Theoretical study of diffusion processes around a non-rotating neutron star

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Abstract. The general relativistic diffusion process on curved space-time manifold around a non-rotating neutron star has been analyzed. The general relativistic diffusion equation of diffusive particles around non-rotating neutron star is derived by constructing phase space in the parametrization of observer time in the hyperbolic coordinate system. This diffusion equation describes the stochastic dynamic of particles around non-rotating neutron stars. In this work we also have studied the diffusion processes around a non-rotating neutron star for asymptotic case.

1. Introduction
Diffusion processes constitute a class of stochastic processes which are characterized by two properties: the Markovian property and the continuity of trajectories. Diffusion process is a solution to a stochastic differential equation. It is a continuous-time Markov process with almost surely continuous sample paths. Some examples of diffusion processes are Brownian motion, reflected Brownian motion and Ornstein-Uhlenbeck processes. A sample path of a diffusion process models the trajectory of a particle embedded in a flowing fluid and subjected to random displacements due to collisions with molecules, which is called Brownian motion. The position of the particle is then random; its probability density function as a function of space and time is governed by an advection-diffusion equation.

The most fundamental diffusion process is Brownian motion or Wiener process. The Brownian motion process was studied originally by English botanist Robert Brown for modeling the motion of a small particle immersed in a liquid subject to molecular collisions. A mathematical model of the process was first derived by Einstein and the underlying theory was subsequently perfected by Fokker, Planck, and Wiener, among others. Furthermore, the Brownian motion as a diffusion process has developed rapidly both physically and mathematically. Research on Brownian motion is also experiencing growth both relativistic and nonrelativistic.

One study of relativistic diffusion process has been performed by Dunkel and Hanggi. In 2005 Dunkel and Hanggi did research on relativistic Brownian motion for 1+1-dimension case \cite{4}. Then, on the same year they did research on relativistic Brownian motion for 3+1-dimension case \cite{5}. Another researcher who studied about relativistic diffusion process is Joachim Herrmann. In 2009 Herrmann studied about diffusion in the special theory of relativity \cite{7}. One year later, he did research about diffusion in the general theory of relativity \cite{8}.
Based on the observations made by astronomers, it has been found that the gas particles which moving toward the compact stars, called as accretion disc. The movement of the particles is determined by the curvature of space-time \[2\]. Movement of gas particles can be viewed as a process of diffusion. To understand the movement of particles which experiencing diffusion around the compact stars or massive objects, it takes a special review of diffusion processes that meet the rules of Einstein’s theory of relativity. That theory give an explanation of the influence of the mass and energy of the curvature of space-time. Higher energy density will cause greater curvature of space-time and vice versa. In other words, the curvature of space-time around objects that have a very high density, for example in the compact stars like white dwarfs, neutron stars, and black holes; is also become very large \[1\].

In this research we will discuss about diffusion process around non-rotating neutron star. Those study refers to research done by Herrmann \[8\]. Herrmann has constructed diffusion equation on the general relativity framework. Then, he applied those equation to describe diffusion process in expanding universe. Whereas, in this research we use diffusion equation from Herrmann’s work to describe diffusion process around a non-rotating neutron star.

2. Diffusion process

Diffusion processes constitute a class of stochastic processes which are characterized by two properties: the Markovian property and the continuity of trajectories. Now, we give a formal definition of diffusion processes. Let \( S \) be a topological space. We sometimes find it convenient to attach an extra point \( \Delta \) to \( S \), either as an isolated point or as point at infinity if \( S \) is locally compact. Thus we set \( S' = S \cup \Delta \). The parameter \( \Delta \) and \( S \) or \( S' \) is called the terminal point and the state space, respectively. Let \( \bar{W}(S) \) be the set of all functions \( w : [0, \infty) \ni t \mapsto w(t) \in S' \) such that there exist \( 0 \leq \zeta(w) \leq \infty \) with the following properties \[9\]:

- \( w(t) \in S \) for all \( t \in [0, \zeta(w)] \) and the mapping \( t \in [0, \zeta(w)] \mapsto w(t) \) is continuous,
- \( w(t) = \Delta \) for all \( t \geq \zeta(w) \).

The parameter \( \zeta(w) \) is called the lifetime of the trajectory \( w \). For convenience, we set \( w(\infty) = \Delta \) for every \( w \in \bar{W}(S) \). A Borel cylinder set in \( \bar{W}(S) \) is defined for some integer \( n \), a sequence \( 0 \leq t_1 < t_2 < \ldots < t_n \) and a Borel subset \( A \) in \( S^n = S' \times S' \times \ldots \times S' \) as \( \pi_{t_1,t_2,...,t_n}^{-1}(A) \). Whereas \( \pi_{t_1,t_2,...,t_n} : \bar{W}(S) \rightarrow S^n \) is given by \( \pi_{t_1,t_2,...,t_n}(w) = (w(t_1), w(t_2), \ldots, w(t_n)) \).

A Borel subset in a topological space is any set in the smallest \( \sigma \)-field containing all open sets. Let \( \mathcal{B}(\bar{W}(S)) \) be the \( \sigma \)-field in \( \bar{W}(S) \) generated by all Borel cylinder sets and let \( \mathcal{B}_t(\bar{W}(S)) \) be that generated by all cylinder sets up to time \( t \), i.e., sets expressed in the form \( \pi_{t_1,t_2,...,t_n}^{-1}(A) \) where \( t_n \leq t \). A family of probabilities \( P_x, x \in S' \) on \( (\bar{W}(S), \mathcal{B}_t(\bar{W}(S))) \) is called Markovian if

- \( P_x w; w(0) = x = 1 \) for every \( x \in S' \),
- \( x \in S \mapsto P_x(A) \) is Borel measurable for each \( A \in \mathcal{B}(\bar{W}(S)) \), and;
- for every \( t > s \geq 0 \), \( A \in \mathcal{B}_s(\bar{W}(S)) \) and a Borel subset \( \Gamma \) in \( S' \),

\[
P_x(A \cap w; w(t) \in \gamma) = \int_A P_{w'(s)}[w; w(t-s) \in \Gamma] P_x(dw')
\]

for every \( x \in S' \).

**Definition 1** \[9\] A family of probabilities \( P_{x \in S'} \) on \( (\bar{W}(S), \mathcal{B}_t(\bar{W}(S))) \) is called a system of diffusion measures, or simply a diffusion if it is a strongly Markovian system.

**Definition 2** \[9\] A stochastic process \( X = X(t) \) on \( S' \) defined on a probability space \( (\Omega, \mathcal{F}, P) \) is called a diffusion process on \( S \) if there exists a system of diffusion measures \( P_{x \in S'} \), such that, for almost all \( \omega \), \( [t \mapsto X(t)] \in \bar{W}(S) \) and the probability law on \( \bar{W}(S) \) of \([t \mapsto X(t)]\) coincides with \( P_{\mu} \) \( \mu(dx) = \int S' P_x(dx) \) where \( \mu \) is the Borel measure on \( S' \) defined by \( \mu(dx) = P_x; X(0, \omega) \in dx \) and is called the initial distribution of \( X \).
If $X = (X(t))$ is a diffusion process and if we set $\zeta(\omega) = \inf t; X(t) = \Delta$, then it is clear that, with probability one, $(0, \zeta) \ni t \mapsto X(t) \in S$ is continuous and $X(t) = \Delta$ for all $t \leq \zeta$. The parameter $\zeta$ is called the life time of the diffusion process $X$. A stochastic process $X$ is called conservative if $\zeta(\omega) = \infty$.

2.1. Brownian motion

Brownian motion is the most fundamental of diffusion processes. Brownian motion is a Markov process with a continuous state space and a continuous index set. Brownian motion is closely linked to the normal distribution. Recall that a random variable $X$ is normally distributed with mean $\mu$ and variance $\sigma^2$ if

$$P \{X > x\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_x^\infty e^{-\frac{(u-x)^2}{2\sigma^2}} \, du, \quad \text{untuk } x \in \mathbb{R}$$

**Definition 3** A real-valued stochastic process $\{B(t) : t \geq 0\}$ is called a (linear) Brownian motion with start in $x \in \mathbb{R}$ if the following holds [11]:

- $B(0) = x$,
- the process has independent increments, i.e. for all times $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$ the increments $B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), \ldots, B(t_2) - B(t_1)$ are independent random variables,
- for all $t \geq 0$ and $h > 0$, the increments $B(t+h) - B(t)$ are normally distributed with expectation zero and variance $h$,
- almost surely, the function $t \mapsto B(t)$ is continuous.

2.2. Langevin equation

Langevin equation is a differential stochastic equation that describes time evolution an observable. Consider a Brownian particle with rest mass $m$, proper time $\tau$, and 4-velocity $u^\beta(\tau)$. The 4-momentum of the particle is given by $p^\alpha = mu^\alpha$, where $u_\alpha u^\alpha \equiv -c^2$. Assume that the particle is surrounded by an isotropic, homogeneous heat bath with constant 4-velocity $U^\beta$ and, additionally, subject to an external 4-force $K^\alpha(x'^\nu, p'^\nu)$ such as, e.g., the Lorentz force. Then, the relativistic Langevin equations of motion read [5]

$$dx^\alpha(\tau) = \frac{p^\alpha}{m} \, d\tau$$

$$dp^\alpha(\tau) = K^\alpha - \kappa^\alpha_\beta [p^\beta - mU^\beta] \, d\tau + w^\alpha(\tau)$$

For an isotropic homogeneous heat bath, the friction tensor $\kappa^\alpha_\beta$ is given by [5]

$$\kappa^\alpha_\beta = \kappa \left( \eta^\alpha_\beta + \frac{u^\alpha u_\beta}{c^2} \right)$$

with $\kappa$ denoting the scalar viscous friction coefficient measured in the rest frame of the particle. Furthermore, the relativistic Wiener increments $w^\alpha(\tau) \equiv dW^\alpha(\tau)$ are distributed according to the probability density [5]

$$P^{1+3} [w^\alpha(\tau)] = \frac{c}{(4\pi D d\tau)^{3/2}} \exp \left[ -\frac{w_\alpha(\tau)w^\alpha(\tau)}{4Dd\tau} \right] \times \delta [u_\alpha w^\alpha(\tau)] \tag{4}$$

where $D$ is the scalar noise amplitude parameter measured in the rest frame of the particle.
2.3. Fokker-Planck equation
The Fokker-Planck equation is a special type of master equation, which is often used as an approximation to the actual equation or as a model for more general Markov processes. The Fokker-Planck equation is a master equation in which $\mathbb{W}$ is a differential operator of second order [12]

$$\frac{\partial P(y,t)}{\partial t} = -\frac{\partial}{\partial y}A(y)P + \frac{1}{2} \frac{\partial^2}{\partial y^2}B(y)P.$$  \hspace{1cm} (5)

Those equation is also called Smoluchowski equation, generalized diffusion equation or second Kolmogorov equation. The first term on the right-hand side is called transport term, convection term, or drift term; the second one diffusion term or fluctuation term. The range of $y$ is necessarily continuous, and for the present time is supposed to be $(-\infty, +\infty)$. The coefficients $A(y)$ and $B(y)$ may be any real differentiate functions with the sole restriction $B(y) > 0$. The equation can be broken up into a continuity equation for the probability density

$$\frac{\partial P(y,t)}{\partial t} = -\frac{\partial J(y,t)}{\partial y},$$  \hspace{1cm} (6)

where $J(y,t)$ is the probability flux, and a constitutive equation

$$J(y,t) = A(y)P - \frac{1}{2} \frac{\partial}{\partial y}B(y)P.$$  \hspace{1cm} (7)

Let $P(y,t|y_1,t_1)$ for $t \geq t_1$ be that solution of equation [5] that at $t_1$ reduces to $\delta(y - y_1)$. Then, we can construct a Markov process whose transition probability is $P(y_2,t_2|y_1,t_1)$ and whose one-time distribution $P_1(y_1,t_1)$ may still be chosen arbitrarily at one initial time $t_0$. If we choose for $P_1$ the stationary solution of [5],

$$P^s(y) = \frac{\text{const.}}{B(y)} \exp\left[2 \int_0^y \frac{A(y')}{B(y')} dy'\right],$$  \hspace{1cm} (8)

the resulting Markov process is stationary. This is only possible, however, when this $P^s$ is integrable so that it can be normalized to represent a probability distribution.

3. Neutron star
Neutron stars play a unique role in physics and astrophysics. On the one hand, they contain matter under extreme physical conditions, and their theories are based on risky and far extrapolations of what we consider reliable physical theories of the structure of matter tested in laboratory. On the other hand, their observations offer the unique opportunity to test these theories. Moreover, neutron stars are important dramatic personae on the stage of modern astrophysics: they participate in many astronomical phenomena. Neutron stars contain the matter of density ranging from a few $g\text{cm}^{-3}$ at their surface, where the pressure is small, to more than $10^{15}g\text{cm}^{-3}$ at the center, where the pressure exceeds $10^{36}\text{dyn cm}^2$. To calculate neutron star structure, one needs the dependence of the pressure on density, the so called equation of state (EOS), in this huge density range, taking due account of temperature, more than $10^9\text{K}$ in young neutron stars, and magnetic fields, sometimes above $10^{15}\text{G}$ [6].

Neutron stars are compact stars which contain matter of supranuclear density in their interiors. They have typical masses $M \sim 1.4M_\odot$, where $M_\odot$ is the Solar mass, and radii $R \sim 10\text{ km}$. Thus, their masses are close to the solar mass $M_\odot = 1.989 \times 10^{33}\text{g}$, but their radii are $10^5$ times smaller than the solar radius $R_\odot = 6.96 \times 10^3\text{km}$. Because of its small size and high density, a neutron star possesses a surface gravitational field about $2 \times 10^{11}$
times that of Earth. Accordingly, neutron stars possess an enormous gravitational energy

\( E_{\text{grav}} \sim GM^2/R \sim 5 \times 10^{53} \text{erg} \sim 0.2Mc^2 \), and surface gravity

\( g \sim GM/R^2 \sim 2 \times 10^{14} \text{cm s}^{-2} \),

where \( G \) is the gravitational constant and \( c \) is the speed of light. Clearly, neutron stars are
very dense. Their mean mass density is \( \bar{\rho} \simeq 3M/4\pi R^3 \simeq 7 \times 10^{14} \text{g cm}^{-3} \sim (2-3)\rho_0 \), where
\( \rho_0 = 2.8 \times 10^{14} \text{g cm}^{-3} \) is the so called normal nuclear density, the mass density of nucleon
matter in heavy atomic nuclei. The central density of neutron stars is even larger, reaching
(1020)\( \rho_0 \). By all means, neutron stars are the most compact stars known in the Universe [6].

A neutron star can be subdivided into the atmosphere and four main internal regions: the
outer crust, the inner crust, the outer core, and the inner core. The atmosphere is a thin plasma
layer, where the spectrum of thermal electromagnetic neutron star radiation is formed. The
outer crust (the outer envelope) extends from the atmosphere bottom to the layer of the density
\( \rho \approx 4 \times 10^{11} \text{g cm}^{-3} \). Its thickness is some hundred meters. Its matter consists
of ions \( Z \) and electrons \( e \). The inner crust (the inner envelope) may be about one kilometer thick. The density
\( \rho \) in the inner crust varies from \( \rho_{ND} \) at the upper boundary to 0.5\( \rho_0 \) at the base. The matter
of the inner crust consists of electrons, free neutrons, and neutron-rich atomic nuclei. The outer
core occupies the density range 0.5\( \rho_0 \leq \rho \leq 2\rho_0 \) and is several kilometers thick. Its matter
consists of neutrons with several per cent admixture of protons, electrons, and possibly muons.
The inner core, where \( \rho \geq 2\rho_0 \), occupies the central regions of massive neutron stars. Its radius
can reach several kilometers, and its central density can be as high as (10-50)\( \rho_0 \). Its composition
and the EOS are very model dependent. Several hypotheses have been put forward, predicting
the appearance of new fermions and/or boson condensates [6].

4. Diffusion process around a neutron star

In this section we will discuss about relativistic diffusion process around a non-rotating neutron
star. We will derive stochastic differential equation which describes the diffusive particles
dynamic around a non-rotating neutron star. The derivation of those equation refers to
Herrmann’s work [8]. Herrmann has generalized the Markovian diffusion theory in the phase
space within the general theory of relativity framework. He has derived the general relativistic
Kramers equation in the phase space both in the parametrization of phase space proper time
and the coordinate time. The general relativistic Kramers equation in the parametrization of
observer time \( x_0 \) derived by Herrmann is given by

\[ N^{-1} \frac{\partial}{\partial x^0} \Phi = -v^M \text{div}_x(e_M \Phi) - \text{div}_v(F \Phi) + \frac{D}{2} \Delta_v \Phi, \quad (9) \]

where \( N = (g_{00})^{-1/2} \), \( \text{div}_x(e_M(x) \Phi) \) is the divergence operator in the position space, \( \text{div}_v(F \Phi) \)
is the divergence operator in the position space, and \( \Delta_v \) is the Laplace-Beltrami operator.
The divergence operator in the position space, the divergence operator in the position space, and
the Laplace-Beltrami operator are expressed in the hyperbolic coordinate system. Then, this
equation is used to construct the diffusion equation to describe the diffusive particles
dynamic around a non-rotating neutron star.

The metric of nonrotating neutron star is given by

\[ ds^2 = c^2 e^{2\nu} dt^2 - e^{2\lambda} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (10) \]

where \( \nu \) and \( \lambda \) are a function with parameter \( r \). The metric coefficients are
\( g_{00} = e^{2\nu}, g_{11} = -e^{2\lambda}, g_{22} = -r^2, g_{33} = -r^2 \sin^2 \theta \), and the others are zero. Then, the Christoffel symbols for this
metric are given by

\[ \Gamma_{00}^1 = \frac{\partial \nu}{\partial r} e^{2\nu - 2\lambda} = \nu' e^{2\nu - 2\lambda} \]
\[ \Gamma_{11}^1 = \frac{\partial \lambda}{\partial r} = \lambda' \]
\[ \Gamma_{22}^1 = -re^{-2\lambda} \]
\[ \Gamma_{33}^1 = -r \sin^2 \theta e^{-2\lambda} \]
\[ \Gamma_{01}^2 = \Gamma_{10}^2 = \frac{\partial \nu}{\partial r} = \nu' \]
\[ \Gamma_{21}^2 = \Gamma_{12}^3 = \Gamma_{31}^3 = \Gamma_{13}^3 = \frac{1}{r} \]
\[ \Gamma_{32}^3 = \Gamma_{33}^3 = \cot \theta, \]

(11)

and the others are 0.

Basis of orthonormal frame vectors are determined by

\[ \eta^{MN} e^\mu_M(x) e^\nu_N(x) = g^{\mu\nu}, \]

(12)

Based on the equation of (12) covariant basis of orthonormal frame vectors are given by

\[ e_0^1 = \sqrt{g^{00}} = \sqrt{e^{2\nu}} = e^{\nu} \]
\[ e_1^1 = \sqrt{g^{11}} = \sqrt{e^{2\lambda}} = e^{\lambda} \]
\[ e_2^2 = \sqrt{g^{22}} = \sqrt{r^2} = r \]
\[ e_3^3 = \sqrt{g^{33}} = \sqrt{r^2 \sin^2 \theta} = r \sin \theta, \]

(13)

whereas the contravariant basis of orthonormal frame vectors are given by

\[ e_0^0 = e^{\nu}; \quad e_1^1 = e^{\lambda} \]
\[ e_2^2 = \frac{1}{r}; \quad e_3^3 = \frac{1}{r \sin \theta}. \]

(14)

Spin connection coefficients are determined by the equation

\[ \Omega^M_{\mu N}(x) = e^\mu_M(x) \Gamma^{\nu}_\mu e^\nu_N(x) + e^M_e(x) \partial_\mu e^\nu_N(x). \]

(15)

By substituting Christoffel symbols and basis to the equation (15) then we get the spin connection coefficients below

\[ \Omega_{00}^1 = \nu' e^{\nu - \lambda}; \quad \Omega_{01}^0 = \nu' e^{\nu - \lambda} \]
\[ \Omega_{21}^2 = -e^{-\lambda}; \quad \Omega_{12}^1 = -e^{-\lambda} \]
\[ \Omega_{31}^3 = -e^{-\lambda} \sin \theta; \quad \Omega_{32}^3 = \cos \theta \]
\[ \Omega_{33}^3 = -e^{-\lambda} \sin \theta; \quad \Omega_{33}^3 = -\cos \theta, \]

(16)

and the others are zero.
The forces of curved space-time are calculated by equation

\[ F_\alpha^g = -\Omega^\alpha_{B}(x)e^\mu_C(x)u^Bu^C, \]  

(17)

where \( \Omega^\alpha_{B} \) is the spin connection coefficients; \( e^\mu_C \) is basis of orthonormal frame vectors; \( u^B \) and \( u^C \) are 4-velocity in the hyperbolic coordinate system. The 4-velocity in the hyperbolic coordinate system are given by

\[
\begin{align*}
u^0 &= \cosh \alpha \\
u^1 &= \sinh \alpha \sin \beta \cos \varphi \\
u^2 &= \sinh \alpha \sin \beta \sin \varphi \\
u^3 &= \sin \alpha \cos \beta.
\end{align*}
\]

(18)

By substituting spin connection coefficients, basis, and 4-velocity to the equation (17) then we get

\[
\begin{align*}
F^1_g &= -\Omega^1_{\mu B}(x)e^\mu_C(x)u^Bu^C \\
&= -\nu' e^{-\lambda} \cosh^2 \alpha + \frac{e^{-\lambda}}{r} \sinh^2 \alpha \sin^2 \beta \sin^2 \varphi + \frac{e^{-\lambda}}{r} \sinh^2 \alpha \cos^2 \beta  \\
F^2_g &= -\Omega^2_{\mu B}(x)e^\mu_C(x)u^Bu^C \\
&= -\frac{e^{-\lambda}}{r} \sinh^2 \alpha \sin^2 \beta \sin \varphi \cos \varphi + \frac{\cot \theta}{r} \sinh^2 \alpha \cos^2 \beta  \\
F^3_g &= -\Omega^3_{\mu B}(x)e^\mu_C(x)u^Bu^C \\
&= -\frac{e^{-\lambda}}{r} \sinh^2 \alpha \sin \beta \cos \beta \cos \varphi - \frac{\cot \theta}{r} \sinh^2 \alpha \sin \beta \cos \beta \sin \varphi
\end{align*}
\]

(19)

In the hyperbolic coordinate system the equation (19) are given by

\[
\begin{align*}
F^\alpha_g &= (\cosh \alpha)^{-1}[\sin \beta(\cos \varphi F^1_g + \sin \varphi F^2_g) + \cos \beta F^3_g] \\
&= -\nu' e^{-\lambda} \cosh \alpha \sin \beta \cos \varphi  \\
F^\beta_g &= (\sinh \alpha)^{-1}[\cos \beta(\cos \varphi F^1_g + \sin \varphi F^2_g) - \sin \beta F^3_g] \\
&= -\nu' e^{-\lambda} \coth \alpha \cosh \alpha \cos \beta \cos \varphi + \frac{e^{-\lambda}}{r} \cosh \alpha \cos \beta \cos \varphi + \frac{\cot \theta}{r} \sinh \alpha \cos \beta \sin \varphi  \\
F^\varphi_g &= (\sinh \alpha)^{-1}(\sin \beta)^{-1}[\sin \varphi F^1_g + \cos \varphi F^2_g] \\
&= \frac{1}{\sinh \alpha \sin \beta} \left\{ \nu' e^{-\lambda} \cosh^2 \alpha \sin \varphi - \frac{e^{-\lambda}}{r} \sinh^2 \alpha \sin \varphi + \frac{\cot \theta}{r} \sinh^2 \alpha \cos^2 \beta \cos \varphi \right\}.
\end{align*}
\]

(20)

The random impact of surrounding particles generally cause two kind of effects, that are, they act as a random driving force leading to a random motion and they give rise to a frictional force. In the non-relativistic theory the friction force is given by \( F^i_f = -\kappa \nu^i \), where \( \kappa \) is the friction coefficient and \( \nu^i \) are the components of the non-relativistic velocity. The relativistic generalization of the friction force requires the introduction of a friction tensor \( \kappa^i_\alpha \) similar to the pressure tensor in the relativity theory. The friction force is expressed as \([5], [8]\)

\[ F^i_f = \kappa^i_\alpha [u^\alpha - V^\alpha] \]

(21)
where $V^\alpha$ is the 4-velocity of the heat bath and $\kappa^i_\alpha$ is friction coefficient tensor. For an isotropic homogeneous heat bath the friction tensor is given by

$$\kappa^i_\alpha = \kappa (\eta^i_\alpha + u^i u_\alpha),$$

(22)

where $\kappa$ denoting the scalar friction coefficient measured in the rest frame of the particles. In the laboratory frame the heat bath is at rest described by $U^\alpha = (1, 0, 0, 0)$. Therefore the friction force is given by $F^\alpha_j = -\kappa u^i u^0$ or in hyperbolic coordinates

$$F^\alpha_j = -\kappa \sinh \alpha; F^\beta_j = 0; F^\varphi_j = 0,$$

(23)

The components of total force of a system in hyperbolic coordinate are given by

$$F^\alpha = F^\alpha_g + F^\alpha_f = \nu' e^{-\lambda} \cosh \alpha \sin \beta \cos \varphi - \nu \sinh \alpha$$

$$F^\beta = F^\beta_g + F^\beta_f = \nu' e^{-\lambda} \coth \alpha \cosh \alpha \cos \beta \cos \varphi + \frac{e^{-\lambda}}{r} \sinh \alpha \cos \beta \sin \varphi$$

$$F^\varphi = F^\varphi_g + F^\varphi_f = \frac{1}{\sinh \alpha \sin \beta} \left\{ \nu' e^{-\lambda} \cosh^2 \alpha \sin \varphi - \frac{e^{-\lambda}}{r} \sinh^2 \alpha \sin \beta \cos \varphi + \frac{\cot \theta}{r} \sinh^2 \alpha \cos^2 \beta \cos \varphi \right\}$$

(24)

Diffusion equation that describes the diffusion of particles around a neutron star can be derived by calculating the divergence in the position space, the divergence in the velocity space, and Laplace-Beltrami operator. The divergence in the velocity space in the hyperbolic coordinate system are given by

$$\text{div}_v(F\Phi) = (\sinh \alpha)^{-2} \frac{\partial}{\partial \alpha}((\sinh \alpha)^2 F^\alpha \Phi) - (\sinh \alpha)^{-1}(\sin \beta)^{-1} \frac{\partial}{\partial \beta}(\sin \beta F^\beta \Phi)$$

$$- (\sinh \alpha)^{-1}(\sin \beta)^{-1} \frac{\partial}{\partial \varphi}(F^\varphi \Phi)$$

$$\text{div}_v(F\Phi) = -2\nu' \Phi e^{-\lambda} \coth \alpha \cosh \alpha \sin \beta \cos \varphi - \nu' \Phi e^{-\lambda} \sinh \alpha \sin \beta \cos \varphi$$

$$- \nu' e^{-\lambda} \frac{\partial \Phi}{\partial \alpha} \cosh \alpha \sin \beta \cos \varphi - 3\nu \Phi \cosh \alpha \sin \beta \cos \varphi - \nu \frac{\partial \Phi}{\partial \alpha} \sinh \alpha$$

$$+ \nu' \Phi e^{-\lambda} \coth^2 \alpha \csc \beta \cos \varphi - \frac{e^{-\lambda} \Phi}{r} \csc \beta \cos \varphi$$

$$- \frac{\Phi}{r} \cot \theta \csc \beta \sin \varphi - 2\nu' \Phi e^{-\lambda} \coth^2 \alpha \sin \beta \cos \varphi + \frac{2\Phi e^{-\lambda}}{r} \sin \beta \cos \varphi$$

$$+ \frac{2\Phi}{r} \cot \theta \sin \beta \sin \varphi + \nu' e^{-\lambda} \frac{\partial \Phi}{\partial \beta} \coth^2 \alpha \cos \beta \cos \varphi$$

$$- \frac{e^{-\lambda} \Phi}{r} \frac{\partial \Phi}{\partial \beta} \cos \beta \cos \varphi - \frac{1}{r} \frac{\partial \Phi}{\partial \beta} \cot \theta \cos \beta \sin \varphi$$

$$- \nu' \Phi e^{-\lambda} \coth^2 \alpha \csc^2 \beta \cos \varphi - \nu' e^{-\lambda} \frac{\partial \Phi}{\partial \varphi} \coth^2 \alpha \csc^2 \beta \sin \varphi$$

$$+ \frac{e^{-\lambda} \Phi}{r} \csc^2 \beta \cos \varphi + \frac{e^{-\lambda} \Phi}{r} \frac{\partial \Phi}{\partial \varphi} \csc^2 \beta \sin \varphi$$

$$- \frac{1}{r} \frac{\partial \Phi}{\partial \varphi} \cot \theta \cot^2 \beta \cos \varphi + \frac{\Phi}{r} \cot \theta \cot^2 \beta \sin \varphi.$$
The divergence in the position space \( \text{div}_x(e_\mathcal{M}(x)\Phi) \) in the hyperbolic coordinate system are given by

\[
-u^M \text{div}_x(e_\mathcal{M}(x)\Phi) = -u^M \frac{\partial}{\sqrt{g} \partial x^i}(\sqrt{g} e_\mathcal{M}(x)\Phi)
\]

\[
= -\nu' e^{-\lambda} \sinh \alpha \sin \beta \cos \varphi \frac{2 e^{-\lambda} \Phi}{r} \sinh \alpha \sin \beta \cos \varphi
- e^{-\lambda} \frac{\partial \Phi}{\partial r} \sinh \alpha \sin \beta \cos \varphi \frac{\Phi}{r} \cot \theta \sinh \alpha \sin \beta \sin \varphi
- \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \sinh \alpha \sin \beta \sin \varphi - \frac{1}{r} \frac{\partial \Phi}{\partial \phi} \csc \theta \sinh \alpha \cos \beta
\]

Laplace-Beltrami operator can be determined by using below equation

\[
\frac{D}{2} \Delta_u \Phi = \frac{D}{2} \left\{ \frac{\partial^2 \Phi}{\partial \alpha^2} + 2 \coth \alpha \frac{\partial \Phi}{\partial \alpha} + \frac{1}{(\sinh \alpha)^2} \left( \frac{\partial^2 \Phi}{\partial \beta^2} + \cot \beta \frac{\partial \Phi}{\partial \beta} \right) \right\}
\]

\[
= \frac{D}{2} \left\{ \frac{\partial^2 \Phi}{\partial \alpha^2} + 2 \coth \alpha \frac{\partial \Phi}{\partial \alpha} + \frac{J(J + 1) \Phi}{(\sinh \alpha)^2} \right\},
\]

where \(-J(J + 1) = \frac{\partial^2}{\partial \beta^2} + \cot \beta \frac{\partial}{\partial \beta} + \frac{1}{(\sin \beta)^2} \frac{\partial^2}{\partial \phi^2}\) and \(J\) is the discrete index with the values \(J = 0, 1, 2, \ldots\).

The general relativistic diffusion equation of diffusive particles around non-rotating neutron star in the parametrization of observer time in the hyperbolic coordinate system are given by

\[
e^{-\nu} \cosh \alpha \frac{\partial \Phi}{\partial x^0} = - \frac{2 e^{-\lambda} \Phi}{r} \sinh \alpha \sin \beta \cos \varphi - e^{-\lambda} \frac{\partial \Phi}{\partial r} \sinh \alpha \sin \beta \cos \varphi
- \frac{\Phi}{r} \cot \theta \sinh \alpha \sin \beta \sin \varphi - \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \sinh \alpha \sin \beta \sin \varphi
- \frac{1}{r} \frac{\partial \Phi}{\partial \phi} \csc \theta \sinh \alpha \cos \beta + 2 \nu' \Phi e^{-\lambda} \coth \alpha \cos \alpha \sin \beta \cos \varphi
\]

\[
+ \nu' e^{-\lambda} \frac{\partial \Phi}{\partial \alpha} \cos \alpha \sin \beta \cos \varphi + 3 \nu' \Phi \cosh \alpha + \nu \frac{\partial \Phi}{\partial \alpha} \sinh \alpha
- \nu' e^{-\lambda} \coth^2 \alpha \csc \beta \cos \varphi + 2 \nu' \Phi e^{-\lambda} \coth \alpha \sin \beta \cos \varphi
\]

\[
+ \frac{\Phi}{r} \cot \theta \csc \beta \sin \varphi + \frac{e^{-\lambda} \Phi}{r} \csc \beta \cos \varphi - \frac{2 e^{-\lambda} \Phi}{r} \sinh \beta \cos \varphi
- \frac{2 e^{-\lambda} \Phi}{r} \cot \theta \sin \beta \sin \varphi - \nu' e^{-\lambda} \frac{\partial \Phi}{\partial \beta} \coth \alpha \cos \beta \cos \varphi
\]

\[
+ \frac{e^{-\lambda} \Phi}{r} \cos \beta \cos \varphi + \frac{1}{r} \frac{\partial \Phi}{\partial \beta} \cot \theta \cos \beta \sin \varphi
+ \nu' \Phi e^{-\lambda} \coth^2 \alpha \csc^2 \beta \cos \varphi - \frac{e^{-\lambda} \Phi}{r} \csc^2 \beta \cos \varphi
\]

\[
+ \nu' e^{-\lambda} \frac{\partial \Phi}{\partial \phi} \coth^2 \alpha \csc^2 \beta \sin \varphi - \frac{e^{-\lambda} \Phi}{r} \csc^2 \beta \sin \varphi
\]

\[
+ \frac{1}{r} \frac{\partial \Phi}{\partial \phi} \cot \theta \cot^2 \beta \cos \varphi - \frac{\Phi}{r} \cot \theta \cot^2 \beta \sin \varphi +
\]

\[
+ \frac{D}{2} \left\{ \frac{\partial^2 \Phi}{\partial \alpha^2} + 2 \coth \alpha \frac{\partial \Phi}{\partial \alpha} + \frac{J(J + 1) \Phi}{(\sinh \alpha)^2} \right\}.
\]
5. The Physical Meaning
In contrast to Herrmann’s work, it’s difficult to determine general solution of our equation \((27)\). However, some of physical meanings of the equation can be concluded qualitatively. Equation \((27)\) can be written as

\[
\frac{\partial \Phi}{\partial x^0} + e^{\nu - \lambda} \frac{\partial \Phi}{\partial r} \tanh \alpha \sin \beta \cos \varphi + \frac{e^{\nu}}{r} \frac{\partial \Phi}{\partial \theta} \tanh \alpha \sin \beta \sin \varphi + \frac{e^{\nu}}{r} \frac{\partial \Phi}{\partial \phi} \csc \theta \tanh \alpha \cos \beta = -2e^{\nu - \lambda} \Phi \tanh \alpha \sin \beta \cos \varphi \frac{e^{\nu}}{r} \cot \theta \tanh \alpha \sin \beta \sin \varphi + 2\nu' e^{\nu - \lambda} \coth \alpha \sin \beta \cos \varphi + \nu' e^{\nu} \frac{\partial \Phi}{\partial \alpha} \sin \beta \cos \varphi + 3\nu e^{\nu} \Phi + e^{\nu} \frac{\partial \Phi}{\partial \alpha} \tanh \alpha - \nu' e^{\nu - \lambda} \coth \alpha \csc \alpha \csc \beta \cos \varphi + 2\nu' \Phi e^{\nu - \lambda} \coth \alpha \csc \alpha \csc \beta \cos \varphi - 2\frac{\Phi}{r} \sech \alpha \sin \beta \cos \varphi - 2\frac{\Phi}{r} \cot \theta \sech \alpha \sin \beta \sin \varphi - \nu' e^{\nu - \lambda} \frac{\partial \Phi}{\partial \beta} \coth \alpha \csc \alpha \cos \beta \cos \varphi + e^{\nu} \frac{\partial \Phi}{\partial \phi} \sech \alpha \cos \beta \cos \varphi + e^{\nu} \frac{\partial \Phi}{\partial \phi} \coth \alpha \csc \alpha \csc \beta \sin \varphi + e^{\nu - \lambda} \frac{\partial \Phi}{\partial \phi} \sech \alpha \csc \alpha \csc \cos \beta \cos \varphi - \nu' e^{\nu - \lambda} \frac{\partial \Phi}{\partial \phi} \coth \alpha \csc \alpha \csc \cos \beta \sin \varphi + \frac{e^{\nu} D}{2} \left\{ \sech \alpha \frac{\partial^2 \Phi}{\partial \alpha^2} + 2 \csc \alpha \frac{\partial \Phi}{\partial \alpha} + \frac{\sech \alpha}{(\sinh \alpha)^2} J(J + 1) \Phi \right\}. \quad (28)
\]

Let compare the equation \((28)\) and \((5)\). The first term in the left hand side of equation \((28)\) is the derivative of probability density function with respect to time coordinate, whereas the others term are the derivative of probability density function with respect to space coordinates. The last term in the right hand side of equation \((28)\) is Laplace-Beltrami operator in the hyperbolic coordinate system where \(D\) is diffusion constant. The others term in the right hand side are predicted as the source of diffusion due to the curvature of spacetime which depend on 4-velocity.

Let consider the equation in the asymptotic case for which \(r \to \infty\) and \(e^\nu = e^\lambda = 1\). In this case, we obtain

\[
\frac{\partial \Phi}{\partial x^0} + \frac{\partial \Phi}{\partial r} \tanh \alpha \sin \beta \cos \varphi = \left( \nu' \sin \beta \cos \varphi + \nu \tanh \alpha \right) \frac{\partial \Phi}{\partial \alpha} - \left( \nu' \coth \alpha \csc \alpha \csc \beta \cos \varphi \right) \frac{\partial \Phi}{\partial \beta} + \left( \nu' \coth \alpha \csc \alpha \csc^2 \beta \sin \varphi \right) \frac{\partial \Phi}{\partial \varphi} + 2\nu' \coth \alpha \csc \alpha \sin \beta \cos \varphi + 3\nu \Phi + 2\nu \Phi \coth \alpha \csc \alpha \sin \beta \cos \varphi - \nu' \Phi \coth \alpha \csc \alpha \csc \beta \cos \varphi + \nu' \Phi \coth \alpha \csc \alpha \csc \cos \beta \cos \varphi + \frac{D}{2} \left\{ \sech \alpha \frac{\partial^\Phi}{\partial \alpha^2} + 2 \csc \alpha \frac{\partial \Phi}{\partial \alpha} + \frac{\sech \alpha}{(\sinh \alpha)^2} J(J + 1) \Phi \right\}. \quad (29)
\]

In the asymptotic case, the diffusion therefore depends only on the radial coordinate as well as the 4-velocity.

6. Conclusion
The diffusion equation of diffusive particles around a non-rotating neutron star is given by \((27)\). The additional terms on the right hand side of equation \((28)\) are predicted as the source of
diffusion due to the curvature of spacetime which depend on 4-velocity. In the asymptotic case, the diffusion depends only on the radial coordinate as well as the 4-velocity.

7. References

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