2], one can then see that both, \( \{\theta_n\} \) and \( \{w_n\}, \) converge.

2. Since

\[
\begin{bmatrix} \Gamma_1 & W_1 \\ \Gamma_2 & W_2 \end{bmatrix} \begin{bmatrix} \theta^* \\ w^* \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},
\] (56)

we have that

\[
h_i(\theta, w) = v_i - \Gamma_i\theta - W_iw = \Gamma_i\theta^* + W_iw^* - \Gamma_i\theta - W_iw = -\Gamma_i(\theta - \theta^*) - W_i(w - w^*).
\] (57)

This establishes (A2).

3. (A3) holds due to \( \mathcal{A}_1 \) in this work.

4. Lastly, in (A4), (i) holds by definition of \( M_{n+1}^{(1)}, M_{n+1}^{(2)} \) (Def. 1); (ii) holds due to \( \mathcal{A}_4; \) (iii) holds due to Def. 1 and the stability condition in Item 1 above; and (iv) holds since \( r_n^{(\theta)} = 0, r_n^{(w)} = 0 \) in our linear case.

\[\square\]

It is now easy to see that Proposition 4 is a consequence of Theorem 13 where the algorithm of choice is one which, in addition to \( \mathcal{A}_1 \) and \( \mathcal{A}_2, \) satisfies \( \mathcal{A}_3. \)
B Preliminaries

B.1 Algebraic Manipulations

Using some easy manipulation on (2) and (3), we get

\[ w_n = -W_2^{-1} \left[ \frac{w_{n+1} - w_n}{\beta_n} \right] + W_2^{-1} [v_2 - \Gamma_2 \theta_n + M_{n+1}^{(2)}] \]  \hspace{1cm} (58)

Substituting this in (1) gives

\[ \theta_{n+1} = \theta_n + \alpha_n \left[ v_1 - \Gamma_1 \theta_n + W_1 W_2^{-1} \left[ \frac{w_{n+1} - w_n}{\beta_n} \right] - W_1 W_2^{-1} [v_2 - \Gamma_2 \theta_n] - W_1 W_2^{-1} M_{n+1}^{(2)} + M_{n+1}^{(1)} \right]. \]  \hspace{1cm} (59)

Recall now from Section 2 that

\[ b_1 = v_1 - W_1 W_2^{-1} v_2 \]  \hspace{1cm} (60)

and

\[ X_1 = \Gamma_1 - W_1 W_2^{-1} \Gamma_2. \]  \hspace{1cm} (61)

Therefore,

\[ \theta_{n+1} = \theta_n + \alpha_n \left[ b_1 - X_1 \theta_n \right] + \alpha_n \left[ W_1 W_2^{-1} \left[ \frac{w_{n+1} - w_n}{\beta_n} \right] \right] + \alpha_n \left[ -W_1 W_2^{-1} M_{n+1}^{(2)} + M_{n+1}^{(1)} \right]. \]  \hspace{1cm} (62)

Next, let

\[ r_n = W_1 W_2^{-1} \left[ \frac{w_{n+1} - w_n}{\beta_n} \right] - W_1 W_2^{-1} M_{n+1}^{(2)} + M_{n+1}^{(1)} \]  \hspace{1cm} (63)

and

\[ \theta^* = X_1^{-1} b_1. \]  \hspace{1cm} (64)

Then we can rewrite (62) as

\[ \theta_{n+1} - \theta^* = \theta_n - \theta^* + \alpha_n \left[ -X_1 (\theta_n - \theta^*) + r_n \right] = (I - \alpha_n X_1) (\theta_n - \theta^*) + \alpha_n r_n. \]  \hspace{1cm} (65)

Rolling out the iterates gives

\[ \theta_{n+1} - \theta^* = \prod_{k=n_0}^{n} (I - \alpha_k X_1) (\theta_{n_0} - \theta^*) + \sum_{k=n_0}^{n} \left[ \prod_{j=k+1}^{n} (I - \alpha_j X_1) \right] \alpha_k r_k. \]  \hspace{1cm} (66)

Similarly, recall that

\[ w^* = W_2^{-1} (v_2 - \Gamma_2 \theta^*). \]  \hspace{1cm} (67)

It is easy to see from (2) that

\[ w_{n+1} - w^* = w_n - w^* + \beta_n [W_2 w^* + \Gamma_2 \theta^* - \Gamma_2 \theta_n - W_2 w_n + M_{n+1}^{(2)}]. \]  \hspace{1cm} (68)

This implies that

\[ w_{n+1} - w^* = (I - \beta_n W_2) (w_n - w^*) + \beta_n (\theta_n - \theta^n) + M_{n+1}^{(2)}. \]  \hspace{1cm} (69)

Setting

\[ s_n = [-\Gamma_2 (\theta_n - \theta^n) + M_{n+1}^{(2)}] \]  \hspace{1cm} (70)

and rolling out the iterates gives

\[ w_{n+1} - w^* = \prod_{k=n_0}^{n} (I - \beta_k W_2) (w_{n_0} - w^*) + \sum_{k=n_0}^{n} \left[ \prod_{j=k+1}^{n} (I - \beta_j W_2) \right] \beta_k s_k. \]  \hspace{1cm} (71)
B.2 Definitions
Recall from (17) that

\[ L_{n+1}^{(\theta)} = \sum_{k=n_0}^{n} \prod_{j=k+1}^{n} [I - \alpha_j X_1] \alpha_k \left[-W_1^2^{-1} M_{k+1}^{(2)} + M_{k+1}^{(1)}\right], \]  

and define

\[ \Delta_{n+1}^{(\theta)} = \prod_{k=n_0}^{n} (I - \alpha_k X_1)(\theta_{n_0} - \theta^*), \]  

\[ R_{n+1}^{(\theta)} = \sum_{k=n_0}^{n} \prod_{j=k+1}^{n} [I - \alpha_j X_1] \alpha_k \left[W_1^2^{-1} \left[\frac{w_{k+1} \theta - w_k}{\beta_k}\right]\right]. \]  

Then, based on (70) and (71),

\[ \theta_{n+1} - \theta^* = \Delta_{n+1}^{(\theta)} + L_{n+1}^{(\theta)} + R_{n+1}^{(\theta)}. \]  

Similarly, recall from (16) that

\[ L_{n+1}^{(w)} = \sum_{k=n_0}^{n} \prod_{j=k+1}^{n} [I - \beta_j X_1] \beta_k M_{k+1}^{(2)}, \]  

and define

\[ \Delta_{n+1}^{(w)} = \prod_{k=n_0}^{n} (I - \beta_k W_2)(w_{n_0} - w^*), \]  

\[ R_{n+1}^{(w)} = -\sum_{k=n_0}^{n} \prod_{j=k+1}^{n} [I - \beta_j W_2] \beta_k \left[\Gamma_2(\theta_k - \theta^*)\right]. \]  

Then, based on (70) and (71),

\[ w_{n+1} - w^* = \Delta_{n+1}^{(w)} + L_{n+1}^{(w)} + R_{n+1}^{(w)}. \]  

Lastly, notice that (74) can also be written as

\[ R_{n+1}^{(\theta)} = \sum_{k=n_0}^{n} \left(\prod_{j=k+1}^{n} [I - \alpha_j X_1]\right) \alpha_k W_1^2 W_2^{-1} \left[\frac{(w_{k+1} - w^*)}{\beta_k}\right]. \]  

From this, we have

\[ R_{n+1}^{(\theta)} = \frac{\alpha_n}{\beta_n} W_1^2 W_2^{-1}(w_{n+1} - w^*) - \left(\prod_{j=n_0+1}^{n} [I - \alpha_j X_1]\right) \frac{\alpha_{n_0}}{\beta_{n_0}} W_1^2 W_2^{-1}(w_{n_0} - w^*) - T_{n+1}, \]  

where

\[ T_{n+1} := \sum_{k=n_0+1}^{n} \left(\prod_{j=k+1}^{n} [I - \alpha_j X_1]\right) \left[\frac{\alpha_k}{\beta_k} I - \frac{\alpha_{k-1}}{\beta_{k-1}} (I - \alpha_k X_1)\right] W_1^2 W_2^{-1}(w_k - w^*). \]  

B.3 Technical Results

Lemma 14. Let \( p \in (0, 1) \) and \( \hat{q} > 0 \). Let \( K_{14} = K_{14}(p, \hat{q}) \geq 1 \) be such that

\[ e^{-\hat{q} \sum_{k=1}^{n-1} (k+1)^{-p}} \leq n^{-p} \]

for all \( n \geq K_{14} \); such an \( K_{14} \) exists as the \( \text{l.h.s.} \) is exponentially decaying. Let

\[ C_{14} \equiv C_{14}(p, \hat{q}) := \max_{1 \leq i \leq K_{14}} i^p e^{-\hat{q} \sum_{k=1}^{n-1} (k+1)^{-p}}. \]

Let \( c_n := \sum_{i=0}^{n-1} [i+1]^{-2p} e^{-2\hat{q} \sum_{k=1}^{n-1} (k+1)^{-p}} \). Then,

\[ c_n \leq C_{14}(p, \hat{q}) e^\hat{q} \frac{n}{q}. \]

Accordingly, \( a_n \leq C_{14, \alpha} n^{-\alpha} \) and \( b_n \leq C_{14, \alpha} n^{-\beta} \) where \( C_{14, \alpha} = C_{14}(\alpha, \hat{q}) e^{\gamma_1} \), \( C_{14, \hat{q}} = C_{14}(\hat{q}, \hat{q}) e^{\gamma_2} \), \( a_n = \sum_{k=0}^{n-1} \alpha_k e^{-2\hat{q} \sum_{j=k+1}^{n-1} \alpha_j} \) and \( b_n = \sum_{k=0}^{n-1} \beta_k e^{-2\hat{q} \sum_{j=k+1}^{n-1} \beta_j} \).
Proof. The bound follows as in (45) from (Dalal et al. 2018a).

Let \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \) of a matrix denote its smallest and largest eigenvalue, respectively. Also, fix

\[
q_1 = \frac{\lambda_{\text{min}}(X_1 + X_1^\top)}{4}
\]

and

\[
q_2 = \frac{\lambda_{\text{min}}(W_2 + W_2^\top)}{4}.
\]

**Lemma 15.** Let \( K_{15,\alpha} \) and \( K_{15,\beta} \) be such that

\[
\alpha_n \leq \frac{\lambda_{\text{min}}(X_1 + X_1^\top) - 2q_1}{\lambda_{\text{max}}(X_1^\top X_1)}, \ n \geq K_{15,\alpha},
\]

and

\[
\beta_n \leq \frac{\lambda_{\text{min}}(W_2 + W_2^\top) - 2q_2}{\lambda_{\text{max}}(W_2^\top W_2)}, \ n \geq K_{15,\beta}.
\]

Then, for \( n \geq K_{15,\alpha} \),

\[
\|I - \alpha_n X_1\| \leq 1,
\]

and, for \( n \geq K_{15,\beta} \),

\[
\|I - \beta_n W_2\| \leq 1.
\]

Proof. Observe that

\[
\|I - \alpha_n X_1\| = \left(\frac{\lambda_{\text{max}}(I - \alpha_n (X_1 + X_1^\top) + \alpha_n^2 (X_1^\top X_1))}{\lambda_{\text{min}}(X_1 + X_1^\top) - 2q_1}\right)^{1/2}.
\]

Let \( \lambda_n := \lambda_{\text{max}}(I - \alpha_n (X_1 + X_1^\top) + \alpha_n^2 (X_1^\top X_1)) \). Then, as in (7) from (Dalal et al. 2018a), we have \( \lambda_n \leq e^{-2q_1 \alpha_n} \leq 1 \) for \( n \geq K_{15,\alpha} \). The desired result is now easy to see. The bound on \( \|I - \beta_n W_2\| \) similarly holds.

**Lemma 16.** For any \( i \leq n \),

\[
\frac{\prod_{k=i}^n \|I - \alpha_k X_1\|}{\prod_{k=i}^n \|I - \beta_k W_2\|} \leq C_{16,\theta} e^{-q_1 \sum_{k=i}^n \alpha_k},
\]

(85)

\[
\frac{\prod_{k=i}^n \|I - \alpha_k X_1\|}{\prod_{k=i}^n \|I - \beta_k W_2\|} \leq C_{16,\omega} e^{-q_2 \sum_{k=i}^n \beta_k}.
\]

(86)

Here, \( C_{16,\theta} = \max\{1, \sqrt{\lambda_{\text{max}}(W_2^\top W_2) / \lambda_{\text{min}}(W_2 + W_2^\top)} \} \) with \( K_{16,1} = \left\lceil \left( \frac{\lambda_{\text{max}}(X_1^\top X_1)}{\lambda_{\text{min}}(X_1 + X_1^\top) - 2q_1} \right)^{1/\alpha} \right\rceil \) and \( \mu_1 = -\lambda_{\text{min}}(X_1 + X_1^\top) + \lambda_{\text{max}}(X_1^\top X_1) \). Similarly, \( C_{16,\omega} = \max\{1, \sqrt{\lambda_{\text{max}}(W_2^\top W_2) / \lambda_{\text{min}}(W_2 + W_2^\top)} \} \) with \( K_{16,2} = \left\lceil \left( \frac{\lambda_{\text{max}}(W_2^\top W_2)}{\lambda_{\text{min}}(W_2 + W_2^\top)} \right)^{1/\beta} \right\rceil \) and \( \mu_2 = -\lambda_{\text{min}}(W_2 + W_2^\top) + \lambda_{\text{max}}(W_2^\top W_2) \).

Proof. For \( K_{16,1} \) given in the statement, \( \alpha_k \leq \frac{\lambda_{\text{min}}(X_1 + X_1^\top) - 2q_1}{\lambda_{\text{max}}(X_1^\top X_1)} \) for all \( k \geq K_{16,1} \). Then, it follows by arguing as in the proof of Lemma 4.1 in (Dalal et al. 2018a) that

\[
\frac{\prod_{k=n_0}^n \|I - \alpha_k X_1\|}{\prod_{k=n_0}^n \|I - \alpha_k X_1 + X_1^\top + \alpha_k^2 X_1^\top X_1\|} \leq C_{16,\theta} e^{-q_1 \sum_{k=n_0}^n \alpha_k}.
\]

(87)

The second part of the statement is proved analogously.

**Lemma 17.** Let \( q_1, q_2, C_{16,\theta}, C_{16,\omega} \) be as defined in Lemma 16. Let \( n \geq n_0 - 1 \geq 0 \). Then,

\[
\|\Delta_{n+1}^{(\theta)}\| \leq C_{16,\theta} e^{-q_1 \sum_{k=n_0}^n \alpha_k} \|\theta_{n_0} - \theta^*\|,
\]

(88)

\[
\|\Delta_{n+1}^{(\omega)}\| \leq C_{16,\omega} e^{-q_2 \sum_{k=n_0}^n \beta_k} \|w_{n_0} - w^*\|.
\]

(89)

Proof. From (73), we have

\[
\|\Delta_{n+1}^{(\theta)}\| \leq \prod_{k=n_0}^n \|I - \alpha_k X_1\| \|\theta_{n_0} - \theta^*\|.
\]

(90)

For \( n = n_0 - 1 \), the desired result follows since \( C_{16,\theta} \geq 1 \), while, for \( n \geq n_0 \), the result holds due to Lemma 16. The second statement follows similarly.
Remark 18. This trivial lemma gives a much stronger convergence rate for \(\|\Delta_n^{\theta}\|\), compared to Lemma 6 in (Mokkadem and Pelletier 2006). It thus raises the following question. On the one hand, in Remark 4 (Mokkadem and Pelletier 2006) the linear case is explained to be easier and does not require using the fact that \(\Delta_n \to 0\). On the other hand, that remark refers in this simplified case to Eqs. 27-28, which are fairly complex and are not decaying exponentially without using sophisticated successive upper bound tricks. These latter Eqs. also recursively consist of \(R_{n+1}^{(\theta)}\). It thus implies that in (Mokkadem and Pelletier 2006) the derivation above can be tightened.

Lemma 19. Let \(B_1, B_3 \geq 0\) with at least one of them being strictly positive, let \(x, y \geq 0\), and let \(B_2 \geq 1\) be some constants. Then, for any \(n \geq 0\) and \(z \geq \max\{x, y\}\)

\[
B_1(n+1)^{-x} \sqrt{\ln(B_2(n+1))} + B_3(n+1)^{-y} \leq \frac{(n+1)^{-z}}{(n+2)^{-z}}.
\]

(91)

Proof. As \(z \geq \max\{x, y\}\), it follows that

\[
\left(\frac{n+2}{n+1}\right)^x \leq \left(\frac{n+2}{n+1}\right)^z.
\]

(92)

Hence, \((n+1)^{-x}(n+2)^{-z} \leq (n+2)^{-x}(n+1)^{-z}\). Similarly, \((n+1)^{-y}(n+2)^{-z} \leq (n+2)^{-y}(n+1)^{-z}\). Therefore, it is easy to see that

\[
B_1(n+1)^{-x}(n+2)^{-z}\sqrt{\ln(B_2(n+1))} + B_3(n+1)^{-y}(n+2)^{-z}
\]

\[
\leq B_1(n+2)^{-x}(n+1)^{-z}\sqrt{\ln(B_2(n+2))} + B_3(n+2)^{-y}(n+1)^{-z}.
\]

(93)

The desired result now follows.

Lemma 20. Let \(z \in [0, 1 - (\alpha - \beta)]\),

\[
K_{20,\alpha}(z) = \max\left\{\left[\left(\frac{q_1}{2(\alpha - \beta + z)}\right)^{1/\alpha}\right], \left[\left(\frac{4(\alpha - \beta + z)}{q_1}\right)^{1/(1-\alpha)}\right]\right\},
\]

and

\[
K_{20,\beta}(z) = \max\left\{\left[\left(\frac{q_2}{\alpha - \beta + z}\right)^{1/\beta}\right], \left[\left(\frac{4(\alpha - \beta + z)}{q_2}\right)^{1/(1-\beta)}\right]\right\}.
\]

(94)

Then,

1. for \(n \geq K_{20,\alpha}(z)\),

\[
\frac{(n+1)^{-z}}{(n+2)^{-z}} \leq \frac{\alpha_{n+1}}{\alpha_n} \frac{\beta_n}{\beta_{n+1}} e^{(q_1/2)\alpha_{n+1}}, \text{ and}
\]

(96)

2. for \(n \geq K_{20,\beta}(z)\),

\[
\frac{(n+1)^{-z}}{(n+2)^{-z}} \leq \frac{\alpha_{n+1}}{\alpha_n} \frac{\beta_n}{\beta_{n+1}} e^{(q_2/2)\beta_{n+2}}.
\]

(97)

Proof. We begin with the first statement. Let us now substitute the stepsizes. Let us write (96) as

\[
1 \leq \frac{(n+2)^{-z}}{(n+1)^{-z}} \left(\frac{n+1}{n+2}\right)^{-z} e^{q_1/2 \alpha_{n+1}} = \left[\left(1 + \frac{1}{n+1}\right)^{\alpha - \beta}\right]^{z} e^{q_1/2 \alpha_{n+1}}.
\]

(98)

Next, we use a first-order approximation for the exponent. Since \(e^{(q_1/2)\alpha_{n+1}} \geq 1 + (q_1/2)(n+2)^{-\alpha}\), to show (98) it is enough to show that \(1 + \frac{1}{n+1}\)^{\alpha - \beta} \leq 1 + \frac{\alpha_{n+1}}{\alpha_n} \frac{\beta_n}{\beta_{n+1}} e^{(q_1/2)\alpha_{n+1}}\).

For this, we shall show that \(1 + \frac{1}{n+1}\)^{\alpha - \beta} \leq 1 + \frac{\alpha_{n+1}}{\alpha_n} \frac{\beta_n}{\beta_{n+1}} e^{(q_1/2)\alpha_{n+1}}\), and later show (99). Denote \(f(x) = (1 + x)^{\alpha - \beta}\); then \(1 + \frac{1}{n+1}\)^{\alpha - \beta} = f(1/(n+1)). From the mean value theorem, \(\exists c \in (0, 1/(n+1))\) s.t. \(f'(c) = \frac{f(1/(n+1)) - f(0)}{1/(n+1)}\). Hence, \((\alpha - \beta + z)(1+c)^{(\alpha - \beta + z)} = (n+1)\left(1 + \frac{1}{n+1}\right)^{\alpha - \beta} - 1\). Therefore, \(1 + \frac{1}{n+1}\)^{\alpha - \beta} = 1 + \frac{\alpha - \beta + z}{(n+1)(1+c)^{(\alpha - \beta + z)} - 1}\). The latter inequality holds because \((1+c)^{1-(\alpha - \beta + z)} \geq 1\), which can be seen from the fact that \(1 - (\alpha - \beta) \geq z\).

Now, we are left to show that

\[
1 + \frac{\alpha - \beta + z}{n+1} \leq 1 + (q_1/2)(n+2)^{-\alpha}.
\]

(99)
meaning that

\[
\frac{2(\alpha - \beta + z)}{q_1} \leq \frac{n + 1}{(n + 2)^\alpha} = (n + 2)^{1-\alpha} - (n + 2)^{-\alpha},
\]

(100)

where the last relation holds by adding and subtracting 1 in the numerator. To show (100), first notice that \((n+2)^{-\alpha} \leq \frac{2(\alpha - \beta + z)}{q_1}\) when \(n \geq \left(\frac{q_1}{2(\alpha - \beta + z)}\right)^{1/\alpha} - 2\). Therefore, (100) holds if \((n+2)^{-\alpha} \geq \frac{4(\alpha - \beta + z)}{q_1}\), which holds for \(n \geq \left(\frac{4(\alpha - \beta + z)}{q_1}\right)^{1/(1-\alpha)} - 2\).

By imposing the condition \(n_0 \geq K_{20,\alpha}(z)\), we obtain (96) and conclude the proof of the first statement.

To show the second statement, it is now enough to show that \(1 + (q_2/2)(n+3)^{-\beta} \geq \left[1 + \frac{1}{n+1}\right]^{\alpha - \beta + z}\). The proof of this is very similar; the main difference is that instead of (100), one obtains that

\[
\frac{2(\alpha - \beta + z)}{q_2} \leq \frac{n + 1}{(n + 3)^\beta} = (n + 3)^{1-\beta} - (n + 3)^{-\beta}.
\]

(101)

holds when \(n \geq \left(\frac{q_2}{2(\alpha - \beta + z)}\right)^{1/\beta} - 3\) and \(n \geq \left(\frac{4(\alpha - \beta + z)}{q_2}\right)^{1/(1-\beta)} - 3\). \(\square\)

**Lemma 21.** Given arbitrary constants \(\alpha > 0\) and \(A > 0\), it holds that \(n^{-\gamma} \log(n^\beta a) \leq A\) for any \(n \geq \left[\frac{2p}{\gamma A} \ln \left(\frac{2p}{\gamma A} n^\alpha / p\right)\right]^{1/\gamma}\). Consequently, for \(\epsilon_n^{(\theta)}, \epsilon_n^{(w)}\) as defined in (14),

\[
\epsilon_n^{(\theta)} \leq \frac{R_{\text{proj}}^\theta}{2}\]

for \(n \geq K_{21,\alpha} := \left[\frac{4d^3 L_w C_{14,\theta} \beta}{\alpha (R_{\text{proj}}^\theta)^2}\right]^{1/\alpha} \left[2 \ln \left(\frac{2d^3 L_w C_{14,\theta} \beta}{\alpha (R_{\text{proj}}^\theta)^2}\right)\right]^{1/\alpha} / \beta\), and

\[
\epsilon_n^{(w)} \leq \frac{R_{\text{proj}}^w}{2}
\]

for \(n \geq K_{21,\beta} := \left[\frac{4d^3 L_w C_{14,\theta} \beta}{\alpha (R_{\text{proj}}^\theta)^2}\right]^{1/\beta} \left[2 \ln \left(\frac{2d^3 L_w C_{14,\theta} \beta}{\alpha (R_{\text{proj}}^\theta)^2}\right)\right]^{1/\beta} / \beta\).

**Proof.** First note that, for any \(C > 0\) it holds that \(C \ln(x) \leq x\) for \(x\) equal to \(2C \ln(2C)\) and, since \(x\) grows faster than \(\ln(x)\), it also holds for any \(x\) larger than that. The first claim of the lemma follows by substituting \(x = [a^{1/p} n]^{\gamma}\) and \(C = \frac{p}{\gamma A} a^{\gamma/p}\).

Then (102) simply follows from this first claim by substituting \(\gamma = \alpha, A = \left(\frac{R_{\text{proj}}^\theta}{2}\right)^2 / d^3 L_w C_{14,\theta}\) and \(\alpha = \frac{4d^2}{\delta}\), and (103) by substituting \(\gamma = \beta, A = \frac{(R_{\text{proj}}^w)^2}{d^3 L_w C_{14,\theta}}\) and \(\alpha = \frac{4d^2}{\delta}\). \(\square\)

### C Proof of Theorem 3

Recall from Section 4 that the analysis is based on Theorem 6. Consequently, what we need to show is that, after the claimed number of iterations, sparse projections ensure \(G_{\text{proj}}^{\theta,0}\). In particular, we show that, after a time, these projections are not needed anymore, as the iterates remain in the close vicinity of \(\theta^*\) and \(w^*\) respectively, and the conclusions of the above theorem take place.

Before we start the proof, we need to analyze briefly the constants in the theorem. Let

\[
A_3 = C_{R}^{w} R_{\text{proj}}^{w},
\]

(104)

\[
A_{4, n_0} = \left[ A_{1, n_0} \sum_{i=0}^{[\alpha n_0 - \beta]} A_2^i \right] \sqrt{d^3 L_w C_{14, w}} + \left[ A_3 A_2^{[\alpha n_0 - \beta]} \right],
\]

(105)

and

\[
A_{5, n_0} = 2 \left[ C_{32, a} \left[ C_{R}^{\theta} R_{\text{proj}}^{\theta} + C_{R}^{w} R_{\text{proj}}^{w} \right] / \epsilon_{n_0 - 1}^{(\theta)} + 1 \right] \sqrt{4d^3 L_w C_{14, w}} + 2 C_{32, b} A_{4, n_0}.
\]

(106)
Lemma 22. Assume $G'_{n_0}$ holds. Let
\[ A_4' = A_{4,C_1} + 1, \quad A_5' = 4 + 2A_{5,C_1} + 2C_{32,b}A_{4,C_1}, \] (107)
where
\[ A_{4,C_1} = d^8 L_w C_{14,w} \left( C_{16,w} \| \gamma_2 \| \left[ R^q_{\text{proj}} + C_{32,a} e^{q_1} \frac{2}{q_{\text{min}}} (R^q_{\text{proj}} + R^w_{\text{proj}}) \right] + C_{16,w} R^w_{\text{proj}} \right) e^{\left\lfloor \frac{\alpha}{2(\alpha - p)} \right\rfloor - 1} \sum_{i=0} \ A^i_2, \] (108)
\[ A_{5,C_1} = C_{32,a} \left[ C^q_R R^q_{\text{proj}} + C^w_R R^w_{\text{proj}} \right]. \] (109)
Then recalling $A_{4,n_0}$ from (105) and $A_{5,n_0}$ from (106).
\[ A_{4,n_0} \leq A_4'(n_0 + 1)^{\beta/2} \left(\ln (4d^2(n_0 + 1)^r/\delta)\right)^{-1/2} \] (110)
\[ A_{5,n_0} \leq A_5'(n_0 + 1)^{\alpha/2} \left(\ln (4d^2(n_0 + 1)^r/\delta)\right)^{-1/2} \] (111)
if
\[ n_0 \geq \max\{K_{22,a}, K_{22,b}\}, \] (112)
where
\[ K_{22,a} = \left[ \frac{p}{\beta(a_{4,c_0})^2} \right]^{1/\beta} \left[ 2 \ln \left( 2 \frac{p}{\beta(a_{4,c_0})^2} \left[ \frac{4d^2}{\delta} \right]^{\beta/p} \right) \right]^{1/\beta}, \] (113)
\[ K_{22,b} = \left[ \frac{p}{\alpha(\min \{C_{32,\theta}, A_{4,c_0}, A_{5,c_0}\})^2} \right]^{1/\alpha} \left[ 2 \ln \left( 2 \frac{p}{\alpha(\min \{C_{32,\theta}, A_{4,c_0}, A_{5,c_0}\})^2} \left[ \frac{4d^2}{\delta} \right]^{\alpha/p} \right) \right]^{1/\alpha}, \] (114)
with
\[ A_{4,c_0} = \left( e + e^2 A_1' \right) \left( \sum_{i=0}^{\left\lfloor \frac{\alpha}{2(\alpha - p)} \right\rfloor - 1} A^i_2 \sqrt{d^8 L_w C_{14,w}} \right) + A_3 A_2 \left[ \frac{\alpha}{\beta(n_0 + 1)^r/\delta} \right] \] (115)
\[ A_{5,c_0} = \sqrt{4d^3 L_0 C_{14,\theta}}. \] (116)

Proof. First, we upper bound $C_{32,c}(n_0)$ based on the definition of $G'_{n_0}$:
\[ C_{32,c}(n_0) = \left[ \beta_{n_0} \| \theta_{n_0} - \theta^* \| + C_{32,a} e^{q_1} \frac{2}{q_{\text{min}}} \| \theta_{n_0} - \theta^* \| + \frac{\alpha_{n_0}}{\beta_{n_0}} \| w_{n_0} - w^* \| \right] \] (117)
\[ \leq \left[ R^q_{\text{proj}} + C_{32,a} e^{q_1} \frac{2}{q_{\text{min}}} (R^q_{\text{proj}} + R^w_{\text{proj}}) \right]. \] (118)
Next, using the definition of $A_{4,n_0}$ from (105), (110) holds if
\[ (n_0 + 1)^{\beta/2} \left(\ln (4d^2(n_0 + 1)^r/\delta)\right)^{-1/2} \geq A_{4,c_0}, \] (119)
Based on Lemma 21, (119) holds if $n_0 \geq K_{22,a}$. This completes the proof of (110).

Next, recall from (106) that
\[ A_{5,n_0} = 2C_{32,a} \sqrt{4d^3 L_0 C_{14,\theta}} \left[ C^q_R R^q_{\text{proj}} + C^w_R R^w_{\text{proj}} \right] / \epsilon_{n_0 - 1} + 2 \sqrt{4d^3 L_0 C_{14,\theta}} + 2C_{32,b} A_{4,n_0} \] (120)
\[ = 2C_{32,a} \sqrt{4d^3 L_0 C_{14,\theta}} \left[ C^q_R R^q_{\text{proj}} + C^w_R R^w_{\text{proj}} \right] (n_0 + 1)^{\alpha/2} \] (121)
\[ + 2 \sqrt{4d^3 L_0 C_{14,\theta}} + 2C_{32,b} A_{4,n_0} \]
\[ = 2 \left[ (n_0 + 1)^{\alpha/2} \left(\ln (4d^2(n_0 + 1)^r/\delta)\right)^{-1/2} A_{5,c_0} + A_{5,c_0} \right] + (n_0 + 1)^{\beta/2} \left(\ln (4d^2(n_0 + 1)^r/\delta)\right)^{-1/2} C_{32,b} A_{4,c_1} + C_{32,b} A_{4,c_0}. \] (122)
Therefore, again based on Lemma 21, (111) holds when $n_0 \geq K_{22,b}$. \qed
Let
\[ K_{3,w} = \left( A_3^w / R_{\text{proj}}^w \right)^{2/\beta} \] (124)
and
\[ K_{3,\theta} = \left( A_3^\theta / R_{\text{proj}}^\theta \right)^{2/\alpha}. \] (125)
Also, define
\[ C_{3,\theta} = A_3^\theta / \nu(N_{3,\alpha}), \] (126)
\[ C_{3,w} = A_3^w / \nu(N_{3,\beta}). \] (127)

We are now ready to prove the theorem.

**Proof of Theorem 3.** Recall that whenever \( n_0 = k^k - 1 \) for some \( k \in \mathbb{Z}_{>0} \), then event \( G'_{n_0} \) holds with probability 1 for the projected iterates. Let \((\theta_n, w_n)_{n \geq n_0}\) be the iterates obtained by running the unprojected algorithm given in (1) and (2) with \( \theta_{n_0} = \theta'_n \) and \( w_{n_0} = w'_n \). Define \( f(x) = x^p \) and note that if we project in round \( n_0 \) then, by definition, \( n_0 = f(k) - 1 \) for some positive integer \( k \), and the next time we project will be in round \( g(n_0) = f(1 + k) - 1 = f(1 + f^{-1}(n_0 + 1)) - 1 \). Therefore,

\[
I := \left\{ \|\theta_j - \theta^*\| \leq R_{\text{proj}}^\theta, \|w_j - w^*\| \leq R_{\text{proj}}^w, \forall j \geq g(n_0) \right\} \] (128)
\[
\subseteq \left\{ \theta_j = \Pi_{j,R_{\text{proj}}^\theta}(\theta_j), w_j = \Pi_{j,R_{\text{proj}}^w}(w_j), \forall j \geq g(n_0) \right\} \] (129)
\[
= \left\{ \theta_j = \Pi_{j,R_{\text{proj}}^\theta}(\theta_j), w_j = \Pi_{j,R_{\text{proj}}^w}(w_j), \forall j \geq n_0 \right\}. \] (130)

Consider the following coupling:
\[
(\tilde{\theta}_n', \tilde{w}_n') := \begin{cases} 
(\theta'_n, w'_n), & \text{for } 0 \leq n < n_0, \\
(\theta_n, w_n), & \text{for } n \geq n_0 \text{ on the event } I, \\
(\theta_n', w_n'), & \text{for } n \geq n_0 \text{ on the complement of the event } I.
\end{cases} \] (131)

Due to (128) - (130), \((\tilde{\theta}_n', \tilde{w}_n')_{n \geq 0}\) and \((\tilde{\theta}_n, \tilde{w}_n)_{n \geq 0}\) are distributed identically. Consequently, it is easy to see that Theorem 6 applies to \((\tilde{\theta}_n', \tilde{w}_n')\) provided we show that the event \( I \) holds, i.e.,
\[
A_{5,n_0}(n+1)^{-\alpha/2} \sqrt{\ln(4d^2(n+1)^p/\delta)} \leq R_{\text{proj}}^\theta \] (132)
\[
A_{4,n_0}(n+1)^{-\beta/2} \sqrt{\ln(4d^2(n+1)^p/\delta)} \leq R_{\text{proj}}^w \] (133)
for all \( n \geq g(n_0) \). In fact, using Lemma 22 together with Theorem 6,
\[
\|\tilde{\theta}'_n - \theta^*\| \leq A_5^\theta / \nu(n_0, \alpha) \nu(n, \alpha) \] (134)
\[
\|\tilde{w}'_n - w^*\| \leq A_4^\theta / \nu(n_0, \beta) \nu(n, \beta) \] (135)
as desired, for \( n \geq n_0 \geq \max\{N_6, K_{22,\alpha}, K_{22,\beta}\} \), provided we show that
\[
\frac{A_5^\theta}{\nu(n_0, \alpha)} \nu(n, \alpha) \leq R_{\text{proj}}^\theta \] (136)
\[
\frac{A_4^\theta}{\nu(n_0, \beta)} \nu(n, \beta) \leq R_{\text{proj}}^w \] (137)
for all \( n \geq g(n_0) \). As we show below, this holds when \( n_0 \geq \max\{K_{3,w}, K_{3,\theta}, e^{1/\alpha}, e^{1/\beta}, (2/\alpha)^{2/\alpha}, (2/\beta)^{2/\beta}\} \).

It is clear that, in order to show that (137) holds, it suffices to show that for \( n = g(n_0) \)
\[
\left( \frac{n+1}{n_0+1} \right)^{\beta} \ln \left[ \left( \frac{4d^2/\delta}{n_0+1} \right)^{\beta/p} (n_0+1)^{\beta} \right] \geq \frac{A_4^\theta}{\nu(n_0, \beta)} \nu(n, \beta) \] (138)
and that \( (n+1)^{\beta} \left( \ln \left[ \left( \frac{4d^2/\delta}{n_0+1} \right)^{\beta/p} (n_0+1)^{\beta} \right] \right)^{-1} \) is monotonically decreasing.
Since \( n_0 = f(k) - 1 \) for some positive integer \( k \) and \( n = f(k + 1) - 1 \), letting \( A = \frac{3}{p} \ln \left[ \frac{4d^2}{\delta} \right] \geq 0 \), we have

\[
\left( \frac{n + 1}{n_0 + 1} \right)^\beta A + \beta \ln(n_0 + 1) = \left( \frac{(k + 1)^{k+1}}{k} \right)^\beta A + \beta k \ln k \\
\geq (k + 1)^\beta \left( \frac{k + 1}{k} \right)^\beta \frac{k \ln k}{(k + 1) \ln(k + 1)}
\]

where (140) follows because \((A + B_1)/(A + B_2) \geq B_1/B_2\) for any \(A \geq 0\) and \(B_2 \geq B_1 > 0\) due to \((A + B_1)B_2 \geq (A + B_2)B_1\), (41) follows because \(\ln(x + 1) - \ln x \leq 1/x\) due to the fact that \(\ln x\) is concave and has derivative \(1/x\), (142) holds when \(k > 1\) due to \(\ln k/n^{\beta} = \frac{k \ln k}{(k + 1) \ln(k + 1)} \geq \frac{k}{1 + k} \), and the last inequality holds when \(k \geq 2/\beta\), or, equivalently when \(n_0 \geq (2/\beta)^{2/\beta} - 1\). Consequently, (138) holds if \(k \geq (\lambda_1/R^w)k^2/\beta\) or, equivalently, if \(n_0 \geq K_{3,w}\).

Showing the monotonicity of \((n + 1)^\beta \left( \ln \left[ \frac{4d^2}{\delta} \right]^{\beta/p} (n + 1)^\beta \right)^{-1}\) goes similarly:

\[
\left( \frac{n + 1}{n} \right)^\beta \ln \left[ \frac{4d^2}{\delta} \right]^{\beta/p} (n + 1)^\beta \frac{n + 1}{n} = \left( \frac{n + 1}{n} \right)^\beta \frac{\beta}{p} \ln \left[ \frac{4d^2}{\delta} \right] + \beta \ln n
\]

which the last inequality holds for \(n \geq e^{1/\beta}\).

Using an argument similar to the above, it is easy to see that (136) holds since \(n_0 \geq \max\{e^{1/\alpha}, (2/\alpha)^{2/\alpha}, K_{3,\alpha}\} \) which is true since \(e^{1/\beta} \geq e^{1/\alpha}\) and \((2/\beta)^{2/\beta} \geq (2/\alpha)^{2/\alpha}\).

Now, substituting \(n_0 = N_3\) in (134) and (135) gives us the desired result.

\[ \square \]

**Remark 23.** The above result introduces double exponential complexity in \(1/\alpha\) and \(1/\beta\) via, e.g., \(K_{3,w}\) and \(K_{3,\alpha}\). One can try and obtain better bounds by increasing the sparsity of the projections. Nevertheless, we argue that, at least for square-summable step-sizes (i.e., for \(\alpha > 1/2\) and \(\beta > 1/2\), this double-exponential bound is not too bad.

### D Details omitted from the proof of Theorem 7

Recalling \(U(n_0)\) from (13) and \(Z_n\) from (22), we have

\[
U^c(n_0) = \bigcup_{n \geq n_0} Z_n^c
\]

\[
= Z_{n_0}^c \cup (Z_{n_0} \cap Z_{n_0+1}^c) \cup ([Z_{n_0} \cap Z_{n_0+1}] \cap Z_{n_0+2}^c) \cup \ldots
\]

This implies that

\[
G_{n_0}^c \cap U^c(n_0) = (G_{n_0}^c \cap Z_{n_0}^c) \cup (G_{n_0}^c \cap A_{n_0} \cap Z_{n_0+1}^c) \cup (G_{n_0}^c \cap A_{n_0+1} \cap Z_{n_0+2}^c) \cup \ldots
\]

Recalling that

\[ C_R^0 = 3 \]
one can see that both are lower bounded by 1, and hence,

\[
\mathcal{G}_{n_0} \cap \mathcal{Z}_{n_0}^c \subseteq \mathcal{G}_{n_0} \cap \mathcal{Z}_{n_0}^c \subseteq \mathcal{G}_{n_0} \cap \left( \{ \| \ell_{n_0+1}^{(\theta)} \| > \epsilon_{n_0}^{(\theta)} \} \cup \{ \| L_{n_0+1}^w \| > \epsilon_{n_0}^w \} \right).
\]

Similarly, observe that

\[
\mathcal{G}_{n_0} \cap \mathcal{A}_n \cap \mathcal{Z}_{n+1}^c = \mathcal{G}_{n_0} \cap \mathcal{A}_n \cap \left( \{ \| \theta_{n+1} - \theta^* \| > C_R^\theta R_{\text{proj}}^\theta \} \cup \{ \| w_{n+1} - w^* \| > C_R^w R_{\text{proj}}^w \} \right)
\]

\[
\cup \left( \{ \| L_{n+2}^{(\theta)} \| > \epsilon_{n+1}^{(\theta)} \} \cup \{ \| L_{n+2}^w \| > \epsilon_{n+1}^w \} \right)
\]

\[
\subseteq \mathcal{G}_{n_0} \cap \mathcal{A}_n \cap \left( \{ \| \theta_{n+1} - \theta^* \| > C_R^\theta R_{\text{proj}}^\theta \} \cup \{ \| w_{n+1} - w^* \| > C_R^w R_{\text{proj}}^w \} \right)
\]

\[
\cup \left( \{ \| L_{n+2}^{(\theta)} \| > \epsilon_{n+1}^{(\theta)} \} \cup \{ \| L_{n+2}^w \| > \epsilon_{n+1}^w \} \right)
\]

\[
\subseteq \mathcal{G}_{n_0} \cap \mathcal{A}_n \cap \left( \{ \| \theta_{n+1} - \theta^* \| > C_R^\theta R_{\text{proj}}^\theta \} \cup \{ \| w_{n+1} - w^* \| > C_R^w R_{\text{proj}}^w \} \right)
\]

Remark 24. One could have obtained an exponentially decaying bound in (35) by defining \( \epsilon_n^{(\theta)} \) to be \( \sqrt{d^3 L_0 C_{14} \theta(n+1)^{-\alpha + \beta}} \ln (4d^3/\delta) \) instead of the current definition given in (14). This bound would then be in the same spirit as that of (Borkar 2009, Chapter 4, Corollary 14) (see the 2nd display there). However, the additional \( (n+1)^\beta \) term means that the new \( \epsilon_n^{(\theta)} \) decays at a slower rate and thereby slows down the rate of convergence of the \( \{ \theta_n \} \) iterates derived in Theorem 3. The same discussion applies for \( \epsilon_n^w \) and the \( \{ w_n \} \) iterates as well.

E A key lemma

Lemma 25. Let \( n \geq n_0 \geq 0 \). Let \( u \in \mathbb{R}^\infty \) be \( \alpha \)-moderate from \( n_0 \) onwards (see Def. 10) and suppose the event \( \mathcal{W}_n(u) \) holds (see (37)). Then,

\[
\| T_{n+1} \| \leq \frac{2e^{\alpha/2}}{q_2} C_{25} \| W_1 W_2^{-1} \| C_{16,\theta} \frac{\epsilon_n}{\beta_n} u_n,
\]

where

\[
C_{25} := \| X_1 \| + 2(\alpha - \beta)[1 + \| X_1 \|].
\]

Proof. From (82), it is easy to see that

\[
\| T_{n+1} \| \leq \sum_{k=n_0+1}^n \left( \prod_{j=k+1}^n \| I - \alpha_j X \| \right) \alpha_k \| U_k \|,
\]

where

\[
U_k := \frac{1}{\beta_k} \left[ I - \frac{\alpha_{k-1}}{\alpha_k} \frac{\beta_k}{\beta_{k-1}} (I - \alpha_k X_1) \right] W_1 W_2^{-1} [w_k - w^*].
\]

Observe that

\[
U_k = \frac{\alpha_k}{\beta_k} \left[ X_1 + \frac{1}{\alpha_k} \left( 1 - \frac{\alpha_{k-1}}{\alpha_k} \frac{\beta_k}{\beta_{k-1}} \right) (I - \alpha_k X_1) \right] W_1 W_2^{-1} [w_k - w^*].
\]

We now show

\[
\left\| X_1 + \frac{1}{\alpha_k} \left( 1 - \frac{\alpha_{k-1}}{\alpha_k} \frac{\beta_k}{\beta_{k-1}} \right) (I - \alpha_k X_1) \right\|
\]

can be bounded by a constant. In particular, it suffices to show that

\[
B_k := \frac{1}{\alpha_k} \left( 1 - \frac{\alpha_{k-1}}{\alpha_k} \frac{\beta_k}{\beta_{k-1}} \right) = (k+1)\left( 1 + \frac{1}{k} \right)^{\alpha - \beta}
\]

is bounded by a constant. To this end, let \( f(x) = (1 + x)^{\alpha - \beta} \). Then, by the mean value theorem, there is a \( c \in (0, 1/k) \) such that

\[
k[f(1/k) - f(0)] = f'(c).
\]
Proof. Let \( C \) where the inequality follows since \( c \in (0, 1/k) \), we obtain
\[
|B_k| = (k + 1)^\alpha |f(0) - f(1/k)| = (k + 1)^\alpha \frac{|f'(c)|}{k} \leq (k + 1)^\alpha \frac{\alpha - \beta}{k} \leq 2(\alpha - \beta).
\]
From this, it follows that
\[
\|U_k\| \leq \frac{\alpha_k}{\beta_k} \|X_1\| + |B_k| \|I - \alpha_k X_1\| \|W_1 W_2^{-1}\| \|w_k - w^*\|
\]
Substituting (171) in (163), we get
\[
\|T_{n+1}\| \leq C_{25} \|W_1 W_2^{-1}\| \sum_{k=n_0+1} \left( \prod_{j=k+1}^{n} \|I - \alpha_j X_1\| \right) \alpha_k \frac{\alpha_k}{\beta_k} \|w_k - w^*\|
\]
where (173) follows from Lemma 16, while in (174) we bound the summation from \( n_0 \) to a summation from 0. For (175), since \( \sup_n \alpha_n \leq 1 \), we have \( \sum_{k=0}^n e^{-q_1/2 \sum_{j=k+1}^n \alpha_j} \alpha_k \leq e^{q_1/2} \sum_{k=0}^n e^{-q_1/2 \sum_{j=k+1}^n \alpha_j} \alpha_k \); hence, by treating this latter sum as a Riemann sum and letting \( t_{n+1} = \sum_{k=0}^n \alpha_k \), we get \( \sum_{k=0}^n e^{-q_1/2 \sum_{j=k+1}^n \alpha_j} \alpha_k \leq e^{q_1/2} e^{-q_1/2 t_{n+1}} \int_0^{t_{n+1}} e^{-(q_1/2) t} dt = 2e^{q_1/2}/2 \), (176) holds due to \( W_n \). Lastly, (177) holds because the terms in the sup argument in (176) monotonically increase with \( k \), since
\[
\frac{\alpha_k}{\beta_k} u_k e^{-q_1/2 \sum_{j=k+1}^n \alpha_j} = \frac{\alpha_{k+1}}{\alpha_k} u_{k+1} e^{-q_1/2 \sum_{j=k+1}^n \alpha_j} \frac{\beta_{k+1}}{\beta_k} u_k \leq \frac{\beta_{k+1}}{\beta_k} u_k \frac{\alpha_{k+1}}{\alpha_k} e^{q_1/2 \alpha_{k+1}}
\]
is upper bounded by 1 due to \( u \) being \( \alpha \)-moderate.

\section*{F Proof of Lemma 9}

Lemma 26. Let \( n_0 \geq K_{20, \alpha}(0) \) (defined in (94)) and \( n \geq n_0 \). Let \( T_{n+1} \) be as in (82). Then, on \( A_n \), we have
\[
\|T_{n+1}\| \leq \frac{2e^{q_1/2}}{q_1} C_{25} \|W_1 W_2^{-1}\| C_{16, \theta} \frac{\alpha_n}{\beta_n} C_{w} R_{\text{proj}}(u)
\]
where \( C_{25} \) is defined in (162) and \( C_{w} R_{\text{proj}} \) is defined in (154).

Proof. Let \( u \in \mathbb{R}^\infty_+ \) be s.t. \( u_n = C_{w}^{R_{\text{proj}}} R(u) \forall n \geq 0 \). Due to Lemma 20 Statement 1 (with \( z = 0 \),
\[
\frac{\alpha_{n+1} \beta_n}{\alpha_n \beta_{n+1}} e^{q_1/2 \alpha_{n+1}} \geq 1 \forall n \geq 0
\]
This implies that \( u \) is \( \alpha \)-moderate from 0 onwards (see Def. 10). Further, because \( A_n \) holds, the event \( \mathcal{W}_n(u) \) holds. The desired result now follows from Lemma 25.
Lemma 27. Let $n \geq n_0$ and suppose the event $G'_{n_0} \cap A_n$ holds. The following statements are true.

1. If $n_0 \geq 0$, then
\[
\| R^{(w)}_{n+1} \| \leq C^o_R \| \Gamma_2 \| C_{16,w} R^o_{proj} e^{q_2}/q_2. \tag{180}
\]

2. If $n_0 \geq K_{21,\beta}$, then
\[
\| L^{(w)}_{n+1} \| \leq R^w_{proj}/2.
\tag{181}
\]

3. If $n_0 \geq K_{15,\beta}$, then
\[
\| \Delta^{(w)}_{n+1} \| \leq R^w_{proj}.
\]

4. Consequently, if $n_0 \geq \max \{ K_{15,\beta}, K_{21,\beta} \}$, then
\[
\| w_{n+1} - w^* \| \leq \frac{3}{2} R^w_{proj} + C^o_R \| \Gamma_2 \| C_{16,w} R^o_{proj} e^{q_2}/q_2.
\]

Proof. Since $A_n$ holds, it follows from (78) that
\[
\| R^{(w)}_{n+1} \| \leq C^o_R \| \Gamma_2 \| R^o_{proj} \sum_{k=n_0}^{n} \left( \prod_{j=k+1}^{n} \| I - \beta_j W_2 \| \right) \beta_k
\leq C^o_R \| \Gamma_2 \| C_{16,w} R^o_{proj} \sum_{k=n_0}^{n} e^{-q_2} \sum_{j=k+1}^{n} \beta_j \beta_k
\leq C^o_R \| \Gamma_2 \| C_{16,w} R^o_{proj} e^{q_2}/q_2,
\]
where the second relation follows by using Lemma 16, while the last one follows by arguing in the same way as we did for (175) above.

The bound on $\| L^{(w)}_{n+1} \|$ follows from the definition of $A_n$ together with Lemma 21. The bound on $\| \Delta^{(w)}_{n+1} \|$ follows from the definition in (77) along with the facts that $\| I - \beta_k W_2 \| \leq 1$ and $\| w_{n_0} - w^* \| \leq R^w_{proj}$, which themselves hold due to Lemma 15 and the event $G'_{n_0}$, respectively.

The last statement of the lemma follows from the first three statements. 

\[\square\]

Lemma 28. Let $n \geq n_0$ and suppose the event $G'_{n_0} \cap A_n$ holds. The following statements are true.

1. If $n_0 \geq \max \{ K_{15,\alpha}, K_{15,\beta}, K_{20,\alpha}(0), K_{21,\beta} \}$, then
\[
\| R^{(\theta)}_{n+1} \| \leq \frac{\alpha_{n_0}}{\beta_{n_0}} \left[ C_{28,\theta} R^0_{proj} + C_{28,w} R^w_{proj} \right],
\]
where
\[
C_{28,\theta} = \| W_1 \| \| W_2^{-1} \| e^{q_2}/2 \| \Gamma_2 \| C_{16,w},
\tag{182}
\]
\[
C_{28,w} = \| W_1 \| \| W_2^{-1} \| \left[ \frac{3}{2} + \frac{2 e^{q_2}}{q_1} C_{25,\theta} C_{25} \right].
\tag{183}
\]

2. If $n_0 \geq K_{21,\alpha}$, then
\[
\| L^{(\theta)}_{n+1} \| \leq R^o_{proj}/2.
\]

3. If $n_0 \geq k'_n$, then
\[
\| \Delta^{(\theta)}_{n+1} \| \leq R^o_{proj}.
\]

4. Consequently, if $n_0 \geq \max \{ K_{15,\alpha}, K_{15,\beta}, K_{20,\alpha}(0), K_{21,\alpha}, K_{21,\beta} \}$, then
\[
\| \theta_{n+1} - \theta^* \| \leq \frac{3}{2} R^o_{proj} + \frac{\alpha_{n_0}}{\beta_{n_0}} \left[ C_{28,\theta} R^0_{proj} + C_{28,w} R^w_{proj} \right].
\]

Proof. On the event $G'_{n_0} \cap A_n$, we have
\[
\| R^{(\theta)}_{n+1} \| \leq \frac{\alpha_n}{\beta_n} \| W_1 \| \| W_2^{-1} \| \| w_{n+1} - w^* \| + \| W_1 \| \| W_2^{-1} \| \frac{\alpha_{n_0}}{\beta_{n_0}} R^w_{proj} + \| T_{n+1} \|.
\tag{184}
\]

\[
\| R^{(\theta)}_{n+1} \| \leq \frac{\alpha_n}{\beta_n} \| W_1 \| \| W_2^{-1} \| \| w_{n+1} - w^* \| + \| W_1 \| \| W_2^{-1} \| \frac{\alpha_{n_0}}{\beta_{n_0}} R^w_{proj} + \| T_{n+1} \|.
\tag{185}
\]
\[ \frac{\alpha_n}{\beta_n} |W_1||W_2^{-1}| \left[ \frac{3}{2} R_{\text{proj}}^w + \frac{\epsilon}{q_2} C_R^\theta \| \Gamma_2 \| C_{16,w} R_{\text{proj}}^\theta \right] \] (186)

\[ + \frac{\alpha_{n_0}}{\beta_{n_0}} |W_1||W_2^{-1}| R_{\text{proj}}^w + \frac{2\epsilon^{3/2}}{q_1} C_{25} |W_1 W_2^{-1}| \frac{\alpha_n}{\beta_n} C_R^\theta R_{\text{proj}}^w \] (187)

\[ \leq \frac{\alpha_{n_0}}{\beta_{n_0}} |W_1||W_2^{-1}| \left[ \frac{5}{2} R_{\text{proj}}^w + \frac{\epsilon}{q_2} C_R^\theta \| \Gamma_2 \| C_{16,w} R_{\text{proj}}^\theta + \frac{2\epsilon^{3/2}}{q_1} C_{25} |W_1 W_2^{-1}| \right] \] (188)

\[ = \frac{\alpha_{n_0}}{\beta_{n_0}} \left( C_{28,\theta} R_{\text{proj}}^\theta + C_{28,w} R_{\text{proj}}^w \right), \] (189)

where the first relation holds due to Definitions (81) and (82) along with the facts that \( |I - \alpha_j X_j| \leq 1 \) and \( |w_{n_0} - w^*| \leq R_{\text{proj}}^w \) which themselves hold because of Lemma 15 and the event \( \mathcal{G}^\theta_{n_0} \), respectively. The second relation holds due to Lemma 27, Statement 4 and Lemma 26. The third relation holds because \( \alpha_n/\beta_n \leq \alpha_{n_0}/\beta_{n_0} \) for \( n \geq n_0 \).

The bound on \( |L_{n+1}^{(w)}| \) follows from the definition of \( A_n \) together with Lemma 21. The bound on \( |\Delta_{n+1}^{(\theta)}| \) follows from the definition in (73) along with the facts that \( |I - \alpha_k X_1| \leq 1 \) and \( \theta_{n_0} - \theta^* \leq R_{\text{proj}}^\theta \), which themselves hold due to Lemma 15 and the event \( \mathcal{G}^\theta_{n_0} \), respectively.

The last statement of the lemma follows from the first three statements.

Let us now define

\[ K_9 = \left[ \frac{2}{3} C_{28,\theta} + \frac{2}{3} C_{28,w} \frac{R_{\text{proj}}^w}{R_{\text{proj}}^w} \right]^{1/(\alpha - \beta)}. \] (190)

Now we are ready to prove Lemma 9.

**Proof of Lemma 9.** To get the desired result, it suffices to show that \( |\theta_{n+1} - \theta^*| \leq C_R^\theta R_{\text{proj}}^w \) and \( |w_{n+1} - w^*| \leq C_R R_{\text{proj}}^\theta \) on the event \( \mathcal{G}^\theta_{n_0} \cap A_n \).

Assume the event \( \mathcal{G}^\theta_{n_0} \cap A_n \) holds. Since \( n_0 \geq K_9 \), we have \( \alpha_{n_0}/\beta_{n_0} \leq \left( \frac{4}{3} C_{28,\theta} + \frac{4}{3} C_{28,w} \frac{R_{\text{proj}}^w}{R_{\text{proj}}^w} \right)^{-1} \). Using this along with the bound on \( |\theta_{n+1} - \theta^*| \) from Lemma 28, item 4, it is easy to see that \( |\theta_{n+1} - \theta^*| \leq C_R^\theta R_{\text{proj}}^w \), as desired. The bound on \( |w_{n+1} - w^*| \) is straightforward from Lemma 27, item 4.

## G Azuma-Hoeffding inequality to bound \( L_{n+1}^{(\theta)} \) and \( L_{n+1}^{(w)} \)

**Lemma 29.** Let \( L_\theta, L_w, \alpha_n \) and \( b_n \) be as in Table 3. Then,

\[ \mathbb{P} \left\{ \mathcal{G}_n, |L_{n+1}^{(\theta)}| \geq \epsilon \left| \mathcal{G}^\theta_{n_0} \right. \right\} \leq 2d^2 \text{exp} \left( -\frac{\epsilon^2}{d^2 L_\theta \alpha_n} \right) \] (191)

and

\[ \mathbb{P} \left\{ \mathcal{G}_n, |L_{n+1}^{(w)}| \geq \epsilon \left| \mathcal{G}^w_{n_0} \right. \right\} \leq 2d^2 \text{exp} \left( -\frac{\epsilon^2}{d^2 L_w b_n} \right). \] (192)

**Proof.** Recall the definitions of \( \mathcal{G}_n \) from (20) and \( L_{n+1}^{(\theta)} \) from (72). Let \( A_{k,n} \equiv \alpha_k \prod_{j=k+1}^n [I - \alpha_j X_j] \). Then

\[ \mathbb{P} \left\{ \mathcal{G}_n, |L_{n+1}^{(\theta)}| \geq \epsilon \left| \mathcal{G}^\theta_{n_0} \right. \right\} \] (193)

\[ = \mathbb{P} \left\{ \mathcal{G}_n, \sum_{k=n_0}^n A_{k,n} \left[ -W_1 W_2^{-1} M_{k+1}^{(2)} + M_{k+1}^{(1)} \right] \geq \epsilon \left| \mathcal{G}^\theta_{n_0} \right. \right\} \] (194)

\[ = \mathbb{P} \left\{ \mathcal{G}_n, \sum_{k=n_0}^n A_{k,n} \left[ -W_1 W_2^{-1} M_{k+1}^{(2)} + M_{k+1}^{(1)} \right] 1_{\mathcal{G}_k \cap \mathcal{G}^\theta_{n_0}} \geq \epsilon \left| \mathcal{G}^\theta_{n_0} \right. \right\} \] (195)

\[ \leq \mathbb{P} \left\{ \sum_{k=n_0}^n A_{k,n} \left[ -W_1 W_2^{-1} M_{k+1}^{(2)} + M_{k+1}^{(1)} \right] 1_{\mathcal{G}_k \cap \mathcal{G}^\theta_{n_0}} \geq \epsilon \left| \mathcal{G}^\theta_{n_0} \right. \right\} \] (196)

\[ \leq \sum_{i=1}^d \sum_{j=1}^d \mathbb{P} \left\{ \sum_{k=n_0}^n A_{ij,k,n} \left[ -W_1 W_2^{-1} M_{k+1}^{(2)} + M_{k+1}^{(1)} \right] 1_{\mathcal{G}_k \cap \mathcal{G}^\theta_{n_0}} \geq \epsilon \left| \mathcal{G}^\theta_{n_0} \right. \right\}, \] (197)
where $x_j$ denotes the $j$-th element of the vector $x$, while $A_{ij}^{(t)}$ is the $ij$-th entry of the matrix $A_{k,n}$. Our arguments for the last inequality are as follows. First, the term within $\| \cdot \|$ in (196) is a vector, call it $\mathcal{X}$; clearly, $\| \mathcal{X} \| \geq \epsilon$ implies $| \mathcal{X}_i | \geq \epsilon / \sqrt{d}$ for at least one coordinate $\mathcal{X}_i$ of $\mathcal{X}$. Next, each $\mathcal{X}_i$ is itself of the form $\sum_{j=1}^{n} \mathcal{X}_{ij}$, where the scalar $\mathcal{X}_{ij} = \sum_{k=n+1}^{n} \mathcal{X}_{ij}^{(t)} \times A_{k,n}$ with $\mathcal{X}_{ij}^{(t)} = a_{ij}^{(t)} \times [-W_1 W_2^{-1} M_{j+k+1}^{(2)} + M_{j+k+1}^{(1)}]$. Consequently, $| \mathcal{X}_i | \geq \epsilon / \sqrt{d}$ implies $| \mathcal{X}_j | \geq \epsilon / (d \sqrt{d})$ for at least one $j$.

Using the union bound, it is now easy to see that (197) holds, as desired.

Let $\mathbb{P}'$ denote the probability measure obtained by conditioning $\mathbb{P}$ on $\mathcal{G}_n$; that is, $\mathbb{P}'(A) = \mathbb{P}(A | \mathcal{G}_n)$. Then, 

$$
\mathbb{P} \left\{ | \mathcal{Y}_j | \geq \frac{\epsilon}{d \sqrt{d}} \middle| \mathcal{G}_n \right\} = \mathbb{P}' \left\{ | \mathcal{Y}_j | \geq \frac{\epsilon}{d \sqrt{d}} \right\}.
$$

We want to bound the RHS using the Azuma-Hoeffding inequality. To this end, let $\mathbb{E}'$ denote the expectation with respect to $\mathbb{P}'$ and $\mathcal{F}_k' := \mathcal{F}_k \cap \mathcal{G}_n$. We now show that $\{ M_k \}_{k \geq n}$ is a martingale difference sequence w.r.t. $\mathcal{F}_k'$ under $\mathbb{P}'$. Observe that, for all $F \in \mathcal{F}_k'$,

$$
\int_F \mathbb{E}'[M_{k+1} | \mathcal{F}_k'] \, d\mathbb{P}' = \mathbb{E}'[1_F M_{k+1} | \mathcal{G}_n] = \mathbb{E}[1_F \cap \mathcal{G}_n M_{k+1} | \mathcal{G}_n] / \mathbb{P}[\mathcal{G}_n]
$$

This implies that

$$
\mathbb{E}'[| \mathcal{Y}_{k+1} | \mathcal{F}_k'] = \mathbb{E}' \left[ A_{ij}^{(t)} \left[ -W_1 W_2^{-1} M_{j+k+1}^{(2)} + M_{j+k+1}^{(1)} \right] | \mathcal{F}_k' \right] = A_{ij}^{(t)} \mathbb{E} \left[ -W_1 W_2^{-1} M_{j+k+1}^{(2)} + M_{j+k+1}^{(1)} \right] | \mathcal{F}_k' = 0.
$$

Further, observe that

$$
| \mathcal{Y}_{k+1} | = | A_{ij}^{(t)} \left[ -W_1 W_2^{-1} M_{j+k+1}^{(2)} + M_{j+k+1}^{(1)} \right] | \mathcal{F}_k' |
$$

where (208) holds because for matrix $A$, $\max_{i,j} | A_{ij}^{(t)} | = \| A \|_{\max} \leq \| A \|_2$. (209) follows from the noise condition (see Defn. 1)

Now, applying the Azuma-Hoeffding inequality to the RHS of (198) and using the fact that $\sum_{k=n+1}^{n} \alpha_k e^{-q_0 \sum_{j=k+1}^{k+1} \alpha_j} \leq \alpha_{k+1}$, we obtain (191).

Repeating the same steps above for $L_{n+1}^{(w)}$ (see (76)), we obtain the bound in (192).

### H Proof of Lemma 12

**Lemma 30.** Fix some $n_0 \in \mathbb{N}$. The following holds for $n \geq n_0 + 1$:

1. $\sum_{k=n_0+1}^{n} e^{-q_0 \sum_{j=k+1}^{k+1} \beta_j} \beta_k e^{-q_0 \sum_{j=n_0}^{n} \alpha_j} \leq \frac{2 q_0 e^{-q_0 \sum_{j=n_0+1}^{n} \alpha_j}}{q_0 \min \{ q_1, q_2 \}}$, for $n_0 \geq K_{30,a}$ where $K_{30,a} = 2^{(1/\alpha - \beta)}$. 


2. For any $u \in \mathbb{R}^{\infty}$ that is $\beta$-moderate from $n_0$ onwards (see Def. 11),
\[
\sum_{k=n_0+1}^{n} e^{-q_2 \sum_{j=k+1}^{n} \beta_j \beta_k \frac{\alpha_{k-1}}{\beta_{k-1}} u_{k-1}} \leq \frac{2e^{q_2/2} \alpha_{n-1}}{\beta_{n-1}} u_{n-1}.
\]

3. \[
\sum_{k=n_0+1}^{n} e^{-q_2 \sum_{j=k+1}^{n} \beta_j \beta_k \epsilon_k^{(j)}} \leq C_{30} \epsilon_n^{(j)} \text{ for } n_0 \geq K_{30,b} \text{ where } K_{30,b} = (3\alpha/q_2)^{1/(1-\beta)} - 2 \text{ and } C_{30} = 2e^{q_2/2}/q_2.
\]

Proof: For the first claim, denote $t_{n+1} = \sum_{j=0}^{n} \alpha_j$ and $s_{n+1} = \sum_{j=0}^{n} \beta_j$. Hence, $t_{n+1} - t_{n_0} = \sum_{j=n_0}^{n} \alpha_j$ and $s_{n+1} - s_{n_0} = \sum_{j=n_0}^{n} \beta_j$. Clearly,

\[
\sum_{k=n_0+1}^{n} e^{-q_2 ((s_{n+1} - s_{n_0}) - (s_{k+1} - s_{k_0})) - q_1 (t_{k+1} - t_{n+1})} \beta_k
\]

(211)

\[
\leq \sum_{k=n_0+1}^{n} e^{-q_{\min} ((s_{n+1} - s_{k+1}) + (t_{k+1} - t_{n+1}))} \beta_k
\]

(212)

\[
= e^{-q_{\min} (t_{n+1} - t_{n_0+1})} \sum_{k=n_0+1}^{n} e^{-q_{\min} ((s_{n+1} - s_{k+1}) - (t_{n+1} - t_{k+1}))} \beta_k
\]

(213)

\[
\leq e^{-q_{\min} (t_{n+1} - t_{n_0+1})} q_{\min}/2 \sum_{k=n_0+1}^{n} e^{-q_{\min} (s_{n+1} - s_k)/2} \beta_k
\]

(214)

\[
\leq e^{-q_{\min} (t_{n+1} - t_{n_0+1})} q_{\min}/2 \sum_{k=n_0+1}^{n} e^{-q_{\min} (s_{n+1} - s_k)/2} \beta_k
\]

(215)

\[
\leq e^{-q_{\min} (t_{n+1} - t_{n_0+1})} q_{\min}/2 \int_{s_{n_0+1}}^{s_{n+1}} e^{-q_{\min} (s_{n+1} - \tau)/2} d\tau
\]

(216)

\[
\leq e^{-q_{\min} (t_{n+1} - t_{n_0+1})} q_{\min}/2 \left[ \frac{1 - e^{-q_{\min} (s_{n+1} - s_{n_0+1})/2}}{q_{\min}/2} \right]
\]

(217)

\[
\leq \frac{2e^{q_{\min}/2}}{q_{\min}} e^{-q_{\min} (t_{n+1} - t_{n_0+1})}
\]

(218)

where (214) holds since, for all $j \geq K_{30,a}$, $(s_{n+1} - s_{k+1}) - (t_{n+1} - t_{k+1}) \geq (s_{n+1} - s_{k+1})/2$ which itself holds because $\beta_j/2 \geq \alpha_j$, and (216) follows by treating the sum as a left Riemann sum.

For the second claim, observe that

\[
\sum_{k=n_0+1}^{n} e^{-q_2 \sum_{j=k+1}^{n} \beta_j \beta_k \frac{\alpha_{k-1}}{\beta_{k-1}} u_{k-1}}
\]

(219)

\[
\leq \left( \sup_{n_0+1 \leq k \leq n} e^{-(q_2/2) \sum_{j=k+1}^{n} \beta_j \beta_k \frac{\alpha_{k-1}}{\beta_{k-1}} u_{k-1}} \right) \sum_{k=n_0+1}^{n} e^{-(q_2/2) \sum_{j=k+1}^{n} \beta_j \beta_k}
\]

(220)

\[
\leq \left( \sup_{n_0+1 \leq k \leq n} e^{-(q_2/2) \sum_{j=k+1}^{n} \beta_j \beta_k \frac{\alpha_{k-1}}{\beta_{k-1}} u_{k-1}} \right) e^{q_2/2} \sum_{k=n_0+1}^{n} e^{-(q_2/2) \sum_{j=k}^{n} \beta_j \beta_k}
\]

(221)

\[
\leq \left( \sup_{n_0+1 \leq k \leq n} e^{-(q_2/2) \sum_{j=k+1}^{n} \beta_j \beta_k \frac{\alpha_{k-1}}{\beta_{k-1}} u_{k-1}} \right) e^{q_2/2} \int_{s_{n_0+1}}^{s_{n+1}} e^{-(q_2/2)(s_{n+1} - \tau)} d\tau
\]

(222)

\[
\leq \left( \sup_{n_0+1 \leq k \leq n} e^{-(q_2/2) \sum_{j=k+1}^{n} \beta_j \beta_k \frac{\alpha_{k-1}}{\beta_{k-1}} u_{k-1}} \right) \frac{2e^{q_2/2}}{q_2}
\]

(223)

where (221) follows because $\beta_k \leq 1$, and (222) follows by treating the sum as a left Riemann sum. In order to obtain the claim, it is now enough to show that the term in the supremum is monotonically increasing. For that we need to show that

\[
e^{-q_2/2} \beta_k \beta_k \frac{\alpha_{k-1}}{\beta_{k-1}} u_{k-1} \leq \frac{\alpha_k}{\beta_k} u_k.
\]

(224)

But this is exactly the $\beta$-moderate behavior, which is assumed true here.

For the third term, observe that

\[
\sum_{k=n_0+1}^{n} \left[ e^{-q_2 \sum_{j=k+1}^{n} \beta_j} \right] \beta_k \epsilon_k^{(j)}
\]

(225)
3.

\[ \sum_{k=n_0+1}^{n} e^{-q_2/2 \sum_{j=k+1}^{n} \beta_j} \leq \left( \sup_{n_0+1 \leq k \leq n} e^{-q_2/2 \sum_{j=k+1}^{n} \beta_j} \right) \sum_{k=n_0+1}^{n} e^{-q_2/2 \sum_{j=k+1}^{n} \beta_j} \beta_k \]

(226)

\[ \leq \left( \sup_{n_0+1 \leq k \leq n} e^{-q_2/2 \sum_{j=k+1}^{n} \beta_j} \right) \frac{2e^{q_2/2}}{q_2}, \]

(227)

where the last relation follows as in the proof of the second claim.

As in the second claim, in order to get the desired result, we show that the terms in the supremum expression are monotonically increasing. For this, we only need to verify if

\[ \frac{\epsilon_{k-1}^{(\theta)}}{\epsilon_k^{(\theta)}} \leq e^{(q_2/2)\beta_{k+1}}. \]

(228)

But this is true since

\[ \frac{\epsilon_{k-1}^{(\theta)}}{\epsilon_k^{(\theta)}} \leq \left( \frac{k+1}{k} \right)^{\alpha/2} \leq e^{\alpha/(2k)} \leq e^{q_2/2\beta_{k+1}/2}. \]

(229)

The first relation follows from (14) by cancelling out the constants and by dropping the ratio of log terms since the latter is bounded from above by 1. The last relation follows from the fact that

\[ \frac{(k+2)^\beta}{k} = \frac{k+2}{k} \frac{1}{(k+2)^{1-\beta}} \leq 3 \frac{1}{(k+2)^{1-\beta}} \leq \frac{q_2}{\alpha \beta}, \]

(230)

in which the rightmost inequality itself is true since \( n_0 \geq [3\alpha/q_2]^{1/(1-\beta)} - 2 \), and \( k \geq n_0 \) together imply \( (k+2)^{1-\beta} \geq [3\alpha/q_2] \).

By exploiting the monotonicity of (228) in (227), the desired result is now easy to see.

Lemma 31. The following statements hold.

1. \( \frac{c_n^{(w)}}{c_{n+1}^{(w)}} \leq e \) for \( n \geq 1 \).

2. Suppose \( u \in \mathbb{R}_{+}^\infty \) is \( \alpha \)-moderate from some \( k_0 \) onwards and \( n \geq k_0 + 1 \). Then, \( \frac{u_{n-1}}{u_{n+1}} \leq e^{q_1} \).

3. \( \frac{\alpha_{n-1}}{\beta_{n-1}} \leq e^{2(\alpha-\beta)} \frac{\alpha_{n+1}}{\beta_{n+1}} \) for \( n \geq 1 \).

Proof. Employing Lemma 19, with \( B_1 = 1, B_2 = [4d^2/\theta]^{1/p}, B_3 = 0, x = -\beta/2, y = 0 \) and \( z = \beta/2 \), we have

\[ \frac{c_n^{(w)}}{c_{n+1}^{(w)}} = \frac{(n+1)^{-\beta/2} \sqrt{\ln[B_2(n+1)]} + (n+1)^{-\beta/2} \sqrt{\ln[B_2(n+2)]}}{(n+2)^{-\beta/2} \sqrt{\ln[B_2(n+2)]}} \leq \left[ 1 + \frac{1}{n+1} \right]^{-\beta/2} \leq e^{\beta/2} \leq e, \]

(231)

where the last relation follows since \( \beta/2 \leq 1 \). This proves the first statement.

Now, consider the second statement. Since \( u \) is \( \alpha \)-moderate, \( \frac{u_{n-1}}{u_{n+1}} \leq 1 \), and \( \sup \beta_k \leq 1 \), we have \( u_{n}/u_{n+1} \leq e^{q_1/2} \) for \( n \geq k_0 \). The desired result is now easy to see.

Finally, since

\[ \frac{\alpha_{n-1}}{\beta_{n-1}} \leq e^{\alpha-\beta}, \]

(232)

it is easy to see that the third statement holds.

Lemma 32. Fix some \( n_0 \in \mathbb{N} \) and let \( n \geq n_0 - 1 \). Let \( u \in \mathbb{R}_{+}^\infty \) be a decreasing sequence that is both \( \alpha \)-moderate and \( \beta \)-moderate from \( n_0 \) onwards (see Defs. 10 and 11). Suppose that the event \( \mathcal{W}_n(u) \) holds (see (37)). Then

\[ \|R_n^{(\theta)}\| \leq \|W_1W_2^{-1}\| \left[ 1 + 2e^{q_1/2} \frac{q_1}{C_{25}C_{16,\theta}} + \frac{\alpha_{n-1}}{\beta_{n-1}} u_{n-1} + \frac{\alpha_{n-1}}{\beta_{n-0}} \|w_{n-0} - w\| C_{16,\theta} e^{-q_1 \sum_{j=n+1}^{\infty} \alpha_j} \right] \]

(233)

if \( n \geq n_0 \). If, additionally, \( \|L_n^{(\theta)}\| \|1_{n \geq n_0 + 1}\| \leq e^{(\theta)} \leq \epsilon_{n-1}, \) then

\[ \|\theta_n - \theta^*\| \leq C_{32,a} \left[ \|\theta_n - \theta^*\| + \frac{\alpha_{n-1}}{\beta_{n-0}} \|w_{n-0} - w\| \right] e^{-q_1 \sum_{j=n+1}^{\infty} \alpha_j} + C_{32,b} \frac{\alpha_{n-1}}{\beta_{n-1}} u_{n-1} + e^{(\theta)} \]

(234)
if \( n \geq n_0 \), and

\[
\| R_{n+1}^{(w)} \| \leq C_{16,w} \| \Gamma_2 \| \left[ C_{32,c}(n_0) e^{-q_n \min \{ n+1, \alpha_j \}} + C_{32,b} \frac{2 e^{q_2/2}}{q_2} \alpha_{n-1} \beta_{n-1} \| u_{n-1} \| + C_{30} \epsilon_{n-1}^{(g)} \right] \tag{235}
\]

if \( n \geq n_0 - 1 \geq \max\{ K_{30,a}, K_{30,b} \} - 1 \). Here, \( K_{30,a}, K_{30,b} \) are defined in the statement of Lemma 30,

\[
C_{32,c}(n_0) = \left[ \beta_{n_0} \| \theta_{n_0} - \theta^* \| + C_{32,a} \| \epsilon_{q_1/2} \| \min \left\{ \| w_{n_0} - w^* \|, \| w_{n_0} - w^* \| \right\} \right] ,
\]

\( C_{32,a} = C_{16,\theta} \max \{ \| W_1 W_2^{-1} \|, 1 \} \) and \( C_{32,b} = \| W_1 W_2^{-1} \| \left( 1 + \frac{2 e^{q_1/2}}{q_1} C_{25} C_{16,\theta} \right) \).

**Remark 33. Difference between Lemma 32 and Lemma 28:**

- In Lemma 32, we assume that \( \| w_k - w^* \| \leq u_k \) for all \( n \geq k \geq n_0 \). Using this, we try and obtain better rates of convergence for \( \| w_k - w^* \| \). In other words, this is part of our inductive proof where we are showing the \( \ell \)th step to be true.
- In Lemma 28, we establish the base case of the above induction. In particular, we try and show that the iterates are bounded with high probability. In order to prove this, we use another induction on the iterate index which reads as: if the iterates are bounded until time \( n \), what is the bound at the \( n + 1 \)th step.

**Proof of Lemma 32.** We first establish (233). Notice from (74) that \( R_{n_0}^{(g)} = 0 \) and hence (233) trivially holds for \( n = n_0 \). As for \( n \geq n_0 + 1 \), from (81) we have

\[
\| R_{n}^{(g)} \| \leq \| W_1 W_2^{-1} \| \left[ \frac{\alpha_{n-1}}{\beta_{n-1}} u_n + C_{16,\theta} \frac{\alpha_{n_0}}{\beta_{n_0}} \| w_{n-1} - w^* \| e^{-q_1 \min \{ n+1, \alpha_j \}} + \| T_n \| \right] \tag{236}
\]

\[
\leq \| W_1 W_2^{-1} \| \left[ \frac{\alpha_{n-1}}{\beta_{n-1}} u_n + C_{16,\theta} \frac{\alpha_{n_0}}{\beta_{n_0}} \| w_{n-1} - w^* \| e^{-q_1 \min \{ n+1, \alpha_j \}} + \frac{2 e^{q_1/2}}{q_1} C_{25} C_{16,\theta} \frac{\alpha_{n-1}}{\beta_{n-1}} u_{n-1} \right] \tag{237}
\]

\[
\leq \| W_1 W_2^{-1} \| \left[ \left( 1 + \frac{2 e^{q_1/2}}{q_1} C_{25} C_{16,\theta} \right) \frac{\alpha_{n-1}}{\beta_{n-1}} u_{n-1} + \frac{\alpha_{n_0}}{\beta_{n_0}} \| w_{n-1} - w^* \| C_{16,\theta} e^{-q_1 \min \{ n+1, \alpha_j \}} \right] , \tag{238}
\]

where the first relation follows using Lemma 16 and the fact that the event \( W_{n}(u) \) holds, the second relation is due to Lemma 25 (recall that \( u_n \) is \( \alpha \)moderate), while the third relation is due to the fact that \( u_n \) monotonically decreases.

We now derive the bound (234) for \( \| \theta_n - \theta^* \| \). Since \( C_{16,\theta} \geq 1 \) implies \( C_{32,a} \geq 1 \), it follows that (234) trivially holds for \( n = n_0 \). As for \( n \geq n_0 + 1 \),

\[
\| \theta_n - \theta^* \| \leq \| \Delta_n^{(g)} \| + \| R_n^{(g)} \| + \| L_n^{(g)} \| \leq C_{16,\theta} \| \theta_{n_0} - \theta^* \| e^{-q_1 \min \{ n_0+1, \alpha_j \}} + \| W_1 W_2^{-1} \| \left[ \left( 1 + \frac{2 e^{q_1/2}}{q_1} C_{25} C_{16,\theta} \right) \frac{\alpha_{n_0}}{\beta_{n_0}} \| w_{n_0} - w^* \| C_{16,\theta} e^{-q_1 \min \{ n_0+1, \alpha_j \}} \right] + \| \epsilon_{n-1}^{(g)} \| \tag{239}
\]

\[
\leq C_{32,a} \left[ \| \theta_{n_0} - \theta^* \| + \frac{\alpha_{n_0}}{\beta_{n_0}} \| w_{n_0} - w^* \| \right] e^{-q_1 \min \{ n_0+1, \alpha_j \}} + C_{32,b} \frac{\alpha_{n-1}}{\beta_{n-1}} u_{n-1} + \| \epsilon_{n-1}^{(g)} \| , \tag{240}
\]

where the first relation follows by (75), the second one holds on account of (88) of Lemma 17, (233), and our assumption that \( \| L_n^{(g)} \| \leq \| \epsilon_{n-1}^{(g)} \| \), while the third relation is obtained by dropping \( \alpha_{n_0} = (n_0 + 1)^{-\alpha} \) term from the exponent multiplying \( \| \theta_{n_0} - \theta^* \| \).

Lastly, for the third statement, (235) trivially holds for \( n = n_0 - 1 \) since \( R_{n_0}^{(w)} = 0 \) by definition (78). Similarly, for \( n = n_0 \), it follows from (78) that

\[
\| R_{n_0+1}^{(w)} \| \leq \beta_{n_0} \| \Gamma_2 \| \| \theta_{n_0} - \theta^* \| . \tag{242}
\]

From this and the fact that \( C_{16,w} \geq 1 \), it is easy to see that (235) holds again.

For \( n \geq n_0 + 1 \), we break the summation in (78) into the first and the rest of the terms; thus,

\[
\| R_{n+1}^{(w)} \| \leq \prod_{j=n_0+1}^{n} \| I - \beta_j W_2 \| \beta_{n_0} \| \Gamma_2 \| \| \theta_{n_0} - \theta^* \| + \sum_{k=n_0+1}^{n} \left[ \prod_{j=k+1}^{n} \| I - \beta_j W_2 \| \right] \beta_k \| \Gamma_2 (\theta_k - \theta^*) \| \tag{243}
\]
This gives the desired result. Hence, (250) to (252) can be bounded by

\[
\leq C_{16,w} \| \Gamma_2 \| \beta_{n_0} \| \theta_{n_0} - \theta^* \| e^{-q_2 \sum_{j=n_0+1}^{n} \beta_j}
\]

(247) follows by applying Lemma 16. (248) follows because \( e^q e^{-q_1 \alpha_j} \geq 1 \), and finally (251) follows recalling the first statement from Lemma 30. Now, using the second and third statements in Lemma 30 (recall that \( u_n \) is \( \beta \)-moderate), it is easy to see that the expression in (252) can be bounded by

\[
C_{16,w} \| \Gamma_2 \| C_{32,b} e^{q_1} \left[ \frac{2 e^{-q_2/2} \alpha_n \beta_n^{-1} u_n - 1 + C_{30} \epsilon_n^{-1}}{q_2} \right].
\]

(253)

Since \( q_2 \geq q_{\text{min}} \) and \( \beta_j \geq \alpha_j \), the term in (250) can be bounded by

\[
C_{16,w} \| \Gamma_2 \| \beta_{n_0} \| \theta_{n_0} - \theta^* \| e^{-q_{\text{min}} \sum_{j=n_0+1}^{n} \alpha_j}.
\]

(254)

Hence, (250) to (252) can be bounded by

\[
C_{16,w} \| \Gamma_2 \| C_{32,b} e^{-q_{\text{min}} \sum_{j=n_0+1}^{n} \alpha_j} + C_{32,b} \left[ \frac{2 e^{-q_2/2} \alpha_n \beta_n^{-1} u_n - 1 + C_{30} \epsilon_n^{-1}}{q_2} \right].
\]

(255)

This gives the desired result.

We define

\[
A_{1,n_0} = e + C_{16,w} \| \Gamma_2 \| C_{32,c} e^{-q_{\text{min}} \sum_{j=n_0+1}^{n} \alpha_j} + C_{32,b} \left[ \frac{2 e^{-q_2/2} \alpha_n \beta_n^{-1} u_n - 1 + C_{30} \epsilon_n^{-1}}{q_2} \right] + e^{2 \epsilon_{n_0} \| \Gamma_2 \| C_{30},
\]

(256)

Proof of Lemma 12. Note that (42) follows immediately from (234). Define now

\[
A'_1 = 1 + \frac{1}{\epsilon_{n_0} \| \Gamma_2 \| C_{32,c} + C_{16,w} \| \theta_{n_0} - \theta^* \|}
\]

(257)
and observe that
\[ A_{1,n_0} = eA_1^* + e^2A_1^* \]  
and
\[ A_2 = e^{q_2/2}(\alpha - \beta) A_2'. \]  

For \( n \geq n_0 \), observe that
\[
\| w_{n+1} - w^* \| \leq \| \Delta_{n+1}^{(w)} + R_{n+1}^{(w)} + I_{n+1}^{(w)} \|
\leq C_{16,w}\| w_{n_0} - w^* \| e^{-q_2 \sum_{j=n_0}^{\infty} \beta_j}
+ C_{16,w} \| \Gamma_2 \| \left[ C_{32,c}(n_0)e^{-q_3\alpha_{n+1}} + C_{32,b} \frac{2e^{q_3/2} \alpha_{n-1}}{q_2} \beta_{n-1} u_{n-1} + C_{30} \right]
+ e_n(w),
\]  

Here, the first relation follows from (79). In the second relation, (263) follows from Lemma 17, while (264) follows from Lemma 32, third statement. As for (265), it follows from our assumption that \( \| I_{n}^{(w)} \| [n \geq n_0 + 1] \leq \epsilon_n(w). \)  

Because of Lemma 35, for \( n \geq n_0 \geq K_{35,b} \), the above relation can be written as:
\[
\| w_{n+1} - w^* \| \leq A_1 \epsilon_n^{(w)} + A_1 \epsilon_n^{-1} + A_2 \frac{\alpha_{n-1}}{\beta_{n-1}} u_{n-1}.
\]  

Again, from Lemma 35 and the fact \( n_0 \geq K_{35,a} + 1 \), we have \( \epsilon_n^{(w)} \leq \epsilon_n(w) \) for any \( n \geq n_0 \). This implies
\[
\| w_{n+1} - w^* \| \leq A_1 \epsilon_n^{(w)} + A_1 \epsilon_n^{(w)} - 1 + A_2 \frac{\alpha_{n-1}}{\beta_{n-1}} u_{n-1}.
\]  

Using Lemma 31, since \( u \) is \( \alpha \)-moderate from \( n_0 - 1 \) onwards, we finally have
\[
\| w_{n+1} - w^* \| \leq A_1 \epsilon_n^{(w)} + A_2 \frac{\alpha_{n-1}}{\beta_{n-1}} u_{n-1},
\]  

Lastly, notice that (43) also holds for \( n = n_0 - 1 \) because
\[
\| w_{n+1} - w^* \| = \| w_{n_0} - w^* \| e_n^{(w)} \leq A_1 \epsilon_n^{(w)} = A_1 \epsilon_n^{(w)} \leq A_1 \epsilon_n^{(w)} + A_2 \frac{\alpha_{n-1}}{\beta_{n-1}} u_{n+1},
\]  

where the second relation holds because \( C_{16,w} \geq 1 \) by definition, and the third relation due to the definitions of \( A_1 \) and \( A_1' \).

### I \( \alpha \)-moderateness, \( \beta \)-moderateness, and monotonicity of \( \{ u_n(\ell) \} \)

**Lemma 34.** Let \( \ell \leq \beta / 2(\alpha - \beta) \) and assume that \( u_n(\ell) \) is as in (44) for some constants \( B_1, B_3 \geq 0 \) and \( B_2 \geq 1 \) (these constants may depend on \( \ell \)), where at least one of \( B_1 \) and \( B_3 \) is strictly positive. Then, \( u_n(\ell) \) is \( \alpha \)-moderate from \( K_{20,a}(\beta / 2) \) onwards and \( \beta \)-moderate from \( K_{20,b}(\beta / 2) \) onwards. Furthermore, \( \{ u_n(\ell) \} \) is monotonically decreasing from \( e^{3/2} / B_2 \) onwards.

**Proof.** It is easy to see from Lemma 19 that, for \( z \geq \max \{ \beta/2, (\alpha - \beta) \ell \} \),
\[
\frac{u_n(\ell)}{u_{n+1}(\ell)} = \frac{B_1(n+1)^{-\beta/2} \sqrt{\ln[B_2(n+1)]} + B_3(n+1)^{-1/2}(\alpha - \beta) \ell}{B_1(n+2)^{-\beta/2} \sqrt{\ln[B_2(n+2)]} + B_3(n+2)^{-1/2}(\alpha - \beta) \ell} \leq \left( \frac{n+1}{n+2} \right)^{-z}.
\]  

Because \( (\alpha - \beta) \ell \leq \beta / 2 \), we can pick \( z = \beta / 2 \). Then, it remains to show that
\[
\left( \frac{n+1}{n+2} \right)^{-\beta/2} \leq \frac{\alpha_{n+1}}{\alpha_n} \frac{\beta_n}{\beta_{n+1}} e^{q_2/2} \alpha_{n+1}.
\]  

But this indeed holds for \( n \geq K_{20,a}(\beta / 2) \) due to Lemma 20, Statement 1, since \( z = \beta / 2 \in [0, 1 - (\alpha - \beta)] \). Hence, \( u_n(\ell) \) is \( \alpha \)-moderate. Similarly, in order to establish that \( u_n(\ell) \) is \( \beta \)-moderate, it suffices to show that
\[
\left( \frac{n+1}{n+2} \right)^{-\beta/2} \leq \frac{\alpha_{n+1}}{\alpha_n} \frac{\beta_n}{\beta_{n+1}} e^{q_2/2} \beta_{n+1}.
\]  

which indeed holds for \( n \geq K_{20,b}(\beta / 2) \) due to Lemma 20, Statement 2.  

For monotonicity, let us first redefine \( u_n(\ell) \) in the form \( B_1 \sqrt{\ln[B_2(n+1)]} + B_3(n+1)^{-1/2}(\alpha - \beta) \ell \), where \( f(x) := x^{-\beta} \ln[B_2x] \). The claimed monotonicity then follows from the monotonicity of \( B_3(n+1)^{-1/2}(\alpha - \beta) \ell \) and the fact that, for \( x \geq e^{3/2} / B_2 \),
\[
f'(x) = -\beta x^{-\beta-1} \ln[B_2x] + x^{-\beta-1} = x^{-\beta-1} (1 - \beta \ln[B_2x]) \leq 0,
\]  

as desired. \( \square \)
Lemma 35. The following statements are true.

1. Let $K_{35,a} := ([L_0 C_{14,b}] / [L_w C_{14,w}])^{1/(\alpha - \beta)}$. For $n \geq K_{35,a}$, $\epsilon_n(\theta) \leq \epsilon_n^{(w)}$.

2. Let $K_{35,b} := \left[1 + \frac{\alpha}{2q_{\min}}\right]^{1/(1-\alpha)}$. For $n \geq n_0 \geq K_{35,b}$,

$$e^{-q_2 \sum_{j=0}^{n_0} \alpha_j} \leq e^{-q_{\min} \sum_{j=n_0+1}^{n} \alpha_j} \leq \min\{\epsilon_n(\theta) / \epsilon_n^{(w)} / \epsilon_n^{(w)}\}.$$  \hspace{1cm} (274)

Proof. The inequality in (1) holds by the definition of $\epsilon_n(\theta)$ and $\epsilon_n^{(w)}$ given in (14).

The first relation in (274) holds trivially since $q_2 \geq q_{\min}, \alpha_{n_0} \geq 0$, and $\alpha_j \leq \beta_j \forall j$.

Consider the second relation in (274). Let $\gamma \in \{\alpha, \beta\}$ (the proof holds for both $\alpha$ and $\beta$). Substituting $\epsilon_n(\theta)$ and $\epsilon_n^{(w)}$ from (14) and making use of the fact that $\log(4d^2(n + 1)/\delta)/\log(4d^2(n_0 + 1)/\delta) \geq 1$, it follows that to prove this second relation we only need to show

$$e^{-q_{\min} \sum_{j=n_0+1}^{\infty} \alpha_j} \leq \left[\frac{n+1}{n_0+1}\right]^{-\gamma/2}, \hspace{1cm} n \geq n_0.$$  \hspace{1cm} (275)

Observe that $\sum_{j=n_0+1}^{\infty} \alpha_j \geq \int_{n_0+1}^{\infty} \frac{1}{(x+1)^\alpha} dx = \frac{1}{1-\alpha}[(n + 2)^{1-\alpha} - (n_0 + 2)^{1-\alpha}]$. Hence, to establish (275), it suffices to show that

$$\left[\frac{n+1}{n_0+1}\right]^{-\gamma/2} \geq \exp \left[-q_{\min}[(n + 2)^{1-\alpha} - (n_0 + 2)^{1-\alpha}] / (1 - \alpha)\right].$$  \hspace{1cm} (276)

Equivalently, it suffices to show that

$$f(x) := \frac{q_{\min}}{1 - \alpha} \left[(x + 2)^{1-\alpha} - (n_0 + 2)^{1-\alpha}\right] - \frac{\gamma}{2} \ln(x + 1) - \ln(n_0 + 1) \geq 0$$  \hspace{1cm} (277)

for $x \geq n_0$. To this end, note that

$$f'(x) = q_{\min} (x + 2)^{-\alpha} - \frac{\gamma}{2(x + 1)}$$  \hspace{1cm} (278)

$$= \frac{q_{\min}}{(x + 1)^\alpha} \left[\frac{x + 1}{(x + 2)^\alpha} - \frac{\gamma}{2q_{\min}}\right]$$  \hspace{1cm} (279)

$$= \frac{q_{\min}}{(x + 1)^\alpha} \left[(x + 2)^{1-\alpha} - \frac{1}{(x + 2)^\alpha} - \frac{\gamma}{2q_{\min}}\right]$$  \hspace{1cm} (280)

$$\geq \frac{q_{\min}}{(x + 1)^\alpha} \left[(x + 2)^{1-\alpha} - 1 - \frac{\gamma}{2q_{\min}}\right],$$  \hspace{1cm} (281)

which is nonnegative when $x \geq K_{35,b}$. This, combined with the facts that $f(n_0) = 0$ and $n_0 \geq K_{35,b}$ implies (277) and, thereby, concludes the proof. \qed

K Applications to Reinforcement Learning: GTD2 and TDC

Here, we show with which constants Corollary 5 can be derived for GTD2 and TDC algorithms. This is done by validating the assumptions required and the constants involved.

K.1 GTD2

The GTD2 algorithm (Sutton et al. 2009) minimizes the objective function

$$J_{\text{MSPBE}}(\theta) = \frac{1}{2}(b - A\theta)^T C^{-1}(b - A\theta).$$  \hspace{1cm} (282)

The update rule of the algorithm takes the form of Equations (1) and (2) with

$$h_1(\theta, w) = A^T w,$$

$$h_2(\theta, w) = b - A\theta - Cw,$$

and

$$M_{n+1}^{(1)} = (\phi_n - \gamma\phi_n')\phi_n^T w_n - A^T w_n,$$

$$M_{n+1}^{(2)} = r_n \phi_n + \phi_n[\gamma\phi_n - \phi_n]^{T} \theta_n - \phi_n \phi_n^T w_n - [b - A\theta_n - Cw_n].$$
For GTD2, the relevant matrices are \( \Gamma_1 = 0, W_1 = -A^\top \), \( v_1 = 0 \), and \( \Gamma_2 = A, W_2 = C, v_2 = b \). Additionally, \( X_1 = \Gamma_1 - W_1 W_1^\top \Gamma_2 = A^\top C^{-1} A \). By our assumptions, both \( W_2 \) and \( X_1 \) are symmetric positive definite matrices, and thus the real part of their eigenvalues are also positive. Additionally,

\[
\|M^{(1)}_{n+1}\| \leq (1 + \gamma + \|A\|)\|w_n\|,
\|M^{(2)}_{n+1}\| = \|\Gamma \phi_n - b + [A + \phi_n(\gamma \phi_n - \phi_n)^\top]\theta_n - [\phi_n \theta_n^\top - C]w_n\|
\leq 1 + \|b\| + (1 + \gamma + \|A\|)\|\theta_n\| + (1 + \|C\|)\|w_n\|.
\]

Consequently, Assumption 1 is satisfied with constants \( m_1 = (1 + \gamma + \|A\|) \) and \( m_2 = 1 + \max(\|b\|, \gamma + \|A\|, \|C\|) \).

### K.2 TDC

The TDC algorithm is designed to minimize (282), just like GTD2. However, its update rule takes the form of Equations (1) and (2) with

\[
h_1 \theta(\theta, w) = b - A \theta + [A^\top - C]w,
h_2 \theta(\theta, w) = b - A \theta - Cw,
\]

and

\[
M^{(1)}_{n+1} = r_n \phi_n + \phi_n [\gamma \phi_n - \phi_n]^\top \theta_n - \gamma \phi_n^\top w_n - b - A \theta_n + [A^\top - C]w_n,
M^{(2)}_{n+1} = r_n \phi_n + \phi_n [\gamma \phi_n - \phi_n]^\top \theta_n - \phi_n \theta_n^\top w_n - b - A \theta_n + Cw_n.
\]

Thus, for TDC the relevant matrices in the update rules are \( \Gamma_1 = A, W_1 = [C - A^\top], v_1 = b \), and \( \Gamma_2 = A, W_2 = C, v_2 = b \). Additionally, \( X_1 = \Gamma_1 - W_1 W_1^\top \Gamma_2 = A - [C - A^\top] C^{-1} A = A^\top C^{-1} A \). By our assumptions, both \( W_2 \) and \( X_1 \) are symmetric positive definite matrices, and thus the real part of their eigenvalues are also positive. Additionally,

\[
\|M^{(1)}_{n+1}\| \leq 2 + (1 + \gamma + \|A\|)\|\theta_n\| + (\gamma + \|A\| + \|C\|)\|w_n\|,
\|M^{(2)}_{n+1}\| = 2 + (1 + \gamma + \|A\|)\|\theta_n\| + (1 + \|C\|)\|w_n\|.
\]

As a result, Assumption 1 is satisfied with constants \( m_1 = (2 + \gamma + \|A\| + \|C\|) \) and \( m_2 = (2 + \gamma + \|A\| + \|C\|) \).
| Constant | Definition | Source |
|----------|------------|--------|
| $q_1$    | $q_1 \in (0, \lambda_{\min}(X_1 + X_1^T)/2)$ | (83) |
| $q_2$    | $q_2 \in (0, \lambda_{\min}(W_2 + W_2^T)/2)$ | (84) |
| $\mu_1$  | $-\lambda_{\min}(X_1 + X_1^T) + \lambda_{\max}(X_1^T X_1)$ | Lemma 16 |
| $\mu_2$  | $-\lambda_{\min}(W_2 + W_2^T) + \lambda_{\max}(W_2^T W_2)$ | Lemma 16 |
| $K_{16,1}$ | $\left(\frac{2\lambda_{\max}(X_1^T X_1)}{\lambda_{\min}(X_1 + X_1^T)}\right)^{1/\alpha}$ | Lemma 16 |
| $K_{16,2}$ | $\left(\frac{2\lambda_{\max}(W_2^T W_2)}{\lambda_{\max}(W_2 + W_2^T)}\right)^{1/\beta}$ | Lemma 16 |
| $C_{16,\theta}$ | $\sqrt{\max_{t_1 \leq t_2 \leq K_{16,1}} \prod_{t_k = t_1}^{t_2} e^{\alpha_k (\mu_1 + 2q_1)}}$ | Lemma 16 |
| $C_{16,w}$ | $\sqrt{\max_{t_1 \leq t_2 \leq K_{16,2}} \prod_{t_k = t_1}^{t_2} e^{\alpha_k (\mu_2 + 2q_2)}}$ | Lemma 16 |
| $K_{14}(p, \hat{q})$ | $\min\{\{e^{-p} \sum_{k=1}^{n-1} \lambda_k (k+1)^{-p} \leq \hat{q}\}$ | Lemma 14 |
| $C_{14}(\alpha, q_1) e^{\alpha_1/q_1}$ | | Lemma 14 |
| $C_{14,w}$ | $C_{14}(\beta, q_2) e^{\beta_2/q_2}$ | Lemma 14 |
| $a_n$    | $\sum_{k=0}^{n-1} \frac{\alpha_k^2}{\gamma_k} e^{-2q_1 \sum_{j=k+1}^{n-1} \alpha_j}$ | Lemma 14 |
| $b_n$    | $\sum_{k=0}^{n-1} \beta_k^2 e^{-2q_2 \sum_{j=k+1}^{n-1} \beta_j}$ | Lemma 14 |
| $C_R^\theta$ | $3$ | (153) |
| $C_R^w$  | $3/2 + (e^{q_2}/q_2) \|R\|_2 \|C_{16,w}\|_{R_{\text{proj}}} C_R^\theta R_{\text{proj}}^\theta$ | (154) |
| $L_\theta$ | $2 \left[ C_{16,\theta} \left( 1 + C_R^w R_{\text{proj}}^w + C_R^\theta R_{\text{proj}}^\theta \right) + \|\theta^*\| + \|w^*\| \right] (m_2 + m_1 \|W_1\| \|W_{2-1}\|)^2$ | Lemma 29 |
| $L_w$    | $2 \left[ C_{16,w} m_2 \left( 1 + C_R^w R_{\text{proj}}^w + C_R^\theta R_{\text{proj}}^\theta \right) + \|\theta^*\| + \|w^*\| \right] (m_2 + m_1 \|W_1\| \|W_{2-1}\|)^2$ | Lemma 29 |
| $p$      | $p \in (1, \infty)$ | Section 4 |
| $K_{15,\alpha}$ | $\min\{i | \alpha_i \leq \frac{\lambda_{\min}(X_1 + X_1^T)}{2\lambda_{\max}(X_1 + X_1^T)} \}$ | Lemma 15 |
| $K_{15,\beta}$ | $\min\{i | \beta_i \leq \frac{\lambda_{\max}(W_2 + W_2^T)}{2\lambda_{\max}(W_2 + W_2^T)} \}$ | Lemma 15 |
| $K_{20,\alpha}(z)$ | $\max \left\{ \left( \frac{q_1}{2(\alpha - \beta + z)} \right)^{1/\alpha}, \left( \frac{4(\alpha - \beta + z)}{q_1} \right)^{1/(1-\alpha)} \right\}$ | Lemma 20 |
| $K_{20,\beta}(z)$ | $\max \left\{ \left( \frac{q_2}{\alpha - \beta + z} \right)^{1/\beta}, \left( \frac{4(\alpha - \beta + z)}{q_2} \right)^{1/(1-\beta)} \right\}$ | Lemma 20 |
| $K_{21,\alpha}$ | $\left[ \frac{4d^3 L_{\alpha C_{14} s} P}{\alpha (R_{\text{proj}}^s)^2} \right]^{1/\alpha} \left[ 2 \ln \left( \frac{4d^3 L_{\alpha C_{14} s} P}{\alpha (R_{\text{proj}}^s)^2} \right)^{\alpha/p} \right]^{1/\alpha}$ | Lemma 21 |
| $K_{21,\beta}$ | $\left[ \frac{4d^3 L_{\beta C_{14} w} P}{\beta (R_{\text{proj}}^s)^2} \right]^{1/\beta} \left[ 2 \ln \left( \frac{4d^3 L_{\beta C_{14} w} P}{\beta (R_{\text{proj}}^s)^2} \right)^{\beta/p} \right]^{1/\beta}$ | Lemma 21 |
| $C_{28,\theta}$ | $\|W_1\| \|W_{2-1}\|^2 \frac{4\delta}{q_2} C_R^\theta \|R\|_{\text{proj}} C_{16,w}$ | Lemma 28 |
| $C_{25}$ | $\|[X_1] + 2(\alpha - \beta)[1 + \|X_1\|]\|$ | Lemma 25 |
\begin{align}
C_{28,w} & \quad \|W_1\|_2^{-1} \left\lfloor \frac{\frac{3}{2} + 2e^{\alpha_1/2} C_{25} C_{16,\theta}}{q_{i_1}} \right\rfloor \\
K_9 & \quad \left( \frac{3}{4} C_{28,\theta} + \frac{2}{3} C_{28,w} \frac{R_{\text{proj}}^w}{R_{\text{proj}}^w} \right)^{1/(\alpha - \beta)} \\
K_{30,a} & \quad 2^{1/(\alpha - \beta)} \\
K_{30,b} & \quad (3\alpha/q_2)^{1/(1-\beta)} - 2 \\
C_{30} & \quad 2e^{q_2^2/2}/q_2 \\
C_{32,a} & \quad C_{16,w} \max\{\|W_1 W_2^{-1}\|, 1\} \\
C_{32,b} & \quad \left\lfloor \|W_1 W_2^{-1}\| \left( 1 + 2e^{\alpha_1/2} C_{25} C_{16,\theta} \right) \right\rfloor \\
C_{32,c}(n_0) & \quad \left( \frac{\beta}{\beta_{n_0}} \right)^{\|\theta_{n_0} - \theta^*\|} + C_{32,a} e^{q_1} \frac{2}{q_{\min}} \left[ \|\theta_{n_0} - \theta^*\| + \frac{\alpha_{n_0}}{\beta_{n_0}} \|w_{n_0} - w^*\| \right] \\
A_{1,n_0} & \quad e + e^{q_1} 2^{\alpha - \beta} C_{16,w} \|\Gamma_2\| C_{32,b} 2e^{q_2^2/2}/q_2 \\
A_2 & \quad e^{q_1} + 2^{\alpha - \beta} C_{16,w} \|\Gamma_2\| C_{32,b} 2e^{q_2^2/2}/q_2 \\
A_3 & \quad C_{R} R_{\text{proj}}^w \\
A_{4,n_0} & \quad A_{1,n_0} \sqrt{d^3 L_{\text{proj}}^w C_{14,w}} \sum_{i=0}^{\left\lfloor \frac{\alpha - \beta}{\gamma} \right\rfloor - 1} A_i^2 + A_{3} A_2 \left( \frac{\beta}{\beta_{n_0}} \right)^{\left\lfloor \frac{\alpha - \beta}{\gamma} \right\rfloor} \\
A_{5,n_0} & \quad \left[ C_{32,a} \left[ C_{R} R_{\text{proj}}^w + C_{R} R_{\text{proj}}^w \right] / \epsilon_{q_1} - 1 \right] \sqrt{4d^3 L_{\text{proj}}^w C_{14,\theta}} + C_{32,b} A_4 \\
A_{4,4} & \quad d^3 L_{\text{proj}}^w C_{14,w} \left( C_{16,w} \|\Gamma_2\| \right) \left[ R_{\text{proj}}^w + C_{32,a} \frac{2}{q_{\min}} \left( R_{\text{proj}}^w + R_{\text{proj}}^w \right) \right] + C_{16,w} R_{\text{proj}}^w \sum_{i=0}^{\left\lfloor \frac{\alpha - \beta}{\gamma} \right\rfloor - 1} A_i^2 \\
A_{5,4} & \quad C_{32,a} \left[ C_{R} R_{\text{proj}}^w + C_{R} R_{\text{proj}}^w \right] \\
A_4' & \quad A_{4,4} + 1 \\
A_5' & \quad 2 + A_{5,4} + C_{32,b} A_{4,4} \\
C_{3,\theta} & \quad A_2 \left( N_3 + 1 \right)^{\alpha/2} / \sqrt{\ln \left( 4d^2 (N_3 + 1)^{p/\delta} \right)} \\
C_{3,w} & \quad A_2 \left( N_3 + 1 \right)^{\beta/2} / \sqrt{\ln \left( 4d^2 (N_3 + 1)^{p/\delta} \right)} \\
K_{22,a} & \quad \left[ \frac{P}{p_p \langle p^{\alpha} c_{\theta} \rangle} \right]^{\frac{1}{\beta}} \left[ 2 \ln \left( \frac{2 \beta p_p \langle p^{\alpha} c_{\theta} \rangle}{4^\alpha} \right)^{\beta/\gamma} \right]^{1/\beta} \\
K_{22,b} & \quad \left[ e^{\alpha (c_{32,\theta} A_4 / c_{32,\theta} A_4 c_{32,\theta})^2} \right]^{\frac{1}{\alpha}} \left[ 2 \ln \left( \frac{2 \alpha (c_{32,\theta} A_4 / c_{32,\theta} A_4 c_{32,\theta})^2}{4^\alpha} \right)^{\beta/\alpha} \right]^{1/\alpha} \\
A_1'' & \quad C_{16,w} \|\Gamma_2\| C_{30} \\
A_{4,4} & \quad A_3 A_2 \left( \frac{\gamma}{\gamma_{\min}} \right) \sum_{i=0}^{\left\lfloor \frac{\alpha - \beta}{\gamma} \right\rfloor - 1} A_i^2 \sqrt{d^3 L_{\text{proj}}^w C_{14,w}} \\
A_{5,4} & \quad \sqrt{4d^3 L_{\text{proj}}^w C_{14,\theta}} \\
K_{35,a} & \quad \left( \frac{\|L_{\theta} C_{14,\theta}\|}{\|L_{\theta} C_{14,\theta}\|} \right)^{1/(\alpha - \beta)} \\
K_{35,b} & \quad \left[ 1 + \alpha/(2q_{\min}) \right]^{1/(1-\alpha)} \\
K_{3,w} & \quad \left( A_4' / R_{\text{proj}}^w \right)^{2/\beta} \left( A_4' / R_{\text{proj}}^w \right)^{2/\beta} \\
K_{3,\theta} & \quad \left( A_5' / R_{\text{proj}}^w \right)^{2/\alpha} \left( A_5' / R_{\text{proj}}^w \right)^{2/\alpha} \\
\end{align}

Table 3: Summary of all constants